Affine Solitons: A Relation Between
Tau Functions, Dressing and Bäcklund Transformations.

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Abstract.
We reconsider the construction of solitons by dressing transformations in the
sine-Gordon model. We show that the \( N \)-soliton solutions are in the orbit
of the vacuum, and we identify the elements in the dressing group which
allow us to built the \( N \)-soliton solutions from the vacuum solution. The
dressed \( \tau \)-functions can be computed in two different ways: either using
adjoint actions in the affine Lie algebra \( \hat{sl}_2 \), and this gives the relation with
the Bäcklund transformations, or using the level one representations of the
affine Lie algebra \( \hat{sl}_2 \), and this directly gives the formulae for the \( \tau \)-functions
in terms of vertex operators.
1 Introduction.

The group of dressing transformations is the classical precursor of the quantum group symmetry of an integrable system. It acts on the phase space, and its most remarkable property is that the action is a Lie-Poisson action i.e. to preserve the Poisson brackets, the group itself has to carry a non trivial Poisson bracket \[^1\]. This is why after quantization the group of dressing transformations becomes a quantum group.

Already at the classical level, this group plays a fundamental role in our understanding of the structure of integrable two dimensionnal field theories \[^2,\ 3,\ 4\]. In particular, it allows to organize the fields into multiplets closed by dressing, and satisfying a classical exchange algebra. In the case of a conformal field theory, e.g. the Liouville theory, these multiplets are precisely the classical analogues of the degenerate primary fields for which one can compute correlation functions.

In the case of sine-Gordon, the group of dressing transformations is infinite dimensional, and there are presumptions that the phase space of the theory is just one orbit of the dressing group. If this were the case, the construction of the quantum sine-Gordon theory would just reduce to an exercise in quantum group theory.

We will be interested in this paper to another aspect of the group of dressing transformations i.e. its relation to soliton solutions of the sine-Gordon model. In the sine-Gordon model, particles are soliton-antisoliton bound states, and therefore solitons are adequate to describe the asymptotic states of the theory \[^5\]. Moreover the symmetries of the theory can be expressed conveniently in the soliton state basis. One example is provided by the use of local conserved charges for implementing the bootstrap program \[^6,\ 7\]. Another important example is the quantum group symmetry of an integrable theory \[^8,\ 9\]. These symmetries can be used for instance to determine the S-matrix for the solitons. There is some hope that correlation functions might also be determined from this quantum symmetry. We therefore expect a deep connection between solitons and dressing transformations, and after quantization, between solitons and quantum group symmetry.

There are at least two methods to generate solitons from the vacuum solution. One is through the Riemann-Hilbert problem and is attached to the Zakharov-Shabat scheme \[^10,\ 11\]. The other one occurs in connexion with \(\tau\)-functions and transformation groups for soliton equations \[^12\]. The
first approach has the advantage that its Lie-Poisson properties are now well understood \[1, 2, 3, 4\] and this is exactly what will become the quantum group symmetry of the quantum theory. In the second approach, solitons are expressed in terms of vertex operators by means of remarkably simple formulae which are deeply rooted to the algebraic structure of the theory.

The aim of this work is to establish the precise relation between these two descriptions of the solitons in the sine-Gordon model.

In the first approach, the dressing group is defined as an algebraic version of the Riemann-Hilbert problem. The fields $\Phi^g$ in the orbit of the vacuum are given by the formula

$$e^{-2\Lambda_{\pm}(\Phi^g)} = \xi_{\text{vac}}^{\pm}, g_{\pm}^{-1} \cdot g_{\mp}^{\pm}$$

where $\xi_{\text{vac}}^{\pm}$ and $\xi_{\text{vac}}^{-}$ characterize the vacuum, $\Lambda_{\pm}$ are the two fundamental highest weights of the affine Lie algebra $\hat{sl}_2$, and $g_{\pm}^{-1}, g_{\mp}$ are triangular elements of the dressing group.

In the second approach, the $\tau$-functions, which form orbits of an infinite dimensional transformation group, are defined as expectation values in the affine group $\hat{SL}_2$:

$$\tau^g_{\pm}(z_-, z_+) = <\Lambda_{\pm}|e^{-mz_+ \mathcal{E}_+} g e^{mz_- \mathcal{E}_-} \Lambda_{\pm}>$$

Here, $\mathcal{E}_\pm$ are special elements of the affine algebra $\hat{sl}_2$, and $g$ is an element of the transformation group.

The two formulae are in fact identical and we have

$$\tau^g_{\pm}(z_-, z_+) = e^{-2\Lambda_{\pm}(\Phi^g)}$$

This shows explicitly that the transformation group of the $\tau$-function is just the group of dressing transformations and $g = g_{\pm}^{-1} g_{\mp}$. Moreover this group is not the affine group. Its composition law is given by

$$h \bullet g = (h g_{\mp})^{-1} h_{\pm} g_{\pm}$$

We will first identify which elements of the dressing group $g_{\pm}^{-1}, g_{\mp}$ generate the $N$-soliton solutions. They are found to be of the form:

$$g_{\pm}^{-1} = g_{\pm}^{-1}(1) \cdots g_{\pm}^{-1}(N)$$

$$g_{\mp} = g_{\mp}(N) \cdots g_{\mp}(1)$$
Each factor is given by

\[ g_-(k) = e^{f_k(0)}e^{\frac{1}{2}h_k(0)} e^{\frac{1}{2}g_k(0)} = e^{f_k(0)}e^{\frac{1}{2}h_k(0)} e^{\frac{1}{2}g_k(0)}K \]

\[ g_+(k) = e^{\frac{1}{2}g_k(0)}K e^{-\frac{1}{2}h_k(0)} = e^{f_k(0)}e^{\frac{1}{2}h_k(0)} e^{\frac{1}{2}g_k(0)}V_k(l) \]

where \( V^\pm \), \( H \), \( K \), are special elements in the affine Lie algebra \( \hat{sl}_2 \). We will then compute the \( \tau \)-functions in two different ways:

- **The first way** consists in commuting \( e^{-mz_+H} \) and \( e^{mz_-H} \) through \( g = g_-g_+ \) using the algebraic structure of \( \hat{sl}_2 \). In doing so, the elements \( g_\pm(k) \) remain of the same form, but \( f_k, h_k, g_k \) become functions depending on \( z_\pm \). These functions are shown to satisfy differential equations which are directly related to Bäcklund transformations. We present an algebraic solution to them.

- **The second way** consists in evaluating the \( \tau \)-functions by computing the expectation values in the level one representation. We use the level one vertex operator representation in the principal gradation constructed from a single \( Z_2 \) twisted free boson. The factorized elements \( g = g_-g_+ \) turn out to be expressible as a product of normal ordered vertex operators. The result is remarkably simple and leads directly to the formula of the Kyoto group.

\[
\tau^{(N)}_\pm(z_+, z_-) = \tau_0(z_+, z_-) < \Lambda^\pm | \prod_{i=1}^{N}(1 + 2X_iV(\mu_i))|\Lambda^\pm >
\]

where \( V_{\mu_i} = V^{(+)}_{\mu_i} + V^{(-)}_{\mu_i} \) is the vertex operator and \( X_i = a_i e^{2m(\mu_{i}z_+ + \mu_{i}^{-1}z_-)} \)

are the parameters of the \( N \) solitons. \( \tau_0 \) is the vacuum \( \tau \)-function.

We do not touch here the question of the relation between the known Poisson structure of the group of dressing transformations and the soliton solutions. The hope being that when \( N \to \infty \) the \( N \)-soliton solutions become dense in the orbit of the vacuum and we would therefore obtain a convenient coordinate system on the group of dressing transformations. We will study these questions elsewhere.

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2 Lie algebra conventions.

Let $E_+, E_-, H$ be the three generators of the Lie algebra $sl_2$

$$\begin{align*}
[H, E_+] &= \pm 2E_+ \\
[E_+, E_-] &= H
\end{align*}$$

We normalize the trace on $sl_2$ to $tr(HH) = 2$, $tr(E_+E_-) = 1$. The loop algebra $\tilde{sl}_2$ is the Lie algebra of traceless $2 \times 2$ matrices with entries which are Laurent polynomials in $\lambda$: $\tilde{sl}_2 = C(\lambda, \lambda^{-1}) \otimes sl_2$. The affine Lie algebra $\hat{sl}_2$ is the central extension of $\tilde{sl}_2$: $\hat{sl}_2 = \tilde{sl}_2 \oplus CK \oplus Cd$, with $K$ the central element and $d$ the derivation $d = \lambda \frac{\partial}{\partial \lambda}$. The bracket reads

$$[\hat{X}, \hat{Y}] = [\tilde{X}, \tilde{Y}] + \frac{1}{2} \oint \frac{d\lambda}{2i\pi} tr(\partial_\lambda \tilde{X}(\lambda) \cdot \tilde{Y}(\lambda)) K$$

We have the Cartan decomposition: $\hat{sl}_2 = \hat{N}_- \oplus \hat{H} \oplus \hat{N}_+$. In the principal gradation we have:

$$\begin{align*}
\hat{H} &= \{H, d, c\} \\
\hat{N}_+ &= \{E_+^{(2n-1)} = \lambda^{2n-1}E_+, E_-^{(2n-1)} = \lambda^{2n-1}E_-, H^{(2n)} = \lambda^{2n}H, n > 0\} \\
\hat{N}_- &= \{E_+^{(2n+1)} = \lambda^{2n+1}E_+, E_-^{(2n+1)} = \lambda^{2n+1}E_-, H^{(2n)} = \lambda^{2n}H, n < 0\}
\end{align*}$$

The explicit form of the commutation relations are:

$$\begin{align*}
[H^{(r)}, H^{(s)}] &= Kr \delta_{r+s,0} \\
[H^{(r)}, E_+^{(s)}] &= \pm 2E_+^{(r+s)} \\
[E_+^{(r)}, E_-^{(s)}] &= H^{(r+s)} + \frac{K}{2} r \delta_{r+s,0}
\end{align*}$$

In particular, the simple roots vectors will be taken to be $E_{\pm \alpha_1} = \lambda^{\pm 1}E_\pm$ and $E_{\pm \alpha_2} = \lambda^{\pm 1}E_\mp$. We need to define

$$\begin{align*}
\mathcal{E}_+ &= \lambda(E_+ + E_-) \\
\mathcal{E}_- &= \lambda^{-1}(E_- + E_+)
\end{align*}$$

The affine $\tilde{sl}_2$ algebra possesses two fundamental highest weights, denoted $\Lambda^-$ and $\Lambda^+$. They are characterized by:

$$\begin{align*}
\Lambda^\pm(H) &= \pm \frac{1}{2} \\
\Lambda^\pm(K) &= 1 \\
\Lambda^\pm(d) &= 0
\end{align*}$$
The affine Sine-Gordon model.

The affine sinh-Gordon model is a Toda model over the affine $\hat{sl}_2$ algebra. Let $z_{\pm} = x \pm t$ be the light cone coordinates, $\partial_{z_{\pm}} = \frac{1}{2}(\partial_x \pm \partial_t)$. Introduce the field $\Phi$ with values in the Cartan subalgebra of $\hat{sl}_2$

$$\Phi = \frac{1}{2} \varphi H + \eta d + \frac{1}{4} \zeta K$$

Following the general construction of Toda systems, we define a Lax connexion $A_{z_{\pm}}$ by

$$A_{z_{+}} = \partial_{z_{+}} \Phi + me^{ad} \mathcal{E}_{+}$$

$$A_{z_{-}} = -\partial_{z_{-}} \Phi + me^{-ad} \mathcal{E}_{-}$$

and write the associated linear system

$$(\partial_{z_{\pm}} + A_{z_{\pm}})T(x, t) = 0$$

where $T(x, t)$ belongs to a group whose Lie algebra is $\hat{sl}_2$. The matrix $T(x, t)$ is called the transfer matrix. We normalized it by imposing $T(0) = 1$. The equations of motion are the compatibility conditions for this linear problem. These are the zero curvature condition

$$F_{z_{+}z_{-}} = \partial_{z_{+}} A_{z_{-}} - \partial_{z_{-}} A_{z_{+}} + [A_{z_{+}}, A_{z_{-}}] = 0$$

which can be worked out using only the Lie algebra structure of $\hat{sl}_2$. We get

$$\partial_{z_{+}} \partial_{z_{-}} \varphi = m^2 e^{2\eta}(e^{2\varphi} - e^{-2\varphi})$$

$$\partial_{z_{+}} \partial_{z_{-}} \eta = 0$$

$$\partial_{z_{+}} \partial_{z_{-}} \zeta = m^2 e^{2\eta}(e^{2\varphi} + e^{-2\varphi})$$

Thanks to the field $\eta$, the above equations are conformally invariant. Moreover they are also Hamiltonian.

