LOW REGULARITY CONSERVATION LAWS FOR THE BENJAMIN-ONO EQUATION

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ABSTRACT. We obtain conservation laws at negative regularity for the Benjamin-Ono equation on the line and on the circle. These conserved quantities control the $H^s$ norm of the solution for $-\frac{1}{2} < s < 0$.

1. INTRODUCTION

We study real-valued solutions to the Benjamin-Ono equation

\[ \frac{d}{dt} q = H q'' + 2qq' \]

on the line $\mathbb{R}$ and the circle $\mathbb{R}/\mathbb{Z}$, where the Hilbert transform $H$ is defined in either setting by

\[ \hat{H}f(\xi) = -i \text{sgn}(\xi) \hat{f}(\xi). \]

This equation is a model for the propagation of long internal waves. For a recent review of the literature on (BO), see [19]. The equation (BO) is known to be completely integrable and to enjoy an infinite hierarchy of conservation laws which control the $H^k$ norms of the solution for $k = 0, 1, \ldots$ (see [4], [3]). The well-posedness of the Cauchy problem for (BO) has been well-studied on both the line ([11], [8], [20], [2]) and the circle ([16], [12]). On both spaces, the lowest regularity for which global well-posedness is known in $H^s$ is $s = 0$ (see [9], [13], [15], [5]). The equation (BO) is also known to be well-posed in the category of $C^0_t H^s_x \cap C^1_t H^1_x$ classical solutions (see [7], [18]); note that a classical $C^0_t H^s_x$ solution is automatically $C^1_t H^1_x$ because it solves (BO). For these results at nonnegative regularity, global well-posedness can be deduced from local well-posedness and the aforementioned hierarchy of conservation laws.

The equation (BO) also enjoys a scaling symmetry, to wit

\[ q \mapsto \lambda q(\lambda^2 t, \lambda x). \]

This symmetry leaves $\|q\|_{H^{-1/2}(\mathbb{R})}$ invariant. This suggests that at least below the critical regularity $s = -1/2$ we should expect (BO) to be ill-posed. Well-posedness on the line in the regime $-\frac{1}{2} \leq s < 0$ appears to be an open question, and we hope to apply the conservation laws obtained in this paper to a future study of this problem. On the other hand, on the circle, for all $s < 0$ the Cauchy problem is known to be ill-posed in the sense that the data-to-solution map fails to be pointwise continuous; see [14]. Our results show that, nevertheless, norm blowup does not occur on the circle for regularities $s > -\frac{1}{2}$. This leaves open the possibility that after some suitable renormalization of the solutions, one can recover well-posedness on the circle, as was done for the cubic Wick-ordered NLS on the circle in [17].
The equation [BO] is related to the Korteweg-de Vries equation

\[
\frac{d}{dt} q = -q''' + 6qq'.
\]

by way of the Intermediate Long Wave equation; [BO] is formally obtained from the Intermediate Long Wave equation in the deep water limit, while [KdV] arises from the shallow water limit. For details, see [1]. Using the integrable structure of [KdV] and in particular the Lax pair, Killip, Visan, and Zhang [10] obtained conservation laws which govern the \(H^s\) norm of the solution for \(s \geq -1\). These same conservation laws were employed in [9] to obtain global well-posedness of [KdV] in the space \(H^{-1}\).

In this note, we follow the method of [10] to obtain low-regularity conservation laws for [BO]. Our principal result is the following:

**Theorem.** Let \(q\) be a classical solution to [BO] on the line or the circle and let \(-\frac{1}{2} < s < 0\), \(1 \leq r < \infty\). Then

\[
(1 + \|q(0)\|_{L^r_{s/2}}^2)\sup_{t \in \mathbb{R}} \|q(t)\|_{H^r_s} \lesssim_{s,r} \|q(0)\|_{B^r_{s/2}} \lesssim_{s,r} (1 + \|q(0)\|_{L^r_{s/2}}^2)^{-s} \inf_{t \in \mathbb{R}} \|q(t)\|_{B^r_{s/2}}.
\]

The particular case of \(r = 2\) is equivalent to the conservation of the Sobolev norm:

\[
\sup_{t \in \mathbb{R}}(1 + \|q(0)\|_{H^r_{s/2}}^2)^{-s} \|q(t)\|_{H^r_s} \lesssim_{s,r} \|q(0)\|_{H^r_s} \inf_{t \in \mathbb{R}}(1 + \|q(0)\|_{H^r_{s/2}}^2)^{-s} \|q(t)\|_{H^r_s}.
\]

This will be proved as Theorem 3.2. See section 3 for the definition of the Besov norms \(\|f\|_{B^r_{s/2}}\).

