Universal Scaling of Decay Rates of Emitter Arrays in the Subradiant States

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Sub-wavelength arrays of emitters support subradiant excitations with decay rates scaled as $N^{-\alpha}$, with $N$ the number of emitters. In this Letter, we study one-dimensional arrays and find $\alpha = s + 1$ for the subradiant states with wave numbers close to an extremum ($k_{\text{ex}}$) of the dispersion relation $\omega_k$, where the $s$-index is the order of the leading term of the expansion $\omega_k - \omega_{k_{\text{ex}}} \propto (k - k_{\text{ex}})^s$. The relation is universal regardless of the specific Hamiltonians. It also implies a new effect in systems with multiple bands: the occurrence of band gap closing (e.g., at topological transitions), changes the $s$-index hence surges the radiation from the subradiant states. This effect is demonstrated in dimerized emitter arrays coupled to free space modes or an ideal waveguide.

To address them, consider a wave packet propagating in a 1D emitter array coupled to some electromagnetic field. The duration for it to be trapped in the array is still limited by the group velocity, even if it does not directly radiate. The group velocity will be slow if the wave number of the excitation is close to an extremum $k_{\text{ex}}$ of the dispersion relation $\omega_k$. Suppose that the order of the extremum is $s$, i.e., we have the expansion $\omega_k - \omega_{k_{\text{ex}}} \approx (k - k_{\text{ex}})^s$ for $k \approx k_{\text{ex}}$. A larger $s$ implies a slower group velocity (also, a flatter band [29]). Thus a relation between $\alpha$ and $s$ is expected, while it is nontrivial to substantiate the universality.

In this Letter, we shall show $\alpha = s + 1$ for the single-excitation states of 1D arrays. This relation is established through a solvable toy model, which is then linked to real systems by the primary perturbation formalism. The universality will be manifested from the approach. This relation implies a new effect: a change of the $s$-index (caused by, e.g., band gap closing at topological transitions) will induce a surge of the radiation of the subradiant states. We demonstrate this effect in dimerized emitter chains in the 3D free $\alpha$ from 3 to 1 (numerically) or coupled to an ideal 1D waveguide (analytically), where a jump of $\alpha$ from 3 to 1 occurs at the critical point of the Su-Schrieffer-Heeger (SSH) type topological transition.

Models. We shall take the emitters arrays in the 3D free space as the exemplary system of this work. Thereof, the Born-Markov approximation works well hence the light field can be traced out. It results in a non-Hermitian Hamiltonian that describes the effective emitter-emitter coupling induced by light [30]:

$$H_{\text{eff}} = -\mu_0 \omega_0^2 \sum_{i,j=1}^{N} d_i^\dagger G_0(x_i - x_j, \omega_0) d_j \sigma_i^g \sigma_j^g,$$  \hspace{1cm} (1)

where $\mu_0$ is the vacuum permeability, $\omega_0$ is the transition frequency between the emitter ground state $\ket{g}$ and excited state $\ket{e}$, $d_i$ and $x_i$ are the transition dipole moment and coordinate of the $i$th emitter, $\sigma_i = \ket{g_i} \bra{e_i}$, and $G_0$ is the dyadic Green’s tensor of the Maxwell’s equations. Dipole-dipole interactions induced by other light fields are obtained by replacing $G_0$ with the specified ones [30]. As depicted in Fig. 1(a), we suppose the emitters are equally separated by $d$ and polarized transverse to the array. We denote the system by DD-T in shorthand. DD-T was noticed to be more irregular because of its $1/r$ long-range interaction [4].

For infinite chains, the Bloch’s theorem reveals a great universality that the eigenstates are always the Bloch states, which is, if written for finite $N$,

$$\ket{k} = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{ikx_m} \sigma_m^g \ket{g_1 g_2 \cdots g_N}$$  \hspace{1cm} (2)

where $k \in [-\pi/d, \pi/d]$ (the first Brillouin Zone). The specified Hamiltonian determines only the eigenvalues. For the non-Hermitian $H_{\text{eff}}$, the eigenvalue is written as $\omega_k - i\gamma_k/2$, where $\gamma_k$ is the decay rate of $\ket{k}$ (with infinite $N$). In Fig. 1(b), $\omega_k$ and $\gamma_k$ of DD-T with the resonant wave number $k_0 = 0.4\pi/d$ are depicted. It is notable that $\gamma_k = 0$ if $|k| > k_0$, because no resonant light modes in vacuum have so large wave numbers [4].

Eigenstates of finite chains would have amplitude profiles that look like standing waves, rather than the Bloch states. Nevertheless, their phase profiles help to associate
them with wave numbers. In this sense, people found the most subradiant states of DD-T near the extremum $k_{ex} = \pi / d$ where $\gamma_k = 0$ \cite{4, 6}.

In Fig. 1(c), we plot the decay rate of the most subradiant state as a function of $N$ for DD-T with three different $k_0$. The curve of $k_0 = 0.3\pi / d$ oscillates but descends roughly like $N^{-3}$, where $\gamma_0$ is the spontaneous emission rate. The curve of $k_0 = 0.55\pi / d$ shows a perfect $N^{-3}$ descending. However, the curve of $k_0 = 0.4828\pi / d$ shows an $N^{-5}$ scaling, which is not seen before. Amplitudes profiles of the first two cases are found to be similar, but visibly different from those of the third, see Fig. 1(d) for a comparison. As a starter for the relation between $\alpha$ and $s$, we show in \cite{31} that for DD-T with $k_0 = k_{(4)} \approx 0.4828\pi / d$, $k = \pi / d$ is an extremum of $s = 4$, while $s = 2$ otherwise.

