C1,1 SOLUTION OF THE DIRICHLET PROBLEM FOR DEGENERATE k-HESSIAN EQUATIONS

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Abstract. In this paper, we prove the existence of C1,1-solution to the Dirichlet problem for degenerate elliptic k-Hessian equations $S_k[u] = f$ under a condition which is weaker than the condition $f^{1/k} \in C^{1,1}(\Omega)$.

1. Introduction

In this work, we study the following Dirichlet problem for the k-Hessian equation

\begin{equation}
\begin{cases}
S_k[u] = f(x) & \text{in } \Omega, \\
u = \varphi(x) & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $S_k[u]$ is defined as follow

\begin{equation}
S_k[u] = \sigma_k(\lambda), \quad k = 1, \ldots, n,
\end{equation}

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i$ is the eigenvalue of the Hessian matrix $(D^2u)$, and

\begin{equation}
\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}
\end{equation}

is the $k$-th elementary polynomial. Note that the case $k = 1$ corresponds to the Possion’s equation, while for $k = n$, that is the Monge-Ampère equation $\det D^2u = f$.

The nonlinear equation of (1.1) is referred to as non-degenerate when the function $f$ is positive, it is degenerate elliptic if $f$ is non-negative and allowed to vanish somewhere in $\Omega$.

The non-degenerate k-Hessian equations were firstly studied by Caffarelli, Nirenberg and Spruck [1]. They proved that if $f \in C^{1,1}(\overline{\Omega})$, $f > 0$, $\partial \Omega$ and $\varphi$ were sufficiently smooth, (1.1) had a unique $C^{3, \alpha}$ $k$-admissible solution. For the degenerate case, Ivochina, Trudinger and Wang [9] studied a class of fully nonlinear degenerate elliptic equations which depended only on the eigenvalues of the Hessian matrix. This kind of equations include the k-Hessian equations. They got the priori estimate with the condition $f^{1/k} \in C^{1,1}(\overline{\Omega})$. In particular, their estimation of second order derivatives was independent with $\inf_{\Omega} f$. Thus, the condition $f^{1/k} \in C^{1,1}(\overline{\Omega})$ implied the existence of $C^{1,1}$-solutions to the degenerate k-Hessian equations. Then, the regularity of the degenerate k-Hessian equations paused at

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For Monge-Ampère equations, Hong, Huang and Wang [6] gave a special condition to the smooth solution for the 2-dimensional Monge-Ampère equation. We can find that even if is analytic, the solution may be not in $C^2$. For degenerate $k$-Hessian equations, some papers concentrated on the convexity of the solutions [7].

In this work, we want to improve these results of $C^1$ regularity with a condition weaker than $f^{1/k} \in C^{1,1}(\Omega)$. To state our results, we set the following condition for the function $f$ which is the right hand side term of the equations.

**Condition (H)**: Assume that $f \in C^{1,1}(\Omega)$, $f \geq 0$ and there exists a constant $C_0 > 0$ such that

$$|Df(x)| \leq C_0 f^{1 - \frac{1}{k}}(x) \quad \forall x \in \Omega,$$

and for any vector $\xi \in S^{n-1}$,

$$f(x)f_{\xi\xi}(x) - (1 - \frac{1}{k})f^2_\xi(x) \geq -C_0 f^{2 - \frac{1}{k}}(x) \quad \forall x \in \Omega,$$

where $f_\xi(x) = \frac{\partial f}{\partial \xi}(x), f_{\xi\xi}(x) = \frac{\partial^2 f}{\partial \xi^2}(x)$.

We will show that Condition (H) is weaker than $f^{1/k} \in C^{1,1}(\Omega)$ in Section 2. Indeed, for the case of 3-dimension we can give an example that $f \geq 0$ is analytic and $f^{1/2}$ is only Lipshitz continuous, while $f$ satisfies Condition (H).

Our main result is stated as follow.

**Theorem 1.1.** Assume that $\Omega$ is a bounded $(k - 1)$-convex domain in $\mathbb{R}^n$ with $C^{3,1}$ boundary $\partial \Omega$, $f \geq 0$, $f$ satisfies Condition (H), and $\varphi \in C^{3,1}(\partial \Omega)$. Then the Dirichlet problem (1.1) has a unique $k$-admissible solution $u \in C^{1,1}(\Omega)$. Moreover,

$$\|u\|_{C^{1,1}(\Omega)} \leq C,$$

where $C$ depends only on $n, k, \Omega, \|f\|_{C^{1,1}(\Omega)}, \|\varphi\|_{C^{3,1}(\partial \Omega)}$ and $C_0$. In particular, $C$ is independent with $\inf_{\Omega} f$.

We will recall the notions of $(k - 1)$-convexity and $k$-admissibility in Section 2.

In the paper [3], for the degenerate Monge-Ampère equations, P. Guan introduced a condition weaker then $f^{1/n} \in C^{1,1}(\Omega)$. So that our condition (H) is an extension of Guan’s condition to $k$-Hessian equation.

In the following Section 2 we will recall some definitions and some known results, then we will give the sketch of the proof to the main theorem. Then, the rest of this paper (Section 3 to Section 5) is to establish the uniform a priori estimates for the approximate solutions.

## 2. Sketch of the proof to the Main Theorem

In this section, we firstly recall some definitions and known results. Then we will present the sketch of the proof of Theorem 1.1.
Preliminaries. Firstly, we recall some definitions about the $k$-Hessian equations.

**Definition 2.1** ([12]). We say a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is $k$-admissible if

$$\lambda(D^2u) \in \Gamma_k,$$

where $\Gamma_k$ is an open symmetric convex cone in $\mathbb{R}^n$, with vertex at the origin, given by

$$\Gamma_k = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \forall j = 1, \ldots, k\},$$

where $\sigma_j(\lambda)$ is defined by (1.2).

The geometry condition for $\Omega \subset \mathbb{R}^n$ is (See [1]),

**Definition 2.2.** We say that $\Omega$ is $(k-1)$-convex if there exists a constant $c > 0$, such that, for any $x \in \partial \Omega$

$$\sigma_j(\kappa)(x) \geq c > 0 \quad j = 1, \ldots, k - 1,$$

where $\kappa = (\kappa_1, \ldots, \kappa_{n-1})$, $\kappa_i(x)$ is the principal curvature of $\partial \Omega$ at $x$. When $k = n$, it is the usual convexity.

The weak solution to the $k$-Hessian equation is defined as follow.

**Definition 2.3** ([11]). A function $u \in C^0(\Omega)$ is called an admissible weak solution of equation (1.1) in the domain $\Omega$, if there exists a sequence $\{u_m\} \subset C^2(\Omega)$ of $k$-admissible functions such that

$$u_m \to u \quad \text{in} \quad C^0(\Omega), \quad S_k(u_m) \to f \quad \text{in} \quad L^1_{\text{loc}}(\Omega).$$

**Condition (H).** For our Condition (H), by simple computation, we have

**Lemma 2.4.** Assume that $f \in C^{1,1}(\overline{\Omega}), f \geq 0$:

(i) if $f^{1/k} \in C^{1,1}(\overline{\Omega})$, then, $f$ satisfies Condition (H).

(ii) if $f \geq \delta_0 > 0$ and $f$ satisfies Condition (H), then, $f^{1/k} \in C^{1,1}(\overline{\Omega})$;

(iii) if $f$ satisfies Condition (H), then for any small $\epsilon > 0$, $f + \epsilon$ satisfies Condition (H) with the same constant $C_0$.

