One-dimensional chaos in a system with dry friction: analytical approach

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Abstract We introduce a new analytical method, which allows to find chaotic regimes in non-smooth dynamical systems. A simple mechanical system consisting of a mass and a dry friction element is considered. The corresponding mathematical model is being studied. We show that the considered dynamical system is a skew product over a piecewise smooth mapping of a segment (the so-called base map). For this base map we demonstrate existence of a domain of parameters where a chaotic dynamics can be observed. We prove existence of an infinite set of periodic points of arbitrarily big period. Moreover, a reduction of the considered map to a compact subset of the segment is semi-conjugated to a shift on the set of one-sided infinite boolean sequences. We find conditions, sufficient for existence of a superstable periodic point of the base map. The obtained result partially solves a general problem: theoretical confirmation of chaotic and periodic regimes numerically and experimentally observed for models of percussion drilling.

Keywords Chaos · Mappings of segments · Dry friction · Filippov systems · Turbulence

1 Introduction

Systems with dry friction form a wide and important class inside discontinuous dynamical systems. They appear in many applications, especially in manufacturing systems: vibrating conveyors, percussion drilling, metal cutting, etc (see [1–15] and references therein for review). Their properties manifest many principle differences with ones of smooth dynamics. For instance, the uniqueness theorem is not valid any more. An approach to study such systems has been developed by A. F. Filippov [16] (see also [1, 10, 11]). He offered to consider piecewise continuous systems of differential equations as families of vector fields, defined on disjoint domains of the phase space. Boundaries of those domains are supposed to be piecewise smooth. Auxiliary tangent flows on boundaries are defined as convex combination of limit values of vector fields taken at points of boundaries. This approach reduces a discontinuous system of ordinary
differential equations to differential inclusions. Moreover, the phase space may become multidimensional e.g. a set of initial data of the full dimension may be transferred to a set of a lower dimension. The theory of discontinuous systems and specific bifurcations is developed by many authors \([1, 2, 5, 10, 11, 14, 15, 17, 18]\). It must be noticed, however, that this theory is very far from being complete or even comparable to one of smooth systems. It is also well-known that chaotic dynamics frequently occurs in such systems \([1–3, 6, 11, 13–15, 19–22]\).

Observe that systems with dry friction are very close to Filippov systems. One of the most common ideas to study Filippov systems involves a reduction of dimension, corresponding to presence of so-called sliding regimes. For some systems with dry friction it is possible to demonstrate that there exists an invariant set of dimension 1 where the attractor resides. A method to find such attractors has been developed by M. Wiercigroch, E. Pavlovskaya and A. Krivtsov in papers \([8, 9]\) and, in its general form, in the paper \([21]\). In our paper we use some ideas of this approach. Another powerful method has been proposed by R. Szalai and H. M. Osinga \([22]\). For a general class of systems with dry friction they have proved that the attractor resides in a polygon type set. It has been demonstrated that a kind of chaotic dynamics is possible there. Later \([23]\) the same authors have shown, using a modification of their method that some complex structures like Arnold tongues can be observed in a neighborhood of the so-called grazing-sliding bifurcation \([1, 2, 18]\).

In our paper we operate with the following idea of chaotic dynamics.

**Definition 1** \([24]\). Let \(I_0 = [0, 1]\), \(T : I_0 \rightarrow I_0\) be a piecewise continuous mapping. We say that mapping \(T\) is 1D-turbulent if there exist two disjoint subsegments \(J_0\) and \(J_1\) of the set \(I\) such that \(T(J_i) \supset J = J_0 \cup J_1\) \((i = 0, 1)\) and there exists an open set \(U \supset J\) such that reduction \(T|_U\) is continuous.

In \([24]\) this phenomenon has has been called turbulence. We use prefix 1D in order to avoid confusions with turbulence in its “hydrodynamical” sense.

**Definition 2** Let a mapping \(T\) be piecewise continuous. It is called robustly 1D-turbulent if it is 1D-turbulent and

\[
T(J_i) \supset J, \quad (i = 0, 1)
\]

where \(J_i\) is the interior of the corresponding segment \(J_i\).

Then there exist \(\varepsilon > 0\) such that any mapping \(T_1 : I_0 \rightarrow I_0\), continuous on \(U\) and such that \(|T_1 - T|_{C^0(U-I_0)} < \varepsilon\) is also 1D-turbulent. Observe that here, unlike to what is common in hyperbolic dynamics, we can consider \(C^0\)- small perturbations, not \(C^1\) small.

Definition 1 uses a construction which is quite common in low-dimensional dynamics. However, 1D-turbulence is not Devaney chaos because the fixed points of \(T\) may be non-isolated. Neither this is a particular case of the Li-Yorke chaos \([25]\) due to the same reason. However, a mapping, 1D-turbulent in the sense of Definition 1 is semi-conjugated to one-side symbolic dynamics (see Sect. 7 for precise definitions and discussion).

We spread Definition 1 to automorphisms of curves, homeomorphic to \(I_0\). In this case such curves are called 1D-turbulent w.r.t. \(T\).

The main aim of this paper is to provide a new method which allows to find 1D-turbulent invariant sets for mappings, corresponding to systems with dry friction. We use a very simple example, first considered be A. Krivtsov and M. Wiercigroch \([8]\). First of all, we show that the considered dynamical system engenders a discontinuous mapping of a segment. Here we use ideas from \([21]\). Then we apply some techniques of one dimensional dynamics \([26, Part 3, Section 15]\) in order to demonstrate robust 1D-turbulence of the obtained map.

The main advantages of the method, we offer in our paper, are the following.

1. We can obtain chaotic sets which, in general, are not attractors.
2. For simple systems with dry friction, the offered method gives coefficient type criteria of chaos.
3. Though we need a presence of a small parameter in our proofs, it is possible to estimate numerically how small this parameter must be.
4. In general, presence of chaotic invariant sets does not correspond to a neighborhood of any bifurcation.
5. The fact of presence of chaotic behavior is robust.
A corresponding invariant measure can be described using techniques of [26, Part 3, Section 15].

The paper is organized as follows. In Sect. 2 we introduce the mathematical model for the considered system and describe possible regimes of motion. In Sect. 3 we define the main object of our investigation: the one-dimensional mapping, corresponding to phases of switching for solutions. For this, we describe all possible scenarios of behavior of solutions. In the next section we describe some properties of the introduced mapping. In Sect. 5 we study how the segments of continuity for the constructed mapping look like. We find out two segments of continuity whose images cover their union. Sections 6 and 7 are technical. We prove existence of periodic points of all possible periods. Also, we find a symbolic pattern for the considered dynamics. Obtained 1D-turbulent invariant dynamics may coexist with superstable fixed points of the considered map. This is discussed in Sect. 8. Main results of the paper are formulated in Sect. 9. Discussion, including remarks on the robustness of the obtained set and some plans on the future research is given in Sect. 10.