To finish the preparations, suppose $H_{eff} = H_{eff}^{Re} - iH_{eff}^{Im}$ where $H_{eff}^{Re}$ and $-iH_{eff}^{Im}$ are the coherent and dissipation parts, respectively. It is simple to prove \cite{31} that the discrete translation symmetry leads to

$$H_{eff}^{Im} = \frac{1}{2} N \int_{\Gamma} d k \frac{\gamma_k}{2\pi} \langle k | k \rangle .$$

Therein, “$\Gamma$” denotes the the closed subset where $\gamma_k \neq 0$. For DD-T, we have $\Gamma = [-k_0, k_0]$. Equation (3) also provides a model of general $H_{eff}^{Im}$. For our purpose, knowledge of $\gamma_k$ is unnecessary. But we require that $k_{ex} \notin \Gamma$.

**Eigenstates.** We deliberately digress from the theme and consider the eigenstates of a toy Hamiltonian

$$H_R = h_0 \mathbb{I} + \sum_{r=1}^{R} \sum_{i=1}^{N-r} (h_r a_i^{\dagger} a_{i+r} + h_r^{*} a_{i+r}^{\dagger} a_i)$$

where $a_i$ ($a_i^{\dagger}$) is the annihilation (creation) operator of the $i$th site, and $R$ is the farthest hopping distance. The leftmost $R$ sites $\partial_l = \{1, 2, \cdots, R\}$ and (similarly) the rightmost $R$ sites $\partial_r$ constitute the boundary set $\partial_l \cup \partial_r$, of which the projector is denoted by $P_0$. Projector of the remaining sites (bulk) is denoted by $P$. Then the eigenstate satisfying $(H_R - E)|\psi\rangle = 0$ must fulfill the bulk equation $P(H_R - E)|\psi\rangle = 0$. If we find out all the solutions to the bulk equation of eigenvalue $E$, which form a linear space, the desired eigenstate must be contained. And it must be the one satisfying also the boundary equation $P_0(H_R - E)|\psi\rangle = 0$.

For the bulk equation, we note that the Bloch state $|k\rangle$, and its generalizations with $k$ being complex numbers, satisfy $P H_R |k\rangle = \omega_R |k\rangle P |k\rangle$, thus also the bulk equation if $\omega_R(k) = E$. Actually, in generic situations all solutions to the bulk equation (a linear space) is spanned by all such $\{|k\rangle\}$. This is mathematically proved by a generalized Bloch’s theorem \cite{32–34}. To proceed, we introduce $z = e^{ikd}$ and re-denote the Bloch state of Eq. (2) by $|z\rangle$. Then $\omega_R(k)$ becomes

$$H_R(z) = h_0 + \sum_{r=1}^{R} (h_r z^r + h_r^{*} z^{-r}).$$

From it we know that $H_R(z) = E$, a polynomial equation, has totally $2R$ solutions. Suppose $k_{ex}$ is an extremum of $\omega_R(k)$ of order $s$ ($s \leq 2R$). Then near $z_{ex} = e^{ikd}$, we have the expansion

$$H_R(z) = H_R(z_{ex}) + a_{s} \frac{1}{(iz_{ex})} (z - z_{ex})^s + \cdots .$$

To choose a reasonable $E$, note that there are totally $N$ eigenstates in the Brillouin Zone of length $2\pi / d$, hence the separation between the neighboring two is $O(N^{-1})$. Then states near $k_{ex}$ have eigenvalues $E = H_R(z_{ex}) + a_s \delta^s$ with $\delta \sim N^{-1}$. Equation (6) implies $s$ solutions to $H_R(z) = E$ distributed close to $z_{ex}$:

$$z_j \approx z_{ex}(1 + i \delta e^{\frac{2\pi i j}{2R}}), \quad j = 1, 2, \cdots s .$$

All solutions are here if $s = 2R$ (always achievable by adjusting $\langle h_r, h_r^{*}\rangle$ of $H_R$, we shall call this special toy model Hamiltonian $H_{ij}/2$).

For the boundary equation, we need to find coefficients $\{c_j\}$ so that $|\psi\rangle = \sum_j c_j |z_j\rangle$ satisfies $P_0(H_R - E)|\psi\rangle = 0$. Equivalently, we consider the boundary matrix $M$ defined with elements $M_{b,j} = \langle b | (H_R - E) | z_j \rangle$, and require $\{c_j\}$ to satisfy $\sum_j M_{b,j} c_j = 0$ for any $b \in \partial_l \cup \partial_r$. The solution exists iff $\det M = 0$. By introducing $\epsilon_j, \eta_j \sim N^{-1}$...
so that \( z_{ex}/z_j = 1 + \epsilon_j = (1 + \eta_j)^{-1} \), we prove in [31] that the boundary equation is equivalent to
\[
\sum_{j=1}^{s} c_j e_j^{(j)} = 0, \quad \sum_{j=1}^{s} c_j z_j^{N+1} \eta_j^{(j)} = 0,
\]
where \( r = 0, 1, \ldots, s/2 - 1 \).

