Convergent star product algebras on “$ax + b$”

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1. Introduction

The notion of convergent star product is generally understood as the data of a one parameter family $\{E_t\}_{t \in I} \subset C^\infty(M)$ of function algebras on a Poisson manifold $(M, \{,\})$. On each of them one is given an associative algebra structure $\star_t$ which respect to which the function space $E_t$ is closed. The family of products $\{\star_t\}$ should moreover define in some sense a deformation of the commutative pointwise product of functions in the direction of the Poisson structure $\{,\}$.

Stable function algebras for the Weyl-Moyal product have been studied in various contexts. For instance, see:

- [7] for such a study in the framework of tempered distributions on a symplectic vector space;
- [9] for a $C^\star$-algebraic study on $\mathbb{R}^d$-manifolds;
- [8] for non-tempered stable function spaces on $\mathbb{C}^n$.

A special feature of the Weyl-Moyal star product—indeed of the functional framework— is its maximal invariance under the group of affine transformations with respect to a flat affine connection. This can be rephrased by saying that the Weyl-Moyal quantization is universal with respect to actions of $\mathbb{R}^d$ [9]. A natural question is then the one of defining universal (convergent) deformations for non-Abelian Lie group actions. In the formal framework, this has been investigated in [9]. In [4, 5], such formulae in the case of $ax + b$ have been studied within the context of Wigner formalism and signal analysis. However, the question of defining an adapted functional framework has not been investigated. In [3, 6], one finds a functional analytic study for solvable Lie group actions and symmetric spaces in the tempered distributions and/or $C^\star$-context.

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It therefore appears quite challenging to investigate the problem of defining non-tempered (e.g. exponential growth) function spaces on such a non-Abelian Lie group which are stable under some left-invariant (convergent) star product. In other words, studying a situation where non-temperedness and non-linear invariance mix. This is what is done in this paper for the particular case of the group $ax + b$. More precisely, we first start by giving a construction of a left-invariant star product on the (symplectic) group manifold underlying the Lie group $ax + b$. This star product is obtained via an equivalence transformation $T$ performed on Moyal’s product (which is not left-invariant). The equivalence $T$ involves two ingredients: a partial Laplace transform and a family \( \{ \phi_{\nu,\gamma} : \mathbb{C} \to \mathbb{C} \}_{\nu,\gamma \in \mathbb{C}} \) of holomorphic maps. For special values of the parameters $\nu$ and $\gamma$ one refinds the functional calculus studied in \([2, 3]\). But for other values, one can define stable function algebras constituted by type-S functions. Using a holomorphic presentation of these spaces, one gets non-commutative algebra structures on spaces of entire functions on $\mathbb{C}^2$. Such a space contains functions of exponential growth, one therefore has a non-tempered invariant calculus.

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2. Formal star products on “$ax + b$”

