Unit conversions and collected numbers in cosmology

Version 2.0

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Abstract. This note collects together useful unit conversions and numerical values from early universe cosmology. It is a quick reference that can be used to make easy order-of-magnitude estimates. Included are tables for unit conversions, the thermal history of the universe, and collected properties of astronomical objects. The note also introduces a modifiable Mathematica package NaturalUnits (newest version 2.0), which makes it easy to convert between natural and physical units.
1 Introduction

This note contains collections of numerical quantities, constants, and unit conversions useful in cosmology and particle physics. It is meant to be an easy-to-use reference for these quantities and for making order-of-magnitude estimates, with an emphasis on early universe phenomenology. Example uses include estimating the masses of primordial black holes and the frequencies of gravitational waves originating from different eras, or estimating the number density of dark matter given the particle mass. The note comes together with an easy-to-use and modifiable Mathematica package NaturalUnits for unit conversions, available for download at www.eemelitomberg.net/NaturalUnits.

The note is organized as follows: Section 2 gives conversion tables for various units, including their conversions into powers of electronvolts in natural units ($c = h = k_B = 1$), a table of useful numbers during the history of the universe, and a table of the properties of some astronomical objects. Section 3 introduces the associated Mathematica package.

This note and the Mathematica package are by no means comprehensive and may be updated in the future. If you have suggestions for improvements, please contact the author.

2 Tables

This section contains tables of unit conversions for energy, mass, distance, time, and others on page 2. The unit on the left is given in terms of the unit at the top. Conversions to electronvolts (eV) are also provided, in natural units where the speed of light $c$, the reduced Planck constant $\hbar$, and the Boltzmann constant $k_B$ are set to one. Units: J = Joule, K = Kelvin, $T_{Pl} = $ Planck temperature $= \sqrt{\hbar c/(Gk_B^2)}$ ($G = $ Newton’s constant), eV = electronvolt, kg = kilogram, $M_\odot = $ the solar mass, $m_{Pl} = $ Planck mass $= \sqrt{\hbar c/G}$, $M_{Pl} = $ reduced Planck mass $= m_{Pl}/\sqrt{8\pi}$, m = meter, pc = parsec, AU = astronomical unit, $l_{Pl} = $ Planck length $= \sqrt{\hbar G/c^3}$, s = second, y = year, $t_{Pl} = $ Planck time $= \sqrt{\hbar G/c^5}$, Hz = Hertz.

The table on page 3 presents the thermal history of the universe. Standard ΛCDM cosmology with quasi-de Sitter slow-roll inflation and instantaneous reheating is assumed, with no visible matter beyond the SM. The numerical values needed to derive these are mostly taken from the textbook by Liddle and Lyth [1]; for other modern textbooks, see [2, 3], and for useful lecture notes, see [4, 5]. More accurate parameter values can be found e.g. in [6, 7]. Explanations of symbols: $\rho = $ energy density, $H = $ Hubble parameter, $k_s = $ comoving scale of the Hubble radius, $T = $ temperature, $z = $ redshift, $g_* = $ the effective number of degrees of freedom, $t = $ the age of the universe, $M = $ mass within one Hubble radius, $\Omega_X = $ the energy density fraction of dark energy ($\Lambda$), cold dark matter (CDM), baryons (b), and radiation (r), $\Lambda = $ cosmological constant.

The table on page 4 lists the properties of some astronomical objects and other observables, with $M = $ mass, $R = $ radius, $N = $ number count, $\rho = $ density, $v = $ velocity, $R_S = $ Schwarzschild radius. For more accurate values, see e.g. [7–9].
|                | J          | K          | $T_{P1}$   | eV        |
|----------------|------------|------------|------------|-----------|
| J              | -          | $7.24 \times 10^{22}$ | $5.11 \times 10^{-10}$ | $6.24 \times 10^{18}$ |
| K              | $1.38 \times 10^{-23}$ | -          | $7.06 \times 10^{-33}$ | $8.62 \times 10^{-5}$ |
| $T_{P1}$       | $1.96 \times 10^9$ | $1.42 \times 10^{32}$ | -          | $1.22 \times 10^{28}$ |
| eV             | $1.6 \times 10^{-19}$ | $1.16 \times 10^4$ | $8.19 \times 10^{-29}$ | -          |

