Abstract
A novel approach for solving a multiple judge, multiple criteria decision making (MCDM) problem is proposed. The presence of multiple criteria leads to a non-total order relation. The ranking of the alternatives in such a framework is done by reinterpreting the MCDM problem as a multivariate statistics one and by applying the concepts in Hamel and Kostner (J Multivar Anal 167:97–113, 2018). A function that ranks alternatives as well as additional functions that categorize alternatives into sets of “good” and “bad” choices are presented. The paper shows that the properties of these functions ensure a reasonable decision making process.

Keywords
Multi criteria decision making · Set optimization · Multivariate statistics · Multivariate quantiles

1 Introduction

The aim of this work is to propose a new method for solving multiple criteria decision making (MCDM) problems. The basic idea is to interpret the MCDM problem as one of multivariate statistics and apply new set optimization methods. More specifically, a modified version of the multiple judge, multiple criteria ranking problem in Buckley (1984) is solved by replacing the fuzzy set approach by a new one based on set-valued quantiles for multivariate random variables. This actually is a new way to deal with the fundamental difficulty—the lack of a natural complete order for multidimensional objects—in MCDM.

Raiffa and Schlaifer (1961) as well as Howard (1968) had an important influence on modern decision theory. Nowadays, MCDM is a sub-discipline of operations research that explicitly evaluates multiple conflicting criteria in decision making, refer to Greco et al. (2016) for a comprehensive survey. More specifically, a decision maker has to choose between different alternatives (options, objects) that are described by multiple...
criteria that are often in conflict with each other, e.g., maximum speed and fuel consumption of a car. As well presented in Grabisch (2016), the fundamental difficulty of this decision problem is that, "there is no natural complete order on multidimensional objects". The last quote has many counterparts in the statistical literature since the lack of a “canonical” quantile (function) in the multivariate case usually is attributed to this “lack of a natural order in higher dimensions”, e.g., Belloni and Winkler (2011, p. 1126). The concepts introduced in Hamel and Kostner (2018) deal with this problem. Moreover, these new notions enjoy basically all the properties of their univariate counterparts and are based on the recent developments in set optimization as surveyed in Hamel et al. (2015).

The decision making problem discussed in Buckley (1984) is the following. The aim is to rank a set of alternatives \( A = \{a_1, a_2, \ldots, a_m\} \) from “best” to “worst”, based on a set of criteria \( \Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_d\} \). This is accomplished by employing judges (advisors, experts): \( J_1, J_2, \ldots, J_n \). The judges use a scale \( \mathcal{L} \) of preference information to assess the criteria’s importance and the criteria of the alternatives. It is assumed that \( \mathcal{L} \) is a well-ordered set.

Each judge \( J_j \) indicates the importance (weight) of criterion \( \gamma_k \) via \( v_j(k) \in \mathcal{L} \). Moreover, each judge \( J_j \) supplies the information of how well alternative \( a_i \) satisfies criterion \( \gamma_k \), \( x_j: \Gamma \times A \rightarrow \mathcal{L} \). The same scale \( \mathcal{L} \) is used for \( x_j(\gamma_k, a_i) \) and \( v_j(\gamma_k) \). Therefore, each judge \( J_j \) has two fuzzy sets: \( (\Gamma, v_j(\gamma_k)) \) and \( (A, x_j(\gamma_k, a_i)) \).

In Buckley (1984), these two sets are denoted as fuzzy sets, even if their membership functions have values in \( \mathcal{L} \) (instead of \([0, 1]\)). They are used to equip the sets of criteria and alternatives with a total order. The paper (Buckley 1984) discusses methods of aggregating all the sets to achieve a set of ranked alternatives. More specifically, it examines when and how to pool the judges’ information as well as how to compute the final ranking.

In the present paper, we propose an alternative approach to a slightly modified problem, where the information on how well the alternatives satisfy each criterion \( x_j(\gamma_k, a_i) \) is provided by a third party. The cone distribution function and the \( C \)-quantiles from Hamel and Kostner (2018) perfectly fit the requirements for solving this problem: the judges’ opinions on the importance of the criteria are modeled via a vector order generated by a cone \( C \) and the alternatives with their criteria ratings are interpreted as realizations of a random vector. The cone distribution can be understood as an analytic tool, namely a ranking function for the alternatives, while the \( C \)-quantiles have a geometric flavor since polyhedral convex sets—clusters of alternatives—are compared with each other.

