On the other side of the bialgebra of chord diagrams

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Abstract

In this paper we describe complexes whose homologies are naturally isomorphic to the first term of the Vassiliev spectral sequence computing (co)homology of the spaces of long knots in $\mathbb{R}^d$, $d \geq 3$. The first term of the Vassiliev spectral sequence is concentrated in some angle of the second quadrant. In homological case the lower line of this term is the bialgebra of chord diagrams (or its superanalog if $d$ is even). We prove in this paper that the groups of the upper line are all trivial. In the same bigradings we compute the homology groups of the complex spanned only by strata of immersions in the discriminant (maps having only self-intersections). We interprete the obtained groups as subgroups of the (co)homology groups of the double loop space of a $(d - 1)$-dimensional sphere. In homological case the last complex is the normalized Hochschild complex of the Poisson or Gerstenhaber (depending on parity of $d$) algebras operad. The upper line bigradings are spanned by the operad of Lie algebras. To describe the cycles in these bigradings we introduce new homological operations on Hochschild complexes. These new operations are in fact the Dyer-Lashof operations induced by the action of the singular chains operad of little squares on Hochschild complexes.

Keywords: knot spaces, discriminant, bialgebra of chord diagrams, operads, Hochschild complexes, Deligne’s conjecture, Dyer-Lashof operations.

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0 Introduction

0.1 Spaces of long knots. Approach of V. Vassiliev

The space of long knots in $\mathbb{R}^d$ is the space of smooth embeddings $\mathbb{R}^1 \hookrightarrow \mathbb{R}^d$ that coincide with a fixed linear map $\mathbb{R}^1 \hookrightarrow \mathbb{R}^d$ outside some compact set (depending on a knot). The long knots form an open everywhere dense subset in the affine space $K \simeq \mathbb{R}^\omega$ of all smooth maps $\mathbb{R}^1 \rightarrow \mathbb{R}^d$ with the same behavior at infinity. The complement $\Sigma$ of this dense subset is called the discriminant space. It consists of the maps having self-intersections and/or singularities. Any cohomology class $\gamma \in H^i(K \setminus \Sigma)$ of the knot space can be realized as the linking coefficient with an appropriate chain in $\Sigma$ of codimension $i + 1$ in $K$. In other words one has the Alexander duality:

\[
\tilde{H}^i(K \setminus \Sigma) \simeq \tilde{H}_{\omega-i-1}(\bar{\Sigma}),
\]

where $\bar{\Sigma}$ designates the one-point compactification of the discriminant $\Sigma$.

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Strictly speaking the isomorphism \(0.1\) has no sense since \(\omega\) is infinity. To define rigorously the right-hand side of \(0.1\) one needs to use finite-dimensional approximations of the space of long knots, i.e. finite-dimensional spaces \(\mathbb{R}^N \subset \mathcal{K}\) that are in general position with the discriminant \(\Sigma\), cf. [34].

**0.2 Vassiliev’s and Sinha’s spectral sequences. Main results**

The main tool of Vassiliev’s approach to computation of the (co)homology of the knot space is *simplicial resolution* \(\sigma\) of the discriminant, cf. [34]. The space \(\sigma\) has a natural filtration
\[
\emptyset = \sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \ldots.
\] (0.2)

Vassiliev conjectures that this filtration homotopically splits, i.e.
\[
\bar{\sigma} \cong \bigoplus_{i=1}^{+\infty} (\bar{\sigma}_i / \bar{\sigma}_{i-1}).
\] (0.3)

This would imply the isomorphism
\[
\tilde{H}_*(\Sigma) \equiv \check{H}_*(\bar{\sigma}) \cong \bigoplus_{i=1}^{+\infty} \check{H}_*(\bar{\sigma}_i / \bar{\sigma}_{i-1}),
\] (0.4)

or in other words that the spectral sequence (called Vassiliev’s main spectral sequence) associated with the filtration (0.2) stabilizes in the first term.

Filtration (0.2) induces an increasing filtration in the homology groups of \(\Sigma\), i.e. in the cohomology of the knot space:
\[
H^*_0(\mathcal{K}\setminus\Sigma) \subset H^*_1(\mathcal{K}\setminus\Sigma) \subset H^*_2(\mathcal{K}\setminus\Sigma) \subset \ldots.,
\] (0.5)

and decreasing dual filtration in the homology groups:
\[
H_0(\Sigma \setminus \mathcal{K}) \supset H_1(\Sigma \setminus \mathcal{K}) \supset H_2(\Sigma \setminus \mathcal{K}) \supset \ldots.
\] (0.6)

For \(d \geq 4\) filtrations (0.5), (0.6) are finite for any dimension *. Vassiliev’s main spectral sequence in this case computes the graded quotient associated with these filtrations.

In the most intriguing case \(d = 3\) almost nothing is clear. However we can say something about the dimension \(* = 0\). The filtration (0.5) does not exhaust the whole cohomology of degree zero. The knot invariants obtained by this method are called the Vassiliev invariants, or invariants of finite type. The dual space to the graded quotient of the space of finite type knot invariants is the *bialgebra of chord diagrams*. The invariants and the bialgebra in question were intensively studied in the last decade, cf. [3]. The completeness conjecture for the Vassiliev knot invariants is the question about the convergence of the filtration (0.6) to zero for \(d = 3, * = 0\). The realization theorem of M. Kontsevich [16] proves that the Vassiliev spectral sequence over \(\mathbb{Q}\) for \(d = 3, * = 0\) computes the corresponding associated quotient (for positive dimensions * in the case \(d = 3\) even this is not for sure) and does stabilize in the first term. The groups of the graded quotient associated to filtration (0.5) in the case \(d = 3, * > 0\) are some quotient groups of the groups calculated by Vassiliev’s main spectral sequence.
The first term of Vassiliev’s spectral sequence is concentrated in some angle of the second quadrant, see Figure 1.

The lower and the upper edges of this angle are determined by the following equations:

\[ q = -(d - 2)p \quad \text{lower line}; \]
\[ q = -(d - 1)p - 1 \quad \text{upper line}. \]  

In the cell \((p, q)\) stands the group \(\tilde{H}^{p-q-1}(\bar{\sigma}_{p-}\bar{\sigma}_{p-1})\). Observe, that \(p\) is equal to minus the filtration: \(p = -\imath\), and \(q\) is defined by the condition that \(p + q\) equals the usual (co)homology dimension.

Up to a changing of grading this first term depends only on the parity of \(d\) (dimension of the ambient space \(\mathbb{R}^d\)). For different \(d_1\) and \(d_2\) of the same parity one has the isomorphism:

\[ d_1E^1_{p,q} \simeq d_2E^1_{p-q-p(d_2-d_1)}, \]
\[ d_1E^1_{p,q} \simeq d_2E^1_{p,q-p(d_2-d_1)}, \]

where \(dE^1_{p,q}\) (cohomological case), \(dE^1_{p,q}\) (homological case) designate the first term of the Vassiliev spectral sequence computing the (co)homology of the space of long knots in \(\mathbb{R}^d\). Over \(\mathbb{Z}_2\) this is also true for dimensions \(d_1\), \(d_2\) of different parities. In the homological case the direct sum of groups standing in the lower line is isomorphic to the bialgebra of chord diagrams, if \(d\) is odd, and to a non-trivial superanalog of this bialgebra, if \(d\) is even.

Actually there is another and absolutely different approach to studying the space of embeddings. Briefly speaking in this approach one "approximates" the space of knots \(\text{Emb} = K \setminus \Sigma\) by means of homotopy limits of diagrams of maps. This approach was initiated by T. Goodwillie and M. Weiss [14, 15], and then developed by D. Sinha [24, 25], and also by I. Volic [37]. In particular for the space of long knots this method provides a spectral sequence whose second term is isomorphic up to a shift of bigradings to the first term of the Vassiliev spectral sequence (this spectral sequence was constructed by D. Sinha [24]):

**Proposition 0.1** (i) The groups of the first term of Vassiliev’s spectral sequence (computing the (co)homology of the space of long knots) are naturally isomorphic up to a shift...
of bigradings to the groups of the second term of Sinha’s spectral sequence (computing the (co)homology of a space homotopy equivalent to the space of long knots).

(ii) Moreover, the first term of Vassiliev’s auxiliary spectral (stabilizing in the second term and computing the first term of the main spectral sequence) is isomorphic as a complex to the first term of Sinha’s spectral.

Assertion (ii) says that the complexes $CT, D^{odd}, CT, D^{even}$ that we define in Section 1.2 is nothing else but the first term of Sinha’s spectral sequence. To prove Proposition 0.1 one needs to compare Section 7 in [24] with Vassiliev’s auxiliary spectral sequence which was defined in the seminal work [34]. For a more explicit description of the auxiliary spectral sequence, see also [30, Chapitre II].

It is also worth to mention the Cattaneo-Cotta-Ramusino-Longoni construction [5, 6] that provides a morphism from a graph-complex to the De Rham complex of the space of knots. In the case of long knots the corresponding graph-complex is quasi-isomorphic to the complex $CT, D$.

In our paper we find the groups of the upper line (0.7). Actually we prove that all these groups are trivial.

We also consider complexes that arise if in the degree zero term we take into account only the summands corresponding to strata assigned to maps having only self-intersections (but not degeneration of the derivative). For these complexes we find the homology groups in the bigradings of the upper line. I. Volic and D. Sinha give a geometrical interpretation of these complexes. In their approach the complexes are the first terms of the spectral sequences computing the cohomology groups of the homotopy fiber of the inclusion of the space of knots $Emb = \mathcal{K} \setminus \Sigma$ to the space of immersions $Imm$, cf. [37, 25]. D. Sinha proved that this homotopy fiber is homotopy equivalent to a direct product $Emb \times \Omega^2 S^{d-1}$, see [25]. Therefore, the non-trivial upper diagonal homology groups that we find in this paper are some subgroups of the (co)homology of $\Omega^2 S^{d-1}$.

One more reason why the above complexes are worth being studied is that in homological case they are isomorphic to the normalized Hochschild complexes of the Poisson algebras operad ($d$ odd) and Gerstenhaber algebras operad ($d$ even), cf. [29, 30, 31]. Their homology groups are the characteristic classes of Hochschild cohomology of Poisson, resp. Gerstenhaber algebras considered as associative algebras. The homology groups of the upper line bigradings are the characteristic classes defined by the Lie-algebra structure in Poisson or Gerstenhaber algebras.

0.3 Plan of the paper

The paper is divided into 4 parts. Each part starts with a brief description of its contents. The results of Parts I and II were given before in the thesis [30]. The main results of Part III were given also (and with more details) in [31]. The results of Part IV and of Appendix E are new.

Part I is cohomological: in this part we define complexes computing the first term of the Vassiliev spectral sequence converging to the cohomology of the space of long knots. We remind that one conjectures that this spectral sequence stabilizes in the first term.

Part II is homological: in this part we define complexes that are dual to those of Part I. We give a hint how these dual complexes were obtained and describe their bases. The obtained complexes are the normalized Hochschild complexes of some operads endowed with a map from the associative algebras operad.
In Part III we study the upper diagonal homology groups of the complexes defined in Parts I-II. In this part we define new homological operations on Hochschild complexes. It will be proven elsewhere that these operations are the Dyer-Lashof homology operations induced by the little square chains action on Hochschild complexes, see also Appendix E. The new results in this part are given by Theorems 10.1, 10.2, 14.4. Theorem 14.4 is given without proof.

The last part contains various appendixes. It is a mixture of explicit demonstration of objects we deal with; results of computer calculations; and reformulation of old results with some nuances that are necessary for our considerations. In Appendix E we show how the homology bialgebra of $\Omega^2 S^{d-1}$ is included in the Hochschild homology of Poisson and Gerstenhaber algebras operads. This result is new: we use there our new "Dyer-Lashof" operations on Hochschild complexes; but it was more logical to put this section after Appendix D.

All along the paper we make a confusion of $\mathbb{Z}$-grading $\deg = p + q$ and corresponding $\mathbb{Z}_2$ supergrading. It is done deliberately to emphasize the fact that the obtained complexes depends only on the parity of dimension $d$ of the ambient space $\mathbb{R}^d$.

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Part I

Differential Hopf algebras of $T_*$-diagrams, $T$-diagrams and $T_0$-diagrams

To compute the first term of the main spectral sequence V. A. Vassiliev introduced an auxiliary filtration on the spaces $\sigma_i \setminus \sigma_{i-1}$. The associated auxiliary spectral sequence stabilizes in the second term since its first term is concentrated on the only row. The degree zero term with the differential on it is a direct sum of tensor products of so-called complexes of connected graphs. The homology groups of the complex of connected graphs on any finite set $M$ are trivial everywhere except the minimal possible dimension. This only non-trivial group is the $\mathbb{Z}$-module $T^-_M$ described in Section 1.1. Complex $CT_*D$ of $T_*$-diagrams that we define in Section 1.2 is exactly the first term of the auxiliary spectral sequence in cohomological case — “cohomological” means relation to the cohomology of the knot space.

In Section 2 we define a base in the space of the complex of $T_*$-diagrams.
In Section 3 we introduce the complexes of $T$-diagrams and of $T_0$-diagrams. The first complex $CTD$ is a quotient-complex of complex $CT_0D$. It is spanned by strata of maps without degeneration of derivative. The complex $CT_0D$ of $T_0$-diagrams is a subcomplex of $CTD$ and $CT_0D$. It is quasi-isomorphic to the complex $CT_0D$ and serves to simplify computations of the homology groups of $CT_0D$: 

\[
\begin{array}{c}
CT_0D \\
\sim \\
CT_0D \\
\end{array}
\]

In Section 3 we describe the differential Hopf algebra structure on the complexes $CT_0D$, $CTD$, $CT_0D$. The corresponding differential Hopf algebras are designated by $DHAT_0D$, $DHATD$, $DHAT_0D$. So, (0.8) is a commutative diagram of differential Hopf algebras morphisms.

1 Z-modules $T^-_M$, $T^+_M$. Complexes of $T_*$-diagrams

1.1 Z-modules $T^-_M$, $T^+_M$

1.1.1 $T^-_M$

Consider a finite set $M$ of some cardinality $\#M$. We will define an orientation of a tree with $\#M$ vertices labelled one-to-one by the elements of $M$ as an ordering of its edges. Consider a $\mathbb{Z}$-module spanned by the oriented trees — changing of order of the edges implies multiplication of such a tree by $(-1)^{|\sigma|}$, where $|\sigma|$ is the parity of the corresponding permutation. An oriented tree can be viewed as a monomial of anticommuting elements $\alpha_{ab} = \alpha_{ba}$ representing edges ($a, b \in M$ and $a \neq b$).

$\mathbb{Z}$-module $T^-_M$ is defined as the quotient-space of the above $\mathbb{Z}$-module by all the 3-term relations of the following type:

\[
\begin{array}{c}
A \\
\begin{array}{c}
1 \\
B \\
2 \\
C \\
\end{array}
\end{array} + 
\begin{array}{c}
A \\
\begin{array}{c}
2 \\
B \\
1 \\
C \\
\end{array}
\end{array} + 
\begin{array}{c}
A \\
\begin{array}{c}
2 \\
B \\
1 \\
C \\
\end{array}
\end{array} = 0
\]

Figure 2: 3-term relations in $T^-_M$

This picture is a sum of 3 trees whose edges are the same except the first two. This relation in terms of monomials can be rewritten as

\[
(\alpha_{ab}\alpha_{bc} + \alpha_{bc}\alpha_{ca} + \alpha_{ca}\alpha_{ab}) \cdot T_A \cdot T_B \cdot T_C = 0,
\]

where $M = A \cup B \cup C$, $a \in A$, $b \in B$, $c \in C$; $T_A$, $T_B$ and $T_C$ are some trees on sets $A$, $B$ and $C$ respectively.
In this form the relations resemble the Arnold’s relations in the cohomology algebra of the configuration spaces of different points in $\mathbb{R}_2^3$, cf. \[2\]. Indeed $T_M^{-}$ is the maximal degree cohomology group of the space of configurations of $\#M$ points (labelled by the elements of $M$).

**Statement 1.1** \[35\] $T_M^{-} \simeq \mathbb{Z}(\#M-1)!$. □

In the case $M = \{1, 2, \ldots, n\}$ the module $T_M^{-}$ will be also denoted by $T_n^{-}$. Note that $T_n^{-}$ is endowed with an action of the symmetric group $S_n$.

1.1.2 $T_M^{+}$

For the same finite set $M$ we define another $\mathbb{Z}$-module $T_M^{+}$ that is also spanned by the trees and quotiented by some 3-term relations. We will orientate trees in another way. By definition an orientation of a tree is an orientation of all its edges. Note we demand no more their ordering. Changing of orientation of one of edges is equivalent to multiplication by $(-1)$. An oriented tree can be viewed as a monomial of commuting elements $\alpha_{ab} = -\alpha_{ba}$ representing oriented edges ($a, b \in M$ and $a \neq b$).

The 3-term relations are as follows:

\[
\begin{align*}
A 
\quad + 
A 
\quad + 
A 
\quad = 
0
\end{align*}
\]

Figure 3: 3-term relations in $T_M^{+}$

In terms of monomials this relation is given by the same formula (1.1).

Let us denote by $\pm \mathbb{Z}$ the one-dimensional sign representation of the symmetric group $S_n$. Let $T_n^{+}$ designate $T_{\{1,2,\ldots,n\}}^{+}$. \[36\]

**Statement 1.2** \[30\] There is a natural isomorphism of the $S_n$-modules $T_n^{+} \simeq T_n^{-} \otimes (\pm \mathbb{Z})$. □

In particular this statement implies $T_M^{+} \simeq \mathbb{Z}(\#M-1)!$.

