Algorithmic properties of first-order modal logics of linear Kripke frames in restricted languages*

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Abstract

We study the algorithmic properties of first-order monomodal logics of frames $\langle N, \leq \rangle$, $\langle N, < \rangle$, $\langle Q, \leq \rangle$, $\langle Q, < \rangle$, $\langle R, \leq \rangle$, $\langle R, < \rangle$, as well as some related logics, in languages with restrictions on the number of individual variables as well as the number and arity of predicate letters. We show that the logics of frames based on $N$ are $\Pi_1^1$-hard—thus, not recursively enumerable—in languages with two individual variables, one monadic predicate letter and one proposition letter. We also show that the logics of frames based on $Q$ and $R$ are $\Sigma_0^0$-hard in languages with the same restrictions. Similar results are obtained for a number of related logics.

1 Introduction

How algorithmically expressive are first-order modal logics? More expressive, it is reasonable to assume, than the classical first-order logic $\text{QCl}$—just as propositional modal logics are, as a rule, computationally harder than the classical propositional logic. (In this context, it is natural to consider logics as sets of validities, rather than as calculi: understood as calculi conservatively extending $\text{QCl}$ with a recursively enumerable set of axioms and finitary rules of inference, first-order modal logics are \textit{a priori} $\Sigma_0^0$-complete.\textsuperscript{1}) Numerous first-order modal logics are, however, just as algorithmically expressive as $\text{QCl}$,

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\textsuperscript{1}The reader in need of a reminder of the basic concepts of computability theory may consult [41].
i.e. $\Sigma^0_1$-complete: some—such as $\mathbf{QK}$, $\mathbf{QT}$, $\mathbf{QD}$, $\mathbf{QKB}$, $\mathbf{QKTB}$, $\mathbf{QS4}$ and $\mathbf{QS5}$—are recursively axiomatizable over $\mathbf{QCl}$; others—such as logics of elementary classes of Kripke frames—are recursively embeddable [47, 43] into $\mathbf{QCl}$ by the standard translation [5, 11], [21, Section 3.12]. This suggests a need for a more fine-grained analysis, which takes into account the algorithmic expressivity not only of full logics but also of their fragments obtained by placing restrictions on the structure of formulas. Such analysis allows us to distinguish $\Sigma^0_1$-complete modal logics from $\mathbf{QCl}$ by algorithmic expressivity: while the monadic fragment of $\mathbf{QCl}$ is decidable [32, 4], the monadic fragments of most $\Sigma^0_1$-complete modal logics are not [31]; while the two-variable fragment of $\mathbf{QCl}$ is decidable [37, 25], the two-variable fragments of most $\Sigma^0_1$-complete modal logics are not [30]. This leads to the study of the algorithmic properties of the fragments of first-order modal logics.

This study is also motivated by an underdevelopment, relative to $\mathbf{QCl}$ [9], of the algorithmic classification problem for first-order modal logics—an effort to identify their maximal decidable and minimal undecidable fragments. Despite extensive literature [31, 34, 36, 38, 16, 3, 19, 61, 30, 46, 49], whose summary can be found in the Introduction to the authors’ earlier article [49], much less is known about the algorithmic properties of the fragments of first-order modal, and closely related superintuitionistic, logics than about the algorithmic properties of the fragments of $\mathbf{QCl}$.

The algorithmic properties of one-variable and two-variable fragments of first-order modal logics are also of interest due to close links between those fragments and, respectively, two-dimensional and three-dimensional propositional modal logics [20, 18, 52].

In the present paper, we attempt to identify the minimal undecidable fragments of the first-order monomodal logics of frames $\langle \mathbb{N}, \leq \rangle$, $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Q}, \leq \rangle$, $\langle \mathbb{Q}, < \rangle$, $\langle \mathbb{R}, \leq \rangle$ and $\langle \mathbb{R}, < \rangle$, as well as of closely related linear orders. The logics of these structures are of interest on at least three counts.

First, the structures themselves are of interest, for at least two reasons. They have long been considered natural models of the flow of time [39, 23, 17]; therefore, their study has been stimulated by the long-standing interest in temporal reasoning. Even though we focus on monomodal languages, since our results are negative, they do apply to more expressive languages with modalities for the past as well as the future. On a more basic level perhaps, the structures based on the naturals, the rationals and the reals are so fundamental to mathematics that the properties of the corresponding logics are of intrinsic mathematical significance: the classical theories of these structures, both first-order and second-order, have been extensively studied; in particular, it has long been known that the monadic second-order theory of $\langle \mathbb{N}, < \rangle$ is decidable [14].

Second, the logics considered here call for techniques substantially different from those used in the previous studies [31, 19, 30, 46, 49] of the algorithmic properties of fragments of monomodal predicate logics. For most such logics, known undecidability proofs for

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2Preliminary results on the logics of $\langle \mathbb{N}, \leq \rangle$ and $\langle \mathbb{N}, < \rangle$ were reported in a conference paper [48]. The present article improves on the earlier paper in two respects. First, we obtain stronger results on the logics of frames based on the naturals by proving $\Sigma^0_1$-hardness for weaker languages; the results reported in this article are plausibly optimal, as discussed in Section 7. Second, we report results on logics of the rationals (and hence on $\mathbf{QS4.3}$, $\mathbf{QK4.3.D.X}$ and $\mathbf{QK4.3}$), the reals and infinite ordinals distinct from $\omega$. 

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fragments with a single monadic predicate letter, a restriction considered here, require a transformation of models that increases their branching factor. The methods the authors used earlier [46, 49] intrinsically rely on increasing the branching factor of models, a feature inherited from the propositional-level techniques [26, 12, 42, 44, 45, 50] those methods are based on. On the other hand, the construction by Blackburn and Spaan [8], which is propositional but, in principle, adaptable to first-order logics, does not seem to be readily applicable to logics of transitive frames since it relies on the use of a modal operator suitable for counting transitions along the accessibility relation of a frame. The techniques used here should, therefore, be of relevance to the study of the algorithmic properties of fragments of monomodal logics of structures with a restricted branching factor, including trees.

Third, the logics of frames \(\langle N, \leq \rangle\) and \(\langle N, < \rangle\) are algorithmically hard—as follows from Theorem 4.2 below, they are \(\Pi_1^1\)-hard. Most research into the algorithmic properties of monomodal, and closely related superintuitionistic, predicate logics has been focused on decidability and undecidability. The only study to date [49], as far as we know, of the algorithmic properties of fragments of not recursively enumerable monomodal predicate logics concerns logics of frames with finite sets of worlds (likewise, very few results [16, Theorem 1, p. 272], [51] are known on algorithmic properties of fragments of not recursively enumerable superintuitionistic predicate logics). While it is natural that decidability and undecidability are the main concern of the Classical Decision Problem [9], the study of the algorithmic properties of modal first-order logics should, we believe, involve identifying minimal fragments that are as hard—in pertinent classes of the arithmetical, or the analytical, hierarchy—as the full logics.

The algorithmic properties of fragments of not recursively enumerable logics have been, however, extensively studied in the context of first-order languages more expressive than monomodal ones considered here—most recently, by Hodkinson, Wolter and Zakharyaschev [28, 61] (for a summary, see [18, Chapter 11]; for earlier work, see [2, 57, 58, 1, 35]). The methods used here have been inspired by those of Wolter and Zakharyaschev [61, Theorem 2.3], who encode a \(\Sigma_1^1\)-hard tiling problem in a first-order language with two modal operators, one corresponding to a basic accessibility relation and the other to its reflexive transitive closure. A similar result [28, Theorem 2] has been obtained by Hodkinson, Wolter and Zakharyaschev for the first-order temporal logic of \(\langle N, \leq \rangle\) in the language with two temporal operators: “next,” corresponding to the immediate successor relation on \(N\), and “always in the future,” corresponding to its reflexive transitive closure, the partial order \(\leq\) (both operators can be expressed with a binary temporal operator “until”). Another similar result [60, Theorem 5.6] has been obtained by Wolter for the first-order logics containing, alongside the individual knowledge operators, the common knowledge operator whose semantics involves the reflexive transitive closure of the union of the accessibility relations for the individual knowledge operators. We extend herein to monomodal logics, which do not have expressive power for capturing the reflexive transitive closures of accessibility relations, techniques developed by Hodkinson, Wolter, and Zakharyaschev [28, 60, 61].

\[^3\text{We touch on such languages in Section 7.}\]
The paper is structured as follows. In Section 2, we introduce preliminaries on first-order modal logic. In Section 3, we prove that satisfiability for the logic of \( \langle N, \leq \rangle \) is \( \Sigma^1_1 \)-hard in languages with two individual variables, a single monadic predicate letter and a single proposition letter. In Section 4, the results of Section 3 are extended to logics of frames \( \langle N, R \rangle \), where \( R \) is a binary relation between \( < \) and \( \leq \), and to frames based on infinite ordinals of a special form. In Section 5, we prove, by modifying the argument of Sections 3 and 4, that satisfiability for logics of \( \langle Q, \leq \rangle \), \( \langle Q, < \rangle \), \( \langle R, \leq \rangle \) and \( \langle R, < \rangle \) is \( \Pi^0_1 \)-hard in languages with the same restrictions. In Section 6, we briefly mention some corollaries of the results proven earlier. We conclude, in Section 7, by discussing first-order temporal logics with modalities “next” and “always in the future,” as well as questions for future study.

## 2 Preliminaries

An unrestricted first-order predicate modal language contains countably many individual variables; countably many predicate letters of every arity, including 0 (nullary predicate letters are also called proposition letters); the propositional constant \( \bot \) (falsity), the binary propositional connective \( \rightarrow \), the unary modal connective \( \Box \) and the quantifier \( \forall \). Formulas as well as the symbols \( \top \), \( \neg \), \( \lor \), \( \land \), \( \leftrightarrow \), \( \exists \) and \( \Diamond \) are defined in the usual way.

We also use the abbreviations \( \Box^0 \varphi = \varphi \), \( \Box^{n+1} \varphi = \Box \Box^n \varphi \) and \( \Diamond^n \varphi = \neg \Box^n \neg \varphi \), for every \( n \in \mathbb{N} \).