**Scaleings.** Now we return to the decay rates and consider \( H_{s/2} - iH_{eff}^{lm} \) (remember that \( k_{ex} \notin \Gamma \)). For eigenstates near \( k_{ex} \), we view \( H_{eff}^{lm} \) as a perturbation to \( H_{s/2} \). Thus to the leading order, the decay rate is obtained by \( 1/\gamma = \langle \psi | H_{eff}^{lm} | \psi \rangle \). The validity of this treatment will be seen later. Firstly, for \( k = 0 \), \( \langle k | \psi \rangle \) equals
\[
\langle k | \psi \rangle = O\left( \frac{1}{N} \right) \sum_{j=1}^{s} c_j z_j e^{-ikd} - \frac{(z_j e^{-ikd})^{N+1}}{1 - z_j e^{-ikd}}.
\]

We split the fraction and evaluate separately:
\[
\frac{1}{\sum_{n=0}^{\infty} (z_{ex} e^{-ikd} - 1)^n} = \frac{1}{\sum_{n=0}^{\infty} \sum_{j=1}^{s} c_j e^{j} (z_{ex} e^{-ikd} - 1)^n}.
\]

By substituting Eq. (8) into Eq. (10), we see that in the series of \( n \), the leading contribution can only come from \( n = R = s/2 \). Thus \( \langle k | \psi \rangle \sim N^{-s/2 - 1} \) and\n\[
\langle \psi | H_{eff}^{lm} | \psi \rangle \sim N \times |N^{-2/2} - 1|^2 \sim N^{-s-1}.
\]

Note that the gaps between eigenstates of \( H_{s/2} \) near \( k_{ex} \) are \( O(N^{-s}) \), which is much larger than \( \langle \psi | H_{eff}^{lm} | \psi \rangle \). Thus the perturbation treatment is valid and we obtain the relation \( \alpha = s + 1 \) for \( H_{s/2} \).

However, coherent part \( H_{Re} \) of a real system such as DD-T does not have finite \( R \), or always has \( 2R \gg s \). Nevertheless, \( H_{s/2} \) well describes the neighbor of \( k_{ex} \), while we can patch \( H_{s/2} \) with a new term \( H_\Omega \) for \( k \in \Omega \), a closed subset of the Brillouin Zone (\( k_{ex} \notin \Omega \)). Then \( H_{s/2} + H_\Omega \) can match \( H_{Re} \) well. \( H_\Omega \) surely has the form same with Eq. (3), because the latter requires only the discrete translation symmetry [31]. Then the same calculation shows that \( H_\Omega \) is also a perturbation to the eigenstates of \( H_{s/2} \) near \( k_{ex} \). Therefore, we conclude that \( \alpha = s + 1 \) is indeed a universal property, as what is already shown in Fig. (1c).

Meanwhile, the eigenstate \( | \psi \rangle \) of \( H_{s/2} \) near \( k_{ex} \) will be perturbed by \( H_\Omega \) and \( -iH_{eff}^{lm} \). Because \( \langle \psi | H_{eff}^{lm} | \psi \rangle \sim N^{-s-1} \) (and so does \( H_\Omega \)), the primary perturbation formalism tells us that\n\[
| \psi \rangle \rightarrow | \psi \rangle \propto | \psi \rangle + O(N^{-1}) | \psi \rangle
\]
where \( \langle \psi | \psi \rangle = 0 \). It leads to the scaling of the infidelities that \( 1 - \langle \psi | \psi \rangle^2 \simeq N^{-s} \).

To verify the infidelities and thus verify the perturbation approach, we consider a toy model \( H_{R=2} \) with positive couplings \( h_1 = h'_1 \) and \( h_2 = h'_2 \), and the dispersion relation \( \omega_2(k) = 2h_1 \cos(kd) + 2h_2 \cos(2kd) \). \( k_{ex} = \pi/d \) is an extremum of \( s = 4 \) if \( h_1/h_2 = 4 \), and a non-degenerate extremum of \( s = 2 \) if \( h_1/h_2 > 4 \). Eigenstates of \( H_{R=2} \) near \( k_{ex} = \pi/d \) are analytically derived in [31]. In Fig. 2, we plot the infidelities between the most subradiant state of DD-T \( (k_0d/\pi = 0.3, 0.55) \), with the eigenstate of \( H_{R=2} \) \((h_1/h_2 = 6) \) that is closest to \( k_{ex} \), and similarly between DD-T \( (k_0 = k_{(4)}) \) and \( H_{s/2} \) \((h_1/h_2 = 4) \). The \( N^{-s} \) scaling is perfectly confirmed for \( k_0d/\pi = k_{(4)} \) and 0.55.

For curves of \( k_0d/\pi = 0.3 \), oscillations are found in both Figs. (1d) and 2. This is caused by degeneracies of \( k_{ex} = \pi/d \). To see it, we plot the second and forth Taylor’s series coefficients \((\omega_{k}k_{(4)} = 0) \) of \( \omega_{k} \) at \( k = \pi/d \) in the upper panel of Fig. 2. Therein, we find \( a_2 > 0 \) and \( a_4 < 0 \) for \( k < k_{(4)} \), hence \( \omega_{k} \) has three extrema of \( s = 2 \), as illustrated in the insertion panel of Fig. 2. So \( k_{ex} = \pi/d \) is degenerate to two Bloch states. Their contributions to \( | \psi_{(s)}^j \rangle \) are enhanced if their wavelengths match certain resonance with the length of the chain (\( Nd \)). We highlight that similar oscillations were also found in subradiant bound states [35] and firstly explained in [29].

\[
\sum_{j=1}^{s} c_j (z_j e^{-ikd})^{N+1} - \frac{1}{z_{ex} e^{-ikd} - 1} \sum_{n=0}^{\infty} \sum_{j=1}^{s} c_j z_j^{N+1} \eta_j^{(j)}.
\]

Figure 2. Upper panel: The 2nd and 4th Taylor’s expansion coefficients \((a_{2,4}) \) of DD-T dispersion relation \( \omega_{k} = \pi/d \), as a function of \( k_0 \). Region of \( k_0 < k_{(4)} \) is shaded. Zoom in panel: fine structure of \( \omega_{k} \) around \( k = \pi/d \) for DD-T with \( k_0 = 0.4826\pi/d \). Lower panel: infidelities (log scale) between the most subradiant states of DD-T with \( k_0d/\pi = 0.3, 0.55 \) and \( k_{(4)} \), and the “fundamental mode” eigenstate (near \( k = \pi/d \)) of \( H_{s/2} \) with \( h_1/h_2 = 6 \) and \( h_1/h_2 = 4 \), respectively. The dashed lines are guides of eyes.
also seen (but more clearly) in the toy model $H_{R=2}$ with $h_1/h_2 < 4$, which is solved analytically in [31].

