ON CERTAIN THREE ALGEBRAS GENERATED BY BINARY ALGEBRAS

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Abstract. This paper’s central theme is to prove the existence of an n-algebra whose multiplication cannot be expressed employing any binary operation. Furthermore, to prove if two algebras are not isomorphic, this property does not hold for 3-algebras corresponding to these two algebras. The proof drives applying some results gotten early applying a new approach for the classification algebras problem, introduced recently, which showed great success in solving many classification algebras problems.
1. Introduction

In 1969 [11], Kurosh introduced the notion of multilinear operator algebra. It is known that such algebraic structures are attractive for their applications to problems of modern mathematical physics. In 1973 [14], Nambu proposed an exciting generalization of classical Hamiltonian mechanics; the Nambu bracket is a generalization of the classical Poisson bracket.

Indeed, the advance of theoretical physics of quantum mechanics and the discovery of Nambu mechanics (see [14]), together with Okubo's work on the Yang-Baxter equation (see [15]), gave impetus to significant development on triple algebra (3-algebras).

Furthermore, Carlsson, Lister, and Loos have studied triple algebra of associative type (see [9, 12, 13]). Hestenes provided the typical and founding example of totally associative triple algebra (see [10]).

In this article, we give basic definitions and examples related to general $n$-algebras, and we shall focus our attention on 3-algebras structures generated by binary algebras presented recently in [1]. Then, we introduce the definition of totally associative 3-algebras with examples, which show. Owing to the large size of the matrices involved in our computations of totally associative 3-algebras, we present only Mathematica's results.

2. Preliminaries

Let $\mathbb{F}$ be any field and the product $A \otimes B$ is the Kronecker product which stands for the matrix with blocks $(a_{ij}B)$, where $A = (a_{ij})$ and $B$ are matrices over $\mathbb{F}$.

Definition 2.1. A vector space $\mathbb{A}$ over $\mathbb{F}$ with multiplication $\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ given by $(u, v) \mapsto u \cdot v$ such that

- $(\alpha u + \beta v) \cdot w = \alpha(u \cdot w) + \beta(v \cdot w)$,
- $w \cdot (\alpha u + \beta v) = \alpha(w \cdot u) + \beta(w \cdot v),$

whenever $u, v, w \in \mathbb{A}$ and $\alpha, \beta \in \mathbb{F}$, is said to be an algebra.

Definition 2.2. Two algebras $\mathbb{A}$ and $\mathbb{B}$ are called isomorphic if there is an invertible linear map $f : \mathbb{A} \to \mathbb{B}$ such that

\begin{equation}
    f(u \cdot_A v) = f(u) \cdot_B f(v) \text{ whenever } u, v \in \mathbb{A}.
\end{equation}

Definition 2.3. A vector space $V$ over $\mathbb{F}$ equipped by a multilinear map $f : V \times V \times \ldots \times V \to V$ is said to be a $n$-algebra, that means:

- $f(x_1, x_2, \ldots, x_i + x'_i, \ldots, x_n) = f(x_1, x_2, \ldots, x_i, \ldots, x_n) + f(x_1, x_2, \ldots, x'_i, \ldots, x_n)$
- $f(x_1, x_2, \ldots, \lambda x_i, \ldots, x_n) = \lambda f(x_1, x_2, \ldots, x_i, \ldots, x_n)$

where $(x_1, x_2, \ldots, x_i, \ldots, x_n) \in V$.

Example 2.4. Let $\mathbb{A} = (V, \mu)$ be an algebra over a field $\mathbb{F}$: Then multilinear map

\begin{equation}
    f(x_1, x_2, \ldots, x_n) = \mu(x_1, \mu(x_2, \ldots, \mu(x_{n-1}, x_n) \ldots))
\end{equation}

defines an $n$-algebra structure on $V$. 


