Quantization of the Type II Superstring in a Curved Six-Dimensional Background

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Abstract

A sigma model action with N=2 D=6 superspace variables is constructed for the Type II superstring compactified to six curved dimensions with Ramond-Ramond flux. The action can be quantized since the sigma model is linear when the six-dimensional spacetime is flat. When the six-dimensional spacetime is $AdS_3 \times S^3$, the action reduces to one found earlier with Vafa and Witten.
1. Introduction

Construction of quantizable superstring actions in $AdS_d \times S^d$ backgrounds with Ramond-Ramond flux is currently of great interest. Although such a construction has not yet been done for the $AdS_5 \times S^5$ case, it has been done for the $AdS_2 \times S^2$ and $AdS_3 \times S^3$ cases. In the $AdS_2 \times S^2$ case [1], this construction was straightforward since a quantizable sigma model action was already known for the superstring in a general four-dimensional background [2]. In the $AdS_3 \times S^3$ case [3], however, guesswork had to be used since the action in a general six-dimensional background had not yet been found.

In this paper, a quantizable sigma model action is constructed for the superstring in a general six-dimensional background. This construction is useful for several reasons. Firstly, it allows quantization of the superstring in six-dimensional curved backgrounds other than $AdS_3 \times S^3$. Secondly, it provides a new and simpler description of the $AdS_3 \times S^3$ action which may be useful for the construction of vertex operators. Thirdly, it provides clues which might be useful for quantization of the superstring in a general ten-dimensional background.

The six-dimensional action in a flat background was constructed in [4] (and reviewed in [3]) using worldsheet variables from the ‘hybrid’ description of the superstring. These hybrid variables are related by a field redefinition to the worldsheet variables of the Ramond-Neveu-Schwarz (RNS) formalism and include spacetime spinors as in the Green-Schwarz (GS) formalism. The hybrid action has an N=2 worldsheet superconformal invariance which replaces $\kappa$ symmetry of the GS action and which is related to a twisted N=2 BRST symmetry of the RNS formalism [5]. Unlike the GS action, quantization is straightforward since the hybrid action in a flat background is quadratic.

In the formalism of [4] [3], only half of the sixteen $\theta$’s of N=2 D=6 superspace are present as fundamental worldsheet fields. Although this preserves manifest SO(5,1) Lorentz invariance, it breaks half of the manifest D=6 supersymmetries. This fact made it difficult to generalize the formalism to arbitrary curved six-dimensional backgrounds.

In this paper, this difficulty will be overcome by adding eight more $\theta$’s (and their conjugate momenta) as fundamental fields in the action, as well as eight first-class “harmonic” constraints [6] [7] which can be used to gauge away these new fields. With these additional fields and constraints, it will be easy to construct a sigma model action for the superstring in an arbitrary curved six-dimensional background.

Like the GS sigma model action, the hybrid action is defined using superspace variables which are extremely convenient for describing backgrounds with Ramond-Ramond flux.
However, unlike the GS sigma model action, the hybrid action reduces to a free quadratic action when the six-dimensional spacetime is flat. By using a normal coordinate expansion, this allows quantization in a curved background. Another difference with the GS action is that the hybrid action contains a coupling of the spacetime dilaton to the worldsheet curvature, as expected from the coupling-constant dependence of scattering amplitudes.

In section 2 of this paper, the hybrid action will be reviewed in a flat six-dimensional background. In section 3, this action will be written in a manifestly N=2 D=6 supersymmetric form by introducing new \( \theta \) variables and new harmonic constraints \([6][7]\). In section 4, the manifestly spacetime-supersymmetric form of the action will be generalized to a curved six-dimensional background. And in section 5, the action will be shown to reduce to that of \([3]\) when the six-dimensional background is chosen to be \( AdS_3 \times S^3 \) with Ramond-Ramond flux.

2. Review of hybrid action in a flat six-dimensional background

2.1. Action and N=2 constraints

In a flat six-dimensional background, the hybrid formalism was developed in reference \([4]\) and was reviewed in \([3]\). Besides \( x^m \) for \( m = 0 \) to 5, the six-dimensional left-moving worldsheet fields consist of eight fermions, \( \theta^\alpha \) and \( p_\alpha \) for \( \alpha = 1 \) to 4, and two chiral bosons, \( \rho \) and \( \sigma \). For the closed superstring, the right-moving worldsheet fields consist of \( \bar{\theta}^\bar{\alpha} \) and \( \bar{p}_{\bar{\alpha}} \) for \( \bar{\alpha} = 1 \) to 4, and two anti-chiral bosons, \( \bar{\rho} \) and \( \bar{\sigma} \). For the Type IIB (or Type IIA) superstring, an up \( \alpha \) index and up (or down) \( \bar{\alpha} \) index transform as 4 representations of SU(4), and a down \( \alpha \) index and down (or up) \( \bar{\alpha} \) transform as \( \bar{4} \) representations. In addition, one has a \( c = 6 \) N=2 superconformal field theory representing the compactification manifold.

In a flat background, the free action is

\[
S = \int d^2 z \left( \frac{1}{2} \partial x^m \partial x_m + p_\alpha \partial \theta^\alpha + \bar{p}_{\bar{\alpha}} \partial \bar{\theta}^{\bar{\alpha}} \right) + S_B + S_C
\]

where \( S_C \) is the action for the compactification variables and \( S_B \) is an action for the chiral and anti-chiral bosons which we will not write explicitly. One also has the following critical N=2 superconformal generators:

\[
T = \frac{1}{2} \partial x^m \partial x_m + p_\alpha \partial \theta^\alpha + \frac{1}{2} \partial \rho \partial \rho + \frac{1}{2} \partial \sigma \partial \sigma + \frac{3}{2} \partial^2 (\rho + i \sigma) + T_C,
\]
where \([T_C, G^+_C, G^-_C, J_C]\) are the \(c = 6\) \(N=2\) generators of the superconformal field theory representing the compactification, \((p)^4 = \frac{1}{24} \epsilon^{\alpha\beta\gamma\delta} p_\alpha p_\beta p_\gamma p_\delta\), \(x^m\) has been written in bispinor notation as \(x^{\alpha\beta} = (\sigma_m)^{\alpha\beta} x^m\), and \((\sigma_m)^{\alpha\beta}\) are the six-dimensional Pauli matrices satisfying