The current paper differs from the research in group decision making as reviewed in Herrera-Viedma et al. (2014) in three fundamental aspects. First, in the latter research stream the judges’ opinions are brought together into a group decision by reaching a consensus. The aim is to attain the consent of the judges for a decision that will be beneficial to the whole group, which does not necessarily mean that each judge is in full personal agreement. In contrast, in this paper the judges opinions as well as all potential compromises between the judges are taken into account in order to find the most conservative (risk-limiting) decision. Second, as the criteria of the alternatives can often only be described qualitatively (linguistically), several approaches in the literature [e.g., most of the references in Herrera-Viedma et al. (2014)] have used fuzzy
set theory, which enables to quantify linguistic assessments. This paper is understood as a theoretical basis for a completely new solution method for group MCDM that can be applied to finance and other disciplines, where the criteria can be assessed by exact data. However, these new concepts can be further developed to deal with imprecise criteria assessments. Third, the paper utilizes set-valued quantiles as a new tool for decision making that enables the clustering of the alternatives into “bad” and “good” choices.

In the next section the cone distribution function and the set-valued quantiles from Hamel and Kostner (2018) are applied to the decision problem. The properties of these functions and their use in the decision making process are analyzed. Simple examples and figures are provided as illustrations.

2 A set optimization approach to MCDM

Our problem is as follows. As in Buckley (1984), the goal is to rank a set of alternatives $A = \{a_1, a_2, \ldots, a_m\}$, taking into account a set of criteria $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_d\}$. However, the information is taken from two different sources. On the one hand, each judge $J_1, J_2, \ldots, J_n$ indicates the relative importance of criterion $\gamma_k$ with respect to the other criteria via a non-negative number, $v_j: \Gamma \rightarrow \mathbb{R}_+$. Therefore, each judge has a vector of weights:

$$v^j = \begin{bmatrix} v_j(\gamma_1) \\ \vdots \\ v_j(\gamma_d) \end{bmatrix} \in \mathbb{R}_+^d \setminus \{0\},$$

called the importance vector. On the other hand, the information of how well $a_i$ satisfies criterion $\gamma_k$ is provided by an external evaluator/source, $x_E: \Gamma \times A \rightarrow \mathbb{R}$. This enables the customization of MCDM to applications in finance, where the external source could be the asset market. Each alternative $a_i$ is assessed via the vector

$$x^E(a_i) = \begin{bmatrix} x_E(\gamma_1, a_i) \\ \vdots \\ x_E(\gamma_d, a_i) \end{bmatrix} \in \mathbb{R}^d$$

whereas the set $\tilde{X}$ stores the evaluation of all alternatives:

$$\tilde{X} = \left\{ x^E(a_1), \ldots, x^E(a_m) \right\} = \left\{ \begin{bmatrix} x_E(\gamma_1, a_1) \\ \vdots \\ x_E(\gamma_d, a_1) \end{bmatrix}, \ldots, \begin{bmatrix} x_E(\gamma_1, a_m) \\ \vdots \\ x_E(\gamma_d, a_m) \end{bmatrix} \right\}$$

Below, $\tilde{X}$ will be understood as realizations of a $d$-dimensional random vector.

Let us recall a few useful concepts from the theory of ordered vector spaces. A preorder is a reflexive and transitive relation, and an antisymmetric preorder is called
a partial order. The set \( H^w = \{ z \in \mathbb{R}^d \mid w^\top z \geq 0 \} \) is called the closed homogeneous halfspace with normal \( w \). A set \( C \subseteq \mathbb{R}^d \) is called a convex cone if \( s > 0, z \in C \) imply \( sz \in C \) and if \( x, y \in C \) implies \( x + y \in C \). A vector preorder \( \leq_C \) is generated by the convex cone \( C \subseteq \mathbb{R}^d \) with \( 0 \in C \) by means of

\[ x \leq_C y \iff y - x \in C. \]

### 2.1 Modeling the judges’ preferences

The data presented in the section above is used to introduce two convex cones. These cones generate a vector order which is crucial for the ranking of the alternatives.

The \( n \) importance vectors \( v^j \) of the judges generate a cone via

\[
K_I = \left\{ \sum_{j=1}^n s_j v^j \mid s_1, \ldots, s_n \geq 0 \right\}.
\]

This convex cone pools the judges’ opinions on the importance of each criterion. It is called the importance cone \( K_I \). There is a minimal number of importance vectors \( v^j \) that generate \( K_I \). These vectors belong to the judges with the most “extreme” opinions. In general, the cone includes all non-negative linear combinations of these “extreme” vectors. From an application standpoint, the inclusion of all non-negative linear combinations means that not only the “extreme judges” views are incorporated, but also all potential compromises between the judges.

