ON TWO NEW MEANS OF TWO ARGUMENTS III

BARKAT ALI BHAYO AND JÓZSEF SÁNDOR

Abstract. In this paper authors establish the two sided inequalities for the following two new means

\[ X = X(a, b) = A e^{G/P - 1}, \quad Y = Y(a, b) = G e^{L/A - 1}. \]

As well as many other well known inequalities involving the identric mean \( I \) and the logarithmic mean are refined from the literature as an application.

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1. Introduction

The study of the inequalities involving the classical means such as arithmetic mean \( A \), geometric mean \( G \), identric mean \( I \) and logarithmic mean \( L \) have been of the extensive interest for several authors, e.g., see \[2, 3, 4, 11, 20, 21, 29, 30, 31, 41\].

In 2011, Sándor \[27\] introduced a new mean \( X(a, b) \) for two positive real numbers \( a \) and \( b \), defined by

\[ X = X(a, b) = A e^{G/P - 1}, \]

where \( A = A(a, b) = (a + b)/2 \), \( G = G(a, b) = \sqrt{ab} \), and

\[ P = P(a, b) = \frac{a - b}{2 \arcsin \left( \frac{a - b}{a + b} \right)}, \quad a \neq b, \]

are the arithmetic mean, geometric mean, and Seiffert mean \[39\], respectively. This paper contains essentially results on the \( X \) mean, where several inequalities involving the \( X \) mean and the refinement of \((1.3)\) and \((1.4)\) are established.

In the same paper, Sándor introduced another mean \( Y(a, b) \) for two positive real \( a \) and \( b \), which is defined by

\[ Y = Y(a, b) = G e^{L/A - 1}, \]

where

\[ L = L(a, b) = \frac{a - b}{\log(a) - \log(y)}, \quad a \neq b, \]
is a logarithmic mean. For two positive real numbers $a$ and $b$, the identric mean and harmonic mean are defined by

$$I = I(a, b) = \frac{1}{e} \left( \frac{a^b}{b^a} \right)^{1/(a-b)}, \quad a \neq b,$$

and

$$H = H(a, b) = \frac{2ab}{a + b},$$

respectively. In 2012, the $X$ mean appeared in [25]. In 2014, $X$ and $Y$ means published in the journal of Notes in Number Theory and Discrete Mathematics [28]. For the historical background and the generalization of these means we refer the reader to see, e.g., [3, 9, 17, 20, 21, 26, 29, 30, 31, 32, 33, 41]. Connections of these means with the trigonometric or hyperbolic inequalities can be found in [5, 27, 28, 31].

In [28], Sándor proved inequalities of $X$ and $Y$ means in terms of other classical means as well as their relations with each other. Some of the inequalities are recalled for the easy reference.

1.1. Theorem. [28] For $a, b > 0$ with $a \neq b$, one has

$$(1) \quad G < \frac{AG}{P} < X < \frac{AP}{2P - G} < P,$$

$$(2) \quad H < \frac{LG}{A} < Y < \frac{AG}{2A - L} < G,$$

$$(3) \quad 1 < \frac{L^2}{IG} < \frac{L \cdot e^{G/L-1}}{G} < \frac{PX}{AG},$$

$$(4) \quad H < \frac{G^2}{I} < \frac{LG}{A} < \frac{G(A + L)}{3A - L} < Y.$$

In [5], a series expansion of $X$ and $Y$ was given and proved the following inequalities.

1.2. Theorem. [5] For $a, b > 0$ with $a \neq b$, one has

$$(1) \quad \frac{1}{e} (G + H) < Y < \frac{1}{2} (G + H),$$

$$(2) \quad G^2 I < IY < IG < L^2,$$

$$(3) \quad \frac{G - Y}{A - L} < \frac{Y + G}{2A} < \frac{3G + H}{4A} < 1,$$

$$(4) \quad L < \frac{2G + A}{3} < X < L(X, A) < P < \frac{2A + G}{3} < I,$$

$$(5) \quad 2 \left(1 - \frac{A}{P}\right) < \log \left(\frac{X}{A}\right) < \left(\frac{P}{A}\right)^2.$$

For $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$, the $p$th power mean $M_p(a, b)$ and $p$th power-type Heronian mean $H_p(a, b)$ are define by
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\[ M_p = M_p(a, b) = \begin{cases} \left( \frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \]

and

\[ H_p = H_p(a, b) = \begin{cases} \left( \frac{a^p + (ab)^{p/2} + b^p}{3} \right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \]

respectively.

