TOTAL POSITIVE CURVATURE AND THE EQUALITY CASE IN THE RELATIVE ISOPERIMETRIC INEQUALITY OUTSIDE CONVEX DOMAINS

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Abstract. We extend a total positive curvature estimate proved by Choe, Ghomi and Ritoré. As an application we settle the case of equality for the relative isoperimetric inequality outside any convex set with not empty interior.

1. Introduction

In [5] Choe, Ghomi and Ritoré proved the following relative isoperimetric inequality outside convex sets, see also [14] for an alternative proof and [13] for a generalization to higher codimension.

Theorem 1.1 ([5]). Let $C \subset \mathbb{R}^N$ be a closed convex set with nonempty interior. For any set of finite perimeter $\Omega \subset \mathbb{R}^N \setminus C$ we have

\[ P(\Omega; \mathbb{R}^N \setminus C) \geq N \left( \frac{\omega_N}{2} \right)^{\frac{1}{N}} |\Omega|^{\frac{N-1}{N}}. \]

Moreover, if $C$ has a $C^2$ boundary and $\Omega$ is a bounded set for which the equality in (1.1) holds, then $\Omega$ is a half ball.

Here and in what follows $P(\Omega; \mathbb{R}^N \setminus C)$ denotes the perimeter of a set $\Omega$ in $\mathbb{R}^N \setminus C$ in the sense of De Giorgi. As observed by the authors in [5] the equality case for general, possibly nonsmooth, convex sets does not follow from their methods as it cannot be handled by a simple approximation argument. However there are many situations in which nonsmooth convex sets naturally appear. For instance, in models of vapor-liquid-solid-grown nanowires the nanotube is often described as a semi-infinite convex cylinder with sharp edges and possibly nonsmooth cross sections. In these models super-saturated liquid droplets correspond to isoperimetric regions for the relative perimeter outside the cylinder or more in general for the capillarity energy, see [12, 17]. Experimentally it is observed that in some regimes preferred configurations are given by spherical caps lying on the top facet of the cylinder. Understanding these phenomena from a mathematical point of view was our first motivation to study the equality cases in (1.1) also for nonsmooth convex obstacles, beside the intrinsic geometric interest of the problem.

The main result of this paper reads as follows.

Theorem 1.2 (The equality case). Let $C \subset \mathbb{R}^N$ be a closed convex set with nonempty interior and let $\Omega \subset \mathbb{R}^N \setminus C$ be a set of finite perimeter such that equality holds in (1.1). Then $\Omega$ is a half ball supported on a facet of $C$.

Observe that, compared to the last part of Theorem 1.1 here we don’t have any restriction on the convex set $C$ and we allow for possibly unbounded competitors. As in [5] an important
tool in order to get the characterization of the equality case in (1.1) is an estimate of the positive total curvature $K^+ (\Sigma)$ of a hypersurface $\Sigma \subset \mathbb{R}^N \setminus C$ when the contact angle between $\partial C$ and $\Sigma$ is larger than or equal to a fixed $\theta \in (0, \pi)$. Here $K^+ (\Sigma)$ denotes, roughly speaking, the measure of the image of the Gauss map restricted to those points where there exists a support hyperplane, see Definition 1.3 below. To state more precisely our result we need to introduce some notation: Given $\theta \in (0, \pi)$ we denote by $S_\theta$ the spherical cap

$$S_\theta := \{ y \in S^{N-1} : y \cdot e_N \geq \cos \theta \}.$$ 

Moreover, given $\Sigma \subset \mathbb{R}^N \setminus C$ and a point $x \in \Sigma$ we denote by $N_x \Sigma$ the normal cone

$$N_x \Sigma = \{ \nu \in S^{N-1} : (y - x) \cdot \nu \leq 0 \text{ for all } y \in \Sigma \},$$

that is the set of (exterior) normals to support hyperplanes to $\Sigma$. We can now recall the definition of total positive curvature.

**Definition 1.3.** Let $C$ be a closed convex set with not empty interior, $\Omega \subset \mathbb{R}^N \setminus C$ a bounded open set and $\Sigma := \partial \Omega \setminus C$. The total positive curvature of $\Sigma$ is given by

$$K^+ (\Sigma) := H^{N-1} \left( \bigcup_{x \in \Sigma \cap C} N_x \Sigma \right).$$

The aforementioned estimate on the total positive curvature is provided by the following theorem, which will be proved in Section 3.

**Theorem 1.4.** Let $C \subset \mathbb{R}^N$ be a closed convex set of class $C^1$, $\Omega \subset \mathbb{R}^N \setminus C$ a bounded open set and $\Sigma := \partial \Omega \setminus C$. Let $\theta_0 \in (0, \pi)$ such that

(1.2) \quad $\nu \cdot \nu_C (x) \leq \cos \theta_0$ \quad whenever $x \in \Sigma \cap C$, $\nu \in N_x \Sigma$,

where $\nu_C (x)$ stands for the outer unit normal to $C$ at $x$. Then,

(1.3) \quad $K^+ (\Sigma) \geq H^{N-1} (S_{\theta_0})$.

Moreover, let $r > 0$ be such that $\Sigma \cap C \subset B_r (0)$. For any $\varepsilon > 0$ there exists $\delta$, depending on $\varepsilon, \theta_0$ and $r$, but not on $C$ or $\Omega$, such that if

(1.4) \quad $\nu \cdot \nu_C (x) \leq \cos \theta_0 + \delta$ \quad whenever $x \in \Sigma \cap C$, $\nu \in N_x \Sigma$,

and

(1.5) \quad $K^+ (\Sigma) \leq H^{N-1} (S_{\theta_0}) + \delta$,

then $\Sigma \cap C$ is not empty, width($\Sigma \cap C$) $\leq \varepsilon$ and more precisely $\Sigma \cap C$ lies between two parallel $\varepsilon$-distant hyperplanes orthogonal to $\nu_C (x)$ for some $x \in \Sigma \cap C$. In particular, if (1.2) is satisfied and the equality in (1.3) holds, then $\Sigma \cap C$ is not empty and lies on a support hyperplane to $C$.

Note that in the previous statement width($\Sigma \cap C$) denotes the distance between the closest pair of parallel hyperplanes which contains $\Sigma \cap C$ in between them, see (3.5). Even though the proof of this theorem follows the general strategy of [4] we are able improve their result in three directions: (1) we consider a general contact angle $\theta_0 \in (0, \pi)$, whereas in [4] only the case $\theta_0 = \pi/2$ is considered; (2) we do not assume any regularity on $\Sigma$ so that the contact angle condition can be expressed in the weak form (1.2); (3) we get a stability estimate on the ‘wet part’ $\Sigma \cap C$ which is independent of the shape of the convex set $C$. As we will explain below (2) and (3) are crucial in the proof of Theorem 1.2.
Figure 1. Both $\Sigma_1 = \partial \Omega_1 \setminus C$ and $\Sigma_2 = \partial \Omega_2 \setminus C$ meet $C$ with contact angle $\pi/4$ and satisfy the equality in (1.3) with $\theta_0 = \pi/4$. Note that $\partial \Omega_1 \cap C$ is a point and $\partial \Omega_2 \cap C$ is a segment.

Observe that the equality case in (1.3) does not imply $\partial \Omega \cap C$ lies on a facet of $C$. In fact it may happen that $\partial \Omega \cap C$ is contained in a convex set of Hausdorff dimension strictly less than $N - 1$, see Figure 1. As a consequence of independent interest of the previous theorem we prove a sharp inequality for the Willmore energy, see Theorem 3.10.

Before outlining our strategy of the proof of Theorem 1.2 we briefly recall how in [5] it is proven that a bounded set $\Omega$ satisfying the equality in (1.1) is a half ball, when $C$ is sufficiently smooth. There the idea is to consider the isoperimetric profile

$$I(m) = \inf \{ P(E; \mathbb{R}^N \setminus C) : E \subset \mathbb{R}^N \setminus C, |E| = m \},$$

defined for all $m \in (0, |\Omega_0|]$, and to show that $I(m) = N(\frac{\omega_N}{2}) \frac{1}{2} m^{\frac{N-1}{N}}$, that is $I(m)$ coincides with the isoperimetric profile $I_{\mathcal{H}}(m)$ of the half space. In particular $I'(|\Omega_0|) = I'_{\mathcal{H}}(|\Omega_0|)$ and this in turn implies that $\Sigma_0 := \partial \Omega_0 \setminus C$ satisfies the equality in (1.3) for $\theta_0 = \pi/2$. However in the latter argument it is crucial that the regular part of $\Sigma_0$ meets $\partial C$ orthogonally and in a $C^2$ fashion, see [5, Lemma 3.1]. Indeed the boundary regularity theory for perimeter minimizers only works if $C$ is sufficiently smooth and cannot be applied in the case of an arbitrary convex set.

In order to deal with this lack of regularity our strategy is to approximate $C$ with more regular convex sets and to take advantage of the fact that the geometric information provided by our Theorem 1.4 is more quantitative. Of course there are also subtle regularity issues to be dealt with in this approximation procedure.

Let us describe the argument more in detail. For $\eta > 0$ sufficiently small we consider the closed $\eta$-neighborhood $\Omega_\eta = C + B_\eta(0)$ and introduce the following approximate constrained isoperimetric profiles

$$(1.6) \quad I_\eta(m) = \min \{ P(E; \mathbb{R}^N \setminus C_\eta) : E \subset \Omega_0 \setminus C_\eta, |E| = m \}$$

for all $m \in (0, |\Omega_0 \setminus C_\eta|]$. Denote by $\Omega_{\eta,m}$ a minimizer of the above problem and set $\Sigma_{\eta,m} := \partial \Omega_{\eta,m} \setminus C_\eta$. An important step in the proof is to show that $I_\eta(m) \to I_{\mathcal{H}}(m)$ as $\eta \to 0$ for every $m \in (0, |\Omega_0|)$. The proof of this fact that relies on the inequality

$$(1.7) \quad K^+(\Sigma_{\eta,m}) \geq \mathcal{H}^{N-1}(S_{\pi/2}) = \frac{1}{2} N \omega_N$$
which in turn holds, thanks to Theorem 1.4, provided
\begin{equation}
\nu \cdot \nu_{C_\eta}(x) \leq 0 \quad \text{whenever } x \in \Sigma_{\eta,m} \cap C_\eta, \quad \nu \in N_x \Sigma_{\eta,m}.
\end{equation}
Note that due to the presence of the obstacle \( \Omega_0 \) with a possibly nonempty singular set, we cannot apply the known boundary regularity results at the points \( x \in \partial \Omega_{\eta,m} \cap \partial C_\eta \cap \partial \Omega_0 \) to prove that at those points \( \Sigma_{\eta,m} \) meets \( \partial C_\eta \) orthogonally and in a sufficiently smooth way. Nevertheless we are able to prove that the weak Young's law (1.8) still holds regardless of the regularity of \( \Sigma_{\eta,m} \) and \( \partial \Omega_0 \). This allows us to apply Theorem 1.4 to deduce (1.7) and, with some extra effort, that \( I_\eta \to I_{\eta,0} \). Note that for this argument to work it is crucial that in our version of the geometric Theorem 1.4 the classical contact angle condition is replaced by the weak form (1.2).

With the convergence of the isoperimetric profiles \( I_\eta \) at hand we can then prove that for a.e. \( m \in (0,|\Omega_0|) \)
\begin{equation}
\mathcal{K}^+(\Sigma_{\eta,m}) \to \frac{1}{2} N \omega_N,
\end{equation}
and thus \( \Sigma_{\eta,m} \) satisfies (1.5) for \( \eta \) sufficiently small. Thanks to the last part of Theorem 1.4 we may then infer that \( \Sigma_{\eta,m} \cap C_\eta \) is almost flat and, passing to the limit, that \( \Sigma_0 \cap C \) lies on a hyperplane, and more precisely that \( \partial \Omega_0 \cap C \) lies on a facet of \( C \). Here it is crucial that the stability estimate on the width of \( \Sigma_{\eta,m} \cap C_\eta \) provided by Theorem 1.4 is independent of the shape of the convex set \( C_\eta \). Having established that the wet part \( \partial \Omega_0 \cap C \) is flat, some extra work is still needed to deduce that \( \Omega_0 \) is a half ball.

2. Preliminaries

Throughout the paper we denote by \( B_r(x) \) the ball in \( \mathbb{R}^N \) of center \( x \) and radius \( r > 0 \). In the following we shall often deal with sets of finite perimeter. For the definition and the basic properties of sets of (locally) finite perimeter we refer to the books [3, 15]. Here we fix some notation for later use. Given \( E \subset \mathbb{R}^N \) of locally finite perimeter and a Borel set \( G \) we denote by \( P(E;G) \) the perimeter of \( E \) in \( G \). The reduced boundary of \( E \) will be denoted by \( \partial^* E \), while \( \partial E \) will stand for the essential boundary defined as
\begin{equation}
\partial^* E := \mathbb{R}^N \setminus (E^{(0)} \cup E^{(1)}),
\end{equation}
where \( E^{(0)} \) and \( E^{(1)} \) are the sets of points where the density of \( E \) is 0 and 1, respectively. Moreover, we denote by \( \nu_E \) the generalized exterior normal to \( E \), which is well defined at each point of \( \partial^* E \), and by \( \mu_E \) the Gauss-Green measure associated to \( E \)
\begin{equation}
\mu_E := \nu_E \mathcal{H}^{N-1} \llcorner \partial^* E.
\end{equation}
In the following, when dealing with a set of locally finite perimeter \( E \), we shall always tacitly assume that \( E \) coincides with a precise representative that satisfies the property \( \partial E = \overline{\partial^* E} \), see [15] Remark 16.11. A possible choice is given by \( E^{(1)} \) for which one may easily check that
\begin{equation}
\partial E^{(1)} = \overline{\partial^* E}.
\end{equation}

We recall the well known notion of perimeter \((\Lambda, r_0)\)-minimizer and the main properties which will be used here.

**Definition 2.1.** Let \( \Omega \subset \mathbb{R}^N \) be an open set. We say that a set of locally finite perimeter \( E \subset \mathbb{R}^N \) is a perimeter \((\Lambda, r_0)\)-minimizer in \( \Omega \), \( \Lambda \geq 0 \) and \( r_0 > 0 \), if for any ball \( B_r(x_0) \subset \Omega \), with \( 0 < r \leq r_0 \) and any \( F \subset \mathbb{R}^N \) such that \( E \Delta F \subset B_r(x_0) \) we have
\begin{equation}
P(E; B_r(x_0)) \leq P(F; B_r(x_0)) + \Lambda |E \Delta F|.
\end{equation}
In order to state a useful compactness theorem for \( \Lambda \)-minimizers we recall that a sequence \( \{C_n\} \) of closed sets converge in the Kuratowski sense to a closed set \( C \) if the following conditions are satisfied:

(i) if \( x_n \in C_n \) for every \( n \), then any limit point of \( \{x_n\} \) belongs to \( C \);
(ii) any \( x \in C \) is the limit of a sequence \( \{x_n\} \) with \( x_n \in C_n \).

One can easily see that \( C_n \to C \) in the sense of Kuratowski if and only if \( \text{dist}(\cdot, C_n) \to \text{dist}(\cdot, C) \) locally uniformly in \( \mathbb{R}^N \). In particular, by the Arzelà-Ascoli Theorem any sequence of closed sets admits a subsequence which converge in the sense of Kuratowski.

Throughout the paper, with a common abuse of notation, we write \( E_h \to E \) in \( L^1(\mathbb{R}^N) \) instead of \( \chi_{E_h} \to \chi_E \) in \( L^1(\mathbb{R}^N) \). Moreover, given a sequence of Radon measures \( \mu_h \) in an open set \( \Omega \), we say that \( \mu_h \rightharpoonup^* \mu \) weakly* in \( \Omega \) in the sense of measures if

\[
\int_\Omega \varphi \, d\mu_h \to \int_\Omega \varphi \, d\mu \quad \text{for all } \varphi \in C_0^\infty(\Omega).
\]

Next theorem is a well known result, see for instance [15, Ch. 21].

**Theorem 2.2.** Let \( \Omega \subset \mathbb{R}^N \) be an open set and \( \{E_n\} \) a sequence of locally finite perimeter sets contained in \( \Omega \) satisfying the following property: there exists \( r_0 > 0 \) such that for every \( n \), \( E_n \) is a perimeter \( (\Lambda_n, r_0) \)-minimizer in \( \Omega \), with \( \Lambda_n \to \Lambda \in [0, +\infty) \). Then there exist \( E \subset \Omega \) of locally finite perimeter and a subsequence \( \{n_k\} \) such that

(i) \( E \) is a \( (\Lambda, r_0) \)-minimizer in \( \Omega \);
(ii) \( E_{n_k} \to E \) in \( L^1(\mathbb{R}^N) \),
(iii) \( \partial E_{n_k} \to C \) in the Kuratowski sense for some closed set \( C \) such that \( C \cap \Omega = \partial E \cap \Omega \);
(iv) \( H^{N-1}(\partial E_{n_k} \cap \Omega) \rightharpoonup^* H^{N-1}(\partial E \cap \Omega) \) weakly* in \( \Omega \) in the sense of measures.

**Remark 2.3.** From the definition of Kuratowski convergence it is not difficult to see that (ii) and (iii) of Theorem 2.2 imply that, up to extracting a further subsequence if needed, \( E_{n_k} \to K \) in the sense of Kuratowski, with \( K \cap \Omega = E \cap \Omega \).

