Optimal Self-Dual $\mathbb{Z}_4$-Codes and a Unimodular Lattice in Dimension 41

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November 16, 2011

Abstract

For lengths up to 47 except 37, we determine the largest minimum Euclidean weight among all Type I $\mathbb{Z}_4$-codes of that length. We also give the first example of an optimal odd unimodular lattice in dimension 41 explicitly, which is constructed from some Type I $\mathbb{Z}_4$-code of length 41.

1 Introduction

Let $\mathbb{Z}_4 (= \{0, 1, 2, 3\})$ denote the ring of integers modulo 4. A $\mathbb{Z}_4$-code $C$ of length $n$ is a $\mathbb{Z}_4$-submodule of $\mathbb{Z}_4^n$. The dual code $C^\perp$ of $C$ is defined as $\{x \in \mathbb{Z}_4^n \mid x \cdot y = 0 \text{ for all } y \in C\}$ under the standard inner product $x \cdot y$. A code $C$ is self-dual if $C = C^\perp$. The Euclidean weight of a codeword $x = (x_1, \ldots, x_n)$ is $n_1(x) + 4n_2(x) + n_3(x)$, where $n_\alpha(x)$ denotes the number of components $i$ with $x_i = \alpha$ ($\alpha = 1, 2, 3$). The minimum Euclidean weight $d_E(C)$ of $C$ is the smallest Euclidean weight among all nonzero codewords of $C$. A self-dual code which has the property that all Euclidean weights are divisible by eight, is called Type II [2] (see also [15]). A self-dual code which is not Type II, is called Type I. A Type II $\mathbb{Z}_4$-code of length $n$ exists if and only if $n \equiv 0 \pmod{8}$ [2], while a Type I $\mathbb{Z}_4$-code exists for every length.

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It was shown in [2] that the minimum Euclidean weight $d_E(C)$ of a Type II code $C$ of length $n$ is bounded by $d_E(C) \leq 8 \lfloor \frac{n}{24} \rfloor + 8$. A Type II code meeting this bound with equality is called extremal. It was also shown in [19] that the minimum Euclidean weight $d_E(C)$ of a Type I code $C$ of length $n$ is bounded by $d_E(C) \leq 8 \lfloor \frac{n}{24} \rfloor + 8$ if $n \not\equiv 23 \pmod{24}$, and $d_E(C) \leq 8 \lfloor \frac{n}{24} \rfloor + 12$ if $n \equiv 23 \pmod{24}$. It is a fundamental problem to determine the largest minimum Euclidean weight among self-dual codes of that length. We denote the largest minimum Euclidean weight among Type I codes of length $n$ by $d_{\text{max},E}(n)$. These values $d_{\text{max},E}(n)$ have been determined in [8] and [18] for $n \leq 24$. We say that a Type I code of length $n$ is optimal or Euclidean-optimal if it has minimum Euclidean weight $d_{\text{max},E}(n)$. We pay attention to the Euclidean weight from the viewpoint of a connection with optimal odd unimodular lattices.

In this paper, we determine the largest minimum Euclidean weight $d_{\text{max},E}(n)$ for lengths $n \leq 47$, $n \neq 37$. To do this, we slightly improve upper bounds on the minimum Euclidean weights for lengths $n = 25, 26, \ldots, 31, 33, 34, 35$ (see the bound (1)), and we construct Type I $\mathbb{Z}_4$-codes meeting the bound (1) with equality. The values $d_{\text{max},E}(n)$ are listed in Table 1. For length 37, our extensive search failed to discover a Type I $\mathbb{Z}_4$-code with minimum Euclidean weight 16. However, we have found a Type I code with minimum Euclidean weight 12. We also give the first explicit example of an optimal odd unimodular lattice in dimension 41, which is constructed from an optimal code of length 41 by Construction A. All computer calculations in this paper were done using MAGMA [4].

