On Integrals, Hamiltonian and Metriplectic Formulations of 3D Polynomial Systems

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Abstract

We apply the Darboux integrability method to determine first integrals and Hamiltonian formulations of three dimensional polynomial systems; namely the reduced three-wave interaction problem, the Rabinovich system, the Hindmarsh-Rose model, and the oregonator model. Additionally, we investigate their Hamiltonian, Nambu-Poisson and metriplectic characters.

Key words: Darboux integrability method, Prelle-Singer method, the reduced three-wave interaction problem, The Rabinovich system, the Hindmarsh-Rose model, Oregonator Model, Metriplectic Structure, Nambu-Poisson Brackets. MSC2010: 37K10, 70G45

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1 Introduction

The problem of solving ordinary nonlinear differential equations is a challenging area in nonlinear dynamics. For a two dimensional system the existence of a first integral completely determines its phase portrait. In these cases chaos cannot arise because of the Poincaré-Bendixson theorem [34] which says that any limit of a 2D system of differential equation is either a fixed point or a cycle. In three dimension this is no longer true. In the case of non planar systems the problem of determining first integrals is a non trivial task in general, and various methods have been introduced for studying the existence of first integrals. However, except for some special cases [36] there are few known satisfactory methods to solve it in general.

Non planar systems are often non-Hamiltonian in character and describe the time evolution of physical processes which are usually dissipative in nature. In general, a Pfaff differential form in n-dimensions

$$F_1(x_1, \cdots, x_n)dx_1 + \cdots + F_n(x_1, \cdots, x_n)dx_n$$

is not exact and therefore integrating factor may not be exist. Earlier a direct method [18] has been used to search for a first integral of three dimensional dynamical systems. This method consists in proposing an ansatz for the invariant which is a polynomial of a given degree in
one coordinates of the phase space of the system. So reader can see immediately that this is a tedious method applied to a very special class of systems. In fact, Grammaticos et al [24] proposed another method, based on the Frobenius integrability theorem, for finding integrals for three-dimensional ordinary differential equations. None of these methods are extremely successful. In a similar programme, Dorizzi et al [12] investigated a three-dimensional Hamiltonian systems with quartic potentials that are even in $x$, $y$, and $z$. They applied reduction method to obtain two new integrable systems and their constants of motion.

One might ask why do we need first integrals. An integral defines an invariant manifold for the flow which can be used to eliminate one degree of freedom. When the system admits an integral of motion, the analysis of its dynamical behaviour, especially in $t \rightarrow \infty$ limit, is greatly simplified. As elucidated by Giacomini and Neukrich [20, 21], the first integrals can be used in the non integrable regimes to build generalized Lyapunov functions and obtain bounds on the chaotic attractors of three-dimensional vector fields and prove the absence of homoclinic orbits. Therefore computing the first integral is an important problem but unfortunately the problem of finding a first integral is mathematically the same problem as solving the original system. Indeed exact first integrals are known only in special cases.

In this paper, we are interested in the integrability of the polynomial differential systems of 3 dimensions. A polynomial system is said to be Darboux integrable if it possesses a first integral or an integrating factor given by Darboux polynomial [11]. In particular, Darboux showed (see for example [9]) that a polynomial system of degree $n$, with at least $n(n + 1)/2 + 1$ invariant algebraic curves, has a first integral which can be expressed by means of these algebraic curves. Note that, the knowledge of algebraic curves can be used to study the topological properties of the system.

The goal of this paper is to obtain the first integrals of some polynomial three dimensional ODE systems, namely the reduced three-wave interaction problem, Rabinovich system, Hindmarsh-Rose model and Oregonator model, using Darboux polynomials. After deriving the first integrals, we shall further investigate the possible Hamiltonian formulations, bi-Hamiltonian representations or/and metriplectic realizations of these systems. We shall derive Poisson tensors, metric tensors for each system explicitly.

In order to achieve these goals, the paper is divided into two main sections. The following section is reserved for the theoretical background on the notions of integrability, Hamiltonian, Nambu Poisson and metriplectic formulations in three dimensional models. The theorem (1) in the first subsection has the prominent role while determining the first integrals using the Darboux polynomials. After finding an integral of a system $\dot{x} = X$, one starts to wonder whether or not that the system is Hamiltonian. In three dimensions, a Hamiltonian system is bi-Hamiltonian and Nambu-Poisson if it is possible to find a Jacobi’s last multiplier $M$ which makes $MX$ divergence free (c.f. see theorem (3)). A dissipative system can not is not Hamiltonian, but it can be written as a metriplectic formulation which is a linear combination of a Poisson and a gradient systems. The third section is for application of the technics presented in the section 2 to the particular models. For several subcases of the reduced three-wave interaction problem, for the Rabinovich system, and for the subcases of the Hindmarsh-Rose model, the first integrals will be constructed. A
bi-Hamiltonian/Nambu metriplectic formulation of these systems will be exhibited. First integrals of the Oregonator model is established and the model written as a Hamiltonian system.

2 Some Theory on 3D Polynomial Systems

2.1 Darboux’ Polynomials

A three dimensional polynomial ODE system is given by the set of equations

$$\dot{x} = P(x), \quad \dot{y} = Q(x), \quad \dot{z} = R(x),$$

(1)

where $P, Q, R$ are real valued polynomials with real coefficients. Here, the boldface $x$ stands for the three tuple $(x, y, z)$. The degree $m$ of a system is the maximum of degrees of the coefficient polynomials. The system (1) defines a polynomial vector field $X = X(x)$ by the identity $\dot{x} = X(x)$.

