Extending partial isometries of antipodal graphs

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Abstract

We prove EPPA (extension property for partial automorphisms) for all antipodal classes from Cherlin’s list of metrically homogeneous graphs, thereby answering a question of Aranda et al. This paper should be seen as the first application of a new general method for proving EPPA which can bypass the lack of automorphism-preserving completions. It is done by combining the recent strengthening of the Herwig–Lascar theorem by Hubička, Nešetřil and the author with the ideas of the proof of EPPA for two-graphs by Evans et al.

Keywords: EPPA, Hrushovski property, metrically homogeneous graph, antipodal space

1. Introduction

Let $G = (V, E)$ be a (not necessarily finite) graph and let $X, Y$ be subsets of $V$. We say that a function $f: X \to Y$ is a partial automorphism of $G$ if $f$ is an isomorphism of $G[X]$ and $G[Y]$, the graphs induced by $G$ on $X$ and $Y$ respectively. This notion naturally extends to arbitrary structures (see Section 2).

In 1992 Hrushovski [18] proved that for every finite graph $G$ there is a finite graph $H$ such that $G$ is an induced subgraph of $H$ and every partial automorphism of $G$ extends to an automorphism of $H$. This property is, in general, called the extension property for partial automorphisms (EPPA):

Definition 1.1. Let $C$ be a class of finite structures. We say that $C$ has the extension property for partial automorphisms (or EPPA), also called the Hrushovski property, if for every $A \in C$ there is $B \in C$ such that $A$ is an (induced) substructure of $B$ and for every isomorphism $f$ of substructures of $A$ there is an automorphism $g$ of $B$ such that $f \subseteq g$. We call such $B$ an EPPA-witness for $A$.

Hrushovski’s proof was group-theoretical, Herwig and Lascar [15] later gave a simple combinatorial proof by embedding $G$ into the complement of a Kneser graph. After this, the quest of identifying new classes of structures with EPPA continued with a series of papers including [4, 8, 9, 13, 14, 15, 17, 20, 21, 22, 25, 30, 33, 34].

Let $G = (V, E)$ be a graph. We say that a (partial) map $f: V \to V$ is distance-preserving if whenever $u, v$ are in the domain of $f$, the distance between $u$ and $v$ is the same as the distance between $f(u)$ and $f(v)$. Clearly, every automorphism is distance-preserving. In 2005, Solecki [33] (and independently also Vershik [34]) proved that the class of all finite graphs has a variant of EPPA for distance-preserving maps. Namely, they proved that for every finite graph $G$ there is a finite graph $H$ satisfying the following:

1. $G$ is an induced subgraph of $H$,
2. whenever $u, v$ are vertices of $G$, then the distance between $u$ and $v$ in $G$ is the same as in $H$, and
3. every partial distance-preserving map of $G$ extends to an automorphism of $H$. 
It is not very convenient to work with distance-preserving maps, because they are relative to a graph and thus a distance-preserving map on a subgraph need not be distance-preserving with respect to a supergraph and vice versa. Given a graph $G = (V, E)$, it is more natural to consider the metric space $M = (V, d)$ where $d(u, v)$ is the number of edges of the shortest path from $u$ to $v$ in $G$ (we will call this the path-metric space of $G$). And this is in fact what Solecki and Vershik did — they proved EPPA for all (integer-valued) metric spaces, which is equivalent to EPPA for graphs with distance-preserving maps.

Vershik’s proof is unpublished, Solecki’s proof uses a complicated general theorem of Herwig and Lascar [15, Theorem 3.2] about EPPA for structures with forbidden homomorphisms. Hubiˇ cka, Neˇ setˇ ril and the author [20] recently gave a simple self-contained proof of Solecki’s result. There is also a group theoretical proof by Sabok [31] using a construction à la Mackey [27].

This paper continues in this direction. Generalising the concept of distance transitivity, we say that a (countable) connected graph $G$ is metrically homogeneous if every partial distance-preserving map of $G$ with finite domain extends to an automorphism of $G$ (so it is, in a sense, an EPPA-witness for itself). Cherlin [6] gave a list of countable metrically homogeneous graphs (which is conjectured to be complete and is provably complete in some cases [1, 7]) in terms of classes of finite metric spaces which embed into the path-metric space of the given metrically homogeneous graph. EPPA and other combinatorial properties of classes from Cherlin’s list were studied by Aranda, Bradley-Williams, Hng, Hubiˇ cka, Karamanlis, Kompatscher, Pawliuk and the author [2, 3, 4] (see also [24]) and in [4] almost all the questions were settled, only EPPA for antipodal classes of odd diameter and bipartite antipodal classes of even diameter (see Section 2.3) remained open. An important step was later done by Evans, Hubiˇ cka, Neˇ setˇ ril and the author [9] who proved EPPA for antipodal metric spaces of diameter 3.

In this paper we combine the results of [4] with the ideas from [9] and the new strengthening of the Herwig–Lascar theorem by Hubiˇ cka, Neˇ setˇ ril and the author [22] (stated here in a weaker form as Theorem 2.5) and prove the following theorem, thereby answering a question of Aranda et al. (Problem 1.3 in [4]) and completing the study of EPPA for classes from Cherlin’s list.

**Theorem 1.2.** Every class of antipodal metric spaces from Cherlin’s list has EPPA.

### 2. Preliminaries

A (not necessarily finite) structure $A$ is homogeneous if every partial automorphism of $A$ with finite domain extends to a full automorphism of $A$ itself (so it is, in a sense, an EPPA-witness for itself). Gardiner proved [11] that the finite homogeneous graphs are precisely disjoint unions of cliques of the same size, their complements, the 5-cycle and the line graph of $K_3$. Lachlan and Woodrow later [26] classified the countably infinite homogeneous graphs. These are disjoint unions of cliques of the same size (possibly infinite), their complements, the Rado graph, the $K_n$-free variants of the Rado graph and their complements.

Every homogeneous structure can be associated with the class of all (isomorphism types of) its finite substructures, which is called its *age*. By the Fraïssé theorem [10], one can reconstruct the homogeneous structure back from this class (because it has the so-called amalgamation property). For more on homogeneous structures see the survey by Macpherson [28].

A graph $G$ is vertex transitive if for every pair of vertices $u, v$ there is an automorphism sending $u$ to $v$, it is edge transitive if every edge can be sent to every other edge by an automorphism and it is distance transitive if for every two pairs of vertices $u, v$ and $x, y$ such that the distance between $u$ and $v$ is the same as the distance between $x$ and $y$ there is an automorphism sending $u$ to $x$ and $v$ to $y$.

Distance transitivity is a very strong condition. For example, there are only finitely many finite 3-regular distance transitive graphs [5] and the full catalogue is available in some other particular cases. However, for larger degrees, the classification is unknown, see e.g. the book by Godsil and Royle [12], largely devoted to the study of distance transitive graphs.

Recall that a connected graph $G$ is metrically homogeneous if every partial distance-preserving map of $G$ with finite domain extends to an automorphism of $G$. This is equivalent to saying...
that the path-metric space of $G$ is homogeneous in the sense of the previous paragraphs. All
closed homogeneous graphs are also metrically homogeneous, because every pair of vertices is
either connected by an edge or by a path of length 2. Finite cycles of size at least 6 are examples
of metrically homogeneous graphs which are not homogeneous.

Remark 2.1. If one checks the known classes with EPPA, they will find out that they all are ages
of homogeneous structures. This is not a coincidence. It is easy to see that if a class of finite
structures $C$ has EPPA and the joint embedding property (for every $A, B \in C$ there is $C \in C$
which contains a copy of both of them), then $C$ is the age of a homogeneous structure provided
that it contains at most countably many members up to isomorphism. This restricts the candidate
classes for EPPA severely and connects finite combinatorics with the study of infinite homogeneous
structures and infinite permutation groups.

In the other direction, EPPA has some implications for the automorphism group (with the
pointwise convergence topology) of the corresponding homogeneous structure, see for example the
paper of Hodges, Hodkinson, Lascar, and Shelah [16].

2.1. $\Gamma_L$-structures

An important feature of the strengthening of the Herwig–Lascar theorem by Hubička, Nešetřil
and the author (Theorem 2.5) is that it allows to also permute the language. Namely, we will work
with categories whose objects are the standard model-theoretic structures (in a given language),
but the arrows are potentially richer, allowing a permutation of the language. The reader is invited
to verify that in the following paragraphs, if the group $\Gamma_L$ consists of the identity, one obtains the
usual notion of model-theoretic $L$-structures with the corresponding maps.

