Entanglement swapping for multi-particle pure states and maximally entangled states

Zhaoxu Ji†, Atta Ur Rahman, Peiru Fan, Huanguo Zhang

Key Laboratory of Aerospace Information Security and Trusted Computing, Ministry of Education, School of Cyber Science and Engineering, Wuhan University, Wuhan 430072 China

†jizhaoxu@whu.edu.cn

Abstract

Entanglement swapping has played an important role in quantum information processing, and become one of the necessary core technologies in the future quantum network. In this paper, we study entanglement swapping for multi-particle pure states and maximally entangled states in qudit systems. We generalize the entanglement swapping of two pure states from the case where each quantum system contains two particles to the case of containing any number of particles, and consider the entanglement swapping between any number of systems. We also generalize the entanglement swapping chain of bipartite pure states to the one of multi-particle pure states. In addition, we consider the entanglement swapping chains for maximally entangled states.

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1 Introduction

Suppose that there are several composite quantum systems in which each composite system contains at least two subsystems, then two new composite systems can be created by measuring some subsystems in each initial composite system simultaneously. This process is called entanglement swapping, which is one of the most fascinating phenomena in quantum mechanics and has attracted extensive attention since it was discovered.

Entanglement swapping is a very useful means for entanglement purification and teleportation [1], and it plays an increasingly important role in quantum information science, including quantum computing and quantum cryptography [1-3]. In addition, entanglement swapping has become one of the necessary core technologies for building the future large-scale and long-distance quantum network [1, 4].

The original idea of entanglement swapping is that Alice and Bob share an entangled state beforehand, and Bob shares another entangled state with Charlie; then Bob performs a Bell measurement on the two particles he holds, which eventually enables Alice and Charlie to share a new entangled state [5, 6]. Afterwards, Bose et al. [7] and Bouda et al. [8] generalized entanglement swapping to multi-particle qubit systems and qudit systems, respectively. Hardy and Song proposed the entanglement swapping chains of d-level bipartite pure states [9]. Later, Karimipour et. al. considered the entanglement swapping between a multi-particle maximally entangled state and a Bell state in d-level quantum systems [10]. Ban et. al. proposed the entanglement swapping for continuous variable systems [11-14].

In this paper, we study the entanglement swapping for any number of multi-particle pure states [15] and maximally entangled states, which are realized by performing measurements with the basis constructed by multi-particle maximally entangled states in d-level quantum systems. We generalize the entanglement swapping chains of bipartite pure states proposed by Hardy and Song [9] to multi-particle case. We also consider entanglement swapping chains for multi-particle maximally entangled states. What is more, we provide the formulas describing the entanglement swapping of d-level multi-particle Greenberger-Horne-Zeilinger (GHZ) states [16], and propose the entanglement swapping between a multi-particle pure state and a maximally entangled state. The structure of the rest of this paper is as follows. In Sec. 2, we first derive the formulas for the entanglement swapping between any number of d-level multi-particle pure states. We then present the formulas describing the entanglement swapping for d-level multi-particle GHZ states, and provide the formula for the entanglement swapping between a multi-particle pure state and a maximally entangled state. In Sec. 3, we consider the multi-particle generalization for the entanglement swapping chains of pure states, and study the entanglement swapping chains for multi-particle maximally entangled states. This paper is summarized in Sec. 4.
2 Swapping any number of multi-particle systems

Let us start by reviewing the entanglement swapping between two bipartite pure states in d-level systems, which was proposed by Hardy and Song [8]. A d-level bipartite general pure state [3, 15] can be expressed as

$$|\mathcal{P}\rangle = \sum_{i=0}^{d-1} a_i |i, i\rangle,$$

where $$\sum_{i=0}^{d-1} |a_i|^2 = 1$$. Suppose that there are two d-level bipartite pure states, marked by

$$|\mathcal{P}_1\rangle = \sum_{i_1=0}^{d-1} a_{1i_1} |i_1, i_1\rangle_{1,2}, |\mathcal{P}_2\rangle = \sum_{i_2=0}^{d-1} a_{2i_2} |i_2, i_2\rangle_{3,4},$$

where the superscripts 1 and 2 denote their orders, and the subscripts (1,2) and (3,4) outside the Dirac notations denote two particles in the two states respectively. Performing a Bell measurement on the particles (2,4), one can get

$$|\phi(u, v)\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} e^{iu} |i, l \oplus v\rangle_{2,4},$$

where $$\zeta = e^{2\pi i/d}$$ and the symbol $$\oplus$$ denotes addition modulo d throughout this paper. Then, the particles 1 and 3 are projected onto

