Entropy vs Gravitational Action: Do Total Derivatives Matter?

Amin Faraji Astaneh\textsuperscript{1}, Alexander Patrushev\textsuperscript{2,3} and Sergey N. Solodukhin\textsuperscript{3}

\textsuperscript{1} Department of Physics, Sharif University of Technology, P.O. Box 11365-9161, Tehran, Iran and School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5531, Tehran, Iran

\textsuperscript{2} Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 6 Joliot-Curie, 141980 Dubna, Russia

\textsuperscript{3} Laboratoire de Mathématiques et Physique Théorique CNRS-UMR 7350, Fédération Denis Poisson, Université François-Rabelais Tours, Parc de Grandmont, 37200 Tours, France

Abstract

The total derivatives in the gravitational action are usually disregarded as non-producing any non-trivial dynamics. In the context of the gravitational entropy, within Wald’s approach, these terms are considered irrelevant as non-contributing to the entropy. On the other hand, the total derivatives are usually present in the trace anomaly in dimensions higher than 2. As the trace anomaly is related to the logarithmic term in the entanglement entropy it is natural to ask whether the total derivatives make any essential contribution to the entropy or they can be totally ignored. In this note we analyze this question for some particular examples of total derivatives. Rather surprisingly, in all cases that we consider the total derivatives produce non-trivial contributions to the entropy. Some of them are non-vanishing even if the extrinsic curvature of the surface is zero. We suggest that this may explain the earlier observed discrepancy between the holographic entanglement entropy and Wald’s entropy.
1  Introduction

Since the inspiring paper of Wald [1] and the subsequent works [2], [3] it became clear that there exists a certain correspondence between the entropy associated to horizons and terms in the gravitational action. This relation by now is very well established and has many important applications and generalizations, see for instance [4] and [5]. In this context the possible total derivatives in the gravitational action are generally neglected as they are thought not to produce any essential contribution to the entropy.

In a wider context the discussed correspondence is important for the calculation of entanglement entropy of a co-dimension two surface. The gravitational action in this case is the quantum effective action which in general can be represented as a certain local or non-local expansion [6] in Riemann curvature. In particular, there has been established [7], [8] a relation between the trace anomaly (obtained as variation of the effective action under conformal rescaling of metric) and the logarithmic term in the entanglement entropy.

The trace anomaly may, in general, contain some total derivatives which originate from the local counter terms that could be added to the effective action. These terms thus are (regularization) scheme dependent and are not universal. The entanglement entropy, on the other hand, is a quantity which appears to be independent of the chosen regularization. A natural question then arises: whether these total derivatives produce any non-trivial contribution to the entropy?

In this note we analyze this question. In dimension $d = 4$ the only total derivative which may contribute to the trace anomaly is $\Box R$. It originates from $R^2$ in the effective action. In dimension $d = 6$ there is much more freedom and there appears a set of possible terms. Generally they are rather complicated for the analysis. Here, for the purposes of simplicity, we focus on some particular terms, $\Box R^2$ and $\Box (R_{\mu\nu}R^{\mu\nu})$, which appear to be among the simplest ones. In all these cases the gravitational entropy is non-vanishing. In particular, for the term $\Box (R_{\mu\nu}R^{\mu\nu})$, the entropy is non-zero even if the surface has no extrinsic curvature. This is especially interesting in the light of the discrepancy first found in [9]. We suggest that this discrepancy may originate from the total derivative terms in the trace anomaly that are normally ignored in the entropy calculation. Below we present our analysis.

2  Regularization method

First we want to explain our method. Consider a general class of metrics of the type

$$ds^2 = e^{2\sigma(x,r)} (dr^2 + r^2 d\tau^2) + (h_{ij}(x) + 2K_{ij}^a(x)n^a r + ..)dx^i dx^j,$$

