Abstract

The instanton representation of Plebanski gravity admits a natural canonical structure where the (densitized) eigenvalues of the CDJ matrix are the basic momentum space variables. Canonically conjugate configuration variables exist for six distinct configurations in the full theory, referred to as quantizable configurations. The CDJ matrix relates to the Petrov classification and principal null directions of spacetime, which we directly correlate to these quantizable degrees of freedom. The implication of this result is the ability to perform a quantization procedure for spacetimes of Petrov Type I, D, and O, using the instanton representation.
1 Introduction

In the full theory of general relativity there currently remain at least three unresolved questions. (i) One question is to find the projection from the full unreduced phase space $\Omega$ of the theory to the physical phase space $\Omega_{\text{Phys}}$, through implementation of the initial value constraints. (ii) The second main question regards the quantization of the theory, which has posed technical difficulties in various approaches. (iii) The third main question is the verification of the quantum theory in terms of quantities which can be measured in the classical limit. The aim of this paper is to provide a preliminary addressal of these questions using the instanton representation of Plebanski gravity (See e.g. paper II and listings therein). The CDJ matrix $\Psi_{ae}$ is a $SO(3,C) \otimes SO(3,C)$-valued matrix which was introduced $[1]$ in order to construct a solution to the diffeomorphism and the Hamiltonian constraints in Ashtekar variables. The instanton representation uses $\Psi_{ae}$ as the fundamental momentum space variable. It so happens that the initial value constraints of GR when written on $\Omega_{\text{Inst}} = (\Psi_{ae}, A^a_i)$, the phase space of the instanton representation, are essentially constraints on $\Psi_{ae}$.\footnote{Here $A^a_i$ is the self-dual Ashtekar connection, which is the configuration space variable. By convention lowercase symbols from the beginning of the Latin alphabet $a, b, c, \ldots$ signify internal $SO(3,C)$ indices, while those from the middle $i, j, k, \ldots$ signify spatial indices.} This feature enables one to readily address question (i) by projecting directly to the reduced momentum space, where it remains to find the physical principle fixing the canonically conjugate configuration variables in the preservation of a cotangent bundle structure on this reduced space.

Further investigation of the physical interpretation of the CDJ matrix $\Psi_{ae}$ reveals that it correlates to the algebraic properties of spacetime which are independent of coordinates and of tetrad frames. In the addressal of question (iii) above, it then suffices to correlate these aspects of spacetime to degrees of freedom which in the instanton representation can be quantized. We show in this paper that such degrees of freedom correspond to the nondegenerate spacetimes, those spacetimes whose self-dual Weyl curvature tensor possess three linearly independent eigenvectors. The eigenvalues of the CDJ matrix for these spacetimes encode their Petrov classification and certain information pertaining to the principal null directions (PND). It is then clear, in the addressal of question (ii) above, that one may formulate a quantum theory of the CDJ matrix which admits a direct link to these PND which are in principle directly measurable in the classical limit.

A closer analysis of the canonical structure of the instanton representation indicates a potential obstruction in that there are no configuration
space variables canonically conjugate to the bare CDJ matrix $\Psi_{ae}$. This is a consequence of the fact that the transformation from the Ashtekar variables into the instanton representation is a noncanonical transformation. On first sight, it may seem that this prevents one from formulating a quantum theory where $\Psi_{ae}$ is the momentum space variable. However, we have found that this obstruction is circumvented by using a densitized CDJ matrix

$$\Psi_{ae} = \Psi_{ae}(\det A)$$

in lieu of $\Psi_{ae}$ as the basic momentum space variable, and projecting to the kinematic phase space $\Omega_{Kin}$. This results in six distinct quantizable configurations of the instanton representation, which correlate to Petrov classifications in the classical limit.

The organization of this paper is as follows. Section 2 transforms vacuum GR from the Ashtekar variables directly into the instanton representation using the CDJ Ansatz, which holds on the space of nondegenerate variables. Additionally, we show how the CDJ matrix facilitates the implementation of the initial value constraints, and how the eigenvalues of $\Psi_{ae}$ emerge as a natural candidate for the momentum space variables. In section 3 we delineate the conditions for which there exist globally holonomic coordinates on the instanton representation kinematic configuration space $\Gamma_{Inst}$ which are canonically conjugate to these (densitized) eigenvalues. This limits one to nondegenerate connections $A^a_{\mu}$ with three degrees of freedom per point, of which there are six distinct configurations corresponding to nondegenerate metrics.

Section 4 elucidates the natural correspondence from these quantizable configurations to the intrinsic $SO(3,C)$ frame, the frame corresponding to the kinematic phase space. While the instanton representation is not canonically related to the Ashtekar variables on the full starting phase space $\Omega_{Inst}$, we show that it is canonically related in the intrinsic $SO(3,C)$ frame. This establishes the reduced phase space of the Ashtekar variables, which corresponds to nondegenerate metrics. Section 5 creates a library of the so-called ‘quantizable’ instanton representation configurations, by explicit construction.

Having demonstrated that the eigenvalues of $\Psi_{ae}$ admit a quantization of the full theory consistent with the implementation of the initial value constraints, we now show the manner in which these eigenvalues directly correlate to aspects of spacetime which are directly measurable. In this section there is some background which is provided regarding the Weyl curvature tensor and its two-component spinor formalism and principal null

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2This is with the exception of the trace of $\Psi_{ae}$, whose canonically conjugate variable is $L_{CS}$, the spatial Chern–Simons Lagrangian.

3This is defined as the phase space of the instanton representation after implementation of the Gauss’ law and diffeomorphism constraints, and prior to the Hamiltonian constraint.

4As shown in Paper II, the instanton representation can also be obtained by implementing the simplicity constraint and eliminating the densitized triad $\tilde{\sigma}^a_i$ directly from the starting Plebanski action.

5Inherently, this limits the results of this paper to quantization to spacetimes of Petrov type I, D and O, where the CDJ matrix is diagonalizable.
directions. We then explicitly relate these quantities to the CDJ matrix, which establishes the direct link from the classical to the quantum theory. In this paper we establish the canonical structure required for quantization. The full quantization procedures, including the Hilbert space structure, is reserved for separate papers.
2 From the Ashtekar variables into the instanton representation

In the Ashtekar description of gravity the basic phase space variables are a self-dual $SU(2)$ connection and a densitized triad $(A^a_i, \tilde{\sigma}_i^a) \in \Omega$. The Ashtekar connection is given by

$$A^a_i = \Gamma^a_i + \beta K^a_i,$$

where $\Gamma^a_i$ is the spin connection compatible with the triad defined by $\tilde{\sigma}_i^a$, $\beta$ is the Immirzi parameter, which we choose to be $-i$, and $K^a_i$ is the triadic form of the extrinsic curvature of 3-space $\Sigma$. The 3+1 decomposition of the resulting action for vacuum general relativity in $M = \Sigma \times R$, where $M$ is a 4-dimensional spacetime manifold foliated by 3-dimensional spatial hypersurfaces $\Sigma$, is given by

$$I_{Ash} = \int dt \int_\Sigma \tilde{\sigma}_i^a \dot{A}^a_i - A^a_0 G_a - N^i H_i - N H.$$

Equation (2) is a canonical one form $\theta_{Ash}$ minus a linear combination of first class constraints smeared by auxiliary fields [2],[3],[4]. The auxiliary fields are $N^i$, $A^a_0$ and $N = N(\det \tilde{\sigma})^{-1/2}$, respectively the shift vector, $SO(3, C)$ rotation angle and lapse density function, and the corresponding constraints $H_i$, $G_a$ and $H$ are the diffeomorphism, Gauss’ law and Hamiltonian constraints. The diffeomorphism constraint is given by

$$H_i = \epsilon_{ijk} \tilde{\sigma}_j^a B^k_a = 0,$$

which signifies invariance under spatial diffeomorphisms in $\Sigma$. The Hamiltonian constraint signifies invariance under deformations normal to $\Sigma$ and is given by

$$H = \Lambda \epsilon^{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j \tilde{\sigma}_c^k + \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j \tilde{\sigma}_c^k B^k_a = 0,$$

where $\Lambda$ is the cosmological constant and $B^k_a = \frac{1}{2} \epsilon_{ijk} F^a_{jk}$ is the magnetic field derived from the curvature of the Ashtekar connection $A^a_i$, where

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6By our convention, lowercase symbols from the beginning part of the Latin alphabet $a, b, c, \ldots$ signify internal $SO(3, C)$ indices, while from the middle of the alphabet $i, j, k, \ldots$ signify spatial indices in 3-space $\Sigma$. For Lorentzian signature one may perform a Wick rotation $N \rightarrow iN$. 

7For $N$ real the action (2) corresponds to a spacetime of Euclidean signature. For Lorentzian signature one may perform a Wick rotation $N \rightarrow iN$. 

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\[ F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + f^{abc} A_b^a A_c^i, \]  
\[ (5) \]

The Gauss’ law constraint, which signifies invariance under left-handed \( SO(3,C) \) rotations on internal indices, is given by

\[ G_a = D_i \bar{\sigma}_a^i = \partial_i \bar{\sigma}_a^i + f_{abc} A_b^a \bar{\sigma}_c^i = 0, \]  
\[ (6) \]

where \( f_{abc} \) are the structure constants for \( SO(3,C) \).

In this paper we would like to find a set of degrees of freedom suitable for quantization, which entails an implementation of the initial value constraints (3), (4) and (6). Let us transform (2) into a new set of variables using the CDJ Ansatz

\[ \bar{\sigma}_a^i = \Psi_{ae} B_e^i \]  
\[ (7) \]

attributed to Riccardo Capovilla, John Dell and Ted Jacobson [1], where \( \Psi_{ae} \in SO(3,C) \otimes SO(3,C) \) is the CDJ matrix.\(^8\) Equation (7) is good as long as \( B_e^i \) is nondegenerate, and has been shown in [1] to allow explicit solution of \( H_\mu = (H,H_i) \) algebraically at the classical level. Under (7) the action (2) becomes

\[ I_{Inst} = \int dt \int d^3x \left[ \Psi_{ae} B_e^j \dot{A}_j^a - (\epsilon_{ijk} N^i B_a^j B_e^k + A_0^a w_e^a) \Psi_{ae} \right. \\
\left. + N (\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + tr \Psi^{-1}) \right]. \]  
\[ (8) \]

We have also defined

\[ w_e^a \{ \Psi_{ae} \} = B_e^j \partial_j \Psi_{ae} + (f_{ghe} \delta_{af} + f_{fae} \delta_{gh}) C_{eh} \Psi_{fg} \equiv w_e^{fg} \{ \Psi_{fg} \}, \]  
\[ (9) \]

which comes from the transformation of the Gauss’ law constraint

\[ G_a = D_i \bar{\sigma}_a^i = D_i (\Psi_{ae} B_e^i) = \Psi_{ae} D_i B_e^i + B_e^i D_i \Psi_{ae}, \]  
\[ (10) \]

where \( C_{ae} = A_e^a B_e^i \) is the magnetic helicity density (See e.g. Paper VI). Upon use of the Bianchi identity \( D_i B_e^i = 0 \), the definition (9) follows from the evaluation of the covariant derivative \( D_i \Psi_{ae} \) in (10) in the tensor representation of the gauge group.

\(^8\)We actually use the inverse of the matrix used in [1], and allow for a nonzero trace.
2.1 Consistency with the algebraic constraints

Let us now demonstrate consistency of the CDJ Ansatz (7) with the initial value constraints, from a different approach to that introduced in [1]. The CDJ matrix $\Psi_{ae}$ can be parametrized by its symmetric and antisymmetric parts, $\lambda_{ae}$ and $a_{ae}$ respectively, which can in turn be parametrized by a polar decomposition

$$\Psi_{ae} = a_{ae} + \lambda_{ae} = \epsilon_{aed} \psi_d + O_{af}(\bar{\theta}) \lambda_f O^T_{fe}(\bar{\theta}),$$

(11)

where $\psi_d = \epsilon_{dae} \Psi_{ae}$ is a $SO(3, C)$ 3-vector.\(^9\) Here, $\bar{\lambda} \equiv (\lambda_1, \lambda_2, \lambda_3)$ are the eigenvalues of $\lambda_{ae} = \Psi_{(ae)}$, while $O_{ae} \in SO(3, C)$ implements a complex orthogonal transformation of $\bar{\lambda}$ parametrized by three complex angles $\bar{\theta} = (\theta_1, \theta_2, \theta_3)$. In exponential form this is given by $O = e^{\bar{\theta} T}$, where $T$ are generators satisfying the $SO(3)$ Lie algebra

$$[T_f, T_g] = i \epsilon_{fgh} T_h.$$ \hspace{1cm} (12)

The diffeomorphism constraint in the instanton representation is given by

$$H_i = \epsilon_{ijk} B^j_a B^k_e \Psi_{ae} = \epsilon_{ijk} B^j_a B^k_e \epsilon_{aed} \psi_d = 0 \hspace{0.5cm} \forall \hspace{0.2cm} x,$$

(13)

which implies that the antisymmetric part of the CDJ matrix vanishes ($\psi_d = 0$), or that CDJ matrix is symmetric $\Psi_{ab} = \Psi_{(ab)}$. The Hamiltonian constraint is given by the last term of (8). Since $(\det B \neq 0$ and $(\det \Psi \neq 0)$, then it suffices that

$$\frac{1}{2} Var \Psi + \Lambda \det \Psi = 0 \hspace{0.5cm} \forall \hspace{0.2cm} x.$$ \hspace{1cm} (14)

Substitution of the parametrization (11) into (14) after dividing by $\det B \neq 0$, which requires nondegeneracy of $B^i_a$, yields

$$H = \frac{1}{2} Var(\Psi_{(ae)}) + \Lambda \det(\Psi_{(ae)}) + (\Lambda \Psi_{(ae)} - \delta_{ae}) \psi_a \psi_e = 0.$$ \hspace{1cm} (15)

The first two terms of (15) can be rewritten explicitly in terms of the eigenvalues of $\lambda_{ae}$ due to the cyclic property of the trace, yielding\(^{10}\)

\(^9\)This decomposition is possible when $\Psi_{ae}$ contains three linearly independent eigenvectors, which as we will see restricts one to spacetimes of Petrov type $I$, $D$ and $O$.

\(^{10}\)See Appendix A for the details of the derivation.
\[ H = (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) + \Lambda \lambda_1 \lambda_2 \lambda_3 \\
+ (\Lambda O_{af} O_{ef} + \delta_{ae}) \psi_a \psi_e = 0. \]  
(16)

Using \( \psi_d = 0 \) from (13), on the space of solutions to \( H \) for nondegenerate \( B_i^a \), the terms in (16) quadratic in \( \psi_d \) vanish along with the \( SO(3, C) \) matrix \( O_{ae} \). The Hamiltonian constraint (16) then reduces to the following algebraic relation amongst the eigenvalues \( \lambda_f \)

\[ \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0 \rightarrow \lambda_3 = -\left(\frac{\lambda_1 \lambda_2}{\Lambda \lambda_1 \lambda_2 + \lambda_1 + \lambda_2}\right). \]  
(17)

Hence for \( \lambda_f \neq 0 \) (17) fixes one eigenvalue \( \lambda_3 \) completely in terms of the remaining two eigenvalues \( \lambda_1 \) and \( \lambda_2 \), with no appearance of the \( SO(3, C) \) angles \( \vec{\theta} \). Observe that the implementation of \( H_\mu = (H, H_i) \) has resulted in a reduction of \( \Psi_{ae} \) by four D.O.F. to its eigenvalues with no corresponding restriction on the Ashtekar connection \( A_i^a \).

We will ultimately choose \( \lambda_1 \) and \( \lambda_2 \) as the physical D.O.F. for the momentum space of the instanton representation. To endow the theory with symplectic structure, we must find two D.O.F. corresponding to the configuration space variables \( \Gamma_{Phys} \) canonically conjugate to \((\lambda_1, \lambda_2)\), which will be one of the main results of this paper.

### 2.2 Consistency with the Gauss' law constraint

While the \( SO(3, C) \) matrix \( O_{ae} \) has been eliminated from the Hamiltonian constraint \( H \), owing to its restriction to the invariants of \( \Psi_{(ae)} \) on the diffeomorphism constraint shell, it will appear explicitly in the Gauss’ law constraint \( G_a \). The unconstrained momentum space D.O.F. have already been reduced to \( \vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \) at the level prior to implementation of \( H \). Therefore \( G_a \) should not reduce these particular D.O.F. any further, and neither does it impose any restrictions on \( A_i^a \in \Gamma_{Inst} \). Hence, \( G_a \) must be viewed as a constraint on the angles \( \vec{\theta} \), which in turn define a special \( SO(3, C) \) frame. The eigenvalues \( \lambda_f \) must then be rotated from the intrinsic frame where \( \vec{\theta} = 0 \) into the \( SO(3, C) \) frame where \( \Psi_{ae} \) becomes annihilated by \( G_a \). This frame, fixed by the correctly chosen \( \vec{\theta} \), should correspond to a solution to the initial value constraints of GR.

