KdV type systems and $\mathcal{W}$-algebras in the Drinfeld-Sokolov approach

László Fehér

Department of Physics, University of Swansea, Singleton Park, Swansea SA2 8PP, UK
and
Institute for Nuclear Study, University of Tokyo, Tanashi, Tokyo 188, Japan

Abstract. The generalized Drinfeld-Sokolov construction of KdV systems is reviewed in the case of an arbitrary affine Lie algebra paying particular attention to Hamiltonian aspects and $\mathcal{W}$-algebras. Some extensions of known results as well as a new interpretation of the construction are also presented.

1 Introduction

In order to set the context of this talk, let us recall (see e.g. the reviews in [1, 2]) that the so called second Hamiltonian structure of the KdV hierarchy, for which the Lax operator is $L = \partial_x^2 + u(x)$, is the Poisson bracket version of the Virasoro algebra. The simplest generalized KdV hierarchies are the $n$-KdV hierarchies, which are based on Lax operators of the form

$$L = \partial_x^n + u_2(x)\partial^{n-2} + \cdots + u_n(x), \quad (1.1)$$

and whose second (Gelfand-Dickey) Hamiltonian structure defines the $\mathcal{W}_n$-algebra.

Drinfeld and Sokolov [3] constructed a set of generalizations of the $n$-KdV hierarchy using a Lie algebraic framework. In their construction the phase space of a generalized KdV system appears as the space of orbits — reduced phase space — of a certain gauge group in a manifold. The manifold in question consists of first order differential operators

$$\mathcal{L} = \partial_x + j(x) + \Lambda, \quad (1.2)$$

where $\Lambda$ is a regular semisimple element of an affine Lie algebra $\mathcal{A}$ with principal grade one. The form of $j(x)$ and that of the gauge group are specified with the aid of the principal grading of $\mathcal{A}$ together with a standard grading of $\mathcal{A}$. The system is Hamiltonian with respect to a Poisson bracket that may be identified as the $\mathcal{W}$-algebra corresponding to the principal $\mathfrak{sl}_2$ embedding in the finite dimensional reductive Lie algebra $\mathcal{A}_0 \subset \mathcal{A}$, where $\mathcal{A}_0$ has zero grade in the relevant standard grading of $\mathcal{A}$. The simplest examples are obtained by taking $\mathcal{A} = \mathcal{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$.

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†Present address. On leave from Theoretical Physics Department of Szeged University, Szeged, Hungary.
for $\mathcal{G}$ a finite dimensional simple Lie algebra and choosing the standard grading to be the homogeneous grading. Then $A_0 = \mathcal{G}$ and the second Hamiltonian structure is the well-known principal (Casimir) $\mathcal{W}$-algebra associated to $\mathcal{G}$. The $W_n$-algebra is recovered for $\mathcal{G} = sl_n$. In the case of $\mathcal{A}$ a twisted loop algebra the Drinfeld-Sokolov KdV systems have only a single Hamiltonian structure in general, which is still often referred to as the “second” one.

Drinfeld and Sokolov also constructed modified KdV systems related to their KdV systems by generalized Miura maps. There is such a modified KdV system for every affine Lie algebra, having a phase space of elements of the form

$$\mathcal{L}_{\text{mod--KdV}} = \partial_x + \theta(x) + \Lambda, \quad \theta(x) \in A^0,$$

where $A^0$ is the abelian subalgebra of $\mathcal{A}$ given by the elements of principal grade zero.

Generalized Drinfeld-Sokolov reduction has received a lot of interest in the last few years. On the one hand, this term has been used to refer to Hamiltonian reductions of current algebras based on simple Lie algebras to $\mathcal{W}$-algebras that possess a finite generating set consisting of a Virasoro field and conformal tensors. The main success here has been the construction of a classical $\mathcal{W}$-algebra to every $sl_2$ embedding into a simple Lie algebra $\mathcal{G}$ [4, 5] and the construction of the corresponding quantum $\mathcal{W}$-algebra by the BRST technique [6, 7]. We also have some classification results [8, 9, 10] pointing to a distinguished role of these $\mathcal{W}$-algebras among classical extended conformal algebras based on finitely generated, freely generated differential rings.

In the literature the $\mathcal{W}_G^S$-algebra [4, 5] is usually considered for an $sl_2$ embedding $S \subset \mathcal{G}$ into a simple Lie algebra $\mathcal{G}$, but the construction extends in an obvious way to any reductive Lie algebra. A reductive Lie algebra, like $gl_n$, decomposes into the direct sum of a semisimple and a central part, and any $sl_2$ embedding is contained in the semisimple factor.

The other sense in which the notion of generalized Drinfeld-Sokolov reduction has been used in the literature concerns the construction of KdV type systems (see refs. [11–20]).

The common feature of Drinfeld-Sokolov reductions in the $\mathcal{W}$-algebra context and in the KdV context is that the existence of “DS gauges” is required in order to obtain a finite set of independent differential polynomial invariants with respect to a non-trivial gauge group. In the KdV context these invariants are the generalized KdV fields themselves. The “second” Poisson bracket algebra is not necessarily a $\mathcal{W}$-algebra in all generalized KdV systems. Conversely, according to present knowledge, not all classical $\mathcal{W}$-algebras resulting from generalized Drinfeld-Sokolov reductions support KdV type hierarchies.

In the rest of this talk we review the construction of modified KdV and KdV type systems, and also review what is known about the classification of such systems and their relationship to $\mathcal{W}$-algebras. To proceed in order of increasing complexity and for logical reasons, it will be convenient to first recall the construction of systems of modified KdV type.

## 2 Generalized modified KdV systems

The subsequent construction is due to Wilson [11] with elaborations and extensions contributed by McIntosh [12] and de Groot et al [13]. It associates a hierarchy of “modified KdV type” to any triplet

$$(\mathcal{A}, d_\sigma, \Lambda),$$

(2.1)
where $\mathcal{A}$ is an affine Lie algebra with vanishing centre, $d_\sigma$ is a $\mathbb{Z}$-grading of $\mathcal{A}$, and $\Lambda$ is a semisimple element of $\mathcal{A}$ which is homogeneous of $d_\sigma$-grade $k > 0$.

The affine Lie algebra $\mathcal{A}$ can be realized as a twisted loop algebra
\[
\mathcal{A} = \ell(\mathcal{G}, \mu) \subset \mathcal{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}]
\]
for a finite dimensional complex simple Lie algebra $\mathcal{G}$ and an automorphism $\mu$ of $\mathcal{G}$ of finite order. To investigate concrete systems it is often convenient to choose such a realization of $\mathcal{A}$, which means fixing $\mu$, but for our purpose it is better to define $\mathcal{A}$ abstractly \[21\] in terms of its Chevalley generators belonging to the simple roots, $e_i, f_i, h_i, i = 0, 1, \ldots, r$.

These satisfy the relations
\[
[e_i, f_j] = \delta_{ij} h_i, \quad [h_i, e_j] = K_{j,i} e_j, \quad [h_i, f_j] = -K_{j,i} f_j,
\]
where $K_{j,i}$ is the Cartan matrix of $\mathcal{A}$, together with the Serre relations and a relation expressing the vanishing of the centre, \[\sum_{i=0}^r n_i h_i = 0\]

with some non-negative integers $n_i$. Without loss of generality, the derivation $d_\sigma : \mathcal{A} \to \mathcal{A}$, whose eigenspaces $\mathcal{A}^n$ fix a grading $\mathcal{A} = \bigoplus_n \mathcal{A}^n$ of $\mathcal{A}$, can be defined by
\[
d_\sigma(e_i) = \sigma_i e_i, \quad d_\sigma(f_i) = -\sigma_i f_i, \quad d_\sigma(h_i) = 0,
\]
where $(\sigma_0, \sigma_1, \ldots, \sigma_r)$ is a set of relatively prime non-negative integers. The assumption that $\Lambda \in \mathcal{A}$ is a graded semisimple element means that it defines a direct sum decomposition
\[
\mathcal{A} = \text{Ker}(\text{ad} \Lambda) + \text{Im}(\text{ad} \Lambda), \quad \text{Ker}(\text{ad} \Lambda) \cap \text{Im}(\text{ad} \Lambda) = \{0\},
\]
and it satisfies $d_\sigma(\Lambda) = k\Lambda$ for some integer $k > 0$.