The following interesting example shows the signification of our Condition (H).

**Example.** Let $\Omega = B_1(0) = \{(x, y, z) \in \mathbb{R}^3; \ |(x, y, z)| < 1\}$ and

$$u = (x^2 + y^2 + z^2)^{3/2} - 1.$$

Then, we have

$$\begin{cases} S_2[u] = 45(x^2 + y^2 + z^2) & \text{in} \ \Omega, \\ u = 0 & \text{on} \ \partial \Omega. \end{cases}$$

Remind that $u \in C^{2,1}(\overline{\Omega})$ and $u \notin C^3(\overline{\Omega})$. In particular, $f = 45(x^2 + y^2 + z^2)$ is analytic and $f^{1/2}$ is only Lipschitz continuous near the origin. However, $f$ satisfies Condition (H). Indeed, $f$ is radially symmetric. Then, we may choose $\xi = (1, 0, \ldots, 0)$, for any positive constant $C_0$,  

$$f(x)f_{\xi\xi}(x) - (1 - \frac{1}{2})f_{\xi}^2(x) = \frac{90^2}{2}(x^2 + y^2 + z^2) - \frac{90^2}{2}x^2 \geq 0 \quad \geq -C_0(45(x^2 + y^2 + z^2))^{3/2} = -C_0f^{2-1/2}. $$
Sketch of the proof to Theorem 1.1. The proof of the Theorem 1.1 is standard by using a non-degenerate approximation and a uniform à priori estimate of approximate solution.

For the non-degenerate equation, we have the following existence result of Caffarelli, Nirenberg, Spruck [4] and Trudinger [10].

**Proposition 2.5.** Assume that \( \Omega \) is a bounded \((k-1)\)-convex domain in \( \mathbb{R}^n \) with \( C^{3,1} \) boundary \( \partial \Omega \), \( f^{1/k} \in C^{1,1}(\Omega) \) and \( f \geq \delta_0 \) for some positive constant \( \delta_0 \), \( \varphi \in C^{3,1}(\partial \Omega) \). Then the Dirichlet problem (1.1) has a unique \( k \)-admissible solution \( u \in C^{3,\alpha}(\Omega) \). Moreover,

\[
\|u\|_{C^{3,\alpha}} \leq C,
\]

where \( \alpha \in (0,1) \), \( C \) depends only on \( n, k, \alpha, \delta_0, \Omega, \|\varphi\|_{C^{3,1}(\partial \Omega)} \), and \( \|f\|_{C^{1,1}(\Omega)} \).

Now we study the following approximate problem with \( \vartheta > 0 \),

\[
\begin{aligned}
S_k[u^\vartheta] &= f + \vartheta & \text{in } \Omega, \\
u^\vartheta &= \varphi & \text{on } \partial \Omega.
\end{aligned}
\]

(2.1)

We will prove the following uniform à priori estimate.

**Theorem 2.6** (à priori estimate). Assume that \( \Omega \) is a bounded \((k-1)\)-convex domain in \( \mathbb{R}^n \) with \( C^{3,1} \) boundary \( \partial \Omega \), \( f \) satisfies Condition (H), \( \varphi \in C^{3,1}(\partial \Omega) \). Let \( u^0 \in C^{3,1}(\overline{\Omega}) \) be a \( k \)-admissible solution to the Dirichlet problem (2.1). Then, we have the following à priori estimate

\[
\|u^0\|_{C^{1,1}(\overline{\Omega})} \leq C,
\]

where \( C \) depends only on \( n, k, \Omega, \|f\|_{C^{1,1}(\overline{\Omega})}, \|\varphi\|_{C^{3,1}(\partial \Omega)} \) and \( C_0 \). In particular, \( C \) is independent with \( \vartheta \).

The proof of the theorem above will be our main task for the rest of this paper. Now we explain how to prove Theorem 1.1 through Theorem 2.6.

**Proof of Theorem 1.1.** As we have said previously, we complete the proof of the existence theorem by an approximation of non-degenerate problems, (See [3] and [4]). Since \( f \) satisfies Condition (H), \( f + \vartheta \geq \vartheta > 0 \). By Lemma 2.4 \((f + \vartheta)^{1/k} \in C^{1,1}(\overline{\Omega}) \). Thus, by Proposition 2.5 Theorem 2.6 and the continuity method in [2] and [3], the equation (2.1) has a \( k \)-admissible solution \( u^0 \) which belongs to \( C^{3,\alpha}(\overline{\Omega}) \) and satisfies the following estimate

\[
\|u^0\|_{C^{1,1}} \leq C
\]

with \( C \) depends only on \( n, k, \Omega, \|f\|_{C^{1,1}(\overline{\Omega})}, \|\varphi\|_{C^{3,1}(\partial \Omega)} \) and \( C_0 \), in particular, \( C \) is independent with \( \vartheta \). Then, the Azelà-Ascoli Theorem implies that the Dirichlet problem (1.1) admits a \( k \)-admissible weak solution \( u \in C^{1,1}(\overline{\Omega}) \). The uniqueness from the comparison principles of fully nonlinear degenerate elliptic equations. \( \Box \)
3. Interior à Priori estimates

Without any difficulties, we can get the $L^\infty$ estimate and the gradient estimate of the approximate solution similar to the papers [1] and [12].

**Proposition 3.1.** Assume that $\Omega$ is a bounded $(k-1)$-convex domain in $\mathbb{R}^n$ with $C^{3,1}$ boundary. $f$ satisfies Condition (H), $\varphi \in C^{3,1}(\partial \Omega)$. Let $u^\vartheta$ be a $k$-admissible solution to the Dirichlet problem (2.1). Then we have

$$\|u^\vartheta\|_{C^1(\overline{\Omega})} \leq C,$$

where $C$ depends only on $\|\varphi\|_{C^1(\partial \Omega)}$, $\|f\|_{C^1(\overline{\Omega})}$, $C_0$ and $\Omega$, in particular, $C$ is independent with $\vartheta$.

Now we consider the second order derivative estimates of $u^\vartheta$ which suffice to prove the following one side estimate,

$$u^\vartheta_{\xi\xi}(x) \leq C \quad \forall (x, \xi) \in \bar{\Omega} \times S^{n-1}.$$

Since we have already known that $\Delta u^\vartheta \geq 0$ by the definition of the $k$-admissibility, then by a rotation, we obtain,

$$|u^\vartheta_{\xi\xi}(x)| \leq nC.$$

Thus, if we proved (3.1), we could finish the whole proof of Theorem 2.6.

Now we give a relationship between the boundary estimate and the interior estimate. That is

**Lemma 3.2.** Assume that $\Omega$ is a $(k-1)$-convex domain in $\mathbb{R}^n$ with $C^{3,1}$ boundary. $f$ satisfies the condition (H), $\varphi \in C^{3,1}(\partial \Omega)$. Let $u^\vartheta$ be a $k$-admissible solution to the Dirichlet problem (2.1). Then we have

$$\sup_{\Omega} u^\vartheta_{\eta\eta} \leq C + \sup_{\partial \Omega} u^\vartheta_{\eta\eta}$$

for any unit vector $\eta \in S^{n-1}$, where $C$ depends only on $\Omega$ and $C_0$, in particular, $C$ is independent with $\vartheta$.