The Gauss’ law constraint is given by

\[ w_e \{ \Psi_{ae} \} = B_i^a D_i \Psi_{ae} = v_e \{ \Psi_{ae} \} + (f_{abf} \delta_{ge} + f_{ebg} \delta_{af}) C_{be} \Psi_{fg} = 0 \]  
(18)

where \( v_e = B_i^a \partial_e \) and \( C_{be} = A_i^b B_e^i \) is defined as the helicity density matrix. Unlike the diffeomorphism and Hamiltonian constraints which are algebraic,
the Gauss' law constraint is a set of differential equations. To solve (18) one may first decompose $Ψ_{ae}$ into a basis of shear (off-diagonal symmetric) and anisotropy (diagonal) elements, using the Cartesian representation

$$Ψ_{ae} = (e^f)_{ae}ϕ_f + (E^f)_{ae}Ψ_f,$$

where we have defined

$$E^1_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \ E^2_{ae} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \ E^3_{ae} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e^1_{ae} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ e^2_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ e^3_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The CDJ matrix (19) in matrix form is given by

$$Ψ_{ae} = \begin{pmatrix} ϕ_1 & ϕ_2 \\ Ψ_3 & Ψ_2 \\ Ψ_2 & Ψ_1 \end{pmatrix}.$$

The Gauss’ law constraint then becomes

$$(e^f)_{ae}w_e\{ϕ_f\} + (E^f)_{ae}w_e\{Ψ_f\}$$

which equivalently is given by $Ψ_f = J^q_fϕ_g$. We have defined the Gauss’ law propagator $J^q_f$ from the anisotropy to the shear elements, given by

$$J^q_f = -((E^f)_{ae}w_e)^{-1}(e^g)_{ab}w_b.$$  

For the purposes of the present paper it suffices to note that $G_a$ reduces $Ψ_{ae}$ by three unphysical D.O.F. $Ψ_f$, leaving remaining $ϕ_f$.¹¹ Within this context, $G_a$ establishes a map from the physical D.O.F. $\vec{λ}$ to the angles

$$λ_f \rightarrow ϕ_f \rightarrow (Ψ_f[\vec{λ}; A^g], \vec{θ}[\vec{λ}; A^g]).$$

¹¹The details of the inversion (21), as well as the explicit solution algorithm for the angles $\vec{θ}$, are treated in Papers VI, VII and VIII. The idea is that one must solve (18) explicitly for $Ψ_f = Ψ[ϕ]$ for each configuration $A^g ∈ Γ$. Certain configurations with yield well-defined $θ$ and other configurations will not. But whatever the configuration chosen, there exists a map from the physical degrees of freedom $λ$ to the angles $θ$.  

8
First one chooses a particular configuration for the Ashtekar connection $A^a_i$ and then finds the Gauss’ law propagator (21) corresponding to that configuration $\hat{J}_f^g = \hat{J}_f^g[A]$. To find the angles $\vec{\theta}$, one then equates the polar representation of $\Psi_{ae}$ to the Cartesian representation

$$\lambda_{ab} = O_{ae}(\vec{\theta}) \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}^{T} O_{fb}(\vec{\theta}) = ((e^g)_{ae} + (E^f)_{ae}\hat{J}_f^g)\varphi_g$$

and then solves for $\vec{\theta}$ explicitly in terms of the eigenvalues $\lambda_f$ and $A^a_i$. One then repeats the procedure for all configurations $A^a_i$, which defines a functional $\vec{\theta} = \vec{\theta}[\lambda_f; A^a_i]$ constituting a solution space for $G_a$. Another method is to write the Gauss’ law constraint as a set of differential equations directly on the angles $\vec{\theta}$, thus bypassing the Cartesian representation, as shown in Paper VIII. Note that finding $\Psi_{ae} \in \text{Ker}\{G_a\}$ does not place any restriction on $A^a_i \in \Gamma$.\textsuperscript{12} To obtain a symplectic structure on the reduced phase space under $G_a$ we must find the appropriate restriction required of $A^a_i$ by alternate means, which brings us to the issue of the existence of holonomic coordinates.

\textsuperscript{12}This is because $A^a_i$ is one of the inputs into Gauss’ law constraint. Certain configurations will yield a well-defined solution and other configurations will not.
3 Globally holonomic configuration space for the instanton representation

When the initial value constraints have been implemented, the phase space of GR should consist of four degrees of freedom per point.\(^{13}\) Denote the kinematic phase space $\Omega_{\text{Kin}}$ as the phase space at the level where the diffeomorphism and the Gauss' law constraints, but not the Hamiltonian constraint, have been implemented. At this stage $\Omega_{\text{Kin}}$ should consist of six D.O.F. per point and we would like to use the three eigenvalues $\lambda_f$ as the momentum space part of these degrees of freedom. Were this to be the case then the canonically conjugate configuration space variables should be determined, if they exist, so that a quantization procedure can be carried out.

The canonical structure of (8) suggests naively that $\Psi_{ae}$ should be canonically conjugate to a variable $X_{ae}$ whose velocity is $\dot{X}_{ae} = B_{i}^{e} A_{i}^{a}$.\(^{14}\) However, $X_{ae}$ does not exist globally as a holonomic coordinate on $\Gamma_{\text{Inst}}$ for arbitrary $A_{a}^{i}$, which by our interpretation constitutes an obstruction to quantization. Equation (7) transforms the Ashtekar canonical one form into

$$\theta_{\text{Ash}} = \int_{\Sigma} d^{3}x \tilde{\sigma}_{e}^{i} \delta A_{i}^{a} = \int_{\Sigma} d^{3}x \Psi_{ae} B_{i}^{e} \delta A_{i}^{a}. \quad (23)$$

We would like to define a configuration variable $X_{ae}$ conjugate to $\Psi_{ae}$ with canonical one form

$$\theta_{\text{Inst}} = \int_{\Sigma} d^{3}x \Psi_{ae}(x) \delta X_{ae}(x). \quad (24)$$

However, although the variations $\delta X_{ae} = B_{i}^{e} \delta A_{i}^{a}$ live in the cotangent space $T^{*}_{X}(\Gamma_{\text{Inst}})$ to the configuration space $\Gamma_{\text{Inst}}$, the coordinates $X_{ae}$ do not in general exist globally on $\Gamma$ (See e.g. [5] and [6]), except for the trace

$$\delta X^{11} + \delta X^{22} + \delta X^{33} = B_{a}^{i} \delta A_{i}^{a} = \delta I_{CS}[A], \quad (25)$$

where $I_{CS}$ is the Chern–Simons functional of the connection $A_{i}^{a}$. Moreover, the symplectic two form in Ashtekar variables is the exact variation of $\theta_{\text{Ash}}$

$$\omega_{\text{Ash}} = \int_{\Sigma} d^{3}x \delta \tilde{\sigma}_{a}^{i} \wedge \delta A_{i}^{a} = \delta \left( \int_{\Sigma} d^{3}x \tilde{\sigma}_{e}^{i} \delta A_{i}^{a} \right) = \delta \theta_{\text{Ash}}, \quad (26)$$

\(^{13}\)This refers to complex phase space degrees of freedom. The implementation of reality conditions is a separate procedure from the initial value constraints.

\(^{14}\)The variations $\delta X_{ae} = B_{i}^{e} \delta A_{i}^{a} \in T^{*}(\Gamma)$ are well-defined in the cotangent space to configuration space $\Gamma$ (See e.g. [5] and [6]). The variables $X_{ae}$ were first discovered by Chopin Soo in [5], due to their natural adaptability to the gauge invariances of GR.
whereas in the instanton representation we have

\[
\delta \left( \int d^3x \Psi_{ae} B^i_e \delta A^a_i \right) = \int d^3x \delta \Psi_{ae} \wedge B^i_e \delta A^a_i + \int d^3x \Psi_{ae} \delta B^i_e \wedge \delta A^a_i. \tag{27}
\]

Equation (27) is not an exact two form on $\Omega_{Inst}$ unless the second contribution on the right hand side vanishes. Let us attempt to deduce the allowed configurations of $A^a_i$ for which this may be the case, using the eigenvalues $\lambda_f$ as the fundamental momentum space variables. There is no loss of generality in taking $\Psi_{ae}$ to be already in diagonal form, hence

\[
\theta_{Kin} = \sum_a \lambda_{aa} B^i_a \delta A^a_i \tag{28}
\]

is the canonical one form at the kinematical level for some $A^a_i$. The coefficients of $\lambda_{aa}$ in (28) can be split into two terms $B^i_a \delta A^a_i = N^a + M^a$ for each $a$, where

\[
N^a = \sum_{i,j,k} \epsilon^{ijk} (\partial_j A^a_k) \delta A^a_i; \quad M^a = \frac{1}{2} \sum_{i,j,k} \epsilon^{ijk} f_{abc} (A^b_j A^c_k) \delta A^a_i \tag{29}
\]

with no summation over $a$. Note that $M^a$ is completely free of spatial gradients of the connection $A^a_i$, while $N^a$ contains spatial gradients. Dynamical variables containing spatial gradients pose a problem for the full theory, when promoting them to operators for quantization. Our definition of minisuperspace requires that dynamical variables be spatially homogeneous, depending only on time. This requires that for a minisuperspace theory, all spatial gradients of the variables must be set to zero.\(^{15}\)

If one could find well-defined configurations in the full theory where all terms containing spatial gradients vanish from the canonical structure even though the spatial gradients are in general nonzero, then one would have the full theory with the advantages of the simplicity of minisuperspace as we have defined it. Focus first on $N^a$, expanding the individual terms

\[
N^a = (\partial_2 A^a_3) \delta A^a_1 - (\partial_3 A^a_2) \delta A^a_1 + (\partial_3 A^a_1) \delta A^a_2 - (\partial_1 A^a_3) \delta A^a_2 + (\partial_1 A^a_2) \delta A^a_3 - (\partial_2 A^a_1) \delta A^a_3. \tag{30}
\]

Now rearrange the terms of (30) into the form

\[
N^a = \left( (\delta A^a_3) \partial_1 - (\delta A^a_1) \partial_3 \right) A^a_2 \\
+ \left( (\delta A^a_1) \partial_2 - (\delta A^a_2) \partial_1 \right) A^a_3 + \left( (\delta A^a_2) \partial_3 - (\delta A^a_3) \partial_2 \right) A^a_1. \tag{31}
\]

\(^{15}\)Note that this is not the usual definition of minisuperspace via Bianchi groups, which absorb all spatial dependence of the theory into invariant one forms and vector fields.
A moment’s reflection of shows that a sufficient condition to make (31) vanish, is to set two out of three elements of the set \((A^a_1, A^a_2, A^a_3)\) to zero for each \(a\). For instance, choosing \(A^a_2 = A^a_3 = 0\), which selects two different elements of the 3 by 3 matrix \(A^a_i\) from the same row, causes \(N_a\) to vanish with no restriction on \(A^a_1\). Performing this for each \(a\) leads to the realization that one is free to set six out of the nine elements of \(A^a_i\) to zero in the full theory, while still causing \((N_1, N_2, N_3)\) to vanish. This leaves remaining three nonzero elements \(A^a_i\) which is just as well, since there should be three configuration space physical degrees of freedom canonically conjugate to \((\lambda_{11}, \lambda_{22}, \lambda_{33})\), in order to have a cotangent bundle structure at the kinematical level. This provides the sought after principle for selecting the configuration space variables needed for quantization.

The question then arises as to which three elements \(A^a_i\) to select for the kinematic configuration space \(\Gamma_{Kin}\). If one selects the three \(A^a_i\) such that no two elements come from the same row \(a\) or from the same column \(i\) then one has that \(\det(A^a_i) \neq 0\), namely that the connection is nondegenerate as a three by three matrix. Hence we have that \(N^a = 0\) and only \(M^a\) contributes to (28). This is given by

\[
M^a = (\det A)(A^{-1})^i_a \delta A^a_i, \tag{32}
\]

where we have used the fact that the structure constants \(f_{abc}\) for the Ashtekar variables are numerically the same as the Cartesian epsilon symbol \(\epsilon_{abc}\) in writing the determinant. The canonical one form then is given by

\[
\theta_{Kin} = \int_\Sigma d^3x \lambda_{aa}(\det A)(A^{-1})^i_a \delta A^a_i = \int_\Sigma d^3x \tilde{\lambda}_a (A^{-1})^i_a \delta A^a_i \tag{33}
\]

where \(\tilde{\lambda}_a = \lambda_{aa}(\det A)\) are the densitized version of the eigenvalues of \(\Psi_{(ae)}\). Since \(A^a_i\) contains three D.O.F., then the configuration space term can be written in the form

\[
(A^{-1})^i_a \delta A^a_i = \frac{\delta A^a_i}{A^a_i} = \delta(\ln A^a_i/a_0) = \delta X^a_i \tag{34}
\]

where a different choice of \(i\) must be made for each \(a\).\(^{16}\) The variables \(X^a_i\) are globally holonomic, hence they form a good set of coordinates on \(\Gamma_{Kin}\) with symplectic two form

\[
\omega_{Kin} = \int_\Sigma d^3x \delta \tilde{\lambda}_a \wedge (A^{-1})^i_a \delta A^a_i, \tag{35}
\]

\(^{16}\)The quantity \(a_0\) is a numerical constant of mass dimension \([a_0] = 1\), needed to make the argument of the logarithm dimensionless.
where $\omega_{K_in} = \delta \theta_{K_in}$. These variables are canonically conjugate to the densitized eigenvalues $\lambda_a$, and serve as a basis for quantization of the full theory for configurations where the CDJ matrix is diagonalizable. In this case one can always perform a $SO(3, C)$ rotation into the diagonal configuration by using the polar decomposition of $\Psi_{ae}$, not including the angles $\vec{\theta}$ as part of the canonical structure.\textsuperscript{17} Hence the condition $N_a = 0$ makes it possible to globally define coordinates corresponding to $M_a$ for the full theory, in direct analogy to minisuperspace.

The quantizable configurations on the kinematic phase space of the instanton representation then imply the following restriction of the configuration space of the Ashtekar variables

$$A_i^a = I_{f_1 f_2 f_3} I_{j_1 j_2 j_3} \left( \delta_{af_1} \delta_{ij_1} A_{j_1}^{f_1} + \delta_{af_2} \delta_{ij_2} A_{j_2}^{f_2} + \delta_{af_3} \delta_{ij_3} A_{j_3}^{f_3} \right),$$

(36)

where $I_{ijk} = 1$ for $i \neq j \neq k$, and zero otherwise. The actual variables which will be quantized are obtained by functional antidifferentiation of (34), which yields

$$X^f = I_{f_1 f_2 f_3} I_{j_1 j_2 j_3} \left( \delta_{ff_1} \delta_{ij_1} \ln \left( \frac{A_{j_1}^{f_1}}{a_0} \right) + \delta_{ff_2} \delta_{ij_2} A_{j_2}^{f_2} \ln \left( \frac{A_{j_2}^{f_2}}{a_0} \right) + \delta_{ff_3} \delta_{ij_3} A_{j_3}^{f_3} \ln \left( \frac{A_{j_3}^{f_3}}{a_0} \right) \right)$$

(37)

for $f = 1, 2, 3$. The ranges of the coordinates are $-\infty < |X^f| < \infty$, corresponding to $0 < |A_i^a| < \infty$, which guarantee nondegeneracy of $A_i^a$. The result is that the instanton representation admits a quantization of the full theory on the kinematic phase space we should have canonical commutation relations

$$[X^f(x, t), \tilde{\lambda}_g(y, t)] = \delta^f_g \delta^{(3)}(x, y).$$

(38)

We have shown that on the quantizable configurations the spatial gradients cancel out of the canonical one form $\theta_{K_in}$ in the full theory. It so happens that precisely on these configurations, the spatial gradients also vanish from the Chern–Simons functional

$$L_{CS} = A^a \wedge dA^a + \frac{2}{3} A \wedge A \wedge A \rightarrow \frac{2}{3} A \wedge A \wedge A,$$

(39)

which can be shown by a similar argument as above using (30) and (31) with $\delta A_i^a$ replaced by $A_i^a$. The result is that when one densitizes $\Psi_{ae}$ by $(\det A)$, one is in fact densitizing $\Psi_{ae}$ by the Chern–Simons Lagrangian evaluated on the quantizable configurations.

\textsuperscript{17}In Paper IV it is proven that the $SO(3, C)$ angles $\bar{\theta}$ are indeed ignorable in the canonical and in the symplectic structures of the instanton representation.
4 Quantization in the intrinsic $SO(3, C)$ frame

The intrinsic $SO(3, C)$ frame is defined as the frame of reference in which the symmetric part of the CDJ matrix $\Psi_{(ae)}$ is diagonalized, and can be associated to the kinematic level of the instanton representation. We will provide additional arguments that it is possible to carry out a quantization of the full theory with respect to this particular frame. Let us start from the CDJ Ansatz

$$\tilde{\sigma}^i_a = \Psi_{ae} B^i_e. \tag{40}$$

Next, re-write (40) using the polar decomposition of $\Psi_{ae}$

$$\tilde{\sigma}^i_a = (e^{\theta \cdot T})_{af} \lambda_f (e^{-\theta \cdot T})_{fe} B^i_e + \epsilon_{aed} B^i_e \psi_d, \tag{41}$$

where $\lambda_f$ are the eigenvalues of $\Psi_{(ae)}$. Next, multiply (41) by $e^{-\theta \cdot T}$, which will have the effect of rotating the index $a$ into the intrinsic $SO(3, C)$ frame

$$(e^{-\theta \cdot T})_{fa} \tilde{\sigma}^i_a = \lambda_f (e^{-\theta \cdot T})_{fe} B^i_e + (e^{-\theta \cdot T})_{fa} \epsilon_{aed} B^i_e \psi_d. \tag{42}$$

Now make the definition

$$B^i_e = (e^{\theta \cdot T})_{eh} b^i_h; \quad \tilde{P}^i_a = (e^{-\theta \cdot T})_{ae} \tilde{\sigma}^i_e, \tag{43}$$

where $b^i_h$ is the magnetic field for a ‘reference’ connection $a^a_i$ associated with the intrinsic $SO(3, C)$ frame. Then $B^i_e$ is a gauge-transformed version of $b^i_h$, which can be parametrized by six degrees of freedom. In the intrinsic $SO(3, C)$ frame we have that

$$\tilde{P}^i_f = \lambda_f b^i_f + (e^{-\theta \cdot T})_{fa}(e^{-\theta \cdot T})_{he} \epsilon_{aed} b^i_h \psi_d. \tag{44}$$

Now multiply (44) by $(b^{-1})^g_i$ and use the complex orthogonal property

$$(e^{-\theta \cdot T})_{fa}(e^{-\theta \cdot T})_{ge}(e^{-\theta \cdot T})_{bh} \epsilon_{ach} = \epsilon_{fbg}, \tag{45}$$

which yields

$$(b^{-1})^g_i \tilde{P}^i_f = \delta_{gf} \lambda_f + \epsilon_{fgh}(e^{-\theta \cdot T})_{bd} \psi_d. \tag{46}$$

At this stage one densitizes (46) by multiplying by the determinant of $A^a_i$, which implies the following Schrödinger representation
\[(\det A)(b^{-1})^q_i \frac{\delta}{\delta a_i^f} = (\det A) \frac{\delta}{\delta X f g} \]
\[= \delta_g^f \frac{\delta}{\delta X f} + \epsilon_{f gb}(e^{-\theta T})_{bd} \frac{\delta}{\delta X [d]}, \quad (47)\]

where \( X^{[d]} \) is the variable conjugate to \( \psi_d \), which may not be well-defined.

