Enhancing the geometric quantum discord in the Heisenberg XX chain by Dzyaloshinsky-Moriya interaction

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We studied the trace distance, the Hellinger distance, and the Bures distance geometric quantum discord (GQDs) for a two-spin Heisenberg XX chain with the Dzyaloshinsky-Moriya (DM) interaction and the external magnetic fields. We found that considerable enhancement of the GQDs can be achieved by introducing the DM interaction, and their maxima were obtained in the limiting case $D \to \infty$. The external magnetic fields and the increase of the temperature can also enhance the GQDs to some extent for certain special cases.

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I. INTRODUCTION

For a long time, entanglement was considered to be the only resource responsible for the advantage of many quantum information processing (QIP) tasks [1–4]. As entanglement exists only in the non-separable states, separable states were also considered to be classically correlated and useless for QIP. But recent studies revealed that the separable states may also possess certain kinds of quantum correlations. For example, the quantum discord (QD) [5], which is a more general quantum correlation measure than that of entanglement, can be nonzero for some separable states [6]. From a practical point of view, it is proposed that the QD is responsible for the power of the QIP tasks such as the deterministic quantum computation with one qubit [7], remote state preparation [8], and quantum locking [9, 10]. The QD is also intimately related to many fundamental problems of quantum mechanics [11–14].

Besides the entropic measure of QD [5], the quantumness of a state can also be characterized from many other perspectives. These measures include the measurement-induced disturbance [15], and the measurement-induced nonlocality [16–18]. Also there are distance-based quantum correlation measures, such as the first proposed geometric QD (GQD) defined via the Bures distance [24] (which may be changed by trivial local actions on the unmeasured party [20]), and its modified version via the trace distance [21] or the Hellinger distance [22, 23]. Moreover, the GQD in a state can also be defined via the Bures distance [24].

The above progress prompted a huge surge of people’s interest in this new field. Particularly, as a potential and costly resource responsible for the advantage of many quantum information processing (QIP) tasks [1–4], as entanglement exists only in the non-separable states, separable states were also considered to be classically correlated and useless for QIP. But recent studies revealed that the separable states may also possess certain kinds of quantum correlations. For example, the quantum discord (QD) [5], which is a more general quantum correlation measure than that of entanglement, can be nonzero for some separable states [6]. From a practical point of view, it is proposed that the QD is responsible for the power of the QIP tasks such as the deterministic quantum computation with one qubit [7], remote state preparation [8], and quantum locking [9, 10]. The QD is also intimately related to many fundamental problems of quantum mechanics [11–14].

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The above progress prompted a huge surge of people’s interest in this new field. Particularly, as a potential and costly resource for QIP, the long-time preservation of QDs remains a main pursuit of people [25–27], and their decay dynamics for various open quantum systems have been studied, with many novel phenomena being observed [28–33]. The QDs in various spin systems [36–43], and its role in detecting quantum phase transition points at finite temperatures have also been revealed [44–47].

We investigate in this work the properties of a two-spin system described by the Heisenberg XX model. Different from those previous studies, we introduced here the Dzyaloshinsky-Moriya (DM) interaction induced by the spin-orbit coupling, whose effects on the properties of entanglement [48, 49] and the entropic QD [43] have already been studied. Here, we concentrate on its effects on the trace distance, the Hellinger distance, and the Bures distance GQDs, which have been demonstrated to be well-defined measures of quantum correlations. We will show that for the considered physical model, the three GQDs can be enhanced obviously by increasing the strength of the DM interaction.

This paper is arranged as follows. In Section II we recall the definitions for three different distance-based quantum correlation measures we adopted. In Section III we give the physical model we considered, and some analytical results obtained for the three GQDs. Section IV is devoted to a discussion of the dependence of the GQDs on the system parameters, through which we show that the GQDs can be enhanced obviously by increasing the DM interaction. We conclude this paper in Section V.

II. BASIC FORMALISM FOR THE GQDS

We recall in this section the definitions and the related formula for three types of the GQDs we adopted in this paper, namely, the trace distance, the Hellinger distance, and the Bures distance GQDs [21–24]. They characterize the quantum correlations of a bipartite state $\rho$ with the density operator $\rho$ from different perspectives, and can be classified as the distance-based measures of quantum correlations as they are all related to certain forms of distance.