By definition, the phase space of the mod-KdV hierarchy associated to the triplet $(\mathcal{A}, d_\sigma, \Lambda)$ is the manifold $\Theta$ of first order differential operators given by
\[
\Theta := \{ L = \partial_x + \theta(x) + \Lambda \mid \theta(x) \in \mathcal{A}^{<k} \cap \mathcal{A}^{\geq 0} \}.
\]
We use the notation $\mathcal{A}^{<k} = \bigoplus_{n<k} \mathcal{A}^n$ etc. Since $\mathcal{A}^{<k} \cap \mathcal{A}^{\geq 0}$ is a finite dimensional space, the mod-KdV field $\theta(x)$ encompasses a finite number of complex valued fields depending on the one-dimensional space variable $x$. There is a natural family of compatible evolution equations on $\Theta$, whose members are labelled by the graded basis elements of the space
\[
\mathcal{C}(\Lambda) := (\text{Cent Ker}(\text{ad} \Lambda))^{\geq 0},
\]
which is the positively graded part of the centre of the Lie algebra $\text{Ker}(\text{ad} \Lambda)$. In order to construct these equations, one makes use of the transformation
\[
L = (\partial_x + \theta(x) + \Lambda) \mapsto e^{\text{ad} F}(L) = \partial_x + h(x) + \Lambda,
\]
where $\text{ad} d_\sigma$ is called a standard grading if $\sigma_i = \delta_{i,j}$ for some $j \in \{0, 1, \ldots, r\}$.
for \( F(x) \) and \( h(x) \) being infinite series

\[
F(x) \in (\text{Im}(\text{ad} \Lambda))^{<0}, \quad h(x) \in (\text{Ker}(\text{ad} \Lambda))^{<k}.
\]  

(2.11)

**Basic Lemma I.** *For arbitrarily given \( \theta(x) \), the above equations have a unique solution \( F(x) = F(\theta(x)), \) \( h(x) = h(\theta(x)). \) The components of \( F(\theta(x)) \) and \( h(\theta(x)) \) are differential polynomials in the components of \( \theta(x) \).*

Thanks to the lemma, which goes back to Drinfeld and Sokolov (see also [12, 13]), for any element \( b \in C(\Lambda) \) one can define

\[
A(b, \theta) := e^{-\text{ad} F(\theta)}(b),
\]

(2.12)

and \( A(b, \theta) \) is a differential polynomial in \( \theta \). Using the grading, \( A(b, \theta) \) is decomposed as

\[
A(b, \theta) = A(b, \theta)^{\geq 0} + A(b, \theta)^{<0}.
\]

(2.13)

The flow equation associated to \( b \in C(\Lambda) \) is given by the vector field \( \frac{\partial}{\partial T_b} \) on \( \Theta \):

\[
\frac{\partial \theta}{\partial T_b} := \left[ A(b, \theta)^{\geq 0}, \mathcal{L} \right] = - \left[ A(b, \theta)^{<0}, \mathcal{L} \right].
\]

(2.14)

Equivalently,

\[
\left[ \frac{\partial}{\partial T_b} - A(b, \theta(x))^\geq, \partial_x + \theta(x) + \Lambda \right] = 0.
\]

(2.15)

The flows corresponding to different elements of \( C(\Lambda) \) commute,

\[
\left[ \frac{\partial}{\partial T_b}, \frac{\partial}{\partial T_{b'}} \right] = 0 \quad \forall b, b' \in C(\Lambda).
\]

(2.16)

To understand the Hamiltonian interpretation of the above flows, it is convenient to first introduce the large space

\[
\tilde{A} := \{ \mathcal{L} = \partial_x + J(x) \mid J(x) \in A \}.
\]

(2.17)

Assuming appropriate smoothness and boundary conditions (e.g. periodic boundary condition) on the field \( J(x) \), the following formula defines a Poisson bracket on \( \tilde{A} \),

\[
\{ f, g \}_\sigma(J) := \int dx \left\langle J, \left[ \frac{\delta f}{\delta J}^{\geq 0}, \frac{\delta g}{\delta J}^{\geq 0} \right] - \left[ \frac{\delta f}{\delta J}^{<0}, \frac{\delta g}{\delta J}^{<0} \right] \right\rangle - \langle \left( \frac{\delta f}{\delta J} \right)^0, \partial_x \left( \frac{\delta g}{\delta J} \right)^0 \rangle,
\]

(2.18)

where \( \langle \ , \ \rangle \) denotes the non-degenerate, invariant, symmetric bilinear form on \( A \), which is unique up to a constant, and the functional derivative \( \frac{\delta f}{\delta J(x)} \in A \) is defined in the natural way.

In fact [22], \( \Theta \subset \tilde{A} \) is a Poisson submanifold of \( (\tilde{A}, \{ \ , \ \}_\sigma) \). The flow in (2.14) is hamiltonian with respect to the induced Poisson bracket on \( \Theta \) and the Hamilton function \( H_b(\theta) \) given by

\[
H_b(\theta) = \int dx \langle b, h(\theta(x)) \rangle.
\]

(2.19)
Drinfeld and Sokolov \(^3\) considered the systems for which \(d_\sigma\) is the principal grading, given by \(\sigma_0 = \sigma_1 = \cdots = \sigma_r = 1\), and \(\Lambda\) is the (essentially unique) grade one regular semisimple element, whose centralizer — disregarding the central extension — is the principal Heisenberg subalgebra of \(A\). Wilson \(^{11}\) proposed the above construction for a general grading and \(\Lambda\) a grade one semisimple element, which is clearly the nicest case in the sense that the number of fields is the smallest for a given grading \(d_\sigma\). This case, including a partial classification, was elaborated by McIntosh \(^{12}\). The extension for \(\Lambda\) having arbitrary positive grade is due to de Groot et al \(^{13, 14, 15}\).

### 3 Construction of KdV type systems

The construction of KdV type systems is much more subtle than that of the mod-KdV type systems since it requires the existence of a finitely generated, freely generated ring of differential polynomial invariants with respect to a non-trivial gauge group. As a consequence, it is not necessarily the case that every system of mod-KdV type has a Miura map which relates it to a system of KdV type\(^4\). The existence of the data given below is a sufficient condition for the existence of a Miura map.

