Chaotic shock waves of a Bose-Einstein condensate

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It is demonstrated that the well-known Smale-horseshoe chaos exists in the time evolution of the one-dimensional (1D) Bose-Einstein condensate (BEC) driven by the time-periodic harmonic or inverted-harmonic potential. A formally exact solution of the time-dependent Gross-Pitaevskii equation (GPE) is constructed, which describes the matter shock waves with chaotic or periodic amplitudes and phases. When the periodic driving is switched off and the number of condensed atoms is conserved, we obtained the exact stationary states and non-stationary states. The former contains the stable ‘non-propagated’ shock wave, and in the latter the shock wave alternately collapses and grows for the harmonic trapping or propagates with exponentially increased shock-front speed for the antitrapping. It is revealed that existence of chaos play a role for suppressing the blast of matter wave. The results suggest a method for preparing the exponentially accelerated BEC shock waves or the stable stationary states.

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A shock wave in a compressible classical fluid is characterized by a steep jump in gas velocity and density due to the collisions of particles. The physics and applications of the shock waves have been thoroughly investigated during the last century in different fields of physics [1, 2, 3]. Very recently, a different type of shock waves in a quantum fluid is explored [4, 5, 6, 7, 8, 9], where the quantum gas is a Bose-Einstein condensate (BEC) governed by the Gross-Pitaevskii equation (GPE). Experiments and numerical simulations [10, 11, 12] have depicted a BEC that exhibits the traveling fronts with steep gradients of shock waves. The corresponding analytical studies [13, 14, 15] have also shown that a shock wave could develop in the attractive or repulsive BECs. However, as a nonlinear Schrödinger equation the driven GPE allows existence of the recognizably chaotic behavior [16, 17, 18]. Therefore, various physical behaviors of the BECs may be affected by the chaos, that leads to the chaotic BEC solitons [19, 20, 21], chaotic atomic populations [22, 23, 24], chaotic quantum tunneling [25, 26], chaotic Bogoliubov excitations [27], and the chaotic BEC collapse [28]. All the above theoretical works are based on the analytical or numerical approximations to the GPE, because of the nonintegrability of the system with external potentials. The aim of this paper is to present an exactly analytical evidence of the chaotic and oscillating shock waves in 1D attractive or repulsive BECs driven by the time-periodic harmonic or inverted-harmonic potentials, containing the most studied harmonically trapped BEC with zero driving.

It is well-known that classical Smale-horseshoe chaos can exist in a parametrically driven Duffing system with the cubic nonlinear and harmonic or inverted-harmonic force [29, 30]. The harmonic potential is a most widely used trapping potential in the investigations of BECs [31, 32]. The expulsive parabolic potential has also been applied to study the time evolution of quantum tunneling [33], deterministic chaos [34, 35] and accelerated bright solitons [36, 37] (e.g. see Fig. 3B of Ref. [36] for the attractive interaction) in BECs. The harmonic and inverted-harmonic potentials can be time-dependent [38, 39, 40]. The presences of such potentials will break the integrability of GPE and bring the difficulties to the exact researches of BECs. Although for different initial and boundary conditions the nonlinear GPE have many solutions, the exact solution of GPE with harmonic or inverted-harmonic potential has not been reported yet even for the non-driving cases. Particularly, we know that a turbulent flow is a fluid regime characterized by chaotic, stochastic property changes, and as an important physical behavior the unstable and irregular quantum turbulence has been studied [41]. Therefore, the chaotic shock wave as a different type of turbulent flow warrants further investigation.

In the present paper, we treat the time evolution of a quasi-1D BEC created initially in a range near the potential center and driven by the time-periodic harmonic or inverted-harmonic potential, and seek the formally exact solution of the time-dependent GPE. By using the exact solution we reveal that the well-known Smale-horseshoe chaos exists in the matter shock waves with unpredictable amplitudes, phases and wave fronts. When we switch off the periodic driving and keep the number of condensed atoms, the exact solution becomes explicit functions of spatiotemporal coordinates which contain the stationary and non-stationary states. The stationary states include the exact stable ‘non-propagated’ shock wave. The non-stationary states describe the exactly controllable shock waves which alternately collapse and grow for the harmonic trapping and repulsive interaction or propagate with exponentially increased shock-front speed for the antitrapping and attractive interaction. The suppression of chaos to the blast of matter wave is revealed. The re-

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sults can be observed experimentally [3,11], and supply a
method for preparing the stable stationary state and the
experimentally accelerated shock waves which are similar to
the accelerated bright soliton [36].