1.2 Complexes of $T_\ast$-diagrams

Now we are ready to describe the complexes of $T_\ast$-diagrams. We denote them $CTD^{odd}$ in the case of odd $d$ and $CTD^{even}$ in the case of even $d$. When we want to treat them simultaneously we will write simply $CTD$. We remind that these complexes compute the first term of the Vassiliev spectral sequence, see Figure 1. The complexes are bigraded. Bigrading $(i,j)$ corresponds to bigrading $(p,q) = (-i,id-j)$ of the Vassiliev spectral sequence.
1.2.1 case of even $d$

Consider a finite set of points on the line $\mathbb{R}^1$. We define a $T_*$-diagram on this set as a number of edges joining its points and also a number of asterisks that are put in these points.

We demand
1) each point contains not more than 1 asterisk;
2) the resulting graph is a disjoint union of trees;
3) if a tree consists of only one point, this point must contain an asterisk.

Figure 4 gives an example of a $T_*$-diagram.

![Figure 4: example of a $T_*$-diagram](image)

The points of a $T_*$-diagrams are split into a partition. The subsets of this partition formed by the vertices of the trees will be called minimal components. For example, the $T_*$-diagram on the Figure 4 has 3 minimal components: $\{1, 3, 5\}$, $\{2, 6\}$ and $\{4\}$.

An orientation of a $T_*$-diagram is by definition an ordering of its orienting set that consists of three types of elements:

1) points; 2) edges; 3) asterisks.

Changing of order is equivalent to multiplication by the sign of the corresponding permutation.

For example, the orientation set of the diagram from Figure 4 has 11 elements:

$\alpha_{13}$, $\alpha_{35}$, $\alpha_{26}$ — representatives of edges;

$\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$, $\beta_5$, $\beta_6$ — representatives of points;

$\gamma_3^*$, $\gamma_4^*$ — representatives of asterisks.

To orient the diagram one needs to order these elements, for example as it is done below:

$\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\alpha_{13}\alpha_{35}\alpha_{26}\gamma_3^*\gamma_4^*$.

We can consider this listing as a monomial of anticommuting elements.

Diagrams obtained one from another by an orientation preserving diffeomorphism of the line are set to be equivalent or equal.

The space of the complex of $T_*$-diagrams is defined as the $\mathbb{Z}$-module spanned by the oriented $T_*$-diagrams and quotiented by all the 3-term relations issued from the relations in the $\mathbb{Z}$-modules $T_M^-$, $M$ being any minimal component. This space is a direct sum of tensor products of modules $T_M^-$. 

Now let us describe the differential in this complex $CT_*, D^{even}$.

First of all note that the space of this complex is bigraded:
1) the first grading \( i \) — we call it *complexity* — is the total number of edges and stars of a diagram;

2) the second grading \( j \) is the number of points of a diagram.

The differential \( \partial \) will be of bigrading \((0, -1)\). It conserves the complexity but diminishes by one the number of points on the line. It means that complex \( CT_i D \) is a direct sum over \( i \) of complexes, each of them being spanned by the diagrams of complexity \( i \).

The differential \( \partial \) of a \( T_* \)-diagram \( D \) with \( j \) points is a sum over all possible \( j - 1 \) gluings of two neighbor points on the line \( \mathbb{R}^1 \). A gluing that does not give zero will be called *admissible*. There are two possibilities:

1) the gluing points \( t_1 \) and \( t_2 \) belong to different minimal components;
2) the gluing points \( t_1 \) and \( t_2 \) belong to the same minimal component.

In the first case the gluing is admissible if and only if there is no asterisk in at least one of these two points. The boundary diagram is obtained from \( D \) by contraction of the segment \([t_1, t_2]\), see Figure 5:

![Figure 5: admissible gluings of points from different minimal components](image)

To obtain the orienting monomial (with some sign) of this boundary diagram one needs to place on the first place of the orienting set of \( D \) the representative of the left point \( t_1 \), and on the second place — representative of the right point \( t_2 \) (this gives a sign) and then replace them by one element — representative of the new point of contraction. All the rest in the orienting monomial do not change.

In the second case (\( t_1 \) and \( t_2 \) from the same minimal component) gluing is admissible if and only if the points \( t_1 \) and \( t_2 \) are joined by an edge and none of them contains an asterisk. The boundary diagram is obtained by contracting the segment \([t_1, t_2]\) and replacing the edge \( \alpha_{t_1, t_2} \) joining \( t_1 \) with \( t_2 \) by new asterisk \( \gamma_t^* \), where \( t \) is the point of contraction, see Figure 6:

![Figure 6: admissible gluing of points from the same minimal component](image)

The orientation monomial (with some sign) of the boundary diagram is obtained analogously.
1.2.2 case of odd \(d\)

Now we will define the complex \(CT_*D^{odd}\) of \(T_*\)-diagram in the case of odd \(d\). The space of this complex is also defined to be spanned by \(T_*\)-diagrams and quotiented by 3-term relations. The difference is that we have another definition of orientation of \(T_*\)-diagrams. The orienting set of a diagram contains now only its points on the line. For example the orienting set of the diagram from Figure 4 consists of 6 elements: \(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\) — representatives of its 6 points; as its orienting monomial we can thus take

\[
\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6.
\]

As in the case of odd \(d\) we consider orienting elements as anticommuting generators. To orient a \(T_*\)-diagram one should also fix orientation of all its edges. Changing of orientation of one of edges is equivalent to multiplication by \((-1)\). For instance, one can orient all edges from left to right similarly to the orientation of \(\mathbb{R}^1\):

3-term relations in this space are issued from those in the \(\mathbb{Z}\)-modules \(T^+_M\), \(M\) being any minimal component. So, the space of complex \(CT_*D^{odd}\) is a direct sum of tensor products of modules \(T^+_M\).

The differential in this complex is defined analogously to the case of even \(d\): as the sum of admissible gluings with appropriate signs. When we take a gluing of two points joined by an edge whose orientation is opposit to that of the line, we should additionally multiply the result of this gluing by minus 1.

2 Basis in the complexes \(CT_*D\). Alternated \(T_*\)-diagrams

2.1 Basis in \(T^-_M, T^+_M\)

There exist three different types of basis for these \(\mathbb{Z}\)-modules: basis of snakes, cf. [35], basis of monotone trees and basis of alternated trees, cf. [30]. To define the first basis one needs to fix one element in \(M\) — the head of the snakes. To define the second and the third bases one needs the elements of \(M\) to be linearly ordered. Actually, only the third type of basis turned out to be useful in the investigation of complex \(CT_*D\) and its derivatives \(CTD\) and \(CT_0D\), defined in the next section. The reason for this is that only this basis is preserved via inversion of the order in \(M\).

Let finite set \(M\) be linearly ordered. We will consider trees, whose vertices are labeled one-to-one by elements of \(M\), as rooted trees with a root being the minimal element of \(M\).

**Definition 2.1** Any rooted tree defines a partial order \(\prec\) on the set \(M\) of its vertices: we say that \(a \prec b\) for two different elements of \(M\) if and only if the (only) path from the root to \(b\) passes through \(a\). □

**Definition 2.2** For a linearly ordered set \((M, \prec)\) a tree, whose vertices are labelled one-to-one by the elements of \(M\), is called alternated if for any its edge \((a, b)\) and for any element \(x \in M\) such that \(a \prec x\), \(b \prec x\), one always has \(a < x < b\) or \(b < x < a\). □
Note that all alternated trees contain the edge joining the extremal vertices.
One can give an equivalent recursive definition:

**Definition 2.3** A tree, whose vertices are linearly ordered, is called *alternated* if it is a trivial tree with only one vertex or it contains the edge joining the extremal vertices and when one removes this edge the remaining two disconnected trees are alternated. □

![Figure 7: an alternated tree](image)

**Lemma 2.4** For any linearly ordered set $M$ of cardinality $\#M$ there are exactly $(\#M - 1)!$ alternated trees on it, and they form a basis in the $\mathbb{Z}$-modules $T_M^-, T_M^+$. □

### 2.2 Alternated $T_\ast$-diagrams

**Definition 2.5** A $T_\ast$-diagram is called *alternated* if all its trees are alternated, where the order on the minimal components is induced by the order on the line $\mathbb{R}^1$. □

### 3 Complexes of $T$-diagrams and $T_0$-diagrams

**Definition 3.1** A $T_\ast$-diagram is called *$T$-diagram* if it does not contain asterisks. □

**Definition 3.2** A $T$-diagram is called *$T_0$-diagram* if it does not have edges joining two neighbor points. □

The subspace spanned by $T$-diagrams forms a quotient-complex of complex $CT_\ast D$. We will call it *complex of $T$-diagrams* or simply $CTD$:

$$CT_\ast D \longrightarrow CTD.$$ 

We quotient by the subspace spanned by the diagrams having asterisks. Alternated $T$-diagrams form a basis in this complex.

**Proposition 3.3** The space spanned by $T_0$-diagrams forms a subcomplex of $CT_\ast D$ (and therefore of $CTD$). This inclusion is a quasi-isomorphism of complexes. The set of alternated $T_0$-diagrams forms a basis in $CT_0D$. □

**Proof:** It is not evident from Definition 3.2 that the space of $T_0$-diagrams forms a subcomplex of $CT_\ast D$. It will be clear from the proof that this space is actually the maximal subspace of $CT_\ast D$ that lies in the space of $T$-diagrams and is invariant with respect to the differential.

Consider the spectral sequence associated to the filtration in $CT_\ast D$ by the number of minimal components. The degree zero differential $d_0$ of this spectral sequence is the sum of
admissible gluings of the second type (gluing of points from the same minimal component), see Section 1.2. If we examine the differential \( d_0 \) in the basis of alternated \( T_* \)-diagrams, we obtain that the first term of this spectral sequence is concentrated on the only row and is spanned by the alternated \( T_0 \)-diagrams. This proves the proposition.

One gets the following commutative diagram of complexes:

\[
\begin{array}{ccc}
CT_0 D & \sim & CT_* D \\
\downarrow & & \downarrow \\
CT D & & CT D
\end{array}
\]  

(3.1)

The upper arrow is a quasi-isomorphism.

Note that precisely this proposition shows that the first term of Vassiliev spectral sequence is bounded by the upper line \( q = -(d-1)p - 1 \). Really, \( p = -i \), \( q = di - j \). For a fixed \( i \) the minimal \( j \) equals \( i + 1 \) in \( CT_0 D \) (note, in \( CT_* D \) the minimal \( j \) is \( \lceil \frac{i+1}{2} \rceil \)). Therefore the maximal \( q \) is \( di - (i + 1) = (d-1)i - 1 = -(d-1)p - 1 \).

The complexes \( CT_* D, CT D, CT_0 D \) are direct sums over \( i \) of finite complexes spanned by diagrams of complexity \( i \) — the grading \( j \) is bounded: \( i + 1 \leq j \leq 2i \) for \( CT_* D \) and \( CT_0 D \); \( \lceil \frac{i+1}{2} \rceil \leq j \leq 2i \) for \( CT_* D \). Let \( \chi_i, \chi_*^i, \chi_0^i \) denote Euler characteristics of the complexity \( i \) components of these complexes. Consider the generating functions:

\[
\chi(t) = \sum_{i=0}^{+\infty} \chi_i t^i, \quad \chi_*(t) = \sum_{i=0}^{+\infty} \chi_*^i t^i, \quad \chi_0(t) = \sum_{i=0}^{+\infty} \chi_0^i t^i.
\]

Proposition 3.4

\[
\chi(t) = \frac{1}{1-t}, \quad \chi_*(t) = \chi_0(t) = \frac{1}{1-t^2}. \quad \square
\]

For complexity \( i = 0 \) we count the diagram without any point — the trivial diagram. This diagram is the lost unity (via the Alexander duality) of the cohomology algebra.

In Appendix A we describe the bases of these complexes for small complexities \( i \).

4 Differential Hopf algebra structure on the complexes \( CT_* D, CT D, CT_0 D \)

Let us define multiplication and comultiplication on complex \( CT_* D \) that together with the differential will define differential Hopf algebra structure on it. The complexes \( CT D, CT_0 D \) will inherit this structure from \( CT_* D \). These differential Hopf algebras will be called differential Hopf algebras of \( T_* \)-diagrams, \( T \)-diagrams and \( T_0 \)-diagrams or simply \( DHAT D, DHAT D, DHAT_0 D \). The morphisms \( 3.1 \) will respect this structure:
Remark 4.1 Multiplication, comultiplication and differential define only differential bialgebra structure, but our bialgebras are connected, therefore the antipode always exists. The formula for the antipode is given in [30]. □

In the sequel we will not make difference for the terms "bialgebra" and "Hopf algebra".

4.1 Multiplication

The product of two $T_\ast$-diagrams $D_1$ and $D_2$ is defined as the shuffle of their points on the line. Orienting monomial for each summand is the product of the orienting monomial of $D_1$ and that of $D_2$.

Example 4.2 Let $d$ be odd. Consider the product of the diagrams $\begin{array}{c} \beta_1\beta_2\alpha_12, \\ \beta_1\beta_3\alpha_13\beta_2\gamma_2^* \end{array}$ and $\begin{array}{c} \beta_1\beta_2\alpha_12, \\ \beta_1\beta_3\alpha_13\beta_2\gamma_2^* \end{array}$. Their shuffle product is:

$\begin{align*}
\beta_1\beta_2\alpha_12\beta_3\gamma_3^* + \\
\beta_1\beta_3\alpha_13\beta_2\gamma_2^* + \\
\beta_2\beta_3\alpha_23\beta_1\gamma_1^*
\end{align*}$

Note that the trivial diagram “$\begin{array}{c} \beta_1\beta_3\alpha_13\beta_2\gamma_2^* \end{array}$” is the unity for this algebra.

4.2 Comultiplication

Comultiplication for these complexes is the map dual to concatenation (one can call it coconcatenation). Consider a $T_\ast$-diagram $D$. We say that a point $t \in \mathbb{R}_1$ is separating for $D$ if

1) it is not a vertex of $D$;
2) $D$ does not have edges whose left vertex belongs to $(-\infty, t)$ and the right one — to $(t, +\infty)$.

Two separating points are said to be equivalent if there is no vertices of $D$ between them.

Any separating point split $D$ into two diagrams: the left subdiagram $L_t(D)$ that is on the left from $t$; and the right subdiagram $R_t(D)$ that is on the right from $t$. The orienting monomials of $L_t(D)$ (resp. $R_t(D)$) are obtained from the orienting monomial of $D$ by removing the orienting elements corresponding to the right (resp. left) subdiagram.

We define the coproduct on $D$ as follows:

$$\Delta D = \sum_{s \in S(D)} (-1)^{\varepsilon_s} L_s(D) \otimes R_s(D),$$

where $S(D)$ is the set of equivalence classes of separating points for $D$. By abuse of the language $L_s(D), R_s(D)$ denote $L_t(D), R_t(D)$ for any point $t \in s$. The sign $(-1)^{\varepsilon_s}$ is obtained
as follows: Consider the product of the orienting monomial of \( L_s(D) \) to the orienting monomial of \( R_s(D) \). This product differs from the orienting monomial of \( D \) by some permutation of the orienting elements. \((-1)^{\varepsilon_s}\) is the sign of this permutation.

For example let 1 denote the trivial diagram “_____” , then \( \Delta 1 = 1 \otimes 1 \). For all the other diagrams coproduct \( \Delta D \) has at least two summands \( D \otimes 1 \) and \( 1 \otimes D \).

**Remark 4.3** The multiplication is graded commutative, but the comultiplication is not graded cocommutative. In [30] it was proven that the comultiplication is graded cocommutative on the homology level for any field of coefficients for \( DHAT_* D_{odd} \) and \( DHAT_* D_{even} \). In the case of \( DHAT_* D_{odd} \), it was proven that this is so over \( \mathbb{Q} \). Theorem 14.4 implies that the homology bialgebra of \( DHAT_* D_{odd} \) is graded bicommutative for any field of coefficients.

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**Part II**

**Complexes dual to** \( CT\, D, \, CT_* D_{even}, \, CT_0 D \). **Hochschild complexes**

In this part we describe the complexes dual to \( CT D_{odd}, \, CT D_{even}, \, CT_* D_{even} \) that are the normalized Hochschild complexes of the Poisson algebras operad, resp. of the Gerstenhaber algebras operad, resp of the Batalin-Vilkovisky algebras operad. The dual of complex \( CT_* D_{odd} \) can not be described in terms of the Hochschild complex of none operad. Its description is given in [30] [31]. We describe also the duals of \( CT_0 D_{odd}, \, CT_0 D_{even} \).

We do not prove that the constructed complexes are dual to \( CT D, \, CT_* D_{even}, \, CT_0 D \). However in the first section we give a hint why this is so — namely we describe the spaces \( B_M^+, \, B_M^- \) dual to \( T_M^+, \, T_M^- \); and we show how this duality is organized.

**5** **Z-modules** \( B_M^+, \, B_M^- \)

First thing to do if we want to describe the dual complexes is to describe the Z-modules dual to \( T_M^+, \, T_M^- \).