\[(\sigma_m)^{\alpha\beta}(\sigma_n)_{\beta\gamma} + (\sigma_n)^{\alpha\beta}(\sigma_m)_{\beta\gamma} = 2 \eta_{mn} \delta^\alpha_\gamma\]

with \((\sigma_m)^{\alpha\beta}\) defined as \((\sigma_m)^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta}(\sigma_m)^{\gamma\delta}\). As described in [3], these worldsheet variables can be obtained from the RNS worldsheet variables by a field redefinition and the constraints of (2.2) are related to the stress tensor, BRST current, \(b\) ghost, and ghost-number current of the RNS formalism.

### 2.2. Massless compactification-independent vertex operators

Since the integrated form of the massless vertex operators appear in the sigma model action, it will be useful to review these operators here, beginning with the simpler case of open string massless vertex operators.

The massless compactification-independent open string vertex operators are described by a superfield \(V(x, \theta, \rho + i\sigma)\) which can be expanded as \(V = \sum_n e^{n(\rho + i\sigma)} V_n(x, \theta)\) where \(V_n(x, \theta)\) is an arbitrary function of the zero modes of \(x^m\) and \(\theta^\alpha\). The chiral bosons only appear in the combination \(\rho + i\sigma\) in order that \(V\) has no poles with \(J\). The integrated form of the vertex operator is given by

\[
U = \int dz G^+ G^- V(x, \theta, \rho + i\sigma)
\]

where \(G^\pm Y\) will always denote the single pole in the OPE of \(G^\pm\) with \(Y\). Using the fact that \(G^- e^{n(\rho + i\sigma)} = 0\) for \(n \leq 0\), one can show that when \(V\) is on-shell,

\[
U = \int dz \left[ -\frac{\epsilon^{\alpha\beta\gamma\delta}}{6} e^{-\rho - i\sigma} p_\alpha (\nabla_\beta \nabla_\gamma \nabla_\delta) + i p_\alpha (\nabla_\beta \partial^{\alpha\beta}) + \frac{i}{2} \partial x^{\alpha\beta} (\nabla_\alpha \nabla_\beta) V_1(x, \theta) \right] - \int dz \frac{\epsilon^{\alpha\beta\gamma\delta}}{6} p_\alpha \nabla_\beta \nabla_\gamma \nabla_\delta V_2(x, \theta)
\]
where \( \nabla_{\alpha} = d/d\theta^\alpha \) and \( \partial^{\alpha\beta} = \sigma^\alpha_{\beta m} d/dx_m \). Note that the only \( \rho + i\sigma \) dependence in the integrated vertex operator of (2.4) is the \( e^{-\rho - i\sigma} \) factor in the first term. All terms proportional to \( e^{n(\rho + i\sigma)} \) for \( n > 0 \) must vanish on-shell in order that \( U \) has no poles with \( G^- \).

The closed massless compactification-independent vertex operator is obtained by taking the “square” of the open vertex operator, i.e.

\[
U = \int dxd\bar{x} \tilde{G}^- G^- G^+ V(x, \theta, \bar{\theta}, \rho + i\sigma, \bar{\rho} + i\bar{\sigma})
\]

(2.5)

where \( V = \sum_{n, \bar{n}} e^{n(\rho + i\sigma) + \bar{n}(\bar{\rho} + i\bar{\sigma})} V_{n, \bar{n}}(x, \theta, \bar{\theta}) \).

3. Hybrid formalism with harmonic constraints

3.1. Action and N=2 constraints

Since the vertex operator of (2.5) depends only on four \( \theta \)’s and four \( \bar{\theta} \)’s, it is not obvious how to relate \( V \) and \( U \) of (2.5) with the standard six-dimensional Type II superfields which depend on eight \( \theta \)’s and eight \( \bar{\theta} \)’s. As will be explained below, this relation will become obvious after introducing worldsheet fields for four new \( \theta \)’s and four new \( \bar{\theta} \)’s (and their conjugate momenta), as well as introducing “harmonic” constraints which allow half of the \( \theta \)’s and \( \bar{\theta} \)’s to be gauged away. Unlike the other worldsheet variables and the N=2 superconformal constraints, these new fermionic variables and harmonic constraints do not seem to come from worldsheet variables or constraints in the RNS formalism.

The starting point will be a free action containing the new fermionic worldsheet variables in addition to the variables of (2.1):

\[
S = \int d^2z \left( \frac{1}{2} \partial x^m \bar{\partial} x_m + p_{\alpha j} \bar{\partial} \theta^{\alpha j} + \bar{p}_{\bar{\alpha} j} \partial \bar{\theta}^{\bar{\alpha} j} \right) + S_B + S_C
\]

(3.1)

where \( j = 1 \) to \( 2 \) so one has twice as many \( \theta \)’s and \( p \)’s. The OPE’s of the free fields in this action are:

\[
x^m(y)x^n(z) \rightarrow \eta^{mn} \log(y - z), \quad \rho(y)\rho(z) \rightarrow -\log(y - z), \quad \sigma(y)\sigma(z) \rightarrow -\log(y - z),
\]

(3.2)

\[
p_{\alpha j}(y)\theta^{\beta k}(z) \rightarrow \delta_{\alpha}^{\beta} \delta_{j}^{k} (y - z)^{-1}.
\]

