SUMS OF POLYNOMIAL-TYPE EXCEPTIONAL UNITS MODULO \( n \)

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Abstract

Let \( f(x) \in \mathbb{Z}[x] \) be a nonconstant polynomial. Let \( n \geq 1, k \geq 2 \) and \( c \) be integers. An integer \( a \) is called an \( f \)-exunit in the ring \( \mathbb{Z}_n \) of residue classes modulo \( n \) if \( \gcd(f(a), n) = 1 \). We use the principle of cross-classification to derive an explicit formula for the number \( N_{k,f,c}(n) \) of solutions \((x_1, \ldots, x_k)\) of the congruence \( x_1 + \cdots + x_k \equiv c \pmod{n} \) with all \( x_i \) being \( f \)-exunits in the ring \( \mathbb{Z}_n \). This extends a recent result of Anand et al. [‘On a question of \( f \)-exunits in \( \mathbb{Z}/n\mathbb{Z} \), Arch. Math. (Basel) 116 (2021), 403–409]. We derive a more explicit formula for \( N_{k,f,c}(n) \) when \( f(x) \) is linear or quadratic.

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1. Introduction

Let \( \mathbb{Z} \) and \( \mathbb{Z}^+ \) stand for the set of integers and the set of positive integers, respectively. For any \( n \in \mathbb{Z}^+ \), we set \( \mathbb{Z}_n = \{0, 1, \ldots, n - 1\} \) to be the ring of residue classes modulo \( n \). Let \( \mathbb{Z}_n^* = \{s \in \mathbb{Z}_n : \gcd(s, n) = 1\} \) be the group of the units in \( \mathbb{Z}_n \). Throughout this paper, let \( k \) be an integer with \( k \geq 2 \). In 1925, Rademacher [8] asked for an explicit formula for the number \( N(k, c, n) \) of solutions \((x_1, \ldots, x_k) \in (\mathbb{Z}_n^*)^k \) of the linear congruence

\[
x_1 + \cdots + x_k \equiv c \pmod{n}
\]

In 1926, Brauer [4] solved this problem by showing that

\[
N(k, c, n) = \frac{\varphi^k(n)}{n} \prod_{p|n, p|c} \left(1 - \frac{(-1)^{k-1}}{(p - 1)^{k-1}}\right) \prod_{p|n, p \nmid c} \left(1 - \frac{(-1)^{k}}{(p - 1)^{k}}\right),
\]

where \( \varphi(n) \) is Euler’s totient function and the products are taken over all prime divisors \( p \) of \( n \). In 2009, Sander [9] gave a new proof of the formula for \( N(2, c, n) \) by using the multiplicativity of \( N(2, c, n) \) with respect to \( n \).

The concept of exceptional units was introduced by Nagell [7] in 1969; he introduced it to solve certain cubic Diophantine equations. For any commutative ring \( R \) with the identity element \( 1_R \), let \( R^* \) denote the multiplicative group of units in \( R \).

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An element \( a \in R \) is said to be an \textit{exceptional unit} if both \( a \in R^* \) and \( 1_R - a \in R^* \). Many types of Diophantine equations including Thue equations [16], Thue–Mahler equations [17], discriminant-form equations [13] and others have been studied by means of exceptional units (for more references, see [6]). On the other hand, with the help of exceptional units, Lenstra [5] introduced a new method to find Euclidean number fields. Exceptional units also have connections with cyclic resultants [14,15] and Lehmer’s conjecture related to Mahler measure [11, 12].

Following Sander’s notation in [10], we use the coinage \textit{exunit} to stand for exceptional unit. As usual, for any integer \( m \) and prime number \( p \), we let \( \nu_p(m) \) stand for the \( p \)-adic valuation of \( m \), that is, \( \nu_p(m) \) is the unique nonnegative integer \( r \) such that \( p^r \mid m \) and \( p^{r+1} \nmid m \). We denote by \( \omega(m) := \sum_{p \text{ prime}, p \mid m} 1 \) the number of distinct prime divisors of \( m \). Yang and Zhao [18] extended Sander’s result [10] by showing that the number of ways to represent each element of \( \mathbb{Z}_n \) as the sum of \( k \) exceptional units is given by

\[
(-1)^{k\omega(n)} \prod_{p \mid n} p^{k\nu_p(n)-\nu_p(n)-k} \left( p^k \sum_{j=0}^{k} \binom{k}{j} (2-p)^k - 2^k \right).
\]

(This corrects an error in the formula in [18, Theorem 1], where the sign factor \((-1)^k\) should read \((-1)^{k\omega(n)}\).)