$$\frac{1}{\sqrt{P}} \frac{1}{\sqrt{d}} \sum_{i_1=0}^{d-1} \beta_1^{l_1} \beta_2^{l_2} e^{-il_1} |i_1, l \oplus v\rangle_{1,3},$$

where $$P$$ can be obtained from normalization (similarly hereinafter), that is

$$\left\| \frac{1}{\sqrt{P}} \frac{1}{\sqrt{d}} \sum_{i_1=0}^{d-1} \beta_1^{l_1} \beta_2^{l_2} e^{-il_1} \right\|^2 = 1,$$

thus

$$P = \frac{1}{d} \sum_{i_1=0}^{d-1} |\beta_1^{l_1}|^2 |\beta_2^{l_2} e^{-il_1}|^2.$$

Let us now derive formulas for the entanglement swapping between multi-particle general pure states. The form of a general m-particle ($$m \geq 2$$) pure state is given by

$$|\mathcal{P}\rangle = \sum_{i=0}^{d-1} \beta_i |i, i, \ldots, i\rangle_{1,2,..,m},$$

where the subscripts 1, 2, ..., $$m$$ denote $$m$$ particles in the state respectively (similarly hereinafter). We would first like to consider the case of swapping two states. Assume that there are two d-level bipartite pure states containing $$m_1$$ and $$m_2$$ particles respectively, where $$m_1, m_2 > 2$$. Let us mark the two states by

$$|\mathcal{P}_1\rangle = \sum_{i_1=0}^{d-1} \beta_1^{l_1} |i_1, i_1, \ldots, i_1\rangle_{1,2,..,m_1}, |\mathcal{P}_2\rangle = \sum_{i_2=0}^{d-1} \beta_2^{l_2} |i_2, i_2, \ldots, i_2\rangle_{1,2,..,m_2}.$$

Let us select the last $$k_1(1 \leq k_1 \leq m_1)$$ particles from $$|\mathcal{P}_1\rangle$$ and the last $$k_2(1 \leq k_2 \leq m_1)$$ particles from $$|\mathcal{P}_2\rangle$$, and then measure them with the basis constructed by the d-level ($$k_1 + k_2$$)-particle maximally entangled states

$$|\phi(u_1, u_2, \ldots, u_{k_1+k_2})\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \zeta^{iu} |l, l \oplus u_2, l \oplus u_3, \ldots, l \oplus u_{k_1+k_2}\rangle,$$

where $$u_1, u_2, \ldots, u_{k_1+k_2} \in \{0, 1, \ldots, d - 1\}$$. Note that the maximally entangled states are complete and orthonormal [8, 10].
Measuring the last $k$ particles respectively, we would now like to consider more general cases, that is, swapping the multi-particle pure states which have the $n$ particles in a composed of $d$-level Hilbert space measurement results $|\psi\rangle$ are projected onto

$$|\psi\rangle = \sum_{\beta_j} \beta_j^* |i_1, i_2, \ldots, i_r\rangle_{1,2,\ldots,m_r}, \quad m_r > 2, r = 1, 2, \ldots, n. \quad (13)$$

As before, performing the measurements on the last $k_r (1 \leq k_r \leq m_r)$ particles in $|\psi\rangle$ with the basis constructed by all the $d$-level $k$-particle ($k = \sum_{r=1}^n k_r$) maximally entangled states, we can get the measurement results

$$|\phi(u, 0, 0, \ldots, 0, v_1, v_1, \ldots, v_1, v_2, v_2, \ldots, v_{n-1}, v_{n-1}, \ldots, v_{n-1})\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \xi_{lu}^* |l, l \oplus v_1, l \oplus v_1, \ldots, l \oplus v_1, l \oplus v_2, l \oplus v_2, \ldots, l \oplus v_{n-1}, l \oplus v_{n-1}, \ldots, l \oplus v_{n-1}\rangle_{1,2,\ldots,m_r}, \quad m_r \geq 2, r = 1, 2, \ldots, n. \quad (14)$$

We would now like to consider more general cases, that is, swapping the multi-particle pure states which have the form

$$|\psi\rangle = \sum_{i_1, i_2, \ldots, i_r = 0}^{d-1} \beta_{i_1, i_2, \ldots, i_r} |i_1, i_2, \ldots, i_r\rangle_{1,2,\ldots,m_r}. \quad (16)$$

where $\sum_{i_1, i_2, \ldots, i_r = 0}^{d-1} \beta_{i_1, i_2, \ldots, i_r}^2 = 1$. As before, assume that there are $n$ such pure states composed of $m_1, m_2, \ldots, m_n$ particles respectively,

$$|\psi'\rangle = \sum_{m_r = 0}^{d-1} \beta_{i_1, i_2, \ldots, i_r} |i_1, i_2, \ldots, i_r\rangle_{1,2,\ldots,m_r}, \quad m_r \geq 2, r = 1, 2, \ldots, n. \quad (17)$$

