GALOIS COHOMOLOGY OF REAL SEMISIMPLE GROUPS

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Abstract. Let $G$ be a connected, compact, semisimple algebraic group over the field of real numbers $\mathbb{R}$. Using Kac diagrams, we describe combinatorially the first Galois cohomology sets $H^1(\mathbb{R}, H)$ for all inner forms $H$ of $G$. As examples, we compute explicitly $H^1$ for all real forms of the simply connected simple group of type $E_7$ (which has been known since 2013) and for all real forms of half-spin groups of type $D_{2k}$ (which seems to be new).

0. Introduction

Let $H$ be a linear algebraic group defined over the field of real numbers $\mathbb{R}$. For the definition of the first (nonabelian) Galois cohomology set $H^1(\mathbb{R}, H)$ see Section 4 below. Galois cohomology can be used to answer many natural questions (on classification of real forms, on the connected components of the set of $\mathbb{R}$-points of a homogeneous space etc.). The Galois cohomology sets $H^1(\mathbb{R}, H)$ of the classical groups are well known. Recently the sets $H^1(\mathbb{R}, H)$ were computed for “most” of the simple $\mathbb{R}$-groups by Adams [A], in particular, for all simply connected simple $\mathbb{R}$-groups by Adams [A] and by Borovoi and Evenor [BE].

Victor G. Kac [K] used what was later called Kac diagrams (see Onishchik and Vinberg [OV2, Sections 3.3.7 and 3.3.11]) to classify the conjugacy classes of automorphisms of finite order of a simple Lie algebra over the field of complex numbers $\mathbb{C}$. Let $G$ be a compact (anisotropic), simply connected, simple algebraic group over $\mathbb{R}$. Write $G_C = G \times_{\mathbb{R}} \mathbb{C}$, $g_C = \text{Lie}(G_C)$. With this notation, Kac classified the conjugacy classes of elements of order $n$ in $\text{Aut} g_C = \text{Aut} G_C$. In particular, he classified the conjugacy classes of elements of order $n$ in the group of inner automorphisms $G^{ad}(\mathbb{C}) \subset \text{Aut} G_C$, where $G^{ad} := G/Z_G$ is the corresponding adjoint group. Equivalently, he classified the conjugacy classes of elements of order $n$ in $G^{ad}(\mathbb{R})$.

Note that the set of conjugacy classes of elements of order $n$ in $G^{ad}(\mathbb{R})$ is in canonical bijection with the first Galois cohomology set $H^1(\mathbb{R}, G^{ad})$, see Serre [S] Section III.4.5, Theorem 6]. Thus Kac computed $H^1(\mathbb{R}, G^{ad})$, the Galois cohomology of the compact, simple, adjoint $\mathbb{R}$-group $G^{ad}$.

In the present paper we use the method of Kac diagrams in order to compute $H^1(\mathbb{R}, G)$, or more generally $H^1(\mathbb{R}, qG)$, where $G$ is a connected, compact, semisimple $\mathbb{R}$-group, not necessarily adjoint, and $qG$ is the inner twisted form of $G$ corresponding to a Kac diagram $q$. This is reduced

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to classifying conjugacy classes of square roots of a given central element $z = z_0 \in G(\mathbb{R})$.

The plan of the paper is as follows. In Section 1 we introduce the necessary notation. In Section 2 we describe, following Bourbaki [Bou], the action of $P^\vee / Q^\vee$ on the extended Dynkin diagram of a root system $R$, where $P^\vee$ is the coroot lattice and $Q^\vee$ is the coweight lattice. The heart of the paper is Section 3, where we prove Theorem 3.4 describing the conjugacy classes of $n$-th roots of a given central element $z$ in a connected semisimple compact Lie group $G$ in terms of certain combinatorial objects called Kac $n$-labelings of the extended Dynkin diagram $\tilde{D}$ of $G$. Using this theorem (in the case $n = 2$) and a result of [B1], in Section 4 we prove Theorem 4.3, which is the main result of this paper. It describes the first Galois cohomology set $H^1(\mathbb{R}, qG)$ of an inner twisted form $qG$ of a connected compact (anisotropic) semisimple $\mathbb{R}$-group $G$ in terms of Kac 2-labelings. As an example, in Section 5 we compute, using Kac 2-labelings, the Galois cohomology sets $H^1(\mathbb{R}, qG)$ for all $\mathbb{R}$-forms of the compact simply connected group $G$ of type $E_7$; these results were obtained earlier by other methods in [A] and [BE], see also Conrad [C, Proof of Lemma 4.9]. As another example, in Section 6 we compute the Galois cohomology sets $H^1(\mathbb{R}, qG)$ for all $\mathbb{R}$-forms of a half-spin compact group of type $D_\ell$ for even $\ell > 4$; these results seem to be new.

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1. Notation

In this paper $G$ always is a connected, compact (anisotropic), semisimple algebraic group over the field of real numbers $\mathbb{R}$. We write $Z_G$ for the center of $G$. Let $G^{ad} = G / Z_G$ denote the corresponding adjoint group, and let $G^{sc}$ denote the universal covering of $G$ (which is simply connected). Let $T \subset G$ be a maximal torus. We denote by $t$ the Lie algebra of $T$, which is a vector space over $\mathbb{R}$. Let $N = N_G(T)$ denote the normalizer of $T$ in $G$. Let $W = N / T$ be the Weyl group, which is a finite algebraic group.

Let $T^{ad} := T / Z_G$ be the image of $T$ in $G^{ad}$, and let $T^{sc}$ denote the preimage of $T$ in $G^{sc}$. Then $T^{ad}$ is a maximal torus in $G^{ad}$, and $T^{sc}$ is a maximal torus in $G^{sc}$. Set

$X = X(T_C) := \text{Hom}(T_C, G_{m, \mathbb{C}}), \quad X^\vee = X^\vee(T_C) := \text{Hom}(G_{m, \mathbb{C}}, T_C),$

where $T_C = T \times \mathbb{R} \mathbb{C}$ and $G_{m, \mathbb{C}}$ is the multiplicative group over $\mathbb{C}$; then $X$ and $X^\vee$ are the character group and the cocharacter group of $T_C$, respectively.

We have a canonical isomorphism of abelian complex Lie groups

$X^\vee \otimes \mathbb{C}^\times \cong \tilde{T}(\mathbb{C}), \quad \chi \otimes u \mapsto \chi(u), \quad \chi \in X^\vee, \ u \in \mathbb{C}^\times = G_{m, \mathbb{C}}(\mathbb{C}).$

Thus we obtain an isomorphism of abelian complex Lie algebras (vector spaces over $\mathbb{C}$)

$X^\vee \otimes \mathbb{C} \cong \text{Lie}(T_C), \quad \chi \otimes v \mapsto d\chi(v), \quad \chi \in X^\vee, \ v \in \mathbb{C},$

$d\chi := d_{\chi}: \mathbb{C} = G_{m, \mathbb{C}} \rightarrow \text{Lie}(T_C).$
We obtain the standard embedding
\[ X^\vee \hookrightarrow X^\vee \otimes \mathbb{C} \xrightarrow{\rho} \text{Lie} T_\mathbb{C}, \quad \chi \mapsto \chi \otimes 1 \mapsto d\chi(1). \]

