The Dynamics Of Vortex And Monopole Production By Quench Induced Phase Separation

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Abstract

Our understanding of the mechanism by which topological defects are formed in symmetry breaking phase transitions has recently changed. We examine the non-equilibrium dynamics of defect formation for weakly-coupled global O(N) theories possessing vortices (strings) and monopoles. It is seen that, as domains form and grow, defects are swept along on their boundaries at a density of about one defect per coherence area (strings) or per coherence volume (monopoles).

I. INTRODUCTION

The formation of topological defects during symmetry breaking phase transitions is generic to many physical systems. In particular we cite the vortices and monopoles of superfluid $^4$He and $^3$He and the vortices (flux-tubes) of high- and low-$T_c$ superconductors. Similar defects, cosmic strings or monopoles, most likely appeared in the early universe at the GUT-scale phase transition. All of these systems are described by some form of quantum field theory and, due to the phase transition, their dynamics is intrinsically non-equilibrium. They therefore provide a good means to test non-equilibrium field-theory experimentally over a wide range of energies.

Roughly, the dynamics of defect formation proceeds as follows [1]. From some initial state, which is not too far from thermal equilibrium, some change in the bulk properties of the system, such as pressure or volume, induces a phase transition. During this transition, the scalar fields which describe the order parameter fall from the false vacuum into the true vacuum, choosing a point on the vacuum manifold at each point in space, subject to the constraint that they must be continuous and single-valued. We shall limit ourselves to weakly first order or continuous transitions, for which this collapse to the true vacuum occurs by spinodal decomposition or phase separation. The resulting field configuration is one of domains within each of which the scalar fields have relaxed to a constant vacuum value. If the theory permits defects, it will sometimes happen that the requirements of continuity and single valuedness force the fields to remain in the false vacuum between some of the domains. For example, in the case of a complex scalar field producing vortices, the phase of the field may change by an integer multiple of $2\pi$ on going round a loop in space. This requires at least one zero of the field within the loop, which signifies the presence of a region of unbroken phase. Each zero has topological stability and characterises a vortex passing through the loop. When the phase transition is complete and there is no longer sufficient thermal energy available for the field to fluctuate into the false vacuum, the topological defects are frozen into the field. From then on, the defect density alters almost entirely by interactions of defects amongst themselves, rather than by fluctuations in the fields, see for example [2].

The major question then, is what fixes the initial defect density and the defect correlations? Only then can the subsequent evolution of defect networks be determined with any accuracy. It was first argued that topological defects should be frozen in at the Ginzberg
temperature \( T_G \), the temperature above which there is sufficient thermal energy available for the field to fluctuate into the false vacuum without cost. If so, the defect number would be strongly fluctuating above the Ginzberg temperature but frozen in below it. In this case, the relevant scale for the initial defect density would be the coherence length, \( \xi(T_G) \), of the Higgs field (or fields) at the Ginzberg temperature. For example, in vortex production for the \( U(1) \) theory mentioned above, the initial vortex density (i.e. the number of vortices passing through unit area) would be \( \kappa/\xi^2(T_G) \), where \( \kappa \) is a constant of order unity. Similarly, in monopole production the monopole density would be expected to be \( \kappa/\xi^3(T_G) \), for similar \( \kappa \). Thereafter, defect forces are assumed to take over.

Recently, however, more compelling pictures of the way in which the initial density of topological defects is fixed have been proposed. While the mechanism outlined initially is almost certainly correct, in general it is unlikely that the Ginzberg temperature is relevant to anything other than a thermally produced population of defects. For the cases of interest, for example an expanding universe, we expect that, as the system is driven from some initial thermal state towards the phase transition there comes a point when the rate at which the transition is driven is too fast for the evolution of the field to keep up. The transition may now be viewed as a quench and it is not clear that either temperature or free energy mean anything at all. At this point any defects within the field are assumed to be frozen in until the transition is complete. Upon completion, the field will try to return to thermal equilibrium. At sufficiently low temperatures, however, the return to equilibrium by thermal processes will be so slow that the evolution of the initial defect density thus produced will be almost entirely by interactions between the defects themselves.

Thus, in this scenario, the vortices cease to be produced, not at the Ginzberg temperature, but when the scalar fields go out of equilibrium. The relevant scale which determines the defect density is the coherence length, \( \xi(t) \) at this time, and for some time onwards, rather than the coherence length at the Ginzberg temperature. The only remaining uncertainty is how good an approximation it is to say that the defect number is frozen into the field from the time when it first goes out of equilibrium until the time when it lies in the vacuum manifold almost everywhere and its evolution can be viewed as the interaction of the defects. It is this question which we address here.

II. THE MODEL

In the following we consider a class of theories where the broken and unbroken symmetries are global, thereby guaranteeing that they pass through a second order transition. [Had they passed through a strongly first order transition, the mechanism for the transition, bubble nucleation, would lead to different consequences from those outlined below, although it might still be a good approximation to say that the defect density is frozen in when the field first goes out of thermal equilibrium]. We assume that the change of symmetry is sufficiently rapid that the fields are unable to respond immediately, but evolve by means of

\footnotesize
\begin{enumerate}
  \item Although thermal equilibrium is mode dependent, this does not matter for the crude argument repeated here.
\end{enumerate}
phase separation or spinodal decomposition and domain formation.