First, we recall the trace distance GQD for a bipartite state $\rho$, which is defined as [21]

$$Q_T(\rho) = \min_{\chi \in P_{CQ}} \| \rho - \chi \|_1,$$

where $\| \rho - \chi \|_1 = \text{Tr} \sqrt{(\rho - \chi)\sqrt{(\rho - \chi)}}$ denotes the trace distance between $\rho$ and $\chi \in P_{CQ}$, and

$$\rho_{CQ} = \sum_k p_k \Pi^A_k \otimes \rho^B_k,$$

where $\Pi^A_k$ and $\rho^B_k$ are the projectors and the density matrices, respectively, and $P_{CQ}$ is the set of all possible pure mixed states.
is the classical-quantum state \[|p_k\rangle\], where \(p_k\) is a probability distribution, and \(\Pi^A_k\) and \(\rho^B_k\) are the orthogonal projector for \(A\) and the density operator for \(B\), respectively.

For the two-qubit \(X\) state \(\rho^X\) which only contains nonzero elements along the main diagonal and anti-diagonal, the calculation of the trace distance GQD can be simplified, with the analytical expression being given by \[50\]

\[Q_T(\rho) = \sqrt{\frac{1}{\Pi^A} ||\sqrt{\rho} - \Pi^A (\sqrt{\rho})||_F^2},\]

where \(\gamma_1, 2 = 2(|\rho^X_{23}| + |\rho^X_{14}|)\), \(\gamma_3 = 1 - 2(\rho^X_{11} + \rho^X_{22})\), and \(x_3 = 2(\rho^X_{11} + \rho^X_{22}) - 1\).

Second, we recall the Hellinger distance GQD. It is defined based on the square root of the density operator \(\rho\), and can be written as \[22\]

\[Q_H(\rho) = 2 \inf_{\Pi^A} ||\sqrt{\rho} - \Pi^A (\sqrt{\rho})||_F^2,\]

where the infimum is taken over all the projection-valued measurements \(\Pi^A = \{\Pi^A_k\}\), with \(\Pi^A = \sum_k \Pi^A_k \sqrt{\rho} \Pi^A_k\), and \(||X|| = \sqrt{trX^\dagger X}\) is the Hilbert-Schmidt distance.

The calculation of \(Q_H(\rho)\) is difficult in general. But for the special case of \(2 \times n\) system, \(Q_H(\rho)\) can be calculated via \[23\]

\[Q_H(\rho) = 1 - \lambda_{\text{max}} \{W_{AB}\},\]

where \(\lambda_{\text{max}}\) denotes the maximum eigenvalue of a \(3 \times 3\) matrix \(W_{AB}\), whose elements are given by

\[(W_{AB})_{ij} = \text{Tr}\{\sqrt{\rho} (\sigma^A_i \otimes I_n) \sqrt{\rho} (\sigma^B_j \otimes I_n)\},\]

with \(\sigma^S_{x,y,z}\) (\(S = A, B\)) the three Pauli operators, and \(I_n\) the \(n\times n\) identity operator.

Finally, the Bures distance GQD we consider is defined via the Bures distance between \(\rho\) and \(\chi \in \rho^{CCQ}\) \[24\], which is of the following form

\[Q_B(\rho) = \sqrt{(2 + \sqrt{2}) \left[ 1 - \max_{\chi \in \rho^{CCQ}} F(\rho, \chi) \right]},\]

where \(F(\rho, \chi) = \left[\text{Tr}\{\sqrt{\rho} \chi \sqrt{\rho} \chi^\dagger\}\}^{1/2}\right]^2\) is the Uhlmann fidelity, and \(Q_B(\rho)\) in Eq. (7) is normalized, namely, it takes the value 1 for the maximally discordant states.

For the case of \(2 \times n\) system, the maximum of \(F(\rho, \chi)\) can be calculated via \[51\]

\[\max_{\chi \in \rho^{CCQ}} F(\rho, \chi) = \frac{1}{2} \max_{|i| = 1} \left( 1 - \text{tr}\Lambda + 2 \sum_{k=1}^n \lambda_k(\Lambda) \right),\]

where \(\Lambda = \sqrt{\rho} (\vec{u} \cdot \vec{\sigma} \otimes I_n) \sqrt{\rho}\), with \(\vec{\sigma} = \{\sigma^x, \sigma^y, \sigma^z\}\) being the vector of the Pauli operators, \(\vec{u}\) is a unit vector in \(\mathbb{R}^3\), and \(\lambda_k(\Lambda)\) denote the eigenvalues of \(\Lambda\) in non-increasing order.