2 Description of the mathematical model

Consider a single degree-of-freedom mechanical system, consisting of a point mass and a delimiter (dry friction element) (Fig. 1) which gives a simple model of percussion drilling. Motion of the mass is controlled by external force $F(t)$ which is a sum of a positive constant component equal to $2b$ and a harmonic component of a positive amplitude $a$ and a period equal to $2\pi$, so

$$F(t) = a \sin t + 2b.$$  

Here $t$ is the time variable. The considered system includes a delimiter which gives a dry friction provided the mass is in contact with it. The maximal value of this dry friction force is $q > 0$. Here and later we always suppose that all considered parameters are non-dimensional.

Let $t_0 < t_1$ be zeros of the function $F(t) - q$ and $t_2 < t_3$ be ones of the function $F(t)$ on $[0, 2\pi]$. Observe that values $t_0$ and $t_1$ are well-defined if $q < a + 2b$. In our proofs, we always assume that the last inequality takes place. However, all our proofs can easily be modified for the case when $q > a + 2b$ i.e. when $t_0$ and $t_1$ do not exist.

Let $x$ be the current position of the mass, $y$ be one of the delimiter and $v$ be the current velocity of the mass.

**Physical assumptions.** Before proceeding to mathematical models, we list the desired properties of motions of the considered system.

1. The mass can never penetrate beyond the delimiter i.e. $y \geq x$.
2. The delimiter can never move up i.e. the value $y$ is non-decreasing.
3. The delimiter is massless. It cannot move separately with respect to the point mass and stands immobile while $y > x$.
4. Consider the value $\theta_0 \in [0, \pi/2]$ such that

$$\pi - \theta_0 = \cot(\theta_0/2).$$

This value is unique: $\theta_0 \approx 0.81047$, $\sin \theta_0 \approx 0.72461$. Let

$$a > 0, \quad b \in (0, a/2), \quad q > a \sin \theta_0.$$  

5. We shall always suppose that $b \ll a$ (which implies $b \ll q$). This means that during our proofs we suppose that the ratio $b/a$ is as small as we need.

Let us discuss physical sense of Eq. (2). Inequality $a > 0$ implies presence of harmonic component of the force; $b > 0$ implies that the mean value of the force is positive that corresponds to drilling; $b < a/2$ implies that the constant component of the force is not to large.
If \( b > a/2 \), the dynamics of the considered system is quite trivial, see Remark 5 at Sect. 9. The assumption \( q > a \sin \varphi_0 \) is technical. Roughly speaking, it means that the friction force is not too small. We need it for our proof, but we do not know exactly what happens if it is violated e.g. if \( q = 0.7 a \).

Finally, we consider \( b/a \) as a small parameter of the system. This means that we fix some values of \( a \) and \( q \), satisfying (2), and prove that there exists a \( b_0(a,q) \) in \((0,a/2)\) such that for all \( b \in (0,b_0(a,q)) \) dynamics of the system is 1D-turbulent (see Definition 1 below).

Consider a four-dimensional phase space \( \mathbb{R}^4_{x,y,v,t} \) and three domains:

\[
P_1 = \{ (x,y,v,t) : x < y \}, \quad P_2 = \{ (x,y,v,t) : x > y, v > 0 \}, \quad P_3 = \{ (x,y,v,t) : x > y, v < 0 \}.
\]

Phase states, corresponding to points of \( P_2 \) and \( P_3 \) are impossible; we need to consider them in order to have a complete mathematical model i.e. a system of differential equations on the whole phase space. Also, we consider three 3D surfaces:

\[
\Sigma_{12} = \{ (x,x,v,t) : v > 0 \} = \partial P_1 \setminus \partial P_2 \setminus \partial P_3; \quad \Sigma_{13} = \{ (x,x,v,t) : v < 0 \} = \partial P_1 \setminus \partial P_3 \setminus \partial P_2; \quad \Sigma_{23} = \{ (x,y,v,t) : x > y \} = \partial P_2 \setminus \partial P_3 \setminus \partial P_1,
\]

and a two-dimensional subspace \( \Theta = \{ (x,x,0,t) \} = \partial P_1 \cap \partial P_2 \cap \partial P_3 \) (Fig. 2). Here the symbol \( \partial \) denotes boundary of a set.

Filippov system. In order to describe dynamics of the considered system with a dry friction we are going to apply some standard techniques of piecewise continuous dynamical systems, using ideas of [16]. We define autonomous systems of ODEs:

\[
\begin{align*}
\dot{x} &= v, & \dot{v} &= F(t), & \dot{y} &= 0, & i &= 1 \\
\text{if } (x,y,v,t) &\in P_1; \\
\dot{x} &= v, & \dot{v} &= F(t) - q = a \sin t + 2c, & \dot{y} &= v, & i &= 1 \\
\text{if } (x,y,v,t) &\in P_2; \\
\dot{x} &= v, & \dot{v} &= F(t) - q = a \sin t + 2c, & \dot{y} &= 0, & i &= 1 \\
\text{if } (x,y,v,t) &\in P_3.
\end{align*}
\]

The so-called sliding regime is observed: solutions that arrive to \( \Sigma_{12} \) from \( P_1 \), do not cross that boundary, they move along it according to Eq. (4) until they reach \( \Theta \). Here we observe that the convex combination of vector fields \( f_1 \) and \( f_2 \) which is tangent to \( \Sigma_{12} \), is \( f_3 \).

For any point \( X \in \Sigma_{13} \), we have \( \langle f_1(X), n_{13} \rangle \neq 0, \quad \langle f_2(X), n_{13} \rangle > 0 \). This corresponds to so-called crossing: all solutions, arriving to \( \Sigma_{13} \) from \( P_3 \), cross the boundary and continue their motion in \( P_1 \).

The behavior of solutions that start at points \( X = (x,y,0,t) \in \Sigma_{23} \) depends on the value of \( t \). If \( t \mod 2\pi \in [0,t_0) \cup [t_1,t_2) \cup [t_3,2\pi) \) we have \( F(t) \in (0,q) \) in a right vicinity of \( t \). So, the corresponding forward motion is described by a convex combination of \( f_2 \) and \( f_3 \) that gives sliding. Namely, it satisfies the following system of differential equations

\[
\dot{x} = 0, \quad \dot{v} = 0, \quad \dot{y} = 0, \quad i = 1.
\]

If \( t \mod 2\pi \in [0,t_0) \cup [t_1,t_2) \cup [t_3,2\pi) \), the corresponding trajectory moves inside \( \Theta \) according to
to Eq. (6); if \( t \) mod \( 2\pi \in [t_0, t_1] \), the trajectory leaves \( \Theta \) for \( \Sigma_{12} \) where a sliding regime, governed by Eq. (4) is observed; if \( t \in [t_2, t_3] \), a trajectory leaves \( \Theta \) for \( \Pi_1 \).

Later on we use notion \((F)\) for the Filippov system, defined by Eqs. (3)–(5).

**Definition 3** We say that a function \( X(t) = (x(t), y, v, t) \) is a solution of Filippov system \((F)\) on an interval \((\tau_0, \tau_1)\) if this interval is a finite union of disjoint intervals \( J_i \) and points \( P_i \). Here every point \( P_i \) is crossing or beginning/end of a sliding regime; on every segment \( J_i \), values of the function \( X(t) \) belong to one of sets \( \Pi_k \), \( \Sigma_{k,l} \) or \( \Theta \) and satisfies there the corresponding system of differential equations.

