On the genericity of pseudo-Anosov braids I: rigid braids

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Abstract

We prove that, in the \( l \)-ball of the Cayley graph of the braid group with \( n \geq 3 \) strands, the proportion of rigid pseudo-Anosov braids is bounded below independently of \( l \) by a positive value.

1 Introduction

A natural question concerning the Nielsen-Thurston classification of braids is: what is the most likely Nielsen-Thurston type of a “long random braid”? Different interpretations can be given to this question, but in this paper we shall use the following setting. We consider the Cayley graph of the braid group \( B_n \) (for a fixed number of strands \( n \)), with generators the set of simple braids – this is the standard generating set when the braid group is studied as a Garside group. A well-known conjecture since the work of Thurston is as follows.

Conjecture. The proportion of pseudo-Anosov braids among all elements in the ball of radius \( l \) in the Cayley graph converges to 1 as \( l \) tends to infinity.

The best known results going into this direction were, to the best of our knowledge, the classical paper of Fathi [9], and the article of Atalan and Korkmaz [1] which deals with the case of three-strand braids. The present paper, together with the article in preparation [4], contains a proof of the above conjecture. In this first part, we introduce some essential tools needed for the proof in [4], and already prove a result of independent interest concerning the proportion of rigid pseudo-Anosov braids (see Corollary 4.9):

Theorem. For sufficiently large \( l \), the proportion of rigid pseudo-Anosov braids in the ball of radius \( l \) in the Cayley graph of \( B_n \) is bounded below by a strictly positive constant which does not depend on \( l \) (but might depend on \( n \)).

The proof is in two steps: we shall see that the proportion of so-called rigid braids is bounded below independently of \( l \), and among rigid braids the proportion of pseudo-Anosov elements converges to 1.

Another possible interpretation of the original question should be pointed out: the work of Maher [12] and Sisto [13] deals with braids obtained by a long random walk in the Cayley graph. They prove that in this setting, as well, the probability of obtaining a pseudo-Anosov braid converges to 1 as the length of the walk tends to infinity.
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2 Definitions

Throughout the article, we fix an integer $n \geq 3$. All the considered braids will be braids
with $n$ strands.

2.1 Garside structure

A general introduction to Garside theory can be found in [7]. The reader can also consult
[8]. We shall only recall some facts which are useful for our purposes.

While the group $B_n$ admits the well-known presentation of groups $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} ; \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ and } \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2 \rangle$, the monoid of positive braids $B_n^+$, which is embedded in $B_n$, is defined by the same presen-
tation, interpreted as a presentation of monoids.

For $i < j \leq n$, we denote by $\Delta_{ij}$ the element of $B_n^+$ defined by

$$\Delta_{ij} = (\sigma_{i} \cdots \sigma_{j-1}) (\sigma_{i} \cdots \sigma_{j-2}) \cdots (\sigma_{i} \sigma_{i+1}) \sigma_{i}$$

and we denote by $\Delta = \Delta_{1n} \in B_n^+$.

The pair $(B_n^+, \Delta)$ defines what we call a Garside structure on $B_n$. Without giving the complete definition, here are some properties of such a structure. The group $B_n$ is endowed
with a partial order $\preceq$ defined by $x \preceq y \iff x^{-1}y \in B_n^+$. If $x \preceq y$, we say that $x$ is a prefix
of $y$. Any two elements $x, y$ of $B_n$ have a unique greatest common prefix.

We also define $\succeq$ by $x \succeq y \iff xy^{-1} \in B_n^+$. Note that $x \succeq y$ is not equivalent to $y \preceq x$.

The elements of the set \{ $x \in B_n, 1 \preceq x \preceq \Delta$ \} are called simple braids.

**Proposition 2.1.** The set of simple braids is in bijection with the set $\mathfrak{S}_n$ of permutations
of $n$ elements, via the canonical projection from $B_n$ to $\mathfrak{S}_n$.

**Definition 2.2** (left-weighting). Let $s_1, s_2$ be two simple braids in $B_n$. We say that $s_1$
and $s_2$ are left-weighted, or that the pair $(s_1, s_2)$ is left-weighted, if there does not exist any generator $\sigma_i$ such that $s_1 \sigma_i$ and $\sigma_i^{-1} s_2$ are both still simple.

**Definition 2.3** (starting set, finishing set). Let $s \in B_n$ be a simple braid. We call starting
set of $s$ the set $S(s) = \{ i, \sigma_i \preceq s \}$ and finishing set of $s$ the set $F(s) = \{ i, s \succeq \sigma_i \}$.