We shall consider the simplest theory, one of \(N\) massive relativistic scalar fields \(\phi_a\), where \(a = 1, \ldots, N\), in \(D\) spatial dimensions, transforming as the fundamental representation of a globally \(O(N)\) invariant theory. Changes in the environment cause the symmetry to be broken to \(O(N-1)\) (i.e. as given by the generalised ‘wine-bottle’ potential) leading to a theory of one massive Higgs boson and \(N-1\) massless Goldstone bosons, with the vacuum manifold \(S^{N-1}\). Since the \(n\)th homotopy group \(\Pi_n\) of the \(n\)-sphere is \(\Pi_n(S^n) = \mathbb{Z}\), the group of integers, the theory possesses global monopoles if \(N=D\) and global strings if \(N=D-1\).

We are primarily interested in \(D=3\) dimensions, for which the \(O(3)\) theory possesses monopoles, and the \(O(2)\) theory possesses strings. However, the vortex production in the \(D=2\) Kosterlitz-Thouless transition has some interest, although we shall not pursue it here.

The transition is realised by the changing environment inducing an explicit time-dependence in the field parameters. Although we have the early universe in mind, we remain as simple as possible, in flat space-time with the \(\phi\)-field action:-

\[
S[\phi] = \int \mathcal{D}^{D+1}x \left( \frac{1}{2} \partial_{\mu} \phi_a \partial^{\mu} \phi_a - \frac{1}{2} m^2(t) \phi_a^2 - \frac{1}{4} \lambda(t) (\phi_a^2)^2 \right).
\]

The \(t\)-dependence of \(m^2(t)\) and \(\lambda(t)\) is assumed given and adjacent \(O(N)\) indices are summed over.

We wish to calculate the evolution of the defect density during the fall from the false vacuum to the true vacuum after a rapid quench from an initial thermal state. The simplest assumption, which we shall adopt, is that the symmetry breaking occurs at time \(t = t_0\), with the sign of \(m^2(t)\) changing from positive to negative at \(t_0\). Further, after some short period \(\Delta t = t - t_0 > 0\), \(m^2(t)\) and \(\lambda(t)\) have relaxed to their final values, denoted by \(m^2\) and \(\lambda\) respectively. The field begins to respond to the symmetry-breaking at \(t = t_0\) but we assume that its response time is greater than \(\Delta t\), again ignoring any mode dependence.

To follow the evolution of the defect density during the fall off the hill involves two problems. The first is how to count the defects and the second is how to follow the evolution of the quantum field. We take these in turn.

### III. COUNTING THE DEFECT DENSITY

To calculate the defect density requires knowledge of \(p_t[\Phi]\), the probability that, the measurement of the field \(\phi(t, x) = (\phi_1, \phi_2, \ldots, \phi_N)\) would yield the result \(\Phi(x) = (\Phi_1, \Phi_2, \ldots, \Phi_N)\). This is obviously a consequence of both the initial conditions and the subsequent dynamics. We follow Halperin [7] in adopting a Gaussian distribution for the field of the form:-

\[
p_t[\Phi] = \mathcal{N} \exp \left(-\frac{1}{2} \int \mathcal{D}^{D}x \mathcal{D}^{D}y \, \Phi_a(x) K_{ab}(x-y; t) \Phi_b(y) \right),
\]

with \(K_{ab} = \delta_{ab} K\) and \(\mathcal{N}\) a normalisation. The circumstances under which a Gaussian is valid will be examined later. For weakly coupled theories we shall see that, for short times after \(t_0\) at least, a Gaussian \(p_t[\Phi]\) will occur. If this is taken for granted it is relatively
straightforward to calculate the number density of defects. Postponing the calculation of $K$ until then, we quote those of Halperin’s results that are relevant to us.

Suppose that the field $\phi(t, x)$ takes the particular value $\Phi(x)$. We count the vortices by identifying them with its zeroes. The only way for a zero to occur with significant probability is at the centre of a topological defect so, but for a set of measure zero, all zeroes are topological defects. $^2$

Consider the $O(D)$ theory in $D$ spatial dimensions, with global monopoles. Although less relevant than strings for the early universe they are slightly easier to perform calculations for. Almost everywhere, monopoles occur at the zeroes of $\Phi(x)$, labelled $x_i, i = 1, 2, \ldots$, at which the orientation $\Phi(x)/|\Phi(x)|$ is ill-defined. A topological winding number $n_i = \pm 1$ can be associated with each zero $x_i$ by the rule:-

$$n_i = \text{sgn det}(\partial_a \Phi_b)|_{x=x_i}.$$  

Monopoles with higher winding number are understood as multiple zeroes of $\Phi(x)$ at which the $n_i$ are additive. The net monopole density is then given by:-

$$\rho_{\text{net}}(x) = \sum_i n_i \delta(x - x_i).$$

The volume integral of this gives the number of monopoles minus the number of antimonopoles. The correlations of $\rho_{\text{net}}$ give us information on monopole-(anti)monopole correlations but, in the first instance, we are interested in the cruder grand totals. The quantity of greater relevance to us, is the total monopole density:-

$$\rho(x) = \sum_i \delta(x - x_i),$$

whose volume integral gives the total number of monopoles plus antimonopoles in the volume of integration.

