RICCI FLOW ON MANIFOLDS WITH BOUNDARY WITH ARBITRARY INITIAL METRIC

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Abstract. In this paper, we study the Ricci flow on manifolds with boundary. In the paper, we substantially improve Shen’s result [16] to manifolds with arbitrary initial metric. We prove short-time existence and uniqueness of the solution, in which the boundary becomes instantaneously totally geodesic for positive time. Moreover, we prove that the flow we constructed preserves natural boundary conditions. More specifically, if the initial metric has a convex boundary, then the flow preserves positive curvature operator and the PIC1, PIC2 conditions. Moreover, if the initial metric has a two-convex boundary, then the flow preserves the PIC condition.

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1. Introduction

In this article, we study the deformation of Riemannian metrics on compact smooth manifolds with boundary by Ricci flow. The study of Ricci flow dates back to the work of Hamilton [9], who applied it to prove that every three dimensional closed compact Riemannian manifold with positive Ricci curvature admits a metric of constant positive sectional curvature. Since then, many significant results on the interaction between geometry and topology were established using Ricci flow. One important geometric property Ricci flow enjoys is that it preserves various conditions curvatures. A crucial ingredient in [9] is the preservation of positive Ricci curvature along Ricci flow on closed compact Riemannian manifolds in dimension three. In dimension four, Hamilton showed that the Ricci flow on closed compact Riemannian manifolds preserves positive curvature operator [10] and positive isotropic curvature [11], which were used in subsequent convergence results of the Ricci flow. In higher dimensions, the natural curvature conditions PIC, PIC1,
PIC2 are preserved by the Ricci flow on closed compact Riemannian manifolds, as observed by Brendle [2], Brendle and Schoen [3], Nguyen [14]. These curvature conditions play vital roles in general convergence of Ricci flow in higher dimensions, as in the results of Brendle [2, 4].

A natural question to ask is whether these results can be generalized to smooth manifolds with boundary. More specifically, we would like to study short-time existence problem for Ricci flow on manifold with boundary. Moreover, we would like the flow to preserve natural curvature conditions as its counterpart in the case of closed manifolds. The challenge to the short-time existence problem is the diffeomorphism invariance of the Ricci curvature. One would need to study carefully the geometry of the boundary in order to set up a well-posed boundary value problem. The challenge to the preservation of curvature conditions is the failure of usual parabolic maximum principle on the boundary, if one asks for an arbitrary initial metric.

The first work in this direction was done by Shen [16], where he proved short-time existence to the Ricci flow on smooth compact manifold with umbilic boundary. One might ask for deformation of a more arbitrary metric. Later on, short-time existence results for Ricci flow on smooth compact manifold with boundary with an arbitrary initial metric were established in the works of Pulemotov [13] and Gianniotis [8]. In particular, Gianniotis [8] has proven both short-time existence and uniqueness results, where he set up the evolution equations by prescribing the mean curvature and the conformal class of the boundary. However, there are no results so far on preserving curvature conditions in this setting. In dimension 2, more results were contributed by Brendle [1], Cortissoz and Murcia [6].

Our work substantially improves the result of [16], where we prove short-time existence and uniqueness of solutions to Ricci flow on manifold with boundary in which the boundary become instantaneously umbilic for positive time. We remark that our result does not require the initial metric to have a umbilic boundary. We approach the problem via doubling of the manifold. Extending the initial metric to the doubled manifold by reflection, we obtain an extended metric which is merely Hölder continuous. We seek a solution to Ricci flow on the doubled manifold with a Hölder continuous initial metric that is smooth for positive time. To that end, we proved:

**Main Theorem 1.** Let $M$ be a closed compact smooth manifold and $g_0 \in \mathcal{C}^\alpha(M)$ be a Riemannian metric for some $\alpha \in (0, 1)$. Let $k \geq 2$, $\gamma \in (0, \alpha)$ and $\beta \in (\gamma, \alpha)$ be given. Then there exists a $\mathcal{C}^{1,\beta}$ diffeomorphism $\psi$ and $T = T(M, \|g_0\|_\alpha)$, $K = K(M, k, \|g_0\|_\alpha)$ such that the following holds:

There is a solution $g(t) \in \mathcal{X}^{k,\beta}_k(M \times [0, T])$ to the Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t)) \quad \text{on} \quad \tilde{M} \times (0, T]$$

such that $g(0) = \psi^*g_0$ and

$$\|g\|_{\mathcal{X}^{k,\beta}_k(M \times [0, T])} \leq K.$$

Here the Banach spaces $\mathcal{X}^{k,\beta}_k(M \times (0, T])$ will be defined in the beginning of section 4. We also show that the above theorem gives a canonical solution to the Ricci flow, in the sense of the following uniqueness theorem:
Main Theorem 2. Let \( \alpha \in (0, 1) \) be given. Let \( M \) be a closed compact smooth manifold and \( g_0 \in C^\alpha(M) \) be a Riemannian metric on \( M \). Suppose that the pairs \( (g^1(t), \psi^1) \) and \( (g^2(t), \psi^2) \) satisfy the conclusion of Main Theorem 1. Then there exists a \( C^{k+1} \) diffeomorphism \( \varphi : M \to M \) such that
\[
g^2(t) = \varphi^*(g^1(t)),
\]
where \( \psi^2 = \psi^1 \circ \varphi \). In particular, \( ((\psi^1)^{-1})^* g^1(t) = ((\psi^2)^{-1})^* g^2(t) \).

Via doubling the above results imply the existence and uniqueness results for Ricci flow on manifolds with boundary:

Main Theorem 3. Let \( (M, g_0) \) be a compact smooth Riemannian manifold with boundary. Let \( k \geq 2 \), \( \beta \in (0, 1) \) and \( \varepsilon \in (0, 1 - \beta) \) be given. Then there exists a \( C^{1+\beta} \) diffeomorphism \( \psi \) and \( T = T(M, \tilde{g}, \|g_0\|_{\beta+\varepsilon}) \), \( K = K(M, k, \tilde{g}, \|g_0\|_{\beta+\varepsilon}) \) such that the following holds:

There is a solution \( g(t) \in X_{1,\beta}^k(M \times [0, T]) \) to the Ricci flow on manifold with boundary
\[
\begin{align*}
\frac{\partial}{\partial t} g(t) &= -2\text{Ric}(g(t)) & \text{on } M \times (0, T) \\
A_g(t) &= 0 & \text{on } \partial M \times (0, T)
\end{align*}
\]
such that \( g(0) = \psi^* g_0 \) and
\[
\|g\|_{X_{1,\beta}^k(M \times [0, T])} \leq K.
\]
Here \( A_g(t) \) stands for the second fundamental form of the boundary with respect to the metric \( g(t) \). For each \( t > 0 \), the metric \( g(t) \) extends smoothly to the doubled manifold \( \tilde{M} \) of \( M \), and the doubled metric lies in \( X_{1,\beta}^k(\tilde{M} \times [0, T]) \). The diffeomorphism \( \psi \) also extends to a \( C^{1+\beta} \) diffeomorphism on the doubled manifold.

Main Theorem 4. Let \( (M, g_0) \) be a compact smooth Riemannian manifold with boundary. Suppose that the pairs \( (g^1(t), \psi^1) \) and \( (g^2(t), \psi^2) \) satisfy the conclusion of Main Theorem 3. Then there exists a \( C^{k+1} \) diffeomorphism \( \varphi : M \to M \) such that \( \varphi \) extends to a \( C^{k+1} \) diffeomorphism on the doubled manifold and
\[
g^2(t) = \varphi^*(g^1(t)),
\]
where \( \psi^2 = \psi^1 \circ \varphi \). In particular, \( ((\psi^1)^{-1})^* g^1(t) = ((\psi^2)^{-1})^* g^2(t) \).

Next, we show that the canonical solution constructed above preserves natural curvature conditions, provided that the geometry of the boundary is controlled. More precisely, we prove that the flow preserves positive curvature operator, PIC1 and PIC2 conditions, provided that the boundary is convex with respect to the initial metric. Moreover, if the initial metric has a two-convex boundary, we prove that the flow preserves the PIC conditions. We recall the definitions of various curvature conditions we are interested in:

**Definition 1.1.**

(i) We say that \( (M, g) \) has positive curvature operator if \( R(\varphi, \varphi) > 0 \) for all nonzero two-vectors \( \varphi \in \Lambda^2 T_p M \) and all \( p \in M \).

(ii) We say that \( (M, g) \) is PIC if \( R(z, w, \bar{z}, \bar{w}) > 0 \) for all linearly independent vectors \( z, w \in T_p M \otimes \mathbb{C} \) such that \( g(z, z) = g(w, w) = g(z, w) = 0 \).
We say that \((M, g)\) is PIC\(1\) if 
\[ R(z, w, \bar{z}, \bar{w}) > 0 \]
for all linearly independent vectors \(z, w \in T_pM \otimes \mathbb{C} \) such that 
\[ g(z, z)g(w, w) - g(z, w)^2 = 0. \]

We say that \((M, g)\) is PIC\(2\) if 
\[ R(z, w, \bar{z}, \bar{w}) > 0 \]
for all linearly independent vectors \(z, w \in T_pM \otimes \mathbb{C} \). That means \((M, g)\) has positive complex sectional curvature.

We now state the result for curvature preservation:

**Main Theorem 5.** Suppose that \(g(t)\) is a canonical solution to the Ricci flow on manifold with boundary on \(M \times [0, T]\) given by the Main Theorem 3. Then the following holds:

If \((M, g_0)\) has a convex boundary, then

(i) \((M, g_0)\) has positive curvature operator \(\implies (M, g(t))\) has positive curvature operator;

(ii) \((M, g_0)\) is PIC\(1\) \(\implies (M, g(t))\) is PIC\(1\);

(iii) \((M, g_0)\) is PIC\(2\) \(\implies (M, g(t))\) is PIC\(2\).

If \((M, g_0)\) has a two-convex boundary, then

(iv) \((M, g_0)\) is PIC \(\implies (M, g(t))\) is PIC.

Moreover, if \((M, g_0)\) has a mean-convex boundary, then

(v) \((M, g_0)\) has positive scalar curvature \(\implies (M, g(t))\) has positive scalar curvature.

It is worth noting that a similar result to the Main Theorem 5 was proved by Schlichting in his Ph.D. thesis [15] which concerns with curvature preservation along the Ricci-DeTurck flow under convexity assumptions on the boundary. Nevertheless, our result allows the boundary to be two-convex for preserving positive isotropic curvature. The proof in Schlichting’s thesis is different from ours, whereas in our proof of Main Theorem 5 additional properties of the approximated metrics constructed in [5] are used.

Since the canonical solution we obtained preserves natural curvature conditions under suitable assumptions of the boundary data and can be extended smoothly to the doubled manifold, many results in Ricci flow on closed compact manifolds can be applied to our case. For examples, if the initial metric is PIC\(1\) and has a convex boundary, our results and [2] then imply that the Ricci flow converges to a metric of constant curvature with a totally geodesic boundary after rescaling. From [4], we can also obtain a topological classification of all compact manifolds with boundary of dimension \(n \geq 12\) which admit metrics that are PIC and have two-convex boundary and do not contain non-trivial incompressible \((n-1)\)-dimensional space forms.

The proof of the Main Theorems will occupy section 6. In section 3, we study a linear parabolic equation on vector bundles over \(M\). This result will be applied to prove the existence and uniqueness of solutions to the Ricci-DeTurck flow and the harmonic map heat flow in section 4 and 5 using the Banach fixed point argument.

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Throughout Sections 2, 3, 4, and 5, we assume, unless said otherwise, that $M$ is a closed compact manifold, $\pi : E \to M$ is a smooth vector bundle, and $\tilde{g}$ a smooth metric on $M$. For sections $\eta, \zeta$ of $E$, the expression $\eta * \zeta$ means a bilinear combination with respect to $\tilde{g}$. We fix a time $T \leq 1$, which we will make small according to our needs. We fix a finite set of coordinate charts which simultaneously serve as trivializations of the vector bundle $\{U_s, \varphi_s, \tilde{\varphi}_s\}_{s=1, \ldots, m}$ so that $\varphi_s : U_s \to \mathbb{R}^n$ is a diffeomorphism and $\tilde{\varphi}_s : \pi^{-1}(U_s) \to U_s \times \mathbb{R}^N$ is a trivialization. We may also assume the charts trivialize the vector bundle $\text{Sym}^2(T^*M)$. Our trivializations give local frames $e_s^r$ for $E$ and local frames $dx^i dx^j$ for $\text{Sym}^2(T^*M)$ on $U_s$ (note the coordinates $x_1, \ldots, x_n$ depend on $s$).

We want to work on the parabolic Hölder spaces of tensors on $M$. Consider any tensor bundle of the form $E = T^{(p,q)}M$. If $S$ is a section of $E$, then in the open set $U_s$ it can be written as $S = \sum_{r=1}^N S^r e^r_s$, where $\{e^r_s\}$ is a local frame on $E|_{\pi^{-1}(U_s)}$. For any $\alpha \in (0,1)$, we define by $C^{\alpha,\alpha/2}(M \times [t_1, t_2]; E)$ the space of maps $\eta : M \times [t_1, t_2] \to E$ such that $\eta(t) \in C^{\alpha,\alpha/2}(M; E)$ is an $\alpha$-Hölder continuous section of $E$ for each $t \in [t_1, t_2]$. Given any map $\eta \in C^{\alpha,\alpha/2}(M \times [t_1, t_2]; E)$, we define the associated $C^0$ norm to be

$$|\eta|_{0,M \times [t_1, t_2]} := \sum_{s=1}^N \sum_{r=1}^N |\eta^r_s|_{0,\varphi_s(U_s) \times [t_1, t_2]},$$

and we also define the associated parabolic Hölder semi-norm to be

$$[\eta]_{\alpha,\alpha/2; M \times [t_1, t_2]} := \sum_{s=1}^N \sum_{r=1}^N [\eta^r_s]_{\alpha,\alpha/2; \varphi_s(U_s) \times [t_1, t_2]}.$$

Subsequently, we define the parabolic Hölder norm on the space $C^{\alpha,\alpha/2}(M \times [t_1, t_2]; E)$ to be

$$||\eta||_{\alpha,\alpha/2; M \times [t_1, t_2]} := |\eta|_{0,M \times [t_1, t_2]} + [\eta]_{\alpha,\alpha/2; M \times [t_1, t_2]}.$$
\textbf{Definition 2.1.} Let $M$ be a smooth closed compact manifold, $E$ a vector bundle on $M$, and $\hat{\sigma}$ a smooth background metric on $M$. Given a nonnegative integer $k \geq 0$, a Hölder exponent $\alpha \in (0, 1)$ and a positive real number $\gamma$, we define a weighted parabolic Hölder space on the space of sections by

\begin{equation}
C^{k,\alpha}_{\gamma}(M \times (0, T]; E) := \{\eta : M \times (0, T) \to E| \eta(t) \in C^k(M; E) \text{ for each } t \in (0, T], \|\eta\|_{C^{k,\alpha}_{\gamma}(M \times (0, T))} < \infty\}.
\end{equation}

We now state some easy consequences from the definition of these spaces:

\textbf{Lemma 2.2.}

(1) $\eta \in C^{k,\alpha}_{\gamma}(M \times (0, T))$ implies $\hat{\nabla}^j \eta \in C^{k-j,\alpha}_{\gamma+j}(M \times (0, T))$ for $j \leq k$;

(2) For any $\delta > 0$, $\|\eta\|_{C^{k,\alpha}_{\gamma}(M \times (0, T))} \leq T^\delta \|\eta\|_{C^{k,\alpha}_{\gamma+j}(M \times (0, T))}$. In particular, this implies that $C^{k,\alpha}_{\gamma}(M \times (0, T)) \subset C^{k,\alpha}_{\gamma+j}(M \times (0, T))$.

(3) Suppose that $\eta \in C^{k,\alpha}_{\gamma}(M \times (0, T))$, then $\eta \in C^{k,\beta}_{\gamma}(M \times (0, T))$ for any $\beta < \alpha$;

(4) Suppose that $\eta \in C^{k,\alpha}_{\gamma}(M \times (0, T))$ and $\zeta \in C^{k,\alpha}_{\delta}(M \times (0, T))$. Then $\eta \ast \zeta \in C^{k,\alpha}_{\gamma+j}(M \times (0, T))$, where $\eta \ast \zeta$ means a bilinear combination with respect to $\hat{\sigma}$. Moreover,

$$
\|\eta \ast \zeta\|_{C^{k,\alpha}_{\gamma+j}(M \times (0, T))} \leq K(\hat{\sigma}) \|\eta\|_{C^{k,\alpha}_{\gamma}(M \times (0, T))} \|\zeta\|_{C^{k,\alpha}_{\delta}(M \times (0, T))}.
$$

\textbf{Proof.} Statements (1) and (2) follow from the definition. Statement (3) follows from the properties of parabolic Hölder spaces. Now we prove statement (4). Fix $\sigma \in (0, T]$. For any $j \leq k$, we have

$$
\hat{\nabla}^j (\eta \ast \zeta) = \sum_{j_1+j_2=j} \hat{\nabla}^{j_1} \eta \ast \hat{\nabla}^{j_2} \zeta.
$$

Let us denote $M_{\sigma} = M \times [\frac{\sigma}{2}, \sigma]$ for simplicity, we have

$$
\sigma^{\gamma+j} \hat{\nabla}^j (\eta \ast \zeta)|_{0,M_{\sigma}} \leq K(\hat{\sigma}) \sum_{j_1+j_2=j} \sigma^{\gamma+j_2} |\hat{\nabla}^{j_1} \eta|_{0,M_{\sigma}} \sigma^{\delta+j_2} |\hat{\nabla}^{j_2} \zeta|_{0,M_{\sigma}}
$$

and

$$
\sigma^{\gamma+j} \hat{\nabla}^{j} (\eta \ast \zeta)|_{\frac{\sigma}{2};M_{\sigma}} \leq K(\hat{\sigma}) \|\eta\|_{C^{k,\alpha}_{\gamma}(M \times (0, T))} \|\zeta\|_{C^{k,\alpha}_{\delta}(M \times (0, T))}.
$$

From these, statement (4) follows. \qed
3. Linear Parabolic Equation with $C^{\alpha}$ Initial Data

Although our goal is to solve a boundary-value problem for PDE on a manifold with boundary, it is equivalent to work with PDE with rough initial data on a closed manifold. Fix a real number $I \in (0, 1)$. Let $w(x, t), t \in [0, I]$ be a continuous family of Riemannian metrics on $M$. Given a section $\eta_0 \in C^{\alpha}(M; E)$, we consider the following parabolic system on vector bundle:

\[
\begin{cases}
\frac{\partial}{\partial t} \eta(x, t) - \text{tr}_w \hat{\nabla}^2 \eta(x, t) = F(x, t) & \text{on } M \times (0, T] \\
\eta(x, 0) = \eta_0(x) & \text{on } M,
\end{cases}
\]

where $T \leq I$ and $F \in \Gamma(M \times (0, T]; E)$, $w \in \Gamma(M \times [0, T]; \text{Sym}^2(T^*M))$. Here $\hat{\nabla}$ is the Levi-Civita connection with respect to the background metric $\hat{g}$. Our goal is to prove solvability of (3.1). To do that, we need the uniform parabolicity assumption on $w$: there is a $\lambda > 0$ such that

\[
\lambda |\xi|^2(x) \geq w^{kl}(x, t) \xi_k(x) \xi_l(x) \geq \frac{1}{\lambda} |\xi|^2(x)
\]

for any $(x, t) \in M \times [0, I]$ and any $\xi \in \Gamma(TM)$.

We now state the main result of this section. It will be utilized to study the existence of Ricci flow on manifold with boundary in the next section.

**Theorem 3.1.** Let $\alpha, \gamma \in (0, 1)$ be given such that $\alpha > \gamma$. Let $k \geq 0$ be an non-negative integer. Suppose that

1. $\eta_0 \in C^\alpha(M; E)$;
2. $w$ satisfies the uniform parabolicity condition (3.2);
3. $\|w\|_{C^{1,\frac{\alpha-\gamma}{2}}(M \times [0, I])} + \|\hat{\nabla} w\|_{C^{k-1,\frac{\gamma}{2}}(M \times [0, I])} \leq A$ when $k \geq 1$;
   or $\|w\|_{C^{\gamma,2}(M \times [0, I])} \leq A$ when $k = 0$.

Then there exits a positive constant $K = K(M, k, \hat{g}, A)$ such that the following holds:

For each $T \leq I$, if $F \in C^{k,\frac{\gamma-\alpha}{2}}(M \times (0, T]; E)$, then there is an unique solution $\eta$ to the system (3.1) such that

$\eta \in C^{\alpha,\frac{\gamma}{2}}(M \times [0, T]; E)$, $\hat{\nabla} \eta \in C^{k+1,\frac{\gamma}{2}}(M \times (0, T]; E)$.

Moreover, $\eta$ satisfies the estimate

$\|\eta\|_{C^{\alpha,\frac{\gamma}{2}}(M \times [0, T])} + \|\hat{\nabla} \eta\|_{C^{k+1,\frac{\gamma}{2}}(M \times (0, T])} \leq K(\|F\|_{C^{k,\frac{\gamma-\alpha}{2}}(M \times (0, T])} + \|\eta_0\|_{C^{\alpha,2}(M)})$.

We will prove this theorem in the remainder of the section.

### 3.1. Formulation of the Proof

For every $T \in (0, I]$, we define the Banach space

$W_k(M \times (0, T]; E) := C^{k,\gamma}(M \times (0, T]; E) \times C^{\alpha}(M; E)$

whose elements are the pairs of sections $h = (F, \eta_0)$, where $F \in C^{k,\frac{\gamma-\alpha}{2}}(M \times (0, T]; E)$ and $\eta_0 \in C^{\alpha}(M; E)$. We equip $W_k$ with the norm

$|h|_{W_k} := |F|_{C^{k,\frac{\gamma-\alpha}{2}}(M \times (0, T])} + \|\eta_0\|_{C^{\alpha,2}(M)}$.

Moreover, we define the norm $\|\cdot\|_{X_k(M \times [0, T]; E)}$ by

$|\eta|_{X_k} := |\eta|_{C^{\alpha,\frac{\gamma}{2}}(M \times [0, T])} + \|\hat{\nabla} \eta\|_{C^{k+1,\frac{\gamma}{2}}(M \times (0, T])}$.
We define the associated Banach space \( \mathcal{X}_k(M \times [0,T]; E) \) by
\[
\mathcal{X}_k(M \times [0,T]; E) := \{ \eta : M \times [0,T] \rightarrow E \mid \| \eta \|_{\mathcal{X}_k} < \infty \}.
\]
Subsequently \( \mathcal{X}_{k+2} \) will serve as the solution space. We basically adapt the method in Chapter IV of [12] to the case of vector bundles. The idea of the proof is as follows:

Let \( H : \mathcal{X}_{k+2} \rightarrow \mathcal{W}_k \) be the linear operator that associates any \( \eta \in \mathcal{X}_{k+2} \) to
\[
H \eta = (L \eta, \eta(\cdot, 0)),
\]
where \( L \eta = \frac{\partial}{\partial t} \eta - tr_w \hat{\nabla}^2 \eta \). Then Theorem [5.1] can be interpreted to the solvability of
\[
H \eta = h
\]
for any \( h \in \mathcal{W}_k(M \times (0,T]; E) \). It is equivalent to prove the existence of a bounded inverse operator \( H^{-1} \). The key is to construct an operator \( R : \mathcal{W}_k \rightarrow \mathcal{X}_{k+2} \) which satisfies
\[
\begin{cases}
H R h = h + Sh \\
R H \eta = \eta + G \eta
\end{cases}
\]
for some bounded operators \( S : \mathcal{W}_k \rightarrow \mathcal{W}_k \) and \( G : \mathcal{X}_{k+2} \rightarrow \mathcal{X}_{k+2} \). If their norms can be controlled such that \( \| S \|, \| G \| < 1 \), then it follows from an elementary argument that \( H^{-1} \) exists.

### 3.2. Construction of an approximated solution.

Let \( h = (F, \eta_0) \in \mathcal{W}_k \) be given. To construct the operator \( R \), we consider a system of PDE on each chart \( U_s \). As in the beginning of section 2, we let \( \tilde{\varphi}_s : \pi^{-1}(U_s) \rightarrow U_s \times \mathbb{R}^N \) be the local trivialization of \( E \) on the open set \( U_s \), and let \( \{ e^r_s \}_{r=1,\ldots,N} \) be the canonical local frame of \( \pi^{-1}(U_s) \) with respect to the trivialization. Then \( F \) can be written as \( F = \sum_{r=1}^N F^r_s e^r_s \), similarly for \( \eta \), where we abbreviate \( F^r_s(x,t) = F^r_s \circ (\varphi^{-1}_s)(x,t) \) for \( x \in \varphi_s(U_s) \equiv \mathbb{R}^n \). On each chart \( U_s \), we consider the following parabolic system:

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{\partial}{\partial t} \eta^r_s(x,t) - w^kl(x,t) \frac{\partial^2}{\partial x^l \partial x^k} \eta^r_s(x,t) = F^r_s(x,t) & \text{on } \mathbb{R}^n \times (0,T], \quad r = 1,\ldots,N \\
\eta^r_s(x,0) = (\eta_0)_s^r(x) & \text{on } \mathbb{R}^n, \quad r = 1,\ldots,N.
\end{array} \right.
\end{align*}
\]

We see that the system is equivalent to \( N \) uncoupled scalar equations, one for each \( \eta^r_s \). We will prove the existences and the uniqueness for the uncoupled problems in this subsection. We begin with an auxiliary lemma for linear parabolic PDEs.

**Lemma 3.2.** Let \( \alpha, \gamma \in (0,1) \) be given such that \( \alpha > \gamma \). Suppose that
\begin{enumerate}
\item \( a_{ij}(x,t) \in C^{\gamma,\gamma}(\mathbb{R}^n \times [0,T]) \) and satisfies the uniform parabolicity condition, i.e. there is \( \lambda > 0 \) such that \( \frac{1}{2} \delta_{ij} - a_{ij}(x,t) < \lambda \delta_{ij} \);
\item \( \| f \|_{C^{\alpha\gamma,\alpha\gamma}(\mathbb{R}^n \times (0,T))} < \infty \) and \( \| u_0 \|_{C^{0,\alpha}} < \infty \).
\end{enumerate}

Then the initial-value problem
\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{\partial}{\partial t} u(x,t) - a_{kl}(x,t) \frac{\partial^2}{\partial x^k \partial x^l} u(x,t) = f(x,t) & \text{on } \mathbb{R}^n \times (0,T] \\
u(x,0) = u_0(x) & \text{on } \mathbb{R}^n
\end{array} \right.
\end{align*}
\]
has a unique solution \( u \), where \( u \in C^{\alpha, \frac{3}{2}}(\mathbb{R}^n \times [0, T]) \) and \( Du \in C^{1, \gamma}(\mathbb{R}^n \times (0, T)) \). Moreover, \( u \) satisfies the estimate

\begin{equation}
\|u\|_{\alpha, \frac{3}{2}, \mathbb{R}^n \times [0, T]} + \|D_x u\|_{\alpha, \frac{3}{2}, \mathbb{R}^n \times (0, T)} \leq K(\|f\|_{\alpha, \frac{3}{2}, \mathbb{R}^n \times (0, T)} + \|u_0\|_{\alpha; \mathbb{R}^n}).
\end{equation}

Here \( K \) is a constant depending only on \( \lambda, \|a_{ij}\|_{\gamma, \frac{3}{2}} \).

