Gravitational resonances on $f(T)$-branes

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In this work, we investigate the gravitational resonances in various $f(T)$-brane models with the warp factor $e^{A(y)} = \tanh (k(y+b)) - \tanh (k(y-b))$. For three kinds of $f(T)$, we give the solutions to the system. Besides, we consider the tensor perturbation of vielbein and obtain the effective potentials by the Kaluza-Klein (KK) decomposition. Then, we analyze what kind of effective potential can produce the gravitational resonances. Effects of different parameters on the gravitational resonances are analyzed. The lifetimes of the resonances could be long enough as the age of our universe in some ranges of the parameters. This indicates that the gravitational resonances might be considered as one of the candidates of dark matter. Combining the current experimental observations, we constrain the parameters for these brane models.

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I. INTRODUCTION

It is known that extra-dimensional theories have been developed for about one hundred years. In 1920s, in order to unify electromagnetic interaction and gravitational interaction, Kaluza and Klein (KK) proposed a five-dimensional spacetime theory by introducing a compact spatial dimension with Planck scale [1, 2]. In 1982, Keichi Akama presented the braneworld model which picture that we live in a dynamically localized 3-brane in higher-dimensional space-time [3]. In 1983, Rubakov and Shaposhnikov proposed the domain wall model in a five-dimensional flat spacetime [4]. In this model, the domain wall is generated by a scalar field with a kink configuration and the extra dimension can be infinite. Fermions can be localized on the domain wall by a Yukawa coupling. However, the four-dimensional effective Newtonian gravity cannot be recovered from this model. Then, more than twenty years ago, in order to solve the huge hierarchical problem between the weak and Planck scales, some brane world models were presented. The two famous ones are the large extra dimension model proposed by Arkani-Hamed, Dimopoulos, and Dvali (ADD) [6] and the warped extra dimension model (RS-I) by Randall and Sundrum (RS) [7]. In these brane models, the sizes of extra dimensions are finite. A great development was achieved in Ref. [8], which shows that the four-dimensional gravity can be recovered on the brane even though the extra dimension is infinite. After that, extra-dimensional theories attracted a lot of interests [9–28].

In this paper, we are interested in thick brane models. In most of these models, branes are generated dynamically by one or more background scalar fields [29–34]. Various matter fields and gravity are fundamentally higher dimensional. Therefore, in order to recover the standard model and the effective four-dimensional Newtonian potential, the zero modes of these matter fields and tensor fluctuations of gravity should be localized on branes [41]. In addition to the zero modes, we will get massive KK modes, which are new particles predicted by these theories. Generally, for the case of a thick brane embedded in five-dimensional asymptotic Anti-de Sitter (AdS) spacetime, the effective potential felt by KK modes along the extra dimension is volcano-like. In this case, the massive KK modes cannot be localized on the brane, but a finite number of massive KK modes could be quasi-localized on the brane. These quasi-localized KK modes are called resonant KK modes. We can judge whether there are resonances by analyzing the shape of the supersymmetric partner potential of the effective potential [42]. In this paper, we focus on the case of gravitational resonant KK modes, which also contribute to the four-dimensional Newtonian potential [43–45]. This provides a possible way to detect the extra dimension. In fact, in the GRS model proposed by Gregory, Rubakov, and Sibiryakov, the four-dimensional Newtonian gravitational force is generated by the quasilocalized gravitons [46]. Furthermore, if the lifetime of the resonance can be long enough as the age of our universe, they might be a candidate of dark matter [48]. Therefore, in braneworld models, the investigation of gravitational resonances is an important topic. In other scenarios, there are also gravitational resonances, such as the quasinormal modes of black hole fluctuation theory [49].