As usual, we set
\[ P = X(T^\text{sc}_\mathbb{C}), \quad Q = X(T^\text{ad}_\mathbb{C}); \]
these are the weight lattice and the root lattice. We set also
\[ P^\vee = X^\vee(T^\text{sc}_\mathbb{C}), \quad Q^\vee = X^\vee(T^\text{ad}_\mathbb{C}); \]
these are the coweight lattice and the coroot lattice. Then
\[ Q \subset X \subset P \quad \text{and} \quad Q^\vee \subset X^\vee \subset P^\vee. \]

Let \( G \) and \( T \) be as above. We write \( G = G(\mathbb{R}) \) for the set of \( \mathbb{R} \)-points of \( G \), and similarly we write \( G^\text{ad} = G^\text{ad}(\mathbb{R}) \), \( G^\text{sc} = G^\text{sc}(\mathbb{R}) \). We write \( T = T(\mathbb{R}) \), and similarly we write \( T^\text{ad} = T^\text{ad}(\mathbb{R}) \), \( T^\text{sc} = T^\text{sc}(\mathbb{R}) \). We write \( N = N(\mathbb{R}) \) and \( W = W(\mathbb{R}) \). We write \( Z_G = Z_G(\mathbb{R}) \) for the center of \( G \).

We define an action of the group \( X^\vee \rtimes W \) on the set \( t \) as follows: an element \( \chi \in X^\vee \subset t_\mathbb{C} \) acts by translation by \( \imath \chi \in t \) (where \( \imath^2 = -1 \)), and \( w \in W \subset \text{Aut} T \) acts on \( t = \text{Lie} T \) as usual, i.e., as \( d_t w : \text{Lie} T \to \text{Lie} T \). It follows that the groups \( Q^\vee \rtimes W \) and \( P^\vee \rtimes W \) act on \( t \).

Let \( R = R(G_C, T_C) \) denote the root system of \( G_C \) with respect to \( T_C \). Let \( \Pi \subset R \) be a basis (a system of simple roots). Let \( D = D(G, T, \Pi) = D(R, \Pi) \) denote the Dynkin diagram; the set of the vertices of \( D \) is \( \Pi \).

Assume that \( G \) is (almost) simple. We write \( \Pi = \{ \alpha_1, \ldots, \alpha_\ell \} \). Let \( \tilde{D} = \tilde{D}(G, T, \Pi) = \tilde{D}(R, \Pi) \) denote the extended Dynkin diagram; the set of vertices of \( \tilde{D} \) is \( \tilde{\Pi} = \{ \alpha_1, \ldots, \alpha_\ell, \alpha_0 \} \), where \( \alpha_1, \ldots, \alpha_\ell \) are the simple roots, and \( \alpha_0 \) is the lowest root. These roots \( \alpha_1, \ldots, \alpha_\ell, \alpha_0 \) are linearly dependent, namely,
\[ m_{\alpha_1} \alpha_1 + \cdots + m_{\alpha_\ell} \alpha_\ell + m_{\alpha_0} \alpha_0 = 0, \]
where the coefficients \( m_{\alpha_j} \) are positive integers for all \( j = 1, \ldots, \ell, 0 \) and where \( m_{\alpha_0} = 1 \). We write \( m_j \) for \( m_{\alpha_j} \). These coefficients \( m_j \) are tabulated in [OV1] Table 6 and in [OV2] Table 3.

Now assume that \( G \) is semisimple, not necessarily simple. Then we have a decomposition \( G = G^{(1)} \cdot G^{(2)} \cdots G^{(r)} \) into an almost direct product of simple groups. Then \( T = T^{(1)} \cdot T^{(2)} \cdots T^{(r)} \) (an almost direct product of tori), where each \( T^{(k)} \) is a maximal torus in \( G^{(k)} \) \( (k = 1, \ldots, r) \). We write \( t = \text{Lie} T, \ t^{(k)} = \text{Lie} T^{(k)} \), then
\[ t = t^{(1)} \oplus \cdots \oplus t^{(r)}. \]

The root system \( R \) decomposes into a “direct sum” of irreducible root systems
\[ R = R^{(1)} \sqcup \cdots \sqcup R^{(r)} \]
(disjoint union), where \( R^{(k)} = R(G^{(k)}_C, T^{(k)}_C) \), and we have
\[ \Pi = \Pi^{(1)} \sqcup \cdots \sqcup \Pi^{(r)}, \]
where each subset \( \Pi^{(k)} \) \( (k = 1, \ldots, r) \) is a basis of \( R^{(k)} \). We have
\[ D = D^{(1)} \sqcup \cdots \sqcup D^{(r)}, \]
where each connected component $D^{(k)}$ ($k = 1, \ldots, r$) is the Dynkin diagram of the irreducible root system $R^{(k)}$ with respect to $\Pi^{(k)}$. Let $\alpha_0^{(k)} \in R^{(k)}$ denote the lowest root of $R^{(k)}$. Let $\tilde{D}^{(k)}$ denote the extended Dynkin diagram of $R^{(k)}$ with respect to $\Pi^{(k)}$, then the set of vertices of $\tilde{D}^{(k)}$ is $\tilde{\Pi}^{(k)} := \Pi^{(k)} \cup \{ \alpha_0^{(k)} \}$. We define the extended Dynkin diagram of $R$ with respect to $\Pi$ to be $\tilde{D} = \tilde{D}^{(1)} \sqcup \cdots \sqcup \tilde{D}^{(r)}$; then the set of vertices of $\tilde{D}$ is $\tilde{\Pi} = \tilde{\Pi}^{(1)} \sqcup \cdots \sqcup \tilde{\Pi}^{(r)} = \Pi \sqcup \tilde{\Pi}_0$, where $\tilde{\Pi}_0 = \{ \alpha_0^{(1)}, \ldots, \alpha_0^{(r)} \}$. For each $k = 1, \ldots, r$, let $(m_\beta)_{\beta \in \tilde{\Pi}^{(k)}}$ be the coefficients of linear dependence $\sum_{\beta \in \tilde{\Pi}^{(k)}} m_\beta \beta = 0$ normalized so that $m_{\alpha_0^{(k)}} = 1$. Then $m_\beta \in \mathbb{Z}$, $m_\beta \geq 0$ for any $\beta \in \tilde{\Pi}$.

2. Action of $P^\vee/Q^\vee$ on the extended Dynkin diagram

First let $G$ be a simple compact $\mathbb{R}$-group. Recall that $t$ denotes the Lie algebra of $T$. Following [OV2, Section 3.3.6], we introduce the barycentric coordinates $x_{\alpha_1}, \ldots, x_{\alpha_\ell}, x_{\alpha_0}$ of a point $x \in t$ by setting $d\alpha_j(x) = ix_{\alpha_j}$ for $j = 1, \ldots, \ell$, $d\alpha_0(x) = i(x_{\alpha_0} - 1)$, where $i^2 = -1$. We write $x_j$ for $x_{\alpha_j}$. By (1) we have

$$0 = \left( \sum_{j=0}^\ell m_j d\alpha_j \right)(x) = i \left( -1 + \sum_{j=0}^\ell m_j x_j \right),$$

hence

$$\sum_{j=0}^\ell m_j x_j = 1. \quad (2)$$

By [Bou, Section VI.2.1] and [Bou, Section VI.2.2, Proposition 5(i)], see also [OV2, Section 3.3.6, Proposition 3.10(2)], the closed simplex $\Delta \subset t$ given by the inequalities

$$x_1 \geq 0, \ldots, x_n \geq 0, x_0 \geq 0$$

is a fundamental domain for the affine Weyl group $Q^\vee \rtimes W$, where $W$ is the usual Weyl group. This means that every orbit of $Q^\vee \rtimes W$ intersects $\Delta$ in one and only one point.

