Ground state, quasi-hole, a pair of quasihole wavefunctions and instability in bilayer quantum Hall systems

Longhua Jiang and Jinwu Ye

Department of Physics, The Pennsylvania State University, University Park, PA, 16802

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Bilayer quantum Hall system (BLQH) differ from its single layer counterparts (SLQH) by its symmetry breaking ground state and associated neutral gapless mode in the pseudo-spin sector. Due to the gapless mode, qualitatively good groundstate and low energy excited state wavefunctions at any finite distance is still unknown. We investigate this important open problem by the Composite Boson (CB) theory developed by one of the authors to study BLQH systematically. We derive the ground state, quasi-hole and a pair of quasihole wavefunctions from the CB theory and its dual action. We find that the ground state wavefunction is the product of two parts, one in the charge sector which is the well known Halperin’s (111) wavefunction and the other in the spin sector which is non-trivial at any finite $d$ due to the gapless mode. So the total groundstate wavefunction differs from the well known (111) wavefunction at any finite $d$. In addition to commonly known multiplicative factors, the quasi-hole and a pair of quasi-holes wavefunctions also contain non-trivial normalization factors multiplying the correct ground state wavefunction. We expect that the quasi-hole and pair wave function not only has logarithmically divergent energy, well localized charge distribution, but also correct interlayer correlations. All the distance dependencies in all the wavefunctions are encoded in the spin part of the ground state wavefunction. The instability encoded in the spin part of the groundstate wavefunction leads to the pseudo-spin density wave formation proposed previously by one of the authors. Some subtleties related to the Lowest Landau Level (LLL) projection of the CB theory are also noted.

I. INTRODUCTION

The Wavefunction approach has been very successfully applied to study single layer quantum Hall (SLQH) systems at Laughlin series $\nu = \frac{1}{2k+1}$ [1] and Jain’s series at $\nu = \frac{p}{2kp+1}$ [2]. One of main reasons for the success of the wavefunction approach in SLQH is that there is a gap in the bulk, suitable wavefunctions [1,2] can describe both the groundstate and low energy excitations quite accurately. Its accuracy can be checked easily by exact diagonalization in a finite size system whose size need only go beyond a few magnetic length. Spherical geometry can be used to get rid of edge state effects quite efficiently. In general, trial wave function approach is a very robust approach to study SLQH and multi-components systems as long as there is a gap in the bulk. The gap protects the many properties of the system such as charge density distributions and energies from being sensitive to some subtle details of wavefunctions.

However, the situation could be completely different in Spin-polarized Bilayer Quantum Hall system (BLQH) at total filling factor $\nu_T = 1$. This system has been under enormous experimental and theoretical investigations over the last decade [3]. When the interlayer separation $d$ is sufficiently large, the bilayer system decouples into two separate compressible $\nu = 1/2$ layers [4]. However, when $d$ is smaller than a critical distance $d_{c1}$, even in the absence of interlayer tunneling, the system undergoes a quantum phase transition into a novel spontaneous interlayer coherent incompressible phase which is an excitonic superfluid state (ESF) in the pseudospin channel [5-7]. In [10], Halperin proposed the (111) wavefunction to describe the groundstate of the ESF state. Starting from the (111) wavefunction, using various methods, several authors [11] discovered a Neutral Gapless Mode (NGM) with linear dispersion relation $\omega \sim \nu k$ and that there is a finite temperature Kosterlitz-Thouless (KT) phase transition associated with this NGM.

By treating the two layer indices as two pseudo-spin indices, Girvin, Macdonald and collaborators mapped the bilayer system into a Easy Plane Quantum Ferromagnet (EPQFM) [3,12] (which is equivalent to the ESF) and explored many rich and interesting physical phenomena in this system.

As first pointed out in [12], the (111) wavefunction may not be qualitatively good at finite $d$, because (111) is a broken symmetry state in a direction in $XY$ plane of isotropic ferromagnet at $d = 0$ instead of a easy-plane ferromagnet at finite $d$. The NGM is a hallmark of the interlayer coherent Quantum Hall state. Its existence is expected to dramatically alter the properties of the wavefunctions of the ground state, quasi-hole and quasi-particle. In [13], G. S. Jeon and one of the authors studied properties of essentially all the known trial wavefunctions of ground state and excitations in bilayer quantum Hall systems at the total filling factor $\nu_T = 1$. The results indicated that qualitatively good trial wave functions for the ground state and the excitations of the interlayer coherent bilayer quantum Hall system at finite $d$ are still not available and searching for them remains an important open problem.
Specifically, they investigated the properties of the quasi-hole wave function, meron wavefunction and a pair of meron wavefunction built on (111) state which have superscripts ”prime” in this paper:

$$\Psi^{'\text{qh}}_{111} = \prod_{i} (z_i) \Psi_{111}, \quad \Psi^{'\text{meron}}_{111} = \prod_{i} \frac{z_i}{|z_i|} \Psi_{111}, \quad \Psi^{'\text{pair}}_{111} = \prod_{i} (z_i - z_0) \prod_{i} (w_i - w_0) \Psi_{111}$$

(1)

where $\Psi_{111}$ is the Halperin’s (111) wavefunction:

$$\Psi_{111}(z, w) = \prod_{i<j} (z_i - z_j) \prod_{i<j} (w_i - w_j) \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} (z_i - w_j) \exp\left(-\frac{1}{4d_0} \sum_{i} |z_i|^2 + |w_i|^2 \right)$$

(2)

where $N_1 = N_2 = N$ in the balanced case and $z$ and $w$ are the coordinates in layer 1 and layer 2 respectively. In the following, we suppress the exponential factor.