**Definition 2.4.** Given a set of locally finite perimeter \( E \), we say that a function \( h \in L^1(\partial^* E) \) is the weak mean curvature of \( E \) if for any vector field \( X \in C^1(\mathbb{R}^N ; \mathbb{R}^N) \) we have

\[
\int_{\partial^* E} \text{div}_E X \, dH^{N-1} = \int_{\partial^* E} h \, X \cdot \nu_E, dH^{N-1},
\]

where \( \text{div}_E X := \text{div} X - (\partial_E X) \cdot \nu_E \) stands for the tangential divergence of \( X \) along \( \partial^* E \). If such an \( h \) exists we will denote it by \( H_{\partial E} \).

Note that if \( \partial E \) is of class \( C^2 \) then \( H_{\partial E} \) coincides with the classical mean curvature, or more precisely with the sum of all principal curvatures. In particular, if \( E \) coincides locally with the subgraph of a function \( u \) of class \( C^2 \) then locally

\[
H_{\partial E} = -\text{div}\left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).
\]

Concerning the above mean curvature operator, we recall the following useful Strong Maximum Principle, see for instance [18, Th. 2.3], which covers a more general class of quasilinear equations.
Theorem 2.5. Let \( \Omega \subset \mathbb{R}^{N-1} \) be an open set and let \( u, v \in C^2(\Omega) \) such that \( u \leq v \) and
\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \lambda = \text{div} \left( \frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right)
\]
for some constant \( \lambda \in \mathbb{R} \). If \( u(x_0) = v(x_0) \) for some \( x_0 \in \Omega \), then \( u \equiv v \).

We recall the following classical regularity result for \( \Lambda \)-minimizers.

Theorem 2.6. Let \( E \) be a perimeter \((\Lambda, r_0)\)-minimizer in some open set \( \Omega \subset \mathbb{R}^N \). Then
(i) \( \partial^* E \cap \Omega \) is a hypersurface of class \( C^{1,\alpha} \) for every \( \alpha \in (0,1) \), relatively open in \( \partial E \cap \Omega \).
Moreover, \( \text{dim}_H((\partial E \setminus \partial^* E) \cap \Omega) \leq N - 8 \), where \( \text{dim}_H \) stands for the Hausdorff dimension;
(ii) \( H_{\partial E} \in L^\infty(\partial^* E \cap \Omega) \), with \( ||H_{\partial E}||_{L^\infty} \leq \Lambda \), and thus \( \partial^* E \cap \Omega \) is of class \( W^{2,p} \) for all \( p \geq 1 \);
(iii) if there exists a \( C^1 \) hypersurface \( \Sigma \) touching \( \partial E \) at \( x \in \Omega \) and lying on one side with respect to \( \partial E \) in a neighborhood of \( x \), then \( x \in \partial^* E \).

Items (i) and (ii) are classical, see for instance Theorems 21.8 and 28.1 in [15] for (i) and Theorem 4.7.4 in [2] for (ii).

Concerning (iii) one can show that under the assumption on \( x \) the minimal cone obtained by blowing up \( E \) around \( x \) is contained in a half space. For the existence of such a minimal cone see Theorem 28.6 in [15]. Since any minimal cone contained in a half space is a half space, see for instance [8, Lemma 3], it follows that \( x \) is a regular point.

The so-called \( \varepsilon \)-regularity theory for \( \Lambda \)-minimizers underlying the proof of the above theorem yields that sequences of \( \Lambda \)-minimizers \( E_h \) converging in \( L^1 \) to a smooth set \( E \) are regular for \( h \) large and in fact converge in a stronger sense. More precisely, we have the following result, which is well known to the experts.

Theorem 2.7. Let \( E_n, E \) be \((\Lambda, r_0)\)-minimizers in an open set \( \Omega \subset \mathbb{R}^N \) such that \( E_n \to E \) in \( L^1_{\text{loc}}(\Omega) \). Let \( x \in \partial^* E \cap \Omega \). Then, up to rotations and translations, there exist a \((N-1)\)-dimensional open ball \( B' \subset \mathbb{R}^{N-1} \), functions \( \varphi_n, \varphi \in W^{2,p}(B') \) for all \( p \geq 1 \), and \( r > 0 \) such that \( x \in B' \times (-r, r) \) and for \( n \) large
\[
\partial E_n \cap (B' \times (-r, r)) = \{(x', \varphi_n(x')) : x' \in B'\},
\]
\[
\partial E \cap (B' \times (-r, r)) = \{(x', \varphi(x')) : x' \in B'\},
\]
\[
\varphi_n \to \varphi \quad \text{in } C^{1,\alpha}(\overline{B'}) \text{ for some } \alpha \in (0,1).
\]
Moreover, \( H_{\partial E_n}(x', \varphi_n(x')) \xrightarrow{\ast} H_{\partial E}(x', \varphi(x')) \) in \( L^\infty(B') \) and thus \( \varphi_n \rightharpoonup \varphi \) in \( W^{2,p}(B') \) for all \( p \geq 1 \).

Properties stated in (2.3) follow from the classical \( \varepsilon \)-regularity theory, see [21, Th. 1.9] (see also the arguments of Lemma 3.6 in [8]). The last part of the statement then easily follows from Theorem 2.6 (ii) combined with the classical Calderón-Zygmund estimates.

3. An estimate of the total positive curvature

This section is mainly devoted to the proof of Theorem 1.4 and to some applications.

We recall that a set \( X \subset S^{N-1} \) is called spherically convex (in short convex) if it is geodesically convex, that is, for any pair of points \( x_1, x_2 \in X \) there exists a distance minimizing geodesic connecting \( x_1 \) and \( x_2 \) contained in \( X \).
If \( x \in S^{N-1} \) and \( \theta \in (0, \pi) \) we denote by \( S_{\theta,x} \) the spherical cap
\[
S_{\theta,x} := \{ y \in S^{N-1} : x \cdot y \geq \cos \theta \}.
\]
If \( x = e_N \) we shall simply write \( S_\theta \) instead of \( S_{\theta,e_N} \). Note that \( S_{\pi-\theta,-x} \) coincides with \((S^{N-1} \setminus S_{\theta,x}) \cup \partial S_{\theta,x}\), where \( \partial S_{\theta,x} \) denotes the relative boundary of \( S_{\theta,x} \) in \( S^{N-1} \). We recall that
\[
\mathcal{H}^{N-1}(S_\theta) = (N-1)\omega_{N-1} \int_0^\theta \sin^{N-2} \sigma \ d\sigma,
\]
and \( \mathcal{H}^{N-1}(S^{N-1}) = N\omega_N \), where \( \omega_N \) is the measure of the unit ball.

The following lemma extends [4, Proposition 3.1] to general angles.

**Lemma 3.1.** Let \( X \subset S^{N-1} \) be spherically convex and closed, with \( \mathcal{H}^{N-1}(X) > 0 \), let \( \theta \in (0, \pi) \) and fix \( x \in X \). Then we have
\[
\mathcal{H}^{N-1}(X \cap S_{\theta,x}) \geq \frac{\mathcal{H}^{N-1}(S_\theta)}{N\omega_N} \mathcal{H}^{N-1}(X).
\]
Moreover, the equality holds if and only if \( -x \in X \). Finally, given \( \theta_0 \in (0, \pi) \), for every \( \varepsilon > 0 \) there exists \( \delta > 0 \), independent of \( X \), such that if \( \theta \in [\theta_0/2, \theta_0] \), then
\[
\mathcal{H}^{N-1}(X \cap S_{\theta,x}) \leq \left( \frac{\mathcal{H}^{N-1}(S_\theta)}{N\omega_N} + \delta \right) \mathcal{H}^{N-1}(X) \quad \text{implies} \quad \text{dist}(-x, X) \leq \varepsilon.
\]

**Proof.** We denote by \( A \) the spherically convex subset of \( S_{\theta,x} \) obtained by taking the union of all the minimal geodesics connecting \( x \) with the points of \( X \cap \partial S_{\theta,x} \). Let \( B := S_{\theta,x} \setminus A \). Similarly denote by \( A^- \) the spherically convex subset of \( S_{\pi-\theta,-x} \) obtained by taking the union of all the minimal geodesics connecting \(-x\) with the points of \( X \cap \partial S_{\theta,x} \), and set \( B^- := S_{\pi-\theta,-x} \setminus A^- \).

Assume first that \( \mathcal{H}^{N-1}(A) > 0 \). We note that
\[
\frac{\mathcal{H}^{N-1}(A^-)}{\mathcal{H}^{N-1}(A)} = \frac{\mathcal{H}^{N-1}(S_{\pi-\theta,-x})}{\mathcal{H}^{N-1}(S_{\theta,x})}.
\]
Thus, we have
\[
\mathcal{H}^{N-1}(X \cap A) = \mathcal{H}^{N-1}(A) = \frac{\mathcal{H}^{N-1}(S_{\theta,x})}{\mathcal{H}^{N-1}(S_{\pi-\theta,-x})} \mathcal{H}^{N-1}(A^-) \geq \frac{\mathcal{H}^{N-1}(S_\theta)}{N\omega_N} \mathcal{H}^{N-1}(X \cap A^-).
\]
Note now that \( X \cap B^- = \emptyset \). Indeed, if \( y \in X \cap B^- \), then the geodesic connecting \( y \) to \( x \) is contained in \( X \) and intersects \( \partial S_{\theta,x} \) at a point \( z \in X \cap \partial S_{\theta,x} \). It follows in turn that \( y \) belongs to the geodesic connecting \( z \) with \(-x\), and thus \( y \in A^- \), which is a contradiction. Therefore,
\[
\mathcal{H}^{N-1}(X \cap S_{\theta,x}) = \mathcal{H}^{N-1}(A) + \mathcal{H}^{N-1}(X \cap B) \geq \mathcal{H}^{N-1}(A)
\]
\[
= \frac{\mathcal{H}^{N-1}(S_{\theta,x})}{\mathcal{H}^{N-1}(S_{\pi-\theta,-x})} \mathcal{H}^{N-1}(A^-)
\]
\[
\geq \frac{\mathcal{H}^{N-1}(S_\theta)}{N\omega_N} \mathcal{H}^{N-1}(X \cap S_{\pi-\theta,-x}).
\]
From this inequality (3.1) follows, recalling that \( \mathcal{H}^{N-1}(S^{N-1}) = N\omega_N \).

If instead \( \mathcal{H}^{N-1}(A) = 0 \), then \( \mathcal{H}^{N-1}(X \setminus S_{\theta,x}) = 0 \) and thus (3.1) holds trivially.

If (3.1) holds with the equality, then \( \mathcal{H}^{N-1}(A) > 0 \) and all the inequalities in (3.2) are equalities. In particular, \( \mathcal{H}^{N-1}(A^-) = \mathcal{H}^{N-1}(X \cap S_{\pi-\theta,-x}) > 0 \). In turn, by closedness and
convexity we deduce that \(-x \in X\). Conversely, if \(-x \in X\) then by spherical convexity we have \(A^- = X \cap S_{\pi - \theta, -x}\) and also \(X \cap B = \emptyset\) since otherwise any geodesic connecting a point \(y \in X \cap B\) to \(-x\) would intersect \(\partial S_{\theta, x} \cap X\), thus implying that \(y\) belongs to \(A\), a contradiction. Therefore all the inequalities in (3.2) are equalities and the conclusion follows.

To establish the last part, we argue by contradiction assuming that there exist \(\varepsilon > 0\), a sequence of closed spherically convex sets \(X_n \ni x\) such that \(\mathcal{H}^{-1}(X_n) > 0\) and a sequence \(\theta_n \in [\theta_0/2, \theta]\) converging to \(\theta'\) such that

\[
\mathcal{H}^{-1}(X_n \cap S_{\theta_n, x}) \leq \left(\frac{\mathcal{H}^{-1}(S_{\theta_n})}{\mathcal{H}^{-1}(X_n) + \frac{1}{n}}\right) \mathcal{H}^{-1}(X_n) \quad \text{but} \quad \text{dist}(-x, X_n) \geq \varepsilon.
\]

We denote by \(A_n\) and by \(A_n^-\) the sets corresponding to \(X_n\) and \(S_{\theta_n, x}\) defined as above. Note that \(X_n = (X_n \cap S_{\theta_n, x}) \cup (X_n \cap A_n^-)\). From (3.3) it follows that \(\mathcal{H}^{-1}(A_n), \mathcal{H}^{-1}(A_n^-) > 0\) for \(n\) large and

\[
\frac{\mathcal{H}^{-1}(X_n \cap S_{\theta_n, x})}{\mathcal{H}^{-1}(X_n \cap A_n^-)} \leq \frac{\mathcal{H}^{-1}(S_{\theta_n})}{\mathcal{H}^{-1}(X_n) + \frac{1}{n}} + O\left(\frac{1}{n}\right).
\]

Since \(\mathcal{H}^{-1}(X_n \cap S_{\theta_n, x}) \geq \mathcal{H}^{-1}(A_n)\) and \(\mathcal{H}^{-1}(X_n \cap A_n^-) \geq \mathcal{H}^{-1}(S_{\pi - \theta_n, -x})\mathcal{H}^{-1}(X_n \cap A_n^-)\), it follows that

\[
\lim_{n \to \infty} \frac{\mathcal{H}^{-1}(X_n \cap S_{\theta_n, x})}{\mathcal{H}^{-1}(X_n \cap S_{\pi - \theta_n, -x})} = \lim_{n \to \infty} \frac{\mathcal{H}^{-1}(A_n)}{\mathcal{H}^{-1}(X_n \cap A_n^-)} = \frac{\mathcal{H}^{-1}(S_{\theta', x})}{\mathcal{H}^{-1}(S_{\pi - \theta', -x})}.
\]

Note that we have

\[
\frac{\mathcal{H}^{-1}(S_{\theta_n, x})}{\mathcal{H}^{-1}(S_{\pi - \theta_n, -x})} = \frac{\mathcal{H}^{-1}(A_n)}{\mathcal{H}^{-1}(X_n \cap A_n^-)} \rightarrow \frac{\mathcal{H}^{-1}(S_{\theta', x})}{\mathcal{H}^{-1}(S_{\pi - \theta', -x})}
\]

and thus, from (3.4) we get

\[
\lim_{n \to \infty} \frac{\mathcal{H}^{-1}(A_n^-)}{\mathcal{H}^{-1}(X_n \cap A_n^-)} = 1,
\]

which clearly contradicts the fact that by the second inequality in (3.3) we easily infer that \(\mathcal{H}^{-1}(A_n^- \setminus X_n) \geq C(\varepsilon)\mathcal{H}^{-1}(A_n^-)\), for a positive constant \(C(\varepsilon)\) depending only in \(\varepsilon\).

Next we adapt to our case [4, Proposition 4.2]. To this aim we recall some preliminary definitions.

**Definition 3.2.** Given a set \(X \subset \mathbb{R}^N\) and \(x \in \mathbb{R}^N\) the unit normal cone of \(X\) at \(x\) is the (possibly empty) set defined as

\[
N_x X := \{\nu \in S^{N-1} : (y-x) \cdot \nu \leq 0 \text{ for all } y \in X\}.
\]

Any hyperplane passing through \(x\) and orthogonal to a direction \(\nu \in N_x X\) is called a support hyperplane for \(X\) with outward normal \(\nu\). In turn, we define the corresponding normal bundle of \(X\) as

\[
N X := \bigcup_{x \in X} N_x X.
\]

Given a map \(\sigma : X \to S^{N-1}\) and \(\theta \in (0, \pi)\) we introduce the following restricted normal cone and restricted normal bundle respectively as

\[
N_{\sigma, x}^\theta X := N_x X \cap S_{\theta, \sigma(x)} \quad \text{and} \quad N_{\sigma, \theta} X := \bigcup_{x \in X} N_{\sigma, x}^\theta X.
\]
Moreover, we say that a point \( x \in X \) is exposed if there exists a support hyperplane \( \Pi \) passing through \( x \) such that \( X \cap \Pi = \{x\} \). Finally, we denote by \( \text{width}(X) \) the distance between the closest pair of parallel hyperplanes which contains \( X \) in between them, i.e.,

\[
(3.5) \quad \text{width}(X) = \inf_{\nu \in S^{N-1}} \left( \sup \{ x \cdot \nu : x \in X \} - \inf \{ x \cdot \nu : x \in X \} \right).
\]

**Lemma 3.3.** Let \( r > 0 \) and let \( X = \{x_1, \ldots, x_k\} \subset B_r(0) \). Let \( \sigma : X \to S^{N-1} \) be such that \( \sigma(x_i) \in N_{x_i}X \) whenever \( N_{x_i}X \) is nonempty. Then

\[
(3.6) \quad \mathcal{H}^{N-1}(N_{\sigma(x)}X) \geq \mathcal{H}^{N-1}(S_\theta).
\]

Moreover, equality holds in (3.6) if and only if \( X \) lies in a hyperplane \( \Pi \) such that \( \sigma(x_i) \perp \Pi \) whenever \( x_i \) is exposed. Finally, given \( \theta_0 \in (0, \pi) \), for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) (depending also on \( r > 0 \) and \( \theta_0 \), but not on \( \sigma \) and not on \( X \)) such that if \( \theta \in [\theta_0/2, \theta_0] \), then

\[
(3.7) \quad \mathcal{H}^{N-1}(N_{\sigma(x)}X) \leq \mathcal{H}^{N-1}(S_\theta) + \delta \quad \text{implies} \quad \text{width}(X) \leq \varepsilon
\]

and more precisely there exist an exposed point \( x \in X \) and two parallel hyperplanes orthogonal to \( \sigma(x) \) with mutual distance equal to \( \varepsilon \) such that \( X \) lies between them.