We can easily observe that for any \( a \in \mathbb{Z}^+ \), \( a \pmod{n} \) is an exunit in \( \mathbb{Z}_n \) if and only if \( \gcd(a(1-a), n) = 1 \). In other words, \( a \pmod{n} \) is an exunit in \( \mathbb{Z}_n \) if and only if \( \gcd(f(a), n) = 1 \) with \( f(x) = x(1-x) \). This observation naturally motivates the concept of \( f \)-exunit as follows. Let \( n \geq 1 \) be an integer and let \( f(x) \in \mathbb{Z}[x] \). An integer \( a \) is an \textit{\( f \)-exunit} in the ring \( \mathbb{Z}_n \) if \( \gcd(f(a), n) = 1 \) (see [1]). We denote by \( E_f(n) \) the set of all \( f \)-exunits in the ring \( \mathbb{Z}_n \). It is clear that

\[ E_f(n) = \{ a \in \mathbb{Z}_n : \gcd(f(a), n) = 1 \}. \]

Throughout, we assume that \( f \) is a nonconstant polynomial and \( c \) is an integer. For any finite set \( S \), we denote by \( \#S \) the number of the elements in \( S \). We set \( N_{k,f,c}(n) \) to be the number of solutions \( (x_1, \ldots, x_k) \) of the congruence \( x_1 + \cdots + x_k \equiv c \pmod{n} \) with \( x_1, \ldots, x_k \in E_f(n) \), that is,

\[
N_{k,f,c}(n) := \#\{(x_1, \ldots, x_k) \in E_f(n)^k : x_1 + \cdots + x_k \equiv c \pmod{n}\}.
\]

For any given prime number \( p \), associated with \( f(x) \in \mathbb{Z}[x] \) and \( c \in \mathbb{Z} \), we define the nonnegative number \( M_{k,f,c}(p) \) by

\[
M_{k,f,c}(p) := \#\{(x_1, \ldots, x_{k-1}) \in \mathbb{Z}_p^{k-1} : f(x_1) \cdots f(x_{k-1})f\left(c - \sum_{i=1}^{k-1} x_i\right) \equiv 0 \pmod{p}\}.
\]

(1.2)
In [2], Anand et al. presented a formula for \( N_{z,f,c}(n) \) which also extends Sander’s theorem [10]. However, it still remains open to give an explicit formula for \( N_{k,f,c}(n) \) when \( k \geq 3 \).

In this paper, we introduce a new method to investigate the number \( N_{k,f,c}(n) \). Actually, we make use of the well-known principle of cross-classification [3] to derive an explicit formula of \( N_{k,f,c}(n) \) for all positive integers \( n \). The first main result of this paper can be stated as follows.

**Theorem 1.1.** Let \( f(x) \in \mathbb{Z}[x] \) be a nonconstant polynomial and let \( c \) be an integer. Then \( N_{k,f,c} \) is a multiplicative function and, for any positive integer \( n \),

\[
N_{k,f,c}(n) = n^{k-1} \prod_{p|n} \left( 1 - \frac{M_{k,f,c}(p)}{p^{k-1}} \right),
\]

where

\[
\delta_p := \begin{cases} 
    p - 1 & \text{if } p \mid (ac + kb), \\
    -1 & \text{if } p \nmid (ac + kb).
\end{cases}
\]  

(1.3)

Theorem 1.1 reduces to the main result of [2] (Theorem 2) if \( k = 2 \). For a linear polynomial \( f(x) \), we have the second main result of this paper generalising (1.1).