Let us review the method of [10] as it applies to our problem. The first thing to note is that [BO] has a Lax pair. We proceed formally, leaving aside considerations of boundedness until we have identified the objects of our study. We follow [21] in presenting the Lax pair as it decomposes along the Hardy spaces \(H^\pm\) of \(L^2\) functions whose Fourier transforms are supported on positive and negative modes, respectively. On the line,

\[
L^2(\mathbb{R}) = H^+(\mathbb{R}) \oplus H^-(\mathbb{R}).
\]

On the circle we must be more careful, because the zero frequency mode contributes positive mass. However, if we restrict to the space \(L^2_0(\mathbb{R}/\mathbb{Z})\) of mean-zero \(L^2\) functions, then

\[
L^2_0(\mathbb{R}/\mathbb{Z}) = H^+(\mathbb{R}/\mathbb{Z}) \oplus H^-(\mathbb{R}/\mathbb{Z}).
\]

Concordantly, for much of this paper we will assume that all our solutions to [BO] on the circle have mean 0. Because the \([BO]\) flow preserves the mean of the data (since its right hand side is a complete derivative), this amounts to requiring the initial data to have mean 0. This assumption will be removed in the end by way of the Galilei transformation [6].

The orthogonal Cauchy projections \(C_\pm : L^2(\mathbb{R}) \to H^\pm(\mathbb{R})\) and \(C_\pm : L^2_0(\mathbb{R}/\mathbb{Z}) \to H^\pm(\mathbb{R}/\mathbb{Z})\) are given by

\[
C_\pm f = \frac{1}{2}(f \pm iHf).
\]

Given a smooth, decaying function \(q(t,x)\), we define operators \(L_\pm, P_\pm\) by

\[
L_\pm(t) \varphi = \pm C_\pm \frac{1}{4} \partial_x^2 \varphi + C_\pm(q(t))C_\pm \varphi, \quad P_\pm(t) \varphi = \pm \frac{1}{2} C_\pm \partial_x^2 \varphi - C_\pm(q(t))C_\pm \varphi - C_\pm(\partial_x^2 q(t)).
\]

Because these operators leave \(H^\pm\) (respectively) invariant, it will not matter whether we understand them to act on \(L^2\) or on \(H^\pm\). Now \(q(t)\) (mean 0 if on the circle) solves [BO] if and only if

\[
\frac{d}{dt} L_\pm = [L_\pm, P_\pm].
\]
Let us restrict our attention to the action on $H^+$. Because of (1.1), the (BO) flow preserves all the spectral properties of $L_+$. Thus, formally, we expect the perturbation determinant (where the determinant is taken over $H^+$)

$$\det((\kappa + L_+)(t)) = \det(\text{id} + C_+q(t)C_+R_\kappa)$$

to be preserved in time if $q$ solves (BO). Here

$$R_\kappa = C_+L_+C_+^{-1}$$

is defined by multiplication on the Fourier side by $\mathbb{I}_{(0,\infty)}(\xi)((\kappa + \xi)^{-1}$.

If $\kappa > 0$, this is a positive definite operator, and hence $\sqrt{R_\kappa}$ makes sense and the symbol of $\sqrt{R_\kappa}$ is the square root of that of $R_\kappa$. Its inverse $R_\kappa^{-1}$ also makes sense, albeit as an unbounded operator.

Taking a logarithm, we find

$$(1.2) \quad - \log \det((\kappa + L_+)(t)) = \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell} \text{tr}\{(C_+q(t)C_+R_\kappa)^\ell\}.$$

It will be convenient to reformulate the above in terms of the operator

$$A(\kappa; q) := \sqrt{R_\kappa}C_+qC_+\sqrt{R_\kappa}$$

which depends linearly on $q$ and is self-adjoint when $q$ is real. Cycling the trace, we may rewrite (1.2) as

$$\sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell} \text{tr}\{A(\kappa; q(t))^\ell\}.$$

This quantity almost makes sense; however, $A(\kappa; q)$ is not a trace-class operator, even if $q$ is Schwartz. On the other hand, considered formally,

$$\text{tr}\{A(\kappa; q(t))\} = \text{tr}\{(\kappa + L_+)(t)R_\kappa - \text{id}\}$$

ought to be preserved by the (BO) flow because of (1.1). Thus we may have some confidence in dropping the $\ell = 1$ term to study the quantity

$$\alpha(\kappa; q) := \sum_{\ell=2}^{\infty} \frac{(-1)^\ell}{\ell} \text{tr}\{A(\kappa; q)^\ell\}.$$

As we shall see, this series makes sense if $q \in H^s$ for any $s > -\frac{1}{2}$ and $\kappa$ is sufficiently large. Although we took the determinant over $H^+$, it does not matter whether we interpret the trace to be taken over $H^+$ or $L^2$, since the difference is a matter of null eigenvectors, and henceforth we shall consider $A(\kappa; q)$ to be an $L^2$ operator.

