Step-by-step dimensional crossover of an optically-trapped quantum gas with disorder

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Dimensionality serves as an indispensable ingredient to any attempt of formulating the low-dimensional physics, and studying dimensional crossover on a fundamental level is challenging. The purpose of this work is to study the step-by-step dimensional crossover from three-dimension (3D) to quasi-2D and then 1D. Our model system consists of a 3D Bose-Einstein condensate (BEC) trapped in an anisotropic 2D optical lattice characterized by the lattice depths of $V_1$ along $x$-direction and $V_2$ along $y$-direction respectively. The gradually dimensional crossover is controlled by the continuous increase of $V_1$ and $V_2$. Then we analyze the combined effects of dimensionality and disorder on both equilibrium and non-equilibrium quantum fluctuations of the model system. Accordingly, the analytical expressions of the ground-state energy, quantum depletion and superfluid density of the system are obtained. Our results show that the step-by-step dimensional crossover induces a characteristic 3D to quasi-2D and 1D step-by-crossover in the behavior of quantum fluctuations. Conditions for possible experimental realization of our scenario are also discussed.

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Introduction.—— Dimensionality plays a fundamental role in determining the properties of the physical many-body systems. The typical examples include the remarkable phenomena of the high-Tc superconductivity \cite{1} and magic-angle graphene \cite{2–4} in two dimensions (2D) and the Tomonaga-Luttinger liquid \cite{5} in 1D respectively. Therefore, there are ongoing interests and great efforts on investigating how the dimensionality affecting the physical properties of many-body systems.

At present, a tightly-confined Bose-Einstein condensate (BEC) \cite{6} has opened the new theoretical and experimental windows for investigating fundamental problems of dimensional effects in a very versatile manner. Particularly, with the state-of-art technology, the depth of an optical lattice can be arbitrarily modified by changing the intensities of laser beams, resulting in realization of quasi-1D \cite{7} and quasi-2D \cite{8, 9} BEC systems almost at will. Therefore, an important direction of investigation consists in studying the properties of a BEC system along the dimensional crossover.

Along this research line, much work has been done. In particular, Refs. \cite{10, 11} have studied that a 2D lattice can induce a characteristic 3D to 1D crossover in the behavior of quantum fluctuations. Refs. \cite{12–14} have investigated the quantum phases along a characteristic 3D to 2D crossover and the visualization of dimensional effects in collective excitations. The strategy of previous research \cite{10–17} on dimensional effects, called by one-step dimensional crossover, is limited on exploring separate 1D or 2D behaviors from 3D by the tight confinement. Instead, the aim of this Letter is to investigate, by the strategy of the step-by-step dimensional crossover, the effects on the equilibrium quantum phases due to the gradual dimensional crossover from the 3D to quasi-2D and then 1D.

The second motivation of this Letter is to extend the research scopes of dimensionality from the equilibrium scenario to non-equilibrium one motivated by recent experimental realizations of a BEC with disorder \cite{18, 19}. For example, superfluidity is a kinetic property of a system and the superfluid density is not an equilibrium quantity but a transport coefficient, and should be determined by the linear response theory. Hence, our present model including disorder, will allow for a comprehensive description of the step-by-step dimensionality effects on both equilibrium and non-equilibrium properties of a many-body quantum system.

In this work, we investigate the dimensionality-induced equilibrium and non-equilibrium properties of a disorder BEC trapped in an anisotropic optical lattice within the help of Green function. Here, the ground-state energy and quantum depletion represent the equilibrium properties and the superfluid density is the non-equilibrium property. Accordingly, we analyze the combined effects of dimensionality and disorder on the ground-state energy, quantum depletion and the superfluid density of the system respectively. Our results show that the lattice induces a characteristic 3D to quasi-2D and 1D crossover in the behavior of ground-state energy. Furthermore, we calculate the normal fluid density induced by the disorder along the step-by-step dimensional crossover.