Now let $G$ be a semisimple (not necessarily simple) compact $\mathbb{R}$-group. We introduce the barycentric coordinates $(x_\beta)_{\beta \in \tilde{\Pi}}$ of $x$ defined by

$$d\beta(x) = ix_\beta \text{ for } \beta \in \Pi, \quad d\beta(x) = i(x_\beta - 1) \text{ for } \beta \in \tilde{\Pi}_0 = \tilde{\Pi} \setminus \Pi,$$

they satisfy

$$\sum_{\beta \in \tilde{\Pi}^{(k)}} m_\beta x_\beta = 1 \quad \text{ for each } k = 1, \ldots, r,$$
Real Galois Cohomology

see (2). Write \( t = \bigoplus_{k=1}^{r} t_k \). For each \( k = 1, \ldots, r \), let \( \Delta^{(k)} \) denote the closed simplex in \( t^{(k)} \) given by the inequalities

\[ x_\beta \geq 0 \quad \text{for} \ \beta \in \tilde{\Pi}^{(k)}. \]

Then the product \( \Delta = \prod_{k=1}^{r} \Delta^{(k)} \) is the closed subset in \( t \) given by the inequalities

\[ x_\beta \geq 0 \quad \text{for} \ \beta \in \tilde{\Pi}, \]

and \( \Delta \) is a fundamental domain for the affine Weyl group \( Q \) and \( \Delta \) is a fundamental domain for the affine Weyl group in one and only one point.

The group \( (X^\vee \rtimes W)/(Q^\vee \rtimes W) = X^\vee/Q^\vee \cong \pi_1(G) \) acts on \( \Delta \). We wish to describe this action. Since \( X^\vee/Q^\vee \subset P^\vee/Q^\vee \), it suffices to describe the action of \( P^\vee/Q^\vee \), and it suffices to consider the case when \( R \) is irreducible.

From now on till the end of this section we assume that \( R \) is an irreducible root system. The action of \( P^\vee/Q^\vee \) on \( \Delta \) is given by permutations of coordinates corresponding to a subgroup of the automorphism group of the extended Dynkin diagram acting simply transitively on the set of vertices \( \alpha_j \) with \( m_j = 1 \). This action is described in [Bou] Section VI.2.3, Proposition 6.

Namely, let \( \omega^\vee_1, \ldots, \omega^\vee_\ell \) denote the set of fundamental coweights, i.e., the basis of \( P^\vee \) dual to the basis \( \alpha_1, \ldots, \alpha_\ell \) of \( Q \). Then the nonzero cosets of \( P^\vee/Q^\vee \) are represented by the fundamental coweights \( \omega^\vee_j \) such that \( i\omega^\vee_j \) belongs to \( \Delta \), i.e., by those \( \omega^\vee_j \) with \( m_j = 1 \). Let \( w_0 \), resp. \( w_j \), denote the longest element in \( W \), resp. in the Weyl group \( W_j \) of the root subsystem \( R_j \) generated by \( \Pi \setminus \{\alpha_j\} \). Then the transformation

\[ x \mapsto w_j w_0 x + i \omega^\vee_j \]

preserves \( \Delta \) whenever \( m_j = 1 \) and gives the action of the respective coset \([\omega^\vee_j]\) in \( P^\vee/Q^\vee \) on \( \Delta \).

Observe that the affine transformation (3) is an isometry of the Euclidean structure on \( t \) given by the restriction of the Killing form. Hence the action of \([\omega^\vee_j]\) preserves the Euclidean polytope structure of the simplex \( \Delta \). In particular, it permutes the vertices of \( \Delta \), which are equal to \( v_i = i\omega^\vee_j/m_i \) (\( i = 1, \ldots, \ell \)) and \( v_0 = 0 \), and the facets \( \Delta_i \) of \( \Delta \), which correspond to the roots \( \alpha_i \in \tilde{\Pi} \) (\( i = 1, \ldots, \ell, 0 \)), preserving the angles between the facets. Hence the action of \([\omega^\vee_j]\) induces a permutation \( \sigma = \sigma_j \) of the set \( \{1, \ldots, \ell, 0\} \) such that the facet \( \Delta_i \) maps to \( \Delta_{\sigma(i)} \), and the opposite vertex \( v_i \) is mapped to \( v_{\sigma(i)} \). In particular, \( \sigma_j \) takes 0 to \( j \).

Since the relative lengths of the roots in \( \tilde{\Pi} \) and the angles between them and between the respective facets of \( \Delta \) are read off from the extended Dynkin diagram \( \tilde{D} \), the permutation \( \sigma \) comes from an automorphism of \( \tilde{D} \). Furthermore, the action of \([\omega^\vee_j]\) permutes the barycentric coordinates \( x_i \) of a point \( x \in \Delta \), because they are determined by the vertices \( v_i \in \Delta \). Namely, any \( x \in \Delta \) is mapped to \( x' \in \Delta \) with coordinates \( x'_i = x_{\sigma^{-1}(i)} \). One obtains an action of \( P^\vee/Q^\vee \) on \( \tilde{D} \), which we describe below explicitly case by case, using [Bou] Planches I-IX, assertion (XII)].
If $G$ is of one of the types $E_8$, $F_4$, $G_2$, then $P^\vee/Q^\vee = 0$. If $G$ is of one of the types $A_1$, $B_\ell$ ($\ell \geq 3$), $C_\ell$ ($\ell \geq 2$), $E_7$, then $P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$, and the nontrivial element $P^\vee/Q^\vee$ acts on $\tilde{D}$ by the only nontrivial automorphism of $\tilde{D}$.

It remains to consider the cases $A_\ell$ ($\ell \geq 2$), $D_\ell$, and $E_6$. In order to describe the action of the group $P^\vee/Q^\vee$ on $\tilde{D}$, it suffices to describe its action on the set of vertices $\alpha_j$ of $\tilde{D}$ with $m_j = 1$. These are the images of $\alpha_0$ under the automorphism group of $\tilde{D}$.