These quasi-hole and meron wavefunctions differ only by ”normalization factors”. As shown in [13], the normalization factor $|z_i|$ is accurate only at long distance limit $|z_i| \to \infty$ limit. Near the origin, the ”meron” and the ”quasi-hole” have similar behaviors. Normalization factors have been shown not to be important in single layer Quantum Hall (SLQH) systems. However, as shown in [13], they make a dramatic difference in BLQH. Although the smallest meron has a localized charge 1/2 and logarithmically divergent energy, the charge of the quasi-hole excitation extends over the whole system and its energy also diverges linearly as the area of the system size. This indicates that the quasi-hole wavefunction is not a good trial wavefunction for any low energy excitations. The meron wavefunction is not a good trial wavefunction either, because it ignores the strong interlayer correlations [13]. It was found the energy of the possible wavefunction of a pair of merons in Eqn.1 increases quadratically $\sim |z_0 - w_0|^2$ instead of logarithmically as the separation of the pair increases. All the results achieved in [13] indicate that qualitatively good trial wave functions in the interlayer coherent bilayer quantum Hall system at finite $d$, both for ground state and excitations, are still unknown and searching for them remains an important open problem. so the wavefunction approach to BLQH is much more difficult and far less powerful in BLQH than in SLQH. Fortunately, effective theory approaches such as EPQFM approach [12,3] and Composite Boson theory approach [11,12,14–16] circumvent this difficulty associated with the unknown wavefunction at any finite $d$ and are very effective to bring out most of the interesting phenomena in the pseudo-spin sector in this system. In fact, all these effective theories start from the insights gained from Halperin’s (111) wavefunction which is exact at $d = 0$.

In a series of papers [14–16], one of the authors developed a systematic composite boson approach to study balanced and im-balanced Bi-Layer Quantum Hall systems in rather details. The theory puts spin and charge degree freedoms in the same footing, explicitly bring out the spin-charge connection and classify all the possible excitations in a systematic way. Then he pushed the theory further to understand novel phases and quantum phase transitions as the distance between the two layers is changed. He found that starting from the well studied excitonic superfluid (ESF) state, as distance increases, the instability driven by magneto-roton minimum collapsing at a finite wavevector in the pseudo-spin channel leads to the formation of a pseudo-spin density wave (PSDW) at some intermediate distances. He constructed a quantum Ginsburg-Landau theory to study the transition from the excitonic superfluid (ESF) to the PSDW and analyze in detail the properties of the PSDW. He showed that a square lattice is the favorite lattice and the correlated hopping of vacancies in the active and passive layers in the PSDW state leads to very large and temperature dependent drag observed in the experimental. In the presence of disorders, the properties of the PSDW are consistent with all the experimental observations [6,8] in the intermediate distances. Further experimental implications of the PSDW are given. Then he extended the Composite Boson theory to study slightly im-balanced BLQH. In the global $U(1)$ symmetry breaking excitonic superfluid side, as the imbalance increases, the system supports continuously changing fractional charges. In the translational symmetry breaking PSDW side, there are two quantum phase transitions from the commensurate PSDW to an in-commensurate PSDW and then to the excitonic superfluid state. These results explained the experimental observations in [9] very nicely. The author found that the theory can be easily extended to study some additional interesting phenomena in trilayer quantum Hall systems [17]. It was concluded in [16] that field theory approaches are much more powerful in BLQH than in SLQH.

Obviously, the CB theory circumvent this difficulty associated with the unknown wavefunction at any finite $d$ and is used to achieve all these interesting and important results at two different distance regimes without knowing the precise wavefunctions for the ground state and excitations. It would be interesting to use the CB theory to address the important and outstanding problem avoided in [13] and in all the other pervious work that finding the good ground state and low energy excited wavefunction for BLQH at any finite $d$. In SLQH, the CB theory developed in [18] was used to re-derive the already well known Laughlin’s wavefunctions for ground state and quasi-hole at $\nu = \frac{2k+1}{2k}$.
previously, the gap in the bulk protects the properties of the system such as charge density distributions and energies from being sensitive to some subtle details of wavefunctions. Here we are facing a more difficult and interesting task: to derive these unknown wavefunctions at finite \(d\).

The rest of the paper is organized as the following. In Sec. II, in order to be self-contained, we review briefly the CB approach and its dual action developed in [16] which are needed to derive the wavefunctions in the following sections. In Sec. III, using the formalism presented in Sec. II, we derive the ground state wavefunction which is different from the \((111)\) wavefunction at any finite \(d\). In Sec. IV, using the dual action presented in Sec. II, we derive the quasi-hole wavefunction and compare it with the "quasi-hole" and "meron" wavefunction built on \((111)\) wavefunction listed in Eqn.1. In Sec. V, we derive a pair of meron wavefunction with charge 1 and compare it with the "pair meron" wavefunction built on \((111)\) wavefunction listed in Eqn. 1. In Sec. VI, we look at the instability in the ground state wavefunction as distance approaches \(d_{e1}\). Finally, we reach conclusions in Sec. VII. Some caveats related to the Lowest Landau Level (LLL) projection of the wavefunctions are also pointed out. We note that there is also an alternative approach in [19].