A function $I = I(t, x, y, z)$ is the first integral if it is constant on any integral curve of the system, that is if the total derivative of $I$ with respect to $t$ vanishes on the solution curves. A second integral $g$ of a system $\dot{x} = X(x)$ is a function satisfying

$$X(g) = \lambda g$$

(2)

for some function $\lambda$ called the cofactor. Polynomial second integrals for the polynomial vector fields are called the Darboux polynomials. The Darboux polynomials simplify the determination of possible first integrals [11]. For example, if there exist two relatively prime Darboux polynomials, say $P_1$ and $P_2$, having a common cofactor then their fraction $P_1/P_2$ is a rational first integral of the polynomial vector field $X$. The inverse of this statement is also true that is, if we have a rational first integral $P_1/P_2$ of a vector field $X$, then $P_1$ and $P_2$ are Darboux polynomials for $X$.

For the case of planar polynomial vector fields, there are more strong tools for the determination of the first integrals. In [53, 54], a semi-algorithm, called Prelle-Singer method, is presented for the determinations of elementary first integrals for planar systems. If we have a certain number of relatively prime irreducible Darboux polynomials, not necessarily having a common cofactor, it is possible to write first integrals using the Darboux polynomials [11, 13, 39, 54]. Unfortunately, this algorithm cannot be applicable for non-planar systems. However, Darboux polynomials are still useful though at times the use of a specific ansatz or a polynomial in one variable (of particular degree) with coefficients depending on the remaining variables remains the only option. One may at times use a variant of the Prelle-Singer/Darboux method to derive what are called quasi-rational first integrals [43]. Now, we state the following observation which enables one to arrive a time dependent first integral of a given system when it possesses autonomous Darboux polynomials.
Theorem 1 If $g_\alpha$’s are Darboux Polynomials for an autonomous system $\dot{x} = X$ and there exist constants $n_\alpha$’s, not all zero, satisfying the equality
\[
\sum_{\alpha=1}^{k} n_\alpha \lambda_\alpha = r,
\] (3)
for some real number $r \in \mathbb{R}$, then the function
\[
I = e^{-rt} \prod_{\alpha=1}^{k} g_\alpha^{n_\alpha}
\] (4)
is a time dependent first integral of the system $\dot{x} = X$.

Proof 1 To prove this assertion, we compute the total derivative of the function $I$ given in (4) as follows
\[
\tilde{X}(I) = \frac{\partial}{\partial t} \left( e^{-rt} \prod_{\alpha=1}^{k} g_\alpha^{n_\alpha} \right) + e^{-rt} X \left( \prod_{\alpha=1}^{k} g_\alpha^{n_\alpha} \right)
\]
\[
= -re^{-rt} \prod_{\alpha=1}^{k} g_\alpha^{n_\alpha} + e^{-rt} \left( \prod_{\alpha=1}^{k} g_\alpha^{n_\alpha-1} \right) \left( \sum_{\beta=1}^{k} (n_\beta g_1 \ldots X(g_\beta) \ldots g_k) \right)
\]
\[
= -re^{-rt} \prod_{\alpha} g_\alpha^{n_\alpha} + e^{-rt} \left( \prod_{\alpha=1}^{k} g_\alpha^{n_\alpha} \right) \left( \sum_{\beta=1}^{k} n_\beta \lambda_\beta \right)
\]
\[
= -re^{-rt} \prod_{\alpha} g_\alpha^{n_\alpha} + re^{-rt} \prod_{\alpha=1}^{k} g_\alpha^{n_\alpha} = 0
\]
where in the first line we assumed that $g_\alpha$ is not explicitly time dependent, we applied product rule in the second line, in the third line we used the fact that $g_\alpha$’s are Darboux’ polynomials by satisfying the equalities (2.1), and finally, in the last line, we applied the equality (3).

To the best of our knowledge, in the literature, the case where $\sum_\alpha n_\alpha \lambda_\alpha \neq r$ is still open.

2.2 Poisson Systems in 3D
Poisson bracket on an $n$-dimensional space is a binary operation $\{\cdot, \cdot\}$ on the space of real-valued smooth functions satisfying the Leibnitz and the Jacobi identities [40, 41, 50, 58]. We define a Poisson bracket of two functions $F$ and $H$ by
\[
\{F, H\} = \nabla F \cdot N \nabla H,
\] (5)
where $N$ is skew-symmetric Poisson matrix, $\nabla F$ and $\nabla H$ are gradients of $F$ and $H$, respectively. A Casimir function $C$ on a Poisson space is the one that commutes with all the other
functions. In order to have a non-trivial Casimir function, the Poisson matrix $N$ must be degenerate. A system of ODEs is Hamiltonian if it can be written in the form of Hamilton’s equation
\[
\dot{x} = \{x, H\} = N \nabla H
\]
for $H$ being a real-valued function, called Hamiltonian function, $\{\bullet, \bullet\}$ being a Poisson bracket and $N$ being the Poisson matrix. A dynamical system is bi-Hamiltonian if it admits two different Hamiltonian structures
\[
\dot{x} = N_1 \nabla H_2 = N_2 \nabla H_1,
\]
with the requirement that the Poisson matrices $N_1$ and $N_2$ be compatible [44, 50].

Space of three dimensional vectors and space of three by three skew-symmetric matrices are isomorphic. Existence of this isomorphism enables us to identify a three by three Poisson matrix $N$ with a three dimensional Poisson vector field $J$ [14, 28]. In this case, the Hamilton’s equation takes the particular form
\[
\dot{x} = J \times \nabla H,
\]
whereas a bi-Hamiltonian system is in form
\[
\dot{x} = J_1 \times \nabla H_2 = J_2 \times \nabla H_1.
\]
and the Jacobi identity turns out to be
\[
J \cdot (\nabla \times J) = 0.
\]

The following theorem establishes form of a general solution of the Jacobi identity. For the proof this theorem we refer [1, 31, 32, 33].