### III. THE MODEL

We consider in this work two spin systems described by the Heisenberg XX model, with the addition of the DM interaction which arises from the spin-orbit coupling being involved. The corresponding Hamiltonian is given by

\[\hat{H} = J (\sigma^+_1 \sigma^-_2 + \sigma^+_2 \sigma^-_1) + B (\sigma^+_1 \sigma^y_2 + \sigma^-_1 \sigma^y_2 - \sigma^y_1 \sigma^y_2) + D (\sigma^+_1 \sigma^z_2 - \sigma^-_1 \sigma^z_2),\]

where \(\sigma^\mu_i (\mu = x, y, z)\) are the familiar Pauli operators acting on the \(n\)-th spin, \(J\) is the coupling constant between the two spins, while \(B\) and \(D\) denote respectively the strengths of the external magnetic field and the DM interaction, both of which along the \(z\) direction. Moreover, \(\hbar = 1\) is assumed here.

For the physical model described in Eq. (9), its eigenvalues and eigenvectors can be derived analytically, which are given respectively by

\[\epsilon_{1,2} = \pm 2\delta, \quad \epsilon_{3,4} = \pm 2B,\]

and

\[|\Psi_{1,2}\rangle = \frac{1}{\sqrt{2}} ([10] \pm e^{i\theta} [01]),\]

\[|\Psi_3\rangle = [00], \quad |\Psi_4\rangle = [11].\]

From the above two equations, one can obtain the state of the system at thermal equilibrium with temperature \(T\), which is given by the Gibb’s density operator \(\rho = Z^{-1} \exp(-\beta H)\), with \(\beta = 1/k_B T\), and \(k_B\) the Boltzman’s constant. Moreover, \(Z = \text{Tr} [\exp(-\beta H)]\) denotes the partition function, which can be obtained explicitly as

\[Z = 2 (\cosh 2\beta\delta + \cosh 2\beta B),\]

while the density operator \(\rho\) is given by

\[\rho = \frac{1}{Z} \begin{pmatrix} e^{-2\beta B} & 0 & 0 & 0 \\ 0 & \cosh 2\beta\delta & -e^{i\theta} \sinh 2\beta\delta & 0 \\ 0 & -e^{-i\theta} \sinh 2\beta\delta & \cosh 2\beta B & 0 \\ 0 & 0 & 0 & e^{2\beta B} \end{pmatrix}.\]

Clearly, \(\rho\) expressed in Eq. (13) is of the X form, and therefore the trace distance GQD can be derived analytically as

\[Q_T(\rho) = \frac{2 \sinh 2\beta\delta}{Z}.\]

Moreover, as the square root of the density operator \(\rho\) is given by

\[\sqrt{\rho} = \frac{1}{\sqrt{Z}} \begin{pmatrix} e^{-\beta B} & 0 & 0 & 0 \\ 0 & 2 \cosh \beta\delta & -2e^{i\theta} \sinh \beta\delta & 0 \\ 0 & -2e^{-i\theta} \sinh \beta\delta & 2 \cosh \beta\delta & 0 \\ 0 & 0 & 0 & e^{2\beta B} \end{pmatrix},\]

the Hellinger distance GQD can be derived analytically as

\[Q_H(\rho) = 1 - \max \{\lambda_1, \lambda_2\},\]

where

\[\lambda_1 = \frac{8}{Z} \cosh \beta\delta \cosh \beta B, \quad \lambda_2 = \frac{1}{Z} (8 + 2 \cosh 2\beta B).\]

Finally, when considering the Bures distance GQD, there is no analytical expression can be obtained, and thus we calculate it via numerical methods.
IV. GQDS IN THE XX CHAIN WITH DM INTERACTION

We discuss in this section the effects of the DM interaction, the external magnetic fields, and the reservoir temperature on the considered GQDs, i.e., the trace distance, the Hellinger distance, and the Bures distance GQDs introduced in Sec. [11]. As $Q_T(\rho)$, $Q_B(\rho)$, and $Q_H(\rho)$ are all symmetric functions with respect to $J = 0$, $B = 0$, and $D = 0$, we consider in the following only the cases of $J \geq 0$, $B \geq 0$, and $D \geq 0$.

We begin with a heuristic analysis of the dependence of the GQDs on $D$ and $B$ for two special cases. First, for the zero absolute temperature case, the system is in its ground state, which is given by $|\Psi_2\rangle$ if $\delta > B$, $|\Psi_4\rangle$ if $\delta < B$, and the equal mixture of $|\Psi_2\rangle$ and $|\Psi_4\rangle$ if $\delta = B$. Therefore, we have

$$Q_\alpha(\rho) = \begin{cases} 
0 & \text{if } \delta < B, \\
c & \text{if } \delta = B, \\
1 & \text{if } \delta > B,
\end{cases} \quad (18)$$

where $c = 0.5$ for $\alpha = \{T, H\}$, and $c \approx 0.5098$ for $\alpha = B$.

