WELL-FOUNDED BOOLEAN ULTRAPowers AS LARGE CARDINAL EMBEDDINGS

JOEL DAVID HAMKINS AND DANIEL EVAN SEABOLD

Abstract. Boolean ultrapowers extend the classical ultrapower construction to work with ultrafilters on any complete Boolean algebra, rather than only on a power set algebra. When they are well-founded, the associated Boolean ultrapower embeddings exhibit a large cardinal nature, and the Boolean ultrapower construction thereby unifies two central themes of set theory—forcing and large cardinals—by revealing them to be two facets of a single underlying construction, the Boolean ultrapower.

Contents

1. A quick review of forcing via Boolean-valued models 2
2. Transforming Boolean-valued models into actual models 6
3. The Boolean Ultrapower 7
4. The naturalist account of forcing 9
5. Well-founded Boolean ultrapowers 10
6. A purely algebraic construction of the Boolean ultrapower 14
7. Direct limits and an extender-like presentation 16
8. Boolean ultrapowers via partial orders 19
9. Subalgebras, iterations and quotients 21
10. Products 24
11. Ideals 29
12. Boolean ultrapowers versus classical ultrapowers 35
13. Boolean ultrapowers as large cardinal embeddings 38
References 40

Boolean ultrapowers generalize the classical ultrapower construction to work with ultrafilters on an arbitrary complete Boolean algebra, rather than only on a power set algebra as is customary with ordinary ultrapowers. The method was introduced by Vopěnka [Vop65] as a presentation of the method of forcing via Boolean-valued models, and Bell [Bel85] remains an excellent exposition of this approach, based on early notes of Dana Scott and Robert Solovay. Quite apart from forcing, Mansfield [Man71] emphasized Boolean ultrapowers as a purely model-theoretic technique, applicable generally to any kind of structure. In this article, we shall focus on the possibility of well-founded Boolean ultrapowers, whose corresponding Boolean ultrapower embeddings exhibit a large cardinal nature. The Boolean ultrapower construction thereby unifies the two central concerns of set theory—forcing and large cardinals—and reveals them to be two facets of a single underlying construction, the Boolean ultrapower.

Our main concern is with the well-founded Boolean ultrapowers and their accompanying large cardinal Boolean ultrapower embeddings, and this material begins in §5. Although the Boolean-valued model approach to forcing has been long known, our investigation depends on some of the subtle details of the method, which are not as well known, and so we provide an efficient but thorough summary of the topic in §11 followed by our introduction of the Boolean ultrapower itself in §12 and §13 and a summary of the naturalist account of forcing in §14. In §16 we give the connection between forcing and the algebraic approach to the Boolean ultrapower, which gives rise to the extender-like characterization in §17. In §18 we explain the subtle issues for the Boolean ultrapower with respect to partial orders versus Boolean algebras. In sections §9, §10 and §11 we
provide the basic interaction of the Boolean ultrapower with respect to subalgebras, quotients, products and ideals. In [12] we characterize the circumstances under which a Boolean ultrapower is a classical ultrapower. And in [13] we begin to explore the extent to which large cardinal embeddings may be realized as Boolean ultrapowers.

1. A quick review of forcing via Boolean-valued models

The concept of a Boolean-valued model is a general model-theoretic notion, having nothing especially to do with set theory or forcing. For example, one can have Boolean-valued groups, rings, graphs, partial orders and so on, using any first-order language. Suppose that \( \mathbb{B} \) is a complete Boolean algebra and \( \mathcal{L} \) is any first-order language. A \( \mathbb{B} \)-valued model \( M \) in the language \( \mathcal{L} \) consists of an underlying set \( M \), whose elements are called \textit{names}, and an assignment of the atomic formulas \( [ s = t ], [ R(s_0, \ldots, s_n) ] \) and \( [ y = f(s_0, \ldots, s_n) ] \), with parameters \( s, t, s_0, \ldots, s_n \in M \), to elements of \( \mathbb{B} \), providing the Boolean value that this atomic assertion holds. These assignments must obey the laws of equality, so that

\[
\begin{align*}
[s = s] & = 1 \\
[s = t] & = [t = s] \\
[\bigwedge_{i < n} s_i = t_i] \wedge [R(\vec{s})] & \leq [R(\vec{t})].
\end{align*}
\]

If the language includes functions symbols, then we also insist on:

\[
\begin{align*}
\bigwedge_{i < n} [s_i = t_i] \wedge [y = f(\vec{s})] & \leq [y = f(\vec{t})] \\
\bigvee_{t \in M} [t = f(\vec{s})] & = 1 \\
[t_0 = f(\vec{s})] \wedge [t_1 = f(\vec{s})] & \leq [t_0 = t_1].
\end{align*}
\]

These requirements assert that with Boolean value one, the equality axiom holds for functions and that the function takes on some unique value. Once the Boolean values of atomic assertions are provided, then one extends the Boolean value assignment to all formulas by a simple, natural recursion:

\[
\begin{align*}
[\varphi \wedge \psi] & = [\varphi] \wedge [\psi] \\
[\neg \varphi] & = \neg[\varphi] \\
[\exists x \varphi(x, \vec{s})] & = \bigvee_{t \in M} [\varphi(t, \vec{s})]
\end{align*}
\]

The reader may check by induction on \( \varphi \) that the general equality axiom now has Boolean value one.

\[
[\vec{s} = \vec{t} \wedge \varphi(\vec{s}) \rightarrow \varphi(\vec{t})] = 1
\]

The Boolean-valued structure \( M \) is \textit{full} if for every formula \( \varphi(x, \vec{x}) \) in \( \mathcal{L} \) and \( \vec{s} \) in \( M \), there is some \( t \in M \) such that \( [/\exists x \varphi(x, \vec{s})] = [\varphi(t, \vec{s})] \).

An easy example is provided by the classical ultrapower construction, which has a Boolean-valued model underlying it, namely, if \( M_i \) is an \( \mathcal{L} \)-structure for each \( i \in I \), then for the set of names we simply take all of the functions used in the ordinary ultrapower construction, \( f \in \Pi, M_i \), and then define the Boolean values \( [\varphi(f_0, \ldots, f_n)] = \{ i \in I \mid M_i \models \varphi(f_0(i), \ldots, f_n(i)) \} \), which makes this a \( \mathbb{B} \)-valued structure for the power set \( \mathbb{B} = P(I) \), a complete Boolean algebra.

In this article, we focus on the Boolean-valued models arising in set theory and forcing, and so our basic language will be the language of set theory \( \{ \in \} \), consisting of one binary relation symbol \( \in \). Later, we will augment this language with a unary predicate for the ground model.

Let us now describe how every complete Boolean algebra gives rise canonically to a Boolean-valued model of set theory. Suppose that \( \mathbb{B} \) is a complete Boolean algebra and let \( V^\mathbb{B} \) denote the universe of all sets. The class of \( \mathbb{B} \)-names, denoted \( V^\mathbb{B} \), is defined by recursion: \( \tau \) is a \( \mathbb{B} \)-name if \( \tau \) is a set of pairs \( \langle \sigma, b \rangle \), where \( \sigma \) is a \( \mathbb{B} \)-name and \( b \in \mathbb{B} \). The atomic Boolean values are defined by double recursion:

\[
\begin{align*}
[\tau \in \sigma] & = \bigvee_{(\eta, b) \in \sigma} [\tau = \eta] \wedge b \\
[\tau = \sigma] & = [\tau = \sigma] \wedge [\sigma \subseteq \tau] \\
[\tau \subseteq \sigma] & = \bigwedge_{\eta \in \text{dom}(\tau)} [\eta \in \tau] \rightarrow [\eta \in \sigma]
\end{align*}
\]

We leave it as an exercise for the reader to verify that these assignments satisfy the laws of equality. One then extends the Boolean value assignment to all assertions \( \varphi(\tau_0, \ldots, \tau_n) \) in the language of set theory by the general recursion given previously.
Lemma 1 (Mixing lemma). If $A \subseteq B$ is an antichain and $\langle \tau_a \mid a \in A \rangle$ is any sequence of names indexed by $A$, then there is a name $\tau$ such that $a \subseteq [\tau = \tau_a]$ for each $a \in A$.

Proof. If $A \subseteq B$ is an antichain and $\tau_a$ is a name for each $a \in A$, then let

$$\tau = \{ \langle \sigma, b \land a \rangle \mid \langle \sigma, b \rangle \in \tau_a \land a \in A \},$$

which looks exactly like $\tau_a$, as far as Boolean values up to $a$ are concerned, and we leave the calculations verifying $a \subseteq [\tau = \tau_a]$ to the reader. □

In the special case that $A$ is a maximal antichain and $[\tau_a \neq \tau_b] = 1$ for $a \neq b$ in $A$, then it is easy to see that $a = [\tau = \tau_a]$ for every $a \in A$, and we shall regard this extra fact as also part of the mixing lemma.

Lemma 2 (Fullness principle). The Boolean-valued model $V^B$ is full.

Proof. Consider the assertion $\exists \bar{x} \varphi(x, \bar{\tau})$, where $\bar{\tau}$ is a finite sequence of names and $\varphi$ is a formula in the language of set theory. By definition, the Boolean value $b = [\exists \bar{x} \varphi(x, \bar{\tau})]$ is the join of the set $S = \{ [\varphi(\sigma, \bar{\tau})] \mid \sigma \in V^B \}$ of Boolean values that arise. Let $A$ be a maximal antichain below $b$, contained in the downward closure of $S$, so that every element of $A$ is below an element of $S$ and $\forall A = b$. For each $a \in A$ choose $\sigma_a$ such that $a \subseteq [\varphi(\sigma_a, \bar{\tau})]$. By the mixing lemma, find a name $\sigma$ such that $a \subseteq [\sigma = \sigma_a]$ for all $a \in A$. It follows by the equality axiom that $a \subseteq [\varphi(\sigma, \bar{\tau})]$ for each $a \in A$, and so $b = \forall A \subseteq [\varphi(\sigma, \bar{\tau})]$. Since $[\varphi(\sigma, \bar{\tau})] \in S$, we must also have $[\varphi(\sigma, \bar{\tau})] \subseteq \forall S = b$, and so $[\exists \bar{x} \varphi(x, \bar{\tau})] = [\varphi(\sigma, \bar{\tau})]$, as desired. □

Theorem 3 (Fundamental theorem of forcing). For any complete Boolean algebra $B$, every axiom of ZFC has Boolean value one in $V^B$.

proof sketch. We merely sketch the proof, as the theorem is widely known. The theorem is actually a theorem scheme, making a separate claim of each ZFC axiom. The axiom of extensionality has Boolean value one because $[\forall \sigma \leftrightarrow \tau \subseteq \sigma \land \sigma \subseteq \tau] = 1$ as a matter of definition. For the pairing, union, power set and infinity axioms, it is an elementary exercise to create suitable names for the desired objects in each case.

To illustrate, we consider the power set axiom in more detail. Suppose that $\tau$ is any $B$-name. We shall prove that $\sigma = \{ \langle \eta, b \rangle \mid \eta \subseteq \text{dom}(\tau) \times B \land b = [\eta \subseteq \tau] \}$ is a name for the power set of $\tau$. For any name $\nu$, observe that $[\nu \in \sigma] = \bigvee_{\langle \eta, b \rangle \in \sigma} [\nu = \eta] \land b = \bigvee_{\langle \eta, b \rangle \in \sigma} [\nu = \eta \land \eta \subseteq \tau] \land [\nu \subseteq \tau]$, and consequently $[\forall \nu \nu \in \sigma \land \nu \subseteq \tau] = 1$. Conversely, for any name $\nu$, consider the name $\nu' = \{ \langle \eta, b \rangle \mid \eta \in \text{dom}(\tau) \land b = [\eta \in \nu \cap \tau] \}$. The reader may check that $[\nu' = \nu \land \nu' \subseteq \nu] = 1$, and consequently $[\nu \subseteq \tau \land \nu = \nu'] = 1$. Since $\nu' \subseteq \text{dom}(\tau) \times B$, it follows that $\nu' \in \text{dom}(\sigma)$ and since $[\nu' \in \sigma] \geq [\nu' \subseteq \tau]$, by the definition of $\sigma$, we conclude $[\nu \subseteq \tau \land \nu \in \sigma] = 1$. So $[\sigma = \text{P}(\tau)] = 1$, as desired.

For the separation axiom, if $\tau$ is any $B$-name and $\varphi(x)$ is any formula in the forcing language, then $\{ \langle \sigma, b \rangle \mid \sigma \in \text{dom}(\tau), b = [\sigma \in \tau \land \varphi(\sigma)] \}$ is a name for the corresponding subset of $\tau$ defined by $\varphi$. For replacement, it suffices to verify collection instead, and it is easy to do so simply by using the name $\{ \langle \tau, 1 \rangle \mid \tau \in V_\alpha \cap V^B \}$ for a sufficiently large ordinal $\alpha$, so that all possible Boolean values realized by witnesses are realized by witnesses in $V_\alpha \cap V^B$, by appealing to collection in $V$. Finally, the foundation axiom can be verified, essentially by choosing a minimal rank name for an element of a given name. □

The map $\varphi \mapsto [\varphi]$ is defined by induction on $\varphi$ in the metatheory, which allows us to verify the forcing relation $p \Vdash \varphi$ in the ground model. Although the forcing slogan is, “the forcing relation is definable in the ground model,” this is not strictly true when one wants to vary the formula. Rather, it is the restriction of the forcing relation to any fixed formula, thought of as a relation on the forcing conditions and the names to be used as parameters in that formula, which is definable in the ground model. By means of universal formulas, one can similarly get the forcing relation in the ground model for formulas of any fixed bounded complexity. But the full forcing relation on all formulas cannot be definable, just as and for the same reasons that Tarski proved that the full first-order satisfaction relation is not definable.

We now elucidate in greater detail the way in which the ground model $V$ sits inside the Boolean-valued model $V^B$. For each set $x \in V$, we recursively define the check names by $\check{x} = \{ \langle \check{y}, 1 \rangle \mid y \in x \}$, and we introduce the predicate $\check{V}$ into the forcing language, defining $[\tau \in \check{V}] = \bigvee_{x \in V} [\tau = \check{x}]$ for any $B$-name $\tau$. This definition arises naturally by viewing $V$ as the class name $\{ \langle \check{x}, 1 \rangle \mid x \in V \}$, and the reader may verify that this predicate obeys the corresponding equality axiom. Note that for any $x \in V$, we have $[\check{x} \in \check{V}] = 1.$
Lemma 4. $V \models \varphi[x_0, \ldots, x_n]$ if and only if $[[\varphi^V(x_0, \ldots, x_n)]] = 1$.

Proof. First, let us explain more precisely what the lemma means. The formula $\varphi^V$ is the relativization of $\varphi$ to the class $\hat{V}$, obtained by binding all quantifiers to this class. Specifically, in the easy cases, $\varphi^V = \varphi$ for $\varphi$ atomic, $(\neg \varphi)^V = \neg \varphi^V$ and $(\varphi \land \psi)^V = \varphi^V \land \psi^V$; for the quantifier case, let $(\exists x \varphi)^V = \exists x \in \hat{V} \varphi^V$, so that $[[\exists x \varphi(x)]] = \bigvee_{x \in \hat{V}} [[\varphi^V(x)]] = \bigvee_{x \in V} [[\varphi^V(x)]]$, where we use the fact that $[[\tau \in V]] = \bigvee_{x \in V} [[\tau = x]]$ to obtain the last equality.

The lemma is clear for atomic formulas, and the induction step for logical connectives is also easy. For the existential case, in the forward direction, suppose that $V \models \exists x \varphi(x, x_0, \ldots, x_n)$. Then $V \models \varphi(x, x_0, \ldots, x_n)$ for some specific $x$, and so $[[\varphi^V(x, x_0, \ldots, x_n)]] = 1$. Since $[[\hat{x} \in \hat{V}]] = 1$, this implies that $[[\exists x \varphi(x, x_0, \ldots, x_n)]] = 1$, as desired. Conversely, suppose that the Boolean value of $[[\exists x \varphi(x, x_0, \ldots, x_n)]]$ is 1. By the remarks of the previous paragraph, this is equivalent to $\bigvee_{x \in V} [[\varphi^V(x, x_0, \ldots, x_n)]] = 1$. So there must be some $x \in V$ with $[[\varphi^V(x, x_0, \ldots, x_n)]] > 0$. By induction, this Boolean value is either 0 or 1, and since it is not 0, we conclude $[[\varphi(x, x_0, \ldots, x_n)]] = 1$. Therefore, $V \models \varphi(x, x_0, \ldots, x_n)$, and so $V \models (\exists x \varphi)(x_0, \ldots, x_n)$, as desired. \qed

The ordinal rank of a set $x$ is defined recursively by $\rho(x) = \sup\{\rho(y) + 1 \mid y \in x\}$; this is the same as the least ordinal $\alpha$ such that $x \in V_{\alpha+1}$. It is easy to see in ZFC that if $\beta$ is an ordinal, then $\rho(\beta) = \beta$.

Lemma 5. If $\tau \in V^B$ and $\rho(\tau) = \alpha$, then $[[\rho(\tau) \leq \alpha]] = 1$.

Proof. The lemma asserts that with Boolean value one, the rank of a set is bounded by the rank of its name. That is, in the assumption $\rho(\tau) = \alpha$ is referring to the rank of the name $\tau$, but in the conclusion, $\rho(\tau) \leq \alpha$ is referring inside the Boolean structure to the rank of the set that $\tau$ names, which could be less. Suppose that this is true for names of rank less than $\alpha$. In particular, if $\eta \in \text{dom}(\tau)$, then $\eta$ has lower rank than $\tau$ and so $[[\rho(\eta) < \alpha]] = 1$. Thus, $\bigwedge_{\eta \in \text{dom}(\tau)}[[\rho(\eta) < \alpha]] = 1$. But $[[\sigma \in \tau]] = V_{\rho(\eta)} [[\sigma = \eta]] \land \beta$, and so $[[\sigma \in \tau]] \leq [[\rho(\sigma) < \alpha]]$. It follows that $[[\forall \sigma \in \tau (\rho(\sigma) < \alpha)]] = 1$ and so $[[\rho(\tau) \leq \alpha]] = 1$, as desired. \qed

The converse implication can fail, because a small set can have a large name. Let us now further explore the nature of $\hat{V}$ inside $V^B$.

Lemma 6. $[\hat{V}$ is a transitive class, containing all ordinals$] = 1$.

Proof. To see that $\hat{V}$ is transitive in $V^B$, consider any two $B$-names $\sigma$ and $\tau$. By the definition of Boolean value in the atomic case, we first observe for any set $x \in V$ that $[[\sigma \in \hat{x}]] = \bigvee_{y \in x} [[\sigma = \hat{y}]]$. Using this, compute

$$[[\sigma \in \tau \in \hat{V}]] = [[\sigma \in \tau]] \land [[\tau \in \hat{V}]]$$
$$= [[\sigma \in \tau]] \land \bigvee_{x \in \hat{V}} [[\tau = \hat{x}]]$$
$$= \bigvee_{x \in \hat{V}} [[\sigma \in \tau \land \tau = \hat{x}]]$$
$$\leq \bigvee_{y \in \hat{V}} [[\sigma \in \hat{y}]]$$
$$= \bigvee_{y \in \hat{V}} [[\sigma = \hat{y}]]$$
$$= [[\sigma \in \hat{V}]].$$

Thus, $[[\sigma \in \tau \in \hat{V} \rightarrow \sigma \in \hat{V}]] = 1$, and consequently, $[[\hat{V}$ is transitive$]] = 1$, as desired. It remains to show that $\hat{V}$ contains all the ordinals with Boolean value one. If $\tau$ is any name, let $\alpha$ be an ordinal larger than $\rho(\tau)$, so that $[[\rho(\tau) < \alpha]] = 1$ by lemma 5. Thus, $[[\tau$ is an ordinal $\rightarrow \tau \in \alpha]] = 1$. By transitivity, we have $[[\tau \in \alpha \rightarrow \tau \in \hat{V}]] = 1$, and so $[[\tau$ is an ordinal $\rightarrow \tau \in \hat{V}]] = 1$, as desired. \qed

If one wanted to build a Boolean-valued sub-model of $V^B$ corresponding to the ground model, one might guess at first that the natural choice would be to use $\{\hat{x} \mid x \in V\}$, which could be denoted $V^2$ for the complete Boolean subalgebra $2 = \{0, 1\} \subseteq B$. After all, we have the atomic Boolean values $[[\tau = \sigma]]$ and $[[\tau \in \sigma]]$ induced from $V^B$, and so this is indeed a Boolean-valued model. Indeed, this model has ZFC with Boolean value one, and it is full. But this is largely because all the Boolean values in this model are 0 or 1, as a consequence of lemma 4. Specifically, it is easy to see by induction that $[[\varphi(x_0, \ldots, x_n)]]$ as computed in $\{\hat{x} \mid x \in V\}$ is the same as $[[\varphi^V(x_0, \ldots, x_n)]]$ in $V^B$, which corresponds simply to the truth in
Lemma 8. The following assertions have 

\[ V \] of \( \varphi(x_0, \ldots, x_n) \) by lemma 4. So the Boolean-valued model arising with \( \{
 x \mid x \in V \} \) is actually a 2-valued model, an isomorphic copy of \( V \).

The more subtle and interesting Boolean-valued model corresponding to the ground model, rather, has domain \( \hat{V} = \{ \tau \mid 1 \in V \} \). (Please note the difference between this notation \( \hat{V} \), pronounced “\( \hat{V} \)-hat”, and the notation \( V \), pronounced “\( V \)-check”.) Of course every \( \hat{x} \) is in \( \hat{V} \), but in general, if \( \mathbb{B} \) is nontrivial, then by the mixing lemma there will be additional names \( \tau \) in \( \hat{V} \) that are not \( \hat{x} \) for any particular \( x \). Since names \( \tau \) in \( \hat{V} \) have \( \bigvee_{x \in V} [\tau = \hat{x}] = 1 \), they constitute a kind of quantum superposition of sets, which have Boolean value one of being some \( \hat{x} \), without necessarily being fully committed to being any particular \( \hat{x} \). We may regard \( \hat{V} \) as a Boolean \( \mathbb{B} \)-valued model using the same atomic values \( [\tau = \sigma] \) and \( [\tau \in \sigma] \) as in \( V^\mathbb{B} \).

And with this interpretation, the model is full.

Lemma 7. Suppose that \( \vec{\tau} = (\tau_0, \ldots, \tau_n) \), where \( \tau_i \in \hat{V} \). Then:

1. \( \left\llbracket \varphi(\vec{\tau}) \right\rrbracket_{\hat{V}} = \left\llbracket \varphi(\vec{\tau}) \right\rrbracket_{V^\mathbb{B}} \).
2. As a \( \mathbb{B} \)-valued model, \( \hat{V} \) is full. That is, for any formula \( \varphi \) with parameters \( \vec{\tau} \) from \( \hat{V} \), there is a name \( \sigma \in \hat{V} \) such that \( \left\llbracket \exists x \varphi(x, \vec{\tau}) \right\rrbracket_{\hat{V}} = \left\llbracket \varphi(\sigma, \vec{\tau}) \right\rrbracket_{\hat{V}} \).

Proof. Statement (1) is clear for atomic formulas, and the inductive step is clear for logical connectives. For the quantifier case, \( \left\llbracket \exists x \varphi(x, \vec{\tau}) \right\rrbracket_{\hat{V}} = \bigvee_{x \in V} \left\llbracket \varphi(x, \vec{\tau}) \right\rrbracket_{\hat{V}} = \bigvee_{x \in V} \left\llbracket \varphi(\hat{x}, \vec{\tau}) \right\rrbracket_{\hat{V}} = \left\llbracket \exists x \varphi(x, \vec{\tau}) \right\rrbracket_{V^\mathbb{B}} \), where the second = relies on the fact that \( \tau \in \hat{V} \) exactly when \( \tau \in V \).

Now consider (2). By the fullness of \( V^\mathbb{B} \), there is some name \( \eta \) such that \( \left\llbracket \exists x \in \hat{V} \varphi(\hat{x}, \vec{\tau}) \right\rrbracket = \left\llbracket \eta \in \hat{V} \land \varphi(\hat{V}(\eta, \vec{\tau})) \right\rrbracket \). Let this Boolean value be \( b \). By the mixing lemma, there is a name \( \sigma \) such that \( b \leq \left\llbracket \eta = \sigma \right\rrbracket \) and \( 1 - b \leq \left\llbracket \sigma = \emptyset \right\rrbracket \). Thus, \( \left\llbracket \sigma \in \hat{V} \right\rrbracket = 1 \), and also \( b \leq \left\llbracket \varphi(\hat{V}(\sigma, \vec{\tau})) \right\rrbracket \leq b \), so \( \left\llbracket \exists x \in \hat{V} \varphi(\hat{x}, \vec{\tau}) \right\rrbracket = \left\llbracket \varphi(\sigma, \vec{\tau}) \right\rrbracket \), as desired.

We now show that the full model \( V^\mathbb{B} \) believes that it is a forcing extension of the class \( \hat{V} \). The canonical name for the generic filter is the name \( \hat{G} = \{ \langle \hat{b}, b \rangle \mid b \in \mathbb{B} \} \).

Lemma 8. The following assertions have \( \mathbb{B} \)-value one in \( V^\mathbb{B} \):

1. \( \hat{G} \) is an ultrafilter on \( \mathbb{B} \).
2. \( \hat{G} \) is a \( \hat{V} \)-generic filter on \( \mathbb{B} \).

Proof. The reader is asked to verify that \( \left\llbracket \hat{b} \in \hat{G} \right\rrbracket = b \) for every \( b \in \mathbb{B} \). It is easy to see that \( \left\llbracket \hat{G} \subseteq \mathbb{B} \right\rrbracket = 1 \), and also \( \left\llbracket \emptyset \notin \hat{G} \right\rrbracket = 1 \). If \( b \leq c \in \mathbb{B} \), then \( \left\llbracket \hat{b} \in \hat{G} \right\rrbracket = b \leq c = \left\llbracket \hat{c} \in \hat{G} \right\rrbracket \), so \( \left\llbracket \hat{b} \in \hat{G} \land \hat{b} \leq \hat{c} \rightarrow \hat{c} \in \hat{G} \right\rrbracket = 1 \). Also, for any \( b, c, \in \mathbb{B} \), we have \( \left\llbracket \hat{b} \in \hat{G} \land \hat{c} \in \hat{G} \right\rrbracket = b \land c = \left\llbracket (b \land c) \in \hat{G} \right\rrbracket \). So with full Boolean value, \( \hat{G} \) is an ultrafilter on \( \mathbb{B} \), establishing (1). For (2), observe that if \( D \in \hat{V} \) is a dense subset of \( \mathbb{B} \), then \( \left\llbracket D \cap \hat{G} \neq \emptyset \right\rrbracket = \left\llbracket \exists x (x \in D \cap \hat{G}) \right\rrbracket = \bigvee_{b \in D} \left\llbracket \hat{b} \in \hat{G} \right\rrbracket = \bigvee_{b \in D} b = 1 \). So with Boolean value one, the filter \( \hat{G} \) meets every dense subset of \( \hat{B} \) in \( \hat{V} \), and consequently \( \forall D \in \hat{V} (\hat{G} \cap \hat{D} \neq \emptyset) \) = 1, as desired.

If \( \tau \) is a \( \mathbb{B} \)-name and \( F \subseteq \mathbb{B} \) is any filter, the value of \( \tau \) with respect to \( F \) is defined recursively by \( \text{val}(\tau, F) = \{ \text{val}(\sigma, F) \mid \langle \sigma, b \rangle \in \tau \text{ for some } b \in F \} \).

Lemma 9. For any \( \mathbb{B} \)-name \( \tau \), we have \( \left\llbracket \tau = \text{val}(\tau, \hat{G}) \right\rrbracket = 1 \). In other words, \( V^\mathbb{B} \) believes that the set named by \( \tau \) is the value of the name \( \tau \) by the canonical generic filter. As a consequence, \( V^\mathbb{B} \) believes with Boolean value one that every set is the interpretation of a \( \mathbb{B} \)-name in \( \hat{V} \) by the filter \( \hat{G} \). In short, \( V^\mathbb{B} \) believes that it is the forcing extension \( \hat{V}[\hat{G}] \).

Proof. To clarify, observe that although \( \tau \) is the name of a set, \( \hat{\tau} \) is the name of a name and \( \hat{G} \) is the name of a filter, so the notation \( \text{val}(\hat{\tau}, \hat{G}) \) is perfectly sensible inside the Boolean brackets. The claim asserts that with Boolean value one, the set named by the name of \( \tau \), when using the generic ultrafilter \( \hat{G} \), is exactly \( \tau \). This is proved by induction on the rank of \( \tau \). If \( \langle \sigma, b \rangle \in \tau \), then \( \left\llbracket \sigma = \text{val}(\sigma, \hat{G}) \right\rrbracket = 1 \) by induction. Thus, \( \left\llbracket \eta \in \text{val}(\hat{\tau}, \hat{G}) \right\rrbracket = \bigvee_{(\sigma, b) \in \tau} \left\llbracket \eta = \sigma \right\rrbracket \land \left\llbracket \hat{b} \in \hat{G} \right\rrbracket = \bigvee_{(\sigma, b) \in \tau} \left\llbracket \eta = \sigma \right\rrbracket \land b \). Since this is the same as \( \left\llbracket \eta \in \tau \right\rrbracket \), it follows that with Boolean value one the sets \( \tau \) and \( \text{val}(\hat{\tau}, \hat{G}) \) have the same members in \( V^\mathbb{B} \), and so the lemma is proved. \( \square \)
2. Tranforming Boolean-valued models into actual models

We now transform the Boolean-valued models into actual classical first-order models by means of the quotient with respect to an ultrafilter on $\mathbb{B}$. This process works with any sort of Boolean-valued model, whether it is a group, a ring, a graph or a partial order, and it is simply the Boolean ultrapower analogue of the quotient process for classical ultrapowers. Let us give the details in the case of the Boolean-valued model of set theory $V^\mathbb{B}$ we have constructed above. Suppose that $U$ is an ultrafilter on $\mathbb{B}$.

There is NO need for the ultrafilter $U$ to be generic in any sense.

The construction works equally well for ultrafilters $U$ in the ground model $V$ as for those not in $V$. Define two relations on the class of $\mathbb{B}$-names:

$$\tau =_U \sigma \iff [\tau = \sigma] \in U$$

$$\tau \in_U \sigma \iff [\tau \in \sigma] \in U,$$

and also define the predicate for the ground model:

$$\tau \in_U \check{V}_U \iff [\tau \in \check{V}] \in U.$$  

We invite the reader to verify that $=_U$ is indeed an equivalence relation on $V^\mathbb{B}$, and it is congruence with respect to $\in_U$ and $\check{V}_U$ as defined above, meaning that these relations are well defined on the corresponding equivalence classes.