Let us define the dual of \( T_M^+ \). Consider the Z-module spanned by the monomials with commuting generators \( \alpha_{ab} = -\alpha_{ba}, \, a, b \in M, \, a \neq b \), that are assigned to trees with vertices labelled one-to-one by \( M \) (but not quotiented by 3-term relations (1.1)), see Section 1.1. This Z-module is self-dual: pairing of two monomials assigned to different trees is zero; pairing of two monomials corresponding to the same tree is equal to 1, if they define the same orientation of the tree, and to \(-1\) if they define opposite orientation. The module \( T_M^+ \) is a quotient-space of the described space. Hence the dual of \( T_M^+ \) is a subspace of that space determined by 3-term equations. This dual is the orthogonal to the space spanned by left-hand sides of Figure 3.

The dual of \( T_M^- \) is defined in the same way, except that we take anticommuting generators \( \alpha_{ab} = \alpha_{ba}, \, a, b \in M, \, a \neq b \).

This definition of the duals of \( T_M^+, \, T_M^- \) is not very easy to manipulate. Fortunately, these duals have another and much simpler description.
Let $\text{Lie}^+(M)$ (resp. $\text{Lie}^-(M)$) be a usual free Lie algebra (resp. free Lie super-algebra with odd bracket) with generators (resp. with even generators) $x_a$, $a \in M$. We define $B_M^+$ (resp. $B_M^-$) as its subspace linearly spanned by the brackets containing each generator exactly once.

**Example 5.1** $B_{(1)}^+$, $B_{(1)}^-$ are spanned by the only element $x_1$.

$B_{(1,2)}^+$, $B_{(1,2)}^-$ are spanned by $[x_1, x_2], [x_2, x_1]$ that are equal up to a sign.

$B_{(1,2,3)}^+$, $B_{(1,2,3)}^-$ are spanned by $[[x_1, x_2], x_3], [[[x_1, x_3], x_2], [x_2, [x_1, x_3]]], \text{ etc.}$

In the case $M = \{1, 2, \ldots, n\}$ we will write $B_n^+$, $B_n^-$ instead of $B_M^+$, $B_M^-$.  

**Remark 5.2** $B_n^+$ is the $n$-th component $\mathcal{LIE}(n)$ of the Lie algebras operad $\mathcal{LIE}$. It is well known that $\mathcal{LIE}(n)$ is isomorphic to $\mathbb{Z}^{[n-1]}$. $B_n^-$ is the $n$-th component of the operad of Lie super-algebras with odd bracket. □

**Lemma 5.3** There is a natural isomorphism of $S_n$-modules $B_n^+ \simeq B_n^- \otimes (\pm \mathbb{Z})$. □

**Statement 5.4** The space $B_M^+(n)$ (resp. $B_M^-(n)$) is the dual $\mathbb{Z}$-module of $T_M^+$ (resp. $T_M^-$). In the case $M = \{1, 2, \ldots, n\}$ this duality is $S_n$-equivariant. □

**Sketch of the proof:** Let us define a map $\Psi$ from $B_M^+$ (resp. $B_M^-$) to the described dual of $T_M^+$ (resp. $T_M^-$). We will define $\Psi$ inductively. If $M = \{a\}$ (cardinality of $M$ equals 1) we set $\Psi(x_a)$ to be the only trivial tree with one vertex $a$. In terms of monomials this is simply 1. Let cardinality $\#M$ be $\geq 2$. Consider any bracket $L \in B_M^+$ (resp. $L \in B_M^-$). Obviously, $L = [A, B]$, where $A$, $B$ are some subbrackets. The set $M$ is split into two subsets $M_A$ and $M_B$, where $M_A$, $M_B$ are elements of $M$ corresponding to generators in $A$, resp. in $B$. We define $\Psi(L) = \Psi([A, B])$ by the following formula:

$$\Psi([A, B]) = \Psi(A) \left( \sum_{a \in M_A, b \in M_B} \alpha_{ab} \right) \Psi(B). \quad (5.1)$$

It can be easily verified that

1) $\Psi$ respects (super)anticommutativity and (super)-Jacobi identities;
2) The image of $\Psi$ belongs to the dual of $T_M^+$ (resp. $T_M^-$).

To see that $\Psi$ is bijective one needs to find a basis in $B_M^+$, $B_M^-$ that is dual to the basis of alternated trees.

**Definition 5.5** Let $M$ be an ordered set. A bracket in $B_M^+$ or $B_M^-$ is said to be monotone if for any its subbracket (including itself) the generator with the minimal index occupies the left-most position and the generator with the maximal index occupies the right-most position. □

For example $[[x_1, x_3], [x_2, x_4]]$ is a monotone bracket.

$\Psi$ of any bracket is a sum of trees. For any monotone bracket in this sum there is only one alternated tree. For instance, for the bracket $[[x_1, x_3], [x_2, x_4]]$ this tree is

Hence the images of monotone brackets form a basis in the spaces dual to $T_M^+$, $T_M^-$, and therefore the monotone brackets form a basis in $B_M^+$, $B_M^-$. Of course, this basis is dual to the basis of alternated trees in $T_M^+$, $T_M^-$. □
6 Lie, pre-Lie and brace algebra structures on the space of any graded linear operad

In this and the next sections we briefly remind algebraic structures that exist on a graded linear operad, cf. [13]. In Section 8 we apply these structures to describe the complexes dual to $CTD$, $CT$, $D^{even}$. For people not familiarized with the notion of operad we recommend to read [31] instead of reading these three sections.

Let $O = \{O(n), n \geq 0\}$ be a graded linear operad. By abuse of the language the space $\bigoplus_{n \geq 0} O(n)$ will be also denoted by $O$. A tilde over an element will always designate its grading. For any element $x \in O(n)$ we put $n_x := n - 1$. The numbers $n$ and 1 here correspond to $n$ inputs and to 1 output respectively.

Define a new grading $|.|$ on the space $O$. For an element $x \in O(n)$ we put $|x| := \tilde{x} + n_x = \tilde{x} + n - 1$. It turns out that $O$ is a graded Lie algebra with respect to the grading $|.|$. Note that the composition operations respect this grading.

Define the following collection of multilinear operations on the space $O$.

$$x\{x_1, \ldots, x_n\} := \sum (-1)^\epsilon x(id, \ldots, id, x_1, \ldots, id, x_n, id, \ldots, id)$$

(6.1)

for $x, x_1, \ldots, x_n \in O$, where the summation runs over all possible substitutions of $x_1, \ldots, x_n$ into $x$ in the prescribed order, $\epsilon := \sum_{p=1}^n n_x r_p + n_x \sum_{q=1}^n \tilde{x}_p + \sum_{p < q} n_x \tilde{x}_q$, $r_p$ being the total number of inputs in $x$ going after $x_p$. For instance, for $x \in O(2)$ and arbitrary $x_1, x_2 \in O$

$$x\{x_1, x_2\} = (-1)^{n_{\tilde{x}_1} + (\tilde{x}_1 + \tilde{x}_2) + n_{\tilde{x}_2}} x(x_1, x_2).$$

(6.2)

By convention:

$$x\{\} := x.$$  

(6.3)

One can check immediately the following identities:

$$x\{x_1, \ldots, x_m\}\{y_1, \ldots, y_n\} =$$

$$\sum_{0 \leq i_1 \leq j_1 \leq \cdots \leq i_m \leq j_m \leq n} (-1)^\epsilon x\{y_1, \ldots, y_{i_1}, x_1\{y_{i_1+1}, \ldots, y_{j_1}\}, y_{j_1+1}, \ldots, y_{i_m}\},$$

$$x_m\{y_{i_m+1}, \ldots, y_{j_m}\}, y_{j_m+1}, \ldots, y_{n}\},$$

(6.3)

where $\epsilon = \sum_{p=1}^m (|x_p| \sum_{q=1}^{i_p} |y_q|)$.

**Definition 6.1**[13] A brace algebra is a graded linear space endowed with a set of multilinear $(n+1)$-ary operations $b_n(x, x_1, \ldots, x_n) = x\{x_1, \ldots, x_n\}$, $n = 0, 1, 2, \ldots$, respecting the grading $|.|$ and satisfying (6.2), (6.3). □

Define a bilinear operation (respecting the grading $|.|$) $\circ$ on the space $O$:

$$x \circ y := x\{y\},$$

(6.4)

for $x, y \in O$. This operation is not associative.
**Definition 6.2** A graded vector space $A$ with a bilinear operation $\circ : A \otimes A \to A$ is called a *pre-Lie algebra*, if for any $x, y, z \in A$ the following holds:

$$(x \circ y) \circ z - x \circ (y \circ z) = (-1)^{|y||z|}((x \circ z) \circ y - x \circ (z \circ y)).$$

□

Any graded pre-Lie algebra $A$ can be considered as a graded Lie algebra with the bracket

$$[x, y] := x \circ y - (-1)^{|x||y|} y \circ x.$$  (6.5)

The description of the operad of pre-Lie algebras is given in [9].

The following lemma is a corollary of the identity (6.3) applied to the case $m = n = 1$.

**Lemma 6.3** The operation (6.4) defines a graded pre-Lie algebra structure on the space $O$. □

In particular this implies that any graded linear operad $O$ can be considered as a graded Lie algebra with the bracket (6.5). This bracket is usually called Gerstenhaber bracket in honor of Murray Gerstenhaber who discovered this operation for the Hochschild cochain complex of an associative algebra, see [12] and also next section.

## 7 Hochschild complexes

Let $O = \bigoplus_{n \geq 0} O(n)$ be a graded linear operad equipped with a morphism $\Pi : \mathcal{ASSOC} \to O$

from the operad $\mathcal{ASSOC}$. This morphism defines the element $m = \Pi(m_2) \in O(2)$, where the element $m_2 = x_1 x_2 \in \mathcal{ASSOC}(2)$ is the operation of multiplication. Note that the elements $m_2$, $m$ are odd with respect to the new grading $|.|$ and $|m| = |m_2| = 1$ and $|m, m| = |\Pi(m_2), \Pi(m_2)| = 2|\Pi(m_2 \circ m_2) - 0$. (One has $m_2 \circ m_2 = -(x_1 \cdot x_2 \cdot x_3 + x_1 \cdot (x_2 \cdot x_3) = -x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_3 = 0.$) Thus $O$ becomes a differential graded Lie algebra with the differential $\partial$:

$$\partial x := [m, x] = m \circ x - (-1)^{|x||m|} x \circ m,$$  (7.1)

for $x \in O$.

We will call complex $(O, \partial)$ *Hochschild complex* of operad $O$. Actually a better name would be *Hochschild complex of the morphism* $\Pi : \mathcal{ASSOC} \to O$ since this complex is in fact the deformation complex of the morphism $\Pi$, see [17] [18]. We preferred the first name for its shortness.

**Example 7.1** If $O$ is the endomorphism operad $\mathcal{END}(A)$ of a vector space $A$, and we have a morphism $\Pi : \mathcal{ASSOC} \to \mathcal{END}(A)$,

that defines an associative algebra structure on $A$, then the corresponding complex $\left(\bigoplus_{n=0}^{\infty} \text{Hom}(A^\otimes A, A), \partial\right)$ is the usual Hochschild cochain complex $C^\ast(A, A)$ of an associative algebra $A$. □
Define another grading
\[ \text{deg} := |. | + 1 \] (7.2)
on the space \( \mathcal{O} \). With respect to this grading the bracket \([., .]\) is homogeneous of degree \(-1\).

It is easy to see that the product \(*\), defined as follows
\[ x * y := (-1)^{|x|} m(x, y) = (-1)^{(\text{deg}(x)+1)} m(x, y), \] (7.3)
for \( x, y \in \mathcal{O} \), together with the differential \( \partial \) defines a differential graded associative algebra structure on \( \mathcal{O} \) with respect to the grading (7.2).

**Theorem 7.2** [12, 13] The multiplication \(*\) and the bracket \([., .]\) induce a Gerstenhaber algebra structure on the homology of the Hochschild complex \((\mathcal{O}, \partial)\). □

For a definition of Gerstenhaber algebras we refer to the next section.

**Proof:** The proof is deduced from the following homotopy formulas.
\[ x * y - (-1)^{\text{deg}(x)\text{deg}(y)} y * x = (-1)^{\text{deg}(x)} (\partial(x \circ y) - \partial x \circ y - (-1)^{\text{deg}(x)-1} x \circ \partial y). \] (7.4)
The above formula proves the graded commutativity of the multiplication \(*\).
\[ [x, y * z] - [x, y] * z - (-1)^{\text{deg}(x)-1}\text{deg}(y)} y * [x, z] = \]
\[ = (-1)^{|x|+|y|}(\partial(x\{y, z\}) - \partial x\{y, z\} - (-1)^{|x|} x(\partial y, z) - (-1)^{|x|+1} y\{x, \partial z\}). \] (7.5)
This formula proves the compatibility of the bracket with the multiplication. □

### 8 Dual complexes

**Definition 8.1** A graded commutative algebra is called *Poisson algebra* if it is endowed with a Lie bracket respecting the grading and compatible with the multiplication:
\[ [x, yz] = [x, y]z + \epsilon (\text{deg}(x) = -1) y[x, z]. \] □
(8.1)

**Definition 8.2** A graded commutative algebra is called *Gerstenhaber algebra* if it is endowed with a Lie bracket of degree \(-1\) and compatible with the multiplication:
\[ [x, yz] = [x, y]z + (-1)^{\text{deg}(y)+1} y[x, z]. \] □
(8.2)

**Definition 8.3** A Gerstenhaber algebra is called *Batalin-Vilkovisky algebra* if it is endowed with an unary linear operation \( \delta \) of degree \(-1\) satisfying:
(i) \( \delta^2 = 0 \);
(ii) \( \delta(ab) = \delta(a)b + (-1)^{\text{deg}(a)}a\delta(b) + (-1)^{\text{deg}(a)}[a, b]. \) □

Note that (i) and (ii) imply
(iii) \( \delta([a, b]) = \delta(a, b) + (-1)^{\text{deg}[a, b]}[a, \delta(b)]. \)

Denote by \( \mathcal{POISS}, \mathcal{GERST}, \mathcal{BV} \) the operads of Poisson, Gerstenhaber and Batalin-Vilkovisky algebras.
Let us describe these operads.

The space \( \mathcal{P} \) of all multilinear \( n \)-ary operations that are induced by a Poisson algebra structure is described as follows. Consider a free Lie algebra \( \mathfrak{L}(x_1, \ldots, x_n) \) with \( n \) generators \( x_1, \ldots, x_n \) and consider the symmetric algebra \( S^n \mathfrak{L}(x_1, \ldots, x_n) \). Any symmetric algebra of a Lie algebra is endowed with a Poisson algebra structure. This one is a free Poisson algebra. The space \( \mathcal{P} \) is a subspace of this algebra linearly spanned by the products of brackets using each generator exactly once. For instance, if \( n = 3 \) we have the following elements: \( x_1 \cdot x_2 \cdot x_3, x_1 \cdot [x_2, x_3], [[x_3, x_1], x_2], \) etc. The space \( \mathcal{P} \) is bigraded: the first bigrading \textit{complexity} \( i \) is the total number of comas in the products of brackets; the second grading \( j = n \) is the component number.

The space \( \mathcal{G} \) is defined analogously. The only difference is that we need to consider a free graded Lie algebra \( \mathfrak{L}_1(x_1, \ldots, x_n) \) with the bracket of degree \(-1\) but also with the generators \( x_1, \ldots, x_n \) of degree zero. The space \( \mathcal{G} \) is also bigraded by complexity and by component number.

To define \( \mathcal{B} \) we need to start from a free graded Lie algebra \( \mathfrak{L}_1(x_1, \ldots, x_n, \delta x_1, \ldots, \delta x_n) \) with the bracket of degree \(-1\) and with the generators \( x_1, \ldots, x_n \) of degree \( 0 \) and \( \delta x_1, \ldots, \delta x_n \) of degree \(-1\). Then we consider the symmetric algebra \( S^n \mathfrak{L}_1(x_1, \ldots, x_n, \delta x_1, \ldots, \delta x_n) \) and take its subspace linearly spanned by the products of brackets using each index \( 1, \ldots, n \) exactly once. For instance, if \( n = 3 \) we have the elements \( x_1 \cdot \delta x_2 \cdot x_3, x_1 \cdot [x_2, \delta x_3], [[\delta x_3, x_1], \delta x_2], \) etc. The grading complexity is the total number of comas and deltas. The second grading is as usual the component number.

Any Poisson (resp. Gerstenhaber, resp. Batalin-Vilkovisky) algebra is a commutative algebra and therefore is an associative algebra. It means that the operads \( \mathcal{P}, \mathcal{G}, \mathcal{B} \) are endowed with a map from the operad \( \mathcal{J} \) of associative algebras. By the previous subsection they form Hochschild complexes that we will denote by \( (\mathcal{P}, \partial), (\mathcal{G}, \partial), (\mathcal{B}, \partial) \). These are not yet the duals of \( CT_{\text{odd}}, CT_{\text{even}}, \mathcal{C} \). There are two types of products of brackets that span \( \mathcal{P}, \mathcal{G}, \mathcal{B} \) or \( CT_{\text{odd}}, CT_{\text{even}}, \mathcal{C} \): in the first group we put those products that have at least one factor \( x_i \) for some \( i \); in the second group — all the others. The point is that both groups span a subcomplex (and therefore each of the complexes \( (\mathcal{P}, \partial), (\mathcal{G}, \partial), (\mathcal{B}, \partial) \) is a direct sum of two complexes). It is easy to show that the first subcomplex is always acyclic and even contractible. This implies that the second one is quasi-isomorphic to \( (\mathcal{P}, \partial), (\mathcal{G}, \partial), (\mathcal{B}, \partial) \) respectively. The second subcomplex will be called \textit{normalized Hochschild complex} and denoted by \( (\mathcal{P}_{\text{Norm}}, \partial), (\mathcal{G}_{\text{Norm}}, \partial), (\mathcal{B}_{\text{Norm}}, \partial) \).