In this paper, we will only be interested in the $\eta = 0$ sector. There, the equations of motion of the field $\varphi$ are those of the sinh-Gordon model. Changing $\varphi$ to $i\varphi$ we obtain the sine-Gordon model but we will not discuss here these reality conditions.
As usual, to any highest weight vector $|\Lambda_{\text{max}}\rangle$ we associate two sets of fields $\xi(x, t)$ and $\overline{\xi}(x, t)$ defined by [14] :

$$
\xi(x, t) = \langle \Lambda_{\text{max}} | e^{-\Phi} T(x, t)
$$

$$
\overline{\xi}(x, t) = T(x, t)^{-1} e^{-\Phi} |\Lambda_{\text{max}}\rangle
$$

These fields, which are the classical analogues of the quantum vertex operators, are chiral:

$$
\partial_{z-} \xi = 0 \quad \partial_{z+} \overline{\xi} = 0
$$

The Toda field can be reconstructed from them; the reconstruction formula reads

$$
e^{-2\Lambda_{\text{max}}(\Phi)} = \xi(z_+) \cdot \overline{\xi}(z_-)
$$

The vacuum solution of the equations of motion is

$$
\varphi_{\text{vac}} = 0 \quad \eta_{\text{vac}} = 0 \quad \zeta_{\text{vac}} = 2m^2 z_+ z_-
$$

One can insert this solution into the linear system and compute the vacuum transfer matrix $T_{\text{vac}}(x, t)$. We get

$$
T_{\text{vac}}(x, t) = e^{-\frac{m^2}{2} z_+ z_- K} e^{-m z_+ \mathcal{E}_+} e^{-m z_- \mathcal{E}_-}
$$

$$
= e^{\frac{m^2}{2} z_+ z_- K} e^{-m z_- \mathcal{E}_-} e^{-m z_+ \mathcal{E}_+}
$$

In the last equation, we used $[\mathcal{E}_+, \mathcal{E}_-] = K$. We can then compute the chiral fields $\xi(z_+)$ and $\overline{\xi}(z_-)$. We find :

$$
\xi_{\text{vac}}(z_+) = \langle \Lambda_{\text{max}} | e^{-m z_+ \mathcal{E}_+}
$$

$$
\overline{\xi}_{\text{vac}}(z_-) = e^{m z_- \mathcal{E}_-} |\Lambda_{\text{max}}\rangle
$$

Notice that the reconstruction formula gives

$$
\xi_{\text{vac}}(z_+), \overline{\xi}_{\text{vac}}(z_-) = \exp\left(-m^2 z_+ z_- \Lambda_{\text{max}}(K)\right)
$$

in agreement with eq.(9).
4 Dressing Transformations

4.1 Generalities.

Dressing transformations are special symmetries of non-linear differential equations. They are defined as gauge transformations acting on the Lax connexion and preserving its form. One of their most remarkable property is that they induce Lie-Poisson actions on the phase space \[ \mathbb{P} \]. As dressing symmetries have already been fully described in \[ \mathbb{P} \mathbb{P} \] we restrict ourselves in giving only the basic facts concerning them that we will need in the following. We therefore describe only the Toda model.

The dressing transformations are associated to a factorization problem \[ \mathbb{P} \mathbb{P} \]. For a Toda field theory over an algebra \( \mathcal{G} \), this problem is a factorization problem in the group \( \mathcal{G} \) whose Lie algebra is \( \mathcal{G} \). In the case of the affine sinh-Gordon model, the group \( \mathcal{G} \) is an infinite dimensional group whose Lie algebra is the affine algebra \( \hat{sl}_2 \); namely \( \mathcal{G} \equiv \hat{SL}_2 \equiv (\exp \hat{H})(\exp \hat{N}_-)(\exp \hat{N}_+) \). The factorization problem needed to define the dressing transformations is as follows. Any element \( g \in \hat{SL}_2 \) is decomposed as :

\[
g = g_{-1}^- g_+ \quad \text{with} \quad g_\pm \in B_\pm = (\exp \hat{H})(\exp \hat{N}_\pm)
\]

and moreover we require that \( g_- \) and \( g_+ \) have inverse components on the Cartan torus. In practice it is given by half splitting the Gaussian decomposition of \( g \). The infinitesimal version of this factorization problem consists in decomposing any element \( X \in \mathcal{G} = \hat{sl}_2 \) as

\[
X = X_+ - X_- \quad \text{with} \quad X_\pm \in (\hat{H} \oplus \hat{N}_\pm)
\]

such that \( X_+ \) and \( X_- \) have opposite components on \( \hat{H} \).

For any element \( g \in \hat{SL}_2 \) with factorization \( g = g_-^- g_+ \), the dressing transformations are defined by the following gauge transformation :

\[
T(x) \rightarrow T^g(x) = \Theta_\pm(x) \ T(x) \ g_\pm^{-1}
\]  

(10)

where \( \Theta_\pm(x) \) are given by factorizing in \( \hat{SL}_2 \) the element \( T(x)gT^{-1}(x) \):

\[
\Theta_-^{-1}(x) \ \Theta_+(x) = T(x) \ g \ T^{-1}(x)
\]  

(11)

Notice that the gauge transformation \( \mathbb{P} \mathbb{P} \) can be implemented either using \( \Theta_- \) and \( g_- \) or using \( \Theta_+ \) and \( g_+ \): the result is the same thanks to eq. \( \mathbb{P} \mathbb{P} \). Also
these transformations preserve the normalization condition $T(0) = 1$. They induce gauge transformations on the Lax connexion: $A_\mu = -(\partial_\mu T) T^{-1}$ is transformed into $A_\mu^g = -(\partial_\mu T^g) T^{g-1}$ with

$$A_\mu^g = \Theta_\pm A_\mu \Theta_\pm^{-1} - \partial_\mu \Theta_\pm \Theta_\pm^{-1}$$

The factorization problem described above is devised precisely in order that the form of the Lax connexion is preserved by these transformations. The proof of this statement was given in [2]. It relies on the fact that the gauge transformations can be implemented using either $\Theta_-$ or $\Theta_+$. One first shows that the degrees of the components of the Toda Lax connexion are preserved by the dressing and then, one verifies that the connexion can be written as in eqs. (4,5).

Because the form of the Lax connexion is preserved by these transformations, the gauge transformations (10) induce transformations of the Toda fields:

$$\Phi(x, t) \rightarrow \Phi^g(x, t)$$

Moreover, since the equations of motion are equivalent to the zero curvature condition and because gauge transformations preserve this condition, the dressing transformations are symmetries of the equations of motion. In other words, a dressing transformation maps a solution of the Toda equations $\Phi(x, t)$ into another solution $\Phi^g(x, t)$ (which, in general, possess non-trivial topological numbers).

The transformations of the Toda fields can be described as follows. Factorize $\Theta_\pm$ as:

$$\Theta_\pm(x) = K^g_\pm(x) M^g_\pm(x)$$

with $M^g_\pm \in \text{exp}\, \hat{N}_\pm$ and $K^g_\pm \in \text{exp}\, \hat{H}$. According to the factorization problem, the components of $\Theta_-$ and $\Theta_+$ on the Cartan torus are inverse: $K^- g K^+_g = 1$. Put

$$K^g_\pm(x) = \text{exp}(\pm \Delta^g(x)) \quad \text{with} \quad \Delta^g \in \hat{H}$$

Then, by looking at the exact expression of the transformed Lax connexion $A^g_\mu$, one deduces that:

$$\Phi^g(x) = \Phi(x) - \Delta^g(x) \quad (12)$$
The relation between $\Phi^g$ and $\Phi$ is non-local because $\Delta^g$ are expressed in a non-local way in terms of $\Phi$. We have:

$$\exp(2\Lambda_{\text{max}}(\Delta^g(x))) = \langle \Lambda_{\text{max}} | T(x)gT^{-1}(x) | \Lambda_{\text{max}} \rangle$$

for any highest weight $\Lambda_{\text{max}}$.

It should be noted that the composition law in the group of dressing transformation is not the composition law in the group $G$ on which the Toda model is defined. Indeed, in the dressing group the product of two elements $(g_-, g_+)$ and $(h_-, h_+)$ is:

$$(g_-, g_+) \cdot (h_-, h_+) = (g_-h_-, g_+h_+)$$

In particular, the plus and minus components commute. The brackets $[X, Y]_R$ in the dressing Lie algebra follows from the composition for infinitesimal group elements $g = g_+^{-1}g_+ \sim 1 + X_+ - X_-$ and $h = h_+^{-1}h_+ \sim 1 + Y_+ - Y_- :$

$$[X, Y]_R = [X_+, Y_+] - [X_-, Y_-]$$

As we already mentionned, the action of dressing transformations is not a symplectic action, but a Lie-Poisson action. This means that the Poisson brackets transform covariantly only if the group of dressing transformations is equipped with a non-trivial Poisson structure. The Poisson structure on the dressing group is given by the Semenov-Tian-Shansky Poisson brackets.

Since the dressing transformations are not symplectic they are not generated by Hamiltonian functions. However there exists a non-Abelian generalization of these Hamiltonians which applies to Lie-Poisson actions. The non-Abelian Hamiltonian generating the dressing transformations is the monodromy matrix $T$. Namely, the variation $\delta_X \Phi(x, t)$, with $X = X_+ - X_- \in \hat{sl}_2$, of the field $\Phi(x, t)$ under an infinitesimal dressing transformation is given by:

$$\delta_X \Phi(x, t) = Tr(\ X T^{-1}\{T, \Phi(x, t)\})$$

where $Tr$ denote the trace on the affine algebra $\hat{sl}_2$ and $\{, \}$ the Poisson brackets on the phase space. The relation (13) is enough to prove that the dressing transformations have a Lie-Poisson action and this was used in \cite{2} to describe the relation between dressing and quantum group symmetries.
4.2 Tau-functions and dressings

In Toda field theories, the number of τ-functions is the rank of the Lie algebra. They are defined as the expectation value of the exponential of the Toda field between the fundamental highest weight vectors $|\Lambda_{\text{max}}^{(i)}\rangle$:

$$\tau_i(x, t) = \langle \Lambda_{\text{max}}^{(i)} | \exp (-2\Phi) | \Lambda_{\text{max}}^{(i)} \rangle = \xi^{(i)}(z_+) \cdot \xi^{(i)}(z_-)$$

In the affine $\widehat{sl}_2$ case, there are two fundamental weights, $\Lambda^+$ and $\Lambda^-$, and therefore two τ-functions: $\tau^+ = \exp (-2\Lambda^+(\Phi))$ and $\tau^- = \exp (-2\Lambda^-(\Phi))$. Explicitly, we have,

$$e^{-\phi} = \frac{\tau^+}{\tau^-}$$

$$e^{-\zeta} = \tau^+ \tau^-$$

In terms of the τ-functions, the equations of motion become Hirota bilinear equations:

$$(D_x^2 - D_t^2) \tau_+ \cdot \tau_+ = -8m^2 e^{2\eta} \tau_+^2$$

$$(D_x^2 - D_t^2) \tau_- \cdot \tau_- = -8m^2 e^{2\eta} \tau_-^2$$

where the Hirota operators $D_{\nu}$ are defined by:

$$D_{\nu}^n (f,g)(x) = \left( \frac{\partial}{\partial u_{\nu}} \right)^n f(x + u)g(x - u) \big|_{u=0}$$

When $\eta = 0$ this is just the Hirota form of sinh-Gordon. For example, the τ-functions of the vacuum solution are:

$$\tau^+ = \tau^- = \tau_0 = \exp [-m^2 z_+ z_-]$$

As we proved in [2], under dressing transformations by an element $g = g_-^{-1} g_+$ of the dressing group, the chiral fields $\xi_{\text{vac}}$ and $\xi_{\text{vac}}$ become $\xi^g$ and $\xi_{\text{vac}}^g$ with:

$$\xi^g(x, t) = \xi_{\text{vac}}(x, t) \cdot g_-^{-1}$$

$$\xi_{\text{vac}}^g(x, t) = g_+ \cdot \xi_{\text{vac}}(x, t)$$

(15)
Notice how the factorization problem in the group $\hat{SL}_2$ is linked with the splitting of the chirality. By dressing the vacuum $\tau$-function $\tau_0$ is transformed into the dressed $\tau$-functions $\tau^g_\pm(x,t)$:

$$
\tau^g_\pm(x,t) = \langle \Lambda^\pm | \exp \left(-2\Phi^g\right) | \Lambda^\pm \rangle = \xi_{vac}^\pm(z_+) g^{-1}_- \cdot g_+ \bar{\xi}_{vac}^\pm(z_-)
$$

Using the exact expression of the vacuum transfer matrix, and hence of the vacuum chiral fields $\xi_{vac}$ and $\bar{\xi}_{vac}$, we find an alternative expression of the dressed $\tau$-functions:

$$
\tau^g_\pm(x,t) = \langle \Lambda^\pm | e^{-mz_+ \mathcal{E}_+} g^{-1}_- g_+ e^{mz_- \mathcal{E}_-} | \Lambda^\pm \rangle = \tau_0(x,t) \langle \Lambda^\pm | e^{-mz_- \mathcal{E}_-} e^{-mz_+ \mathcal{E}_+} g^{-1}_- g_+ e^{mz_+ \mathcal{E}_+} e^{mz_- \mathcal{E}_-} | \Lambda^\pm \rangle
$$

Hence, by acting with the dressing group we generate new $\tau$-functions which form the orbit of the vacuum $\tau$-function under the dressing group. Since the dressing transformations are symmetries of the equations of motion, all the $\tau$-functions $\tau^g_i$ are solutions of the Hirota equations.