(1)

where $n^1 = \cos \tau$ and $n^2 = \sin \tau$. The entangling surface $\Sigma$ is defined by condition $r = 0$, $h_{ij}(x)$ is the intrinsic metric on $\Sigma$ and $K_{ij}^a(x), a = 1,2$ is the extrinsic curvature of the surface. Now,
in order to compute the entropy associated to surface $\Sigma$ we use the replica trick, for a review see [10]. It consists in making a periodicity $2\pi n$, $n$ is an integer, for the coordinate $\tau$ and then taking the limit $n \to 1$. This procedure introduces an angle deficit $2\pi(1-n)$ and thus produces a conical singularity. The entropy then is obtained by differentiating the gravitational action with respect to $(n-1)$ and taking the limit $n \to 1$. If extrinsic curvature of $\Sigma$ is non-vanishing then the singularity is the squashed conical singularity studied in [11]. In order to compute the corresponding curvature invariants we introduce some regularization. This regularization consists of two parts. First, we smooth the conical singularity by replacing $g_{rr} \to g_{rr} f_n(r)$ with the regularization function

$$f_n(r) = \frac{r^2 + b^2 n^2}{r^2 + b^2},$$

where $b$ is the regularization parameter later to be taken to zero. This regularization was introduced in [4] so that we shall call it FS regularization. It should be applied every time we have a conical singularity. If the conical singularity is squashed (i.e. the extrinsic curvature of $\Sigma$ is non-vanishing) FS regularization alone does not lead to everywhere regular space with a finite curvature. Thus it should be supplemented by yet another regularization: replace $K^a_{ij}(x)n^a r^n$ by $K^a_{ij}(x)n^a r^n$ in the metric. We stress that the terms in the metric that remain there if $\Sigma$ is a Killing horizon should not be regularized. Otherwise we would get deviations from Wald’s entropy calculation even for the Killing horizons. This second regularization is introduced in [11] and we shall call it FPS regularization. It should be used if the singular surface has a non-trivial extrinsic curvature. The final result for the entropy thus can be considered as coming from both FS and FPS regularizations.

3 Regularized metric and scalar curvature

With these explanations we consider the following regularized metric

$$ds^2 = e^{2\sigma(x,r)}[f_n(r)dr^2 + r^2 d\tau^2] + g_{ij}(x, r, \tau)dx^i dx^j,$$

where

$$\sigma(x, r) = \sigma_0(x) + \frac{1}{2}\sigma_2(x)r^2 + \cdots,$$

$$g_{ij}(x, r, \tau) = h_{ij}(x) + 2K^a_{ij}(x)n^a r^n + (K^a K^b)_{ij}n^a n^b r^{2n} + g^{(2)}_{ij}(x)r^2 + \cdots.$$ (4)

According to our prescription in [4] we changed the power of $r$ only for terms which are due to extrinsic curvature keeping the power of $r$ in all other terms unchanged.
If $n = 1$ we have the following relations for the metric (3):

$$\sigma_2 = -\frac{1}{4}e^{2\sigma_0}(R_{abab} + 2(\nabla_\Sigma \sigma_0)^2),$$

$$g^{(2)}_{ij} = \frac{1}{2}e^{2\sigma_0}R_{ajaj} - e^{2\sigma_0}(\nabla_i \nabla_j \sigma_0 + \nabla_i \sigma_0 \nabla_j \sigma_0),$$

$$\text{Tr} g^{(2)} = \frac{1}{2}e^{2\sigma_0}(R_{abab} - R_{aa} - 2\Delta_\Sigma \sigma_0 - 2(\nabla_\Sigma \sigma_0)^2).$$  (5)

If $n \neq 1$ the scalar curvature of regularized metric (4) reads

$$R = R_\Sigma - 4\Delta_\Sigma \sigma_0 - 6(\nabla_\Sigma \sigma_0)^2 - 4(\sigma_2 + \text{Tr} g^{(2)})e^{-2\sigma_0}f^{-1}_n(r) - r^{-1}\partial_r f^{-1}_n(r)e^{-2\sigma_0}$$

$$+ C_1(r)\text{Tr} \kappa^{a} n^{a} r^{n-2} + C_2(r)\text{Tr} (\kappa^{a} \kappa^{b}) n^{a} n^{b} r^{2(n-1)} + C_3(r)\text{Tr} \kappa^{a} \text{Tr} \kappa^{b} n^{a} n^{b} r^{2(n-1)}$$

$$- e^{-2\sigma_0}[(\text{Tr} \kappa)^2 - \text{Tr} \kappa^2] r^{2(n-1)}. $$  (6)

where

$$C_1(r) = 2(1 - n^2/f_n(r)) + n r \partial_r f_n(r) f^{-2}_n(r),$$

$$C_3(r) = (1 - n^2/f_n(r)), \quad C_2(r) = -C_1(r) - C_3(r).$$  (7)

