Exact Equilibria of a Stellar System in a Linearised Tidal Field

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Abstract
We study the motion of stars in a star cluster which revolves in a circular orbit about its parent galaxy. The star cluster is modelled as an ellipsoid of uniform spatial density. We exhibit two 2-parameter families of self-consistent equilibrium models in which the velocity at each point is confined to a line in velocity space. We exhibit the link between this problem and that of a uniform rotating ellipsoidal galaxy. With minimal adaptation, Freeman’s bar models yield a third family.

Key words: stellar dynamics - globular clusters: general - open clusters and associations: general

1 INTRODUCTION

In a star cluster moving in a circular orbit about the axis of symmetry of a galaxy, a star is subject to the self-gravity of the cluster and the tidal field of the galaxy. If studied in a rotating, accelerating frame following the orbital motion of the cluster, it is also subject to centrifugal and Coriolis forces. If the spatial distribution of stars is uniform within an ellipsoid, and if the usual linear approximation of the tidal field is adopted, the accelerations inside the ellipsoid are all linear in the spatial coordinates. There are then two normal modes of oscillation in planes orthogonal to the axis of rotation. Fellhauer & Heggie (2005) studied this problem in the case when one of the normal frequencies is imaginary. Taking for convenience the case of an ellipsoid of revolution, they showed that it was possible to choose the axial ratio so that it was equal to that of the orbits in the remaining normal mode. By a very simple orbit superposition it was possible to construct a self-consistent model.

This problem closely resembles that of constructing a model of a uniform, rotating ellipsoidal bar (Freeman 1966a,b,c; Hunter 1974, 1975). This goal has been achieved in three cases: (i) elliptical cylinders, in which the semi-major axis parallel to the axis of rotation is infinite; (ii) so-called “balanced” systems, in which the centrifugal and gravitational forces are equal along the major axis; and (iii) disks with the surface density that is obtained by projecting a uniform ellipsoid onto one plane.

In this paper we set out the connection between these two problems, and give a more systematic treatment of the models which Fellhauer & Heggie stumbled upon. In principle there are three dimensionless parameters: two axial ratios, and one parameter which measures the density of the system. The requirement of choosing one axial ratio appropriately reduces the family to a two-parameter family. Fellhauer & Heggie chose to work with axisymmetric models, which simplifies consideration to a one-parameter family. In cases where both normal frequencies are real, however, it is clear that there is a possibility of a second two-parameter family. Finally, when this second eigenfrequency is zero, the problem is equivalent to that of Freeman’s “balanced” bar.

We begin in the next section with a summary of the equations of motion and their solution. Then section 3 surveys the models based on a single normal mode, and those which exploit the idea behind Freeman’s bar model. The paper concludes with some final discussion, and an Appendix describes the models in terms of distribution functions.

2 ORBITAL THEORY

We use the rotating, accelerating frame in which the centre of the ellipsoidal star cluster is at the origin, the $x$-axis points to the centre of the galaxy, and the $y$-axis points in the direction of orbital motion of the system. (Fellhauer & Heggie took the $y$-axis in the opposite direction, but we switch in order to increase the resemblance with the theory as set out by Freeman.) Then the equations of motion are

\[
\begin{align*}
\ddot{x} + 2\Omega \dot{y} + (\kappa^2 - 4\Omega^2)x &= -A^2 x \\
\ddot{y} - 2\Omega \dot{x} &= -B^2 y \\
\ddot{z} + \nu^2 z &= -C^2 z,
\end{align*}
\]

where $\Omega$ is the angular velocity about the galaxy, $\kappa$ is the epicyclic frequency, $\nu$ is the frequency of small motions orthogonal to the galactic orbit of the cluster (if the cluster itself is neglected), and $A^2, B^2, C^2$ are coefficients of $x^2/2, y^2/2$ and $z^2/2$ in the expression for the gravitational potential inside a uniform ellipsoid with semi-axies $a, b, c$. (They are functions of the density ($\rho$) and the axial ratios $b/a, c/a$.) For $a > b > c$, these expressions are given in Freeman (1966b), eqs.(4)-(12), which we would refer to henceforth as FII.4-12, with corresponding abbreviations for equations in his two other papers. Equivalent formulæ will be found in Table 2-1 of Binney & Tremaine (1987), along with the special forms.