The set of data to which a “KdV type system” can be associated \(^{12, 13}\) is a quadruplet

\[
(A, d_\sigma, \Lambda, d_\tau)
\]

(3.1)

subject to certain conditions that include those required previously for the triplet \((A, d_\sigma, \Lambda)\). That is \(\Lambda \in A\) is a semisimple element of \(d_\sigma\)-grade \(k > 0\). Here \(d_\tau\) is another \(\mathbb{Z}\)-grading of \(A\), which is compatible with \(d_\sigma\) in the sense that there exist Chevalley generators of \(A\) in terms of which \(d_\sigma\) is given as in (2.6) and \(d_\tau\) is given similarly in the same basis,

\[
d_\tau(e_i) = \tau_i e_i, \quad d_\tau(f_i) = -\tau_i f_i, \quad d_\tau(h_i) = 0,
\]

(3.2)

where \((\tau_0, \tau_1, \ldots, \tau_r)\) is a set of relatively prime non-negative integers. It is also assumed that

\[
\sigma_i = 0 \implies \tau_i = 0.
\]

(3.3)

The two compatible gradings together define a bi-grading of \(A\),

\[
A = \bigoplus_{m,n} A^m_n, \quad A^m_n := \{ X \in A \mid d_\sigma(X) = nX, \ d_\tau(X) = mX \},
\]

(3.4)

so that \(A^m_n\) has \(d_\sigma\)-grade \(n\) and \(d_\tau\)-grade \(m\). The relation in (3.3) together with \(\tau_i \sigma_i \geq 0\) means that \(d_\tau\) is “coarser” than \(d_\sigma\),

\[
A^0 \subseteq A_0.
\]

(3.5)

In \(^{13}\) the notation \(\tau \preceq \sigma\) was used to denote the relation between the two gradings, and the following useful consequences of this relation were also noted,

\[
A_{\geq 0} \subseteq A^{> 0}, \quad A_{< 0} \subseteq A^{< 0}, \quad A^{> 0} \subseteq A_{\geq 0}, \quad A^{< 0} \subseteq A_{< 0}.
\]

(3.6)

\(^3\)Our notion of KdV type system is defined by the subsequent construction. In the literature these systems are often called “partially modified KdV systems” reserving the term “KdV system” for the special case of \(d_\tau\) in (3.1) being a standard grading of \(A\).
Here and below superscripts denote $d_\sigma$ grades and subscripts $d_\tau$ grades as in (3.4). Finally, we require the “non-degeneracy condition” on the quadruplet given by

$$\text{Ker}(\text{ad } \Lambda) \cap \mathcal{A}_0^{<0} = \{0\}. \quad (3.7)$$

This is a non-trivial condition if $\mathcal{A}_0^{<0} \neq \{0\}$, i.e. if $\mathcal{A}_0 \neq \mathcal{A}^0$, which we also assume.

By definition, the phase space of the KdV system associated to the quadruplet $(\mathcal{A}, d_\sigma, \Lambda, d_\tau)$ is the factor space

$$\mathcal{M}_{\text{red}} = \mathcal{M}_c / \mathcal{N}, \quad (3.8)$$

where

$$\mathcal{M}_c = \{ \mathcal{L} = \partial x + j(x) + \Lambda \mid j(x) \in \mathcal{A}_{\geq 0}^{<k} = \mathcal{A}^{<k} \cap \mathcal{A}_{\geq 0} \} \quad (3.9)$$

and $\mathcal{N}$ is the group of “gauge transformations” $e^\alpha$ acting on $\mathcal{M}_c$ according to

$$e^\alpha : \mathcal{L} \mapsto e^{\text{ad}_\alpha} (\mathcal{L}) = e^\alpha \mathcal{L} e^{-\alpha}, \quad \mathcal{L} \in \mathcal{M}_c, \quad \alpha(x) \in \mathcal{A}_0^{<0}. \quad (3.10)$$

That is the Lie algebra of $\mathcal{N}$ is given by $\tilde{\mathcal{A}}_0^{<0} := \{ \alpha \mid \alpha(x) \in \mathcal{A}_0^{<0} \}$ (with some smoothness and boundary conditions for $\alpha(x)$).

Let $V \subset \mathcal{A}_{\geq 0}^{<k}$ be a $d_\sigma$-graded vector space defining a direct sum decomposition

$$\mathcal{A}_{\geq 0}^{<k} = [\Lambda, \mathcal{A}_0^{<0}] + V. \quad (3.11)$$

Define $\mathcal{M}_V \subset \mathcal{M}_c$ by

$$\mathcal{M}_V := \{ \mathcal{L} = \partial x + j_V(x) + \Lambda \mid j_V(x) \in V \}. \quad (3.12)$$

Due to the non-degeneracy condition (3.7) and the grading assumptions, the action of $\mathcal{N}$ on $\mathcal{M}_c$ is a free action and we have

**Basic lemma II.** The submanifold $\mathcal{M}_V \subset \mathcal{M}_c$ is a global cross section of the gauge orbits defining a one-to-one model of $\mathcal{M}_{\text{red}} = \mathcal{M}_c / \mathcal{N}$. When regarded as functions on $\mathcal{M}_c$, the components of $j_V(x) = j_V(j(x))$ are differential polynomials, which thus provide a free generating set of the ring $\mathcal{R}$ of gauge invariant differential polynomials on $\mathcal{M}_c$.

Since this lemma also goes back to Drinfeld and Sokolov, we refer to $\mathcal{M}_V$ as a *DS gauge* (see also [3, 10]). The construction of commuting local flows on $\mathcal{M}_c$ given in [12, 13] is based on the variant of “Basic lemma I” which says that for any $\mathcal{L} = (\partial x + j + \Lambda) \in \mathcal{M}_c$ there is a unique differential polynomial $F(j(x)) \in (\text{Im}(\text{ad } \Lambda))^{<0}$ such that

$$e^{\text{ad}_F(j)}(\mathcal{L}) = \partial x + h(j) + \Lambda, \quad \text{with} \quad h(j(x)) \in (\text{Ker}(\text{ad } \Lambda))^{<k}. \quad (3.13)$$

Using this, for any $b \in \mathcal{C}(\Lambda)$ in (2.9) one can define the commuting vector fields $\frac{\partial}{\partial b}$ on $\mathcal{M}_c$ by

$$\frac{\partial j}{\partial t_b} = [A(b, j)_{\geq 0}, \mathcal{L}] = -[A(b, j)_{<0}, \mathcal{L}], \quad \mathcal{L} = \partial x + j + \Lambda, \quad (3.14)$$

where $A(b, j) = e^{\text{ad}_F(j)}(b)$ and the splitting $A(b, j) = A(b, j)_{<0} + A(b, j)_{\geq 0}$ is now given by $d_\tau$.

It can be shown that the equations in (3.14) have a gauge invariant meaning. Algebraically, this means that any $\frac{\partial}{\partial t_b}$ gives rise to an evolutionary derivation of the ring $\mathcal{R}$ of $\mathcal{N}$-invariant
differential polynomials in $j$. Geometrically, letting $\pi : \mathcal{M}_c \to \mathcal{M}_c/\mathcal{N} \simeq \mathcal{M}_V$ denote the natural mapping, the statement is that the vector field $\frac{\partial}{\partial \pi}$ on $\mathcal{M}_c$ has a well-defined projection $\pi_* \left( \frac{\partial}{\partial \pi} \right)$ on $\mathcal{M}_V$. The flow induced on $\mathcal{M}_V$ has an equation of the form

$$\pi_* \left( \frac{\partial}{\partial b} \right) j_V = [A(b, j_V)_{\geq 0} + \eta(j_V), \partial_x + j_V + \Lambda],$$

(3.15)

where $\eta(j_V(x)) \in \mathcal{A}_0^2$ is a uniquely determined differential polynomial in $j_V$.

