THE REPRESENTATION-THEORETIC RANK OF THE DOUBLES OF QUASI-QUANTUM GROUPS

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ABSTRACT. We compute the representation-theoretic rank of a finite dimensional quasi-Hopf algebra $H$ and of its quantum double $D(H)$, within the rigid braided category of finite dimensional left $D(H)$-modules.

1. Introduction

The definition of a quasi-bialgebra $H$ ensures that the category of left $H$-modules $H\mathcal{M}$ is a monoidal category, and for a quasi-Hopf algebra $H$ the definition ensures that $H\mathcal{M}^{fd}$, the category of finite dimensional left $H$-modules, is a monoidal category with duality. Moreover, a quasi-Hopf algebra is called quasi-triangular (ribbon) if the monoidal category $H\mathcal{M}$ is braided (ribbon, at least in the finite dimensional case). So, in general, the study of quasi-Hopf algebras is strictly connected to the study of monoidal, or braided (ribbon) categories. Consequently, when we want to define some classes of quasi-Hopf algebras the first thing we should think about is to reword at a categorical level the corresponding definitions given in the classical Hopf case. If it is possible, then we can come back to the quasi-Hopf case. For example, this was the case in [3], where using the categorical interpretation of a factorizable Hopf algebra (due to Majid [27]), we were able to define and study the class of factorizable quasi-Hopf algebras. But sometimes this point of view cannot be followed. For further use we choose as an example the cosemisimple notion.

It is well known that a Hopf algebra $H$ is cosemisimple if the category of left or right $H$-comodules is cosemisimple. In the quasi-Hopf case we cannot consider $H$-comodules, because the quasi-Hopf algebra $H$ is not coassociative; thus in this case we have to look at some other objects. One of these objects could be the quantum double $D(H)$ associated to a finite dimensional quasi-Hopf algebra $H$. In the Hopf case we know that $D(H)$ is semisimple if and only if $H$ is semisimple and cosemisimple (see [31]). But, once again, at this moment we cannot follow this path because in the quasi-Hopf case we do not know the form of an integral in $D(H)$. However, in the Hopf case, the Maschke-type theorem asserts that $H$ is cosemisimple if and only if there exists a left or right integral $\lambda$ in $H^*$ such that $\lambda(1) = 1$. Now by [1] this is equivalent to the existence of a bilinear form $\sigma \in (H \otimes H)^*$ such that $h_1 \sigma(h_2, h') = \sigma(h, h'_1)h'_2$ and $\sigma(h_1, h_2) = \varepsilon(h)$, for all $h, h' \in H$. This approach was

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used by Hausser and Nill in [21] for the quasi-Hopf algebra setting. They proved that for a finite dimensional quasi-Hopf algebra $H$ there is a one-to-one correspondence between left cointegrals $\lambda \in H^*$ (see the definition below) satisfying the normalized condition $\lambda(S^{-1}(\alpha)\beta) = 1$ (here $\alpha$ and $\beta$ are the elements which occur in the definition of the antipode $S$ of $H$), and certain bilinear forms $\sigma \in (H \otimes H)^*$ satisfying properties which generalize the ones described above. This is why we will say that a finite dimensional quasi-Hopf algebra $H$ is cosemisimple if $H$ admits a left cointegral $\lambda$ obeying $\lambda(S^{-1}(\alpha)\beta) = 1$. Furthermore, we believe that an integral in $D(H)$ has the form $\beta \leftarrow \lambda \Rightarrow r$, so if it is the case then $D(H)$ is semisimple if and only if $H$ is semisimple and the left cointegral $\lambda$ satisfies $\lambda(S^{-1}(\alpha)\beta) = 1$ (here $r$ is a right integral in $H$). Comparing this with the Hopf algebra case we will land to the same definition for a finite dimensional cosemisimple quasi-Hopf algebra.

The starting point of this paper was the intention to generalize some important results concerning semisimple cosemisimple Hopf algebras to quasi-Hopf algebras. Namely, a Hopf algebra over a field of characteristic zero is semisimple if and only if it is cosemisimple, if and only if it is involutory, this means $S^2 = id_H$. The result was proved by Larson and Radford in [23, 24], answering in positive, in characteristic zero, the fifth conjecture of Kaplansky. They have also proved that in characteristic $p$ sufficiently large a semisimple cosemisimple Hopf algebra is involutory. Afterwards, using this result and a lifting theorem, Etingof and Gelaki prove in [18] that the antipode of a semisimple cosemisimple Hopf algebra over any field is an involution.

Trying to generalize the above results for quasi-Hopf algebras, the first problem which occur is: what could be an involutory quasi-Hopf algebra? We believe that we cannot keep the same definition as in the Hopf case because, in general, $S^2$ is not a coalgebra morphism, while $id_H$ is. So one of the purposes of this paper is to find a plausible definition for this notion. Toward this end we will use a categorical point of view due to Majid [26]. More exactly, he has observed that $\text{Tr}(S^2)$, the trace of $S^2$, is an important invariant of any finite dimensional Hopf algebra. In fact, he has shown that $\text{Tr}(S^2)$ arises in a very natural way as the representation-theoretic rank of the Schrödinger representation of $H$, $\dim(H)$, or as the representation-theoretic rank of the canonical representation of the quantum double, $\dim(D(H))$. Correlating this with the trace formula obtained by Radford in [32] we get that

$$\dim(H) = \dim(D(H)) = \text{Tr}(S^2) = \varepsilon(r)\lambda(1),$$

where $\lambda$ is a left integral in $H^*$ and $r$ is a right integral in $H$ such that $\lambda(S(r)) = 1$. By the Larson-Radford-Etingof-Gelaki results we conclude that

$$\dim(H) = \dim(D(H)) = \begin{cases} 0, & \text{if } H \text{ is not semisimple or cosemisimple} \\ \dim(H), & \text{if } H \text{ is both semisimple and cosemisimple}. \end{cases}$$

The aim of this paper is to generalize some of the results presented above for quasi-Hopf algebras by computing the representation-theoretic rank of a finite dimensional quasi-Hopf algebra $H$ and of its quantum double $D(H)$. We hope that the point of view presented here will open the way for solving the remaining ones. The paper is organized as follows. In Section 8 we compute the Schrödinger representation associated to a finite dimensional quasi-Hopf algebra $H$. In fact, we will transfer the associated algebra structure of $H$ within the category of left Yetter-Drinfeld modules constructed in [9, 8] to the category of left $D(H)$-modules, through some monoidal isomorphisms explicitly constructed in [4] and [12]. Now following
In any braided rigid monoidal category $C$ we can compute the representation-theoretic rank of an object $V$ of $C$. Considering $C = D(H)\mathcal{M}^{fd}$, the category of finite dimensional left $D(H)$-modules, we will compute in Section 3 the representation-theoretic rank of $H$ and $D(H)$ within $C$, $\dim(D(H))$ and $\dim(D(H))$, respectively. After some technical and complicated computations we will find that

$$\dim(H) = \dim(D(H)) = \text{Tr}(h \mapsto S^{-2}(S(\beta)\alpha h \beta S(\alpha))).$$

Therefore, we call a quasi-Hopf algebra $H$ involutory if $h \mapsto S^{-2}(S(\beta)\alpha h \beta S(\alpha)) = id_H$. Firstly, because, just as in the Hopf case, the above representation-theoretic ranks reduce to the classical dimension of $H$, provided $H$ involutory. Secondly, because $g = \beta S(\alpha)$ is invertible with $g^{-1} = S(\beta)\alpha$ and $g^{-1}$ defines both $S^2$ as an inner automorphism of $H$ and (assuming $k$ algebraically closed of characteristic zero) that unique pivotal structure in [17, Propositions 8.24 and 8.23]. More explicitly, if $k$ is an algebraically closed field of characteristic zero then $g^{-1}$ gives rise to the unique pivotal structure of $H\mathcal{M}^{fd}$ with respect to which the categorical dimensions of simple objects coincide with their usual dimensions. (Complete proofs for the above facts, examples, properties and results on involutory (dual) quasi-Hopf algebras can be found in [13].) Furthermore, specializing the above equality for $H = H^*_v$, the quasi-Hopf algebra considered in [29], we obtain that $\text{dim}(D^v(H)) = \text{dim}(H)$, where $D^v(H)$ is the quasi-triangular quasi-Hopf algebra constructed in [11], and we should stress the fact that in this particular case both $H^*_v$ and $D^v(H)$ are involutory in the quasi-Hopf sense mentioned above.

Finally, in Section 5 we prove a trace formula for quasi-Hopf algebras. Specializing it for the endomorphism $h \mapsto S^{-2}(S(\beta)\alpha h \beta S(\alpha))$ we get that

$$\text{Tr}(h \mapsto S^{-2}(S(\beta)\alpha h \beta S(\alpha))) = \varepsilon(r)(\lambda(S^{-1}(\alpha))\beta),$$

where $\lambda$ is a left cointegral in $H$ and $r$ is a right integral in $H$ such that $\lambda(r) = 1$. Combining the results in the last two Sections we conclude that $\text{dim}(H) = \text{dim}(D(H)) = \varepsilon(r)(\lambda(S^{-1}(\alpha))\beta)$, so this scalar is non-zero if and only if $H$ is both semisimple and cosemisimple.

In view of these results we believe that a semisimple cosemisimple quasi-Hopf algebra is always involutory and therefore, in this case, $\text{dim}(H) = \text{dim}(D(H)) = \text{dim}(H)$, the classical dimension of $H$. In this direction we do not know if the techniques used in [23, 24, 18] can be generalized for quasi-Hopf algebras. But without doubt it is an interesting problem which is worthwhile to study.

2. Preliminaries

2.1. Quasi-Hopf algebras. We work over a commutative field $k$. All algebras, linear spaces etc. will be over $k$; unadorned $\otimes$ means $\otimes_k$. Following Drinfeld [16], a quasi-bialgebra is a four-tuple $(H, \Delta, \varepsilon, \Phi)$ where $H$ is an associative algebra with unit, $\Phi$ is an invertible element in $H \otimes H \otimes H$, and $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ are algebra homomorphisms satisfying the identities

$$\begin{align*}
(1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi)(\Phi \otimes 1) &= (id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi), \\
(id \otimes \Delta)(\Phi) &= 1 \otimes 1 \otimes 1.
\end{align*}$$
The map $\Delta$ is called the coproduct or the comultiplication, $\varepsilon$ the counit and $\Phi$ the reassociator. As for Hopf algebras we denote $\Delta(h) = h_1 \otimes h_2$, but since $\Delta$ is only quasi-coassociative we adopt the further convention (summation understood):

$$(\Delta \otimes \text{id})(\Delta(h)) = h_{1(1)} \otimes h_{1(2)} \otimes h_2, \quad (\text{id} \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{2(1)} \otimes h_{2(2)},$$

for all $h \in H$. We will denote the tensor components of $\Phi$ by capital letters, and the ones of $\Phi^{-1}$ by small letters, namely

$$\Phi = X^1 \otimes X^2 \otimes X^3 = T^1 \otimes T^2 \otimes T^3 = V^1 \otimes V^2 \otimes V^3 = \ldots$$

$$\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = t^1 \otimes t^2 \otimes t^3 = v^1 \otimes v^2 \otimes v^3 = \ldots$$

$H$ is called a quasi-Hopf algebra if, moreover, there exists an anti-morphism $S$ of the algebra $H$ and elements $\alpha, \beta \in H$ such that, for all $h \in H$, we have:

$$S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad h_1 \beta S(h_2) = \varepsilon(h)\beta,$$

$$X^1 \beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad S(x^1)\alpha x^2 \beta S(x^3) = 1.$$

Our definition of a quasi-Hopf algebra is different from the one given by Drinfeld in the sense that we do not require the antipode to be bijective. Nevertheless, in the finite dimensional or quasi-triangular case this condition can be deleted because it follows from the other axioms, see [10] and [11].

Together with a quasi-Hopf algebra $H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ we also have $H^{\text{cop}}$ and $H^{\text{cop}}$ as quasi-Hopf algebras, where "cop" means opposite multiplication and "cop" means opposite comultiplication. The quasi-Hopf structures are obtained by putting $\Phi_{\text{cop}} = \Phi^{-1}$, $\Phi_{\text{cop}} = (\Phi^{-1})^{221}$, $S_{\text{cop}} = S^S_{\text{cop}} = S^{-1}$, $\alpha_{\text{cop}} = S^{-1}(\alpha)$ and $\beta_{\text{cop}} = S^{-1}(\beta)$.