As we all know, general relativity (GR) is a theory of gravity with only curvature. In 1928, Einstein established a gravitational theory with only torsion in spacetime in order to unify gravitational interaction and electromagnetic interaction, called Teleparallel equivalence of general relativity (TEGR) [50, 51]. It is in fact equivalent to GR, based on the fact that torsion scalar differs from the Ricci scalar only by a boundary term. Al-
though the field equation of TEGR is equivalent to GR, the space-time geometry described by TEGR is different from GR. In TEGR, the dynamic field is vielbein, which is defined in the tangent space at each point in space-time. Inspired by \( f(R) \) gravity theory, Bengochea and Ferraro firstly proposed the generalization of teleparallel gravity, \( f(T) \) gravity, to explain the acceleration of the universe [52]. Subsequently, in cosmology, \( f(T) \) gravity theory has been widely investigated [33–57].

In 2012, the thick brane model in \( f(T) \) gravity was firstly constructed in Ref. [58]. They gave the solutions of \( f(T) \) brane world and investigated the localization of fermions. After that, using the superpotential method [59], more thick brane solutions in \( f(T) \) theory were obtained [60]. The tensor perturbations of the vielbein of \( f(T) \) brane and the stability of this system were studied in Ref. [61]. It was found that the zero mode of the perturbation is localized on the brane. Then, in 2018, the braneworld model of \( f(T) \) gravity with non-canonical scalar matter field (K-fields) was studied in Ref. [62]. More \( f(T) \)-brane related studies can be found in Refs. [63–67]. In this paper we investigate the effects of torsion on thick branes structure and the KK gravitation resonance spectrum. The spacetime torsion will cause the thick brane to split, making the thick brane appear more abundant internal structure. Therefore, the gravitational resonance may appear. This provides an experimental way to detect the extra dimension if we can find the KK resonance in the collider. In addition, by studying the resonance spectrum of KK gravitons, it is also possible to reflect the structure of the extra dimension. In this paper, we constructed some new \( f(T) \)-brane solutions. Based on these solutions, we will study the effects of different parameters on the effective potential and the gravitational resonance. We will also give the region where the resonance exists in the parameter space. Besides, the effects of the torsion will also be studied by comparing different kinds of \( f(T) \). More importantly, we will investigate the possibility of the first resonance to be a candidate of dark matter.

The organization of this paper is as follows. In Sec. II, we will review \( f(T) \)-brane and its tensor perturbations [61], and get the zero mode normalization condition. In Sec. III, we will construct some thick \( f(T) \)-brane models. Then we will investigate the possibility of the first resonance in these models to be a candidate of dark matter. Finally, in Sec. IV, we come to the conclusions and discussions.

II. BRANE WORLD MODEL IN \( f(T) \) GRAVITY

Firstly, we give a brief review of the teleparallel gravity. This gravity theory was proposed by Einstein as an attempt of a unified theory of electromagnetism and gravity on the mathematical structure of distant parallelism. We usually use the vielbein fields \( e_A(x^M) \) instead of the metric field \( g_{MN} \) to describe the dynamics and structure of the spacetime. This is done in the tangent space associated with a spacetime point in the manifold, instead of the coordinate basis. These vielbein fields form an orthonormal basis of the tangent space at each point in the manifold with spacetime coordinates \( x^M \). The relation between the spacetime metric and the vielbein fields is given by

\[
g_{MN} = e^A_M e^B_N \eta_{AB},
\]

where \( \eta_{AB} = \text{diag}(−1, 1, 1, 1, 1) \) is the Minkowski metric (in this paper, we focus on five dimensions). In this paper, capital Latin indices \( A, B, C, \cdots = 0, 1, 2, 3, 5 \) label tangent space coordinates, while \( M, N, O, \cdots = 0, 1, 2, 3, 5 \) label spacetime ones. In this gravity theory, the spacetime is characterized by a curvature-free linear connection, i.e., the Weitzenböck connection \( \Gamma^P_{MN} \) which is defined in terms of the vielbein fields:

\[
\Gamma^P_{MN} \equiv e^A_P \partial_N e^M_A = -e^A_M \partial_N e^P_A.
\]