Now consider an ensemble of systems in which the fields $\Phi$ are distributed according to $p_t[\Phi]$ at time $t$. Then, on average, the total monopole density is:-

$$\langle \rho(t) \rangle = \left\langle \sum_i \delta(x - x_i) \right\rangle_t,$$

where the triangular brackets denote averaging with respect to $p_t[\Phi]$. That is:-

$$\langle F[\Phi] \rangle_t = \int D\Phi F[\Phi] p_t[\Phi],$$

with $p_t[\Phi]$ normalised so that $\int D\Phi p_t[\Phi] = 1$. The translational invariance of the Gaussian kernel of the probability density ensures that $\rho(t)$ is translationally invariant.

In terms of the fields $\Phi_a$, vanishing at $x_i$, $\rho(t)$ can be re-expressed as:-

$$\langle \rho(t) \rangle = \langle \delta^D[\Phi_a(x)] \mid \det (\partial_a \Phi_b(x)) \mid \rangle_t.$$

$^2$This counting procedure alone gives no information about the length distribution of the defects.
The second term in the brackets is just the Jacobian of the transformation from \( \mathbf{x} \) to \( \Phi(\mathbf{x}) \). It follows from the Gaussian form of the probability density that the \( \Phi_a \) individually and independently Gaussian distributed with zero mean, as

\[
\langle \Phi_a(\mathbf{x}) \Phi_b(\mathbf{x}) \rangle_t = \delta_{ab} W(|\mathbf{x} - \mathbf{y}|; t),
\]

where \( W(|\mathbf{x} - \mathbf{y}|; t) = K^{-1}(\mathbf{x} - \mathbf{y}; t) \). So also are the first derivatives of the field \( \partial_a \Phi_b \), which are independent of the field:

\[
\langle \Phi_c(\mathbf{x}) \partial_a \Phi_b(\mathbf{y}) \rangle_t = 0
\]
due to the fact that \( W \) is dependent only on the magnitude of \( \mathbf{x} - \mathbf{y} \).

Thus, the total defect density may be separated into two independent parts:

\[
\langle \rho(t) \rangle = \langle \delta^D[\Phi(\mathbf{x})] \rangle_t \langle \det(\partial_a \Phi_b) \rangle_t.
\]

The first factor is easy to calculate, the second less so.

Consider first the delta-distribution factor:-

\[
\langle \delta[\Phi(\mathbf{x}_0)] \rangle_t = \int D\Phi \delta[\Phi(\mathbf{x}_0)] \exp \left\{ -\frac{1}{2} \int d^D x d^D y \Phi(\mathbf{x}) K(\mathbf{x} - \mathbf{y}; t) \Phi(\mathbf{y}) \right\}
\]

\[
= \int \phi \alpha \int D\Phi \exp \left\{ i\alpha \Phi(\mathbf{x}_0) - \frac{1}{2} \int d^D x d^D y \Phi(\mathbf{x}) K(\mathbf{x} - \mathbf{y}; t) \Phi(\mathbf{y}) \right\},
\]

where \( O(N) \) indices and integrals over spatial variables have been suppressed and \( \phi \alpha = d\alpha/2\pi \). On defining \( \phi \alpha = \alpha(\mathbf{x} - \mathbf{x}_0) \alpha \), we find:

\[
\langle \delta[\Phi(\mathbf{x}_0)] \rangle_t = \int \phi \alpha \int D\Phi \exp \left\{ \int d^D x \left( i\alpha(\mathbf{x}) \Phi(\mathbf{x}) - \frac{1}{2} \int d^D x d^D y \Phi(\mathbf{x}) K(\mathbf{x} - \mathbf{y}; t) \Phi(\mathbf{y}) \right) \right\} \delta(\mathbf{x} - \mathbf{x}_0)
\]

\[
= \int \phi \alpha \exp \left\{ -\frac{1}{2} \int d^D x d^D y \Phi(\mathbf{x}) W(|\mathbf{x} - \mathbf{y}|; t) \Phi(\mathbf{y}) \right\}
\]

\[
= \int \phi \alpha \exp \left\{ -\frac{1}{2} \int d^D x d^D y \delta(\mathbf{x} - \mathbf{x}_0) \alpha W(|\mathbf{x} - \mathbf{x}'; t) \alpha \delta(\mathbf{x}' - \mathbf{x}_0) \right\}
\]

\[
= \frac{1}{2\pi} \left( \frac{1}{\sqrt{K^{-1}}} \right)^2 = \frac{1}{2\pi} \langle \Phi \Phi \rangle = \frac{1}{2\pi} W(0; t)
\]

Consider now the second factor. Writing out the determinant explicitly for \( N = D = 2 \), and exploiting the fact that the field is Gaussian, we have:

\[
\langle \det(\partial_a \phi_b(\mathbf{x})) \rangle_t^2 = \left\langle \left[ \det(\partial_a \phi_b(\mathbf{x})) \right]^2 \right\rangle_t
\]

\[
= \left\langle (\partial_1 \phi_1 \partial_2 \phi_2)^2 + (\partial_1 \phi_2 \partial_2 \phi_1)^2 - 2\partial_1 \phi_2 \partial_2 \phi_1 \partial_1 \phi_1 \partial_2 \phi_2 \right\rangle_t
\]