The crux of the method is to show, as the foregoing discussion suggests, that $\alpha(\kappa; q)$ is conserved by the (BO) flow (section 2) and that it controls the relevant norm(s) of the solution (section 3). In our case and unlike in [10], the main term of $\alpha(\kappa; q)$ is not directly comparable to any Sobolev norm of $q$. Therefore, it will be necessary to “build” a proxy for the $H^s$ norm of $q$ out of $\alpha(\kappa; q)$ for various scales $\kappa$. The materials of our construction being conserved, it will follow that the (proxy) norm is also conserved.
1.1. Notation and Preliminaries. We write $A \lesssim B$ to mean that $A \leq CB$ for an absolute constant $C$; if the value of $C$ depends on parameters $a, b, \ldots$ then we will instead write $A \lesssim_{a, b, \ldots} B$. We write $A \lesssim B^\gamma \pm$ to mean that, for any $\varepsilon > 0$, $A \lesssim_{\varepsilon, \gamma} B^\gamma \pm \varepsilon$.

In this paper our conventions for the Fourier transform are

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx, \quad \hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(\xi) d\xi$$

for functions on the line and

$$\hat{f}(\xi) = \int_0^1 e^{-ix\xi} f(x) dx, \quad \hat{f}(x) = \sum_{\xi \in 2\pi\mathbb{Z}} e^{ix\xi} f(\xi)$$

for functions on the circle. We define

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi, \quad \|f\|_{H^s(\mathbb{R}/\mathbb{Z})}^2 = \sum_{k \in 2\pi\mathbb{Z}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2$$

and let $H^0_s(\mathbb{R}/\mathbb{Z})$ denote the subspace of $H^s(\mathbb{R}/\mathbb{Z})$ functions with $\hat{f}(0) = 0$, i.e. mean zero.

Because our problem is translation-invariant, we may avoid any functional-analytic subtleties by working entirely on the Fourier side. If $T$ is a linear operator given on the Fourier side by

$$\hat{T}\varphi(\xi) = \int_{\mathbb{R}} K(\xi, \eta) \varphi(\eta) d\eta$$

then we may define the Hilbert-Schmidt norm of $T$ by

$$\|T\|_{HS}^2 = \iint_{\mathbb{R}^2} |K(\xi, \eta)|^2 d\eta d\xi.$$ Similarily, if $n \geq 2$ and $T_1, \ldots, T_n$ are Hilbert-Schmidt operators with Fourier kernels $K_1, \ldots, K_n$, then we say $T_1 \cdots T_n$ is trace class and define the trace

$$\text{tr}\{T_1 \cdots T_n\} = \int_{\mathbb{R}^n} K_1(\xi_1, \xi_2) \cdots K_n(\xi_n, \xi_1) d\xi_1 \cdots d\xi_n.$$ In this formulation, cycling the trace amounts to an application of Fubini’s theorem.

By the Cauchy-Schwarz inequality, $\alpha(\kappa; q)$ is a sub-geometric series with a common ratio $\lesssim \|A(q)\|_{HS}$. The following lemma gives sufficient conditions for this series to converge and submit to term-by-term differentiation and ensures that $\alpha(\kappa; q)$ is comparable to its first term.

**Lemma 1.1.** Let $t \mapsto A(t)$ define a $C^1$ curve in $\mathcal{F}_2$. Suppose for some $t_0$ we have

$$\|A(t_0)\|_{\mathcal{F}_2} < \frac{1}{3}.$$ Then there is a closed interval $I$ containing $t_0$ on which the series

$$\alpha(t) := \sum_{\ell=2}^{\infty} (-1)^\ell \text{tr}\{A(t)^\ell\}$$

converges uniformly and defines a $C^1$ function which can be differentiated term by term:

$$\frac{d}{dt} \alpha(t) = \sum_{\ell=2}^{\infty} (-1)^\ell \text{tr}\{A(t)^{\ell-1} \frac{d}{dt} A(t)\}.$$ If $A(t)$ is self-adjoint, then

$$\frac{1}{3} \|A(t)\|_{\mathcal{F}_2}^2 \leq \alpha(t) \leq \frac{2}{3} \|A(t)\|_{\mathcal{F}_2}^2.$$
For a proof of this lemma, see [10], Lemma 1.5.

