On gravitational couplings in D-brane action

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ABSTRACT: We compute the two closed string graviton – two open string scalar superstring scattering amplitude on the disc to show that there is no second-derivative curvature–scalar coupling term $RX^2$ in the low-energy effective action of a D-brane in curved space.

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1. Introduction

The structure of gravitational coupling terms in the action for a D-brane probe moving in a curved space is of interest in many contexts, in particular, in brane-world constructions and gauge theory – string theory duality.

The standard Born-Infeld [1, 2] action for a D-brane [3, 4] in flat space [5] has a direct generalization to curved ambient space

\[ S_{DBI} = -T(p) \int d^{p+1} \sigma \ e^{-\phi(X)} \sqrt{\det \left [ \left ( G_{\mu \nu}(X) + B_{\mu \nu}(X) \right ) \partial _{\alpha} X^{\mu} \partial _{\beta} X^{\nu} + F_{\alpha \beta} \right ] + ...} \]  

(1.1)

Dots stand for various higher-derivative corrections (present already in flat space [6]). We shall ignore the well-known WZ-type couplings to R-R potentials and concentrate on the parity-even part of the action.

One may wonder still if the action (1.1) correctly describes the gravitational couplings of the D-brane even to the lowest order in derivatives. A way to determine the precise form of the low-energy D-brane effective action is to compare the corresponding vertices with the open–closed string S-matrix on the disc. Several of such studies checking the consistency of (1.1) and extending it to the next (4-th) derivative order were carried out in the past [7, 8, 9, 10, 11, 12, 13]. It was found (in agreement with expectation based on the leading gravitational interaction term on the disc in type I string theory being $R^2$ [14]) that there is no Einstein-type $R$-term in the D-brane action in type II superstring case\(^2\) but there are $O(\alpha'^2)$ correction terms [11, 13, 12]

\[ L^{(p)} = T(p) \sqrt{G} e^{-\phi} \left ( 1 - \frac{1}{24} \left ( \frac{4\pi^2 \alpha'}{32\pi^2} \right )^2 \left [ (R_T)_{\alpha \beta \gamma \delta} (R_T)_{\alpha \beta \gamma \delta} - 2(R_T)_{\alpha \beta} (R_T)_{\alpha \beta} + 2 \bar{R}_{ij} \bar{R}^{ij} \right ] \right ). \]  

(1.2)

\(^1\)We absorb $2\pi \alpha'$ into the $U(1)$ gauge field strength.

\(^2\)R-term does appear in the bosonic string D-brane action [15, 16].
Here \( \alpha, \beta, \ldots = 0, 1, \ldots, p \) are the “parallel” and \( i, j, \ldots = 1, \ldots, 9 - p \) are the “transverse” indices and the tensors \( R_T \) and \( R_N \) are constructed from the world-volume and normal bundle connections and involve the second fundamental form (see [11, 13] for details). Note that (1.2) cannot be written just in terms of the curvature of the induced metric.

The analysis of [11, 13] excluded the standard \( R \) term but it did not address the possible presence of a Brans-Dicke type \( R f(X) \) terms, e.g., \( RX^2 \), where \( X \) stands for the “transverse” \( (X^i) \) scalar components and \( R \) is the curvature in the directions parallel to the brane.\(^3\)

One may actually rule out the presence of similar terms with explicit \( X \)-dependence on quite general grounds. For example, given a supergravity soliton, its effective action may be derived in a static gauge. Since \( X^i \) is a goldstone boson reflecting the breaking of translational invariance, the action should depend on it only through \( \partial X^i \).\(^4\) Also, from the string-theory side, the \( U(1) \) gauge invariance combined with T-duality relation between \( F_{\alpha \beta} \) and \( \partial_\alpha X^i \) implies that the effective action reconstructed from string amplitudes should depend only on derivatives of the embedding coordinates. For example, we may start with the D9-brane action which does not contain transverse \( X \)-scalars; its action is
\[
\int d^{10}\sigma \ e^{-\phi} \left[ \sqrt{\det(G_{\alpha \beta} + F_{\alpha \beta})} + \text{higher-derivative terms} + \frac{\alpha'}{2} R^2 \text{-terms} + \ldots \right].
\]
Assuming that the space-time metric is flat in some \( 9 - p \) (toroidal) directions we may apply T-duality in these directions. That will effectively convert the corresponding components of \( A_\alpha \) into \( X^i \); we should then finish with a Dp-brane action in a curved metric in “parallel” directions, and there is no way it can contain \( R f(X) \) terms.

At this point one may wonder how the absence of the \( RX^2 \) term in D3-brane action is consistent with the AdS/CFT correspondence. One needs the standard conformal coupling term \( \frac{1}{6} g_{\alpha \beta} X^i X^i \) for the 6 scalars of the \( N = 4 \) SYM theory to define the conformal stress tensor operator for the scalars, so one would expect this term to be present in a D3-brane probe action. This apparent puzzle was implicit already in [17, 18, 19, 20].\(^5\)

This puzzle was effectively resolved in [23]. As was explained there, starting with a negative curvature Einstein space with a conformal boundary with an arbitrary curved boundary metric \( g_{\alpha \beta} \) (e.g., asymptotically \( AdS_{p+2} \) space), and considering a Dp-brane probe placed close and parallel to the boundary and described by the standard DBI action (1.1), one finds the effective conformal coupling term \( \frac{p-2}{4(p-1)} R(g) r^2 \) for the normal-direction scalar \( r \) \( (r^2 = X^i X^i \text{ in the } AdS_5 \times S^5 \text{ case}) \). In more detail, expanding the metric \( g_{\alpha \beta} \) in

\(^3\) One might wonder whether the previous analysis of curvature corrections was somehow incomplete. In [13] it was shown that the results of [11] had several ambiguities which had to be fixed before one could conclude that [12] is indeed the correct action to order \( \alpha'^2 \). The case of the \( RX^2 \) term has a similar ambiguity as we shall discuss below.

