Non-Uniqueness of Quantized Yang-Mills Theories

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Abstract. - We consider quantized Yang-Mills theories in the framework of causal perturbation theory which goes back to Epstein and Glaser. In this approach gauge invariance is expressed by a simple commutator relation for the S-matrix. The most general coupling which is gauge invariant in first order contains a two-parametric ambiguity in the ghost sector - a divergence- and a coboundary-coupling may be added. We prove (not completely) that the higher orders with these two additional couplings are gauge invariant, too. Moreover we show that the ambiguities of the n-point distributions restricted to the physical subspace are only a sum of divergences (in the sense of vector analysis). It turns out that the theory without divergence- and coboundary-coupling is the most simple one in a quite technical sense. The proofs for the n-point distributions containing coboundary-couplings are given up to third or fourth order only, whereas the statements about the divergence-coupling are proven in all orders.

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1. Introduction

1.1 The Model

In a recent series of papers [1-5] non-abelian gauge invariance has been studied in the framework of causal perturbation theory [6,7]. This approach, which goes back to Epstein and Glaser [6], has the merit that one works exclusively with free fields, which are mathematically well-defined, and performs only justified operations with them.

In causal perturbation theory one makes an ansatz for the S-matrix as a formal power series in the coupling constant

$$ S(g_0, g_1, \ldots, g_l) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1, \ldots, i_n=0}^{l} \int d^4x_1 \ldots d^4x_n T_{n}^{i_1 \ldots i_n}(x_1, \ldots, x_n) g_{i_1}(x_1) \ldots g_{i_n}(x_n). \quad (1.1) $$

The indices $i \in \{0, 1, \ldots, l\}$ label different couplings $T_i$, which are switched by different test functions $g_i \in S(\mathbb{R}^4)$. The operator-valued distribution $T_{n}^{i_1 \ldots i_n}(x_1, \ldots, x_n)$ has a vertex of the type $T_i$ at $x_s$ ($1 \leq s \leq n$). The $T_i$’s are constructed inductively from the given first order (see appendix A). In our model the $i = 0$-coupling

$$ T_0^A(x) \overset{\text{def}}{=} T_1^A(x) + T_1^u(x), \quad (1.2) $$

is the usual three-gluon coupling

$$ T_1^{0A}(x) \overset{\text{def}}{=} \frac{i}{2} f_{abc} \, : A_{\mu a}(x) A_{\nu b}(x) F_{c}^{\mu \nu}(x) :, \quad (1.3) $$

plus the usual ghost coupling

$$ T_1^{0u}(x) \overset{\text{def}}{=} - ig f_{abc} \, : A_{\mu a}(x) u_b(x) \partial^\nu \tilde{u}_c(x) :, \quad (1.4) $$

Herein $g$ is the coupling constant and $f_{abc}$ are the structure constants of the group SU(N). The gauge potentials $A_{\mu}^a$, $F_{\mu \nu}^a \overset{\text{def}}{=} \partial^\mu A_{\nu}^a - \partial^\nu A_{\mu}^a$, and the ghost fields $u_a$, $\tilde{u}_a$ are massless and fulfil the wave equation. (We work throughout in the Feynman gauge $\lambda = 1$.)

**Gauge invariance** means roughly speaking that the commutator of the $T_n^{0,0}$-distributions with the gauge charge

$$ Q \overset{\text{def}}{=} \int_{t=\text{const.}} d^3x \left( \partial_\nu A_\nu^a \partial_0 u_a \right) \quad (1.5) $$

is a (sum of) divergence(s) (in the sense of vector analysis). In first order this holds true

$$ [Q, T_1^{0}(x)] = i \partial_\nu T_1^{1\nu}(x), \quad (1.6) $$

where

$$ T_1^{1\nu}(x) \overset{\text{def}}{=} ig f_{abc} : A_{\mu a}(x) u_b(x) F_{c}^{\nu \mu}(x) :- \frac{1}{2} : u_a(x) u_b(x) \partial^\nu \tilde{u}_c(x) :. \quad (1.7) $$

We choose this expression to be the $i = 1$-coupling in (1.1) and call it a Q-vertex. Note that $[Q, T_1^{0A}]$ alone is not a divergence. In order to have gauge invariance in first order, we
are forced to introduce the ghost coupling \( T^0_{u_1} \) (1.4). However, the latter coupling is not uniquely fixed by this procedure. The present paper deals with these ambiguities. We define gauge invariance in arbitrary order \([2]\) by

\[
[Q, T^0_{n}(x_0, \ldots, x_n)] = i \sum_{l=1}^{n} \partial^l \phi \partial_0^0 T^0_{n}(x_0, \ldots, x_n), \tag{1.8}
\]

where the upper index 1 in \( T^0_{n} \) is at the \( l \)-th position. The divergences on the r.h.s. of (1.8) are precisely specified: \( T^0_{n}(x_0, \ldots, x_n) \) is the \( T_n \)-distribution of (1.1) which has a \( Q \)-vertex (1.7) at \( x_l \) and all other vertices are \( T^0_1 \)-couplings (1.2). Gauge invariance (1.8), which has been proven in all orders \([1-5]\), implies the invariance of the S-matrix \( S(g, 0, \ldots, 0) \) (1.1) with respect to simple gauge transformations of the free fields \([5]\). These transformations are the free field version of the famous BRS-transformations \([8]\). Moreover unitarity on the physical subspace \([4]\) can be proven by means of gauge invariance (1.8). The C-number identities expressing (1.8) imply the Slavnov-Taylor identities \([9]\). Finally we mention that the four-gluon interaction is a normalization term in second order, which is uniquely fixed by gauge invariance (see \([1, 5]\) and (2.59)).

Let us turn to the mentioned non-uniqueness in the ghost sector. The most popular method to derive the ghost-coupling is the one of Faddeev and Popov. However, this method of quantization contains loopholes (even in perturbation theory) \([10]\). Therefore, Beaulieu \([10]\) determined the quantum Lagrangian from the requirement of its full BRS-invariance. We proceed in an analogous way. Our aim is to work out the most general Yang-Mills theory which is gauge invariant (1.8) in all orders and to investigate the physical and technical implications of the ambiguities.

1.2 Most General Coupling which is Gauge Invariant in First Order

In order to simplify the notations we define an operator \( d_Q \) by means of our gauge charge \( Q \) (1.5)

\[
d_Q A \overset{\text{def}}{=} QA - (-1)^{Qg} A(-1)^{Qg} Q, \tag{1.9}
\]

where \( Qg \) is the ghost charge operator \([11, 12]\)

\[
Q_g = i \int_{t = \text{const.}} d^3 \bar{\psi} \partial_0 u_a \partial_0 u_a ; \quad [Q_g, u_a] = -u_a, \quad [Q_g, \bar{u}_a] = \bar{u}_a. \tag{1.10}
\]

and \( A \) is a suitable operator on the Fock space such that (1.9) makes sense. If the ghost charge of \( A \) is an integer, \( [Q_g, A] = zA, z \in \mathbb{Z} \), the expression (1.9) is the commutator or anticommutator of \( Q \) with \( A \). Note the product rule

\[
d_Q(AB) = (d_Q A)B + (-1)^{Qg} A(-1)^{Qg} d_Q B. \tag{1.11}
\]

One easily verifies \([1]\)

\[
Q^2 = 0, \tag{1.12}
\]

which implies

\[
(d_Q)^2 = 0. \tag{1.13}
\]

Because \( d_Q \) is nilpotent, it can be interpreted as coboundary-operator in the framework of a homological algebra \([11]\). (The gradiation is given by the ghost charge (1.10).) Therefore, we call an element of the range (kernel) of \( d_Q \) a coboundary (cocycle).
Let us add a coboundary
\[ \beta_1 d_Q K_1(x), \quad \beta_1 \in \mathbb{R} \text{ arbitrary}, \] (1.14)
with
\[ K_1(x) \overset{\text{def}}{=} gf_{abc} : u_a(x) \bar{u}_b(x) \bar{u}_c(x) :, \] (1.15)
to \( T^1_1(x) \). Due to (1.13), gauge invariance in first order (1.6) remains true with the same Q-vertex \( T^1_1 \) (1.7). Moreover, we add a divergence
\[ \beta_2 \partial_\mu K_2^\mu(x), \quad \beta_2 \in \mathbb{R} \text{ arbitrary}, \] (1.16)
with
\[ K_2^\mu(x) \overset{\text{def}}{=} igf_{abc} : A^\mu_a(x) u_b(x) \bar{u}_c(x) :, \] (1.17)
to \( T^1_1(x) \). Adding simultaneously \( \beta_2 d_Q K_2^\mu(x) \) to \( T^1_1 \) (x), our gauge invariance (1.6) is obviously preserved. Are there further couplings which are gauge invariant in first order? The answer is ‘no’ [11,13], if the following, physically reasonable requirements are additionally imposed:

(A) The coupling is a combination of at least three free field operators.
(B) The coupling has mass-dimension \( \leq 4 \). This guarantees (re)normalizability of the theory, if the fundamental (anti)commutators have singular order \( \omega([A^\mu_a, A^\nu_b]) = -2 \) and \( \omega([u_a, \bar{u}_b]) = -2 \) (see appendix A and [2]).
(C) Lorentz covariance.
(D) SU(N)-invariance.
(E) The coupling has ghost charge zero: \( [Q_g, T^1_1] = 0 \).
(F) Invariance with respect to the discrete symmetry transformations \( P, T \) and \( C \).
(G) Pseudo-unitarity \( S_1(g_0^1, 0, ..., 0)^K = S_1(g_0, 0, ..., 0)^{-1} \) forces \( \beta_1, \beta_2 \) to be real. \( S_1 \) is the first order \( n = 1 \) of (1.1) and \( K \) is a conjugation which is related to the adjoint [4,12].)

Remarks: (1) The self-interaction of the gauge bosons \( T^A_1 \) (1.3) is unique. There is only an ambiguity in the ghost coupling.
(2) In [5] the coupling to fermionic matter fields in the fundamental representation was studied in detail. It is easy to see that the above requirements fix this coupling uniquely. Therefore, we do not consider matter fields in this paper.