We use the Weitzenböck connection rather than the Levi-Civita connection \( \Gamma^P_{MN} \) to define the associated tensors. The torsion tensor is constructed from the Weitzenböck connection as

\[
T^P_{MN} = \Gamma^P_{NM} - \Gamma^P_{MN} = e^A_P (\partial_N e^M_A - \partial_M e^N_A).
\]

The difference between the Weitzenböck connection and the Levi-Civita connection is given by

\[
K^P_{MN} = \Gamma^P_{MN} - \Gamma^P_{MN} = \frac{1}{2} (T^P_{MN} + T^P_{NM} - T^P_{MN}).
\]

It is useful to define another tensor \( S^P_{MN} \)

\[
S^P_{MN} \equiv \frac{1}{2} (K^{MN,P} - \delta^P_Q T^{QM}_{MN} + \delta^M_P T^{QN}_{MN}).
\]

So the torsion scalar \( T \) is given by

\[
T \equiv S^P_{MN} T^{PMN}.
\]

The Lagrangian of the teleparallel gravity can be written as

\[
L_T = -\frac{M_5^3}{3} e T,
\]

where \( e \) is the determination of the vielbein \( e_A(x^M) \) and \( M_5 \) is the five-dimensional mass scale, which is set to \( M_5 = 1 \) in this paper. It is known that the teleparallel gravity is equivalent to general relativity and hence is also called as the teleparallel equivalent of general relativity since \( R = -T - 2 \nabla^N T_{MN} \).

In \( f(T) \) gravity the torsion scalar is replaced with \( f(T) \), a function of \( T \). When \( f(T) = T \), it goes back to teleparallel gravity, and hence is equivalent to GR. In a five-dimensional \( f(T) \) gravity, the action is given by

\[
S = -\frac{1}{4} \int d^5x \, f(T) + \int d^5x L_M,
\]
where \( \mathcal{L}_M \) denotes the Lagrangian density of the matter. After varying the action with respect to the vielbein, we can get the field equation:

\[
e^{-1} f_T g_{NP} \partial_Q \left( e^S_M P^Q \right) + f_{TT} S_{MN}^Q \partial_Q T = T_{MN}, \quad (9)
\]

where \( f_T \equiv \delta f(T) \), \( f_{TT} \equiv \frac{\delta^2 f(T)}{\delta t^2} \), and \( T_{MN} \) is the energy-momentum tensor of the matter field.

Now, we would like to consider thick brane models in \( f(T) \) gravity theory. The metric of the flat brane with codimension one is given by

\[
ds^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad (10)
\]

where \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) is the four-dimensional Minkowski metric and \( e^{2A(y)} \) is the wrap factor. The bulk vielbein is

\[
e^A_M = \text{diag}(e^{A(y)}, e^{A(y)}, e^{A(y)}, e^{A(y)}, 1), \quad (11)
\]

and the torsion scalar is

\[
T = -12 A'(y)^2. \quad (12)
\]

We choose the matter Lagrangian density as

\[
\mathcal{L}_M = e \left( -\frac{1}{2} \partial^M \phi \partial_M \phi - V(\phi) \right), \quad (13)
\]

where \( \phi \) is the background scalar field only depending on the extra dimension \( y \), and \( V(\phi) \) is the potential of the scalar field \( \phi \). For such setup, the explicit equations of motion are given by

\[
6A'^2 f_T + \frac{1}{4} f = -V + \frac{1}{2} \phi'^2, \quad (14)
\]

\[
\frac{1}{4} f + \left( \frac{3}{2} A'' + 6A'^2 \right) f_T - 36A'^2 A'' f_{TT} = -V - \frac{1}{2} \phi'^2, \quad (15)
\]

\[
\phi'' + 4A' \phi' = \frac{dV}{d\phi}. \quad (16)
\]