\(^4\) There is also a functional dependence of the background fields on \( X = \hat{X} + \tilde{X} \), e.g., \( G_{\mu \nu}(X) \simeq G_{\mu \nu}(\hat{X}) + X^i \partial_i G_{\mu \nu}(\hat{X}) + \ldots \), but it results only in terms with normal derivatives of the background fields which we are not interested in.

\(^5\) One needs this curvature coupling term to argue [19] that the moduli space is lifted when SYM theory is defined on a sphere. This term is also crucial for conformally-invariant coupling of SYM theory to external conformal supergravity sources [21] needed to define the partition function form [24, 22] of the AdS/CFT correspondence.
geodesic distance from the boundary and plugging the expansion into the induced-metric determinant part of the DBI action for the brane, we get indeed the $Rr^2$ coupling between the boundary curvature and the normal scalar. It should be stressed that this curvature coupling term originates not from an additional contribution to the DBI action but rather from a specific choice of (the expansion of) the bulk metric, which is curved in both the boundary and the normal directions. The Weyl invariance of the normal scalar coupling to the curved boundary metric is again a consequence of the specific embedding of the brane in the AdS-type space with a conformal boundary.7

To complement the above (already convincing) arguments, we decided to explicitly verify the absence of the $RX^2$ term in the D-brane action by a first-principle – string S-matrix – computation. The previous discussions based on comparing the 3-point (one graviton $h$ – two scalar $X$) amplitudes are not sufficient for this purpose. Here we are interested in the case when the gravitons are polarized along the D-brane world-volume directions.8 In this case the $RX^2$ term gives vanishing contribution to $hXX$ amplitude.

The direct way to rule out the $RX^2$ term is then to compute the 2-graviton – 2-scalar ($hhXX$) 4-point superstring amplitude on the disc (with the Dp-brane boundary conditions) and to compare it with the field-theory amplitude predicted by the sum of the bulk Einstein action and the DBI action (1.1). This is what we are going to do below. We shall start with the superstring computation in sections 2 and 3 and then show in section 4 that the result is in complete agreement with the DBI action at the second-derivative order.

For completeness, let us briefly mention some other work on gravitational couplings on the brane. While there is no tree-level $R$-term on D-branes of type II superstring theories, such term may be induced at 1-loop string level in the case of reduced amount of supersymmetry (see [24, 25]). In the case of D-branes in bosonic string (and non-BPS branes in superstring) in addition to tree-level $R$-term one expects to find $Rf(T)$ couplings for the open string tachyon $T$ [15]. In the case of branes in AdS case (in the RS set-up [26]) one generically finds (by performing zero-mode analysis [27, 28] which should be related to the discussion in [23]) Brans-Dicke type terms on the 3-brane.

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6It is crucial here that eq. (3.6) in [23] holds for the specific choices of the embedding and the metric. Then the radial coordinate $r$ can be identified with the Riemann normal coordinate at the point we expand, and the expansion of the metric near the boundary will have the form $g_{\alpha\beta}(x,r) = g_{\alpha\beta}(x,0) + \frac{1}{3}r^2 R_{\alpha\beta} + \ldots$.

7In more general backgrounds or for different orientations of the brane one may end up with other effective couplings which may not be Weyl-invariant.

8In general, in the case of the “transversely” polarized gravitons, there is an ambiguity of adding terms like $(D\Omega_{\alpha\beta})\Omega^{\alpha\beta}X^2$ and $(D^\mu D^\nu \xi^i)\Pi_{\mu\nu}X^2$ where we have used the normal frame $\xi^i$ and the second fundamental form $\Omega_{\alpha\beta} (\Pi_{\mu\nu} \equiv \partial_\mu X^\nu \partial_\nu X^\mu)$. The bulk covariant derivatives are projected to the normal and tangent bundles using the projectors $\delta^i_\alpha \xi^\alpha_\nu$ and $\Pi_{\mu\nu}$. Such terms with appropriate coefficients can cancel the contribution of $RX^2$ term to the $hXX$ amplitude without any influence on the four-scalar amplitude.

9Notice also that the possible ambiguities mentioned in the previous footnote give individually vanishing contributions and do not lead to any contradiction.
2. Preliminaries

As explained above, we intend to compute the tree-level (disc) scattering amplitude involving two closed-string modes – gravitons polarized along the brane – and two open string modes – scalars $X$ describing transverse fluctuations of the brane. This computation is aimed at determining whether a world-volume coupling $RX^2$ is present in the D-brane action. Notice that the contribution from such a term to the one graviton – two scalar amplitude is vanishing as a result of the lowest order equations of motion.

D-branes are non-perturbative string states [4]. This means that they are not part of the ordinary string spectrum and at weak coupling have infinite mass compared to string modes. In space-time D-branes are represented as static p-dimensional defects. As usual, that means that we must impose different (Neumann and Dirichlet) boundary conditions on the tangent and normal directions to the D-brane,

$$\partial_\perp X^\alpha |_{\partial\Sigma} = 0$$

$$X^i |_{\partial\Sigma} = 0$$  \hspace{1cm} (2.1)

The Greek indices ($\alpha = 0, 1 \ldots, p$), correspond to the coordinates parallel to the brane (we shall call them “world-volume” directions) and the Latin ones ($i = p + 1, \ldots, 9$) – to the coordinates in the directions normal to the brane.

