Estimates for moments of supremum of reflected fractional Brownian motion

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Abstract

Let $B_H(\cdot)$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1]$. Motivated by applications to maximal inequalities for fractional Brownian motion, in this note we derive bounds for

$$K_T(H, \gamma) := \mathbb{E} \left[ \sup_{t \in [0,T]} |B_H(t)|^\gamma \right],$$

with $\gamma, T > 0$.

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1 Introduction

In this note we study properties of

$$K_T(H, \gamma) := \mathbb{E} \left[ \sup_{t \in [0,T]} |B_H(t)|^\gamma \right]$$

for $\gamma, T > 0$, where $B_H(\cdot)$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1]$, i.e., a centered Gaussian process with stationary increments and variance function $\text{var}(B_H(t)) = t^{2H}$.

Constants $K_T(H, \gamma)$ appear in the context of analyzing maximal inequalities for fractional Brownian motion; see e.g. [5]. The aim of this paper is to give bounds for $K_T(H, \gamma)$. To our best knowledge the exact values of $K_T(H, \gamma)$ are not known.

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Let
\[ \beta(s) = \sum_{k=0}^{\infty} (-1)^k (2k + 1)^{-s}, \]
s > 0, be the Dirichlet beta function. By \( E_n(x) \), \( n = 0, 1, \ldots \), we denote the Euler polynomials; \( E_n := 2^n E_n(1/2) \), \( n = 0, 1, \ldots \), stand for Euler numbers; see [1], page 804. Additionally, let \( \Psi(t) = \mathbb{P}(N > t) \) where \( N \sim \mathcal{N}(0,1) \).

The use of technique based on comparison of Gaussian processes yields the following theorem.

**Theorem 1.1** Let \( \gamma > 0 \).

(i) If \( H < 1/2 \), then
\[ K_T(H, \gamma) \geq T^{\gamma H} \frac{1}{\sqrt{\pi}} 2^{\frac{3}{2}} \Gamma \left( \frac{\gamma + 1}{2} \right). \]

(ii) If \( H \geq 1/2 \), then
\[ T^{\gamma H} \frac{1}{\sqrt{\pi}} 2^{\frac{3}{2}} \Gamma \left( \frac{\gamma + 1}{2} \right) \leq K_T(H, \gamma) \leq T^{\gamma H} \frac{1}{\sqrt{\pi}} 2^{\frac{3}{2} + 1} \Gamma \left( \frac{\gamma + 1}{2} \right). \]

In the following proposition we calculate exact value of \( K_T(1/2, \gamma) \) and \( K_T(1, \gamma) \).

**Proposition 1.2** Let \( \gamma > 0 \). Then

(i) \( K_T(1/2, \gamma) = \frac{1}{\sqrt{\pi}} 2^{1+\frac{3}{2}} \Gamma \left( \frac{\gamma + 1}{2} \right) \beta(\gamma) T^{\gamma} ; \)

(ii) \( K_T(1, \gamma) = \frac{1}{\sqrt{\pi}} 2^{\frac{3}{2}} \Gamma \left( \frac{\gamma + 1}{2} \right) T^{\gamma} . \)

The detailed proofs of Proposition 1.2 and Theorem 1.1 are deferred to Section 2.

As an immediate consequence of Proposition 1.2 and view of [1], page 805, we have
\[ K_1(1/2, 2n + 1) = \sqrt{\frac{\pi}{2}} \left( \frac{\pi^2}{2} \right)^n \frac{n!}{(2n)!} |E_{2n}| \] \hspace{1cm} (2)

\[ K_1(1/2, 2n) = \frac{(-1)^n}{(n-1)!} \left( \frac{\pi^2}{2} \right)^n \int_0^1 E_{2n-1}(x) sec(\pi x) dx, \]

for \( n = 1, 2, \ldots \). In particular \( K_1(1/2, 1) = \sqrt{\pi/2} \) and \( K_{1/2}(1/2, 2) \) is the Catalan’s constant.

The comparison of the upper bound for \( K_1(1/2, 2n + 1) \) given in Theorem 1.1 with (2) enables us to recover the known inequality for Euler numbers
\[ |E_{2n}| \leq \frac{4^{n+1} (2n)!}{\pi^{2n+1}} ; \]

see, e.g., [1], page 805. We refer to [4] for other results that relate moments of functionals of Brownian motion with number theory.
2 Proofs

In this section we present complete proofs of Proposition 1.2 and Theorem 1.1. We frequently use the fact that the property of self-similarity of fractional Brownian motion enables us to write

\[ K_T(H, \gamma) = K_1(H, \gamma) T^{\gamma H}. \]

We start with an auxiliary result which is also of independent interest.