It can be shown that only two of the above three equations are independent. We need to give two of the four variables to solve these equations. From Eq. (14) and Eq. (15), we can obtain that

\[
V = -\frac{1}{4} f - \left( \frac{3}{2} A'' + 6A'^2 \right) f_T + 18A'^2 A'' f_{TT}, \quad (17)
\]

\[
\phi'^2 = 36A'^2 A'' f_{TT} - \frac{3}{2} A'^2 f_T. \quad (18)
\]

Next, we consider the linear transverse-traceless tensor perturbation of the metric corresponding to the vielbein, which was investigated firstly in Ref. [61]. The perturbed vielbein can be written as

\[
e^A_M = \left( e^{A(y)}(\delta^\mu_a + h^\mu_a) 0 \right), \quad (19)
\]

where the Latin letters \( a, b, \ldots \) denote the tangent space coordinates on the brane, and the Greek letters \( \mu, \nu, \ldots \) denote the spacetime coordinates on the brane. And then, the tensor perturbation of the background metric can be written as

\[
g_{MN} = \begin{pmatrix} e^{2A(y)}(\eta_{\mu\nu} + \gamma_{\mu\nu}) & 0 \\ 0 & 1 \end{pmatrix}, \quad (20)
\]

where

\[
\gamma_{\mu\nu} = (\delta^\mu_a h^\nu_b + \delta^\nu_a h^\mu_b) \eta_{ab}, \quad (21)
\]

satisfies the transverse-traceless condition

\[
\partial_\mu \gamma^\mu \nu = 0 = \eta^\mu\nu \gamma_{\mu\nu}. \quad (22)
\]

Considering the above conditions, we obtain the main equation of the tensor perturbation [61]:

\[
\begin{align*}
\left( e^{-2A} \Box^{(4)} \gamma_{\mu\nu} + \gamma''_{\mu\nu} + 4A' \gamma'_{\mu\nu} \right) f_T \\
- 24A' \gamma'' f_{TT} = 0,
\end{align*}
\]

where \( \Box^{(4)} = \eta^\mu\nu \partial_\mu \partial_\nu \). With the coordinate transformation

\[
dz = e^{-A} dy, \quad (24)
\]

Eq. (23) becomes

\[
\left( \partial_z^2 + 2H \partial_z + \Box^{(4)} \right) \gamma_{\mu\nu} = 0, \quad (25)
\]

where

\[
H = \frac{3}{2} \partial_z A + 12e^{-2A} \left( (\partial_z A)^3 - \partial_z^2 A \partial_z A \right) \frac{f_{TT}}{f_T}. \quad (26)
\]

Now, we introduce the KK decomposition

\[
\gamma_{\mu\nu}(x^\alpha, z) = \epsilon_{\mu\nu}(x^\alpha) F(z) \psi(z), \quad (27)
\]

where

\[
F(z) = e^{-\frac{1}{2} A(z)} f K(z) dz, \quad (28)
\]

\[
K(z) = 12e^{-2A} \left( \partial_z^2 A \partial_z A - (\partial_z A)^3 \right) \frac{f_{TT}}{f_T}. \quad (29)
\]

Substituting Eq. (27) into Eq. (25), we get two equations: the Klein-Gordon equation for the four-dimensional KK gravitons \( \epsilon_{\mu\nu} \):

\[
\left( \Box^{(4)} + m^2 \right) \epsilon_{\mu\nu}(x^\alpha) = 0, \quad (30)
\]

and the Schrödinger-like equation for the extra-dimensional profile:

\[
-\partial_z^2 + U(z) \psi = m^2 \psi, \quad (31)
\]

where \( m \) is the mass of the KK graviton and the effective potential is given by [61]

\[
U(z) = H^2 + \partial_z H. \quad (32)
\]
The Schrödinger-like equation (31) can be factorized as
\[
(\partial_z + H)(-\partial_z + H)\psi = m^2\psi,
\]
which ensures that the eigenvalues \(m^2\) are non-negative, so there is no tensor tachyon mode with \(m^2 < 0\). That is to say any brane solution of \(f(T)\) gravity theory is stable under the transverse-traceless tensor perturbation. The solution of the graviton zero mode (the four-dimensional massless graviton) is
\[
\psi_0 = N_0 e^{2A - fK(z)dz},
\]
where \(N_0\) is the normalization coefficient. Note that, in order to recover the four-dimensional Newtonian potential on the brane, the zero mode of graviton should satisfy the following normalization condition
\[
\int dz \psi_0^2(z) < \infty.
\]

