Exact non-equilibrium current from the partition function
for impurity transport problems

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We study the partition functions of quantum impurity problems in the domain of complex applied bias for its relation to the non-equilibrium current suggested by Fendley, Lesage and Saleur (cond-mat/9510055). The problem is reformulated as a certain generalization of the linear response theory that accommodates an additional complex variable. It is shown that the mentioned relation holds in a rather generic case in the linear response limit, or under certain condition out of equilibrium. This condition is trivially satisfied by the quadratic Hamiltonians and is rather restrictive for the interacting models. An example is given when the condition is violated.

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1. INTRODUCTION

Strongly correlated quantum systems out of equilibrium is one of the lesser understood domains in theoretical physics, mainly because such systems reveal much more complex behavior compared to the case of equilibrium. Correspondingly, there are not that many non-trivial examples known today that are solved exactly out of equilibrium, and extension of powerful field-theoretical methods to non-equilibrium situations is desirable.

Situation is somewhat simpler in the stationary non-equilibrium regime. In a remarkable work, Ref. [1], the authors put forward a novel idea of how to calculate the current for integrable systems out of equilibrium. In particular, they obtained, based on the standard Bethe ansatz technique, a non-perturbative expression for the backscattering current through a single impurity in the quantum Hall bar with \( \nu = \frac{1}{3} \). Their result, cast into appropriate form [2], reads:

\[
I_B = \frac{e}{h} i \pi T \nu \lambda_{bs} \frac{d}{d \lambda_{bs}} \log \left[ \frac{Z_{imp}(i \mu_N \rightarrow \mu)}{Z_{imp}(-i \mu_N \rightarrow \mu)} \right],
\]

where \( T \) stands for temperature, \( \nu = \frac{1}{3} \) is a filling fraction, \( \lambda_{bs} \) is the effective tunneling strength of the impurity, \( Z_{imp} \) is impurity’s partition function defined as the ratio of the equilibrium partition function of the full system to that of the “pure” system without impurity (the leads), and analytic continuation from the discrete imaginary to real chemical potential of the edges \( \mu \) is assumed. It is worth emphasizing that Eq. (1) was meant to have a truly non-equilibrium nature, unlike similar expressions for persistent currents [3], where the current \( I = \partial F / \partial \Phi \) is a purely equilibrium phenomenon, a consequence of the incident flux \( \Phi \) on the structure of energy levels.

Relation (1) was obtained through several intermediary steps, rather than derived directly. (a) First, the “fusion” of Bethe ansatz with the Boltzmann rate equation was implemented as follows [1]. The quantum Hall bar with impurity, described by the Luttinger theory with point-like backscattering term \( \lambda_{bs} \delta(x)(\Psi^+_L \Psi_R + \Psi^+_R \Psi_L) \) was reformulated in terms of the boundary sine-Gordon model

\[
H_{BSG} = \frac{1}{2} \int_0^\infty dx \left[ \Pi^2(x) + (\partial_x \Phi)^2 \right] + \lambda_{bs} \cos \frac{\sqrt{8 \pi} \nu}{2} \Phi(0),
\]

which is exactly solvable in equilibrium and can be diagonalized by the Bethe ansatz. Transport of Laughlin \((e/3)\) quasiparticles through impurity [3] maps onto a scattering of sine-Gordon quasiparticles (kinks, antikinks and breathers) off the boundary. All the interaction in (2) is now at the boundary, which behaves like a non-elastic scatterer: kink can be bounced as an antikink and vice versa, which changes the charge of the system. However, it is a bare Hamiltonian where the bulk interactions are absent. As a result of diagonalization, sine-Gordon quasiparticles interact also in the bulk by a pairwise point-like interaction that adds merely a phase-shift with momentum of each quasiparticle preserved – a consequence of the peculiar conservation laws of [3], and the distribution function of the gas of quasiparticles differs from the usual Fermi one. Under these circumstances, the current was calculated on the basis of the standard probability arguments which are known to lead to the (classical) Boltzmann rate equation (1):

\[
I_B = \frac{e}{h} \int_0^\infty dp \left[ n_+ (p, \mu, T) - n_- (p, \mu, T) \right] |S^\dagger_T (p, \lambda_{bs})|^2.
\]

This expression was referred in literature to as an exact result because of the exact quantum nature of all the entries of expression (3) used in (1). In particular, the densities of states \( n_{\pm} \) of quasiparticles in the presence of the external bias \( \mu \int_0^\infty \partial_x \Phi \) were found from the thermodynamic Bethe ansatz (TBA) [3,4], whereas for the
transition probability $W = |(\pm \pm \pm)^2$ out of equilibrium it was taken the kink-antikink equilibrium scattering matrix element $S_{\pi}$, obtained in [3].

(b) Second, the exact impurity partition function of [3] was obtained either as a Coulomb gas expansion, or based on the relation to the Kondo model in a magnetic field and using TBA for the latter [3]. Eventually, formula [3] was checked by comparing series expansions of both sides to a high order [3].