2. CONSERVATION OF THE PERTURBATION DETERMINANT

In light of Lemma 1.1, our first task is to understand \( \|A(q(t))\|_2 \). Our next result is most conveniently formulated in terms of the linear operator \( T_\kappa \) given by the Fourier multiplier

\[
\hat{T_\kappa f}(\xi) = \frac{\log(2 + |\xi|/\kappa)}{\sqrt{\kappa^2 + |\xi|^2}} \hat{f}(\xi).
\]

**Theorem 2.1.** If \( q \in H^s(\mathbb{R}) \) or \( q \in H^s_0(\mathbb{R}/\mathbb{Z}) \) for \(-\frac{1}{2} < s < 0\), then for \( \kappa \geq 1 \)

\[
\|A(\kappa; q)\|_2^2 \sim \langle q, T_\kappa q \rangle \lesssim s \kappa^{-1-2s} \|q\|_{H^s}^2.
\]

**Proof.** We first consider the case of the line. We compute

\[
\|A(\kappa; q(t))\|_2^2 = \int_{\xi \geq 0} \int_{\eta \geq 0} (\kappa + \xi)^{-1}(\kappa + \eta)^{-1}|\hat{q}(\xi - \eta)|^2 d\eta d\xi
\]

\[
= \int_{-\infty}^{\infty} \int_{\eta \geq \max(0,-\xi)} (\kappa + \xi + \eta)^{-1}(\kappa + \eta)^{-1}|\hat{q}(\xi)|^2 d\eta d\xi
\]

\[
= \int_0^\infty \frac{1}{\xi} \log \left( 1 + \frac{\xi}{\kappa} \right) |\hat{q}(\xi)|^2 d\xi - \int_{-\infty}^0 \frac{1}{\xi} \log \left( 1 - \frac{\xi}{\kappa} \right) |\hat{q}(\xi)|^2 d\xi
\]

\[
= \int_{-\infty}^{\infty} \log(1 + \frac{|\xi|}{\kappa}) |\hat{q}(\xi)|^2 d\xi
\]

\[
\sim \int_{-\infty}^{\infty} \frac{\log(2 + |\xi|/\kappa)}{\sqrt{\kappa^2 + |\xi|^2}} |\hat{q}(\xi)|^2 d\xi
\]

where the implicit constant in the last line is absolute. This proves the first inequality. The second inequality follows from the fact that

\[
\log(2 + |\xi|/\kappa)(\kappa^2 + |\xi|^2)^{-1/2} \lesssim s \kappa^{-1} \left( 1 + \left( \frac{\xi}{\kappa} \right)^2 \right)^s \leq \kappa^{-1-2s}(1 + \xi^2)^s
\]

for any \(-\frac{1}{2} < s < 0\), \( \kappa \geq 1 \).

In the case \( q \in H^s_0(\mathbb{R}/\mathbb{Z}) \), a similar computation to the above may be repeated, although the analogue of the third equality holds only within the bounds of multiplicative constants, rather than exactly. \( \square \)

**Theorem 2.2.** Let \( q \) be a \( C^1_tH^3_x \cap C^1_tH^1_x \) solution to (BO) on the line or the circle, having mean 0 if on the circle. For any \( t \in \mathbb{R} \) and \( s > -\frac{1}{2} \), there exists a constant \( C = C(s) \) such that for all \( \kappa \geq 1 + C\|q(t)\|_{H^s} \),

\[
\frac{d}{dt} g(\kappa; q(t)) = 0.
\]

**Proof.** We choose \( C \) large enough that Theorem 2.1 ensures that

\[
\|A(\kappa; q(t))\|_2 \lesssim \frac{1}{3}
\]
whenever $\kappa \geq 1 + \|q(t)\|_{H^2}^\alpha + \|q(t)\|_{H^\alpha}$. We then apply Lemma \[1\] to conclude that $\alpha(\kappa; q)$ converges on a neighborhood of $t$ and
\[
\frac{d}{dt} \alpha(\kappa; q(t)) = \sum_{\ell=2}^{\infty} (-1)^{\ell} \text{tr} \left\{ A(\kappa; q)^{\ell-1} A(\kappa; q) \right\}
\]
\[
= \sum_{\ell=2}^{\infty} (-1)^{\ell} \left\{ A(\kappa; q)^{\ell-1} A(\kappa; Hq'' + 2qq') \right\}.
\]
By Theorem \[2\] $A(\kappa; q)$ is a Hilbert-Schmidt operator, as is $A(\kappa; Hq'' + 2qq')$ if $q \in H^3$, so we may cycle a copy of $A(\kappa; q)$ in the trace to obtain
\[
\frac{d}{dt} \alpha(\kappa; q(t)) = \sum_{\ell=2}^{\infty} (-1)^{\ell} \text{tr} \left\{ A(\kappa; q)^{\ell-2} A(\kappa; Hq'') A(\kappa; q) \right\} + \sum_{\ell=2}^{\infty} (-1)^{\ell} \text{tr} \left\{ A(\kappa; q)^{\ell-1} A(\kappa; 2qq') \right\},
\]
which we rearrange slightly to give a telescoping series:
\[
\frac{d}{dt} \alpha(\kappa; q(t)) = \text{tr} \left\{ A(\kappa; q) A(\kappa; Hq'') \right\} + \sum_{\ell=2}^{\infty} (-1)^{\ell} \left[ 2 \text{tr} \left\{ A(\kappa; q)^{\ell-1} A(\kappa; qq') \right\} - \text{tr} \left\{ A(\kappa; q)^{\ell-1} A(\kappa; Hq'') A(\kappa; q) \right\} \right].
\]
Evidently it suffices to show that
\[
\text{tr} \left\{ A(\kappa; q) A(\kappa; Hq'') \right\} = 0
\]
and
\[
2 \text{tr} \left\{ A(\kappa; q)^{\ell-1} A(\kappa; qq') \right\} = \text{tr} \left\{ A(\kappa; q)^{\ell-1} A(\kappa; Hq'') A(\kappa; q) \right\} \]
for all $\ell \geq 2$.