In this section, we briefly recall results appearing in \([2, 3]\). Let $\mathcal{G}$ denote the Lie algebra of the group of affine transformations of the real line. Formally, one has $\mathcal{G} = \text{span}_\mathbb{R} \{ A, E \}$ with table \([A, E] = 2E\). Consider the linear map $\lambda : \mathcal{G} \to C^\infty(\mathbb{R}^2)$ : $X \mapsto \lambda_X$ defined by $\lambda_A(a, l) = 2l$; $\lambda_E(a, l) = e^{-2a}$, where $\mathbb{R}^2 = \{(a, l)\}$. One then checks that the map $\lambda$ is a homomorphism of Lie algebras when $C^\infty(\mathbb{R}^2)$ is endowed with the symplectic Poisson bracket $\{, \} := \partial_a \wedge \partial_l$. Moreover, if $\star^M_\nu$ denotes the formal Moyal star product on $C^\infty(\mathbb{R}^2)[[\nu]]$ (i.e. $u \star^M_\nu v = u \exp(\nu \partial_a \wedge \partial_l) v, u, v \in C^\infty(\mathbb{R}^2)[[\nu]]$), one has $[\lambda_A, \lambda_E]_\nu = 2\nu \lambda_A, \lambda_E \} (where $[u, v]_\nu := u \star^M_\nu v - v \star^M_\nu u$). In particular, the formula
\[
\rho_\nu(X)u := \frac{1}{2\nu} [\lambda_X, u]_\nu \quad X \in \mathcal{G}, u \in C^\infty(\mathbb{R}^2)[[\nu]]
\]
defines a homomorphism of Lie algebras
\[
\rho_\nu : \mathcal{G} \to \text{Der}(C^\infty(\mathbb{R}^2)[[\nu]], \star^M_\nu).
\]
Explicitly, one has $\rho_\nu(A)u = -\partial_a u; \quad \rho_\nu(E)u = -\frac{e^{-2a}}{\nu} \sinh(\nu \partial_l) u$. Intertwining the representation $\rho_\nu$ by a transformation of the type
\[
\mathcal{L}(u)(a, z) := \int_\mathbb{R} e^{-z l} u(a, l) \, dl,
\]
one gets
\[ \hat{\rho}_\nu(A)\mathcal{L}(u) := \mathcal{L}(\rho_\nu(A)u) = -\partial_
u \mathcal{L}(u); \]
\[ \hat{\rho}_\nu(E)\mathcal{L}(u) := \mathcal{L}(\rho_\nu(E)u) = -\frac{e^{-2a}}{\nu} \sinh(\nu z) \mathcal{L}(u), \]
where we assumed \( u(a, \pm\infty) = 0 \). Now, set formally
\[ Z_\nu(u)(a, z) := \int_\mathbb{R} \exp \left( \frac{1}{\nu} \sinh(\nu z) l \right) u(a, l) dl, \]
and
\[ f \circ_\nu g := Z_\nu(Z_\nu^{-1} f \circ Z_\nu^{-1} g) \quad (\gamma \in \mathbb{C}). \]

PROPOSITION 2.1. For all \( X \in \mathcal{G} \), \( \hat{\rho}_\nu(X) \) is a derivation of the commutative product \( \circ_\nu \).

In other words, the associative formal product \( u \circ_\nu v := T^{-1}(Tu \circ_\nu^M Tv) \) where \( T = \mathcal{L}^{-1}Z_\nu \) is invariant under the infinitesimal action \( \mathcal{G} \rightarrow \Gamma T(\mathbb{R}^2) : X \mapsto \hat{\nu}(d\lambda_X) \) where \( \hat{\nu}(d\lambda_X) \) denotes the Hamiltonian vector field associated (via the symplectic structure) to the function \( \lambda_X \). This action of \( \mathcal{G} \) turns out to exponentiate as a global simply transitive symplectic action of the group \( G = \{ ax + b \} \) on \( \mathbb{R}^2 \), providing an identification of the group manifold underlying \( G \) with \( \mathbb{R}^2 \).

The integral form of the transformation \( Z_\nu \) allows to define specific functions algebras on \( \mathbb{R}^2 \) (as opposed to power series algebras) stable under the product \( \circ_\nu \), where the formal Moyal product \( \circ_\nu^M \) is replaced by its "convergent" version: the Weyl product. The case where \( \nu \in i\mathbb{R}, \ \gamma = 1, \ z \in i\mathbb{R} \) has been studied in [3, 4]. In what follows, we are concerned with the general case where \( \nu \in i\mathbb{R}, \ \gamma \in U(1), \ z \in \mathbb{C} \). We end up this section by observing that the intertwiner \( T = \mathcal{L}^{-1}Z_\nu \) can be expressed as
\[ T = \mathcal{L}^{-1} \circ (\phi_{\nu,\gamma})^* \circ \mathcal{L} \]
where we set \( \phi_{\nu,\gamma}(z) := \frac{2}{\nu} \sinh(\nu z) \). The map \( \phi_{\nu,\gamma} \) will be referred in the sequel as the "twisting map" (see Section 5).

