QUANTUM PROPERTIES OF THE DUAL
MATRICES IN $GL_q(1|1)$

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Abstract

In this paper, we give the quantum analogue of the dual matrices for the quantum supergroup $GL_q(1|1)$ and discuss these properties of the quantum dual supermatrices.
1. INTRODUCTION

An explicit quantum deformation of the supergroup $GL(1|1)$ with two even and two odd generators was given by Corrigan et al in [1]. The properties of the 2x2-supermatrices in $GL_q(1|1)$ was investigated by Schwenk et al in [2]. In this work, we consider the dual supermatrices in $GL(1|1)$ and discuss the properties of quantum dual supermatrices.

Let us begin with some remarks. We know that the supergroup $GL(1|1)$ can be deformed by assuming that the linear transformations in $GL(1|1)$ are invariant under the action of the quantum superplane and its dual [3]. Consider a quantum superplane and its dual,

$$V = \begin{pmatrix} x \\ \xi \end{pmatrix} \quad \text{and} \quad \hat{V} = \begin{pmatrix} \eta \\ y \end{pmatrix}$$

satisfying

$$x\xi - q\xi x = 0, \quad \xi^2 = 0,$$

$$\eta^2 = 0, \quad y\eta - q\eta y = 0$$

(1.2a, 1.2b)

where latin and greek letters denote even and odd elements respectively. Taking

$$M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$$

(1.3)

as a supermatrix in $GL(1|1)$, we demand that the relations (1.2) are preserved under the action of $M$ on the quantum superplane $V$ and its dual $\hat{V}$

$$MV = V' \quad \text{and} \quad M\hat{V} = \hat{V}'$$

(1.4)

We assume that even generators commute with everything and odd generators anticommute among themselves. Then we obtain the following $q$-commutation relations [1] (also see [2])

$$a\beta = q\beta a, \quad d\beta = q\beta d,$$

$$a\gamma = q\gamma a, \quad d\gamma = q\gamma d,$$

$$\beta\gamma + \gamma\beta = 0, \quad \beta^2 = 0 = \gamma^2,$$

$$ad - da = (q - q^{-1})\gamma\beta.$$  

(1.5)

These relations will be used in sec. 4. Note that if $M \in GL_q(1|1)$ then $M^n \in GL_{q^n}(1|1)$. This is proved in [2].
2. QUANTUM DUAL SUPERMATRICES IN $GL_q(1|1)$

In this section we give the $q$-commutation relations which the matrix elements of a dual supermatrix satisfy. Let $\hat{M}$ be a dual supermatrix in $GL_q(1|1)$, namely,

$$\hat{M} = \begin{pmatrix} \alpha & b \\ c & \delta \end{pmatrix}$$

with its generators (anti)commuting with the coordinates of $V$ and $\hat{V}$. Then, the transformations

$$V \rightarrow \hat{M}V = \hat{V}' \quad \text{and} \quad \hat{V} \rightarrow \hat{M}\hat{V} = V'$$

impose the following bilinear product relations among the generators of $\hat{M}$:

$$\begin{align*}
\alpha b &= q^{-1}b\alpha, \quad \alpha c = q^{-1}c\alpha, \\
\delta b &= q^{-1}b\delta, \quad \delta c = q^{-1}c\delta, \\
\alpha\delta + \delta\alpha &= 0, \quad \alpha^2 = 0 = \delta^2, \\
bc - cb &= (q - q^{-1})\delta\alpha
\end{align*}$$

and $q^2 - 1 \neq 0$. From these relations one obtains

$$\begin{align*}
\alpha b^{-1} &= qb^{-1}\alpha, \quad \alpha c^{-1} = qc^{-1}\alpha, \\
\delta b^{-1} &= qb^{-1}\delta, \quad \delta c^{-1} = qc^{-1}\delta, \\
bc^{-1} - c^{-1}b &= (q - q^{-1})\alpha c^{-1}\delta c^{-1}
\end{align*}$$

provided $b$ and $c$ are invertible.