Model.—— At zero temperature, an optically-trapped BEC can be well described by the $N$-body Hamiltonian \cite{10–13}

$$\hat{H} - \mu \hat{N} = \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \left[ -\frac{\hbar^2 \nabla^2}{2m} - \mu + V_{\text{opt}}(\mathbf{r}) + V_{\text{ran}}(\mathbf{r}) + \frac{g}{2} \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \right] \hat{\Psi}(\mathbf{r}), \quad (1)$$

where $\hat{\Psi}(\mathbf{r})$ is the field operator for bosons with mass $m$, $\mu$ is the chemical potential, $\hat{N} = \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r})$ is the number operator, and $g = 4\pi\hbar^2a_{3D}/m$ is the coupling

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constant with \(a_{3D}\) being the 3D scattering length \([20]\).

In Hamiltonian (1), \(V_{\text{opt}}(r)\) and \(V_{\text{ran}}(r)\) represent the anisotropic 2D optical lattice and the external random potential respectively.

Physically, low-dimensional crossovers are characterized by hierarchical access to new energy by using an optical lattice as tight confinement tool. In more details, a 3D Bose gas becomes quasi-2D when energetic restriction to freeze \(x\)-direction excitations is reached. Next, by further freezing kinetic energy along the \(y\)-direction, the quasi-2D is supposed to enter into the quasi-1D regime. In this end, our strategy is to adopt the anisotropic 2D optical lattice in Hamiltonian (1), reading \([6]\)

\[
V_{\text{opt}}(r) = E_R[V_1 \sin^2(q_B x) + V_2 \sin^2(q_B y)],
\]

where \(V_{1,2}\) are laser intensities and \(E_R = \hbar^2 q_B^2 / 2m\) is the recoil energy, with \(\hbar q_B\) being the Bragg momentum and \(m\) the atomic mass. The lattice period is fixed by \(d = \pi / q_B\). Atoms are unconfined in the \(z\) direction.

The disorder potential with the form of \(V_{\text{ran}}(r) = \sum_i N_{\text{imp}} v(r - r_i)\) in Hamiltonian (1) can be produced by the random potential associated with quenched impurities. In this work, we restrict ourselves to the conditions of a dilute BEC system in the presence of a very small concentration of disorder \([18, 19]\). Thereby, \(v(r)\) can be approximated by an effective pseudopotential in the form \(v(r) = g_{\text{imp}} \delta(r)\), with \(g_{\text{imp}} = 2 \pi \hbar^2 \tilde{b} / m\) being the effective coupling constant of an impurity-boson pair and \(\tilde{b}\) being the effective scattering length accounting for the presence of a 2D optical lattice \([21, 22]\).

We are interested in the case where the lattice depths \(V_{1}\) and \(V_{2}\) in Eq. (2) are relatively large that the inter-well barriers are significantly higher than the chemical potential \(\mu\). Meanwhile, because of the quantum tunneling, the overlap between the wave functions of two consecutive wells are still sufficient to ensure full coherence even in the presence of disorder. In the tight-binding approximation, the lowest Bloch band of the model system can be described in terms of Wannier functions as \(\phi_k(x)\phi_k(y)\) with \(\phi_k(x_i) = \sum_i e^{ikx_i} w(x_i - ld)\) and \(w(x_i) = \exp[-x_i^2 / 2 \sigma_k^2] / \pi^{1/4} \sigma_k^{1/2}\) and \(d/\sigma_i = \pi V_i^{1/4} \exp(-1/4 \sqrt{V_i})\) (\(i = 1, 2\) and \(x_1 = x, x_2 = y\)).

Directly following Refs. [11, 13], we proceed to expand the field operators of Hamiltonian (1) by the expression \(\Psi(r) = \sum_k \hat{a}_k e^{-ik \cdot r} \phi_k(x)\phi_k(y)\) as follows

\[
H = -\mu N + \sum_k (\varepsilon_k^0 - \mu) \hat{a}_k^\dagger \hat{a}_k + \frac{\tilde{g}}{2V} \sum_{k,q,k'} \hat{a}_k^\dagger \hat{a}_{k+q} \hat{a}_{k'-q} \hat{a}_k \hat{a}_{k'},
\]

with

\[
\varepsilon_k^0 = \frac{\hbar^2 k_x^2}{2m} + 2[t - t_1 \cos k_x - t_2 \cos k_y],
\]

being the free energy dispersion of the non-interacting case, \(t_1\) and \(t_2\) being the tunneling rates along the \(x\)- and \(y\)-direction optical lattices and \(t = t_1 + t_2\), \(V\) is volume and \(\tilde{g}\) being the re-normalized coupling constant due to the optical lattice. The \(V_k = 1/V \int e^{ik \cdot r} V_{\text{ran}} d\mathbf{r}\) in Eq. (3) is the Fourier transform of disorder potential.