### 4.3 The dressing problem.

Let us spell out how the computation of solutions obtained from dressing the vacuum solution is related to a factorization problem. As eq.(16) made it clear, the $\tau$-functions can be written as:

$$
\frac{\tau^g_\pm(x,t)}{\tau_0(x,t)} = \langle \Lambda^\pm | g^{-1}_-(z_+, z_-) \cdot g_+(z_+, z_-) | \Lambda^\pm \rangle
$$

where the factors $g_\pm(z_+, z_-)$ are solution of the following factorization problem:

$$
g^{-1}_-(z_+, z_-) \cdot g_+(z_+, z_-) = e^{-mz_- \mathcal{E}_-} e^{-mz_+ \mathcal{E}_+} g^{-1}_- g_+ e^{mz_+ \mathcal{E}_+} e^{mz_- \mathcal{E}_-}
$$

Once again, the factorization problem (18) is the one specified in section (4.1) : $g_\pm(z_+, z_-)$ belong to $B_\pm$ and have opposite component on the Cartan torus. Notice that, since $\Lambda^\pm$ is a highest weight, only the components on the Cartan torus of $g_\pm(z_+, z_-)$ contribute to eq. (17) i.e once the factorization problem has been solved, the $\tau$-functions are known explicitly. In view of eq.(18), what we have to do is to commute $e^{-mz_+ \mathcal{E}_+}$ to the right and $e^{mz_- \mathcal{E}_-}$ to the left of $g^{-1}_- g_+$. 

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4.4 The Dressing Group and the sinh-Gordon Hierarchy.

Before examining the factorization problem, let us notice that we can embed the vacuum equations of motion \( \partial_z \zeta = 2m^2 \) into a larger hierarchy. Introduce the variables \( z^{(r)} \) for \( r \) odd, and consider the following connexion (generalizing eqs. (4,5) evaluated at \( \Phi_{\text{vac}} \))

\[
A_{z^+(r)} = \frac{1}{4} \partial_{z^+(r)} \zeta K + m \mathcal{E}^{(r)}_+ \\
A_{z^-(r)} = -\frac{1}{4} \partial_{z^-(r)} \zeta K + m \mathcal{E}^{(r)}_-
\]

where

\[
\mathcal{E}^{(r)}_\pm = \lambda^{\pm r}(E_+ + E_-), \quad \text{with} \quad r \text{ odd}
\]

Since \( [\mathcal{E}^{(r)}_+, \mathcal{E}^{(s)}_-] = Kr \delta_{rs} \), the vanishing of the curvature of this connexion reduces to

\[
\partial_{z^+(r)} \partial_{z^-(s)} \zeta_{\text{vac}} = 2m^2 r \delta_{rs}
\]

A solution is

\[
\zeta_{\text{vac}} = 2m^2 \sum_r r z^{(r)}_+ z^{(r)}_-
\]

In the same way as before, we can calculate \( T_{\text{vac}}(z^{(r)}_+, z^{(r)}_-) \)

\[
T_{\text{vac}}(z^{(r)}_+, z^{(r)}_-) = e^{-m \sum_r \partial^{(r)}_+ z^{(r)}_- K e^{\partial^{(r)}_+ z^{(r)}_- e^{-m \sum_r \partial^{(r)}_+ z^{(r)}} e^{-m \sum_r \partial^{(r)}_- z^{(r)}}}}
\]

By dressing, the vacuum connexion will become

\[
A_{z^{(r)}_ \pm} = \pm \partial_{z^{(r)}_ \pm} \Phi + \cdots + me^{\pm \Phi} \mathcal{E}^{(r)}_\pm
\]

The \( \tau \)-functions \( \tau^{(r)}_\pm \), eqs. (10), are easily generalized to the multi-variable \( \tau \)-functions depending on all the coordinates \( z^{(r)}_\pm \):

\[
\left( \frac{\tau^{(r)}_\pm}{\tau_0} \right) (z^{(r)}_\pm) = \langle \Lambda^\pm e^{-m \sum_r \partial^{(r)}_- z^{(r)}_-} e^{-m \sum_r \partial^{(r)}_+ z^{(r)}_+} g_- g_+ e^m \sum_r \partial^{(r)}_+ z^{(r)}_+ e^m \sum_r \partial^{(r)}_- z^{(r)}_- | \Lambda^\pm \rangle (19)
\]
where the vacuum $\tau$-function, $\tau_0$, is given by

$$\tau_0(z^{(r)}) = e^{-m^2 \sum_r r^2 z^{(r)}_+ z^{(r)}_-}$$

The factorization problem involved in the computation of these $\tau$-functions is the subject of the following sections.

5 Solitons.

The aim of this section consists in proving that the $N$-soliton solutions of the affine sinh-Gordon model are in the orbit of the vacuum solution under the dressing group.

The $N$-soliton solution of the sine-Gordon model is well known. The $\tau$-functions are given by [15]

$$\frac{\tau^{(N)}_\pm}{\tau_0} = \det(1 \pm V) \quad (20)$$

where $V$ is a $N \times N$ matrix with elements

$$V_{ij} = 2 \sqrt{\mu_i \mu_j} \sqrt{X_i X_j}$$

and

$$X_i = a_i \exp \left[ 2m (\mu_i z_+ + \mu_i^{-1} z_-) \right] \quad (21)$$

The parameters $\mu_i$ are interpreted as the rapidities of the solitons and $a_i$ are related to their positions. Expanding the determinants provides the explicit expression:

$$\frac{\tau^{(N)}_\pm}{\tau_0} = 1 + \sum_{p=1}^N (\pm)^p \sum_{k_1<k_2<\ldots<k_p} X_{k_1} \cdots X_{k_p} \prod_{k_i<k_j} \left( \frac{\mu_{k_i} - \mu_{k_j}}{\mu_{k_i} + \mu_{k_j}} \right)^2 \quad (22)$$

According to the general theory of dressing transformation, Cf. section 4, our strategy for proving that the N-solitons can be generated by dressing the vacuum will be to look for gauge transformations $g_{\pm}(x, t)$, respectively
upper and lower triangular, both relating $A_{z\pm}(\Phi_{sol})$ to $A_{z\pm}(\Phi_{vac})$. For the transfer matrix this means:

$$T_{sol}(x, t) = g_\pm(x, t)T_{vac}(x, t)g_\pm^{-1}(0)$$

where $T_{sol}$ and $T_{vac}$ are the transfer matrices for the soliton and vacuum solutions. By compatibility we must have

$$g_\pm^{-1}(x, t)g_\pm(x, t) = T_{vac}(x, t)g_\pm^{-1}(0)g_\pm(0)T_{vac}^{-1}(x, t)$$

(23)
i.e we solve the factorization problem. The element of the dressing group which map the vacuum solution into the $N$-soliton solution is then identified to be $g_\pm = g_\pm(0)$.

5.1 Determination of $g_\pm$ for the one-soliton solution.

The one soliton solution to eqs(6,7,8) reads

$$e^{-\varphi} = \frac{1 + X}{1 - X}, \quad \zeta = \zeta_{vac} - \log(1 - X^2)$$

(24)

where

$$X = a \exp \left[2m(\mu z_+ + \mu^{-1}z_-)\right]$$

The independent parameters are $\mu, a$. The $\tau$-functions are

$$\tau_\pm = \tau_0(1 \pm X)$$

where $\tau_0$ is the vacuum $\tau$-function.

Proposition. There exists two gauge transformations $g_\pm(x, t)$ relating the Lax connexion $A_{z\pm}(\Phi_{1\ sol})$ to $A_{z\pm}(\Phi_{vac})$. They are given by

$$g_\pm^{-1}(x) = e^{f(x) V^{(-)}_\mu} e^{\log(1 - X^2) \frac{K}{4}}$$

$$g_\pm(x) = e^{\log(1 - X^2) \frac{K}{4}} e^{f(x) V^{(+)}_\mu}$$

where $f(x) = \log\left(\frac{1 + X}{1 - X}\right)$ and $V^{(\pm)}_\mu$ are special elements of the affine Lie algebra $\hat{sl}_2$ defined by:

$$V^{(\pm)}_\mu = \frac{1}{2} H + \frac{(\lambda/\mu)^{\pm 2}}{1 - (\lambda/\mu)^{\pm 2}} H + \frac{(\lambda/\mu)^{\pm 1}}{1 - (\lambda/\mu)^{\pm 2}} (E_+ - E_-)$$
Proof.
Here is the outline of the proof. In the Appendix A, we describe the computation in the affine group. Since \( g_\pm(x, t) \) are triangular elements in the affine group, we can restrict ourselves to the loop group once the central terms have been determined. The gauge transformation \( g_-(x, t) \) is determined by the equations

\[
\begin{align*}
g_-(x, t) \left[ \partial_{z_+} + \partial_{z_+} \Phi_{\text{vac}} + m \varepsilon_+ \right] g_-^{-1}(x, t) &= \partial_{z_+} + \partial_{z_+} \Phi_{\text{sol}} + me^{a\Phi_{\text{sol}}} \varepsilon(25) \\
g_-(x, t) \left[ \partial_{z_-} - \partial_{z_-} \Phi_{\text{vac}} + m \varepsilon_- \right] g_-^{-1}(x, t) &= \partial_{z_-} - \partial_{z_-} \Phi_{\text{sol}} + me^{-a\Phi_{\text{sol}}} \varepsilon(26)
\end{align*}
\]

To solve this equation, we first factorize the central terms according to eq.(12):

\[
g_-(x, t) = e^{\frac{1}{2}(\zeta_{\text{sol}} - \zeta_{\text{vac}}) K} \tilde{g}_-(x, t)
\]

One can then simply forget the central extension and work in the loop group. The equation for \( \tilde{g}_- \) in the loop group reads

\[
\left[ \partial_{z_+} + m \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \tilde{g}_-^{-1} = \tilde{g}_-^{-1} \begin{pmatrix} \frac{1}{2} \partial_{z_+} \varphi_{\text{sol}} & me^{\varphi_{\text{sol}}} \\ m\lambda e^{-\varphi_{\text{sol}}} & -\frac{1}{2} \partial_{z_+} \varphi_{\text{sol}} \end{pmatrix} \tag{27}
\]

The \( N \)-solitons solutions correspond to solutions having poles at \( \lambda = \pm \mu_i \) where the \( \mu_i \) are the rapidities of the solitons:

\[
\tilde{g}_-^{-1} = D^{-1} + \sum_i \left( \frac{P_{\mu_i}}{\lambda - \mu_i} + \frac{P_{-\mu_i}}{\lambda + \mu_i} \right)
\]

with \( D \) the diagonal matrix: \( D = \exp(-\frac{1}{2} \varphi_{\text{sol}} H) \) and one should understand the above formula as an expansion at \( \lambda \to \infty \). Parametrizing the \( P_{\pm \mu_i} \) as follows:

\[
P_{\pm \mu_i} = \frac{\tau_0}{\sqrt{\tau_+ \tau_-}} \begin{pmatrix} \pm A_{\mu_i} & B_{\mu_i} \\ -A_{\mu_i} & \mp B_{\mu_i} \end{pmatrix}
\]

we find

\[
\begin{align*}
\tau_- \partial_{z_+} A_{\mu_i} - \partial_{z_+} \frac{\tau_-}{\tau_0} A_{\mu_i} &= m\mu_i \left( \frac{\tau_+}{\tau_0} B_{\mu_i} + \frac{\tau_-}{\tau_0} A_{\mu_i} \right) \\
\tau_+ \partial_{z_-} B_{\mu_i} - \partial_{z_-} \frac{\tau_+}{\tau_0} B_{\mu_i} &= m\mu_i \left( \frac{\tau_-}{\tau_0} A_{\mu_i} + \frac{\tau_+}{\tau_0} B_{\mu_i} \right)
\end{align*}
\]
These equations are of the Hirota type. For the one soliton case, the solution is easily found: $A_{\mu} = B_{\mu} = \mu X$. The factor $\mu$ in front of $X$ is fixed by the affine group computation of the Appendix A. Including the central term, we get

$$g^{-1}(x, t) = e^{\frac{1}{2} \log(1-X^2) K \left( \sqrt{\frac{1+X}{1-X}} + \frac{2X}{\sqrt{1-X^2}} \frac{(\lambda/\mu)^{-2}}{1-(\lambda/\mu)^{-2}} \right)}$$

We can rewrite $g_-(x, t)$ as the exponential of something. We find

$$g^{-1}(x, t) = e^{\frac{1}{2} \log(1-X^2) K} e^{-\varphi_{\text{sol}} \left[ \frac{1}{2} H + \frac{(\lambda/\mu)^{-2}}{1-(\lambda/\mu)^{-2}} H + \frac{(\lambda/\mu)^{-1}}{1-(\lambda/\mu)^{-2}} (E_+ - E_-) \right]}$$

where the element in the exponential is to be interpreted as an expansion at $\lambda = \infty$. So, we find $g_- = g_- (0)$ by replacing $X \rightarrow a$ in eq. (28). One can compute similarly $g_+$. 

\[\square\]

5.2 One-soliton solution as a dressing problem.

We have found the gauge transformations $g^{-1}$ and $g_+$ relating the one soliton solution to the vacuum solution. We know that they are indeed solutions of the factorization problem defining the dressing transformations, cf eq. (23). We now reconsider this factorization problem in a more algebraic setting. We show that it is implied by some very remarkable commutation relations between the elements $V_{\mu}^{(\pm)}$ that appeared in the gauge transformations and $E_{\pm}$. They are described in Appendix B. Once the algebraic content of this one soliton solution has been clarified, the generalization to N-solitons is easy.