For any $\tau \in V^\mathbb{B}$, let $[\tau]_U$ be the restricted equivalence class of $\tau$, namely, the set of all those names $\sigma$, having minimal rank, such that $\tau =_U \sigma$. Using only the names of minimal rank is analogous to Scott’s trick in the case of ordinary ultrapowers, and ensures that the restricted equivalence class is a set, rather than a proper class. The quotient structure $V^\mathbb{B}/U = \{ [\tau]_U \mid \tau \in V^\mathbb{B} \}$, is the class of restricted equivalence classes. Since the relation $=_U$ is a congruence with respect to $\in_U$ and $\check{V}_U$, those relations are well-defined on the quotient structure, and we ambiguously use the notation $V^\mathbb{B}/U$ to represent the full structure $(V^\mathbb{B}/U, \in_U, \check{V}_U)$, as well as its domain.

Lemma 10 (Łoś lemma). If $\mathbb{B}$ is a complete Boolean algebra and $U$ is any ultrafilter on $\mathbb{B}$, then $V^\mathbb{B}/U \models \varphi[\check{\tau}_0, \ldots, \check{\tau}_n]$ if and only if $[[\varphi(\tau_0, \ldots, \tau_n)] \in U$.

Proof. This is the classical Łoś’s lemma for this context. For atomic formulas, it was arranged by definition in $V^\mathbb{B}/U$. For logical connectives, the inductive argument is straightforward, and the quantifier case amounts to the fullness principle.

In particular, since the axioms of ZFC hold with $\mathbb{B}$-value one by theorem $\mathbb{B}$ it follows that $V^\mathbb{B}/U$ is a first-order model of ZFC. In order to prove that a certain set-theoretic statement $\psi$ is consistent with ZFC, therefore, it suffices to find a Boolean algebra $\mathbb{B}$ such that $[[\psi]] \neq 0$; for having done so, one then selects any ultrafilter $U$ on $\mathbb{B}$ with $[[\psi]] \in U$ and observes that $V^\mathbb{B}/U \models \text{ZFC + } \psi$ by lemma $\mathbb{B}$. This method provides a way to prove relative consistency results by forcing, without ever needing to consider countable transitive models or to construct filters that are in any way generic or even pseudo-generic over a particular structure. (Indeed, we constructed $V^\mathbb{B}/U$ and proved all the fundamental facts about it without even mentioning generic filters.) In this way, one can perform forcing over $V$, without ever leaving $V$, for the quotient structure $V^\mathbb{B}/U$ is constructed inside $V$, and provides a (class) model of the desired theory. From any set model $M$ of ZFC, one constructs inside it the model $M^\mathbb{B}/U$, which satisfies ZFC + $\psi$, and this is exactly what one seeks in a general method for consistency and independence proofs.

We are getting closer to the Boolean ultrapower. To help us arrive there, let us investigate more thoroughly how the ground model class $\check{V}$ interacts with the quotient procedure of $V^\mathbb{B}/U$. For any ultrafilter $U$ on $\mathbb{B}$, let $\check{V}_U = \{ [\tau]_U \mid [\tau] \in \check{V} \} \in U$. Although $[\check{x}]_U \in \check{V}_U$ for every $x \in V$, we emphasize that $\check{V}_U$ is not necessarily the same as the class $\{ [x]_U \mid x \in V \}$, since in the case that the filter $U$ is not $V$-generic, there will be some names $\tau$ that are mixtures of various $\check{x}$ via antichains not met by $U$. Indeed, this very point will be the key to understanding the nature of the Boolean ultrapower in the next section. In light of the following lemma, $\check{V}_U$ is exactly identical to the quotient of the Boolean-valued model $\check{V}$ by $U$, and so we could use $\check{V}_U$ and $\check{V}_U$ interchangeably.
Lemma 11. The following are equivalent:

1. \([\tau]_U \in \hat{V}_U\); that is, \([\tau]_U = [\sigma]_U\) for some \(\sigma\) with \([\sigma]_U \in U\).
2. \(\tau \in \hat{V}\) \(\in U\).
3. \([\tau]_U = [\sigma]_U\) for some \(\sigma\) with \([\sigma]_U \in \hat{V}\) = 1.

Proof. For (1) \(\rightarrow\) (2), suppose \([\tau]_U \in \hat{V}_U\). By definition, there is a name \(\sigma\) such that \([\sigma]_U \in U\) and \([\sigma]_U = [\tau]_U\). By the equality axiom, therefore, \([\tau]_U \in \hat{V}\) \(\in U\) as well. For (2) \(\rightarrow\) (3), suppose that \([\tau]_U \in \hat{V}\) \(\in U\). Let \(b = [\tau]_U\), and apply the mixing lemma to build a name \(\sigma\) such that \([\sigma]_U = [\tau]_U\) \(\in U\) and \([\sigma]_U \in \hat{V}\) \(\in U\). Thus, \([\sigma]_U \in \hat{V}\) \(\Rightarrow\) \(\tau \in \hat{V}\) \(\in U\). But since \(b \in \hat{V}\) \(\in U\), we have \(\sigma =_U \tau\) and so \([\tau]_U = [\sigma]_U\), as desired. The converse implications (3) \(\rightarrow\) (2) \(\rightarrow\) (1) are clear. \(\square\)

Lemma 12. \(\hat{V}_U \models \phi([\tau_0]_U, \ldots, [\tau_n]_U)\) if and only if \([\phi^U(\tau_0, \ldots, \tau_n)]_U\) \(\in U\).

Proof. This is true for atomic \(\phi\) as a matter of definition, and the induction proceeds smoothly through logical connectives and the forward implication of the quantifier case. For the converse implication, suppose \([[(\exists x \phi(x, \tau_0, \ldots, \tau_n)]_U = 1\) \(\in U\). By fullness, there is a name \(\tau\) such that this Boolean value is exactly \([\tau \in V \wedge \phi^U(\tau, \tau_0, \ldots, \tau_n)]_U\) \(\in U\). So \([\tau]_U \in \hat{V}_U\) and \(\hat{V}_U \models \phi([\tau]_U, [\tau_0]_U, \ldots, [\tau_n]_U)\) by induction. \(\square\)

3. The Boolean Ultrapower

We are now ready to define the Boolean ultrapower.

Definition 14. The Boolean ultrapower of the universe \(V\) by the ultrafilter \(U\) on the complete Boolean algebra \(\mathbb{B}\) is the structure \(\hat{V}_U = \{[\tau]_U | \{\tau \in \hat{V}\} \in U\}\), under the relation \(\varepsilon_U\), with the accompanying Boolean ultrapower map

\(j_U : V \rightarrow \hat{V}_U\)

defined by \(j_U : x \mapsto [x]_U\). The Boolean ultrapower is also accompanied by the full Boolean extension \(\hat{V}_U \subseteq V_U[G] = V^\mathbb{B}/U\). The Boolean ultrapower of any other structure \(M\) is simply the restriction of the map to that structure, namely, \(j_U \upharpoonright M : M \rightarrow j_U(M)\).

Theorem 15. For any complete Boolean algebra \(\mathbb{B}\) and any ultrafilter \(U\) on \(\mathbb{B}\), the Boolean ultrapower embedding \(j_U : V \rightarrow \hat{V}_U\) is an elementary embedding.

Proof. If \(V \models \phi(x_0, \ldots, x_n)\), then by lemma \(\mathbb{1}\) we know \([\phi(x_0, \ldots, x_n)]_U = 1\). By lemma \(\mathbb{12}\) this implies \(\hat{V}_U \models \phi([x_0]_U, \ldots, [x_n]_U)\), which means \(V_U \models \phi(j_U(x_0), \ldots, j_U(x_n))\), and so \(j_U\) is elementary. \(\square\)

Theorem 16. Suppose that \(U\) is an ultrafilter on the complete Boolean algebra \(\mathbb{B}\) (with \(U\) not necessarily in \(V\)). Then the following are equivalent:

1. \(U\) is \(V\)-generic.
2. The Boolean ultrapower \(j_U\) is trivial, an isomorphism of \(V\) with \(\hat{V}_U\).
Proof. Suppose that $U$ is a $V$-generic filter on $\mathbb{B}$, and that $[\tau]_U \in \check{V}_U$. Thus, the Boolean value $b = [\tau \in \check{V}]$ is in $U$. Since $b = \bigvee_{x \in V} [\tau = \check{x}]$, and these Boolean values are incompatible, the set $\{ [\tau = \check{x}] \mid x \in V \}$ is a maximal antichain below $b$ in $V$. By the genericity of $U$, there must be some $x_0 \in V$ such that $[\tau = \check{x_0}] \in U$, and so $[\tau]_U = [\check{x_0}]_U = j_U(x_0)$. Thus, $j_U$ is surjective from $V$ to $\check{V}_U$. Since it is also injective and $\in$-preserving, $j_U$ is an isomorphism of $V$ with $\check{V}_U$. Conversely, suppose that $j_U$ is an isomorphism and $A \subseteq B$ is a maximal antichain in $V$. By mixing, let $\tau$ be a name such that $a = [\tau = \check{a}]$ for every $a \in A$. Thus, $[\tau \in \check{A}] = \bigvee_{\alpha < \kappa} [\tau = \check{\alpha}] = \bigvee A = 1$. In particular, $[\tau]_U \in \check{V}_U$ and consequently, since we assumed $j_U$ is surjective, $[\tau]_U = [\check{a}]_U$ for some $a \in V$. Necessarily, $a \in A$ since $[\tau \in \check{A}] = 1$. It follows that $a = [\tau = \check{a}] \in U$, and so $U$ meets $A$. Thus, $U$ is $V$-generic, as desired.

Set theorists customarily view the principal ultrafilters, on the one hand, and the generic ultrafilters, on the other, as extreme opposite kinds of ultrafilter, for the principal ultrafilters are completely trivial and the generic ultrafilters are (usually) highly nontrivial. For the purposes of the Boolean ultrapower, however, theorem 16 brings these two extremes together: the principal ultrafilters and the generic ultrafilters both are exactly the ultrafilters with a trivial ultrapower. Thus, the generic ultrafilters become the trivial case. This may be surprising at first, but it should not be too surprising, because in the power set case we already knew that the kinds of ultrafilters were identical, for in any atomic Boolean algebra, the generic filters are precisely the principal filters. Another way to view it is that non-principality for an ultrafilter exactly means that it does not meet the maximal antichain of atoms, and therefore it is explicitly an assertion of non-genericity. Since theorem 16 shows more generally that for any Boolean algebra it is exactly the non-generic ultrafilters that give rise to nontrivial embeddings, the summary conclusion is that:

non-generic is the right generalization of non-principal.

We define that the critical point of $j_U$ is the least ordinal $\kappa$ such that $j_U$ is not an isomorphism of the predecessors of $\kappa$ to the $\in_U$ predecessors of $j_U(\kappa)$. This is the same as the least ordinal $\kappa$ such that $j_U(\kappa)$ does not have order type $\kappa$ under $\in_U$, so either $j_U(\kappa)$ is a larger well-order than $\kappa$ or it is not well ordered at all.

Theorem 17. The critical point of the Boolean ultrapower $j_U : V \rightarrow \check{V}_U$ is the cardinality of the smallest maximal antichain in $V$ not met by $U$, if either exists.

Proof. Suppose that $U$ does not meet the maximal antichain $A \subseteq B$. Enumerate $A = \{ a_\alpha \mid \alpha < \kappa \}$, where $\kappa = |A|$. By the mixing lemma, let $\tau$ be a name such that $a_\alpha = [\tau = \check{\alpha}]$ for every $\alpha < \kappa$. Since $U \cap A = \emptyset$, we have $[\tau]_U \neq [\check{a}]_U$ for all $\alpha < \kappa$. But $[\tau \in \check{\kappa}] = \bigvee_{\alpha < \kappa} [\tau = \check{\alpha}] = \bigvee A = 1$, and so $[\tau]_U \in_U [\check{\kappa}]_U = j_U(\kappa)$. So $j_U \upharpoonright \kappa$ is not onto the $\in_U$ predecessors of $j_U(\kappa)$, and consequently, $j_U$ has critical point $\kappa$ or smaller.

Conversely, suppose that $j_U$ has critical point $\kappa$. Thus, the $\in_U$ predecessors of $j_U(\kappa)$ are not exhausted by $[\check{a}]_U$ for $\alpha < \kappa$. So there is a name $\tau$ such that $[\tau]_U \in_U j(\kappa)$ but $[\tau]_U \neq [\check{\alpha}]_U$ for all $\alpha < \kappa$. This means that $[\tau \in \check{\kappa}] \in U$, but $[\tau = \check{\alpha}] \notin U$ for all $\alpha < \kappa$. Since $[\tau \in \check{\kappa}] = \bigvee_{\alpha < \kappa} [\tau = \check{\alpha}]$, this means that $U$ does not meet the maximal antichain $\{ [\tau = \check{\alpha}] \mid \alpha < \kappa \} \cup \{ [\neg \tau \in \check{\kappa}] \}$, which has size $\kappa$.

We define the degree of genericity of an ultrafilter $U$ on a complete Boolean algebra $\mathbb{B}$ to be the cardinality of the smallest maximal antichain in $\mathbb{B}$ not met by $U$, if such an antichain exists, and Ord otherwise. Thus, the generic filters are exactly those with the largest possible degree of genericity. One can see from theorem 16 that the Boolean ultrapower $\check{V}_U$, for an ultrafilter $U$ on a complete Boolean algebra $\mathbb{B}$, is well-founded up to the degree of genericity of $U$. Theorem 23 improves upon this, for the case $U \in V$, by showing that if the Boolean ultrapower $\check{V}_U$ has a standard $\omega$, then it is fully well-founded.

Corollary 18. The degree of genericity of a non-generic ultrafilter $U$ in $V$ on a complete Boolean algebra $\mathbb{B}$ is either $\aleph_0$ or a measurable cardinal.

Proof. Let $\kappa$ be the degree of genericity of $U$, that is, the size of the smallest maximal antichain not met by $U$. This is the same as the critical point of the Boolean ultrapower $j_U : V \rightarrow \check{V}_U$. Pick any $a < j_U(\kappa)$ with $j_U(\alpha) < a$ for all $\alpha < \kappa$, and define a measure $\mu$ on $\kappa$ by $X \in \mu \iff a \in j_U(X)$. It is easy to see that $\mu$ is a $\kappa$-complete ultrafilter on $\kappa$, and so $\kappa$ is either $\aleph_0$ or a measurable cardinal. 
For an alternative argument, suppose that \( A = \{ a_\alpha \mid \alpha < \kappa \} \subseteq B \) has minimal size \( \kappa \), such that it is not met by \( U \). Define \( X \subseteq \mu \Rightarrow X \subseteq \kappa \) and \( \bigvee_{\alpha \in X} a_\alpha \in U \); using the minimality of \( \kappa \), it is easy to see that \( \mu \) is a \( \kappa \)-complete nonprincipal ultrafilter on \( P(\kappa) \), and so \( \kappa \) is a measurable cardinal, or \( \omega \).

**Theorem 19.** Suppose that \( B \) is a complete Boolean algebra in \( V \) and \( A \subseteq B \) is a maximal antichain of size \( \kappa \), a regular cardinal in \( V \). In a forcing extension of \( V \), there is an ultrafilter \( U \subseteq B \) not meeting \( A \), but meeting all smaller maximal antichains, of size less than \( \kappa \). The corresponding Boolean ultrapower \( j_U : V \to \tilde{V}_U \), therefore, has critical point \( \kappa \).

**Proof.** For the purposes of this proof, let us regard a set as small if it is in \( V \) and has size less than \( \kappa \) in \( V \). Fix the maximal antichain \( A \) of size \( \kappa \), a regular cardinal in \( V \). Let \( I \) be the ideal of all elements of \( A \) that are below the join of a small subset of \( A \). Using the regularity of \( \kappa \), it is easy to see that \( I \) is \( \kappa \)-closed, meaning that \( I \) contains the join of any subset of \( I \) of size less than \( \kappa \). Let \( G \subseteq B/I \) be \( V \)-generic for this quotient forcing, and let \( U = \cup G \) be the corresponding ultrafilter on \( B \). Since \( A \subseteq I \), it follows that \( U \) does not meet \( A \). But if \( B \subseteq B \) is any small antichain in \( B \), then \( \{ [b]_I \mid b \in B \setminus I \} \) is a maximal antichain in \( B/I \) (one uses the facts that \( |B| < \kappa \) and \( I \) is \( \kappa \)-complete to see that this antichain in maximal). Thus, \( U \subseteq B \) is an ultrafilter in \( B \) that meets all small antichains in \( B \), but misses \( A \). Consequently, the Boolean ultrapower \( V \to \tilde{V}_U \) has critical point \( \kappa \), as desired. \( \square \)

Theorem 19 will show that the ultrapower in the proof of theorem 19 is necessarily ill-founded below \( j_U(\kappa) \). Let us close this section with the following observation.

**Theorem 20.** Suppose that \( j : V \to \tilde{V} \subseteq \tilde{V}[G] \) is the Boolean ultrapower by an ultrafilter \( U \subseteq B \). Then \( \tilde{V}[G] = \text{ran}(j)[G] \), because every element of \( \tilde{V}[G] \) has the form \( \text{val}(j(\tau), G) \), the interpretation of a name in \( \text{ran}(j) \) by \( G \).

**Proof.** Lemma 19, the extension \( \tilde{V}[G] \), which is the same as \( V^B/U \), believes that \( [\tau]_U = \text{val}(\tilde{\tau}_U, \tilde{G}[U]) \), which is the same as \( \text{val}(j(\tau), G) \). Thus, everything in \( \tilde{V}[G] \) is the value of a name in \( \text{ran}(j) \) by the filter \( G \), and so \( \tilde{V}[G] = \text{ran}(j)[G] \).

This theorem is exactly analogous to the classical form theorem for classical ultrapowers \( j : V \to V^I/U \), for which every element of the ultrapower is represented as \( [f]_U \), which is equal to \( j(f)([\text{id}]_U) \), an observation that is the basis of seed theory for ultrapowers. In particular, the classical ultrapower is the Skolem hull of \( \text{ran}(j) \) and \([\text{id}]_U \). A refinement of this idea will resurface in theorem 37, providing the extender representation for Boolean ultrapowers.

4. **The naturalist account of forcing**

We now present what we call the naturalist account of forcing, first in a syntactic form, and then in a semantic form.

**Theorem 21** (Naturalist account of forcing). If \( V \) is the universe of set theory and \( B \) is a notion of forcing, then there is in \( V \) a definable class model of the theory expressing what it means to be a forcing extension of \( V \). Specifically, in the forcing language with \( \in \), constant symbols \( \check{x} \) for every element \( x \in V \), a predicate symbol \( \check{V} \) to represent \( V \) as a ground model, and constant symbol \( \check{G} \), the theory asserts:

1. The full elementary diagram of \( V \), relativized to the predicate \( \check{V} \), using the constant symbols for elements of \( V \).
2. The assertion that \( \check{V} \) is a transitive proper class in the (new) universe.
3. The assertion that \( \check{G} \) is a \( \check{V} \)-generic ultrafilter on \( \check{B} \).
4. The assertion that the (new) universe is \( \check{V}[\check{G}] \), and ZFC holds there.

**Proof.** This is really a theorem scheme, since \( V \) cannot have access to its own elementary diagram by Tarski’s theorem on the non-definability of truth. Rather, in the theorem we define a particular class model, and then claim as a scheme that this class satisfies all the desired properties. The class model is simply the quotient model \( V^B/U \), as constructed in section 2. The point is that the results of that section show that theory mentioned in the theorem holds with Boolean value one in \( V^B \), using the check names as the constant symbols and using \( \check{V} \) as the predicate for the ground model, and consequently actually hold in the quotient model \( V^B/U \). \( \square \)
The theorem legitimates actual set-theoretic forcing practice. The typical set-theorist is working in a set-theoretic universe, called $V$ and thought of as the entire universe, but then for whatever reason he or she wants to move to a forcing extension of this universe and so utters the phrase

“Let $G$ be $V$-generic for $\mathbb{B}$. Argue in $V[G]$ . . .”

The effect is to invoke theorem 21 for one now works precisely in the theory of theorem 21 dropping the decorative accents: all previous claims about $V$ in the argument become claims about $V$ as the ground model of the new model $V[G]$, which is a forcing extension of $V$, and new claims about $V[G]$ are available as a result of $G$ being $V$-generic and every set in $V[G]$ having the form $\text{val}(\tau, G)$ for some name $\tau$ in $V$. One may then iteratively invoke theorem 21 by proceeding to further forcing extensions, and so on. In this way, the naturalist account of forcing shows that this common informal usage of forcing is completely legitimate and fully rigorous, and there is no need to introduce supplemental meta-theoretic explanations involving the reflection theorem or countable transitive models and so on, to explain what is “really” going on. Indeed, in the use of the theorem—as opposed to its proof—there is no need to mention Boolean-valued models or the Boolean ultrapower or any other technical foundation of forcing; rather, the theorem allows one simply to assume that $G$ is a $V$-generic filter and that $V[G]$ is the new larger universe and then to proceed naturally in $V[G]$ to manipulate $G$ in some useful or interesting way.

The naturalist account of forcing has the following semantic form:

**Theorem 22.** For any notion of forcing $\mathbb{B}$, a complete Boolean algebra, the set-theoretic universe $V$ has an elementary extension to a structure $(\bar{V}, \bar{\in})$, a definable class in $V$, for which there is in $V$ a $\bar{V}$-generic filter $G$ for $\mathbb{B}$ (the image of $\mathbb{B}$).

$$V \not\subset \bar{V} \subseteq \bar{V}[G]$$

In particular, the entire extension $\bar{V}[G]$ and embedding is a definable class in $V$.

**Proof.** This is exactly what the Boolean ultrapower provides. For any ultrafilter $U$ on $\mathbb{B}$ in $V$, let $\bar{V} = \check{V}_U$ be the corresponding Boolean ultrapower, with relation $\bar{\in} = \in_U$. The corresponding Boolean ultrapower embedding $j : V \to \bar{V}$ is a definable class in $V$ (using parameter $U$), and by lemma 13 the full extension $V^B/U$, which is also a definable class in $V$, is exactly the forcing extension $\bar{V}[G]$, where $G = [\check{G}]_U$ is $V$-generic for $\mathbb{B} = j(\mathbb{B})$, as desired.

This theorem provides a way for a model of set theory to view forcing extensions of itself as actual first-order structures. Of course, we know it is impossible for $V$ to build an actual $V$-generic filter for nontrivial forcing, since the complement of the generic filter would be a dense set that is missed. This method slips by that obstacle by replacing the ground model with an elementary extension, so that the complement of the filter $G$ is not in $\bar{V}$, even though it is in $V$. This account therefore relies on the dual nature of the relationship between $V$ and $\bar{V}$, by which $V$ is smaller than $\bar{V}$, in the sense that the embedding $V \not\subset \bar{V}$ places a copy of the former in the latter, but $V$ is larger than $\bar{V}$, in the sense that $\bar{V}$ is a definable class of $V$. This same dichotomy is often exploited with large cardinal embeddings $j : V \to M$, where $V$ is smaller than $M$ in that it is isomorphic to $\text{ran}(j) \subseteq M$, but larger than $M$ in that $M$ is a definable class of $V$.

5. **Well-founded Boolean ultrapowers**

We have finally arrived at one of the main goals of our investigation, namely, the possibility that the Boolean ultrapower $\check{V}_U$ may be well-founded. In this case, the corresponding Boolean ultrapower embedding $j_U$ would be a large cardinal embedding, whose nature we should like to study. We shall begin with several characterizations of well-foundedness. Let us say that an ultrafilter $U$ on a complete Boolean algebra $\mathbb{B}$ is well-founded if the corresponding Boolean ultrapower $\check{V}_U$ is well-founded.

Theorem 10 showed that the Boolean ultrapower $\check{V}_U$ by a $V$-generic ultrafilter $U$ is well-founded, since it is isomorphic to $V$, and so we may view well-foundedness as a weak form of genericity. It is not difficult to see that if $G \subseteq \mathbb{B}$ is a $V$-generic ultrafilter, then it is $V$-complete, that is, $\kappa$-complete over $V$ for all $\kappa$, meaning that for any sequence $\{a_\alpha \mid \alpha < \kappa\} \subseteq V$ of any length $\kappa$, with $a_\alpha \in G$ for all $\alpha < \kappa$, we have $\bigwedge_{\alpha} a_\alpha \in G$. (The reason is that the set of $b \in \mathbb{B}$ that are either incompatible with some $a_\alpha$ or below $\bigwedge_{\alpha} a_\alpha$ is dense.) In particular, this meet is not zero. It is not difficult to prove conversely that this $V$-completeness property characterizes genericity. A slight but natural change of terminology back in the early days of forcing, therefore, would now have us all referring to $V$-complete ultrafilters instead of $V$-generic ultrafilters in our
forcing arguments. Since well-foundedness, of course, has to do with countable completeness, we arrive at the following characterization of the well-founded ultrafilters in $\mathcal{V}$ as exactly the ultrafilters meeting all the countable maximal antichains in $\mathcal{V}$, an appealing kind of semi-genericity.

**Theorem 23.** If $U$ is an ultrafilter in $\mathcal{V}$ on the complete Boolean algebra $\mathbb{B}$, then the following are equivalent:

(1) The Boolean ultrapower $\mathcal{V}_U$ is well-founded.

(2) The Boolean ultrapower $\mathcal{V}_U$ is an $\omega$-model; that is, it has only standard natural numbers.

(3) $U$ meets all countable maximal antichains of $\mathcal{B}$ in $\mathcal{V}$. That is, if $A \subseteq \mathcal{B}$ is a countable maximal antichain in $\mathcal{V}$, then $A \cap A \neq \emptyset$.

(4) $U$ is countably complete over $\mathcal{V}$. That is, if $a_n \in U$ for all $n < \omega$ and $\langle a_n \mid n < \omega \rangle \in \mathcal{V}$, then $\bigwedge_n a_n \in U$.

(5) $U$ is weakly countably complete over $\mathcal{V}$. That is, if $a_n \in U$ for all $n < \omega$ and $\langle a_n \mid n < \omega \rangle \in \mathcal{V}$, then $\bigwedge_n a_n \neq 0$.

If the ultrafilter $U$ is not in $\mathcal{V}$, but in some larger set-theoretic universe $\mathcal{V} \supseteq \mathcal{V}$, then nevertheless statements 2 through 5 remain equivalent, and each of them is implied by statement 1.

**Proof.** First, we shall prove that 2 through 5 are equivalent, and implied by 1, without assuming that $U \in \mathcal{V}$.

(1 $\rightarrow$ 2) Immediate.

(2 $\rightarrow$ 3) Suppose that $\mathcal{V}_U$ is an $\omega$-model, and consider any countable maximal antichain $A = \{ a_n \mid n < \omega \} \subseteq \mathcal{B}$ in $\mathcal{V}$. By the mixing lemma, there is a name $\tau$ such that $[\tau = n] = a_n$, and consequently $[\tau \in \omega] = 1$. Thus, $[\tau_U] \in [\omega]_U$. Since $\mathcal{V}_U$ is an $\omega$-model, all $\in_U$-elements of $[\omega]_U$ have the form $[n]_U$ for some $n < \omega$. Thus, $[\tau]_U = [n]_U$ for some $n < \omega$. Consequently, $a_n = [\tau = n] \in U$, and so $U$ meets $A$.

(3 $\rightarrow$ 4) Suppose that $U$ meets all countable maximal antichains in $\mathcal{V}$ and that $a_n \in U$ for all $n < \omega$, with $\langle a_n \mid n < \omega \rangle \in \mathcal{V}$, but $a_\omega = \bigwedge_n a_n \notin U$. Let $b_n = a_0 \land \cdots \land a_n$, so that $b_n \in U$ for all $n$, but also $b_0 \geq b_1 \geq \cdots$ is descending and $\bigwedge_n b_n = \bigwedge_n a_n = a_\omega \notin U$. It follows that $\{-b_0\} \cup \{b_n - b_{n+1} \mid n < \omega\} \cup \{a_\omega\}$ is a countable maximal antichain. But $\neg b_0 \notin U$ since $b_0 \in U$, and $b_n - b_{n+1} \notin U$ since $b_{n+1} \in U$. Since $U$ meets the antichain, the only remaining possibility is $a_\omega \in U$, as desired.

(4 $\rightarrow$ 5) Immediate.

(5 $\rightarrow$ 1) Suppose that $\mathcal{V}_U$ is not an $\omega$-model. Thus, $[\omega]_U$ has nonstandard $\in_U$-elements, and so there is some name $\tau$ such that $[\tau]_U \in [\omega]_U$, but $[\tau]_U \neq [n]_U$ for any $n < \omega$. By mixing if necessary, we may assume without loss of generality that $[\tau \in \omega] = 1$. Let $a_n = [\tau = n]$, and observe that $\neg a_n \in U$ for each $n < \omega$, and also $\langle a_n \mid n < \omega \rangle \in \mathcal{V}$. Since $\bigvee_n a_n = [\tau \in \omega] = 1$, it follows that $\bigwedge_n \neg a_n = 0$, contrary to $U$ being weakly countably complete over $\mathcal{V}$.

Lastly, we prove that (5 $\rightarrow$ 1) under the assumption that $U \in \mathcal{V}$. Suppose that $U \in \mathcal{V}$ is countably complete, but $\mathcal{V}_U$ is not well-founded. Thus, there is a sequence of names $\tau_n$ such that $[\tau_{n+1}]_U \in [\tau_n]_U$. Thus, the Boolean value $a_n = [\tau_{n+1} \in \tau_n] \in U$ for every $n$. By countable completeness, we know that $\bigwedge_n a_n \in U$ as well. Let $\sigma$ be the name assembling the sets $\tau_n$ into an $\omega$-sequence, so that $[\sigma \in [\omega]_U] = 1$ and $[\sigma(n) = \tau_n] = 1$ for every $n < \omega$. Observe that the Boolean value $\bigvee \langle n < \omega \sigma(n+1) \in \sigma(n) \rangle = \bigwedge_n a_n \in U$. This contradicts the axiom of foundation holds in $\mathcal{V}_U$. \qed

In the general case that $U \notin \mathcal{V}$, theorem 58 will show that statement 1 of theorem 23 is no longer necessarily equivalent to statements 2 through 5. Nevertheless, we now provide in theorem 24 an additional critical characterization of well-foundedness, which applies whether $U \in \mathcal{V}$ or not and which will form the basis of many of our subsequent constructions. Before this, a clarification of the connection between $\mathcal{V}_U$ and $\mathcal{V}^\mathcal{B}/U$ will be useful.

