Super heat kernel and one-loop divergence of super Yang–Mills theory in conformal supergravity

Ka-Hei Leung

Department of Physics and Theoretical Physics Group, Lawrence Berkeley National Laboratory, University of California, Berkeley, California 94720, USA
E-mail: kaheileung@berkeley.edu

Received May 18, 2019; Revised August 21, 2019; Accepted August 22, 2019; Published October 10, 2019

We consider super Yang–Mills (SYM) theory in \( N = 1 \) conformal supergravity. Using the background field method and the Faddeev–Popov procedure, the quantized action of the theory is presented. Its one-loop effective action is studied using the heat kernel method. We shall develop a non-iterative scheme, generalizing the non-supersymmetric case, to obtain the super heat kernel coefficients. In particular, the first three coefficients, which govern the one-loop divergence, will be calculated. We shall also demonstrate how to schematically derive the higher-order coefficients. The method presented here can be readily applied to various quantum theories. We shall, as an application, derive the full one-loop divergence of SYM in conformal supergravity.

Subject Index B11, B16

1. Introduction

Supergravity has been a subject of interest over the past thirty years. It is believed that it could represents the low-energy effective model of a more fundamental theory of quantum gravity, for instance string theory. Therefore, despite not being renormalizable in a similar way to its non-supersymmetric counterpart, understanding supergravity at the quantum level may provide insights on how quantum gravity behaves away from the Planck scale. In particular, supergravity at the one-loop level has been extensively studied as it captures the majority of the features of quantum supergravity. A considerable amount of work has been done on topics like one-loop divergences, regularization, and anomalies, some examples being Refs. [1–4].

A natural extension of supergravity is to introduce conformal symmetry, and the class of models with this extra symmetry is naturally called conformal supergravity. Historically it was introduced by Kaku, Townsend, van Nieuwenhuizen, Ferrara, and Grirasu [5–8]. Conformal supergravity was initially developed in the component approach. But Butter [9] later discovered it can be described as a theory with superfields in conformal superspace. It is known that one can reduce the theory to the ordinary Poincaré supergravity, or the \( U(1)_K \) supergravity developed by Binétruy, Girardi, and Grimm [10], by choosing a special gauge. Hence, by studying conformal supergravity, which seemed to be a restricted class of theories, one can also understand supergravity in general.

In the following we shall consider super Yang–Mills theory in conformal supergravity, which by gauge fixing covers the case of usual Poincaré supergravity as well. We will examine its one-loop effective action via the heat kernel method first developed by de Witt [11]. The central idea is to have
a series expansion of the quantum heat kernel, with certain coefficients of the expansion encoding one-loop divergences of the theory. It is possible to evaluate such coefficients starting from the lowest order by either a recursive method, as was done originally by de Witt, or via non-recursive means, for example the methods of Avramidi [12]. However, most of the work done has been on non-supersymmetric theories. In this work we shall describe how to extend Avramidi’s method to the case of supersymmetric theories. This allows us to calculate super heat kernel coefficients up to any order non-iteratively, more effectively than, for instance, the recursive method in Ref. [13], or the Fourier transform method in Refs. [14,15].

This work is divided into two parts. In the first we shall discuss the quantization of the super Yang–Mills (SYM) theory in conformal supergravity. The quantized quantum action, which will be crucial in one-loop studies, will be derived. The second part will be devoted to studying the one-loop effective action and divergence of the theory. By using the heat kernel method, we shall ultimately calculate the full one-loop divergence of the SYM theory.

2. Conformal supergravity

In the following the superspace formulation of $\mathcal{N} = 1$ conformal supergravity discovered by Butter [9] will be reviewed, which can be shown to be equivalent to the original component approach. A detailed comparison of the two formalisms can be found in Refs. [16,17]. We shall closely follow the notation and convention of Butter’s original paper.

2.1. Superconformal algebra and conformal superspace

We start by constructing the superspace from the superconformal algebra. Consider first the case without supersymmetry; the Poincaré algebra is

\[
[M_{ab}, M_{cd}] = \eta_{bc} M_{ad} + \eta_{ad} M_{bc} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac},
\]

\[
[M_{ab}, P_c] = P_a \eta_{bc} - P_b \eta_{ac},
\]

\[
[P_a, P_b] = 0.
\]  

(1)

Note that the definition of the generators is so that they are anti-Hermitian, and the structure constants of the algebra are real. For example, the differential representation for $P_a$ is $P_a = \partial_a$. One can extend the algebra to the conformal one by adding the dilatation operator $D$ and the special conformal transformation $K_a$, the extra commutation relations being:

\[
[D, P] = P, \quad [D, K_a] = -K_a, \quad [D, M_{ab}] = 0,
\]

\[
[M_{ab}, K_c] = K_a \eta_{bc} - K_b \eta_{ac},
\]

\[
[K_a, P_b] = 2 \eta_{ab} D - M_{ab}.
\]  

(2)

Now let us introduce $\mathcal{N} = 1$ supersymmetry by adding the fermionic operator $Q_\alpha$ and its conjugate $\bar{Q}^{\dot{\alpha}}$. The well-known graded commutation relation is, with $(\sigma^{\alpha\beta}_{ab})^{\dot{\alpha}}_{\dot{\beta}}$ being the Lorentz generator in the spinor representation,

\[
[Q_\alpha, \bar{Q}^{\dot{\beta}}] = -2i \sigma^{\alpha\beta}_{ab} P_\alpha,
\]

\[
[M_{ab}, Q_\alpha] = (\sigma^{\alpha\beta}_{ab})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^{\dot{\beta}},
\]

\[
[P_\alpha, Q_\alpha] = 0.
\]  

(3)
Here, the grading will be implicit as in Ref. [9], and thus [ , ] is actually the graded commutator, 
\[ [A, B] = AB - (-1)^{AB}BA, \]
with a handy notation being adopted: the exponent in \((-1)^A\) is the fermion number of the generator \(A\), which equals 0 if \(A\) is bosonic and 1 if fermionic. The conformal extension requires three extra operators, the chiral rotation generator \(A\) and the fermionic counterparts of \(K_a\), \(S_a\) and \(\bar{S}^\dot{a}\). The commutation relations will be:

\[
\begin{align*}
[D, Q_a] &= \frac{1}{2} Q_a, & [D, \bar{Q}^\dot{a}] &= \frac{1}{2} \bar{Q}^{\dot{a}}, \\
[D, S_a] &= -\frac{1}{2} S_a, & [D, \bar{S}^\dot{a}] &= -\frac{1}{2} \bar{S}^{\dot{a}}, \\
[A, Q_a] &= -iQ_a, & [A, \bar{Q}^\dot{a}] &= i\bar{Q}^{\dot{a}}, \\
[A, S_a] &= iS_a, & [A, \bar{S}^\dot{a}] &= -i\bar{S}^{\dot{a}}, \\
[K_a, Q_a] &= i\sigma_{aa'\dot{b'}} \bar{S}^{\dot{b'}}, & [K_a, \bar{Q}^\dot{a}] &= i\bar{\sigma}^{\dot{a}b} S_b, \\
[M_{ab}, S_a] &= (\sigma_{ab})^{\dot{a}b} S_b, & [S_a, \bar{S}^\dot{a}] &= 2i\bar{\sigma}^{\dot{a}a} K_a, \\
[S_a, P_a] &= i\sigma_{aa'b} \bar{Q}^{\dot{b}}, & [\bar{S}^{\dot{a}}, P_a] &= i\bar{\sigma}^a_{\dot{a}} \bar{Q}_{\dot{b}}, \\
[S_a, Q_{\beta}] &= (2D - 3i\delta) \epsilon_{\alpha\beta} - 2M_{ab}, & [\bar{S}^{\dot{a}}, \bar{Q}^{\dot{b}}] &= (2D + 3i\delta) \epsilon^{\dot{a}\dot{b}} - 2M^{\dot{a}\dot{b}}. 
\end{align*}
\]

Other commutation relations not shown above are just zero. Here, \(M_{ab} = (\sigma^{ba} \epsilon)_{\alpha\beta} M_{ab}\) and \(M^{\dot{a}\dot{b}} = (\bar{\sigma}^{ba} \epsilon)_{\dot{a}\dot{b}} M_{ab}\); roughly speaking, they are projections of \(M_{ab}\) that transform only undotted spinors and dotted spinors respectively.

The conformal superspace can be constructed by gauging the above algebra. The space has the collection of coordinates \(z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})\). Associated with each operator \(X_A\) in the algebra is a gauge connection field \(W_M^A\):

\[
W_M^A X_A = E_M^A P_A + \frac{1}{2} \phi_{ab} M_{ab} + B_M D + A_M A + f_M^A K_A
= E_M^A P_A + h_M^A X_A;
\]

(5)

\(E_M^A\) is the supervielbein, and \(X_A\) denotes all generators except \(P_A\). \(h_M^A\) will be the connection corresponding to the generator, for instance \(B_M\) is the connection field for \(D\). An important fact is that a connection encodes the infinitesimal gauge transformation \(\delta_g\) of a field:

\[
\mathcal{L}_\xi \Phi(x) = \delta_g (\xi^M W_M^A X_A) \Phi(x);
\]

(6)

\(\mathcal{L}_\xi\) is the Lie derivative or, equivalently, an infinitesimal general coordinate transformation with parameter \(\xi_M\): \(\mathcal{L}_\xi = \delta_{GC}(\xi)\). If \(\Phi\) is a scalar without any Einstein indices, we have \(\mathcal{L}_\xi \Phi = \xi^M \partial_M \Phi\), and thus

\[
\partial_M \Phi = E_M^A P_A \Phi + h_M^A X_A \Phi.
\]

(7)

This in particular implies that \(E_M^A P_A\) acts as the covariant derivative:

\[
E_M^A P_A \Phi = \nabla_M \Phi = (\partial_M - h_M^A X_A) \Phi,
\]

(8)
and inverting the vielbein gives

\[ P_A \Phi = E_A^M \nabla_M \Phi = \nabla_A \Phi. \]  \hspace{1cm} (9)

This certainly makes sense as \( P_A \) should represent translations.

To introduce curvature to the space, one deforms the conformal algebra such that \([P_A, P_B]\) develops a non-zero commutator, while retaining other commutation relations:

\[ [P_A, P_B] = -R_{AB}^C X_A \]
\[ = -T_{AB}^C P_C - \frac{1}{2} R_{AB}^{dc} M_{cd} - H_{AB} D - F_{ABA} - R(K)_{AB}^C K_C. \]  \hspace{1cm} (10)

Objects like \( H_{AB}, F_{AB} \) will be the curvature field of the corresponding generator. These curvature terms can be expressed in terms of the connection fields in Eq. (5). Notice that the connection fields transform under an infinitesimal transformation, with parameter \( g = g^A X_A \), as

\[ \delta_g W_M^A = \delta_{MS}^B X_B + W_M^B g^C f_{CB}^A. \]  \hspace{1cm} (11)

\( f_{CB}^A \) here are the structure constants of the algebra:

\[ [X_C, X_B] = -f_{CB}^A X_A. \]  \hspace{1cm} (12)

Recall that Eq. (6) relates gauge transformations to coordinate transformations; requiring the consistency relation \( [\delta_G (\xi), \delta_G (\chi)] = -\delta_G ([\xi, \chi]) \) results in

\[ \partial_M W_N^A - \partial_N W_M^A - W_M^B W_N^C f_{CB}^A = 0. \]  \hspace{1cm} (13)

In other words, the gauge curvature of the connection one-form \( W^A \) vanishes. By splitting the generators into \( P_A \) and \( X_A \), we have an expression for \( R_{AB}^A \) as

\[ R_{MN}^A = \partial_M W_N^A - \partial_N W_M^A - (E_M^B h_N^C - E_N^B h_M^C) f_{CB}^A - h_M^B h_N^C f_{CB}^A, \]  \hspace{1cm} (14)
\[ R_{AB}^A = E_B^N E_A^M R_{MN}^A. \]  \hspace{1cm} (15)

Alternatively, in differential form notation the expression is

\[ R^A = dW^A - E^B \wedge h^C_{CB} f_{CB}^A - \frac{1}{2} h^B_{CB} \wedge h^C_{CB} f_{CB}^A. \]  \hspace{1cm} (16)

2.2. Constraints

The curvature introduced certainly contains too many degrees of freedom to make a sensible theory. Similar to the case of Poincaré supergravity, one imposes constraints to reduce the number of dynamical fields appearing. For example, demanding chiral fields exist will impose \( \{ \tilde{\nabla}_{\dot{\alpha}}, \tilde{\nabla}_{\dot{\beta}} \} = 0 \). The following are the constraints imposed:

(1) Torsion constraints:

\[ T_{\alpha\beta}^C = T_{\dot{\alpha}\dot{\beta}}^C = 0, \]
\[ T_{\alpha}\dot{\beta}^c = 2i \sigma_{\alpha\dot{\beta}}^c, \]
\[ T_{\alpha}\dot{\beta}^c = T_{\dot{\alpha}}\dot{\beta}^c = 0, \]  \hspace{1cm} (17)

where \( \dot{\beta} \) means either \( \beta \) or \( \dot{\beta} \).
(2) Lorentz curvature constraints:

\[ R_{\alpha\beta}^{ab} = 0. \]  

(18)

(3) Chiral curvature and dilatation curvature constraints:

\[ F_{\alpha\beta} = F_{\alpha b} = 0, \]
\[ H_{\alpha\beta} = H_{\alpha b} = 0. \]  

(19)

(4) Special conformal curvature constraints:

\[ R(K)_{\alpha\beta}^{C} = 0. \]  

(20)

Note that the constraints above show that the covariant derivatives satisfy

\[ \{\nabla_\alpha, \nabla_\beta\} = \{\bar{\nabla}_{\dot{\alpha}}, \bar{\nabla}_{\dot{\beta}}\} = 0, \]
\[ \{\nabla_\alpha, \bar{\nabla}_{\dot{\beta}}\} = -2i\nabla_\alpha \dot{\beta}. \]  

(21)

This is analogous to the Yang–Mills case of flat superspace.

One has to verify that the above sets of constraints are valid by solving the Bianchi identities. In this case one finds that the curvature can be expressed by a gaugino-like field \( W_\alpha = W_\alpha^{A}X_{A} \):

\[ R_{\alpha,\beta\dot{\beta}} = 2i\epsilon_{\alpha\beta\dot{\beta}}W_\beta, \]
\[ R_{\dot{\alpha},\dot{\beta}\beta} = 2i\epsilon_{\dot{\alpha}\dot{\beta}\beta}W_\beta, \]
\[ R_{\alpha\dot{\alpha},\beta\dot{\beta}} = -\epsilon_{\alpha\dot{\beta}}\{\nabla_\alpha, W_\beta\} - \epsilon_{\dot{\alpha}\dot{\beta}}\{\bar{\nabla}_{\dot{\alpha}}, W_\dot{\beta}\}. \]  

(22)

Here, \( R_{\alpha,\beta\dot{\beta}}^{A} = \sigma_{\alpha\dot{\beta}} R_{\alpha\beta\dot{\beta}}, \)
\( R_{\dot{\alpha},\dot{\beta}\beta}^{A}, \) and \( R_{\alpha\dot{\alpha},\beta\dot{\beta}}^{A} \) are similarly defined.\(^1\) These fields satisfy the following conditions:

\[ \{\nabla_\alpha, W_\beta\} = \{\bar{\nabla}_{\dot{\alpha}}, W_\dot{\beta}\} = 0, \]
\[ \{\nabla_\alpha, W_\dot{\beta}\} = \{\bar{\nabla}_{\dot{\alpha}}, W_\beta\}. \]  

(23)

It also turns out that \( W_\alpha \) has no \( P_A, D, \) or \( A \) components:

\[ W_\alpha = \frac{1}{2}W(M)_{\alpha}{}^{cb}M_{bc} + W(K)_{\alpha}{}^{A}K_{A}, \]  

(24)

and it can be expressed in terms of one single symmetric chiral field \( W_{\alpha\beta\gamma} \), the details of which are omitted here. The field \( W_{\alpha\beta\gamma} \) satisfies

\[ \bar{\nabla}_{\dot{\alpha}}W_{\alpha\beta\gamma} = 0, \]
\[ DW_{\alpha\beta\gamma} = \frac{3}{2}W_{\alpha\beta\gamma}, \]
\[ AW_{\alpha\beta\gamma} = iW_{\alpha\beta\gamma}, \]
\[ K_{A}W_{\alpha\beta\gamma} = 0, \]
\[ \nabla_\beta \nabla_\gamma \dot{\alpha}W_{\gamma}\beta\alpha = -\nabla_\beta \nabla_\gamma \dot{\alpha}W_{\gamma}\beta\dot{\alpha}. \]  

(25)

All curvature terms can be expressed in terms of \( W_{\alpha\beta\gamma} \) and its conjugate, and the details can be found in the original reference, Ref. [9]. Thus, \( W_{\alpha\beta\gamma} \) is the only dynamical field encoding the geometry of the conformal superspace.

\(^1\) One can consult Appendix D of Ref. [9] for the detailed derivation.
2.3. Conformal supergravity action

Matter in conformal superspace is described by primary superfields. These fields satisfy the following conditions:

\[ M_{ab}\Phi = S_{ab}\Phi, \quad D\Phi = \Delta\Phi, \quad A\Phi = iw\Phi, \quad K_A\Phi = 0. \]  

(26)

\( S_{ab}\) are Lorentz generators in the appropriate representation, \( \Delta \) and \( w \) are respectively called scaling and chiral weight. Note that \( K_A\Phi = 0 \) is a non-trivial condition that has to be carefully dealt with; for instance, \( \Phi \) being primary does not guarantee that \( K_A\nabla B\Phi = 0 \). A primary field is chiral if in addition \( \bar{\nabla} \Phi = 0 \). Chiral fields have only undotted indices, just like the usual supergravity case, and must have \( 2\Delta = 3w \). For example, \( W_{\alpha\beta\gamma} \) appearing in the curvature terms is a primary chiral superfield with weights \( (\Delta, w) = (3/2, 1) \).