**Theorem 1.2.** Let \( c \) and \( n \) be integers with \( n \geq 1 \). Let \( f(x) = ax + b \in \mathbb{Z}[x] \) with \( \gcd(a,n) = 1 \). Then

\[
N_{k,f,c}(n) = n^{k-1} \prod_{p|n} \frac{(p - 1)^k + (-1)^k \delta_p}{p^k},
\]

where

\[
\delta_p := \begin{cases} 
    p - 1 & \text{if } p \mid (ac + kb), \\
    -1 & \text{if } p \nmid (ac + kb).
\end{cases}
\]  

(1.3)

For a quadratic polynomial \( f(x) \), one can deduce from Theorem 1.1 the third main result of this paper, which extends the Yang–Zhao theorem [18].

**Theorem 1.3.** Let \( c \) and \( n \) be integers with \( n \geq 1 \). Let \( f(x) = (a_1x - a_2)(b_1x - b_2) \) with \( a_1, a_2, b_1, b_2 \in \mathbb{Z} \) and \( \gcd(a_1,n) = \gcd(b_1,n) = \gcd(a_1b_2 - a_2b_1,n) = 1 \). Then

\[
N_{k,f,c}(n) = (-1)^{\omega(n)} \prod_{p|n} p^{k\nu_p(n) - \nu_p(n) - k} \sum_{j=0}^{k} \binom{k}{j} + (2 - p)^k - 2^k.
\]

This paper is organised as follows. In Section 2 we present several lemmas that are needed in the proofs of Theorems 1.1 and 1.3. Section 3 is devoted to the proof of Theorem 1.1 and Sections 4 and 5 to the proofs of Theorems 1.2 and 1.3, respectively.

2. Preliminary lemmas

In this section, we supply several lemmas that will be needed in the proofs of Theorems 1.1 and 1.3. We begin with the celebrated principle of cross-classification.

**Lemma 2.1** (Principle of cross-classification; [3, Theorem 5.31]). Let \( R \) be any given finite set. For a subset \( T \) of \( R \), we denote by \( \overline{T} = R \setminus T \) the set of those elements of \( R \)
which are not in $T$. If $R_1, \ldots, R_{m-1}$ and $R_m$ are m arbitrary given distinct subsets of $R$, then

$$\# \bigcap_{i=1}^{m} R_i = \#R + \sum_{i=1}^{m} (-1)^i \sum_{1 \leq i_1 < \cdots < i_t \leq m} \# \bigcap_{j=1}^{t} R_{i_j}.$$  

The next result can be proved by using the Chinese remainder theorem.

**Lemma 2.2** [3, Theorem 5.28]. Let $r \in \mathbb{Z}^+$ and $g(x_1, \ldots, x_r) \in \mathbb{Z}[x_1, \ldots, x_r]$ and let $m_1, \ldots, m_k$ be pairwise relatively prime positive integers. For any integer $i$ with $1 \leq i \leq k$, let $N_i$ be the number of zeros of $g(x_1, \ldots, x_r) \equiv 0 \pmod{m_i}$ and let $N$ denote the number of zeros of $g(x_1, \ldots, x_r) \equiv 0 \pmod{\prod_{i=1}^{k} m_i}$. Then $N = \prod_{i=1}^{k} N_i$.

**Lemma 2.3.** Let $n, r, m \in \mathbb{Z}^+$ with $m \mid n$ and let $g(x_1, \ldots, x_r) \in \mathbb{Z}[x_1, \ldots, x_r]$. Then

\[ \# \{ (x_1, \ldots, x_r) \in \mathbb{Z}_n^r : g(x_1, \ldots, x_r) \equiv 0 \pmod{m} \} = \left( \frac{n}{m} \right)^r \# \{ (x_1, \ldots, x_r) \in \mathbb{Z}_m^r : g(x_1, \ldots, x_r) \equiv 0 \pmod{m} \}. \]

**Proof.** Let $(x_1, \ldots, x_r)$ be any $r$-tuple of integers with all $x_i$ in the set $\{0, 1, \ldots, m-1\}$ such that

\[ g(x_1, \ldots, x_r) \equiv 0 \pmod{m}. \]