We introduce following notions for motions and crossings/switchings of the considered Filippov system \((F)\).

1. **No contact (free) motion** \((f)\) This motion takes place if and only if \( X \in \Pi_1 \) i.e. the mass and the delimitter do not interact. It is governed by Eq. (3).
2. **Contact with progression** \((p)\) This regime corresponds to sliding in \( \Sigma_{12} \): a friction is already there. The motion is defined by Eq. (4).
3. **Stop** \((s)\) This is a codimension 2 sliding in \( \Theta \), described by Eq. (6).
4. **Instantaneous stop** \((is)\) This happens if Condition (6) is satisfied for a fixed instant of time but is not true in its small neighborhood. So, this happens if the system switches from or to free motion or from/to contact with progression.
5. **Instantaneous transition from no contact regime to contact with progression** \((fp)\).

Formally, motions and crossings inside \( \Pi_2, \Pi_3 \) and \( \Sigma_{23} \) are possible for Filippov system \((F)\). However, we do not need to deal with them. As we see later (Lemma 1), starting from a point of \( \Pi_1 \), a solution cannot enter the open half-space \( \Pi_2 \cup \Pi_3 \cup \Sigma_{23} \).

We always suppose that free motion, contact with progression and stop regime are observed at open intervals of time. This allows us to classify all instants of transition.

Later on we consider the phase \( \varphi = t \) mod \( 2\pi \). Here \( \varphi \in S^1 = \mathbb{R}/2\pi \mathbb{Z} \).

Solutions of Eqs. (3), (4) and (6) can easily be written down. If \( x(t_0) = x_0, \dot{x}(t_0) = x_1 \) we have

\[
x(t) = -a \sin t + b(t - \theta_0)^2 + (x_1 + a \cos \theta_0)(t - \theta_0) + x_0 + a \sin \theta_0
\]

for Eq. (3) (free motion) and

\[
x(t) = -a \sin t + c(t - \theta_0)^2 + (x_1 + a \cos \theta_0)(t - \theta_0) + x_0 + a \sin \theta_0
\]

for contact with progression. For stop regime we always have \( x_1 = 0 \) and \( x(t) \equiv x_0 \).

### 3 Reduction to dimension 1

First of all, we prove a result on existence and forward uniqueness for solutions of Filippov system \((F)\).

**Lemma 1** For any \( (x_0, y_0, v_0, \theta_0) \in \Pi_1 \) there exists a \( \theta_1 > 0 \) such that the corresponding solution of system \((F)\) is correctly defined and unique on any subsegment of \((\theta_0, +\infty)\). It can never enter \( \Pi_2, \Pi_3 \) or \( \Sigma_{23} \). Moreover, the behavior of the corresponding motion is described by one of following six scenarios.

1. **Scenario A:** \((f) \rightarrow (fp) \rightarrow (p) \rightarrow (is) \rightarrow (f) \rightarrow \ldots \). Switching \((is)\) corresponds to instant \( t \in [t_2 + 2\pi k, t_3 + 2\pi k) \).
2. **Scenario B:** \((f) \rightarrow (fp) \rightarrow (p) \rightarrow (is) \rightarrow (s) \rightarrow (is) \rightarrow (f) \rightarrow \ldots \), where the first switching \((is)\) happens for \( t \in [t_1 + 2\pi k, t_2 + 2\pi k) \).
3. **Scenario C:**

   \[
   (f) \rightarrow (fp) \rightarrow (p) \rightarrow (is) \rightarrow (s) \rightarrow (is) \rightarrow (p) \rightarrow (is) \rightarrow (s) \rightarrow (is) \rightarrow (f) \rightarrow \ldots .
   \]

   In this case the first transition \((is)\) happens at one of segments \( [t_3 + 2\pi k, t_0 + 2\pi(k + 1)] \) and a new contact with progression starts at \( t_0 + 2\pi(k + 1) \) with initial velocity equal to zero. Parameters of the system must be selected so that this second contact with progression stops before \( t_2 + 2\pi(k + 1) \) otherwise the next scenario is observed.

4. **Scenario D:**

   \[
   (f) \rightarrow (fp) \rightarrow (p) \rightarrow (is) \rightarrow (s) \rightarrow (is) \rightarrow (p) \rightarrow (is) \rightarrow (*).
   \]

   Here \((*)\) implies any sequence of regimes except \((s) \rightarrow (is) \rightarrow (f) \rightarrow \ldots \), corresponding to Scenario C.
Starting from the free motion, the mass will move carefully. Observe that Fig. 3 does not reflect all possible motions and transitions. For example, transitions (f) may take place anytime, so we do not mark them on the figure.

**Proof** Starting from the free motion, the mass will always return to the same regime e.g. (f) → (is) → (s). This does not hurt to equations of motion. However, such stops play an important role since they correspond to discontinuities of stroboscopic mappings. Later on (Sect. 4) we study them more carefully. Observe that Fig. 3 does not reflect all possible motions and transitions. For example, transitions (fp) may take place anytime, so we do not mark them on the figure.

**Scenario C’:**

\[
\begin{align*}
(f) & \rightarrow (is) \rightarrow (s) \rightarrow (is) \rightarrow (p) \rightarrow (is) \rightarrow (s) \\
& \rightarrow (is) \rightarrow (f) \rightarrow \ldots;
\end{align*}
\]

**Scenario D’:**

\[
\begin{align*}
(f) & \rightarrow (is) \rightarrow (s) \rightarrow (is) \rightarrow (p) \rightarrow (is) \rightarrow (*).
\end{align*}
\]

(see Fig. 2, 3 for illustration).

In Contact with Progression regime, the derivative \( \dot{x} \) vanishes soon or later since \( c < 0 \), see (7). If this happens when \( \varphi \in [t_2, t_3] \) we immediately proceed to free motion, i.e. Scenario A takes place. Otherwise, the mass stops. If this corresponds to the phase \( \varphi \in [t_1, t_2] \) the mass stops until \( \varphi = t_2 \) and then switches to free motion; this is Scenario B.

Notice that contact with progression cannot be stopped while

\[
\varphi = t \mod 2\pi \in [t_0, t_1).
\]

So, if Scenarios A and B fail, contact with progression stops when \( \varphi \in [t_3, t_0) \); the stop regime is observed until the next instant \( t_0 + 2\pi k \); then the system starts moving according to Eq. (4), that is contact with progression. For that motion, we have \( \theta_0 = t_0 + 2\pi k \) and \( \tau_1 = 0 \) in Eq. (7). Consequently,

\[
\dot{x}(t) = -a(\cos t - \cos t_0) + 2c(t - t_0 - 2\pi k).
\]  (8)

This function increases (and, therefore, cannot vanish) until \( t = t_1 + 2\pi k \). However, since \( b \ll q \), we may say that \( \cos t_1 > \cos t_0 \) and the right hand side of Eq. (8) is negative for \( t = t_3 + 2\pi k \). So, the motion stops at \( t \in [t_1 + 2\pi k, t_3 + 2\pi k] \). The stop regime may be finished by a transition to free motion at \( t = t_2 + 2\pi k \) (Scenario (C)) or by transition to contact with progression at \( t = t_0 + 2\pi k \) (Scenario D).