To check eq. (23), we have to show that

$$g^{-1}(x, t)g_+(x, t) = e^{-mz_{-} E_{-}} e^{-mz_{+} E_{+}} g^{-1}(0)g_+(0) e^{mz_{+} E_{+}} e^{mz_{-} E_{-}}$$

This means that the dependence in $x$ and $t$ of the soliton field is recovered by commuting the factor $e^{-mz_{-} E_{-}} e^{-mz_{+} E_{+}}$ through $g^{-1}(0)g_+(0)$. Let us consider first the commutation with $E_{+}$.

Proposition. We have the following relations:

$$e^{-mz_{+} E_{+}} e^{f(0) V_{\mu}^{(-)}} = e^{f(z_{+}) V_{\mu}^{(-)}} e^{g(z_{+}) \frac{d}{dz}} Y_{+}(z_{+})$$

$$e^{f(0) V_{\mu}^{(+)}} e^{mz_{+} E_{+}} = Y_{+}^{-1}(z_{+}) e^{f(z_{+}) V_{\mu}^{(+)}}$$

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where
\[
\tanh \left( \frac{f(z_+)}{2} \right) = \tanh \left( \frac{f(0)}{2} \right) e^{2m\mu z_+}; \quad \frac{dg}{dz_+} = -2m\mu [\cosh(f) - 1]
\]

and \( Y_+ \) is determined by the equation
\[
\frac{d}{dz_+} Y_+ \cdot Y_+^{-1} = -m \cosh(f) E_+ + m\mu \sinh(f) \left( 1 - \frac{1}{2\mu} \text{ad} E_+ \right) H \quad (31)
\]

Proof.
In eq.(29), the existence of the matrix \( Y_+(z_+) \) is ensured by Gaussian decomposition of the left hand side. The remarkable fact is that the component on \( B_- \) of the right hand side is proportional to \( V_\mu(-) \). This property relies on the special commutation relations between \( V_\mu(-) \) and \( E_+ \). First consider eq.(29).

Let \( F = e^{-mz_+ E_+} e^{f(0) V_\mu(-)} \). We have
\[
\frac{d}{dz_+} F = -m E_+ F = \frac{d}{dz_+} (e^{f V_\mu(-)} e^{g K/2} Y_+)
\]
so that
\[
-m e^{-f \text{ad} V_\mu(-) E_+} = \left( \frac{df}{dz_+} \right) V_\mu(-) + \left( \frac{dg}{dz_+} Y_+ \right) + \left( \frac{dg}{dz_+} \right) K/2
\]
Calculating the left hand side with the help of eq.(64) of Appendix B, gives the conditions
\[
\frac{df}{dz_+} = 2m\mu \sinh(f) \quad (32)
\]
\[
\frac{dg}{dz_+} = -2m\mu [\cosh(f) - 1]
\]
\[
\frac{d}{dz_+} Y_+ \cdot Y_+^{-1} = -m \cosh(f) E_+ + m\mu \sinh(f) \left( 1 - \frac{1}{2\mu} \text{ad} E_+ \right) H
\]
Next, Considering eq.(30), we find
\[
Y_+ \frac{d}{dz_+} Y_+^{-1} = -\left( \frac{df}{dz_+} \right) V_\mu(+) + me^{-f \text{ad} V_\mu(+) E_+}
\]
Therefore, to prove eq. (30), we have to show that this relation is compatible with eq. (31). Using eq. (64) of Appendix B, we get

\[ Y_+ \frac{d}{dz_+} Y_+^{-1} = \left( -\frac{df}{dz_+} + 2m\mu \sinh(f) \right) V_\mu^{(+)} + m \cosh(f)\mathcal{E}_+ - m\mu \sinh(f) \left( 1 - \frac{1}{2\mu} \text{ad} \mathcal{E}_+ \right) H \]

The coefficient of \( V_\mu^{(+)} \) vanishes if eq. (32) is satisfied. Using \( \frac{d}{dz} Y_+^{-1} = -Y_+^{-1} \left( \frac{d}{dz} Y_+ \right) Y_+^{-1} \), we see that the equation for \( Y_+ \) is identical to eq. (31).

\[ \blacksquare \]

A similar analysis can be done for the commutation of \( e^{-mz_+ \mathcal{E}_-} \):

**Proposition.** We have:

\[ e^{-mz_+ \mathcal{E}_-} e^{f(0) V_\mu^{(-)}} = e^{f(z_-) V_\mu^{(-)}} Y_-^{(z_-)^{-1}} \]

\[ e^{f(0) V_\mu^{(+)} e^{mz_+ \mathcal{E}_-}} = Y_-^{(z_-)} e^{f(z_-) V_\mu^{(+)} e^{g(z_-) \mathcal{E}_-}}} \quad (33) \quad (34) \]

where

\[ \tanh \left( \frac{f(z_-)}{2} \right) = \tanh \left( \frac{f(0)}{2} \right) e^{2m\mu z_-}; \quad \frac{dg}{dz_-} = -2m\mu^{-1} [\cosh(f) - 1] \]

and \( Y_- \) is determined by the equation

\[ \frac{d}{dz_-} Y_+ \cdot Y_+^{-1} = -m \cosh(f)\mathcal{E}_- - \frac{m}{\mu} \sinh(f) \left( 1 - \frac{\mu}{2} \text{ad} \mathcal{E}_- \right) H \quad (35) \]

As a result, when we commute \( e^{-mz_+ \mathcal{E}_+} \) and \( e^{-mz_- \mathcal{E}_-} \) through \( g_-^{-1}(0) g_+(0) \), the matrices \( Y_+ \) and \( Y_- \) cancel and the parameter \( a \) is replaced by

\[ a \rightarrow X = ae^{2m(\mu z_+ + \mu^{-1} z_-)} \]

as it should be. Moreover, the central term evolves according to \( \log(1 - a^2) \rightarrow \log(1 - a^2 e^{4m(\mu z_+ + \mu^{-1} z_-)}) \). Hence, we recover the one-soliton solution from this dressing.
5.3 N-solitons

In the previous section, we showed that the one-soliton solution was in the orbit of the vacuum solution under the dressing group. We did it in two different ways: first, we proved that there exist gauge transformations \( g_{\pm} \), mapping the vacuum Lax connexion \( A_{\text{vac}} \) into the one-soliton Lax connexion \( A_{\text{sol}} \); then, we demonstrated that these same gauge transformations could be found by solving a dressing problem.

In this section, we show that the N-soliton solutions are also in the orbit of the vacuum under the dressing group. We will do it by solving a dressing problem using only the algebraic properties of the affine Lie algebra \( \hat{sl}_2 \).

According to the discussion of section 4, this means that, as we will show, there exist elements \( g = g_{-1}g_+ \) in the affine group \( \hat{SL}_2 \) such that the N-soliton solution \( \Phi_{N-\text{sol}} \) is given by:

\[
e^{-2\Lambda \pm (\Phi_{N-\text{sol}} - \Phi_{\text{vac}})} = \left( \frac{\tau^{(N)}_{\pm}}{\tau_0} \right) (z_+, z_-)
\]

\[
= \langle \Lambda^{\pm} | g^{-1}(z_+, z_-) \cdot g_+(z_+, z_-) \mid \Lambda^{\pm} \rangle
\]

where \( g_{\pm}(z_+, z_-) \) are defined from the following factorization problem:

\[
g^{-1}(z_+, z_-) \cdot g_+(z_+, z_-) = e^{-m z_- \varepsilon_-} e^{-m z_+ \varepsilon_+} g^{-1} g_+ e^{m z_+ \varepsilon_+} e^{m z_- \varepsilon_-}
\]

The elements \( g_{\pm} \) of the affine group \( \hat{SL}_2 \) which connect the N-soliton solution to the vacuum solution are products of N factors:

\[
g^{-1} = g^{-1}(1) \cdots g^{-1}(N)
g_+ = g_+(N) \cdots g_+(1)
\]

where each factor is given by:

\[
g^{-1}(k) = e^{f_k(0)\nu_k(-)} e^{\frac{1}{2} h_k(0) H} e^{\frac{1}{2} y_k(0) K}
g_+(k) = e^{\frac{1}{2} y_k(0) K} e^{-\frac{1}{2} h_k(0) H} e^{f_k(0)\nu_k(+)}
\]

Here we choose \( h_N \), which cancels in eq.(36), such that \( \sum_k h_k = 0 \). This ensures that \( g_+ \) and \( g_- \) have opposite components on the Cartan torus. The dressing problem is solved by the following
Proposition. Consider the factorization problem eq.(36) where the elements $g_{\pm}$ are given by eqs.(37-39).

(a) Its solution $g_{\pm}(z_{+}, z_{-})$ is of the form:

$$
g_{-}(z_{+}, z_{-})^{-1} = g_{-}^{-1}(1)(z_{+}, z_{-}) \cdots g_{-}^{-1}(N)(z_{+}, z_{-})$$

$$
g_{+}(z_{+}, z_{-})^{-1} = g_{+}^{-1}(1)(z_{+}, z_{-}) \cdots g_{+}^{-1}(N)(z_{+}, z_{-})$$

Each factor $g_{\pm}(k)(z_{+}, z_{-})$, has the same form as the corresponding factor $g_{\pm}(k)$ but with $z_{+}$ and $z_{-}$ dependent functions $f_{k}$, $g_{k}$ and $h_{k}$.

(b) These functions satisfy:

$$
\partial_{z+} f_{k} = 2m\mu_{k} \sinh (\phi_{k} + f_{k}) - 2m\mu_{k} \sinh (\phi_{k})
$$

$$
\partial_{z+} g_{k} = -2m\mu_{k} \cosh (\phi_{k} + f_{k}) + 2m\mu_{k} \cosh (\phi_{k})
$$

$$
\partial_{z+} h_{k} = 2m\mu_{k} \sinh(\phi_{k} + f_{k}) - 2m\mu_{k+1} \sinh (\phi_{k+1})
$$

$$
\phi_{k+1} = \phi_{k} + f_{k} + h_{k}
$$

and

$$
\partial_{z-} f_{k} = -2m\mu_{k}^{-1} \sinh (\bar{\phi}_{k} - f_{k}) + 2m\mu_{k}^{-1} \sinh (\bar{\phi}_{k})
$$

$$
\partial_{z-} g_{k} = -2m\mu_{k}^{-1} \cosh (\bar{\phi}_{k} - f_{k}) + 2m\mu_{k}^{-1} \cosh (\bar{\phi}_{k})
$$

$$
\partial_{z-} h_{k} = 2m\mu_{k}^{-1} \sinh(\bar{\phi}_{k} - f_{k}) - 2m\mu_{k+1}^{-1} \sinh (\bar{\phi}_{k+1})
$$

$$
\bar{\phi}_{k+1} = \bar{\phi}_{k} - f_{k} + h_{k}
$$

(c) The N-soliton $\tau$-functions are then given by:

$$
\tau_{\pm}^{(N)}(z_{+}, z_{-}) = \exp \left( \pm \frac{1}{2} \sum_{k} f_{k}(z_{+}, z_{-}) + \sum_{k} g_{k}(z_{+}, z_{-}) \right)
$$

Proof.

The commutation of $e^{-mz_{+}E_{+}}$ through $g_{-}^{-1}$ is done in $N$ steps. The first step is given by eq.(29). In the second step, we have to commute $Y_{+}(z_{+})$ through $g_{-}^{-1}(2)$, etc ... The general pattern is as follows. There exists elements $Y_{+}(k, z_{+})$ such that

$$
Y_{+}(k, z_{+}) e^{f_{k}(0)V_{\mu_{k}}^{(-)} h_{k}(0)H} e^{g_{k}(0)K} = e^{f_{k}(z_{+})V_{\mu_{k}}^{(-)} h_{k}(z_{+})H} e^{g_{k}(z_{+})K} Y_{+}(k + 1, z_{+})
$$

The important point is that they are determined by the simple equations (compare with eq.(31))

$$
\partial_{z+} Y_{+}(k, z_{+}) \cdot Y_{+}(k, z_{+})^{-1} = -m \cosh \phi_{k} E_{+} + 2m\mu_{k} \sinh \phi_{k} U_{\mu_{k}}^{(+)}
$$
where the element $U^{(\mu)}$ is defined in the Appendix B. The proposition is a consequence of this observation. The details are given in Appendix C. Just as in the one-soliton case, we have:

$$e^{-mz} g^{-1} = g^{-1}(z) Y(N+1; z)$$

$$g e^{mz} = Y(N+1; z) g(z)$$

Hence, the matrix $Y(N+1; z)$ cancels out when commuting $e^{-mz} g$ through $g = g^{-1} g$. Similarly with $e^{-mz} g$.

\[\square\]

6 Relation with the Bäcklund transformations.

In this section we show how the dressing problem solved in the previous section is related to the so-called Bäcklund transformations, See e.g. [16] and references therein. More precisely, we will map the differential equations (40,41) into a set of Bäcklund transformations and use this relation to present an algebraic solution to them.

