MORITA EQUIVALENCE OF NEST ALGEBRAS

G. K. ELEFTHERAKIS

Abstract. Let $\mathcal{N}_1$ (resp. $\mathcal{N}_2$) be a nest, $A$ (resp. $B$) be the corresponding nest algebra, $A_0$ (resp. $B_0$) be the subalgebra of compact operators. We prove that the nests $\mathcal{N}_1, \mathcal{N}_2$ are isomorphic if and only if the algebras $A, B$ are weakly-∗ Morita equivalent if and only if the algebras $A_0, B_0$ are strongly Morita equivalent. We characterize the nest isomorphisms which implement stable isomorphism between the corresponding nest algebras.

1. Introduction

Rieffel, introduced the idea of Morita equivalence in Operator Theory developing the theory of Morita equivalence for $C^*$ and $W^*$ algebras [17]. After the advent of the theory of operator spaces and operator algebras a parallel Morita theory for non-selfadjoint algebras was developed by Blecher, Muhly and Paulsen [6], [2]. We call this equivalence strong Morita equivalence.

Recently, two approaches have been suggested for the Morita equivalence of dual operator algebras. The one was introduced [11] by the author of this article and it is equivalent to the notion of stable isomorphism of dual operator algebras [13]. We call this equivalence $\Delta$-equivalence. The other was introduced by Blecher and Kashyap [3], [14] and it is strictly weaker than $\Delta$-equivalence. This equivalence is called weak-∗ Morita equivalence. It is interesting that if $A$ and $B$ are strongly Morita equivalent approximately unital operator algebras then the second dual operator algebras $A^{**}, B^{**}$ are weakly-∗ Morita equivalent [3]. New results on weak-∗ Morita equivalence and $\Delta$-equivalence can be found in [4].

In this paper we prove that strong and weak-∗ Morita equivalence is a lattice property for nest algebras. Particularly we prove that if $A$ and $B$ are nest algebras and $A_0, B_0$ are the subalgebras of compact operators then $A_0$ and $B_0$ are strongly Morita equivalent if and only if $A$ and $B$ are weakly-∗ Morita equivalent if and only if the nests $\text{Lat}(A), \text{Lat}(B)$ are isomorphic. The main tool of the proof is that if $\theta : \text{Lat}(A) \rightarrow \text{Lat}(B)$ is a nest isomorphism we can construct a dual operator $A - B$ bimodule $Y$ and a dual operator $B - A$ bimodule $X$ such that the identity operator of $A$ is the limit in strong operator topology of a net of finite rank contractions $(f_\lambda)$ where every $f_\lambda$ is the norm limit of a sequence $(y_\lambda x_\lambda^*)_{i \in \mathbb{N}}$, where $y_\lambda^*$ is a contractive row operator with finite entries from $Y$ and $x_\lambda$ is a contractive column operator with finite entries from $X$.
entries from $X$. Similarly we can decompose the identity of the algebra $B$. This can be considered as a generalization of the Erdos density Theorem for nest algebras [7].

In section 3 we prove that two nest algebras are weakly-\v{S} Morita equivalent if and only if they are spatially Morita equivalent (definition 3.1). Also we prove that every spatially Morita equivalent dual operator algebra with a nest algebra is weakly-\v{S} Morita equivalent with this nest algebra. It is interesting that this does not happen for the more general class of operator algebras, the CSL algebras.

In section 4 we present a measure-theoretic result which describes when two separably acting nest algebras are stably isomorphic. As it was pointed out in [3] the [12, example 3.7] is an example of weak-\v{S} Morita equivalent algebras which are not stably isomorphic. Using the results of this paper we give a new proof of the fact that weak-\v{S} Morita equivalence is strictly weaker than $\Delta$-equivalence.

In section 5 we present a counterexample which states that the second duals of two unital strongly Morita equivalent algebras are not necessarily stably isomorphic.

In what follows we describe the notions we use in this paper. Since we use extensively the basics of Operator Space Theory, we refer the reader to the monographs [5], [9], [15] and [16] for further details. A (normal) representation of a (dual) operator algebra $A$ is a (w$^*$-continuous) completely contractive homomorphism $\alpha : A \rightarrow B(H)$ on a Hilbert space $H$. In the case $A$ is unital, we assume that $\alpha$ is unital.

Let $H,K$ be Hilbert spaces and $A \subset B(H)$ be an algebra. A subspace $X \subset B(K,H)$ is called a left module over $A$ if $AX \subset X$. Similarly we can define the right modules over $A$. A left and right module over $A$ is called a bimodule over $A$. An abstract left (right) operator module over an abstract operator algebra $A$ is an operator space $Y$ such that there exist a completely contractive bilinear map $A \times Y \rightarrow Y$ ($Y \times A \rightarrow Y$). A left and right operator module over $A$ is called an operator bimodule over $A$.

If $A$ is a dual operator algebra and $Y$ is a dual operator space we say that $Y$ is a left (right) dual operator module if the above completely contractive bilinear map is separately $w^*$-continuous. A left and right dual operator module over $A$ is called a dual operator bimodule over $A$.

Two operator bimodules $Y$ and $Z$ over an operator algebra $A$ are called isomorphic as operator bimodules if there exists a completely isometric and onto $A$-module map $\pi : Y \rightarrow Z$. We denote $Y \cong Z$ as operator bimodules. In the case $A$ is a dual operator algebra and $Y,Z$ are dual operator bimodules we denote $Y \cong Z$ as dual operator bimodules if the above completely isometric and onto $A$-module map $\pi$ is $w^*$-(bi)continuous.
If \( Y \) is a right operator module over an operator algebra \( A \) and \( X \) is a left operator module over \( A \) we denote by \( Y \otimes^{h} A X \) the balanced Haagerup tensor product of \( Y \) and \( X \) which linearizes the completely bounded \( A \)-balanced bilinear maps \([5, 3.4]\). If \( Y \) (resp. \( X \)) is a left (resp. right) operator module over an operator algebra \( B \) then \( Y \otimes^{h} A X \) is also a left (resp. right) operator module over \( B \), \([6] \) Lemma 2.4].

If \( Y \) is a right operator module over a dual operator algebra \( A \) and \( X \) is a left dual operator module over \( A \) we denote by \( Y \otimes^{\sigma h} A X \) the balanced normal Haagerup tensor product of \( Y \) and \( X \) which linearizes the separately \( w^{*} \)-continuous completely bounded \( A \)-balanced bilinear maps \([13]\). In the case \( Y \) (resp. \( X \)) is a left (resp. right) dual operator module over a dual operator algebra \( B \) then \( Y \otimes^{\sigma h} A X \) is also a left (resp. right) dual operator module over \( B \), \([13]\).

We give now the two definitions of Morita equivalence using in this paper:

**Definition 1.1.** \([6]\) The operator algebras \( A, B \) are called **strongly Morita equivalent** if there exist an \( A - B \) operator module \( X \) and a \( B - A \) operator module \( Y \) such that \( A \cong X \otimes^{h} B Y \) and \( B \cong Y \otimes^{h} A X \) as \( A \) and \( B \) operator bimodules respectively.

**Definition 1.2.** \([3]\) The dual operator algebras \( A, B \) are called **weakly-\(*\) Morita equivalent** if there exist an \( A - B \) dual operator module \( X \) and a \( B - A \) dual operator module \( Y \) such that \( A \cong X \otimes^{\sigma h} B Y \) and \( B \cong Y \otimes^{\sigma h} A X \) as \( A \) and \( B \) dual operator bimodules respectively.

If \( X \) is a subspace of \( B(H, K) \), where \( H \) and \( K \) are Hilbert spaces, we denote by \( R^{\text{fin}}_{\infty}(X) \) (resp. \( C^{\text{fin}}_{\infty}(X) \)) the space of operators \((x_{1}, x_{2}, ...) : H^{\infty} \rightarrow K \) (resp. \((x_{1}, x_{2}, ...)^{T} : H \rightarrow K^{\infty} \)) such that \( x_{i} \in X \) for all \( i \) and there exists \( n_{0} \in \mathbb{N} \) such that \( x_{n} = 0 \) for all \( n \geq n_{0} \).

If \( s_{1} = (s_{1}^{1}, s_{1}^{2}, ..., s_{n_{1}}^{1}, 0, 0, ...), s_{1}^{1} \neq 0 \) and \( s_{2} = (s_{2}^{1}, s_{2}^{2}, ..., s_{n_{2}}^{2}, 0, 0, ...), s_{2}^{2} \neq 0 \) are operators in \( R^{\text{fin}}_{\infty}(X) \) we denote by \( (s_{1}, s_{2}) \) the operator

\[
(s_{1}^{1}, s_{1}^{2}, ..., s_{n_{1}}^{1}, s_{1}^{2}, ..., s_{n_{2}}^{2}, 0, 0, ...)
\]

which also belongs to \( R^{\text{fin}}_{\infty}(X) \). In the same way if \( s_{1}, s_{2}, ..., s_{n} \in R^{\text{fin}}_{\infty}(X) \) we define the operator \( (s_{1}, s_{2}, ..., s_{n}) \in R^{\text{fin}}_{\infty}(X) \). Similarly if \( t_{1}, t_{2}, ..., t_{n} \in C^{\text{fin}}_{\infty}(X) \) we define the operator \( (t_{1}, t_{2}, ..., t_{n})^{T} \in C^{\text{fin}}_{\infty}(X) \).

A **nest** \( \mathcal{N} \) is a totally ordered set of projections of a Hilbert space \( H \) containing the zero and identity operators which is closed under arbitrary intersections and closed spans. The corresponding **nest algebra** is

\[
\text{Alg}(\mathcal{N}) = \{ x \in B(H) : N^{\perp} x N = 0 \ \forall \ N \in \mathcal{N} \}.
\]

If \( N \in \mathcal{N} \) we denote by \( N_{-} \) the projection onto the closed span of the union \( \bigcup_{M \in \mathcal{N}} (M(H)) \). If \( N_{-} < N \) we call the projection \( N \ominus N_{-} \) an **atom**. If the nest
has not atoms is called a continuous nest. If the atoms span the identity operator the nest is called a totally atomic nest. An order preserving 1-1 and onto map between two nests is called a nest isomorphism.

If \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) are nests acting on the Hilbert spaces \( H_1, H_2 \) respectively and \( \theta : \mathcal{N}_1 \to \mathcal{N}_2 \) is a nest isomorphism we denote by \( \text{Op}(\theta) \) the space of operators \( x \in B(H_1, H_2) \) satisfying \( \theta(N) \downarrow x N = 0 \) for all \( N \in \mathcal{N}_1 \). Observe that \( \text{Op}(\theta) \) is an \( \text{Alg}(\mathcal{N}_2) - \text{Alg}(\mathcal{N}_1) \) bimodule.