Let $D$ be of type $A_\ell$, $\ell \geq 2$. The generator $[\omega_1^\vee]$ of $P^\vee/Q^\vee$ acts on $\tilde{D}$ as the cyclic permutation $0 \mapsto 1 \mapsto \ldots \mapsto \ell - 1 \mapsto \ell \mapsto 0$:

Let $D$ be of type $D_\ell$, $\ell \geq 4$ is even. We have $P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and the classes $[\omega_1^\vee]$ and $[\omega_{\ell - 1}^\vee]$ are generators of $P^\vee/Q^\vee$. These generators act on $\tilde{D}$ as follows: $[\omega_1^\vee]$ acts as $0 \leftrightarrow 1$, $\ell - 1 \leftrightarrow \ell$, and $[\omega_{\ell - 1}^\vee]$ acts as $0 \leftrightarrow \ell - 1$, $1 \leftrightarrow \ell$:

Let $D$ be of type $D_\ell$, $\ell \geq 5$ is odd. We have $P^\vee/Q^\vee \simeq \mathbb{Z}/4\mathbb{Z}$, and the class $[\omega_{\ell - 1}^\vee]$ is a generator of $P^\vee/Q^\vee$. This generator acts on $\tilde{D}$ as the 4-cycle $0 \leftrightarrow \ell - 1 \leftrightarrow 1 \leftrightarrow \ell \mapsto 0$:

Let $D$ be of type $E_6$. The generator $[\omega_1^\vee] \in P^\vee/Q^\vee$ acts as the 3-cycle $0 \leftrightarrow 1 \leftrightarrow 5 \mapsto 0$:
3. n-TH ROOTS OF A CENTRAL ELEMENT

Let $G$ a compact semisimple $\mathbb{R}$-group, not necessarily simple. Let $T, G, T, X, D, \tilde{D}$, etc. be as in Section 1.

Let $z \in Z_G$ and let $n$ be a positive integer. We consider the set of $n$-th roots of $z$ in $G$

$$G^n_z := \{g \in G \mid g^n = z\}.$$ 

In particular, $G_n := G^n_1$ is the set of $n$-th roots of 1 in $G$, i.e., the set of elements of order dividing $n$ in $G$.

The group $G$ acts on $G^n_z$ on the left by conjugation $g * a = gag^{-1}$ ($g \in G, a \in G^n_z$). We wish to compute the set $G^n_z/\sim$ of $n$-th roots of $z$ modulo conjugation.

Consider the set $T^n_z \subset G^n_z$ (note that $z \in Z \subset T$). The group $W$ acts on $T^n_z$ on the left by

$$w * t = ntn^{-1},$$

where $w = nT \in W, n \in N, t \in T$. It is easy to see that the embedding $T^n_z \hookrightarrow G^n_z$ induces a bijection $T^n_z/W \cong G^n_z/\sim$. Thus we wish to compute $T^n_z/W$.

We describe the set $T^n_z/W$ in terms of Kac $n$-labelings of $\tilde{D}$.

**Definition 3.1.** A Kac $n$-labeling of an extended Dynkin diagram $\tilde{D} = \tilde{D}^{(1)} \sqcup \cdots \sqcup \tilde{D}^{(r)}$, where each $\tilde{D}^{(k)}$ is connected for $k = 1, \ldots, r$, is a family of nonnegative integer numerical labels $p = (p_{\beta})_{\beta \in \Pi} \in \mathbb{Z}_{\geq 0}^\Pi$ at the vertices $\beta \in \tilde{\Pi}$ of $\tilde{D}$ satisfying

$$\sum_{\beta \in \Pi^{(k)}} m_{\beta}p_{\beta} = n \quad \text{for each } k = 1, \ldots, r.$$ 

Note that a Kac $n$-labeling $p$ of $\tilde{D} = \tilde{D}^{(1)} \sqcup \cdots \sqcup \tilde{D}^{(r)}$ is the same as a family $(p^{(1)}, \ldots, p^{(r)})$, where each $p^{(k)}$ is a Kac $n$-labeling of $\tilde{D}^{(k)}$.

Let $z \in Z_G \subset T$. We write

$$z = \exp 2\pi i \zeta, \quad \text{where } \zeta \in t_\mathbb{C}.$$ 

For $\lambda \in X$ consider $d\lambda(\zeta) \in \mathbb{C}$. We have

$$\exp 2\pi i d\lambda(\zeta) = \exp d\lambda(2\pi i \zeta) = \lambda(\exp 2\pi i \zeta) = \lambda(z).$$

Since $z$ is an element of finite order in $T$, we see that $\lambda(z)$ is a root of unity, hence by (1) $d\lambda(\zeta) \in \mathbb{Q}$, and it follows from (1) that the image of $d\lambda(\zeta)$ in $\mathbb{Q}/\mathbb{Z}$ depends only on $z$, and not on the choice of $\zeta$. Note that if $\lambda \in \mathbb{Q} \subset X$, then $\lambda(z) = 1$, hence $d\lambda(\zeta) \in \mathbb{Z}$.
Notation 3.2. We denote by $\mathcal{K}_n$ the set of Kac $n$-labelings of $\tilde{D}$, i.e., the set of $p = (p_\beta) \in \mathbb{Z}_{\geq 0}^\Pi$ satisfying (5). We denote by $\mathcal{K}_{n,\mathbb{R}}$ the set of families $p = (p_\beta) \in \mathbb{R}_+^\Pi$ satisfying (4), i.e., the set of tuples of barycentric coordinates of points in $n\Delta$. For $z \in Z_G$, we denote by $\mathcal{K}_n^z$ the set of Kac $n$-labelings $p \in \mathcal{K}_n$ of $\tilde{D}$ satisfying
\begin{align}
\sum_{\alpha \in \Pi} c_\alpha p_\alpha \equiv d\lambda(\zeta) \pmod{\mathbb{Z}}
\end{align}
for any generator $[\lambda]$ of $X/Q$ with $\lambda = \sum_{\alpha \in \Pi} c_\alpha \alpha$,

where $\zeta$ is as in (6). Condition (8) does not depend on the choice of $\zeta$ satisfying (6). We have $\mathcal{K}_n^z \subset \mathcal{K}_n \subset \mathcal{K}_{n,\mathbb{R}}$. The group $X^\vee/Q^\vee$ acts on $\mathcal{K}_{n,\mathbb{R}}$ and $\mathcal{K}_n$ via the action on $\tilde{D}$. We shall see below that the subset $\mathcal{K}_n^z$ of $\mathcal{K}_n$ is $X^\vee/Q^\vee$-invariant.

Construction 3.3. Let $p = (p_\beta) \in \mathcal{K}_{n,\mathbb{R}}$. Set
\[ x = (x_\beta)_{\beta \in \Pi} := (p_\beta/n)_{\beta \in \Pi} \in \mathcal{K}_{1,\mathbb{R}}, \]
then there exists a point $x \in \Delta \subset t$ with barycentric coordinates $(x_\beta)_{\beta \in \Pi}$. We set
\[ \varphi(p) = e(x) := \exp 2\pi x \in T. \]

The following theorem gives a combinatorial description of the set $T^*_n/W$ in terms of Kac $n$-labelings. It generalizes a result of Kac [K], who described, in particular, the set $T^*_n/W$ in the case when $G$ is an adjoint group.

Theorem 3.4. Let $G$ be a compact semisimple $\mathbb{R}$-group, $T \subset G$ be a maximal torus, $R = R(G_C, T_C)$ be the corresponding root system, $\Pi$ be a basis of $R$, $\tilde{D} = \tilde{D}(G, T, \Pi)$ be the corresponding extended Dynkin diagram. Let $n$ be a positive integer. Let $z \in Z_G$ be a central element. Then the subset $\mathcal{K}_n^z \subset \mathcal{K}_n$ is $X^\vee/Q^\vee$-invariant, and the map $\varphi: \mathcal{K}_{n,\mathbb{R}} \to T$ of Construction 3.3 induces a bijection
\[ \varphi_*: \mathcal{K}_n^z/(X^\vee/Q^\vee) \sim T^*_n/W \]
between the set of $X^\vee/Q^\vee$-orbits in $\mathcal{K}_n^z$ and the set of $W$-orbits in $T^*_n$.

