1. Introduction

The scalar curvature of a Riemannian manifold \((M^n, g)\) is the sum of all sectional curvatures. A complete metric \(g\) on \(M\) is said to have positive scalar curvature if the scalar curvature is positive at all points in \(M\). There are several topological obstructions for a smooth manifold to have a complete metric with positive scalar curvature.

For example, the proof of Thurston’s Geometrization conjecture [Per02a, Per02b, Per03] shows that a closed 3-manifold admits a metric with positive scalar curvature if and only if it is a connected sum of spherical 3-manifolds and some copies of \(S^1 \times S^2\). Recently, Besson, Bessi`eres and Maillot [BBM11] showed a similar result for (non-compact) 3-manifolds with bounded geometry.

A fundamental and important question one may ask is how to classify non-compact 3-manifolds with positive scalar curvature. There is a few related result except a series of works by Gromov-Lawson [GL83] and by Schoen-Yau [SY79, SY82].

The topological structure of compact 3-manifolds is fully understood (see Thurston’s Geometrization Conjecture [MT07, BBB+10]). However, the topological structure of non-compact 3-manifolds is much more complicated. For example, the Whitehead manifold is a contractible 3-manifold but not homeomorphic to \(\mathbb{R}^3\) (see [Whi35]).

One of the approaches to the classifying question is to study whether the positivity of scalar curvature implies some topological rigidity for non-compact 3-manifolds. The simplest case is that of contractible 3-manifolds.
For example, $\mathbb{R}^3$ admits a complete metric $g_0$ with positive scalar curvature,

$$g_0 = \sum_{i=1}^{3} dx_i^2 + \left( \sum_{i=1}^{3} x_i dx_i \right)^2.$$  

It is the only known contractible 3-manifold which has a complete positive scalar curvature metric. This suggests the rigidity question:

**Question:** Is any complete contractible 3-manifold with positive scalar curvature homeomorphic to $\mathbb{R}^3$?

Gromov-Lawson [GL83] and Chang-Weinberger-Yu [CWY10] showed that a complete contractible 3-manifold with uniformly positive scalar curvature (i.e. the scalar curvature is bounded away from zero) is homeomorphic to $\mathbb{R}^3$. Recently, we showed that a complete contractible 3-manifold, with positive scalar curvature and with trivial fundamental group at infinity, is homeomorphic to $\mathbb{R}^3$ (see [Wan19a, Wan19b]).

In this paper, we give a full and positive answer and show that

**Theorem 1.1.** Any complete contractible 3-manifold with positive scalar curvature is homeomorphic to $\mathbb{R}^3$.

We combine [Kaz82] (see Theorem 8.2 in [Wan19a]) to show that

**Corollary 1.2.** Any complete contractible 3-manifold with non-negative scalar curvature is homeomorphic to $\mathbb{R}^3$.

### 1.1. Simply-connectedness at infinity vs. Triviality at infinity.

Our main focus of this paper is on the understanding of how the positivity of scalar curvature influences the properties at infinity of manifolds. We are interested not only in the topology at infinity but also in the geometry at infinity.

In order to describe the topology at infinity, let us introduce the *simply-connectedness at infinity*.

- A manifold $X$ is simply-connected for a compact set $K \subset X$, if there is a compact set $K' \subset X$ containing $K$, so that the induced map $\pi_1(X \setminus K') \to \pi_1(X \setminus K)$ is trivial (i.e. any circle in $M \setminus K'$ is contractible in $M \setminus K$).

- A manifold $X$ is simply-connected at infinity if it is simply-connected for any compact subset of $X$.

The only contractible and simply-connected at infinity 3-manifold is $\mathbb{R}^3$ (see [Sta72, HP71] and the Poincaré conjecture [BBB+10, MT07]). For example, the Whitehead manifold is not simply-connected at infinity [Whi35].

The simply-connectedness at infinity has great influence on the geometry of a complete contractible 3-manifold $(M, g)$. Especially, if $M$ is not simply-connected at infinity, there is a lamination (a disjoint union of embedded surfaces) whose leaves are stable minimal planes (see Sections 3 and 5).

Generally, the topological structure of contractible 3-manifolds is characterized by the simply-connectedness at infinity. This notion has a delicate
but fruitful connection with minimal planes. Their relationship is embodied by the triviality at infinity of minimal planes.

Let \( L \subset (M, g) \) be a complete properly embedded minimal plane (i.e. homeomorphic to \( \mathbb{R}^2 \)).

- \( L \) is trivial for a compact set \( K \subset M \) if there is a compact set \( K' \subset M \) containing \( K \) so that the induced map \( \pi_1(L \setminus K') \to \pi_1(M \setminus K) \) is trivial.
- \( L \) is trivial at infinity if it is trivial for any compact subset of \( M \).

Notice that there exists a non-properly embedded minimal plane in some contractible 3-manifold.

**Remark.** Let \( (M, g) \) be a complete contractible 3-manifold and \( L \) a properly embedded minimal plane.

- If \( L \) is not trivial for some compact set \( K \subset M \), then \( M \) is not simply-connected for \( K \).
- If \( L \) is not trivial at infinity, then \( M \) is not simply-connected at infinity.

Roughly speaking, if a complete contractible 3-manifold \( (M, g) \) is not simply-connected for some compact set \( K \subset M \) (at infinity), there is a stable minimal plane \( L \subset (M, g) \) which is not trivial for \( K \) (at infinity).

When considering the above rigidity question, it becomes necessary and crucial to study the triviality at infinity and its relationship with positive scalar curvature. In this article, we suggest their relationship as follows:

**Theorem 1.3.** Let \( (M, g) \) be a complete contractible 3-manifold with positive scalar curvature. Then each complete embedded stable minimal plane in \( (M, g) \) is trivial at infinity.

It gives us an intriguing circle involving the topology at infinity, minimal planes and positive scalar curvature. That is, the simply-connectedness at infinity gives a family of stable minimal planes. Then the positivity of scalar curvature implies the triviality at infinity of minimal planes. The triviality at infinity shows the simply-connectedness at infinity.

1.2. **Idea of the proof of Theorem 1.1.** We explain the idea of the proof of Theorem 1.1. Our main strategy pertains to the same philosophy as in [GL83] and [SY82].

Suppose to the contrary that a complete contractible 3-manifold \( (M, g) \) with positive scalar curvature is not homeomorphic to \( \mathbb{R}^3 \) (i.e. it is not simply-connected at infinity). It is an increasing union of handlebodies \( \{N_k\} \) (see Theorem 2.5).

By the definition of simply-connectedness at infinity, we may assume that \( M \) is not simply-connected for \( N_0 \).

Before constructing minimal surfaces, let us introduce a topological notation. An embedded circle \( \gamma \subset \partial N \) is a **meridian curve** of a handlebody \( N \) if it is contractible in \( N \) but non-contractible in \( \partial N \).
Choose a meridian curve $\gamma_k$ of $N_k$, which is not homotopically trivial in $M \setminus N_0$. Its existence is ensured by the fact that $M$ is not simply-connected for $N_0$ (see Lemma 2.14).

We could find an area-minimizing disc $D_k \subset N_k$ with boundary $\gamma_k$. Its existence is ensured by the result of Meeks and Yau [MY80], when the boundary $\partial N_k$ is mean convex (i.e. the mean curvature is non-negative).

If $D_k$ converges to a minimal plane $L$, the contradiction comes from Theorem 1.3. Since $\gamma_k$ is not homotopically $M \setminus N_0$ for $k > 0$, we show that $L$ is not trivial for $N_0$. It is in contradiction with the fact that $L$ is trivial at infinity (see Theorem 1.3).

However, in general, the sequence $\{D_k\}$ sub-converges to a lamination $L$ instead of a single surface (see Theorem 5.5). It may have infinitely many components. Each component is a complete stable minimal plane.

In this article, we use the positivity of scalar curvature to show that each component of $L$ is trivial at infinity (see Theorem 1.3). That is to say, there is an integer $k_0 > 0$ so that the induced map $\pi_1(L \setminus \text{Int} N_{k_0}) \to \pi_1(M \setminus N_0)$ is trivial for each $L \in L$ (see Corollary 5.13).

However, since $\gamma_k$ is not homotopically trivial in $M \setminus N_0$, we can find a non-contractible circle $c_k \subset D_k \cap \partial N_{k_0}$ in $M \setminus N_0$ (see Lemma 2.7). Roughly speaking, we show that, under the Hausdorff topology, these non-contractible circles sub-converges to a circle in $L \cap \partial N_{k_0}$ which is not null-homotopic in $M \setminus N_0$, as $D_k$ sub-converges to $L$. Namely, the map $\pi_1(L \setminus \text{Int} N_{k_0}) \to \pi_1(M \setminus N_0)$ is non-trivial for some leaf $L$ in $L$, which is in contradiction with the last paragraph.

If $\partial N_k$ is not mean convex, we modify the metric near $\partial N_k$ so that it is mean convex for the new metric. The disc $D_k$ is stable minimal for the new metric and for the original one away from $\partial N_k$ (near $N_{k-1}$ for example) which is sufficient for our proof.

Actually, the convergence of the circles, $\{c_k\}$, in the Hausdorff topology comes from the convergence with finite multiplicity of subsurfaces of $\{D_k\}$, which requires a local area estimate for $\{D_k\}$ (stated in Theorem 4.9). However, the area-minimizing property could not provide such an estimate.

For overcoming it, we modify the construction of $D_k$ so that the sequence $\{D_k\}$ not only has the area-minimizing property but also satisfies the so-called disjointness property (see Section 3.1). These two properties could give the required area estimate (see Theorem 4.9 and Remark 4.10).

The existence of such a sequence of $\{D_k\}$ is a combination of the maximum principle for minimal surfaces and a geometric version of loop lemma due to Meeks and Yau [MY82].

The paper is organized as follows. In Section 2, we recall the background and describe the topological structure of contractible 3-manifolds. In Sections 3 and 4, we construct the sequence $\{D_k\}$ and show the local area...
estimate for \( \{D_k\} \). In Section 5, we consider the limit of \( \{D_k\} \) and introduce the triviality at infinity. In Sections 6 and 7, we show the absence of non-trivial minimal plane at infinity and use it to complete the proof of Theorem 1.1.

This project was developed from my Ph.D project supported by ERC Advanced Grant 320939, GETOM. We are grateful to Gérard Besson for suggesting this question and for many helpful comments that improved the exposition of this paper. We would like to thank Misha Gromov, Blaine Lawson and Bernhard Hanke for their interest. This research is partially supported by the Special Priority Program SPP 2026, Geometry at infinity.

2. Contractible 3-manifolds and Handlebodies

In this section, our main focus is on the topological structure of contractible 3-manifolds. We begin with the simply-connectedness at infinity. Then we introduce the handlebody and explain the topological structure of contractible 3-manifolds.

2.1. Simply-connectedness at infinity.

Definition 2.1. A path-connected topological space \( M \) is simply-connected for a compact set \( K \subset M \), if there is a compact set \( K' \subset M \) containing \( K \) so that the induced map \( \pi_1(M \setminus K') \rightarrow \pi_1(M \setminus K) \) is trivial (i.e. its image is a trivial group).

The space \( M \) is simply-connected at infinity if it is simply-connected for any compact subset of \( M \).

For example, the Whitehead manifold is a contractible 3-manifold but not homeomorphic to \( \mathbb{R}^3 \). It is not simply-connected at infinity (see [Whi35]).

The Poincaré conjecture (see [Per02a, Per02b, Per03]) shows that a contractible 3-manifold is irreducible (i.e. any embedded 2-sphere in the 3-manifold bounds a 3-ball). A classical result (see [Sta72, HP71]) implies that the only contractible and simply-connected at infinity 3-manifold is \( \mathbb{R}^3 \).

Remark 2.2. If a contractible 3-manifold \( M \) is not homeomorphic to \( \mathbb{R}^3 \), it is not simply-connected at infinity. That is to say, it is not simply-connected for some compact set \( K \subset M \). Further, \( K \) is not contained in a 3-ball of \( M \).

Consider a properly embedded plane in a contractible 3-manifold. We have that

Lemma 2.3. Let \( M \) be a contractible 3-manifold and \( L \subset M \) a properly embedded plane. Then \( L \) cuts \( M \) into two contractible 3-manifolds.

Remark that there is a non-properly embedded plane in some contractible 3-manifold.
Proof. First, we use Mayer-Vietoris sequence to show that \( H_0(M \setminus L) \cong \mathbb{Z}^2 \).
That is to say, \( L \) cuts \( M \) into two components, \( M_1 \) and \( M_2 \).

Then, Van-Kampen’s Theorem shows that each \( M_i \) is simply-connected.
Mayer-Vietoris sequence implies that \( H_j(M_i) = \{1\} \) for \( i = 1, 2 \) and \( j \geq 1 \).
We inductively use Hurewicz’s theorem (see [Theorem 4.32, Page 366] in [Hat05]) to have that \( \pi_j(M_i) = \{1\} \) for \( i = 1, 2 \) and \( j \geq 1 \).
Whitehead’s Theorem (see [Theorem 4.5, Page 346] in [Hat05]) implies that each \( M_i \) is contractible.

\[ \square \]

2.2. Handebody.

**Definition 2.4.** (see Page 59 in [Rol03]) A closed handebody is any space obtained from the closed 3-ball \( D^3 \) (0-handle) by attaching \( g \) distinct copies of \( D^2 \times [-1, 1] \) (1-handle) with the homeomorphisms identifying the \( 2g \) discs \( D^2 \times \{ \pm 1 \} \) to \( 2g \) disjoint 2-disks on \( \partial D^3 \), all to be done in such a way that the resulting 3-manifold is orientable. The integer \( g \) is called the genus of the handebody.

Let us remark that a handebody of genus \( g \) is homeomorphic to a boundary connected sum of \( g \) solid tori. Therefore, its boundary is a compact surface of genus \( g \). (See Page 59 in [Rol03])

We use a result of McMillan [McM61] and the Poincaré conjecture (see [BBB+10, MT07]) to have that

**Theorem 2.5.** ([Theorem 1, Page 511] in [McM01]) Any contractible 3-manifold can be written as an ascending union of handebodies.

**Definition 2.6.** Let \( \{c_i\}_{i \in I} \) be a finite set of pairwise disjoint circles in the disc \( D^2 \) and \( D_i \subset D^2 \) the unique disc with boundary \( c_i \). Consider the set \( \{D_i\}_{i \in I} \) and define a partially ordered relation induced by the inclusion. \( (\{D_i\}_{i \in I}, \subset) \) is a partially ordered set. For each maximal element \( D_j \) in \( (\{D_i\}_{i \in I}, \subset) \), its boundary \( c_j \) is defined as a maximal circle of \( \{c_i\}_{i \in I} \) in \( D \).

**Lemma 2.7.** Let \( M \) be a 3-manifold and \( N \subset M \) a closed handlebody. Assume that the circle \( \gamma \subset M \setminus N \) bounds an immersed disc \( D \). If any closed circle in \( D \cap \partial N \) is homotopically trivial in \( M \setminus K \) where \( K \subset \text{Int} N \) is a compact set, then \( \gamma \) is also null-homotopic in \( M \setminus K \).

**Proof.** We may assume that \( D \) intersects \( \partial N \) transversally. The intersection \( D \cap \partial N \) has finitely many components, \( \{\gamma_i\}_{i \in I} \). Each component \( \gamma_i \) bounds a disc \( D_i \subset D \). Since each \( \gamma_i \) is homotopically trivial in \( M \setminus K \), we find an immersed disc \( D'_i \subset M \setminus K \) with boundary \( \gamma_i \).

Choose the set \( C^{max} \) consisting of all maximal circles of \( \{\gamma_i\}_{i \in I} \) in \( D \) (see Definition 2.5). The maximality shows that \( \{D_i\}_{\gamma_i \in C^{max}} \) are disjoint. The set \( D \setminus \bigcup_{\gamma_i \in C^{max}} D_i \) is contained in \( M \setminus N \). Consider the immersed disc

\[ D' = (D \setminus \bigcup_{\gamma_i \in C^{max}} D_i) \bigcup (\bigcup_{\gamma_i \in C^{max}} D'_i) \subset M \setminus K \]

with boundary \( \gamma \). Therefore, \( \gamma \) is null-homotopic in \( M \setminus K \). \[ \square \]
Corollary 2.8. Let $M$ be a contractible 3-manifold and $N \subset M$ a closed handlebody. If $M$ is not simply-connected for some compact set $K \subset \text{Int} \ N$, then the induced map $\pi_1(\partial N) \rightarrow \pi_1(M \setminus K)$ is not a trivial map.

Proof. Suppose that the induced map $\pi_1(\partial N) \rightarrow \pi_1(M \setminus K)$ is trivial (i.e. any circle in $\partial N$ is homotopically trivial in $M \setminus K$).

First, any embedded circle $\gamma \subset M \setminus N$ bounds an immersed disc $D \subset M$ which comes from the solution to Plateau problem (see [Mor09], [Gul73] and [Oss70]). Then, we use the same proof as in Lemma 2.7 to show that $\gamma$ is contractible in $M \setminus K$.

We can conclude that $\pi_1(M \setminus N) \rightarrow \pi_1(M \setminus K)$ is trivial, a contradiction with the fact that $M$ is not simply-connected for $K$. \hfill \Box

2.3. Meridian curves.

Definition 2.9. (See [Definition 3.1, Page 7] in [Wan19b]) A meridian curve $\gamma \subset \partial N$ of the handlebody $N$ is an embedded closed curve which is homotopically trivial in $N$ but non-contractible in $\partial N$.