II. COMPOSITE BOSON APPROACH AND ITS DUAL ACTION IN BLQH

In this section, we briefly review the formalism developed in [16] which is needed to derive the wavefunctions in the following sections. Consider a bi-layer system with \(N_1\) ( \(N_2\) ) electrons in left ( right ) layer and with interlayer distance \(d\) in the presence of magnetic field \(\vec{B} = \nabla \times \vec{A}\) (Fig.1):

\[
H = H_0 + H_{\text{int}}
\]

\[
H_0 = \int d^2x c^\dagger_\alpha(\vec{x}) \frac{-i\hbar \vec{\nabla} + \frac{e}{2} \vec{A}(\vec{x})}{2m} c_\alpha(\vec{x})
\]

\[
H_{\text{int}} = \frac{1}{2} \int d^2x'^2 \delta \rho_\alpha(\vec{x}) V_{\alpha \beta}(\vec{x} - \vec{x}') \delta \rho_\beta(\vec{x}')
\]

where electrons have bare mass \(m\) and carry charge \(-e\), \(c_\alpha, \alpha = 1, 2\) are electron operators in top and bottom layers, \(\delta \rho_\alpha(\vec{x}) = c^\dagger_\alpha(\vec{x}) c_\alpha(\vec{x}) - n_\alpha\), \(\alpha = 1, 2\) are normal ordered electron densities on each layer. The intralayer interactions are \(V_{11} = V_{22} = e^2/\epsilon r\), while interlayer interaction is \(V_{12} = V_{21} = e^2/\epsilon \sqrt{r^2 + d^2}\) where \(\epsilon\) is the dielectric constant.

Performing a singular gauge transformation \([14,16]\):

\[
\phi_\alpha(\vec{x}) = e^i \int d^2 x' \phi(\vec{x} - \vec{x}') \rho(\vec{x}') c_\alpha(\vec{x}')
\]

where \(\phi(\vec{x} - \vec{x}') = \text{arg}(\vec{x} - \vec{x}')\) is the angle between the vector \(\vec{x} - \vec{x}'\) and the horizontal axis. \(\rho(\vec{x}) = c^\dagger_1(\vec{x}) c_1(\vec{x}) + c^\dagger_2(\vec{x}) c_2(\vec{x})\) is the total density of the bi-layer system. Note that this transformation treats both \(c_1\) and \(c_2\) on the same footing. This is reasonable only when the distance between the two layers is sufficiently small. It can be shown that \(\phi_\alpha(\vec{x})\) satisfies all the boson commutation relations. We can transform the Hamiltonian Eqn.3 into the Lagrangian in Coulomb gauge:

\[
\mathcal{L} = \phi^\dagger_\alpha(\partial_r - ia_0)\phi_\alpha + \phi^\dagger_\alpha(\vec{x}) \frac{-i\hbar \vec{\nabla} + \frac{e}{2} \vec{A}(\vec{x}) - \hbar \vec{\alpha}(\vec{x})}{2m} \phi_\alpha(\vec{x})
\]

\[
+ \frac{1}{2} \int d^2 x' \delta \rho(\vec{x}) V_+(\vec{x} - \vec{x}') \delta \rho(\vec{x})
\]

\[
+ \frac{1}{2} \int d^2 x' \delta \rho_-(\vec{x}) V_-(\vec{x} - \vec{x}') \delta \rho_-(\vec{x}) - \frac{i}{2\pi} a_0(\nabla \times \vec{a})
\]

where \(V_\pm = \frac{V_{11} \pm V_{12}}{2}\) and \(V_{11} = V_{22} = \frac{2\pi e^2}{\epsilon q}, V_{12} = \frac{2\pi e^2}{\epsilon q} e^{-qd}\). The Chern-Simon gauge field is \(\vec{\alpha} = \int d^2 \rho \phi(\vec{x} - \vec{x}') \rho(\vec{x}')\).

In Coulomb gauge, integrating out \(a_0\) leads to the constraint: \(\nabla \times \vec{a} = 2\pi \phi^\dagger \phi\). Note that if setting \(V_- = 0\), then the above equation is identical to a single layer with spin in the absence of Zeeman term, so the Lagrangian has a \(SU(2)\) pseudo-spin symmetry. The \(V_-\) term breaks the \(SU(2)\) symmetry into \(U(1)\) symmetry. In the BLQH at finite \(d\), \(V_- > 0\), so the system is in the Easy-plane limit.

We can write the two bosons in terms of magnitude and phase
\[ \phi_a = \sqrt{\rho_a + \delta \rho_a e^{i\theta_a}} \]  

(6)

The boson commutation relations imply that \(|\delta \rho_a(x), \theta_b(x)\rangle = i\hbar \delta_{ab} \delta(x - x')\). After absorbing the external gauge potential \(\vec{A}\) into \(\vec{a}\), we get the Lagrangian in the Coulomb gauge:

\[
\begin{align*}
\mathcal{L} &= i\hbar \rho^+ \left( \frac{1}{2} \partial_\mu \theta^+ - a_\mu \right) + \frac{\rho}{2m} \left[ \frac{1}{2} \nabla \theta_+ + \frac{1}{2} (\nu_1 - \nu_2) \nabla \theta_+ - \vec{a} \right] + \frac{1}{2} \delta \rho^+ V_+(\vec{q}) \delta \rho^+ - \frac{i}{2\pi} a_0 (\nabla \times \vec{a}) \\
&+ \frac{i}{2} \rho^- \partial_\kappa \theta^- + \frac{\rho}{2m} \left[ \frac{1}{2} \nabla \theta_- \right]^2 + \frac{1}{2} \delta \rho^- V_- \left( \vec{q} \right) \delta \rho^- - h_z \delta \rho^- 
\end{align*}
\]

(7)

where \(f = 4\nu_1\nu_2\) which is equal to 1 at the balanced case and \(h_z = V_- \partial_{-} = V_-(\hat{p}_1 - \hat{p}_2)\) plays a role like a Zeeman field.

Performing the duality transformation on Eqn.7 leads to the dual action in terms of the vortex degree of freedoms \(J^\pm_\mu\) where \(A_\mu\) into \(\vec{a}\), we find the corresponding Hamiltonian in the charge sector:

\[
\begin{align*}
\mathcal{H}_c &= \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\mu J^\pm_\nu \partial_\lambda J^\mp_\mu + \frac{m}{2\rho f} \left( \partial_\alpha b^\pm_\alpha - \partial_0 b^\pm_0 \right)^2 + \frac{1}{2} (\nabla \times \vec{b}^\pm)^2 \mathcal{V}_+(\vec{q}) (\nabla \times \vec{b}^\pm) \\
&+ \frac{1}{2} \rho^- \left( \partial_\beta \theta_+ - \partial_0 \theta_+ \right) \left( \partial_\beta \theta_+ - \partial_0 \theta_+ \right) - 1 \frac{1}{4m} \theta_+ \left( \vec{q} \right) \theta_+ \left( \vec{q} \right) 
\end{align*}
\]

(8)

where \(A^\pm_\mu = A^1_\mu \pm A^2_\mu\) are the two source fields. It is useful to stress that the dual CS term only appears in the charge sector.