Finally, if \( X \) is a normed space we denote by \( \text{Ball}(X) \) the unit ball of \( X \) and by \( X^\ast \) its dual space. If \( H_1, H_2 \) are Hilbert spaces and \( \xi \in H_2, \eta \in H_1 \) are vectors we denote by \( \xi \otimes \eta^* \) the rank 1 operator sending every \( \omega \in H_1 \) to \( \langle \omega, \eta \rangle \xi \in H_2 \), where \( \langle \cdot, \cdot \rangle \) is the inner product of \( H_1 \). Also we symbolize the strong operator topology by SOT.

2. Morita equivalence for nest algebras

In this section we fix nests \( \mathcal{N}_1, \mathcal{N}_2 \) acting on the Hilbert spaces \( H_1, H_2 \) respectively, and a nest isomorphism \( \theta : \mathcal{N}_1 \to \mathcal{N}_2 \). We denote \( A = \text{Alg}(\mathcal{N}_1), B = \text{Alg}(\mathcal{N}_2), X = \text{Op}(\theta), Y = \text{Op}(\theta^{-1}) \). If \( Z \) is a space of operators we denote its subspace of compact operators by \( Z_0 \). Observe that
\[
AYB \subset Y, \ BXA \subset X, \ YX \subset A, \ XY \subset B,
\]
\[
A_0 Y_0 B_0 \subset Y_0, \ B_0 X_0 A_0 \subset X_0, \ Y_0 X_0 \subset A_0, \ X_0 Y_0 \subset B_0.
\]

The main result of this section is Theorem 2.9. In particular we are going to prove that
\[
A_0 \cong Y_0 \otimes_{b_0} X_0, \ B_0 \cong X_0 \otimes_{A_0}^h Y_0, \ A \cong Y \otimes_{B}^h X, \ B \cong X \otimes_{A}^h Y.
\]
Suppose that \( p = \vee \{ N \in \mathcal{N}_- : N \in \mathcal{N}_1 \} \). The following lemmas are used in Theorem 2.5 where we are going to prove a variant of the Erdos density Theorem for nest algebras: There exists a net of finite rank contractions \( (f_\lambda) \subset A \) converging in SOT topology to the identity operator of \( H_1 \), where every \( f_\lambda \) is the norm limit of a sequence \( (y_\lambda^i x_\lambda^i)_{i \in \mathbb{N}} \) where \( y_\lambda^i \in \text{Ball}(R^f_{\infty}(Y_0)), x_\lambda^i \in \text{Ball}(C^f_{\infty}(X_0)) \) for all \( i, \lambda \).

**Lemma 2.1.** There exists a net \( (l_\lambda) \) of finite rank contractions converging in SOT topology to the projection \( p \) such that \( l_\lambda = s_\lambda t_\lambda \) where \( s_\lambda \in \text{Ball}(R^f_{\infty}(Y_0)), t_\lambda \in \text{Ball}(C^f_{\infty}(X_0)) \) for all \( \lambda \).

**Proof** Suppose that \( p = \vee_{k \in J} p_k \) where \( p_k = N_k \ominus (N_k)_- \) for \( N_k \in \mathcal{N}_1, k \in J \). Choose a net of finite rank contractions \( (f_i)_{i \in I} \) converging in SOT topology to the identity operator of \( H_1 \). If \( \mathcal{F} = \{ F : F \text{ finite subset of } J \} \) the family \( (g_{F,i})_{(F,i)} \) indexed by \( \mathcal{F} \times I \) where \( g_{F,i} = \sum_{k \in F} p_k f_i p_k \) is a net. Observe that every \( g_{F,i} \) is a finite rank contraction belonging to \( A \). We can easily check that \( SOT - \lim_{i \to \infty} g_{F,i} = \vee_k p_k = p \).
Let \( f = p_k f_i p_k \) for some \( k \in I, i \in I \) with polar decomposition \( f = u |f| \).

Suppose that

\[
|f| = \sum_{j=1}^{n} \lambda_j \xi_j \otimes \xi_j^*
\]

for \( \lambda_j \geq 0 \) and \( \xi_j \) orthogonal vectors of \( p_k(H_1) \). Choose a unit vector \( \eta \) in \( (\theta(N_k) \otimes \theta(N_k))(H_2) \). Now we have

\[
|f| = \sum_{j=1}^{n} \lambda_j \xi_j \otimes \eta^* \cdot \eta \otimes \xi_j^* = y y^*
\]

where \( y = (\sqrt{\lambda_1} \xi_1 \otimes \eta^*, ..., \sqrt{\lambda_n} \xi_n \otimes \eta^*) \). Observe that \( f = u y y^* \) and \( u y \in Ball(R_{\infty}^{fin}(Y_0)), y^* \in Ball(C_{\infty}^{fin}(X_0)) \).

Suppose now that \( F = \{ j_1, ..., j_n \} \subset I, i \in I \) and \( g_{F,i} = \sum_{k=1}^{n} p_{jk} f_i p_{jk} \).

By the above arguments \( p_{jk} f_i p_{jk} = s_k t_k \) where \( s_k \in Ball(R_{\infty}^{fin}(Y_0)), t_k \in Ball(C_{\infty}^{fin}(X_0)) \). So \( g_{F,i} = s t \) where

\[
s = (s_1, ..., s_n) \in R_{\infty}^{fin}(Y_0), \quad t = (t_1, ..., t_n)^T \in C_{\infty}^{fin}(X_0).
\]

Also, since the projections \( (p_k)_{k \in J} \) are pairwise orthogonal and \( \|s_k s_k^*\| \leq 1 \) for all \( k \) we have that

\[
\|s\|^2 = \| \sum_{k=1}^{n} s_k s_k^* \| = \| \sum_{k=1}^{n} p_{jk} s_k s_k^* p_{jk} \| \leq 1.
\]

Similarly we can prove \( \|t\| \leq 1 \) and this completes the proof.

\[ \square \]

**Lemma 2.2.** Suppose that \( p^\perp \neq 0, \xi, \eta \in Ball(H_1) \) and \( N \in \mathcal{N}_1 \) such that \( \xi = p^\perp N(\xi), \eta = p^\perp N^\perp(\eta) \). There exist rank 1 operators \( (s_n)_{n \in N} \subset Ball(Y), (t_n)_{n \in N} \subset Ball(X) \), such that the operator \( \xi \otimes \eta^* \) is the norm limit of the sequence \( (s_n t_n)_{n \in N} \).

**Proof** We define the continuous order preserving map

\[
\phi : p^\perp \mathcal{N}_1 \to [0, \|\xi\|^2] : p^\perp M \mapsto \|p^\perp M(\xi)\|^2.
\]

The nest \( p^\perp \mathcal{N}_1 \) is continuous, so \( \phi \) is onto \( [0, \|\xi\|^2] \). Choose a strictly increasing sequence \( (\lambda_n) \) such that \( \lambda_n \to \|\xi\|^2 \). Choose \( N_n \in \mathcal{N}_1 \) such that \( \phi(p^\perp N_n) = \lambda_n \).

It follows that \( N_n < N_{n+1} < N \) for all \( n \in N \) and \( p^\perp N_n(\xi) \to \xi \). Similarly we can find a sequence \( (M_n)_{n \in N} \) such that \( N < M_{n+1} < M_n \) for all \( n \in N \) and \( p^\perp (I - M_n)(\eta) \to \eta \). For every \( n \in N \) we choose \( \omega_n \in H_2 \) such that \( \|\theta(M_n) \otimes \theta(N_n)(\omega_n)\| = 1 \). The operator

\[
s_n = p^\perp N_n(\xi) \otimes (\theta(M_n) \otimes \theta(N_n)(\omega_n))^*
\]

satisfies \( s_n = N_n s_n \theta(N_n)^\perp \) and so \( s_n \in Ball(Y_0) \). Similarly the operator

\[
t_n = (\theta(M_n) \otimes \theta(N_n)(\omega_n)) \otimes p^\perp (I - M_n)(\eta)^*
\]
Lemma 2.3. Suppose that $p^\perp \neq 0$, $\xi \in H_1$ such that $\|p^\perp(\xi)\| = 1$ and $q$ is the projection onto the space $p^\perp N_1^p \xi$. There exists a sequence of finite rank contractions $(r_n)_{n \in \mathbb{N}} \subset A$ converging in SOT topology to the projection $q$ such that $r_n = \| \cdot \| \lim_{i \in \mathbb{N}} s_i^n t_i^n$ where $s_i^n \in Ball(R_{0}^{fin}(Y_0)), t_i^n \in Ball(C_{0}^{fin}(X_0))$ for all $i, n \in \mathbb{N}$.

Proof. We define the continuous order preserving map

$$\phi : N_1 p^\perp \to [0, 1], \phi(N p^\perp) = \| N p^\perp(\xi) \|^2.$$ 

Since the nest $N_1 p^\perp$ is continuous $\phi$ is onto $[0, 1]$. Choose $N_{k,n} p^\perp$ the least element in $N_1 p^\perp$ such that $\phi(N_{k,n} p^\perp) = \frac{k}{2^n}, k = 0, 1, \ldots, 2^n$.

We denote

$$E_{k,n} = (N_{k,n} \ominus N_{k-1,n}) p^\perp, \quad \xi_{k,n} = \frac{2^n}{2^n} E_{k,n}(\xi), \quad r_n = \sum_{k=2}^{2^n} f_{k,n}$$

where $f_{k,n} = \xi_{k-1,n} \ominus \xi_{k,n}$.

As in [7, Lemma 3.9] we can prove that $\| r_n \| \leq 1$ and the sequence $(r_n)_{n \in \mathbb{N}}$ converges in SOT topology to the operator $q$.

By the above lemma there exist sequences of rank 1 operators $(s_i^{k,n})_{i \in \mathbb{N}} \subset Ball(Y_0), (t_i^{k,n})_{i \in \mathbb{N}} \subset Ball(X_0)$, such that $s_i^{k,n} t_i^{k,n} \overset{\| \cdot \|}{\to} f_{k,n}, i \to \infty$ for all $k, n$.

We denote

$$s_i^n = (s_{i}^{2,n}, s_{i}^{3,n}, \ldots, s_{i}^{2^n,n}), \quad t_i^n = (t_{i}^{2,n}, t_{i}^{3,n}, \ldots, t_{i}^{2^n,n})^T$$

and we have $r_n = \| \cdot \| - \lim_i s_i^n t_i^n$. Also

$$\| s_i^n \|^2 = \| \sum_{k=2}^{2^n} s_i^{k,n}(s_i^{k,n})^* \|.$$ 

We may assume that $s_i^{k,n} = E_{k-1,n} s_i^{k,n}$ so

$$\| s_i^n \|^2 = \| \sum_{k=2}^{2^n} E_{k-1,n} s_i^{k,n}(s_i^{k,n})^* E_{k-1,n} \|.$$ 

Since $\| s_i^{k,n} \| \leq 1$ and the projections $(E_{k-1,n})_k$ are pairwise orthogonal we have $\| s_i^n \| \leq 1$. Similarly we can prove $\| t_i^n \| \leq 1$.