**Remark 2.4.** Two simple braids $s_1$ and $s_2$ are left-weighted if and only if $S(s_2) \subset F(s_1)$.
Remark 2.5. Let $s$ be a simple braid, and $\pi$ be the permutation associated to $s$. Then $i \in S(s)$ if and only if $\pi(i) > \pi(i+1)$, and $i \in F(s)$ if and only if $\pi^{-1}(i) > \pi^{-1}(i+1)$.

Proposition 2.6. Let $x \in \mathcal{B}_n$. There exists a unique decomposition $x = \Delta^p x_1 \cdots x_r$ such that $x_1, \ldots, x_r$ are simple braids, distinct from $\Delta$ and 1, and such that $x_i$ and $x_{i+1}$ are left-weighted for all $i = 1, \ldots, r-1$.

Definition 2.7 (left normal form). In the previous proposition, the writing $x = \Delta^p x_1 \cdots x_r$ is called the left normal form of $x$, $p$ is called the infimum of $x$ and is denoted by $\inf x$, $p + r$ is the supremum of $x$ and is denoted by $\sup x$, and $r$ is called the canonical length of $x$.

Furthermore, if $r \geq 1$, we denote by $\iota(x) = \Delta^p x_1 \Delta^{-p}$ the initial factor of $x$ ($\iota(x) = x_1$ if $p$ is even, $\iota(x) = \Delta x_1 \Delta^{-1}$ if $p$ is odd), and $\phi(x) = x_r$ its final factor.

Definition 2.8 (rigidity). A braid $x$ of positive canonical length is said to be rigid if $\phi(x)$ and $\iota(x)$ are left-weighted.

2.2 Braids and mapping class group of the punctured disk

Definition 2.9 (Mapping class group of the punctured disk). Let $D_n$ be the closed unit disk in $\mathbb{C}$, with $n$ punctures regularly spaced on the real axis. The mapping class group of $D_n$, denoted $\text{Mod}(D_n)$, is the group of homeomorphisms of $D_n$, modulo the isotopy relation. We also denote $\text{Mod}(D_n, \partial D_n)$ the group of homeomorphisms of $D_n$ fixing pointwise the boundary $\partial D_n$ of $D_n$, modulo the isotopy relation.

The Artin braid group with $n$ strands is isomorphic to the group $\text{Mod}(D_n, \partial D_n)$.

Recall that the classification theorem of Nielsen and Thurston states that a mapping class $f \in \text{Mod}(D_n)$ is exactly one of the following: periodic, or reducible non-periodic, or pseudo-Anosov. A braid $x \in \text{Mod}(D_n, \partial D_n)$ can be projected on an element of $\text{Mod}(D_n)$. We call Nielsen-Thurston type of $x$ the Nielsen-Thurston type of its projection. The definition of periodicity is then transformed as follows: a braid $x \in \mathcal{B}_n$ is periodic if and only if there exist nonzero integers $m$ and $l$ such that $x^m = \Delta^l$, where $\Delta = (\sigma_1 \cdots \sigma_{n-1}) (\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \sigma_1$. (Geometrically $\Delta$ corresponds to the half-twist around the boundary of the disk).

2.3 Round curves and almost round curves

Let us consider a braid as a mapping class in the mapping class group $\text{Mod}(D_n, \partial D_n)$.

Definition 2.10 (curve). We call closed curve in $D_n$ the image of the circle $\mathbb{S}^1$ by a continuous map with values in $D_n$. The curve is said to be simple if this map is injective. It is said to be non degenerated if it is neither homotopic to a point, nor to the boundary of the disk, and it bounds a least two punctures.
In the following, we simply call curve a homotopy class of non degenerate simple closed curves.

**Definition 2.11 (round curve).** A curve is said to be *round* if it is represented by a circle in $D_n$.

**Definition 2.12 (almost round curve).** A curve is said to be *almost round* if it is not round, and is the image by a simple braid of a round curve.

### 3 Properties of the left-weighting graph

**Definition 3.1 (Left-weighting graph).** We call *left-weighting graph*, denote by $G_{lw}$, the following finite oriented graph. The vertices are indexed by the simple braids except $1$ and $\Delta$, and there is an edge from the vertex $x_1$ to the vertex $x_2$ if and only if the pair $(x_1, x_2)$ is left-weighted.

We call *path* a sequence $(x_1 \to x_2 \to \cdots \to x_l)$ such that there is an edge from the vertex $x_i$ to the vertex $x_{i+1}$, and the *length* of such a path means the number of edges in the path.