Before presenting details of our calculation, let us review the basic formalism of string vertex operators and their expectation values on the disc. We follow closely the review of [8] and references there, in particular [10, 7].

The string operators for an NS-NS massless closed string have the following general form

$$V(z, \bar{z}) = \epsilon_\mu_\nu : V^\mu (z) \, : \, : V^\nu (\bar{z}) :$$  \hspace{1cm} (2.2)

where $\mu = 0, 1, \ldots, 9$ and $s = 0, -1$ denotes the superghost charge or equivalently the picture in which the operator is defined. The total superghost charge on the disk is required to be $Q_{sg} = -2$ as a consequence of the super-diffeomorphism invariance. The holomorphic parts of the in the pictures 0 and 1 are given by:

$$V^\mu_0 (p, z) = (\partial X^\mu (z) + ip \cdot \bar{\psi}(z) \bar{\psi}(z)) e^{ip \cdot X(z)}$$

$$V^\mu_1 (p, z) = e^{-\phi(z)} \psi^\mu (z) e^{ip \cdot X(z)}$$  \hspace{1cm} (2.3)

The Green’s functions on the disc are found using the method of image charges on a two dimensional surface [31]. Each string vertex inserted at position $z$ on the disc has an image at $\frac{1}{z}$. Imposing Neumann or Dirichlet boundary conditions we find the correlators on the disc [28, 3, 30, 13]:

$$\langle \partial_\perp X^\mu (z) e^{i(Dp) \cdot X(\bar{w})} \rangle = \frac{i(Dp)^\mu \bar{w}}{1 - \bar{z} \bar{w}}$$

$$\langle \partial_\perp X^\mu (\bar{z}) e^{ip \cdot X(w)} \rangle = \frac{i(Dp)^\mu w}{1 - \bar{z} \bar{w}}$$

$$\langle \partial_\perp X^\mu (z) \partial_\perp X^\nu (\bar{w}) \rangle = \frac{\eta^\mu \nu}{(1 - \bar{z} \bar{w})^2}$$
$$\langle \psi^\mu(z) \psi^\nu(w) \rangle = -\frac{\eta^\mu\nu}{z-w}$$ (2.4)

$$\langle \psi^\mu(z) \bar{\psi}^\nu(\bar{w}) \rangle = iD^\mu\nu$$

$$\langle c(z_1)c(z_2)c(z_3) \rangle = C_{D_2}^{\text{ghost}}(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)$$

$$\langle c(z_1)c(z_2)\bar{c}(\bar{z}_3) \rangle = C_{D_2}^{\text{ghost}}(z_1 - z_2)(1 - z_1\bar{z}_3)(1 - z_2\bar{z}_3)$$

$$\langle e^{-\phi(z)}e^{-\phi(\bar{w})} \rangle = 1$$

where $D^\nu_\mu$ reverses signs of the fields with Dirichlet conditions and is defined in \[9\] ($D^\lambda\mu D^\lambda\nu = \eta_{\mu\nu}$).

We shall specialize to the case of a scattering involving only “world-volume polarized” gravitons. As we shall see in section 4 the field theory computation simplifies considerably for such a choice of polarizations, mainly because in such a case the tadpole diagram of a graviton with a “transverse” scalar is vanishing. We have the following conditions on the momenta and polarizations due to lowest order equations of motion and momentum conservation:

$$k_1 + k_2 + p + q = 0 \, , \quad \text{Tr}(\epsilon_1) = \text{Tr}(\epsilon_2) = 0$$
$$\epsilon_1 \cdot p = 0 \, , \quad \epsilon_2 \cdot q = 0$$
$$\zeta_n \cdot p = \zeta_n \cdot q = \zeta_n \cdot k_n = 0 \, .$$ (2.5)

Here $n = 1, 2$ labels the two scalars with momenta $k_n^\mu$ and polarizations $\zeta^\mu_n$ while graviton momenta are $p$ and $q$ and their traceless polarization tensors have non-zero components only along the world-volume directions of the brane. All momenta are assumed to have only “world-volume” components being non-vanishing.

### 3. String theory amplitude

The correlator we need to compute is:

$$A_{\text{string}} \propto \int_{|z| \leq 1} dz \int_{|z| = 1} dz' \frac{d^2w}{(2\pi)^2} \langle c(z)c(z')V^\mu\nu_{(-1,-1)}(z',z')V^\rho\sigma_{(0,0)}(z,\bar{z}) \rangle$$

$$\langle c(x) - c(\bar{x}) \rangle V^i_{(0)}(x)V^j_{(0)}(\bar{x}) \rangle (\epsilon_1 D)_{\mu\nu}(\epsilon_2 D)_{\rho\sigma} \zeta^1_1 \zeta^2_2 \rangle .$$ (3.1)

Here we have fixed the $SL(2, R)$ Mobius symmetry gauge by setting the position of one of the two gravitons to be at the center of the disc $z' = \bar{z}' = 0$ and the positions of the two open string scalars to be at the complex conjugate points $(x, \bar{x})$ at the boundary of the disc. 10