Imposing $n = 1$ in (6) we find the Gauss-Codazzi relation

$$R = R_\Sigma - 4\sigma_2 e^{-2\sigma_0} - 4\Delta_\Sigma \sigma_0 - 6(\nabla_\Sigma \sigma_0)^2 - 4e^{-2\sigma_0} \text{Tr} g^{(2)} - e^{-2\sigma_0}[(\text{Tr} \kappa)^2 - \text{Tr} \kappa^2].$$  (8)

With the help of relations (3) it takes the usual form

$$R = R_\Sigma + 2R_{aa} - R_{abab} + \text{Tr} \kappa^2 - (\text{Tr} \hat{\kappa})^2, $$  (9)

where $\hat{\kappa}^a_{ij} = e^{-\sigma_0} \kappa^a_{ij}$, $a = 1, 2$ is the extrinsic curvature.

### 4 Integrals over a squashed cone

In this section we want to compute the contribution of a total derivative due to a conical singularity. In fact, an obvious geometric quantity which is a total derivative is the scalar curvature in two dimensions. This was the first case analyzed in [4] in order to illustrate the distributional nature of the curvature due to a conical singularity. The procedure considered in [4] was the following. Let us first take a disk of a fixed radius $r_0$ in the plane $(r, \tau)$. Then we consider the integral of the scalar curvature $R$ for the FS regularized metric. Formally the radial integral can be taken from $r = 0$ to $r = r_0$. However, the term at $r = 0$ vanishes for the FS regularized metric. So that only term at $r = r_0$ is important. One decomposes this term in powers of $(1 - n)$ to linear order and then takes the limit when the regularization parameter $b \to 0$ provided the value of $r_0$ is kept fixed. The result of this procedure is finite and independent of $r_0$. 

5
Now we want to repeat this procedure and compute the integral of \( \Box R \) over a regularized squashed cone and extract the contribution which is due to the conical singularity. As above we consider a disk of radius \( r_0 \), where \( r_0 \) is small but finite. Then the integral reduces to two boundary terms, at \( r = 0 \) and \( r = r_0 \),

\[
\int_{M_n} \Box R = \int_{r=r_0} \frac{\sqrt{g}}{\sqrt{f}} r \partial_r R - \int_{r=0} \frac{\sqrt{g}}{\sqrt{f}} r \partial_r R,
\]

where integration goes over \( \tau \) (from 0 to \( 2\pi n \)) and \( x^i \), and \( g \) is determinant for the metric (4).

We notice that for curvature (6), in the limit of small \( r \) and provided that \( b \) is kept finite and \( n > 1 \), we have that \( r \partial_r R \sim r^{2n-2} \) vanishes at \( r = 0 \). Thus there is no “internal boundary” in (10) and the integral reduces to the boundary term at \( r = r_0 \). We expand the first term in (10) in powers of \( (1 - n) \) and then take the limit \( b \to 0 \) while keeping \( r_0 \) small but finite. The result of this procedure is:

\[
\int_{M_n} \Box R = n \int_{M_{n-1}} \Box R + 4\pi (n-1) \int_{\Sigma} \text{Tr} \hat{K}^2.
\]

Similarly, for the integral of \( \Box R^2 \) we find that

\[
\int_{M_n} \Box R^2 = n \int_{M_{n-1}} \Box R^2 + 8\pi (n-1) \int_{\Sigma} R \text{Tr} \hat{K}^2,
\]

where in the r.h.s. of this equation the scalar curvature \( R \) takes the form (8) (or, equivalently, (9)). Interestingly, the surface term in (11) is non-zero even if the spacetime is flat. This makes it similar (but not identical) to the famous Gibbons-Hawking term.