Proof. Consider a $W$-orbit $[a]$ in $T/W$, where $a \in T$. Write $a = e(x)$ for some $x \in t$. The map $e: t \to T$ is $W$-equivariant. The group $X^\vee$ acts on the set $t$ by translations, and the map $e$ induces a bijection $t/X^\vee \sim T$, hence it induces a bijection
\[ t/(X^\vee \times W) \sim T/W. \]
Since $\Delta$ is a fundamental domain of the normal subgroup $Q^\vee \times W \subset X^\vee \times W$ (see Section 2), after changing the representative $a \in T$ of $[a] \in T/W$ we may choose $x$ lying in $\Delta$, and such $x$ is unique up to the action of the quotient group $(X^\vee \times W)/(Q^\vee \times W) = X^\vee/Q^\vee$. We see that the map $e$ induces a bijection
\[ \Delta/(X^\vee/Q^\vee) \sim T/W. \]
The map
\[ \mathcal{K}_{n,\mathbb{R}} \to \Delta, \quad p \mapsto x = p/n \mapsto x \]
is a $P^\vee/Q^\vee$-equivariant bijection, hence it induces a bijection
\[ K_{n,R}/(X^\vee/Q^\vee) \xrightarrow{\sim} \Delta/(X^\vee/Q^\vee). \]
We see that the map $\varphi: K_{n,R} \to T$ induces a bijection
\[ K_{n,R}/(X^\vee/Q^\vee) \xrightarrow{\sim} T/W. \]
In particular, two tuples $p, p' \in K_{n,R}$ are in the same $X^\vee/Q^\vee$-orbit if and only if $\varphi(p), \varphi(p') \in T$ are in the same $W$-orbit.

Now we wish to describe $p = (p_\beta) \in K_{n,R}$ such that $\varphi(p) \in T^n_\pi$, i.e., $\varphi(p)^n = z$. For $x \in \Delta$ obtained from $p \in K_{n,R}$ as in (10), the assertion that $e(x)^n = z$ is equivalent to the condition
\[ \lambda(\exp 2\pi nx) = \lambda(\exp 2\pi i \zeta) \]
for all $\lambda \in X$, which in turn is equivalent to
\[ -i\text{nd} \lambda(x) \equiv d\lambda(\zeta) \quad (\text{mod } \mathbb{Z}). \]
We write $\lambda = \sum_{\alpha \in \Pi} c_\alpha \alpha$ and obtain
\[ -i\text{nd} \sum_{\alpha \in \Pi} c_\alpha d\alpha(x) \equiv d\lambda(\zeta) \quad (\text{mod } \mathbb{Z}). \]
Since $d\alpha(x) = ix_\alpha$ for $\alpha \in \Pi$, and $nx_\alpha = p_\alpha$, we obtain
\[ \sum_{\alpha \in \Pi} c_\alpha p_\alpha = n \sum_{\alpha \in \Pi} c_\alpha x_\alpha \equiv d\lambda(\zeta) \quad (\text{mod } \mathbb{Z}). \]
Thus $\varphi(p) \in T^n_\pi$ if and only if
\[ \sum_{\alpha \in \Pi} c_\alpha p_\alpha \equiv d\lambda(\zeta) \quad (\text{mod } \mathbb{Z}) \quad \text{for any } \lambda \in X \text{ with } \lambda = \sum_{\alpha \in \Pi} c_\alpha \alpha. \]

Assume that $\varphi(p) \in T^n_\pi$, then (12) holds. Observe that for $\lambda = \alpha \in \Pi$, condition (12) means that $p_\alpha \equiv \mathbb{Z}$, because $d\alpha(\zeta) \in \mathbb{Z}$. Since $p_\alpha \in \mathbb{Z}$ for all $\alpha \in \Pi$, by [5] we have $p_\beta \in \mathbb{Z}$ for any $\beta \in \Pi_0 = \Pi \setminus \Pi$, because $m_\beta = 1$. Thus $p \in K_{n}$. Condition (8) is a special case of (12). We conclude that $p \in K_{n}^\pi$.

Conversely, assume that $p \in K_{n}^\pi \subset K_n$, then condition (12) holds for $\lambda = \alpha$ for any $\alpha \in \Pi$. Since condition (12) is additive in $\lambda$ (i.e., it holds for any integer linear combination of two weights $\lambda, \lambda' \in P$ whenever it holds for $\lambda$ and $\lambda'$), it holds for any $\lambda \in Q$, because $\Pi$ generates $Q$ as an abelian group. Now condition (8) implies that (12) holds for all $\lambda \in X$. We conclude that $\varphi(p) \in T^n_\pi$.

Thus $\varphi(p) \in T^n_\pi$ if and only if $p \in K_{n}^\pi$. Since the subset $T^n_\pi \subset T$ is $W$-invariant, we conclude that the subset $K_{n}^\pi \subset K_{n,R}$ is $X^\vee/Q^\vee$-invariant. Bijection (11) induces (9), which proves the theorem. \qed

We need another version of Theorem 3.4. We start from a Kac $n$-labeling $q = (q_\beta) \in K_n$ of $\bar{D}$. Set $z = \varphi(q)^n$. It follows from the proof of Theorem 3.4 that $z \in Z_G$. 

REAL GALOIS COHOMOLOGY
Corollary 3.5. With the assumptions and notation of Theorem 3.4 let \( q \) be an \( n \)-labeling of \( \tilde{D} \). Set \( z = \varphi(q)^n \in Z_G \). Then the subset \( \mathcal{K}_n(q) \subset \mathcal{K}_n \) consisting of Kac \( n \)-labelings \( p \in \mathcal{K}_n \) of \( \tilde{D} \) satisfying

\[
\sum_{\alpha \in \Pi} c_{\alpha} p_{\alpha} \equiv \sum_{\alpha \in \Pi} c_{\alpha} q_{\alpha} \pmod{\mathbb{Z}}
\]

for any generator \([\lambda]\) of \( X/Q \) with \( \lambda = \sum_{\alpha \in \Pi} c_{\alpha} \alpha \),

is \( X^\vee/Q^\vee \)-invariant, and the map \( \varphi \) of Construction 3.3 induces a bijection between \( \mathcal{K}_n(q)/(X^\vee/Q^\vee) \) and \( T_2^\vee/W \).

Indeed, by Theorem 3.4 we have \( q \in \mathcal{K}_n \), hence \( \mathcal{K}_n(q) = \mathcal{K}_n \), and the corollary follows from the theorem.

4. Real Galois cohomology

We denote by \( H^1(\mathbb{R}, H) \) the first (nonabelian) Galois cohomology set of an \( \mathbb{R} \)-group \( H \). By definition, \( H^1(\mathbb{R}, H) = Z^1(\mathbb{R}, H)/\sim \), where \( Z^1(\mathbb{R}, H) = \{ c \in H(\mathbb{C}) \mid c \bar{c} = 1 \} \), and \( c \sim c' \) if there exists \( h \in H(\mathbb{C}) \) such that \( c' = h^{-1} c \). We say that \( c \in Z^1(\mathbb{R}, H) \) is a cocycle.