It can also be shown that the vector field in (3.15) is Hamiltonian with respect to the Hamilton function

$$H_b(j_V) = \int dx \langle b, h(j_V(x)) \rangle,$$

(3.16)

where $h(j)$ is defined in (3.13), and the Poisson bracket $\{ , \}_2$ on $\mathcal{M}_V$ is given as follows. In order to define $\{ , \}_2$ we identify the space of functions on $\mathcal{M}_V \simeq \mathcal{M}_c/\mathcal{N}$ with the space of gauge invariant functions on $\mathcal{M}_c$. Then, for arbitrary such functions $f, g$, we have

$$\{ f, g \}_2(j) := \int dx \left\langle j_0 + \Lambda_0, \left[ \frac{\delta f}{\delta j_0}, \frac{\delta g}{\delta j_0} \right] \right\rangle - \left\langle \frac{\delta f}{\delta j_0}, \frac{\partial}{\partial x} \frac{\delta g}{\delta j_0} \right\rangle - \left\langle j_{> 0} + \Lambda_{> 0}, \left[ \frac{\delta f}{\delta j_{> 0}}, \frac{\delta g}{\delta j_{> 0}} \right] \right\rangle,$$

(3.17)

where $\frac{\delta f}{\delta j_0(x)} \in \mathcal{A}_0$, $\frac{\delta f}{\delta j_{> 0}(x)} \in \mathcal{A}_{< 0}$ are defined with the aid of the scalar product $\langle , \rangle$ on $\mathcal{A}$ and the decomposition $j = j_0 + j_{> 0}$ of $j(x) \in \mathcal{A}_0^2$ for which $j_0(x) \in \mathcal{A}_0$, $j_{> 0}(x) \in \mathcal{A}_{> 0}$, $\Lambda = \Lambda_0 + \Lambda_{> 0}$.

If $d_\tau = d_\sigma$ then the KdV system constructed above using the quadruplet $(A, d_\sigma, \Lambda, d_\tau)$ becomes the mod-KdV system of the previous section based on the triplet $(A, \Lambda, d_\sigma)$. In general $\Theta \subset \mathcal{M}_c$. If the gauge group $\mathcal{N}$ is non-trivial the canonical mapping

$$\pi|_\Theta : \Theta \to \mathcal{M}_V$$

(3.18)

provides a generalized Miura map — that is a local Poisson map having a non-local, non single-valued inverse — from the modified KdV system $(A, d_\sigma, \Lambda)$ to the KdV system $(A, d_\sigma, \Lambda, d_\tau)$. The above construction of KdV type systems was proposed by de Groot et al in [13, 14, 15]. Independently, the same construction was proposed earlier by McIntosh in [12] using more restrictive conditions on the quadruplet $(A, d_\sigma, \Lambda, d_\tau)$. In [12] it was assumed that $\Lambda$ has $d_\sigma$-grade one and $d_\tau$-decomposition of the form $\Lambda = \Lambda_0 + \Lambda_1$. The “second” Hamiltonian structure (3.17) was described in [14] for the case $\mathcal{A} = \mathcal{G} \otimes \mathbb{C}[\Lambda, \lambda^{-1}]$.

In the above exposition we took the logical path whereby one first constructs the evolution equations by an algebraic method and then, almost as an afterthought, one finds their Hamiltonian properties by direct computations. This was the approach taken in refs. [3, 11, 12, 13]. There is another approach, advocated in special cases (with grade one $\Lambda$) in [18] where the Hamiltonian properties are clear from the very beginning and there is essentially nothing to prove with regard to the commutativity of the flows or the Jacobi identity of the Poisson bracket. In the next section this phase space reduction approach will be explained concentrating on the origin of formula (3.17) for the “second” Poisson bracket.

For a fixed KdV system $(A, d_\sigma, \Lambda, d_\tau)$, the subsequent interpretation relies on the Poisson bracket $\{ , \}_\tau$ on the space $\mathcal{A}$ in (2.17), which is given analogously to (2.18) by the formula

$$\{ f, g \}_\tau(J) := \int dx \left\langle J, \left[ \frac{\delta f}{\delta J}, \frac{\delta g}{\delta J} \right]_{> 0} \right\rangle - \left\langle \left( \frac{\delta f}{\delta J} \right)_{< 0}, \left( \frac{\delta g}{\delta J} \right)_{< 0} \right\rangle - \left\langle \left( \frac{\delta f}{\delta J} \right)_0, \partial_x \left( \frac{\delta g}{\delta J} \right)_0 \right\rangle,$$

(3.19)
where the splitting of $\frac{\partial f}{\partial \tau}$ is defined by $d_\tau$. It will be useful to decompose $\Lambda$ as

$$\Lambda = \Lambda_0 + \Lambda_{>0}, \quad \Lambda_0 \in \mathcal{A}_0, \quad \Lambda_{>0} \in \mathcal{A}_{>0},$$

by means of the $d_\tau$ grading, since the two components play different roles. We introduce the component $\Lambda_{>0}$ by defining the submanifold $\mathcal{M} \subset \tilde{\mathcal{A}}$,

$$\mathcal{M} := \left\{ \mathcal{L} = \partial_x + J_0(x) + J^{\leq k}_{>0}(x) + \Lambda_{>0} \mid J_0(x) \in \mathcal{A}_0, \quad J^{\leq k}_{>0}(x) \in \mathcal{A}^{\leq k} \cap \mathcal{A}_{>0} \right\}. \quad (3.21)$$

It is important to notice that $\mathcal{M}$ is a Poisson submanifold of $(\tilde{\mathcal{A}}, \{ \, , \}_\tau)$. To see this we first write down the Hamiltonian vector field $X(J) := \{ J, f \}_\tau$ generated by an arbitrary function $f$ on $\tilde{\mathcal{A}}$. Using the $d_\tau$-grading and the notation $Y(J) := \frac{\partial f}{\partial \tau}$, from (3.11) we obtain

$$X(J)_{<0} = [Y(J)_{>0}, J_{<0}]_{<0},$$
$$X(J)_{0} = [Y(J)_{0}, J_{0}] - \partial_x Y(J)_{0} + [Y(J)_{>0}, J_{<0}]_{0},$$
$$X(J)_{>0} = -[Y(J)_{<0}, J_{>0}]_{>0}. \quad (3.22)$$

Inserting $J = (J_0 + J^{\leq k}_{>0} + \Lambda_{>0})$ into (3.22), it follows — since $Y(J)_{<0} \in \mathcal{A}_{<0} \subseteq \mathcal{A}^{<0}$ by (3.6) — that the restriction of $X(J)$ to $\mathcal{M} \subset \tilde{\mathcal{A}}$ is tangent to $\mathcal{M}$, i.e. $X(J = J_0 + J^{\leq k}_{>0} + \Lambda_{0}) \in \mathcal{A}_0 + \mathcal{A}^{\leq k}_{>0}$. This proves that $\mathcal{M}$ is a Poisson submanifold. The hamiltonian interpretation of constraining $\mathcal{M}$ to $\mathcal{M}_c$ and that of the gauge group will be given shortly.

### 4 KdV construction as phase space reduction

In the standard case of Drinfeld and Sokolov there is a well-known interpretation of the “second” Poisson bracket of the generalized KdV systems as the reduced Poisson bracket obtained by imposing first class constraints on the manifold $\mathcal{M}$ and factoring by the gauge group generated by those constraints. However, this interpretation is not tenable in the general case, since the action of $\mathcal{N}$ is not generated by the constraints defining $\mathcal{M}_c \subset \mathcal{M}$ as is easily seen. In order to see what happens, a general phase space reduction procedure will be described next, which includes the KdV construction as a particular example.