**Proof.** Denote

$$F[D^2 u^\vartheta] = \left(S_k(u^\vartheta)\right)^{1/k} = (f + \vartheta)^{1/k}.$$

Then, differentiating (3.3) in direction $\eta \in S^{n-1}$ twice, one can verify that

$$F_{ij} \vartheta_{ij\eta\eta} + F^i_{ij, st} \vartheta_{ij\eta} u^\vartheta_{st\eta} = \frac{1-k}{k^2} (f + \vartheta)^{1/k - 2} f_{\eta\eta}^2 + \frac{1}{k} (f + \vartheta)^{1/k - 1} f_{\eta\eta},$$

where

$$F_{ij} = F_{u_{ij}} = \frac{\partial F(u^\vartheta)}{\partial u_{ij}^\vartheta}, \quad F^i_{ij, st} = F_{u_{ij}^\vartheta u_{st}^\vartheta} = \frac{\partial^2 F(u^\vartheta)}{\partial u_{ij}^\vartheta \partial u_{st}^\vartheta}.$$

From [1] and [12], we have known that $F$ is a concave operator, thus,

$$F_{ij} \vartheta_{ij\eta\eta} \geq \frac{1-k}{k^2} (f + \vartheta)^{1/k - 2} f_{\eta\eta}^2 + \frac{1}{k} (f + \vartheta)^{1/k - 1} f_{\eta\eta}.$$

That is

$$\vartheta_{\eta\eta} \geq \frac{1-k}{k} f_{\eta\eta}^2 + (f + \vartheta) f_{\eta\eta},$$
where $\mathcal{L} = (f + \vartheta) \cdot S_k^{ij}[u^\vartheta] \partial_i \partial_j$. Next, let $w = \frac{1}{2}a|x|^2$, we have, by Maclaurin’s inequality
\[
\mathcal{L}w = a(n + 1 - k)\sigma_k - 1 [\lambda(D^2u^\vartheta)] \cdot (f + \vartheta)^{2-1/k},
\]
where $C_{n,k}$ is a constant depends only on $n$ and $k$. By Condition (H), we choose $a$ so large that
\[
a \cdot C_{n,k} \geq C_0,
\]
then,
\[
\mathcal{L}[w + u^\vartheta_{\eta\eta}] \geq C_0(f + \vartheta)^{2-1/k} + \frac{1 - k}{k} f^2 + (f + \vartheta)f_{\eta\eta} \geq 0
\]
holds. By the classical weak maximum principle,
\[
\sup_{\Omega}(w + u^\vartheta_{\eta\eta}) \leq \sup_{\partial\Omega}(w + u^\vartheta_{\eta\eta}).
\]
That is (3.2), proof is done.

4. The Weakly Interior Estimate

Since we have completed the double tangent derivative estimate and the mix type derivative estimate by Proposition 3.3, we just need to study the double normal derivative estimate. We can assume that
\[
\max\{\sup_{\partial\Omega}\|u^\vartheta\|, \sup_{\partial\Omega}\|Du^\vartheta\|, \sup_{\partial\Omega}\|D^2u^\vartheta\|\} = \sup_{x \in \partial\Omega} u^\vartheta_{\gamma\gamma}(x),
\]
where $\gamma \in \mathbb{S}^{n-1}$ is normal vector of $\partial\Omega$ at $x \in \partial\Omega$. If (4.1) did not hold, we should have finished the proof by Proposition 3.3. Besides, we also assume $\sup_{x \in \partial\Omega} u^\vartheta_{\gamma\gamma}(x) > 0$.

The double normal derivative estimate will be established by two steps. The first step is the following weakly interior estimate.
Lemma 4.1. Assume that function \( u^\theta \in C^{3,1}(\Omega) \) satisfies (2.1), \( \Omega \) is \((k - 1)\)-convex, \( \partial \Omega \in C^{3,1}, \varphi \in C^{3,1}(\partial \Omega) \), \( f \) satisfies Condition (H). Then we have a weakly interior estimate. That is, for some sufficiently small constant \( \epsilon \) and some \( \delta > 0 \), there exists a constant \( C_{\epsilon, \delta} \) such that, for any \( \xi \in \mathbb{R}^n \),

\[
\sup_{x \in \Omega_\delta} u^\theta_{\xi\xi}(x) \leq \epsilon \sup_{x \in \partial \Omega} u^\theta_{\gamma\gamma}(x) + C_{\epsilon, \delta} |\xi|^2,
\]

where \( \Omega_\delta = \{ x \in \Omega | \text{dist}(x, \partial \Omega) \geq \delta \} \), \( \gamma \) is the unit inner normal vector, \( C_{\epsilon, \delta} \) depends only on \( n, \varphi, \partial \Omega, C_0 \) and \( |f|_{C^{1,1}(\Omega)} \), in particular, \( C_{\epsilon, \delta} \) is independent with \( \vartheta \).

To prove the proposition above, we will use the following maximum principle.

Proposition 4.2 (H). Let \( A = A(x, \hat{\xi}) \) be a \((n + N) \times (n + N)\) matrix which is positive definite, \( b = b(x, \hat{\xi}) \) be a \((n + N)\)-dimensional vector in \( Q \), where \( Q =: \Omega \times \mathbb{R}^N, x \in \Omega, \hat{\xi} \in \mathbb{R}^N \), such that

\[
L[w] := \text{tr}(AD^2w) + <b, Dw> \tag{4.3}
\]

is an elliptic operator in \( Q \), where \( \text{tr}(AD^2w) \) is the trace of matrix \( (AD^2w), <, > \) is the inner product. Assume that \( g, h \in C^2(Q), h > 0 \), and \( g, h \) are \( p \)-homogeneous in \( \hat{\xi} \) for some \( p > 1 \). If there exists positive constants \( \mu, \nu \) such that

\[
L[g] \geq -\mu |\hat{\xi}|^p \quad \forall (x, \hat{\xi}) \in \Omega \times \mathbb{R}^N, \tag{4.4}
\]

\[
L[h] \leq -\nu |\hat{\xi}|^p \quad \forall (x, \hat{\xi}) \in \Omega \times \mathbb{R}^N. \tag{4.5}
\]

Then, we have

\[
\sup_{\Omega \times \{ |\xi| = 1 \}} \frac{g}{h} \leq \frac{\mu}{\nu} + \sup_{\partial \Omega \times \{ |\xi| = 1 \}} \frac{g}{h}.
\]

Next, we construct the operator \( L \) which has the form (1.9), the functions \( g \) and \( h \) which are applicable to Proposition 4.2. Denote \( \hat{\xi} = (\xi, \xi_0), \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n, \xi_0 \in \mathbb{R} \). That is \( Q \ni (x, \xi) = \Omega \times \mathbb{R}^N =: \Omega \times \mathbb{R}^{n+1} \). The matrix \( A \) and vector \( b \) are given by the following formula:

\[
A = \begin{pmatrix} A & rA & Aq \\ rA & r^2A & rrAq \\ q^*A & rq^*A & q^*Aq \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ -2Aq \\ 0 \end{pmatrix},
\]

where

\[
A = \frac{1}{\text{tr}(F)} F = (a^{ik}), (F^{ij})_{n \times n} = F,
\]

and

\[
r = \frac{\alpha \psi}{\psi + \beta},
\]

\[
q = \frac{1}{\psi + \beta} \left( \frac{\xi}{4} + \frac{\alpha \psi}{\psi + \beta} D\psi \right),
\]
0 < \alpha < 1/2, 0 < \beta \ll 1 and 0 < \bar{\varepsilon} \ll \beta^4. \psi is a special auxiliary function (see Section 2 of [1] and the Lemma 3.3 of [8]) which is non-negative and $C^\infty$ on $\Omega$ such that

$$|D\psi| \geq n \text{ on } \partial\Omega, \quad \text{tr}(AD^2\psi) \leq -n \text{ in } \Omega.$$  