Let us now apply a counting argument of the degrees of freedom.\(^{18}\) At the unconstrained level, starting from the full unconstrained phase space \( \Omega_{Inst} \), we have \( Dim(\Psi_{ae}, A^a_i) = (9, 9) \). The phase space variables can both be written in terms of a polar decomposition\(^{19}\)

\[\Psi_{ae} = (e^{\theta T})_{af} \lambda_f(e^{-\theta T})_{fe} + \epsilon_{aed} \psi_d;\]
\[A^a_i = (e^{\theta T})_{af} a_f U_{fi}[\theta] - \frac{1}{2} \epsilon_{abc}(e^{\theta T})_{fb} \partial_i(e^{\theta T})_{fc}. \quad (48)\]

Implementation of the diffeomorphism constraint sets \( \psi_d = 0 \), with no corresponding reduction of the configuration space, yielding \( Dim(\Psi_{ae}, A^a_i)_{diff} = (6, 9) \). At this point the rotation into the intrinsic \( SO(3, C) \) frame absorbs the angles \( \vec{\theta} \) into the definition of the variables, which corresponds to a reduction both of configuration space and momentum space to \( Dim(\Psi_{ae}, A^a_i)_{Kin} = (3, 6) \). In order to have a cotangent bundle structure at this stage, we need to eliminate three D.O.F. from the configuration space. By setting three elements of \( A^a_i \) to zero such that \( (\det A) \neq 0 \), we obtain the required structure with \( Dim(\Psi_{ae}, A^a_i) \equiv Dim(\lambda_f, a_f) = (3, 3) \) and globally holonomic coordinates. This is tantamount to setting \( \vec{\phi} = 0 \) in (48), and forms the starting point for a quantization of the physical degrees of freedom of the theory.\(^{20}\)

Note for the diagonal elements \( f = g \), that we have

\[(\det A)(b^{-1})^f_i \frac{\delta}{\delta a_i^f} = \frac{\delta}{\delta X f}. \quad (49)\]

---

\(^{18}\)The following notation \( Dim(p, q) = (a, b) \) signifies that the complex degrees of freedom per point respectively in the momentum space \( p \) and the configuration space \( q \) are respectively \( a \) and \( b \).

\(^{19}\)For the second line of (48) we have adapted for GR the polar representation of \( SU(2) \) Yang–Mills gauge fields presented in [21]. There are two complex orthogonal matrices, \( e^{\vec{\theta} T} \) which rotates the internal index, hence a gauge transformation, and the other matrix \( U_{fi}[\vec{\theta}] \) which rotates the spatial index. The latter is parametrized by three angles \( \vec{\phi} = (\phi^1, \phi^2, \phi^3) \). The physical degrees of freedom are encoded in the three diagonal components \( a_f \). Hence there are a total of nine complex degrees of freedom in \( A^a_i \).

\(^{20}\)Is is known from Paper XIV that \( X^{[d]} \) in (47) cannot be defined as coordinates on configuration space \( \Gamma \), since they are not integrable, and that there are no D.O.F. conjugate to the angles \( \vec{\theta} \). This implies that the \( SO(3, C) \) frame is ignorable in the canonical structure of the instanton representation, and suggests that not more than six phase space variables on \( \Omega_{Inst} \) are quantizable.
which is globally holonomic on configuration space.

The intrinsic frame can be achieved directly from the level of the instanton representation action, starting from

\[
I_{\text{Inst}} = \int dt \int d^3x \Psi_{ae} B_i^e \dot{A}_i^a + A_0^a B_i^e D_i \Psi_{ae} - \epsilon^{ijk} N^i B^j_a B^k_e \Psi_{ae} - N (\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}).
\]

Integrating by parts and separating \(\Psi_{ae}\) into its symmetric and its antisymmetric parts, we have

\[
I_{\text{Inst}} = \int dt \int d^3x (B_i^e F^a_{0i} - \epsilon^{ijk} N^i B^j_a B^k_e) \Psi_{ae} - N (\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}).
\]

Implementation of the diffeomorphism constraint eliminates the second line of (51) since \(\Psi_{[ae]} = 0\) from (13). Using the fact that \(\Psi_{ae} = \Psi_{(ae)}\) is symmetric in \(a\) and \(e\), we can now write (51) as

\[
I_{\text{Inst}} = \int d^4x \left( \frac{1}{8} \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - N (\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}) \right),
\]

Equation (44) effectively appends the Hamiltonian constraint to an object which resembles a topological \(F \wedge F\) term, with \(\Psi_{ae}\) replacing the Cartan–Killing form. Next we will implement the Gauss’ law constraint, using the polar decomposition of the CDJ matrix and the fact that the Hamiltonian constraint for symmetric \(\Psi_{ae}\) is \(SO(3, C)\) invariant. First, note that the first term of (52) can be written as

\[
\frac{1}{8} \int_M d^4x \lambda_f (e^{-\theta \cdot T}) f_a (e^{-\theta \cdot T}) f_e F^a_{\mu\nu} [A] F^e_{\rho\sigma} [A],
\]

such that each curvature is rotated in its internal index. This rotation corresponds to the \(SO(3, C)\) gauge transformation of \(F^a_{\mu\nu} [A]\) into a new curvature \(f^a_{\mu\nu} [a]\) for some connection \(a^a_{\mu} dx^\mu\), which is just \(A^a = A^a_{\mu} dx^\mu\) in another gauge. The relation is given by

\[
a = (e^{-\theta \cdot T})(A + d)(e^{\theta \cdot T}).
\]

Hence, the rotation of (52) into this \(SO(3, C)\) frame yields
\[ I_{\text{Inst}} = \int_M d^4x \left( \frac{1}{8} \lambda f f' [a] f f' [a] \epsilon^{\mu \nu \rho \sigma} \right) - N (\det b)^{1/2} \sqrt{\lambda_1 \lambda_2 \lambda_3} \left( A + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \). \] (55)

As shown in Paper II, we can implement the Gauss’ law constraint (149) and choose the gauge \( a_0 = 0 \), which puts (55) into canonical form. From this point we can implement the Hamiltonian constraint and use (43) to construct a Hamilton–Jacobi functional.

### 4.1 Canonical equivalence to the Ashtekar variables

We will now show that the phase space of the instanton representation on globally holonomic configurations is equivalent to the physical phase space. The commutation relations for the Ashtekar variables are given by

\[ [A^a_i (x), \tilde{\sigma}^b_j (y)] = \delta^a_b \delta^i_j \delta^{(3)} (x, y), \] (56)

where we have omitted the time dependence to avoid cluttering up the notation. Let us now substitute the CDJ Ansatz \( \tilde{\sigma}^i_a = \psi_{ae} B^i_e \) into (56)

\[ [A^a_i (x), \psi_{be} (y) B^i_j (y)] = \delta^a_o \delta^i_j \delta^{(3)} (x, y). \] (57)

We will now multiply (57) by \( A^c_j (y) \) in the following form

\[ [A^a_i (x), \psi_{be} (y) B^i_j (y) A^c_j (y)] = \delta^a_o A^c_i (y) \delta^{(3)} (x, y), \] (58)

which is allowed since \( [A^a_i, A^c_j] = 0 \) for the Ashtekar connection. Define the magnetic helicity density matrix \( C_{ce} = A^b_c B^i_e \), written in component form as

\[ C_{ce} = \epsilon^{ijk} A^c_i \partial_j A^e_k + \delta_{ce} (\det A), \] (59)

which has a diagonal part free of spatial gradients and an off-diagonal part containing spatial gradients. Then the commutation relations read

\[ [A^a_i (x), \psi_{be} (y) C_{ce} (y)] = \delta^a_o A^c_i (y) \delta^{(3)} (x, y). \] (60)

The kinematic configuration space \( \Gamma_{K_{\text{in}}} \) must have three degrees of freedom per point.\(^{21}\) Let us choose, without loss of generality, for these D.O.F. to

\(^{21}\)This is nine total degrees of freedom, minus three corresponding to \( G_a \), and minus three corresponding to \( H_i \).
be the three diagonal elements $A^a_i = \delta^a_i A_a$. Then we can set $a = i$ in (60) to obtain

$$[A^a_i(x), \Psi_{be}(y)C_{ce}(y)] = \delta^a_b A^e_b(y)\delta^{(3)}(x,y).$$ \hspace{1cm} (61)

Since $A^a_i$ is diagonal by supposition, then the only nontrivial contribution to (61) occurs for $a = c$. Since $a = b$ also is the only nontrivial contribution, it follows that $b = c$ as well. Hence the commutation relations for diagonal connection are given by

$$[A^a_i(x), \Psi_{be}(y)C_{be}(y)] = \delta^a_b \delta A^e_b(y)\delta^{(3)}(x,y).$$ \hspace{1cm} (62)

Substituting (59) subject to a diagonal connection into (62) we have

$$\sum_{e=1}^3 [A^a_i(x), \Psi_{be}(y)C_{be}(y)] + \sum_{e=1}^3 [A^a_i(x), \Psi_{be}(y)\epsilon^{bje}A^e_b \partial_j A^e_c] = \delta^a_b A^e_b(y)\delta^{(3)}(x,y),$$ \hspace{1cm} (63)

which has split up into two terms. We have been explicit in putting in the summation symbol to indicate that $e$ is a dummy index, while $a$ and $b$ are not. There are two cases to consider, $e = b$ and $e \neq b$. For $e \neq b$ the first term of (63) vanishes, leaving remaining the second term. Since the right hand side stays the same, then this would correspond to the commutation relations for a CDJ matrix whose diagonal components are zero. For the second possibility $e = b$ the second term of (63) vanishes while the first term survives, with the right hand side the same as before. This case occurs only if the CDJ matrix $\Psi_{ae}$ is diagonal. Let us choose $\Psi_{ae} = \text{Diag}(\lambda_1, \lambda_2, \lambda_3)$ as the diagonal matrix of eigenvalues, then (63) reduces to

$$[A^a_i(x), \lambda_b(y)(\det A(y))] = \delta^a_b A^a_b(y)\delta^{(3)}(x,y).$$ \hspace{1cm} (64)

The conclusion is that in order for (64) to have arisen from (56), that: (i) The antisymmetric part of $\Psi_{ae}$ must be zero, namely, the diffeomorphism constraint must be satisfied. (ii) The symmetric off-diagonal part of $\Psi_{ae}$ is not part of the commutation relations on the diffeomorphism invariant phase space $\Omega_{diff}$. Given the eigenvalues $\lambda_f$ on this space, the Gauss’ law constraint can be solved separately from the quantization process. The choice of diagonal $A^a_a$ is consistent with the implementation of the kinematic constraints, which means that only the Hamiltonian constraint is necessary to obtain the physical phase space $\Omega_{phys}$. 

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Equation (64) is not canonical owing to the field-dependence on the right hand side, however it implies canonical relations according to the following

\[
\ln\left(\frac{A^a(x)}{a_0}\right), \Pi_b(y) \right] = (A^{-1}(x))^a\left[A^a(x), \Pi_b(y)\right] = (A^{-1}(y))^a\left[A^a(x), \Pi_b(y)\right], \tag{65}
\]

where we have defined \(\Pi_b = \lambda_b(\det A)\). The first step of (65) follows from the chain rule, and the second step follows from the fact that the only nontrivial contribution comes from \(x = y\). Comparison of (65) with (64) implies that the canonical version of (64) is given by

\[
\left[X^a(x), \Pi_b(y)\right] = \delta^a_b \delta^{(3)}(x, y), \tag{66}
\]

where we have defined \(X^a = \ln(A^a/a_0)\). We have shown that \(\Omega_{Kin}\) of the instanton representation admits a cotangent bundle structure with diagonal connection \(A^a(x)\). It happens from (56) that \(A^a(x)\) is canonically conjugate to \(\tilde{\sigma}^a(x)\). Since the instanton representation maps to the Ashtekar formalism and vice versa on the unreduced phase space for nondegenerate \(B^a\), it follows that (66) corresponds as well to the kinematic phase space of the Ashtekar variables for \((\det A) \neq 0\), six total degrees of freedom per point, where the variables are diagonal. The bonus is that all the kinematic constraints have been implemented, leaving behind the Hamiltonian constraint which in the instanton representation is easy to solve.

We have previously shown that each nondegenerate \(A^a_i\) with six out of nine elements set to zero admits globally holonomic coordinates in the instanton representation. Since \(A^a_i\) serves also as the configuration variable for the Ashtekar phase space \(\Omega_{Ash}\), it follows that on this subspace the densitized triad must also be nondegenerate. Hence

\[
\left[A^f_i(x, t), \tilde{\sigma}^g_j(y, t)\right] = \delta^f_g \delta^{(3)}(x, y). \tag{67}
\]

The result is that the kinematic phase space of the instanton representation corresponds to nondegenerate triads, in the original Ashtekar variables, at the level prior to implementation of the Hamiltonian constraint.

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22While (64) are not canonical commutation relations, they are affine commutation relations which serve as an intermediate step in the formulation of canonical commutation relations. Affine commutation relations have been used by Klauder in [22] in the affine quantum gravity programme, and are viable as well in the instanton representation.

23This is because the inverse is the same as the reciprocal for a diagonal connection. The coordinate ranges are \(\infty < X < \infty\), which corresponds to \(0 < A^a_i < \infty\), which is a subset of the latter. To utilize the full range of \(A^a_i\), which includes the degenerate cases, one may instead use (64).
5 Verification of the quantizable configurations for the instanton representation

We will now demonstrate the results of the previous subsection by explicitly computing the allowed configurations which may be used globally as configuration space variables for quantization of the instanton representation. This will lead us to six quantizable configurations in the full theory. Let us define $\phi_i^a \equiv A_i^a$ as the resulting spatial connection, in terms of which we will derive the canonical structure. It is convenient for bookkeeping purposes, starting from the Ashtekar magnetic field

$$B_a^i = \epsilon^{ijk} \partial_j A_k^a + \frac{1}{2} \epsilon^{ijk} f_{abc} A_j^b A_k^c,$$  \hspace{1cm} (68)

to write out explicitly the individual components and group into

$$B_1^1 = \partial_2 A_3^1 - \partial_3 A_2^1 + A_2^2 A_3^3 - A_2^2 A_2^3;$$
$$B_2^2 = \partial_3 A_1^2 - \partial_1 A_3^2 + A_3^3 A_1^1 - A_3^3 A_1^3;$$
$$B_3^3 = \partial_1 A_2^3 - \partial_2 A_1^3 + A_1^1 A_2^2 - A_1^1 A_2^2;$$  \hspace{1cm} (69)

for the diagonal components, and

$$B_1^1 = \partial_2 A_3^2 - \partial_3 A_2^2 + A_2^2 A_3^3 - A_2^2 A_2^3;$$
$$B_2^2 = \partial_3 A_1^3 - \partial_1 A_3^3 + A_3^3 A_1^1 - A_3^3 A_1^3;$$
$$B_3^3 = \partial_1 A_2^1 - \partial_2 A_1^1 + A_1^1 A_2^2 - A_1^1 A_2^2;$$  \hspace{1cm} (70)

and

$$B_1^2 = \partial_3 A_1^1 - \partial_1 A_3^1 + A_3^3 A_1^3 - A_3^3 A_1^3;$$
$$B_2^2 = \partial_1 A_2^2 - \partial_2 A_1^2 + A_1^1 A_2^1 - A_1^1 A_2^1;$$
$$B_3^3 = \partial_2 A_3^3 - \partial_3 A_2^3 + A_2^2 A_3^2 - A_2^2 A_3^2;$$  \hspace{1cm} (71)

for the off-diagonal components. Using (69), (70) and (71) we will explicitly determine the configurations that yield the desired canonical structure.