Lemma 2.4. Suppose that $p^\perp \neq 0$. There exists a net $(g_\lambda)$ of finite rank contractions in $A$ converging in SOT topology to $p^\perp$ such that $g_\lambda = \| \cdot \| - \lim_{i \in \mathbb{N}} s_i^\lambda t_i^\lambda$ for all $\lambda$ where $s_i^\lambda \in Ball(R_{0}^{fin}(Y_0)), t_i^\lambda \in Ball(C_{0}^{fin}(X_0))$ for all $i \in \mathbb{N}$. 


Proof Using Zorn’s Lemma we find a family of vectors $\xi_k : k \in L$ such that the projections $q_k$ onto $p_n^{1/2} N_1^{f_n} \xi_k$, $k \in L$ are pairwise orthogonal and they span $p^+$. We assume that $\|p^+(\xi_k)\| = 1$ for all $k \in L$. From Lemma 2.2 there exist finite rank contractions $(r^k_n)_{n \in \mathbb{N}}$ such that $q_k = SOT - \lim_{n \in \mathbb{N}} r^k_n$ and $r^k_n = \| \cdot \| - \lim_{i \in \mathbb{N}} s_i^{n,k} t_i^{n,k}$ for sequences

$$(s_i^{n,k})_{i \in \mathbb{N}} \subset Ball(R_{\infty}^{fin}(Y_0)), \quad (t_i^{n,k})_{i \in \mathbb{N}} \subset Ball(C_{\infty}^{fin}(X_0)).$$

We define $\mathcal{F} = \{ F : F \text{ finite subset of } L \}$. If $F \in \mathcal{F}$ and $n \in \mathbb{N}$ we define the finite rank contraction $g_{n,F} = \sum_{k \in F} r^k_n$. The family $(g_{n,F})_{n,F}$ indexed by $\mathbb{N} \times \mathcal{F}$ is a net. Fix $\xi \in H_1$.

Observe that for all $n \in \mathbb{N}$

$$\| r^k_n(\xi) - q_k(\xi) \|^2 = \| q_k(r^k_n - I_{H_1})q_k(\xi) \|^2 \leq 2 \| q_k(\xi) \|^2$$

and so

$$\sum_{k \in L} \| r^k_n(\xi) - q_k(\xi) \|^2 \leq 2 \sum_{k \in L} \| q_k(\xi) \|^2 < \infty.$$ 

If $n \in \mathbb{N}$ and $F \in \mathcal{F}$ we have

$$\begin{align*}
(2.1) \quad \| g_{n,F}(\xi) - p^+(\xi) \|^2 &= \| g_{n,F}(\xi) - \sum_{k \in L} q_k(\xi) \|^2 \\
&= \sum_{k \in F} \| r^k_n(\xi) - q_k(\xi) \|^2 + \| p^+(\xi) \|^2 - \sum_{k \in F} \| q_k(\xi) \|^2 \\
&\leq \sum_{k \in L} \| r^k_n(\xi) - q_k(\xi) \|^2 + \| p^+(\xi) \|^2 - \sum_{k \in F} \| q_k(\xi) \|^2
\end{align*}$$

Since $\lim_{n \in \mathbb{N}} \| r^k_n(\xi) - q_k(\xi) \|^2 = 0$ by the Theorem of dominated convergence we have

$$\lim_{n \in \mathbb{N}} \sum_{k \in L} \| r^k_n(\xi) - q_k(\xi) \|^2 = 0.$$ 

It follows now from (2.1) that $\lim_{(n,F)} \| g_{n,F}(\xi) - p^+(\xi) \|^2 = 0$. We proved that $SOT - \lim_{(n,F)} g_{n,F} = p^+$.

If $F = \{ k_1, \ldots, k_r \} \subset L$ then $g_{n,F} = \sum_{m=1}^r r^k_m$ where $r^k_m = \| \cdot \| - \lim_{i \in \mathbb{N}} s_i^{n,k_m} t_i^{n,k_m}$. 

So $g_{n,F} = \| \cdot \| - \lim_{i \in \mathbb{N}} s_i^{n,F} t_i^{n,F}$ where $s_i^{n,F} = (s_i^{n,k_1}, \ldots, s_i^{n,k_r})$, $t_i^{n,F} = (t_i^{n,k_1}, \ldots, t_i^{n,k_r})^T$.

Since $s_i^{n,k} = q_k s_i^{n,k}$, $t_i^{n,k} = t_i^{n,k} q_k$, and the projections $(q_k)$ are pairwise orthogonal we conclude that $s_i^{n,F} \in Ball(R_{\infty}^{fin}(Y_0)), t_i^{n,F} \in Ball(C_{\infty}^{fin}(X_0))$ for all $(n,F)$. This completes the proof. □
\textbf{Theorem 2.5.} There exists a net of finite rank contractions \((f_\lambda)_{\lambda \in \Lambda}\) converging in SOT topology to the identity operator \(I_{H_1}\) such that \(f_\lambda = \| \cdot \| - \lim_{i \in \mathbb{N}} v_i^\lambda u_i^\lambda\) where \((v_i^\lambda)_{i \in \mathbb{N}} \subset \text{Ball}(R_\infty^{fin}(Y_0)), (u_i^\lambda)_{i \in \mathbb{N}} \subset \text{Ball}(C_\infty^{fin}(X_0))\) for all \(\lambda \in \Lambda\).

\textbf{Proof} If \(p^\perp = 0\) the conclusion comes from Lemma 2.1. So we may assume that \(p^\perp \neq 0\). From Lemmas 2.1, 2.4 there exists a net \((f_\lambda)_{\lambda \in \Lambda}\) of finite rank contractions converging in SOT topology to the projection \(p\) such that \(l_\lambda = s_\lambda t_\lambda\) where \(s_\lambda \in \text{Ball}(R_\infty^{fin}(Y_0)), t_\lambda \in \text{Ball}(C_\infty^{fin}(X_0))\) for all \(\lambda \in \Lambda\), and a net \((g_\lambda)_{\lambda \in \Lambda}\) of finite rank contractions converging in SOT topology to \(p^\perp\) such that \(g_\lambda = \| \cdot \| - \lim_{i \in \mathbb{N}} y_i^\lambda x_i^\lambda\) for all \(\lambda \in \Lambda\) where
\[
y_i^\lambda \in \text{Ball}(R_\infty^{fin}(Y_0)), \quad x_i^\lambda \in \text{Ball}(C_\infty^{fin}(X_0))
\]for all \(i \in \mathbb{N}\).

We denote \(f_\lambda = l_\lambda + g_\lambda, \quad v_i^\lambda = (v_\lambda, y_i^\lambda), \quad u_i^\lambda = (t_\lambda, x_i^\lambda)^T\) for all \(\lambda \in \Lambda, i \in \mathbb{N}\). Observe that \(I_{H_1} = \text{SOT} - \lim_{\lambda \in \Lambda} f_\lambda\) and \(f_\lambda = \| \cdot \| - \lim_{i \in \mathbb{N}} v_i^\lambda u_i^\lambda\). Now we have
\[
\|v_i^\lambda\|^2 = \|s_\lambda s_\lambda^* + y_i^\lambda(y_i^\lambda)^*\|
= \|ps_\lambda s_\lambda^* + p^\perp y_i^\lambda(y_i^\lambda)^* p^\perp\| \leq 1.
\]
Similarly \(\|u_i^\lambda\| \leq 1\) for all \(\lambda \in \Lambda, i \in \mathbb{N}\). \(\square\)

\textbf{Theorem 2.6.} The algebras \(A_0, B_0\) are strongly Morita equivalent. Particularly \(A_0 \cong Y_0 \otimes_{B_0}^H X_0, \quad B_0 \cong X_0 \otimes_{A_0}^H Y_0\) as operator modules.

\textbf{Proof} We define the bilinear map \(Y_0 \times X_0 \to A_0 : (y, x) \to yx\). This map is completely contractive and \(B_0\)-balanced, so induces a completely contractive \(A_0\)-module map \(\pi : Y_0 \otimes_{B_0}^- X_0 \to A_0 : y \otimes x \to yx\). We shall prove that \(\pi\) is completely isometric. It suffices to prove that if
\[
z_{i,j} = \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_{B_0} x_k^{i,j}, \quad i,j = 1, \ldots, n
\]
then
\[
\|(z_{i,j})_{i,j}\| \leq \left\| \left( \sum_{k=1}^{m_{i,j}} y_k^{i,j} x_k^{i,j} \right)_{i,j} \right\|.
\]
We recall the contractions \((f_\lambda, (v_\lambda^s)_{s \in \mathbb{N}}, (u_\lambda^s)_{s \in \mathbb{N}}), \lambda \in \Lambda\) from Theorem 2.3.

If \(x\) is a compact operator then \(x = \| \cdot \| - \lim_{\lambda} xf_\lambda\) ([7, Proposition 1.18]). It follows that \(z_{i,j} = \| \cdot \| - \lim_{\lambda} \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_{B_0} (x_k^{i,j} f_\lambda)\). If \(\epsilon > 0\) there exists \(\lambda \in \Lambda\) such that
\[
\|(z_{i,j})_{i,j}\| - \epsilon < \left\| \left( \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes_{B_0} (x_k^{i,j} f_\lambda) \right)_{i,j} \right\| - \epsilon.
\]
Since $x_k^{i,j} f_\lambda = \| \cdot \| - \lim_{s \in \mathbb{N}} x_k^{i,j} v_s^\lambda u_s^\lambda$ there exists $s \in \mathbb{N}$ such that

$$\left\| \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes B_0 \left( x_k^{i,j} f_\lambda \right) \right\|_{i,j} - \frac{\epsilon}{2} < \left\| \left( \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes B_0 \left( x_k^{i,j} v_s^\lambda u_s^\lambda \right) \right) \right\|_{i,j}.$$  

Since $x_k^{i,j} v_s^\lambda \in R_{\infty}(B_0)$ we have

$$\left\| \left( \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes B_0 \left( x_k^{i,j} v_s^\lambda \right) \right) \right\|_{i,j} \leq \left\| \left( \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes B_0 \left( x_k^{i,j} v_s^\lambda \right) \right) \right\|_{i,j} \leq \left\| \left( \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes B_0 \left( x_k^{i,j} v_s^\lambda \right) \right) \right\|_{i,j} \leq \left\| \left( \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes B_0 \left( x_k^{i,j} v_s^\lambda \right) \right) \right\|_{i,j}.$$

Since $\epsilon$ was arbitrary we have $\| (z_{i,j})_{i,j} \| \leq \left\| \left( \sum_{k=1}^{m_{i,j}} y_k^{i,j} \otimes B_0 \left( x_k^{i,j} \right) \right) \right\|_{i,j}$. We proved that $\pi$ is completely isometric. It remains to prove that $\pi$ is onto $A_0$. It suffices to prove that the space $Im \pi$ is dense in $A_0$.