Let $(\mathcal{M}, \{ \, , \})$ be a Poisson manifold. (For ease of understanding, one might wish to think of $\mathcal{M}$ as being finite dimensional.) Suppose that we have a non-trivial left action of a connected Lie group $\mathcal{N}_0$ on $\mathcal{M}$ which is a Hamiltonian action with respect to $\{ \, , \}$. This means that the infinitesimal action of the Lie algebra of $\mathcal{N}_0$ on $\mathcal{M}$ is given by the Hamiltonian vector fields

$$X_{\phi_i} = -\{ \phi_i, \}, \quad i = 1, \ldots, \text{dim}(\mathcal{N}_0), \quad (4.1)$$

of some functions $\phi_i$ on $\mathcal{M}$, which are the components of a momentum map $\phi$. Suppose also that $\mathcal{N}_0$ is a subgroup of a connected Lie group $\mathcal{N}$, which acts on $\mathcal{M}$ as a group of Poisson maps. That is the action of $\mathcal{N}$ on $\mathcal{M}$ is a symmetry of $\{ \, , \}$, but it is not necessarily generated by a momentum map, except for the sugroup $\mathcal{N}_0$. In practice this means that the action of $\mathcal{N}$ is not required to preserve the symplectic leaves in $\mathcal{M}$.

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4This was pointed out to me by J.L. Miramontes. See also [17].
Impose now first class constraints of the form

$$\phi_i = c_i, \quad c_i \text{ some constant},$$

(4.2)
on the symmetry generators of the $N_0$-action, that is fix the value of the momentum map to an infinitesimal character of the Lie algebra of $N_0$. The constrained manifold $M_c \subset M$ is given by

$$M_c := \{ p \in M \mid \phi_i(p) = c_i \}. \quad (4.3)$$

The constraints being first class means that $M_c$ is mapped to itself by the action of $N_0$ on $M$.

Let us now make the additional assumption that $M_c \subset M$ is mapped to itself by the action of $N$ on $M$. Then we can talk about the space of $N$-orbits in $M_c$ denoted as

$$M_{\text{red}} := M_c/N. \quad (4.4)$$

In a sufficiently regular situation, $M_{\text{red}}$ is a manifold and the space of smooth functions on $M_{\text{red}}$ can be identified with the space of smooth $N$-invariant functions on $M_c$. If the embedding of $M_c$ in $M$ and the action of $N$ on $M_c$ meet certain regularity conditions, the following proposition is verified in a standard manner.

**Proposition I.** The Poisson bracket $\{ , \}$ on $M$ induces a Poisson bracket $\{ , \}_{\text{red}}$ on $M_{\text{red}}$. The induced Poisson bracket is determined by the extension-computation-restriction algorithm. That is if $f$ and $g$ are $N$-invariant functions on $M_c$ then the $N$-invariant function $\{f, g\}_{\text{red}}$ on $M_c$ is given by

$$\{f, g\}_{\text{red}}(p) := \{\tilde{f}, \tilde{g}\}(p) \quad \forall p \in M_c, \quad (4.5)$$

where $\tilde{f}$, $\tilde{g}$ are arbitrary (locally defined, smooth) extensions of $f$ and $g$ to $M$.

If $N_0$ is the trivial group, the above reduction procedure is just standard Poisson reduction defined by factoring $M = M_c$ by the group of Poisson maps $N$. The other extreme case is $N = N_0$, when it becomes reduction by first class constraints, which can be also viewed as Marsden-Weinstein reduction applied to the symplectic leaves in $M$. The general case interpolates between these two well-known reduction procedures, and for this reason we will refer to it by means of the term “hybrid reduction”.

We can specialize the general notion of hybrid reduction to recover the KdV situation as follows. First, the role of $(M, \{, \})$ is played by $(\mathcal{M}, \{, \}_\tau)$, where the manifold $\mathcal{M}$ is defined in (3.21) and the Poisson bracket $\{, \}_\tau$ on $\mathcal{M}$, the restriction of (3.19), can be written

$$\{f, g\}_\tau = \int dx \left( J_0 \left[ \delta f, \frac{\delta g}{\delta J_0} \right] + \frac{\delta f}{\delta J_0} \partial_x \frac{\delta g}{\delta J_0} \right) - \left( J_{>0}^{<k} + \Lambda_{>0}, \left[ \frac{\delta f}{\delta J_{>0}^{<k}}, \frac{\delta g}{\delta J_{>0}^{<k}} \right] \right), \quad (4.6)$$

where $\frac{\delta f}{\delta J_0(x)} \in \mathcal{A}_0$ and $\frac{\delta f}{\delta J_{>0}^{<k}(x)} \in \mathcal{A}_{<0}^{<k}$ for any smooth function $f$ on $\mathcal{M}$. The role of the group $N_0$ is played by the group $\mathcal{N}_0$ whose Lie algebra is by definition $\mathcal{A}_{<0}^{<k} := \{ \alpha \mid \alpha(x) \in \mathcal{A}_0^{<k} \}$, with appropriate smoothness and boundary conditions for $\alpha(x)$. The group $\mathcal{N}_0$ acts on the manifold $\mathcal{M}$ via

$$e^\alpha : (\partial_x + J_0 + J_{>0}^{<k} + \Lambda_{>0}) \mapsto e^\alpha (\partial_x + J_0 + J_{>0}^{<k} + \Lambda_{>0}) e^{-\alpha} = e^\alpha (\partial_x + J_0) e^{-\alpha} + J_{>0}^{<k} + \Lambda_{>0}. \quad (4.7)$$
where the last equality follows from the relation $\mathcal{A}^{<0}_0 = \{0\}$, which is a consequence of (3.6). Combining (4.7) and (4.6) we see that the action of $\mathcal{N}_0$ on $\mathcal{M}$ is indeed a Hamiltonian action. The corresponding momentum map $\phi : \mathcal{M} \to (\mathcal{A}^{<k}_0)^* \simeq \mathcal{A}^{\geq k}_0 := \{ \beta | \beta(x) \in \mathcal{A}^{\geq k}_0 \}$ is given by

$$
\phi : \mathcal{L} = (\partial_x + J_0 + J^{<k}_{>0} + \Lambda_{>0}) \mapsto J^{\geq k}_0 \quad \text{with} \quad J_0 = J^{\geq k}_0 + J^{<k}_0.
$$

(4.8)

We define first class constraints by setting the current component $J^{\geq k}_0$ generating the $\mathcal{N}_0$ action equal to the constant $\Lambda_0 \in \mathcal{A}^k_0$. This leads to the constrained manifold $\mathcal{M}_c \subset \mathcal{M}$,

$$
\mathcal{M}_c = \left\{ \mathcal{L} = \partial_x + J^{<k}_0(x) + J^{<k}_{>0}(x) + \Lambda | J^{<k}_0(x) \in \mathcal{A}^{<k}_0, J^{<k}_{>0}(x) \in \mathcal{A}^{<k}_0 \right\}.
$$

(4.9)

By the identification $j := J^{<k}_0 + J^{<k}_{>0}$, $\mathcal{M}_c$ in (4.9) the same as the constrained manifold defined in (3.9). Finally, it is clear that the role of the group $N$ in the hybrid reduction is played by the group $\mathcal{N}$ acting on $\mathcal{M}$ according to

$$
e^\alpha : \mathcal{L} \mapsto e^{\text{ad}_\alpha} (\mathcal{L}) = e^{\alpha} \mathcal{L} e^{-\alpha}, \quad \forall \mathcal{L} \in \mathcal{M}, \quad \alpha(x) \in \mathcal{A}^{<0}_0.
$$

(4.10)

This is a Poisson action preserving $\mathcal{M}_c \subset \mathcal{M}$. Since the action of $\mathcal{N}$ on $\mathcal{M}_c$ admits DS gauges, $\mathcal{M}_{\text{red}} = \mathcal{M}_c/\mathcal{N} \simeq \mathcal{M}_V$ is a manifold and the induced Poisson bracket $\{ , \}_{\text{red}}$ of Proposition I is just the “second” Poisson bracket of the KdV system given in (3.17).