The axioms for a quasi-Hopf algebra imply that $\varepsilon \circ S = \varepsilon$ and $\varepsilon(\alpha)\varepsilon(\beta) = 1$, so, by rescaling $\alpha$ and $\beta$, we may assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$. The identities (2.2), (2.3) and (2.4) also imply that

$$(\varepsilon \otimes \text{id} \otimes \varepsilon)(\Phi) = (\text{id} \otimes \varepsilon \otimes \varepsilon)(\Phi) = 1 \otimes 1 \otimes 1.$$

It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following statement: there exists an invertible element $f \in H \otimes H$ such that $(\varepsilon \otimes \text{id})(f) = (\text{id} \otimes \varepsilon)(f) = 1$ and

$$f \Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{\text{cop}}(h)),$$

for all $h \in H$, where $\Delta^{\text{cop}}(h) = h_2 \otimes h_1$. $f$ can be computed explicitly. First set

$$A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (\Phi \otimes 1)(\Delta \otimes \text{id} \otimes \text{id})(\Phi^{-1}),$$

$$B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes \text{id} \otimes \text{id})(\Phi)(\Phi^{-1} \otimes 1)$$

and then define $\gamma, \delta \in H \otimes H$ by

$$\gamma = S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \quad \text{and} \quad \delta = B^1 \beta S(B^4) \otimes B^2 \beta S(B^3).$$

$f$ and $f^{-1}$ are then given by the formulas

$$f = (S \otimes S)(\Delta^{\text{cop}}(x^1))\gamma \Delta(x^2 \beta S(x^3)),$$

$$f^{-1} = \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\text{cop}}(x^3)).$$

Moreover, $f = f^1 \otimes f^2$ and $f^{-1} = g^1 \otimes g^2$ satisfy the following relations:

$$f \Delta(\alpha) = \gamma, \quad \Delta(\beta)f^{-1} = \delta,$$

$$\Delta(\beta)(f^{-1})(f^{-1})(f^{-1} \otimes 1) = S(X^3) \otimes S(X^2) \otimes S(X^1),$$

$$f^1 \beta S(f^2) = S(\alpha), \quad g^1 S(g^2 \alpha) = \beta, \quad (S f^1)f^2 = \alpha.$$
In a Hopf algebra $H$, we obviously have the identity
\[ h_1 \otimes h_2 S(h_3) = h \otimes 1, \text{ for all } h \in H. \]

We will need the generalization of this formula to quasi-Hopf algebras. Following [19, 20], we define
\[ p_R = p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3), \quad q_R = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3)X^2, \]
\[ p_L = \tilde{p}^1 \otimes \tilde{p}^2 = X^2 S^{-1}(X^1 \beta) \otimes X^3, \quad q_L = \tilde{q}^1 \otimes \tilde{q}^2 = S(x^1) \alpha x^2 \otimes x^3. \]

For all $h \in H$, we then have
\[ (2.17) \quad \Delta(h_1)p_R(1 \otimes S(h_2)) = p_R(h \otimes 1) \]
\[ (2.18) \quad (S(h_1) \otimes 1)q_L \Delta(h_2) = (1 \otimes h)q_L. \]

Furthermore, the following relations hold
\[ (2.19) \quad (1 \otimes S^{-1}(p^2))q_R \Delta(p^1) = 1 \otimes 1 \]
\[ (2.20) \quad \Delta(q^1)p_R(1 \otimes S(q^2)) = 1 \otimes 1 \]
\[ (2.21) \quad (S(\tilde{p}^1) \otimes 1)q_L \Delta(\tilde{p}^2) = 1 \otimes 1 \]
\[ \Phi(\Delta \otimes id)(p_R)(p_R \otimes id) = (id \otimes \Delta)(\Delta(x^1)p_R)(1 \otimes f^{-1})(1 \otimes S(x^3) \otimes S(x^2)) \]
\[ (2.22) \quad (q_R \otimes 1)(\Delta \otimes id)(q_R)\Phi^{-1} = (1 \otimes S^{-1}(f^2X^3) \otimes S^{-1}(f^1X^2))(id \otimes \Delta)(q_R \Delta(X^1)), \]
\[ (2.23) \quad (1 \otimes q_L)(id \otimes \Delta)(q_L)\Phi = (S(x^2) \otimes S(x^1) \otimes 1)(f \otimes 1)(\Delta \otimes id)(q_L \Delta(x^3)). \]

2.2. Quasi-triangular quasi-Hopf algebras and the quantum double. Recall that a quasi-Hopf algebra $H$ is quasi-triangular if there exists an element $R \in H \otimes H$ such that
\[ (2.25) \quad (\Delta \otimes id)(R) = \Phi_{312}R_{13}\Phi^{-1}_{132}R_{23}\Phi, \]
\[ (2.26) \quad (id \otimes \Delta)(R) = \Phi_{231}R_{13}\Phi_{213}R_{12}\Phi^{-1}, \]
\[ (2.27) \quad \Delta^{op}(h)R = R\Delta(h), \text{ for all } h \in H, \]
\[ (2.28) \quad (\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = 1. \]

Here we use the following notation. If $\sigma$ is a permutation of $\{1, 2, 3\}$, we set $\Phi_{\sigma(1)\sigma(2)\sigma(3)} = X^{\sigma^{-1}(1)} \otimes X^{\sigma^{-1}(2)} \otimes X^{\sigma^{-1}(3)}$, and $R_{ij}$ means $R$ acting non-trivially in the $i^{th}$ and $j^{th}$ positions of $H \otimes H \otimes H$.

In [7] it is shown that $R$ is invertible, and that the element
\[ (2.29) \quad u = S(R^2p^2)\alpha R^1p^1 \]
(with $p_R = p^1 \otimes p^2$ defined as in (2.15)) is invertible in $H$ and satisfies for all $h \in H$ the following relation
\[ (2.30) \quad S^2(h) = uh^{-1}. \]

As in the Hopf algebra theory the most important example of quasi-triangular quasi-Hopf algebra is produced by the double construction.

From [20, 5], we recall the definition of the quantum double $D(H)$ of a finite dimensional quasi-Hopf algebra $H$. Let $\{e_i\}_{i=1, \ldots, n}$ be a basis of $H$, and $\{e^i\}_{i=1, \ldots, n}$ the corresponding dual basis of $H^*$. We can easily see that $H^*$, the linear dual of $H$,
is not a quasi-Hopf algebra. But $H^*$ has a dual structure coming from the initial structure of $H$. So $H^*$ is a coassociative coalgebra, with comultiplication

$$\hat{\Delta}(\varphi) = \varphi_1 \otimes \varphi_2 = \sum_{i,j=1}^{n} \varphi(e_i e_j) e^i \otimes e^j,$$

or, equivalently,

$$\hat{\Delta}(\varphi) = \varphi_1 \otimes \varphi_2 \leftrightarrow \varphi(h h') = \varphi_1(h) \varphi_2(h'), \quad \forall \ h, h' \in H.$$

$H^*$ is also an $H$-bimodule, by

$$\langle h \rightarrow \varphi, h' \rangle = \varphi(h' h), \quad \langle \varphi \leftarrow h, h' \rangle = \varphi(h h').$$

The convolution is a multiplication on $H^*$; it is not associative, but only quasi-associative:

$$[\varphi \psi] \xi = (X^1 \rightarrow \varphi \leftarrow x^1)(X^2 \rightarrow \psi \leftarrow x^2)(X^3 \rightarrow \xi \leftarrow x^3), \quad \forall \ \varphi, \psi, \xi \in H^*.$$

We also introduce $\overline{S} : H^* \rightarrow H^*$ as the coalgebra antimorphism dual to $S$, this means $\langle \overline{S}(\varphi), h \rangle = \langle \varphi, S(h) \rangle$, for all $\varphi \in H^*$ and $h \in H$.

Now consider $\Omega \in H^\otimes 5$ given by

$$\Omega = \Omega^1 \otimes \Omega^2 \otimes \Omega^3 \otimes \Omega^4 \otimes \Omega^5$$

$$\quad \quad = X^{1,(1,1)}_{x^1} x^1 \otimes X^{1,(2,2)}_{x^1} y^1 x^2 \otimes X^{1,2}_{x^2^3} x^2 \otimes S^{-1}(f^1 X^2 x^3) \otimes S^{-1}(f^2 X^3),$$

where $f \in H \otimes H$ is the element defined in (2.10). We define the quantum double $D(H) = H^* \rtimes H$ as follows: as a $k$-linear space, $D(H)$ equals $H^* \otimes H$, and the multiplication is given by

$$\langle \varphi \triangleright h \rangle \langle \psi \triangleright h' \rangle = ([\Omega^1 \rightarrow \varphi \leftarrow \Omega^5]([\Omega^2 \rightarrow \psi \leftarrow \Omega^4]) \triangleright \Omega^3([\overline{S}^{-1}(\psi_1) \rightarrow h) \leftarrow \psi_3])h' $$

$$\quad \quad = ([\Omega^1 \rightarrow \varphi \leftarrow \Omega^5]([\psi^2, h_{(1,1)} \rightarrow \psi \leftarrow S^{-1}(h_2)\Omega^4]) \triangleright \Omega^3 h_{(1,2)} h'. $$

From [19][20] we have that $D(H)$ is an associative algebra with unit $\varepsilon \triangleright 1$, and $H$ is a unital subalgebra via the morphism $i_D : H \rightarrow D(H)$, $i_D(h) = \varepsilon \triangleright h$. Moreover, $D(H)$ is a quasi-triangular quasi-Hopf algebra with the following structure:

$$\Delta_D(\varphi \triangleright h) = (\varepsilon \triangleright X^1 Y^1)(p^1_1 x^1 \rightarrow Y^2 S^{-1}(p^2 \triangleright p^1_2 x^2 y^1))$$

$$\otimes (X^2_1 \rightarrow \varphi_1 \leftarrow S^{-1}(X^3) \triangleright X^2_2 Y^2 x^3 x^2 y^2)$$

$$\varepsilon_D(\varphi \triangleright h) = \varepsilon(\varphi) \varepsilon(S^{-1}(\alpha))$$

$$\Phi_D = (i_D \otimes i_D \otimes i_D)(\Phi)$$

$$S_D(\varphi \triangleright h) = (\varepsilon \triangleright S(h) f^1)(p^1_1 U^1 \rightarrow \overline{S}^{-1}(\varphi) \leftarrow f^2 S^{-1}(p^2) \triangleright p^1_2 U^2)$$

$$\alpha_D = \varepsilon \triangleright \alpha, \quad \beta_D = \varepsilon \triangleright \beta$$

$$R_D = \sum_{i=1}^{n} (\varepsilon \triangleright S^{-1}(p^2) e_i p^1_1) \otimes (e^i \triangleright p^2).$$

Here $p_R = p^1 \otimes p^2$ and $f = f^1 \otimes f^2$ are the elements defined by (2.15) and (2.10), respectively, and $U = U^1 \otimes U^2 \in H \otimes H$ is the following element

$$U = U^1 \otimes U^2 = g^1 S(q^2) \otimes g^2 S(q^1),$$

where $f^{-1} = g^1 \otimes g^2$ and $q_R = q^1 \otimes q^2$ are the elements defined by (2.11) and (2.15), respectively.
2.3. The center construction and the Yetter-Drinfeld modules. If $H$ is a quasi-bialgebra then the category of left $H$-modules, denoted by $\mathcal{H}\mathcal{M}$, is a monoidal category and, moreover, if $H$ is quasi-triangular then $\mathcal{H}\mathcal{M}$ is braided (the reader is invited to consult [22 XI.4] or [27 IX.1] for the complete definition of a monoidal or (pre) braided category, and also for the notion of a monoidal, respectively (pre) braided, functor between them). The tensor product $\otimes$ is given via $\Delta$, for $U,V,W \in \mathcal{H}\mathcal{M}$ the associativity constraint on $\mathcal{H}\mathcal{M}$ is given by

$$a_{U,V,W}((u \otimes v) \otimes w) = X^1 \cdot u \otimes (X^2 \cdot v \otimes X^3 \cdot w),$$

the unit is $k$ as a trivial $H$-module and the left and right unit constraints are the usual ones. When $H$ is quasi-triangular we have the following braiding $c$ on $\mathcal{H}\mathcal{M}$:

$$c_{U,V}(u \otimes v) = R^2 \cdot v \otimes R^1 \cdot u.$$

To any monoidal category $\mathcal{C}$ we can associate two (pre) braided monoidal categories, namely the (weak) left and right centers $(\mathcal{W}_l(\mathcal{C}))$, $(\mathcal{W}_r(\mathcal{C}))$ of $\mathcal{C}$. For the (weak) left center construction the reader is invited to consult [25] for the right (weak) center construction [22 XIII.4], and for the connection between them [3], respectively.