In \cite{10}, Chu et al. proved that the following double inequality

\begin{equation}
M_p < X < M_q
\end{equation}

holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( p \leq 1/3 \) and \( q \geq \log(2)/(1 + \log(2)) \approx 0.4093 \).

Recently, Zhou et al. \cite{42} proved that for all \( a, b > 0 \) with \( a \neq b \), the following double inequality

\begin{equation}
H_\alpha < X < H_\beta
\end{equation}

holds if and only if \( \alpha \leq 1/2 \) and \( \beta \geq \log(3)/(1 + \log(2)) \approx 0.6488 \).

This paper is organized as follows: In Section 1, we give the introduction. Section 2 consists of main results and remarks. In Section 3, some connections of \( X, Y \) and other means are given with trigonometric and hyperbolic functions. Some lemmas are also proved in this section which will be used in the proof of main result. Section 4 deals with the proof of the main result and corollaries.

2. Main result and motivation

Making contribution to the topic, authors refine some previous results appeared in the literature \cite{1, 2, 5, 10, 42, 28} as well as establish new results involving the \( X \) mean.

2.1. Theorem. For \( a, b > 0 \), we have

\begin{equation}
\alpha G + (1 - \alpha)A < X < \beta G + (1 - \beta)A,
\end{equation}

with best possible constants \( \alpha = 2/3 \approx 0.6667 \) and \( \beta = (e - 1)/e \approx 0.6321 \), and

\begin{equation}
A + G - \alpha_1 P < X < A + G - \beta_1 P,
\end{equation}

with best possible constants \( \alpha_1 = 1 \) and \( \beta_1 = \pi(e - 1)/(2e) \approx 0.9929 \).

2.4. Remark. In \cite{28} Theorem 2.7, Sándor proved that for \( a \neq b \),

\begin{equation}
X < A \left[ \frac{1}{e} + \left( 1 - \frac{1}{e} \right) \frac{G}{P} \right],
\end{equation}
and
\[(2.6) \quad Y < G \left[ \frac{1}{e} + \left(1 - \frac{1}{e} \right) \frac{L}{A} \right].\]

As \(A/P > 1\), the right side of (2.2) gives a slight improvement to (2.5). From (2.6), as clearly \(G \cdot L/A < A\), we get a similar inequality. The second inequality in (2.3) could be a counterpart of the inequality \(L + G - A < Y\) studied in [5, Theorem 20].

H. Alzer [1] proved the following inequalities:
\[(2.7) \quad 1 < \frac{(A + G)}{e} < \frac{(L + I)}{2} < \frac{(A + G)}{2},\]
where the constants 1 and 2/e are best possible. The following result improves among others the right side of (2.7).

2.8. Theorem. For \(a \neq b\), one has
\[(2.9) \quad \frac{(A + G)}{e} < X < M_q < \frac{(L + I)}{2} < \frac{(A + G)}{2},\]
where \(q = \log(2)/(1 + \log(2)) \approx 0.4094\) is the best possible constant.

2.10. Remark. Particularly, (2.9) implies that
\[(2.11) \quad X < \frac{(L + I)}{2},\]
which is new. Since \(L < X < I\) (see Theorem 1.1 and 1.2), \(X\) is below the arithmetic mean of \(L\) and \(I\). In fact, by left side of (1.3), and by \(L < M_{1/3}\) and \(L < I < M_{2/3}\), we get also
\[(2.12) \quad L < M_{1/3} < X < M_q < \frac{(L + I)}{2} < I < M_{2/3}\]