**Lemma 24.** If $\mathcal{V}_U$ is well-founded, then so is $\mathcal{V}^\mathcal{B}/U$. In this case, the Mostowski collapse of $\mathcal{V}^\mathcal{B}/U$ is the forcing extension $\mathcal{V}[G]$ of the transitive model $\mathcal{V} \models \text{ZFC}$ arising as the collapse of $\mathcal{V}_U$, as in the following
commutative diagram,

\[ \begin{array}{ccc}
\tilde{V}_U & \subseteq & V^B/U \\
\pi \upharpoonright \tilde{V}_U & \downarrow & \downarrow \pi \\
V & \subseteq & V[G]
\end{array} \]

where \( G = \pi([\tilde{G}]_U) \) and \( \pi \) is the Mostowski collapse of \( V^B/U \), which agrees with the Mostowski collapse of \( \tilde{V}_U \).

**Proof.** Since \([\tilde{V}]_{\text{contains all ordinals}} = 1\), it follows that \( \tilde{V}_U \) has the same ordinals as \( V^B/U \), and so if either of these models is well-founded, so is the other. Since \( \tilde{V}_U \) is a transitive subclass of \( V^B/U \) by lemma \( \text{[13]} \) it follows that the restriction of the Mostowski collapse of \( V^B/U \) to \( \tilde{V}_U \) is the same as the Mostowski collapse of \( \tilde{V}_U \). Since lemma \( \text{[13]} \) also shows that \( V^B/U \) is the forcing extension \( (V_U)/[\tilde{G}]_U \), this carries over via the isomorphism \( \pi \) to show that the Mostowski collapse of \( V^B/U \) is the same as \( V[G] \).  

**Theorem 25.** If \( U \) is an ultrafilter (not necessarily in \( V \)) on a complete Boolean algebra \( B \), then the following are equivalent.

1. The Boolean ultrapower \( \tilde{V}_U \) is well-founded.
2. There is an elementary embedding \( j : V \rightarrow M \) into a transitive class \( M \), and there is an \( M \)-generic filter \( G \subseteq j(B) \) such that \( j^* U \subseteq G \).

**Proof.** Suppose that \( U \) is well-founded, with the corresponding Boolean ultrapower map \( j_U : V \rightarrow \tilde{V}_U \). By lemma \( \text{[21]} \) the Mostowski collapse of \( V^B/U \) has the form \( \pi : V^B \cong M[G] \), where \( \pi \upharpoonright \tilde{V}_U \cong M \) is the Mostowski collapse of \( \tilde{V}_U \) and \( G = \pi([\tilde{G}]_U) \) is \( M \)-generic. Let \( j = \pi \circ j_U \), so that \( j : V \rightarrow M \) is an elementary embedding of \( V \) into the transitive class \( M \). Since \([\tilde{b}]_U = b \) it follows that if \( b \in U \), then \([\tilde{b}]_U \in U [\tilde{G}]_U \) and hence \( j(b) \in G \). Thus, \( j^* U \subseteq G \), as desired.

Conversely, suppose that \( j : V \rightarrow M \) is an elementary embedding of \( V \) into a transitive class \( M \), for which there is an \( M \)-generic filter \( G \) on \( j(B) \) such that \( j^* U \subseteq G \). We want to show that the Boolean ultrapower \( \tilde{V}_U \) is well-founded. We will do this by mapping it into \( M \). The typical element of \( \tilde{V}_U \) has the form \([\tau]_U \), where \([\tau] \in \tilde{V} \] = 1. Define \( k : \tilde{V}_U \rightarrow M \) by \( k : ([\tau]_U) \mapsto \text{val}(\tau(G), G) \). This makes sense, because \( j(\tau) \) is a \( j(B) \)-name in \( M \), and so it has a value by the filter \( G \) in \( M[G] \). The map is well defined, because if \([\tau]_U = [\sigma]_U \), then \([\tau] = [\sigma] \in U \) and so \([j(\sigma)]_U = [\sigma]_U \) \( \upharpoonright \tilde{V}_U \), leading to \( \text{val}(j(\sigma), G) = \text{val}(\sigma(G), G) \). Although at first glance it appears that \( k \) is mapping \( \tilde{V}_U \) into \( M[G] \), actually the situation is that since \([\tau] \in \tilde{V} \] = 1, we have \([j(\tau)]_U = \pi( [\tilde{G}]_U \upharpoonright M[G] ) = 1 \), and so \( \text{val}(j(\tau), G) \in M \). Furthermore, the map \( k \) is elementary because if \( \tilde{V}_U \models \phi([\tau]_U) \), then \([\phi^\tilde{V}(\tau)] \in U \) by lemma \( \text{[11]} \) and so \([\phi^M](\text{val}(\tau, G)) \)]_U \) \( = 1 \) by the elementarity of \( j \), leading to \( M[G] \models \phi^M(\text{val}(\sigma(G), G)) \) and consequently \( M \models \phi(\text{val}(\tau, G)) \). And notice also that \( k(j_U(x)) = k([\tilde{x}]_U) = \text{val}(j(\tilde{x}), G) = j(x) \), and so \( k \circ j_U = j \). So \( j_U \) is a factor of \( j \). In particular, \( \tilde{V}_U \) maps elementarily into \( M \) and is therefore well-founded.

The proof establishes the following.

**Corollary 26.** If \( j : V \rightarrow M \) is an elementary embedding into a transitive class \( M \) and there is an \( M \)-generic filter \( G \subseteq j(B) \) for which \( j^* U \subseteq G \), then the Boolean ultrapower embedding \( j_U \) is a factor of \( j \) as follows:

\[ \begin{array}{ccc}
V & \xrightarrow{j} & M \\
\tilde{V}_U \subseteq [\tilde{G}]_U = V^B/U & \xrightarrow{k \upharpoonright \tilde{V}_U} & M[G]
\end{array} \]

**Proof.** As in theorem \( \text{[25]} \) define \( k : V^B/U \rightarrow M[G] \) by \( k : [\tau]_U \mapsto \text{val}(\tau(G), G) \). This is well defined and elementary on the forcing extension because \( V^B/U \models \phi([\tau]_U) \) \( \iff \phi([\tilde{\tau}]_U) \in U \rightarrow M \models \phi(\text{val}(\tau, G)) \] \([\text{[18]} \in G \leftrightarrow M[G] \models \phi(\text{val}(\tau, G)) \). The restriction is elementary from \( \tilde{V}_U \) to \( M \) as in theorem \( \text{[25]} \).  

\[ \square \]
It turns out that one needs $G$ only to be ran$(j)$-generic, rather than fully $M$-generic, for much of the previous, and this is established in theorem 39.

**Theorem 27.** Every infinite complete Boolean algebra $\mathcal{B}$ admits non-well-founded ultrafilters.

**Proof.** Every infinite Boolean algebra has a countably infinite maximal antichain $A$. For any infinite maximal antichain $A$, the set $\{ -a \mid a \in A \}$ has the finite intersection property, and so it may be extended to an ultrafilter $U$ on $\mathcal{B}$. Since $U$ misses $A$, a countable maximal antichain, it follows by theorem 25 that $\check{V}_U$ is not well-founded.

When the Boolean ultrapower is well-founded, we shall simplify notation by combining the Boolean ultrapower embedding with the Mostowski collapse, in effect identifying the Boolean ultrapower $\check{V}_U$ and the full extension $V^\mathcal{B}/U$ with the transitive collapses of these structures, just as classical well-founded ultrapowers are commonly replaced with their collapses. Thus, we shall refer to the well-founded Boolean ultrapower $j : V \to M \subseteq M[G] \subseteq V$, where $M$ is the collapse of $\check{V}_U$ and $M[G]$ is the collapse of the full Boolean extension $V^\mathcal{B}/U$, and where $G = [\mathcal{G}]_U$.

Next, we generalize to well-founded Boolean ultrapowers one of the standard and useful facts about the well-founded classical ultrapowers, namely, that they always exhibit closure up to their critical points. It is well known, for example, that if $j : V \to M$ is the ultrapower by a measure $\mu$ on a measurable cardinal cardinal $\kappa$, then $^\kappa M \subseteq M$. More generally, it is also true that if $j : V \to M$ is the ultrapower by any countably complete ultrafilter $\mu$ on any set $Z$ and $\kappa = \text{cp}(j)$, then $^\kappa M \subseteq M$. An even greater degree of closure, of course, is the defining characteristic of supercompactness embeddings. The general phenomenon is that if $j : V \to M$ is a classical ultrapower by a measure $\mu$ on a set $Z$, then $^\theta M \subseteq M$ if and only if $j'' \theta \in M$. This equivalence does not hold generally for non-ultrapower embeddings, however, such as various extender embeddings, including the $\omega$-iteration of a normal measure on a measurable cardinal. For Boolean ultrapowers, which includes this $\omega$-iteration case, what we prove is not that $^\theta M \subseteq M$, but rather $^\theta M[G] \subseteq M[G]$, where $M[G] = V^\mathcal{B}/U$ is the full Boolean extension. This does generalize the power set ultrapower case, on the technicality that if $\mathcal{B}$ is a power set and hence trivial as a forcing notion, then $M = M[G]$. After the theorem, corollary 29 shows how to obtain $^\theta M \subseteq M$, provided that $\mathcal{B}$ is sufficiently distributive. (And if $\mathcal{B}$ is a power set, then it is trivially $\delta$-distributive for every $\delta$.)

**Theorem 28.** Suppose that $j : V \to M \subseteq M[G] \subseteq V$ is the well-founded Boolean ultrapower by an ultrafilter $U \subseteq \mathcal{B}$ in $V$. Then for any ordinal $\theta$ we have $^\theta M[G] \subseteq M[G]$ if and only if $j'' \theta \in M[G]$. In particular, if $\kappa = \text{cp}(j)$, then definitely $^\kappa M[G] \subseteq M[G]$.

**Proof.** As explained above, we mean that $M$ is the transitive collapse of $\check{V}_U$ and $M[G]$ is the transitive collapse of $V^\mathcal{B}/U$. The theorem is referring to $\theta$-sequences in these classes in $V$. The forward implication is immediate. Conversely, suppose that $j'' \theta \in M[G]$. Fix any $\theta$-sequence $\langle z_\alpha \mid \alpha < \theta \rangle \in ^\theta M[G]$. Thus, $z_\alpha = [r_\alpha]_U$ for some name $r_\alpha$. It is an easy exercise in name manipulation to build a $\mathcal{B}$-name $\sigma$ amalgamating these names into a sequence, so that $[\sigma]$ is a $\theta$-sequence] = 1 and for each $\alpha < \theta$ we have $[\sigma(\delta) = r_\alpha] = 1$. If $s = [\sigma]_U$, then $s$ is a sequence of length $[\theta]_U = j(\theta)$ in $M[G]$, and $s(j(\alpha)) = z_\alpha$. Since $j'' \theta \in M[G]$, the sequence $s' \upharpoonright j'' \theta$ is isomorphic in $M$ to $\langle z_\alpha \mid \alpha < \theta \rangle$, by simply collapsing the domain $j'' \theta$ to $\theta$. Thus, $\langle z_\alpha \mid \alpha < \theta \rangle \in M[G]$, as desired. For the final claim, if $\kappa = \text{cp}(j)$, then of course $j'' \kappa = \kappa$, and so we conclude $^\kappa M \subseteq M$, as desired.

**Corollary 29.** If $\mathcal{B}$ is $<\delta$-distributive and $j : V \to M \subseteq M[G] \subseteq V$ is the well-founded Boolean ultrapower by $U \subseteq \mathcal{B}$ in $V$, then for any $\theta < j(\delta)$, we have $^\theta M \subseteq M$ if and only if $j'' \theta \in M[G]$. In particular, if $\kappa = \text{cp}(j)$ and $\mathcal{B}$ is $<\kappa$-distributive, then $^\kappa M \subseteq M$.

**Proof.** The forward direction again is immediate. Conversely, suppose that $j'' \theta \in M[G]$. By theorem 28 it follows that $^\theta M[G] \subseteq M[G]$, and consequently $^\theta M \subseteq M[G]$. But since $\theta < j(\delta)$, we know that the forcing $G \leq j(\mathcal{B})$ adds no new $\theta$-sequences over $M$, and so $^\theta M \subseteq M$, as desired. The final claim is a special case, since $\kappa < j(\kappa)$ and $j'' \kappa = \kappa \in M$. \qed

Perhaps one should view theorem 28 as similar to the proof of theorem 27 where we argued that if the Boolean ultrapower $\check{V}_U$ is an $\omega$-model, then it is well-founded. The reason is that if it is an $\omega$-model, then $j_U'' \omega$ is represented in the Boolean ultrapower, and so the Boolean ultrapower is closed under $\omega$-sequences.
It follows that it is well-founded. Later, in theorem 66 we will prove a better closure theorem for the generic Boolean ultrapower arising via the quotient by an ideal with the disjointifying property.

Before continuing our investigation of the well-founded Boolean ultrapowers, we shall first develop in the next several sections a sufficient general theory of Boolean ultrapowers.

6. A purely algebraic construction of the Boolean ultrapower

The Boolean ultrapower construction admits of a purely algebraic or model-theoretic presentation, without any reference to forcing or names, in a manner naturally generalizing the usual power set ultrapower construction. Such a presentation is used in Main77, Can87 and OR98, as opposed to the forcing-based approach of Vop65 and Bell85. In this section, we prove the two approaches are equivalent.

Let us give the model-theoretic account. If $B$ is a complete Boolean algebra, $U$ is an ultrafilter on $B$ and $M$ is a structure in a first-order language, then the functional presentation of the Boolean ultrapower of $M$ by $U$ on $B$ will consist of certain equivalence classes of functions $f : A \to M$, where $A$ is a maximal antichain in $B$. We compare two spanning functions $R$ and $C$ of all spanning functions, using any maximal antichain. If $A$ and $B$ are maximal antichains in $B$, then we say $B$ refines $A$ if for every element $b \in B$ there is some $a \in A$ with $b \leq a$ (note that this $a$ is unique). In this case, if $f : A \to M$ is a function, then the reduction of $f$ to $B$ is the function $f \downarrow B : B \to M$ such that $(f \downarrow B)(b) = f(a)$ when $b \leq a$. Note that any two maximal antichains have a common refinement. We compare two spanning functions $f : A \to M$ and $g : B \to M$ by finding a common refinement $C$ of $A$ and $B$ and comparing $f \downarrow C$ and $g \downarrow C$. Specifically, we say $f \equiv_U g$ if and only if $\forall \{ c \in C \mid (f \downarrow C)(c) = (g \downarrow C)(c) \} \in U$. This is an equivalence relation on $M^{IB}$, and it does not depend on the choice of $C$. The functional presentation of the Boolean ultrapower is the quotient $M^{IB}$, consisting of all equivalence classes $[f]_U = \{ g \in M^{IB} \mid f \equiv_U g \}$. For any relation symbol $R$ of $M$, we define $R([f_0]_U, \ldots, [f_n]_U)$ in $M^{IB}$ if $\forall \{ c \mid M \models R((f_0 \downarrow C)(c), \ldots, (f_n \downarrow C)(c)) \} \in U$, where $C$ is any common refinement of $\text{dom}(f_0), \ldots, \text{dom}(f_n)$. Similarly, for any function symbol $r$, we define $r([f_0]_U, \ldots, [f_n]_U) = [f]_U$ in $M^{IB}$, where again $C$ is a suitable common refinement and $f(c) = r((f_0 \downarrow C)(c), \ldots, (f_n \downarrow C)(c))$. One can easily establish the Los theorem for this presentation, so that

$M^{IB} \models \varphi([f_0]_U, \ldots, [f_n]_U)$

if and only if there is a common refinement $C$ of $\text{dom}(f_0), \ldots, \text{dom}(f_n)$ such that

$\forall \{ c \in C \mid M \models \varphi((f_0 \downarrow C)(c), \ldots, (f_n \downarrow C)(c)) \} \in U$.

It follows that the map $j : x \mapsto [c_x]_U$, where $c_x : 1 \to x$ is the constant function with domain $\{1\}$, is an elementary embedding $j : M \to M^{IB}$.

**Theorem 30.** The two presentations of the Boolean ultrapower of the universe are isomorphic. Specifically, there is an isomorphism $\pi : V^{IB}_U \cong \hat{V}_U$ making the following diagram commute.

![Diagram](image)

**Proof.** The point is that the use of spanning functions $f : A \to V$ corresponds exactly to an application of the mixing lemma to the values in $\text{ran}(f)$ using the antichain $A$. Specifically, for any spanning function $f : A \to V$, we may apply the mixing lemma to produce a $B$-name $\tau_f$ such that that $a \leq [\tau_f = f(a)]$ for every $a \in A$. If $B$ refines $A$, then $[\tau_f = \tau_{f \downarrow B}] = 1$ because $\tau_{f \downarrow B}$ mixes the same value $f(a)$ on the portion of $B$ below $a$ that $\tau_f$ has with value $a$. It follows that $f \equiv_U g$ if and only if $[\tau_f = \tau_g] \in U$, which is to say, if and only if $[\tau_f \downarrow U = \tau_g \downarrow U]$. A similar argument shows that $f \equiv_U g$ in $V^{IB}_U$ if and only if $[\tau_f \in \tau_g] \in U$. So let us define $\pi : V^{IB}_U \to \hat{V}_U$ by $\pi : [f]_U \mapsto [\tau_f]_U$. To see that this is onto, we use that every element of $\hat{V}_U$ has the form $[\tau]_U$, where $[\tau]_U = 1$. For such a $\tau$, consider the values $x$ for which $[\tau = \tilde{x}] \neq 0$. The corresponding set $A$ of these Boolean values is a maximal antichain. If $f : A \to V$ is
the function mapping \( \lfloor \tau = x \rfloor \) to \( x \), it follows that \( \lfloor \tau = \tau f \rfloor = \forall A = 1 \), and so \( \lfloor \tau \rfloor_U = \pi((f)_U) \). So \( \pi \) is an isomorphism of \( V^\mathcal{B}_U \) with \( V_U \). The diagram commutes because \( \tau_{c_x} \) is a name such that \( \lfloor \tau_{c_x} = \hat{x} \rfloor = 1 \), and so \( \pi \circ j(x) = \pi([(c_x)_U]) = [\tau_{c_x}]_U = [\hat{x}]_U = j_U(x) \).

Thus, the two accounts of the Boolean ultrapower are equivalent. The equivalence is true generally, for any first-order structure \( \mathcal{M} \), but the forcing and name presentation seems to be the most illuminating in the case of a Boolean ultrapower of the set-theoretic universe \( V \). As we see it, the advantage of the forcing and name viewpoint, for those who know forcing, is that it places the Boolean ultrapower \( V_U \) into the broader context of the full extension \( V^\mathcal{B}_{/U} \), where the existence of the canonical generic object can be illuminating. This broader context will be helpful in our later investigation of Boolean ultrapowers as large cardinal embeddings. Next, we prove that Boolean ultrapowers include all instances of the classical ultrapower by an ultrafilter on a set.

**Theorem 31.** Boolean ultrapowers generalize the usual ultrapowers. Specifically, if \( U \) is an ultrafilter on a set \( Z \), meaning that it is an ultrafilter on the power set Boolean algebra \( \mathcal{B} = P(Z) \), then the usual ultrapower by \( U \) is the same as (isomorphic to) the Boolean ultrapower by \( U \).

**Proof.** In the case of the power set Boolean algebra \( \mathcal{B} = P(Z) \), every spanning function \( f : A \to V \) has a maximal refinement down to the maximal antichain consisting of the atoms \( A = \{ \{ z \} \mid z \in Z \} \). Thus, for the purposes of the Boolean ultrapower \( V^\mathcal{B}_U \), it suffices to consider functions \( f : A \to V \). Any such function naturally corresponds to a function \( f : Z \to V \), by stripping off one layer of set braces, and this correspondence is an isomorphism of \( V^\mathcal{B}_U \) with \( V^Z_{/U} \). Furthermore, this correspondence respects constant functions, and so the ultrapower embeddings themselves are also isomorphic. \( \square \)

We now investigate a method to limit the size of the antichains needed to represent spanning functions in the Boolean ultrapower. Of course, the chain condition of \( \mathcal{B} \) provides a crude upper bound, for if \( \mathcal{B} \) is \( \delta \)-c.c., then all maximal antichains have size less than \( \delta \). The concept of *descents*, however, refines this idea. Specifically, for any ultrafilter \( U \) on a complete Boolean algebra \( \mathcal{B} \), a *descent* through \( U \) is a continuous descending sequence from 1 through \( U \) with meet 0, that is, a sequence \( \{ b_\alpha \mid \alpha < \kappa \} \) beginning with \( b_0 = 1 \), having \( b_\alpha \in U \) for all \( \alpha \), ascending \( \alpha < \beta \to b_\alpha \geq b_\beta \), continuous \( b_\beta = \bigwedge_{\alpha < \beta} b_\alpha \) at limit ordinals \( \lambda \), and with meet zero \( \bigwedge_{\alpha < \kappa} b_\alpha = 0 \). The descent is *strict* if \( b_\alpha > b_{\alpha + 1} \) for all \( \alpha < \kappa \). The *descent spectrum* of \( U \) is the collection of all \( \kappa \) for which there is a descent through \( U \) of order type \( \kappa \). For any such descent, the corresponding *difference antichain* has elements \( d_\alpha = b_\alpha - b_{\alpha + 1} \), and these form a maximal antichain \( \{ d_\alpha \mid \alpha < \kappa \} \) (allowing 0 in the non-strict case). Every element \( b_\alpha \) in the descent is simply the join of all subsequent differences \( b_\alpha = \bigvee \{ d_\beta \mid \beta \geq \alpha \} \). If \( U \) is \( \kappa \)-complete but not \( \kappa^+ \)-complete, then of course there is a descent of order type \( \kappa \), and this is the critical point of the corresponding Boolean ultrapower. More generally, we have:

**Theorem 32.** An ultrafilter \( U \) on a complete Boolean algebra admits a descent of order type \( \kappa \) if and only if the Boolean ultrapower \( j_U \) is discontinuous at \( \kappa \).

**Proof.** Suppose that \( \{ b_\alpha \mid \alpha < \kappa \} \) is a descent through \( U \), with corresponding difference antichain consisting of \( d_\alpha = b_\alpha - b_{\alpha + 1} \). By the mixing lemma, there is a name \( \tau \) with \( \lfloor \tau = \hat{\alpha} \rfloor = d_\alpha \). It follows that \( \lfloor \tau < \hat{\beta} \rfloor = \bigvee_{\alpha < \beta} \lfloor \tau = \hat{\alpha} \rfloor = \bigvee_{\alpha < \beta} d_\alpha \). If \( \beta < \kappa \), then this is disjoint from \( b_\beta \), which is in \( U \), and so \( j_U(\beta) \leq [\tau]_U \). But \( [\tau]_U < j_U(\kappa) \) since \( \lfloor \tau < \hat{\kappa} \rfloor = 1 \), and thus \( sup(j_U(\kappa)) \leq [\tau]_U < j_U(\kappa) \), and so \( j_U \) is discontinuous at \( \kappa \).

Conversely, suppose that \( j_U \) is discontinuous at \( \kappa \), so there is some \( [\tau]_U \) with \( j_U(\alpha) < [\tau]_U < j_U(\kappa) \) for all \( \alpha < \kappa \). We may choose such a name \( \tau \) such that also \( \lfloor \tau < \hat{\kappa} \rfloor = 1 \). Let \( b_\alpha = \lfloor \tau \geq \hat{\alpha} \rfloor \) and observe that \( b_0 \geq b_1 \geq \cdots \) is descending through \( U \), and \( \bigwedge_{\alpha < \kappa} b_\alpha = 0 \) because \( [\tau < \hat{\kappa}] = 1 \). Thus, \( U \) admits a descent of order type \( \kappa \). \( \square \)

Principal ultrafilters, of course, have no descents. More generally, \( V \)-generic ultrafilters have no descents in \( V \), by the discussion before theorem 23. In general, however, the descent spectrum of an ultrafilter can be interesting. To give one example, if \( \mu \) is an ultrafilter on \( \omega \) and \( \nu \) is a measure on a measurable cardinal \( \kappa \), then the usual product measure \( \mu \times \nu \) is an ultrafilter on \( P(\omega \times \kappa) \), whose spectrum consists exactly of the ordinals of cofinality \( \omega \) or \( \kappa \).

15
Lemma 33. Every element of the Boolean ultrapower $V^{|\mathbb{B}|}_U$ is either in the range of $j_U$ or has the form $[f]_U$ for some one-to-one spanning function $f : A \to V$ on a maximal antichain arising from the differences in a strict descent, whose order type is a cardinal.

Proof. Suppose that $f : A \to V$ is any spanning function, where $A$ has minimal cardinality among all spanning functions $U$-equivalent to $f$. If $A$ is finite, then $U$ must concentrate on a member of $A$, so by minimality $A = \{1\}$ and $f$ is constant, which puts $[f]_U$ into the range of $j_U$. Otherwise, $A$ is infinite, and we enumerate it $A = \{\alpha_\beta \mid \alpha < \kappa\}$, where $\kappa = |A|$. By amalgamating together elements of $A$ giving the same value via $f$, we easily find an equivalent one-to-one function, on a smaller antichain; so we may assume without loss of generality that $f$ is one-to-one. By the minimality of $\kappa$, it follows that $\bigvee_{\alpha < \beta} d_\alpha$ is not in $U$ for any $\beta < \kappa$. Consequently, the negation $b_\beta = \bigvee_{\alpha \geq \beta} d_\alpha$ is in $U$. Furthermore, it is easy to see that $b_0 = 1$, that $\alpha < \beta \to b_\alpha > b_\beta$, that $b_\lambda = \bigwedge_{\alpha < \lambda} b_\alpha$ and that $\bigwedge_{\alpha < \kappa} b_\alpha = 0$. That is, $\langle b_\alpha \mid \alpha < \kappa \rangle$ is a strict descent through $U$ of order type $\kappa$, and $A$ is the difference antichain, as desired. \hfill \Box

A modification to the functional presentation avoids the need to consider refinements of maximal antichains or indeed, maximal antichains at all, and this will sometimes be convenient. Specifically, let us define that an open dense spanning function is a function $f : D \to V$ such that $D \subseteq \mathbb{B}$ is an open dense subset of $\mathbb{B}$, such that $b \leq c \in D$ implies $f(b) = f(c)$. If $f : A \to V$ is a spanning function on a maximal antichain $A \subseteq \mathbb{B}$, then the corresponding open dense spanning function $\tilde{f} : D \to V$ is obtained by taking $D$ to be all elements $b$ below an element $a$ in $A$, and defining $\tilde{f}(b) = f(a)$. This is the same as taking the union $f = \bigcup \{f \downarrow B \mid B$ refines $A\}$ of all reductions of $f$ to finer antichains. For two open dense spanning functions $f : D \to V$ and $g : D' \to V$, one defines $f \equiv_U g$ if $\bigvee\{b \in D \cap D' \mid f(b) = g(b)\} \in U$. By associating elements $[f]_U \in V^{|\mathbb{B}|}_U$, for a spanning function $f : A \to V$, with the corresponding equivalence class $[\tilde{f}]_U$ for the open dense spanning functions, one arrives at an isomorphic representation of the Boolean ultrapower.

7. Direct limits and an extender-like presentation

Generalizing the previous section, we shall now prove that every Boolean ultrapower is the direct limit of a certain induced directed system of classical power set ultrapowers. Suppose that $\mathbb{B}$ is a complete Boolean algebra. If $U$ is an ultrafilter on $\mathbb{B}$ and $A \subseteq \mathbb{B}$ is a maximal antichain, we define $U_A$ to be the corresponding ultrafilter on the power set of $A$, induced by $X \in U_A$ if and only if $X \subseteq A$ and $\forall X \in U$. Let $j_{U_A} : V \to V^A/U_A$ be the corresponding ultrapower by $U_A$.

Lemma 34. Suppose that $\mathbb{B}$ is a complete Boolean algebra with ultrafilter $U$ and $A$ is a maximal antichain in $\mathbb{B}$. Then the Boolean ultrapower $j_U$ factors through $j_{U_A}$. There is a natural map $\pi_A : V^A/U_A \to \hat{V}_U$ making the following diagram commute.

![Diagram]

Proof. Every element of $V^A/U_A$ has the form $[f]_{U_A}$ for some function $f : A \to V$. Define $\pi_A : V^A/U \to V^{|\mathbb{B}|}_U$ by $\pi_A : [f]_{U_A} \mapsto [f]_U$. This is clearly well defined, since $f \equiv_{U_A} g$ implies $f \equiv_U g$ in $V^{|\mathbb{B}|}_U$. The diagram commutes, because $\pi_A$ maps the equivalence class of a constant function to the equivalence class of a constant function. The map is elementary, because $V^A/U_A \models \varphi([f]_{U_A})$ if and only if $\{a \in A \mid V \models \varphi(f(a))\} \in U_A$ by Łoś’s theorem, and this holds if and only if $\bigvee\{a \in A \mid V \models \varphi(f(a))\} \in U$ by the definition of $U_A$, and this holds if and only if $V^{|\mathbb{B}|}_U \models \varphi([f]_U)$ by Łoś’s theorem for Boolean ultrapowers in the functional presentation. To complete the proof, we simply replace $V^{|\mathbb{B}|}_U$ with its isomorphic copy $\hat{V}_U$ via theorem \textbf{30}. \hfill \Box

Lemma 35. Suppose that $\mathbb{B}$ is a complete Boolean algebra with ultrafilter $U$ and maximal antichains $A$ and $B$. If $B$ refines $A$, then there is an elementary embedding $\pi_{A,B} : V^A/U_A \to V^B/U_B$ making the following
Theorem 37. As the identity function. This conforms with theorem 36, because in this case we know \( B \) essentially is the identity map as a direct limit of a directed system of identity maps. Second, if the Boolean further refined. All threads therefore terminate in an antichain. 

\[
\text{Proof. Let us consider } V \Rightarrow V \quad \text{in place of } \tilde{V}, \text{as these are isomorphic. Theorem 36 shows that every element of } V_U^{LB} \text{ has the form } [f]_{U_A} \text{ for some } A \subseteq \mathbb{B}, \text{ and so the maps } \pi_A \text{ are collectively onto } V_U^{LB}. \text{ Since the diagram of lemma 35 commutes, the association of } [f]_{U_A} \in V^A/U_A \text{ with } [f]_U \text{ is well defined on the threads. Since this association is also onto (and preserves } \in) \text{, it provides an isomorphism of the Boolean ultrapower with the direct limit.}
\]

Two degenerate instances of this direct limit phenomenon help to shed light on its nature. First, in the case that \( U \) is \( V \)-generic, then we know by theorem 16 that the Boolean ultrapower \( j_U \) is (isomorphic to) the identity function. This conforms with theorem 36 because in this case we know \( U \) meets every maximal antichain \( A \), and so every \( U_A \) is a principal ultrafilter. Thus, in the case of generic filters, what we have essentially is the identity map as a direct limit of a directed system of identity maps. Second, if the Boolean algebra \( \mathbb{B} \) is a power set algebra, the power set of some set \( Z \), then there is a terminal node in the directed system \( I \), corresponding to the antichain of singletons \( A = \{ \{ a \} \mid a \in Z \} \), the atoms of \( \mathbb{B} \), which cannot be further refined. All threads therefore terminate in \( V^A/U_A \) and the direct limit \( j_U \) is consequently the same as \( j_{U_A} \), just as in theorem 33.