**Proof.** In the sequel of the proof, \( K \) will denote a constant depending only on \( \lambda, \|a_{ij}\|_{\gamma, \frac{3}{2}}, \alpha, \gamma \) unless otherwise specified.

**Step 1:** We use the single layer potential method to construct a unique solution \( u \) to (3.4). Given \( (\xi, \tau) \in \mathbb{R}^n \times [0, T] \), let \( \Gamma(x, t; \xi, \tau) \) to be the fundamental solution to

\[ \frac{\partial}{\partial t} u - a_{kl}(x, t)\frac{\partial^2}{\partial x_k \partial x_l} u = 0 \]

on \( \mathbb{R}^n \times (\tau, T] \) such that \( \Gamma(x, t; \xi, \tau) \to \delta(x - \xi) \) as \( t \to \tau \) in the sense of distribution. We claim that the formula

\begin{equation}
(3.6) \quad u(x, t) = -\int_0^\tau \int_{\mathbb{R}^n} \Gamma(x, t; \xi, \tau)f(\xi, \tau)d\xi d\tau + \int_{\mathbb{R}^n} \Gamma(x, t; \xi, 0)u_0(\xi)d\xi
\end{equation}

gives a solution to the system (3.4), and uniqueness would follow from the maximum principle. It is well known from [7] that the formula (3.6) gives a unique solution to (3.4) provided that \( f \) and \( u_0 \) are Hölder-continuous on \( \mathbb{R}^n \times [0, T] \). In our case, the assumption \( \|f\|_{\alpha, \frac{3}{2}, \mathbb{R}^n \times (0, T)} < \infty \) implies that \( f \) is Hölder-continuous on \( \mathbb{R}^n \times [\sigma/2, \sigma] \) for any \( \sigma \in (0, T] \), and that

\[ \tau^{-\frac{1}{2}}|f(\cdot, \mathbb{R}^n \times [\tau/2, \tau]| + \tau^{-\frac{1}{2}}|f(\cdot, \mathbb{R}^n \times [\tau/2, \tau]| \leq K \]

for each \( \tau \in (0, T] \). Hence

\begin{align*}
\|f(\cdot, \tau)\|_{\gamma; \mathbb{R}^n} &= |f(\cdot, \tau)|_{0; \mathbb{R}^n} + |f(\cdot, \tau)|_{\gamma; \mathbb{R}^n} \\
&\leq |f(\cdot, \mathbb{R}^n \times [\tau/2, \tau]| + |f(\cdot, \mathbb{R}^n \times [\tau/2, \tau]| \\
&\leq K\tau^{-\frac{1}{2}} + K\tau^{-\frac{1}{2}} \\
&\leq K\tau^{-\frac{1}{2}} - \frac{s}{2}
\end{align*}

for each \( \tau \in (0, T] \). In particular, the elliptic Hölder bound for \( f(\cdot, \tau) \) is integrable over \( \tau \in (0, t) \) for any \( t \in (0, T] \). This implies that the proof of Theorem 9 in Chapter 1 of [7] still works so that (3.6) satisfies the evolution equation in (3.3) for every \( t > 0 \). To show that (3.6) gives the correct initial condition, it suffices to show that the first integral in RHS of (3.6) tends to zero as \( t \to 0 \).

Let us write (3.6) as \( u := u_1 + u_2 \), where \( u_1 \) stands for the first term in RHS of (3.6), and \( u_2 \) stands for the second term in RHS of (3.6). We recall the estimates for the fundamental solution

\begin{equation}
(3.7) \quad \left| D_x^r D_t^s \Gamma(x, t; \xi, \tau) \right| \leq K(t - \tau)^{-\frac{\alpha + r + s}{2}} \exp \left( -\frac{|x - \xi|^2}{K(t - \tau)} \right), \quad 2r + s \leq 2
\end{equation}
given in page 376 of \cite{12}. For any \((x, t) \in \mathbb{R}^n \times [0, T]\), we estimate

\begin{equation}
\|u_1(x, t)\| \leq \int_0^t \int_{\mathbb{R}^n} |\Gamma(x, t; \xi, \tau)f(\xi, \tau)|d\xi d\tau \\
\leq K \int_0^t \int_{\mathbb{R}^n} (t - \tau)^{-\frac{\alpha}{2}} \exp\left(-\frac{|x - \xi|^2}{K(t - \tau)}\right) |f(\xi, \tau)| d\xi d\tau \\
\leq K \int_0^t \int_{\mathbb{R}^n} (t - \tau)^{-\frac{\alpha}{2}} \exp\left(-\frac{|x - \xi|^2}{K(t - \tau)}\right) \tau^{-1+\frac{\alpha}{2}} \|f\|_{c^{0,\gamma}_1(M \times (0, T))} d\xi d\tau \\
\leq K t^\frac{\alpha}{2} \|f\|_{c^{0,\gamma}_1(M \times (0, T))}.
\end{equation}

Since \(u_2(x, t)\) converges to \(u_0(x)\) as \(t \to 0\), this shows that \(u(x, t)\) converges to \(u_0(x)\) as \(t \to 0\), and that \(u\) is continuous on \(\mathbb{R}^n \times [0, T]\). For the integral \(u_2\), it is easy to see that

\begin{equation}
|u_2(x, t)| \leq K \sup_{\mathbb{R}^n} |u_0|
\end{equation}

for any \((x, t) \in \mathbb{R}^n \times [0, T]\). Thus we have obtained a \(C^0\) estimate for \(u(x, t)\):

\begin{equation}
\|u\|_{0, \mathbb{R}^n \times [0, T]} \leq K (T^\frac{\alpha}{2} \|f\|_{c^{0,\gamma}_1(M \times (0, T))} + \|u_0\|_{0, \mathbb{R}^n}).
\end{equation}

**Step 2:** In this step, we are going to derive

\begin{equation}
\|u\|_{0, \mathbb{R}^n \times [0, T]} \leq K (\|f\|_{c^{0,\gamma}_1(M \times (0, T))} + \|u_0\|_{0, \mathbb{R}^n}).
\end{equation}

We first bound the Hölder semi-norm for \(u_1\). That is, we seek the following inequalities for any \(x, y \in \mathbb{R}^n\) and \(s, t \in [0, T]\):

\begin{equation}
\begin{cases}
|u_1(x, t) - u_1(y, t)| \leq K|x - y|^{\alpha}\|f\|_{c^{0,\gamma}_1(M \times (0, T))} \\
|u_1(x, t) - u_1(x, s)| \leq K|t - s|^{\frac{\alpha}{2}} \|f\|_{c^{0,\gamma}_1(M \times (0, T))}
\end{cases}
\end{equation}

To derive the first inequality in (3.12), we divide \(\mathbb{R}^n\) into \(A_1 = \{\xi \in \mathbb{R}^n : |x - \xi| > 2|x - y|\}\) and \(A_2 = \mathbb{R}^n - A_1\). Using the estimates for fundamental solutions \(\Phi\), we obtain

\begin{equation}
|u_1(x, t) - u_1(y, t)| \\
\leq \int_0^t \int_{A_1} \sup_{z \in \mathbb{R}^n} |D_z \Gamma(z, t; \xi, \tau)||x - y||f(\xi, \tau)|d\xi d\tau + \int_0^t \int_{A_2} (|\Gamma(x, t; \xi, \tau)| + |\Gamma(y, t; \xi, \tau)||f(\xi, \tau)|d\xi d\tau \\
\leq K|x - y| \int_0^t \int_{|x - \xi| > 2|x - y|} (t - \tau)^{-\frac{\alpha + 1}{2}} \sup_{z \in \mathbb{R}^n} \exp\left(-\frac{|x - \xi|^2}{K(t - \tau)}\right) |f(\xi, \tau)|d\xi d\tau \\
+ K \int_0^t \int_{|x - \xi| < 2|x - y|} (t - \tau)^{-\frac{\alpha}{2}} \left(\exp\left(-\frac{|x - \xi|^2}{K(t - \tau)}\right) + \exp\left(-\frac{|y - \xi|^2}{K(t - \tau)}\right)\right) |f(\xi, \tau)|d\xi d\tau.
\end{equation}

Note that if \(\xi \in A_1\) and \(z\) is a point on the segment \(\overline{xy}\), then \(|z - \xi| > \frac{1}{4}|x - \xi|\). Thus

\begin{equation}
\sup_{z \in \mathbb{R}^n} \exp\left(-\frac{|x - \xi|^2}{K(t - \tau)}\right) \leq \exp\left(-\frac{|x - \xi|^2}{4K(t - \tau)}\right).
\end{equation}
We observe that for any $m \geq \alpha$, we have

\begin{equation}
(l - \tau)^{-\frac{\alpha}{2}} \exp\left(-\frac{|x - \xi|^2}{K(l - \tau)}\right) = \frac{1}{|l - \tau|^{\frac{\alpha}{2}} |x - \xi|^{m - \alpha}} \exp\left(-\frac{|x - \xi|^2}{K(l - \tau)}\right) \leq \frac{K}{|l - \tau|^{\frac{\alpha}{2}} |x - \xi|^{m - \alpha}}.
\end{equation}

Using the above estimate with $m = n + 1$ for the first term and $m = n$ for the second term respectively in the last inequality in (3.13), we proceed as in (3.8) to obtain that

\begin{equation}
|u_1(x, t) - u_1(y, t)|
\leq K|x - y| \int_0^t \int_{|x - \xi| > 2|x - y|} \frac{1}{|l - \tau|^{\frac{\alpha}{2}} |x - \xi|^{n + 1 - \alpha}} |f(\xi, \tau)| d\xi d\tau
+ K \int_0^t \left( \int_{|x - \xi| < 2|x - y|} \frac{1}{|l - \tau|^{\frac{\alpha}{2}} |x - \xi|^{n - \alpha}} |f(\xi, \tau)| d\xi + \int_{|y - \xi| < 3|x - y|} \frac{1}{|l - \tau|^{\frac{\alpha}{2}} |y - \xi|^{n - \alpha}} |f(\xi, \tau)| d\xi d\tau \right)
\leq K|x - y|^{\alpha} \|f\|_{C^{0, \gamma\frac{\alpha}{2}}(M \times (0, T))},
\end{equation}

where we have used the inequality

$$
\int_0^t \frac{\tau^{\frac{\alpha}{2} - 1}}{(l - \tau)^{\frac{\alpha}{2}}} d\tau \leq K(\alpha)
$$

in deriving the last line. Next, to derive the second inequality in (3.12), we assume without loss of generality that $s < t$ and we divide $\mathbb{R}^n$ into $A_3 = \{\xi \in \mathbb{R}^n : |x - \xi| > \sqrt{t - s}\}$ and $A_4 = \mathbb{R}^n - A_3$. Then

\begin{equation}
|u_1(x, t) - u_1(x, s)|
\leq \int_0^t \int_{A_3} \sup_{\xi \in [s, t]} |D_{\xi} \Gamma(x, \xi, \tau)||t - s||f(\xi, \tau)| d\xi d\tau + \int_s^t \int_{A_3} |\Gamma(x, t; \xi, \tau)||f(\xi, \tau)| d\xi d\tau
+ \int_0^t \int_{A_4} |\Gamma(x, t; \xi, \tau)||f(\xi, \tau)| d\xi d\tau + \int_0^t \int_{A_4} |\Gamma(x, s; \xi, \tau)||f(\xi, \tau)| d\xi d\tau
\end{equation}

For instance, using (3.7) and (3.14) with $m = n + 1$, we can estimate the second term in the RHS of the above inequality

\begin{align*}
\int_s^t \int_{A_3} |\Gamma(x, t; \xi, \tau)||f(\xi, \tau)| d\xi d\tau
&\leq K \int_s^t (t - \tau)^{\frac{\alpha}{2}} \int_{A_3} (t - \tau)^{-\frac{\alpha}{2} + \frac{1}{2}} \exp\left(-\frac{|x - \xi|^2}{K(l - \tau)}\right) |f(\xi, \tau)| d\xi d\tau \\
&\leq K \int_s^t (t - s)^{\frac{\alpha}{2}} \int_{|x - \xi| > \sqrt{t - s}} \frac{1}{(t - \tau)^{\frac{\alpha}{2}} |x - \xi|^{n + 1 - \alpha}} |f(\xi, \tau)| d\xi d\tau
\leq K|t - s|^{\frac{\alpha}{2}} \|f\|_{C^{0, \gamma\frac{\alpha}{2}}(M \times (0, T))} \int_0^t \frac{\tau^{\frac{\alpha}{2} - 1}}{(l - \tau)^{\frac{\alpha}{2}}} d\tau.
\end{align*}
We can estimate the other terms in the RHS of (3.16) similarly. Hence we obtain
\[ |u_1(x, t) - u_1(x, s)| \leq K|t - s|^\frac{\alpha}{2} \|f\|_{\mathbb{L}^q_t(M \times (0, T))} \left( \int_0^t \frac{\tau^{\frac{\alpha}{q} - 1}}{|t - \tau|^\frac{\alpha}{q}} d\tau + \int_0^s \frac{\tau^{\frac{\alpha}{q} - 1}}{|s - \tau|^\frac{\alpha}{q}} d\tau \right) \leq K|t - s|^\frac{\alpha}{2} \|f\|_{\mathcal{V}^{0, \frac{\alpha}{q}}(M \times (0, T))}. \]
From which (3.12) follows.

In the remainder of this step, we derive the following estimate for the Hölder semi-norm of \( u_2 \):
\[ [u_2]_{\alpha; \mathbb{R}^n \times [0, T]} \leq K [u_0]_{\alpha; \mathbb{R}^n}. \]
That is, for any \( x, y \in \mathbb{R}^n \) and \( s, t \in [0, T] \), we claim that:
\[ (3.17) \]
\[ \left\{ \begin{array}{ll}
|u_2(x, t) - u_2(y, t)| & \leq K|x - y|^\alpha [u_0]_{\alpha; \mathbb{R}^n} \\
|u_2(x, t) - u_2(x, s)| & \leq K|t - s|^\frac{\alpha}{2} [u_0]_{\alpha; \mathbb{R}^n}.
\end{array} \right. \]
To derive the first inequality, we break \( u_2(x, t) - u_2(y, t) \) into two integrals as follows:
\[ u_2(x, t) - u_2(y, t) = \int_{\mathbb{R}^n} (\Gamma(x, t; \xi, 0) - \Gamma(y, t; \xi, 0))(u_0(\xi) - u_0(x)) d\xi + u_0(x) \int_{\mathbb{R}^n} (\Gamma(x, t; \xi, 0) - \Gamma(y, t; \xi, 0)) d\xi := J_1 + J_2. \]
For the integral \( J_1 \), we use (3.7) to estimate
\[ (3.18) \]
\[ |J_1| \leq \int_{A_1} \sup_{z \in \mathbb{R}^n} |D_y \Gamma(z, t; \xi, 0)||x - y||u_0(x) - u_0(\xi)| d\xi + \int_{A_2} ([\Gamma(x, t; \xi, 0)] + [\Gamma(y, t; \xi, 0)])|u_0(x) - u_0(\xi)| d\xi \leq K|x - y| \int_{(|x - \xi| > 2|x - y|)} (t - \tau)^{-\frac{\alpha}{2} + \delta} \exp \left( -\frac{|x - \xi|^2}{4K(t - \tau)} \right) |x - \xi|^\alpha [u_0]_{\alpha; \Omega, \Omega} d\xi + K \int_{(|x - \xi| < 2|x - y|)} (t - \tau)^{-\frac{\alpha}{2} + \delta} \left( \exp \left( -\frac{|x - \xi|^2}{K(t - \tau)} \right) + \exp \left( -\frac{|y - \xi|^2}{K(t - \tau)} \right) \right) |x - \xi|^\alpha[u_0]_{\alpha; \Omega, \Omega} d\xi. \]
Similar to the derivation in (3.13) to (3.15), we apply (5.14) with \( \alpha = 0 \) to above, and we subsequently obtain
\[ |J_1| \leq K|x - y|^\alpha [u_0]_{\alpha; \mathbb{R}^n}. \]
On the other hand, we see that the second integral \( J_2 \) vanishes since the fundamental solution satisfies \( \int_{\Omega} \Gamma(x, t; \xi, 0) d\xi = \int_{\Omega} \Gamma(y, t; \xi, 0) d\xi = 1. \)
Next, we preform the same procedure on the term \( u_2(x, t) - u_2(x, s) \). We break it into two integrals:
\[ u_2(x, t) - u_2(x, s) = \int_{\mathbb{R}^n} (\Gamma(x, t; \xi, 0) - \Gamma(x, s; \xi, 0))(u_0(\xi) - u_0(x)) d\xi + u_0(x) \int_{\mathbb{R}^n} (\Gamma(x, t; \xi, 0) - \Gamma(x, s; \xi, 0)) d\xi := J_3 + J_4. \]
Again, the integral $J_4$ vanishes. Using a similar argument as in (3.16) and (3.18), we have

$$ |J_3| \leq \int_{A_3} \sup_{t \in [s, \bar{t}]} |D_t \Gamma(x, \bar{t}; \xi, 0)||t - s||u_0(x) - u_0(\xi)|d\xi $$

$$ + \int_{A_3} (|\Gamma(x, t; \xi, 0)| + |\Gamma(x, s; \xi, 0)|)|u_0(x) - u_0(\xi)|d\xi $$

$$ \leq K|t - s|^{\frac{3}{2}} [u_0]_{\alpha, \Omega}. $$

From this, (3.17) follows.

Therefore we summarize that

$$ [u]_{\alpha, \frac{3}{2}; \mathbb{R}^n \times [0, T]} \leq K(\|f\|_{C^0, \gamma, \frac{3}{2}}(\mathbb{R}^n \times (0, T)) + [u_0]_{\alpha, \mathbb{R}^n}). $$

From (3.10) and (3.20), the estimate (3.11) follows.

**Step 3:** In this step we claim that the solution $u(x, t)$ satisfies the estimates

$$ \|D_x u\|_{C^{0, \gamma}}(\mathbb{R}^n \times (0, T)) \leq K(\|f\|_{C^0, \gamma, \frac{3}{2}}(\mathbb{R}^n \times (0, T)) + [u_0]_{\alpha, \mathbb{R}^n}). $$

Fix a point $z \in \mathbb{R}^n$ and $\sigma \in (0, T]$, we consider the parabolic cylinder

$$ P_\sigma(z) := B_\sqrt{\sigma}(z) \times [0, \sigma]. $$

We define a function $v$ on the parabolic cylinder $P_1(0) = B_1(0) \times [0, 1]$ by scaling:

$$ v(y, s) := u(z + \sigma^\frac{1}{2} y, \sigma s). $$

Let $\chi(s)$ be a three times continuously differentiable cutoff function on $[0, 1]$ such that

$$ \chi(s) = \begin{cases} 
1, & \text{if } s \in [\frac{1}{4}, 1] \\
0, & \text{if } s \in [0, \frac{1}{4}] 
\end{cases} $$

and

$$ |D^j \chi(s)| \leq C, \quad j = 0, 1, 2. $$

We now define a function $\tilde{v}$ on $P_1(0)$ by

$$ \tilde{v}(y, s) := \chi(s)(v(y, s) - v(0, 1)). $$

Then $\tilde{v}$ satisfies

$$ \begin{cases} 
\frac{\partial \tilde{v}}{\partial s} - a_{ij}(z + \sigma^\frac{1}{2} y, \sigma s)\frac{\partial^2 \tilde{v}}{\partial y_i \partial y_j} = \tilde{f}(y, s) & \text{on } B_1(0) \times (0, 1) \\
\tilde{v}(y, 0) = 0 & \text{on } B_1(0), 
\end{cases} $$

where $\tilde{f}(y, s) = \chi'(s)(v(y, s) - v(0, 1)) + \sigma \chi(s)f(z + \sigma^\frac{1}{2} y, \sigma s)$.

Note that $\tilde{f} \in C^{1, \frac{3}{2}}(B_1(0) \times [0, 1])$ after change of variables since by the previous step we have $u \in C^{\gamma, \frac{3}{2}}(B_\sqrt{\sigma}(z) \times [0, \sigma])$ and $\alpha \geq \gamma$. Hence by standard parabolic Schauder interior estimate $\tilde{v} \in C^{2+\gamma, \frac{3}{2}}(B_\frac{1}{2}(0) \times [\frac{1}{2}, 1])$ and we have the following estimate:

$$ |\tilde{v}|_{2+\gamma, \frac{3}{2}, B_{1/2}(0) \times [1/2, 1]} \leq K(\|a_{ij}\|_{0, \gamma, \frac{3}{2}}(\mathbb{R}^n \times (0, 1)) + |\chi f\|_{0, \gamma, \frac{3}{2}, B_1(0) \times [0, 1]} + |\chi' v - v(0, 1)|_{0, \gamma, \frac{3}{2}, B_1(0) \times [0, 1]} + |\sigma \chi f\|_{0, \gamma, \frac{3}{2}, B_1(0) \times [0, 1]}). $$
Then it follows from the rescaling \( u(x, t) = v(\sigma^{-\frac{1}{\gamma}} (x - z), \sigma^{-1} t) \) that

\[
(3.28) \quad \sum_{i=1}^{2} \sigma^{\frac{\alpha}{2}} |D^i u|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] + \sum_{i=1}^{2} \sigma^{\frac{\alpha}{2} + \frac{1}{2}} |D^i u|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma]
\leq K(\|a_{ij} \|_{\gamma, \frac{\alpha}{2}, C}) \left( |u - u(z, \sigma)|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] + \sigma^{\frac{\alpha}{2}} |u - u(z, \sigma)|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma]
+ \sigma|f|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] + \frac{\alpha+1}{2} |f|_{\gamma, \frac{\alpha}{2}, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] \right)
\]

Observe that \([f]_{\gamma, \frac{\alpha}{2}, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma]\) can be controlled by norms of \(f\) on \([\sigma, \frac{\sigma + 1}{2}, \sigma]\) and \([\sigma, \frac{\sigma + 1}{2}, \sigma]\). Indeed, for any \(x \in B \gamma \mathcal{H}_2(z)\) and \(\tau, t \in [\sigma, \sigma]\), we have

\[
\frac{|f(x, t) - f(x, \tau)|}{|t - \tau|^{\frac{\alpha}{2}}} \leq \frac{|f(x, t) - f(x, \frac{\sigma + 1}{2})|}{|t - \frac{\sigma + 1}{2}|^{\frac{\alpha}{2}}} + \frac{|f(x, \tau) - f(x, \frac{\sigma + 1}{2})|}{|\tau - \frac{\sigma + 1}{2}|^{\frac{\alpha}{2}}}
\]

It suffices to consider the case where \(\tau \leq \frac{\sigma + 1}{2} \leq t\), then

\[
\frac{|f(x, t) - f(x, \tau)|}{|t - \tau|^{\frac{\alpha}{2}}} \leq \frac{|f(x, t) - f(x, \frac{\sigma + 1}{2})|}{|t - \frac{\sigma + 1}{2}|^{\frac{\alpha}{2}}} + \frac{|f(x, \tau) - f(x, \frac{\sigma + 1}{2})|}{|\tau - \frac{\sigma + 1}{2}|^{\frac{\alpha}{2}}}
\]

This implies

\[
\frac{\sigma^{\frac{\alpha}{2}}}{\frac{\alpha}{2} + \frac{1}{2} + \frac{\alpha}{2}} |f|_{\gamma, \frac{\alpha}{2}, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] \leq \frac{\sigma^{\frac{\alpha}{2}}}{\frac{\alpha}{2} + \frac{1}{2} + \frac{\alpha}{2}} |f|_{\gamma, \frac{\alpha}{2}, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma].
\]

Putting this inequality into (3.29), we obtain

\[
(3.29) \quad \sum_{i=1}^{2} \sigma^{\frac{\alpha}{2}} |D^i u|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] + \sum_{i=1}^{2} \sigma^{\frac{\alpha}{2} + \frac{1}{2}} |D^i u|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma]
\leq K(\|a_{ij} \|_{\gamma, \frac{\alpha}{2}, C}) \left( |u - u(z, \sigma)|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] + \sigma^{\frac{\alpha}{2}} |u - u(z, \sigma)|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma]
+ \sigma|f|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] + \frac{\alpha+1}{2} |f|_{\gamma, \frac{\alpha}{2}, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] \right)
\]

Now, note that the Hölder estimate for \(|u|_{\alpha, \frac{\alpha}{2}}\) in (3.20) implies that

\[
|u - u(z, \sigma)|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] \leq K \sigma^{\frac{\alpha}{2}} |u|_{\alpha, \frac{\alpha}{2}, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma]
\]

and

\[
|u - u(z, \sigma)|_{\gamma, \frac{\alpha}{2}, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] \leq K \sigma^{\frac{\alpha}{2} - \frac{\alpha}{2}} |u|_{\alpha, \frac{\alpha}{2}, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma]
\]

Putting these facts into (3.30), we obtain

\[
(3.30) \quad \sum_{i=1}^{2} \sigma^{\frac{\alpha}{2}} |D^i u|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma] + \sum_{i=1}^{2} \sigma^{\frac{\alpha}{2} + \frac{1}{2}} |D^i u|_{0, B \gamma \mathcal{H}_2(z)} \times [\sigma, \sigma]
\leq K(\|a_{ij} \|_{\gamma, \frac{\alpha}{2}, C}) \left( |f|_{c^{\alpha, \frac{\alpha}{2}}_1 (M \times (0, T))} + \|u_0\|_{\alpha, R^n} \right).
\]
Since $z \in \mathbb{R}^n$ and $\sigma \in (0, T]$ are arbitrary, the desired estimate follows. Putting (3.11) and (3.21) together, the lemma is thus proved.

\[ \square \]

Using a standard bootstrap argument, we can improve the regularity of $u$. We have the following auxiliary lemmas for higher order regularity:

**Lemma 3.3.** Let $\alpha, \gamma \in (0, 1)$ be given such that $\alpha > \gamma$. Suppose that

1. $a_{ij}(x, t) \in C^{\gamma, 2}(\mathbb{R}^n \times [0, T])$ and satisfies the uniform parabolicity condition. i.e. there is $\lambda > 0$ such that $\frac{1}{\lambda} \delta_{ij} < a_{ij}(x, t) < \lambda \delta_{ij}$;
2. $\|D_x a_{ij}\|_{C^{\alpha-1, \gamma}(\mathbb{R}^n \times (0, T))} < \infty$ if $k \geq 1$;
3. $\|f\|_{C^{k+1, \gamma}(\mathbb{R}^n \times (0, T))} < \infty$ and $\|u_0\|_{C^{\alpha, \gamma}} < \infty$.

Then the initial-value problem

\[
\begin{align*}
\frac{\partial}{\partial t} u(x, t) &= a_{ij}(x, t) \frac{\partial^2}{\partial x_i \partial x_j} u(x, t) = f(x, t) & \text{on} & \quad \mathbb{R}^n \times (0, T) \\
u(x, 0) &= u_0(x) & \text{on} & \quad \mathbb{R}^n 
\end{align*}
\]

has a unique solution $u$, where $u \in C^{\alpha, \frac{\gamma}{2}}([0, T])$ and $D u \in C^{k+1, \gamma}(\mathbb{R}^n \times (0, T))$, such that

\[
\|u\|_{C^{\alpha, \frac{\gamma}{2}}}, \mathbb{R} \times [0, T] + \|D_x u\|_{C^{k+1, \gamma}(\mathbb{R}^n \times (0, T))} \leq K(\|f\|_{C^{k+1, \gamma}(\mathbb{R}^n \times (0, T))} + \|u_0\|_{C^{\alpha, \gamma}}).
\]

Here $K$ is a constant depending only on $k, \mathbb{R}^n, \|a_{ij}\|_{C^{\alpha, \frac{\gamma}{2}}}$ and $\|a_{ij}\|_{C^{k+1, \gamma}}$.