The objective of this section is to study some properties of the graph $G_{lw}$, especially some asymptotic properties of the number of paths of length $l$, with $l$ tending to infinity. We introduce the following notations, for all $l \in \mathbb{N}^*$:

- $N(l)$ is the number of paths $(x_1 \to x_2 \to \cdots \to x_{l+1})$ of length $l$ in $G_{lw}$;
- $N_o(l)$ is the number of loops of length $l + 1$, with marked base vertex, in $G_{lw}$. The quantity $N_o(l)$ can also be seen as the number of paths of length $l$, such that there is an edge from the last to the first vertex.
- Let $w$ be a path of length $k \in \mathbb{N}^*$ in $G_{lw}$. We denote by $N^{(w)}(l)$ the number of paths of length $l$ in $G_{lw}$ that do not pass through $w$ (ie that do not contain $w$ as a subpath), and $N_o^{(w)}(l)$ the number of loops of length $l + 1$ with marked base vertex in $G_{lw}$, that do not pass through $w$.

Furthermore, if $(u_l)$ and $(v_l)$ are two sequences of real numbers, we write $u_l = \Theta(v_l)$ if and only if there exist constants $c_1, c_2 > 0$ such that for all large enough $l$, $c_1 v_l < u_l < c_2 v_l$. We say that $u_l$ *is of the order of* $v_l$.

We also use the usual notations $u_l \sim v_l$ when $u_l$ is equivalent to $v_l$, that is when for all $\varepsilon > 0$, there exists an integer $L$ such that for all $l > L$, $|u_l - v_l| < \varepsilon|v_l|$, and $u_l = O(v_l)$ when there exists $c_2 > 0$ such that for all large enough $l$, $u_l < c_2 v_l$.

We will prove some properties of the left-weighting graph by using the notion of adjacency matrix. For more details on graph theory and adjacency matrices, the reader can consult [11]. We recall the following definition and proposition, together with the theorem of Perron-Frobenius.
Proof. Let $G$ be an oriented finite graph, whose vertices are numbered. We call adjacency matrix of $G$ the matrix whose $(i,j)$-entry contains the number of edges from the vertex $i$ to the vertex $j$.

Proposition 3.3. Let $G$ be an oriented finite graph and $A$ its adjacency matrix. Let $l \in \mathbb{N}$. The $(i,j)$-entry of the matrix $A^l$ contains the number of paths of length $l$ in $G$ linking the vertex $i$ to the vertex $j$.

Theorem (Perron-Frobenius). Let $A$ be a matrix such that there exists $k \in \mathbb{N}^*$, such that all entries of $A^k$ are positive. Then the spectral radius of $A$ is positive, is a simple eigenvalue of $A$, and is the unique eigenvalue of maximal module.

Lemma 3.4. Each pair of vertices in $G_{lw}$ is linked by at least one path of length exactly 5.

Proof. Let us recall that two simple braids $s$ and $t$ are left-weighted if and only if $S(t) \subset F(s)$. Let $s_1$ and $s_2$ be two simple braids distinct from 1 and $\Delta$. There exist $i_1$ and $i_2$ in $\{1, \ldots, n-1\}$ such that $F(s_1) \supset \{i_1\}$ and $S(s_2) \subset \{1, \ldots, n-1\}\{i_2\}$. We will construct some simple braids $x_1, x_2, x_3, x_4$ satisfying:

- $S(x_1) = \{i_1\}$,
- $F(x_1) = S(x_2) = \{\lfloor \frac{n}{2} \rfloor\}$,
- $F(x_2) = S(x_3) = \{1, 3, \ldots, 2\lfloor \frac{n}{2} \rfloor - 1\}$ (the set of odd numbers between 1 and $n-1$),
- $F(x_3) = S(x_4) = \{1, \ldots, n-1\}\{\lfloor \frac{n}{2} \rfloor\}$,
- $F(x_4) = \{1, \ldots, n-1\}\{i_2\}$.

Thus, $s_1 \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow s_2$ will be a path of length 5 in the graph $G_{lw}$.

Here is how we choose the braids $x_1, x_2, x_3, x_4$. We set $x_1 = \sigma_{i_1} \cdots \sigma_{\lfloor \frac{n}{2} \rfloor}$. The simple braid $x_2$ is the braid corresponding to following permutation:

$$\pi_2 = \begin{pmatrix} 1 & 2 & \cdots & \lfloor \frac{n}{2} \rfloor & \lfloor \frac{n}{2} \rfloor + 1 & \lfloor \frac{n}{2} \rfloor + 2 & \cdots & n \\ 2 & 4 & \cdots & 2 \lfloor \frac{n}{2} \rfloor & 1 & 3 & \cdots & 2 \lfloor \frac{n}{2} \rfloor - 1 \end{pmatrix}$$

As to the braid $x_3$, it is equal to $\tilde{x}_2 \Delta_{1, \lfloor \frac{n}{2} \rfloor} \Delta_{\lfloor \frac{n}{2} \rfloor + 1, n}$, where $\tilde{x}_2$ is the simple braid of permutation $\pi_2^{-1}$. Finally, $x_4 = \Delta \sigma_{\lfloor \frac{n}{2} \rfloor}^{-1} \cdots \sigma_{i_2}^{-1}$ is the left complement of $\sigma_{i_2} \cdots \sigma_{\lfloor \frac{n}{2} \rfloor}$. The braids $x_1$ to $x_4$ are represented for $n = 6$ in Figure 1.