For simplicity, we only consider the balanced case. Putting \(\nu_1 = \nu_2 = 1/2\) and \(h_z = 0\) into Eqn.7, we get the Lagrangian in the balanced case where the symmetry is enlarged to \(U(1)_L \times U(1)_G \times Z_2\).

### III. GROUND STATE WAVEFUNCTION

In this section, we derive the ground state wavefunction from the formalism reviewed in the last section. From Eqn.7, we can find the corresponding Hamiltonian in the charge sector:

\[
\mathcal{H}_c = \frac{1}{2} \sum_q \left[ 4 \Pi_+ (-\vec{q}) (V_+ (\vec{q}) + \frac{4\pi^2 \rho}{q^2} \Pi_+ (\vec{q}) + \frac{\rho q^2}{4m} \theta_+ (-\vec{q}) \theta_+ (\vec{q}) \right] 
\]

(9)

where \(\Pi_+ (\vec{q}) = \delta \rho^+ / 2\) and \([\theta_+ (\vec{q}), \Pi_+ (\vec{q})] = i\hbar \delta (\vec{q} + \vec{q}')\).

Representing \(\theta_+ (\vec{q})\) by \(-i \frac{\rho}{\hbar} \Pi_+ (-\vec{q})\), in the long wavelength limit, neglecting the Coulomb interaction \(V_+ (\vec{q}) \sim 1/q\) which is less singular than \(1/q^2\), we find the ground state wavefunction in the charge sector:

\[
\Psi^{b}_{\phi_0} [\Pi_+ (\vec{q})] = \exp \left[ - \frac{1}{2} \sum_q \frac{2\pi}{q^2} \delta \rho_+ (\vec{q}) \delta \rho_+ (\vec{q}) \right] 
\]

(10)

In the following, we use \(\vec{x}\) to stand for the complex coordinate \(x + iy\). Using \(\delta \rho_+ (x) = \sum \delta (x - z_i) + \sum \delta (x - w_i) - \rho\) and transforming to the position space, it is simply the modulus of the (111) wavefunction in the balanced case \(\Psi^{b}_{\phi_0} = |\Psi_{111}|\). This is the wavefunction in the bosonic picture. In order to get the wavefunction in the original fermionic picture, we need to perform the inverse of the SGT in Eqn.4 on the bosonic wavefunction. In the first quantization form, the inverse is:

\[
U_0 = \prod_{i < j} \frac{(z_i - z_j)}{|z_i - z_j|} \prod_{i < j} \frac{(w_i - w_j)}{|w_i - w_j|} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} |z_i - w_j|
\]

(11)

Performing the inverse transformation on the modulus leads to the (111) wavefunction in the fermionic coordinates:

\[
\Psi_{\phi_0} = U_0 \Psi^{b}_{\phi_0} = \Psi_{111} (z, w)
\]

(12)

In contrast to the SLQH, there is also an additional pseudo-spin sector in the BLQH which contains the most interesting physics. From Eqn.7, we can find the corresponding Hamiltonian in the spin sector is:
\[ \mathcal{H}_s = \frac{1}{2} \sum_q [4V \Pi_\cd(-\vec{q})\Pi_\cd(-\vec{q}) + \frac{\rho Eq^2}{4} \theta_\cd(-\vec{q})\theta_\cd(-\vec{q})] \]  \hspace{1cm} (13)

where \( \Pi_\cd(\vec{q}) = \delta \rho / 2 \) and \( [\theta_\cd(-\vec{q}), \Pi_\cd(-\vec{q})'] = i \hbar (\vec{q} + \vec{q}') \); \( \rho E = \rho/m \) is the spin stiffness [21]. At small \( q \), \( V_\cd(q) = a - bq + c q^2 \) [21] where \( a \sim d^2, b \sim d^2 \) and \( c \) remains a constant at small distances [12,16]. It is important to stress that this form of \( V_\cd(q) \) has the shape displayed in the Fig.1, it not only has a phonon part near \( q = 0 \), also has a roton part near \( q = q_0 \sim 1/l_B \) where \( l_B \) is the magnetic length.

Representing \( \theta_\cd(-\vec{q}) \) by \( -i \sum \frac{\vec{d}}{\vec{q} - \vec{q}} \), we find the ground state wavefunction in the spin sector:

\[ \Psi_{s0}[\Pi_\cd(-\vec{q})] = \exp[-\frac{1}{2} \sum_q \frac{\sqrt{V_\cd(q)/\rho E}}{q} \delta \rho_\cd(-\vec{q})\delta \rho_\cd(-\vec{q})] \]  \hspace{1cm} (14)

It is easy to see that the above equation make sense only when \( V_\cd(q) \) is positive for all \( q \).