**Proof.** We intend to prove the lemma by induction on $k$. The case $k = 0$ follows from Lemma 3.2. We suppose that the conditions (2) and (3) in the lemma holds with $k$ replaced by $k + 1$. That is,

- $\|D_x a_{ij}\|_{C^{\alpha-1, \gamma}(\mathbb{R}^n \times (0, T))} < \infty$;
- $\|f\|_{C^{k+1, \gamma}(\mathbb{R}^n \times (0, T))} < \infty$ and $\|u_0\|_{C^{\alpha, \gamma}} < \infty$.

Moreover, the induction hypothesis implies that the estimate

\[
\|u\|_{C^{\alpha, \frac{\gamma}{2}}}, \mathbb{R} \times [0, T] + \|D_x u\|_{C^{k+1, \gamma}(\mathbb{R}^n \times (0, T))} \leq K(\|f\|_{C^{k+1, \gamma}(\mathbb{R}^n \times (0, T))} + \|u_0\|_{C^{\alpha, \gamma}})
\]

holds. Let us fix $\sigma \in (0, T]$. Let $\chi(t)$ be a three times continuously differentiable cutoff function on $[0, \sigma]$ such that

\[
\chi(t) = \begin{cases} 
1, & \text{if } s \in [\frac{\sigma}{2}, \sigma] \\
0, & \text{if } s \in [0, \frac{\sigma}{4}]
\end{cases}
\]

and

\[
|D^h \chi(t)| \leq C \sigma^{-h}, \quad h = 0, 1, 2.
\]

It follows that the $\frac{\gamma}{2}$-Hölder norm for $\chi$ and $\chi'$ has estimates

\[
|\chi|_{\frac{\gamma}{2}, [0, \sigma]} \leq C \sigma^{-\frac{\gamma}{2}} \quad \text{and} \quad |\chi'|_{\frac{\gamma}{2}, [0, \sigma]} \leq C \sigma^{-1-\frac{\gamma}{2}}.
\]

On $\mathbb{R}^n \times (0, \sigma]$, we define $\bar{u}(x, t) := \chi(t)D^k u(x, t)$, then $\bar{u}$ is the solution to the system

\[
\begin{align*}
\frac{\partial}{\partial t} \bar{u}(x, t) - a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \bar{u}(x, t) &= \tilde{f}(x, t) & \text{on} & \quad \mathbb{R}^n \times (0, \sigma] \\
\bar{u}(x, 0) &= 0 & \text{on} & \quad \mathbb{R}^n,
\end{align*}
\]
where $f$ is given by

$$
(3.35) \quad f = \chi D_x^{k+1}u + \chi D^{k+1}f + \chi \sum_{r=0}^{k} \left( \frac{k+1}{r} \right) D^{k+1-r}a_{ij}D^r D_y^2 u.
$$

Note that by the induction hypothesis we have $f \in C^\gamma_x(\mathbb{R}^n \times [0, \sigma])$. Then we apply Theorem 5.1 in [12] to obtain $D_x^{k+1}u \in C^{2+\gamma, \frac{3+\gamma}{2}}(\mathbb{R}^n \times [\varphi, \sigma])$ and the estimate

$$
\sigma[D^{k+3}u][0,\mathbb{R}^n \times \sigma] + \sigma^{1+\gamma/2}[D^{2+k+3}u][\gamma, \mathbb{R}^n \times (0, \sigma)] \leq K \left( \sigma[f][0,\mathbb{R}^n \times [0, \sigma]] + \sigma^{1+\gamma/2}[f][\gamma, \mathbb{R}^n \times (0, \sigma)] \right).
$$

Hence,

$$
(3.36) \quad \sigma^{3/2 + \frac{3+\gamma}{2}} \left[ \chi D_x^{k+1}u \right] \gamma, \mathbb{R}^n \times (\varphi, \sigma) \leq K \left( \sigma^{3/2 + \frac{3+\gamma}{2}} \left[ D_x^{k+1}u \right] \gamma, \mathbb{R}^n \times (\varphi, \sigma) \right) + \sigma^{1+\gamma/2 + \frac{3+\gamma}{2}} \left[ D^{k+1}f \right] \gamma, \mathbb{R}^n \times (0, \sigma) \right) \leq K \left( \sigma^{3/2 + \frac{3+\gamma}{2}} \left[ D_x^{k+1}u \right] \gamma, \mathbb{R}^n \times (\varphi, \sigma) \right) + \sigma^{1+\gamma/2 + \frac{3+\gamma}{2}} \left[ D^{k+1}f \right] \gamma, \mathbb{R}^n \times (0, \sigma) \right)
$$

We can check that (3.36) implies the estimate (3.32). For instance, we can check for the Hölder-semi norm term of $f$ in the last line of (3.36). We have

$$
\sigma^{3/2 + \frac{3+\gamma}{2}} \left[ \chi D^{k+1}f \right] \gamma, R^n \times (0, \sigma) \leq K \left( \sigma^{3/2 + \frac{3+\gamma}{2}} \left[ D^{k+1}f \right] \gamma, R^n \times (0, \sigma) \right) + \sigma^{1+\gamma/2 + \frac{3+\gamma}{2}} \left[ D^{k+1}f \right] \gamma, R^n \times (0, \sigma) \right)
$$

Also, for each integer $r \in [0, k]$ we have

$$
\sigma^{3/2 + \frac{3+\gamma}{2}} \left[ \chi D^{k+1-r}a_{ij}D^r u_{ij} \right] \gamma, \mathbb{R}^n \times (\varphi, \sigma) \leq K \left( \sigma^{3/2 + \frac{3+\gamma}{2}} \left[ D^{k+1-r}a_{ij}D^r u_{ij} \right] \gamma, \mathbb{R}^n \times (\varphi, \sigma) \right) + \sigma^{1+\gamma/2 + \frac{3+\gamma}{2}} \left[ D^{k+1-r}a_{ij}D^r u_{ij} \right] \gamma, \mathbb{R}^n \times (0, \sigma) \right)
$$

From these, we have

$$
(3.37) \quad \sigma^{3/2 + \frac{3+\gamma}{2}} \left[ f \right] \gamma, \mathbb{R}^n \times (\varphi, \sigma) \leq K(a_{ij}) \left( \left\| f \right\|_{C^{k+1,\gamma}(\mathbb{R}^n \times (0, \sigma))} \right) + \left\| u_0 \right\|_{C^\infty(\mathbb{R}^n)}.
$$

And similarly we get

$$
(3.38) \quad \sigma^{3/2 + \frac{3+\gamma}{2}} \left[ f \right] \gamma, \mathbb{R}^n \times (0, \sigma) \leq K(a_{ij}) \left( \left\| f \right\|_{C^{k+1,\gamma}(\mathbb{R}^n \times (0, \sigma))} \right) + \left\| u_0 \right\|_{C^\infty(\mathbb{R}^n)}.
$$

Therefore we have completed the induction. \(\square\)
3.3. Completion of the proof.

Now we are on the ground of constructing the operator \( R : W_k \to \mathcal{X}_{k+2} \). To begin with, let \( \{\rho_s\} \) be the partition of unity subordinate to the charts \( \{U_s\} \). On the basis of Lemma 3.2 and Lemma 3.3, there is a unique solution set \( \{\eta_s^r\} \) to the system \((3.3)\) such that for each \((r, s)\),

\[
\eta_s^r \in C^{\alpha, \frac{\gamma}{2}}(\mathbb{R}^n \times [0, T]), \quad D_s \eta_s^r \in C^{k+1, \gamma}(\mathbb{R}^n \times (0, T))
\]

and

\[
(3.39)\]

\[
\|\eta_s^r\|_{C^{\alpha, \frac{\gamma}{2}}(\mathbb{R}^n \times [0, T])} + \|D_s \eta_s^r\|_{C^{k+1, \gamma}(\mathbb{R}^n \times (0, T))} \leq K\|\eta_0\|_{C^{\alpha, \frac{\gamma}{2}}(\mathbb{R}^n \times [0, T])} + \|\eta_0\|_{C^{\alpha, \frac{\gamma}{2}}(\mathbb{R}^n \times [0, T])}.
\]

We then define \( R_s : W_k \to \Gamma(U_s \times [0, T], \pi^{-1}(U_s)) \) by

\[
(3.40) \quad (R_s h)(p, t) = \rho_s(p) \sum_r \eta_s^r(\varphi_s(p), t)e_s^r(p)
\]

for any \((p, t) \in U_s \times [0, T]\). Moreover, by setting \( R_s h \) to be the zero section outside the support of \( \rho_s \), we can extend \( R_s h \) to a section of \( \Gamma(M \times [0, T], E) \). Now, we define a map \( R : W_k \to \Gamma(M \times [0, T], E) \) by

\[
(3.41) \quad R h = \sum_s R_s h.
\]

From the construction of \( R \) it is clear that \( Rh \in \mathcal{X}_{k+2} \) for any \( h \in W_k \). By \((3.39), R h \) satisfies the following estimates:

\[
(3.42) \quad \|Rh\|_{\mathcal{X}_{k+2}(M \times [0, T])} \leq K\|h\|_{W_k(M \times [0, T])}.
\]

Here \( K \) is a constant depending only on \( k, M, A \). Recall that \( A \) is the positive constant such that \( \|w\|_{C^{k+\gamma, \frac{\gamma}{2}}(M \times [0, T])} + \|\nabla w\|_{C^{k+\gamma, \frac{\gamma}{2}}(M \times [0, T])} \leq A \) provided in the assumption of the theorem.

Next, we define \( S : W_k \to W_k \) and \( G : \mathcal{X}_{k+2} \to \mathcal{X}_{k+2} \) by

\[
(3.43) \quad \begin{cases} 
Sh = HRh - h \\
G\eta = RH\eta - \eta.
\end{cases}
\]

**Lemma 3.4.** The operators defined above in \((3.34)\) satisfies

\[
\|Sh\|_{W_k} \leq KT^2\|h\|_{W_k}
\]

and

\[
\|G\eta\|_{\mathcal{X}_{k+2}} \leq KT^2\|\eta\|_{\mathcal{X}_{k+2}}.
\]

Here \( K \) is a constant depending only on \( k, M, \tilde{g}, A \).

**Proof.** Firstly, note that on the intersection \( U_\mu \cap U_\nu \), the transition map for the vector bundle \( E \) is given by

\[
\tilde{\varphi}_{\mu \nu} := \tilde{\varphi}_{\mu} \circ (\tilde{\varphi}_{\nu})^{-1} : (U_\mu \cap U_\nu) \times \mathbb{R}^N \to (U_\mu \cap U_\nu) \times \mathbb{R}^N.
\]

We denote by \( \Phi_{\mu \nu} : (U_\mu \cap U_\nu) \to \text{GL}_N\mathbb{R} \) the induced isomorphism which is given by

\[
\tilde{\varphi}_{\mu \nu}(p, V) := (p, \Phi_{\mu \nu}(x)V), \quad \forall p \in U_\mu \cap U_\nu, \ V \in \mathbb{R}^N.
\]

Then with respect to the canonical basis \( \{e_s^r\} \) and \( \{e_s^r\} \) of \( \pi^{-1}(U_\mu) \) and \( \pi^{-1}(U_\nu) \), the matrix components for the map \( \Phi_{\mu \nu} \) are given by

\[
e_s^r = (\Phi_{\mu \nu})_j^i(e_s^r).
\]
With this notation, we can write the components of $R_\mu h$ on $U_\mu \cap U_\nu$ with respect to the basis $\{e_\mu^i\}$ as

$$(R_\mu h)^i_\nu = \sum_i \rho_\nu \eta^i_\nu (\Phi_{\nu\mu})^j_i.$$  

Thus we can write the component $(Rh)^i_\mu$ on $U_\mu$ as

$$(3.44) \quad (Rh)^i_\mu = \sum_\nu (R_\nu h)^i_\mu = \sum_\nu \rho_\nu (\Phi_{\nu\mu})^j_i \eta^j_\nu.$$  

Now, let us prove the first inequality in the lemma. For any $h = (F, \eta_0) \in \mathcal{W}_k$, we have

$$Sh = (LRh - F, 0).$$  

Let us write $v = Rh$. Then on each chart $U_\mu$, we have $v = \sum_i v^i_\mu e_\mu^i$ satisfying

$$v^i_\mu = \sum_\nu \rho_\nu (\Phi_{\nu\mu})^j_i \eta^j_\nu,$$

where $\{\eta^j_\nu\}$ denotes the unique solution of the system (3.3). By (3.4), we have

$$\frac{\partial}{\partial t} v^i_\mu = \sum_\nu \rho_\nu (\Phi_{\nu\mu})^j_i (\frac{\partial}{\partial t} \eta^j_\nu),$$

and

$$w^{kl} D^2_{kl} v^i_\mu = \sum_\nu \rho_\nu (\Phi_{\nu\mu})^j_i \left( w^{kl} D^2_{kl} \eta^j_\nu + 2 w^{kl} D_k (\rho_\nu (\Phi_{\nu\mu})^j_i) \cdot D_l (\eta^j_\nu) + w^{kl} D^2_{kl} (\rho_\nu (\Phi_{\nu\mu})^j_i) \cdot \eta^j_\nu \right).$$

This implies that $\{v^i_\mu\}$ satisfies the system

$$\begin{cases}
\frac{\partial}{\partial t} v^i_\mu (x, t) - w^{kl} D^2_{kl} v^i_\mu (x, t) = \tilde{F}^i_\mu (x) & \text{on } \varphi_\mu (U_\mu) \times (0, T], \quad i = 1, \ldots, N, \\
v^i_\mu (x, 0) = 0 & \text{on } \varphi_\mu (U_\mu), \quad i = 1, \ldots, N,
\end{cases}$$

where

$$(3.45) \quad \tilde{F}^i_\mu = \sum_\nu \rho_\nu (\Phi_{\nu\mu})^j_i \left( F^j_\nu + 2 w^{kl} D_k (\rho_\nu (\Phi_{\nu\mu})^j_i) \cdot D_l (\eta^j_\nu) + w^{kl} D^2_{kl} (\rho_\nu (\Phi_{\nu\mu})^j_i) \cdot \eta^j_\nu \right).$$

Note that $v$ satisfies the estimate

$$\|v\|_{\alpha, \cdot, \mu, \nu; \mathcal{M} \times [0, T]} + \|\nabla v\|_{c^{k+1, \gamma}} (\mathcal{M} \times (0, T)) \leq K \|h\| \mathcal{W}_k (\mathcal{M} \times (0, T)).$$

by (3.42). And by $F^i_\mu = \sum_\nu \rho_\nu F^i_\nu = \sum_\nu \rho_\nu (\Phi_{\nu\mu})^j_i F^j_\nu$, we have the estimate

$$(3.46) \quad \|\tilde{F}^i_\mu - F^i_\mu\|_{c^{k, \gamma}} (\mathbb{R}^N \times (0, T)) \leq K \sum_\nu \left( \|D_x \eta^j_\nu\|_{c^{k, \gamma}} (\mathbb{R}^N \times (0, T)) + \|\eta^j_\nu\|_{c^{k, \gamma}} (\mathbb{R}^N \times (0, T)) \right) + \|\tilde{F}^i_\mu\|_{c^{k, \gamma}} (\mathbb{R}^N \times (0, T)) \leq K T^2 \left( \|v\|_{\alpha, \cdot, \nu, \mathcal{M} \times [0, T]} + \|\nabla v\|_{c^{k+1, \gamma}} (\mathcal{M} \times (0, T)) \right).$$
Hence, we have
\[
LRh - F = \sum_r \left( \frac{\partial}{\partial t} \nu_i^r(x, t) - w^{kl} \tilde{\nabla}_k \tilde{\nabla}_l \nu_i^r(x, t) - F_i^r(x, t) \right) e_i^r
\]
\[
= \sum_r \left( (\partial \hat{\Gamma} + \hat{\Gamma} \ast_w \hat{\Gamma} \ast_w v + \hat{\nabla} v + \hat{F}_i^r) e_i^r \right),
\]
where \(\hat{\Gamma}\) are the connection terms with respect to \(\hat{g}\). Thus, using Lemma 2.2 and (3.40) we obtain the estimate
\[
(3.47)
\]
\[
\|Sh\|_{W_h(M \times (0, T))} = \|LRh - F\|_{C^{1, \gamma}_{\tilde{\psi}+\hat{g}}(M \times (0, T))}
\]
\[
\leq K \left( \|v\|_{C^{k, \gamma}_{\tilde{\psi}+\hat{g}}(M \times (0, T))} + \|\tilde{\nabla}v\|_{C^{k, \gamma}_{\tilde{\psi}+\hat{g}}(M \times (0, T))} + \sum_{i, \mu} \|\hat{F}_i^r - F_i^r\|_{C^{k, \gamma}_{\tilde{\psi}+\hat{g}}(\partial N \times (0, T))} \right)
\]
\[
\leq KT^{\frac{1}{2}} (\|v\|_{\tilde{\psi}+\hat{g}}(M \times [0, T]) + \|\tilde{\nabla}v\|_{\tilde{\psi}+\hat{g}}(M \times (0, T)))
\]
\[
\leq KT^{\frac{1}{2}} \|h\|_{W_h(M \times (0, T))},
\]
where \(K = K(k, M, \hat{g}, A)\).

In the next step, we derive the second inequality in the lemma. For any \(\eta \in \mathcal{A}_{k+2}(M \times [0, T]; E)\), we denote by \((F, \eta_0) := (L \eta, \eta(\cdot, 0)) = H \eta\) and \(\zeta := RH \eta\). Then on each chart \(U_\mu\), we see that \((F, \eta_0) = (\sum_{r=1}^N F_i^r e_i^\mu, \eta(\cdot, 0))\) satisfies
\[
F_i^\mu(x, t) = \frac{\partial}{\partial \mu} \eta_i^\mu(x, t) - w^{kl} \tilde{\nabla}_k \tilde{\nabla}_l \eta_i^\mu(x, t) \quad \text{on} \quad \varphi_\mu(U_\mu) \times (0, T), \quad r = 1, \ldots, N.
\]

Now, we denote by \(\{\eta_i^\mu\}\) the unique solution of the system
\[
\begin{cases}
\frac{\partial}{\partial \mu} \eta_i^\mu(x, t) - w^{kl} \tilde{\nabla}_k \tilde{\nabla}_l \eta_i^\mu(x, t) = F_i^\mu(x, t) & \text{on} \quad \varphi_\mu(U_\mu) \times (0, T), \quad r = 1, \ldots, N \\
\eta_i^\mu(x, 0) = \eta_0_i^\mu(x) & \text{on} \quad \varphi_\mu(U_\mu), \quad r = 1, \ldots, N.
\end{cases}
\]

Then by (3.34), \(RH \eta = \zeta = \sum_{r=1}^N \zeta_i^\mu e_i^\mu\) satisfies
\[
\zeta_i^\mu = \sum_{j, \nu} \rho_{\nu}(\Phi_{\nu, \mu})_j^i \eta_i^\nu,
\]
and in particular
\[
\zeta_i^\mu(x, 0) = \sum_{j, \nu} \rho_{\nu}(\Phi_{\nu, \mu})_j^i \eta_0_i^\nu(x, 0) = (\eta_0)_i^\mu(x, 0).
\]

Moreover, the components \(\{\zeta_i^\mu\}\) satisfy the system
\[
\begin{cases}
\frac{\partial}{\partial \mu} \zeta_i^\mu(x, t) - w^{kl} D^2_{\nu, \mu} \zeta_i^\mu(x, t) = \tilde{F}_i^\mu(x, t) & \text{on} \quad \varphi_\mu(U_\mu) \times (0, T), \quad i = 1, \ldots, N \\
\zeta_i^\mu(x, 0) = \eta_0_i^\mu(x, 0) & \text{on} \quad \varphi_\mu(U_\mu), \quad i = 1, \ldots, N,
\end{cases}
\]
where $\tilde{F}^i_{\mu}$ is again defined by (3.45) with $\eta$ replaced by $\tilde{\eta}$. Similar to (3.46), we have the estimate

\[(3.48)\]

\[\|\tilde{F}^i_{\mu} - F^i_{\mu}\|_{C^{1+1,\gamma}_x(\mathbb{R}^N \times (0, T))} \leq K \sum_{j, \nu} \left\|\tilde{F}^i_{\mu} \right\|_{C^{1+1,\gamma}_x(\mathbb{R}^N \times (0, T))} + \|\tilde{\eta}^i_{\nu}\|_{C^{1+1,\gamma}_x(\mathbb{R}^N \times (0, T))}\]

where the second last inequality follows from Lemma 3.3. This implies that $G\eta = RH\eta - \eta = \sum_{r=1}^{N} (C^i_{\mu} - \eta^i_{\nu})C^i_{1} \eta^i_{\nu}$ satisfies

\[\left\{ \begin{array}{l}
\frac{\partial}{\partial \tau} (C^i_{\mu} - \eta^i_{\nu}) - w^k \frac{\partial}{\partial \tau} (C^i_{\mu} - \eta^i_{\nu}) = (\partial \tilde{\Gamma} + \tilde{\Gamma} \ast w \tilde{\Gamma} + \tilde{\Gamma} \ast w \tilde{\nabla} \eta + \tilde{F}^i_{\mu} - F^i_{\mu}) & \text{on } \varphi_{s}(U_s) \times (0, T) \\
(C^i_{\mu} - \eta^i_{\nu})(x, 0) = 0 & \text{on } \varphi_{s}(U_s).
\end{array} \right.\]

Then (3.48) and Lemma 3.3 imply that

\[(3.49)\]

\[\|G\eta\|_{H^{k+2}(\mathbb{R}^N \times (0, T))} \leq K \left( \left\|\tilde{\Gamma} + \tilde{\Gamma} \ast w \tilde{\Gamma} + \tilde{\Gamma} \ast w \tilde{\nabla} \eta + \tilde{F}^i_{\mu} - F^i_{\mu}\right\|_{C^{1+1,\gamma}_x(\mathbb{R}^N \times (0, T))} + \sum_{i, \mu} \left\|\tilde{F}^i_{\mu} - F^i_{\mu}\right\|_{C^{1+1,\gamma}_x(\mathbb{R}^N \times (0, T))} \right) \]

We thereby proved the Lemma.

To complete the proof of Theorem 3.1, we choose $T^*$ sufficiently small so that the constant given in Lemma 3.3 satisfies $KT^* \leq \frac{1}{2}$. Then for each $T \in (0, T^*], \text{ Lemma 3.3}$ implies

\[\|Sh\|_{W_k} \leq \frac{1}{2} \|h\|_{W_k}, \quad \|G\eta\|_{H^{k+2}} \leq \frac{1}{2} \|\eta\|_{H^{k+2}}.\]

Then on the basis of contraction mapping principle and the Fredholm alternative the operators $Id_{W}^* + S$ and $Id_{X} + G$ have bounded inverses. From this and (3.48) we conclude that the operator $H$ has bounded inverse such that

\[(3.50)\]

\[R(Id_{W}^* + S)^{-1} = (Id_{X} + G)^{-1} R = H^{-1}.\]
Consequently we have
\begin{equation}
\|\eta\|_{\alpha, \frac{3}{2}, M \times [0,T]} + \|\nabla^k \eta\|_{C^{k+1,\gamma} \left( M \times (0,T) \right)}
\end{equation}
\begin{align*}
&= \|\eta\|_{\alpha, k+2, M \times [0,T]} \\
&= \|H^{-1} h\|_{\alpha, k+2, M \times [0,T]} \\
&\leq K \|h\|_{W_k} \\
&\leq K \left( \|F\|_{C^{k,\gamma} \left( M \times (0,T) \right)} + \|\eta_0\|_{\alpha, M} \right)
\end{align*}
for each $T \in (0,T^*)$. Here $K$ is a constant depending only on $k, M, \hat{g}, A$. Having established the theorem on a small time interval $[0, \min\{T^*, I\}]$, we can prove the theorem on an arbitrary interval $[0,T]$ for each $T \leq I$ by standard parabolic theory. Since for $t > 0$, we have $\eta(\cdot, t) \in C^{k+2,\gamma}(M)$ and $F, w \in C^{k+\gamma} \left( M \times [t,T] \right)$. Therefore we have established Theorem 3.1.
4. Ricci Flow with H"older Continuous Initial Metrics

Let $M$ be a smooth, compact Riemannian manifold with boundary $\partial M$. Let $g_0$ be a smooth Riemannian metric on $M$. Recall that the goal of this paper is to prove short time existence to Ricci flow on manifold with boundary in the following sense:

\begin{equation}
\begin{aligned}
\frac{\partial}{\partial t} g(t) &= -2\text{Ric}(g(t)) & \text{on } M \times (0, T], \\
A_{g(t)} &= 0 & \text{on } \partial M \times (0, T],
\end{aligned}
\end{equation}

where $A_{g(t)}$ is the second fundamental form of $\partial M$ in $(M, g(t))$. In [16], Shen proved short time existence to the above system provided that the initial metric $g_0$ is totally geodesic. We remark here that we do not impose any condition on the boundary second fundamental form $A_{g_0}$ for the initial metric $g_0$.

Equivalently, we can prove short time existence to the above boundary value problem via doubling the manifold and solving the corresponding Ricci flow on the doubled manifold but with rough initial data. Let $M$ be the double of $M$. More precisely, we define $\tilde{M} = M_1 \cup M_2 / \sim$, where $M_1$ and $M_2$ are identical copies of $M$ and $p_1 \sim p_2$ if $p \in \partial M$. Fix a smooth background metric $\hat{g}$ on $M$ such that in a small collar neighborhood of $\partial M$ the metric $\hat{g}$ is isometric to a product $\partial M \times [0, \epsilon)$. Note that $\hat{g}$ extends to a smooth metric on the doubled manifold $\tilde{M}$ via reflection about $\partial M$, which we would still denote it as $\hat{g}$. Next, we extend $g_0$ to a metric $\tilde{g}_0$ on the doubled manifold $\tilde{M}$ via reflection about $\partial M$. Then $\tilde{g}_0$ is a Lipschitz metric on $\tilde{M}$. In particular, we have $\tilde{g}_0 \in C^\alpha(M; \text{Sym}^2(T^*M))$ for all $\alpha \in (0, 1)$.

We consider the Ricci flow on $\tilde{M}$:

\begin{equation}
\frac{\partial}{\partial t} \tilde{g}(t) = -2\text{Ric}(\tilde{g}(t)) & \text{on } \tilde{M} \times (0, T].
\end{equation}

Note that solving (4.1) is equivalent to solving (4.2) with initial metric $\tilde{g}_0$. Namely if $\tilde{g}(t)$ is a smooth solution to (4.2) on $\tilde{M} \times (0, T]$, then $\tilde{g}(t)$ preserves the $\mathbb{Z}_2$ symmetry of $\tilde{g}_0$ and therefore $\tilde{g}(t)|_{M_i}$ is a smooth solution to (4.1) on $M_i \times (0, T]$ such that $(M_i, \tilde{g}(t)|_{M_i})$ has totally geodesic boundary.