1.3 Outline of the Paper

The paper yields the following results:
(A) The higher orders with divergence- or coboundary-coupling (1.14-17) are gauge invariant for all values of \( \beta_1, \beta_2 \in \mathbb{R} \) (sects.2.2, 2.4). (For the coboundary-coupling this will be proven up to third order only.) The analogous result for the full BRS-symmetry in the usual Lagrangian approach is known in the literature, see e.g. [10]. However, only a one-parametric ambiguity is studied there. This difference will be discussed in sect.2.7, remark (4).
(B) We will show that the \( T^n \)’s with divergence-coupling are divergences with respect to their divergence-vertices (sect.2.2). The \( T^n \)’s (\( 1 \leq n \leq 4 \)) with coboundary-coupling are divergences too, if they are restricted to the physical subspace [4] (sect.2.8). This will be an immediate consequence of a representation of these \( T^n \)’s, which will be proven in sect.2.4.
(C) The results in higher orders about the divergence-coupling and partly the results about the coboundary-coupling are independent on the explicit expressions (1.2-4) and (1.14-17) of the couplings (sect.2.5). They apply to any gauge invariant quantum field theory.
(D) Gauge invariance for second order tree diagrams requires normalization terms, namely the usual four-gluon interaction and a four-ghost interaction (sect.2.7). (The latter appears only for $\beta_1, \beta_2 \neq (0,0).$) By studying these normalization terms we will find a criterion which reduces the freedom in the choice of $\beta_1, \beta_2 \in \mathbb{R}$ to a one-parametric set (sects.2.7-8). We will mention a second, quite technical criterion which gives another restriction of $\beta_1, \beta_2$ (sect.2.8). Together we will see that the theory with $\beta_1 = 0 = \beta_2$ is the most simple one.

(E) The Q-vertex is not uniquely fixed by gauge invariance in first order (1.6). In order to prove gauge invariance in higher orders of the theory $(T_1^0 + \beta_1 d_Q K_1 + \beta_2 \partial_\mu K_2^\mu)$, $\beta_1, \beta_2 \in \mathbb{R}$ ((1.2-4), (1.14), (1.16)), it is not necessary to modify the above introduced Q-vertex ((1.7) plus $\beta_2 d_Q K_2^\nu$). Therefore, the ambiguity of the Q-vertex is not very interesting. Nevertheless we show in sect.2.3 that the possible modifications of the Q-vertex do not destroy gauge invariance in higher orders.

(F) In appendix (C) we assume certain identities to hold true. They concern exclusively the starting-coupling $T_0^0$ (1.2-4), its Q-vertex $T_1^0$ (1.7) and its 'Q-Q-vertex' $T_1^5$ introduced below (2.5), and are a kind of generalization of gauge invariance (1.8). A special case of this assumption is verified in appendix (B). By means of these identities we will be able to prove the results about the coboundary-coupling in all orders.

2. Divergence- and Coboundary-Coupling in Higher Orders

2.1 Preparations

In order to study the $T_n$’s with a divergence- (1.16) and/or a coboundary-coupling (1.14) in higher orders $n \geq 2$, we define a big theory which contains these couplings and some auxiliary vertices

$$S_1(g_0, g_1, \ldots, g_7) \equiv \int d^4 x \{T_1^0(x) g_0(x) + T_1^{1\nu}(x) g_{1\nu}(x) + T_2^0(x) g_2(x) + T_1^{3\nu}(x) g_{3\nu}(x) +$$

$$+ T_1^{4\nu}(x) g_{4\nu}(x) + T_1^{5\nu\mu}(x) g_{5\nu\mu}(x) + T_1^{6\nu}(x) g_6(x) + T_1^{7\nu}(x) g_7(x)\}, \quad (2.1)$$

where $T_1^0, T_1^{1\nu}$ are given by (1.2-4) and (1.7), furthermore

$$T_1^{1\nu}(x) \overset{\text{def}}{=} \beta_2 K_2^{\nu}(x), \quad (2.2)$$

$$T_1^{2\nu}(x) \overset{\text{def}}{=} \partial_\nu T_1^{4\nu}(x) = \beta_2 \partial_\nu K_2^{\nu}(x), \quad (2.3)$$

$$i T_1^{3\nu}(x) \overset{\text{def}}{=} d_Q T_1^{4\nu}(x) = \beta_2 d_Q K_2^{\nu}(x), \quad (2.4)$$

$$T_1^{5\nu\mu}(x) \overset{\text{def}}{=} \frac{ig_7 f_{abc}}{2} : u_a(x) u_b(x) F_{c\nu}^{\nu\mu}(x) := -T_1^{5\nu\mu}(x), \quad (2.5)$$

$$T_1^{6\nu}(x) \overset{\text{def}}{=} \beta_1 K_1(x), \quad (2.6)$$

$$T_1^{7\nu}(x) \overset{\text{def}}{=} \beta_1 K_1(x).$$
For technical reasons the divergence-coupling \(T_1^7\) and \(T_1^7\) are not directly added to the appearance of the vertex \(T_1^7\). Therefore, we sometimes call \(T_1^5\) 'Q-Q-vertex'. Furthermore note that \(T_1^{-5}\) is a cocycle

\[
d_Q T_1^{1\nu}(x) = i \partial_\nu T_1^{5\mu\nu}(x). \tag{2.8}
\]

Therefore, we sometimes call \(T_1^5\) 'Q-Q-vertex'. Furthermore note that \(T_1^{5\mu\nu}\) is a cocycle

\[
d_Q T_1^{5\mu\nu}(x) = 0. \tag{2.9}
\]

The vertices \(T_1^{1\nu}\), \(T_1^{3\nu}\) and \(T_1^{6}\) are fermionic, all other vertices are bosonic. The first ones give rise to some additional minus-signs in the inductive construction of the \(T_n\)'s, but there is no serious complication (see the appendix of [3]). We are interested in the physically relevant theory

\[
T_n(x_1, ..., x_n) \overset{\text{def}}{=} \sum_{i_1, ..., i_n \in \{0, 2, 7\}} T_n^{i_1 ... i_n}(x_1, ..., x_n), \tag{2.10}
\]

which corresponds to the choice \(g \overset{\text{def}}{=} g_0 = g_2 = g_7 \neq 0\) and \(g_1 = 0, g_{3\nu} = 0, g_{4\nu} = 0, g_{5\mu\nu} = 0\) and \(g_6 = 0\) in the \(n\)-th order S-matrix \(S_n(g_0, g_1, ..., g_7)\). Gauge invariance in the sense (1.8) of this theory is formulated in terms of the Q-vertices \(T_1^{1\nu}, T_1^{3\nu}\) and \(T_1^{6} \overset{\text{def}}{=} 0\). This means in first order

\[
d_Q T_1^0 = i \partial_\nu T_1^{1\nu}, \tag{2.11}
\]

\[
d_Q T_1^2 = i \partial_\nu T_1^{3\nu}, \tag{2.12}
\]

\[
d_Q T_1^7 = 0, \tag{2.13}
\]

and in arbitrary order \(n\)

\[
d_Q T_n^{i_1 ... i_n} = i \sum_{l=1}^n \partial_{\nu_l} T_n^{i_1 ... i_{l-1} i_l+1 i_{l+1} ... i_n \nu}, \tag{2.14}
\]

where \(i_1, ..., i_n \in \{0, 2, 7\}\) and

\[
T_n^{i_1 ... i_n \nu} \overset{\text{def}}{=} 0. \tag{2.15}
\]

We shall often use that \(T_n^{0 ... 0} \) is gauge invariant (1.8) \([1-5]\).

### 2.2 Higher Orders with Divergence-Coupling

We are going to prove

**Proposition 1:** Choosing suitable normalizations, the relations

\[
F_n^{2 ... 20 ... 0}(x_1, ..., x_n) = \partial_{\mu_1}^1 ... \partial_{\mu_r}^r F_n^{4 ... 40 ... 0\mu_1 ... \mu_r}(x_1, ..., x_n), \tag{2.16}
\]

\[
F_n^{32 ... 20 ... 0\nu}(x_1, ..., x_n) = \partial_{\mu_2}^2 ... \partial_{\mu_r}^r F_n^{34 ... 40 ... 0\mu_2 ... \mu_r}(x_1, ..., x_n), \tag{2.17}
\]

\[
F_n^{2 ... 210 ... 0\nu}(x_1, ..., x_n) = \partial_{\mu_1}^1 ... \partial_{\mu_r}^r F_n^{4 ... 410 ... 0\mu_1 ... \mu_r \nu}(x_1, ..., x_n). \tag{2.18}
\]
hold true for all $F = A', R', R'^n, D, A, R, T', T, \tilde{T}$ and in all orders $n$.

Remarks: (1) The assertions (2.16-18) are generalizations of (2.3) to arbitrary orders and mean that the divergence-structure of $T_n^2$ can be maintained by constructing the higher orders.

(2) Due to the symmetrization (A.14) the $T_{n-1}^\ldots$, $\tilde{T}_{n-1}^\ldots$ fulfill

$$T_{n}^{1\ldots i\ldots n}(x_1, \ldots, x_n) = (-1)^f(\pi) T_{n}^{i\ldots i\pi n}(x_{\pi 1}, \ldots, x_{\pi n}), \quad \forall \pi \in S_n,$$  \hspace{1cm} (2.19)

where the Lorentz indices are permuted, too, and $f(\pi)$ is the number of transpositions of fermionic vertices in $\pi$. Therefore, the equations (2.16-18) remain true for $T_n$, $\tilde{T}_n$, if the indices are permuted according to (2.19).

(3) We will see in the proof that the $T_{n-4\ldots}^\ldots$ on the r.h.s. can be normalized in an arbitrary symmetrical way. (A normalization is said to be symmetrical if the corresponding $T_n$ satisfies (2.19).) But the normalization of the $T_{n-2\ldots}^\ldots$ on the l.h.s. is uniquely fixed by the normalization of the $T_{n-4\ldots}^\ldots$.

Proof: We show that (2.16-18) can be maintained in the inductive step $(n - 1) \to n$ described in appendix A. Obviously there are only two operations in this step which need an investigation, namely (A) the construction of the tensor products in $A'_n, R'_n, R''_n$ (A.1-3) and (B) the distribution splitting $D_n = R_n - A_n$ (A.7).