When parentheses are omitted, unary connectives and quantifiers are assumed to bind tighter than \( \land \) and \( \lor \), which are assumed to bind tighter than \( \rightarrow \) and \( \leftrightarrow \). We usually write atomic formulas in prefix notation; for some predicate letters we, however, use infix.

A normal predicate modal logic is a set of formulas containing the validities of the classical first-order predicate logic \( QCl \), as well as the formulas of the form \( \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \), and closed under predicate substitution, modus ponens, generalisation and necessitation.\(^4\)

In this paper, we are interested in predicate logics defined using the Kripke semantics.\(^5\)

A Kripke frame is a tuple \( \mathfrak{F} = \langle W, R \rangle \), where \( W \) is a non-empty set of possible worlds and \( R \) is a binary accessibility relation on \( W \); if \( wRv \), we say that \( v \) is accessible from \( w \) and that \( w \) sees \( v \).

A predicate Kripke frame with expanding domains is a tuple \( \mathfrak{F}_D = \langle W, R, D \rangle \), where \( \langle W, R \rangle \) is a Kripke frame and \( D \) is a function from \( W \) into the set of non-empty subsets of some set, the domain of \( \mathfrak{F}_D \); the function \( D \) is required to satisfy the condition that \( wRw' \) implies \( D(w) \subseteq D(w') \). The set \( D(w) \), also denoted by \( D_w \), is the domain of \( w \). We also consider predicate frames satisfying the stronger condition that \( D(w) = D(w') \), for every \( w, w' \in W \); such frames are predicate frames with a constant domain.\(^6\)

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\(^4\)The reader wishing a reminder of the definition of these closure conditions may consult [21, Definition 2.6.1]; for a detailed discussion of predicate substitution, consult [21, §2.3, §2.5].

\(^5\)For Kripke semantics for predicate modal logics, see [53, 55, 29, 15, 22, 10, 24], [21, §3.1].

\(^6\)More precisely, such predicate frames are known as predicate frames with globally constant domains. For connected predicate frames, the global constancy condition given above is equivalent to the local
for every world \( w \) of \( \phi \) and \( M \)

A Kripke model is a tuple \( \mathfrak{M} = \langle W, R, D, I \rangle \), where \( \langle W, R, D \rangle \) is a predicate Kripke frame and \( I \), the interpretation of predicate letters with respect to worlds in \( W \), is a function assigning to a world \( w \in W \) an \( n \)-ary predicate letter \( P \) an \( n \)-ary relation \( I(w, P) \) on \( D(w) \)—i.e., \( I(w, P) \subseteq D_w^n \). In particular, if \( p \) is a proposition letter, then \( I(w, p) \subseteq D_w^0 = \{ \langle \rangle \} \); thus, we can identify truth with \( \{ \langle \rangle \} \) and falsity with \( \emptyset \). We often write \( P^{f,w} \) instead of \( I(w, P) \). We say that a model \( \langle W, R, D, I \rangle \) is based on the frame \( \langle W, R, D \rangle \) and is based on the predicate frame \( \langle W, R, D, I \rangle \).

We use the standard notation for binary relations: the \( n \)-fold, for each \( n \in \mathbb{N}^+ \), composition of a binary relation \( R \) is denoted by \( R^n \); if \( R \) is a binary relation on a non-empty set \( W \) and \( w \in W \), then \( R(w) = \{ v \in W : wRv \} \).

An assignment in a model is a function \( g \) associating with every individual variable \( x \) an element \( g(x) \) of the domain of the underlying predicate frame. We write \( g' \models g \) if assignment \( g' \) differs from assignment \( g \) in at most the value of \( x \).

The truth of a formula \( \varphi \) at a world \( w \) of a model \( \mathfrak{M} \) under an assignment \( g \) is defined recursively:

- \( \mathfrak{M}, w \models^g P(x_1, \ldots, x_n) \) if \( \langle g(x_1), \ldots, g(x_n) \rangle \in P^{f,w} \), where \( P \) is an \( n \)-ary predicate letter;
- \( \mathfrak{M}, w \not\models^g \perp \);
- \( \mathfrak{M}, w \models^g \varphi_1 \rightarrow \varphi_2 \) if \( \mathfrak{M}, w \models^g \varphi_1 \) implies \( \mathfrak{M}, w \models^g \varphi_2 \);
- \( \mathfrak{M}, w \models^g \Box \varphi_1 \) if \( \mathfrak{M}, w' \models^g \varphi_1 \), for every \( w' \in R(w) \);
- \( \mathfrak{M}, w \models^g \forall x \varphi_1 \) if \( \mathfrak{M}, w \models^{g'} \varphi_1 \), for every \( g' \) such that \( g' = g \) and \( g'(x) \in D_w \).

Observe that, if \( \mathfrak{M} = \langle W, R, D, I \rangle \) is a Kripke model, \( w \in W \) and \( I_w(P) = I(w, P) \), then \( \mathfrak{M}_w = \langle D_w, I_w \rangle \) is a classical model, or structure.

We shall often use the following notation. Let \( \mathfrak{M} = \langle W, R, D, I \rangle \) be a model, \( w \in W \), and \( a_1, \ldots, a_n \in D_w \); let also \( \varphi(x_1, \ldots, x_n) \) be a formula whose free variables are among \( x_1, \ldots, x_n \) and \( g \) an assignment with \( g(x_1) = a_1, \ldots, g(x_n) = a_n \). Then, we write \( \mathfrak{M}, w \models \varphi(a_1, \ldots, a_n) \) instead of \( \mathfrak{M}, w \models^g \varphi(x_1, \ldots, x_n) \). This notation is unambiguous since the languages we consider lack constants and the truth value of \( \varphi(x_1, \ldots, x_n) \) does not depend on the values of variables other than \( x_1, \ldots, x_n \).

A formula \( \varphi \) is true at a world \( w \) of a model \( \mathfrak{M} \) (in symbols, \( \mathfrak{M}, w \models \varphi \), or simply \( w \models \varphi \) if \( \mathfrak{M} \) is clear from the context) if \( \mathfrak{M}, w \models^g \varphi \), for every \( g \) assigning to free variables of \( \varphi \) elements of \( D_w \). A formula \( \varphi \) is true in a model \( \mathfrak{M} \) (in symbols, \( \mathfrak{M} \models \varphi \)) if \( \mathfrak{M}, w \models \varphi \), for every world \( w \) of \( \mathfrak{M} \). A formula \( \varphi \) is valid on a predicate frame \( \mathfrak{F}_D \) if \( \varphi \) is true in every model based on \( \mathfrak{F}_D \). A formula \( \varphi \) is valid on a frame \( \mathfrak{F} \) (in symbols, \( \mathfrak{F} \models \varphi \) if \( \varphi \) is valid on every predicate frame \( \langle \mathfrak{F}, D \rangle \)). These notions, and the corresponding notation, can be extended to sets of formulas, in a natural way.

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constancy condition requiring that \( D(w) = D(w') \) whenever \( wRw' \). Since the frames we consider are rooted, and therefore connected, the distinction between global and local constancy is immaterial for the purposes of this paper.
We shall often rely on the following observation: if a model \( \mathcal{M} \) is based on a predicate frame with a constant domain, then \( \mathcal{M}, w \models \varphi \) if, and only if, \( \mathcal{M}, w \models^g \varphi \), for every assignment \( g \).

Let \( \mathcal{C} \) be a class of Kripke frames. The set of formulas valid on every frame in \( \mathcal{C} \) is a predicate modal logic, which we denote by \( \mathbf{L}(\mathcal{C}) \); we write \( \mathbf{L}(W, R) \) instead of \( \mathbf{L}(\{\langle W, R \rangle\}) \). The set of formulas valid on every predicate frame with a constant domain based on some frame in \( \mathcal{C} \) also is a predicate modal logic, which we denote by \( \mathbf{L}_c(\mathcal{C}) \); we write \( \mathbf{L}_c(W, R) \) instead of \( \mathbf{L}_c(\{\langle W, R \rangle\}) \).

3 The first-order logic of \( \langle \mathbb{N}, \leq \rangle \)

In this section, we prove that satisfiability for \( \mathbf{L}(\mathbb{N}, \leq) \) is \( \Sigma^1_1 \)-hard—hence, \( \mathbf{L}(\mathbb{N}, \leq) \) is \( \Pi^1_1 \)-hard, and therefore not recursively enumerable—in languages with two individual variables, one monadic predicate letter and one proposition letter.

3.1 Reduction from a tiling problem

We do so by encoding the following \( \Sigma^1_1 \)-complete [27, Theorem 6.4] \( \mathbb{N} \times \mathbb{N} \) recurrent tiling problem. We are given a set of tiles, a tile \( t \) being a 1 \( \times \) 1 square, with a fixed orientation, whose edges are colored with \( \text{left}(t), \text{right}(t), \text{up}(t) \) and \( \text{down}(t) \). A tile type is a quadruplet of edge colors. Each tile has a type from the set \( T = \{t_0, \ldots, t_s\} \), tiles of each type being in an unlimited supply. A tiling is an arrangement of tiles on the rectangular \( \mathbb{N} \times \mathbb{N} \) grid so that the edge colors of the adjacent tiles match, both horizontally and vertically. We are to determine whether there exists a tiling of the grid in which a tile of type \( t_0 \) occurs infinitely often in the leftmost column, i.e., whether there exists a function \( f : \mathbb{N} \times \mathbb{N} \to T \) such that, for every \( n, m \in \mathbb{N} \),

\[
(T_1) \quad \text{right}(f(n, m)) = \text{left}(f(n + 1, m)); \\
(T_2) \quad \text{up}(f(n, m)) = \text{down}(f(n, m + 1)); \\
(T_3) \quad \text{the set } \{m \in \mathbb{N} : f(0, m) = t_0\} \text{ is infinite.}
\]

The idea of the encoding we use is based on the work of Hodkinson, Wolter and Zakharyaschev [28, Theorem 2] (also see [18, Theorem 11.1]; similar constructions have been used elsewhere [56, 33, 61, 30]), but the encoding itself is more involved since our language lacks the “next” operator available to them (we touch on languages with “next” in Section 7). To make the underlying idea clearer, we construct, in the initial encoding, a formula of two individual variables without regard for the number of predicate letters involved; subsequently, we reduce the formula thus obtained to a formula with a single monadic and a single proposition letter.