To see (2.1), we compute the trace directly on the line:
\[
\text{tr} \left\{ A(\kappa; q) A(\kappa; Hq'') \right\}
\]
\[
= - \int_{\xi \geq 0} \int_{\eta \geq 0} (k + \xi)^{-1} \hat{q} (\xi - \eta) (k + \eta)^{-1} \hat{H} (\eta - \xi) (\eta - \xi)^2 \hat{q} (\eta - \xi) d\eta d\xi
\]
\[
= -i \int_{\xi \geq 0} \int_{\eta \geq 0} \frac{\text{sgn}(\xi - \eta)(\xi - \eta)^2}{(k + \xi)(k + \eta)} |\hat{q}(\eta - \xi)|^2 d\eta d\xi.
\]
This integral converges absolutely when $q \in H^2$. The integrand is odd with respect to $\xi = \eta$, so the integral evaluates to 0. The computation on the circle is similar.

To reduce the number of derivatives on $q$ in the right hand side of (2.2), we require a Leibniz rule for the derivative operator $R^{-1}_\kappa$. If $f \in H^3$, we write
\[
C_+ f' C_+ = iC_+[C_+ (\kappa - i\partial_x) C_+ f] C_+ = iC_+[R^{-1}_\kappa f] C_+
\]
and so, commuting $C_+$ and $R_\kappa$ as needed,
\[
A(\kappa; q) A(\kappa; f') A(\kappa; q) = i \sqrt{R_\kappa} C_+ q R_\kappa C_+ [R^{-1}_\kappa f] C_+ R_\kappa C_+ q \sqrt{R_\kappa}
\]
\[
= i \sqrt{R_\kappa} C_+ q C_+ (f R_\kappa - R_\kappa f) C_+ q C_+ \sqrt{R_\kappa}
\]
\[
= i \sqrt{R_\kappa} C_+ q C_+ f C_+ \sqrt{R_\kappa} A(\kappa; q) - i A(\kappa; q) \sqrt{R_\kappa} C_+ C_+ q C_+ \sqrt{R_\kappa}.
\]
Because $R^{-1}_\kappa$ is an unbounded operator, the first equality above holds only on the domain of $R^{-1}_\kappa$, which is a dense subset of $L^2$. However, $A(\kappa; q) \in L^2_0$ and $\sqrt{R_\kappa} C_+ f C_+ g \sqrt{R_\kappa} \in L^2_0$ when $f, g \in H^2$. This suffices to conclude

$$A(\kappa; q) A(\kappa; f') A(\kappa; q) = i \sqrt{R_\kappa} C_+ q C_+ f C_+ \sqrt{R_\kappa} A(\kappa; q) - i A(\kappa; q) \sqrt{R_\kappa} C_+ f C_+ q C_+ \sqrt{R_\kappa}$$

with equality as operators on $L^2$.