6.1 Bäcklund transformations.

The Bäcklund transformations are also symmetries of non-linear differential equations but of different kind than the dressing transformations. For the sinh-Gordon model, the Bäcklund transformation can be defined as follows: Let $\varphi$ be a solution of the sinh-Gordon model, its image under a Bäcklund transformation with spectral parameter $\mu$ is the field $\hat{\varphi} \equiv B_\mu \cdot \varphi$ implicitly defined by the following differential equations:

$$\partial_{z_+} (\hat{\varphi} + \varphi) = 2m \mu \sinh(\hat{\varphi} - \varphi)$$

$$\partial_{z_-} (\hat{\varphi} - \varphi) = \frac{2m}{\mu} \sinh(\hat{\varphi} + \varphi)$$

(42)

If $\varphi$ solves the sinh-Gordon equation, so does the transformed field $\hat{\varphi}$. This can be seen by remarking that eqs.(42) implies:

$$\partial_{z_+} \partial_{z_-} (\varphi + \hat{\varphi}) = 2m^2 \left[ \sinh(2\varphi) + \sinh(2\hat{\varphi}) \right]$$

In general the relation between $\varphi$ and $\hat{\varphi}$ is non-local. However, when expanding $\varphi$ in formal power of either $\mu$ or $1/\mu$ each term of the infinite
series is a local function of $\varphi$. This remark can be used to deduce an infinite set of local conserved currents in the sinh-Gordon model. Indeed, from the defining relations (42) of the Bäcklund transformation, it follows that the current $J_{z+}, J_{z-}$ with components:

\[
J_{z+} = \mu \cosh(\varphi - \tilde{\varphi}) \\
J_{z-} = -\frac{1}{\mu} \cosh(\varphi + \tilde{\varphi})
\]

is a conserved current: $\partial_{z-} J_{z+} + \partial_{z+} J_{z-} = 0$. Expanding it in power series of either $\mu$ or $1/\mu$ gives two infinite series of local conserved currents.

A remarkable property of the Bäcklund transformations is that we have the commutative diagram (see e.g. [16]):

\[
\begin{array}{ccc}
\varphi_0 & \xrightarrow{\mu_1} & \varphi_1 \\
\downarrow{\mu_2} & & \downarrow{\mu_2} \\
\varphi_2 & \xrightarrow{\mu_1} & \varphi_3 = B_{\mu_1} \cdot \varphi_2 = B_{\mu_2} \cdot \varphi_1
\end{array}
\] (43)

A consequence of this diagram is that the four solutions $\varphi_0, \varphi_1, \varphi_2$ and $\varphi_3$ are linked by purely algebraic relations:

\[
\tanh\left(\frac{\varphi_3 - \varphi_0}{2}\right) = \left(\frac{\mu_1 + \mu_2}{\mu_1 - \mu_2}\right) \tanh\left(\frac{\varphi_2 - \varphi_1}{2}\right)
\] (44)

We called this rule the “tangent rule”. It is proved starting from the relation:

\[
\partial_{z+}(\varphi_3 + \varphi_1) - \partial_{z+}(\varphi_1 + \varphi_0) = \partial_{z+}(\varphi_3 + \varphi_2) - \partial_{z+}(\varphi_2 + \varphi_0)
\]

and similarly with $z_-$, using the defining relation (42) of the Bäcklund transformation.

By repeated uses of this rule, one can construct in a purely algebraic manner an infinite set of solutions of the sinh-Gordon equation.

The ‘tangent rule’ can be expressed in terms of the $\tau$-functions. If $\tau_{\pm}(k)$ with $k = 0, \cdots, 3$, are $\tau$-functions of the fields $\varphi_k$, then we have:

\[
\tau_{+}(3)\tau_{-}(0) + \tau_{+}(0)\tau_{-}(3) = \tau_{+}(1)\tau_{-}(2) + \tau_{+}(1)\tau_{-}(2) \quad (45)
\]

\[
\beta_{12}[\tau_{+}(3)\tau_{-}(0) - \tau_{+}(0)\tau_{-}(3)] = \tau_{+}(2)\tau_{-}(1) - \tau_{+}(1)\tau_{-}(2) \quad (46)
\]
where, for later convenience, we put:

\[
\beta_{12} = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}
\]

In this form the ‘tangent rule’ also applies to the affine sinh-Gordon model as well.

### 6.2 From Dressing to Bäcklund transformations.

We now map the relations (40,41) into a set of successive Bäcklund transformations. First, let us introduce the following change of variables:

\[
\begin{align*}
  f_k &= -\varphi_k + \varphi_{k-1} \\
  h_k &= \rho_k - \rho_{k-1} \\
  g_k &= -\frac{1}{2}(\zeta_k - \zeta_{k-1})
\end{align*}
\]

with the initial condition \( \varphi_0 = \rho_0 = \zeta_0 = 0 \). In terms of these new functions, the N-soliton \( \tau \)-functions are:

\[
\frac{\tau_{\pm}^{(N)}}{\tau_0} = \exp\left(\mp\frac{1}{2}\varphi_N - \frac{1}{2}\zeta_N\right)
\]

In particular, the functions \( \varphi_N \) and \( \zeta_N \) are the N-solitons solutions of the affine sinh-Gordon model.

In these new variables, the relations (40,41) involving \( f_k \) and \( h_k \) become:

\[
\begin{align*}
  \partial_{z^+}(\varphi_k + \rho_{k-1}) &= 2m\mu_k \sinh(\varphi_k - \rho_{k-1}) \\
  \partial_{z^+}(\varphi_k - \rho_{k-1}) &= 2m\mu_k \sinh(\varphi_k + \rho_{k-1})
\end{align*}
\]

and

\[
\begin{align*}
  \partial_{z^-}(\rho_k + \varphi_k) &= -2m\mu_{k+1} \sinh(\rho_k - \varphi_k) \\
  \partial_{z^-}(\rho_k - \varphi_k) &= -2m\mu_{k+1} \sinh(\rho_k + \varphi_k)
\end{align*}
\]

Comparing with eqs. (12) shows that the fields \( \varphi_k \) and \( \rho_k \) are connected through Bäcklund transformations as follows:

\[
0 \xrightarrow{B_{\mu_1}} \varphi_1 \xrightarrow{B_{-\mu_2}} \rho_1 \xrightarrow{B_{\mu_2}} \varphi_2 \xrightarrow{B_{-\mu_3}} \cdots \xrightarrow{B_{\mu_{k-1}}} \varphi_{k-1} \xrightarrow{B_{-\mu_k}} \rho_{k-1} \xrightarrow{B_{\mu_k}} \varphi_k \cdots (53)
\]
These Bäcklund transformations are for the fields satisfying the sinh-Gordon model. But, since the auxiliary fields $\zeta$ in the affine sinh-Gordon model, eqs. (6, 7, 8), are determined from the field $\varphi$, the Bäcklund transformations (53) can be lifted to the affine model. Notice that in order to find the $N$-soliton $\tau$-function, we have to apply $2N$ Bäcklund transformations, i.e. we apply successively $B_{-\mu_k}$ and $B_{\mu_k}$ $N$ times. This is in contrast with the standard computation.

We now use the relation (53) to solve for the $N$-soliton $\tau$-functions.

**Proposition.** The $N$-soliton $\tau$-functions $\tau^{(N)}_{\pm}$ as defined in eq. (48) are polynomials in the variables $X_k$, $k = 1, \ldots, N$, with:

$$X_k = a_k \exp(2m(\mu_k z_+ + \mu_k^{-1} z_-))$$

They satisfy the following recursion relation:

$$\tau^{(N)}_{\pm}(X_k) = \tau^{(N-1)}_{\pm}(X_k) \pm X_N \tau^{(N-1)}_{\pm}(\beta_{Nk}^2 X_k)$$

This determines them uniquely. The solution of this recursion relations are the $\tau$-functions (22).

We prove it in the Appendix D using the recursive Bäcklund transformations (53). As an intermediate step of the proof, we show that the fields $\rho_k$ correspond to the $k$-soliton solutions $\varphi_k$ but with a rescaling of the variables $X_j$. Namely:

$$\rho_k(X_j) = \varphi_k(\beta_{k+1,j} X_j)$$

Finally, notice that it is rather intriguing that the dressing transformations which, by essence, are non-local and Lie-Poisson actions are mapped into a set of Bäcklund transformations which are more naturally connected to the local conserved currents.

### 7 Relation with the Vertex operators.

Another well known construction of the solitons solutions was given by the Kyoto school. It is ultimately expressed in terms of vertex operators. We show in the following sections that the previous analysis directly leads to the Japanese formulae [12] when we use the level one vertex operator representation of the affine $sl_2$ algebra.
7.1 Vertex operator representations of \( \hat{sl}_2 \).

We begin by a description of the level one representations of \( \hat{sl}_2 \). Following \cite{17}, we introduce oscillators \( p_n \) for \( n \) odd, such that

\[
[p_m, p_n] = m\delta_{n+m,0}
\]

Assume \( p_n^+ = p_{-n} \). The vacuum is defined by

\[ p_n|0> = 0 \quad n > 0 \]

The associated normal ordering is defined by putting \( p_n, n > 0 \) to the right. Let

\[ Z(\lambda) = -i\sqrt{2} \sum_{n \text{ odd}} \frac{\lambda^n}{n} p_{-n} \]

We have

\[ < Z(\lambda)Z(\mu) > = \log \left( \frac{\lambda + \mu}{\lambda - \mu} \right) \quad |\lambda| > |\mu| \]

The vertex is defined by

\[ V(r, \lambda) = \frac{1}{2} : e^{irZ(\lambda)} : \]

We have

\[ V(r, \lambda)V(s, \mu) = \left( \frac{\lambda - \mu}{\lambda + \mu} \right)^{rs} : V(r, \lambda)V(s, \mu) : \quad |\lambda| > |\mu| \]

The level one vertex operator representations of the Lie algebra \( \hat{sl}_2 \) are obtained as follows:

\[
\sum_{n \text{ odd}} \lambda^{-n}(E^n_+ + E^n_-) = \frac{i}{\sqrt{2}} \lambda \frac{d}{d\lambda} Z(\lambda)
\]

\[
\sum_{n \text{ even}} \lambda^{-n}H^n + \sum_{n \text{ odd}} \lambda^{-n}(E^n_+ - E^n_-) = \pm V(\lambda)
\]

where \( V(\lambda) \) denotes the vertex

\[ V(\lambda) = V(-\sqrt{2}, \lambda) = \frac{1}{2} : e^{-i\sqrt{2}Z(\lambda)} : = \frac{1}{2} : e^{i\sqrt{2}Z(-\lambda)} : \]

26
The representation with highest weight $\Lambda^+$ ($\Lambda^-$) corresponds to the plus (minus) sign in eq. (57). It is then a standard calculation to verify that we have obtained a representation of the affine $\mathfrak{sl}_2$ algebra at level one in the principal gradation. In particular we have

$$[E^+_n + E^-_n, V(\lambda)] = -2\lambda^n V(\lambda) \quad (58)$$

Finally, remark that $V(\lambda)^2 = 0$ inside any expectation value.

### 7.2 $\tau$-functions and Vertex operators.

We have shown in the previous sections that the element $g^{-1}g_+$ has very special commutation properties with $e^{-mz_+\mathcal{E}_+}$ and $e^{-mz_-\mathcal{E}_-}$. In fact, commuting these factors simply reintroduce the $z_\pm$ evolution of the solitons.

$$a_i \rightarrow X_i = a_i e^{2m(\mu_i z_+ + \mu_i^{-1} z_-)}$$

In view of eq. (58) we see that the vertex operator has the same property

$$e^{-mz_-\mathcal{E}_-} e^{-mz_+\mathcal{E}_+} V(\mu_i) e^{mz_+\mathcal{E}_+} e^{mz_-\mathcal{E}_-} = e^{2m(\mu_i z_+ + \mu_i^{-1} z_-)} V(\mu_i)$$

So, we expect that $g = g^{-1}g_+$ has a simple expression in terms of $V$. This is indeed the case.

