Covering lattice points by subspaces and counting point-hyperplane incidences

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For $d \in \mathbb{N}$, let $S$ be a collection of subsets in $\mathbb{R}^d$ and let $P$ be a set of points from $\mathbb{R}^d$. We say $S$ covers $P$ if every point from $P$ lies in some set from $S$. For $n \in \mathbb{N}$, what is the minimum number of lines needed to cover $n \times n$ lattice? What if all the lines have to contain the origin?
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Introduction

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Problem 1 (Brass, Moser, Pach, 2005)
What is the minimum number of $k$-dimensional linear subspaces needed to cover the $d$-dimensional $n \times \cdots \times n$ lattice?

For affine subspaces the answer is $\Theta(n^{d-k})$.

Covering by linear subspaces is more difficult. Bárány, Harcos, Pach, Tardos (2001) solved the problem for hyperplanes containing the origin, i.e., for $k = d - 1$.
They showed that the answer is $\Theta(n^{d/(d-1)})$.

Their proof works in the following more general setting.
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- Their proof works in the following more general setting.
Lattices and symmetric convex bodies

For linearly independent vectors \( b_1, \ldots, b_d \in \mathbb{R}^d \), the \( d \)-dimensional lattice \( \Lambda \) with basis \( \{b_1, \ldots, b_d\} \) is the set
\[
\Lambda = \{a_1 b_1 + \cdots + a_d b_d : a_1, \ldots, a_d \in \mathbb{Z}\}.
\]

A convex body \( K \) is symmetric about 0 if \( K = -K \).

Let \( L_d \) be the set of \( d \)-dimensional lattices and \( K_d \) be the set of \( d \)-dimensional compact convex bodies in \( \mathbb{R}^d \) that are symmetric about 0.
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Successive minima

For $\Lambda \in \mathbb{L}_d$ and $K \in \mathbb{K}_d$, what is the minimum number of $k$-dimensional linear subspaces needed to cover $\Lambda \cap K$?

How to measure $|\Lambda \cap K|$?

For $i = 1, \ldots, d$, the $i$th successive minimum of $\Lambda$ and $K$ is

$$\lambda_i(\Lambda, K) = \inf \{ \lambda \in \mathbb{R} : \dim(\Lambda \cap (\lambda \cdot K)) \geq i \}.$$ 

The successive minima are achieved and $0 < \lambda_1 \leq \cdots \leq \lambda_d$. 


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For $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$ with $\lambda_d \leq 1$, the set $\Lambda \cap K$ can be covered with at most

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- We consider Generalized problem 1 for general \(k\).
Our results – covering by linear subspaces

Theorem 1
For $k$ with $1 \leq k \leq d - 1$, $\Lambda \in L_d$, and $K \in K_d$ with $\lambda_d \leq 1$, we can cover $\Lambda \cap K$ with $O\left(\alpha_d - k\right)$ $(k)$-dimensional linear subspaces, where $\alpha = \min_{1 \leq j \leq k} \left(\lambda_j \cdots \lambda_d\right)^{-1} / (d - j)$.

Using probabilistic method, we can also show the following lower bound.

Theorem 2
For $k$ with $1 \leq k \leq d - 1$, $\Lambda \in L_d$, $K \in K_d$ with $\lambda_d \leq 1$, and $\epsilon \in (0, 1)$, we need at least $\Omega\left((1 - \lambda_d) \beta \right) \left(\left(\frac{1}{d - k - \epsilon}\right)\right)$ $(k)$-dimensional linear subspaces to cover $\Lambda \cap K$, where $\beta = \min_{1 \leq j \leq d - 1} \left(\lambda_j \cdots \lambda_d\right)^{-1} / (d - j)$.