**Theorem 8.4** The complexes \( (\mathcal{P}_{\text{Norm}}, \partial), (\mathcal{G}_{\text{Norm}}, \partial), (\mathcal{B}_{\text{Norm}}, \partial) \) are dual to \( CT_{\text{odd}}, CT_{\text{even}}, \mathcal{C} \). □

**Idea of the proof:** It is a direct check. To prove one needs to use the duality described in Secion 5. □

These complexes are differential Hopf algebras (since they are dual to differential Hopf algebras), they also inherit pre-Lie, Lie and brace algebra structures.

To define the complexes dual to \( CT_{\text{odd}}, CT_{\text{even}} \) one needs to quotient the complexes \( (\mathcal{P}_{\text{Norm}}, \partial), (\mathcal{G}_{\text{Norm}}, \partial), (\mathcal{B}_{\text{Norm}}, \partial) \) by the \textit{neighbor commutativity relations}. Namely, instead of taking free (graded) Lie algebras \( \mathfrak{L}(x_1, \ldots, x_n) \) or \( \mathfrak{L}_1(x_1, \ldots, x_n) \) one should
take Lie algebras with the same generators but satisfying the relations \([x_i, x_{i+1}] = 0\), for \(i = 1, \ldots, n - 1\). And then we proceed as before. We denote these complexes by \((\text{POISS}^{\text{zero}}, \partial), (\text{GERST}^{\text{zero}}, \partial)\). These complexes are as before differential Hopf algebras, but they have no more pre-Lie, Lie or brace algebra structure.

9 Bases of monotone bracket diagrams

In Appendix A we describe bases of the complexes \((\text{POISS}^{\text{Norm}}, \partial), (\text{GERST}^{\text{Norm}}, \partial), (\text{BV}^{\text{Norm}}, \partial), (\text{POISS}^{\text{zero}}, \partial), (\text{GERST}^{\text{zero}}, \partial)\) for small complexities \(i\). This provides an illustration of the given below considerations.

Products of brackets that span the complexes \((\text{POISS}^{\text{Norm}}, \partial), (\text{GERST}^{\text{Norm}}, \partial)\) will be called **bracket diagrams**. If none of the factors of a bracket diagram contains subbracket \([x_i, x_{i+1}]\) for some \(i\), then this bracket will be called **bracket zero-diagram**. For example, \([x_1, x_3] \cdot [x_2, x_4, x_5]\) is not bracket zero-diagram because it contains subbracket \([x_4, x_5]\).

If all the factors of a bracket diagram are monotone brackets, see Section 5, then this diagram is called **monotone bracket diagram**. Monotone bracket diagrams (resp. monotone bracket zero-diagrams) form bases in the complexes \((\text{POISS}^{\text{Norm}}, \partial), (\text{GERST}^{\text{Norm}}, \partial)\) (resp. \((\text{POISS}^{\text{zero}}, \partial), (\text{GERST}^{\text{zero}}, \partial)\)) that are dual to the bases of alternated \(T\)-diagrams (resp. \(T_0\)-diagrams).

Let us make a few remarks about how these bases behave with respect to the algebraic operations in the corresponding complexes.

Note first that the bases of alternated \(T/T_*/T_0\)-diagrams are invariant with respect to multiplication and comultiplication in the sense that all the summands in the shuffle product and in the formula (4.2) of comultiplication will be basis elements (if we apply multiplication or comultiplication to basis elements). It means that we do not need to use basis decomposition to compute these operations. On the contrary we do need to use basis decomposition to compute the differential of these complexes.

We have the same for the dual complexes: multiplication and comultiplication are invariant with respect to the bases of monotone bracket (zero)-diagrams, but the differential is not.

Neither pre-Lie product, nor brace algebra operations respect the basis. However in one case these operations \(x \circ y, x\{x_1, \ldots, x_n\}\) do respect the basis. Namely, if \(y\), resp. \(x_1, \ldots, x_n\) consist of only one minimal component — only one bracket. Really, in this case all the summands of \((6.1)\) will be also monotone bracket diagrams. We will use this fact in Sections 11 and 13.
Part III

Upper diagonal of the Vassiliev spectral sequence and of the related complexes

In Section 10 we prove that the upper diagonal homology groups of complex $CT_0D$ are trivial (Theorem 10.1). Theorem 10.2 describes the upper diagonal homology groups of complexes $CTD_{\text{odd}}$, $CTD_{\text{even}}$. This theorem is not completely proven in this section.

In Section 11 we introduce new homological operations on Hochschild complexes. These operations are defined in finite characteristic. It will be proven elsewhere that they are related to Dyer-Lashof operations (see Theorem E.3). By means of these operations we find cycles in the dual complexes and this completes the proof of Theorem 10.2.

In Section 12 we prove some composition formulas for the operations defined in previous section.

By Theorem 10.2 any upper diagonal diagram of $CTD$ is a cycle homologous to standard diagram (10.1) with some coefficient. In Section 13 we use the duality of Section 5 and also the formulas for dual cycles (from Section 11) of dual complexes in order to determine the above coefficients.

In Section 14 we describe the relation between homology bialgebra of $CTD$ and that of $CT_0D$. Main Theorem 14.4 is given without proof.

10 Upper diagonal of the Vassiliev spectral sequence

We have seen that for a fixed complexity $i$ the minimal $j$ equals $i + 1$: in this case the diagrams have only one minimal component. The homology groups of the complex $CT_0D$ in the bigradings $(i, i + 1)$ are the upper diagonal elements of the first term of the Vassiliev spectral sequence.

**Theorem 10.1** For all $i \in \mathbb{N}$ the homology groups in the bigradings $(i, i + 1)$ are trivial for complex $CT_0D$, and are cyclic for complex $CTD$: if the corresponding group is untrivial, as a generator one can take the one diagram cycle:

![Diagram](image)

(10.1)

**Proof:** We will consider both complexes $CT_0D$ and $CTD$ simultaneously.

Since there is no diagrams of complexity $i$ with less then $i + 1$ points, any diagram of bigrading $(i, i + 1)$ always defines a cycle. We will prove that any such cycle is homologous to diagram (10.1) with some coefficient\(^1\). But diagram (10.1) does not belong to the space of $T_0$-diagrams, therefore this coefficient must be zero if we treat complex $CT_0D$.

\(^1\)In Section 12 we determine these coefficients.
We will say that an alternated $T$-diagram (resp. $T_0$-diagram) is of $r$-type, where $2 \leq r \leq i + 1$, if it has the edges $(1, r), (1, r + 1), \ldots, (1, i + 1)$ and for the points $r + 1, r + 2, \ldots, i + 1$ there is no any other incident edges.

Figure 8: diagram $D$ of 4-type. $i = 5$.

Any alternated diagram of bigrading $(i, i + 1)$ is of $(i + 1)$-type. The diagram (10.1) is the only one of 2-type.

Let us prove that any alternated diagram $D$ of $r$-type ($r \geq 3$) is homologous to a sum of alternated diagrams of $(r - 1)$-type. Consider a diagram $D_\star$ of bigrading $(i, i + 2)$ that is obtained from $D$ by splitting the point $r$ into two points: left one $r_-$ and right one $r_+$. The edges that were not incident to $r$ stay without any change. The edge $(1, r)$ becomes $(1, r_+)$; all the other edges incident to $r$ become incident to $r_-$ (if there were no such edges the diagram $D$ was already of type $r - 1$), see the example below — diagram $D_\star$ is obtained from the one of the Figure 8.

Figure 9: diagram $D_\star$

The diagram $D_\star$ is also alternated $T$-diagram (resp. $T_0$-diagram). Consider the differential of this diagram $\partial D_\star$, see Figure 10.

Figure 10: $\partial D_\star$

The last admissible gluing — gluing of the points $r_-$ and $r_+$ — gives the diagram $D$. All the other boundary diagrams have the edges $(1, r_+), (1, r + 1), (1, r + 2), \ldots, (1, i + 1)$ and they are the only incident for the points $r_+, r + 1, r + 2, \ldots, i + 1$ respectively. For any of these boundary diagrams its subdiagram on the first $r$ points is a tree. It means that the decomposition of the sum of these diagrams in our basis is a sum of alternated diagrams of type $r - 1$. This completes the proof of the theorem. □
**Theorem 10.2** For complex $CTD^{even}$ the homology groups of the bigrading $(i, i+1)$ are

$$
\begin{cases}
    \mathbb{Z}, & i = 1; \\
    \mathbb{Z}_p, & i = p^k, \text{ where } p \text{ is any prime, } k \in \mathbb{N}; \\
    0, & \text{otherwise}.
\end{cases}
$$

(10.2)

For complex $CTD^{odd}$ these groups are

$$
\begin{cases}
    \mathbb{Z}, & i = 1, 2; \\
    \mathbb{Z}_p, & i = 2p^k, \text{ where } p \text{ is any prime, } k \in \mathbb{N}; \\
    0, & \text{otherwise}.
\end{cases}
$$

(10.3)

**First part of the proof:** In this section we will prove that the homology groups are some quotient-groups of the described groups. To finish the proof we will present the dual cycles in the dual complexes tensored to the corresponding cyclic group (10.2), (10.3), see Section 11.

**Definition 10.3** A divided product $\langle D_1, D_2 \rangle$ of two diagrams $D_1$ and $D_2$ is the sum of those elements in the shuffle product $D_1 \ast D_2$ that have the left-most point of $D_1$ on the left from the left-most point of $D_2$. □

Note $D_1 \ast D_2 = \langle D_1, D_2 \rangle + (-1)^{\deg D_1 \deg D_2} \langle D_2, D_1 \rangle$, where $\deg D_i$ is the number of orienting elements of $D_i$, or, what is the same modulo 2, the corresponding cohomology degree of the knot space, i.e. $p + q = i(d - 1) + j$.

![Figure 11: Example of a divided product](image)

Denote the diagram (10.1) by $Z_i$.

In the case of odd $d$ this diagram has $i + 1$ orienting elements, each of them is a representative of a point. We fix its orientation by the monomial:

$$\beta_1 \beta_2 \ldots \beta_{i+1}.$$  

We orient the edges all from left to right:

![Diagram](image)

(10.4)

In the case of even $d$ we fix an orientation of $Z_i$ by the orienting monomial:

$$\beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3 \ldots \alpha_1(i+1) \beta_{i+1}.$$
Consider divided product \( \langle Z_k, Z_l \rangle \), \( k, l \geq 1 \).

For even \( d \) one has:
\[
\partial \langle Z_k, Z_l \rangle = \binom{k + l}{k} Z_{k+l},
\]
(10.5)

For odd \( d \) one has
\[
\partial \langle Z_k, Z_l \rangle = (-1)^k \binom{k + l}{k} Z_{k+l},
\]
(10.6)

where
\[
\binom{k + l}{k}^{-1} = \begin{cases} 
0, & k \text{ and } l \text{ are odd;} \\
\left[ \frac{k + l}{2} \right], & \text{otherwise,}
\end{cases}
\]
see Appendix C. For instance,
\[
\partial \langle Z_1, Z_{2r} \rangle = -Z_{2r+1}.
\]

Really, all the gluings in \( \partial \langle Z_k, Z_l \rangle \) will be compensated with each other except the gluing of the left-most point of \( Z_k \) with the left-most point of \( Z_l \). Such gluing always gives \( Z_{k+l} \) (with some sign). This situation occurs \( \binom{k+l}{k} \) times since the points 2, 3, \ldots, \( k + 1 \) of \( Z_k \) can be arbitrarily shuffled with the points 2, 3, \ldots, \( l + 1 \) of \( Z_l \). In the case of even \( d \) all these gluings give the same orientation of \( Z_{k+l} \). In the case of odd \( d \) one needs to apply the combinatorics of shuffles of anticommuting elements, see Appendix C.

The following lemma completes the first part of the proof of the theorem.

**Lemma 10.4** The greatest common diviser of the numbers \( \binom{i}{1}, \binom{i}{2}, \dots, \binom{i}{i-1} \) (resp. \( \binom{i}{1}^{-1}, \frac{i}{2}, \binom{i}{2}^{-1}, \dots, \binom{i}{i-1}^{-1} \)) is equal to 0 if \( i = 1 \) (resp. \( i = 1, 2 \)), to \( p \) if \( i = p^k \) (resp. \( i = 2p^k \)), \( p \) being any prime, and to 1 in all the other cases. \( \square \)

### 11 End of the proof of Theorem 10.2

We will consider Hochschild complexes \( (\mathcal{O}, \partial) \), see Section 7, over some commutative ring \( k \). We are mostly interested in the cases when \( k \) is \( \mathbb{Z} \), \( \mathbb{Q} \) or any finite field \( \mathbb{Z}_p \).

Let \( \varphi \in \mathcal{O} \). We will define \( [n] \)-operation of \( \varphi \) as follows:
\[
\varphi^{[n]} : = \underbrace{(\ldots((\varphi \circ \varphi) \circ \varphi) \ldots) \circ \varphi}_{n \text{ times}} \quad (11.1)
\]

One has \( \varphi^{[1]} = \varphi \), \( \varphi^{[n+1]} = \varphi^{[n]} \circ \varphi \).

Remind that we defined two gradings \( \text{deg} \) and \( |.| \) on the complexes \( (\mathcal{O}, \partial) \), and \( \text{deg} = |.| + 1 \), see formula (7.2). In the cases \( \mathcal{O} \) is \( \text{POISS} \) or \( \text{GERST} \) the grading \( \text{deg} \) is the main grading: modulo 2 it equals to the corresponding homology degree of the knot spaces. In our formulas we will use both gradings. \( \text{deg}(\varphi^{[n]}) \equiv (n \cdot \text{deg}(\varphi) \pm n - 1) \mod 2; |\varphi^{[n]}| \equiv n \cdot |\varphi| \mod 2. \)

Theorem 10.2 will follow from the proposition:

**Proposition 11.1** Let \( \varphi \in \mathcal{O} \) and \( \partial \varphi = 0 \).

If \( \text{deg}(\varphi) \) is odd then
\[
\partial(\varphi^{[n]}) = -\sum_{i=1}^{n-1} \binom{n}{i} \varphi^{[i]} \ast \varphi^{[n-i]}.
\]
(11.2)
If \( \text{deg}(\varphi) \) is even then
\[
\partial(\varphi[n]) = \sum_{i=1}^{n-1} (-1)^{i-1} \binom{n}{i}^{-1} \varphi[i] \ast \varphi[n-i].
\]  
\[ (11.3) \]

\[ \square \]

**Proof of Proposition 11.1** The proof is based on two formulas. The first one is \[ (7.4) \]. It can be rewritten as:
\[
\partial(x \circ y) = \partial x \circ y - (-1)^{\text{deg}(x)-1} x \circ \partial y + (-1)^{\text{deg}(x)} (x \ast y - (-1)^{\text{deg}(x)} \text{deg}(y) y \ast x). \]  
\[ (11.4) \]

The second one is given by the following lemma:

**Lemma 11.2**
\[ (x \ast y) \circ z = x \ast (y \circ z) + (-1)^{\text{deg}(y) \cdot |z|} (x \circ z) \ast y. \]  
\[ (11.5) \]

**Proof of Lemma 11.2** Due to \[ (7.3) \] and \[ (6.3) \], one has
\[
(x \ast z) \circ y = (-1)^{|x|} m\{x, y\}\{z\} = (-1)^{|x|} m\{x, y\{z\}\} + (-1)^{|x| + |y| \cdot |z|} m\{x\{z\}, y\} \\
= x \ast (y \circ z) + (-1)^{|x| + |y| \cdot |z|} (x \circ z) \ast y = x \ast (y \circ z) + (-1)^{\text{deg}(y) \cdot |z|} (x \circ z) \ast y. \]  
\[ \square \]

The formulae \[ (11.4) \], \[ (11.5) \] together with the combinatorial formula
\[
\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1},
\]
and its super-analog (see Appendix C):
\[
\binom{n+1}{i}^{-1} = \binom{n}{i}^{-1} + (-1)^{n-i+1} \binom{n}{i-1}^{-1},
\]
prove Proposition 11.1. \[ \square \]

Proposition 11.1 and Lemma 10.4 imply that \([n] \)-operation is a homology operation in the following cases:

1) \( k = \mathbb{Z} \) or any other commutative ring: if \( \text{deg}(\varphi) \) is even and \( n = 2 \) — the resulting element has odd grading \( \text{deg} \).

2) \( k = \mathbb{Z}_2 \): for any grading \( \text{deg} \), \( n = 2^k \).

3) \( k = \mathbb{Z}_p \), \( p \) being any odd prime:
   - if \( \text{deg}(\varphi) \) is odd and \( n = p^k \) — the resulting element has odd grading \( \text{deg} \);
   - if \( \text{deg}(\varphi) \) is even and \( n = 2p^k \) — the resulting element has odd grading \( \text{deg} \).