A meridian disc $(D, \partial D) \subset (N, \partial N)$ is an embedded disc whose boundary is a meridian curve of $N$.

Let us recall the definition of push-out in the category of groups. Let $A$, $B$ and $C$ be groups and let $\phi : A \rightarrow B$ and $\psi : A \rightarrow C$ be (not necessarily injective) homomorphisms. The push-out is a triple $(D, i, j)$ with $i \circ \phi = j \circ \psi$ and with the following universal property: for every triple $(P, i', j')$ with $i' \circ \phi = j' \circ \psi$, there is a unique homomorphism $u : D \rightarrow P$ making the digram commute.

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow{\psi} & & \downarrow{\psi} \\
C & \xrightarrow{i} & D
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow{\psi} & & \downarrow{\psi} \\
C & \xrightarrow{j} & D
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow{\psi} & & \downarrow{\psi} \\
C & \xrightarrow{i'} & D
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow{\psi} & & \downarrow{\psi} \\
C & \xrightarrow{j'} & D
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow{\psi} & & \downarrow{\psi} \\
C & \xrightarrow{u} & P
\end{array}
$$

In the category of groups, the push-out always exists (see [Theorem 11.58, Page 395] of [Rot12]). It is unique up to isomorphisms.

Remark 2.10. If $i'$ and $j'$ are both trivial (i.e. the images are isomorphic to \{1\}) for any triple $(P, i', j')$ with $i' \circ \phi = j' \circ \psi$, then $D$ is a trivial group.

Proposition 2.11. Let

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow{\psi} & & \downarrow{(\ast)} \\
C & \xrightarrow{\ast} & \{1\}
\end{array}
$$

be a push-out in the category of groups. Assume that $\psi$ is surjective and $G$ is a normal subgroup of $B$. If $\phi^{-1}(G)$ contains $\ker(\psi)$, then $A = \phi^{-1}(G)$, $B = G$ and $C = \psi(\phi^{-1}(G))$. 
Proof. The group $\phi^{-1}(G)$ is a normal subgroup of $A$. Because $\psi$ is surjective, then $\psi(\phi^{-1}(G))$ is a normal subgroup of $C$.

Since $\ker(\psi) \subset \phi^{-1}(G)$, the map $\bar{\psi} : A/\phi^{-1}(G) \to C/\psi(\phi^{-1}(G))$ is injective. In addition, the map $\bar{\phi} : A/\phi^{-1}(G) \to B/G$ is also injective.

We have $i \circ \bar{\phi} = \bar{j} \circ \bar{\psi}$. Further, the digram $(\ast)$ is a push-out in the category of groups. The reason is as follows:

$$
\begin{array}{ccc}
A/\phi^{-1}(G) & \xrightarrow{\bar{\phi}} & B/G \\
\downarrow{\bar{\psi}} & & \downarrow{i} \\
C/\psi(\phi^{-1}(G)) & \xrightarrow{\bar{j}} & \{1\} \\
\downarrow{\bar{j}'} & & \downarrow{\bar{\psi}} \\
P & & \\
\end{array}
$$

For any triple $(P, \tilde{i}', \tilde{j}')$ with $\tilde{i}' \circ \bar{\phi} = \tilde{j}' \circ \bar{\psi}$, consider the map $\tilde{i}' : B \to B/G \to P$, the composition of the quotient map and $\tilde{i}'$. Similarly, we have the map $\tilde{j}' : C \to C/\psi(\phi^{-1}(G)) \to P$ and $\tilde{i}' \circ \phi = \tilde{j}' \circ \psi$.

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow{\psi} & & \downarrow{i} \\
C & \xrightarrow{j} & \{1\} \\
\downarrow{j'} & & \downarrow{\tilde{i}'} \\
C/\psi(\phi^{-1}(G)) & \xrightarrow{j'} & P \\
\end{array}
$$

Since the digram $(\ast)$ is a push-out in the category of groups, there is a map $u : \{1\} \to P$ making the digram commute. Thus, $\tilde{i}'$ and $\tilde{j}'$ are both trivial. By Remark 2.10, the digram $(\ast)$ is a push-out in the category of groups.

Since $\bar{\phi}$ and $\bar{\psi}$ are both injective, we use [Theorem 11.67, Page 404] in [Rot12] to show that $\bar{i}$ and $\bar{j}$ are also injective. Namely, $B = G$ and $C = \psi(\phi^{-1}(G))$. In consequence, $A/\phi^{-1}(G)$ is trivial (i.e. $A = \phi^{-1}(G)$).

Let $M$ be a contractible 3-manifold and $N \subset M$ a closed handlebody. Van-Kampen’s theorem (see [Theorem 11.60, Page 396] in [Rot12]) gives the following push-out in the category of groups:

$$
\begin{array}{ccc}
\pi_1(\partial N) & \xrightarrow{\phi} & \pi_1(M \setminus N) \\
\downarrow{\psi} & & \downarrow{\bar{i}} \\
\pi_1(N) & \xrightarrow{j} & \{1\} \\
\end{array}
$$

Assume that $K \subset \text{Int } N$ is a compact set and $G \subset \pi_1(M \setminus N)$ is the kernel of the map $\pi_1(M \setminus N) \to \pi_1(M \setminus K)$. Then we have that

- $\phi^{-1}(G) = \ker(\pi_1(\partial N) \to \pi_1(M \setminus K))$;
the map $\psi$ is surjective, since $N$ is a handlebody.

**Remark 2.12.** If $M$ is not simply-connected for $K$, then the map $\pi_1(\partial N) \to \pi_1(M \setminus K)$ is non-trivial (see Corollary 2.8). Namely, $\phi^{-1}(G)$ is a proper subgroup of $\pi_1(\partial N)$.

We can conclude that $\ker(\psi)$ is not a subset of $\phi^{-1}(G)$. (If not, Proposition 2.11 shows that $\phi^{-1}(G) = \pi_1(\partial N)$, a contradiction with the last paragraph.)

In the following, we use the loop lemma to find a required meridian curve.

**Lemma 2.13.** ([Theorem 4.2, Page 39] in [Hem04]) Let $M$ be a 3-manifold and $F \subset \partial M$ a connected 2-manifold. If $G_0$ is a normal subgroup of $\pi_1(F)$ and if $\ker(\pi_1(F) \to \pi_1(M)) \setminus G_0 \neq \emptyset$, then there is an embedding $f : (D^2, \partial D^2) \to (M, F)$ such that $[f|_{\partial D^2}] \notin G_0$.

**Lemma 2.14.** Let $M$ be a contractible 3-manifold and $N \subset M$ a closed handlebody with $g(N) > 0$. If $M$ is not simply-connected for some compact subset $K \subset \text{Int } N$, then there is a meridian curve of $N$ which is not homotopically trivial in $M \setminus K$.

**Proof.** Let $\phi$, $\psi$ and $G$ be assume as above. The group $\phi^{-1}(G)$ is the kernel of the map $\pi_1(\partial N) \to \pi_1(M \setminus K)$, a normal subgroup of $\pi_1(\partial N)$.

Since $M$ is not simply-connected for $K$, we use Remark 2.12 to show that $\ker(\pi_1(\partial N) \to \pi_1(N)) \setminus \phi^{-1}(G)$ is not empty. Lemma 2.13 gives an embedding $f : (D^2, \partial D^2) \to (N, \partial N)$ with $[f|_{\partial D^2}] \notin \phi^{-1}(G)$. That is to say, $f(\partial D^2)$ is a meridian curve of $N$ but non-contractible in $M \setminus K$. \qed

### 3. Minimal surfaces

It is classical that minimal surfaces can give topological information about 3-manifolds. This fact appeared in Schoen-Yau's works [SY79b, SY79a, SY82] as well as in Gromov-Lawson’s [GL83] and various other authors.

In this section, we use the topological property of contractible 3-manifolds to construct a sequence of stable minimal discs. Then we study the properties of these discs, such as the area-minimizing property and the disjointness property.

#### 3.1. Construction of the required discs.

Consider a complete contractible 3-manifold $(M, g)$. From Theorem 2.15 it is an increasing union of handlebodies $\{N_k\}_{k=0}^{\infty}$.

Our first step is to use 3-dimensional topology to find an embedded minimal disc in each $N_k$. Meeks and Yau [MYS2] provide a geometric version of the loop lemma to construct such discs.

**Theorem 3.1.** (See [MYS2] and [Theorem 6.28, Page 224] in [CM11]) Let $(X^3, g)$ be a compact Riemannian 3-manifold whose boundary is mean convex and $\gamma$ a simple closed curve in $\partial X$ which is null-homotopic in $X$. Then, $\gamma$ bounds an area-minimizing disc and any such least area disc is embedded.
However, Meeks-Yau’s results require a geometric condition, mean convex boundary. For overcoming it, we find a new metric $g_k$ over $M$ such that

- $g_k \geq g$ and $g_k$ is equal to $g$ on $N_{k-1}$;
- the mean curvature of the boundary of $(N_k, g_k|_{N_k})$ is positive.

The metric $g_k$ is constructed as below:

Let $h_k(t)$ be a smooth function on $\mathbb{R}$ so that $h_k(t) \geq 1$ and $h_k(t)|_{\mathbb{R} \setminus [-\epsilon, \epsilon]} = 1$. Consider the function $f_k(x) := h_k(d(x, \partial N_k))$ and the metric $g_k := f_k^2 g|_{N_k}$. Under $(N_k, g_k|_{N_k})$, the mean curvature $\hat{H}(x)$ of $\partial N_k$ is

$\hat{H}(x) = h_k^{-1}(0)(H(x) + 2h'_k(0)h_k^{-1}(0))$

Choosing $\epsilon$ small enough and a function $h_k$ with $h_k(0) = 2$ and $h'_k(0) > 2 \max_{x \in \partial N_k} |H(x)| + 2$, one gets the metric $g_k$ which is the required candidate in the assertion.

In the following, we assume that $M$ is not homeomorphic to $\mathbb{R}^3$. Namely, $M$ is not simply-connected for some compact set. We may assume that $M$ is not simply-connected for $N_0$. Then, the genus $g(N_k)$ is greater than zero for $k \geq 1$ (see Remark 2.2).

Lemma 2.11 gives a meridian curve $\gamma_k \subset \partial N_k$ of $N_k$ which is not homotopically trivial in $M \setminus N_0$ for $k \geq 1$.

We now construct a sequence $\{D_k\}_{k=0}^\infty$ of discs satisfying

- each $D_k \subset N_k$ is an embedded disc with boundary $\gamma_k$;
- The intersection $D_k \cap N_{k-1}$ consists of finitely many smooth surfaces $\{\Sigma^j_k\}_{j \in J_k}$ and each $\Sigma^j_k$ is stable minimal for the original metric $g$;
- (Disjointness) $\Sigma^j_{k'} \subset \Sigma^j_k$ or $\Sigma^j_{k'} \cap \Sigma^j_k = \emptyset$ where $k' < k$, $j \in J_k$ and $j' \in J_{k'}$;
- (Area-minimizing) $D_k$ is an area-minimizing disc with boundary $\gamma_k$ in the closure of the manifold $(W_k, g_k|_{W_k})$, where $W_k = N_k \cup \bigcup_{l=1}^{k-1} D_l \cap N_{l-1}$.

**Remark 3.2.** (1) The set $\bigcup_{l=1}^{k} D_l \cap N_{l-1}$ is a disjoint union of some surfaces in $\{\Sigma^j_k\}_{k \leq j \in J_k}$.

(2) The closure of the manifold $(W_k, g_k|_{W_k})$ is not equal to $N_k$. The boundary consists of two parts. One part is $\partial N_k$ while another part is the union of some surfaces $(N_{l-1} \cap D_l)^+$ and $(N_{l-1} \cap D_l)^-$ for $l \leq k-1$, where $(N_{l-1} \cap D_l)^\pm$ comes from the same minimal surface $N_{l-1} \cap D_l$.

Since the mean curvature $\partial N_k$ is positive for $g_k$, then the closure of $(W_k, g_k|_{W_k})$ has mean convex boundary.

**Remark 3.3.** The construction of $\{D_k\}_k$ is much more complicated than the construction in [Wan19a, Wan19b], because we expect to get the disjointness and area-minimizing properties. These two properties are crucial and necessary when considering the local area estimate for $D_k$ (see Remark 3.5 and Theorem 4.9).

Generally, the area-minimizing property could not give us the required area estimate. However, the disjointness property makes such an estimate available (see Section 4).
Let us describe the inductive construction of such a sequence.

When $k = 1$, the manifold $(N_1, g_1|_{N_1})$ has mean convex boundary. Theorem [3.1] gives an embedded disc $D_1 \subset N_1$ with boundary $\gamma_1$. It is an area-minimizing disc in $(N_1, g_1|_{N_1})$.

We may assume that $D_1$ intersects $\partial N_0$ transversally. The intersection $D_1 \cap N_0$ has finitely many components $\{\Sigma^j_1\}_{j \in J_1}$. Each component is smooth and stable minimal for the original metric $g$ (since $g_1|_{N_0} = g|_{N_0}$).

Suppose that there are $k_0$ embedded discs $\{D_k\}_{k=1}^{k_0}$ so that for $1 \leq k \leq k_0$

- each $D_k \subset N_k$ is an embedded disc with boundary $\gamma_k$;
- the intersection $D_k \cap N_{k-1}$ consists of finitely many smooth surfaces $\{\Sigma^j_k\}_{j \in J_k}$ and each $\Sigma^j_k$ is stable minimal for the original metric $g$;
- (Disjointness) $\Sigma^j_k' \subset \Sigma^j_k$ or $\Sigma^j_k \cap \Sigma^j_k' = 0$ where $k' < k$, $j \in J_k$ and $j' \in J_{k'}$;
- (Area-minimizing) $D_k$ is an area-minimizing disc with boundary $\gamma_k$ in the closure of the manifold $(W_k, g_k|_{W_k})$, where $W_k = N_k \setminus \bigcup_{l=1}^{k-1} D_l \cap N_{l-1}$.

When $k = k_0 + 1$, we consider the closure of the manifold

$$(W_{k_0+1}, g_{k_0+1}|_{W_{k_0+1}})$$

where $W_{k_0+1} := N_{k_0+1} \setminus \bigcup_{l=1}^{k_0} D_l \cap N_{l-1}$. By Remark [3.2] the closure of $(W_{k_0+1}, g_{k_0+1}|_{W_{k_0+1}})$ is a compact 3-manifold with mean convex boundary.

**Remark 3.4.** The circle $\gamma_{k_0+1}$ is also homotopically trivial in the closure of $(W_{k_0+1}, g_{k_0+1}|_{W_{k_0+1}})$. The reason is described as below:

Since $D_1 \subset \text{Int} \ N_{k_0+1}$ is an embedded disc, Van-Kampen's theorem gives that $\pi_1(N_{k_0+1}) \cong \pi_1(N_{k_0+1} \setminus D_1)$. Namely, $\gamma_{k_0+1}$ is contractible in $N_{k_0+1} \setminus D_1$. Hence, it is also contractible in $N_{k_0} \setminus D_1 \cap N_0$.

We repeat the above argument and have that $\gamma_{k_0+1}$ is contractible in the closure of $(W_{k_0+1}, g_{k_0+1}|_{W_{k_0+1}})$.

We use Theorem [3.1] to find an embedded disc $D_{k_0+1}$ in the closure of $(W_{k_0+1}, g_{k_0+1}|_{W_{k_0+1}})$. It is an area-minimizing disc with boundary $\gamma_{k_0+1}$.

We may assume that $D_{k_0+1}$ intersects $\partial N_{k_0}$ transversally. (Otherwise, we deform $\partial N_{k_0}$ in $N_{k_0}$ so that it intersects $D_{k_0+1}$ transversally.) The intersection $N_{k_0} \cap D_{k_0+1}$ has finitely many components, $\{\Sigma^j_{k_0+1}\}_{j \in J_{k_0+1}}$. Each $\Sigma^j_{k_0+1}$ is smooth and stable minimal for the original metric $g$.

Recall that $\Sigma^j_k$ is stable minimal for $g$, where $k' < k_0 + 1$ and $j' \in J_{k'}$. If $\Sigma^j_{k_0+1}$ meets the boundary of the closure of $(W_{k_0+1}, g_{k_0+1}|_{W_{k_0+1}})$ coming from some $\Sigma^j_{k'}$, the maximum principle (see [Corollary 1.28, Page 37] in [CMII]) shows that $\Sigma^j_{k'}$ is a subset of $\Sigma^j_{k_0+1}$. Otherwise, the intersection $\Sigma^j_{k_0+1} \cap \Sigma^j_{k'}$ is empty.

We finish the inductive construction and get the required candidate in our assertion.
Remark 3.5. For each $k_0$, there is a positive constant $C_{k_0}$, such that for $k > k_0$, the area of each component of $D_k \cap N_{k_0}$ is not greater than $C_{k_0}$. (See Theorem 4.9)

The local area estimate play a crucial role in the proof of Theorem 1.1. We will prove the estimate in Section 4.

3.2. The filling lemma. The area-minimizing property for $\{D_k\}$ implies a filling lemma (see Lemma 3.10). We use it to show the area estimate (stated in Remark 3.5 and Theorem 4.9) for some special cases.

All proofs in this subsection only depends on the area-minimizing property and the disjointness property.