The halfspace with normal \( v^j \)

\[
H^+(v^j) = \left\{ z \in \mathbb{R}^d \mid v^j\top z \geq 0 \right\}
\]

is the set of “non-negative” alternatives with respect to the judge’s opinion on the importance of the criteria \( (v^j) \). It can be interpreted as the set of accepted alternatives (by judge \( j \)) in the sense that judge \( j \) prefers the alternatives in its interior over the zero alternative and is indifferent between the alternatives on its boundary and the zero alternative. The halfspace \( H^+(v^j) \) generates the total preorder

\[ x \leq_{H^+(v^j)} y, \]

which ranks the criteria evaluations of the alternatives \( x^{E}(a_i) \) depending on the judge’s importance vector \( v^j \). By taking the intersection over such halfspaces with the \( n \) vectors \( v^j \) as normals another polyhedral convex cone can be formed:

\[
K_A = \bigcap_{j=1}^n H^+(v^j).
\]

This cone incorporates the view of all judges. It represents the positions with respect to the criteria (evaluations) that are unanimously accepted—perceived as not worse.
than zero—by all judges. It is called the \textit{acceptance cone $K_A$} and it generates a vector preorder on the set $\tilde{X}$:

$$x \leq_{K_A} y.$$

In summary, $K_A$ represents the accepted positions in the criteria and $K_I$ the relative importance given to each criterion.

In order to understand the relation between the concepts in Hamel and Kostner (2018) and their application to MCDM, it needs to be clarified that the cones (generating vectors) given as primary input data are different. On the one hand, the goal in Hamel and Kostner (2018) is to incorporate a vector preorder $\leq_C$ given by a cone $C$ [input in Hamel and Kostner (2018)] into the statistical analysis of multivariate data. The dual cone of $C$ comes into play from a technical standpoint, as $\leq_C$ can be represented as intersection of total preorders generated by the closed halfspaces $H^+(w)$ for $w \in C^+ \setminus \{0\}$. On the other hand, in this paper the judges’ opinions on the relevance of each criterion $v_j$ constitute the cone $K_I$ (primary input). However, the order relation that ranks the criteria assessments of the alternatives $x^E(a_i)$ is given by the acceptance cone $K_A$.

The cone $K_A$ resembles the solvency cone, an important concept in financial mathematics, see Hamel and Heyde (2010). $K_I$ is a finitely generated convex cone and hence closed. Since $v_j \in \mathbb{R}^d_+$ it follows that $K_I \subseteq \mathbb{R}^d_+$. Moreover, due to the bipolar theorem the importance cone $K_I$ and the acceptance cone $K_A$ are dual to each other. To summarize the discussion, we give an abstract definition.

\textbf{Definition 2.1} An importance cone $K_I$ is any closed convex cone such that $\{0\} \subsetneq K_I \subseteq \mathbb{R}^d_+$, whereas an acceptance cone $K_A$ is any closed convex cone such that $\mathbb{R}^d_+ \subseteq K_A \subsetneq \mathbb{R}^d$.

\textbf{Example 2.2} In order to illustrate the intuition behind these new concepts, a decision making example with two criteria is used. Figure 1 shows a two-dimensional importance vector $v^1 = (2, 1)\top$. Moreover it depicts a closed homogeneous halfspace $H^+(v^1)$ (blue area). This includes on its boundary alternatives that the judge would not deem as a loss, whereas the alternatives in its interior are the “better” perceived the further away from the boundary. Therefore, $H^+(v^1)$ can be seen as the acceptance set of the judge.

Now, if there are several judges all their opinions should be incorporated. Figure 2a shows the values given (dotted vectors) to the criteria by three judges: $v^1 = (2, 1)\top$, $v^2 = (1, 1)\top$ and $v^3 = (1, 2)\top$. Again, these vectors are normals to the acceptance sets (blue halfspaces) of the judges. In order to derive an acceptance set that complies with all judges, the intersection (2b) over all their acceptance sets is taken. From this follows the acceptance cone $K_A$ (light yellow cone in 2b) that represents the positions in the criteria accepted by all judges. As illustrated in Fig. 2c the importance cone (yellow opaque) is only generated by $v^1$ and $v^3$, as $v^2$ is a non-negative linear combination of the former two.

\textbf{Example 2.3} An interesting illustration on how the cones reflect the judges’ views is shown in Fig. 3. The more the “extreme” judges differ from each other, the wider the
Fig. 1 The importance vector $v^1$ and its acceptance set $H^+(v^1)$. (Color figure online)

Fig. 2 From the individual importance vectors and acceptance sets to the cones $K_I$ and $K_A$. (Color figure online)

importance cone $K^1_I \supseteq K^2_I$ and the smaller the acceptance cone $K^1_A \subseteq K^2_A$. Further differing opinions on the criteria imply that a greater compromise has to be made, which takes shape in a smaller acceptance cone $K_A$.