Now we show (2.2). We write

$$H q'' = \frac{1}{2} q''_+ - \frac{1}{2} q''_- = (\frac{1}{2} q'_+ - \frac{1}{2} q'_-)',$$

where $\varphi_\pm$ denotes the projection of $\varphi$ onto $H^\pm$. Letting $f = \frac{1}{2} q'_+ - \frac{1}{2} q'_-$ in (2.3), we find

$$\text{tr} \{ A(\kappa; q)^{-1} A(\kappa; H q'') A(\kappa; q) \} = \text{tr} \left\{ A(\kappa; q)^{-1} \sqrt{R_\kappa} C_+ q C_+ q C_+ \sqrt{R_\kappa} \right\} - \text{tr} \left\{ A(\kappa; q)^{-1} \sqrt{R_\kappa} C_+ q C_+ q C_+ \sqrt{R_\kappa} \right\} + \text{tr} \left\{ A(\kappa; q)^{-1} \sqrt{R_\kappa} C_+ q C_+ q C_+ \sqrt{R_\kappa} \right\}.$$

We pass to the penultimate line above by cycling a copy of $A(\kappa; q)$ in two of the trace terms. Adding and subtracting $A + D$ yields

$$\text{tr} \{ A(\kappa; q)^{-1} A(\kappa; H q'') A(\kappa; q) \} = 2(A + D) - A - B - C - D.$$

We exploit some identities of the Cauchy projections in order to simplify the above expressions. If $f \in L^2(\mathbb{R})$ or $f \in L^2_0(\mathbb{R}/\mathbb{Z})$, then $C_+ f_+ C_+ = f_+ C_+$ and $C_+ f_- C_+ = C_+ f_-$. Thus

$$A = \text{tr} \left\{ A(\kappa; q)^{-1} \sqrt{R_\kappa} C_+ q C_+ q C_+ \sqrt{R_\kappa} \right\}, \quad D = \text{tr} \left\{ A(\kappa; q)^{-1} \sqrt{R_\kappa} C_+ q C_+ q C_+ \sqrt{R_\kappa} \right\}.$$

Applying the identity $f_+ + f_- = f$, we find

$$A + D = \text{tr} \left\{ A(\kappa; q)^{-1} A(\kappa; q') \right\}.$$

Thus to show (2.2) and complete the proof of the theorem, it suffices to show $A + B + C + D = 0$. By the same identity, we may simplify

$$A + C = \text{tr} \left\{ A(\kappa; q)^{-1} \sqrt{R_\kappa} C_+ q C_+ q C_+ \sqrt{R_\kappa} \right\}$$

and

$$B + D = \text{tr} \left\{ A(\kappa; q)^{-1} \sqrt{R_\kappa} C_+ q C_+ q C_+ \sqrt{R_\kappa} \right\}.$$
Because $\sqrt{\mathcal{R}_\kappa} C_+ q C_+ q C_+ q C_+ \sqrt{\mathcal{R}_\kappa} \in \mathcal{I}_2$, we may substitute this into the trace and cycle a copy of $A(\kappa; q)$ to obtain
\[ A + B + C + D = \text{tr}\{X\} = 0. \]

In the case $\ell = 2$, we do not have two copies of $A(\kappa; q)$ to place around the commutator, so we cannot apply the Leibniz rule as an operator identity. Instead we apply the same idea at the level of the integrals:
\[
(A + C) + (B + D) = \int_{\xi \geq 0} \int_{\eta \geq 0} \int_{\nu \geq 0} \frac{i(\nu - \xi)}{\kappa + \xi} \hat{q}(\xi - \eta) \hat{q}(\eta - \nu) \hat{q}(\nu - \xi) d\nu d\eta d\xi 
+ \int_{\xi \geq 0} \int_{\eta \geq 0} \int_{\nu \geq 0} \frac{i(\eta - \nu)}{\kappa + \xi} \hat{q}(\xi - \eta) \hat{q}(\eta - \nu) \hat{q}(\nu - \xi) d\nu d\eta d\xi 
= \int_{\xi \geq 0} \int_{\eta \geq 0} \int_{\nu \geq 0} \frac{i(\nu - \xi)}{\kappa + \xi} \hat{q}(\xi - \eta) \hat{q}(\eta - \nu) \hat{q}(\nu - \xi) d\nu d\eta d\xi 
- i \int_{\xi \geq 0} \int_{\eta \geq 0} \int_{\nu \geq 0} \frac{1}{\kappa + \xi} \hat{q}(\xi - \eta) \hat{q}(\eta - \nu) \hat{q}(\nu - \xi) d\nu d\eta d\xi.
\]
The above integrals converge by Cauchy-Schwarz. Cycling the variables $\xi \mapsto \nu \mapsto \eta \mapsto \xi$ in the second integral, we see that the two integrals in the last identity are equal. This completes the proof. \qed

Because $\alpha$ is comparable to its first term, as a corollary to this result we obtain uniform in time control of $\|A(\kappa; q(t))\|_{\mathcal{I}_2}$.