Existence of valid generic relations like Eq. (1) would be rather helpful. Indeed, a partition function is much easier to find and can be tackled by a variety of techniques: Feynmann integration in imaginary time, Monte Carlo simulations, Bethe ansatz etc. Then the proper an-
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We study in detail only the latter case here. We argue that: a) equation (2) holds for free systems (quadratic couplings), and b) Eq. (2) holds for the interacting sys-
systems under certain condition for the self-energy, or as a
generic formula in the linear response limit. The results

can be easily generalinized to the case of transport through

II. PARTITION FUNCTION vs. NON-EQUILIBRIUM CURRENT

Consider the following Hamiltonian, which describes the leads $(\psi)$, the impurity $(d)$ located at $x = 0$, and possible interactions $(H_{\text{int}})$:

$$H = H_0 + \lambda \sum_{m=L,R} (\psi_m^\dagger(0)d + d^\dagger \psi_m(0)) + \epsilon_0 d^\dagger d + H_{\text{int}}(\{d\}), \quad (5)$$

Operator $d^\dagger$ creates an electron on the impurity site at the energy $\epsilon_0$. For this model, an analogue of relation (1) to be shown reads:

$$J = \frac{e}{\hbar} i \pi T \frac{\partial}{\partial \lambda} \log Z_{\text{imp}}^+, \quad (6)$$

where $\Gamma \equiv \lambda^2 [12]$. Define the impurity partition function as

$$Z_{\text{imp}}(i\mu_L, i\mu_R) = \frac{\text{Tr} e^{-(H+i\mu_L N_L+i\mu_R N_R)/T}}{\text{Tr} e^{-(H_0+i\mu_L N_L+i\mu_R N_R)/T}}, \quad (7)$$

where $i\mu_{L(R)}$ take the values on the bosonic Matsubara frequencies. The key point in Eq. (7) is analytic continuation in the chemical potential. To perform it carefully, we proceed as follows. From the definition of $Z_{\text{imp}}$ and Eq. (7) one has:

$$\frac{\partial}{\partial \lambda} \log Z_{\text{imp}} = - T^{-1} \sum_{m=L,R} \langle \psi_m^\dagger d + d^\dagger \psi_m \rangle_{x=0} \quad (8)$$

$$= -2\lambda \sum_{\omega_n} (G_{0L} + G_{0R}) G_{dd}^L(\omega_n), \quad (9)$$

quantum current can be derived rigorously in the spirit of [1].

To be specific, we consider three types of coupling terms: point-contact, resonant-level and Kondo. Inter-
actions can be introduced in the free (quadratic) mod-
els (a) by moving away from the Toulouse point in the
Kondo model; (b) by taking leads to be the Luttinger li-
quid instead of free electrons in the point-contact model
and (c) by considering a larger space of interacting states
$\{d_n\}$ instead of one state in the resonant-level model.
We study in detail only the latter case here. We argue
that:

a) equation (2) holds for free systems (quadratic couplings), and

b) Eq. (2) holds for the interacting sys-

ths under certain condition for the self-energy, or as a
generic formula in the linear response limit. The results

can be easily general

In this paper we undertake further studies of the re-
lations of the form of Eq. (1) without appealing to the
knowledge of Bethe ansatz, but rather on the basis of standard Green functions technique. We focus on
the transport properties of one-dimensional electrons pass-
ing through an impurity or an interacting region, the
quantity of interest being an I-V characteristic. Ex-
perimentally, such systems are realized, e. g., as quantum
dots, quenched nanoconstrictions, quantum wires with
an impurity or the edges of quantum Hall liquid. On
the theoretical level, we start with a hypothetical model
of two reservoirs, $L$ and $R$, of free 1D electrons with a
linearized spectrum, kept at different constant chemical
potentials $\mu_L$ and $\mu_R$, and described by the Hamiltonian

$$H_0 = H_L + H_R = \sum_{m=L,R} \int_{-\infty}^{+\infty} dx \psi_m^\dagger \partial_x \psi_m, \quad (4)$$

where we have chosen left-moving branch for both $L$ and
$R$ species and set $v_F = 1$ [10] (similar starting point
of view was taken in [11]); the reservoirs being the edge
channels or kinks-antikinks in the sine-Gordon formula-
tion). Each of the reservoirs (referred to as leads below)
is initially in a well-defined equilibrium state. Then, one
allows for a current to flow from $L$ to $R$ by adiabati-
cally switching on some coupling term. Since the amount
of electrons in reservoirs is infinite in the infinite band-

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where \( G_{dd}^U = -\frac{1}{\hbar} \frac{1}{\beta} d\tau e^{i\omega_n \tau} \langle T_\tau d \psi_L^+(0) \rangle \) is the full Matsubara propagator for the impurity,

\[
G_{dd}^U = \frac{1}{i\omega_n - \epsilon_0 - \frac{\Sigma'(\omega_n)}{\beta}}.
\]

(10)

\( \omega_n \) are the fermionic Matsubara frequencies, and to obtain (9) the Dyson equations were used, which follow from the quadratic nature of coupling between the impurity and the leads:

\[
\begin{align*}
\langle T_\tau \psi_L \rangle_{x=0} &= -\lambda G_{0L} G_{dd}^U, \\
\langle T_\tau d \psi_L^+ \rangle_{x=0} &= -\lambda G_{0R} G_{dd}^U,
\end{align*}
\]

(11a)

where

\[
\begin{align*}
G_{0L(R)} &= -\frac{1}{\hbar} \frac{1}{\beta} d\tau e^{i\omega_n \tau} \langle T_\tau \psi_L(\tau)\psi_L^+(0) \rangle_0.
\end{align*}
\]

(12)

are free Matsubara Green functions of the disconnected leads (\( \lambda = 0 \)) at \( x = 0 \). Substituting (13) one has:

\[
\begin{align*}
\frac{\partial}{\partial \tau} \log Z_{\text{imp}} &= i \sum_{\omega_n > 0} \left[ G_{dd}^U(i\omega_n + i \max[\mu_L, \mu_R]) \\
&\quad - G_{dd}^U(-i\omega_n + i \min[\mu_L, \mu_R]) \right],
\end{align*}
\]