General actions in conformal superspace are the familiar D-term and F-term integrations. A D-term expression is given by:

\[ S_D = \int d^4x d^4\theta \ E V; \]  

(27)

\( E = \det E_A^M \) is the (super)-determinant of the vielbein, and one can easily show

\[ AE = K_A E = 0, \quad DE = -2E. \]  

(28)

This implies that \( V \) must be a real primary field with weights \( (\Delta, w) = (2, 0) \), and obviously it also has to be a Lorentz scalar. An F-term will be of the form

\[ S_F = \int d^4x d^2\theta \ E W, \]  

(29)

with the integral evaluated at the subspace \( \bar{\theta} = 0 \); here, \( E = \det E_A^M \), and \( E^M_A \) is \( E_M^A \) but with dotted indices omitted. \( W \) ought to be a primary chiral field with weights \( (\Delta, w) = (3, 2) \); again, it has to be a Lorentz scalar.

One can convert a D-term action to F-term via the chiral projector, similar to the role of \(-1/4(\bar{D}^2 - 8R)\) in ordinary supergravity. The expression is surprisingly simple in the conformal superspace:

\[ \mathcal{P}[V] = -\frac{1}{4} \bar{\nabla}^2 V. \]  

(30)

The D-term to F-term conversion formula is

\[ \int d^4x d^4\theta \ E V = \frac{1}{2} \int d^4x d^2\theta \ E \mathcal{P}[V] + \frac{1}{2} \int d^4x d^2\theta \bar{\mathcal{P}}[V]. \]  

(31)

Suppose we have multiple chiral superfields \( \Phi_I \); the most general action one can write down is

\[ S = [-3Z(\Phi_I, \Phi_I)]_D + ([P(\Phi_I)]_F + \text{h.c.}) \]  

(32)

and we can reasonably assume that \( Z \) is non-negative for stability. By a suitable redefinition of the fields, one can have all but one chiral field, \( \Phi^I \), have vanishing weights, and the remaining one, \( \Phi_0 \), have weights \( (\Delta, w) = (1, 2/3) \).\(^2\) This implies that one can write

\(^2\) Recall that primary chiral fields have a fixed ratio for the weights.
\[ Z = \Phi_0 \Phi_0 \exp \left( -\frac{K(\Phi_i, \Phi^i)}{3} \right), \quad P = \Phi_0^3 W(\Phi^i), \] (33)

and we arrive at the supergravity action in the conformal compensator formalism appearing in Ref. [18]:

\[ S = -3 \int d^4xd^4\theta \, \Phi_0 \Phi_0 e^{-K/3} + \left( \int d^4xd^2\theta \, \mathcal{E} \Phi_0^3 W + \text{h.c.} \right). \] (34)

Note that there is a natural Kähler structure in this model, as the action is invariant under the following Kähler transformation:

\[ K \to K + F + \bar{F}, \quad \Phi_0 \to \Phi_0 \exp(F/3), \quad W \to \exp(-F)W, \] (35)

where \( F \) is a holomorphic function of the fields \( \Phi^i \). This in particular implies that the supergravity Lagragian depends only on the combination \( G = K + \log |W|^2 \), as in the case of Poincaré supergravity.

It is easy to also introduce the Yang–Mills interaction to conformal supergravity [17]. Suppose we have an internal symmetry group with generators \( \{X_{(r)}\} \) and commutation relations

\[ [X_{(r)}, X_{(s)}] = -f_{(r)(s)}^{(t)} X_{(t)}; \] (36)

one just extends the super-conformal algebra to include these extra generators. We have the extra gauge connection fields

\[ W_M^{A} X_A = E_M^A P_A + h_M^A X_A + A_M^{(r)} X_{(r)}, \] (37)

with the corresponding curvature being

\[ [P_A, P_B] = -R_{AB}^A X_A \\
= -T_{AB}^C P_C - R_{AB}^A X_A - \mathcal{F}^{(r)}_{AB} X_{(r)}. \] (38)

The Yang–Mills curvature is given by the familiar expression:

\[ \mathcal{F}^{(r)}_{MN} = \partial_M A_N^{(r)} - \partial_N A_M^{(r)} + A_M^{(s)} A_N^{(t)} f_{(s)(t)}^{(r)}. \] (39)

We impose the following constraint on this curvature:

\[ \mathcal{F}^{(r)}_{\alpha\beta} = 0, \] (40)

to guarantee that we maintain the derivative relations \( \{\nabla_\alpha, \nabla_\beta\} = \{\bar{\nabla}_{\dot{\alpha}}, \bar{\nabla}_{\dot{\beta}}\} = 0 \) and \( \{\nabla_\alpha, \bar{\nabla}_{\dot{\beta}}\} = -2i\nabla_{\alpha\dot{\beta}} \). Therefore, we can express the remaining curvature in terms of gaugino superfields \( \mathcal{W}_{(r)}^{(\alpha\beta)} \):

\[ \mathcal{F}^{(r)}_{\alpha\beta\dot{\beta}} = 2i\epsilon_{\alpha\beta\dot{\beta}} \mathcal{W}_{(r)}^{(\alpha\beta)}, \]
\[ \mathcal{F}^{(r)}_{\dot{\alpha}\beta\dot{\beta}} = 2i\epsilon_{\dot{\alpha}\beta\dot{\beta}} \mathcal{W}_{(r)}^{(\alpha\beta)}, \]
\[ \mathcal{F}^{(r)}_{\alpha\dot{\alpha}\dot{\beta}} = -\epsilon_{\dot{\alpha}\beta\dot{\beta}} \{\nabla_\alpha, \mathcal{W}_{(r)}^{(\beta\dot{\beta})}\} - \epsilon_{\alpha\beta\dot{\beta}} \{\bar{\nabla}_{\dot{\alpha}}, \mathcal{W}_{(r)}^{(\beta\dot{\beta})}\}. \] (41)
The gaugino $\mathcal{W}_\alpha^{(r)}$ is a primary chiral superfield with weights $(\Delta, w) = (3/2, 1)$, and satisfies the condition

$$\nabla^\alpha \mathcal{W}_\alpha^{(r)} = \bar{\nabla}_\beta \mathcal{W}_\beta^{(r)\dot{\beta}}. \quad (42)$$

The Yang–Mills action has the usual form,

$$S_{\text{YM}} = \frac{1}{4} \int d^4x d^2\theta \epsilon f_{(r)(s)} \mathcal{W}_\alpha^{(r)\alpha} \mathcal{W}_\alpha^{(s)} + \text{h.c.}, \quad (43)$$

$f_{(r)(s)}$ being the gauge kinetic function, which is symmetric in $(r)$ and $(s)$ and transforms under gauge transformation as a symmetric product of two adjoint representations.

Combining Eqs. (34) and (43), we have the conformal supergravity/matter/Yang–Mills system. The Yang–Mills generators acts as a Kähler isometry on $/\Phi_1$, that is, like a holomorphic Killing vector field:

$$X_{(r)} = V_{(r)}^i(\Phi) \frac{\partial}{\partial \Phi_i}; \quad (44)$$

the action on the conjugate field is similar. Such an action is generated by a Killing potential:

$$V_{(r)}^i K_i = -iG_{(r)}, \quad K_i = \frac{\partial K}{\partial \Phi_i}, \quad (45)$$

a relation that can be inverted to obtain $V_{(r)}^i$.\footnote{More technical details on Kähler isometry can be found in Ref. [10], for example.}

## 2.4. Reduction to Poincaré supergravity

We have introduced the conformal supergravity model coupled to chiral and Yang–Mills matter. It is natural to ask how it compares to the ordinary Poincaré supergravity. Since the conformal case is a theory with a bigger gauged isometry group, one might expect Poincaré supergravity to appear as a certain gauge-fixed version of conformal supergravity. This is indeed the case, and we shall describe below how this is achieved.

One key observation is that the gauge field $B_M$ corresponding to $D$ is rotated by the $K_A$ transformation, and it is actually possible, by a suitable choice of transformation parameter, to obtain the $K_A$-gauge condition:

$$B_M = 0. \quad (46)$$

One immediate consequence will be that $D$ drops out from the covariant derivative. Since we fix the $K_A$ gauge and thus it is no longer a symmetry, its gauge field $f_M^A$ becomes an auxiliary field instead. In particular, if we consider the constraint for the $D$-curvature, $H_{\alpha\beta\gamma} = 0$, under this gauge choice, we have

$$f_{\alpha\beta} = -\epsilon_{\alpha\beta} R, \quad f_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} R, \quad f_{\alpha\dot{\beta}} = -f_{\dot{\alpha}\beta} = -\frac{1}{2} G_{\alpha\dot{\beta}}. \quad (47)$$
These fields turn out to be exactly the auxiliary fields appearing in ordinary supergravity. We also redefine the curvatures such that no $f_M A$ dependence appears:

$$\tilde{R}_{AB}^A = R_{AB}^A + R(f)_{AB}^A,$$  \hspace{1cm} (48)

where $\tilde{R}_{AB}^A$ denotes the curvature before gauge fixing, and $R(f)_{AB}^A$ means the terms in the original curvature that contain $f_M A$. The gauge-fixed curvature $R_{AB}^A$ will have a new set of constraints, and can be expressed in terms of the auxiliary $f_M A$. The details can be found in Ref. [9], the net result being that all these curvature relations are exactly those of $U(1)$-superspace as introduced in Ref. [10]. Therefore, the gauge choice in Eq. (46) reduces the conformal superspace to the $U(1)$-superspace.

The next step is to map the conformal supergravity action to the one in $U(1)$K supergravity. Recall the supergravity action in Eq. (34); one gets the correct normalization of the Einstein–Hilbert term by imposing

$$\Phi_0 = \tilde{\Phi}_0 = \exp \left( \frac{K}{6} \right).$$  \hspace{1cm} (49)

This is actually possible by choosing the appropriate $D$ and $A$ gauge. Since a gauge for $A$ is chosen, its gauge field is now auxiliary. The conformal chiral condition for $\Phi_0$, $\dot{\nabla}_\alpha \exp \left( \frac{K}{6} \right) = 0$, and its conjugate give

$$A_\alpha = \frac{i}{4} D_\alpha K = \frac{i}{4} K_i D_\alpha \Phi^i, \quad A_{\dot{\alpha}} = -\frac{i}{4} D_{\dot{\alpha}} K = -\frac{i}{4} K_i D_{\dot{\alpha}} \Phi^i.$$  \hspace{1cm} (50)

Here, $D_\alpha$ is the $U(1)$ superspace covariant derivative, with only Lorentz, Yang–Mills, and $A$ generators appearing, and $K_i = \partial_{\Phi^i} K$. After some algebra with constraints, one also gets

$$A_{\alpha \dot{\alpha}} = \frac{i}{4} \left( K_{ij} D_{ij} \Phi^i - K_{ij} D_{ij} \Phi^i \right) + \frac{1}{4} K_{ij} D_\alpha \Phi^i \dot{D}_{\dot{\alpha}} \Phi^j - \frac{3}{2} G_{\alpha \dot{\alpha}},$$  \hspace{1cm} (51)

where $K_{ij}$ is the Kähler metric: $K_{ij} = \partial_{\Phi^i} \partial_{\Phi^j} K$. These expressions are in perfect agreement with those in the $U(1)$ superspace. Hence we have shown the equivalence between the gauge-fixed conformal supergravity and $U(1)_K$ supergravity.

Of course, one can also consider a different gauge choice, a reasonable one being $\Phi_0 = 1$. This will give another commonly seen supergravity action, albeit with a non-canonical Einstein–Hilbert normalization. In practice, one employs a specific rescaling of the fields such that the normalized Einstein–Hilbert term is recovered. In our language, such rescaling is nothing but a $D$ and $A$ gauge transformation that changes the gauge condition to the one in Eq. (49).

### 3. Quantization of super Yang–Mills theory in conformal supergravity

Supersymmetric Yang–Mills theory in conformal supergravity will be our main subject to examine, which turns out to contain both crucial ingredients of an $N = 1$ supersymmetric theory: the quanta of super Yang–Mills theory itself form a vector superfield, and chiral fields appear from the gauge-fixing procedure. In the following we shall consider its quantization, in preparation for studying the one-loop effective action.

#### 3.1. Prepotential and the background field method

In conformal supergravity with Yang–Mills interaction we have the covariant derivative algebra

$$\{\nabla_\alpha, \nabla_\beta\} = \{\dot{\nabla}_\dot{\alpha}, \dot{\nabla}_\dot{\beta}\} = 0,$$
\[
\begin{align*}
\{\nabla_\alpha, \bar{\nabla}_{\bar{\beta}}\} &= -2i \nabla_{\alpha \bar{\beta}}, \\
[\nabla_\alpha, \nabla_\beta] &= -2i \epsilon_{\alpha \beta} \mathcal{W}, \\
[\nabla_\alpha, \nabla_{\bar{\beta}}] &= -2i \epsilon_{\alpha \bar{\beta}} \mathcal{W}. 
\end{align*}
\] (52)

This set of equations has an identical form to the SYM theory in flat superspace, except that the gaugino \(\mathcal{W}_\alpha\) has components from the conformal algebra, apart from the usual gauge ones. Hence, with some modification from the flat case, it is not difficult to handle the SYM theory in conformal superspace.

The derivative algebra in Eq. (52) imposes constraints that can be solved by introducing the Yang–Mills prepotential.4 The spinor covariant derivatives are given by

\[
\nabla_\alpha = S^{-1} \nabla_{0\alpha} S, \quad \nabla_{\dot{\alpha}} = T^{-1} \nabla_{0\dot{\alpha}} T. 
\] (53)

Here, \(\nabla_0\) with a 0 subscript denotes the covariant derivatives without Yang–Mills interactions. These prepotentials possess the following transformation freedom:

\[
S \rightarrow PSg^{-1}, \quad T \rightarrow QTg^{-1}, 
\] (54)

with \(g\) the gauge transformation parameter, and \(P, Q\) respectively anti-chiral and chiral with respect to \(\nabla_0\).

In the flat scenario one may make use of the above freedom to arrive at the chiral representation, in which the dotted spinor derivative remains unchanged when the Yang–Mills interaction is turned on: \(\nabla_{\dot{\alpha}} = \nabla_{0\dot{\alpha}}\). In this setup one needs only to consider the chiral part, but the drawback is that Hermiticity is lost: the covariant derivatives are no longer Hermitian. Alternatively, we shall consider the picture in which the chiral and anti-chiral parts are on an equal footing, and Hermiticity is manifest.

Define the gauge-invariant object \(U = ST^{-1}\), which transforms under chiral transformation as

\[
U \rightarrow PUQ^{-1}. 
\] (55)

The Hermitian representation is such that

\[
\nabla_\alpha = U^{-1/2} \nabla_{0\alpha} U^{1/2}, \quad \nabla_{\dot{\alpha}} = U^{1/2} \nabla_{0\dot{\alpha}} U^{-1/2}. 
\] (56)

This can be achieved by choosing an appropriate \(g\). This \(U\) is connected to the more familiar gauge vector superfield:

\[
U = \exp(-2i V); 
\] (57)

the \(-i\) factor is from the convention that the gauge generators are chosen to be anti-Hermitian. Note that \(V\) is Hermitian and the Yang–Mills gauge connection and gaugino can be expressed in terms of \(V\) as in the literature; see, for instance, Ref. [19].

In the following we shall employ the background field method, which has the advantage that our background expansion has the desired invariance throughout. The treatment in conformal supergravity is very similar to the flat case [20,21], which we closely follow. The background quantum splitting we are using will be of the form

\[
S = S_B S_Q, \quad T = T_B T_Q. 
\] (58)

4 Some discussion on this can be found, for example, in Ref. [10].
The transformation law for $S$ and $T$ in Eq. (54) can be interpreted in two ways. One can view it as a background transformation:

$$
S_B \rightarrow PS_B g^{-1}, \quad S_Q \rightarrow gS_Q g^{-1}, \quad T_B \rightarrow QT_B g^{-1}, \quad T_Q \rightarrow gT_Q g^{-1},
$$

(59)

where one can see the quantum prepotentials transform covariantly under $g$. The same transformation can alternatively be treated as a quantum transformation:

$$
S_B \rightarrow S_B, \quad T_B \rightarrow T_B, \quad S_Q \rightarrow P_Q S_Q g^{-1}, \quad T_Q \rightarrow Q_Q T_Q g^{-1},
$$

(60)

where $P_Q = S_B^{-1} P S_B$, $Q_Q = T_B^{-1} Q T_B$ are respectively background anti-chiral and chiral, that is, with respect to the background derivatives $\nabla_B$.

We might, similarly to the pre-split case, define $U_Q = S_Q T_Q^{-1}$, which transforms as

$$
U_Q \rightarrow P_Q U_Q Q_Q^{-1}.
$$

(61)

Again, by choosing a suitable $g$, we may go between different quantum representations for the covariant derivatives. One of the most common and useful representations will be the quantum-chiral representation, in which $T_Q = 1$, thus

$$
\nabla_\alpha = U_Q^{-1} \nabla_B U_Q, \quad \nabla_\dot{\alpha} = \nabla_B \dot{\alpha}.
$$

(62)

The most notable feature here is that the dotted derivative is identical to the background one. Another important one will be the quantum-Hermitian representation:

$$
\nabla_\alpha = U_Q^{-1/2} \nabla_B U_Q^{1/2}, \quad \nabla_\dot{\alpha} = U_Q^{1/2} \nabla_B \dot{\alpha} U_Q^{-1/2},
$$

(63)

where the chiral part and the anti-chiral part are treated equally.

The quanta for the SYM is the vector superfield $V_Q$, given by

$$
U_Q = \exp(-2iV_Q).
$$

(64)

It is a Hermitian (if we are in background-Hermitian representation) conformally primary superfield with vanishing weights, and as usual transforms as the adjoint representation of the Yang–Mills gauge group. $V_Q$ will be the only quantum object we will care about here; hence, from now on, for notational simplicity, we shall, unless otherwise specified, drop the subscript distinguishing background and quantum: $V$ is always $V_Q$, and other objects are always the background ones.