The first configuration, where $A_i^a = \delta_{ai} A_i^a$ is diagonal, is given by

$$A_i^a = \begin{pmatrix} A_1^1 & 0 & 0 \\ 0 & A_2^2 & 0 \\ 0 & 0 & A_3^3 \end{pmatrix}; \hspace{1cm} B_a^i = \begin{pmatrix} A_1^1 A_2^2 & -\partial_3 A_2^2 & \partial_2 A_3^3 \\ \partial_3 A_1^1 & A_3^3 A_1^1 & -\partial_1 A_3^3 \\ -\partial_2 A_1^1 & \partial_1 A_2^2 & A_1^1 A_2^2 \end{pmatrix};$$
\[
B_i^e \dot{A}_i^e = \begin{pmatrix}
A_2^3 A_3^3 \dot{A}_1^1 & - (\partial_3 A_2^3) \ddot{A}_2^2 & (\partial_2 A_3^3) \ddot{A}_3^3 \\
(\partial_3 A_1^1) \dot{A}_1^1 & A_3^3 A_1^1 \dot{A}_2^2 & - (\partial_1 A_3^3) \ddot{A}_3^3 \\
- (\partial_2 A_1^1) \dot{A}_1^1 & (\partial_1 A_2^2) \dot{A}_2^2 & A_1^1 A_2^2 \ddot{A}_3^3
\end{pmatrix}
\]

Upon contraction with a diagonal CDJ matrix \( \Psi_{ae} = \delta_{ae} \lambda_{ee} \) this leads to the canonical structure

\[
\lambda_{11} A_2^2 A_3^3 \dot{A}_1^1 + \lambda_{22} A_3^3 A_1^1 \dot{A}_2^2 + \lambda_{33} A_1^1 A_2^2 \ddot{A}_3^3 \\
= (A_1^1 A_2^2 A_3^3) \left[ \lambda_{11} \left( \frac{\dot{A}_1^1}{A_1^1} \right) + \lambda_{22} \left( \frac{\dot{A}_2^2}{A_2^2} \right) + \lambda_{33} \left( \frac{\ddot{A}_3^3}{A_3^3} \right) \right],
\]

where \( \det A = A_1^1 A_2^2 A_3^3 \). Note in (72) that all spatial gradients of \( A_i^a \) have been cancelled out. Making the definitions

\[
\Pi_f = \lambda_{ff} (\det A); \quad X^1 = \ln \left( \frac{A_1^1}{a_0} \right); \quad X^2 = \ln \left( \frac{A_2^2}{a_0} \right); \quad X^3 = \ln \left( \frac{A_3^3}{a_0} \right),
\]

this yields the symplectic two form

\[
\Omega_{Kin} = \int \Sigma \delta \Pi_f (x) \wedge \delta X^f (x) = \delta \left( \int \Sigma \Pi_f (x) \delta X^f (x) \right) = \delta \theta_{Kin},
\]

which is the exact functional variation of the canonical one form \( \theta_{Kin} \) on the kinematic phase space. This implies canonical commutation relations

\[
[X^f (x, t), \Pi_g (y, t)] = \delta^f_g \delta^{(3)} (x, y).
\]

Hence we have obtained globally holonomic coordinates on the kinematic phase space \( \Omega_{Kin} \) of the instanton representation, even though such coordinates do not exist on the full phase space \( \Omega \). Equation (73) provides the degrees of freedom which can be used for quantization of the full theory, and ranges of the coordinates are

\[
-\infty < |X^f| < \infty; \quad 0 < |A_f^a| < \infty.
\]

Note that the canonical commutation relations (75) can also be written in the form

\[
[A_f^a (x, t), \Pi_g^b (y, t)] = \delta^a_g A_f^b \delta^{(3)} (x, y),
\]

which are affine commutation relations analogous to the type introduced in [22]. We will refer to such configurations, where \( A_i^a \) is nondegenerate and
has three nonzero entries, as quantizable. In (77) we have treated Π_f, the densitized eigenvalues, as the fundamental momentum space variable. But it is really a composite variable, and (77) can be written in self-adjoint form as

\[
[A^f_j(x, t), \frac{1}{2}(\dot{\lambda}_g(y, t)(\det A) + (\det A)\dot{\lambda}_g(y, t))] = \delta^f_g \delta^{(3)}(x, y).
\]

(78)

In the original Ashtekar variables this corresponds to canonical commutation relations

\[
\begin{bmatrix}
A^1_1(x, t) & 0 & 0 \\
0 & A^2_2(x, t) & 0 \\
0 & 0 & A^3_3(x, t)
\end{bmatrix}
\begin{bmatrix}
\bar{\sigma}^1_1(y, t) & 0 & 0 \\
0 & \bar{\sigma}^2_2(y, t) & 0 \\
0 & 0 & \bar{\sigma}^3_3(z, t)
\end{bmatrix}
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \delta^{(3)}(x, y),
\]

which involves only the corresponding nondegenerate components of the densitized triad on the kinematical phase space. The conclusion is that upon implementation of the kinematic constraints the instanton representation can be quantized in the full theory, and maps to the corresponding quantization on the Ashtekar phase space evaluated on nondegenerate triads.

### 5.1 Second quantizable configuration

The second case is given by

\[
A^i_i = \begin{bmatrix}
A^1_i & 0 & 0 \\
0 & A^2_i & 0 \\
0 & 0 & A^3_i
\end{bmatrix};
B^i_a = \begin{bmatrix}
-A^2_i A^3_i & \partial_2 A^3_i & -\partial_3 A^3_i \\
\partial_3 A^1_i & -\partial_1 A^2_i & -A^1_i A^3_i \\
-\partial_2 A^1_i & -A^2_i A^3_i & \partial_1 A^3_i
\end{bmatrix};
\]

\[
B^i_c A^i_c = \begin{bmatrix}
-A^3_i A^2_i A^1_i & -\partial_3 A^2_i A^3_i & (\partial_2 A^3_i)^2 A^1_i \\
(\partial_3 A^1_i) A^1_i & \partial_2 A^3_i A^3_i & -\partial_1 A^3_i A^3_i \\
-(\partial_2 A^1_i) A^1_i & (\partial_3 A^2_i) A^2_i & -A^2_i A^2_i A^3_i
\end{bmatrix}.
\]

Upon contraction with a diagonal CDJ matrix \(\Psi_{ae} = \delta_{ae} \lambda_e\) this leads to the canonical structure

\[
-\lambda_{11} A^2_i A^3_i A^1_i - \lambda_{22} A^1_i A^2_i A^3_i - \lambda_{33} A^3_i A^3_i A^1_i
\]

\[
= -(A^2_i A^3_i A^1_i) \left[ \lambda_{11} \left( \frac{A^1_i}{A^1_i} \right) + \lambda_{22} \left( \frac{A^2_i}{A^2_i} \right) + \lambda_{33} \left( \frac{A^3_i}{A^3_i} \right) \right].
\]

(79)

where \(\det A = A^2_i A^3_i A^1_i\). Making the definitions

\[
\Pi_f = \lambda_{ff}(\det A); \quad X^1 = \ln \left( \frac{A^1_i}{a_0} \right); \quad X^2 = \ln \left( \frac{A^2_i}{a_0} \right); \quad X^3 = \ln \left( \frac{A^3_i}{a_0} \right),
\]

(80)
This can also be written in the form degrees of freedom which can be used for quantization of the full theory.

\[ \{X^f(x, t), \Pi_f(y, t)\} = \delta^f_y \delta^{(3)}(x, y). \quad (81) \]

Hence we have obtained globally holonomic coordinates on the kinematic phase space \( \Omega_{Kin} \) of the instanton representation, even though such coordinates do not exist on the full phase space \( \Omega \). Equation (80) provides the degrees of freedom which can be used for quantization of the full theory. This can also be written in the form

\[ \{A^f_i(x, t), \Pi_i(y, t)\} = \delta^f_y A^i_3 \delta^{(3)}(x, y), \quad (82) \]

which are affine commutation relations. We will refer to such configurations, where \( A^i_3 \) is nondegenerate and has three nonzero entries, as quantizable. In the original Ashtekar variables this corresponds to canonical commutation relations

\[
\begin{pmatrix}
A^1_1(x, t) & 0 & 0 \\
0 & 0 & A^2_2(x, t) \\
0 & A^2_2(x, t) & 0
\end{pmatrix}, 
\begin{pmatrix}
\bar{\sigma}^1_1(y, t) & 0 & 0 \\
0 & 0 & \bar{\sigma}^2_2(y, t) \\
0 & \bar{\sigma}^2_2(y, t) & 0
\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \end{pmatrix} \delta^{(3)}(x, y),
\]

**5.2 Third quantizable configuration**

The third case is given by

\[ A^i_3 = \begin{pmatrix} 0 & 0 & A^3_1 \\ 0 & A^2_3 & 0 \\ A^1_3 & 0 & 0 \end{pmatrix}; \quad B^i_a = \begin{pmatrix} \partial_2 A^1_3 & -\partial_1 A^2_3 & -A^1_3 A^2_3 \\ -\partial_3 A^1_3 & -A^2_3 A^1_3 & -A^3_3 A^1_3 \\ -A^2_3 A^1_3 & \partial_1 A^3_3 & -\partial_2 A^3_3 \end{pmatrix}; \]

\[ B^i_e A^e_3 = \begin{pmatrix} -A^1_3 A^2_3 A^3_1 & -(\partial_3 A^2_3) A^2_1 \\ (\partial_3 A^2_3) A^1_3 & -A^1_3 A^2_3 A^2_1 \\ -(\partial_2 A^3_1) A^1_3 & (\partial_2 A^3_1) A^1_3 \end{pmatrix} \]

Upon contraction with a diagonal CDJ matrix \( \Psi_{ae} = \delta_{ae} \lambda_{ee} \) this leads to the canonical structure

\[ -\lambda_{11} A^1_3 A^2_3 A^3_1 - \lambda_{22} A^3_1 A^1_3 A^2_3 - \lambda_{33} A^2_3 A^3_1 A^3_3 = -(A^1_3 A^2_3 A^3_1) \left[ \lambda_{11} \left( \frac{A^3_1}{A^1_3} \right) + \lambda_{22} \left( \frac{A^2_3}{A^2_3} \right) + \lambda_{33} \left( \frac{A^1_3}{A^3_3} \right) \right], \quad (83) \]

where \( \det A = A^1_3 A^2_3 A^3_1 \). Making the definitions
\[ \Pi_f = \lambda_{ff}(\det A); \quad X^1 = \ln \left( \frac{A_3^3}{a_0} \right); \quad X^2 = \ln \left( \frac{A_2^2}{a_0} \right); \quad X^3 = \ln \left( \frac{A_1^1}{a_0} \right), \quad (84) \]

This yields the canonical commutation relations

\[ [X^f(x, t), \Pi_f(y, t)] = \delta^f_y \delta^{(3)}(x, y). \quad (85) \]

Hence we have obtained globally holonomic coordinates on the kinematic phase space \( \Omega_{Kin} \) of the instanton representation, even though such coordinates do not exist on the full phase space \( \Omega \). Equation (84) provides the degrees of freedom which can be used for quantization of the full theory. This can also be written in the form

\[ [A^f_j(x, t), \Pi_j(y, t)] = \delta^f_y A^q_y \delta^{(3)}(x, y), \quad (86) \]

which are affine commutation relations. We will refer to such configurations, where \( A^q \) is nondegenerate and has three nonzero entries, as quantizable. In the original Ashtekar variables this corresponds to canonical commutation relations

\[ \left[ \begin{array}{ccc}
0 & 0 & A_1^1(x, t) \\
0 & A_2^2(x, t) & 0 \\
A_1^1(x, t) & 0 & 0
\end{array} \right], \left[ \begin{array}{ccc}
0 & 0 & \bar{\sigma}_2^1(y, t) \\
0 & \sigma_2^1(y, t) & 0 \\
\bar{\sigma}_1^1(y, t) & 0 & 0
\end{array} \right] = \left[ \begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array} \right] \delta^{(3)}(x, y), \]

### 5.3 Fourth quantizable configuration

The fourth case, which concludes the configurations containing at least one diagonal connection element, is given by

\[ A_q^i = \begin{pmatrix} 0 & A_1^1 & 0 \\ A_2^2 & 0 & 0 \\ 0 & 0 & A_3^3 \end{pmatrix}; \quad B_a^i = \begin{pmatrix}
-\partial_3 A_2^1 & -A_3^3 A_1^1 & 0 \\
-A_1^1 A_2^2 & \partial_3 A_1^1 & A_1^3 \\
\partial_1 A_2^2 & -\partial_2 A_1^2 & -A_2^2 A_3^3
\end{pmatrix}; \]

\[ B_e^i \dot{A}_q^i = \begin{pmatrix}
-A_3^3 \dot{A}_2^1 & -\partial_3 A_2^1 \dot{A}_2^1 & (\partial_2 A_3^3) \dot{A}_3^3 \\
(\partial_3 A_2^1) \dot{A}_2^1 & -A_2^2 A_3^3 \dot{A}_3^3 & -\partial_1 A_3^3 \dot{A}_3^3 \\
-\partial_2 A_3^3 \dot{A}_1^1 & (\partial_1 A_3^3) \dot{A}_2^2 & -A_2^2 A_3^3 \dot{A}_3^3
\end{pmatrix}. \]

Upon contraction with a diagonal CDJ matrix \( \Psi_{ae} = \delta_{ae} \lambda_{ee} \) this leads to the canonical structure

\[ \lambda_{11} A_3^3 A_2^1 A_1^1 \dot{A}_3^3 - \lambda_{22} A_1^1 A_3^3 \dot{A}_2^2 - \lambda_{33} A_2^2 A_1^1 \dot{A}_3^3 \\
= -A_3^3 \dot{A}_2^1 \left[ \lambda_{11} \left( \frac{\dot{A}_2^1}{A_1^1} \right) + \lambda_{22} \left( \frac{\dot{A}_2^2}{A_1^1} \right) + \lambda_{33} \left( \frac{\dot{A}_3^3}{A_1^1} \right) \right]. \quad (87) \]
where \( \det A = A_3^2 A_2^1 A_1^2 \). Making the definitions

\[
\Pi_f = \lambda_f f(\det A); \quad X^1 = \ln \left( \frac{A_3^1}{a_0} \right); \quad X^2 = \ln \left( \frac{A_2^2}{a_0} \right); \quad X^3 = \ln \left( \frac{A_1^3}{a_0} \right),
\]

this yields the canonical commutation relations

\[
[X^f(x, t), \Pi_f(y, t)] = \delta^f_y \delta^{(3)}(x, y).
\]

Hence we have obtained globally holonomic coordinates on the kinematic phase space \( \Omega_{Kin} \) of the instanton representation, even though such coordinates do not exist on the full phase space \( \Omega \). Equation (88) provides the degrees of freedom which can be used for quantization of the full theory. This can also be written in the form

\[
[A_f^i(x, t), \Pi_g(y, t)] = \delta^i_g A^g \delta^{(3)}(x, y),
\]

which are affine commutation relations. We will refer to such configurations, where \( A_i^a \) is nondegenerate and has three nonzero entries, as quantizable. In the original Ashtekar variables this corresponds to canonical commutation relations

\[
\begin{pmatrix}
0 & A_2^1(x, t) & 0 \\
A_2^1(x, t) & 0 & 0 \\
0 & 0 & A_3^3(x, t)
\end{pmatrix}
\begin{pmatrix}
0 & \tilde{\sigma}_1^1(y, t) & 0 \\
\tilde{\sigma}_1^1(y, t) & 0 & 0 \\
0 & 0 & \tilde{\sigma}_3^3(y, t)
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \delta^{(3)}(x, y),
\]

**5.4 Fifth quantizable configuration**

The fifth case, involving the even permutations, is given by

\[
A_i^a = \begin{pmatrix}
0 & A_2^1 & 0 \\
0 & 0 & A_3^2 \\
A_1^3 & 0 & 0
\end{pmatrix}; \quad B_a^i = \begin{pmatrix}
-\partial_3 A_2^1 & \partial_2 A_3^2 & A_2^1 A_3^2 \\
A_3^2 A_1^3 & -\partial_1 A_3^2 & \partial_3 A_1^3 \\
\partial_1 A_2^1 & A_1^3 A_2^2 & -\partial_2 A_1^3
\end{pmatrix};
\]

\[
B_a^i \dot{A}_i^a = \begin{pmatrix}
A_1^3 A_2^2 A_3^1 & -(\partial_3 A_1^3) \dot{A}_2^1 & (\partial_2 A_1^3) \dot{A}_3^2 \\
(\partial_3 A_2^2) \dot{A}_1^3 & A_2^2 A_3^1 \dot{A}_1^3 & -(\partial_1 A_2^2) \dot{A}_3^2 \\
-(\partial_2 A_3^1) \dot{A}_1^3 & (\partial_1 A_3^1) \dot{A}_2^2 & A_3^1 A_2^2 A_1^3
\end{pmatrix}.
\]

Upon contraction with a diagonal CDJ matrix \( \Psi_{ae} = \delta_{ae} \lambda_{ee} \) this leads to the canonical structure

\[
\lambda_{11} A_2^1 A_3^2 \dot{A}_3^1 + \lambda_{22} A_3^2 A_1^3 \dot{A}_2^1 + \lambda_{33} A_1^3 A_2^2 \dot{A}_3^2
\]

\[
= (A_2^1 A_3^2 A_1^3) \left[ \lambda_{11} \left( \frac{A_3^1}{A_1^3} \right) + \lambda_{22} \left( \frac{A_2^1}{A_2^2} \right) + \lambda_{33} \left( \frac{A_3^2}{A_3^3} \right) \right].
\]
where \( \det A = A_1^2 A_2^2 A_3^2 \). Making the definitions

\[
\Pi_f = \lambda_f (\det A); \quad X^1 = \ln \left( \frac{A_3^3}{a_0} \right); \quad X^2 = \ln \left( \frac{A_2^2}{a_0} \right); \quad X^3 = \ln \left( \frac{A_1^2}{a_0} \right),
\]

this yields the canonical commutation relations

\[
[X^f(x, t), \Pi_f(y, t)] = \delta^f_y \delta^{(3)}(x, y).
\]