**Proof.** The proof is essentially the same as for [4, Proposition 4.2], using Lemma 3.1 in place of [4, Proposition 3.1]. We give the argument for the sake of completeness. Owing to the compactness of \( X \), for every \( \nu \in S^{N-1} \) there exists a support hyperplane to \( X \) with outward normal equal to \( \nu \). Thus, \( N X = S^{N-1} \). Observe also that \( \nu \in \text{int}_{S^{N-1}}(N_{x_i}X) \) if and only if the hyperplane orthogonal to \( \nu \) and passing through \( x_i \) is a support hyperplane intersecting \( X \) only at \( x_i \) (and thus \( x_i \) is exposed). In turn, if \( i \neq j \) we have

\[
\text{int}_{S^{N-1}}(N_{x_i}X) \cap \text{int}_{S^{N-1}}(N_{x_j}X) = \emptyset.
\]

Since by [4, Lemma 4.1] every \( N_{x_i}X \) with nonvanishing \( \mathcal{H}^{N-1} \)-measure is spherically convex, we may invoke Lemma 3.1 to conclude that

\[
(3.8) \quad \mathcal{H}^{N-1}(N_{\sigma(x)}X) = \sum_{i=1}^{k} \mathcal{H}^{N-1}(N_{x_i}X) \geq \frac{\mathcal{H}^{N-1}(S_\theta)}{N \omega_N} \sum_{i=1}^{k} \mathcal{H}^{N-1}(N_{x_i}X) = \mathcal{H}^{N-1}(S_\theta),
\]

thus establishing (3.6).

If equality holds in (3.6), then the above inequality is an equality and in particular

\[
(3.9) \quad \mathcal{H}^{N-1}(N_{x_i}X) = \frac{\mathcal{H}^{N-1}(S_\theta)}{N \omega_N} \mathcal{H}^{N-1}(N_{x_i}X)
\]

whenever \( \mathcal{H}^{N-1}(N_{x_i}X) > 0 \), that is whenever \( x_i \) is exposed. Therefore, by Lemma 3.1 \( N_{x_i}X \) contains both \( \sigma(x_i) \) and \( -\sigma(x_i) \) and thus \( X \) lies in the hyperplane orthogonal to \( \sigma(x_i) \) and passing through \( x_i \). Conversely, if \( X \) lies in a hyperplane orthogonal to \( \sigma(x_i) \), for every \( x_i \) exposed, then also \( -\sigma(x_i) \in N_{x_i}X \) and thus by Lemma 3.1 (3.9) holds for all \( x_i \) exposed. And thus equality holds also in (3.8).

To prove (3.7) and the last part of the lemma, let \( X_n = \{x^n_1, \ldots, x^n_k\} \subset B_r(0) \) and let \( \sigma_n : X_n \to S^{N-1} \), with \( \sigma_n(x^n_i) \in N_{x^n_i}X_n \) whenever \( x^n_i \) is exposed, \( \theta_n \in [\theta_0/2, \theta_0] \) be such that

\[
\mathcal{H}^{N-1}(N_{\sigma_n(x^n_i)X_n}) - \mathcal{H}^{N-1}(S_{\theta_n}) \to 0.
\]
Arguing as for (3.8) we then have, in particular, that for every $n \in \mathbb{N}$ there exists $i_n \in \{1, \ldots, k_n\}$ such that
\[
\frac{\mathcal{H}^{N-1}(N_{x_{i_n}^n, \theta} X_n)}{\mathcal{H}^{N-1}(N_{x_{i_n}^n} X_n)} - \frac{\mathcal{H}^{N-1}(S_{\theta_n})}{N \omega_N} \to 0.
\]
By Lemma 3.1 this implies that $\text{dist}(\sigma_n(x_{i_n}^n), N_{x_{i_n}^n} X_n) \to 0$. From this, owing to the equi-boundedness of the $X_n$’s it follows that for every $k \in \mathbb{N}$ and for $n$ large enough $X_n$ lies between the two parallel hyperplanes orthogonal to $\sigma_n(x_{i_n}^n)$ and passing through the points $x_{i_n}^n$ and $x_{i_n}^n - \frac{1}{k} \sigma_n(x_{i_n}^n)$. In particular, $\text{width}(X_n) \to 0$. \hfill \Box

Next proposition extends the previous lemma to the case of a general compact set $X$ and a continuous map $\sigma$.

**Proposition 3.4.** Let $X \subset B_r(0)$ be a compact set. Let $\sigma : X \to \mathbb{S}^{N-1}$ be a continuous map such that $\sigma(x) \in N_x X$ for all $x \in X$ such that $N_x X \neq \emptyset$. Then,
\[
\mathcal{H}^{N-1}(N^{\sigma, \theta} X) \geq \mathcal{H}^{N-1}(S_{\theta})
\]
and if equality holds, then $X$ lies in a hyperplane $\Pi$ which is orthogonal to $\sigma(x)$ for some $x \in X$. Moreover, given $\theta_0 \in (0, \pi)$ and $\varepsilon > 0$ there exists $\delta_0 > 0$ (depending also on $r$ and $\theta_0$, but not on $\sigma$ and not on $X$) such that if $\theta \in [\theta_0/2, \theta_0]$, then
\[
\mathcal{H}^{N-1}(N^{\sigma, \theta} X) \leq \mathcal{H}^{N-1}(S_{\theta}) + \delta_0 \text{ implies \ width}(X) \leq \varepsilon
\]
and more precisely there exist $x \in X$ and two parallel hyperplanes orthogonal to $\sigma(x)$, with mutual distance equal to $\varepsilon$ such that $X$ lies between them.

**Proof.** Let $\{X_i\}_{i \in \mathbb{N}}$ be an increasing sequence of discrete subsets of $X$ such that $X_i \to X$ in the Hausdorff sense. We claim that
\[
\chi_{N^{\sigma, \theta} X} \geq \limsup_i \chi_{N^{\sigma, \theta} X_i} \text{ pointwise in } \mathbb{S}^{N-1}.
\]
To this aim let $\nu \notin N^{\sigma, \theta} X$ and assume by contradiction that (3.12) does not hold at $\nu$ and thus that there exist a subsequence $\{x_n\}$ such that $\nu \in N_{x_n}^{\sigma, \theta} X_n$. Passing to a further (not relabelled) subsequence if needed, we may assume that $x_n \to \bar{x} \in X$. Observe that by the continuity of $\sigma(\cdot)$, $\nu \in S_{\theta, \sigma(\bar{x})}$. Fix now any $x \in X$ and due to the Hausdorff convergence find $y_n \in X_{i_n}$ such that $y_n \to x$. Since for every $n$, $(y_n - x_n) \cdot \nu \leq 0$ passing to the limit we get $(x - \bar{x}) \cdot \nu \leq 0$. Due to the arbitrariness of $x$, we have shown that $\nu \in N_{\bar{x}}^{\sigma, \theta} X$ and thus $\nu \in N^{\sigma, \theta} X$, a contradiction.

Using the first part of Lemma 3.3 (with $X$ replaced by $X_i$), (3.12) and Fatou’s Lemma we get
\[
\mathcal{H}^{N-1}(N^{\sigma, \theta} X) \geq \limsup_i \mathcal{H}^{N-1}(N^{\sigma, \theta} X_i) \geq \liminf_i \mathcal{H}^{N-1}(N^{\sigma, \theta} X_i) \geq \mathcal{H}^{N-1}(S_{\theta}).
\]
Assume now that the first inequality (3.11) holds for some $\theta \in [\theta_0/2, \theta_0]$, with $\delta_0 = \frac{\delta}{2}$, where $\delta$ is the constant provided by Lemma 3.3. Then the previous inequality yields for $i$ sufficiently large, depending on $\theta$,
\[
\mathcal{H}^{N-1}(N^{\sigma, \theta} X_i) \leq \mathcal{H}^{N-1}(S_{\theta}) + \delta
\]
and thus, thanks to second part of Lemma 3.3 we infer that there exists $x_i \in X_i$ and two parallel hyperplanes orthogonal to $\sigma(x_i)$ with mutual distance equal to $\varepsilon$ such that $X_i$ lies between them. By a compactness argument and the continuity of $\sigma$, letting $i \to \infty$ we get
that there exist \( x \in X \) and two parallel hyperplanes orthogonal to \( \sigma(x) \) with mutual distance equal to \( \varepsilon \) such that \( X \) lies between them. Thus, in particular \( \text{width}(X) \leq \varepsilon \). This establishes (3.11), which in turn, again by a compactness argument and the continuity of \( \sigma \), yields the conclusion in the equality case. \( \square \)

Next we prove a result in the spirit of [4, Theorem 1.1]. In the following \( C, \Omega \) and \( \Sigma \) will be as in Definition 1.3. Moreover if \( x \in \Sigma \) is a point where the tangent hyperplane to \( \Sigma \) exists we denote by \( \nu_\Sigma(x) \) the normal to this hyperplane pointing outward with respect to \( \Omega \). We give the following definition.

**Definition 3.5.** We denote by \( \Sigma^+ \) the set of points in \( \Sigma \setminus C \) such that there exists a support hyperplane \( \Pi_x \) with the property that \( \Pi_x \cap \Sigma = \{x\} \).

We recall the following result, see [19, Theorem 2.2.9]:

**Theorem 3.6.** Let \( K \subset \mathbb{R}^N \) be a compact convex set. Then for \( \mathcal{H}^{N-1} \)-almost every \( \nu \in \mathbb{S}^{N-1} \) the support hyperplane for \( K \) orthogonal to \( \nu \) intersects \( K \) at a single point.

**Corollary 3.7.** Let \( C \) and \( \Sigma \subset \mathbb{R}^N \) be as in Definition 1.3. With the notation above, we have that
\[
K^+(\Sigma) = \mathcal{H}^{N-1}\left( \bigcup_{x \in \Sigma^+} N_x \Sigma \right),
\]
where \( K^+(\Sigma) \) is the total positive curvature defined in Definition 1.3.

**Proof.** Let \( K \) denote the convex hull of \( \Sigma \). By Theorem 3.6 we have that for \( \mathcal{H}^{N-1} \)-a.e. direction \( \nu \in \bigcup_{x \in \Sigma \setminus C} N_x \Sigma \) the corresponding support plane for \( K \) intersects \( K \) at a single point that necessarily belongs to \( \Sigma \setminus C \) and thus to \( \Sigma^+ \). \( \square \)

**Proof of Theorem 1.4** Observe that if \( \Sigma \cap C = \emptyset \) then
\[
\bigcup_{x \in \Sigma \setminus C} N_x \Sigma = \mathbb{S}^{N-1},
\]
hence (1.3) trivially holds.

Hence in the following we may assume that \( \Sigma \cap C \neq \emptyset \).

We denote by \( \nu_C \) the outward normal to \( C \). We start by proving (1.3). Let us define \( \sigma : \Sigma \cap C \to \mathbb{S}^{N-1} \) as \( \sigma(x) := \nu_C(x) \). Note that since \( C \) is convex the direction \( \sigma(x) \) belongs to \( N_x C \) and thus to \( N_x (\Sigma \cap C) \) for every \( x \in \Sigma \cap C \).

Given \( \nu \in \mathbb{S}^{N-1} \), we denote by \( \nu^\perp \) the hyperplane orthogonal to \( \nu \) and passing through the origin and we set
\[
\bar{t} := \max\{ t \in \mathbb{R} : (t\nu + \nu^\perp) \cap \Sigma \neq \emptyset \}.
\]
Clearly, by definition for every \( \nu \in \mathbb{S}^{N-1} \) the hyperplane \( \bar{t}\nu + \nu^\perp \) is a support hyperplane for \( \Sigma \). Fix \( \theta \in (0, \theta_0) \). We claim that for every \( x \in \Sigma \cap C \)
\[
(3.13) \quad \nu \in N_x (\Sigma \cap C) \cap S_{\theta, \sigma(x)} \quad \text{implies} \quad \bar{t}\nu + \nu^\perp \cap \Sigma \subset \Sigma \setminus C.
\]

Let \( t_0 \in \mathbb{R} \) be such that \( x + \nu^\perp = t_0 \nu + \nu^\perp \) and observe that since \( \nu \cdot \sigma(x) \geq \cos \theta > \cos \theta_0 \) then by assumption (1.2) \( \nu \notin N_x \Sigma \), hence the hyperplane \( t_0 \nu + \nu^\perp \) enters \( \Omega \). Thus it easily follows that \( \bar{t} > t_0 \). Let \( y \in \bar{t}\nu + \nu^\perp \cap \Sigma \). Then \( y \notin \Sigma \cap C \), since otherwise this would contradict the
fact that \( t_0 \nu + \nu^\perp \) is a support hyperplane for \( \Sigma \cap C \). This establishes (3.13). From (3.13) it follows that

\[
N^{\sigma, \theta}(\Sigma \cap C) \subset \bigcup_{x \in \Sigma \setminus C} N_x \Sigma.
\]

Recall that by Definition 1.3

\[
\mathcal{H}^{N-1} \left( \bigcup_{x \in \Sigma \setminus C} N_x \Sigma \right) = K^+(\Sigma).
\]

Combining the equality above with (3.14), the inequality (1.3) follows from (3.10) with \( X = \Sigma \cap C \), letting \( \theta \to \theta_0^- \).

Given \( \varepsilon > 0 \), let \( \delta_0 \) be the constant provided by Proposition 3.4 and let \( \theta \in \left( \theta_0 / 2, \theta_0 \right) \) such that

\[
\mathcal{H}^{N-1}(S_{\theta_0}) \leq \mathcal{H}^{N-1}(S_{\theta}) + \frac{\delta_0}{2}.
\]

Assume that (1.4) and (1.5) for some \( \delta \in (0, \delta_0 / 2) \) such that \( \cos \theta_0 + \delta < \cos \theta \). Then, using the assumption (1.4), the same argument as before yields (3.13), hence (3.14). Thus, from (1.5) and (3.15) we have in particular

\[
\mathcal{H}^{N-1}(N^{\sigma, \theta}(\Sigma \cap C)) \leq K^+(\Sigma) \leq \mathcal{H}^{N-1}(S_{\theta_0}) + \delta \leq \mathcal{H}^{N-1}(S_{\theta}) + \delta_0.
\]

The conclusion follows from Proposition 3.4. \( \Box 

\textbf{Remark 3.8.} Note that if \( x \in \Sigma \cap C \) is a point where \( \nu_\Sigma(x) \) exists and belongs to \( N_x \Sigma \), then \( \nu \cdot \nu_\Sigma(x) \leq \nu(x) \cdot \nu_C(x) \) for every \( \nu \in N_x \Sigma \). Therefore in this case it suffices to check (1.2) for \( \nu = \nu_\Sigma(x) \).

\textbf{Remark 3.9.} Observe that if equality holds in (1.3) and \( \mathcal{H}^{N-1}(\partial \Omega \cap C) > 0 \) then \( \partial \Omega \cap C \) lies on a facet of \( C \). In fact, since \( \partial C(\partial \Omega \cap C) \subset \Sigma \cap C \) and \( \Sigma \cap C \) lies on a support hyperplane \( \Pi \) to \( C \), by convexity \( \partial \Omega \cap C \) lies on the same hyperplane.

It is well known that for surfaces \( \Sigma \subset \mathbb{R}^3 \) without boundary the following inequality holds

\[
\int_{\Sigma} |H_\Sigma|^2 \, dH^2 \geq 16\pi.
\]

with equality achieved if and only if \( \Sigma \) is a sphere. We now apply Theorem 1.4 to extend this inequality to the following extension of the Willmore energy in \( N \)-dimensions

\[
\int_{\Sigma \setminus C} |H_\Sigma|^{N-1} \, dH^{N-1},
\]

for \( C^{1,1} \) hypersurfaces with boundary supported on convex sets and with contact angle larger than a given \( \theta_0 \in (0, \pi) \). Note that in the next theorem we do not assume any regularity on the convex set \( C \).

\textbf{Theorem 3.10} (A Willmore type inequality.). Let \( C, \Omega \) and \( \Sigma \) be as in Definition 1.3 and let \( \theta_0 \in (0, \pi) \). Assume that \( \Sigma \setminus C \) is of class \( C^{1,1} \). Set \( H_\Sigma := \text{div}_\Sigma \nu_\Sigma \) (where \( \nu_\Sigma \) is the unit normal to \( \Sigma \) pointing outward with respect to \( \Omega \)). Assume also

\[
\nu \cdot \nu' \leq \cos \theta_0 \quad \text{whenever} \ x \in \Sigma \cap C, \ \nu \in N_x \Sigma \quad \text{and} \quad \nu' \in N_x C.
\]
Then,
\begin{equation}
(3.17) \quad \int_{\Sigma \setminus C} |H_{\Sigma}|^{N-1} d\mathcal{H}^{N-1} \geq (N-1)^{N-1} \mathcal{H}^{N-1}(S_{\theta_0}).
\end{equation}

Moreover, if equality holds in (3.17) and $H_{\Sigma} \neq 0$ a.e., then $\Sigma \setminus C$ coincides, up to a rigid motion, with an omothetic of $S_{\theta_0}$ sitting on a facet of $C$.