Let \( \prec \) be a binary predicate letter, \( M \) and \( P_t \)—for every \( t \in T \)—monadic predicate letters and \( p \) a proposition letter.
Given a formula $\varphi$ in such a language, define

$$\Diamond \varphi = \Diamond(p \land \Diamond(\neg p \land \varphi));$$
$$\Diamond^0 \varphi = \varphi; \quad \Diamond^{n+1} \varphi = \Diamond \Diamond^n \varphi,$$
for every $n \in \mathbb{N}$.

The operator $\Diamond$ forces a transition to a different world when evaluating a formula $\Diamond \varphi$ in a reflexive model, just as $\Diamond$ does in an irreflexive one. To make this explicit, we define, given a model based on the frame $\langle \mathbb{N}, \leq \rangle$, a binary relation $R_{\Diamond}$ on $\mathbb{N}$ by

$$w R_{\Diamond} v \iff v \not\models p \text{ and, for some } u \in \mathbb{N}, \text{ both } w \leq u \leq v \text{ and } u \models p.$$ 

Thus, $R_{\Diamond}$ is irreflexive and transitive.

We also define

$$U(x) = \bigwedge_{t \in T} \neg P_t(x).$$

In such a language, define (for brevity, in formulas we write $l$, $r$, $u$ and $d$ instead of left, right, up and down)

$$A_0 = \exists x \Diamond U(x);$$
$$A_1 = \exists x (\neg U(x) \land M(x));$$
$$A_2 = \forall x \exists y (x \prec y);$$
$$A_3 = \forall x \forall y (x \prec y \rightarrow \Diamond(\exists x M(x) \rightarrow x \prec y));$$
$$A_4 = \forall x \forall y (x \prec y \rightarrow \Diamond(M(x) \leftrightarrow \neg p \land \Diamond M(y) \land \neg \Diamond^2 M(y));$$
$$A_5 = \forall x \forall y \Diamond \bigwedge_{t \in T} (M(x) \land P_t(y) \rightarrow \Diamond(M(x) \rightarrow P_t(y)));$$
$$A_6 = \forall x \Diamond \bigwedge_{t \in T} (P_t(x) \rightarrow \bigwedge_{t' \neq t} \neg P_{t'}(x));$$
$$A_7 = \forall x \forall y \Diamond \bigwedge_{t \in T} (x \prec y \land P_t(x) \rightarrow \bigvee_{r(t) = l(t')} P_{t'}(y));$$
$$A_8 = \forall x \forall y \Diamond \bigwedge_{t \in T} (M(x) \land P_t(y) \rightarrow \Diamond(\exists y (x \prec y \land M(y)) \rightarrow \bigvee_{u(t) = d(t')} P_{t'}(y)));$$
$$A_9 = \forall x (M(x) \rightarrow \Diamond \Diamond P_{t_0}(x)).$$

Let $A$ be the conjunction of $A_0$ through $A_9$. Observe that $A$ contains only two individual variables.

The relation $\prec$ can be thought of as the immediate successor relation on the domain $D_0$ of the world $0$ where $A$ is being evaluated. An element $a \in D_0$ such that $w \models M(a)$ can be thought of as marking, or labelling, world $w$; thus, we say that $a$ is a mark of $w$. Then, $A_2$ asserts that every element of $D_0$ has an immediate successor, while $A_3$ asserts that the immediate successor relation persists throughout the part of the frame where worlds are marked by elements of $D_0$. Given that, $A_1$ and $A_4$ imply the existence of an infinite
sequence \( a_0 \triangleq a_1 \triangleq a_2 \triangleq \ldots \) of elements of \( D_0 \) such that every world refuting \( p \) is marked, as we shall see uniquely, by some element of the sequence; they also imply that the order of the marks of successive, with respect to \( \leq \), worlds agrees with the relation \( \triangleq \). This, as we shall see, gives us an \( \mathbb{N} \times \mathbb{N} \) grid whose rows correspond to the worlds of \( \langle \mathbb{N}, \leq \rangle \) and whose columns correspond to the elements of the sequence \( a_0 \triangleq a_1 \triangleq a_2 \triangleq \ldots \). Building on this, \( A_5 \) through \( A_9 \) describe a sought tiling of thus obtained grid. (The element of \( D_0 \) whose existence is asserted by \( A_0 \) is not part of the tiling—its presence shall be relied upon in a subsequent reduction.)

**Lemma 3.1** There exists a recurrent tiling of \( \mathbb{N} \times \mathbb{N} \) satisfying \( (T_1) \) through \( (T_3) \) if, and only if, \( \langle \mathbb{N}, \leq \rangle \not\models \neg A \).

**Proof.** ("if") Suppose \( \mathfrak{M}, \mathfrak{w}_0 \models A \), for some model \( \mathfrak{M} = \langle \mathbb{N}, \leq, D, I \rangle \) and some world \( \mathfrak{w}_0 \in \mathbb{N} \). Since truth of formulas is preserved under taking generated submodels,\(^7\) we may assume \( \mathfrak{w}_0 = 0 \).

Since \( 0 \models A_1 \), there exists \( a_0 \in D_0 \) such that \( 0 \not\models U(a_0) \) and \( 0 \models M(a_0) \). Since \( 0 \models A_2 \), we obtain an infinite sequence \( a_0, a_1, a_2, \ldots \) of elements of \( D_0 \) such that \( a_0 \triangleq a_1 \triangleq \ldots \). Since \( 0 \models A_3 \), we obtain that \( a_0 \triangleq a_1 \triangleq \ldots \), for every \( w \in \mathbb{N} \) such that \( w \models \exists x. M(x) \).

Since \( 0 \models A_4 \), we obtain, for every \( w, n \in \mathbb{N} \),

\[
 w \models M(a_n) \iff \begin{cases} w \not\models p; \\ w' \models M(a_{n+1}), \text{ for some } w' \in R_\Diamond(w); \\ w'' \in R\Diamond_{\nabla}(w) \text{ implies } w'' \not\models M(a_{n+1}). \end{cases} \tag{1}
\]

Thus, a mark changes from \( a_n \) to \( a_{n+1} \) once we pass through a world, or an unbroken non-empty sequence of worlds, satisfying \( p \) to a world refuting \( p \).

We now show that a mark remains unchanged until we have reached a world satisfying \( p \), i.e., that for every \( u, u', n \in \mathbb{N} \),

\[
 \text{if } u \models M(a_n), \text{ } u \not\models p, \text{ } u' \not\models p, \text{ and no } v \text{ with } u \leq v \leq u' \text{ or } u' \leq v \leq u \text{ satisfies } v \models p, \text{ then } u' \models M(a_n). \tag{2}
\]

Assume that \( u \models M(a_n), \text{ } u \not\models p, \text{ } u' \not\models p, \text{ and that no } v \text{ with } u \leq v \leq u' \text{ or } u' \leq v \leq u \text{ satisfies } v \models p \). Then, by (1), there exists \( w' \in R_\Diamond(u) \) such that \( w' \models M(a_{n+1}) \), and \( w'' \not\models M(a_{n+1}) \), for every \( w'' \in R\Diamond_{\nabla}(u) \). Let us fix the said \( w' \). It follows immediately from the assumption that, for every \( w, n \in \mathbb{N} \),

\[
 w \in R\Diamond_{\nabla}(u) \iff w \in R\Diamond_{\nabla}(u').
\]

Therefore, \( w' \in R_\Diamond(u') \), and \( w'' \in R\Diamond_{\nabla}(u') \) implies \( w'' \not\models M(a_{n+1}) \), for every \( w'' \in \mathbb{N} \). Since by assumption \( u' \not\models p \), we obtain, by (1), that \( u' \models M(a_n) \).

---

\(^7\)The notions of generated subframe and generated submodel for predicate modal logics are straightforward extensions of the respective notions [7, Section 2.1] for propositional modal logics.
We next show that a mark of every world is unique, i.e. for every \( w, n \in \mathbb{N} \) and every \( j \in \mathbb{N}^+ \),
\[
w \models M(a_n) \text{ implies } w \not\models M(a_{n+j}). \tag{3}
\]
Assume \( w \models M(a_n) \). By (1), there exists \( w' \in R^{j+1}_\varnothing(w) \) such that \( w' \models M(a_{n+j+1}) \). Since \( j \geq 1 \) and \( R_\varnothing \) is transitive, \( w' \in R^j_\varnothing(w) \). Therefore, by (1), \( w \not\models M(a_{n+j}) \).

We next show that every element \( a_n \) is tiled at every world marked by some element \( a_m \), i.e. that for every \( w, m, n \in \mathbb{N} \),
\[
w \models M(a_n) \text{ implies } w \models P_t(a_n), \text{ for some } t \in T. \tag{4}
\]
We proceed by induction on \( m \).

As we have seen, \( 0 \not\models U(a_0) \), i.e., \( 0 \not\models P_t(a_0) \), for some \( t \in T \). Since \( 0 \models A_7 \), for every \( n \in \mathbb{N} \), there exists \( t \in T \) such that \( 0 \models P_t(a_n) \). Since \( 0 \models A_5 \), for every \( w, v, m, n \in \mathbb{N} \) and every \( t \in T \),
\[
w \models M(a_m), v \models M(a_m) \text{ and } w \models P_t(a_n) \text{ imply } v \models P_t(a_n). \tag{5}
\]
Therefore, (4) holds for \( m = 0 \).

Assume (4) holds for \( m \geq 0 \) and suppose \( w \models M(a_{m+1}) \). We claim that, then, there exists \( w' \) such that \( w' < w \) and \( w' \models M(a_m) \). To prove the claim we, first, observe that there exists \( w' \) such that \( w' \not\models p \) and \( w \in R_\varnothing(w') \); otherwise, by (2), \( w \models M(a_0) \), in contradiction with (3). Fix the said \( w' \). We next show that \( w'' \in R^2_\varnothing(w') \) implies \( w'' \not\models M(a_{m+1}) \). Assume \( w'' \in R^2_\varnothing(w') \). Then, \( w'' \in R_\varnothing(w) \). Since \( w \models M(a_{m+1}) \), by (1) and (2), \( w'' \models M(a_{m+2}) \) and thus, by (3), \( w'' \not\models M(a_{m+1}) \). Last, since \( w' \not\models p \), we obtain, by (1), \( w' \models M(a_m) \), thereby proving the claim.