Let us conclude this section by explaining how the direct limit provides an extender-like representation of the Boolean ultrapower.

Theorem 36. The Boolean ultrapower \( j_U : V \to \tilde{V} \), where \( U \) is any ultrafilter on a complete Boolean algebra \( \mathbb{B} \), is isomorphic to \( \text{dir lim}(V^A/U_A | j_{U_A}, \pi_{A,B}) \), the direct limit of the induced commutative system of power set ultrapowers \( j_{U_A} : V \to V^A/U_A \), indexed by maximal antichains in \( \mathbb{B} \) ordered by refinement.

\[
\text{Proof. Let us consider } V_U^{LB} \text{ in place of } \tilde{V}, \text{ as these are isomorphic. Theorem 36 shows that every element of } V_U^{LB} \text{ has the form } [f]_{U_A} = \pi_A([f]_{U_A}) \text{ for some } A \subseteq \mathbb{B}, \text{ and so the maps } \pi_A \text{ are collectively onto } V_U^{LB}. \text{ Since the diagram of lemma 35 commutes, the association of } [f]_{U_A} \in V^A/U_A \text{ with } [f]_U \text{ is well defined on the threads. Since this association is also onto (and preserves } \in) \text{, it provides an isomorphism of the Boolean ultrapower with the direct limit.}
\]

Two degenerate instances of this direct limit phenomenon help to shed light on its nature. First, in the case that \( U \) is \( V \)-generic, then we know by theorem 16 that the Boolean ultrapower \( j_U \) is (isomorphic to) the identity function. This conforms with theorem 36 because in this case we know \( U \) meets every maximal antichain \( A \), and so every \( U_A \) is a principal ultrafilter. Thus, in the case of generic filters, what we have essentially is the identity map as a direct limit of a directed system of identity maps. Second, if the Boolean algebra \( \mathbb{B} \) is a power set algebra, the power set of some set \( Z \), then there is a terminal node in the directed system \( I \), corresponding to the antichain of singletons \( A = \{ \{ a \} \mid a \in Z \} \), the atoms of \( \mathbb{B} \), which cannot be further refined. All threads therefore terminate in \( V^A/U_A \) and the direct limit \( j_U \) is consequently the same as \( j_{U_A} \), just as in theorem 33.

Let us conclude this section by explaining how the direct limit provides an extender-like representation of the Boolean ultrapower.

Theorem 37. If \( j : V \to \overline{V} \) is the Boolean ultrapower by the ultrafilter \( U \) on the complete Boolean algebra \( \mathbb{B} \), then \( \overline{V} = \{ j(f)(b) \mid f : \mathbb{B} \to V, b \in j(\mathbb{B}) \} \). More precisely, \( \overline{V} = \{ j(f)(b_A) \mid A \subseteq \mathbb{B} \text{ maximal antichain}, f : A \to V \} \), where \( b_A \) is the unique member of \( j(A) \) in the \( \overline{V} \)-generic filter \( G \) on \( j(\mathbb{B}) \) arising from \( [G]_U \).

\[
\text{Proof. Every element of the classical ultrapower } V^A/U_A, \text{ of course, has the form } [f]_{U_A}, \text{ where } f : A \to V \text{ is a function on a maximal antichain } A \subseteq \mathbb{B}, \text{ and furthermore this is the same as } j_{U_A}(f)([\text{id}]_{U_A}). \text{ Thus, every element of } \overline{V} \text{ has the form } \pi_{A,\infty}(j_{U_A}(f)([\text{id}]_{U_A})) = \pi_{A,\infty}(j_{U_A}(f))(\pi_{A,\infty}([\text{id}]_{U_A}) = j(f)(b), \text{ where}
\]

\[
\text{Diagram commute.}
\]
Suppose that \( b = \pi_{A,\infty}(\text{id}_{U_A}) \), which is in \( j(\mathbb{B}) \) because \( \text{id}_{U_A} \) is an element of \( j_{U_A}(A) \), which is a subset of \( j_{U_A}^2(\mathbb{B}) \). Note that \( b \) depends only on \( A \), and not on \( f \), so we may call it \( b_A \). The identity function \( \text{id} \) on \( A \) corresponds via theorem 33 to the name \( \tau_A \) such that \( [\tau_A] = a \) for all \( a \in A \). If \( G \) is the canonical name of the generic filter, then \( [\hat{a} \in \hat{G}] = a \), and so \( \hat{A} \cap \hat{G} = \{\tau_A\} \) = 1. Thus, \( j(A) \cap G = \{b_A\} \) in \( \nabla[G] \), as desired. \( \square \)

We also get a converse characterization.

**Theorem 38.** Suppose that \( \mathbb{B} \) is a complete Boolean algebra and \( j : V \rightarrow \nabla \) is an embedding, such that there is a filter \( F \subseteq j(\mathbb{B}) \) that is ran(\( j \))-generic and \( \nabla = \{ j(f)(b_A) \mid A \subseteq \mathbb{B} \text{ maximal antichain}, f : A \rightarrow V \} \), where \( b_A \) is the unique member of \( F \cap j(A) \). Then \( j : V \rightarrow \nabla \) is isomorphic to the Boolean ultrapower by \( U = j^{-1}F \subseteq \mathbb{B} \). In this case, \( F \) is actually \( \nabla \)-generic and \( \nabla[F] \) is isomorphic to \( V^B / U \).

Proof. Notice that \( U = j^{-1}F \) is an ultrafilter on \( \mathbb{B} \), since \( F \) meets the maximal antichain \( \{j(b), j(-b)\} \). We define an isomorphism \( \pi : V^B_{U} \cong \nabla \) by \( \pi : [f]_U \mapsto j(f)(b_A) \) for any spanning function \( f : A \rightarrow V \). This is well defined, since if \( [f]_U = [g]_U \) for two spanning functions \( f : A \rightarrow V \) and \( g : B \rightarrow V \), then we may find a common refinement \( C \subseteq A, B \) such that \( \nabla \{ c \in C \mid (f \downarrow C)(c) = (g \downarrow C)(c) \} \subseteq U \), which means that \( j \) of this value is in \( F \). Since \( F \) is ran(\( j \))-generic, it selects a unique element \( b_C \) from \( j(C) \), and so this means that \( j(f \downarrow C)(b_C) = j(g \downarrow C)(b_C) \). Since \( C \) refines \( A \) and \( B \), it follows that \( b_C \) must be below \( b_A \) and \( b_B \), so this implies \( j(f)(b_A) = j(g)(b_B) \), as desired. By replacing \( = \) with \( \in \) in this argument, we see that \( \pi \) is also \( \in \)-preserving. Since \( \pi \) is surjective by assumption, it is an isomorphism. Observe that if \( j = \pi \circ j_U \), since for constant functions \( c_x : A \rightarrow V \) we have \( \pi : [c_x]_U \mapsto j(c_x)(b_A) = j(x) \). So \( j \) is isomorphic to the Boolean ultrapower.

We claim next that \( F \) is actually \( \nabla \)-generic. Suppose that \( D \subseteq j(\mathbb{B}) \) is a dense subset of \( j(\mathbb{B}) \) in \( \nabla \). Thus, \( D = j(D)(b_A) \) for some spanning function \( D = \{a \mid a \in A \} \), and we may as well assume that \( D_a \subseteq \mathbb{B} \) is dense for each \( a \in A \). Let \( \tilde{D} = \bigcup_{a \in A} D_a \cap \mathbb{B} \), where \( \mathbb{B} \subseteq \{b \in \mathbb{B} \mid b \in a \} \). This is clearly dense, since any \( b \in \mathbb{B} \) can be refined below such an \( a \in A \), and then into \( D_a \), which is dense. Since \( F \) is ran(\( j \))-generic, it follows that there is some \( b \in F \cap j(D) \). Necessarily, \( b \subseteq b_A \), since every element of \( \tilde{D} \) is below an element of \( A \). Thus, by elementarity and the definition of \( \tilde{D} \), it follows that \( b \in j(D)(b_A) = D \). So \( F \cap D \neq \emptyset \), and so \( F \) is \( \nabla \)-generic, and consequently also an ultrafilter on \( j(\mathbb{B}) \).

Finally, we argue that \( V^B / U \cong \nabla[F] \). To see this, let \( \pi : [\tau]_U \mapsto \text{val}(\tau, F) \). This is well defined, because if \( [\tau]_U = [\sigma]_U \), then \( [\pi = \sigma]_U \in U \), so \( [\pi]_U = [\tau = \sigma]_U \in F \), which implies \( \text{val}(\tau, F) = \text{val}(\pi, F) \). By replacing \( = \) with \( \in \), we see that \( \pi \) preserves \( \in \). And finally, every element of \( \nabla[F] \) has the form \( \text{val}(\tau, F) \) for some \( \tau \subseteq \nabla[F] \). But \( \tau = j(\tilde{D})(b_A) \) for some spanning function \( \tilde{D} = \{a \mid a \in A \} \). By the mixing lemma, there is a name \( \sigma \) such that \( [\sigma = \tau]_U \geq a \). Applying \( j \), we conclude that \( [\pi = \tau]_U \geq b_A \), which means \( j(\tau) = j(\sigma)(b_A) \). \( j(\tau) \in \nabla[F] \), since \( \text{val}(\tau, F) = \text{val}(\tau, F) \), and so \( \tau \subseteq \nabla[F] \). Thus, \( \text{val}(\tau, F) = \text{val}(\tau, F) \), and so \( \tau \) is surjective. Thus, \( \pi \) is an isomorphism, as desired. Since \( \pi : [\tilde{x}]_U \mapsto \text{val}(\tilde{x}, F) = j(\tilde{x}) \), this isomorphism respects the Boolean ultrapower map. So \( j : V \rightarrow \nabla \subseteq \nabla[F] \) is isomorphic to the Boolean ultrapower by \( U \).

Similarly, we may weaken the genericity hypothesis on \( G \) in theorem 36 to mere ran(\( j \))-genericity.

**Theorem 39.** Suppose that \( U \subseteq \mathbb{B} \) is an ultrafilter on a complete Boolean algebra \( \mathbb{B} \) and \( j : V \rightarrow M \) is an elementary embedding, with \( M \) not necessarily well-founded. Then the following are equivalent:

1. \( j_U \) is an elementary factor of \( j \), as in the diagram below.

\[
\begin{array}{ccc}
V & \xrightarrow{j} & \nabla \\
\downarrow{j_U} & & \downarrow{j} \\
\mathbb{U} & \xrightarrow{k} & M
\end{array}
\]

2. There is a ran(\( j \))-generic filter \( H \subseteq j(\mathbb{B}) \) extending \( j^*U \). That is, \( j^*U \subseteq H \) and \( H \cap j(A) \neq \emptyset \) for all maximal antichains \( A \subseteq \mathbb{B} \).

Proof. Suppose that \( j_U \) is a factor of \( j \) for some elementary embedding \( k \) as in the diagram. Let \( G = \hat{G}|_U \) be the canonical \( \hat{V}_U \)-generic filter determined by \( U \), which is \( V_U \)-generic. Let \( H \) be the filter generated by \( k^*G \). If \( A \subseteq \mathbb{B} \) is any maximal antichain in \( V \), then \( G \cap j_U(A) \neq \emptyset \), and so \( H \cap j(A) \neq \emptyset \), as desired for (2).
Conversely, suppose that there is a filter \( H \) as in (2). For each maximal antichain \( A \subseteq B \), let \( b_A \) be the unique element of \( j(A) \) selected by \( H \). Note that if a maximal antichain \( C \) refines \( A \), then \( b_C \leq b_A \). To define \( k \), it will be convenient to use the functional presentation of the Boolean ultrapower. Specifically, define \( k : V_U^{\mathbb{B}} \to M \) by \( k : f \mapsto j(f)(b_A) \) for any spanning function \( f : A \to V \). This is well defined, since if \( f = g \) for two spanning functions \( f : A \to V \) and \( g : B \to V \), then for any common refinement \( C \) of \( A \) and \( B \) we have \( \{ c \in C \mid (f \upharpoonright C)(c) = (g \upharpoonright C)(c) \} \in U \), which implies that \( j \) of this join is in \( j^n U \), and consequently that \( \{ c \in j(C) \mid (j(f) \upharpoonright j(C))(c) = (j(g) \upharpoonright j(C))(c) \} \in H \). Since \( H \cap j(C) = \{ b_C \} \), this implies \( (j(f) \upharpoonright j(C))(b_C) = (j(g) \upharpoonright j(C))(b_C) \). Since \( j(C) \) refines \( j(A) \), the unique element of \( j(A) \) above \( b_C \) is \( b_A \), as these are both in \( H \) and hence compatible, while all members of \( j(A) \) not above \( b_C \) will be incompatible with \( b_C \). By the corresponding fact with \( j(B) \), it follows that \( j(f)(b_A) = j(g)(b_A) \), and so \( k \) is well defined. A similar argument now shows that \( k \) is elementary. To illustrate, if \( V_U^{\mathbb{B}} \models \varphi([f]_U) \) for a spanning function \( f : A \to V \), then \( \{ a \in A \mid V \models \varphi(f(a)) \} \in U \), which implies \( \{ a \in j(A) \mid M \models \varphi([j(f)(a)] \} \in j^n U \subseteq H \). Since \( H \) selects exactly one member \( b_A \) from \( j(A) \), this means \( M \models \varphi([j(f)(b_A)] \), and so \( k \) is elementary, as desired.

\[ \Box \]

8. Boolean ultrapowers via partial orders

Set theorists often prefer to understand forcing in terms of partial orders or incomplete Boolean algebras, rather than complete Boolean algebras, and this works fine when the ultrafilters being considered are generic. When the focus is as it is here on non-generic ultrafilters, however, then subtle issues arise. The principal initial difficulty is that without a certain degree of genericity, an ultrafilter on a dense subset of a Boolean algebra simply may not generate an ultrafilter on the whole Boolean algebra. If \( P \) is a partial order, then let us denote by \( B = B(P) \) the Boolean algebraic completion of \( P \), which is isomorphic to the collection of regular open subsets of \( P \), the sets that are the interior of their closure. It is easy to see that \( B \) is a complete Boolean algebra, and \( P \) maps in an order preserving manner to a dense subset of \( B \). In the case that \( P \) is separative, we may regard \( P \subseteq B \) as a dense set itself. It is well known that forcing with \( P \) is equivalent to forcing with \( B \), because every \( V \)-generic filter on \( P \) generates a \( V \)-generic filter on \( B \) and vice versa. Without genericity, however, the connection breaks down:

**Theorem 40.** If \( P \) is an incomplete Boolean algebra, then there is an ultrafilter \( U \subseteq P \) that does not generate an ultrafilter on the completion \( B = B(P) \).

**Proof.** We may regard \( P \subseteq B \). Fix any \( a \in B \setminus P \). Thus also \( \neg a \in B \setminus P \). Let \( F = \{ p \in P \mid a \leq p \) or \( \neg a \leq p \} \). We use the fact that \( a \notin P \) to see that \( F \) has the finite intersection property: if \( a \leq p_1, \ldots, p_n \) and \( \neg a \leq q_1, \ldots, q_m \), then let \( p = p_1 \wedge \cdots \wedge p_n \) and \( q = q_1 \wedge \cdots \wedge q_m \) and observe that because \( a \notin P \), it must be that \( a < p \) and \( \neg a < q \). Thus, the difference \( p - a = p \wedge \neg a \) is strictly above 0. Since \( \neg a < q \), this implies \( 0 < p \wedge q \), and so \( F \) has the finite intersection property. Thus, there is an ultrafilter \( U \subseteq P \) with \( F \subseteq U \). Note that \( U \) contains no elements below \( a \), since \( p < a \) implies \( \neg a < \neg p \) and so \( \neg p \in F \subseteq U \). Similarly, \( U \) contains no elements below \( \neg a \) since \( q < \neg a \) implies \( a < \neg q \) and so \( \neg q \in F \subseteq U \). Therefore, the filter generated by \( U \) in \( B \), namely \( \bar{U} = \{ b \in B \mid \exists x \in U \ x \leq b \} \), contains neither \( a \) nor \( \neg a \), and consequently is not an ultrafilter.

Nevertheless, a mild genericity condition on \( U \) will ensure that the generated filter \( \bar{U} \) is an ultrafilter. We define that a 2-split maximal antichain is a partition of a maximal antichain \( A = A_0 \cup A_1 \) into two disjoint nonempty pieces. A filter \( U \) in \( P \) weakly decides this split antichain if there is some condition \( p \in U \) such that \( p \perp A_0 \) or \( p \perp A_1 \), meaning that either \( p \) is incompatible with every element of \( A_0 \) or \( p \) is incompatible with every element of \( A_1 \). This is a weak form of genericity, because if \( U \) actually contains an element of \( A \), then this element will of course weakly decide the antichain, since it is incompatible with all other elements of \( A \).

**Theorem 41.** Suppose that \( U \) is an ultrafilter on a separative partial order \( P \), regarded as a suborder of \( B \), the regular open algebra of \( P \). Then the following are equivalent:

1. The filter \( \bar{U} \) generated by \( U \) in \( B \) is an ultrafilter.
2. \( U \) weakly decides all 2-split maximal antichains in \( P \).
Proof. Let \( \bar{U} = \{ b \in \mathbb{B} \mid \exists p \in U \ b \geq p \} \) be the filter generated by \( U \) in \( \mathbb{B} \). For any element \( a \in \mathbb{B} \), let \( A_0 \subseteq \mathbb{P} \) be a maximal antichain in \( \mathbb{P} \) below \( a \), and let \( A_1 \subseteq \mathbb{P} \) be a maximal antichain in \( \mathbb{P} \) below \( \neg a \). It follows that \( A = A_0 \cup A_1 \) is a 2-split maximal antichain in \( \mathbb{P} \). If \( a \in \bar{U} \), then there is some \( p \leq a \) with \( p \in U \). It follows that \( p \perp \neg a \) and consequently \( p \perp A_1 \). Conversely, if \( p \perp A_1 \), then in \( \mathbb{B} \) it must be that \( p \perp \forall A_1 = \neg a \), and so \( p \leq a \). Thus, \( a \in \bar{U} \) if and only if there is some \( p \in U \) with \( p \perp A_1 \). It follows that \( \bar{U} \) contains \( a \) or \( \neg a \) if and only if \( U \) weakly decides \( a \) or \( \neg a \).

A generalization of this idea provides a characterization of well-foundedness. If \( \mathbb{P} \) is a partial order, then we define that a countably split maximal antichain is a partition \( A = \bigsqcup_n A_n \) of a maximal antichain \( A \) into countably many sub-antichains, allowing \( A_n = \emptyset \) to handle finite partitions. A filter \( U \) weakly decides such a split antichain, if there is a condition \( p \in U \) and a natural number \( n \) such that \( p \perp A_k \) for all \( k \neq n \). Thus, the condition \( p \) rules out all the other partitions except \( A_n \).

**Theorem 42.** Suppose that \( \mathbb{P} \) is a partial order and \( U \) is an ultrafilter on \( \mathbb{P} \) in \( V \). Then the following are equivalent:

1. The filter \( \bar{U} \) generated by \( U \) in \( \mathbb{B} = \text{RO}(\mathbb{P}) \) is a well-founded ultrafilter.
2. \( U \) weakly decides all countably split maximal antichains in \( \mathbb{P} \).

Proof. Suppose that the filter \( \bar{U} \) generated by \( U \) in \( \mathbb{B} \) is an ultrafilter and well-founded, and suppose \( A = \bigsqcup_n A_n \) is a countably split maximal antichain in \( \mathbb{P} \). In \( \mathbb{B} \), let \( a_n = \forall A_n \). If \( k \neq n \), then every element of \( A_k \) is incompatible with every element of \( A_n \), and so \( a_k \land a_n = 0 \). Observe that \( \forall_n a_n = \forall (\bigsqcup_n A_n) = \forall A = 1 \), and so \( \{ a_n \mid n < \omega \} \) is a countable maximal antichain in \( \mathbb{B} \). Since \( \bar{U} \) is well-founded, it follows by theorem 23 that \( a_0 \in \bar{U} \) for some \( n < \omega \). Thus, there is some \( p \in U \) with \( p \leq a_n \). If \( k \neq n \), then since \( a_k \perp a_n \), it follows that \( p \perp A_k \), and so \( U \) weakly decides the countably split maximal antichain, as desired.

Conversely, suppose that \( U \) weakly decides all countably split maximal antichains. In particular, this means that \( U \) weakly decides all 2-split maximal antichains, and so by theorem 11 the filter \( \bar{U} \) generated by \( U \) in \( \mathbb{B} \) is an ultrafilter. To see that \( \bar{U} \) is well-founded, it suffices by theorem 23 to verify that \( \bar{U} \) meets all countable maximal antichains in \( \mathbb{B} \). If \( \{ a_n \mid n < \omega \} \subseteq \mathbb{B} \) is such a countable maximal antichain, then since \( \mathbb{P} \) is dense in \( \mathbb{B} \) there are antichains \( A_n \subseteq \mathbb{P} \) such that \( a_n = \forall A_n \) in \( \mathbb{B} \). (That is, every element of \( \mathbb{B} \) is the join of an antichain in \( \mathbb{P} \).) Thus, \( A = \bigsqcup_n A_n \) is a partition as in (2). Thus, there is some \( p \in U \) and \( n < \omega \) such that \( p \perp A_k \) for all \( k \neq n \). It follows that \( p \land a_k = 0 \) for all \( k \neq n \), and so \( p \leq a_n \). Thus, \( a_n \in \bar{U} \), and so \( \bar{U} \) meets the antichain, as desired.

Set theorists are quite used to a generic filter \( G \subseteq \mathbb{B} \) being generated by a linearly ordered set. This occurs, for example, when forcing with a tree, a common situation, for in this case the generic filter is determined by a path through this tree. Nevertheless, we prove next that nonprincipal ultrafilters in the ground model can never be generated in this way by a linearly ordered set.

**Lemma 43.** If \( U \subseteq \mathbb{B} \) is a nonprincipal ultrafilter on a complete Boolean algebra, then \( U \) is not generated by any linearly ordered collection of elements. In particular, no nonprincipal ultrafilter on a complete Boolean algebra is generated by a countable set.

Proof. This lemma refers to \( U \) in \( V \) on a complete Boolean algebra \( \mathbb{B} \) in \( V \). If \( U \) is generated by a linearly ordered subset \( L \subseteq U \), in the sense that \( b \in U \iff \exists c \in L \ c \leq b \), then we may choose a cofinal descending sequence \( \langle c_\alpha \mid \alpha < \gamma \rangle \) such that \( c_0 \geq c_1 \geq \cdots \geq c_\alpha \geq \cdots \) and \( U \) is generated by \( \{ c_\alpha \mid \alpha < \gamma \} \). We may assume without loss of generality that \( c_\alpha = 1 \) and \( c_\lambda = \bigwedge_{\alpha < \lambda} c_\alpha \) for limit ordinals \( \lambda < \gamma \). Since \( U \) is nonprincipal, it follows that \( \bigwedge_{\alpha < \gamma} c_\alpha = 0 \). The difference elements \( d_\alpha = c_\alpha - c_{\alpha+1} \), therefore, constitute a maximal antichain \( \{ d_\alpha \mid \alpha < \gamma \} \). Let \( x = \bigvee \{ d_\alpha \mid \alpha \text{ is even} \} \), the join of the even differences. It follows that the negation \( \neg x = \bigvee \{ d_\alpha \mid \alpha \text{ is odd} \} \) is the join of the odd differences. Note that every \( c_\alpha \) has nonzero meet with both \( x \) and \( \neg x \), since \( c_\alpha = \bigvee_{\alpha < \beta} d_\beta \), which includes both even and odd differences beyond \( \alpha \). Thus, no \( c_\alpha \) is below \( x \) or \( \neg x \), and so the filter generated by the \( c_\alpha \) includes neither \( x \) or \( \neg x \), a contradiction. So \( U \) cannot have been generated by a linearly ordered set. This implies, in particular, that \( U \) cannot be generated by a countable set \( \{ b_n \mid n < \omega \} \), for then it would be generated by the linearly ordered set \( b_0 \geq (b_0 \land b_1) \geq (b_0 \land b_1 \land b_2) \geq \cdots \), which is impossible.
If one forces to add a V-generic ultrafilter \( G \subseteq B \) that is generated by a linearly ordered set in the extension, which as we said is often the case, then this lemma shows that \( G \) does not generate an ultrafilter on the new completion of \( B \), meaning the regular open algebra of \( B \) as computed in \( V[G] \).

We conclude this section with an intriguing example that illustrates some of the subtleties of working with partial orders rather than complete Boolean algebras.

**Example 44.** Consider the forcing \( \text{Add}(\omega_1,1) \), the complete Boolean algebra corresponding to the the partial order \( P = 2^{<\omega_1} \), which adds a subset to \( \omega_1 \) by initial segments. This forcing \( P \) is a tree. Suppose that \( U \) is an ultrafilter on \( P \) in \( V \), meaning that \( U \) is a maximal filter. It is easy to see that \( U \) is generated by the branch \( b = \bigcup U \), which has order type \( \omega_1 \). Every countable maximal antichain \( A \) in \( P \) is bounded and therefore refined by a maximal antichain consisting of any level of the tree above this bound. Since \( b \) passes through all the levels of \( P \), it follows that \( U \) meets \( A \). So we have established that \( U \) meets every countable maximal antichain in \( P \). A naive reading of theorem 23 would suggest, consequently, that the Boolean ultrapower by \( U \) must be well-founded. But this is absurd, since the critical point of the ultrapower would therefore have to be a measurable cardinal, even though we made no large cardinal assumption, and anyway the size of this forcing is far below the least measurable cardinal. So it is impossible for this Boolean ultrapower to be well-founded. What is going on? The answer is that although \( U \) is an ultrafilter on \( P \), it follows by lemma 43 that it does not generate an ultrafilter on the corresponding Boolean algebra, so there is no such thing as the Boolean ultrapower by \( U \). Although one can extend \( U \) to an ultrafilter on the Boolean algebra, it is not possible to do this in such a way so as to meet all the countable maximal antichains there. The Boolean algebra simply has many more countable maximal antichains than the partial order does, and many of these do not arise from countable maximal antichains in \( P \).

---

**9. Subalgebras, iterations and quotients**

In this section, we investigate the interaction of the Boolean ultrapowers arising from complete subalgebras, iterations and the corresponding quotients of complete Boolean algebras. A Boolean algebra \( B \) is a subalgebra of another \( C \), if \( B \subseteq C \) and the Boolean operations \( \wedge, \vee, \neg \) for elements in \( B \) are computed the same in \( B \) as they are in \( C \). This subalgebra is complete if the infinitary meets and joins are also computed the same in \( B \) as in \( C \). It follows that any maximal antichain in \( B \) remains maximal in \( C \) (and this property alone is sufficient for a subalgebra to be complete). In order to avoid confusion when comparing the Boolean ultrapowers by different Boolean algebras, we introduce the notation \( \dot{V}(B) \) and \( \dot{V}(C) \) to indicate the respective Boolean-valued ground models of \( V^B \) and \( V^C \), built from \( B \)-names and \( C \)-names, respectively.

**Theorem 45.** Suppose that \( B \) is a complete subalgebra of \( C \). Then:

1. The ground model \( \dot{V}(B) \) is a Boolean elementary submodel of \( \dot{V}(C) \), meaning for any formula \( \varphi \) and \( B \)-names \( \tau_i \) with \( \{ \tau_i \in \dot{V} \}^B = 1 \) that

\[
[\varphi^{\dot{V}}(\tau_0, \ldots, \tau_n)]^B = [\varphi^{\dot{V}}(\tau_0, \ldots, \tau_n)]^C.
\]

2. The forcing extension \( V^B \) is a Boolean elementary submodel of the extension \( \dot{V}[\dot{G} \cap \dot{B}] \) in \( V^C \), meaning for any formula \( \varphi \) and \( B \)-names \( \tau_i \) that

\[
[\varphi(\tau_0, \ldots, \tau_n)]^B = [\varphi^{\dot{V}}[\dot{\sigma} \cap \dot{B}](\tau_0, \ldots, \tau_n)]^C.
\]

3. The forcing extension \( V^C \) believes with Boolean value one that it is the forcing extension of \( \dot{V}[\dot{G} \cap \dot{B}] \) by the quotient forcing \( \dot{G} \subseteq C/(\dot{G} \cap \dot{B}) \).