**Proof.** Without loss of generality we may assume that
\[
\int_{\Sigma \setminus C} |H_{\Sigma}|^{N-1} d\mathcal{H}^{N-1} < \infty.
\]

Set for any $\eta > 0$ sufficiently small $C_\eta := C + B_\eta(0)$ and $\Sigma_\eta := \partial \Omega \setminus C_\eta$. Observe that $C_\eta$ satisfies both an outer and inner uniform ball condition and thus is of class $C^{1,1}$, see [16 7]. Note also that there exists $\theta_\eta \in (0, \theta_0)$ such that
\begin{equation}
(3.18) \quad \nu \cdot \nu_{C_\eta}(x) \leq \cos \theta_\eta \quad \text{whenever } x \in \Sigma_\eta \cap C_\eta, \quad \nu \in N_x \Sigma_\eta,
\end{equation}

with $\theta_\eta \to \theta_0$ as $\eta \to 0^+$. Indeed, if not, there would exist a sequence $\eta_k \to 0$, a sequence of points $x_k \in \Sigma_{\eta_k} \cap C_{\eta_k}$ and a sequence $\nu_k \in N_{x_k} \Sigma_{\eta_k}$, such that $\nu_k \cdot \nu_{C_{\eta_k}}(x_k) \geq \cos \theta'$ for some $\theta' \in (0, \theta_0)$. We may assume that $x_k \to x \in \Sigma \cap C$, $\nu_k \to \nu$ and $\nu_{C_{\eta_k}}(x_k) \to \nu'$. Clearly $\nu \in N_x \Sigma$, $\nu' \in N_x C$ and $\nu \cdot \nu_C(x) \geq \cos \theta'$, a contradiction to (1.2).

We set
\[
\tilde{\Sigma} = \{ x \in \Sigma \setminus C : N_x \Sigma \neq \emptyset \}, \quad \tilde{\Sigma}_\eta = \{ x \in \Sigma \setminus C_\eta : N_x \Sigma_\eta \neq \emptyset \}.
\]

We claim that
\begin{equation}
(3.19) \quad \chi_{\tilde{\Sigma}_\eta} \to \chi_{\tilde{\Sigma}} \quad \text{pointwise in } \Sigma \setminus C.
\end{equation}

First of all note that $\tilde{\Sigma} \setminus C_\eta \subset \tilde{\Sigma}_\eta$ for all $\eta$, whence
\[
\chi_{\tilde{\Sigma}} = \lim_{\eta \to 0^+} \chi_{\tilde{\Sigma} \setminus C_\eta} \leq \liminf_{\eta \to 0^+} \chi_{\tilde{\Sigma}_\eta} \quad \text{pointwise in } \Sigma \setminus C.
\]

If otherwise $x \not\in \tilde{\Sigma}$, we show that $x \not\in \tilde{\Sigma}_\eta$ for $\eta$ small. Indeed, assume by contradiction that there exist $\nu_k \in N_x (\Sigma_{\eta_k})$, for a sequence $\eta_k \to 0$. Then, passing to a subsequence, if needed, $\nu_k \to \nu \in N_x \Sigma$, a contradiction. This proves that
\[
\chi_{\tilde{\Sigma}} \geq \limsup_{\eta \to 0^+} \chi_{\tilde{\Sigma}_\eta} \quad \text{pointwise in } \Sigma \setminus C
\]
and thus (3.19) holds.

Let $(\Sigma_\eta)^+$ the subset of $\tilde{\Sigma}_\eta$ defined as in Definition 3.5 with $\Sigma$ replaced by $\Sigma_\eta$. Denote by $K_\Sigma$ the Gaussian curvature of $\Sigma \setminus C$ and observe that on $\tilde{\Sigma}_\eta$ we have $H_{\Sigma} \geq 0$. By the arithmetic-geometric mean inequality $(N-1)^N K_\Sigma \leq H_{\Sigma}^{N-1}$ on $\tilde{\Sigma}_\eta$. Then by Theorem 1.4 we get
\begin{equation}
(3.20) \quad \int_{\Sigma \setminus C_\eta} |H_{\Sigma}|^{N-1} d\mathcal{H}^{N-1} \geq \int_{\tilde{\Sigma}_\eta} H_{\Sigma}^{N-1} d\mathcal{H}^{N-1} \geq (N-1)^{N-1} \int_{\tilde{\Sigma}_\eta} K_\Sigma \; d\mathcal{H}^{N-1}
\end{equation}
\[
\geq (N-1)^{N-1} \int_{(\Sigma_\eta)^+} K_\Sigma \; d\mathcal{H}^{N-1} = (N-1)^{N-1} \int_{(\Sigma_\eta)^+} \det(D\nu_C) \; d\mathcal{H}^{N-1}
\]
\[
= (N-1)^{N-1} K^+(\Sigma_\eta) \geq (N-1)^{N-1} \mathcal{H}^{N-1}(S_{\theta_0}).
\]
where in the second equality we have used Corollary 3.7 and the area formula, since \( \nu_\Sigma \) is a Lipschitz map in a neighborhood of \( \Sigma^+ \). Then, letting \( \eta \to 0 \) and recalling (3.19) and the fact that \( \theta_\eta \to \theta_0 \), we get
\begin{equation}
\int_{\Sigma \setminus C} |H_\Sigma|^{N-1} d\mathcal{H}^{N-1} \geq \int_\Sigma H_\Sigma^{N-1} d\mathcal{H}^{N-1} \\
\geq (N-1)^{N-1} \int_\Sigma K_\Sigma d\mathcal{H}^{N-1} \geq (N-1)^{N-1} \mathcal{H}^{N-1}(S_{\theta_0}).
\end{equation}
(3.21)

In particular (3.17) follows.

If equality holds in (3.17) holds, from (3.20) we have that \( \mathcal{K}^+(\Sigma_\eta) - \mathcal{H}^{N-1}(S_{\theta_\eta}) \to 0 \). In turn this yields that \( \nu_\eta \to 0 \) and more precisely that \( \Sigma_\eta \cap C_\eta \) lies between two parallel hyperplanes orthogonal to \( \nu_{\Sigma_\eta}(x_\eta) \) for some \( x_\eta \in \Sigma_\eta \cap C_\eta \) with mutual distance going to zero. Passing to the limit by a simple compactness argument we infer that \( \Sigma \cap C \) lies on a support hyperplane to \( C \).

Note also that in the equality case, if \( H_\Sigma \neq 0 \) \( \mathcal{H}^{N-1} \)-a.e., then (3.21) implies that \( \Sigma \setminus C = \Sigma \). In turn this yields that every \( x \in \Sigma \cap C \) has a support hyperplane to \( \Sigma \). Moreover, (3.21) yields also that \( H_\Sigma^{N-1} = (N-1)^{N-1} K_\Sigma \). In turn this implies that \( \Sigma \) is umbilical and thus, by a classical result, see for instance [10, Prop. 4, Ch. 3], each connected component \( \Sigma_i \) of \( \Sigma \) is contained in a sphere. Since \( \Sigma \cap C \) is contained in a hyperplane \( \Pi \) tangent to \( C \), each \( \Sigma_i \) is either a spherical cap supported on \( \Pi \) and satisfying (3.16) with \( \Sigma \) replaced by \( \Sigma_i \), or a sphere not intersecting \( C \).

In either case, since (3.16) is satisfied at every point in \( \Sigma \cap C \) (recall that \( \Sigma = \Sigma \)), we may apply (3.17) to infer that for every connected component \( \Sigma_i \) we have
\[ \int_{\Sigma \setminus C} H_{\Sigma_i}^{N-1} d\mathcal{H}^{N-1} \geq (N-1)^{N-1} \mathcal{H}^{N-1}(S_{\theta_0}). \]

In particular, since we are in the equality case, there must be only one connected component. Thus \( \Sigma \) is a spherical cap homothetic to \( S_{\theta_0} \) up to a rigid motion. Finally \( \Sigma \cap C \) by convexity must lie on a facet of \( C \).

\section{The equality case in the relative isoperimetric inequality outside a convex set}

In this section we give the proof of Theorem 1.2. Throughout this proof we will denote by \( \mathcal{H} \) the half space
\begin{equation}
\mathcal{H} := \{(x', x_N) \in \mathbb{R}^N : x_N > 0\}.
\end{equation}
(4.1)

We will also need the following notions of \((\Lambda, r_0)\)-minimizer and restricted \((\Lambda, r_0)\)-minimizer for the relative perimeter, which extend the standard notion of perimeter \((\Lambda, r_0)\)-minimizer recalled in Definition 2.1.

\textbf{Definition 4.1.} Let \( C \subset \mathbb{R}^N \) be a closed convex set with nonempty interior and let \( \Lambda, r_0 > 0 \). We say that a set of finite perimeter \( E \subset \mathbb{R}^N \setminus C \) is a \((\Lambda, r_0)\)-minimizer of the relative perimeter \( P(\cdot; \mathbb{R}^N \setminus C) \) if for any \( F \subset \mathbb{R}^N \setminus C \) such that \( \text{diam}(E \Delta F) \leq r_0 \) we have
\[ P(E; \mathbb{R}^N \setminus C) \leq P(F; \mathbb{R}^N \setminus C) + \Lambda |E \Delta F|. \]

Moreover, we say that \( E \subset \mathbb{R}^N \setminus C \) is a restricted \((\Lambda, r_0)\)-minimizer if the above inequality holds for every set \( F \subset \mathbb{R}^N \setminus C \) such that \( \text{diam}(E \Delta F) \leq r_0 \) and \( \partial^* F \cap C \subset \partial^* E \cap C \) up to a \( \mathcal{H}^{N-1} \)-negligible set.
Proof of Theorem 1.2. Let $m_0 > 0$ be a given mass and let $\Omega_0$ be a minimizer of the perimeter outside $C$ such that $|\Omega_0| = m_0$ and

\begin{equation}
(4.2) \quad P(\Omega_0; \mathbb{R}^N \setminus C) = N\left(\frac{\omega_N}{2}\right)^\frac{1}{N} m_0^\frac{N-1}{N}.
\end{equation}

Since $\Omega_0$ solves the isoperimetric problem we have that $\Omega_0$ is a $(\Lambda_0, r_0)$-minimizer of the relative perimeter in $\mathbb{R}^N \setminus C$, see Definition 4.1 for some $\Lambda_0, r_0 > 0$, depending on $\Omega_0$, see for instance the argument of [15, Example 21.3]. In turn by Proposition 5.2 $\Omega_0$ satisfies uniform volume density estimates and thus it easily follows that $\Omega_0$ is bounded.

We fix a sufficiently large ball $B_R(0)$ containing $\Omega_0$. Note that by standard argument, see also the argument of Step 1 below, $\Omega_0$ solves the following penalized minimum problem

\[
\min \{ P(E; \mathbb{R}^N \setminus C) + \Lambda_0 |E| - m | : E \subset B_R \setminus C \},
\]

for a possibly larger $\Lambda_0$. In particular we have

\begin{equation}
(4.3) \quad P(\Omega_0; \mathbb{R}^N \setminus C) \leq P(E; \mathbb{R}^N \setminus C) + \Lambda_0 |\Omega_0 \Delta E| \quad \text{for all } E \subset B_R \setminus C.
\end{equation}

Observe that by Theorem 2.6 we may assume that $\Omega_0$ is an open set and that $\partial \Omega_0 \setminus C$ coincides with the reduced boundary $\partial^* \Omega_0 \setminus C$ up to an $\mathcal{H}^{N-1}$-negligible set. Let us show that $\Omega_0$ is connected. Indeed, if otherwise $\Omega_0 = \Omega_1 \cup \Omega_2$, with $\Omega_1$ and $\Omega_2$ open, $\Omega_1$ a connected component of $\Omega_0$ with $0 < |\Omega_1| < m_0$, we have by Theorem 1.1

\[
P(\Omega_0; \mathbb{R}^N \setminus C) = P(\Omega_1; \mathbb{R}^N \setminus C) + P(\Omega_2; \mathbb{R}^N \setminus C)
\]

\[\geq N\left(\frac{\omega_N}{2}\right)^\frac{1}{N} |\Omega_1|^{\frac{N-1}{N}} + N\left(\frac{\omega_N}{2}\right)^\frac{1}{N} |\Omega_2|^{\frac{N-1}{N}} = N\left(\frac{\omega_N}{2}\right)^\frac{1}{N} m_0^{\frac{N-1}{N}},\]

which is a contradiction to (4.2).

For every $\eta \geq 0$ we set $C_\eta = C + B_\eta(0)$ and, for $\eta \in [0, \bar{\eta}]$ we set $m_\eta := |\Omega_0 \setminus C_\eta|$, where $\bar{\eta} > 0$ is such that $|\Omega_0 \setminus C_{\bar{\eta}}| > 0$. Correspondingly, we set for $m \in (0, m_\eta)$

\begin{equation}
(4.4) \quad I_\eta(m) = \min\{ P(E; \mathbb{R}^N \setminus C_\eta) : E \subset \Omega_0 \setminus C_\eta, |E| = m \}
\end{equation}

and denote by $\Omega_{\eta, m}$ any minimizer of the above problem. Note that $\Omega_{0, m_0} = \Omega_0$. Observe also that

\begin{equation}
(4.5) \quad \sup_{\eta \in [0, \bar{\eta}]} \sup_{m \in (0, m_\eta)} P(\Omega_{\eta, m}) < \infty.
\end{equation}

Indeed, given $\eta \in [0, \bar{\eta}]$ and $0 < m \leq m_\eta$ there exists $\eta' \geq \eta$ such that $|\Omega_0 \setminus C_{\eta'}| = m$. Thus

\[
P(\Omega_{\eta, m}) \leq P(\Omega_{\eta, m}; \mathbb{R}^N \setminus C_\eta) + P(C_\eta; B_R) \leq P(\Omega_0 \setminus C_{\eta'}) + P(C_\eta; B_R)
\]

\[\leq P(\Omega_0; \mathbb{R}^N \setminus C) + 2 \sup_{s \geq 0} P(C_s; B_R) \leq P(\Omega_0; \mathbb{R}^N \setminus C) + 2N \omega_N R^{N-1}.
\]

Let us fix $m', m'' \in (0, m_0)$, with $m' < m''$. We claim that there exists $\bar{\eta} \in (0, \bar{\eta}]$ such that

\begin{equation}
(4.6) \quad \text{if } \eta \in [0, \bar{\eta}] \text{ and } U \text{ is a connected component of } \Omega_0 \setminus C_\eta, \text{ then } |U| \notin [m', m''].
\end{equation}

Note that this property implies in particular that

\begin{equation}
(4.7) \quad \partial \Omega_{\eta, m} \cap (\Omega_0 \setminus C_\eta) \neq \emptyset \quad \text{for all } \eta \in [0, \bar{\eta}] \text{ and } m \in [m', m''].
\end{equation}

\footnote{Note that in [15, Example 21.3] it is proved that a mass constrained minimizer $E$ of the relative perimeter in an open set $A$ is a perimeter $(\Lambda_0, r_0)$-minimizer in $A$ according to Definition 2.1. However an inspection of the proof shows that $E$ is also a $(\Lambda_0, r_0)$-minimizer of the relative perimeter $P(\cdot, A)$ according to Definition 4.1.}
To prove (4.6) we fix \( x_0 \in \Omega \) and for every \( \eta \) we denote by \( U_\eta \) the connected component of \( \Omega \setminus C_\eta \) containing \( x_0 \). Note that \( U_\eta \) increases as \( \eta \) becomes smaller. Given any other point \( x \in \Omega \) there exists a path connecting \( x_0 \) and \( x \) contained in \( \Omega \), thus \( x \in U_\eta \) for \( \eta \) small enough. Hence \( |U_\eta| \to m_0 \), and the claim follows.

We split the remaining part of the proof in several steps. Some of the long technical claims contained in these steps will be proved in Appendix B so as not to break the line of reasoning.

**Step 1.** (Equivalence with a volume penalized problem). Fix \( 0 < m' < m'' < m_0 \) and let \( 0 < \tilde{\eta} \leq \eta \) be as in (4.6). We claim that there exists \( \Lambda > 0 \) with the following property: for every \( \eta \in [0, \tilde{\eta}] \) and \( m \in [m', m''] \) we have that \( \Omega_{\eta, m} \) is a minimizer of the following problem

\[
(4.8) \quad \min \{ P(E; \mathbb{R}^N \setminus \mathcal{C}_\eta) + \Lambda' ||E| - m| : E \subset \Omega \setminus \mathcal{C}_\eta \}.
\]

The proof of this claim will be given in the Appendix B.