Additionally, for fixed $N$, as $k_0$ approaches to $k_{(4)}$ from below, the separation between the three extrema will be $O(N^{-1})$. They might provide totally six $z_j \approx 2\omega_0$ thus lead to an $N^{-7}$ scaling, given that we keep adjusting $k_0$ according to the increasing $N$ [9].

Radiation surge in topological transition. In systems supporting multiple bands, the topological transition is usually accompanied by band gap closing at some point of the Brillouin Zone. If that point is an extremum ($k_{\text{ex}} \notin \Gamma$) before the gap closes, the transition may decreases its $s$-index discontinuously. Although the above theory directly covers only the simple arrays with one band, it inspires us that the shift of $s$ would induce a surge of radiation, if the system is prepared in subradiant states with wave number close to $k_{\text{ex}}$.

We numerically demonstrate this effect in a dimerized DD-T depicted in Fig. 3(a), where a unit cell has two emitters separated by $d_1$ and the period of the array is $d$. The dimerized chain has two bands, which are topologically trivial (absence of topology protected boundary states) if $d_1 < d_2$ where $d_2 = d - d_1$, and nontrivial if $d_1 > d_2$. An SSH type topological transition occurs at the critical point $d_1 = d_2$ [36, 37]. In Fig. 3(b), part of the dispersion relations (for $k \notin \Gamma$) of the dimerized DD-T with $k_0 = 0.8\pi/d$ and $d_1/d = 0.47, 0.5, 0.53$ are plotted, respectively. We find that $k_{\text{ex}} = \pi/d$ is an extremum of $s = 2$ if $d_1/d \neq 0.5$. But at the critical point $d_1/d = 0.5$, the band gap closes and the dispersion relation is linearized. Then we focus on the subradiant states associated with wave numbers closest to $k = \pi/d$, in both the upper and the lower band. Their decay rates are plotted in Fig. 3(c). The $N^{-3}$-scaling is seen for both bands when $d_1/d = 0.47$ (topological trivial) and 0.53 (topologically nontrivial), but $N^{-5}$-scaling instead, at the critical point $d_1/d = 0.5$. So the numerical results confirm the expected surge of radiation.

Next, we consider dimerized chains coupling to an ideal 1D waveguide where the Hamiltonian is [38]

\begin{equation}
H_{1D} = -i\frac{\Gamma_0}{2} \sum_{i,j} e^{ik_0|x_i-x_j|}\sigma_i^+\sigma_j^., \tag{13}
\end{equation}

This model is dissipative for only $k = \pm k_0$. Interestingly, being directly inspired by Ref. [29], we notice that in the subspace of single-excitation states the inverse of Eq. (13), $H_{1D}^{-1}$, is the original SSH model where the staggered nearest-neighbor-couplings are $J = 1/\sin(k_0d_1)$ and $J' = 1/\sin(k_0d_2)$. The only difference is that the two emitters sitting at the two chain ends are dissipative [31]. The critical points of this model are $d_1 = d_2$ and $d_1 = d_2 \pm \pi/k_0$. To be different from DD-T, we consider the latter case where the band gap closes at $k = 0$, an extremum of $s = 2$ if the system is away from the critical point. Subradiant states associated with wave numbers close to $k_{\text{ex}} = 0$ have $\alpha = 3$. However, at the topological transition, their decay rates become (assuming $\sin(k_0d_1) > 0$):

\begin{equation}
\gamma = \frac{\Gamma_0}{4N} \cot(k_0d_1) \ln\left(\frac{1 + \sin k_0d_1}{1 - \sin k_0d_1}\right), \tag{14}
\end{equation}

for states in both bands [31]. Same with DD-T, this is also a jump of $\alpha$ from 3 to 1.

Conclusions and Discussions. In this Letter, we have established a universal equality $\alpha = s + 1$, where $s$ is the order of an extremum ($k_{\text{ex}}$) of the dispersion relation, and $\alpha$ is the exponent of the decay rates scaling ($N^{-\alpha}$) of subradiant states with wave numbers close to $k_{\text{ex}}$. It answers the question that what determines $\alpha$, and implies a surge of radiation of the subradiant states induced by band gap closing. We demonstrated this effect by a jump from the $N^{-3}$ scaling to $N^{-1}$ that numerically found in dimerized emitter arrays coupled to 3D free space modes, and analytically in the case of ideal 1D waveguide.

To substantiate the universality, we developed an approach independent to specific Hamiltonians. We believe that our method can be extended to arrays of more complex lattices and higher dimensions. In 2D and 3D, geometry of the lattices and energy bands will have much more varieties. Thus a more sophisticated relation might be expected. For other possible future studies, we note that this work reveals the intrinsic connection between subradiant states and flatter bands (larger $s$), as what highlighted in [29]. Thus applications relying on subradiant states may refer to systems having flatter bands. Photonic flat bands [39], if being coupled to quantum emit-
ters, may provide new controllable platforms for subradiant states, and for strongly-correlated manybody physics such as the fractional Hall effect that relevant to flat bands [40].

