FRACTIONAL $\mathcal{Q}$-EDGE-COLORING OF GRAPHS

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Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphism. Let $\mathcal{Q}$ be an additive hereditary property of graphs. A $\mathcal{Q}$-edge-coloring of a simple graph is an edge coloring in which the edges colored with the same color induce a subgraph of property $\mathcal{Q}$. In this paper we present some results on fractional $\mathcal{Q}$-edge-colorings. We determine the fractional $\mathcal{Q}$-edge chromatic number for matroidal properties of graphs.

Keywords: fractional coloring, graph property.

2010 Mathematics Subject Classification: 05C15, 05C70, 05C72.

1. Introduction

Our terminology and notation will be standard. The reader is referred to [1, 11] for undefined terms. All graphs considered in this paper are simple, i.e. they have no loops or multiple edges.

† Peter Mihók passed away on March 27, 2012.
We denote the class of all finite simple graphs by $\mathcal{I}$. A graph property $Q$ is a non-empty isomorphism-closed subclass of $\mathcal{I}$. We also say that a graph $G$ has property $Q$ whenever $G \in Q$. The fact that $H$ is a subgraph of $G$ is denoted by $H \subseteq G$ and the disjoint union of two graphs $G$ and $H$ is denoted by $G \cup H$. A property $Q$ is called additive if $G \cup H \in Q$ whenever $G \in Q$ and $H \in Q$. A property $Q$ is called hereditary if $G \in Q$ and $H \subseteq G$ implies $H \in Q$. The set of all additive hereditary properties will be denoted by $\mathcal{L}$.

We list several well-known additive hereditary properties

$$
\mathcal{D}_k = \{ G \in \mathcal{I} : \text{each subgraph of } G \text{ contains a vertex of degree at most } k \},
$$

$$
\mathcal{I}_k = \{ G \in \mathcal{I} : G \text{ does not contain } K_{k+2} \},
$$

$$
\mathcal{J}_k = \{ G \in \mathcal{I} : \chi'(G) \leq k \},
$$

$$
\mathcal{O}_k = \{ G \in \mathcal{I} : \text{each component of } G \text{ has at most } k + 1 \text{ vertices} \},
$$

$$
\mathcal{S}_k = \{ G \in \mathcal{I} : \Delta(G) \leq k \},
$$

$$
\mathcal{B} = \{ G \in \mathcal{I} : G \text{ is a bipartite graph} \},
$$

where $K_{k+2}$ denotes the complete graph on $k + 2$ vertices, $\chi'(G)$ is the edge chromatic number (chromatic index) and $\Delta(G)$ is the maximum degree of the graph $G$.

Generalized colorings of edges or/and vertices of graphs under restrictions given by graph properties have recently attracted much attention, see e.g. [2, 3, 4, 6, 7, 8, 10] and references therein.

By using the class of additive hereditary properties, there is the following generalization of edge coloring. Let $Q \in \mathcal{L}$ and let $t$ be a positive integer. A $t$-fold $Q$-edge-coloring of a graph is an assignment of $t$ distinct colors to each edge such that each color class induces a subgraph of property $Q$. The smallest number $k$ such that $G$ admits a $t$-fold $Q$-edge-coloring with $k$ colors is the $(t, Q)$-chromatic index of $G$, denoted by $\chi'_{t,Q}(G)$. Clearly, a 1-fold $O_1$-edge-coloring is a usual proper edge coloring and hence $\chi'_{1,O_1}(G) = \chi'(G)$.

Another generalization of edge coloring is fractional edge coloring. The fractional chromatic index of a graph $G$ is defined in the following way: $\chi'_{f}(G) = \lim_{t \to \infty} \frac{\chi'_{t,O_1}(G)}{t}$. If we replace the property $O_1$ by any other additive hereditary graph property $Q$ in the definition of the fractional chromatic index, then we obtain the fractional $Q$-chromatic index of a graph $G$ and we denote it $\chi'_{f,Q}(G)$.