The following notions are taken from [22], sometimes stated in a more special form which is
sufficient for our purposes. Many of them were introduced by Hubička and Nešetřil [23] in the
context of structural Ramsey theory (e.g. homomorphism-embeddings or completions).

Let $L = L_R \cup L_F$ be a language with relational symbols $R \in L_R$, each having associated arities
denoted by $a(R)$ and function symbols $F \in L_F$. All functions in this paper are unary and have
arity range. Let $\Gamma_L$ be a permutation group on $L$ such that each $\alpha \in \Gamma_L$ preserves the partition
$L = L_R \cup L_F$ (that is, maps relations to relations and functions to functions) and the arities of all
symbols. We will say that $\Gamma_L$ is a language equipped with a permutation group.

A $\Gamma_L$-structure $A$ is a structure with vertex set $A$, functions $F_A: A \to A$ for every $F \in L_F$ and
relations $R_A \subseteq A^{a_R}$ for every $R \in L_R$. We will write structures in bold and their corresponding
vertex sets in normal font. If $\Gamma_L$ is trivial, we will often talk about $L$-structures instead of $\Gamma_L$-
structures.

If the set $A$ is finite we call $A$ a finite structure. If the language $L$ contains no function symbols,
we call $L$ a relational language and say that a $\Gamma_L$-structure is a relational $\Gamma_L$-structure.

A homomorphism $f: A \to B$ is a pair $f = (f_L, f_A)$ where $f_L \in \Gamma_L$ and $f_A$ is a mapping $A \to B$
such that for every $R \in L_R$ and $F \in L_F$ we have:

\begin{enumerate}
\item $(x_1, x_2, \ldots, x_{a(R)}) \in R_A \implies (f_A(x_1), f_A(x_2), \ldots, f_A(x_{a(R)})) \in f_L(R)_B$, and
\item $f_A(F_A(x)) = f_L(F)_B(f_A(x))$.
\end{enumerate}

For brevity, we will also write $f(x)$ for $f_A(x)$ in the context where $x \in A$ and $f(S)$ for $f_L(S)$
where $S \subseteq L$. For a subset $A' \subseteq A$ we denote by $f(A')$ the set $\{f(x) : x \in A'\}$ and by $f(A)$ the
homomorphic image of a structure $A$. Note that we write $f: A \to B$ to emphasize that $f$ respects
the structure.

If $f_A$ is injective then $f$ is called a monomorphism. A monomorphism $f$ is an embedding if for
every $R \in L_R$ we have the equivalence in the definition, that is,

\[(x_1, x_2, \ldots, x_{a(R)}) \in R_A \iff (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in f(R)_B.\]

If the inclusion $A \subseteq B$ together with the identity of $\Gamma_L$ form an embedding, we say that $A$ is a
substructure of $B$ and often denote it as $A \subseteq B$. For an embedding $f: A \to B$ we say that $f(A)$
is a copy of A in B. If f is an embedding where f_A is onto, then f is an isomorphism and an isomorphism A → A is called an automorphism.

Note that from the previous paragraph it follows that when L contains functions, not every subset of vertices induces a substructure. Namely, every substructure needs to be closed on functions. For example, if L consists of one unary function F, Γ_L contains only the identity and B is a Γ_L-structure with vertex set B = \{b_1, b_2\} such that F(b_1) = b_2 and F(b_2) is not defined, then there is a substructure of B on the set \{b_2\}, but the smallest substructure of B containing b_1 is B itself. Generalising this example, we say that for a Γ_L-structure B and a set A which is a subset of B, the closure of A in B, denoted by Cl_B(A), is the smallest substructure of B containing A. For x \in B, we will also write Cl_B(x) for Cl_B(\{x\}).

Generalising the notion of a graph clique, we say that a Γ_L-structure A is irreducible if for every pair of distinct vertices x, y \in A there is a relation R \in L and a tuple \bar{r} \in A^{(R)} containing both x and y such that \bar{r} \in R_A. Note that the definition of irreducibility from [22] is more general than this one (making more structures irreducible in general languages with functions), but stating it would need some more preliminary definitions and moreover they are equivalent for structures which we will consider in this paper.

Example 1. If the language only contains unary relations, irreflexive symmetric binary relations and unary functions (which will always be true in this paper), a structure is irreducible if and only if the union of the binary relations is a complete graph.

A homomorphism f : A → B is a homomorphism-embedding if the restriction f|_C is an embedding whenever C is an irreducible substructure of A.

2.2. EPPA for Γ_L-structures

We next state the main result of [22] for which we need the following definitions, which are mostly variants of the definitions needed for the Hubička–Nešetřil theorem [23].

A partial automorphism of a Γ_L-structure A is an isomorphism f : C → C’ where C and C’ are substructures of A (remember that it also includes a permutation of the language which is not partial). We say that a class C of finite Γ_L-structures has the extension property for partial automorphisms (EPPA) if for every A ∈ C there is B ∈ C such that A is a substructure of B and every partial automorphism of A extends to an automorphism of B. We call B with such a property an EPPA-witness for A. If B is an EPPA-witness for A, we say that it is irreducible-structure faithful if for every irreducible substructure C of B there exists an automorphism g of B such that g(C) ⊆ A. We say that a class C of finite Γ_L-structures has EPPA if there is an EPPA-witness B ∈ C for every A ∈ C. We say that C has irreducible-structure faithful EPPA if the witness can always be chosen to be irreducible-structure faithful.

Example 2.

1. Let A be the graph on vertices u, v, w containing a single edge uw (here, the language consists of one binary relation and the permutation group is trivial). Then a possible (irreducible-structure faithful) EPPA-witness for A is the graph B on vertices u, v, w, x with edges uv and wx.

2. To see an example of a non-trivial permutation group, let L be the language consisting of unary relations R^i, where 1 ≤ i ≤ 10 and let Γ_L consist of all permutation of L which fix R^{10}. Let A be the Γ_L structure on one vertex v such that R^1_A = \{v\} and R^i_A = \emptyset for every i ≥ 2. Then every EPPA-witness B for A must contain vertices v_2, . . . , v_9 such that v_i ∈ R^i_B for every 2 ≤ i ≤ 9, because B needs to extend all partial automorphism f^i, 2 ≤ i ≤ 9, such that f^i_A is the empty function and f^i_B ∈ Γ_L sends R^i to R^i.

Definition 2.2. Let C be a Γ_L-structure. A Γ_L-structure C’ is a completion of C if there exists an injective homomorphism-embedding f : C → C' which fixes every symbol of the language. We say that C' is an automorphism-preserving completion of C, if C ⊆ C', the inclusion together with the identity from Γ_L give a homomorphism-embedding, for every α ∈ Aut(C) there is β ∈ Aut(C') such that α ⊆ β and moreover the map α → β is a group homomorphism Aut(C) → Aut(C').
In this paper, the languages will contain only unary and binary relations and unary functions, and moreover, whenever $C'$ will be a completion of $C$, it will always hold that $C' = C$ and that the identity is a homomorphism-embedding. In such a case, for every relation $R \in L$ we have $R_C \subseteq R_{C'}$ with equality for unary $R$. For binary $R$ it holds that if $(u,v) \in R_{C'} \setminus R_C$, then for every binary $R^0 \in L$ we have $(u,v) \notin R^0_C$. Furthermore, $C'$ is an automorphism-preserving completion of $C$ if and only if $\text{Aut}(C') = \text{Aut}(C)$.

**Example 3.** Consider the class $C_N$ of all finite integer-valued metric spaces understood as structures in a binary symmetric relational language $L$ with a relation for every nonzero distance (the fact that $d(x,x) = 0$ is implicit). In Figure 1 we see the following:

(a) An $L$-structure which has an automorphism-preserving completion in $C$,
(b) one such completion, and
(c) an $L$-structure which has no completion in $C$.

**Definition 2.3.** Let $L$ be a finite language with relations and unary functions equipped with a permutation group $\Gamma_L$. Let $\mathcal{E}$ be a class of finite $\Gamma_L$-structures and let $\mathcal{K}$ be a subclass of $\mathcal{E}$ consisting of irreducible structures. We say that $\mathcal{K}$ is a *locally finite subclass* of $\mathcal{E}$ if for every $A \in \mathcal{K}$ and every $B_0 \in \mathcal{E}$ there is a finite integer $n = n(A,B_0)$ such that every $\Gamma_L$-structure $B$ has a completion $B' \in \mathcal{K}$ provided that it satisfies the following:

1. For every vertex $v \in B$ we have that $\text{Cl}_{B}(v)$ lies in a copy of $A$,
2. there is a homomorphism-embedding from $B$ to $B_0$, and
3. every substructure of $B$ with at most $n$ vertices has a completion in $\mathcal{K}$.