Let \( H(\mathbb{R})_2 \subset H(\mathbb{R}) \) denote the subset of elements of order dividing 2. If \( b \in H(\mathbb{R})_2 \), then

\[
b^2 = 1,
\]

hence \( b \) is a cocycle. Thus \( H(\mathbb{R})_2 \subset Z^1(\mathbb{R}, H) \).

Let \( G \) be a connected, compact (anisotropic), semisimple algebraic group over the field of real numbers \( \mathbb{R} \). Let \( T \subset G \) be a maximal torus. We use the notation of Section 1.

Theorem 4.1. Let \( G \) be a connected, compact, semisimple algebraic \( \mathbb{R} \)-group. There is a canonical bijection between the set of \( P^\vee/Q^\vee \)-orbits in the set \( \mathcal{K}_2 \) of Kac 2-labelings of the extended Dynkin diagram \( \tilde{D} = \tilde{D}(G, T, \Pi) \) and the first Galois cohomology set \( H^1(\mathbb{R}, G^\text{ad}) \).

We specify the bijection. Consider the map \( \varphi^\text{ad} : \mathcal{K}_2 \rightarrow T^\text{ad} \) of Construction 3.3 for \( G^\text{ad} \), it sends \( \mathcal{K}_2 \subset \mathcal{K}_2 \) to \( (T^\text{ad})_2 \), where \( (T^\text{ad})_2 \) denotes the set of elements of order dividing 2 in \( T^\text{ad} \). The bijection of the theorem sends the \( P^\vee/Q^\vee \)-orbit of \( p \in \mathcal{K}_2 \) to the cohomology class \([\varphi(p)] \in H^1(\mathbb{R}, G^\text{ad})\) of \( \varphi(p) \in (T^\text{ad})_2 \subset Z^1(\mathbb{R}, G^\text{ad}) \).

This result goes back to Kac [K]. In the last sentence of [K] Kac notes that his results yield a classification of real forms of simple Lie algebras. Inner real forms of a compact simple group \( G \) (or of its Lie algebra \( \mathfrak{g} \)) are classified by the orbits of the group \( \text{Aut} \tilde{D} = (P^\vee/Q^\vee) \rtimes \text{Aut} D \) in the set \( \mathcal{K}_2 \) of Kac 2-labelings of \( \tilde{D} \). Those orbits and the corresponding real forms are listed in [OV1], Table 7, Types I and II.

Proof. By Theorem 3.4 for the adjoint group \( G^\text{ad} \), the map \( \varphi^\text{ad} \) induces a bijection \( \mathcal{K}_2/(P^\vee/Q^\vee) \sim (T^\text{ad})_2/W \). By [S] Section III.4.5, Example (a)] the map sending an element \( t^\text{ad} \in (T^\text{ad})_2 \subset Z^1(\mathbb{R}, G^\text{ad}) \) to its cohomology
class \([t^\mathrm{ad}] \in H^1(\mathbb{R}, G^\mathrm{ad})\) induces a bijection \((T^\mathrm{ad})_2/W \xrightarrow{\sim} H^1(\mathbb{R}, G^\mathrm{ad})\), and the theorem follows. \(\square\)

Let \(\mathcal{G}\) be an inner twisted form of a compact semisimple \(\mathbb{R}\)-group \(G\), where \(c \in Z^1(\mathbb{R}, G^\mathrm{ad})\). By Theorem 4.3, the cocycle \(c\) is equivalent to a cocycle of the form \(t^\mathrm{ad} = \varphi^\mathrm{ad}(q) \in (T^\mathrm{ad})_2 \subset Z^1(\mathbb{R}, G^\mathrm{ad})\) for some Kac 2-labeling \(q = (q_\beta)_{\beta \in \Pi}\) of \(\widetilde{D}\). We have \(t^\mathrm{ad} = \exp 2\pi y\), where \(y \in \Delta\) has barycentric coordinates \(y_\beta = q_\beta/2\) for \(\beta \in \Pi\). It follows that \(t^\mathrm{ad}\) is determined by the equations
\[
\alpha(t^\mathrm{ad}) = (-1)^{\gamma_\alpha} \quad \text{for } \alpha \in \Pi.
\]
We can twist \(G\) using \(t^\mathrm{ad}\); we denote the obtained twisted form by \(qG\), then \(\mathcal{G} \simeq qG\). Note that there is a canonical isomorphism between \(T\) and the twisted torus \(qT\), because the inner automorphism of \(G\) defined by \(t^\mathrm{ad}\) acts on \(T\) trivially. It follows that \(T\) canonically embeds into \(qG\), in particular, \(T_2 \subset qG(\mathbb{R})_2 \subset Z^1(\mathbb{R}, qG)\).

We compute \(H^1(\mathbb{R}, qG)\). Set \(t = \varphi(q) \in T\), where \(\varphi: K_{2, \mathbb{R}} \to T\) is the map of Construction 3.3. Then the image of \(t\) in \(T^\mathrm{ad}\) is \(t^\mathrm{ad}\). Since \((t^\mathrm{ad})^2 = 1\), we see that \(t^2 \in Z_G\). Set \(z = t^2\), then \(t \in T_2^z\).

**Lemma 4.2.** There is a bijection \(T_2^z/W \xrightarrow{\sim} H^1(\mathbb{R}, qG)\) that sends the \(W\)-orbit of \(a \in T_2^z\) to the cohomology class of \(at^{-1} \in T_2 \subset Z^1(\mathbb{R}, qG)\).

**Proof.** Recall that we have the standard left action \(*\) of \(W\) on \(T_2^z\) given by formula (1). We define the \(t^\mathrm{ad}\)-twisted left action \(*_{\mathrm{t^ad}}\) of \(W\) on \(T_2\) as follows: let \(w = nT \in W\), \(n \in N\), \(b \in T_2\), then
\[
w *_{\mathrm{t^ad}} b = nb t n^{-1} t^{-1}.
\]

We define a bijection
\[(14) \quad a \mapsto at^{-1}; T_2^z \to T_2
\]
(which takes \(t\) to 1). We have
\[(w * a) t^{-1} = nan^{-1} t^{-1} = n(at^{-1}) t n^{-1} t^{-1} = w *_{\mathrm{t^ad}} (at^{-1}),\]
hence, the standard left action \(*\) of \(W\) on \(T_2^z\) is compatible with the \(t^\mathrm{ad}\)-twisted left action \(*_{\mathrm{t^ad}}\) of \(W\) on \(T_2\) with respect to bijection (14). We obtain a bijection \(T_2^z/W = T_2^z/ * W \xrightarrow{\sim} T_2/ *_{\mathrm{t^ad}} W\) between the sets of \(W\)-orbits.

By [B1, Theorem 1], see also [B2, Theorem 9], the map sending \(b \in T_2 \subset Z^1(\mathbb{R}, qG)\) to its cohomology class \([b] \in H^1(\mathbb{R}, qG)\) induces a bijection \(T_2/ *_{\mathrm{t^ad}} W \xrightarrow{\sim} H^1(\mathbb{R}, qG)\).