(14)

where it was used that \( i\mu_{L(R)} \) are restricted to the bosonic Matsubara frequencies, \( 2\pi n_N T \), which allows to bring it in the argument of the Green functions. Emergence of the non-analytic functions “max” and “min” at this stage of calculation is an important feature. Expression (14) can be viewed as a function of two complex variables \( \mu_L \) and \( \mu_R \) with a cut at \( \mu_L = \mu_R = 1/2 \). Because of the cut, one can define two separate analytic continuations: once from the domain \( \mu_L > \mu_R \), and once from \( \mu_R > \mu_L \) to get two different functions, related to \( Z_{\text{imp}}^+ \) and \( Z_{\text{imp}}^- \) (this procedure is analogous to obtaining retarded and advanced Green functions from the Matsubara Green function by continuation in \( \omega \)). Subtracting one function from another and substituting \( i\mu_{L(R)} \rightarrow \mu_{L(R)} \), one obtains:

\[
\begin{align*}
\frac{\partial}{\partial \tau} \log Z_{\text{imp}} &= i \sum_{\omega_n > 0} \left[ G_{dd}^U(i\omega_n + \mu_L) - G_{dd}^U(i\omega_n + \mu_R) \\
&\quad + G_{dd}^U(-i\omega_n + \mu_L) - G_{dd}^U(-i\omega_n + \mu_R) \right],
\end{align*}
\]

(15)

where the continuation \( i\mu \rightarrow \mu \) is implicitly assumed in (13) also in the second argument, \( \{\mu\} \), which is omitted here for brevity, and the expression in the square brackets contains now only analytic functions over real \( \mu \). Note that the sum over \( \omega_n \) in (14) diverges and hence an analytic continuation in \( \mu \) might be not unique. We assume the sum in (14) to be regularized in the standard way by inserting \( \exp(i\omega_n \tau) \). Then, since expression in the right-hand side of (14) is now a function of \( \mu \) which vanishes when \( |\mu| \rightarrow \infty \) (by the reason similar to as a Green function vanishes when \( \omega_n \rightarrow \infty \)), the analytic continuation in \( \mu \) to the appropriate domain in the complex \( \mu \) plane becomes unique by the well-known mathematical theorem. One may also notice that, without the preliminary regularization, the divergence in \( Z^+ \) cancels against the same divergence in \( Z^- \) and the sum in (15) is convergent, which also defines a certain way of regularization. The correctness of this procedure can be checked against the non-interacting case.

Next, consider an interacting non-equilibrium system in the real time. By definition,

\[
J_L = \frac{\partial \rho_L}{\partial t} = i \hbar \{ [H, \rho_L],
\]

(16)

where \( \rho_L = \psi_L^+ \psi_L \) is the density of charge in the left lead. Opening the commutator in (16) and taking the average, one obtains

\[
J_L = \frac{i e^L}{\hbar} \left( \langle \psi_L(t,0) \psi_L^+(t,0) \rangle - \langle \psi_L^+(t,0) \psi_L(t,0) \rangle \right).
\]

(17)

Using Dyson equations (Eq. (B13) from appendix B), in analogy with the Matsubara calculation above, correlators in (17) can be expressed in terms of the free lead propagators and the full d-level retarded Green function to obtain (14):

\[
J = \frac{1}{2} (J_L - J_R) = \frac{e^L}{\hbar} \int_{-\infty}^{\infty} d\omega [f_L - f_R] \text{Im} G_{dd}^R(\omega),
\]

(18)

where \( f_{L,R} = f(\omega - \mu_{L,R}) \) are the Fermi functions. Eq. (13) was derived without any reference to the nature of impurity interactions \( H_{\text{int}} \), in the case when \( d \) has additional quantum numbers, trace operation must be performed (16). Full \( d-d \) propagator,

\[
G_{dd}^R = \frac{1}{\omega - \epsilon_0 + i\Gamma - \Sigma^R(\omega, \mu)},
\]

(19)

can be obtained by the Keldysh technique (17) if one uses (13) with \( \Sigma^R = 0 \) as a zeroth-order approximation for the impurity Green function and then adiabatically switches on interactions. Using that \( 2i \text{Im} G^R = G^R - G^A \), and that \( G^R (G^A) \) is analytic in the upper (lower) complex half-plane, we split the expression for current (13) into two parts, one containing \( G^R \) and another \( G^A \) and close the contour in the upper/lower complex half-planes respectively. Then, by the residue theorem, one has:

\[
J = \frac{e\pi T}{2} \sum_{\omega_n > 0} \left[ G_{dd}^R(\mu_R + i\omega_n) - G_{dd}^R(\mu_L + i\omega_n) \\
+ G_{dd}^A(\mu_R - i\omega_n) - G_{dd}^A(\mu_L - i\omega_n) \right]
\]

(20)
To this formula is related our first observation: finding the exact quantum current is equivalent to evaluating the non-equilibrium retarded Green function in the complex ω plane at the fermionic Matsubara frequencies shifted from the imaginary axis by the amount of the chemical potential. Now it is convenient to compare the real time calculation with the Matsubara calculation, Eq. (3), since both expressions are defined over the same set of points in the complex ω plane (μL(R) + iωn). For the non-interacting case self-energy vanishes, so that one has an exact expression

\[ G_{dd}^R = \frac{1}{\omega - \epsilon_0 + i\Gamma}. \]  

(21)

Note that in the absence of interactions the retarded non-equilibrium Green function does not depend on the chemical potentials of the leads and coincides with the equilibrium retarded Green function. With interactions, the self-energy (and thus \( G_{dd}^R \)) appears to be in general bias-dependent. For the current case one also has an exact result, given in Appendix A. Comparing (3) with (21) for the free correlators (\( \Sigma^M = \Sigma^R = 0 \)) we see that they agree term by term, hence Eq. (3) holds. We checked also that the analogue of Eq. (1) holds for other free models, e.g. the anisotropic two-channel Kondo model in the Toulouse limit (8) or the point-contact model (see Appendix A):

\[ H_{pc} = H_0 + \lambda \delta(x)(\psi_L^+ \psi_R + \psi_R^+ \psi_L). \]  