We say that $\mathcal{K}$ is a *locally finite automorphism-preserving subclass* of $\mathcal{E}$ if in the condition above, the completion of $B$ can always be chosen to be automorphism-preserving.

**Remark 2.4.** While in Definition 2.3 we promise that $\text{Cl}_B(v)$ lies in a copy of $A$, the definition of local finiteness for the Hubička–Nešetřil theorem (Definition 2.4 from [23], similarly also the definition of local finiteness from [22]) promises that every irreducible substructure of $B$ comes from $\mathcal{K}$. The difference here is due to the fact that the definition of irreducibility is simplified in this paper and does not work well for general languages with functions. In the applications, both conditions are used to ensure that closures behave well in $B$.

**Example 4.** Let us observe that the class $C_N$ from Example 3 is a locally finite automorphism-preserving subclass of the class $\mathcal{E}$ consisting of all finite $L$-structures (for $L$ from Example 3), where all relations are symmetric and irreflexive and every pair of vertices is in at most one relation. Fix $A \in \mathcal{K}$ and $B_0 \in \mathcal{E}$. The assumption on $\mathcal{E}$ justifies defining a symmetric partial function $d_{B_0} : B_0^2 \to \mathbb{N}$ where $d(u,u) = 0$ and $d(u,v) = \ell$ if and only if $u$ and $v$ are in the relation corresponding to $\ell$ in $B_0$. Let $S$ be the set of all integers $\ell$ for which there are vertices $u,v \in B_0$ such that $d(u,v) = \ell$. Since $B_0$ is finite, $S$ is also finite. Put $n = \max_{u,v \in S} \lceil \frac{a}{b} \rceil$. 

![Figure 1: Completions](image)
Let \( B \) be an \( L \)-structure satisfying the conditions of Definition 2.3 (since \( L \) contains no functions, condition 1 is satisfied trivially). The existence of a homomorphism-embedding from \( B \) to \( B_0 \) implies that all the relations in \( B \) are also symmetric and irreflexive and every pair of vertices of \( B \) is in at most one relation, hence we can analogously define a partial function \( d_B : B^2 \to \mathbb{N} \). Moreover, since there is a homomorphism-embedding from \( B \) to \( B_0 \), we also get that the only non-empty distance relations in \( B \) are those representing distances from \( S \).

Next we define function \( d' : B^2 \to \mathbb{N} \) by

\[
d'(x, y) = \min_{P \text{ is a path } x \to y \text{ in } B} \|P\|,
\]

where by a path \( x \to y \) we mean a sequence of distinct vertices \( x = p_1, \ldots, p_k = y \) such that \( d_B(p_i, p_{i+1}) \) is defined for every \( i \) satisfying \( 1 \leq i < k \), and we define \( \|P\| = \sum_{i=1}^{k-1} d_B(p_i, p_{i+1}) \). It is easy to verify that \( (B, d') \) is a metric space with distances from \( \mathbb{N} \) and that \( d_B \subseteq d' \) if and only if \( B \) contains no non-metric cycles, that is, sequences of vertices \( v_1, \ldots, v_k \) such that \( d_B(v_i, v_{i+1}) \) is defined for every \( 1 \leq i \leq k \) (we identify \( v_{k+1} = v_1 \)) and \( d_B(v_1, v_k) > \sum_{i=1}^{k-1} d_B(v_i, v_{i+1}) \).

Since, clearly, non-metric cycles do not have a completion in \( C_N \), it follows from the definition of \( n \) that \( B \) contains no non-metric cycles and hence has a completion \( B' = (B, d') \) in \( C_N \) as requested. Moreover, from the canonicity of the definition of \( d' \) it follows that this completion is automorphism-preserving and hence we have proved that \( C_N \) is an automorphism-preserving completion of \( B \).

This construction of \( B' \) is called the shortest-path completion in [23] and was already used by Solecki [33] to prove EPPA for the class of all finite metric spaces and by Nešetřil [29] to find a Ramsey expansion of the class of all finite metric spaces.

The main theorem of [22] can be stated as follows.

**Theorem 2.5 ([22]).** Let \( L \) be a finite language with relations and unary functions equipped with a permutation group \( \Gamma_L \), let \( \mathcal{E} \) be a class of finite \( \Gamma_L \)-structures which has irreducible-structure faithful EPPA and let \( \mathcal{K} \) be a hereditary locally finite automorphism-preserving subclass of \( \mathcal{E} \) with the strong amalgamation property, which consists of irreducible structures. Then \( \mathcal{K} \) has EPPA.

Here \( \mathcal{K} \) is hereditary if whenever \( B \in \mathcal{K} \) and \( A \subseteq B \), then also \( A \in \mathcal{K} \). We will not define what the strong amalgamation property is (see [22]), but all classes for which we will use Theorem 2.5 will have this property.

Since Theorem 2.5 has a form of implication, we will need the following theorem from [22] to supply us with the base EPPA class \( \mathcal{E} \).

**Theorem 2.6 ([22]).** Let \( L \) be a finite language with relations and unary functions equipped with a permutation group \( \Gamma_L \). Then the class of all finite \( \Gamma_L \)-structures has irreducible-structure faithful EPPA.

Note that combining Example 4 with Theorems 2.5 and 2.6 gives a proof of Solecki's result that finite metric spaces have EPPA (to prove that \( \mathcal{E} \) has EPPA, one has to use Theorem 2.6 for a finite fragment of \( L \) to get an EPPA-witness for a given \( A \in \mathcal{E} \)). Both Theorems 2.5 and 2.6 are proved by an application of the method of valuation functions, a variant of which we will also use in this paper. More precisely, Theorem 2.6 is proved by giving an explicit construction of EPPA-witnesses. Theorem 2.5 iteratively applies the method of valuation functions to produce, given \( A \in \mathcal{K} \) and its irreducible-structure faithful EPPA-witness \( B_0 \in \mathcal{E} \), an EPPA-witness \( B \) satisfying the conditions of Definition 2.3, the automorphism-preserving completion \( B' \) of \( B \) is then the desired EPPA-witness for \( A \) in \( \mathcal{K} \).

### 2.3. Metrically homogeneous graphs

Most of the details of Cherlin's metric spaces are not important for this paper. We only give the necessary definitions and facts and refer the reader to [6], [4] or [24].

All the metric spaces we will work with have distances from \( \{0, 1, \ldots, \delta\} \) for some integer \( \delta \). Therefore, we will view them interchangeably as pairs \((A, d)\) where \( d \) is the metric, as complete
graphs with edges labelled by \(\{1, \ldots, \delta\}\) (we will call these complete \([\delta]\)-edge-labelled graphs) where the labels of every triangle satisfy the triangle inequality, and as relational structures with trivial \(\Gamma_e\) and binary symmetric irreflexive relations \(R^1, \ldots, R^\delta\) (distance 0 is not represented) such that every pair of vertices is in exactly one relation and the triangle inequality is satisfied. The middle point of view works best with the notion of completions: Given a (not necessarily complete) \([\delta]\)-edge-labelled graph \(G\), a \([\delta]\)-edge-labelled graph \(G'\) is a completion of \(G\) if \(G\) is a non-induced subgraph of \(G'\) and the labels are preserved.

We will say that two vertices are at distance \(a\) and that they are connected by an edge of length \(a\) interchangeably. In particular, when we talk about an edge of a \([\delta]\)-edge-labelled graph, we mean a pair of vertices such that their distance is defined, it does not necessarily mean that they are at distance 1.

A major part of Cherlin’s list of the classes of finite metric spaces which embed into the path-metric of a countably infinite metrically homogeneous graph consists of certain 5-parameter classes \(\mathcal{A}_{K_1, K_2, C_0, C_1}\). These are classes of metric spaces with distances \(\{0, 1, \ldots, \delta\}\) (we call \(\delta\) the diameter of such spaces) omitting certain families of triangles (e.g. triangles of short odd perimeter or triangles of long even perimeter).