Of course $S(x_1) = \{i_1\}$ and $F(x_1) = \{\lfloor \frac{n}{2} \rfloor\}$. For $x_2$, the permutation $\pi_2$ is increasing on $\{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and on $\{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$, and we have $\pi_2(\lfloor \frac{n}{2} \rfloor + 1) < \pi_2(\lfloor \frac{n}{2} \rfloor)$, so $S(x_2) = \{\lfloor \frac{n}{2} \rfloor\}$. On the other hand, $\pi_2^{-1}(i) > \pi_2^{-1}(i+1)$ if and only if $i$ is odd, hence $F(x_2) = \{1, 3, \ldots, 2 \lfloor \frac{n}{2} \rfloor - 1\}$. The permutation $\pi_3$ associated with $x_3$ first applies $\pi_2^{-1}$, then reverses the order, on the one hand, of the elements from 1 to $\lfloor \frac{n}{2} \rfloor$, and on the other hand, of $\lfloor \frac{n}{2} \rfloor + 1$ to $n$. It follows that $\pi_3(i) > \pi_3(i+1)$ if and only if $i$ is odd, and that $\pi_3^{-1}(i) > \pi_3^{-1}(i+1)$.
Figure 1: Braids $x_1$ to $x_4$

for all $i$ except $i = \lfloor \frac{n}{2} \rfloor$. So $S(x_3) = \{1, 3, \ldots, 2\lfloor \frac{n}{2} \rfloor - 1\}$ and $F(x_3) = \{1, \ldots, n-1\}\backslash\{\lfloor \frac{n}{2} \rfloor\}$.

Finally, $x_4$ is the left complement of $\sigma_{i_2} \cdots \sigma_{\lceil \frac{n}{2} \rceil}$, and thus satisfies $S(x_4) = \{1, \ldots, n-1\}\backslash\{n-\lceil \frac{n}{2} \rceil\} = \{1, \ldots, n-1\}\backslash\{\lfloor \frac{n}{2} \rfloor\}$ and $F(x_4) = \{1, \ldots, n-1\}\backslash\{i_2\}$.

Lemma 3.5. The following properties are true.

(i) There exists a constant $\lambda$ such that $N_o(l) \sim \lambda^{l+1}$.

(ii) We have $N(l) = \Theta(\lambda^l)$. In particular, for large enough $l$, the proportion $N_o(l)/N(l)$ is bounded below, independently of $l$, by a positive constant.
(iii) For all path \( w \), there exists a constant \( \mu(w) < \lambda \) such that \( N(w)(l) = O(\mu_w^l) \) and \( N^0(w)(l) = O(\mu_w^l) \).

The reader can also consult [6], which contains results and proofs similar to those of this lemma.

**Proof of Lemma 3.3.** Let \( A \) be the adjacency matrix of the graph \( G_{lw} \). According to Proposition 3.3, \( N_0(l) = \text{tr}(A^{l+1}) \) and \( N(l) = |A^l|_1 \), where \( |·|_1 \) is the sum of all entries in the matrix.

(i) According to Lemma 3.4, the matrix \( A^5 \) has positive entries. So we can apply the Perron-Frobenius theorem to \( A \), and deduce that \( A \) has a unique eigenvalue of maximal module. This value is real and positive, and the associated eigenspace has dimension 1. We denote by \( \lambda \) this eigenvalue, and by \( \lambda_i \) (\( i = 1, \ldots, n! - 3 \)) the others (not necessarily distinct and not necessarily real). We have \( \text{tr}(A^{l+1}) = \lambda^{l+1} + \lambda_1^{l+1} + \cdots + \lambda_{n!-3}^{l+1} \), hence \( N_0(l) \sim \lambda^{l+1} \) when \( l \) tends to infinity.

(ii) There exists an invertible matrix \( P \) such that \( PA^LP^{-1} \) is in Jordan normal form, and we can calculate

\[
|PA^LP^{-1}|_1 = \lambda^l + \sum p_i(\lambda_i)
\]

where the \( p_i \) are some polynomials of degree \( l \). We deduce again the equivalence \( |PA^LP^{-1}|_1 \sim \lambda^l \). Furthermore, \( |PA^LP^{-1}|_1 = \Theta(|A^l|_1) \), so \( N(l) = \Theta(\lambda^l) \).

We deduce that \( N_0(l)/N(l) = \Theta(1) \), and in particular, that for large enough \( l \), this ratio is bounded below, independently of \( l \), by a positive constant.