(22)

Hamiltonian (22) corresponds to a particular case of the quantum Hall bar with an impurity (1) at \( \nu = 1 \), when the Luttinger liquid on edges reduces to free fermions, and thus it provides a direct check of Eq. (1). However, in order to satisfy Eq. (3) in the presence of interactions, one needs, e.g.,

\[ \Sigma^R(i\omega_n + \mu_L|\{\mu\}) = \Sigma^M(i\omega_n + i\mu_L|\{i\mu\})_{\mu \rightarrow -\mu}. \]  

(23)

for \( \omega_n > 0 \), or, equivalently,

\[ \Sigma^R(i\omega_n|\{\mu\}) = \Sigma^M(i\omega_n|\{i\mu\})_{\mu \rightarrow -\mu}. \]  

(24)

Equation (23) states that, out of equilibrium, if the retarded self-energy is equal to the Matsubara self-energy analytically continued both in \( \omega \) and in \( \mu \), then Eq. (3) holds, and it is very likely that for generic interactions this statement works both ways. Thus, Eq. (3) represents a very interesting conjecture. It is a very simple generalization of the analogous result in the linear response theory. Indeed, to obtain the current in the linear response limit, \( \mu_L, \mu_R \rightarrow 0 \) (20), one can substitute for the retarded Green function in Eq. (8) its equilibrium value, and then Eq. (23) becomes just a familiar relation from the linear response theory. Hence, Eq. (3) holds in the linear response limit.

\[ III. \ \text{PERTURBATIVE ANALYSIS OUT OF EQUILIBRIUM} \]

In strongly interacting 1D systems a naive perturbation series without some self-consistent resummations often do not reveal the correct physical properties of the system. However, such a series, if convergent, are quite suitable for the purposes of comparison of the mathematical structure of both sides of (23).

To see that in general out of equilibrium Eq. (23) is not satisfied, consider the simplest interacting model with \( \{d_n\} = \{d^+_\uparrow, d^+_\downarrow\} \) and interaction of the form

\[ H_{int} = Ud^+_\uparrow d^+_\downarrow d^\downarrow d^\uparrow, \]  

(25)

(correspondingly, electrons in the leads also carry spin and interactions are such that the total spin is preserved). This is known as the Anderson model (21), and it was studied in the \( U = \infty \) (24) as well as small \( U/\Gamma \) limits (23) out of equilibrium for its connection with quantum dots. In the equilibrium, Anderson model is exactly solvable by the Bethe ansatz (26), and it is a potential candidate to be solved exactly out of equilibrium. It can be shown (27) that, e.g., susceptibility in equilibrium can be expanded into the power series in \( U/\Gamma \) which converges absolutely for \( |U/\Gamma| < 1 \) and rapidly attains the asymptotic form. It is natural to assume that such convergence properties remain true also out of equilibrium. Thus, if Eq. (23) holds, then it must be satisfied for every term in the \( U \)-perturbation expansion of both sides, in analogy with the standard linear response theory (28). We checked Eq. (23) for the first few terms. In the first order in \( U \) the self-energy depends on the occupation of the levels at \( U = 0 \). The occupation \( \langle n \rangle \) obtained from the analytic continuation of the Matsubara Green function of the non-interacting system,

\[ \Sigma^M_{(1)} = UG^0\left(\tau, \tau + 0\right)_{\mu \rightarrow -\mu} = \frac{U}{2} - \frac{U}{\pi} \text{Im} \Psi\left[ \frac{1}{2} + i\frac{\epsilon_0}{2\pi T} \right] + \frac{U}{2\pi} \int \Psi\left( \frac{1}{2} + \frac{1}{2\pi T} \right) - \Psi\left( \frac{1}{2} - \frac{1}{2\pi T} \right) \]  

(26)

\[ \Psi\left( \frac{1}{2} + i\frac{\epsilon_0}{2\pi T} \right) - \Psi\left( \frac{1}{2} - i\frac{\epsilon_0}{2\pi T} \right) \]  

\[ + \Psi\left( \frac{1}{2} + i\frac{\epsilon_0 - \mu}{2\pi T} \right) - \Psi\left( \frac{1}{2} - i\frac{\epsilon_0 - \mu}{2\pi T} \right) \]  

does not in general agree with the exact non-equilibrium occupation of the same system,

\[ \Sigma^R_{(1)} = -iUG^{-+}(t, t) = \frac{U}{2} - \frac{U}{2\pi} \text{Im} \int \Psi\left( \frac{1}{2} + \frac{1}{2\pi T} + i\frac{\epsilon_0 + \mu}{2\pi T} \right) + \Psi\left( \frac{1}{2} + \frac{1}{2\pi T} + i\frac{\epsilon_0 - \mu}{2\pi T} \right) \]  

(27)

\[ \Psi\left( \frac{1}{2} + \frac{1}{2\pi T} + i\frac{\epsilon_0 + \mu}{2\pi T} \right) + \Psi\left( \frac{1}{2} + \frac{1}{2\pi T} + i\frac{\epsilon_0 - \mu}{2\pi T} \right) \]  

where \( \Psi(x) \) is the derivative of logarithm of gamma-function and we have taken \( \mu = \mu_L = -\mu_R > 0 \). We do not exclude, however, that for some special choice of
parameters Eq. (23) might hold. Indeed, from the above explicit expressions in the first order we see that, except for the equilibrium point $\mu = 0$, the two expressions agree also at the symmetric point $\epsilon_0 = 0$ out of equilibrium, where $\Sigma^{R}_{(1)} = \Sigma^{M}_{(1)} = U/2$. Of course, one cannot rely merely on the first order result. It just indicates that Eq. (23) could hold in the true symmetric point merely on the first order result. It just indicates that Eq. (23) could hold in the true symmetric point $\epsilon_0 = 0$ out of equilibrium, where all the energy levels ($\mu, \epsilon_0, \epsilon_0 + U$) are symmetric with respect to zero and an additional particle-hole symmetry is present. For $\epsilon_0 = -U/2$ one treats the $\epsilon_0 d^d$ term as a perturbation, and the first order contribution to self-energy vanishes: $\Sigma^{R}_{(1)} = \Sigma^{M}_{(1)} = 0$, thus the non-trivial check for the symmetric point can be provided by the second order in $U$. As can be shown, only one diagram, Fig. 1, gives a non-vanishing contribution up to this order.

\[ \phi = i \sqrt{\frac{\omega_0}{2}} (b - b^+). \]  

The first non-trivial correction to the $d$-electron self-energy is given by the one-loop diagrams shown in Fig. 2.