### 3.2. Quantum field action

We are ready to find the quantum action with the background field splitting discussed. We shall consider first the case with the simplest gauge coupling, the Yang–Mills action being

$$
S_{YM} = \frac{1}{4g^2} \int d^4xd^2\theta \, \mathcal{E} \, \text{tr} (\mathcal{W}_{YM}^\alpha \mathcal{W}_{YM\dot{\alpha}}) + \text{h.c.},
$$

(65)

which corresponds to a canonical gauge kinetic function $f_{(r)(s)} = g^{-2} \delta_{(r)(s)}$. The case with a more general kinetic function will not be discussed here, as it turns out that the quantization procedure is very similar.

For quantization and the one-loop effective action it is sufficient to expand this action up to second order in $V$. One useful observation is that the final result must be representation independent; in
particular, one finds that the quantum-chiral representation is the most convenient, which we shall adopt. Note that the derivative algebra in Eq. (52) tells us that we can express the gaugino field in terms of covariant derivatives:

$$W_\alpha = \frac{1}{8} [\nabla_\alpha, \{\nabla^\beta, \nabla_\beta\}]^\cdot. \quad (66)$$

In the quantum-chiral representation the dotted derivatives have no $V_Q$ dependence, and using Eqs. (62) and (64) one easily finds that the quantum SYM gaugino is given by

$$W_{YM, Q\alpha} = W_{YM, B\alpha} - i\frac{1}{4} \tilde{\nabla}^2 \nabla_\alpha V + \frac{1}{4} \tilde{\nabla}^2 [V, \nabla_\alpha V] + O(V^3). \quad (67)$$

One can actually go further and obtain a closed-form expression:

$$W_{YM, Q\alpha} = W_{YM, B\alpha} - i\frac{1}{4} \tilde{\nabla}^2 \left[ \sum_{n=0}^{\infty} \frac{(L_{2iV})^n}{(n+1)!} \nabla_\alpha V \right]$$

$$= W_{YM, B\alpha} - i\frac{1}{4} \tilde{\nabla}^2 \frac{1 - \exp(L_{2iV})}{L_{-2iV}} \nabla_\alpha V, \quad (68)$$

where $L_X$ is the commutator: $L_X Y = [X, Y]$. But in practical calculations, one can choose a Wess–Zumino type of gauge and then only terms up to second order in $V$ will be important.

Next we have to expand the SYM action, sorted by orders of $V$. The zeroth order is just the classical action of Eq. (65). The first-order term gives us the equation of motion:

$$\nabla^\alpha W_{YM\alpha} + \text{h.c.} = 0. \quad (69)$$

The Bianchi identity on $W_{YM\alpha}$ actually implies that the two terms in Eq. (69) vanish individually. The result here is exactly the same as the flat superspace case. Now the second-order term is given by

$$S^{(2)}_{YM} = \frac{1}{16g^2} \int d^4xd^4\theta \ E \text{ tr } \left( \nabla^\alpha V \tilde{\nabla}^2 \nabla_\alpha V - 4W_{YM\alpha}^\alpha [V, \nabla_\alpha V] \right) + \text{h.c.} \quad (70)$$

Here, the chiral projector $P = -\tilde{\nabla}^2/4$ is used to convert chiral integrals into the full superspace integration. To properly quantize the theory, one has to eliminate the remaining gauge degree of freedom, and this shall be considered next.

Note that if one wants to develop Feynman rules for this theory, the above expression, after the gauge-fixing procedure, will give us the gauge field propagator, and the terms of higher order in $V$ will become the interaction vertices. Combined with the rules for ghost fields, which we will discuss shortly, it is possible to perform amplitude calculations graphically, just like the known case in the flat limit [20].

3.3. **Gauge fixing and the ghost action**

Recall that the quantum vector superfield $V$ was defined from the quantum prepotential, which has the transformation law

$$U = \exp(-2iV), \quad U \rightarrow PUQ^{-1}, \quad (71)$$
where the background-quantum splitting is already applied, and the subscript $Q$ is dropped for notational simplicity. These extra gauge degrees of freedom must be fixed in the quantization procedure, and in the case of conformal supergravity this is not much different from the flat space setting [21], with only minor modifications required.

The above gauge freedom has two free parameters $P$ and $Q$, which are respectively background anti-chiral and chiral, thus one can impose a chiral and an anti-chiral gauge condition. In the flat case, one sets the gauge-fixing function to be

$$\bar{\nabla}^2 V - f = 0, \quad \nabla^2 V - \bar{f} = 0. \tag{72}$$

However, in conformal superspace such a condition is not desirable because of one subtlety: while $\bar{\nabla}^2 V$ is chiral, it is not conformal primary. One easy way to see this is that a primary chiral field with weights $(\Delta, w)$ must have a fixed ratio for the weights, $2\Delta = 3w$; however, as $V$ has zero weights, $\bar{\nabla}^2 V$ will have conformal weights being $(1, 2)$, certainly not satisfying the primary condition.

To fix this it is necessary to introduce the compensator, a well-known object in conformal supergravity where it was first developed in component approaches. For our case in particular we introduce the superfield $X$, which is primary and has conformal weights $(2, 0)$. In the usual conformal supergravity model, $X$ can just be the expression of the D-term action:

$$X = \Phi_0 \Phi_0 e^{-K/3}. \tag{73}$$

One use of this $X$ is to construct the associated derivatives $D_A$, which map conformal primary objects to conformal primary ones, that is, $D_A f$ is automatically primary if $f$ is primary, a property that usual covariant derivatives do not have. This was originally discussed by Kugo and Uehara [23] in the component approach and was also considered in superspace [22]. Although these derivatives are practically very useful, such machinery is not needed here and will not be discussed further.

The introduction of this new superfield allows us to fix the problem of being not conformal primary, now $\bar{\nabla}^2 (X V)$ is a primary chiral field. Hence we shall adopt the gauge choice

$$\bar{\nabla}^2 (X V) - f = 0, \quad \nabla^2 (X V) - \bar{f} = 0. \tag{74}$$

Note that in practical applications one will compare conformal supergravity to the usual Poincaré one by choosing a suitable conformal gauge. In that case we will have $X \to 1$, and it can be shown straightforwardly that in this limit we have

$$\bar{\nabla}^2 (X V) \to (\bar{D}^2 - 8R) V; \tag{75}$$

the familiar chiral projection $\bar{D}^2 - 8R$ appearing is certainly a pleasing feature.

In ordinary Yang–Mills theory without supersymmetry, a covariant quantization with proper gauge fixing requires the famous Faddeev–Popov procedure, with ghost fields having opposite statistics showing up as a result and having non-trivial effects in loop calculations. With supersymmetry we also have the Faddeev–Popov procedure, but in superfield language [21]. The details are shown in Appendix A.

With the gauge condition we are using, the final gauge-fixing action will be

$$S_{g.f.} = \frac{1}{8g^2}\xi \int d^8 z \, E X^{-2} [\bar{\nabla}^2 (X V) \nabla^2 (X V)] + S_{gh}, \tag{76}$$
with $\xi$ being a constant to be determined. The ghost contribution is

$$S_{gh} = \text{tr} \int d^8z E \left\{ X(c' + \bar{c}') \mathcal{L}_{V/2}[c - \bar{c} + \coth(\mathcal{L}_{V/2})(c + \bar{c})] + X^{-2}b\bar{b} \right\}. \quad (77)$$

The field $b$ is the so-called Nielsen–Kallosh ghost, which has abnormal statistics. Note that as $f$, the gauge-fixing functional, is primary chiral from the gauge condition, $b$ is as well. This ghost is normally absent in the regular Faddeev–Popov procedure, and its role is to ensure that additional gauge-fixing terms added to the action are properly normalized. Its appearance here is due to two reasons, one being the factor $X^{-2}$ in the action, which is included to make the Lagrangian a proper $D$-term with the correct behavior under dilation. This makes the averaging factor from the smearing process not strictly a Gaussian, as opposed to the typical case. The second reason is that the gauge condition, and thus $f$, has a non-trivial background field dependence, therefore the term with $f$ alone cannot be normalized.

The Nielsen–Kallosh ghost, as with the usual ghost fields, only takes part in loop calculations and has no classical significance. In fact, in the case of flat space SYM with the background field method, where the compensator $X$ is not needed at all, it only has an effect at the one-loop level. This is because without the $X^{-2}$ factor its action is nothing but that of a free chiral field. In conformal supergravity, if $X$ is kept classical and if we go to the conformal gauge $X = 1$, we have the identical behavior that $b$ is a free field. However, $X$ typically has a Kähler potential dependence in it, so applying the background field expansion of $X$ will generate couplings between $b$ and other chiral fields. But this only gives terms with at least three quantum fields and thus does not affect the one-loop effective action.

In Eq. (77), $c$ and $c'$ are the famous Faddeev–Popov ghosts. In particular, their second-order action will be

$$S_{FP}^{(2)} = \int d^8z EX \text{tr}(c' + \bar{c}')(c + \bar{c}). \quad (78)$$

Note that without the factor $X$, one can remove the pure chiral or pure anti-chiral factors $c'c$ and $\bar{c}'\bar{c}$ via chiral projections, the result being a simple action equivalent to two free chiral ghosts. Similar to the Nielsen–Kallosh ghost, as long as only the one-loop action is of concern, we might treat $X$ as classical and set $X = 1$ in the conformal gauge, and thus the action simplifies as described above. But in general, when $X$ cannot be ignored, one can at least treat such a term as an effective Kähler potential term.

Note that we have three types of chiral ghost, similar to the usual SYM case, and they are all conformal primary. The complete action combined with the original Yang–Mills term allows one to develop the full Feynman rules for graphical treatment, similar to what was historically done for the flat space SYM, for example in Ref. [20].

### 3.4. Simplifying the second-order action

The previous gauge-fixing procedure gives us the full quantized action, with a vector superfield as the quanta and three types of chiral ghosts. The ghost term is simple enough, in the sense that no derivatives occur, so one can easily write down the action up to any order in quantum fields. It is thus not too difficult to perform practical calculations involving ghost fields, say with a graphical method. However, the vector multiplet action requires some extra work to analyze, even if we consider only
the one-loop level. We would like to study its one-loop effective action, and what we shall aim at below is to rewrite the second-order action in the form
\[
S_{YM}^{(2)} = \frac{1}{2} \int d^8 z \, E \, \text{tr}(V \mathcal{O} V),
\]  
(79)
and obtain the key object $\mathcal{O}$.

Let us start with the original second-order action,
\[
S_{YM}^{(2)} = \frac{1}{16g^2} \int d^8 z \, E \left( \nabla^a V \tilde{\nabla}^2 \nabla_a V + \nabla_a V \nabla^2 \nabla^\alpha V - 4\mathcal{V}^\alpha[V, \nabla_a V] + 4\mathcal{V}^{\alpha \beta \gamma}[V, \nabla^\alpha \nabla^\beta \nabla^\gamma V] + S_{\text{g.f.}}^{(V)} \right),
\]
(80)
where $S_{\text{g.f.}}^{(V)}$ is the gauge-fixing term containing $V$ only and without the ghosts. The tricky part is the first two terms, with four spinor derivatives appearing. Usually in the quantization procedure we would like to have terms with two derivatives or less, since otherwise the propagator will be much more difficult to compute and to use. This is precisely where the specific gauge choice comes in. It turns out we can eliminate terms with four derivatives and we are left with those with at most two.

In fact, we shall see that the final result will involve the d’Alembertian $\Box$ in the leading term, which governs the spacetime propagation of the quantum field $V$.

Let us start with the first problematic term, $\nabla^a V \tilde{\nabla}^2 \nabla_a V$. The idea is to manipulate this so that we have the form $V \mathcal{O} V$ for some derivative operator $\mathcal{O}$, and hope that some undesired terms with too many derivatives can be eliminated by the gauge-fixing action. This can be achieved by integration by parts. However, we have to be cautious when using it, as integration by parts is non-trivial in many derivatives.

Let us consider the term $\nabla^a V \tilde{\nabla}^2 \nabla_a V$. We have
\[
\nabla^a V \tilde{\nabla}^2 \nabla_a V = \nabla^a (V \tilde{\nabla}^2 \nabla_a V) - V \nabla^a \tilde{\nabla}^2 \nabla_a V \\
\approx -f^{\alpha \beta \gamma} K_B (V \tilde{\nabla}^2 \nabla_a V) - V \nabla^a \tilde{\nabla}^2 \nabla_a V,
\]
(81)
where $\approx$ denotes equal up to a surface term that can be integrated out under appropriate boundary conditions, which we always assume, and we shall for simplicity not distinguish $\approx$ and $=$ from now on.

To find the effect of $K_B = (K_b, S_\beta, \tilde{S}_\dot{\beta})$ we need an important fact that will be used multiple times: If $V$ is a primary scalar field with vanishing conformal weights, $(\Delta, w) = (0, 0)$ or equivalently $DV = AV = 0$, then $\nabla_a V$ and $\nabla_\dot{a} V$ are also conformal primary. To prove this, it is sufficient to show that both derivatives are annihilated by $S_\beta$ and $\tilde{S}_\dot{\beta}$, as $
abla_a V, \nabla_\dot{a} V$ anti-commute with $\nabla_a V$, so automatically annihilates $\nabla_a V$. We also have
\[
\{S_\beta, \nabla_a\} = (2D - 3iA) \epsilon_{\beta a} - 2M_{\beta a},
\]
(82)
the right-hand side of which vanishes when acting on $V$, and thus $S_\beta$ also annihilates $\nabla_a V$. The case for $\nabla_\dot{a} V$ is similar.

Now we have to find $K_b \tilde{\nabla}^2 \nabla_a V$. For $S_\beta$ it is easy, as $S_\beta$ and $\nabla_\dot{a}$ anti-commute and thus $S_\beta \tilde{\nabla}^2 \nabla_a V = \tilde{\nabla}^2 S_\beta \nabla_a V = 0$, using the fact above. Next, note that for spinors we have $\chi^a \xi_a = -\chi_a \xi^a$, and similarly for the dotted ones. We then have
\[
\tilde{S}_\dot{\beta} \tilde{\nabla}^2 \nabla_a V = (-\{\tilde{S}_\dot{\beta}, \nabla_\dot{a}\} \nabla_a - \nabla_a \{\tilde{S}_\dot{\beta}, \nabla_\dot{a}\} + \tilde{\nabla}^2 \tilde{S}_\dot{\beta}) \nabla_a V \\
\quad = [2M^\dot{\beta \dot{a}} - (2D + 3iA) \epsilon^\dot{\beta \dot{a}}] \nabla_\dot{a} \nabla_a V,
\]
15/40
\[ + \nabla_\alpha [2M^{\dot{b}\dot{\alpha}} - (2D + 3i\lambda)\epsilon^{\dot{b}\dot{\alpha}}] \nabla_\alpha V + 0 \]
\[ = (2M^{\dot{b}\dot{\alpha}} - 2\epsilon^{\dot{b}\dot{\alpha}}) \nabla_\alpha V - 4\nabla_\alpha \epsilon^{\dot{b}\dot{\alpha}} \nabla_\alpha V \]
\[ = 2M^{\dot{b}\dot{\alpha}} \nabla_\alpha V - 6\nabla^{\dot{b}} \nabla_\alpha V. \quad (83) \]

Recall that \( M^{\dot{b}\dot{\alpha}} = (\tilde{\sigma}^{\dot{b}a} \epsilon)^{\dot{b}\dot{\alpha}} M_{ab} \) acts on dotted indices only, and one can show that
\[ M^{\dot{b}\dot{\alpha}} \nabla_\alpha V = (\nabla \dot{\delta} \dot{\delta} \nabla_\alpha V + \epsilon \dot{\delta} \dot{\delta} \nabla_\alpha V) \nabla_\alpha V = 3\nabla \dot{\delta} \dot{\delta} \nabla_\alpha V. \quad (84) \]

This leads to \( \bar{S}^{\dot{b}} \bar{\nabla}^2 \nabla_\alpha V = 0 \), which further implies \( K_b \bar{\nabla}^2 \nabla_\alpha V = 0 \) via the relation \( \{S_\beta, \bar{S}_{\dot{\beta}}\} = 2i\sigma^{\beta \dot{\beta}} K_b \), and so \( \bar{\nabla}^2 \nabla_\alpha V \) is primary. This is actually not a big surprise, as the term \( \bar{\nabla}^2 \nabla_\alpha V \) is derived from the background field expansion of the Yang–Mills gaugino, which is conformal primary. To conclude,
\[ \nabla^\alpha V \bar{\nabla}^2 \nabla_\alpha V = -V \nabla^\alpha \bar{\nabla}^2 \nabla_\alpha V, \quad (85) \]

which is the same as the naive result, but we will see that corrections are necessary for the gauge-fixing term. Similarly, we also have
\[ \nabla_\dot{\alpha} V \nabla^2 \nabla^\dot{\alpha} V = -V \nabla_\dot{\alpha} \nabla^2 \nabla^\dot{\alpha} V. \quad (86) \]

Let us now first turn to the gauge-fixing term, as the terms with the Yang–Mills gaugino are easily dealt with. Let us use cyclicity of traces to split the term into two, treating \( \nabla^2 \) and \( \nabla^2 \) symmetrically:
\[ S^{(V)}_{g.f.} = \frac{1}{16g^2 \xi} \text{tr} \int d^8 z \mathcal{E} X^{-2} [\bar{\nabla}^2 (XY) \nabla^2 (XY) + \nabla^2 (XY) \bar{\nabla}^2 (XY)]; \quad (87) \]

the reason behind this will be clear soon. We expand out the derivatives and write
\[ Y = X^{-1} \bar{\nabla}^2 (XY) = (\bar{\nabla}^2 - 8R) V + 2\nabla_\dot{\alpha} V \nabla^\dot{\alpha} \log X, \]
\[ \bar{Y} = X^{-1} \nabla^2 (XY) = (\nabla^2 - 8\bar{R}) V + 2\nabla^\alpha V \nabla_\alpha \log X. \quad (88) \]

We have defined, as in Ref. [22],
\[ R = -\frac{1}{8X} \bar{\nabla}^2 X, \quad \bar{R} = -\frac{1}{8X} \nabla^2 X. \quad (89) \]

It is not difficult to show that when we choose the conformal gauge \( X \to 1 \) these fields reduce to the auxiliary fields with the same name in Poincaré supergravity. Also in that case, the terms \( \nabla^\alpha \log X \) and \( \nabla^\alpha \log X \) will vanish and so we get back the usual chiral projectors of the ordinary supergravity on the right-hand side of Eq. (88).