It remains to consider the case when the first hits takes place at \( t = t_3 + 2\pi n \) and \( \dot{x}(t_3 + 2\pi k) = 0 \) i.e. the initial free flight motion transfers directly to stop regime. We can prove, similarly to what we have done above that one of Scenarios C’ or D’ is observed in this case.

**Lemma 2** There exists a \( b_0(q) > 0 \) such that if \( b < b_0(q) \) the following statement is true. Starting with progression at the instant \( t_0 \) with an initial velocity equal to zero, the motion must stop at the instant \( t_4 \in [t_1, t_2] \). Consequently, Scenarios D and D’ are impossible for such motions.

**Proof** If \( x(t) \) is a solution of Eq. (4) with \( \dot{x}(t_0) = 0 \), we have

\[
\dot{x}(t_2) = -a \cos t_2 + a \cos t_0 + 2c(t_2 - t_0).
\]  (9)

In order to prove that this \( \dot{x}(t) \) vanishes somewhere at \( [t_0, t_2] \) it suffices to prove that the right hand side of
Eq. (9) is negative. Instead of this one could prove that $a + a \cos t_0 - (q - 2b)(\pi - t_0) < 0$. Here we replaced $t_2 > \pi$ with $\pi$ in (9) and respected the fact that $2c = 2b - q$. If we demonstrate for a fixed $q$ that $a + a \cos t_0 - q(\pi - t_0) < 0$ then there exists a $b_0(q) > 0$ such that if $b < b_0(q)$ then the estimate (9) is true.

Observe that

$$a + a \cos t_0 - q(\pi - t_0) \leq a(1 + \cos t_0 - \sin t_0(\pi - t_0)).$$

(10)

The right hand side of inequality (10) is negative if $\cot(t_0/2) < \pi - t_0$ which is true if $t_0 > \vartheta_0$ (see Eq. (1)) or, equivalently if $q > \sin \vartheta_0 a$. This follows from Eq. (2).

So, wherever and whenever a free motion starts, it transfers to contact with progression or to stop regime and, after some transitions, returns to free flight regime.

Take an instant $\theta \in [t_2, t_3]$ where a free motion starts so that

$$x(\theta) = y(\theta) = \dot{x}(\theta) = \dot{y}(\theta) = 0$$

(11)

The value $\theta$ uniquely defines the farther dynamics of system (F).

The system is initially moving in free regime and, after several transitions, switches to free regime once again. Let $\tilde{T}(\theta) > \theta$ be the first moment of such switching, $T(\theta) = \tilde{T}(\theta) \bmod 2\pi$. Both these values are uniquely defined by $\theta$.

So, we may consider the 1D mapping $T : [t_2, t_3) \to \mathbb{R}$. Fix $\theta \in [t_2, t_3)$, consider the motion with initial conditions (11) and take

$$T_1(\theta) = \theta_1 = \inf\{\tau > \theta : x(\tau) = y(\tau)\}.$$  

Notice that the image of $T_1$ can be greater than $2\pi$.

The value $\theta_1$ corresponds to the minimal zero of the equation

$$G_1(\theta, t) := \frac{b(t - \theta)^2}{a} \cos \theta(\pi - t - \theta) - a \sin t + a \cos \theta(t - \theta) + a \sin \theta = 0$$

(12)

such that $t > \theta$. We rewrite Eq. (12) in the form

$$\frac{b}{a}(t - \theta)^2 = \sin t - \cos \theta(t - \theta) - \sin \theta.$$  

(13)

The left hand side of Eq. (13) is always positive and proportional to the small parameter $b/a$. The right hand side is positive in a right vicinity of zero and grows faster than the left hand side (its first derivatives vanishes at $t = \theta$, but its second derivative of the right hand side is greater since $\sin \theta < -2b/a$).

Geometrically, the right hand side of (13) is the distance between the graph of sine function and the tangent line to it, drawn at $\theta$ (Fig. 4).

If $\theta \in (3\pi/2, 2\pi)$ i.e. $\cos \theta > 0$, the graph and its tangent line intersect once again on $[\theta, \theta + 2\pi)$ and, therefore $\theta_1 - \theta < 2\pi$. Otherwise, they do not intersect. Then there exist positive constants $b$ and $C$ which do not depend on $\theta$ and such that if $b < b$ then

$$\theta_1 - \theta \geq Ca \cos \theta/b.$$  

(14)

### 5 Points of discontinuity

The mapping $T_1$ is, in general, discontinuous. All possible discontinuity points correspond to the case when $\theta_1$ is not a simple zero of (12). In this case derivative $\partial G_1/\partial t$ vanishes for $t = \theta_1$ which means that the following condition is satisfied

$$-a \cos \theta_1 + 2b(\theta_1 - \theta) + a \cos \theta = a\left(2\sin \frac{\theta_1 - \sin \theta}{\theta_1 - \theta} - \cos \theta - \cos \theta_1\right) = 0.$$

(15)
Lemma 3 There exists a \( \epsilon > 0 \) such that if \( b/a < \epsilon \), intersection of the set of discontinuity points of the mapping \( T_1 \) with the segment \( L_0 = [101\pi/100, 3\pi/2] \) is finite.

Remark 1 We could take any value \( \pi + \delta, \delta > 0 \) instead of \( 101\pi/100 \). However, smaller is \( \delta \), smaller value of \( \epsilon \) must be chosen.

Proof Take \( \epsilon \) so small that \( t_2 < 101\pi/100 \). Notice that if
\[
0^1 < 0^2, \quad 0^{1,2} \in L_0,
\]
then \( T_1(0^1) > T_1(0^2) \). Indeed, the derivative \( \partial G_1/\partial \theta \) of the function \( G_1 \) defined by Eq. (12), equals to
\[
-2b + a \sin(\theta)(t - \theta). \]
This value is positive if \( \theta \in (t_2, t_3) \) and \( t > 0 \). If \( \theta^0 \) and \( \theta^2 \) satisfy (16) and \( t^1 \) is such that \( G_1(\theta^0, t^1) = 0 \) then \( G_1(\theta^2, t^1) > 0 \) and the function \( G_1(\theta^2, t) \), negative in a right neighborhood of \( t = \theta^2 \), must have a zero on \( (\theta^2, t^1) \).

So the function \( T_1 \) is monotonous. Notice that if \( \theta \in L_0 \) is a point of discontinuity of \( T_1 \) and \( \theta_1 = T_1(\theta) \) then
\[
\frac{\partial G_1(\theta, t)}{\partial t} = 0, \quad \frac{\partial^2 G_1(\theta, t)}{\partial t^2} = 2b + a \sin \theta_1.
\]
It follows from Eq. (15) that for any discontinuity point \( \theta \) of the mapping \( T_1 \)
\[
\cos \theta_1 = -\cos \theta + O((\theta_1 - \theta)^{-1}). \tag{17}
\]
So, there exists a \( \rho > 0 \) such that the absolute value of second derivative of the function \( G_1(\theta, T_1(\theta)) \) is greater than \( \rho \) if \( \theta \in L_0 \) is a point of discontinuity for \( T_1 \). Consequently, distance between \( \theta_1 \) and the next zero of \( G(\theta, \cdot) \) that is “jump” \( T_1(\theta_0 - \theta) - T_1(\theta_0 + \theta) \) is greater than a fixed positive value. This proves that the number of discontinuity points on \( L_0 \) is finite. \( \square \)

Grace to Eqs. (14) and (17) we may take \( b/a \) so small that \( \theta \in [t_2, 3\pi/4] \) then \( \theta_1 = T(\theta) \in [3\pi/2, t_3] \).