Proposition 3.6. Let $\{D_k\}$ and $\{(W_k, g_k|_{W_k})\}$ be assumed as in Section 3.1. Assume that the circle $\gamma \subset \Sigma_{k_1}^{j_1}$ bounds an embedded disc $D \subset N_{k_0} - 1$ in the closure of $(W_{k_0}, g_{k_0}|_{W_{k_0}})$, where $k_1 \geq 1$ and $j_1 \in J_{k_1}$. Then, for each $\epsilon > 0$, $\gamma$ bounds an embedded disc $D' \subset N_{k_0}$ in the closure of $(W_{k_0+1}, g_{k_0+1}|_{W_{k_0+1}})$ satisfying that

$$\text{Area}_g(D') \leq \text{Area}_g(D) + \epsilon.$$

Remark. (1) The integer $k_1$ may be greater than $k_0$. In this case, $\Sigma_{k_1}^{j_1}$ is not a subset of $N_{k_0}$.

(2) Since $\gamma$ is a subset of $\Sigma_{k_1}^{j_1}$, it can be considered as a circle in the closure of $(W_k, g_k|_{W_k})$ for $k \geq k_0$. In addition, the set $W_{k_0+1} \cap N_{k_0}$ is equal to $W_{k_0} \setminus (D_{k_0} \cap N_{k_0-1})$.

Proof. Recall that $D_{k_0} \subset N_{k_0}$ is an area-minimizing disc with boundary $\gamma_{k_0}$ in the closure of $(W_{k_0}, g_{k_0}|_{W_{k_0}})$. The disjointness property (see Section 3.1) shows that $\gamma$ is a subset of $D_{k_0}$ or $D_{k_0} \cap \gamma$ is empty.

Case (I): If $\gamma$ is a subset of $D_{k_0}$, there is a disc $D' \subset D_{k_0}$ with boundary $\gamma$. The area-minimizing property of $D_{k_0}$ tells that

$$\text{Area}_{g_{k_0}}(D') \leq \text{Area}_{g_{k_0}}(D) = \text{Area}_g(D).$$

The last equality comes from the fact that $D \subset N_{k_0-1}$ and $g = g_{k_0}$ on $N_{k_0-1}$.

Since $g_{k_0} \geq g$, we have that $\text{Area}_g(D') \leq \text{Area}_{g_{k_0}}(D') \leq \text{Area}_g(D)$. In addition, $D' \subset N_{k_0}$ is embedded in the closure of $(W_{k_0+1}, g_{k_0+1}|_{W_{k_0+1}})$. Hence, it is the required candidate in our assertion.

Case (II): If $\gamma \cap D_{k_0}$ is empty, we deform $D$ to find an embedded disc Int $D'' \subset N_{k_0-1} \cap \text{Int} W_{k_0}$ with boundary $\gamma$ satisfying that

- $D''$ intersects $D_{k_0}$ transversally;
- $\text{Area}_g(D'') \leq \text{Area}_g(D) + \epsilon/2$.

The intersection $D'' \cap D_{k_0}$ has finitely many components, $\{c_i\}_i$. Each $c_i$ is a circle and bounds a disc $\hat{D}_i' \subset D'' \subset N_{k_0-1}$. It also bounds a disc $\hat{D}_i \subset D_{k_0} \subset N_{k_0}$. The area-minimizing property for $D_{k_0}$ implies that

$$\text{Area}_{g_{k_0}}(\hat{D}_i) \leq \text{Area}_{g_{k_0}}(\hat{D}_i').$$
Let $C^{\text{max}}$ be the set of all maximal circles of $\{c_i\}_i$ in $D''$. The discs $\{\hat{D}_i\}_{c_i \in C^{\text{max}}}$ are disjoint. Consider the disc $\hat{D}'' := (D'' \setminus \Pi_{c_i \in C^{\text{max}}} \hat{D}_i) \cup (\bigcup_{c_i \in C^{\text{max}}} \hat{D}_i) \subset N_{k_0}$, in the closure of $(W_{k_0+1}, g_{k_0+1}|_{W_{k_0+1}})$ with boundary $\gamma$. We have that

$$\text{Area}_{g_{k_0}}(\hat{D}'') = \text{Area}_{g_{k_0}}(D'' \setminus \Pi_{c_i \in C^{\text{max}}} \hat{D}_i) + \sum_{c_i \in C^{\text{max}}} \text{Area}_{g_{k_0}}(\hat{D}_i) \leq \text{Area}_{g_{k_0}}(D'' \setminus \Pi_{c_i \in C^{\text{max}}} \hat{D}_i) + \sum_{c_i \in C^{\text{max}}} \text{Area}_{g_{k_0}}(\hat{D}_i) = \text{Area}_{g_{k_0}}(D'') = \text{Area}_g(D'') \leq \text{Area}_g(D) + \epsilon/2.$$

Furthermore, $\text{Area}_g(\hat{D}'') \leq \text{Area}_{g_{k_0}}(\hat{D}'') \leq \text{Area}_g(D) + \epsilon/2$ (since $g_{k_0} \geq g$).

However, $\hat{D}''$ is piecewise smooth, not smooth. In addition, $\hat{D}''$ may be not embedded, because $\hat{D}_i$ may be a subset of some disc $\hat{D}_i$.

To overcome it, we deform $\hat{D}''$ along the normal direction of $\hat{D}_i$ to find an embedded disc $D' \subset N_{k_0}$ in the closure of $(W_{k_0+1}, g_{k_0+1}|_{W_{k_0+1}})$, with boundary $\gamma$. In addition, $\text{Area}_g(D') \leq \text{Area}_g(\hat{D}'') + \epsilon/2$. Hence, $\text{Area}_g(D') \leq \text{Area}_g(D) + \epsilon$.

The disc $D'$ is a required candidate in our assertion.

We inductively use Lemma 5.6 to get that

**Corollary 3.7.** Let $\{D_k\}$ and $\{(W_k, g_k|_{W_k})\}$ be assumed as in Section 3.1. Assume that the circle $\gamma \subset \Sigma_{k_1}$ bounds an embedded disc $D \subset N_{k_0-1}$ in the closure of $(W_k, g_k|_{W_k})$ for some $k_1$ and $j_1 \in J_{k_0}$. Then, for each $\epsilon > 0$ and $k \geq k_0$, $\gamma$ bounds an embedded disc $D' \subset N_{k_1} \subset N_{k-1}$ in the closure of $(W_k, g_k|_{W_k})$ satisfying that

$$\text{Area}_g(D') \leq \text{Area}_g(D) + \epsilon.$$

**Remark.** (1) Such a circle $\gamma$ can be considered as an embedded circle in the closure of $(W_k, g_k|_{W_k})$ for each $k > k_0$.

(2) Since $D' \subset N_{k-1}$ and $g = g_k$ on $N_{k-1}$, then $\text{Area}_{g_k}(D') = \text{Area}_g(D')$.

**Definition 3.8.** Let $(W_k, g_k|_{W_k})$ be assumed as in Section 3.1 and $\gamma$ an embedded circle in the closure of $(W_k, g_k|_{W_k})$. Define

$$\text{Fill}_k(\gamma) := \inf_{\partial D = \gamma} \{\text{Area}_{g_k}(D) \mid D \text{ is a disc in the closure of } (W_k, g_k|_{W_k})\}.$$

Remark that if $\text{Fill}_k(\gamma) < \infty$, we use the solution to the Plateau Problem to find an immersed disc $D'$ in the closure of $(W_k, g_k)$ with boundary $\gamma$ and with $\text{Fill}_k(\gamma) = \text{Area}_{g_k}(D')$ (see [Mor09], [Gul73] and [Oss70]). In addition, if $\gamma$ is not homotopically trivial in the closure of $(W_k, g_k|_{W_k})$, then $\text{Fill}_k(\gamma) = \infty$.

**Remark 3.9.** Let $D_k$ and $(W_k, g_k|_{W_k})$ be defined as in Section 3.1.

(1) $\text{Fill}_k(\gamma_k) = \text{Area}_{g_k}(D_k)$. 
(2) Let \( D \subset D_k \) be an embedded disc. Then, \( \text{Fill}_k(\partial D) = \text{Area}_{g_k}(D) \).

(3) Let \( \Sigma \subset W_k \) be a disc with finitely many closed discs removed. If \( \partial \Sigma \) is the disjoint of the circles, \( \{c\} \coprod \{c_i\}_{i=1}^n \), we have that

\[
\text{Fill}_k(c) \leq \text{Area}_{g_k}(\Sigma) + \sum_{i=1}^n \text{Fill}_k(c_i).
\]

We use Corollary 3.7 and the above remark to get the filling lemma.

**Lemma 3.10.** Let \( \{D_k\} \) and \( \{(W_k, g_k|_{W_k})\} \) be assumed as in Section 3.1. Assume that the circle \( \gamma \subset \Sigma_{k_1} \) bounds a disc \( D \subset N_{k_0-1} \) in the closure of \( (W_{k_0}, g_{k_0}|_{W_{k_0}}) \) for some \( k_1 \) and \( j_1 \in J_{k_1} \). Then for \( k \geq k_0 \), one has that

\[
\text{Fill}_k(\gamma) \leq \text{Area}_{g}(D).
\]

We give two corollaries of lemma 3.10, which show the local area estimate (see Remark 3.5 and Theorem 4.9) for some special cases.

**Corollary 3.11.** Let \( \{N_k\} \) and \( \{D_k\} \) be assumed as in Section 3.1. If an embedded circle \( \gamma \subset D_{k_1} \cap \partial N_{k_0} \) bounds a disc \( D' \subset \partial N_{k_0} \), where \( k_1 \geq k_0 + 1 \), then one has that for \( k \geq k_0 + 1 \)

\[
\text{Fill}_k(\gamma) \leq \text{Area}_{g}(D').
\]

**Proof.** The surface \( \partial N_{k_0} \subset N_{k_0} \) is a subset of the closure of \( (W_{k_0+1}, g_{k_0+1}|_{W_{k_0+1}}) \). Lemma 3.10 implies that \( \text{Fill}_k(\gamma) \leq \text{Area}_{g}(D') \) for \( k \geq k_0 + 1 \). \(\square\)

**Corollary 3.12.** Let \( \{N_k\} \) and \( \{D_k\} \) be assumed as in Section 3.1. If the circle \( \gamma \subset D_{k_1} \) bounds a disc \( D \subset D_{k_1} \), then for \( k \geq k_1 + 1 \), one has that

\[
\text{Fill}_k(\gamma) \leq \text{Area}_{g}(D).
\]

**Proof.** The disc \( D \subset D_{k_1} \subset N_{k_1} \) can be considered as a disc in the closure of \( (W_{k_1+1}, g_{k_1+1}|_{W_{k_1+1}}) \). Lemma 3.10 shows that for \( k \geq k_1 + 1 \), \( \text{Fill}_k(\gamma) \leq \text{Area}_{g}(D) \). \(\square\)

4. The local area estimate

In this section, we study the local area estimate for the discs, \( \{D_k\} \), constructed in Section 3.1. These discs satisfy the area-minimizing property and the disjointness property.

The area-minimizing property gives the filling lemma (see Lemma 3.10, Corollary 3.11 and Corollary 3.12). The disjointness property implies the finiteness of homotopy classes (see Corollary 3.6). We combine the filling lemma and the finiteness of homotopy classes to get the local area estimate (see Theorem 4.9).
4.1. Finiteness of homotopy classes. We begin with a topological property for surfaces.

**Proposition 4.1.** Let \( F \) be a compact orientable surface (possibly with boundary) and \( \gamma \subset \text{Int} \ F \) an embedded circle.

- If \( F \setminus \gamma \) is connected, then the genus \( g(F \setminus \gamma) \) is less than \( g(F) \);
- If \( \gamma \) cuts \( F \) into two components, \( F_1 \) and \( F_2 \), then

\[
g(F) = g(F_1) + g(F_2).
\]

The proof is the direct application of Mayer-Vietoris sequence.

**Definition 4.2.** Let \( \{c_i\}_{i \in I} \) be a family of disjoint embedded circles which are not null-homotopic in a closed orientable surface \( F \). Define the set of homotopy classes

\[
H(\{c_i\}_{i \in I}) := \{[c_i] \in \pi_1(F) \mid i \in I\}.
\]

**Lemma 4.3.** Let \( \{c_i\}_{i \in I} \) be a family of disjoint embedded circles which are not null-homotopic in a closed orientable surface \( F \). Then the set \( H(\{c_i\}_{i \in I}) \) has (at most) finitely many elements.

**Proof.** Suppose that there is a sequence \( \{c_i\}_{i = 1}^{\infty} \subset \{c_i\}_{i \in I} \) so that \( [c_i] \neq [c_j] \) in \( \pi_1(F) \), for \( i \neq j \).

**Step 1:** Inductively construct a decreasing family \( \{F_j\} \) of subsurfaces of \( F \).

When \( j = 1 \), define the subsurface \( F_1 \subset F_0 := F \) as one component of \( F \setminus c_1 \) containing infinitely many elements in \( \{c_i\}_{i = 1}^{\infty} \), where \( c_{i_0} := c_1 \).

Suppose that there are \( k \) subsurfaces \( \{F_j\}_{j = 1}^{k} \) and \( k \) circles, \( \{c_i\}_{j = 0}^{k - 1} \subset \{c_i\}_{i = 1}^{\infty} \) satisfying that for \( j \leq k - 1 \), \( F_{j+1} \subset F_j \) is one component of \( F_j \setminus c_i \) containing infinitely many elements in \( \{c_i\}_{i = 1}^{\infty} \).

When \( j = k + 1 \), choose a circle \( c_{i_k} \subset \text{Int} \ F_k \) in \( \{c_i\}_{i = 1}^{\infty} \) and the subsurface \( F_{k+1} \) as the component of \( F_k \setminus c_{i_k} \) which has infinitely many elements in \( \{c_i\}_{i = 1}^{\infty} \). The existence of \( c_{i_k} \) is ensured by the fact that \( F_k \) has infinitely many elements in \( \{c_i\}_{i = 1}^{\infty} \).

Therefore, there is a decreasing family \( \{F_j\}_{j = 1}^{\infty} \) of subsurfaces of \( F \) and a subsequence \( \{c_{i_j}\}_{j = 0}^{\infty} \) so that \( F_{j+1} \) is one component of \( F_j \setminus c_{i_j} \).

**Step 2:** Get a contradiction.

Proposition 4.1 shows that for each \( j \), \( g(F_{j+1}) \leq g(F_j) \) and \( g(F_j) \leq g(F) \). Hence, there is an integer \( j_0 > 0 \) satisfying that \( g(F_j) = g(F_{j+1}) \) for \( j \geq j_0 \).

Therefore, \( c_{i_j} \) cuts \( F_j \) into two components, \( F_{j+1} \) and \( F'_{j+1} \) for \( j \geq j_0 \). (If not, Proposition 4.1 implies that \( g(F_{j+1}) < g(F_j) \). It is in contradiction with the last paragraph.)

Proposition 4.1 gives that \( g(F_j) = g(F_{j+1}) + g(F'_{j+1}) \) for \( j \geq j_0 \). Hence, \( g(F'_{j+1}) = 0 \) for \( j \geq j_0 \).
Consider the number, $b(F_j)$, of components of $\partial F_j$. We have that for $j \geq j_0$,

$$b(F_j) + 2 = b(F_{j+1}) + b(F'_{j+1}).$$

We can conclude that $b(F'_{j+1}) \geq 3$ for $j \geq j_0$. The reason is as follows:

If not, $b(F'_{j+1})$ is equal to 1 or 2 for some $j \geq j_0$. From above, $g(F'_{j+1}) = 0$. That is to say, $F'_{j+1}$ is a disc or an annulus. Because $c_{ij} \subset \partial F'_{j+1}$ is non-contractible in $F$, $F'_{j+1}$ is an annulus, not a disc.

The boundary $\partial F'_{j+1}$ consists of $c_{ij}$ and $c_{ij'}$ for some $j' < j$. That is to say, $[c_{ij}] = [c_{ij'}]$ in $\pi_1(F)$, which contradicts the assumption ($[c_i] \neq [c_j]$ in $\pi_1(F)$ for $i \neq j$).

From the above facts, we have that for $j \geq j_0$

$$b(F_{j+1}) < b(F_j).$$

Therefore, $\{b(F_j)\}_{j \geq j_0}$ is a strictly decreasing sequence of integers and $b(F_{j_0}) < \infty$, which contradicts the fact that $b(F_j) \geq 0$ for $j \geq j_0$.

We finish the proof of the lemma.

**Lemma 4.4.** Let $F$ be an orientable surface (possibly with boundary and non-compact). Assume that two disjoint embedded circles, $\alpha$ and $\alpha'$, are not null-homotopic in $F$. If these two circles are homotopic in $F$, then there is an annulus $A \subset F$ with boundary $\partial A = \alpha \cup \alpha'$.

See the proof in Appendix A.

**Remark 4.5.** Let $A$ be an annulus and $\gamma \subset \text{Int } A$ an embedded circle. Then, $\gamma$ bounds a disc in $A$ or $\gamma$ cuts $A$ into two annuli. The reason is as follows:

The boundary $\partial A$ is a disjoint union of two circles, $c_1$ and $c_2$. Consider the disc $D_1 := A \cup_{c_2} D_2$ where $D_2$ is a disc. The circle $\gamma$ bounds a disc $D' \subset D_1$. We have that $D' \subset A$ or $D_2 \subset D'$.