|                | kg         | $M_\odot$  | $m_{P1}$   | $M_{P1}$  | eV        |
|----------------|------------|------------|------------|-----------|-----------|
| kg             | -          | $5.03 \times 10^{-31}$ | $4.59 \times 10^7$ | $2.3 \times 10^8$ | $5.61 \times 10^{45}$ |
| $M_\odot$      | $1.99 \times 10^{30}$ | -          | $9.14 \times 10^{37}$ | $4.58 \times 10^{38}$ | $1.12 \times 10^{66}$ |
| $m_{P1}$       | $2.18 \times 10^{-8}$ | $1.09 \times 10^{-38}$ | -          | -          | $5.01 \times 10^{28}$ |
| $M_{P1}$       | $4.34 \times 10^{-9}$ | $2.18 \times 10^{-39}$ | $0.20$     | -          | $2.44 \times 10^{27}$ |
| eV             | $1.78 \times 10^{-36}$ | $8.96 \times 10^{-67}$ | $8.19 \times 10^{-29}$ | $4.11 \times 10^{-28}$ | -          |

|                | m          | pc         | AU         | $l_{P1}$  | eV$^{-1}$ |
|----------------|------------|------------|------------|-----------|-----------|
| m              | -          | $3.24 \times 10^{-17}$ | $6.68 \times 10^{-12}$ | $6.19 \times 10^{34}$ | $5.07 \times 10^6$ |
| pc             | $3.09 \times 10^{16}$ | -          | $2.06 \times 10^5$ | $1.91 \times 10^{51}$ | $1.56 \times 10^{23}$ |
| AU             | $1.5 \times 10^{11}$ | $4.85 \times 10^{-6}$ | -          | $9.26 \times 10^{45}$ | $7.58 \times 10^{17}$ |
| $l_{P1}$       | $1.62 \times 10^{-35}$ | $5.24 \times 10^{-52}$ | $1.08 \times 10^{-46}$ | -          | $8.19 \times 10^{-29}$ |
| eV$^{-1}$      | $1.97 \times 10^{-7}$ | $6.39 \times 10^{-24}$ | $1.32 \times 10^{-18}$ | $1.22 \times 10^{28}$ | -          |

|                | s          | y          | $l_{P1}$   | eV$^{-1}$ |
|----------------|------------|------------|------------|-----------|
| s              | -          | $3.17 \times 10^{-8}$ | $1.85 \times 10^{43}$ | $1.52 \times 10^{15}$ |
| y              | $3.16 \times 10^7$ | -          | $5.85 \times 10^{30}$ | $4.79 \times 10^{22}$ |
| $l_{P1}$       | $5.39 \times 10^{-44}$ | $1.71 \times 10^{-51}$ | -          | $8.19 \times 10^{-29}$ |
| eV$^{-1}$      | $6.58 \times 10^{-16}$ | $2.09 \times 10^{-23}$ | $1.22 \times 10^{28}$ | -          |

|                | Hz         | Mpc$^{-1}$  | $M_{P1}$   | GeV      |
|----------------|------------|------------|------------|----------|
| Hz             | -          | $1.03 \times 10^{14}$ | $2.7 \times 10^{-43}$ | $6.58 \times 10^{-25}$ |
| Mpc$^{-1}$     | $9.72 \times 10^{-15}$ | -          | $2.63 \times 10^{-57}$ | $6.39 \times 10^{-39}$ |
| $M_{P1}$       | $3.7 \times 10^{42}$ | $3.81 \times 10^{56}$ | -          | $2.44 \times 10^{18}$ |
| GeV            | $1.52 \times 10^{24}$ | $1.56 \times 10^{38}$ | $4.11 \times 10^{-19}$ | -          |
Cosmic inflation (r=tensor-to-scalar ratio)
\[ \rho_{1/4} = \left( \frac{r}{0.08} \right)^{1/4} \times 1.7 \times 10^{16} \text{ GeV} \]
\[ H = \left( \frac{r_0}{r_0} \right)^{1/2} \times 7 \times 10^{13} \text{ GeV} = \left( \frac{r_0}{r_0} \right)^{1/2} \times 10^{52} \text{ Mpc}^{-1} \]
\[ k_\pi = 0.05 \text{ Mpc}^{-1} = 5 \times 10^{-16} \text{ Hz (CMB scale)} \]

Reheating
\[ z = 10^{29+1/4 \log_{10} \frac{r_0}{r_0}} = e^{67+1/4 \log_{10} \frac{r_0}{r_0}} \text{ Hz} \]
\[ k_\pi = 10^{23+1/4 \log_{10} \frac{r_0}{r_0}} \text{ Mpc}^{-1} = 10^{9+1/4 \log_{10} \frac{r_0}{r_0}} \text{ Hz} \]

Radiation domination: \( a \propto t^{1/2}, \ H = 1/(2t) \propto a^{-2} \)