The action of (3.1) can be written in manifestly N=2 D=6 supersymmetric notation as

\[
S = \int d^2z \left( \frac{1}{2} \Pi_x \Pi_{\bar{x}} + B_{\text{flat}}^{MN} \partial y^M \bar{\partial} y^N + d_{\alpha j} \bar{\partial} \theta^{\alpha j} + \bar{d}_{\bar{\alpha} j} \partial \bar{\theta}^{\bar{\alpha} j} \right) + S_B + S_C
\]

(3.3)
where $\Pi^m = \partial x^m - \frac{i}{2} \epsilon_{jk} (\sigma^m_{\alpha \beta} \theta^{\alpha j} \partial \theta^{\beta k} + \sigma^m_{\alpha \beta} \bar{\theta}^{\beta j} \partial \bar{\theta}^{\alpha k})$, $y^M = (x^m, \theta^{\alpha j}, \bar{\theta}^{\alpha j})$, $B^{flat}_{MN} \partial y^M \partial y^N$ is the same Wess-Zumino term as in the Green-Schwarz action in a flat background, and

$$d_{\alpha j} = p_{\alpha j} - \frac{i}{2} \epsilon_{jk} \theta^{\beta k} \partial x_{\alpha \beta} + \frac{1}{8} \epsilon_{\alpha \beta \gamma \delta} \epsilon_{jk} \epsilon_{lm} \theta^{\beta k} \theta^{\gamma l} \partial \theta^{\delta m} + \ldots, \quad (3.4)$$

$$\bar{d}_{\bar{\alpha} j} = \bar{p}_{\bar{\alpha} j} - \frac{i}{2} \epsilon_{jk} \bar{\theta}^{\beta k} \partial x_{\bar{\alpha} \beta} + \frac{1}{8} \epsilon_{\bar{\alpha} \bar{\beta} \gamma \delta} \epsilon_{jk} \epsilon_{lm} \bar{\theta}^{\beta k} \bar{\theta}^{\gamma l} \partial \bar{\theta}^{\delta m} + \ldots$$

where ... signifies terms which vanish using the equations of motion (e.g. terms involving $\partial \theta^{\alpha j}$ or $\partial \bar{\theta}^{\alpha j}$). Note that the non-linear terms in $d_{\alpha j}$ and $\bar{d}_{\bar{\alpha} j}$ cancel the cubic and quartic terms in (3.3) coming from $\Pi^m \Pi \bar{z} m$ and from the Wess-Zumino term. In ten dimensions, a similar action to (3.3) was constructed by Siegel in [8], and in four dimensions, a similar action was constructed in [2]. The first two terms of (3.3) is the standard six-dimensional GS action in a flat background.

It is easy to check that the equations of motion imply that $d_{\alpha j}$, $\Pi^m$, and $\partial \theta^{\alpha j}$ are holomorphic, commute with the spacetime-supersymmetry generators, and satisfy the OPE’s [8]

$$d_{\alpha j} (y) d_{\beta k} (z) \to -i (y - z)^{-1} \epsilon_{jk} (\sigma_m)_{\alpha \beta} \Pi_z^m (y). \quad (3.5)$$

$$d_{\alpha j} (y) \Pi_z^m (z) \to -i (y - z)^{-1} \epsilon_{jk} (\sigma_m)_{\alpha \beta} \partial \theta^{\beta k} (y),$$

$$d_{\alpha j} (y) \bar{\partial} \theta^{\beta k} (z) \to (y - z)^{-2} \delta^k_j \delta^\beta_\alpha, \quad \Pi_z^m (y) \Pi_z^n (z) \to (y - z)^{-2} \eta^{mn}.$$

To make (3.3) equivalent to the original action of (2.1), one now imposes the following eight first-class constraints:

$$D_\alpha \equiv d_{\alpha 2} - e^{-\rho - i\sigma} d_{\alpha 1} = 0, \quad \bar{D}_{\bar{\alpha}} \equiv \bar{d}_{\bar{\alpha} 2} - e^{-\bar{\rho} - i\bar{\sigma}} \bar{d}_{\bar{\alpha} 1} = 0. \quad (3.6)$$

It is interesting to note that similar constraints were used in [4] to describe the $D = 6$ superparticle with worldline supersymmetry. Since $\{D_\alpha, \theta^{\beta 2}\} = \delta^\beta_\alpha$, the first-class constraints of (3.6) can be used to gauge-fix $\theta^{\alpha 2} = \bar{\theta}^{\alpha 2} = 0$. In this gauge, the action of (3.3) reduces to the action of (2.1) where $\theta^{\alpha 1}$ is identified with $\theta^\alpha$ and $p_{\alpha 1}$ is identified with $p_\alpha$.

The N=2 superconformal generators for (3.1) are modified from those of (2.2) to:

$$T = \frac{1}{2} \Pi_z^m \Pi_z^m + d_{\alpha 1} \partial \theta^{\alpha 1} + e^{-\rho - i\sigma} d_{\alpha 1} \partial \theta^{\alpha 2} + \frac{1}{2} \partial \rho \partial \rho + \frac{1}{2} \partial \sigma \partial \sigma + \frac{3}{2} \partial^2 (\rho + i\sigma) + T_C,$$

$$G^+ = -e^{-2\rho - i\sigma} (d_1)^4 + \frac{i}{2} e^{-\rho} (d_{\alpha 1} d_{\beta 1} \Pi_z^{\alpha \beta} - 2i \partial (\rho + i\sigma) d_{\alpha 1} \partial \theta^{\alpha 2} + id_{\alpha 1} \partial^2 \theta^{\alpha 2}) + e^{i\sigma} (\frac{1}{2} \Pi_z^m \Pi_z^m + \frac{1}{2} \delta (\rho + i\sigma) \partial (\rho + i\sigma) - \frac{1}{2} \partial^2 (\rho + i\sigma)) + G_C^+ \quad (3.7)$$
\[ G^- = e^{-i\sigma} + G^-_C, \]
\[ J = \partial(\rho + i\sigma) + J_C \]

where \((d_1)^4 = \frac{1}{24} \epsilon^{\alpha\beta\gamma\delta} d_\alpha d_\beta d_\gamma d_\delta\). When \(\theta^{a2} = \bar{\theta}^{a2} = 0\) and \([\theta^{a1}, p_\alpha]\) are identified with \([\theta^a, p_\alpha]\), these constraints reduce to those of (2.2). So (2.2) can be interpreted as a gauge-fixed version of (3.7).