If $D_2$ is a subset of $D'$, then $D_1 \setminus D'$ and $D' \setminus D_2$ are two annuli. That is to say, $\gamma$ cuts $A$ into two annuli.

4.2. **The local area estimate.** In the subsection, we first recall the properties of $\{D_k\}$, constructed in Section 3.1. Then, we use Lemma 4.3 to show the finiteness of homotopy classes. Finally, we combine the filling lemma (see Lemma 3.10) to get the required estimate for $D_k$.

All proofs in this subsection only depends on the area-minimizing and disjointness properties.

Let $(M, g)$ be a complete contractible 3-manifold which is not homeomorphic to $\mathbb{R}^3$. From Theorem 2.5, it is an increasing union of handlebodies, $\{N_k\}_k$. For each $k$, we find a metric $g_k$ on $M$ so that

(a) $g_k \geq g$ and $g_k|_{N_{k-1}} = g|_{N_{k-1}}$;

(b) $(N_k, g_k|_{N_k})$ has mean convex boundary.
As in Section 3.1, we may assume that $M$ is not simply-connected for $N_0$ (see Remark 2.2). Lemma 2.14 gives a meridian curve $\gamma_k$ of $N_k$ which is not null-homotopic in $M \setminus N_0$. There is a family of discs $\{D_k\}$ (see Section 3.1) satisfying that

1. each $D_k \subset N_k$ is an embedded disc with boundary $\gamma_k$;
2. The intersection $D_k \cap N_{k-1}$ consists of finitely many smooth surfaces $\{\Sigma^j_k\}_{j \in J_k}$ and each $\Sigma^j_k$ is stable minimal for the original metric $g$;
3. (Disjointness) $\Sigma^j_k \subset \Sigma^l_k$ or $\Sigma^j_k \cap \Sigma^l_k = \emptyset$ where $k' < k$; $j \in J_k$ and $j' \in J_{k'}$;
4. (Area-minimizing) $D_k$ is an area-minimizing disc with boundary $\gamma_k$ in the closure of the manifold $(W, g_k | W)$, where $W_k = N_k \setminus \cup_{i=1}^{k-1} D_i \cap N_{i-1}$.

We now use the disjointness property to get the finiteness of homotopy classes.

**Corollary 4.6.** Let $(M, g)$ and $\{D_k\}$ be assumed as in Section 3.1. Then for each integer $k_0 > 0$, there is an integer $k' > k_0$ so that for $k > k_0 + 1$, each embedded circle in $D_k \cap \partial N_{k_0}$ is null-homotopic in $\partial N_{k_0}$ or homotopic to some circle of $\cup_{i=k_0+1}^{k'} D_i \cap \partial N_{k_0}$ in $\partial N_{k_0}$.

The set, $\cup_{i=k_0+1}^{k'} D_i \cap \partial N_{k_0}$ is a subset of a disjoint union of some surfaces in $\{\Sigma^j_i\}_{k_0 < i \leq k'}$, $j \in J_i$.

**Proof.** Suppose that there is a strictly increasing sequence $\{k_i\}$ of integers satisfying that

1. $D_{k_i}$ has a non-contractible embedded circle $c_i \subset D_{k_i} \cap \partial N_{k_0}$ in $\partial N_{k_0}$;
2. (for each $i \neq j$, $[c_i] \neq [c_j]$ in $\pi_1(\partial N_{k_0})$.

Let $\Sigma^j_{k_i}$ be the component of $D_{k_i} \cap N_{k_i-1}$ containing $c_i$, where $j_i \in J_{k_i}$.

Recall that $D_{k_i}$ intersects $\partial N_{k_i-1}$ transversally. If $\Sigma^j_{k_i} \subset \Sigma^j_{k_{i'}}$, where $i' > i$, then $\Sigma^j_{k_{i'}}$ is one component of $\Sigma^j_{k_{i'}} \cap N_{k_{i}-1}$. Hence, $\Sigma^j_{k_{i'}}$ is a subset of $\Sigma^j_{k_{i}}$.

**Claim:** for each $i$, there are finitely many circles in $\{c_i\}_{i=1}^\infty$ intersecting $\Sigma^j_{k_{i}}$.

If some $c_{i'}$ intersects $\Sigma^j_{k_{i}}$ where $i' > i$, then $\Sigma^j_{k_{i'}} \cap \Sigma^j_{k_{i}} \neq \emptyset$. The disjointness property shows that $\Sigma^j_{k_{i'}} \subset \Sigma^j_{k_{i'}}$. Thus, $c_{i'}$ is contained in one of components of $\Sigma^j_{k_{i'}} \cap N_{k_{i}-1}$. Hence, $c_{i'}$ is a subset of $\Sigma^j_{k_{i}}$.

If there are infinitely many such circles in $\{c_i\}_i$, then $\Sigma^j_{k_{i}}$ contains infinitely many homotopy classes in $\{[c_i]\} \subset \pi_1(\partial N_{k_0})$. We may assume that $\Sigma^j_{k_{i}}$ intersects $\partial N_{k_0}$ transversally. (Otherwise, we deform $\partial N_{k_0}$ so that it intersects $\Sigma^j_{k_{i}}$ transversally.) The intersection $\partial N_{k_0} \cap \Sigma^j_{k_{i}}$ is the disjoint union of finitely many circles. It contradicts the fact that $\Sigma^j_{k_{i}}$ contains infinitely many homotopy classes in $\{[c_i]\} \subset \pi_1(\partial N_{k_0})$. The proof of the claim is complete.

We use the claim to find a strictly increasing sequence of integers $\{i_l\}$ so that for each $l$, $c_{i_l} \cap (\cup_{i_{l-1}}^{i_l} \Sigma^j_{k_{i_l}}) = \emptyset$. Therefore, the circles, $\{c_{i_l}\}$ are
Proposition 4.7. Let $\gamma \subset D_k \cap \partial N_{k_0}$ be an embedded circle and $D \subset D_k$ a disc with boundary $\gamma$, where $k \geq k_0 + 1$. If $\gamma$ bounds an embedded disc $D' \subset \partial N_{k_0}$, then one has that
\[
\text{Area}_g(D) \leq \text{Area}_g(D') \leq \text{Area}_g(\partial N_{k_0}).
\]
Moreover, $\text{Fill}_k(\gamma) \leq \text{Area}_g(D')$.

Proof. Corollary 3.11 shows that $\text{Fill}(\gamma) \leq \text{Area}_g(D')$. That is to say, $\text{Area}_g(D) \leq \text{Area}_g(D')$. Since $g_k \geq g|N_k$, then $\text{Area}_g(D) \leq \text{Area}_g(D')$. \hfill \square

We now study the case when $\gamma$ is non-contractible in $\partial N_{k_0}$.

Proposition 4.8. For each $k_0 > 0$, there is a positive constant $C$, so that
\[
\text{Area}_g(D) \leq C,
\]
for $k \geq k_0 + 1$, any embedded circle $\gamma \subset D_k \cap \partial N_{k_0}$ bounds a disc $D \subset D_k$ and $\text{Area}_g(D) \leq C$.

Proof. Proposition 4.7 shows that if $\gamma$ is homotopically trivial in $\partial N_{k_0}$, then $\text{Fill}_k(\gamma) \leq \text{Area}_g(\partial N_{k_0})$. It remains to study the case when $\gamma$ is not homotopically trivial in $\partial N_{k_0}$.

Step 1: Find the constant $C$ and Reduce the proof.

Corollary 4.6 gives an integer $k' > 0$, so that $\gamma$ is homotopic to some circle $c \subset \partial N_{k_0} \cap D_{k_1}$ in $\partial N_{k_0}$ for some $k_1 \leq k'$. Choose the constant $C_1 := \sum_{l \leq k'} \text{Area}_g(D_l)$. We have that
\begin{itemize}
  \item For $k \leq k'$, $\text{Area}_g(D') \leq \text{Area}_g(D_k) \leq C_1$;
  \item Corollary 3.12 implies that $\text{Fill}_k(c) \leq C_1$ for $k > k'$.
\end{itemize}

It is sufficient to show that $\text{Area}_g(D) \leq C_1 + \text{Area}_g(\partial N_{k_0})$ for $k > k'$.

If $c \subset \Sigma^j_{k_1}$ and $\gamma \subset \Sigma^j_k$ intersects, where $k_0 < k_1 \leq k' < k$ and $j \in J_{k_1}, j \in J_k$, the disjointness property shows that $\Sigma^j_{k_1} \subset \Sigma^j_k$. Since $D_{k_1}$ intersects $\partial N_{k_1 - 1}$ transversally, $\Sigma^j_{k_1}$ is one component of $\Sigma^j_k \cap N_{k_1 - 1}$. Because $\gamma \subset \Sigma^j_k \cap N_{k_1 - 1}$ intersects $\Sigma^j_{k_1}$, it is a subset of $\Sigma^j_{k_1} \subset D_{k_1}$. Corollary 3.12 shows that $\text{Fill}_k(\gamma) \leq \text{Area}_g(D_{k_1}) \leq C_1$. The area-minimizing property of $D_k$ implies that $\text{Area}_g(D) \leq \text{Area}_g(D_k) = \text{Fill}_k(\gamma) \leq C_1$.

It remains to show the case when $c$ and $\gamma$ are disjoint. Lemma 4.4 gives an open annulus $A \subset \partial N_{k_0}$ with boundary $\gamma \cap c$.

Step 2: Classify the circles in $A \cap (\cup_{l=k_0+1}^{k-1} D_l)$ and show two key inequalities.

We may assume that each $D_l$ intersects $\partial N_{k_0}$ transversally for $k_0 < l \leq k - 1$. From the disjointness property of $D_k$, the set $\cup_{l=k_0+1}^{k-1} D_l \cap \partial N_{k_0}$ is a disjoint union of finitely many circles, $\{c_i\}_{i \in I'}$.

Consider the set $I_{\text{non}} := \{ i \in I \mid c_i \subset A \text{ is non-contractible in } A \}$. (It may be empty). For our convenience, we write $\{c_i\}_{i=1}^m$ for $\{c_i\}_{i \in I_{\text{non}}}$.
We repeatedly use Remark 4.5 to show that the circles \( \{c_i\}_{i=1}^m \) cut \( A \) into finitely many open annuli \( \{A_i\}_{i=1}^{m+1} \), where \( \partial A_i = c_{i-1} \cup c_i \), \( c_0 := \gamma \) and \( c_{m+1} := c \) (see Figure 1).

Notice that if \( I_{\text{non}} \) is empty, we choose that \( A_1 := A \) and \( c_1 := c \).

Let \( I_j := \{i \in I \mid c_i \subset A_j\} \) (it may also be empty). We have that

- \( A \cap ( \bigcup_{l=k+1}^{l=k+1} D_l ) \) is the disjoint union, \( \{c_i\}_{i \in I_{\text{non}}} \cup \{c_i\}_{i \in I_j} \).
- Each component of \( A_j \setminus \bigcup_{i \in I_j} c_i \) is a subset of \( W_k \).

Since \( I_{\text{non}} \cap I_j = \emptyset \) for each \( j \), each \( c_i \) is contractible in \( A \) for \( i \in I_j \). Then, we have that

(A) For each \( j \) and \( i \in I_j \), \( c_i \) bounds the disc \( \hat{D}_i \subset A_j \) and \( \text{Fill}_k(c_i) \leq \text{Area}_g(\hat{D}_i) \);

(B) For \( 1 \leq j \leq m+1 \), \( \text{Fill}_k(c_{j-1}) \leq \text{Fill}_k(c_j) + \text{Area}_g(A_j) \).

Notice that for each \( j \) and \( i \in I_j \), \( c_i \) is contractible in \( A \). Namely, it bounds the disc \( \hat{D}_i \subset A \). The item (A) follows from Proposition 4.7.

We now present the proof of the item (B).

Consider the set \( \{\hat{D}_i\}_{i \in I_j} \) and define the partially ordered relation induced by the inclusion. Let \( \{\hat{D}_i\}_{i \in I_j} \) be the set of all maximal elements in \( (\{\hat{D}_i\}_{i \in I_j}, \subset) \). The discs \( \{\hat{D}_i\}_{i \in I_j} \) are disjoint.

The set \( A_j \setminus \bigcup_{i \in I_j} \hat{D}_i \subset W_k \) is homeomorphic to a disc with finite punctures. Its boundary is the disjoint union of \( c_{j-1} \), \( c_j \) and \( \{c_i\}_{i \in I_j} \). We use Remark 3.9 to show that

\[
\text{Fill}_k(c_{j-1}) \leq \text{Area}_g(A_i) \setminus \bigcup_{i \in I_j} \text{Area}_g(\hat{D}_i) + \sum_{i \in I_j} \text{Fill}_k(c_i) + \text{Fill}_k(c_j) \leq \text{Area}_g(A_i) \setminus \bigcup_{i \in I_j} \text{Area}_g(\hat{D}_i) + \sum_{i \in I_j} \text{Area}_g(\hat{D}_i) + \text{Fill}_k(c_j). \text{ From the item(A)}
\]

\[
= \text{Area}_g(A_i) + \text{Fill}_k(c_j).
\]

Note that \( A_i \subset \partial N_{k_0} \subset \text{Int} N_{k-1} \).

**Step 3:** Finish the proof
We use the item (B) to have that
\[ \text{Area}_g(D) \leq \text{Area}_{g_k}(D) = \text{Fill}_k(\gamma) \leq \text{Fill}_k(c_{m+1}) + \sum_{i=1}^{m+1} \text{Area}_g(A_i) \]
\[ \leq C_1 + \text{Area}_g(\partial N_{k_0}). \]
where \( c_0 = \gamma \) and \( c = c_{m+1} \). The last inequality follows from the fact that \( c \subset D_{k_1} \) and \( \text{Fill}_k(c) \leq C_1 \), where \( k_1 \leq k' \).

**Theorem 4.9.** For each integer \( k_0 \), there is a positive constant \( C_{k_0} \), so that for each \( k \geq k_0 + 1 \), the area of each component of \( D_k \cap \text{Int} \ N_{k_0} \) is not greater than \( C_{k_0} \).

**Proof.** We may assume that \( D_k \) intersects \( \partial N_{k_0} \) transversally. The intersection is a disjoint union of finitely many circles.

Let \( \Sigma \) be a component of \( D_k \cap \text{Int} \ N_k \). The boundary is a disjoint union of circles \( \{c_i\}_{i \in I} \). Each \( c_i \) bounds a disc \( D_i' \subset D_k \).

We have that \( \{c_i\}_{i \in I} \) has a unique maximal circle in \( D_k \).

If not, there are (at least) two maximal circles, \( c_{i_0} \) and \( c_{i_1} \), of \( \{c_i\}_{i \in I} \) in \( D_k \). From the maximality, \( D_{i_0}' \cap D_{i_1}' = \emptyset \). However, \( \Sigma \cap D_{i_0}' \) is nonempty for \( l = 0, 1 \), which implies that \( \Sigma \) is not connected. It is in contradiction with the connectedness of \( \Sigma \).

Let \( c_i \) be the unique maximal circle of \( \{c_i\}_{i \in I} \). The disc \( D_i' \subset D_k \) with boundary \( c_i \) contains \( \Sigma \). From Proposition 4.8, there is a positive constant \( C_{k_0} \) so that
\[ \text{Area}_g(D_i') \leq C_{k_0}. \]
Therefore, we have that \( \text{Area}_g(\Sigma) \leq \text{Area}_g(D_i') \leq C_{k_0} \).

**Remark 4.10.** All proofs, in Sections 3.2 and 4.2, only depend on the area-minimizing property and the disjointness property. That is to say, Theorem 4.9 can be generalized to the following case:

Let \( \{V_i\}_i \) be an increasing sequence of compact and connected sets in \( (M, g) \) and \( \tilde{g}_i \) a metric on \( V_i \) with \( \tilde{g}_i \geq g|_{V_i} \) and \( \tilde{g}_i|_{V_{i-1}} = g|_{V_{i-1}} \). The embedded disc \( (\Omega_i, \partial \Omega_i) \subset (V_i, \partial V_i) \) has the following properties:

- the intersection \( \Omega_i \cap V_{i-1} \) consists of finitely many smooth surfaces \( \{\Omega_i^{j_i}\}_{j \in J_i} \);
- (Disjointness) \( \Omega_i^{j_i} \subset \Omega_i^{j_i'} \) or \( \Omega_i^{j_i} \cap \Omega_i^{j_i'} = \emptyset \) where \( i' < i, j \in J_i \) and \( j' \in J_{i'} \);
- (Area-minimizing) \( \Omega_i \) is an area-minimizing disc with boundary \( \partial \Omega_i \) in the closure of the manifold \( (V_i \setminus \bigcup_{i=1}^{i-1} \Omega_i \cap V_{i-1}, \tilde{g}_i|_{V_i \setminus \bigcup_{i=1}^{i-1} \Omega_i \cap V_{i-1}}}) \).

Then for each \( i_0 \), there is a positive constant \( C_{i_0} \) so that for \( i > i_0 \), the area of each component of \( \Omega_i \cap V_{i_0} \) is not greater than \( C_{i_0} \).

5. **Properties of Limit Surfaces and the Triviality at Infinity**

In this section, we study the limit of \( \{D_k\}_k \), constructed in Section 3.1, and its relationship with the topology of contractible 3-manifolds.
First, we recall the convergence theory for minimal surfaces and show that the sequence \( \{D_k\} \) sub-converges towards a lamination (a disjoint union of some embedded surfaces.)

