SymAR: Symmetry Abstractions and Refinement for Accelerating Scenarios with Neural Network Controllers Verification

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ABSTRACT
We present a Symmetry-based abstraction refinement algorithm SymAR that is directed towards safety verification of large-scale scenarios with complex dynamical systems. The abstraction maps modes with symmetric dynamics to a single abstract mode and refines recursively split the modes when safety checks fail. We show how symmetry abstractions can be applied effectively to closed-loop control systems, including non-symmetric deep neural network (DNN) controllers. For such controllers, we transform their inputs and outputs to enforce symmetry and make the closed loop system amenable for abstraction. We implemented SymAR in Python and used it to verify paths with 100s of segments in 2D and 3D scenarios followed by a six dimensional DNN-controlled quadrotor, and also a ground vehicle. Our experiments show significant savings, up to 10x in some cases, in verification time over existing methods.

1 INTRODUCTION
The set of states that can be reached by a hybrid or dynamical system over a given time horizon is called the reachable set. Computing or approximating the reachable set, known as reachability analysis, is a fundamental subroutine for formal verification, controller synthesis, and monitoring. Notwithstanding the undecidability results [13], and the curse of dimensionality plaguing the early algorithms, over the last decade, the hybrid systems community has pushed the reach of reachability analysis tools from small, academic, linear models to realistic nonlinear models [1, 4, 6, 11, 18], linear models with millions of dimensions [3, 12], and deep neural networks (DNN) and DNN-controlled systems [8, 14, 21]. This remarkable progress has been accomplished by and large by focusing on one half of the hybrid reachability problem—namely the the continuous part.

Specifically, (a) exploiting ideas from control theory and real analysis for approximating solutions of dynamical systems [4, 11], and (b) invention of clever new data structures like zonotopes [1], support functions [1, 12] and generalized star sets [3], for representing and manipulating sets of solutions of dynamical systems. The problem studied in this paper addresses challenges arising from reachability analysis of models with complex continuous and discrete dynamics.

Consider the scenario verification problem illustrated in Figure 1a: An autonomous or semi-autonomous vehicle system driving in a complex environment. There are many obstacles in the map. The vehicle has nonlinear-dynamics. It may be using a complicated controller function for tracking waypoints, for example, DNNs. A high-level planner (MP) generates a set of possible paths or waypoint sequences using a motion planning algorithm like Rapidly Expanding Random Trees (RRT) [16] that does not consider the dynamics of the vehicle nor the state estimation errors. Then, the verification task for this system is to check which parts of this plan can be safely executed by the vehicle, without running into the obstacles. Such a problem arises in online planning where many options or paths have to be evaluated quickly. Naively viewing this system as a hybrid automaton evolving in the vehicle’s state space with the modes defined by the segments of the planner, will lead to a hybrid model with hundreds or even thousands of modes and transitions. Indeed, many of these modes will be geometric transformations of each other, but current hybrid reachability tools cannot take advantage of this structure. Analyzing automaton models with large number of modes and transitions is challenging as (a) the error in over-approximations grows with the number of transitions, and (b) the size of the data structures that have to be maintained to track reachable sets grow combinatorially with the length of the mode sequences, and (c) the time needed to compute the reachable set grows with the number of transitions and the over-approximation error. An example of this is shown in Figure 1b.

In this paper, we show how symmetry abstractions can be used to tackle scenario verification problems with existing reachability analysis tools. The notion of symmetry abstractions was introduced in [20]. There, it was shown that these abstractions significantly reduce the number of modes and edges of a given automaton \( \mathcal{A} \). These abstractions achieve this reduction by grouping all modes of \( \mathcal{A} \) that share symmetric continuous dynamics in a single representative mode of the abstract automaton \( \mathcal{A}_s \). It was further shown how the reachable set \( \text{Reach}_{\mathcal{A}} \) of \( \mathcal{A} \) can be transformed using symmetry maps to get the reachable set \( \text{Reach}_{\mathcal{A}_s} \) of \( \mathcal{A}_s \). This method accelerated the computation of \( \text{Reach}_{\mathcal{A}} \) since the computation of \( \text{Reach}_{\mathcal{A}_s} \) reaches a fixed point faster than the former, and transforming reachable sets is cheaper than computing them. However, expectedly, this approach for \( \text{Reach}_{\mathcal{A}_s} \) computation results in larger over-approximation errors than computing it directly using existing methods, which limits its usability in safety verification. An example is shown in Figure 1c. Therefore, there is a need for a method that culminates the acceleration benefits of abstraction without trading off accuracy. Another limitation of symmetry abstractions is their requirement for symmetric continuous dynamics. For closed-loop control systems, although the open-loop dynamics might be symmetric, as that of a quadrotor or a car, their controllers might not be. In particular, vehicle controllers are increasingly developed using machine learning approaches like DNNs, which are nonlinear functions resulting from optimization over non-structured data with no symmetry guarantees. How can symmetry abstractions be employed for such DNN-based control systems?
In this paper, we present SymAR, a refinement algorithm for symmetry-based abstractions of hybrid automata for fast and accurate safety verification (Algorithm 2). We further propose a method to enforce symmetric properties on non-symmetric controllers by transforming their inputs and outputs with well-chosen maps that match the symmetry of the open-loop dynamics.

Our refinement happens recursively in a depth first search (DFS) manner over the mode graph of the abstraction \(A\) of \(\mathcal{A}\). When an abstract mode \(p_i\) is being visited, its reachable set is computed using any of the existing reachability tools (e.g. Verisig [14], Sherlock [8], DryVR [10]). If it is unsafe, \(p_i\) is split into two modes, each representing half of the modes of \(\mathcal{A}\) that were represented by \(p_i\). If that split is not possible, i.e. \(p_i\) represents a single mode of \(\mathcal{A}\), the parent of \(p_i\) in DFS gets split instead.

The splitting of modes during DFS changes \(\mathcal{A}\) and its mode graph, complicating the search problem. For example, a cache saving the reachable sets computed for each mode of \(\mathcal{A}\) usually gets maintained for fixed point checking purposes. Upon a mode split, some of the cached reachable sets of its parent and siblings modes in the DFS graph need to be carefully deleted and recomputed. We tackled this and similar problems in SymAR while avoiding unnecessary repetitions of computations.

We implemented our abstraction refinement-based safety verification method SymAR in Python along with the transformations for making a non-symmetric controller symmetric. We tested it on the NN-controlled quadrotor presented in [14] and also a car that we trained ourselves. We considered complex 2D and 3D scenarios with paths consisting of 100s of segments. Our results show significant acceleration in verification time over existing methods, as that of Figure 1d, while maintaining the accuracy of the result.

2 MODEL AND PROBLEM STATEMENT

Notations. We denote by \(\mathbb{N}, \mathbb{R}\), and \(\mathbb{R}^0\) the sets of natural numbers, real numbers and non-negative reals. Given a finite set \(S\), its cardinality is denoted by \(|S|\). Given a function \(\gamma : \mathbb{R}^k \rightarrow \mathbb{R}^k\) and a set \(\mathcal{S} \subseteq \mathbb{R}^k\), we abuse notation and define \(\gamma \mathcal{S} = \{\gamma(x) \mid x \in \mathcal{S}\}\).

2.1 Hybrid automaton

In this section, we present the general definition of hybrid automata that we will use in this paper [2, 15, 17], along with the safety verification problem.

**Definition 2.1.** A hybrid automaton is a tuple \(\mathcal{A} := (\mathcal{X}, \mathcal{P}, \Theta, p_{init}, E, \text{guard}, \text{reset}, f)\), where

\[(a) \mathcal{X} \subseteq \mathbb{R}^n \text{ is the continuous state space, or simply the state space, and } \mathcal{P} \subseteq \mathbb{R}^d \text{ is the discrete state space, which we call the parameter or mode space,}
(b) \(\Theta, p_{init}\) is a pair of a compact set of possible initial states and an initial mode,
(c) \(E \subseteq \mathcal{P} \times \mathcal{P}\) is a set of edges that specify possible transitions between modes,
(d) \(\text{guard} : E \rightarrow 2^\mathcal{P}\) defines the set of states at which a mode transition over an edge is possible,
(e) \(\text{reset} : \mathcal{P} \times E \rightarrow 2^\mathcal{P}\) defines the possible updates of the state after a mode transition over an edge, and
(f) \(f : \mathcal{X} \times \mathcal{P} \rightarrow \mathcal{X}\) is the dynamic function that define the continuous evolution of the state in each mode. It is Lipschitz continuous in the first argument.

A trajectory \(\xi : \mathcal{X} \times \mathcal{P} \times \mathbb{R}^0 \rightarrow \mathcal{X}\) of \(\mathcal{A}\) starting from initial state \(x\) in mode \(p\) should satisfy the differential equation:

\[
\frac{d\xi}{dt} = f(\xi, p, t, p),
\]

for any \(t \in \mathbb{R}^0\) and \(\xi, x, p, 0 = x\). For a time bounded trajectory, we denote its duration by \(\text{dur}_\xi\), its initial state by \(\xi\text{-state}\) and last one by \(\xi\text{-state}\).

An execution \(\sigma = \xi_0, p_0, \xi_1, p_1, \ldots\) of \(\mathcal{A}\) is a sequence of pairs of trajectories and modes that have the following properties:

1. \(\xi_0\text{-state} \in \Theta, p_0 \in p_{init}, p_1 \in E\),
2. \(\xi_i\text{-state} \in \text{guard}_p, p_{i+1}\), and
3. \(\xi_i\text{-state} \in \text{reset}_p, p_{i+1}\).