Note that \(\Pi^m_m \Pi^m_m = \frac{1}{24} \epsilon^{\alpha\beta\gamma\delta} D_\alpha (D_\beta (D_\gamma (D_\delta (e^{2\rho+3i\sigma})))) + G^+_C\)

where \(D_\alpha(Y)\) denotes the contour integral of \(D_\alpha\) around \(Y\). Furthermore, the generators of (3.7) still form a \(c = 6\) N=2 superconformal algebra. This algebra is guaranteed since the constraints of (3.7) are invariant under the gauge transformations generated by (3.6) and in the gauge \(\theta^{a2} = \bar{\theta}^{a2} = 0\), they reduce to the N=2 constraints of (2.2). So the free action of (3.1), together with the constraints of (3.6) and (3.7), still describes a critical N=2 superconformal field theory.

### 3.2. Massless compactification-independent vertex operators

To see how the harmonic constraints of (3.6) affect the massless open string vertex operator of (2.3), consider a function \(V\) which depends of the zero modes of \((x^m, \theta^{\alpha j}, \rho + i\sigma)\) and which satisfies

\[ (\nabla_{\alpha 2} - e^{-\rho-i\sigma} \nabla_{\alpha 1})V = 0, \tag{3.9} \]

i.e. \(V\) has no poles with (3.6) where \(\nabla_{\alpha j} = \frac{\partial}{\partial \theta^{\alpha j}} - \frac{i}{2} \epsilon_{j\beta} \theta^{\beta k} \partial_{\alpha k}\). Defining

\[ \hat{x}^m = x^m + \frac{i}{4} \sigma^m_{\alpha \beta} (e^{\rho+i\sigma} \theta^{a1} \theta^{\beta 1} - e^{-\rho-i\sigma} \theta^{a2} \theta^{\beta 2}), \tag{3.10} \]

\[ \theta^{\alpha -} = \theta^{a1} - e^{-\rho-i\sigma} \theta^{a2}, \quad \theta^{\alpha +} = \theta^{a1} + e^{-\rho-i\sigma} \theta^{a2}, \]

(3.10) implies that \(V\) is a function of the zero modes of \((\hat{x}^m, \theta^{\alpha +}, \rho + i\sigma)\), but is independent of the zero modes of \(\theta^{\alpha -}\). So the component fields of \(V\) can be related to the component fields of \(V\) in (2.3) by identifying \(\hat{x}\) with \(x\) and \(\theta^{\alpha +}\) with \(\theta^\alpha\).
Applying $G^- G^+$ on $V$ to obtain the integrated version of the vertex operator, one gets

$$ U = \int dz \left[ -\frac{\epsilon^{\alpha\beta\gamma\delta}}{6} e^{-\rho - i\sigma} d_{\alpha_1} (\nabla_{\beta_1} \nabla_{\gamma_1} \nabla_{\delta_1}) + i d_{\alpha_1} (\nabla_{\beta_1} \partial^{\alpha_1}) + \frac{i}{2} \Pi^\alpha_2 (\nabla_{\alpha_1} \nabla_{\beta_1}) \right] (3.11) $$

$$ + \partial \theta^{\alpha_2} \nabla_{\alpha_1} V_1(x, \theta) - \int dz \frac{\epsilon^{\alpha\beta\gamma\delta}}{6} d_{\alpha_1} \nabla_{\beta_1} \nabla_{\gamma_1} \nabla_{\delta_1} V_2(x, \theta). $$

Using (3.9) to relate $\nabla_{\alpha_1} V_1 = \nabla_{\alpha_2} V_0$ and $\nabla_{\alpha_1} V_2 = \nabla_{\alpha_2} V_1$, $U$ can be written in a more symmetric form in terms of $V_0$ as

$$ U = \int dz \left[ -\frac{\epsilon^{\alpha\beta\gamma\delta}}{6} (e^{-\rho - i\sigma} d_{\alpha_1} (\nabla_{\beta_1} \nabla_{\gamma_1} \nabla_{\delta_2}) + d_{\alpha_1} (\nabla_{\beta_2} \nabla_{\gamma_2} \nabla_{\delta_1})) \right] (3.12) $$

$$ + \frac{i}{4} \Pi^\alpha_2 [\nabla_{\alpha_1}, \nabla_{\beta_2}] - \frac{1}{2} \partial \theta^{\alpha_1} \nabla_{\alpha_1} + \frac{1}{2} \partial \theta^{\alpha_2} \nabla_{\alpha_2} V_0(x, \theta) $$

where we have subtracted the surface term $\frac{1}{2} \int dz [\Pi^m_z \partial_m + \partial \theta^{\alpha_j} \nabla_{\alpha_j}] V_0 = \frac{1}{2} \int dz \partial V_0$. Replacing $e^{-\rho - i\sigma} d_{\alpha_1}$ with $d_{\alpha_2}$ using (3.6), the vertex operator of (3.12) closely resembles the four-dimensional massless vertex operator of [8]. It is interesting to note that both the four and six-dimensional massless vertex operators are of the form proposed by Siegel in [8] for the ten-dimensional vertex operator, $U = \int dz [d_\alpha W^\alpha + \partial y^M A_M]$ where $W^\alpha$ is the super-Yang-Mills field-strength and $A_M$ are the superspace gauge fields.