A special case of these classes are the antipodal classes, where the five parameters have only two degrees of freedom. Here we will denote the antipodal classes as \(\mathcal{A}_{K}^\delta\), where \(1 \leq K \leq \frac{\delta}{2}\), or \(K = \delta\). \(\mathcal{A}_{K}^\delta\) is defined as the class of all finite metric spaces with distances \(\{0, 1, \ldots, \delta\}\) such that they contain no triangle with distances \(a, b, c\) for which at least one of the following holds:

1. \(a + b + c > 2\delta\),
2. \(a + b + c\) is odd and \(a + b + c < 2K\), or
3. \(a + b + c\) is odd and \(a + b + c > 2(\delta - K) + 2\min(a, b, c)\).

However, for our purposes, we need only the following fact:

**Fact 2.7 (Antipodal spaces).** The following holds in every class \(\mathcal{A}_{K}^\delta\) of antipodal metric spaces from Cherlin’s list:

1. The edges of length \(\delta\) form a matching (that is, for every vertex there is at most one vertex at distance \(\delta\) from it) and for every \(A \in \mathcal{A}_{K}^\delta\) there is a unique \(B \in \mathcal{A}_{K}^\delta\) such that \(A \subseteq B\), the edges of length \(\delta\) form a perfect matching in \(B\) and every edge of length \(\delta\) in \(B\) has at least one endpoint from \(A\).
2. For every pair of vertices \(u, v\) such that \(d(u, v) = \delta\) and for every vertex \(w\) we have \(d(u, w) + d(v, w) = \delta\).
3. If one selects exactly one vertex from each edge of length \(\delta\), the metric space they induce belongs to a special (non-antipodal) class of diameter \(\delta - 1\) which we will call \(\mathcal{B}_{K}^\delta\). And the other way around, one can get an antipodal metric space from every metric space \(M \in \mathcal{B}_{K}^\delta\) by taking two disjoint copies of \(M\), connecting every vertex to its copy by an edge of length \(\delta\) and using point 2 to fill-in the missing distances.

There are two kinds of antipodal classes with different combinatorial behaviour — those that come from a countable bipartite metrically homogeneous graph (they correspond to the case \(K = \delta\)) and those that come from a non-bipartite one. We will call the first the bipartite classes (their members have the property that they contain no triangles, or more generally cycles, of odd perimeter) and we will call the others the non-bipartite ones. This is slightly misleading, because some of the finite metric subspaces of the path-metric of a non-bipartite metrically homogeneous graph are surely bipartite, but it should not cause any confusion in this paper. The non-bipartite

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1. If \(K \neq \delta\), the other parameters are then defined as \(K_1 = K, K_2 = \delta - K, C_0 = 2\delta + 2\) and \(C_1 = 2\delta + 1\), if \(K = \delta\), then \(K_1 = \infty\) and the other parameters are as before.

2. It is in fact \(\mathcal{A}_{K_1, K_2, C_0, C_1}\) for \(K_1 = K, K_2 = \delta - K, C_0 = 2\delta + 2\) and \(C_1 = 2\delta + 1\).
class of antipodal metric spaces of diameter 3 is closely connected to switching classes of graphs and to two-graphs (see [9]).

The following fact summarizes results from [4] about the non-bipartite odd diameter antipodal classes.

**Fact 2.8.** Let $\mathcal{A}^3_K$ be a non-bipartite class of antipodal metric spaces of odd diameter $\delta$. Let $A$ be a $[\delta]$-edge-labelled graph such that the edges of length $\delta$ of $A$ form a perfect matching and furthermore for every $u,v,w \in A$ such that $d_A(u,v) = \delta$ and $w \neq u,v$, either $w$ is not connected by an edge to either of $u,v$, or $d_A(u,w) + d_A(v,w) = \delta$. Suppose furthermore that $A$ contains none of the finitely many cycles forbidden in $B^3_K$.

Let $f : (\frac{\delta}{2}) \to \{0,1\}$ be a mapping satisfying the following.

1. Whenever $uv$ is an edge of $A$, then $f(uv) \equiv d_A(u,v) \mod 2$.
2. Let $u_1v_1$ and $u_2v_2$ be two different edges of length $\delta$ of $A$. Then $f(u_1v_2) = f(v_1u_2)$, and $f(u_1u_2) \neq f(u_1v_2)$.

Then there is $\bar{A} \in \mathcal{A}^3_K$ such that the following holds.

1. $\bar{A}$ is a completion of $A$ with the same vertex set,
2. for every edge $uv$ of $\bar{A}$ it holds that $f(uv) \equiv d_{\bar{A}}(u,v) \mod 2$, and
3. Every automorphism of $A$ which preserves values of $f$ is also an automorphism of $\bar{A}$.

Such $\bar{A}$ can be constructed by picking one vertex from each edge of length $\delta$, considering this auxiliary metric space of diameter $\delta - 1$, completing it using Theorem 4.9 from [4] (see also Lemma 4.18 from the same paper, or [19]) and then pulling this completion back using $f$ to decide parities of the edges. The proof then uses the observation that the completion procedure for $\mathcal{A}^3_{K_1,K_2,C_0,C_1}$ from [4] preserves the equivalence “$a \sim \delta - a$”. That is, we say that two $[\delta - 1]$-edge-labelled graphs $G$ and $G'$ are equivalent if they share the same vertex set and the same edge set and every edge has either the same label in both $G$ and $G'$, or it has label $a$ in $G$ and $\delta - a$ in $G'$.

The completion procedure then produces equivalent graphs whenever given equivalent graphs.

3. The odd diameter non-bipartite case

EPPA for the even diameter non-bipartite case was proved in [4]. In this section we prove the following proposition.

**Proposition 3.1.** Let $\mathcal{A}_K^3$ be a non-bipartite class of antipodal metric spaces of odd diameter. Then for every $A \in \mathcal{A}_K^3$ there is $B \in \mathcal{A}_K^3$ which is an EPPA-witness for $A$.

Proposition 3.1 extends the results of [9] where it was proved for diameter 3.

3.1. Motivation

We first give some motivation and intuition behind Proposition 3.1, as its proof is a bit technical. Consider the class $\mathcal{A}_K^3$. It consists of all finite complete $[3]$-edge-labelled graphs which omit triangles with distances $(1,1,3)$, $(2,2,3)$ and $(3,3,a)$, where $1 \leq a \leq 3$. In other words, the edges of length 3 form a matching (and by Fact 2.7 we can assume that it is a perfect matching), and if $u,v,w,x$ are pairwise distinct vertices such that $d(u,v) = d(w,x) = 3$, then they form an antipodal quadruple, which means that $d(u,w) = d(v,x), d(u,x) = d(v,w)$, and either $d(u,w) = 1$ and $d(u,x) = 2$, or $d(u,w) = 2$ and $d(u,x) = 1$ (see Figure 2).

Suppose that we want to find an EPPA-witness for a single edge of length 3 using Theorem 2.5. To do it, we in particular need to show that $\mathcal{A}_K^3$ is a locally finite automorphism-preserving subclass of the class $E$ of all $[3]$-edge-labelled graphs, which has irreducible-structure faithful EPPA by Theorem 2.6.

However, this does not hold. Take the disjoint union of two edges of length 3. This clearly has a completion in $\mathcal{A}_K^3$ (the antipodal quadruple), but it has no automorphism-preserving completion, because one has to pick which edges have length 1 and which edges have length 2.
Enumerate the edges of loss of generality assume that for every vertex \( d \) and \( T \) forbidden in perfect matching. Given asked to complete in fact comes from enough to observe that the conditions of Definition 2.3 imply that every such following hold:

1. \( E \) and \( E^+ \) share the same vertices and \( R^i_E = R^i_{E^+} \) for every \( 1 \leq i \leq 3 \) (that is, \( E^+ \) and \( E \) also share the distance relations),
2. \( M_{E^+}(u) = v \) if and only if \( (u, v) \in R^3_{E^+} \) (we need this for the strong amalgamation property),
3. every vertex of \( E^+ \) is in precisely one of \( T_{E^+} \) and \( B_{E^+} \),
4. if \( u, v \in E^+ \) are connected by an edge of an odd length, then precisely one of \( \{u, v\} \) is in \( T_{E^+} \) and the other is in \( B_{E^+} \), and
5. if \( u, v \in E^+ \) are connected by an edge of length 2, then either \( \{u, v\} \subseteq T_{E^+} \), or \( \{u, v\} \subseteq B_{E^+} \).

Note that not every \( E \in \mathcal{E} \) has a suitable expansion, however, \( A \) has two of them and both preserve all partial automorphisms of \( A \) (for this, we need the transposition \((T\ B)\)).