We use the DeTurck’s trick to relate the system (4.2) to a modified system which is strictly parabolic. In the sequel of this section, we will work on the doubled manifold $(\tilde{M}, \tilde{g}_0)$. For the sake of notation simplicity we will still denote the doubled Riemannian manifold $(\tilde{M}, \tilde{g}_0)$ as $(M, g_0)$ if no confusions would be made. $\hat{\nabla}$ will denote the covariant derivative with respect to the background metric $\hat{g}$. We consider the following Ricci-DeTurck system on the doubled manifold $(M, g_0)$:

\begin{equation}
\begin{aligned}
\frac{\partial}{\partial t} g &= -2\text{Ric}(g) + L_W g & \text{on } M \times (0, T] \\
g(x, 0) &= g_0 & \text{on } M.
\end{aligned}
\end{equation}

Here the vector field $W$ is defined as $W^k = g^{ij}(\Gamma^k_g)_{ij} - (\hat{\Gamma}_\hat{g})^k_{ij}$. Moreover, we have

\begin{align*}
-2\text{Ric}(g) + L_W g &= g^{kl} \hat{\nabla}_k \hat{\nabla}_l g + Q(g, \hat{\nabla} g),
\end{align*}

where $Q(g, \hat{\nabla} g)$ is defined by

\begin{align*}
Q(g, \hat{\nabla} g)_{ij} &= -g^{kl} g^{pq} \hat{R}_{kjiql} - g^{kl} g^{pq} \hat{\hat{R}}_{kjiql} \\
&\quad + \frac{1}{2} g^{kl} g^{pq} (\hat{\nabla} g_{ip} \hat{\nabla}_j g_{ql} + 2 \hat{\nabla} g_{ip} \hat{\nabla}_j g_{ql} - 2 \hat{\nabla} g_{ip} \hat{\nabla}_j g_{ql}) \\
&\quad - 4 \hat{\nabla} g_{ip} \hat{\nabla}_j g_{ql}).
\end{align*}

We remark that the initial metric $g_0$ is merely H"older-continuous. We will use the Banach fixed point theorem to prove existence of a short time solution to (4.3).
To begin with, we define a suitable Banach space for the solutions. Let \( \alpha, \gamma \in (0, 1) \) be given such that \( \alpha > \gamma \) and let \( k \geq 1 \), for \( \eta \in \Gamma(M \times [0, T]; E) \) we define a norm \( \| \eta \|_{X^k_{\alpha, \gamma}(M \times [0, T])} \) by
\[
(4.4) \quad \| \eta \|_{X^k_{\alpha, \gamma}(M \times [0, T])} := \| \eta \|_{\alpha, \gamma; M \times [0, T]} + \| \nabla \eta \|_{C^{k-1, \gamma}(M \times [0, T])}.
\]
Moreover we define the Banach space
\[
(4.5) \quad X^k_{\alpha, \gamma}(M \times [0, T]; E) := \{ \eta : M \times [0, T] \to E | \| \eta \|_{X^k_{\alpha, \gamma}(M \times [0, T])} < \infty \}.
\]

4.1. Formulation of the existence result to the Ricci-DeTurck Flow.

Let \( \alpha, \gamma \in (0, 1) \) be given such that \( \alpha > \gamma \) and \( g_0 \in C^\infty(M) \). Denote by \( E = \text{Sym}^2(T^* M) \) the tensor bundle of symmetric \((0,2)\)-tensors on the closed manifold \( M \). The system (4.3) is equivalent to
\[
\begin{aligned}
\frac{\partial}{\partial t} g(x, t) - tr g \nabla^2 g(x, t) &= Q(g, \hat{\nabla} g)(x, t) \quad \text{on} \ M \times (0, T] \\
g(x, 0) &= g_0 \quad \text{on} \ M.
\end{aligned}
\]
The result in section 3 will help us to apply the Banach fixed point theorem.

Let \( w \in X^k_{\alpha, \gamma}(M \times [0, T]; E) \) be given such that \( w(\cdot, t) \) is a family of Riemannian metrics on \( M \times [0, T] \), we consider the following linear system:
\[
(4.7) \quad \begin{cases}
\frac{\partial}{\partial t} \eta(x, t) - tr \nabla \nabla^2 \eta(x, t) = tr \nabla \nabla w(x, t) - tr \nabla \nabla^2 w(x, t) + Q(w, \hat{\nabla} w)(x, t) \quad \text{on} \ M \times (0, T] \\
\eta(x, 0) = g_0(x) \quad \text{on} \ M.
\end{cases}
\]
Note that if a solution \( \eta \) to (4.7) satisfies \( \eta = w \), then \( \eta \) solves the nonlinear system (4.3).

**Proposition 4.1.** Consider the linear system (4.7). Suppose that
\begin{enumerate}
\item \( w(x, 0) = g_0(x) \);
\item \( w(\cdot, t) \) is a family of Riemannian metrics on \( M \times [0, T] \) such that \( \Lambda g_0(x) \geq w(x, t) \geq \frac{1}{\Lambda} g_0(x) \) for any \( (x, t) \in M \times [0, T] \).
\item \( \| w \|_{X^k_{\alpha, \gamma}(M \times [0, T])} \leq A. \)
\end{enumerate}

Then there is a unique solution \( \eta \in X^k_{2, \gamma}(M \times [0, T]) \) to the system (4.7) and positive constants \( K_1 = K_1(M, \hat{\nabla} g_0) \), \( K_2 = K_2(M, \Lambda, \hat{\nabla} g_0, A) \) such that
\[
\| \eta \|_{X^k_{\alpha, \gamma}(M \times [0, T])} \leq K_1 \left( K_2 T^2 + \| g_0 \|_{\alpha, M} \right).
\]
Moreover, there exists \( T^* = T^*(M, \Lambda, \hat{\nabla} g_0, A) \) such that \( \eta(\cdot, t) \) is a family of \( \Lambda g_0(x) \geq \eta(x, t) \geq \frac{1}{\Lambda} g_0(x) \) for any \( (x, t) \in M \times [0, T] \).
Proof. In the sequel, $K_1$ will denote a constant depending only on $M, \hat{g}, \|g_0\|_{\alpha, M}$ and $K_2$ will denote a constant depending only on $M, \Lambda, \hat{g}, \|g_0\|_{\alpha, M, A}$. We first consider the term
\[
Q = w^{-1} \ast w \ast \hat{R} + w^{-1} \ast w^{-1} \ast \nabla w \ast \nabla w.
\]
By assumption (2) and the matrix identity $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$, we have
\[
|w^{-1}|_{\alpha, M \times [0, T]} \leq K_2(\Lambda, \hat{g}) \quad \text{and} \quad |w^{-1}|_{\alpha, \hat{\Phi}, M \times [0, T]} \leq K(|w^{-1}|_{\alpha, M \times [0, T]}, |w|_{\alpha, \hat{\Phi}, M \times [0, T]}) \leq K_2(\Lambda, \hat{g}, A).
\]
This implies
\[
(4.8)
\]
\[
\|Q(w, \nabla^+ w)\|_{\mathcal{C}^{1,\gamma}_{\frac{1}{4} + \delta}(M \times (0, T))} \leq K_2 \left( \|w^{-1} \ast w\|_{\mathcal{C}^{1,\gamma}_{\frac{1}{4} + \delta}(M \times (0, T))} + \|w^{-1} \ast w^{-1} \ast \nabla w \ast \nabla w\|_{\mathcal{C}^{1,\gamma}_{\frac{1}{4} + \delta}(M \times (0, T))} \right)
\]
\[
\leq K_2 \left( T^\frac{1}{2} \|w^{-1}\|_{\alpha, \hat{\Phi}, M \times [0, T]} + T\|\nabla w\|_{\alpha, \hat{\Phi}, M \times [0, T]} \|\nabla^+ w\|_{\alpha, \hat{\Phi}, M \times [0, T]} \right)
\]
\[
\leq K_2 T^\frac{1}{2}.
\]
Similarly, using the facts $\alpha > \gamma$ and $|w - g_0(x, t)| \leq T^\frac{1}{2} \|w - g_0\|_{\alpha, \hat{\Phi}, M \times [0, T]}$, and the matrix identity $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$, we have
\[
(4.9)
\]
\[
\|tr_w \nabla^2 w - tr_{g_0 \nabla^2 w}\|_{\mathcal{C}^{1,\gamma}_{\frac{1}{4} + \delta}(M \times (0, T))} \leq K_2 \left( \|w^{-1} \ast w\|_{\mathcal{C}^{1,\gamma}_{\frac{1}{4} + \delta}(M \times (0, T))} \right)
\]
\[
\leq K_2 \left( T^\frac{1}{2} \|w^{-1}\|_{\alpha, \hat{\Phi}, M \times [0, T]} \right)
\]
\[
\leq K_2 T^\frac{1}{2}.
\]
From (4.8), (4.9) and Theorem 3.1, we conclude that there is a unique solution $\eta \in \mathcal{X}^{(\alpha)}_{\gamma}(M \times [0, T])$ to the linear system (4.7) and a positive constant $K_1 = K_1(M, \hat{g}, \|g_0\|_{\alpha})$ such that
\[
\|\eta\|_{\mathcal{X}^{(\alpha)}_{\gamma}(M \times (0, T))} \leq K_1 \left( \|tr_w \nabla^2 w - tr_{g_0 \nabla^2 w} + Q(w, \nabla w)\|_{\mathcal{C}^{1,\gamma}_{\frac{1}{4} + \delta}(M \times (0, T))} + \|g_0\|_{\alpha, M} \right)
\]
\[
\leq K_1(K_2 T^\frac{1}{2} + \|g_0\|_{\alpha, M}).
\]
Next, note that the bound for the semi-Hölder norm $[\eta]_{\alpha, \hat{\Phi}, M \times [0, T]}$ implies that
\[
\|\eta(x, t) - g_0(x)\|_{\alpha, M \times [0, T]} \leq K(M, \Lambda, \hat{g}, \|g_0\|_{\alpha, A}) T^\frac{1}{2}.
\]
Thus if $T^* = T^*(M, \Lambda, \hat{g}, \|g_0\|_{\alpha, A})$ is sufficiently small, then the second conclusion of the proposition also holds.
\[
\square
\]
4.2. Short time existence and uniqueness to Ricci-DeTurck flow.

We now prove the short time existence for the Ricci-DeTurck flow (4.3) by employing the Banach fixed point theorem. Let $A > 2$ and $A > 10AK_1\|g_0\|_{\alpha,M}$ be large constants, where $K_1$ is the constant given in Proposition 4.1.1 We define a closed subset in $X_{2,\gamma}^\alpha(M \times [0, T])$ by

$$B := \{ w \in X_{2,\gamma}^\alpha \mid w|_{t=0} = g_0, \ A_0(t) \geq w(\cdot, t) \geq \frac{1}{A} g_0(\cdot), \ \|w\|_{X_{2,\gamma}^\alpha} \leq \Lambda \}.$$ 

The subset $B$ is non-empty by our choice of $A$ provided that $T = T(M, \Lambda, \hat{g}, \|g_0\|_{\alpha, A})$ is chosen sufficiently small. We next define an operator $R : B \to X_{2,\gamma}^\alpha$ by

$$\eta := R(w),$$

where $\eta$ is the unique solution to the system (4.1) in $X_{2,\gamma}^\alpha$. The operator $R$ is well defined by Proposition 4.1. Moreover, we can further set $T = T(M, \Lambda, \hat{g}, \|g_0\|_{\alpha, A})$ sufficiently small such that $\|\eta\|_{X_{2,\gamma}^\alpha} \leq A$ and $A_0(t) \geq \eta(t) \geq \frac{1}{A} g_0(\cdot)$ by Proposition 4.1. Consequently we have $R(B) \subset B$ by our choice of $T$.

**Proposition 4.2.** If $T = T(M, \Lambda, \hat{g}, \|g_0\|_{\alpha, A})$ is chosen sufficiently small, the operator $R$ is a contraction mapping.

**Proof.** In the sequel, $K$ will denote a constant depending only on $M, \Lambda, A, \hat{g}, \|g_0\|_{\alpha, M}$. Let $w_1, w_2 \in B$ and write $\eta_i := R(w_i)$ for $i = 1, 2$. Then $\eta := \eta_1 - \eta_2$ solves the following system:

$$(4.11)\begin{cases}
\frac{\partial}{\partial t} \eta - tr_{g_0} \hat{\nabla}^2 \eta = \hat{Q} & \text{on } M \times (0, T) \\
\eta(x, 0) = 0 & \text{on } M,
\end{cases}$$

where $\hat{Q} := tr_{w_i} \hat{\nabla}^2 w_1 - tr_{w_2} \hat{\nabla}^2 w_2 - tr_{g_0} \hat{\nabla}^2 (w_1 - w_2) + Q(w_1, \hat{\nabla} w_1) - Q(w_2, \hat{\nabla} w_2)$. Then Theorem 4.1 with $\eta_0 = 0$ asserts that

$$\|\eta\|_{X_{2,\gamma}^\alpha} \leq K \|\hat{Q}\|_{C_1(\hat{\nabla}^2 \Omega_1(0, T))}.$$ 

We first derive the estimate for the term $Q(w_1, \hat{\nabla} w_1) - Q(w_2, \hat{\nabla} w_2)$. Recall that $Q(w, \hat{\nabla} w) = w^{-1} w \hat{R} + w^{-1} \hat{\nabla} w - \hat{\nabla} w$, thus we can write the difference as

$$Q(w_1, \hat{\nabla} w_1) - Q(w_2, \hat{\nabla} w_2)$$

$$= (w_1^{-1} - w_2^{-1}) * w_1 \hat{R} + w_2^{-1} * (w_1 - w_2) * \hat{R}$$

$$+ (w_1^{-1} - w_2^{-1}) * w_1^{-1} \hat{\nabla} w_1 + w_2^{-1} * (w_1^{-1} - w_2^{-1}) * \hat{\nabla} w_1$$

$$+ w_2^{-1} * w_2^{-1} * (\hat{\nabla} w_1 - \hat{\nabla} w_2) * \hat{\nabla} w_1 + w_2^{-1} * w_2^{-1} * \hat{\nabla} w_2 * (\hat{\nabla} w_1 - \hat{\nabla} w_2).$$

Using the matrix identity $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$ and the fact that $\|w_i\|_{X_{2,\gamma}^\alpha} \leq A$, we have for each $\sigma \in (0, T)$,

$$\|Q(w_1, \hat{\nabla} w_1) - Q(w_2, \hat{\nabla} w_2)\|_{0, M \times [\sigma, \tau]}$$

$$\leq K \left( |w_1 - w_2|_{0, M \times [\sigma, \tau]} + \sigma^{\alpha - 1} |w_1^{-1} - w_2^{-1}|_{0, M \times [\sigma, \tau]} + \sigma^{\alpha - \frac{1}{2}} \|\hat{\nabla} w_1 - \hat{\nabla} w_2\|_{0, M \times [\sigma, \tau]} \right)$$

$$\leq K \left( \sigma^{\alpha - 1} |w_1 - w_2|_{0, M \times [\sigma, \tau]} + \sigma^{\alpha - \frac{1}{2}} \|\hat{\nabla} w_1 - \hat{\nabla} w_2\|_{0, M \times [\sigma, \tau]} \right).$$
This implies

\[ \sigma^{-1} \hat{\mathbf{w}} [Q(w_1, w_1^{-1}, \nabla w_1) - Q(w_2, w_2^{-1}, \nabla w_2)]_{0, M \times \hat{\mathbf{D}}} \leq K \sigma^{-1} \hat{\mathbf{w}} \sup_{\sigma \in [0, T]} |w_1 - w_2|_{0, M \times \hat{\mathbf{D}}} + \sigma^{\alpha - 1} \hat{\mathbf{w}} |\nabla w_1 - \nabla w_2|_{0, M \times \hat{\mathbf{D}}} \leq KT \hat{\mathbf{w}} \|w_1 - w_2\|_{\chi^2(\alpha)}.
\]

On the other hand,

\[ [Q(w_1, \nabla w_1) - Q(w_2, \nabla w_2)]_{\gamma, \hat{\mathbf{D}} \times \hat{\mathbf{D}}} \leq K \left( |w_1 - w_2|_{\gamma, \hat{\mathbf{D}} \times \hat{\mathbf{D}}} + |w_1 - w_2|_{0, M \times \hat{\mathbf{D}}} + \sigma^{\alpha - 1} |w_1 - w_2|_{\gamma, \hat{\mathbf{D}} \times \hat{\mathbf{D}}} \right) \]

where \( \alpha > \gamma \), the Hölder semi-norm \( |w_1 - w_2|_{\gamma, \hat{\mathbf{D}} \times \hat{\mathbf{D}}} \) is controlled by \( |w_1 - w_2|_{0, \hat{\mathbf{D}} \times \hat{\mathbf{D}}} \) and \( \sigma \in (0, T) \). Putting (4.13) and (4.14) together we obtain

\[ \|Q(w_1, \nabla w_1) - Q(w_2, \nabla w_2)\|_{C_{\alpha, \gamma}^M (0, T)} \leq K \hat{\mathbf{w}} \|w_1 - w_2\|_{\chi^2(\alpha)}.
\]

Next, we derive the estimate for the term \( tr_{w_1} \nabla^2 w_1 - tr_{w_2} \nabla^2 w_2 - tr_{g_0} \nabla^2 (w_1 - w_2) \). We write it as

\[ tr_{w_1} \nabla^2 w_1 - tr_{w_2} \nabla^2 w_2 - tr_{g_0} \nabla^2 (w_1 - w_2) = (w_1^{-1} - w_2^{-1}) \ast \nabla^2 w_1 + (w_2^{-1} - g_0^{-1}) \ast \nabla^2 (w_1 - w_2).
\]

For the first term in (4.16), we use the matrix identity \( A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1} \) to estimate

\[ |(w_1^{-1} - w_2^{-1}) \ast \nabla^2 w_1|_{0, M \times \hat{\mathbf{D}}} \leq K |w_1 - w_2|_{0, M \times \hat{\mathbf{D}}} \|\nabla^2 w_1|_{0, M \times \hat{\mathbf{D}}} \]

\[ \leq K \sigma^{-1} \hat{\mathbf{w}} |w_1 - w_2|_{0, M \times \hat{\mathbf{D}}}, \]

for any \( \alpha \). Consequently,

\[ \sigma^{-1} \hat{\mathbf{w}} [(w_1^{-1} - w_2^{-1}) \ast \nabla^2 w_1]_{\gamma, \hat{\mathbf{D}} \times \hat{\mathbf{D}}} \leq K \sigma^{-1} \hat{\mathbf{w}} \sup_{\tau \in (0, T)} \tau^{-1} |w_1 - w_2|_{0, M \times \hat{\mathbf{D}}} \]

\[ \leq K \sigma^{-1} \hat{\mathbf{w}} \|w_1 - w_2\|_{\chi^2(\alpha)}.
\]

for any \( \sigma \in (0, T) \). On the other hand,

\[ [(w_1^{-1} - w_2^{-1}) \ast \nabla^2 w_1]_{\gamma, \hat{\mathbf{D}} \times \hat{\mathbf{D}}} \]

\[ \leq K \left( |w_1 - w_2|_{0, M \times \hat{\mathbf{D}}} |\nabla^2 w_1|_{\gamma, \hat{\mathbf{D}} \times \hat{\mathbf{D}}} + |w_1 - w_2|_{\gamma, \hat{\mathbf{D}} \times \hat{\mathbf{D}}} |\nabla^2 w_1|_{0, M \times \hat{\mathbf{D}}} \right) \]

\[ \leq K \left( \sigma^{-1} \hat{\mathbf{w}} |w_1 - w_2|_{0, M \times \hat{\mathbf{D}}} + \sigma^{-1} \hat{\mathbf{w}} |w_1 - w_2|_{\gamma, \hat{\mathbf{D}} \times \hat{\mathbf{D}}} \right).
\]
for any $\sigma \in (0, T]$. By an argument similar to (4.17) and the fact $\alpha > \gamma$, we have

(4.18)\[
\sigma^{0,1} \hat{\xi} \left( [w_1^{0,1} - w_2^{0,1}] \right) * \hat{\nabla}^2 w_1 |_{\gamma, M \times \{ z \_ \rho \}} \leq K \left( \sigma^{0,1} \hat{\xi} \left( w_1 - w_2 \right) \right)_{[0, M \times \{ z \_ \rho \}_{\rho}]} + \sigma^{0,1} \hat{\xi} \left( w_1 - w_2 \right)_{[\gamma, \alpha, \beta, M \times \{ z \_ \rho \}_{\rho}]} \leq K \sigma^{0,1} \left\| w_1 - w_2 \right\|_{\chi^{0,1}_{2,\gamma}}
\]

for any $\sigma \in (0, T]$. For the second term in (4.16), we have

(4.19)\[
\sigma^{0,1} \hat{\xi} \left( w_2^{0,1} - g_0^{0,1} \right) * \hat{\nabla}^2 (w_1 - w_2) |_{0, M \times \{ z \_ \rho \}} \leq K \sigma^{0,1} \hat{\xi} \left( w_2 - g_0 \right)_{[0, M \times \{ z \_ \rho \}_{\rho}]} |\hat{\nabla}^2 (w_1 - w_2)|_{[0, M \times \{ z \_ \rho \}_{\rho}]} \leq K \sigma^{0,1} \sup_{\tau \in [0, T]} \tau^{1,0,1} \hat{\xi} \left( w_1 - w_2 \right)_{[0, M \times \{ z \_ \rho \}_{\rho}]} \leq K \sigma^{0,1} \left\| w_1 - w_2 \right\|_{\chi^{0,1}_{2,\gamma}}.
\]

(4.20)\[
\sigma^{0,1} \hat{\xi} \left( w_2^{0,1} - g_0^{0,1} \right) * \hat{\nabla}^2 (w_1 - w_2) |_{0, M \times \{ z \_ \rho \}} \leq K \sigma^{0,1} \hat{\xi} \left( w_2 - g_0 \right)_{[0, M \times \{ z \_ \rho \}_{\rho}]} |\hat{\nabla}^2 (w_1 - w_2)|_{[0, M \times \{ z \_ \rho \}_{\rho}]} \leq K \sigma^{0,1} \sup_{\tau \in [0, T]} \tau^{1,0,1} \hat{\xi} \left( w_1 - w_2 \right)_{[0, M \times \{ z \_ \rho \}_{\rho}]} \leq K \sigma^{0,1} \left\| w_1 - w_2 \right\|_{\chi^{0,1}_{2,\gamma}}.
\]

Putting (4.16) to (4.19) together, we obtain that

(4.21)\[
\left\| \tau w_1 \right\|_{\nu, \gamma, M} \leq K T^{\hat{\xi}} \left\| w_1 - w_2 \right\|_{\chi^{0,1}_{2,\gamma}}
\]

Hence, putting (4.12), (4.15) and (4.20) together, we obtain

(4.22)\[
\left\| \eta_1 - \eta_2 \right\|_{\chi^{0,1}_{2,\gamma}} \leq K T^{\hat{\xi}} \left\| w_1 - w_2 \right\|_{\chi^{0,1}_{2,\gamma}}.
\]

If $T = T(M, A, \hat{g}, g_0, A)$ is chosen sufficiently small, we then have

\[
\left\| R(w_1) - R(w_2) \right\|_{\chi^{0,1}_{2,\gamma}} \leq \frac{1}{2} \left\| w_1 - w_2 \right\|_{\chi^{0,1}_{2,\gamma}}.
\]

This proves the proposition. \(\square\)

Therefore, by the Banach fixed point theorem we have proved

**Theorem 4.3.** There exist $K = K(M, \hat{g}, g_0, A, M)$ and $T = T(M, \hat{g}, g_0, A, M)$ such that the following holds:

There is a unique solution $g \in \chi^{0,1}_{2,\gamma}(M \times [0, T])$ to the Ricci-DeTurck flow (4.7) such that

- $g(t, \cdot)$ is a family of Riemannian metrics;
- $\left\| g \right\|_{\chi^{0,1}_{2,\gamma}(M \times [0, T])} \leq K$.

Next, we can improve the regularity of $g$ by bootstrapping.
Corollary 4.4. Let $k \geq 2$ be given. There exist $K = K(M, k, \hat{g}, \|g_0\|_{\alpha, M})$ and $T = T(M, \hat{g}, \|g_0\|_{\alpha, M})$ such that the following holds:

There is a unique solution $g \in \mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])$ to the Ricci-DeTurck flow (4.3) such that

- $g(\cdot, t)$ is a family of Riemannian metrics;
- $\|g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} \leq K$.

Proof. We will prove the corollary by induction on $k$. It is clear from Theorem 3.1 that the assertion holds when $k = 2$. Let us assume that $g \in \mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])$ for some $k \geq 2$. In the sequel, $K$ will denote a constant depending only on $M, k, \hat{g}, \|g_0\|_{\alpha, M}, \|g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}}$. Firstly, note that by the matrix identity $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$, the term $\|g^{-1}\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])}$ is controlled by $\|g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])}$. Moreover, we observe that

$$\|g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} = \|g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} + \|\hat{\nabla} g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])}.$$

Since $\alpha > \gamma$,

$$\|g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} \leq K \|g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])},$$

and it follows from Lemma 2.2 that

$$\|\hat{\nabla} g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} \leq KT^{\frac{2}{3}} \|\hat{\nabla} g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])},$$

we have

$$\|g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} \leq K.$$

Since $g$ solves the Ricci-DeTurck system (4.3) on $M \times (0, T)$, by Lemma 2.2 the induction hypothesis, and the above estimates, we have

$$\|g^{-1} \ast g^{-1} \ast \hat{\nabla} g \ast \hat{\nabla} g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} \leq K T^{\frac{4}{3}} \|g^{-1}\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} \|\hat{\nabla} g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} \leq K$$

and

$$\|g^{-1} \ast g \ast \hat{R}\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} \leq K T^{1 - \frac{2}{3}} \|g^{-1}\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} \|\hat{R}\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} \leq K.$$

Theorem 4.3 then implies that

$$\|g\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} \leq K (\|Q(g, \hat{\nabla} g)\|_{\mathcal{X}_{k, \gamma}^{(\alpha)}(M \times [0, T])} + \|g_0\|_{\alpha}) \leq K.$$

From these, the assertion follows.
4.3. Short time existence to the Ricci flow.

We will prove the short time existence of solution to the Ricci flow with Hölder-continuous initial metrics in this subsection. More specifically, we show that any solution of the Ricci-DeTurck flow with Hölder-continuous initial data gives rise to a solution of the Ricci flow with Hölder-continuous initial data.