Then, for arbitrary integers $i_1, \ldots, i_r$ with $0 \leq i_1, \ldots, i_r < n/m$,

\[ 0 \leq x_1 + i_1 m, \ldots, x_r + i_r m \leq n - 1 \]

and

\[ g(x_1 + i_1 m, \ldots, x_r + i_r m) \equiv 0 \pmod{m}. \]

Every such $i_j$ (1 $\leq j \leq r$) has $n/m$ choices. So, one can immediately deduce the assertion in the lemma. \qed

In the following, let $p$ be a prime number. For any integer $a$ coprime to $p$, let $a^{-1}$ stand for an integer satisfying $aa^{-1} \equiv 1 \pmod{p}$. For any $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ with $\gcd(a_1a_2, p) = 1$, let $f_1(x) = (a_1x - b_1)(a_2x - b_2)$ and $f_2(x) = (x - a_1^{-1}b_1)(x - a_2^{-1}b_2)$. Then $f_1(t) = a_1a_2f_2(t)$ for any integer $t$. It follows that

\[ E_{f_1}(p) = \{ t \in \mathbb{Z}_p : \gcd(f_1(t), p) = 1 \} = \{ t \in \mathbb{Z}_p : \gcd(f_2(t), p) = 1 \} = E_{f_2}(p). \]

So, for our purpose, if $f(x)$ is a reducible quadratic polynomial with no multiple zeros, then we can assume that $f(x) = (x - a)(x - b) \in \mathbb{Z}[x]$. Let us now compute $\mathcal{M}_{k,f,c}(p)$.

**Lemma 2.4.** Let $a, b \in \mathbb{Z}_p$ with $a \neq b$ and let $f(x) = (x - a)(x - b)$. Then

\[ \mathcal{M}_{k,f,c}(p) = p^{k-1} - \frac{(-1)^k}{p} \sum_{j=0}^{k} \binom{k}{j} + (2 - p)^k - 2^k. \]
PROOF. Since \( f(x) = (x - a)(x - b) \in \mathbb{Z}[x] \),

\[
\mathcal{M}_{k,c}(p) = \#\left\{ (x_1, \ldots, x_{k-1}) \in \mathbb{Z}_p^{k-1} : f(x_1) \cdots f(x_{k-1}) f\left(c - \sum_{i=1}^{k-1} x_i\right) \equiv 0 \pmod{p}\right\}
\]

\[
= \#\mathbb{Z}_p^{k-1} - \#\left\{ (x_1, \ldots, x_{k-1}) \in \mathbb{Z}_p^{k-1} : f(x_1) \cdots f(x_{k-1}) f\left(c - \sum_{i=1}^{k-1} x_i\right) \not\equiv 0 \pmod{p}\right\}
\]

\[
= p^{k-1} - \#\left\{ (x_1, \ldots, x_{k-1}) \in \mathbb{Z}_p^{k-1} : (c - \sum_{i=1}^{k-1} x_i - a)(c - \sum_{i=1}^{k-1} x_i - b) \prod_{i=1}^{k-1} (x_i - a)(x_i - b) \not\equiv 0 \pmod{p}\right\}
\]

\[
= p^{k-1} - \#\left\{ (x_1, \ldots, x_k) \in (\mathbb{Z}_p \setminus \{a, b\})^k : \sum_{i=1}^k x_i \equiv c \pmod{p}\right\}.
\]

Moreover,

\[
\#\left\{ (x_1, \ldots, x_k) \in (\mathbb{Z}_p \setminus \{a, b\})^k : \sum_{i=1}^k x_i \equiv c \pmod{p}\right\}
\]

\[
= \frac{1}{p} \sum_{y=0}^{p-1} \sum_{(x_1, \ldots, x_k) \in (\mathbb{Z}_p \setminus \{a, b\})^k} \exp\left(\frac{2\pi iy(x_1 + \cdots + x_k - c)}{p}\right)
\]

\[
= \frac{1}{p} \sum_{y=0}^{p-1} \left( \sum_{x \in \mathbb{Z}_p \setminus \{a, b\}} \exp\left(\frac{2\pi ixy}{p}\right)\right)^k \exp\left(\frac{2\pi i(-cy)}{p}\right)
\]