Now, let \( n \in \mathbb{N} \) be given. By inductive hypothesis, there exists \( t \) such that \( w' \models P_t(a_n) \). Since \( 0 \models A_8 \), this implies that \( w \models P_{t'}(a_n) \), for some \( t' \in T \). Thus, (4) is proven.

In view of (5), for every \( m \in \mathbb{N} \), we may pick an arbitrary world marked by \( a_m \in D_0 \) to be part of the sought tiling. For definiteness, let, for every \( m \in \mathbb{N} \),
\[
w_m = \min\{w \in \mathbb{N} : w \models M(a_m)\}.
\]

By (4), for every \( n, m \in \mathbb{N} \), there exists \( t \in T \) such that \( w_m \models P_t(a_n) \); it follows from \( 0 \models A_6 \) that such \( t \) is unique. We can, therefore, define a function \( f : \mathbb{N} \times \mathbb{N} \to T \) by
\[
f(n, m) = t \text{ whenever } w_m \models P_t(a_n).
\]

We next show that \( f \) satisfies \((T_1)\) through \((T_3)\).

Since \( 0 \models A_7 \), the condition \((T_1)\) is, evidently, satisfied.

To see that \((T_2)\) is satisfied, assume \( f(n, m) = t \). Then, \( w_m \models P_t(a_n) \), by definition of \( f \). From the definition of \( w_m \) we know that \( w_m \models M(a_m) \). Since \( 0 \models A_8 \), if \( v \geq w_m \) and \( v \models M(a_{m+1}) \), then \( v \models P_{t'}(a_n) \), for some \( t' \) with \( up(t) = \text{down}(t') \). We next show that \( w_{m+1} \geq w_m \). Assume otherwise: let \( w_{m+1} < w_m \). From the definition of \( w_{m+1} \) we know that \( w_{m+1} \models M(a_{m+1}) \). Therefore, by (2) and (3), \( w_m \in R_\varnothing(w_{m+1}) \). Since \( w_m \models M(a_m) \), there exists, by (1), \( w' \in R^{2}_\varnothing(w_m) \) such that \( w' \models M(a_{m+2}) \). Since \( R_\varnothing \) is transitive,
Figure 1: Model \( \mathfrak{M}_0 \)

\[
\begin{array}{ccccccc}
  & 7 & p & & & & \\
 6 & M(3) & P_f(0,3)(0) & P_f(1,3)(1) & P_f(2,3)(2) & P_f(3,3)(3) & \cdots \\
 5 & p & & & & & \\
 4 & M(2) & P_f(0,2)(0) & P_f(1,2)(1) & P_f(2,2)(2) & P_f(3,2)(3) & \cdots \\
 3 & p & & & & & \\
 2 & M(1) & P_f(0,1)(0) & P_f(1,1)(1) & P_f(2,1)(2) & P_f(3,1)(3) & \cdots \\
 1 & p & & & & & \\
 0 & M(0) & P_f(0,0)(0) & P_f(1,0)(1) & P_f(2,0)(2) & P_f(3,0)(3) & \cdots \\
\end{array}
\]

\[0 < 1 < 2 < 3 \cdots\]

\[w' \in R_\diamondsuit^2(w_{m+1}),\text{ in contradiction with the third clause of (1). Thus, we have shown that } w_{m+1} \geq w_m. \text{ Hence, } w_{m+1} \models P_r(a_n), \text{ for some } t' \text{ with } up(t) = down(t'). \text{ Therefore, } (T_2) \text{ is satisfied.}\]

It remains to show that \((T_3)\) is satisfied. Since \(0 \models M(a_0)\) and \(0 \models A_9\), the set \(\{ w \in \mathbb{N} : w \not\models p \text{ and } w \models P_0(a_0) \}\) is infinite. It follows from \(0 \models M(a_0), \text{ (1) and (2) that, for every } w \in \mathbb{N}, \text{ if } w \not\models p, \text{ then there exists } m \in \mathbb{N} \text{ such that } w \models M(a_m). \text{ Therefore, by (5), the set } \{ w_m : m \in \mathbb{N} \text{ and } w_m \models P_0(a_0) \}\) is infinite. Hence, \((T_3)\) is satisfied.

Thus, \(f\) is a required function.

("only if") Suppose \(f\) is a function satisfying \((T_1)\) through \((T_3)\). We obtain a model based on \(\langle \mathbb{N}, \leq, A \rangle\) satisfying \(A\).

Let \(D = \mathbb{N} \cup \{-1\}\) and \(D(w) = D, \text{ for every } w \in \mathbb{N}\).

Let \(\mathfrak{M}_0 = \langle \mathbb{N}, \leq, D, I \rangle\) be a model such that, for every \(w \in \mathbb{N}\) and every \(a, b \in D,\)

\[
\begin{align*}
\mathfrak{M}_0, w \models a < b & \implies w \text{ is even and } b = a + 1; \\
\mathfrak{M}_0, w \models p & \implies w \text{ is odd}; \\
\mathfrak{M}_0, w \models M(a) & \implies w = 2a; \\
\mathfrak{M}_0, w \models P_1(a) & \implies \text{ for some } m \in \mathbb{N}, \text{ both } w = 2m \text{ and } f(a, m) = t.
\end{align*}
\]

It is straightforward to check that \(\mathfrak{M}_0, 0 \models A\), so we leave this to the reader. \(\square\)

Thus, in the proof of the "if" part of Lemma 3.1, we obtained a grid for the tiling by treating the worlds of model \(\mathfrak{M}\) as rows and elements \(a_0, a_1, a_2, \ldots\) of the domain \(D_0\) of the world 0 satisfying \(A\) as columns.
3.2 Elimination of the binary predicate letter

We next eliminate, following ideas of Kripke’s [31], the binary predicate letter $\triangleleft$ of formula $A$, without increasing the number of individual variables in the resultant formula.

From now on, we assume, for ease of notation, that $A$ contains monadic predicate letters $P_0, \ldots, P_s$—rather than $P_t$, for each $t \in \{t_0, \ldots, t_s\}$—to refer to the tile types.

Recall that Kripke’s construction [31] transforms a model $\mathcal{M}$ satisfying, at world $w$, a formula containing a binary predicate letter, and no modal connectives, so that, for every pair of elements of the domain of $w$, a fresh world accessible from $w$ is introduced to $\mathcal{M}$. This construction cannot be applied here in a straightforward manner, for two reasons.

First, since we are working with the frame $\langle N, \le \rangle$, we may not introduce fresh worlds to a model satisfying $A$; we, rather, have to use the worlds from $N$ to simulate $\triangleleft$. Second, since $\triangleleft$ occurs within the scope of the modal connective in $A$, we need to simulate the interpretation of $\triangleleft$ not just at the world satisfying $A$, but at every world accessible from it.

We resolve these difficulties by working with the model $\mathcal{M}_0$ defined in the “only if” part of the proof of Lemma 3.1, rather than with an arbitrary model satisfying $A$, and relying on $\mathcal{M}_0$ being based on a frame with a constant domain and on the interpretation of $\triangleleft$ being identical at every world of $\mathcal{M}_0$.

Let $P_{s+1}$ and $P_{s+2}$ be monadic predicate letters distinct from $M, P_0, \ldots, P_s$ and from each other, and let $\cdot'$ be the function substituting $\Diamond(P_{s+1}(x) \land P_{s+2}(y))$ for $x \triangleleft y$.

**Lemma 3.2** There exists a recurrent tiling of $N \times N$ satisfying $(T_1)$ through $(T_3)$ if, and only if, $\langle N, \le \rangle \not\models \neg A'$.

**Proof.** ("if") Suppose $\mathcal{M}, w_0 \models A'$, for some model $\mathcal{M} = \langle N, \le, D, I \rangle$ and some world $w_0$, which can be assumed to be 0.

The argument is essentially the same as in the proof of the “if” part of Lemma 3.1. The only, inconsequential, difference is that $\Diamond(P_{s+1}(x) \land P_{s+2}(y))$ now plays the role of $x \triangleleft y$: for every $w \in N$, the relation $I(w, \triangleleft) \subseteq D_w \times D_w$ is replaced by the relation

$$\{\langle a, b \rangle \in D_w \times D_w : \mathcal{M}, w \models \Diamond(P_{s+1}(a) \land P_{s+2}(b))\}.$$ 

Since $\mathcal{M}, 0 \models A'$, the two relations are indistinguishable, for every $w \in N$, with respect to the properties we rely on in the proof.

("only if") Suppose $f$ is a function satisfying $(T_1)$ through $(T_3)$. Let $\mathcal{M}_0 = \langle N, \le, D, I \rangle$ be the model defined in the “only if” part of the proof of Lemma 3.1. As we have seen, $\mathcal{M}_0, 0 \models A$. We use $\mathcal{M}_0$ to obtain a model satisfying $A'$.

Let $\alpha$ be the infinite sequence

$$0, 0, 1, 0, 1, 2, 0, 1, 2, 3, 0, 1, 2, 3, 4, \ldots$$

and let $\alpha_k$, for each $k \in N$, be the $k$th element of $\alpha$. 

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Let $\mathcal{M}_0 = \langle \mathbb{N}, \leq, D, I' \rangle$ be a model such that, for every $w, c \in \mathbb{N}$,

$\mathcal{M}_0, w \models P_{s+1}(c) \iff$ for some $m \in \mathbb{N}$, both $w = 2m$ and $c = \alpha_m$;

$\mathcal{M}_0, w \models P_{s+2}(c) \iff$ for some $m \in \mathbb{N}$, both $w = 2m$ and $c = \alpha_m + 1$,

and for every $w \in \mathbb{N}$ and every $S \in \{P_0, \ldots, P_s, M, p\}$,

$I'(w, S) = I(w, S)$.

We show that $\mathcal{M}_0, 0 \models A'$.