There are some different methods for the reductions
from original 3D GPE to quasi-1D GPE. The resulting
equations contain the nonpolynomial version [42]
and the cubic nonlinear one [41] which has been used
in the regime of shock wave [7,4,12]. Assuming the
transverse wave function to be the ground state of a
harmonic oscillator of frequency \(\omega_r\), the governing
longitudinal GPE becomes the cubic nonlinear equation
\[ i\hbar \psi_t = -\frac{\hbar^2}{2m} \psi_{xx} + |V_0'(x,t) + g_{1d}|\psi|^2|\psi, \]
where \(m\) is atomic mass, \(g_{1d} = m\omega_r g_0/(2\pi\hbar) = 2\hbar \omega_r a_s\) is the quasi-
1D atom-atom interaction intensity with \(a_s\) being the s-
wave scattering length, \(V_0'(x,t) = (\alpha \frac{1}{2} \omega x^2 + V_1 \cos \omega t)x^2\)
denotes the harmonic \((\alpha = 1)\) and inverted-harmonic
\((\alpha = -1)\) potentials of frequency \(\omega_r\) with the driving
of strength \(V_1\) and frequency \(\omega\) [40]. Taking an experimentally suitable frequency \(\omega_0\) as the units of frequencies
\(\omega_r, \omega\) and normalizing the time, space and wave function
with \(\omega_0^{-1}, L_0 = \sqrt{\hbar/(m \omega_0)}\) and \(1/\sqrt{L_0}\), the GPE becomes the dimensionless one
\begin{align}
iv_x = & \frac{1}{2} \psi_{xx} + \left| \kappa_a(t)x^2 + g_{1d}|\psi|^2|\psi, \\
\kappa_a(t) = & \left( \frac{1}{2} \omega x_1 + V_1 \cos \omega t \right). \tag{1}
\end{align}

Here the interaction is reduced to \(g_{1d} = 2\omega_r a_s/(\omega_0 L_0)\)
and the potential strength \(V_1\) is normalized by \(\hbar \omega_0\).
Throughout the paper we take \(|\kappa_a| = 0.4\) which means
\(\hbar \omega_0 = 25m\omega_r^2 a_s^2\) with \(m, \omega_r \) and \(a_s\) determined by ex-
periment.

Noticing that Eq. (1) can describe a symmetry BEC
system in the transformation \(x \rightarrow -x\), we consider a
real physical process in which a BEC is created experiment-
ally with the symmetry profile between positions
\(\pm x_0 = \pm L\) at initial time \(t_0\), then the condensed atoms
propagate along \(+x\) directions such that the bound-
ary coordinates \(\pm x_0(t)\) are time-dependent. Letting the
number of condensed atoms conserve as \(N\), we face the
definite problem with the initial data [3]
\begin{align}
\psi(x,t_0) = & \left\{ \begin{array}{ll}
\psi(x,t_0) & \text{for } |x| \leq L, \\
0 & \text{for } |x| > L
\end{array} \right. \tag{2}
\end{align}

and the boundary-dependent normalization condition
\[ \int_{-x_0(t)}^{x_0(t)} |\psi(x,t)|^2 dx = N \text{ for any } t. \]
For a practical BEC the boundary density \(|\psi(\pm L, t_0)|^2|\) could be made nonzero
by using Feshbach resonances [12] that leads to the steep
jumps in atomic density and flow velocity, which charac-
terize the feature of shock wave [3] with the shock front
\(\pm x_0(t)\) at which the front gradients \(\psi_x(\pm x_0, t)\) is dis-
continuous. Although the initial data of \(\psi(\pm L, t_0)\) and
\(\psi_x(\pm L, t_0)\) cannot be measured accurately, in mathemat-
ics, they are corresponded with an unique solution of
Eq. (1) strictly. Hence, if we construct an exact solution
\(\psi(x,t)\) which obeys Eq. (2) and the normalization condi-
tion experimentally, it will describe the exact shock wave
of the system. For other initial and boundary conditions,
of course, one could obtain different type of solutions. In
fact, the accelerated bright soliton of the system has been
reported for the initial soliton state [30].