2.13. Theorem. For \(a \neq b\), one has
\[(2.14) \quad A + G - P < X < \frac{P^2}{A} < \frac{(A + G)}{2}.\]

2.15. Remark. The right side of (2.14) offers another refinement to \(X < \frac{(A + G)}{2}\). An improvement of \(P^2 > XA\) appears in [28, Theorem 2.9]:
\[P^2 > \left(\frac{(A + G)}{2}\right)^{4/3} > AX,\]
so (2.14) could be further refined. For the following inequalities
\[(2.16) \quad L < \frac{2G + A}{3} < A + G - P < X < \sqrt{PX} < \frac{A + G}{2} < \frac{P + X}{2} < P < \frac{2A + G}{3} < I,\]
one can see that the first inequality is Carlson’s inequality, while the second written in the form \(P < \frac{(2A + G)}{3}\) is due to Sándor [32]. The third inequality is Theorem 2.10 in [28], while the fourth, written as \(PX < ((A + G)/2)^2\) is Theorem 2.11 of [28]. The inequality \((P + X)/2 < P\) follows by \(X < P\), while the last two inequalities are due to Sándor (32, 30).
2.17. Theorem. For $a \neq b$, one has
\begin{equation}
M_{1/2} < (P + X)/2 < M_k,
\end{equation}
where $k = (5 \log 2 + 2)/(6(\log 2 + 1)) \approx 0.5380$.

2.19. Remark. One has
\begin{equation}
L < \frac{2G + A}{3} < X < \frac{L + I}{2} < \frac{A + G}{2} < \frac{P + X}{2} < P < \frac{2A + G}{3} < I.
\end{equation}
\begin{equation}
\sqrt{AG} < \sqrt{PX} < \frac{A + G}{2}.
\end{equation}
Relation (2.21) shows that $\sqrt{PX}$ lies between the geometric and arithmetic means of $A$ and $G$; while (2.16) shows among others that $(A + G)/2$ lies between the geometric and arithmetic means of $P$ and $X$.

2.22. Theorem. One has
\begin{equation}
M_p \leq M_{1/3} < (2G + A)/3 < X, \quad \text{for} \quad p \leq 1/3,
\end{equation}
\begin{equation}
H_{\alpha} \leq H_{1/2} < (2G + A)/3 < X, \quad \text{for} \quad \alpha \leq 1/2.
\end{equation}

2.25. Theorem. For $a \neq b$, one has
\begin{equation}
(AX)^{1/\alpha} < P < (AX^{\beta_2})^{1/(1+\beta_2)},
\end{equation}
with best possible constants $\alpha_2 = 2$ and $\beta_2 = \log(\pi/2)/\log(2e/\pi) \approx 0.8234$.

3. Preliminaries and lemmas

The following result by Biernacki and Krzyż [8] will be used in studying the monotonicity of certain power series.

3.1. Lemma. For $0 < R \leq \infty$. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $C(x) = \sum_{n=0}^{\infty} c_n x^n$ be two real power series converging on the interval $(-R, R)$. If the sequence $\{a_n/c_n\}$ is increasing (decreasing) and $c_n > 0$ for all $n$, then the function $A(x)/C(x)$ is also increasing (decreasing) on $(0, R)$.

For $|x| < \pi$, the following power series expansions can be found in [13. 1.3.1.4 (2)–(3)],
\begin{equation}
x \cot x = 1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n},
\end{equation}
\begin{equation}
\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1},
\end{equation}
and
\begin{equation}
\coth x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1},
\end{equation}
where $B_{2n}$ are the even-indexed Bernoulli numbers (see [12, p. 231]). We can get the following expansions directly from (3.3) and (3.4),

$$\frac{1}{(\sin x)^2} = -(\cot x)' = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n} |(2n - 1)x^{2n-2}|,$$

(3.5)

$$\frac{1}{(\sinh x)^2} = -(\coth x)' = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} (2n - 1)|B_{2n}|x^{2n-2}.$$

(3.6)

For the following expansion formula

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n},$$

(3.7) see [15].

For easy reference we recall the following lemma from [5, 6].

3.8. Lemma. For $a > b > 0$, $x \in (0, \pi/2)$ and $y > 0$, one has

$$\begin{align*}
\frac{P}{A} &= \frac{\sin(x)}{x}, \quad \frac{G}{A} = \cos(x), \quad \frac{H}{A} = \cos(x)^2, \quad \frac{X}{A} = e^{x \cot(x)-1}, \\
\frac{L}{G} &= \frac{\sinh(y)}{y}, \quad \frac{L}{A} = \tanh(y), \quad \frac{H}{G} = \frac{1}{\cosh(y)}, \quad \frac{Y}{G} = e^{\tanh(y)/y-1}.
\end{align*}$$

$$\log \left( \frac{I}{G} \right) = \frac{A}{L} - 1, \quad \log \left( \frac{Y}{G} \right) = \frac{L}{A} - 1.$$