The bounds are not tight. The lower bound can be improved?
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For $k$ with $1 \leq k \leq d - 1$, $\Lambda \in \mathcal{L}^d$, and $K \in \mathcal{K}^d$ with $\lambda_d \leq 1$, we can cover $\Lambda \cap K$ with $O(\alpha^{d-k})$ $k$-dimensional linear subspaces, where

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For $k$ with $1 \leq k \leq d - 1$, $\Lambda \in \mathcal{L}^d$, $K \in \mathcal{K}^d$ with $\lambda_d \leq 1$, and $\varepsilon \in (0, 1)$, we need at least $\Omega(((1 - \lambda_d)\beta)^{d-k-\varepsilon})$ $k$-dimensional linear subspaces to cover $\Lambda \cap K$, where

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Our results – covering by linear subspaces

**Theorem 1**

For $k$ with $1 \leq k \leq d - 1$, $\Lambda \in \mathcal{L}^d$, and $K \in \mathcal{K}^d$ with $\lambda_d \leq 1$, we can cover $\Lambda \cap K$ with $O(\alpha^{d-k})$ $k$-dimensional linear subspaces, where

$$\alpha = \min_{1 \leq j \leq k} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}.$$

- Using probabilistic method, we can also show the following lower bound.

**Theorem 2**

For $k$ with $1 \leq k \leq d - 1$, $\Lambda \in \mathcal{L}^d$, $K \in \mathcal{K}^d$ with $\lambda_d \leq 1$, and $\varepsilon \in (0, 1)$, we need at least $\Omega(((1 - \lambda_d)\beta)^{d-k-\varepsilon})$ $k$-dimensional linear subspaces to cover $\Lambda \cap K$, where

$$\beta = \min_{1 \leq j \leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}.$$

- The bounds are not tight. The lower bound can be improved?
Our results – covering by affine subspaces

The bounds are sufficient to nearly settle Problem 1:

Corollary

For $k$ with $1 \leq k \leq d - 1$ and $n \in \mathbb{N}$, the $n \times \cdots \times n$ lattice can be covered with $O\left(\frac{n^d}{d-1} \right)^k$-dimensional linear subspaces and for every $\varepsilon > 0$ we need at least $\Omega\left(\frac{n^d}{d-1} - \varepsilon\right)^k$-dimensional linear subspaces to cover it.

We also consider the problem of covering $\Lambda \cap K$ with affine subspaces.

Theorem 3

For $k$ with $1 \leq k \leq d - 1$, $\Lambda \in \mathbb{L}_d$, and $K \in \mathbb{K}$ with $\lambda d \leq 1$, the set $\Lambda \cap K$ can be covered with $O\left(\left(\lambda^k + \cdots + \lambda^d\right) - 1\right)^k$-dimensional affine subspaces and this is tight.
Our results – covering by affine subspaces

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**Corollary**

For $k$ with $1 \leq k \leq d - 1$ and $n \in \mathbb{N}$, the $n \times \cdots \times n$ lattice can be covered with $O(n^{d(d-k)/(d-1)})$ $k$-dimensional linear subspaces and for every $\varepsilon > 0$ we need at least $\Omega(n^{d(d-k)/(d-1)-\varepsilon})$ $k$-dimensional linear subspaces to cover it.
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**Theorem 3**

For $k$ with $1 \leq k \leq d - 1$, $\Lambda \in \mathcal{L}^d$, and $K \in \mathcal{K}^d$ with $\lambda_d \leq 1$, the set $\Lambda \cap K$ can be covered with

$$O((\lambda_{k+1} \cdots \lambda_d)^{-1})$$

$k$-dimensional affine subspaces and this is tight.
Sketch of the proof of Theorem 1

We want to cover \( \Lambda \cap K \) with \( O(\alpha^d - k) \) \( k \)-dimensional linear subspaces, where \( \alpha = \min_{1 \leq j \leq k} (\lambda_j \cdots \lambda_d)^{-1/(d-j)} \).

We show the result for \( K \) being the unit ball \( B^d \). The result for general \( K \) then follows by John's Lemma.

Using Second Minkowski's Theorem, we show that \( O((\lambda_k \cdots \lambda_d)^{-1}) \) \( k \)-dimensional linear subspaces are sufficient (i.e., prove the case \( j = k \)).