REMARK 2.2. An alternative simple way for obtaining an explicit formula of an invariant star product on \( ax + b \) is based on the following observation [1, 5]. The symplectic group manifold underlying \( ax + b \) can be seen as an open coadjoint orbit \( \mathcal{O} \) in \( \mathcal{G}^* \). Quantizing the Poisson manifold \( \mathcal{G}^* \) via the universal enveloping algebra product and then restricting to \( \mathcal{O} \) yields an invariant star product on \( \mathcal{O} \) hence a left-invariant one on \( ax + b \). It is classical that an oscillatory integral formula for this product can be written down in terms of the Campbell-Baker-Haussdorff function (see [1] for explicit computation: Formulae 5.8 and
5.12). Our product \(*_\nu\) described above is different from the universal enveloping algebra product. Indeed, their invariance diffeomorphism groups do not coincide \([2]\).

3. Fundamental spaces of type \(S\)

In this section, we follow Chapter IV of I.M. Guelfand’s book \([4]\). We will denote by \(\mathcal{O}(\mathbb{C}^m)\) the space of holomorphic (entire) functions on \(\mathbb{C}^m\).

DEFINITION 3.1. Let \(\alpha, \beta \in \mathbb{R}^m\). The fundamental space \(S^\beta_\alpha(m)\) is defined as the space of holomorphic functions \(\varphi \in \mathcal{O}(\mathbb{C}^m)\) such that there exists \(a, b \in (\mathbb{R}^+)^m\) and \(C > 0\) with

\[
|\varphi(x + iy)| \leq C \exp\left(-a|x|^\frac{1}{\alpha} + b|y|^{1-\beta}\right),
\]

where we adopt the usual notations: \(a|x|^e = \sum_j a_j |x_j|^{e_j}\) (\(a, x, e \in \mathbb{R}^m\)); \(\frac{1}{\alpha} = (\frac{1}{\alpha_1}, ..., \frac{1}{\alpha_m})\); \(1 - \beta = (1 - \beta_1, ..., 1 - \beta_m)\).

Every element \(\varphi \in S^\beta_\alpha(m)\) is entirely determined by its restriction the “real axis” \(\varphi(x)\) \(x \in \mathbb{R}^m\). We will often identify the space \(S^\beta_\alpha(m)\) with the subspace \((S^\beta_\alpha(m))_\mathbb{R}\) of \(C^\infty(\mathbb{R}^m)\) constituted by the restrictions.

In order to consider only non-trivial spaces, we will assume \(\alpha + \beta \geq 1; \alpha > 0; \beta > 0\). We will denote by \(\mathcal{F}(u)(\xi)\) the Fourier transform of the function \(u \in L^1(\mathbb{R}^m)\):

\[
\mathcal{F}(u)(\xi) := \int_{\mathbb{R}^m} e^{i\xi.x} u(x) \, dx,
\]

where \(\xi.x\) denotes the canonical dot product on \(\mathbb{R}^m\). For even \(m = 2n\), we will denote by \(J\) the endomorphism of \(\mathbb{R}^{2n}\) defined by the matrix

\[
[J] := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}
\]

where \(I_n\) is the \(n \times n\) identity matrix. We denote by \(\omega\) the bilinear symplectic structure on \(\mathbb{R}^{2n}\) defined by \(\omega(x, y) := x.Jy\).

DEFINITION 3.2. We define the symplectic Fourier transform of the function \(u \in L^1(\mathbb{R}^{2n})\) as

\[
S\mathcal{F}(u)(y) := \int_{\mathbb{R}^{2n}} e^{i\omega(x, y)} u(x) \, dx \quad (y \in \mathbb{R}^{2n}).
\]
Equivalently, one has $S \mathcal{F} = J^* \circ \mathcal{F}$, which yields (see [4])

**LEMMA 3.3.** One has

$$S \mathcal{F}(S^\beta_\alpha(2n)) = S^{\sigma(\alpha)}(2n),$$

where for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{2n} (\alpha_i \in \mathbb{R}^n)$, we set $\sigma(\alpha) := (\alpha_2, \alpha_1)$.