The first term may be factorised into a product of two Gaussian variables and calculated as follows:

\[
\langle (\partial_1 \phi_1 \partial_2 \phi_2)^2 \rangle_t = \langle (\partial_1 \phi_1)^2 \rangle_t \langle (\partial_2 \phi_2)^2 \rangle_t
\]

\[
= [-\delta_{11} \partial_1 \partial_1 W(\mathbf{x}; t)] [-\delta_{22} \partial_2 \partial_2 W(|\mathbf{x}|; t)] = [\partial_1 \partial_1 W(|\mathbf{x}|; t)]^2,
\]

\[
\langle (\partial_1 \phi_2 \partial_2 \phi_1)^2 \rangle_t = \langle (\partial_2 \phi_2)^2 \rangle_t \langle (\partial_1 \phi_1)^2 \rangle_t
\]

\[
= \langle (\partial_2 \phi_2)^2 \rangle_t \langle (\partial_1 \phi_1)^2 \rangle_t
\]

\[
= [\partial_2 \partial_2 W(|\mathbf{x}|; t)]^2
\]

\[
\langle (\partial_1 \phi_1 \partial_2 \phi_2 \partial_2 \phi_1 \partial_1 \phi_1 \partial_2 \phi_2)^2 \rangle_t
\]

\[
= \langle (\partial_1 \phi_2 \partial_2 \phi_1 \partial_1 \phi_1 \partial_2 \phi_2)^2 \rangle_t
\]

\[
= [\partial_1 \partial_1 W(|\mathbf{x}|; t)]^2
\]

\[
\langle (\partial_1 \phi_2 \partial_2 \phi_1 \partial_1 \phi_1 \partial_2 \phi_2)^2 \rangle_t
\]

\[
= \langle (\partial_1 \phi_2 \partial_2 \phi_1 \partial_1 \phi_1 \partial_2 \phi_2)^2 \rangle_t
\]

\[
= \langle (\partial_1 \phi_2 \partial_2 \phi_1 \partial_1 \phi_1 \partial_2 \phi_2)^2 \rangle_t
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\[
= \langle (\partial_1 \phi_2 \partial_2 \phi_1 \partial_1 \phi_1 \partial_2 \phi_2)^2 \rangle_t
\]

\[
= \langle (\partial_1 \phi_2 \partial_2 \phi_1 \partial_1 \phi_1 \partial_2 \phi_2)^2 \rangle_t
\]
where \( W(|x|; t) \) is as before. Fourier transforming the two-point function, we find:

\[
\langle (\partial_1 \phi_1 \partial_2 \phi_2)^2 \rangle_t = \left( \partial_1 \partial_1 \int e^{i k \cdot x} \tilde{W}(k; t) \, d^3 k \right) \left( \partial_2 \partial_2 \int e^{i k \cdot x} \tilde{W}(k; t) \, d^3 k \right)
\]

\[
= \left( \int k^2 \cos^2(\theta) W(k; t) \, d^3 k \right)^2
\]

\[
= (\nabla^2 W(0; t))^2
\]

A similar result applies for the second term whereas the third term vanishes. The final result is:

\[
\langle \rho(t) \rangle = C_N \left| \frac{W''(0; t)}{W(0; t)} \right|^{N/2}
\]

where the second derivative in the numerator is with respect to \( x = |x| \). \( C_N \) is \( 1/\pi^2 \) for \( N = D = 3 \) and \( 1/2\pi \) for \( N = D = 2 \) (had we performed the calculation in \( D = 2 \) dimensions), the difference coming entirely from the determinant factor.

Let us now consider the case of global strings in \( D = 3 \) dimensions that arise from the \( O(2) \) theory. Strings are identified with lines of zeroes of \( \phi(t, x) = \Phi(x) \) and the net vortex density (vortices minus antivortices) in a plane perpendicular to the \( i \)-direction is:

\[
\rho_{\text{net},i}(x) = \delta^2[\Phi(x)] \epsilon_{ijk} (\partial_j \Phi_1)(\partial_k \Phi_2),
\]

with obvious generalisations to \( N = D - 1 \), for all \( N \) in terms of the Levi-Cevita symbol \( \epsilon_{i_1,i_2,...,i_D} \). As before, the total vortex density is of more immediate use. On a surface perpendicular to the \( i \)-direction this is:

\[
\rho_{\text{net},i}(x) = \delta^2[\Phi(x)] |\epsilon_{ijk} (\partial_j \Phi_1)(\partial_k \Phi_2)|.
\]

in analogy with the monopole case. The expectation value of this total density, when calculated as before reproduces the same expression:

\[
\langle \rho(t) \rangle = C_N \left| \frac{W''(0; t)}{W(0; t)} \right|^{N/2},
\]

but for \( N = D - 1 \). For the case of interest, \( N = 2 \) and \( C_2 = 1/2\pi \). Thus, whether we are concerned about global monopole or global string density, once we have calculated \( K(x; t) \) we can find the total string density. Similar results apply for the correlations between net densities, which are important in determining the subsequent evolution of the defect network.

**IV. EVOLUTION OF THE QUANTUM FIELD**

During a symmetry breaking phase transition, the dynamics of the quantum field is intrinsically non-equilibrium. The normal techniques of equilibrium thermal field theory are therefore inapplicable. Out of equilibrium, one typically proceeds using a functional Schrödinger equation or using the closed time path formalism of Mahanthappa, Schwinger...
and Keldysh \[8\] [10]. Here, we employ the latter, following closely the work of Boyanovsky, de Vega and coauthors [11,12].