(iii) We construct from \( G_{lw} \) a graph \( G_{lw}^{(k)} \) (where, we recall, \( k \) is the length of the path \( w \)) as follows: the vertices in \( G_{lw}^{(k)} \) are the paths of length \( k - 1 \) in \( G_{lw} \), and two paths \( w_1 = (s_1 \rightarrow \cdots \rightarrow s_k) \) and \( w_2 = (t_1 \rightarrow \cdots \rightarrow t_k) \) are linked by an edge if and only if \( s_2 = t_1, s_3 = t_2, \ldots, s_k = t_{k-1} \). Thus, the edges of \( G_{lw}^{(k)} \) correspond to the paths of length \( k \) in \( G_{lw} \). We denote by \( A_k \) the adjacency matrix of \( G_{lw}^{(k)} \).

If \( s \) and \( t \) are two vertices in \( G_{lw} \), a path of length \( l \geq k \) from \( s \) to \( t \) in \( G_{lw} \) corresponds to a path of length \( l - k + 1 \) from \( (s \rightarrow s_2 \rightarrow \cdots \rightarrow s_k) \) to \( (t_1 \rightarrow \cdots \rightarrow t_k \rightarrow t) \) in \( G_{lw}^{(k)} \) for some \( s_2, \ldots, s_{k-1}, t_1, \ldots, t_{k-2} \). This leads to the following consequences. As each pair of vertices in \( G_{lw} \) is linked by a path of length 5 (Lemma 3.4), each pair of vertices in \( G_{lw}^{(k)} \) is linked by a path of length exactly \( k + 4 \). Furthermore, as the number of paths of length \( l \) in \( G_{lw}^{(k)} \) is a \( \Theta(\lambda^l) \), it is the same for the number of paths of length \( l \) in \( G_{lw}^{(k)} \). As \( A_k^{k+4} \) has positive entries, \( A_k \) satisfies the hypothesis of the Perron-Frobenius theorem, and we deduce, as in (ii), that the number of paths of length \( l \) in \( G_{lw}^{(k)} \) is a \( \Theta(\lambda_k^l) \) where \( \lambda_k \) is the spectral radius of \( A_k \). The two asymptotic estimates obtained ensure that \( \lambda_k(k) = \lambda \).

Moreover, avoiding a path of length \( k \) in \( G_{lw} \) is equivalent to avoiding an edge in \( G_{lw}^{(k)} \). Let \( \tilde{G}_{lw}^{(k)} \) be the graph obtained from \( G_{lw}^{(k)} \) by removing the edge \( a_w \) corresponding to \( w \). We denote by \( A_k \) its adjacency matrix, and by \( \mu(w) \) the spectral radius of this matrix.
Then, the number of paths of length \( l - k + 1 \) in \( G_{lw}^{(k)} \) is a \( O(\mu_1^{l(w)}) \): indeed, as in (ii), there exists an invertible matrix \( Q \) such that \( |Q \tilde{A}_k^{l-1} Q^{-1}|_1 \) is a sum of polynomials of degree \( l - k + 1 \) in the eigenvalues of \( \tilde{A}_k \). As these eigenvalues are, in module, not greater than the spectral radius \( \mu_1^{(w)} \), we deduce that \( |Q \tilde{A}_k^{l-1} Q^{-1}|_1 = O(\mu_1^{l-1}) = O(\mu_1^{l(w)}) \), and then, that \( N^{(w)}(l) = |\tilde{A}_k^{l-1}|_1 = O(\mu_1^{l(w)}) \).

As for the number of loops of length \( l + 1 \) with marked base point in \( G_{lw} \), their number is not greater than the number of paths of length \( l \), and so we have also \( N_0^{(w)}(l) = O(\mu_1^{l(w)}) \).

It remains to prove that \( \mu_1^{(w)} \) is not greater than the number of paths of length \( l \). Indeed, as in (ii), there exist strictly more paths of length \( l \) than in \( G_{lw}^{(k)} \). That is to say, the matrix \( \tilde{A}_k^l - \tilde{A}_k^0 \) has positive entries. Let \( \varepsilon > 0 \) be such that \( \tilde{A}_k^0 - (\tilde{A}_k^0 + \varepsilon I) \) has still positive entries (where \( I \) is the identity matrix). The spectral radius of \( \tilde{A}_k^0 \) is \( \lambda^0 \), the one of \( \tilde{A}_k^0 + \varepsilon I \) is \( \mu_1^{l(w)} + \varepsilon \). Recall that the spectral radius of a matrix \( B \) is the limit of \( \| B_k^k \|^{\frac{1}{k}} \) when \( k \) tends to infinity, where \( \| B \| \) is any matrix norm. By choosing for \( \| \cdot \| \), for example, the infinity-norm, we deduce that, as the entries of \( \tilde{A}_k^0 \) are all greater than those of \( \tilde{A}_k^0 + \varepsilon I \), we have \( \lambda^0 \geq \mu_1^{l(w)} + \varepsilon \), and thus \( \lambda > \mu_1^{l(w)} \).