Measuring the last $k_r$ particles in $|\psi'\rangle$, we can get

$$|\phi(u, v_1, v_1, \ldots, v_1, v_2, v_2, \ldots, v_1, v_1, v_2, \ldots, v_{n-1}, v_{n-1}, \ldots, v_{n-1})\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \xi_{lu}^* |l, l \ominus v_1, l \ominus v_1, \ldots, l \ominus v_1, l \ominus v_2, l \ominus v_2, \ldots, l \ominus v_{n-1}, l \ominus v_{n-1}, \ldots, l \ominus v_{n-1}\rangle_{1,2,\ldots,m_r}, \quad m_r \geq 2, r = 1, 2, \ldots, n. \quad (18)$$

Then the remaining particles are projected onto

$$\frac{1}{\sqrt{P}} \sum_{\beta_{i_1, i_2, \ldots, i_r} = 0}^{d-1} \beta_{i_1, i_2, \ldots, i_r} \sum_{l=0}^{d-1} \sum_{l=0}^{d-1} \cdots \sum_{l=0}^{d-1} \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \sum_{l=0}^{d-1} \cdots \sum_{l=0}^{d-1} \alpha_{i_1, i_2, \ldots, i_r, m_1, m_2, \ldots, m_n} |i_1, i_2, \ldots, i_r\rangle_{1,2,\ldots,m_r} \quad (19)$$
In what follows we will provide the formulas for the entanglement swapping between any number of the $d$-level multi-particle GHZ state \([\phi]\)

\[
|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i_0}^{d-1} |i_0, i_1, \ldots, i_l\rangle_{1,2,\ldots,m}.
\]  

(20)

The GHZ state is a maximally entangled state because we can get it from the forms of the maximally entangled states. In the $d$-level $m$-particle maximally entangled states $|\phi(u_1, u_2, \ldots, u_m)\rangle$ (see Eq. 9), we set $u_i = 0$ $\forall i = 1, 2, \ldots, m$, then we can get the GHZ states

\[
|\phi(0, 0, \ldots, 0)\rangle = \frac{1}{\sqrt{d}} \sum_{i_0}^{d-1} |i_0, i_1, \ldots, i_l\rangle_{1,2,\ldots,m}.
\]  

(21)

Therefore, the formulas for the entanglement swapping of the GHZ states can be derived from that of maximally entangled states. Admittedly, it seems insignificant to derive such formulas, but it may be instructive to show these explicit formulas in terms of finding the application of the entanglement swapping in quantum information processing. The entanglement swapping between any number of multi-particle maximally entangled states, proposed by Bouda et al. [8] and organized in our recent work [1], is given by

\[
\bigotimes_{r=1}^{n} |\phi(u'_1, u'_2, \ldots, u'_m)\rangle
\]

where these states contain $m_1, m_2, \ldots, m_n$ particles, respectively. We can now derive the formulas for the entanglement swapping between the GHZ states directly. Similarly, assuming that there are $n$ GHZ states, $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle$, which contain $m_1, m_2, \ldots, m_n$ particles respectively, we can get

\[
\bigotimes_{r=1}^{n} |\phi_r\rangle
\]

\[
= \frac{1}{d^{n-1}} \sum_{i_1\ldots i_{n-1} = 0}^{d-1} |i_1, i_2, \ldots, i_n \rangle_{1,2,\ldots,m} |i_2, i_2, \ldots, i_2 \rangle_{1,2,\ldots,m} \otimes \cdots \otimes |i_n, i_n, \ldots, i_n \rangle_{1,2,\ldots,m}
\]

\[
= \frac{1}{d^{n-1}} \sum_{i_1\ldots i_{n-1} = 0}^{d-1} |i_1, i_1, \ldots, i_1, i_2, i_2, \ldots, i_2, \ldots, i_n, i_n, \ldots, i_n \rangle_{1,2,\ldots,m} \otimes \cdots \otimes |i_n, i_n, \ldots, i_n \rangle_{1,2,\ldots,m}
\]

\[
= \frac{1}{d^{n-1}} \sum_{i_1\ldots i_{n-1} = 0}^{d-1} \sum_{v_1=0}^{d-1} |\phi(-v_1, 0, 0, \ldots, 0, v_2, v_2, \ldots, v_2, v_3, v_3, \ldots, v_3, v_n, v_n, \ldots, v_n)\rangle_{1,2,\ldots,m} \otimes \cdots \otimes |\phi(v_1, 0, 0, \ldots, 0, v_2, v_2, \ldots, v_2, v_3, v_3, \ldots, v_3, v_n, v_n, \ldots, v_n)\rangle_{1,2,\ldots,m}
\]