We now give the formally exact solution of Eq. (1) as
\[ \psi(x,t) = |A(t)x + iB(t)|e^{i[\alpha(t) + b(t)x^2]}, \tag{3}\]
where \(A(t), B(t), \alpha(t), b(t)\) are real functions of time
and obey the coupled equations
\[ \dot{A} + g_{1d}B^2 = 0, \quad \dot{b} + 2b^2 = -g_{1d}A^2 - k_a(t), \]
\[ A + 3Ab = 0, \quad B + Bb = 0. \tag{4}\]

The solution (3) can be easily proved by inserting Eqs.
(3) and (4) into Eq. (1) directly. The atomic density
and flow velocity are associated with the macroscopic wave function
\[ \psi(x,t) = \sqrt{\rho} e^{i\varphi} \]
through
\[ \rho(x,t) = |\psi(x,t)|^2 = A^2x^2 + B^2, \]
\[ v(x,t) = \frac{\hbar}{m} \theta_x = \frac{\hbar}{2m} \left( 2bx - \frac{AB}{\rho} \right). \tag{5}\]

In order to evidence the solution (3), we have to solve
Eq. (4) as follows. From the third equation of Eq. (4)
we get \(b = -A/(3A), \quad b = A_2/(3A_2^2) - A/(3A)\).
Applying these formulas to the second one of Eq. (4) produces
the decoupled equation
\[ \dot{A} = 3\kappa_a(t)A + 3g_{1d}A^3 + \frac{5A^2}{3A}. \tag{6}\]

This is just the parametrically driven Duffing equation
with a ‘quadratic damping’ term. It is well known that
according to the Melnikov chaos criterion for a certain pa-
rameter region the Smale-horseshoe chaos exists in such a
system [29,30]. To confirm the existence of chaos nu-
merically, we adopt the MATHEMATICA code
\begin{align}
T = 2\pi/\omega;[\{\text{Anew}_-, \text{vnew}_-\}] := \{\text{A}[T], \text{v}[T]\}/. \text{Flatten}[
\{\text{NDSolve}[\{\text{A}'[t] == \text{v}[t], \text{v}'[t] == 5/3 \cdot \text{v}[t]^2/A[t]
+3(0.5\omega x_2 + V_1 \cos \omega t)\text{A}[t] + 3g_{1d}\text{A}[t]^3, \text{A}[0] == \text{Anew},
\text{v}[0] == \text{vnew}\}, \{\text{A}, \text{v}\}, \{t, 0, T\})]; \text{Dol[]cft = ListPlot}\n\{\text{Drop}[\text{Nestlist}[\{\text{Random}[\text{Real}, \{0.2, 1.5\}],
\text{Random}[\text{Real}, \{-0.8, 0.8\}]\}, 6010], 10], \{i, 1, 20\}\}
\end{align}

with the parameters \(g_{1d} = -0.4, \quad 3V_1 = 20.2\) and random
initial conditions \(\{A(0) \in [0.2, 1.5], \quad A(0) \in [-0.8, 0.8]\}\)
to plot two groups of Poincaré sections on the ‘phase space’ \((A, \dot{A})\) respectively for \((a) \omega x_2 = 0.8, \quad \omega = 6.0\)
and \((b) \omega x_2 = -0.8, \quad \omega = 5.4\). We find some periodic and
chaotic orbits from each group consisting of 20 Poincaré
sections. Here any Poincaré section is a discrete set of the
phase space points at every period of the external poten-
tial. In Fig. 1a and Fig. 1b, we show the superpositions
of 20 Poincaré sections for each group respectively, where
Fig. 1a with \(a = 1\) means the harmonic potential case
and Fig. 1b with \(a = -1\) the case of inverted-harmonic
From the third and fourth equations of Eq. (4) we know the relation $A = A_0 B^3$ with constant $A_0$ such that Eq. (7) infers the formula $2 A_0^3 B^3 (t) x_0^3 + 6 B^2 (t) x_0 = 3 N$, namely $B^2 (t) x_0 = |A(t)/A_0|^2/3 x_0 = \lambda$ is a constant and the coordinate $x_0(t)$ reads

$$x_0(t) = \lambda [A_0/A(t)]^{2/3},$$ (8)

where $\lambda$ is fixed by the algebraic equation $2 A_0^2 \lambda^3 + 6 \lambda = 3 N$ and $A_0$ is determined by the initial condition $x_0(t_0) = L$ in Eq. (2). It is interesting noting that function $A(t)$ in Eq. (8) may be periodic or chaotic so that the shock wave front $x_0(t)$ may periodically or chaotically oscillate. For the periodic solution $A(t)$, the driven BEC breathes in the spatiotemporal evolutions. In the chaotic solution case, the shock wave front $x_0(t)$, BEC density $\rho(x,t)$ and flow velocity $v(x,t)$ become unpredictable.