Since $\mathcal{M}_0, 0 \models A$, it suffices to prove that, for every $m \in \mathbb{N}$ and every $a, b \in D$,

$\mathcal{M}_0, 2m \models a \triangleleft b \iff \mathcal{M}_0, 2m \models \Box((P_{s+1}(a) \land P_{s+2}(b)))$.

Assume $\mathcal{M}_0, 2m \models a \triangleleft b$. Then, $b = a + 1$, by definition of $\mathcal{M}_0$. Choose $k \in \mathbb{N}$ so that $k > m$ and $\alpha_k = a$; by definition of $\alpha$, such a number $k$ certainly exists. By definition of $\mathcal{M}_0$, both $\mathcal{M}_0, 2k \not\models p$ and $\mathcal{M}_0, 2k \models P_{s+1}(a) \land P_{s+2}(b)$. By the same definition, $\mathcal{M}_0, 2k - 1 \models p$. Hence, $\mathcal{M}_0, 2m \models \Box((P_{s+1}(a) \land P_{s+2}(b)))$.

Conversely, assume $\mathcal{M}_0, 2m \models \Box((P_{s+1}(a) \land P_{s+2}(b)))$. Then, for some $v > 2m$, both $\mathcal{M}_0, v \not\models p$ and $\mathcal{M}_0, v \models P_{s+1}(a) \land P_{s+2}(b)$. By definition of $\mathcal{M}_0$, we have $\mathcal{M}_0, v \not\models p$; hence $v = 2k$, for some $k > m$. Also by definition of $\mathcal{M}_0$, both $a = \alpha_k$ and $b = \alpha_k + 1$; hence, $b = a + 1$. Therefore, $\mathcal{M}_0, 2m \models a \triangleleft b$, by definition of $\mathcal{M}_0$.

3.3 Elimination of monadic predicate letters

We lastly simulate the occurrences of letters $p, M, P_0, \ldots, P_{s+2}$ in $A'$ with one monadic and one proposition letter, without increasing the number of individual variables in the resultant formula.

Let $P$ be a monadic letter distinct from $M, P_0, \ldots, P_{s+2}$, and let $q$ be a proposition letter distinct from $p$.

For a formula $\varphi$ in the language containing $P$ and $q$, define

$\Box \varphi = \Box(\forall x P(x) \land \Box(\neg \forall x P(x) \land \varphi))$;

$\Box^0 \varphi = \varphi$; $\Box^{n+1} \varphi = \Box \Box^n \varphi$, for every $n \in \mathbb{N}$.

Define, for every $n \in \{0, \ldots, s+2\}$,

$\beta_n(x) = \exists y (\Box^{s+4}(q \land P(y)) \land \neg \Box^{s+5}(q \land P(y)) \land \Box(\Box^{n+1}(q \land P(y)) \land \neg \Box^{n+2}(q \land P(y)) \land P(x)))$;

$\beta_n(y) = \exists x (\Box^{s+4}(q \land P(x)) \land \neg \Box^{s+5}(q \land P(x)) \land \Box(\Box^{n+1}(q \land P(x)) \land \neg \Box^{n+2}(q \land P(x)) \land P(y)))$.

Let $\cdot^*$ be the function replacing

- $P_n(x)$ with $\beta_n(x)$, for every $n \in \{0, \ldots, s+2\}$;

- $P_n(y)$ with $\beta_n(y)$, for every $n \in \{0, \ldots, s+2\}$;
• $P_n(y)$ with $\beta_n(y)$, for every $n \in \{0, \ldots, s+2\}$;

• $M(x)$ with $q \land P(x)$;

• $M(y)$ with $q \land P(y)$.

Let $A_i^*$, for each $i$ with $0 \leq i \leq 8$ and $i \neq 4$, be the result of applying the function $\cdot^*$ to $A_i'$. Also, let

$$A_i^* = \forall x \forall y \left( \Box (\beta_{s+1}(x) \land \beta_{s+2}(y)) \rightarrow \Box(q \land P(x)) \land \Box^{*+4}(q \land P(y)) \land \neg \Box^{*+5}(q \land P(y)) \right),$$

and

$$A_9^* = \forall x (q \land P(x) \rightarrow \Box \beta_0(x)).$$

Lastly, let $A^*$ be the conjunction of $A_0^*$ through $A_9^*$. Observe that $A^*$ contains only two individual variables, a monadic letter $P$ and a proposition letter $q$.

We shall show that $A^*$ is satisfiable if, and only if, there exists a recurrent tiling satisfying $(T_1)$ through $(T_3)$.

To obtain a model satisfying $A^*$, we “stretch out” the model $\mathcal{M}_0$ defined in the “only if” part of the proof of Lemma 3.2 to include “additional” worlds whose sole purpose is to simulate the interpretation of letters $P_0, \ldots, P_{s+2}$ at worlds of $\mathcal{M}_0$. We “insert” $s + 3$ worlds between worlds $m$ and $m + 1$ to simulate the interpretation of letters $P_0, \ldots, P_{s+2}$ at $m$. The interpretation of $P_n$, for each $n \in \{0, \ldots, s+2\}$, at $m$ is simulated by the interpretation of letter $P$ at a newly inserted world “$n$ steps away from” $m + 1$. To be able to step through the newly defined model, we also “insert” extra worlds satisfying $\forall x P(x)$; these play the same role the worlds satisfying $p$ played in $\mathcal{M}_0$. The proposition letter $q$ marks off the “old” worlds from $\mathcal{M}_0$. The resultant model is depicted in Figure 2, where $\beta_{f(a,b)}(x)$ stands for $\beta_n(x)$, where $n$ is such that $f(a, b) = t_n$.

**Lemma 3.3** There exists a recurrent tiling of $\mathbb{N} \times \mathbb{N}$ satisfying $(T_1)$ through $(T_3)$ if, and only if, $(\mathbb{N}, \leq) \not\models \neg A^*$.

**Proof.** ("if") Suppose $\mathcal{M}, w_0 \models A^*$, for some model $\mathcal{M} = (\mathbb{N}, \leq, D, I)$ and some world $w_0$, which can be assumed to be 0.

The argument is essentially the same as in the proof of the “if” part of Lemma 3.2, the only difference being that we use

• $\beta_n(x)$ instead of $P_n(x)$;

• $\beta_n(y)$ instead of $P_n(y)$;

• $q \land P(x)$ instead of $M(x)$;

• $q \land P(y)$ instead of $M(y)$. 

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Figure 2: Model $\mathcal{M}_0^*$
(“only if”) Suppose \( f \) is a function satisfying \((T_1)\) through \((T_3)\). Let \( \mathcal{M}_0 = (\mathbb{N}, \leq, D, I') \) be the model defined in the “only if” part of the proof of Lemma 3.2. As we have seen, \( \mathcal{M}_0, 0 \models A' \). We use \( \mathcal{M}_0 \) to obtain a model satisfying \( A^* \).

We think of the worlds from \( \mathbb{N} \) as being labeled, in the ascending order,

\[
\begin{align*}
& w_0, \bar{w}_0, v_0^0, \bar{v}_0^0, \ldots, v_0^0, \bar{v}_0^0, \\
& w_1, \bar{w}_1, v_1^0, \bar{v}_1^0, \ldots, v_1^0, \bar{v}_1^0, \\
& w_2, \bar{w}_2, \ldots,
\end{align*}
\]

i.e., we put \( w_0 = 0, \bar{w}_0 = 1, v_0^0 = 2, \) etc.

Let \( \mathcal{M}_0 = (\mathbb{N}, \leq, D, I^*) \) be a model such that, for every \( u \in \mathbb{N} \)

\[
\mathcal{M}_0, u \models q \iff u = w_m, \text{ for some } m \in \mathbb{N},
\]

and for every \( u \in \mathbb{N} \) and every \( a \in D \), the relation \( \mathcal{M}_0, u \models P(a) \) holds if, and only if, one of the following conditions is satisfied:

- \( u = w_m \) and \( \mathcal{M}_0, 2m \models M(a) \), for some \( m \in \mathbb{N} \);
- \( u = v_n^m \) and \( \mathcal{M}_0, 2m \models P_n(a) \), for some \( m \in \mathbb{N} \) and some \( n \in \{0, \ldots, s + 2\} \);
- \( u = \bar{w}_m \), for some \( m \in \mathbb{N} \);
- \( u = \bar{v}_m^n \), for some \( m \in \mathbb{N} \) and some \( n \in \{0, \ldots, s + 2\} \).

Thus, by definition of \( \mathcal{M}_0 \),

\[
\mathcal{M}_0, w_m \models q \land P(a) \iff a = m.
\] (6)

We now prove that \( \mathcal{M}_0, w_0 \models A^* \).

First, we show that

\[
\mathcal{M}_0, u \models \forall x P(x) \iff u \in \{\bar{w}_m : m \in \mathbb{N}\} \cup \{\bar{v}_m^n : m \in \mathbb{N}, 0 \leq n \leq s + 2\}.
\] (7)

The right-to-left implication is immediate from the definition of \( \mathcal{M}_0 \).

For the converse, assume \( u \notin \{\bar{w}_m : m \in \mathbb{N}\} \cup \{\bar{v}_m^n : m \in \mathbb{N}, 0 \leq n \leq s + 2\} \).

We have four cases to consider.

Case \( u = w_m \): The definition of \( \mathcal{M}_0 \) implies that \( \mathcal{M}_0, w_m \not\models P(c) \), for every \( c \in D \setminus \{m\} \). Since \( D \setminus \{m\} \neq \emptyset \), we obtain \( \mathcal{M}_0, w_m \not\models \forall x P(x) \).

Case \( u = v_m^{s+1} \): The definition of \( \mathcal{M}_0 \) implies that \( \mathcal{M}_0, v_m^{s+1} \models P(a) \) if, and only if, \( \mathcal{M}_0, 2m \models P_{s+1}(a) \), which by definition of \( \mathcal{M}_0 \), holds if, and only if, \( \mathcal{M}_0, 0 \models a \land b \) and \( \alpha_m = a \). Therefore, \( \mathcal{M}_0, v_m^{s+1} \not\models P(c) \), for every \( c \in D \setminus \{\alpha_m\} \). Since \( D \setminus \{\alpha_m\} \neq \emptyset \), we obtain \( \mathcal{M}_0, v_m^{s+1} \not\models \forall x P(x) \).