3.9. Remark. It is well known that many inequalities involving the means can be obtain from the classical inequalities of trigonometric functions. For example, the following inequality

$$e^{(x/\tanh(x)-1)/2} < \frac{\sinh(x)}{x}, \quad x > 0,$$

recently appeared in [7, Theorem 1.6], which is equivalent to

$$\frac{\sinh(x)}{x} > e^{x/\tanh(x)-1} \frac{x}{\sinh(x)}.$$

(3.10)

By Lemma 3.8 this can be written as

$$\frac{L}{G} > \frac{I}{G} \cdot \frac{G}{L} = \frac{I}{L},$$

or

$$\frac{L}{G} > \frac{I}{G} \cdot \frac{G}{L} = \frac{I}{L},$$

(3.11)

$$L > \sqrt{IG}.$$
\[
\begin{aligned}
\exp\left(\frac{1}{2}\left(\frac{x}{\tan x} - 1\right)\right) &< \frac{\sin x}{x} < \exp\left((\log \frac{\pi}{2})\left(\frac{x}{\tan x} - 1\right)\right) \quad x \in (0, \pi/2), \\
&\sqrt{AX} < P < A\left(\frac{x}{2}\right)^{\log(\pi/2)}.
\end{aligned}
\]

The second mean inequality in (3.12) was also pointed out by Sándor (see [28, Theorem 2.12]).

**3.13. Lemma.** [4, Theorem 2] For $-\infty < a < b < \infty$, let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, and differentiable on $(a, b)$. Let $g'(x) \neq 0$ on $(a, b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on $(a, b)$, then so are $f(x) - f(a)/g(x) - g(a)$ and $f(x) - f(b)/g(x) - g(b)$.

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

**3.14. Lemma.** The following function

\[ h(x) = \frac{\log(x/ \sin(x))}{\log(e^{1-x/\tan(x)} \sin(x)/x)} \]

is strictly decreasing from $(0, \pi/2)$ onto $(\beta_2, 1)$, where $\beta_2 = \log(\pi/2)/\log(2e/\pi) \approx 0.8234$. In particular, for $x \in (0, \pi/2)$ we have

\[ \left(\frac{e^{1-x/\tan(x)} \sin(x)}{x}\right)^{\beta_2} < \frac{x}{\sin(x)} < \left(\frac{e^{1-x/\tan(x)} \sin(x)}{x}\right). \]

**Proof.** Let

\[ h(x) = \frac{h_1(x)}{h_2(x)} = \frac{\log(x/ \sin(x))}{\log(e^{1-x/\tan(x)} \sin(x)/x)}, \]

for $x \in (0, \pi/2)$. Differentiating with respect to $x$, we get

\[ \frac{h_1'(x)}{h_2'(x)} = \frac{1 - x/ \tan(x)}{(x/ \sin(x))^2 - 1} = \frac{A_1(x)}{B_1(x)}. \]

Using the expansion formula we have

\[ A_1(x) = \sum_{n=1}^{\infty} \frac{2^{2n}2n}{(2n)!} |B_{2n}| x^{2n} = \sum_{n=1}^{\infty} a_n x^{2n} \]

and

\[ B_1(x) = \sum_{n=1}^{\infty} \frac{2^{2n}2n}{(2n)!} |B_{2n}|(2n-1) x^{2n} = \sum_{n=1}^{\infty} b_n x^{2n}. \]

Let $c_n = a_n/b_n = 1/(2n-1)$, which is the decreasing in $n \in \mathbb{N}$. Thus, by Lemma 3.1, $h_1'(x)/h_2'(x)$ is strictly decreasing in $x \in (0, \pi/2)$. In turn, this implies by Lemma...
3.13 that $h(x)$ is strictly decreasing in $x \in (0, \pi/2)$. Applying l’Hôpital rule, we get $\lim_{x \to 0} h(x) = 1$ and $\lim_{x \to \pi/2} h(x) = \beta_2$. This completes the proof. □

3.16. Remark. It is observed that the inequalities in (3.15) coincide with the trigonometric inequalities given in (3.12). Here Lemma 3.14 gives a new and an optimal proof for these inequalities.