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This Supplemental Material is arranged as following: In Sec. S-I, we show that DD-T has an extremum of $s = 4$ when $k_0 = k_{(4)}$; In Sec. S-II, we show that Hamiltonians having the discrete translation symmetry, including $H_{\text{eff}}^{\text{Im}}$ and $H_{\Omega}$, can be written into the form of Eq. (3) of the main text; In Sec. S-III, we derive the equivalent forms of the boundary equation and Eq. (8) of the main text; In Sec. S-IV, we analytically derive the eigenstates of the toy model $\mathbf{H}_{R=2}$ for cases of $h_1 = 4h_2$ ($s = 4$) and $h_1 > 4h_2$ ($s = 2$, $k_{\text{ex}}$ is non-degenerate), and $h_1 < 4h_2$ ($s = 2$, $k_{\text{ex}}$ is degenerate). In Sec. S-V, we show the inverse of $H_{1D}$ and analytically derive Eq. (13) of the main text.

S-I. DD-T: THE $s$-INDEX OF $k_{\text{ex}} = \pi/d$

The dispersion relation of DD-T is given by [S1]

$$\omega_k = \frac{3\gamma_0}{4} \sum_{z=\pm 1} \sum_{\xi=1}^3 i\left(\frac{i}{k_0 d}\right)^\xi \text{Li}_\xi[e^{(k_0 + zk)d}],$$

where $\text{Li}_\xi$ denotes the polylogarithm of order $\xi$ $\text{Li}_\xi(z) = \sum_{n=1}^\infty z^n n^{-\xi}$. The series does not converge if $\xi < 1$. In this case, we can use $\text{Li}_1(z) = -\ln(1 - z)$ and calculate them via the relation

$$\frac{d}{dz} \text{Li}_\xi(z) = \frac{1}{z} \text{Li}_{\xi-1}(z).$$

For DD-T, the parity symmetry ($\omega_k = \omega_{-k}$) makes all the odd order derivatives of $\omega_k$ at $k = \pi/d$ vanish. Then the second order derivative is found to be

$$\partial^2_{kd}\omega_{kd=\pi} = \frac{3}{2k_0d} \left[ \ln(2\cos\frac{k_0d}{2}) + \frac{k_0d}{2} \tan\left(\frac{k_0d}{2}\right) - \left(\frac{k_0d}{2}\right)^2 \frac{1}{\cos^2\frac{k_0d}{2}} \right].$$

Numerical calculation shows that it vanishes at $k_0 = k_{(4)} \approx 0.48280076\pi/d$.

S-II. DERIVATION OF EQ. (3)

Consider $H = \sum_{i,j=1}^N H_{ij} |i\rangle \langle j|$, where $H_{ij}$ depends only on $i - j$ (discrete translation symmetry). The dispersion relation $\omega_k$ is obtained from the Fourier transformation of $H_{ij}$

$$H_{ij} = \int dk/(2\pi) \omega_k e^{ik(x_i-x_j)} \quad \text{(S1)}$$

so that

$$H = \sum_{i,j=1}^N \int_{\text{B.Z.}} \frac{dk}{2\pi} \omega_k e^{ik(x_i-x_j)} |i\rangle \langle j|$$

$$= N \int_{\text{B.Z.}} \frac{dk}{2\pi} \omega_k |k\rangle \langle k|,$$

where “B.Z.” refers to the first Brillouin zone. In the model of $\mathbf{H}_{\text{eff}}^{\text{Im}}$, the corresponding “dispersion relation” is just $\gamma_k/2$ for $k \in \Gamma$, a subset of the Brillouin zone.

For DD-T, we showed in Ref. [S2] that the effective Hamiltonian of DD-T can be written as

$$H_{\text{eff}} = -i\frac{3\gamma_0}{4k_0} \int_0^{k_0} \frac{d\tilde{k}}{2\pi} \rho_+^{\tilde{k}} \sum_{m,n=1}^N e^{i|\tilde{k}|z_m-z_n} \sigma_m^\dagger \sigma_n$$

$$- \frac{3\gamma_0}{4k_0} \int_0^{+\infty} \frac{d\tilde{k}}{2\pi} \rho_-^{\tilde{k}} \sum_{m,n=1}^N e^{-i|\tilde{k}|z_m-z_n} \sigma_m^\dagger \sigma_n,$$

where $\rho_\pm^{\tilde{k}}(\tilde{k}) = \pi(1 \pm \tilde{k}^2/k_0^2)$. Therefore in the single-excitation sector, we have

$$H_{\text{eff}}^{\text{Im}} = N \frac{3\gamma_0}{4k_0} \int_{-k_0}^{k_0} \frac{d\tilde{k}}{4\pi} \rho_+^{\tilde{k}} |\tilde{k}\rangle \langle \tilde{k}|.$$

So DD-T Hamiltonian can be modeled by Eq. (3) of the main text.

S-III. EQUIVALENT FORMS OF THE BOUNDARY MATRIX

The boundary equation $P_b(\mathbf{H}_R - E) |z_j\rangle = 0$ can be formulated through the boundary matrix $M$. For convenience, here we use the notation that

$$|z_j\rangle = \sum_{m=1}^N z_j^m |m\rangle.$$

Then the element $M_{b,j} \equiv \langle b|(\mathbf{H}_R - E)|z_j\rangle$ reads

$$M_{b,j} = (h_0 - E)z_j^b + \sum_{r=1}^R (\theta_{b+r} z_j^{b+r} + \theta_{b-r} z_j^{b-r}). \quad \text{(S3)}$$

Therein, we introduced the notation $\theta_x$, which equals 1 if $x \in \partial R \cup \partial_\text{ex}$ and equals 0 otherwise. The bulk equation $H(z_j) = E$ is written as

$$h_0 + \sum_{r=1}^R (h_r z_j^r + h'_r z_j^{-r}) = E.$$
By substituting the above equation into Eq. (S3), we can eliminate \( h_0 - E \) and obtain
\[
M_{b,j} = z_j^b \sum_{r=1}^{R} [(1 - \theta_{b+r})h_r z_j^r + (1 - \theta_{b-r})h_r' z_j^{-r}]
\]
To write them explicitly, for \( b \in \partial_l = \{1, 2, \cdots R\} \), we have
\[
\begin{align*}
M_{R,j} &= h_R', \\
M_{R-1,j} &= h_R' z_j^{-1} + h_{R-1}', \\
\vdots \\
M_{1,j} &= h_{R} z_j^{-(R-1)} + \cdots + h_{1}' z_j^{-1} + h_{1}'.
\end{align*}
\] (S4)
Expressions for \( b \in \partial_r = \{N, N-1, \cdots, N-R+1\} \) will be given below.

