Resummation of diagrammatic series with zero convergence radius for strongly correlated fermions

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We demonstrate that summing up series of Feynman diagrams can yield unbiased accurate results for strongly-correlated fermions even when the convergence radius vanishes. We consider the unitary Fermi gas, a model of non-relativistic fermions in three-dimensional continuous space. Diagrams are built from partially-dressed or fully-dressed propagators of single particles and pairs. The series is resummed by a conformal-Borel transformation that incorporates the large-order behavior and the analytic structure in the Borel plane, which are found by the instanton approach. We report highly accurate numerical results for the equation of state in the normal phase, which is necessary to incorporate exactly the two-particle scattering problem. This is done most naturally by using a conformal-Borel transformation that incorporates the large-order behavior and the knowledge of the analytical structure standing behind the series. Combining this new resummation method with diagrammatic Monte Carlo evaluation up to order 9, we obtain new results for the equation of state in the normal phase, which agree with the ultracold-atom experimental data from [25, 26], except for the 4-th virial coefficient for which our data point to the theoretically conjectured value of [34].

In order to have a well-defined diagrammatic framework for the contact interaction in continuous space, it is necessary to incorporate exactly the two-particle scattering problem. This is done most naturally by using the sum of all ladder diagrams \( \Gamma_0 \) as an effective interaction vertex between \( \uparrow \) and \( \downarrow \) fermions, or equivalently, a partially dressed pair-propagator. Diagrammatically,

\[
\Gamma_0 = G_{0,\uparrow} + \cdots + G_{0,\downarrow} + \cdots
\]

where the \( \bullet \) denotes the bare coupling constant and \( G_0 \) is the free fermion propagator, given by \( G_{0,\sigma}(p, \omega) = (i\omega + \mu_\sigma - p^2/2m)^{-1} \) in the momentum Matsubara-frequency representation. Here \( \sigma \in \{\uparrow, \downarrow\} \), \( \mu_\sigma \) is the chemical potential, \( m \) the fermion mass, and \( \omega = 2\nu + \cdots \)
1)π/β with β = 1/(k_B T) the inverse temperature. Γ_0 is well-defined for the continuous-space zero-range interaction (without momentum cutoff) and only depends on the s-wave scattering length a (apart from μ_1, μ_4, β, and external momentum-frequency). The same property holds for higher-order diagrams built from G_0 and Γ_0. This “ladder scheme” is suited to describe the crossover between Fermi and Bose gases, and its lowest-order approximation (left diagram in Fig. 1) is widely used [33, 35, 36]. A diagrammatic Monte Carlo algorithm [13] allows us to stochastically evaluate all Feynman diagrams up to order 9 (see Fig. 1).

An intensive quantity Q, such as pressure or self-energy, can be formally written as a diagrammatic series \( \sum_{N=0}^{\infty} a_N \). Here a_N is a sum of connected diagrams of order N (see Fig. 1). As we shall see this diagrammatic series is divergent, and it is not obvious how to give a meaning to the formal expansion \( Q = \sum_{N=0}^{\infty} a_N \). To do so, we introduce a function \( Q(z) \) whose Taylor series is \( \sum_{N=0}^{\infty} a_N z^N \), and such that \( Q(z = 1) \) is the desired exact physical result. Here z is a formal parameter playing the role of an effective coupling constant. A non-perturbative construction of \( Q(z) \) is realised by introducing the action [37]

\[
S(z) = -\int d^3r \int_0^\beta d\tau \left[ \sum_{\sigma = \uparrow, \downarrow} \bar{\varphi}_\sigma G_{0,\sigma}^{-1} \varphi_\sigma + \bar{\eta} \Gamma_0^{-1} \eta - z \bar{\eta} \Pi_0 \eta + \sqrt{z}(\bar{\eta} \varphi_\uparrow \varphi_\uparrow + \bar{\varphi}_\uparrow \bar{\varphi}_\uparrow \eta) \right] \tag{2}
\]

where \( \varphi_\sigma \) are fermionic Grassmann fields, \( \eta \) is a bosonic complex field, and \( \Pi_0 \) is the particle-particle bubble \( (G_0 \Pi_0 G_0) \), which cancels out all diagrams containing particle-particle bubbles, as required to avoid double-counting. For example, for the pressure we simply have

\[
Q(z) = \lim_{V \to \infty} \frac{1}{\beta V} \ln \int D\varphi D\eta e^{-S(z)[\varphi, \eta]} \tag{3}
\]

with \( V \) the volume.