The hybrid reduction interpretation is useful not only for understanding the origin of the “second” Poisson bracket, but it also sheds light on the commuting Hamiltonians of the KdV system. Namely, the commuting Hamiltonians can be explained combining this reduction procedure with the $r$-matrix (Adler-Kostant-Symes) scheme, which tells us that the monodromy invariants of $\mathcal{L}$ provide commuting Hamiltonians on $\mathcal{M}$.

**Remarks.**

a) If $[\mathcal{A}^{<0}_0, \Lambda_{>0} + \mathcal{A}^{<k}_0] \neq \{0\}$ then the action of $\mathcal{N}$ on $\mathcal{M}$ is not a Hamiltonian action. This is the case in examples for which $\Lambda$ is not a semisimple element of minimal positive $d_\sigma$-grade.

b) If the $d_\sigma$-grade $k$ of $\Lambda$ is large enough then $\Lambda_0 = 0$ and $\mathcal{N}_0$ is trivial. In this case there are no first class constraints, $\mathcal{M}_c = \mathcal{M}$, and we are dealing with Poisson reduction.

c) If $k = 1$ then $\mathcal{N}_0 = \mathcal{N}$ and the hybrid reduction becomes reduction by first class constraints.

d) The space $(\hat{\mathcal{A}}, \{ , \}_{\hat{\tau}})$ can be naturally embedded as a Poisson submanifold in the dual of a Lie algebra having a Lie-Poisson bracket engendered by an $r$-matrix (see [22]). In the periodic case, the Lie algebra in question is given by $\ell(\hat{\mathcal{G}}, \hat{\tau}) \subset \hat{\mathcal{G}} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$, where $\hat{\mathcal{G}}$ is the central extension of the loop algebra $C^\infty(S^1, \mathcal{G})$, and $\hat{\tau}$ is the automorphism of $\hat{\mathcal{G}}$ naturally induced from the automorphism $\hat{\tau}$ of $\mathcal{G}$ that corresponds to the grading $d_\tau$ of $\mathcal{A}$ by Kac’s theorem [21]. $\mathcal{A}$ can be realized as $\mathcal{A} = \ell(\hat{\mathcal{G}}, \hat{\tau})$ and $d_\tau$ becomes the homogeneous grading in this realization. The action of $\hat{\tau}$ on $\hat{\mathcal{G}}$ is defined by extending the action of $\hat{\tau}$ to $C^\infty(S^1, \mathcal{G})$ in a pointwise fashion and acting as the identity on the centre of $\hat{\mathcal{G}}$. The $r$-matrix corresponds to the splitting of $\ell(\hat{\mathcal{G}}, \hat{\tau})$ according to negative and non-negative powers of $\lambda$. Observe that the two loop parameters $x \in S^1$ and $\lambda$ play completely different roles in the construction.

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Equation (4.10) defines a Hamiltonian action of $\mathcal{N}$ on $\mathcal{M}$, with momentum map $J_0 \mapsto J^{>0}_0$, if $\mathcal{A}^{<k}_0 = \{0\}$.  

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10
5 \( \mathcal{W} \)-algebras related to KdV systems

Notice from formula \((4.6)\) of the Poisson bracket \(\{ \cdot, \cdot \}_\tau\) on \(\mathcal{M}\) that the component \(J_0\) of \(J = (J_0 + J_{>0})\) satisfies the standard current algebra Poisson bracket associated to the reductive Lie algebra \(\mathcal{A}_0\), with the central extension being defined by means of the restriction of the invariant scalar product of \(\mathcal{A}\) to \(\mathcal{A}_0\). Going to the constrained manifold \(\mathcal{M}_c\), this ensures that we can talk about the Poisson subalgebra of the gauge invariant functions depending only on invariant scalar product of polynomials depending only on \(A\).

Notice from formula \((4.6)\) of the Poisson bracket \(\{ \cdot, \cdot \}\) of the Poisson bracket algebra of the generators of the subring \(J = (\mathcal{W}_\alpha)\) of \(\mathcal{A}\) generated differential ring and the corresponding subalgebra of the “second” Poisson bracket algebra is a \(\mathcal{W}\)-algebra of the standard type.

Using the decomposition \(\Lambda = \Lambda_0 + \Lambda_{>0}\), suppose that \(\Lambda_0 \neq 0\) and the KdV quadruplet \((\mathcal{A}, d_\sigma, \Lambda, d_\tau)\) satisfies the “strong non-degeneracy condition”

\[
\text{Ker}(\text{ad} \Lambda_0) \cap A_0^{<0} = \{0\}. \tag{5.1}
\]

This implies the original non-degeneracy condition in \((3.7)\). If condition \((5.1)\) is satisfied, we can choose the complementary space \(V\) in \((3.11)\) so that

\[
V = V_0 + V_{>0} \quad \text{with} \quad A_0^{<k} = [\Lambda_0, A_0^{\leq 0}] + V_0, \quad V_{>0} = A_0^{\geq k}. \tag{5.2}
\]

Note that \(V_0 \subset A_0^{\geq 0}\) since \([\Lambda_0, A_0^{<0}] \cap A_0^{>0}] = A_0^{<0} \cap A_0^{>0}\) on account of \((5.1)\). The action of the gauge group \(\mathcal{N}\) on \(\mathcal{M}_c\) given in \((5.10)\) can be written

\[
e^\alpha : j_0 \mapsto e^\alpha(\partial_x + j_0 + \Lambda_0)e^{-\alpha} - \Lambda_0 - \partial_x, \quad j_{>0} \mapsto e^\alpha(j_{>0} + \Lambda_{>0})e^{-\alpha}, \quad \forall \alpha(x) \in A_0^{\leq 0}, \tag{5.3}
\]

so it does not explicitly mix the components \(j_0\) and \(j_{>0}\) of \(j = (j_0 + j_{>0})\). Combining \((5.1)\), \((5.2)\) and \((5.3)\), we see that the DS gauge fixing can be performed purely at the \(d_\tau\)-grade zero level. This means that the gauge transformation \(e^\alpha\) that brings \(j = (j_0 + j_{>0})\) to the DS normal form \(j_V = j_{V_0} + j_{>0}\) corresponding to \(V\) in \((5.2)\) depends only on \(j_0, \alpha = \alpha(j_0)\). As a consequence, the ring \(\mathcal{R}\) of gauge invariant differential polynomials in \(j\) is generated by the components of \(j_{V_0} = j_{V_0}(j_0)\) and \(j_{>0} = j_{>0}(j_0, j_0)\). This immediately implies the first statement of the following proposition.

**Proposition II.** Suppose that \(\Lambda_0 \neq 0\) and \((5.7)\) is satisfied. Then the subring \(\mathcal{R}_0 \subset \mathcal{R}\) of \(\mathcal{N}\)-invariant differential polynomials in \(j_0\) is freely generated by the components of \(j_{V_0}(j_0)\) for any DS normal form \(j_V = j_{V_0} + j_{>0}\) associated to a decomposition of the type in \((5.2)\). There exists a generating set of \(\mathcal{R}_0\) whose induced Poisson brackets satisfy the \(\mathcal{W}_S^{A_0}\)-algebra for the \(\mathfrak{sl}_2\) embedding \(S = \text{span}\{I_-, I_0, I_+\} \subset \mathcal{A}_0, \{I_0, I_{\pm}\} = I_{\pm}, \{I_+, I_-\} = 2I_0\), defined by \(I_+ := \Lambda_0\).