Then, we study the geometric and topological properties of the limit (see Theorem 5.6, Corollary 5.12 and Corollary 5.13).

5.1. Limit of minimal surfaces. Let us recall the classical convergence theory for minimal surfaces.

**Definition 5.1.** In a complete Riemannian 3-manifold \((M, g)\), a sequence \( \{\Sigma_n\} \) of immersed minimal surfaces converges smoothly with finite multiplicity (at most \( m \)) to an immersed minimal surface \( \Sigma \), if for each point \( p \) of \( \Sigma \), there is a disc neighborhood \( D \) in \( \Sigma \) of \( p \), an integer \( m \) and a neighborhood \( U \) of \( D \) in \( M \) (consisting of geodesics of \( M \) orthogonal to \( D \) and centered at the points of \( D \)) so that for \( n \) large enough, each \( \Sigma_n \) intersects \( U \) in at most \( m \) connected components. Each component is a graph over \( D \) in the geodesic coordinates. Moreover, each component converges to \( D \) in \( C^{2,\alpha} \)-topology as \( n \) goes to infinity.

Note that when each \( \Sigma_n \) is embedded, the surface \( \Sigma \) is also embedded. The multiplicity at \( p \) is equal to the number of connected components of \( \Sigma_n \cap U \) for \( n \) large enough. It remains constant on each component of \( \Sigma \).

**Remark 5.2.** Let \( \{\Sigma_n\}_n \) be a family of properly embedded minimal surfaces converging to the minimal surface \( \Sigma \). Fix a compact simply-connected subset \( D \subset \Sigma \). Let \( U \) be the tubular neighborhood of \( D \) in \( M \) with radius \( \epsilon \) and \( \pi : U \to D \) the projection from \( U \) onto \( D \). It follows that the restriction \( \pi|_{\Sigma_n \cap U} : \Sigma_n \cap U \to D \) is a \( m \)-sheeted covering map for \( \epsilon \) small enough and \( n \) large enough, where \( m \) is the multiplicity.

Therefore, the restriction of \( \pi \) to each component of \( \Sigma_n \cap U \) is also a covering map. Hence, since \( D \) is simply-connected, it is bijective. Therefore, each component of \( \Sigma_n \cap U \) is a normal graph over \( D \).

**Theorem 5.3.** (See [Compactness Theorem, Page 96] of [And85] and [Theorem 4.37, Page 49] of [MRR02]) Let \( \{\Sigma_k\}_{k \in \mathbb{N}} \) be a family of properly embedded minimal surfaces in a 3-manifold \( M^3 \) satisfying (1) each \( \Sigma_k \) intersects a given compact set \( K_0 \); (2) for any compact set \( K \) in \( M \), there are three constants \( C' = C'(K) > 0 \), \( C'' = C''(K) > 0 \) and \( j_0 = j_0(K) \in \mathbb{N} \) such that for each \( k \geq j_0 \), it holds that

\[
(1) \ |A_{\Sigma_k}|^2 \leq C' \text{ on } K \cap \Sigma_k, \quad \text{where } |A_{\Sigma_k}|^2 \text{ is the square length of the second fundamental form of } \Sigma_k \n\]

\[
(2) \ \text{Area}_g(\Sigma_k \cap K) \leq C''. \n\]

Then, after passing to a subsequence, \( \Sigma_k \) converges to a properly embedded minimal surface with finite multiplicity in the \( C^\infty \)-topology.

In general, we could not get the area estimate for \( \{D_k\} \), stated in Theorem 5.3. The sequence \( \{D_k\} \) may not sub-converges with finite multiplicity. To overcome it, we consider the convergence toward a lamination.
Definition 5.4. (See Appendix B, Page 609-612 in CM04) A codimension one lamination in a 3-manifold $M^3$ is a collection $\mathcal{L}$ of smooth disjoint surfaces (called leaves) such that $\bigcup_{L \in \mathcal{L}} L$ is closed in $M^3$. Moreover, for each $x \in M$ there exists an open neighborhood $U$ of $x$ and a coordinate chart $(U, \Phi)$, with $\Phi(U) \subset \mathbb{R}^3$ so that in these coordinates the leaves in $\mathcal{L}$ pass through $\Phi(U)$ in slices form $\mathbb{R}^2 \times \{t\} \cap \Phi(U)$.

A minimal lamination is a lamination whose leaves are minimal. Finally, a sequence of laminations is said to converge if the corresponding coordinate maps converge.

For example, $\mathbb{R}^2 \times \Lambda$ is a lamination in $\mathbb{R}^3$, where $\Lambda$ is a closed set in $\mathbb{R}$.

Now we consider the limit of the discs $\{D_k\}$ constructed in Section 3.1.

Theorem 5.5. (See Theorems 5.7 and 5.8 in Wan19a) If $(M,g)$ has positive scalar curvature, then after passing to a subsequence, $\{D_k\}$ converges to a minimal lamination $\mathcal{L}$. Furthermore, each leaf is a complete stable minimal plane.

The positivity of the scalar curvature shows that each leaf is homeomorphic to $\mathbb{R}^2$ (see SY82). The geometry of leaves is influenced by the extrinsic Cohn-Vossen inequality:

\textbf{Theorem 5.6.} (See Theorem 5.10 in Wan19a) Let $(M,g)$ be a complete 3-manifold with positive scalar curvature ($\kappa > 0$) and $L \subset (M,g)$ a complete (non-compact) stable minimal surface. Then one has

$$\int_L \kappa dv \leq 2\pi,$$

where $dv$ is the volume form of the induced metric on $L$. Moreover, if $L$ is an embedded surface, it is properly embedded.

\textbf{Corollary 5.7.} Let $(M,g)$ be a complete non-compact 3-manifold with positive scalar curvature ($\kappa > 0$) and $\{L_i\}$ a sequence of complete (non-compact) stable minimal surfaces in $M$. If each $L_i$ intersects a compact set $K_0$, then $L_i$ sub-converges with finite multiplicity.

\textbf{Proof.} Since $L_i$ is a stable minimal surface, we use [Theorem 3, Page122] of Sch83 to have that for any compact set $K \subset M$, there is a constant $C := C(K,M,g)$ such that

$$|A_{L_i}|^2 \leq C \text{ on } K \cap L_i,$$

where $|A_{L_i}|^2$ is the squared norm of the second fundamental form of $L_i$.

Theorem 5.6 gives $\int_{L_i} \kappa(x) dv \leq 2\pi$. Hence,

$$\text{Area}(K \cap L_i) \leq 2\pi (\inf_{x \in K} \kappa(x))^{-1}.$$

From Theorem 5.3, $\{L_i\}$ subconverges with finite multiplicity. \qed

22 JIAN WANG
5.2. Triviality at infinity. First, we study the triviality at infinity and its relationship with the simply-connectedness at infinity. Then, we discuss the geometric and topological properties of the trivial minimal plane at infinity.

**Definition 5.8.** Let \( L \subset (M,g) \) be a complete properly embedded minimal plane (i.e. homeomorphic to \( \mathbb{R}^2 \)).

- \( L \) is trivial for a compact set \( K \subset M \) if there is a compact set \( K' \subset M \) containing \( K \) so that the map \( \pi_1(L \setminus K') \to \pi_1(M \setminus K) \) is trivial.
- \( L \) is trivial at infinity if it is trivial for each compact subset of \( M \).

**Remark 5.9.** Let \( K \subset K' \) be two compact subsets of \( M \). If a stable minimal plane is not trivial for \( K \), it is not trivial for \( K' \).

The triviality at infinity is deeply linked with relationship with the simply-connectedness at infinity.

**Remark 5.10.** Let \((M,g)\) be a complete contractible Riemannian manifold and \( L \) a properly embedded minimal plane.

(A) If \( L \) is not trivial for some compact set \( K \subset M \), then \( M \) is not simply-connected for \( K \).

(B) If \( L \) is not trivial at infinity, then \( M \) is not simply-connected at infinity.

(C) If \( L \) is not trivial for \( K \), then the closure \( M_j \) is not simply connected for \( K \cap M_j \) for \( j = 1, 2 \), where \( L \) cuts \( M \) into two contractible spaces, \( M_1 \) and \( M_2 \).

Roughly speaking, if a complete contractible 3-manifold \((M,g)\) is not simply-connected for some compact set \( K \), then there is a properly embedded stable minimal plane \( L \subset (M,g) \) which is not trivial for \( K \).

The item (C) implies that \( M_j \) is not simply-connected at infinity for \( j = 1, 2 \). In Section 7, we will use the construction in Section 3.1 to find a lamination in \( M_j \) for \( j = 1, 2 \).

**Proposition 5.11.** Let \((M,g)\) be a complete contractible 3-manifold with positive scalar curvature and \( \{L_i\} \) a sequence of embedded stable minimal planes converging with finite multiplicity. If each component of the limit is trivial for a compact set \( K \subset M \), then there is a positive integer \( i_0 \) and a compact set \( K' \) containing \( K \) satisfying that for \( i \geq i_0 \)

\[
\text{the induced map } \pi_1(L_i \setminus K') \to \pi_1(M \setminus K) \text{ is trivial.}
\]

**Proof.** Let \( \mathcal{L}_1 \) be the limit of \( \{L_i\} \). Theorem 5.6 shows that each \( L_i \) is properly embedded. The limit \( \mathcal{L}_1 \) is also properly embedded (it may have infinitely many components). There are finitely many components, \( \mathcal{L}_1^i := \bigcap_{s=1}^{m} \hat{L}_s \) of \( \mathcal{L} \) intersecting \( K \).

Since each \( \hat{L}_s \) is trivial for \( K \), there is a handlebody \( \text{Int } N \subset M \) containing \( K \) so that for each \( s \), the map \( \pi_1(\hat{L}_s \setminus \text{Int } N) \to \pi_1(M \setminus K) \) is trivial (see Theorem 2.3 and Definition 5.8). In addition, each circle in \( \mathcal{L}_1 \setminus \mathcal{L}_1^i \) is also homotopically trivial in \( M \setminus K \). Therefore, we can conclude that the map
embedded, \( \pi_1(L_1 \setminus \text{Int } N) \rightarrow \pi_1(M \setminus K) \) is trivial. Namely, each circle in \( L_1 \cap \partial N \) is null-homotopic in \( M \setminus K \).

We assume that \( L_1 \) intersects \( \partial N \) transversally. Since \( L_1 \) is properly embedded, \( L_1 \cap \partial N \) is a disjoint union of finitely many circles. Choose a constant \( \epsilon > 0 \) so that \( B^{\partial N}(L_1 \cap \partial N, \epsilon) \) is a disjoint union of some annuli, where \( B^{\partial N}(L_1 \cap \partial N, \epsilon) \) is the tubular neighborhood of \( L_1 \cap \partial N \) in \( \partial N \) with radius \( \epsilon \). The last paragraph shows that

\[
\text{the map } \pi_1(B^{\partial N}(L_1 \cap \partial N, \epsilon)) \rightarrow \pi_1(M \setminus K) \text{ is trivial.}
\]

Choose a handlebody \( N' \) with \( N \subset \text{Int } N' \) and a constant \( \epsilon_0 > 0 \) so that the intersection \( \partial N \cap B(L_1 \cap N', \epsilon_0) \) is a subset of \( B^{\partial N}(L_1 \cap \partial N, \epsilon) \), where \( B(L_1 \cap N', \epsilon_0) \) is the tubular neighborhood of \( L_1 \cap N' \) with radius \( \epsilon_0 \).

Since \( L_i \) converges to \( L_1 \), there is an integer \( i_0 > 0 \) so that for \( i \geq i_0 \),

\[
L_i \cap N' \subset B(N' \cap L_1, \epsilon_0).
\]

From the above fact, each circle in \( L_i \cap \partial N \) is homotopically trivial in \( M \setminus K \) for \( i \geq i_0 \).

A closed curve \( \gamma \subset L_i \setminus N \) bounds a disc \( D \subset L_i \). Hence, each circle in \( D \cap \partial N \) is null-homotopic in \( M \setminus K \) for \( i \geq i_0 \). Lemma 2.7 shows that \( \gamma \) is contractible in \( M \setminus K \). That is to say, the map \( \pi_1(L_i \setminus N) \rightarrow \pi_1(M \setminus K) \) is trivial for \( i \geq i_0 \). \( \square \)

**Corollary 5.12.** Let \((M, g)\) be a complete contractible 3-manifold with positive scalar curvature and \( K \subset M \) a compact set. Assume that each complete stable minimal plane \( L_i \) is not trivial for \( K \). If the sequence \( \{L_i\} \) converges with finite multiplicity, then the limit has a component which is not trivial for \( K \).

We use Proposition 5.11 to show that

**Corollary 5.13.** Let \((M, g)\) be a complete contractible 3-manifold with positive scalar curvature and \( \mathcal{L} \) a lamination whose leaves are complete stable minimal planes. If each leaf is trivial for a compact set \( K \subset M \), then there is a compact set \( K' \) containing \( K \) so that the map \( \pi_1(L \setminus K') \rightarrow \pi_1(M \setminus K) \) is trivial for each leaf \( L \) in \( \mathcal{L} \).

**Proof.** Suppose that there is a sequence \( \{L_i\}_{i=1}^{\infty} \subset \mathcal{L} \) satisfying that

- each \( L_i \) has a closed curve \( c_i \) which is non-contractible in \( M \setminus K \).
- the circles \( \{c_i\} \) go to infinity

Remark that each \( L_i \) intersects \( K \).

From Corollary 5.11 \( \{L_i\} \) subconverges with finite multiplicity. From now on, we write \( \{L_i\}_i \) for a convergent sequence. The limit is a subset of \( \mathcal{L} \).

We use Proposition 5.11 to find an integer \( i_0 > 0 \) and a compact set \( K' \) containing \( K \) satisfying that for each \( i \geq i_0 \), the map \( \pi_1(L_i \setminus K') \rightarrow \pi_1(M \setminus K) \) is trivial.
However, the circles \( \{ c_i \} \) go to infinity. We can find a circle \( c_{i_1} \subset L_{i_1} \setminus K' \) which is non-contractible in \( M \setminus K' \) for \( i_1 \) large enough, which is in contradiction with the last paragraph. \( \square \)

6. Proof of Theorem \[1.1\]

In this section, we complete the proof for Theorem \[1.1\]. We begin with a proposition.

**Proposition 6.1.** Let \((M, g)\) be a complete contractible 3-manifold of positive scalar curvature and \(\{ \Omega_k \}_k\) a family of embedded discs satisfying that
- for each \(k\), \(\partial \Omega_k\) is a subset of \(M \setminus \text{Int} N_k\) and \(\Omega_k \cap N_{k-1}\) is stable minimal;
- the sequence \(\{ \Omega_k \}_k\) converges to a lamination \(\mathcal{L}_0\) whose leaves are complete stable minimal planes;
- for each \(k_0\), there is a constant \(C_{k_0} > 0\) and an integer \(k'_0 > k_0\) so that for each \(k > k'_0\), the area of each component of \(\Omega_k \cap N_{k_0}\) is not greater than \(C_{k_0}\).

If each leaf in \(\mathcal{L}_0\) is trivial for a compact set \(K \subset M\), then there is a positive integer \(k'\) so that for \(k \geq k'\), \(\partial \Omega_k\) is homotopically trivial in \(M \setminus K\).

**Proof.** Suppose that there is an increasing sequence \(\{ k_i \}_{i=1}^\infty\) of integers so that \(\partial \Omega_{k_i}\) is non-contractible in \(M \setminus K\). Since each leaf in \(\mathcal{L}_0\) is trivial for \(K\), we use Corollary \[5.13\] to find a handlebody \(N_{k_0-2}\) containing \(K\) so that the map \(\pi_1(L \setminus N_{k_0-2}) \to \pi_1(M \setminus K)\) is trivial for each leaf \(L\) of \(\mathcal{L}_0\).

Since \(\partial \Omega_{k_i}\) is non-contractible in \(M \setminus K\), there is a non-contractible circle \(\hat{\gamma}_i \subset \Omega_i \cap \partial N_{k_0-1}\) in \(M \setminus K\). If not, Lemma \[2.7\] shows that \(\partial \Omega_{k_i}\) is homotopically trivial in \(M \setminus K\), which is in contradiction with our assumption.

Let \(\Sigma_i\) be the component of \(\Omega_i \cap N_{k_0}\) containing \(\hat{\gamma}_i\). It is stable minimal for \(g\). We use [Theorem 3, Page122] of [Sch83] to find a constant \(C' := C'(N_{k_0}, M, g)\) such that for each \(k_i > k_0\)

\[ |A_{\Sigma_i}|^2 \leq C' \text{ on } \Sigma_i, \]

where \(|A_{\Sigma_i}|^2\) is the squared norm of the second fundamental form of \(\Sigma_i\).

From the assumption, there is a constant \(C_{k_0} > 0\) and an integer \(k'_0 > k_0\) so that for \(k_i > k'_0\),

\[ \text{Area}_g(\Sigma_i) \leq C_{k_0}. \]

Theorem \[5.3\] tells that \(\Sigma_i\) sub-converges with finite multiplicity.

From now on, we abuse the notation and write \(\{ \Sigma_i \}\) for a convergent sequence. The limit is a subset of \(\mathcal{L}_0 \cap N_{k_0}\). Since the limit is properly embedded, then the limit is a subset of a disjoint union of finite leaves, \(\{ L_s \}_{s=1}^{s_0} \subset \mathcal{L}_0\). We have that \(\pi_1(L_s \setminus N_{k_0-2}) \to \pi_1(M \setminus K)\) is trivial for each \(s\).