\[
= \frac{1}{p} \left( \sum_{y=0}^{p-1} \left( - \exp\left(\frac{2\pi iay}{p}\right) - \exp\left(\frac{2\pi iby}{p}\right)\right)^k \exp\left(\frac{2\pi i(-cy)}{p}\right) + (p-2)^k \right)
\]

\[
= \frac{1}{p} \left( (-1)^k \sum_{y=1}^{p-1} \left( \sum_{j=0}^{k-1} \binom{k}{j} \exp\left(\frac{2\pi i(aj + (k-j)b)}{p}\right)\right) \exp\left(\frac{2\pi i(-cy)}{p}\right) + (p-2)^k \right)
\]

\[
= \frac{1}{p} \left( (-1)^k \sum_{j=0}^{k} \binom{k}{j} \sum_{y=1}^{p-1} \exp\left(\frac{2\pi i(aj + bk - bj - c)}{p}\right) + (p-2)^k \right)
\]

\[
= \frac{(-1)^k}{p} \sum_{j=0}^{k} \binom{k}{j} \left( (p-1) - \sum_{j=0}^{k} \binom{k}{j} + (2 - p)^k \right)
\]

\[
= \frac{(-1)^k}{p} \sum_{j=0}^{k} \binom{k}{j} + (2 - p)^k - 2^k.
\]
We then deduce that
\[
\mathcal{M}_{k,f,c}(p) = p^{k-1} - \frac{(-1)^k}{p} \sum_{j=0}^{k} \binom{k}{j} + (2 - p)^k - 2^k. 
\]
as desired. Lemma 2.4 is proved. □

3. Proof of Theorem 1.1

In this section, we use Lemmas 2.1, 2.2 and 2.3 to show Theorem 1.1.

PROOF OF THEOREM 1.1. First of all,
\[
\mathcal{N}_{k,f,c}(n) = \#\{(x_1, \ldots, x_k) \in E_f(n)^k : x_1 + \cdots + x_k \equiv c \pmod{n}\} 
\]
(3.1)
\[
= \#\{(x_1, \ldots, x_k) \in \mathbb{Z}_n^k : x_1 + \cdots + x_k \equiv c \pmod{n}, \gcd(f(x_i), n) = 1, 1 \leq i \leq k\}
\]
\[
= \#\{(x_1, \ldots, x_{k-1}) \in \mathbb{Z}_n^{k-1} : \gcd\left(f\left(c - \sum_{i=1}^{k-1} x_i\right), n\right) = 1, \gcd(f(x_i), n) = 1, 1 \leq i \leq k - 1\}. 
\]

Let \(n = p_1^{r_1} \cdots p_s^{r_s}\) be the standard prime factorisation of \(n\). In Lemma 2.1, let \(R = \mathbb{Z}_n^{k-1}\) and, for any integer \(i\) with \(1 \leq i \leq s\), let
\[
R_i = \{(x_1, \ldots, x_{k-1}) \in R : f(x_1) \cdots f(x_{k-1}) f\left(c - \sum_{i=1}^{k-1} x_i\right) \equiv 0 \pmod{p_i}\}. 
\]

Then
\[
\bar{R}_i = \{(x_1, \ldots, x_{k-1}) \in R : f(x_1) \cdots f(x_{k-1}) f\left(c - \sum_{i=1}^{k-1} x_i\right) \not\equiv 0 \pmod{p_i}\} 
\]
\[
= \{(x_1, \ldots, x_{k-1}) \in R : \gcd\left(f\left(c - \sum_{i=1}^{k-1} x_i\right), p_i\right) = 1, \gcd(f(x_j), p_i) = 1, 1 \leq j \leq k - 1\}. 
\]
(3.2)