Denote by \( \mathcal{E}^+ \) the class of all suitable expansions of structures from \( \mathcal{E} \) and similarly define \( A_{1^+}^3 \). Theorem 2.6 implies that \( \mathcal{E}^+ \) has irreducible-structure faithful EPPA.

In order to prove that \( A_{1^+}^3 \) is a locally finite automorphism-preserving subclass of \( \mathcal{E}^+ \), it is enough to observe that the conditions of Definition 2.3 imply that every such \( B \) which we are asked to complete in fact comes from \( \mathcal{E}^+ \), and if we pick \( n = 6 \), we get that it contains no triangles forbidden in \( A_1^3 \). It then suffices to define the missing distances according to the unary relations \( T \) and \( B \): If \( uv \) is not an edge of \( B \), we put \( d(u, v) = 2 \) if they are in the same unary relation and \( d(u, v) = 1 \) otherwise.

In order to prove Proposition 3.1, we now generalise the construction above for larger diameters and arbitrary \( A \in A_{1^+}^3 \). For the rest of this section, fix \( A \in A_{1^+}^3 \). Using Fact 2.7, we can without loss of generality assume that for every vertex \( v \in A \) there is a vertex \( w \in A \) such that \( d_A(v, w) = \delta \). Enumerate the edges of \( A \) of length \( \delta \) as \( e_1, \ldots, e_m \) and let \( D = \{1, 2, \ldots, m\} \) be their indices, that is, \( |D| = \frac{|A|}{2} \) (we will sometimes treat \( D \) also as the set \( \{e_1, \ldots, e_m\} \) itself using the natural bijection). We furthermore denote \( e_i = \{x_i, y_i\} \), where \( x_i \) and \( y_i \) are vertices of \( A \).

### 3.2. The expanded language

We will say that a function \( \chi : D \to \{0, 1\} \) is a valuation function. For a set \( F \subseteq D \), we denote by \( \chi^F \) the flip of \( \chi \), that is, the function \( D \to \{0, 1\} \) defined as

\[
\chi^F(i) = \begin{cases} 
1 - \chi(i) & \text{if } i \in F \\
\chi(i) & \text{otherwise},
\end{cases}
\]

![Figure 2: Two possible (isomorphic) antipodal quadruples](image-url)
and for a permutation \( \psi \) of \( D \) we denote by \( \chi_\psi \) the function satisfying \( \chi_\psi(i) = \chi(\psi^{-1}(i)) \). If \( \chi \) is a valuation function, \( \psi \) is a permutation of \( D \) and \( F \subseteq D \), then by \( \chi^F_\psi \) we will mean \((\chi^F_\psi)^{-1}\), that is, we first apply the flip and then the permutation.

Let \( L \) be the language consisting of binary symmetric irreflexive relations \( R^1, \ldots, R^\delta \) representing the distances, a unary function \( M \), and unary relations \( U_i^\chi \) for every \( 1 \leq i \leq m \) and for every valuation function \( \chi \). If \( A \) is an \( L \)-structure and \( v \) is a vertex of \( A \) such that \( v \in U_i^\chi \), we will say that \( v \) has a unary mark \( U_i^\chi \). As in Section 3.1, the function \( M \) will ensure that the edges of length \( \delta \) form a matching, the relations \( U_i^\chi \) are generalisations of the relations \( T \) and \( B \).

Let \( F \subseteq D^2 \) be such that if \((i,j) \in F\), then also \((j,i) \in F\) (\( F \) is symmetric). For every \( 1 \leq i \leq m \) let \( F_i \subseteq D \) be the set \( \{ j \in D : (i,j) \in F \} \). We denote by \( \alpha F \) the permutation of \( L \) sending \( U_i^\chi \rightarrow U_i^{\chi \psi(i)} \), which fixes \( M \) and \( R^1, \ldots, R^\delta \) pointwise. In other words, \( \alpha F \) “flips” the mutual valuations of pairs from \( F \).

For a permutation \( \psi \) of \( D \), we denote by \( \alpha \psi \) the permutation of \( L \) sending \( U_i^\chi \rightarrow U_i^{\chi \psi(i)} \), which fixes \( M \) and \( R^1, \ldots, R^\delta \) pointwise. Now we can define \( \Gamma_L \) as the group generated by

\[
\{ \alpha F : F \subseteq D^2 \text{ and } F \text{ is symmetric} \} \cup \{ \alpha \psi : \psi \text{ is a permutation of } D \}.
\]

**Lemma 3.2.** For every member \( g \in \Gamma_L \) there is a permutation \( \psi \) of \( D \) and a symmetric subset \( F \subseteq D^2 \) such that \( g = \alpha \psi \alpha F \).

**Proof.** Put

\[
S = \{ \alpha F : F \subseteq D^2 \text{ and } F \text{ is symmetric} \} \cup \{ \alpha \psi : \psi \text{ is a permutation of } D \}.
\]

We first show three claims:

**Claim 3.3.** For every \( \alpha F, \alpha F' \in S \) it holds that \( \alpha F \alpha F' = \alpha F'' \), where \( F'' \) is the symmetric difference of \( F \) and \( F' \) (that is, \((i,j) \in F'' \text{ if and only if it is in exactly one of } F \text{ and } F' \)). Consequently, \( \alpha F \alpha F = 1 \).

Follows directly from the definitions of \( \alpha F \) and \( \chi F \).

**Claim 3.4.** For every \( \alpha \psi, \alpha \psi' \in S \) it holds that \( \alpha \psi \alpha \psi' = \alpha \psi'' \), where \( \psi'' = \psi \psi' \). Consequently, \( \alpha \psi \alpha \psi^{-1} = 1 \).

Again follows directly from the definitions of \( \alpha \psi \) and \( \chi \psi \).

**Claim 3.5.** For every \( \alpha \psi, \alpha F \in S \) there is \( \alpha F' \in S \) such that \( \alpha F \alpha \psi = \alpha \psi \alpha F' \).

Put \( F' = \psi^{-1}(F) \), that is, \( F' = \{ (\psi^{-1}(i), \psi^{-1}(j)) : (i,j) \in F \} \). The rest is straightforward verification.

We are now ready to prove the statement of this lemma. By definition, every member \( g \in \Gamma_L \) can be written as a word consisting of members of \( S \) and their inverses. Using Claims 3.3 and 3.4, we can replace the inverses by members of \( S \), using Claim 3.5 we can ensure that the word can be split into two subwords, first consisting only of \( \alpha \psi \)'s and the second consisting of \( \alpha F \)'s. From Claims 3.3 and 3.4 it follows that there are \( \alpha \psi, \alpha F \in S \) such that indeed \( g = \alpha \psi \alpha F \).

**Proof.** Put \( F' = \psi^{-1}(F) \), that is, \( F' = \{ (\psi^{-1}(i), \psi^{-1}(j)) : (i,j) \in F \} \). The rest is straightforward verification.

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From now on we will thus denote members of \( \Gamma_L \) by \( \alpha \psi \), where

\[
\alpha \psi(U_i^\chi) = \alpha \psi(U_i^{\chi \psi(i)}).
\]

and \( \alpha \psi \) is the identity on \( \{ M, R^1, \ldots, R^\delta \} \). In other words, \( \alpha \psi \) first “flips” the mutual valuations of pairs from \( F \) and then permutes the set \( D \).

For notational convenience, whenever \( C \) is a \( \Gamma_L \)-structure, \( U_i^\xi \subseteq L \) and \( u \in C \) is a vertex such that \( u \in U_i^\xi \) and \( u \) has no other unary mark, we will denote by \( \pi(u) = i \) its projection and by
\(\chi(u) = \xi\) its valuation. If \(u\) does not have precisely one unary mark, we leave \(\pi(u)\) and \(\chi(u)\) undefined.

The following (easy) observation says that the unary marks \(U_i^\chi\) indeed generalise the construction from Section 3.1.

**Observation 3.6.** Let \(C\) be a \(\Gamma_L\)-structure such that every vertex of \(C\) has precisely one unary mark, let \(g\) be an automorphism of \(C\) and let \(u, v \in C\) be arbitrary vertices of \(C\). Then we have

\[
\chi(u)(\pi(v)) = \chi(v)(\pi(u))
\]

if and only if

\[
\chi(g(u))(\pi(g(v))) = \chi(g(v))(\pi(g(u))).
\]

This implies that the function \(f: (C_2^\chi) \to \{0, 1\}\), defined by \(f(uv) = 0\) if \(\chi(u)(\pi(v)) = \chi(v)(\pi(u))\) and \(f(uv) = 1\) otherwise, is invariant under \(g\) and consequently under all automorphisms of \(C\).