**Proposition.** In the level one representations, we have:

$$g = (1 + 2a_1 V(\mu_1))(1 + 2a_2 V(\mu_2)) \cdots (1 + 2a_N V(\mu_N)) \quad (59)$$

where $g = g^{-1}g_+$ is defined in eqs. (37, 38). As a consequence, we get the following alternative expression for the $\tau$-functions

$$\frac{\tau_{\pm}^{(N)}}{\tau_0} (z^{(s)}_{\pm}) = \langle \Lambda^\pm | \prod_{i=1}^N (1 + 2X_i V(\mu_i)) |\Lambda^\pm \rangle$$

where

$$X_i = a_i e^{2m \left( \sum \mu_i z_+^{(s)} + \mu_i^{-1} z_-^{(s)} \right)}$$

**Proof.**

To prove these formulae, we remark that the meaning of the elements $V^{(\pm)}_\mu$
discovered in our study of the solitons in the sinh-Gordon model is now clear. They are just the projections of the vertex operator $V(\mu)$ on $B_{\pm}$ respectively.

$$V_{\mu}^{(-)} + V_{\mu}^{(+)} = V(\mu)$$

moreover the projections of $V_{\mu}^{(\pm)}$ on $\mathcal{H}$ are equal. In other words, $V_{\mu}^{(\pm)}$ are the solutions of the factorization of $V(\mu)$ corresponding to the dressing problem. Therefore

$$V_{\mu}^{(-)} = \frac{1}{2} \oint_{|\lambda|>|\mu|} \frac{d\lambda}{2i\pi} \frac{\lambda + \mu}{\lambda - \mu} V(\lambda)$$
$$V_{\mu}^{(+)} = -\frac{1}{2} \oint_{|\lambda|<|\mu|} \frac{d\lambda}{2i\pi} \frac{\lambda + \mu}{\lambda - \mu} V(\lambda)$$

These operators satisfy the remarkable identity:

$$e^{fV_{\mu}^{(-)}} e^{fV_{\mu}^{(+)}} = \cosh \frac{f}{2} + 2 \sinh \frac{f}{2} V(\mu) \quad (60)$$

We prove it in the Appendix E. It relies on the relation:

$$V_{\mu}^{(+) 2} + 2V_{\mu}^{(-)} V_{\mu}^{(+)} + V_{\mu}^{(-)} 2 = \frac{1}{4}$$

which is satisfied in the level one vertex operator representations.

Our claim in the one-soliton case immediately follows from this formula. Remember that for the one-soliton case we had:

$$g_{-}^{-1} g_{+} = e^{\frac{f}{2} \log(1-a^{2})} e^{fV_{\mu}^{(-)}} e^{fV_{\mu}^{(+)}}$$

with $f = \log(1 + a)/(1 - a)$. Therefore, since $K = 1$,

$$g_{-}^{-1} g_{+} = 1 + 2a V(\mu) \quad (61)$$

In the general $N$-soliton case, we have to evaluate,

$$g_{-}^{-1} g_{+} = g_{-}^{-1}(1) g_{-}^{-1}(2) \cdots g_{-}^{-1}(N) \cdot g_{+}(N) g_{+}(N-1) \cdots g_{+}(1)$$

where all the factors are defined in eq.(38,39). The middle terms can be computed from the identity (60), we have:

$$g_{-}^{-1}(N) \cdot g_{+}(N) = \langle E(N)|W(N)\rangle e^{g_{N}(0)K}$$

28
where we have introduced a convenient bra-ket notation by denoting \( \langle E(k) \rangle \) the line vector with operator entries
\[
\langle E(k) \rangle = \left( 1, V(\mu_k), V(-\mu_k), \frac{i}{\sqrt{2}}\mu_k \frac{d}{d\mu_k} Z(\mu_k) \right)
\]
and \( |W(k)\rangle \) the column vector
\[
|W(k)\rangle = \begin{pmatrix} \cosh \frac{f_k}{2} \\ 2\sinh \frac{f_k}{2} \\ 0 \\ 0 \end{pmatrix}
\]
Next one has to commute \( g_+(N-1) \) through \( \langle E(N) \rangle \). In the Appendix F, we prove the relations :
\[
\langle E(k) \rangle g_+(j) = g_+(j) \langle E(k) \rangle R(j, k) \tag{62}
\]
with \( R(j, k) \) a \( C \)-number matrix explicitly computed in the Appendix F. As a result, the commutation of \( g_+(N-1) \) gives the factors :
\[
\langle E(N-1) |W(N-1)\rangle \langle E(N) |R(N-1, N)|W(N)\rangle e^{(g_{N-1}^{(0)}+g_{N}^{(0)})K}
\]
The general case is now clear, we obtain :
\[
g = g^{-1}g_+ = g(1)g(2) \cdots g(N)
\]
where
\[
g(k) = \langle E(k) \rangle \left( \prod_{j=1}^{k-1} R(j, k) \right) |W(k)\rangle e^{g_k^{(0)}K}
\]
The next step consists in showing that the \( f_k \) and \( h_k \) are such that:
\[
R(1, k)R(2, k) \cdots R(k - 1, k)|W(k)\rangle = \Delta_k \begin{pmatrix} 1 \\ 2X_k \\ 0 \\ 0 \end{pmatrix} \tag{63}
\]
where $\Delta_k$ are some coefficients and $X_k$ is the variable eq.(21) of the $k$th-soliton. We relegate the proof of this assertion in the Appendix G. As a consequence, we get for $z_+ = z_- = 0$:

$$g = \left( \prod_{k=1}^{N} \Delta_k e^{\eta_k(0) K} \right) (1 + 2a_1 V(\mu_1))(1 + 2a_2 V(\mu_2)) \cdots (1 + 2a_N V(\mu_N))$$

We also find the consistency relation $\prod_k \Delta_k e^{\eta_k(0)} = 1$ which can be checked directly. It is now trivial to recalculate the $\tau$-functions from this expression. We of course get back eq.(22).

\[ \square \]

### 8 Appendix A: Determination of $g_\pm$ in the affine group.

Here we determine the gauge transformation $g_\pm(x, t)$ in the affine group for the one-soliton case. In the course of this computation we will find the first few constraints characterizing the vacuum orbit. Let us consider $g_-(x, t)$. It is determined by the equations (25) and (26). We look for $g_-(x, t)$ as a graded product

$$g_-(x, t) = g_0 g_{-1} g_{-2} \cdots$$

where the lower indices refer to the degree in the affine group in the principal gradation. Comparing the terms of degree one in eq.(25) and of degree zero in eq.(24), we find $g_0 = e^{\Phi_{sol} - \Phi_{vac}}$ in agreement with the definition of the factorization problem. Next, we set $g_{-1} = e^{X_{-1}}$, with degree$(X_{-1}) = -1$. Comparing the terms of degree zero in eq.(25) we obtain the condition for $X_{-1}$:

$$m[X_{-1}, E_+] = 2\partial_{z_+} (\Phi_{sol} - \Phi_{vac})$$

Its solution is $X_{-1} = X_{-\alpha_1} E_{-\alpha_1} + X_{-\alpha_2} E_{-\alpha_2}$ with:

$$X_{-\alpha_1} = -\frac{1}{2m} \partial_{z_+} (\varphi_{sol} + \zeta_{sol} - \zeta_{vac})$$

$$X_{-\alpha_2} = -\frac{1}{2m} \partial_{z_+} (-\varphi_{sol} + \zeta_{sol} - \zeta_{vac})$$
The next level is more interesting. We set \( g_{-2} = e^{X_{-2}}, \) with degree\( (X_{-2}) = -2. \) The equation determining \( X_{-2} \) is

\[
m[X_{-2}, \mathcal{E}_+] = \partial_{z_+} X_{-1} - [X_{-1}, \partial_{z_+} \Phi_{sol}]
\]

There is only one root at level \(-2\), therefore, \( X_{-2} = X_{-\alpha_1 - \alpha_2}[E_{-\alpha_1}, E_{-\alpha_2}]. \) The equation for \( X_{-\alpha_1 - \alpha_2} \) reads

\[
2m X_{-\alpha_1 - \alpha_2}(E_{-\alpha_1} - E_{-\alpha_2}) = (\partial_{z_+} X_{-\alpha_1} - \partial_{z_+} \varphi_{sol} X_{-\alpha_1})E_{-\alpha_1} + (\partial_{z_+} X_{-\alpha_2} + \partial_{z_+} \varphi_{sol} X_{-\alpha_2})E_{-\alpha_2}
\]

These are two equations for one unknown. Therefore, we get the constraint

\[
(\partial_{z_+} \varphi_{sol})^2 - \partial_{z_+}^2 (\zeta_{sol} - \zeta_{vac}) = 0
\]

This constraint can easily be checked for one and two solitons. This is the first of an infinite set of constraints (which will appear at every even degree), characterizing the orbit of the vacuum. To proceed along this line would rapidly becomes untractable. Fortunately, at this point the computation can be done in the loop group. Indeed, we notice that all central terms occur with \( g_{-1} \) and they have been taken care of. Defining

\[
g_-(x, t) = e^{\frac{i}{2}(\zeta_{sol} - \zeta_{vac})K} \tilde{g}_-(x, t)
\]

one can simply forget the central extension and work in the loop group. The equation for \( \tilde{g}_- \) in the loop group reads

\[
\tilde{g}_-(x, t) \left[ \partial_{z_+} + m \mathcal{E}_+ \right] \tilde{g}_-^{-1}(x, t) = \frac{1}{2} \partial_{z_+} \varphi_{sol} H + me^{\frac{i}{2} \varphi_{sol} ad H} \mathcal{E}_+
\]

or more explicitly eq.(27). At this point, we make contact with the computation in the loop group as explained in section(5.1).

9 Appendix B. The elements \( V^{(+)}_\mu \) and \( V^{(-)}_\mu \).

We list some formulae concerning the commutation relations between \( V^{(\pm)}_\mu \) and \( \mathcal{E}_\pm \). Let

\[
U^{(\pm)}_\mu = \frac{1}{2} \left( 1 - \frac{\mu^{\mp 1}}{2} ad \mathcal{E}_\pm \right) H
\]
then
\[
\text{ad } V_{\mu}^{(\pm)} \cdot \mathcal{E}_{\pm} = 2\mu^{\mp 1} \left( V_{\mu}^{(\pm)} - U_{\mu}^{(\pm)} \right)
\]
\[
\text{ad } V_{\mu}^{(\pm)} \cdot \mathcal{E}_{\mp} = 2\mu^{\mp 1} \left( V_{\mu}^{(\pm)} + U_{\mu}^{(\pm)} \right)
\]

Therefore, defining \( V_{\mu} = V_{\mu}^{(+)} + V_{\mu}^{(-)} \), we see that \( \text{ad } V_{\mu} \cdot \mathcal{E}_{\pm} = 2\mu^{\mp 1} V_{\mu} \). See also \([18]\). Strictly speaking, in the loop representation the element \( V_{\mu} \) is not defined, but it is well defined for instance in the level one representation.

Next, one has
\[
\text{ad } V_{\mu}^{(\pm)} \cdot U_{\mu}^{(\pm)} = -\frac{\mu^{\mp 1}}{2} \mathcal{E}_{\mp} \pm \frac{K}{2}
\]

From this we get
\[
e^{-fad} V_{\mu}^{(\pm)} \mathcal{E}_{\mp} = \cosh(f) \mathcal{E}_{\mp} - 2\mu^{\mp 1} \sinh(f) \left( V_{\mu}^{(\pm)} - U_{\mu}^{(\pm)} \right) \pm \mu^{\mp 1} \cosh(f) - 1 \right] K
\]
\[
e^{-fad} V_{\mu}^{(\pm)} \mathcal{E}_{\pm} = \cosh(f) \mathcal{E}_{\pm} - 2\mu^{\mp 1} \sinh(f) \left( V_{\mu}^{(\pm)} + U_{\mu}^{(\pm)} \right)
\]

also
\[
e^{-fad} V_{\mu}^{(\pm)} U_{\mu}^{(\pm)} = -\frac{\mu^{\mp 1}}{2} \sinh(f) \mathcal{E}_{\mp} + (\cosh(f) - 1) V_{\mu}^{(\pm)} + \cosh(f) U_{\mu}^{(\pm)} \pm \sinh(f) \frac{K}{2}
\]
\[
e^{-fad} V_{\mu}^{(\pm)} U_{\mp}^{(\pm)} = \frac{\mu^{\mp 1}}{2} \sinh(f) \mathcal{E}_{\pm} - (\cosh(f) - 1) V_{\mu}^{(\pm)} + \cosh(f) U_{\mu}^{(\pm)}
\]

10 Appendix C: Solution of the N-soliton dressing problem.

We look for elements \( Y_{+}(k, z_{+}) \) such that
\[
Y_{+}(k, z_{+}) e^{f_{k}(0) V_{\mu}^{(-)} e^{-\frac{1}{2} h_{k}(0) H} e^{\frac{1}{2} g_{k}(0) K}} = e^{f_{k}(z_{+}) V_{\mu}^{(-)} e^{\frac{1}{2} h_{k}(z_{+}) H} e^{\frac{1}{2} g_{k}(z_{+}) K}} Y_{+}(k + 1, z_{+})
\]
or equivalently
\[
\partial_{z_{+}} Y_{+}(k + 1, z_{+}) Y_{+}(k + 1, z_{+})^{-1} = e^{-\frac{1}{2} h_{k} \text{ad}} H \left\{ e^{-f_{k} \text{ad}} V_{\mu}^{(-)} \left[ \partial_{z_{+}} Y_{+}(k, z_{+}) Y_{+}(k, z_{+})^{-1} \right] - \partial_{z_{+}} f_{k} V_{\mu}^{(-)} - \frac{1}{2} \partial_{z_{+}} h_{k} H - \frac{1}{2} \partial_{z_{+}} g_{k} K \right\}
\]
We now make the crucial remark that the above equations can be satisfied if we take the $Y_+(k, z_+)$'s such that (compare with eq. (30))

$$\partial_{z_+} Y_+(k, z_+) Y_+(k, z_+)^{-1} = -m \cosh \phi_k \mathcal{E}_+ + 2m \mu_k \sinh \phi_k \quad U_\mu_k^{(+)} \equiv Y_+(\phi_k, \mu_k)(67)$$