**Proof.** The relation \(d_\sigma(\Lambda_0) = k\Lambda_0\) implies that \(\Lambda_0\) is a nilpotent element of \(\mathcal{A}_0\) and thus \(I_+ := \Lambda_0\) determines an \(\mathfrak{sl}_2\) subalgebra \(S \subset \mathcal{A}_0\) which is unique up to conjugation \([23]\). The generator \(I_-\) of \(S \subset \mathcal{A}_0\) can be chosen so as to also have \(d_\sigma(I_-) = -kI_-\). With this \(I_-\), define

\[
V_0 := \text{Ker}(\text{ad} I_-). \tag{5.4}
\]
It is not hard to verify that $V_0$ in (1.4) satisfies (1.2). By the definition of $\mathcal{W}^{A_0}_S \ [4, 8]$, the components of $j_{V_0}$ generate the $\mathcal{W}^{A_0}_S$-algebra with respect to the Dirac bracket determined by the second class constraints that reduce the $A_0$ valued current $J_0$ to $(j_{V_0} + \Lambda_0)$. We wish to show that the Dirac brackets of the components of $j_{V_0}$ coincide with the induced Poisson brackets of the $R_0$ generators given by the components of $j_{V_0}(j_0)$. For this, let us consider the subalgebra $\Gamma \subset A_0^{<0}$,

$$\Gamma := A_0^{< -k/2} + P^{-k/2}, \quad (5.5)$$

where $P^{-k/2}$ is defined to be a maximal subspace of $A_0^{-k/2}$ for which $\langle \Lambda_0, [X, Y] \rangle = 0$ for any $X, Y \in P^{-k/2}$. It can be checked that the constraints given by

$$\phi_i(J_0) = \langle \gamma_i, J_0 - \Lambda_0 \rangle = 0, \quad (5.6)$$

where $\{\gamma_i\}$ is a basis of $\Gamma$, are a set of first class constraints on $J_0$, which upon gauge fixing lead to the same second class constraints that restrict $J_0$ to have the form of $(j_{V_0} + \Lambda_0)$. Indeed, the $\Gamma$-constrained current, $J^\Gamma_0$, has the form

$$J^\Gamma_0 = (j^\Gamma_0 + \Lambda_0) \quad \text{with} \quad j^\Gamma_0(x) \in \Gamma^\perp, \quad (5.7)$$

where $\Gamma^\perp \subset A_0$ is the annihilator of $\Gamma$ with respect to the scalar product $\langle \ , \ \rangle$, and we have

$$\Gamma^\perp = [\Lambda_0, \Gamma] + V_0 \quad (5.8)$$

with the same complementary space $V_0$ as in (5.3). Then we can use the fact that $j_{V_0}(j_0)$ commutes with the first class constraints associated to $\Gamma$, as $\Gamma \subset A_0^{<0}$, to conclude from the standard formula for the Dirac bracket that the Dirac brackets of the components of $j_{V_0}(j_0)$ coincide with their induced Poisson brackets determined by hybrid reduction. Q.E.D.

Proposition II is due to Miramontes et al. [16]. We note that the symplectic halving part of the above proof is not contained in [14], where the case of $A = \mathcal{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ was considered.

Remarks. e) Under the strong non-degeneracy condition, the components of $j_{V_{>0}}(j_{>0}, j_0)$ commute with those of $j_{V_0}(j_0)$ with respect to the “second” Poisson bracket [17].

f) If $\Lambda$ has $d_\sigma$-grade one then $[\Lambda_{>0}, A_0^{<0}] \subset A_0^{<2} = \{0\}$, and so the non-degeneracy condition in (3.7) is equivalent to (5.4). In this case the “second” Poisson bracket algebra of the KdV system associated to the quadruplet $(\mathcal{A}, d_\sigma, \Lambda, d_\tau)$ is just the $\mathcal{W}^{A_0}_S$-algebra.

The strong non-degeneracy condition is a sufficient but not a necessary condition for the presence of a $\mathcal{W}$-subalgebra in the second Poisson bracket algebra of a KdV system [17]. An instructive set of examples is obtained by taking $\mathcal{A} = \mathfrak{sl}_n \otimes \mathbb{C}[\lambda, \lambda^{-1}], \ d_\sigma$: the principal grading, $d_\tau$: the homogeneous grading, and $\Lambda := (\Lambda_n)^l$ for some $1 < l < n$ relatively prime to $n$. Here $\Lambda_n := (\lambda e_{n,1} + \sum_{i=1}^{n-1} e_{i,i+1})$ is the grade one regular element from the principal Heisenberg subalgebra. For these examples (3.7) holds but the strong non-degeneracy condition (5.1) is not satisfied unless $l = 2$ and $n$ is odd. Nevertheless, the presence of a $\mathcal{W}^{A_0}_S$-subalgebra can be exhibited in the second Poisson bracket algebra [24]. The case $l = (n - 1)$ is treated in [17] by a different method.

h) In the general case, it is known [17] that if $\Lambda_0 \neq 0$ then the “second” Poisson bracket of the KdV system $(\mathcal{A}, d_\sigma, \Lambda, d_\tau)$ enjoys a conformal invariance property given by a (non-unique) Poisson action of the conformal group on the phase space. The existence of a corresponding Virasoro density, not to mention a standard or new $\mathcal{W}$-algebra, is not clear at present.

\footnote{We are using here a version of “symplectic halving” [3]. If $k$ is odd then $P^{-k/2} = \{0\}$.}
6 Classification of mod-KdV and KdV type systems

We have seen that a modified KdV type system can be associated to any triplet \((A, d_\sigma, \Lambda)\) and that a KdV type system can be associated to any quadruplet \((A, d_\sigma, \Lambda, d_\tau)\) subject to \(\tau \preceq \sigma\), \(A_0^{<0} \neq \{0\}\) and the non-degeneracy condition in (3.7). The classification of modified KdV type (KdV type) systems requires listing all triplets (quadruplets) modulo an equivalence relation taking care of possible isomorphisms of hierarchies corresponding to different triplets (quadruplets). We are very far from having a complete classification, but some progress has been made.

In [12] various conditions on the gradings admitting a grade one semisimple element were described. It was explained that \(\sigma_i \in \{0, 1\}\) for all \(i = 0, 1, \ldots, r\) is a necessary condition for the grading \(d_\sigma\) in (2.6) to admit a grade one semisimple element. A sufficient condition (which was conjectured to be necessary too) was also given. The sufficient condition of [12] is a constructive one, and for instance in the case of \(A = sl_2 \otimes \mathbb{C}[^{\lambda, \lambda^{-1}}]\) it is satisfied for any choice of \(\sigma_i \in \{0, 1\}\). The question of possible isomorphisms of modified KdV systems associated to different triplets was also studied in [12], and the interested reader is urged to consult this reference.

The series of papers in [13–17] contains the assumption that \(\Lambda \in A\) belongs to a graded Heisenberg subalgebra (a maximal abelian subalgebra of semisimple elements) of \(A\) and, except in [13], it was also assumed that \(A\) is a non-twisted loop algebra. Although these assumptions are not essential, as we have seen, it would be important to know whether every graded semisimple element is contained in a graded Heisenberg subalgebra. If the answer is positive, then the known classification of the inequivalent graded Heisenberg subalgebras of the affine Lie algebras, which is described in [23] in the non-twisted case, would yield a partial classification of the generalized (modified) KdV systems. This classification would still be incomplete for several reasons, including the fact that in general there exist different gradings of \(A\) that reduce to the grading of a given Heisenberg subalgebra of \(A\), which itself is unique up to a constant.