A hypergraph $\mathcal{H}$ is a pair $(S, X)$, where $S$ is a finite set and $X$ is a family of subsets of $S$. The elements of $X$ are called hyperedges. A $t$-fold covering of a hypergraph $\mathcal{H}$ is a collection (multiset) of hyperedges which includes every element of $S$ at least $t$ times. The smallest cardinality of such a multiset is called the $t$-fold covering number of $\mathcal{H}$ and is denoted $k_t(\mathcal{H})$. The fractional covering number of $\mathcal{H}$ is defined as $k_{f}(\mathcal{H}) = \lim_{t \to \infty} \frac{k_t(\mathcal{H})}{t}$.
For given simple graph \( G = (V, E) \) and additive hereditary property \( Q \), let \( \mathcal{H}_G = (E_G, Q_G) \) denote the hypergraph whose vertex set is the edge set of \( G \) and the hyperedges are those subsets of \( E(G) = E_G \) which induce a graph of property \( Q \). Since \( Q \) is hereditary, we can formulate the \((t, Q)\)-chromatic index of the graph \( G \) as the \( t \)-fold covering number of the hypergraph \( \mathcal{H}_G \). There is a natural one-to-one correspondence between the color classes of \( G \) and the hyperedges of \( \mathcal{H}_G \). Therefore the following assertion holds.

**Claim 1.** The fractional \( Q \)-chromatic index of the graph \( G \) is equal to the fractional covering number of the hypergraph \( \mathcal{H}_G = (E_G, Q_G) \).

A matroid \( M = (S, I) \) is a hypergraph which satisfies the following three conditions:
1. \( \emptyset \in I \),
2. if \( X \in I \) and \( Y \subseteq X \), then \( Y \in I \),
3. if \( X, Y \in I \) and \(|X| > |Y|\), then there is an \( x \in X \setminus Y \) such that \( Y \cup \{x\} \in I \).

In [12] the fractional covering number of matroids is determined. Let \( X \) be a subset of the ground set \( S \) of a matroid \( M \). The rank of \( X \), denoted \( \rho(X) \), is defined as the maximum cardinality of an independent subset of \( X \) (a subset of \( X \) which belongs to \( I \)).

**Theorem 2** [12]. If \( M = (S, I) \) is a matroid, then

\[
k_f(M) = \max_{X \subseteq S, X \neq \emptyset} \frac{|X|}{\rho(X)}.
\]

In this paper, by combining Claim 1 and Theorem 2, we give a general formula for the fractional \( Q \)-chromatic index. Afterwards, by this formula and with other results from the literature, we determine the exact values of \( \chi'_f, Q(G) \) for so-called \( Q \)-matroidal graphs.

2. Results

Let \( G = (V, E) \) be a graph and let \( Q \) be an additive hereditary property. If the hypergraph \( (E_G, Q_G) \) is a matroid, then \( G \) is called \( Q \)-matroidal. Let \( Q^M \) denote the set of all \( Q \)-matroidal graphs. A property \( Q \) is called \textit{matroidal} if every graph \( G \) is \( Q \)-matroidal. Schmidt [13] proved the existence of uncountably many matroidal properties.

A subset of the edge set of a graph is called \( Q \)-independent if it induces a graph of property \( Q \). For a graph \( H \) let \( Q(H) \) denote the maximum cardinality of a \( Q \)-independent subset of \( E(H) \).
Lemma 3. Let \( a_i, b_i > 0 \) for \( i = 1, \ldots, n \). Then \( \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \max_i \left\{ \frac{a_i}{b_i} \right\} \).

Proof. By induction on \( n \). \( \square \)

Theorem 4. Let \( Q \in \mathbb{L} \) and let \( G \in Q^M \). Then

\[
\chi'_{f,Q}(G) = \max \frac{|E(H)|}{Q(H)},
\]

where the maximum is taken over all connected nontrivial subgraphs \( H \) of \( G \).

Proof. Since \( G \) is \( Q \)-matroidal, the hypergraph \( \mathcal{H}_G = (E_G, Q_G) \) is a matroid.

Claim 1 with Theorem 2 imply that

\[
\chi'_{f,Q}(G) = \max_{X \subseteq E_G: X \neq \emptyset} \frac{|X|}{\rho(X)} = \max \frac{|E(H)|}{Q(H)},
\]

where the maximum is taken over all nontrivial subgraphs \( H \) of \( G \).