The correlators of the ghosts and superghosts are (using (2.4)):

$$\langle (c(x) - c(\bar{x}))c(0)\bar{c}(0) \rangle = C_{D_2}^{\text{ghost}}(x - \bar{x})$$

$$\langle e^{-\phi(0)}e^{-\phi(0)} \rangle = 1$$

10We thank Ashoke Sen for pointing out an error in $SL(2)$ gauge fixing in the original version of this paper.
Since the scalars have polarizations only in the transverse direction and all momenta are in world-volume directions the computation simplifies considerably. From $V_{(0)}^i V_{(0)}^j$ we need only to retain the following correlators:

$$\langle \partial X^i \partial X^j \rangle = -\frac{\eta^{ij}}{(x-x')^2}, \quad \langle \psi^i \psi^j \rangle = -\frac{\eta^{ij}}{(x-x')},$$ \hspace{1cm} (3.2)  

with all other correlators (coming from the cross product of world-sheet scalars and fermions in the $V_{(0),V_{(0)}^j}$ vertices) producing vanishing contributions. It is easy to check that given the symmetries of the $\epsilon_1, \epsilon_2$ tensors, the part in $V_{(0)}^\mu V_{(0)}^\rho$ multiplying $\langle \partial X^i \partial X^j \rangle$ in (3.2) is vanishing. It remains then to compute the following correlator

$$\langle k_1 \cdot \psi(x)k_2 \cdot \psi(x)\rangle \mu(0) \psi^\beta(0)$$

$$\langle \partial X^\gamma \partial X^\delta + i \partial X^\gamma q \cdot \bar{\psi} \psi^\delta + i \partial X^\delta q \cdot \psi \bar{\psi}^\gamma - q \cdot \psi \bar{\psi}^\gamma q \cdot \bar{\psi}^\delta \rangle$$

$$\langle e^{2ik_1 X} e^{2ik_2 X} e^{ipX} e^{ipX} e^{iqX} e^{iqX} \rangle$$ \hspace{1cm} (3.3)  

which should be multiplied with $\langle \psi^i \psi^j \rangle$ in (3.2) and $(\epsilon_1 D)_{\mu\nu} \rightarrow (\epsilon_1)_{\alpha\beta}$, $e_{\mu\nu} \rightarrow (\epsilon_2)_{\gamma\delta}$.

After some tedious computations and using the symmetries of the polarization tensors and the symmetry under interchange of the two scalars we arrive at the following result for the integrand in (3.3) (we ignore overall numerical coefficient and isolate the polarization tensor factors $\epsilon_1, \epsilon_2, \zeta^1, \zeta^2$):

$$A_{\text{string}} \sim i |1 - \bar{x}z|^4 q^k |1 - x\bar{z}|^4 q^j |z|^2 |x - \bar{x}|^4 q^k \eta^{ij}$$

$$\left[ k_1^\alpha k_2^\beta (k_1^\gamma k_1^\delta I_1 + k_1^\gamma k_2^\delta I_2) - k_1^\alpha k_1^\gamma (k_2 q \eta^\beta - k_2 q^\beta) I_3 - \frac{k_1^\gamma k_2^\alpha}{2} (k_1 q \eta^\beta - k_2 q^\beta) I_4 - \frac{k_1^\gamma k_2^\alpha}{2} (k_1 q \eta^\beta - k_2 q^\beta) I_5 + \frac{1}{2} (k_1 q \eta^\beta - k_2 q^\beta)(k_1 q \eta^\beta - k_2 q^\beta) I_6 \right] + (1 \leftrightarrow 2)$$ \hspace{1cm} (3.4)  

Here $(1 \leftrightarrow 2)$ stands for symmetrization under interchange of the two scalars and the two graviton polarizations and momenta, and

$$I_1 = \frac{|1 + \bar{x}z|^2}{|1 - \bar{x}z|^2 |z|^2}, \quad I_2 = \frac{(1 - |z|^2)^2 + |z|^2(x^2 + \bar{x}^2) - (\bar{x}^2 + z^2)}{|1 - \bar{x}z|^2 |1 - \bar{x}z|^2 |z|^2},$$

$$I_3 = \frac{(1 + \bar{x}z)(1 - x\bar{z})^2}{(x - \bar{x})^2 |1 - x\bar{z}|^2 |1 - x\bar{z}|^2 |z|^2} \text{ c.c.}, \quad I_4 = \frac{(1 - x\bar{z})(1 + x\bar{z})\bar{x} - \text{ c.c.}}{|x - \bar{x}|^2 |1 - \bar{x}z|^2 |z|^2}$$

$$I_5 = \frac{(1 - x\bar{z})(1 + x\bar{z})\bar{x} - \text{ c.c.}}{(x - \bar{x})^2 |1 - x\bar{z}|^2 |z|^2}, \quad I_6 = \frac{(1 - |z|^2)^2}{|1 - x\bar{z}|^2 |1 - \bar{x}z|^2 |z|^2}.$$  

Next, we should perform the integration over the world-sheet coordinates $x$ and $z, \bar{z}$. Since we are interested only in determining the $RX^2$ contribution, we do not actually need to compute the whole integral: it is sufficient to extract the terms with only two powers of the momenta. Given that all the terms in (3.4) have four powers of momenta we should
just extract the residues of the poles on the disc. The integrand has various poles. The integral over \( z \) has poles at \( z = 0, x, \bar{x} \). There are no poles when \( x \to \bar{x} \) as we can easily infer from (3.4), (3.3) except possibly when \( z \to x \to \bar{x} \to \pm 1 \). This latter case will be discussed later in this section.