**DEFINITION 3.4.** For $u, v \in L^1(\mathbb{R}^{2n})$, one defines their twisted convolution by

$$u \times_q v(x) := \int_{\mathbb{R}^{2n}} e^{i\varphi(x,y)} u(y) v(x - y) \, dy \quad (q \in \mathbb{R}_0).$$

**LEMMA 3.5.** One has

$$S^\beta_\alpha(m) S^{\beta'}_{\alpha'}(m) \subset S^{\max(\beta, \beta')}_{\min(\alpha, \alpha')}(m);$$

where we set $\max(\beta, \beta') := (\max(\beta_1, \beta'_1), \ldots, \max(\beta_m, \beta'_m))$.

**Proof.** For simplicity, we assume $m = 1$. Let $\varphi \in S^\beta_\alpha$ and $\varphi' \in S^{\beta'}_{\alpha'}$. Then, for $z = x + iy$, one has

$$|\varphi \varphi'(z)| \leq C \exp \left( -a|x|^{\frac{1}{\alpha}} - a'|x|^{\frac{1}{\alpha'}} \right)$$

which is lower than $C' \exp \left( -a''|x|^{\max(\frac{1}{\alpha}, \frac{1}{\alpha'})} + b''|y|^{\max(\frac{1}{\beta}, \frac{1}{\beta'})} \right)$ for some $C', a'', b''$. But, $\max(\frac{1}{\alpha}, \frac{1}{\alpha'}) = \frac{1}{\min(\alpha, \alpha')}$ and $\max(\frac{1}{\beta}, \frac{1}{\beta'}) = 1 - \max(\beta, \beta')$.

**LEMMA 3.6.** Let $u, v \in S^\beta_\alpha(2n)$. Then $u \times_q v \in S^{\sigma(\alpha)}(2n)$.

**Proof.** Changing the variables following $y \mapsto -y$, one gets $u \times_q v = \left[ d_q^*_u S \mathcal{F}(\hat{u} \alpha_x v) \right]$ where $\hat{u}(x) := u(-x)$, $(\alpha_x v)(y) := v(y + x)$ and where $d_q$ denotes the dilation in $\mathbb{R}^{2n} : x \mapsto qx$ $(q \in \mathbb{R})$. Hence $u \times_q v \in S \mathcal{F} \left( S^{\max(\beta, \beta')}(2n) \right) = S^{\sigma(\alpha)}(2n)$.

**DEFINITION 3.7.** (see e.g. [3]) The Weyl product between $u$ and $v$ in $L^1(\mathbb{R}^{2n})$ is defined by

$$u \ast_q^W v := S \mathcal{F} \left[ S \mathcal{F}(u) \times_q S \mathcal{F}(v) \right].$$

**PROPOSITION 3.8.** Let $u, v \in S^\beta_\alpha(2n)$, Then $u \ast_q^W v \in S^{\sigma(\alpha)}(2n)$. In particular, the space $S^{\sigma(\alpha)}(2n)$ is stable under the Weyl product.
REMARK 3.9. The space $S_\alpha^{\sigma(\alpha)}(2n)$ is stable under the pointwise multiplication as well.

4. Laplace Transformation

In this section we follow L. Schwartz’ book [10]. We adopt the following notations. We denote by $\mathcal{D}$ the space of compactly supported smooth functions on $\mathbb{R}$ endowed with the topology of test functions. We denote by $\mathcal{D}'$ the space of distributions on $\mathbb{R}$. Also, if $\Omega$ is an open domain in $\mathbb{C}$, we set $\mathcal{O}(\Omega)$ for the space of holomorphic functions on $\Omega$.