It follows from (3.1) and (3.2) that
\[
\mathcal{N}_{k,f,c}(n) = \# \bigcap_{i=1}^{s} \bar{R}_i 
\]
(3.3)
and, for arbitrary integers \(i_1, \ldots, i_t\) with \(1 \leq i_1 < \cdots < i_t \leq s\),
\[
\bigcap_{j=1}^{t} R_{i_j} = \{(x_1, \ldots, x_{k-1}) \in R : f(x_1) \cdots f(x_{k-1}) f\left(c - \sum_{i=1}^{k-1} x_i\right) \equiv 0 \pmod{\prod_{j=1}^{t} p_{i_j}}\}. 
\]
On the other hand, by Lemmas 2.2 and 2.3,
\[ \# \big( \bigcap_{j=1}^{t} R_{ij} \big) = \# \left\{ (x_1, \ldots, x_{k-1}) \in R : f(x_1) \cdots f(x_{k-1}) f \left( c - \sum_{i=1}^{k-1} x_i \right) \equiv 0 \pmod{\prod_{j=1}^{t} p_{ij}} \right\} \]
\[ = \left( \frac{n}{\prod_{j=1}^{t} P_{ij}} \right)^{k-1} \# \left\{ (x_1, \ldots, x_{k-1}) \in \mathbb{Z}^{k-1}_{p_{ij}} : \right. \]
\[ f(x_1) \cdots f(x_{k-1}) f \left( c - \sum_{i=1}^{k-1} x_i \right) \equiv 0 \pmod{\prod_{j=1}^{t} p_{ij}} \} \]
\[ = n^{k-1} \prod_{j=1}^{t} \frac{1}{p_{ij}^{k-1}} \# \left\{ (x_1, \ldots, x_{k-1}) \in \mathbb{Z}^{k-1}_{p_{ij}} : f(x_1) \cdots f(x_{k-1}) f \left( c - \sum_{i=1}^{k-1} x_i \right) \equiv 0 \pmod{p_{ij}} \right\} \]
\[ = n^{k-1} \prod_{j=1}^{t} \frac{M_{k, f, c}(p_{ij})}{p_{ij}^{k-1}}. \tag{3.4} \]

It then follows from Lemma 2.1, (3.3) and (3.4) that
\[ N_{k, f, c}(n) = \# \left( \bigcap_{i=1}^{s} \mathbb{R}_{i} \right) = \# R + \sum_{i=1}^{s} (-1)^{i} \sum_{1 \leq i_{1} < \cdots < i_{s}} \# \big( \bigcap_{j=1}^{t} R_{ij} \big) \]
\[ = n^{k-1} + \sum_{i=1}^{s} (-1)^{i} \sum_{1 \leq i_{1} < \cdots < i_{s}} n^{k-1} \prod_{j=1}^{t} \frac{M_{k, f, c}(p_{ij})}{p_{ij}^{k-1}} \]
\[ = n^{k-1} \left( 1 + \sum_{i=1}^{s} (-1)^{i} \sum_{1 \leq i_{1} < \cdots < i_{s}} \prod_{j=1}^{t} \frac{M_{k, f, c}(p_{ij})}{p_{ij}^{k-1}} \right) \]
\[ = n^{k-1} \prod_{p | n} \left( 1 - \frac{M_{k, f, c}(p)}{p^{k-1}} \right), \]

as required. This concludes the proof of Theorem 1.1. \qed

4. Proof of Theorem 1.2

In this section, we present the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Choose a prime \( p \) with \( p \mid n \), so that \( \gcd(a, p) = 1 \). By (1.2),
\[ M_{k, f, c}(p) = \# \left\{ (x_1, \ldots, x_{k-1}) \in \mathbb{Z}^{k-1}_{p} : \left( a \left( c - \sum_{i=1}^{k-1} x_i \right) + b \right) \prod_{i=1}^{k-1} (ax_i + b) \equiv 0 \pmod{p} \right\} \]
\[ = \# \left\{ (x_1, \ldots, x_{k-1}) \in \mathbb{Z}^{k-1}_{p} : \right. \]
\[ \left( c - \sum_{i=1}^{k-1} (x_i + ba^{-1}) + kba^{-1} \right) \prod_{i=1}^{k-1} (x_i + ba^{-1}) \equiv 0 \pmod{p} \}. \]
Letting $y_i = x_i + ba^{-1}$ for $1 \leq i \leq k - 1$ gives