For any $\alpha \in (0, 1)$ and $k \geq 0$, we define by $C^{k,\alpha}(M; M)$ the space of $C^{k,\alpha}$ maps $f : M \to M$. Recall in section 2 that $\{U_{\mu}, \varphi_{\mu}\}_{\mu=1,\ldots,m}$ is a fixed set of coordinate charts on $M$. On the coordinate chart $U_{\mu}$, a map $f : M \to M$ has components $(f_{\mu}^{1}, \ldots, f_{\mu}^{n})$. The Hölder norm for maps $f \in C^{k,\alpha}(M; M)$ is measured with respect to the fixed charts $\{U_{\mu}, \varphi_{\mu}\}$. More precisely, given $f \in C^{k,\alpha}(M; M)$, we define the associated $C^{k,\alpha}$ norm to be

$$\|f\|_{C^{k,\alpha}(M; M)} = \sum_{\mu} \sum_{r=1}^{n} |f_{\mu}|_{C^{k,\alpha}(\varphi_{\mu}(U_{\mu}))}.$$

In the remainder of this subsection, we will consider the ODE

$$\begin{cases}
\frac{\partial}{\partial t} \psi_t = -W(\psi_t, t), \\
\psi_0 = \text{id},
\end{cases} \tag{4.23}$$

where $W$ is the DeTurck vector field defined by $W^k = g^{ij}(\Gamma_{g}^k)_{ij} - (\hat{\Gamma}_{g}^k)_{ij}$, and $g \in \mathcal{X}_{k,\gamma}^{(\alpha)}$ is the solution to the Ricci-DeTurck flow on $M \times (0, T)$. Note that the vector field $W$ is undefined at $t = 0$. Nevertheless, we can show that the one-parameter family of maps $\{\psi_t\}$ generated by (4.23) can be extended to a map $\psi_0$ at $t = 0$. In the sequel of the subsection, $K$ will denote a constant depending only on $M, \alpha, g, \|g_0\|_{\alpha, M}, \|g\|_{\mathcal{X}_{k,\gamma}^{(\alpha)}}$.

Lemma 4.5. Let $g \in \mathcal{X}_{k,\gamma}^{(\alpha)}$ be the solution to the Ricci-DeTurck flow on $M \times (0, T)$, and $W$ be the DeTurck vector field defined by $W^k = g^{ij}(\Gamma_{g}^k)_{ij} - (\hat{\Gamma}_{g}^k)_{ij}$. Then the one-parameter family of maps $\{\psi_t\}$ generated by (4.23) can be extended to a map $\psi_0$ at $t = 0$ in $C^{1,\beta}(M; M)$ for any $\beta < \alpha$ and satisfies

$$\|\psi_t\|_{C^{1,\beta}(M; M)} \leq K$$

for each $t \in [0, T]$. Moreover, $\psi_t$ remains a diffeomorphism for $t \in [0, T]$ if $T$ is sufficiently small.

Proof. Since $W = g^{-1} * \nabla g$, we have the estimate $\|W\|_{C^{k-1,\gamma}((M \times (0, T))} \leq K$. Note that the ODE (4.23) is uniquely solvable on $M \times (0, T)$, we would like to show that the solution $\psi_t$ can be extended to $M \times [0, T]$ in $C^{1,\beta}(M; M)$ for any $\beta < \alpha$.

We now work in local coordinates. Suppose that $\psi_t$ has components $\psi_t^k$ in a chart $U$, we see that $\partial_t \psi_t^k$ satisfies the ODE

$$\frac{\partial}{\partial t} \partial_t \psi_t^k = -\partial_t W^k(\psi_t, t) \partial_t \psi_t^l, \quad i = 1, \ldots, n \tag{4.24}$$

for each $t \in (0, T)$. Let $\beta \in (0, \alpha)$ be given and define $\beta' := \frac{\beta}{2}$. We claim that $\|\psi_t\|_{C^{1,\beta'}(M; M)}$ has a uniform bound for every $t \in (0, T)$. Define

$$f(t) := \|\psi_t\|_{C^{1,\alpha}(M; M)},$$
then from (4.24) we have
\[(4.25)\]
\[
\frac{\partial}{\partial t} \partial_t \psi^k_t \leq K \ t^{\frac{2}{3}} - f(t).
\]
To integrate the above inequality, it is possible that the integral curves pass into different charts. Fix \(x \in M\) and \(t \in (0, T)\). Suppose that \(\psi T(x) \in U\) and \(\psi t(x) \in V\). Since there are only finitely many coordinate charts, we may assume without loss of generality that \(U \cap V \neq \emptyset\) and there is \(s \in (t, T)\) such that \(\psi T(x) \in U\) for all \(\tau \in [s, T]\) and \(\psi t(x) \in V\) for all \(\tau \in [t, s]\). By integrating the inequality (4.24) from \(s\) to \(T\), and by noting that \(\partial_t \psi^k_T = \delta^k_t\), we have
\[(4.26)\]
\[
|\partial_t \psi^k_s(x) - \delta^k_t| \leq \int_s^T K \tau^{\frac{2}{3}} - f(\tau) \ d\tau.
\]
Moreover, on the intersection \(U \cap V\) we may write \(\partial_t \psi^k_T = \partial_{\tilde{\psi}^k_T} \frac{\partial y}{\partial x}\), where \((y_k)\) and \((z_i)\) stand for the local coordinates on the charts \(U\) and \(V\) respectively, and \(\tilde{\psi}^k_T\) stand for the components of \(\psi T\) on \(V\). Since \(\partial_t \tilde{\psi}^k_T\) satisfies (4.25), it also satisfies a similar inequality as above by integrating (4.25) from \(t\) to \(s\). This implies that
\[(4.27)\]
\[
|\partial_t \tilde{\psi}^k_t(x) - \partial_t \tilde{\psi}^k_s(x)| \leq \int_t^s K \tau^{\frac{2}{3}} - f(\tau) \ d\tau.
\]
From (4.26) and (4.27) we obtain
\[(4.28)\]
\[
|\partial_t \tilde{\psi}^k_t(x)| \leq K + \int_t^T K \tau^{\frac{2}{3}} - f(\tau) \ d\tau.
\]
We can similarly obtain bounds for \(|\tilde{\psi}^k_t(x)|\) by integrating the ODE (4.23). Since \(x \in M\) is arbitrary, we obtain
\[(4.29)\]
\[
f(t) \leq K + \int_t^T K \tau^{\frac{2}{3}} - f(\tau) \ d\tau.
\]
Grönewall’s lemma then implies that
\[(4.30)\]
\[
\|\tilde{\psi}_t\|_{C^1,\alpha(M, M)} \leq K \exp(KT^{\frac{2}{3}})
\]
for any \(t \in (0, T)\).

Next, let \(x, y \in M\) and fix \(t \in (0, T)\). To bound the \(C^{1,\alpha'}\) norm for \(\tilde{\psi}_t\), we may assume without loss of generality that \(\tilde{\psi}_t(x)\) stays in the same chart with \(\tilde{\psi}_t(y)\) for \(\tau \in [t, T]\) using the uniform \(C^1\) bound (4.30). Note that it is still possible that the integral curves \(\psi T(x)\) and \(\psi T(y)\) may pass into different charts. By (4.24) we have
\[(4.31)\]
\[
\frac{\partial}{\partial \tau} \left( \frac{\partial_t \psi^k_t(x) - \partial_t \psi^k_t(y)}{|x - y|^{\beta'}} \right) = -\frac{\partial_t W^k(\psi T(x), \tau) - \partial_t W^k(\psi t(y), \tau)}{|x - y|^{\beta'}} \partial_t \psi^k_t(x) - \partial_t \psi^k_t(y)\]
for each \(\tau \in (t, T)\). By Lemma 2.2 we have \(\|W\|_{\frac{1}{2}, \frac{\beta}{2}'(M \times (0, T))} \leq K\). Thus the bound for \(\check{\nabla} W\) and (4.30) imply
\[
|\partial_t W^k(\psi T(x), \tau) - \partial_t W^k(\psi t(y), \tau)| \leq K \tau^{r^{1+\frac{\beta - \alpha'}{3}}} |\psi T(x) - \psi t(y)|^{\beta'}
\]
\[
\leq K \tau^{r^{1+\frac{\beta - \alpha'}{3}}} |x - y|^{\beta'}
\]
and
\[
|\partial_t W^k(\psi t(y), \tau)| \leq K \tau^{-r^{1+\frac{\beta - \alpha'}{3}}}.
\]
Hence
\[
\frac{\partial}{\partial \tau} \left( \frac{\partial \psi^k_t(x) - \partial \psi^k_t(y)}{|x-y|^{\sigma'}} \right) \leq K \tau^{-1+\frac{2-\alpha'}{\alpha'}} |\partial_t \psi^k_t(x)| + K \tau^{-1+\frac{2-\alpha'}{\alpha'}} |\partial_t \psi^k_t(y)|
\]
\[
\leq K \tau^{-1+\frac{2-\alpha'}{\alpha'}} + K \tau^{-1+\frac{2-\alpha'}{\alpha'}} \|\psi_t\|_{C^{1,\alpha'}(M;M)}
\]
for each \( \tau \in (t, T) \). Subsequently similar to the argument in deriving the \( C^1 \) bound (4.30), we can integrate the above inequality to obtain
\[
\frac{\partial \psi^k_t(x) - \partial \psi^k_t(y)}{|x-y|^{\sigma'}} \leq \int_t^T \left( K \tau^{-1+\frac{2-\alpha'}{\alpha'}} + K \tau^{-1+\frac{2-\alpha'}{\alpha'}} \|\psi_t\|_{C^{1,\alpha'}(M;M)} \right) d\tau
\]
\[
\leq KT \frac{2-\alpha'}{\alpha'} + \int_t^T K \tau^{-1+\frac{2-\alpha'}{\alpha'}} \|\psi_t\|_{C^{1,\alpha'}(M;M)} d\tau.
\]
Since \( x, y \in M \) are arbitrary, we can combine the above inequality and the inequality (4.29) to obtain
\[
\|\psi_t\|_{C^{1,\alpha'}(M;M)} \leq K + KT \frac{2-\alpha'}{\alpha'} + \int_t^T K \tau^{-1+\frac{2-\alpha'}{\alpha'}} \|\psi_t\|_{C^{1,\alpha'}(M;M)} d\tau.
\]
Then Grönwall’s lemma implies
\[
(4.32) \quad \|\psi_t\|_{C^{1,\alpha'}(M;M)} \leq K'(T)
\]
for any \( t \in (0, T] \), which is uniform for each \( t \in (0, T] \). Since bounded subsets in \( C^{1,\beta'}(M;M) \) are precompact in \( C^{1,\beta}(M;M) \) as \( \beta' > \beta \), upon passing to a subsequence we have \( \psi_t \to \psi_0 \) in \( C^{1,\beta}(M;M) \) as \( t \to 0 \).

Lastly, \( \psi_t \) remains a diffeomorphism on \( (0, T] \) provided \( T \) is sufficiently small by the inverse function theorem. It remains to show that \( \psi_0 \) is also a diffeomorphism. We need a uniform lower bound for the differential \( d\psi_t \). Fix \( x, y \in M \) and \( t \in (0, T] \).

Let \( c : [0, L] \to M \) be a curve such that \( c(0) = x \) and \( c(L) = y \). Define
\[
h_c(\tau) = \int_0^L (\psi^* \hat{g})(c'(s), c'(s)) ds.
\]
Then \( \|\psi_t(x) - \psi_t(y)\|_{\hat{g}} = \inf_{c} h_c(\tau) \). Moreover, from the fact that \( \sigma^{1-\frac{2}{\alpha'}} \|W\|_{0,M \times [\hat{g}, \sigma]} \leq K \), we have
\[
h'_c(\tau) = \int_0^L (\psi^* \hat{\nabla} W(\tau))(c'(s), c'(s)) ds
\]
\[
\leq K \|\hat{\nabla} W(\tau)\|_{0,M} h_c(\tau)
\]
\[
\leq K \tau^{-\frac{2}{\alpha'}} h_c(\tau)
\]
for each \( \tau \in (t, T) \). This implies
\[
h_c(T) \leq h_c(t) \exp \left( \int_t^T K \tau^{-\frac{2}{\alpha'}} d\tau \right) \leq Kh_c(t).
\]
Similarly,
\[
h_c(T) \geq \frac{1}{K} h_c(t).
\]
In particular, this implies the uniform estimate
\[
(4.33) \quad \frac{1}{K} \|x - y\|_{\hat{g}} \leq \|\psi_t(x) - \psi_t(y)\|_{\hat{g}} \leq K \|x - y\|_{\hat{g}}
\]
for any \( t \in [0, T] \) and any \( x, y \in M \). Consequently both \( \psi_0 \) and \( D\psi_0 \) are injective. Thus \( \psi_0 \) is a diffeomorphism to its image. Now we show that \( \psi_0 \) is surjective. Take any \( y \in M \) and choose a sequence \( s_j \to 0 \). Define a sequence of points by \( x_j := \psi^{-1}_{s_j}(y) \). Since \( M \) is compact, after passing to a subsequence we have \( x_j \to x_\infty \in M \).
The uniform estimate (4.33) then implies that \(|\psi_{x_j}(x_j) - \psi_{x_j}(x_{\infty})| \to 0\). On the other hand, we have \(|\psi_{x_j}(x_{\infty}) - \psi_0(x_{\infty})| \to 0\). Hence \(|\psi_{x_j}(x_j) - \psi_0(x_{\infty})| \to 0\) and we get \(\psi_0(x_{\infty}) = y\).

\[\]

For the family of diffeomorphisms \(\{\psi_t\}_{t \in [0, T]}\), it is possible that the curve \(t \mapsto \psi_t(x)\) does not stay in the same chart for all \(t \in [0, T]\). Hence if \(\tilde{\psi} : M \times [0, T] \to M\) is a map defined by \(\tilde{\psi}(x, t) = \psi_t(x)\), we cannot measure its parabolic Hölder norm the way we did for the elliptic Hölder norm without any modification. However, our final goal is to show that the pullback metric \(\psi^*_tg_t\) satisfies the Ricci flow equation and lies in the weighted space \(X^{\alpha}_{k, \gamma}\) provided that \(g \in X^{\alpha}_{k, \gamma}\) is a solution to the Ricci-DeTurck flow.

Now, consider the section \(d\psi_t : M \to T^*M \otimes (\psi_t)^*(T^*M)\). We introduce the multi-index notations

\[\mu = (\mu_1, \mu_2).\]

If \((x^\mu_i)_{i=1,...,n}\) and \((y^\mu_j)_{j=1,...,n}\) are local coordinates on the charts \(U_{\mu_1}\) and \(U_{\mu_2}\) respectively, then

\[e^\mu_{ij}(t) := dx^\mu_i \otimes \frac{\partial}{\partial(y^\mu_j \circ \psi_t)}, \quad i_1, i_2 = 1, ..., n\]

form a local frame on \(T^*M \otimes (\psi_t)^*(T^*M)|_{U_{\mu_1} \cap \psi_t^{-1}(U_{\mu_2})}\). Moreover, if \(x \in U^\mu_i := U_{\mu_1} \cap \psi_t^{-1}(U_{\mu_2})\), then locally around \(x\) the section \(d\psi_t\) can be expressed as

\[d\psi_t = (d\psi_t)^\mu_i e^\mu_{ij}(t), \quad \text{where} \quad (d\psi_t)^\mu_i = \partial_i(\psi_t)^\mu_i.\]

**Definition 4.6.** With the above notations, let \(\{\rho_{\mu_2}\}\) be the partition of unity subordinate to the chart \(U_{\mu_2}\), we define functions \(\Psi^\mu_{ij}(x, t)\) by

\[\Psi^\mu_{ij}(x, t) = \begin{cases} \rho_{\mu_2}(\psi_t(x))(d\psi_t)^\mu_i(x) & \text{if} \ \psi_t(x) \in \supp(\rho_{\mu_2}) \\ 0 & \text{if} \ \psi_t(x) \notin \supp(\rho_{\mu_2}) \end{cases}.\]

Note that in this way, although the component maps \(t \mapsto (d\psi_t)^\mu_i(x)\) may not be defined for all \(t \in [0, T]\), the maps \(\Psi^\mu_{ij}(x, \cdot)\) are well defined for all \(t \in [0, T]\).

**Lemma 4.7.** Let \(g \in X^{\alpha}_{k+2, \gamma}\) be the solution to the Ricci-DeTurck flow on \(M \times [0, T]\), and \(\{\psi_t\}_{t \in [0, T]}\) be the one-parameter family of diffeomorphisms generated by (4.23). Then given any \(\beta < \alpha\), the functions \(\Psi^\mu_{ij}\) which are defined by Definition 4.6 satisfy

\[\]

1. \(\Psi^\mu_{ij} \in C^{\beta, \frac{k}{2}}(M \times [0, T]; \mathbb{R})\) and \(\nabla \Psi^\mu_{ij} \in C^{\beta - 2, \gamma}(M \times [0, T]; \mathbb{R})\);

2. \(\|\Psi^\mu_{ij}\|_{\beta, \frac{k}{2}; M \times [0, T]} + \|\nabla \Psi^\mu_{ij}\|_{\beta - 2, \gamma}(M \times [0, T]) \leq K(M, k, \hat{g}, \|g_0\|_\alpha, \|g\|_{X^{\alpha}_{k+2, \gamma}})\).

**Proof.** Firstly, Lemma 4.5 has already shown that \(\Psi^\mu_{ij}(\cdot, t) \in C^{0, \beta}(M; \mathbb{R})\) for each \(t \in [0, T]\) with an uniform upper bound for the elliptic Hölder norm. Thus to show that \(\Psi^\mu_{ij} \in C^{\beta, \frac{k}{2}}(M \times [0, T]; \mathbb{R})\), it remains to show that \(\Psi^\mu_{ij}(x, \cdot)\) is also Hölder-continuous for \(t \in [0, T]\) for each \(x \in M\). We rewrite the equation (4.24) in the new notations as

\[\frac{\partial}{\partial t}(d\psi_t)^\mu_i = -\partial_t W^j(\psi_t, t) (d\psi_t)^\mu_i.\]

This gives

\[\frac{\partial}{\partial t} \Psi^\mu_{ij} = -\langle D\rho_{\mu_2}(\psi_t), W \rangle (d\psi_t)^\mu_i - \rho_{\mu_2}(\psi_t) \partial_t W^j(\psi_t, t) (d\psi_t)^\mu_i.\]
Using the $C^1$ bound \((4.30)\) and the fact that \(\|W\|_{C^{k+1,\gamma}(\overline{M} \times (0,T))} \leq K\), we have
\[
\frac{\partial}{\partial t} \Psi_{ij}^\mu \leq K t^{-1+\frac{\gamma}{2}}
\]
for each \(t \in (0,T)\). Hence
\[
|\Psi_{ij}^\mu(x,t) - \Psi_{ij}^\mu(x,s)| \leq \int_s^t K t^{-1+\frac{\gamma}{2}} \, dt
\leq K (t^{\frac{\gamma}{2}} - s^{\frac{\gamma}{2}})
\leq K |t-s|^{\frac{\gamma}{2}}
\]
for \(t, s \in [0,T]\). This gives \(\sup_{t \neq s} \frac{|\Psi_{ij}^\mu(x,t) - \Psi_{ij}^\mu(x,s)|}{|t-s|^{\frac{\gamma}{2}}} \leq K\) for any \(x \in M\). Hence, we conclude that
\[
\|\Psi_{ij}^\mu\|_{\overline{M}^{\frac{\gamma}{2}}; C^{k,\gamma}(\overline{M} \times [0,T])} \leq K.
\]

In the next step, we prove the remaining assertions. Heuristically as \(g \in C^{\gamma_0}_k\),
we have \(W \in C^{k+1,\gamma}_0\), and so the highest regularity we can achieve for \(\nabla^\gamma W\)
is \(C^{k-1,\gamma-\frac{\gamma}{2}}\). On the other hand, we can easily control the Hölder semi-norm
\[
|\nabla^{m-1} W|_{M \times [0,T]} \leq K \quad \text{by the higher order } C^0 \text{ norm } |\nabla^m \Psi|_{0,M \times [0,T]}.
\]
And since we have the freedom of choosing \(k\) to any large integer, it doesn’t hurt if we omit
controlling the highest order Hölder semi-norm \(|\nabla^k \Psi|_{M \times [0,T]}\). Nevertheless,
we will first control the \(C^0\) norm of \(|\nabla^m \Psi|_{0,M \times [0,T]}\) up to the highest order. More
precisely, we first prove the estimate
\[
(4.34)
\]
by induction for \(2 \leq m \leq k + 1\). For notation simplicity, we abbreviate the
components \((d\psi_t)_{ij}^\mu\) of the differential \(d\psi_t\) simply by \(\partial \psi_t\), the partial derivatives
\(\partial^{m-1} (d\psi_t)_{ij}^\mu\) and \(\partial^m W\) by \(\partial^m \psi_t\) and \(\partial^m \psi\) respectively, and the functions \(\Psi_{ij}^\mu\) by
\(\Psi\). Moreover, with this abbreviation, for each \(t \in (0,T]\) the \(C^0\) norm for \(\partial^m \psi_t\) is
given by
\[
\|\partial^m \psi_t\|_{0,M} = \sum_{\mu_2} \sum_{i,j} \|\partial^{m-1} (d\psi_t)_{ij}^\mu\|_{C^0(U_{\mu_2}; \mathbb{R})}.
\]
Now suppose that the estimate \((4.34)\) holds for all \(2 \leq j \leq m\). Thus the induction hypothesis implies that
\[
\|\partial^j \psi_t\|_{0,M} \leq K t^{\frac{\gamma}{2} - \frac{\gamma}{2} - \frac{\gamma}{2}}
\]
for all \(2 \leq j \leq m\) and \(t \in [0,T]\). Using the Francesco Faà di Bruno formula for
higher-order chain rule, the following ODE is satisfied by \(\partial^{m+1} \psi_t\):
\[
(4.35)
\]
Here \((\partial^j \psi)^{\ast j}\) stands for \(\partial^j \psi \ast \cdots \ast \partial^j \psi\) and the summation sums over all non-negative integer solutions of \(j_1 + 2j_2 + \cdots + mj_m = m + 1\).
Now observe that for any \( t \in [\frac{T}{2}, T] \), the \( C^0 \) bound for \( \partial^1 W, \partial \psi_t \), and the induction hypothesis imply

\[
(4.36) \quad \left| (\partial^{i_1 + \cdots + i_j} W)(\psi_t, t) \cdot (\partial \psi_t)^{s_{i_1}} \cdots (\partial^m \psi_t)^{s_{i_j}} \right| \\
\leq K t^{\frac{j}{2} + \frac{m+1}{2}} \prod_{r=2}^m \| \partial^r \psi_t \|_{0, M} \\
\leq K t^{\frac{j}{2} + \frac{m+1}{2}} \prod_{r=2}^m (t^{\frac{r}{2} + \frac{1}{2} - \frac{3}{2}})^{i_r} \\
= K t^{\frac{j}{2} + \frac{m+1}{2} + \frac{1}{2} \sum_{r=2}^m i_r} \\
\leq K t^{\frac{j}{2} + \frac{m+1}{2}} 
\]

for each \( \{j_1, \ldots, j_m\} \) satisfying \( j_1 + 2j_2 + \cdots + mj_m = m + 1 \).

Let us define \( f(t) := \| \partial^{m+1}_x \psi_t \|_{0, M} \). Since \( (d \psi_T)^j(x) = \delta^j_1 \) for all \( x \in \psi_t^{-1}(U_{\rho_2}) \), we have \( f(T) = 0 \). By \((4.34), (4.35), (4.36)\) and the fundamental theorem of calculus, we have

\[
f(t) \leq K \int_t^T \tau^{\frac{j}{2} + \frac{m+1}{2}} d\tau + K \int_t^T \tau^{\frac{j}{2} + 1} f(\tau) d\tau \\
\leq K t^{\frac{j}{2} + \frac{m+1}{2}} + K \int_t^T \tau^{\frac{j}{2} + 1} f(\tau) d\tau
\]

for each \( t \in [\frac{T}{2}, T] \). Hence Grönwall’s lemma implies

\[
(4.37) \quad \| \partial^{m+1}_x \psi_t \|_{0, M} \leq K t^{\frac{j}{2} + \frac{m+1}{2}} \exp(KT^{\frac{j}{2}})
\]

for each \( t \in [\frac{T}{2}, T] \). Now, since \( \Psi = (\rho \circ \psi_t) \ast (\partial \psi_t) \), where we have abbreviated the function \( \rho_{\rho_2} \) as \( \rho \). Using the Francesco Faà di Bruno formula again, we obtain

\[
\partial^l (\rho \circ \psi_t) = \sum_{i_1 + 2i_2 + \cdots + li_l = l} \rho^{(i_1 + \cdots + i_l)}(\psi_t) \ast (\partial \psi_t)^{s_{i_1}} \ast \cdots \ast (\partial^l \psi_t)^{s_{i_l}}
\]

for \( 1 \leq l \leq m + 1 \). By the induction hypothesis and \((4.37)\), we have the estimate

\[
(4.38) \quad |\partial^l (\rho \circ \psi_t)| \leq K \sum_{i_1 + 2i_2 + \cdots + li_l = l} \left( \prod_{r=2}^l \| \partial^r \psi_t \|_{0, M} \right) \\
\leq K \sum_{i_1 + 2i_2 + \cdots + li_l = l} \left( \prod_{r=2}^l (t^{\frac{r}{2} + \frac{1}{2} - \frac{3}{2}})^{i_r} \right) \\
\leq K \sum_{i_1 + 2i_2 + \cdots + li_l = l} t^{\frac{1}{2} \sum_{r=2}^l i_r} \\
\leq K t^{\frac{1}{2} \sum_{r=2}^l i_r} \\
\leq K t^{\frac{j}{2} + \frac{m+1}{2}} 
\]

for any \( 1 \leq l \leq m + 1 \) and \( t \in [\frac{T}{2}, T] \). Then we observe that \( \partial^{m+1}_x \Psi \) satisfies

\[
\partial^m \Psi = \sum_{j_1 + j_2 = m} \partial^{j_1} (\rho \circ \psi_t) \ast \partial^{j_2+1} \psi_t.
\]

Using the estimate \((4.38)\), the induction hypothesis and \((4.37)\), we subsequently obtain

\[
|\partial^m \Psi(x, t)| \leq K \sum_{j_1 + j_2 = m} t^{\frac{j_1}{2} + \frac{j_2}{2} - \frac{3}{4}} \leq K t^{\frac{j}{2} + \frac{m+1}{2}}.
\]
for each $t \in [\frac{\sigma}{2}, T]$. Therefore we conclude that
\[
\sigma \frac{2}{\tau} - \frac{\tau}{2} + \frac{m-2}{m-1} |\nabla^m \psi_i^j|_{0, M \times [\frac{\sigma}{2}, T]} \leq K.
\]
Note that in the RHS of (4.37) the exponent of $\sigma$ is negative for any $m \geq 1$. This is why we cannot hope for higher powers of $\nabla^m \psi_i^j$ to converge for any $m \geq 1$ as $t \to 0$. Lastly, to verify that (4.34) holds for $m = 2$, it suffices to consider the following ODE satisfied by $\partial^2 \psi_i$:
\[
\frac{\partial}{\partial t}(\partial^2 \psi_i) = - \partial^2 W(\psi_t, t) * \partial \psi_t * \partial \psi_t - \partial W(\psi_t, t) * \partial^2 \psi_t.
\]
For each $t \in [\frac{\sigma}{2}, T]$, the $C^0$ bound for $\partial^2 W$ and $\partial \psi_t$ imply
\[
|\partial^2 W(\psi_t, t) * \partial \psi_t * \partial \psi_t| \leq K t^{\frac{\sigma}{2} - \frac{\tau}{2}}.
\]
Now we can proceed as before to obtain (4.34) for the case $m = 2$. This proves (4.34) by induction. In particular, this implies that
\[
(4.39) \quad \sigma \frac{2}{\tau} - \frac{\tau}{2} + \frac{m-2}{m-1} |\nabla^m \psi_i^j|_{0, M \times [\frac{\sigma}{2}, \sigma]} \leq K
\]
for $2 \leq m \leq k + 1$.