**Theorem 2** General solution of the Jacobi identity (10) is
\[
J = \frac{1}{M} \nabla H_1
\]
for arbitrary functions $M$ called the Jacobi’s last multiplier, and $H_1$ called as the Casimir.

Existence of the scalar multiple $1/M$ in the solution is a manifestation of the conformal invariance of Jacobi identity. In the literature, $M$ is called Jacobi’s last multiplier [23, 37, 38, 59]. The potential function $H_1$ in Eq.(11) is a Casimir function of the Poisson vector field $J$. Any other Casimir of $J$ has to be linearly dependent to the potential function $H_1$ since the kernel is one dimensional. Substitution of the general solution (11) of $J$ into the Hamilton’s equations (8) results with
\[
\dot{x} = \frac{1}{M} \nabla H_1 \times \nabla H_2.
\]
While writing a non-autonomous system in form of the Hamilton’s equations (8), inevitably, one of the two, Poisson vector or Hamiltonian function, must depend explicitly on the time variable \( t \). The calculation

\[
\frac{d}{dt} H(x, t) = \nabla H(x, t) \cdot \dot{x} + \frac{\partial}{\partial t} H(x, t) = \nabla H \cdot (J \times \nabla H) + \frac{\partial}{\partial t} H(x, t) = \frac{\partial}{\partial t} H(x, t),
\]

shows that if the time parameter appears only in the Poisson vector, then the Hamiltonian is a constant of the motion, if the time parameter appears in the Hamiltonian, then the Hamiltonian fails to be an integral invariant of the system.

### 2.3 Nambu-Poisson Systems in 3D

In [49], a ternary operation \( \{\bullet, \bullet, \bullet\} \), called Nambu-Poisson bracket, is defined on the space of smooth functions satisfying the generalized Leibnitz identity

\[
\{F_1, F_2, FH\} = \{F_1, F_2, F\} H + F \{F_1, F_2, H\}
\]

and the fundamental (or Takhtajan) identity

\[
\{F_1, F_2, \{H_1, H_2, H_3\}\} = \sum_{k=1}^{3} \{H_1, \ldots, H_{k-1}, \{F_1, F_2, H_k\}, H_{k+1}, \ldots, H_3\},
\]

for arbitrary functions \( F, F_1, F_2, H, H_1, H_2 \), see [55]. A dynamical system is called Nambu-Hamiltonian with Hamiltonian functions \( H_1 \) and \( H_2 \) if it can be recasted as

\[
\dot{x} = \{x, H_1, H_2\}.
\]

By fixing the Hamiltonian functions \( H_1 \) and \( H_2 \), we can write Nambu-Hamiltonian system (15) in the bi-Hamiltonian form

\[
\dot{x} = \{x, H_1\}^{H_2} = \{x, H_2\}^{H_1}
\]

where the Poisson brackets \( \{\bullet, \bullet\}^{H_2} \) and \( \{\bullet, \bullet\}^{H_1} \) are defined by

\[
\{F, H\}^{H_2} = \{F, H, H_2\} \quad \{F, H\}^{H_1} = \{F, H_1, H\},
\]

respectively [27].

In 3D, we define a Nambu-Poisson bracket of three functions \( F, H_1 \) and \( H_2 \) as the triple product

\[
\{F, H_1, H_2\} = \frac{1}{M} \nabla F \cdot \nabla H_1 \times \nabla H_2
\]

of their gradient vectors. Note that, the Hamilton’s equation (12) is Nambu-Hamiltonian (15) with the bracket (18) having the Hamiltonian functions \( H_1 \) and \( H_2 \) [27, 56]. If the function \( F \) in (18) is taken as the coordinate functions, then it becomes the Lie-Poisson
bracket on $\mathbb{R}^3$ of two functions $H_1$ and $H_2$ identified with $(\mathbb{R}^3)^*$ using the dot product [4], that is

$$\{H_1, H_2\}_{LP} = \frac{1}{M} \mathbf{x} \cdot \nabla H_1 \times \nabla H_2.$$  

The following theorem establishes the link between the existence of the Hamiltonian structure of a dynamical system and the existence of the Jacobi’s last multiplier. For the proof of the assertion we cite [14, 17].

**Theorem 3** A three dimensional dynamical system $\dot{\mathbf{x}} = \mathbf{X}$ having a time independent first integral is Hamiltonian, bi-Hamiltonian hence Nambu Hamiltonian if and only if there exist a Jacobi’s last multiplier $M$ which makes $M\mathbf{X}$ divergence free.

### 2.4 Metriplectic Systems in 3D

Let $G$ be a positive semi definite symmetric matrix on an Euclidean space, and consider the symmetric bracket of two functions

$$(F, S) = \nabla F \cdot G \nabla S.$$  

In terms of this symmetric bracket, we define a metric or a gradient system by

$$\dot{\mathbf{x}} = (\mathbf{x}, S) = G \nabla S. \quad (19)$$

The generating function, usually called the entropy, is not a conserved quantity for the system instead we have $\dot{S} = (S, S) \geq 0$, see [16].

The representation of a dynamical system as a metriplectic system requires two geometrical structures namely a Poisson structure $N$ and a metric structure $G$. Metriplectic bracket is the sum of the two brackets

$$\{\{F, E\}\} = \{F, E\} + \lambda \{F, E\} = \nabla F \cdot N \nabla E + \lambda \nabla F \cdot G \nabla E,$$

for any scalar $\lambda$. There are extensive studies on the metriplectic systems see, for example, [2, 4, 5, 16, 26, 48, 46, 42]. The metriplectic structures also called with the name GENERIC [25]. The metriplectic structure satisfies the Leibnitz identity for each entry hence it is an example of a Leibnitz bracket [51]. We refer [47] for a brief history of metriplectic structures and more.