Now we invoke the integration by parts formula on the term
\[ \bar{\nabla}^2 V \bar{Y} \]
\[ = \nabla_\dot{\alpha} (\nabla^\dot{\alpha} V \bar{Y}) - \nabla_\dot{\alpha} V \nabla^\dot{\alpha} \bar{Y} \]
\[ = -\nabla_\dot{\alpha} V \nabla^\dot{\alpha} \bar{Y}, \quad (90) \]
as both $\nabla^{\hat{a}} V$ and $\bar{Y}$ are primary. Applying integration by parts once more we get
\begin{align}
\tilde{\nabla}^2 V \bar{Y} = - \nabla^{\hat{a}} V \nabla^{\hat{a}} \bar{Y} \\
= - \nabla^{\hat{a}} (V \nabla^{\hat{a}} \bar{Y}) + V \tilde{\nabla}^2 \bar{Y} \\
= f^{\hat{a}}_\beta V K_B \nabla^{\hat{a}} \bar{Y} + V \tilde{\nabla}^2 \bar{Y}.
\end{align}
\tag{91}

Finding $K_B \nabla^{\hat{a}} \bar{Y}$ is just a routine application of the superconformal algebra, from $\{S_\beta, \nabla^{\hat{a}} \}$ = 0 and $[K_B, \nabla^{\hat{a}}] = i \bar{\sigma}^{\hat{a} \hat{b}} S_\beta$ we immediately know that applying $S_\beta$ or $K_B$ will give zero. Using the anti-commutation relation $\{\tilde{S}^{\hat{b}}, \nabla^{\hat{a}} \} = (2D + 3iA) \epsilon^{\hat{b} \hat{a}} - 2M^{\hat{b} \hat{a}}$ results in
\begin{align}
\tilde{S}^{\hat{b}} \nabla^{\hat{a}} \bar{Y} = 8 \epsilon^{\hat{b} \hat{a}} \bar{Y}.
\end{align}
\tag{92}

The final result is
\begin{align}
\tilde{\nabla}^2 V \bar{Y} = 8 f^{\hat{a}}_\alpha V \bar{Y} + V \tilde{\nabla}^2 \bar{Y} \\
= V \tilde{\nabla}^2 \nabla^{\hat{a}} \bar{Y} + 8 f^{\hat{a}}_\alpha V \bar{Y} \\
+ V \left(2 \nabla^a \log X \tilde{\nabla}^2 \bar{Y} - 8 \bar{R} \tilde{\nabla}^2 - 4 \nabla^{\hat{a}} \nabla^{\hat{a}} \log X \nabla^{\hat{a}} \nabla^\alpha \right. \\
+ \frac{16}{3} \nabla^\alpha \nabla^\alpha - 16 \nabla^{\hat{a}} \bar{R} \nabla^{\hat{a}} - 8 \tilde{\nabla}^2 \bar{R} \right) V.
\end{align}
\tag{93}

Here, we defined $X_\alpha = \frac{3}{8} \tilde{\nabla}^2 \nabla_\alpha \log X, \quad X^{\hat{a}} = \frac{3}{8} \tilde{\nabla}^2 \nabla^{\hat{a}} \log X;
\tag{94}

these reduce, under the conformal gauge, to their $U(1)$ supergravity counterparts, just like the previously defined $R$ and $\bar{R}$.

Next, we have the term $2 \nabla^{\hat{a}} V \nabla^{\hat{a}} \log X \bar{Y}$. One integration by parts gives
\begin{align}
2 \nabla^{\hat{a}} V \nabla^{\hat{a}} \log X \bar{Y} = - 2 f^{\hat{a}}_\beta K_B (V \nabla^{\hat{a}} \log X \bar{Y} - 2 V \nabla^{\hat{a}} (\nabla^{\hat{a}} \log X \bar{Y})) \\
= - 2 f^{\hat{a}}_\beta V \bar{Y} - 2 V (\tilde{\nabla}^2 \log X) \bar{Y} - 2 V \nabla^{\hat{a}} \log X \nabla^{\hat{a}} \bar{Y} \\
= - 2 f^{\hat{a}}_\beta V \bar{Y} - 2 V (\tilde{\nabla}^2 \log X) (\nabla^2 + 2 \nabla^a \log X \nabla^a - 8 \bar{R}) V \\
- 2 V \nabla^{\hat{a}} \log X (\nabla^{\hat{a}} \nabla^2 - 2 \nabla^a \log X \nabla^{\hat{a}} \nabla^a + 2 \nabla^{\hat{a}} \nabla^a \log X \nabla^a \\
- 8 \bar{R} \nabla^{\hat{a}} - 8 \tilde{\nabla}^2 \bar{R}) V.
\end{align}
\tag{95}

The correction term can be found by using
\begin{align}
\tilde{S}^{\hat{b}} (\nabla^{\hat{a}} \log X) = X^{-1} \tilde{S}^{\hat{b}} (\nabla^{\hat{a}} X) \\
= X^{-1} [(2D + 3iA) \epsilon^{\hat{b} \hat{a}} - 2M^{\hat{b} \hat{a}}] X \\
= 4 \epsilon^{\hat{b} \hat{a}},
\end{align}
\tag{96}
and we see that the correction here cancels with that of $\tilde{V}^2 V \tilde{Y}$. Also note that $\tilde{V}^2 \log X = -8R - \nabla_\alpha \log X \nabla^{\hat{\alpha}} \log X$. Combining Eqs. (93), (95), and $-8RV \tilde{Y} = -8RV (\nabla^2 + 2\nabla^{\hat{\alpha}} \log X \nabla_\alpha - 8\tilde{R}) V$, we get

\[
Y \tilde{Y} = V \tilde{V}^2 \nabla^2 V + 2V (U^\alpha \tilde{V}^2 \nabla_\alpha - \nabla_\alpha \nabla^{\hat{\alpha}} \nabla^2) V \\
+ V \left[ (8R + 2U_\alpha \nabla^{\hat{\alpha}}) \nabla^2 - 8\tilde{R} \tilde{V}^2 + (4U^{\hat{\alpha}} \nabla^{\hat{\alpha}} - 4U^{\hat{\alpha}} U^\alpha) \nabla_\alpha \nabla_\alpha \nabla^{\hat{\alpha}} \nabla_\alpha \right] V \\
+ V \left( \frac{16}{3} X^\alpha - 16U^{\hat{\alpha}} \nabla^{\hat{\alpha}} - 8G^{\hat{\alpha}} \nabla^{\hat{\alpha}} U_\alpha + 4U_\alpha U^{\hat{\alpha}} U^\alpha - 4U^{\hat{\alpha}} U^{\hat{\alpha}} \right) \nabla_\alpha V \\
+ 16V (R U_\alpha - \nabla_\alpha \tilde{R}) \nabla^{\hat{\alpha}} V - 8V (\tilde{V}^2 \tilde{R} + 8R \tilde{R} + 2U_\alpha U^{\hat{\alpha}} \tilde{R} - 2U^{\hat{\alpha}} \nabla^{\hat{\alpha}} \tilde{R}) V.
\]  

(97)

We have ordered the terms by the number of derivatives involved, and used a handy notation for derivatives of log $X$: $U_\alpha = \nabla_\alpha \log X$, $U^{\hat{\alpha}} = \nabla^{\hat{\alpha}} \log X$, and so on. It is important to note that the first derivative $U_\alpha$ will vanish in the conformal gauge, and higher-order derivatives can be expressed in terms of $R$, $\tilde{R}$, $X_\alpha$, $X^{\hat{\alpha}}$, and also

\[
G_{\alpha \hat{\alpha}} = -\frac{1}{4} (U^{\hat{\alpha}} U_\alpha - U_\alpha U^{\hat{\alpha}}) - \frac{1}{2} U_\alpha U^{\hat{\alpha}}.
\]  

(98)

This $G_{\alpha \hat{\alpha}}$ shares a characteristic with the other four objects mentioned: it coincides with the familiar $G_{\alpha \hat{\alpha}}$ in the Poincaré limit.

Now, $\tilde{Y} Y$ is just the conjugate of Eq. (97), and hence we arrive at the expression

\[
Y \tilde{Y} + \tilde{Y} Y = V \tilde{V}^2 \nabla^2 + \nabla^2 \tilde{V}^2 V + 2V (U^\alpha \tilde{V}^2 \nabla_\alpha + \nabla_\alpha \nabla^{\hat{\alpha}}) V \\
+ V \left[ 2U_\alpha U^{\hat{\alpha}} \nabla^2 + 2U^{\hat{\alpha}} U_\alpha \tilde{V}^2 + (8G^{\hat{\alpha}} + 8U^{\hat{\alpha}} U^\alpha) \nabla_\alpha \nabla_\alpha \nabla^{\hat{\alpha}} \nabla_\alpha \right] V \\
+ V \left( \frac{16}{3} X^\alpha - 16U^\alpha \nabla^\alpha - 32R U^\alpha + 4U_\alpha U^{\hat{\alpha}} U^\alpha - 4U^{\hat{\alpha}} U^{\hat{\alpha}} \right) \nabla_\alpha V \\
+ V \left( \frac{16}{3} X_\alpha - 16\tilde{V}^{\hat{\alpha}} + 4U^{\hat{\alpha}} U_\alpha \tilde{R} + 4U^{\hat{\alpha}} U^{\hat{\alpha}} U_\alpha - 4U^{\hat{\alpha}} U^{\hat{\alpha}} \right) \nabla^{\hat{\alpha}} V \\
+ 16V U_\alpha \nabla_\alpha V - 8V (\tilde{V}^2 \tilde{R} + \nabla^2 R + 2U^{\hat{\alpha}} U^{\hat{\alpha}} R + 2U^{\hat{\alpha}} U^{\hat{\alpha}} R \\
- 2U^{\hat{\alpha}} \nabla_\alpha R - 2U^{\hat{\alpha}} \nabla_\alpha \tilde{R}) V.
\]  

(99)

To further simplify, we use the following identity, which is Eq. (3.27) of Ref. [16]:

\[
\nabla^2 \tilde{V}^2 + \tilde{V}^2 \nabla^2 - \nabla^{\alpha} \tilde{V} \nabla^{\alpha} \nabla^{\hat{\alpha}} \nabla^{\hat{\alpha}} = 16\Box + 8\nabla^{\alpha} \nabla^{\hat{\alpha}} - 8\nabla_\alpha \nabla^{\hat{\alpha}}.
\]  

(100)

This can be proved by showing first that

\[
[\tilde{V}^2, \nabla_\alpha] = \nabla_\beta (\nabla^{\beta} \nabla_\alpha) - (\nabla_\beta, \nabla_\alpha) \nabla^{\beta} \\
= 2i\nabla^{\beta} \nabla^{\alpha} \nabla_\beta + 2i\nabla^{\alpha} \nabla_\beta \nabla^{\beta} \\
= 2i([\nabla^{\beta}, \nabla^{\alpha}] + \nabla_\alpha \nabla^{\beta}) + 2i\nabla^{\alpha} \nabla_\beta \nabla^{\beta}
\]

18/40
These two equations give Eq. (3.26) of the quoted reference:

\[ 2\xi = 4i\nabla_{\alpha\beta} \nabla^\beta + 2i(-2\delta^\beta_\beta \mathcal{W}_\alpha) \]
\[ = 4i\nabla_{\alpha\beta} \nabla^\beta + 8\mathcal{W}_\alpha, \quad (101) \]

and similarly

\[ \left[ \nabla^2, \nabla^\alpha \right] = \nabla^\beta \{ \nabla_\beta, \nabla^\alpha \} - \{ \nabla^\alpha, \nabla_\beta \} \nabla_\beta \]
\[ = 2i\nabla^{\beta\alpha} \nabla_\beta + 2i\nabla^\beta \nabla^{\beta\alpha} \]
\[ = 4i\nabla^{\beta\alpha} \nabla_\beta - 8\mathcal{W}_\alpha. \quad (102) \]

These two equations give Eq. (3.26) of the quoted reference:

\[ \nabla^2 \bar{\nabla}^2 = \nabla^{\alpha\beta} \nabla^\alpha \nabla^\beta + 8\Box - 2i\nabla^{\beta\alpha} \{ \nabla_\alpha, \nabla_\beta \} - 8\mathcal{W}\nabla_\alpha, \]
\[ \bar{\nabla}^2 \nabla^2 = \nabla^{\alpha\beta} \bar{\nabla}_\alpha \nabla^\beta + 8\Box + 2i\nabla^{\beta\alpha} \{ \nabla_\alpha, \nabla_\beta \} + 8\mathcal{W}_\alpha \nabla_\alpha, \quad (103) \]

and these immediately give the desired result. Now, from Eq. (100) it is clear that we should choose \( \xi = 1 \) in the gauge-fixing action of Eq. (87); this removes the term with four derivatives in Eq. (99). Note that the total gaugino fields \( \mathcal{W} \) in Eq. (100) have only \( M_{ab}, K_4 \), and Yang–Mills components; when acting on \( V \) only the Yang–Mills terms survive, and thus we can safely replace \( \mathcal{W}_a \) by \( \mathcal{W}_a^{YM} \) and \( \mathcal{W}_YM,\alpha \).

Also, using Eqs. (101) and (102) one can convert the terms with three derivatives in Eq. (99) into a term with fewer derivatives. This shows that after gauge fixing we can remove all terms with more than two derivatives, which is what we want to achieve.

Finally, there are two more terms in the action: \( -4\mathcal{W}_YM[V, \nabla_\alpha V] \) and \( 4\mathcal{W}_YM,\alpha [V, \nabla^{\alpha\beta} V] \). Using the cyclicity of traces, integration by parts, and the Bianchi identity \( \nabla^{\alpha\beta} \mathcal{W}_YM,\alpha = \nabla_\alpha \mathcal{W}_YM^{\alpha\beta} \), we obtain

\[ -4\mathcal{W}_YM^{\alpha\beta} [V, \nabla_\alpha V] + 4\mathcal{W}_YM,\alpha [V, \nabla^{\alpha\beta} V] = 8V(\mathcal{W}_YM^{\alpha\beta} \nabla_\alpha - \mathcal{W}_YM,\alpha \nabla^{\alpha\beta}) V. \quad (104) \]

Combining all terms, we finally come to the conclusion that the second-order action is

\[ S^{(2)}_{YM} = \frac{1}{2} \text{tr} \int d^8z \left( \frac{2}{g^2} \right) V \mathcal{O}_V V, \quad (105) \]

where \( 2/g^2 \) is just an irrelevant constant, and the crucial second-order differential operator \( \mathcal{O}_V \) is

\[ \mathcal{O}_V = \Box + \frac{1}{2} G^{\alpha\beta} \{ \nabla_\alpha, \nabla_\beta \} + \left( \frac{X^{\alpha\beta}}{3} - \nabla^{\alpha\beta} R + \mathcal{W}_YM^{\alpha\beta} \right) \nabla_\alpha \]
\[ + \left( \frac{X^{\alpha\beta}}{3} - \nabla^{\alpha\beta} R - \mathcal{W}_YM^{\alpha\beta} \right) \nabla_\beta - \frac{1}{2} (\bar{\nabla} \nabla R + \nabla^2 R + 16R) \]
\[ + \frac{i}{4} U^{\alpha\beta} (\nabla^{\beta\alpha} \nabla_\alpha + \nabla_\alpha \nabla^{\beta\alpha}) + \frac{i}{4} U^{\alpha\beta} (\nabla^{\beta\alpha} \nabla_\beta + \nabla_\beta \nabla^{\beta\alpha}) \]
\[ + \frac{1}{8} \left( U^{\alpha\beta} \nabla^2 + U^{\alpha\beta} U_\alpha \bar{\nabla}^2 + 4U^{\alpha\beta} U^{\alpha\beta} \nabla_\alpha, \nabla_\beta \right) \]
\[ + \frac{1}{4} \left( 8RU^{\alpha\beta} + U^{\alpha\beta} U^{\alpha\beta} - U^{\alpha\beta} U^{\alpha\beta} \right) \nabla_\alpha \]

19/40
\[
+ \frac{1}{4} \left( 8\tilde{R}U_{\dot{a}} + U^\alpha U_\alpha U_{\dot{a}} - U^\alpha U_{\dot{a}a} \right) \nabla \dot{a}
+ U^\alpha \nabla_a + \left( U^\alpha \nabla_a R + U_{\dot{a}} \nabla \dot{a} \tilde{R} - U^\alpha U_\alpha R - U_{\dot{a}} U_{\dot{a}} \tilde{R} \right).
\]

(106)

We have divided the expression into two parts: the first two lines remain non-zero in the Poincaré limit, and the rest contain \( U_\dot{a} \) which instead vanishes.

One might compare this result to the literature with similar calculations in ordinary supergravity, for instance the case of an Abelian vector multiplet as in Ref. [13]. One can easily reproduce most of the terms there by carefully considering how each term above reduces to its Poincaré counterpart; for example, one can show that \( \bar{\nabla}^2 \bar{R} \rightarrow (\bar{D}^2 - 16R)\bar{R} \) instead of the naive guess \( (\bar{D}^2 - 8R)\bar{R} \). Some discrepancy arises as we are actually going down from conformal to \( U(1) \) supergravity, rather than the so-called minimal supergravity. For the latter, \( X_\alpha \) and \( \dot{X}_\dot{a} \) do not exist and the bosonic derivatives \( \bar{D}_a \) are defined differently.