**Large-order behavior.** We now turn to the crucial problem of computing the large-N behavior of a_N. In the pioneering works [3 9], the large-order behavior for \( \phi^4 \) theory was obtained from a saddle point of the functional integral. To study the large-order behavior of fermionic theories, it was found essential to integrate out fermionic fields, which leads to a purely bosonic functional integral, whose integrand \( e^{-S_{\text{eff}}[\eta]} \) can be estimated in the large-field limit using a Thomas-Fermi (i.e. quasi-local) approximation [33 42]. In our problem, this procedure can be justified by showing that this integrand is an entire function of \( z \). We find that the bosonic action \( S_{\text{eff}}[\eta] \) scales as \( z^{5/4} \int d^3r d\tau |\eta(r, \tau)|^{5/2} \) for large \( |\eta| \). The saddle-point method then gives

\[
a_N = \lim_{N \to \infty} \Gamma(N/5) A^{-N} \text{Re} \exp \left[ i 4 \pi N/5 - U_1 e^{i \pi/5} N^{4/5} + O(N^{3/5}) \right] \tag{4}
\]

where \( U_1 = 5^{1/5} A \) and

\[
A := \frac{1}{\pi^2} \left( \frac{4}{5 \Gamma(3/4)^4} \right)^{15/8} \min_{\eta} \frac{-\langle \eta | \Gamma_0^{-1} | \eta \rangle}{\int d^3r d\tau |\eta(r, \tau)|^{5/2}}. \tag{5}
\]

The fact that \( a_N \) is of order \( (N!)^{1/5} \) immediately implies that the radius of convergence is zero. This raises a fundamental question: Can the exact physical result still be constructed in a unique way from the set \{a_N\}?**

**Resummation.** Given the above asymptotic behavior, it is natural to introduce the generalized Borel transform defined by

\[
B(z) := \sum_{N=0}^{\infty} a_N z^N, \quad |z| < A \tag{6}
\]

\[
\mu_N := \int_0^\infty dt \int t^4 e^{-t^4-b t^4-c t^3} t^N. \tag{7}
\]

Note that \( \mu_N \sim \Gamma(N/5) \exp[-b(N/5)^{4/5}] \) for \( N \to \infty \). The corresponding inverse Borel transformation reads

\[
Q_B(z) := \int_0^\infty dt \int t^4 e^{-t^4-b t^4-c t^3} B(z t) \tag{8}
\]

where \( b \) and \( c \) are free parameters at this stage.

The answer to the above unicity question is then given by the following theorem due to Neveu-Fliess [15 47 69 70]. Let \( W := \{ z \in \mathbb{C} \mid 0 < |z| < R, |\arg z| < \pi/10 + \epsilon \} \), for some \( R > 0 \) and \( \epsilon > 0 \). If

1. \( Q(z) \) is analytical for \( z \in W \)
2. \( \exists A \) and \( C \) such that \( |d^N Q(z)/dz^N|/N! < C A^{-N} (N!)^{1/5} \) for all \( N \geq 0 \) and \( z \in W \)
3. \( a_N = \lim_{z \to 0, z \in W} d^N Q(z)/dz^N / N! \)

then

- \( B(z) \) can be analytically continued for \( z \in \mathbb{R}^+ \)
- \( \exists R' > 0 \) such that \( Q_B(z) = Q(z) \) for \( z \in [0, R'] \).
The hypotheses of this theorem hold in our situation for the following reasons: Hypothesis 1 follows from the functional integral representation \( Q \) and the fact that the integrand, after integrating out the fermions, is an entire function of \( z \) that can be bounded in the large-\( \eta \) limit using the Thomas-Fermi result. Hypothesis 2 can be obtained in a similar way to the large-order behavior of \( a_N \). Hypothesis 3 is plausible given that the functional integral for \( z \in W \) is absolutely convergent.

The problem of resummation is thereby reduced to the one of analytical continuation of the Borel transform \( B(z) \) to the whole real positive axis. To this end, it is essential to know the analytical structure of the one of analytical continuation of the Borel transformation, see Fig. 3. The final resummation procedure to these diagrammatic series. The resulting resummed \( \Sigma \) and \( \Pi \) are then plugged into the Dyson equations to obtain new propagators \( G \) and \( \Gamma \). This cycle is repeated until convergence.

We note that on approach to the superfluid transition, \( \lambda \to 0 \) so that the series becomes increasingly hard to resum, while in the high-temperature limit, \( \lambda \to \infty \) for both ladder and bold schemes) so that the series divergence becomes weaker.

**Numerical results.** In this paper we focus on the central point of the BEC-BCS crossover, the unitary limit, where the dimer binding energy vanishes and the scattering length diverges. This unitary Fermi gas is strongly correlated since the scattering cross-section is on the order of the squared interparticle distance. We report results for the Equation of State (EoS) in the normal phase, restricting for now to the unpolarized gas, \( \mu = \mu_\uparrow = \mu_\downarrow \).