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corresponding to axisymmetrical and spherical cases. Thus
\[ \ddot{x} + 2\Omega \dot{y} + A_1^2 x = 0, \]
\[ \ddot{y} - 2\Omega \dot{x} + B_1^2 y = 0, \]
\[ \ddot{z} + C_1^2 z = 0, \]  
(2)

where
\[ A_1^2 = \kappa^2 - 4\Omega^2 + A^2, \]  
(3)
\[ B_1^2 = B^2, \]  
and  
(4)
\[ C_1^2 = \nu^2 + C^2 \]  
(5)

\[ \text{(cf. FI.8, FI.16).} \]

For \( \alpha \neq 0 \) the solutions of these equations are
\[ x = A_\alpha \sin(\alpha t + \epsilon_\alpha) + A_\beta \sin(\beta t + \epsilon_\beta), \]
\[ y = k_\alpha A_\alpha \cos(\alpha t + \epsilon_\alpha) - k_\beta A_\beta \cos(\beta t + \epsilon_\beta) \]  
(6)
\[ z = A_\gamma \sin(\gamma t + \epsilon_\gamma) \]  
(7)

(FI.9, FI.17c), where \( A_\alpha, A_\beta, A_\gamma \) are arbitrary constants, which we can take positive, \( \epsilon_\alpha, \epsilon_\beta, \epsilon_\gamma \) are arbitrary constants, \( \gamma = C_1, \alpha, \beta \) are the two roots for \( \xi \) of the quartic
\[ \xi^4 - (A_1^2 + B_1^2 + 4\Omega^2)\xi^2 + A_1^2 B_1^2 = 0, \]  
(8)
\[ k_\alpha = \frac{A_1^2 - \alpha^2}{2\alpha\Omega}, \]  
(9)
\[ k_\beta = \frac{\beta^2 - A_1^2}{2\beta\Omega} \]  
(10)

(FI.10-11). We adopt Freeman’s convention that \( \alpha < \beta \) when both frequencies are real; when one frequency is imaginary (a case not considered by Freeman) we denote the single real frequency by \( \beta \). Then both \( k_\alpha \) and \( k_\beta \) are positive. Note also that, for a given galactic potential, they are functions of \( \rho, b/a \) and \( c/a \).

3 THREE FAMILIES OF MODELS

3.1 \( \beta \) models

We first summarise the family of self-consistent models which generalises those found by Fellhauer & Heggie [2005]. We take \( A_\alpha = 0 \) in eqs. (6). The models are then built from the normal mode with frequency \( \beta \), and so we refer to them as \( \beta \)-models. We can construct self-consistent models provided that \( k_\beta = b/a \). When the frequencies \( \beta \) and \( \gamma \) are incommensurable, each orbit fills the curved surface of an elliptical cylinder. The upper end of this cylinder is the ellipse \( x^2 + y^2/k_\beta^2 = A_1^2, z = A_\gamma \), which lies on the surface of the ellipsoid \( x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \) if \( A_\beta^2/a^2 + A_\gamma^2/c^2 = 1 \). Thus in general
\[ A_\beta^2 \leq A_{\gamma, \max}^2 = c^2(1 - A_\beta^2/a^2), \]  
(11)
provided that \( A_\beta^2 \leq a^2 \).

For given \( A_\beta, A_\gamma \), the distribution of \( z \) is \( f(z|A_\gamma) = \frac{1}{\pi A_\gamma^2 - z^2} \leq A_\gamma \). Given \( A_\beta \), we choose the distribution of \( A_\gamma \) according to
\[ f(A_\gamma) = \frac{A_\gamma}{A_{\gamma, \max}^2 - A_\gamma^2} \quad (0 < A_\gamma < A_{\gamma, \max}) \]  
(12)

(For the case \( a = c \), considered by Fellhauer & Heggie, this is equivalent to their eq.(19).) With this choice we see easily that \( z \) is uniformly distributed in \( |z| \leq A_{\gamma, \max} \). In order to generate a spatially uniform density, we choose the distribution of \( A_\beta \) so that \( f(A_\beta) dA_\beta \) is proportional to the mass inside the cluster in the cylindrical volume between \( x^2 + y^2/k_\beta^2 = A_1^2, (A_\beta + dA_\beta)^2 \), i.e.
\[ f(A_\beta) = \frac{3}{a^2 A_\beta} \sqrt{1 - A_\beta^2/a^2}, A_\beta < a \]  
(c.f. Fellhauer & Heggie, eq.(18)). While these formulae are convenient for constructing a model, we also (section A2) provide a more conventional definition in terms of a distribution function in phase space.

Our purpose now is to sketch the range of models which satisfy the essential requirement \( k_\beta = b/a \). Since \( k_\beta \) is a function of \( \rho, b/a \) and \( c/a \), this equation may be solved for \( \rho \) as a function of \( b/a \) and \( c/a \). This can be exhibited by plotting contours of \( \rho \) in the plane \( b/a, c/a \). Results of a numerical survey are shown in Fig.1, where for definiteness we consider the case of an isothermal galaxy potential, in which \( \kappa = \sqrt{2}\Omega \).