There are two types of metriplectic systems in the literature. One of them is the one governed by so called a generalized free energy $F$ which is the difference of a Hamiltonian function $H$ and a entropy function $S$. In this case, we require that $\nabla S$ lives in the kernel of $N$ and $\nabla H$ lives in the kernel of $G$, that is

$$N \nabla S = 0, \quad G \nabla H = 0. \quad (20)$$

The equation of motion is given by

$$\dot{\mathbf{x}} = \{\{\mathbf{x}, F\}\} = \{\mathbf{x}, F\} + (\mathbf{x}, F) = \{\mathbf{x}, H\} - (\mathbf{x}, S).$$
Note that, for the dynamics governed by the metriplectic bracket, we have the conservation law \( \dot{H} = \{\{H, F\}\} = 0 \) and the dissipation \( \dot{S} = \{\{S, F\}\} \leq 0 \). We note that a weaker version of the condition (20) can be given by

\[ N \nabla S + G \nabla H = 0. \]  

(21)

The second type of the metriplectic systems is generated by a single function, say \( H \), and written as

\[ \dot{x} = \{\{x, H\}\} = \{x, H\} + \lambda(x, H) \]  

(22)

without any restriction on \( H \) as given in (20) or (21).

If the Hamiltonian (reversible) part of the dynamics can be written in the terms of Nambu-Poisson bracket we may rewrite the system as

\[ \dot{x} = \{x, H_1, H_2\} + (x, S) = \frac{1}{M} \nabla H_1 \times \nabla H_2 - G \nabla S, \]  

(23)

where \( M \) is the Jacobi’s last multiplier \([3]\). In this case, one may take \( S \) equals to \( H_1 \) or \( H_2 \).

3  Examples

3.1  Reduced three-wave interaction problem

The reduced three-wave interaction model \([23, 52]\) is given by the system of ODEs

\[
\begin{align*}
\dot{x} &= -2y^2 + \gamma x + z + \delta y \\
\dot{y} &= 2xy + \gamma y - \delta x \\
\dot{z} &= -2xz - 2z.
\end{align*}
\]  

(24)

where three quasisynchronous waves interact in a plasma with quadratic nonlinearities. In \([6]\), this model is studied by means of Painlevé method. In \([18]\), the existence of first integrals for this and other systems were investigated by proposing an ansatz for the first integral which explicitly involves a pre-set dependence on a particular phase space coordinate. We show how their results can be obtained in a more simplified manner using Darboux polynomials. We, additionally, present bi-Hamiltonian and metriplectic realizations of the model.

Proposition 1 The three dimensional reduced three-wave interaction problem (24) has the following first integrals.

1. If \( \delta = \) arbitrary, \( \gamma = 0 \), then \( I = e^{2t} y(y - \delta/2) \).
2. If \( \delta = \) arbitrary, \( \gamma = -1 \), then \( I = e^{2t}(x^2 + y^2 + z) \).
3. If \( \delta = \) arbitrary, \( \gamma = -2 \), then \( I = e^{4t}(x^2 + y^2 + 2/\delta yz) \).
4. If $\delta = 0$, $\gamma$ = arbitrary, then $I = e^{2-\gamma yz}$.

5. If $\delta = 0$, $\gamma = -1$, then $I_1 = e^{2t(x^2 + y^2 + z)}$, $I_2 = e^{3t}yz$.

In order to prove this assertion, we recall the eigenvalue problem (2.1) associated with the system (24) where $g$ is a second degree polynomial of the form

$$g = Ax^2 + By^2 + Cz^2 +Exy + Fxz + Gyz + Jx + Ky + Lz.$$  \(25\)

Equating coefficients then leads to the following set of equations

$$A = B, \quad E = F = C = 0$$  \(26\)

\[
\begin{align*}
2A\gamma - E\delta &= \lambda A, \\
2B\gamma +&E\delta - 2J &= \lambda B, \\
F - 4C &= \lambda C, \\
2A\delta + 2E\gamma - 2B\delta + 2K &= \lambda K, \\
2A + (\gamma - 2)F - G\delta - 2L &= \lambda F, \\
E + F\delta + (\gamma - 2)G &= \lambda G \\
J\gamma - K\delta &= \lambda J, \\
J\delta + K\gamma &= \lambda K, \\
J - 2L &= \lambda L.
\end{align*}
\]  \(27\)

for the third order, the second order, and the linear terms, respectively. We distinguish a number of cases following from the solutions of the system (26)-(28) for specific parameter values. These cases will determine the integrals of the reduced systems by following the theorem (1). In the first three cases, $\delta$ is arbitrary, and we study on three different values of $\gamma$, namely 0, -1 and -2. For the remaining cases wherein $\delta = 0$ one can identify explicitly Darboux functions of the associated vector field, with associated eigenpolynomials which are not of degree zero.