2.2 Cone distribution functions

In this section, the MCDM problem is translated into a multivariate statistics framework. The set of alternatives $A = \{a_1, a_2, \ldots, a_m\}$ is understood as the sample set $\Omega$ of a vector-valued random variable $X$. More precisely, $(\Omega, \mathcal{F}, \Pr)$ is the probability space with $\Omega = A$, $\mathcal{F} = \mathcal{P}(A)$ and $\Pr$ is a probability measure (distribution), whereas $X: \Omega \rightarrow \mathbb{R}^d$ is a $d$-dimensional random variable with the image set $\tilde{X}$.

The recently introduced cone distribution function (see Hamel and Kostner 2018) is used in order to rank the alternatives. Therefore, it is reinterpreted as a ranking
function of the elements in $\tilde{X}$, which associates to a $x^E(a_i) \in \tilde{X}$ a rank $p$ from 0 to 1. From now on the parameter $p$ is interpreted as a rank (order) indicator and denoted as rank. The cone distribution function of $X$ is defined in a two step procedure.

First, fix $v \in K_I \setminus \{0\}$ and consider $F_{X,v}(z) : \mathbb{R}^d \to [0, 1]$ defined by

$$F_{X,v}(z) = F_{v^\top X}(v^\top z) = \Pr\left( v^\top X \leq v^\top z \right),$$

where the random vector $X$ and the function’s input $z$ are scalarized by the importance vector $v$. In fact, $v^\top X$ is a random variable and $F_{X,v}(z)$ is its cumulative distribution function taken at $v^\top z$. This distribution function can be reformulated as the probability of $X$ being in the affine halfspace $z - H^+(v)$:

$$F_{X,v}(z) = \Pr\left( v^\top X \leq v^\top z \right) = \Pr\left( X \in z - H^+(v) \right).$$

If $\tilde{X} = \{x^E(a_1), \ldots, x^E(a_m)\}$ is the set of the criteria evaluations of all alternatives and $z = x^E(a)$ is the criteria assessments of the alternative $a$, then $F_{\tilde{X},v}(z)$ quantifies how many $x^E(a_i) \in \tilde{X}$ are ranked below $x^E(a)$ by a judge with the importance vector $v$.

The scalar distribution function is applied on the finite sample set $\tilde{X}$:

$$F_{\tilde{X},v}(z) = \frac{1}{m} \# \left\{ i \in \{1, \ldots, m\} \mid x^E(a_i) \in z - H^+(v) \right\}.$$
Second, all the judges’ importance vectors are considered jointly to rank the criteria evaluations of the alternatives $x^E(a_i) \in \tilde{X}$. This can be done by taking the infimum of $F_{\tilde{X},v}(z)$ over all $v \in K_I$. The use of $K_I$ implies that not only the judges’ importance vectors $v$ are considered but also their non-negative linear combinations. This has the beneficial effect that the optimal solution does not have to compel to a single judge, but can also be a compromise between judges. It follows that the vector $v \in K_I$ that assigns to $x^E(a)$ the lowest rank via $F_{\tilde{X},v}(x^E(a))$ is chosen. Hence, a conservative/risk-limiting ranking of the assessed alternatives is ensured. This is exactly what the cone distribution function $F_{\tilde{X},C} : \mathbb{R}^d \rightarrow [0, 1]$ in Definition 3 of Hamel and Kostner (2018) with $C = K_A$ and applied to $\tilde{X}$ does:

$$F_{\tilde{X},K_A}(z) = \inf_{v \in K_I \setminus \{0\}} \frac{1}{m} \# \left\{ i \in \{1, \ldots, m\} \mid x^E(a_i) \in z - H^+(v) \right\}.$$  

The properties of the function $F_{\tilde{X},K_A}(z)$ (see Hamel and Kostner 2018) which have the most interesting meaning for the multiple criteria decision making problem are reviewed.

(a) **Affine equivariance**, i.e., if $b \in \mathbb{R}^d$ and $B \in \mathbb{R}^{d \times d}$ is an invertible matrix, then

$$\forall \ z \in \mathbb{R}^d : F_{B\tilde{X} + b,BK_A}(Bz + b) = F_{\tilde{X},K_A}(z).$$

On one side, if the scale of the criteria has a shift for all criteria ($b$), then the ranking of the assessed alternatives $x^E(a_i)$ is not compromised. On the other side, if only some criteria undergo a change in scale ($B$), then this must be reflected in the acceptance cone, respectively in the importance cone.