The set of all executions of \(\mathcal{A}\) is denoted by \(\text{Execs}_\mathcal{A}\). The set of reachable states \(\text{Reach}_\mathcal{A}\) of \(\mathcal{A}\) is the set of states reachable by \(\mathcal{A}\). Formally, \(\text{Reach}_\mathcal{A} = \{x \in \mathcal{X} \mid \sigma \in \text{Execs}_\mathcal{A}, \sigma\text{-state} = x\}\). When we restrict the states to be those visited in mode \(p\), we denote the reachset by \(\text{Reach}_\mathcal{A}_p\).

Given an unsafe map \(O : \mathcal{P} \rightarrow 2^\mathcal{X}\), that defines the unsafe set of states in each mode, the safety verification problem is to check if \(\forall p \in \mathcal{P}\),

\[
\text{Reach}_\mathcal{A}_p \cap O(p) = \emptyset.
\]

2.2 NN-controlled quadrotor case study

In this section, we will describe a case study of a scenario having a planner, NN controller, and a quadrotor and model it as a hybrid automaton. We use the quadrotor model that was presented in [14] along, its trained NN controller, and a RRT planner to construct its reference trajectories, independent of its dynamics.
The dynamics of the quadrotor are as follows:

\[
q := \begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix} = \begin{bmatrix}
v_1^q \\
v_2^q \\
v_3^q \\
v_4^q
\end{bmatrix}, \quad w := \begin{bmatrix}
b_x \\
b_y \\
b_z
\end{bmatrix},
\]

(3)

where \( q \) and \( w \) are the states of the quadrotor and the planner reference trajectory representing their position and velocity vectors in the 3D physical space, respectively. The variables \( \theta, \phi, \) and \( \tau \) represent the control inputs pitch, roll, and thrust, respectively, provided by the NN controller. The input to the NN controller is the difference \( \Delta q \) of the quadrotor state and the planner trajectory.

Definition 3.4. Given three maps \( \beta : U \to U, \gamma : \mathcal{X} \to \mathcal{X} \), and \( \rho : \mathcal{P} \to \mathcal{P} \). We call the control function \( h, \beta, \gamma, \rho \)-symmetric, if for all \( x \in \mathcal{X} \) and \( p \in \mathcal{P} \), \( \beta h_x, p = h\gamma x, \rho p \).

Such a control might violate the differentiability of \( f \) in equation (2), however the solutions still exist and are unique.

3 SYMMETRY AND EQUIVARIANT DYNAMICAL SYSTEMS

Symmetries of dynamical systems are maps acting on their state spaces that transform their trajectories to other trajectories, reducing their analysis complexity. In this section, we present the basic definitions and theorems on symmetries of parameterized dynamical systems. Moreover, we present conditions that feedback controllers should satisfy for a closed loop system to possess certain symmetry.

3.1 Symmetries of dynamical systems

First, let \( \Gamma \) be a group of smooth maps acting on \( \mathcal{X} \).

Definition 3.1 (Definition 2 in [19]). We say that \( \gamma \in \Gamma \) is a symmetry of system (2) if its differentiable, invertible, and for any solution \( \xi_{x_0}, p, \cdot, \cdot \) is also a solution.

Coupled with the notion of symmetries, is the notion of equivariant dynamical systems.

Definition 3.2 ([19]). The dynamic function \( f : \mathcal{X} \times \mathcal{P} \to \mathcal{X} \) is said to be \( \gamma \)-equivariant if for any \( \gamma \in \Gamma \), there exists \( \rho : p \to p \) such that,

\[
\forall x \in \mathcal{X}, \forall p \in \mathcal{P}, \frac{\partial \gamma}{\partial x} f x, p = f (\gamma x, \rho p).
\]

(4)

The following theorem draws the relation between symmetries and equivariance definitions.

Theorem 3.3 (Theorem 10 in [19]). If \( f \) is \( \gamma \)-equivariant, then all maps in \( \Gamma \) are symmetries of (2). Moreover, for any \( \gamma \in \Gamma \), map \( \rho : \mathcal{P} \to \mathcal{P} \) that satisfies equation (4), \( x_0 \in \mathcal{X}, \) and \( p \in \mathcal{P}, \gamma_{x_0}^\rho p, \cdot = \xi_{x_0}, p, \cdot \).

That means that one can get the trajectory of the system starting from an initial state \( \gamma_{x_0} \) in mode \( p \) by transforming its trajectory starting from \( x_0 \) in mode \( p \) using \( \gamma \).

3.2 Symmetries of closed loop control systems

In this section, we will consider the special case when the dynamical system in equation (2) is a closed loop control system similar to that of the quadrotor formalized in part (f) of its hybrid automaton definition in Section 2.2. Mainly, we discuss the property that the controller should satisfy for the system to be symmetric.

Fix an input space \( U \subseteq \mathbb{R}^m \) and consider a right hand side of equation (2) of the form:

\[
f : x, p := gx, hx, p,
\]

(5)

where \( g : \mathcal{X} \times U \to \mathcal{X} \) and \( h : \mathcal{X} \times \mathcal{P} \to U \) are Lipschitz continuous functions with respect to both of their arguments.

In order to retain symmetry for such systems, we update the notion of equivariance of dynamic functions. But first, let us define symmetric controllers.

Definition 3.4. Given three maps \( \beta : U \to U, \gamma : \mathcal{X} \to \mathcal{X}, \) and \( \rho : \mathcal{P} \to \mathcal{P} \). We call the control function \( h, \beta, \gamma, \rho \)-symmetric, if for all \( x \in \mathcal{X} \) and \( p \in \mathcal{P} \), \( \beta h_x, p = h\gamma x, \rho p \).
Definition 3.4 means that if the input of the controller, the state and the mode, using the maps $\gamma$ and $\rho$, respectively, then its output gets transformed with the map $\beta$. Such a property formalizes intuitive assumptions about controllers in general. For example, translating the position of the quadrotor and the planned trajectory by the same vector should not change the controller output. The NN controller discussed in Section 2.2 indeed satisfies this property since its input is the relative state $q-w$. We update the notion of equivariance for closed-loop systems to account for the controller in the following definition.

**Definition 3.5.** We call the control system dynamic function $f_c$ of equation (5) $\Gamma$-equivariant if for any $\gamma \in \Gamma$, there exist $\rho : \mathcal{P} \rightarrow \mathcal{P}$ and $\beta : \mathbb{U} \rightarrow \mathbb{U}$ such that $h$ is $\beta, \gamma, \rho$-symmetric and
\[
\forall x \in \mathcal{X}, \forall u \in \mathbb{U}, \quad \frac{\partial \gamma}{\partial x} gx, u = g \gamma x, \beta u. \tag{6}
\]

The following theorem repeats the results of Theorem 3.3 for the closed loop control system.

**Theorem 3.6.** If $f_c$ of equation (5) is $\Gamma$-equivariant, then all maps in $\Gamma$ are symmetries. Moreover, for any $\gamma \in \Gamma$, maps $\rho : \mathcal{P} \rightarrow \mathcal{P}$ and $\beta : \mathbb{U} \rightarrow \mathbb{U}$ that satisfy equation (6), $x_0 \in \mathcal{X}$, and $p \in \mathcal{P}$,
\[
\gamma \xi_{x_0}, p, \cdot = \xi \gamma x_0, p, \cdot , \quad \text{where } \xi \text{ is the trajectory of the dynamical system with RHS equation (5)}.
\]

**Proof.** Fix an initial state $x_0 \in \mathcal{X}$, a mode $p \in \mathcal{P}$, and $\gamma \in \Gamma$ with its corresponding maps $\rho$ and $\beta$ that satisfy Definition 3.2 per the assumption of the theorem. For any $t \geq 0$, let $x = \xi_x x_0, p, t$ and $y = \gamma x$. Then,
\[
\frac{dy}{dt} = \frac{\partial \gamma}{\partial x} dx + \frac{\partial \gamma}{\partial p} dp, \quad \text{using the chain rule,}
\]
\[
= \frac{\partial \gamma}{\partial x} gx, h x, p, \quad \text{using equation (5),}
\]
\[
= g \gamma x, \beta h x, p, \quad \text{using equation (6),}
\]
\[
= g \gamma x, h \gamma x, pp, \quad \text{using Definition 4.3},
\]
\[
= gy, hy, pp, \quad \text{by substituting } \gamma x \text{ with } y,
\]
\[
= f_c y, pp. \tag{7}
\]

Hence, $\gamma \xi_{x_0}, p, t$ also satisfies equation (5) and thus a valid solution of the system. Therefore, $\gamma$ is a symmetry per Definition 3.1. Moreover, $y$ is a solution starting from $\gamma x_0$ in mode $pp$. This proof is similar to that of Theorem 10 in [19] with is the difference of having a controller $h$, which requires the additional assumption that $h$ is symmetric. $\square$

In Section 5, we discuss how to make non-symmetric controllers symmetric, and apply that to the NN-controller of the quadrotor to make it rotation symmetric.

4 REFINEMENT OF SYMMETRY-BASED ABSTRACTION OF HYBRID AUTOMATA

In this section, we present our refinement algorithm SymAR of symmetry abstractions.