At \( d = 0, a = b = 0, V_\cd(q) = c q^2 \), then Eqn.14 becomes:

\[ \Psi_{s0}[\Pi_\cd(-\vec{q}); d = 0] = \exp[-\frac{1}{2} \sum_q \sqrt{c/\rho E} \delta \rho_\cd(-\vec{q})\delta \rho_\cd(-\vec{q})] = \text{const.} \]  \hspace{1cm} (15)

At any finite distance, as long as \( d < d_{c1} \sim l_B \) in the Fig.1, so the roton has a large gap, we can neglect the contributions from the roton part in Fig.1 and only focus on the phonon contributions. In the long wavelength limit \( q \ll q_0 \sim 1/l_B, V_\cd(q) \to a \sim d^2 \). Using \( \delta \rho_\cd(x) = \sum \delta(x - z_i) - \sum \delta(x - w_i) \) and transforming to the coordinate space, it is

\[ \Psi_{s0}(z, w) = \exp[-2\sqrt{1/\rho E}\sum_{i<j} \frac{d}{|z_i - z_j|} - \sum_{i,j} \frac{d}{|z_i - w_j|} + \sum_{i<j} \frac{d}{|w_i - w_j|}] \]  \hspace{1cm} (16)

Obviously, the above equation only holds at small distance \( d < d_{c1} \) and in the long distance limit \( |z_i - w_j| \gg l_B \) [22].

The total wavefunction is:

\[ \Psi_0(z, w) = \Psi_{111}(z, w)\Psi_{s0}(z, w) \]  \hspace{1cm} (17)

It is easy to see that the total wavefunction coincides with the (111) wavefunction only in \( d \to 0 \) limit. At any finite \( d \), it has an extra factor from the gapless spin sector \( \Psi_{s0}(z, w) \). Note that this extra spin factor Eqn.16 is not in the LLL, this should not be too worrisome, because similar to the meron wavefunction listed in Eqn.1, modulus of the coordinates could appear in the long distance limit where Eqn.16 hold.

IV. QUASI-HOLE WAVE FUNCTIONS

By inserting one static vortex at the origin in layer 1 by setting \( J_0^{u,v} = J_0^{-u,v} = \delta(x) \) or layer 2 by setting \( J_0^{u,v} = -J_0^{-u,v} = \delta(x) \), from the dual action Eqn.8, we will first try to derive the quasi-hole wavefunction and compare this quasi-hole wavefunction with the known "quasi-hole" wavefunction and "meron" wavefunction written down in [13].

In order to derive the quasi-hole wavefunction, we have to resort to the dual action Eqn.8 in the balanced case where the last term vanishes. Setting the two sources \( A^\pm_{\mu} = 0 \), in the Coulomb gauge \( \nabla \cdot b^\pm_{\alpha} = 0 \), Eqn.8 becomes:

\[ \mathcal{L}_d = -i2\pi b^+_{\alpha} \epsilon_{\alpha\beta} \partial_\beta b^-_{\beta} + i \pi b^+_{\alpha} J^{+\alpha}_0 + i \pi b^+_{\alpha} J^{\alpha}_0 + \frac{m}{2\rho} (\partial_\alpha b^+_{\alpha})^2 + \frac{m}{2\rho} (\partial_\alpha b^-_{\alpha})^2 + \frac{1}{2} (\nabla \times \vec{b}^+) V_+(\vec{q}) (\nabla \times \vec{b}^+) \] 

\[ + i \pi b^-_{\alpha} J^{-\alpha}_0 + i \pi b^-_{\alpha} J^{\alpha}_0 - \frac{m}{2\rho} (\partial_\alpha b^-_{\alpha})^2 + \frac{m}{2\rho} (\partial_\alpha b^+_{\alpha})^2 + \frac{1}{2} (\nabla \times \vec{b}^-) V_-(\vec{q}) (\nabla \times \vec{b}^-) \]  \hspace{1cm} (18)

Note the absence of CS term in the spin sector.

We only consider static vortices, so \( J^{+\alpha}_0 = J^{-\alpha}_0 = 0 \). Integrating out \( b^+_{\alpha} \) and \( b^-_{\alpha} \) and transforming into the coordinate space lead to:

\[ \mathcal{L}_d = -\frac{1}{4\pi} \frac{\rho}{2m} (\pi J^+_{\alpha}_0 - 2 \pi \epsilon_{\alpha\beta} \partial_\beta b^-_{\beta}) ln|x - y| (\pi J^{-\alpha}_0 - 2 \pi \epsilon_{\alpha\beta} \partial_\beta b^+_{\beta}) + \frac{m}{2\rho} (\partial_\alpha b^+_{\alpha})^2 + \frac{1}{2} (\nabla \times \vec{b}^+) V_+(\vec{q}) (\nabla \times \vec{b}^+) \] 

\[ + \frac{m}{2\rho} (\partial_\alpha b^-_{\alpha})^2 - \frac{1}{4\pi m} \frac{\rho}{4m} (\pi J^+_{\alpha}_0) ln|x - y| (\pi J^{-\alpha}_0) + \frac{1}{2} (\nabla \times \vec{b}^-) V_-(-\vec{q}) (\nabla \times \vec{b}^-) \]  \hspace{1cm} (19)
The corresponding Hamiltonian is:

\[
\mathcal{H} = \frac{1}{2m}b^2_{q+}(-\vec{q})\theta_{+}(\vec{q}) + \frac{\rho}{2m}(\pi J_{0}^{++}(\vec{q}) - 4\pi \Pi_{+}(-\vec{q})) + \frac{1}{q^2}(\pi J_{0}^{++}(\vec{q}) - 4\pi \Pi_{+}(\vec{q})) + 2\Pi_{+}(-\vec{q})V_{+}(\vec{q})\Pi_{+}(\vec{q})
\]

\[
+ \frac{1}{2m}b^2_{q-}(-\vec{q})\theta_{-}(\vec{q}) + \frac{\rho}{2m}(\pi J_{0}^{-+}(\vec{q}) - 4\pi \Pi_{-}(\vec{q})) + \frac{1}{q^2}(\pi J_{0}^{-+}(\vec{q}) - 4\pi \Pi_{-}(\vec{q})) + \frac{1}{2}\Pi_{-}(-\vec{q})V_{-}(\vec{q})\Pi_{-}(\vec{q})
\]

(20)

where \([\theta_{\pm}(\vec{q}), \Pi_{\pm}(\vec{q})] = i\hbar\delta(\vec{q} + \vec{q}').\)

From the above Hamiltonian in the bosonic representation, we can see the charge sector and spin sector remain "decoupled" [20]. Due to the absence of the C-S term in the spin sector, the inserted vortex only shifts the total density variable in the charge sector, but does not couple to the relative density in the spin sector, so the Hamiltonian in the spin sector remains the same as the ground state one Eqn.13. All the effects of the inserted vortex are encoded in the charge density variable in the charge sector, but does not couple to the relative density in the spin sector, so the Hamiltonian "decoupled" [20].