Scale invariance implies that the rescaled density \( n\lambda^3 \) is a universal function of \( \beta\mu \), with \( \lambda = \sqrt{2\pi\hbar^2/\beta m} \) the thermal wavelength.

In the moderately degenerate regime, we find very good convergence of the series as a function of the maximal diagram order \( N_{\text{max}} \) after resummation by the new conformal-Borel transformation, see Fig. 3. The final results for ladder and bold schemes agree within their error bars which are below 0.1%. The value measured at MIT is 2% higher, a deviation within the experimental uncertainty.

Here and in what follows we empirically fixed the free parameter \( b \) such that \( \theta(b) = \pi/10 \) (i.e. \( b = -5^{1/5}U_1 \)). We observed consistent results for different values of the free parameter \( c \), and we adjusted it to optimise the convergence. In Fig. 3 the conformal-Borel transformation was applied to \( Q(z) = n(z) \) with \( c = 12 \) for the ladder scheme, and \( Q(z) = \Sigma(z)/z \) resp. \( \Pi(z)/z \) with \( c = 10 \) for the bold scheme. The error bars shown at each \( N_{\text{max}} \) include the statistical noise coming from the Monte Carlo simulation.

FIG. 2: Conformal mapping: the singularities of the Borel transform (in color) are mapped onto the unit circle. The two points \( z_+ \) are mapped onto \( w_+ \) and the real positive axis is mapped onto the segment \( [0,1] \).
FIG. 3: Resummed density vs. maximal diagram order at $\beta\mu = 0$ ($T/T_F \approx 0.6$). The ladder and bold diagrammatic schemes agree with each other and with experiment.

FIG. 4: Density vs. maximal diagram order at $\beta\mu = 2$ ($T/T_F \approx 0.2$). The bold diagrammatic series is resummed by three variants of the conformal-Borel transformation (see text).

Carlo, and for the bold scheme also the error due to the finite number of iterations. Our final error bars also include errors due to finite $N_{\text{max}}$ and to cutoffs and discretizations in the numerics, so that all sources of errors are taken into account.

At lower temperatures, the ladder scheme is not applicable (due to a pole in $\Gamma_0$) but we still observe convergence of the bold scheme, as shown in Fig. 4 where we cross-check three variants of the conformal-Borel resummation: $Q(z) = \Sigma(z)/z$ resp. $\Pi(z)/z$ with $c = 13$ (circles), the same $Q(z)$ with $c = 60$ (diamonds), and $Q(z) = \Sigma(z)$ resp. $\Pi(z)$ with $c = 60$ (squares). Our final result agrees with the MIT measurement up to a 3% deviation consistent with the experimental uncertainty.

In the related earlier work [49], much simpler resummation methods such as the Lindelöf method were used, assuming that the diagrammatic series has a non-zero convergence radius. This assumption is invalidated by the large-order behavior $|a_N| \sim (N!)^{1/5}$ found here. Hence the results of [49] contained a systematic error. Nevertheless, they deviate from the new results reported here by less than 2 %, which is likely related to the smallness of the exponent 1/5.

The sub-factorial scaling $|a_N| \sim (N!)^{1/5}$ also implies that for a given order $N$, the sum $a_N$ of all diagrams is much smaller than the number $\sim N!$ of diagrams. This is a manifestation of the massive cancellation between different diagrams due to the fermionic sign.

Finally we turn to the higher-temperature regime, where our new high-accuracy data sheds light on a controversy. In the limit $T \gg T_F$, the EoS admits a virial expansion $n_{\text{virial}}^{(3)} = 2 \sum_{j=0}^3 b_j \zeta^j$ in powers of the fugacity $\zeta = e^{\beta\mu}$. The virial coefficient $b_j$ is determined by the $j$-body problem, and is known exactly for $j = 2$ [23, 50] and $j = 3$ [51, 52]. In Figure 5 we subtract the known virial-3 result from our EoS data so that the result tends to $b_4$ in the non-degenerate limit $\zeta \rightarrow 0$. Accordingly we display at $\zeta = 0$ several values reported for $b_4$: The value obtained by Endo and Castin [51] (based on a physically motivated mathematical conjecture) deviates from the values reported by experimentalists from ENS [24] and MIT [20]. The dedicated Path Integral Quantum Monte Carlo result of Yan and Blume [53] has an error bar too large to resolve the discrepancy. Our data suggest that the Endo-Castin result is correct, but requires sufficiently small $\zeta$ to be extracted, and correspondingly high accuracy to resolve the difference $n - n_{\text{virial}}^{(3)} \propto \zeta^4$ (at $\zeta \approx 0.2$ our error on $n\lambda^3$ is $< 0.01\%$), while extrapolations from $\zeta \gtrsim 0.6$ lead to the overestimated $b_4$ values reported in [24, 26]. In other words, at $\zeta \approx 0.6$ ($T/T_F \approx 1$) the unitary Fermi gas is still so strongly correlated that it cannot be reduced to a 4-body problem.
In summary, we found that for the unitary Fermi gas, a strongly correlated fermion model without small expansion parameter, diagrammatic series built on partially or fully dressed propagators can be Borel-resummed and yield accurate unbiased results, even though the convergence radius is zero.