We need to find the kernel of the boundary matrix \( M \), i.e., to find \( \{c_j\}_j \) satisfying \( \sum_j M_{b,j}c_j = 0 \). For this purpose, we are free to manipulate \( M \) by row operations: to divide one row by some number, and to sum one row with another row multiplied by some factor. The first step is
\[
M_{R,j} \rightarrow M_{R,j}/h_R' = 1.
\]
Next, we work on the second row:
\[
M_{R-1,j} \rightarrow (M_{R-1,j} - h_{R-1}' M_{R,j})/h_R' = z_j^{-1}.
\]
Similarly for the remaining rows. Finally, we obtain a normal form of \( M_{b,j} \) (\( b \in \partial_l \)):
\[
\begin{align*}
M_{R,j} &= 1, \\
M_{R-1,j} &= z_j^{-1}, \\
\vdots \\
M_{1,j} &= z_j^{-(R-1)}.
\end{align*}
\] (S5)
For \( M_{b,j} \) (\( b \in \partial_r \)), we have the normal form
\[
\begin{align*}
M_{N-R+1,j} &= z_j^{N+1}, \\
M_{N-R+2,j} &= z_j^{N+2}, \\
\vdots \\
M_{N,j} &= z_j^{N+R}.
\end{align*}
\] (S6)
Based on that, it is more convenient to denote rows of \( b \in \partial_l \) as \( M^L \) and those of \( b \in \partial_r \) as \( M^R \), and relabel the matrix elements as
\[
\begin{align*}
M^L_{r,j} &= z_j^{1-r}, & M^{R}_{r,j} &= z_j^{N+r},
\end{align*}
\] (S7)
where \( r = 1, 2, \cdots R \). The equation \( \sum_j M_{b,j}c_j = 0 \) for every \( b \) is equivalent to require
\[
\sum_j z_j^{1-r}c_j = 0, \quad \sum_j z_j^{N+r}c_j = 0.
\] (S8)
for every \( r \in \{1, 2, \cdots R\} \).

The boundary matrix elements can be reformulated by \( c_j \) and \( h_j \). Firstly we multiply the \( r \)-th row of \( M^L \) by \( z_{ex}^{-r} \) so that \( M^L_{r,j} = (1 + c_j)z_{ex}^{-r} \), or
\[
M^L_{r+1,j} = \sum_{k=0}^{r} C^k_r c^k_j,
\]
where \( C^k_r \) is the binomial coefficient. Then we sequentially apply the row operations
\[
M^L_{r+1,j} \rightarrow M^L_{r+1,j} - \sum_{m=0}^{r-1} C^m_r M^L_{m,j}
\]
by the order of \( r = 1, 2, \cdots \), to reshape the rows into
\[
M^L = \begin{pmatrix}
1 & \cdots & 1 \\
\epsilon_1 & \cdots & \epsilon_s \\
\vdots & \vdots & \vdots \\
\epsilon^{R-1}_1 & \cdots & \epsilon^{R-1}_s \\
\end{pmatrix}_{R \times s}
\] (S9)
As to \( M^R \), firstly we multiply \( M^R_{R+1,j} \) by \( z_{ex}^{-r} \) so that
\[
M^R_{r+1,j} = z_j^{N+1}(1 + \eta_j)^{r} = z_j^{N+1} \sum_{k=0}^{r} C^k_r \eta_j^k.
\]
Following the similar row manipulations, we have
\[
M^R = \begin{pmatrix}
z_1^{N+1} & \cdots & z_s^{N+1} \\
z_1^{N+1} \eta_1 & \cdots & z_s^{N+1} \eta_s \\
\vdots & \vdots & \vdots \\
z_1^{N+1} \eta_1^{R-1} & \cdots & z_s^{N+1} \eta_s^{R-1} \\
\end{pmatrix}_{R \times s}
\] (S10)
Then we obtain Eq. (8) of the main text.

S-IV. ANALYTICAL RESULTS OF THE TOY MODEL \( H_{R=2} \)

We consider the case of \( s = 2 \) and \( s = 4 \) in the toy model \( (R = 2) \) of the Hamiltonian
\[
H_2 = h_1 \sum_{i=1}^{N-1} a_i^\dagger a_{i+1} + h_2 \sum_{i=1}^{N-2} a_i^\dagger a_{i+2}.
\] (S11)
Therein, we assume \( h_1 \) and \( h_2 \) are both positive numbers. The dispersion relation of this model is given by
\[
\omega_2(k) = 2h_1 \cos(k) + 2h_2 \cos(2k).
\]
The point \( k = \pi/d \) is an extremum of \( s = 4 \) if \( h_1 = 4h_2 \) (\( 2R = s \), essential solution), and a non-degenerate extremum of \( s = 2 \) if \( h_1 < 4h_2 \).
To derive the eigenstates near \( k_{ex} \), firstly we study the bulk equation
\[
H_2(z) = h_1(z + \frac{1}{z}) + h_2(z^2 + \frac{1}{z^2}) = E.
\] (S12)
Therein, we suppose $E = \omega^2(k_{ex}) + e$ with $e > 0$ and $e \ll h_1$. The equation $H_2(z) = E$ leads to

$$z + \frac{1}{z} = -\frac{h_1}{2h_2} \pm \frac{1}{2h_2} \sqrt{(h_1 - 4h_2)^2 + 4h_2e}.$$  \hspace{1cm} (S13)