Combining these two bijections, we obtain the bijection of the lemma. \(\square\)

The following theorem is the main result of this paper. It gives a combinatorial description of the first Galois cohomology set \(H^1(\mathbb{R}, qG)\) of an inner twisted form \(qG\) of a compact semisimple \(\mathbb{R}\)-group \(G\) in terms of Kac 2-labelings of the extended Dynkin diagram of \(G\).

**Theorem 4.3.** Let \(G\) be a connected, compact, semisimple algebraic \(\mathbb{R}\)-group. Let \(T \subset G\) be a maximal torus and \(\Pi\) be a basis of the root system \(R = R(G_{\mathbb{C}}, T_{\mathbb{C}})\). Let \(q\) be a Kac 2-labeling of the extended Dynkin diagram \(\widetilde{D} = \widetilde{D}(G, T, \Pi)\). Then the subset \(K_2^q \subset K_2\) of Kac 2-labelings \(p\) of \(\widetilde{D}\)
satisfying condition (13) of Corollary 3.5 is $X/Q$-invariant, and there is a bijection between the set of orbits $K_2^{(q)}/(X/Q)$ and the first Galois cohomology set $H^1(\mathbb{R}, qG)$.

We specify the bijection of the theorem. It is induced by the map sending a Kac 2-labeling $p \in K_2$ satisfying (13) to the cocycle $\exp 2\pi u \in T_2 \subset Z^1(\mathbb{R}, qG)$, where $u \in t$ is the element with barycentric coordinates $u_\alpha = (p_\alpha - q_\alpha)/2$ for $\alpha \in \Pi$. In particular, this bijection sends the $X/Q$-orbit of $q$ to the neutral element of $H^1(\mathbb{R}, qG)$.

**Proof of Theorem 4.3.** By Corollary 3.5 there is a bijection between the set of orbits of $X/Q$ in the set of Kac 2-labelings $p \in K_2$ of $\tilde{D}$ satisfying (13) and the set $T_2/W$, which sends the $X/Q$-orbit of $p$ to the $W$-orbit of $\exp 2\pi x \in T_2$, where $x \in t$ is the element with barycentric coordinates $x_\beta = p_\beta/2$ for $\beta \in \tilde{\Pi}$. By Lemma 4.2 there is a bijection $T_2/W \rightarrow H^1(\mathbb{R}, qG)$, which sends the $W$-orbit of an element $a \in T_2$ to the cohomology class of $at^{-1} \in T_2 \subset Z^1(\mathbb{R}, qG)$. We compose these two bijections. Since $t = \exp 2\pi y$, where $y \in t$ is the element with barycentric coordinates $y_\beta = q_\beta/2$ for $\beta \in \tilde{\Pi}$, the composite bijection sends the $X/Q$-orbit of a Kac 2-labeling $p$ satisfying (13) to the cohomology class of

$$\exp 2\pi x \cdot (\exp 2\pi y)^{-1} = \exp 2\pi(x - y) = \exp 2\pi u \in T_2 \subset Z^1(\mathbb{R}, qG),$$

where $u := x - y \in t$ has barycentric coordinates $u_\alpha = (p_\alpha - q_\alpha)/2$ for $\alpha \in \Pi$. Clearly this composite bijection sends $p = q$ to the cohomology class of $1 \in Z^1(\mathbb{R}, qG)$, thus to the neutral element of $H^1(\mathbb{R}, qG)$. □

5. Example: forms of $E_7$

Let $G$ be the simply connected compact group $G$ of type $E_7$. Since $G$ is simply connected, we have $X = P$.

Below in the left hand side we give the extended Dynkin diagram $\tilde{D}$ of $G_C$ with the numbering of vertices of [OV1] Table 1], and in the right hand side we give $\tilde{D}$ with the coefficients $m_j$ from [OV1] Table 6], see [11]. We have $X/Q = P/Q \simeq \mathbb{Z}/2\mathbb{Z}$, and there is $\lambda \in X \setminus Q$ with

$$\lambda = \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_7),$$

see e.g. [OV1] Table 3]. In the left-hand side diagram below we mark in black the roots appearing (with non-integer half-integer coefficients) in formula (15):

![Diagram of E_7 extended Dynkin diagram](image-url)
The Kac 2-labelings of $\tilde{D}$ are:

$q^{(1)} = 000002$
$q^{(2)} = 200000$
$q^{(3)} = 100001$
$q^{(4)} = 010000$
$q^{(5)} = 000010$
$q^{(6)} = 00001$.

The real forms of $E_7$ correspond to elements of $H^1(\mathbb{R}, G^{ad})$, and by Theorem 4.1 to the orbits of $P^\vee /Q^\vee$ in the set $K_2$ of Kac 2-labelings of $\tilde{D}$. These orbits are:

$\{ q^{(1)}, q^{(2)} \}$, $\{ q^{(3)} \}$, $\{ q^{(4)}, q^{(5)} \}$, $\{ q^{(6)} \}$,

hence $\#H^1(\mathbb{R}, G^{ad}) = 4$.

Concerning $H^1(\mathbb{R}, qG)$, condition (13) defining $K_2^{(q)}$ reads

$$\frac{1}{2}(p_1 + p_3 + p_7) \equiv \frac{1}{2}(q_1 + q_3 + q_7) \pmod{\mathbb{Z}},$$

which is equivalent to

$$p_1 + p_3 + p_7 \equiv q_1 + q_3 + q_7 \pmod{2}.$$ 

We say that a 2-labeling $p \in K_2$ is even (resp., odd) if the sum over the black vertices

$$p_1 + p_3 + p_7$$

is even (resp., odd). Then $K_2^{(q)}$ is the set of labelings $p \in K_2$ of the same parity as $q$. Since $G$ is simply connected, we have $X^\vee = Q^\vee$, and by Theorem 4.3 the first Galois cohomology set $H^1(\mathbb{R}, qG)$ is in a bijection with the set $K_2^{(q)}$.

For $qG = E_7$ (the compact form) we take $q = q^{(1)}$, then $q_1 + q_3 + q_7 = 0$, hence $q$ is even. For $qG = EVI$ we take $q = q^{(4)}$, see [OV1, Table 7]. We have $q_1 + q_3 + q_7 = 0$, so again $q$ is even. We see that in both cases the set $K_2^{(q)}$ is the set of all even 2-labelings of $\tilde{D}$:

(16) $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

The set $H^1(\mathbb{R}, qG)$ is in a bijection with the set (16). In particular, $\#H^1(\mathbb{R}, qG) = 4$ in both the compact case and EVI.

For $qG = EV$ (the split form) we take $q = q^{(6)}$, see [OV1, Table 7]. We have $q_1 + q_3 + q_7 = 1$, hence $q$ is odd. For $qG = EVII$ (the Hermitian form) we take $q = q^{(3)}$, see [OV1, Table 7]. Again we have $q_1 + q_3 + q_7 = 1$, and again $q$ is odd. In both cases the set $K_2^{(q)}$ is the set of all odd 2-labelings of $\tilde{D}$:

(17) $\begin{pmatrix} 100001 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

The set $H^1(\mathbb{R}, qG)$ is in a bijection with the set (17). In particular, $\#H^1(\mathbb{R}, qG) = 2$ in both cases EV and EVII.

In each case the element $q \in K_2^{(q)}$ corresponds to the neutral element of $H^1(\mathbb{R}, qG)$. 