3. THE INVERSE OF $\hat{M}$

To obtain the inverse of $\hat{M}$, we introduce $\Delta_1$ and $\Delta_2$ in the form

$$\Delta_1 = bc - q\delta\alpha \quad \text{and} \quad \Delta_2 = cb - q\alpha\delta.$$  \hspace{1cm} (3.1)

just as in [4]. Then one can write

$$\hat{M}_L^{-1} = \begin{pmatrix} -q\Delta_1^{-1}\delta & \Delta_1^{-1}b \\ \Delta_2^{-1}c & -q\Delta_2^{-1}\alpha \end{pmatrix}$$

as the left inverse of $\hat{M}$. After some calculations one obtains
\[ \Delta_1 b = b \Delta_1, \quad \Delta_2 c = c \Delta_2, \]
\[ \Delta_k \alpha = q^2 \alpha \Delta_k, \quad \Delta_k \delta = q^2 \delta \Delta_k, \quad k = 1, 2 \tag{3.3} \]

and also
\[ b^2 \Delta_1^{-1} = bc^{-1} - \alpha c^{-1} \delta c^{-1}, \]
\[ c^2 \Delta_2^{-1} = cb^{-1} - \delta b^{-1} \alpha b^{-1}. \tag{3.4} \]

Note that it is easy to verify that \( b^2 \Delta_1^{-1} \) and \( c^2 \Delta_2^{-1} \) commute with everything. Therefore the matrix \( \hat{M}_L^{-1} \) in (3.2) may be written as
\[ \hat{M}_L^{-1} = \left( \begin{array}{cc}
-c^{-1} \delta c^{-1} & b^{-1} \\
c^{-1} & -b^{-1} \alpha b^{-1}
\end{array} \right) \left( \begin{array}{cc}
c^2 \Delta_2^{-1} & 0 \\
0 & b^2 \Delta_1^{-1}
\end{array} \right) \tag{3.5} \]

which shows that \( \hat{M}_L^{-1} = \hat{M}^{-1} \) after some calculations along the lines of [2], sec. 3. Thus one can define the quantum dual superdeterminant as follows:
\[ s \hat{D}_q(\hat{M}) = b^2 \Delta_1^{-1} = bc^{-1} - \alpha c^{-1} \delta c^{-1}. \tag{3.6} \]

Note that the inverse of a dual supermatrix \( \hat{M} \) can be also obtained from the decomposition
\[ \hat{M} = \left( \begin{array}{cc}
\alpha & b - \alpha c^{-1} \delta \\
c & 0
\end{array} \right) \left( \begin{array}{cc}
1 & c^{-1} \delta \\
0 & 1
\end{array} \right). \tag{3.7} \]

Finally we note that the product of two dual supermatrices is not a dual supermatrix, i.e., the matrix elements of a product \( \hat{M} = \hat{M} \hat{M}' \) do not satisfy (2.6) but they satisfy (1.5) if \( \hat{M} \) and \( \hat{M}' \) are two dual supermatrices and \((b, c) ((\alpha, \delta)) \) pairwise commute (anti-commute) with \((b', c') ((\alpha', \delta'))\). This interesting property will show as the way to the contents of the next section.

4. PROPERTIES OF \( \hat{M}^n \)

From sec. 3 we know that the matrix elements of a product matrix \( \hat{M} \hat{M}' \) obey the relations (1.5). Therefore we must consider the matrix elements of \( \hat{M} \) with respect to even and odd values of \( n \). Let the \((2n - 1)\)-th power of \( \hat{M} \) be
\[ \hat{M}^{2n-1} = \left( \begin{array}{cc}
A_{2n-1} & B_{2n-1} \\
C_{2n-1} & D_{2n-1}
\end{array} \right), \quad n \geq 1. \tag{4.1} \]
After some algebra, one obtains

\[
A_{2n-1} = \{ [n]_q \alpha + q[n-1]_q \delta \} (bc)^{n-1},
\]

\[
B_{2n-1} = \{ bc + q[n-1]_q^2 \alpha \delta \} (bc)^{n-2} b,
\]

\[
C_{2n-1} = \{ cb + q[n-1]_q^2 \delta \alpha \} (cb)^{n-2} c,
\]

\[
D_{2n-1} = \{ [n]_q \delta + q[n-1]_q \alpha \} (cb)^{n-1},
\]

where

\[
[n]_q = \frac{1 - q^{2n}}{1 - q^2}
\]