4. Super heat kernel coefficients

4.1. Heat kernel method

Consider a superfield \( \Phi \), with its quantum quadratic action being

\[
S^{(2)} = \frac{1}{2} \text{tr} \int d^8z E \Phi \mathcal{O} \Phi;
\]

we can quickly generalize the non-supersymmetric scenario and conclude that the one-loop effective action is given by the analogous expression

\[
\Gamma^{(1)} = \frac{i}{2} \text{Tr}_\tau \log \mathcal{O}.
\]

(108)

Here, the trace is taken over the superspace \( \{z = (x, \theta, \bar{\theta})\} \); in other words we are taking the supertrace.

It is noted that numerous methods employed in the bosonic case can also be used to analyze the supersymmetric effective action, with only minimal modifications required. In particular, Schwinger’s proper-time technique [11], which was originally developed for non-supersymmetric theory, may be applied. We define the super heat kernel \( K(z, z', \tau) \) via the differential equation

\[
\left( \mathcal{O} + i \frac{\partial}{\partial \tau} \right) K(z, z', \tau) = 0,
\]

(109)

with the boundary condition being

\[
\lim_{\tau \to 0^+} K(z, z', \tau) = E^{-1} \delta^8(z - z').
\]

(110)

Equivalently, \( K \) is defined by the operator expression

\[
K(z, z', \tau) = e^{i\tau \mathcal{O}} E^{-1} \delta^8(z - z'),
\]

(111)

where \( \mathcal{O} \) acts on the primed variable \( z' \).

Similar to the bosonic case, this \( K \) encodes information about the Green’s function and one-loop effective action of the theory. Integrating \( K \) over \( \tau \) gives us the Green’s function \( G(z, z') \):

\[
G(z, z') = i \int_0^\infty d\tau \ K(z, z'; \tau), \quad \mathcal{O}G(z, z') = -E^{-1} \delta^8(z - z').
\]

(112)
Also, the coincidence limit \( z' \to z \) is related to the effective action:

\[
\Gamma_1 = -\frac{i}{2} \int_0^\infty \frac{d\tau}{\tau} K(\tau), \quad K(\tau) = \int d^8z E K(z, z; \tau).
\] (113)

If \( \mathcal{O} \) is a second-order differential operator that contains the d’Alembertian as the leading term, \( \mathcal{O} = \Box + \cdots \), we may expand the heat kernel into power series containing De Witt heat kernel coefficients. The starting point is the simplest case: with the superspace being flat and \( \mathcal{O} = \Box \), the heat kernel of this theory is simply

\[
K(z, z'; \tau) = \frac{-i}{(4\pi \tau)^2} \exp \left( \frac{y^a y_a}{4\tau} \right) (\theta - \theta')^2 (\bar{\theta} - \bar{\theta}')^2,
\] (114)

where \( y^a = (x - x')^a - i(\theta - \theta')\sigma^a \bar{\theta}' + i\theta' \sigma^a (\bar{\theta} - \bar{\theta}') \). This \( y^a \) is simply the integral of the vielbein \( E^a \) along the straight line connecting \( z \) and \( z' \).

For a more general quadratic operator \( \mathcal{O} \), we can write it as

\[
\mathcal{O} = X^{AB} \nabla_A \nabla_B + Y A \nabla_A + Z = \mathcal{O}' + Z.
\] (115)

Here, \( \mathcal{O}' \) is the part of \( \mathcal{O} \) containing derivatives, so \( \mathcal{O}' \) annihilates constants. We require that \( X^{ab} = \eta^{ab} \), so that the leading term is indeed the d’Alembertian. Using the covariant derivative algebra, we can, without losing generality, further assume that the tensor \( X^{AB} \) is graded symmetric, \( X^{AB} = (-1)^{AB} X^{BA} \). From now on we shall always employ the implicit grading scheme and the graded-symmetric condition is simply \( X^{AB} = X^{BA} \). With this \( X^{AB} \), we may construct a bilinear product of two scalar functions,

\[
\langle f, g \rangle = X^{AB} (\nabla_A f)(\nabla_B g);
\] (116)

the symmetry of \( X^{AB} \) implies that this product is actually symmetric.

Following the non-supersymmetric case, we propose that the heat kernel is of the form

\[
K(z, z'; \tau) = \frac{-i}{(4\pi \tau)^2} \exp \left( \frac{i\sigma}{2\tau} \right) \Delta^{1/2}F(z, z'; \tau).
\] (117)

\( \sigma (z, z') \) is a two-variable function symmetric in \( z \) and \( z' \), a supersymmetric analog of the geodesic interval between \( z \) and \( z' \) in superspace. It corresponds roughly to one half of the distance squared between \( z \) and \( z' \). However, it is well known that the \( N = 1 \) superspace has no natural metric defined, so strictly speaking the concept of distance makes no sense in superspace. Nevertheless, one can treat \( \sigma (z, z') \) as the curved space extension of the flat space object \( y^a y_a/2 \). The boundary conditions for \( \sigma \) are such that it reduces to the appropriate flat limit; the details can be found in Ref. [13]. \( \Delta \) is another scalar function, the supersymmetric version of the Van Vleck–Morrette determinant, which only arises if the space is curved; in particular, \( \Delta \) is identically 1 for a flat superspace. We also impose that \( \Delta(z, z) = 1 \).

Now let us substitute this expression into the differential equation in Eq. (109) to get

\[
\frac{1}{4\tau^2} (2\sigma - \langle \sigma, \sigma \rangle) \Delta^{1/2}F + \frac{i}{2\tau} (\mathcal{O}' \sigma + \langle \sigma, \log \Delta \rangle - 4) \Delta^{1/2}F
+ \frac{i}{\tau} \Delta^{1/2} \langle \sigma, F \rangle + i \Delta^{1/2} \frac{\partial}{\partial \tau} F + \mathcal{O} \Delta^{1/2}F = 0.
\] (118)
If we require \( F \) to be analytic, then the \( 1/\tau^2 \) term must be identically zero. This implies that \( \sigma \) must satisfy

\[
\langle \sigma, \sigma \rangle = 2\sigma. \tag{119}
\]

There is a further simplification if we demand that

\[
\mathcal{O}' \sigma + \langle \sigma, \log \Delta \rangle = 4. \tag{120}
\]

The final result is then

\[
\frac{1}{i} \frac{\partial}{\partial \tau} F + \frac{1}{i\tau} \langle \sigma, F \rangle = \hat{\mathcal{O}} F, \tag{121}
\]

where \( \hat{\mathcal{O}} \) is the operator \( \hat{\mathcal{O}} = \Delta^{-1/2} \mathcal{O} \Delta^{1/2} \). We rewrite \( F \) into a power series in \( \tau \):

\[
F(z, z'; \tau) = \sum_{n=0}^{\infty} a_n \frac{(i\tau)^n}{n!}. \tag{122}
\]

The \( \{a_n\} \) are the De Witt coefficients of the super heat kernel. In terms of these coefficients, Eq. (121) becomes an iterative equation:

\[
a_n + \frac{1}{n} \langle \sigma, a_n \rangle = \hat{\mathcal{O}} a_{n-1} \quad (n > 0), \tag{123}
\]

\[
\langle \sigma, a_0 \rangle = 0. \tag{124}
\]

As in the non-supersymmetric setup, the first coefficient \( a_0 = \delta^2(\theta - \theta')\delta^2(\bar{\theta} - \bar{\theta}) \mathcal{I}(z, z') \) contains the \textit{parallel displacement propagator} \( \mathcal{I}(z, z') \), which has the useful property [24]

\[
[\nabla_{(A_1)} \nabla_{(A_2)} \cdots \nabla_{(A_k)} \mathcal{I}] = 0, \tag{125}
\]

where \([ \ldots \] denotes the coincidence limit \( z' \rightarrow z \), and \(( \ldots )\) means the graded symmetrization of the bracketed indices. This property theoretically allows us to obtain \([a_n]\) iteratively, by repeatedly applying Eq. (123) multiple times. Calculations for finding the first three coefficients of some model in this way can be found, for example, in Ref. [13]. Similar to the original calculation by De Witt [11] for the non-supersymmetric case, this procedure quickly becomes very tedious and thus impractical beyond the first few coefficients.

As the coincidence limit of the heat kernel is closely related to the one-loop effective action, \([a_n]\) will naturally be objects of interest. In fact, analogous to the non-supersymmetric regime, the first three coefficients are significant in that they give the divergence of the theory. Using a cutoff scheme to regulate Eq. (113),

\[
\Gamma^\Lambda_{(1)} = \frac{1}{32\pi^2} \int d^8 z E \int_{\Lambda^{-2}}^{\infty} \frac{d(i\tau)}{(i\tau)^3} F(z, z; \tau). \tag{126}
\]

Writing \( F \) in terms of \( \{a_n\} \) gives the one-loop divergence,

\[
\Gamma^\Lambda_{(1)\text{div}} = \frac{1}{32\pi^2} \int d^8 z E \left( \frac{\Lambda^4}{2} [a_0] + \Lambda^2 [a_1] + \frac{1}{2} \log \Lambda^2 [a_2] \right). \tag{127}
\]

Of course, supersymmetry implies that the quartic divergence must vanish, \([a_0] = 0\), and thus the one-loop divergence is governed by the coincidence limit of \( a_1 \) and \( a_2 \).
4.2. Super heat kernel in conformal supergravity

In conformal supergravity, extra complications arise due to the presence of the dilation operator $D$. For example, in the case of super Yang–Mills theory, the quadratic action is

$$S_{V}^{(2)} = \frac{1}{2} \text{tr} \int d^8z EV O_{V} V,$$  \hspace{1cm} (128)

where $O_{V}$ carries a non-zero $D$ charge: $[D, O_{V}] = 2 O_{V}$. Because of this non-trivial charge, it is technically not appropriate to exponentiate $O_{V}$, as in Eq. (111), to define the heat kernel. To resolve this, let us consider the quantum functional integral

$$Z[V] = \int \mathcal{D}V e^{i \frac{1}{2} \text{tr} \int d^8z EV O_{V} V} Z_{\text{free}},$$  \hspace{1cm} (129)

where $Z_{\text{free}}$ is the functional integral for the free theory of $V$, which serves to normalize the path integral measure. In usual quantum field theory, one takes the free action to be a Gaussian:

$$Z_{\text{free}} = \int \mathcal{D}\phi e^{i \frac{1}{2} \text{tr} \int d^8z E\phi^2}.$$ \hspace{1cm} (130)

However, this is not possible in conformal supergravity, as $V^2$ is not a valid action, lacking the correct $D$-charge. To fix the problem, we have to use the compensator $X$ and set

$$Z_{\text{free}} = \int \mathcal{D}V e^{i \frac{1}{2} \text{tr} \int d^8z EX V^2}.$$ \hspace{1cm} (131)

This implies that the one-loop effective action is actually a difference of two supertraces:

$$\Gamma_{(1)} = \frac{i}{2} (\text{Tr} \log O_{V} - \text{Tr} \log X);$$ \hspace{1cm} (132)

now this expression is perfectly $D$-invariant.

By inspecting the one-loop action above, we may now define the heat kernel of $O_{V}$ by temporarily breaking the $D$-symmetry and choose the $D$-gauge $X = 1$ so that $\text{Tr} \log X = 0$.\footnote{From $DX = 2X$ this is always possible by a local $D$ transformation.} In this gauge we can forget about the $D$-charge and proceed normally; one can calculate the heat kernel of $O_{V}$, in particular the heat kernel coefficients. To restore the $D$-symmetry, we may just insert powers of $X$ in various expressions such that we get the correct quantum number. If we are considering the one-loop action and its divergence, this procedure gives the correct result.

Of course, this is just one of the ways to regulate the $D$-symmetry. For instance, one can alternatively just consider the heat kernel of $X^{-1}O_{V}$, which is $D$-invariant. Different results may appear for different schemes, but various methods should be equivalent as long as we consider the theory on-shell.

4.3. Non-iterative method for super heat kernel coefficients

While it is in theory possible to compute the heat kernel coefficients $[a_n]$ up to arbitrary order via the recursive method, the computational complexity escalates so quickly that it is not practical to do so for higher-order coefficients. It would be useful to develop non-iterative techniques to effectively compute heat kernel coefficients, and we shall describe below a supersymmetric generalization of
a method developed by Avramidi [12]. A short discussion of the original technique is given in Appendix C.

There are a few restrictions that will be imposed in order to apply such a method. We shall suppose that:

1. The trace of the torsion vanishes:

\[ T_{AB}^B = 0. \]  \hspace{1cm} (133)

2. For the operator \[ \mathcal{O} = X^{AB} \nabla_A \nabla_B + Y^A \nabla_A + Z, \] we require

\[ X^{aa} = X^{2a} = Y^a = 0. \]  \hspace{1cm} (134)

The first condition is crucial for the integration by parts formula to be true, which is very reasonable to assume, and in particular this is valid for familiar types of supergravity theory. The second condition translates to the statement that the bosonic derivative \( \nabla_a \) only appears in the d'Alembertian and nowhere else. This can be achieved by, for example, redefining the covariant derivatives, choosing certain gauges, and so on. At worst we may ignore the extra terms temporarily and treat them as a perturbation later.

In the following we shall adopt a special coordinate system, the normal coordinate system in superspace, \( y^M = (y^m, \dot{y}^\mu, y_{\dot{\mu}}) \), developed in Ref. [25]. This is a straightforward supersymmetric extension of Riemann normal coordinates. We shall also choose a supersymmetric Schwinger–Fock gauge [26] for all the gauged symmetries, which implies that

\[ y^M h_{M}^{A} = 0 \]  \hspace{1cm} (135)

for all gauge connections \( h^{A} \), which includes the Lorentz connections, Yang–Mills connections, and others.

When defining the Schwinger heat kernel, we require that the bilinear \( \sigma(z, z') \) satisfies \( \langle \sigma, \sigma \rangle = 2\sigma \) near the point \( z \), which we shall assume to be the superspace origin from now on; \( \sigma \) has the simple expression

\[ \sigma = \frac{y^a y_a}{2}, \]  \hspace{1cm} (136)

where \( y^a = y^M E_{M}^a \). This can be shown by the properties of the normal coordinate system [25]:

\[ y^M E_M^A = y^M \delta_{M}^{A}, \quad y^M \partial_{M} = y^M \nabla_{M}. \]  \hspace{1cm} (137)

The object \( \Delta \) also simplifies in this coordinate system, to become simply

\[ \Delta = E^{-1} = \det(E_{A}^{M}). \]  \hspace{1cm} (138)

The covariant derivative of \( \Delta \) is given by

\[ \nabla_A \Delta = E_{M}^{B} \nabla_A E_{B}^{M} = E_{M}^{B} \nabla_B E_{A}^{M} - T_{AB}^{M} E_{M}^{B} = \nabla_M E_{A}^{M}; \]  \hspace{1cm} (139)

note that the vanishing torsion trace is used here. Using this identity and some algebra, we can show that

\[ \mathcal{O}'\sigma + \langle \sigma, \log \Delta \rangle \]
\[ = \nabla_M (X^{AB} \nabla_A \sigma E_B^M) \]
\[ = \partial_M (X^{aB} y_a E_B^M) = 4. \quad (140) \]

Remember that \( O' = X^{AB} \nabla_A \nabla_B E + Y^A \nabla_A \) is the part of \( O \) without the non-derivative term, and note that the constraints imposed on \( O \) imply that \( O' \sigma = \Box \sigma \). Thus, \( \Delta \) has the desired property.

Now we have \( (\sigma, a_n) = y^m \partial_m a_n \) thanks to the property in Eq. (137), so the De Witt recursion equation becomes
\[ (1 + \frac{y^m \partial_m}{n}) a_n = \tilde{O} a_{n-1}. \quad (141) \]

To compute \( a_n \), in fact the coincidence limit of the coefficients \( [a_n] \), we shall expand spacetime functions with respect to the following basis, similar to the non-supersymmetric case:
\[ |n = (a, b, c) = |a\rangle |\mu, b\rangle |\dot{\mu}, c\rangle \quad (a \geq 0, \quad 2 \geq b, c \geq 0), \]
\[ |0 \rangle = 1, \quad |a\rangle = \frac{1}{a!} y^{m_1} y^{m_2} \ldots y^{m_a} \quad (a > 0), \]
\[ |\mu, 0 \rangle = 1, \quad |\mu, 1 \rangle = y^{\mu}, \quad |\mu, 2 \rangle = y^{\mu} y^{\mu}, \]
\[ |\dot{\mu}, 0 \rangle = 1, \quad |\dot{\mu}, 1 \rangle = y^{\dot{\mu}}, \quad |\dot{\mu}, 2 \rangle = y^{\dot{\mu}} y^{\dot{\mu}}. \quad (142) \]

Their corresponding bras are defined by
\[ \langle n | = \langle a | \langle \mu, b | \langle \dot{\mu}, c |, \]
\[ \langle a | = \partial_{m_1} \partial_{m_2} \ldots \partial_{m_a}, \]
\[ \langle \mu, 0 | = 1, \quad \langle \mu, 1 | = \partial_{\mu}, \quad \langle \mu, 2 | = \frac{1}{4} \partial_{\mu} \partial_{\mu}, \]
\[ \langle \dot{\mu}, 0 | = 1, \quad \langle \dot{\mu}, 1 | = \partial_{\dot{\mu}}, \quad \langle \dot{\mu}, 2 | = \frac{1}{4} \partial_{\dot{\mu}} \partial_{\dot{\mu}}. \quad (143) \]

The inner product is given by taking the coincidence limit \( y^M \to 0 \) after taking the derivatives, for example
\[ \langle a, 1, 0 | f \rangle = \partial_{m_1} \partial_{m_2} \ldots \partial_{m_a} \partial_{\mu} f \mid_{y^M \to 0}. \quad (144) \]

The basis \( |n \rangle \) is actually a complete basis; its completeness can be seen by using the supersymmetric covariant Taylor series [24], which is simple in normal coordinates:
\[ f(z') = I(z', z) \sum_{n=0}^{\infty} \frac{1}{n!} y^{M_1} y^{M_2} \ldots y^{M_n} \nabla_{M_n} \nabla_{M_{n-1}} \ldots \nabla_{M_1} f(w) \mid_{w=z} \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} y^{M_1} y^{M_2} \ldots y^{M_n} \partial_{M_n} \partial_{M_{n-1}} \ldots \partial_{M_1} f (0). \quad (145) \]

Note that the parallel displacement operator is just the identity, \( I(z', z) = 1 \), in normal coordinates.