Second, for the limiting case of $D \to \infty$, we have $\rho_{11,44} = 0$, $\rho_{22,33} = 0.5$, $\rho_{23} = \rho_{32} = -0.5i$, and therefore

$$Q_\alpha(\rho) = 1, \quad (19)$$

for $\alpha \in \{T, H, B\}$, i.e., they all achieved their maxima in the limit of $D \to \infty$. For finite but large enough DM interaction $D$, it is also natural to hope that one can achieve considerable enhancement of the three GQDs. We show in the following that this is indeed the case. Moreover, we remark here that $T$ in all the following figures is plotted in units of the Boltzman’s constant $k_B$.

A. D dependence of the GQDs

We now discuss the general behaviors of the GQDs at finite temperature $T$. We first consider their dependence on the DM interaction. Our numerical simulations show that $Q_T(\rho)$, $Q_H(\rho)$, and $Q_B(\rho)$ display qualitatively the similar behaviors under different temperatures. In Fig. [1] we presented an exemplified plot of their dependence on $D$ at finite temperature $T = 0.5$ with different strengths of the external magnetic fields. From this plot, one can note that for any given magnitude of $B$, the trace distance GQD $Q_T(\rho)$ always increases monotonously with the increasing value of $D$, and this can be understood from Eq. (14), which yields

$$\frac{\partial Q_T(\rho)}{\partial \delta} = \frac{8\beta}{Z^2}(1 + \cosh 2\beta \delta \cosh 2\beta B) > 0. \quad (20)$$

Similarly, the Hellinger distance GQD $Q_H(\rho)$ can also be increased monotonously by increasing $D$. But as displayed in Fig. [1]b, it exhibits sudden change behaviors for strong magnetic fields, and this is caused by the process of maximization in Eq. (19), as before the sudden change point denoted by $D_c$, we have $\max\{\lambda_1, \lambda_2\} = \lambda_2$ and $Q_H(\rho) = 1 - \lambda_2$, while after $D_c$, $\max\{\lambda_1, \lambda_2\} = \lambda_1$ and $Q_H(\rho) = 1 - \lambda_1$.

The Bures distance GQD $Q_B(\rho)$ for the relative weak external magnetic fields case also increases monotonously with the increase of $D$. But when an strong magnetic field is applied, $Q_B(\rho)$ exhibits sudden change behavior at a critical point $D_{c1}$. Particularly, different from that of $Q_H(\rho)$, $Q_B(\rho)$ is decreased after $D_{c1}$ [see, Fig. [1]c], and this tendency of decrease will continue until another critical point $D_{c2}$ is reached, after which it turns out to be increased again and finally approaches the asymptotic value 1.

B. B dependence of the GQDs

In Figs. [2] and [3] we displayed the $B$ dependence of the three GQDs under different DM interactions, with the reservoir temperatures being given by $T = 0.5$ and $T = 1.5$, respectively. At a first glance, one can note that they all approach the asymptotic value 0 in the limit of $B \to \infty$. This happens because for this special case, the density operator reduces to $\rho = |11\rangle\langle 11|$, which is a product state and therefore there is no correlations between $A$ and $B$.

Another general behavior which can be observed from Figs. [2] and [3] is that the trace distance GQD $Q_T(\rho)$ always decreases with the increasing strength of the external magnetic fields $B$. This phenomenon is obvious because from Eq. (14) one can see that the partition function $Z$ is an increasing function of $B$, and therefore $Q_T(\rho)$ always decreases with the increase of $B$.

The Hellinger distance GQD $Q_H(\rho)$ behaves quite differ-
C. $T$ dependence of the GQDs

Finally, we discuss the temperature dependence of the three GQDs. We will show that while the increase of $T$ can in general destroy the coherence of the system, the GQDs may also

different from that of $Q_T(\rho)$. First, there are sudden change behaviors which are caused by the maximization process appeared in Eq. (16), and second, its change with the variation of $B$ is temperature dependent. As displayed in Figs. 2(b) and 3(b), $Q_H(\rho)$ always decays with the increasing strength of $B$ at relative low temperature region, while at high temperature region with weak DM interactions, $Q_H(\rho)$ may be increased by increasing $B$ before the sudden change point $B_c$. But after $B_c$, $Q_H(\rho)$ still decays to zero monotonously.