**Step 2.** (\( \Omega_{\eta, m} \) is a restricted \( \Lambda \)-minimizer). Fix \( 0 < m' < m'' < m_0 \) and let \( 0 < \tilde{\eta} \leq \eta \) be as in (4.6). We claim that there exists \( \Lambda \geq \Lambda_0 \) such that for every \( \eta \in [0, \tilde{\eta}] \) and \( m \in [m', m''] \) we have that \( \Omega_{\eta, m} \) is a restricted \( \Lambda \)-minimizer under the constraint that \( \partial^* E \cap C_\eta \subset \partial^*(\Omega \setminus C_\eta) \cap C_\eta \). More precisely, for every set of finite perimeter \( E \subset B_R(0) \setminus C_\eta \) such that \( \partial^* E \cap C_\eta \subset \partial^*(\Omega \setminus C_\eta) \cap C_\eta \) up to a \( \mathcal{H}^{N-1} \)-negligible set

\[
(4.9) \quad P(\Omega_{\eta, m}; \mathbb{R}^N \setminus \mathcal{C}_\eta) \leq P(E; \mathbb{R}^N \setminus \mathcal{C}_\eta) + \Lambda \|E \setminus \Omega_{\eta, m}\|_E.
\]

In particular \( \Omega_{\eta, m} \) is a restricted \((\Lambda, r_0)\)-minimizer according to Definition 4.1, choosing for instance \( r_0 := \text{dist}(\Omega_{\eta, m}, \partial B_R(0)) \).

To prove the claim, we take \( \Lambda = \max\{\Lambda', \Lambda_0\} \) where \( \Lambda' \) is as in Step 1. Given \( E \) as above, from Step 1 we get

\[
(4.10) \quad P(\Omega_{\eta, m}; \mathbb{R}^N \setminus \mathcal{C}_\eta) \leq P(E \cap \Omega_{\eta, m}; \mathbb{R}^N \setminus \mathcal{C}_\eta) + \Lambda' \|E \cap \Omega_{\eta, m}\|_E,
\]

\[
+ \mathcal{H}^{N-1}(\partial^* E \cap \Omega_{\eta, m} \cap \{\nu_E = \nu_{\Omega_{\eta, m}}\} \setminus \mathcal{C}_\eta) + \Lambda \|\Omega_{\eta, m} \cap \partial E \setminus \Omega_{\eta, m}\|_E.
\]

Then, using (4.3) and the condition \( \partial^* E \cap C_\eta \subset \partial^*(\Omega \setminus C_\eta) \cap C_\eta \), we have

\[
\mathcal{H}^{N-1}(\{\partial^* \Omega_{\eta, m} \cap C_\eta \} \setminus \mathcal{C}) + \mathcal{H}^{N-1}(\partial^* \Omega_{\eta, m} \setminus \{\nu_E = \nu_{\Omega_{\eta, m}}\} \setminus \mathcal{C})
\]

\[
\leq P(E \cup \Omega_{\eta, m}; \mathbb{R}^N \setminus \mathcal{C}) + \Lambda_0 |E \setminus \Omega_{\eta, m}|.
\]

Simplifying the above inequality, we get

\[
\mathcal{H}^{N-1}(\partial^* \Omega_{\eta, m} \cap \{\nu_E = -\nu_{\Omega_{\eta, m}}\} \setminus \mathcal{C}) \leq \mathcal{H}^{N-1}(\partial^* E \setminus \Omega_{\eta, m}) + \Lambda_0 |E \setminus \Omega_{\eta, m}|.
\]

Combining this inequality with (4.10) we conclude that

\[
P(\Omega_{\eta, m}; \mathbb{R}^N \setminus \mathcal{C}_\eta) \leq P(E; \mathbb{R}^N \setminus \mathcal{C}_\eta)
\]

\[
+ \mathcal{H}^{N-1}(\partial^* E \setminus \Omega_{\eta, m}) + \Lambda \|E \cap \Omega_{\eta, m}\| + \Lambda_0 |E \setminus \Omega_{\eta, m}|.
\]

so that the claim is proved.

**Step 3.** (Lipschitz equicontinuity of the isoperimetric profiles). Fix \( 0 < m' < m'' < m_0 \) and let \( 0 < \tilde{\eta} \leq \eta \) be as in (4.6). We claim that there exists a constant \( L \), such that for \( \eta \in [0, \tilde{\eta}] \) the function \( I_\eta \) defined in (4.4) is \( L \)-Lipschitz in \([m', m'']\).
We postpone the proof to Appendix B.

**Step 4.** (A formula for \( I_{\eta} \)). Fix \( 0 < m' < m'' < m_0 \) and let \( 0 < \tilde{\eta} \leq \eta \) be as in (4.6). For \( m \in [m', m''] \) and \( \eta \in [0, \tilde{\eta}] \) we set \( \Sigma_{\eta,m} := \partial\Omega_{\eta,m} \setminus \mathbf{C}_\eta \) and denote by \( \Sigma_{\eta,m}^* \) the regular free part of \( \Sigma_{\eta,m} \), that is \( \Sigma_{\eta,m}^* := \partial^* \Omega_{\eta,m} \setminus (\partial \Omega_0 \cup \mathbf{C}_\eta) \). Observe that by (4.7) \( \Sigma_{\eta,m}^* \) is nonempty. We recall that by a standard first variation argument \( \Sigma_{\eta,m}^* \) is a constant mean curvature manifold. We denote by \( H_{\Sigma_{\eta,m}}^* \) such a mean curvature.

We claim that at any point \( m \in (m', m'') \) of differentiability for \( I_{\eta}, \eta \in [0, \tilde{\eta}] \), we have

\[
I_{\eta}'(m) = H_{\Sigma_{\eta,m}}^*.
\]

To this end we fix \( x \in \Sigma_{\eta,m}^* \) and a ball \( B_r(x) \subset \Omega_0 \setminus \mathbf{C}_\eta \) such that \( \Sigma_{\eta,m}^* \cap B_r(x) = \partial \Omega_{\eta,m} \cap B_r(x) \). Let \( X \) be a smooth vector field compactly supported in \( B_r(x) \) such that

\[
\int_{\Sigma_{\eta,m}} X \cdot \nu_{\Omega_{\eta,m}} d\mathcal{H}^{N-1} \neq 0.
\]

Consider now the flow associated with \( X \), that is the solution in \( \mathbb{R}^N \times \mathbb{R} \) of

\[
\begin{cases}
\frac{\partial \Phi}{\partial t}(x, t) = X(\Phi(x, t)) \\
\Phi(x, 0) = x
\end{cases}
\]

and set \( \Omega_{\eta,m}(t) := \Phi(\Omega_{\eta,m}, t) \). Clearly, \( P(\Omega_{\eta,m}(t)); \mathbb{R}^N \setminus \mathbf{C}_\eta) \geq I_{\eta}(|\Omega_{\eta,m}(t)|) \), with the equality at \( t = 0 \). Therefore

\[
\frac{d}{dt} \left( P(\Omega_{\eta,m}(t)); \mathbb{R}^N \setminus \mathbf{C}_\eta \right) \bigg|_{t=0} = \frac{d}{dt} \left( I_{\eta}(|\Omega_{\eta,m}(t)|) \right) \bigg|_{t=0}.
\]

Note that

\[
\frac{d}{dt} \left( P(\Omega_{\eta,m}(t)); \mathbb{R}^N \setminus \mathbf{C}_\eta \right) \bigg|_{t=0} = H_{\Sigma_{\eta,m}}^* \int_{\Sigma_{\eta,m}^*} X \cdot \nu_{\Omega_{\eta,m}} d\mathcal{H}^{N-1},
\]

\[
\frac{d}{dt} \left( I_{\eta}(|\Omega_{\eta,m}(t)|) \right) \bigg|_{t=0} = I_{\eta}'(m) \frac{d}{dt} \left( |\Omega_{\eta,m}(t)| \right) \bigg|_{t=0} = I_{\eta}'(m) \int_{\Sigma_{\eta,m}^*} X \cdot \nu_{\Omega_{\eta,m}} d\mathcal{H}^{N-1},
\]

where we have used the well known formulas for the first variation of the perimeter and the volume, see for instance [15] Chap. 17. Thus (4.11) follows.

**Step 5.** (A weak Young’s law). Fix \( 0 < m' < m'' < m_0 \) and let \( 0 < \tilde{\eta} \leq \eta \) be as in (4.6). We claim that the if \( \eta \in [0, \tilde{\eta}] \) and \( m \in [m', m''] \), the following weak Young’s law holds:

\[
\nu \cdot \nu_{C_{\eta}}(x) \leq 0 \quad \text{whenever} \quad x \in \Sigma_{\eta,m} \cap \mathbf{C}_{\eta} \quad \text{and} \quad \nu \in N_x \Sigma_{\eta,m}.
\]

Let \( x \in \Sigma_{\eta,m} \cap \mathbf{C}_{\eta} \) and \( \nu \in N_x \Sigma_{\eta,m} \). Without loss of generality, by rotating the coordinate system if needed, we may assume that \( x = 0 \), \( \nu_{C_{\eta}}(0) = e_N \) and \( \nu = (\nu_1, 0, \ldots, 0, \nu_N) \) with \( \nu_1 \leq 0 \). Note that (4.12) will be proven if we show that

\[
\nu_N \geq 0 \quad \text{implies that} \quad \nu_N = 0.
\]

Set \( E_h = \frac{1}{h} \Omega_{\eta,m}, \ h \in \mathbb{N} \) and \( \mathbf{C}_{h,\eta} = \frac{1}{h} \mathbf{C}_{\eta} \) and observe that, since \( \nu_1 \leq 0 \) and \( \nu_N \geq 0 \), \( E_h \subset \{ x_1 \geq 0 \} \). Note also that by (4.9) we have that

\[
P(E_h; \mathbb{R}^N \setminus \mathbf{C}_{\eta,h}) \leq P(G; \mathbb{R}^N \setminus \mathbf{C}_{\eta,h}) + \frac{1}{h} \Delta|E_h\Delta G|
\]

for all sets \( G \subset B(hR(0) \setminus \mathbf{C}_{\eta,h}) \) such that \( \partial^* G \cap \mathbf{C}_{\eta,h} \subset \partial^* E_h \cap \mathbf{C}_{\eta,h} \) up to a \( \mathcal{H}^{N-1} \)-negligible set. Using the density estimate proved in Proposition 5.2 and passing possibly to a not relabelled subsequence we may assume that \( E_h \) converge in \( L^1_{\text{loc}}(\mathbb{R}^N) \) to some set \( E \subset \mathcal{H} \cap \{ x_1 > 0 \} \) (see
of locally finite perimeter and that $\mu_{E_h} \rightharpoonup^* \mu_E$ as Radon measures in $\mathbb{R}^N$, see (2.1) for the definition of $\mu_E$. Finally, given $r > 0$, from the volume density estimate in Proposition 5.2 we get that for $h$ large enough $|E_h \cap B_r(0)| \geq cr^{-N}$ and thus, passing to the limit, we have $|E \cap B_r(0)| \geq cr^{-N}$ for all $r > 0$. This in turn implies that $0 \in \partial E \subset \partial E$. Since each $E_h$ is a $\frac{A}{h}$-minimizer, by Theorem 2.2 we have that $E$ is a 0-minimizer that is

$$P(E; B_r(x_0)) \leq P(F; B_r(x_0)) \text{ for any } F, B_r(x_0) \text{ s.t. } E \Delta F \subset B_r(x_0) \subset \mathcal{H}.$$  

We claim that also the minimality with respect to inner perturbations passes to the limit. We now denote by $\hat{E}$ $E \cup R(E)$ where $R$ denotes the reflection map $R(x',x_N) = (x',-x_N)$. From (4.16) one can easily check that given an open set with Lipschitz boundary $V \subset \mathcal{H}$

$$H^{N-1}(\partial E \cap \partial V \cap \mathcal{H}) = 0 \text{ implies } P(E; \mathcal{H} \cap Q_r(0)) \leq P(E \setminus V; \mathcal{H} \cap Q_r(0)).$$

We postpone the proof of this claim to Appendix B.

We now denote by $\hat{E} = E \cup R(E)$ where $R$ denotes the reflection map $R(x',x_N) = (x',-x_N)$. From (4.16) one can easily check that given an open set with Lipschitz boundary $V \subset \mathcal{H}$ such that $H^{N-1}(\partial \hat{E} \cap \partial V) = 0$ we have

$$P(\hat{E}; Q_r(0)) \leq P(\hat{E} \setminus V; Q_r(0)).$$

We claim that the connected component $\Gamma$ of $\partial \hat{E}$ containing 0 coincides with $\{x_1 = 0\}$. In turn this implies (4.13). To see this assume first that $\Gamma$ intersects $\{x_1 = 0\} \setminus \{x_N = 0\}$ at some point $x_0$. Then, by Theorem 2.6(iii) $\Gamma$ is a smooth minimal surface in a neighborhood of $x_0$. In turn, by the Strong Maximum Principle Theorem 2.5 it coincides with the hyperplane $\{x_1 = 0\}$ in a neighborhood of $x_0$. The same argument shows that $\Gamma \cap \{x_1 = 0\}$ is both relatively closed and open in $\{x_1 = 0\}$ and therefore $\Gamma = \{x_1 = 0\}$.

Observe that $\partial \hat{E} \cap \{x_1 = 0\} \subset \{x_1 = 0\} \cap \{x_N = 0\}$ and thus in particular $H^{N-1}(\partial \hat{E} \cap \{x_1 = 0\}) = 0$. We may then apply Lemma 5.3 to conclude that $0 \notin \partial \hat{E}$, thus getting a contradiction.

**Step 6.** (Convergence of the isoperimetric profiles) We claim that

$$\lim_{\eta \to 0} I_\eta(m) = I_0(m) \text{ for all } m \in [0,m_0) \text{ and } \lim_{\eta \to 0} I_\eta(m_0) = I_0(m_0).$$

Let $\eta_n$ be a sequence converging to zero such that $I_{\eta_n}(m) \to \liminf_{\eta \to 0} I_\eta(m)$. Since the perimiters of $\Omega_{\eta_n,m}$ are equibounded, see (4.5), up to a subsequence we may assume that $\Omega_{\eta_n,m}$ converge in $L^1$ to a set of finite perimeter $E \subset \Omega_0$ with $|E| = m$. Thus, by lower semicontinuity,

$$I_0(m) \leq P(E; \mathbb{R}^N \setminus C) \leq \liminf_{\eta \to 0} P(\Omega_{\eta,m}; \mathbb{R}^N \setminus C_{\eta_n}) = \liminf_{\eta \to 0} I_\eta(m).$$

Recall that $\Omega_{0,m}$ denotes a minimizer for the problem defining $I_0(m)$.

$$I_\eta(m - |\Omega_{0,m} \cap C_{\eta_n}|) \leq P(\Omega_{0,m}; \mathbb{R}^N \setminus C_{\eta_n}) \leq I_0(m),$$

using the equilipschitz continuity of $I_\eta$ proved in Step 3, by letting $\eta$ tend to 0 in the previous inequality and recalling (4.18) we obtain the first equality in (4.17). The second one follows simply from the fact that $\Omega_{\eta,m} = \Omega_0 \setminus C_{\eta_n}$.

Note that the above argument shows in particular that if $m \in (0,m_0)$, $\eta_n \to 0$ and $\Omega_{\eta_n,m}$ is a sequence converging in $L^1$ to a set $E$, then $E$ coincides with a minimizer $\Omega_{0,m}$. 


Step 7. \(I_0 = I_{\mathcal{H}}\). We set

\[
I_{\mathcal{H}}(m) = N\left(\frac{\omega N}{2}\right)^{\frac{1}{N}} m^{\frac{N-1}{N}},
\]

that is the isoperimetric profile of half spaces. We claim that

\[
I_0(m) = I_{\mathcal{H}}(m) \quad \text{for all } m \in [0, m_0].
\]

To this end we fix \(0 < m' < m'' < m_0\) and let \(0 < \tilde{\eta} \leq \tilde{\eta}\) be as in (4.6). Recall that by Step 2 for all \(\eta \in [0, \tilde{\eta}]\), \(\Omega_{\eta,m}\) is a restricted \((\Lambda, r_0)\)-minimizer for all \(m \in [m', m'']\). We claim that for any such \(\eta\) if \(x_0 \in \Sigma_{\eta,m}^+\) then \(\Sigma_{\eta,m}\) is of class \(C^{1,1}\) in a neighborhood of \(x_0\). Here \(\Sigma_{\eta,m}\) is defined as in Definition 3.5 with \(\Sigma\) and \(C\) replaced by \(\Sigma_{\eta,m}\) and \(C_{\eta}\). Indeed, observe first that if \(x_0 \in \Sigma_{\eta,m}^+\) then by Theorem 2.6 \(\Sigma_{\eta,m}\) is of class \(C^{1,1}\) in a neighborhood of \(x_0\). Moreover, if \(x_0 \in \Omega_0\) then, since \(H_{\Sigma_{\eta,m}}\) is constant in a neighborhood of \(x_0\), we have that in fact \(\Sigma_{\eta,m}\) is analytic in such a neighborhood.

If instead \(x_0 \in \partial\Omega_0\), since \(\Omega_0\) is a \((\Lambda, r_0)\)-minimizer and \(\partial\Omega_0\) lies on one side with respect to \(\Sigma_{\eta,m}\) which is of class \(C^{1,\alpha}\) in a neighborhood of \(x_0\), again by Theorem 2.6 we infer that \(\partial\Omega_0\) is of class \(C^{1,\alpha}\), hence analytic in a neighborhood of \(x_0\). The claim then follows from Proposition 5.5.