Now note that \( y^m \partial_m |a\rangle = a |a\rangle \) is an eigenvector by definition, so we have \( y^m \partial_m |a, b, c\rangle = a |a, b, c\rangle \), and the zeroth-order coefficients in normal coordinates are simply \( a_0 = I(z, z') \delta^2 (\theta - \theta') \delta^2 (\dot{\theta} - \dot{\theta}') = \)

25/40
There are only a few choices of \( n \). This is because \( a \) is a quadratic operator and thus \( \langle a_i, b_i, c_i | \tilde{O} | a_{i+1}, b_{i+1}, c_{i+1} \rangle \) is non-zero, after taking the coincidence limit, only if

\[
a_i + b_i + c_i + 2 \geq a_{i+1} + b_{i+1} + c_{i+1},
\]

otherwise there are not enough derivatives to annihilate the factors of \( y^M \) in \( |a_{i+1}, b_{i+1}, c_{i+1}\rangle \).

4.4. The first three coefficients and an algorithm to compute higher-order coefficients

Let us compute the first few coefficients. The first coefficient is zero:

\[
[a_0] = \langle 0, 0, 0 | 0, 2, 2 \rangle = 0,
\]

as expected by supersymmetry. \( [a_1] \) is also trivial:

\[
[a_1] = \langle 0, 0, 0 | \tilde{O} | 0, 2, 2 \rangle = 0,
\]

from the condition in Eq. (147).

To compute \([a_2]\), note that the imposed constraints and graded symmetry of \( X^{AB} \) imply that

\[
\tilde{O} = \Box + A \nabla^2 + B \nabla^2 + V_0 \nabla^2 \nabla \alpha + \cdots,
\]

with \( V_0 = \langle \tilde{a} \rangle_{a \tilde{a}} V_a \) for some bosonic vector \( V_a \). It is clear that \( \tilde{O} = \Delta^{-1/2} \tilde{O} \Delta^{1/2} \) and \( \tilde{O} \) share the same quadratic part. Now, using Eq. (146),

\[
[a_2] = \sum_n (1 + a)^{-1} \langle 0, 0, 0 | \tilde{O} | n \rangle \langle n | \tilde{O} | 0, 2, 2 \rangle.
\]

There are only a few choices of \( n \) to give a non-zero product; direct inspection shows that

\[
[a_2] = \langle 0, 0, 0 | \tilde{O} | 0, 1, 1 \rangle \langle 0, 1, 1 | \tilde{O} | 0, 2, 2 \rangle + \langle 0, 0, 0 | \tilde{O} | 0, 2, 0 \rangle \langle 0, 2, 0 | \tilde{O} | 0, 2, 2 \rangle + \langle 0, 0, 0 | \tilde{O} | 0, 0, 2 \rangle \langle 0, 0, 2 | \tilde{O} | 0, 2, 2 \rangle = 16 V_0 \tilde{a} V_a + 16 AB + 16 BA = -32 V_0 V_a + 32 AB.
\]

We shall outline schematically how to obtain all the non-zero terms in the summation of Eq. (146) that contribute to \([a_2]\). By examining the structure of Eq. (146), each non-zero product of brackets is characterized by a chain of triplets:

\[
n_0 = (0, 0, 0) \rightarrow n_1 = (a_1, b_1, c_1) \rightarrow n_2 = (a_2, b_2, c_2) \rightarrow \cdots \rightarrow n_k = (0, 2, 2),
\]

with \( a_i \geq 0 \) and \( 2 \geq b_i, c_i \geq 0 \). Denote \( s_i = a_i + b_i + c_i \) and \( \Delta s_i = s_i - s_{i-1} \); the chain will have to satisfy the additional properties

\[
\Delta s_i \leq 2, \quad \Delta a_i \leq 2, \quad \Delta b_i + \Delta c_i \leq 2.
\]
Also, if $\Delta b_i > 0$ or $\Delta c_j > 0$, then $\Delta a_i \leq 0$. To find all these chains, first obtain the chain of $s_i$: $0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_k = 4$, such that $\Delta s_i \leq 2$. Then, from $s_i$, obtain all pairs $b_i$ and $c_i$ with $\Delta b_i + \Delta c_i \leq 2$, and then check that $\Delta a_i \leq 2$ and $\Delta a_i \leq 0$ if $\Delta b_i > 0$ or $\Delta c_i > 0$. This is effectively just a combinatorial problem.

Let us consider $[a_3]$ as an example. We need to find all chains $0 \rightarrow s_1 \rightarrow s_2 \rightarrow 4$ such that each step increases by at most two. There are only a handful of possibilities:

$$0 \rightarrow 0 \rightarrow 2 \rightarrow 4,$$
$$0 \rightarrow 1 \rightarrow 2 \rightarrow 4,$$
$$0 \rightarrow 1 \rightarrow 3 \rightarrow 4,$$
$$0 \rightarrow 2 \rightarrow 2 \rightarrow 4,$$
$$0 \rightarrow 2 \rightarrow 3 \rightarrow 4,$$
$$0 \rightarrow 2 \rightarrow 4 \rightarrow 4. \quad (155)$$

Next, we have to split $s_i$ into the triplet $(a_i, b_i, c_i)$. These are listed below:

1. $0 \rightarrow 0 \rightarrow 2 \rightarrow 4$ branch:
   (a) $(0, 0, 0) \rightarrow (0, 0, 0) \rightarrow (0, b, c) \rightarrow (0, 2, 2)$, with $(b, c)$ one of $(2, 0)$, $(0, 2)$, or $(1, 1)$.

2. $0 \rightarrow 1 \rightarrow 2 \rightarrow 4$ branch:
   (a) $(0, 0, 0) \rightarrow (a, b, c) \rightarrow (0, b', c') \rightarrow (0, 2, 2)$, with $(a, b, c)$ one of $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$, $(b', c')$ one of $(2, 0)$, $(0, 2)$, or $(1, 1)$.

3. $0 \rightarrow 1 \rightarrow 3 \rightarrow 4$ branch:
   (a) $(0, 0, 0) \rightarrow (0, b, c) \rightarrow (0, b', c') \rightarrow (0, 2, 2)$, with $(b, c)$ either $(1, 0)$ or $(0, 1)$, $(b', c')$ either $(2, 1)$ or $(1, 2)$;
   (b) $(0, 0, 0) \rightarrow (1, 0, 0) \rightarrow (1, b', c') \rightarrow (0, 2, 2)$, with $(b', c')$ one of $(2, 0)$, $(0, 2)$, or $(1, 1)$.

4. $0 \rightarrow 2 \rightarrow 2 \rightarrow 4$ branch:
   (a) $(0, 0, 0) \rightarrow (0, b, c) \rightarrow (0, b', c') \rightarrow (0, 2, 2)$, with $(b, c)$ and $(b', c')$ one of $(2, 0)$, $(0, 2)$, or $(1, 1)$;
   (b) $(0, 0, 0) \rightarrow (2, 0, 0) \rightarrow (0, b', c') \rightarrow (0, 2, 2)$, with $(b', c')$ one of $(2, 0)$, $(0, 2)$, or $(1, 1)$.

5. $0 \rightarrow 2 \rightarrow 3 \rightarrow 4$ branch:
   (a) $(0, 0, 0) \rightarrow (0, b, c) \rightarrow (0, b', c') \rightarrow (0, 2, 2)$, with $(b, c)$ one of $(2, 0)$, $(0, 2)$, or $(1, 1)$, $(b', c')$ either $(2, 1)$ or $(1, 2)$;
   (b) $(0, 0, 0) \rightarrow (2, 0, 0) \rightarrow (1, b', c') \rightarrow (0, 2, 2)$, with $(b', c')$ one of $(2, 0)$, $(0, 2)$, or $(1, 1)$.

6. $0 \rightarrow 2 \rightarrow 4 \rightarrow 4$ branch:
   (a) $(0, 0, 0) \rightarrow (0, b, c) \rightarrow (0, 2, 2) \rightarrow (0, 2, 2)$, with $(b, c)$ one of $(2, 0)$, $(0, 2)$, or $(1, 1)$;
   (b) $(0, 0, 0) \rightarrow (0, b, c) \rightarrow (2, b, c) \rightarrow (0, 2, 2)$, with $(b, c)$ one of $(2, 0)$, $(0, 2)$, or $(1, 1)$;
   (c) $(0, 0, 0) \rightarrow (2, 0, 0) \rightarrow (2, b', c') \rightarrow (0, 2, 2)$, with $(b', c')$ one of $(2, 0)$, $(0, 2)$, or $(1, 1)$.

After obtaining all the chains of triplets, one can immediately write down the coefficient $[a_k]$ in terms of various brackets, according to Eq. (146). It remains to compute each of the terms $\langle a_{i-1}, b_{i-1}, c_{i-1} \rangle \langle \tilde{O} [a_i, b_i, c_i] \rangle$; for the case of $\Delta s_i \geq 0$ only one term of $\tilde{O}$ will contribute, the one that

---

This is from the fact that the quadratic part of $\tilde{O}$ satisfies $X^{\alpha_2} = X^{\alpha_3} = 0$.
With the machinery developed, we are now ready to calculate the super heat kernel coefficients of problematic terms being torsion is clear for conformal supergravity; however, the second constraint is not satisfied, with the order to apply the non-recursive method we have to satisfy the constraints. The vanishing trace of \(\hat{H}\) actives that act on \(\tilde{Y}\) in conformal supergravity. First, looking at the operator \(O\) derived previously in Eq. (106), the lowest-order results for the vielbein are presented. For example, one can express the Yang–Mills connection in terms of the derivatives of the field strength \([24,26]\]. More details for the case of ordinary supergravity can be found, for instance, in Refs. [25,27], where the algorithm to compute the series expansion up to any other is shown, and thus all the terms above are eliminated. This choice of gauge implies that we have to break the \(D\) and \(K_A\) symmetry; in other words, we are reducing to the \(U(1)\) supergravity regime. To ensure

\[
\langle a, b, c | \tilde{O} | a, b, c \rangle = [Z'],
\]

(156)

where \(Z'\) is the non-derivative part of \(\tilde{O}\): \(\tilde{O} = X^{AB} \nabla_A \nabla_B + Y^{iA} \nabla_A + Z'\). Another example will be

\[
\langle 1, 0, 0 | \tilde{O} | 0, 2, 0 \rangle = \langle 1, 0, 0 | A \nabla^2 | 0, 2, 0 \rangle = [\partial_a A] = [4 \nabla_a A],
\]

(157)

as we want to annihilate \(y^\mu y_\mu\) appearing in \(|0, 2, 0\rangle\). Note that the we can replace the partial derivative of \(A\) by the covariant derivative, as all the connection fields vanish at the origin in the normal coordinate system.

In general, whenever we have a decrease of \(a_i, b_i,\) or \(c_i\) in the bracket \(\langle a_{i-1}, b_{i-1}, c_{i-1} | \tilde{O} | a_i, b_i, c_i \rangle\), some derivatives appearing in \(\langle a_{i-1}, b_{i-1}, c_{i-1} | \tilde{O} | a_i, b_i, c_i \rangle\) will have to act on \(\tilde{O}\) instead of the ket \(|a_i, b_i, c_i\rangle\). Hence we are required to compute the coincidence limit of derivatives of various objects in \(\tilde{O}\). The convenient way to do so is to use the normal coordinate expansion, in which one can expand the vielbein and other connections as a power series of \(y^M\), with the coefficients expressed in terms of the field strength of the connections, in other words the torsion, Riemann curvature, and so on.

For example, one can express the Yang–Mills connection in terms of the derivatives of the field strength \([24,26]\). More details for the case of ordinary supergravity can be found, for instance, in Refs. [25,27], where the algorithm to compute the series expansion up to any other is shown, and the lowest-order results for the vielbein are presented.

Observing the derivative structure of \(\langle a_{i-1}, b_{i-1}, c_{i-1} | \tilde{O} | a_i, b_i, c_i \rangle\) shows that the number of derivatives that act on \(\tilde{O}\) is roughly \(2 + s_{i-1} - s_i = 2 - \Delta s_i\). Hence, to compute the bracket we shall need normal coordinate expansion up to this order. For \(|a_k\rangle\), the maximum of \(2 - \Delta s_i = 2k - 4\), which occurs when all but one of \(\Delta s_i = 2\). Thus we need the normal expansion of different objects up to \(2k - 4\) order, which is always possible.

For example, to compute \(|a_2\rangle\) we have to calculate the terms in the normal coordinate expansion up to order \(2(2) - 4 = 0\); in other words we do not need the expansion at all, as demonstrated in Eq. (152). For \(|a_3\rangle, 2k - 4 = 2\) and thus we need the second-order expansion, which is not difficult to obtain as in Refs. [25,27]. Indeed, looking at the list of terms above, the bracket \(|2, 0, 0 | \tilde{O} | 0, b, c \rangle\) in the \(0 \to 2 \to 2 \to 4\) branch does require the second-order normal expansion. This also happens in the \(0 \to 2 \to 4 \to 4\) branch.

5. One-loop divergences of the super Yang–Mills theory

5.1. Heat kernel coefficients for the Yang–Mills vector superfield

With the machinery developed, we are now ready to calculate the super heat kernel coefficients of SYM in conformal supergravity. First, looking at the operator \(O_V\) derived previously in Eq. (106), in order to apply the non-recursive method we have to satisfy the constraints. The vanishing trace of torsion is clear for conformal supergravity; however, the second constraint is not satisfied, with the problematic terms being

\[
O_V = \frac{i}{4} U^a (\nabla^\hat{\beta} \nabla_{\alpha} + \nabla_{\alpha} \nabla^\hat{\beta}) + \frac{i}{4} U_\phi (\nabla^\hat{\beta} \nabla_{\beta} + \nabla_{\beta} \nabla^\hat{\beta}) + U^a \nabla_a + \cdots .
\]

(158)

To resolve this one can choose the gauge \(\nabla_A X = 0\), which implies the \(U_A = \nabla_A \log X = 0\) and thus all the terms above are eliminated. This choice of gauge implies that we have to break the \(D\) and \(K_A\) symmetry; in other words, we are reducing to the \(U(1)\) supergravity regime. To ensure
consistency, we have to make sure that the final result must be $D$ and $K_A$ invariant, that is, having the correct scaling dimension and being conformal primary.

Alternatively, one can employ the associated derivatives \cite{22,23}, which is a set of modified covariant derivatives briefly mentioned before. The primary feature of this tool is that the $D$ and $K_A$ invariance are treated somewhat as a hidden symmetry, the effect of which is equivalent to choosing the gauge $\nabla_A X = 0$. One can show that by rewriting $O_V$ in terms of these derivatives, the undesired terms will be absorbed via the redefinition of derivatives. Then one can proceed normally using such new covariant derivatives. Of course, the two routes discussed will give the same result; for the sake of simplicity we shall choose the former by temporarily breaking some symmetries and manually checking the gauge invariance afterwards.

Since we are explicitly breaking the $D$ and $K_A$ symmetry, we have to redefine the covariant derivatives by removing the relevant terms. The new derivative will be exactly the covariant derivatives $D_A$ in $U(1)$ supergravity. Such a redefinition of course cannot alter the quadratic part of the operator $O_V$, as the redefinition is a linear shift, $D_A = \nabla_A + \cdots$, and thus the quadratic part of $O_V$ after gauge fixing is

$$O_V = \Box + \frac{1}{2} G^{a\dot{a}} [D_a, D_{\dot{a}}] + \cdots. \quad (159)$$

Using the calculations of the previous section, Eq. (152), we immediately arrive at the result:

$$[a_0] = [a_1] = 0, \quad [a_2] = 16 \left( \frac{G^{a\dot{a}}}{2} \right) \left( \frac{G_{a\dot{a}}}{2} \right) + 0 = -8 G_a G^a. \quad (160)$$

Note that direct verification shows that $G_{a\dot{a}} = -\frac{1}{4} (U_{a\dot{a}} - U_{\dot{a}a}) - \frac{1}{2} U_a U_{\dot{a}}$ is conformal primary, and thus the expression for $[a_2]$ above is $K_A$ invariant. Also, $DG^{a\dot{a}} = G_{a\dot{a}}$, so $[a_2]$ has the correct $D$-charge too. Combined with the fact that $\nabla_A X$ is not conformal primary, and thus cannot appear in $[a_2]$ before gauge fixing, we conclude that $[a_2] = -8 G_a G^a$ holds identically in conformal supergravity, with $D$ and $K_A$ symmetries manifest.

It is interesting to note that $[a_2]$, and thus the logarithmic divergence, does not depend on the constant term of the operator $O_V$. Typically in supergravity models the Yang–Mills vector multiplet in general acquires a mass from the background field expansion of the Kähler potential term $\exp(K/3)$. The implication is that the mass of $V$ will not contribute to the one-loop divergence, a very special feature that most other theories, for instance a theory with only chiral superfields, do not have.

One can compare this result with the case of an Abelian vector multiplet in minimal supergravity \cite{13}; the results indeed agree apart from a factor of two, which originates from different normalizations of heat kernel coefficients. The result is also consistent with the fact that in flat superspace, for pure SYM theory with Faddeev–Popov gauge fixing, the Yang–Mills vector multiplet is UV finite and logarithmic divergence arises only from the ghost fields \cite{29}; thus $[a_2] = 0$ in this case, as we can just set the auxiliary field $G_{\alpha} = 0$ in the case of flatness.