Hence we have obtained globally holonomic coordinates on the kinematic phase space \( \Omega_{Kin} \) of the instanton representation, even though such coordinates do not exist on the full phase space \( \Omega \). Equation (92) provides the degrees of freedom which can be used for quantization of the full theory. This can also be written in the form

\[
[A^f(x, t), \Pi_g(y, t)] = \delta^f_g A^g \delta^{(3)}(x, y),
\]

which are affine commutation relations. We will refer to such configurations, where \( A^i \) is nondegenerate and has three nonzero entries, as quantizable. In the original Ashtekar variables this corresponds to canonical commutation relations

\[
\begin{pmatrix}
A_1^1(x, t) & 0 & 0 \\
0 & A_2^2(x, t) & 0 \\
A_3^3(x, t) & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & \tilde{\sigma}_1^1(y, t) & 0 \\
0 & 0 & \tilde{\sigma}_2^2(y, t) \\
\tilde{\sigma}_3^3(y, t) & 0 & 0
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix} \delta^{(3)}(x, y),
\]

5.5 Sixth quantizable configuration

The sixth case, involving the odd permutations, is given by

\[
\begin{pmatrix}
0 & 0 & A_3^1 \\
A_1^2 & 0 & 0 \\
0 & A_2^3 & 0
\end{pmatrix};
\begin{pmatrix}
\partial_2 A_1^1 & A_2^2 A_1^3 & -\partial_2 A_1^3 \\
-\partial_3 A_1^1 & \partial_3 A_1^2 & A_1^3 A_1^3 \\
A_1^2 & -\partial_2 A_1^2 & \partial_1 A_1^3
\end{pmatrix};
\begin{pmatrix}
A_1^2 A_3^3 \hat{A}_1^2 & -\partial_3 A_1^3 \hat{A}_1^2 \\
(\partial_1 A_1^1) \hat{A}_1^2 & A_1^3 A_2^2 \hat{A}_1^3 & -\partial_1 A_1^3 \hat{A}_1^3 \\
-\partial_2 A_1^3 \hat{A}_1^1 & (\partial_3 A_1^2) \hat{A}_1^3 & A_1^2 A_3^3 \hat{A}_1^3
\end{pmatrix}.
\]

Upon contraction with a diagonal CDJ matrix \( \Psi_{ae} = \delta_{ae} \lambda_{ee} \) this leads to the canonical structure

\[
\lambda_{11} A_1^2 A_1^3 \hat{A}_1^2 + \lambda_{22} A_2^2 A_1^2 \hat{A}_2^2 + \lambda_{33} A_1^2 A_3^2 \hat{A}_3^2
\]

\[
= (A_2^2 A_3^3 A_1^1) \left[ \lambda_{11} \left( \frac{A_2^2}{A_1^2} \right) + \lambda_{22} \left( \frac{A_3^3}{A_2^2} \right) + \lambda_{33} \left( \frac{A_1^1}{A_3^3} \right) \right],
\]
where \( \det A = A_3^2 A_3^1 A_3^2 \). Making the definitions

\[
\Pi_f = \lambda_f (\det A); \quad X^1 = \ln \left( \frac{A_2^2}{a_0} \right); \quad X^2 = \ln \left( \frac{A_3^2}{a_0} \right); \quad X^3 = \ln \left( \frac{A_1^1}{a_0} \right),
\]

this yields the canonical commutation relations

\[
[X^f(x,t), \Pi_f(y,t)] = \delta_y^f \delta^{(3)}(x,y).
\]

Hence we have obtained globally holonomic coordinates on the kinematic phase space \( \Omega_{Kin} \) of the instanton representation, even though such coordinates do not exist on the full phase space \( \Omega \). Equation (92) provides the degrees of freedom which can be used for quantization of the full theory. This can also be written in the form

\[
[A^f(x,t), \Pi_g(y,t)] = \delta_g^f A^g \delta^{(3)}(x,y),
\]

which are affine commutation relations. We will refer to such configurations, where \( A_i^c \) is nondegenerate and has three nonzero entries, as quantizable. In the original Ashtekar variables this corresponds to canonical commutation relations

\[
\left\lfloor \begin{array}{ccc}
0 & 0 & A_3^1(x,t) \\
A_1^1(x,t) & 0 & 0 \\
0 & A_2^2(x,t) & 0
\end{array} \right\rfloor, \quad \left( \begin{array}{ccc}
0 & 0 & \tilde{\sigma}_3^1(y,t) \\
\tilde{\sigma}_1^2(y,t) & 0 & 0 \\
0 & \tilde{\sigma}_2^3(y,t) & 0
\end{array} \right) = \left( \begin{array}{ccc}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array} \right) \delta^{(3)}(x,y).
\]

We have identified six distinct configurations of the configuration space \( \Gamma_{Inst} \) for which a well-defined canonical structure can be defined. Note, while the spatial gradients do not appear in these configurations, that they correspond to the full theory and not minisuperspace.\(^{24}\) Note that these are not simply repetitions of the same configuration, since they correspond to different specific combinations of Ashtekar connection components \( A_i^a \) preferentially selected by the momentum space variables \( \lambda_f \). This allows \( X^{ae} \) to be well-defined as a canonical variable, and will form the basis for quantization of the full theory.\(^{25}\)

\(^{24}\)Hence for each of these configurations the components of the connection \( A_i^c = A_i^c(x) \) can be chosen differently at each spatial point, defining three continuous functions of position. Upon quantization, one would be quantizing the infinite dimensional spaces of field theory and not quantum mechanics, as in minisuperspace.

\(^{25}\)As \( \Psi_{ae} \equiv \delta_{af} \lambda_f \) will correspond to the diagonal matrix of eigenvalues, we will restrict our quantization to spacetimes of Petrov type \( I, D \) and \( O \).
Another interesting configuration containing three D.O.F. is given by an Ashtekar connection $A^a_i$ where the diagonal elements are zero

$$A^a_i = \begin{pmatrix} 0 & A^1_2 & A^1_3 \\ A^2_1 & 0 & A^2_3 \\ A^3_1 & A^3_2 & 0 \end{pmatrix};$$

$$B^i_a = \begin{pmatrix} \partial_2 A^3_3 - \partial_3 A^3_2 - A^2_2 A^3_3 \\ -\partial_1 A^3_3 + A^2_2 A^1_3 \\ \partial_1 A^3_1 + A^1_2 A^2_3 \end{pmatrix}, \quad \begin{pmatrix} \partial_2 A^3_3 - \partial_3 A^3_2 - A^2_2 A^3_3 \\ -\partial_1 A^3_3 + A^2_2 A^1_3 \\ \partial_1 A^3_1 + A^1_2 A^2_3 \end{pmatrix} = \begin{pmatrix} \partial_2 A^3_3 - \partial_3 A^3_2 - A^2_2 A^3_3 \\ -\partial_1 A^3_3 + A^2_2 A^1_3 \\ \partial_1 A^3_1 + A^1_2 A^2_3 \end{pmatrix} + \begin{pmatrix} \partial_2 A^3_3 - \partial_3 A^3_2 - A^2_2 A^3_3 \\ -\partial_1 A^3_3 + A^2_2 A^1_3 \\ \partial_1 A^3_1 + A^1_2 A^2_3 \end{pmatrix};$$

Each component of $B^i_a$ contains spatial gradients, which is problematic for the quantization. One may circumvent this by restricting oneself to spatially homogeneous connections, which yields a magnetic field of

$$B^i_a = \begin{pmatrix} -A^2_3 A^3_3 & A^2_3 A^3_1 & A^1_3 A^3_3 \\ A^2_3 A^3_1 & -A^2_3 A^3_1 & A^1_3 A^3_1 \\ A^3_3 A^3_1 & A^3_3 A^3_1 & -A^2_3 A^3_1 \end{pmatrix}.$$  

The corresponding symplectic structure for the instanton representation is given by

$$B^i_a \dot{A}^a_i = \begin{pmatrix} A^3_3 A^1_3 A^2_3 + A^3_3 A^1_3 A^3_3 & -A^3_3 A^2_3 A^1_3 + A^3_3 A^3_1 A^2_3 & -A^3_3 A^2_3 A^3_1 + A^3_3 A^3_1 A^3_1 \\ -A^2_3 A^3_3 A^1_1 + A^2_3 A^2_3 A^1_1 & -A^3_3 A^1_3 A^2_1 + A^3_3 A^3_1 A^2_1 & -A^3_3 A^2_3 A^3_1 + A^3_3 A^3_1 A^3_1 \\ A^3_3 A^1_1 A^2_1 - A^2_3 A^1_1 A^3_1 & A^3_3 A^3_1 A^2_1 + A^3_3 A^2_3 A^3_1 & -A^3_3 A^2_3 A^3_1 + A^3_3 A^3_1 A^3_1 \end{pmatrix}.$$  

There is no obvious way to obtain a canonical structure using the off-diagonal terms. However, for symmetric connections $A^a_i$, namely

$$A^1_2 = A^2_1; \quad A^2_3 = A^3_2; \quad A^3_1 = A^1_3,$$

the diagonal terms in $B^i_a \dot{A}^a_i$ contain total time derivatives. Upon contraction with a diagonal CDJ matrix we obtain

$$\lambda_{11} A^3_3 \frac{d}{dt} (A^3_3 A^1_2) + \lambda_{22} A^3_1 \frac{d}{dt} (A^3_1 A^2_3) + \lambda_{33} A^3_2 \frac{d}{dt} (A^3_2 A^3_1)$$

$$= (A^3_2 A^2_3 A^3_1) \left[ \lambda_{11} \left( \frac{d}{dt} (A^3_3 A^1_2) \right) + \lambda_{22} \left( \frac{d}{dt} (A^3_1 A^2_3) \right) + \lambda_{33} \left( \frac{d}{dt} (A^3_2 A^3_1) \right) \right].$$

where $\det A = A^1_2 A^2_3 A^3_1$. Hence, even though $A^a_i$ is nondiagonal, it induces diagonal canonical variables which can be used for quantization albeit in minisuperspace when one defines variables

$$28$$
\begin{equation}
X^1 = A^3_1 A^1_1; \quad X^2 = A^1_2 A^2_3; \quad X^3 = A^2_3 A^3_1; \quad \Pi_f = \lambda_{ff}(\det A),
\end{equation}
where \( \det A = A^1_2 A^2_3 A^3_1 \). Then the following relation holds \([X^f, \Pi_g] = \delta^f_g \) for minisuperspace.

### 5.7 Relation to the metric theory

The phase space in metric variables is given by \((h_{ij}, \pi^{ij})\), which at the unconstrained level consists of twelve phase space degrees of freedom. Upon implementation of the diffeomorphism constraint \( H_i \), we should have a cotangent bundle structure with six phase space degrees of freedom. This constitutes the analogue of the kinematic phase space \( \Omega_{Kin} \) of the instanton representation, where the dynamics of the Hamiltonian constraint can be implemented to yield the physical phase space \( \Omega_{Phys} \). The Dirac method to quantize the metric representation would be to write the canonical commutation relations on the full unconstrained phase space

\begin{equation}
[h_{ij}(x, t), \pi^{mn}(y, t)] = \delta^m_i \delta^n_j \delta^{(3)}(x, y).
\end{equation}

On the reduced phase space corresponding to invariance under spatial diffeomorphisms, only three degrees of freedom from (102) should be quantized. On this space \( \Omega_{Kin} \) the metric \( h_{ij} \) should be diagonal,\(^{26}\) which in turn implies that the conjugate momentum \( \pi^{ij} \) must also be diagonal in order to preserve a cotangent bundle structure. Equation (102) is invariant under \( SO(3) \) transformations, an observation which we can exploit in obtaining \( \Omega_{Kin} \). For nondegenerate variables we can write

\begin{equation}
[O_{ik} h_k O^T_{kj}, U^{m'l'} \pi^l (U^T)^{ln}],
\end{equation}

where \( O \) and \( U \) are orthogonal matrices. It is shown in Paper IV that the \( SO(3, C) \) angles used to diagonalize the CDJ matrix \( \Psi_{ae} \) can at the canonical level be considered ignorable. The analogue for (103) would be to multiply by \( O^T_{ij} (U^T)^{m'm} O_{jj'} U^{nn'} \). If \( O = U \), then the metric and its conjugate momentum are diagonalized by the same \( SO(3) \) transformation, which transforms the relations into

\begin{equation}
\delta_{\nu'k} \delta_{j'k} \delta^{\alpha' \ell'} \delta^{m' \ell} [h_k(x, t), \pi^j(y, t)] = \delta^{m'}_{\nu'} \delta^{\ell'}_{j'} \delta^{(3)}(x, y).
\end{equation}

\(^{26}\)This is the only way to obtain a Riemannian metric, of signature \((1, 1, 1)\) on three dimensional space, which incidentally corresponds to a spacetime metric of signature \((-1, 1, 1)\).
Since only the diagonal configurations contribute, then the Kronecker deltas can be dropped and the canonical commutation relations on the kinematic phase space can be written as

\[ [h_k(x, t), \pi^l(y, t)] = \delta^l_k \delta^{(3)}(x, y). \] (105)

The diagonal metric can be mapped directly to the quantizable instanton configurations, which yields the six possibilities on the kinematic phase space

\[ h^{(1)}_{ij} = \begin{pmatrix} \tilde{\sigma}_1^1 \tilde{\sigma}_1^1 & 0 & 0 \\ 0 & \tilde{\sigma}_2^2 \tilde{\sigma}_2^2 & 0 \\ 0 & 0 & \tilde{\sigma}_3^3 \tilde{\sigma}_3^3 \end{pmatrix}; \quad h^{(2)}_{ij} = \begin{pmatrix} \tilde{\sigma}_1^1 \tilde{\sigma}_1^1 & 0 & 0 \\ 0 & \tilde{\sigma}_3^3 \tilde{\sigma}_3^3 & 0 \\ 0 & 0 & \tilde{\sigma}_2^2 \tilde{\sigma}_2^2 \end{pmatrix}; \]

\[ h^{(3)}_{ij} = \begin{pmatrix} \tilde{\sigma}_2^2 \tilde{\sigma}_2^2 & 0 & 0 \\ 0 & \tilde{\sigma}_3^3 \tilde{\sigma}_3^3 & 0 \\ 0 & 0 & \tilde{\sigma}_1^1 \tilde{\sigma}_1^1 \end{pmatrix}; \quad h^{(4)}_{ij} = \begin{pmatrix} \tilde{\sigma}_2^2 \tilde{\sigma}_2^2 & 0 & 0 \\ 0 & \tilde{\sigma}_1^1 \tilde{\sigma}_1^1 & 0 \\ 0 & 0 & \tilde{\sigma}_3^3 \tilde{\sigma}_3^3 \end{pmatrix}; \]

\[ h^{(5)}_{ij} = \begin{pmatrix} \tilde{\sigma}_3^3 \tilde{\sigma}_3^3 & 0 & 0 \\ 0 & \tilde{\sigma}_1^1 \tilde{\sigma}_1^1 & 0 \\ 0 & 0 & \tilde{\sigma}_2^2 \tilde{\sigma}_2^2 \end{pmatrix}; \quad h^{(6)}_{ij} = \begin{pmatrix} \tilde{\sigma}_3^3 \tilde{\sigma}_3^3 & 0 & 0 \\ 0 & \tilde{\sigma}_2^2 \tilde{\sigma}_2^2 & 0 \\ 0 & 0 & \tilde{\sigma}_1^1 \tilde{\sigma}_1^1 \end{pmatrix}. \]

The question then arises as to whether a diagonal 3-metric on the kinematic phase space excludes off-diagonal configurations for the spacetime metric \( g_{\mu \nu} \). The answer is no, since any off-diagonal parts of \( g_{\mu \nu} \) can be attributed to the existence of a nonvanishing shift vector \( N^i \). The purpose of the diffeomorphism constraint is to eliminate this contribution in bringing us to the kinematic phase space. Such off-diagonal terms, as we have shown, should not be on the same footing as the diagonal terms corresponding to the intrinsic \( SO(3, C) \) frame.
6 Self-dual Weyl curvature tensor

One of the outstanding issues in quantum gravity is to get a handle on the manifestation of the quantum theory in the classical limit in terms of measurable quantities. We have shown that there exist configurations on the kinematic phase space of the instanton representation which it is possible to quantize. The purpose of the next two sections will be to provide the interpretation of what will be quantized, in terms of directly measurable quantities of physical significance for general relativity in the semiclassical limit. This requires some introductory material on the Weyl curvature tensor and its relation to the physical degrees of freedom.

The instanton representation of Plebanski gravity implies that the physical degrees of freedom of gravity may be encoded within the self dual part of the Weyl curvature tensor, denoted \( W_{\mu\nu\rho\sigma} \). The Weyl curvature tensor \( C_{\mu\nu\rho\sigma} \) is the traceless part of the Riemann curvature \( R_{\mu\nu\rho\sigma} \), given by

\[
R_{\mu\nu\rho\sigma} = \frac{1}{6} (g_{\rho\nu} g_{\mu\sigma} - g_{\rho\mu} g_{\nu\sigma}) R + C_{\mu\nu\rho\sigma} + \frac{1}{2} (g_{\rho\mu} R_{\nu\sigma} - g_{\rho\nu} R_{\mu\sigma} - g_{\sigma\mu} R_{\nu\rho} + g_{\sigma\nu} R_{\mu\rho}).
\] (106)

The Weyl curvature tensor describes the nonlocal effects of radiation on curvature not including matter fields and encodes the algebraic classification of spacetime. Equation (106) can be decomposed into electric and magnetic parts \( E_{\mu\nu} \) and \( B_{\mu\nu} \) with respect to an observer with 4-velocity \( u^\mu \) tangent to a congruence of timelike integral curves as

\[
Q_{\mu\rho} = (C_{\mu\nu\rho\sigma} + i^* C_{\mu\nu\rho\sigma}) u^\nu u^\sigma.
\] (107)

Note that \( Q_{\mu\nu} u^\nu = 0 \), namely that \( Q_{\mu\nu} \) lives in the three dimensional space orthogonal to \( u^\mu \). For \( u^\mu = \delta_0^\mu \) the tensor \( Q_{\mu\nu} \) is purely spatial, and can be written as a symmetric traceless three by three matrix \( Q_{ij} = E_{ij} + iB_{ij} \).