III. GRAVITATIONAL RESONANCES IN VARIOUS \(f(T)\)-BRANE MODELS

In this section, we will give some solutions of braneworld and investigate the gravitational resonances in thick \(f(T)\)-braneworld models. Because \(f(T)\) is an arbitrary function of the torsion scalar \(T\), so different functional forms can give different solutions.

In this paper, we consider the following warp factor
\[
A(y) = \ln \left[ \tanh (k(y + b)) - \tanh (k(y - b)) \right].
\]
Here the parameter \(k\) has mass dimension one. The parameter \(b\) has length dimension one and denotes the distance of two sub-branes. For convenience, we define the dimensionless scaled distance \(\bar{y} = kw\). The shape of the warp factor is shown in Fig. 1(a), from which we can see that there is a platform near \(\bar{y} = 0\) for large \(b\). When \(\bar{y} \to \pm \infty, A(y) \to -2k|y|\), so the spacetime is asymptotically AdS\(_5\).

A. Model 1: \(f(T) = T\)

We first consider \(f(T) = T\), which is equivalent to GR. From Eqs. (17) and (18) we get the solution
\[
\phi(y) = -i\sqrt{3}\text{sech}(bk)\left[ \cosh(2bk)F(iky; \tanh^2(bk) + 1) - 2\sinh^2(bk)\Pi(\text{sech}^2(bk); iky; \tanh^2(bk) + 1) \right],
\]
\[
V(y) = \frac{3}{4}k^2 \left[ -4\left( \tanh(k(y - b)) + \tanh(k(b + y)) \right)^2 + \text{sech}^2(k(y - b)) + \text{sech}^2(k(b + y)) \right],
\]
where \(F(y; q)\) and \(\Pi(y; q; p)\) are the first and third kind elliptic integrals, respectively. Plots of the scalar field are shown in Fig. 1(b). It can be seen that the scalar field has the configuration of a single kink for small \(b\). With the increase of \(b\), the single kink becomes a double kink, and the value of \(|\phi(\pm \infty)|\) increases accordingly. In particular, when \(b \to 0\), \(\phi(\pm \infty) \to \pm \frac{3\sqrt{3}}{2} \approx \pm 2.72\); when \(b \to \infty\), \(\phi(\pm \infty) \approx \pm 3.85\). In general, the appearance of the double kink means that the brane splits into two sub-branes. The distance between the two sub-branes is \(b\). Because the solution of the scalar field is more complicated, we have not inversely solved the expression of \(V(\phi)\). Numerically, we plot the shape of the scalar potential \(V(\phi)\) in Fig. 1(c). It can be seen that as \(b\) increases, \(V(\phi)\) splits at \(\phi = 0\).

The effective potential (32) in the coordinate \(y\) is given by
\[
U(z(y)) = -\frac{3}{8}k^2\text{sech}^2(k(b - y))\text{sech}^2(k(b + y)) \times \left( \tanh(k(b - y)) + \tanh(k(b + y)) \right)^2
\times \left( -5\cosh(4ky) + 2\cosh(2k(b - y)) + 2\cosh(2k(b + y)) \right).
\]

We plot the shape of the above effective potential in Fig. 2(a). It can be seen that the depth of the potential well decreases with the parameter \(b\), and the height of the potential barrier increases with \(b\). Both changes become smaller with \(b\). On the other hand, it can be seen that the width of the effective potential increases with \(b\). With the increase of \(b\), the potential splits from one well to two wells. We will see later that the appearance of the two-well structure will result in gravitational resonances.