3. Classification approach of m-dimensional 3-algebras

Let $\mathbb{A}$ be $m$-dimensional 3-algebra over $\mathbb{F}$ and $e = (e^1, e^2, ..., e^m)$ its basis. Then the multilinear map $\cdot$ is represented by a matrix $A = (\alpha_{ijk}^l) \in M(m \times m^3; \mathbb{F})$ as follows

\[(3.1) \quad u \cdot v \cdot w = eA(u \otimes v \otimes w),\]

for $u = eu, v = ev, w = ew$, where $u = (u_1, u_2, ..., u_m)^T, v = (v_1, v_2, ..., v_m)^T$ and $w = (w_1, w_2, ..., w_m)^T$ are column coordinate vectors of $u, v, \text{ and } w$, respectively. The matrix $A \in M(m \times m^3; \mathbb{F})$ defined above is called the matrix of structural constants (MSC) of $\mathbb{A}$ with respect to the basis $e$. Further we assume that a basis $e$ is fixed and we do not make a difference between the algebra $\mathbb{A}$ and its MSC $A$.

If $e' = (e'^1, e'^2, ..., e'^m)$ is another basis of $\mathbb{A}$, $e'g = e$ with $g \in G = GL(m; \mathbb{F})$, and $A'$ is MSC of $\mathbb{A}$ with respect to $e'$ then it is known that

\[(3.2) \quad A' = gA(g^{-1})^\otimes 3 \]

is valid (see [5]). Thus, we can reformulate the isomorphism of algebras as follows.

**Definition 3.1.** Two m-dimensional 3-algebras $\mathbb{A}, \mathbb{B}$ over $\mathbb{F}$, given by their matrices of structure constants $A, B$, are said to be isomorphic if $B = gA(g^{-1})^\otimes 3$ holds true for some $g \in GL(m; \mathbb{F})$.

Further we consider only the case $m = 2$ then $\mathbb{A}$ can be represented by its matrix of structural constants (MSC) $A = (\gamma^l_{ijk}) \in M(2 \times 2^3; \mathbb{F})$ where $i, j, k, l = 1, 2$ and $\gamma^l_{ijk} \in \mathbb{F}$, as follows:

\[A = \left( \begin{array}{cccc} \gamma_{111} & \gamma_{112} & \gamma_{121} & \gamma_{122} \\ \gamma_{211} & \gamma_{212} & \gamma_{221} & \gamma_{222} \end{array} \right) \]

(for more information refer to [2]).

4. 3-algebras generated by binary algebras

Due to [1] we have the following classification theorems according to $Char(\mathbb{F}) \neq 2, 3$.