**Proof.** This theorem can be viewed as a generalization of lemma 4 since \( \{0,1\} \) is a complete subalgebra of any Boolean algebra. We prove (1) by induction on \( \varphi \). The atomic case is proved by sub-induction on the names appearing in \( \{ \tau \in \sigma \} \) and \( \{ \tau = \sigma \} \). The point is that for \( B \)-names, the calculation of this Boolean value in \( B \) or \( C \) will be the same, precisely because the supremum of a set of values in \( B \) is the same whether computed in \( B \) or in \( C \). Proceeding with the induction on \( \varphi \), one easily handles Boolean combinations. For the quantifier case, observe that

\[
[\exists x \varphi(x, \bar{\tau})]^{\dot{V}} = \bigvee_{\bar{x} \in \dot{V}} [\varphi^{\dot{V}}(\bar{x}, \bar{\tau})]^{\dot{V}} = \bigvee_{\bar{x} \in \dot{V}} [\varphi^{\dot{V}}(\bar{x}, \bar{\tau})]^{\dot{C}} = [\exists x \varphi(x, \bar{\tau})]^{\dot{V}}
\]

using the induction hypothesis in the second equality and the fact that \( \bar{x} \) is the same whether computed with \( B \) or \( C \).
We prove (2) by induction on $\varphi$. The atomic case follows from the proof of (1), and Boolean combinations are handled easily as before. Before handling quantifiers, we first observe that if $\sigma \in V^B$, then because $\sigma$ refers only to Boolean values in $B$, it follows that $\lbrack \text{val}(\dot{\sigma}, \dot{G}) = \text{val}(\dot{\sigma}, \dot{G} \cap B) \rbrack^C = 1$, and consequently $\lbrack \sigma \in \dot{V}[\dot{G} \cap \dot{B}] \rbrack^C = 1$. More generally, for any C-name $\tau$ we have $\lbrack \tau \in \dot{V}[\dot{G} \cap \dot{B}] \rbrack^C = \bigvee_{\sigma \in V^B} \lbrack \tau = \sigma \rbrack^C$, since any such $\tau$, to the extent that it is in $\dot{V}[\dot{G} \cap \dot{B}]$, will be a mixture of such names $\sigma \in V^B$. We may now observe
\[
\begin{align*}
\lbrack \exists x \varphi(x, \vec{\tau}) \rbrack^B &= \bigvee_{\sigma \in V^B} \lbrack \varphi(\sigma, \vec{\tau}) \rbrack^B \\
&= \bigvee_{\sigma \in V^B} \lbrack \varphi(\dot{\varphi}(\dot{G} \cap \dot{B}), (\sigma, \vec{\tau})) \rbrack^C \\
&= \lbrack \exists x \in \dot{V}[\dot{G} \cap \dot{B}] \varphi(\dot{G} \cap \dot{B}, (x, \vec{\tau})) \rbrack^C \\
&= \lbrack \exists x \varphi(x, \vec{\tau}) \rbrack^C,
\end{align*}
\]
as desired for (2).

Statement (3) amounts to the standard fact about quotient forcing. Specifically, suppose that $B \subseteq C$ is a complete subalgebra of complete Boolean algebras and $G \subseteq C$ is $V$-generic. It follows easily that $G_0 = G \cap B$ is $V$-generic for forcing over $B$. In $V[G_0]$, one may form the quotient partial order $C/G_0$, consisting of all $c \in C$ that are compatible with every element of $G_0$, for which we now argue that $G$ is $V[G_0]$-generic. Suppose that $D \subseteq C/G_0$ is open dense, and $D \in V[G_0]$. Choose a $B$-name $\dot{D}$ such that $D = \text{val}(\dot{D}, G_0)$ and $\dot{D}$ is forced by $1$ over $B$ to be open dense in $C/(G \cap B)$. Since $\dot{D}$ is a $B$-name, it is also a $C$-name, and one may easily establish $\text{val}(\dot{D}, G_0) = \text{val}(\dot{D}, G)$. It follows that $1$ forces over $C$ that $\dot{D}$ is open dense in $C/(G \cap B)$. In $V$, let $E = \{ c \in C \mid c \cap C \in \dot{D} \}$. To see that $E$ is dense, fix any $d \in C$. Clearly, $d$ forces $\dot{d} \in \dot{C}/(G \cap B)$, and so there is some stronger $c \leq d$ forcing over $C$ that some particular $c \leq d$ has $\dot{c} \in \dot{D}$. But if $c$ forces $\dot{c} \in \dot{D}$, then it must be that $c$ and $\dot{c}$ are compatible, since $\dot{D}$ was forced to be contained in $C/(G \cap B)$. By strengthening $c$, we may assume without loss of generality that $c \leq e$. In this case, since $\dot{D}$ was forced to be open, we see that $c$ forces $\dot{c} \in \dot{D}$ also, so $c \in E$ below $d$. So $E$ is dense, and so there is $c \in E \cap G$, meaning that $c \in D \cap G$, as desired. Thus, $G$ is $V[G \cap B]$-generic for the quotient forcing $C/(G \cap B)$. Since we’ve established this as a general consequence of ZFC, it holds with Boolean value one in $V^C$, as stated in (3).

One should not in general expect, of course, that $V^B$ is a Boolean elementary submodel of $V^C$, because forcing with $B$ may result in a different theory than forcing with $C$. For example, adding one Cohen real preserves CH, while adding many Cohen reals will force $\neg$CH, even though the forcing $\text{Add}(\omega, 1)$ is a complete subalgebra of $\text{Add}(\omega, \theta)$ for any $\theta$. More generally, if the quotient forcing $C/(G \cap B)$ is nontrivial, then $V^C$ will have a $V^B$-generic filter for it, but $V^B$ will not, and so except in trivial cases, $V^B$ is not a $\Sigma_1$-elementary Boolean submodel of $V^C$.

Having theorem 45 we may now take the quotient by an ultrafilter.

**Theorem 46.** Suppose that $B$ is a complete subalgebra of $C$ and $U$ is an ultrafilter on $C$. Then $U_0 = U \cap B$ is an ultrafilter on $B$, whose corresponding Boolean ultrapower $j_{U_0} : V \to \bar{V} = \bar{V}_0/U_0$ is an elementary factor of the Boolean ultrapower $j_U : V \to \bar{V} = \bar{V}(C)/U$ as in the following commutative elementary diagram.

\[
\begin{array}{cccc}
V & \xrightarrow{j_U} & \bar{V} & \subseteq \bar{V}[G \cap j_U(B)] \subseteq \bar{V}[G] \\
\xrightarrow{j_{U_0}} & & \xleftarrow{V_0} & \xleftarrow{k} & V_0[U_0] \subseteq V_0[G_0] \\
\end{array}
\]

The full Boolean extensions are $V_0[G_0] = V^B/U_0$ and $\bar{V}[G] = V^C/U$, with generic filters $G_0 = [\dot{G}]_{U_0} \subseteq j_{U_0}(B)$ and $G = [\dot{G}]_{U} \subseteq j_{U}(C)$, and $k(G_0) = G \cap j_{U}(B)$.

**Proof.** It is clear that $U_0 = U \cap B$ is an ultrafilter on $B$. So we may form the ultrapowers $j_U : V \to \bar{V}$ and $j_{U_0} : V \to V_0$, and we will have $\bar{V} \subseteq \bar{V}[G]$ and $V_0 \subseteq V_0[G_0]$, using the canonical generic filters arising from $U$ and $U_0$ as in lemma 13 so that $V_0[G_0] = V^B/U_0$ and $\bar{V}[G] = V^C/U$. To define the factor embedding, let $k : [\tau]_{U_0} \mapsto [\tau]_{U}^C$ for any $B$-name $\tau$, where the superscript indicates the Boolean algebra that is used. This is well defined since if $\tau \equiv_U \sigma$, then $[\tau = \sigma]^B \in U_0 = U \cap B$, and so $[\tau = \sigma]^C \in U$ by theorem 45 and consequently $\tau =_U \sigma$ in $V^C/U$. Note that if $[\tau \in \bar{V}]^B = 1$, then also $[\tau \in \bar{V}]^C = 1$, and so $k$...
carries $V_0$ to $\nabla$. The resulting restriction $k \upharpoonright V_0$ is elementary precisely because of theorem 13, since for $\mathbb{B}$-names $\tau$ with $[\tau \in V] = 1$, we have $V_0 \models \varphi(\tau\upharpoonright U_0)$ if and only if $[\varphi^\mathbb{V}(\tau)]^\mathbb{B} \in U_0$, which is equivalent to $[\varphi^\mathbb{V}(\tau)]^\mathbb{C} \in U$, since these Boolean values are the same. Similarly, the full embedding $k$ is elementary from $V_0[G_0]$ to $\nabla[G \cap j_U(\mathbb{B})]$ precisely because of theorem 14, since $V_0[G_0] = \mathbb{V}/U_0 \models \varphi(\tau\upharpoonright U_0)$ if and only if $[\varphi(\tau)]^\mathbb{C} \in U$, as these Boolean values are the same, and this holds if and only if $\nabla[G \cap j_U(\mathbb{B})] \models \varphi(\tau\upharpoonright U)$. Since $G_0 = [\hat{G}(\mathbb{B})]_{U_0}$, using the canonical name $\hat{G}(\mathbb{B})$ for the generic filter on $\mathbb{B}$, it follows that $k(G_0) = [\hat{G}(\mathbb{B})]_U$. But $[\hat{G}(\mathbb{B})]_U = [\hat{G}(\mathbb{B})] \cap \mathbb{B}^0 = 1$, so this means that $k(G_0) = G \cap j_U(\mathbb{B})$ in $\nabla[G]$. The diagram commutes, because $k$ takes $[\hat{x}]_{U_0}$ to $[\hat{x}]_U$. \hfill \square

We turn now to iterations. Suppose that $\mathbb{B}_0$ is a complete Boolean algebra and $\mathbb{B}_1$ is a full $\mathbb{B}_0$-name of a complete Boolean algebra. We define the iteration algebra $\mathbb{B}_0 \ast \mathbb{B}_1 = \text{RO}(\mathbb{P})$ to be the regular open algebra of the partial pre-order $\mathbb{P} = \{(b, \hat{c}) \mid b \in \mathbb{B}_0^+, \hat{c} \in \text{dom} \mathbb{B}_1, b \Vdash \hat{c} \in \mathbb{B}_1^\mathbb{P} \}$, where $\mathbb{B}_0^+$ denotes the collection of nonzero elements in any Boolean algebra $\mathbb{B}$, and where as usual the order is $(b, \hat{c}) \leq (d, \hat{\ell}) \iff b \leq d$ and $b \Vdash \hat{c} \leq \hat{\ell}$. For convenience, we regard elements of $\mathbb{P}$ as being members of $\mathbb{B}_0 \ast \mathbb{B}_1$, although technically it is the regular open set generated by the lower cone of the element that is in $\mathbb{B}_0 \ast \mathbb{B}_1$. If $U_0 \subseteq \mathbb{B}_0$ is an ultrafilter on $\mathbb{B}_0$ and $U_1 \subseteq \mathbb{B}_1$ is an ultrafilter on the resulting complete Boolean algebra $\mathbb{B}_1 = [\mathbb{B}_1]_{U_0}$ in $\mathbb{V}^{\mathbb{B}_0}/U_0$, then we define the iteration ultrafilter $U_0 \ast U_1$ by

$$U_0 \ast U_1 = \{ (b, \hat{c}) \in \mathbb{P} \mid b \in U_0, [\hat{c}]_{U_0} \in U_1 \}.$$

By the next theorem, this is an ultrafilter on $\mathbb{B}_0 \ast \mathbb{B}_1$.

**Theorem 47.** If $U_0 \subseteq \mathbb{B}_0$ is an ultrafilter on the complete Boolean algebra $\mathbb{B}_0$ and $U_1 \subseteq \mathbb{B}_1$ is an ultrafilter on the complete Boolean algebra $\mathbb{B}_1 = [\mathbb{B}_1]_{U_0}$ in the Boolean extension $\mathbb{V}^{\mathbb{B}_0}/U_0$, where $\mathbb{B}_1$ is a full $\mathbb{B}_0$-name for a complete Boolean algebra, then $U_0 \ast U_1$ is an ultrafilter on $\mathbb{B}_0 \ast \mathbb{B}_1$, whose Boolean ultrapower obeys the following commutative diagram, where $V[G_0] = \mathbb{V}^{\mathbb{B}_0}/U_0$ and $\nabla[G_1] = \mathbb{V}^{\mathbb{B}_0 \ast \mathbb{B}_1}/U_0 \ast U_1$.

\[
\begin{array}{ccc}
V & \xrightarrow{j_{U_0} \ast j_{U_1}} & \nabla \\
\downarrow{\nabla[G_0]} & & \downarrow{\nabla[G_1]} \\
V[G_0] & \subseteq & \nabla[G_0] \subseteq \nabla[G_1]
\end{array}
\]

Conversely, every ultrafilter $U \subseteq \mathbb{B}_0 \ast \mathbb{B}_1$ has the form $U = U_0 \ast U_1$ for some such ultrafilters $U_0$ and $U_1$.

**Proof.** Because $U_0$ is an ultrafilter on $\mathbb{B}_0$ and $U_1$ is an ultrafilter on $\mathbb{B}_1 = [\mathbb{B}_1]_{U_0}$ in $V[G_0]$, we may construct the Boolean ultrapowers $j_{U_0} : V \to V_0 \subseteq V[G_0] = \mathbb{V}^{\mathbb{B}_0}/U_0$ and $j_{U_1} : V[G_0] \to \nabla[G_0] \subseteq \nabla[G_0][G_1] = (V[G_0])^U_{[\hat{G}(\mathbb{B})]}_{U_1}$, where $G_0 = j_1(G_0)$ and $G_1 = [\hat{G}]_{U_1}$, using the canonical name $\hat{G}$ for the generic filter for $\mathbb{B}_1$ over $V[G_0]$. To simplify notation, we denote $j_{U_0}$ and $j_{U_1}$ simply by $j_0$ and $j_1$, respectively. Let $j = j_1 \circ j_0$. Since $G_0$ is $\nabla$-generic and $G_1$ is $\nabla[G_0]$-generic, we may view the forcing extension as $\nabla[G_0][G_1] = \nabla[G_0 \ast G_1]$, arising by forcing over $V$ with $j(\mathbb{B}_0 \ast \mathbb{B}_1)$, where $G_0 \ast G_1$ is the (ultra)filter generated by the set of pairs $(x, \hat{y}) \in j(\mathbb{B}_0 \ast \mathbb{B}_1)$ with $x \in G_0$ and $\text{val}(\hat{y}, G_0) \in G_1$.

We claim now that $U_0 \ast U_1 = j^{-1}(G_0 \ast G_1)$. Since $j_0$ is the Boolean ultrapower of $V$ by $U_0$, we know that $U_0 = j_0^{-1}G_0$. Similarly, since $j_1$ is the Boolean ultrapower of $V[G_0]$ by $U_1$, we know $U_1 = j_1^{-1}G_1$. Combining these facts, observe that $(b, \hat{c}) \in U_0 \ast U_1$ if and only if $b \in U_0$ and $[\hat{c}]_{U_0} \in U_1$, which holds if and only if $j_0(b) \in G_0$ and $\text{val}(j_0(\hat{c}), G_0) \in U_1$. Applying $j_1$, this is equivalent to $j(b) \in G_0$ and $\text{val}(j(\hat{c}), G_0) \in G_1$, which means $(j(b), j(\hat{c})) \in G_0 \ast G_1$, establishing that $U_0 \ast U_1 = j^{-1}(G_0 \ast G_1)$, as we claimed. Since the pre-image of an ultrafilter is an ultrafilter, we conclude that $U_0 \ast U_1$ is an ultrafilter on $\mathbb{B}_0 \ast \mathbb{B}_1$.

Since $\nabla[G_0]$ is the Boolean ultrapower of $V[G_0]$ by $U_1 \subseteq \mathbb{B}_1$, it follows that every element of $\nabla$ has the form $x = [\tau]\upharpoonright_U$, where $\tau$ is a $\mathbb{B}_1$-name in $V[G_0]$ for an element of $V_0$. Since $V[G_0]$ is the Boolean extension $\mathbb{V}^{\mathbb{B}_0}/U_0$, it follows that $\tau = [\tau_0]\upharpoonright_U = \text{val}(j_0(\tau_0), G_0)$, where $\tau_0$ is a $\mathbb{B}_0$-name in $V$. Since $\tau$ is a $\mathbb{B}_1$-name for an element of $V_0$, we may assume without loss of generality that the $\mathbb{B}_0$-Boolean value that $\tau_0$ names a $\mathbb{B}_1$-name for an element of the ground model is 1. That is, $[\tau_0] = [\tau]\upharpoonright_U = [\nu]\upharpoonright_U = \nu = 1$, where $\nu$ is the $\mathbb{B}_0$-name for the $\mathbb{B}_1$-name for the ground model $V$. In short, $x = ([\tau_0]\upharpoonright_U]_{U_1}$. In $V$, the object $\tau_0$ is a $\mathbb{B}_0$-name of a $\mathbb{B}_1$-name for an element of $V$. So any condition $(r, s) \in \mathbb{P} \subseteq \mathbb{B}_0 \ast \mathbb{B}_1$ can be strengthened to
a condition \((b, \dot{c}) \leq (r, \dot{s})\) such that for some \(z \in V\) we have that \(b\) forces over \(B_0\) that \(\dot{c}\) forces over \(\dot{B}_1\) that the object named by \(\tau_0\) is \(z\). Succinctly, \(b \forces_{B_0} (\dot{c} \forces_{\dot{B}_1} \tau_0 \dot{=} \dot{z})\), where \(\dot{z}\) is the \(B_0\) check name of the \(\dot{B}_1\) check name for \(z\). Let’s denote this \(z\) by \(z_{b, \dot{c}}\). We have argued that the collection \(D \subseteq B_0 \ast \dot{B}_1\) consisting of all \((b, \dot{c})\) that decide \(\tau_0\) in this way is dense. Let \(A \subseteq D\) be a maximal antichain in \(V\), and let \(f : A \to V\) be the spanning function defined by \(f(b, \dot{c}) = z_{b, \dot{c}}\). Since \(j(A) \subseteq (B_0 \ast \dot{B}_1)\) is a maximal antichain in \(\check{V}\) and \(\check{G}_0 \ast \check{G}_1\) is \(\check{V}\)-generic, there is a unique condition \(b_A = (d, \dot{e}) \in j(A) \cap \check{G}_0 \ast \check{G}_1\). Since this is in \(j(D)\), it means that there is some \(z = j(f)(d, \dot{e})\) such that \(d\) forces over \(j(B_0)\) that \(\dot{e}\) forces over \(j(\dot{B}_1)\) that the object named by \(j(\tau_0)\) is \(z\). But the object named by \(j(\tau_0)\) is \(\text{val}(\text{val}(j(\tau_0), G_0), \check{G}_1) = \text{val}(\text{val}(j_1(j_0(\tau_0)), j_1(G_0)), \check{G}_1) = \text{val}(j_1(\text{val}(j_0(\tau_0), G_0)), \check{G}_1) = \text{val}(j_1(\text{val}(\check{G}_0, \check{G}_1)), \check{G}_1) = \text{val}(\check{G}_0, \check{G}_1) = x\), our original arbitrary object from \(\check{V}\). So we have established that \(j(f)(b_A) = x\), so every element of \(\check{V}\) has the form \(j(f)(b_A)\) for some spanning function \(f : A \to V\) in \(\check{V}\) on some maximal antichain \(\check{A} \subseteq B_0 \ast \dot{B}_1\). From this, it follows by theorem \(46\) that \(j : V \to \check{V} \subseteq \check{V}[\check{G}_0 \ast \check{G}_1]\) is the Boolean ultrapower of \(V\) by \(U_0 \ast U_1 \subseteq B_0 \ast \dot{B}_1\), as desired.

Conversely, suppose that \(U\) is any ultrafilter on \(B_0 \ast \dot{B}_1\). It is easy to see that the projection \(U_0 = \{b \in B_0 \mid \langle b, 1\rangle \in U\}\) onto the first coordinate is an ultrafilter on \(B_0\). Thus, we may form the corresponding Boolean ultrapower \(j_{U_0} : V \to V_0[G_0] = V^{\check{B}_0} / U_0\) and consider the complete Boolean algebra \(B_1 = [\check{B}_1]_{U_0}\) in \(V_0[G_0]\). It is easy to see that \(U_1 = [\check{c}]_{U_0} \mid \check{b}(b, \dot{c}) \in U\) is an ultrafilter on \(B_1\). Note that \((b, \dot{d}) \in U\) if and only if \(b \in U_0\) and \([\check{d}]_{U_0} \in U_1\), so \(U\) and \(U_0 \ast U_1\) agree on \(\check{B}_1\), and consequently \(U = U_0 \ast U_1\). \(\square\)

It now follows that the factor embedding \(k\) of theorem \(46\) is isomorphic to the Boolean ultrapower of \(V_0[G_0]\) by the quotient forcing \(B_1 = \text{RO}(j_{U_0}(C)/G_0)\) using the ultrafilter \(U_1 = k^{-1}G_1 / (G \cap j_U(B))\), and \(\check{V}[G]\) there is isomorphic to \(V_0[G_0] \check{B}_1 / U_1\). The reason is that whenever \(\check{B}\) is a complete subalgebra of \(C\), then \(C\) is isomorphic to \(\check{B} \ast \dot{B}_1\), where \(\dot{B}_1\) is a \(\check{B}\)-name for the complete Boolean algebra in \(V^{\check{B}}\) corresponding to the quotient \(\check{C} / \check{G}\). Thus, any ultrafilter \(U \subseteq \check{C}\) is isomorphic to an ultrafilter \(\check{U} \subseteq \check{B} \ast \dot{B}_1\), which by theorem \(47\) has the form \(U_0 \ast U_1\), where \(U_0 = U \cap \check{B}\) and \(U_1\) is an ultrafilter in the quotient forcing \(j_0(C)/G_0\) in \(V_0[G_0]\). The map \(j_U\) sends \(x \mapsto [\check{x}]_{U_1}\), using the \(\check{B}\)-name \(\check{x}\). If \(x = [\tau_0]_{U_0}\) for a \(\check{B}\)-name \(\tau_0\), then this \(\check{x}\) is the \(\check{B}_1\)-name for the same object named by \(\tau\) in \(V^{\check{B}}\). The canonical \(C\)-name for this object is, of course, \(\tau\) itself, which as a \(\check{B}\) name is also a \(C\)-name for the same object. So \(j_U\) has mapped \([\tau_0]_{U_0}\) to (the isomorphic copy of) \([\tau_0]_{U_1}\), which is precisely how we defined \(k\).

10. Products

We now consider products of complete Boolean algebras \(B_0\) and \(B_1\). The first step is to dispense with an incorrect choice for the product, namely the direct sum Boolean algebra \(B_0 \oplus B_1 = \{ (b, c) \mid b \in B_0, c \in B_1 \}\), with coordinate-wise operations. Although this is a complete Boolean algebra, it is not the right choice for product forcing or for product Boolean ultrapowers, because the maximal antichain \(\{(1, 0), (0, 1)\}\) in \(B_0 \oplus B_1\) means that any ultrafilter concentrates in effect on only one factor; so this algebra corresponds to what is known as side-by-side forcing or the lottery sum of \(B_0\) and \(B_1\). Instead, for product forcing what one wants is

\[
B_0 \times B_1 = \text{RO}(B_0^+ \times B_1^+).
\]

It is easy to see that any ultrafilter \(U \subseteq B_0 \times B_1\) projects to ultrafilters \(U_0 \subseteq B_0\) and \(U_1 \subseteq B_1\) in each factor, and these give rise to the following commutative diagram.
Theorem 48. For any ultrafilter $U \subseteq \mathcal{B}_0 \times \mathcal{B}_1$, projecting to ultrafilters $U_0 \subseteq \mathcal{B}_0$ and $U_1 \subseteq \mathcal{B}_1$, there are elementary maps $k_0$ and $k_1$ making the following diagram commute (dotted lines indicate forcing extensions).

In the diagram, $V_0 = \hat{V}_{U_0} \subseteq V_0[G_0] = V^{\mathcal{B}_0}/U_0$ is the Boolean ultrapower by $U_0$, with $G_0 = [\hat{G}]_{U_0}$; $V_1 = \hat{V}_{U_1} \subseteq V_1[G_1] = V^{\mathcal{B}_1}/U_1$ is the Boolean ultrapower by $U_1$, with $G_1 = [\hat{G}]_{U_1}$; and $V = \hat{V}_U \subseteq \hat{V}[G_0 \times G_1] = V^{\mathcal{B}_0 \times \mathcal{B}_1}/U$ is the ultrapower by $U$, with $G_0 \times G_1 = [\hat{G}]_U$.

Proof. Since $\mathcal{B}_0$ and $\mathcal{B}_1$ are each completely embedded in $\mathcal{B}_0 \times \mathcal{B}_1$, this is a simple consequence of theorem 46. The diagram here consists simply of two copies of the diagram in theorem 46, one above and one below the central axis. \qed

We now consider to what extent an ultrafilter $U$ on the product $\mathcal{B}_0 \times \mathcal{B}_1$ is determined by its factors $U_0$ and $U_1$. When $U$ is a generic filter, then a basic fact about product forcing is that the corresponding projections $U_0$ and $U_1$ are mutually generic and the rectangular product $U_0 \boxtimes U_1 = \{ (b,c) \mid b \in U_0, c \in U_1 \}$ generates $U$. Corollary 55 shows, however, that when $U$ is not generic, then $U_0 \boxtimes U_1$ does not always generate an ultrafilter. Indeed, even in the case of power set algebras, one does not usually expect to measure all subsets of the plane with rectangles. Rather, one defines the product measure, which slices every subset of the plane into vertical strips, demanding that a large number of them is large. This idea generalizes naturally to the context of arbitrary complete Boolean algebras as follows. Since the elements of $\mathcal{B}_0 \times \mathcal{B}_1$ are exactly the regular open subsets $X \subseteq \mathbb{B}_0^+ \times \mathbb{B}_1^+$, we use the notation $X_b = \{ c \mid (b,c) \in X \}$ and define the product filter $U_0 \times U_1$ by

$$X \in U_0 \times U_1 \quad \text{if and only if} \quad \forall \{ b \in \mathcal{B}_0 \mid \forall X_b \in U_1 \} \in U_0$$

It is easy to see that $U_0 \times U_1$ is indeed a filter on $\mathcal{B}_0 \times \mathcal{B}_1$. In the classical power set context, of course, where $U_0 \subseteq P(X)$ and $U_1 \subseteq P(Y)$, then $\mathcal{B}_0 \times \mathcal{B}_1$ is isomorphic with $P(X \times Y)$ and $U_0 \times U_1$ is the familiar product measure, an ultrafilter measuring all subsets of the plane. But in the general context of complete Boolean algebras, unfortunately, $U_0 \times U_1$ need not be an ultrafilter (see corollary 51), even when $U_0$ and $U_1$ are ultrafilters. In the power set case, the ultrapower by the product measure $U_0 \times U_1$ can be expressed either as the composition of the ultrapower by $U_0$ followed by the ultrapower by $j_0(U_1)$, or as the (external) composition $j_0 \circ j_1$. That is, $j_{U_0 \times U_1} = j_{j_0(U_1)} \circ j_0 = j_0 \circ j_1$. In the general setting of arbitrary complete Boolean algebras, however, since $U_0 \times U_1$ may not be an ultrafilter, we have a somewhat more complex picture:

Theorem 49. If $U$ is an ultrafilter on $\mathcal{B}_0 \times \mathcal{B}_1$ extending $U_0 \times U_1$, then there are embeddings $k_0$ and $k_1^*$ making the following diagram commute, where $j_0 : V \to V_0 \subseteq V_0[G_0]$ and $j_1 : V \to V_1 \subseteq V_1[G_1]$ are the Boolean ultrapowers by $U_0$ and $U_1$, respectively, $j_1^* : V_0 \to V^* \subseteq V^*[G_1^*]$ is the ultrapower by $j_0(U_1)$ as...
computed in $V_0$ and $j_U : V \to V \subseteq [\bar{G}_0 \times \bar{G}_1]$ is the ultrapower by $U$.

Proof. Suppose that $U$ is an ultrafilter on $\mathbb{B}_0 \times \mathbb{B}_1$ projecting to $U_0 \subseteq \mathbb{B}_0$ and $U_1 \subseteq \mathbb{B}_1$ and that $U_0 \times U_1 \subseteq U$. Let $j_U : V \to V_0 \subseteq V_0[G_0]$ and $j_1 : V \to V_1 \subseteq V_1[G_1]$ be the corresponding Boolean ultrapowers by $U_0$ and $U_1$, and let $j_U : V \to \tilde{V} \subseteq [\bar{G}_0 \times \bar{G}_1]$ be the Boolean ultrapower by $U$. By theorem [S] we have the elementary embedding $k_0 : V_0[G_0] \to \tilde{V}$, defined by $k_0 : [\tau]_{V_0} \mapsto [\tau^*]_U$, where $\tau$ is any $\mathbb{B}_0$-name and $\tau^*$ is the natural translation into a $\mathbb{B}_0 \times \mathbb{B}_1$ name via the complete embedding $b \mapsto (b, 1)$. Furthermore, that theorem shows $j_U = k_0 \circ j_0$, and so the lower part of the diagram commutes.

Consider $j_0(U_1)$, which is an ultrafilter on $j_0(\mathbb{B}_1)$ in $V_0$. We claim that $k_0^* j_0(U_1) \subseteq G_1$. To see this, suppose that $x$ is any element of $j_0(U_1)$. Thus, $x = [\bar{x}]_{U_0}$ for some $\mathbb{B}_0$-name $\bar{x}$ with $[\bar{x}]_{U_1} = 1$ and hence $\bigvee_{c \in U_1} [\bar{x} = \bar{c}] = 1$. Let $X = \{(b, c) \in \mathbb{B}_0 \times \mathbb{B}_1 \mid b \parallel \bar{c} \leq \bar{x}\}$. This is clearly open, and it is regular, because if $(b, c) \notin X$, then there is some strengthening $(b_1, c_1) \leq (b, c)$ such that $b_1 \parallel \bar{c}_1 \perp \bar{x}$, which means the cone below $(b_1, c_1)$ avoids $X$, establishing that $X$ is regular. If $b \parallel \bar{c} = \bar{x}$, then clearly $\forall x_b = c$. Since there is a maximal antichain of $b$ forcing that $\bar{x} = \bar{c}$ for some $c \in U_1$, it follows that $\forall \{ b \mid \forall X_b \in U_1 \} \subseteq U_0$ and so $X \subseteq U_0 \times U_1$, and consequently $X \subseteq U$. It follows that $j(X)$ contains an element $(b^*, c^*) \in \bar{G}_0 \times \bar{G}_1$, with the consequence that $b^*$ forces that $\bar{c}^* \leq \bar{x}$, and so $\text{val}(j(\bar{x}), G_0) \subseteq G_1$.

But $\text{val}(j_{\bar{x}}, G_0) = k_0(\text{val}(j_0(\bar{x}), G_0)) = k_0([\bar{x}]_{U_0}) = k_0(\bar{x})$, establishing that $k_0^* j_0(U_1) \subseteq \bar{G}_1$, as we claimed.

Let $j_U : V \to V^* \subseteq V^*[G_1^*]$ be the Boolean ultrapower as computed in $V_0$ using the ultrafilter $j_0(U_1) \subseteq j_0(\mathbb{B}_1)$. Every element of $V^*[G_1^*]$ has the form $[\tau]_{j_0(U_1)}$ (as a set in $V_0$), where $\tau$ is a $j_0(\mathbb{B}_1)$ name in $V_0$. Let $k^* : [\tau]_{j_0(U_1)} \mapsto \text{val}(k_0(\tau), G_1)$. This is well defined and elementary precisely because $k_0^* j_0(U_1) \subseteq G_1$. Furthermore, $k^*$ carries $V^*$ to $V$, since if $\tau$ names an element of the ground model in $V_0$, then $k_0(\tau)$ has this feature in $\tilde{V}$. Note that for $x \in V$, then $j_U(x) = [\bar{x}]_{j_0(U_1)}$, using the $j_0(\mathbb{B}_1)$-name $\bar{x}$ for $x$, and this object maps by $k^*$ to $\text{val}(k_0(\bar{x}), G_1) = k_0(x)$. This shows that $k^* \circ j_U = k_0 \circ j_0$, and so this part of the diagram commutes.