5.2. Chiral heat kernel and one-loop divergence from ghost fields

So far we have derived the heat kernel coefficients for the Yang–Mills vector multiplet $V$, but there are still ghost fields $c, c', \text{ and } b$. They are chiral superfields and thus we can examine them using the chiral heat kernel, which is just a chiral analog of the heat kernel in the full superspace. One can simply replace various full superspace quantities by their chiral subspace counterparts. For example,
the heat kernel for chiral fields is defined as

\[ K(z, z'; \tau) = e^{i \tau \mathcal{O}} \mathcal{E}^{-1} \delta^6(z - z'), \] (161)

with \( z \) now the coordinates of the chiral subspace \( \theta' = 0 \). Any superspace integration, for instance when taking the supertrace to get the one-loop effective action, must be replaced by an integration over the chiral subspace; in other words we have an \( F \)-term action instead. Any techniques to compute the super heat kernel and its coefficients can be easily applied to the chiral case, with some technical, but manageable, modifications in order to respect the chirality. One example is that one must use a special set of normal coordinates for the chiral subspace \[15\], such that the coordinate functions \( y^m \) and \( y^{\mu} \) are chiral.\(^7\) In particular, it is straightforward to generalize the algorithm discussed in the previous section to calculate the chiral heat kernel coefficients.

The one-loop divergence for chiral fields in conformal supergravity was discussed and calculated in Ref. \[28\]; the results therein will be applied here for the Yang–Mills ghost fields. Let us start with the Nielsen–Kallosh ghost, with its action being

\[ S_b = \text{tr} \int d^8 z E X^{-2} b \bar{b}. \] (162)

This is close to a free action, except that there is an extra factor \( X^{-2} \). To deal with this we can rewrite the action as \( X^{-2} b \bar{b} b = b \exp(-2 \log X) \bar{b} \), which is very similar to how super Yang–Mills theory couples to chiral fields: \( \phi \exp(2V) \phi \). Thus we can introduce an artificial \( U(1) \) gauge symmetry, with \( \log X \) taking the role of its “vector multiplet.” The \( \exp(-2 \log X) \) factor can be absorbed using a new covariant derivative, by adding the corresponding term for this \( U(1) \). This extra \( U(1) \) will have the “gaugino” field\(^8\):

\[ \frac{1}{8} \bar{\nabla}^2 e^{2 \log X} \nabla_\alpha e^{-2 \log X} = -\frac{2}{3} X_\alpha. \] (163)

Hence, we can simply replace the Yang–Mills gaugino, whenever it shows up, by the shifted version:

\( \mathcal{W}_\alpha^a \rightarrow \mathcal{W}_\alpha^a - \frac{2}{3} X^a \). Therefore, from Ref. \[28\], the one-loop divergence is given by

\[ \Gamma^b_{(1)\text{div}} = \frac{\Lambda^2}{32 \pi^2} [(1 - 4 V_{YM}) X]_D + \frac{\log \Lambda^2}{96 \pi^2} S_{\chi} \]

\[ - \frac{\log \Lambda^2}{64 \pi^2} \left( \left( \mathcal{W}_\alpha^a X^a - \frac{2}{3} X^a \right)^2 + \frac{2}{3} W_{\alpha \beta \gamma} W_{\gamma \beta \alpha} \right)_F + \text{h.c.}, \] (164)

where

\[ S_{\chi} = [G^a G_a + 2 R R]_D + \left( \left[ \frac{1}{12} X^a X_a + \frac{1}{2} W_{\alpha \beta \gamma} W_{\gamma \beta \alpha} \right]_F + \text{h.c.} \right) \] (165)

is a topological invariant. Note that we have an extra minus sign from the abnormal statistics of ghost fields, and \( \log \epsilon = - \log \Lambda^2 \).

\(^7\) This is not the case for full superspace normal coordinates.

\(^8\) Recall that \( X_\alpha = \frac{1}{8} \bar{\nabla}^2 \nabla_\alpha \log X \).
Next, we turn to the Faddeev–Popov ghost $c$ and $c'$, the relevant one-loop action being

$$S_{FP}^{(2)} = \text{tr} \int d^8z \, EX (c' + \bar{c}')(c + \bar{c})$$

$$= \text{tr} \int d^8z \, EX (c' \bar{c} + \bar{c}'c) + \text{tr} \int d^6z \, E \bar{X}c'c + \text{tr} \int d^6z \, E \bar{X}c'\bar{c}. \quad (166)$$

Here, the chiral projector is used to produce the $F$-terms by using the definition of $R$: $-\frac{1}{3} \bar{\nabla}^2 X = 2RX$. Similar to the case of the Nielsen–Kallosh ghost we absorb the factor of $X$ in the action by introducing by hand an extra $U(1)$ gauge sector; this time the shift of the gaugino is given by $\mathcal{W}_\alpha^\alpha \to \mathcal{W}_\alpha^\alpha + \frac{1}{3} X^\alpha$. Now 2$R$ and its conjugate can be treated as a mass term for the fields. Again using the result of Ref. [28], we have

$$\Gamma_{FP}^{(1)} = \frac{\Lambda^2}{16\pi^2} \left[ \text{tr}(1 - 4V_{YM})X \right]_D + \frac{\log \Lambda^2}{48\pi^2} \left[ S_X \right] - \frac{\log \Lambda^2}{16\pi^2} \left[ 4R \bar{R} \right]_D$$

$$- \frac{\log \Lambda^2}{32\pi^2} \left[ \left( \mathcal{W}_\alpha^\alpha + \frac{1}{3} X^\alpha \right)^2 + \frac{2}{3} W^{\alpha \beta \gamma} W_{\gamma \beta \alpha} \right]_F + \text{h.c.}, \quad (167)$$

where we have multiplied the divergence by a factor of $-2$ as there are two sets of ghosts with abnormal statistics.

### 5.3. Full one-loop divergence of super Yang–Mills theory

Finally, combining the result for the Yang–Mills vector field, the Faddeev–Popov ghosts, and the Nielsen–Kallosh ghost, and taking the Yang–Mills traces, we present here the full one-loop divergence of the super Yang–Mills theory in conformal supergravity:

$$\Gamma_{SYM}^{(1)} = \frac{3\Lambda^2}{32\pi^2} \left[ \text{tr}(1 - 4V_{YM})X \right]_D + \frac{NG \log \Lambda^2}{8\pi^2} S_X - \frac{NG \log \Lambda^2}{8\pi^2} \left[ G^\alpha G_\alpha + 2R \bar{R} \right]_D$$

$$- \frac{\log \Lambda^2}{64\pi^2} \left[ 2 \text{tr} \left( \mathcal{W}_\alpha^\alpha + \frac{1}{3} X^\alpha \right)^2 + \text{tr} \left( \mathcal{W}_\alpha^\alpha - \frac{2}{3} X^\alpha \right)^2 \right]_F + \text{h.c.}, \quad (168)$$

where $NG = \text{tr}1$ is the rank of the Yang–Mills gauge group. There are some cancellations for the logarithmic divergence: by substituting the definition of $S_X$, we get

$$\Gamma_{\log} = - \frac{3NG \log \Lambda^2}{32\pi^2} \left[ G^\alpha G_\alpha + 2R \bar{R} \right]_D$$

$$- \frac{\log \Lambda^2}{64\pi^2} \left[ 3 \text{tr} \mathcal{W}_\alpha^\alpha \mathcal{W}_\alpha^\alpha + \frac{NG}{2} X^\alpha X_\alpha + NG W^{\alpha \beta \gamma} W_{\gamma \beta \alpha} \right]_F + \text{h.c.}, \quad (169)$$

a somewhat surprisingly simple result. It is easily checked that this expression is the same as the calculation in the special case of supersymmetric quantum electrodynamics in minimal supergravity [13]. Of course, the term with the trace of $\mathcal{W}_{YM}$ can be interpreted as a renormalization of the original Yang–Mills action, the pre-factor here being consistent with the known beta function, at least in the flat superspace case.
Let us consider the presented results in terms of component fields. For practical purposes we shall apply the conformal gauge $X = 1$; in other words, we are reducing to the $U(1)$ supergravity. Let us start with the quadratic divergence in Eq. (168). We have a term proportional to $[1]_D$, which corresponds to a renormalization of the supergravity multiplet action; its component form can be looked up from Eq. (4.5.6) of Ref. [10]:

$$[1]_D = \int d^4x \left[ -\frac{1}{2} \mathcal{R} + \frac{1}{2} e^{mnpq} (\bar{\psi}_m \sigma_n \nabla_p \psi_q - \psi^m \sigma_n \nabla_p \bar{\psi}_q) - \frac{1}{3} M \bar{M} + \frac{1}{3} b^a b_a + D_{\text{matter}} \right].$$

Here, $\mathcal{R}$ is the Ricci scalar, $\psi$ is the gravitino, $M$ and $b^a$ are the auxiliary fields of the multiplet, and $D_{\text{matter}}$ is the matter contribution, which depends on the Kähler potential. Next we have the term $[V_{YM}]_D$, which is helicity-odd. In fact, it induces a divergent Fayet–Iliopoulos term which can be cancelled by introducing a local counterterm [28]. Thus we will not discuss this term further here.

We turn to the logarithmic divergence in Eq. (169). As mentioned previously there is a renormalization term for the SYM action and the full component expression is quite lengthy (it can be found in Ref. [10]). The remaining divergence is a linear combination of

$$\Delta_1 = [G^a G_a + 2 R \bar{R}]_D + \left( \frac{1}{12} [X^a X_a]_F + \text{h.c.} \right)$$

and

$$\Delta_2 = [W^{a\beta\gamma} W_{\beta\gamma a}]_F + \text{h.c.}$$

These two expressions are actually related to topological invariants, namely the Gauss–Bonnet and the Pontryagin invariants. Such supergravity invariants are discussed in, for instance, Refs. [30,31].

By using the general technique in Sect. 4 of Ref. [10], some calculation shows that the bosonic components of Eqs. (171) and (172) are given by

$$\Delta_1 = \int d^4x e \left( -\frac{1}{8} R_{mn} R_{mn} + \frac{1}{96} \mathcal{R}^2 - \frac{1}{6} F_{mn} F_{mn} \right)$$

and

$$\Delta_2 = \int d^4x e \left( \frac{1}{8} W^{mnpq} W_{mnpq} + \frac{1}{3} F_{mn} F_{mn} \right).$$

where $R_{mn}$ is the Ricci tensor, $W^{mnpq}$ is the Weyl tensor, and $F_{mn}$ is the chiral $U(1)$ curvature. Apart from the $U(1)_R$ field strength, the expressions do resemble the well-known curvature-squared invariants. Note that only the bosonic components are presented here as the remaining parts are less interesting in comparison, and one can uniquely recover those by supersymmetry. For discussions involving the fermionic components in the minimal supergravity formalism one can consult, for example, Ref. [32].

The appearance of such invariants is anticipated. It can be shown [13] that the one-loop trace of the energy–momentum tensor $T$, which directly measures the superconformal anomaly, is related to
the super heat kernel coefficient \([a_2]\) and thus the logarithmic divergence:\(^9\)

\[
\langle T \rangle = \frac{1}{64\pi^2} [a_2]. ~ (175)
\]

By analyzing the super-Weyl cohomology \([33]\), it can be shown that in the absence of background
matter the superconformal anomaly must be constructed from the Gauss–Bonnet and Pontryagin
invariants. Our result here is in agreement with this statement, as the original analysis was performed
in minimal supergravity in which the \(U(1)_R\) curvature is absent. It is worth noting that in each of
the individual logarithmic divergences of the various fields, Eqs. (160), (164), and (167), they all
contain non-topological invariant terms. Only when we add up the contributions to obtain the total
divergence in Eq. (169) do the problematic terms combine nicely into a multiple of \(\Delta_1\). This provides
a strong consistency check for our calculations.

6. Conclusion

We have derived the one-loop divergence of the super Yang–Mills theory in conformal supergravity
by first quantizing the theory with the background field method and obtaining the second-order action.
The main tool employed is the heat kernel method, applied to superspaces. We have described a non-
recursive technique to compute the heat kernel coefficients for the theory, and explicitly computed the
first three coefficients. The method presented here theoretically allows us to compute the coefficients
up to any order, as demonstrated.

The technique developed can be readily applied to other supersymmetric theories, and it will be
interesting to apply such machinery not just to the Yang–Mills theory, but to different interactions
with distinct field contents. It is hoped that one can derive the one-loop divergence of various
theories using such a method. Also, one may consider a more general version of Yang–Mills theory
characterized by a non-trivial gauge kinetic function. Such a generalization typically arises from
different phenomenological models, for instance from string theory models. The study of such a
general class of super Yang–Mills theory will be the subject of future work.

Acknowledgements

The author would like to thank Mary K. Gaillard for inspiring discussions, helpful comments, and encour-
agement for this work. This work was supported in part by the Director, Office of Science, Office of High
Energy and Nuclear Physics, Division of High Energy Physics, of the US Department of Energy under Contract
DE-AC02-05CH11231, and in part by the National Science Foundation under grant PHY-1316783.

Funding

Open Access funding: SCOAP³.

Appendix A. Faddeev–Popov procedure in superspace and derivation of the ghost
action

In the following we shall derive the gauge-fixed action of the super Yang–Mills theory with ghost
fields. Let us start by considering the functional integral for the field \(V\):

\[
Z = \int \mathcal{D} V \, e^{iS_{YM}}. ~ (A.1)
\]

\(^9\) A difference of a factor of 2 between here and Ref. [13] is due to a different normalization.
To impose the gauge condition of Eq. (74) one introduces a delta functional to the above path integral, which becomes

\[ Z = \int \mathcal{D}V \Delta_{FP}^{-1} \delta(\tilde{\nabla}^2(XV) - f) \delta(\nabla^2(XV) - \tilde{f}) e^{iS_{YM}}, \tag{A.2} \]

with \( \Delta_{FP} \) the famous Faddeev–Popov determinant, which will be computed later. As in the usual case, one may average over \( f \) and \( \tilde{f} \) in the gauge-fixing function with certain weight. The standard one is the Gaussian smearing, but with a slight twist here for our scenario. Instead, we insert into the functional integral the factor

\[ 1 = \int \mathcal{D}f \mathcal{D}\tilde{f} \mathcal{D}b \mathcal{D}\tilde{b} \exp \left[ i \frac{1}{8g^2\xi} \int d^8z E X - 2 \operatorname{tr}(f\tilde{f} + b\tilde{b}) \right], \tag{A.3} \]

with \( b \) having opposite statistics to \( f \) to normalize the factor; this contributes to the action

\[ S_{GF} = \frac{1}{8g^2\xi} \int d^8z EX^{-2} \operatorname{tr}(f\tilde{f} + b\tilde{b}). \tag{A.4} \]

By rescaling \( b \) we get the Nielsen–Kallosh ghost action in Eq. (76). We then turn to the computation of the Faddeev–Popov determinant. To do so, it is necessary to know how the gauge-fixing function in Eq. (74) changes under a gauge transformation. First, let us rewrite the transformation law in Eq. (71) by defining

\[ P = \exp \left[ -2i\frac{1}{\Lambda} \right] \quad \text{and} \quad Q = \exp \left[ 2i\frac{1}{\Lambda} \right]; \]

we have the familiar expression

\[ e^{V'} = e^{\bar{\Lambda}} e^V e^{\Lambda}. \tag{A.5} \]

It is well known that one can obtain the infinitesimal change in closed form:

\[ \delta V = \mathcal{L}_{V/2}[\Lambda - \bar{\Lambda} + \coth(\mathcal{L}_{V/2})(\Lambda + \bar{\Lambda})] = \Lambda + \bar{\Lambda} + O(V). \tag{A.6} \]

This will be relevant in the ghost action.

We can now write the Faddeev–Popov determinant as

\[ \Delta_{FP} = \int \mathcal{D}\Lambda \mathcal{D}\bar{\Lambda} \delta(F - f) \delta(\tilde{F} - \tilde{f}), \tag{A.7} \]

with \( F \) and \( \tilde{F} \) the gauge functions: \( F = \tilde{\nabla}^2(XV), \tilde{F} = \nabla^2(XV) \). Here the path integral is over the gauge group parameter space. As in the usual covariant quantization procedure, one uses gauge invariance and properties of the delta function to rewrite the delta functionals. We then obtain

\[ \Delta_{FP} = \int \mathcal{D}\Lambda \mathcal{D}\bar{\Lambda} \left. \left[ \delta F \right|_{\delta \Lambda} \Lambda + \left. \delta \bar{F} \right|_{\delta \bar{\Lambda}} \bar{\Lambda} \right] \left. \left[ \delta \tilde{F} \right|_{\delta \Lambda} \Lambda + \left. \delta \tilde{\bar{F}} \right|_{\delta \bar{\Lambda}} \bar{\Lambda} \right], \tag{A.8} \]

the stroke | denoting evaluation at the “origin” \( \Lambda = \bar{\Lambda} = 0 \). The next step is to recast the delta functionals using their integral representation, introducing the new fields \( \Lambda' \) and \( \bar{\Lambda}' \) with the obvious chirality:

\[ \Delta_{FP} = \int \mathcal{D}\Lambda \mathcal{D}\bar{\Lambda} \mathcal{D}\Lambda' \mathcal{D}\bar{\Lambda}' \exp \left[ i \operatorname{tr} \int d^4x d^2\theta \mathcal{E} \Lambda' \left. \left[ \delta F \right|_{\delta \Lambda} \Lambda + \left. \delta \bar{F} \right|_{\delta \bar{\Lambda}} \bar{\Lambda} \right] \right] \]

\[ + i \operatorname{tr} \int d^4x d^2\bar{\theta} \mathcal{E} \bar{\Lambda}' \left. \left[ \delta \bar{F} \right|_{\delta \Lambda} \Lambda + \left. \delta \tilde{\bar{F}} \right|_{\delta \bar{\Lambda}} \bar{\Lambda} \right] \right]. \tag{A.9} \]
What we desire is the reciprocal of $\Delta_{FP}$. As in the functional integral $Z$, this can be achieved by replacing the fields appearing in $\Delta_{FP}$ by ghost fields with opposite statistics, which at the end introduces the Faddeev–Popov action:

$$S_{FP} = \text{tr} \int d^4 x d^2 \theta \varepsilon c' \left( \frac{\delta F}{\delta c} c + \frac{\delta \tilde{F}}{\delta \tilde{c}} \tilde{c} \right) + \text{tr} \int d^4 x d^2 \bar{\theta} \tilde{\varepsilon} \tilde{c}' \left( \frac{\delta F}{\delta c} c + \frac{\delta \tilde{F}}{\delta \tilde{c}} \tilde{c} \right)$$

$$= \text{tr} \int d^4 x d^2 \theta \varepsilon c' \sqrt{2} (X \delta V) + \text{tr} \int d^4 x d^2 \bar{\theta} \tilde{\varepsilon} \tilde{c}' \sqrt{2} (X \delta V)$$

$$= \int d^8 z E X \left[ c + \bar{c} + \text{coth}(X/2) \right]. \quad (A.10)$$

Here we have used the chiral projection to convert chiral integrals into $D$-terms, and rescaled the fields to remove the numerical pre-factor that appears. Apart from the compensator $X$, this is the same as in the case of flat space.