In non-degenerate scenarios (A–C) at the moment \( t = \theta_1 \) contact with progression regime starts. The initial velocity of the motion is \( x_1 = -a \cos \theta_1 + 2b(\theta_1 - \theta) + a \cos \theta \). Dynamics of this velocity is described by the formula
\[
\dot{x}(t) = x_1 + 2c(t - \theta_1) - a \cos t + a \cos \theta_1.
\]

Progression regime stops as soon as this derivative vanishes and the next transition \( \theta_2 = \theta_2(\theta) \) to free flight or to the stop may be found from equations
\[
G_2(\theta, \theta_1, \theta_2) := 2b(\theta_1 - \theta) + a \cos \theta + 2c(\theta_2 - \theta_1) - a \cos \theta_2 = 0. \tag{18}
\]

Lemma 4 The map \( T \) is such that \( T(\theta) = \theta_2 \mod 2\pi \) provided \( \theta_2 \mod 2\pi \in [t_2, t_3] \). Otherwise, a motion with stop is observed (Scenarios B and C) and \( T(\theta) = t_2 \).

Proof If \( \theta_2 \in [t_2, t_3] \) then the motion, with initial conditions (11) corresponds to the free regime, so \( T = \theta_2 \). If \( \theta_2 \in [t_3, t_0] \), the mass stops before \( t \) reaches the value \( t_0 \) then starts moving in contact with progression regime until \( t = t_4 \) (see Lemma 2), stops until \( t_2 \) and proceeds to a free regime. If \( \theta_2 \in [t_1, t_2] \) the motion stops until \( t = t_2 \) and also proceeds to the free regime. Since \( \theta_2 \) cannot belong to \( (t_0, t_1) \), we obtain the desired statement. \( \square \)

To finish our proof, we need the following lemma.

Lemma 5 There exist two disjoint subsegments \( J_0 \) and \( J_1 \) of the segment \( [t_2, 3\pi/2] \) such that \( T(J_i) \supset [t_2, 3\pi/2] \) and \( T \) is continuous on both segments \( J_i \).

Remark 2 We may claim without loss of generality that \( T(J_i) = [t_2, 3\pi/2] \).

Proof Let \( L_1 \) be the arc \( [197\pi/100, t_3] \) of the unit circle and \( L_2 \) be the arc \( [101\pi/100, 51\pi/50] \).

The first arc is correctly defined if the ratio \( b/a \) is sufficiently small.

Let \( \theta \in L_1 \). Consider the solution \( x(t) \) such that \( x(\theta) = y(\theta) = 0, v(\theta) = 0 \). Then
\[
x(t) = -a \sin t + b(t - \theta)^2 + a \cos \theta(t - \theta) + a \sin \theta. \tag{19}
\]

Let \( b = 0 \). Then the first zero of \( x(t) \) on \( (\theta, +\infty) \) satisfies the equation.
\[ H(t, \theta) := \sin \theta - \sin t + \cos \theta(t - \theta) = 0. \]

Let \( \hat{i} \) is the first zero of \( H(\cdot, \theta) \) on \((\theta, \infty)\). This is the first intersection of the graph of the sine function and the tangent line, drawn at \( \theta \). Then \( \cos \hat{i} < \cos \theta \) since \( \sin t \) grows faster than \( \sin \theta + \cos \theta(t - \theta) \) while \( t \in [0, 4\pi - \theta] \). So, \( \partial H/\partial t|_{t=\hat{i}} = \cos \theta - \cos \hat{i} > 0 \).

Observe that

\[ \partial H/\partial \theta|_{t=\hat{i}} = \sin \theta(\theta - \hat{i}) > 0. \]

Then \( \partial \hat{i}/\partial \theta < 0 \).

Evidently, the maximal value \( \hat{i}_{\text{max}} \) of \( \hat{i} \) corresponds to \( \theta = 197\pi/100 \) and \( \hat{i}_{\text{max}} - 2\pi \approx 0.16329 \).

Write down derivative \( \dot{x}(\hat{i}) = a(\cos \theta - \cos \hat{i}) \); we have observed that this value can never be negative for \( \theta \in L_1 \). Now we consider contact with progression which starts at \( \hat{i} \). The corresponding function \( \dot{x}(t) \) may be found by formula:

\[ \dot{x}(t) = -a \cos t + a \cos \theta - q(t - \hat{i}). \]

So, for \( t = t_0 + 2\pi \) and any \( \theta \in L_1 \) we have

\[ \dot{x}(t) < -a \cos t_0 + a \cos \theta - q(t_0 + 2\pi - \hat{i}_{\text{max}}) \leq a(1 - \cos \theta_0 - \sin \theta_0(2\pi - \hat{i}_{\text{max}})) \approx -0.1581a < 0 \]

This means that for \( b = 0 \) \( \dot{x}(t) \) must vanish somewhere between \( \hat{i} \) and \( t_0 + 2\pi \). So, contact with progression regime stops before \( t = t_0 + 2\pi \). So, \( \dot{x}(t_0 + 2\pi) = 0 \).

Due to continuous dependence of the solution on the parameter \( b \), the same is true provided the ratio \( b/a \) is sufficiently small. In this case, as we have already proved, \( T(\theta) = t_2 \).

Let \( z_1 < z_2 < \ldots < z_n (n \geq 0) \) be discontinuity points of the map \( T_1 \) inside \( L_2 \). Denote \( z_0 = 101\pi/100 \), \( z_{n+1} = 51\pi/50 \).

Suppose that \( n < 3 \). Then there exists \( i \in \{0, \ldots, n\} \) such that there is a subsegment \( \mathcal{I} \subset (z_i, z_{i+1}) \) of the length not less than \( \pi/500 \). Let \( \theta \in \mathcal{I} \), \( \theta_1 = T_1(\theta) \).

It follows from Eq. (13) that

\[ \theta_1 - \theta = \frac{a}{b} \left( -\cos \theta + \frac{\sin \theta_1 - \sin \theta}{\theta_1 - \theta} \right). \]

Consequently, if \( b/a \) is sufficiently small, the derivative \( \partial \theta_1/\partial \theta \) is so big on \( \mathcal{I} \) that values \( \{\theta_1 \mod 2\pi : \theta \in \mathcal{I}\} \) cover \([0, 2\pi)\). Since \( T_1 \) is continuous on \( \mathcal{I} \), due to (15) we have

\[ 2 \sin \theta_1 - \sin \theta \frac{\sin \theta_1 - \sin \theta}{\theta_1 - \theta} - (\cos \theta_1 + \cos \theta) \neq 0 \quad (20) \]

everywhere on \( \mathcal{I} \). However, due to Eq. (14), the maximum of the left hand side of inequality (20) is positive while the minimum is negative. So this inequality cannot hold true everywhere.