Finally, observe that by applying $j_0$ to the entire embedding $j_1 : V \to V_1 \subseteq V_1[G_1]$, which is the Boolean ultrapower of $V$ by $U_1 \subseteq \mathbb{B}_1$, we get by elementarity the Boolean ultrapower of $V_0$ by $j_0(U_1) \subseteq j_0(\mathbb{B}_1)$ as computed in $V_0$, which is exactly how we defined $j^* : V_0 \to V^* \subseteq V^*[G_1^*]$. In short, $j^* = j_0(j_1)$, $V^* = j_0(V_1)$ and $V^*[G_1^*] = j_0(V_1[G_1])$. It follows that if $j_1(x) = y$, then $j_0(j_1)(j_0(x)) = j_0(y)$, which is to say, $j^* \circ j_0 = j_0 \circ j_1$, and so the entire diagram commutes.

Let us now turn to the possibility that either $U_0 \equiv U_1$ or $U_0 \times U_1$ does not generate an ultrafilter on $\mathbb{B}_0 \times \mathbb{B}_1$. We begin with a necessary and sufficient condition for $U_0 \times U_1$ to generate an ultrafilter.

Theorem 50. Suppose that $U_0 \subseteq \mathbb{B}_0$ and $U_1 \subseteq \mathbb{B}_1$ are ultrafilters in $V$ on complete Boolean algebras in $V$. Then the following are equivalent:

1. $U_0 \times U_1$ is an ultrafilter in $\mathbb{B}_0 \times \mathbb{B}_1$.
2. $\bar{U}_1$ generates an ultrafilter in $\text{RO}(\mathbb{B}_1)$.

And in this case, the Boolean ultrapower by $U_0 \times U_1$ is isomorphic to the Boolean ultrapower by $U_0 \ast (j_{U_0}(U_1))$ on the iteration algebra $\mathbb{B}_0 \ast \text{RO}(\mathbb{B}_1)$.  

26
Proof. (1 → 2) We prove the contrapositive. If (2) fails, then there is a \( \mathbb{B}_0 \)-name \( \dot{Y} \) such that \( \dot{Y} \subseteq \dot{\mathbb{B}}_1 \) is regular open, but \( \dot{Y}, \neg \dot{Y} \) contain no elements of \( U_1 \) in \( U_0 \), where \( \neg \dot{Y} \) denotes the negation of \( \dot{Y} \) in the sense of \( \text{RO}(\dot{\mathbb{B}}_1) \). Let \( X = \{ (b,c) \mid b \vdash c \in \dot{Y} \} \). Since \( \dot{Y} \) names an open set, \( X \) is also open. Observe that if \( (b,c) \notin X \), then \( b \not\vdash c \in \dot{Y} \), and so there is some \( b_0 \leq b \) and \( c_0 \leq c \) such that \( b_0 \vdash c_0 \in \neg \dot{Y} \). It follows that the entire cone below \( (b_0,c_0) \) in \( \mathbb{B}_0^+ \cong \mathbb{B}_1^+ \) avoids \( X \). So we have proved that every neighborhood of any point not in \( X \) contains a neighborhood of the complement of \( X \), and so \( X \) is regular open. Thus, \( X \subseteq \mathbb{B}_0 \times \mathbb{B}_1 \). For any \( b \in \mathbb{B}_0 \), since \( X_b \) is a regular open subset of \( \mathbb{B}_1 \), which is complete, it follows that \( X_b \) is simply the cone below \( \forall X_b \subseteq \mathbb{B}_1 \). Since \( b \vdash X_b \subseteq \dot{Y} \), it follows that \( \forall X_b \not\subseteq U_1 \), for otherwise \( b \) would force that \( \dot{Y} \) in the filter generated by \( U_1 \), contrary to our assumption. The set \( \{ b \mid \forall X_b \in U_1 \} \), therefore, is empty, and consequently \( X \not\subseteq U_0 \times U_1 \). Now consider the negation \( \neg X = \{ (b,c) \mid \text{the cone below } (b,c) \text{ avoids } X \} = \{ (b,c) \mid b \vdash c \in \neg \dot{Y} \} \), which is the interior of the complement of \( X \). Since \( b \vdash \neg X_b \subseteq \neg \dot{Y} \), it follows again that \( \forall (\neg X)_b \not\subseteq U_1 \), and consequently \( \neg X \not\subseteq U_0 \times U_1 \). So \( U_0 \times U_1 \) is not an ultrafilter.

(2 → 1) Suppose that 2 holds. Let \( j_0 : V \rightharpoonup V_0 \subseteq V_0[G_0] \) be the Boolean ultrapower by \( U_0 \). Applying \( j_0 \) to statement 2, it follows that \( j_0(U_1) \) generates an ultrafilter \( U_1^* \subseteq \text{RO}(j_0(\mathbb{B}_1)) \) in \( V_0[G_0] \), with corresponding Boolean ultrapower \( j_1 \), giving rise to the following commutative diagram, as in theorem \( 17 \) where \( j \) is the ultrapower by the ultrafilter \( U_0 \times U_1^* \) on the iteration algebra \( \mathbb{B}_0 \times \text{RO}(\dot{\mathbb{B}}_1) \). The filters \( [\mathbb{G}_0]_j \cong [\mathbb{G}_1]_j \) are \( V \)-generic for the forcing \( j(\mathbb{B}_0 \times \text{RO}(\dot{\mathbb{B}}_1)) \). But this forcing is equivalent to the product forcing \( j(\mathbb{B}_0^+ \times \mathbb{B}_1^+) \), so by genericity \( [\mathbb{G}_0 \times \mathbb{G}_1]_j \) is the same as \( [\mathbb{G}_0 \times \mathbb{G}_1]_j \), with \( [\mathbb{G}_0 \times \mathbb{G}_1]_j \subseteq j(\mathbb{B}_0^+ \times \mathbb{B}_1^+) \). Suppose that \( X \subseteq B_0 \times \mathbb{B}_1 \), a regular open subset of \( \mathbb{B}_0^+ \cong \mathbb{B}_1^+ \). We now observe that

\[
X \in U_0 \times U_1 \iff \forall b \in U_0 \forall X_b \in U_1 \subseteq U_0 \\
\iff j_0(\forall b \in U_0 \forall X_b \in U_1) \subseteq G_0 \\
\iff \exists b^* \in G_0 \cap j_0(\forall b \in U_0 \forall X_b \in U_1) \\
\iff \exists b^* \in G_0 \text{ such that } \forall b^* \in j_0(U_1) \\
\iff \exists b^* \in G_0 \text{ such that } j_1(b^* \in j_0(U_1)) \\
\iff \exists b^* \in G_0 \text{ such that } c^* \in G_1 \text{ such that } j_1(c^* \in j_0(U_1)) \\
\iff \exists b^* \in G_0 \text{ such that } c^* \in G_1 \text{ such that } j_1(c^* \in j_0(U_1)) \\
\iff \exists b^* \in G_0 \text{ such that } c^* \in G_1 \text{ such that } j_1(c^* \in j_0(U_1))
\]

Since \( j_1(G_0) = G_1 \), this means that \( U_0 \times U_1 \) is precisely the pre-image by \( j \) of the filter generated by \( G_0 \times G_1 = G_0 \times G_1 \) in \( j(\mathbb{B}_0 \times \mathbb{B}_1) \). Since this latter filter is \( V \)-generic, it is an ultrafilter, and so \( U_0 \times U_1 \) is the pre-image of an ultrafilter, and hence also an ultrafilter, as desired.

Because we know that the product forcing \( \mathbb{B}_0 \times \mathbb{B}_1 \) is forcing equivalent to the iteration \( \mathbb{B}_0 \times \text{RO}(\dot{\mathbb{B}}_1) \), it follows that these complete Boolean algebras are canonically isomorphic. What we have argued above in essence is that hypothesis 2 ensures that the image of \( U_0 \times U_1 \) under this isomorphism is the same as \( U_0 \times U_1^* \). Thus, the Boolean ultrapower by \( U_0 \times U_1 \) is isomorphic to the Boolean ultrapower by \( U_0 \times U_1^* \). □

It follows quite generally that \( U_0 \times U_1 \) need not be an ultrafilter.

**Corollary 51.** For any atomless complete Boolean algebra \( \mathbb{B}_1 \) there is a complete Boolean algebra \( \mathbb{B}_0 \) such that for any ultrafilters \( U_0 \subseteq \mathbb{B}_0 \) and \( U_1 \subseteq \mathbb{B}_1 \) in \( V \), the product filter \( U_0 \times U_1 \) on \( \mathbb{B}_0 \times \mathbb{B}_1 \) is not an ultrafilter.

Proof. Suppose that \( \mathbb{B}_1 \) is a given atomless complete Boolean algebra. Let \( \mathbb{B}_0 \) be any complete Boolean algebra necessarily forcing that \( \mathbb{B}_1 \) becomes countable (this is much more than required). Suppose that \( U_0 \subseteq \mathbb{B}_0 \) and \( U_1 \subseteq \mathbb{B}_1 \) are ultrafilters in \( V \). In any forcing extension \( V[H] \) by a \( V \)-generic filter \( H \subseteq \mathbb{B}_0 \), the filter \( U_1 \) becomes countable. In particular, the filter in \( \text{RO}(\mathbb{B}_1)_[V[H]] \) generated by \( U_1 \) is generated by a countable set, and so by lemma \( 43 \) it is not an ultrafilter. In short, \( U_1 \) does not generate an ultrafilter in \( \text{RO}(\mathbb{B}_1) \) in \( V[H] \). By theorem \( 50 \) it follows that \( U_0 \times U_1 \) is not an ultrafilter. □

27
Theorem 52. Suppose that $U_0 \subseteq \mathbb{B}_0$ and $U_1 \subseteq \mathbb{B}_1$ are ultrafilters in $V$ on complete Boolean algebras in $V$, and that $j_0 : V \rightarrow V_0$ is the Boolean ultrapower by $U_0$. Then the following are equivalent:

1. $U_0 \boxtimes U_1$ generates the product filter $U_0 \times U_1$ in $\mathbb{B}_0 \times \mathbb{B}_1$.
2. $j_0 \upharpoonright U_1$ generates the ultrafilter $j_0(U_1)$ in $V_0$.

Proof. (1 $\rightarrow$ 2) Suppose that $U_0 \boxtimes U_1$ generates $U_0 \times U_1$, and consider any $x \in j_0(U_1)$. This object is represented by a $\mathbb{B}_0$-name $\dot{x}$ such that $x = [\dot{x}]_{U_0}$ and $[\dot{x} \in U_1]_{U_0} = 1$. Let $X = \{ (b,c) \in \mathbb{B}_0^+ \times \mathbb{B}_1^+ \mid b \Vdash \dot{c} \leq \dot{x} \}$. This is clearly open, and it is regular because if $(b,c) \notin X$, then $b \not\Vdash \dot{c} \leq \dot{x}$, so there are $b^* \leq b$ and $c^* \leq c$ such that $b^* \Vdash \dot{c} \leq \dot{x}$, which means the cone below $(b^*,c^*)$ avoids $X$. For each $c \in U_1$, let $b_c = [\dot{x} = \dot{c}]_{U_0}$, and observe that $\bigwedge_{c \in U_1} b_c = [\dot{x} \in U_1]_{U_0} = 1$. Note that $(b,c) \in X$ for every $c$ for which $b_c \neq 0$. In particular, $c \leq \exists X_b \in U_1$ in this case, and so $\forall \{ b \mid \exists X_b \in U_1 \} \supseteq \bigwedge_{c \in U_1} b_c = 1$. Thus, $X \subseteq U_0 \times U_1$. By 1, this means that this is some $b \in U_0$ and $c \in U_1$ with $(b,c) \in X$, and so $b \Vdash \dot{c} \leq \dot{x}$. This implies $[\dot{c}]_{U_0} \leq [\dot{x}]_{U_0}$ in $[\mathbb{B}_1]_{U_0}$, or in other words, $j_0(c) \leq x$ in $j_0(\mathbb{B}_1)$, as desired.

(2 $\rightarrow$ 1) Conversely, suppose that $j_0 \upharpoonright U_1$ generates $j_0(U_1)$ in $j_0(\mathbb{B}_1)$ in $V_0$. Consider any $X \subseteq U_0 \times U_1$, so that $X$ is a regular open subset of $\mathbb{B}_0^+ \times \mathbb{B}_1^+$ and $\forall \{ b \mid \exists X_b \in U_1 \} \in U_0$. Since this outer join is in $U_0$, it follows that $j_0(\forall \{ b \mid \exists X_b \in U_1 \}) \in G_0$. Since $G_0$ is $V_0$-generic for $j_0(\mathbb{B}_0)$, there is some $b^* \in G_0$ with $\forall j_0(\exists X_b) \in j_0(U_1)$. By 2, there is some $c \in U_1$ with $j_0(c) \leq j_0(X_b)$. Thus, $(b^*,j_0(c)) \in j_0(X)$, and so $j_0(\forall X^c) \in G_0$ and so $\forall X^c \in U_0$. Let $b = \forall X^c$. Since $X$ is regular, it follows that $b \in X^c$ and so $(b,c) \in X$. So $X$ contains an element of $U_0 \boxtimes U_1$, as desired.

The following theorem is an analogue of mutual genericity of generic filters, although the statements in the theorem can hold for non-regular filters. Note that the previous theorem only required $j_0 \upharpoonright U_1$ to generate an ultrafilter in $V_0$, here we ask that it generate an ultrafilter on the completion of $j_0(\mathbb{B}_1)$ in $V_0[G_0]$. In the below, $U_0 \times U_1$ refers to the dual product, defined by $X \subseteq U_0 \times U_1$ if and only if $\forall \{ c \mid \exists X^c \in U_0 \} \in U_1$, where $X^c = \{ b \mid (b,c) \in X \}$. This amounts merely to taking the factors in the other order, and so $U_0 \times U_1 \cong U_1 \times U_0$.

Theorem 53. Suppose that $U_0 \subseteq \mathbb{B}_0$ and $U_1 \subseteq \mathbb{B}_1$ are ultrafilters in $V$ on complete Boolean algebras in $V$, with corresponding Boolean ultrapowers $j_0 : V \rightarrow V_0 \subseteq V_0[G_0] = V^\mathbb{B}_0/U_0$ and $j_0 : V \rightarrow V_1 \subseteq V_1[G_1] = V^\mathbb{B}_1/U_1$. Then the following are equivalent:

1. $U_0 \times U_1$ is an ultrafilter in $\mathbb{B}_0 \times \mathbb{B}_1$.
2. $j_0 \upharpoonright U_1$ generates an ultrafilter on $\text{RO}(j_0(\mathbb{B}_1))$ in $V_0[G_0]$.
3. $j_0 \upharpoonright U_1$ generates an ultrafilter on $\text{RO}(j_0(\mathbb{B}_1))$ in $V_1[G_1]$.

And in this case, the ultrafilter generated by $U_0 \boxtimes U_1$ is precisely $U_0 \times U_1$, which in this case is equal to $U_0 \times U_1$.

Proof. We argue first that (1 $\leftrightarrow$ 2). Since $U_0 \boxtimes U_1$ is contained within the filter $U_0 \times U_1$, it is clear that $U_0 \boxtimes U_1$ generates an ultrafilter on $\mathbb{B}_0 \times \mathbb{B}_1$ if and only if $U_0 \boxtimes U_1$ generates $U_0 \times U_1$ and $U_0 \times U_1$ generates an ultrafilter on $\mathbb{B}_0 \times \mathbb{B}_1$. By theorem 52, the former requirement is equivalent to the assertion that $j_0 \upharpoonright U_1$ generates $j_0(U_1)$ on $j_0(\mathbb{B}_1)$ in $V_0$. By theorem 53, the latter requirement is equivalent to the assertion that $j_0(U_1)$ in turn generates an ultrafilter on $\text{RO}(j_0(\mathbb{B}_1))$ in $V_0[G_0]$. Altogether, therefore, we see that $U_0 \boxtimes U_1$ generates an ultrafilter if and only if $j_0 \upharpoonright U_1$ generates an ultrafilter on $\text{RO}(j_0(\mathbb{B}_1))$ in $V_0[G_0]$, establishing (1 $\leftrightarrow$ 2). Because statement 1 is respected by the natural automorphism of $\mathbb{B}_0 \times \mathbb{B}_1$ with $\mathbb{B}_1 \times \mathbb{B}_0$, the dual equivalence (1 $\leftrightarrow$ 3) follows simply by swapping the order of the factors. Finally, if $U_0 \boxtimes U_1$ generates an ultrafilter on $\mathbb{B}_0 \times \mathbb{B}_1$, then because it is included in the filters $U_0 \times U_1$ and $U_0 \times U_1$, these must all be the same ultrafilter.

An equivalent formulation of (2) asserts that whenever $\dot{X}$ is a $\mathbb{B}_0$-name for which $[\dot{X} \subseteq \mathbb{B}_1]$ is regular open $^0_{\mathbb{B}_0} \neq 1$, then there is some $c \in U_1$ with $[\dot{c} \in X]^0_{\mathbb{B}_0} \in U_0$ or $[\dot{c} \in \neg X]^0_{\mathbb{B}_0} \in U_0$.

Let us now give an analysis of whether $U_0 \boxtimes U_1$ generates an ultrafilter in terms of the descent spectrums of $U_0$ and $U_1$.

Theorem 54. If $U_0 \subseteq \mathbb{B}_0$ and $U_1 \subseteq \mathbb{B}_1$ are ultrafilters on complete Boolean algebras whose descent spectrums have a common order type $\kappa$, then $U_0 \boxtimes U_1$ does not generate an ultrafilter on $\mathbb{B}_0 \times \mathbb{B}_1$. Indeed, it does not even generate $U_0 \times U_1$. 28
Proof. Suppose that $U_0$ and $U_1$ admit descents $\langle b_\alpha \mid \alpha < \kappa \rangle$ and $\langle c_\alpha \mid \alpha < \kappa \rangle$, respectively, of common order type $\kappa$. By moving to cofinal subsequences, we may assume that $\kappa$ is regular and that these are strict descents. As in theorem 53, the difference elements $d_\alpha = b_\alpha - b_{\alpha+1}$ and $e_\alpha = c_\alpha - c_{\alpha+1}$ form maximal antichains $D = \{ d_\alpha \mid \alpha < \kappa \} \subseteq \mathcal{B}_0$ and $E = \{ e_\alpha \mid \alpha < \kappa \} \subseteq \mathcal{B}_1$. It is easy to see that $D \times E = \{ (d_\alpha, e_\beta) \mid \alpha, \beta < \kappa \}$ is a maximal antichain in $\mathcal{B}_0 \times \mathcal{B}_1^\kappa$. In the completion $\mathcal{B}_0 \times \mathcal{B}_1$, let $x = \lor \{ (d_\alpha, e_\beta) \mid \alpha < \beta \}$, which corresponds to the upper left triangle. Since $D \times E$ was a maximal antichain, it follows that $\neg x = \lor \{ (d_\alpha, e_\beta) \mid \alpha \geq \beta \}$, the lower right triangle. We claim that neither $x$ nor $\neg x$ are in the filter generated by $U_0 \equiv U_1$. Note that every $b \in U_0$ has nonzero meet with every $b_\beta = \lor_{\alpha \geq \beta} d_\alpha$, and consequently has nonzero meet with unboundedly many difference elements $d_\alpha$. Similarly, every $c \in U_1$ has nonzero meet with unboundedly many difference elements $e_\beta$. Thus, every pair $(b, c) \in U_0 \times U_1$ has nonzero meet with both $x$ and $\neg x$. Consequently, the filter generated by $U_0 \times U_1$ in $\mathcal{B}_0 \times \mathcal{B}_1$ does not decide $\{ x, \neg x \}$, and so it is not an ultrafilter. To prove the final claim of the theorem, we observe that for every $\alpha$, the join $\lor \{ e_\beta \mid \alpha < \beta \}$ is precisely $c_{\alpha+1}$, which is $U_1$, meaning that all the vertical slices in the set representing $x$ are large, and so $x \in U_0 \times U_1$.

Corollary 55. For any pair $\mathcal{B}_0$ and $\mathcal{B}_1$ of infinite complete Boolean algebras, there are ultrafilters $U_0 \subseteq \mathcal{B}_0$ and $U_1 \subseteq \mathcal{B}_1$ such that $U_0 \not\equiv U_1$ does not generate an ultrafilter on $\mathcal{B}_0 \times \mathcal{B}_1$.

Proof. If $\mathcal{B}_0$ and $\mathcal{B}_1$ are infinite, then there are countable maximal antichains $\{ a_n \mid n < \omega \} \subseteq \mathcal{B}_0$ and $\{ b_n \mid n < \omega \} \subseteq \mathcal{B}_1$, from which it is easy to construct countable descending sequences $c_m = \lor_{n \geq m} a_n$ and $d_m = \lor_{n \geq m} b_n$ with $\land_m c_m = 0$ and $\land_m d_m = 0$. Any ultrafilters $U_0$ and $U_1$ with $c_m \in U_0$ and $d_m \in U_1$ for all $m < \omega$ will therefore have descents of common order type $\omega$. By theorem 54, therefore, $U_0 \not\equiv U_1$ does not generate an ultrafilter in $\mathcal{B}_0 \times \mathcal{B}_1$.

We have a partial converse:

Theorem 56. If the cardinals in the descent spectrum of $U_0$ all lie completely below the descent spectrum of $U_1$, then $U_0 \equiv U_1$ generates $U_0 \times U_1$.

Proof. Suppose that the cardinals in the descent spectrum of $U_0$ all lie completely below the descent spectrum of $U_1$, and let $j_0 : \mathcal{B} \to \mathcal{B}_0$ be the Boolean ultrapower by $U_0$. Consider any element $x$ in $j_0(U_1) \subseteq j_0(\mathcal{B}_1)$. By lemma 33, we may represent $x = [f]_{U_0}$ using a spanning function $f : A \to \mathcal{B}$ on a maximal antichain $A$, whose cardinality is in the descent spectrum of $U_0$. We may assume $f(a) \in U_1$ for all $a \in A$. Since $|A|$ lies below any member of the descent spectrum of $U_1$, it follows that $U_1$ is sufficiently complete that $c = \land_{a \in A} f(a)$ is in $U_1$. Thus, $j_0(c) \leq [f]_{U_0} = x$, and so we have established that $j_0 \circ U_1$ generates $j_0(U_1)$ in $\mathcal{B}_0$. By theorem 52, this means that $U_0 \equiv U_1$ generates $U_0 \times U_1$.

The question of whether or not $U_0 \equiv U_1$ generates an ultrafilter, or whether or not it generates $U_0 \times U_1$, appears to be related to the descent spectrums.

11. Ideals

In this section, we generalize the concept of precipitous ideal and other notions from the case of power sets to arbitrary complete Boolean algebras. Suppose that $\mathcal{B}$ is a complete Boolean algebra. An ideal $I$ on $\mathcal{B}$ is a subset $I \subseteq \mathcal{B}$ such that $0 \in I$, $1 \notin I$, closed under join $a, b \in I \to a \lor b \in I$ and lower cones $a \leq b \in I \to a \in I$. To each ideal there is an associated equivalence relation on $\mathcal{B}$, where $a = b$ if and only if $a \triangle b \in I$, where $a \triangle b = (a - b) + (b - a)$ is the symmetric difference. The induced order $a \leq_I b \leftrightarrow a - b \in I$ is well defined on the equivalence classes $[a]_I = \{ b \mid a = b \}$, and the quotient $\mathcal{B}/I = \{ [a]_I \mid a \in \mathcal{B} \}$ is a Boolean algebra, although it is not necessarily complete. If $F$ is a filter on $\mathcal{B}/I$, then the corresponding induced filter on $\mathcal{B}$ is $\cup F = \{ b \in \mathcal{B} \mid [b]_I \in F \}$ on $\mathcal{B}$. It is easy to see that the induced filter of an ultrafilter is also an ultrafilter.

Definition 57. An ideal $I$ on a complete Boolean algebra $\mathcal{B}$ is precipitous if whenever $G \subseteq \mathcal{B}/I$ is a $\mathcal{V}$-generic filter, then the Boolean ultrapower $\mathcal{V}_U$ by the induced ultrapower $U = \cup G$ on $\mathcal{B}$ is well-founded.

This definition directly generalizes the usual Jech-Prikry definition of precipitous ideal in the case that $\mathcal{B}$ is a power set Boolean algebra. (The word precipitous was chosen by Jech, in honor of Prikry, whose name reportedly has that meaning in Hungarian.) Technically, the trivial ideal $I = \{ 0 \}$ is precipitous, since if $G \subseteq \mathcal{B}/I$ is $\mathcal{V}$-generic, then $U = \cup G \subseteq \mathcal{B}$ is also $\mathcal{V}$-generic, and so $\mathcal{V}_U$ is isomorphic to $\mathcal{V}$, which is

29
well-founded. Consequently, we are primarily interested in the case of nontrivial ideals \( I \), meaning that \( I \) contains a maximal antichain from \( \mathcal{B} \), or in other words, that \( \forall I = 1 \), since it is exactly this condition which ensures that the resulting filter \( U = \bigcup G \) is not \( V \)-generic and consequently has a nontrivial Boolean ultrapower \( j_U : V \to \hat{V}_U \). This requirement therefore generalizes the concept of a non-principal ideal from the power set context.

If \( \mathcal{B} \) is a complete Boolean algebra and \( A \subseteq \mathcal{B} \) is a maximal antichain of size \( \kappa \), then the small ideal relative to \( A \) is the ideal of all elements of \( \mathcal{B} \) below the join of a small subset of \( A \), that is, the ideal \( I = \{ b \in B \mid \exists a_0 \subseteq A | A_0 < \kappa \text{ and } b \leq \vee A_0 \} \).

**Theorem 58.** The small ideal, relative to any maximal antichain \( A \subseteq \mathcal{B} \) of infinite regular size \( \kappa \), is not precipitous. If \( G \subseteq \mathcal{B}/I \) is \( V \)-generic, then the Boolean ultrapower \( j_U \) by the induced ultrafilter \( U = \cup G \) is well-founded up to \( \kappa \), but ill-founded below \( j_U(\kappa) \).

**Proof.** In theorem 19, we already established that the induced ultrafilter \( U = \cup G \) meets all small maximal antichains in \( \mathcal{B} \) in \( V \), but not \( A \), and so the Boolean ultrapower map \( j_U : V \to \hat{V}_U \) has critical point \( \kappa \). The ultrapower \( \hat{V}_U \) is consequently well-founded up to \( \kappa \).

We now show that \( \hat{V}_U \) is necessarily ill-founded below \( j_U(\kappa) \). To do so, we adapt the proof from the classical power set context. Enumerate \( A = \{ a_\alpha \mid \alpha < \kappa \} \) in a one-to-one manner. For every \( X \subseteq \kappa \) of size \( \kappa \), let \( a_X = \bigvee_{\alpha \in X} a_\alpha \) and define \( f_X : A \to \kappa \) by \( f_X(a_\alpha) = \text{ot}(X \cap \alpha) \). Observe that \( f_X \) is a spanning function on domain \( A \), with value less than \( \kappa \) at every point, but constant only on small sets. Thus, \( [f_X]_U \) will represent an ordinal less than \( j_U(\kappa) \) in the (functional presentation of the) Boolean ultrapower, but not any ordinal less than \( \kappa \). If \( Y \) consists of the successor elements of \( X \), in the canonical enumeration of \( X \), then it is easy to see that \( f_Y(a_\alpha) < f_X(a_\alpha) \) for any \( \alpha \in Y \). Thus, provided that \( a_Y \in U \), we will have \( [f_Y]_U < [f_X]_U \).

This is one step of how we shall build our descending sequence in the Boolean ultrapower. Specifically, we build a sequence of \( \mathcal{B}/I \)-names \( f_n \) for spanning functions \( f_n : A \to \kappa \) in \( V \). We will specify a sequence of maximal \( I \)-partitions \( A \subseteq I^+ \), with every element of \( A \) having the form \( a_X \) for some \( X \subseteq \kappa \) of size \( \kappa \).

Assumed to each such \( a_X \) have the spanning function \( f_X : A \to \kappa \), and by mixing these objects along the partition, we arrive at a \( \mathcal{B}/I \)-name \( f \) for a spanning function on \( A \) in \( V \), with \( a_X \) forcing \( f = f_X \). We begin with \( A_0 = \{ 1 \} \), resulting in the name \( f_0 \) for the spanning function \( f_0 : a_0 \to \alpha \). Given \( f_n \) and \( A_n \), let us say that \( Y \subseteq \kappa \) of size \( \kappa \) is good, if there is some \( X \subseteq \kappa \) with \( a_X \in A_n \) and for all \( \alpha \in Y \), \( f_Y(a_\alpha) < f_X(a_\alpha) \). The argument we gave above established that the collection of good \( Y \subseteq \kappa \) is dense in \( P(\kappa) \) modulo the ideal of small sets. By using the canonical complete embedding of \( P(\kappa) \) into \( \mathcal{B} \) via \( P(A) \), this means that the set \( D = \{ a_Y \mid Y \subseteq \kappa \text{ is good } \} \) is predense in \( I^+ \). Thus, we may select a maximal \( I \)-partition \( A_{n+1} \subseteq D \), each having the form \( a_Y \) for a good \( Y \subseteq \kappa \). By mixing the corresponding \( f_Y \), this gives rise to the desired name \( f_{n+1} \), with \( a_Y \) forcing \( f_{n+1} = f_Y \) for each \( a_Y \in A_{n+1} \).

Assume now that \( G \subseteq \mathcal{B}/I \) is \( V \)-generic, and let \( U = \bigcup G \subseteq \mathcal{B} \) be the corresponding ultrafilter on \( \mathcal{B} \). For each \( n < \omega \), the filter \( G \) selects a unique element \( a_{X_n} \) from \( A_n \), which therefore decides \( f_n = \text{val}(f_n, G) = f_{X_n} \). By construction, consequently, \( [f_{n+1}]_U < [f_n]_U \) in the Boolean ultrapower, and so the Boolean ultrapower is ill-founded below \( j_U(\kappa) \), as desired.