(b) **Monotone non-decreasing** function of $z$ with respect to $\leq_{CA}$, i.e., if $y \leq_{CA} z$, then $F_{\tilde{X},K_A}(y) \leq F_{\tilde{X},K_A}(z)$. This property states that $F_{\tilde{X},K_A}$ ranks with respect to the preorder given by the acceptance cone.

(c) **Monotone non-increasing** function of $\tilde{X}$ with respect to $\leq_{CA}$, i.e., if $\tilde{X} \leq_{CA} \tilde{Y}$, then $F_{\tilde{X},K_A}(z) \geq F_{\tilde{Y},K_A}(z)$ for all $z \in \mathbb{R}^d$. The rank of an element depends on how the element compares against the others. If there are two sets of evaluated alternatives $\tilde{X}$ and $\tilde{Y}$, where $\tilde{Y}$ includes better rated elements with respect to the judges tastes, then an element included in both sets gets a lower ranking with the set including the better elements $\tilde{Y}$.

(d) If $\mathbb{R}^d_+ \subseteq K_A^1 \subseteq K_A^2 \subseteq \mathbb{R}^d$ are two closed convex cones, then $F_{\tilde{X},K_A^1}(z) \leq F_{\tilde{X},K_A^2}(z)$ for all $z \in \mathbb{R}^d$. The judges’ views have an important effect on the ranking. As illustrated in Fig. 3, the more apart the opinions of the judges on the criteria are, the wider is the importance cone, consequently the narrower is the acceptance cone—with the latter cone implying a bigger compromise to be made. This effort of accommodating all the judges’ opinions is reflected by the cone distribution function that assigns a lower rank to the same alternative when the acceptance cone gets narrower.

From the discussion above, it is clear that the cone distribution function $F_{\tilde{X},K_A}(z)$ is an appropriate tool for ordering a set of alternatives by integrating the opinions of multiple judges.
Fig. 4 Five alternatives with two criteria

Table 1 \( F_{\tilde{X},K_A}(z) \) versus \( F_{\tilde{X},v}(z) \)

| \( z = x^E(a_i) \) | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( a_5 \) |
|-----------------|-----|-----|-----|-----|-----|
| \( F_{\tilde{X},K_A}(z) \) | 0.4 | 0.2 | 0.2 | 0.4 | 1   |
| \( F_{\tilde{X},v_1}(z) \)  | 0.4 | 0.4 | 0.6 | 0.8 | 1   |
| \( F_{\tilde{X},v_2}(z) \)  | 0.8 | 0.4 | 0.4 | 0.8 | 1   |
| \( F_{\tilde{X},v_3}(z) \)  | 0.8 | 0.6 | 0.4 | 0.4 | 1   |

**Example 2.4** In Fig. 4 a simple example with two criteria and the criteria assessments of five alternatives helps to understand the new method. The goal is to order the alternatives \( A = \{a_1, a_2, a_3, a_4, a_5\} \) by giving a rank to their criteria evaluations \( \{x^E(a_1), x^E(a_2), x^E(a_3), x^E(a_4), x^E(a_5)\} \). The five points in Fig. 4 are ranked with help of the cone distribution function and compared to the individual ranking of each judge, where \( v^1 = (2, 1)^T \), \( v^2 = (1, 1)^T \) and \( v^3 = (1, 2)^T \).

By comparing the ranking derived from the cone distribution function \( F_{\tilde{X},K_A}(z) \) against the “scalarized CDFs” \( F_{\tilde{X},v}(z) \), three things stand out (see Table 1). First, \( F_{\tilde{X},K_A}(z) \) assigns the lowest values. This is simply due to the infimum in its definition, where the vector \( v \in K_I \) that minimizes \( F_{\tilde{X},v}(z) \) is chosen. Therefore, the cone distribution function for \( x^E(a_2) \) and \( x^E(a_3) \) is equal to 0.2, as there is a \( v \in K_I \) for that \( F_{\tilde{X},v}(z) = 0.2 \). Second, in all cases \( x^E(a_5) \) has a rank equal to one. This is inherent in the definition of \( F_{\tilde{X},v}(z) \), as \# \( \{i \in \{1, \ldots, m\} \mid x^E(a_i) \in z - H^+(v) \} = m, \forall v \in K_I \). Third, for \( F_{\tilde{X},K_A}(z) \) the elements \( x_E(a_1) \) and \( x_E(a_4) \), respectively \( x_E(a_2) \)
and $x_{E}(a_3)$ have the same position. These alternatives with equal ranks are actually non-comparable with respect to the vector preorder $\leq_{K_A}$.