This is given for vanishing cosmological constant by

\[
E_{ij} = R_{ij} - \pi^k_i \pi_{kj} + (\text{tr} \pi) \pi_{ij}; \quad B_{ij} = \epsilon^k_i \nabla_k \pi_{ij},
\] (108)

where \( (h_{ij}, \pi^{ij}) \) are the 3-metric of \( \Sigma \) and its conjugate momentum, where \( \Sigma \) represents 3-space and \( \nabla_i \) is the three dimensional Levi–Civita connection for \( h_{ij} \), with Ricci curvature \( R_{ij} u^j = -2 \nabla_i \nabla_j u^j \) and Ricci scalar \( R = R_{ij} h^{ij} \). When \( Q_{ij} \) is diagonalizable, it can be written in a canonical frame such that it is diagonal

\[
Q_{ij} = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}.
\]
The eigenvalues of $Q_{ij}$ define the algebraic properties of spacetime which are invariant under coordinate transformations and the choice of a tetrad frame. At most two eigenvalues are independent due to the tracefree condition

$$Q_i^i = \lambda_1 + \lambda_2 + \lambda_3 = 0.$$  \hspace{1cm} (109)

The Petrov classification distinguishes between algebraically general (Petrov Type I) and algebraically special spacetimes (Petrov types II, III, N, D, O) according to the degeneracy of eigenvalues and eigenvectors of $Q_{ij}$. One defines invariants $I$ and $J$, given by [11]

$$I = \frac{1}{2} \text{tr} Q^2 = \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2); \quad J = \frac{1}{6} \text{tr} Q^3 = \frac{1}{6}(\lambda_1^3 + \lambda_2^3 + \lambda_3^3),$$  \hspace{1cm} (110)

and finds the eigenvalues from the characteristic equation for $Q_{ij}$

$$\lambda^3 - I\lambda + 2J = 0.$$  \hspace{1cm} (111)

The real and the imaginary parts of $I$ are given by [12]

$$\text{Re}[I] = \frac{1}{2}(E_{ij}E^{ij} - B_{ij}B^{ij}); \quad \text{Im}[I] = \frac{1}{2}E_{ij}B^{ij},$$  \hspace{1cm} (112)

which is reminiscent of the radiative invariants $\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B}$ and $\vec{E} \cdot \vec{B}$ for electromagnetism. From the invariants $I$ and $J$ can be defined a specialty index $S$, given by

$$S = \frac{27J^2}{I^3},$$  \hspace{1cm} (113)

where $S = 1$ for algebraically special spacetimes. For Petrov types III, N and O the invariants $I = J = 0$, and for Petrov types II and D they are nontrivial.

### 6.1 Two component spinor SL(2,C) formalism

To place the CDJ matrix $\Psi_{ae}$ into context, it is instructive to establish its relation to the $SL(2,C)$ formalism of GR [13]. Define two component spinors

$$\eta^A \text{ and } \eta'^A,$$

and their complex conjugates $\bar{\eta}^A$, where $A$ and $A'$ respectively denote left handed and right handed $SL(2,C)$ spinorial indices.\(^27\) These indices are raised and lowered by the two dimensional Levi–Civita symbol $\epsilon_{AB}$, where

\(^27\)Spinorial indices take on the values 0 and 1 for both primed and unprimed indices. Note that $SL(2,C)$ is the covering group for $SO(3,C)$, and we regard the CDJ matrix $\Psi_{ae}$ as taking values in two copies of the left-handed $SO(3,C)$.  

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\[
\eta^A = \epsilon^{AB} \eta_B; \quad \eta_B = \eta^A \epsilon_{AB}; \quad \bar{\eta}^{A'} = \epsilon^{A'B'} \eta_{B'}; \quad \eta_{B'} = \eta^{A'} \epsilon_{A'B'}.
\]

(114)

To connect the internal spinor space to objects containing world indices \(\mu\), one can define \(\forall x \in M\) a set of soldering forms \(\sigma^\mu_{AA'}\), which in a Cartesian coordinate system may take on the matrix form

\[
\sigma^0_{AA'} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma^1_{AA'} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};
\]

\[
\sigma^2_{AA'} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3_{AA'} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The soldering forms \(\sigma^\mu_{AA'}\) define an isomorphism between 4-vectors and spinorial pairs in the \((\frac{1}{2}, \frac{1}{2})\) representation at each point of \(M\), and a 4-vector \(V_\mu\) decomposes as

\[
V_{AA'} = V_\mu \sigma^\mu_{AA'}; \quad V^\mu = \sigma^\mu_{AA'} V^{AA'}.
\]

(115)

Tensors of multiple rank decompose by generalization of (115) as

\[
V_{A_1 A_2 A'_1 A'_2 \ldots A_n A'_n} = V_{\mu_1 \mu_2 \ldots \mu_n} \sigma^\mu_{A_1 A'_1} \sigma^\mu_{A_2 A'_2} \ldots \sigma^\mu_{A_n A'_n}
\]

(116)

and similarly

\[
V_{\mu_1 \mu_2 \ldots \mu_n} = \sigma_{\mu_1 A_1 A'_1} \sigma_{\mu_2 A_2 A'_2} \ldots \sigma_{\mu_n A_n A'_n} V_{A_1 A_2 A'_1 A'_2 \ldots A_n A'_n}.
\]

(117)

Any pair of left handed null two component spinors \(n_A\) and \(l_A\) in \(M\) satisfying the normalization conditions

\[
n^A n_A = l^A l_A = 0; \quad n^A l_A = 1,
\]

(118)

in conjunction with the soldering form \(\sigma^\mu_{AA'}\) defines a null tetrad

\[
l^\mu = \sigma^\mu_{AA'} l^{A'A'}; \quad n^\mu = \sigma^\mu_{AA'} n^{A'A'}; \quad m^\mu = \sigma^\mu_{AA'} m^{A'A'}; \quad \overline{m}^\mu = \sigma^\mu_{AA'} l^{A'A'}
\]

(119)

such that

\[
l^\mu l_\mu = m^\mu m_\mu = \overline{m}^\mu \overline{m}_\mu = n^\mu n_\mu = 0;
\]

\[
l_\mu m^\mu = l^\mu \overline{m}_\mu = n_\mu n^\mu = n_\mu \overline{m}_\mu = 0;
\]

\[
l_\mu n^\mu = -m^\mu \overline{m}_\mu = 1.
\]

(120)
The null vectors $l^\mu$ and $n^\nu$ are real and span a time-like 2-plane in $T_p(M)$, the tangent space at each point of spacetime $M$. The null vectors $m^\mu$ and $\overline{m}^\mu$ are complex, and span the orthogonal space-like 2-plane in $T_p(M)$. The tetrad $(l^\mu, n^\mu m^\mu, \overline{m}^\mu)$ is useful in the Penrose approach to GR [14], which is suited to characterizing the radiation properties of spacetime [15].

The spinors $(n_A, l_A)$ induce a basis $\eta^a_{AB}$ in spin space [13], one such basis given by

$$\eta^1_{AB} = \sqrt{2} l(A n_B); \quad \eta^2_{AB} = \frac{i}{\sqrt{2}} (l_A l_B + n_A n_B); \quad \eta^3_{AB} = \frac{i}{\sqrt{2}} (l_A l_B - n_A n_B),$$  \hspace{1cm} (121)$$

where

$$\eta^3_{AB} \eta^A_f = \delta^3_f.$$

These objects define an isomorphism between internal indices $a = (1, 2, 3)$ and symmetric $SU(2)$ index pairs $(00)$, $(01)$ and $(10)$. Any dyad can be expressed in the basis (121) as

$$\phi_{AB} = \sum_{m=1}^{3} \chi_m (\eta^m)_{AB},$$  \hspace{1cm} (123)$$

where $\chi_m$ are the components. A $SL(2, C)$ transformation $g$, acting on the column vector $(l_A, n_A)$, given by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad ad - bc = 1,$$

induces a transformation of the basis and the corresponding components

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \\ \phi'_3 \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & bc + ad & bd \\ c^2 & 2cd & d^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$  \hspace{1cm} (124)$$

The basis (121) also induces an orthonormal basis of completely symmetric four-spinors, given by [13]

$$\eta^0_{ABCD} = \frac{1}{\sqrt{2}} (l_A l_B l_C l_D + n_A n_B n_C n_D);$$
$$\eta^1_{ABCD} = \sqrt{2} i (l_A l_B l_C n_D) + l(A n_B n_C n_D));$$
$$\eta^2_{ABCD} = \sqrt{2} i (l_A l_B l_C n_D);$$
$$\eta^3_{ABCD} = \sqrt{2} (l_A l_B l C n_D) - l(A n_B n_C n_D));$$
$$\eta^4_{ABCD} = \frac{i}{\sqrt{2}} (l_A l_B l_C l_D - n_A n_B n_C n_D),$$  \hspace{1cm} (124)$$

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satisfying orthonormality relations

\[ \eta^\alpha_{ABCD}(\eta^\beta)^{ABCD} = \delta^\alpha\beta. \]  (125)

6.2 Principal null directions of spacetime

By application of (116) and (117), one may decompose the Riemann curvature tensor into \( SL(2, \mathbb{C}) \) indices as

\[ R_{\mu\nu\rho\sigma}^\mu \sigma^\nu_{AA'} \sigma^\rho_{BB'} \sigma^\sigma_{CC'} \sigma^\sigma_{DD'} = \epsilon_{A'B'C'D'} \psi_{ABCD} + \ldots, \]  (126)

where \( \psi_{ABCD} = \psi_{(ABCD)} \) is Weyl, the self-dual part of the Weyl curvature tensor, and the dots signify the remaining components which will not concern us in this paper. Using the basis (124), \( Weyl \) can be written as

\[ \psi_{ABCD} = 2 \sum_{f=0}^{4} \Psi^\alpha_f \eta^\alpha_{ABCD}, \]  (127)

where \( \Psi^\alpha_f \) are defined as the Weyl scalars. In a suitable adapted frame, the Weyl scalars may be shown to admit the following physical interpretations in vacuum spacetimes: (i) \( \Psi_0 \) and \( \Psi_4 \) are transverse components of gravitational radiation propagating in the \( l^\mu \) and the \( n^\mu \) directions respectively. (ii) \( \Psi_1 \) and \( \Psi_3 \) are longitudinal components propagating in the \( l^\mu \) and the \( n^\mu \) directions respectively. (iii) \( \Psi_2 \) is a Coulombic component.

The principal null directions of spacetime can be computed directly by performing an \( SL(2, \mathbb{C}) \) transformation to eliminate \( \Psi_4(\Psi_0) \), which leaves \( l^\mu(n^\mu) \) as the principal null direction. For example a null rotation which keeps \( l^A \) invariant transforms \( \Psi_4 \) into

\[ \Psi'_4 = \Psi_4 + 4\Psi_3 z + 6\Psi_2 z^2 + 4\Psi_1 z^3 + \Psi_0 z^4. \]  (128)

The condition for \( l^\mu \) to be a principal null direction is that \( \Psi'_4 = 0 \). This yields a quartic polynomial equation in \( z \), which is also given by

\[ \Psi_{ABCD} \xi^A \xi^B \xi^C \xi^D = 0. \]  (129)

Equation (128) in general has four roots, and the multiplicity of each principal null direction is the same as the multiplicity of the corresponding root. The roots \( z_i \) for \( i = 1, \ldots, 4 \) can be parametrized as [10]

\[ z_i = \tan(\theta_i/2) e^{-i\phi_i}, \]  (130)
which can be put in one-to-one correspondence with points on the two-sphere by stereographic projection of \( z_i \). The principal null directions are given by

\[
P^a_{(i)} = \cos \theta_i \hat{z}^a + \sin \theta_i \cos \phi_i \hat{x}^a + \sin \theta_i \sin \phi_i \hat{y}^a,
\]

(131)

where \((\theta_i, \phi_i)\) coordinatize angular position on the two-sphere. The number and multiplicity of principal null directions determines the Petrov classification of \( Weyl \). The Petrov classification scheme then is as follows

\[
Type I \ (1, 1, 1, 1); \ Type II \ (2, 1, 1); \ Type D \ (2, 2);
\]

\[
Type III \ (3, 1); \ Type N \ (4); \ Type O (Conformally flat).
\]

(132)

In brackets we have indicated the multiplicity of PNDs within each category. From \( \psi_{ABCD} \) one can form the invariants

\[
I = \frac{1}{2} \psi_{ABCD} \psi^{ABCD}; \quad J = \frac{1}{6} \psi_{ABCD} \psi^{CDEF} \psi^{AB}_{EF},
\]

(133)

which in direct analogy to (113) define a specialty index \( S \) given by

\[
S' = \frac{I^3}{J^2} - 27.
\]

(134)

### 6.3 Relation to the CDJ matrix

We will now establish a direct correspondence from the principal null directions of spacetime to the quantizable degrees of freedom in the instanton representation. Using the basis (121), \( Weyl \) can also be decomposed into the following form

\[
\psi_{ABCD} = \sum_{a,e=1}^{3} \psi_{ae} \eta^{a}_{AB} \eta^{e}_{CD},
\]

(135)

which defines a symmetric and traceless matrix \( \psi_{ae} \). The relation between \( \psi_{ae} \) and the Weyl scalars in this basis \( \Psi_\alpha \) is given by [13]

\[
\psi_{ae} = \begin{pmatrix}
-2\Psi_2 & i(\Psi_1 + \Psi_3) & (\Psi_3 - \Psi_1) \\
i(\Psi_1 + \Psi_3) & \frac{i}{2}(2\Psi_2 + \Psi_0 + \Psi_4) & \frac{i}{2}(\Psi_0 - \Psi_4) \\
(\Psi_3 - \Psi_1) & \frac{i}{2}(\Psi_0 - \Psi_4) & \frac{1}{2}(2\Psi_2 - \Psi_0 - \Psi_4)
\end{pmatrix}.
\]

The invariants of \( \psi_{ae} \) are given by
\[2J = \text{tr} \psi^2 = 2\Psi_0 \Psi_4 - 8\Psi_1 \Psi_3 + 6\Psi_2^2;\]
\[6J = \det \Psi = 2(\Psi_0 \Psi_2 \Psi_4 + \Psi_0 \Psi_2 \Psi_3 - \Psi_4 \Psi_1^2 + 2\Psi_1 \Psi_2 \Psi_3 - \Psi_2^3)\]  \hspace{1cm} (136)

which imply the characteristic equation

\[r^3 - Ir + 2J = 0.\]  \hspace{1cm} (137)

A Weyl spinor \(\psi_{ABCD}\) of Petrov Type I, has four distinct PNDs at any point, which in a certain frame form the vertices of a disphenoid [17]. This disphenoid represents the intersection of a spacelike plane with \(S^+\), the cone of null directions at that point. It has been elaborated in [18] the relation of this disphenoid to the geometry of the roots of (137). One takes one PND from (130) as the north pole of \(S^+\) and projects stereographically the other three PND onto the extended Argand plane. The shape of the PND on \(S^+\) mirrors the pattern of eigenvalues of \(\psi_{ae}\), fixed by (137), which are the vertices of a triangle in the complex plane.

The roots \(r_1, r_2\) and \(r_3\) of (137) depend explicitly on \(I\) and \(J\) from (136), and the CDJ matrix \(\Psi_{ae}\) is defined by the addition of a spin 0 part to \(\psi_{ae}\), hence the relation

\[\Psi_{ae}^{-1} - \delta_{ae} \varphi = \psi_{ae},\]  \hspace{1cm} (138)

where \(\varphi = \frac{1}{3}(\text{tr} \Psi^{-1})\). Equation (138) can be inverted to yield \(\Psi_{ae} = (\delta_{ae} \varphi + \psi_{ae})^{-1}\). Therefore, since \(\psi_{ae} = \psi_{ae}(I, J)\) encodes the algebraic classification of the spacetime, it follows that the CDJ matrix \(\Psi_{ae} = \Psi_{ae}(I, J)\) also encodes this algebraic classification.

The decomposition of Weyl into electric and magnetic parts (108) in spacetime \(M\) is generally known. But also in four spacetime dimensions, there is a three dimensional vector space \(W^-\) spanned by the triple of self-dual \(SO(3, C)\) two forms \(\Sigma^a\) [20]. Expansion of (106) with respect to \(W^-\) yields a symmetric and traceless three by three matrix \(\psi_{ae}\), related to the self-dual part of Weyl by

\[C_{\mu\nu\rho\sigma} = \psi_{ae} \Sigma^a_{\mu\rho} \Sigma^e_{\nu\sigma},\]  \hspace{1cm} (139)

which can also be seen from (135). Using (107), the following relation can be written

\[Q_{\mu\nu} = \psi_{ae} \Sigma^a_{\mu\rho} \Sigma^e_{\nu\sigma} u^\rho u^\sigma,\]  \hspace{1cm} (140)
which relates the spacetime object $Q_{\mu\nu}$ to the purely internal object $\psi_{ae}$.