To prove (4.20) observe that the very same argument of (3.20) (with \(\tilde{\Sigma}_{\eta}\) replaced by \(\Sigma_{\eta,m}^+\) and \(\theta_{\eta}\) replaced by \(\pi/2\)) yields that

\[
\int_{\Sigma_{\eta,m}^+ \setminus C_{\eta}} H_{\Sigma_{\eta,m}}^{N-1} \, dH^{N-1} \geq (N - 1)^{N-1} \frac{\omega N}{2}.
\]

Indeed this argument only requires that \(\Sigma_{\eta,m}\) is of class \(C^{1,1}\) in a neighborhood of \(\Sigma_{\eta,m}^+\) and that (3.18) holds. Recall that the latter condition with \(\theta_{\eta} = \pi/2\) is ensured by Step 5. Observe also that if \(\Sigma_{\eta,m}^+\) intersects \(\partial\Omega_0\) in a set of positive \(H^{N-1}\) measure then for \(H^{N-1}\)-a.e. \(x\) on such a set

\[
H_{\Sigma_{\eta,m}}(x) = H_{\partial\Omega_0} \leq H_{\Sigma_{\eta,m}}^*,
\]

where \(\Sigma_{\eta,m}^*\) is the regular free part defined in Step 4 and the inequality follows from Proposition 5.5. Here, with a slight abuse of notation, we denote by \(H_{\partial\Omega_0}\) the constant curvature of \(\partial^*\Omega_0 \setminus C\). Therefore the previous inequality, (4.21) and (4.11) imply in particular that for a.e. \(m \in (m', m'')\) and for all \(\eta \in [0, \tilde{\eta}]\)

\[
I_{\eta}(m)(I_{\eta}(m))^{N-1} = P(\Omega_{\eta,m}; \mathbb{R}^N \setminus C_{\eta})H_{\Sigma_{\eta,m}}^{-1}
\]

\[
\geq (N - 1)^{N-1} \frac{\omega N}{2} = I_{\mathcal{H}}(m)(I_{\mathcal{H}}(m))^{N-1},
\]

where the last equality follows from (4.19). Recalling that \(I_{\eta}\) is Lipschitz in \([m', m'']\) and thus absolutely continuous, raising the above inequality to the power \(\frac{1}{N-1}\) and integrating in \([m, m'']\), for any \(m \in (m', m'')\) we get

\[
I_{\eta}(m'^{N-1}) - I_{\eta}(m)^{N-1} \geq I_{\mathcal{H}}(m'^{N-1}) - I_{\mathcal{H}}(m)^{N-1},
\]

for all \(\eta \in [0, \tilde{\eta}]\). Passing to the limit as \(\eta \to 0\) and using Step 6 we get

\[
I_0(m'^{N-1}) - I_0(m)^{N-1} \geq I_{\mathcal{H}}(m'^{N-1}) - I_{\mathcal{H}}(m)^{N-1},
\]

for all \(0 < m < m'' < m_0\). Observe now that \(\lim_{m'' \to m_0} I_0(m'') = I_0(m_0)\) (this follows by a simple semicontinuity argument and by the fact that \(I_0\) is increasing). Thus, passing to
the limit in (4.24) as $m'' \to m_0$, recalling that by assumption $I_0(m_0) = I_m(m_0)$ and that by
Theorem 1.1, $I_0(m) \geq I_m(m)$, we get $I_0(m) = I_m(m)$ for all $m \in (0, m_0)$, as claimed.

**Step 8.** $(\partial \Omega_0 \cap C$ is flat). In this step we prove that $\partial \Omega_0 \cap C$ lies on a hyperplane $\Pi$.

To this aim we start by showing that

$$(4.25) \quad (I_{\eta}^{N})'(m) \to (I_{\eta}^{\frac{N}{2}})' \quad \text{in } L^1_{loc}(0, m_0).$$

Indeed, given $0 < m' < m'' < m_0$ from (4.23) and the fact that $I_\eta \to I_m$, we have that for a.e. $m \in (m', m'')$ and $\eta \in [0, \tilde{\eta}]$, $(I_{\eta}^{N})'(m) \geq (I_{\eta}^{\frac{N}{2}})'(m)$ and

$$\int_{m'}^{m''} (I_{\eta}^{\frac{N}{2}})'(t) dt \to \int_{m'}^{m''} (I_{\eta}^{\frac{N}{2}})'(t) dt \quad \text{as } \eta \to 0.$$ Hence, (4.25) follows.

Returning to the proof of the flatness of $\partial \Omega_0 \cap C$, observe that by a simple diagonal argument we can construct two sequences $m_n \to m_0$ and $\eta_n \to 0$ such that $\Omega_{\eta_n,m_n}$ is a $\Lambda_n$-minimizer for some $\Lambda_n > 0$ (possibly going to $+\infty$) and

$$I_{\eta_n}(m_n) \to I_0(m_0), \quad (I_{\eta_n}^{\frac{N}{2}})'(m_n) \to (I_{\eta}^{\frac{N}{2}})'(m_0) = N \left( \frac{N \omega_N}{2} \right)^{\frac{1}{N-1}}.$$ This is possible thanks to Step 2, Step 6 and (4.25). Given $\varepsilon > 0$, let $\delta > 0$ be as in Theorem 1.4 with $\theta_0 = \pi/2$. Recall that $\delta$ depends only on $\varepsilon$ and on $\text{diam}(\Omega_0)$. Recall also that $\Sigma_{\eta_n,m_n}$ is of class $C^{1,1}$ in a neighborhood of $\Sigma_{\eta_n,m_n}$, thanks to Step 7. Then from the above convergence, arguing as in the proof of (3.20) with $\Sigma_\eta$ replaced by $\Sigma_{\eta_n,m_n}$, and recalling that the weak Young inequality (4.12) holds for $\Sigma_{\eta_n,m_n}$, we have that for $n$ large

$$\frac{N \omega_N}{2} \leq K^+(\Sigma_{\eta_n,m_n}) \leq (N - 1)^{1-N} \int_{\Sigma_{\eta_n,m_n}} H_{\Sigma_{\eta_n,m_n}}^{N-1} dH_{\Sigma_{\eta_n,m_n}}^{N-1}$$

$$\leq (N - 1)^{1-N} P(\Omega_{\eta_n,m_n}; \mathbb{R}^N \setminus C_{\eta_n}) H_{\Sigma_{\eta_n,m_n}}^{N-1}$$

$$= \left[ \frac{1}{N} (I_{\eta_n}^{\frac{N}{2}})'(m_n) \right]^{N-1} < \frac{N \omega_N}{2} + \delta.$$ Note that in the third inequality above we have used (4.22). Thus from Theorem 1.4 we get that for $n$ sufficiently large width$(\Sigma_{\eta_n,m_n} \cap C_{\eta_n}) \leq \varepsilon$ and that there exists $x_n \in \partial C_{\eta_n}$ such that

$$(4.26) \quad \Sigma_{\eta_n,m_n} \cap C_{\eta_n} \subset \{ x : -\varepsilon \leq (x - x_n) \cdot \nu_{C_{\eta_n}}(x_n) \leq 0 \}.$$ We claim that

$$(4.27) \quad \partial \Omega_{\eta_n,m_n} \cap C_{\eta_n} \subset \{ x : -\varepsilon \leq (x - x_n) \cdot \nu_{C_{\eta_n}}(x_n) \leq 0 \}.$$ To prove this claim let $y \in \partial \Omega_{\eta_n,m_n} \cap C_{\eta_n}$ such that

$$(y - x_n) \cdot \nu_{C_{\eta_n}}(x_n) = \min\{ (z - x_n) \cdot \nu_{C_{\eta_n}}(x_n) : z \in \partial \Omega_{\eta_n,m_n} \cap C_{\eta_n} \}.$$ Assume by contradiction that $(y - x_n) \cdot \nu_{C_{\eta_n}}(x_n) < -\varepsilon$. Then consider a sequence of balls $B_{r_i}(y)$ with $r_i \to 0$ and $B_{r_i}(y) \subset \{ x : (x - x_n) \cdot \nu_{C_{\eta_n}}(x_n) < -\varepsilon \}$. Note that if $B_{r_i}(y) \setminus C_{\eta_n} \subset \Omega_{\eta_n,m_n} \setminus C_{\eta_n}$ then (owing to the fact that the hyperplane $\{ x : (x - y) \cdot \nu_{C_{\eta_n}}(x_n) = 0 \}$ is not tangent to $C_{\eta_n}$ at $y$) there would exist $z \in B_{r_i}(y) \cap \partial \Omega_{\eta_n,m_n} \cap C_{\eta_n}$ such that $(z - x_n) \cdot \nu_{C_{\eta_n}}(x_n) < (y - x_n) \cdot \nu_{C_{\eta_n}}(x_n)$, which is impossible by the minimality of $y$. Therefore for every $i$ there exists $z_i \in (\mathbb{R}^N \setminus \Omega_{\eta_n,m_n} \cap B_{r_i}(y)) \setminus C_{\eta_n}$. Observe that either $z_i \in \partial \Omega_{\eta_n,m_n} \setminus C_{\eta_n}$, or
z_i \in (\mathbb{R}^N \setminus \Omega_{\eta_n,m_n}) \setminus C_{\eta_n}. In the latter case, we may find a continuous path \gamma \subset B_{r_i}(y) \setminus C_{\eta_n} connecting z_i with a point in B_{r_i}(y) \cap C_{\eta_n,m_n}. Therefore we may conclude that in any case there exists a point \hat{z}_i \in (\partial\Omega_{\eta_n,m_n} \setminus C_{\eta_n}) \cap B_{r_i}(y). Hence, y \in (\partial\Omega_{\eta_n,m_n} \setminus C_{\eta_n}) \cap C_{\eta_n} = \Sigma_{\eta_n,m_n} \cap C_{\eta_n}, which contradicts (4.26). Thus (4.27) is established.

Observe that since $C_{\eta_n} \cap B_R(0)$ is a sequence of convex sets converging to the convex set $C \cap B_R(0)$ in the sense of Kuratowski then $P(C_{\eta_n} \cap B_R(0)) \to P(C \cap B_R(0))$. This in turn yields that $H^{N-1}(\partial(C_{\eta_n} \cap B_R(0))) \searrow H^{N-1}(\partial(C \cap B_R(0)))$ and in particular that

$$
(4.28) \quad H^{N-1}(\partial(C_{\eta_n} \cap B_R(0))) \searrow H^{N-1}(\partial C) \quad \text{in } B_R(0).
$$

Assume now, passing possibly to a further subsequence, that $\partial\Omega_{\eta_n,m_n} \cap C_{\eta_n} \to K$ in the sense of Kuratowski, for some $K \subset \partial\Omega \cap C$. Note that from (4.27) width($K$) \leq \varepsilon for every \varepsilon > 0 and that $K$ lies on a hyperplane tangent to $C$. We claim that

$$
(4.29) \quad \limsup_n H^{N-1}(\partial\Omega_{\eta_n,m_n} \cap C_{\eta_n}) \leq H^{N-1}(K)
$$

To this aim set $K_\sigma = K + B_\sigma(0) \subset B_R(0)$ for $\sigma > 0$ sufficiently small. Then for $n$ sufficiently large $\partial\Omega_{\eta_n,m_n} \cap C_{\eta_n} \subset K_\sigma \cap \partial C_{\eta_n}$, hence

$$
H^{N-1}(\partial\Omega_{\eta_n,m_n} \cap C_{\eta_n}) \leq H^{N-1}(K_\sigma \cap \partial C_{\eta_n}).
$$

From this inequality we then have

$$
\limsup_n H^{N-1}(\partial\Omega_{\eta_n,m_n} \cap C_{\eta_n}) \leq \limsup_n H^{N-1}(K_\sigma \cap \partial C_{\eta_n}) \leq H^{N-1}(K_\sigma \cap \partial C),
$$

where in the last inequality we have used (4.28). Then (4.29) follows letting $\sigma \to 0$.

To conclude this step we observe that since $\Omega_{\eta_n,m_n} \to \Omega_0$ in $L^1$, by the lower semicontinuity of perimeter and (4.29)

$$
P(\Omega_0) = I_0(m_0) + H^{N-1}(\partial^* \Omega_0 \cap C) \leq \liminf_n P(\Omega_{\eta_n,m_n})
$$

$$
= \liminf_n \left[ I_{\eta_n}(m_n) + H^{N-1}(\partial\Omega_{\eta_n,m_n} \cap C_{\eta_n}) \right]
$$

$$
\leq I_0(m_0) + H^{N-1}(K).
$$

Recall that by the volume estimate Proposition 5.2(ii) $\partial^* \Omega_0 \cap C$ coincides $H^{N-1}$-a.e. with $\partial\Omega_0 \cap C$. Thus the above inequality implies that $K$ coincides $H^{N-1}$-a.e. with $\partial\Omega_0 \cap C$. Hence, the claim follows.

**Step 9. (Conclusion).** In this final step we show that $\Omega_0$ is a half ball.

To this aim we fix $m \in (0,m_0)$ and a sequence $\eta_n \to 0$ such that

$$
(4.30) \quad \frac{1}{m} \to I_0(m) = I_{\mathcal{F}}(m), \quad (I_N^{\eta_n})'(m) \to (I_N^{\mathcal{F}})'(m) = N\left(\frac{N}{2}\right)^{\frac{1}{N-1}}.
$$

Owing to Steps 6-8 we can find such a sequence for a.e. $m \in (0,m_0)$. Thanks to Step 2, we may assume that there exists $\Lambda > 0$ such that $\Omega_{\eta_n,m}$ is a $\Lambda$-minimizer for all $n$. By Theorem 2.6(ii) this implies in particular that $|H_{\Sigma_{\eta_n,m}}| \leq \Lambda H^{N-1}$-a.e. on $\partial^* \Omega_{\eta_n,m} \setminus C_{\eta_n}$. 

Arguing as in the previous step, see also the proof of (3.20), we have then
\[
\frac{N\omega_N}{2} \leq K^+_{\Sigma_{\eta_n,m}} = \int_{\Sigma_{\eta_n,m}^+} K_{\Sigma_{\eta_n,m}} d\mathcal{H}^{N-1} \leq (N - 1)^{1-N} \int_{\Sigma_{\eta_n,m}^+} H_{\Sigma_{\eta_n,m}}^{N-1} d\mathcal{H}^{N-1}
\]
(4.31)
where we recall $K_{\Sigma_{\eta_n,m}}$ is the Gaussian curvature of $\Sigma_{\eta_n,m}$. We start by observing that, since $H_{\Sigma_{\eta_n,m}}(x) \leq H_{\Sigma_{\eta_n,m}}$ for $\mathcal{H}^{N-1}$ a.e. $x \in \Sigma_{\eta_n,m}$, from the third inequality in (4.31) we have in particular that
\[
\lim_n \mathcal{H}^{N-1}(\Sigma_{\eta_n,m}) = \lim_n P(\Omega_{\eta_n,m}; \mathbb{R}^N \setminus C_{\eta_n}) = \mathcal{H}^{N-1}(\partial^*\Omega_{0,m} \setminus C).
\]
Note that indeed $H_{\Sigma_{\eta_n,m}}$ may only take the constant values $H_{\partial\Omega_0}$ or $H_{\Sigma_{\eta_n,m}}$. Then, again from (4.31), it follows that
\[
\text{either } H_{\partial\Omega_0} = H_{\Sigma_{\eta_n,m}} \text{ or } \mathcal{H}^{N-1}(\partial(\partial\Omega_{\eta_n,m} \cap \partial\Omega_0) \setminus C_{\eta_n}) \to 0.
\]
Fix now $x \in \partial^*\Omega_{0,m}$. Since $\Omega_{\eta_n,m} \to \Omega_{0,m}$ in $L^1$ and $P(\Omega_{\eta_n,m}; \mathbb{R}^N \setminus C_{\eta_n}) \to P(\Omega_{0,m}; \mathbb{R}^N \setminus C)$ thanks to the first condition in (4.30), we have that $\mathcal{H}^{N-1}(\partial\Omega_{\eta_n,m} \cap \partial\Omega_0) \to 0$ in $\mathbb{R}^N \setminus C$. In turn, by Theorem 2.7 it follows that, up to rotations and translations, there exist a $(N - 1)$-dimensional ball $B^p \subset \mathbb{R}^{N-1}$, functions $\varphi_n, \varphi \in W^{2,p}(B^p)$, and $r > 0$ such that $x \in B^p \cap (-r, r)$ and
\[
\begin{align*}
\partial\Omega_{\eta_n,m} \cap (B^p \cap (-r, r)) &= \{(x', \varphi_n(x')) : x' \in B^p\}, \\
\partial\Omega_{0,m} \cap (B^p \cap (-r, r)) &= \{(x', \varphi(x')) : x' \in B^p\}, \\
\varphi_n \to \varphi & \text{ in } W^{2,p}(B^p) \text{ for all } p \geq 1, \\
H_{\Sigma_{\eta_n,m}}(x', \varphi_n(x')) &\to H_{\Sigma_{0,m}}(x', \varphi(x')) \text{ in } L^p(B^p) \text{ for all } p \geq 1,
\end{align*}
\]
Recalling (4.33) the fourth condition above implies that
\[
H_{\Sigma_{\eta_n,m}}(x', \varphi_n(x')) \to H_{\Sigma_{0,m}}(x', \varphi(x')) \equiv H_{\Sigma_{0,m}}^{N-1}
\]
strongly in $L^p(B^p)$ for all $p \geq 1$. In turn, see for instance [11 Lemma 7.2], this implies
\[
\varphi_n \to \varphi \text{ strongly in } W^{2,p}(B^p) \text{ for all } p \geq 1.
\]
Note also that, since from (4.32) $\mathcal{H}^{N-1}(\Sigma_{\eta_n,m} \setminus \Sigma^+_{\eta_n,m}) \to 0$, we have that for every $y \in (B^p \cap (-r, r)) \cap \Sigma_{0,m}$ there exists a sequence $y_n \in (B^p \cap (-r, r)) \cap \Sigma^+_{\eta_n,m}$ such that $y_n \to y$. Therefore, using the $L^1$ convergence of $\Omega_{\eta_n,m}$ to $\Omega_{0,m}$ we conclude that the tangent hyperplane to $\partial\Omega_{0,m}$ at $y$ is also a support hyperplane. Thus we have shown that all principal curvatures at any point in $(B^p \cap (-r, r)) \cap \partial\Sigma_{0,m}$ are nonnegative. Thus, from the second inequality in (4.31), recalling (4.32) and (4.34) we may conclude that
\[
K_{\Sigma_{0,m}} = (N - 1)^{1-N} H_{\Sigma_{0,m}}^{N-1} = (N - 1)^{1-N} H_{\Sigma_{0,m}}^{N-1} \text{ on } (B^p \cap (-r, r)) \cap \Sigma_{0,m}.
\]
The equality above implies that $\Sigma_{0,m} \cap (B^p \cap (-r, r))$ is umbilical. Hence $\partial^*\Omega_{0,m} \setminus C$ is umbilical, thus each connected component of $\partial^*\Omega_{0,m} \setminus C$ lies on a sphere of radius $R_m = (N - 1)/H_{\Sigma_{0,m}}$. Consider the unique unbounded connected component of $U := \mathbb{R}^N \setminus \Omega_{0,m}$. Then $\partial U \setminus C$ is contained in a sphere of radius $R_m$ intersecting $C$ on $\Pi$, thus $\partial U \setminus C$ is a spherical cap and $\Omega_{0,m}$ is contained in the region enclosed by $\partial U \setminus C$ and $\Pi$. In particular $\Omega_{0,m}$
is contained in the half space $\Pi^+$ determined by $\Pi$ not containing $C$. Since $P(\Omega_0,m;\Pi^+)=P(\Omega_0,m;\mathbb{R}^N \setminus C)=N(\frac{d}{d+1})^\frac{d}{d+1} m_\infty$ by Theorem 19.21 in [15] for a.e. $m$ we conclude that for such $m$ $\Omega_{0,m}$ is a half ball. Since the argument above can be carried out for a.e. $m \in (0,m_0)$, in particular there exists a sequence $m_n \to m_0$ such that $\Omega_{0,m_n}$ is a half ball. Hence $\Omega_0$ is a half ball.