6. Example: half-spin groups

Let $G$ be the compact group of type $D_\ell$ with even $\ell = 2k \geq 4$ with the cocharacter lattice

$$X^\vee = \langle Q^\vee, \omega_{\ell-1}^\vee \rangle.$$  

This compact group is neither simply connected nor adjoint, and it is isomorphic to $SO_{2\ell}$ only if $\ell = 4$. It is called a half-spin group.

We show that the character lattice $X$ is generated by $Q$ and the weight

$$\lambda := (\alpha_1 + \alpha_3 + \cdots + \alpha_{\ell-3} + \alpha_\ell)/2.$$  

Indeed, $\lambda$ is orthogonal to $\omega_{\ell-1}^\vee$ and $\langle \lambda, \alpha^\vee \rangle = 0, 1, -1 \in \mathbb{Z}$ for any $\alpha \in \Pi$. We see that $\lambda \in X$. Since $\lambda \notin Q$ and $[X : Q] = 2$, we conclude that $X = \langle Q, \lambda \rangle$.

Below in the left hand side we give the extended Dynkin diagram $\widetilde{D}$ of $G_C$ with the numbering of vertices of [OV1, Table 1] (which coincides with the labeling of Bourbaki [Bou]). We mark in black the roots that appear (with non-integer half-integer coefficients) in the formula (18) for $\lambda$. In the right hand side we give $\widetilde{D}$ with the coefficients $m_j$ from [OV1, Table 6], see [1]:

Let $p$ be a Kac 2-labeling of the extended Dynkin diagram $\widetilde{D}$. We say that $p$ is even (resp., odd), if the sum over the black vertices

$$p_1 + p_3 + \cdots + p_{\ell-3} + p_\ell$$  

is even (resp., odd). If $q \in K_2$ is a Kac 2-labeling of $\widetilde{D}$, then $K_2^{(q)}$ is the set of Kac 2-labelings $p$ of the same parity as $q$.

The group $X^\vee/Q^\vee = \{0, [\omega_{\ell-1}^\vee]\}$ acts on $\widetilde{D}$ and on the set $K_2$ of Kac 2-labelings of $\widetilde{D}$. The nontrivial element $\sigma := [\omega_{\ell-1}^\vee] \in X^\vee/Q^\vee$ acts as the reflection with respect to the vertical axis of symmetry of $\widetilde{D}$, see Section 2 and clearly preserves the parity of labelings. We say that a $\sigma$-orbit in $K_2$ is even (resp., odd), if it consists of even (resp., odd) 2-labelings.

Let $q$ be a 2-labeling of $\widetilde{D}$. By Theorem 1.3 the cohomology set $H^1(\mathbb{R}, qG)$ is in a bijection with the set $K_2^{(q)}/(X^\vee/Q^\vee)$, i.e., with the set of $\sigma$-orbits in $K_2$ of the same parity as $q$. Thus in order to compute $H^1(\mathbb{R}, qG)$ for all 2-labelings $q$ of $\widetilde{D}$, it suffices to compute the sets $\text{Orb}_{\text{even}}(D_\ell)$ and $\text{Orb}_{\text{odd}}(D_\ell)$ of the even and odd $\sigma$-orbits, respectively. We compute also the cardinalities

$$h_{\text{even}}(D_\ell) = \#\text{Orb}_{\text{even}}(D_\ell) \quad \text{and} \quad h_{\text{odd}}(D_\ell) = \#\text{Orb}_{\text{odd}}(D_\ell).$$  

We compute $\text{Orb}_{\text{even}}(D_\ell)$. Recall that $\ell = 2k$. For representatives of even $\sigma$-orbits we take

$$1 \ 0 \cdots 1 \ 0 \cdots 0 \ 2 \ 0 \cdots 0 \ 2 \ 0 \cdots 0$$

and on the set $K_2$ of Kac 2-labelings of $\widetilde{D}$. The nontrivial element $\sigma := [\omega_{\ell-1}^\vee] \in X^\vee/Q^\vee$ acts as the reflection with respect to the vertical axis of symmetry of $\widetilde{D}$, see Section 2 and clearly preserves the parity of labelings. We say that a $\sigma$-orbit in $K_2$ is even (resp., odd), if it consists of even (resp., odd) 2-labelings.

Let $q$ be a 2-labeling of $\widetilde{D}$. By Theorem 1.3 the cohomology set $H^1(\mathbb{R}, qG)$ is in a bijection with the set $K_2^{(q)}/(X^\vee/Q^\vee)$, i.e., with the set of $\sigma$-orbits in $K_2$ of the same parity as $q$. Thus in order to compute $H^1(\mathbb{R}, qG)$ for all 2-labelings $q$ of $\widetilde{D}$, it suffices to compute the sets $\text{Orb}_{\text{even}}(D_\ell)$ and $\text{Orb}_{\text{odd}}(D_\ell)$ of the even and odd $\sigma$-orbits, respectively. We compute also the cardinalities

$$h_{\text{even}}(D_\ell) = \#\text{Orb}_{\text{even}}(D_\ell) \quad \text{and} \quad h_{\text{odd}}(D_\ell) = \#\text{Orb}_{\text{odd}}(D_\ell).$$  

We compute $\text{Orb}_{\text{even}}(D_\ell)$. Recall that $\ell = 2k$. For representatives of even $\sigma$-orbits we take

$$1 \ 0 \cdots 1 \ 0 \cdots 0 \ 2 \ 0 \cdots 0 \ 2 \ 0 \cdots 0$$
and for each integer $j$ with $0 < 2j \leq k$, the 2-labeling with 1 at $2j$. Thus
\[ h^{\text{even}}(D_{2k}) = \left\lfloor k/2 \right\rfloor + 4. \]

We compute $\text{Orb}^{\text{odd}}(D_k)$. For representatives of odd $\sigma$-orbits we take
\[
\begin{pmatrix}
1 & \cdots & 0 & 1 & \cdots & 0 \\
1 & 0 & 0 & 0 & 1
\end{pmatrix}
\] and for each integer $j$ with $1 < 2j + 1 \leq k$, the 2-labeling with 1 at $2j + 1$. Thus
\[ h^{\text{odd}}(D_{2k}) = \left\lceil k/2 \right\rceil + 1. \]

As an example, we give a list of representatives of even and odd orbits for $D_6$:
\begin{align*}
\text{Orb}^{\text{even}}(D_6) : & \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & & & & \\
0 & 0 & & & &
\end{pmatrix}, \\
\text{Orb}^{\text{odd}}(D_6) : & \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.
\end{align*}

Note that if $\ell > 4$, our compact half-spin group $G$ has no outer automorphisms, hence all its real forms are inner forms, and we have computed the Galois cohomology for all the forms of $G$.

Note also that for the compact half-spin group $G$ we have
\[ \#H^1(\mathbb{R}, G) = h^{\text{even}}(D_{2\ell}) = \left\lfloor \ell/4 \right\rfloor + 4. \]

For comparison, $\#H^1(\mathbb{R}, \text{SO}_{2\ell}) = \ell + 1$. We have $\lfloor \ell/4 \rfloor + 4 = \ell + 1$ for an even natural number $\ell$ if and only if $\ell = 4$. (In this case, because of triality, our half-spin group $G$ is isomorphic to $\text{SO}_8$.)

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