Since for a quasi-bialgebra $H$ the category $\mathcal{H}\mathcal{M}$ is monoidal it makes sense to consider $\mathcal{W}_l(\mathcal{H}\mathcal{M})$ or $\mathcal{W}_r(\mathcal{H}\mathcal{M})$. In [25] Majid computed the left weak center $\mathcal{W}_l(\mathcal{H}\mathcal{M})$. The objects are identified with the so called left Yetter-Drinfeld modules, i.e. left $H$-modules $M$ (denote the action by $h \otimes m \mapsto h \cdot m$) together with a $k$-linear map $\lambda_M : M \rightarrow H \otimes M$, $\lambda_M(m) := m \cdot_{(-1)} \otimes m \cdot_{(0)}$, such that $\varepsilon(m \cdot_{(-1)})m \cdot_{(0)} = m$ and for all $h \in H$ and $m \in M$ the following relations hold:

$$X^1 m \cdot_{(-1)} \otimes (X^2 \cdot m \cdot_{(0)}) 
\otimes X^3 \otimes (X^2 \cdot m \cdot_{(0)}) \cdot_{(0)}$$

$$= X^1 \cdot Y^1 \cdot m \cdot_{(-1)} \otimes Y^2 \otimes X^2 \cdot Y^1 \cdot m \cdot_{(-1)} \otimes Y^3 \otimes X^3 \cdot Y^1 \cdot m \cdot_{(0)},$$

$$h_1 m \cdot_{(-1)} \otimes h_2 \cdot m \cdot_{(0)} = (h_1 \cdot m) \cdot_{(-1)} h_2 \otimes (h_1 \cdot m) \cdot_{(0)}.$$

The category of left Yetter-Drinfeld modules and $k$-linear maps that preserve the $H$-action and $H$-coaction is denoted by $\mathcal{H}^l\mathcal{Y}\mathcal{D}$.

The prebraided monoidal structure on $\mathcal{W}_l(\mathcal{H}\mathcal{M})$ induces a prebraided monoidal structure on $\mathcal{H}^l\mathcal{Y}\mathcal{D}$. This structure is such that the forgetful functor $\mathcal{H}^l\mathcal{Y}\mathcal{D} \rightarrow \mathcal{H}\mathcal{M}$ is monoidal, and the coaction on the tensor product $M \otimes N$ of two left Yetter-Drinfeld modules $M$ and $N$ is given by

$$\lambda_M \otimes_N (m \otimes n) = X^1 (x^1 Y^1 \cdot m) \cdot_{(-1)} x^2 (Y^2 \cdot n) \cdot_{(-1)} Y^3 \otimes X^2 \cdot (x^1 Y^1 \cdot m) \cdot_{(0)} \otimes X^3 x^3 \cdot (Y^2 \cdot n) \cdot_{(0)}.$$

For any $M, N \in \mathcal{H}^l\mathcal{Y}\mathcal{D}$ the braiding $c_{M,N} : M \otimes N \rightarrow N \otimes M$ is given by

$$c_{M,N}(m \otimes n) = m \cdot_{(-1)} \cdot n \otimes m \cdot_{(0)},$$

for all $m \in M$ and $n \in N$. Moreover, if $H$ is a quasi-Hopf algebra then $c_{M,N}$ is invertible (see [3]) and therefore $\mathcal{H}^l\mathcal{Y}\mathcal{D}$ is a braided category.

We notice that the right weak center $\mathcal{W}_r(\mathcal{H}\mathcal{M})$ was computed in [3]: it is isomorphic to the category of left-right Yetter-Drinfeld modules (see the definition below).
3. The Schrödinger representation

Let $H$ be a finite dimensional Hopf algebra. It is well known that $H$ is a left $D(H)$-module algebra via the action $(\varphi \in H^*, h, h' \in H)$:

$$(\varphi \triangleleft h) \bullet h' = \langle \varphi, S^{-1}((h \triangleright h')_1)\rangle(h \triangleright h')_2.$$ 

Here and also in the rest of the paper, $h \triangleright h' := h_1 h'S(h_2)$, for all $h, h' \in H$.

The aim of this section is to compute a similar structure for a finite dimensional quasi-Hopf algebra $H$. This fact is absolutely necessary in order to compute the representation-theoretic rank (or quantum dimension) of $H$ within the braided category of left $D(H)$-modules. Toward this end, we will use the following three results:

1) To any quasi-Hopf algebra $H$ we can associate an algebra, denoted by $H_0$, in the category of left Yetter-Drinfeld modules $H\mathcal{YD}$, cf. [8]. More precisely, we denote by $H_0$ the $k$-vector space $H$ with the new multiplication $\circ$ defined by

$$(3.1) \quad h \circ h' = X^1 h S(x^1 X^2) \alpha x^2 X_1^3 h' S(x^3 X_2^3),$$

for all $h, h' \in H$. From [9] we know that $H_0$ is a left $H$-module algebra, this means an algebra in $H\mathcal{M}$. The unit of $H_0$ is $\beta$ and $H_0$ is an object of $H\mathcal{M}$ via the left adjoint action $\triangleright$, i.e. for all $h, h' \in H$,

$$(3.2) \quad h \triangleright h' = h_1 h'S(h_2).$$

Moreover, $H_0$ becomes an algebra in $H\mathcal{YD}$ with the additional structure $\lambda_{H_0} : H_0 \to H \otimes H_0$, given by

$$(3.3) \quad (H_0, \triangleright) := X^1 Y^1 h_1 Y h S(q^2 Y^2_2) Y^3 \otimes X^2 Y^2_2 h_2 Y_2 S(X^3 q^1 Y^2_1),$$

for all $h \in H$, where $q_R = q^1 \otimes q^2$ is the element defined by (2.15).

2) There is a braided isomorphism between $H\mathcal{YD}$ and $H\mathcal{YD}^M$, cf. [4]. First of all recall that the category of left-right Yetter-Drinfeld modules over a quasi-bialgebra $H$, denoted by $H\mathcal{YD}^H$, has as objects left $H$-modules $M$ (denote the action by $h \otimes m \mapsto h \cdot m$) for which $H$ coacts on the right (denote the right $H$-coaction by $M \ni m \mapsto m_0 \otimes m_1 \in M \otimes H$) such that $\varepsilon(m_1) m_0 = m$ and for all $m \in M$ and $h \in H$ the following relations hold

$$(3.4) \quad (x^2 \cdot m_0)_0 \otimes (x^2 \cdot m_0)_1 x^1 \otimes x^3 m_1$$

$$(3.5) \quad h_1 \cdot m_0 \otimes h_2 m_1 = (h_2 \cdot m)_0 \otimes (h_2 \cdot m)_1 h_1.$$ 

The morphisms are left $H$-linear, right $H$-colinear maps.

Since $H\mathcal{YD}^H$ can be identified with the right weak center of $H\mathcal{M}$ (see [4]) we find that $H\mathcal{YD}^H$ has the following prebraided structure: the right $H$-coaction on the tensor product $M \otimes N$ of $M, N \in H\mathcal{YD}^H$ is the following:

$$(3.6) \quad \rho_{M \otimes N}(m \otimes n) = x^1 X^1 \cdot (y^2 \cdot m)_0 \otimes x^2 \cdot (X^3 y^3 \cdot n)_0 \otimes x^3 (X^3 y^3 \cdot n)_1 X^2 (y^2 \cdot m)_1 y^1,$$

for all $m \in M, n \in N$, and the functor forgetting the $H$-coaction is monoidal, so

$$(3.7) \quad h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n.$$
The braiding $\epsilon$ on $_H\mathcal{YD}^H$ is defined by

$$\epsilon_{M,N}(m \otimes n) = n_{(0)} \otimes n_{(1)} \cdot m,$$

for $m \in M$ and $n \in N$. Furthermore, if $H$ is a quasi-Hopf algebra the braiding $\epsilon$ is invertible. The inverse braiding is given by

$$\epsilon^{-1}_{M,N}(m \otimes n) = q_1^1 x_1 S(q_2^2 x_3 (\tilde{p}_2^2 \cdot n_{(1)}) \tilde{p}_1^1) \cdot m \otimes q_2^1 x_2 \cdot (\tilde{p}_2^2 \cdot n_{(0)}),$$

where $q_R = q^1 \otimes q^2$ and $p_L = \tilde{p}_1 \otimes \tilde{p}_2$ are the elements defined in (2.15) and (2.16), respectively. Finally, for a quasi-Hopf algebra $H$, $_HYD^H$ will be our notation for the category of left-right Yetter-Drinfeld modules endowed with the braided structure given by (3.6-3.8), and $_H\mathcal{YD}^{H^\text{in}}$ will be our notation for the category $_HYD^H$ with monoidal structure (3.6-3.7) and the mirror reversed braiding $\epsilon_{M,N} = \epsilon^{-1}_{N,M}$.

Now, by [4] there is a monoidal isomorphism between $_H\mathcal{YD}$ and $_HYD^H$ produced by the following functor $\mathfrak{g}$. If $M \in _H\mathcal{YD}$ then $\mathfrak{g}(M) = M$ as left $H$-modules and with the right $H$-coaction defined by

$$\rho_{F(M)}(m) = q_1^2 x_2 \cdot (p_1 \cdot m)_{(0)} \otimes q_2^1 x_3 S^{-1}(q_1^1 X_1 (p_1 \cdot m)_{(-1)} \tilde{p}_2),$$

for all $m \in M$. The functor $\mathfrak{g}$ acts as identity on morphisms. Moreover, $\mathfrak{g}$ provides a braided isomorphism between $_H\mathcal{YD}$ and $_HYD^{H^\text{in}}$, see [4] for more details.

3) The category $_HYD^H$ is braided isomorphic to $_{D(H)}\mathcal{M}$.

Indeed, from [20] [12] we know that the above categories are isomorphic. The isomorphism is the following. To any left-right Yetter-Drinfeld module $M$ we can associate a left $D(H)$-module structure given by

$$(\varphi \otimes h) \cdot m = \langle \varphi, q^2 (h \cdot m)_{(1)} \rangle q^1 \cdot (h \cdot m)_{(0)},$$

for all $\varphi \in H^*$, $h \in H$ and $m \in M$, where, as usual, $q_R = q^1 \otimes q^2$ is the element defined in (2.15). Moreover, a morphism between two left-right Yetter-Drinfeld modules becomes in this way a morphism between two left $D(H)$-modules, so we have a well defined functor $\mathcal{F} : _HYD^H \rightarrow _{D(H)}\mathcal{M}$. If $H$ is a finite dimensional quasi-Hopf algebra then $\mathcal{F}$ is an isomorphism (for the explicit description of the inverse of $\mathcal{F}$ see [12]). Also, it is not hard to see that the functor $\mathcal{F}$ is monoidal; the functorial isomorphism $\Psi_{M,N} : \mathcal{F}(M) \otimes \mathcal{F}(N) \rightarrow \mathcal{F}(M \otimes N)$ is the identity morphism. Moreover, the next result asserts that it is a braided isomorphism.

**Proposition 3.1.** Let $H$ be a finite dimensional quasi-Hopf algebra. Then the categories $_HYD^H$ and $_{D(H)}\mathcal{M}$ are braided isomorphic.

**Proof.** By the previous comments, we only have to check that the functor $\mathcal{F}$ defined above is braided, this means that for any two left-right Yetter-Drinfeld modules $M$ and $N$ we have

$$\mathcal{F}(\epsilon_{M,N}) \circ \Psi_{\mathcal{F}(M), \mathcal{F}(N)} = \Psi_{\mathcal{F}(N), \mathcal{F}(M)} \circ \epsilon_{\mathcal{F}(M), \mathcal{F}(N)}.$$
Indeed, for all \( m \in M \) and \( n \in N \) we compute
\[
\Psi_{F(M),F(M)} \circ c_{F(M),F(N)}(m \otimes n) = \sum_{i=1}^{n} (e^i \otimes p_1^i) \rightarrow n \otimes (\varepsilon \otimes S^{-1}(p^2)e_ip_1^i) \rightarrow m
\]
\[
= (e^i, q^2(p_1^i \cdot n)(1))q^1 \cdot (p_1^i \cdot n)_{(0)} \otimes S^{-1}(p^2)e_ip_1^i \cdot m
\]
\[
= q^1 \cdot (p_1^i \cdot n)_{(0)} \otimes S^{-1}(p^2)q^2(p_1^i \cdot n)(1)p_1^i \cdot m
\]
\[
= q^1 p_1^i \cdot n_{(0)} \otimes S^{-1}(p^2)q^2 p_1^i n_{(1)} \cdot m
\]
\[
= n_{(0)} \otimes n_{(1)} \cdot m = F(c_{M,N}) \circ \Psi_{F(M),F(N)}(m \otimes n),
\]
as needed, so the proof is complete.