Take \( t = t_0 \) as our starting time. Suppose that, at this time, the system is in a pure state, in which the measurement of \( \phi \) would give \( \Phi_0(x) \). That is:

\[
\hat{\phi}(t_0, x)|\Phi_0, t_0 \rangle = \Phi_0|\Phi_0, t_0 \rangle.
\]

The probability \( p_{tf}[\Phi_f] \) that, at time \( t_f > t_0 \), the measurement of \( \phi \) will give the value \( \Phi_f \) is

\[
p_{tf}[\Phi_f] = |c_f|^2,
\]

where:

\[
c_f = \int_{\phi(t_0) = \Phi_0} C \phi \exp\left\{ iS[\phi] \right\},
\]

in which \( C \phi = \prod_a D\phi_a \) and spatial labels have been suppressed. It follows that \( p_{tf}[\Phi_f] \) can be written in the closed time-path form:

\[
p_{tf}[\Phi_f] = \int_{\phi(t_0) = \Phi_0} C \phi \exp\left\{ i \left( S[\phi_+] - S[\phi_-] \right) \right\}.
\]

Instead of separately integrating \( \phi_\pm \) along the time paths \( t_0 \leq t \leq t_f \), the integral can be interpreted as time-ordering of a field \( \phi \) along the closed path \( C_+ \oplus C_- \) where \( \phi = \phi_+ \) on \( C_+ \) and \( \phi = \phi_- \) on \( C_- \).
FIGURES

\[ \text{FIG. 1. The closed time path contour } C_+ \oplus C_- \text{.} \]

It is convenient to extend the contour from \( t_f \) to \( t = \infty \). Either \( \phi_+ \) or \( \phi_- \) is an equally good candidate for the physical field, but we choose \( \phi_+ \):-

\[ \text{FIG. 2. Extending the integration contour.} \]

With this choice and suitable normalisation, \( p_{t_f} \) becomes:

\[
p_{t_f}[\Phi_f] = \int_{\phi_+ (t_0) = \Phi_0} D\phi_+ D\phi_- \delta[\phi_+ (t) - \Phi_f] \exp \left\{ i \left( S[\phi_+] - S[\phi_-] \right) \right\},
\]

where \( \delta[\phi_+ (t) - \Phi_f] \) is a delta functional, imposing the constraint \( \phi_+ (t, x) = \Phi_f (x) \) for each \( x \).

The choice of a pure state at time \( t_0 \) is too simple to be of any use. The one fixed condition is that we begin in a symmetric state with \( \langle \phi \rangle = 0 \) at time \( t = t_0 \). Otherwise, our ignorance is parametrised in the probability distribution that at time \( t_0 \), \( \phi(t_0, x) = \Phi(x) \). If we allow for an initial probability distribution \( P_{t_0}[\Phi] \) then \( p_{t_f}[\Phi_f] \) is generalised to:

\[
p_{t_f}[\Phi_f] = \int D\Phi P_{t_0}[\Phi] \int_{\phi_+ (t_0) = \Phi} D\phi_+ D\phi_- \delta[\phi_+ (t) - \Phi_f] \exp \left\{ i \left( S[\phi_+] - S[\phi_-] \right) \right\}.
\]

At this stage, we have to begin to make approximations. So that \( p_{t_f}[\Phi_f] \) shall be Gaussian, it is necessary to take \( P_{t_0}[\Phi] \) to be Gaussian also, with zero mean. All the cases that we might wish to consider are encompassed in the assumption that \( \Phi \) is Boltzmann distributed at time \( t_0 \) at an effective temperature of \( T_0 = \beta_0^{-1} \) according to a quadratic Hamiltonian \( H_0[\Phi] \). That is:

\[
P_{t_0}[\Phi] = \langle \Phi, t_0 | e^{-\beta H_0} | \Phi, t_0 \rangle = \int_{\phi_3 (t_0) = \Phi = \phi_3 (t_0 - i\beta_0)} D\phi_3 \exp \left\{ i S_0[\phi_3] \right\},
\]
for a corresponding action \( S_0[\phi_3] \), in which \( \phi_3 \) is taken to be periodic in imaginary time with period \( \beta_0 \). We take \( S_0[\phi_3] \) to be quadratic in the \( O(N) \) vector \( \phi_3 \) as:

\[
S_0[\phi_3] = \int d^{D+1}x \left[ \frac{1}{2} (\partial_\mu \phi_3 a)(\partial^\mu \phi_3 a) - \frac{1}{2} m_0^2 \phi_3^2 \right].
\]

We stress that \( m_0 \) and \( \beta_0 \) parametrise our uncertainty in the initial conditions. The choice \( \beta_0 \to \infty \) corresponds to choosing the \( p_t[\Phi] \) to be determined by the ground state functional of \( H_0 \), for example. Whatever, the effect is to give an action \( S_3[\phi] \) in which we are in thermal equilibrium for \( t < t_0 \) during which period the mass \( m(t) \) takes the constant value \( m_0 \) and, by virtue of choosing a Gaussian initial distribution, \( \lambda(t) = 0 \) for \( t < t_0 \).