(A) Let us consider (2.17) for $A_{n-2}^\ldots$ (A.2)

$$A_n^{2\ldots 20\ldots 0\nu}(x_1, \ldots, x_n) = \sum_{X,Y, (x_1 \in X)} \hat{T}_{k}^{2\ldots 20\ldots 0\nu}(X)T_{n-k}^{2\ldots 0\nu}(Y, x_n) + \sum_{X,Y, (x_1 \in Y)} \hat{T}_{k}^{2\ldots 20\ldots 0}\nu(X)T_{n-k}^{32\ldots 20\ldots 0\nu}(Y, x_n).$$  \hspace{1cm} (2.20)

Inserting the induction hypothesis (2.16-17) in lower orders $k, n - k$, we obtain

$$\begin{align*}
(2.20) &= \sum_{(x_1 \in X)} \partial^2_{\mu_2} \ldots \partial^2_{\mu_s} \hat{T}_{k}^{\ldots 34\ldots 40\ldots 0\mu_2\ldots \mu_s}(x)\partial^1_{\mu_{s+1}} \ldots \partial^1_{\mu_{r}} T_{n-k}^{\ldots 34\ldots 40\ldots 0\mu_{s+1}\ldots \mu_{r}}(Y, x_n) + \\
&\quad + \sum_{(x_1 \in Y)} \partial^1_{\mu_1} \ldots \partial^1_{\mu_s} \hat{T}_{k}^{\ldots 34\ldots 40\ldots 0\mu_1\ldots \mu_s}(x)\partial^2_{\mu_{s+2}} \ldots \partial^2_{\mu_{r}} T_{n-k}^{\ldots 34\ldots 40\ldots 0\mu_{s+2}\ldots \mu_{r}}(Y, x_n) = \\
&\quad = \partial^2_{\mu_2} \ldots \partial^2_{\mu_{r}} A_n^{3\ldots 40\ldots 0\mu_2\ldots \mu_{r}}(x_1, \ldots, x_n).
\end{align*}$$  \hspace{1cm} (2.21)

The other verifications of (2.16-18) for $A'_n$, $R'_n, R''_n$ are completely analogous.

(B) According to (A) the $D_n$’s (A.4) fulfill (2.16-18). Let $R_n^{3\ldots 40\ldots 0\mu_2\ldots \mu_{r}}$ be an arbitrary splitting solution of $D_n^{3\ldots 40\ldots 0\mu_2\ldots \mu_{r}}$. Then the definition

$$R_n^{3\ldots 20\ldots 0\nu}(x_1, \ldots, x_n) \overset{\text{def}}{=} \partial^2_{\mu_2} \ldots \partial^2_{\mu_{r}} R_n^{3\ldots 40\ldots 0\mu_2\ldots \mu_{r}}(x_1, \ldots, x_n),$$  \hspace{1cm} (2.22)

yields a splitting solution of $D_n^{3\ldots 20\ldots 0\nu}$, because $R_n^{3\ldots 20\ldots 0\nu}$ (2.22) has its support in $\Gamma_{-1}^+(x_n)$ (A.6) and $R_n^{3\ldots 20\ldots 0\nu} = D_n^{3\ldots 20\ldots 0\nu}$ on $\Gamma_{-1}^+(x_n) \setminus \{(x_1, \ldots, x_n)\}$. The procedure for (2.16), (2.18) is similar. □

Applying $d_Q$ to (2.16) we see that $d_Q T_n^{2\ldots 20\ldots 0}$ is a divergence

$$d_Q T_n^{2\ldots 20\ldots 0}(x_1, \ldots, x_n) = \partial^1_{\mu_1} \ldots \partial^1_{\mu_{r}} d_Q T_n^{\ldots 34\ldots 40\ldots 0\mu_1\ldots \mu_{r}}(x_1, \ldots, x_n),$$  \hspace{1cm} (2.23)
if there is at least one divergence-vertex $T^2_n$. However, the divergences on the r.h.s. of (2.23) are derivatives with respect to the divergence-vertices and generally not to the Q-vertices. Consequently, (2.23) means not gauge invariance of $T^{2...20}_n$ in the sense of (1.8) resp. (2.14). In order to obtain the latter we will prove

**Proposition 2:** Starting with arbitrary symmetrical normalizations of $T^{4...40}_n$, $T^{4...410}_n$, ..., $T^{4...430}_n$, there exists a symmetrical generalization of (2.4) and (2.11) to higher orders.

In order to obtain the latter we will prove

The total Q-vertex $T^{1ν}_n$ of the theory (2.10) is not uniquely fixed by gauge invariance in first order $d QT^{1ν}_n + T^{3ν}_n = i∂νT^{1ν}_n$, one has the freedom to replace $T^{1ν}_{1/1}$ by

$$T^{ν}_B = T^{ν}_{1/1} + γB^ν,$$

if $γ ∈ C$, arbitrary.

(2.27)
if \( \partial_{,\nu} B^{\nu} = 0 \). Requiring additionally\( B^{\nu} \) to fulfil the properties (A), (B), (C) and (D) listed in sect.1.2, and to have ghost charge \(-1\), there remains only one possibility

\[ B^{\nu}(x) = \partial_{\mu} D^{\mu\nu}(x) \quad (2.28) \]

with

\[ D^{\mu\nu}(x) \overset{\text{def}}{=} igf_{abc} : u_a(x) A^c_{\nu}(x) A^b_{\mu}(x) := -D^{\mu\nu}(x). \quad (2.29) \]

This is proven in [11,13]. The\( T_n \)-distribution with a modified Q-vertex\( T^\nu_{1/1, B} \) (resp. an original Q-vertex\( T^\nu_{1/1} \) / resp. a vertex\( B^{\nu} \) / resp.\( D^{\nu\mu} \)) at\( x_l \) and with all other vertices being a\( T_1 \overset{\text{def}}{=} (T_1^7 + T_2^7 + T_3^7) \)-coupling is denoted by\( T^\nu_{n/l, B}(x_1, ..., x_l, ..., x_n) \) (resp.\( T^\nu_{n/l}(x_1, ..., x_n) \) / resp.\( B^{\nu}_{n/l}(x_1, ..., x_n) \) / resp.\( D^{\nu\mu}_{n/l}(x_1, ..., x_n) \)). The relation\( D^{\nu\mu} = -D^{\mu\nu} \) can be maintained in the inductive construction of the\( T_n \)'s

\[ D^{\nu\mu}_{n/l} = -D^{\mu\nu}_{n/l}. \quad (2.30) \]

This is evident for the tensor products (A.1-3) and for the steps (A.4), (A.13-15). Concerning the splitting (A.7) note that the antisymmetrization (in\( \nu \leftrightarrow \mu \)) of an arbitrary splitting solution yields again a splitting solution. Due to proposition 1 (2.16), there exists a symmetrical normalization of\( B^{\nu}_{n/l} \) which fulfils

\[ B^{\nu}_{n/l} = \partial_{\mu} D^{\mu\nu}_{n/l}. \quad (2.31) \]

Moreover the normalizations can be chosen such that (2.27) propagates to higher orders

\[ T^\nu_{n/l, B} = T^\nu_{n/l} + \gamma B^\nu_{n/l}. \quad (2.33) \]

We conclude

\[ \partial_{\nu} T^\nu_{n/l, B} = \partial_{\nu} T^\nu_{n/l}. \quad (2.33) \]

Assuming\( T_n, T^\nu_{n/l} \) (\( l = 1, ..., n \)) to be gauge invariant (i.e. to fulfil (1.8)), there exists a symmetrical normalization of the distributions\( T^\nu_{n/l, B} \), such that\( T_n, T^\nu_{n/l, B} \) are gauge invariant, too. The modification (2.27) of the Q-vertex does not destroy gauge invariance in higher orders.

### 2.4 Higher Orders with Coboundary-Coupling

The results of this subsection are summarized in

**Proposition 4:** Choosing suitable symmetrical normalizations the following statements hold true for all\( F = A', R', R'^{\prime}, D, T, \tilde{T} \):

In orders\( 1 \leq n \leq 4 \) the\( F_n \)'s with coboundary-coupling have the representation

\[ F^7_{n, ... 79} = \frac{1}{\rho} \left( d_Q F^7_{n, ... 79} + d_Q F^7_{n, ... 70} + ... + d_Q F^7_{n, ... 0} \right) + \frac{i}{\rho} \sum_{i=r+1}^{n} \partial_{\nu} \left( F^7_{n, ... 010} + F^7_{n, ... 010} + ... + F^7_{n, ... 0} \right), \quad (2.34a) \]

\[ + \frac{i}{\rho} \sum_{i=r+1}^{n} \partial_{\nu} \left( F^7_{n, ... 010} + F^7_{n, ... 010} + ... + F^7_{n, ... 0} \right), \quad (2.34b) \]
and they are gauge invariant (2.14) in orders $1 \leq n \leq 3$

$$d_Q F^7_{\nu} \cdots F^{0\nu} = i \sum_{l=r+1}^{n} \partial_{\nu} \partial_{\mu} F^7_{\nu} \cdots F^{0\nu},$$  \hspace{1cm} (2.35)$$

where each $F^{\nu}_{\mu\cdots\mu}$ has $r$ upper indices 7 or 6, $1 \leq r \leq n$, and the upper index 1 is always at the $l$-th position.

These equations (2.34-35), gauge invariance (1.8) of $T^{0\nu}_{\nu}$ ($n \in \mathbb{N}$) and the second order identities

$$d_Q F^5_{\nu} = F^5_{\nu},$$  \hspace{1cm} (2.37)$$

$$d_Q F^7_{\nu} = i \partial_{\mu} F^5_{\nu},$$  \hspace{1cm} (2.38)$$

$$d_Q F^{10}_{\nu} = i \partial_{\mu} F^5_{\nu} - i \partial_{\mu} F^8_{\nu},$$  \hspace{1cm} (2.39)$$

they all can be fulfilled simultaneously.

Remarks: (1) Replacing $F^{\nu}_{\mu\cdots\mu}(x_1, x_2, \ldots, x_n)$ by $T^{\nu}_{\mu\cdots\mu}(x_1)$ and applying (1.11), (1.13), (2.7-9), (2.11) and (2.13), the equations (2.34-39) are obviously fulfilled - this is the intuition.

(2) Due to (2.19), similar equations with permuted upper indices hold true for $T^\nu_{\nu}, \tilde{T}^\nu_{\nu}$.