**Corollary 59.** If one allows \( U \not\in V \), then statement 1 of theorem 25 is no longer necessarily equivalent to statements 2 through 5.

What we did in the previous argument essentially was to adapt the classical proof that the ideal \( I_0 \) of small subsets of \( \kappa \) is not precipitous. In corollary 78, we will show that all the classical generic ultrapowers obtained by forcing over the quotient of a power set \( P(\kappa)/I \) can also be obtained as generic Boolean ultrapowers obtained by forcing over the quotient of \( \mathcal{B}/I^+ \), for any complete Boolean algebra having an antichain of size \( \kappa \). Because of this, the generic Boolean ultrapowers we consider in this section generalize the classical generic embeddings.

We define that a quotient \( \mathcal{B}/I \) of a complete Boolean algebra \( \mathcal{B} \) by an ideal \( I \) has the disjointifying property if for every maximal antichain \( A \subseteq \mathcal{B}/I \), there is a choice of representatives \( b_a \in \mathcal{B} \) such that \( a = [b_a]_I \) for each \( a \in A \) and \( \{ b_a \mid a \in A \} \) forms a maximal antichain in \( \mathcal{B} \). For such a disjointification, it is not difficult to see that if \( A_0 \subseteq A \), then \( \forall A_0 = [\bigvee_{\alpha \in A} b_\alpha]_I \). That is, the join in \( \mathcal{B}/I \) of a subset of \( A \) is simply the equivalence class of the join in \( \mathcal{B} \) of the corresponding collection of representatives (without disjointification and maximality, this is not typically the case). Consequently, the quotient \( \mathcal{B}/I \) is a complete Boolean algebra.
Theorem 60. Suppose that \( U_0 \) is an ultrafilter on the quotient \( \mathbb{B}/I \) of a complete Boolean algebra \( \mathbb{B} \) by an ideal \( I \) with the disjointifying property. Then the Boolean ultrapower by the induced ultrafilter \( U = \bigcup U_0 \) on \( \mathbb{B} \) factors through \( U_0 \) as follows:

\[
\begin{array}{c}
V \\
\downarrow j_U \\
\mathbb{B}/I \\
\uparrow k_U \\
\hat{V}_U
\end{array}
\]

Proof. We use the functional representation of \( \hat{V}_U \) as \( V_{U_0}^{\mathbb{B}/I} \). The typical element of \( V_{U_0}^{\mathbb{B}/I} \) is \([f]_{U_0}\), where \( f : A \to V \) is a spanning function on a maximal antichain \( A \subseteq \mathbb{B}/I \). By the disjointifying property, we may select representatives \( b_a \in \mathbb{B} \) for each \( a \in A \) such that \( A^* = \{ b_a \mid a \in A \} \) forms a maximal antichain in \( \mathbb{B} \). Let \( f^* : A^* \to V \) be the corresponding spanning function on \( A^* \), with \( f^*(b_a) = f(a) \). We define \( k : [f]_{U_0} \mapsto [f^*]_U \).

We argue first that \( k \) is well defined. To begin, we check that the particular choice of disjoint representatives \( b_a \) does not matter. If \( c_a \) is some other way of choosing disjoint representatives, leading to the maximal antichain \( A^{**} = \{ c_a \mid a \in A \} \) in \( \mathbb{B} \), then we compare the function \( f^* : b_a \mapsto f(a) \) with \( f^{**} : c_a \mapsto f(a) \). Let \( d_a = b_a \cap c_a \), and observe that \( |d_a| = |b_a| = |c_a| \) and also that the \( d_a \) are disjoint for different \( a \). If \( x \) is disjoint from \( d_a \) for all \( a \in A \), then \( [x]_I \) would be incompatible in \( \mathbb{B}/I \) with \( \{ d_a \}_I \), which is to say, with every element of \( A \), and so \( x \in I \). It follows that \( \bigvee_{a \in A} d_a \in U \). Extend \( \{ d_a \mid a \in A \} \) to a maximal antichain \( D \subseteq \mathbb{B} \) refining both \( A^* \) and \( A^{**} \). Note that \( (f^* \downarrow D)(d_a) = f(a) = (f^{**} \downarrow D)(d_a) \), and so \( \bigvee \{ d \in D \mid (f^* \downarrow D)(d) = (f^{**} \downarrow D)(d) \} \geq \bigvee_{a \in A} d_a \in U \), and so \( [f^*]_U = [f^{**}]_U \), as desired. Next, we show that \( k \) is well defined for spanning functions with the same domain. Suppose \( [f]_{U_0} = [g]_{U_0} \) for two spanning functions \( f : A \to V \) and \( g : A \to V \) in \( \mathbb{B}/I \), both with domain \( A \). Thus, \( \bigvee \{ a \in A \mid f(a) = g(a) \} \in U_0 \). By the observation just before the theorem, this means that \( \bigvee \{ d_a \mid a \in A, f(a) = g(a) \}_I \in U_0 \) and so \( \bigvee \{ b \in A^* \mid f^*(b) = g^*(b) \} \in U_0 \), which means \( [f^*]_U = [g^*]_U \). Finally, we show that \( k \) is well defined for the reduction of a spanning function \( f : A \to V \) to a finer domain \( B \subseteq A \). Fix any disjoint selection of representatives \( x_b \in \mathbb{B} \) for \( b \in B \) such that \( [x_b]_B = b \), and let \( B^* = \{ x_b \mid b \in B \} \) be the corresponding antichain. Observe that \( y_a = \bigvee \{ x_b \mid b \leq a, b \in B \} \) is a disjoint selection of representatives for \( A \), with corresponding \( A^* = \{ y_a \mid a \in A \} \) and corresponding spanning functions \( f^* : A^* \to V \) and \( f^{**} : B^* \to V \) defined by \( f^*(y_a) = f(a) \) and \( f^{**}(x_b) = (f \downarrow B)(b) \). Since \( (f \downarrow B) = f(a) \) for the unique element \( a \in A \) above \( b \), it follows that \( (f^* \downarrow B^*)(x_b) = f^*(y_a) \) for every \( x_b \in B^* \), which implies that \( [f^*]_U = [f^{**}]_U \), as desired. So altogether, \( k \) is well defined.

We now show that \( k \) is fully elementary by induction on formulas. The atomic case and Boolean connectives are straightforward, as is the upward direction of the quantifier case. Suppose now that \( V_{U_0}^{\mathbb{B}/I} \models \exists x \varphi(x,[f^*]_U) \), where \( f^* : A^* \to V \) arises from the spanning function \( f : A \to V \) for a maximal antichain \( A \subseteq \mathbb{B}/I \). Define a spanning function \( g : A \to V \) by \( g(b) = \varphi(g(b),f^*(b)) \), if there is any such witness. By the L"os Theorem, it follows that \( V_{U_0}^{\mathbb{B}/I} \models \varphi([g]_U,[f^*]_U) \), and we have found a witness using the very same antichain \( A^* \). Define \( g_0 : A \to V \) by \( g_0(a) = g(b_a) \), where \( b_a \) is the unique representative of \( a \) in \( A^* \). It follows that \( g = g_0 \) and so \( [g]_U = k([g_0]_{U_0}) \). Thus, \( V_{U_0}^{\mathbb{B}/I} \models \varphi([g_0]_{U_0},[f]_{U_0}) \), as desired. So \( k \) is fully elementary.

If \( I \subseteq \mathbb{B} \) is an ideal, then we say that a subset \( A \subseteq I^+ \) is an (maximal) antichain modulo \( I \) if distinct elements of \( A \) are not equivalent modulo \( I \) and \( \{ [a]_I \mid a \in A \} \) is a (maximal) antichain in the quotient \( \mathbb{B}/I \). A tree of maximal antichains modulo \( I \) in \( \mathbb{B} \) is a sequence \( (A_n \mid n < \omega) \) of maximal antichains modulo \( I \) such that \( A_{n+1} \) refines \( A_n \) modulo \( I \), meaning that for every element \( a \in A_{n+1} \) there is \( b \in A_n \) such that \( [a]_I \leq [b]_I \) in \( \mathbb{B}/I \). A Boolean algebra \( \mathbb{B} \) is countably distributive if as a forcing notion it adds no new \( \omega \)-sequences of ordinals. This is equivalent to the countable distributivity rule: \( \bigwedge_n \forall A_n = \forall \{ \bigwedge_n f(n) \mid f : \omega \to \mathbb{B}, f(n) \in A_n \} \).
Theorem 61. Suppose that $\mathbb{B}$ is a countably distributive complete Boolean algebra and $I \subseteq \mathbb{B}$ is an ideal. Then the following are equivalent:

1. $I$ is precipitous.
2. For every tree of maximal antichains $A_n \subseteq \mathbb{B}$, every $a_0 \in A_0$ can be continued to a sequence $\langle a_n \mid n < \omega \rangle$ with $a_n \in A_n$, $a_{n+1} \leq_I a_n$ and $\bigwedge_n a_n \neq 0$.

Proof. Suppose that $I$ is precipitous. Fix the tree of maximal antichains $A_n \subseteq \mathbb{B}$, for $n < \omega$, and $a_0 \in A_0$. Let $G \subseteq \mathbb{B}/I$ be $V$-generic below $a_0$. Since $I$ is precipitous, this means that the Boolean ultrapower of $V$ by the corresponding filter $U = \bigcup G \subseteq \mathbb{B}$ is well-founded. Let $j : V \to M = V_U$ be the corresponding ultrapower map into the transitive class $M$. By theorem there is in $V[G]$ an $M$-generic ultrafilter $H \subseteq j(\mathbb{B})$ with $j^{-1} U \subseteq H$. Since $H$ is $M$-generic, there is a unique condition $b_n \in j(A_n) \cap H$. In fact, $b_n$ is exactly the element of $j(\mathbb{B})$ corresponding to the name $\langle \dot{a}, a \rangle \mid a \in A_n$. Since $A_{n+1}$ refines $A_n$ modulo $I$, it follows that $b_{n+1} \leq j(I) b_n$. The sequence $\langle b_n \mid n < \omega \rangle$ exists in $M[H]$, and consequently, by distributivity, in $M$. It follows that $\bigwedge_n b_n \in H$, since $H$ is $M$-generic and hence $M$-complete by the remarks before theorem . Consequently, $\bigwedge_n b_n \neq 0$. Thus, in $M$ we have found a path as desired through the tree of maximal antichains $\langle j(A_n) \mid n < \omega \rangle = j(\langle A_n \mid n < \omega \rangle)$ in $j(\mathbb{B})$ with respect to $j(I)$. By elementarity, it follows that there is such a sequence in $V$ for $\langle A_n \mid n < \omega \rangle$ in $\mathbb{B}$ with respect to $I$, as desired.

Conversely, suppose that $I$ is not precipitous. Thus, there is a condition $a_0 \in I^+$ forcing via $\mathbb{B}/I$ that $\dot{\check{V}}_{G/I}$ is ill-founded. Thus, there are $\mathbb{B}/I$-names $f_n$ and $D_n$ such that $a_0$ forces via $\mathbb{B}/I$ that $\check{f}_n : D_n \to \text{Ord}$ are open dense spanning functions in $V$ and that $\check{f}_{n+1} \leq I \check{f}_n$. We shall make use of the following lemma.

Lemma 61.1. If $I$ is an ideal on a complete Boolean algebra $\mathbb{B}$ and $[b]_I \models_{\mathbb{B}/I} “f : \check{D} \to \check{V}”$ is an open dense spanning function in $\check{V}^+$, then there is $\check{A} \subseteq \mathbb{B}$, a maximal antichain modulo $I$ below $b$, and for each $a \in \check{A}$ a function $\check{f}_a : \check{B}_a \to \check{V}$, where $\check{B}_a$ is the cone below $a$, such that $[a]_I \models_{\mathbb{B}/I} \check{f}_a \models \check{f}_a = \check{f}_a$.

Proof. Since $b$ forces that $\check{D}$ is an open dense subset of $\mathbb{B}$ in $\check{V}$, there is a maximal antichain $A$ of conditions $a$ deciding exactly which open dense subset in $\check{V}$ it is. By further refining inside these dense sets, we obtain a maximal antichain $\dot{A}$ of conditions $a$ forcing that the cone $\check{B}_a$ below $a$ is contained in $\check{D}$. By the same reasoning, since $b$ forces that $\check{f}$ is a function on $\check{D}$ in $\check{V}$, we may also assume that the conditions in $\check{A}$ also decide which function from $\check{V}$ $\check{f}$ is. Thus, we obtain functions $\check{f}_a : \check{B}_a \to \check{V}$ such that $[a]_I \models_{\mathbb{B}/I} \check{f}_a \models \check{f}_a = \check{f}_a$, as desired.

We now iteratively apply the lemma to build a tree of maximal antichains $A_n \subseteq \mathbb{B}$ modulo $I$ such that for every $a \in A_n$ there are functions $\check{f}_a : \check{B}_a \to \text{Ord}$ on the cone $\check{B}_a$ below $a$, such that $[a]_I \models_{\mathbb{B}/I} \check{f}_a \models \check{f}_a \in \check{f}_a$. Furthermore, we can arrange that every element of $A_{n+1}$ lies below an element of $A_n$, and lastly, since the sequence was forced to be descending, that $\check{f}_{a_{n+1}}(b) < \check{f}_{a_n}(b)$ for all $b \leq a$, whenever $a \leq a'$ with $a \in A_{n+1}$ and $a' \in A_n$. Suppose now that there is a path $\langle a_n \mid n < \omega \rangle$ through this tree of antichains in the sense of (2), so that $a_n \in A_n$ and $z = \bigwedge_n a_n \neq 0$. Since $z \leq a_n$ for all $n$, it follows that $z \in \check{B}_a$ for all $n$, and so our observations ensure $\check{f}_{a_{n+1}^I}(z) < \check{f}_{a_n}(z)$. This is an infinite descending sequence of ordinals in $\check{V}$, a contradiction. So there can be no such path as in (2), and the theorem is proved.

We emphasize that we have not assumed in the theorem above that the quotient forcing $\mathbb{B}/I$ is countably distributive, but only that the underlying Boolean algebra $\mathbb{B}$ itself is. This generalizes the classical power set situation, where one forces with the quotient $P(\mathbb{Z})/I$ of a power set, since the underlying Boolean algebra there is the power set $P(\mathbb{Z})$ itself, which is trivial as a forcing notion, of course, and consequently fully distributive. Thus, the previous theorem generalizes the classical characterization of precipitous ideals. The question remains whether we actually need distributivity in theorem.

In the classical power set context, every saturated ideal is disjointifying, and every disjointifying ideal is precipitous. We can easily generalize these implications to the case of countably distributive complete Boolean algebras.

Theorem 62. If $I$ is a $<\kappa$-complete ideal in a complete Boolean algebra $\mathbb{B}$ and $\mathbb{B}/I$ is $\kappa^+$-c.c., then $\mathbb{B}/I$ is a complete Boolean algebra with the disjointifying property.

Proof. This can be proved just as in the classical power set context. If $A \subseteq \mathbb{B}/I$ is any maximal antichain, then by $\kappa^+$-saturation, we know $A$ has size at most $\kappa$, so we may enumerate $A = \{ a_\alpha \mid \alpha < |A| \}$. Choose
any representatives \(x_\alpha\) such that \(a_\alpha = [x_\alpha]_I\), and then disjointify them: let \(b_\alpha = x_\alpha - \bigvee_{\beta < \alpha} x_\beta\). Clearly, \(b_\alpha \leq x_\alpha\) and \(b_\alpha \wedge b_\beta = 0\) if \(\alpha \neq \beta\). Observe that \(x_\alpha - b_\alpha = \bigvee_{\beta < \alpha} (x_\alpha - x_\beta)\), which is a small join of elements of \(I\) and hence in \(I\), by the \(<\kappa\)-completeness of \(I\), and so \([b_\alpha]_I = [x_\alpha]_I = a_\alpha\). Thus, we have found a disjoint selection of representatives. Since the join of a subset of a maximal antichain in \(B/I\) is the equivalence class of the join of these disjoint representatives, it follows that \(B/I\) is a complete Boolean algebra. \(\square\)

**Theorem 63.** If \(I\) is a countably complete ideal with the disjointifying property on a countably distributive complete Boolean algebra \(B\), then \(I\) is precipitous.

**Proof.** We use theorem 61. Suppose that \((A_n \mid n < \omega)\) is a tree of maximal antichains. This means that each \(A_n \subseteq B\) is a maximal antichain modulo \(I\), and every element of \(A_{n+1}\) is below an element of \(A_n\) modulo \(I\). Because \(I\) has the disjointifying property, we may systematically disjointify these antichains, replacing each element with a slightly smaller element but \(I\)-equivalent member, in such a way that each \(A_n\) becomes an antichain in \(B\), not merely an antichain in \(B/I\) modulo \(I\). Let \(F\) be the dual filter to \(I\). Since \(A_n\) is a maximal antichain modulo \(I\), it follows that \(\bigvee A_n \in F\). Since \(I\) is countably complete, it follows that \(\bigwedge_n (\bigvee A_n) \in F\) as well. By the countable distribution law in \(B\), it follows that \(\bigwedge_n (\bigvee A_n) = \bigvee \{ \bigwedge_n a_n \mid (a_0, a_1, \ldots) \in B^n, a_n \in A_n\}\). In particular, there is some such sequence \((a_0, a_1, \ldots)\) with \(a_n \in A_n\) and \(\bigwedge_n a_n \neq 0\). Since every element of \(A_{n+1}\) lies below an element of \(A_n\), and is incompatible with the other elements of \(A_n\), it follows that this sequence is descending, and fulfills the criterion of theorem 61 as desired. \(\square\)

**Corollary 64.** If \(I\) is a \(<\kappa\)-complete ideal in a complete, countably distributive Boolean algebra \(B\) and \(B/I\) is \(\kappa^+\)-c.c., for some uncountable cardinal \(\kappa\), then \(I\) is precipitous.

Next, we generalize Solovay’s observation from the power set context that disjointification allows for a convenient anticipation of names for spanning functions in the ground model.

**Lemma 65.** Suppose that \(B/I\) has the disjointifying property and that \(\dot{f}\) is a \(B/I\)-name for which \(b = [\dot{f}]\) is a spanning function in \(V[G]/I \neq 0\). Then there is an actual spanning function \(g : A \rightarrow V\) in \(V\) such that \(b\) forces that \(\dot{f}\) and \(\dot{g}\) are equivalent modulo \(\dot{\bigcup}G\).

**Proof.** It is somewhat more convenient to use open dense spanning functions, so let us assume that \(\dot{f}\) is defined not merely on an antichain, but is extended to the open dense set of conditions below elements of an antichain, by copying the value on the antichain to the entire lower cone of that element. By lemma 61, we may find a maximal antichain \(A \subseteq B\) modulo \(I\) below \(b\) such that for each \(a \in A\) there is a function \(f_a : B_a \rightarrow V\), where \(B_a\) is the cone of nonzero elements below \(a\), such that \([a]_I\) forces that \(\dot{f} \upharpoonright B_a = f_a\). Since \(I\) has the disjointifying property, we may replace the elements of \(a\) with smaller elements but still equivalent modulo \(I\), so that \(A\) is an antichain in \(B\). In this case, the cones \(B_a\) and \(B_b\) for \(a \neq b\) in \(A\) do not overlap, and so \(g = \bigcup\{ f_a \mid a \in A\}\) is a function. Every \(a \in A\) has \([a]_I\) forcing that \(\dot{f} \upharpoonright B_a = \dot{g} \upharpoonright B_a\), and since \(A\) is a maximal antichain modulo \(I\) below \(b\), this means that \(b\) forces that \(\dot{f}\) is equivalent to \(\dot{g}\) modulo \(\dot{\bigcup}G\), as desired. \(\square\)

The next theorem generalizes theorem 28 by obtaining closure of the Boolean ultrapower in the quotient forcing extension \(V[G]\), rather than merely in \(V\) as before.

**Theorem 66.** Suppose that \(j : V \rightarrow M \subseteq M[H] \subseteq V[G]\) is the generic Boolean ultrapower arising from the induced filter \(U = \bigcup G\) by a \(V\)-generic filter \(G \subseteq B/I\), where \(B/I\) has the disjointifying property. Then \((^0 M[H])^{V[G]} \subseteq M[H]\) if and only if \(j^* \theta \in M[H]\). In particular, if \(\kappa = \text{cp}(j)\), then \((^0 M[H])^{V[G]} \subseteq M[H]\).

**Proof.** To be clear, we assume here that \(M = V_U\) and \(M[H] = V^B/U\), where \(H = [H]_U\), using the canonical \(B\)-name \(\dot{H}\) for the generic filter. Since \(I\) was disjointifying, the ultrapower is well-founded and we identify the elements of the ultrapower with their Mostowski collapse. The forward implication is immediate, since \(j\) is definable in \(V[G]\). Conversely, suppose that \(j^* \theta \in M[H]\) and consider any \((z_\alpha \mid \alpha < \theta) \in (^0 M[H])^{V[G]}\). Since these are ZFC models, it suffices to consider sequences of ordinals, so we may assume \(z_\alpha \in \text{Ord}\). Let \(\dot{z}\) be a \(B/I\)-name for this sequence, and we may assume that \([\dot{z} \in \text{Ord}] = 1\). From \(\dot{z}\), we may construct names \(\dot{z}_\alpha\) such that \([\dot{z}_\alpha = \dot{z}(\alpha)] = 1\). Since each \(\dot{z}_\alpha\) is forced to be an ordinal, there is a \(B/I\)-name \(\dot{f}_\alpha\) for the spanning function representing \(\dot{z}_\alpha\). That is, \(1\) forces that \(\dot{z}_\alpha = [\dot{f}_\alpha]_{\dot{\bigcup}G}\). By lemma 65, we may find...
an actual spanning function \( g_\alpha : A_\alpha \to \text{Ord} \) in \( V \) such that \( 1 \) forces that \( \dot{z}_\alpha \) is represented by \( [g_\alpha]_{\|G}\). Let \( \tau_\alpha \) be the \( B \)-name arising by mixing the values of \( g_\alpha \) along \( A_\alpha \). Thus, in the Boolean ultrapower we have \( z_\alpha = [\tau_\alpha]_{\|J} \), and the sequence of names \( (\tau_\alpha)_{\alpha < \theta} \) is in \( V \). As in theorem 28, construct a \( B \)-name \( \sigma \) for the \( B \)-sequence, so that \( [\sigma] = [\tau_\alpha]_{\|J} \) is in \( B \) and for each \( \alpha < \theta \) we have \( [\sigma(\alpha)]_{\|J} = z_\alpha \). Since \( j^* \theta \in M[H] \), we may restrict \( s \) to \( j^* \theta \) and collapse to the domain to see that \( \{z_\alpha \mid \alpha < \theta\} \in M[H] \), as desired. The final claim, about \( \kappa = \text{cp}(j) \), is a special case since \( j^* \kappa = \kappa \in M \).

\[ \square \]

**Corollary 67.** In the context of the previous theorem, if \( B < \delta \)-distributive, then for any \( \theta < j(\delta) \) we have \( (\theta M)^{V[G]} \subseteq M \) if and only if \( j^* \theta \in M[H] \). In particular, if \( \kappa = \text{cp}(j) \) and \( B < \delta \)-distributive, then \( (\kappa M)^{V[G]} \subseteq M \).

**Proof.** This follows from theorem 66 just as corollary 29 followed from theorem 28. Namely, if \( j^* \theta \in M[H] \), then we know \( \theta M \subseteq \theta M[H] \subseteq M[H] \) by theorem 66. But since the forcing \( H \subseteq j(B) \) is \( \delta \)-distributive, this means that \( \theta M \subseteq M \), as desired. \( \square \)

We turn now to what we find to be an exciting new possibility in the general setting of complete Boolean algebras, namely, the notion of relative precipitousness and disjointification.

**Definition 68.** Suppose that \( I \) and \( J \) are ideals in a complete Boolean algebra \( B \), the quotients \( B/I \) and \( B/J \) are also complete, and \( I \subseteq J \). We define that \( I \) is **precipitous over** \( J \) if whenever \( G \subseteq B/I \) is \( V \)-generic, then the Boolean ultrapower by the induced ultrafilter \( U = \{ [a]_J \mid a \in G \} \) on \( B/J \) is well-founded. We say that \( I \) is **disjointifying over** \( J \), if for every maximal antichain modulo \( I \), there is a choice of representatives that form an antichain modulo \( J \).

It is clear that every ideal is precipitous over itself, and an ideal is precipitous if and only if it is precipitous over the zero ideal \( \{0\} \). Similarly, every ideal is disjointifying over itself, and an ideal is disjointifying if and only if it is disjointifying over the zero ideal. If \( I \subseteq J \subseteq B \) are ideals, then \( I/J = \{ [a]_J \mid a \in I \} \) is an ideal in \( B/J \). Furthermore, \( (B/J)/(I/J) \) is isomorphic to \( B/I \) by the canonical map \( [a]_J \mapsto [a]_I \).

**Theorem 69.** Suppose that \( B \) is a complete Boolean algebra, \( I \subseteq J \) are ideals and \( B/I \) and \( B/J \) are both complete. Then:

1. \( I \) is precipitous over \( J \) in \( B \) if and only if \( I/J \) is precipitous in \( B/J \).
2. \( I \) is disjointifying over \( J \) if and only if \( I/J \) is disjointifying in \( B/J \).

**Proof.** (1) The point here is that \( B/I \) is canonically isomorphic to \( (B/J)/(I/J) \) by the map \( [a]_I \mapsto [a]_J \). Therefore, forcing over either of these Boolean algebras results in \( V \)-generic filters \( G \subseteq B/I \) and \( \tilde{G} = \{ [a]_J \mid [a]_I \in G \} \subseteq (B/J)/(I/J) \). The filter induced on \( B/J \) in either case is \( U = \{ [a]_J \mid [a]_I \in G \} = \{ [a]_J \mid [a]_J \in \tilde{G} \} \). The ideal \( I \) is precipitous over \( J \) if and only if the Boolean ultrapower by \( U \) is necessarily well-founded, since \( U \) is the filter on \( B/J \) induced by \( G \subseteq B/I \). At the same time, \( I/J \) is precipitous in \( B/J \) if and only if the Boolean ultrapower by \( U \) is necessarily well-founded, since \( U \) is the filter on \( B/J \) induced by \( \tilde{G} \) on \( (B/J)/(I/J) \). Thus, \( I \) is precipitous over \( J \) if and only if \( I/J \) is precipitous in \( B/J \), as desired.

(2) Suppose that \( I \) is disjointifying over \( J \). This means that every antichain modulo \( I \) in \( B \) has a choice of representatives that are disjoint modulo \( J \). Suppose now that \( \{ [a_\alpha]_J \mid \alpha < \gamma \} \) is an antichain modulo \( I/J \) in \( B/J \). Thus, if \( \alpha \neq \beta \), we have that \( [a_\alpha]_I \wedge [a_\beta]_I \in I/J \). This means \( a_\alpha \wedge a_\beta \in I \), and so \( [a_\alpha]_I \mid \alpha < \gamma \) is an antichain modulo \( I \). Thus, by our assumption, we can find representatives \( b_\alpha \) such that \( a_\alpha = b_\alpha \) and \( b_\alpha \wedge b_\beta \in J \) for \( \alpha \neq \beta \). It follows that \( [a_\alpha]_J \) is equivalent to \( [b_\alpha]_J \) modulo \( I/J \) in \( B/J \), and the \( [a_\alpha]_J \) form an actual antichain in \( B/J \). So \( I/J \) is disjointifying in \( B/J \), as desired.

Conversely, suppose that \( I/J \) is disjointifying in \( B/J \) and \( [a_\alpha]_I \mid \alpha < \gamma \) is an antichain modulo \( I \) in \( B \). Thus, \( \{ [a_\alpha]_J \mid \alpha < \gamma \} \) is an antichain modulo \( I/J \) in \( B/J \), and so by our assumption we may find \( [b_\alpha]_J \equiv [a_\alpha]_J \) such that \( [b_\alpha]_J \wedge [b_\beta]_J = 0 \) for \( \alpha \neq \beta \). This means exactly that \( a_\alpha \) is equivalent to \( b_\alpha \) modulo \( I \) and \( b_\alpha \wedge b_\beta \in J \). Thus, we have found a choice of representatives that are disjoint modulo \( J \), as desired. \( \square \)

It is easy to see that relative disjointification is a transitive relation.
Lemma 70. If $I$ is disjointifying over $J$ and $J$ is disjointifying over $K$, then $I$ is disjointifying over $K$.

Proof. Suppose that $I$ is disjointifying over $J$ and $J$ is disjointifying over $K$. Any antichain modulo $I$ in $\mathbb{B}$ has a choice of representatives that forms an antichain modulo $J$, and this has a choice of representatives forming an antichain modulo $K$, as desired. \hfill \Box

It is not clear to what extent this holds for relative precipitousness. With a bit of disjointification, we have the following transitivity converse:

Theorem 71. Suppose that $K \subseteq J \subseteq I$ are ideals in a complete Boolean algebra $\mathbb{B}$ and all the quotients are complete. If $I$ is precipitous over $K$ and $J$ is disjointifying over $K$, then $I$ is precipitous over $J$.

Proof. Suppose that $I$ is precipitous over $K$ and $J$ is disjointifying over $K$ in $\mathbb{B}$. We want to show that $I$ is precipitous over $J$. Suppose that $G \subseteq \mathbb{B}/I$ is $V$-generic, and consider the induced ultrafilter $U = \{ [a]_J \mid [a]_I \in G \}$ on $\mathbb{B}/J$. Since $\mathbb{B}/J$ is canonically isomorphic to $(\mathbb{B}/K)/(J/K)$, the ultrafilter $U$ is canonically isomorphic to $\hat{U} = \{ [[a]_K]_{J/K} \mid [a]_J \in U \}$ on $(\mathbb{B}/K)/(J/K)$. Since $J$ is disjointifying over $K$, it follows by theorem 69 that $J/K$ is disjointifying in $\mathbb{B}/K$. Consequently, by theorem 60 the Boolean ultrapower by $\hat{U}$ is a factor of the Boolean ultrapower by the induced ultrafilter $U^* = \cup \hat{U} = \{ [a]_K \mid [[a]_K]_{J/K} \in \hat{U} \} = \{ [[a]_K]_{J/K} \mid [a]_I \in U \}$ on $\mathbb{B}/K$. But this $U^*$ is precisely the ultrafilter induced by $G$ on $\mathbb{B}/K$. Since $I$ is precipitous over $K$, the ultrapower by $U^*$ is consequently well-founded. Since the ultrapower by $\hat{U}$ is a factor of it, the ultrapower by $\hat{U}$ is also well-founded. And since $\hat{U}$ is isomorphic to $U$, we conclude that the ultrapower by $U$ is also well-founded. So $I$ is precipitous over $J$, as desired. \hfill \Box

Corollary 72. Suppose that $J \subseteq I$ are ideals in a complete Boolean algebra $\mathbb{B}$, whose quotients are complete. If $I$ is precipitous and $J$ is disjointifying, then $I$ is precipitous over $J$.