5 Holographic entanglement entropy

As first application of our finding we consider the generalization of the holographic proposal [7] for the entanglement entropy. In the holographic duality the AdS gravity may be described by an action which includes terms quartic in derivatives. The general structure of such an action then includes also a total derivative term,

\[
I = -\int_{M^{d+1}} \sqrt{g} d^{d+1} x \left[ \frac{R}{16\pi G_{(d+1)}} + 2\Lambda + \lambda_1 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \lambda_2 R_{\mu\nu} R^{\mu\nu} + \lambda_3 R^2 + \lambda_4 \Box R \right].
\]

Respectively, the generalized holographic entropy is a combination of the proposal made in [11] and our finding (11)

\[
S(\mathcal{H}) = \frac{A(\mathcal{H})}{4G_{(d+1)}} + 4\pi \int_{\mathcal{H}} \left[ 2\lambda_1 (R_{iijj} - \text{Tr} k^2) + \lambda_2 (R_{ii} - \frac{1}{2} k^2) + 2\lambda_3 R - \lambda_4 \text{Tr} k^2 \right],
\]

\footnote{In the first version of this paper we incorrectly took the limit \( r_0 \to 0 \).}
where $\mathcal{H}$ is a co-dimension 2 surface which bounds the entangling surface $\Sigma$, $k$ is the extrinsic curvature of $\mathcal{H}$. The surface $\mathcal{H}$ is supposed to be a minimizer of the functional \((14)\). If $d = 4$ the surface $\mathcal{H}$ has dimension three and the holographic entropy \((14)\) is supposed to reproduce the entanglement entropy of a conformal field theory with general conformal charges.

6 Conformal anomaly and entanglement entropy in four dimensions

In four dimensions the trace anomaly is a combination of the following terms

$$
\langle T \rangle = -aE_4 + bW^2 + c\Box R,
$$

$$
E_4 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2,
$$

$$
W^2 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2,
$$

\(15\)

where $W$ is the Weyl tensor and $E_4$ is the Euler density in four dimensions. The first two terms in \((15)\) are universal. They come from a conformal variation of the non-local part of the CFT effective action. On the other hand, the last term originates from a conformal variation of a local term $R^2$ which could be added to the effective action. This term depends on the regularization scheme and it is not universal. Respectively, the $c$-term in \((15)\) is not universal.

On the other hand, the trace anomaly integrated over a conical space $\mathcal{M}_n$

$$
\int_{\mathcal{M}_n} \langle T \rangle = n \int_{\mathcal{M}_{n-1}} \langle T \rangle + (1 - n)s_0/2,
$$

\(16\)

is related to the logarithmic term in the entanglement entropy,

$$
S = \frac{N_s A(\Sigma)}{48\pi \epsilon^2} + s_0 \ln \epsilon.
$$

\(17\)

The term $s_0$ in \((16)\) is then given by a surface integral

$$
s_0 = 16\pi \int_{\Sigma} \left( aR_{\Sigma} - bK_{\Sigma} + \frac{c}{2}Tr\dot{K}^2 \right),
$$

\(18\)

where $R_{\Sigma}$ is intrinsic curvature of surface $\Sigma$, and we define

$$
K_{\Sigma} = R_{abab} - R_{aa} + \frac{1}{3}R - \frac{1}{2}(Tr\dot{K}^2 - \frac{1}{2}(Tr\dot{K})^2).
$$

\(19\)

The $a$- and $b$-contributions to logarithmic term \((18)\) have been obtained earlier, see [7] and [8].
7 Some puzzles

The $c$-term in (18) is new. The existence of this term is a direct consequence of (11). However, its presence in the logarithmic term of entanglement entropy is rather puzzling. In a conformal field theory $s_0$ is expected to be conformally invariant and indeed the $a$- and $b$-terms in (18) are conformal invariants. However, the $c$-term is not invariant under conformal transformations. Indeed, under Weyl rescaling $g_{\mu\nu} \rightarrow g_{\mu\nu} e^{-2\omega}$, $n^a_\mu \rightarrow e^{-\omega} n^a_\mu$ it changes as

$$
\int_{\Sigma} \text{Tr} \hat{K}^2 \rightarrow \int_{\Sigma} \text{Tr} \hat{K}^2 + \int_{\Sigma} \text{Tr} \hat{K}^a n^a_\alpha \partial_\alpha \omega
$$

and is invariant only if normal derivative of $\omega$ vanishes on $\Sigma$. So that the $c$-term breaks conformal invariance down to transformations which preserve the extrinsic curvature.