2 Preliminaries

2.1 Self-dual $\mathbb{Z}_4$-codes

Every $\mathbb{Z}_4$-code $C$ of length $n$ has two binary codes $C^{(1)}$ and $C^{(2)}$ associated with $C$:

$$C^{(1)} = \{ c \mod 2 \mid c \in C \} \text{ and } C^{(2)} = \{ c \mod 2 \mid c \in \mathbb{Z}_4^n, 2c \in C \}.$$ 

The binary codes $C^{(1)}$ and $C^{(2)}$ are called the residue and torsion codes of $C$, respectively. If $C$ is a self-dual $\mathbb{Z}_4$-code, then $C^{(1)}$ is a binary doubly even code with $C^{(2)} = C^{(1) \perp}$ [5]. It is easy to see that

$$\min\{d(C^{(1)}), 4d(C^{(2)})\} \leq d_E(C) \leq 4d(C^{(2)}),$$

for all $C$. We denote the minimum Euclidean weight among Type II codes of length $n$ by $d_{\text{max},E}(n)$. These values $d_{\text{max},E}(n)$ have been determined in [8] and [18] for $n \leq 24$. We say that a Type I code of length $n$ is optimal or Euclidean-optimal if it has minimum Euclidean weight $d_{\text{max},E}(n)$. We pay attention to the Euclidean weight from the viewpoint of a connection with optimal odd unimodular lattices.

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| Length | \(n\) | \(d_{\text{max},E}^I(n)\) | Reference | | Length | \(n\) | \(d_{\text{max},E}^I(n)\) | Reference |
|--------|-------|-----------------|----------|--------|-------|-----------------|----------|
| 25     | 8     | \text{sub}(C_{26}) |          | 37     | 12   | 16             | [9]      |
| 26     | 12    | \(C_{26}\)     |          | 38     | 16   |                | [11]     |
| 27     | 12    | \(C_{27}\)     |          | 39     | 16   |                | [12]     |
| 28     | 12    | \(C_{28}\)     |          | 40     | 16   | \(C_{41}\)    |          |
| 29     | 12    | \(C_{29}\)     |          | 41     | 16   | \(C_{41}\)    |          |
| 30     | 12    | \(\text{sub}(C_{32})\) |   | 42     | 16   | [9]            |          |
| 31     | 12    | \(\text{sub}(C_{32})\) |   | 43     | 16   | \(C_{43}\)    |          |
| 32     | 16    | \(C_{32}\)     |          | 44     | 16   | \(C_{44}\)    |          |
| 33     | 12    | \(C_{33}\)     |          | 45     | 16   | \(C_{45}\)    |          |
| 34     | 12    | \(C_{34}\)     |          | 46     | 16   | [13]           |          |
| 35     | 12    | \(\text{sub}(C_{36})\) |   | 47     | 16   | [13]           |          |
| 36     | 16    | \(C_{36}\)     |          |        |      |                |          |

where \(d(C^{(i)})\) denotes the minimum weight of \(C^{(i)}\) \((i = 1, 2)\).

Codes differing by only a permutation of coordinates are called permutation-equivalent. Any self-dual \(\mathbb{Z}_4\)-code of length \(n\) is permutation-equivalent to a code \(C\) with generator matrix of the standard form

\[
\begin{pmatrix}
I_{k_1} & A & B_1 + 2B_2 \\
O & 2I_{k_2} & 2D
\end{pmatrix},
\]

where \(A, B_1, B_2\) and \(D\) are \((1, 0)\)-matrices, \(I_k\) denotes the identity matrix of order \(k\), and \(O\) denotes the zero matrix [5]. The residue code \(C^{(1)}\) of \(C\) is an \([n, k_1]\) code with generator matrix \((I_{k_1} \ A \ B_1)\), and the torsion code \(C^{(2)}\) is an \([n, k_1 + k_2]\) code with generator matrix \((I_{k_1} \ A \ B_1)\).

### 2.2 Unimodular lattices and upper bounds

A (Euclidean) lattice \(L\) in dimension \(n\) is \textit{unimodular} if \(L = L^*\), where the dual lattice \(L^*\) of \(L\) is defined as \(\{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\}\) under the standard inner product \((x, y)\). The norm of a vector \(x\) is \((x, x)\). Two lattices \(L\) and \(L'\) are \textit{isomorphic}, denoted \(L \cong L'\), if there exists an orthogonal matrix \(A\) with \(L' = L \cdot A = \{xA \mid x \in L\}\). The minimum norm of \(L\) is the smallest norm among all nonzero vectors of \(L\). The theta series \(\theta_L(q)\) of \(L\) is the formal
power series $\theta_L(q) = \sum_{x \in L} q^{(x,x)} = \sum_{m=0}^{\infty} N_m q^m$, where $N_m$ is the number of vectors of norm $m$. The kissing number is the second nonzero coefficient of the theta series, that is, the number of vectors of $L$ with minimum norm.