Let $a \in Ball(A_0)$ and $\epsilon > 0$. Since $a = \| \cdot \| - \lim_{\lambda} f_\lambda a$ there exists $\lambda \in \Lambda$ such that $\| a - f_\lambda a \| < \frac{\epsilon}{2}$. Since $f_\lambda = \| \cdot \| - \lim_{s} v_s^\lambda u_s^\lambda$ there exists $s \in \mathbb{N}$ such that

$$\| f_\lambda - v_s^\lambda u_s^\lambda \| < \frac{\epsilon}{2}.$$  

It follows that $\| a - v_s^\lambda u_s^\lambda a \| < \epsilon$. But $v_s^\lambda u_s^\lambda a = \pi(v_s^\lambda \otimes B_0 (u_s^\lambda a))$ and this completes the proof. Similarly we can prove that $B_0 \cong X_0 \otimes_{A_0} Y_0$. \hfill $\square$

We define the bilinear map $Y \times X \rightarrow A : (y, x) \rightarrow yx$. This map is completely contractive $B-$balanced and separately $w^*-$continuous, so induces a completely contractive $w^*-$continuous map $\rho : Y \otimes_B^{th} X \rightarrow A : y \otimes_B x \rightarrow yx$ which is also an $A-$module map. We shall prove that the restriction of $\rho$ on the space $Y \otimes_B^{h} X$ is completely isometric and we shall use this fact in Theorem 2.9 to prove that $A \cong Y \otimes_B^{h} X$.

**Lemma 2.7.** The restriction of $\rho$ on the space $Y \otimes_B^{h} X$ is completely isometric.
Proof. It suffices to prove that if
\[ z_{i,j} = \sum_{k=1}^{m_{i,j}} y_{k}^{i,j} \otimes B x_{k}^{i,j}, \quad i, j = 1, \ldots, n \]
then
\[ \| (z_{i,j})_{i,j} \| \leq \left\| \left( \sum_{k=1}^{m_{i,j}} y_{k}^{i,j} x_{k}^{i,j} \right)_{i,j} \right\| . \]

We recall the contractions \( f_{\lambda}, (v_{s}^{\lambda})_{s \in \mathbb{N}}, (u_{s}^{\lambda})_{s \in \mathbb{N}}, \lambda \in \Lambda \) from Theorem 2.5. Fix \( \lambda \in \Lambda \). If \( \epsilon > 0 \) there exists \( s \in \mathbb{N} \) such that
\[
\left\| \left( \sum_{k=1}^{m_{i,j}} y_{k}^{i,j} \otimes B (x_{k}^{i,j} f_{\lambda}) \right)_{i,j} \right\| - \epsilon < \left\| \left( \sum_{k=1}^{m_{i,j}} y_{k}^{i,j} \otimes B (x_{k}^{i,j} v_{s}^{\lambda} u_{s}^{\lambda}) \right)_{i,j} \right\|
\]
\[
= \left\| \left( \sum_{k=1}^{m_{i,j}} (y_{k}^{i,j} x_{k}^{i,j} v_{s}^{\lambda}) \otimes B u_{s}^{\lambda} \right)_{i,j} \right\| \leq \left\| \left( \sum_{k=1}^{m_{i,j}} (y_{k}^{i,j} x_{k}^{i,j}) (v_{s}^{\lambda} \otimes B u_{s}^{\lambda}) \right)_{i,j} \right\|
\]
\[
\leq \left\| \left( \sum_{k=1}^{m_{i,j}} y_{k}^{i,j} x_{k}^{i,j} \right)_{i,j} \right\| .
\]
It follows that
\[
\left\| \left( \sum_{k=1}^{m_{i,j}} y_{k}^{i,j} \otimes B (x_{k}^{i,j} f_{\lambda}) \right)_{i,j} \right\| \leq \left\| \left( \sum_{k=1}^{m_{i,j}} y_{k}^{i,j} x_{k}^{i,j} \right)_{i,j} \right\|
\]
for all \( \lambda \in \Lambda \). Since
\[
\left( \sum_{k=1}^{m_{i,j}} y_{k}^{i,j} \otimes B x_{k}^{i,j} \right)_{i,j} = w^{*} - \lim_{\lambda} \left( \sum_{k=1}^{m_{i,j}} y_{k}^{i,j} \otimes B (x_{k}^{i,j} f_{\lambda}) \right)_{i,j}
\]
we have
\[
\left\| \left( \sum_{k=1}^{m_{i,j}} y_{k}^{i,j} \otimes B x_{k}^{i,j} \right)_{i,j} \right\| \leq \left\| \left( \sum_{k=1}^{m_{i,j}} y_{k}^{i,j} x_{k}^{i,j} \right)_{i,j} \right\|. \quad \square
\]

The second dual operator space \( A_{0}^{**} \) of the operator algebra \( A_{0} \) is also an operator algebra with product describing in \([5\text{ section 2.5}]\). The product on \( A_{0}^{**} \) extends the product on \( A_{0} \). With this we mean that if \( \iota : A_{0} \rightarrow A_{0}^{**} \) is the canonical embedding then \( \iota(ab) = \iota(a) \iota(b) \) for all \( a, b \in A_{0} \).

Lemma 2.8. The operator algebra \( A \) (resp. \( B \)) is isomorphic as dual operator algebra with \( A_{0}^{**} \) (resp. \( B_{0}^{**} \)).
The following are equivalent:

**Theorem 2.9.**

By [7, section 16] the maps are surjective isometries. We define the isometry \( \phi \) for all \( a, b \in A_0 \) and \( \phi \) is \( w^* \)-continuous. This map satisfies \( \phi(a) = \iota(a) \) for all \( a \in A_0 \). Since \( \iota(ab) = \iota(a)\iota(b) \) for all \( a, b \in A_0 \) and \( \phi \) is \( w^* \)-continuous we mean that \( \phi \) is \( B(H_1)_* - A_0^* \) continuous.

If \( n \in \mathbb{N} \) the algebra \( M_n(A) \) is also a nest algebra, so by the above arguments, there exists a \( w^* \)-continuous isometry

\[
\tilde{\phi}: M_n(A) \to (M_n(A)_0)^{**} = M_n(A_0)^{**}
\]

such that

\[
\tilde{\phi} (\{a_{i,j}\}) = \iota (\{a_{i,j}\})
\]

for all \( \{a_{i,j}\} \in M_n(A_0) \), where \( \iota: M_n(A_0) \to M_n(A_0)^{**} \) is the canonical embedding. By [3, 1.4.11] there exists a \( w^* \)-continuous isometry \( \tau: M_n(A_0)^{**} \to M_n(A_0^*) \) such that \( \tau(\tilde{\phi} (\{a_{i,j}\}) = \iota(a_{i,j}) \) for all \( (a_{i,j}) \in M_n(A_0) \). So we have a \( w^* \)-continuous isometry \( \tau \circ \phi: M_n(A) \to M_n(A_0^*) \) satisfying

\[
\tau \circ \phi (\{a_{i,j}\}) = \iota(a_{i,j})
\]

for all \( \{a_{i,j}\} \in M_n(A_0) \). But the map

\[
\phi_n: M_n(A) \to M_n(A_0^*) : (b_{i,j}) = (\phi(b_{i,j}))
\]

is a \( w^* \)-continuous map satisfying

\[
\phi_n (a_{i,j}) = (\phi(a_{i,j})) = \iota(a_{i,j})
\]

for all \( \{a_{i,j}\} \in M_n(A_0) \). So \( \phi_n \) is equal to \( \tau \circ \phi \) in \( M_n(A_0) \). Since \( M_n(A_0) / w^* = M_n(A) \) we have \( \phi_n = \tau \circ \phi \) so \( \phi_n \) is isometry for all \( n \in \mathbb{N} \). We proved that \( \phi \) is a completely isometric map and this completes the proof.  

\[ \square \]

We are now ready to present the main theorem of this paper:

**Theorem 2.9.** A. The following are equivalent:

(i) The nests \( N_1, N_2 \) are isomorphic.

(ii) The algebras \( A_0, B_0 \) are strongly Morita equivalent.

(iii) The algebras \( A, B \) are weakly*-Morita equivalent.

B. If \( \theta : N_1 \to N_2 \) is a nest isomorphism, \( X = Op(\theta), Y = Op(\theta^{-1}) \) then:

(i) \( A_0 \cong Y_0 \otimes_{B_0}^h X_0 \), \( B_0 \cong X_0 \otimes_{A_0}^h Y_0 \), as operator modules,

(ii) \( A \cong Y \otimes_{B}^h X, \quad B \cong X \otimes_{A}^h Y \), as dual operator modules.
Proof
A.(i) ⇒ (ii)
This is Theorem 2.6.
(ii)⇒(iii)
If $A_0$ and $B_0$ are strongly Morita equivalent then the operator algebras $A_0^{**}$ and $B_0^{**}$ are weakly−∗ Morita equivalent, [3, section 3] . So by Lemma 2.8 $A$ and $B$ are weakly−∗ Morita equivalent.
(iii)⇒(iv)
Let $(A,B,V,U)$ be a weak−∗ Morita context [3]. It follows that there exist completely contractive separately $w^*$−continuous bilinear maps $(\cdot,\cdot) : V \times U \to A$ which is $A$−module and $B$−balanced map and $[\cdot,\cdot] : U \times V \to B$ which is $B$−module and $A$−balanced map satisfying

$$(y, x)y' = y[x, y], \quad x'(y, x) = [x', y]x \quad \forall x, x' \in U, \quad y, y' \in V,$$

and $A = \overline{\text{span}}^{w^*}(\{(y, x) : x \in U, y \in V\}), B = \overline{\text{span}}^{w^*}(\{[x, y] : x \in U, y \in V\})$.

If $N \in \mathcal{N}_1$ we define $\theta(N)$ the projection onto the space generated by vectors of the form $[xN, y](\omega), x \in U, y \in V, \omega \in H_2$. Since $b[xN, y] = [bxN, y]$ for all $b \in B$ we have $\theta(N)^\perp B\theta(N) = 0$ so $\theta(N) \in \mathcal{N}_2$. Also if $N_1 \leq N_2$ then $\theta(N_1) \leq \theta(N_2)$ and so $\theta$ is an order preserving map from $\mathcal{N}_1$ into $\mathcal{N}_2$.

Similarly if $M \in \mathcal{N}_1$ we define $\sigma(M)$ the projection onto the space generated by vectors of the form $(yM, x)(\xi), x \in U, y \in V, \xi \in H_1$. The map $\sigma : \mathcal{N}_1 \to \mathcal{N}_2$ is an order preserving map.