Moreover, for a sphere $\Sigma = S_2$ in flat spacetime, we have that

$$
R_{\Sigma} = \text{Tr} \hat{K}^2,
$$

so that the $c$-term takes exactly same form as the $a$-term. In this case the $b$-term disappears and the whole contribution to the logarithmic terms is due to the Euler number of the sphere multiplied by $(2a + c)$,

$$
s_0(S_2) = 32\pi^2(2a + c) .
$$

Usually one considers the logarithmic term in the entropy for a round sphere as a simple way to identify the $a$-charge of the CFT (see for instance [12], [13]). However, we see that, in general, this term may also have a part that depends on $c$.

The dependence of entanglement entropy on $c$ should mean that the entropy depends on the regularization. So far no indication of such a dependence has been found neither in direct lattice type calculations of the entropy nor in the numerous holographic calculations that use the prescription of Ryu-Takayanagi [7]. The absence of $c$-term in the holographic analysis however may have a simple explanation. As shows the analysis in original paper [14] the total derivative term vanishes ($c = 0$) in the holographic trace anomaly in four dimensions. However, the total derivatives appear in the holographic trace anomaly in six dimensions.

\footnote{However, we were informed by Christopher Eling about his earlier unpublished work [17] on resolution of the mismatch in the logarithmic term in the entropy of gauge fields first observed by Dowker [19]. Eling uses the previous results of [18]. According to [18] in the trace anomaly due to the gauge spin-1 field the parameter $c$ is non-zero. Using these results Eling obtains $(4a + 2c)\ln c$ for the logarithmic term in the entanglement entropy in agreement with our eq. (22) (he uses different normalization for $a$ and $c$). Then, for $a = 62n_1/360$ and $c = -n_1/6$ as in [18] he finds that $a + c/2 = 32n_1/360$ in agreement with Dowker. This relates the mismatch to the presence of $c$-term in the trace anomaly.}
8 Earlier observed discrepancy in six dimensions

Curiously enough, paper [9] does observe some discrepancy between the holographic calculation of entanglement entropy (using Jacobson-Myers functional for the holographic minimal surface) and the CFT trace anomaly calculation if one uses Wald’s prescription to compute the entropy in $d = 6$. This discrepancy has not yet been explained in the literature. As observed in [9] the discrepancy comes from the conformal invariant $I_3$, the only conformal invariant in $d = 6$ which contains total derivatives, and it is conceivable that it originates entirely from the total derivative terms present in $I_3$. $\Box R^2$ (12) is one of such terms. The discrepancy found in [9] is for surfaces without $O(2)$ symmetry but with vanishing extrinsic curvature. So that (12) can not give the required contribution to explain the discrepancy. However, there are more total derivative terms in $I_3$ (see for instance [20]) and some of them may have the required properties. In support to these expectations we shall consider a particular example of the total derivative which does appear in the trace anomaly in six dimensions and has the required property.

9 More general metric

First we need to generalize the metric (3), (4). Indeed, it was assumed in (3), (4) that to second order in $r$ the $\tau$-dependence of the metric may appear only due to the extrinsic curvature $(K^aK^b)n^an^b r^2$ so that $g^{(2)}$ does not depend on $\tau$. This, however, is not the most general situation. Indeed, in the examples considered in [9] the extrinsic curvature of the entangling surface is zero. However the surface is not $O(2)$ invariant in the transverse subspace due to $\tau$-dependent terms in the $r^2$ order of the metric when expanded near the surface.