### 2.3 Cone quantiles

The set-valued quantile functions as introduced in Hamel and Kostner (2018) are transferred into a decision making tool. In particular, these quantiles admit to cluster the alternatives’ evaluations $x_{E}(a_i)$ based on the rank $p$ and the judge’s preferences via $K_I$ and $K_A$, respectively.

Initially, the paper (Hamel and Kostner 2018) defines quantiles based on the scalarization of the random vector $X$ with $w \in C^+\setminus \{0\}$. These quantiles categorize the values of $X$ with respect to a $w \in C^+\setminus \{0\}$. Therefore, these functions can be adapted to extract elements from a set of alternatives based on the individual opinion of a judge. Here are those functions adapted to a single judge decision making, where $v \in K_I \setminus \{0\}$. The function $Q^-_{\tilde{X},v} : (0, 1) \rightarrow \mathcal{P}(\mathbb{R}^d)$ defined by

$$Q^-_{\tilde{X},v}(p) = \left\{ z \in \mathbb{R}^d \mid F_{\tilde{X},v}(z) \geq p \right\} = \left\{ z \in \mathbb{R}^d \mid \# \left\{ i \in \{1, \ldots, m\} \mid x_{E}(a_i) \in z - H^+(v) \right\} \geq mp \right\},$$

is called the lower $v$-quantile function of $\tilde{X}$, and the function $Q^+_{\tilde{X},v} : (0, 1) \rightarrow \mathcal{P}(\mathbb{R}^d)$ defined by

$$Q^+_{\tilde{X},v}(p) = \left\{ z \in \mathbb{R}^d \mid \# \left\{ i \in \{1, \ldots, m\} \mid x_{E}(a_i) \in z - \text{int } H^+(v) \right\} \leq mp \right\},$$

is called the upper $v$-quantile function of $\tilde{X}$. Both functions have convex values, in particular $Q^-_{\tilde{X},v}$ has upward directed (with respect to $v$) halfspaces and $Q^+_{\tilde{X},v}$ has downward directed halfspaces.

It turns out that the two sets are also useful concepts from a decision making point of view. Since, $Q^-_{\tilde{X},v}(p)$ includes all $x_{E}(a_i)$ that for a given rank $p$ and an importance vector $v$ have a rank higher or equal to $p$, whereas $Q^+_{\tilde{X},v}(p)$ includes all $x_{E}(a_i)$ that have a rank lower or equal to $p$. Therefore, $Q^-_{\tilde{X},v}(p)$ is the set of better ranked ("good") elements and $Q^+_{\tilde{X},v}(p)$ the set of “bad” elements, with respect to the rank $p$ and the importance vector $v$. Moreover, the higher the $p$ the better ranked and the fewer the elements in $Q^-_{\tilde{X},v}(p)$. The question is: what are good choices for $p$? This depends on the available alternatives as well as on the judges’ views on the criteria.

**Example 2.5** Figures 5 and 6 illustrate the $v$-quantile functions for $v^1 = (2, 1)^T$, $p = 0.8$ and the uniform random vector $X$ that has as realizations the elements in Fig. 4. Figure 5 shows the graph of the cumulative distribution function for the scalarized random vector $v^1 \top X$, $F_{\tilde{X},v^1}(z)$. Figure 6 has on its axes the criteria values and depicts the quantiles as halfspaces.
As shown in Fig. 5, the “scalar” quantile of order \( p = 0.8 \) for the random variable \( v^1 \top X \) is the assessed alternative \( x^E(a_4) \) scalarized by \( v^1 \).

From a decision making perspective one would always choose the set \( Q^-_{X,v^1}(0.8) \) over the set \( Q^+_{X,v^1}(0.8) \), given that the alternatives should have a ranking of at least \( p = 0.8 \).