Next, we will investigate the gravitational resonances in this model. Inspired by the investigation of Ref. [68],...
Almeida et al. first proposed a method to find the fermion resonances by using large peaks in the distribution of the normalized squared wavefunction \cite{28}. But this method is only applicable to even functions, and it is no longer valid when the solutions are odd. In order to find all resonances, Liu et al. proposed another method by defining the relative probability \cite{69}

\[ P(m^2) = \frac{\int_{z_{min}}^{z_{max}} |\psi(z)|^2 dz}{\int_{z_{min}}^{z_{max}} |\psi(z)|^2 dz}, \]

where \( \psi(z) \) is the solution of Eq. (31), \( z_{b} \) is approximately the width of the brane, and \( z_{max} = 10 z_{b} \). Here \( |\psi(z)|^2 \) can be explained as the probability density. If the relative probability \( P(m^2) \) has a peak around \( m = m_n \) and this peak has a full width at half maximum, we can say that there exists a resonant mode with mass \( m_n \). Since the potentials considered in this paper are symmetric, the wave functions are either even or odd. Hence, we can use the following boundary conditions to solve the differential equation (31) numerically:

\[
\begin{align*}
\psi_{even}(0) &= 1, \quad \partial_z \psi_{even}(0) = 0; \\
\psi_{odd}(0) &= 0, \quad \partial_z \psi_{odd}(0) = 1,
\end{align*}
\]

where \( \psi_{even} \) and \( \psi_{odd} \) denote the even and odd modes of \( \psi(z) \), respectively. Then, substituting the effective potential (39) into Eq. (31) we can obtain the solution of the extra-dimensional profile \( \psi(z) \) with mass \( m \) and hence the relative probability \( P(m^2) \). According to the supersymmetric quantum mechanics, the supersymmetric partner potentials will share the same spectrum of massive excited states. So, we can judge whether there are resonances by analyzing the shape of the supersymmetric partner potential. The dual potential corresponding to the effective potential (32) is \( U^{(dual)}(z) = H^2 - \partial_z H \). If there is no well or quasi-well in the dual potential, then there is no resonances \cite{47}. Thus, for \( f(T) = T \), only for \( b > 1 \), there might exist resonances. Just as said before, the width of the potential barrier increases with \( b \), which indicates there are more resonances for larger \( b \), which can be seen from Figs. 2(b), 2(c), 2(d). Furthermore, we can obtain the corresponding lifetime \( \tau \) of the gravitational resonances by the width \( \Gamma \) at half maximum of the peak, i.e., \( \tau = \frac{\Gamma}{2} \). For convenience, we define the dimensionless scaled mass \( \bar{m}_n = m_n/k \) and the dimensionless scaled lifetime \( \bar{\tau} = \frac{\tau}{m} \). The relations of the scaled mass \( \bar{m}_n \) and the scaled lifetime \( \bar{\tau}_n \) of the first resonance with the parameter \( \bar{b} \) are shown in Fig. 3. It can be seen that, the scaled mass \( \bar{m}_1 \) of the first resonance decreases with \( \bar{b} \), while the scaled lifetime \( \bar{\tau}_1 \) of the first resonance increases with \( \bar{b} \). This behaviour means that if the parameter \( \bar{b} \) is large enough, the lifetime of the first resonance with very light mass can be long enough as the age of our universe.

So, we can consider such first gravitational resonance as one of the candidates of dark matter \cite{48}.