The emphasis and value of the present work are to investigate the step-by-step crossover from 3D to quasi-2D and then 1D of an optically-trapped Bose gas in a highly controllable way. In the first part of the paper, we focus on the dimensionality effects on the equilibrium properties by calculating the ground state energy and quantum depletion. Note that the previous studies \([11, 13]\) have shown that the effects on the ground state energy and quantum depletion due to disorder are somehow trivial energy shifts. In order to keep our physical model as simple as possible, we ignore the disorder potential by setting \(V_{\text{ran}} = 0\) in this part. In the highly contrast, the second part of the paper is to investigate how the dimensionality affect the non-equilibrium properties by calculating the superfluid density. Here, the existence of disorder is the key point and we plan to include the disorder potential by setting \(V_{\text{ran}} \neq 0\).

**Ground state energy and quantum depletion.** — For an optically-trapped Bose gas described by Hamiltonian (1), the ground state energy \(E_n\) and quantum depletion \(\eta_{ex}\) can be calculated by the single-particle Green’s function...
$G(k, \omega)$ [23] as follows

$$
\frac{E_g}{V} = \frac{\tilde{g}n_0^2}{2} + \frac{1}{(2\pi)^3d^2} \int dk d^2k \int d\omega \frac{2\pi}{2\pi} e^{-i\omega t} E_k G(k, \omega) \tag{5}
$$

$$
n_{ex} = \frac{i}{(2\pi)^4d^2} \int dk d^2k \int d\omega e^{-i\omega t} G(k, \omega), \tag{6}
$$

with $E(k)$ being the excitation energy. In Eqs. (5) and (6), the $G(k, \omega)$ is the Fourier transformation of Green function $G(k, t-t')$ in the Heisenberg representation defined as

$$
G(k, t-t') = -i \langle T\hat{a}_k(t)\hat{a}^\dagger_k(t') \rangle, \tag{7}
$$

where $T$ denotes the chronological product.

By applying the Bogoliubov theory [10–14] to the Hamiltonian (1) and proceeding in the standard fashion, the analytical expression of $G(k, \omega)$ can be calculated as

$$
G(k, \omega) = \frac{\hbar \omega + \varepsilon^0_k + \tilde{g}n_0}{\hbar^2\omega^2 - E_k^2 + i0}, \tag{8}
$$

with $n_0$ being the condensate density of model system, $E_k = \sqrt{\varepsilon^0_k + 2\tilde{g}n_0}$ and $\varepsilon^0_k$ being defined in Eq. (4).

Now, we are ready to investigate how dimensional crossover from 3D to quasi-2D and quasi-1D can affect the quantum fluctuation based on Eqs. (5) and (6). By plugging Eq. (8) into Eqs. (5) and (6) respectively, the ground state energy $E_g$ and quantum depletion of the model system can be obtained as follows

$$
E_g = \frac{1}{2} \tilde{g}n_0^2 - \sqrt{2m\tilde{g}n_0}\varepsilon^0_k f \left( \frac{2t}{\tilde{g}n_0} \right), \tag{9}
$$

and

$$
n_{ex} = \frac{\sqrt{2m\tilde{g}n_0}}{4\pi \hbar^2} h \left( \frac{2t}{\tilde{g}n_0} \right), \tag{10}
$$

where the functions of $f(x)$ in Eq. (9) and $h(x)$ in Eq. (10) are respectively given as follows

$$
f(x) = \frac{\pi}{2} \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \frac{1}{\sqrt{x^2 + \gamma^2}} \gamma \left[ \frac{1}{\gamma} \gamma \right], \tag{11}
$$

and

$$
h(x) = \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \int_0^{\infty} dy \frac{1}{\sqrt{y}} \left[ \frac{y + x + \gamma + 1}{\sqrt{(y + x + \gamma)(y + x + \gamma + 2)}} - 1 \right]. \tag{12}
$$

In both Eqs. (11) and (12), the variable $x$ stands for $x = x_1 + x_2 = 2(t_1 + t_2)/\tilde{g}n_0$ which can be controlled by the strength of optical lattice in Eq. (2) and $\gamma = 1 - (t_1/t)\cos k_x - (t_2/t)\cos k_y$. The function $\gamma \left[ \frac{1}{\gamma} \gamma \right]$ in Eq. (11) is referred to as the hypergeometric function.