Now we show that we may restrict our attention to connected \( H \). Suppose that the maximum on the right-hand side of (1) is achieved for a graph \( H \) with more than one component. Let \( H = H_1 \cup \cdots \cup H_n \), where \( H_i \) are the components of \( H \). If one of these components, say \( H_j \), is an empty graph (set of isolated vertices), then

\[
\frac{|E(H)|}{Q(H)} = \frac{|E(H - H_j)|}{Q(H - H_j)}. \]

Thus we can assume that each component has at least one edge. Then

\[
\frac{|E(H)|}{Q(H)} = \frac{|E(H_1)| + \cdots + |E(H_n)|}{Q(H_1) + \cdots + Q(H_n)} \leq \max_i \left\{ \frac{|E(H_i)|}{Q(H_i)} \right\}. \quad \square
\]

We can now determine the fractional \( Q \)-chromatic index for \( Q \)-matroidal graphs. The following question arises: Which graphs are \( Q \)-matroidal for given properties \( Q \)?

Each hereditary property \( Q \) can be determined by the set of minimal forbidden subgraphs \( F(Q) = \{ G \in \mathbb{L}: G \not\in Q \text{ but } G \setminus \{ e \} \in Q \text{ for each } e \in E(G) \} \). For example: \( F(O_k) = \{ H; H \text{ is a tree on } k+2 \text{ vertices } \} \); \( F(I_k) = \{ K_{k+2} \} \). Simões-Pereira [14] proved that if \( F(Q) \) is finite, then \( Q \) is not matroidal.

In [9] there is the following characterization of \( Q \)-matroidal graphs.

Theorem 5 [9]. A graph \( G = (V, E) \) is \( Q \)-matroidal if and only if for each \( Q \)-independent set \( I \subseteq E \) and for each edge \( e \in E \setminus I \) the graph \( G[I \cup \{ e \}] \) induced by \( I \cup \{ e \} \) contains at most one minimal forbidden subgraph of \( Q \).

By Theorem 5 each graph \( G \) which contains either at most one minimal forbidden subgraph of \( Q \) or only edge-disjoint minimal forbidden subgraphs of \( Q \) is \( Q \)-matroidal.

Lemma 6 [9]. The property \( Q^M \) belongs to \( \mathbb{L} \) for every \( Q \in \mathbb{L} \).
By Lemma 6 we can characterize the structure of $Q$-matroidal graphs by describing the set of minimal forbidden subgraphs $F(Q^M)$.

For any two given graphs $G_1$ and $G_2$ with a common induced subgraph $H$ we construct the graph $G = (G_1; H; G_2)$ by amalgamation of $G_1$ and $G_2$ with respect to $H$ so that $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$ and $H = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$.

In the following $P_n$ and $C_n$ will denote the path and the cycle on $n$ vertices, respectively. $D_n$ will denote the complement of $K_n$.

**Theorem 7** [9]. Let $G$ be a graph and let $k \geq 1$. Then

- $G \in F(O_k^M)$ if and only if $G \in T \setminus \{K_{1,k+2}; C_{k+2}\}$, where $T$ is the set of all trees on $k + 3$ vertices and all unicyclic graphs on $k + 2$ vertices,
- $G \in F(S_k^M)$ if and only if $G = (K_{1,k+1}; K_2 \cup D_p; K_{1,k+1})$ for some $0 \leq p \leq k$ and $k \geq 2$, where $K_2$ joins the central vertices of the stars,
- $G \in F(T_k^M)$ if and only if $G = (K_{k+2}; K_r; K_{k+2})$ for some $2 \leq r \leq k + 1$,
- $G \in F(B^M)$ if and only if $G = (C_{2p+1}; P_q; C_r)$ for some $p \geq 1$, $q \geq 2$ and $r \geq 3$.

The seminal result on fractional edge colorings is due to Edmonds [5]. For a graph $G$ we define $\Gamma(G) = \max \frac{2|E(H)|}{|V(H)| - 1}$, where the maximization is over every induced subgraph $H$ of $G$ with $|V(H)| \geq 3$ and $|V(H)|$ odd.