Let $\mu_{\max}^O(n)$ denote the largest minimum norm among odd unimodular lattices in dimension $n$. We say that an odd unimodular lattice is optimal if it has the largest minimum norm $\mu_{\max}^O(n)$. These values $\mu_{\max}^O(n)$ have been determined for $n \leq 47, n \neq 37, 41$ (see [6] and [16]). For $25 \leq n \leq 47$, the following values are known

\begin{equation}
\mu_{\max}^O(n) = \begin{cases} 
2 & (n = 25), \\
3 & (n = 26, 27, \ldots, 31, 33, 34, 35), \\
4 & (n = 32, 36, 38, 39, 40, 42, \ldots, 47), \\
3 \text{ or } 4 & (n = 37, 41).
\end{cases}
\end{equation}

In this paper, we give the first example of an odd unimodular lattice in dimension 41 having minimum norm 4. Hence, we have $\mu_{\max}^O(41) = 4$.

Let $C$ be a Type II (resp. Type I) $\mathbb{Z}_4$-code of length $n$ and minimum Euclidean weight $d_E$. Then the following lattice

$$A_4(C) = \frac{1}{2}\{(x_1, \ldots, x_n) \in \mathbb{Z}^n | (x_1 \mod 4, \ldots, x_n \mod 4) \in C\}$$

is an even (resp. odd) unimodular lattice with minimum norm $\min\{4, d_E/4\}$ [2]. Let $C$ be a Type I code of length $n$ and minimum Euclidean weight $d_E$. Since $A_4(C)$ is an odd unimodular lattice with minimum norm $\min\{4, d_E/4\}$, $\mu_{\max}^O(n)$ in (3) implies upper bounds, which improve the upper bounds described in Section [1] for $25 \leq n \leq 35, n \neq 32$. Hence, we have the following bounds

\begin{equation}
d_{\max,E}(n) \leq \begin{cases} 
8 & (n = 25), \\
12 & (n = 26, 27, \ldots, 31, 33, 34, 35), \\
16 & (n = 32, 36, 37, \ldots, 47).
\end{cases}
\end{equation}

## 3 Optimal Type I $\mathbb{Z}_4$-codes

In this section, we determine the largest minimum Euclidean weights $d_{\max,E}(n)$. This is done by constructing Type I codes meeting the bound (4) with equality for lengths $n = 25, \ldots, 29, 31, \ldots, 36, 41, 43, 44, 45$. 

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• Lengths 26, . . . , 29, 33, 34, 36, 41, 43, 44, 45:

We have found a binary doubly even code $B_n$ of length $n$ with $d(B_n) = 12$ and $d(B_n^+) \geq 3$ for $n = 26, \ldots, 29, 33, 34$, and $d(B_n) = 16$ and $d(B_n^+) \geq 4$ for $n = 36, 41, 43, 44, 45$. Note that there is a self-dual $\mathbb{Z}_4$-code $C$ with $C^{(1)} = B$ for any given binary doubly even code $B$ (see [17]). It follows from (1) that there is a Type I code meeting the bound (4) with equality for these lengths.

The method of construction of a self-dual $\mathbb{Z}_4$-code $C$ with $C^{(1)} = B$ was given in [17, Section 3]. Using this method, we explicitly have found an optimal Type I code $C_n$ with $C_n^{(1)} = B_n$ for these lengths. In order to save space, instead of listing generator matrices, we only list in Figure 1 the $k_1 \times (n-k_1)$ matrix

$$M_n = \begin{pmatrix} A & B_1 + 2B_2 \end{pmatrix},$$

in standard form (2), since the lower part of (2) can be obtained from $M_n$ for each code $C_n$. The minimum Euclidean, Lee, Hamming weights $d_E, d_L, d_H$ of $C_n$ (see e.g. [2] for the definition of $d_L$) are listed in Table 2. The parameters $[n, k, d]$ and the orders $\# \text{Aut}$ of the automorphism groups of their residue codes $C_n^{(1)}$ are also listed in Table 2. Note that the minimum Hamming weight of a self-dual $\mathbb{Z}_4$-code $C$ is the same as $d(C^{(2)})$ [18].

• Length 32:

Let $L$ be a unimodular lattice in dimension $n$ and let $k$ be a positive integer. A set $\{f_1, \ldots, f_n\}$ of $n$ vectors $f_1, \ldots, f_n$ in $L$ with $(f_i, f_j) = k\delta_{ij}$ is called a $k$-frame of $L$, where $\delta_{ij}$ is the Kronecker delta. It is known that an even (resp. odd) unimodular lattice $L$ contains a 4-frame if and only if there is a Type II (resp. Type I) $\mathbb{Z}_4$-code $C$ with $A_4(C) \cong L$.