As a remark, the Faddeev–Popov ghosts have vanishing conformal weights, and the Nielsen–Kallosh ghost has weights $(\Delta, w) = (3, 2)$. Having dimension 3 may sound awkward and one might think this will pose technical difficulties. It turns out that its action is simple enough that this will not be a concern, especially for one-loop calculations.

**Appendix B. Integration by parts in conformal supergravity**

In usual supergravity theory, if one has a superfield $v^A$ one can easily show that the term

$$\int d^8 z E \nabla_A v^A \quad (B.1)$$

is a surface term that vanishes given the appropriate boundary conditions, and hence one can safely treat them as zero. Here, $A$ can be either a vector or spinor index, $a, \alpha, \dot{\alpha}$. However, in conformal superspace such total derivative terms actually may not vanish, since there are extra generators, in particular the special conformal ones that act non-trivially on $v^A$. In fact, special conformal curvature terms will appear, as we will derive here.

Let us start with the covariant derivative

$$\nabla_M (EE_A M v^A) = \partial_M (EE_A M v^A) - h_M^A X_A (EE_A M v^A). \quad (B.2)$$

The first term of the right-hand side is the surface term that can be neglected. For the second term, only the special conformal curvature $K_A$ may give a non-zero result,

$$h_M^A X_A (EE_A M v^A) = E f_A^B K_B v^A, \quad (B.3)$$

where $f_A^B$ is the special conformal gauge field. Note that $K_A$ does not commute with the covariant derivatives, so even if $\nabla_A v^A$ is conformal primary, $K_B \nabla_A v^A = 0$, it does not imply that $v^A$ is also primary. This is exactly the reason that the usual integration by parts has to be modified in conformal superspace.

Now consider the left-hand side: $\nabla_M$ acting on $v^A$ gives the desired term $E \nabla_A v^A$. There is one more term showing up when $\nabla_M$ acts on the vielbein, which can be computed by mimicking the calculation in ordinary supergravity, or in $U(1)$ supergravity [10]. The resulting expression is identically the same as the aforementioned cases, with torsion coefficients appearing:

$$\nabla_M (EE_A M v^A) = ET_{AB} v^A + E \nabla_A v^A. \quad (B.4)$$
In conformal supergravity one can easily check that the torsion term has vanishing trace similar to the ordinary scenario. We then arrive at the integration by parts formula:

\[ \nabla_A v^A \approx -f_A^B K_B v^A, \]

with \( \approx \) denoting equal up to a surface term that can effectively be set to zero.

One notational remark is that since the implicit grading is used throughout here, one may need to insert terms like \((-1)^d\) if such implicit grading is lifted, and such factors do appear in the usual supergravity integration by parts formula.

### Appendix C. Avramidi's method for non-supersymmetric theories

We shall present an efficient technique for deriving heat kernel coefficients developed by Avramidi [12]. For simplicity, we shall only consider operators of the form

\[ \mathcal{O} = \nabla_\mu \nabla^\mu + Q(x). \]  

First, it will be convenient to work in special coordinates, the Riemann normal coordinates [34]. Near the point \( x' \), define a coordinate system \( \{y^m\} \) whose coordinate function satisfies

\[ y^m = -\delta^m_\sigma e^\sigma_\mu (x') \nabla^\mu \sigma. \]  

Here we have introduced a moving frame with vielbein \( e^a_\mu \). The recursion relation in this case is

\[ \left( 1 + \frac{D}{n} \right) a_n = \tilde{\mathcal{O}} a_{n-1} \quad (n > 0), \]

\[ Da_0 = 0, \quad a_0(x,x) = 1, \]

where

\[ D = \nabla_\mu \sigma \nabla^\mu = y^m \nabla_m = y^m \delta_m, \quad \tilde{\mathcal{O}} = \Delta^{-\frac{1}{2}} \nabla_\mu \nabla^\mu \Delta^{\frac{1}{2}} + Q. \]

Note that \( y^m \nabla_m = y^m \delta_m \) is a consequence of using Riemann normal coordinates. The Van Vleck–Morette determinant also simplifies in this coordinate system:

\[ \Delta = \frac{1}{\sqrt{-g(x)}} = e(x)^{-1}. \]

The coefficient \( a_0 \) is a known quantity, the so-called parallel displacement operator,

\[ a_0 = \mathcal{I}(x,x'), \]

which parallel transports a field \( \phi \) at \( x' \) to the point \( x \), and is just the identity for scalars. It satisfies the following key properties:

\[ [\mathcal{I}] = 1, \quad [\nabla_{(\mu_1} \nabla_{\mu_2} \ldots \nabla_{\mu_k)} \mathcal{I}] = 0. \]

Here, the square bracket means we are taking the coincidence limit:

\[ [f(x,x')] (y) = f(y,y); \]

this common convention will appear often from now on.
We are ready to evaluate the coefficients $a_n$ by formally solving Eq. (C.4):

$$a_n = \left(1 + \frac{D}{n}\right)^{-1} \tilde{O} \left(1 + \frac{D}{n-1}\right)^{-1} \tilde{O} \cdots (1 + D)^{-1} \tilde{O} I.$$  \hspace{1cm} (C.10)

To compute this, notice that

$$D y^m = y^m$$  \hspace{1cm} (C.11)

implies that $y^m$ is the eigenvector of $D$ with eigenvalue 1. This allows us to define the eigenvector $|n\rangle$ with eigenvalue being any positive integer $n$, using symmetric products:

$$|0\rangle = 1, \quad |n\rangle = \frac{1}{n!} y^m_1 y^m_2 \cdots y^m_n \quad (n > 0).$$  \hspace{1cm} (C.12)

The dual $\langle n|\phi \rangle$ will be defined via

$$\langle n|\phi \rangle = \partial_m \partial_{m_2} \cdots \partial_{m_n} \phi \mid_{y^m=0}.$$  \hspace{1cm} (C.13)

The orthonormal relation is easily seen to be satisfied. We also have the completeness relation:

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = 1.$$  \hspace{1cm} (C.14)

To show this, we fix the point $x'$ and consider $x$ close to $x'$. The covariant Taylor series [35] for a scalar function around $x'$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \nabla^\mu_1 \sigma \nabla^\mu_2 \sigma \cdots \nabla^\mu_n \sigma \nabla_{\mu_1} \nabla_{\mu_2} \cdots \nabla_{\mu_n} f(z) \mid_{z=x'}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} y^m_1 y^m_2 \cdots y^m_n \partial_{m_1} \partial_{m_2} \cdots \partial_{m_n} f(z) \mid_{z=x'},$$  \hspace{1cm} (C.15)

which immediately gives the desired formula in Eq. (C.14). Note that if the object being acted on is not a scalar, which will not concern us here, we have instead:

$$\sum_{n=0}^{\infty} \langle n| \sum_{n=0}^{\infty} |n\rangle = 1.$$  \hspace{1cm} (C.16)

Also note that the coincidence limit is just given by a simple bracket,

$$[\phi] = \langle 0|\phi \rangle,$$  \hspace{1cm} (C.17)

as $y^m = 0$ is equivalent to $x = x'$.

Using the tools just introduced, the operator inverse appearing in Eq. (C.10) can be written as

$$\left(1 + \frac{D}{n}\right)^{-1} = \sum_{m} \left(1 + \frac{I}{k}\right)^{-1} |l\rangle \langle l|. \hspace{1cm} (C.18)$$
It is useful to note that it commutes with $\mathcal{I}$ as $D\mathcal{I} = 0$. Now we have
\[
[a_n] = \sum_{l_1,l_2,\ldots,l_{n-1}} \left(1 + \frac{l_{n-1}}{n}\right)^{-1} \left(1 + \frac{l_{n-2}}{n-1}\right)^{-1} \cdots \left(1 + l_1\right)^{-1} \times \langle 0|\mathcal{P}|l_{n-1}\rangle \langle l_{n-1}|\mathcal{P}|l_{n-2}\rangle \cdots \langle l_1|\mathcal{P}|0\rangle,
\]
where we have used $\langle 0|\mathcal{I}|k\rangle = \delta_{k0}$ and
\[
\mathcal{P} = \mathcal{I}^{-1} \hat{\mathcal{O}} \mathcal{I}.
\]
One important fact is that since $\mathcal{P}$ is a quadratic differential operator, $\langle k|\mathcal{P}|l\rangle$ is non-zero only if $l \leq k + 2$, and hence the summation is actually finite. It is convenient to decompose $\mathcal{P}$ into the following form:
\[
\mathcal{P} = X^{mn} \partial_m \partial_n + Y^m \partial_m + Z,
\]
where $\phi^m$ is the connection, $\nabla_m = \partial_m + \phi_m$, and
\[
B = \log \Delta = \log e^{-1}.
\]
Note that in normal coordinates $\mathcal{I}$ is just unity, $\mathcal{I} = 1$.

Let us compute the coefficients $[a_0], [a_1],$ and $[a_2]$. $[a_0]$ is trivial: $[a_0] = 1$. For $[a_1]$, it is simply
\[
[a_1] = \langle 0|\mathcal{P}|0\rangle = [Z] = Q + [\partial_m \phi^m] + [\phi_m \phi^m] + \frac{1}{2} [\partial_m \partial^m B] - \frac{1}{4} [\partial_m B] [\partial^m B].
\]
To proceed, we need the expansion for the connection $\phi_m$. Note that $y^m \nabla_m = y^m \partial_m$ implies that
\[
y^m \phi_m = 0.
\]
This is analogous to the Fock–Schwinger gauge in gauge theory, and hence the connection will have an expansion [36]:
\[
\phi_m = \sum_{k=0}^{\infty} \frac{\nabla_y)^k}{k!(k+2)} y^m F_{nm}, \quad \nabla_y = y^m \nabla_m.
\]
$F_{nm}$ is the field strength for the connection. Hence,
\[
[\phi_m \phi^n] = 0, \quad [\partial_m \phi^m] = \frac{1}{2} F^m_m = 0.
\]
We also use the vielbein expansion in normal coordinates [34],
\[
\epsilon^a_m(x) = \delta^a_m + \frac{1}{3!} y^m y^p \delta^b_m R_{mba}(x') + \cdots.
\]
and the formula for determinant:

\[ B = \log \det (e^a_m)^{-1} = - \text{tr} \log e^a_m = - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \text{tr}(e^a_m - \delta^a_m)^k. \] (C.31)

We see that

\[ [\partial_m B] = 0, \quad [\partial_m \partial^m B] = \frac{1}{3} R, \] (C.32)

with \( R \) the Ricci scalar. Thus,

\[ [a_1] = Q + \frac{R}{6}. \] (C.33)

The calculation for \([a_2]\) is slightly more tedious. It is given by

\[ [a_2] = \sum_l (1 + l)^{-1} \langle 0 | \mathcal{P} | l \rangle \langle l | \mathcal{P} | 0 \rangle \]
\[ = \langle 0 | \mathcal{P} | 0 \rangle^2 + \frac{1}{2} \langle 0 | \mathcal{P} | 1 \rangle \langle 1 | \mathcal{P} | 0 \rangle + \frac{1}{3} \langle 0 | \mathcal{P} | 2 \rangle \langle 2 | \mathcal{P} | 0 \rangle \]
\[ = [Z]^2 + \frac{1}{2} [y^m] [\partial_m Z] + \frac{1}{3} [X^{mn}] [\partial_m \partial_n Z] \]
\[ = \left( Q + \frac{R}{6} \right)^2 + [\phi^m][\partial_m Z] + \frac{1}{3} [g^{mn}][\partial_m \partial_n Z]. \] (C.34)

The second term vanishes as \([\phi^m] = 0\); for the last term,

\[ [g^{mn}][\partial_m \partial_n Z] = [\partial_m \partial^m Z] = \Box Q + [\partial_m \partial^m \phi^m] + [\partial_n \partial^m \phi^m] \]
\[ + \frac{1}{2} [\partial_n \partial^m \partial^m B] - \frac{1}{4} [\partial_n \partial^m (\partial_m B \partial^m B)]. \] (C.35)

From Eq. (C.28), we know that

\[ [\partial_n \partial^m \phi^m] = 2[\partial_n \phi_m][\partial^m \phi] = \frac{1}{2} F_{nm} F^{nm}, \quad [\partial_n \partial^m \phi^m] = 0. \] (C.36)

The second equation comes from the fact that the gauge condition is equivalent to any symmetrized partial derivatives of \( \phi^m \) vanishing:

\[ \partial_{(n_1} \ldots \partial_{n_k} \phi_m) = 0. \] (C.37)

The normal coordinate expansion of the vielbein gives the following result [37]:

\[ [\partial_n \partial^m \partial^m B] = \frac{2}{5} \Box R + \frac{2}{45} R_{mn} R^{mn} + \frac{1}{15} R_{mpnq} R^{mpnq}, \] (C.38)

\[ [\partial_n \partial^m (\partial_m B \partial^m B)] = 2[\partial_n \partial_m B][\partial^m B] = \frac{2}{5} R_{mn} R^{mn}. \] (C.39)
Combining everything, we have the final result:

\[
[a_2] = \left( Q + \frac{R}{6} \right)^2 + \frac{1}{3} \Box Q + \frac{1}{6} F_{mn} F^{mn} + \frac{1}{15} \Box R
- \frac{1}{30} R_{mn} R^{mn} + \frac{1}{30} R_{mn pq} R^{mn pq}.
\]  
(C.40)

The above result agrees with De Witt’s original calculation. In principle, it is possible to compute higher-order coefficients using the same method, as Avramidi calculated \([a_3]\) and \([a_4]\) with this machinery.

References

[1] M. K. Gaillard and V. Jain, Phys. Rev. D 49, 1951 (1994).
[2] M. K. Gaillard, V. Jain, and K. Saririan, Phys. Rev. D 55, 883 (1997).
[3] M. K. Gaillard, Phys. Rev. D 61, 084028 (2000).
[4] D. Butter and M. K. Gaillard, Phys. Rev. D 91, 025015 (2015).
[5] M. Kaku, P. K. Townsend, and P. van Nieuwenhuizen, Phys. Rev. D 17, 3179 (1978).
[6] M. Kaku and P. K. Townsend, Phys. Lett. B 76, 54 (1978).
[7] P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D 19, 3166 (1979).
[8] S. Ferrara, M. T. Grisaru, and P. Van Nieuwenhuizen, Nucl. Phys. B 138, 430 (1978).
[9] D. Butter, Ann. Phys. 325, 1026 (2010).
[10] P. Binétruy, G. Girardi, and R. Grimm, Phys. Rept. 343, 255 (2001).
[11] B. S. De Witt, Dynamical Theory of Groups and Fields (Gordon and Breach, New York, 1964).
[12] I. G. Avramidi, Nucl. Phys. B 355, 712 (1991); 509, 557 (1998) [erratum].
[13] I. L. Buchbinder and S. M. Kuzenko, Nucl. Phys. B 274, 653 (1986).
[14] I. N. McArthur, Phys. Lett. B 128, 194 (1983).
[15] I. N. McArthur, Class. Quantum Grav. 1, 245 (1984).
[16] T. Kugo, R. Yokokura, and K. Yoshioka, Prog. Theor. Exp. Phys. 2016, 073B07 (2016).
[17] T. Kugo, R. Yokokura, and K. Yoshioka, Prog. Theor. Exp. Phys. 2016, 093B03 (2016).
[18] T. Kugo and S. Uehara, Nucl. Phys. B 226, 49 (1983).
[19] J. Wess and J. Bagger, Supersymmetry and Supergravity (Princeton University Press, Princeton, NJ, 1992).
[20] M. T. Grisaru, W. Siegel, and M. Roček, Nucl. Phys. B 159, 429 (1979).
[21] S. J. Gates, M. T. Grisaru, M. Roček, and W. Siegel, Front. Phys. 58, 1 (1983).
[22] D. Butter, Nucl. Phys. B 828, 233 (2010).
[23] T. Kugo and S. Uehara, Prog. Theor. Phys. 73, 235 (1985).
[24] S. M. Kuzenko and I. N. McArthur, J. High Energy Phys. 0305, 015 (2003).
[25] I. N. McArthur, Class. Quantum Grav. 1, 233 (1984).
[26] T. Ohrndorf, Nucl. Phys. B 268, 654 (1986).
[27] M. T. Grisaru, M. E. Knutt-Wehlau, and W. Siegel, Nucl. Phys. B 523, 663 (1998).
[28] D. Butter, arXiv:0911.5426 [hep-th] [Search INSPIRE].
[29] T. Ohrndorf, Phys. Lett. B 176, 421 (1986).
[30] P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D 19, 3592 (1979).
[31] G. Girardi and R. Grimm, Phys. Lett. B 260, 365 (1991).
[32] S. Ferrara and M. Villasante, J. Math. Phys. 30, 104 (1989).
[33] L. Bonora, P. Pasti, and M. Tonin, Nucl. Phys. B 252, 458 (1985).
[34] E. Poisson, A. Pound, and I. Vega, Living Rev. Rel. 14, 7 (2011).
[35] A. O. Barvinsky and G. A. Vilkovisky, Phys. Rept. 119, 1 (1985).
[36] M. A. Shifman, Nucl. Phys. B 173, 13 (1980).
[37] A. E. M. van de Ven, Class. Quantum Grav. 15, 2311 (1998).