Due to the absence of the C-S term in the spin sector, the inserted vortex only shifts the total "quasi-hole" and the "meron" carry total charge 1 in the layer 1 in the charge sector and spin sector remain "decoupled" [20]. Again, neglecting the Coulomb interaction \(V_{\pm}(\vec{q})\) in the long wavelength limit, we find that the wavefunction in the charge sector is:

\[
\Psi_{\text{qh}}^b = \exp\left[\frac{1}{2} \sum_{\vec{q}} \left( \frac{1}{2} J_{0}^{++}(\vec{q}) - \delta \rho_{+}(\vec{q}) \right) \right]
\]

(21)

Transforming to the coordinate space and setting \(J_{0}^{++}(x) = J_{0}^{+-}(x) = -\delta(x)\) [20] lead to:

\[
\Psi_{\text{qh}}^b = \exp\left[\frac{1}{2} \int dx dy \left( \frac{1}{2} \delta(\vec{x}) + \left( \sum_i \delta(\vec{x} - \vec{z}_i) + \delta(\vec{x} - \vec{w}_i - \vec{p}) \right) \right)\ln|x - y| \right]
\]

(22)

The SGT for the quasi-hole could be different from that for the groundstate. If one inserts a vortex at the origin at the layer 1 in the boson Lagrangian Eqn.5, in order to recover the original electronic Hamiltonian Eqn.3, \(U_0\) in Eqn.11 is needed to remove the C-S term, an additional SGT \(U_{v1}\) is needed to remove the effects of the inserted vortex. In the first quantization, it is easy to show that [23]:

\[
U_{v1} = e^{\sum_i \text{arg} \frac{z_i}{|z_i|}} = e^{\sum_i \ln \frac{z_i}{|z_i|}} = \prod_i \frac{z_i}{|z_i|}
\]

(23)

The total SGT for the quasi-hole at the layer 1 is \(U_{\text{qh}} = U_0U_{v1}\).

Performing the SGT on Eqn.22, we get the quasi-hole wavefunction,

\[
\Psi_{\text{qh}}(z, w) = \left( \prod_{i=1}^{N_i} \frac{z_i}{|z_i|} \right) \prod_{i=1}^{N_i} \left| w_i \right|^2 \Psi_0(z, w)
\]

(24)

where \(\Psi_0(z, w)\) is the ground state wavefunction Eqn.17 and \(N_1 = N_2 = N\) in the balanced case. Note that there is no singularity at the origin.

Note that even in the \(d \rightarrow 0\) limit, Eqn.24 differs from both the "quasi-hole" and the "meron" wavefunction listed in Eqn.1. This should not cause any problem. It is known that \((111)\) wavefunction is the exact wave function in the \(d \rightarrow 0\) limit. But both both the "quasi-hole" and the "meron" wavefunction listed in Eqn.1 make sense only at finite distances. It is known that the lowest energy excitation at \(d = 0\) is a skyrmion carrying charge 1, while the "quasi-hole" and the "meron" carry total charge 1/2, so they are not valid excitations anymore at \(d = 0\). We conclude the quasi-hole Eqn.24 make sense only at finite \(d\). It is not interesting to take \(d \rightarrow 0\) limit to this equation [24].

Compared to the "quasi-hole" and the "meron" wavefunction listed in Eqn.1, we can see that there are two modifications in the Eqn.24: (1) The ground state wavefunction is the correct one Eqn.17 instead of the \((111)\) wavefunction. (2) The prefactor is different from both the "quasi-hole" and the "meron". We expect this prefactor takes care of the strong interlayer correlations. All the wavefunctions in Eqn.1 are built upon the \((111)\) wavefunction. As suggested in [13] and explicitly shown in this paper, \((111)\) wavefunction is not even qualitatively correct at any finite \(d\). As shown in [13], although prefactors are not important in SLQH due to the gap in the bulk, they maybe crucial in BLQH due to the gapless mode in the interlayer correlations. The two factors maybe responsible for the
"quasi-hole"'s charge distribution spreading over the whole system and its energy diverges linearly with the area of the system. Although, the "meron" wavefunction’s energy is only logarithmically divergent, it ignores the strong interlayer correlations, so it is not a good trial wavefunction either. We propose that the quasi-hole wave function Eqn.24 not only has logarithmically divergent energy, well localized charge distribution, but also correct interlayer correlations.