We may assume that each \(L_s\) intersects \(\partial N_{k_0-1}\) transversally. The intersection \(\Pi_s L_s \cap \partial N_{k_0-1}\) has finitely many components, \(\{ c_l \}_l\). From the last paragraph, each \(c_l\) is homotopically trivial in \(M \setminus K\).
Choose a constant $\varepsilon_1 > 0$ so that the tubular neighborhood $B^{\partial N_{k_0}^{-1}}(\bigcup_i c_i, \varepsilon_1) \subset \partial N_{k_0}^{-1}$ is a disjoint union of some annuli. We have that
the map $\pi_1(B^{\partial N_{k_0}^{-1}}(\bigcup_i c_i, \varepsilon_1)) \to \pi_1(M \setminus K)$ is also trivial. \hfill (\star)

The remaining is the same as the proof of Proposition 5.11
Since $\Sigma_i$ converges to a subset of $\bigcup_s L_s$, we have that for $i$ large enough,
$\Sigma_i \cap \partial N_{k_0}^{-1}$ is a subset of $B^{\partial N_{k_0}^{-1}}(\bigcup_i c_i, \varepsilon_1)$. (See the detail in the proof for
Proposition 5.11)
The fact (\star) shows that each circle in $\Sigma_i \cap \partial N_{k_0}^{-1}$ is contractible in $M \setminus K$.
However, the circle $\hat{\gamma}_i \subset \Sigma_i \cap \partial N_{k_0}^{-1}$ is non-contractible in $M \setminus K$, which
leads to a contradiction. We complete the proof of the proposition.

Now we use the Proposition 6.1 and Theorem 1.3 to finish the proof of
Theorem 1.1. We will show the proof of Theorem 1.3 in Section 7.

\textit{Proof.} Suppose that a complete contractible 3-manifold $(M, g)$ with positive
scalar curvature is not homeomorphic to $\mathbb{R}^3$, where $M$ is an increasing union
of handlebodies $\{N_k\}$.

We may assume that $M$ is not simply-connected for $N_0$. Lemma 2.14
gives a meridian curve $\gamma_k \subset \partial N_k$ which is non-contractible in $M \setminus N_0$.

Each $\gamma_k$ bounds an embedded disc $D_k \subset N_k$, constructed in Section 3.1.
The intersection $D_k \cap N_{k-1}$ is stable minimal for $g$. We then use Theorem
4.9 to show that for each $k_0$, there is a positive constant $C_{k_0}$, so that for
each $k > k_0$, the area of each component of $D_k \cap N_k$ is not greater than
$C_{k_0}$.

Theorem 5.5 shows that $\{D_k\}$ sub-converges to a lamination $\mathcal{L}$. Each
leaf is a complete stable minimal plane.

From now on, we abuse the notation and write $\{D_k\}$ for a convergent
sequence. Hence, the sequence $\{D_k\}$ satisfies the properties described in
Proposition 6.1.

From Theorem 1.3, each leaf in $\mathcal{L}$ is trivial at infinity. More precisely,
each leaf is trivial for $N_0$. We use Proposition 6.1 to find a positive integer
$k'$ so that for each $k > k'$, $\gamma_k := \partial D_k$ is homotopically trivial in $M \setminus N_0$. It
is in contradiction with the fact that $\gamma_k$ is non-contractible in $M \setminus N_0$.

We complete the proof of Theorem 1.1. \hfill \Box

7. Absence of non-trivial minimal plane at infinity

In this section, we use the positivity of scalar curvature to show the
absence of non-trivial minimal plane at infinity.

\textbf{Theorem 1.3} Let $(M, g)$ be a complete contractible 3-manifold with positive
scalar curvature. Then each complete embedded stable minimal plane in
$(M, g)$ is trivial at infinity.

In the rest of the section we will use the following notations.
• We let $B(D, \epsilon)$ denote the tubular neighborhood of the embedded disc $D$ with radius $\epsilon > 0$, consisting of geodesics of $M$ orthogonal to $D$ and centered at the points of $D$.
• The annulus $A(\partial D, \epsilon)$ consists of geodesics of $M$ orthogonal to $D$ and centered at the points of $\partial D$, whose lengths are equal to $2\epsilon$.

Remark that $A(\partial D, \epsilon)$ is a subset of the boundary of $B(D, \epsilon)$.

7.1. Scheme of the proof of Theorem [1.3] We now explain the scheme of the proof of Theorem [1.3]. Our main strategy is to argue by contradiction.

Suppose that there is a complete stable minimal plane in $(M, g)$ which is not trivial at infinity. That is to say, there is a compact and connected set $K \subset M$ so that a complete stable minimal plane is not trivial for $K$ (see Definition 5.8 and Remark 5.9). Our proof follows these three steps:

Step 1: Find a complete embedded stable minimal plane $L \subset (M, g)$ satisfying that
• it is not trivial for $K$
• each complete stable minimal plane in $M'$ is trivial for $K$,
where $M'$ is one of the components of $M \setminus L$. (See Lemma [7.3])

Step 2: Use the non-triviality of $L$ to find a family $\{\hat{\Omega}_i\}$ of embedded discs in $M'$ which converges to a lamination in $\overline{M'}$ and satisfies the property described in Proposition [6.1] (See Lemma [7.5])

Step 3: Modify $\hat{\Omega}_i$ to get an embedded disc $\Omega_i$ which converges to a lamination $L \subset M'$. Proposition [6.1] will lead to a contradiction.

7.2. Find a required minimal plane $L$. Before constructing the required plane $L$, we introduce a notation.

Definition 7.1. Let $\{P_i\}_{i=1}^2$ be two properly embedded planes in $(M, g)$ and $K \subset M$ a compact and connected set. We define
$$d_K(P_1, P_2) := d^K(P_1 \cap K, P_2 \cap K),$$
where $d^K$ is the distance function induced by $(K, g|_K)$.

Remark 7.2. Let $\{P_i\}_{i=0}^2$ be three disjoint and properly embedded planes in $(M, g)$. The plane $P_0$ cuts $M$ into two components, $M_1$ and $M_2$. If $P_i$ is a subset of $M_i$ for $i = 1, 2$, then one has that
$$\hat{d}_K(P_1, P_0) + \hat{d}_K(P_2, P_0) \leq \hat{d}_K(P_1, P_2).$$

The reason is as follows:

There is a curve $c : [0, 1] \to K$ whose length is equal to $\hat{d}_K(P_1, P_2)$, where $c(0) \in P_1$ and $c(1) \in P_2$. It intersects $P_0$. Namely, there is a point $c(t) \in P_0$ for some $0 < t < 1$. We have that $\hat{d}_K(P_1, P_0) \leq \text{Length}(c|_{[0, t]})$ and $\hat{d}_K(P_2, P_0) \leq \text{Length}(c|_{[t, 1]})$, which implies the above inequality.

We now use the function $\hat{d}_K$ to show the existence of the required plane.

Lemma 7.3. Let $(M, g)$ be a complete contractible 3-manifold with positive scalar curvature and $K \subset M$ a compact and connected set. If a complete
stable minimal plane is not trivial for $K$, then there is a complete embedded stable minimal plane $L \subset (M, g)$ satisfying that

- $L$ is not trivial for $K$;
- any complete stable minimal plane in $M'$ is trivial for $K$.

where $M'$ is one of the components of $M \setminus L$

Proof. Let $L_0 \subset (M, g)$ be a complete embedded stable minimal plane which is not trivial for $K$. Theorem 5.6 shows that it is properly embedded. By Lemma 2.3, $L_0$ cuts $M$ into two components, $M_0$ and $M'_0$.

If each stable minimal plane in $M_0$ is trivial for $K$, $L_0$ is the required candidate in our assertion.

If not, we consider the set $\mathcal{C}$ of all complete embedded stable minimal planes in $M_0$ which are not trivial for $K$. It is non-empty. We define

$$T := \max\{d_K(L_0, L') \mid L' \in \mathcal{C}\}.$$

Since each plane in $\mathcal{C}$ intersects the compact and connected set $K$, then $T$ is finite.

Let $\{L_i\}$ be a sequence of stable minimal planes in $\mathcal{C}$ with $\lim_{i \to \infty} d_K(L_0, L_i) = T$. Each plane $L_i$ intersects $K$. From Corollary 5.7, $\{L_i\}$ subconverges with finitely multiplicity.

From now on, we write $\{L_i\}_i$ for a convergent sequence. For each component $L$ of the limit, $d(\hat{L}, L_0) \geq T$. Therefore, the limit is in $M_0 \subset M \setminus L_0$.

Corollary 5.12 shows that one component $L_0 \subset M_0$ of the limit is not trivial for $K$ (since each $L_i$ is not trivial for $K$). Hence, $d(L_0, L) \leq T$. The last paragraph shows that $d_K(L, L_0) = T$.

We can conclude that $L$ is the required candidate in the assertion.

If not, $M'$ has a complete stable minimal plane $L'$ which is not trivial for $K$, where $M' \subset M_0$ is one component of $M \setminus L$. It belongs to $\mathcal{C}$. Then, $d_K(L_0, L') \leq T$.

However, $L_0 \subset M \setminus M'$. Remark 7.2 shows that

$$d_K(L_0, L) + d_K(L, L') \leq d_K(L', L_0) \leq T.$$

Hence, $d_K(L, L') = 0$. Namely, $L$ intersects $L'$, which is in contradiction with the fact that $L' \subset M' \subset M \setminus L$. \qed

7.3. Construction of the sequence $\{\hat{\Omega}_i\}$. We begin with a deformation lemma.
Lemma 7.4. Let \((M, g)\) be a complete 3-manifold and \(L \subset (M, g)\) a complete properly embedded minimal surface. Assume that the subset \(K \subset M\) is compact and connected. Then for each \(\epsilon > 0\), after a small deformation on \(\partial K\), there is a complete metric \(g'\) on \(M\) satisfying that

- \(g' \geq g\) and \(g'\) is equal to \(g\) on \(M \setminus B(\partial K, \epsilon)\);
- the surface \(L\) is also minimal for \(g'\);
- the mean curvature of \(\partial K\) is positive for \(g'\).

See the proof in Appendix B.

Lemma 7.5. Let \((M, g), L, M'\) and \(K\) be assumed as in Lemma 7.3. Then there is a family \(\{c_i\}\) of circles in \(M'\) so that

(a) the circles \(\{c_i\}\) go to infinity and each \(c_i\) is non-contractible in \(M \setminus K\);
(b) each \(c_i\) bounds an embedded disc \(\hat{\Omega}_i \subset M'\) which converges to a lamination \(\mathcal{L}_0 \subset M'\) whose leaves are complete stable minimal planes;
(c) for each positive integer \(k_0\), there is an integer \(i_0 > 0\) and a constant \(C_{k_0} > 0\), so that for \(i \geq i_0\), the area of each component of \(N_{k_0} \cap \hat{\Omega}_i\) is not greater than \(C_{k_0}\).

Remark that the lamination may contain the plane \(L\).

Remark 7.6. Since \(M = \cup_k N_k\), we can find a positive integer \(k_0\) with \(K \subset N_{k_0}\). The set \(L \cap N_{k_0}\) is compact in \(L\). Then, there is a disc \(D_0 \subset L\) with \(L \cap N_{k_0} \subset D_0\). Further, there is an integer \(k_1\) with \(D_0 \subset N_{k_1}\).

We repeat the above process and get a subsequence \(\{N_{k_i}\}\) of \(\{N_k\}\) with the following properties:

- \(N_{k_0}\) contains the compact set \(K\);
- the intersection \(N_{k_i} \cap L\) is a subset of a disc \(\hat{D}_i \subset L \cap N_{k_{i+1}}\).
Proof. Let \( \{ N_{k_i} \} \) and \( \{ \hat{D}_i \} \) be assumed as in Remark 7.6. Then, \( \partial \hat{D}_i \) is non-contractible in \( M \setminus K \) for \( i \geq 1 \). (If not, the map \( \pi_1(L \setminus \hat{D}_i \cup K) \to \pi_1(M \setminus K) \) is trivial. Namely, \( L \) is trivial for \( K \), a contradiction.)

Consider the tubular neighborhood \( B(\hat{D}_i, \epsilon_i) \) and the annulus \( A(\partial \hat{D}_i, \epsilon_i) \). Choose a closed curve \( c_i \subset A(\partial \hat{D}_i, \epsilon_i) \cap M' \) which is homotopic to \( \partial \hat{D}_i \) in \( A(\partial \hat{D}_i, \epsilon_i) \). From the last paragraph, it is not homotopically trivial in \( M \setminus K \).

In the following, we use the construction, described in Section 3, to construct the embedded disc \( \hat{\Omega}_i \) with boundary \( c_i \).

Let \( \hat{V}_i := N_{k_i} \cup B(\hat{D}_i, \epsilon_i) \subset N_{k_i+1} \). It is compact and connected. By Theorem 5.6 \( L \) is properly embedded. We use Lemma 7.4 to find a metric \( \hat{g}_i \) on \( M \) so that
- \( \hat{g}_i \geq g \) and \( \hat{g}_i \) is equal to \( g \) on \( N_{k_i-1} \);
- the mean curvature of \( \partial \hat{V}_i \) is positive for \( \hat{g}_i \) and \( L \) is minimal for \( \hat{g}_i \).

Let \( V_i := (\hat{V}_i \setminus L) \cap M' \) be the component of \( \hat{V}_i \setminus L \) containing \( c_i \). (Remark that \( N_{k_i} \cap M' \subset V_i \).) Consider the closure of the Riemannian manifold \((V_i, \hat{g}_i|_{V_i})\).

Its boundary comes from \( L \) and \( \partial \hat{V}_i \). The mean curvature of the boundary is non-negative for \( \hat{g}_i \). Therefore, the closure of \((V_i, \hat{g}_i|_{V_i})\) has mean convex boundary.

Each \( c_i \subset \partial V_i \) is homotopically trivial in \( V_i \). We use the inductive construction described in Section 3.1, to have that
- each \( c_i \) bounds an embedded disc \( \hat{\Omega}_i \subset V_i \);
- the intersection \( \hat{\Omega}_i \cap V_{i-1} \) consists of finitely many smooth surfaces \( \{ \hat{\Omega}_j^i \}_{j \in J_i} \)
  and each \( \hat{\Omega}_j^i \) is stable minimal for the original metric \( g \);
- (Disjointness) \( \hat{\Omega}_j^i \subset \hat{\Omega}_j^i \) or \( \hat{\Omega}_j^i \cap \hat{\Omega}_j^i = \emptyset \) where \( i' < i, j \in J_i \) and \( j' \in J_{i'} \);
- (Area-minimizing) \( \hat{\Omega}_i \) is an area-minimizing disc with boundary \( c_i \) in the closure of the manifold \((V_i \setminus \cup_{j=1}^{i-1} \hat{\Omega}_j \cap V_{i-1}, \hat{g}_i|_{V_i \setminus \cup_{j=1}^{i-1} \hat{\Omega}_j \cap V_{i-1}})\).

Since \( \hat{\Omega}_i \) is minimal for \( \hat{g}_i \), we can conclude that \( \hat{\Omega}_i \) is a subset of \( M' \). If not, \( L \cap \hat{\Omega}_i \) is non-empty. Since \( L \) is minimal for \( \hat{g}_i \), the maximum principle (see [Corollary 1.27, Page 37] in [CM11]) shows that \( \hat{\Omega}_i \) is a subset of \( L \). However, \( c_i \subset M' \subset M \setminus L \), a contradiction.

Theorem 4.9 and Remark 4.10 shows that for each \( i_0 \), there is a constant \( C_{i_0} > 0 \), so that for \( i > i_0 \), the area of each component of \( \hat{\Omega}_i \cap V_{i_0} \) is not greater than \( C_{i_0} \). Hence, the area of each component of \( \hat{\Omega}_i \cap N_{k_{i_0}} \) is not greater than \( C_{i_0} \) for \( i > i_0 \) (since \( N_{k_i} \cap M' \subset V_i \) and \( \hat{\Omega}_i \subset M' \)).

Because \( c_i \) is non-contractible in \( M \setminus K \), \( \hat{\Omega}_i \) intersects \( K \). The intersection \( \hat{\Omega}_i \cap N_{k_{i-1}} \) is stable minimal for \( g \). By Theorem 5.5 \( \hat{\Omega}_i \) sub-converges to a lamination whose leaves are complete stable minimal planes. \( \square \)
Roughly speaking, $\hat{\Omega}_i$ is deformed from $\hat{D}_i$ and the sequence $\hat{\Omega}_i$ converges with finite multiplicity.

7.4. Modify the disc $\hat{\Omega}_i$ and complete the proof of Theorem 1.3.

We begin with the existence of the embedded minimal annulus and then use it to give an improved version of Lemma 7.5.

**Lemma 7.7.** Let $(M, g)$, $L$, $M'$ and $K$ be assumed as in Lemma 7.3 and let $c \subset \partial L \setminus K$ be a non-contractible circle in $M \setminus K$. Assume that $c$ bounds an embedded disc $D \subset \partial L$ and the sets, $K$ and $D$, are the subsets of Int $N_k$ for some $k$. Then, there is a constant $\epsilon > 0$ so that if we have that

- a circle $c'$ bounds an embedded minimal disc $D' \subset \text{Int} N_{k+1} \cap M'$,
- $c$ and $c'$ bounds an annulus $\text{Int} A_1 \subset M' \setminus D'$, and $\text{Area}(A_1) \leq \epsilon$,
- $d(c', K \cup \partial N_k) \geq 1/2 d(c, K \cup \partial N_k)$,

then there is an embedded minimal annulus $\text{Int} A_0 \subset M' \setminus D'$ with boundary $c \cup c'$.