Motivated by these examples we consider the following generalization of the (not yet regularized) metric

$$ds^2 = e^{2\sigma(x,r)}[dr^2 + r^2d\tau^2] + g_{ij}(x,r,\tau)dx^idx^j,$$

$$g_{ij}(x,r,\tau) = h_{ij}(x) + 2K^a_{ij}(x)n^ar + r^2((K^aK^b)_{ij}n^an^b + H_{ij}^{ab}(x)n^an^b) + \cdots,$$

$$\sigma(x,r) = \sigma_0(x) + \frac{1}{2}\sigma_2(x)r^2 + \cdots.$$

This metric is obviously $2\pi n$ periodic if $n$ is an integer. It should be noted that the trace part of $H_{ij}^{ab}$ in (23) is identical to what we called $g^{(2)}_{ij}(x)$ in metric (4). The transverse components of the Ricci tensor of this metric read

$$R_{rr} = -2\sigma_2 - e^{2\sigma_0}[\Delta_\Sigma\sigma_0 + 2(\nabla_\Sigma\sigma_0)^2] - Tr H^{ab}n^an^b,$$

$$r^{-2}R_{\phi\phi} = -2\sigma_2 - e^{2\sigma_0}[\Delta_\Sigma\sigma_0 + 2(\nabla_\Sigma\sigma_0)^2] + Tr H^{ab}n^an^b - Tr H^{aa},$$

$$R_{r\phi} = Tr H^{ab}n^ae_bc_n^c r.$$
In what follows we assume that the extrinsic curvature of the surface vanishes, \( K_{ij}^a = 0 \), but the term \( H_{ij}^{ab} \) is non-vanishing. So that vector \( \xi = \partial_r \) is locally a Killing vector, \( \mathcal{L}_\xi g_{\mu\nu} = O(r^2) \). Thus, the surface at \( r = 0 \) in the metric (23) is some sort of generalized horizon. In this case the conical singularity does not appear to be “squashed” although it is not \( O(2) \) invariant either.

If we use the FS regularization only, i.e. replace \( g_{rr} \to f_n(r) g_{rr} \), then the regularized metric has everywhere finite curvature. However, the gradient of the curvature is divergent at \( r = 0 \). Therefore, we need to use additionally the FPS regularization in order to make the derivatives of the curvature finite. The analysis shows that the divergence is due to the traceless part of \( H_{ij}^{ab} \) in the metric (23). Therefore, only this part needs to be regularized while the part of the metric due to the trace of \( H_{ij}^{ab} \) is independent of \( \tau \) and thus it does not need to be regularized. This is the prescription advocated in [15]. With this prescription we replace

\[
H_{ij}^{ab}(x) n^a n^b r^2 \to \frac{1}{2} H_{ij}(x) r^2 + \left( H_{ij}^{ab}(x) - \frac{1}{2} \delta^{ab} H_{ij}(x) \right) n^a n^b r^{2n},
\]

where \( H_{ij}(x) = H_{ij}^{ab} \delta^{ab} \), in the metric (23). If the traceless part of \( H_{ij}^{ab} \) vanishes then metric (23) (provided \( K_{ij}^a \) vanishes as well) possesses the Killing symmetry and describes a Killing horizon at \( r = 0 \). For this metric Wald’s calculation of entropy is applicable and we do not expect any modifications of this calculation. This explains why we did not modify the power of \( r \) in front of \( H_{ij}(x) \) in (25).

### 10 Entropy calculation

As an example of a total derivative term we shall consider \( \Box(R_{\mu\nu} R^{\mu\nu}) \). The respective integral over the conical space then reduces to a boundary term at \( r = r_0 \),

\[
\int_{\mathcal{M}_n} \Box(R_{\mu\nu} R^{\mu\nu}) = \int_{r=r_0} \sqrt{g} r \partial_r (R_{\mu\nu} R^{\mu\nu}) ,
\]

in which we have to expand in powers of \((n-1)\) and take the limit \( b \to 0 \). The result of this calculation for the metric (23) regularized as we just explained is rather simple and it depends only on \( H_{ij}^{ab} \),

\[
\int_{\mathcal{M}_n} \Box(R_{\mu\nu} R^{\mu\nu}) = 8\pi(1-n) \int_{\Sigma} \left( \text{Tr} H^{ab} \text{Tr} H^{ab} - \frac{1}{2} (\text{Tr} H^{aa})^2 \right) ,
\]

where the trace is defined with respect to intrinsic metric \( h_{ij}(x) \) of the surface. Not surprisingly, the result (27) depends only on the traceless part of \( H_{ij}^{ab} \).