We now have the explicit form for \( p_t[\Phi_f] \):

\[
p_{t_f}[\Phi_f] = \int D\Phi \int_{\phi_3(t_0) = \Phi} D\phi_3 e^{iS_0[\phi_3]} \int_{\phi_\pm(t_f) = \Phi} D\phi_+ D\phi_- e^{i(S[\phi_+] - S[\phi_-])} \delta[\phi_+(t_f) - \Phi_f] = \int_B D\phi_3 D\phi_+ D\phi_- \exp \left\{ iS_0[\phi_3] + i(S[\phi_+] - S[\phi_-]) \right\} \delta[\phi_+(t_f) - \Phi_f],
\]

where the boundary condition \( B \) is \( \phi_\pm(t_0) = \phi_3(t_0) = \phi_3(t_0 - i\beta_0) \). This can be written as the time ordering of a single field:

\[
p_{t_f}[\Phi_f] = \int_B D\phi e^{iS_C[\phi]} \delta[\phi_+(t_f) - \Phi_f],
\]

along the contour \( C = C_+ \oplus C_- \oplus C_3 \), extended to include a third imaginary leg, where \( \phi \) takes the values \( \phi_+ \), \( \phi_- \) and \( \phi_3 \) on \( C_+ \), \( C_- \) and \( C_3 \) respectively, for which \( S_C \) is \( S[\phi_+] \), \( S[\phi_-] \) and \( S_0[\phi_3] \).

![FIG. 3. A third imaginary leg](image)

We stress again that although \( S_0[\phi] \) may look like the quadratic part of \( S[\phi] \), its role is solely to encode the initial distribution of configurations \( \Phi \) and need have nothing to do with the physical action. Henceforth we drop the suffix \( f \) on \( \Phi_f \) and take the origin in time from which the evolution begins as \( t_0 = 0 \).

We perform one final manoeuvre with \( p_t[\Phi] \) before resorting to further approximation. This will enable us to avoid an ill-defined inversion of a two-point function later on. Consider the generating functional:

\[
Z[j_+, j_-, j_3] = \int_B D\phi \exp \left\{ iS_C[\phi] + j \int \phi \right\},
\]
where $\int j\phi$ is a short notation for:

$$\int j\phi \equiv \int_0^\infty dt \left[ j_+(t)\phi_+(t) - j_-\phi_-(t) \right] + \int_0^{-i\beta} j_3(t)\phi_3(t) dt,$$

omitting spatial arguments. Then introducing $\alpha_a(x)$ where $a = 1, \ldots, N$, we find:

$$p_{tf}[\Phi] = \int \mathcal{D}\alpha \int_B \mathcal{D}\phi \exp\left\{iS_C[\phi]\right\} \exp\left\{i \int d^4x \alpha_a(x)[\phi_+(t_f, x) - \Phi(x)]_a\right\}$$

$$= \int \mathcal{D}\alpha \exp\left\{-i \int \alpha_a\Phi_a\right\} Z[\alpha, 0, 0],$$

where $\alpha$ is the source $\alpha(t, x) = \alpha(x)\delta(t - t_f)$. As with $\mathcal{D}\phi$, $\mathcal{D}\alpha$ denotes $\prod_1^N \mathcal{D}\alpha_a$.

We have seen that analytic progress can only be made insofar as $p_{tf}[\Phi]$ is itself Gaussian, requiring in turn that $Z[\alpha, 0, 0]$ be Gaussian in the source $\alpha$. In order to treat the fall from the false into the true vacuum, at best this means adopting a self-consistent or variational approach i.e. a Hartree approximation or a large N-expansion. \[\text{[3]}\] However, if we limit ourselves to small times $t$ then $p_{tf}[\Phi]$ is genuinely Gaussian since the field has not yet felt the upturn of the potential. That is, we may treat the potential as an inverted parabola until the field begins to probe beyond the spinodal point. The length of time for which it is a good approximation to ignore the upturn of the potential is greatest for weakly coupled theories which, for the sake of calculation we assume, but physically, we expect that if the defect counting approximation is going to fail, then it will do so in the early part of the fall down the hill.

V. EVOLUTION OF THE DEFECT DENSITY

The onset of the phase transition at time $t = 0$ is characterised by the instabilities of long wavelength fluctuations permitting the growth of correlations. Although the initial value of $\langle \phi \rangle$ over any volume is zero, the resulting phase separation or spinodal decomposition will lead to domains of constant $\langle \phi \rangle$ whose boundaries will evolve so that ultimately, the average value of $\phi$ in some finite volume, will be non-zero. That is, the relativistic system has a non-conserved order parameter. In this sense, the model considered here is similar to those describing the $\lambda$ transition in liquid helium or transitions in a superconductors.