We need to keep in mind now that when we are integrating poles at \( x, \bar{x} \), residues should be taken with factor 1/2 since they are at the boundary of the integration region. As a first step we expand each of the integrands in (3.5) in some small region around each pole of \( z \). The residue is extracted using the standard rule (see, e.g., [32], eq. (4.7)):

\[
\int dy \ y^{-1+\alpha' k_i k_j} \to -\frac{1}{\alpha' k_i k_j}.
\] (3.6)

For example, let us consider \( I_1 \) in (3.3) which has poles at \( z = 0, x = \frac{1}{\bar{x}} \). Expanding around a given pole both \( I_1 \) and the terms in the first line of (3.4) we get:

\[
\int \frac{dx}{x} (x - \bar{x})^{4k_1 k_2 + 1} \int_{0}^{2\pi} d\theta \int_{r>\epsilon} dr \ r^{2pq - 2} \to \frac{8\pi}{2pq},
\] (3.7)

\[
\int \frac{dx}{x} (x - \bar{x})^{4k_1 k_2 + 4q - 2} \int_{0}^{\varphi} d\theta \int_{r>\epsilon} dr \ r |1 + 1|^{2, 4q - 1} \to \frac{8\pi}{2q \cdot k_1}.
\]

We have introduced an UV cut-off \( \epsilon \) which we take to zero when we extract the poles. We take the limit \( \alpha' \to 0 \), i.e. expand in powers of momenta, for all exponents which do not contain singular terms.\(^{12}\) We have also accounted for the relative factor \( \frac{1}{2} \) between poles at the boundary and in the interior by integrating \( \theta \) only from \([0, \pi]\) in the second of (3.7). Notice that the \( x = e^{i\varphi} \) integral has decoupled in the pole regions from the one over \( z \) and is equal to \( \int_{0}^{\pi} d\varphi \sin \varphi = 2 \) for all the cases in (3.7).\(^{13}\)

One may worry that there also poles at \( z \to \bar{x} \) or \( x \to \bar{x} \) in the expression above. These potential singularities could appear at the points of the disc where the three operators approach each other, \( z = x = \bar{x} = \pm 1 \). As we argue in Appendix A there are no massless poles in these cases other than those considered in (3.7).

Following the same procedure for all terms in the amplitude and symmetrizing with respect to the two scalars we get:

\[
A_{\text{string}} \sim (k_1 \epsilon_1 k_2) \left[(k_1 \epsilon_2 k_1) + (k_1 \epsilon_2 k_2) \frac{qk_2}{4(qk_1pq)} + (k_1 \epsilon_2 k_2) \frac{1}{2pq} \right]
\]

\[
- \frac{1}{4pq} \left[(k_1 \epsilon_1 k_2) + (k_1 \epsilon_2 k_2) \right] \left[(k_2 \epsilon_1 k_1) + (k_2 \epsilon_2 k_2) \right] + \frac{1}{4pq} \left[(k_2 \epsilon_1 k_1) + (k_2 \epsilon_2 k_2) \right] + \frac{1}{4pq} \left[(k_2 \epsilon_1 k_1) + (k_2 \epsilon_2 k_2) \right] + (1 \leftrightarrow 2)
\] (3.8)

where \((1 \leftrightarrow 2)\) stands for the remaining symmetrization in graviton polarization and momenta. The apparent “double-pole” momentum factors \( \frac{1}{(qk_1pq)} \), etc. here can be eliminated by using momentum conservation.

\(^{11}\)There is a possibility that we have residues of poles which do not diverge as we take the limit \( \alpha' \to 0 \). These lead to higher order contact terms and can be ignored.

\(^{12}\)Keeping these contributions would produce only higher order corrections.

\(^{13}\)The integration region is chosen to be \([0, \pi]\) since the remaining region up to \(2\pi\) corresponds to exchange of the open string states which was taken into account explicitly in (3.4).
As a check, one may demonstrate that the result is gauge invariant under \((\epsilon_i)_\mu p_\nu + (\xi)_\nu p_\mu\) (the first and fourth lines of (3.8) each are separately gauge invariant, while the second and third lines combine into a gauge-invariant expression).

Using conservation of momentum to eliminate some \(q\)-momenta in terms of \(k_1\) and \(k_2\), and symmetrizing in the graviton polarizations, we arrive at a much simpler expression:

\[
A_{\text{string}} \sim 2(\zeta_1 \zeta_2) \left[ (k_1 \epsilon_1 k_2)(k_1 \epsilon_2 k_2) \frac{k_{1q}}{2(pq)} + (k_2 \epsilon_1 \epsilon_2 k_1) \frac{k_{2q}}{2(pq)} \right. \\
+ (k_1 \epsilon_1 k_1)(k_2 \epsilon_2 k_2) \frac{k_{1q}}{4(k_2 q)(pq)} + (k_2 \epsilon_1 k_2)(k_1 \epsilon_2 k_1) \frac{k_{2q}}{4(k_1 q)(pq)} + \text{Tr}(\epsilon_1 \epsilon_2) \frac{(k_{2q})(k_{1q})}{4(pq)} \]  

(3.9)

4. D-brane field theory amplitude

Let us now compare the string-theory result with field theory amplitude that follows from graviton–scalar interaction terms in D-brane action.