The following commutation relations hold

$$e^{-\frac{1}{2} h_{k \mu}^d} \mathcal{Y}_+(\phi_k, \mu_k) = \mathcal{Y}_+(\phi_k + f_k, \mu_k) + 2m \mu_k [\sinh(\phi_k + f_k) - \sinh(\phi_k)] V_{\mu_k}^{(-)}$$

$$e^{-\frac{1}{2} h_{k \mu}^d} H \mathcal{Y}_+(\phi_k + f_k, \mu_k) = \mathcal{Y}_+(\phi_k + f_k + h_k, \mu_k)$$

Using the first of these relations, eq. (68) becomes

$$\partial_{z_+} Y_+(k + 1, z_+) Y_+(k + 1, z_+)^{-1} = e^{-\frac{1}{2} h_{k \mu}^d} H \left\{ \mathcal{Y}_+(\phi_k + f_k, \mu_k) - \frac{1}{2} \partial_{z_+} h_k H \right\}$$

$$- \left[ \partial_{z_+} f_k + 2m \mu_k \sinh (\phi_k) - 2m \mu_k \sinh (\phi_k + f_k) \right] V_{\mu_k}^{(-)}$$

$$- \frac{1}{2} \left( \partial_{z_+} g_k + 2m \mu_k [\cosh(\phi_k + f_k) - \cosh(\phi_k)]\right) K$$

Projecting this equation on $\mathcal{G}_-$ and on the central element, we get

$$\partial_{z_+} f_k = 2m \mu_k [\sinh(\phi_k + f_k) - \sinh(\phi_k)] \quad (68)$$

$$\partial_{z_+} g_k = -2m \mu_k [\cosh(\phi_k + f_k) - \cosh(\phi_k)]$$

while the projection on $\mathcal{G}_+$ gives

$$\partial_{z_+} Y_+(k + 1, z_+) Y_+(k + 1, z_+)^{-1} = e^{-\frac{1}{2} h_{k \mu}^d} H \left\{ \mathcal{Y}_+(\phi_k + f_k, \mu_k) - \frac{1}{2} \partial_{z_+} h_k H \right\}(69)$$

Next, using the commutation relations between $H$ and $\mathcal{Y}_+$, eq. (69) becomes

$$\partial_{z_+} Y_+(k + 1) Y_+(k + 1)^{-1} = \mathcal{Y}_+(\phi_k + f_k + h_k, \mu_k)$$

$$+ \left[ m \mu_k \sinh(\phi_k + f_k) - m \mu_{k+1} \sinh(\phi_k + f_k + h_k) - \frac{1}{2} \partial_{z_+} h_k \right] H$$

Choosing the $z_+$ evolution of $h_k$ as

$$\partial_{z_+} h_k = 2m \mu_k \sinh(\phi_k + f_k) - 2m \mu_{k+1} \sinh(\phi_k + f_k + h_k) \quad (70)$$
we recover an expression of the form eq.(67) with
\[ \phi_{k+1} = \phi_k + f_k + h_k \]  \hspace{1cm} (71)

Eq.(68,71,70) describe the solution of the dressing problem.

Next, we relate these equations to the Bäcklund transformation s. Let us set:
\[ f_k = \varphi_{k-1} - \varphi_k \hspace{1cm} \varphi_0 = 0 \]
\[ h_k = \rho_k - \rho_{k-1} \hspace{1cm} \rho_0 = 0 \]

Eq.(71) gives \( \phi_k = \rho_{k-1} - \varphi_{k-1} \) and therefore \( \phi_k + f_k = \rho_{k-1} - \varphi_k \). The recursion equations (68,70) become
\[ \partial z_+ (\varphi_{k-1} - \varphi_k) = 2m\mu_k [\sinh(\rho_{k-1} - \varphi_k) - \sinh (\rho_{k-1} - \varphi_{k-1})] \]
\[ \partial z_+ (\rho_k - \rho_{k-1}) = 2m\mu_k \sinh(\rho_{k-1} - \varphi_k) - 2m\mu_{k+1} \sinh(\rho_k - \varphi_k) \]
from which we deduce
\[ \partial z_+ (\varphi_{k-1} + \rho_k) + 2m\mu_k \sinh(\rho_{k-1} - \varphi_{k-1}) = \partial z_+ (\varphi_k + \rho_k) + 2m\mu_{k+1} \sinh(\rho_k - \varphi_k) \]
\[ \partial z_+ (\varphi_k + \rho_{k-1}) + 2m\mu_k \sinh(\rho_{k-1} - \varphi_k) = \partial z_+ (\varphi_{k+1} + \rho_k) + 2m\mu_{k+1} \sinh(\rho_k - \varphi_{k+1}) \]
or
\[ \partial z_+ (\varphi_k + \rho_k) = 2m\mu_{k+1} \sinh(\varphi_k - \rho_k) \]  \hspace{1cm} (72)
\[ \partial z_+ (\varphi_k + \rho_{k-1}) = 2m\mu_k \sinh(\varphi_k - \rho_{k-1}) \]  \hspace{1cm} (73)

Similarly, one shows that the factorization problem for \( g_+ \) is solved in the same way. The \( z_- \) dependence can also be treated in a similar way.

11 Appendix D: Determination of the \( \tau \)-functions.

In order to solve the successive set of Bäcklund transformations eqs.(49, 50,51,52) by using the ‘tangent rule’, we have to introduce a larger set of transformations. Before plunging into the general case, let us first consider the two-soliton case. Starting from the zero-soliton solution \( \Phi_0 \) and from two one-soliton solutions, \( \Phi_1 \) and \( \Phi_2 \), with parameter \( a_i, \mu_i, i = 1, 2 \), we consider
the following Bäcklund transformations:

\[ \Phi_0 \xrightarrow{\mu_1} \Phi_1 \xleftarrow{-\mu_2} \Phi_0 \]
\[ \Phi_0 \xrightarrow{-\mu_2} \Phi_1 \xleftarrow{\mu_1} \Phi_0 \]
\[ \Phi_0 \xrightarrow{-\mu_1} \Phi_1 \xleftarrow{\mu_2} \Phi_0 \]
\[ \Phi_0 \xrightarrow{\mu_2} \Phi_1 \xleftarrow{-\mu_1} \Phi_0 \]

The upper line is formed with the Bäcklund transformations involved in the dressing problem, eq. (53). The lower line is the same but with the exchange of the indices 1 and 2. Using the ‘tangent rule’ applied to the \( \tau \)-functions, we first compute \( \Phi^{(2)}_1 \) from the knowledge of \( \Phi_0 \) and \( \Phi_1 \). By exchange of \( \mu_1 \) and \( \mu_2 \), it also gives \( \Phi^{(1)}_2 \). Then we compute \( \Phi_{12} \) from \( \Phi_0, \Phi^{(2)}_1 \) and \( \Phi^{(1)}_2 \).

We find:

\[ \tau_\pm(\Phi^{(2)}_1) = 1 \pm \beta_{21}X_1 \]
\[ \tau_\pm(\Phi_{12}) = 1 \pm X_1 \pm X_2 + \beta^2_{12}X_1X_2 \]
as it should be.

We now proceed by proving by induction that:

\[ \tau(\Phi_{1\ldots N}) = \tau_{1\ldots N}(X_j) \]
\[ \tau(\Phi^{(a)}_{1\ldots N}) = \tau_{1\ldots N}(\beta_{a,j}X_j) \]
\[ \tau(\Phi^{(a,b)}_{1\ldots N}) = \tau_{1\ldots N}(\beta_{bc,j}\beta_{a,j}X_j) \quad (74) \]

The lower indices on \( \Phi \) refer to the number of soliton parameters \( X_j \), and the upper indices refer to the shift of the factors \( X_j \) in the \( \tau \)-functions. We
assume that the fields are connected via Bäcklund transformations as follows:

\[ \cdots \Phi_{1\ldots N-1} \]
\[ \downarrow -\mu_N \]
\[ \Phi^{(N)}_{1\ldots N-1} \]
\[ \cdots \Phi^{(1;N)}_{2\ldots N-1} \]
\[ \downarrow \mu_1 \]
\[ \Phi^{(1)}_{1\ldots N} \]
\[ \downarrow -\mu_N \]
\[ \Phi^{(N+1)}_{1\ldots N-1} \]
\[ \cdots \Phi^{(1;N+1)}_{2\ldots N-1} \]
\[ \downarrow \mu_1 \]
\[ \Phi^{(1)}_{2\ldots N-1} \]
\[ \downarrow -\mu_N \]
\[ \Phi^{(N+1)}_{2\ldots N-1} \]
\[ \cdots \Phi^{(2;N+1)}_{2\ldots N-1} \]
\[ \downarrow \mu_2 \]
\[ \Phi^{(2)}_{2\ldots N+1} \]
\[ \cdots \Phi^{(3)}_{3\ldots N+1} \]
\[ \downarrow -\mu_2 \]

As the recursion hypothesis we assume that the relations (74) have been proved for all the fields in (75) but the last four ones: \( \Phi^{(1;N+1)}_{2\ldots N} \), \( \Phi^{(N+1)}_{2\ldots N} \), \( \Phi^{(1)}_{2\ldots N+1} \) and \( \Phi_{1\ldots N+1} \). Then, we have to prove that the relations (74) hold also for these four fields. The field \( \Phi^{(2;N+1)}_{2\ldots N+1} \) is computed from \( \Phi^{(1;N+1)}_{2\ldots N} \), \( \Phi^{(1)}_{2\ldots N+1} \) and \( \Phi^{(N+1)}_{2\ldots N+1} \) using the ‘tangent rule’. Then \( \Phi^{(1;N+1)}_{1\ldots N} \) is computed from \( \Phi^{(1)}_{2\ldots N} \), \( \Phi^{(1)}_{1\ldots N+1} \) and \( \Phi^{(2;N+1)}_{2\ldots N+1} \), and similarly for \( \Phi^{(N+1)}_{2\ldots N+1} \); Finally, \( \Phi_{1\ldots N+1} \) is computed from \( \Phi^{(1;N+1)}_{2\ldots N} \), \( \Phi^{(N+1)}_{1\ldots N+1} \) and \( \Phi^{(1)}_{2\ldots N+1} \). In all these successive computations, proving the relations (74) reduces in proving the two following identities:

\[
\tau^{(N)}_+(\beta_{a;k}\beta_{b;k}X_k)\tau^{(N)}_-(X_k) + \tau^{(N)}_-(\beta_{a;k}\beta_{b;k}X_k)\tau^{(N)}_+(X_k) = \tau^{(N)}_+(\beta_{a;k}X_k)\tau^{(N)}_-(\beta_{b;k}X_k) + \tau^{(N)}_-(\beta_{a;k}X_k)\tau^{(N)}_+(\beta_{b;k}X_k) \tag{76}
\]

\[
\beta_{ab}\left[\tau^{(N)}_+(\beta_{a;k}\beta_{b;k}X_k)\tau^{(N)}_-(X_k) - \tau^{(N)}_-(\beta_{a;k}\beta_{b;k}X_k)\tau^{(N)}_+(X_k)\right] = \tau^{(N)}_+(\beta_{a;k}X_k)\tau^{(N)}_-(\beta_{b;k}X_k) - \tau^{(N)}_-(\beta_{a;k}X_k)\tau^{(N)}_+(\beta_{b;k}X_k) \tag{77}
\]
We demonstrate them also by induction assuming that the $\tau$-functions satisfy the following recursion relation:

$$\tau_{\pm}^{(N)}(X_k) = \tau_{\pm}^{(N-1)}(X_k) \pm X_N \tau_{\pm}^{(N-1)}(\beta_{N;k} X_k)$$

First let us start with (76). We consider the left hand side as a function of $\mu$, denoted $F(\mu)$. It has simple poles for $\mu = -\mu_k$ and is finite at $\mu_a \to \infty$. Therefore, as a function of $\mu_a$

$$F(\mu_a) = \sum_{k=1}^{N} \frac{Res_k}{\mu_a + \mu_k} + F(\mu_a = \infty)$$

Since $\beta_{a;k} \to 1$ for $\mu_a \to \infty$, the relation (74) is obviously satisfied at $\mu_a = \infty$. Hence, we only have to show that the residues at $\mu_a = -\mu_k$ of the left hand side and of the right hand side coincide. By symmetry, we can choose to only pick up the residue at $\mu_a = -\mu_N$. Using the fact that:

$$\beta_{a;k}|_{\mu_a=-\mu_N} = \beta_{N;k}^{-1}$$

and the recursion formula for the $\tau$-functions, the equality of the residue at $\mu_a = -\mu_N$ is equivalent to:

$$\beta_{b;N} \left[ \tau_{+}^{N-1}(\beta_{b;k} \beta_{N;k} X_k) \tau_{-}^{N}(X_k) - \tau_{-}^{N-1}(\beta_{b;k} \beta_{N;k} X_k) \tau_{+}^{N}(X_k) \right] = \tau_{-}^{N-1}(\beta_{N;k} X_k) \tau_{+}^{N}(\beta_{b;k} X_k) - \tau_{+}^{N-1}(\beta_{N;k} X_k) \tau_{-}^{N}(\beta_{b;k} X_k)$$

(78)

Next, using once again the recursion formula for the $\tau$-functions, we expand the relation (78) in $X_N$. It is linear in $X_N$. The terms linear in $X_N$ are (after a rescaling $X_k \to \beta_{N;k}^{-1} X_k$):

$$\tau_{+}^{N-1}(\beta_{b;k} \beta_{N;k} X_k) \tau_{-}^{N}(X_k) + \tau_{-}^{N-1}(\beta_{b;k} \beta_{N;k} X_k) \tau_{+}^{N}(X_k)$$

$$= \tau_{+}^{N-1}(\beta_{b;k} X_k) \tau_{-}^{N-1}(\beta_{N;k} X_k) + \tau_{-}^{N}(\beta_{b;k} X_k) \tau_{+}^{N-1}(\beta_{N;k} X_k)$$

(79)

The terms independent in $X_N$ are:

$$\beta_{b;N} \left[ \tau_{+}^{N-1}(\beta_{b;k} \beta_{N;k} X_k) \tau_{-}^{N-1}(X_k) - \tau_{-}^{N-1}(\beta_{b;k} \beta_{N;k} X_k) \tau_{+}^{N-1}(X_k) \right] = \tau_{+}^{N-1}(\beta_{b;k} X_k) \tau_{-}^{N-1}(\beta_{N;k} X_k) - \tau_{-}^{N}(\beta_{b;k} X_k) \tau_{+}^{N-1}(\beta_{N;k} X_k)$$

(80)

These two last relations are the recursion relations (74) and (77) but one step before, i.e. for the $(N-1)$-soliton $\tau$-functions. The relation (74) is proved in the same way. Hence, the recursion is proved : the $\tau$-functions which satisfy the recursion formula solve the Bäcklund transformations (73).