The functions $g$ and $h$ are defined as follows

$$g = u_\xi^\theta + 2\xi_0 u_\xi^\theta + \xi_0^2 u^\theta,$$

$$h = \frac{1}{4\alpha}(\psi + \beta)^{1-\alpha}|\xi|^2 + \frac{\psi_\xi^2}{(\psi + \beta)^\alpha} + \bar{\varepsilon}(\psi + \beta)\xi_0^2.$$ Immediately, $g$ and $h$ are 2-homogeneous in $\tilde{\xi}$. In order to apply Proposition 4.2, we need to confirm assumptions (4.4) and (4.5). Those are the following two lemmas.

For (4.4), we have

Lemma 4.3. To the function $g$ defined as (4.7), we have (4.4) holds, that is

$$\mathcal{L}[g] \geq -\mu|\tilde{\xi}|^2 \quad \forall (x, \tilde{\xi}) \in \Omega \times \mathbb{R}^N,$$

where $\mu$ can be $\beta^{-5}$.

Proof. We have

$$\mathcal{L}[g] = a^{ij}[u^\theta_{ij\xi\xi} + (4r + 2\xi_0)u^\theta_{ij\xi} + (2r^2 + 4r\xi_0 + \xi_0^2)u^\theta_{ij} + 4ru_i^\theta q_j + 2q_i q_j u^\theta]$$

$$= a^{ij}(\tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5).$$

For each term of $\mathcal{L}[g]$, we have

$$a^{ij}\tilde{I}_1 = \text{tr}(AD^2u^\theta_{\xi\xi}) \geq -K|\xi|^2,$$

$$|a^{ij}\tilde{I}_2| \leq |4rK + 2K\xi_0||\xi| \leq K(|\xi|^2 + \xi_0^2) + \frac{C(\alpha, K, c_0)}{\beta}|\xi|^2,$$

where

$$c_0 = \|\psi\|_{C^3(\Omega)}, \quad K = \max\{|f|^{1/k}_{C^{1,1}(\Omega)}, C_0\}.$$ For $a^{ij}\tilde{I}_3$, since what we need is the lower bound, we only need to estimate $4r\xi_0 u^\theta_{ij} a^{ij}$, that is

$$4r\xi_0 u^\theta_{ij} a^{ij} \leq \frac{C(\alpha, K, c_0)}{\beta}(|\xi|^2 + \xi_0^2).$$

For the left terms, one can verify that

$$|a^{ij}\tilde{I}_4| \leq 4a^{ij} u_i^\theta \frac{\alpha\psi_\xi}{(\psi + \beta)^2} \left(\frac{\xi^j}{4} + \frac{\alpha\psi_j\psi_\xi}{\psi + \beta}\right)$$

$$\leq \frac{C(n, \alpha, c_0, \|u\|_{L^\infty})}{\beta^3}|\xi|^2,$$

and

$$|a^{ij}\tilde{I}_5| \leq |2u^\theta \cdot q_i^2| \leq \frac{2\|u^\theta\|_{L^\infty}}{\beta^2} \left(\frac{1}{16} + \frac{n^2\alpha^2}{\beta^2}c_0^4 + \frac{nan^2c_0^2}{2\beta^3}\right)|\xi|^2$$

$$\leq \frac{C(n, \alpha, c_0, \|u\|_{L^\infty})}{\beta^4}|\xi|^2.$$
Thus,
\[
\mathcal{L}[g] \geq -(2K + \frac{C(\alpha, K, c_0)}{\beta} + \frac{C(n, \alpha, c_0, \|u\|_{L^\infty})}{\beta^3} + \frac{C(n, \alpha, c_0, \|u\|_{L^\infty})}{\beta^4})|\xi|^2 - (K + \frac{C(\alpha, K, c_0)}{\beta})\xi_0^2. 
\]
Since $\beta$ is sufficiently small, we may choose
\[(4.9) \quad \mu = \beta^{-5}.
\]
Thus, we have
\[
\mathcal{L}[g] \geq -\mu(|\xi|^2 + \xi_0^2) = -\mu|\tilde{\xi}|^2.
\]
Proof is done. \(\square\)

Then, we state (4.5) as follow.

**Lemma 4.4.** To the function $h$ defined as (4.8), we have
\[
\mathcal{L}[h] \leq -\nu|\tilde{\xi}|^2 \quad \forall (x, \tilde{\xi}) \in \Omega \times \mathbb{R}^N,
\]
where
\[
\nu = \min \left\{ \frac{1}{16\alpha(K_0 \delta + \beta)^\alpha}, \frac{\xi}{2} \right\}.
\]

**Proof.** We divide $h$ into two parts: $\hat{h}$ and $h - \hat{h}$. $\hat{h}$ is defined as
\[
\hat{h} = \frac{1}{4\alpha} (\psi + \beta)^{1-\alpha} \xi^2 + \frac{\psi^2}{(\psi + \beta)^\alpha}.
\]
We have
\[
\mathcal{L}[\hat{h}] = tr(A \cdot D^2 \hat{h}) + < b, D\hat{h} >
\]
\[
= \frac{1}{N_0(\psi + \beta)^{\alpha+1}} \left\{ F^{ij}(\psi + \beta)I_1 + F^{ij}I_2 + \frac{F^{ij}}{(\psi + \beta)}I_3 
+ F^{ij}(\psi + \beta)(-\frac{\xi_i \xi_j}{4\alpha}) + F^{ij}(-\alpha - 1)\psi_i \psi_j \xi_j \right\},
\]
where
\[
I_1 = \frac{1 - \alpha}{4\alpha} \psi_{ij} |\xi|^2 + 2\psi_i \cdot \psi_j + 2\psi_i \psi_{ij},
I_2 = 3\alpha \psi_{ij} \cdot \psi^2 - \frac{1 - \alpha}{4} \psi_i \psi_j |\xi|^2 + \frac{\alpha^2}{2} \psi^2 \delta_{ij},
I_3 = -3\alpha \psi_i \psi_j \psi^2 - \alpha^2 \psi_i \psi_j \psi^2.
\]
To $F^{ij}I_1$, we have, by (4.6),
\[
\frac{1 - \alpha}{4\alpha} F^{ij} \psi_{ij} |\xi|^2 \leq \frac{1 - \alpha}{4\alpha} N_0 |\xi|^2, \quad 2F^{ij} \psi_i \psi_j \xi_j \leq 2N_0 c_0 |\xi|^2,
2F^{ij} \psi_i \psi_j \xi_j \leq 2n^2 N_0 \|D^3 \psi\|_{L^\infty} |\xi|^2 \leq 2n^2 N_0 c_0^2 |\xi|^2.
\]
Thus, we have
\[(4.10) \quad F^{ij}I_1 \leq \left[ -\frac{1 - \alpha}{4\alpha} + (2n^2 + 2)c_0^2 \right] N_0 |\xi|^2.
\]
To $F^{ij}I_2$, one can verify that, by (4.6)

$$3\alpha F^{ij}\psi_{ij}\psi_\xi^2 + \frac{\alpha}{2}\psi_\xi^2\delta_{ij} F^{ij} \leq (-3n\alpha\psi_\xi^2 + \frac{n\alpha}{2}\psi_\xi^2)N_0 \leq 0,$$