The closed massless vertex operator is the “square” of (3.12), i.e.

$$ U = \int d^2 z \left[ -\frac{\epsilon^{\alpha\beta\gamma\delta}}{6} (d_{\alpha_2} (\nabla_{\beta_1} \nabla_{\gamma_1} \nabla_{\delta_2}) + d_{\alpha_1} (\nabla_{\beta_2} \nabla_{\gamma_2} \nabla_{\delta_1})) \right] (3.13) $$

$$ + \frac{i}{4} \Pi^\alpha_2 [\nabla_{\alpha_1}, \nabla_{\beta_2}] - \frac{1}{2} \partial \theta^{\alpha_1} \nabla_{\alpha_1} + \frac{1}{2} \partial \theta^{\alpha_2} \nabla_{\alpha_2} V_0,0(x, \theta, \bar{\theta}). $$

4. Hybrid action in a curved background

Given the N=2 D=6 spacetime-supersymmetric form of the action in a flat background in (3.3) and the closed massless vertex operators of (3.13), it is easy to guess the action in a curved background if one ignores the Fradkin-Tseytlin term which couples the dilaton to the worldsheet curvature. This action is given by

$$ S_0 = \frac{1}{\alpha'} \int d^2 z \left( \frac{1}{2} \Pi^\alpha_2 \Pi_{\alpha_2} + B_{MN} \partial y^M \partial y^N + d_{\alpha_j} \Pi^\alpha_2 + \bar{d}_{\alpha_j} \Pi^\alpha_2 + d_{\alpha_j} \bar{d}_{\beta_k} P^{\alpha_j \beta_k} \right) + S_B + S_C $$

(4.1)
where \( E^A_M \) is the super-vierbein with \( A = (c, \alpha j, \bar{\alpha} j) \) and \( M = (m, \mu j, \bar{\mu} j) \), \( \Pi^c = E^c_M \partial y^M \) is the vector current, \( \Pi^{\alpha j} = E^{\alpha j}_M \partial y^M \) and \( \Pi^{\bar{\alpha} j} = E^{\bar{\alpha} j}_M \partial y^M \) are the spinor currents, and \( P^{\alpha j} \bar{\beta} k \) is the superfield whose lowest components are the bispinor Ramond-Ramond field strengths. The necessity of including the term proportional to \( d_{\alpha j} \bar{d}_{\bar{\alpha} k} \) can be seen from the massless vertex operator of (3.13).

To reproduce the expected coupling-constant dependence of scattering amplitudes, one needs a Fradkin-Tseytlin term which couples the spacetime dilaton to the worldsheet curvature. As in the four-dimensional sigma model of [2], this term is constructed by coupling spacetime compensator superfields to the N=(2,2) worldsheet supercurvature. The N=(2,2) worldsheet supercurvature is described by a chiral and twisted-chiral worldsheet superfield \( \Sigma_c \) and \( \Sigma_{tc} \), and their complex conjugates \( \Sigma_c^* \) and \( \Sigma_{tc}^* \). In the four-dimensional sigma model, the worldsheet supercurvature coupled to spacetime superfields \( \Phi_c \) and \( \Phi_{tc} \) which compensated a \( U(1) \times U(1) \) subgroup of the \( U(1) \times SU(2) \) R-transformations of N=2 D=4 supergravity. In the six-dimensional sigma model, the worldsheet supercurvature superfields couple to spacetime superfields \( \Phi_c \) and \( \Phi_{tc} \) which compensate a \( U(1) \times U(1) \) subgroup of the \( SU(2) \times SU(2) \) R-transformations of N=2 D=6 supergravity. In both the four and six-dimensional actions, the coupling term is defined as

\[
S_{FT} = \int d^2 z [G^- G^- (\Phi_c \Sigma_c) + G^- G^+ (\Phi_{tc} \Sigma_{tc}) + G^+ G^+ (\Phi_c^* \Sigma_c^*) + G^+ G^- (\Phi_{tc}^* \Sigma_{tc}^*)]
\]  

(4.2)

where \( \Phi_c^* \) and \( \Phi_{tc}^* \) are the complex conjugates of \( \Phi_c \) and \( \Phi_{tc} \). The complete sigma model action is therefore \( S = S_0 + S_{FT} \).

The spacetime compensator superfields \( \Phi_c \) and \( \Phi_{tc} \) are functions of the zero modes of \((x^m, \theta^{\mu j}, \bar{\theta}^{\bar{\mu} j}, \rho + i\sigma, \bar{\rho} + i\bar{\sigma})\) which satisfy the chirality and twisted-chirality constraints

\[
G^+ \Phi_c = \bar{G}^+ \Phi_c = 0, \quad G^- \Phi_c^* = \bar{G}^- \Phi_c^* = 0,
\]

\[
G^+ \Phi_{tc} = \bar{G}^- \Phi_{tc} = 0, \quad G^- \Phi_{tc}^* = \bar{G}^+ \Phi_{tc}^* = 0,
\]

(4.3)

in addition to the constraints implied by (3.6) that

\[
(\nabla_{\alpha 1} - e^{-\rho - i\sigma} \nabla_{\alpha 2} ) \Phi = (\nabla_{\alpha 1} - e^{-\rho - i\sigma} \nabla_{\alpha 2} ) \Phi^* = 0,
\]

(4.4)

\[
(\bar{\nabla}_{\bar{\alpha} 1} - e^{-\bar{\rho} - i\bar{\sigma}} \bar{\nabla}_{\bar{\alpha} 2} ) \Phi = (\bar{\nabla}_{\bar{\alpha} 1} - e^{-\bar{\rho} - i\bar{\sigma}} \bar{\nabla}_{\bar{\alpha} 2} ) \Phi^* = 0.
\]