For $u^\mu = \delta^\mu_0$, the decomposition reads

$$Q_{ij} = \psi_{ae} \Sigma^a_{0i} \Sigma^e_{0j},$$

which involves the temporal components $\Sigma^a_{0i}$ of the two forms. Hence it is clear that $Q_{\mu\nu}$ contains the same number of D.O.F. as does $\psi_{ae}$, since $\Sigma^a_{\mu\nu} u^\mu u^\nu = 0$. We will regard $\psi_{ae}$ as being the fundamental object, with $Q_{ij}$ being derived upon specification of a self-dual two form and a choice of Lorentz observer.

### 6.4 Quantizable degrees of freedom

For spacetimes of Petrov type $I$, $D$ and $O$, $\psi_{ae}$ contains three linearly independent eigenvectors and can be diagonalized according to [16]. For these spacetimes, one can perform a $SO(3,C)$ transformation of (138), putting the symmetric part of the CDJ matrix $\Psi_{ae}$ into diagonal form

$$\Psi_{(ae)} = (e^{\theta T})_{ab} \left( \begin{array}{ccc} \lambda & 0 & 0 \\ 0 & \lambda + \alpha & 0 \\ 0 & 0 & \lambda + \beta \end{array} \right) (e^{-\theta T})_{ce}$$

for some $\lambda$, $\alpha$ and $\beta$. This is a polar decomposition of $\Psi_{ae}$ using a complex orthogonal transformation parametrized by three complex angles $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$. The Hamiltonian constraint (17) is a relation amongst the eigenvalues which is independent of the $SO(3,C)$ frame, and can be written as

$$\Lambda + \text{tr}\Psi^{-1} = \Lambda + \frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta} = 0.$$  \hspace{1cm} (142)

Equation (142) for $\Lambda \neq 0$ implies the cubic equation

$$\lambda(\lambda + \alpha)(\lambda + \beta) + \frac{3}{\Lambda}(\lambda^2 + \frac{2}{3}(\alpha + \beta)\lambda + \frac{1}{3}\alpha \beta) = 0 \longrightarrow \lambda = \lambda_{\alpha,\beta}$$ \hspace{1cm} (143)

where $\lambda_{\alpha,\beta}$ are the roots. One may use the same $SO(3,C)$ transformation to diagonalize both sides of (138), obtaining the relations

$$\frac{1}{\lambda_{\alpha,\beta} + \alpha} + \frac{\Lambda}{3} = r_1(I,J); \quad \frac{1}{\lambda_{\alpha,\beta} + \beta} + \frac{\Lambda}{3} = r_2(I,J); \quad \frac{1}{\lambda_{\alpha,\beta}} + \frac{\Lambda}{3} = r_3(I,J)$$ \hspace{1cm} (144)

Equation (144) is a system of three equations in two unknowns which should provide $\alpha = \alpha(I,J)$, $\beta = \beta(I,J)$ and $\lambda = \lambda(I,J)$ explicitly through the roots of (137). These roots must satisfy the relation $r_1 + r_2 + r_3 = 0$, which is
equivalent to the tracelessness of $\psi_{ae}$. By inverting (144) we can express $\alpha$, $\beta$ and $\lambda$, which are directly related to the eigenvalues of $\Psi_{ae}$ directly in terms of $r_1$, $r_2$ and $r_3$

$$\begin{align*}
\lambda_{\alpha,\beta} &= \left(\frac{1}{r_3} - \frac{1}{3}\right) = -\left(\frac{1}{r_1 + r_2 + \frac{4}{3}}\right); \\
\alpha &= \frac{1}{r_1 - \frac{1}{3}} + \frac{1}{r_1 + r_2 + \frac{4}{3}}; \\
\beta &= \frac{1}{r_2 - \frac{1}{3}} + \frac{1}{r_1 + r_2 + \frac{4}{3}}. 
\end{align*}$$

(145)

In a solution to the initial value constraints the $SO(3,C)$ angles $\tilde{\theta}$ are not arbitrary, but must satisfy the Gauss’ law constraint

$$w_{\alpha f e}\{ (e^{\theta T})_{af} \lambda_f (e^{-\theta T})_{fe} \} = 0$$

(146)

for $\tilde{\theta} = \bar{\theta} [\bar{\lambda}; A]$ where $\lambda_f = (\lambda_{\alpha,\beta}, \lambda_{\alpha,\beta} + \alpha, \lambda_{\alpha,\beta} + \beta)$ satisfying (142). Each configuration $A^e_\alpha$ defines an equivalence class of $SO(3,C)$ frames corresponding to the eigenvalues thus chosen. Since $\lambda_f$ are coordinate independent, then it follows that the coordinate-dependent information in the principal null directions must be encoded in $A^e_\alpha$.

One can formulate a quantum theory of the algebraic classification of spacetime by quantizing the eigenvalues of the CDJ matrix $\Psi_{ae}$, regarded as the physical degrees of freedom of the momentum space. We have proven the existence of configuration space variables which are canonically conjugate to the eigenvalues of $\Psi_{ae}$. Since the aim of quantization is to construct quantum states corresponding to the CDJ matrix, then we need a prescription for relating its invariants to the invariants of spacetime. Starting from the CDJ matrix

$$\Psi_{ae}^{-1} = \delta_{ae} \varphi + \psi_{ae},$$

(147)

where $\varphi = -\frac{\Lambda}{3}$, we have the following library of terms

$$\begin{align*}
\text{tr} \Psi^{-1} &= 3 \varphi; \\
\text{tr} (\Psi^{-1} \Psi^{-1}) &= 3 \varphi^2 + \text{tr} \psi^2 = 3 \varphi^2 + 2 I \equiv M; \\
\det \Psi^{-1} &= (\det \Psi)^{-1} = \varphi^3 - I \varphi + 2 J \equiv Q. 
\end{align*}$$

(148)

For type $N$, $II$ and $III$ spacetimes $\psi_{ae}$ is not diagonalizable, and so we defer the quantization of such spacetimes for future study.

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28 It is shown in Paper IV that the angles $\tilde{\theta}$, at the canonical level, are not independent degrees of freedom and therefore should not be quantized.
We now provide a prescription for computing the principal null directions directly from the state labels, as follows. For the given algebraic type that one is in one must first choose a quantum state labelled by \((\alpha,\beta)\), and then use these labels to determine the eigenvalues \(\lambda = \lambda_f(\alpha,\beta)\) of Weyl as in (167) for Type D spacetimes, or (172) for Type I spacetimes. Next, compute Weyl in the \(SO(3,C)\) frame where the Gauss’ law constraint is satisfied. The Gauss’ law constraint is a condition on the CDJ matrix

\[
\mathbf{w} = \{(\lambda_f)_{\alpha,\beta}(e^{-\bar{\theta} T})_{fa}(e^{-\bar{\theta} T})_{fe}\} = 0, \tag{149}
\]

which reduces to a condition on the three complex \(SO(3,C)\) rotation angles \(\bar{\theta} \equiv (\theta^1,\theta^2,\theta^3)\). On solutions to (149) the rotation angles are given by \(\bar{\theta} = \theta[A;A] = \theta_{\alpha,\beta}[A]\), which is labelled by the labels \((\alpha,\beta)\) and by the choice of connection \(A_a^i\);\(^{29}\) and Weyl is given by

\[
\psi_{ae} = (\psi_{\alpha,\beta}[\bar{\gamma}])_{ae} = (e^{\bar{\theta}_{\alpha,\beta}[\bar{\gamma}] T})_{af}(\lambda_f)_{\alpha,\beta}(e^{-\bar{\theta}_{\alpha,\beta}[\bar{\gamma}] T})_{fe}. \tag{150}
\]

One then constructs the Weyl scalars from the elements of (150), of which there should be five

\[
\Psi_I = (\Psi_{\alpha,\beta}[\bar{\gamma}])_I = T^{ae}_{I}(\psi_{\alpha,\beta}[\bar{\gamma}])_{ae}. \tag{151}
\]

\(^{29}\)For the purposes of the Gauss’ law constraint one may regard the connection \(A_a^i\) as a derived quantity from the magnetic field \(B_a^i\). The latter is is turn derived from the vector fields \(v_a = B_a^i \partial_i\) tangent to three congruences of integral curves \(\bar{\gamma}\) which fill 3-space \(\Sigma\). As shown in papers VI, VII and VIII we regard the integral curves \(\bar{\gamma}\) as fundamental, with the magnetic field \(B_a^i\) derived upon making a choice of coordinates \(x^i\). In this sense the coordinate-dependent information in the PNDS resides within \(\bar{\gamma}\).
We have proven the existence of globally holonomic configuration space variables in the full theory for the instanton representation, which correspond to the existence of quantizable configurations. The next thing is to show explicitly which observable characteristics of spacetime correspond to the quantizable degrees of freedom. The basic momentum space variables are the densitized eigenvalues of the CDJ matrix $\Pi_f = \lambda_f (\det A)$, and the undensitized versions $\lambda_f$ directly encode the algebraic classifications of the spacetimes via the invariants $I$ and $J$. These invariants in turn fix the principal null directions of spacetime. Therefore in a sense, the quantization of the instanton representation should correspond to a quantization of the principal null directions, and more fundamentally a quantization of the algebraic classification of spacetime. The degrees of freedom designated for quantization are given by

$$\Psi_{ae} = (\delta_{ae} \lambda + \alpha (e^2)_{ae} + \beta (e^3_{ae})) (\det A)^{-1}, \quad (152)$$

which we will relate directly to these classifications. We will treat all cases covering first the nondegenerate spacetimes, where $\Psi_{ae}$ has three independent eigenvectors. The eigenvalues of such spacetimes must satisfy the Hamiltonian constraint

$$\frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta} = 3 \varphi, \quad (153)$$

irrespective of the degeneracy of the eigenvalues, where $\varphi = -\frac{\Lambda}{3} (\det A)^{-1}$.

### 7.1 Type O spacetimes

We will first consider spacetimes where the CDJ matrix has three linearly independent eigenvectors, with all three eigenvalues equal $\lambda_1 = \lambda_2 = \lambda_3$. These are spacetimes of algebraic type O, which include DeSitter spacetime. The eigenvalues for $Weyl$, the self-dual Weyl curvature $\psi_{ae}$, as well as the Weyl scalars $\Psi_{\alpha}$, are zero for this case

$$\psi_{ae} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \Psi_{\alpha} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
\[ \Psi_{ae} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda + \alpha & 0 \\ 0 & 0 & \lambda + \beta \end{pmatrix} = \begin{pmatrix} \varphi^{-1} & 0 & 0 \\ 0 & \varphi^{-1} & 0 \\ 0 & 0 & \varphi^{-1} \end{pmatrix}, \]

which implies that the elements of the deviation from isotropy vanish

\[ \alpha = \beta = 0. \quad (154) \]

Since one would like to be able to deduce the properties of the spacetime directly from the state we have, one may alternatively start from the Hamiltonian constraint, which implies that

\[ \frac{3}{\lambda_{0,0}} = 3\varphi \rightarrow \lambda_{0,0} = \frac{1}{\varphi}. \quad (155) \]

Hence we have that

\[ \lambda_1 = \lambda_2 = \lambda_3 = 0; \quad I = J = 0. \quad (156) \]

### 7.2 Type D spacetimes

Spacetimes with three linearly independent eigenvectors where there are two independent eigenvalues \( \lambda_1 = \lambda_2 \neq \lambda_3 \) for Weyl, are of algebraic type D. The eigenvalues are related by

\[ \psi_{ae} = \begin{pmatrix} -2\lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}; \quad \Psi_{\alpha} = \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]

For \( \alpha = \beta \) the CDJ matrix is given by

\[ \Psi_{ae} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda + \alpha & 0 \\ 0 & 0 & \lambda + \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{\varphi - 2\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\varphi + \lambda_1} & 0 \\ 0 & 0 & \frac{1}{\varphi + \lambda_1} \end{pmatrix}, \]

whence one reads off that

\[ \alpha = \beta = \frac{1}{\varphi + \lambda_1} - \frac{1}{\varphi - 2\lambda_1}. \quad (157) \]
Since one would like to use the quantum state to make predictions about the spacetime properties directly from the labels of the quantum state, one alternatively may start from \( \alpha \) and use the Hamiltonian constraint

\[
\frac{1}{\lambda} + \frac{2}{\lambda + \alpha} = 3\varphi.
\]  

(158)

This leads to the quadratic equation

\[3\varphi \lambda^2 + 3(\alpha\varphi - 1)\lambda - \alpha = 0\]  

(159)

with roots

\[
\lambda_{\alpha,\alpha}(\varphi) = \frac{1}{2\varphi} \left(1 - \alpha\varphi \pm \sqrt{(\alpha\varphi)^2 - \frac{2}{3}(\alpha\varphi) + 1}\right).
\]  

(160)

The roots of Weyl are then given by

\[
\lambda_3 = -\varphi + \frac{1}{\lambda_{\alpha,\alpha}(\varphi)}; \\
\lambda_1 = \lambda_2 = -\varphi + \frac{1}{\alpha + \lambda_{\alpha,\alpha}(\varphi)},
\]  

(161)

from which one may directly compute the radiation invariants

\[
I = I(\alpha, \varphi) = \frac{1}{2} \left[ (-\varphi + \frac{1}{\lambda_{\alpha,\alpha}(\varphi)})^2 + 2\left( -\varphi + \frac{1}{\alpha + \lambda_{\alpha,\alpha}(\varphi)} \right)^2 \right]; \\
J = J(\alpha, \varphi) = \frac{1}{6} \left[ (-\varphi + \frac{1}{\lambda_{\alpha,\alpha}(\varphi)})^3 + 2\left( -\varphi + \frac{1}{\alpha + \lambda_{\alpha,\alpha}(\varphi)} \right)^3 \right].
\]  

(162)

as a function of the state label \( \alpha \).

For \( \alpha \neq \beta = 0 \) the CDJ matrix is given by

\[
\Psi_{ae} = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda + \alpha
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\varphi + \lambda_1} & 0 & 0 \\
0 & \frac{1}{\varphi + \lambda_1} & 0 \\
0 & 0 & \frac{1}{\varphi - 2\lambda_1}
\end{pmatrix},
\]  

whence one reads off that

\[
\alpha = \frac{1}{\varphi - 2\lambda_1} - \frac{1}{\varphi + \lambda_1}.
\]  

(163)

The Hamiltonian constraint in this case is given by
\[ \frac{2}{\lambda} + \frac{1}{\lambda + \alpha} = 3\varphi, \quad (164) \]

which leads to the quadratic equation

\[ 3\varphi \lambda^2 + 3(\alpha\varphi - 1)\lambda - 2\alpha = 0 \quad (165) \]

with roots

\[ \lambda_{\alpha,0}(\varphi) = \frac{1}{2\varphi} \left( 1 - \alpha\varphi \pm \sqrt{(\alpha\varphi)^2 + \frac{2}{3}(\alpha\varphi) + 1} \right). \quad (166) \]

The roots of Weyl are then given by

\[ \lambda_1 = \lambda_2 = -\varphi + \frac{1}{\lambda_{\alpha,0}(\varphi)}; \]
\[ \lambda_3 = -\varphi + \frac{1}{\alpha + \lambda_{\alpha,0}(\varphi)} \quad (167) \]

from which one may directly compute the radiation invariants

\[ I = I(\alpha, \varphi) = \frac{1}{2} \left[ 2 \left(-\varphi + \frac{1}{\lambda_{\alpha,0}(\varphi)}\right)^2 + \left(-\varphi + \frac{1}{\alpha + \lambda_{\alpha,0}(\varphi)}\right)^2 \right]; \]
\[ J = J(\alpha, \varphi) = \frac{1}{6} \left[ 2 \left(-\varphi + \frac{1}{\lambda_{\alpha,0}(\varphi)}\right)^3 + \left(-\varphi + \frac{1}{\alpha + \lambda_{\alpha,0}(\varphi)}\right)^3 \right]. \quad (168) \]

as a function of the state label \( \alpha \).