(23)

Let us finally consider the entanglement swapping between $d$-level multi-particle pure states and maximally entangled states. Suppose that there are a $m_1$-particle pure state and a $m_2$-particle maximally entangled state,

\[
|\psi\rangle = \sum_{i_1, i_2, \ldots, i_{m_1} = 0}^{d-1} \beta_{i_1, i_2, \ldots, i_{m_1}} |i_1, i_2, \ldots, i_{m_1} \rangle_{1,2,\ldots,m_1} \otimes |\phi(u_1, u_2, \ldots, u_{m_2})\rangle = \frac{1}{\sqrt{d}} \sum_{i_0}^{d-1} \xi_{i_0} |i, l \oplus u_2, l \oplus u_3, \ldots, l \oplus u_{m_2}\rangle
\]

(24)

and suppose that the measurements are performed on the last $k_1$ and $k_2$ particles in the pure state and maximally entangled state, respectively. Let us mark the measurement results as $|\phi(v_1, v_2, \ldots, v_{k_1+k_2})\rangle$, then the remaining particles are projected onto

\[
\frac{1}{\sqrt{d}} \sum_{i_1, i_2, \ldots, i_{m_1-k_1}, l, i, i \oplus u_2, l \oplus u_3, \ldots, l \oplus u_{m_2-k_2}} \beta_{i_1, i_2, \ldots, i_{m_1-k_1}} |i_1, i_2, \ldots, i_{m_1-k_1}, i, i \oplus u_2, l \oplus u_3, \ldots, l \oplus u_{m_2-k_2}\rangle_{1,2,\ldots,m_1+ m_2-k_1-k_2}
\]

(25)
3 Multi-particle generalization for entanglement swapping chains

Entanglement swapping chain was proposed by Hardy and Song [9], which is realized by performing d-level Bell measurements at all the intermediate locations of more than two d-level bipartite pure states. Suppose that there are \( n \) pure states, marked by

\[
|\psi^{ (1) \rangle} = \sum_{i=0}^{d-1} a_{1i}^1 |i_1, i_1\rangle_1, |\psi^{ (2) \rangle} = \sum_{i=0}^{d-1} a_{2i}^2 |i_2, i_2\rangle_2, \ldots, |\psi^{ (n) \rangle} = \sum_{i=0}^{d-1} a_{ni}^n |i_n, i_n\rangle_{2n-1, 2n},
\]

where the superscripts 1, 2, \ldots, \( n \) denote their orders, and the subscripts (1, 2), (3, 4), \ldots, \((2n - 1, 2n)\) outside the Dirac notations denote two particles in these states respectively. If performing Bell measurements on the particles (2, 3), (4, 5), \ldots, \((2n - 2, 2n - 1)\) in turn and marking the measurement results by

\[
|\phi(u_1, v_1)\rangle, |\phi(u_2, v_2)\rangle, \ldots, |\phi(u_{n-1}, v_{n-1})\rangle,
\]

then the remaining particles marked by 1 and 2 are projected onto

\[
\frac{1}{\sqrt{d}} \frac{1}{\sqrt{d^{n-1}}} \sum_{i=0}^{d-1} \beta_{1i}^1 \beta_{2i}^2 \cdots \beta_{ni}^n e^{\frac{i}{d} \sum_{r=1}^{n-1} v_r} |i_1, i_2, \ldots, i_n, v_1, v_2, \ldots, v_{n-1}, 1, 2\rangle.
\]

In what follows, we will generalize entanglement swapping chains to multi-particle case. Assume that there are \( 2m \) particles and the last \( m \) particles are projected onto \( |\psi^{ (1) \rangle}, |\psi^{ (2) \rangle}, \ldots, |\psi^{ (2m-1) \rangle} \). Then, then the remaining particles marked by 1 and 2 are projected onto

\[
\frac{1}{\sqrt{d}} \frac{1}{\sqrt{d^{m-1}}} \sum_{i=0}^{d-1} \beta_{1i}^1 \beta_{2i}^2 \cdots \beta_{ni}^n e^{\frac{i}{d} \sum_{r=1}^{m-1} v_r} |i_1, i_2, \ldots, i_n, v_1, v_2, \ldots, v_{m-1}, 1, 2\rangle.
\]