So, \( n \geq 3 \). Then it suffices to prove that \( T([z_i, z_{i+1}]) \supset [t_2, 3\pi/2] \) for all \( i = 1, \ldots, n - 1 \).

Observe that estimate (17) implies that \( T_1(z_i) > 3\pi/2 \) if \( b/a \) is small. Since \( x(T_1(z_i)) = 0 \), the corresponding motion proceeds to the free flight immediately after \( t = T_1(z_i) \) and, consequently, \( T(z_i) = T_1(z_i) \in L_1 \).

On the other hand, \( T_1(\theta) \rightarrow T_1(T_1(z_{i+1})) \) as \( \theta \rightarrow z_{i+1} - 0 \) (in the limit case, we have a motion, which touches the delimiter with zero velocity i.e. \( x(t) = y(t), v(t) = 0 \)). Since \( T_1(z_{i+1}) \in L_1 \) the corresponding motion resides in the stop regime at \( t = t_0 + 2\pi \) and, consequently, \( T(\theta) = t_2 \) for all \( \theta \) in a left vicinity of \( z_{i+1} \). This finishes the proof.

\section*{6 Infinite set of periodic points}

We have obtained two disjoint segments \( J_0 \) and \( J_1 \) which are subsets of the arc \([t_2, 3\pi/2]\) of the unit circle such that for both \( i = 0, 1 \) mappings \( T|_J \) are continuous and \( T(J) \supset J_0 \cup J_1 \). Let us prove the following lemma.

\textbf{Lemma 6} For any \( m \in \mathbb{N} \) the mapping \( T \) has a point of the period \( m \), which is not periodic for all maps \( T^k \), \( k = 1, \ldots, m - 1 \). There exist at least \( 2^m \) fixed points for the map \( T^m \).

\textbf{Proof} Take a sequence \( \{\sigma_k \in \{0, 1\} : k \in \mathbb{Z}_+\} \). First of all, we notice that there exists a point \( p \in J_0 \) such that

\[ T^k(p) \in J_{\sigma_k} \]

for any \( k \in \mathbb{N} \).

There exists a segment \( J_{\sigma_0} \subset J_0 \) such that \( T(J_{\sigma_0}) = J_{\sigma_1} \). Then, we may find a segment \( J_{\sigma_0 \sigma_1} \subset J_{\sigma_1} \) such that \( T^2(J_{\sigma_0 \sigma_1}) = J_{\sigma_2} \). Repeating this procedure, we obtain a nested sequence of segments

\[ J_{\sigma_0} \supset J_{\sigma_0 \sigma_1} \supset J_{\sigma_0 \sigma_1 \sigma_2} \supset \ldots \]

The corresponding intersection is non-empty and, consequently, contains a desired point \( p \) which may be non-unique.
7 Symbolical patterns

Inclusion (19) implies much more than existence of infinite set of periodic points.

Lemma 7 The reduction of the map T to the union $J_0 \cup J_1$ is topologically semi-conjugated to the shift of one-side sequences of boolean values.

Proof Let $\Sigma = \{ \sigma = \{ \sigma_k \in \{0, 1\} : k \in \mathbb{Z}_+ \} \}$. Introduce the metrics $d$ on the set $\Sigma$ by the formula

$$d(\sigma, \zeta) = \sum_{k=0}^{\infty} 2^{-k} |\sigma_k - \zeta_k|.$$ 

Let $\tilde{J}$ be the set of all points $p \in J_0 \cup J_1$ such that $T^k(p) \in J_0 \cup J_1$ for all $k \in \mathbb{N}$. Clearly, this set is non-empty and compact. For any $p \in \tilde{J}$ we may introduce the sequence $H(p) = \{ \sigma_k \} \in \Sigma$ where values $\sigma_k$ are uniquely defined by Eq. (21).

The map $H : J \rightarrow \Sigma$ is continuous since all iterations of the map $T_{\tilde{J}}$ are continuous. We have already proved that $H(\tilde{J}) = \Sigma$ (see first two paragraphs of Lemma 6).

If $S$ is the left shift of sequences of $\Sigma$, defined by the formula

$$S(\sigma) = \zeta \iff \zeta_k = \sigma_{k+1} \forall k \in \mathbb{Z}_+,$$

the maps $T_{\tilde{J}}$ and $S$ are semi-conjugated: $H \circ T_{\tilde{J}} = S \circ H$.

8 Superstable fixed points

Here we discuss sufficient conditions for existence of a stable fixed point of the map $T$. Namely, this will be the point $t_2$.

Take $T_1(t_2) > t_2$ i.e. the first zero of the equation $G_1(t_2, t) = 0$ which is a particular case of Eq. (12) and $\theta_2(t_2) > \theta_1$ is the first zero of the equation $G_2(t_2, T_1(t_2), t) = 0$ which is a particular case of Eq. (18).

Let one of inclusions

$$\theta_2(t_2) \text{ mod } 2\pi \in (t_1, t_2) \quad (22)$$

or

$$\theta_2(t_2) \text{ mod } 2\pi \in (t_1, 2\pi) \bigcup [0, t_0) \quad (23)$$

be satisfied. Then, due to Lemma 4 there exists $\varepsilon > 0$ such that

$$T([t_2, t_2 + \varepsilon)) = \{ t_2 \}$$

Hence $t_2$ is a superstable fixed point of $T$ i.e. a neighborhood of this point in $[t_2, t_3)$ is mapped to $t_2$.

Due to Implicit Function Theorem conditions (22) and (23) are robust with respect to small variations of parameters of the considered system if the following conditions are satisfied:

$$\frac{\partial G_1}{\partial \theta}(t_2, 0)|_{\theta = \theta_1(t_2)} \neq 0; \quad \frac{\partial G_1}{\partial \theta}(T_1(t_2), 0)|_{\theta = \theta_2(t_2)} \neq 0$$

Remark 4 If inclusion (22) is true, we do not need to assume that $q > a \sin \theta_0$ (see Eq. (2)). Moreover, we do not need the ratio $b/a$ to be small in both cases. We need weaker assumptions

$$a > 0, \quad b > 0, \quad q > 0$$

(24)

instead.

The obtained superstable periodic solution may coexist with the chaotic invariant set, described in previous sections.

9 Conclusion

Let us formulate principle results of the paper as theorems. Recall that the external force $F(t)$ equals $a \sin t + 2b$, $t_2$ and $t_3$ are zeros of $F(t)$ inside $[0, 2\pi)$,
$q \in (0, a + 2b)$ is the maximal value of the dry friction force, $t_0$ and $t_1$ are zeros of $F(t) - q$.