This can be re-written in terms of the Ricci tensor projected on the transverse subspace, \( R_{ab} = R_{\mu\nu} n^\mu_a n^\nu_b \), where \( n^\mu_a, a = 1, 2 \) is a pair of normal vectors to \( \Sigma \). For metric (23) we have

\[3\text{We thank Joan Camps for pointing this out to us.}\]
that $n_1^2 = e^{-\sigma_0(x)}$ and $n_2^2 = r^{-1}e^{-\sigma_0(x)}$. Then, using (24), we have that
\[ \int_{M_n} \Box (R_{\mu\nu}R^{\mu\nu}) = 8\pi (1 - n) \int_\Sigma (R_{ab} - \frac{1}{2}\delta_{ab}R_{cc})^2. \] (28)
Together with equations (11) and (12) this is our main result.

The entropy which follows from (28) is
\[ S = -8\pi \int_\Sigma \left( R_{ab}^2 - \frac{1}{2}(R_{aa})^2 \right). \] (29)
It has the structure that resembles the one proposed in [9] in terms of the Weyl tensor. However, in order to see whether there is a complete agreement we need to know the respective entropy which comes from all possible total derivative terms that appear in the trace anomaly. Work in this direction is in progress.

11 General expression for entropy and some tests

We can advance a bit more in our attempt to explain the discrepancy of [9]. It is clear from the analysis above that the possible contributions of the total derivative terms in the trace anomaly in six dimensions should be a combination of invariants constructed from matrix $H_{ij}^{ab}$. There are in general four such invariants so that the respective entropy is a linear combination
\[ S = \int_\Sigma \left( \alpha_1 \text{Tr} H_{ab} \text{Tr} H^{ab} + \alpha_2 (\text{Tr} H_{aa})^2 + \beta_1 H_{ij}^{ab}H^{ab,ij} + \beta_2 H_{ij}^a H^{aa,ij} \right). \] (30)
It is natural to expect that only the traceless part of $H_{ij}^{ab}$ contributes to the missing entropy. Then we have $\alpha_2 = -\alpha_1/2$ and $\beta_2 = -\beta_1/2$.

In the examples considered in [9] the matrix $H_{ij}^{ab}$ has only one non-vanishing component, $H^{11}$. Therefore among these four invariants there are only two independent
\[ S = \int_\Sigma \left( \alpha (\text{Tr} H^{11})^2 + \beta H_{ij}^{11} H^{11,ij} \right), \] (31)
where $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$.

The analysis of [9] can be summarized as follows. The mismatch in the entropy that they have found takes the form
\[ \Delta S = -\pi B_3 g A(\Sigma) \ln \epsilon, \] (32)
\footnote{In an alternative regularization considered in [5] and [10] one effectively regularizes all the terms in the metric including the trace part of $H_{ij}^{ab}$. Provided we follow this procedure the formulae (27) and (29) would be modified by the trace part of $H_{ij}^{ab}$. For instance the entropy (29) would get the corrections: (29) + $16\pi \int_\Sigma (\frac{1}{4}R_{aa}^2 + \frac{1}{4}R_{aiaj}^2)$.
These corrections do not vanish in the case when $\Sigma$ is a black hole horizon and thus lead to deviations from Wald’s entropy even for the Killing horizons.}
where $B_3$ is the central charge which corresponds to conformal invariant $I_3$. In [9] they considered four cases (we use their notations and set all radii to 1):

a). $R^1 \times S^2 \times S^3$ with $\Sigma = S^1 \times S^3$. [9] finds that in this case $g = 6$. We find that $(\text{Tr } H^{11})^2 = 1$ and $H_{ij}^{11} H^{11,ij} = 1$ for this geometry.

a’). $R^1 \times S^2 \times S^3$ with $\Sigma = S^2 \times S^2$. [9] finds $g = 8$. Respectively we find that $(\text{Tr } H^{11})^2 = 4$ and $H_{ij}^{11} H^{11,ij} = 2$.

b). $R^3 \times S^3$ with $\Sigma = S^2 \times R^2$. [9] finds $g = 8$. We find $(\text{Tr } H^{11})^2 = 4$ and $H_{ij}^{11} H^{11,ij} = 2$.

c). $R^2 \times S^4$ with $\Sigma = S^3 \times R^1$. [9] finds $g = 6$. We find $(\text{Tr } H^{11})^2 = 9$ and $H_{ij}^{11} H^{11,ij} = 3$.