Remark 3.6. By similar arguments, we obtain finer results, on the number of paths that do not contain \( w \) in a more localized area of the path. More precisely, if \( \beta \) is a path of length \( l \), and if \( a_1, a_2, a_3 \) are functions of \( l \) taking values in \( \mathbb{N} \), with \( a_1 + a_3 \) and \( a_2 \) nondecreasing functions that tend to infinity when \( l \) tends to infinity, and such that \( a_1(l) + a_2(l) + a_3(l) = l \), we can cut the path \( \beta \) into three path \( \beta_1, \beta_2 \) and \( \beta_3 \) of respective lengths \( a_1(l), a_2(l) \) and \( a_3(l) \). The number of paths \( \beta \) of length \( l \) whose “middle part” \( \beta_2 \) does not contain the path \( w \) is a \( \Theta(\mu_1^{(w)} \lambda^{a_1(l) + a_3(l)}) = \Theta(\mu_1^{a_2(l)} \lambda^{l - a_2(l)}) \).

4 Genericity of pseudo-Anosov braids

4.1 Proportion of rigid braids

Proposition 4.1. Let \( l \in \mathbb{N}^* \). Among the braids \( \beta \) such that \( \inf \beta = 0 \) and \( \sup \beta = l \), the proportion of rigid braids is bounded below independently of \( l \) by a positive constant.

Proof. According to the unicity of the left normal form of a braid, the set of all braids \( \beta \) such that \( \inf \beta = 0 \) and \( \sup \beta = l \) is in bijection with the set of paths of length \( l \) in the left-weighting graph \( G_{lw} \). The set of rigid braids of infimum 0 and supremum \( l \) is in bijection with the set of paths of length \( l \), for which there is an edge from the last to the first vertex. Hence, the proposition is a corollary of Lemma 3.5 (ii). \( \square \)
4.2 Proportion of non pseudo-Anosov braids with infimum 0

The aim of this section is to show that, among the rigid braids of some fixed infimum and canonical length \( l \), the proportion of non pseudo-Anosov braids tends to 0 when \( l \) tends to infinity. For this, we can use the following theorem, due to González-Meneses and Wiest [10] (Theorem 5.16):

**Theorem 4.2.** Let \( \beta \) be a non-periodic, reducible braid which is rigid. Then there is some positive integer \( k \leq n \) such that one of the following conditions holds:

1. \( \beta^k \) preserves a round curve, or
2. \( \inf(\beta^k) \) and \( \sup(\beta^k) \) are even, and either \( \Delta^{-\inf(\beta^k)} \beta^k \) or \( \beta^{-k} \Delta^{\sup(\beta^k)} \) is a positive braid which preserves an almost round curve whose corresponding interior strands do not cross.

Let us also state the following theorem of Bernadete, Gutiérrez and Nitecki (Theorem 5.7 in [2]) as given in [3] (Theorem 1):

**Proposition 4.3.** Let \( x \in \mathcal{B}_n \), seen as a mapping class in \( \text{Mod}(D_n, \partial D_n) \), with left normal form \( x = \Delta^p x_1 \cdots x_r \). Let \( C \) be a round curve in \( D_n \). If \( x(C) \) is round, then \( \Delta^p x_1 \cdots x_m(C) \) is round for all \( m = 1, \ldots, r \).

**Notation 4.4.** In what follows we shall use the following two braids, written in normal form as follows:

\[
\gamma_1 = \sigma_1 \sigma_3 \cdots \sigma_{\lfloor \frac{n}{2} \rfloor - 1} \cdot \sigma_1 \sigma_3 \cdots \sigma_{\lfloor \frac{n}{2} \rfloor - 1} \sigma_2 \sigma_4 \cdots \sigma_{\lceil \frac{n}{2} \rceil - 2} \quad \text{(length 2)}
\]

\[
\gamma_2 = \Delta_{2,n} \sigma_1 \cdot \sigma_1 \cdot \sigma_1 \sigma_2 \cdots \sigma_{n-1} \cdot \sigma_{n-1} \quad \text{(length 4)}
\]

(see Figures 2 and 3.)

Figure 2: A braid sending no round curve to a round curve

**Proposition 4.5.** A rigid braid whose normal form contains both \( \gamma_1 \) and \( \gamma_2 \) as subwords is pseudo-Anosov.
Figure 3: A braid where each pair of strands crosses in some factor, and does not cross in some other factor

Proof. Let us study a rigid braid $\beta$, denoting $\inf(\beta) = \epsilon$ and the canonical length of $\beta$ as $l$.