Thus, we have

(4.11) \quad $F^{ij}I_2 \leq 0.$

For the left terms, we have

(4.12) \quad $- (3\alpha + \alpha^2)\frac{F^{ij}}{\psi + \beta}\psi_{ij}\psi_\xi^2 - F^{ij}(\psi + \beta)\frac{\xi_{ij}}{4\alpha} - (1 + \alpha)F^{ij}\psi_{ij}\xi_j \leq 0.$

In sum of the computation (4.10), (4.11) and (4.12),

$$\mathcal{L}[\hat{h}] \leq \frac{1}{(\psi + \beta)^{\alpha}} \left[ \frac{1}{4\alpha} - (2n^2 + 2)c_0^2 \right] |\xi|^2.$$

We can find a constant $\alpha_0$ which depends on $c_0$ such that, for $0 < \alpha \leq \alpha_0$

$$- \frac{1}{4\alpha} + (n^2 + 2)c_0^2 \leq - \frac{1}{8\alpha}.$$

Thus, we have

(4.13) \quad $\mathcal{L}[\hat{h}] \leq - \frac{|\xi|^2}{8\alpha(K_0\delta + \beta)^{\alpha}}.$

For the left terms, we have

$$\mathcal{L}[\hat{\varepsilon}(\psi + \beta)\xi_0^2] \leq \hat{\varepsilon} \left[ -\xi_0^2 + \frac{C(n, \alpha, c_0)}{\beta^2 m}\xi_0^2 + \frac{C(n, \alpha, c_0)m}{\beta^2} |\xi|^2 \right],$$

where $m$ is sufficiently large.

We choose $m$ so large that

$$-1 + \frac{C(n, \alpha, c_0)}{\beta^2 m} \leq - \frac{1}{2}.$$

Still, we can choose $\hat{\varepsilon}$ sufficiently small such that

$$\hat{\varepsilon} \frac{C(n, \alpha, c_0)m}{\beta^2} \leq \frac{1}{16\alpha(K_0\delta + \beta)^{\alpha}}.$$

Thus, we claim, by (4.13),

$$\mathcal{L}[\hat{h}] \leq - \frac{1}{16\alpha(K_0\delta + \beta)^{\alpha}} |\xi|^2 - \frac{\hat{\varepsilon}^2}{2c_0^2}.$$

We denote

(4.14) \quad $\nu = \min \left\{ \frac{1}{16\alpha(K_0\delta + \beta)^{\alpha}} \frac{\hat{\varepsilon}}{2}, \frac{\hat{\varepsilon}^2}{2c_0^2} \right\},$

we may choose $\nu = \beta^3$, hence

$$\mathcal{L}[\hat{h}] \leq -\nu(|\xi|^2 + \xi_0^2).$$

By Lemma 4.3 and Lemma 4.4, we can establish the proof of our weakly interior estimate.
Proof of Lemma 4.1. By the virtue of Proposition 4.2

\begin{equation}
\sup_{\Omega \times \{\xi = 1\}} \frac{g}{h} \leq \sup_{\partial \Omega \times \{\xi = 1\}} \frac{g}{h} + \frac{\mu}{\nu}.
\end{equation}

As we have proved, \(\xi\) depends on \(\|f\|_{C^{1,1}}, C_0, \|\varphi\|_{L^\infty}, \|\psi\|_{C^3}, \delta\) and \(\|\rho\|_{C^{2,1}}\).

Next, we estimate \(\sup_{\partial \Omega \times \{\xi = 1\}} (g/h)\).

Let \(z = (x, \xi, \xi_0) \in \partial \Omega \times \{\xi = 1\}\), \(\xi^2 + \xi_0^2 = 1\) such that

\begin{equation}
\frac{g}{h}(z) \geq \frac{1}{2} \sup_{\partial \Omega \times \{\xi = 1\}} \frac{g}{h}.
\end{equation}

Denote \(\hat{\theta} = \langle \gamma, \xi \rangle = |\xi| \cos \theta\), where \(\gamma\) is the unit normal of \(\partial \Omega\) at \(x\). Thus, at \(z\) we have, there exists a unit tangent vector \(\tau\) such that

\[\xi = \gamma \cdot |\xi| \cos \theta + \tau \cdot |\xi| \sin \theta.\]

Then,

\begin{equation}
u_{\xi \xi} = u_{\gamma \gamma} |\xi|^2 \cos^2 \theta + u_{\gamma \tau} |\xi|^2 \sin^2 \theta + 2 u_{\gamma \tau} |\xi|^2 \sin \theta \cdot \cos \theta
\end{equation}

\begin{equation}
\leq 2 \sup_{\partial \Omega} u_{\gamma \gamma} \cdot \hat{\theta}^2 + 2 ||u^0||_{C^1}(1 - \hat{\theta}^2),
\end{equation}

Thus, we have

\[g \leq 2 \hat{\theta}^2 \sup_{\partial \Omega} u_{\gamma \gamma} + n ||u^0||_{C^1}(1 - \theta^2).\]

Still, at \(z\), \(\psi = 0\) and \(D \psi\) is in parallel with \(\gamma\), if \(\hat{\varepsilon} \leq \beta^{1-\alpha}/4\alpha\),

\[h \geq \hat{\varepsilon} ||\xi||^2 + \hat{\varepsilon} \cdot \beta^{-\alpha} + \hat{\varepsilon} \beta \xi_0^2 \geq \hat{\varepsilon} (1 - \theta^2) + \hat{\varepsilon} \cdot \beta^{-\alpha}.\]

\(\hat{\varepsilon}\) may change from line to line, but all of them are smaller than \(\beta^{1-\alpha}/4\alpha\), we can choose \(\hat{\varepsilon} = \beta^3\). Hence we can obtain

\begin{equation}
\sup_{\partial \Omega \times \{\xi = 1\}} \frac{g}{h} \leq \beta^3 \sup_{\partial \Omega} u_{\gamma \gamma} + \frac{n ||u^0||_{L^\infty}}{\hat{\varepsilon}} \leq \beta^3 \cdot \sup_{\partial \Omega} u_{\gamma \gamma} + \hat{c}_0 \beta^{-3},
\end{equation}

where \(\hat{c}_0\) depends on \(||u^0||_{L^\infty}\). Then, by the virtue of Proposition 4.2 and (4.15), for any \(z \in \Omega \times \{\xi = 1\}\)

\[g(z) \leq h(z) \sup_{\Omega \times \{\xi = 1\}} \frac{g}{h} \leq h(z) \left( \sup_{\partial \Omega \times \{\xi = 1\}} \frac{g}{h} + \frac{\mu}{\nu} \right)
\]

\[\leq h(z)(\beta^3 \sup_{\partial \Omega} u_{\gamma \gamma} + \hat{c}_0 \beta^{-3} + \beta^{-8}).\]