### 7.3 Type I spacetimes

Type I spacetimes are algebraically general, and posses three independent eigenvalues \( \lambda_1 \neq \lambda_2 \neq \lambda_3 \) for Weyl and three linearly independent eigenvectors. The eigenvalues are related by

\[ \psi_{ae} = \begin{pmatrix} -\lambda_1 - \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}; \quad \Psi_{\alpha} = \frac{\lambda_1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{\lambda_2}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \]

The CDJ matrix is given by
\[
\Psi_{ae} = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda + \alpha & 0 \\
0 & 0 & \lambda + \beta
\end{pmatrix} = \begin{pmatrix}
\frac{\varphi - \lambda_1 - \lambda_2}{\varphi + \lambda_1} & 0 & 0 \\
0 & \frac{1}{\varphi + \lambda_1} & 0 \\
0 & 0 & \frac{1}{\varphi + \lambda_2}
\end{pmatrix},
\]
whence one reads off that
\[
\alpha = \frac{1}{\varphi + \lambda_1} - \frac{1}{\varphi - \lambda_1 - \lambda_2}; \quad \beta = \frac{1}{\varphi + \lambda_2} - \frac{1}{\varphi - \lambda_1 - \lambda_2}. \quad (169)
\]
We would like to read off the properties of the spacetime directly from the quantum state, for any pair \((\alpha, \beta)\). Hence the Hamiltonian constraint reads
\[
\frac{1}{\lambda} + \frac{1}{\lambda + \alpha} + \frac{1}{\lambda + \beta} = 3\varphi. \quad (170)
\]
Equation (170) leads to the cubic equation
\[
\lambda^3 + \left(\alpha + \beta - \frac{1}{3\varphi}\right)\lambda^2 + \left(\alpha\beta - \frac{2}{3\varphi}(\alpha + \beta)\right)\lambda - \frac{\alpha\beta}{3\varphi} = 0, \quad (171)
\]
with three roots \(\lambda_{\alpha,\beta}\) labelled by \(\alpha\) and \(\beta\).\(^{30}\) Then one solves for the eigenvalues of \(\text{Weyl}\), given by
\[
\lambda_3 = -\varphi + \frac{1}{\lambda_{\alpha,\beta}(\varphi)}; \\
\lambda_1 = -\varphi + \frac{1}{\alpha + \lambda_{\alpha,\beta}(\varphi)}; \\
\lambda_2 = -\varphi + \frac{1}{\beta + \lambda_{\alpha,\beta}(\varphi)}, \quad (172)
\]
from which one can directly compute the invariants
\[
I = I(\alpha, \beta) = \frac{1}{2} \left( -\varphi + \frac{1}{\lambda_{\alpha,\beta}(\varphi)} \right)^2; \\
+ \left( -\varphi + \frac{1}{\alpha + \lambda_{\alpha,\beta}(\varphi)} \right)^2 + \left( -\varphi + \frac{1}{\beta + \lambda_{\alpha,\beta}(\varphi)} \right)^2; \\
J = J(\alpha, \beta) = \frac{1}{6} \left[ \left( -\varphi + \frac{1}{\lambda_{\alpha,\beta}(\varphi)} \right)^3 \\
+ \left( -\varphi + \frac{1}{\alpha + \lambda_{\alpha,\beta}(\varphi)} \right)^3 \right] + \left( -\varphi + \frac{1}{\beta + \lambda_{\alpha,\beta}(\varphi)} \right)^3 \quad (173)
\]
as a function of the state labels.
\(^{30}\)We do not display the explicit solution for the roots here, but they can be found by the method of Cardano.
7.4 Degenerate spacetimes

For the nondegenerate cases it was straightforward to identify the quantizable degrees of freedom because \( \Psi_{ae} \) could be taken to be already in diagonal form, known as the intrinsic \( SO(3, C) \) frame. We will see for the degenerate cases that a intrinsic \( SO(3, C) \) frame cannot be defined since \( \Psi_{ae} \) is not diagonalizable. This is due to the fact that the number of linearly independent eigenvectors is less than the rank of the matrix.\(^{31}\) We will now consider each case in turn.

Type N spacetimes have one independent eigenvalue \( \lambda_1 = \lambda_2 = 0 \) for Weyl, and two linearly independent eigenvectors. For this case Weyl is given by

\[
\psi_{ae} = k \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & i \\
0 & i & -1
\end{pmatrix};
\Psi_{a} = \begin{pmatrix}
2 \\
0 \\
0 \\
0
\end{pmatrix}
\]

for some numerical constant \( k \). The CDJ matrix is given by

\[
\Psi_{ae} = \begin{pmatrix}
\frac{1}{\phi} & \frac{1}{\phi} & 0 \\
0 & \frac{1}{\phi} - \frac{k}{\phi^2} & -i \frac{k}{\phi^2} \\
0 & -i \frac{k}{\phi^2} & \frac{1}{\phi} + \frac{k}{\phi^2}
\end{pmatrix}.
\]

The attempt to diagonalize this leads to the matrix

\[
\Psi_{ae} = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda + \alpha & 0 \\
0 & 0 & \lambda + \beta
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\phi} & \frac{1}{\phi} - \frac{k}{\phi^2} e^{-2i\phi} & 0 \\
0 & 0 & \frac{1}{\phi} + \frac{k}{\phi^2} e^{-2i\phi}
\end{pmatrix};
\tan 2\phi = -i.
\]

which is ill-defined. One may nevertheless attempt to read off \( \alpha \) and \( \beta \), given by

\[
\alpha = -\frac{k}{\phi^2} e^{-2i\phi}; \quad \beta = \frac{k}{\phi^2} e^{-2i\phi}
\]

which are also ill-defined.

Type III spacetimes have one independent eigenvalue for Weyl, and one linearly independent eigenvector. The eigenvalue is zero for this case

\(^{31}\)One could determine the diagonalizable subspace of \( \Psi_{ae} \), and attempt to perform a quantization restricted to this subspace. We relegate such treatments for future study.
\[ \psi_{ae} = k \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 0 \end{pmatrix}; \quad \Psi_{\alpha} = k \begin{pmatrix} 1 \\ -\frac{i}{2} \\ 0 \\ -\frac{i}{2} \\ -1 \end{pmatrix}. \]

for some numerical constant \( k \). The CDJ matrix is given by

\[ \Psi_{ae} = \begin{pmatrix} \frac{1}{\varphi} + \frac{k^2}{\varphi^2} & -\frac{k^2}{\varphi^2} & i \frac{k^2}{\varphi^2} \\ \frac{1}{\varphi^2} & \frac{1}{\varphi} & i \frac{k}{\varphi^2} \\ i \frac{k}{\varphi^2} & i \frac{k}{\varphi^2} & \frac{1}{\varphi} - \frac{k^2}{\varphi^2} \end{pmatrix}. \]

Type N spacetimes have two independent eigenvalues for Weyl, and two linearly independent eigenvectors. The eigenvalue is zero for this case

\[ \psi_{ae} = k \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \Psi_{\alpha} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \]

for some numerical constant \( k \). The CDJ matrix is given by

\[ \Psi_{ae} = (\varphi + \lambda_2)^{-2} \begin{pmatrix} \frac{(\varphi + \lambda_2)^2}{\varphi^2 - 2\lambda_2} & 0 & 0 \\ 0 & \varphi + \lambda_2 + k & -ik \\ 0 & ik & \varphi + \lambda_2 - k \end{pmatrix}. \]

Attempts to diagonalize this yield

\[ \Psi_{ae} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda + \alpha & 0 \\ 0 & 0 & \lambda + \beta \end{pmatrix} \]

\[ = (\varphi + \lambda_2)^{-2} \begin{pmatrix} \frac{(\varphi + \lambda_2)^2}{\varphi^2 - 2\lambda_2} & 0 & 0 \\ 0 & \varphi + \lambda_2 + ke^{-2i\theta} & 0 \\ 0 & 0 & \varphi + \lambda_2 - ke^{-2i\theta} \end{pmatrix}; \quad \tan(2\theta) = -i. \]

From this one may read off \( \alpha \) and \( \beta \), given by

\[ \alpha = (\varphi + \lambda_2)^{-2} \left( \varphi + \lambda_2 + ke^{-2i\theta} - \frac{(\varphi + \lambda_2)^2}{\varphi^2 - 2\lambda_2} \right); \]

\[ \beta = (\varphi + \lambda_2)^{-2} \left( \varphi + \lambda_2 - ke^{-2i\theta} - \frac{(\varphi + \lambda_2)^2}{\varphi^2 - 2\lambda_2} \right). \quad (175) \]

Equation (175) are ill-defined, as with the rest of the degenerate cases.
8 Conclusion

The main result of this paper is as follows. We have shown that there exists a natural canonical structure associated with the Petrov classification of spacetime for spacetimes of Petrov type $I$, $D$ and $O$. This canonical structure corresponds to the kinematic phase space of the instanton representation of Plebanski gravity, the phase space at the level of implementation of the kinematic constraints. The Hamiltonian constraint at this level fixes the algebraic classification of the spacetime through the eigenvalues of the CDJ matrix $\Psi_{ae}$, and enables one to quantize the theory in an intrinsic $SO(3,C)$ frame. The aforementioned canonical structure provides globally holonomic coordinates on the instanton representation configuration space, which requires the use of a densitized version of the CDJ matrix as the basic momentum space variable. Once the elements of $\Psi_{ae}$ have been densitized by the Chern–Simons Lagrangian, then the implementation of the initial value constraints yields a kinematic phase space space where this canonical structure is realized. In essence, the instanton representation admits a canonical structure which allows for the possibility to quantize gravity and obtain a classical limit in terms of the Petrov classification of spacetime, which is in principle directly measurable in this limit. The next step is the construction of a Hilbert space for such a quantum theory, which is the topic of Papers XV and XVIII.
9 Appendix A. Hamiltonian constraint in polynomial form

While the Hamiltonian constraint is nonpolynomial in the instanton representation, we will see that it is convenient to extract the polynomial part when quantizing the theory. The smeared form of the Hamiltonian constraint at the classical level can be written

\[ H[N] = \int_\Sigma d^3x N (\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}) = \int_\Sigma d^3x N \sqrt{\frac{\det B}{\det \Psi}} \left( \frac{1}{2} \text{Var} \Psi + \Lambda \det \Psi \right) \]

where we have used the characteristic equation

\[ \text{tr} M^{-1} = \frac{\text{Var} M}{2 \det M} \] (177)

for a three by three matrix \( M \), where \( \text{Var} M = (\text{tr} M)^2 - \text{tr} M^2 \).

Since we are restricting to nondegenerate configurations \( \det B \neq 0 \) and \( \det \Psi \neq 0 \), we can then focus on the part

\[ \frac{1}{2} \text{Var} \Psi + \Lambda \det \Psi, \] (178)

where \( \text{Var} \Psi = (\text{tr} \Psi)^2 - \text{tr} \Psi^2 \).

The CDJ matrix \( \Psi_{ae} \) can be parametrized by its symmetric and antisymmetric parts \( \lambda_{ae} \) and \( a_{ae} \) such that

\[ \Psi_{ae} = \lambda_{ae} + a_{ae} = \lambda_{ae} + \epsilon_{aed} \psi^d \] (179)

for some arbitrary \( SU(2) \) valued 3-vector \( \psi^d \). The ingredients of the Hamiltonian constraint are then given by

\[ \det \Psi = \frac{1}{6} \epsilon_{abc} \epsilon_{efg} (\lambda_{ae} + \epsilon_{aed} \psi^d_1) (\lambda_{bf} + \epsilon_{bdf} \psi^d_2) (\lambda_{cg} + \epsilon_{cgd} \psi^d_3) \]

\[ = \det(\lambda_{ae}) + \det(\epsilon_{aed} \lambda^d) + \frac{1}{2} \epsilon_{abc} \epsilon_{efg} (\epsilon_{cgd}\lambda_{ae}\lambda_{bf} \psi^d + \epsilon_{bdf} \epsilon_{cgd} \lambda_{ae} \psi^d \psi^d') \] (180)

Using the fact that the determinant of an antisymmetric matrix of odd rank vanishes, and the annihilation of symmetric on antisymmetric quantities, we end up with

\[ \det \Psi = \det \lambda + \frac{1}{2} (\epsilon_{abc} \epsilon_{efd}) (\epsilon_{efg} \epsilon_{edg'}) \psi^d \psi^d' \lambda_{ae} \]

\[ = \det \lambda + \frac{1}{2} \lambda_{ae} (\delta_{af} \delta_{cd} - \delta_{ad} \delta_{cf}) (\delta_{ec} \delta_{fd} - \delta_{cd} \delta_{fc}) \psi^d \psi^d' \]

\[ = \det \lambda + \lambda_{fg} \psi^f \psi^g \] (181)
where we have made use of epsilon symbol identities. Likewise, one may compute the variance

$$Var\Psi = (\text{tr}\lambda)^2 - (\lambda_{ae} + \epsilon_{aed}\psi^d)(\lambda_{ea} - \epsilon_{aed}\psi^d) = Var\lambda - 2\delta_{fg}\psi_f\psi_g. \quad (182)$$

The Hamiltonian constraint (178) then is given by

$$H = Var\lambda + \Lambda \det\lambda + (\lambda_{fg} - 2\delta_{fg})\psi^f\psi^g. \quad (183)$$

The symmetric part of the CDJ matrix $\Psi_{ae}$ can be re-written as a complex orthogonal ($SO(3, C)$ transformation parametrized by three complex angles $O = e^{\mathbf{\theta} \cdot T}$, where $\mathbf{\theta} \equiv (\theta^1, \theta^2, \theta^3)$ and $T \equiv (T_1, T_2, T_3)$ are the generators of the $so(3,c)$ algebra in the adjoint representation.\(^{32}\) This is given by

$$\lambda_{ae} = (e^{\mathbf{\theta} \cdot T})_{af}\lambda_f(e^{\mathbf{\theta} \cdot T})^T_{fe}. \quad (184)$$

where $\lambda_f \equiv (\lambda_1, \lambda_2, \lambda_3)$ are the eigenvalues of the symmetric part. Since the first two terms of the Hamiltonian constraint (183) depend upon the invariants of $\lambda_{ae}$, then the matrix $O_{ae}$ cancels out and we obtain

$$H = 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) + \Lambda \lambda_1\lambda_2\lambda_3 + (\lambda_f - 2)\psi^f\psi^f, \quad (185)$$

where $\psi_f' = (e^{\mathbf{\theta} \cdot T})_{d\ell}\psi^d$ is the Lorentz transformation of $\psi^d$ into a new Lorentz frame.

We would rather like to interpret the Hamiltonian constraint as being independent of the Lorentz frame, and consider the angles $\mathbf{\theta}$ as not being independent physical degrees of freedom. The most direct way to do this is to require that $\psi^d = 0$, which implies that the antisymmetric part of the CDJ matrix $\Psi_{[ae]}$ vanish.

References

[1] Richard Capovilla, Ted Jacobson, John Dell ‘General Relativity without the Metric’ Class. Quant. Grav. Vol 63, Number 21 (1989) 2325-2328

[2] Abhay Ashtekar. ‘New perspectives in canonical gravity’, (Bibliopolis, Napoli, 1988).

\(^{32}\)This has the interpretation of a Lorentz transformation into a new frame, where $Im[\mathbf{\theta}]$ and $Re[\mathbf{\theta}]$ represent rotations and boosts respectively.
[3] Abhay Ashtekar ‘New Hamiltonian formulation of general relativity’
Phys. Rev. D36(1987)1587

[4] Abhay Ashtekar ‘New variables for classical and quantum gravity’ Phys.
Rev. Lett. Volume 57, number 18 (1986)

[5] Chopin Soo and Lay Nam Chang ‘Superspace dynamics and perturba-
tions around ”emptiness”’ Int. J. Mod. Phys. D3 (1994) 529-544

[6] Chopin Soo and Lee Smolin ‘The Chern–Simons invariant as the natural
time variable for classical and quantum cosmology’ Nucl. Phys. B449
(1995) 289-316

[7] Eyo Ita ‘Proposed solution for the initial value constraints of four di-
mmensional vacuum general relativity: Part II’ In preparation

[8] Charles W. Misner, Kip S. Thorne and John Archibald Wheeler ‘Grav-
itation’ W. H. Freeman and Company, New York (1973)

[9] Hans Stephani, Dietrich Kramer, MacClimacCallum, Cornelius
Hoenselaers, and Eduard Herlt ‘Exact Solutions of Einstein’s Field
Equations’ Cambridge University Press

[10] Laurens Gunnarsen, Hisa-Aki Shinkai and Kei-Ichi Maeda ‘A ‘3+1’
method for finding principal null directions’ Class. Quantum Grav. 12
(1995) 133-140

[11] Donato Bini, Christian Cherubini and Robert T Jantzen ‘The speciali-
ty index and the Lifshitz–Khalatnikov Kasner index parametrization’
Class. Quantum Grav. (2007) 5622-5636

[12] W. B. Bonnor ‘The electric and magnetic Weyl tensors’ Class. Quantum
Grav. 12 (1995) 499-502

[13] Moshe Carmeli ‘Group theory and general relativity.’ Imperial College
Press. Copyright 2000

[14] Ezra Newman and Roger Penrose ‘An approach to gravitational radia-
tion by a method of spin coefficients’ J. Math Phys. Vol.1, No.3 (1962

[15] Ezra Newman and Roger Penrose ‘An approach to gravitational radia-
tion by a method of spin coefficients’ J. Math. Phys. Vol 3, No. 3,
1962

[16] Asher Peres ‘Diagonalization of the Weyl tensor’ Phys. Rev. D18, Num-
ber 2 (1978)

[17] R. Penrose and W. Rindler ‘Spinors and space-time’ Cambridge Mono-
graphs in Mathematical Physics
[18] R Arianrhod and C. B. G. McIntosh ‘Principal null directions of Petrov type I Weyl spinor: geometry and symmetry’ Class. Quantum Grav. 9 (1992) 1969-1982

[19] R Arianrhod and C. B. G. McIntosh ‘Degenerate’ non-degenerate space-time metrics’ Class. Quantum Grav. 7 (1990) L213-216

[20] Ingemar Bengtsson ‘Clifford algebra of 2 forms, conformal structures and field equations’ arXiv:gr-qc/921000

[21] Y.A. Simonov Sov. J. Nucl. Phys. 41 (1985), 835

[22] J. R. Klauder ‘The affine quantum gravity programme’ Class. Quantum Grav. 19, 817-826 (2002)

[23] A.M. Khevedelidze and H.P. Pavel ‘Unconstrained Hamiltonian formulation of SU(2) gluodynamics’ Phys. Rev. D 59, 105017 (1999)

[24] A.M. Khevedelidze, D.M. Mladenov, H.P. Pavel and G. Ropke ‘Unconstrained SU(2) Yang–Mills theory with a topological term in the long-wavelength approximation’ Phys. Rev. D67, 105013 (2003)

[25] Guillermo A Mena Marugan ‘Is the exponential of the Chern–Simons action a normalizable physical state?’ Class. Quantum Grav. 12(1995) 435-441

[26] Eyo Eyo Ita III ‘Finite states in four dimensional quantized gravity’ Class. Quantum Grav. 25 (2008) 125001