(4.3)

Now it is easy to show that the following relations are satisfied.

\[
A_{2n-1}B_{2n-1} = q^{-2(n-1)} B_{2n-1} A_{2n-1}
\]

\[
A_{2n-1}C_{2n-1} = q^{-2(n-1)} C_{2n-1} A_{2n-1}
\]

\[
D_{2n-1}B_{2n-1} = q^{-2(n-1)} B_{2n-1} D_{2n-1}
\]

\[
D_{2n-1}C_{2n-1} = q^{-2(n-1)} C_{2n-1} D_{2n-1},
\]

\[
A_{2n-1}D_{2n-1} + D_{2n-1} A_{2n-1} = 0, 
\]

\[
A_{2n-1}^2 = 0 = D_{2n-1}^2;
\]

\[
B_{2n-1} C_{2n-1} - C_{2n-1} B_{2n-1} = (q^{2n-1} - q^{-(2n-1)}) A_{2n-1} D_{2n-1}.
\]

Then \( \hat{M}^{2n-1} \) is a dual supermatrix with deformation parameter \( q^{2n-1} \).

Similarly, if we write for the matrix \( \hat{M}^{2n} \), the \((2n)\)-th power of \( \hat{M} \) as

\[
\hat{M}^{2n} = \begin{pmatrix} A_{2n} & B_{2n} \\ C_{2n} & D_{2n} \end{pmatrix}, \quad n \geq 1
\]

(4.5)

where (after some calculations)

\[
A_{2n} = \{ bc + q \frac{1 - q^2}{1 + q^2} [n]_q [n-1]_q \alpha \delta \} (bc)^{n-1},
\]

\[
B_{2n} = [n]_q \{ \alpha + q \delta \} b (cb)^{n-1},
\]

\[
C_{2n} = [n]_q \{ \delta + q \alpha \} c (bc)^{n-1},
\]

\[
D_{2n} = \{ bc + q \frac{1 - q^2}{1 + q^2} [n]_q [n-1]_q \delta \alpha \} (cb)^{n-1},
\]

(4.6)
then the elements of $\widehat{M}^{2n}$ obey the following relations

\begin{align*}
A_{2n}B_{2n} &= q^{2n}B_{2n}A_{2n} \\
A_{2n}C_{2n} &= q^{2n}C_{2n}A_{2n} \\
D_{2n}B_{2n} &= q^{2n}B_{2n}D_{2n} \\
D_{2n}C_{2n} &= q^{2n}C_{2n}D_{2n}, \\
B_{2n}C_{2n} + C_{2n}B_{2n} &= 0, \\
B_{2n}^2 &= 0 = C_{2n}^2, \\
A_{2n}D_{2n} - D_{2n}A_{2n} &= (q^{2n} - q^{-2n})C_{2n}B_{2n}.
\end{align*}

(4.7)

Thus the matrix $\widehat{M}^{2n}$ is a supermatrix in the form (1.3).

Equations (4.4) and (4.7) can be proved using the relation (2.3).

5. CONCLUSIONS

We have given the $q$-commutation relations which the matrix elements of a dual supermatrix in $GL_q(1|1)$ satisfy and obtained the (dual) quantum superinverse and (dual) quantum superdeterminant of a dual quantum supermatrix. Finally we have shown that it must consider the matrix elements of a dual supermatrix with respect to even and odd values of $n$. And so we discussed the properties of the $n$-th power of a dual supermatrix.

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REFERENCES

[1] Corrigan, E., Fairlie, B., Fletcher, P. and Sasaki, R., J. Math. Phys. 31, 776, 1990.
[2] Schwenk, J., Schmidke, B. and Vokos, S., Z. Phys. C 46, 643, 1990.
[3] Manin, Yu I., Commun. Math. Phys. 123, 163, 1989.
[4] Celik, S., Celik, S. A., On the quantum supergroup $SU_{p,q}(1|1)$ and quantum oscillators, Preprint MSUMB - 95/1, 1995.