(3) Applying $d_Q$ to (2.34) we obtain

$$d_Q F^7_{\nu} \cdots F^{0\nu} = \frac{i}{r} \sum_{l=r+1}^{n} \partial_{\nu} \{d_Q F^7_{\nu} \cdots F^{0\nu} + \ldots + d_Q F^7_{\nu} \cdots F^{0\nu}\}. \hspace{1cm} (2.40)$$

However, this is not gauge invariance in the sense of Q-vertices (2.14). The latter is given by (2.35).

(4) By means of (2.34-35) the list (2.36-39) of second order identities, which are a kind of gauge invariance equations, can be extended

$$d_Q F^7_{\nu} = i \partial_{\mu} F^7_{\nu},$$  \hspace{1cm} (2.41)$$

$$d_Q F^7_{\nu} = 0,$$  \hspace{1cm} (2.42)$$

$$d_Q F^7_{\nu} = F^7_{\nu} - i \partial_{\mu} F^6_{\nu}. \hspace{1cm} (2.43)$$

$$\frac{1}{2}(d_Q F^7_{\nu} + d_Q F^7_{\nu}) = F^7_{\nu}.$$

\hspace{1cm} (2.44)$$

Proof of proposition 4: (A) Outline: The proof of (2.34-35) is by induction on the order $n$. However, we will see that the proof of (2.35) in order $n$ needs identities of the type (2.36), (2.38-39) in lower orders $k \leq n - 1$. But (2.39) cannot be proven by means of the general, elementary inductive methods of this section, it needs an explicit proof which uses the actual couplings (1.2-4), (1.7) and (2.5). This proof, which is given in appendix B, is similar to the proof of gauge invariance (1.8) of $T^0_{\nu}$. To prove an identity analogous to (2.39) in higher orders (see (2.50a) below), requires a huge amount of work (compare [2-5]), which is not done in this paper. Therefore, the inductive proof of gauge invariance (2.35) stops at $n = 3$. Moreover the proof of (2.34) in order $n$ needs (2.35) in lower orders $k \leq n - 1$. Consequently, the representation (2.34) of $F^7_{\nu} \cdots F^{0\nu}$ will be proven for $n \leq 4$ only.
(B) Proof of (2.34) by means of (2.34-35) in lower orders: We start with (A.2)

\[ A_n^{7\ldots0\ldots0}(x_1, \ldots; x_n) = \sum_{X,Y} \frac{s}{r} \tilde{T}_k^{7\ldots0\ldots0}(X)T_{n-k}^{7\ldots0\ldots0}(Y, x_n) + \]

\[ + \frac{r-s}{r} \tilde{T}_k^{7\ldots0\ldots0}(X)T_{n-k}^{7\ldots0\ldots0}(Y, x_n), \]  

where \( \tilde{T}_k^{7\ldots0\ldots0} \) (resp. \( T_{n-k}^{7\ldots0\ldots0} \)) has \( s \) (resp. \( r-s \)) upper indices 7. Next we insert the induction hypothesis (2.34) for \( \tilde{T}_k^{7\ldots0\ldots0} \) into (2.45a) (resp. (2.34) for \( T_{n-k}^{7\ldots0\ldots0} \) into (2.45b)). Then we apply (1.11) to the terms with a \( d_Q \)-operator and obtain

\[ \frac{s}{r} \tilde{T}_k^{7\ldots0\ldots0}(X)T_{n-k}^{7\ldots0\ldots0}(Y, x_n) = \frac{1}{r} [d_Q(\tilde{T}_k^{67\ldots0\ldots0}(X)T_{n-k}^{7\ldots0\ldots0}(Y, x_n)) + \]

\[ + \tilde{T}_k^{67\ldots0\ldots0}(X)d_QT_{n-k}^{7\ldots0\ldots0}(Y, x_n) + \ldots + \]

\[ + i \sum_{l=s+1}^{k} \{ (\partial^{4}_{\nu} \tilde{T}_k^{7\ldots0\ldots0\nu10\ldots0}(X))T_{n-k}^{7\ldots0\ldots0}(Y, x_n) + \ldots \}, \]

and similar for (2.45b). The next step is to insert the induction hypothesis (2.35) or gauge invariance (1.8) (the latter in the special case \( r-s=0 \)) into \( d_QT_{n-k}^{7\ldots0\ldots0}(Y, x_n) \) in (2.46b). Then we see that the \( A_n^{r\nu} \)-distributions fulfil (2.34): The terms of type (2.46a) add up to (2.34a); (2.46b) and (2.46c) can be combined and all terms of this type give together (2.34b). Similarly one proves that the \( R_{n}^{r\nu} \)- and, therefore, the \( D_{n}^{r\nu} \)-distributions satisfy (2.34).

We turn to the splitting (A.7). Let \( R_{n}^{67\ldots0\ldots0}, R_{n}^{67\ldots0\ldots0\nu}, R_{n}^{67\ldots0\ldots0\nu10\ldots0\nu}, \ldots \) be arbitrary splitting solutions of the corresponding \( D_{n}^{r\nu} \)-distributions. By means of the definition

\[ R_{n}^{7\ldots0\ldots0\nu} \stackrel{df}{=} \frac{1}{r} \{ d_QR_{n}^{67\ldots0\ldots0\nu} + d_QR_{n}^{76\ldots0\ldots0\nu} + \ldots \} + \]

\[ + \frac{i}{r} \sum_{l=r+1}^{n} \partial^{4}_{\nu} \{ R_{n}^{67\ldots0\ldots0\nu10\ldots0\nu} + R_{n}^{76\ldots0\ldots0\nu10\ldots0\nu} + \ldots \}, \]  

we obtain a splitting solution of \( D_{n}^{r\nu} \), analogously to (2.22), (2.26). Obviously (2.34) is maintained in the remaining steps - the construction of \( T_{n}^{r\nu}, T_{n}^{\tau\nu}, \) and \( \tilde{T}_{n}^{r\nu} \) (A.13-15).

(C) Proof of (2.35) by means of (2.34) in the same order \( n \), and by means of (2.34-35) and identities of the type (2.36), (2.38-39) in lower orders: One easily verifies (by inserting (2.35) and (1.8) in lower orders) that the \( A_{n}^{r\nu}, R_{n}^{r\nu}, R_{n}^{\tau\nu} \)-distributions fulfil (2.35). Therefore, as usual gauge invariance (2.35) can be violated in the distribution splitting only.

However, to prove that this violation can be avoided by choosing a suitable normalization, is a completely non-trivial business [1-5]. Moreover the normalization of \( T_{n}^{r\nu} \) is restricted by (2.34'). Therefore, we go another way to prove (2.35) for \( T_{n}^{r\nu}, T_{n}^{\tau\nu} \). We show that the r.h.s. of (2.40) agrees with the r.h.s. of (2.35), if a suitable symmetrical normalization of \( T_{n}^{r\nu} \), \( 1 \leq r \leq n-1 \), is chosen. (The case \( r=n \) is trivial.) For this purpose we consider

\[ A_{n}^{7\ldots0\ldots0\nu10\ldots0\nu} \stackrel{df}{=} \frac{1}{r} \{ d_QA_{n}^{67\ldots0\ldots0\nu10\ldots0\nu} + \ldots + d_QA_{n}^{76\ldots0\ldots0\nu10\ldots0\nu} \}, \]
where the upper index 1 is always at the l-th position. We insert the definition (A.2) of the $A'_n$-distributions. Similarly to (2.25) we then apply (1.11) and the induction hypothesis, i.e. we insert (2.7-8), (2.11) and (2.13) if $n = 2$, and additionally (2.36), (2.38-39), (2.41-44) if $n = 3$. In this way we obtain

\begin{equation}
(2.47) = \frac{i}{r} \left\{ \sum_{j=r+1}^{n} \left[ \pm \partial_{\mu} A_{n}^{7 \ldots 0 \ldots 0} + \cdots + \partial_{\mu} A_{n}^{7 \ldots 6 \ldots 760 \ldots 0} \right] \right\},
\end{equation}

(2.48a)

\begin{equation}
+ \partial_{\mu} A_{n}^{6 \ldots 7 \ldots 0 \ldots 0} + \cdots + \partial_{\mu} A_{n}^{7 \ldots 6 \ldots 760 \ldots 0} \right\}. \tag{2.48b}
\end{equation}

In (2.48a) the two upper indices 1 are at the j-th and l-th position, and we have a plus (resp. a minus) if $j < l$ (resp. $j > l$). One proves (2.47)=(2.48) for the $R'_n$, $R''_n$-distributions in a similar way.

Analogously to (2.30), the antisymmetry $T^{5 \mu}_{4 \nu} = -T^{5 \mu}_{4 \nu}$ (2.5) can be preserved in the inductive construction of the $T_n$’s. Starting with arbitrary splitting solutions $R_{n}^{6 \ldots 0 \ldots 0} = -R_{n}^{6 \ldots 0 \ldots 0}$, etc., we may (similar to (2.34)) define $R_{n}^{7 \ldots 0 \ldots 0}$ by the equation (2.47)=(2.48) (with $A'_n$, everywhere replaced by $R'_n$). This equation is not destroyed in the construction of $T_n$, $T_n$, and $T_n$. Summing up we have proven

\begin{equation}
\begin{aligned}
F_n^{7 \ldots 0 \ldots 0} &= \frac{1}{r} \{ dQ F_n^{6 \ldots 7 \ldots 0 \ldots 0} + \cdots + dQ F_n^{7 \ldots 6 \ldots 760 \ldots 0} \} + \\
&= \frac{i}{r} \left\{ \sum_{j=r+1}^{n} \left[ \pm \partial_{\mu} F_n^{7 \ldots 0 \ldots 0} + \cdots + \partial_{\mu} F_n^{7 \ldots 6 \ldots 760 \ldots 0} \right] \right\} + \\
&+ \partial_{\mu} F_n^{6 \ldots 7 \ldots 0 \ldots 0} + \cdots + \partial_{\mu} F_n^{7 \ldots 6 \ldots 760 \ldots 0} \right\} \tag{2.49}
\end{aligned}
\end{equation}

for all $F = A', R', R''$, $D, A, R, T', T, T$ and for $n \leq 3$, $1 \leq r \leq n - 1$. We insert this equation into

\begin{equation}
\sum_{l=r+1}^{n} \partial_{\nu} \left\{ F_n^{7 \ldots 0 \ldots 0} \right\} - \frac{i}{r} \{ dQ F_n^{6 \ldots 7 \ldots 0 \ldots 0} + \cdots + dQ F_n^{7 \ldots 6 \ldots 760 \ldots 0} \} \right\} \tag{2.50}
\end{equation}

for $F = T, T$. Taking the different signs of the $(j, l)$- and the $(l, j)$-term in $\sum_{j,l} (j \neq l) \pm \partial_{j} \partial_{l}$ $F_n^{0 \ldots 0}$ and $F_n^{0 \ldots \nu}$ into account, we see that (2.50) vanishes. This is the desired result.