DEFINITION 4.1. Let $\Gamma$ be an open interval in $\mathbb{R}$. The fundamental space $\mathcal{S}_\Gamma$ is defined as the space of distributions $T \in \mathcal{D}'$ such that for all $\xi \in \Gamma$, the distribution $\exp(-\xi x)T_x$ is tempered. We denote by $\mathcal{O}_\Gamma$ the space of holomorphic functions $F \in \mathcal{O}(\Gamma + i\mathbb{R})$ such that for all compact set $K \subset \Gamma$, the restriction $F|_{K + i\mathbb{R}}(\xi + i\eta)$ is bounded by a polynomial in $\eta$.

PROPOSITION 4.2. One defines the Laplace transform of an element $T \in \mathcal{S}_\Gamma$ as the Fourier transform of $e^{-\xi x}T_x$

$$\mathcal{L}(T)(\xi + i\eta) := (\mathcal{F}_x [\exp(-\xi x)T_x])(\eta).$$

Then, setting $z = \xi + i\eta \in \Gamma + i\mathbb{R}$, one has a linear isomorphism

$$\mathcal{L} : \mathcal{S}_\Gamma \to \mathcal{O}_\Gamma.$$

REMARK 4.3. Provided the following integrals make sense, one has

$$\mathcal{L}(T)(z) = \int_{\mathbb{R}} e^{-zx}T_x \, dx \quad \text{and} \quad \mathcal{L}^{-1} F(x) = \int_{c+i\mathbb{R}} e^{zx}F(z) \, dz,$$

where $c$ is any element of $\Gamma$. Indeed, if $F = \mathcal{L}T$ one has for all $c \in \Gamma$:

$$\int_{c+i\mathbb{R}} e^{zx}F(z) \, dz = e^{xc} \int_{\mathbb{R}} F(c + it)e^{ixt} \, dt = e^{xc} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it(x-y)} (T_ye^{-yc}) \, dy \, dt = e^{xc} \delta_x (T_ye^{-yc}) = T_x.$$

Now we assume $m = 1$ and set $S_\alpha^\beta(1) =: S_\alpha^\beta$.

LEMMA 4.4. For all $\xi \in \mathbb{R}$ the function $e^{-\xi x}$ is a multiplier in $S_\alpha^\beta$ ($\alpha < 1$).
Proof. Let \( \varphi \in \mathcal{S}_\alpha^\beta \). Then the function \( f(z) := e^{-\xi z} \varphi(z) \) is entire and one has
\[
|f(z)| \leq C \exp \left( -a|x|^\frac{1}{\alpha} + b|y|^{1-\beta} \right) e^{\xi|z|}
\]
which is lower than \( C'|\exp \left( -a'|x|^\frac{1}{\alpha} + b'|y|^{1-\beta} \right) \) as soon as \( \alpha < 1 \).

In particular, one has \( \mathcal{S}_\alpha^\beta \subset \mathcal{S}'_\mathbb{R} = \cap \mathcal{S}'_\Gamma \) as soon as \( \alpha < 1 \).

**PROPOSITION 4.5.** The Laplace transformation yields a linear isomorphism
\[
\mathcal{L} : \left( \mathcal{S}_\alpha^\beta \right)_x \to J^* \mathcal{S}_\beta^x
\]
where \( J : \mathbb{C} \to \mathbb{C} \) denotes the multiplication by \( i = \sqrt{-1} \).

**Proof.** Let \( \varphi \in \mathcal{S}_\alpha^\beta \). Then \( (J^* \mathcal{L}(\varphi))(x + iy) = \int_{\mathbb{R}} e^{xt} e^{-yt} \varphi(t) \, dt \). Hence \( (J^* \mathcal{L}(\varphi))|_{x} (x) = (\mathcal{F} \varphi)(x) \in \mathcal{S}_\beta^x \). Thus \( \mathcal{L} \varphi \in J^* \mathcal{S}_\beta^x \). Now let \( f \in J^* \mathcal{S}_\beta^x \). One has \( |f(x + iy)| \leq C \exp \left( -a|y|^\frac{1}{\alpha} + b|x|^{1-\beta} \right) \) which guarantees that \( f \in \mathcal{O}_\mathbb{R} \). Therefore \( \mathcal{L}^{-1} f = e^{xt} \int_{\mathbb{R}} e^{yt} f(c + it) \, dt \) \( (c \in \mathbb{R}) \). Choosing \( c = 0 \), one gets \( \mathcal{L}^{-1} f = \mathcal{F} J^* f \in \mathcal{S}_\beta^x \). One therefore has an isomorphism \( \mathcal{L}^{-1} : J^* \mathcal{S}_\alpha^\beta \to \mathcal{S}_\alpha^\beta \).