Suppose we write the boundary matrix $M$ in the normal form of Eqs. (S5) and (S6). Then the determinant of $M$ is given by

$$\det M \propto \sum_{j_1 < j_2, j_3 < j_4} \epsilon_{j_1 j_2 j_3 j_4} (z_{j_2} - z_{j_1})(z_{j_4} - z_{j_3}) \times (z_{j_3} z_{j_4})^{N+2}. \hspace{1cm} (S14)$$

S-IV.A. Case of $2R = s = 4$ ($h_1 = 4h_2$)

Suppose $\sqrt{e/h_2} = \delta$ so that $z + 1/z = -2 + \delta$. Then the four solutions to the bulk equation are

$$z_1 = z_2^{-1} \approx -e^{-i\sqrt{3}}, \hspace{1cm} z_3 = z_4^{-1} \approx -e^{-i\sqrt{3}}.$$  \hspace{1cm} (S15)

Directly substituting them into Eq. (S14) yields

$$\cos[(N + 2)\sqrt{3}] \cosh[(N + 2)\sqrt{3}] = 1,$$

which leads to

$$\sqrt{3} \approx \frac{\zeta \pi}{N + 2}, \hspace{1cm} \zeta = 1.5, 2.5, 3.5 \ldots \hspace{1cm} (S16)$$

In the above formula, the approximation is exponentially precise. A notable feature is that the eigenstates are labelled by half integers ($\zeta$).

Then we calculate the superposition coefficients. Suppose that $\zeta = \xi + 0.5$. Then we have

$$-\frac{c_4}{c_3} = \frac{e^{-\xi\pi} + (-1)^{\xi}}{e^{\xi\pi} - (-1)^{\xi}}. \hspace{1cm} (S17)$$

If we let $c_3 = e^{\xi\pi} + (-1)^{\xi}$, the other two coefficients will be given by

$$c_1 = (-1)^{\xi} - \sinh(\xi\pi) + i \cosh(\xi\pi),$$

$$c_2 = (-1)^{\xi} - \sinh(\xi\pi) - i \cosh(\xi\pi). \hspace{1cm} (S18)$$

Note that the four coefficients have comparable magnitudes.

S-IV.B. Case of $2R > s = 2$ ($h_1 > 4h_2$)

We expand the square root in Eq. (S13) to first order of $e$ and get

$$\sqrt{(h_1 - 4h_2)^2 + 4h_2e} = |h_1 - 4h_2| + \frac{2h_2e}{|h_1 - 4h_2|}.$$  \hspace{1cm} (S19)

We use the shorthand expression $h_1/h_2 - 4 = a (a > 0)$ and $e/(ah_2) = \delta$. Then we have

$$z + 1/z = -2 + \delta, \hspace{1cm} \tilde{z} + 1/\tilde{z} = -(a + 2) - \delta.$$  \hspace{1cm} (S20)

The four solutions are $z_{1,2}$ and $\tilde{z}_{3,4}$ [these two solutions have magnitudes different with 1 by $O(1)$, the corresponding $|\tilde{z}_{3,4}|$ are boundary states]:

$$z_1 = 1/z_2 = 1 + i\sqrt{\delta} \approx -e^{-i\sqrt{3}},$$

$$\tilde{z}_3 = 1/\tilde{z}_4 = 1 - x = \frac{a}{2} + 1.\hspace{1cm} \text{(S21)}$$

is the essential solution of $s = 2$ (obtained in the toy model of $R = 1$). Solutions given by Eq. (S19) deviate from the essential solution by $O(N^{-2})$. The relative deviations of $c_j$ and $\eta_j$ are thus $O(N^{-1})$.

For the superposition coefficients, we have

$$\frac{c_1}{c_3} = (-1)^{N + \xi} z_3^{N+1}, \hspace{1cm} (S22)$$

which shows the balance between them [note that the norm of $|z_3|$ scales like $O(|z_3|^N)$ while that of $|\tilde{z}_4|$ scales as $O(1)$]. Coefficients $\{c_j\}$ are

$$c_1 = -1 - x - i\sqrt{\delta + \beta\delta} \frac{c_4}{2\sqrt{\delta}} c_4, \hspace{1cm} (S23)$$

where we have neglected the exponentially small contribution from $c_3$. So we see that the eigenstates satisfy Eq. (12) of the main text.

S-IV.C. Case of $k_{ex}$ is degenerate ($h_1 < 4h_2$)

Now $k_{ex} = \pi/d$ is a maximum. Thus we consider eigenvalues $E = \omega^2(k_{ex}) + e$ where $e > 0$. Similar to previous sections, we define $4 - h_1/h_2 = a$ and $e/(ah_2) = \delta$ so that

$$z + 1/z = -2 + \delta, \hspace{1cm} \tilde{z} + 1/\tilde{z} = -(a + 2) - \delta.$$  \hspace{1cm} (S24)

The solutions are $z_{1,2} = 1/z_2 \approx -e^{-i\sqrt{3}}$, and

$$\tilde{z}_3 = \frac{1}{\tilde{z}_4} = 1 - a - \frac{\delta}{2} + \frac{i}{2} \sqrt{4a - a^2} + \frac{(a - 2)\delta}{\sqrt{4a - a^2}}.$$  \hspace{1cm} (S25)
Compared with the previous section, here $|\tilde{z}_3| = |\tilde{z}_4| = 1$. Thus we can introduce $\theta$ so that $\tilde{z}_3 = e^{i\theta}$. The dependence to $\delta$ is expressed as $\theta = \theta_0 + \delta'$, where $\delta' \sim \delta$. Then det $M = 0$ leads to

$$\sqrt{\delta} = \frac{\xi}{N+2} (1 + \eta),$$

where $\xi = 1, 2, \ldots$ and the higher order correction

$$\eta = -\frac{\tan(\theta_0/2)}{N+2} \frac{(-1)^{\xi+N} \cos[\theta_0(N+2)]}{(-1)^{\xi+N} \sin[\theta_0(N+2)]}.$$

(S23)

Comparing the above equation with Eq. (S19). We see that now $\eta$ has trigonometric functions of $\theta_0(N+2)$, which lead to the oscillations depending on $N$.