**Lemma 2.1** Let \( \{X(t) : t \geq 0\} \) be a centered Gaussian process with stationary increments and continuous and strictly increasing variance function \( \sigma^2_X(\cdot), X(0) = 0 \) a.s.

(i) If \( \sigma^2_X(\cdot) \) is concave, then

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} X(t) \right]^\gamma \geq \left( \sigma^2_X(T) \right)^{\gamma} \frac{1}{\sqrt{\pi}} \frac{2^{\gamma}}{\Gamma \left( \frac{\gamma + 1}{2} \right)}.
\]

(ii) If \( \sigma^2_X(\cdot) \) is convex, then

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} X(t) \right]^\gamma \leq \left( \sigma^2_X(T) \right)^{\gamma} \frac{1}{\sqrt{\pi}} \frac{2^{\gamma}}{\Gamma \left( \frac{\gamma + 1}{2} \right)}.
\]

**Proof** Since the proof of (ii) is analogous to the proof of (i), we focus on the argument that justifies (i). Assume that \( \sigma^2_X(\cdot) \) is concave. Observe that for \( Y(t) := B_{\frac{1}{2}} \left( \sigma^2_X(t) \right) \) we have

\[
\mathbb{V}ar \left( Y(t) \right) = \mathbb{V}ar \left( B_{\frac{1}{2}} \left( \sigma^2_X(t) \right) \right) = \sigma^2_X(t) = \mathbb{V}ar \left( X(t) \right)
\]

for all \( t \in [0, T] \) and, due to concavity of \( \sigma^2_X(\cdot) \),

\[
\mathbb{V}ar \left( Y(t) - Y(s) \right) = \mathbb{V}ar \left( B_{\frac{1}{2}} \left( \sigma^2_X(t) \right) - B_{\frac{1}{2}} \left( \sigma^2_X(s) \right) \right) = \sigma^2_X(t) - \sigma^2_X(s) \leq \sigma^2_X(t - s) = \mathbb{V}ar \left( X(t) - X(s) \right)
\]

for all \( t > s \) and \( s, t \in [0, T] \). Thus, using Slepian inequality (see, e.g., Theorem 2.1 in Adler [2]),

\[
P \left( \sup_{t \in [0,T]} X(t) > x \right) \geq P \left( \sup_{t \in [0,T]} B_{\frac{1}{2}} \left( \sigma^2_X(t) \right) > x \right)
\]
for all $x \geq 0$. Since $P\left(\sup_{t \in [0,T]} B_{\frac{1}{2}}(\sigma_X^2(t)) > x\right) = P\left(\sup_{t \in [0,\sigma_X^2(T)]} B_{\frac{1}{2}}(t) > x\right)$, then we get

$$E\left[\sup_{t \in [0,T]} X(t)\right]^\gamma \geq E\left[\sup_{t \in [0,\sigma_X^2(T)]} B_{\frac{1}{2}}(t)\right]^\gamma.$$  

Due to self-similarity of Brownian motion we have

$$E\left[\sup_{t \in [0,\sigma_X^2(T)]} B_{\frac{1}{2}}(t)\right]^\gamma = E\left[\sup_{t \in [0,1]} B_{\frac{1}{2}}(\sigma_X^2(T) t)\right]^\gamma = \left(\sigma_X^2(T)\right)^{\frac{\gamma}{2}} E\left[\sup_{t \in [0,1]} B_{\frac{1}{2}}(t)\right]^\gamma.$$  

Finally, using that $P\left(\sup_{t \in [0,1]} B_{\frac{1}{2}}(t) > t\right) = 2P\left(\mathcal{N} > t\right)$, we get

$$E\left[\sup_{t \in [0,1]} B_{\frac{1}{2}}(t)\right]^\gamma = \int_0^\infty \gamma x^{\gamma-1} 2\psi(x) \, dx = \frac{1}{\sqrt{\pi}} 2^{\frac{\gamma}{2}} \Gamma\left(\frac{\gamma+1}{2}\right).$$  

Combination of (4) with (5) completes the proof of (i). \hfill \square

### 2.1 Proof of Theorem 1.1

Following (3) we consider only the case of $T = 1$. Note that

$$P\left(\sup_{t \in [0,1]} B_H(t) > x\right) \leq P\left(\sup_{t \in [0,1]} |B_H(t)| > x\right) \leq 2P\left(\sup_{t \in [0,1]} B_H(t) > x\right).$$  

The combination of the above with Lemma 2.1 completes the proof of (i) and the upper bound in (ii).