If $x, x' \in U, y \in V$ and $N \in \mathcal{N}_1$ then

$$\theta(N)^\perp [xN, y] = 0 \Rightarrow [\theta(N)^\perp xN, y] = 0 \Rightarrow [\theta(N)^\perp xN, y]x' = 0 \Rightarrow [\theta(N)^\perp xN(y, x')] = 0.$$

Since the operators $(y, x')$ span the algebra $A$ we have

(2.2) \quad $\theta(N)^\perp xN = 0 \Rightarrow xN = \theta(N)xN \quad \forall x \in U, \ N \in \mathcal{N}_1$.

Similarly

(2.3) \quad $yM = \sigma(M)yM \quad \forall \ y \in V, \ M \in \mathcal{N}_2$.

If $x, x' \in U, y, y' \in V, N \in \mathcal{N}_1$ we have

$$[xN^\perp, y][x'N, y'] = [[xN^\perp, y]x'N, y']$$

$$=[[xN^\perp(y, x')N, y'] = 0 \quad \text{because} \ (y, x') \in \text{Alg}(\mathcal{N}_1).$$

It follows that $[xN^\perp, y]\theta(N) = 0 \Rightarrow [x, N^\perp y\theta(N)] = 0$ for all $x \in U, y \in V$, and so

(2.4) \quad $N^\perp y\theta(N) = 0 \Rightarrow y\theta(N) = N y\theta(N), \quad \forall \ y \in V, \ N \in \mathcal{N}_1$.

Similarly we can prove
\begin{equation}
(2.5) \quad x\sigma(M) = Mx\sigma(M) \quad \forall \ x \in U, \ M \in \mathcal{N}_2
\end{equation}

If \( N \in \mathcal{N}_1 \) and \( x \in U, y \in V \) then \( (y, x)N = (y, xN) = (y, \theta(N)xN) \) because of \( (2.2) \). The last operator is equal to \( (y\theta(N), xN) = \sigma(\theta(N))(y, x)N \) because of \( (2.3) \). It follows that \( N \leq \sigma(\theta(N)) \).

Similarly \( (y, x)^*N^\perp = (N^\perp(y, x))^* = (N^\perp y, x)^* = (N^\perp y\theta(N)^\perp, x)^* \) because of \( (2.4) \). The last operator is equal to \( (y\theta(N)^\perp x)^* = (N^\perp y, \theta(N)^\perp x)^* \) because of \( (2.5) \).

Since \( I_{H_i} = w^* - \lim_i \sum_{k=1}^n (y_k^i, x_k^i)^* \) for \( (y_k^i) \subset V, (x_k^i) \subset U \) we have \( N^\perp \leq \sigma(\theta(N))^\perp \) and so \( N = \sigma(\theta(N)) \).

Similarly we can prove that \( M = \theta(\sigma(M)) \) for all \( M \in \mathcal{N}_2 \). This completes the proof of the fact that \( \theta \) is a nest isomorphism.

B. Let \( \theta : \mathcal{N}_1 \to \mathcal{N}_2 \) be a nest isomorphism and \( X = Op(\theta), Y = Op(\theta^{-1}) \). Claim B-(i) follows from Theorem 2.6.

Let \( \rho : Y \otimes_B^h X \to A \) be the map which was defined above Lemma 2.7. Let 
\[ z \in \text{Ball}(M_n(Y \otimes_B^h X)) \]
for a fixed \( n \in \mathbb{N} \). By [3, Corollary 2.8] there exists a net \( (z_i) \subset \text{Ball}(M_n(Y \otimes_B^h X)) \) converging in \( w^* \) topology to \( z \). It follows that \( \rho(z_i) \xrightarrow{w^*} \rho(z) \) in \( M_n(A) \). If \( f, g \) are finite rank operators in \( A \) denote \( f^n = f \oplus f \oplus \ldots \oplus f \) and similarly for \( g^n \). We have that 
\[ f^n \rho(z_i)g^n \xrightarrow{w} f^n \rho(z)g^n \Rightarrow \rho(f^n z_i g^n) \xrightarrow{w} \rho(f^n z g^n). \]

From Lemma 2.7 it follows that \( f^n z g^n \in M_n(Y \otimes_B^h X) \) and \( \|\rho(f^n z g^n)\| = \|f^n z g^n\| \) for all finite rank operators \( f, g \) in \( A \). We recall the finite rank contractions \( (f_\lambda)_{\lambda \in \Lambda} \) from Theorem 2.5. For \( \lambda, \mu \in \Lambda \) we have 
\[ \|f_\lambda^* z f_\mu^n\| = \|\rho(f_\lambda^* z f_\mu^n)\| = \|f_\lambda^* \rho(z) f_\mu^n\| \leq \|\rho(z)\|. \]

Since \( z f_\mu^n = w^* - \lim_\lambda f_\lambda^* z f_\mu^n \) we have \( \|z f_\mu^n\| \leq \|\rho(z)\| \) for all \( \mu \in \Lambda \). Now taking the \( w^* \)-limit of \( (z f_\mu^n)_{\mu \in \Lambda} \) we obtain \( \|z\| \leq \|\rho(z)\| \). We proved that the map \( \rho : Y \otimes_B^h X \to A \) is a complete isometry. From Theorem 2.6 and its proof we have that \( A_0 = \text{span}^w(Y_0 X_0) \). Since \( A = \overline{\text{span}}^w(Y X) \) we have
\[ A = \overline{\text{span}}^w(Y X) = \overline{\text{span}}^w(\{\rho(y \otimes_B x) : y \in Y, x \in X\}). \]

By the Krein-Smulian Theorem the space \( \text{Im} \rho \) is \( w^* \)-closed and so \( \rho \) is onto \( A \). Similarly we can prove that \( B \cong X \otimes_A^h Y \), as dual operator modules. \( \square \)
3. Spatial Morita equivalence and nest algebras

In this section we shall investigate the relation between weak−∗ and spatial Morita equivalence for nest algebras. We give the definition of spatial Morita equivalence:

**Definition 3.1.** (I. G. Todorov) Let \( C, D \) be \( w^* \)-closed algebras acting on the Hilbert spaces \( K_1, K_2 \) respectively. We say that \( C \) and \( D \) are **spatially Morita equivalent** if there exists a \( D \)-\( C \) bimodule \( V \subset B(K_1, K_2) \) and a \( C \)-\( D \) bimodule \( U \subset B(K_2, K_1) \) such that \( C = \overline{\text{span}}_{w^*}(UV) \), \( D = \overline{\text{span}}_{w^*}(VU) \).

We also need the following notions. If \( L \) is a set of projections acting on the Hilbert space \( H \) the set

\[
\text{Alg}(L) = \{ x \in B(H) : p^\perp xp = 0, \forall \ p \in L \}
\]

is an algebra. An algebra \( A \) is called reflexive if there exists a set of projections \( L \) such that \( A = \text{Alg}(L) \). In the special case where \( L \) is a complete lattice of commuting projections containing the zero and identity operators the algebra \( \text{Alg}(L) \) is called a CSL algebra and the lattice \( L \) is called a CSL lattice. Obviously, nest algebras are CSL algebras. If \( A \) is an algebra acting on the Hilbert space \( H \) the set

\[
\{ p \in \text{pr}(B(H)) : p^\perp xp = 0, \forall \ x \in A \}
\]

is called the lattice of \( A \) and we denote it by \( \text{Lat}(A) \). If \( L \) is a CSL lattice then \( \text{Lat}(\text{Alg}(L)) = L \) [1], [8].

Two spatially Morita equivalent algebras are not always weakly−∗ Morita equivalent even in the case one of them is a CSL algebra:

**Example 3.1.** Let \( C \) be a nest algebra. We denote the algebras \( A = C \oplus C \) and

\[
B = \left\{ \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix} : a, b \in C \right\}.
\]

Observe that \( A \) is a CSL algebra whose lattice is

\[
\text{Lat}(A) = \{ p \oplus q : p, q \in \text{Lat}(C) \}.
\]

Since the center of \( C \) is trivial [7, Corollary 19.5] the center of \( A \) is \( Z(A) = \mathbb{C} \oplus \mathbb{C} \) and the center of \( B \) is

\[
Z(B) = \left\{ \begin{pmatrix} \lambda & \mu - \lambda \\ 0 & \mu \end{pmatrix} : \lambda, \mu \in \mathbb{C} \right\}.
\]

We also denote the spaces

\[
X = \left\{ \begin{pmatrix} a & -a \\ 0 & b \end{pmatrix} : a, b \in C \right\}, \quad Y = \left\{ \begin{pmatrix} a & b \\ 0 & b \end{pmatrix} : a, b \in C \right\}.
\]
We can check that $X$ is an $A - B$ bimodule, $Y$ is a $B - A$ bimodule and $XY = A, YX = B$. So the algebras $A, B$ are spatially Morita equivalent. If $A$ and $B$ were weakly-∗ Morita equivalent by [3, Theorem 3.7] they would have isomorphic centers through a completely isometric homomorphism. This is a contradiction because $Z(A)$ is a von Neumann algebra and $Z(B)$ is a non-selfadjoint algebra.

Despite the above example, in [10] we proved that two CSL algebras are spatially Morita equivalent if and only if their lattices are isomorphic, so by Theorem 2.9 we conclude the following theorem:

**Theorem 3.2.** Two nest algebras are spatially Morita equivalent if and only if they are weakly-∗ Morita equivalent.

Also despite the example 3.1 we have the following theorem:

**Theorem 3.3.** Let $A$ be a nest algebra, $B$ be a unital dual operator algebra and $β$ be a completely isometric normal representation of $B$ such that $A$ and $β(B)$ are spatially Morita equivalent. It follows that $A$ and $B$ are weakly-∗ Morita equivalent.

**Proof** By [10, Theorem 4.1, remark 4.2] $β(B)$ is a nest algebra whose nest is isomorphic with the nest of $A$. The conclusion comes from Theorems 2.9 and 3.2. □

**Theorem 3.4.** (Blecher-Kashyap) If $A, B$ are weakly-∗ Morita equivalent unital dual operator algebras, for every completely isometric normal representation $α$ of $A$ there exists a completely isometric normal representation $β$ of $B$ such that the algebras $α(A), β(B)$ are spatially Morita equivalent.