First, we remark that there is no periodic rigid braid except $\Delta^\epsilon$. Indeed, if a braid $\beta$ is rigid and has canonical length at least 1, then its left normal form is of the shape

$$\beta = \Delta^\epsilon s_1 s_2 \cdots s_l$$

where $(s_i, s_{i+1})$ ($i = 1, \ldots, l - 1$) and $(s_l, \tau^\epsilon s_1)$ are left-weighted. Therefore, the normal form of a power of this braid is of the shape

$$\beta^k = \Delta^{k \epsilon} s_1^{(1)} s_2^{(1)} \cdots s_1^{(2)} s_2^{(2)} \cdots s_1^{(k)} s_2^{(k)} \cdots s_l^{(k)}$$

where $s_i^{(j)} = \tau^{(k-j)\epsilon}(s_i)$, which is never a power of $\Delta$ when $l \geq 1$.

Let us now deal with the possibility that $\beta$ might be reducible. According to Theorem 4.2, there are three possible cases.

The first case correspond to the case (1) of the theorem. A power of $\beta$ preserves a round curve. The rigidity of $\beta$ implies that the normal form of a power of $\beta$ contains the normal form of $\beta$ (except the initial factors $\Delta$) as a subword. According to Proposition 4.3, we deduce that there exists a round curve whose image by $\beta$ is still a round curve.

The second case is the case where a $k$-th power of $\beta$ is such that $\Delta^{-\inf(\beta^k)} \beta^k = \Delta^{-k \epsilon} \beta^k$ preserves an almost round curve whose interior strands do not cross. If the normal form of $\beta$ is $\Delta^\epsilon s_1 s_2 \cdots s_l$, then, as before, $\Delta^{-k \epsilon} \beta^k$ has normal form

$$\Delta^{-k \epsilon} \beta^k = s_1^{(1)} s_2^{(1)} \cdots s_1^{(2)} s_2^{(2)} \cdots s_1^{(k)} s_2^{(k)} \cdots s_l^{(k)}$$

This word has two strands that never cross, and hence so does the word $s_1 s_2 \cdots s_l$ representing $\Delta^{-\epsilon} \beta$. 

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Let us look at the third case. This time, it is the braid $\beta^{-k} \Delta^{\sup(\beta^k)} = \beta^{-k} \Delta^{k(l+\epsilon)}$ which has two strands that do not cross. Note that this braid has infimum $0$ and supremum $k \cdot l$. Therefore in the braid

$$\Delta^{k \cdot l} \cdot \left( \beta^{-k} \Delta^{k(l+\epsilon)} \right)^{-1} = \Delta^{-k \epsilon} \beta^k$$

(whose normal form was given in the previous paragraph) there are two strands which cross in every single factor. Hence the same is true for the word $s_1 s_2 \cdots s_l$ representing $\Delta^{-\epsilon} \beta$: it has two strands which cross in every factor.

Now, a braid $\beta$ whose normal form contains $\gamma_1$ cannot send any round curve to a round curve. The reason for this is that no round curve is sent to a round curve by this sequence of two simple braids (see Figure 2), and according to Proposition 4.3, this is also the case for the whole braid $\beta$. Similarly, a braid containing $\gamma_2$ cannot contain two strands that do not cross at all, or that cross in every single factor (see Figure 3). This completes the proof.

We now restrict our attention temporarily to the case of braids with infimum $0$.

**Lemma 4.6.** The number of braids of infimum $0$ and supremum $l$, which are rigid and pseudo-Anosov, is a $\Theta(\lambda^l)$, where $\lambda$ is the constant of Lemma 3.5.

**Proof.** Let us denote by $\Omega$ the set of rigid braids of infimum $0$ and supremum $l$. We also denote by $E_1 \subset \Omega$ the subset of the braids that do not contain, in their normal form, the normal form of $\gamma_1$ as a subword. We denote by $E_2 \subset \Omega$ the subset of the braids that do not contain, in their normal form, the normal form of $\gamma_2$ as a subword.

According to Lemma 3.5, with the same notations, the cardinality $\#(\Omega)$ is equivalent to $\lambda^{l+1}$.

Still from Lemma 3.5, we also have estimates $\#(E_1) = O(\mu_{(\gamma_1)}^l)$ where $\mu_{(\gamma_1)} < \lambda$ and $\#(E_2) = O(\mu_{(\gamma_2)}^l)$ where $\mu_{(\gamma_2)} < \lambda$. Thus the cardinality of the set $E_1 \cup E_2$, whose cardinality is less than $c(\mu_{(\gamma_1)}^l + \mu_{(\gamma_2)}^l)$ for a suitable constant $c > 0$, and this set contains all rigid braids of infimum $0$ and supremum $l$ which are non pseudo-Anosov.