Two limiting cases can be discerned:

(i) Low density: If \( \rho = 0 \) then \( \beta = \kappa \) and \( k_\beta = \sqrt{2} / 2 \), by eqs. (7), (9). This is the limit at the extreme right of Fig.1.

(ii) Axisymmetric models: If \( \rho \) is very large and \( a = b \), we have \( A = B, \beta = A + \Omega \) and \( k_\beta \approx 1 \), by the same pair of equations. This limit is reached at the extreme left of Fig.1.

The models discussed by Fellhauer & Heggie [2005] lie on the line \( c/a = 1 \) in Fig.1. (This subfamily is also depicted in Fig.4.) We now see that they are part of a two-parameter family.

3.2 \( \alpha \) models

For sufficiently low densities, the value of \( \alpha^2 \) is negative, and orbital motions are unstable. For higher densities, however, \( \alpha^2 \geq 0 \), and so we have another family of self-consistent models where now \( A_\beta = 0 \) and \( k_\alpha = b/a \). These we refer to as \( \alpha \)-models. Their distribution in parameter space is shown in Fig.2. Again there are axisymmetrical models of arbitrarily high density. The construction of these models is exactly analogous to that of \( \beta \)-models. Again the subfamily on which \( c/a = 1 \) is depicted in Fig.4.

3.3 Freeman’s Bars

In F.II, Freeman described a series of three-dimensional ellipsoidal models of uniform density for the case in which \( A_1 = 0 \). In the
galactic context this was referred to as the condition for a “balanced” ellipsoid. For star clusters the corresponding interpretation is that of a system which is marginally stable against tidal disruption, as $\alpha = 0$. They appear to exist only for $G\rho/\Omega^2 > 0.3$ approximately, and their distribution is shown in Fig.3.

For the case considered here, eqs. (6a,b) are replaced by
\begin{align*}
x &= A_\alpha + A_\beta \cos(\beta t + \epsilon_\beta) \\
y &= k_\beta A_\beta \sin(\beta t + \epsilon_\beta)
\end{align*}
(FII.17, with some changes of notation), where now $\beta^2 = 4\Omega^2 + B_1^2$, $k_\beta = \beta/(2\Omega)$, and the constant term $A_\alpha$ represents the zero-frequency mode. The effect of this term is that epicycles can be placed with guiding centre at any point along the $x$-axis. This freedom allows the construction of models with an axial ratio different from that of the epicycles. A little thought, however, shows that the axial ratio of the epicycles ($k_\beta$) must exceed that of the cluster ($b/a$) in order that all parts of the cluster are accessible by epicycles lying entirely within it. This condition (which is best thought of as an upper limit to $b/a$) gives the near-vertical curve at about $b = 1.2$ in Fig.3; realisable models must lie to the left of this line. This curve represents the intersection of the families of Freeman and beta models, as can be seen in the cross section in Fig.4.

An attractive special case is the spherical model $a = b = c$. Then $A = B = C = 4\pi G\rho/3$, and so eq. (3), with $A_1 = 0$, yields $\rho = \frac{3}{4\pi G}(4\Omega^2 - \kappa^2)$. A typical orbit is shown in Fig.5.

For Freeman models it was found convenient to generate initial conditions from the distribution function, which is described in FII.

\section*{4 CONCLUSIONS AND FINAL REMARKS}

In this paper we have considered the problem of constructing an exact self-consistent equilibrium of a star cluster in a tidal field. The usual linear approximation of the tidal field is adopted, and we assume that the motion of the cluster in the galaxy is uniform and circular.

We have found three distinct families of models, in all of which the cluster is an ellipsoid of uniform density. Each is a two-parameter family, the two parameters being the axial ratios of the ellipsoid. Additional free parameters are the angular frequency of the motion around the galaxy and the radius of the cluster.
All three models are built of epicyclic motions, of the kind which are familiar in galactic dynamics, though the details of the orbits are modified by the action of the cluster potential. In two families of models, one of which was partially explored by Fellhauer & Heggie (2005), the space density is built up by epicycles whose guiding centre lies at the centre of the cluster and whose axial ratio (in the $x, y$ directions) coincides with that of the ellipsoid. We call these $\alpha$- and $\beta$-models. In a third family the guiding centre of the epicycles may be displaced at various points along the $x$-axis, which points to the galactic centre. This family closely resembles Freeman’s family of “balanced” models for barred galaxies (Freeman 1966b). One of the families has models for all values of the space density (for suitable choice of axial ratio), while the other two exist only for densities above a certain minimum value.