4.1 Abstraction definition

The abstraction in [20] requires a set of symmetry maps which they call the virtual map, defined as follows:

**Definition 4.1 (Definition 4 in [20]).** Given a hybrid automaton $\mathcal{A}$, a virtual map is a set
\[
\Phi = \{ \gamma_p, \rho_p \}_{p \in \mathcal{P}}. \tag{8}
\]

where for every $p \in \mathcal{P}$, $\gamma_p : \mathcal{X} \rightarrow \mathcal{X}$, $\rho_p : \mathcal{P} \rightarrow \mathcal{P}$, and they satisfy equation (4).

Given a virtual map $\Phi$, they define a new map $r_v : \mathcal{P} \rightarrow \mathcal{P}$ as follows: $r_v p = \rho_p p$. It maps modes of the original automaton $\mathcal{A}$ to their corresponding ones in the abstract one $\mathcal{A}'$, defined next.

The idea of the abstraction is to group modes of $\mathcal{A}$ that share similar behavior in symmetry terms together in the same mode in the abstract automaton. The trajectories of such modes can be obtained by transforming the trajectories of the corresponding abstract mode using symmetry maps.

**Definition 4.2 (Definition 5 in [20]).** Given a hybrid automaton $\mathcal{A}$, and a virtual map $\Phi$, the resulting abstract (virtual) hybrid automaton is:
\[
\mathcal{A}' = (\mathcal{X}', \mathcal{P}', \Theta_1, \Theta_2, \mathcal{E}, \gamma_{\mathcal{E}} \circ \gamma_{\mathcal{R}} \circ \gamma_{\Phi}) (\gamma_{\mathcal{R}} \circ \gamma_{\Phi}) (\gamma_{\mathcal{R}} \circ \gamma_{\Phi}),
\]

where (a) $\Phi = \{ \gamma_p, \rho_p \}_{p \in \mathcal{P}}$, (b) $r_v p = \rho_p p$, (c) $\mathcal{E}_v = r_v \mathcal{E}$, (d) $\forall e_v \in \mathcal{E}_v$, $\gamma_{\mathcal{E}} e_v = \gamma_{\mathcal{E}} \circ \gamma_{\Phi} \circ \gamma_{\mathcal{R}} (\gamma_{\mathcal{E}} \circ \gamma_{\Phi} \circ \gamma_{\mathcal{R}})$.

4.2 Quadrotor abstract automaton

An example virtual map of the quadrotor in Section 2.2 would be the one that has for any mode $r$, i.e. waypoint segment, $p_r$, which maps it to the mode $r$, that is of same length as $r$ but aligned with the $y$-axis and has its end point at the origin. Additionally, it would have $\gamma_r$ being the map that transforms the state to the new coordinate system of the physical space. Such a map would result in an abstract automaton where all segments of $\mathcal{R}$ having the same length and same relative $z$-distance between their end points being mapped to the same abstract mode. For more elaborate construction of similar abstractions, the reader is referred to [20]. In this section, we assumed that the controller is symmetric, which is not, and we will tackle this issue in Section 5.

4.3 Forward simulation relations

For an automaton $\mathcal{A}_1$ to be an abstraction of another one $\mathcal{A}_2$, a forward simulation relation (FSR) is usually defined to show that for any execution of $\mathcal{A}_2$, there is a corresponding one of $\mathcal{A}_1$ that represents it. The following is a formal definition of FSR.

**Definition 4.3 (1151).** A forward simulation relation from hybrid automaton $\mathcal{A}_1$ to another one $\mathcal{A}_2$, is a relation $\mathcal{R} \subseteq \mathcal{X}_1 \times \mathcal{P}_1 \times \mathcal{X}_2 \times \mathcal{P}_2$, such that

(a) for any initial state $x_1 \in \Theta_1$, there exists a state $x_2 \in \Theta_2$, such that $x_1, p_{init} \in \mathcal{R} x_2, p_{init,2}$.
Then, $R$ as input the mode to be split as well. The converse is not necessarily true since previous theorem. Section 4.6. We define for every $\sigma_1$, and $p_1, p_2, p' \in P_2 \times P_2$ such that $x_2, p_2 \rightarrow x'_2, p'_2$ is a discrete transition of $\sigma_2$ and $x'_1, p'_1 \Rightarrow x'_2, p'_2$, and

c for any solution $x_1, p_1 \rightarrow x'_1, p'_1$, of $\sigma_1$ and pair $x_2, p_2, p' \in P_2 \times P_2$, such that $x_1, p_1 \Rightarrow x'_2, p'_2$, there exists a solution $x_2, p_2, p' \in P_2 \times P_2$, with $dur x_1 = dur x_2$ and $\xi, lstate, p_1, P_2, lstate, p_2$.

The implication of the existence of a FSR on the relation between the trajectories of the two automata is formalized in the following theorem from [20], which is a modified version of Corollary 4.23 of [15] for the hybrid automaton of Definition 2.1.

**Theorem 4.4 ([15]).** If there exists a forward simulation relation $R \rightarrow \sigma_1 \rightarrow \sigma_2$, then for every execution $\sigma_1$, there exists a corresponding execution $\sigma_2$ of $\sigma_2$ such that

(a) $\sigma_1\.len = \sigma_2\.len$.
(b) $\forall i \in \sigma_1\.len, dur x_{2,i} = dur x_{2,i}$, and
(c) $\forall i \in \sigma_1\.len, \xi, lstate, p_1, \xi, lstate, p_2, i$.

If there exists a FSR $R \rightarrow \sigma_1 \rightarrow \sigma_2$, we say that $\sigma_1 \leq_R \sigma_2$. A FSR has been suggested in [20] for the automaton $\sigma$ of Definition 2.1 to $\sigma$, of Definition 4.2.

**Theorem 4.5 (Theorem 3 in [20]).** Consider the relation $R_{\sigma_1} \subseteq \sigma_1 \times \sigma \times \sigma_1 \times \sigma_1$, defined as $x, p \sigma_1, x, p \sigma_1$, if and only if:

(a) $x_1 = x_2$, and
(b) $p_1 = p_2$.

Then, $R_{\sigma_1}$ is a forward simulation relation from $\sigma_1$ to $\sigma_2$.

### 4.4 Safety verification of the abstract automaton

In this paper, we map the unsafe map $O$ of $\sigma_0$ to the abstract automaton $\sigma_0'$. It will be used later in the safety verification algorithm in Section 4.6. We define for every $p_1 \in P$,

$$O_0, p_0 := \cup_{p_0 = rvp} O_0.$$  (9)

The following theorem on the relation between the safety of $\sigma$ with respect to $O$ and $\sigma_0'$ with respect to $O_0$ follows from the FSR in the previous theorem.

**Theorem 4.6.** Fix any $p \in P$ and let $p_0 = rvp$. If Reach$_{\sigma, p} \cap O \neq \emptyset$, then Reach$_{\sigma, p} \cap O_0 \neq \emptyset$.

The theorem above shows that if $\sigma$ is unsafe, then $\sigma_0'$ is unsafe as well. The converse is not necessarily true since $\sigma_0'$ might have executions that do not correspond to ones of $\sigma'$.

### 4.5 Abstraction refinement by splitting modes

In this section, we describe splitMode, the algorithm used to split an abstract mode $p_1^*$ and update the abstract automaton $\sigma_0'$. Then, we prove that its result $\sigma_0'$ is an abstraction of $\sigma_0'$ and that $\sigma_0'$ is an abstraction of $\sigma_0'$, using two forward simulation relations.

#### 4.5.1 splitMode description

The procedure splitMode takes as input the mode to be split $p_1^*$, the map $rv$ that maps original modes to abstract ones, the original and abstract automata $\sigma$ and $\sigma_0'$, and the original and abstract unsafe maps $O$ and $O_0$. It outputs the two resulting modes $p_1$ and $p_2$ of the splitting of $p_1^*$, the updated map $rv$, updated abstract automaton $\sigma_0'$, the updated abstract unsafe map $O_0'$, and a boolean indicating if the splitting was successful.

It starts by obtaining the set of original modes $P$ of $\sigma$ represented by $p_1^*$ in line 2. If the number of these modes is less than two, then the algorithm returns the given input along with a False flag to indicate failure to split in line 3. Otherwise, it initializes the output variables $rv'$, $\sigma_0'$, and $O_0'$ by copying $rv$, $\sigma_0'$, and $O_0$. It then creates two new abstract modes $p_{1,1}$ and $p_{1,2}$ and adds them to $P_1$ in line 5. Then, it decomposes $P$ into two disjoint sets $P_1$ and $P_2$ in line 6. After that, it updates the map $rv$ to map modes in $P_1$ to $p_{1,1}$ and those of $P_2$ to $p_{1,2}$ in line 7. It then updates the initial mode of $\sigma_0'$ in case the split mode was the root.

Now that it created the new modes, it proceeds into updating the edges of $\sigma_0'$ and their guard, and reset, annotations. It iterates over the edges that connect $p_1^*$ with its parents, which may include $p_1^*$ itself, and create for each such edge, two edges connecting that parent with both $p_{1,1}$ and $p_{1,2}$ in line 9. It repeats the same process but for the edges that connect $p_1^*$ with its children in line 11. Finally, it checks if $p_1^*$ had an edge connecting it to itself, and if that is the case, creates two edges connecting each of $p_{1,1}$ and $p_{1,2}$ to themselves.