Conway and Sloane [6] showed that there are exactly five odd unimodular lattices in dimension 32 having minimum norm 4, up to isomorphism. In addition, such a lattice $\Lambda$ contains vectors of the form

$$\frac{1}{\sqrt{8}}(\pm 4, \pm 4, 0, \ldots, 0), \ldots, \frac{1}{\sqrt{8}}(0, \ldots, 0, \pm 4, \pm 4).$$

Hence, $\Lambda$ contains a 4-frame. This means that there is a Type I $\mathbb{Z}_4$-code $C_{32}$ of length 32 with $A_4(C_{32}) \cong \Lambda$. Since $\Lambda$ has minimum norm 4, $C_{32}$ has minimum Euclidean weight 16, that is, $C_{32}$ is optimal.
Figure 1: Generator matrices
Let \( C \) be a self-dual code of length \( n \) \((n \geq 2)\). Then the following code
\[
\text{sub}(C) = \{(x_2, \ldots, x_n) \mid (x_1, x_2, \ldots, x_n) \in C, x_1 \in \{0, 2\}\}
\]
is a self-dual code of length \( n - 1 \). The codes \( \text{sub}(C_{26}) \), \( \text{sub}(C_{32}) \) and \( \text{sub}(C_{36}) \) are self-dual codes of lengths 25, 31 and 35, respectively. Moreover, from (4), the codes \( \text{sub}(C_{26}), \text{sub}(C_{32}) \) and \( \text{sub}(C_{36}) \) have minimum Euclidean weights 8, 12 and 12, respectively, since \( C_{26}, C_{32} \) and \( C_{36} \) have minimum Euclidean weight 12, 16 and 16, respectively.

Since there are self-dual codes of lengths 8, 17 and minimum Euclidean weight 8, the direct sum of the codes is also a self-dual code of length 25 and minimum Euclidean weight 8.

By constructing Type I codes meeting the bound (4) with equality for lengths \( n = 25, \ldots, 47 \), we determine the largest minimum Euclidean weight \( d_{\text{max},E}^I(n) \) for \( n \leq 47 \), \( n \neq 37 \), as follows.

**Proposition 1.** Let \( d_{\text{max},E}^I(n) \) denote the largest minimum Euclidean weight among Type I \( \mathbb{Z}_4 \)-codes of length \( n \). Then \( d_{\text{max},E}^I(25) = 8 \), \( d_{\text{max},E}^I(n) = 12 \) if \( n = 26, \ldots, 31, 33, 34, 35 \), and \( d_{\text{max},E}^I(n) = 16 \) if \( n = 32, 36, 38, \ldots, 47 \).
Table 2: Optimal Type I $\mathbb{Z}_4$-codes

| Code | $d_E$ | $d_L$ | $d_H$ | $[n, k, d]$ | $\# \text{Aut}$ |
|------|--------|--------|--------|-------------|----------------|
| $C_{26}$ | 12 | 6 | 3 | $[26, 6, 12]$ | 120 |
| $C_{27}$ | 12 | 6 | 3 | $[27, 7, 12]$ | 240 |
| $C_{28}$ | 12 | 8 | 4 | $[28, 7, 12]$ | 10752 |
| $C_{29}$ | 12 | 6 | 3 | $[29, 7, 12]$ | 1 |
| $C_{33}$ | 12 | 6 | 3 | $[33, 9, 12]$ | 1 |
| $C_{34}$ | 12 | 6 | 3 | $[34, 10, 12]$ | 1 |
| $C_{36}$ | 16 | 8 | 4 | $[36, 7, 16]$ | 1451520 |
| $C_{41}$ | 16 | 8 | 4 | $[41, 10, 16]$ | 1 |
| $C_{43}$ | 16 | 8 | 4 | $[43, 10, 16]$ | 1 |
| $C_{44}$ | 16 | 8 | 4 | $[44, 11, 16]$ | 11 |
| $C_{45}$ | 16 | 8 | 4 | $[45, 9, 16]$ | 9 |

For length 37, our extensive search failed to discover a Type I $\mathbb{Z}_4$-code with minimum Euclidean weight 16. However, we have found a Type I code with minimum Euclidean weight 12. Hence, $d_{\text{max},E}(37) = 12$ or 16 (see Table 1).