Writing \( \Phi = \sum_{n, \bar{n}} e^{n(\rho + i\sigma) + \bar{n}(\bar{\rho} + i\bar{\sigma})} \Phi^{n, \bar{n}} \), (4.3) and (4.4) imply that \( \Phi^{n, \bar{n}} = 0 \) when either \( n < 0 \) or \( \bar{n} < 0 \) and \( \Phi_{tc}^{n, \bar{n}} = 0 \) when either \( n < 0 \) or \( \bar{n} > 0 \). To see that \( G^+ \Phi_c = 0 \)
implies that $\Phi^{\vec{n}, \vec{n}} = 0$ when $n < 0$, observe that (3.8) and (4.4) imply that $\Phi^c$ has no poles with $e^{2\rho + 3i\sigma}$. The complex conjugate superfields $\Phi^c^*$ and $\Phi^{tc^*}$ are related to $\Phi^c$ and $\Phi^{tc}$ by defining $(\theta^{\alpha j})^* = \epsilon^{jk} \theta^{ak}$, $(\bar{\theta}^{\bar{\alpha} j})^* = \epsilon^{jk} \bar{\theta}^{\bar{a}k}$, $(e^{\rho + i\sigma})^* = -e^{-\rho - i\sigma}$ and $(e^{\bar{\rho} + i\bar{\sigma}})^* = -e^{-\bar{\rho} - i\bar{\sigma}}$. One can check that this definition of complex conjugation implies that $\Phi^c^*$ and $\Phi^{tc^*}$ satisfy (4.3) and (4.4).

The N=2 constraints for the superstring in a curved six-dimensional background are given by

\[
T = T_0 + \partial^2(\Phi^c + \Phi^c^* + \Phi^{tc} + \Phi^{tc^*}),
\]

\[
G^+ = G^+_0 + G^+ \partial(\Phi^c^* + \Phi^{tc^*}), \quad G^- = G^-_0 + G^- \partial(\Phi^c + \Phi^{tc}),
\]

\[
J = J_0 + \partial(\Phi^c - \Phi^c^* + \Phi^{tc} - \Phi^{tc^*}),
\]

where $[T_0, G^+_0, G^-_0, J_0]$ are the N=2 constraints of (3.7) after replacing $\Pi^m_z$ with $\Pi^c_z$ and $\partial \theta^{\alpha j}$ with $\Pi^{\alpha j}$. The Fradkin-Tseytlin contribution to the constraints of (4.5) are analogous to those discussed in [2] and [10] for the four-dimensional background.

5. Relation to $PSU(2|2)$ action for $AdS_3 \times S^3$

In reference [3], an action based on the $PSU(2|2)$ supergroup was constructed for the superstring in an $AdS_3 \times S^3$ background with Ramond-Ramond flux. Although the $(z, \bar{z})$ symmetric part of this action was the usual group action for $PSU(2|2)$, the $(z, \bar{z})$ anti-symmetric part of this action contained complicated dependence on the chiral and anti-chiral bosons $\rho + i\sigma$ and $\bar{\rho} + i\bar{\sigma}$. The N=2 worldsheet superconformal generators also depended in a complicated way on these bosons. When the paper was written, it was unclear how to give a geometrical interpretation to this $(\rho + i\sigma, \bar{\rho} + i\bar{\sigma})$ dependence.

Using the results of the previous section, it will now be shown how the rather complicated $(\rho + i\sigma, \bar{\rho} + i\bar{\sigma})$ dependence in the action of [3] can be derived from a relatively simple action whose only dependence on $\rho + i\sigma$ and $\bar{\rho} + i\bar{\sigma}$ comes from the harmonic constraints of (3.9).

Since the $AdS_3 \times S^3$ background of interest has a constant dilaton, the Fradkin-Tseytlin term of (1.2) contributes the usual Euler number dependence which will be ignored. The remaining part of the action is given by (4.1) where $E^A_M$, $B_{MN}$ and $P^{\alpha j} \tilde{\beta} k$ take values determined by the $AdS_3 \times S^3$ metric and by the NS-NS and R-R three-form flux.
For convenience, only backgrounds with pure Ramond-Ramond flux will be discussed although it should be easy to generalize the discussion to include backgrounds with NS-NS flux. In the presence of a three-form R-R flux with values\footnote{This choice of three-form R-R flux is slightly more convenient that that of \cite{3} since it preserves a diagonal SU(2) $R$-symmetry.}

\begin{equation}
H_{jk}^{012} = H_{jk}^{345} = N \epsilon_{jk},
\end{equation}

the superfield $P^{\alpha j} \tilde{\beta}^k$ satisfies

\begin{equation}
P^{\alpha j} \tilde{\beta}^k = N \lambda \delta^{\alpha \tilde{\beta}} \epsilon^{jk}
\end{equation}

where the coupling constant $\lambda$ appears in (5.2) because there is no $\lambda^{-2}$ factor in front of the Ramond-Ramond $H^{mn}H_{np}$ kinetic term in the action. Furthermore, in the $AdS_3 \times S^3$ background, one can choose the only non-zero values of $B_{AB} = E_A^M E_B^N B_{MN}$ to be \footnote{This choice of three-form R-R flux is slightly more convenient that that of \cite{3} since it preserves a diagonal SU(2) $R$-symmetry.}

\begin{equation}
B_{\alpha j} \tilde{\beta} k = B_{\tilde{\beta} k} \alpha j = -\frac{1}{4} (N \lambda)^{-1} \epsilon_{jk} \delta_{\alpha \tilde{\beta}}.
\end{equation}

Using $H_{ABC} = \nabla_{[AB} B_{BC]} + T_{[AB} D_{BC]} D$ and the torsion constraints

\begin{equation}
T_c^{}{}_{\alpha j} \tilde{\gamma}^k = (\sigma_c)^{}{}_{\alpha \beta} \epsilon^{jl} \tilde{P}^l_j \tilde{\gamma}^k, \quad T_c^{}{}_{\alpha j} \gamma^k = -(\sigma_c)^{}{}_{\alpha \beta} \epsilon^{jl} P^l_j \gamma^k \tilde{\beta}^l,
\end{equation}

it is easy to show that $H_c^{}{}_{\alpha j} \tilde{\beta} k = \frac{1}{4} (\sigma_c)^{}{}_{\alpha \beta} \epsilon_j^k$ and $H_{\alpha j}^{}{}_{\tilde{\beta} k} = -\frac{1}{4} (\sigma_c)^{}{}_{\alpha \beta} \epsilon_j^k$ as desired.