**Proof.** Assume that \(g = (\alpha_F^\psi, g_C)\) and put \(F_u = \{j \in D: (\pi(u), j) \in F\}\) and \(F_v = \{j \in D: (\pi(v), j) \in F\}\). Since \(g\) is an automorphism, we have

\[
g(u) \in \alpha_F^\psi(U_\pi(u)^\chi(u)),
\]

hence \(\chi(g(u)) = \chi(u)F_\psi^\chi\) and \(\pi(g(u)) = \psi(\pi(u))\) and similarly \(\chi(g(v)) = \chi(v)F_\psi^\chi\) and \(\pi(g(v)) = \psi(\pi(v))\).

It follows that

\[
\chi(g(u))(\pi(g(v))) = \chi(u)F_\psi^\chi(\psi(\pi(v))) = \begin{cases} 1 - \chi(u)(\pi(v)) & \text{if } \pi(v) \in F_u \\ \chi(u)(\pi(v)) & \text{otherwise,} \end{cases}
\]

and similarly for \(\chi(g(v))(\pi(g(u)))\). Since \(F\) is symmetric, we have that \(\pi(v) \in F_u\) if and only if \(\pi(u) \in F_v\) and thus the claim follows. \(\square\)

### 3.3. The class \(K\) and completion to it

Let \(C \in A_K^L\). We say that a \(\Gamma_L\)-structure \(C^+\) is a suitable expansion of \(C\) if the following hold:

1. \(C\) and \(C^+\) share the same vertex set,
2. for every \(1 \leq i \leq \delta\) we have that \(R_iC^+ = R_iC^+\),
3. \(M_{C^+}(u) = v\) if and only if \(d_{C^+}(u, v) = \delta\),
4. every vertex of \(C^+\) has precisely one unary mark,
5. if \(d_{C^+}(u, v) = \delta\) and \(u \in U_i^\chi\) in \(C^+\), then \(v \in U_i^{1-\chi}\), where \((1 - \chi)(j) = 1 - \chi(j)\), and
6. in \(C^+\) it holds that \(\chi(u)(\pi(v)) \neq \chi(v)(\pi(u))\) if and only if \(d_{C^+}(u, v)\) is odd.

Denote by \(K\) the class of all suitable expansions of all \(C \in A_K^L\) where the edges of length \(\delta\) form a perfect matching (Fact 2.7 says that this is without loss of generality; one can always uniquely and canonically add vertices so that this condition is satisfied). Note that it is possible that there is no suitable expansion of a given \(C \in A_K^L\).

**Proposition 3.7.** \(K\) is a locally finite automorphism-preserving subclass of \(E\), the class of all finite \(\Gamma_L\)-structures.

**Proof.** Let \(n\) be a large enough integer (say, at least 4 and at least twice the number of vertices of the largest forbidden cycle in \(B_{\infty}^L\)) and let \(A \in K\) and \(B\) be as in Definition 2.3. Note that there is an unfortunate notational clash, this \(A\) is different from the structure \(A\) which we fixed at the beginning of this section.

The fact that for every \(v \in B\) one has that \(\text{Cl}_B(v)\) lies in a copy of \(A\) implies that \(M_B(u) = v\) if and only if \(d_B(u, v) = \delta\) and furthermore the edges of length \(\delta\) form a perfect matching in \(B\) (because this holds in \(A\)).

The fact that every substructure of \(B\) on at most \(n\) vertices has a completion in \(K\) (which is promised by Definition 2.3) implies the following:
1. Every pair of vertices is in at most one distance relation $R^i$ and these relations are symmetric and irreflexive,
2. every vertex of $B$ is in precisely one unary relation, and
3. if $d_B(u, v) = \delta$ then $v \in U^1_{\pi(u)}$.

We can assume that if $d_B(u, v) = \delta$ and $u \neq v$ is a vertex of $B$ such that at least one of $d_B(u, w)$, $d_B(v, w)$ is defined, then in fact both distances are defined and furthermore $d_B(u, w) + d_B(v, w) = \delta$, because there is a unique way to complete it. It also follows that whenever $u, v$ are vertices such that their distance is defined, then $\chi(u)(\pi(v)) \neq \chi(v)(\pi(u))$ if and only if $d_B(u, v)$ is odd.

Finally, from the definition of $n$ it also follows that $B$ contains no cycles forbidden in $B_K^\phi$ (we needed $n$ to be twice the number of vertices because of Theorem 2.3 talks about substructures and for the need to be closed for functions). Hence if we define the function $f : (\hat{B}_K^\phi) \to \{0, 1\}$ as $f(uv) = 0$ if $\chi(u)(\pi(v)) = \chi(v)(\pi(u))$ and $f(uv) = 1$ otherwise, Fact 2.8 gives us an automorphism-preserving way to add the remaining non-$\delta$ distances, which is exactly what we need for a completion to $K$.

Let us remark that $K$ is hereditary: Whenever $B$ is a substructure of $C \in \mathcal{A}_K^\phi$, such that the edges of length $\delta$ form a perfect matching in both $B$ and $C$, we have that if $C^+$ is a suitable expansion of $C$, then the substructure of $C^+$ induced on the vertex set $B$ is a suitable expansion of $B$.

### 3.4. Constructing the witness

Recall that at the beginning of this section, we fixed $A \in \mathcal{A}_K^\phi$ and enumerated its edges of length $\delta$ as $e_1 = \{x_1, y_1\}, \ldots, e_m = \{x_m, y_m\}$.

For $1 \leq i \leq m$, we define $\chi_i : D \to \{0, 1\}$ by putting

$$\chi_i(j) = \begin{cases} 1 & \text{if } i > j \text{ and } d_A(x_i, x_j) \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

We define a suitable expansion $A^+ \in K$ of $A$ by putting, for every $1 \leq i \leq m$, $M_{A^+}(x_i) = y_i$, $M_{A^+}(y_i) = x_i$, $x_i \in U^1_{\chi_i}$ and $y_i \in U^{1-\chi_i}$. Next we use Theorems 2.5 and 2.6 with Proposition 3.7 to get $B^+ \in K$ which is an EPPA-witness for $A^+$ (so, in particular, $A^+ \subseteq B^+$). Finally, we put $B$ to be the reduct of $B^+$ forgetting all the unary marks and the function $M$. Then indeed, $B \in \mathcal{A}_K^\phi$. And since $A^+ \subseteq B^+$, we also have $A \subseteq B$.

### 3.5. Extending partial automorphisms

We will show that $B$ extends all partial automorphisms of $A$. Fix a partial automorphism $\varphi$ of $A$. Without loss of generality we can assume that whenever $d_A(u, v) = \delta$ and $u \in \text{Dom}(\varphi)$, then also $v \in \text{Dom}(\varphi)$ (because there is a unique way of extending $\varphi$ to $v$). Let $\psi : D \to D$ be an arbitrary permutation of $D$ extending the action of $\varphi$ on the edges of length $\delta$ of $A$.

We now define a set $F \subseteq D^2$ of flipping pairs. We put $(i, j)$ and $(j, i)$ in $F$ if $x_i \in \text{Dom}(\varphi)$ and $\chi(\varphi(x_i)) = \chi(x_i)$ in $A^+$. Note that if both $x_i$ and $x_j$ are in the domain of $\varphi$ then the outcome is the same if we consider $x_j$ instead of $x_i$, because $\varphi$ is an automorphism and therefore preserves the parity of $d_A(x_i, x_j)$ and thus also the (non)-equality of the corresponding valuations. Note also that if we considered $y_i$ instead of $x_i$, the outcome would still be the same.

What remains is to verify that the pair $(\alpha^F_\psi, \varphi)$ is a partial ($\Gamma_i$-)automorphism of $A^+$. Indeed, assuming that it is the case, we get that it extends to an automorphism $(\theta_L, \theta)$ of $B^+$, where $\theta_L = \alpha^F_\psi$ and $\varphi \subseteq \theta$. But this means that $\theta$ is an automorphism of $B$ extending $\varphi$ and hence $B$ is an EPPA-witness for $A$. In the rest of this section we verify that $(\alpha^F_\psi, \varphi)$ is a partial automorphism of $A^+$. It amounts to (technical) checking that our construction does what it is supposed to do.