Next, we show that
\[
(4.40) \quad \sigma \frac{2}{\tau} - \frac{\tau}{2} + \frac{m-2}{m-1} |\nabla^m \psi_i^j|_{0, M \times [\frac{\sigma}{2}, \sigma]} \leq K
\]
for any $2 \leq m \leq k$. We first observe that the following equation is satisfied by $\partial^m \psi$:
\[
\frac{\partial}{\partial t}(\partial^m \psi) = \sum_{j_1 + j_2 = m-1} \left\{ \partial^{j_1+1} \psi_t * \sum_{l_1 + l_2 = j_2} (\partial^{l_1}(\rho \circ \psi_t) * \frac{\partial}{\partial t}(\partial^{j_2} \psi_t)) + \frac{\partial}{\partial t}(\partial^{j_1+1} \psi_t) * \partial^{j_2}(\rho \circ \psi_t) \right\}.
\]
Using (4.35) to (4.37) and the $C^0$ norm for $\nabla W$, we obtain
\[
\left| \frac{\partial}{\partial t}(\partial^m \psi_t) \right| \leq K t^{\frac{\sigma}{2} - \frac{l+1}{2}} + K t^{\frac{\sigma}{2} - \frac{l+1}{2}}
\leq K t^{\frac{\sigma}{2} - \frac{l+1}{2}} + K t^{\frac{\alpha - l+1}{2}}
\leq K t^{\frac{\sigma}{2} - \frac{l+1}{2}}
\]
for all $0 \leq l \leq k + 1$. Combining the above estimate, (4.34) and (4.38), we obtain
\[
\left| \frac{\partial}{\partial t}(\partial^m \psi_t)(x, t) \right| \leq K \sum_{j_1 + j_2 = m-1} \left\{ t^{\frac{\sigma}{2} - \frac{l}{2}} * \sum_{l_1 + l_2 = j_2} t^{-\frac{l}{2}} \cdot L^{\frac{\sigma}{2} - \frac{l+2}{2}} \cdot t^{\frac{l+2}{2}} \right\}
\leq K \sum_{j_1 + j_2 = m-1} \left\{ t^{\alpha - \frac{l+1}{2}} + t^{\frac{\sigma}{2} - \frac{l+1}{2}} + t^{\frac{\sigma}{2} - \frac{l+1}{2}} \right\}
\leq K t^{\frac{\sigma}{2} - \frac{\alpha}{2}}.
\]

Now, we fix a point $z \in M$, and let $(x, t), (y, s) \in B_{\sqrt{\tau}}(z) \times [\frac{\sigma}{2}, \sigma]$. We may assume without loss of generality that $\psi_t(x), \psi_s(y) \in \text{supp}(\rho_{jz_2})$, and $t > s$ are close
enough such that \( \psi_\tau(y) \in \text{supp}(\rho_{\mu_2}) \) for \( \tau \in [s, t] \). Using \((4.35), (4.36)\) and \((4.37)\) we have
\[
\frac{|\partial^{m-1}_x \Psi^{ij}_\mu(x, t) - \partial^{m-1}_x \Psi^{ij}_\mu(y, s)|}{|x - y|^\gamma + |t - s|^\frac{\sigma}{2}} \leq \frac{|\partial^{m-1}_x \Psi^{ij}_\mu(x, t) - \partial^{m-1}_x \Psi^{ij}_\mu(y, t)|}{|x - y|^\gamma} + \frac{|\partial^{m-1}_x \Psi^{ij}_\mu(y, t) - \partial^{m-1}_x \Psi^{ij}_\mu(y, s)|}{|t - s|^\frac{\sigma}{2}} \leq K \left| \frac{\partial^{m-1}_x \Psi^{ij}_\mu|_{0, M \times [\frac{\sigma}{2}, \sigma]} |x - y|^{1-\gamma} + K |t - s|^{-\frac{\sigma}{2}} \int_s^t \left| \frac{\partial}{\partial \tau} \left( \partial^{m-1}_x \Psi^{ij}_\mu \right)(y, \tau) \right| \, d\tau \right|
\]
\[
\leq K \sigma^{-\frac{\mu}{2}} - \frac{\mu - 1}{2} + K|t - s|^{-\frac{\sigma}{2}} \int_s^t \tau^{-\frac{\mu - 1}{2}} \, d\tau \leq K \sigma^{-\frac{\mu}{2}} - \frac{\mu - 1}{2} + K|t - s|^{-\frac{\sigma}{2}} (t^{-\frac{\mu - 1}{2}} - s^{-\frac{\mu - 1}{2}}).
\]
Now, we can write \( \frac{\sigma}{2} - \frac{\mu - 1}{2} = \left( \frac{\alpha}{2} - \frac{\beta}{2} - \frac{\mu - 1}{2} \right) + \left( \frac{\alpha - \beta}{2} + \frac{\mu}{2} \right) \). Using the fact that \( \alpha > \beta \) and \( t, s \in \left[ \frac{\sigma}{2}, \sigma \right] \), we obtain
\[
|\partial^{m-1}_x \Psi^{ij}_\mu(x, t) - \partial^{m-1}_x \Psi^{ij}_\mu(y, s)| \leq K \sigma^{-\frac{\mu}{2}} - \frac{\mu - 1}{2} + \frac{\beta}{2} (t^{-\frac{\mu - 1}{2}} - s^{-\frac{\mu - 1}{2}})
\]
for any \((x, t), (y, s) \in B_{\sqrt{\sigma}}(z) \times \left[ \frac{\sigma}{2}, \sigma \right] \). Since \( z \in M \) is arbitrary, we have proved \((4.40)\). ☐

**Proposition 4.8.** Let \( k \geq 2 \) and \( \beta, \gamma \in (0, \alpha) \) be given. There exist a \( C^{1, \beta} \) diffeomorphism \( \psi : M \to M \) and \( K = K(M, k, \tilde{g}, \|g_0\|_{\alpha, M}) \), \( T = T(M, \tilde{g}, \|g_0\|_{\alpha, M}) \) such that the following holds: There is a solution \( \tilde{g} \in A^{(\beta)}_{k, \gamma}(M \times [0, T]) \) to the Ricci flow such that
\[
\tilde{g}(\cdot, 0) = \psi^* g_0 \quad \text{and} \quad \|\tilde{g}\|_{A^{(\beta)}_{k, \gamma}(M \times [0, T])} \leq K.
\]

**Proof.**

By Corollary \((4.4)\) we can find \( T = T(M, \tilde{g}, \|g_0\|_{\alpha}) > 0 \) sufficiently small such that \( g(t) \in A^{(\alpha)}_{k+3, \gamma}(M \times [0, T]) \) is the unique solution to the Ricci-DeTurck system such that
\[
\|g\|_{A^{(\alpha)}_{k+3, \gamma}} \leq K.
\]
By Lemma \((4.3)\) and Lemma \((4.7)\) we can find a one-parameter family of diffeomorphisms \( \{\psi_t\}_{t \in [0, T]} \) which is generated by
\[
\begin{cases}
\frac{\partial}{\partial \tau} \psi_t = -W(\psi_t, t) \\
\psi_T = \text{id}
\end{cases}
\]
such that if the functions \( \Psi^{ij}_\mu \) are defined by Definition \((4.6)\) then they satisfy
1. \( \Psi^{ij}_\mu \in C^{3, \frac{\mu}{2}}(M \times [0, T]; \mathbb{R}) \) and \( \nabla \Psi^{ij}_\mu \in C^{3-1, \gamma}_{2-\frac{\mu}{2}}(M \times (0, T]; \mathbb{R}) \).
\[(2) \|\Psi_t^j\|_{L^1,\frac{2k}{2-k}; M \times [0,T]} + \|\nabla^k \Psi_t^j\|_{L^{k-1,\gamma}; (M \times (0,T))} \leq K(M, k, \hat{g}, \|g_0\|, \|g\|_{C^{k-1,\gamma}}).\]

Equivalently, we have \(\|\nabla^k \Psi_t^j\|_{L^{k-1,\gamma}; (M \times (0,T))} \leq K\).

We then define \(\tilde{g}(t) := (\psi_t)^*g(t)\). Thus \(\tilde{g}(0) = \psi_0^*g_0\) where \(\psi_0\) is a \(C^{1,\beta}\) diffeomorphism. We have
\[
\frac{\partial}{\partial t} \tilde{g} = (\psi_t)^*(-2Ric(g(t)) + L_Wg(t)) - L_W \tilde{g}(t) = -2Ric(\tilde{g}(t)).
\]

Hence \(\tilde{g}\) is a solution to the Ricci flow on a \(M \times (0,T]\). In the sequel, let us denote \(g(t)\) by \(g_t\). Next, we fix a chart \(U\). Let \(x \in U\) and \(\psi_t(x) \in V\) for some chart \(V\). Let \(\{x_i\}\) and \(\{\tilde{x}_i\}\) be the coordinates on the charts \(U\) and \(V\) respectively. Moreover, we denote by \(h_{\mu} : U_\mu \cap V \rightarrow GL_n(\mathbb{R})\) the induced transition maps between trivializations of the tangent bundle \(TM\); and by \(H_{\alpha} : U_\alpha \cap V \rightarrow GL_N(\mathbb{R})\) the induced transition maps between trivializations of the bundle \(\text{Sym}^2(T^*M)\). So on the intersection \(U_\mu \cap V\), we have
\[
(d\psi_t)^i_j = (h_{\mu})_i^k (d\psi_t)^k_j \quad \text{and} \quad g_{\mu}^{ab} = (H_{\alpha})_{ab}^{kl} \cdot g_{\mu}^{kl}
\]
where \((d\psi_t)^i_j = \partial_i \psi_t^j\) are the components of \(d\psi_t\) on \(U \cap \psi_t^{-1}(V)\), and \(g_{\mu}^{ab}\) are the components of \(g\) on \(U_\mu\). Then
\[(4.41)
\]
\[
\tilde{g}(x,t)_{ij} = \psi_t^*g_t(x)_{ij}
\]
\[
= g_t(\psi_t(x))((d\psi_t)(x)^j_i, (d\psi_t)(x)^k_i)
\]
\[
= g_t(\psi_t(x))((d\psi_t)^i_k, (d\psi_t)^j_l)
\]
\[
= (g_t \circ \psi_t)_{kl}(d\psi_t)^i_j
\]
\[
= \sum_{\alpha} \rho_\alpha(\psi_t^\alpha)(g_t \circ \psi_t)_{kl}. \sum_{\beta} \mu_\beta(\psi_t^\beta)(d\psi_t)^i_j. \sum_{\gamma} \nu_\gamma(\psi_t)(d\psi_t)^i_j
\]
\[
= \sum_{\mu, \nu} \rho_\mu(\psi_t)(g_t \circ \psi_t)_{ab}^\mu (H_{\alpha}^\mu)^i_j. \sum_{\nu} \rho_\nu(\psi_t)(h_{\mu}^\nu)^j_i (d\psi_t)^i_j
\]
\[
= \sum_{\mu, \nu} \rho_\mu(\psi_t) (g_t \circ \psi_t)_{ab}^\mu . \Psi^\phi(\psi_t)^j_i . (h_{\mu}^\nu)^j_i (H_{\alpha}^\mu)^i_j.
\]

Now, since \(g \in C^\alpha, \frac{\alpha}{2}(M \times [0,T]), \Psi \in C^{\beta, \frac{\beta}{2}}(M \times [0,T])\) and \(\alpha > \beta\), we have \(\tilde{g} \in C^{\beta, \frac{\beta}{2}}(M \times [0,T])\) from (4.41). Next, for \(1 \leq m \leq k\), we see from (4.41) that
\[
\nabla^i \tilde{g} = \sum_{j_1+j_2+j_3+j_4=1} \nabla^{j_1} (g_t \circ \psi_t) \ast \nabla^{j_2} (g_t \circ \psi_t) \ast \nabla^{j_3} \Psi \ast \nabla^{j_4} \Psi.
\]

Note that we have \(\|\nabla^i \tilde{g}\|_{L^{k-1,\gamma}; (M \times (0,T))} \leq K\). Since \(\psi_t\) is a \(C^{1,\beta}\) diffeomorphism, \(g_t \circ \psi_t\) has the same regularity as \(g_t\), and so \(\|\nabla^i (g_t \circ \psi_t)\|_{L^{k-1,\gamma}; (M \times (0,T))} \leq \|\nabla^i (g_t \circ \psi_t)\|_{L^{k-1,\gamma}; (M \times (0,T))} \leq K\). By the definition of \(\Psi\), \(\rho \circ \psi_t\) has at least the regularity
of $\Psi$, thus $\|\hat{\nabla}(\rho \circ \psi_t)\|_{C^{\frac{k-1}{2},\frac{\gamma}{2}-\beta}(M \times [0,T])} \leq K$. Moreover, by Lemma 2.2, there is
\[
\|\psi_t\|_{C^{\frac{k}{2},\frac{\gamma}{2}}(M \times (0,T))} \geq K\left(\|\Psi\|_{C^{\frac{k}{2},\frac{\gamma}{2}}(M \times (0,T))} + \|\nabla\Psi\|_{C^{\frac{k-1}{2},\frac{\gamma}{2}}(M \times (0,T))}\right)
\leq K.
\]
Furthermore, by Lemma 2.2, there is
\[
\|\hat{\nabla}\psi_t\|_{C^{\frac{k-1}{2},\frac{\gamma}{2}}(M \times (0,T))} \leq K\left(\|\Psi\|_{C^{\frac{k}{2},\frac{\gamma}{2}}(M \times (0,T))} + \|\nabla\Psi\|_{C^{\frac{k-1}{2},\frac{\gamma}{2}}(M \times (0,T))}\right)
\leq K.
\]
Similarly,
\[
\|g_t \circ \psi_t\|_{C^{\frac{k-1}{2},\frac{\gamma}{2}}(M \times (0,T))} \leq K \quad \text{and} \quad \|\rho \circ \psi_t\|_{C^{\frac{k-1}{2},\frac{\gamma}{2}}(M \times (0,T))} \leq K.
\]
Putting everything together, we obtain
\[
\|(\rho \circ \psi_t) * (g_t \circ \psi_t) * \Psi * \hat{\nabla}\Psi\|_{C^{\frac{k-1}{2},\frac{\gamma}{2}}(M \times (0,T))}
\leq K\left(\|(\rho \circ \psi_t)\|_{C^{\frac{k}{2},\frac{\gamma}{2}}(M \times (0,T))} \|\nabla g_t \circ \psi_t\|_{C^{\frac{k-1}{2},\frac{\gamma}{2}}(M \times (0,T))} \|\Psi\|_{C^{\frac{k-1}{2},\frac{\gamma}{2}}(M \times (0,T))} \|\nabla\Psi\|_{C^{\frac{k-1}{2},\frac{\gamma}{2}}(M \times (0,T))}\right)
\leq K.
\]
We can similarly derive the bounds for the other terms in the summation of (4.41). Therefore we conclude that
\[
\|\hat{\nabla}\tilde{g}\|_{C^{\frac{k-1}{2},\frac{\gamma}{2}}(M \times (0,T))} \leq K.
\]
This proves that $\tilde{g} \in \mathcal{X}_{k,\gamma}^{(\beta)}(M \times [0,T])$ and the assertion follows. \(\Box\)
5. Short Time Existence and Uniqueness to the Harmonic Map Heat Flow

In this subsection, we assume that a solution to the Ricci flow is given. To construct a solution to the Ricci-Deturck flow, we prove the short time existence and the uniqueness to the associated harmonic map heat flow.

Throughout this subsection, let $\alpha \in (0, 1)$ be given such that $g_0 \in C^\alpha(M)$. Let $\gamma \in (0, \alpha)$ be given and we fix $\beta \in (\gamma, \alpha)$. Moreover, let $g(t) \in \mathcal{X}_{k,\gamma}(M \times [0, T])$ be a solution to the Ricci flow on $M \times (0, T]$ and $\psi$ be a $C^{1,\beta}$ diffeomorphism such that

- $g(0) = \psi^*g_0$;
- $\|\psi\|_{C^{1,\gamma}(M; M)} \leq C$;
- $\|g\|_{\mathcal{X}_{k,\gamma}(M \times [0, T])} = \|g\|_{C^{1,\gamma}(M \times [0, T])} + \|\nabla g\|_{C^{1,\gamma}} \leq C.$

for some constant $C > 0$.

Associated with $g(t)$, we consider the harmonic map heat flow

\[
\begin{cases}
\frac{d\phi_t}{dt} = \Delta_{g(t), \hat{g}} \phi_t, & \text{on } M \times (0, T] \\
\phi_0 = \psi, & \text{on } M.
\end{cases}
\]

We seek short time existence and uniqueness to the initial value problem \((5.1)\). To do that, we reformulate the problem into an equivalent equation on $TM$ via the exponential map. Since $\psi \in C^{1,\beta}(M)$, we can find a $C^\infty$ map $\hat{\psi} : (M, g) \to (M, \hat{g})$ such that $d_{\hat{g}}(\hat{\psi}(x), \hat{\psi}(x)) < \frac{1}{4}\text{inj}(M, \hat{g})$. Here $d_{\hat{g}}$ is the Riemannian distance with respect to the metric $\hat{g}$. Thus we can write

\[
\psi(x) = \exp_{\hat{\psi}(x)}(U(x))
\]

for some $C^{1+\beta}$ vector field $U(x)$. The exponential map is taken with respect to the metric $\hat{g}$. If we assume that $T$ is sufficiently small so that $d_{\hat{g}}(\phi_t(x), \hat{\psi}(x)) < \frac{1}{4}\text{inj}(M, \hat{g})$, then we can write the harmonic map heat flow $\phi_t(x)$ in the form

\[
\phi_t(x) = \exp_{\hat{\psi}(x)}(V(x, t))
\]

for some vector field $V(x, t)$. Note that the assumptions on the injectivity radius ensures that $V(x, t)$ is well-defined. Now the idea is to transform the initial value problem \((5.1)\) into an equivalent PDE for the vector field $V(x, t)$ with initial condition $U(x)$. The following lemma gives such an equivalence.

**Lemma 5.1.** If $\phi_t(x) \in \Gamma(M \times [0, T]; M)$ is a solution to the harmonic map heat flow \((5.1)\), then the vector field $V(x, t)$ defined by $\phi_t(x) = \exp_{\hat{\psi}(x)}(V(x, t))$ is a solution in $\Gamma(M \times [0, T]; \hat{\psi}^*(TM))$ to the initial value problem

\[
\begin{cases}
\frac{d}{dt}V = \mathcal{P}(V, \hat{\nabla}V), & \text{on } M \times (0, T] \\
V(x, 0) = U(x) & \text{on } M,
\end{cases}
\]

provided that $T$ is sufficiently small, where

\[
\mathcal{P}(V, \hat{\nabla}V)^a := g^{ij}(\Gamma_{\hat{\gamma}} - \Gamma_\gamma)_{ij}^k (\hat{\nabla}_k V^a + Z_a^k(V)) + g^{ij}S_{ij}^a(V, \hat{\nabla}V),
\]

$Z(V)$ and $S(V, \hat{\nabla}V)$ are sections of $T^*M \otimes \hat{\psi}^*TM$ and $T^*M \otimes T^*M \otimes \hat{\psi}^*TM$ respectively. Here $Z_a^k(V)$ is a smooth function in $V$, whereas $S_{ij}^a(V, \hat{\nabla}V)$ is a smooth function in $V$ and a polynomial of degree 2 in $\hat{\nabla}V$. Moreover, the converse is also true.
Proof. Let \( \{x^i\}, \{y^a\} \) and \( \{z^c\} \) be local coordinates around \( x, \psi(x) \) and \( \phi_t(x) \) respectively. We first show that

\[
(5.6) \quad d\phi_t(x)\left( \frac{\partial}{\partial x^i} \right) = (d\exp_{\psi(x)^0})_{V(x,t)} \left( \hat{\nabla}_i V(x,t) + Z_i(V(x,t)) \right),
\]

where \( Z_i = Z_i(x,V(x,t)) \in T_{\psi(x)} M \) is a vector field depending smoothly on \( x \) and \( V \).

Fix \( x \) and \( t \), let \( \lambda(t) := \exp_{\psi(x)^0}(\tau V(x,t)) \) be a geodesic. Let \( \gamma(s) \) be a curve such that \( \gamma(0) = x \) and \( \gamma'(0) = \frac{\partial}{\partial s} \). Let \( F(s, \tau) := \exp_{\psi(\gamma(s))^0}(\tau V(\gamma(s), t)) \) be a variation of \( \lambda \) through geodesics. Then

\[
J(\tau) := \left. \frac{\partial}{\partial s}(s, \tau) \right|_{s=0}
\]
is a Jacobi field with initial conditions \( J(0) = d\psi(x)\left( \frac{\partial}{\partial s} \right) \), \( \hat{\nabla}_i J(0) = \hat{\nabla}_i V(x,t) \) such that \( J(1) = d\phi_t(x)\left( \frac{\partial}{\partial s} \right) \). Let us decompose the Jacobi field \( J := J_1 + J_2 \) in a way that \( J_1(0) = 0 \), \( \hat{\nabla}_i J_1(0) = \hat{\nabla}_i V(x,t) \) and \( J_2(0) = d\psi(x)\left( \frac{\partial}{\partial s} \right) \), \( \hat{\nabla}_i J_2(0) = 0 \). Then we have

\[
J_1(1) = (d\exp_{\psi(x)^0})_{V(x,t)} \left( \hat{\nabla}_i V(x,t) \right).
\]

Moreover, the vector \( J_2(1) \) depends smoothly on \( x \) and \( V(x,t) \). To see that, let \( \sigma(u) := \exp_{\psi(x)^0}(u J_2(0)) \) be a geodesic, and let \( H(u, \tau) := \exp_{\sigma(u)}(\tau P_{\sigma(u)}(V(x,t))) \) be another geodesic variation of \( \lambda(t) \) where \( P_{\sigma(u)}(V(x,t)) \) is the parallel transport of \( V(x,t) \) through \( \sigma(u) \). Note that the Jacobi field \( J_2(\tau) \) arises from the variation \( H \) since

\[
\left. \frac{\partial H}{\partial u}(u, \tau) \right|_{u=0, \tau=0} = J_2(0) \quad \text{and} \quad \left. D_\tau \frac{\partial H}{\partial u}(u, \tau) \right|_{u=0, \tau=0} = 0.
\]

Then

\[
\hat{Z}_i(x, V(x,t)) := J_2(1) = \left. \frac{\partial H}{\partial u}(u, 1) \right|_{u=0} \in T_{\exp_{\psi(x)^0}(V)} M
\]
is a vector field depending smoothly on \( x \) and \( V(x,t) \). We then define the vector field \( \hat{Z}_i(x, V(x,t)) \in T_{\psi(x)} M \) by

\[
(d\exp_{\psi(x)^0})_{V(x,t)}(Z_i(x, V(x,t))) = \hat{Z}_i(x, V(x,t)).
\]

Combining the above results, we obtain the identity \( (5.6) \).

Next, let \( \hat{\nabla} \) be the connection on \( T^*M \otimes \phi_t^* (TM) \) induced by \( \nabla_{g(t)} \) and \( \phi_t^* \hat{\nabla} \). It is worth noting that although we are using the connection \( \nabla = \nabla_{g(t)} \otimes 1 + 1 \otimes \phi_t^* \hat{\nabla} \) which applies to sections of the form \( dx^i \otimes \phi_t^* \partial_{x^i} \), we will be using the basis \( \{(d\exp_{\psi(x)^0})_{V(x,t)}(\partial\gamma^a)\} \) on the fibre \( T_{\phi_t(x)} M \). Moreover, we denote by \( \omega_{ab}^c \) the connection 1-forms for the connection \( \phi_t^* \hat{\nabla} \) with respect to the basis \( \{(d\exp_{\psi(x)^0})_{V(x,t)}(\partial\gamma^a)\} \), thus

\[
\hat{\nabla}_{(d\exp_{\psi})_{V(\partial\gamma^a)}}(d\exp_{\psi})_{V}(\partial\gamma^a) = \omega_{ab}^c \cdot (d\exp_{\psi})_{V}(\partial\gamma^a).
\]
Note that $\omega^c_{ab}$ depend smoothly on $V(x, t)$. Then (5.6) implies
\[
\nabla d\phi_t \left( \frac{\partial}{\partial x^i_x}, \frac{\partial}{\partial x^j_x} \right)
= (\phi_t^*) \nabla \left( d\phi_t \left( \frac{\partial}{\partial x^i_x} \right) \right) - d\phi_t \left( \frac{\partial}{\partial x^i_x} \right)
\]
\[
= (\phi_t^*) \nabla \left( (\nabla_j V^b + Z^b_j) (d\exp_{\psi}) V \left( \frac{\partial}{\partial y^b} \right) \right) - d\phi_t \left( (\Gamma_g)_j^k \frac{\partial}{\partial x^k} \right)
\]
\[
= \left( \nabla_j V^b + \nabla_i Z^b_j \right) (d\exp_{\psi}) V \left( \frac{\partial}{\partial y^b} \right)
\]
\[
- (\Gamma_g)_j^k \nabla_k V^c + Z^c_k \right) (d\exp_{\psi}) V \left( \frac{\partial}{\partial y^c} \right)
\]
\[
= \left( \nabla_j V^c + (\Gamma_g - \Gamma_g)^k \nabla_k V^c + Z^c_k \right) \cdot \omega^c_{ab} \left( \frac{\partial}{\partial y^c} \right)
\]
This means that
\[
(5.7)
\]
\[
\nabla d\phi_t \left( \frac{\partial}{\partial x^i_x}, \frac{\partial}{\partial x^j_x} \right)
\]
\[
= \left( \nabla_j \nabla_i V^a + (\Gamma_g - \Gamma_g)^k \nabla_k V^a + Z^a_k \right) \cdot \omega^a_{bc} + \omega^c_{bc} \cdot \omega^a_{bc} \right) \cdot \omega^c_{bc}
\]
is a smooth function in $V(x, t)$ and a polynomial of degree 2 in $\nabla V(x, t)$. The Laplacian $\Delta_{g(t), \phi_t}$ is thus given by
\[
\Delta_{g(t), \phi_t} = \nabla \cdot d\phi_t \left( \frac{\partial}{\partial x^i_x}, \frac{\partial}{\partial x^j_x} \right)
\]
\[
= \left( \nabla_j \nabla_i V^a + (\Gamma_g - \Gamma_g)^k \nabla_k V^a + Z^a_k \right) \cdot \omega^a_{bc} + \omega^c_{bc} \cdot \omega^a_{bc} \right) \cdot \omega^c_{bc}
\]
On the other hand, it is easy to see that
\[
\nabla V^a = \left( \frac{\partial}{\partial t} V^a - \nabla^2 V^a - g^{ij} (\Gamma_g - \Gamma_g)^k \nabla_k V^a + Z^a_k \right) \cdot \omega^a_{bc} + \omega^c_{bc} \cdot \omega^a_{bc} \right) \cdot \omega^c_{bc}
\]
Therefore, we conclude that
\[
(5.8)
\]
\[
\frac{\partial}{\partial t} \phi_t = \left( \frac{\partial}{\partial t} V^a - \nabla^2 V^a - g^{ij} (\Gamma_g - \Gamma_g)^k \nabla_k V^a + Z^a_k \right) \cdot \omega^a_{bc} + \omega^c_{bc} \cdot \omega^a_{bc} \right) \cdot \omega^c_{bc}
\]
The assertion then follows since the exponential map $\exp$ is a smooth diffeomorphism with respect to the metric $\hat{g}$. \qed
Similar to the proof of the short time existence and uniqueness to the Ricci De-Turck flow, the short time existence and uniqueness to (5.3) can be obtained by applying the Banach fixed point theorem to the following linear system:

\[
\begin{cases}
  \left( \frac{\partial}{\partial t} - \text{tr}_g \tilde{\nabla}^2 \right) V = \mathcal{P}(W, \tilde{\nabla} W) & \text{on } M \times (0, T) \\
  V(x, 0) = U(x) & \text{on } M,
\end{cases}
\]

where

\[
\mathcal{P}(W, \tilde{\nabla} W)^a := g^{ij} (\Gamma^a_{ij} - \Gamma^a_g) \left( \tilde{\nabla}_k W^a + Z^a_k(W) \right) + g^{ij} S^a_{ij}(W, \tilde{\nabla} W).
\]

Moreover, we note that the fact \( U \in C^{1, \beta}(M) \) implies that \( U \in C^\delta(M) \) for any exponent \( \delta \in (0, 1) \) which can be arbitrarily close to 1.