In the non-driving case with $V_1 = 0$ and $k_\alpha = a \frac{1}{2} \omega^2$, Eq. (6) becomes integrable. Applying the test formula $A^2 = C_1 A^2 + C_2 A^4$ to Eq. (6) and integrating this equation, we obtain the undetermined constants $C_1 = - \frac{9}{2} k_\alpha, C_2 = 9 g_{1d}$ and the two exact solutions

$$A_\alpha(t) = \sqrt{\frac{k_\alpha}{2 g_{1d}}} \sech \left[ 3 \sqrt{1/2} k_\alpha (t - t_0) \right]$$ (9)

respectively for $g_{1d} < 0$, $\alpha = -1$ and $g_{1d} > 0$, $\alpha = 1$, where $t_0$ is an integration constant. The exact solutions of Eq. (9) can be directly proved by substituting them into Eq. (6) with constant $k_\alpha$. The $A_{-1}(t)$ of Eq. (9) denotes a sec-shaped solution for the attractive interaction and inverted-harmonic potential case. It is just the homoclinic solution which is associated with the Smale-horseshoe chaos of Melnikov criterion [29, 30] when the periodic perturbation appears. This agrees with the numerical result shown in Fig. 1. Noticing sech$^2$ = sec, the $A_1(t)$ means a sec-shaped one for the repulsive interaction and harmonic potential case. Given Eq. (9), from Eq. (4) we easily write the undetermined functions

$$B_\alpha(t) = \left( \frac{A_\alpha}{A_0} \right)^{1/3}, \quad b_\alpha(t) = - \frac{A_\alpha}{3 A_\alpha},$$

$$a_\alpha(t) = - g_{1d} \int \left( \frac{A_\alpha}{A_0} \right)^{2/3} dt, \quad \alpha = \pm 1$$ (10)

with $A_0$ being an integration constant adjusted by the initial conditions. Inserting Eq. (9) into Eq. (10), a simple calculation can give the explicit forms of the functions $B_\alpha(t), a_\alpha(t)$ and $b_\alpha(t)$. Combining these functions with Eq. (3), we obtain the corresponding explicit forms of the exact non-stationary solutions immediately.

Applying Eq. (9) to Eq. (8), we find that for the sech-shaped solution $A_{-1}(t)$ the shock wave front $x_0(t)$ is proportional to $\cos^{1/3} \left[ \frac{3}{2} \omega_c (t - t_0) \right]$ which increases exponentially fast. However, for the sech-shaped solution $A_1(t)$ the shock wave front $x_0(t)$ is proportional to $\cos^{1/3} \left[ \frac{3}{2} \omega_c (t - t_0) \right]$ which oscillates periodically. The zero points of the periodic $x_0(t)$ are just the singular points of $A(t)$, where the BEC density $\rho(x,t) = A_1^2(t) x^2 + |A_1(t)/A_0|^2/3$ becomes infinity. By inserting Eq. (3) into Eq. (1) and

\[ \begin{align*}
\text{FIG. 1: The Poincaré sections on the dimensionless ‘phase space’} (A, A) \text{ for the parameters} & \text{g}_{1d} = -0.4, 3 V_1 = 20.2 \text{ and (a) } \omega^2 = 0.8, \omega = 6.0, \text{ (b) } \omega^2 = -0.8, \omega = 5.4. \text{ Each figure consists of 20 Poincaré sections associated with different initial conditions.}
\end{align*} \]
applying Eq. (4) to the resulting equation, we find that at the singular points of \( A(t) \) the original GPE (1) is fulfilled, since the infinite terms are offsetted each other. For a finite time the infinite density means the escape of solution (3) and describes the alternate collapses and growths of the blast matter wave physically [5, 11]. In the both cases, the traveling fronts have the steep gradients of shock waves \([10, 11, 12]\), since normalization condition (7) implies that for a finite time \( \psi(x = \pm x_0(t), t) = \) complex constants and \( |\psi(x)| > |x_0(t)|, \) such that the front gradients \( \psi_x(x = \pm x_0(t), t) \) seem to be steep. No chaos exists for the non-driving case.