5. **Twisting maps**

**DEFINITION 5.1.** Let \( q \in \mathbb{R} \) and \( \theta \in [0, 2\pi[. \) We define the twisting map \( \phi_{q, \theta} : \mathbb{C} \to \mathbb{C} \) by
\[
\begin{align*}
\phi_{q, \theta}(z) &= \frac{e^{i\theta}}{q} \sinh(iqz) \quad \text{if } q \neq 0 \\
\phi_{0, \theta}(z) &= z.
\end{align*}
\]

**LEMMA 5.2.** The twisting map \( \phi_{q, \theta} \) \( (q \neq 0) \) establishes a biholomorphic diffeomorphism
\[
\phi_{q, \theta} : S_q := \{ z \in \mathbb{C} | |Re(z)| < \frac{\pi}{2q} \} \to \mathbb{C} - \{ \pm i \frac{1}{q}, \infty \}.
\]

**Proof.** In coordinates \( z = x + iy \), the twisting map is
\[
\phi_{q, \theta}(x, y) = \left( \frac{1}{q} \sinh(qy) \cos(qx), \frac{1}{q} \cosh(qy) \sin(qx) \right).
\]
In particular, the imaginary axis \( x = 0 \) is sent onto the real axis \( y = 0 \), while the image of the vertical line \( x = \pm \frac{\pi}{2q} \) under \( \phi_{q, \theta} \) is the half imaginary line \( \pm i \frac{1}{q}, \infty \). At last, for \( c \in [0, \frac{\pi}{2q}[, \) the image of the vertical line \( x = \pm \frac{\pi}{2q} \) is the branch of hyperbola \( \{ -(\frac{q}{\cos \epsilon})^2 + (\frac{q}{\sin \epsilon})^2 = 1 \} \cap \{ \pm y > 0 \} \). For \( \epsilon \in ]0, \frac{\pi}{2q}[ \), one therefore gets a holomorphic diffeomorphism between the strip \(- \frac{\pi}{q} \leq y \leq \frac{\pi}{q} + i \mathbb{R} \) and the region \( \{ -(\frac{q}{\cos \epsilon})^2 + (\frac{q}{\sin \epsilon})^2 < 1 \} \).
As explained in Section 2, we are interested in considering transformations of the type $L^{-1} \circ \phi_{q,\theta}^* \circ L$ or $L^{-1} \circ (\phi_{q,\theta}^{-1})^* \circ L$. Let $F$ be a function defined on some domain $\Omega$ of $\mathbb{C}$. In order to define $L^{-1} (\phi_{q,\theta}^{-1})^* F$, we want $(\phi_{q,\theta}^{-1})^* F \in \mathcal{O}_\Gamma$. In particular we want $(\phi_{q,\theta}^{-1})^* F$ to be defined on vertical lines. This imposes to $\theta$ to be an integral multiple of $\frac{\pi}{2}$.

**Proposition 5.3.** Let $\alpha, \beta \in ]0, 1[$ be such that $\alpha + \beta \geq 1$.

(i) For all open interval $I$ of positive numbers ($I \subset \mathbb{R}_0^+$), one has the injection $(\phi_{q,0}^{-1})^* : J^* \mathcal{S}_\alpha^\beta \to \mathcal{O}_I \cap \mathcal{O}_-I$.

(ii) For all open interval $I \subset ]-\frac{\pi}{2}, \frac{\pi}{2}[$, one has the injection $(\phi_{q,\pi/2}^{-1})^* : J^* \mathcal{S}_\beta^\alpha \to \mathcal{O}_I$.