**Theorem 8** [5]. Let $G$ be a graph. Then

$$\chi'_{f,J_1}(G) = \chi'_{f,S_1}(G) = \chi'_{f,C_1}(G) = \chi'_{f}(G) = \max \{\Delta(G), \Gamma(G)\}.$$ 

**Lemma 9.** Every graph is $D_1$-matroidal.

**Proof.** Clearly, $F(D_1)$ is a set of cycles. Moreover, if we add an edge to a tree (forest) we obtain exactly (at most) one cycle. So the claim follows from Theorem 5.

Although all graphs are $D_1$-matroidal, for $k \geq 2$ the characterization of $D_k$-matroidal graphs seems to be difficult.

**Theorem 10.** Let $G$ be a graph. Then

$$\chi'_{f,D_1}(G) = \max \frac{|E(H)|}{|V(H)| - 1},$$

where the maximum is taken over all connected nontrivial subgraphs $H$ of $G$. 


Proof. From Lemma 9 it follows that $G$ is $D_1$-matroidal. Any spanning tree of a connected graph $H$ on $n$ vertices has $n-1$ edges, therefore $D_1(H) = |V(H)| - 1$. Theorem 4 implies
\[
\chi'_{f, D_1}(G) = \max_{H \subseteq G} \frac{|E(H)|}{D_1(H)} = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)| - 1}.
\]

Corollary 11. Let $G$ be a graph and let $Q \in \mathcal{L}$ such that $D_1 \subseteq Q$. Then
\[
\chi'_{f, Q}(G) \leq \max_{H \subseteq G} \left| E(H) \right| - H_{k+2},
\]
where the maximization is over all connected nontrivial subgraphs $H$ of $G$.

Lemma 12. Let $k \geq 1$. The graph $G$ is $I_k$-matroidal if and only if any two complete graphs on $k + 2$ vertices are edge-disjoint in $G$.

Proof. Assume that $G$ contains two complete graphs on $k + 2$ vertices which have $r \geq 2$ vertices in common. These $r$ vertices induce $K_r$, hence $G$ contains $(K_{k+2}; K_r; K_{k+2})$ as a subgraph. So $G \notin I_k^M$ since $(K_{k+2}; K_r; K_{k+2}) \in F(I_k^M)$ (see Theorem 7).

If $G \notin I_k^M$, then $G$ contains a forbidden subgraph $(K_{k+2}; K_r; K_{k+2})$ for some $2 \leq r \leq k + 1$, thus it contains two complete graphs on $k + 2$ vertices which share an edge.

Let $H_{k+2}$ denote the number of complete graphs on $k + 2$ vertices in the graph $H$.

Theorem 13. Let $G$ be an $I_k$-matroidal graph, $k \geq 1$. Then
\[
\chi'_{f, I_k}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|E(H)| - H_{k+2}},
\]
where the maximum is taken over all connected nontrivial subgraphs $H$ of $G$.

Proof. From Theorem 4 it follows that $\chi'_{f, I_k}(G) = \max_{H \subseteq G} \frac{|E(H)|}{I_k(H)}$. So it is sufficient to show that $I_k(H) = |E(H)| - H_{k+2}$.

Lemma 12 implies that any two complete graphs on $k + 2$ vertices are edge-disjoint in every subgraph $H$ of $G$. Hence, if we remove less than $H_{k+2}$ edges from $H$, then the obtained graph still contains at least one $K_{k+2}$. Therefore $I_k(H) \leq |E(H)| - H_{k+2}$.

On the other hand, if we remove one edge from each $K_{k+2}$, then the remaining edges form an $I_k$-independent set, hence $I_k(H) \geq |E(H)| - H_{k+2}$.

Lemma 14. Let $k \geq 2$. The graph $G$ is $S_k$-matroidal if and only if no two vertices of degree at least $k + 1$ are incident in $G$. 
**Proof.** Let $uv$ be an edge of $G$ such that its endvertices have degree at least $k+1$. Let $G_1$ be a subgraph of $G$ which contains only the edges incident with $u$ or $v$. Clearly, $G_1$ contains a subgraph $G_2$ in which the vertices $u$ and $v$ are joined by an edge and they have degree $k+1$. Let $p$ denote the number of common neighbors of $u$ and $v$ in $G_2$. Observe that $G_2 = (K_{1,k+1}; K_2 \cup D_p; K_{1,k+1})$, consequently $G_2 \in F(S^M_k)$. So $G$ cannot be $S^k$-matroidal.