Then we proceed by induction on \( d-k = 1, \ldots, d-1 \).

We use the fact that the larger \( \|z\| \) is, the sparser (\( \Lambda \cap H(z) \)) \( \cap B^d \) is.
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\[
\begin{align*}
d &= 3 \\
k &= 1
\end{align*}
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\[ H(z) \]

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An incidence between an \( n \)-point set \( P \subseteq \mathbb{R}^d \) and a set of \( m \) hyperplanes \( H \) in \( \mathbb{R}^d \) is a pair \((p, H)\) such that \( p \in P \), \( H \in H \), and \( p \in H \).

What is the maximum number of incidences between \( P \) and \( H \) in \( \mathbb{R}^d \)?

In the plane, the Szemerédi–Trotter Theorem says that it is at most \( O((mn)^{2/3} + m + n) \) for all \( P \) and \( H \). Moreover, this is tight.

For \( d \geq 3 \) it is trivially at most \( mn \) and this is tight!

To avoid this, we forbid \( K_r^r \), for some fixed \( r \) in the incidence graph. Then the maximum number of incidences is at most \( O((mn)^{1 - 1/(d+1)} + m + n) \) (Chazelle, 1993).
Application: bounds for point-hyperplane incidences

- An incidence between an \( n \)-point set \( P \subseteq \mathbb{R}^d \) and a set of \( m \) hyperplanes \( \mathcal{H} \) in \( \mathbb{R}^d \) is a pair \((p, H)\) such that \( p \in P \), \( H \in \mathcal{H} \), and \( p \in H \).
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- Then the maximum number of incidences is at most \( O \left( (mn)^{1 - 1/(d+1)} + m + n \right) \) (Chazelle, 1993).
Our results – counting point-hyperplane incidences

Theorem (Brass and Knauer, 2003)

For $d \geq 3$, $\epsilon > 0$ there is an $r$ such that for all $n$ and $m$ there is a set $P$ of $n$ points in $\mathbb{R}^d$ and a set $H$ of $m$ hyperplanes in $\mathbb{R}^d$ with no $K_r$, $r$ in the incidence graph and with the number of incidences at least $\Omega\left(\left(\frac{mn}{1 - 2/d + \epsilon}\right)\right)$ if $d$ is odd and $d > 3$, $\Omega\left(\left(\frac{mn}{1 - 2(d+1)/(d+2) - \epsilon}\right)\right)$ if $d$ is even, $\Omega\left(\left(\frac{mn}{7/10}\right)\right)$ if $d = 3$.

For $d \geq 4$, we improve these bounds to $\Omega\left(\left(\frac{mn}{1 - (2d+3)/(d+2)(d+3) - \epsilon}\right)\right)$ if $d$ is odd, $\Omega\left(\left(\frac{mn}{1 - (2d^2 + d - 2)/(d+2)(d^2 + 2d - 2) - \epsilon}\right)\right)$ if $d$ is even.
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Our results – counting point-hyperplane incidences

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**Theorem (Brass and Knauer, 2003)**

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\Omega \left( (mn)^{1-2/(d+3)-\varepsilon} \right) \quad \text{if } d \text{ is odd and } d > 3,
\]

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Final remarks

The gap in the exponents is of order $\Theta(1/d)$ and the improvement is of order $\Theta(1/d^2)$. It is the first improvement in the last 13 years. It provides the best known lower bound for so-called Semialgebraic Zarankiewicz’s problem.

Open problems:
Close the gap between estimates from Theorem 1 and Theorem 2.
For $1 < k < d-1$, some fixed $r \in \mathbb{N}$, and an arbitrarily large $n \in \mathbb{N}$, construct a set $R \subseteq \mathbb{Z}^d \cap [-n,n]^d$ of size $\Omega(n^d(d-k)/(d-1))$ such that no $k$-dimensional linear subspace contains $r$ points from $R$.

Improve the bounds for the maximum number of point-hyperplane incidences.

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