Electroweak phase transition
\[ T = 100 \text{ GeV} \quad z = 10^{15} = e^{35} \quad H = 10^{-5} \text{ eV} = 10^{24} \text{ Mpc}^{-1} \]
\[ g_* = 106.75 \quad t = 20 \text{ ps} \quad M = 10^{-6} M_\odot = 10^{28} \text{ g} \]
\[ k_\pi = 10^6 \text{ Mpc}^{-1} = 10^{-5} \text{ Hz} \]

QCD phase transition
\[ T = 150 \text{ MeV} \quad z = 10^{12} = e^{28} \quad H = 10^{-11} \text{ eV} = 10^{18} \text{ Mpc}^{-1} \]
\[ g_* = 17.25 \quad t = 30 \mu s \quad M = 5 M_\odot = 10^{34} \text{ g} \]
\[ k_\pi = 10^6 \text{ Mpc}^{-1} = 10^{-8} \text{ Hz} \]

Big Bang Nucleosynthesis
\[ T = 100 \text{ keV} \quad z = 10^8 = e^{20} \quad H = 10^{-18} \text{ eV} = 10^{11} \text{ Mpc}^{-1} \]
\[ g_* \approx 3 \quad t = 100 \text{ s} = 2 \text{ min} \quad M = 10^7 M_\odot = 10^{41} \text{ g} \]
\[ k_\pi = 10^3 \text{ Mpc}^{-1} = 10^{-11} \text{ Hz} \]

Matter-radiation equality
\[ T = 0.8 \text{ eV} = 9000 \text{ K} \quad z = 3 \times 10^3 = e^8 \quad H = 30 \text{ Mpc}^{-1} \]
\[ t = 60 \text{ 000 y} = 2 \times 10^{12} \text{ s} \quad M = 10^{17} M_\odot = 10^{51} \text{ g} \]
\[ k_\pi = 10^{-2} \text{ Mpc}^{-1} = 10^{-16} \text{ Hz} \]

Matter domination: \( a \propto t^{2/3}, \ H = 2/(3t) \propto a^{-3/2} \)

Recombination
\[ T = 0.25 \text{ eV} = 3000 \text{ K} \quad z = 1050 = e^7 \quad H = 6 \text{ Mpc}^{-1} \]
\[ t = 400 \text{ 000 y} = 10^{13} \text{ s} \quad M = 10^{18} M_\odot \]
\[ k_\pi = 5 \times 10^{-3} \text{ Mpc}^{-1} = 5 \times 10^{-17} \text{ Hz} \]

Today
\[ T = 2.7 \text{ K} = 0.23 \text{ meV} \quad t = 13.8 \times 10^9 \text{ y} = 4.35 \times 10^{17} \text{ s} \quad M = 10^{22} M_\odot \]
\[ \Omega_\Lambda = 0.72 \quad \Omega_{\text{CDM}} = 0.23 \quad \Omega_b = 0.05 \quad \Omega_\gamma = 8 \times 10^{-5} \]
\[ H = 70 \text{ km/s/Mpc} = 2 \times 10^{-4} \text{ Mpc}^{-1} = 1/(4000 \text{ Mpc}) = 10^{-33} \text{ eV} = 10^{-61} M_{\text{Pl}} \]
\[ \rho = 9 \times 10^{-27} \text{ kg/m}^3 = 5 \text{ GeV/m}^3 = 10^{-120} M_{\text{Pl}}^4 \]
\[ \Lambda = 10^{-120} M_{\text{Pl}}^4 = 10^{-52} \text{ m}^{-2} = 1/(3000 \text{ Mpc})^2 \]
| System          | Mass $M$ (in $M_{\odot}$) | Radius $R$ (in kpc) | Number of stars $N_{stars}$ |
|-----------------|---------------------------|---------------------|-----------------------------|
| Milky Way       | $10^{12}$                 | $10^2$              | $10^{11}$                   |
| Local dark matter halo | $0.3$ GeV/cm$^3$   | $200$ km/s          |                             |
| Sun             | $2 \times 10^{30}$        | $7 \times 10^8$ m   | $3$ km                      |
| Earth           | $6 \times 10^{24}$        | $6400$ km           | $9$ mm                      |
| Moon            | $7 \times 10^{22}$        | $1700$ km           | $0.1$ mm                    |
3 Mathematica package NaturalUnits, version 2.0

The Mathematica package for unit conversions, NaturalUnits, can be downloaded from the author’s web page, www.eemelitomberg.net/NaturalUnits. This section briefly describes the functionality of the package’s version 2.0; more examples and tutorials can be found on the web page.