Plugging these values for the background fields into (4.1), one obtains

\begin{equation}
S = \frac{1}{\alpha'} \int d^2 z \left[ \frac{1}{2} \Pi^c_z \Pi^c_{\bar{z} c} - \frac{1}{4} (N \lambda)^{-1} \delta_{jk} \delta_{\alpha \tilde{\beta}} (\Pi^c_z \bar{\Pi}^k_{\bar{z}} - \bar{\Pi}^c_z \Pi^k_z) \right. \end{equation}

\begin{equation}
\left. + \epsilon_{jk} \delta_{\alpha \tilde{\beta}} N \lambda d_{\alpha j} \bar{d}_{\tilde{\beta} k} + S_B + S_C. \right]
\end{equation}

After rescaling $E^c_M \rightarrow (N \lambda)^{-1} E^c_M$, $E^\alpha_j M \rightarrow (N \lambda)^{-\frac{1}{2}} E^\alpha_j M$, $E^\tilde{\alpha} j M \rightarrow (N \lambda)^{-\frac{1}{2}} E^\tilde{\alpha} j M$, $d_{\alpha j} \rightarrow (N \lambda)^{-\frac{2}{3}} d_{\alpha j}$, $\bar{d}_{\alpha j} \rightarrow (N \lambda)^{-\frac{2}{3}} \bar{d}_{\alpha j}$, the $(N \lambda)$ dependence of (5.4) simplifies to

\begin{equation}
S = \frac{1}{\alpha' N^2 \lambda^2} \int d^2 z \left[ \frac{1}{2} \Pi^c_z \Pi^c_{\bar{z} c} - \frac{1}{4} \epsilon_{jk} \delta_{\alpha \tilde{\beta}} (\Pi^c_z \bar{\Pi}^k_{\bar{z}} - \bar{\Pi}^c_z \Pi^k_z) \right. \end{equation}

\begin{equation}
\left. + \epsilon_{jk} \delta_{\alpha \tilde{\beta}} d_{\alpha j} \bar{d}_{\tilde{\beta} k} + S_B + S_C. \right]
\end{equation}

The first line of (5.3) is precisely the GS action for the $AdS_3 \times S^3$ background which was studied in \cite{1} and which is based on the $AdS_5 \times S^5$ action of \cite{2}. However, the second line of (5.3) is crucial for quantization and is absent from the action of [1].
As discussed in \([\text{11}]\), the currents \([\Pi^c, \Pi^\alpha j, \Pi^{\bar{\alpha} j}]\) can be identified with the currents of the supergroup coset \(PSU(2|2) \times PSU(2|2)/SU(2) \times SU(2)\). If \(g\) takes values in the supergroup \(PSU(2|2) \times PSU(2|2)\), the left-invariant one-forms \(g^{-1} \partial g\) can be defined as \((S^\alpha j, K^{\alpha \beta})\) and \((\tilde{S}^\alpha j, \tilde{K}^{\alpha \beta})\) which generate the Lie algebra
\[
[K^{\alpha \beta}, K^{\gamma \delta}] = \delta^{\alpha \gamma} K^{\beta \delta} - \delta^{\alpha \delta} K^{\beta \gamma} - \delta^{\beta \gamma} K^{\alpha \delta} + \delta^{\beta \delta} K^{\alpha \gamma},
\]
\[
[K^{\alpha \beta}, S^{\gamma j}] = \delta^{\alpha \gamma} S^{\beta j} - \delta^{\beta \gamma} S^{\alpha j},
\]
\[
\{S^{\alpha j}, S^{\beta k}\} = \frac{1}{2} \epsilon^{j k} \epsilon^{\alpha \beta \gamma \delta} K^{\gamma \delta},
\]
and similarly for the tilded currents. (The untilded and tilded currents commute with each other.) Defining \(K^{\alpha \beta} + \tilde{K}^{\alpha \beta}\) to be the one-forms which are absent from the action to provide the local \(SU(2) \times SU(2)\) invariance, the remaining six bosonic currents and sixteen fermionic currents are related to \([\Pi^c, \Pi^\alpha j, \Pi^{\bar{\alpha} j}]\) as
\[
\Pi^c = \sigma^c_{\alpha \beta}(K^{\alpha \beta} - \tilde{K}^{\alpha \beta}), \quad \Pi^{\alpha j} = S^{\alpha j} + i \tilde{S}^{\alpha j}, \quad \Pi^{\bar{\alpha} j} = S^{\bar{\alpha} j} - i \tilde{S}^{\bar{\alpha} j}
\]
where \((x, \theta, \bar{\theta})\) parameterize the \(PSU(2|2) \times PSU(2|2)/SU(2) \times SU(2)\) coset supermanifold.

To further simplify the action of \((5.3)\), one has two options. One option is to integrate out all \(d_{\alpha j}\) and \(d_{\bar{\alpha} j}\) worldsheet fields, producing the action
\[
S = \frac{1}{\alpha' N^2 \lambda^2} \int d^2 z \left[ \frac{1}{2} \Pi^c \Pi^{\bar{c}} - \epsilon_{j k} \delta_{\alpha \beta} \left( \frac{1}{4} \Pi^{\alpha j} \Pi^{\bar{k}} - \frac{3}{4} \Pi^{\bar{k}} \Pi^{\bar{\alpha} j} \right) \right] + S_B + S_C. \tag{5.8}
\]
Using the identification of \((5.7)\), the above action can be interpreted as a sigma model action with WZ term for the supercoset \(PSU(2|2) \times PSU(2|2)/SU(2) \times SU(2)\). In fact, precisely this sigma model action was considered in \([\text{1}]\) and shown to be one-loop conformally invariant. Of course, this action needs to be supplemented by the constraints of \((3.6)\) and \((3.7)\) to remove the unphysical degrees of freedom.