![FIG. 1. First non-vanishing contribution to the self-energy of the symmetric Anderson model.](image)

Unfortunately, in the second order the calculation is significantly more cumbersome and we were not able to complete it and obtain an explicit analytical result. We hope, thus, that Eq. (23) may hold for the special choice of the value of $\epsilon_0$ in the Anderson model, leaving it as an open question for future studies. Note also that according to the Luttinger-Ward procedure for calculating the ground state energy of interacting Fermi gas the first order correction to the self-energy (which is due to instantaneous self-interaction) cancels against the correction to chemical potential. This result does not apply to our case, since the chemical potential is kept fixed in our problem, while the total number of particles, being in fact infinite, is allowed to vary.

As another example, which allows to perform a reliable check of Eq. (23) avoiding above difficulties, we consider model (1) with the dissipation on impurity. Namely, let $H_{\text{int}}$ describes the effects of phonons (photons) emission and absorption on the impurity [23].

\[ H_{\text{int}} = \gamma d^d \phi, \]  

where $\phi$ is a real phonon field and $\gamma$ is the electron-phonon coupling constant. For simplicity we consider a toy model where the spectrum of phonons consists of a single mode, $\omega_0(k) = \omega_0 > 0$. The phonons are assumed to be in thermodynamic equilibrium initially, with the energy

\[ H_{\text{ph}} = \omega_0 b^+ b. \]  

The field $\phi$ is quantized in terms of the bosonic operator $[b, b^+] = 1$ as follows:

\[ \phi = i \sqrt{\frac{\omega_0}{2}} (b - b^+). \]  

The corresponding non-equilibrium retarded self-energy (Fig. 3) is given by

\[ \Sigma^{R}_{(1)}(\omega, \mu) = i \gamma^2 \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \left[ D^- (\omega_1) G_{dd^-}^* (\omega_1 + \omega) - D^+ (\omega_1) G_{dd^+}^* (\omega_1 + \omega) \right], \]  

where the Keldysh correlators $D^-, D^+$ are given in Appendix B. Substituting corresponding bare propagators and evaluating (31), (33) explicitly one can see that the only case when Eq. (23) holds is the case of equilibrium, $\mu = 0$.

![FIG. 2. The leading contributions to the self-energy of d-electron interacting with phonons (dashed line).](image)

![FIG. 3. Leading contribution to the retarded self-energy of d-electron interacting with phonons, $\Sigma^R = \Sigma^{+-} + \Sigma^{-+}$, non-equilibrium case.](image)

**IV. CONCLUSION**

To summarize, a generalization of the linear response theory to the non-equilibrium case was discussed. This
generalization is based on the certain analytic continuation in the additional complex parameter which plays a role of external bias. The complex chemical potential is formally introduced in the equilibrium density matrix and analytically continued to the real values after averaging over all the states is performed. Thermodynamic averages turn out to have non-analytic dependence on the complex bias, with the cut along the plane $\text{Im} \mu_{\text{L}} = \text{Im} \mu_{\text{R}}$, as a result of divergences in the summation over infinite-particle states. It was shown that formula for the current (1) is valid in the linear response limit for systems of the type (3) and is valid also out of equilibrium if condition (23) holds. This condition, which is trivially satisfied by the free systems, indicates that there are no general coupling constants is hopeless. The fact that Eq. (1) holds for the free systems as well. One may proceed, e. g., from the Lehmann spectral representation (31) to obtain a relation resembling real non-equilibrium propagators. Hence, it would be interesting to derive an analogue of structure and resemble real non-equilibrium propagators. Formula (1) as it is formally introduced in the equilibrium density matrix and analytically continued to the real values after averaging over infinite-particle states. It was shown that formula for the current (1) is valid in the linear response limit for systems of the type (3) and is valid also out of equilibrium if condition (23) holds. This condition, which is trivially satisfied by the free systems, is not satisfied in general by the interacting systems. This does not mean, however, that the very idea of obtaining non-equilibrium current by exploiting analytic properties of the thermodynamic averages in the space of complex coupling constants is hopeless. The fact that Eq. (1) holds for the free systems indicates that there are no general grounds for ruling out this idea. As the perturbative analysis indicates, the propagators obtained by an analytic continuation from the Matsubara ones have similar structure and resemble real non-equilibrium propagators. Hence, it would be interesting to derive an analogue of Eq. (1) based on the analytic continuation in the complex bias which would be true for the generic interacting systems as well. One may proceed, e. g., from the Lehmann spectral representation (31) to obtain a relation between the equilibrium partition function and the non-equilibrium current by controlling two complex parameters, frequency and chemical potential. Formula (1) as it is can be used, nevertheless, to obtain the exact transport through Anderson impurity in the linear response limit (32), if the exact partition function is supplied by the Bethe ansatz technique. It would be interesting to understand also in the Green functions language why Eq. (1) works for the quantum Hall bar with a constriction.