3.1.1 Case 1: $\delta$ is arbitrary and $\gamma = 0$

The choices of $\delta$ is arbitrary and $\gamma = 0$ reduce the system of equations (26)-(28) to the following list

$$A = B = C = E = F = K = J = 0, \quad L = -\frac{\delta}{2}G, \quad \lambda = -2$$

where $G$ is arbitrary function. Additionally, by choosing $G = 1$, we obtain the eigenfunction

$$g = zy - \frac{\delta}{2}z.$$  \(29\)

The condition (3) translates to the following requirement $-r + n\lambda = 0$. For $r = -1$, we have $n = 1/2$, so that an integral of the motion equals to $e^t(zy - \frac{\delta}{2}z)^{\frac{3}{2}}$. As any function of this integrating factor is also a first integral we write the integral as

$$I = e^{2t} \left( zy - \frac{\delta}{2}z \right)^{\frac{3}{2}}.$$  \(29\)
We change the dependent variable \( z \) by \( w \) according to
\[
w = e^{2t}z.
\]
In this case, the system (24) turns out to be a non-autonomous system
\[
\begin{align*}
\dot{x} &= -2y^2 + we^{-2t} + \delta y \\
\dot{y} &= 2xy - \delta x \\
\dot{w} &= -2xw
\end{align*}
\]  
whereas the integral \( I \) in (29) becomes time independent Hamiltonian of the system given by
\[
H_1 = wy - \frac{\delta}{2}w.
\]
This reduced system is divergence free, hence, according to the theorem (3), it is bi-Hamiltonian (7) and Nambu-Poisson (15). To exhibit these realizations, we need to introduce a second time dependent Hamiltonian function
\[
H_2 = x^2 + y^2 + e^{-2t}w
\]
of the system (30). Note that, the system is divergce free, hence Jacobi’s last multiplier for the system is a constant function, say \( M = 1 \). So that, the system is in the form of cross product of two gradients
\[
(\dot{x}, \dot{y}, \dot{w})^T = \nabla H_1 \times \nabla H_2 = J_1 \times \nabla H_2 = J_2 \times \nabla H_1
\]
in the form of bi-Hamiltonian and Nambu-Poisson forms (12) with Poisson vector fields \( J_1 = \nabla H_1 \) and \( J_2 = -\nabla H_2 \), respectively. Since the first Hamiltonian is autonomous, the second one has to, evidently, be time dependent. Note that, this second time dependent Hamiltonian \( H_2 \) can not be observed as a consequence of the theorem (1), because it is not an integral invariant of the system.

At this point, we make a break to the cases and discuss the metriplectic structure of the system (24) starting and inspiring from the bi-Hamiltonian/Nambu formulation of its particular case (30). The proof of the following assertion is a matter of direct calculation

**Proposition 2** The reduced three-wave interaction problem (24)is in bi-Hamiltonian/Nambu metriplectic formulation (23) given by
\[
(\dot{x}, \dot{y}, \dot{z})^T = \nabla H_1 \times \nabla H_2 - G \nabla H_2.
\]  
where the Hamiltonian functions are \( H_1 = zy - \frac{\delta}{2}z \), and \( H_2 = x^2 + y^2 + e^{-2t}z \), and the metric tensor is
\[
G = \begin{pmatrix}
-\gamma/2 & 0 & 0 \\
0 & -\gamma/2 & 0 \\
0 & 0 & 2ze^{2t}
\end{pmatrix}.
\]

In (31), the metriplectic structure in of the second kind. Note that, by replacing the roles of \( H_1 \) and \( H_2 \) in (31), up to some modifications in the definition of the metric, we may also generate the system (24) by the Hamiltonian \( H_1 \) as well. This case will be presented in the case 5.
3.1.2 Case 2: $\delta$ is arbitrary and $\gamma = -1$

In the case $\delta$ is arbitrary and $\gamma = -1$, the system of equations (26)-(28) becomes

$$C = E = F = G = K = J = 0, \quad A = B = L, \quad \lambda = -2$$

so that the eigenfunction becomes $A(x^2 + y^2 + z)$. Hence, the condition for $I$ to be a first integral, namely $-r + n\lambda = 0$ implies $n = \frac{1}{2}$ and $r = -1$. The corresponding first integral is then given by

$$I = e^{2t}(x^2 + y^2 + z). \tag{32}$$

We make the change of dependent variables

$$u = xe^t, \quad v = ye^t, \quad w = z e^{2t}$$

and rescale the time variable by $\bar{t} = et$, then we arrive the non-autonomous system

$$\begin{cases} 
\dot{u} = -2v^2 + w + \delta v \bar{t} \\
\dot{v} = 2uv - \delta u \bar{t} \\
\dot{w} = -2uw
\end{cases} \tag{33}$$

where prime denotes the derivative with respect to the new time variable $\bar{t} = et$. In this coordinates, the integral (32) is autonomous

$$H_1 = u^2 + v^2 + w.$$ 

Note that, the system (33) is divergence free, hence we can take the Jacobi’s last multiplier $M$ as the unity. Hence, we argue that, there exist a second Hamiltonian which enables us to write the system (33) in bi-Hamiltonian/Nambu formulation. After a straight forward calculation, we arrive a non-autonomous Hamiltonian

$$H_2 = vw + \frac{\delta v^2}{2} \bar{t} - \frac{\delta w^2}{2} \bar{t}$$

which enables us to write the system (33) as a bi-Hamiltonian (7) and Nambu-Poisson (15) system

$$(\dot{u}, \dot{v}, \dot{w})^T = \nabla H_1 \times \nabla H_2 = J_1 \times \nabla H_2 = J_2 \times \nabla H_1$$

where the Poisson vectors are $J_1 = \nabla H_1$ and $J_2 = -\nabla H_2$, respectively.