The full field-theory action contains the standard bulk supergravity action (written in the Einstein frame)

\[
S_{10} = -\frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[ R - \frac{1}{2}(\partial \phi)^2 + \ldots \right],
\]

(4.1)

and the D-brane action. The low energy dynamics of a Dp-brane in a curved space is encoded in the DBI action. The D-brane action contains the world-volume massless scalar fields \(X^\mu(\sigma)\) (the embedding coordinates of the p-brane in the ambient space-time) and the \(U(1)\) gauge fields \(A_\alpha(\sigma)\) coupled to the bulk supergravity fields (\(\sigma^\alpha\) are the world-volume coordinates). In addition, there are supersymmetric partners of these fields but they will be irrelevant for our present discussion. We shall also ignore the WZ type term describing well-known coupling of D-brane to the R-R potentials. In terms of the Einstein-frame bulk metric \(g_{\mu\nu}\) the DBI action may be written as

\[
S_{\text{DBI}} = -T_p \int d^{p+1}\sigma \ e^{\frac{\phi}{2}} \sqrt{-\det[\tilde{g}_{\alpha\beta}(X) + e^{-\phi/2} \tilde{B}_{\alpha\beta}(X) + e^{-\phi/2} F_{\alpha\beta}]} ,
\]

(4.2)

where \(\tilde{g}_{\alpha\beta} = g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu\), \(\tilde{B}_{\alpha\beta} = B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu\) are the pull-backs on the brane of the corresponding bulk tensors.

Our aim is to compute the 2-graviton \((h)\)– 2-scalar \((X)\) tree amplitude and to compare it with string-theory result, to see if we need to introduce some additional second-derivative interactions to the DBI action like \(RX^2\).

The field-theory amplitude will contain contact \(hhXX\) contribution from the DBI action, a scalar exchange between the two \(hXX\) vertices, and also a graviton exchange between the closed string vertex \(hhh\) present in the bulk supergravity action and the \(hXX\) vertex present in the DBI action (see Fig.1). Since there is no bulk coupling for the dilaton of the form \(\phi hh\) there will be no dilaton exchange diagram, i.e. the dilaton (and \(B_{\mu\nu}\)) contributions may be ignored in the present case. The scalar–gauge field couplings are also irrelevant, i.e. we may set \(A_\alpha = 0\). An even more significant simplification is that our choice of momenta and polarizations for the graviton excludes mixing vertices like
$\partial^\alpha X^i h_{\alpha i}$. These "tadpole" diagrams would have complicated considerably the field theory computation since we would have to include diagrams with two and even three intermediate states.

Expanding the induced metric in (4.2) near the flat space and using the static gauge $X^\alpha = \sigma^\alpha$ we get $(\alpha = 0, \ldots, p; \ i = 1, \ldots, 9 - p)$

$$
\tilde{g}_{\alpha\beta} = g_{\alpha\beta}(\sigma, X) + 2g_{i(\alpha}(\sigma, X)\partial_{\beta)}X^i + g_{ij}(\sigma, X)\partial_\alpha X^i\partial_\beta X^j
$$

(4.3)

$$
g_{\alpha\beta} = \eta_{\alpha\beta} + 2\kappa h_{\alpha\beta}(\sigma, X), \quad g_{ij} = \delta_{ij} + 2\kappa h_{ij}(\sigma, X), \quad g_{\alpha i} = 2\kappa h_{\alpha i}(\sigma, X). \quad (4.4)
$$

In what follows we shall assume that the graviton is polarized parallel to the brane, i.e. $h_{ij} = 0 = h_{\alpha i}$.

![Figure 1](image)

Figure 1: This figure shows three types of contributions to the scattering amplitude, both in string-theory (upper raw) and field-theory (lower raw) representation. In string diagrams darker dots stand for the graviton vertices and lighter dots for the scalar vertices; the double line is an open or closed string propagator. In field-theory diagrams the wiggly lines are graviton and straight lines are scalar propagators. The first contribution is a contact term (coming from the region when all 4 points on the disc are close to each other). The second one is the s-channel exchange contribution originating from the factorization of the disc into two discs connected by a scalar propagator (corresponding to the region where points come close pairwise). The third diagram corresponds to the region where the points of graviton insertions are close to each other so that the amplitude factorizes to a sphere and a disc connected by a graviton propagator.

To expand the square root of the determinant in (4.2) we use formula

$$
\sqrt{\det(\delta^\alpha_\beta + M^\alpha_\beta)} = 1 + \frac{1}{2}M^\alpha_\alpha - \frac{1}{4}M^\alpha_\beta M^\beta_\alpha + \frac{1}{8}(M^\alpha_\alpha)^2 + \frac{1}{8}M^\alpha_\beta M^\beta_\gamma M^\gamma_\alpha - \frac{1}{8}M^\alpha_\beta M^\beta_\alpha M^\gamma_\gamma + \frac{1}{82}((M^\alpha_\alpha)^3 + \ldots \quad (4.5)
$$

Using Fourier representation for the fluctuations $h_{\alpha\beta}$ and $X^i$ and expanding (4.2) we get the following scalar propagator and $hXX$ and $hhXX$ vertices (multiplying them by $i$)

$$
P^i_j = -\frac{i\delta^i_\gamma}{k^2} \quad (4.6)$$
The contribution from the exchange of a scalar field (s-channel) is:

\[ V_{X\chi}^{h} = -2i\kappa T_{\rho}(\zeta_{1}\zeta_{2})(k_{1}\epsilon k_{2}) \]  \hspace{1cm} (4.7)

\[ V_{X\chi}^{hh} = 4ik^{2}T_{\rho}(\zeta_{1}\zeta_{2})[(k_{1}\epsilon e_{2}k_{2}) + (k_{2}\epsilon e_{1}k_{1}) - \frac{1}{2}(k_{1}k_{2})Tr(\epsilon_{1}\epsilon_{2})] . \]  \hspace{1cm} (4.8)