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APPENDIX A: FREE CASE

To understand how the analytic continuation in the complex chemical potential works, we consider the following simple point-contact model (33):

$$H_{pc} = H_0 + \lambda \delta(x)(\psi^+_L \psi_R + \psi^+_R \psi_L).$$  \hspace{1cm} (A1)

This model can be viewed also as a particular case of the quantum Hall bar with an impurity (1) at $\nu = 1$, when the Luttinger liquid on edges reduces to free fermions, and thus it provides a direct check of Eq. (1). By definition,

$$\dot{J}_L = \frac{i e \lambda}{\hbar} \left( \langle \psi^+_R \psi_L \rangle - \langle \psi^+_L \psi_R \rangle \right).$$  \hspace{1cm} (A2)

The latter expression involves equal time correlators at $x = 0$, which can be easily found by using Dyson equations and the Keldysh technique (17). Eventually, one derives

$$J_L = \frac{e \lambda^2}{\hbar} \int_{-\infty}^{\infty} d\omega [f_L - f_R] \left\{ \frac{2 \text{Im} g^R}{1 - (\lambda g^R)^2} \right\} \hspace{1cm} (A4)$$

$$= \frac{e}{\hbar} \Gamma \frac{\mu_R - \mu_L}{(\mu_R - \mu_L)^2},$$  \hspace{1cm} (A5)

where $f_{L,R} = f(\omega - \mu_{L,R})$ are the Fermi functions, $\Gamma = \lambda^2$ and $g^R$ is the retarded Green function of either of the disconnected leads ($\lambda = 0$) at $x = 0$ (14).

$$g^R(\omega) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{\omega + p + i\eta} = -\frac{i}{2}.$$  \hspace{1cm} (A6)

Partition function of (A3) can be easily found. For the complex chemical potentials $i\mu_{L}$ and $i\mu_{R}$, according to (1) one has

$$\frac{\partial}{\partial T} \log Z_{\text{imp}}(i\mu_{L}, i\mu_{R}) = \sum_{\omega_n} \frac{G_{0L}(\omega_n) G_{0R}(\omega_n)}{\bar{G}_{0L}(\omega_n) G_{0R}(\omega_n)} \hspace{1cm} (A7)$$

$$= \frac{|\mu_L - \mu_R|}{2\pi T} \frac{1}{\bar{\Gamma} - 4} + \sum_{\omega_n > \max(\mu_L, \mu_R), \omega_n < \min(\mu_L, \mu_R)} \frac{1}{\bar{\Gamma} + 4},$$

where $G_{0L,R}(\omega) = -\int_{0}^{1/T} d\tau e^{i\omega_\tau \tau} (T_{\tau} \psi_{L,R}(\tau) \psi^+_L(0))$ \hspace{1cm} (A8)

are the Matsubara Green functions for the disconnected leads ($\lambda = 0$) at $x = 0$, e. g.,

$$G_{0L}(\omega_n) = \int_{-\infty}^{\infty} dp \frac{1}{2\pi} \frac{1}{i\omega_n + p - i\mu_L}$$

$$= -\frac{i}{2} \text{sign}(\omega_n - \mu_L),$$  \hspace{1cm} (A9)
and \( \omega_n \) are the fermionic Matsubara frequencies. When passing from the first line to the second in (A7) it was used that \( \mu_L = 2\pi NT \) and \( \mu_R = 2\pi MT \), where \( N, M \) are integers. The infinite sum in (A7) needs to be properly regularized:

\[
\sum_{\omega_n > \max[\mu_L, \mu_R]} \frac{1}{\Gamma + 4} = -\frac{1}{2\pi i T} \frac{1}{\Gamma + 4} \lim_{\epsilon \to 0} \int_{C} \frac{e^{i\omega \tau}}{\omega^{\Gamma + 4} + 1} d\omega
\]

where contour \( C \) consists of two parts: one goes along \( \text{Im} \omega = \max[\mu_L, \mu_R] \) and an infinite semicircle above it, the other along \( \text{Im} \omega = \min[\mu_L, \mu_R] \) and a semicircle below it, both counter-clockwise.

Considering \( \mu_L - \mu_R \) as one variable \( y \), one has a function \( |y| \) defined on the imaginary \( y \)-axis. We have to continue it analytically to the real axis. Since it has a cut at \( y = 0 \), one can define two separate analytic continuations, to the upper and lower half-planes: \( f_x = -iz \), \( f_- = iz \). Then, one continues from the positive imaginary axis to the real axis, \( f_x(x) = -ix \), and from the negative imaginary axis, \( f_-(x) = ix \), and subtracts one from another to get \( f_+ - f_- = -2i(\mu_L - \mu_R) \) [23]. Comparing to (A3), one has

\[
J = \frac{e}{h} 8\pi iT \frac{4(1 - \Gamma)}{4 + \Gamma} \partial \log \frac{Z_{\text{imp}}}{Z_{\text{imp}}}.
\]

(A11)

Note that the prefactor \((4 - \Gamma)/(4 + \Gamma)\) in (A11) is not universal and depends on the regularization scheme.

2. Resonant-level model

Next, instead of the point-contact scattering we introduce a single dynamical impurity at \( x = 0 \) described by an additional state at the energy \( \epsilon_0 \). Operator \( d^+ \) creates an electron on the impurity. The simplest model to write is a resonant-level model:

\[
H_{RL} = H_0 + \epsilon_0 d^+ d + \lambda \sum_{m=L,R} (\psi_m^+ (0) d + d^+ \psi_m (0)).
\]

(A12)

Note that the impurity can be effectively removed from the action by integrating out impurity degrees of freedom, and one gets certain generalization of the point-contact model with time-dependent coupling. Then, in the limit \( \epsilon_0, \lambda \to \infty \) one recovers (A1). A dynamical impurity, however, suggests a solid way of regularizing the divergences encountered in Eq. (A7).