3.1.3 Case 3: $\delta$ is arbitrary and $\gamma = -2$

For the above choice of parameters $\delta$ is arbitrary and $\gamma = -2$, it may be verified that, the system of equations (26)-(28) turn out to be

$$C = E = F = J = K = L = 0, \quad A = B, \quad G = \frac{2}{\delta} A, \quad \lambda = -4.$$
This leads to the eigenfunction $A(x^2 + y^2 + \frac{\delta}{2}yz)$, so that choosing $A = 1$ we get the following first integral
\[
I = e^{4t}(x^2 + y^2 + \frac{2}{\delta}yz).
\] (34)

To arrive the Hamiltonian form of this system, we first make the substitutions $u = xe^{2t}$, $v = ye^{2t}$, $w = ze^{2t}$ which results with the non-autonomous divergence free system
\[
\begin{align*}
\dot{u} &= -2v^2e^{-2t} + w + \delta v \\
\dot{v} &= 2uve^{-2t} - \delta u \\
\dot{z} &= -2uve^{-2t}
\end{align*}
\] (35)

Actually, the system (35) is a bi-Hamiltonian (7) and Nambu-Poisson (15) system with the introductions of Hamiltonian functions
\[
H_1 = \frac{\delta}{2}(u^2e^{-2t} + v^2e^{-2t} + w), \quad H_2 = u^2 + v^2 + \frac{2}{\delta}vw,
\]
where the second Hamiltonian is the integral (34).

### 3.1.4 Case 4: $\delta = 0$ and $\gamma$ is arbitrary

It is a straightforward matter to verify that the following functions $g_\alpha$ ($\alpha = 1, 2$) are Darboux polynomials whose associated eigenpolynomials $\lambda_\alpha$’s are
\[
g_1 = y, \quad \lambda_1 = 2x - 1, \quad \text{and} \quad g_2 = z, \quad \lambda_2 = -2x - 2, \quad (36)
\]
if $\delta = 0$ and $\gamma$ is arbitrary. The condition (3) now leads to
\[
0 = -r + \sum_\alpha n_\alpha g_\alpha \Rightarrow -r + n_1(2x + \gamma) + n_2(-2x - 2) = 0.
\]
Setting $r = -1$ we obtain the following equations:
\[
n_1 - n_2 = 0, \quad \gamma n_1 - 2n_2 + 1 = 0
\]
leading to $n_1 = n_2 = \frac{1}{2-\gamma}$. The corresponding first integral is
\[
I = e^{(2-\gamma)t}yz.
\] (37)

In order to exhibit the Hamiltonian formulation of the system, we define $u = xe^{-\gamma t}$, $v = ye^{-\gamma t}$, $w = ze^{2t}$ then we have a non-autonomous divergence free system
\[
\begin{align*}
\dot{u} &= -2v^2e^{\gamma t} + we^{-(2+\gamma)t} \\
\dot{v} &= 2uve^{\gamma t} \\
\dot{z} &= -2uve^{\gamma t}
\end{align*}
\] (38)

with the Hamiltonian $H_2 = vw$. The bi-Hamiltonian (7) and Nambu-Poisson (15) structure of the system can be realized after the introduction of the second (time dependent) Hamiltonian
\[
H_1 = u^2e^{\gamma t} + v^2e^{\gamma t} + e^{-(2+\gamma)t}w.
\]
3.1.5 Case 5: $\delta = 0$ and $\gamma = -1$

For this case, in addition to $g_1, g_2$ given in (36), we have another Darboux polynomial

$$g_3 = x^2 + y^2 + z, \quad \lambda_3 = -2.$$ 

The condition (3) becomes

$$2(n_1 - n_2) - (n_1 + 2n_2 + 2n_3) = r.$$ 

We make the standardization $r = -1$ and obtain the following set of equations

$$n_1 = n_2$$

or, in other words, $3n_1 + 2n_3 = 1$ which leads to the following subcases: (a) $n_3 = 0$ and $n_1 = n_2 = \frac{1}{3}$, and (b) $n_1 = n_2 = 0$ and $n_3 = \frac{1}{2}$. So that, we have two time dependent integrals of the motion

$$I_1(x, y, z) = e^t(yz)$$

and

$$I_2(x, y, z) = e^t(x^2 + y^2 + z).$$

We make the change of variables $u = xe^t$, $v = ye^t$, and $w = ze^{2t}$ and rescale the time variable by $\bar{t} = e^t$, then arrive the autonomous system

$$\begin{cases}
\dot{u} = -2v^2 + w \\
\dot{v} = 2uv \\
\dot{w} = -2uw
\end{cases}$$

where prime denotes the derivative with respect to the new time variable $\bar{t} = e^t$. Note that, this system is divergence free, hence we can take the Jacobi’s last multiplier as the unity. In the new coordinate system, the integrals of the system (39) become the Hamiltonian functions of the system given by

$$H_1 = vw, \quad H_2 = u^2 + v^2 + w.$$  

This enables us to write the system (40) in bi-Hamiltonian (7) and Nambu-Poisson (15) form.

Note that, as a particular case of the proposition (2), we show how the reduced three-wave interaction model (24) with $\delta = 0$ and $\gamma = -1$ given by

$$\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix} = \begin{pmatrix}
-2y^2 + z \\
2xy \\
-2xz
\end{pmatrix} + \begin{pmatrix}
-x \\
y \\
-2z
\end{pmatrix}$$

can be put in a metricplectic realization of the second kind (22). Note that, the first term at the right hand side is the conservative part of the system with two Hamiltonian functions $H_1 = yz$ and $H_2 = x^2 + y^2 + z$ inspired from the ones in (41). This enables us to write the system (42) in two different ways. In the first one, we take $H_2$ as the Casimir function.
of the system and $H_1$ as the Hamiltonian system with Poisson vector $J_2 = -\nabla H_2$. Hence, the second term on the right hand side can be described by a dissipative term by taking the metric two-form as

$$G = \begin{pmatrix} 0 & \frac{x}{z} & 0 \\ \frac{x}{z} & 0 & 1 \\ 0 & 1 & \frac{z}{y} \end{pmatrix}$$

where $\lambda = -1$. In this case the reduced three wave interaction model (42) can be written as

$$\dot{x} = J_2 \times \nabla H_1 - G \nabla H_1.$$

### 3.2 Rabinovich system

This is described by the following system of equations:

$$\begin{cases}
\dot{x} = h y - \nu_1 x + y z \\
\dot{y} = h x - \nu_2 y - x z \\
\dot{z} = -\nu_3 z + x y,
\end{cases} \tag{43}$$

where $h$ and $\nu_i$ are real constants. We shall very briefly illustrate how the results of [18] for this system may be derived by the Darboux integrability method. In addition, we will show that the Rabinovich system (43) can be written as a bi-Hamiltonian/Nambu metriplectic form.