Equations (9) and (10) are the key results of this work. Note that the functions of $f(x)$ and $h(x)$ are plotted in Figs. 1 and 2. A tight confinement along one or two directions described by the parameter of $x = 2(t_1 + t_2)/\tilde{g}n_0$ will introduce a dimensional crossover from anisotropic 3D to low-dimensional regimes when energetic restrictions to freeze axial excitations is reached. In the limit of $x \to \infty$ or $x \to 0$, the model system is anisotropic 3D or 1D respectively. In order to study the step-by-step dimensionality crossover controlled by the continuous decrease of $x$ from $\infty$ to 0, we devise two scenarios: first, we fixed the lattice strength of $V_1$ and increase the $V_2$ (or decrease the $x_2$ to be near zero), under which the model system is supposed to become from 3D to quasi-2D; second, we further increase the value of $V_1$ with the fixed $V_2$ until the value of $x = x_1 + x_2$ becomes to be almost zero. In such, the model system will experience a step-by-step dimensionality crossover from quasi-2D to 1D. In what follows, we outline the detailed calculations:

First, we investigate the scenario corresponding to the dimensional crossover from 3D to quasi-2D by plotting the functions of $f(x)$ and $h(x)$ in Eqs. (9) and (10) into Figs. 1 and 2 (see the solid curves) as follows:

(i). We first check whether our analytical results in Eqs. (9) and (10) in the limit of $x \to \infty$ can recover the well-known 3D results of Bose gases. In the limit of $x \to \infty$ corresponding to the anisotropic 3D regime, we find $f(x) \simeq 1.43 / \sqrt{x} - 32 \sqrt{2}/(15\pi x)$ in Eq. (11) and $h(x) \simeq 8/3\pi \sqrt{2} x$ as shown by the black circled curves in Figs. 1 and 2 respectively, which can exactly recover the 3D results of the quantum ground state energy and quantum depletion in Ref. [13].

(ii). Note that there are different ways of realizing a 2D quantum system. One way adopted by Ref. [13] is to add a 1D optical lattice and then increase the lattice depth. Finally, a pure 2D quantum system can be...
and set $x_2 = 0.01$. In the right side, we fix $x_1 = 2$ and set $x_2 = [0.01, 2]$. The BEC behaves from 1D-like to quasi-2D like, and finally to 3D-like, as $x$ increases. The black dotted line in the right side denotes the 3D asymptotic behavior.

reached. The other way adopted in this work is to realize the quasi-2D quantum system. In order to compare these two methods, we plot the related results of realizing pure 2D results in Ref. [13] into the red curves in Fig. 1. Thus our model is different from that in Ref. [13] in that we realize a quasi-2D quantum system and further reach the dimensional crossover from quasi-2D to quasi-1D.

Second, in the scenario corresponding to the dimensional crossover from quasi-2D to quasi-1D, we then increase the value of the $V_1$ with the fixed lattice depth of $V_2$. In particular, note that the function $f(x)$ shown in Fig. 1 approaches to be the exact value of $3\sqrt{2}/4$ in the limit of $x \to 0$, corresponding to the Lieb-Liniger result of 1D Bose gas in Ref. [10]. For the quantum depletion shown in Fig. 2, the function $h(x)$ diverges as $h(x) \sim -\ln(1.35x)/\sqrt{2}$. This signals that in the absence of tunneling there is no real BEC in agreement with the general theorems in one dimension.

Finally, we remark that our results in Eqs. (9) and (10) together with the corresponding ones in Refs. [10–13] gives a complete description of the dimensional crossover. More elaborate theoretical treatments beyond the Bogoliubov approximation are beyond the scope of this work.