To compute the graviton exchange contribution we will need also the hXX vertex with off-shell graviton (which we shall denote by \( H_{\mu\nu} \), with polarization tensor \( E_{\mu\nu} \))

\[ V_{X\chi}^{H} = -2i\kappa T_{\rho}(\zeta_{1}\zeta_{2})(k_{1}E_{k_{2}})(\zeta_{1}\zeta_{2}) - \frac{1}{2}(\zeta_{1}\zeta_{2})(k_{1}k_{2})E_{\mu} - (k_{1}k_{2})(\zeta_{1}E_{\zeta_{2}}) . \]  \hspace{1cm} (4.9)

The graviton propagator corresponding to the bulk Einstein action is

\[ (P_{H})_{\mu\nu,\lambda\rho} = -\frac{i}{2p^{2}}(\eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda} - \frac{1}{4}\eta_{\mu\nu}\eta_{\lambda\rho}) , \]  \hspace{1cm} (4.10)

while the vertex for the two on-shell gravitons and one off-shell one is:

\[ V(H, h_{1}, h_{2}) = -4\kappa(H_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}H_{X}^{X})(h_{\rho\sigma}R_{\mu\rho\sigma
u}^{(1)} - \frac{1}{4}h_{\rho\sigma,\mu}h_{\rho\sigma,\nu} + \frac{1}{2}h_{\rho\mu,\sigma}(h_{\sigma\nu,\rho} - h_{\mu,\sigma})] , \]  \hspace{1cm} (4.11)

The relevant field-theory amplitude contributions are shown in Fig. 1. The contact contribution is given by the \( V_{X\chi}^{hh} \) term in (4.8):

\[ (A_{c})_{X\chi}^{hh} = i\kappa^{2}T_{\rho}(\zeta_{1}\zeta_{2})\left[4(k_{1}\epsilon e_{1}e_{2}k_{2}) + 4(k_{2}\epsilon e_{1}e_{2}k_{1}) - 2(k_{1}k_{2})Tr(\epsilon_{1}\epsilon_{2})\right] . \]  \hspace{1cm} (4.12)

The contribution from the exchange of a scalar field (s-channel) is:

\[ (A_{s})_{X\chi}^{hh} = (V_{s})_{X\chi}^{h}P_{i}(V_{j})_{X\chi}^{h} = 2i\kappa^{2}T_{\rho}\left[(k_{1}\epsilon k_{1})(k_{2}\epsilon k_{2})\frac{1}{k_{2}q} + (k_{1}\epsilon k_{2})(k_{2}\epsilon k_{1})\frac{1}{k_{1}q}\right] , \]  \hspace{1cm} (4.13)

where we have symmetrized in both scalars and gravitons. Finally, it is long but straightforward to compute the contribution from the exchange of a graviton (t-channel) using (4.9),(4.10) and (4.11):\(^{14}\)

\[ (A_{t})_{X\chi}^{hh} = (V_{t})_{X\chi}^{H}(P_{H})_{\mu\nu,\lambda\rho}V^\lambda_{\nu\rho}(H, h_{1}, h_{2}) = i\kappa^{2}T_{\rho}\left[-2(k_{1}\epsilon e_{1}e_{2}k_{2})(\frac{k_{1}q}{p_{q}} + 1)\right. \right.

\[ -2(k_{2}\epsilon e_{1}e_{2}k_{1})(\frac{k_{2}q}{p_{q}} + 1) - Tr(\epsilon_{1}\epsilon_{2})\frac{(k_{1}q)(k_{2}q)}{p_{q}} - (k_{1}\epsilon k_{1})(k_{2}\epsilon k_{2})\frac{k_{1}q}{(k_{2}q)(p_{q})}) \right. \]

\[ \left. - (k_{1}\epsilon k_{2})(k_{2}\epsilon k_{1})\frac{k_{2}q}{(k_{1}q)(p_{q})}) - \frac{1}{p_{q}}(k_{1}\epsilon e_{2}k_{2})(k_{1}\epsilon k_{1}) + 2(k_{1}\epsilon k_{2}) + (k_{2}\epsilon k_{2})) \right. \]

\[ \left. + \frac{1}{p_{q}}(k_{1}\epsilon k_{2})(2(k_{2}e_{2}k_{2}) + (k_{1}\epsilon k_{2})) + (\epsilon_{1} \leftrightarrow \epsilon_{2}) \right] . \]  \hspace{1cm} (4.14)

Combining (4.12),(4.13) and (4.14) and using the symmetry under \( \epsilon_{1} \leftrightarrow \epsilon_{2} \) in \( (A_{t})_{X\chi}^{hh} \) to simplify some terms, it is possible to show that the field theory amplitude reproduces the

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\(^{14}\)It turns out that the last term in \( V_{X\chi}^{H} \) does not actually contribute due to the structure of the three-graviton vertex.
string-theory amplitude (3.9). This rules out the presence of an extra $RX^2$ term in the DBI action.