**Proof** Suppose that $(A, B, X, Y)$ is a weakly-∗ Morita context [3]. We use now arguments from the beginning of the 4th section of [3]. If $α$ is a completely isometric normal representation of $A$ on the Hilbert space $H$ the tensor product $K = Y \otimes_A^σH$ with its norm is a Hilbert space on which $B$ is represented through the $w^*$—continuous complete isometry $β$ given by

$$β(b)(y \otimes h) = (by) \otimes h \, \forall \, b \in B, \, y \in Y, \, h \in H.$$ 

Also Blecher and Kashyap prove that the maps $ϕ : Y → B(H, K), ψ : X → B(K, H)$ given by $ϕ(y)(h) = y \otimes h$ and $ψ(x)(y \otimes h) = α((x, y))(h)$ are $w^*$—continuous complete isometries. See in [3] for the properties of the bilinear map $(\cdot, \cdot) : X \times Y → A$. We can easily check that $ψ(X)$ is an $α(A) - β(B)$ bimodule, $ϕ(Y)$ is a $β(B) - α(A)$ bimodule and

$$α(A) = \text{span}^{w^*}(ψ(X)ϕ(Y)), \, \, β(B) = \text{span}^{w^*}(ϕ(Y)ψ(X)).$$ 

**Corollary 3.5.** If $A$ is a nest algebra and $B$ is a unital dual operator algebra which are weakly-∗ Morita equivalent then there exists a completely isometric normal representation $β$ of $B$ such that $β(B)$ is a nest algebra.
Proof. By the above Theorem there exists a completely isometric normal representation $\beta$ of $B$ such that the algebras $A$ and $\beta(B)$ are spatially Morita equivalent. From [10, remark 4.1] the algebra $\beta(B)$ is reflexive and from [10, Theorem 4.2] the lattice of $\beta(B)$ is isomorphic with the nest of $A$. So $\beta(B)$ is a nest algebra. □

Corollary 3.6. If $A$ is a CSL algebra which is not a nest algebra then $A$ is not weakly--* Morita equivalent with anyone nest algebra.

Proof. By the above corollary if $A$ was weakly--* Morita equivalent with a nest algebra then it would have a normal completely isometric representation $\alpha$ such that $\alpha(A)$ is a nest algebra. This is a contradiction because as we can easily check $\alpha(\text{Lat}(A)) = \text{Lat}(\alpha(A))$. □

4. A stable isomorphism theorem for nest algebras

In this section we are going to present a new theorem which characterizes the stable isomorphism of separably acting nest algebras.

Definition 4.1. Two dual operator algebras $C,D$ are called stably isomorphic if there exists a Hilbert space $H$ and a completely isometric, $w^*$-bicontinuous isomorphism from the algebra $C \otimes B(H)$ onto the algebra $D \otimes B(H)$, where $\otimes$ is the normal spatial tensor product.

We give two relevant definitions:

Definition 4.2. [10] Let $C,D$ be $w^*$-closed algebras acting on Hilbert spaces $H_1$ and $H_2$ respectively. If there exists a TRO $M \subset B(H_1,H_2)$, i.e. a subspace satisfying $MM^*M \subset M$, such that $C = \text{span}^{w^*}(M^*DM)$ and $D = \text{span}^{w^*}(MCM^*)$ we write $C \sim^M D$. We say that the algebras $C,D$ are TRO equivalent if there exists a TRO $M$ such that $C \sim^M D$.

Definition 4.3. [11] Let $C,D$ be abstract dual operator algebras. These algebras are called $\Delta$-equivalent if they have completely isometric normal representations $\phi,\psi$ such that the algebras $\phi(C),\psi(D)$ are TRO-equivalent.

In [13] we proved the following theorem:

Theorem 4.1. Two unital dual operator algebras are stably isomorphic if and only if they are $\Delta$-equivalent.

$\Delta$-equivalence implies weak--* Morita equivalence [3, section 3]. The converse does not hold. The counterexample is [12] example 3.7. We shall give a new proof of this fact in Theorem [4.7].

[12] Theorem 3.2] implies the following corollary:
Corollary 4.2. Two nest algebras are Δ-equivalent if and only if they are TRO-equivalent.

In what follows if X is a subset of B(H) where H is a Hilbert space we denote by X' the commutant of X and by X'' the algebra (X')'. In [10] we proved the following criterion of TRO-equivalence for reflexive algebras:

Theorem 4.3. Two reflexive algebras C, D are TRO-equivalent if and only if there exists a ∗-isomorphism δ : (C ∩ C∗)' → (D ∩ D∗)' such that δ(Lat(C)) = Lat(D).

Comparing Theorems 4.1, 4.3 and Corollary 4.2 we take the following:

Corollary 4.4. The nest algebras Alg(N_1), Alg(N_2) are stably isomorphic if and only if there exists a ∗-isomorphism δ : N_1'' → N_2'' such that δ(N_1) = N_2.

In the rest of this section we fix two nests N_1, N_2 acting on the separable Hilbert spaces H_1, H_2 respectively and we denote A = Alg(N_1), B = Alg(N_2). We use now extensively notions from [7, section 7]. If ξ (resp. ω) is a unit separating vector for the algebra N_1'' (resp. N_2'') we define the order isomorphism φ_ξ (resp. ψ_ω) from N_1 (resp. N_2) onto a closed subset of the interval [0, 1] given by φ_ξ(N) = ∥N(ξ)∥^2 (resp. ψ_ω(M) = ∥M(ω)∥^2).

Suppose that [0, 1] \ φ_ξ(N_1) = ∪_n(l_n, r_n) and [0, 1] \ ψ_ω(N_2) = ∪_n(t_n, s_n). If m is the Lebesgue measure we define the measures μ_ξ, ν_ω given by

$$μ_ξ(S) = m(S \cap φ_ξ(N_1)) + \sum_{r_n \in S} (r_n - l_n)$$

$$ν_ω(S) = m(S \cap ψ_ω(N_2)) + \sum_{s_n \in S} (s_n - t_n),$$

for every Borel subset S of [0, 1]. We denote M_1 (resp. M_2) the nest M_s : 0 ≤ s ≤ 1 ⊂ B(L^2([0, 1], μ_ξ)) (resp. N_s : 0 ≤ s ≤ 1 ⊂ B(L^2([0, 1], ν_ω))) where M_s (resp. N_s) is the projection onto the space L^2([0, s], μ_ξ) (resp. L^2([0, s], ν_ω)).

The algebra N_1'' is ∗-isomorphic with the algebra L^∞([0, 1], μ_ξ) (resp. L^∞([0, 1], ν_ω)) acting on the Hilbert space L^2([0, 1], μ_ξ) (resp. L^2([0, 1], ν_ω)) through an isomorphism mapping the nest N_1 (resp. N_2) onto M_1 (resp. M_2).

We denote by AbsHom([0, 1]) the set of order homeomorphisms α : [0, 1] → [0, 1] which satisfy the property m(S) = 0 ⇒ m(α(S)) = 0. The theorem below describes when two separably acting nest algebras are stably isomorphic.

Theorem 4.5. The algebras A, B are stably isomorphic if and only if there exist separating unit vectors ξ for N_1'', ω for N_2'' and α ∈ AbsHom([0, 1]) such that α(φ_ξ(N_1)) = ψ_ω(N_2).
Suppose that the algebras $A, B$ are stably isomorphic. From Corollary [4.4] there exists a $*$-isomorphism $\delta : \mathcal{N}_1'' \to \mathcal{N}_2''$ such that $\delta(\mathcal{N}_1) = \mathcal{N}_2$. Fix separating unit vectors $\xi$ for $\mathcal{N}_1''$, and $\omega$ for $\mathcal{N}_2''$. Taking compositions we obtain a $*$-isomorphism

$$\tilde{\delta} : L^\infty([0, 1], \mu_\xi) \to L^\infty([0, 1], \nu_\omega)$$

such that $\tilde{\delta}(\mathcal{M}_1) = \mathcal{M}_2$. Every isomorphism between maximal abelian self-adjoint algebras is implementing by a unitary. So the nests $\mathcal{M}_1, \mathcal{M}_2$ are unitarily equivalent. By [7, Theorem 7.23] there exists $\alpha \in \text{AbsHom}([0, 1])$ such that $\alpha(\phi_\xi(\mathcal{N}_1)) = \psi_\omega(\mathcal{N}_2)$.

Conversely if there exist such $\xi, \omega$ and $\alpha$, by the same theorem there exists a unitary $u \in B(L^2([0, 1], \mu_\xi), L^2([0, 1], \nu_\omega))$ such that $u^* \mathcal{M}_2 u = \mathcal{M}_1$. It follows that $L^\infty([0, 1], \mu_\xi) = u^* L^\infty([0, 1], \nu_\omega) u$. Taking compositions we take a $*$-isomorphism $\delta : \mathcal{N}_1'' \to \mathcal{N}_2''$ such that $\delta(\mathcal{N}_1) = \mathcal{N}_2$. Again from Corollary 4.4 we conclude that the algebras $A$ and $B$ are stably isomorphic. □

Remark 4.6. If there exist separating unit vectors $\xi$ for $\mathcal{N}_1''$, $\omega$ for $\mathcal{N}_2''$ and $\alpha \in \text{AbsHom}([0, 1])$ such that $\alpha(\phi_\xi(\mathcal{N}_1)) = \psi_\omega(\mathcal{N}_2)$ then for all separating unit vectors $\xi_1$ for $\mathcal{N}_1''$ and $\omega_1$ for $\mathcal{N}_2''$ there exists $\alpha_1 \in \text{AbsHom}([0, 1])$ such that $\alpha_1(\phi_{\xi_1}(\mathcal{N}_1)) = \psi_{\omega_1}(\mathcal{N}_2)$. This is a consequence of [7] Proposition 7.22.

We give a new proof of the following result:

Theorem 4.7. Weak-$\ast$ Morita equivalence is strictly weaker than $\Delta$-equivalence.

Proof Let $C$ be the Cantor set, $\gamma$ be an order homeomorphism of $[0, 1]$ such that $m(\gamma(C)) > 0$. Suppose that $[0, 1] \setminus C = \cup_n (l_n, r_n)$ and $[0, 1] \setminus \gamma(C) = \cup_n (t_n, s_n)$. We denote by $\mu$ the measure

$$\mu(S) = \sum_{r_n \in S} (r_n - l_n)$$

and by $\nu$ the measure

$$\nu(S) = m(S \cap \gamma(C)) + \sum_{s_n \in S} (s_n - t_n).$$

We denote $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) the nest $\{M_s : 0 \leq s \leq 1\} \subset B(L^2([0, 1], \mu))$ (resp. $\{N_s : 0 \leq s \leq 1\} \subset B(L^2([0, 1], \nu))$) where $M_s$ (resp. $N_s$) is the projection onto the space $L^2([0, s], \mu)$ (resp. $L^2([0, s], \nu)$).

The map $\theta : \mathcal{M}_1 \to \mathcal{M}_2 : M_s \to N_{\gamma(s)}$ is a nest isomorphism so by Theorem 2.9 the algebras $A = \text{Alg}(\mathcal{M}_1), B = \text{Alg}(\mathcal{M}_2)$ are weakly-$\ast$ Morita equivalent. If the algebras $A, B$ were $\Delta$-equivalent by Theorem 4.5 there would exist unit vectors $\xi$ for $\mathcal{M}_1''$, $\omega$ for $\mathcal{M}_2''$ and $\alpha \in \text{AbsHom}([0, 1])$ such that $\alpha(\phi_\xi(\mathcal{M}_1)) = \psi_\omega(\mathcal{M}_2)$. From [7] Proposition 7.22 we have that $m(\phi_\xi(\mathcal{M}_1)) =$
exists a completely bounded operator on a separable infinite dimensional Hilbert space $H$.