Using these braided isomorphisms we will transfer the algebra structure of \( H_0 \) in \( H^H\mathcal{YD} \) to \( D(H)\mathcal{M} \). In this way we will associate to any finite dimensional quasi-Hopf algebra \( H \) a left \( D(H) \)-module algebra structure. As in the classical Hopf algebra case, the obtained representation will be called the Schrödinger representation.

First we shall compute the algebra structure of \( H_0 \) in \( H^H\mathcal{YD} \), and then its left \( D(H) \)-module algebra structure.

**Proposition 3.2.** Let \( H \) be a quasi-Hopf algebra. Then \( H_0 \) is an algebra in the monoidal category \( H^H\mathcal{YD} \) with the left \( H \)-module structure defined in (3.12) and with the right \( H \)-coaction \( \rho_{H_0} : H_0 \rightarrow H_0 \otimes H \) given for all \( h \in H \) by
\[
(3.12) \quad \rho_{H_0}(h) = h_{(0)} \otimes h_{(1)} = x^1 q^2 y_2^2 h_2 g^2 S(x^2 y_1^3 \otimes x^3 y_2^3 S^{-1}(q^1 y_1^3 h_1 g^1))y_1^1,
\]
where \( q_L = q^1 \otimes q^2 \) and \( f^{-1} = g^1 \otimes g^2 \) are the elements defined in (2.10) and (2.11), respectively. Moreover, \( H_0 \) is a left \( D(H) \)-module algebra via the action
\[
(3.13) \quad \phi \triangleright h = \langle \phi, q^2 x^3 y_2^3 S^{-1}(q^1 y_1^3 (h \triangleright h')) \rangle y_1^1
\]
\[
q_1^1 x^1 q^2 y_2^2 (h \triangleright h') g^2 S(q_2^2 y_2^3 y_1^1),
\]
for all \( \phi \in H^* \) and \( h, h' \in H \), where \( q_R = q^1 \otimes q^2 \) is the element defined in (2.16).

**Proof.** Since the functor \( \mathfrak{F} \) described in (3.10) is monoidal it carries algebras to algebras. Moreover, the isomorphisms \( \Psi_{M,N} : \mathfrak{F}(M) \otimes \mathfrak{F}(N) \rightarrow \mathfrak{F}(M \otimes N), M, N \in H^H\mathcal{YD} \), which define the monoidal structure of the functor \( \mathfrak{F} \) are trivial, so if \( A \) is an algebra in \( H^H\mathcal{YD} \) then \( \mathfrak{F}(A) \) is an algebra in \( H^H\mathcal{YD} \) with the same multiplication and unit. Now, \( \mathfrak{F} \) acts as identity on objects at the level of actions. Thus \( \mathfrak{F}(H_0) = H_0 \) as left \( H \)-module algebras, so we only have to show that that corresponding right \( H \)-action on \( H_0 \) through the functor \( \mathfrak{F} \) is the one claimed in (3.12). For this we need the following relations
\[
(3.14) \quad X^1 p_1^i \otimes X^2 p_1^i \otimes X^3 p_2^2 = x^1 \otimes x_2^1 p_1^i \otimes x_2^2 p_2^2 S(x^3),
\]
\[
(3.15) \quad q^1 X^1 \otimes q_2^2 X^2 \otimes q_2^2 X^3 = S(x^1) q^1 x^2_2 \otimes q^2 x^2 \otimes x^3,
\]
\[
(3.16) \quad f^{-1} = \Delta(S(h_1)) U(h_2 \otimes 1) = U(1 \otimes S(h)), \quad \forall h \in H.
\]

Indeed, (3.14) and (3.15) follow easily from (2.3), (2.5) and from the definitions of \( p_R \) and \( q_L \), respectively. The relation (3.16) is an immediate consequence of (2.8) and (2.19), and the formula in (3.17) can be found in [21].
Finally, by (2.8) and (2.35) we get the following second formula for the left 
$H$-coaction on $H_0$ defined in (3.3)
\[ \lambda_{H_0}(h) = h_{(-1)} \otimes h_{(0)} = (X^1 \otimes X^2) \Delta(Y^1 h S(Y^2)) U(Y^3 \otimes S(X^3)). \]

Now, for any $h \in H$ we calculate
\[ \rho_{H_0}(h) = q_1^3 Z^2 \triangleright (p_1 \triangleright h)_{(0)} \otimes q_2^3 Z^3 S^{-1}(q_1^1 Z^1 (p_1 \triangleright h)_{(-1)} p_2^2) \]
\[ = q_1^3 Z^2 \triangleright [X^2 (Y^1 (p_1 \triangleright h) S(Y^2)) U^2 S(X^3)] \]
\[ \otimes q_2^3 Z^3 S^{-1}(q_1^1 Z^1 Y^1 (p_1 \triangleright h) S(Y^2)) U^1 Y^3 p_2^2) \]
\[ = q_1^3 Z^2 \triangleright [X^2 x_2^2 h_2 S(x_1^2 p_1) U^2 S(X^3)] \]
\[ \otimes q_2^3 Z^3 x_1^3 S^{-1}(q_1^1 Z^1 X_1^1 h_1 S(x_1^2 p_1) U^1 x_2^2 p_2^2) \]
\[ = q_1^3 \triangleright [Z^2 x_2^2 h_2 S(p_1) U^2 S(Z_2^2 X^3 x_1^3)] \]
\[ \otimes q_2^3 Z^3 x_1^3 S^{-1}(q_1^1 Z^1 X_1^1 h_1 S(p_1) U^1 p_2^2) \]
\[ = q_{(2,1)}^2 q_{(1,1)}^1 X^2 h_2 g^2 S(q_2^2 x^2 x_1^3) \otimes q_2^1 x_1^3 S^{-1}(q_1^1 X_1^1 h_1 g^1) \]
\[ = x^1 q^2 y_2 h_2 g^2 S(x_2 y_1^1) \otimes x^3 y_3 S^{-1}(q_1^1 y_1^1 h_1 g^1) y_1^1, \]
as needed. The last assertion is a consequence of (3.11) and (3.12), the details are 
left to the reader. \qed

Remark 3.3. Let $H$ be a quasi-triangular quasi-Hopf algebra. Under this condition 
it was proved in [8] that $H_0$ is a braided Hopf algebra in $\text{H \text{YD}}$. Using the functor 
$\otimes$ described above we obtain that $H_0$ has also a braided Hopf algebra structure in 
$\text{H \text{YD}}^{\text{Hin}}$; note that the left $H$-coaction of $H_0$ in $\text{H \text{YD}}$ (viewed as a braided Hopf 
algabra) is different from the coaction defined in (3.2), so the braided Hopf algebra 
structure of $H_0$ within $\text{H \text{YD}}^{\text{Hin}}$ is not induced by the algebra structure of $H_0$ 
which was defined in Proposition 3.2. Furthermore, if we want to associate to $H$ a braided 
Hopf algebra in $\text{H \text{YD}}^H$ (and therefore in $\text{D(H)}$ when $H$ is finite dimensional), 
is sufficient to consider $H_0^{\text{op}}$ (or $H_0^{\text{coop}}$), the opposite (the cooposite, respectively) 
braided Hopf algebra associated to $H_0$. We leave the verification of the details to 
the reader.

4. The representation-theoretic rank

Let $\mathcal{C}$ be a braided category which is left rigid (the definition of a left rigid 
category can be found in [22 XIV.2] or [27 IX.3]). If $V$ is an object of $\mathcal{C}$ and $ev_V$ 
and $coev_V$ are the evaluation and coevaluation maps associated to $V$, then following 
[26] we define the representation-theoretic rank (or quantum dimension) of $V$ as 
follows:
\[ \dim(V) = ev_V \circ c_V \circ V \circ coev_V. \]

If $H$ is a quasi-Hopf algebra then the category $\text{Mfd}$ of finite dimensional modules 
over $H$ is left rigid. For $V \in \text{M}$, its left dual is $V^* = \text{Hom}(V, k)$, with left 
$H$-action $(h \cdot \varphi, v) = (\varphi, S(h) \cdot v)$. The evaluation and coevaluation maps are 
given for all $\varphi \in V^*$ and $v \in V$ by
\[ ev_V(\varphi \otimes v) = \varphi(\alpha \cdot v), \quad coev_V(1) = \sum_i \beta \cdot v_i \otimes v_i, \]
where $\{v_i\}_i$ is a basis in $V$ with dual basis $\{v^i\}_i$ in $V^*$. 
Therefore, if \( H \) is a quasi-triangular quasi-Hopf algebra and \( V \) a finite dimensional left \( H \)-module it makes sense to consider the representation-theoretic rank of \( V \). If \( R = R^1 \otimes R^2 \) is an \( R \)-matrix for \( H \) then by [10] we have that

\[
\text{dim}(V) = \sum_i v^i (S(R^2)\alpha R^1 \beta \cdot v_i) = \text{Tr}(\eta),
\]

where \( \eta := S(R^3)\alpha R^1 \beta \). (Here \( \text{Tr}(\eta) \) is the trace of the linear endomorphism of \( V \) defined by \( v \mapsto \eta \cdot v \).)

Let \( u \) be the element defined in (2.29). By [2, 7] we have that \( S(R^2)\alpha R^1 = S(\alpha)u \), so by (2.30) we obtain

\[
\eta = S(S(\beta)\alpha)u = uS^{-1}(\alpha)\beta.
\]

In the rest of this section \( H \) will be a finite dimensional quasi-Hopf algebra, and \( \{ e_i \}_{i=1}^n \) a basis in \( H \) with dual basis \( \{ e^i \}_{i=1}^n \) in \( H^* \). Our goal is to compute \( \text{dim}(H) \) and \( \text{dim}(D(H)) \) within the braided rigid category \( D(H)_\mathcal{M}^{id} \). To this end we shall compute for \( D(H) \) the corresponding elements \( u \) and \( \eta \), denoted in what follows by \( u_D \) and \( \eta_D \), respectively.

**Proposition 4.1.** Let \( H \) be a finite dimensional quasi-Hopf algebra, and \( u_D \) and \( \eta_D \) the corresponding elements \( u \) and \( \eta \) for \( D(H) \), the quantum double of \( H \). Then

\[
u_D = \sum_{i=1}^n \beta \rightarrow S^{-1}(e^i) \otimes e_i \quad \text{and} \quad \eta_D = \sum_{i=1}^n \beta \rightarrow S^{-1}(e^i) \otimes e_i S^{-1}(\alpha)\beta.
\]

**Proof.** Let us start by noting that (2.13), (2.14) and (2.5) imply

\[
f_1^1 p^1 \otimes f_2^2 p^2 S(f^2) = g^1 S(q^2) \otimes g^2 S(q^1).
\]

Secondly, observe that the definition (2.30) of the antipode \( S_D \) of \( D(H) \) can be reformulated as follows:

\[
S_D(\varphi \triangleright h) = (\varepsilon \triangleright S(h)) \left( (f_1^1 p^1)_1 U^1 \rightarrow S^{-1}(\varphi) \leftarrow f_2^2 S^{-1}(f_2^2 p^2) \otimes (f_1^1 p^1)_2 U^2 \right) \right.
\]

\[
(\varepsilon \triangleright S(h)) \left( g_1^1 S(q^2)_1 U^1 \rightarrow S^{-1}(\varphi) \leftarrow q_1^1 S^{-1}(g^2) \otimes g_2^2 S(q^2)_2 U^2 \right)
\]

\[
(\varepsilon \triangleright S(h)) \left( g_1^1 G^1 S(q^2 q^2_2) \rightarrow S^{-1}(\varphi) \leftarrow q_1^1 S^{-1}(g^2) \otimes g_2^1 G^2 S(q^1 q^2_1) \right),
\]

where we denoted by \( G^1 \otimes G^2 \) another copy of \( f^{-1} \). Now, we claim that

\[
S_D(R^2)\alpha D^1 R^1 = \sum_{i=1}^n \beta \rightarrow S^{-1}(e^i) \leftarrow \alpha \triangleright e_i,
\]

where \( R_D = R^1 \otimes R^2 \) is the \( R \)-matrix of \( D(H) \) defined in (2.35). Indeed, we can easily check that

\[
S^{-1}(h \rightarrow \varphi) = S^{-1}(\varphi) \leftarrow S(h) \quad \text{and} \quad S^{-1}(\varphi \leftarrow h) = S(h) \rightarrow S^{-1}(\varphi),
\]
for all $\varphi \in H^*$ and $h \in H$. Now, we calculate:

$$S_D(R^2)\alpha_D R^1$$

(2.38) \[ \sum_{i=1}^{n} S_D(e^i \bowtie p_1^1)(\varepsilon \bowtie \alpha)(\varepsilon \bowtie S^{-1}(p^2)e_i p_1^1) \]

(2.39) \[ \sum_{i=1}^{n} (S(p_1^1)g^1)_1 G^1 S(q^2_1 q_2^2) \rightarrow S^{-1}(e^i) \leftarrow q^1 S^{-1}(S(p_1^2)g_2^2) \]