**Proof.** Let $F \in J^* \mathcal{S}_\beta^\alpha$ and set for simplicity $\phi = \sinh(iz)$. Let $I \subset ]0, \infty[$ and $K$ be a compact set in $I$. Then $\phi^{-1}$ is well defined on the strip $K + i\mathbb{R}$ and $\phi^{-1}(K + i\mathbb{R}) \subset ]-\frac{\pi}{2}, \frac{\pi}{2}[ + i\mathbb{R}$ (cf. Lemma 5.2). Now, consider the integral $I_K := \int_{\phi^{-1}(K + i\mathbb{R})} |(\phi^{-1})^* F(z)| dz \wedge \overline{dz}$. If $I_K < \infty$ for all $K$, then $(\phi^{-1})^* F \in \mathcal{O}_I$. Changing the variables $z = \phi(w)$, one gets $I_K = \int_{\phi^{-1}(K + i\mathbb{R})} |F(w)| |\text{Jac}_\phi(w)||dw \wedge d\overline{w}|$ where $\text{Jac}_\phi$ is the Jacobian determinant of $\phi$. A computation yields $\text{Jac}_\phi(w) = \cos^2(x) + \sinh^2(y)$ ($w = x + iy$) and, since $\phi^{-1}(K + i\mathbb{R}) \subset ]-\frac{\pi}{2}, \frac{\pi}{2}[ + i\mathbb{R}$, one gets $I_K \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\infty}^{\infty} \exp \left(-a|y|^{\frac{1}{\beta}} + b|x|^{\frac{1}{\alpha}}\right) \left(\cos^2(x) + \cosh^2(y)\right) dy \, dx$ which is lower than $2b \exp \left[\left(\frac{\pi}{2}\right)^{\frac{1}{\alpha}}\right] \int_{-\infty}^{\infty} \exp \left(-a|y|^{\frac{1}{\beta}}\right) (1 + \cosh^2(y)) dy$, thus finite as soon as $\beta < 1$. The exact same argument works for $I \subset ]-\frac{\pi}{2}, 0[$, hence $(\phi^{-1})^* F \in \mathcal{O}_I \cap \mathcal{O}_-I$. A similar argument yields item (ii).
6. Twisted Weyl product

6.1. The product formula

Let us consider the fundamental space $S^{σ(α_1, α_2)}_0(2)$ ($α_1, α_2) ∈ ℝ^2$ (cf. Section 3). Let $φ ∈ S^{σ(α_1, α_2)}_0(2)$ and consider the partial function $φ_{x_1} : x → φ(x_1, x)$. For all $x_1 ∈ ℝ$, the function $φ_{x_1}$ belongs to $S^{σ(α_1, α_2)}_0(2)$. Therefore provided some restrictions on $(α_1, α_2)$, the function $L^{-1}(φ^{-1}_{q, k_2})^*L(φ_{x_1})$ ($k = 0, 1$) is well defined as an element of $S'_i$ (cf. Proposition 5.3).

**DEFINITION 6.1.** We define the linear map

$S^{σ(α_1, α_2)}_0(2) → C^∞(ℝ^2)$ ($k = 0, 1$)

by

$τ_q^{(k)} := id_{x_1} ⊗ L^{-1}(φ^{-1}_{q, k_2})^*L_{x_2}$

We denote by $E^{(k)}_{(α_1, α_2)}$ its range in $C^∞(ℝ^2)$. The inverse map $id_{x_1} ⊗ (L^{-1}(φ^{-1}_{q, k_2})^*L_{x_2})_{E^{(k)}_{(α_1, α_2)}}$ will be denoted by $T_q^{(k)}$. It yields a linear isomorphism $T_q^{(k)} : E^{(k)}_{(α_1, α_2)} → S^{σ(α_1, α_2)}_0(2)$.

**PROPOSITION 6.2.** The formula

$u *_q^{(k)} v := τ_q^{(k)} (T_q^{(k)} u * W T_q^{(k)} v)$

defines an associative $ℝ$-algebra structure $*_q^{(k)}$ on $E^{(k)}_{(α_1, α_2)}$.