**Theorem 4.1.** Over an algebraically closed field $\mathbb{F}$ ($Char(\mathbb{F}) \neq 2$ and $3$), any non-trivial 2-dimensional algebra is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_1(c) = \left( \begin{array}{cccc} \alpha_1 & \alpha_2 & 0 & 0 \\ \beta_1 & \beta_2 & 1 - \alpha_1 & 0 \end{array} \right)$, where $c = (\alpha_1, \alpha_2, 0, 0) \in \mathbb{F}^4$,
- $A_2(c) = \left( \begin{array}{cccc} \alpha_1 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & 1 - \alpha_1 & 0 \end{array} \right) \simeq \left( \begin{array}{cccc} \alpha_1 & 0 & 0 & 0 \\ -\beta_1 & \beta_2 & 1 - \alpha_1 & 0 \end{array} \right)$, where $c = (\alpha_1, \beta_1, \beta_2) \in \mathbb{F}^3$,
- $A_3(c) = \left( \begin{array}{cccc} 0 & 0 & 0 & 1 \\ \beta_1 & 1 & 0 & 0 \end{array} \right)$, where $c = (\alpha_1, \beta_2) \in \mathbb{F}^2$,
- $A_4(c) = \left( \begin{array}{cccc} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & \beta_2 & 0 \end{array} \right)$, where $c = (\alpha_1, \beta_2) \in \mathbb{F}^2$,
- $A_5(c) = \left( \begin{array}{cccc} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & \beta_2 & 0 \end{array} \right)$, where $c = (\alpha_1, \beta_2) \in \mathbb{F}^2$,
- $A_6(c) = \left( \begin{array}{cccc} \alpha_1 & 0 & 0 & 0 \\ \beta_1 & 1 - \alpha_1 & 0 & 0 \end{array} \right) \simeq \left( \begin{array}{cccc} \alpha_1 & 0 & 0 & 0 \\ -\beta_1 & 1 - \alpha_1 & -\alpha_1 & 0 \end{array} \right)$, where $c = (\alpha_1, \beta_1) \in \mathbb{F}^2$,
- $A_7(c) = \left( \begin{array}{cccc} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 1 & 0 \end{array} \right)$, where $c = \beta_1 \in \mathbb{F}$,
- $A_8(c) = \left( \begin{array}{cccc} \alpha_1 & 0 & 0 & 0 \\ 0 & 1 - \alpha_1 & -\alpha_1 & 0 \end{array} \right)$, where $c = \alpha_1 \in \mathbb{F}$,
Example 4.2. In example 2.4, if $V$ is 2-dimensional vector space with a fixed basis $\{e_1, e_2\}$ and $(V, f)$ is 3-algebra structure on $V$. Then $f$ and $\mu$ can be expressed by their structure constants as follows:

$$f(e_i, e_j, e_k) = \gamma_{ijk}^1 e_1 + \gamma_{ijk}^2 e_2$$

and

$$\mu(e_r, e_s) = \eta_{rs}^1 e_1 + \eta_{rs}^2 e_2.$$ 

Then due to (2.2) we get the system of equations

\begin{align*}
\gamma_{ijk}^1 &= \eta_{jk}^1 \eta_{i1}^1 + \eta_{jk}^2 \eta_{i2}^1 \\
\gamma_{ijk}^2 &= \eta_{jk}^1 \eta_{i1}^2 + \eta_{jk}^2 \eta_{i2}^2
\end{align*}

we get $2^4 = 16$ equations for $2^3 = 8$ unknowns (the coefficients $\eta_{ij}^k$), which cannot be solved in general, except maybe for some very special cases.

Indeed, using (4.1), we can find the 3-algebras corresponding to all algebras presented in [1] under this procedure (see Table 1).

From the table we can see

- $(1 0 0 0 0 0 0 0)$ is a 3-algebra which cannot be expressed by any algebras in the above theorem and it is not isomorphic to any $B_i$ where $i = 1, \ldots, 11$.
- On the other hand, $A_4(1/3, -1/3)$ and $A_5(1/3)$ are not isomorphic algebras, but from these two non-isomorphic algebras, we get one 3-algebra

$$\left( \begin{array}{ccccccc}
\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}
\end{array} \right).$$

- Also $A_4(1, -1)$ and $A_4(1, 1)$ are not isomorphic algebras but from these two algebras we get one 3-algebra

$$\left( \begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array} \right).$$

5. Totally associative 3-algebras

In this section, we introduce the following definition of totally associative 3-algebra using its MSC.

**Definition 5.1.** A 3-algebra $\mathbb{A}$ is a totally associative 3-algebra if

\begin{equation}
(u \cdot v \cdot w) \cdot x \cdot y = u \cdot (v \cdot w \cdot x) \cdot y = u \cdot v \cdot (w \cdot x \cdot y)
\end{equation}

for all $u, v, w, x, y \in \mathbb{A}$.

**Example 5.2.** Let $\{e_1, e_2\}$ be a basis of a 2-dimensional 3-algebra $\mathbb{A}$, the multilinear map "\cdot" given by:

$$e_1 \cdot e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_1 \cdot e_2 = e_2$$

defines a totally associative 3-algebra.