**Proof.** First, we choose the positive constant $\epsilon$. Then, we use the choice of $\epsilon$ to construct the minimal annulus.

**Step 1:** Choose the constant $\epsilon$.

Define two constants, $c_0 := \max_{x \in N_{k+1}} |K_M(x)|$ and $i_0 := \min_{x \in N_{k+1}} \text{Inj}_M(x)$, where $K_M(x)$ is the sectional curvature and $\text{Inj}_M(x)$ is the injective radius at $x$.

Let $r_0 := \frac{1}{4} \min\{i_0, d(c, K \cup \partial N_k), d(\partial N_{k+1}, \partial N_k)\}$. Then, $2r_0 \leq d(c', K \cup \partial N_k)$. We use [Lemma 1, Page 445] in [MY80] to find a positive constant $C := C(c_0, i_0, r_0)$, so that if $\Sigma$ is a minimal surface in $N_{k+1}$ and $d(x, \partial \Sigma) \geq r_0$ for some $x \in \Sigma$, then

$$\text{Area}(\Sigma \cap B(x, r_0)) \geq C.$$ 

We choose $\epsilon := C/2$. In the following, we construct the embedded minimal annulus, $A_0$.

**Step 2:** Construct the required minimal surface.

By Theorem 5.6, $L$ is properly embedded. We use Lemma 7.4 to find a metric $\tilde{g}_{k+2}$ on $M$ so that

- $\tilde{g}_{k+2} \geq g$ and $\tilde{g}_{k+2}$ is equal to $g$ on $N_{k+1}$;
- $(N_{k+2}, \tilde{g}_{k+2}|_{N_{k+2}})$ has mean convex boundary.
- $L$ is also minimal for $\tilde{g}_{k+2}$

The disc $D' \subset N_k$ is also minimal for $\tilde{g}_{k+2}$. Let $W$ be the component of $M' \cap N_{k+2}$ containing $D'$. The sets, $A_1$ and $D$, are both in $W$. Consider the closure of the manifold

$$W \setminus D', \tilde{g}_{k+2}|_{W \setminus D'}.$$ 

The boundary comes from $\partial N_{k+2}$, $L$ and $D'$. The mean curvature of these surfaces are both non-negative for the metric $\tilde{g}_{k+2}$. Therefore, the closure of $(W \setminus D', \tilde{g}_{k+2}|_{W \setminus D'})$ has mean convex boundary.
The circles, \( c \) and \( c' \) can be viewed as the circles in the boundary of the closure of \((W \setminus D', \tilde{g}_{k+2}|_{W \setminus D'})\). We use [Theorem 5, Page 433] in [MYS2] to find an embedded surface \( \text{Int} A_0 \) in the closure of \((W \setminus D', \tilde{g}_{k+2}|_{W \setminus D'})\) with boundary \( c' \amalg c \). In addition, we have

- the surface \( A_0 \) is an embedded annulus or two disjoint discs;
- the surface \( A_0 \) is stable minimal for \( \tilde{g}_{k+2} \);
- \( \text{Area} \tilde{g}_{k+2}(A_0) \leq \text{Area} \tilde{g}_{k+2}(A_1) = \text{Area} g(A_1) \leq \epsilon \).

**Step 3:** Show that \( A_0 \) is a minimal annulus for the original metric \( g \).

First, we have that \( A_0 \) is a subset of \( N_k \). The reason is described as below:

If not, we find a point \( x_1 \in A_0 \cap \partial N_k \). Since \( B(x_1, r_0) \subset N_{k+1} \) and \( \tilde{g}_{k+2}|_{N_{k+1}} = g|_{N_{k+1}} \), the surface \( A_0 \cap B(x_1, r_0) \) is minimal for \( g \). In addition, \( d(x_1, \partial(A_0 \cap B(x_1, r_0))) \geq r_0 \).

The choice of \( C \) tells that \( \text{Area} g(A_0 \cap B(x_1, r_0)) \geq C \). However, \( \text{Area} g(B(x_1, r_0) \cap A_0) \leq \text{Area} \tilde{g}_{k+2}(A_0) \leq \epsilon = C/2 \), which leads to a contradiction.
Since $A_0 \subset N_k$ and $\bar{g}_{k+2}|N_k = g|N_k$, then $A_0$ is minimal for the original metric $g$. Recall that $A_0$ is an annulus or a disjoint union of two discs.

If $A_0$ is not an annulus, it is a disjoint union of two discs $D'_1$ and $D'_2$ with $\partial D'_1 = c$. Since $c$ is not contractible in $M \setminus K$, $D'_1 \cap K$ is non-empty. Choose $x_2 \in K \cap D'_1$. Hence, $d(x_2, \partial D'_1) \geq r_0$.

The choice of $C$ shows that $\text{Area}_g(D'_1 \cap B(x_2, r_0)) \geq C$. However, $\text{Area}_g(D'_1 \cap B(x_2, r_0)) \leq \text{Area}_g(A_0) \leq \epsilon = C/2$, a contradiction.

We can conclude that $A_0$ is a minimal annulus with boundary $c\Pi c'$. The maximum principle for minimal surfaces (see [Corollary 1.28 , Page 37] in [CMII]) shows that $L = A_0$ is embedded in $M' \setminus D'$. We complete the proof. \(\square\)

We use the existence of minimal annulus to give an improved version of Lemma 7.5.

**Lemma 7.8.** Let $(M, g)$, $L$, $M'$, $K$ and $\{c_i\}$ be assumed as in Lemma 7.5. Then each $c_i$ bounds an embedded disc $\Omega_i \subset M'$ so that

(a) the sequence $\Omega_i \subset M'$ converges to a lamination $\mathcal{L}_1 \subset M'$ whose leaves are complete stable minimal planes;

(b) for each positive integer $k_0$, there is an integer $i_0 > 0$ and a constant $C_{k_0} > 0$, so that for $i \geq i_0$, the area of each component of $N_{k_0} \cap \Omega_i$ is not greater than $C_{k_0}$

Remark that since each complete stable minimal plane in $M'$ is trivial for $K$ (see Lemma 7.3), each leaf of $\mathcal{L}_1$ is trivial for $K$.

**Proof.** Lemma 7.3 gives a family $\{c_i\}$ of circles in $M'$ so that

- the circles $\{c_i\}$ go to infinity and each $c_i$ is non-contractible in $M \setminus K$;
- each $c_i$ bounds an embedded disc $\Omega_i \subset M'$ which converges to a lamination $\mathcal{L}_0$ whose leaves are complete stable minimal planes;
- for each positive integer $k_0$, there is an integer $i_0 > 0$ and a constant $C_{k_0} > 0$ so that for $i \geq i_0$, the area of each component of $N_{k_0} \cap \Omega_i$ is not greater than $C_{k_0}$

We may assume that $\mathcal{L}_0$ intersects $L$. (If not, $\mathcal{L}_0 \subset M'$. The sequence $\{\Omega_i\}$ is the required candidate in our assertion.) The maximum principle for minimal surface shows that $L$ is one of the components of $\mathcal{L}_0$.

In the following, we first find a minimal annulus. Then we use it to find a new sequence $\{\Omega_i\}$ of embedded discs whose limit is in $M'$.

**Step 1:** Find a circle $c' \subset M'$ which bounds an embedded minimal disc $D'$.

Since $L$ is properly embedded (see Theorem 5.6), we can find a disc $D \subset L$ with $L \cap K \subset D$. It is a subset of $\text{Int } N_k$ for some $k$. The circle $c := \partial D$ is non-contractible in $M \setminus K$. (If not, $\pi_1(L \setminus (D \cup K)) \to \pi_1(M \setminus K)$ is trivial. Namely, $L$ is trivial for $K$, a contradiction.)

We use Lemma 7.7 to find a constant $\epsilon > 0$ (which satisfies the property in Lemma 7.7). Choose a positive constant $\epsilon_1 > 0$ so that

- $\text{Area}_g(A(\partial D, \epsilon_1)) \leq \epsilon$ and $B(D, \epsilon_1) \subset N_k$;
\[ d(A(\partial D, \epsilon_1), \partial N_k \cup K) \geq 1/2d(c, \partial N_k \cup K). \]

Since \( L \) is a subset of the limit of \( \Omega_i \), there is an integer \( i_0 > 0 \) so that for each \( i \geq i_0 \), \( \Omega_i \cap B(D, \epsilon_1) \) has a component \( \Sigma_i \) which converges to \( D \). Each \( \Sigma_i \) is stable minimal for \( g \).

By Remark 5.2, the restriction \( \pi|_{\Sigma_i} : \Sigma_i \to D \) is a cover map for \( i \) large enough, where \( \pi \) is the projection from \( B(D, \epsilon_1) \) to \( D \). Since \( \pi_1(D) = \{1\} \), then \( \Sigma_i \) is a disc. Namely, the circle \( c' := \partial \Sigma_i \subset A(D, \epsilon_1) \cap M' \) bounds an embedded minimal disc \( D' := \Sigma_i \subset \Omega_i \cap N_k \) for some \( i \). In addition, \( c \cup c' \) bounds an annulus \( A_1 \subset A(\partial D, \epsilon_1) \) whose area is not greater than \( \epsilon \).

**Step 2:** Construct the manifolds with mean convex boundary.

We use the choice of \( \epsilon \) and Lemma 7.5 to find an embedded minimal annulus \( \text{Int} A_0 \subset M' \setminus D' \) with boundary \( c \cup c' \). Consider the embedded 2-sphere \( D \cup A_0 \cup c \). Since \( M' \) is contractible, it is irreducible (see the Poincaré conjecture [BBB+10, MT07]). Therefore, the sphere bounds a 3-ball \( B \subset M' \cap N_k \).

Let \( V_i, \hat{g}_i \) and \( \hat{V}_i \) be assumed as in the proof of Lemma 7.5. We may assume that \( k_i \geq k_0 > k \) for each \( i \). Namely, \( B \subset N_k \cap M' \subset V_i \) for each \( i \). The surfaces \( D, A_0 \) and \( D' \) are both minimal for \( \hat{g}_i \).

Consider the closure of the Riemannian manifold \( (V_i \setminus B, \hat{g}_i|_{V_i \setminus B}) \), for \( i > 0 \). The boundary comes from \( L, \partial \hat{V}_i, A_0 \) and \( D' \). The mean curvature of these surfaces are both non-negative for \( \hat{g}_i \) (see the choice of \( \hat{g}_i \) in the proof).
of Lemma 7.5). The closure of \((V_i \setminus B, \hat{g}_i|_{V_i \setminus B})\) has mean convex boundary for \(i > 0\).

**Step 3:** Find the disc \(\Omega_i\).

Since \(\pi_1(V_i) \cong \pi_1(V_i \setminus B)\), \(c_i\) is also contractible in \(\overline{V_i \setminus B}\) for \(i > 0\). We use the inductive construction, described in Section 3.1, to have that
- each \(c_i\) bounds an embedded disc \(\Omega_i\) in the closure of \(V_i \setminus B\);
- the intersection \(\Omega_i \cap \overline{V_{i-1} \setminus B}\) consists of finitely many smooth surfaces \(\{\Omega^j_i\}_{j \in J_i}\) and each \(\Omega^j_i\) is stable minimal for the original metric \(g\);
- (Disjointness) \(\Omega^j_i \cap \Omega^{j'}_i = \emptyset\) where \(i' \leq i, j \in J_i\) and \(j' \in J_{i'}\);
- (Area-minimizing) \(\Omega_i\) is an area-minimizing disc with boundary \(c_i\) in the closure of the manifold \((\overline{(V_i \setminus B) \cup \bigcup_{l=1}^{i-1} \Omega_l \cap V_{l-1}}, \hat{g}_i|_{(V_i \setminus B) \cup \bigcup_{l=1}^{i-1} \Omega_l \cap V_{l-1}})\).

As in the proof of Lemma 7.5, for each \(i_0\), there is a positive constant \(C_{i_0}\), so that for \(i > i_0\), the area of each component of \(\Omega_i \cap N_{k_{i_0}}\) is not greater than \(C_{i_0}\).

Each \(\Omega_i\) is a subset of \(M'\) and \(\{\Omega_i\}\) sub-converges to a lamination \(\mathcal{L}_1 \subset \overline{M' \setminus B}\) whose leaves are complete stable minimal planes. The reason is the same as in the proof of Lemma 7.5. From now on, we write \(\{\Omega_i\}_i\) for a convergent sequence.

We can conclude that \(\mathcal{L}_1\) is a subset of \(M'\). If not, some leaf in \(\mathcal{L}_1\) intersects \(L\). The maximum principle tells that \(L\) is a subset of \(\mathcal{L}_1 \subset \overline{M' \setminus B}\). However, \(L\) is not a subset of \(\overline{M' \setminus B}\), a contradiction.

Therefore, the sequence \(\{\Omega_i\}\) is the required candidate in our assertion. \(\Box\)

In the following, we complete the proof of Theorem 1.3.

**Proof.** Suppose that a complete stable minimal plane in \((M, g)\) is non-trivial at infinity. Namely, we can find a compact and connected set \(K \subset M\) so that a complete stable minimal plane in \((M, g)\) is not trivial for \(K\).

We use Lemma 7.3 to find a complete stable minimal plane \(L\) so that
1) it is non-trivial for \(K\);
2) each complete embedded stable minimal plane in \(M'\) is trivial for \(K\), where \(M'\) is one component of \(M \setminus L\).

Lemma 7.5 and Lemma 7.8 give a family \(\{c_i\}\) of circles in \(M'\) so that
- the circles \(\{c_i\}\) go to infinity and each \(c_i\) is non-contractible in \(M \setminus K\);
- each \(c_i\) bounds an embedded disc \(\Omega_i \subset M'\) which converges to a lamination \(\mathcal{L}_1 \subset M'\) whose leaves are complete stable minimal planes;
- for each positive integer \(k_0\), there is an integer \(i_0 > 0\) and a constant \(C_{k_0} > 0\) so that for \(i \geq i_0\), the area of each component of \(N_{k_0} \cap \Omega_i\) is not greater than \(C_{k_0}\).

The sequence \(\{\Omega_i\}\) satisfies the properties described in Proposition 6.1.
Furthermore, we have that each leaf of $\mathcal{L}_1 \subset M'$ is trivial for $K$ (since each stable minimal plane in $M'$ is trivial for $K$). By Proposition 6.1, $c_i = \partial \Omega_i$ is homotopically trivial in $M \setminus K$ for $i$ large enough. It leads to a contradiction with the fact that $c_i$ is non-contractible in $M \setminus K$. □

APPENDIX A: Topological lemmas for surfaces

Lemma 4.4 Let $F$ be an orientable surface (possibly with boundary and non-compact). Assume that two disjoint embedded circles, $\alpha$ and $\alpha'$, are not null-homotopic in $F$. If these two circles are homotopic in $F$, then there is an annulus $A \subset F$ with boundary $\alpha \amalg \alpha'$.

Proof. First, we show the case when $\alpha$ cuts $F$ into two components. Then, we reduce the proof to the splitting case.

Step 1: Consider the case when $\alpha$ cuts $F$ into two parts, $F_1$ and $F_2$.

We may assume that the component $F_1$ contains the circle $\alpha'$. Consider the 2-manifold $F' := F_1 \cup_{\alpha} D$, where $D$ is a unit disc in $\mathbb{R}^2$. Therefore, $\alpha'$ is contractible in $F'_1$. It bounds a disc $D' \subset F'_1$.

We can conclude that $D \subset F'_1$ is a subset of $D'$. (If not, $D \cap D'$ is empty. Namely, $D'$ is a subset of $F_1 \subset F$. However, $\alpha'$ is non-contractible in $F$, a contradiction.)

The annulus $D' \setminus D$ is a subset of $F_1 \subset F$ with boundary $\alpha \amalg \alpha'$, which is the required candidate in our assertion.

Step 2: Prove the case when $F \setminus \alpha$ is connected.

We assume that $F \setminus \alpha$ is connected. Let $\alpha^+$ and $\alpha^-$ be two boundaries of $F \setminus \alpha$ coming from $\alpha$. Thus, we have a triple $(F \setminus \alpha, \alpha^+, \alpha^-)$. Form countably many copies, denoted $(\hat{F}_i, \alpha_i^+, \alpha_i^-)$ for $i \in \mathbb{Z}$. Choose $\alpha_i' \subset \hat{F}_i$ as the image of $\alpha' \subset F \setminus \alpha$ in $\hat{F}_i$.

Consider the surface $\hat{F}_\infty$ obtained by gluing $\hat{F}_i$ and $\hat{F}_{i+1}$ along $\alpha_i^+$ and $\alpha_{i+1}^-$ for each $i \in \mathbb{Z}$. It is a covering space of $F$. We use the homotopy lifting property (see [Proposition 1.30, Page 60] in [Hat05]) to show that either $\alpha_i^+$ or $\alpha_i^-$ is homotopic to $\alpha_i'$ in $\hat{F}_\infty$ for each $i$.

We may assume that $\alpha_i^+$ and $\alpha_i'$ are homotopic in $\hat{F}_\infty$. Since $\alpha_i^+$ cuts $\hat{F}_\infty$ into two components, we use Step 1 to find an annulus $A_i \subset \hat{F}_\infty$ with boundary $\alpha_i^+ \amalg \alpha_i'$.