3.17. Lemma. \textit{The following function}

$$f(x) = \frac{1 - e^{x/\tan(x) - 1}}{1 - \cos(x)}$$

\textit{is strictly decreasing from $(0, \pi/2)$ onto $((e-1)/e, 2/3)$ where $(e-1)/e \approx 0.6321$. In particular, for $x \in (0, \pi/2)$, we have}

$$\frac{1}{\log(1 + (e-1)\cos(x))} < \frac{\tan(x)}{x} < \frac{1}{1 + \log(1 + 2\cos(x))/3}.$$ 

\textit{Proof. Write} $f(x) = f_1(x)/f_2(x)$, \textit{where} $f_1(x) = 1 - e^{x/\tan(x) - 1}$ \textit{and} $f_2(x) = 1 - \cos(x)$ \textit{for all} $x \in (0\pi/2)$. \textit{Clearly,} $f_1(x) = 0 = f_2(x)$. \textit{Differentiating with respect to} $x$, \textit{we get}

$$f'_1(x)/f'_2(x) = \frac{e^{x/\tan(x) - 1}}{\sin(x)^3} \left( \frac{x}{\sin(x)^2} - \frac{\cos(x)}{\sin(x)} \right) = f_3(x).$$

\textit{Again}

$$f'_3(x) = -\frac{e^{x/\tan(x) - 1}}{\sin(x)^3} (c(x) - 2),$$

\textit{where}

$$c(x) = x \left( \frac{\cos(x)}{\sin(x)} + \frac{x}{\sin(x)^2} \right).$$

\textit{In order to show that} $f'_3 < 0$, \textit{it is enough to prove that}

$$c(x) > 2,$$

\textit{which is equivalent to}

$$\frac{\sin(x)}{x} < \frac{x + \sin(x) \cos(x)}{2 \sin(x)}.$$ 

\textit{Applying the Cusa-Huygens inequality}

$$\frac{\sin(x)}{x} < \frac{\cos(x) + 2}{3},$$

\textit{we get}

$$\frac{\cos(x) + 2}{3} < \frac{x + \sin(x) \cos(x)}{2 \sin(x)},$$

\textit{which is equivalent to} $(\cos(x) - 1)^2 > 0$. \textit{Thus} $f'_3 > 0$, \textit{clearly} $f'_1/f'_2$ \textit{is strictly decreasing in} $x \in (0, \pi/2)$. \textit{By Lemma 3.13} we conclude that the function $f(x)$ is strictly decreasing in $x \in (0, \pi/2)$. \textit{The limiting values follows easily. This completes the proof of the lemma.} □
3.18. Lemma. The following function
\[ f_4(x) = \frac{\sin(x)}{x(\cos(x) - e^{x\cot(x)} - 1)} \]
is strictly increasing from \((0, \pi/2)\) onto \((1, c)\), where \(c = 2e/(\pi(e - 1)) \approx 1.0071\). In particular, for \(x \in (0, \pi/2)\) we have
\[ 1 + \cos(x) - e^{x/\tan(x)} < \frac{\sin(x)}{x} < c(1 + \cos(x) - e^{x/\tan(x)}). \]

Proof. Differentiating with respect to \(x\) we get
\[ f'_4(x) = \frac{e(x - \sin(x)) (e \cos(x) - (x + \sin(x)) e^{x \cot(x)} \csc(x) + e)}{x^2 (e \cos(x) - e^{x \cot(x)} + e)^2}. \]
Let
\[ f_5(x) = \log \left((x + \sin(x)) e^{x \cot(x)} / \sin(x) \right) - \log(\cos(x) + e), \]
for \(x \in (0, \pi/2)\). Differentiation yields
\[ f'_5(x) = \frac{2 - x (\cot(x) + x \csc^2(x))}{x + \sin(x)}, \]
which is negative by the proof of Lemma 3.17 and \(\lim_{x \to 0} f_5(x) = 0\). This implies that \(f'_5(x) > 0\), and \(f_4(x)\) is strictly increasing. The limiting values follows easily. This implies the proof. \(\square\)