4  Optimal odd unimodular lattices

4.1 Dimension 41

By Construction A, optimal Type I $\mathbb{Z}_4$-codes $C_n$ constructed in the previous section give optimal odd unimodular lattices $A_4(C_n)$. In particular, the first explicit example of an optimal odd unimodular lattice $A_4(C_{41})$ in dimension 41 can be constructed from $C_{41}$.

**Proposition 2.** There is an odd unimodular lattice in dimension 41 having minimum norm 4.

We consider the theta series of optimal odd unimodular lattices in dimension 41. Conway and Sloane [6] show that if the theta series of an odd
unimodular lattice $L$ in dimension $n$ is written as

$$\theta_L(q) = \sum_{j=0}^{\lfloor n/8 \rfloor} a_j \theta_3(q)^{n-8j} \Delta_8(q)^j,$$

then the theta series of the shadow $S$ (see [6] for the definition) is written as

$$\theta_S(q) = \sum_{j=0}^{\lfloor n/8 \rfloor} \frac{(-1)^j}{16^j} a_j \theta_2(q)^{n-8j} \theta_4(q^2)^{8j} = \sum_i B_i q^i \text{ (say)},$$

where $\Delta_8(q) = q \prod_{m=1}^{\infty} (1 - q^{2m-1})^8 (1 - q^{4m})^8$ and $\theta_2(q), \theta_3(q)$ and $\theta_4(q)$ are the Jacobi theta series [7]. As the additional conditions, it holds that there is at most one nonzero $B_r$ for $r < (\mu + 2)/2$; $B_r = 0$ for $r < \mu/4$; and $B_r \leq 2$ for $r < \mu/2$, where $\mu$ is the minimum norm of $L$.

In the case $n = 41$, since minimum norm $\mu$ is 4, $a_0, \ldots, a_3$ in (5) and (6) are determined as follows: $a_0 = 1$, $a_1 = -82$, $a_2 = 1476$ and $a_3 = -3280$. Since the coefficients in the shadow must be non-negative integers, $a_4$ is divisible by $2^7$ and $a_5$ is divisible by $2^{19}$. Thus, we put $a_4 = 2^7\alpha$ and $a_5 = 2^{19}\beta$. Then we have the possible theta series $\theta_L$ and $\theta_S$ of an optimal odd unimodular lattice $L$ in dimension 41 and its shadow $S$:

$$\theta_L = 1 + (15170 + 128\alpha)q^4 + (1226720 - 1792\alpha - 524288\beta)q^6 + \cdots$$
$$+ (42928640 + 8192\alpha + 19922944\beta)q^8 + \cdots$$

and

$$\theta_S = \beta q^{1/4} + (\alpha - 79\beta)q^{9/4} + (104960 - 55\alpha + 3040\beta)q^{17/4} + \cdots,$$

respectively, where $\beta = 0$ or $\alpha = 79\beta$ by the above additional conditions. By calculating the kissing number of $A_4(C_{41})$ and the minimum norm of its shadow, we determine the theta series of the lattice $A_4(C_{41})$ as follows:

$$1 + 15426q^4 + 1223136q^5 + 42945024q^6 + 867179520q^7$$
$$+ 11719744560q^8 + 1165321216256q^9 + 909236984832q^{10} + \cdots.$$

### 4.2 Minimum norms and kissing numbers

In Table 3, we list the minimum norms $\mu(L)$ and the kissing numbers $N(L)$ of optimal odd unimodular lattices $L = A_4(C_n)$ constructed from $C_n$ given in Table 2.
Table 3: Minimum norms and kissing numbers

| $L$        | $\mu(L)$ | $N(L)$ | $L$        | $\mu(L)$ | $N(L)$ |
|------------|----------|--------|------------|----------|--------|
| $A_4(C_{26})$ | 3        | 3120   | $A_4(C_{30})$ | 4        | 51032  |
| $A_4(C_{27})$ | 3        | 2664   | $A_4(C_{41})$ | 4        | 15426  |
| $A_4(C_{28})$ | 3        | 1728   | $A_4(C_{43})$ | 4        | 9286   |
| $A_4(C_{29})$ | 3        | 1856   | $A_4(C_{44})$ | 4        | 8392   |
| $A_4(C_{33})$ | 3        | 752    | $A_4(C_{45})$ | 4        | 7866   |
| $A_4(C_{34})$ | 3        | 528    |             |          |        |