Consider small amplitude fluctuations of $\phi_a$, at the top of the parabolic potential hill described by $V(\phi) = \frac{1}{2}m^2(t)\phi_a^2$. At $t < 0$, $m^2(t) > 0$ and, for $t > 0$, $m^2(t) < 0$. However, by $t \approx \Delta t$, $m^2(t)$ and $\lambda$ have achieved their final values, namely $-\mu^2$ and $\lambda$. Long wavelength fluctuations, for which $|k|^2 < -m^2(t)$, begin to grow exponentially. If their growth rate $\Gamma_k \approx \sqrt{-m^2(t) - |k|^2}$ is much slower than the rate of change of the environment which is causing the quench, then those long wavelength modes are unable to track the quench. For the case in point, this requires $m\Delta t \ll 1$. We take this to be the case. To exemplify the growth of domains and the attendant dispersal of defects, it is sufficient to take the idealised

\[\text{[3]}\text{In the latter case, given the relationship between } N \text{ and spatial dimension } D = N, \text{ this corresponds to a large-dimension expansion}\]
case, $\Delta t = 0$ in which the change of parameters at $t = 0$ is instantaneous. That is, $m^2(t)$ satisfies:

$$m^2(t) = \begin{cases} m_0^2 > 0 & \text{if } t < 0, \\ -\mu^2 < 0 & \text{if } t > 0 \end{cases}$$

where for $t < 0$, the field is in thermal equilibrium at inverse temperature $\beta_0$. As for $\lambda(t)$, for $t < 0$, it has already been set to zero, so that $p_{\alpha_0} [\Phi]$ be Gaussian. For small $t$, when the amplitude of the field fluctuations is small, the field has yet to experience the upturn of the potential and we can take $\lambda(t) = 0$ then as well. At best, this can be valid until the exponential growth $|\phi| \approx \mu e^{\mu t}$ in the amplitude reaches the point of inflection $|\phi| \approx \mu/\sqrt{\lambda}$, that is $\mu t \approx O(\ln(1/\lambda))$. The smaller the coupling then, the longer this approximation is valid. As noted earlier, it should be possible to perform more sophisticated calculations with the aim of evolving the defect density right through the transition. For our present purposes, however, the small time or Gaussian approximation is adequate.

We are now in a position to evaluate $p_t[\Phi]$, identify $K$ and calculate the defect density accordingly. $S_C[\phi]$ becomes $S_0[\phi_3]$ on segment $C_3$ so setting the boundary condition $\phi_+(0, x) = \phi_3(0, x) = \phi_3(-i\beta_0, x)$ and we have:

$$S[\phi_+] = \int d^{D+1}x \left[ \frac{1}{2} (\partial_{\mu} \phi_a)(\partial^\mu \phi_a) + \frac{1}{2} \mu^2 \phi_a^2 \right],$$

on $C_+$. The Gaussian integrals can now be performed to give:

$$p_t[\Phi] = \int D\alpha \exp\left\{-i \int d^Dx \alpha_a \Phi_a \right\} \exp\left\{ i \int d^Dx d^Dy \alpha_a(x)G(x - y; t, t)\alpha_b(y) \right\},$$

where $G(x - y; t, t)$ is the equal time correlation, or Wightman, function with thermal boundary conditions. Because of the time evolution there is no time translation invariance in the double time label. As this is not simply invertible, we leave the $\alpha$ integration unperformed. The form is then a mnemonic reminding us that $K^{-1} = G$.

In fact, there is no need to integrate the $\alpha$ since from the previous equation it follows that the characteristic functional $\langle \exp \{ i \int j_a \Phi_a \} \rangle_t$ is directly calculable as:

$$\langle \exp \{ i \int j_a \Phi_a \} \rangle_t = \int D\Phi p_t[\Phi] \exp \{ i \int j_a \Phi_a \} = \exp \left\{ i \int d^Dx d^Dy j_a(x)G(x - y; t)j_a(y) \right\}.$$

Thus for example, the first factor in the monopole density $\rho(t)$ is:

$$\langle \delta^D[\Phi(x)] \rangle_t = \left\langle \int dj \exp(i\Phi_a(x)j_a) \right\rangle_t = \int dj \exp \left\{ \frac{1}{2} ij_a^2 G(0; t, t) \right\} = [-iG(0; t, t)]^{-D/2},$$

with suitable normalisation, without having to invert $G(0; t, t)$. Thus, on identifying $-iG(x; t, t)$ with $W(x, t)$ as defined earlier, $\rho(t)$ becomes:
\[ \langle \rho(t) \rangle = C_N \left| \frac{-iG''(0; t, t)}{-iG(0; t, t)} \right|^{N/2} \]

where \(-iG(x; t, t)\) has to be calculated from the equations of motion, subject to the initial condition.

Details are given by Boyanovsky et al., [12] and we quote their results, which give \(-iG(x; t, t)\) as the real, positive quantity:

\[-iG(x; t, t) = \int \frac{d^Dk}{2\omega_<(k)} e^{i\mathbf{k}\cdot\mathbf{x}} \coth(\beta_0 \omega_<(k)/2) \left\{ 1 + A_k \left( \cosh(2W(k)t) - 1 \right) \right\} \Theta(\mu^2 - |\mathbf{k}|^2)\]

with:

\[ \omega^2_<(k) = |\mathbf{k}|^2 + m_0^2 \]
\[ \omega^2_>(k) = |\mathbf{k}|^2 - \mu^2 \]
\[ W^2_<(k) = \mu^2 - |\mathbf{k}|^2 \]
\[ A_k = \frac{1}{2} \left( 1 + \frac{\omega^2_>(k)}{W^2(k)} \right) \]
\[ \alpha_k = \frac{1}{2} \left( 1 - \frac{\omega^2_>(k)}{\omega^2_<(k)} \right). \]