As to the superposition coefficients, suppose $c_1 = 1$. Then we have $c_2 = e^{i\phi} \approx -1 + i\phi$, where

$$\phi \approx \frac{\sqrt{\delta}}{\sin \theta} \frac{1 + e^{-i\theta}}{2 \sin \theta} \cdot \frac{1}{\cos \theta} \cdot \phi$$

and $\tilde{c}_4 \approx \tilde{c}_3 + i\phi$. Thus the relation $c_1(2) = O(N) \tilde{c}_3(4)$ is still valid and the eigenstates satisfy Eq. (12) of the main text.

**S-V. DIMERIZED IDEAL 1D WAVEGUIDE MODEL**

We consider a chain of $N$ unit cells, hence totally $2N$ emitters. The inverse of the $2N \times 2N$ effective Hamiltonian is expressed as

$$H_{1D}^{-1} = \frac{1}{\Gamma_0} \left( \begin{array}{cccccc}
\alpha & J & 0 & 0 & 0 & \cdots \\
J & e_0 & J' & 0 & 0 & \cdots \\
0 & J' & e_0 & J & 0 & \cdots \\
0 & 0 & J & e_0 & J' & \cdots \\
& \ddots & & \ddots & \ddots & \ddots
\end{array} \right)$$

(S24)

where the parameters are

$$\alpha = i - \cot(k_0d_1), J' = \frac{1}{\sin(k_0d_2)},$$

$$J = \frac{1}{\sin(k_0d_1)}, \quad e_0 = -\frac{\sin(k_0d)}{\sin(k_0d_1) \sin(k_0d_2)}.$$
\[ h_k = \frac{1}{i\sqrt{2}} \Gamma_0 e^{i(k+k_0)(z_N+d)} \left( u^+_{-k_0} |n_k \rangle \right). \]

The eigenstates are superpositions of two Bloch states \(|\pm k \rangle\) which satisfy
\[ g_k h_{-k} = g_{-k} h_k. \] (S29)

**S-V.A. the upper band**

Equation (S29) is expanded as
\[ e^{-2iNdk} \frac{\sin^2 \frac{k_0+k}{2} d}{\sin^2 \frac{k_0-k}{2} d} = \frac{1 - \sin(k_0d_1 + \frac{k_0d}{2})}{1 + \sin(k_0d_1 - \frac{k_0d}{2})}. \] (S30)

For those \( k \approx 0 \), we substitute the ansatz
\[ k = \epsilon \frac{\pi}{Nd} (1 + \frac{1}{N} \delta) \]
into Eq. (S30) and obtain
\[ e^{-i2\pi \epsilon} = \frac{1 - \sin k_0d_1}{1 + \sin k_0d_1}, \]
\[ i\delta = \cot \left( \frac{k_0d_1}{2} \right) + \frac{1}{2} \tan(k_0d_1). \]

The solution is not unique and we have
\[ \epsilon \xi = \xi + i \frac{1}{2\pi} \ln \frac{1 - \sin k_0d_1}{1 + \sin k_0d_1}, \]
where \( \xi = 1, 2, 3 \cdots \). The imaginary parts of \( \epsilon \xi \) measures the biased mixture of the two components of \(|\pm k_+ \rangle\).

Substituting the solutions of \( k \) into \( \omega_+(k) \), we obtain the frequency shift and the decay rates
\[ \omega_+(\xi) = \frac{\Gamma_0/2}{1 - \cos k_0d} \left( \sin k_0d + \xi \frac{\pi}{N} \sin k_0d_1 \right), \] (S32a)
\[ \gamma_+ = -\frac{\Gamma_0}{4N} \cot(k_0d_1) \ln \frac{1 - \sin k_0d_1}{1 + \sin k_0d_1}. \] (S32b)

Note that \( \gamma_+ \) is uniform for all \( \xi \), and \( \gamma_+ \propto N^{-1} \).

**S-V.B. the lower band**

Now Eq. (S29) yields
\[ e^{-2iNdk} \frac{\sin^2 \frac{k_0+k}{2} d}{\sin^2 \frac{k_0-k}{2} d} = \frac{1 + \sin(k_0d_1 + \frac{k_0d}{2})}{1 - \sin(k_0d_1 - \frac{k_0d}{2})}. \]

Therefore the solutions would be
\[ \epsilon \xi = \xi + i \frac{1}{2\pi} \ln \frac{1 + \sin k_0d_1}{1 - \sin k_0d_1}, \]
And the eigenvalues are given by
\[ \omega_-(\xi) = \frac{\Gamma_0/2}{1 - \cos k_0d} \left( \sin k_0d - \xi \frac{\pi}{N} \sin k_0d_1 \right), \] (S33a)
\[ \gamma_- = \frac{\Gamma_0}{4N} \cot(k_0d_1) \ln \frac{1 + \sin k_0d_1}{1 - \sin k_0d_1}. \] (S33b)

[S1] A. Asenjo-Garcia, M. Moreno-Cardoner, A. Albrecht, H. J. Kimble, and D. E. Chang, Phys. Rev. X 7, 031024 (2017).
[S2] Y.-X. Zhang and K. Mølmer, Phys. Rev. Lett. 122, 203605 (2019).