The only potential caveat could be the following. Since we have not been careful to include the normalization factors in the string amplitude, the agreement is only up to the overall coefficient. One could imagine that $RX^2$ term could produce a contribution which is also proportional to the full string amplitude (3.9). However, this possibility is excluded as $RX^2$ cannot give an s-channel contribution present in the string amplitude.\footnote{It may still contribute to the t-channel since it modifies the $V_{X^XX}$ vertex.}

One may also try to add some other terms which could account for an additional s-channel contribution; the only candidate with the right number of momenta is $X_i\Omega^i_{\alpha\beta}$, where $\Omega^i_{\alpha\beta}$ is the second fundamental form [11, 13]. Such term, however, is proportional to the lowest-order equations of motion and therefore can be removed by a field redefinition.

We conclude that the standard DBI action (4.2) is in full agreement with the string S-matrix at the second derivative order.

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**Appendix A  On singularities of the string amplitude**

Our aim here will be to show that the contribution to the amplitude from the region in the integration space where the three vertex operators (of graviton at point $z$ and two scalars at points $x$ and $\bar{x}$) approach each other at the same time is finite. We will study the integral from the $I_1$ term in (3.3):

$$\int \frac{dx}{x} \int d^2z |1 - \bar{x}z|^{4q-k_1} |1 - x\bar{z}|^{4q-k_2} |z|^{2p-\bar{q}(x - \bar{x})^{4k_1\cdot k_2+1}} \frac{|1 + \bar{x}z|^2}{|1 - \bar{x}z|^2 |z|^2}$$

\[\text{A.1}\]

where $x = e^{i\varphi}$. We want to examine the integral for $z \to x$. Let us expand the integrand around $z = x + \rho e^{i\theta}$ where $\rho$ is radial distance from the boundary of the disc and $\theta \in [0, \pi]$. We want the leading contribution for $\alpha' \to 0$, i.e. to lowest order in momenta, so we take the momenta to zero for any term in the integrand as long as this does not produce a singularity. We get this way:

$$- 4 \int_0^\pi d\varphi \int_0^\pi d\theta \int_{\varphi \neq \theta} d\rho \rho^{4q-k_1-1}(\sin \varphi)^{4k_1\cdot k_2+1} |1 - e^{2i\varphi} + \rho e^{i\varphi+i\theta}|^{4q-k_2}$$

\[\text{A.2}\]
where \( \epsilon \) is a radial cut-off with the upper limit of the radial integration not specified but assumed to be small enough for our approximation to make sense. Now we can split the integration over \( \phi \) into two regions. One far from \( \phi = 0 \) and another close to it. The latter corresponds to the limit where \( x \to \bar{x} \) which we wish to examine and we will cut-off the singular region by the same parameter \( \epsilon \) as for the radial direction. It is more rigorous to split the integration as:

\[
\int_{\epsilon}^{\frac{\pi}{2}} d\phi \int_0^\pi d\theta \int_{\rho>\epsilon} d\rho \rho^{4q_1} k_1^{-1} (\sin \phi)^{4k_1^1 - k_2 + 1} \to 8\pi \cos A \left( \frac{1}{4q \cdot k_1} \right) + \log \epsilon \sim 8\pi \left( \frac{1}{4q \cdot k_1} \right) + \log \epsilon \tag{A.3}
\]

where in the last expression we expanded for small \( A \) and kept only the \( A \)-independent terms. For the second region we can expand the integrand in (A.2) for small \( \phi \) as well:

\[
\int_{\epsilon}^{A} d\phi \int_0^\pi d\theta \int_{\rho>\epsilon} d\rho \rho^{4q_1} k_1^{-1} (\sin \phi)^{4k_1^1 - k_2 + 1} \to 8\pi \cos A \left( \frac{1}{4q \cdot k_1} \right) + \log \epsilon \sim 8\pi \left( \frac{1}{4q \cdot k_1} \right) + \log \epsilon \tag{A.4}
\]

Now we can change coordinates from “cartesian” \((\rho, \phi)\) to polar \((\lambda, \omega)\) through the transformation:

\[
\rho = \epsilon (1 + \lambda \sin \omega), \quad \phi = \epsilon (1 + \lambda \cos \omega).
\]

The first set of coordinates has the range \((\rho > \epsilon, \phi > \epsilon)\) and the second one\(^{16}\) \((\lambda > 0, \omega \in [0, \frac{\pi}{2}])\). Doing this coordinate transformation in (A.4) and expanding in powers of the cut-off \( \epsilon \) we get:

\[
\epsilon^2 \int_0^{\frac{\pi}{2}} d\omega \int_0^\pi d\theta \int_0^\infty d\lambda \left( \epsilon^{4q_2 - k_2 + 4q_1 - k_1} (1 + \lambda \sin \omega)^{4q_1 - k_1 - 1} (1 + \lambda \cos \omega)^{4k_1^1 - k_2 + 1} \right) \tag{A.5}
\]

where we took the zero momentum limit in the last term of (A.4) after the change of variables. Using momentum conservation we can show that the exponent of \( \epsilon \) is equal to two. The integral will be regular and cut-off independent in the limit \( \epsilon \to 0 \). We can therefore take safely \( A = 0 \) since all the singular terms come from (A.3).\(^{17}\) Adding the contribution from the \( \phi \in \left[ \frac{\pi}{2}, \pi \right] \) region of integration we arrive to the expression from (3.7).

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\(^{16}\)We ignore once again (since this is sufficient for our purposes) some dependence of the region of integration on \( A \) and on higher powers of the cut-off \( \epsilon \).

\(^{17}\)As we pointed out above, had we kept all \( A \) dependence in various formulas leading to (A.3) we should have found a contribution cancelling the \( A \)-dependent terms in the full integral.
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