Proof of (2.36-39): The first identity (2.36) is the case $n = 2$, $r = 1$ of (2.49). All these equations (2.36-39) are easily verified for the $A'_n$-distributions etc. and, therefore, can be violated in the splitting only. The latter is no problem for (2.37), since we may define $R^{5 \nu}_{2} \equiv dQ R^{5 \nu}_{2}$ for an arbitrary splitting solution $R^{5 \nu}_{2}$. Applying $dQ$ to (2.36), we obtain (2.38) by means of (2.37). It remains (2.39), which is proven in appendix (B) by explicitly inserting the actual couplings. It turns out that there exists a normalization of $T^{3 \nu}_{2}(x_1, x_2) = T^{3 \nu}_{2}(x_2, x_1)$ such that (2.39) and gauge invariance (1.8) (in second order) are satisfied simultaneously. One easily verifies that this is the only problem of compatibility in (2.34-39) and (1.8). For example in second order the distributions $T^{5 \nu}_{2} = -T^{5 \nu}_{2}$, $T^{61}_{2}$, $T^{62}_{2}$, $T^{57}_{2}$ can be normalized in an arbitrary symmetrical way. Then the normalizations of $T^{17}_{2}$, $T^{57}_{2}$, $T^{70}_{2}$, $T^{77}_{2}$ are uniquely fixed by (2.36-37), (2.43-44), and all
identities (2.36-38) and (2.41-44) are fulfilled. The remaining distributions $T_{2}^{00}$, $T_{2}^{10}$, $T_{2}^{50}$ and $T_{2}^{11}$ appear in (1.8) and (2.39) only.

If the identities $(F = T, T)$

$$d_{Q}F_{n}^{5\ldots51\ldots10\ldots0} = i \sum_{j=t+1}^{t+s} (-1)^{(j-t-1)} \partial^{j} F_{n}^{5\ldots51\ldots151\ldots10\ldots0} +$$

$$+ i(-1)^{s} \sum_{j=t+s+1}^{n} \partial^{j} F_{n}^{5\ldots51\ldots10\ldots010\ldots0}, \quad n \in \mathbb{N}, \quad 0 \leq t, s \leq n, \quad t + s \leq n \quad (2.50a)$$

hold true (where $F_{n}^{5\ldots51\ldots10\ldots0}$ on the l.h.s. has $t$ indices 5, $s$ indices 1 and all derivatives on the r.h.s. are divergences, the Lorentz indices are omitted), one can prove the representation (2.34) and gauge invariance (2.35) in all orders. This is shown in appendix C by a generalization of this proof here. Unfortunately an inductive proof of (2.50a) by means of the simple technique of this section fails because of the splitting (A.7) - there is no term in (2.50a) which has neither a $d_{Q}$-operator nor a derivative. We emphasize that the identities (2.50a) do not depend on the explicit form (1.14-15) of the coboundary coupling (no upper indices 6 or 7 appear in (2.50a)). These identities concern solely the starting-coupling $T_{1}^{0}$, its Q-vertex $T_{1}^{1}$ and its Q-Q-vertex $T_{5}^{5}$.

Remark: The compatibility of (2.39) and gauge invariance (1.8) in second order is remarkable in the tree sector: Each of this two identities fixes the normalization of $T_{2}^{10}_{\text{tree}}$ uniquely and these two normalizations agree in fact (see appendix B and sect.3.2 of [5]). This is a further hint that our gauge invariance (1.8) relies on a deeper (cohomological?) structure. The knowledge of the latter would presumably shorten the proof of (1.8) and would be an excellent tool to prove the missing identities (2.50a).

2.5 Generality of the Results

In the preceeding subsections 2.2 and 2.4 the explicit structures of the starting-theory $T_{1}^{0}$ (1.2), of the corresponding Q-vertex $T_{1}^{1\nu}$ (1.7), of the divergence-coupling (1.16-17) and of the coboundary-coupling (1.14-15) have not been needed. We have solely used the following properties:

(i) The starting-theory $T_{1}^{0}$ is gauge invariant with respect to the Q-vertex $T_{1}^{1\nu}$ in all orders which are considered.

(ii) There exists a Q-Q-vertex $T_{1}^{5\nu \mu}(x)$ which fulfis

$$T_{1}^{5\nu \mu} = -T_{1}^{5\mu \nu}, \quad d_{Q}T_{1}^{5\nu \mu} = 0 \quad \text{and} \quad d_{Q}T_{1}^{1\nu}(x) = i \partial_{\mu}T_{1}^{5\nu \mu}(x). \quad (2.51)$$

(iii) The second order identity (2.39) holds true and is compatible with gauge invariance (1.8) of $T_{2}^{50}$.

Only (i) is needed in sect.2.2. Therefore, the results about the divergence-coupling apply to any gauge invariant quantum field theory, e.g. to quantum gravity [14]. This holds also true for (2.34) in second order, i.e. (2.43-44).

If additionally (ii) is fulfilled ($d_{Q}T_{1}^{5\nu \mu} = 0$ is not needed for the following statement), gauge invariance (2.35) is proven in second order (i.e. (2.41-42) are valid), and this implies the identities (2.34) up to third order. Note that the modified Q-vertex $T_{1/1 B}^{\nu}$ (2.27) satisfies (ii), too,

$$d_{Q}T_{1/1 B}^{\nu} = i \partial_{\mu}T_{1 B}^{5\nu \mu} \quad (2.52)$$
with

\[ T_{1B}^{5\nu\mu} \overset{\text{def}}{=} T_1^{5\nu\mu} - i\gamma d_Q D^\nu{}^\mu = -T_{1B}^{5\mu\nu}, \quad d_Q T_{1B}^{5\nu\mu} = 0. \]  

(2.53)

For a model which satisfies (i), (ii) and all identities (2.50a) (2.39) is a special case of the latter) also the statements (2.34-35) about the coboundary-coupling are proven in all orders.

2.6 n-Point Distributions with Divergence- and Coboundary-Coupling

The general case (2.10) of \( T_n \) containing the ordinary Yang-Mills coupling \( T_1^0 \), the divergence- and the coboundary-coupling can easily be traced back to the results of the preceeding sections 2.2 and 2.4-5. We replace \( T_1^0 \) by

\[ T_1^0 \overset{\text{def}}{=} T_1^0 + T_2^0 = T_1^0 + \beta_2 \partial_\nu K_2^\nu \]  

(2.54)

and \( T_1^{1\nu} \) by

\[ \bar{T}_1^{1\nu} \overset{\text{def}}{=} T_1^{1\nu} + T_2^{1\nu} = T_1^{1\nu} - i\beta_2 d_Q K_2. \]  

(2.55)

Due to corollary 3, the \( \bar{T}_1^0 \)-theory is gauge invariant with respect to the Q-vertex \( \bar{T}_1^{1\nu} \) in all orders, i.e. property (i) of subsect.2.5 is fulfilled. Obviously property (ii) holds also true with the old \( T_1^0 \)-vertex (2.5): \( d_Q \bar{T}_1^{1\nu} = i\partial_\mu T_1^{5\mu\nu} \). It would be very suprising if (2.39) would be wrong for the \( (T_1^0, T_1^{1\nu}, T_1^{5\nu\mu}) \)-couplings. By means of proposition 4 we conclude that the general n-point distributions (2.10) (with coboundary- and divergence-coupling) are gauge invariant in second and most probably third order, and we obtain the representation (2.34) with respect to the coboundary-vertices up to third (resp. fourth) order.

Let us describe an alternative way. We replace \( T_1^0 \) by

\[ \bar{T}_1^0 \overset{\text{def}}{=} T_1^0 + T_2^0 = T_1^0 + \beta_1 d_Q K_1. \]  

(2.56)

The Q-vertex (1.7) needs no change: \( d_Q \bar{T}_1^0 = i\partial_\nu T_1^{1\nu} \). Proposition 4 (2.35) tells us that the \( \bar{T}_1^0 \)-theory is gauge invariant up to third order. Applying corollary 3 we obtain gauge invariance (2.14) of the general \( T_n \)'s (2.10) up to third order. Moreover, due to proposition 1, these distributions are divergences with respect to their divergence-vertices in any order.

2.7 Gauge Invariant Normalization of Second Order Tree Diagrams

We only consider the tree sector and start with the following normalization of \( T_2(x_1, x_2) \)

(2.10) \( T_2 \overset{\text{def}}{=} T_2^{10} + T_2^{20} + T_2^{02} + T_2^{22} + T_2^{50} + T_2^{70} + T_2^{77} + T_2^{27} + T_2^{72} + T_2^{22} \): The C-number distributions of \( T_{20}\)tree (the lower index 0 indicates this special normalization) are

\[ t_\mathcal{O}(x_1 - x_2) \sim D^F(x_1 - x_2), \quad \partial^\mu D^F(x_1 - x_2), \quad \partial^\mu \partial^\nu D^F(x_1 - x_2), \]  

(2.57)

they have no local terms. The singular order \( \omega \) of \( t_\mathcal{O} \) (resp. the number of derivatives on \( D^F \) in (2.57)) can be computed from the combination \( \mathcal{O} \) of the four external free field operators (see \( \omega(\mathcal{O}) \) in (A.17)) and is \( \omega(\mathcal{O}) = -2, -1, 0 \). For each four-legs combination \( \mathcal{O} \) with \( \omega(\mathcal{O}) = 0 \) we may add a local term

\[ N_\mathcal{O}(x_1 - x_2) = C_\mathcal{O} \delta(x_1 - x_2) : \mathcal{O}(x_1 - x_2) : \]  