Case \( u = v_m^{s+2} \): The definition of \( \mathcal{M}_0 \) implies that \( \mathcal{M}_0, v_m^{s+2} \models P(a) \) if, and only if, \( \mathcal{M}_0, 2m \models P_{s+2}(a) \), which by definition of \( \mathcal{M}_0 \), holds if, and only if, \( a = \alpha_m + 1 \). Therefore, \( \mathcal{M}_0, v_m^{s+2} \not\models P(c) \), for every \( c \in D \setminus \{\alpha_m + 1\} \). Since \( D \setminus \{\alpha_m + 1\} \neq \emptyset \), we obtain \( \mathcal{M}_0, v_m^{s+2} \not\models \forall x P(x) \).
Case $u = v^n_m$, where $n \in \{0, \ldots, s\}$: By definitions of $\mathcal{M}_0^*$ and $\mathcal{M}_0'$,
\[
\mathcal{M}_0^*, v^n_m \models P(a) \iff \mathcal{M}_0', 2m \models P_n(a) \iff \mathcal{M}_0, 2m \models P_n(a).
\]
The definition of $\mathcal{M}_0$ implies that $\mathcal{M}_0, 2m \not\models P_n(-1)$. Therefore, $\mathcal{M}_0^*, v^n_m \not\models P(-1)$; hence, $\mathcal{M}_0^*, v^n_m \not\models \forall x P(x)$.
Thus, $\mathcal{M}_0^*, u \not\models \forall x P(x)$, and so (7) is proven.

Next, we show that, for every $m \in \mathbb{N}$, every $n \in \{0, \ldots, s + 2\}$ and every $a \in \mathcal{D}$,
\[
\mathcal{M}_0^*, w_m \models \beta_n(a) \iff \mathcal{M}_0', 2m \models P_n(a). \tag{8}
\]

First, define a binary relation $R_\Diamond$ on $\mathbb{N}$ by
\[
w R_\Diamond v \iff v \not\models \forall x P(x) \text{ and, for some } w \in \mathbb{N}, \text{ both } w \leq u \leq v \text{ and } u \models \forall x P(x).
\]
Now, assume $\mathcal{M}_0', 2m \models P_n(a)$.

By (6), $\mathcal{M}_0', w_{m+1} \models q \land P(m+1)$. By (7) and the definition of $\mathcal{M}_0^*$,
\begin{itemize}
  \item $w_{m+1} \in R_\Diamond^{s+4}(w_m) - R_\Diamond^{s+5}(w_m)$;
  \item $w_m < v^n_m$;
  \item $w_{m+1} \in R_\Diamond^{n+1}(v^n_m) - R_\Diamond^{n+2}(v^n_m)$;
  \item $\mathcal{M}_0^*, v^n_m \models P(a)$.
\end{itemize}

Therefore, $\mathcal{M}_0^*, w_m \models \beta_n(a)$.

Conversely, assume $\mathcal{M}_0^*, w_m \models \beta_n(a)$.

Then,
\[
\mathcal{M}_0^*, w_m \models \exists y (\Diamond^{s+4}(q \land P(y)) \land \neg \Diamond^{s+5}(q \land P(y))).
\]

Hence, there exist $u \in \mathbb{N}$ and $b \in \mathcal{D}$ such that
\[
\mathcal{M}_0^*, u \models q \land P(b) \text{ and } u \models R_\Diamond^{s+4}(w_m) - R_\Diamond^{s+5}(w_m).
\]

By definition of $\mathcal{M}_0^*$ and by (6), the only choices for $u$ and $b$ are, respectively, $w_{m+1}$ and $m+1$. Hence,
\[
\mathcal{M}_0^*, w_m \models \Diamond(\Diamond^{n+1}(q \land P(m+1)) \land \neg \Diamond^{n+2}(q \land P(m+1)) \land P(a)).
\]

Thus, by definition of $\mathcal{M}_0^*$, we obtain $\mathcal{M}_0^*, v^n_m \models P(a)$ and, hence, $\mathcal{M}_0', 2m \models P_n(a)$. Thus, (8) is proven.

From (6), (7) and (8), we obtain $\mathcal{M}_0^*, w_0 \models A_i^*$, for each $i$ with $0 \leq i \leq 8$ and $i \neq 4$. Furthermore, based on (6), (7) and (8), it is straightforward to check that $\mathcal{M}_0^*, w_0 \models A_4^*$ and $\mathcal{M}_0^*, w_0 \models A_5^*$.

Thus, $\mathcal{M}_0^*, w_0 \models A^*$. \hfill \square

From Lemma 3.3 we immediately obtain the following result:

**Theorem 3.4** Satisfiability for $\mathbf{L}(\mathbb{N}, \leq)$ is $\Sigma_1^1$-hard in languages with two individual variables, one monadic predicate letter and one proposition letter.
4 Logics of discrete linear orders

We now generalise Theorem 3.4 to logics of discrete linear orders other than $\langle \mathbb{N}, \leq \rangle$.

We first consider logics of frames based on $\mathbb{N}$. Let $R$ be a binary relation on $\mathbb{N}$ between $<$ and $\leq$. Define

$$\Box^+ \varphi = \varphi \land \Box \varphi.$$  

Then, the relation $\leq$ is the reflexive closure of $R$ and, hence, it is the accessibility relation associated with the operator $\Box^+$: for every model $\mathfrak{M} = \langle \mathbb{N}, R, D, I \rangle$, every $w \in \mathbb{N}$ and every assignment $g$,

$$\mathfrak{M}, w \models^g \Box^+ \varphi \iff \mathfrak{M}, w' \models^g \varphi, \text{ for every } w' \in \mathbb{N} \text{ such that } w \leq w'.$$

Let $A^+$ to be the formula obtained from $A^*$ by replacing every occurrence of $\Box$ with an occurrence of $\Box^+$. The noted correspondence between $\Box^+$ and $\leq$, as well as their connection with, respectively, $\Box$ and $R$, give us the following analogue of Lemma 3.3:

**Lemma 4.1** There exists a recurrent tiling of $\mathbb{N} \times \mathbb{N}$ satisfying $(T_1)$ through $(T_3)$ if, and only if, $\langle \mathbb{N}, R \rangle \not\models \neg A^+$.

From Lemma 4.1, we obtain the analogue of Theorem 3.4 for $L(\mathbb{N}, R)$. Moreover, since none of the arguments made so far depend on the assumption of properly expanding domains, we obtain the following generalisation of Theorem 3.4:

**Theorem 4.2** Let $R$ be a binary relation on $\mathbb{N}$ between $<$ and $\leq$, and let $L$ be a logic such that $L(\mathbb{N}, R) \subseteq L \subseteq L_c(\mathbb{N}, R)$. Then, satisfiability for $L$ is $\Sigma^1_1$-hard in languages with two individual variables, one monadic predicate letter and one proposition letter.

As we next observe, Theorem 4.2 covers countably many logics, countably many pairs of which are incompatible. First, note that $L(\mathbb{N}, <)$ and $L(\mathbb{N}, \leq)$ are incompatible. Let

$$Z = \Box(\Box p \rightarrow p) \rightarrow (\Diamond \Box p \rightarrow \Box p);$$  

$$\text{ref} = \Box p \rightarrow p.$$
It is well known [23] that \( \langle \mathbb{N}, \leq \rangle \models Z \), but \( \langle \mathbb{N}, \leq \rangle \not\models Z \); hence, \( L(\mathbb{N}, <) \not\subseteq L(\mathbb{N}, \leq) \). It is also clear that \( \langle \mathbb{N}, \leq \rangle \models ref \), but \( \langle \mathbb{N}, < \rangle \not\models ref \); hence, \( L(\mathbb{N}, \leq) \not\subseteq L(\mathbb{N}, <) \).

Generalising this observation, we obtain countably many logics, countably many pairs of which are incompatible. Let

\[
\boxtimes \varphi = (q \land \Box(\neg q \rightarrow \varphi)) \lor (\neg q \land \Box(q \rightarrow \varphi)).
\]

Let \( \mathcal{G}_n \) be the reflexive chain \( 0, \ldots, n - 1 \), followed by the infinite reflexive chain \( n, n+1, \ldots \), shown in Figure 3 on the left. Dually, let \( \mathcal{H}_n \) be the reflexive chain \( 0, \ldots, n - 1 \), followed by the infinite reflexive chain \( n, n+1, \ldots \), shown in Figure 3 on the right.

We show that \( L(\mathcal{G}_k) \neq L(\mathcal{G}_m) \) and \( L(\mathcal{H}_k) \neq L(\mathcal{H}_m) \) provided \( k \neq m \). Indeed, \( k > m \) implies \( L(\mathcal{G}_k) \subseteq L(\mathcal{G}_m) \): if \( \mathcal{G}_m, s \models \varphi \) then \( \mathcal{G}_k, s \not\models \varphi \) since \( \mathcal{G}_m \) is a generated subframe of \( \mathcal{G}_k \). Also, \( \mathcal{G}_n \models \Box^n ref \), but \( \mathcal{G}_{n+1}, 0 \not\models \Box^n ref \). Hence, \( L(\mathcal{G}_k) \neq L(\mathcal{G}_m) \) if \( k \neq m \). A similar argument, using the formula \( \boxtimes^n Z \) to distinguish \( L(\mathcal{H}_{n+1}) \) from \( L(\mathcal{H}_n) \), shows that \( L(\mathcal{H}_k) \neq L(\mathcal{H}_m) \) if \( k \neq m \).

Thus, we have infinitely many logics \( L(\mathcal{G}_n) \) and infinitely many logics \( L(\mathcal{H}_n) \). Note that, for every \( k, m \in \mathbb{N} \), logics \( L(\mathcal{G}_k) \) and \( L(\mathcal{H}_m) \) are incompatible since, for every \( k, m \in \mathbb{N} \), both \( \Box^{k+m} ref \in L(\mathcal{G}_k) - L(\mathcal{H}_m) \) and \( \boxtimes^{k+m} Z \in L(\mathcal{H}_m) - L(\mathcal{G}_k) \).