The next step is to categorize a set of alternatives by complying with all judges’ opinions as well as their compromises. This is done intuitively by taking the inter-
section of the sets $Q_{X,v}^{-}(p)$, and respectively $Q_{X,v}^{+}(p)$, over all directions $v$ of the importance cone $K_I$. This results in quantiles with the order relation $\leq_{K_A}$. As for the cone distribution function, this has the positive consequence that the sets do not have to compel to single judges but can accommodate also compromises between judges. On the one hand, the lower $K_A$-quantile $Q_{X,K_A}^{-} : (0, 1) \rightarrow \mathcal{P}(\mathbb{R}^d)$ is defined as

$$Q_{X,K_A}^{-} (p) = \bigcap_{v \in K_I \setminus \{0\}} Q_{X,v}^{-} (p),$$

on the other hand, the upper $K_A$-quantile $Q_{X,K_A}^{+} : (0, 1) \rightarrow \mathcal{P}(\mathbb{R}^d)$ is defined as

$$Q_{X,K_A}^{+} (p) = \bigcap_{v \in K_I \setminus \{0\}} Q_{X,v}^{+} (p).$$

The lower $K_A$-quantile can be also defined with respect to the cone distribution function $F_{\tilde{X},K_A}(z)$:

$$Q_{X,K_A}^{-} (p) = \left\{ z \in \mathbb{R}^d \mid F_{\tilde{X},K_A}(z) \geq p \right\}.$$

On the one side, lower quantile $Q_{X,K_A}^{-} (p)$ is directed “upwards” with respect to the order relation $\leq_{K_A}$. This means that the lower quantile maps into a collection of sets, which are directed upwards with respect to the preorder generated by the cone $K_A$, refer to Hamel and Kostner (2018) for further details. On the other side, the upper quantile $Q_{X,K_A}^{+} (p)$ is directed “downwards”. This has an important implication for the decision making process. The lower quantile is the set to consider in order to extract alternatives that are “recommended” by the judges, whereas the upper quantile defines an area of “non-advisable” elements.

The potential of these quantiles as decision making tools is shown by interpreting their most relevant properties from a decision making perspective. It is enough to discuss the properties of the lower $K_A$-quantile $Q_{X,K_A}^{-} (p)$ (see Hamel and Kostner 2018), as one can easily transfer these properties into those for the upper quantiles.

(a) For all $b \in \mathbb{R}^d$ and all invertible matrices $B \in \mathbb{R}^{d \times d}$ it holds

$$\forall p \in (0, 1) : Q_{B\tilde{X}+b,K_A}^{-} (p) = BQ_{\tilde{X},K_A}^{-} (p) + b.$$

If the scale for the criteria changes, the quantile can be easily adapted for proportional variations in all criteria $(b)$ or/and changes in single criteria $(B)$.

(b) If $p_1, p_2 \in (0, 1), p_1 \geq p_2$, then $Q_{\tilde{X},K_A}^{-} (p_1) \subseteq Q_{\tilde{X},K_A}^{-} (p_2)$. By increasing the rank $p$ the quantile gets more selective and excludes more alternatives. Therefore, the higher the parameter $p$, the less and the better ranked the elements in $Q_{\tilde{X},K_A}^{-} (p)$. 

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(c) If \( \tilde{X} \leq K_A \tilde{Y} \), then \( Q^-_{\tilde{X}, K_A}(p) \supseteq Q^-_{\tilde{Y}, K_A}(p) \) for all \( p \in (0, 1) \). \( \tilde{Y} \leq K_A \tilde{Y} \) means that \( \tilde{Y} \) has better--with respect to the judges’ perspectives--choices than \( \tilde{X} \). For a given rank \( p \) and importance cone \( K_I \), the quantile \( Q^-_{\tilde{X}, K_A}(p) \) of the set with the inferior choices \( \tilde{X} \) includes the quantile \( Q^-_{\tilde{Y}, K_A}(p) \) of the “better” set \( \tilde{Y} \). This can be interpreted as \( Q^-_{\tilde{Y}, K_A}(p) \) being more selective due to its derivation from a “better” set.

(d) If \( \mathbb{R}^d_+ \subseteq K^1_A \subseteq K^2_A \subset \mathbb{R}^d \) are two closed convex cones, then \( Q^-_{\tilde{X}, K_A}(p) \subseteq Q^-_{\tilde{X}, K_A}(p) \) for all \( p \in (0, 1) \). As discussed above in property d of the cone distribution, a narrower acceptance cone \( K_A \) implies a bigger compromise to be made, due to a wider variety in judges’ opinions represented by a wider importance cone \( K_I \). This effort of accommodating all the judges’ opinions, is reflected in a “smaller” quantile, subsequently in one that includes less elements.

The following two examples reveal further interesting insights for the decision making process. Example 2.6 illustrates the classification of the alternatives via the quantiles, whereas Example 2.7 shows the ability of the quantile to reveal the fit between the set of alternatives and the judges’ opinions.

**Example 2.6** Figure 7 depicts the lower \( K_A \)-quantile \( Q^-_{\tilde{X}, K_A}(p) \) and the upper \( K_A \)-quantile \( Q^+_{\tilde{X}, K_A}(p) \) for different values of \( p \), for the set of assessed alternatives \( \tilde{X} \) in Fig. 4 and the importance cone \( K_I \) generated by the judges \( v^1 \) and \( v^3 \).