**Proposition 5.2.** Let \( k \geq 2 \) and \( \delta \in (0, 1) \) be given such that \( \delta \geq \beta \). Suppose that

- \( W(x, 0) = U(x) \);
- \( \|W\|_{X^\delta_{k, \gamma}(M \times [0, T])} \leq B \).

Then there exists a unique solution \( V \in X_{k+1, \gamma}^{(\delta)}(M \times [0, T]; \tilde{\nabla}^*(TM)) \) to the linear system (5.9) and there are positive constants \( K_1 = K_1(\hat{g}, M, \|g\|_{X^\delta_{k, \gamma}}) \), \( K_2 = K_2(\hat{g}, M, \|g\|_{X^\delta_{k, \gamma}}, B) \) such that

\[
\|V\|_{X_{k+1, \gamma}^{(\delta)}(M \times [0, T])} \leq K_1(K_2 T^{\frac{\delta}{2}} + \|U\|_{C^{1, \beta}(M)}).
\]

**Proof.**

We first claim that the assumption \( \|W\|_{X^\delta_{k, \gamma}(M \times [0, T])} \leq B \) implies

\[
\|\mathcal{P}(W, \tilde{\nabla} W)\|_{C^{k-1, \gamma}_{1-\frac{\delta}{2}}(M \times [0, T])} \leq K(\hat{g}, \|g\|_{X^\delta_{k, \gamma}}, B).
\]

To see that, we estimate

\[
\left\| g^{ij} (\Gamma^a_{ij} - \Gamma^a_g) \left( \tilde{\nabla}_k W^a + Z^a_k(W) \right) \right\|_{C^{k-1, \gamma}_{1-\frac{\delta}{2}}(M \times [0, T])}
\]

\[
\leq K(\hat{g}) \left\| g^{-1} \right\|_{C^{k-1, \gamma}_{1-\frac{\delta}{2}}(M \times [0, T])} \left\| \Gamma^a_{ij} - \Gamma^a_g \right\|_{C^{k-1, \gamma}_{1-\frac{\delta}{2}}(M \times [0, T])} \left\| \tilde{\nabla} W + Z \right\|_{C^{k-1, \gamma}_{1-\frac{\delta}{2}}(M \times [0, T])}
\]

\[
\leq K(\hat{g}, \|g\|_{X^\delta_{k, \gamma}}, B)
\]

and

\[
\left\| g^{ij} S^a_{ij}(W, \tilde{\nabla} W) \right\|_{C^{k-1, \gamma}_{1-\frac{\delta}{2}}(M \times [0, T])}
\]

\[
\leq K(\hat{g}) T^{\frac{\delta}{2}} + g^{-1} \left\| \tilde{\nabla} W \right\|^2_{C^{k-1, \gamma}_{1-\frac{\delta}{2}}(M \times [0, T])}
\]

\[
\leq K(\hat{g}, \|g\|_{X^\delta_{k, \gamma}}, B) T^{\frac{\delta}{2}} + g^{-1} \left\| \tilde{\nabla} W \right\|^2_{C^{k-1, \gamma}_{1-\frac{\delta}{2}}(M \times [0, T])}.
\]

Hence the estimate (5.11) is established. By Lemma 2.2, this in particular implies that

\[
\|\mathcal{P}(W, \tilde{\nabla} W)\|_{C^{k-1, \gamma}_{1-\frac{\delta}{2}}(M \times [0, T])} \leq K(\hat{g}, \|g\|_{X^\delta_{k, \gamma}}, B) T^{\frac{\delta}{2}}.
\]

Since the initial condition in (5.9) satisfies \( U \in C^\delta(M; \tilde{\nabla}^*(TM)) \), then by Theorem 5.1 there exists a unique solution \( V \) to the linear system (5.9) such that
where $S$ will denote a constant depending only on $\hat{g}, M, \|g\|_{X^{(\delta)}_{k,\gamma}}$ and $K_2 = K_2(\hat{g}, M, \|g\|_{X^{(\delta)}_{k,\gamma}}, B)$. \hfill $\Box$

Now, we choose $B > 2K_1\|U\|_{C^{1,\alpha}(M)}$ to be a large positive constant. For $k \geq 2$, we define a closed subset $W$ in $X^{(\delta)}_{k,\gamma}(M \times [0, T])$ by

$$W := \{W \in X^{(\delta)}_{k,\gamma}(M \times [0, T]) \mid \|W\|_{X^{(\delta)}_{k,\gamma}(M \times [0, T])} \leq B\}.$$ 

Next, we define an operator $S : W \to X^{(\delta)}_{k,\gamma}(M \times [0, T])$ by

$$V := S(W),$$

where $V$ is the unique solution to the system (5.13) in $X^{(\delta)}_{k,\gamma}(M \times [0, T])$. By Proposition 5.2 and our choice of $B$, we can make $\|V\|_{X^{(\delta)}_{k,\gamma}(M \times [0, T])} \leq B$ provided that $T$ is sufficiently small. Consequently $S(W) \subset W$.

**Proposition 5.3.** If $T = T(\hat{g}, M, \|g\|_{X^{(\delta)}_{k,\gamma}}, B)$ is chosen sufficiently small, the operator $S$ is a contraction mapping.

**Proof.** In the sequel, $K$ will denote a constant depending only on $\hat{g}, M, \|g\|_{X^{(\delta)}_{k,\gamma}}$, $B$.

Let $W_1, W_2 \in W$ and write $V_i := S(W_i)$ for $i = 1, 2$. Then $V = V_1 - V_2$ solves the system

$$(5.13) \quad \begin{cases} \frac{\partial}{\partial t} - tr_g \hat{\nabla}^2 V = \mathcal{P}(W_1, \hat{\nabla}W_1) - \mathcal{P}(W_2, \hat{\nabla}W_2) & \text{on } M \times (0, T) \\
V|_{t=0} = 0 & \text{on } M. \end{cases}$$

We recall that

$$\mathcal{P}(W, \hat{\nabla}W)^a = g^{ij}(\Gamma_{ab} - \Gamma_{aj})_{\hat{g}}^{b} (\hat{\nabla}_i W^a + Z_K^a(W)) + g^{ij} S^a_{ij}(W, \hat{\nabla}W).$$

Now, we define the tensors $Z(s) \in \Gamma(M; T^* M \otimes \hat{\psi}^* TM)$ and $S(s) \in \Gamma(M; T^* M \otimes T^* M \otimes \hat{\psi}^* TM)$ by

$$Z(s) := Z(sW_1 + (1-s)W_2)$$

and

$$S(s) := S(sW_1 + (1-s)W_2, \hat{\nabla}W_1 + (1-s)\hat{\nabla}W_2).$$

Note that $(1-s)W_1 + sW_2 \in W$ for all $s \in [0, 1]$. From Lemma 5.1 we observe that

- $\nabla_q Z^a_K(x, q)$ is a smooth function in $x$ and $q$;
- $\nabla_q S^a_{ij}(x, q, A)$ is a smooth function in $x$ and $q$, and a polynomial of degree two in $A$;
- $\nabla_A S^a_{ij}(x, q, A)$ is a smooth function in $x$ and $q$, and a polynomial of degree one in $A$. 

$V \in C^8(M \times [0, T])$ and $\hat{\nabla}V \in C^{k,\gamma}_{\frac{3}{2} - \frac{j}{2}}(M \times (0, T))$. Moreover, $V$ satisfies the estimate

$$(5.12) \quad \|V\|_{\delta, \frac{3}{2}M \times [0, T]} + \|\hat{\nabla}V\|_{C^{k,\gamma}_{\frac{3}{2} - \frac{j}{2}}(M \times (0, T))} \leq K_1 \left( \|\mathcal{P}(W, \hat{\nabla}W)\|_{C^{k,\gamma}_{\frac{3}{2} - \frac{j}{2}}(M \times (0, T))} + \|U\|_{\delta, M} \right) \leq K_1 (K_2 T^2 + \|U\|_{C^{1,\alpha}(M)}),$$

where $K_1 = K_1(\hat{g}, M, \|g\|_{X^{(\delta)}_{k,\gamma}})$ and $K_2 = K_2(\hat{g}, M, \|g\|_{X^{(\delta)}_{k,\gamma}}, B)$.
Let \( \{dx^i \otimes \frac{\partial}{\partial y^k}\} \) be the local frame of \( T^*M \otimes \tilde{\psi}^*TM \) on the chart \( U_{ija} \), then from the above observations and Lemma 2.2, we have

\[
\begin{align*}
\frac{\partial}{\partial s} \|Z(s)\|_{C^{k-2,\gamma}_0(M \times (0,T))} &= \sum_{i,o} \frac{\partial}{\partial s} \|Z(s)\|_{C^{k-2,\gamma}_0(U_{ia} \times (0,T))} \\
&= \sum_{i,o} \|\langle D_q Z(s)\rangle^a_{ij} \|_{C^{k-2,\gamma}_0(U_{ia} \times (0,T))} \\
&\leq \sum_{i,o} K \sup_{s \in [0,1]} \|D_q Z(s)\|_{C^{k-2,\gamma}_0(U_{ia} \times (0,T))}\|W_1 - W_2\|_{C^{k-2,\gamma}_0(U_{ia} \times (0,T))} \\
&\leq K\|W_1 - W_2\|_{\mathcal{X}^{(s)}_{k,\gamma}(M \times [0,T])}.
\end{align*}
\]

Hence

\[
(5.14)
\]

\[
\|Z(W_1) - Z(W_2)\|_{C^{k-2,\gamma}_0(M \times (0,T))} \leq \|\frac{\partial}{\partial s} Z(s)\|_{C^{k-2,\gamma}_0(M \times (0,T))} \leq K\|W_1 - W_2\|_{\mathcal{X}^{(s)}_{k,\gamma}(M \times [0,T])}.
\]

Let \( \{dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^k}\} \) be the local frame of \( T^*M \otimes T^*M \otimes \tilde{\psi}^*TM \) on the chart \( U_{ija} \), then we similarly have

\[
\frac{\partial}{\partial s} \|S(s)\|_{C^{k-2,\gamma}_0(M \times (0,T))} \\
= \sum_{i,j,o} \frac{\partial}{\partial s} \|S(s)\|_{C^{k-2,\gamma}_0(U_{ija} \times (0,T))} \\
\leq \sum_{i,j,o} \|\langle D_q S(s)\rangle^a_{ij} \|_{C^{k-2,\gamma}_0(U_{ija} \times (0,T))} \|W_1 - W_2\|_{C^{k-2,\gamma}_0(U_{ija} \times (0,T))} \\
\leq K T^2 \frac{\sup}{s \in [0,1]} \|\langle D_q S(s)\rangle^a_{ij} \|_{C^{k-2,\gamma}_0(U_{ija} \times (0,T))}\|W_1 - W_2\|_{C^{k-2,\gamma}_0(U_{ija} \times (0,T))} \\
+ K T^2 \frac{\sup}{s \in [0,1]} \|D_A S(s)\|_{C^{k-2,\gamma}_0(U_{ija} \times (0,T))}\|\tilde{\nabla} W_1 - \tilde{\nabla} W_2\|_{C^{k-2,\gamma}_0(U_{ija} \times (0,T))} \\
\leq K T^2 \|W_1 - W_2\|_{\mathcal{X}^{(s)}_{k,\gamma}(M \times [0,T])}.
\]

This gives

\[
(5.15)
\]

\[
\|S(W_1, \tilde{\nabla} W_1) - S(W_2, \tilde{\nabla} W_2)\|_{C^{k-2,\gamma}_0(M \times (0,T))} \leq \|\frac{\partial}{\partial s} S(s)\|_{C^{k-2,\gamma}_0(M \times (0,T))} \\
\leq K\|W_1 - W_2\|_{\mathcal{X}^{(s)}_{k,\gamma}(M \times [0,T])}.
\]
From (5.14), (5.15) and Lemma 2.2 we obtain

\[ \| \mathcal{P}(W_1, \nabla W_1) - \mathcal{P}(W_2, \nabla W_2) \|_{C^{k-\gamma}(M \times (0,T))} \]

\[ \leq g^{-1} * (\Gamma_g - \Gamma_g) * (\nabla(W_1 - W_2)) \|_{C^{k-\gamma}(M \times (0,T))} \]

\[ + g^{-1} * (\Gamma_g - \Gamma_g) * (Z(W_1) - Z(W_2)) \|_{C^{k-\gamma}(M \times (0,T))} \]

\[ + g^{-1} * (\mathcal{S}(W_1, \nabla W_1) - \mathcal{S}(W_2, \nabla W_2)) \|_{C^{k-\gamma}(M \times (0,T))} \]

\[ \leq K \left( T^\frac{4}{5} \| \Gamma_g - \Gamma_g \|_{C^{k-2\gamma}(M \times (0,T))} \| \nabla(W_1 - W_2) \|_{C^{k-2\gamma}(M \times (0,T))} \right. \]

\[ \left. + T^\frac{1+\frac{2}{5}-\frac{1}{2}}{5} \| \Gamma_g - \Gamma_g \|_{C^{k-\gamma}(M \times (0,T))} \| Z(W_1) - Z(W_2) \|_{C^{k-\gamma}(M \times (0,T))} \right) \]

\[ + \| \mathcal{S}(W_1, \nabla W_1) - \mathcal{S}(W_2, \nabla W_2) \|_{C^{k-\gamma}(M \times (0,T))} \right) \]

\[ \leq K T^\frac{4}{5} \| W_1 - W_2 \|_{X^{k,\gamma}(M \times (0,T))}. \]

Theorem 3.1 then implies

\[ \| V \|_{L^2(M \times [0,T])} + \| \nabla V \|_{C^{k-1,\gamma}(M \times (0,T))} \leq K T^{\frac{4}{5}} \| W_1 - W_2 \|_{X^{k,\gamma}(M \times (0,T))}. \]

Therefore,

\[ \| V_1 - V_2 \|_{X^{k,\gamma}(M \times [0,T])} \leq K T^{\frac{4}{5}} \| W_1 - W_2 \|_{X^{k,\gamma}(M \times [0,T])}. \]

This proves the proposition. \( \square \)

From the previous proposition and Lemma 5.1, the harmonic map heat flow (5.1) has a unique solution \( \phi_t \) such that \( \phi_t \in C^{k,\gamma}(M \times [0,T]) \) and \( d\phi_t \in C^{k-1,\gamma}(M \times (0,T)) \) provided that \( g \in X_{k,\gamma}^{(\delta)}(M \times [0,T]) \) and that \( T = T(\hat{g}, M, \| g \|_{X_{k,\gamma}^{(\delta)}}, \| \psi \|_{C^1(M)}) \) is chosen sufficiently small. Furthermore, as the initial condition satisfies \( \psi \in C^{1,\beta}(M) \), we can choose \( \delta \) to be arbitrarily close to 1. It turns out that the regularity of \( \phi_t \) can be improved.

**Theorem 5.4.** Let \( \lambda \in (\gamma, \beta) \) be given. If \( T = T(\hat{g}, M, \| g \|_{X_{k,\gamma}^{(\delta)}}, \| \psi \|_{C^1(M)}) \) is chosen sufficiently small, then there exists a unique solution \( V(x,t) \) to the initial value problem (5.4) such that

\[ (1) \quad V \in C^{1+\lambda, \frac{1}{1-\frac{k}{2}}}(M \times [0,T]) \quad \text{and} \quad \nabla^2 V \in C^{k-1,\gamma}(M \times (0,T)); \]

\[ (2) \quad \| V \|_{1+\lambda, \frac{1}{1-\frac{k}{2}}(M \times [0,T])} + \| \nabla^2 V \|_{C^{k-1,\gamma}(M \times (0,T))} \leq K(\hat{g}, M, \| g \|_{X_{k,\gamma}^{(\delta)}}, \| \psi \|_{C^1(M)}). \]

**Proof.** In the sequel of the proof, \( K \) will always denote a constant depending only on \( g, M, \| g \|_{X_{k,\gamma}^{(\delta)}}, \| \psi \|_{C^1(M)}. \)

Let us take \( \delta = 1 + \lambda - \beta \) in the definition of \( X_{k,\gamma}^{(\delta)}(M \times [0,T]; \hat{\psi}^\ast(TM)) \). Note that \( \lambda \in (\gamma, \beta) \) implies \( \delta < 1 \). By Proposition 5.3, we can find a unique solution \( V \in X_{k,\gamma}^{(\delta)}(M \times [0,T]) \) to the initial value problem (5.4) if \( T \) is chosen sufficiently small. Since \( \| \nabla U \|_{\lambda; M} \leq K \), we claim that

\[ \lim_{t \to 0^+} \nabla U(x,t) = \nabla U(x). \]
For instance, let us consider an arbitrary chart \( \varphi : U \to \mathbb{R}^n \), so that on this chart \( V = V^a \frac{\partial}{\partial y^a} \) and we abbreviate \( V^a(x, t) = V^a(\varphi^{-1}(x), t) \) for \( x \in \varphi(U) \). Then \( V^a \) solves the equation

\[
\begin{align*}
\frac{\partial}{\partial t} V^a - g^{kl} D_k^a D_l V^a &= \tilde{F}^a(V, \nabla V) \quad &\text{on} \quad \varphi(U) \times (0, T], \\
V^a(x, 0) &= U^a(x) \quad &\text{on} \quad \varphi(U),
\end{align*}
\]

where \( \tilde{F}^a(V, \nabla V) = F^a(V, \nabla V) + (\partial \Gamma + \tilde{\Gamma}_\ast g \tilde{\Gamma}) \ast g V + \tilde{\Gamma}_\ast g \nabla V \). Since \( \|V\|_{\mathcal{C}^{s, \gamma}(M \times [0, T])} \leq K \), the estimate \((5.11)\) implies that

\[(5.18) \quad \|\tilde{F}^a(V, \nabla V)\|_{\mathcal{C}^{s-1, \gamma}(U \times (0, T))} \leq K.\]

Moreover, the uniqueness part of Lemma \((8.26)\) and \((3.6)\) imply that \( V^a \) has the form

\[
(5.19) \quad D_l V^a(x, t) = -\int_0^t \int_{\mathbb{R}^n} D_l \Gamma(x, t; \xi, \tau) \tilde{F}^a(V, \nabla V)(\xi, \tau) d\xi d\tau + \int_{\mathbb{R}^n} D_l \Gamma(x, t; \xi, 0) U^a(\xi) d\xi.
\]

Using the estimates of fundamental solution \((5.7)\) and \((5.18)\), we have

\[
(5.20) \quad \int_0^t \int_{\mathbb{R}^n} |D_l \Gamma(x, t; \xi, \tau) \tilde{F}^a(\xi, \tau)| d\xi d\tau \\
\leq K \int_0^t \int_{\mathbb{R}^n} (t - \tau)^{-\frac{\rho + 1}{2}} \exp \left( -\frac{|x - \xi|^2}{K(t - \tau)} \right) |\tilde{F}^a(\xi, \tau)| d\xi d\tau \\
\leq K \int_0^t \int_{\mathbb{R}^n} (t - \tau)^{-\frac{\rho + 1}{2}} \exp \left( -\frac{|x - \xi|^2}{K(t - \tau)} \right) |\tilde{F}^a(\xi, \tau)| d\xi d\tau \\
\leq K \int_0^t \int_{\mathbb{R}^n} (t - \tau)^{-\frac{\rho + 1}{2}} \exp \left( -\frac{1}{K \rho^2} \right) d\xi d\tau \\
\leq K t^{\rho}. \]

For the second integral in \((5.19)\), we note that by \([12, (11.13)]\) the fundamental solution \( \Gamma(x, t; \xi, 0) \) can be written in the form

\[
\Gamma(x, t; \xi, 0) = \Gamma_0(x - \xi, t; \xi, 0) + \Gamma_1(x, t; \xi, 0),
\]

where the function \( \Gamma_0(x - \xi, t; \xi, 0) \) is defined in \([12, (11.2)]\) which is the fundamental solution obtained by freezing the operator \( \frac{\partial}{\partial t} - g^{kl} D_k^a D_l \) at the point \( (\xi, 0) \). Moreover, \( \Gamma_0 \) also satisfies the estimates \((5.7)\) by \([12, (11.3)]\). On the other hand, since \( g^{kl} \in C^{\beta, \frac{\beta}{2}}(\mathbb{R}^n \times [0, T]) \), the minor term \( \Gamma_1(x, t; \xi, 0) \) satisfies the estimate

\[(5.21) \quad \left| D_2 \Gamma_1(x, t; \xi, 0) \right| \leq K t^{-\frac{\rho + 1 - \beta}{2}} \exp \left( -\frac{|x - \xi|^2}{K t} \right).\]
by [12, P.377]. Since \(\int_{\mathbb{R}^n} D_x \Gamma_0(z; t, \xi, 0) dz = 0\) for any fixed \(\xi\) by [12, (11.5)], we also have \(\int_{\mathbb{R}^n} D_x \Gamma_1(x; t, \xi, 0) dx = 0\). Hence \(\text{[24]}\) implies

\[
(5.22) \quad \left| \int_{\mathbb{R}^n} D_1 \Gamma_1(x; t, \xi, 0) U^n(\xi) dx \right| \\
= \left| \int_{\mathbb{R}^n} D_1 \Gamma_1(x; t, \xi, 0) (U^n(\xi) - U^n(x)) dx \right| \\
\leq K \int_{\mathbb{R}^n} t^{-\frac{n+\beta}{2}} \exp \left( -\frac{|x - \xi|^2}{K t} \right) \| \nabla U \|_{0, M} |x - \xi| \, dx \\
\leq K \int_0^\infty t^{\frac{\beta}{2}} \rho^n \exp \left( -\frac{1}{K} \rho^2 \right) \, d\rho \\
\leq K t^{\frac{\beta}{2}}.
\]

We next write

\[
(5.23) \quad \int_{\mathbb{R}^n} D_i \Gamma_0(x - \xi; t, \xi, 0) U^n(\xi) dx \\
= \int_{\mathbb{R}^n} D_i \Gamma_0(x - \xi; t, x, 0) U^n(\xi) dx \\
+ \int_{\mathbb{R}^n} (D_i \Gamma_0(x - \xi; t, \xi, 0) - D_i \Gamma_0(x - \xi; t, x, 0)) (U^n(\xi) - U^n(x)) dx \\
:= I_1 + I_2.
\]

For the second term \(I_2\), we apply the estimate [12, (11.4)] to obtain

\[
(5.24) \quad |I_2| \leq K \int_{\mathbb{R}^n} |x - \xi|^\beta t^{-\frac{n+\beta}{2}} \exp \left( -\frac{|x - \xi|^2}{K t} \right) |U^n(x) - U^n(\xi)| \, dx \\
\leq K \int_{\mathbb{R}^n} |x - \xi|^{\beta} t^{-\frac{n+1}{2}} \exp \left( -\frac{|x - \xi|^2}{K t} \right) \| \nabla U \|_{0, M} \, dx \\
\leq K \int_0^\infty t^{\frac{\beta}{2}} \rho^n \exp \left( -\frac{1}{K} \rho^2 \right) \, d\rho \\
\leq K t^{\frac{\beta}{2}}.
\]

For the first term \(I_1\), by noting that \(D_i \Gamma_0(x - \xi; t, x, 0) = -D_\xi \Gamma_0(x - \xi; t, x, 0)\), we have

\[
(5.25) \quad I_1 = \int_{\mathbb{R}^n} \Gamma_0(x - \xi; t, x, 0) D_i U^n(\xi) dx.
\]

Now, by putting (5.20) into (5.19), we obtain

\[
\left| \int_{\mathbb{R}^n} D_i V^n(x, t) - D_i \Gamma_0(x - \xi; t, x, 0) U^n(\xi) dx \right| \leq K t^{\frac{\beta}{2}}.
\]

Note that \(\Gamma_0(x - \xi; t, x, 0) \to \delta(x - \xi)\) as \(t \to 0^+\) in the sense of distribution. Hence by taking \(t \to 0^+\), we obtain (5.17).

Next, we observe that \(\nabla_i V(x, t)\) solves the system

\[
(5.26) \quad \begin{cases}
\left( \frac{\partial}{\partial t} - t \rho_0 \nabla^2 \right) (\nabla_i V) = \tilde{P}(V, \nabla V, \nabla^2 V) & \text{on } M \times (0, T] \\
\nabla_i V(x, 0) = \nabla_i U(x) & \text{on } M,
\end{cases}
\]

where

\[
\tilde{P}(V, \nabla V, \nabla^2 V) = \nabla g^{-1} \ast \nabla^2 V + g^{-1} \ast \nabla^* V + g^{-1} \ast \nabla^* R \ast V + \nabla_i (P(V, \nabla V))
\]

and the initial condition makes sense in view of (5.17).
Since \( \|V\|_{X^{(s)}_{k,\gamma}(M \times [0,T])} \leq K \), the estimate (5.11) implies that

\[
\|\tilde{\nabla}_t (\mathcal{P}(V, \tilde{\nabla}V))\|_{C^{k-2,\gamma}_{1,\frac{1}{2}}(M \times (0,T))} \leq K.
\]

It is also easy to see that

\[
g^{-1} \ast \tilde{R} \ast \tilde{\nabla}V + g^{-1} \ast \tilde{\nabla} \tilde{\nabla}R \ast V \|_{C^{k-2,\gamma}_{1,\frac{1}{2}}(M \times (0,T))} \leq K.
\]

Moreover, \( \|V\|_{X^{(s)}_{k,\gamma}(M \times [0,T])} \leq K \) implies that \( \|\tilde{\nabla}^2 V\|_{C^{k-2,\gamma}_{1,\frac{1}{2}}(M \times (0,T))} \leq K \). This gives

\[
\|\tilde{\nabla}^{-1} \ast \tilde{\nabla}^2 V\|_{C^{k-2,\gamma}_{1,\frac{1}{2}}(M \times (0,T))} \leq K\|\tilde{\nabla}^{-1}\|_{C^{k-2,\gamma}_{1,\frac{1}{2}}(M \times (0,T))} \quad \leq K'.
\]

As \( 1 - \frac{\lambda}{\beta} = \frac{3}{2} - \frac{\sigma}{2} - \frac{\rho}{2} \) and \( \lambda < \beta \), we obtain by putting (5.21) to (5.29) that

\[
\|\tilde{\nabla}V\|_{\lambda,M} \leq K \quad \text{to} \quad \text{obtain}
\]

\[
\|\tilde{\nabla}V\|_{\lambda,M} \times (0, T) + \|\tilde{\nabla}^2 V\|_{C^{k-1,\gamma}_{1,\frac{1}{2}}(M \times (0,T))} \leq K.
\]

From this, the assertion follows.