To use the package, download NaturalUnits.m and place it into the same folder with your Mathematica notebook. Load the package and set up the natural units in the notebook with the commands

\begin{verbatim}
In: SetDirectory[NotebookDirectory[]];
   "NaturalUnits.m"
   NaturalUnitsSetup[NewUnitSystem->CosmologyUnits, NewNatUnits->{eV}];
Out: NaturalUnits 2.0
\end{verbatim}

The last line sets up the natural unit system used in this note, $c = \hbar = k_B = 1$, with electronvolts as the default base unit. In version 2.0, other unit systems are also supported, and custom systems can be set up by giving the defining equations, e.g. NewUnitSystem->{c==G==1} in the above for geometrized units.

To do unit conversions, call the function NaturalConvert. For example,

\begin{verbatim}
In: NaturalConvert[10^-4 MSolar, kg]
Out: 1.989*10^26 kg
\end{verbatim}

The first parameter is the quantity to be converted, with units represented by Mathematica symbols. The second parameter is the target unit. If the second parameter is left empty, a conversion to a power of the base units is made:

\begin{verbatim}
In: NaturalConvert[10^-4 MSolar]
Out: 1.11575*10^62 eV
\end{verbatim}

If the two units don’t match in their natural-unit form (powers of electronvolts), a conversion to the correct power is attempted for the target unit. Each input to NaturalConvert must be proportional to a single power of the natural units, otherwise the conversion fails.

The package supports many units useful for cosmology, together with prefixes such as GeV = $10^9$ eV, and also knows some common constants of nature, corresponding to standard symbols when possible\footnote{For less obvious special cases, MSolar, MEarth, and MMoon are the masses of the Sun, the Earth and the Moon, mElectron and mProton are the electron and proton masses, and the Planck units are given by mP (Planck mass), MP (reduced Planck mass), tP (Planck time), lP (Planck length), TP (Planck temperature), and EP (Planck energy).}. A full list of symbols is contained in $\$AllUnits$, and can be printed together with unit definitions with NaturalUnitsList[]. Adding custom units is simple: open NaturalUnits.m and add the unit to the list $\$UnitRules$ in terms of the previously defined units and constants, like the other entries in the list. New units can also be added during a notebook session with the command NaturalUnitsAdd, e.g.

\begin{verbatim}
In: NaturalUnitsAdd[mile->1.609344 km]
NaturalConvert[10 mile/h, m/s]
Out: 4.4704 m/s
\end{verbatim}

For more advanced functionality, the second parameter to NaturalConvert can be a list of units \{x, y, z, \ldots\}, and the function then tries to convert the quantity into a product of powers of these units, $x^{n_1} \times y^{n_2} \times z^{n_3} \ldots$. Option KeepUnit tells whether the units should be kept in the result (KeepUnit->True, default), kept in a non-simplified form (KeepUnit->Defer), or dropped (KeepUnit->False), for example:

\begin{verbatim}
In: NaturalConvert[200 ly/rad, {pc, 180 deg}, KeepUnit->Defer]
Out: 192.639 pc/(180 deg)
\end{verbatim}
The default value of the target unit is stored in the option \texttt{ToUnitDefault} (its starting value is \texttt{NoUnit}, which produces the above-described results in powers of eV).

Beware: since many simple symbols are used as units, there may be overlap with other packages and custom code. The package is best used on its own for simple number manipulation; otherwise, the namespace label \texttt{NaturalUnits}’ should be used in front of the unit symbols.

**Examples of use.** The package can be used to easily calculate quantities that would otherwise require cumbersome unit conversions. For example, the Schwarzschild radius of a 10 solar mass black hole:

\begin{verbatim}
In: NaturalConvert[2*10 MSolar*G, km]
Out: 29.5399 km
\end{verbatim}

The typical frequency of CMB radiation, starting from temperature:

\begin{verbatim}
In: NaturalConvert[2.7 K, Hz]
Out: 3.53485\times10^{11} \text{ Hz}
\end{verbatim}

Typical mass of a primordial black hole formed when the temperature of the universe was 100 MeV (that is, mass of a sphere with radius \(H^{-1}\), with \(\rho \sim T^4, H^2 M^2_{\text{Pl}} \sim \rho\)):

\begin{verbatim}
In: NaturalConvert[(T^4*(T^2/MP)^{-3}) /. T \rightarrow 100 \text{ MeV}, MSolar]
Out: 1.29459 \text{ MSolar}
\end{verbatim}

More examples can be found at \url{www.eemelitomberg.net/NaturalUnits}.

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