Comparison between the driving and non-driving cases reveals that existence of chaos could play a role for suppressing the escape of solution and the blast of matter wave. In fact, from Fig. 1 we can see that in the driving case with chaos, the function \( A(t) \) is finite and does not vanish for most of the orbits. These imply the corresponding shock front \( x_0(t) \) and density \( \rho(x, t) \) having no singular points.

For the time-independent ‘spring constant’ \( k_\alpha \), we have also found that Eq. (4) has the simple solutions

\[
a = -\mu t, \ b = 0, \ A^2 = -\alpha \frac{\omega^2}{2g_{1d}}, \ B^2 = \frac{\mu}{g_{1d}}. \quad (11)
\]

Substituting Eq. (11) into Eq. (3) yields the exact stationary state wave functions

\[
\psi_\alpha(x, t) = \left[ \pm \sqrt{\frac{-\alpha \omega^2}{2g_{1d}}} \right] x \pm i \frac{\mu}{g_{1d}} e^{-i\mu t}. \quad (12)
\]

Here the harmonic potential with \( \alpha = 1 \) is associated with the attractive interaction \( g_{1d} < 0 \) and negative chemical potential \( \mu \), and the inverted-harmonic potential with \( \alpha = -1 \) corresponds to the repulsive interaction \( g_{1d} > 0 \) and positive chemical potential. The second spatial derivative of Eq. (12) vanishes that implies the zero kinetic energy of BEC and the ‘non-propagated’ shock wave with the steep front gradients. The constant front \( x_0 \) is determined by the algebraic equation \( -\alpha \omega^2 x_0^2 + 6\mu x_0 = 3g_{1d}N \) from the normalization. Such a non-propagated shock wave can be realized for the time-independent potential \( V = \alpha \frac{\omega^2}{2} x^2 \) in the region \( |x| \leq x_0 \) and \( V = \alpha \frac{\omega^2 x_0^2}{2} \) outside [13]. Under such a finite potential, the stability of BEC can be determined by the known criterion. For \( \alpha > 0 \) and \( g_{1d} > 0 \), the stability criterion reads [14] \( \mu = \mu_s = V(x_0) + g_{1d}|\psi(x_0)|^2 \). This implies that for the harmonic potential and repulsive interaction the stationary state of Eq. (12) is stable, since the \( \psi(x_0) \) meets the stability criterion.

In conclusion, for an atomic BEC created initially in a range near the potential center and driven by the time-periodic harmonic or inverted-harmonic potential, we have demonstrated that the classical Smale-horseshoe chaos certainly exists in the time evolutions of the system. The formally exact solution of the time-dependent GPE, whose amplitude and phase depend on the solutions of the famous Duffing equation with periodic driving and quadratic damping, has been constructed, which describes the matter shock waves with chaotic or periodic amplitude and phase. When the periodic driving is switched off and the number of condensed atoms is conserved, we arrive at the most studied BEC systems with parabolic potentials. Then the exact solutions become the explicit functions of spatiotemporal coordinates and govern the exact non-stationary states or the exact stable stationary states. The stationary states are called the ‘non-propagated’ shock wave. The non-stationary shock waves alternately collapse and grow for the harmonic trapping and repulsive interaction or propagate with exponentially increased shock-front speed for the antitrapping and attractive interaction. It is revealed that existence of chaos play a role for suppressing the blast of matter wave. The results can be observed experimentally and suggest a useful method for preparing the exponentially accelerated matter shock wave or the stable stationary state of the BEC system.

The well-known criterion for the onset of temporal chaos is the Melnikov criterion based on the nonlinear ordinary differential equations. In order to apply such a criterion to demonstrate the chaotic behaviors of Eq. (1), we have to consider the ansatz (3) governed by the reduced ordinary differential equations (4) and (6). In general, the investigation of spatiotemporal chaos also can be performed directly from the nonlinear partial differential equation (1), by using the Deissler-Kaneko criterion [15], which relies on the determination of the time evolution of a function defined by the integral of the square modulus of the difference between wave functions with nearby initial conditions [28].

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