(2.58)
to $T_{20}$, where $C_{Q}$ is a free normalization constant (A.12). Gauge invariance (2.14) fixes the values of $C_{Q}$ uniquely [1,5,13]. In $T_{2}^{00}$ the normalization term

$$N_{AAAA}(x_{1} - x_{2}) = -\frac{1}{2}g^{2}f_{abcd}f_{cdrs}\delta(x_{1} - x_{2}): A_{\mu a}A_{\nu b}A_{\rho c}^{\nu}A_{\sigma d}^{\sigma}:$$  

is required [1,5]. This is the four-gluon interaction, which propagates to higher orders in the inductive construction of the $T_{n}$'s (sect.4.2 of [15]). The normalization terms (2.58) of $T_{2}^{20}$, ..., $T_{2}^{72}$ which are needed for gauge invariance (2.14) can quickly be calculated by using our results. We have proven that $T_{2}^{20}$, $T_{2}^{22}$, $T_{2}^{70}$, $T_{2}^{77}$ and $T_{2}^{72}$ are gauge invariant with the normalizations given by proposition 1 (2.16), r.s. proposition 4 (2.34). (In the case of $T_{2}^{72}$ we do the replacement (2.54-55) (or alternatively (2.56)) before applying (2.34) (r.s. (2.16))). Therefore, we simply have to pick out the local terms in $\partial^{4}_{\mu}T_{40}^{00} (= T_{2}^{20})$, $\partial^{3}_{\mu}\partial^{2}_{\nu}T_{27}^{44\mu\nu}(= T_{2}^{22})$, $d_{Q}T_{60}^{20} + i\partial_{T_{2}^{20}}^{27\nu}(= T_{2}^{70})$ and in $\frac{1}{2}(d_{Q}T_{60}^{27} + d_{Q}T_{60}^{27})(= T_{2}^{77})$. In the tree sector there are no local terms in $T_{2}^{20}$, $T_{2}^{44}$, $T_{2}^{60}$, $T_{2}^{61}$, $T_{2}^{67}$ (their normalization is unique) and, therefore, neither in $d_{Q}T_{2}^{60}$, $d_{Q}T_{2}^{67}$. All local terms are generated by the divergences in $\partial^{4}_{\mu}T_{40}^{00}$, $\partial^{3}_{\mu}\partial^{2}_{\nu}T_{27}^{44\mu\nu}$ or $i\partial_{T_{2}^{20}}^{27\nu}$, due to $DF(x_{1} - x_{2}) = \delta(x_{1} - x_{2})$. It turns out that all these local terms are four-ghost interactions, which add up to

$$N_{uu\bar{u}\bar{d}}(x_{1} - x_{2}) = -ig^{2}(\frac{(\beta_{2})^{2}}{2} + \beta_{1} - 2\beta_{1}\beta_{2})f_{abcd}f_{cdrs}\delta(x_{1} - x_{2}): u_{a}u_{b}\bar{u}_{c}\bar{u}_{d}:$$  

in agreement with the much longer calculation in [13].

Remarks: (1) The powers of $\beta_{1}$, $\beta_{2}$ in (2.60) tell us the origin of the corresponding term. For example the term $\sim \beta_{1}\beta_{2}$ comes from $T_{2}^{27} + T_{2}^{72}$.

(2) We have seen that on the tree sector the normalizations of $T_{2}^{20}$, ..., $T_{2}^{72}$ are uniquely fixed by (2.16) or (2.34). However, this does not imply that gauge invariance fixes the normalization of $T_{20}^{20}|_{\text{tree}}, ..., T_{2}^{72}|_{\text{tree}}$ uniquely. The latter statement is a by-product of the calculation in [13].

(3) In agreement with our observations in first order (see remark (1) in sect.1.2), there is no ambiguity in the four-gluon interaction (2.59) - it is independent on $\beta_{1}$, $\beta_{2}$.

(4) The most general coupling which is gauge invariant (2.14) in all orders (this is not proven completely for the coboundary-coupling) has been given. It can be compared with the most general Lagrangian (written in terms of interacting fields) which is invariant under the full BRS-transformations of the interacting fields - see formula (3.13) of [10]. For this purpose we must choose the Feynman gauge $\lambda = 1$ in this formula. Then one easily verifies that the terms $\sim g$ and $\sim g^{2}$ in the interaction part of this Lagrangian agree with $(T_{1}^{0} + \beta_{1}d_{Q}K_{1} + \beta_{2}\partial_{\mu}K_{1}^{\mu}) \sim g$ and with $N_{AAAA}$, $N_{uu\bar{u}\bar{d}} \sim g^{2}$, if we set $\beta_{2} = 2\beta_{1}$ and identify the free parameter $\alpha$ of [10] with $\beta_{2} = 2\beta_{1}$. There is only one parametric freedom in [10] which is given by adding to the Lagrangian $\alpha s(...)$.

The latter is a coboundary with respect to the BRS-operator $s$. In doing so the Lagrangian remains $s$-invariant, due to the nilpotency of $s$. This seems to be analogous to our coboundary-coupling $\beta_{1}d_{Q}K_{1}$ (1.14). However, we see from $\alpha = 2\beta_{1} = \beta_{2}$ that there is not a complete correspondence - a change of $\alpha$ means also the addition of a divergence $\beta_{2}\partial_{\mu}K_{2}^{\mu}$ (1.16). Since in our framework the interaction is switched off by $g \in \mathcal{S}(\mathbb{R}^{4})$, our gauge invariance is not $[Q, T_{n}] = 0$ but $[Q, T_{n}] = (\text{divergences})$, and, therefore, we have the freedom of adding a divergence-coupling (1.16) to $T_{1}$. This explains the fact that we have a two-parametric freedom and not only a one-parametric one.

(5) We call a normalization term $N_{Q}$ (2.58) 'natural', if there is a corresponding non-vanishing non-local term, more precisely if $T_{20}|_{\text{tree}}$ (2.57) contains a non-vanishing C-number.
distribution \( t_\Omega \) (with the same \( \Omega \)). \( N_{AAAA} \) (2.59) is of this kind. It can be generated by replacing

\[
\partial^\mu \partial^\nu D^F(x_1 - x_2) \quad \text{by} \quad [\partial^\mu \partial^\nu D^F(x_1 - x_2) - \frac{1}{2} g^\mu\nu \delta(x_1 - x_2)] \quad (2.61)
\]

in \( t_{AAAA} \) [1,5]. The other normalization terms are called 'unnatural', since they do not naturally arise in the inductive construction of the \( T_n \)'s - the numerical distribution \( d_\Omega = 0 \) is splitted in \( d_\Omega(x_1 - x_2) = \delta^{(4)}(x_1 - x_2) - \delta^{(4)}(x_1 - x_2) \). \( N_{uu\tilde{u}\tilde{u}} \) is unnatural, because in the corresponding diagram \( \partial_\mu A^\mu_u(x_1) \) and \( \partial_\nu A^\nu_{\tilde{u}}(x_2) \) are contracted, which gives \(-i \delta_{ab} g^{\mu\nu} \partial_\mu \partial_\nu D_0^b(x_1 - x_2) = 0 \). \( (D_0^b(x_1 - x_2) \) is the positive frequency part of the massless Pauli-Jordan distribution.) Note that the proof of gauge invariance (1.8) in higher orders \( n \geq 3 \) [2-5] uses normalizations which could be unnatural in an analogous sense.

### 2.8 Non-Uniqueness of Quantized Yang-Mills Theories

To simplify the discussion we assume (2.34) and (2.35) to hold true in any order. Then the ambiguities of quantized Yang-Mills theories, which are given by the free choice of the parameters \( \beta_1, \beta_2 \in \mathbb{R} \) (1.14), (1.16), are not restricted by gauge invariance in higher orders, due to corollary 3 and (2.35). The freedom is reduced to a one-parametric set, if we admit only natural normalization terms for second order tree diagrams

\[
N_{uu\tilde{u}\tilde{u}} = 0 \iff \beta_1 = \frac{(\beta_2)^2}{4\beta_2 - 2}, \quad \beta_2 \neq \frac{1}{2}. \quad (2.62)
\]

This prescription agrees partially with the Faddeev-Popov procedure: The exponentiation of a determinant can generate only terms quadratic in the ghosts. Therefore, the Faddeev-Popov method cannot yield a four-ghost interaction.

There is a more technical criterion which gives another restriction of the ambiguities and roughly speaking requires that the cancellations in the gauge invariance equation (2.14) are simple. To be more precise let us consider this equation for second order tree diagrams. In the natural operator decomposition [5] the terms \( \sim \partial^\mu \delta(x_1 - x_2) \) cancel completely iff

\[
\beta_2 = 0. \quad (2.63)
\]

(For \( \beta_2 \neq 0 \) the terms \( \sim \partial \delta : \Omega \) must be combined with terms \( \sim \delta : \Omega' \); where the difference of the two operator combinations \( \Omega' \) and \( \Omega \) is that \( \Omega' \) has one derivative more.) Let us assume that one can prove C-number identities (called 'Cg-identities' [2-5]) which express gauge invariance (2.14). Then the transition from the natural operator decomposition of (2.14) to the Cg-operator decomposition (i.e. the operator, in which the Cg-identities hold true) is more complicated for \( \beta_2 \neq 0 \) than for \( \beta_1 = 0 = \beta_2 [5] \). We see from (2.62-63) that the theory with \( \beta_1 = 0 = \beta_2 \) is the most simple one. However, this does not exclude the other values of \( \beta_1, \beta_2 \), since we can construct a Lorentz-, SU(N)- and P-, T-, C-invariant, (re)normalizable, gauge invariant and pseudo-unitary S-matrix for any choice of \( \beta_1, \beta_2 \in \mathbb{R} \).

We turn to the physical consequences of the freedom in the choice of \( \beta_1, \beta_2 \). For this purpose we consider \( PT_n(x_1, ..., x_n)P \), where \( T_n \) is given by (2.10) and \( P \) is the projector on the physical subspace [4]. By means of \( d_Q A^\mu_u = i \partial^\mu u_a, \quad d_Q u_a = 0, \quad d_Q \tilde{u}_a = -i \partial_\nu A^\nu_{\tilde{u}} \) and the fact that \( \partial^\mu u_a \) and \( \partial_\nu A^\nu_{\tilde{u}} \) are unphysical fields, we conclude

\[
P d_Q F_n(x_1, ..., x_n)P = 0, \quad (2.64)
\]
where \( F = A', R', R'', D, A, R, T', T, \tilde{T} \). Together with propositions 1 and 4 (2.16), (2.34) we obtain
\[
PT_n(x_1, \ldots, x_n)P = T_n^{0 \ldots 0}(x_1, \ldots, x_n) + \text{(sum of divergences)}.
\] (2.65)

On the r.h.s. the dependence on \( \beta_1, \beta_2 \) is exclusively in the divergences. But the infrared behavior of Yang-Mills theories is not under control. Therefore, we cannot conclude that the divergences in (2.65) vanish in the adiabatic limit \( g \to 1 \).