In appropriate definitions in sections 1, 3 and 4. If stable isomorphism and we shall consider nest and CSL algebras. See the contradiction.

$\Box$

5. A counterexample in Morita equivalence

In this section we shall use the notions of TRO equivalence, of $\Delta-$equivalence, of stable isomorphism and we shall consider nest and CSL algebras. See the appropriate definitions in sections 1, 3 and 4. If $C$ and $D$ are unital operator algebras which are strongly Morita equivalent then for every $\epsilon > 0$ there exists a completely bounded isomorphism from $C \otimes_{\min} K$ onto $D \otimes_{\min} K$ with $\|\rho\|_{cb} < 1 + \epsilon$ and $\|\rho^{-1}\|_{cb} < 1 + \epsilon$, where $K$ is the $C^*$-algebra of compact operators on a separable infinite dimensional Hilbert space $H$ and $\otimes_{\min}$ is the spatial tensor product [6 Corollary 7.10]. It follows that for every $\epsilon > 0$ there exists a completely bounded $w^*$-continuous isomorphism $\sigma$ from $C^{**} \otimes B(H)$ onto $D^{**} \otimes B(H)$ with $\|\sigma\|_{cb} < 1 + \epsilon$ and $\|\sigma^{-1}\|_{cb} < 1 + \epsilon$, where $\otimes$ is the normal spatial tensor product. One can wonder now, if the operator algebras $C^{**}$ and $D^{**}$ are stably isomorphic.

In this section we give a negative answer to this question. We present a counterexample of unital strongly Morita equivalent algebras $C$ and $D$ whose second duals are not stably isomorphic. Also for the algebras $C^{**}$ and $D^{**}$ there exist normal completely isometric representations $\phi$ and $\psi$ respectively such that for every $\epsilon > 0$ there exists an invertible bounded operator $T_\epsilon$ satisfying $\|T_\epsilon\| < 1 + \epsilon$, $\|T_\epsilon^{-1}\| < 1 + \epsilon$, $\phi(C^{**}) = T_\epsilon^{-1}\psi(D^{**})T_\epsilon$ and $\phi(C) = T_\epsilon^{-1}\psi(D)T_\epsilon$.

Two nests $\mathcal{N}, \mathcal{M}$ acting on the separable Hilbert spaces $H, K$ respectively are called similar if there exists an order isomorphism $\theta : \mathcal{N} \to \mathcal{M}$ which preserves dimension of intervals. We say that an invertible operator $S \in B(H, K)$ implements $\theta$ if $\theta(N)$ is the projection onto $SN(H)$ for all $N \in \mathcal{N}$. In what follows if $C$ is an operator algebra, $\Delta(C)$ is its diagonal $C \cap C^*$.

We fix similar nests $\mathcal{N}, \mathcal{M}$ as above with corresponding nest algebras $A = \text{Alg}(\mathcal{N})$ and $B = \text{Alg}(\mathcal{M})$ such that $\Delta(A)$ is a totally atomic maximal abelian selfadjoint algebra (masa in sequel) and $\Delta(B)$ is a masa with a nontrivial continuous part, [7 example 13.15]. Suppose that $\theta : \mathcal{N} \to \mathcal{M}$ is an order isomorphism implementing similarity for $\mathcal{N}, \mathcal{M}$. We denote by $A_0$ (resp. $B_0$) the algebra of compact operators belonging to $A$ (resp. $B$) and by $A_1$ (resp. $B_1$) the operator algebra $A_0 + \mathbb{C}I_H$ (resp. $B_0 + \mathbb{C}I_K$). We denote by $X$ the space $Op(\theta)$ and by $Y$ the space $Op(\theta^{-1})$.

**Theorem 5.1.** [7 Theorem 13.20] (Davidson) For every $\epsilon > 0$ there exists an invertible bounded operator $S_\epsilon$ which implements $\theta$ such that $\|S_\epsilon\| < 1 + \epsilon$, $\|S_\epsilon^{-1}\| < 1 + \epsilon$. (Observe that $S_\epsilon \in X$ and $S_\epsilon^{-1} \in Y$ for all $\epsilon > 0$.)
Suppose that \( j : A_1 \to A_1^{**} \) is the canonical embedding. We denote by \( J_A \) the space \( j(A_0)^{w^*} \).

**Lemma 5.2.**

(i) \( A_1^{**} = J_A + \mathbb{C} I \)

(ii) \( J_A \cap \mathbb{C} I = 0. \)

**Proof**

(i) Since \(|\lambda| \leq \|a + \lambda I_H\|\) for all compact operators \( a \) the functional

\[ \rho : A_1 \to \mathbb{C} : a + \lambda I_H \to \lambda \]

belongs to \( A_1^* \). If \( x \in A_1^{**} \) by the Goldstine Theorem there exists a net \((a_i + \lambda_i I_H) \in A_0 + \mathbb{C} I \) converging in \( w^* \)-topology to \( x \). Since \((\lambda_i)\) converges to \( \rho(x) \) we have that \((a_i)\) converges to \( a \in J_A \) and so \( x = a + \rho(x) \in J_A + \mathbb{C} I \).

(ii) Since \( \rho|_A_0 = 0 \) if \( \lambda I \in J_A \) then \( \lambda = 0 \). So \( J_A \cap \mathbb{C} I = 0. \)

Suppose that \( \iota : A_0 \to A_0^{**} \) is the canonical embedding. In lemma 2.3 we have proved that there exists a \( w^* \)-continuous completely isometric onto homomorphism \( \phi : A \to A_0^{**} \) extending \( \iota \).

The map \( \phi|_{A_1} : A_1 \to A_0^{**} \) extends to a \( w^* \)-continuous completely contractive map \( \hat{\phi} : A_1^{**} \to A_0^{**} \) satisfying \( \hat{\phi} (j(a)) = \phi(a) \) for all \( a \in A_1 \). Also the completely contractive map \( j|_{A_0} : A_0 \to A_1^{**} \) extends to a \( w^* \)-continuous completely contractive map \( \hat{\kappa} : A_0^{**} \to A_1^{**} \) such that \( \hat{\kappa} (\iota(a)) = j(a) \) for all \( a \in A_0 \). So the map \( \hat{\phi} \circ \hat{\kappa} : A_0^{**} \to A_0^{**} \) satisfies

\[ \hat{\phi} \circ \hat{\kappa} (\iota(a)) = \hat{\phi} (j(a)) = \phi(a) = \iota(a) \]

for all \( a \in A_0 \). It follows that \( \hat{\phi} \circ \hat{\kappa} = id_{A_1^{**}} \). Therefore \( \hat{\kappa} \) is a complete isometry.

We denote by \( \theta \) the \( w^* \)-continuous completely isometric homomorphism \( \hat{\kappa} \circ \phi : A \to A_1^{**} \). Observe that

\[ \theta(A) = \hat{\kappa} (\phi(A)) = \hat{\kappa} (\iota(A_0)^{w^*}) = j(A_0)^{w^*} = J_A. \]

Suppose that \( p \) is the projection \( \theta(id_A) \). Lemma 5.2 implies that \( p^\perp \neq 0 \) and \( A_1^{**} = J_A \oplus \mathbb{C} p^\perp \).

**Lemma 5.3.** The algebra \( A_1^{**} \) is completely isometric and \( w^* \)-continuously isomorphic with the algebra \( A \oplus \mathbb{C} \) acting on the Hilbert space \( H \oplus \mathbb{C} \).

**Proof** We define the map \( \theta \) and the projection \( p \) as in the above discussion. We define the completely isometric normal representation

\[ \pi : A_1^{**} = J_A \oplus \mathbb{C} p^\perp \to B(H \oplus \mathbb{C}) : a \oplus \lambda p^\perp \to \theta^{-1}(a) \oplus \lambda \]

which is onto \( A \oplus \mathbb{C}. \) \( \square \)
For every $\epsilon > 0$ we denote by $T_\epsilon$ the bounded invertible operator $S_\epsilon \oplus id_C \in B(H \oplus \mathbb{C}, K \oplus \mathbb{C})$. Also we denote the spaces $U = X \oplus \mathbb{C} \subset B(H \oplus \mathbb{C}, K \oplus \mathbb{C})$ and $V = Y \oplus \mathbb{C} \subset B(K \oplus \mathbb{C}, H \oplus \mathbb{C})$. Observe that $U$ is a $B \oplus \mathbb{C} - A \oplus \mathbb{C}$ bimodule and $V$ is an $A \oplus \mathbb{C} - B \oplus \mathbb{C}$ bimodule.

By the above lemma $\pi(A_1^{**}) = A \oplus \mathbb{C}$. If $j : A_1 \to A_1^{**}$ is the canonical embedding we have $\pi(j(a)) = a \oplus 0$ for all $a \in A_0$ and $\pi(j(id_{A_1})) = id_{H \oplus \mathbb{C}}$. So

$$\pi(j(A_1)) = \text{span}\{a \oplus 0, id_{H \oplus \mathbb{C}}, a \in A_0\}.$$

Similarly if $j_2 : B_1 \to B_1^{**}$ is the canonical embedding there exists a normal completely isometric onto homomorphism $\rho : B_1^{**} \to B \oplus \mathbb{C}$ such that

$$\rho(j_2(B_1)) = \text{span}\{b \oplus 0, id_{K \oplus \mathbb{C}}, b \in B_0\}.$$

Since $S^{-1}_\epsilon B_0 S_\epsilon = A_0$ and $S^{-1}_\epsilon B S_\epsilon = A$ we have that

$$T_\epsilon^{-1} \rho(j_2(B_1)) T_\epsilon = \pi(j(A_1)), \quad T_\epsilon^{-1} \rho(B_1^{**}) T_\epsilon = \pi(A_1^{**})$$

for all $\epsilon > 0$.

In the following lemmas 5.4, 5.5 we identify the algebra $A_1^{**}$ with $A \oplus \mathbb{C}$, the algebra $B_1^{**}$ with $B \oplus \mathbb{C}$, the algebra $A_1$ with $\pi(j(A_1))$ and the algebra $B_1$ with $\rho(j_2(B_1))$.