Since the description of all possible triplets \((A, d_\sigma, \Lambda)\) is a prerequisite for finding the list of the quadruplets \((A, d_\sigma, \Lambda, d_\tau)\), we know even less about the latter problem. As for the possible auxiliary gradings \(d_\tau\) admitted for a given triplet \((A, d_\sigma, \Lambda)\), no general result is available. There is no difficulty in choosing \(d_\tau\) so that \(\tau \preceq \sigma\) and \(A_0^{<0} \neq \{0\}\), but the non-degeneracy condition in (3.7) places a non-trivial restriction on \(d_\tau\). However, this restriction disappears in the important special case for which the graded semisimple element \(\Lambda\) is regular. By definition, for \(\Lambda\) a regular semisimple element \(\text{Ker}(\text{ad} \Lambda)\) is an abelian subalgebra of \(A\) (a graded Heisenberg subalgebra if \(\Lambda\) has definite grade). In the regular case \(\text{Ker}(\text{ad} \Lambda)\) consists of semisimple elements, and thus it cannot intersect \(A_0^{<0}\), which consists of nilpotent elements.

In the regular case more classification results are available than in the general case. We now explain how this comes about concentrating, for simplicity, on the non-twisted affine algebras. Let \(\sigma\) denote the finite order inner automorphism of \(\mathcal{G}\) corresponding to the grading \(\sigma\) of \(A = \mathcal{G} \otimes \mathbb{C}[^{\lambda, \lambda^{-1}}]\). If \(\Lambda(\lambda) \in A\) is a semisimple element of definite \(d_\sigma\)-grade, then its “projection” \(E \in \mathcal{G} \rightarrow E \in \mathcal{G}\) is obtained from \(\Lambda(\lambda) \in A\) by replacing the formal parameter \(\lambda\) by 1, that is \(E := \Lambda(\lambda = 1)\) — is a semisimple eigenvector of \(\sigma\). Moreover, \(\Lambda(\lambda) \in A\) is regular if and only if \(E \in \mathcal{G}\) is regular in the usual finite dimensional sense. But if \(E\) is a regular semisimple eigenvector of \(\sigma\), then \(\sigma\) preserves the Cartan subalgebra \(\mathcal{H} \subset \mathcal{G}\) defined by \(\mathcal{H} := \text{Ker}(\text{ad} E)\), and thus it gives rise to an element of the Weyl group \(W(\mathcal{G})\) acting on
\( \mathcal{H} \). On the other hand, the conjugacy classes in \( \mathbf{W}(\mathcal{G}) \) whose representatives admit a regular eigenvector — themselves called regular conjugacy classes — are all known \([20]\). Reversing the above arguments, all of the graded regular semisimple elements of \( \mathcal{A} \) can be constructed\(^7\) out of the regular eigenvectors of representatives of the regular conjugacy classes in \( \mathbf{W}(\mathcal{G}) \). There is a “lifting ambiguity” involved in the construction since non-conjugate inner automorphisms of \( \mathcal{G} \) may give rise to the same Weyl transformation. This ambiguity has not been settled yet, but a particularly nice lift is given by the following result.

**Proposition III.** Let \( w \in \mathbf{W}(\mathcal{G}) \) be a representative of a regular conjugacy class of the Weyl group acting on the Cartan subalgebra \( \mathcal{H} \subset \mathcal{G} \). Then there exists a finite order automorphism \( \hat{w} = \exp(2i\pi a I_0/m) \) of \( \mathcal{G} \) that reduces to \( w \) on \( \mathcal{H} \), where \( m \) is the order of \( w \) and the following statements are valid:

i) \( I_0 \in \mathcal{G} \) is the “defining vector” \([27]\) of an \( sl_2 \) subalgebra of \( \mathcal{G} \).

ii) The largest eigenvalue \( j_{\text{max}} \) of \( a I_0 \) equals \( (m - 1) \) except for some cases in \( F_4 \) and \( E_{6,7,8} \).

iii) The order of \( \hat{w} \) is \( m \) (resp. \( 2m \)) if \( a I_0 \) has only integral (resp. also half-integral) eigenvalues.

iv) \( \hat{w} \) has a regular eigenvector \( E = (C_1 + C_{-(m-1)}) \in \mathcal{H} \) with eigenvalue \( e^{2i\pi/m} \), \([I_0, C]\) = \( IC_1 \).

v) For \( I_+ := C_1 \) appearing in \( E \) above, there exists \( I_\pm \in \mathcal{G} \) so that \([I_0, I_\pm] = I_\pm, [I_+, I_-] = 2I_0\).

Proposition III generalizes the celebrated relation between the Coxeter class in \( \mathbf{W}(\mathcal{G}) \) and the principal \( sl_2 \) subalgebra of \( \mathcal{G} \) due to Kostant \([28]\). It was verified in \([19]\) putting together isolated results of \([14, 20, 30]\). The \( sl_2 \) embeddings associated by Proposition III to the regular conjugacy classes in the Weyl group are given explicitly in \([19]\) in terms of the standard classifications of the \( sl_2 \) subalgebras of \( \mathcal{G} \) \([27]\) and the conjugacy classes of \( \mathbf{W}(\mathcal{G}) \) \([31]\).

Finally, we wish to mention an application of Proposition III to KdV systems. Define \( \Lambda \in \mathcal{A} = \mathcal{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \) by \( \Lambda := (C_1 + \lambda C_{-(m-1)}) \) using \( E = \Lambda(\lambda = 1) \) occurring in property iv). It follows that \( \Lambda \) is a regular semisimple element of grade \( \nu \) with respect to \( d_\sigma := \nu \left(m \lambda \frac{d}{d\lambda} + a I_0\right) \), where \( \nu \in \{1, 2\} \) equals the order of \( \hat{w} \) divided by the order of \( w \). In fact, the homogeneous grading \( d_\sigma := \lambda \frac{d}{d\lambda} \) satisfies the relation \( \tau \leq \sigma \) if and only if \( j_{\text{max}} = (m - 1) \). (In the exceptional cases for which this does not hold one has \( j_{\text{max}} = m \).) This means that in the cases for which \( j_{\text{max}} = (m - 1) \), the quadruplet \( (\mathcal{A}, d_\sigma, \Lambda, d_\tau) \) can be used to construct a KdV type system. Then \( \mathcal{A}_0 = \mathcal{G} \) and the strong non-degeneracy condition of \([5.1]\) is satisfied as a consequence of \( d_\sigma \) being defined in terms of the \( sl_2 \) generator \( I_0 \) and the equality \( \Lambda_0 = I_+ \) by property v). The subring \( \mathcal{R}_0 \) mentioned in Proposition II exhausts \( \mathcal{R} \) for these cases, and the second Hamiltonian structure of the corresponding KdV system is just the \( \mathcal{W}_G^\mathcal{G} \) algebra for the \( sl_2 \) embedding \( \mathcal{S} \) defined by \( I_0 \). A detailed investigation of these systems, focused on developing a Gelfand-Dickey type description analogous to that of the \( n \)-KdV system in \([11]\), is given in \([19, 20]\). For \( \mathcal{G} \) a classical Lie algebra, a Gelfand-Dickey type pseudo-differential operator model was found for about half of these systems, which were shown to also arise from reductions of KdV systems associated to \( gl_n \) by means of involutive discrete symmetries. The interested reader may consult the papers in \([19, 20]\) for details.

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\(^7\)Using arbitrary Cartan preserving automorphisms of \( \mathcal{G} \) of finite order instead of just inner ones, an analogous result applies to general affine Lie algebras.
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