We have given enough details of the distribution functions associated with these models to allow explicit $N$-body realisations of them to be constructed. Code is available from the authors for the construction of the $\alpha$- and $\beta$-models, and for the spherical Freeman model.

An interesting extension of the study of these models would be consideration of their dynamical evolution, under both collisionless and collisional processes. For a one-dimensional subfamily of the $\beta$-models this was considered by Fellhauer & Heggie (2005), with particular emphasis on low-density models. They found that the models are unstable, and that the time scale of the instability increases as the density decreases. Though they did not explore high-density models in such detail, their results indicated that the lifetime increases again at the highest end of the range they studied.

The question arises whether it is possible to construct more general models along these lines. Though no further analytic models have been found yet, it has been pointed out to us by J.P. Ostriker that this problem seems well suited to Schwarzschild’s orbit-superposition method (Schwarzschild 1979).

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APPENDIX A: DISTRIBUTION FUNCTIONS

A1 Integrals of the motion

We invoke Jeans’ Theorem and express the distribution function as a function of the isolating integrals of motion. As shown in FI and FII these are

$$E_\alpha = \frac{\alpha \beta k_\alpha}{2 \pi} A_\alpha^2 (\dot{y} - \beta k_\beta z)^2 + \frac{\sigma \alpha}{2 k_\alpha} \left( \dot{x} + \beta k_\beta \right)^2$$

$$E_\beta = \frac{\beta k_\beta}{2 \pi} A_\beta^2 (\dot{y} + \alpha k_\alpha x)^2 + \frac{\sigma \beta}{2 k_\beta} \left( \dot{x} - \alpha k_\alpha \right)^2$$

$$E_\gamma = \frac{1}{2} \gamma^2 A_\gamma^2 = \frac{1}{2} z^2 + \frac{1}{2} \gamma^2 z^2,$$

where $\frac{1}{\alpha} = \alpha k_\alpha + \beta k_\beta$. These are in fact the energies of the three normal modes, and the total energy is $E = E_\alpha + E_\beta + E_\gamma$.

By combining results in FI.41 and FII.27, we see that the expression of the surface of the model is $J = 1$, where, in general,

$$J = 2 \pi (k_\beta - k_\gamma) \left\{ \frac{E_\alpha}{\alpha k_\alpha (a^2 k_\beta^2 - b^2)} + \frac{E_\beta}{\beta k_\beta (b^2 - a^2 k_\gamma^2)} \right\} - \frac{2 E_\gamma}{\gamma^2 c^2}.$$  (A2)

This expression is very useful for constructing distribution functions vanishing outside the surface of the model. When $\alpha$ is imaginary only the $\beta$-models exist, and then $E_\alpha = 0$. When $\alpha = 0$ we have the case of Freeman models, and then we find somewhat different expressions for $J$ and the quantities it contains (FII). In general, FI.12–19 are a useful resource of handy identities for checking some of the expressions in the following subsections.

A2 $\alpha$- and $\beta$-models

We consider here only the $\beta$-models, as the distribution function for $\alpha$-models can then be constructed by obvious substitutions. When $A_\alpha = E_\alpha = 0$ and $b/\alpha = k_\beta$, eq. (A3) becomes simply

$$J = \frac{2 \pi E_\beta}{\beta \alpha k_\beta} + \frac{2 E_\alpha}{\gamma^2 c^2}.$$  (A3)

Our search for the distribution function can be guided by eq.(12). Substituting the first part of eq.(1b) into eq.(10), we see that $A_{\gamma,\text{max}} = c^2 \left( 1 - \frac{2 \alpha}{\beta \alpha k_\beta} E_\beta \right)$. From the first part of eq.(1b), and eqs.(1) and (A3), it follows that the probability density function of $E_\gamma$ is $f(E_\gamma) = \frac{1}{\gamma^2 A_\gamma} f(A_\gamma) \propto \frac{1}{\sqrt{1 - J}}$. In fact it can be seen that this expression, with a $\delta$-function to set $E_\gamma = 0$ and with suitable normalisation, gives the full distribution function:

$$f(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{\rho \sigma \alpha}{2 \pi^2 \gamma c} \frac{\delta(E_\alpha)}{\sqrt{1 - \frac{2 \pi E_\beta}{\beta \alpha^2 k_\beta} - \frac{2 E_\gamma}{\gamma^2 c^2}}}.$$  (A4)

It is easy to check that this yields the correct density, by integrating over the velocity: for the integration with respect to $\dot{x}$ and $\dot{y}$ we convert to suitable polar coordinates, which leads to an integral of the form $\int_0^\infty \delta(r^2)2rdr$, and for this we adopt the value 1.

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