A second option is to first use the eight \((3.6)\) constraints and the six local \(SU(2) \times SU(2)\) invariances to gauge-fix to the identity the tilded \(PSU(2|2)\) group parameters. This gauge-fixing can be inserted directly into the action of \((5.3)\) since the group parameters transform without derivatives under these gauge transformations. The resulting action (after imposing the \((3.6)\) constraints to solve for \(d_{\alpha j}\) and \(d_{\bar{\alpha} j}\) in terms of \(d_{\alpha 2}\) and \(d_{\bar{\alpha} 2}\)) is
\[
S = \frac{1}{\alpha' N^2 \lambda^2} \int d^2 z \left[ \frac{1}{8} \xi_{\alpha \beta \gamma \delta} K_{\alpha \beta}^{\gamma \delta} \right]
\]
\[
+ d_{\alpha 1}(S_{\alpha 1}^2 + e^{-\rho - i \sigma} S_{\bar{\alpha} 1}^2) + d_{\bar{\alpha} 2}(S_{\bar{\alpha} 2}^2 + e^{\bar{\rho} + i \bar{\sigma}} S_{\bar{\alpha} 1}^1) + (1 - e^{-\rho - i \sigma + \bar{\rho} + i \bar{\sigma}}) d_{\alpha 1} \bar{d}_{\alpha 2}] + S_B + S_C. \tag{5.9}
\]
Integrating out \( d_{\alpha 1} \) and \( \bar{d}_{\alpha 2} \), one obtains

\[
S = \frac{1}{\alpha' N^2 \lambda^2} \int d^2 z \left[ \frac{1}{8} \epsilon_{\alpha \beta \gamma \delta} K^\alpha_2 K^\beta_2 K^\gamma_2 K^\delta_2 \right] \\
-(1-x)^{-1} (S^{\alpha 1}_z + e^{-\rho - i \sigma} S^{\alpha 2}_z) (S^{\alpha 2}_z + e^{\bar{\rho} + i \bar{\sigma}} S^{\alpha 1}_z) + S_B + S_C \\
= \frac{1}{\alpha' N^2 \lambda^2} \int d^2 z \left[ \frac{1}{8} \epsilon_{\alpha \beta \gamma \delta} K^\alpha_2 K^\beta_2 K^\gamma_2 K^\delta_2 - \frac{1}{2} \delta_{jk} S^{\alpha j}_z S^{\alpha k}_z \right] \\
+(1-x)^{-1} (e^{\bar{\rho} + i \bar{\sigma}} S^{\alpha 1} \wedge S^{\alpha 1} + e^{-\rho - i \sigma} S^{\alpha 2} \wedge S^{\alpha 2} + 2 S^{\alpha 1}_z \wedge S^{\alpha 2}_z) + S_B + S_C,
\]

where \( x = e^{-\rho - i \sigma + \bar{\rho} + i \bar{\sigma}} \) and a term proportional to \( \int d^2 z \ S^{\alpha 1} \wedge S^{\alpha 2} \) (with no \( \rho + i \sigma \) factors) has been dropped since it is a total derivative \([3]\).

This action is the same as that of \([3]\) if one redefines \( \rho \rightarrow \rho + \log(2) \), \( \bar{\rho} \rightarrow - \bar{\rho} - \log(2) \), \( \sigma \rightarrow \sigma \) and \( \bar{\sigma} \rightarrow - \bar{\sigma} \). The N=2 constraints of (4.5) can be related to those of \([3]\) by inserting the equation of motion for \( d_1 \) into (4.3) in the gauge where all tilded currents vanish. The result is

\[
T = \frac{1}{8} \epsilon_{\alpha \beta \gamma \delta} K^\alpha_2 K^\beta_2 K^\gamma_2 K^\delta_2 - \frac{1}{2} \epsilon_{jk} S^{\alpha j}_z S^{\alpha k}_z + \frac{1}{2} \partial \rho \partial \rho + \frac{1}{2} \partial \sigma \partial \sigma + \frac{3}{2} \partial^2 (\rho + i \sigma) + T_C, \tag{5.12}
\]

\[
G^+ = -e^{-2 \rho - i \sigma} (1-x)^{-1} (S^{\alpha 1} + e^{\bar{\rho} + i \bar{\sigma}} S^{\alpha 1})^4 \tag{5.12}
\]

\[
+ \frac{i}{2} e^{-\rho} [(1-x)^{-2} (S^{\alpha 2} + e^{\bar{\rho} + i \bar{\sigma}} S^{\alpha 1}) (S^{\beta 2} + e^{\bar{\rho} + i \bar{\sigma}} S^{\beta 1}) K^\alpha_2 \]

\[
-2 i \partial (\rho + i \sigma) e^{\bar{\rho} + i \bar{\sigma}} (1-x)^{-1} S^{\alpha 1} S^{\alpha 2} + i (1-x)^{-1} (S^{\alpha 2} + e^{\bar{\rho} + i \bar{\sigma}} S^{\alpha 1}) \partial S^{\alpha 2} \]

\[
+ e^{i \sigma} \left[ \frac{1}{8} \epsilon_{\alpha \beta \gamma \delta} K^\alpha_2 K^\beta_2 K^\gamma_2 + (1-x)^{-1} S^{\alpha 2} S^{\alpha 1} + \frac{1}{2} \partial (\rho + i \sigma) \partial (\rho + i \sigma) - \frac{1}{2} \partial^2 (\rho + i \sigma) \right] + G^+_C,
\]

\[
G^- = e^{-i \sigma} + G^-_C,
\]

\[
J = \partial (\rho + i \sigma) + J_C,
\]

which matches the N=2 constraints of \([3]\) up to normal-ordering ambiguities (after redefining \( \rho \rightarrow \rho + \log(2) \), \( \bar{\rho} \rightarrow - \bar{\rho} - \log(2) \), \( \sigma \rightarrow \sigma \) and \( \bar{\sigma} \rightarrow - \bar{\sigma} \).

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