In line 15, it annotates the created edges with their guards and resets in the same way Definition 4.2 defined them, but using the update map $rv'$. In lines 17 and 18, it sets the dynamic function of both modes to be the same as that of $p_1^*$. Finally, it deletes $p_1^*$ with all edges connected to it in line 19. It then initializes the unsafe maps for the newly created modes by decomposing the unsafe set of $p_1^*$ into those of the two modes. It returns in line 22 the new modes, new $rv'$ and $\sigma_0'$, the unsafe map $O_0'$, and the flag True to indicate successful splitting.

**Algorithm 1 splitMode**

1. **input**: $p_1^*$, $rv$, $\sigma$, $\sigma_0'$, $O$, $O_0$
2. $p_1^* := rv^{-1}p_1^*$
3. if $|P| < 2$ then return: $\bot, \bot, rv, \sigma_0'$, $O_0'$, False.
4. Create copies of $rv$, $\sigma_0'$, and $O_0'$, and name them $rv'$, $\sigma_0'$, and $O_0'$.
5. Create two new virtual modes $p_{1,1}$ and $p_{1,2}$ and add them to $P_1$.
6. Split $P$ in half to two sets $P_1$ and $P_2$.
7. $rv P_1 := p_{1,1}$, $rv P_2 := p_{1,2}$.
8. $p_1^* := rv P_{\text{out}}$
9. for $e_1 \in E_1$ such that $e_1.dest = p_{1,1}$ do
10. Create two new edges $e_{1,src} p_{1,1}$ and $e_{1,src} p_{1,2}$.
11. for $e_1 \in E_1$ such that $e_1.src = p_{1,1}$ do
12. Create two new edges $e_{1,src} p_{1,1}$ and $e_{1,src} p_{1,2}$.
13. if $\exists e_1 \in E_1$ such that $e_1,src = p_{1,1}$ and $e_1,dest = p_{1,2}$ then
14. Create two new edges $p_{1,1} e_{1,src} p_{1,1}$ and $p_{1,2} e_{1,src} p_{1,2}$.
15. Define the guards and resets of new edges using the virtual map $rv'$.
16. Remove added edges that have empty guards.
17. Set $f_{1,1} := p_{1,1}$.
18. Set $f_{1,2} := p_{1,2}$.
19. Remove $p_1^*$ from $P_1' \text{ and } O_1'$, and remove all attached edges from $E_1'$.
20. $O, p_1 := \cup_{p \in P} O, p$
21. $O, p_2 := \cup_{p \in P_1} O, p$
22. return: $p_{1,1}, p_{1,2}, rv', \sigma_0', O_0'$, True

### 4.5.2 Correctness guarantees of splitMode

In this section, we show that the resulting automaton $\sigma_0'$ from splitMode is still a valid abstraction of $\sigma_0'$, but it is a tighter one than $\sigma_0'$ by showing that $\sigma_0'$ is an abstraction of $\sigma_0'$. |
Consider $R_{rv}^\prime$, the same relation as $R_{rv}$ defined in Theorem 4.5, but using $rv'$ instead of $rv$. Formally, $R_{rv}^\prime \subseteq X \times P \times X \times P$ defined as $x, p \in R_{rv}^\prime$, $x'$, $p'$. If and only if:

(a) $x' = x \gamma_3 x$ and 
(b) $p' = rv' p$

Let us refer to $R_{rv}^\prime$ by $\mathcal{R}_1$ and let $R_2 \subseteq P \times P \times P \times P$ be defined as: $x, p, x', p'$. If and only if:

(a) $x = x'$, and 
(b) 
$$p = \{ p', \text{ if } p' \notin \{p_{v,1}, p_{v,2}\} \}, \quad (10)$$

The following theorem shows that these two relations are forward simulation relations between $\mathcal{A}$ and $\mathcal{A}'$, $\mathcal{A}'_1$, and $\mathcal{A}'_2$, respectively.

**Theorem 4.7.** Fix any abstract mode $p^v_1 \in \mathcal{A}_1$ of $\mathcal{A}_1$ and let $rv'$. $\mathcal{A}'_1 = \text{splitMode}_{rv}^*, rv, \mathcal{A}_1$. Then, the resulting relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are FSRs from $\mathcal{A}$ to $\mathcal{A}'_1$ and $\mathcal{A}'_2$ to $\mathcal{A}'_2$, respectively, and $\mathcal{A}'_1 \subseteq \mathcal{A}_1$.

**Proof.** Let us prove the first half first: that $\mathcal{R}_1$ is a FSR from $\mathcal{A}$ to $\mathcal{A}'_1$. We do that by showing that $\mathcal{A}'_1$ is the result of following Definition 4.2 to create an abstraction of $\mathcal{A}$ using a slightly modified version $\Phi'$ of the virtual map $\Phi$, where $\Phi'$ itself is another virtual map for $\mathcal{A}$.

By definition, $\forall p \in \mathcal{A}$, $rv p = rv p$, where $\forall p, p \in \Phi$. Let $\Phi'$ be equal to $\Phi$ for all $p \notin P$, where $P$ is as in line 2 of splitMode. For any $p \in P$, let $p' \in P$, if $p \in P$, and $p' \in P$, otherwise. Moreover, as in lines 17 and 18, define the continuous dynamics $f_r'$ to be equal to $f_r$ for all $p \in \mathcal{A} \setminus \{ p_{v,1}, p_{v,2} \}$, and to be equal to $f_r$, otherwise. Then, $\Phi'$ is a virtual map of $f_r'$ since any $\gamma_3, p' \in \Phi'$ satisfies equation (4) for $f_r'$, because the corresponding $\gamma_3, p' \in \Phi$ satisfies it for $f_r$. The map $rv'$ is just the result of $\Phi'$ as $rv$ is the result of $\Phi$.

The edges created in $\text{splitMode}$ for $p_{v,1}$ and $p_{v,2}$ in $\mathcal{A}'_1$ are a decomposition of the edges connected to $p^v_1$ in $\mathcal{A}$, including self edges. Hence, the output of $\text{splitMode}$ $\mathcal{A}'_1$ is indeed the result of following Definition 4.2 to construct an abstraction of $\mathcal{A}$ using $rv'$. It follows from Theorem 4.5, that $\mathcal{A}'_1$ is an abstraction of $\mathcal{A}$ and $\mathcal{R}_1 = \mathcal{R}_{rv}^\prime$ is a corresponding FSR.

Now we prove the second half of the theorem: that $\mathcal{R}_2$ is a FSR from $\mathcal{A}'_1$ to $\mathcal{A}'_2$. We follow similar steps of the proof of the first half in defining a new map, which we name $\Phi_2$, and prove that it is a virtual map of $\mathcal{A}'_1$. Let $\Phi_2 = \{ \gamma_2, p' \}$, where $\gamma_2, x' = x'$, is the identity map and $\gamma_3, p' = p'$. If $p \notin \{p_{v,1}, p_{v,2}\}$, and $\gamma_3, p' = p'$. Otherwise, because of lines 17 and 18, $\gamma_3, p' \gamma_3, p'$ satisfy equation (4) with the RHS dynamic function being $f_r'$. Finally, notice that $\mathcal{A}_2$ can be retrieved from $\mathcal{A}'_1$ by following Definition 4.2 using $\Phi_2$. It follows from Theorem 4.5, that $\mathcal{R}_2$ is a FSR from $\mathcal{A}'_1$ to $\mathcal{A}_2$. Thus, $\mathcal{A}_2 \subseteq \mathcal{A}_1$, $\mathcal{A}'_1 \subseteq \mathcal{A}_2$, $\mathcal{A}'_2$.

**Remark 1.** If in $\text{splitMode}$, instead of removing $p^v_1$ from $\mathcal{A}$, with all of its connected edges, $p^v_1$ is kept with only its outgoing edges, it will not be visited by any execution of $\mathcal{A}'_1$, and the correctness guarantees of $\text{splitMode}$ would not be altered. This version will be used in the safety verification algorithm in Section 4.6. If $p^v_1$ has a self edge $e$, we delete it and create two outgoing edges from $p^v_1$ to $p_{v,1}$ and $p_{v,2}$, with guards and resets being the parts of guard and reset of $e$, that correspond to modes in $P_1$ and $P_2$.

**4.5.3 Usefulness of splitMode in safety verification.** We discuss now the benefits of splitting a mode $p^v_1$, where $\text{Reach}_{\mathcal{A}} p^v_1 \in \mathcal{O}_p \neq \emptyset$. The non-empty intersection with the unsafe set can mean either that:

1. genuine counterexample: $\exists p \in \mathcal{A}$ such that $\text{Reach}_{\mathcal{A}} p \in \mathcal{O}_p$, and thus $\mathcal{A}$ is unsafe, or
2. spurious counterexample: there exists an execution of $\mathcal{A}$ that does not correspond to one of $\mathcal{A}$ that is intersecting $\mathcal{O}_p$, and thus the intersection is a result of the abstraction, and not a correct counterexample.

Spurious counter examples could happen because of the guards and resets of the edges incoming to $p^v_1$ being too large that the initial set of states for that mode is being larger than it should. Remember from Definition 4.2, that the guard and reset of any of these edges $e$, is the union of all the transformed guards and resets of the edges in $E$ that get mapped to $e$. If too many of the original edges are mapped to $e$, its guard and reset will get larger, causing more transitions and larger initial set of $p^v_1$ in $\mathcal{A}'$. This might increase the possibility of spurious counter example. Moreover, from equation (9), $\mathcal{O}_p$ is the union of the unsafe sets of all the modes that are mapped to $p^v_1$ under $rv$. The more the original modes that get mapped to $p^v_1$, the larger is the unsafe set of $p^v_1$ and the higher is the chance of a spurious counter example.