Expression for the current was derived in Sec. I:

\[
J = \frac{1}{2}(J_L - J_R) = \frac{e}{h} \pi T \int_{-\infty}^{\infty} d\omega [f_L - f_R] \text{Im} G_{dd}^R(\omega),
\]

(A13)

where for the retarded Green function \( G_{dd}^R \) one has in this case an exact expression

\[
G_{dd}^R = \frac{1}{\omega - \epsilon_0 + i\Gamma}.
\]

(A14)

Substituting (A14) into (A13) one obtains:

\[
J = \frac{e}{h} \pi T \Psi \left[ \frac{1}{2} + \frac{\Gamma}{2\pi T} + i \frac{\epsilon_0 - \mu_L}{2\pi T} \right] - \Psi \left[ \frac{1}{2} + \frac{\Gamma}{2\pi T} + i \frac{\epsilon_0 - \mu_R}{2\pi T} \right],
\]

(A15)

where \( \Psi \) is the standard psi (digamma) function (compare to Eq. (27)).

The partition function involves gaussian Feynmann integral and can be obtained immediately:

\[
\frac{\partial}{\partial \Gamma} \log Z_{\text{imp}}(i\mu) = - \sum_{\omega_n > 0} \frac{G_{0L}(\omega_n) + G_{0R}(\omega_n)}{i\omega_n - \epsilon_0 - \Gamma(G_{0L} + G_{0R})}
\]

\[
= \sum_{\omega_n > 0} \left[ \frac{i}{\omega_n - \epsilon_0 - \Gamma(G_{0L} + G_{0R})} + \frac{i}{\omega_n - \epsilon_0 - \Gamma(G_{0L} + G_{0R})} \right]
\]

(A16)

where, while passing from the first line to the second above, Eq. (A9) was used and the fact that \( i\mu_{L,R} \) have the form of bosonic Matsubara frequencies, which allows to bring \( i\mu_{L,R} \) into the denominators. Expression (A10), viewed as a function of two complex variables \( \mu_L \) and \( \mu_R \), has a cut at \( \mu_L = \mu_R \). The analytic continuation is performed from the 2D plane of purely imaginary \( \mu_L \) and \( \mu_R \) with the cut at \( \mu_L = \mu_R \). Once one continues analytically from the domain \( \mu_L > \mu_R \), and once from \( \mu_R > \mu_L \) to get two functions. Subtracting one from another and substituting \( i\mu \to \mu \), one gets by comparing with (A13)

\[
J = \frac{e}{h} \pi T \frac{\partial}{\partial \Gamma} \log \frac{Z_{\text{imp}}}{Z_{\text{imp}}}.
\]

(A17)

3. Kondo model

Consider an impurity spin \( S = \frac{1}{2} \) which couples \( L \) and \( R \) leads by the \( s - d \) exchange (Kondo) interaction [18]:

\[
H_K = H_0 + \sum_{\lambda=1}^{3} \sum_{ab=L,R} J_{ab}^\lambda \psi_{\lambda a}^\dagger (0) \sigma_{\lambda \sigma} \psi_{\sigma b} (0) S^\lambda,
\]

(A18)

where \( \sigma^\lambda \) are Pauli matrices and \( J_{ab}^\lambda \) are coupling constants. Electrons here have additional spin index \( \sigma \). As it was shown in [13] following the ideas of [18], under the assumptions that \( J_{sL}^L = J_{sL}^R = 0, J_x = J_y = J_z = J_{zL} = J_{zL} = 0 \),
$J_{RR}, J_{LR} = J_{RL}, J_{LL} = J_{RR} = 2\pi$ model (A18) is equivalent to a quadratic model with the Hamiltonian

$$H_K = \sum_{m=1,2} i \int dx \psi_m^\dagger \partial_x \psi_m + J_1 \delta(x)(\psi_1^\dagger + \psi_1)(d^+ - d)$$

$$+ J_2 \delta(x)(\psi_2^\dagger - \psi_2)(d^+ + d), \quad (A19)$$

where $J_1 \sim J_{LL}, J_2 \sim J_{LR}$, and $\{d^+, d\} = 0$. Indexes 1 and 2 correspond to the spin-flavor and flavor channels of the original model (A18) [5], while only second channel has now non-zero chemical potential $\mu$ if one sets initially $\mu_L = -\mu_R = \frac{\mu}{2}$. When calculating $Z_{imp}$ it is convenient to change variables in the Feynmann integral from $d, d^+$ to the real and imaginary parts $a = (d + d^+)/\sqrt{2}, b = i(d - d^+)/\sqrt{2}$, where $a$ and $b$ are classical Majorana fermions.

After integrating out $\psi_{1,2}$ the action factorizes as $S = S(a, J_2^L, \mu) + S(b, J_2^R)\mu$, and only the $\mu$-dependent part is of interest to us. Since $S$ is gaussian, $Z_{imp}$ can be easily computed. The result coincides with Eq. (A14) at $\epsilon_0 = 0, \mu_L = -\mu_R$. Thus, Kondo interactions reduce to the case of resonant-level model studied above for the special point in the couplings space (an analogue of Toulouse point for the single-channel Kondo problem).