$$M_{k,f,c}(p) = \#\{ (y_1, \ldots, y_{k-1}) \in \mathbb{Z}_p^{k-1} : y_1 \cdots y_{k-1} (c + kba^{-1} - \sum_{i=1}^{k-1} y_i) \equiv 0 \pmod{p} \}$$

$$= \#\{ (y_1, \ldots, y_{k-1}) \in \mathbb{Z}_p^{k-1} : y_1 \cdots y_{k-1} (c + kba^{-1} - \sum_{i=1}^{k-1} y_i) \neq 0 \pmod{p} \}$$

$$= \#\{ (y_1, \ldots, y_{k-1}) \in (\mathbb{Z}_p)^{k-1} : c + kba^{-1} - \sum_{i=1}^{k-1} y_i \equiv 0 \pmod{p} \}$$

$$= p^{k-1} - (p-1)^{k-1} + \#\{ (y_1, \ldots, y_{k-1}) \in (\mathbb{Z}_p)^{k-1} : \sum_{i=1}^{k-1} y_i \equiv c + kba^{-1} \pmod{p} \}$$

$$= p^{k-1} - (p-1)^{k-1} + N(k-1, c + kba^{-1}, p). \quad (4.1)$$

But (1.1) tells us that

$$N(k-1, c + kba^{-1}, p) = \begin{cases} \frac{((p-1)^{k-1} + (-1)^{k-1}(p-1))/p}{(p-1)^{k-1} + (-1)^{k}/p} & \text{if } p \mid (c + kba^{-1}), \\ \frac{((p-1)^{k-1} + (-1)^{k})/p}{(p-1)^{k-1} + (-1)^{k}/p} & \text{if } p \nmid (c + kba^{-1}). \end{cases} \quad (4.2)$$

Combining (4.2) with (4.1),

$$M_{k,f,c}(p) = p^{k-1} - \frac{(p-1)^{k} + (-1)^{k}\delta_p}{p} \quad (4.3)$$

with $\delta_p$ being given as in (1.3). Finally, by Theorem 1.1 and (4.3),

$$N_{k,f,c}(n) = n^{k-1} \prod_{p \mid n} \left( 1 - \frac{M_{k,f,c}(p)}{p^{k-1}} \right) = n^{k-1} \prod_{p \mid n} \left( \frac{p^{k-1} - (p-1)^k + (-1)^k\delta_p}{p^k} \right).$$

This finishes the proof of Theorem 1.2. \qed

5. Proof of Theorem 1.3

In this section, we use Theorem 1.1 and Lemma 2.4 to prove Theorem 1.3.

**Proof of Theorem 1.3.** Choose a prime $p$ with $p \mid n$ and let $h(x) = (x-a)(x-b)$ with $a = a_2a_1^{-1}$ and $b = b_2b_1^{-1}$ taken modulo $p$. The inverses exist and $a \neq b \pmod{p}$ because $\gcd(a_1, p) = \gcd(b_1, p) = \gcd(a_1b_2 - a_2b_1, p) = 1$ and for $t \in \mathbb{Z}$, $f(t) \equiv 0 \pmod{p}$ if and only if $h(t) \equiv 0 \pmod{p}$. Applying Lemma 2.4 to
Theorem 1.3.

\[ \mathcal{N}_{k,f,c}(n) = n^{k-1} \prod_{p\mid n} \left( 1 - \frac{\mathcal{M}_{k,f,c}(p)}{p^{k-1}} \right) \]
\[ = (-1)^{\omega(n)} \prod_{p\mid n} p^{k\nu_p(n) - \nu_p(n) - k} \left( p \sum_{j=0}^{k} \binom{k}{j} (2 - p)^j - 2^k \right) \]
\[ = (-1)^{\omega(n)} \prod_{p\mid n} p^{k\nu_p(n) - \nu_p(n) - k} \left( \sum_{j=0}^{k} \binom{k}{j} (2 - p)^j - 2^k \right) \]

as expected. This completes the proof of Theorem 1.3. □

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