From Theorem 4.2, we obtain the following:

**Corollary 4.3** Satisfiability for \( L_c(\mathbb{N}, \leq) \), \( L(\mathbb{N}, <) \) and \( L_c(\mathbb{N}, <) \) is \( \Sigma^1_1 \)-hard in languages with two individual variables, one monadic predicate letter and one proposition letter.

The frame \( \langle \mathbb{N}, < \rangle \) is isomorphic to the structure \( \langle \omega, < \rangle \), where \( \omega \) is the least infinite ordinal and \( < \), as for all ordinals, is the membership relation on \( \omega \). We next generalise Theorem 4.2 to logics of frames based on infinite ordinals of a special form, which include \( \omega \).

**Theorem 4.4** Let \( \alpha = \omega \cdot m + k \), for some \( m \in \mathbb{N} \) with \( 1 \leq m < \omega \) and some \( k < \omega \), let \( R \) be a binary relation on \( \alpha \) between \( < \) and its reflexive closure \( \leq \), and let \( L = L(\alpha, R) \). Then, satisfiability for \( L \) is \( \Sigma^1_1 \)-hard in languages with two individual variables, one monadic predicate letter and one proposition letter.

**Proof.** The proof is similar to that of Theorem 4.2. We only comment on how to obtain an analogue of Lemma 3.1, an encoding of the recurrent tiling problem in \( L(\alpha, \leq) \). (For the general case of an arbitrary relation between \( < \) and \( \leq \), we use \( \Box^+ \) instead of \( \Box \).)

Since the frame \( \langle \alpha, \leq \rangle \) may contain a world that does not see another world, we need to define a variant of the formula \( A_0 \) suitable for such a situation:

\[
A^*_0 = \forall x \left( (M(x) \rightarrow \Box(\exists y M(y)) \rightarrow \Diamond(\exists y M(y) \rightarrow P_{u_0}(x))) \right).
\]

Let \( A^* \) be the conjunction of formulas \( A_0 \) through \( A_n \) from Section 3, as well as \( A^*_0 \). We claim that there exists a recurrent tiling satisfying \( (T_1) \) through \( (T_3) \) if, and only if, \( \langle \alpha, \leq \rangle \not\models \neg A^* \).

Assume \( \mathfrak{M}, u_0 \models A^* \), for some model \( \mathfrak{M} = \langle \alpha, \leq, D, I \rangle \) and some world \( u_0 \in \alpha \). Then, there exists in \( \alpha \) a last copy of \( \omega \) that has the following property: it contains a world \( w \)
marked by an element, say $a_k$, of the sequence $a_0 \triangleleft^{I,u_0} a_1 \triangleleft^{I,u_0} a_2 \triangleleft^{I,u_0} \ldots$ of elements of $D(u_0)$ whose existence follows from $\mathfrak{M}, u_0 \models A_2$. Then, a tiling can be obtained from the said copy of $\omega$, similarly to the way it was done in the proof of the “if” part of Lemma 3.1: columns are simulated by elements $a_0, a_1, a_2, \ldots$ of $D(u_0)$; rows are simulated by worlds $w_k, w_{k+1}, w_{k+2}, \ldots$ such that

- $w_k = w$,
- $w_k < w_{k+1} < w_{k+2} < \ldots$,
- $w_{k+n} \models M(a_{k+n})$, for every $n \in \mathbb{N}$;

and a tiling function $f: \mathbb{N} \times \mathbb{N} \to T$ is defined by

$$f(n, m) = t \quad \text{whenever} \quad w_{k+m} \models P_t(a_n).$$

Thus defined $f$ clearly satisfies $(T_1)$ and $(T_2)$. Also, since $\mathfrak{M}, u_0 \models A^\bullet$, the set

$$\{w_{k+m} : m \in \mathbb{N} \text{ and } w_{k+m} \models P_{t_0}(a_0)\}$$

is infinite; hence, $f$ satisfies $(T_3)$ and so is a required function.

For the converse, we use the first copy of $\omega$ contained in $\alpha$ for the satisfaction of $A^\bullet$: first, we define the interpretation of all the letters occurring in $A^\bullet$ on the said copy of $\omega$ as in the proof of the “only if” part of Lemma 3.1; second, we define the interpretation of letters $p, M, P_t$, for each $t \in T$, and $\triangleleft$ to be empty at every world not belonging to the said copy of $\omega$; last, we define $U$ to be an arbitrary non-empty subset of the domain of every world not belonging to the said copy of $\omega$. Then, $A^\bullet$ is true at the least, with respect to $\leq$, world of $\alpha$; hence, it is satisfiable. \hfill $\square$

## 5 Logics of dense and continuous linear orders

It is not clear whether $\Sigma^1_1$-hardness results analogous to Theorem 4.4 can be obtained for logics of linear orders distinct from those mentioned there. Perhaps the most significant logics of linear orders not covered by Theorem 4.4 are logics of the rationals and the reals with natural partial and strict orders, i.e. $L(\mathbb{Q}, \leq), L(\mathbb{Q}, <), L(\mathbb{R}, \leq)$ and $L(\mathbb{R}, <)$.

The proof of Lemma 3.1 does not carry over to either $L(\mathbb{Q}, \leq)$ or $L(\mathbb{R}, \leq)$ since we cannot ensure, given a model of the formula $A$ based on either $\langle \mathbb{Q}, \leq \rangle$ or $\langle \mathbb{R}, \leq \rangle$, that the tiling defined as in the proof of Lemma 3.1 satisfies $(T_3)$. In the case of $L(\mathbb{Q}, \leq)$, no such tiling exists: $L(\mathbb{Q}, \leq)$ is recursively enumerable [13] and hence $\Sigma^0_1$-complete. The case of $L(\mathbb{R}, \leq)$ might turn out to be similar as it is not known whether $L(\mathbb{R}, \leq)$ is distinct from $L(\mathbb{Q}, \leq)$. (The superintuitionistic logics of $\langle \mathbb{Q}, \leq \rangle$ and $\langle \mathbb{R}, \leq \rangle$ coincide [54, p. 701] and are $\Sigma^0_1$-complete [59, Theorem 1]; on the other hand, the superintuitionistic, and hence modal, logics of predicate frames with constant domains over $\langle \mathbb{Q}, \leq \rangle$ and $\langle \mathbb{R}, \leq \rangle$ differ [59, Theorem 2].)
A slight modification of the proof of Lemma 3.1 shows, however, that satisfiability for \( L(Q, \leq) \) and \( L(R, \leq) \) is \( \Pi^0_1 \)-hard—hence, \( L(Q, \leq) \) and \( L(R, \leq) \) are \( \Sigma^0_1 \)-hard—in languages with two variables, one monadic predicate letter and one proposition letter: simply leaving out the argument for \( (T_3) \), we obtain a reduction to satisfiability for \( L(Q, \leq) \) and \( L(R, \leq) \) in appropriate languages of the \( \Pi^0_1 \)-complete \([6], [9, Appendix A.4]\) \( \mathbb{N} \times \mathbb{N} \) tiling problem whose solution is required to satisfy \( (T_1) \) and \( (T_2) \), but not \( (T_3) \). We do, rather, establish a more general result.

Define \( B \) to be the conjunction of formulas \( A_0 \) through \( A_8 \) (i.e., leave out \( A_9 \) from the formula \( A \) defined in Section 3).

**Lemma 5.1** Let \( \langle W, \leq \rangle \) be a partial linear order containing an infinite ascending chain of pairwise distinct elements of \( W \). Then, there exists a tiling of \( \mathbb{N} \times \mathbb{N} \) satisfying \( (T_1) \) and \( (T_2) \) if, and only if, \( \langle W, \leq \rangle \not\models \neg B \).

**Proof.** ("if") The proof is identical to that of Lemma 3.1, except that we leave out the argument for \( (T_3) \).

("only if") Suppose \( f \) is a function satisfying \( (T_1) \) and \( (T_2) \). We obtain a model based on \( \langle W, \leq \rangle \) satisfying \( B \).

Let \( D_w = \mathbb{N} \cup \{-1\} \), for every \( w \in W \). To define the interpretation function \( I \) on \( \langle W, \leq, D \rangle \), we use elements of the infinite ascending chain \( w_0 \leq w_1 \leq w_2 \leq \ldots \) of worlds from \( W \) that exists by assumption: we define \( I \) so that, for every \( k \in \mathbb{N} \) and every \( a, b \in D \),

\[
\begin{align*}
\mathfrak{M}, w_k \models a \circ b & \iff k \text{ is even and } b = a + 1; \\
\mathfrak{M}, w_k \models p & \iff k \text{ is odd}; \\
\mathfrak{M}, w_k \models M(a) & \iff k = 2a; \\
\mathfrak{M}, w_k \models P_i(a) & \iff k = 2m \text{ and } f(a, m) = t, \text{ for some } m \in \mathbb{N},
\end{align*}
\]

and, for every \( v \not\in \{w_i : i \in \mathbb{N}\} \) and every predicate letter \( S \) of \( B \),

\[
I(v, S) = I(w_m, S), \text{ where } m = \min\{k \in \mathbb{N} : v \leq w_k\}.
\]

It is straightforward to check that \( \mathfrak{M}, w_0 \models B \), so we leave this to the reader. \( \Box \)

Using a modification of the formula \( B \) obtained by replacing every occurrence of \( \Box \) by that of \( \Box^+ \), we can prove the following analogue of Theorem 4.2 (the proof uses Lemma 5.1 in the same way Theorem 4.2 used Lemma 3.1):

**Theorem 5.2** Let \( \langle W, \prec \rangle \) be a strict linear order containing an infinite ascending chain of pairwise distinct elements of \( W \). Let \( \leq \) be the reflexive closure of \( \prec \) and \( R \) a binary relation between \( \prec \) and \( \leq \). Let \( L \) be a logic such that \( L(W, R) \subseteq L \subseteq L_c(W, R) \). Then, satisfiability for \( L \) is \( \Pi^0_1 \)-hard—hence, \( L \) is \( \Sigma^0_1 \)-hard—in languages with two individual variables, one monadic predicate letter and one proposition letter.