(2.4) \[ \varepsilon_{\bowtie \alpha_D}(S(p_1^2)g_2^2) G^2 S(q^1_1 g_2^1) \alpha S^{-1}(p^2)e_i p_1^1 \]

(2.5) \[ \sum_{i=1}^{n} g^1_i G^1 S(q^2_1 p_1^2 q_2^2) \rightarrow S^{-1}(e^i) \leftarrow q^1 S^{-1}(p^2) \]

(2.8) \[ \varepsilon_{\bowtie \alpha_D} g^1_i G^2 S(q^1_1 p_1^1) \alpha S^{-1}(p^2)e_i \]

(2.9) \[ \sum_{i=1}^{n} g^1_i G^1 S(\varepsilon_{\bowtie \alpha_D}) \rightarrow S^{-1}(e^i) \leftarrow q^1 S^{-1}(g^2) \]

(2.10) \[ \varepsilon_{\bowtie \alpha_D} g^1_i G^2 S(q^1_1 g_2^1) \alpha S^{-1}(p^2) \]

(2.11) \[ \sum_{i=1}^{n} g^1_i G^1 S(q^2_1 q_2^2) \rightarrow S^{-1}(e^i) \leftarrow q^1 S^{-1}(g^2) \]

(2.12) \[ \varepsilon_{\bowtie \alpha_D} g^1_i G^2 S(q^1_1 g_2^1) \alpha S^{-1}(g^2) \]

We are now able to calculate the element $u_D$. Since $H$ can be viewed as a quasi-Hopf subalgebra of $D(H)$ via the morphism $i_D$ it follows that the corresponding element $p_H$ for $D(H)$ is $(p_H)_D = p_D^1 \otimes p_D^2 = \varepsilon \bowtie p^1 \otimes \varepsilon \bowtie p^2$. Therefore:

$$u_D$$

(2.29) \[ S_D(R^2)\alpha_D R^1 p_1^1 = (\varepsilon \bowtie S(p^2)) S_D(R^2)\alpha_D R^1(\varepsilon \bowtie p^1) \]

(4.3) \[ \sum_{i=1}^{n} S(p^2)_{(1,1)} \beta \rightarrow S^{-1}(e^i) \leftarrow \alpha S^{-1}(S(p^2)) \]

(4.3) \[ \sum_{i=1}^{n} \varepsilon_{\bowtie \alpha_D} S(p^2)_{(1,2)} S^{-1}(\alpha S^{-1}(S(p^2)) \varepsilon_{\bowtie p^1} \alpha S^{-1}(S(p^2)) S(p^2)_{(1,1)} \beta) p^1 \]

(2.3) \[ \sum_{i=1}^{n} \varepsilon_{\bowtie \alpha_D} S^{-1}(p^1) \alpha p^2 e_i \beta \]

(2.3) \[ \sum_{i=1}^{n} \varepsilon_{\bowtie \alpha_D} S^{-1}(e_i \beta) = \sum_{i=1}^{n} \beta \rightarrow S^{-1}(e^i) \bowtie e_i, \]

as claimed. It is clear now that the above equality and (4.3) imply the expression of $\eta_D$ in (4.4), so our proof is complete.
4.1. The representation-theoretic rank of $H$. We start to compute the representation-theoretic rank (or quantum dimension) of $H$ within the braided rigid category $D_H \mathcal{M}^{fd}$. Let us start by noting that the action $\varphi \otimes h$ obtained in (3.13) can be rewritten as follows:

\[
\varphi \otimes h \mapsto h' = (\varphi, S^{-1}(Y^3)q^2Y^2_y y^3_2 S^{-1}(\bar{q}^1 y^2(h \triangleright h')_1 g^1) y^1) \]

\[
Y^1 q^2 y^2_2 (h \triangleright h')_2 g^2 S(q^1 Y^2_y y^3_1) \]

\[
\varphi \otimes S^{-1}(Y^3)q^2Y^2_y y^3_2 S^{-1}(\bar{q}^1 y^2(h \triangleright h')_1 S(Y^2_y^3)) \]

\[
Y^1 q^2(y^2(h \triangleright h') S(Y^2_y^3))_1 U^1 Y^3 \]

\[
\bar{q}^2(Y^2_y^3(h \triangleright h') S(Y^2_y^3))_2 U^2. \]

Hence we have showed that for all $\varphi \in H^*$ and $h, h' \in H$ we have

\[
(\varphi \otimes h) \mapsto h' = (\varphi, S^{-1}(Y^3)q^2Y^2_y y^3_2 S^{-1}(\bar{q}^1 y^2(h \triangleright h')_1 S(Y^2_y^3)) \]

\[
Y^1 q^2(Y^2_y^3(h \triangleright h') S(Y^2_y^3))_1 U^1 Y^3 \]

(4.8)

\[
\bar{q}^2(Y^2_y^3(h \triangleright h') S(Y^2_y^3))_2 U^2.
\]

So this action defines on $H$ a left $D(H)$-module structure, and on $H_0$ a left $D(H)$-module algebra structure.

In order to "simplify" the computation for $\text{dim}(H)$ we need the following formulas.

**Lemma 4.2.** Let $H$ be a finite dimensional quasi-Hopf algebra and $\{e_i\}_i$ a basis in $H$ with dual basis $\{e^i\}$. Then for all $h, h', h'' \in H$ the following relations hold:

\[
\sum_{i=1}^n \langle e^i, S^{-1}(\beta)S^{-2}(\bar{Q}^1(e_i)_1 h') h q^2 \bar{Q}^2_2(e_i)_2(2,2) h'' S^{-1}(\bar{q}^1 \bar{Q}^2_1(e_i)(2,1)) \rangle
\]

\[
\sum_{i=1}^n \langle e^i, S^{-1}(\beta)S^{-2}(\bar{Q}^1(e_i)_1 h') h q^2 \bar{Q}^2_2(e_i)_2(2,2) S^{-1}(\bar{q}^1 \bar{Q}^2_1(e_i)(2,1)) h \rangle
\]

\[
\sum_{i=1}^n \langle e^i, S^{-1}(\beta)S^{-2}(\bar{Q}^1(e_i)_1 h') h q^2 \bar{Q}^2_2(e_i)_2(2,2) X^3 p^2 S(h_2) h'' \]

\[
\times S^{-1}(\bar{q}^1 \bar{Q}^2_1(e_i)(2,1) X^2 p_2 h_1) \rangle = \sum_{i=1}^n \langle e^i, S^{-1}(\beta)S^{-2}(\bar{Q}^1(e_i)_1 h') h q^2 \bar{Q}^2_2(e_i)_2(2,2) X^3 p^2 S(h_2) h'' \]

\[
\times S^{-1}(\bar{q}^1 \bar{Q}^2_1(e_i)(2,1) X^2 p_2 h_1) \rangle \]

(4.10)

where we denoted $q_L = q^1 \otimes q^2 = \bar{Q}^1 \otimes \bar{Q}^2$ and $p_R = p^1 \otimes p^2$.

**Proof.** In order to prove (4.9) we shall apply (2.18) twice, and then the properties of dual bases and (2.9). Explicitly,

\[
\sum_{i=1}^n \langle e^i, S^{-1}(\beta)S^{-2}(\bar{Q}^1(e_i)_1 h') h q^2 \bar{Q}^2_2(e_i)_2(2,2) S^{-1}(\bar{q}^1 \bar{Q}^2_1(e_i)(2,1)) \rangle
\]

\[
= \sum_{i=1}^n \langle e^i, S^{-1}(\beta)S^{-2}(\bar{Q}^1(e_i)_1 h') h q^2(h_2 \bar{Q}^2_2(e_i)(2,2) h'' \]

\[
\times S^{-1}(\bar{q}^1(h_2 \bar{Q}^2_2)(e_i)(2,1) h_1) \rangle
\]
We finally need the following formula
\[ S^{-1}(\beta)S^{-2}(S(h_{(2,1)})\hat{Q}^{1}(h_{(2,2)}e_{i}h')\hat{q}^{2}\hat{Q}^{2}_{2}(h_{(2,2)}e_{i})(2,2)h'') \]
\[ \times S^{-1}(\hat{q}^{1}\hat{Q}^1_{2}(h_{(2,2)}e_{i})(2,1))h_{1}) \]
\[ = \sum_{i=1}^{n} \langle e^{i}, S^{-1}(\beta)S^{-2}(S(h_{(2,1)})\hat{Q}^{1}(e_{i})h')\hat{q}^{2}\hat{Q}^{2}_{2}(e_{i})(2,2)h'' \]
\[ \times S^{-1}(\hat{q}^{1}\hat{Q}^1_{2}(e_{i})(2,1))h_{1}) \]
\[ = \sum_{i=1}^{n} \langle e^{i}, S^{-1}(\beta)S^{-2}(\hat{Q}^{1}(e_{i})h')\hat{q}^{2}\hat{Q}^{2}_{2}(e_{i})(2,2)h'' \]
\[ \times S^{-1}(\hat{q}^{1}\hat{Q}^1_{2}(e_{i})(2,1))h_{1}) \]
\[ = \sum_{i=1}^{n} \langle e^{i}, S^{-1}(\beta)S^{-2}(\hat{Q}^{1}(e_{i})h')\hat{q}^{2}\hat{Q}^{2}_{2}(e_{i})(2,2)h'' \]
\[ \times S^{-1}(\hat{q}^{1}\hat{Q}^1_{2}(e_{i})(2,1))h_{1}) \].

In a similar manner we can prove (4.10). It follows applying (4.9), dual basis, (2.1) and (2.17), we leave the details to the reader. □

Now, equation (2.9) shows by using (2.3) and (2.5) that
\[ \gamma = \gamma^{1} \otimes \gamma^{2} = S(x^{1}X^{2})\alpha x^{3}X^{3} \otimes S(X^{1})\alpha x^{3}X^{3}, \]
\[ \delta = \delta^{1} \otimes \delta^{2} = x^{1}S(x^{3}X^{3}) \otimes x^{2}X^{1}S(x^{3}X^{3}). \]

We finally need the following formula
\[ p_{R} = \Delta(S(p^{1}))U(\hat{p}^{2} \otimes 1), \]
which can be found in [21]. We are now able to compute \( \dim(H) \).

**Proposition 4.3.** Let \( H \) be a finite dimensional quasi-Hopf algebra. Then the representation-theoretic rank of \( H \) is
\[ \dim(H) = \text{Tr} \left( h \mapsto S^{-2}(S(\beta)\alpha h S(\alpha)) \right). \]

**Proof.** We know from Proposition 4.1 that in the quantum double case the element \( \eta_{D} \) is given by
\[ \eta_{D} = \sum_{i=1}^{n} \gamma^{1} \otimes \gamma^{2} = \sum_{i=1}^{n} (e^{i} \otimes S^{-1}(\alpha e^{i})) \otimes \beta \].