Consider the vector field $X$ generating the Rabinovich system (43). We note that application of $X$ to the function $g_1 = y^2 + z^2$ yields

$$X(g_1) = 2hxy - 2(\nu_2 y^2 + \nu_3 z^2). \tag{44}$$

Consequently $g_1$ becomes a Darboux polynomial when $h = 0, \nu_2 = \nu_3$. In this case, the eigenpolynomial being of degree zero viz $\lambda = -2\nu_3$. We are lead to the first integral

$$I_1 = e^{2\nu_3 t} (y^2 + z^2) \tag{45}$$

of the system (43) when $h = 0, \nu_2 = \nu_3$ with $\nu_1$ and $\nu_3$ being arbitrary. The application of the vector field $X$ generating the Rabinovich system (43) on the polynomial $g_2 = x^2 + y^2$ results with

$$X(g_2) = 4hxy - 2(\nu_1 x^2 + \nu_2 y^2).$$

Consequently, $g_2$ becomes a Darboux polynomial when $h = 0, \nu_1 = \nu_2$. In this case, the eigenpolynomial being of degree zero viz $\lambda = -2\nu_1$. We are lead to the first integral

$$I_2 = e^{2\nu_1 t} (x^2 + y^2) \tag{46}$$

of the system (43) when $h = 0, \nu_1 = \nu_2$ with $\nu_1, \nu_3$ being arbitrary.
Let us transform the Rabinovich system (43) in a form where we can write it as a bi-Hamiltonian/Nambu system. For the case of \( \nu_1 = \nu_2 = \nu_3 = v \), we have two integrals \( I_1 \) and \( I_2 \) of the system (43). In this case, we apply a coordinate change

\[
\begin{align*}
u_1 &= xe^{vt}, \\
v_2 &= ye^{vt}, \\
v_3 &= ze^{vt}
\end{align*}
\]

with the time rescaling \( t = \frac{1}{v} e^{vt} \) with \( v \neq 0 \), then the system turns out to be a divergence free system

\[
\begin{align*}
u &= vw, \\
\dot{v} &= -uw, \\
\dot{w} &= uv.
\end{align*}
\]

In this case the integrals of motion given in (45) and (46) become the Hamiltonian functions of the system, namely

\[
H_1 = \frac{1}{2}(v^2 + w^2), \quad H_2 = \frac{1}{2}(u^2 + v^2).
\]

Hence we can write (47) as in the form of bi-Hamiltonian (7) and Nambu-Poisson (15) form

\[
(\dot{u}, \dot{v}, \dot{w})^T = \nabla H_1 \times \nabla H_2
\]

with Jacobi’ last multiplier being the unity, see also [10]. For another discussion on the case where \( h \) is nonzero and \( \nu_1 = \nu_2 = \nu_3 = 0 \), we refer [57].

In the following proposition, inspiring from the bi-Hamiltonian/Nambu form (48) of the transformed system (47), we are, now, exhibiting a metriplectic realization of the Rabinovich system (43).

**Proposition 3** The Rabinovich system (43) is in bi-Hamiltonian/Nambu metriplectic formulation (23) given by

\[
(\dot{x}, \dot{y}, \dot{z})^T = \nabla H_1 \times \nabla H_2 - G \nabla H_1
\]

where the Hamiltonian functions are \( H_1 = \frac{1}{2}(x^2 + y^2) \), and \( H_2 = \frac{1}{2}(y^2 + z^2) \), and the metric tensor is

\[
G = \begin{pmatrix}
\nu_1 & -h & 0 \\
-h & \nu_2 & \frac{z\nu_3}{y} \\
0 & \frac{z\nu_3}{y} & 0
\end{pmatrix}.
\]

The metriplectic formulation (49) of the Rabinovich system (43) is of the second kind. As in the case of the reduced three-wave interaction problem, one may generate (43) by the Hamiltonian \( H_2 \) instead of \( H_1 \) by adopting a new metric.

### 3.3 Hindmarsh-Rose model

The Hindmarsh-Rose model of the action potential which is a modification of Fitzhugh model was proposed as a mathematical representation of the bursting behaviour of neurones, and was expected to simulate the repetitive, patterned and irregular activity seen in molluscan
neurons [35]. The Hindmarsh-Rose model consists of a system of three autonomous differential equations, with mild nonlinearities for modelling neurons that exhibit triggered firing. The usual form of the equations are

\[
\begin{align*}
\dot{x} &= y + \phi(x) - z - C \\
\dot{y} &= \psi(x) - y \\
\dot{z} &= r(s(x - x_R) - z)
\end{align*}
\] (50)

where \(\phi(x) = ax^2 - x^3\) and \(\psi(x) = 1 - bx^2\). Here \(C\) is a control parameter, while of the remaining five parameters \(s\) and \(x_R\) are usually fixed. We re-write them in the following form appending two extra parameters

\[
\begin{align*}
\dot{x} &= y - z - ax^3 + bx^2 + \alpha \\
\dot{y} &= \beta - dx^2 - y \\
\dot{z} &= px - rz - \gamma
\end{align*}
\] (51)

Here \(\alpha, \beta, \gamma, a, b, d, p, r\) are parameters. Unfortunately we have not found a first integral with \(a \neq 0\), which is the dominant nonlinear term here.