**Remark 11.3** To show that some operation is a homology operation one should prove that the operation does not depend on a representative of a cycle: if we take a homologous element the result will be homologous. A priori it is not evident. In the article we do not prove this for the above operations. \[ \square \]
Now, let $O$ be $GERST$ or $POISS$ and $\varphi = [x_1, x_2]$. $\varphi$ defines a nontrivial cycle for any ring $k$ of coefficients since it is the only diagram of complexity $i = 1$. $deg(\varphi)$ is odd in the case of the operad $GERST$ and is even in the case of the operad $POISS$. Suppose we are in one of the situations where we want to prove non-triviality of the corresponding cyclic group, see the statement of Theorem $10.2$, and $k$ is the corresponding cyclic group. Obviously, $\varphi^{[i]}$ has the complexity $i$. On the other hand, basis decomposition of $\varphi^{[i]}$ has the monotone diagram $[\ldots, [x_1, x_2], x_3, \ldots, x_{i+1}]$ with coefficient $\pm 1$ (since operation $\circ$ respects the basis, see Section $9$). Therefore this element provides a non-trivial pairing with the diagram $10.1$ denoted by $Z_i$ and this proves a non-triviality of the corresponding homology group. Thus Theorem $10.2$ is proved. □

12 Composition of operations

In the previous section we defined $[n]$-operations that are homology operations in the following cases:

1) over $\mathbb{Z}_2$: $[2^k]$-operations for both even and odd elements. The resulting elements are always odd.

2) over $\mathbb{Z}_p$: $[p^k]$-operation for odd elements and $[2p^k]$-operation for even element. The resulting elements are also odd.

In this section we will prove two lemmas:

**Lemma 12.1** If $deg(\varphi)$ is odd, then

$$\varphi^{[pn]} \equiv (\varphi^{[p]})^{[n]} \mod p$$

for any prime $p$ and $n \in \mathbb{N}$. □

**Lemma 12.2** If $deg(\varphi)$ is even then

$$\varphi^{[2n]} = (\varphi^{[2]})^{[n]}, \quad (12.1)$$

for any $n \in \mathbb{N}$. □

Note that equality $(12.1)$ is over $\mathbb{Z}$.

These lemmas imply that

1) Over $\mathbb{Z}_2$: $[2^k]$-operation is $k$ times applied $[2]$-operation.

2) Over $\mathbb{Z}_p$: $[p^k]$-operation (on odd elements) is $k$ times applied $[p]$-operation; $[2p^k]$-operation (on even elements) is $[2]$-operation and then $k$ times applied $[p]$-operation.

This explains why we have powers of $p$. Note, the above assertions are true already for the chains.

**Proof of Lemma 12.1** Before giving a general proof let us see what is going on in the simplest case: $deg(\varphi)$ is odd, $p = 2$ and $n = 2$. We do the following computations over $\mathbb{Z}$ and then compare the results:

$$\varphi^{[4]} = ((\varphi \circ \varphi) \circ \varphi) \circ \varphi = \varphi(\varphi(\varphi(\varphi(\varphi(\varphi)))) = \varphi(\varphi(\varphi(\varphi(\varphi)))) + 2\varphi(\varphi(\varphi(\varphi)))) \{\varphi\}$$

$$= \varphi(\varphi(\varphi(\varphi))) + 2\varphi(\varphi(\varphi(\varphi))) \{\varphi\}$$

$$= \varphi(\varphi(\varphi(\varphi))) + \varphi(\varphi(\varphi(\varphi))) + \varphi(\varphi(\varphi(\varphi))) + \varphi(\varphi(\varphi(\varphi))) + 2\varphi(\varphi(\varphi(\varphi))) + 2\varphi(\varphi(\varphi(\varphi))) + 2\varphi(\varphi(\varphi(\varphi))) + 6\varphi(\varphi(\varphi(\varphi)))$$

$$= \varphi(\varphi(\varphi(\varphi))) + 3\varphi(\varphi(\varphi(\varphi))) + 3\varphi(\varphi(\varphi(\varphi))) + 6\varphi(\varphi(\varphi(\varphi)));$$

(12.2)
\[
(\varphi[2])[2] = (\varphi \circ \varphi)(\varphi \circ \varphi) = \varphi(\varphi(\varphi \{\varphi\})) = \varphi(\varphi(\varphi(\varphi))) + \varphi(\varphi(\varphi(\varphi))) + \varphi(\varphi(\varphi(\varphi))).
\] (12.3)

In the computations we used the brace algebra identities. The results are the same modulo 2.

To visualize better the result of (12.2) we will draw it as a sum of trees, see Figure 12.

![Figure 12: \(\varphi[4] = \varphi(\varphi(\varphi(\varphi))) + 3\varphi(\varphi(\varphi(\varphi))) + 3\varphi(\varphi(\varphi(\varphi))) + 6\varphi(\varphi(\varphi(\varphi)))\)](image)

The numbers 1, 2, 3 and 4 are assigned in the order of appearance of \(\varphi\) in the element of our sum. For instance, for the element \(\varphi(\varphi, \varphi(\varphi))\): 1 is assigned to \(\varphi(\varphi(\varphi))\), 2 is assigned to \(\varphi(\varphi, \varphi(\varphi))\), 3 is assigned to \(\varphi(\varphi, \varphi(\varphi))\), 4 is assigned to \(\varphi(\varphi, \varphi(\varphi))\). This enumeration corresponds to the standard recursion algorithm of enumeration of vertices in a tree.

Let us consider \(\varphi^{[n]}\) for arbitrary \(n\), deg(\(\varphi\)) being odd. By means of the formula (6.3) we decompose \(\varphi^{[n]}\) as a sum of similar trees with some coefficients. By tree we mean a rooted tree. The set of child vertices for each vertex is supposed to be ordered. All the trees will have \(n\) vertices. Consider any such tree \(T\) and find its coefficient \(O(T)\) in the decomposition. Let us label its vertices from 1 to \(n\) as it was described before correspondingly to the standard recursion algorithm of enumeration of vertices. The structure of a rooted tree induces a partial order on the set of vertices: the minimal element is the root that is always 1; one says \(i \prec j\) if the path from 1 to \(j\) passes through \(i\), see Definition 2.1. It can be easily seen that the coefficient \(O(T)\) is equal to the number of complete orders that respect the partial order \(\prec\).

The number \(O(T)\) can be easily found. Denote by \(ch_1(i), ch_2(i), \ldots, ch_k(i)\), \(i = 1 \ldots n\), the child vertices of vertex \(i\); by \(#i\) — the number of vertices that are greater or equal to vertex \(i\) with respect to the partial order \(\leq\), we call this number \(\#i\) cardinality of vertex \(i\). Also we set

\[
\{n_1, n_2, \ldots, n_k\} := \frac{(n_1 + n_2 + \cdots + n_k)!}{n_1!n_2!\cdots n_k!}. \tag{12.4}
\]

It is the number of shuffles of \(k\) sets of cardinalities \(n_1, n_2, \ldots, n_k\).

This expression is symmetric and satisfies the following identity:

\[
\{n_1, \ldots, n_i, \ldots, n_k\} \cdot \{n_{i1}, \ldots, n_{im}\} = \{n_1, \ldots, n_{i-1}, n_{i1}, n_{i2}, \ldots, n_{im}, n_{i+1}, \ldots, n_k\}, \tag{12.5}
\]

where \(n_i = n_{i1} + n_{i2} + \cdots + n_{im}\).
One has
\[
O(T) = \prod_{i=1}^{n} O_i(T) = \prod_{i=1}^{n} \{\#ch_1(i), \#ch_2(i), \ldots, \#ch_{k_i}(i)\}. \tag{12.6}
\]
If the set of child vertices is empty for some vertex \(i\), one sets \(O_i(T) := 1\).

Let us prove the assertion of the lemma.

Denote by \(D_p(N)\) the sum of digits of the representation of \(N\) in the base \(p\) number system. It is easy to see that \(\{n_1, n_2, \ldots, n_k\} \neq 0 \mod p\) if and only if \(D_p(n_1+n_2+\cdots+n_p) = D_p(n_1) + D_p(n_2) + \cdots + D_p(n_k)\).

Now, consider \(\varphi^{[p]}\). Consider any tree \(T\) with \(pn\) vertices. We want to prove that \(O(T)\) is equal modulo \(p\) to the coefficient of this tree in the decomposition of \((\varphi^{[p]})^n\). By the previous remark \(O(T)\) is not zero modulo \(p\) if and only if

\((*)\) For any vertex \(i = 1, 2, \ldots, pn\) we always have
\[
D_p(\#ch_1(i) + \cdots + \#ch_{k_i}(i)) = D_p(\#ch_1(i)) + \cdots + D_p(\#ch_{k_i}(i)).
\]

In our proof we will need to examine only the last digit.

Remind that \(pn\) vertices of our tree \(T\) are enumerated from 1 to \(pn\). We will also index them by numbers from 1 to \(n\) as follows. We pass all the vertices accordingly to their enumeration. We assign \(1\) to the root. If for vertex \(i\) we have cardinality \(\#i\) is not a number divisible by \(p\), we assign to this vertex the same index as for its parent vertex, if \(\#i\) is divisible by \(p\) we assign the next number that was not yet used. In the case tree \(T\) satisfies the condition \((*)\) we will use exactly \(n\) numbers (otherwise this number can be less), see Figure 13.

If \(O(T) \neq 0 \mod p\), then for any number \(j = 1 \ldots n\) the vertices with the same index \(j\) form a subtree with \(p\) vertices. Denote this subtree by \(T_j\). We draw the edges of trees \(T_j\) by continuous lines, all the other edges — by dotted lines. Consider quotient-tree \(H\) obtained from \(T\) by contraction of continuous edges, see Figure 13. Tree \(H\) has \(n\) vertices. We will prove that
\[
O(T) \equiv O(H) \cdot O(T_1) \cdot O(T_2) \cdot \ldots \cdot O(T_n) \mod p. \tag{12.7}
\]
Evidently, this will prove the lemma: tree \(H\) and coefficient \(O(H)\) correspond to tree decomposition of \([n]\)-operation; trees \(T_1, \ldots, T_n\) and coefficients \(O(T_1), \ldots, O(T_n)\) correspond to tree decomposition of \([p]\)-operation.

![Figure 13: Assignment of indexes: \(p = 3, n = 2\). Subtrees and quotient-tree.](image)

Consider any vertex \(i = 1 \ldots pn\) indexed by some \(j = 1 \ldots n\), suppose its number in the tree \(T_j\) is \(s\). It has child vertices of two types: those that are indexed by \(j\), suppose their cardinalities are \(a_1p + b_1, a_2p + b_2, \ldots, a_Lp + b_L\); and those that have another index, suppose their cardinalities are \(a_{\ell+1}p, a_{\ell+2}p, \ldots, a_{\ell+L}p\). The numbers \(a_1, \ldots, a_L, b_1, \ldots, b_L\) are positive and satisfy \(A := a_1 + a_2 + \cdots + a_L < n\), \(B := b_1 + b_2 + \cdots + b_L < p\). 28
Note that \( \{mp, b\} \equiv 1 \mod p \), if \( b < p \).

By the previous remark and by propriety (12.5) we get:

\[
O_i(T) = \{a_1p + b_1, a_2p + b_2, \ldots, a_lp + b_l, a_{l+1}p, \ldots, a_{Lp}\}
\equiv \{a_1p, b_1, a_2p, b_2, \ldots, a_lp, b_l, a_{l+1}p, \ldots, a_{Lp}\}
\equiv \{Ap, B\} \cdot \{a_1p, a_2p, \ldots, a_{Lp}, b_1, b_2, \ldots, b_l\}
\equiv \{a_1, a_2, \ldots, a_l\} \cdot \{b_1, b_2, \ldots, b_l\}
\equiv \{a_1, a_2, \ldots, a_L\} \cdot O_i(T_j) \mod p.
\]

The next to last equality was obtained due to the fact that \{a_1p, a_2p, \ldots, a_{Lp}\} \equiv \{a_1, a_2, \ldots, a_L\} \mod p.

If we take the product of expressions \{a_1, a_2, \ldots, a_L\} for all vertices of the tree \( T_j \) then we obtain exactly \( O_j(H) \) (we will need to apply many times the identity (12.5)). This proves (12.7) and thus the lemma. □

Proof of Lemma 12.2.

First we need to understand how \( \varphi^{[n]} \) is decomposed as a sum of trees if \( \deg(\varphi) \) is even. More precisely we need to find the appropriate coefficients \( O^{-}(T) \) of all trees \( T \) with \( n \) vertices in this case. Combinatorically we will have the same number of summands that give the same tree \( T \) (the number of complete orders that respect the partial order of vertices induced by the structure of a tree). But in this situation not all of the summands are "1" — they are \( \pm 1 \). The sign is the sign of the permutation corresponding to a complete order (remind that we have the fixed enumeration, with respect to this enumeration one defines permutations).

Denote by

\[
\{n_1, n_2, \ldots, n_k\}_{-1}
\]

the superanalog of the expression (12.8), see Appendix C. This number is the number of shuffles of sets of cardinalities \( n_1, \ldots, n_k \), that provide even permutations of elements, minus the number of shuffles of the same sets providing odd permutations. We will call it the number of shuffles of sets with anticommuting elements. This number is

\[
\{n_1, n_2, \ldots, n_k\}_{-1} = \begin{cases} 
0, & \text{if at least two numbers } n_i \text{ and } n_j, \{i, j\} \subset \{1, \ldots, k\}, \text{ are odd;} \\
\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor, \ldots, \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor, & \text{otherwise,}
\end{cases}
\]

(12.9)

see Appendix C. We see from this formula that expression (12.8) is symmetric: it does not depend on the order of numbers \( n_1, \ldots, n_k \).

Analogously to (12.5) we have:

\[
O^{-}(T) = \prod_{i=1}^{n} O^{-}(T_i) = \prod_{i=1}^{n} \{\#ch_1(i), \#ch_2(i), \ldots, \#ch_k_{_i}(i)\}_{-1}.
\]

(12.10)

We are ready to prove the lemma. Consider any tree \( T \) with \( 2n \) vertices. By (12.4) inequality \( O^{-}(T) \neq 0 \) implies

\((**)\) for any vertex \( i = 1, \ldots, 2n \) all the cardinalities \( \#ch_1(i), \#ch_2(i), \ldots, \#ch_k_{_i}(i) \) (of its child vertices) except probably one are even.
Suppose our tree \( T \) satisfies this condition (**). The vertices of \( T \) are enumerated from 1 to \( 2n \). By means of the algorithm described in the proof of the previous lemma we will index them by numbers \( 1, 2, \ldots, n \): We assign 1 to the root. If for vertex \( i \) we have cardinality \( \#i \) is odd, we assign to this vertex the same index as for its parent vertex, if \( \#i \) is even we assign the next number that was not yet used. In the case tree \( T \) satisfies the condition (**), we will use exactly \( n \) numbers (otherwise this number will be less), see Figure 14. Finally we obtain subtrees \( T_1, \ldots, T_n \) with 2 vertices — they are all \( \bigcirc \), and a quotient tree \( H \) with \( n \) vertices, see Figure 14.

![Figure 14: Assignment of indexes. Subtrees and quotient-tree.](image)

Consider any subtree \( T_j \), \( j = 1, \ldots, n \). It has two vertices: the upper one \( i_1 \) and its child \( i_2 \), where \( i_1, i_2 \in \{1, \ldots, 2n\} \). Let us compute \( O_{i_1}^0(T), O_{i_2}^0(T) \). Suppose cardinalities of the child vertices of \( i_1 \) are \( 2a_1 + 1, 2a_2, 2a_3, 2a_4, \ldots, 2a_t \) (the only odd cardinality is that of \( i_2 \)). Respectively for \( i_2 \) this cardinalities are \( 2a_{i_1}, 2a_{i_2}, \ldots, 2a_{i_t} \) (where \( 2a_{i_1} + 2a_{i_2} + \cdots + 2a_{i_t} = 2a_1 \)). Thus

\[
O_{i_1}^0(T) \cdot O_{i_2}^0(T) = \{2a_1 + 1, 2a_2, 2a_3, 2a_4, \ldots, 2a_t\}_{-1} \cdot \{2a_{i_1}, 2a_{i_2}, \ldots, 2a_{i_t}\}_{-1} = \{a_1, a_2, a_3, \ldots, a_t\} \cdot \{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\} = O_j(H) = O_{j}(H) \cdot O^{-}(T_j).
\]

The second equality follows from \([12]\). The last equality is due to \( O^-(\bigcirc) = 1 \).

As a consequence we obtain

\[
O^-(T) = O(H) = O(H) \cdot O^-(T_1) \cdot O^-(T_2) \cdot \ldots \cdot O^-(T_n).
\]

This proves the lemma — tree \( H \) and coefficient \( O(H) \) correspond to \([n]\)-operation, trees \( T_1, \ldots, T_n \) and coefficients \( O^-(T_1), \ldots, O^-(T_n) \) correspond to \([2]\)-operation. \( \square \)

### 13 Pairing

Any diagram of bigrading \((i, i + 1)\) is a cycle in complex \( CTD \) homologous to diagram \([11]\) taken with some coefficient, see Theorem \([10] \). In this section we find out this coefficient.

Consider any alternated \( T \)-diagram \( A \) of complexity \((i, i + 1)\). It has only one minimal component, \( i.e. \) only one alternated tree. By abuse of the language we designate this alternated tree also by \( A \). Let us assign to \( A \) a tree in the sense of previous section. This
tree $T(A)$ will have $i$ vertices. We will do it recursively. If $i = 1$ there is only one diagram $\bigcirc$. We assign to it one vertex tree: $T(\bigcirc) = \bullet$. Let $i > 1$. Since $A$ is alternated it always contains the edge $(1, i + 1)$ joining the extremal vertices. If we remove this edge we obtain whether two disconnected non-trivial alternated trees $A_1$ and $A_2$, whether an alternated tree $A_1$ and a single point. In the first situation we set

$$T(A) = T(A_1) \cup T(A_2).$$

In the second situation we set

$$T(A) = T(A_1).$$

For instance, in the following situation this algorithm gives:

$$T\left( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right) = \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc.$$

**Proposition 13.1** Any alternated $T$-diagram $A$ of bigrading $(i, i + 1)$ is homologous to zero if it has intersecting edges. If $A$ has no intersecting edges, it is homologous to diagram $\bullet$.