**Corollary 5.3** Logics \( L(Q, \leq), L_c(Q, \leq), L(Q, \prec), L_c(Q, \prec), L(R, \leq), L_c(R, \leq), L(R, \prec) \) and \( L_c(R, \prec) \) are \( \Sigma^0_1 \)-hard in languages with two individual variables, one monadic predicate letter and one proposition letter.
Well-known axiomatically defined predicate modal logics coincide with some logics mentioned in Corollary 5.3. Let $K$ be the minimal propositional normal modal logic and, for a set of formulas $\Gamma$ and a formula $\varphi$, let $\Gamma \oplus \varphi$ be the closure of $\Gamma \cup \{ \varphi \}$ under modus ponens, necessitation and propositional substitution. Recall the following definitions of propositional modal logics:

$$\text{S4.3} = K \oplus \text{ref} \oplus \Box p \rightarrow \Box \Box p \oplus \Box(\Box p \rightarrow q) \vee \Box(\Box q \rightarrow p);$$

$$\text{K4.3.D.X} = K \oplus \Box p \rightarrow \Box \Box p \oplus \Box(\Box p \rightarrow q) \vee \Box(\Box q \rightarrow p) \oplus \Diamond \top \oplus \Box \Box p \rightarrow \Box p.$$ 

For a propositional modal logic $L$, denote by $Q_L$ the minimal predicate modal logic containing $Q_{\text{Cl}} \cup L$. It follows from the definitions of $\text{S4.3}$ and $\text{K4.3.D.X}$ given above that logics $QS4.3$ and $QK4.3.D.X$ are finitely axiomatizable and, hence, recursively enumerable, i.e., they are in $\Sigma_1^0$.

It is well known [13] that $QS4.3 = L(Q, \leq)$ and $QK4.3.D.X = L(Q, <)$. We, therefore, obtain the following result:

**Corollary 5.4** The logics $QS4.3$ and $QK4.3.D.X$ are $\Sigma_0^0$-complete in languages with two individual variables, one monadic predicate letter and one proposition letter.

Thus, $L(Q, \leq)$ and $L(Q, <)$ are $\Sigma_1^0$-complete in languages with two individual variables, one monadic predicate letter and one proposition letter.

### 6 Some other logics

In this section, we note some corollaries of Theorem 5.2 other than those mentioned in Corollary 5.3. We also note that a straightforward modification of the proof of Theorem 5.2 establishes $\Sigma_1^0$-completeness of the logic $QK4.3$ in the languages we consider.

The first corollary concerns logics of infinite ordinals: as before, for ordinals, by $<$ and $\leq$ we mean, respectively, the relation $\in$ and its reflexive closure.

**Corollary 6.1** Let $\alpha$ be an infinite ordinal. Then, $L(\alpha, <)$, $L_c(\alpha, <)$, $L(\alpha, \leq)$ and $L_c(\alpha, \leq)$ are $\Sigma_1^0$-hard in languages with two individual variables, one monadic predicate letter and one proposition letter.

The second concerns logics of non-standard models of the elementary theories of some of the structures considered thus far (the elementary theory of the structure $\mathfrak{A}$ is denoted by $\text{Th}(\mathfrak{A})$):

**Corollary 6.2** Let $\mathfrak{A}$ be one of the structures $\langle \mathbb{N}, \leq \rangle$, $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Q}, \leq \rangle$, $\langle \mathbb{Q}, < \rangle$, $\langle \mathbb{R}, \leq \rangle$ and $\langle \mathbb{R}, < \rangle$, and let $\mathfrak{F}$ be a non-standard classical first-order model of $\text{Th}(\mathfrak{A})$. Then, $L(\mathfrak{F})$ and $L_c(\mathfrak{F})$ are $\Sigma_1^0$-hard in languages with two individual variables, one monadic predicate letter and one proposition letter.
Lastly, a slight modification of the argument of Section 5 gives us the following result on $\text{QK}4.3$, the logic of strict linear orders [13]. We recall that

$$K4.3 = K \oplus \Box p \rightarrow \Box p \oplus \Box (\Box^+ p \rightarrow q) \lor \Box (\Box^+ q \rightarrow p).$$

Thus, $\text{QK}4.3$ is finitely axiomatizable and, hence, recursively enumerable.

**Corollary 6.3** The logic $\text{QK}4.3$ is $\Sigma_1^0$-complete in languages with two individual variables, one monadic predicate letter and one proposition letter.

**Proof.** One can show that the formula $A^+$ defined in Section 4 is satisfiable in a model based on a $\text{QK}4.3$-frame if, and only if, there exists a tiling of $\mathbb{N} \times \mathbb{N}$ satisfying $(T_1)$ and $(T_2)$. We only notice that, in the proof of the “only if” part, we need to show that the model satisfying $A^+$ is infinite—this readily follows by $A_1$, $A_2$ and $A_4^+$. □

7 Discussion

We now discuss some questions arising out of the present work.

The first question is whether our main result, Theorem 4.2, can be strengthened to languages with two variables and a single monadic predicate letter: is the “extra” proposition letter necessary?

For the majority of natural predicate modal—and closely related superintuitionistic—logics similar results have been obtained [46, 49, 51] for languages with a single monadic predicate letter (the number of variables—two [46] or three [49, 51]—depends on the logic). Those results do not, however, cover some notable logics—the known results for the predicate counterparts of propositional modal logics $\text{GL}3$, $\text{Grz}3$ and $\text{S5}$ involve “extra” proposition letters [46, Discussion]. The propositional modal logics of frames $\langle \mathbb{N}, \leq \rangle$ and $\langle \mathbb{N}, < \rangle$—in common with $\text{GL}3$, $\text{Grz}3$ and $\text{S5}$—are NP-complete, i.e., not as computationally hard (provided $\text{PSPPSACE} \neq \text{NP}$) as PSPACE-hard propositional logics whose first-order counterparts are known to be undecidable in languages with a few variables and a single monadic predicate letter. Whether this observation points to a genuine connection is unclear; the hypothesis, however, seems to be worth investigating. It seems at least plausible that predicate logics of $\langle \mathbb{N}, \leq \rangle$ and $\langle \mathbb{N}, < \rangle$ are decidable in languages with two variables and a single monadic letter.

We note that a stronger result, $\Sigma_1^1$-hardness of satisfiability for languages with two variables and a single monadic letter, is relatively easily obtainable for the logics of the naturals in the more expressive language containing, alongside $\Box$, the unary operator $\Diamond$ (“next”) with the truth condition $\mathcal{M}, n \models^g \Diamond \varphi$ if $\mathcal{M}, n + 1 \models^g \varphi$. The resultant logic is a notational variant of the first-order quantified linear time temporal logic $\text{QLTL}$ with temporal operators $\Box$ (interpreted as “always in the future”) and $\Diamond$—even without the more expressive binary operator “until.” It is well known [28, Theorem 2] that satisfiability for $\text{QLTL}(\Box, \Diamond)$ is $\Sigma_1^1$-hard in languages with two variables and only monadic predicate letters: the proof, which inspired the proofs presented above, is a reduction of
the recurrent \( \mathbb{N} \times \mathbb{N} \) tiling problem described in Section 3. Since the number of tile types in the recurrent tiling problem is unbounded, [28, Theorem 2] establishes \( \Sigma^1_1 \)-hardness for languages with an unlimited supply of monadic predicate letters. The proof of [28, Theorem 2] can, however, be modified to establish \( \Sigma^1_1 \)-hardness for languages with a single monadic predicate letter. Perhaps the easiest way is to modify the formulas used in the original proof [28] so that the satisfying model has equal-seized gaps (non-empty sequences of worlds) between the worlds corresponding to the tiling; the size of the gaps is proportional to the number of tile types. The binary letter can then be modelled as in the proof of Lemma 3.2 above. To model all the monadic letters with a single one, we use the following observation. If \( f \) is a tiling function satisfying \((T_1)\) through \((T_3)\), then for every \( m \in \mathbb{N} \), there do not exist \( t, t' \in T \) such that \( t \neq t' \) and, for every \( n \in \mathbb{N} \), both \( f(n, m) = t \) and \( f(n, m) = t' \): an entire row cannot be tiled simultaneously with tiles of two distinct types. Therefore, a construction similar to the one used in the proof of Lemma 3.3 above would not produce a model where two successive worlds satisfy \( \forall x P(x) \). Hence, we can use \( \Diamond \forall x P(x) \land \Diamond \forall x P(x) \) in place of the proposition letter \( q \) in the final reduction. This gives us the following result:

**Theorem 7.1** Satisfiability for QLTL(\( \Box, \Diamond \)) is \( \Sigma^1_1 \)-hard in languages with two individual variables and a single monadic predicate letter.

The second question arising out of the present work is whether stronger lower bounds are obtainable for the logics of the reals—provided they are distinct from the logics of the rationals. One approach would be to attempt to adapt techniques developed by Reynolds and Zakharyashev [40] for proving \( \Sigma^1_1 \)-hardness of products of two propositional modal logics of linearly ordered frames. In particular, Reynolds and Zakharyashev establish [40, Theorem 6.1] \( \Sigma^1_1 \)-hardness of product logics satisfying two conditions: first, the product logic admits a frame with infinite ascending chains along both accessibility relations; second, the order relation associated with one of the factor logics is Dedekind complete. Even though this setup appears similar to predicate modal logics of the reals, it is not immediately clear how to apply the techniques of Reynolds and Zakharyashev [40] in our circumstances since it is not obvious how an infinite linear partial or strict order, which has to be transitive, can be defined on domains of a predicate Kripke model using formulas with only two variables.

The third question is whether our results are tight. We are not aware of upper-bound results for logics considered here, the exception being the logics of the rationals, which are, as mentioned in Section 5, \( \Sigma^0_1 \)-complete. Thus, a search for upper bounds appears to be an interesting topic of future study.

The final question we mention is whether analogous results can be obtained for super-intuitionistic logic of \( \langle \mathbb{N}, \leq \rangle \). Whether this can be done is unclear to us: the techniques used here appear unsuitable for superintuitionistic logics given the difficulty of modelling the changing values of tile types on a linear frame with a hereditary valuation.
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