The first term is the contribution of the unstable long wavelength modes, which relax most quickly; the second is that of the short wavelength stable modes which provide the noise. The first term will dominate for large times and even though the approximation is only valid for small times, there is a regime, for small couplings, in which \(t\) is large enough for \(\cosh(2\mu t) \approx \frac{1}{2} e^{2\mu t}\) and yet \(\mu t\) is still smaller than the time \(O(\ln 1/\lambda)\) at which the fluctuations sample the deviation from a parabolic hill. In these circumstances the integral at time \(t\) is dominated by a peak in the integrand \(k^{D-1} e^{2W(k)t} \) at \(k\) around \(k_c\), where:-

\[ tk_c^2 = \frac{(D-1)}{2} \mu \left( 1 + O\left( \frac{1}{\mu t} \right) \right). \]

The effect of changing \(\beta_0\) is only visible in the \(O(1/\mu t)\) term. In the region \(|x| < \sqrt{t/\mu}\) the integral is dominated by the saddle-point at \(k_c\), to give:-

\[- - iG(x; t, t) = W(x; t) \approx W(0; t) \exp\left( -\frac{\mu x^2}{8t} \right) \sinc\left( \frac{|x|}{\sqrt{t/\mu}} \right), \]

for \(D = 3\), where:-

\[ W(0; t) \approx C e^{2\mu t} (\mu t)^{3/2}, \]

for some \(C\), which we don’t need to know. The exponential growth of \(G(0; t)\) in \(t\) reflects the way the field amplitudes fall off the hill \(\langle \Phi \rangle = 0\). It is sufficient for our purposes to retain \(D = 3\) only.
After symmetry breaking to $O(N-1)$ the mass of the Higgs is $m_H = \sqrt{2} \mu$ with cold correlation length $\xi(0) = m_H^{-1}$. On identifying $e^{-\mu x^2/8t}$ as $e^{-x^2/\xi^2(t)}$ we interpret:

$$\xi(t) = (8\sqrt{2})^{1/2} \sqrt{t \xi(0)},$$

as the size of Higgs field domains. This $t^{1/2}$ growth behaviour at early times is characteristic of relativistic systems (with a double time derivative) with a non-conserved order parameter.

To calculate the number density of defects at early times we have to insert this expression for $-iG$ or $W$ into the equations derived earlier. Expanding $W(x; t)$ as:

$$W(x; t) = W(0; t) \exp\left(-\frac{x^2}{\xi^2(t)}\right) \left(1 - \frac{4}{3} \frac{x^2}{\xi^2(t)} + O\left(\frac{x^2}{\xi^2(t)}\right)\right),$$

and substituting in () we find:

$$\langle \rho(t) \rangle = \frac{1}{\pi^2} \left(\frac{\sqrt{14/3}}{\xi(t)}\right)^3 \approx \frac{1.02}{\xi^3},$$

for an $O(3)$ theory with monopoles in three dimensions and:

$$\langle \rho(t) \rangle = \frac{1}{2\pi} \left(\frac{\sqrt{14/3}}{\xi(t)}\right)^2 \approx \frac{0.74}{\xi^2},$$

for an $O(2)$ theory with strings in three dimensions. The first observation is that the dependence of the density on time $t$ is only through the correlation length $\xi(t)$. As the domains of coherent field form and expand, the interdefect distance grows accordingly. This we would interpret as the domains carrying the defects along with them on their boundaries. Secondly, there is roughly one defect per coherence size, a long held belief for whatever mechanism. However, in this case the density is exactly calculable.

It is also possible to use Halperin’s results to calculate defect-defect correlation functions. For example, the monopole-monopole correlation function on scales larger than a coherence length is found to be:

$$\langle \rho_{\text{net}}(x) \rho_{\text{net}}(0) \rangle_t = \langle \rho(t) \rangle_t \delta(x) + g(x)_t,$$

where $g(x)_t$ is a measure of the screening of a monopole at the origin. Explicit calculation yields:

$$g(x)_t = -\frac{3\sqrt{2} \exp(-3x^2/\xi^2)}{8\pi^3 x^3 (2\sqrt{2} x/\xi)} \sin^3(2\sqrt{2} x/\xi).$$

VI. CONCLUSIONS

Under the conditions of a symmetry breaking phase transition, from $O(N)$ to $O(N-1)$, which proceeds by a rapid quench, we have derived expressions for the evolution of the defect density during the early part of the fall from the false vacuum to the true vacuum. Our
results confirm that the defects are, indeed, frozen into the field when it first goes out of thermal equilibrium, at least for strongly coupled theories. Further, there is approximately one defect per correlation volume for the time during which the approximation is valid. The \( t^{1/2} \) time-dependence of the correlation-length \( \xi(t) \) that we have seen above is specific to a non-conserved order parameter in a theory with a double time derivative, but we expect the qualitative features to be similar for all defect production by quenched symmetry breaking phase transitions.

Thus, generically, we expect the defect density to follow the correlation length during the early part of the fall from the hill. Further, we expect the correlation length to grow more during this fall for weakly coupled theories, since the regime before which the fields feel the upturn in potential and slow down is longer. For very weakly coupled theories the correlation length can grow significantly in the time interval \( t = O(ln(1/\lambda)) \) available. In almost all physically realistic scenarios except inflation, however, the coupling is not small enough for this growth to be significant. For more strongly coupled theories we know less. Whatever, it is quite reasonable to say that the defect density is fixed at the time when the scalar field first goes out of equilibrium.

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\(^4\)although see the self-consistent calulations of Boyanovsky et al.
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