\[ \square \]

5. Appendix A: Some auxiliary results

In this section we collect some auxiliary results needed in the proof of Theorem 1.2.

5.1. Density estimates. Density estimates for $(\Lambda, r_0)$-minimizers are well known. However for the sake of completeness we give the proof of the proposition below showing that such density estimates are independent of the convex obstacle.

Lemma 5.1. Let $C$ be a closed convex set with nonempty interior and $F \subset \mathbb{R}^N \setminus C$ a bounded set of finite perimeter. Then

\[ P(F; \partial C) \leq P(F; \mathbb{R}^N \setminus C) \]

Proof. Let $B$ a ball such that $F \subseteq B$ and let $H_i$ be a sequence of closed half spaces such that $C = \bigcap_{i=1}^\infty H_i$. Since $C = (C \cup F) \cap \bigcap_{i=1}^\infty H_i$ we have

\[ P(C; B) \leq \lim inf \ P\left((C \cup F) \cap \bigcap_{i=1}^n H_i; B\right) \leq P(C \cup F; B), \]

where the last inequality follows by applying repeatedly the inequality $P(G \cap H_i; B) \leq P(G; B)$ where $G$ is a set of finite perimeter. Since $P(C \cup F; B) = \mathcal{H}^{N-1}(\partial C \cap F(0) \cap B) + \mathcal{H}^{N-1}(\partial^* F \setminus C)$, the conclusion follows observing that $P(C; B) = \mathcal{H}^{N-1}(\partial C \cap F(0) \cap B) + \mathcal{H}^{N-1}(\partial C \cap \partial^* F).$ \[ \square \]

Proposition 5.2. Let $C$ be a closed convex set with nonempty interior and let $E \subset \mathbb{R}^N \setminus C$ be a restricted $(\Lambda, r_0)$-minimizer of the relative perimeter $P(\cdot; \mathbb{R}^N \setminus C)$ according to Definition 4.1. Then there are positive constants $c_1 = c_1(N)$ and $C_1 = C_1(N)$ independent of $C$ such that for all $r \in (0, \min\{r_0, N/(4\Lambda)\})$ we have:

(i) for all $x \in \mathbb{R}^N \setminus \text{int}(C)$

\[ P(E; B_r(x)) \leq C_1 r^{N-1}, \]

(ii) for all $x \in \partial^* E$

\[ |E \cap B_r(x)| \geq c_1 r^N. \]

Moreover $E$ is equivalent to an open set $\Omega$ such that $\partial \Omega = \partial^* \Omega$, hence $\mathcal{H}^{N-1}(\partial \Omega \setminus \partial^* \Omega) = 0$, and (ii) holds at any point $x \in \partial \Omega$.

Proof. Given $x \in \mathbb{R}^N \setminus \text{int}(C)$ and $r < \min\{r_0, N/(4\Lambda)\}$, we set $m(r) := |E \cap B_r(x)|$. Recall that for a.e. such $r$ we have $m'(r) = \mathcal{H}^{N-1}(E(1) \cap \partial B_r(x))$ and $\mathcal{H}^{N-1}(\partial^* E \cap \partial B_r(x)) = 0$. For any such $r$ we set $F := E \setminus B_r(x)$. Then, using Definition 4.1 we have

\[ P(E; B_r(x) \setminus C) \leq \mathcal{H}^{N-1}(\partial B_r(x) \cap E(1)) + \Lambda |E \cap B_r(x)| \leq C_1 r^{N-1} \]
for a suitable constant $C_1$. In turn
\[
P(E; B_r(x)) \leq P(E; B_r(x) \setminus C) + \mathcal{H}^{N-1}(\partial(C \cap B_r(x))) \leq C_1 r^{N-1} + \mathcal{H}^{N-1}(\partial B_r(x)),
\]
where in the last inequality we estimated the perimeter $C \cap B_r(x)$ with the perimeter of the larger convex set $B_r(x)$. Thus (i) follows by taking $C_1$ larger.

Observe now that by Lemma 5.1
\[
P(E \cap B_r(x); \partial C) \leq P(E \cap B_r(x); \mathbb{R}^N \setminus C).
\]
Thus, using also (5.1), we have
\[
P(E \cap B_r(x)) = P(E \cap B_r(x); \mathbb{R}^N \setminus C) + P(E \cap B_r(x); \partial C)
\leq 2P(E \cap B_r(x); \mathbb{R}^N \setminus C) = 2P(E; B_r(x) \setminus C) + 2m'(r)
\leq 4m'(r) + 2\Lambda m(r).
\]
In turn, using the isoperimetric inequality and the fact that $2\Lambda r < N/2$ we get
\[
N\omega_N^{-\frac{1}{N}} m(r)^{\frac{N-1}{N}} \leq P(E \cap B_r(x)) \leq 4m'(r) + 2\Lambda m(r)
\leq 4m'(r) + 2\Lambda r\omega_N^{-\frac{1}{N}} m(r)^{\frac{N-1}{N}} \leq 4m'(r) + \frac{N}{2} \omega_N^{-\frac{1}{N}} m(r)^{\frac{N-1}{N}}.
\]
Then from the previous inequality we get
\[
\frac{N}{2} \omega_N^{-\frac{1}{N}} m(r)^{\frac{N-1}{N}} \leq 4m'(r).
\]
Observe now that if in addition $x \in \partial^* E$, then $m(r) > 0$ for all $r$ as above. Thus, we may divide the previous inequality by $m(r)^{\frac{N-1}{N}}$, and integrate the resulting differential inequality thus getting
\[
|E \cap B_r(x)| \geq c_1 r^N,
\]
for a suitable positive constant $c_1$ depending only on $N$.

We show that $\partial^* E \subset \partial^* E$. To this aim note that (ii) holds for every $x \in \partial^* E$. Thus, if $x \in \mathbb{R}^N \setminus C$, since both $E$ and $\mathbb{R}^N \setminus E$ are $\Lambda$-minimizers in a neighborhood of $x$ we have that $|E \setminus B_r(x)| \geq c_1 r^N$ for $r$ small. Thus $x \not\in (E^{(0)} \cup E^{(1)})$, that is $x \in \partial^* E$. If $x \in \partial C \cap \partial^* E$ then there exists a constant $c_2 > 0$, depending on $x$ such that for $r$ small $|C \cap B_r(x)| \geq c_2 r^N$. This estimate, together with (ii) again implies that $x \in \partial^* E$. Hence $\mathcal{H}^{N-1}(\partial^* E \setminus \partial^* E) = \mathcal{H}^{N-1}(\partial^* E \cap \partial^* E) = 0$, where the last equality follows from Theorem 16.2 in [13].

Set now $\Omega = E^{(1)} \setminus \partial E^{(1)}$. Recalling that $\partial E^{(1)} = \partial^* E$, see (2.2), we have that $\Omega$ is an open set equivalent to $E$ such $\partial\Omega = \partial E^{(1)}$. Hence the conclusion follows. \hfill $\Box$

5.2. **A maximum principle.** Next result is essentially the strong maximum principle proved in [9] Lemma 2.13]. However, we have to apply it in a slightly different situation and therefore we indicate the little changes needed in the proof.

**Lemma 5.3.** Let $E \subset \{x_1 > 0\}$ be a set of locally finite perimeter such that
\[
\mathcal{H}^{N-1}((\partial E \setminus \partial^* E) \setminus \{x_N = 0\}) = 0
\]
satisfying the following minimality property: for every $r > 0$ and every open set with Lipschitz boundary $V \subset Q_r(0)$ such that $\mathcal{H}^{N-1}(\partial E \cap \partial V) = 0$ we have
\[
P(E; Q_r(0)) \leq P(E \setminus V; Q_r(0)).
\]
Assume also that $\mathcal{H}^{N-1}(\partial E \cap \{x_1 = 0\}) = 0$. Then $0 \notin \partial E^{2}$.

The proof of lemma above is in turn based on the following variant of [9, Lemma 2.12]. To this aim, given $r > 0$ we set $C_r := (0, r) \times D_r$, where $D_r := \{x' \in \mathbb{R}^{N-1} : \|x'\| < r\}$.

**Lemma 5.4.** Let $E$ be as in Lemma 5.3, let $\tilde{r} > 0$ and let $u_0 \in C^{2}(D_r) \cap \text{Lip}(D_r)$ with $0 < u_0 < \tilde{r}$ on $\partial D_r$. Assume also that

$$E^{(1)} \cap ((0, \tilde{r}) \times \partial D_r) \subset \{(x_1, x') \in (0, \tilde{r}) \times \partial D_r : x_1 \geq u_0(x')\},$$

and

$$\mathcal{H}^{N-1}(\partial E \cap \partial \{(x_1, x') \in C_\tilde{r} : x_1 < u_0(x')\}) = 0.$$  

Then,

$$E^{(1)} \cap C_\tilde{r} \subset \{(x_1, x') \in C_\tilde{r} : x_1 \geq u_0(x')\}.$$  

**Proof.** The proof goes exactly as the one of Lemma 2.12 in [9] as it is based on the comparison with the competitor $F = E \setminus V$, where $V = \{(x_1, x') \in C_\tilde{r} : x_1 < u_0(x')\}$. Observe that assumption (5.4) guarantees that such a competitor satisfies $\mathcal{H}^{N-1}(\partial E \cap \partial V) = 0$, which is required in order (5.3) to hold. □

**Proof of Lemma 5.3.** For reader’s convenience we reproduce the proof of Lemma 2.13 in [9] with the small changes needed in our case.

We choose $\tilde{r} > 0$ so that $\mathcal{H}^{N-1}(\partial E \cap \partial C_\tilde{r}) = 0$ and $\mathcal{H}^{N-2}(\partial E \cap \partial D_r) = 0$, where with a slight abuse of notation $\partial D_r$ stands for the relative boundary of $D_r$ in $\{x_1 = 0\}$. Note that a.e. $r > 0$ satisfies these conditions thanks to (5.2) and to the assumption $\mathcal{H}^{N-1}(\partial E \cap \{x_1 = 0\}) = 0$.

Define now a function $w_{E} : \overline{D_r} \to [0, \infty]$ by setting

$$w_{E}(x') = \inf\{x_1 \in \mathbb{R} : (x_1, x') \in \overline{C_\tilde{r}} \cap \partial E\}.$$  

Observe that $w_{E}$ is nonnegative and lower semicontinuous on $\overline{D_r}$, with the property that

$$E^{(1)} \cap C_\tilde{r} \subset \{(x_1, x') : x' \in D_r, x_1 \geq w_{E}(x')\}.$$  

Recalling that $\mathcal{H}^{N-2}(\partial E \cap \partial D_r) = 0$, we have that $w_{E} > 0 \mathcal{H}^{N-2}$-a.e. on $\partial D_r$. Therefore there exists a family $(\varphi_{t})_{t \in (0, 1)} \subset C^{\infty}(\partial D_r)$ such that

$$0 \leq \varphi_{t_1} \leq \varphi_{t_2} \leq \min \left\{ w_{E}, \frac{\tilde{r}}{2} \right\}, \quad \varphi_{t_1} \neq \varphi_{t_2} \quad \text{for all } 0 < t_1 < t_2 < 1.$$  

By Lemma 2.11 in [9] for every $t \in (0, 1)$ there exists $u_t \in C^{\infty}(D_r) \cap \text{Lip}(D_r)$ such that

$$\begin{cases} 
\text{div} \left( \frac{\nabla u_t}{\sqrt{1 + |\nabla u_t|^2}} \right) = 0 & \text{in } D_r, \\
u_t = \varphi_t & \text{on } \partial D_r.
\end{cases}$$  

Note that by the Strong Maximum Principle Theorem [2.5] we have that $0 < u_{t_1} < u_{t_2} < \tilde{r}/2$ in $D_r$ for every $0 < t_1 < t_2 < 1$. Therefore the graphs $\Gamma_{t}$ of $u_{t}$ are mutually disjoint in $\lambda \overline{E}$ and so $\mathcal{H}^{N-1}(\Gamma_{t} \cap \partial E) = 0$ for all but countably many $t \in (0, 1)$. In particular there exists $t$ such that (5.4) holds with $u_0$ replaced by $u_t$. Therefore we may apply Lemma 5.4 to conclude that $E^{(1)} \cap C_\tilde{r} \subset \{(x_1, x') \in C_\tilde{r} : x_1 \geq u_t(x')\}$ so that in particular $w_{E}(0) \geq u_t(0) > 0$, hence $0 \notin \partial E$. □
5.3. A regularity result. The following proposition is a slight variant of a result contained in [20].

Proposition 5.5. Let \( \Omega \subset \mathbb{R}^N \) be a bounded open set and let \( x_0 \in \partial \Omega \) be such that \( \partial \Omega \) is of class \( C^2 \) in a neighborhood \( U \) of \( x_0 \). Let \( E \subset \Omega \) satisfy

\[
P(E) \leq P(F) \quad \text{for all } F \subset \Omega, \quad |F| = |E|, \quad \text{s.t. } E \Delta F \subset \subset U.
\]

If there exists a support hyperplane \( \Pi \) to \( E \) at \( x_0 \) such that \( \partial E \cap \Pi = \{x_0\} \), then \( \partial E \) is of class \( C^{1,1} \) in a neighborhood \( V \) of \( x_0 \). Moreover if \( \partial^* E \cap \Omega \cap V \neq \emptyset \), then for \( H^{N-1} \)-a.e. \( x \in \partial E \cap \partial \Omega \cap V \)

\[
H_{\partial \Omega}(x) \leq H,
\]

where \( H \) denotes the constant curvature of \( \partial E \cap \partial \Omega \cap V \).