Appendix A: Inductive Construction of the \( T_n \)'s according to Epstein and Glaser

The input of the inductive construction of the \( T_n \)'s (1.1) are the \( T_1 \)'s (e.g. (1.2-4), (1.7), (2.2-7)) in terms of free fields. The couplings \( T_1 \) are roughly speaking given by the interaction Lagrangian densities. Let us summarize the inductive step as a recipe. For the derivation of this construction from causality and translation invariance (only these two requirements are needed) we refer the reader to [6,7]. In analogy to (1.1) we denote the \( n \)-point distributions of the inverse S-matrix \( S(g_0, \ldots, g_l)^{-1} \) by \( \tilde{T}_n(x_1, \ldots, x_n) \). Having constructed all \( T_n, \tilde{T}_n \) in lower orders \( k \leq n - 1 \), we can define the operator-valued distributions \( R_n', A_n', R_n'' \), which are sums of tensor products,
\[
R_n'(x_1, \ldots; x_n) \overset{\text{def}}{=} \sum_{X,Y} T_n-k(Y, x_n)\tilde{T}_k(X),
\] (A.1)
\[
A_n'(x_1, \ldots; x_n) \overset{\text{def}}{=} \sum_{X,Y} \tilde{T}_k(X)T_n-k(Y, x_n),
\] (A.2)
\[
R_n''(x_1, \ldots; x_n) \overset{\text{def}}{=} \sum_{X,Y} T_k(X)\tilde{T}_n-k(Y, x_n),
\] (A.3)
where \( X \overset{\text{def}}{=} \{x_{i_1}, \ldots, x_{i_k}\} \), \( Y \overset{\text{def}}{=} \{x_{i_{k+1}}, \ldots, x_{i_{n-1}}\} \), \( X \cup Y = \{x_1, \ldots, x_{n-1}\} \) and the sum is over all partitions of this kind with \( 1 \leq k \equiv |X| \leq n - 1 \). In order to simplify the notations, the Lorentz indices and the upper indices \( i_s \) denoting the kind of vertex \( T_1^{i_s}(x) \) (see e.g. (2.1-7)) are omitted. This gives no confusion since \( i_s \) is strictly coupled to the time-space argument \( x_s \). One can prove that \( D_n \overset{\text{def}}{=} R_n' - A_n' \)
\] (A.4)
has causal support
\[
\text{supp } D_n(x_1, \ldots; x_n) \subset (\Gamma^+_{n-1}(x_n) \cup \Gamma^-_{n-1}(x_n)),
\] (A.5)
where
\[
\Gamma^\pm_{n-1}(x_n) \overset{\text{def}}{=} \{(x_1, \ldots, x_n) \in \mathbb{R}^{4n} | x_j \in x_n + \vec{V}^\pm, \forall j = 1, \ldots, n - 1\}.
\] (A.6)
The crucial step in the inductive construction is the correct distribution splitting of \( D_n \)
\[
D_n = R_n - A_n,
\] (A.7)
with
\[ \text{supp } R_n(x_1;...;x_n) \subset \Gamma_{n-1}^+(x_n) \quad \text{and} \quad \text{supp } A_n(x_1;...;x_n) \subset \Gamma_{n-1}^-(x_n). \]  
(A.8)

For this purpose we expand the operator-valued distributions in normally ordered form
\[ F_n(x_1,...,x_n) = \sum_{\mathcal{O}} f_{\mathcal{O}}(x_1 - x_n, ..., x_{n-1} - x_n) : \mathcal{O}(x_1,...,x_n) :, \]  
(A.9)

where \( F = R', A', D, R, A, T, \hat{T} \) and \( \mathcal{O}(x_1,...,x_n) \) is a combination of the free field operators. The coefficients \( f_{\mathcal{O}} \) are C-number distributions. Due to translation invariance, they depend on the relative coordinates only and, therefore, are responsible for the support properties. Consequently, the splitting must be done in these C-number distributions. Obviously, the critical point for the splitting is the UV-point
\[ \Gamma_{n-1}^+(x_n) \cap \Gamma_{n-1}^-(x_n) = \{(x_1,...,x_n) \in \mathbb{R}^{4n} | x_1 = x_2 = ... = x_n\}. \]  
(A.10)

In order to measure the behavior of the C-number distribution \( f \) in the vicinity of this point, one defines an index \( \omega(f) \), which is called the singular order of \( f \) at \( x = 0 \) [6,7]. We will need the following example: Let \( D^a, a \overset{\text{def}}{=} (a_1, ..., a_m) \), be a partial differential operator. Then
\[ \omega(D^a \delta^{(m)}(x_1,...,x_m)) = |a| = a_1 + ... + a_m. \]  
(A.11)

If \( \omega(d_{\mathcal{O}}) < 0 \), the splitting of \( d_{\mathcal{O}} \) is trivial and uniquely given by multiplication with a step function [6,7].

If \( \omega(d_{\mathcal{O}}) \geq 0 \), one must do the splitting more carefully [6,7]. Moreover it is not unique. One has an undetermined polynomial which is of degree \( \omega(d_{\mathcal{O}}) \) (the degree cannot be higher since renormalizability requires \( \omega(r_{\mathcal{O}}) = \omega(d_{\mathcal{O}}) \)),
\[ r_{\mathcal{O}}(x_1 - x_n, ..., x_{n-1} - x_n) = r_{\mathcal{O}}^0(\ldots) + \sum_{|a|=0}^{\omega(d_{\mathcal{O}})} C_a D^a \delta^{(4(n-1))}(x_1 - x_n, ..., x_{n-1} - x_n), \]  
(A.12)

where \( r_{\mathcal{O}}^0 \) is a special splitting solution and \( C_a \) are the undetermined normalization constants. If one does the splitting also in this case by multiplying with a step function, one obtains the usual, UV-divergent Feynman rules. But this procedure is mathematically inconsistent. The correct distribution splitting saves us from UV-divergences.

From \( R_n \) one constructs
\[ T'_n \overset{\text{def}}{=} R_n - R'_n \]  
(A.13)

and \( T_n \) is obtained by symmetrization of \( T'_n \)
\[ T_{n \pi_n \ldots \pi_1}(x_1,...,x_n) = \sum_{\pi \in S_n} \frac{1}{n!} T'_n(x_{\pi_1},...,x_{\pi_n}). \]  
(A.14)

In order to finish the inductive step we must construct
\[ \hat{T}_n \overset{\text{def}}{=} -T_n - R'_n - R''_n. \]  
(A.15)
One can prove that (A.14-15) are the correct n-point distributions of $S(g_0, \ldots, g_l)$ (1.1) resp. $S(g_0, \ldots, g_l)^{-1}$, fulfilling the requirements of causality and translation invariance. Note

$$\omega \equiv \omega(t_\mathcal{O}) = \omega(r_\mathcal{O}) = \omega(d_\mathcal{O}).$$

(A.16)

The undetermined local terms (A.12) go over from $r_\mathcal{O}$ to $t_\mathcal{O}$. The normalization constants $C_a$ are restricted by Lorentz- and SU(N)-invariance, the permutation symmetry (2.19), discrete symmetries, pseudo-unitarity and gauge invariance (compare with sect.1.2). The latter restriction plays an important role in this paper.

In our Yang-Mills model one can prove by means of scaling properties [7]

$$\omega \leq \omega(\mathcal{O}) \overset{\text{def}}{=} 4 - b - g - d,$$

(A.17)

where $b$ is the number of gauge bosons $(A, F)$, $g$ the number of ghosts $(u, \tilde{u})$ and $d$ the number of derivatives $(F, \partial u, \ldots)$ in $\mathcal{O}$. The proof of (A.17) in [2] is written for $T_0^{\text{tree}}$, and $T_1^{\text{tree}}$ have mass-dimension 3 instead of 4. Therefore, there exists a lower upper bound $\omega(\mathcal{O})$ for the singular order $\omega$ of diagrams with at least one vertex $T_1^{\text{tree}}$ or $T_1^{\text{tree}}$. $\omega \leq \omega(\mathcal{O}) < \omega(\mathcal{O}) = 4 - b - g - d$. The fact that $\omega$ is bounded in the order $n$ of the perturbation series (here it is even independent on $n$), is the (re)normalizability of the model.

Appendix B: Proof of (2.39)

Since (2.39) is a gauge invariance equation, it can be violated only in the splitting (A.7) and solely by local terms. No vacuum diagrams appear in (2.39).

B.1 Tree Diagrams

We work with the technique of [1]. The splitting $D_2 \overset{\text{tree}}{\rightarrow} R_{20}^{\text{tree}}$ is done by replacing everywhere $D_0(x_1 - x_2)$ (which is the mass zero Pauli-Jordan distribution) by its retarded part $D_0^{\text{ret}}(x_1 - x_2)$. As in (2.57) the lower index 0 in $R_{20}^{\text{tree}}$ and in $T_2^{\text{ret}} = R_{20}^{\text{tree}} - R_{20}^{\text{tree}}$ (A.13-14) indicates this special normalization in the tree sector. Note $\square D_0^{\text{ret}} = \delta^{(4)}$, in contrast to $\Box D_0 = 0$. This is the reason for the appearance of local terms $A^\nu$ which destroy (2.39)

$$d_4 R_{20}^{\text{tree}} \overset{\text{tree}}{\rightarrow} i\partial_\mu R_{20}^{\text{tree}} - i\partial_\mu R_{20}^{\text{tree}} - A^\nu.$$  

(B.1)