As $\mu_{(\gamma_1)} < \lambda$ and $\mu_{(\gamma_2)} < \lambda$, the number of braids of infimum $0$ and supremum $l$ which are rigid and pseudo-Anosov, is still of the order of $\lambda^l$. \qed

### 4.3 Arbitrary infimum

Let us consider the Cayley graph of the braid group, with generators the simple braids. The following lemma, which is an immediate consequence of Lemma 3.1 in [5], gives the possible left normal forms for a braid that is at distance $l$ from the neutral element in this graph.

**Lemma 4.7.** Let $\beta$ be a braid at distance $l$ from the neutral element in the Cayley graph. Then the left normal form of $\beta$ has one of the following shapes:
(i) $\beta = \Delta^{-l}s_1 \cdots s_k$, $k \in \{0, \ldots, l-1\}$,
(ii) $\beta = \Delta^{-k}s_1 \cdots s_l$, $k \in \{0, \ldots, l\}$,
(iii) $\beta = \Delta^k s_1 \cdots s_{l-k}$, $k \in \{1, \ldots, l\}$.

The following theorem is a generalization of the results previously obtained in the particular case of a zero infimum.

**Theorem 4.8.** For large enough $l$, among all braids at distance $l$ from the neutral element in the Cayley graph, the proportion of rigid pseudo-Anosov braids is bounded below by a positive constant.

**Proof.** First, let us make a remark: a braid $\beta$ is pseudo-Anosov if and only if $\Delta^2\beta$ is pseudo-Anosov. The same is true when we replace “pseudo-Anosov” by “rigid”. Thus, a braid with left normal form $\Delta^p s_1 \cdots s_r$ with $p$ even is pseudo-Anosov (respectively rigid) if and only if $s_1 \cdots s_r$ is.

According to Lemma 4.6, there exists a constant $c_1 > 0$ such that for all large enough $l$, the number of rigid pseudo-Anosov braids of the form $s_1 \cdots s_l$ is bounded below by $c_1 \lambda^l$. Consequently, the number of rigid pseudo-Anosov braids of the form $\Delta^{-k}s_1 \cdots s_l$ with $k \in \{0, \ldots, l\}$ and $k$ even is bounded below by $c_1 \lambda^l$.

Furthermore, let us bound above the total number of braids at distance $l$ of the neutral element. According to Lemma 3.5, there exists a constant $c_2$ such that the number of braids with normal form $s_1 \cdots s_k$ is bounded above by $c_2 \lambda^k$. So:

(i) the number of braids with normal form $\Delta^{-l}s_1 \cdots s_k$ ($0 \leq k < l$) is bounded above by $c_2(1 + \cdots + \lambda^{l-1})$,
(ii) the number of braids with normal form $\Delta^{-k}s_1 \cdots s_l$ ($0 \leq k \leq l$) is bounded above by $c_2 l \lambda^l$,
(iii) the number of braids with normal form $\Delta^k s_1 \cdots s_{l-k}$ ($0 < k \leq l$) is bounded above by $c_2(1 + \cdots + \lambda^{l-1})$.

As $c_2(1 + \cdots + \lambda^{l-1}) \sim \frac{c_2}{\lambda-1} \lambda^l$, if we replace $c_2$ by an even larger constant, we can suppose that, in the cases (i) and (iii), the number of braids is bounded above by $\frac{c_1}{\lambda-1} \lambda^l$. Finally, the proportion of rigid pseudo-Anosov braids among all braids of length $l$ is bounded below by

$$\frac{c_1 \frac{1}{2} \lambda^l}{\frac{c_2}{\lambda-1} \lambda^l + c_2 l \lambda^l + \frac{c_2}{\lambda-1} \lambda^l} = \frac{c_1}{2c_2} \cdot \frac{1}{1 + \frac{2}{l(\lambda-1)}} \geq \frac{c_1}{2c_2} \cdot \frac{1}{1 + \frac{2}{\lambda-1}} > 0,$$

which completes the proof. \qed

**Corollary 4.9.** For large enough $l$, in the $l$-ball of the Cayley graph, the proportion of rigid pseudo-Anosov braids is bounded below independently of $l$ by a positive constant.
Proof. The number of braids in the $k$-sphere is of the order of $k\lambda^k$, and the $l$-ball is the union of the $k$-spheres for $k \leq l$. We deduce that the number of braids in the $l$-ball is of the order of $l\lambda^l$, that is to say, of the order of the number of braids in the $l$-sphere. So the proportion of rigid pseudo-Anosov braids remains of the order of a constant. 

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