Proof. Observe that by a standard argument (5.5), together with the assumption that \( \partial \Omega \) of class \( C^2 \), implies that \( E \) is a \((\Lambda, r_0)\)-minimizer in a possibly smaller neighborhood \( U' \) of \( x_0 \). Hence, since there exists a support hyperplane to \( \partial E \) at \( x_0 \), by Theorem 2.6 \( \partial E \) is of class \( C^{1,\alpha} \) in a neighborhood of \( x_0 \). Moreover, up to a change of coordinate system, we may assume that the support hyperplane to \( \partial E \) at \( x_0 \) is the horizontal hyperplane \( \{x_N = 0\} \) and \( E \subset \{x_N > 0\} \). Since \( \{x_N = 0\} \cap \partial E = \{x_0\} \), there exists \( \varepsilon > 0 \) sufficiently small such that \( E \cap \{x_N = \varepsilon\} \) is an \((N - 1)\)-dimensional relatively open set, denoted by \( \omega \), and there exists \( \beta \in C^2(\omega), \ u \in C^{1,\alpha}(\omega) \) whose graphs coincide with \( \partial \Omega \cap (\omega \times (-r, r)) \) and \( \partial E \cap (\omega \times (-r, r)) \) respectively, for some \( r > 0 \), with \( u = 0 \) on \( \omega \) and \( \beta \leq u \leq 0 \). The \( C^{1,1} \) regularity of \( \partial E \) then follows arguing exactly as in the proof at page 658 of [20]. Finally, inequality (5.6) is also a byproduct of the same proof, see (3.5) in [20]. \( \square \)

6. Appendix B: some steps of the proof of Theorem 1.2

6.1. Proof of the claim of Step 1. We argue by contradiction assuming that there exist a sequence \( \Lambda_h \to +\infty, \eta_h \in [0, \hat{\eta}], \eta_h \to \eta_0, \ m_h \in [m', m''] \) converging to some \( m \), and a sequence \( E_h \subset \Omega_0 \setminus C_{\eta_h} \) such that each \( E_h \) is a minimizer of (4.8) with \( \Lambda \), \( m \) and \( \eta \) replaced by \( \Lambda_h, m_h \) and \( \eta_h \) respectively, and \( |E_h| \neq m_h \). Since \( P(E_h; \mathbb{R}^N \setminus C_{\eta_h}) \leq P(\Omega_{\eta_h, m_h}; \mathbb{R}^N \setminus C_{\eta_h}) \), from (4.5) we have that the perimeters of \( E_h \) are equibounded perimeters. Therefore, without loss of generality we may assume that \( E_h \) converges in \( L^1 \) to some set \( F \subset \Omega_0 \setminus C_{\eta_0} \) such that \( |F| = m \). We assume also that \( |E_h| < m_h \) for all \( h \), the other case being analogous. Note also that, since \( \Lambda_h \to +\infty \) we have \( m_h - |E_h| \to 0 \).

Observe now that (4.6) implies that there exists a point \( x \in \partial^* F \cap (\Omega_0 \setminus C_{\eta_0}) \). Arguing as in Step 1 of Theorem 1.1 in [11], given \( \varepsilon > 0 \) sufficiently small, we can find nearby \( x \) a point \( x' \) and \( r > 0 \) such that \( B_r(x') \subset \Omega_0 \setminus C_{\eta_0} \) and

\[
|F \cap B_{r/2}(x')| < \varepsilon r^N, \quad |F \cap B_r(x')| > \frac{\omega_N}{2^{N+2}} r^N.
\]

Therefore, for \( h \) sufficiently large, we also have

\[
|E_h \cap B_{r/2}(x')| < \varepsilon r^N, \quad |E_h \cap B_r(x')| > \frac{\omega_N}{2^{N+2}} r^N.
\]

We can now continue as in the proof of [11] Theorem 1. We recall the main construction for the reader’s convenience. For a sequence \( 0 < \sigma_h < 1/2^N \) to be chosen, we introduce the
contradiction proves that $H(6.1)$.

$\Pi C_t \pi, m$ increasing in $[0$ for all $x \in \pi \Omega$. From this inequality it is clear that we can choose $\sigma > 0$. With this choice of $\sigma$, it follows from (6.1) that $P(E_h; \mathbb{R}^N \setminus C_{\eta}) + \Lambda_h ||E_h|| - m_h \leq P(E_h; \mathbb{R}^N \setminus C_{\eta}) + \Lambda_h ||E_h|| - m_h + 2^N N \sigma h \mathcal{H}^{N-1}(-\partial^* E_h \setminus C_{\eta}) - \Lambda_h c \frac{\sigma h r^N}{2}$

for suitable universal constants $c, C > 0$. If we fix $\varepsilon$ so that the negative term in the brackets does not exceed half the positive one, then we have

(6.2) \[ |E_h| - |E_h| \geq \frac{c}{2} \sigma h r^N. \]

In particular from this inequality it is clear that we can choose $\sigma$ so that $|E_h| = m_h$; this implies $\sigma_h \to 0$. With this choice of $\sigma_h$, it follows from (6.1) and (6.2) that

\[ P(E_h; \mathbb{R}^N \setminus C_{\eta}) + \Lambda_h ||E_h|| - m_h \leq P(E_h; \mathbb{R}^N \setminus C_{\eta}) + \Lambda_h ||E_h|| - m_h + 2^N N \sigma_h \mathcal{H}^{N-1}(-\partial^* E_h \setminus C_{\eta}) - \Lambda_h c \frac{\sigma h r^N}{2} \]

for $h$ large, thus contradicting the minimality of $E_h$.

6.2. Proof of the claim of Step 3. We start by showing that the functions $I_\eta$ are strictly increasing in $[0, m_\eta]$ for all $\eta \in [0, \bar{\eta}]$. To this end we fix $m \in (0, m_\eta]$ and a point $x \in \pi_{C_\eta}(\Omega_{\eta, m})$, where $\pi_{C_\eta}$ is the orthogonal projection on $C_\eta$. Let $\Pi$ be the tangent hyperplane to $C_\eta$ at $x$. Define $\Pi_\bar{t} = \Pi + t v_{C_\eta}(x)$ for $t \in \mathbb{R}$ and set

$\bar{t} = \max\{t \geq 0 : \Pi_t \cap \Omega_{\eta, m} \neq \emptyset\}.$

Note that $\bar{t} > 0$ and that $\Pi_\bar{t}$ is a support hyperplane for $\Omega_{\eta, m}$ with $\text{dist}(\Pi_\bar{t}, C_\eta) = \bar{t}$. For all $t \in (0, \bar{t})$ we denote by $\Pi_{t, m}$ the intersection of $\Omega_{\eta, m}$ with the half space with boundary $\Pi_t$ containing $C_\eta$. Then $I_\eta(\Omega_{\eta, m, t}) \leq P(\Omega_{\eta, m, t}; \mathbb{R}^N \setminus C_\eta) \leq P(\Omega_{\eta, m}; \mathbb{R}^N \setminus C_\eta) = I_\eta(m)$. Since the function $t \to |\Omega_{\eta, m, t}|$ is increasing and continuous in a left neighborhood of $\bar{t}$ and $|\Omega_{\eta, m, t}| < |\Omega_{\eta, m}|$ if $t < \bar{t}$, it follows that

(6.3) \[ \text{for every } m \in (0, m_\eta) \text{ there exists } \varepsilon > 0 \text{ s.t. } I_{\eta}(s) < I_{\eta}(m) \text{ for all } s \in (m - \varepsilon, m). \]

Let $I = \{0 < s < m : I_\eta(s) \leq I_\eta(m) \text{ for all } \sigma \in [s, m]\}$. We claim that $I = (0, m)$. Indeed if $m = \inf I > 0$, then there exist $m_n \in I$, with $m_n \to m$. Since the minimizers $\Omega_{\eta, m_n}$ are equibounded sets with equibounded perimeters, see (4.5), up to a subsequence we may assume that $\Omega_{\eta, m_n}$ converge to a set $E \subset \Omega_0 \setminus C_\eta$ with $|E| = m$. Then, by the lower semicontinuity of the perimeter we conclude that $I_\eta(m) \leq P(E; \mathbb{R}^N \setminus C_\eta) \leq \liminf_n I_\eta(m_n) \leq I_\eta(m)$. In turn, (6.3) implies that there exists a left neighborhood $(\bar{m} - \varepsilon, \bar{m})$ such that $I_{\eta}(s) < I_{\eta}(\bar{m}) \leq I_{\eta}(m)$ for all $s \in (\bar{m} - \varepsilon, \bar{m})$ which is a contradiction to the fact that $\bar{m} = \inf I$. This contradiction proves that $I_\eta$ is increasing. The strict monotonicity now follows from (6.3).
We now show that $I_\eta$ is Lipschitz on $[m',m'']$. From Step 2 and the uniform density estimates for restricted $\Lambda$-minimizers proved in Proposition 5.2 it follows that there exist $c_1, r_0 > 0$ (depending on $\Lambda$ and thus on $m',m''$) such that $|\Omega_{\eta,m} \cap B_{r_0}(x)| \geq c_1$ for all $x \in \partial \Omega_{\eta,m} \setminus C_0$ and for every $m \in [m',m'']$ and $\eta \in [0,\bar{\eta}]$. Then we fix $m \in [m',m'']$ and choose $x$ such that the tangent hyperplane $\Pi_0$ to $\partial \Omega_{\eta,m}$ at $x$ is also a support hyperplane for $\Omega_{\eta,m}$ parallel to a support hyperplane to $C_\eta$. Upon changing the coordinate system we may assume that $x = 0$ and that $\nu_{\Omega_{\eta,m}}(0) = e_N$ and so $\Pi_0$ is horizontal. Then we consider the family of parallel hyperplanes $\Pi_0^t = \Pi_0 + te_N$ for $t < 0$. Set
\[
J = \{ t \in (-r_0,0) : \mathcal{H}^{N-1}(\Omega_{\eta,m} \cap \Pi_0^t) \geq c_1/2r_0 \}.
\]
Let $d := \text{diam}(\Omega_0)$. Therefore we have
\[
c_1 \leq \int_{-r_0}^0 \mathcal{H}^{N-1}(\Omega_{\eta,m} \cap \Pi_0^t) \, dt \leq |J| \omega_{N-1} d^{N-1} + (r_0 - |J|) \frac{c_1}{2r_0},
\]
from which we get $|J| \geq c_2 > 0$, with $c_2$ depending only on $m'$ and $m''$ and $d$. On the other hand from the coarea formula
\[
\int_J \mathcal{H}^{N-2}(\partial \Omega_{\eta,m} \cap \Pi_0^t) \, dt \leq \mathcal{H}^{N-1}(\partial^* \Omega_{\eta,m}) \leq I_\eta(m_{\eta}) + \mathcal{H}^{N-1}(\Pi_0 \cap \partial C_\eta) \leq P(\Omega_0; \mathbb{R}^N \setminus C) + \sup_{0 \leq \eta \leq \bar{\eta}} \mathcal{H}^{N-1}(\Pi_0 \cap \partial C_\eta) =: C(\Omega_0).
\]
Therefore we may find $\bar{t} \in J$ such that $\mathcal{H}^{N-2}(\partial \Omega_{\eta,m} \cap \Pi_{\bar{t}}^0) \leq C(\Omega_0)/c_2$. We denote by $\Omega_{\eta,m}^-$ the intersection of $\Omega_{\eta,m}$ with the half space with boundary $\Pi_{\bar{t}}^0$ containing $C_\eta$ and by $\Omega_{\eta,m}^+$ its complement in $\Omega_{\eta,m}$.

Given $s > 0$ sufficiently small, we define the competitor set $F_s := \Omega_{\eta,m}^- \cup C(\bar{t},h) \cup (h e_N + \Omega_{\eta,m}^+)$ where $C(\bar{t},h)$ is the vertical cylinder with base $\Omega_{\eta,m} \cap \Pi_{\bar{t}}^0$ and height $h = s/\mathcal{H}^{N-1}(\Omega_{\eta,m} \cap \Pi_{\bar{t}}^0)$). By construction $|F_s| = m + s$. Moreover, if $s$ is sufficiently small $F_s \subset B_R(0)$. Thus
\[
I_\eta(m + s) \leq P(F_s; \mathbb{R}^N \setminus C_\eta) = P(\Omega_{\eta,m}; \mathbb{R}^N \setminus C_\eta) + h \mathcal{H}^{N-2}(\partial \Omega_{\eta,m} \cap \Pi_{\bar{t}}^0) \leq I_\eta(m) + h \frac{C(\Omega_0)}{c_2} = I_\eta(m) + \frac{C(\Omega_0) s}{c_2 \mathcal{H}^{N-1}(\Omega_{\eta,m} \cap \Pi_{\bar{t}}^0)} \leq I_\eta(m) + \frac{2r_0 C(\Omega_0)}{c_1 c_2} s,
\]
where in the last inequality we used the fact that $\bar{t} \in J$. In conclusion we proved that for every $m \in [m',m'']$ there exists $\delta_m > 0$ such that for all $s \in [0,\delta_m]$
\[
I_\eta(m + s) - I_\eta(m) \leq \frac{2r_0 C(\Omega_0)}{c_1 c_2} s.
\]
Recalling that $I_\eta$ is increasing, a simple compactness argument shows that $I_\eta$ is Lipschitz continuous in $[m',m'']$ with a Lipschitz constant independent of $\eta$.

6.3. Proof of claim 4.16. Let us start by assuming also that
\[
(6.4) \quad \mathcal{H}^{N-1}(\partial E_h \cap \partial V \cap \mathcal{H}) = 0 \quad \text{for all } h \in \mathbb{N}.
\]
To this aim we fix $\delta > 0$ and set $\mathcal{H}_\delta := \{ x \in \mathcal{H} : x_N > \delta \}$ and $(E)_\delta = E + B_\delta(0)$. Then we denote by $\Phi_h : Q_{\gamma}(0) \cap \mathcal{H} \to Q_{\gamma}(0) \setminus \mathcal{C}_{\eta,h}$ a sequence of $C^1$ diffeomorphisms converging in $C^1$ to the identity map as $h \to 0$ with the property that $\Phi_h(\partial \mathcal{H} \cap Q_{\gamma}(0)) = \partial \mathcal{C}_{\eta,h} \cap Q_{\gamma}(0)$.
and \( \Phi_h(x) = x \) if \( x \in \mathcal{H}_\delta \). Recalling the \( \Lambda \)-minimality property \((4.14)\), we have using \((6.4)\) and observing that \( \Phi_h(V) \subsetneq Q_r(0) \) for \( h \) sufficiently large

\[
P(E_h; Q_r(0) \setminus C_{\eta,h}) \leq P(E_h \setminus \Phi_h(V); Q_r(0) \setminus C_{\eta,h}) + \frac{\Lambda}{R} |\Phi_h(V)|
\]

\[
\leq P(E_h; (Q_r(0) \setminus C_{\eta,h}) \setminus \Phi_h(V)) + P(\Phi_h(V); (Q_r(0) \setminus C_{\eta,h}) \cap E_h)
\]

\[
+ \mathcal{H}^{N-1}(\partial \Phi_h(V) \cap \partial E_h \cap \{x_N \leq \delta\} \cap (Q_r(0) \setminus C_{\eta,h})) + \frac{\Lambda}{R} |\Phi_h(V)|.
\]

Since

\[
P(E_h; Q_r(0) \setminus C_{\eta,h}) \geq P(E_h; (Q_r(0) \setminus C_{\eta,h}) \setminus \Phi_h(V)) + P(E_h; (Q_r(0) \setminus C_{\eta,h}) \cap \Phi_h(V)),
\]

and using the fact that \( \mathcal{H}_\delta \cap V = \mathcal{H}_\delta \cap \Phi_h(V) \subset (Q_r(0) \setminus C_{\eta,h}) \cap \Phi_h(V) \), the inequality above yields

\[
P(E_h; \mathcal{H}_\delta \cap V) \leq P(\Phi_h(V); (Q_r(0) \setminus C_{\eta,h}) \cap E_h)
\]

\[
+ \mathcal{H}^{N-1}(\partial \Phi_h(V) \cap \{x_N \leq \delta\} \cap (Q_r(0) \setminus C_{\eta,h})) + \frac{\Lambda}{R} |\Phi_h(V)|
\]

\[
\leq P(V; Q_r(0) \cap \mathcal{H}_\delta \cap E_h)
\]

\[
+ 2\mathcal{H}^{N-1}(\partial \Phi_h(V) \cap \{x_N \leq \delta\} \cap (Q_r(0) \setminus C_{\eta,h})) + \frac{\Lambda}{R} |\Phi_h(V)|
\]

\[
\leq P(V; Q_r(0) \cap \mathcal{H}_\delta \cap (E)_{\delta})
\]

\[
+ 2(\text{Lip}(\Phi_h))^{N-1}P(V; \{0 < x_N \leq \delta\}) + \frac{\Lambda}{R} |\Phi_h(V)|,
\]

where in the last inequality we used the fact that \( \Phi_h^{-1}((Q_r(0) \setminus C_{\eta,h}) \cap \{x_N \leq \delta\}) = Q_r(0) \cap \{0 < x_N \leq \delta\} \) and the fact that \( E_h \) converge in the Kuratowski sense to \( E \) in \( \mathcal{H}_\delta \), see Remark 2.3.

By the lower semicontinuity of the perimeter, passing to the limit in \((6.5)\)

\[
P(E; \mathcal{H}_\delta \cap V) \leq P(V; Q_r(0) \cap \mathcal{H}_\delta \cap (E)_{\delta}) + 2P(V; \{0 < x_N \leq \delta\}).
\]

In turn, by letting \( \delta \to 0 \) we have

\[
P(E; \mathcal{H} \cap V) \leq P(V; Q_r(0) \cap E),
\]

which is equivalent to \((4.16)\) thanks to first condition in \((4.16)\). To remove \((6.4)\) it is enough to consider a sequence of smooth sets \( V_j \subsetneq Q_r(0) \), \( V \subsetneq V_j \), satisfying the first condition in \((4.16)\) and \((6.4)\), and such that \( V_j \to V \) in \( L^1 \) and \( P(V_j; Q_r(0)) \to P(V; Q_r(0)) \). The conclusion then follows by applying \((4.16)\) with \( V \) replaced by \( V_j \) and passing to the limit thanks to the first condition in \((4.16)\).

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