According to (5.1) and (3.1), we can reformulate Definition 5.1 as follows.
Table 1. 2-dimensional 3-algebras generated by binary algebras (where \( \mathbf{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \))
**Definition 5.3.** A 2-dimensional 3-algebra \( A \) with multiplication \( \cdot \) over a field \( \mathbb{F} \) is said to be a totally associative 3-algebra if all of the following conditions are met:

\[
\begin{align*}
(5.2a) & \quad A (A \otimes I \otimes I - I \otimes A \otimes I) = 0, \\
(5.2b) & \quad A (A \otimes I \otimes I - I \otimes I \otimes A) = 0, \\
(5.2c) & \quad A (I \otimes A \otimes I - I \otimes I \otimes A) = 0,
\end{align*}
\]

where \( I \) is the identity \( 2 \times 2 \) matrix.

Using a computation program (here, we use Mathematica), it is easy to verify that the 3-algebras from the list in Table (1) satisfying the system (5.2) are:

(i) \( B_{2}(\alpha_{1}, \beta_{1}, \beta_{2}) \) when

- \( \alpha_{1} = 0, \beta_{1} = 0, \beta_{2} = 0, \)
- \( \alpha_{1} = \frac{1}{2}, \beta_{1} = 0, \beta_{2} = -\frac{1}{2}, \)
- \( \alpha_{1} = \frac{1}{2}, \beta_{1} = 0, \beta_{2} = \frac{1}{2}. \)

(ii) \( B_{4}(\alpha_{1}, \beta_{2}) \) when

- \( \alpha_{1} = 0, \beta_{2} = 0, \)
- \( \alpha_{1} = \frac{1}{2}, \beta_{2} = -\frac{1}{2}, \)
- \( \alpha_{1} = \frac{1}{2}, \beta_{2} = \frac{1}{2}, \)
- \( \alpha_{1} = 1, \beta_{2} = -1, \)
- \( \alpha_{1} = 1, \beta_{2} = 0, \)
- \( \alpha_{1} = 1, \beta_{2} = 1. \)

That means from the above list we get the following totally associative 3-algebras:

(i) \( B_{2}(0,0,0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \)

(ii) \( B_{2}(\frac{1}{2},0,-\frac{1}{2}) = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \)

(iii) \( B_{2}(\frac{1}{2},0,\frac{1}{2}) = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}, \)

(iv) \( B_{4}(\frac{1}{2},-\frac{1}{2}) = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \end{pmatrix}, \)

(v) \( B_{4}(\frac{1}{2},0) = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \end{pmatrix}, \)

(vi) \( B_{4}(\frac{1}{2},\frac{1}{2}) = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \end{pmatrix}, \)

(vii) \( B_{4}(1,-1) = B_{4}(1,1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \)

(viii) \( B_{4}(1,0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \)

According to (\[5\]), over an algebraically closed field, \( \mathbb{F} \) of characteristic, not 2, 3 every nontrivial 2-dimensional associative algebra is isomorphic to only one algebra listed below by

(i) \( A_{2}(\frac{1}{2},0,\frac{1}{2}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \)
(ii) \( A_4(1,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \),

(iii) \( A_4(1,1) = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \end{pmatrix} \),

(iv) \( A_4(1,1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \),

(v) \( A_4(1,2) = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{pmatrix} \),

(vi) \( A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \).

We conclude that there are totally associative 3-algebras, \( B_2(0,0,0) \), \( B_2(1/2,0,-1/2) \), \( B_4(1/2,-1/2) \), \( B_4(1,-1) \), are generated by non-associative algebras, \( A_2(0,0,0) \), \( A_2(1/2,0,-1/2) \), \( A_4(1/2,-1/2) \), \( A_4(1,-1) \), respectively.

Conclusion

Depending on the approach introduced in [8] and applied in [1], one can study the classification of \( n \)-algebras and then study some identities of \( n \)-algebras (refer to [2, 4, 5, 6, 7]).

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