**Proof.** One has $T_q^{(k)} u, T_q^{(k)} v ∈ S^{σ(α_1, α_2)}_0(2)$ provided $u, v ∈ E^{(k)}_{(α_1, α_2)}$. We know that $S^{σ(α_1, α_2)}_0(2)$ is stable under Weyl’s product $*^W_q$ (cf. Proposition 5.8). Hence $u *_q^{(k)} v$ is well defined as an element of $E^{(k)}_{(α_1, α_2)}$. The associativity is obvious since $*_q^{(k)}$ is nothing else than the transportation of Weyl’s product via the isomorphism $T_q^{(k)} : E^{(k)}_{(α_1, α_2)} → S^{σ(α_1, α_2)}_0(2)$.

**DEFINITION 6.3.** The product $*_q^{(k)}$ on $E^{(k)}_{(α_1, α_2)}$ will be referred as the twisted Weyl product.
6.2. Observables of exponential type

Given a convergent star product, an important question is the one of existence of non-tempered observables in the domain of the star product. This question has been studied for the case of the Moyal-Weyl product in [8].

Let \( A = (\alpha_1, \alpha_2) \in [0, 1]^2 \) with \( \alpha_1 + \alpha_2 \geq 1 \). Set \( S_A := S^{(\alpha_1, \alpha_2)}(2) \).

Viewing \( S_A = (S_A)_x \ (x \in \mathbb{R}^2) \) as a subspace of \( C^\infty(\mathbb{R}^2) \), we consider the following alternate presentation of \( S_A \).

Consider the sequence of maps

\[
(S_A)_x \rightarrow O(\mathbb{C}^2) \rightarrow C^\infty(\mathbb{R}^2)
\]

\( f(x_1, x_2) \mapsto f(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2) \mapsto f(iy_1, x_2) \).

The function \( \hat{f}(y_1, x_2) := f(iy_1, x_2) \) determines completely \( f \). So that we have an injection \( (S_A)_x \rightarrow C^\infty(\mathbb{R}^2) : f \mapsto \hat{f} \). Remark that the space \( \hat{S}_A \) contains elements of exponential growth. For example, one has \( f = e^{-(x_1^2 + x_2^2)} \in S^{(\frac{1}{2}, \frac{1}{2})} \), which yields \( |\hat{f}(y_1, x_2)| = e^{y_1^2 + x_2^2} \).

We denote by \( \hat{\star}_q \) the product on \( \hat{S}_A \) obtained by transporting Weyl’s product on \( (S_A)_x \) via \( f \mapsto \hat{f} \). Observe that \( \hat{S}_A \) is still stable under the pointwise multiplication whose \( \hat{\star}_q \) is a non-commutative deformation of. Observe also that for every \( \psi \in \hat{S}_A \) and \( y_1 \in \mathbb{R} \) the partial function \( x_2 \mapsto \psi(y_1, x_2) \) is in \( S^{(\alpha_1)}_{\alpha_2} \). Therefore the transformations \( \tau_q^{(k)} \) and \( T_q^{(k)} \) are well defined on \( \hat{S}_A \).

**Proposition 6.4.** Set \( \widehat{E}_{(\alpha_1, \alpha_2)}^{(k)} := \tau_q^{(k)}(\hat{S}_A) \). Then for all \( a, b \in \widehat{E}_{(\alpha_1, \alpha_2)}^{(k)} \), the formula

\[
a \hat{\star}_q^{(k)} b := \tau_q^{(k)} \left( T_q^{(k)} a \hat{\star}_q^{(k)} T_q^{(k)} b \right)
\]

defines an associative \( \mathbb{R} \)-algebra structure on \( \widehat{E}_{(\alpha_1, \alpha_2)}^{(k)} \). The space \( \widehat{E}_{(\alpha_1, \alpha_2)}^{(k)} \) contains elements of exponential growth.

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