(i) in complex $CTD_{even}$: with coefficient $\pm O(T(A));$

(ii) in complex $CTD_{odd}$: with coefficient $\pm O^-(T(A)).$

The sign depends on the orientation of $A$. The coefficient must be taken modulo the order of the corresponding homology group. □

Precision: we say a diagram has intersecting edges if it has vertices $a < b < c < d$ on the line $\mathbb{R}^1$, and has edges $(a, c), (b, d)$. For the definition of numbers $O(T), O^-(T)$ see the previous section.

**Proof of Proposition 13.1**. Immediately follows from the duality of the basis of alternated trees to that of monotone brackets and also from the considerations of the previous section. □

### 14 How the homology bialgebra of $DHATD$ is related to that of $DHAT_0D$

In this section we will completely describe the relation between the homology bialgebra of $DHATD$ and that of $DHAT_0D$ for any ring $k$ of coefficient. The results of this section (namely Theorem 14.3) are given without proof. The proof will be given elsewhere.

First we need to fix some new objects and notations.

The following definition is a generalization of 10.3.
**Definition 14.1** A divided product \( \langle D_1, D_2, \ldots, D_n \rangle \) of \( T_*/T/T_0 \)-diagrams \( D_1, D_2, \ldots, D_n \) is the sum of those elements in the shuffle product \( D_1 \ast D_2 \ast \ldots \ast D_n \) that have the left-most point of \( D_i \) on the left from the left-most point of \( D_{i+1} \) for all \( i = 1, \ldots, n-1 \). □

We extend these operations as multilinear operations on the space of \( T_* \)-diagrams (resp. \( T \)-diagrams and \( T_0 \)-diagrams).

We will denote by \( \langle \rangle \) the trivial diagram — the unity of algebras \( DHAT_* D, DHAT D \) and \( DHAT_0 D \).

**Lemma 14.2**

\[
\Delta(\langle D_1, D_2, \ldots, D_n \rangle) = \sum_{i=0}^{n}(\langle D_1, D_2, \ldots, D_i \rangle \otimes \langle D_{i+1}, D_{i+2}, \ldots, D_n \rangle).
\]

\[
\langle D_1, D_2, \ldots, D_n \rangle \ast \langle D_{n+1}, D_{n+2}, \ldots, D_{n+m} \rangle = \sum_{\sigma \in S(n,m)} (-1)^{s(\sigma)} \langle D_{\sigma(1)}, D_{\sigma(2)}, \ldots, D_{\sigma(n+m)} \rangle,
\]

where \( S(n,m) \) is a subset in the symmetric group \( S_{n+m} \) whose elements are shuffles of \( 1,2,\ldots,n \) with \( n+1, n+2, \ldots, n+m \); \( (-1)^{s(\sigma)} \) is the sign of the induced permutation of odd elements among \( D_1, D_2, \ldots, D_{n+m} \). □

Proof of this lemma is a direct check.

Consider subspaces \( Z_{\text{even}} \subset CTD_{\text{even}}, Z_{\text{odd}} \subset CTD_{\text{odd}} \) that are spanned by the elements \( \langle Z_{k_1}, Z_{k_2}, \ldots, Z_{k_l} \rangle \).

Remind, \( Z_i \) is the diagram \( 10.1 \). By the previous lemma these subspaces are Hopf subalgebras of \( DHAT_{\text{even}}, \) resp. \( DHAT_{\text{odd}} \). The following lemma generalizes \( 10.5, 10.6 \) and shows that \( Z_{\text{even}}, Z_{\text{odd}} \) are differential Hopf subalgebras:

**Lemma 14.3** For even \( d \) one has

\[
\partial(\langle Z_{k_1}, Z_{k_2}, \ldots, Z_{k_l} \rangle) = \sum_{i=1}^{l-1} (-1)^{i-1} (k_{k_i}+k_{k_{i+1}}) \langle Z_{k_1}, Z_{k_2}, \ldots, Z_{k_{i-1}}, Z_{k_{i+1}}, Z_{k_{i+2}}, \ldots, Z_{k_l} \rangle.
\]

(14.1)

For odd \( d \) one has

\[
\partial(\langle Z_{k_1}, Z_{k_2}, \ldots, Z_{k_l} \rangle) = \sum_{i=1}^{l-1} (-1)^{i-1+k_1+k_2+\ldots+k_i} (k_{k_i}+k_{k_{i+1}}) \langle Z_{k_1}, Z_{k_2}, \ldots, Z_{k_{i-1}}, Z_{k_{i+1}}, Z_{k_{i+2}}, \ldots, Z_{k_l} \rangle.
\]

(14.2)

Relation between \( DHAT D \) and \( DHAT_0 D \) is given by the following theorem.

**Theorem 14.4** The morphisms

\[
\mu : Z_{\text{even}} \otimes DHAT_0 D_{\text{even}} \to DHAT D_{\text{even}},
\]

(14.3)

\[
\mu : Z_{\text{odd}} \otimes DHAT_0 D_{\text{odd}} \to DHAT D_{\text{odd}}
\]

(14.4)

are quasi-isomorphisms of differential Hopf algebras (\( \mu \) is the shuffle multiplication in \( DHAT D \)). □
Without proof. The proof of this theorem is more or less geometrical and is absolutely different from the considerations of this article. This theorem together with Theorem 10.1 provide another proof of Theorem 10.2.

For instance if our ring of coefficients $k$ is a field, then this theory says that the homology bialgebra of $D\text{HAT}_D$ is a tensor product of homology bialgebra of $D\text{HAT}_0D$ to that of $\mathbb{Z}$.

Complexes $Z_{\text{even}}$, $Z_{\text{odd}}$ are well known, see [11, 32, 33, 19]. In Appendix D we give a summary describing the structure of their homology bialgebras.

For a fixed complexity $i$, complex $Z_{\text{odd}}$ (resp. $Z_{\text{even}}$) computes cohomology groups $H^*(Br(i),\mathbb{Z})$ of the braid group with $i$ strings (resp. cohomology groups $H^*(Br(i),\pm\mathbb{Z})$) of this braid group with coefficients in its sign representation $\pm\mathbb{Z})$. Indeed, classifying space of the braid group $Br(i)$ is the configuration space $B(\mathbb{C}^1,i)$ (space of cardinality $i$ subsets of $\mathbb{C}^1$). By Poincaré duality one has

$$H^*(B(\mathbb{C}^1,i),\mathbb{Z}) \simeq \tilde{H}_{2i-\bullet}(B(\mathbb{C}^1,i),\mathbb{Z}),$$
$$H^*(B(\mathbb{C}^1,i),\pm\mathbb{Z}) \simeq \tilde{H}_{2i-\bullet}(B(\mathbb{C}^1,i),\pm\mathbb{Z}).$$

($\tilde{H}_{\bullet}(\cdot,L)$ designates locally finite homology groups with coefficients in a local system $L$.) One point compactification $\overline{B}(\mathbb{C},i)$ of the configuration space has a natural cell decomposition that is defined as follows. Let $\xi = \{z_1, z_2, \ldots, z_i\}$ be a point of $B(\mathbb{C}^1,i)$. We will assign to $\xi$ its index — system of numbers $(k_1, k_2, \ldots, k_l)$ satisfying $k_1 + k_2 + \cdots + k_l = i$, where $k_1$ is the number of elements of $\xi$ with the minimal value of the real component $\Re(z)$; $k_2$ is the number of elements of $\xi$ with next value of $\Re(z)$, and so on... Points with the same index $(k_1, k_2, \ldots, k_l)$ form a cell, that we denote by $e(k_1, k_2, \ldots, k_l)$.

![Figure 15: Point of the cell $e(3, 4, 1)$ of $\overline{B}(\mathbb{C}, 8)$.

All such cells together with the infinite point provide a cell decomposition of $\overline{B}(\mathbb{C}^1,i)$. These cells bound to each other exactly by the rule [147, 148], see [32] ($(Z_{k_1}, \ldots, Z_{k_l})$ should be replaced by $e(k_1, \ldots, k_l)$).

The differential of the cells in the local system $\pm\mathbb{Z}$ is described by the rule [147, 148], see [32].

If we consider complexes $Z_{\text{even}}$, $Z_{\text{odd}}$ as differential Hopf algebras and fix as only grading $deg = p + q = (d-1)i - j$, then each complex computes the cohomology bialgebra $H^*(\Omega^2(S^{d-1}),\mathbb{Z})$ of the double loop space of $(d-1)$-dimensional sphere. To see this one can use Vassiliev’s approach of discriminants, see [36]. Connection with the homology of braid groups is a particular case of the well known Snaith decomposition formula, see [25]:

$$H^i(\Omega^mS^n) \simeq \bigoplus_{i=1}^{\infty} H^{t-i(n-m)}(B(\mathbb{R}^m,i),(\pm\mathbb{Z})^\otimes(n-m))$$

for $m = 2$, $n = d - 1$. ($B(\mathbb{R}^m,i)$ denotes the space of cardinality $i$ subsets of $\mathbb{R}^m$.)
Theorem 14.4 has a very simple geometrical meaning. The space $Emb = \mathcal{K} \setminus \Sigma$ of long knots is a subspace of the space $Imm$ of long immersions — immersions with a fixed behavior at infinity:

$$Emb \hookrightarrow Imm.$$ 

The homotopy fiber $Emb^+$ of this inclusion can be regarded as a Serre fibration over $Emb$:

$$\Pi : Emb^+ \rightarrow Emb.$$ 

Preimage $\Pi^{-1}(f)$ of any knot $f \in Emb$ is the space $\Omega(Imm; f, l)$ of pathes in $Imm$ that start in $f$ and end in the fixed linear embedding $l$. Obviously, $\Pi^{-1}(f)$ is homotopy equivalent to the loop space $\Omega(Imm)$. On the other hand, the space $Imm$ is homotopy equivalent to $\Omega(\mathbb{R}^d \setminus \{0\}) \simeq \Omega S^{d-1}$ (we consider the value of derivative $f'(t)$, $f \in Imm$, to obtain a map in one direction). Thus we have $\Pi^{-1}(k) \simeq \Omega^2(S^{d-1})$. Actually, complex $DHATD$ (resp. $DHAT_0D$) computes the second term of Sinha’s spectral sequence converging to the (co)homology of the space $Emb^+$ (resp. $Emb$), cf. [37, 25]. A crucial point is that $\Pi$ is a trivial fiber bundle, i.e. $Emb^+ \simeq Emb \times \Omega^2S^{d-1}$, since the inclusion $Emb \hookrightarrow Imm$ is a contractible map, see [25]. Therefore, Theorem 14.4 confirms the conjecture that Sinha’s spectral sequence stabilizes at the second term.

Appendixes

In Appendix A we describe bases of complexes $CTD$, $(POISS_{Norm}, \partial)$, $CT_iD$, $(BV_{Norm}, \partial)$, $CT_0D$, $(POISS_{zero}, \partial)$ for small complexities $i$.

Appendix B contains results of computer calculations of the homology of $CTD$ and $CT_0D$.

Appendix C gives shuffle combinatorial formulas (like binomial) for anticommuting elements.

In Appendix D we describe the homology of differential bialgebra $Z$. This bialgebra is well known and its homology bialgebra is $H^*(\Omega^2S^{d-1})$, $d \geq 4$.

Appendix E describes homology bialgebra $H_*(\Omega^2S^{d-1})$ and provides explicit formulas for inclusion of $H_*(\Omega^2S^{d-1})$ in Hochschild homology of Poisson or Gerstenhaber algebras operad.

A Complexes of $T_*/T/T_0$-diagrams for small complexities $i$

Complexes of $T_*/T/T_0$-diagrams are finite for each complexity $i$. The aim of this section is to describe the bases of these complexes for small $i$. In particular, this demonstrates how complex $CT_0D$ simplifies computations of the homology groups of $CT_iD$. We describe also the dual bases in the dual complexes.
A.1 Alternated $T$-diagrams of complexities $i = 1, 2$

In complexity $i = 1$ one has only one alternated $T$-diagrams:

\[ \text{The dual element is } [x_1, x_2]. \]

In complexity $i = 2$ one has five alternated $T$-diagrams:

\[ \text{The dual monotone bracket diagrams are } [x_1, x_4] \cdot [x_2, x_3], \ [x_1, x_3] \cdot [x_2, x_4], \ [x_1, x_2] \cdot [x_3, x_4] \]

and $[x_1, [x_2, x_3]], \ [[x_1, x_2], x_3]$ respectively.

A.2 Alternated $T_\ast$-diagrams of complexities $i = 1, 2$

In complexity $i = 1$ one has two alternated $T_\ast$-diagrams:

\[ \text{The dual elements in the BV}_N^{Norm} \text{ are } [x_1, x_2] \text{ and } \delta x_1. \]

In complexity $i = 2$ one has eleven alternated $T_\ast$-diagrams:

\[ \text{The dual elements in } BV_N^{Norm} \text{ are:} \]

\[ [x_1, x_4] \cdot [x_2, x_3], \ [x_1, x_3] \cdot [x_2, x_4], \ [x_1, x_2] \cdot [x_3, x_4] \]

\[ [x_1, [x_2, x_3]], \ [[x_1, x_2], x_3] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]

\[ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]], \ [x_1, [x_2, x_3]] \]
A.3 Alternated $T_0$-diagrams of complexities $i = 1, 2, 3$

There is no alternated $T_0$-diagrams of complexity $i = 1$.

In complexity $i = 2$ one has only one alternated $T_0$-diagram:

The dual bracket diagram is $[x_1, x_3] : [x_2, x_4]$.

In complexity $i = 3$ one has twelve alternated $T_0$-diagrams:

The dual bracket diagrams are

\[
\begin{align*}
& [x_1, x_6] : [x_2, x_4] : [x_3, x_5] & [x_1, [x_2, x_4]] : [x_3, x_5] \\
& [x_1, x_5] : [x_2, x_4] : [x_3, x_6] & [[x_1, x_3], x_5] : [x_2, x_4] \\
& [x_1, x_4] : [x_2, x_6] : [x_3, x_5] & [x_1, [x_3, x_5]] : [x_2, x_4] \\
& [x_1, x_4] : [x_2, x_5] : [x_4, x_6] & [[x_1, x_3], x_4] : [x_2, x_5] \\
& [x_1, x_3] : [x_2, x_5] : [x_4, x_6] & [x_1, x_4] : [[x_2, x_4], x_5] \\
& & [x_1, x_4] : [x_2, [x_3, x_5]] \\
\end{align*}
\]

B Computer calculations

We give here results of computations of the homology bialgebra of the 4 complexes:

$CT_0D^{even}, CT_0D^{odd}, CTD^{even}, CTD^{odd}$. These complexes are graded commutative differential Hopf algebras.

I remind the grading $i$ is the complexity (number of edges) of the corresponding diagrams, the grading $j$ is the number of points. The numbers $i$ and $j$ correspond to the coordinates $p$ and $q$ of the Vassiliev spectral sequence by the following formulae:

\[
p = -i, \quad q = di - j.
\]

Hence $p + q = i(d - 1) - j$ is the corresponding cohomology degree of the space of long knots in $\mathbb{R}^d$, $d \geq 3$.

Note that the grading $j$ is always odd, but the grading $i$ is odd in the case of $d$ even and is even in the case of $d$ odd.

In the tables below one puts nothing in a cell if there are no diagrams of this bigrading in the corresponding complex. A question symbol "?" without anything else in a cell means that
my computer did not manage to compute this homology group. A question symbol with some extra information in a cell means that this information is not sure — big numbers appeared in the computations.

The groups from the bialgebra of chord diagrams are underlined. Their ranks are known till the complexity $i \leq 12$.

The results in tables 5-6 follow from the results of tables 1-4.