**Corollary 2.3.** Let $s > -\frac{1}{2}$ and let $q$ be a $C^0_t H^2_x \cap C^1_t H^1_x$ solution to (BO) on the line or the circle, having mean 0 if on the circle. Then there exists a constant $C = C(s)$ such that for all $\kappa \geq 1 + C\|q(0)\|_{H^s}^{\frac{2}{s-1}}$,
\[
\sup_{t \in \mathbb{R}} \|A(\kappa; q(t))\|_{\mathcal{I}_2}^2 \leq 2 \|A(\kappa; q(0))\|_{\mathcal{I}_2}^2 < \frac{1}{9}
\]
and therefore, by Theorem 2.2
\[
\langle q(t), T_\kappa q(t) \rangle \lesssim \langle q(0), T_\kappa q(0) \rangle.
\]

**Proof.** We may choose $C$ sufficiently large that $\|A(\kappa; q(0))\|_{\mathcal{I}_2}^2 < \frac{1}{18}$. By Lemma 1.1 and Theorem 2.2 there exists a neighborhood $I$ of 0 on which
\[
\|A(\kappa; q(t))\|_{\mathcal{I}_2}^2 \leq 3\alpha(\kappa; q(t)) = 3\alpha(\kappa; q(0)) \leq 2 \|A(\kappa; q(0))\|_{\mathcal{I}_2}^2 < \frac{1}{9}.
\]
Since $\|A(\kappa; q(t))\|_{\mathcal{I}_2} < \frac{1}{3}$, Lemma 1 implies that (2.4) is an open condition, and the theorem follows by a continuity argument. \qed

### 3. Conservation of Norms

Because of the logarithmic factor, $\langle q, T_\kappa q \rangle$ is not comparable to any $H^s$ norm of $q$; it behaves like $\|q\|_{H^{-1/2}}^2$ at frequencies $\lesssim \kappa$ and like $\|\log(|\nabla|)(|\nabla|^{1/2} q\|^2_{L^2})$ at frequencies $\gg \kappa$. This difficulty is avoided if we “build” $\|q\|_{H^s}$ for $-\frac{1}{2} < s < 0$ one frequency scale at a time, using the contribution of $\langle q, T_\kappa q \rangle$ at the frequency scale $\kappa$ where it behaves like a pure Sobolev norm.
This is naturally expressed in terms of the Besov norms
\[
\|f\|_{B^s_{r,2}} = \left( \|\hat{f}(\xi)\|^r_{L^2(|\xi| \leq 1)} + \sum_{N>1} N^{rs} \|\hat{f}(\xi)\|^r_{L^2(N \leq |\xi| < 2N)} \right)^{1/r}
\]
where the sum is taken over dyadic \( N = 2, 4, 8, \ldots \) and with the usual interpretation in the case \( r = \infty \). The following lemma (the analogue of Lemma 3.2 in [10]) relates this norm to (the leading term of) \( \alpha(k;q) \).

**Lemma 3.1.** Fix \(-\frac{1}{2} < s < 0, 1 \leq r \leq \infty, \kappa_0 \geq 1.\) For any \( H^2 \) function \( f \),
\[
\|f\|_{B^s_{r,2}} \lesssim \sum_{N \in 2^n} N^{rs} \left( \kappa_0 N \langle f, T_{\kappa_0 N} f \rangle \right)^{r/2}
\]
and
\[
\sum_{N \in 2^n} N^{rs} \left( \kappa_0 N \langle f, T_{\kappa_0 N} f \rangle \right)^{r/2} \lesssim \kappa_0^{-rs} \|f\|_{B^s_{r,2}}.
\]

**Proof.** The inequality (3.1) follows easily from the estimate
\[
\left\| \hat{f}(\xi) \right\|^2_{L^2(|\xi| \leq N)} \leq \frac{2}{\log 2} \int \frac{\kappa_0 N \log(2 + |\xi| N)}{\sqrt{\kappa_0^2 N^2 + \xi^2}} |\hat{f}(\xi)|^2 d\xi.
\]

To control the other direction, we decompose
\[
\int \frac{\kappa_0 N \log(2 + |\xi| N)}{\sqrt{\kappa_0^2 N^2 + \xi^2}} |\hat{f}(\xi)|^2 d\xi \\
\leq \log(3) \|\hat{f}(\xi)\|^2_{L^2(|\xi| \leq 1)} + \sum_{M \in 2^n} \kappa_0 N \log(2 + \frac{2M}{\kappa_0 N}) \|\hat{f}(\xi)\|^2_{L^2(M < |\xi| \leq 2M)} \\
\leq \left( \sqrt{\log(3)} \|\hat{f}(\xi)\|_{L^2(|\xi| \leq 1)} + \sum_{M \in 2^n} \left( \kappa_0 N \log(2 + \frac{2M}{\kappa_0 N}) \right)^{1/2} \|\hat{f}(\xi)\|_{L^2(M < |\xi| \leq 2M)} \right)^2.
\]