By the virtue of (4.8), (4.9) and (4.11),

\[g(z) \leq 2 ||\psi||_{C^{2,1}}^2 \beta^2 \left(\psi + \beta\right)^{2\alpha} \sup_{\partial \Omega} u_{\gamma \gamma} + 2 ||\psi||_{C^{2,1}}^2 \hat{c}_0 \beta^{-3} + \beta^{-8},\]

where \(\beta \in (0, \beta_0)\), \(\beta_0\) depends on \(||\psi||_{C^{2,1}}, ||u^0||_{C^1}, ||f||_{C^{2,1}}, C_0\) and \(||\rho||_{C^{2,1}}\). As \(\xi^2 + \xi_0^2 = 1\), we choose \(\xi_0 = 0, 2 \beta^2 = \varepsilon\) and \(C_\varepsilon = \frac{2 \hat{c}_0 \beta^{-3}}{\beta^{3\alpha}} \hat{c}_0 \beta^{-3} + \beta^{-8}\). Thus, for any unit vector \(\xi\),

\begin{equation}
u_{\xi \xi} \leq \frac{\hat{c}_0 \beta^{3\alpha}}{\beta^{3\alpha}} \sup_{\partial \Omega} u_{\gamma \gamma} + C_\varepsilon.
\end{equation}
For general $\xi \in \mathbb{R}^n$, by timing $|\xi|^2$ with respect to both sides of (4.16), we have

$$u^{\vartheta}_{\xi\xi} \leq \varepsilon \frac{\|\psi\|^2_{C^1(\Omega)}}{(\psi + \beta)^{\alpha}} \sup_{\partial\Omega} u^{\vartheta}_{\gamma\gamma} + C_\varepsilon |\xi|^2.$$ 

That is (4.2), our proof is done. \hfill \Box

5. Double Normal Derivative Estimate

Now we prove Theorem 2.6. We need several steps. Our approach is still the barrier.

Firstly, we study the second order derivative estimates on the boundary. For any point $x_0$ on the boundary, by a rotation and a translation, we can take $x_0$ as the origin. Then, choosing the principal coordinates system at the origin, the boundary $\partial\Omega$ is represented by $x_n = \rho(x')$ near the origin, where $x' = (x_1, \ldots, x_{n-1})$, $\rho$ is smooth as the smoothness of $\partial\Omega$.

Let $T = (T_{ij})$ be a skew-symmetric matrix, $\tau = (\tau_1, \ldots, \tau_n)$ be a vector field in $\Omega$ given by

$$\tau_i = T_{ij}x_j + a_i, \quad i = 1, \ldots, n,$$

where $a_i$ is a constant. We set

$$u^{\vartheta}_{\tau}(\tau) =: (u^{\vartheta}_{\tau})_{\tau} =: \tau_i \tau_j u^{\vartheta}_{ij} + (\tau_i)_{\tau} \tau_j u^{\vartheta}_{ij},$$

where $(u^{\vartheta}_{\tau})_{\tau} = \frac{\partial u^{\vartheta}_{\tau}}{\partial x}, (\tau_i)_{\tau} = \frac{\partial \tau_i}{\partial x}.$

We have the following relationship.

**Proposition 5.1** (Lemma 2.1 in [9]).

(5.1) \quad $F[u^{\vartheta}_{\tau}] = (F[u^{\vartheta}_{\tau}])_{\tau}, \quad F^{ij}(u^{\vartheta}_{\tau})(\tau)_{ij} \geq (F[u^{\vartheta}_{\tau}])_{\tau}(\tau)_{ij}.$

**Lemma 5.2.** For any $\partial\Omega \ni x = (x_1, \ldots, x_n) = (x', x_n)$ in the neighborhood of the origin, we have

(5.2) \quad $u^{\vartheta}_{\tau}(\tau)(x) - u^{\vartheta}_{\tau}(\tau)(0) \leq \tilde{c}_0 (|x'|^2 + M|x'|^4) \quad \text{in} \ B_{r_0}(0) \cap \partial\Omega,$

where $\tau$ is the linear combination of $\eta^i$, $\tau = \alpha_i \eta^i$, $\sum \alpha_i^2 = 1$, $\eta^i$ is the annular vector field in the sense of the principal coordinates system at the origin,

$$\eta^i = \eta^i(x) = (1 - \kappa_i(0)x_n) \partial_i + \kappa_i(0)x_i \partial_n, \quad i = 1, \ldots, n - 1,$$

and

$$M = \sup_{x \in \partial\Omega} u^{\vartheta}_{\gamma\gamma}(x),$$

$\tilde{c}_0$ depends only on $\|\rho\|_{C^{2.1}}, \|u^{\vartheta}\|_{C^{1.1}(\Omega)}$, and the bound of the second order mixed type derivatives, in particular, $\tilde{c}_0$ is independent with $\vartheta.$
Proof. Our proof will need several steps. Since
\[ u_{\tau(\tau)}(x) = \sum_{i,j,m,l} u_{ij}^\delta (\alpha_m \eta_i^m)(\alpha_l \eta_j^l) + u_{ij}^\delta (\alpha_m \eta_i^m)(\alpha_l \eta_j^l) \]
\[ = \left[ \sum_{i,j,m} \alpha_m^2 u_{ij}^\delta \eta_i^m \eta_j^m + \alpha_m^2 u_{ij}^\delta (\eta_i^m)_j \eta_j^m \right] \]
\[ + \sum_{m \neq l} \sum_{i,j} [\alpha_m \alpha_l u_{ij}^\delta \eta_i^m \eta_j^l + \alpha_m \alpha_l u_{ij}^\delta (\eta_i^m)_j \eta_j^l] \]
\[ = P(x) + Q(x). \]

Setting
\[ \omega = u_{\tau(\tau)}(x) - u_{\tau(\tau)}(0). \]

Thus, we have
\[ \omega(x) = P(x) - P(0) + Q(x) - Q(0). \]

For some fixed \( m \), let
\[ \omega^m(x) = u_{\tau(\tau)}(x) - u_{\tau(\tau)}(0). \]

Thus,
\[ P(x) - P(0) = \sum_m \alpha_m^2 \omega^m(x). \]

Firstly, we estimate the \( P(x) - P(0) \). Recall the result in [9]. For each \( \omega^m(x) \), we have
\[ \omega^m(x) \leq \tilde{c}_0(|x'|^2 + M|x'|^4), \]
where \( \tilde{c}_0 \) depends only on \( \|\rho\|_{C^2,1} \) and \( \|u\|_{C^1} \). Since \( \sum \alpha_m^2 = 1 \), we have
\[ P(x) - P(0) \leq \tilde{c}_0(|x'|^2 + M|x'|^4). \]