In addition, $A_i$ is a subset of $\hat{F}_i$. That is to say, there is an annulus $A \subset F \setminus \alpha$ which bounds $\alpha^+ \amalg \alpha'$.

□

APPENDIX B: Deformation of surfaces

Theorem B.1. Let $L$ be a complete properly embedded surface in a complete 3-manifold $M$ and $K \subset M$ a compact and connected set. For each $\epsilon > 0$, there is a positive constant $\epsilon_0$ and a smooth embedding map $f : (-\epsilon_0, \epsilon_0) \times \partial K \rightarrow B(\partial K, \epsilon)$ satisfying that
• The map \( f(t, \bullet) : \partial K \to B(\partial K, \epsilon) \) is embedded for any \( t \);
• The image \( \Sigma_t \) of \( f(t, \bullet) \) is isotopic to \( \partial K \);
• The disjoint union \( \bigcup_1 \Sigma_t \) is an open neighborhood of \( \Sigma_0 \) in \( M \);
• \( \langle \frac{\partial f(t, \bullet)}{\partial t} \vert_{t=0}, \mathbf{n}_{\Sigma_0}(x) \rangle > 0 \) for any \( x \in \Sigma_0 \);
• For each point \( x \in L \cap \Sigma_t \), \( \langle \frac{\partial f(t, \bullet)}{\partial t}, \mathbf{n}_L(x) \rangle = 0 \).

where \( B(\partial N, \epsilon) \) is the tubular neighborhood of \( \partial N \) with radius \( \epsilon \) in \( M \).

Proof. We deform \( \partial K \) to obtain a smooth surface \( \Sigma_0 \subset B(\partial K, \epsilon/4) \) which intersects \( L \) transversally. Because of the properness of \( L \), \( L \cap \Sigma_0 \) has finitely many components.

In order to simplify the proof, we assume that \( L \cap \Sigma_0 \) has a unique component, \( \alpha \).

Let \( \mathbf{n}_L \) be the unit normal vector of \( L \) and \( V_0 \subset TL_{\alpha} \) a field vector orthogonal to \( \alpha'(t) \). We have that

\[ \langle \mathbf{n}_{\Sigma_0}, V_0 \rangle(\alpha(t)) \neq 0 \text{ for each } t. \]

If not, \( V_0_{\alpha(t)} \) belongs to \( T_{\alpha(t)} \Sigma_0 \) for some \( t \). Namely, the tangent space \( T_{\alpha(t)} \Sigma_0 \) is spanned by \( \alpha'(t) \) and \( V_0_{\alpha(t)} \). Hence, \( T_{\alpha(t)} \Sigma_0 = T_{\alpha(t)}L \), which contradicts the fact that \( L \) intersects \( \Sigma_0 \) transversally.

We may assume that \( \langle \mathbf{n}_{\Sigma_0}, V_0 \rangle(\alpha(t)) > 0 \) for each \( t \). Otherwise, we use \(-V_0\) to replace \( V_0 \).

**Step 1:** Extend the vector field \( V_0 \) over \( \alpha \) to be the vector field \( V \) over the tubular neighborhood of \( \alpha \) in \( M \).

Choose a tubular neighborhood \( L_0 \subset L \) of \( \alpha \) in \( L \) so that \( V_0 \) can be extended to be a vector field \( V_1 \subset TL_0 \) (i.e. \( V_1_{\alpha} = V_0 \)). Consider the map

\[ i : [\epsilon_1, \epsilon_1] \times L_0 \rightarrow M \]

\[ t_1 \times x \mapsto \exp_x(t_1 \mathbf{n}_L(x)) \]

Choose \( \epsilon_1 \) small enough so that the map \( i \) is an inclusion and the image is a tubular neighborhood \( B(L_0, \epsilon_1) \) of \( L_0 \) in \( M \). Define

\[ V := \frac{\partial}{\partial t_2} \bigg|_{t_2=0} \exp_x(t_1 \mathbf{n}_L(x) + t_2 V_1(x)). \]

It is a vector field over \( B(L_0, \epsilon_1) \) and \( V \bigg|_{L_0} = V_1 \subset TL_0 \).

By a similar argument, we can find a vector field \( N \) over \( B(\Sigma_0, \epsilon') \) with \( N \vert_{\Sigma_0} = \mathbf{n}_{\Sigma_0} \), where \( \epsilon' > 0 \)

**Step 2:** Consider the required map \( f \).

Recall that \( \langle V, \mathbf{n}_{\Sigma_0}(x) \rangle = \langle V_0, \mathbf{n}_{\Sigma_0}(x) \rangle > 0 \) for \( x \in \alpha := L \cap \Sigma_0 \). Choose \( \epsilon_2 < \epsilon_1 \) so that for each \( x \in \Sigma_0 \cap B(L_0, \epsilon_2) \), one has that

\[ \langle V, \mathbf{n}_{\Sigma_0}(x) \rangle > 0. \]
Choose \( \epsilon_3 < \min\{\epsilon'/2, \epsilon/4\} \) so that \( B(\Sigma_0, \epsilon_3) \cap L \subset L_0 \). Consider the ODE on \( B(\Sigma_0, \epsilon_3) \subset B(\partial K, \epsilon) \):
\[
\begin{cases}
\frac{\partial \Sigma_t}{\partial t} = (1 - \phi(x)) N(x) + \phi(x) V(x); \\
\Sigma_t|_{t=0} = \Sigma_0,
\end{cases}
\]
where \( \phi(x) \in C^\infty(B(\Sigma_0, \epsilon_3), \mathbb{R}) \) is supported in \( B(L_0, \epsilon_2) \) with \( 0 \leq \phi \leq 1 \) and \( \phi(x) = 1 \) on \( B(L_0, \epsilon_2/2) \).

There is a constant \( \epsilon_4 > 0 \) and a family \( \{\Sigma_t\}_{|t| \leq \epsilon_4} \) of closed surfaces so that \( \{\Sigma_t\}_{|t| \leq \epsilon_4} \) is a smooth solution to the above ODE and each \( \Sigma_t \) is contained in \( B(\Sigma_0, \epsilon_3) \subset B(\partial K, \epsilon) \). In addition, we have that
\[
\langle \frac{\partial \Sigma_t}{\partial t}, n_{\Sigma_0} \rangle|_{t=0} = (1 - \phi(x)) + \phi(x) \langle V, n_{\Sigma_0} \rangle(x) > 0
\]
Choose \( \epsilon_0 < \epsilon_4 \) so that for \( |t| < \epsilon_0 \), \( \langle \frac{\partial \Sigma_t}{\partial t}, n_{\Sigma_t} \rangle(x) > 0 \) for \( x \in \Sigma_t \). The surfaces \( \{\Sigma_t\}_{|t| < \epsilon_0} \) are disjoint and the union is an open neighborhood of \( \Sigma_0 \) in \( M \).

We have that \( \frac{\partial \Sigma_t}{\partial t} \big|_L = V \subset TL \). Hence, \( \langle \frac{\partial \Sigma_t}{\partial t}, n_L \rangle(x) = 0 \) for \( x \in L \cap \Sigma_t \). The proof is complete. \( \square \)

**Lemma.** \( \text{[7.3]} \) Let \( (M, g) \) be a complete 3-manifold and \( L \subset (M, g) \) a complete properly embedded minimal surface. Assume that the subset \( K \subset M \) is compact and connected. Then for each \( \epsilon > 0 \), after a small deformation on \( \partial K \), there is a complete metric \( g' \) on \( M \) satisfying that

- \( g' \geq g \) and \( g' \) is equal to \( g \) on \( M \setminus B(\partial K, \epsilon) \);
- the plane \( L \) is also minimal for \( g' \);
- the mean curvature of \( \partial K \) is positive for \( g' \).

**Proof.** We use Theorem [B.1] to find a constant \( \epsilon_1 > 0 \) and a family \( \{\Sigma_t\}_{t \in (-\epsilon_1, \epsilon_1)} \) of disjoint closed surfaces in \( B(\partial K, \epsilon) \) satisfying that

- each \( \Sigma_t \) is is isotopic to \( \partial K \) in \( B(\partial K, \epsilon) \);
- the disjoint union \( \bigcup \Sigma_t \) is an open neighborhood of \( \Sigma_0 \);
- \( \langle \frac{\partial \Sigma_t}{\partial t}, n_{\Sigma_0} \rangle(x) > 0 \) for each \( x \in \Sigma_0 \);
- for each \( x \in L \cap \Sigma_t \), then \( \langle \frac{\partial \Sigma_t}{\partial t}, n_L \rangle(x) = 0 \).

Remark that for \( x \in \Sigma_t \), \( T_xM \) is spanned by \( T_x\Sigma_t \) and \( \frac{\partial \Sigma_t}{\partial t}(x) \).

In the following, we replace \( \partial K \) by \( \Sigma_0 \) and explain the construction of \( g' \).

Choose a smooth function \( h : \mathbb{R} \to \mathbb{R}_+ \) satisfying that (1) \( h(t) \geq 1 \); (2) \( h(t) = 1 \) for \( t \in \mathbb{R} \setminus (-\epsilon_1/2, \epsilon_1/2) \). Define
\[
\hat{h}(x) = \begin{cases} 
    h(t), & x \in \Sigma_t; \\
    1, & \text{otherwise}.
\end{cases}
\]
It is smooth and \( \hat{h} = 1 \) on \( M \setminus B(\partial K, \epsilon_0) \). In addition, we have that
\[
\nabla \hat{h}(x) = \begin{cases} 
    h'(t) \cdot \left| \frac{\partial \Sigma}{\partial t} \right|^{-2} \cdot \frac{\partial \Sigma}{\partial t}(x), & x \in \Sigma_t; \\
    0, & \text{otherwise}.
\end{cases}
\]
where \(|\frac{\partial \Sigma_t}{\partial t}|^2(x)| \) is the squared length of the vector \(\frac{\partial \Sigma_t}{\partial t} \) at \(x\). The properties of \(\{\Sigma_t\} \) show that

\[
\langle \nabla \hat{h}, \hat{\nu} \rangle(x) = \frac{h'(0)}{h(0)} \cdot \left| \frac{\partial \Sigma_t}{\partial t} \right|_{t=0}^{-2} \left| \frac{\partial \Sigma_t}{\partial t} \right|_{t=0} \cdot \hat{\nu}(x), \quad \text{for} \quad x \in \partial K
\]

\[
\langle \nabla \hat{h}, \nu_L \rangle(x) = 0 \quad \text{for} \quad x \in L \cap \Sigma_t.
\]

Consider the new metric \(g' = \hat{h}^2g\). Under this metric, the mean curvature of \(\partial K\) is

\[
\tilde{H}\partial K(x) = \frac{1}{h(0)} \left( H\partial K + \frac{2h'(0)}{h(0)} \cdot \left| \frac{\partial \Sigma_t}{\partial t} \right|_{t=0}^{-2} \left| \frac{\partial \Sigma_t}{\partial t} \right|_{t=0} \cdot \hat{\nu}(x) \right)
\]

Since \(\left| \frac{\partial \Sigma_t}{\partial t} \right|_{t=0}^{-2} \left| \frac{\partial \Sigma_t}{\partial t} \right|_{t=0} \cdot \hat{\nu}(x) \) is a positive smooth function on \(\partial K\), it is bounded below by a positive constant \(C_1\) (i.e. \(\left| \frac{\partial \Sigma_t}{\partial t} \right|_{t=0}^{-2} \left| \frac{\partial \Sigma_t}{\partial t} \right|_{t=0} \cdot \hat{\nu}(x) \geq C_1\)). We choose \(h(0) = 2\) and \(h'(0) > 2/C_1 \max_{x \in \partial K} |H\partial K(x)| + 1\). Hence, the mean curvature \(\tilde{H}\partial K\) is positive.

For the new metric \(g'\), the mean curvature of \(L\)

\[
\tilde{H}_L(x) = \frac{1}{h(t)} \left( H_L(x) + \frac{2}{h(t)} \langle \hat{\nu}_L, \nabla \hat{h} \rangle(x) \right) = 0.
\]

The last equality comes from the fact that \(\langle \nabla \hat{h}, \hat{\nu}_L \rangle(x) = 0\) and \(H_L = 0\) for \(x \in L\). The plane \(L\) is still minimal for the new metric \(g'\). \(\square\)

References

[And85] Michael Anderson. Curvature estimates for minimal surfaces in 3-manifolds. In *Annales scientifiques de l’École Normale Supérieure*, volume 18, pages 89–105. Elsevier, 1985.

[BBB+10] Laurent Bessières, Gérard Besson, Michel Boileau, Sylvain Maillot, and Joan Porti. *Geometrisation of 3-manifolds*, volume 13. European Mathematical Society, 2010.

[BBM11] Laurent Bessières, Gérard Besson, and Sylvain Maillot. Ricci flow on open 3-manifolds and positive scalar curvature. *Geom. Topol.*, 15(2):927–975, 2011.

[CM04] Tobias Colding and William Minicozzi. The space of embedded minimal surfaces of fixed genus in a 3-manifold. iv: Locally simply connected. *Ann. Math. (2)*, 160(2):573–615, 2004.

[CM11] Tobias Colding and William Minicozzi. *A course in minimal surfaces*, volume 121. American Mathematical Soc, 2011.

[CWY10] Stanley Chang, Shmuel Weinberger, and Guoliang Yu. Taming 3-manifolds using scalar curvature. *Geometriae Dedicata*, 148(1):3–14, 2010.

[GL83] Mikhail Gromov and Blaine Lawson. Positive scalar curvature and the dirac operator on complete riemannian manifolds. *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, 58(1):83–196, 1983.

[Gul73] Robert Gulliver. Regularity of minimizing surfaces of prescribed mean curvature. *Annals of Mathematics*, pages 275–305, 1973.

[Hat05] Allen Hatcher. *Algebraic topology*. Tsinghua University Press, 2005.

[Hem04] John Hempel. *3-Manifolds*, volume 349. American Mathematical Soc., 2004.

[HP71] LS Husch and TM Price. Addendum to: Finding a boundary for a 3-manifold. *Annals of Mathematics*, pages 486–488, 1971.

[Kaz82] Jerry Kazdan. Deformation to positive scalar curvature on complete manifolds. *Mathematische Annalen*, 261(2):227–234, 1982.
[McM61] DR McMillan. Cartesian products of contractible open manifolds. *Bulletin of the American Mathematical Society*, 67(5):510–514, 1961.

[Mor09] Charles Morrey. *Multiple integrals in the calculus of variations*. Springer Science & Business Media, 2009.

[MRR02] William Meeks, Antonio Ros, and Harold Rosenberg. *The global theory of minimal surfaces in flat spaces*. Springer, 2002.

[MT07] John Morgan and Gang Tian. *Ricci flow and the Poincaré conjecture*, volume 3. American Mathematical Soc., 2007.

[MY80] William Meeks and Shing-Tung Yau. Topology of three dimensional manifolds and the embedding problems in minimal surface theory. *Annals of Mathematics*, 112(3):441–484, 1980.

[MY82] William H Meeks and Shing-Tung Yau. The classical plateau problem and the topology of three-dimensional manifolds: The embedding of the solution given by douglas-morrey and an analytic proof of dehn’s lemma. *Topology*, 21(4):409–442, 1982.

[Oss70] Robert Osserman. A proof of the regularity everywhere of the classical solution to plateau’s problem. *Annals of Mathematics*, pages 550–569, 1970.

[Per02a] Grisha Perelman. The entropy formula for ricci flow and its geometric applications. *arXiv preprint math/0211159*, 2002.

[Per02b] Grisha Perelman. Ricci flow with surgery on three-manifolds. *arXiv preprint math/0303109*, 2002.

[Per03] Grisha Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds. *arXiv preprint math/0307245*, 2003.

[Rol03] Dale Rolfsen. *Knots and links*, volume 346. American Mathematical Soc., 2003.

[Rot12] Joseph Rotman. *An introduction to the theory of groups*, volume 148. Springer Science & Business Media, 2012.

[Sch83] Richard Schoen. Estimates for stable minimal surfaces in three dimensional manifolds. In *Seminar on minimal submanifolds*, volume 103, pages 111–126. Princeton University Press Princeton, NJ, 1983.

[Sta72] John Stallings. *Group theory and three-dimensional manifolds*. Yale University Press, 1972.

[SY79a] Richard Schoen and Shing-Tung Yau. Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature. *Annals of Mathematics*, 110(1), 1979.

[SY79b] Richard Schoen and Shing-Tung Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta mathematica*, 28(1-3):159–183, 1979.

[SY82] Richard Schoen and Shing Tung Yau. Complete three-dimensional manifolds with positive ricci curvature and scalar curvature. In *Seminar on Differential Geometry*, volume 102, pages 209–228. Princeton Univ. Press Princeton, NJ, 1982.

[Wan19a] Jian Wang. Contractible 3-manifolds and positive scalar curvatures (I). *arXiv:1901.04605*, 2019.

[Wan19b] Jian Wang. Contractible 3-manifolds and positive scalar curvatures (II). *arXiv:1906.04128*, 2019.

[Whi35] J.H.C. Whitehead. A certain open manifold whose group is unity. *The Quarterly Journal of Mathematics*, 6(1):268–279, 1935.

Universität Augsburg, Institut für Mathematik, Lehrstuhl für Differentialgeometrie, Universitätsstr. 14, 86159 Augsburg, Germany