If $G \notin S^M_k$, then it contains a minimal forbidden subgraph $(K_{1,k+1}; K_2 \cup D_p; K_{1,k+1})$ for some $0 \leq p \leq k$. The central vertices of these stars are joined by an edge and they have degree $k+1$.

**Theorem 15.** Let $G$ be an $S^k$-matroidal graph, $k \geq 2$. Then

$$\chi'_{S^k}(G) = \max \frac{|E(H)|}{|E(H)| - \sum_{v \in V(H)} (\deg_H(v) - k)},$$

where the maximum is taken over all connected nontrivial subgraphs $H$ of $G$.

**Proof.** Let $H$ be a subgraph of $G$. If for every vertex $v$ of $H$ of degree at least $k+1$ we remove $\deg_H(v) - k$ edges incident with it, then we obtain a graph whose maximum degree is at most $k$. Therefore

$$S^k_k(H) \geq |E(H)| - \sum_{v \in V(H)} (\deg_H(v) - k).$$

The opposite inequality follows from the fact that no two vertices of degree at least $k+1$ are incident in $G$, thus neither in $H \subseteq G$ (see Lemma 14). Therefore the claim follows from Theorem 4.

**Lemma 16.** The graph $G$ is $B$-matroidal if and only if no odd cycle of $G$ shares an edge with any other cycle of $G$.

**Proof.** $G \notin B^M$ if and only if $G$ contains a minimal forbidden subgraph $(C_{2p+1}; P_q; C_r)$ for some $p \geq 1$, $q \geq 2$ and $r \geq 3$. Equivalently, $G$ contains an odd cycle which shares an edge with an other cycle.

**Corollary 17.** If $G \in B^M$, then the odd cycles of $G$ are edge-disjoint.

Let $oc(G)$ denote the number of odd cycles in the graph $G$.

**Theorem 18.** Let $G$ be a $B$-matroidal graph. Then

$$\chi'_{B}(G) = \max \frac{|E(H)|}{|E(H)| - oc(H)},$$

where the maximum is taken over all connected nontrivial subgraphs $H$ of $G$. 

Proof. Let \( H \) be a subgraph of \( G \). If we remove one edge from every odd cycle of \( H \), then the remaining edges induce a bipartite graph, hence \( \mathcal{B}(H) \geq |E(H)| - oc(H) \).

The odd cycles in \( H \) are edge-disjoint (see Corollary 17), thus we must remove at least \( oc(H) \) edges from \( E(H) \) to obtain a \( \mathcal{B} \)-independent set. Therefore \( \mathcal{B}(H) \leq |E(H)| - oc(H) \).

Consequently, \( \mathcal{B}(H) = |E(H)| - oc(H) \) and hence the assertion follows from Theorem 4.

Lemma 19. Let \( k \geq 1 \). The graph \( G \) is \( \mathcal{O}_k \)-matroidal if and only if \( G \) either belongs to \( \mathcal{O}_k \) or it is isomorphic to \( K_{1,p} \), \( p \geq k + 1 \), to \( C_{k+2} \) or to a tree on \( k + 2 \) vertices.

Proof. \( G \) is \( \mathcal{O}_k \)-matroidal if and only if it does not contain any subgraph from \( F(\mathcal{O}_k^M) \). So the claim follows from Theorem 7.

Clearly, if \( G \in \mathcal{O}_k \), then its fractional \( \mathcal{O}_k \)-edge chromatic number equals one. If \( G \in \mathcal{O}_k^M \setminus \mathcal{O}_k \), then it has \( k + 2 \) vertices or it is a star on at least \( k + 3 \) vertices.

Theorem 20. Let \( G \in \mathcal{O}_k^M \setminus \mathcal{O}_k \) and let \(|V(G)| = k + 2, k \geq 2\). Then

\[
\chi'_{\mathcal{O}_k}(G) = \frac{|E(G)|}{|E(G)| - \lambda(G)},
\]

where \( \lambda(G) \) is the edge-connectivity of \( G \).