Then, by this result, we finish the proof of Lemma 5.2. Recall (5.3). Thus, we need to estimate the left terms \( Q(x) - Q(0) \). For fixed \( m \) and \( l \), where \( m \neq l \), we take \( \xi^m \) and \( \xi^l \) as the projections of \( \eta^m \) and \( \eta^l \). We denote \( \cos \theta = \langle \hat{\eta}^m, \hat{\xi}^m \rangle \), \( \sin \theta = \langle \hat{\eta}^m, \hat{\xi}^l \rangle \), where \( \hat{\xi}^m = \xi^m/|\xi^m| \), \( \hat{\eta}^l = \eta^l/|\eta^l| \). Still, we have
\[ u^\delta_{(\tau)}(x) - u^\delta_{(\tau)}(0) \]
\[ \leq (1 + 4\|\rho\|_{C^2,1}|x'|^2)(\varphi^\delta(x) - \varphi^\delta(x)\rho^\delta x^l) - u^\delta_{ml}(0) + H(x') \]
\[ \leq \varphi^\delta_{(\tau)}(x) + (\varphi^\delta(x) - u^\delta_{(\tau)}(x))\rho^\delta x^l - \varphi^\delta_{(\tau)}(0) + H(x') \]
\[ \leq H(x'), \]
where \( H \) denotes a function satisfying
\[ H(x') \leq a|x'| + C|x'|^2 + CM|x'|^4. \]

Thus, by subtracting a linear function, we have
\[ Q(x) - Q(0) \leq \tilde{c}_0(|x'|^2 + M|x'|^4). \]

Accordingly, by (5.3), we have
\[ \omega(x) \leq \tilde{c}_0(|x'|^2 + M|x'|^4), \]
where \( \tilde{c}_0 \) may change from line to line, but all of them depend only on \( \|\rho\|_{C^{2,1}}, \|u^\vartheta\|_{C^1(\overline{\Omega})} \) and the bound of the second order mixed type derivatives, in particular \( \tilde{c}_0 \) is independent with \( \vartheta \).

Next we extend Lemma 5.2 to the points in the neighborhood of the origin. That is

**Lemma 5.3.** For any \( x \in \Omega \) near the origin

\[
\omega(x) \leq C(|x'|^2 + M|x'|^4) + CM|x_n - \rho(x')|,
\]

where \( C \) depends on \( \|f\|_{C^{1,1}}, r_0, C_0 \) and \( \tilde{c}_0 \), in particular, \( C \) is independent with \( \vartheta \).

**Proof.** Let \( \Omega_r = \{x \in \Omega | \rho(x') < x_n < \rho(x') + r^4, |x'| < r^4 \} \), and

\[
v = CK[(x_n - \rho(x'))^2 + \beta(\rho(x') - x_n) - \frac{\tilde{c}_0}{M}|x'|^2 - \tilde{c}_0|x'|^4],
\]

where \( C_K \geq 1 \) is a constant. We choose \( r \) as the highest infinitesimal among \( r, \frac{1}{M}, \beta \). By the \((k - 1)\)-convexity of \( \Omega \), \( v \) is \( k \)-admissible. Thus, we can choose \( C_K \) which depends on \( C_0 \) so large that

\[
F[v] \geq \delta_0 + \sup_{\overline{\Omega}} f^{1/k}
\]

for some positive constant \( \delta_0 \geq C_0 \). By the concavity of \( F \) and Condition (H), one can verify

\[
L[v] \geq F[u^\vartheta + v] - F[u^\vartheta] \geq F[u^\vartheta] - \sup_{\overline{\Omega}} F^{1/k} \geq \delta_0 \geq C_0 \geq -(F[D^2u^\vartheta])_{\tau\tau},
\]

where \( L[v] = F^{ij} \cdot v_{ij} \).

Then by (5.1), we have

\[
L(Mv) \geq -(F[D^2u^\vartheta])_{(\tau)(\tau)} \geq -F^{ij}(u^\vartheta)_{(\tau)(\tau)}i_j = L(-\omega).
\]

The boundary \( \partial \Omega_r \) consists of three parts: \( \partial \Omega_r = \partial_1 \Omega_r \cup \partial_2 \Omega_r \cup \partial_3 \Omega_r \), where \( \partial_1 \Omega_r \) and \( \partial_3 \Omega_r \) are respectively the graph parts of \( \rho \) and \( \rho + r^4 \), \( \partial_2 \Omega_r \) is a portion of \( \{ |x'| = r \} \). \( \partial_1 \Omega_r, \partial_2 \Omega_r \) part, that is \( \rho(x') = x_n \) and \( x_n = \rho(x') + r^4 \). We have

\[
v \leq -\tilde{c}_0 C_K(\frac{|x'|^2}{M} + |x'|^4).
\]

\( \partial_3 \Omega_r \) part, \( |x'| \leq r \), we have

\[
v \leq C_K [C_1r^8 + C_2r^4 - \frac{\tilde{c}_0}{M}r^2 - \tilde{c}_0r^4] = -\frac{C_K\tilde{c}_0r^2}{M},
\]

where \( C_1, C_2 \) are constants depend on \( \beta \) and \( \Omega \). Thus, we can choose \( C_K \) which depends on \( \|f\|_{C^{1,1}}, C_0 \) and \( \Omega \) so large that \( v \leq -2 \).

In sum, by (5.2), we have

\[
Mv \leq -\omega \quad \text{on} \quad \partial \Omega_r.
\]

By the virtue of the classical weak maximum principle and (5.5), we have

\[
Mv \leq -\omega \quad \text{in} \quad \Omega_r.
\]

That is (5.4).
In order to find the barrier function, we still need a third order derivative estimate.

**Lemma 5.4.** For any given $\sigma > 0$ which is sufficiently small, we can find a positive constant $C_\sigma$ such that

\[(5.6) \quad (u^{\psi}_{(\tau)(\tau)})_n(0) \leq \sigma M + C_\sigma,\]

where $C_\sigma$ depends on $\tilde{c}_0$, $\|f\|_{C^{1,1}}$, $C_0$ and the constant $C_{\epsilon,\delta}$ in (4.2) with $\delta = \sigma^4$ and $\epsilon = \sigma^8$.

**Proof.** Let

\[\tilde{v}(x) = \tilde{C}_K \left((x_n - \rho(x'))^2 + \beta(\rho(x') - x_n) - \beta_1 |x'|^2\right),\]

where $\beta, \beta_1 > 0$ are sufficiently small, $\tilde{C}_K > 1$ is sufficiently large. Then for a sufficiently small $r > 0$ which is different from which in Lemma 5.3. $\tilde{v}$ is $k$-admissible in $\Omega_r$. We have

\[F[D^2\tilde{v}] \geq f_0 \quad \text{in } \Omega_r,\]

where $f_0 \geq \delta_0 + \sup_{\Omega} f^{1/k}$.

We want to explain that, if $r > 0$ is sufficiently small and $\tilde{C}_K > 1$ is sufficiently large, $(rM + C_r)\tilde{v}$ is a sub-barrier of $\omega$, where $C_r = r^{-4}C_{\epsilon,\delta}$. We have

\[(5.7) \quad (rM + C_r)\tilde{v} \leq -\omega \quad \text{on } \partial \Omega_r.\]

Indeed, on $\partial_1 \Omega_r$, we have

\[\tilde{v}(x) \leq -\beta_1 \tilde{C}_K |x'|^2,\]

and hence

\[(rM + C_r)\tilde{v} \leq (rM + r^{-4}C_{\epsilon,\delta})(-\beta_1 \tilde{C}_K |x'|^2) \leq -\beta_1 \tilde{C}_K M r |x'|^2 - \beta_1 \tilde{C}_K C_{\epsilon,\delta} \frac{|x'|^2}{r^4} \leq -\omega\]

provided $\tilde{C}_K$ is sufficiently large.