From the fact that $\varphi$ is a partial automorphism of $A$ we get that $d_{B^+}(u, v) = d_{B^+}(\varphi(u), \varphi(v))$ whenever $u, v \in \text{Dom}(\varphi)$. This, together with the assumption that whenever $d_A(u, v) = \delta$ and
$u \in \text{Dom}(\varphi)$, then also $v \in \text{Dom}(\varphi)$, implies that if $v \in \text{Dom}(\varphi)$ then $M_{B^+}(\varphi(v)) = \varphi(M_{B^+}(v))$, or in other words, $\varphi$ respects the function $M$.

It remains to verify that for every $v \in \text{Dom}(\varphi)$ and for every $U_i^\delta$ we have $v \in U_i^\delta$ if and only if $\varphi(v) \in \alpha^\delta(U_i^\delta)$, or in other words, $\pi(\varphi(v)) = \psi(\pi(v))$ and $\chi(\varphi(v)) = \chi(\psi(\pi(v)))$, where $F_i = \{j \in D : (\pi(v), j) \in F\}$. Since $\psi$ extends the action of $\varphi$ on the edges of length $\delta$, and since for every $v \in A$ it holds that $\pi(v) = i$ if and only if $v \in e_i$, it follows that for every $v \in \text{Dom}(\varphi)$ we have $\pi(\varphi(v)) = \psi(\pi(v))$.

Analogously, from the definition of $F$ we have that $(i, j)$ and $(j, i)$ are in $F$ if and only if $x_i \in \text{Dom}(\varphi)$ and $\chi(\varphi(x_i))(\psi(j)) \neq \chi(x_i)(j)$. From the construction it follows that this happens if and only if $y_i \in \text{Dom}(\varphi)$ and $\chi(\varphi(y_i))(\psi(j)) \neq \chi(y_i)(j)$. We can summarize these two equivalences as follows: For every $v \in \text{Dom}(\varphi)$ and for every $j \in D$ we have $(\pi(v), j) \in F$ if and only if $\chi(\varphi(v))(\psi(j)) \neq \chi(v)(j)$.

By the definition of $\alpha^F$, for every $v \in \text{Dom}(\varphi)$ and for every $j \in D$ we have that $\chi(\varphi(F)(j)) \neq \chi(v)(F)(j)$ if and only if $(\pi(v), j) \in F$. Consequently, $\chi(\varphi(F)(\psi(j))) \neq \chi(v)(F)(\psi(j))$ if and only if $(\pi(v), j) \in F$, which happens if and only if $\chi(\varphi(F)(\psi(j))) \neq \chi(v)(j)$. It follows that $\chi(\varphi(F)(\psi(j))) = \chi(\varphi(v))$ which concludes the proof of Proposition 3.1.

3.6. Remarks

1. If we extended the action of $\varphi$ on the edges of length $\delta$ coherently (say, in an order-preserving way), we would get coherent EPPA (see [32]) as in [9].

2. The same strategy would also work for proving EPPA for antipodal metric spaces of even diameter when each of $\delta - a$ is in $O$ for every $a \in \{0, 1, \ldots, \delta\}$ and replace each occurrence of “odd distance” by “distance from $O$” and “even distance” by “distance from $O$”.

3. Cherlin also allows to forbid certain sets of $\{1, \delta - 1\}$-valued metric spaces (he calls them Henson constraints). We chose not to include these classes in order to avoid further technical complications, but using irreducible-structure faithfulness and the fact that the completion from Fact 2.8 does not create distances $1$ and $\delta - 1$ gives EPPA also in this case.

4. The even diameter bipartite case

The odd diameter bipartite case was done in [4] (because every edge of length $\delta$ has one endpoint in each part of the bipartition and thus there is a unique way of determining parities of the distances, which implies that such classes admit automorphism-preserving completions), so it suffices to deal with the even diameter case. We prove the following proposition.

Proposition 4.1. Let $A_K^\delta$ be a bipartite class of antipodal metric spaces of even diameter. Then for every $A \in A_K^\delta$ there is $B \in A_K^\delta$ which is an EPPA-witness of $A$.

The structure of the proof will be very similar to the odd non-bipartite case. We will also introduce some facts from [4] about completions, add unary functions and unary marks which will help us decide how to fill-in the missing distances while preserving all necessary automorphisms. We have to be a bit more careful in dealing with the bipartiteness (edges of length $\delta$ now lie inside the parts), so we need to make $\psi$ preserve the bipartition, there are also infinitely many forbidden cycles — the odd perimeter ones), but the general structure is identical.

For the rest of the section, fix $A \in A_K^\delta$. We can without loss of generality assume that every vertex $v \in A$ has some vertex $w \in A$ such that $d_A(v, w) = \delta$. Consider the set $\{e_1, \ldots, e_m\}$ of edges of $A$ of length $\delta$ and let $D = \{1, 2, \ldots, m\}$ be their indices, that is, $|D| = \frac{|A|}{2}$. We denote $e_i = \{x_i, y_i\}$, where $x_i$ and $y_i$ are vertices of $A$. Since $A$ is bipartite, we have that the relation “vertices $u$ and $v$ are at an even distance” is an equivalence relation on $A$ which has two equivalence classes. Because $\delta$ is even, we can assume that $D = D_1 \cup D_2$, where $D_1$ consists of
the indices of edges with both endpoints in one part and \( D_2 \) consists of the indices of edges with both endpoints in the other part.

We also assume without loss of generality that \(|D_1| = |D_2|\) (otherwise we can add more vertices to \( A \), and if this larger structure has an EPPA-witness \( B \), then it is also an EPPA-witness of the original \( A \)).

We will need the following analogue of Fact 2.8.

**Fact 4.2.** Let \( \mathcal{A}_K \) be a bipartite class of antipodal metric spaces. Let \( A \) be a \([\delta]\)-edge-labelled graph such that the edges of length \( \delta \) of \( A \) form a perfect matching and furthermore for every \( u, v, w \in A \) such that \( d_A(u, v) = \delta \) and \( u \neq v, v \), either \( w \) is not connected by an edge to either of \( u, v \), or \( d_A(u, w) + d_A(v, w) = \delta \). Suppose furthermore that \( A \) contains no odd-perimeter cycles and none of the finitely many even-perimeter cycles forbidden in \( B_K \).

Let \( O \subset \{0, 1, \ldots, \delta\} \) be a set such that \( \delta \in O \) and exactly one of \( a, \delta - a \) is in \( O \) for every \( a \in \{0, 1, \ldots, \delta\} \) and denote by \( \delta - O \) the set \( \{\delta - a : a \in O\} \).

Let \( f : (\frac{\delta}{2}) \to \{0, 1\} \) be a mapping satisfying the following.

1. Whenever \( uv \) is an edge of \( A \), then \( f(uv) = 1 \) implies that \( d_A(u, v) \in O \) implies that \( d_A(u, v) \in \delta - O \).\(^3\)
2. Let \( u_1v_1 \) and \( u_2v_2 \) be two different edges of length \( \delta \) of \( A \). Then \( f(u_1v_2) = f(v_1v_2) \), \( f(u_1v_2) = f(u_1v_1) \) and \( f(u_1v_2) \neq f(u_1v_1) \).

Then there is \( \bar{A} \in \mathcal{A}_K^\delta \) such that the following holds.

1. \( \bar{A} \) is a completion of \( A \) with the same vertex set,
2. for every edge \( uv \) of \( \bar{A} \) it holds that \( f(uv) = 1 \) implies that \( d_{\bar{A}}(u, v) \in O \) and \( f(uv) = 0 \) implies that \( d_{\bar{A}}(u, v) \in \delta - O \), and
3. Every automorphism of \( A \) which preserves values of \( f \) is also an automorphism of \( \bar{A} \).

**4.1. The expanded language**

As in the odd non-bipartite case, we will call a function \( \chi : D \to \{0, 1\} \) a valuation function, adopt the same notions of flips \( \chi^\beta \) and permutations \( \chi^\psi \). We also let \( L \) be the same language as in Section 3, adding a unary function \( M \) and unary relations \( U^\xi_{i} \).

In contrast to Section 3, we put \( \Gamma_L \) to be the group generated by

\[
S = \{ \alpha^F : F \subseteq D^2 \text{ and } F \text{ is symmetric} \} \cup \{ \alpha_{\psi} : \psi \text{ is a partition-preserving permutation of } D \},
\]

where \( \psi \) is partition-preserving if either \( \psi(D_1) = D_1 \) and \( \psi(D_2) = D_2 \), or \( \psi(D_1) = D_2 \) and \( \psi(D_2) = D_1 \). Analogously to Lemma 3.2 it follows that every element of \( \Gamma_L \) can be written as the product \( \alpha_{\psi}\alpha^{F} \), where \( \alpha_{\psi}, \alpha^{F} \in S \). We will denote \( \alpha^{F}_{\psi} = \alpha_{\psi}\alpha^{F} \).