We set \( p_{R} = p^{1} \otimes p^{2} = P^{1} \otimes P^{2} \), \( q_{L} = q^{1} \otimes q^{2} = \hat{Q}^{1} \otimes \hat{Q}^{2} \) and \( f = f^{1} \otimes f^{2} = F^{1} \otimes F^{2} \). Then by (1.2) and the above expression of \( \eta_{D} \) we have:
\[ \dim(H) \]
\[ = \sum_{i,j=1}^{n} \langle e^{i}, (e^{i} \otimes S^{-1}(\alpha e^{i})) \otimes e^{j} \rangle \]
\[ = \sum_{i,j=1}^{n} \langle e^{i}, S^{-1}(\hat{q}^{1}(Y_{1}^{2}y^{2}(S^{-1}(\alpha e^{i})) \beta \otimes e_{j}S(Y^{2}y^{3})), Y^{1}U^{Y^{3}})Y_{1}^{1}y^{1} \rangle \]
\[ \langle e^{i}, \hat{q}^{2}(Y_{2}^{2}y^{2}(S^{-1}(\alpha e^{i})) \beta \otimes e_{j}S(Y^{2}y^{3})), U^{2} \rangle \]
\[ = \sum_{i,j,k=1}^{n} \langle e^{i}, Y_{1}^{1}y^{1} \rightarrow e^{i}, S^{-1}(\hat{q}^{1}(e_{k}))U^{1}Y^{3} \rangle \langle e^{i}, \hat{q}^{2}(e_{k})U^{2} \rangle \]
\[
\sum_{i,k=1}^{n} \langle e^k, Y_1^i y^2 (S^{-1}(\alpha e_i Y_1^i y^1 \beta) \triangleright \tilde{q}^2(e_k) U^2) S(Y^2 y^3) \rangle \\
\langle e^i, S^{-1}(\tilde{q}^1(e_k) U^1 Y^3) \rangle \\
\sum_{i,k=1}^{n} \langle e^k, Y_1^i y^2 S^{-1}(f_2^1 Y_1^i (1,2) y_2^1 \delta_2^2) (S^{-1}(\alpha e_i) \triangleright \tilde{q}^2(e_k) U^2) \rangle \\
x f_1^1 Y_1^i (1,1) y_1^1 \delta_1^1 S(Y^2 y^3) \langle e^i, S^{-1}(\tilde{q}^1(e_k) U^1 Y^3) \rangle \\
\sum_{i,k=1}^{n} \langle e^k, Y_1^i S^{-1}(f_2^1 Y_1^i (1,2) p_2^1) (S^{-1}(\alpha e_i) \triangleright \tilde{q}^2(e_k) U^2) f_1^1 p_1^1 \rangle \\
\langle e^i, S^{-1}(\tilde{q}^1(e_k) U^1 p_2^1) \rangle \\
\sum_{i,k=1}^{n} \langle e^k, S^{-1}(f_2^1 p_2^1) (S^{-1}(\alpha S^{-1}(\tilde{q}^1(e_k) 1 P^1)) \triangleright \tilde{q}^2(e_k) U^2) f_1^1 p_1^1 \rangle \\
\langle e^i, S^{-1}(\tilde{q}^1(e_k) U^1 p_2^2) \rangle \\
\sum_{k=1}^{n} \langle e^k \triangleright x^3, S^{-1}(f_2^1 S^{-1}(F_1^1 q_1^1(e_k) (1,1) p_1^1 g_1^1)x^2 \beta) \rangle \\
x (S^{-1}(\alpha) \triangleright \tilde{q}^2(e_k) U^2) f_1^1 S^{-1}(F_2^1 q_2^1(e_k) (1,2) p_2^1 g_2^1)x^1 \rangle \\
\sum_{k=1}^{n} \langle e^k, S^{-1}(f_2^1 S^{-1}(F_1^1 q_1^1 x_1^3 (1,1) (e_k) (1,1) p_1^1 g_1^1)x^2 \beta) \rangle \\
x (S^{-1}(\alpha) \triangleright \tilde{q}^2(e_k) U^2) f_1^1 S^{-1}(F_2^1 q_2^1 x_1^3 (1,2) (e_k) (1,2) p_2^1 g_2^1)x^1 \rangle \\
\sum_{k=1}^{n} \langle e^k, S^{-1}(\gamma^2 S^{-1}(\tilde{Q}^1 X^1 (e_k) (1,1) p_1^1 g_1^1) \beta) \rangle \\
x \beta_1 \tilde{q}^2 \tilde{Q}^1 X^3 (e_k) U^2 S(\beta_2) \gamma^1 S^{-1}(\tilde{q}^1 \tilde{Q}^2 X^2 (e_k) (1) p_2^1 g_2^2) \rangle \\
\sum_{k=1}^{n} \langle e^k, S^{-1}(\gamma^2 S^{-1}(\tilde{Q}^1 (e_k) X^1 p_1^1 g_1^1) \beta) \rangle \\
x \tilde{q}^2 \tilde{Q}^2 (e_k) (2,2) X^3 g^2 S(\beta_2) \gamma^1 S^{-1}(\tilde{q}^1 \tilde{Q}^2 (e_k) (2,1) X^2 g^2) \rangle \\
\sum_{k=1}^{n} \langle e^k, S^{-1}(\gamma^2 S^{-1}(\tilde{Q}^1 (e_k) X^1 p_1^1 \delta^1) \beta) \rangle \\
x \tilde{q}^2 \tilde{Q}^2 (e_k) (2,2) X^3 g^2 S(\gamma^1 S^{-1}(\tilde{q}^1 \tilde{Q}^2 (e_k) (1) X^2 p_2^1 g_2^2) \rangle \\
\sum_{k=1}^{n} \langle e^k, S^{-1}(\beta) S^{-2}(\tilde{Q}^1 (e_k) X^1 p_1^1 \tilde{Y}_1^1 X^1 \beta S(S(Z^1) \alpha g^3 Z_2^3 Y^3) \rangle \\
x \tilde{q}^2 \tilde{Q}^2 (e_k) (2,2) X^3 g^2 S(\gamma^1 S^{-1}(\tilde{q}^1 \tilde{Q}^2 (e_k) (1) X^2 p_2^1 \delta^1) \beta) \rangle
so the proof is finished.

4.2. the trace formula for quasi-Hopf algebras.

In Section 5 we will see that the representation-theoretic rank of $H$ can be expressed in terms of integrals in $H$ and $H^*$. This result is strictly connected to the trace formula for quasi-Hopf algebras.

4.2. The representation-theoretic rank of $D(H)$. Let $H$ be a finite dimensional quasi-triangular quasi-Hopf algebra. Then $H$ is an object in its own category of finite dimensional representations via the left regular action, so it makes sense to consider $\text{dim}(H)$. The purpose of this subsection is to compute $\text{dim}(D(H))$. As we will see the computation is harder than the one for $\text{dim}(H)$ but the result will be the same. Again, we need some preliminary work.

Recall that $t \in H$ is called a left (respectively right) integral in $H$ if $ht = \varepsilon(h)t$ (respectively $th = \varepsilon(h)t$), for all $h \in H$. We denote by $\int_l$ and $\int_r$ the space of left and right integrals in $H$. When $H$ is finite dimensional we have that $\text{dim}(\int_l) = \text{dim}(\int_r) = 1$, $S(\int_l) = \int_r$ and $S(\int_r) = \int_l$ (see [21, 4]). In addition, if we define

$$\mathfrak{P}(h) = \sum_{i=1}^n \langle e^i, \beta S^2(q^2(e_i)h)q^1(e_i) \rangle, \quad \forall \ h \in H,$$

so the proof is finished. \hfill \Box
then by Proposition 4.5 we have that \( \tilde{\Psi}(h) \in \int_f \), for all \( h \in H \), and \( \tilde{\Psi}(t) = t \) for any \( t \in \int_f \).
Therefore, \( \tilde{\Psi} \) defines a projection from \( H \) to \( \int_f \). Replacing the quasi-Hopf algebra \( H \) by \( H^{\text{cop}} \) we obtain a second projection onto the space of left integrals, denoted in what follows by \( \tilde{\Psi} \). Since in \( H^{\text{cop}} \) we have \( (q_R)_{\text{cop}} = \tilde{q}^2 \otimes \tilde{q}^1 \) we obtain

\[
\tilde{\Psi}(h) = \sum_{i=1}^{n} \langle e^1, S^{-1}(\beta)S^{-2}(\tilde{q}^1(e_i))h \rangle \tilde{q}^2(e_i) \in \int_f, \forall h \in H.
\]

We finally need the following formulas.

**Lemma 4.4.** In a quasi-Hopf algebra \( H \) the following relations hold:

\[
\begin{align*}
\Omega^1_1 \delta^1 S^2(\Omega^1) \otimes \Omega^1_1 \delta^2 g^1 S(\Omega^3) \otimes \Omega^1_2 \delta^2 g^2 S(\Omega^2) \otimes \Omega^5 \\
= X^1 p_1^1 P^1 S(f^1 p^1) \otimes X^2 p_2^2 P^2 \otimes X^3 p^2 \otimes S^{-1}(f^2 p^2), \\
\gamma^1 X^1 \otimes f^1_1 \gamma^2_2 X^2 \otimes f^2_2 \gamma^2_1 X^3 = S(X^3) f^1_1 \gamma^1_1 \otimes S(X^2) f^2_2 \gamma^2_2 \otimes S(X^1) \gamma^2, \\
q_1 x_1^1 \otimes S^{-1}(x^3) q_2^2 x_2^1 \otimes x^3 = X^1 \otimes S^{-1}(\tilde{q}^1 X^1) X^2 \otimes \tilde{q}^2 X^2.
\end{align*}
\]

Here \( \Omega = \Omega^1_1 \otimes \cdots \otimes \Omega^5, \delta = \delta^1 \otimes \delta^2, \gamma = \gamma^1 \otimes \gamma^2, f = f^1 \otimes f^2, f^{-1} = g^1 \otimes g^2, q_R = q^1 \otimes q^2, p_R = p^1 \otimes p^2 = P^1 \otimes P^2 \) and \( q_L = \tilde{q}^1 \otimes \tilde{q}^2 \) are the elements defined in (2.31), (2.11), (2.17) and (2.16), respectively.

**Proof.** Observe that the element \( \delta \) in (2.31) can be rewritten as

\[
\delta = Y^1 p^1_1 \beta S(Y^3) \otimes Y^2 p^2 S(Y^2).
\]

Now, using the above description for \( \delta \) and the definition of \( \Omega \) we compute:

\[
\begin{align*}
\Omega^1_1 \delta^1 S^2(\Omega^1) \otimes \Omega^1_1 \delta^2 g^1 S(\Omega^3) \otimes \Omega^1_2 \delta^2 g^2 S(\Omega^2) \otimes \Omega^5 \\
= X^1_1 \gamma_1^1 Y^1_1 p_1^1 \beta S(f^1_1 X^3) \otimes X^1_1 \gamma_1^1 Y^1_1 p_1^1 \beta S(X^1_1 y^3) \\
\otimes X^1_1 \gamma_1^1 \gamma_2^1 Y^1_1 p_1^1 \beta S(X^1_1 y^3) \otimes X^1_1 \gamma_1^1 \gamma_2^1 S^{-1}(f^2_1 X^3) \\
= Y^1 ( (X^1_1)_1 p^1_1 ) P^1_1 \beta S(f^1_1 X^3) \otimes Y^2 ( (X^1_1)_1 p^1_1 ) P^2_1 \beta S(X^1_2) \\
\otimes Y^2 ( (X^1_1)_2 p^2_2 ) \beta S((X^1_1)_2) \otimes S^{-1}(f^2_1 X^3) \\
= Y^1 p^1_1 P^1_1 S(f^1_1 p^1_1) \otimes Y^2 p^2_1 P^2_1 \otimes Y^3 p^3_1 \beta S^{-1}(f^2_1 p^2_1),
\end{align*}
\]

so the equality in (4.15) is proved. The relation in (4.17) follows more easily since

\[
\begin{align*}
\gamma^1_1 X^1 \otimes f^1_1 \gamma^2_2 X^2 & \otimes f^2_2 \gamma^2_1 X^3 \\
= F^1 \alpha_1 X^1 \otimes f^1_1 \gamma^2_2 X^2 \otimes f^2_2 X^3 \\
= S(X^3) f^1_1 \alpha_1 \otimes S(X^2) f^2_2 \alpha_2 \\
= S(X^3) f^1_1 \gamma^1_1 \otimes S(X^2) f^2_2 \gamma^2_2 \otimes S(X^1) \gamma^2,
\end{align*}
\]

where we denoted by \( F^1 \otimes F^2 \) another copy of \( f \). Finally, (4.18) is an immediate consequence of (2.31) and (2.32).

\[ \square \]

We can now compute the representation-theoretic rank of \( D(H) \). The next result generalizes Proposition 2.1.

**Proposition 4.5.** Let \( H \) be a finite dimensional quasi-Hopf algebra and \( D(H) \) its quantum double. Then

\[
\dim(D(H)) = \dim(H) = \text{Tr} \left( h \mapsto S^{-2}(S(\beta) \alpha h \beta S(\alpha)) \right).
\]
Proof. We set \( p_R = p^1 \otimes p^2 = P^1 \otimes P^2 \), \( q_R = q^1 \otimes q^2 = Q^1 \otimes Q^2 \) and \( f = f^1 \otimes f^2 = F^1 \otimes F^2 = F^1 \otimes F^2 \). In what follows, we shall not perform all the computations but we shall point out the relations which are used in every step.