Let us now consider entanglement swapping chains for maximally entangled states. Similarly, let us assume that there are \( n \) d-level 2m-particle maximally entangled states, and mark them by

\[
|\phi(v_1^1, v_1^2, \ldots, v_1^{2m})\rangle, |\phi(v_2^1, v_2^2, \ldots, v_2^{2m})\rangle, \ldots, |\phi(v_{n-1}^1, v_{n-1}^2, \ldots, v_{n-1}^{2m})\rangle,
\]

then the remaining particles marked by 1 and 2 are projected onto

\[
\frac{1}{\sqrt{d}} \frac{1}{\sqrt{d^{n-1}}} \sum_{i=0}^{d-1} \beta_{1i}^1 \beta_{2i}^2 \cdots \beta_{ni}^n e^{\frac{i}{d} \sum_{r=1}^{n-1} v_r} |i_1, i_2, \ldots, i_n, 1, 2\rangle.
\]

4 Conclusion

We have generalized the entanglement swapping between two d-level bipartite pure states to the case of any number of multi-particle pure states. We have also generalized the entanglement swapping chain for bipartite pure states to multi-particle case. We have proposed the entanglement swapping between a multi-particle pure state and maximally entangled state. We have provided the straightforward formula describing the entanglement swapping between any number of d-level multi-particle GHZ states, and considered entanglement swapping chains.
References

[1] Ji, Z. X., Fan, P. R., & Zhang, H. G. (2019). Entanglement swapping for Bell states and Greenberger-Horne-Zeilinger states in qubit systems. arXiv preprint arXiv:1911.09875.
[2] Galindo, A., & Martin-Delgado, M. A. (2002). Information and computation: Classical and quantum aspects. Reviews of Modern Physics, 74(2), 347.
[3] Ji, Z. X., Zhang H. G., Wang, H. Z., Wu, F. S., Jia, J. W., & Wu, W. Q. (2019). Quantum protocols for secure multi-party summation. Quantum Information Processing, 18(6), 168.
[4] Lu, C., Yang, T., & Pan, J. (2009). Experimental multiparticle entanglement swapping for quantum networking. Physical Review Letters, 103(2), 020501.
[5] Zukowski, M., Zeilinger, A., Horne, M. A., & Ekert, A. K. (1993). “Event-ready-detectors” Bell experiment via entanglement swapping. Physical Review Letters, 71, 4287-4290.
[6] Pan, J., Bouwmeester, D., Weinfurter, H., & Zeilinger, A. (1998). Experimental Entanglement Swapping: Entangling Photons That Never Interacted. Physical Review Letters, 80(18), 3891-3894.
[7] Bose, S., Vedral, V., & Knight, P. L. (1998). Multiparticle generalization of entanglement swapping. Physical Review A, 57(2), 822.
[8] Bouda, J., & Buzek, V. (2001). Entanglement swapping between multi-qudit systems. Journal of Physics A: Mathematical and General, 34(20), 4301-4311.
[9] Hardy, L., & Song, D. D. (2000). Entanglement-swapping chains for general pure states. Physical Review A, 62(5), 052315.
[10] Karimipour, V., Bahraminasab, A., & Bagherinezhad, S. (2002). Entanglement swapping of generalized cat states and secret sharing. Physical Review A, 65(4).
[11] Ban, M. (2004). Continuous variable entanglement swapping. Journal of Physics A: Mathematical and General, 37(31), 385.
[12] Polkinghorne, R. E. S., & Ralph, T. C. (1999). Continuous variable entanglement swapping. Physical review letters, 83(11), 2095.
[13] Takeda, S., Fuwa, M., Van Loock, P., & Furusawa, A. (2015). Entanglement swapping between discrete and continuous variables. Physical Review Letters, 114(10), 100501-100501.
[14] Marshall, K., & Weedbrook, C. (2015). Continuous-Variable Entanglement Swapping. Entropy, 17(5), 3152-3159.
[15] Thapliyal, A. V. (1999). Multipartite pure-state entanglement. Physical Review A, 59(5), 3336.
[16] Ryu, J., Lee, C., Yin, Z., Rahaman, R., Angelakis, D. G., Lee, J., & Zukowski, M. (2014). Multisetting greenberger-horne-zeilinger theorem. Physical Review A, 89(2), 024103.