On $\partial_2 \Omega_r$, $\tilde{v} < -\frac{1}{2} \beta \tilde{C}_K r^4$. By Lemma 4.1

\[-\omega(x) \geq -r^8 M - C_{r,s_r} r^4 \geq -r^8 - \frac{C_{\epsilon,\delta}}{r^8}.\]

Since $r = o(\beta)$, (5.7) holds.

On $\partial_3 \Omega_r$, we have

\[\tilde{v} = \tilde{C}_K \left((x_n - \rho(x') - \frac{\beta}{2})^2 - \frac{\beta^2}{4} - \beta_1 |x'|^2\right) < -\beta_1 \tilde{C}_K r^2.\]

By (5.4),

\[-\omega \geq -C(1 + r^4 M),\]

hence (5.7) holds. In $\Omega_r$, we have $L(\tilde{v}) \geq \delta_0$. We still use the approach as before. One can verify that

\[L[(rM + C_r)\tilde{v}] \geq C_0.\]

Thus, by the virtue of (5.1) and Condition (H)

\[L[(rM + C_r)\tilde{v}] \geq -L(\omega) \quad \text{in } \Omega_r.\]
In sum,
\[(rM + C_\tau)\hat{v} \leq -\omega \quad \text{in } \Omega_r.\]

Thus, we have
\[(u^\theta_{(\tau)(\tau)})_{\partial\Omega}(0) \leq (rM + C_\tau)|\hat{v}|_{\partial\Omega}(0) | \leq C_{\epsilon_\Omega}(rM + C_\tau).\]

Then, let \(\sigma = C_{\epsilon_\Omega}r\), that is \((\ref{5.6}).\)

Finally, we finish the proof of Theorem 2.6. Thus, we complete the whole proof.

**End of the Proof to Theorem 2.6**

For any boundary point \(x_0\) and arbitrary tangential unit vector field \(\xi\) on \(\partial\Omega\) near \(x_0\). We set \(x_0\) to be the origin, Then, \(\partial\Omega\) is locally given by \(x_n = \rho(x')\), \(\tau\) and \(\eta^m\) are defined as which in Lemma 5.2. At the origin,
\[\tau(0) = \alpha_m\partial_m.\]

Since \(\sum_m \alpha_m^2 = 1\), \(\tau(0)\) is a tangent vector. Thus, we can choose some appropriate \(\alpha_m\) such that \(\xi = \tau\) at \(x_0\). One can verify that at \(x_0\)
\[u^\theta_{n(\xi)(\xi)} = u^\theta_{n(\tau)(\tau)} = u^\theta_{nij}(\tau)\tau_j + (\tau_i)\tau_j u^\theta_{nij},\]
\[(u^\theta_{(\tau)(\tau)})_n = u^\theta_{nij}(\tau)\tau_j + u^\theta_{nij}(\tau)\tau_j + ((\tau_i)\tau_j)_{\partial\Omega}u^\theta_{ijn} + (\tau_i)\tau_j u^\theta_{ijn}.\]

At the origin, \(\tau_n = 0\), \(u_{ij}(0)\) is bounded when \(i, j = 1, \ldots, n - 1\). Then
\[|u^\theta_{ij}(\tau_i\tau_j)_n + ((\tau_i)\tau_j)_n u^\theta_{ijn}| \leq C_\sigma.\]

Thus, we have
\[u^\theta_{n(\xi)(\xi)}(0) = (u^\theta_{(\tau)(\tau)})_n - (u^\theta_{ij}(\tau_i\tau_j)_n + ((\tau_i)\tau_j)_n u^\theta_{ijn}) \leq \sigma M + C_\sigma.\]

Since this result is independent with the choice of \(x_0\), we have, for any tangent vector field \(\xi = \sum_{i=1}^{n-1} \alpha_i(\partial_i + \rho_i(x')\partial_n), \sum_{i=1}^{n-1} \alpha_i^2 = 1\)
\[(5.8) \quad u^\theta_{n(\xi)(\xi)}(x) \leq \sigma M + C_\sigma \quad \text{on } \partial\Omega.\]

Choosing a new coordinates system, we suppose the maximum \(M\) is attained at the origin \(0 \in \partial\Omega\). Then near the origin, we set \(G(x') = u^\theta_{n}(x', \rho(x'))\) defined on \(\partial\Omega\), by the Taylor expansion on the boundary, for \(h = (h_1, \ldots, h_n) = (h', h_n) \in \partial\Omega\) near the origin
\[G(h') = G(0) + \sum_{i=1}^{n-1} h_i \partial_i G(0) + \int_0^1 (1 - t)(\sum_{i=1}^{n-1} h_i \partial_i)^2 G(th')dt.\]

One can verify that
\[G(0) = u^\theta_{n}(0),\]
\[\sum_{i=1}^{n-1} h_i \partial_i G(x') = \sum_{i=1}^{n-1} h_i(\partial_i + \rho_i(x')\partial_n)u^\theta_{n}(x', \rho(x')).\]

If we choose \(\alpha_i = h_i/|h'|\), where \(|h'| = (\sum_{m=1}^{n-1} h_m^2)^{1/2}\), we have
\[\sum_{i=1}^{n-1} h_i \partial_i G(x') = |h'| u^\theta_{n\xi}(x', \rho(x')).\]
Still, we have
\[
\left( \sum_{i=1}^{n-1} h_i \partial_i \right)^2 G(t h') = u_\partial^0(t, p(t h')) |h'|^2.
\]
Then, by (5.8), we have
\[
u_\partial^0(h) \leq u_\partial^0(0) + a |h'| + (\sigma M + C_\sigma) |h'|^2,
\]
where \(a\) is the bound of the mixed second order derivatives. By subtracting a linear function and the gradient estimate, we can obtain
\[
L[(\sigma M + C_\sigma) \tilde{\nu}] \geq L(-u_\partial^0) \quad \text{in } \Omega \cap B_r,
\]
\[
(\sigma M + C_\sigma) \tilde{\nu} \leq -(u_\partial^0(x) - u_\partial^0(0)) \quad \text{on } \partial(\Omega \cap B_r).
\]
Thus, we have
\[
-(u_\partial^0(x) - u_\partial^0(0)) \geq (\sigma M + C_\sigma) \tilde{\nu}(x) \quad \text{in } \Omega \cap B_r,
\]
namely,
\[
M = u_\partial^0(0) \leq -\tilde{\nu}(0) (\sigma M + C_\sigma) \leq \tilde{C}_K \sigma (\sigma M + C_\sigma).
\]
Let \(\sigma^2 < 1/2\), one can verify that
\[
M \leq 2 \tilde{C}_K \sigma C_\sigma \leq \frac{C_\sigma}{\sigma}
\]
holds for any \(\sigma \in (0, \sigma_0)\). From the computation above
\[
\frac{C_\sigma}{\sigma} = O(\sigma^{-m}),
\]
where \(m\) is a positive constant. Thus,
\[
\sup_{\partial \Omega} u_{\partial}^0 \leq \frac{C_{\sigma_0}}{\sigma_0}
\]
where \(\sigma_0\) depends only on \(\|\psi\|_{C^3(\Omega)}, \Omega, C_0, \|\rho\|_{C^{2,1}}\) and \(\|\varphi\|_{C^{3,1}}\). In particular, \(C_{\sigma_0}\) and \(\sigma_0\) are independent with \(\vartheta\). Proof is done.

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