The expression of \( \eta_D \) in Proposition 4.1 allows us to compute:

\[
\dim(D(H)) = \sum_{i,j=1}^{n} \langle e_i \otimes e^j, \eta_D(e^i \otimes e^j) \rangle
\]

\[
= \sum_{i,j,k=1}^{n} \langle e_i \otimes e^j, (\beta \rightarrow \Theta_i^{-1}(e^k) \leftarrow S(\beta) \otimes e_k)(e^i \otimes e^j) \rangle
\]

\[
= \sum_{i,j,k=1}^{n} \langle (\Theta_i^{-1}(e^k), S(\beta) \otimes (e_i) \otimes (e_j)^r) \rangle
\]

\[
\langle e^i, S^{-1}(e(k)) \Omega^4(e_i) \Omega^2(e(k))_{(1,1)} \rangle
\]

\[
= \sum_{i,j=1}^{n} \langle e^i, \Omega^3 S^{-1}(S(\beta_2) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(1,1)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\times S^{-1}(F^2 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(2,2)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\langle e^i, S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(1,1)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\times S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(2,2)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\langle e^i, S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(1,1)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\times S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(2,2)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\langle e^i, S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(1,1)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\times S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(2,2)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\langle e^i, S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(1,1)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\times S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(2,2)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\langle e^i, S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(1,1)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\times S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(2,2)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\langle e^i, S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(1,1)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\times S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(2,2)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\langle e^i, S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(1,1)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\times S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(2,2)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\langle e^i, S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(1,1)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]

\[
\times S^{-1}(f^1 S(\beta_1) \gamma^2 S^{-1}(F^2 \bar{p}^2)_{(2,2)}(e_i) \otimes (e_j)^{r^2}) \rangle
\]
\( (e^i, (\beta_1)_{2S^{-1}}(S(Y^2)f^2\gamma_2^1(e_1)_2(\beta_1)_{1,2}P^2)e_j) \)

2.17.1.2
\[ \sum_{i,j=1}^n \langle e^i, \gamma^1 S^{-2} \left( S(Y\gamma^2)j^1(e_1)_1 P^1 \delta^1 S(S(Y^1)\gamma^2) \right) \rangle \]
\[ \langle e^i, S^{-1}(S(Y^2)f^2(e_1)_2P^2)e_j \rangle \]

2.9.1.12
\[ \sum_{i,j=1}^n \langle S^{-2}(e^i), S(Y^1)X^1 \alpha x^3 y_2^2 Z^3 S^{-1}(f^1 P^1 y^1 \beta)(e_1)_2 Y^3 y^2 Z^1 \beta \]
\[ \times S(S(Y^3)\alpha X^3 x^3 y_2^1 Z^3 Y^2(x_2^1)) \langle e^i, S^{-1}(f^2 P^2)(e_1)_1 Y^2 e_j \rangle \]

4.7.2.16
\[ \sum_{i,j=1}^n \langle S^{-2}(e^i), S(q^1 Y^1)\alpha x^3 y_2^3 Z^3 S^{-1}(f^1 P^1 y^1 \beta)(e_1)_2 x^1(1) Y^3 \]
\[ \times y^2 Z^1 \beta S(S(q^2 x^2 y_2^1 Z^2)) \langle e^i, S^{-1}(f^2 P^2)(e_1)_1 x^1(2,1) Y^2 e_j \rangle \]

2.17.1.2.8
\[ \sum_{i,j=1}^n \langle S^{-2}(e^i), S(q^1 Y^1)\alpha x^3 y_2^3 Z^3 S^{-1}(f^1 x^1(1) P^1 y^1 \beta)(e_1)_2 Y^3 x^1 \]
\[ \times y^2 Z^1 \beta S(S(q^2 x^2 y_2^1 Z^2)) \langle e^i, (x_1^1)_2 S^{-1}(f^2 x^1(1) P^2)(e_1)_1 Y^2 e_j \rangle \]

4.7.1.2.8
\[ \sum_{i,j=1}^n \langle S^{-2}(e^i), S(q^1 Y^1)\alpha S^{-1}(f^1 P^1 p^1 \beta)(e_1)_2 Y^3 p^2 S^2(q^2) \]
\[ \langle e^i, S^{-1}(f^2 P^2)(e_1)_1 Y^2 e_j \rangle \]

4.7.1.2.8
\[ \sum_{i,j=1}^n \langle S^{-2}(e^i), S(q^1 Q^1 x^1(1,1)) S^{-1}(f^1 P^1 y^1 \beta)(e_1)_2 S^{-1}(x^2 y^1) \]
\[ \times x^2(x_2^1 P^2) \langle e^i, S^{-1}(f^2 P^2)(e_1)_1 S^{-1}(x^3 y^2) \rangle q^2 Q^1 x^1(1,2) e_j \]

2.8
\[ \sum_{i,j=1}^n \langle S\gamma^1(e^i), S(q^1 Q^1 x^1(1,1)) S^{-1}(x^2(e_1)_1 P^1 p^1 \beta)Q^2 x^2 p^2 \]
\[ \langle e^i, S^{-1}(x^3(e_1)_2 P^2)q^2 Q^1 x^1(1,2) e_j \rangle \]

4.7
\[ \sum_{i,j=1}^n \langle S\gamma^1(e^i), S(q^1 Q^1 x^1(1,1)) S^{-1}(x^2(e_1)_1 q^1(1) Q^1 x^1(1,1) P^1 p^1 \beta)Q^2 x^2 p^2 \]
\[ \langle e^i, q^2(Q^1 x^1(1,2) S^{-1}(x^3(e_1)_2) q^2(Q^1 x^1(1,2) P^2) e_j \rangle \]

2.17.1.2.20
\[ \sum_{i,j=1}^n \langle S\gamma^1(e^i), S^{-1}(x^2(e_1)_1 Q^1 x^1(1,y^1 \beta)Q^2 x^1 P^2) \rangle \langle e^i, S^{-1}(x^3(e_1)_2) e_j \rangle \]

4.13.1.14
\[ \sum_{i,j=1}^n \langle S\gamma^1(e^i), S^{-1}(q^1(e_1)_1 X^1 p^1 \beta)X^2 p^2 S(X^2) \rangle \langle e^i, S^{-1}(q^2 e_1)_2 e_j \rangle \]

2.16.1.15
\[ \sum_{j=1}^n \langle e^i, S^{-1} \left( \Psi(S^{-2}(\beta S(\alpha))) \right) e_j \rangle \]
\[ = \varepsilon \left( \Psi(S^{-2}(\beta S(\alpha))) \right) \]

4.15
\[ \sum_{i=1}^n \langle e^i, S^{-2}(S(\beta)\alpha e_i \beta S(\alpha)) \rangle = \text{Tr} \left( h \mapsto S^{-2}(S(\beta)\alpha h \beta S(\alpha)) \right) , \]
where in the last but one equality we used the fact that $S^{-1}\left(\tilde{\Omega}(S^{-2}(\beta S(\alpha)))\right)$ is a right integral in $H$. So the proof is complete. \hfill $\Box$

We will end this Section by computing the representation-theoretic rank of $D^\omega(H)$, the quasi-triangular quasi-Hopf algebra constructed in [11].

Let $H$ be a finite dimensional cocommutative Hopf algebra and $\omega : H \otimes H \otimes H \to k$ a normalized 3-cocycle on $H$, this means a convolution invertible map satisfying the conditions:

\[
\omega(a, b, c, d) = \omega(a, b) = \omega(a, 1, b) = \omega(a, b, 1) = \varepsilon(a)\varepsilon(b),
\]

for all $a, b, c, d \in H$. Identifying $(H \otimes H \otimes H)^*$ with $H^* \otimes H^* \otimes H^*$ we can regard $\omega$ and its convolution inverse $\omega^{-1}$ as elements of $H^* \otimes H^* \otimes H^*$. Then the commutative Hopf algebra $H^*$ has a non-trivial quasi-Hopf algebra structure by keeping the usual multiplication, unit, comultiplication, counit and antipode of $H^*$, and defining the reassociator $\Phi = \omega^{-1}$ and the elements $\alpha = \varepsilon, \beta(h) = \omega(h_1, S(h_2), h_3), h \in H$. We shall denote by $H^*_\omega$ the quasi-Hopf algebra structure on $H^*$ defined above.

Now, roughly speaking, the quasi-Hopf algebra $D^\omega(H)$ can be identified as a quasi-triangular quasi-Hopf algebra with $D(H^*_\omega)$, the quantum double associated to the finite dimensional quasi-Hopf algebra $H^*_\omega$. Note that this point of view was given in [29], the initial construction of $D^\omega(H)$ being presented earlier in [11] as a generalization of the Dijkgraaf-Pasquier-Roche quasi-Hopf algebra $D^\omega(G)$ constructed in [15] (here $G$ is a finite group and $\omega$ is a normalized 3-cocycle on $G$).

Having this description for $D^\omega(H)$ and the result in Proposition 4.5 we can easily compute its representation-theoretic rank. Note that, one of the goals in [10] was to compute this rank but at that moment only a partial answer was given.

**Proposition 4.6.** Let $H$ be a finite dimensional cocommutative Hopf algebra and $\omega$ a normalized 3-cocycle on $H$. Then $\dim(D^\omega(H)) = \dim(H)$.

**Proof.** By Proposition 4.5 we have that

\[
\dim(D^\omega(H)) = \text{Tr}\left(\varphi \mapsto \tilde{\Omega}(S^{-2}(\beta S(\alpha)))\right).
\]

Since $H^*$ is commutative we have that $\tilde{\Omega}^2 = id_{H^*}$. Moreover, $\alpha = \varepsilon$ and from [11] we know that $\beta$ is convolution invertible with $\beta^{-1} = \tilde{\Omega}(\beta)$. Therefore, the above formula comes out explicitly as

\[
\dim(D^\omega(H)) = \text{Tr}(\varphi \mapsto \varphi) = \text{Tr}(id_{H^*}) = \dim(H^*) = \dim(H),
\]

and this ends the proof. \hfill $\Box$

5. The Trace Formula for Quasi-Hopf Algebras

When $H$ is an ordinary Hopf algebra the formula in Proposition 4.5 reduces to $\dim(H) = \dim(D(H)) = \text{Tr}(S^{-2}) = \text{Tr}(S^2)$. As we has already explained in Introduction, using the Radford and Larson results [23, 24] on one hand, and the Etingof and Gelaki result [18] on the other hand, we obtain that

\[
\dim(H) = \dim(D(H)) = \begin{cases}
0 & \text{if } H \text{ is neither semisimple or cosemisimple} \\
\dim(H) & \text{if } H \text{ is both semisimple and cosemisimple}.
\end{cases}
\]
In this Section we will generalize to the quasi-Hopf algebra setting the first result. Even if a quasi-Hopf algebra is not a coassociative coalgebra, as we have seen in Introduction we can define the cosemisimple notion. Let us explain this more precisely.

Let $H$ be a finite dimensional quasi-Hopf algebra and $t$ a non-zero right integral in $H$. Since $\int_l$ is a two-sided ideal of $H$, it follows from the uniqueness of the integrals in $H$ that there exists $\mu \in H^*$ such that

$$th = \mu(h)t, \; \forall \; t \in \int_l \text{ and } h \in H.$$ 

Note that $\mu$ is an algebra map; as in the Hopf case we will call $\mu$ the distinguished group-like element of $H^*$. We notice that $\mu = \epsilon$ if and only if $H$ is unimodular, this means if and only if $\int_l = \int_r$.

Now, following [21], a left cointegral in $H$ is an element $\lambda \in H^*$ such that

$$\lambda(V^2h_2U^2)V^1h_1U^1 = \mu(x^1)\lambda(hS(x^2))x^3, \; \forall \; h \in H,$$

where $U = U^1 \otimes U^2$ is the element defined in (2.33) and

$$V = V^1 \otimes V^2 := S^{-1}(f^2p^2) \otimes S^{-1}(f^1p^1).$$

We will say that a left cointegral $\lambda$ is normalized if $\lambda(S^{-1}(\alpha)\beta) = 1$ and we will call a finite dimensional quasi-Hopf algebra $H$ cosemisimple if $H$ has a normalized left cointegral.