V. A PAIR OF QUASIHOLE EXCITATIONS WITH CHARGE 1

Now we put two vortices into the BLQH system. One is on the top layer at $z_0$ and the other is on the bottom layer at $w_0$. The only change from the quasi-hole calculation in the last section is $J_0^\pm = \delta(x - z_0) \pm \delta(x - w_0)$ [20]. Again, due to the lack of the CS term in the spin sector, the Hamiltonian in the spin sector remains the same as the ground state one Eqn.13, so the wavefunction is not affected at all by the insertion of the two vortices and remains the same as the ground state one in the spin sector Eqn.14. All the effects of the inserted two vortices are encoded in Eqn.25.

$$\Psi_{z_0} = \exp\left(\frac{1}{2} \int d\vec{x} d\vec{y} \right) [\frac{1}{2}(\delta(\vec{x} - z_0) + \delta(\vec{x} - w_0)) + (\sum_i \delta(\vec{x} - z_i) + \delta(\vec{x} - w_i) - \bar{\rho})] |n\rangle|x - y\rangle$$

$$\Psi_{w_0} = \exp\left(\frac{1}{2} \int d\vec{x} d\vec{y} \right) [\frac{1}{2}(\delta(\vec{y} - z_0) + \delta(\vec{y} - w_0)) + (\sum_i \delta(\vec{y} - z_i) + \delta(\vec{y} - w_i) - \bar{\rho})]$$

$$= \prod_i \left|z_i - z_0\right|^\frac{1}{2} \prod_i \left|z_i - w_0\right|^\frac{1}{2} \prod_i \left|w_i - z_0\right|^\frac{1}{2} \prod_i \left|w_i - w_0\right|^\frac{1}{2}$$

(25)

It is easy to see that in the bosonic picture, the above wavefunction in the charge sector is symmetric under $z_i \leftrightarrow w_i$ or $z_0 \leftrightarrow w_0$ separately. This is under expectation, because the two layers are completely symmetric in the charge sector.

Just like deriving $U_{v1}$ for the quasi-hole, we can get $U_{v2}$ for a pair of vortices inserted at $z_0$ in top layer and $w_0$ at the bottom layer:

$$U_{v2} = \prod_i \left(\frac{z_i - z_0}{|z_i - z_0|}\frac{w_i - w_0}{|w_i - w_0|}\right)$$

(26)

The total SGT for the meron pair is $U_{pair} = U_{0}U_{v2}$.

Performing the SGT on Eqn.25, we get the wavefunction for a pair of quasihole:

$$\Psi_{pair}(z, w; z_0, w_0) = \prod_i (z_i - z_0)(w_i - w_0) \prod_i \left(\frac{z_i - w_0}{|z_i - w_0|}\left|z_i - z_0\right|\left|w_i - z_0\right|\left|w_i - w_0\right|\right)^{1/2} \Psi_0(z, w)$$

(27)

where $\Psi_0(z, w)$ is the ground state wavefunction Eqn.17 and $N_1 = N_2 = N$ in the balanced case. Note that the pair wavefunction is not symmetric under $z_i \leftrightarrow w_i$ or $z_0 \leftrightarrow w_0$ separately anymore. This is because the SGT $U_{v2}$ Eqn.26 is not. Of course, it is still symmetric under $z_i \leftrightarrow w_i$, $z_0 \leftrightarrow w_0$ simultaneously.

If we insert the two vortices at the same point, namely, putting $z_0 = w_0 = 0$ in the above equation, as expected, we get:

$$\Psi_{pair}(z, w, 0) = \prod_i z_i w_i \Psi_0(z, w)$$

(28)

This corresponds to insert a single vortex through the two layers. In contrast to the quasi-hole excitation Eqn.24, Eqn.28 carries charge 1 and remains a valid wavefunction even at $d = 0$. Indeed, in the $d \rightarrow 0$ limit, it recovers the "meron pair" wavefunction listed in Eqn.1. If one splits the whole vortex, it will evolve into the pair wavefunction Eqn.27.

The "pair meron" wavefunction built on (111) is listed in 1( essentially Eqn.(110) in Ref.10 ). It was shown that its energy $E_{pair} \sim |z_0 - w_0|^2$ instead of logarithmically as naively expected, because the charges are extended between $z_0$ and $w_0$. Similar to the quasi-hole wavefunction Eqn.24, there are two modifications in the Eqn.27: (1) The ground state wavefunction is the correct one Eqn.17 (2) The prefactor is different. We expect this prefactor takes care of the strong interlayer correlations between the two vortices, the pair wave function not only has logarithmically divergent energy, well localized charge distribution, but also correct interlayer correlations.
VI. INSTABILITY IN THE WAVEFUNCTION AS THE DISTANCE INCREASES

When the distance is sufficiently small, the BLQH is in the ESF phase, we expect the ground state, quasi-hole and pair wavefunctions Eqs.17, 24, 27 only hold in the ESF phase. When the distance becomes sufficiently large, the two layers become two weakly coupled Fermi liquid (FL) layers. All these wavefunctions completely break down. New set of wavefunctions are needed. Although the ESF phase and FL phase at the two extreme distances are well established, the picture of how the ESF phase evolves into the two weakly-coupled FL states was not clear, namely, the nature of the intermediate phase at \( d_{c1} < d < d_{c2} \) was still under debate. Recently, starting from the well studied excitonic superfluid (ESF) state, as distance increases, one of the authors found [14] that the instability driven by magneto-roton minimum collapsing at a finite wavevector in the pseudo-spin channel leads to the formation of a pseudo-spin density wave (PSDW) at some intermediate distances. He constructed a quantum Ginsburg-Landau theory to study the transition from the ESF to the PSDW and analyze in detail the properties of the PSDW. He showed that a square lattice is the favorite lattice.

As shown in [14], it is the original instability in \( V_{-}(q) = a - bq + cq^2 \) which leads to the magneto-roton minimum in the Fig.1a. By looking at the two conditions \( V_{-}(q)|_{q=q_0} = 0 \) and \( \frac{dV_{-}(q)}{dq}|_{q=q_0} = 0 \), it is easy to see that \( V_{-}(q) \) indeed has the shape shown in Fig.1b. When \( b \sim d^2 < b_c = 2\sqrt{ac} \sim d \), the minimum of \( V_{-}(q) \) at \( q = q_0 = \sqrt{a/c} \sim d \) has a gap, the system is in the ESF state, this is always the case when the distance \( d \) is sufficiently small. However, when \( b = b_c \), the minimum collapses and \( S(q) \) diverges at \( q = q_0 \), which signifies the instability of the ESF to the pseudo-spin density wave (PSDW) formation. When \( b \sim d^2 > b_c = 2\sqrt{ac} \sim d \), the minimum drops to negative, the system gets to the PSDW state, this is always the case when the distance \( d \) is sufficiently large.