This shows that the left-hand side of (3.2) is bounded by
\[
\left\| \sqrt{\log(3)} N^s \|\hat{f}(\xi)\|_{L^2(|\xi| \leq 1)} + \sum_{M \in 2^n} \left( \kappa_0 N^{1+2s} M^{-2s} \log(2 + \frac{2M}{\kappa_0 N}) \right)^{1/2} M^s \|\hat{f}(\xi)\|_{L^2(M < |\xi| \leq 2M)} \right\|^r_{\ell^r(N \in 2^n)}
\]
which reduces our task to estimating the operator norm of a certain \( \ell^r \rightarrow \ell^r \) matrix. To do this, we apply Schur’s test. The row sums of this operator are bounded by
\[
\sqrt{\log(3)} N^s + \sum_{M \in 2^n} \left( \kappa_0 N^{1+2s} M^{-2s} \log(2 + \frac{2M}{\kappa_0 N}) \right)^{1/2} \lesssim s 1 + \kappa_0^{-s}
\]
uniformly in \( N \), while the column sums are bounded by
\[
\sum_{N \in 2^n} \sqrt{\log(3)} N^s \lesssim s, \quad \sum_{N \in 2^n} \left( \kappa_0 N^{1+2s} M^{-2s} \log(2 + \frac{2M}{\kappa_0 N}) \right)^{1/2} \lesssim s \kappa_0^{-s}.
\]
uniformly in $M$. Note that to make these estimates we require the condition $-\frac{1}{2} < s < 0$. This proves (3.2). \hfill \square

Our main result now follows easily from the foregoing lemma and Corollary 2.3.

**Theorem 3.2.** Let $q$ be a $C^1_t H^3_x \cap C^1_t H^1_x$ solution to BO on the line or the circle and let $-\frac{1}{2} < s < 0, 1 \leq r \leq \infty$. Then

$$(1 + \|q(0)\|_{B_t^{r,2}}^{\frac{2}{s}})^s \sup_{t \in \mathbb{R}} \|q(t)\|_{B_t^{r,2}} \lesssim_{s,r} \|q(0)\|_{B_t^{r,2}} \lesssim_{s,r} (1 + \|q(0)\|_{B_t^{r,2}}^{\frac{2}{s(r+2)}})^{-s} \inf_{t \in \mathbb{R}} \|q(t)\|_{B_t^{r,2}}.$$ 

**Proof.** On the circle, we first assume that $q$ has mean 0. By Hölder’s inequality, we have an embedding $B_t^{s_1,2} \hookrightarrow B_t^{s_2,2} = H^{s_2}$ for any $s_2 < s_1$. Let

$$\kappa_0 = 1 + C\|q(0)\|_{B_t^{r,2}}^{\frac{2}{s}} \gtrsim_{s} 1 + C\|q(0)\|_{B_t^{r,2}}^{\frac{2}{s}}$$

for a sufficiently large constant $C$, so that we may apply Corollary 2.3. Then, for any time $t$, Lemma 3.1 implies

$$\|q(t)\|_{B_t^{r,2}}^{r} \lesssim \sum_{N \in \mathbb{Z}} N^rs (\kappa_0 N \langle q(t), T_{\kappa_0 N} q(t) \rangle)^{r/2}$$

$$\lesssim \sum_{N \in \mathbb{Z}} N^rs (\kappa_0 N \langle q(0), T_{\kappa_0 N} q(0) \rangle)^{r/2}$$

$$\lesssim_{s} \kappa_0^{-r}s \|q(0)\|_{B_t^{r,2}}^{r}$$

$$= (1 + \|q(0)\|_{B_t^{r,2}}^{\frac{2}{s}})^{-r}s \|q(0)\|_{B_t^{r,2}}^{r}.$$ 

This proves the first inequality. By time translation symmetry, we then also obtain

$$\|q(0)\|_{B_t^{r,2}} \lesssim_{s} (1 + \|q(t)\|_{B_t^{r,2}}^{\frac{2}{s(r+2)}})^{-r}s \|q(t)\|_{B_t^{r,2}}$$

and applying the first inequality to the quantity in parentheses produces the second inequality.

To remove the mean zero assumption on the circle, we employ Galilean invariance: if $q$ solves BO, then so does

$$(3.3) \quad \tilde{q}(t, x) = q(t, x + 2\mu t) + \mu.$$ 

The estimate

$$\|\tilde{q}(t)\|_{B_t^{r,2}(\mathbb{R}/\mathbb{Z})}^2 \sim \|q(t)\|_{B_t^{r,2}(\mathbb{R}/\mathbb{Z})}^2 + \mu^2$$

then implies the general theorem. \hfill \square

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