**Proposition 4** The reduced Hindmarsh-Rose system

\[
\begin{align*}
\dot{x} &= y - z + bx^2 + \alpha \\
\dot{y} &= \beta - dx^2 - y \\
\dot{z} &= px - rz - \gamma
\end{align*}
\] (52)

has the following first integrals.

1. If \(p = 0\), then the first integral of the system (52) is \(I = e^{rt}(rz + \gamma)\).
2. If \(d = 0\) then \(I = e^t(y - \beta)\).
3. If \(d, \beta, \gamma\) are arbitrary, \(b = -d, p = -2, \alpha = \beta + \gamma\) and \(r = 1\), then \(I = e^{2t}(x - y + z)\).
4. If \(\alpha, \gamma, p\) and \(b\) are arbitrary, and when \(d = 2b, r = -(p + 1), \beta = 2(\frac{2}{p} - \alpha)\) then \(I = e^{-t}(2x + y + \frac{2z}{p})\).
5. If \(\beta, \gamma, r, b, d\) are arbitrary, and

\[
\alpha = -\frac{b(\gamma d + \beta d - b \beta + r \beta b)}{d(d - b + br)} \quad \text{and} \quad p = \frac{(b - d)(d - b + br)}{b^2}
\]

then the first integral becomes

\[
I = e^{\frac{2(b-d)}{b^2}}(Ax^2 + By^2 + Cz^2 + Exy + Fxz + Gyz)
\]
where the coefficients of the polynomial are given by

\[
A = -\frac{(b - d)(d - b + br)}{b(-d + 2b + br)}, \quad B = -\frac{b(b - d)(d - b + br)}{d^2(-d + 2b + br)},
\]
\[
C = -\frac{b(b - d)}{(d - b + br)(-d + 2b + br)}, \quad E = -\frac{2(b - d)(d - b + br)}{d(-d + 2b + br)},
\]
\[
F = 2\frac{b - d}{-d + 2b + br}, \quad G = 2\frac{b(b - d)}{d(-d + 2b + br)}.
\]

6. If \( p = 0 \), \( b = d \), and \( \beta, \gamma, r \) are arbitrary and \( \alpha = -\frac{\beta r + \gamma}{r} \) then \( I = rx + ry - z \). When, additionally, \( r = -1 \), then \( I = x + y + z \).

To prove these assertions, one may take the total time derivatives of the integrals and show that they are zero. Starting with the integrals presented in the previous proposition, we are achieving to write the Hindmarsh-Rose model (51) in a metriplectic form of the second kind in the following proposition.

**Proposition 5** The Hindmarsh-Rose model (51) (with \( r = -1 \) and \( \alpha = \beta - \gamma \)) is in bi-Hamiltonian/Nambu metriplectic formulation (23) given by

\[
(\dot{x}, \dot{y}, \dot{z})^T = \nabla H_1 \times \nabla H_2 - G\nabla H_1. \quad (53)
\]

where the Hamiltonian functions are \( H_1 = x + y + z \), and \( H_2 = yz - \gamma y - \beta z \), and the metric tensor is

\[
G = \begin{pmatrix}
ax^3 - bx^2 & 0 & 0 \\
0 & dx^2 & 0 \\
0 & 0 & -px
\end{pmatrix}.
\]

### 3.4 Oregonator model

The Oregonator model was developed by Field and Noyes [15] to illustrate the mechanism of the Belousov-Zhabotinsky oscillatory reaction. The model can be expressed in terms of coupled three ordinary differential equations

\[
\left\{
\begin{array}{l}
\dot{x} = \frac{1}{\epsilon}(x + y - qx^2 - xy) \\
\dot{y} = -y + 2hz - xy \\
\dot{z} = \frac{1}{p}(x - z). \\
\end{array}
\right. \quad (54)
\]

that describe the complex dynamics of the reaction process. In the physical model considered, all the parameters \( \epsilon, q, p, h \) are positive. However, from a purely mathematical point of view, allowing the parameters to be negative, we have obtained a first integral

\[
I = e^{2t}(x + y + z), \quad (55)
\]

for the parameters \( q = 0, \epsilon = p = -1 \) and \( h = -\frac{3}{2} \) as may be easily verified.
We will write Oregonator model in the Hamiltonian formulation as follows. At first, we change the coordinates according to

\[ u = xe^{2t}, \quad v = ye^{2t}, \quad w = e^{2t}z. \]

which enables us to write the system (54) as the following nonautonomous form

\[
\begin{align*}
\dot{u} &= u - v + uv e^{-2t} \\
\dot{v} &= v - 3w - uv e^{-2t} \\
\dot{w} &= 3w - u
\end{align*}
\]

with a time independent first integral \( H = u + v + w \). Then we introduce the non-autonomous Poisson matrix

\[
N = \begin{pmatrix}
0 & uv e^{-2t} - v & u \\
v - uv e^{-2t} & 0 & -3w \\
-u & 3w & 0
\end{pmatrix}
\]

then the system (54) is in form Hamilton’s equation (6) given by \( \dot{u} = P \nabla H \).

4 Conclusions

In this paper, we have reviewed some technical details of the integrability and Hamiltonian representations of the 3D systems. Then, we have applied these theoretical results, especially the Darboux polynomials, to derive the first integrals of 3D polynomial systems the reduced three-wave interaction problem, Rabinovich system, Hindmarsh-Rose model and Oregonator model. Then we have achieved to exhibit Hamiltonian, and metriplectic realizations of the systems.

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