By $L$ we denote the space of left cointegral in $H$. Then the map

$$\nu : L \otimes H \rightarrow H^*, \; \nu(\lambda \otimes h)(h') = \lambda(h'S(h)) \; \forall \; \lambda \in L \text{ and } h,h' \in H,$$

is an isomorphism of right quasi-Hopf bimodules (the definition of a right quasi-Hopf $H$-bimodule can be found in [21]; roughly speaking it is a right $H$-comodule within the monoidal category of $H$-bimodules). Here $L \otimes H$ and $H^*$ are right quasi-Hopf $H$-bimodules via the structures

$$L \otimes H : \left\{ \begin{array}{l}
h' \cdot (\lambda \otimes h) \cdot h'' = \mu(h'_1)\lambda \otimes h'_2hh'' \\
\lambda \otimes h \mapsto \mu(x^1)\lambda \otimes x^2h_1 \otimes x^3h_2,
\end{array} \right.$$ 

and

$$H^* : \left\{ \begin{array}{l}
\langle h' \rightarrow \varphi \leftarrow h'' \rangle, \\
\varphi \mapsto \sum_{i=1}^{n} e^i \ast \varphi \otimes e_i,
\end{array} \right.$$ 

for all $\lambda \in L$, $h,h',h'' \in H$ and $\varphi \in H^*$, where we denoted by $\ast$ the non-associative multiplication on $H^*$ defined for all $\varphi, \psi \in H^*$ and $h \in H$ by

$$\langle \varphi \ast \psi, h \rangle := \langle \varphi, V^1h_1U^1 \rangle \langle \psi, V^2h_2U^2 \rangle.$$

It follows from the above that $\dim(L) = 1$, and that for a fixed non-zero left cointegral $\lambda$ in $H$ the isomorphism $\nu$ defined in (5.1) induces a right $H$-linear isomorphism

$$\tilde{\nu} : H \rightarrow H^*, \; \tilde{\nu}(h)(h') = \lambda(h'S(h)) \; \forall \; h,h' \in H.$$ 

(Here $H$ and $H^*$ are right $H$-modules via the right regular representation and $(\varphi \leftarrow h)(h') = \varphi(h'S(h))$, respectively.) In particular, there is an unique $r \in H$ such that $\tilde{\nu}(r) = \epsilon$, this means $\lambda(hS(r)) = \epsilon(h)$, for all $h \in H$. As in the Hopf case we can show that $r$ is a non-zero integral with the property that $\lambda(S(r)) = 1$. Indeed, the fact that $\tilde{\nu}$ is right $H$-linear implies:

$$\tilde{\nu}(rh) = \tilde{\nu}(r) \leftarrow h = \epsilon \leftarrow h = \epsilon(h) \epsilon = \tilde{\nu}(\epsilon(h)r),$$

and

$$\tilde{\nu}(r) = \epsilon \leftarrow h = \epsilon \leftarrow h = \epsilon(h) \epsilon = \tilde{\nu}(\epsilon(h)r).$$
for all \( h \in H \). Since \( \tilde{\nu} \) is bijective we conclude that \( rh = \varepsilon(h)r \), for all \( h \in H \), i.e. \( r \in \int r \). Now, \( \tilde{\nu}(r) = \varepsilon \) implies \( \lambda(hS(r)) = \varepsilon(h) \) for all \( h \in H \), and this is equivalent to \( \lambda(S(r)) = 1 \).

As we will see the pair \((\lambda, r)\) described above plays an important role in the trace formula for quasi-Hopf algebras. In particular, we will obtain an important result characterizing semisimple cosemisimple quasi-Hopf algebras in terms of the trace of the “square” of the antipode. Recall that a semisimple quasi-Hopf algebra is a quasi-Hopf algebra which is semisimple as an algebra.

**Theorem 5.1.** Let \( H \) be a finite dimensional quasi-Hopf algebra, \( \mu \) the distinguished group-like element of \( H^* \), \( \lambda \) a non-zero left cointegral in \( H \) and \( r \) a right integral in \( H \) such that \( \lambda(S(r)) = 1 \). Then:

i) For any endomorphism \( \chi \) of \( H \) we have that
\[
\text{Tr}(\chi) = \mu(q_1^1x^1)\lambda(\chi(q_1^2x^2r_2p_2^2)S(q_2^1x^2r_1p_1^1)).
\]

ii) \( \text{Tr} (h \mapsto \beta S(\alpha)S^2(h)S(\beta)\alpha) = \varepsilon(r)\lambda(S^{-1}(\alpha)\beta). \) In particular, \( H \) is semisimple and cosemisimple if and only if \( \text{Tr} (h \mapsto \beta S(\alpha)S^2(h)S(\beta)\alpha) \neq 0 \).

**Proof.** For any linear morphism \( \chi : H \to H \) we denote by \( \chi^* : H^* \to H^* \) the dual morphism of \( \chi \). We also denote by \( \eta : H^* \otimes H \to \text{End}(H^*) \) the linear map defined for all \( \varphi, \psi \in H^* \) and \( h \in H \) by
\[
\eta(\varphi \otimes h)(\psi) = \psi(h)\varphi.
\]
Then, exactly as in [14, Section 7.4], one can easily see that
\[
\eta(\varphi \otimes h) \circ \chi^* = \eta(\varphi \otimes \chi(h)),
\]
\[
\text{Tr}(\eta(\varphi \otimes h)) = \varphi(h),
\]
for all \( \varphi \in H^* \), \( h \in H \) and \( \chi \in \text{End}(H) \).

i) The fact that \( \nu \) is right \( H \)-colinear shows by using of \([5.2, 5.3]\) that
\[
\varphi(V^1h_1U^1)\lambda(V^2h_2U^2S(h')) = \mu(x^1)\varphi(x^2h'_2)\lambda(hS(x^2h'_1)),
\]
for all \( \varphi \in H^* \) and \( h, h' \in H \). If we write the above equation for \( h' = r \) and use the fact that \( S(r) \in \int r \) such that \( \lambda(S(r)) = 1 \), we obtain
\[
\varphi(S^{-1}(\beta)h) = \mu(x^1)\varphi(x^2r_2)\lambda(hS(x^2r_1)),
\]
for all \( \varphi \in H^* \) and \( h \in H \). In particular, we have that
\[
\langle q^2 \mapsto \varphi \leftarrow q^2, S^{-1}(\beta)S^{-1}(q^1)hS(p^1)\alpha \rangle = \mu(x^1)\langle q^2 \mapsto \varphi \leftarrow q^2, x^3r_2 \rangle\lambda(S^{-1}(q^1)hS(p^1)S(x^2r_1)),
\]
and this comes out explicitly as
\[
\varphi(h) = \mu(q_1^1x^1)\varphi(q_2^1x^3r_2p_2^2)\lambda(hS(q_2^1x^2r_1p_1^1)),
\]
for all \( \varphi \in H^* \) and \( h \in H \), where we used the formula
\[
\lambda(S^{-1}(h)h') = \mu(h_1)\lambda(h'S(h_2)), \ \forall \ h, h' \in H,
\]
which can be found in [6, Lemma 3.3]. In other words we have obtained
\[
\eta(\lambda \leftarrow q_2^1x^2r_1p_1^1 \otimes \mu(q_1^1x^1)q_2^1x^3r_2p_2^2) = \text{id}_{H^*}.
\]
Now, using (5.4), (5.5) and the fact that \( \text{Tr}(\chi) = \text{Tr}(\chi^*) \) we conclude that
\[
\text{Tr}(\chi) = \text{Tr}(\chi^*) = \text{Tr}(\text{id}_H \circ \chi^*) = \text{Tr}(\eta(\lambda \rightarrow q_1^2 x^2 r_1 p_1 \otimes \mu(q_1^2 x^2) q^2 x^3 r_2 p_2^2) \circ \chi^*)
\]
\[
= \text{Tr}(\eta(\lambda \leftarrow q_1^2 x^2 r_1 p_1 \otimes \mu(q_1^2 x^2) \lambda(q^2 x^3 r_2 p_2^2)) = \mu(q_1^2 x^2) \lambda(\chi(q^2 x^3 r_2 p_2^2) S(q_1^2 x^2 r_1 p_1)).
\]

ii) One can easily see that (2.20) and \( r \in \mathcal{F}_r \) imply:
\[
r_1 \otimes r_2 = r_1 q_1^2 p_1 \otimes r_2 q_2^2 p_2 S(q^2) = r_1 p_1 \otimes r_2 p_2^2 \alpha.
\]
Also, by (2.17) we have
\[
r_1 p_1 h \otimes r_2 p_2 = (r h_1)_1 p_1 \otimes (r h_1)_2 p_2 S(h_2) = r_1 p_1 \otimes r_2 p_2^2 S(h),
\]
for any \( h \in H \). Combining the two relations above we obtain
\[
(5.8) \quad r_1 \otimes r_2 = r_1 p_1 \otimes r_2 p_2^2 \alpha = r_1 p_1 S^{-1}(\alpha) \otimes r_2 p_2^2.
\]

Now, by part i) we have
\[
\text{Tr}(h \mapsto \beta S(\alpha) S^2(h) S(\beta) \alpha)
\]
\[
= \mu(q_1^2 x^2) \lambda(\beta S(\alpha) S^2(q^2 x^3 r_2 p_2^2) S(\beta) \alpha S(q_1^2 x^2 r_1 p_1))
\]
\[
= \mu(q_1^2 x^2) \lambda(\beta S(\alpha) S(q_1^2 x^2 r_1 \beta S(q^2 x^3 r_2)))
\]
\[
= \varepsilon(r) \mu(q_1^2 p_1) \lambda(\beta S(\alpha) S(q_2^2 p_2^2 S(q^2)))
\]
\[
= \varepsilon(r) \lambda(\beta S(\alpha)).
\]

Next, we claim that \( \varepsilon(r) \lambda(\beta S(\alpha)) = \varepsilon(r) \lambda(S^{-1}(\alpha) \beta) \). Indeed, if \( H \) is not semisimple then by (30) we have that \( \varepsilon(r) = 0 \) and therefore \( \varepsilon(r) \lambda(\beta S(\alpha)) = \varepsilon(r) \lambda(S^{-1}(\alpha) \beta) = 0 \). On the other hand, if \( H \) is semisimple then by the same result in (30) we have that \( \varepsilon(\mathcal{F}_r) = \varepsilon(\mathcal{F}_r) \neq 0 \). In this situation, applying similar arguments as in the Hopf algebra case we can prove that \( H \) is unimodular, so \( \mu = \varepsilon \). Finally, by (5.6) we get
\[
\lambda(S^{-1}(\alpha) \beta) = \mu(\alpha_1) \lambda(\beta S(\alpha_2)) = \varepsilon(\alpha_1) \lambda(\beta S(\alpha_2)) = \lambda(\beta S(\alpha)),
\]
as claimed. Thus the proof is finished. \( \square \)

As a consequence of Proposition 4.5 and Theorem 5.1 we obtain the following formula for the representation-theoretic ranks of \( H \) and \( D(H) \).

**Theorem 5.2.** Let \( H \) be a finite dimensional quasi-Hopf algebra, \( \lambda \) a left cointegral in \( H \) and \( r \) a right integral in \( H \) such that \( \lambda(r) = 1 \). Then
\[
\dim(H) = \dim(D(H)) = \varepsilon(r) \lambda(S^{-1}(\alpha) \beta) = \varepsilon_D(\beta \rightarrow \lambda \otimes r).
\]

In particular, if \( H \) is not semisimple or cosemisimple then
\[
\dim(H) = \dim(D(H)) = 0.
\]

**Proof.** By \( \lambda_{op} \) we denote a left cointegral in \( H_{op} \). It is straightforward to check that in \( H_{op} \) we have \( \mu_{op} = \mu^{-1} := \mu \circ S \), and that the roles of \( U \) and \( V \) interchange. So \( \lambda_{op} \) is an element of \( H^* \) satisfying
\[
\lambda_{op}(V^2 h_2 U^2) V^1 h_1 U^1 = \mu^{-1}(X^1) \lambda_{op}(S^{-1}(X^2) h) X^3, \quad \forall \, h \in H.
\]
Note that, if $H$ is unimodular then $\mu = \varepsilon$ and therefore a left cointegral in $H^{\text{op}}$ is nothing else than a left cointegral in $H$.

Applying now Theorem 5.1 to the quasi-Hopf algebra $H^{\text{op}}$ we obtain

$$\text{Tr} \left( h \mapsto S^{-2}(S(\beta)\alpha h S(\alpha)) = \varepsilon(t)\lambda_{\text{op}}(S^{-1}(\alpha)\beta), \right)$$

where $t$ is a left integral in $H$ such that $\lambda_{\text{op}}(S^{-1}(t)) = 1$. If we denote $r = S^{-1}(t)$ we get that $r$ is a right integral in $H$ such that $\lambda_{\text{op}}(r) = 1$. It follows that $\varepsilon(t) = \varepsilon(r)$, and that

$$\dim(H) = \dim(D(H)) = \text{Tr} \left( h \mapsto S^{-2}(S(\beta)\alpha h S(\alpha)) = \varepsilon(r)\lambda_{\text{op}}(S^{-1}(\alpha)\beta). \right)$$

Finally, we apply the same trick as in the proof of the above Theorem. Namely, if $H$ is not semisimple then $\varepsilon(r) = 0$ and we are done. If $H$ is semisimple then it is unimodular. In this case we have seen that $\lambda_{\text{op}}$ is a cointegral in $H$ and since $\lambda_{\text{op}}(r) = 1$ the above equality finishes the proof.

\begin{proof}

Remark 5.3. It is conjectured in [21] that $\beta \rightarrow \lambda \bowtie r$ is a left integral in $D(H)$. If it is the case then by the Maschke’s theorem proved in [30] we obtain that $\dim(H) = \dim(D(H)) \neq 0$ if and only if $D(H)$ is a semisimple quasi-Hopf algebra.

Now, we conjecture that $D(H)$ is semisimple if and only if $H$ is both semisimple and cosemisimple, if and only if $h \mapsto S^{-2}(S(\beta)\alpha h S(\alpha)) = \text{id}_H$. If it is true then the scalar $\dim(H) = \dim(D(H))$ has the same value as in the Hopf algebra case.

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