When considering the Bures distance GQD, as displayed in Fig. 2(c) which depicts the relative low temperature case, $Q_B(\rho)$ is increased with increasing $B$ only in a very narrow region of $B$, while out of this region, it decays monotonously with $B$. For the case of high temperature with relative weak DM interaction, as displayed in Fig. 3(c), $Q_B(\rho)$ is increased before a sudden change point $B_c$, and decreased after $B_c$, and this phenomenon is somewhat similar to that of $Q_H(\rho)$. But when one enlarges the DM interaction, the $B$ dependence of $Q_B(\rho)$ is changed. It initially decreases to a minimum value, and then turns out to be increased before a sudden change point is arrived, after which it becomes decreasing with $B$ again, and approaches zero in the infinite $B$ limit.

FIG. 2: (Color online) $Q_T(\rho)$, $Q_H(\rho)$, and $Q_B(\rho)$ versus $B$ with $J = 1$ and $T = 0.5$. Here, the black, red, blue, green, magenta, and cyan curves (from left to right) correspond to the cases of $D = 0, 0.5, 1, 1.5, 2, and 3$, respectively.

FIG. 3: (Color online) $Q_T(\rho)$, $Q_H(\rho)$, and $Q_B(\rho)$ versus $B$ with $J = 1$ and $T = 1.5$. Here, the black, red, blue, green, magenta, and cyan curves (from bottom to top) correspond to the cases of $D = 0, 0.5, 1, 1.5, 2$, and $3$, respectively.

FIG. 4: (Color online) $Q_T(\rho)$, $Q_H(\rho)$, and $Q_B(\rho)$ versus $T$ with $J = 1$ and $B = 1.5$. Here, the black, red, blue, green, magenta, and cyan curves (from bottom to top) correspond to the cases of $D = 0, 0.5, \sqrt{5}/2, 1.5, 2$, and $3$, respectively.
be increased by increasing $T$ for certain specific system parameters. For this purpose, we showed in Fig. 3 the $T$ dependence of $Q_T(\rho)$, $Q_H(\rho)$, and $Q_B(\rho)$ with fixed $B = 1.5$ and different DM interactions, from which some general behaviors can be observed.

First, when $\delta < B$ (e.g., $D = 0$ and 0.5 shown in Fig. 4), $Q_T(\rho)$, $Q_H(\rho)$, and $Q_B(\rho)$ initially increase from zero to certain maximum values, and then decrease to zero gradually. Therefore, for this case the increase of $T$ can enhance the GQDs to some extent. Second, when $\delta = B$ (e.g., $D = \sqrt{5}/2$ shown in Fig. 4), the GQDs decrease monotonously from the initial value 0.5 [for $Q_T(\rho)$ and $Q_H(\rho)$] or 0.5098 [for $Q_B(\rho)$] in the whole region of $T$, and arrives at the asymptotic value zero in the infinite temperature limit. Thirdly, when $\delta > B$ (e.g., $D = 1.5$, 2, and 3 shown in Fig. 4), all the three GQDs decrease from the maximum value 1 (when $T = 0$) to the minimum value 0 in the infinite temperature limit. Thus, one can see that for the latter two cases, the increase of temperature always degrades the GQDs.

V. SUMMARY

In summary, we have investigated properties of the GQDs for two spins in thermal equilibrium. The corresponding physical system we considered is described by the Heisenberg $XX$ model, with the DM interaction induced by the spin-orbit coupling being involved, and an external magnetic field is also applied. Moreover, the three GQDs we considered were defined based on the trace distance, the Hellinger distance, and the Bures distance, respectively. By analyzing their dependence on the system parameters, we found that the DM interaction plays a positive role in improving the GQDs. To be explicitly, we found that the GQDs can be enhanced apparently by introducing the DM interaction. Particularly, they all approach the maximum 1 in the infinite limit of $D$.

On the other hand, the applied external magnetic fields always degrade the trace distance GQD, while its effects on the Hellinger distance and the Bures distance GQDs are temperature dependent. For the low temperature case, $Q_H(\rho)$ is always decreased, while $Q_B(\rho)$ may be enhanced in a narrow region of $B$. For the high temperature case with weak DM interaction, both $Q_H(\rho)$ and $Q_B(\rho)$ can also be enhanced in the weak magnetic field region.

Finally, the temperature dependence of the three GQDs are determined by the relative magnitudes of $\delta$ and $B$. When $\delta < B$, the GQDs may be enhanced to some extent by increasing $T$ at the low temperature region, and when $\delta \geq B$, they are degraded in the whole temperature region.

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