3.19. Lemma. For \(a \neq b\), one has
\[ M_{1/3} < (2G + A)/3. \]

Proof. Let \(G = G(a, b)\), etc. Divide both sides with \(b\) and put \(a/b = x\). Then inequality (3.20) becomes the following:
\[ \left(\frac{x^{1/3} + 1}{2}\right)^3 < 4(x + 4\sqrt{x} + 1). \]
Let \(x = t^6\), where \(t > 1\). Then raising both sides of (3.21) to 3th power, after elementary transformations we get,
\[ t^6 - 9t^4 + 16t^3 - 9t^2 + 1 > 0, \]
which can be written as \((t - 1)^4(t^2 + 4t + 1) > 0\), so it is true. Thus (3.21) and (3.20) are proved. \(\square\)

Since \(L < M_{1/3}\), by (3.20) we get a new proof, as well as a refinement of Carlson’s inequality \(L < (2G + A)/3\).

3.22. Lemma. For \(a \neq b\), one has
\[ H_{1/2} < (2G + A)/3. \]
Proof. By definition of \( H_\alpha \) one has
\[
H_{1/2} = \left( (\sqrt{a} + (ab)^{1/4} + \sqrt{b})/3 \right)^2 = \left( \sqrt{2(A + G)} + \sqrt{G} \right)^2 / 9,
\]
by remarking that \( \sqrt{a} + \sqrt{b} = \sqrt{2(A + G)} \). Therefore, (2) can be written equivalently as
\[
(3.24) \quad (2(A + G) + 2\sqrt{2G(A + G) + G})/9 < (2G + A)/3.
\]
Now, it is immediate that (3.24) becomes, after elementary computations
\[
(3.25) \quad A + 3G > 2\sqrt{2G(A + G)},
\]
or by raising both sides to the 2th power:
\[
A^2 + 6AG + 9G^2 > 8AG + 8G^2,
\]
which become \((A - G)^2 > 0\), true. Thus (3.25) and (3.24) are proved, and (3.23) follows. \( \square \)

4. Proof of main result

Proof of Theorem 2.1. It follows from Lemma 3.17 that
\[
\frac{e - 1}{e} < \frac{1 - 1/e^{1-x/\tan(x)}}{\cos(x)/e^{1-x/\tan(x)} - 1/e^{1-x/\tan(x)}} < \frac{2}{3},
\]
Now we get the proof of (2.2) by utilizing the Lemma 3.8. The proof of (2.3) follows easily from Lemmas 3.8 and 3.17. \( \square \)

Proof of Theorem 2.8. The second inequality of (2.9) is right side of relation (1.3). In [2], Alzer and Qiu proved the third inequality of (2.9). The last inequality is the left side of (2.7). By [10] and [2], \( q \) is best possible constant in both sides.

Now we shall prove the first inequality of (2.9). By using Lemma 3.8 is is easy to see that, this becomes equivalent with \( 1 + \cos(x) < e^{x \cot(x)} \), or
\[
(4.1) \quad \log(1 + \cos(x)) < x \cot(x), \quad x \in (0, \pi/2).
\]
Now, by the classical inequality \( \log(1 + t) < t \; (t > 0) \), applied to \( t = \cos(x) \), we get \( \log(1 + \cos(x)) < \cos(x) \). Now \( \cos(x) < x \cot(x) = x \cos(x)/\sin(x) \) is true by \( \sin(x) < x \). The proof of (4.1) follows. \( \square \)

One has the following relation, in analogy with relation (2.7) of Theorem 1.2 for the mean \( Y \):

4.2. Corollary. One has
\[
(A + G)/e < X < (A + G)/2,
\]
where the constants \( e \) and \( 2 \) are best possible.
The inequalities \((A + G)/e < X\) and \((2G + A)/3 < X\) are not comparable.

**Proof of Theorem 2.13.** The second inequality of (2.14) appeared in [25] in the form \(P^2 > AX\). The last inequality follows by \(P < (2A + G)/3\). Indeed, one has \(((2A + G)/3)^2 < A(A + G)/2\) becomes \(2G^2 < A^2 + AG\), and this is true by \(G < A\).