Now, associated with the differential \( d\phi_t \) of the harmonic map flow \( \phi_t \), we define the functions

\[
\Phi^{i,j}_\mu : M \times [0,T] \to \mathbb{R}
\]

as in Definition 4.6. Then Theorem 5.3 and Lemma 5.1 imply

\[
\|\Phi^{i,j}_\mu\|_{\lambda,\frac{1}{\mu}, M \times [0,T]} + \|\tilde{\nabla} \Phi^{i,j}_\mu\|_{C^{k-1,\gamma}_{1,\frac{1}{2}}(M \times (0,T))} \leq K(\tilde{g}, M, \|g\|_{X^{(s)}_{k,\gamma}}, \|\psi\|_{C^{1,\gamma}(M)})
\]

for \( \lambda \in (\gamma, \beta) \).

Next, we show that a solution to the Ricci flow gives rise to a solution to the Ricci-DeTurck flow via the harmonic map heat flow.

**Proposition 5.5.** Let \( \lambda \in (\gamma, \beta) \) be given. Let \( \{\phi_t\}_{t \in [0,T]} \) be a one-parameter family of diffeomorphisms which evolves under the harmonic map flow (4.7). For each \( t \in [0,T] \), we define a metric \( \tilde{g}(t) \) by \( \tilde{g}(t) := (\phi_t)^{-1}(g(t)) \). Then, \( \tilde{g}(t) \in X^{(s)}_{k,\gamma}(M \times [0,T]) \) solves the Ricci-DeTurck system (4.9). Moreover, \( \tilde{g} \) satisfies the estimate

\[
\|\tilde{g}\|_{X^{(s)}_{k,\gamma}(M \times [0,T])} \leq K(\tilde{g}, M, \|g\|_{X^{(s)}_{k,\gamma}}, \|\psi\|_{C^{1,\gamma}(M)}).
\]

**Proof.** Since

\[
\Delta_{g(t), \phi_t} \phi_t = \Delta_{g(t), \phi_t} \phi_t = \Delta_{g(t), \phi_t} \phi_t = -W(\phi_t) \quad \text{for all} \quad (p, t) \in M \times (0,T),
\]

we have

\[
\frac{\partial}{\partial t} \left| \frac{\partial}{\partial s} \tilde{g}(t) \right|_p = \left( \frac{\partial}{\partial t} \tilde{g}(t) \right)_p + \frac{d}{ds} \left| \frac{\partial}{\partial s} \tilde{g}(t) \right|_p = \left( \frac{\partial}{\partial t} \tilde{g}(t) \right)_p - L_W \tilde{g}(t).
\]
But since $\frac{\partial}{\partial t} g(t) = -2 \text{Ric}(g(t))$, by the diffeomorphism invariance of Ricci curvature we obtain
\[
\frac{\partial}{\partial t} \tilde{g}(t) - L_W \tilde{g}(t) = -2 \text{Ric}(\tilde{g}(t)).
\]
Hence $\tilde{g}$ satisfies the Ricci-DeTurck system with initial condition $\tilde{g}(0) = g_0$. Lastly, since Theorem 5.4 and Lemma 5.1 imply
\[
\|\Phi_{ij}^{\mu \nu}\|_{\lambda, \frac{1}{2}, M \times [0, T]} + \|\nabla \Phi_{ij}^{\mu \nu}\|_{C^{1, \gamma}(M \times [0, T])} \leq K(\tilde{g}, M, \|g\|_{X^{(s)}}, \|\psi\|_{C^{1, \gamma}(M)}),
\]
we have $\tilde{g} \in C^{\lambda, \frac{1}{2}}$, and we can then proceed as in Proposition 4.8 to obtain that $\tilde{g} \in X^{(k, \gamma)}_{k, \gamma}(M \times [0, T])$ and the desired estimates. \qed
6. Proof of the Main Theorems

We first prove the existence of a canonical solution to Ricci flow on the doubled manifold with $C^\alpha$ initial metric. The following result implies Main Theorem 1 and Main Theorem 2.

**Theorem 6.1.** Let $\tilde{M}$ be a closed compact smooth manifold and $\tilde{g}_0 \in C^\alpha(\tilde{M})$ be a Riemannian metric for some $\alpha \in (0, 1)$. Let $k \geq 2$, $\gamma \in (0, \alpha)$ and $\beta \in (\gamma, \alpha)$ be given. Then there exists a $C^{1,\beta}$ diffeomorphism $\psi$ and $T = T(\tilde{M}, \tilde{g}, \|\tilde{g}_0\|_\alpha)$, $K = K(\tilde{M}, k, \tilde{g}, \|\tilde{g}_0\|_\alpha)$ such that the following holds:

There is a solution $g(t) \in \mathcal{X}_{k,\gamma}^{(\beta)}(\tilde{M} \times [0, T])$ to the Ricci flow

$$\frac{\partial}{\partial t}g(t) = -2\text{Ric}(g(t)) \quad \text{on} \quad \tilde{M} \times (0, T]$$

such that $g(0) = \psi^* \tilde{g}_0$ and

$$\|g\|_{\mathcal{X}_{k,\gamma}^{(\beta)}(\tilde{M} \times [0, T])} \leq K.$$

Moreover, the solution is canonical in the sense that if $\tilde{g}(t) \in \mathcal{X}_{k,\gamma}^{(\beta)}(\tilde{M} \times [0, T])$ is another solution to the Ricci flow and $\tilde{\psi} \in C^{1,\beta}(\tilde{M})$ is a diffeomorphism such that $\tilde{g}(0) = \psi^* \tilde{g}_0$, then there is a $C^{k+1}$ diffeomorphism $\phi$ such that $\tilde{g}(t) = \phi^* g(t)$ which satisfies $\tilde{\psi} = \psi \circ \phi$.

**Proof.** By Proposition 4.18 we can find $T = T(\tilde{M}, \tilde{g}, \|\tilde{g}_0\|_\alpha) > 0$ sufficiently small and $K = K(\tilde{M}, k, \tilde{g}, \|\tilde{g}_0\|_\alpha)$ and a $C^{1,\beta}$ diffeomorphism $\psi$ such that there is a solution $g(t)$ to the Ricci flow satisfying the estimate

$$\|g\|_{\mathcal{X}_{k,\gamma}^{(\beta)}(\tilde{M} \times [0, T])} \leq K.$$

and $g(0) = \psi^* \tilde{g}_0$.

Now, suppose that $\tilde{g}(t) \in \mathcal{X}_{k,\gamma}^{(\beta)}(\tilde{M} \times [0, T])$ is another solution to the Ricci flow and $\tilde{\psi} \in C^{1+\beta}(\tilde{M})$ is a diffeomorphism such that $\tilde{g}(0) = \psi^* \tilde{g}_0$. Let $\phi_t$ denote the solution to the harmonic map heat flow starting from $\phi_0 = \psi$ with respect to $g(t)$ and $\tilde{g}$. Let $\tilde{\phi}_t$ denote the solution to the harmonic map heat flow starting from $\tilde{\phi}_0 = \psi$ with respect to $\tilde{g}(t)$ and $\tilde{g}$. By Proposition 5.20 these solutions exist and are unique. Subsequently Theorem 5.3 and Proposition 5.2 imply that

$$h(t) := (\tilde{\phi}_t^{-1})^* g(t) \quad \text{and} \quad \tilde{h}(t) := (\phi_t^{-1})^* \tilde{g}(t)$$

are both solutions to the Ricci-DeTurck flow (1.5) such that $h, \tilde{h} \in \mathcal{X}_{k,\gamma}^{(\lambda)}(M \times [0, T])$ for some $\lambda \in (\gamma, \beta)$ and $h(0) = \tilde{h}(0) = \tilde{g}_0$. Since the solution to the Ricci-DeTurck flow in $\mathcal{X}_{k,\gamma}^{(\lambda)}$ are unique by Theorem 4.19 we have $h(t) = \tilde{h}(t)$ on $M \times [0, T]$. Now observe that

$$\Delta_{g(t), \tilde{g}} \phi_t |_{\psi^{-1}(p)} = \Delta_{h(t), \tilde{g}} \phi_t |_{\phi_t \circ \psi^{-1}(p)},$$

hence $\phi_t \circ \psi^{-1}$ is a solution to the ODE

$$\begin{cases}
\frac{\partial}{\partial t} \phi_t \circ \psi^{-1}(p) = \Delta_{h(t), \tilde{g}} \phi_t |_{\phi_t \circ \psi^{-1}(p)} \\
\phi_0 \circ \psi^{-1} = \text{id}.
\end{cases}
$$

(6.1)
Analogously, $\tilde{\phi}_t \circ \tilde{\psi}^{-1}$ satisfies
\begin{equation}
\begin{cases}
\frac{\partial}{\partial t} \tilde{\phi}_t \circ \tilde{\psi}^{-1}(p) = \Delta h(t, \tilde{\phi}_t) \tilde{id} \big|_{\tilde{\phi}_t \circ \tilde{\psi}^{-1}(p)} \\
\tilde{\phi}_0 \circ \tilde{\psi}^{-1} = \text{id}.
\end{cases}
\end{equation}

Since we know that $h(t) = \hat{h}(t)$, it follows that $\phi_t \circ \psi^{-1}$ and $\tilde{\phi}_t \circ \tilde{\psi}^{-1}$ satisfy the same ODE with the same initial condition. Consequently $\phi_t \circ \psi^{-1} = \tilde{\phi}_t \circ \tilde{\psi}^{-1}$ on $M \times [0, T]$. In other words, $\phi_t^{-1} \circ \phi_t = \psi^{-1} \circ \tilde{\psi}$ is constant in $t$. Let us take the desired diffeomorphism $\phi$ to be $\phi := \phi_t^{-1} \circ \tilde{\phi}_t$. Then from Theorem 5.4 we know that $\phi$ is a $C^{k+1}$ diffeomorphism satisfying $\tilde{\psi} = \psi \circ \phi$ and
\[ \hat{g}(t) = (\phi_t^{-1} \circ \tilde{\phi}_t)^* g(t) = \phi^* g(t). \]

The previous theorem implies Main Theorem 3 and Main Theorem 4 via doubling:

**Theorem 6.2** (Main Theorem 3). Let $(M, g_0)$ be a compact smooth Riemannian manifold with boundary. Let $k \geq 2$, $\beta \in (0, 1)$, $\gamma \in (0, \beta)$ and $\epsilon \in (0, 1 - \beta)$ be given. Then there exists a $C^{1, \beta}$ diffeomorphism $\psi$ and $T = T(M, \hat{g}, \|\hat{g}_0\|_{\beta+\epsilon})$, $K = K(M, k, \hat{g}, \|\hat{g}_0\|_{\beta+\epsilon})$ such that the following holds:

There is a solution $g(t) \in X_{k, \gamma}^{(\beta)}(M \times [0, T])$ to the Ricci flow on manifold with boundary
\begin{equation}
\begin{cases}
\frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t)) & \text{on } M \times (0, T) \\
A_{g(t)} = 0 & \text{on } \partial M \times (0, T)
\end{cases}
\end{equation}

such that $g(0) = \psi^* g_0$ and
\[ \|g\|_{X_{k, \gamma}^{(\beta)}(M \times [0, T])} \leq K. \]

For each $t > 0$, the metric $g(t)$ extends smoothly to the doubled manifold $\tilde{M}$ of $M$, and the doubled metric lies in $X_{k, \gamma}^{(\beta)}(\tilde{M} \times [0, T])$. The diffeomorphism $\psi$ also extends to a $C^{1, \beta}$ diffeomorphism on the doubled manifold.

**Proof.** Let $\tilde{M}$ be the doubling of $M$, and $\tilde{g}_0$ be the extension of $g_0$ in $\tilde{M}$ via reflection about the boundary $\partial M$. Since $\tilde{g}_0$ is Lipschitz, we can find $\beta + \epsilon \in (\beta, 1)$ such that $\tilde{g}_0 \in C^{3, \beta+\epsilon}(M)$. Fix a smooth background metric $\hat{g}$ on $M$ such that in a small collar neighborhood of $\partial M$ the metric $\hat{g}$ is isometric to a product $\partial M \times [0, \epsilon]$.

By Proposition 4.5 we can find $T = T(\hat{M}, \hat{g}, \|\hat{g}_0\|_{\beta+\epsilon}) > 0$ sufficiently small and $K = K(M, k, \hat{g}, \|\hat{g}_0\|_{\beta+\epsilon})$ and a $C^{1, \beta}$ diffeomorphism $\psi$ such that there is a solution $\tilde{g}(t)$ to the Ricci flow on $\tilde{M} \times (0, T]$ satisfying the estimate
\[ \|\tilde{g}\|_{X_{k, \gamma}^{(\beta)}(\tilde{M} \times [0, T])} \leq K. \]

and $\tilde{g}(0) = \tilde{\psi}^* \tilde{g}_0$.

Note that by uniqueness and diffeomorphism invariance the solution to the Ricci–DeTurck flow on the doubled manifold $\tilde{M}$ with initial metric $\tilde{g}_0$ is invariant under the natural natural $\mathbb{Z}_2$ action given by the reflection about $\partial M$ which switches the two halves of $\tilde{M}$. Thus the diffeomorphism $\tilde{\psi}$ we obtained by solving the ODE (4.19) is equivariant under this $\mathbb{Z}_2$ action. This implies that the solution $\hat{g}(t)$ to the Ricci flow is also invariant under the $\mathbb{Z}_2$ action. In particular, it follows from the $\mathbb{Z}_2$ symmetry that $g = \hat{g}|_M \in X_{k, \gamma}^{(\beta)}(M \times [0, T])$ has totally geodesic boundary.
Lemma 6.4. Let \( \alpha, \beta \in (0, 1) \) be given such that \( \beta < \alpha \). Moreover, let \( \gamma \in (0, \alpha) \) and \( \delta \in (0, \beta) \) be given such that \( \delta < \gamma \). Then the bounded subsets in the space \( C^{k+\gamma,\delta}_{\frac{1}{2}-\frac{\gamma}{4}}(M \times (0, T)] \) are precompact in the space \( C^{k,\gamma}_{\frac{1}{2}-\frac{\gamma}{4}}(M \times (0, T]) \).

Proof. Note that by Lemma 2.2 we have the inclusion \( C^{k,\gamma}_{\frac{1}{2}-\frac{\gamma}{4}} \subset C^{k,\delta}_{\frac{1}{2}-\frac{\delta}{4}} \). Let \( \{\eta_j\} \) be a bounded sequence in \( C^{k,\gamma}_{\frac{1}{2}-\frac{\gamma}{4}}(M \times (0, T]) \) so that \( \|\eta_j\|_{C^{k,\gamma}_{\frac{1}{2}-\frac{\gamma}{4}}} \leq L \). In particular,

\[
\|\nabla^k \eta_j\|_{0, M \times [\sigma/2, \sigma]} \leq L \quad \text{and} \quad [\nabla^k \eta_j]_{1, M \times [\sigma/2, \sigma]} \leq L
\]

for any \( \sigma \in (0, T] \). Thus \( \{\nabla^k \eta_j\} \) is equicontinuous on \( M \times [\sigma/2, \sigma] \) and uniformly bounded. By Arzela-Ascoli it contains a subsequence \( \{\tilde{\eta}_j\} \) such that \( \{\nabla^k \tilde{\eta}_j\} \) is uniformly convergent on compact subsets of \( M \times [\sigma/2, \sigma] \). By the same argument we can also proceed to find a subsequence of \( \{\tilde{\eta}_j\} \) which is uniformly convergent on bounded subsets of \( M \times [\sigma/2, \sigma] \) together with its lower order \( x \)-derivatives. Moreover, by a standard diagonal argument we can further find a subsequence which is uniformly convergent on bounded subsets of \( M \times (0, T] \). We still denote this subsequence by \( \{\tilde{\eta}_j\} \) and its limit by \( \eta \). Thus \( \eta \) satisfies the same bounds as \( \eta_j \) given above. Define \( \zeta_j = \tilde{\eta}_j - \eta \). To finish the proof we will show that \( \zeta_j \to 0 \) in \( C^{k,\gamma}_{\frac{1}{2}-\frac{\gamma}{4}}(M \times (0, T]) \). For any \( 0 \leq r \leq k \), we have

\[
\sigma^{\frac{1}{2}+\frac{\gamma}{4}+\frac{r}{4}} \|\nabla^r \zeta_j\|_{0, M \times [\sigma/2, \sigma]} \leq T^{\frac{1-m}{2}} \sigma^{\frac{1}{2}+\frac{\gamma}{4}+\frac{r}{4}} \|\nabla^r \zeta_j\|_{0, M \times [\sigma/2, \sigma]}
\]

Moreover,

\[
\sigma^{\frac{1}{2}+\frac{\gamma}{4}+\frac{r}{4}} [\nabla^r \zeta_j]_{1, M \times [\sigma/2, \sigma]} \leq \|\zeta_j\|_{C^{k,\gamma}_{\frac{1}{2}-\frac{\gamma}{4}}(M \times (0, T])} \leq 2L
\]

for any \( \sigma \in (0, T] \). Now, we choose a local chart \( U \) in \( M \) and let \( x, y \in U \) so that the \((0,2)\)-tensors \( \zeta_j \) are represented in local coordinates as \( \zeta_j = \sum_{m=1}^{N} \zeta_{jm} \epsilon_m \). Let \( \epsilon > 0 \) be an arbitrary small number, we may assume without loss of generality that \( \epsilon < \sigma \).
Moreover, if \( g^{\hat{\cdot}} \) \((\ref{6.3})\) implies
\[
\sigma^{-\frac{\beta-2}{2}+\frac{\beta}{2}} \frac{\left| \nabla^\tau_{\zeta} \zeta_{jm}(x, t) - \nabla^\tau_{\zeta} \zeta_{jm}(y, s) \right|}{d((x, t), (y, s))} \\
= \sigma^{-\frac{\beta-2}{2}+\frac{\beta}{2}} \frac{\left| \nabla^\tau_{\zeta} \zeta_{jm}(x, t) - \nabla^\tau_{\zeta} \zeta_{jm}(y, s) \right|}{d((x, t), (y, s))} \cdot d((x, t), (y, s))^{\gamma - \delta} \sigma^{\frac{\beta-2}{2}+\frac{\beta}{2} - \frac{\beta}{2}} \\
\leq 2LT^{-\frac{\beta-2}{2}+\frac{\beta}{2} - \frac{\beta}{2}} \epsilon^{-\frac{\beta}{2}}.
\]
If \( d((x, t), (y, s)) \geq \epsilon \), then
\[
\sigma^{\frac{\beta-2}{2}+\frac{\beta}{2}} \frac{\left| \nabla^\tau_{\zeta} \zeta_{jm}(x, t) - \nabla^\tau_{\zeta} \zeta_{jm}(y, s) \right|}{d((x, t), (y, s))} \leq K(T)\|\nabla^\tau_{\zeta} \zeta\|_{0, M \times [\sigma/2, \sigma]} \epsilon^{-\delta}.
\]
Combining \((6.6)\) and \((6.7)\), we obtain
\[
\sigma^{\frac{\beta-2}{2}+\frac{\beta}{2}} \left| \nabla^\tau_{\zeta} \zeta_{jm}(x, t) - \nabla^\tau_{\zeta} \zeta_{jm}(y, s) \right|_{[\delta, \lambda]_{M \times [\sigma/2, \sigma]} \leq K(T)(2L\epsilon^{-\frac{\beta}{2}} + \|\nabla^\tau_{\zeta} \zeta\|_{0, M \times [\sigma/2, \sigma]} \epsilon^{-\delta}).
\]
By taking \( j \to \infty \) and then \( \epsilon \to 0 \) in \((6.8)\), we obtain
\[
\lim_{j \to \infty} \sigma^{\frac{\beta-2}{2}+\frac{\beta}{2}} \left| \nabla^\tau_{\zeta} \zeta \right|_{[\delta, \lambda]_{M \times [\sigma/2, \sigma]} = 0.
\]
for each \( \sigma \in (0, T] \). Therefore, \((6.2)\) and \((6.3)\) conclude that
\[
\lim_{j \to \infty} \|\zeta\|_{C^{1, \alpha}_{\frac{\beta-2}{2}+\frac{\beta}{2}} (\times (0, T)) = 0
\]
\[\square\]

**Theorem 6.5 (Main Theorem 5).** Suppose that \( g(t) \) is a canonical solution to \((6.3)\) on \( M \times [0, T] \) given by Theorem 6.2. Then the following holds:

If \( (M, g_0) \) has a convex boundary, then

(i) \( (M, g_0) \) has positive curvature operator \( \implies \) \( (M, g(t)) \) has positive curvature operator;

(ii) \( (M, g_0) \) is PIC1 \( \implies \) \( (M, g(t)) \) is PIC1;

(iii) \( (M, g_0) \) is PIC2 \( \implies \) \( (M, g(t)) \) is PIC2.

If \( (M, g_0) \) has a two-convex boundary, then

(iv) \( (M, g_0) \) is PIC \( \implies \) \( (M, g(t)) \) is PIC.

Moreover, if \( (M, g_0) \) has a mean-convex boundary, then

(v) \( (M, g_0) \) has positive scalar curvature \( \implies \) \( (M, g(t)) \) has positive scalar curvature.

**Proof.** By Theorem 2 in [5], there is a family of smooth Riemannian metrics \( \{g_\lambda\}_{\lambda > \lambda^*} \) on \( M \) which converges to \( g_0 \) in \( C^\alpha \) for any \( \alpha \in [0, 1) \) and satisfies:

(i) \( (M, g_\lambda) \) has a totally geodesic boundary.

(ii) If \( (M, g_0) \) has a convex boundary, then

- \( (M, g_0) \) has positive curvature operator \( \implies \) \( (M, \check{g}_\lambda) \) has positive curvature operator;

- \( (M, g_0) \) is PIC1 \( \implies \) \( (M, \check{g}_\lambda) \) is PIC1;

- \( (M, g_0) \) is PIC2 \( \implies \) \( (M, \check{g}_\lambda) \) is PIC2.

(iii) If \( (M, g_0) \) has a two-convex boundary, then

- \( (M, g_0) \) is PIC \( \implies \) \( (M, \check{g}_\lambda) \) is PIC.

(iv) If \( (M, g_0) \) has a mean-convex boundary, then

- \( (M, g_0) \) has positive scalar curvature \( \implies \) \( (M, \check{g}_\lambda) \) has positive scalar curvature.
Note that by Corollary 8 in [5] the positivity conditions in the above statement are uniform in all sufficiently large \( \lambda \).

Let \( \tilde{M} \) be the doubled manifold of \( M \), and fix a background metric \( \tilde{g} \) such that in a small collar neighborhood of \( \partial M \) the metric is isometric to \( \partial M \times [0, \varepsilon] \). We extend the metrics \( g_0 \) to \( \tilde{M} \) via reflection about the boundary \( \partial M \). Moreover, from the construction in [5] in a neighborhood of the boundary \( \tilde{g}_\lambda \) has the form of a product metric, thus the metric \( \tilde{g}_\lambda \) can be extended smoothly to the doubled manifold \( \tilde{M} \). Then \( g_0 \) is a Lipschitz metric on \( \tilde{M} \) and \( \tilde{g}_\lambda \) is a smooth metric on \( \tilde{M} \).

Now, we assume that \( (M, g_0) \) has convex boundary. If \( (M, g_0) \) has positive curvature operator/ PIC1/ PIC2, then \( (\tilde{M}, \tilde{g}_\lambda) \) has positive curvature operator/ PIC1/ PIC2 for all sufficiently large \( \lambda > 0 \) and these positivity conditions are uniformly bounded below for \( \lambda \) large. Let \( \tilde{g}_\lambda(t) \) be the solution to Ricci-DeTurck flow on \( \tilde{M} \times [0, T] \) starting from \( \tilde{g}_\lambda(0) = \tilde{g}_\lambda \). Let \( \phi_t \) solves the harmonic map heat flow so that \( \tilde{g}(t) = (\phi_t^{-1})^* \tilde{g}(t) \) is a solution to Ricci-DeTurck flow on \( \tilde{M} \times [0, T] \) starting from \( \tilde{g}(0) = g_0 \). Proposition 5.5 then implies that \( \tilde{g}(t) \in \mathcal{X}^{(\alpha)}_{2, \gamma}(M \times [0, T]) \) for some exponent \( \alpha \in (0, 1) \) and \( \gamma \in (0, \alpha) \). Next we apply Theorem 4.3 to obtain the estimates

\[
\|\tilde{g}_\lambda(t)\|_{\mathcal{X}^{(\alpha)}_{2, \gamma}(M \times [0, T])} \leq K(M, \tilde{g}, \|g_0\|_{0, M}).
\]

Since \( \tilde{g}_\lambda \to g_0 \) in \( C^\alpha(M) \), we can find a uniform constant \( K(M, \tilde{g}, \|g_0\|_{0, M}) \) such that

\[
\|\tilde{g}_\lambda(t)\|_{\mathcal{X}^{(\alpha)}_{2, \gamma}(M \times [0, T])} \leq K(M, \tilde{g}, \|g_0\|_{0, M}).
\]

Next, we pick some \( \beta \in (0, \alpha) \) and \( \delta \in (0, \gamma) \) such that \( \delta < \beta \). Since bounded subsets in \( C^{\alpha, \beta}(M \times [0, T]) \) are precompact in \( C^{\beta, \gamma}(M \times [0, T]) \), Lemma 6.4 then implies that bounded subsets in the Banach space \( \mathcal{X}^{(\alpha)}_{2, \gamma}(M \times [0, T]) \) are precompact in \( \mathcal{X}^{(\beta)}_{2, \delta}(M \times [0, T]) \). Now, by the estimate (6.11) the metrics \( \tilde{g}_\lambda \) are uniformly bounded in \( \mathcal{X}^{(\alpha)}_{2, \gamma}(M \times [0, T]) \) for all large \( \lambda \). Upon passing to a subsequence we have \( \tilde{g}_\lambda \to \tilde{g}_\infty \) in \( \mathcal{X}^{(\beta)}_{2, \delta}(M \times [0, T]) \) as \( \lambda \to \infty \). By the continuity of coefficients in the Ricci-DeTurck flow, \( \tilde{g}_\infty(t) \) is also a solution to the Ricci-DeTurck flow with \( \tilde{g}_\infty(0) = g_0 \) such that

\[
\|\tilde{g}_\infty(t)\|_{\mathcal{X}^{(\alpha)}_{2, \gamma}(M \times [0, T])} \leq K(M, \tilde{g}, \|g_0\|_{0, M}).
\]

Since \( \delta < \beta \), Theorem 1.3 implies that such a solution is unique, therefore we have \( \tilde{g}_\infty(t) = \tilde{g}(t) \). On the other hand, since \( \tilde{g}_\lambda \) are smooth metrics, the curvature conditions are preserved under the Ricci flow, thus \( \tilde{g}_\lambda(t) \) also has positive curvature operator/ PIC1/ PIC2 for all sufficiently large \( \lambda \) by diffeomorphism invariance of curvatures. This in particular implies that \( \tilde{g}(t) = \tilde{g}_\infty(t) \) also has positive curvature operator/ PIC1/ PIC2 for each \( t > 0 \). Therefore, since \( g(t) = (\phi_t)^* \tilde{g}(t) \), by diffeomorphism invariance of curvatures these curvature conditions also hold for \( g(t) \) for each \( t > 0 \). This proves statements (i) to (iii) in the theorem. Statements (iv) and (v) can be proved similarly. The theorem then follows.

\[ \Box \]

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