1. Homology of complex $CT_0D^{even}$

| Complexity $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|----------------|---|---|---|---|---|---|
| $j$            |   |   |   |   |   |   |
| 1              |   |   |   |   |   |   |
| 2              |   |   |   |   |   |   |
| 3              |   |   |   |   |   |   |
| 4              | $\mathbb{Z}$ | 0 |   |   |   |   |
| 5              | $\mathbb{Z}$ | 0 |   |   |   |   |
| 6              | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 |   |   |   |
| 7              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 |   |   |
| 8              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | ? |   |   |
| 9              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | ? |   |   |
| 10             | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | ? |   |   |
| 11             |   |   |   |   |   |   |
| 12             |   |   |   |   |   |   |

2. Homology of complex $CT_0D^{odd}$

| Complexity $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|----------------|---|---|---|---|---|---|
| $j$            |   |   |   |   |   |   |
| 1              |   |   |   |   |   |   |
| 2              |   |   |   |   |   |   |
| 3              |   |   |   |   |   |   |
| 4              | $\mathbb{Z}$ | 0 |   |   |   |   |
| 5              | $\mathbb{Z}$ | 0 |   |   |   |   |
| 6              | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 |   |   |   |
| 7              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 |   |   |
| 8              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | ? |   |   |
| 9              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | ? |   |   |
| 10             | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | ? |   |   |
| 11             |   |   |   |   |   |   |
| 12             |   |   |   |   |   |   |

3. Homology of complex $CTD^{even}$

| Complexity $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|----------------|---|---|---|---|---|---|
| $j$            |   |   |   |   |   |   |
| 1              |   |   |   |   |   |   |
| 2              | $\mathbb{Z}$ |   |   |   |   |   |
| 3              | $\mathbb{Z}$ | $\mathbb{Z}$ |   |   |   |   |
| 4              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 5              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 6              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 7              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 8              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 9              |   |   |   |   |   |   |
| 10             |   |   |   |   |   |   |

4. Homology of complex $CTD^{odd}$

| Complexity $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|----------------|---|---|---|---|---|---|
| $j$            |   |   |   |   |   |   |
| 1              |   |   |   |   |   |   |
| 2              | $\mathbb{Z}$ |   |   |   |   |   |
| 3              | $\mathbb{Z}$ | $\mathbb{Z}$ |   |   |   |   |
| 4              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 5              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 6              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 7              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 8              | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 9              |   |   |   |   |   |   |
| 10             |   |   |   |   |   |   |
5. Primitive generators of the homology bialgebra of $DHATD_{\text{even}}$

| Complexity $i$ | $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---------------|----|---|---|---|---|---|---|---|
| 1             |    |   |   |   |   |   |   |   |
| 2             |    |   |   |   |   |   |   |   |
| 3             |    |   |   |   |   |   |   |   |
| 4             |    |   | 0 |   |   |   |   |   |
| 5             |    |   | 1 | 0 |   |   |   |   |
| 6             |    | 1 | 0 | 0 |   |   |   |   |
| 7             |    | 0 | 1 | 0 |   |   |   |   |
| 8             |    | 0 | 0 | ? |   |   |   |   |
| 9             |    | 1 | ? | ? |   |   |   |   |
| 10            |    | 2 | ? | ? |   |   |   |   |
| 11            |    |   |   |   |   |   |   |   |
| 12            |    |   |   |   |   |   |   |   |

6. Primitive generators of the homology bialgebra of $DHATD_{\text{odd}}$

| Complexity $i$ | $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---------------|----|---|---|---|---|---|---|---|
| 1             |    |   |   |   |   |   |   |   |
| 2             |    |   |   |   |   |   |   |   |
| 3             |    |   | 1 |   |   |   |   |   |
| 4             |    |   | 1 | 0 |   |   |   |   |
| 5             |    |   | 1 | 0 |   |   |   |   |
| 6             |    | 1 | 0 | 0 |   |   |   |   |
| 7             |    | 2 | 0 | 0 |   |   |   |   |
| 8             |    | 2 | 0 | ? |   |   |   |   |
| 9             |    | 3 | ? |   |   |   |   |   |
| 10            |    | 3 | ? | ? |   |   |   |   |
| 11            |    |   |   |   |   |   |   |   |
| 12            |    |   |   |   |   |   |   |   |

To obtain the table of primitive generators of the homology bialgebra of the complex of $T_0$-diagrams (in both even and odd cases) one should remove from the above tables the content of two cells $i = 1, j = 2$ and $i = 2, j = 3$. The content of all the other cells will be the same.

7. Homology ranks of complex $CT_0D_{\mathbb{Z}_2}$

| Complexity $i$ | $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---------------|----|---|---|---|---|---|---|---|
| 1             |    |   |   |   |   |   |   |   |
| 2             |    |   |   |   |   |   |   |   |
| 3             |    |   |   |   |   |   |   |   |
| 4             |    | 1 | 0 |   |   |   |   |   |
| 5             |    | 1 | 0 |   |   |   |   |   |
| 6             |    | 1 | 1 | 0 |   |   |   |   |
| 7             |    | 3 | 1 | 0 |   |   |   |   |
| 8             |    | 3 | 1 | 1 | 0 |   |   |   |
| 9             |    | 4 | 3 | ? |   |   |   |   |
| 10            |    | 4 | 7 | ? |   |   |   |   |
| 11            |    | 13 | ? |   |   |   |   |   |
| 12            |    | 9 | ? |   |   |   |   |   |
| 13            |    |   |   |   |   |   |   |   |
| 14            |    |   |   |   |   |   |   |   |

8. Homology ranks of complex $CTD_{\mathbb{Z}_2}$

| Complexity $i$ | $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---------------|----|---|---|---|---|---|---|---|
| 1             |    |   |   |   |   |   |   |   |
| 2             |    |   |   |   |   |   |   |   |
| 3             |    |   | 1 |   |   |   |   |   |
| 4             |    |   | 2 | 0 |   |   |   |   |
| 5             |    |   | 2 | 1 |   |   |   |   |
| 6             |    |   | 2 | 2 | 0 |   |   |   |
| 7             |    |   | 6 | 2 | 0 |   |   |   |
| 8             |    |   | 6 | 4 | 2 |   |   |   |
| 9             |    |   | 11 | ? |   |   |   |   |
| 10            |    |   | 10 | ? |   |   |   |   |
| 11            |    |   |   |   |   |   |   |   |
| 12            |    |   |   |   |   |   |   |   |
| 13            |    |   |   |   |   |   |   |   |
| 14            |    |   |   |   |   |   |   |   |
## 9. Ranks of complex $CT_0D$

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|
| 1   |   |   |   |   |   |   |   |
| 2   |   |   |   |   |   |   |   |
| 3   |   |   |   |   |   |   |   |
| 4   |   |   |   |   |   |   |   |
| 5   | 6 | 6 | 34 |   |   |   |   |
| 6   | 68 | 284 | 216 |   |   |   |   |
| 7   | 182 | 711 | 1566 |   |   |   |   |
| 8   | 830 | 20614 | 20624 |   |   |   |   |
| 9   | 329 | 14835 | 100154 |   |   |   |   |
| 10  | 11940 | 237840 |   |   |   |   |   |
| 11  | 3655 | 297620 |   |   |   |   |   |
| 12  | 188720 |   |   |   |   |   |   |
| 13  | 47844 |   |   |   |   |   |   |

## 10. Ranks of complex $CTD$

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|
| 1   |   |   |   |   |   |   |   |
| 2   |   |   |   |   |   |   |   |
| 3   |   |   |   |   |   |   |   |
| 4   |   |   |   |   |   |   |   |
| 5   | 8 | 6 | 34 |   |   |   |   |
| 6   | 45 | 130 | 120 |   |   |   |   |
| 7   | 210 | 924 | 720 |   |   |   |   |
| 8   | 105 | 2380 | 7308 | 5040 |   |   |   |
| 9   | 2520 | 26432 | 64224 |   |   |   |   |
| 10  | 945 | 44100 | 303660 |   |   |   |   |
| 11  | 34650 | 705320 |   |   |   |   |   |
| 12  | 10395 | 866250 |   |   |   |   |   |
| 13  | 540540 |   |   |   |   |   |   |
| 14  | 135135 |   |   |   |   |   |   |

### C $q$-combinatorics

Let us denote by \( \binom{n + m}{n}_q \) the following polynomial over \( q \):\[
\binom{n + m}{n}_q = \sum_{s \in S(n, m)} q^{\alpha(s)},
\]
where \( S(n, m) \) is the set of all shuffles of a cardinality \( n \) set \( x, x, \ldots, x \) with a cardinality \( m \) set \( y, y, \ldots, y \); \( \alpha(s) \) is the minimal number of transpositions that one needs to obtain shuffle \( s \) from the initial shuffle \( x, x, \ldots, x \), \( y, y, \ldots, y \).

It can be easily verified that\[
\binom{n + m + 1}{n}_q = \binom{n + m}{n}_q + q^{m+1} \binom{n + m}{n-1}_q;
\]
\[
\sum_{i=0}^{n} \binom{n}{i}_q (\frac{q^{i+1}}{2}) x^i = (1 + x)(1 + qx) \ldots (1 + q^{n-1}x).
\]

Denote by \( n_q \) the polynomial \( 1 + q + q^2 + \cdots + q^{n-1} \), and by \( n_q! = 1_q \cdot 2_q \cdot \ldots \cdot n_q \).

One has\[
\binom{n + m}{n}_q = \frac{(n + m)_q}{n_q! \cdot m_q!}.
\]

In this paper we use only two situations \( q = \pm 1 \). In the case \( q = -1 \) the last formula can not be applied because of division by zero. To find numbers \( \binom{n + m}{n-1}_q \) one should use the formula (C.3). If \( n \) is even:\[
\sum_{i=0}^{n} \left(\begin{array}{l}n \\ i\end{array}\right)_{-1} (-1)^{(n+1)} x^i = (1 + x)(1 - x) \ldots (1 + x)(1 - x) =
\]
\[
= (1 - x^2)^\frac{q}{2} = \sum_{j=0}^{\frac{q}{2}} (-1)^j \left(\begin{array}{l}\frac{q}{2} \\ j\end{array}\right) x^{2j}.
\]
If $n$ is odd:

$$
\sum_{i=0}^{n} \binom{n}{i} (-1)^{\frac{i(i-1)}{2}} x^i = (1+x)(1-x)\cdots(1+x)(1-x)(1+x) =
= (1-x^2)^{n-1}(1+x) = \sum_{i=0}^{n} (-1)^{\left\lfloor \frac{i}{2} \right\rfloor} \binom{n-1}{\frac{i}{2}} x^i. \quad (C.6)
$$

(C.5), (C.6) imply:

$$
\binom{n+m}{n} = \begin{cases} 
0, & n \text{ and } m \text{ are odd; } \\
\left(\left\lfloor \frac{n+m}{2} \right\rfloor \right), & \text{otherwise.} 
\end{cases} \quad (C.7)
$$

Denote by $\{n_1, n_2, \ldots, n_k\}_q$ the following polynomial over $q$

$$
\{n_1, n_2, \ldots, n_k\}_q = \sum_{s \in S(n_1, n_2, \ldots, n_k)} q^{\alpha(s)}, \quad (C.8)
$$

where $S(n_1, n_2, \ldots, n_k)$ is the set of all shuffles of sets $X_1 = \{x_1, x_1, \ldots, x_1\}$, $n_1$ times; $\{x_2, x_2, \ldots, x_2\}, \ldots, \{x_k, x_k, \ldots, x_k\}; \alpha(s)$ is the minimal number of transpositions that one needs to obtain shuffle $s$ from the initial shuffle $x_1, x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_k$.

This expression is a generalization of (C.1):

$$
\{n, m\}_q = \binom{n+m}{n}_q.
$$

One has

$$
\{n_1, n_2, \ldots, n_k\}_q = \frac{(n_1 + n_2 + \cdots + n_k)_q!}{(n_1)_q!(n_2)_q! \cdots (n_k)_q!}. \quad (C.9)
$$

This expression is symmetric and satisfies the following identity:

$$
\{n_1, n_2, \ldots, n_i, \ldots, n_k\}_q \cdot \{n_{i+1}, n_{i+2}, \ldots, n_{im}\}_q =
\{n_1, n_1, \ldots, n_{i-1}, n_{i+1}, n_{i+2}, \ldots, n_{im}, n_{i+1}, \ldots, n_k\}_q, \quad (C.10)
$$

where $n_i = n_{i+1} + n_{i+2} + \cdots + n_{im}$.

The formula (C.9) can not be used in the case $q = -1$. But in this case (C.10) and (C.7) imply:

$$
\{n_1, n_2, \ldots, n_k\}_{-1} = \begin{cases} 
0, & \text{if at least two numbers } n_i, \text{ and } n_{j}, \text{ } \{i, j\} \subset \{1, \ldots, k\}, \text{ are odd; } \\
\left\lfloor \frac{n_i}{2} \right\rfloor, \left\lfloor \frac{n_j}{2} \right\rfloor, \ldots, \left\lfloor \frac{n_k}{2} \right\rfloor, & \text{otherwise.} 
\end{cases} \quad (C.11)
$$
D Homology bialgebras of $Z^{odd}$, $Z^{even}$

In this section we describe the homology bialgebras of $Z^{odd}$, $Z^{even}$ over $\mathbb{Z}_p$, $p$ being any prime, and over $\mathbb{Q}$. These results are well known, see [11, 32, 33, 19]. Calculation of the Bochschtein homomorphism shows that $Z$ homology groups of $Z^{odd}$, $Z^{even}$ have no higher torsions $\mathbb{Z}_p^k$ with $k \geq 2$ ($p$ being any prime), see [11, 32, 33, 19]. This permit to find the corresponding homology groups for any ring $k$ of coefficients. Complexes $Z^{odd}_k \otimes Z^{even}_k$ will be denoted by $\mathcal{Z}^{odd}_k \otimes Z^{even}_k$. Since $Z^{odd}_2 \cong Z^{even}_2$, this bialgebra will be denoted by $\mathcal{Z}_2$.

The homology bialgebras of $Z^{odd}_k \otimes Z^{even}_k$, $\mathcal{Z}_k$ will be denoted by $H^{odd}_k \otimes H^{even}_k$, $\mathcal{H}_k$. Since $Z^{odd}_2 \cong Z^{even}_2$, this bialgebra will be denoted by $\mathcal{Z}_2$.

First, we give necessary definitions. Most part of them are standard.

**Definition D.1** A bialgebra is called *polynomial* if it is polynomial as algebra and its generators are primitive (if $2 \neq 0$ in $k$, all generators must be of even degree).

Note that over $\mathbb{Z}_p$, the $p$-th power $x^p$ of any primitive element $x$ is always a primitive element: $\Delta x^p = x^p \otimes 1 + 1 \otimes x^p$.

**Definition D.2** A bialgebra is called *exterior* if it is exterior as algebra and all its generators are primitive (if $2 \neq 0$ in $k$, all generators must be of odd degree).

**Definition D.3** We call *graded polynomial bialgebra* any tensor product of a polynomial bialgebra with an exterior bialgebra.

Special case is when the ground ring is of characteristic 2. In this situation generators of a polynomial (resp. exterior) bialgebra can be odd (resp. even) elements. Actually this case makes us emphasise the difference between polynomial and exterior bialgebras.

**Definition D.4** A bialgebra $\Gamma$ that is dual to a polynomial bialgebra $A$ is called *bialgebra of divided powers*. The space in $\Gamma$ dual to the space of generators of $A$ will be called *space of divided powers generators*.

As example consider a bialgebra $\Gamma_k(x)$ of divided powers that is dual to a polynomial bialgebra with only one generator, cf. [11]. $k$ designates the ground ring. The space of $\Gamma_k(x)$ is linearly spanned by the elements $1 = x(0), x = x(1), x(2), x(3), \ldots$. If $k \neq \mathbb{Z}_2$ all these elements are even. Multiplication and comultiplication of the elements are given as follows:

\[
\begin{align*}
x^{(k)} \cdot x^{(l)} &= \binom{k + l}{k} x^{(k+l)}, & (D.1) \\
\Delta(x^{(k)}) &= \sum_{i=0}^{k} x^{(i)} \otimes x^{(k-i)}. & (D.2)
\end{align*}
\]

If $k = \mathbb{Z}_p$, $p$ being any prime, $\Gamma_k(x)$ is a commutative algebra with generators $y_k = x^{(p^k)}$, $k = 0, 1, 2, \ldots$, the only relations are $(y_k)^p = 0$. Note that only the generator $y_1 = x^{(1)}$ is primitive.

Bialgebra $\Gamma_Z(x)$ is isomorphic to a $\mathbb{Z}$-subbialgebra of the polynomial bialgebra $\mathbb{Q}[x]$ $\mathbb{Z}$-spanned by the elements $x^{(k)} = \frac{x^{(k)}}{k!}$. Obviously, $\Gamma_k(x) \simeq \Gamma_Z(x) \otimes k$. Note also that $\Gamma_0(x) \simeq \mathbb{Q}[x]$.

A bialgebra dual to an exterior bialgebra is also an exterior bialgebra.
**Definition D.5** A bialgebra that is dual to a graded polynomial bialgebra is called *graded bialgebra of divided powers*. □

A graded bialgebra of divided powers is a tensor product of a bialgebra of divided powers with an exterior bialgebra.

For any element \( x \) in \( \mathbb{Z}^{\text{odd}} \) or in \( \mathbb{Z}^{\text{even}} \) one can define its *divided powers*:

\[
x^{(n)} := \langle x, x, \ldots, x \rangle \quad \text{n times}, \quad n = 0, 1, 2, \ldots
\]

In the cases

1) \( k = \mathbb{Z}_2 \),
2) \( \deg(x) \) is even and \( k \) being an arbitrary ring of coefficients, multiplication and comultiplication identities \((D.1), (D.2)\) are satisfied by the elements \( x^{(n)} \), \( n = 0, 1, 2, \ldots \) and the following formula holds:

\[
\partial x^{(n)} = \partial x \cdot x^{(n-1)}.
\]

Due to the last formula, divided powers are homological operations in the described situations. Note, this is also true for \( \text{DHAT}_D, \text{DHAT}_0D, \text{DHAT}_sD \). In fact the latter bialgebras are so called *divided systems*, cf. [4, Chapter V, p. 124].

The element \( Z_1 \in \mathbb{Z}^{\text{odd}} \) is even. Denote by \( \iota \) the map \( \iota : \Gamma_{\mathbb{Z}}(x) \to \mathbb{Z}^{\text{odd}} \), that sends

\[
\iota : x^{(k)} \mapsto (Z_1)^{(k)}.
\]

This map is a morphism of differential Hopf algebras (we define the differential in \( \Gamma_{\mathbb{Z}}(x) \) as zero).