Upon splitting $p^v_1$ into two abstract modes, the guards of the edges incoming to $p^v_1$, i.e. where $p^v_1$ is a destination, will have their guards divided between the edges to $p_{v,1}$ and $p_{v,2}$. Additionally, the unsafe sets of $p^v_1$ will be divided between $p_{v,1}$ and $p_{v,2}$. This will make the over-approximation of the behaviors of $\mathcal{A}$ by $\mathcal{A}$ get tighter and safety checking less conservative. We will discuss how he will use this in a faster, but accurate, solving of the safety verification problem of $\mathcal{A}$ in the next section.

**4.6 Abstraction refinement based safety verification algorithm.**

In [20], it has been suggested to compute the reachset of $\mathcal{A}$, of Definition 4.2, without checking safety, while caching the per-mode reachsets in a dictionary. Once a fixed point is reached, they use the symmetry maps in the virtual map $\Phi$ to transform the per-mode reachsets of $\mathcal{A}$ in the dictionary to get back the per-mode reachsets of $\mathcal{A}$. Finally, the resulting reachset of $\mathcal{A}$ is intersected with the unsafe set for safety verification. Such an algorithm lacks the flexibility in modifying the abstraction to eliminate the spurious counterexamples automatically, but rather limited by the virtual map given at hand. In this section, we describe how to use $\text{splitMode}$ to refine $\mathcal{A}$ in the process of the safety verification of $\mathcal{A}$.

Our pseudo code is shown in Algorithm 2, which we denote by SymAR. The idea of the algorithm SymAR is to compute the reachset of $\mathcal{A}$, in the usual way of doing a DFS over the mode graph and computing the reachset of each visited mode till a fixed point is reached. At any particular mode $p^v_1$ being visited, if the computed reachset of $p^v_1$, or any of the modes in the search subtree starting at $p^v_1$, is unsafe, SymAR calls $\text{splitMode}$ to split $p^v_1$ into two modes, to eliminate the possibility of spurious counterexample. If $p^v_1$ cannot be refined, SymAR would undo the computations done in
that subtree and returns that subtree is unsafe. That would lead one of $p^*_v$ ancestors to be refined, if any could, resulting in a tighter reachset so that when $p^*_v$ is visited again, less spurious counterexample would be possible.

4.6.1 SymAR description. The algorithm SymAR is a recursive function that do depth first search (DFS) over the graph of modes and edges of $\mathcal{A}_v$. It takes as input $\mathcal{A}_v$, a mode $p^*_v \in \mathcal{P}_v$, from which we need to continue the reachset computation of $\mathcal{A}_v$, the parent mode, parent, from which $p^*_v$ has been called, the reachset $R_v$ of parent that has been computed before calling $p^*_v$, a dictionary storing the per-mode reachsets of all modes in $\mathcal{A}_v$ and their corresponding initial sets, and the unsafe map $O_v$. It outputs a new abstract automaton $\mathcal{A}'_v$ that is a refined version of $\mathcal{A}_v$, a corresponding refined version of the unsafe set $O'_v$, an updated dictionary $Cache'_v$ of Cache that store the reachsets computed for all modes of $\mathcal{A}_v$, a dictionary refine that maps all modes of $\mathcal{A}_v$ to their corresponding refined modes in $\mathcal{A}'_v$, an indicator fixedpoint if a fixed point has been reached, and an indicator issafe showing if $\mathcal{A}_v$ starting from $p^*_v$ with parent parent of reachset $R_v$ while the per-mode reachsets in the computation of the reachset of $\mathcal{A}_v$ are as stored in Cache is safe with respect to the unsafe map $O_v$.

It starts with computing the initial set $K_v$ of $p^*_v$ at this visit in lines 2 to 5. If $p^*_v$ is the root, i.e., the initial mode of $\mathcal{A}_v$, then $K_v = \Omega_v$, where $\Omega_v$ is a part of $\mathcal{A}_v$ definition. Otherwise, the parent reachset $R_v$ is intersected with the guard of the edge connecting $p^*_v$'s parent to $p^*_v$, then reseted using reset, of the same edge to get $K_v$. The reachset starting from $K_v$ in mode $p^*_v$ has been computed before and saved in Cache, then no need to explore this subtree again and the flag of fixed point fixedpoint, is set to True, and SymAR returns safe in line 7. Otherwise, reachset computation should be done.

After that, it starts with initializing the refine flag to False, and the output automaton $\mathcal{A}'_v$, output unsafe map $O'_v$, and reachsets and initial sets dictionary $Cache'_v$, to the input ones $\mathcal{A}_v$, $O_v$, and $Cache$, respectively (lines 8 and 9). It initializes the dictionary refine to be the modes themselves in line 11. Then, in line 12, the reachset new$R_v$ is computed using the method computeReachset, which is a call to any of the existing tools (e.g. Verisig[14], Sherlock [8], NVN[21], DryVR [10], if the system has a NN controller, or flow* [5], C2E2 [7], DryVR [10] otherwise). If the computed reachset intersects the unsafe set, the refine is set to True and it jumps to line 31. Otherwise, both new$R_v$ and $K_v$ are added to the entry of $p^*_v$ in $Cache'$ in lines 16 and 17. Such an addition is not permanent, it is on-hold till making sure that all the subtree starting from $p^*_v$ is safe as well. The fixedpoint flag is initialized to True in line 18. If any of the children of $p^*_v$ did not reach fixed point yet, this flag will be set to False again in line 23.

At this point, SymAR iterates over all children of $p^*_v$ in $\mathcal{A}_v$, and not in $\mathcal{A}'_v$ in line 19. For each child mode, SymAR iterates over its set of refined modes in refine in line 20. Such a loop accounts for the case if that child mode has been refined by one of its siblings that has been visited before it by the outer loop. For each mode $p'^*_v$ looped over in the inner loop, SymAR is called using: the most updated version $\mathcal{A}'_v$ that includes all the refinements of the siblings, and their subtrees, done so far, along with the corresponding refined unsafe map $O'_v$, and the most updated $Cache'$ that contains all safe reachsets computed so far, with $p'^*_v$, parent being $p^*_v$ and $p^*_v$'s reachset being new$R_v$. All refinements done in that SymAR call, are added to those in refine. That means that for each mode $p'^*_v$, in $\mathcal{A}'_v$, refine,$p'^*_v$ will have all the modes that has been created by splitting $p^*_v$ and its refined modes. If the call of SymAR on $p'^*_v$ returned unsafe, the dictionary $Cache'$, along with $\mathcal{A}'_v$, $O'_v$, $p'^*_v$, and refine gets reset to the input ones in lines 26, 27, and 28, undoing all computations done since line 12, including those done by the siblings. This is to make sure that $Cache'$ only stores reachsets that are safe and lead to safe execution after $p^*_v$. After resetting, the refine flag is set to True and the loop is exited to line 31. If instead all calls to SymAR for $p^*_v$ refined children modes returned safe, SymAR would return safe along with the refined automaton, unsafe map, dictionary of refined modes, and the fixed point flag in line 49.

If refine is True at line 31, SymAR proceeds into refining $p^*_v$. In line 32, splitMode is called to split $p^*_v$. Note that as in Remark 1, we keep $p^*_v$ in $\mathcal{A}'_v$ with only its outgoing edges, only deleting the self edge and creating instead two outgoing edges with corresponding guards and resets. If the split was not successful, SymAR returns unsafe in line 34. Otherwise, it updates refine next by replacing $p^*_v$ with $p_{v,1}$ and $p_{v,2}$ in the entry of $p^*_v$ in line 35. It copies the reachset and initial sets dictionary Cache of $p^*_v$ into those of the refined modes $p_{v,1}$ and $p_{v,2}$ in lines 36 and 37. If $p^*_v$ was the root, i.e. parent = ⊥, SymAR is called with the new root mode of $\mathcal{A}'_v$ obtained after refinement using $\mathcal{A}'_v$ and it returns the result in line 39. Otherwise, it iterates over the two refined modes of $p^*_v$, and call SymAR with the same input parent parent and reachset $R_v$ as $p^*_v$. If any returned unsafe, it would return unsafe as well.

4.6.2 SymAR soundness and completeness guarantees. Given a hybrid automaton $\mathcal{A}_v$, an unsafe map $O_v$, a virtual map $\Phi$ with its corresponding map $rv$, the initial call to SymAR would be:

$$\text{SymARrv}(\mathcal{A}, \Phi, \mathcal{P}_{aut,v}, \bot, O_v, Cache, O_v),$$

where $\mathcal{A}_v$ is the result of Definition 4.2, $\mathcal{P}_{aut,v}$ is its initial mode, $O_v$ is as equation (9), and $Cache$ is a dictionary mapping every mode in $\mathcal{A}_v$ to an empty set. Two key guarantees SymAR provides: (1) it keeps the correctness of the safety verification algorithm that does not use abstraction intact, i.e. soundness and (2) it provides the same result as it is without using abstraction: if that returns safe, then it returns safe and vice versa.

It does that by keeping the invariant that the refined automata are abstractions of $\mathcal{A}_v$ and that each time an automaton is generated by refinement, it is an abstraction for the one it got generated from. More importantly, it keeps a well defined interaction between the different abstractions by making the loop in line 19 iterates over the non-refined modes and let refine handle the boundary between the two abstractions. Finally, for any mode with a reachset intersecting the unsafe set, it keeps refining that mode and its ancestors till safety can be proven or no more refinements are allowed. In the worst case scenario, $\mathcal{A}_v$ can be refined completely to become $\mathcal{A}$ again, resulting in the completeness guarantee. More formal proof will be provided in the longer version of the paper.