**Theorem 1**  For all $a$ and $q$, satisfying inequalities (2) there exists a

$$b_0 = b_0(a, q) > 0$$

such that for all $b \in (0, b_0)$ the mechanical system, described by Eqs. (3), (4) and (6) is chaotic in the following sense. The phase of transition to free motion uniquely defines the phase of the next transition. This defines a discontinuous mapping $T$ from the segment $[t_2, t_3]$ into itself. There exist two disjoint segments $J_0$ and $J_1$ of $[t_2, 3\pi/2]$ such that $T(J_i) \supset [t_2, 3\pi/2]$ and $T$ is continuous on both segments $J_i$. Particularly, there exists an infinite set $P$ of periodic points of the mapping $T$. Minimal periods of points of $P$ are unbounded. Moreover, there exists a compact subset $\tilde{J} \subset J_0 \cup J_1$ such that the map $T|_{\tilde{J}}$ is continuously semi-conjugated with one-sided symbolic dynamics.

**Theorem 2**  Let inequalities (2) and (23) or inequalities (22) and (24) be satisfied. Then the point $t_2$ is superstable i.e. there exists $\varepsilon > 0$ such that $T([t_2, t_2 + \varepsilon]) = \{t_2\}.$

**Remark 5**  We always assumed that $b < a/2$. If $b \geq a/2$, any motion eventually does not have free regimes. Then we cannot define map $T$. However, in this case the dynamics of the considered system is very simple. If $q \geq a + 2b$ any solution eventually resides in the stop regime. If $q \leq 2b$ there exists an instant when an “eternal” contact with progression starts. If $q \in (2b - a, a + 2b)$ any motion eventually behaves in one of following ways (depending on parameters of the system but not on initial conditions): either it always moves in contact with progression regime or contact with progression starting at $t_0 + 2\pi k$ ($k \in \mathbb{Z}$) stops somewhere between $t_1 + 2\pi k$ and $t_0 + 2\pi (k + 1)$, then contact with progression starts at $t_0 + 2\pi (k + 1)$ and so on.

**10 Discussion and plans**

First of all, let us observe that the obtained chaotic dynamics is robust. Of course, we cannot say anything about stability of points of the set $P$. Every particular point of this set may appear or disappear if we slightly change parameters $a$, $b$ and $q$. Cardinality of the set $P$ may be equal to continuum for some values of parameters, while this set is countable for other values. Neither, the method we offer does not specify the topological structure of the set $P$ and one of its closure.

However, we can select a family of segments $J_0$ and $J_1$ from the statement of Theorem 1 so that boundary points of these segments locally continuously depend on parameters $a$, $b$ and $q$. This can be proved similarly to Lemma 3. The fact of presence of the one dimensional turbulence in the considered system is robust. The same is true for the fact of existence of an infinite set $P$. Moreover, for all fixed values $a$, $b$ and $q$, satisfying inequalities (2), there exists an $\varepsilon > 0$ such that for any $C^2$ smooth function $G(t, x, \dot{x}) : S^1 \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$|G(t, x, \dot{x})| < \varepsilon, \quad \frac{\partial G}{\partial (t, x, \dot{x})} < \varepsilon$$

for all $t, x$ and $\dot{x}$ an analog of Theorem 1 is true for the system, where Eq. (3) is replaced with

$$\ddot{x} = F(t) + G(t, x, \dot{x}); \quad \dot{y} = 0,$$

Eq. (4) is replaced with

$$\ddot{x} = F(t) + G(t, x, \dot{x}) - q; \quad \dot{y} = 0.$$

and Eq. (6) is the same.

Particularly, the presence of the considered chaotic dynamics must be observed in simulations and experiments. Fortunately, this work has been already done by M. Wiercigroch, A. Wojewoda and A. Krivtsov in their paper [28]. They have considered the same mathematical model. We literally a paragraph, adjusting notions to ones of our paper. Particularly, we observe that our parameter $b$ is twice less than one of the quoted paper. Displacement $d$ in Fig. 5 equals to $(x(2\pi) - x(0))/q$ in notions of our paper.

Numerical analysis of the system shows that when the static force $2b$ is large enough only periodic motion with the excitation frequency occurs. For $2b = a$ above the first subcritical period of doubling bifurcation located at $2b = a \approx 0.27$, the system responds with a period one motion, as shown in Fig. 5. All bifurcation diagrams presented here were obtained for $a/q = 0.12$ and for the static force, $2b$ varying from zero up to the value of the
dynamic component, \( a \). Figure 5 shows that, after the second subcritical period doubling, a narrow region of chaotic motion occurs, followed by period three and then period six motions. Then again a narrow chaotic region is seen, which is followed by period four and period eight motions. Thus, after each chaotic region, multiplicity of the motion increases by one period. It is clear from Fig. 5 that for this system chaotic regions are quite narrow whilst compared to periodic ones. A similar behavior has been reported for two-dimensional piece-wise smooth maps, e.g. Refs. [29, 30].

**Plans.** We plan to use developed methods for more general systems with a dry friction (see [21] as an example) and provide for these “real life systems” theoretical results accompanied with simulations and experimental data.

Recalling properties of 1D-turbulent mappings of a segment, compare chaotic dynamics, provided by Theorem 1 with Devaney chaos [31], which can be observed for Smale horseshoes, appearing in neighborhood of homoclinic points.

1. Devaney chaos implies that the considered mapping is smooth and invertible in a neighborhood of the invariant set. Moreover, the invariant set must be hyperbolic. Definition 1 (turbulence) implies that mapping \( T \) is non-invertible and may be non-smooth.

2. Hyperbolic invariant sets including Devaney chaos are robust with respect to \( C^1 \) – small variations of the diffeomorphism. Namely, for such variations there exist invariant sets, where dynamics is topologically conjugated to one on the initial invariant set (see [26] for precise definitions). A \( C^0 \) – small perturbation of the diffeomorphism may destroy topological structure of the invariant set. Operating with turbulence, we do not use the idea of smoothness. So, varying coefficients of the system, we always have infinitely many periodic points and a symbolic pattern but we never know topological structure of the corresponding invariant set. Anyway, varying parameters of the system, one observes chaotic band without windows of periodicity.

3. A typical example of Devaney chaos – the famous Smale horseshoe, can be modeled by two-side symbolic dynamics. For turbulence, we have obtained a one-side symbolic dynamics.

4. There is a canonical ergodic invariant probability measure \( \mu \) on the set \( \Sigma \). We take a \( \sigma \) – algebra \( \mathcal{A} \), engendered by sets

\[ \Sigma_{k,\beta} = \{ \zeta \in \Sigma : \zeta_k = \beta \} : \quad k \in \mathbb{N} \bigcup 0, \beta \in \{0,1\}, \]

set \( \mu(\Sigma_{k,\beta}) = 1/2 \) for all \( k \) and \( \beta \) and say that all sets \( \Sigma_{k,\beta} \), corresponding to distinct \( k \), are independent in the probabilistic sense. There exists a measure \( \nu \), defined on the \( \sigma \) – algebra \( H^{-1}(\mathcal{A}) \) and such that \( \nu(H^{-1}(A)) = \mu(A) \). Consider completion of the measure \( \nu \), still denoted as \( \nu \). This is a Borel probability measure, invariant with respect to \( T \). It is ergodic and \( T \) strongly mixing i.e. for any measurable sets \( A \) and \( B \) we have

\[ \lim_{n \to +\infty} \nu(T^{-n}(A) \bigcap B) = \nu(A)\nu(B). \]

5. In Definition 1 a one-dimensional mapping is considered. However, it is easy to to define turbulence in any dimensions. For this it suffices to replace segments with \( n \) – dimensional balls. Observe that such “\( n \) – dimensional” turbulence does imply existence of infinitely many periodic points; it suffices to replace Weierstrass principle with Brauer’s Fixed Point
Theorem in the proof of Lemma 6. It is not difficult to construct mappings, that have a high-dimensional turbulence; it is much more interesting to find them in applications. This is what we are going to do.