Denote by \( I_{\mathbb{Z}} : \mathbb{Z}^{\text{even}} \to \mathbb{Z}^{\text{odd}} \), the map that sends

\[
I_{\mathbb{Z}} : \langle Z_{k_1}, Z_{k_2}, \ldots, Z_{k_\ell} \rangle \mapsto \langle Z_{2k_1}, Z_{2k_2}, \ldots, Z_{2k_\ell} \rangle.
\]

\( I_{\mathbb{Z}} \) is a morphism of differential Hopf algebras. To see this one should compare formulas \((14.1)\) and \((14.2)\).

We will formulate several assertions in order to describe the homology bialgebras of \( \mathbb{Z}^{\text{even}}, \mathbb{Z}^{\text{odd}} \). The most part of these lemmas are whether well known or are reformulations of well known results.

**Lemma D.6** The composition map

\[
\begin{array}{c}
\Gamma_{\mathbb{Z}}(x) \otimes \mathbb{Z}^{\text{even}} \xrightarrow{\iota \otimes I_{\mathbb{Z}}} \mathbb{Z}^{\text{odd}} \otimes \mathbb{Z}^{\text{odd}} \xrightarrow{\mu} \mathbb{Z}^{\text{odd}} \\
\end{array}
\]

is a quasi-isomorphism of differential Hopf algebras. □
Corollary D.7 Since (D.3) is a quasi-isomorphism over \( \mathbb{Z} \), it stays quasi-isomorphism for any ring \( k \) of coefficients. In the special case \( k = \mathbb{Z}_2 \) quasi-isomorphism (D.3) implies the quasi-isomorphism:

\[
\Gamma_{\mathbb{Z}_2}(x) \otimes \mathbb{Z}_2 \xrightarrow{\mu(v_{\mathbb{Z}_2} \otimes I_{\mathbb{Z}_2})} \mathbb{Z}_2,
\]

where \( v_{\mathbb{Z}_2} = v \otimes \mathbb{Z}_2, I_{\mathbb{Z}_2} = I_{\mathbb{Z}} \otimes \mathbb{Z}_2 \).

Applying this quasi-isomorphism infinitely many times we obtain:

Theorem D.8 Differential Hopf algebra \( \mathbb{Z}_{\mathbb{Z}_2} \) is formal. The homology bialgebra \( H_{\mathbb{Z}_2} \) of \( \mathbb{Z}_{\mathbb{Z}_2} \) is a bialgebra of divided powers. Divided powers generators of \( H_{\mathbb{Z}_2} \) are elements \( \mathbb{Z}_{\mathbb{Z}_2} \), \( k = 0, 1, 2, \ldots \). As algebra \( H_{\mathbb{Z}_2} \) is generated by \( x^k; n = (\mathbb{Z}_{\mathbb{Z}_2})^{\langle 2 \rangle} \), \( k, n = 0, 1, 2, \ldots \). The only relations are \((y^k; n)^2 = 0\) for all possible \( k \) and \( n \). □

A differential Hopf algebra is said formal if it is quasi-isomorphic to its differential Hopf subalgebra, such that the restriction of the differential to this subalgebra is zero.

For any ring \( k \) of coefficients Lemma D.6 implies that \( H_{\mathbb{Z}_2} \simeq H_{\mathbb{Z}_2} \otimes \Gamma_k(x) \). So, let us concentrate on complex \( \mathbb{Z}_{\mathbb{Z}_2} \).

Suppose \( k = \mathbb{Z}_p \), \( p \) being any prime. Denote by \( \mathbb{Z}_{<p}^{\mathbb{Z}_2} \) a subspace of \( \mathbb{Z}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \) linearly spanned by all the elements \( \langle Z_{k_1}, Z_{k_2}, \ldots, Z_{k_l} \rangle \), such that \( k_i < p, i = 1, 2, \ldots, l \). It can be easily verified that \( \mathbb{Z}_{<p}^{\mathbb{Z}_2} \) is a differential Hopf subalgebra of \( \mathbb{Z}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \). Really, multiplication and comultiplication preserve this space. Since \( \binom{a+b}{a} \equiv 0 \mod p \) if \( a, b < p \) and \( a + b \geq p \), the differential does also preserve \( \mathbb{Z}_{<p}^{\mathbb{Z}_2} \). By abuse of the language the inclusion \( \mathbb{Z}_{<p}^{\mathbb{Z}_2} \hookrightarrow \mathbb{Z}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \) will be denoted by \( \iota \).

Denote by

\[
I_{\mathbb{Z}_p} : \mathbb{Z}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \rightarrow \mathbb{Z}_{\mathbb{Z}_2}^{\mathbb{Z}_2}
\]

the map that sends

\[
I_{\mathbb{Z}_p} : \langle Z_{k_1}, Z_{k_2}, \ldots, Z_{k_l} \rangle \mapsto \langle Z_{pk_1}, Z_{pk_2}, \ldots, Z_{pk_l} \rangle.
\]

\( I_{\mathbb{Z}_p} \) is a morphism of differential Hopf algebras. To see this, note that \( \binom{pa+pb}{pa} \equiv \binom{a+b}{a} \mod p \) for any \( a, b \in \mathbb{N} \).

Lemma D.9 For any prime \( p \), the composition map

\[
\mathbb{Z}_{<p}^{\mathbb{Z}_2} \otimes \mathbb{Z}_{<p}^{\mathbb{Z}_2} \xrightarrow{\iota \otimes I_{\mathbb{Z}_p}} \mathbb{Z}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \otimes \mathbb{Z}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \xrightarrow{\mu} \mathbb{Z}_{\mathbb{Z}_2}^{\mathbb{Z}_2}
\]

is a quasi-isomorphism of differential Hopf algebras. □

\( \mu \) in (D.5) designates as usual the shuffle multiplication.

In the case \( p = 2 \) Lemma D.9 is equivalent to Corollary D.7.

From now on let \( p > 2 \).
Lemma D.10 Any cycle in $Z_{<p}^{even}$, $p$ being any odd prime, is homologous up to a coefficient to one of the cycles:

\[
\langle Z_1, Z_{p-1}, \ldots, Z_1, Z_{p-1} \rangle, \text{ } n \geq 0, \tag{D.6}
\]

\[
\langle Z_1, Z_{p-1}, \ldots, Z_1, Z_{p-1}, Z_1 \rangle, \text{ } n \geq 0. \tag{D.7}
\]

□

Remark D.11

\[
\langle Z_1, Z_{p-1}, \ldots, Z_1, Z_{p-1} \rangle = (Z_1, Z_{p-1})^{(n)}; \tag{D.8}
\]

\[
\langle Z_1, Z_{p-1}, \ldots, Z_1, Z_{p-1}, Z_1 \rangle = (Z_1, Z_{p-1})^{(n)} \ast Z_1. \tag{D.9}
\]

* designates the shuffle product. □

Applying Lemmas [D.9] [D.10] and Remark [D.11] infinitely many times and also applying Lemma [D.6] we obtain:

Theorem D.12

1) The homology bialgebra $H_{Z_{<p}}^{even}$ of $Z_{<p}^{even}$, $p$ being any odd prime, is a graded bialgebra of divided powers. Its divided powers generators are $y_k = \langle Z_{p^k}, Z_{(p-1)p^k} \rangle$, $k = 0, 1, 2, \ldots$, its exterior generators are $z_k = Z_{p^k}$, $k = 0, 1, 2, \ldots$. As algebra $H_{Z_{<p}}^{even}$ is a graded commutative algebra generated by even elements $y_{k,n} = (y_k)^{(p^n)}$, $k, n = 0, 1, 2, \ldots$ and odd elements $z_k$, $k = 0, 1, 2, \ldots$. The only relations are $(y_{k,n})^p = 0$ for all possible $k$ and $n$.

2) The homology bialgebra $H_{Z_{<p}}^{odd}$ of $Z_{<p}^{odd}$, $p$ being any odd prime, is a graded bialgebra of divided powers. Its divided powers generators are $x = Z_1$ and $y_k = \langle Z_{2^p}, Z_{2(p-1)p^k} \rangle$, $k = 0, 1, 2, \ldots$, its exterior generators are $z_k = Z_{2^p}$, $k = 0, 1, 2, \ldots$. As algebra $H_{Z_{<p}}^{odd}$ is a graded commutative algebra generated by even elements $x_k = x^{(p^n)}$, $k = 0, 1, 2, \ldots$, $y_{k,n} = (y_k)^{(p^n)}$, $k, n = 0, 1, 2, \ldots$ and odd elements $z_k$, $k = 0, 1, 2, \ldots$. The only relations are $(x_k)^p = 0$, $(y_{k,n})^p = 0$ for all possible $k$ and $n$. □

Finally we formulate a theorem describing the situation $k = Q$:

Theorem D.13

1) The homology bialgebra $H_{Q}^{even}$ of $Z_{Q}^{even}$ is an exterior bialgebra, its only generator is $Z_1$.

2) The homology bialgebra $H_{Q}^{odd}$ of $Z_{Q}^{odd}$ is a graded polynomial bialgebra with one even generator $Z_1$ and one odd generator $Z_2$. □

E Dyer-Lashof operations and explicit formulas for inclusion of $H_*(\Omega^2 S^{d-1})$ in Hochschild homology of $\mathcal{POISS}, \mathcal{GERST}$

In previous section we described how the cohomology bialgebra of $\Omega^2 S^{d-1}$ is contained in the first term of Vassiliev cohomological long knots spectral sequence. The aim of this section
is to explicitly describe the dual situation: how $H_*(\Omega^2S^{d-1})$ is included in the homological Vassiliev spectral sequence, i.e. in the homology of the Hochschild complexes $(\text{POLSS}, \partial)$, $(\text{GERST}, \partial)$.

It is already well known that Hochschild cochain complex of an associative algebra can be endowed with an action of an operad quasi-isomorphic to the chain operad of small squares. This statement is named “Deligne’s conjecture”. Over $\mathbb{Z}$ this result is due to J.E. McClure and J.H. Smith, cf. [21], in characteristic zero there are several proofs, cf. [17, 27, 28, 38]. Actually the operad considered by J.E. McClure and J.H. Smith consists of brace operations, see Definition 6.1, and of an associative multiplication that satisfy some composition and differential relations. It means that their proof works as well in the case of Hochschild complex $(\mathcal{O}, \partial)$ of an operad $\mathcal{O}$, see Section 7. This result implies that in Hochschild homology one can define the same homological operations as for double loop spaces. Homological operations for the iterated loop spaces are well known, cf. [10, 7]: In the case of double loops, one has Pontriagin multiplication, Browder operator — degree one bracket (we will designate it $[\ldots, \cdot]$), and also two non-trivial Dyer-Lashof operations (following F.Cohen we designate them $\xi_1$ and $\zeta_1$):

Over $\mathbb{Z}_2$:
$$\xi_1 : H_k(\Omega^2X, \mathbb{Z}_2) \to H_{2k+1}(\Omega^2X, \mathbb{Z}_2).$$

Over $\mathbb{Z}_p$, $p$ being any odd prime:
$$\xi_1 : H_{2k-1}(\Omega^2X, \mathbb{Z}_p) \to H_{2pk-1}(\Omega^2X, \mathbb{Z}_p),$$
$$\zeta_1 : H_{2k-1}(\Omega^2X, \mathbb{Z}_p) \to H_{2pk}(\Omega^2X, \mathbb{Z}_p).$$

$\xi_1$ and $\zeta_1$ are related via Bochstein homomorphism $\beta$:
$$\zeta_1\varphi = \beta\xi_1\varphi - (\text{ad}^{p-1}\varphi)(\beta\varphi). \quad (E.1)$$

Operator $\text{ad}_\varphi$ is the adjoint action $[\varphi, \cdot]$.

Let $\varphi \in \mathcal{O}$ be a cycle of a Hochschild complex $(\mathcal{O}, \partial)$. Suppose the ground ring $k = \mathbb{Z}_2$ or $\text{deg}(\varphi)$ is odd. By abuse of the language define
$$\xi_1(\varphi) := \varphi^{[p]}.$$

(E.2)

Suppose $k = \mathbb{Z}_p$. Define
$$\zeta_1(\varphi) := -\sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} \varphi^{[i]} \ast \varphi^{[p-i]} \equiv \sum_{i=1}^{p-1} \frac{(-1)^i}{i} \varphi^{[i]} \ast \varphi^{[p-i]} \text{ mod } p. \quad (E.3)$$

Modulo Remark [11.3] Proposition [11.4] implies that operations (E.2), (E.3) are homological operations.

Proposition E.1 Operations $\xi_1$, $\zeta_1$ satisfy property (E.1). □

In this proposition $\beta$ is the Bochstein homomorphism, $\text{ad}_\varphi = [\varphi, \cdot]$ is the adjoint Gerstenhaber bracket action of $\varphi$. 

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Proof of Proposition [E.1]. Essentially one needs to prove the following identity:

\[
\phi, \varphi, \ldots, [\varphi, \beta \varphi] \ldots] \equiv \sum_{i=1}^{p} (\cdots ((\varphi \circ \varphi) \circ \cdots) \beta \varphi) \circ \cdots) \circ \varphi \text{ mod } p. \quad \text{(E.4)}
\]

The \(i\)-th summand of the right-hand-side of (E.4) is obtained by substitution of the \(i\)-th \(\varphi\) by \(\beta \varphi\) in \(\cdots ((\varphi \circ \varphi) \circ \cdots) \circ \varphi\).

Identity (E.4) follows from the following combinatorial fact:

Lemma E.2 Let \(T\) be any tree with \(p\) vertices, \(a\) and \(b\) be its two vertices, then

\[O(T_a) \equiv (-1)^{d(a,b)}O(T_b) \text{ mod } p,\]

where \(T_a\) (resp. \(T_b\)) is tree \(T\) with a chosen root \(a\) (resp. \(b\)), \(d(a,b)\) is the number of edges of the only path joining vertex \(a\) to vertex \(b\). □

Proposition E.1 confirms the following result.

Theorem E.3 Homology operations \(\xi_1\) and \(\zeta_1\) are the Dyer-Lashof operations induced by the little disc chain action on the Hochschild complex. □

To prove this theorem one needs to analyse in detail the McClure-Smith construction, that is beyond the capabilities of this paper. The proof of Theorem E.3 will be given by the author elsewhere. Note, this problem was already solved for \(p = 2\) in [39].

Operations \(\xi_1, \zeta_1\) and also the bracket permit to give explicit formulas for the inclusion of \(\Omega^2 S^{d-1}\) homology classes as homology classes of Hochschild complexes (POLSS, \(\partial\)) (case of odd \(d\)) and (GERST, \(\partial\)) (case of even \(d\)).

Let \(\varphi \in H_{d-3}(\Omega^2 S^{d-1}, k)\) denote the image of a generator of \(\pi_{d-3}(\Omega^2 S^{d-1}) \simeq Z\) via Hurewicz homomorphism. The following theorems describe the homology bialgebra of \(\Omega^2 S^{d-1}\), cf. [11][12][22][23][20], and as consequence of Theorems 10.2, 14.4, D.8, D.12, D.13 Proposition 11.1 and Lemmas 12.1, 12.2 provide the desired inclusion formulas (we set \(\varphi = [x_1, x_2]\)).

Theorem E.4 For any even \(d \geq 4\), bialgebra \(H_*(\Omega^2 S^{d-1}, Q)\) is an exterior bialgebra with the only generator \(\varphi\).

For any odd \(d \geq 5\) bialgebra \(H_*(\Omega^2 S^{d-1}, Q)\) is a graded polynomial bialgebra with two generators: even generator \(\varphi\) and odd generator \(\frac{1}{2}[\varphi, \varphi]\). □

Theorem E.5 For any \(d \geq 4\), bialgebra \(H_*(\Omega^2 S^{d-1}, Z_2)\) is a graded polynomial bialgebra with generators \(\xi_1^k \varphi \in H_{2^k(d-2)-1}(\Omega^2 S^{d-1}, Z_2)\), \(k = 0, 1, 2, \ldots\). □

Theorem E.6 For any even \(d \geq 4\), bialgebra \(H_*(\Omega^2 S^{d-1}, Z_p)\), \(p\) being any odd prime, is a graded polynomial bialgebra with exterior generators \(\xi_1^k \varphi \in H_{p^k(d-2)-1}(\Omega^2 S^{d-1}, Z_p)\), \(k = 0, 1, 2, \ldots\), and polynomial generators \(\zeta_1 \xi_1^k \varphi = \beta \xi_1^{k+1} \varphi \in H_{p^{k+1}(d-2)-2}(\Omega^2 S^{d-1}, Z_p)\), \(k = 0, 1, 2, \ldots\). □

Theorem E.7 For any odd \(d \geq 5\), bialgebra \(H_*(\Omega^2 S^{d-1}, Z_p)\), \(p\) being any odd prime, is a graded polynomial bialgebra with exterior generators \(\xi_1^k \psi \in H_{2p^k(d-2)-1}(\Omega^2 S^{d-1}, Z_p)\), \(k = 0, 1, 2, \ldots\), where \(\psi = \frac{1}{2}[\varphi, \varphi]\), and polynomial generators \(\varphi \in H_{d-3}(\Omega^2 S^{d-1}, Z_p)\) and \(\zeta_1 \xi_1^k \psi = \beta \xi_1^{k+1} \psi \in H_{2p^{k+1}(d-2)-2}(\Omega^2 S^{d-1}, Z_p)\), \(k = 0, 1, 2, \ldots\). □
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