Again, for a vertex \( u \) in a \( \Gamma_L \)-structure which has precisely one unary mark \( U^\xi_{i} \), we define \( \pi(u) = i \) and \( \chi(u) = \xi \) and we have the same observation with the same proof as before.

**Observation 4.3.** Let \( C \) be a \( \Gamma_L \)-structure such that every vertex of \( C \) has precisely one unary mark, let \( g \) be an automorphism of \( C \) and let \( u, v \in C \) be arbitrary vertices of \( C \). Then we have

\[
\chi(u)(\pi(v)) = \chi(v)(\pi(u))
\]

if and only if

\[
\chi(g(u))(\pi(g(v))) = \chi(g(v))(\pi(g(u))).
\]

This implies that the function \( f : (\frac{\delta}{2}) \to \{0, 1\} \), defined by \( f(uv) = 0 \) if \( \chi(u)(\pi(v)) = \chi(v)(\pi(u)) \) and \( f(uv) = 1 \) otherwise, is invariant under \( g \) and consequently under all automorphisms of \( C \).

Note that the edge-labelled graph formed by the distance relations in \( C \) may contain odd cycles.

\[^{3}\text{This seemingly sloppy statement is necessary in order to deal with } \frac{\delta}{2} \text{ being in both } O \text{ and } \delta - O \text{ for even } \delta.\]
4.2. The class \( \mathcal{K} \) and completion to it

Now we also have to ensure that structures from \( \mathcal{K} \) are bipartite. Let \( \mathbf{C} \in \mathcal{A}_K^\delta \). Since \( \mathbf{C} \) is bipartite, we can denote by \( Q_1, Q_2 \) its parts (that is, \( Q_1 \cup Q_2 = C \) and each of \( Q_1, Q_2 \) is an equivalence class of the relation “vertices \( u \) and \( v \) are at an even distance from each other”). We say that a \( \Gamma_L \)-structure \( \mathbf{C}^+ \) is a suitable expansion of \( \mathbf{C} \) if the following hold:

1. \( \mathbf{C} \) and \( \mathbf{C}^+ \) share the same vertex set,
2. for every \( 1 \leq i \leq \delta \) we have that \( R_i^\mathbf{C} \equiv R_i^{\mathbf{C}^+} \),
3. \( M_{\mathbf{C}^+}(u) = v \) if and only if \( d_{\mathbf{C}^+}(u, v) = \delta \),
4. every vertex of \( \mathbf{C}^+ \) has precisely one unary mark,
5. if \( d_{\mathbf{C}^+}(u, v) = \delta \) and \( u \in U_1^\chi \) in \( \mathbf{C}^+ \), then \( v \in U_1^{1-\chi} \),
6. in \( \mathbf{C}^+ \) it holds that if \( \chi(u)(\pi(v)) \neq \chi(v)(\pi(u)) \) then \( d_{\mathbf{C}^+}(u, v) \in O \) and if \( \chi(u)(\pi(v)) = \chi(v)(\pi(u)) \) then \( d_{\mathbf{C}^+}(u, v) \in \delta - O \), and
7. let \( P_1 = \{ v \in C : \pi(v) \in D_1 \} \) and \( P_2 = \{ v \in C : \pi(v) \in D_2 \} \) (where \( \pi \) is taken with respect to \( \mathbf{C}^+ \)). Then either \( P_1 = Q_1 \) and \( P_2 = Q_2 \), or \( P_1 = Q_2 \) and \( P_2 = Q_1 \).

Denote by \( \mathcal{K} \) the class of all suitable expansions of all \( \mathbf{C} \in \mathcal{A}_K^\delta \) where the edges of length \( \delta \) form a perfect matching.

**Proposition 4.4.** \( \mathcal{K} \) is a locally finite automorphism-preserving subclass of \( \mathcal{E} \), the class of all finite \( \Gamma_L \)-structures.

**Proof.** Let \( n \) be a large enough integer (say, at least 4 and at least twice the number of vertices of the largest even-perimeter forbidden cycle in \( B^n_k \)) and let \( \mathbf{A} \in \mathcal{K} \) and \( \mathbf{B} \) be as in Definition 2.3. (Again, this is not the \( \mathbf{A} \) which we fixed at the beginning of this section.)

As for the odd non-bipartite case we get the following:

1. Every vertex of \( \mathbf{B} \) is in precisely one unary relation,
2. every pair of vertices is in at most one distance relation \( R_i \) (and these relations are symmetric),
3. \( M_{\mathbf{B}}(u) = v \iff d_{\mathbf{B}}(u, v) = \delta \),
4. the edges of length \( \delta \) form a perfect matching in \( \mathbf{B} \),
5. if \( d_{\mathbf{B}}(u, v) = \delta \) then \( v \in U_1^{1-\chi(u)} \),
6. without loss of generality, we can assume that if \( d_{\mathbf{B}}(u, v) = \delta \) and \( w \neq u \) is a vertex of \( \mathbf{B} \) such that at least one of \( d_{\mathbf{B}}(u, w) \) or \( d_{\mathbf{B}}(v, w) \) is defined, then both distances are defined and furthermore \( d_{\mathbf{B}}(u, w) + d_{\mathbf{B}}(v, w) = \delta \).
7. let \( u, v \) be vertices such that their distance is defined. Then \( \chi(u)(\pi(v)) \neq \chi(v)(\pi(u)) \) implies \( d_{A^+}(u, v) \in O \) and \( \chi(u)(\pi(v)) = \chi(v)(\pi(u)) \) implies \( d_{A^+}(u, v) \in \delta - O \).

Furthermore, from the last condition for a suitable expansion we also get that two vertices \( u, v \) of \( \mathbf{B} \) are at an even distance, if and only if there is \( i \in \{ 1, 2 \} \) such that \( \pi(u), \pi(v) \in D_i \). Note that this implies that \( \mathbf{B} \) contains no cycles of odd perimeter (each cycle has to contain an even number of odd edges).

Finally, from the definition of \( n \) it also follows that \( \mathbf{B} \) contains no even-perimeter cycles forbidden in \( B^n_k \). Hence if we define the function \( f : \binom{2}{2} \to \{ 0, 1 \} \) as \( f(uv) = 0 \) if \( \chi(u)(\pi(v)) = \chi(v)(\pi(u)) \) and \( f(uv) = 1 \) otherwise, Fact 4.2 gives us an automorphism-preserving way to add the remaining distances, which is exactly what we need for a completion to \( \mathcal{K} \).

\[ \square \]

Let us again remark that \( \mathcal{K} \) is hereditary.
4.3. Constructing the witness

This is completely the same as for the odd diameter non-bipartite case. We define a $\Gamma_L$-structure $A^+$ which is a suitable expansion of $A$, and use Theorems 2.5 and 2.6 with Proposition 4.4 to get $B^+ \in \mathcal{K}$ which is an EPPA-witness for $A^+$. Finally, we put $B$ to be the reduct of $B^+$ forgetting all unary marks and all functions $M$.

4.4. Extending partial automorphisms

Again, this is completely the same as before with the exception that the permutation $\psi$ of $D$ has to preserve the bipartition $D = D_1 \cup D_2$ (it can exchange $D_1$ and $D_2$). Every partial automorphism $\varphi$ of $A$ respects the bipartition, and since we assumed that $|D_1| = |D_2|$, it is always possible to extend $\varphi$ to a full permutation $\psi$ as needed.

Let us remark that if one is a bit more careful, the same strategy again gives coherent EPPA.

5. Conclusion

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. In [4], EPPA is proved for non-bipartite classes of even diameter and bipartite classes of odd diameter. Proposition 3.1 proves EPPA for non-bipartite classes of odd diameter and Proposition 4.1 proves EPPA for bipartite classes of even diameter, hence Theorem 1.2 is proved.

We think of this paper as the first example of a more general method for bypassing the lack of an automorphism-preserving completion, namely using the method of valuation functions to add more information to the structures (and thus restrict automorphisms) while preserving all partial automorphisms of one given structure $A$, and then plugging this expanded class into the existing machinery. A similar trick can be done also for structures with higher arities, using higher-arity valuation functions (cf. [22]). However, there are still classes where this method does not work, for example the class of tournaments which poses a long-standing important problem in this area.

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