**APPENDIX B: GREEN FUNCTIONS**

We collect here the definitions of different Green functions used throughout this Letter, following the notations of Ref. [7]. All the Green functions are taken at $x = 0$. The subscript $R$ for “right” should not be confused with superscript $R$ for “retarded”. First, consider the point-contact model. Define:

$$i\langle \psi_R^\dagger(t) \psi_L(t) \rangle = G^R_{RL}(0) \quad (B1)$$

$$i\langle \psi_L^\dagger(t) \psi_R(t) \rangle = G^R_{LR}(0) \quad (B2)$$

Dyson equations yield:

$$G^{++}_{LR} = \lambda g_{RR}^- G^{--}_{LL} - \lambda g_{RR}^- G^{++}_{LL}, \quad (B3a)$$

$$G^{--}_{RR} = \lambda g_{LL}^- G^{++}_{LL} - \lambda g_{RR}^- G^{--}_{LL} \quad (B3b)$$

where

$$-i g_{L(R)}^{++}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \psi_{R(L)}^+ (0) \psi_{L(R)}(t) \rangle$$

$$= f(\omega - \mu_{L(R)}), \quad (B4)$$

$$i g_{L(R)}^{+--}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \psi_{R(L)}^+ (t) \psi_{L(R)}^+ (0) \rangle$$

$$= [1 - f(\omega - \mu_{L(R)})] \quad (B5)$$

$$i g_{L(R)}^{-+}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle T \psi_{L(R)}^+ (t) \psi_{L(R)}^+ (0) \rangle \quad (B6)$$

Green functions of free (decoupled, $\lambda = 0$) leads, $\hat{T}$ stands for anti time ordering operation, and by capital letters $G_{LL}, G_{RR}$ are denoted corresponding full propagators of the interacting system ($\lambda \neq 0$). The latter can be found from the system of four coupled Dyson equations. The solution reads, e. g.,

$$G^{-+}_{LL} = \frac{g_{LL}^- + \lambda^2 g_{LL}^+ g_{RR}^-}{|1 - \lambda^2 g_{LL}^+ g_{RR}^-|} \quad (B9)$$

$$G^{++}_{LL} = \frac{g_{LL}^+ + \lambda^2 g_{LL}^- g_{RR}^+}{|1 - \lambda^2 g_{LL}^- g_{RR}^+|} \quad (B10)$$

For the resonant-level model there are very similar definitions. For example,

$$i\langle d^+(t) \psi_L(t) \rangle = G^{dd^+}_{dd}(0) \quad (B11)$$

$$i\langle \psi_L^+(t) d(t) \rangle = G^{-d^+}_{L}(0) \quad (B12)$$

Dyson equations express these correlators through the full propagator of the impurity:

$$G^{dd^+}_{dd} = \lambda g_{L}^- G^{--}_{dd} - \lambda g_{L}^- G^{++}_{dd}, \quad (B13a)$$

$$G^{-d^+}_{L} = \lambda G^{++}_{dd} g_{L}^+ - \lambda G^{--}_{dd} g_{L}^+ \quad (B13b)$$

Four Green functions $++, --, +-, --$ in general are related to each other by

$$G^{++} + G^{--} = G^{+-} + G^{-+} \quad (B14)$$

and to the retarded (advanced) Green functions

$$G^R = G^{--} - G^{+-}, \quad (B15)$$

$$G^A = G^{++} - G^{-+}, \quad (B16)$$

The impurity Green functions of an isolated non-interacting impurity read:

$$G^{dd^+}_{dd}^{(0)} = i\pi \delta(\omega - \epsilon_0) \quad (B17)$$

$$G^{-d^+}_{L}^{(0)} = -i\pi \delta(\omega - \epsilon_0) \quad (B18)$$

$$G^{R(0)}_{dd} = \frac{1}{\omega - \epsilon_0 + i0} \quad (B19)$$

For the resonant-level model without on-site interactions one can write four Dyson equations for the impurity correlators and easily solve them. Relevant for us is the retarded Green function,

$$G^R_{dd} = \frac{G^{R(0)}_{dd}}{1 - G^{R(0)}_{dd} \Sigma^{R}} \quad (B20)$$
where $\Sigma^R = \Sigma^{-} + \Sigma^{+}$ is the retarded self-energy, given in the free case by

$$\Sigma^R(0) = \lambda^2 (g_L^R + g_R^R)$$

(B21)

For the calculation of interacting Green functions one needs also the following zeroth-order Green propagators:

$$G^{++}_{dd} = \frac{G^{++}_{dd}(0) + \lambda^2 G^{R(0)}_{dd} G^{A(0)}_{dd} (g_L^R + g_R^R)}{1 - \lambda^2 G^{R(0)}_{dd} (g_L^R + g_R^R)^2}$$

(B22)

$$G^{+-}_{dd} = \frac{G^{+-}_{dd}(0) + \lambda^2 G^{R(0)}_{dd} G^{A(0)}_{dd} (g_L^R + g_R^R)}{1 - \lambda^2 G^{R(0)}_{dd} (g_L^R + g_R^R)^2}$$

(B23)

$$G^{--}_{dd} = \frac{G^{--}_{dd}(0) + \lambda^2 G^{R(0)}_{dd} G^{A(0)}_{dd} (g_L^R + g_R^R)}{1 - \lambda^2 G^{R(0)}_{dd} (g_L^R + g_R^R)^2}$$

(B24)

Zeroth order correlators of the phonon field are defined as

$$iD^{+-}(t_1, t_2) = \langle \phi(t_2) \phi(t_1) \rangle$$

(B25)

$$iD^{--}(t_1, t_2) = \langle \phi(t_1) \phi(t_2) \rangle$$

(B26)

Explicitly,

$$D^{+-}(\omega) = -i \pi \omega \left[ N \delta(\omega - \omega_0) + (1 + N) \delta(\omega + \omega_0) \right]$$

(B27)

$$D^{--}(\omega) = -i \pi \omega \left[ N \delta(\omega + \omega_0) + (1 + N) \delta(\omega - \omega_0) \right]$$

(B28)

$$D^{R}(\omega) = \frac{\omega_0}{2} \left[ \frac{1}{\omega - \omega_0 + i0} - \frac{1}{\omega + \omega_0 + i0} \right]$$

(B29)

with

$$N = \frac{1}{e^{\omega_0/T} - 1}$$

(B30)

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