Superfluid density. — In the second part of this paper, we apply the linear response theory to investigate the effects of disorder on the superfluid density of a BEC trapped in a 2D optical lattice. We emphasize that, unlike the ground state energy and quantum depletion which are the equilibrium properties, superfluidity is a kinetic property of a system and the superfluid density is not an equilibrium quantity but a transport coefficient, and should be determined by the response of moment density to an externally imposed velocity field. In a disordered BEC, static current-current response function consists in the low-frequency, long-wavelength longitudinal $\chi_L(k)$ and transverse $\chi_T(k)$ responses of the system as follows: $\chi_{ij}(k) = \frac{k_i k_j}{E^2(k)} \chi_L(k) + (\delta_{ij} - \frac{k_i k_j}{E^2(k)}) \chi_T(k)$, see details of the definition of $\chi_{ij}$ in Refs. [11, 13]. The transverse response of a BEC is only due to the normal fluid, since the superfluid component can only participate in irrotational flow.

For a disordered BEC trapped in a 2D optical lattice described by Hamiltonian (1), the rotational symmetry is broken, giving rise to an anisotropic system. Consequently, the response function along the unconfined $z$ direction is different from that in the confined $x$-$y$ plane. In this work, we are interested in considering a slow rotation with respect to the $z$ axis. We therefore take the transverse response function along the $z$ direction, given by

$$\rho_n = \lim_{k \to 0} \chi_T(k) = \chi_{zz}(0,0) = \frac{2n}{m} \sum_p \frac{p^2 \epsilon_p}{E^4(k)} |V_k|^2$$

$$= \frac{\tilde{R} \sqrt{2m n_0}}{16 \hbar a^2} I(x),$$  \hspace{1cm} (13)

with $\tilde{R} = n_{imp} a^2/n_0 a_{2D}^2$ and $I(x)$ being a function of a variable $x = x_1 + x_2 = \frac{2(t_1 + t_2)}{g n}$ in the form

$$I(x) = \int_{-\pi}^{\pi} \frac{d^2 k}{(2\pi)^2} \frac{2}{\sqrt{x \gamma + 2(x \gamma + 1 + \sqrt{x \gamma (x \gamma + 2)})}}.$$  \hspace{1cm} (14)

Equation (13) can be interpreted as the second-order term in the perturbation expansion of the normal fluid density in terms of weak disorder $V_k$.

The result of Eq. (13) is plotted in Fig. 3. In the asymptotic 3D limit, one finds $I(x) \simeq 4\sqrt{2}/3 x$, corresponding to the dotted curve in Fig. 3. In such a situation, Equation (13) can recover the corresponding result of 3D Bose gases in Ref. [13]. Equation (13) presents another key result of this paper, providing an analytical expression for the normal fluid density in a Bose fluid in an anisotropic two-dimensional optical lattice with the presence of weak disorder.

Conclusion.— We investigate a 3D disordered BEC trapped in an anisotropic 2D optical lattice characterized by the lattice depths of $V_1$ along $x$-direction and $V_2$ along $y$-direction respectively. Accordingly, the analytical expressions of the ground-state energy, quantum depletion and superfluid density of the system are obtained. Our results show that the step-by-step dimensional crossover along the lattice still induces a characteristic 3D to quasi-2D and 1D step-by-step crossover in the behavior of both equilibrium and non-equilibrium quantum fluctuations. The physics of the step-by-step dimensional crossover is captured by the interplay among three quantities: the strength of an optical lattice, the interaction between bosonic atoms, and the strength of disorder.
All these quantities are experimentally controllable using state-of-the-art technologies. In particular, the depth of an optical lattice can be changed from $0E_R$ to $32E_R$ almost at will [6]. Therefore, the phenomena discussed in this paper should be observable within the current experimental capability. Observing this step-by-step dimensional effect directly would present an important step in revealing the interplay between dimensionality and quantum fluctuations in quasi-low dimension. We remark here that the present work stands in the framework of the Bogoliubov theory. Further consideration of the theoretical framework consists in including the treatment of the system properties for whole range of interatomic interaction strength, from zero to infinity, as well as arbitrarily strong disorder, which are beyond the scope of this work.

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Note added.—Before submitting our work, we notice that a similar work [17] has studied the 2D-1D dimensional crossover. In contrast, our work has focused on the gradual 3D-2D-1D dimensional crossover.

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