**Theorem 4.8 (Soundness).** If SymAR returned safe for the call in equation (11), then $\mathcal{A}_v$ is safe with respect to $O_v$.

**Theorem 4.9 (Completeness).** Fix $\mathcal{A}_v$ and $O_v$. If $\mathcal{A}_v$ is safe with respect to $O_v$, and that can be proven by SymAR if it is given
Algorithm 2 SymAR

1: input: \( r_v, \Theta_0, p_v^0, \) parent, \( R_v, \) Cache, \( O_v \)
2: if parent = \( \bot \) then \( K_v \leftarrow \Theta_v \)
3: else
4: \( e_v \leftarrow \text{parent}.p_v^0 \)
5: \( \text{grdinter} \leftarrow R_v \cap \text{guard}, \text{e}_v; \) \( K_v \leftarrow \text{reset}, \text{grdinter}, \text{e}_v \)
6: if \( K_v \subseteq \text{Cache}^p_0, \) initset then
7: return: \( \phi_v, O_v, \) Cache, \( \{ \}, \) True, safe
8: refine \( \leftarrow \text{False} \)
9: \( \phi''_v, O'_v, r_v' \leftarrow \text{copy}, \phi_v, O_v, r_v \)
10: \( \text{Cache}' \leftarrow \text{copyCache} \)
11: \( \text{refree} \leftarrow \{ p_v : \{ p_v \} \}_{p_v \in \mathcal{P}} \)
12: \( \text{newR}_v \leftarrow \text{computeReachset}, \phi_v, p_v^0 \)
13: if \( \text{newR}_v \cap O_v, \ _p_v^0 \neq \bot \) then
14: refine \( \leftarrow \text{True} \)
15: else
16: \( \text{Cache}^p_v, \) reach \( \leftarrow \text{Cache}^p_v, \) reach \( \cup \) newR_v
17: \( \text{Cache}^p_v, \) initset \( \leftarrow \text{Cache}^p_v, \) initset \( \cup K_v \)
18: fixedpoint \( \leftarrow \text{True} \)
19: for \( e_v \in E_v \), such that \( e_v, s_r = p_v^0 \) do
20: for \( p_v^{1,2,} \in \text{refreee}, \) dest do
21: \( r_v', \phi''_v, O'_v \leftarrow \text{copy}, e_v, \text{curfix}, \text{curSAFE} \)
22: \( \text{refree} \leftarrow \{ p_v : \{ p_v \} \}_{p_v \in \mathcal{P}} \)
23: if \( \text{curfix} = \text{False} \) then fixedpoint \( \leftarrow \text{False} \)
24: \( \text{refree} \leftarrow \text{refree} \cup \text{curfree} \)
25: if \( \text{curSAFE} = \text{unsafe} \) then
26: \( \text{Cache}' \leftarrow \text{copyCache} \)
27: \( \phi''_v, O'_v, r_v' \leftarrow \text{copy}, e_v, \text{curfix}, \text{curSAFE} \)
28: \( \text{refree} \leftarrow \{ p_v : \{ p_v \} \}_{p_v \in \mathcal{P}} \)
29: \( \text{refree} \leftarrow \text{True} \)
30: break (both loops)
31: if \( \text{refree} \) then
32: \( p_v^{1,2}, r_v', \phi''_v, \text{didrefine} \leftarrow \text{splitMode}^p_v, r_v', \phi''_v \)
33: if not \( \text{didrefine} \) then
34: return: \( r_v', \phi''_v, O'_v, \) Cache', \( \text{refree}, \text{False}, \text{unsafe} \)
35: \( \text{refreee} \leftarrow \{ p_v^{1,2} \} \)
36: \( \text{Cache}' \leftarrow \text{copyCache}^p_v \)
37: \( \text{Cache}' \leftarrow \text{copyCache}^p_v \)
38: if parent = \( \bot \) then
39: return: \( \text{SymAR}, r_v', \phi''_v, r_v', \text{parent}, \bot, \emptyset, \text{Cache}' , O'_v \)
40: else
41: fixedpoint \( \leftarrow \text{True} \)
42: for \( p_v^{1,2} \in \text{refreee} \) do
43: \( \phi''_v, O'_v, \) Cache', \( \text{curfix}, \text{curSAFE} \)
44: \( \leftarrow \text{SymAR}, r_v', \phi''_v, p_v^{1,2}, \text{parent}, R_v, \text{Cache}' , O'_v \)
45: if \( \text{curfix} = \text{False} \) then fixedpoint \( \leftarrow \text{False} \)
46: \( \text{refree} \leftarrow \text{refree} \cup \text{curfree} \)
47: if \( \text{curSAFE} = \text{unsafe} \) then
48: return: \( r_v', \phi''_v, O'_v, \) Cache', \( \text{refree}, \text{False}, \text{unsafe} \)
49: return: \( r_v', \phi''_v, O'_v, \) Cache', \( \text{refree}, \text{fixedpoint}, \text{safe} \)

\( \mathcal{A} \) instead of \( \mathcal{A} \), i.e. no abstraction used and no refinements, then
SymAR will output safe for the call in equation (11).

5 SYMMETRY WITH NON-SYMMETRIC CONTROLLERS

We assumed so far that in the case of a control system, the controller \( h \) is symmetric. In some cases, \( h \) might not be symmetric. For example, the NN controller of the quadrotor in Section 2.2 is not symmetric with respect to rotations in the xy-plane. We show a counter example in Figure 2a.

The NN input is the relative state \( q = w \), which as we mentioned before, makes it symmetric to translations of the state and the reference trajectory. But, if we rotate the coordinate system of the physical xy-plane, i.e. rotate \( p_v^1, p_v^2, q_v^1, q_v^2, b_x, b_y \) in equation (3), there is no guarantee that the NN will change its outputs \( \Theta \) and \( \phi \), such that the RHS of \( q_v^1, q_v^2, g\tan \theta \) and \( -g\tan \phi \), are rotated accordingly.

Such non-symmetric controllers will prevent the dynamics from being equivariant. Equivariance is a desirable, and expected, property of certain dynamical systems. For example, vehicles dynamics are expected to be translation and rotation invariant in the xy-plane. Thus, non-symmetric controllers violate intuition about systems dynamics. Such controllers may not be feasible to abstract using Definition 4.2.

Next, we will suggest a way to make any controller, including NN ones, such as that of the quadrotor, symmetric, leading to better controllers and retrieving the ability to construct abstractions.

Consider again the closed loop control system of equation (5). Let \( \Phi = \{ \upsilon_{p', P} \}_{p' \in \mathcal{P}} \) be a set of maps that we want it to be a virtual map of system (5). As before let us define \( rv: \mathcal{P} \rightarrow \mathcal{P} \) by \( rvp = p_0p \), for all \( p \in \mathcal{P} \). Assume that for every \( p \in \mathcal{P} \), there exists \( \beta_{p} : U \rightarrow U \), such that the open loop dynamic function \( g \) is symmetric in the sense that it satisfies equation (6). Moreover, assume that \( \gamma_{rvp}, \beta_{rvp}, \) and \( \rho_{rvp} \) are identity maps for any \( p \in \mathcal{P} \). This assumption means that applying the same symmetry transformation twice would not change the state nor the mode. Now, let us define a new controller \( h' : \mathcal{P} \times \mathcal{P} \rightarrow U \), that is shown in Figure 2b, as follows:

\[
h'x, p = \beta^{-1}_p \gamma_{p,x} rvp. \quad (12)
\]

**Theorem 5.1.** For any \( p \in \mathcal{P} \), the controller \( h' \) is \( \beta_p, \gamma_p, rv \)-symmetric.
We implemented the symmetry abstraction refinement based safety verification method SymAR of Section 4.6 in Python3. We used DryVR [10] as a segment reachset computation tool in line 12 of SymAR. We ran several experiments on different complex 2D and 3D scenarios having the quadrotor of Section 2.2 and a simple car with NN controllers to demonstrate the effectiveness of our approach in accelerating their safety verification.

6.1 Implementation details

The inputs to our implementation are scenarios and dynamics files defining the hybrid automaton $A$ along with the unsafe map $O$. Its output is the boolean $issafe$ indicating the safety verification result of the scenario, the refined abstract automaton $A_r$, the corresponding unsafe map $O_r$, and the reachset of $A_r$.

The scenario file specifies: the initial set of states $Θ_r$, the planner output represented as a list of waypoints along with a list of edges defining the plan graph, a list of time horizons specifying the maximum time spent following any segment in the plan graph, and a list of boxes specifying the area around each waypoint at which it is considered reached, and a list of unsafe sets representing the obstacles in the environment that the agent is supposed to avoid. For quadrotor scenarios, we chose the time horizon for any edge to be $T = 5 \times seg\_length$ and for the car, to be $T = seg\_length$.

The dynamics files specify the simulation function that simulates the agent forward from a specific initial state following a certain edge in the plan graph, the virtual map $Φ$ as a list of three functions that implement $γ_p$, $γ_p^{-1}$, and $rv$. In case the controller is not symmetric as in the case of the quadrotor, $β_p$ and $β_p^{-1}$ are also provided to make the NN controller symmetric following Section 5.