For dimensions up to 28, optimal odd unimodular lattices have been classified (see [7, p. xliii–xliv]). Borcherds [3] showed that there is a unique optimal odd unimodular lattice $S_{26}$ in dimension 26 (see [7, p. xliii]). Hence, the lattice $A_4(C_{26})$ gives an alternative construction of $S_{26}$. We list the symmetrized weight enumerator $swe_{26}$ (see [5] for the definition) of $C_{26}$ at the end of this section. Bacher and Venkov [1] showed that there are three (resp. 38) non-isomorphic optimal odd unimodular lattices in dimension 27 (resp. 28) (see [7, p. xlv]). By comparing the kissing numbers and the automorphism groups, we have that $A_4(C_{27}) \cong R_{27,1}(\emptyset)$ in [1, Table 4], and $A_4(C_{28}) \cong R_{28,38e}(\emptyset)$ in [1, Table 5]. Hence, these lattices $S_{26}$, $R_{27,1}(\emptyset)$ and $R_{28,38e}(\emptyset)$ contain a 4-frame. We remark that the lattices have no 3-frame.

For other dimensions, since the lattices $A_4(C_n)$ ($n = 33, 36, 44$) have different kissing numbers than those of the known lattices in [16] and no example was given in [16] for dimension 30, the lattices $A_4(C_n)$ ($n = 30, 33, 36, 44$) provide other examples of optimal odd unimodular lattices.

Since an odd unimodular lattice in dimension 41 having minimum norm 4 has been constructed, the largest minimum norm $\mu_{\text{max}}^{O}(41)$ is 4. The smallest dimension $n$ for which the largest minimum norm $\mu_{\text{max}}^{O}(n)$ has not been determined is 37. Hence, it is worthwhile to determine if there is a Type I $\mathbb{Z}_4$-code of length 37 and minimum Euclidean weight 16. At dimension 48, the largest minimum norm $\mu_{\text{max}}^{O}(48)$ is exactly 5 ([10], [14] and [19]). No Type I $\mathbb{Z}_4$-code constructs an odd unimodular lattice with minimum norm 5 by Construction A. It seems that the connection between self-dual $\mathbb{Z}_4$-codes and unimodular lattices is no longer useful at this point.
\[ swe_{26} = a^{26} + 30a^{23}c^{3} + 255a^{22}c^{4} + 1100a^{21}c^{5} + 3571a^{20}c^{6} + 9990a^{19}c^{7} + 24330a^{18}c^{8} \\
+ 49680a^{17}c^{9} + 83237a^{16}c^{10} + 119004a^{15}c^{11} + 2880a^{14}b^{12} + 150750a^{13}c^{12} \\
+ 40320a^{13}c^{12} + 164680a^{12}c^{13} + 262080a^{12}c^{12} + 150750a^{12}c^{12} + 1048320a^{11}b^{12}c^{3} \\
+ 119004a^{11}c^{15} + 17408a^{10}b^{16} + 2882880a^{10}b^{12}c^{4} + 83237a^{10}c^{16} + 174080a^{9}b^{16}c \\
+ 5765760a^{9}b^{12}c^{5} + 49680a^{9}c^{17} + 783360a^{8}b^{16}c^{2} + 8648640a^{8}b^{12}c^{6} + 24330a^{8}c^{18} \\
+ 2088960a^{7}b^{16}c^{3} + 9884160a^{7}b^{12}c^{7} + 9990a^{7}c^{19} + 16384a^{6}b^{20} + 3655680a^{6}b^{16}c^{4} \\
+ 8648640a^{6}b^{12}c^{8} + 3571a^{6}c^{20} + 98304a^{5}b^{20}c + 4386816a^{5}b^{16}c^{5} + 5765760a^{5}b^{12}c^{9} \\
+ 1100a^{5}c^{21} + 245760a^{4}b^{20}c^{2} + 3655680a^{4}b^{16}c^{6} + 2882880a^{4}b^{12}c^{10} + 255a^{4}c^{22} \\
+ 327680a^{3}b^{20}c^{3} + 2088960a^{3}b^{16}c^{7} + 1048320a^{3}b^{12}c^{11} + 30a^{3}c^{23} + 245760a^{2}b^{20}c^{4} \\
+ 783360a^{2}b^{16}c^{8} + 262080a^{2}b^{12}c^{12} + 98304a^{2}b^{20}c^{5} + 174080a^{2}b^{16}c^{9} + 40320a^{2}b^{12}c^{13} \\
+ 16384a^{2}b^{16}c^{10} + 174080a^{2}b^{16}c^{10} + 2880b^{12}c^{14} + c^{26} \]

**Acknowledgments.** The author would like to thank the anonymous referees for their helpful comments.

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