Picking out all local terms - they all are generated in the divergences on the r.h.s. due to $\square D_0^{\text{ret}} = \delta^{(4)}$ - one finds

$$A^\nu(x_1, x_2) = -g^2 f_{a b r} f_{c d t} \left\{ \frac{1}{2} \delta(x_1 - x_2) : u_a u_b A_{\mu c} F_{d}^{\mu \nu} : + \right\}$$

$$+ \frac{1}{2} \partial^\mu \delta(x_1 - x_2) : u_a(x_1) u_b(x_1) A_{\mu c}(x_2) A_d^\nu(x_2) : +$$

$$+ [\tilde{g}^\mu \partial^\nu \delta(x_1 - x_2) - g^\mu \partial^\nu \delta(x_1 - x_2)] : A_{\tau a}(x_1) u_b(x_1) A_{\mu c}(x_2) u_d(x_2) : \right\} =$$
Due to (B.1-2), these verifies that (2.39) fixes the normalization of (Note that a term \( \sim \) is uniquely determined by gauge invariance (1.8) in second order (see sect.3.2 of [5]) the operator part in \( a, b, c \) with

\[
T_{10} \quad \text{and} \quad T_{10} \quad \text{appears also in (B.8), we have some information about antisymmetric in } \nu \text{ the } C\text{-number identities expressing (B.8)} \quad \text{[2]}, \text{ namely}
\]

\[
F, f \quad \text{for (B.7) and (B.8)) agree exactly.}
\]

On the other hand the normalization in the tree sector of \( T_{10}^{20}(x_1, x_2) = T_{10}^{11}(x_1, x_2) = -T_{10}^{11}(x_2, x_1) \) are preserved in the finite renormalizations

\[
T_{10}^{20} \overset{\text{def}}{=} T_{10}^{11} - B^{\nu\mu},
\]

\[
T_{10}^{11} \overset{\text{def}}{=} T_{10}^{20} + B^{\nu\mu}.
\]

and

\[
T_{10}^{10} \overset{\text{def}}{=} T_{10}^{20} + N^{\nu}.
\]

Due to (B.1-2), these \( T_{10}^{20} \)-distributions (B.5-7) satisfy (2.39) on tree level, and one easily verifies that (2.39) fixes the normalization of \( T_{10}^{20} \text{tree} \) uniquely.

On the other hand the normalization in the tree sector of \( T_{10}^{20}(x_1, x_2) = T_{10}^{11}(x_2, x_1) \) is uniquely determined by gauge invariance (1.8) in second order (see sect.3.2 of [5])

\[
d_{Q}T_{20}^{00} = i\partial_{\nu}^{2}T_{20}^{10} + i\partial_{\nu}^{2}T_{20}^{01},
\]

where \( T_{20}^{00} \text{tree} \) is normalized by (2.59) (four-gluon interaction). These two normalizations of \( T_{20}^{10} \text{tree} \) ((B.7) and (B.8)) agree exactly.

**B.2 Two-Legs Diagrams**

We denote the numerical two-legs distributions in the following way

\[
F_{2}^{10}(x_1, x_2)|_{2\text{-legs}} = f_{10}(x_1, x_2) : u_{a}(x_1)A_{\mu}(x_2) : + \quad f_{20}(x_1, x_2) : A_{\mu}(x_1)u_{a}(x_2) : + \quad \text{... : } uF : + \quad \text{... : } Fu : ,
\]

\[
F_{2}^{50}(x_1, x_2)|_{2\text{-legs}} = f_{50}^{u}(x_1, x_2) : u_{a}(x_1)u_{a}(x_2) : ;
\]

\[
F_{2}^{11}(x_1, x_2)|_{2\text{-legs}} = f_{11}^{u}(x_1, x_2) : u_{a}(x_1)u_{a}(x_2) ;
\]

for \( (F, f) = (T, t), (D, d), \ldots \). Again we choose a normalization of \( T_{20}^{50} \text{2-legs} \) which is antisymmetrical in \( \nu \leftrightarrow \mu \). Together with the fact that there exists no Lorentz covariant, antisymmetric tensor of second rank which depends on one Lorentz vector only, we conclude

\[
i_{50}^{u} = 0.
\]

Since \( T_{20}^{10} \) appears also in (B.8), we have some information about \( i_{10}^{u} \) from the C-number identities expressing (B.8) [2], namely

\[
i_{10}^{u} = -i_{10}^{u} \quad \text{and therefore} \quad t_{10}^{u} = 0,
\]

\[
\partial_{\nu}^{10} = 0,
\]
where \( t^{00\mu}(x_1 - x_2) \) is the C-number distribution which belongs to the operators: \( A_{\mu\nu}(x_1) \) \( A_{\mu\nu}(x_2) \) in \( T_2^{00}(x_1, x_2) \). Note that \( d_{uu}^{11\mu} \) has exactly the same (amputated) diagrams as \( d_{uu}^{10\mu} \); consequently \( d_{uu}^{11\mu} = d_{uu}^{10\mu} \). If we split \( d_{uu}^{11\mu} \) in the same way as \( d_{uu}^{10\mu} \), we obtain

\[
t_{uu}^{11\mu} = t_{uu}^{10\mu}.
\]

Obviously (B.12-16) hold true for \( t \) replaced by \( \tilde{t} \). Inserting (B.16) into (2.39) we see that (2.39) is fulfilled also on the two-legs sector. \( \Box \)

Appendix C: Coboundary-Coupling in arbitrary Order

To shorten the notations we shall omit the Lorentz indices and define

\[
S_r F_n^{67...7_1r+\ldots i_n} \text{ def } \frac{1}{r!} [F_n^{67...7_1r+\ldots i_n} + F_n^{76...7_1r+\ldots i_n} + \ldots + F_n^{76...7_1r+\ldots i_n}], \quad (C.1)
\]

where \( F = T, \tilde{T} \).

**Proposition 5**: Assuming the identities (2.50a) to hold true, the following equations are simultaneously fulfilled in all orders \( n \in \mathbb{N} \) for \( F = T, \tilde{T} \), if suitable symmetrical normalizations are chosen:

\[
d_Q F_n^{7...7_1r+\ldots i_n} = i \sum_{j=r+t+1}^{r+t+s} (-1)^{j-r-t-1} \partial^j F_n^{7...7_1r+\ldots i_n}, \quad 0 \leq r, t, s \leq n, \quad r + t + s \leq n \quad (C.2)
\]

and

\[
d_Q S_r F_n^{67...7_1r+\ldots i_n} = i \sum_{j=r+t+1}^{r+t+s} (-1)^{j-r-0} \partial^j S_r F_n^{67...7_1r+\ldots i_n} + \partial S_r F_n^{7...7_1r+\ldots i_n}, \quad 1 \leq r \leq n, \quad 0 \leq t, s \leq n, \quad r + t + s \leq n \quad (C.3)
\]

where \( F_n^{7...7_1r+\ldots i_n} \) and \( F_n^{67...7_1r+\ldots i_n} \) on the l.h.sides have \( t \) indices 5, \( s \) indices 1 and \( r \) indices 7, r.s. (\( r - 1 \)) indices 7 and one index 6. All derivatives on the r.h.sides are divergences.

Note that (C.2) is a generalization of gauge invariance (2.35) and (1.8); the representation (2.34) and (2.49) are special cases of (C.3). The indices may be permuted in (C.2-3) according to (2.19).
Proof: The reasoning runs essentially along the same lines as the proof of prop. 4. Therefore, we only sketch it. First we consider (C.3). We start with (A.2)

\[ dQ A_1^{101010101010101010} = \sum (dQ \hat{T}_k) T_{n-k}^{101010101010101010}. \quad (C.4) \]

The upper indices of \( \hat{T}_k \) and \( T_{n-k} \) on the r.h.s. are arbitrary many indices 7,5,1,0 and at most one index 6. Consequently, we can insert the induction hypothesis (C.2-3) for \( dQ \hat{T}_k \) and \( dQ T_{n-k} \) and obtain (C.3) for the \( A'_p \)-distributions and similar for \( R_{n-1}^0, R_{n}^0 \). Therefore, we may define the normalization of \( R_{n-1}^0 \ldots 51 \ldots 10 \ldots 0 \) by (C.3). This procedure conserves (C.3) in the splitting (A.7), and the remaining steps do not destroy it either.

We turn to (C.2). The case \( r = 0 \) is the assumption (2.50a). For \( 1 \leq r \leq n \) we apply \( dQ \) to (C.3) and use \( (dQ)^2 = 0 \)

\[ dQ F_1^{101010101010101010} = -i \sum_{j=r+t+1}^{r+t+s} (-1)^{(j-r-7)} \partial^j dQ S_r F_1^{101010101010101010} \quad (C.5a) \]

\[ -i(-1)^{s+1} \sum_{j=r+t+s+1}^{n} \partial^j dQ S_r F_1^{101010101010101010}. \quad (C.5b) \]

Next we insert again (C.3) into both terms on the r.h.s

\[ (C.5a) = -i \sum_{j=r+t+1}^{r+t+s} (-1)^{(j-r-7)} \partial^j \left\{ i \sum_{l=r+t+1}^{r+t+s} \pm \partial^l S_r F_1^{101010101010101010} + \right\} \quad (C.6a) \]

\[ +i(-1)^s \sum_{l=r+t+s+1}^{n} \partial^l S_r F_1^{101010101010101010} \quad (C.6b) \]

\[ + F_1^{101010101010101010} \] \quad (C.6c)

\[ (C.5b) = -i(-1)^{s+1} \sum_{j=r+t+s+1}^{n} \partial^j \left\{ i \sum_{l=r+t+1}^{r+t+s} (-1)^{(l-r-7)} \partial^l S_r F_1^{101010101010101010} + \right\} \quad (C.7a) \]

\[ +i(-1)^s \partial^l S_r F_1^{101010101010101010} \quad (C.7b) \]

\[ +i(-1)^{s+1} \sum_{l=r+t+s+1}^{n} \pm \partial^l S_r F_1^{101010101010101010} \quad (C.7c) \]

\[ + F_1^{101010101010101010} \] \quad (C.7d)

(C.6b) and (C.7a) cancel. Similar to the reasoning after (2.50), the terms (C.7b) and (C.7c) vanish because of \( F_1^{101010101010101010} \pm u \) and the different signs of the \( (j,l) \)- and the \( (l,j) \)-term in \( \sum_{j,l} \frac{(-1)^{(l-j)}}{2} \pm \partial^l F_1^{101010101010101010} \). The latter argument applies to (C.6a), too. (Due to (1.11) the \( \pm \) in (C.6a) is a factor \( (-1)^{(l-r-7)} \) if \( l < j \), and a sign \( (-1)^{(l-r-7-1)} \) for \( l > j \).) It remains \( dQ F_1^{101010101010101010} = (C.6c) + (C.7d) \), which is the assertion (C.2).
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