Our implementation constructs an automaton $A$, as in Section 2.2 for the plan graph that is produced by the planner. The set of modes $P$ considered is the set of segments of the plan, and the set of edges $E$ represents the connected segments in the plan. The guards are constructed from the provided boxes around the waypoints and the resets are considered to be the identity function. It then uses the provided symmetry maps to construct the abstract automaton $A$.

6.2 Scenarios and metrics

In addition to the quadrotor of Section 2.2 that we borrowed from [14], we trained a NN controller, a symmetric one, for a simple car with dynamics: $x = v \cos θ$; $y = v \sin θ$; $θ = δ$. The details of the training and the controller are in the appendix.

We tested the car on three 2D complex plans in different environments with different obstacles, initial sets, goal sets, and paths. Those scenarios are in addition to the city like scenario in Figure 1. We tested the quadrotor on the same 2D environments as the car and three other 3D scenarios. All of the plans are in the appendix.

We experimented with two different virtual maps: (a) translation symmetry (SymT), where for any given mode, where a mode is a segment in the plan, the symmetry maps the states and the segment to the coordinate system centered at the end waypoint of the segment, and (b) rotation and translation symmetry (SymTR), where instead of just translation of the origin of the coordinate system, we rotate the $xy$-plane so that the segment is aligned with the $y$-axis.

Finally, and most importantly, we experiment with different refinement thresholds, where we allow SymAR to refine only a limited number of times to check the benefits of having refinement when using symmetry based abstractions for safety verification, especially when the automaton under consideration has complex dynamics because of a NN controller.

Next section, we compare different experiments according to the following statistics: time (min), being the total verification time, $# rd$, is the number of refinements done, $# rt$, is the number of refinements threshold, $|P|$, $|E|$, $|P_r|$, $|E_r|$, $# co$, is the number of calls to the reachtube computation engine DryVR, $# tr$, is the number of reachsets that had to be transformed from $\mathcal{A}$ to construct the reachset $A_r$, and $issafe$, to indicate the safety verification result.

The different colors used in the figures correspond to the reachsets of different abstract modes. That is the reason why in the NoS case, the whole reachset has the same color, while the number of colors increase with number of refinements as in Figures 1 and 3. The corresponding abstract reachsets for both figures are in the appendix.

6.3 Results and analysis

To have a comprehensive look at our experimental results, please check tables 5 and 6 in the appendix. Here, we select few results to highlight few observations regarding the benefits of refinement and the choice of virtual map used to construct $A_r$.

**Choice of refine thresholds.** To show the effect of refinements on the ability of the usage of abstractions to accelerate safety verification, we show the results of running our implementation on two scenarios using SymTR and different refine thresholds in Table 1.
Table 1: Comparison between different refine thresholds.

| Agent Quadrotor | quadrotor-comp2D-1 | quadrotor-comp2D-2 |
|-----------------|---------------------|---------------------|
| stats           | SymTR               | NoS                 | SymTR               | NoS                 |
| time            | 1.47                | 9.21                | 6.01                | 37.12               | 3.49                | 26.83               | 25.29               | 25.29               | 248.16              |
| $|P|$            | 140                 | 140                 | 140                 | 48                  | 458                 | 458                 | 458                 | 457                 | 457                 | 264                 |
| $|E|$            | 139                 | 139                 | 139                 | 47                  | 457                 | 457                 | 457                 | 457                 | 457                 | 263                 |
| # co            | 1                   | 12                  | 10                  | 96                  | 1                   | 11                  | 16                  | 16                  | 16                  | 264                 |
| issafe          | False               | False               | True                | True                | False               | False               | True                | True                | True                | True                |
| # rt            | 0                   | 2                   | 10                  | NA                  | 0                   | 2                   | 10                  | NA                  | 10                  | NA                  |
| # rd            | 0                   | 2                   | 3                   | NA                  | 0                   | 2                   | 5                   | NA                  | 3                   | NA                  |
| $|P_r|$          | 1                   | 3                   | 5                   | NA                  | 1                   | 3                   | 6                   | NA                  | 1                   | NA                  |
| $|E_r|$          | 19                  | 6                   | 12                  | NA                  | 1                   | 6                   | 16                  | NA                  | 1                   | NA                  |
| # tr            | 140                 | 420                 | 175                 | NA                  | 458                 | 1145                | 831                 | NA                  | 458                 | 1145                | 831                 |

NoS vs SymT vs SymTR. The result of running our implementation on two different scenarios with NoS, SymT, SymTR is shown in Table 2. For the presented results, we can see that with symmetry, our implementation is significantly faster than the NoS case. With SymT, for the comp2D-1 scenario, the running time reduced by 87.9% and for comp2D-2 scenario, the running time reduced by 39.0%. With SymTR, for the comp2D-1 scenario, the running time reduced by 83.8% and for comp2D-2 scenario, the running time reduced by 89.8%. In addition, we can see that the number of calls to the reachset computer while using symmetry decreased comparing with not using symmetry. When comparing between using SymT and SymTR, we observed that for comp2D-1 scenario, using SymTR is slower then using SymT. Since with both translation and rotation symmetry, the computed reachset can be more conservative and therefore, may require a higher number of refinements as the example illustrates. Thus, for the comp2D-1 scenario, our tool has to make more calls to DryVR while using SymTR compared with using SymT.

However, for the comp2D-2 scenario, using SymTR is actually faster then using SymT. Although, using SymT still requires less number of refinement compared with using SymTR. Less number of tubes are actually computed and transformed. Therefore, causing the tool to run faster with SymTR.

Table 2: NoS vs SymT vs SymTR.

| stats \ comp2D-1 | 1 | 2 | 1 | 2 | 1 | 2 |
|------------------|---|---|---|---|---|---|
| time             | 6.01| 25.29| 4.50| 151.26| 19.25| 248.16|
| $|P|$            | 140| 458| 140| 458| 48| 264|
| $|E|$            | 139| 457| 139| 457| 47| 263|
| # co            | 10| 16| 8| 27| 48| 264|
| issafe          | True| True| True| True| True| True|
| # rt            | 10| 10| 10| 10| NA| NA|
| # rd            | 3| 5| 0| 3| NA| NA|
| $|P_r|$          | 5| 6| 7| 10| NA| NA|
| $|E_r|$          | 175| 831| 344| 3825| NA| NA|

7 CONCLUSIONS

We presented an algorithm to refine symmetry-based abstractions in the context of safety verification. We showed how the algorithm selects abstract modes to split based on their role in causing a safety violation. Moreover, we extended this type of abstractions to closed-loop control systems, showing the properties that the controller should satisfy for the abstraction to be feasible. We further showed how to append non-symmetric controllers such as NN with symmetry transformations of their inputs and outputs so that symmetry-abstractions would be possible. Finally, we showed significant savings in safety verification times of NN-controlled quadrotor and car following complex paths in 2D and 3D environments.
A ADDITIONAL EXPERIMENTAL RESULTS

A.1 Car NN controller

The controller is trained to drive the car to follow a straight reference trajectory aligned with positive y-direction towards the origin. The controller follows method explained in Section 5 and any given reference segment to be aligned with the y-axis and compute corresponding control inputs.

The inputs to the controller are the relative position of the car in the plane with respect to the reference position in the segment being followed and the cosine and sine of the relative orientation of the car with respect to the orientation of the segment being followed. The outputs from the controller are the speed v of the vehicle and the steering angle δ of the vehicle. The neural network have 1 hidden layer with 100 hidden neurons.

The training data for this controller is obtained by random sampling positions in range \( x \in [-3, 3] \) and \( y \in [-2.5, 1.5] \) and orientation in range \( \theta \in [0, \pi] \). During the training process, the vehicle will start from the sampled position and orientation and drive towards the origin \( [x, y, \theta] = [0, 0, \frac{\pi}{2}] \) which is the reference state.

A.2 Long segments vs Short segments without using symmetry

To further utilize the symmetry property of the scenario, we will split mode that contains longer segments into several modes that contains shorter segments with equal length. In table 3 shows the statistics for verifying the comp2D-1 scenario for quadrotor without using symmetry. For experiment run comp2D-1-1, modes that contain segments with length larger than 3 is split into mode that contain segments with length smaller than or equal to 3. For experiment run comp2D-1, the modes are not split. The experiment is to compare the effect of splitting longer segments into short segments while not using symmetry.

From the collected data, we can see that although number of modes and number calls to reachset computer increase, the amount of time it takes for verification of the scenarios is not influenced. Therefore, for the rest of the experiments, while not using symmetry, we are verifying scenarios without splitting modes.

Table 3: Results showing that without using symmetry, splitting modes with longer segments to modes with shorter ones won’t influence performance.

| Experiment run | complex2D-1 | complex2D-1-S |
|----------------|-------------|---------------|
| # original modes | 48          | 140           |
| # virtual modes | 47          | 139           |
| # computed tubes | 96          | 280           |
| # segs computed | 48          | 140           |
| time            | 37.12       | 36.45         |

A.3 Effect of model dynamics

The stats in table 4 compare the influence of agent dynamics have on the performance of verification. In this experiment, we are trying...
to verify the comp2D-1 scenario with different agents and compare the performance. For both agents, we will run verification with both no symmetry and symmetry with translation and rotation. While running with SymTR, we will also try different refinement thresholds.