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12 Appendix E. The one soliton vertex operator.

We prove eq.(60). We have

\[ e^{fV(-)} e^{fV(+)} = \sum_{n=0}^{\infty} \frac{f^n}{n!} N(V(-) + V(+))^n \]

where the symbol \(N()\) means writing \(V(-)\) on the left. Let

\[ \mathcal{I} = \frac{1}{4i\pi} \frac{d\lambda_1}{2i\pi \lambda_1} \frac{d\lambda_2}{2i\pi \lambda_2} \frac{\lambda_1 + \mu}{\lambda_1 - \mu} \frac{\lambda_2 + \mu}{\lambda_2 - \mu} \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right)^2 : V(\lambda_1)V(\lambda_2) : \]

We have

\[ V(-)^2 \quad = \quad \oint_{|\lambda_1| > |\lambda_2| > |\mu|} \mathcal{I} \]
\[ V(+)^2 \quad = \quad \oint_{|\mu| > |\lambda_1| > |\lambda_2|} \mathcal{I} \]
\[ V(-)V(+) \quad = \quad -\oint_{|\lambda_1| > |\mu| > |\lambda_2|} \mathcal{I} \]

We reduce all the integrals to the contour \(|\mu| > |\lambda_1| > |\lambda_2|\). Then, we have

\[ V(-)^2 \quad = \quad -V(-)V(+) + \frac{1}{2} \oint_{|\lambda| > |\mu|} \frac{d\lambda}{2i\pi \lambda} \frac{\lambda - \mu}{\lambda + \mu} : V(\lambda)V(\mu) : \]
\[ V(-)V(+) \quad = \quad -V(+)^2 - \frac{1}{2} \oint_{|\mu| < |\lambda|} \frac{d\lambda}{2i\pi \lambda} \frac{\lambda - \mu}{\lambda + \mu} : V(\lambda)V(\mu) : \]

It follows that

\[ V(+)^2 + 2V(-)V(+) + V(-)^2 = \frac{1}{4} \]

Now, one has

\[ N(V(-) + V(+))^n+1 = V(-)N(V(-) + V(+))^n + N(V(-) + V(+))^nV(+) \]

and therefore

\[ N(V(-) + V(+))^n \quad = \quad 2^{-n} \quad n \text{ even} \]
\[ N(V(-) + V(+))^n \quad = \quad 2^{-n+1}(V(-) + V(+)) \quad n \text{ odd} \]

and the result follows.
Appendix F: The matrices $R(j, k)$.

Next, we prove eq. (62). By direct computation, we have

$$e^{\frac{1}{2}hH}V(\mu)e^{-\frac{1}{2}hH} = \frac{1}{2}(\cosh h + 1)V(\mu) - \frac{1}{2}(\cosh h - 1)V(-\mu) + \frac{i}{\sqrt{2}}\sinh h \frac{d}{d\mu}Z(\mu)$$

$$e^{\frac{1}{2}hH}i \frac{d}{d\mu}Z(\mu)e^{-\frac{1}{2}hH} = \cosh h i \frac{d}{d\mu}Z(\mu) + \frac{1}{2} \sinh h[V(\mu) - V(-\mu)]$$

or

$$e^{\frac{1}{2}hH}E(\mu_k)|e^{-\frac{1}{2}hH} = |E(\mu_k)|R_1(j, k)$$

with

$$R_1(j, k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh^2 \frac{h_j}{2} - \sinh^2 \frac{h_j}{2} & -\frac{1}{2} \sinh h_j & 0 \\ 0 & -\sinh^2 \frac{h_j}{2} \cosh^2 \frac{h_j}{2} & -\frac{1}{2} \sinh h_j & 0 \\ 0 & \sinh h_j & -\sinh h_j & \cosh h_j \end{pmatrix}$$

also

$$ad V^{(+)}_{\mu_j}.V(\mu_k) = \frac{i}{\sqrt{2}} \frac{\mu_j - \mu_k}{\mu_j + \mu_k} \frac{\mu_k}{d\mu_k}Z(\mu_k) - \frac{\mu_j \mu_k}{(\mu_j + \mu_k)^2} |\mu_j > |\mu_k|$$

$$ad V^{(+)}_{\mu_j}.\frac{d}{d\mu_k}Z(\mu_k) = -\frac{i}{\sqrt{2}} \left[ \frac{\mu_j + \mu_k}{\mu_j - \mu_k}V(\mu_k) - \frac{\mu_j - \mu_k}{\mu_j + \mu_k}V(-\mu_k) \right] |\mu_j > |\mu_k|$$

This can be rewritten in a more compact notation

$$ad V^{(+)}_{\mu_j}.E(\mu_k) = |E(\mu_k)| \left( \begin{array}{cccc} 0 & -\frac{\mu_j + \mu_k}{\mu_j + \mu_k} & \frac{\mu_j + \mu_k}{(\mu_j - \mu_k)^2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \frac{\mu_j + \mu_k}{\mu_j - \mu_k} \\ 0 & 0 & 0 & -\frac{1}{2} \frac{\mu_j + \mu_k}{\mu_j - \mu_k} \\ 0 & \frac{\mu_j - \mu_k}{\mu_j + \mu_k} & -\frac{\mu_j + \mu_k}{\mu_j - \mu_k} & 0 \end{array} \right)$$

By exponentiation, we get

$$e^{f_jV^{(+)}_{\mu_j}}E(k)|e^{f_jV^{(+)}_{\mu_j}} = |E(k)|R_2(j, k)$$
with
\[
R_2(j, k) = \begin{pmatrix}
1 & \frac{1}{4}(1 - \beta_{jk}^{-2}) \sinh f_j & \frac{1}{4}(1 - \beta_{jk}^{-2}) \sinh f_j & \frac{1}{2}(\beta_{jk} - \beta_{jk}^{-1}) \sinh^2 \frac{f_j}{2} \\
0 & \cosh^2 \frac{f_j}{2} & -\beta_{jk}^{-2} \sinh^2 \frac{f_j}{2} & -\frac{1}{2} \beta_{jk}^{-1} \sinh f_j \\
0 & -\beta_{jk} \sinh^2 \frac{f_j}{2} & \cosh^2 \frac{f_j}{2} & \frac{1}{2} \beta_{jk} \sinh f_j \\
0 & -\beta_{jk} \sinh f_j & \beta_{jk}^{-1} \sinh f_j & \cosh f_j
\end{pmatrix}
\]

From the definition of \( g_+(k) \) in eq.(39), we immediately get eq.(62) with
\[
R(j, k) = R_2(j, k) R_1(j, k)
\]

### 14 Appendix G.

Finally, we prove eq.(63). We notice that we can write
\[
\tilde{v}(j, k) = \left( \frac{1}{4} (1 - \beta_{jk}^{-2}) \cosh f_j, \frac{1}{4} (1 - \beta_{jk}^{-2}) \sinh f_j, -\frac{1}{2} (\beta_{jk} - \beta_{jk}^{-1}) \sinh^2 \frac{f_j}{2} \right)
\]

and \( \tilde{R} \) is an explicitly known 3x3 matrix. With these notations, eq.(63) is equivalent to the following set of two equations
\[
\tilde{R}(1, k) \tilde{R}(2, k) \cdots \tilde{R}(k - 1, k) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \Delta_k X_k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

and
\[
\cosh \frac{f_k}{2} + 2 \sinh \frac{f_k}{2} \sum_{j=1}^{k-1} \tilde{v}(j, k) \tilde{R}(j, k) \tilde{R}(j + 1, k) \cdots \tilde{R}(k - 1, k) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \Delta_k
\]

Eq.(82) is an eigenvector problem. To show its consistency, we remark that each \( \tilde{R}(j, k) \) is a matrix of the form
\[
\tilde{R} = \begin{pmatrix}
a^2 & -b^2 & ab \\
-b^2 & d^2 & -cd \\
2ac & -2bd & 1 + 2bc
\end{pmatrix}
\]

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This form and the condition \(ad - bc = 1\) are preserved by product. On the coefficients \(a, b, c, d\) the product rule is just the matrix product for the 2x2 matrices

\[
\begin{pmatrix}
ar(j, k) & b(j, k) \\
c(j, k) & d(j, k)
\end{pmatrix}
\]

we have

\[
\begin{pmatrix}
cosh \frac{f_j}{2} & -\beta^{-1}_{jk} \sinh \frac{f_j}{2} \\
-\beta_{jk} \sinh \frac{f_j}{2} & \cosh \frac{f_j}{2}
\end{pmatrix}
\begin{pmatrix}
cosh \frac{h_j}{2} & \sinh \frac{h_j}{2} \\
\sinh \frac{h_j}{2} & \cosh \frac{h_j}{2}
\end{pmatrix}
\]

Then eq.(82) is equivalent to

\[
r(1, k)r(2, k) \cdots r(k - 1, k) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha_k \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

with

\[
\alpha_k^2 = \frac{\Delta_k X_k}{\sinh \frac{f_k}{2}}
\]

Eq.(84) reduces to the single equation

\[
[r(1, k)r(2, k) \cdots r(k - 1, k)]_{21} = 0
\]

This equation determines recursively \(\tanh(h_{k-1}/2)\). Therefore, for such \(h_k\), eq.(82) is consistent. The eigenvalue is

\[
\alpha_k = [r(1, k)r(2, k) \cdots r(k - 1, k)]_{11}
\]

It depends only on the fields \(f_j\), \(j \leq k - 1\). Let \(\gamma_k\) be the C-number defined by

\[
\gamma_k = \sum_{j=1}^{k-1} \tilde{v}(j, k) \tilde{R}(j, k) \tilde{R}(j + 1, k) \cdots \tilde{R}(k - 1, k) \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

Again, this depends only on the fields \(f_j\) \(j \leq k - 1\). Then eq.(83) gives

\[
\tanh \frac{f_k}{2} = \frac{X_k}{\alpha_k^2 - 2\gamma_k X_k}
\]
and this determines $f_k$ in terms of the $f_j$, $j \leq k - 1$ and the extra variable $X_k$.

Consistency of the whole construction ensures that the fields $h_k$ and $f_k$ so determined are the same as those of eq.(49-52).

Let us do it for the two soliton case. The condition $r(1,2)|_{21} = 0$ gives

$$\tanh \frac{h_1}{2} = \beta_{12} \tanh \frac{f_1}{2}$$

We recognize the ‘tangent rule’. Then we calculate

$$\alpha_2 = \frac{\cosh \frac{h_1}{2}}{\cosh \frac{f_1}{2}} = \sqrt{\frac{1 - X_1^2}{1 - \beta_{12}^2 X_1^2}}$$

$$\gamma_2 = \frac{1}{4} (1 - \beta_{12}^2) \sinh f_1 \frac{\cosh^2 \frac{h_1}{2}}{\cosh^2 \frac{f_1}{2}} = \frac{1}{2} (1 - \beta_{12}^2) \frac{X_1}{1 - \beta_{12}^2 X_1^2}$$

From eq.(86), we get

$$e^{f_2} = \frac{1 - X_1}{1 + X_1} \cdot \frac{1 + X_1 + X_2 + \beta_{12}^2 X_1 X_2}{1 - X_1 - X_2 + \beta_{12}^2 X_1 X_2} = \frac{\tau_-(1)}{\tau_+(1)} \cdot \frac{\tau_+(2)}{\tau_-(2)}$$

Finally, we can compute

$$\Delta_2 = \frac{\tau_0}{\sqrt{\tau_+(2) \tau_-(2)}}$$

as it should be.

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