**Proof of Theorem 2.17.** By [28, Theorem 2.10], one has \(P + X > A + G\), and remarking that \((A + G)/2 = M_{1/2}\), the left side of (2.18) follows. For the right side of (2.18), we will use \(P < M_t\) with \(t = 2/3\) (see [32]), and \(X < M_q\) (10), where \(q = (\log 2)/(\log 2 + 1)\). On the other hand the function \(f(t) = M_t\) is known to be strictly log-concave for \(t > 0\) (see [34]). Particularly, this implies that \(f(t)\) is strictly concave. Thus \((M_t + M_q)/2 < M_{(t+q)/2}\). As \((t+q)/2 = k \approx 0.5380\), the result follows.

**4.3. Corollary.** One has the following two sets of inequalities:

1. \(PX > PL > AG\),
2. \(IL > PL > AG\).

**Proof.** The first inequality of (1) follows by \(X > L\), while the second appears in [32]. The first inequality of (2) follows by \(I > P\), while the second one is the same as the second one in (1).

**4.4. Remark.** Particularly in Corollary 4.3, (2) improves Alzer’s inequality \(IL > AG\). Inequality (1) improves \(PX > AG\), which appears in [28].

**4.5. Corollary.** One has

1. \(X > A(P + G)/(3P - G) > (2G + A)/3 > L\).
2. \(P^2/A > X > (P + G)/2\).

**Proof.** The first two inequalities of (1) appear in [28, Theorem 2.5 and Remark 2.3]. The second inequality of (2) follows by the first inequality of (1) and the remark that \(A/(3P - G) > 1/2\). Since this is \(P < (2A + G)/3\); while the first one is \(P^2 > AX\) (25).

**4.6. Remark.** Since it is known that \(P > (2/\pi)A\) (due to Seiffert, see [32]). By \(X > (P + G)/2\) we get the inequality \(X > [(2/\pi)A + G]/2\), which is not comparable with \((A + G)/e < X\).

**Proof of Theorem 2.22.** The first inequality of (2.23) follows, since the function \(f(t) = M_t\) is known to be strictly increasing. The second inequality follows by [32], while the third one can be found in Theorem 1.2.

It is known that \(H_p\) is an increasing function of \(p\). Therefore, the proof of (2.24) follows by [32].

**4.7. Corollary.** For \(a, b > 0\) with \(a \neq b\), we have

\[
\frac{I}{L} < \frac{L}{G} < 1 + \frac{G}{H} - \frac{I}{G}.
\]
Proof. The first inequality is due to Alzer [3], while the second inequality follows from the fact that the function
\[ x \mapsto \frac{1 - e^{x/\tanh(x)} - 1}{1 - \cosh(x)} : (0, \infty) \to (0, 1) \]
is strictly decreasing. The proof of the monotonicity of the function is the analogue to the proof of Lemma 3.14. □

The right side of (4.8) may be written as \( L + I < G + A \) (by \( H = G^2/A \)), and this is due to Alzer (see [2, 29] for history of early results).

Proof of Theorem 2.25. The proof follows easily from Lemma 3.14. □

In [38], Seiffert proved that
\[ (4.9) \quad \frac{2}{\pi} A < P, \]
for all \( a, b > 0 \) with \( a \neq 0 \). As a counterpart of the above result we give the following inequalities.

4.10. Corollary. For \( a, b > 0 \) with \( a \neq b \), the following inequalities
\[ \frac{1}{e} A < \frac{\pi}{2e} P < X < P \]
holds true.

Proof. The first inequality follows from (4.9). For the proof of the second inequality we write by Lemma 3.8
\[ f_5'(x) = \frac{X}{P} = \frac{xe^{x/\tan(x)} - 1}{\sin(x)} = f_5(x) \]
for \( x \in (0, \pi/2) \). Differentiation gives
\[ \frac{e^{x/\tan(x)} - 1}{\sin(x)} \left( 1 - \frac{x^2}{\sin(x)^2} \right) < 0. \]
Hence the function \( f_5 \) is strictly decreasing in \( x \), with
\[ \lim_{x \to 0} f_5(x) = 1 \quad \text{and} \quad \lim_{x \to \pi/2} f_5(x) = \pi/(2e) \approx 0.5779. \]
This implies the proof. □

We finish this paper by giving the following open problem and a conjecture.

Open problem. What are the best positive constants \( a \) and \( b \), such that
\[ M_a < (P + X)/2 < M_b. \]

Conjecture. For \( a \neq b \), one has
\[ PX > IL. \]
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