Proof. Let \( H \) be a connected subgraph of \( G \). If \( E(H) \) is not \( \mathcal{O}_k \)-independent, then either \(|E(H)| = k + 2\) or \(|E(H)| = k + 1\). In the first case \( H = C_{k+2} \), hence \( \mathcal{O}_k(H) = |E(H)| - 2 \). In the second case \( H \) is a tree, therefore \( \mathcal{O}_k(H) = |E(H)| - 1 \). Thus the claim follows from Theorem 4.

Theorem 21. Let \( G \in \mathcal{O}_k^M \setminus \mathcal{O}_k \) and let \(|V(G)| = k + i, k \geq 2, i \geq 3\). Then

\[
\chi'_{\mathcal{O}_k}(G) = \frac{|E(G)|}{|E(G)| - i + 1} = \frac{k + i - 1}{k}.
\]

Proof. It follows from Theorem 4 and from the fact that \( G \) is a star.

3. Examples

Example 22. Let \( K_{2,3} \) denote the complete bipartite graph on \( 2 + 3 \) vertices. We will show that \( \chi'_{\mathcal{O}_k}(K_{2,3}) = \frac{3}{2} \).
Solution 1.
From Lemma 14 it follows that $K_{2,3} \in S_M^2$. From Theorem 15 we have $\chi'_{f,S_2}(K_{2,3})$ 

\[ = \max \left\{ \frac{|E(H)|}{|E(H)| - \sum_{v \in V(H)} \deg_H(v) = 3} \right\}, \]

where the maximum is taken over all connected nontrivial subgraphs $H$ of $G$.

If $H$ is a connected subgraph of $G$, then either $H \in S_2$ or it is a graph from Figure 1. So $\chi'_{f,S_2}(K_{2,3}) = \max \{1, \frac{3}{2}, \frac{3}{2}, \frac{5}{4}, \frac{3}{2} \} = \frac{3}{2}$.

Figure 1. Connected subgraphs of $K_{2,3}$ which are not in $S_2$.

Solution 2.
Fractional $Q$-edge-colorings may be viewed in several ways. We present an equivalent definition. Let $r, s$ be positive integers with $r \geq s$. An $(r, s)$-fractional $Q$-edge-coloring of $G$ is an assignment of $s$-element subsets of $\{1, \ldots, r\}$ to the edges of $G$ such that each color class induces a graph of property $Q$. Then the fractional $Q$-edge chromatic number of $G$ is defined as

\[ \chi'_{f,Q}(G) = \inf \left\{ \frac{r}{s} : G \text{ has an } (r, s)\text{-fractional } Q\text{-edge-coloring} \right\}. \]

Note that in this definition we can replace the infimum by the minimum.

For each $(r, s)$-fractional $S_2$-edge-coloring of $K_{2,3}$ and for each color $i \in \{1, \ldots, r\}$ the following holds: at most four edges are colored with sets containing the color $i$. On the other hand, every edge is assigned with an $s$-element color set. This implies that $4r \geq 6s$, hence $\chi'_{f,S_2}(K_{2,3}) \geq \frac{3}{2}$.

To prove the inequality $\chi'_{f,S_2}(K_{2,3}) \leq \frac{3}{2}$ we construct a $(3, 2)$-fractional $S_2$-edge-coloring of $K_{2,3}$, see Figure 2.

Figure 2. A $(3, 2)$-fractional $S_2$-edge-coloring of the graph $K_{2,3}$.

The following results immediately follows from Theorems 13, 15 and 18.
Example 23. If $k \geq 1$, then
\[
\chi'_{f,I}(K_{k+2}) = \left(\frac{k+2}{2}\right)^2 - 1, \quad \chi'_{f,S}(K_{1,k+1}) = \frac{k+1}{k}
\]
and
\[
\chi'_{f,B}(C_{2k+1}) = \frac{2k+1}{2k}.
\]

Acknowledgment
The authors would like to thank anonymous referees for many helpful comments and constructive suggestions.

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Received 3 November 2011
Revised 29 May 2012
Accepted 29 May 2012