How does this relate to other fermionic theories? For QED, the situation is opposite: Large-order behavior and Borel-summability are still open problems [40, 41, 54, 55] but no resummation is needed in practice because the coupling constant is small. QCD combines both difficulties: It is non-perturbative and probably not Borel-summable [56, 57], which calls for new ideas [58]. The present approach may however be directly generalisable to other continuous-space strongly correlated fermion problems, such as nuclear matter or the electron gas.

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[69] The parameters $b$ and $c$ in the Borel transform are absent from Nevanlinna’s formulation, but we expect that the theorem remains valid in presence of these parameters, since this does not change the leading large-order behavior.
[70] One of the original hypotheses in [45, 47] follows from our hypothesis 2 thanks to Taylor’s theorem with Lagrange remainder [48, 49].
[71] For $\phi^4$ theory, $Q(z)$ is expected to have only one branch cut, namely the real negative axis, with a discontinuity $\sim e^{-A/|z|}$ (the corresponding branch cut for the Borel transform is $(-\infty, -A]$, as used in the conformal mapping of [5]), and in one-dimensional $\phi^4$ theory there is a clear physical interpretation in terms of the tunneling of a quantum particle through the barrier $x^2 + zx^4$ for $z < 0$, whose rate is given by the classical action $A/|z|$. For $\phi^4$ theory in 2 and 3 dimensions, Borel summability was even proven fully rigorously [64, 65].
[72] Restricting to the lowest-order diagram for $\Sigma$ and $\Pi$ ($N_{\text{max}} = 1$ in Figs. 3 and 4) is equivalent to the self-consistent T-matrix approximation of Refs. [66, 67].
[73] Here, $\Sigma(z)$ and $\Pi(z)$ have the Taylor series $\sum_{N=1}^{\infty} \Sigma^N z^N$ and $\sum_{N=1}^{\infty} \Pi^N z^N$, where $\Sigma^N$ and $\Pi^N$ are defined in [48].
[74] Our data for the equation of state is available in the Supplemental Material.
Supplemental Material

The equation of state can be expressed as density vs. chemical potential and temperature, \( n(\mu, T) \). Thanks to scale invariance it reduces to a dimensionless function \( n \lambda^3 = f(\beta \mu) \). Our results are given in Table S1. They were obtained using the ladder scheme for \( \beta \mu < 0 \) and the bold scheme for \( \beta \mu \geq 0 \). For cross-check, we also obtained \( n \lambda^3 = 0.53346(45) \) at \( \beta \mu = -1.5 \) using the bold scheme, and \( n \lambda^3 = 2.9033(26) \) at \( \beta \mu = 0 \) using the ladder scheme.

We went up to diagram orders \( N_{\max} = 10 \) at \( \beta \mu \leq -1 \), \( N_{\max} = 8 \) at \( \beta \mu = 1.5 \), and \( N_{\max} = 9 \) in all other cases.

The conformal-Borel transformation was applied to \( Q(z) = \frac{[n(z) - n(0)]}{z} \) for the ladder scheme, and to \( Q(z) = \Sigma(z)/z \) resp. \( \Pi(z)/z \) for the bold scheme. The values used for the free parameter \( c \) were \( c = 10 \) for the bold scheme at \( 0 \leq \beta \mu \leq 1 \), \( c = 15 \) at \( \beta \mu = -0.5 \) and \( \beta \mu = 1.5 \), \( c = 13 \) at \( \beta \mu = 2 \), \( c = 20 \) at \( \beta \mu = -1 \), and \( c = 12 \) in all other cases.

| \( \beta \mu \) | \( n \lambda^3 \) |
|-----------|----------|
| -1.5      | 0.533477(45) |
| -1       | 0.94442(26)  |
| -0.5     | 1.6735(8)   |
| 0        | 2.9049(26)  |
| 0.5      | 4.821(15)   |
| 1        | 7.54(4)     |
| 1.5      | 11.15(10)   |
| 2        | 15.60(12)   |
| 2.25     | 18.28(22)   |

TABLE S1: Density equation-of-state.