![Fig.1: The zero temperature phase diagram in the balanced case as the distance between the two layers increases. ESF where \( \langle \psi \rangle \neq 0, \langle n_{G}^{-} \rangle = 0 \) stands for excitonic superfluid, PSDW where \( \langle \psi \rangle = 0, \langle n_{G}^{-} \rangle \neq 0 \) stands for pseudo-spin density wave phase, FL stands for Fermi Liquid. (a) Energy dispersion relation \( \omega(q) \) in these phases. (b) \( V_{E}(q) \) in these phases. The cross in the PSDW means the negative minimum value of \( V_{E}(q) \) is replaced by the PSDW. The two order parameters were defined in [16]. In reality, the instability happens before the minimum collapses.](image)

We can easily see the instability from the ground state wavefunction Eqn.17. The only distance dependence in the charge sector is encoded in the \( V_{+}(q) \) in the bosonic Hamiltonian in the charge sector Eqn.9, but this dependence is ignored in the \((111)\) wavefunction in the charge sector. The \( d \) dependence in \( V_{+} \) is smooth anyway. Essentially all the distance dependence is encoded in the ground state wavefunction in the spin sector Eqn.14. As can be seen from Fig.1, the instability happens at \( q = q_0 \) where \( V_{-} \) becomes negative, but the spin stiffness \( \rho_E \) remains non-critical through the ESF to PSDW transition. As \( d \to d_{c1} \), the sum over \( q \) in Eqn.14 becomes dominated by the regime \( q \sim q_0 \). When \( d_{c1} < d < d_{c2} \), Eqn.14 breaks down. A new wavefunction to describe the translational symmetry breaking PSDW state is needed. Some trial wavefunctions are proposed in [25]. It would be interesting to derive the new wavefunction of the PSDW state from the CB theory.

VII. CONCLUSIONS

BLQH differs from the SLQH by its symmetry breaking ground state and associated neutral gapless mode in the pseudo-spin sector. Due to the gapless mode in the bulk, the groundstate wavefunctions could be considerably different from the well known \((111)\) wave function [12,13]. The low energy excited states could also be sensitive to details such
as normalization factors. One important problem is to find good trial wavefunctions for the ground state and low energy excited states. We investigated this important open problem from the CB theory developed previously in [14,16] to study BLQH systematically. We derived the ground state, quasi-hole and a pair of quasi-hole wavefunctions from CB theory and its dual action by the following procedures: We first performed the singular gauge transformation Eqn.4 to transform a fermionic problem into a bosonic problem, then found that all the wavefunctions in the bosonic picture are always the product of two parts, one part in the charge sector and the other in the spin sector. All the distance dependence are encoded in the spin part, while all the excitations only happen in the charge sector. After transforming back to the original electron picture by proper inverse SGT’s, we get the final wavefunctions in the electron coordinates. We found that the inverse SGT’s are different in ground state, meron and a pair of merons. By considering the differences carefully, we derived all these wavefunctions in the original electronic picture a systematical way.

At any finite $d$, the ground state wavefunction in the charge sector is the same as the (111) wavefunction, while that in the spin sector is highly nontrivial due to the gapless mode. So the total groundstate wavefunction differs from the well known (111) wavefunction at any finite $d$. In the bosonic picture, when inserting vortices in the ground state, the spin part remains the same due to the lack of CS term in this sector, while the charge part changes accordingly. However, due to the insertion of vortices, in order to recover the original electronic problem, the inverse SGT differs from that in the ground state. After transforming back to the original electron problem by the inverse SGT’s, we showed that the quasi-hole and a pair of quasi-holes wavefunctions contain non-trivial normalization factors as shown in 24,27. We expect that the quasi-hole and pair wave function not only has logarithmically divergent energy, well localized charge distribution, but also correct interlayer correlations. It is important to test these trial wavefunctions by QMC simulations performed in [13] for the states listed in Eqn. 1. We also investigated the instability encoded in the spin sector which leads to the PSDW solid formation proposed in [14,16]. Because the CB field theory has been used to describe the tri-layer quantum Hall systems very successfully, the analysis in this paper can be easily extended to derive the wavefunctions in the TLQH [17].

It is well known that CB approach is not a Lowest Landau Level (LLL) approach [12,16], it is very difficult to incorporate the LLL projection into the CB approach. This may be partially responsible for the spin part of the ground state wavefunction Eqn. 16 not in the LLL level. But as explained below Eqn.17, this should not be too worrisome, because Eqn.16 works only in long distance anyway. As shown in [13] and listed in Eqn.1, in the long wavelength limit, the meron wavefunction’s normalization factor contains modulus which is not in the LLL either. How to getting precise short distance behaviors of these wavefunctions from the CB theory remains an open problem.

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there is a spin-charge separation only in the ground state. However, in any excited states such as meron excitations, there
are spin-charge connection which leads to the correct meron fractional charge and polarization listed in Table 1 in [16].
It is this spin-charge connection which leads to the constraint of the vortices in the two layers $J^{x}_{0}^{+}(x) = J^{x}_{0}^{-}(x) = -\delta(x)$
used in this section and $J^{\pm}_{0} = \delta(x - z_{0}) \pm \delta(x - w_{0})$ used in the next section.
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this very subtle and important point was not explicitly spelled out in any previous work.
[24] It was shown in [13] that although the” quasi-hole “ and the ” meron “ wavefunction listed in Eqn.1 are completely different
at any finite distance, they have finite comparable expectation energies at $d = 0$. So we expect the quasi-hole’s expectation
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