According to our proposal, $gA(\Sigma)$ should be identified with (30)-(31). There are two observations which may serve as some tests on this proposal. First, the cases a’) and b) are characterized by same $H$-invariants. Therefore, if we are right then we expect that their entropy mismatch should be the same. This is indeed the case! Then, the three independent cases give us three equations on parameters $\alpha$ and $\beta$:

\[
\begin{align*}
\alpha + \beta &= 6, \\
4\alpha + 2\beta &= 8, \\
9\alpha + 3\beta &= 6.
\end{align*}
\]

(33)

The first two equations have solution: $\alpha = -2$ and $\beta = 8$. With these values, the third equation in (33) holds automatically! This is second test on our proposal.

In fact, provided the missing entropy (31) depends only on the traceless part $\tilde{H}_{ij}^{ab} = H_{ij}^{ab} - \frac{1}{2} \delta^{ab} H_{ij}^{cc}$ we are now able to get the complete expression

\[
S = \int_{\Sigma} \left( -4 \text{Tr } \tilde{H}^{ab} \text{Tr } \tilde{H}^{ab} + 16 \tilde{H}_{ij}^{ab} \tilde{H}^{ab,ij} \right).
\]

(34)

It should be noted however that the full resolution of the discrepancy may be a rather complicated problem. The reason is the following. The total derivatives may appear on both sides of the holographic relation. On the CFT side they appear in the holographic trace anomaly as derived in [14], [20]. On the other hand, this should be compared to the holographic entropy which itself may be modified by the presence of the total derivative terms in the AdS gravitational action. An example of this modification we have seen in section 5. If terms of 6th order in derivative are allowed, the structure of possible total derivative terms is much richer and some of them may produce contributions to the holographic entropy that do not disappear
even if the surface is minimal. Since not all these contributions are at the moment known we
can not yet bring together all pieces of the puzzle and fully resolve the problem.

On the other hand, our proposal (30), (34) is perhaps an easier way to attack the problem. It represents the total missing entropy which may come from both sides of the holographic relation. It is simple and can be easily checked for new examples of surfaces for which the mismatch in the entropy is found.

12 Conclusions

We have analyzed the possibility that the total derivative terms in the gravitational action may lead to some non-trivial contributions to the entropy. Rather surprisingly, in the examples of total derivative terms which we consider in this note we have found that such a contribution does exist. This observation may have many applications. We have briefly discussed the relevance of our finding to the logarithmic term in entanglement entropy and its relation to the conformal anomaly. In four dimensions the anomaly may in general contain a total derivative which originates from a local term in the quantum effective action. Then we predict that this term would contribute to the logarithmic term in a particular way which sometimes (for any sphere in Minkowski spacetime) mimics the contribution of the $a$-charge.

In six dimensions the structure of total derivative terms in the trace anomaly is much richer. We analyze some of them. In particular, we have found that they may produce contributions which do not disappear when the surface has no extrinsic curvature. The corresponding entropy is not of Wald’s type. We suggest that the entropy which comes from the total derivative terms in the trace anomaly is the source for the discrepancy between the holographic entanglement entropy and Wald’s entropy earlier observed in [9]. In a wider context this should mean that the total derivative terms in the gravitational action can not be neglected and may lead to some non-trivial gravitational entropy. This entropy may manifest itself for time-dependent metrics with a generalized horizon. Further implications for the thermodynamics of horizons of this type are worth exploring.

These conclusions are made under assumption of the correspondence between the gravitational action and the entropy as suggested by the application of the conical singularity method. It would be interesting to verify in an independent way whether the predicted contributions (such as $c$-term in four dimensions) do appear in entanglement entropy. The negative answer to this question would possibly impose certain restrictions on the entropy/action correspondence. These restrictions (once identified) would provide us with the useful information on the applicability of the method of conical singularity.
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