We have a number of questions to ask about the nature of relative precipitousness and disjointification, such as the following:

Questions 73.

(1) If $I$ is precipitous over a precipitous ideal $J$, must $I$ be precipitous?

(2) If $I$ is precipitous over $J$, must $I$ be precipitous over all the relevant intermediate ideals $K$, with $J \subseteq K \subseteq I$?

(3) To what extent can it happen that an ideal $I$ is precipitous over a strictly smaller ideal $J$, but is not precipitous?

(4) What are the large cardinal strengths of the hypotheses that various particular ideals on particular Boolean algebras are disjointifying or precipitous?

(5) What are the large cardinal strengths of the hypotheses that various particular ideals on particular Boolean algebras are disjointifying or precipitous over a strictly smaller ideal?

(6) For example, what is the large cardinal strength of the hypothesis that the nonstationary ideal $\text{NS}$ on $P(\omega_1)$ is precipitous or disjointifying over a strictly smaller ideal $J \subsetneq \text{NS}$?

We shall leave these questions for a subsequent project.

12. Boolean ultrapowers versus classical ultrapowers

In this section, we use the concept of relative genericity to characterize exactly when the Boolean ultrapower by an ultrafilter $U$ on a complete Boolean algebra $\mathbb{B}$ is isomorphic to a classical ultrapower by an ultrafilter on a power set algebra. Suppose that $\mathbb{B}$ is a complete Boolean algebra, $\mathcal{U}$ is an ultrafilter on $\mathbb{B}$ and $A \subseteq \mathbb{B}$ is a maximal antichain. If $C$ is a maximal antichain refining $A$, then we define that $U$ meets $C$ relative to $A$, if for each $a \in A$ there is a smaller element $c_a \in C$ with $c_a \leq a$ such that $\bigvee_{a \in A} c_a \in U$. We define that $U$ is generic relative to $A$ if $U$ meets all such $C$ relative to $A$. The idea is that while generic ultrafilters
select a single point from every maximal antichain, here we select single points below every element of $A$, whose aggregate sum is in $U$. It is easy to see, for example, that if $U$ meets $C$ relative to $A$, then $U$ meets $C$ if and only if $U$ meets $A$. Consequently, if $U$ is $V$-generic relative to $A$, then $U$ is $V$-generic if and only if $U$ meets $A$. Also, a filter $U$ is $V$-generic if and only if it is $V$-generic relative to the trivial $\{1\}$, since in this case only one choice is made; and a filter $U$ is $V$-generic if and only if it is generic relative to every maximal antichain.

**Theorem 74.** Suppose that $U$ is an ultrafilter on the complete Boolean algebra $\mathbb{B}$. Then the following are equivalent:

1. The Boolean ultrapower $j_U : V \to \tilde{V}_U$ is isomorphic to a classical power set ultrapower by a measure $U^*$ on some set $Z$.
2. The ultrafilter $U$ is $V$-generic relative to some maximal antichain $A \subseteq \mathbb{B}$.

In this case, one can take $Z = A$ and the Boolean ultrapower $j_U$ is isomorphic to the power set ultrapower $j_{U_A}$.

**Proof.** Suppose that $j_U : V \to \tilde{V}_U$ is isomorphic to $j_{U^*} : V \to V^Z/U^*$. Thus, using the functional presentation of the Boolean ultrapower, there is an isomorphism $\pi$ making the following diagram commute.

$$
\begin{array}{ccc}
V & \xrightarrow{\pi} & V^Z/U^* \\
\downarrow{j_U} & & \downarrow{j_{U^*}} \\
\tilde{V}_U & \xrightarrow{\sim} & \tilde{V}_U \\
\end{array}
$$

Since $U^*$ is an ultrafilter on a power set Boolean algebra $P(Z)$, it follows that every element of $V^Z/U^*$ has the form $[f]_{U^*} = j_{U^*}(f)(s^*)$, where $s^* = [\mathrm{id}]_{U^*}$ and $f : Z \to V$ with $f \in V$. Applying the isomorphism, it follows that every element of $V^Z/U^*$ has the form $j_U(f)(s)$, where $\pi(s) = s^*$. Since $V^Z/U^*$ is the direct limit of $V^A/U_A$ for maximal antichains $A$, it follows that $s = \pi_{A,\infty}(s_A)$ for some maximal antichain $A \subseteq \mathbb{B}$ and some $s_A \in V^A/U_A$. If $C \subseteq \mathbb{B}$ is any maximal antichain refining $A$ and $x \in V^C/U_C$, then $\pi_{C,\infty}(x) \in V^Z/U^*$, and so $\pi_{C,\infty}(x) = j_U(f)(s)$ for some function $f : Z \to V$ in $V$. Unwrapping this by the commutativity of the directed system, we observe $\pi_{C,\infty}(x) = j_U(f)(s) = \pi_{A,\infty}(j_{U_A}(f)(s_A)) = j_{U_A}(f)(s_A))$. By peeling off the outer $\pi_{C,\infty}$, we conclude that $x = \pi_{A,C}(j_{U_A}(f)(s_A))$. Since $x$ was arbitrary, it follows that $\pi_{A,C}$ is surjective and hence an isomorphism. (From this, it follows that $j_U$ is isomorphic to $j_{U_A}$, since the direct limit system has only isomorphisms below the antichain $A$.) We now argue that $U$ is generic relative to $A$. Consider the identity function $\mathrm{id} : C \to C$. Since $\pi_{A,C}$ is surjective, it follows that $[\mathrm{id} \mid C]_{U_C} = \pi_{A,C}([f]_{U_A})$ for some spanning function $f : A \to V$. This means that $\mathrm{id} \mid C \equiv_U f \downarrow C$, and so $\forall Y \in U$, where $Y = \{ c \in C \mid c = (f \downarrow C)(c) \}$. Note that for each $a \in A$, there is at most one $c \in Y$ below it, since $c = (f \downarrow C)(c) = f(a)$. Let $c_a$ be this value of $c$ below $a$, if $a \leq a$ and $c \in Y$, otherwise $c_a$ is arbitrary. Thus, $\forall a \in A \ c_a \geq \forall Y \in U$. We have therefore proved that $U$ is generic relative to $A$, as desired.

Conversely, suppose that $U$ is generic relative to $A$. We will show that $j_U : V \to \tilde{V}_U$ is isomorphic to the classical ultrapower $j_{U_A} : V \to V^A/U_A$ by the induced measure $U_A$, defined by $X \in U_A$ if and only if $X \subseteq A$ and $\forall X \in U$. Since the Boolean ultrapower is the direct limit of the classical ultrapowers $V^C/U_C$ for maximal antichains $C$ refining $A$ as in lemmas and Theorem 9, it suffices for us to show that $\pi_{A,C}$ is an isomorphism for all such $C$. For this, it suffices to argue that $\pi_{A,C}$ is surjective. Consider any spanning function $g : C \to V$. By genericity below $A$, there is for each $a \in A$ a selection $c_a \in C$ with $c_a \leq a$ such that $\forall a \in A \ c_a \in U$. Define $f : A \to V$ by $f(a) = g(c_a)$, and observe that $(f \downarrow C) \equiv_U g$, precisely because $\forall a \in A \ c_a \in U$. Thus, $\pi_{A,C}([f]_{U_A}) = [g]_{U_C}$, and so $\pi_{A,C}$ is surjective. Thus, the direct limit presentation of the Boolean ultrapower is the the identity on the antichains refining $A$, and so $j_U$ is isomorphic to $j_{U_A}$.

Canjar [Can97] proved that not every Boolean ultrapower is isomorphic to a power set ultrapower.

Suppose that $A$ is an infinite maximal antichain in a complete Boolean algebra $\mathbb{B}$. Recall that the small ideal relative to $A$ in $\mathbb{B}$ is the ideal consisting of all elements $b \in \mathbb{B}$ that are below the join of a small subset of $A$, namely $b \leq \bigvee A_0$ for some $A_0 \subseteq A$ and $\|A_0\| < \kappa = |A|$. If $\kappa$ is regular, this ideal is $\kappa$-complete.
Theorem 75. Suppose that $I$ is the small ideal relative to an infinite maximal antichain $A$ in a complete Boolean algebra $\mathcal{B}$. If $G \subseteq \mathcal{B}/I$ is $V$-generic, then the induced ultrafilter $U = \bigcup G$ on $\mathcal{B}$ does not meet $A$, but is $V$-generic relative to $A$.

Proof. It follows from the conclusion, of course, that the corresponding Boolean ultrapower $j_U$ is isomorphic to the induced power set ultrapower by $U_A$ on $P(A)$, which has critical point $\kappa$.

Since singletons are small, it follows that $A \subseteq I$ and hence $U$ contains no elements from $A$. If $C$ is any maximal antichain refining $A$, then let us write $f : A \setminus C$ to indicate that $f$ is a regressive function from $A$ to $C$, meaning that $f(a) \leq a$ and $f(\alpha) \in C$ for all $a \in A$. Such a function chooses, for each element of $A$, an element of $C$ below it. For any such regressive function, let $b_f = \bigvee_{a \in A} f(a)$ be the join of the choices made by $f$. For $U$ to be $V$-generic relative to $A$, we must find some $f$ such that $b_f \in U$. Let $D = \{ b_f \mid f : A \setminus C \}$. We claim that this is predense in $I^+$. To see this, suppose that $b \in I^+$. Thus, $b \cap a \neq 0$ for $\kappa$ many $a \in A$. Since each $a \in A$ has $a = \bigvee \{ c \in C, c \leq a \}$, this means that for $\kappa$ many $a$, there is $c_a \in C$ with $c_a \leq a$ and $b \cap c_a \neq 0$. Let $f : A \setminus C$ have $f(a) = c_a$ for these $a$. Thus, $b_f$ is above all the $c_a$, for $\kappa$ many $a$, and consequently, $b \cap b_f$ is not below the join of any small subset of $A$. Thus, $b$ is not incompatible with $b_f$ modulo $I$, and so $D$ is predense in $I^+$. It follows that $\{ [b_f]_I \mid f : A \setminus C \}$ is predense in $\mathcal{B}/I$, and so there is some $f$ with $[b_f]_I \in G$, and consequently $b_f \in U$, as desired.

This argument can be somewhat generalized beyond the small ideal. Suppose that $A \subseteq \mathcal{B}$ is a maximal antichain. If $J \subseteq P(A)$ is an ideal on the power set, then $J$ naturally induces an ideal $I = \{ b \in \mathcal{B} \mid \exists a \in J b \leq \bigvee a \}$ on $\mathcal{B}$. Let us say that such ideals $I$ are local to $A$, being generated by an ideal on the power set of $A$. Of course, if $A \subseteq \mathcal{B}$ is a maximal antichain, then the power set $P(A)$ embeds completely into $\mathcal{B}$ via the map $X \mapsto \bigvee X$ for $X \subseteq A$. As a result, we may regard $P(A)$ as a complete subalgebra of $\mathcal{B}$. This relationship extends to the quotients:

Lemma 76. If $I$ is the local ideal on $\mathcal{B}$ induced by $J$ on $P(A)$, then $P(A)/J$ complete embeds into $\mathcal{B}/I$ by the natural map $\pi : [X]_J \mapsto [\bigvee X]_I$.

Proof. This is clearly a Boolean algebra homomorphism. For completeness, suppose that $[X_\alpha]_J$ is a maximal antichain in $P(A)/J$, for $\alpha < \gamma$, and consider $[\bigvee X_\alpha]_I$. Suppose that $[b]_I \neq 0$ is incompatible with all of them, so that $b \notin I$ and $b \cap \bigvee X_\alpha \in I$ for all $\alpha$. Thus, for each $\alpha$, there is $Y_\alpha \in J$ with $b \cap \bigvee X_\alpha \leq \bigvee Y_\alpha$. Let $X = \{ a \in A \mid b \cap a \neq 0 \}$. Since $b \notin J$, it follows that $X \notin J$. Since $b \cap \bigvee X_\alpha \leq \bigvee Y_\alpha$, it follows that $X \cap X_\alpha \subseteq Y_\alpha \in J$. Thus, $[X]_J$ is incompatible with every $[X_\alpha]_J$, contradicting our assumption that this was a maximal antichain in $P(A)/J$.

Theorem 77. Suppose that $I$ is an ideal in $\mathcal{B}$ local to a maximal antichain $A$, and that $I$ contains the small ideal. If $G \subseteq \mathcal{B}/I$ is $V$-generic, then the induced ultrafilter $U = \bigcup G$ on $\mathcal{B}$ is $V$-generic relative to $A$, but does not meet $A$.

Proof. Suppose that $I$ is induced by the ideal $J \subseteq P(A)$ on the power set of $A$. Since $I$ contains the small ideal, it follows that $A \subseteq I$ and consequently $U$ misses $A$. If $C$ is a maximal antichain refining $A$, then consider the collection $D = \{ b_f \mid f : A \setminus C \}$ as before. We claim this is predense in $I^+$. If $b \in I^+$, then since $b = \bigvee a \in A b \cap a$, it follows that $E = \{ a \in A \mid b \cap a \neq 0 \} \in J^+$. If $b \cap a \neq 0$, then there is some $c_a \leq a$ with $c_a \in C$ and $b \cap c_a \neq 0$. Let $f : A \setminus C$ be such that $f(a) = c_a$ for these choices. Now observe that $b \cap b_f$ is at least as big as $\bigvee a \in E c_a$. If this were below $\bigvee F$ for some $F \in J$, then it would have to be that $E \subseteq F$, since $c_a$ is incompatible with all elements of $A$ except $a$. This contradicts our earlier observation that $E \in J^+$. Thus, $b$ is not incompatible with $b_f$ modulo $I$, and so $D$ is predense. Thus, there is some $[b_f]_I \in G$ and consequently $b_f \in U$. So $U$ is $V$-generic relative to $A$, as desired.

A generic Boolean ultrapower by an ideal $I$ on a complete Boolean algebra $\mathcal{B}$ is the Boolean ultrapower by the filter $\bigcup G$ on $\mathcal{B}$ induced by a $V$-generic filter $G \subseteq \mathcal{B}/I$ on the quotient. The classical situation occurs when $\mathcal{B} = P(Z)$ is a power set, and in this case we refer to the resulting embedding as a generic power set ultrapower.

Corollary 78. Every generic Boolean ultrapower by a local ideal containing the small ideal is isomorphic to a generic power set ultrapower. Specifically, if $\mathcal{B}$ is a complete Boolean algebra with an antichain $A$ and $I$ is the local ideal generated by an ideal $J$ on the power set $P(A)$, then the generic ultrapowers arising from
We saw above that if $I$ is precipitous on $P(A)/J$, then every Boolean algebra $B$ isomorphic to those arising from $P(A)/J$. In particular, $J$ is precipitous on $P(A)$ if and only if $I$ is precipitous on $B$.

**Proof.** In the situation of the previous theorem, the Boolean ultrapower by $U$ will be isomorphic to the ultrapower by the ultrafilter $U_A = \{ X \subseteq A \mid \forall X \in U \}$ on the power set $P(A)$. This is because theorem 77 shows that the ultrafilter $U$ is $V$-generic relative to $A$, fulfilling the condition of theorem 74 that the Boolean ultrapower is classical.

**Corollary 79.** If there is a precipitous ideal on a power set $P(\kappa)$, then every Boolean algebra $B$ with an antichain of size $\kappa$ has a precipitous ideal.

**Proof.** If there is a precipitous ideal $J$ on $P(\kappa)$, then by restricting to the smallest $J$-positive set, we may assume that $J$ contains the small ideal. If $A \subseteq B$ has size $\kappa$, we may view $J$ as being a precipitous ideal on $P(A)$. Corollary 78 shows that the induced ideal $I$ on $B$ gives rise to exactly the same generic embeddings, and so $I$ also is precipitous.

### 13. Boolean Ultrapowers as Large Cardinal Embeddings

We will now investigate several instances in which various large cardinal embeddings can be realized as well-founded Boolean ultrapowers. By theorem 31 we know that every large cardinal embedding that is the ultrapower by a measure on a set can also be viewed as a Boolean ultrapower by an ultrafilter on the corresponding power set Boolean algebra. Thus, the large cardinal embeddings witnessing that a cardinal $\kappa$ is measurable, strongly compact, supercompact, huge and so on, are all instances of well-founded Boolean ultrapowers.

Let us mention briefly how this meshes with theorem 25. If $j : V \rightarrow M$ is the ultrapower by a measure $\mu$ on the set $Z$, then the corresponding Boolean algebra is $B = P(Z)$, and $j(B)$ is the power set of $j(Z)$ in $M$. We know that $X \in \mu \iff [id]_\mu \in j(X)$, and so $j^* \mu \subset F$, where $F$ is the principal ultrafilter generated by $[id]_\mu \in j(B)$. Since $j(B)$ is an atomic Boolean algebra, the principal ultrafilter $F$ generated by $[id]_\mu$ is indeed $M$-generic, and $j^* \mu \subset F$, fulfilling theorem 25. Note that if $\mu$ is not principal, then $\mu$ itself is definitely not $V$-generic for $B$, since it misses the maximal antichain of atoms $\{ \{z\} \mid z \in Z \}$.

Let us now investigate a few circumstances under which other kinds of Boolean algebras can give rise to large cardinal embeddings. Theorem 25 will allow us to construct examples of well-founded ultrafilters that are not generic.

**Theorem 80.** If $\kappa$ is a measurable cardinal and $2^\kappa = \kappa^+$ in $V$, then there are many well-founded ultrafilters in $V$ on the Boolean algebra of the forcing $Add(\kappa, 1)$.

**Proof.** Suppose that $\kappa$ is a measurable cardinal, witnessed by the normal ultrapower embedding $j : V \rightarrow M$, and suppose $2^\kappa = \kappa^+$. Let $B$ be the regular open algebra of $Add(\kappa, 1)$, and consider $j(B)$, which corresponds to the forcing $Add(j(\kappa), 1)^M$. This forcing is $\leq \kappa$-closed in $M$. The number of subsets of $Add(\kappa, 1)$ is $\kappa^+$, and so the number of subsets of $Add(j(\kappa), 1)^M$ in $M$ is $j(\kappa^+)$, which has size $\kappa^+$ in $V$. We may therefore assemble the dense subsets for this forcing from $M$ into a $\kappa^+$-sequence in $V$, and proceed to diagonalize against this list, using the closure of the forcing and the fact that $M^c \subseteq M$ at limit stages, to produce in $V$ an $M$-generic filter $G \subseteq j(B)$. Let $U = j^{-1}G$. Thus, $U$ is an ultrafilter on $B$ and $j^* U \subset G$, where $G$ is $M$-generic for $j(B)$. By theorem 25 it follows that $U$ is a well-founded ultrafilter on $B$ in $V$.

Further, the corresponding Boolean ultrapower embedding $j_U$ is a factor of $j$.

**Corollary 81.** If $\kappa$ is a measurable cardinal with $2^\kappa = \kappa^+$ and $\mu$ is a normal measure on $\kappa$, then there is a well-founded ultrafilter $U$ on the Boolean algebra corresponding to $Add(\kappa, 1)$ such that the Boolean ultrapower by $U$ is the same as the ultrapower by $\mu$.

**Proof.** We saw above that if $j_\mu : V \rightarrow M$ is the ultrapower by $\mu$, then we may find an ultrafilter $U$ on the Boolean algebra $B$ corresponding to $Add(\kappa, 1)$ such that $j^* U \subset G$ for some $M$-generic filter $G \subseteq j(B)$. By corollary 25 it follows that $j_U$ is a factor of $j_\mu$. Since $\mu$ was a normal measure, it is minimal in the Rudin-Kiesler order on embeddings, so it has no proper factors. Consequently, $j_U = j_\mu$, as desired.

The proof of theorem 80 used the closure of the forcing notion, in a manner similar to many lifting arguments in large cardinal set theory. But we now show that if $\kappa$ is strongly compact, then we can produce well-founded Boolean ultrafilters on any $<\kappa$-distributive complete Boolean algebra.
**Theorem 82.** For any infinite regular cardinal $\kappa$, the following are equivalent:

1. Every $\kappa$-complete filter $F$ on a $<\kappa$-distributive complete Boolean algebra $B$ is contained in a $\kappa$-complete ultrafilter $U \subseteq B$.
2. $\kappa$ is strongly compact.

**Proof.** Assume (1). Fix any ordinal $\theta$ and let $B$ be the power set of $P_\kappa \theta$, which is trivial as a notion of forcing and is consequently $<\kappa$-distributive. Let $F$ be the fineness filter on $P_\kappa \theta$, generated by the sets $A_\alpha = \{ \sigma \in P_\kappa \theta \mid \alpha \in \sigma \}$. This filter is easily seen to be $\kappa$-complete in $B$. By (2), there is a $\kappa$-complete ultrafilter $U \subseteq B$ with $F \subseteq U$. Thus, there is a $\kappa$-complete fineness measure on $P_\kappa \theta$, and so $\kappa$ is strongly compact.

Conversely, suppose $\kappa$ is strongly compact and $F$ is a $\kappa$-complete filter on $B$, a $<\kappa$-distributive complete Boolean algebra. Let $D$ be the collection of open dense subsets of $B$ and let $\theta = |D|$. Let $j : V \to M$ be a $\theta$-strong compactness embedding for $\kappa$, so $\text{cp}(j) = \kappa$ and $j(\kappa) > \theta$, and every size $\theta$ subset of $M$ is covered by an element of $M$ of size less than $j(\kappa)$ in $M$. Thus, there is a set $E \subseteq M$ with $j^{-1}(D) \subseteq E$ and $|E|^M < j(\kappa)$. Thus, $E$ consists of fewer than $j(\kappa)$ many open dense subsets of $j(B)$ in $M$. Since $j(B)$ is $<j(\kappa)$-distributive in $M$, the intersection of these open dense sets is still open and dense. By the strong compactness cover property again, there is a set $s$ with $j^{-1}(F) \subseteq s \in M$ and $|s|^M < j(\kappa)$. We may assume $s \subseteq j(F)$, and so by the completeness of $j(F)$ in $M$, it follows that $s \not\in j(F)$. Putting the two facts together, we may find $p^* \in j(B)$ such that $p^* \in j(D)$ for all $D \in D$ and $p^* \leq j(a)$ for all $a \in F$. If $H \subseteq j(B)$ is any filter containing $p^*$, it follows that $H$ is random generic. Let $U = j^{-1}H$ be the inverse image, which is an ultrafilter on $B$, because $H$ selects an element of $j(\{b, -b\})$. Since $p^* \leq j(a)$ for all $a \in F$, it follows that $F \subseteq U$. And since $j^{-1}(U) \subseteq H$, it follows by theorem 23 that the Boolean ultrapower $jU$ is a factor of $j$. In particular, the Boolean ultrapower by $U$ is well-founded, as desired. \qed

Let us now sharpen this result in a way that will assist in applications, such as those in [AGH12]. We define that a forcing notion $P$ is $<\kappa$-friendly if for every $\gamma < \kappa$, there is a nonzero condition $p \in P$ below which the restricted forcing $P \upharpoonright p$ adds no subsets to $\gamma$.

**Theorem 83.** If $\kappa$ is a strongly compact cardinal and $P$ is a $<\kappa$-friendly notion of forcing, then there is a well-founded ultrapower $U$ on the Boolean algebra completion $B(P)$. In this case, there is an inner model $W$ satisfying every sentence forced by $P$ over $V$. Indeed, there is an elementary embedding of the universe $j : V \to \mathcal{V} \subseteq \mathcal{V}[G] = W$ into a transitive class $\mathcal{V}$, such that in $V$ there is a $\mathcal{V}$-generic ultrafilter $G \subseteq j(P)$.

**Proof.** Let $B$ be the Boolean algebra completion of $P$, and observe that $B$ also is $<\kappa$-friendly. We will find an ultrafilter $U \subseteq B$ for which the Boolean ultrapower $V^B/U$ is well-founded. Consider any $\theta \geq |B|$ and let $j : V \to M$ be a $\theta$-strongly compactness embedding, so that $j^{-1}B \subseteq s \in M$ for some $s \in M$ with $|s|^M < j(\kappa)$. Since $j(B)$ is $<j(\kappa)$-friendly in $M$, there is a nonzero condition $p \in j(B)$ such that forcing with $j(B) \upharpoonright p$ over $M$ adds no new subsets to $\lambda = |s|^M$. Thus, $j(B) \upharpoonright p$ is $(\lambda, 2)$-distributive in $M$. Applying this, it follows in $M$ that

$$p = p \land 1 = p \land \bigwedge_{b \in s}(b \lor \neg b) = \bigwedge_{b \in s}(p \land b) \lor (p \land \neg b) = \bigvee_{f \in 2^s} \bigwedge_{b \in s}(p \land (f(b)(b)),$$

where $(-)^0b = b$ and $(-)^1b = \neg b$, and where we use distributivity to deduce the final equality. Since $p$ is not 0, it follows that there must be some $f$ with $q = \bigwedge_{b \in s}p \land (f(b)(b)) \neq 0$. Note that $f(b)$ and $(\neg b)$ must have opposite values. Using $q$ as a seed, define the ultrafilter $U = \{ a \in B \mid q \leq f(a) \}$, which is the same as $\{ a \in B \mid f(j(a)) = 0 \}$. This is easily seen to be a $\kappa$-complete filter using the fact that $\text{cp}(j) = \kappa$ (just as in the powerset ultrafilter cases known classically). It is an ultrafilter precisely because $s$ covers $j^B$, so either $f(j(a)) = 0$ or $f(\neg j(a)) = 0$, and so either $a \in U$ or $\neg a \in U$, as desired. So by theorem 23 the Boolean ultrapower $V^B/U$ is well-founded, and the Mostowski collapse of this model is an inner model satisfying any sentence forced by $B$ over $V$, as desired. The Boolean ultrapower embedding $j : V \to \mathcal{V} \subseteq \mathcal{V}[G]$ has the properties stated in the theorem. \qed

Theorem 83 is technically a theorem scheme, asserting separately for each sentence forced by $P$ that $V^B/U$ satisfies it. It follows that when there are sufficient large cardinals, then one may construct transitive inner models of ZFC satisfying a variety of statements ordinarily obtained by forcing. For example, [AGH12] uses this method to show that if there is a supercompact cardinal, then there is a transitive inner model of ZFC.
with a Laver indestructible supercompact cardinal, and furthermore one can ensure for this cardinal \( \kappa \) either that \( 2^\kappa = \kappa^+ \) or that \( 2^\kappa = \kappa^{++} \), or any of a number of other properties; the method is really quite flexible.

Certain instances of the phenomenon of theorem \( \text{SS} \) are familiar in large cardinal set theory. For example, consider Prikry forcing with respect to a normal measure \( \mu \) on a measurable cardinal \( \kappa \), which is \( \text{\textlt}\kappa \)-friendly because it adds no bounded subsets to \( \kappa \). If \( V \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \) is the usual iteration of \( \mu \), with a direct limit to \( j_\omega : V \rightarrow M_\omega \), then the critical sequence \( \kappa_0, \kappa_1, \kappa_2, \ldots \) is well known to be \( M_\omega \)-generic for the corresponding Prikry forcing at \( j_\omega(\kappa) \) using \( j_\omega(\mu) \). This is precisely the situation occurring in theorem \( \text{SS} \) where we have an embedding \( j : V \rightarrow \mathcal{V} \) and a \( \mathcal{V} \)-generic filter \( G \subseteq j(\mathcal{P}) \) all inside \( V \). Thus, theorem \( \text{SS} \) generalizes this classical aspect about Prikry forcing to all friendly forcing under the stronger assumption of strong compactness.

**Question 84.** Which kinds of large cardinal embeddings are fruitfully realized as Boolean ultrapower embeddings?

While preparing earlier drafts of this article, we had asked whether the elementary embedding \( j_\omega : V \rightarrow M_\omega \) arising from the \( \omega \)-iteration of a normal measure \( \mu \) on a measurable cardinal \( \kappa \) is a Boolean ultrapower. This question is now answered by Fuchs and Hamkins \( \text{FH} \), who proved that indeed it is the Boolean ultrapower for Prikry forcing, and similar facts are true much more generally for other iterations, using generalized Prikry forcing. This case offers many very suggestive features, including the Bukovský-Dehornoy phenomenon, for which the forcing extension \( M_\omega[s] \), where \( s \) is the critical sequence that is \( M_\omega \)-generic for Prikry forcing, is the same as the intersection \( M_\omega[s] = \bigcap_n M_n \) of the finite iterates of \( \mu \). Another way to express this is that the full Boolean ultrapower \( V^B/U \) is the intersection of the induced power set ultrapowers \( V^A/U_A \) for maximal antichains \( A \).

**Question 85.** To what extent does the Bukovský-Dehornoy phenomenon hold for Boolean ultrapowers generally?

To our way of thinking, the intriguing situation here is that in the case of an iterated normal measure, the critical sequence was independently known to be a highly canonical useful object, and it also happens to be identical to the canonical generic object of the Boolean ultrapower \( V_U = M_\omega \). If we should be able to realize other large cardinal embeddings as Boolean ultrapowers, such as many of the extender embeddings that are used, then the hope is that their associated canonical generic objects, perhaps currently hidden, would prove to be similarly fruitful.

**References**

[AGH12] Arthur Apter, Victoria Gitman, and Joel David Hamkins. Inner models with large cardinal features usually obtained by forcing. *Archive for Mathematical Logic*, 51:257–283, 2012. 10.1007/s00153-011-0264-5.

[Bel85] J. L. Bell. *Boolean-valued models and independence proofs in set theory*, volume 12 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, second edition, 1985. With a foreword by Dana Scott, first edition 1977.

[Can87] R. Michael Canjar. Complete Boolean ultraproducts. *J. Symbolic Logic*, 52(2):530–542, 1987.

[FH] Gunter Fuchs and Joel David Hamkins. The Bukovský-Dehornoy phenomenon for boolean ultrapowers. manuscript in preparation.

[Man71] Richard Mansfield. The theory of Boolean ultrapowers. *Ann. Math. Logic*, 2(3):297–323, 1970/71.

[OR98] P. Ouwehand and H. Rose. Filtral powers of structures. *J. Symbolic Logic*, 63(4):1239–1254, 1998.

[Vop65] Petr Vopěnka. On \( \neg\neg \)-model of set theory. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 13:267–272, 1965.