**Lemma 5.4.** The algebras $A_1^{**}$ and $B_1^{**}$ are weakly-*$*$ Morita equivalent.

**Proof.** Let $U, V$ and $T_\epsilon, \epsilon > 0$ be as in the above discussion. The completely contractive bilinear map $V \times U \to A_1^{**} : (v, u) \to vu$ is separately $w^*$-continuous, $B_1^{**}$-balanced and $A_1^{**}$-module map. So induces the $w^*$-continuous completely contractive and $A_1^{**}$-module map

$$\tau : V \otimes_{B_1^{**}} U \to A_1^{**} : v \otimes_{B_1^{**}} u \to vu.$$

We shall prove that $\tau$ is isometric: If $(v_i) \subset V, (u_i) \subset U$ and $\epsilon > 0$ we have:

$$\left\| \sum_{i=1}^{n} v_i \otimes_{B_1^{**}} u_i \right\| = \left\| \sum_{i=1}^{n} (T_\epsilon^{-1} T_\epsilon v_i) \otimes_{B_1^{**}} u_i \right\|.$$

Since $T_\epsilon v_i \in UV \subset B_1^{**}$ the last norm is equal with

$$\left\| \sum_{i=1}^{n} T_\epsilon^{-1} \otimes_{B_1^{**}} (T_\epsilon v_i u_i) \right\| = \left\| (T_\epsilon^{-1} \otimes_{B_1^{**}} T_\epsilon) (\sum_{i=1}^{n} v_i u_i) \right\| \leq \|T_\epsilon^{-1}\| \|T_\epsilon\| \left\| \sum_{i=1}^{n} v_i u_i \right\| \leq (1 + \epsilon)^2 \left\| \sum_{i=1}^{n} v_i u_i \right\|.$$

We let $\epsilon \to 0$ and we have that

$$\left\| \sum_{i=1}^{n} v_i \otimes_{B_1^{**}} u_i \right\| = \left\| \sum_{i=1}^{n} v_i u_i \right\|.$$
Similarly we can prove that $\tau$ is completely isometric. Since $A = \text{span}^*$ $(YX)$ we have that $A_1^{**} = \text{span}^*(VU)$ and so by the Krein-Smulian Theorem $\tau$ is onto $A_1^{**}$. The proof of the fact $B_1^{**} \cong U \otimes_{A_1^{**}}^\sigma V$ is similar. \hfill $\square$

**Lemma 5.5.** The algebras $A_1$ and $B_1$ are strongly Morita equivalent.

**Proof** It suffices to prove that they have equivalent categories of left operator modules [2]. If $C$ is an operator algebra we denote by $\mathcal{C} \text{-mod}$ the category of left operator modules over $C$. We assume that every $Z \in \mathcal{C} \text{-mod}$ is essential, i.e. the linear span of $CZ$ is dense in $Z$. If $Z_1, Z_2 \in \mathcal{C} \text{-mod}$ the space of morphisms $\text{Hom}_C(Z_1, Z_2)$ is the space of completely bounded maps $F : Z_1 \to Z_2$ which are $C-$module maps.

We fix an operator $T = T_{c_0}$ for $c_0 > 0$. If $Z \in A_1 \text{-mod}$ then $Z^{**}$ is a left dual operator module over $A_1^{**}$ in a canonical way [5, 3.8.9]. We denote by $\mathcal{F}(Z)$ the subspace of $U \otimes_{A_1^{**}}^\sigma Z^{**}$

$$\mathcal{F}(Z) = \overline{\text{span}}(Ta \otimes_{A_1^{**}} z : a \in A_1, z \in Z).$$

Since $U \otimes_{A_1^{**}}^\sigma Z^{**}$ is a left operator module over $B_1^{**}$ and

$$b(Ta \otimes_{A_1^{**}} z) = (bTa) \otimes_{A_1^{**}} z = T(T^{-1}bTa) \otimes_{A_1^{**}} z$$

with $T^{-1}bTa \in A_1$ for all $b \in B_1$, $\mathcal{F}(Z)$ is a left operator $B_1-$module.

If $W \in B_1 \text{-mod}$ we denote by $\mathcal{G}(W)$ the subspace of $V \otimes_{B_1^{**}}^\sigma W^{**}$

$$\mathcal{G}(W) = \overline{\text{span}}(at^{-1} \otimes_{B_1^{**}} w : a \in A_1, w \in W).$$

Since $V \otimes_{B_1^{**}}^\sigma W^{**}$ is a left operator module over $A_1^{**}$, clearly $\mathcal{G}(W) \in A_1 \text{-mod}$. Now

$$\mathcal{G}(\mathcal{F}(Z)) = \overline{\text{span}}(a_2 T^{-1} \otimes_{B_1^{**}} Ta_1 \otimes_{A_1^{**}} z : a_1, a_2 \in A_1, z \in Z)$$

is a left operator module over $A_1$ and subspace of the space $V \otimes_{B_1^{**}}^\sigma U \otimes_{A_1^{**}}^\sigma Z^{**}$. The $w^*-$Morita equivalence $A_1^{**} \cong V \otimes_{B_1^{**}}^\sigma U, B_1^{**} \cong U \otimes_{A_1^{**}}^\sigma V$ induces ([3, Theorem 3.5]) a complete isometry

$$V \otimes_{B_1^{**}}^\sigma U \otimes_{A_1^{**}}^\sigma Z^{**} \to Z^{**} : v \otimes_{B_1^{**}}^\sigma u \otimes_{A_1^{**}}^\sigma z \to vu z$$

which restricts to a completely isometric map

$$R_Z : \mathcal{G}(\mathcal{F}(Z)) \to Z : a_2 T^{-1} \otimes_{B_1^{**}} Ta_1 \otimes_{A_1^{**}} z \to a_2 a_1 z$$

for all $a_1, a_2 \in A_1, z \in Z$. This map is clearly onto $Z$.

Every morphism $F \in \text{Hom}_{A_1}(Z_1, Z_2)$ can be extended to a morphism $\hat{F}$ belonging to $\text{Hom}_{A_1^{**}}(Z_1^{**}, Z_2^{**})$, the space of $w^*-$continuous completely bounded $A_1^{**}-$module maps. (Use for example [5, 1.4.8]).
The weak—* Morita equivalence \( A_{1}^{**} \cong V \otimes_{B_{1}^{**}}^{\sigma_{h}} U \), \( B_{1}^{**} \cong U \otimes_{A_{1}^{**}}^{\sigma_{h}} V \), generates [3, Theorem 3.5] a normal completely contractive functor \( \hat{\mathcal{F}} \) between the left dual operator modules of \( A_{1}^{**} \) and \( B_{1}^{**} \) such that

\[
\hat{\mathcal{F}}(F) : U \otimes_{A_{1}^{**}}^{\sigma_{h}} Z_{1}^{**} \to U \otimes_{A_{1}^{**}}^{\sigma_{h}} Z_{2}^{**} : u \otimes_{A_{1}^{**}} z \to u \otimes_{A_{1}^{**}} \hat{F}(z).
\]

Since

\[
\hat{\mathcal{F}}(\hat{\mathcal{F}})(Ta \otimes_{A_{1}^{**}} z) = Ta \otimes_{A_{1}^{**}} F(z)
\]

for all \( a \in A_{1}, z \in Z_{1} \) the operator \( \hat{\mathcal{F}}(\hat{\mathcal{F}}) \) maps \( \mathcal{F}(Z_{1}) \) into \( \mathcal{F}(Z_{2}) \). So we can define

\[
\mathcal{F}(F) = (\hat{\mathcal{F}}(\hat{\mathcal{F}}))|_{\mathcal{F}(Z_{1})} : \mathcal{F}(Z_{1}) \to \mathcal{F}(Z_{2}).
\]

We can easily check that \( \mathcal{F}(F) \in \text{Hom}_{B_{1}}(\mathcal{F}(Z_{1}), \mathcal{F}(Z_{2})) \).

In this way we define functors \( \mathcal{F} : A_{1}^{mod} \to B_{1}^{mod} \) and \( \mathcal{G} : B_{1}^{mod} \to A_{1}^{mod} \). Using the above complete isometries \( \{ R_{Z} : Z \in A_{1}^{mod} \} \) we can prove that the functor \( \mathcal{G} \mathcal{F} \) is equivalent to the identity functor \( 1_{A_{1}^{mod}} \) and the functor \( \mathcal{F} \mathcal{G} \) is equivalent to the identity functor \( 1_{B_{1}^{mod}} \).

**Theorem 5.6.** Strong Morita equivalence of unital operator algebras does not imply \( \Delta- \)equivalence of the second dual operator algebras.

**Proof** We recall the unital operator algebras \( A_{1}, B_{1} \) which are strongly Morita equivalent by the above lemma. We shall prove that the algebras \( A_{1}^{**}, B_{1}^{**} \) are not \( \Delta- \)equivalent. Suppose that they are \( \Delta- \)equivalent. We define the completely isometric normal representation (see Lemma 5.3)

\[
\pi : A_{1}^{**} \to B(H \oplus \mathbb{C}) : a \oplus \lambda p_{-} \to \theta^{-1}(a) \oplus \lambda.
\]

The algebra \( \pi(A_{1}^{**}) = A \oplus \mathbb{C} \) is a CSL algebra with lattice

\[
\{ N \oplus 0, N \oplus \mathbb{C} : N \in \mathcal{N} \}.
\]

Suppose that \( B_{1}^{**} = J_{B} \oplus C q_{-} \) where \( q \) is the identity of the algebra \( J_{B} \) and \( J_{B} \) is isomorphic with the algebra \( B \). By [12, Theorem 2.7] there exists a completely isometric normal representation \( \sigma \) of \( B_{1}^{**} \) on a Hilbert space \( K_{1} \oplus K_{2} \) of the form \( \sigma(\xi) = \sigma_{1}(\xi) \oplus \lambda I_{K_{2}} \) for all \( \xi \in J_{B}, \lambda \in \mathbb{C} \) such that the algebras \( \pi(A_{1}^{**}), \sigma(B_{1}^{**}) \) are TRO equivalent. Since \( \pi(A_{1}^{**}) \) is a CSL algebra, \( \sigma(B_{1}^{**}) \) is also a CSL algebra, [10, Remark 5.5]. So the algebra \( \sigma(B_{1}^{**}) \) contains a masa. It follows that \( \text{dim} K_{2} = 1 \). So we may assume that \( \sigma(B_{1}^{**}) \) is a CSL algebra acting on \( K_{1} \oplus \mathbb{C} \).

Since \( \Delta(A) \) (resp. \( \Delta(B) \)) is a masa, then \( \Delta(\pi(A_{1}^{**})) \) (resp. \( \Delta(\sigma(B_{1}^{**})) \)) is also a masa. The algebras \( \Delta(\pi(A_{1}^{**})), \Delta(\sigma(B_{1}^{**})) \) are TRO equivalent [10, Proposition 2.5]. But TRO equivalence between masas is a unitary equivalence (use for example [10, Theorem 3.2]). This is a contradiction because \( \Delta(\pi(A_{1}^{**})) = \Delta(A) \oplus \mathbb{C} \) is a totally atomic masa and the masa
\[ \Delta(\sigma(B_1^{**})) \cong \Delta(B) \oplus \mathbb{C} \] has a nontrivial continuous part. So the algebras \( A_1^{**}, B_1^{**} \) are not \( \Delta \)-equivalent. \( \square \)

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Department of Mathematics, University of Athens, Panepistimioupolis 157 84, Athens, Greece.

E-mail address: gelefth@math.uoa.gr