To prove the lower bound in (ii) we use that

$$E\left[\sup_{t \in [0,1]} |B_H(t)|\right]^\gamma \geq E|B_H(1)|^\gamma = E|\mathcal{N}|^\gamma = \frac{1}{\sqrt{\pi}} 2^{\frac{\gamma}{2}} \Gamma\left(\frac{\gamma+1}{2}\right).$$  

This completes the proof. \hfill \square
2.2 Proof of Proposition 1.2

Ad (i). Assume that $H = \frac{1}{2}$. Following (3) it suffices to analyze $K_1(1/2, \gamma)$. Using formula 1.1.4 in [3], we get

$$K_1(1/2, \gamma) = \mathbb{E} \left[ \sup_{t \in [0,1]} \left| B_{1/2}(t) \right| \right]^{\gamma}$$

$$= \gamma \int_0^\infty x^{\gamma-1} \mathbb{P} \left( \sup_{t \in [0,1]} \left| B_{1/2}(t) \right| > x \right) dx$$

$$= \gamma \int_0^\infty x^{\gamma-1} \sum_{k=-\infty}^{\infty} \left\{ (-1)^k \text{sign}((2k+1)x) \frac{2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{2}}(2k+1)}^{\infty} e^{-s^2} ds \right\} dx. \tag{6}$$

Let

$$f_n(x) := x^{\gamma-1} \sum_{k=-n-1}^{n} \left\{ (-1)^k \text{sign}((2k+1)x) \frac{2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{2}}(2k+1)}^{\infty} e^{-s^2} ds \right\}$$

$$= 2x^{\gamma-1} \sum_{k=0}^{n} \left\{ (-1)^k \frac{2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{2}}(2k+1)}^{\infty} e^{-s^2} ds \right\}$$

and observe that for each $n \geq 0$ and $x \in (0, 1]$

$$0 \leq f_n(x) \leq f_0(x) \leq 2x^{\gamma-1}. \tag{7}$$

Additionally, for each $n \geq 0$ and $x > 1$, we have

$$|f_n(x)| \leq x^{\gamma-1} \sum_{k=0}^{n} \frac{1}{\sqrt{2\pi}} \int_{\frac{x}{2}(2k+1)}^{\infty} e^{-s^2} ds = 4x^{\gamma-1} \sum_{k=0}^{n} \Psi(x(2k+1))$$

$$\leq \frac{4}{\sqrt{2\pi}} x^{\gamma-1} \sum_{k=0}^{n} \frac{1}{x(2k+1)} e^{-\frac{x^2(2k+1)^2}{2}}$$

$$\leq \frac{4}{\sqrt{2\pi}} x^{\gamma-2} e^{-\frac{x^2}{2}} \sum_{k=0}^{\infty} \left( e^{-x^2} \right)^k \tag{8}$$

$$= \frac{4}{\sqrt{2\pi}} x^{\gamma-2} e^{-\frac{x^2}{2}} \frac{1}{1 - e^{-x^2}}, \tag{9}$$

where (8) follows from the fact that $\Psi(t) \leq \frac{1}{\sqrt{2\pi t}} \exp\left( -\frac{t^2}{2} \right)$ for each $t \geq 0$. The combination of (7) with (9) implies that $|f_n(\cdot)|$ is bounded by an integrable function, and hence, by Lebesgue’s dominated convergence theorem, we can rewrite (6) in
the following form

\[ K_1(1/2, \gamma) = \gamma \frac{4}{\sqrt{\pi}} \sum_{k=0}^{\infty} \left\{ (-1)^k \int_0^\infty \left\{ x^{\gamma-1} \int_0^{\frac{x}{\sqrt{2k+1}}} e^{-s^2} ds \right\} dx \right\} . \]

The change of the order of integration in \( I_k \) leads to

\[ I_k = \frac{1}{\gamma} \left( \frac{\sqrt{2}}{2k+1} \right)^\gamma \int_0^\infty s^{\gamma-2} ds = \frac{1}{2\gamma} \left( \frac{\sqrt{2}}{2k+1} \right)^\gamma \int_0^\infty e^{-t^{\frac{\gamma-1}{2}}} dt \]

\[ = \frac{1}{2\gamma} \left( \frac{\sqrt{2}}{2k+1} \right)^\gamma \Gamma \left( \frac{\gamma+1}{2} \right), \]

which implies that

\[ K_1(1/2, \gamma) = \frac{1}{\sqrt{\pi}} 2^{\frac{\gamma+1}{2}} \Gamma \left( \frac{\gamma+1}{2} \right) \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{2k+1} \right)^\gamma = \frac{1}{\sqrt{\pi}} 2^{\frac{\gamma+1}{2}} \Gamma \left( \frac{\gamma+1}{2} \right) \beta(\gamma). \]

This completes the proof of \( (i) \).

**Ad (ii).** Let \( H = 1 \). Since \( B_1(t) = d\mathcal{N}t \), where \( \mathcal{N} \sim \mathcal{N}(0,1) \), then \( \mathbb{P}\left( \sup_{t\in[0,1]} |B_1(t)| > x \right) = 2\Psi(x) \). Standard integration completes the proof. \( \square \)

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