Table 4: Comparison between different agent dynamics.

| stats    | car-comp2D-1 | quadrotor-comp2D-1 |
|----------|--------------|--------------------|
|          | SymTR        | NoS                | SymTR        | NoS                |
| time     | 4.58         | 21.23              | 44.66        | 1.47               | 9.21              | 6.01             | 37.12           |
| |P|     | 140           | 140               | 140           | 140               | 140             | 140             | 48              |
| |E|     | 139           | 139               | 139           | 139               | 139             | 139             | 47              |
| # co     | 6            | 84                | 135           | 96                | 1                | 12               | 10              | 96              |
| issafe   | F            | F                 | T             | T                 | F                | F               | T               | T               |
| # rt     | 0            | 10                | 50           | NA                | 0                | 2                | 10              | NA              |
| # rd     | 0            | 10                | 17           | NA                | 0                | 2                | 3               | NA              |
| |P_v|    | 1             | 11               | 18           | NA                | 1                | 3                | 5               | NA              |
| |E_v|    | 1             | 29               | 47           | NA                | 19               | 6                | 12              | NA              |
| # tr     | 840          | 761               | 1132         | NA                | 140              | 420              | 175             | NA              |

From the statistics, we can see that without symmetry, the amount of time required for verify the scenario with car or quadrotor is relatively close. However, with both translation and rotation symmetry, since the quadrotor have more stable controller and more symmetric dynamics, the number of refinements required for verifying the quadrotor scenario is much less then the number of refinements required for verifying the car scenario. In addition, since the dynamics of quadrotor is more symmetric than the car, it is able to reach the fixed point earlier than the car. Therefore, it takes much less time to verify the scenario with quadrotor than verify the scenario with the car.
Table 5: Results for quadrotor.

| tool \ agent model | Quadrotor |
|--------------------|-----------|
|                    | comp2D-1  | comp2D-2  | comp2D-3  | comp3D-1  | comp3D-2  |
| SymTR              |           |           |           |           |           |
| # rt               | 0         | 2         | 10        | 0         | 2         | 10        | 0         | 1         | 10        | 0         |
| # rd               | 0         | 2         | 3         | 0         | 2         | 5         | 0         | 2         | 4         | 0         | 1         | 2         | 0         |
| time               | 1.47      | 9.21      | 6.01      | 3.49      | 26.83     | 25.29     | 8.16      | 19.47     | 25.114    | 11.66     | 38.30     | 41.34     | 24.75     |
| # rd               | 0         | 2         | 3         | 0         | 2         | 5         | 0         | 2         | 4         | 0         | 1         | 2         | 0         |
| [P]                | 140       | 140       | 140       | 458       | 458       | 458       | 520       | 520       | 520       | 96        | 96        | 96        | 188       |
| [E]                | 139       | 139       | 139       | 457       | 457       | 457       | 519       | 519       | 519       | 95        | 95        | 95        | 187       |
| [P]                | 1         | 3         | 5         | 1         | 3         | 6         | 1         | 3         | 5         | 4         | 5         | 6         | 3         |
| [E]                | 19        | 6         | 12        | 1         | 6         | 16        | 1         | 6         | 13        | 9         | 17        | 24        | 7         |
| # co               | 1         | 12        | 10        | 1         | 11        | 16        | 1         | 6         | 13        | 9         | 17        | 24        | 7         |
| # tr               | 140       | 140       | 140       | 458       | 458       | 458       | 520       | 520       | 520       | 96        | 96        | 96        | 188       |
| # segs computed    | 1         | 5         | 9         | 1         | 5         | 11        | 1         | -5        | 9         | 4         | 8         | 12        | 3         |
| # tubes per segs   | 1         | 2         | 0         | 1         | 2         | 0         | 2         | 4         | 0         | 4         | 5         | 6         | 3         |
| issafe             | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      |
| SymT               |           |           |           |           |           |           |           |           |           |           |           |           |           |
| # rt               | 0         | 2         | 10        | 0         | 2         | 10        | 0         | 2         | 10        | 0         | 1         | 10        | 0         |
| # rd               | 0         | 0         | 0         | 0         | 2         | 3         | 0         | 0         | 2         | 0         | 0         | 0         | 0         |
| time               | 7.19      | 7.49      | 4.50      | 34.96     | 86.76     | 151.26    | 23.43     | 24.02     | 59.01     | 117.51    | 117.94    | 116.74    | 99.17     |
| # rd               | 0         | 0         | 0         | 0         | 2         | 3         | 0         | 0         | 2         | 0         | 0         | 0         | 0         |
| [P]                | 140       | 140       | 140       | 458       | 458       | 458       | 520       | 520       | 520       | 96        | 96        | 96        | 188       |
| [E]                | 139       | 139       | 139       | 457       | 457       | 457       | 519       | 519       | 519       | 95        | 95        | 95        | 187       |
| [P]                | 7         | 7         | 7         | 7         | 9         | 10        | 7         | 7         | 9         | 11        | 11        | 11        | 10        |
| [E]                | 19        | 19        | 19        | 30        | 46        | 53        | 30        | 30        | 48        | 42        | 42        | 42        | 57        |
| # co               | 9         | 15        | 8         | 15        | 33        | 27        | 20        | 20        | 34        | 32        | 32        | 32        | 35        |
| # tr               | 385       | 572       | 344       | 1151      | 2220      | 3825      | 1842      | 1842      | 3480      | 3284      | 3284      | 3284      | 2470      |
| # segs computed    | 12        | 9         | 9         | 10        | 16        | 29        | 10        | 10        | 16        | 24        | 24        | 24        | 20        |
| # tubes per segs   | 140       | 140       | 140       | 458       | 458       | 458       | 520       | 520       | 520       | 96        | 96        | 96        | 188       |
| issafe             | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      |
| NoS                |           |           |           |           |           |           |           |           |           |           |           |           |           |
| time               | 37.12     | 248.16    | 288.54    | 34.32     | 99.92     | 122       | 122       | 122       | 122       | 122       | 122       | 122       | 122       |
| [P]                | 48        | 264       | 105       | 56        | 122       | 122       | 122       | 122       | 122       | 122       | 122       | 122       | 122       |
| [E]                | 47        | 263       | 104       | 55        | 121       | 121       | 121       | 121       | 121       | 121       | 121       | 121       | 121       |
| [P]                | 96        | 264       | 210       | 112       | 244       | 244       | 244       | 244       | 244       | 244       | 244       | 244       | 244       |
| [E]                | 48        | 264       | 105       | 56        | 122       | 122       | 122       | 122       | 122       | 122       | 122       | 122       | 122       |
| # tubes per segs   | 2.0       | 1.0       | 2.0       | 2.0       | 2.0       | 2.0       | 2.0       | 2.0       | 2.0       | 2.0       | 2.0       | 2.0       | 2.0       |
| issafe             | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      | True      |
| tool \ agent model | Car |
|--------------------|-----|
|                    | stats | comp2D-1 | comp2D-3 |
| SymTR-DryVR        |        |          |          |
| # rt               | 0      | 10       | 50       | 0       | 50 |
| # rd               | 0      | 10       | 17       | 0       | 10 |
| time               | 4.58   | 21.23    | 39.5     | 26.54   | 192.32 |
| # co               | 6      | 84       | 135      | 11      | 125 |
| # segs computed    | 2      | 30       | 41       | 4       | 43 |
| # tubes per segs   | 3.0    | 2.8      | 3.59     | 2.75    | 6.02 |
| issafe             | False  | False    | True     | False   | True |
| SymT-DryVR         |        |          |          |
| # rt               | 0      | 10       | 50       | NA      | NA |
| # rd               | 0      | 6        | 0        | NA      | NA |
| time               | 23.80  | 26.02    | 22.43    | NA      | NA |
| # co               | 29     | 84       | 25       | NA      | NA |
| # segs computed    | 14     | 35       | 14       | NA      | NA |
| # tubes per segs   | 3.36   | 26.02    | 2.93     | NA      | NA |
| issafe             | True   | True     | True     | NA      | NA |
| NoSym-DryVR        |        |          |          |
| time               | 44.66  |          |          | 195.71  |      |
| # co               | 96     |          |          | 210     |      |
| # tr               | 0      |          |          | 0       |      |
| # segs computed    | 48     |          |          | 105     |      |
| # tubes per segs   | 2.0    |          |          | 2.0     |      |
| issafe             | True   |          |          | True    |      |
Figure 4: The figures shows the scenarios and planned path that we run the experiment with.
Figure 5: The figures shows the reachset for the comp2D-1 scenario for car with different refine threshold. Figure 5a shows the reachset for the scenario without using symmetry and we can see clearly that the result is safe. Figure 5b shows the original reachset for the scenario without using refinement. We can see that the reachset overbloats. The virtual reachsets for the scenario are plotter in figure 5c. Figure 5c shows the original reachset for the scenario with refine threshold 10. The reachset is more conservative compared with not using refinement, but the result is still different from that from not using symmetry. The virtual reachsets for the scenario are plotter in figure 5f. Figure 5d shows the original reachset for the scenario with refine threshold 50 and the actual number of refinement is 17. With enough refinement, the verification result is safe, which is the same as not using symmetry. The virtual reachsets for the scenario are plotter in figure 5g.
Figure 6: Safety verification of a NN-controlled quadrotor in scenario comp2D-1 using SymAR.
Figure 7: Safety verification of a NN-controlled quadrotor in scenario comp2D-2 using SymAR.

Figure 8: Safety verification of a NN-controlled quadrotor in scenario comp2D-3 using SymAR.
Figure 9: Safety verification of a NN-controlled quadrotor in scenario comp2D-3 using SymAR.