On the one side, the intersection between \( Q^-_{\tilde{X}, K_A}(p) \) and \( Q^+_{\tilde{X}, K_A}(p) \) can be empty as for \( p \in (0, 0.4) \), \( p \in (0.4, 0.6) \) and \( p \in (0.6, 0.8) \). On the other side, the intersection between the quantiles can be a single element as for \( p = 0.4 \) and \( p \in (0.8, 1) \); a line segment as for \( p = 0.6 \); or even a set with non empty interior as for \( p = 0.8 \). It follows that a criteria evaluation of an alternative can be in one of the quantiles, in both quantiles or in neither of them. The assessed alternative included in only one of the quantiles can either be “recommendable” or “non-recommendable”; the element included in both quantiles can be seen as “neutral” element, as it is neither good nor bad; for the element not included in the quantiles no clear statement can be given from a decision making standpoint.

**Example 2.7** The Fig. 8 illustrates the case when the judges’ views do not really match the set of assessed alternatives \( \tilde{X}_I \). The element \( x^E(a_5) \) of Fig. 4 is decreased in value for both criteria, \( x^E(a_5) = (3, 3) \). This implies that \( x^E(a_5) \) is not preferable to \( x^E(a_1) \) and \( x^E(a_4) \), with regards to \( \leq K_A \). Consequently, the lower \( C \)-quantile for \( p \in (0.8, 1) \) does not include any of the given elements in \( \tilde{X}_I \). There are no elements in \( \tilde{X}_I \) with a rank higher than 0.8, which means that \( \tilde{X}_I \) includes elements that only weakly satisfy the judges’ views. Hence, the quantile can be used as indicator on how well the set of elements fit the judges’ opinions.
Fig. 7 $Q_{\tilde{X}, K_A}(p)$ and $Q_{\tilde{X}, K_A}(p)$ for $p \in (0, 1)$ with the importance cone $K_I$ generated by the judges $v^1$ and $v^3$.

Fig. 8 $Q_{\tilde{X}_I, K_A}(p)$ and $Q_{\tilde{X}_I, K_A}(p)$ for $p \in (0.8, 1)$, where $\tilde{X}_I$ is equal to $\tilde{X}$ with the exception that $x^{E(a5)} = (3, 3)$. 

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The question is how to find a sufficiently large parameter $p$. The answer depends on how the alternatives match the judges’ views. If the set of alternatives includes a lot of elements whose criteria match the judges’ opinions, then one can be more selective and choose a higher $p$. However, also the opposite can be true as seen in Example 2.7.

One could also ask how well the alternatives match the judges’ opinions. This is done by finding the lowest $p$ for which the lower quantile $Q^{-}_{X,K_{A}}(p)$ is empty. The higher the $p$, the better the match.

From the discussion above, the utility of the $C$-quantiles for the multiple judge, multiple criteria decision making problem is apparent. It has the ability to derive a set of alternatives that conforms to the judges’ opinions with the parameter $p$ indicating how selective the choice is. Moreover, the capability to distinguish between bad choices (upper quantile) and good choices (lower quantile) is very useful for effective decision making.

3 Conclusion

A novel approach in solving the multiple judge, multiple criteria decision making problem is proposed. Before a choice between alternatives can be made, those alternatives need to be ranked. On the one side, if the alternatives are valued based on a single criterion, they can be clearly ranked. On the other side, if the alternatives are valued based on multiple criteria, it gets complicated to rank them. From a mathematical standpoint, this is due to the non-totalness of the underlying ordered space of non-comparable alternatives. The set-valued quantile deals with this issue by introducing an order relation based on convex cones. The fundamental idea in this paper is that the cones represent the judges’ opinions. Based on that, the cone distribution can be implemented as a ranking function and the set-valued quantiles as functions that categorize alternatives into different sets of “good” and “bad” choices. Moreover, the paper shows that the properties of these functions ensure a reasonable decision making process. A computational procedure for the set-valued quantiles is under development. It is based on ideas from computational geometry (see Rousseeuw and Hubert 2015). Due to $X$ being an empirical distribution, the method of choice would be solving linear vector optimization problems, which can be solved by tools closely related to Löhne and Weißing (2016) and the references therein. A further advancement is to conceptualize a multi-criteria recommender system (see Manouselis and Costopoulou 2007) based on the developments in this work. This recommender system could be designed in a way to adjust more closely to the users’ preferences and, consequently to give targeted and accurate suggestions.

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