In a nutshell, the main idea of this paper is the following. Turbulence and related chaotic dynamics can be observed for piecewise smooth systems, particularly, for simple models of systems with dry friction.

A similar approach may be applied to general Filippov systems. The simplest case is a three-dimensional Filippov system with a smooth two-dimensional smooth discontinuity surface. Suppose that inside this surface there exists a smooth curve $I$, which separates regions where sliding and crossing are observed. It seems to be natural that existence of a locally defined Poincaré mapping from a subsegment of $I$ to $I$ is non-degenerate. It might be interesting to formulate sufficient conditions for existence of turbulence in such systems and spread, if possible, obtained results to higher dimensions.

Filippov systems are not only examples of dynamical systems where a backward non-uniqueness appears. One can consider systems with hysteresis [32, 33] or even timescale systems [34] (we hope to use such ideas to study systems like ones considered in [35]).

Finally, we hope to consider a more precise model of dry friction, offered in [21]. It is also based on Filippov systems but involves much more technical troubles.

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References

1. di Bernardo M, Budd ChJ, Champneys AR, Kowalczyk P, Nordmark AB, Tost GO, Piireinen PT (2008) Bifurcations in nonsmooth dynamical systems. SIAM Rev 50:629–701
2. di Bernardo M, Kowalczyk P, Nordmark AB (2003) Sliding bifurcations: a novel mechanism for a sudden onset of chaos in dry friction oscillators. Int J Bifurc Chaos 13:2935–2948
3. Blazejczyk-Okoleswka B, Kapitaniak T (1996) Dynamics of impact oscillator with dry friction. Chaos Solitons Fractals 7:1455–1459
4. Casapulla C, Portioli F, Maione A, Landolfo R (2013) A macro-block model for in-plane loaded masonry walls with non-associative Coulomb friction. Meccanica 48:2107–2126
5. Csernák G, Stepán G, Shaw SW (2007) Sub-harmonic resonant solutions of a harmonically excited dry friction oscillator. Nonlinear Dyn 50:93–109
6. Feeny B, Moon FC (1994) Chaos in a forced dry-friction oscillator: experiments and numerical modelling. J Sound Vib 170:303–323
7. Kiseleva M (2013) Oscillations of dynamical systems applied in drilling: analytical and numerical methods. PhD Thesis, Jyväskylä University Printing House
8. Krivtsov AM, Wiercigroch M (1999) Dry friction model of percussive drilling. Meccanica 34:425–434
9. Krivtsov AM, Wiercigroch M (2000) Penetration rate prediction for percussive drilling via dry friction model. Chaos Solitons Fractals 11:2479–2485
10. Kowalczyk P, Piireinen PT (2008) Two-parameter sliding bifurcations of periodic solutions in a dry-friction oscillator. Phys D Nonlinear Phenom 237:1053–1073
11. Makarenkov O, Lamb JSW (2012) Dynamics and bifurcations of nonsmooth systems: a survey. Phys D Nonlinear Phenom 241:1826–1844
12. Pugno NM, Qifang Yin, Xinghua Shi, Capozza R (2013) A generalization of the Coulombs friction law: from graphene to macroscale. Meccanica 48:1845–1851
13. Stefański A, Wojewoda J, Wiercigroch M, Kapitaniak T (2003) Chaos caused by non-reversible dry friction. Chaos Solitons Fractals 16:661–664
14. Wiercigroch M, de Kraker A (eds) (2000) Applied nonlinear dynamics and chaos of mechanical systems with discontinuities. World Scientific, Singapore, New Jersey, London, Hong Kong
15. Wojewoda J, Kapitaniak T, Barron R, Brindley J (1993) Complex behaviour of a quasi-periodically forced experimental system with dry friction. Chaos Solitons Fractals 3:35–46
16. Filippov AF (1998) Differential equations with discontinuous right hand sides. Kluwer Academic Publishers, Dordrecht
17. Hogan SJ, Higham L, Griffin TCL (2007) Dynamics of a piecewise linear map with a gap Proc. R Soc A 463:49–65
18. Simpson DJW, Meiss JD (2010) Aspects of bifurcation theory for piecewise-smooth, continuous systems. arXiv: 1006.4123v1
19. Awrejecevicz J (1988) Chaotic motion in a non-linear oscillator with friction. KSME J 2:104–109
20. Galvanetto U (2005) Unusual chaotic attractors in nonsmooth dynamic systems. Int J Bifurc Chaos 15:4081–4086
21. Pavlovskaia EM, Wiercigroch M (2007) Low-dimensional maps for piecewise smooth oscillators. J Sound Vib 305:750–771. doi:10.1063/1.2904774
22. Szalai R, Osinga HM (2008) Invariant polygons in systems with grazing–sliding. Chaos 28:023121
23. Szalai R, Osinga HM (2009) Arnold tongues arising from a grazing–sliding bifurcation. SIAM J Appl Dyn Syst 8:1434–1461
24. Block LS, Coppel, WA (1992) Dynamics in one dimension. Lecture notes in mathematics, 1513. Springer-Verlag, Berlin, 1992. viii+249 pp. ISBN 3-540-55309-6
25. Li TY, Yorke JA (1975) Period three implies chaos. Am Math Mon 82:49–68
26. Katok A, Hasselblatt B (1995) Introduction to the modern theory of dynamical systems. Cambridge University Press, Cambridge
27. Sharkovskii OM (1964) Co-existence of cycles of a continuous mapping of a line onto itself. ukranian Math Z 16:61–71
28. Wiercigroch M, Wojewoda AJ, Krivtsov AM (2005) Dynamics of ultrasonic percussive drilling of hard rocks. J Sound Vib 280:739–757
29. Banerjee S, Grebogi C (1999) Border collision bifurcations in two-dimensional piecewise smooth maps. Physical Rev E 59:4052–4061
30. Chin W, Ott E, Nusse HN, Grebogi C (1999) Grazing bifurcations in impact oscillators. Physical Rev E 50:4427–4444
31. Devaney RL (1987) An introduction to chaotic dynamical systems. Addison-Wesley, Redwood City
32. Mayergoyz ID (2003) Mathematical models of hysteresis and their applications: second edition (Electromagnetism). Academic Press. ISBN 978-0-12-480873-7
33. Krasnosel’skii M, Pokrovskii A (1989) Systems with hysteresis. Springer-Verlag, New York. ISBN 978-0-387-15543-2
34. Mease KD, Bharadwaj S, Iravanchy S (2003) Timescale analysis for nonlinear dynamical systems. J Guid Control Dyn 26:318–330
35. Litak G, Arkadiusz S, Rusinek R, Sen Asok K (2013) Intermittency and multiscale dynamics in milling of fiber reinforced composites. Meccanica 48:783–789