The scaled mass \( \bar{m}_1 \) and the scaled lifetime \( \bar{\tau}_1 \) can be fitted as two functions of the parameter \( \bar{b} \), and the fit functions are given by

\[
\begin{align*}
\bar{m}_1 &= \frac{3.323}{\bar{b}}, \\
\log(\bar{\tau}_1) &= -0.159 + 1.183 \ln(\bar{b}).
\end{align*}
\]

In the brane world theory considered in this paper, the relation between the effective four-dimensional Planck scale \( M_{Pl} \) and the fundamental five-dimensional scale \( M_5 \) is given by:

\[
M_{Pl}^2 = M_5^2 \int_{-\infty}^{\infty} dy e^{2 A(y)} f_T,
\]

for \( f(T) = T \), Eq. (44) becomes

\[
M_{Pl}^2 = \frac{8b \coth(2b) - 4}{k} M_5^2.
\]

According to the recent experiment of the Large Hadron Collider (LHC), the collision energy is 13 TeV and the result shows that the quantum effect of gravity can be ignored. Theoretically, the quantum effect of gravity will appear if the energy scale is larger than the five-dimensional fundamental scale \( M_5 \). Thus, from the condition \( M_5 > 13 \) TeV and Eq. (45), we can give the constraint on the parameter \( k \) in the Natural System of Units:

\[
k > (12b \coth(2b) - 6) \times 10^{-17} \text{ eV}.
\]

On the other hand, it is known that the age of our universe is of about 13.8 billion years, i.e., \( 4.35 \times 10^{17} \) s. So, if we consider the first resonance as one of candidates of dark matter, its lifetime should be larger than the age of universe, i.e., \( \tau_1 \geq 4.35 \times 10^{17} \) s, or in the Natural System of Units,

\[
\tau_1 = \bar{\tau}_1/k \geq 6.6 \times 10^{32} \text{ eV}^{-1}.
\]

Thus, the restriction of the parameter \( k \) can be expressed as

\[
k \lesssim 1.5 \times 10^{-33} \tau_1 \text{ eV} \simeq 0.724 \times 10^{-33} \text{ eV}.
\]

By combining the fit function (42) and the two conditions (46), (48), the restricted expressions of the mass of the first resonance \( m_1 \) with the combination parameter \( \bar{b} \) can be obtained

\[
m_1 > \frac{3.323}{\bar{b}} (12b \coth(2b) - 6) \times 10^{-17} \text{ eV},
\]

\[
m_1 \lesssim 3.4566 \times 10^{-33} \text{ eV}.
\]

The shadow regions of Fig. 4 show the available ranges of the parameters \( k \) and \( m_1 \), respectively. From Fig. 4(a), we can see that if only if \( \bar{b} > 7.9 \times 10^9 \), the two restricted conditions (46) and (48) of \( k \) could be satisfied, which means that the parameter \( \bar{b} \) has a lower bound. And the corresponding constraint of the parameter \( k \) is \( k \geq 9.5 \times 10^{-5} \) eV. From Fig. 4(b), we can see that the first
we deviate the squared inverse law at very large distance. Therefore, we consider the effective four-dimensional Newtonian potential when we fit the functions \((\bar{m} \equiv m_{1, odd}) + 2 \times 288 \alpha k^2 \cosh(4ky) + \cosh(4bk))

\[
\bar{m} = \frac{3.323}{5}
\]

\[
\log(\tau_1) = \frac{-0.159 +1.183 \ln(b)}{b}
\]

FIG. 2: The influence of the parameter \(b\) on the effective potential \((39)\) and the relative probability \(P\) of \(f(T) = T\) for the odd-parity (red lines) and even-parity (blue dashed lines) massive KK modes.

FIG. 3: The relations of the scaled mass \(\bar{m}_1\) and the scaled lifetime \(\tau_1\) of the first resonance with the parameter \(b\) for \(f(T) = T\). The dots are calculated values while the red and blue lines are fit functions.

resonance mass \(m_1\) has a lower bound, i.e., \(m_1 \gtrsim 4 \times 10^{-16}\) eV. But there are some problems here. First, when we fit the functions \((42)\) and \((43)\), the value of \(b\) is much smaller than \(7.9 \times 10^9\). When the value of \(b\) is very large, we cannot guarantee that the fit function is still valid. Second, such a large \(b\) value means that the thickness \(b\) of the brane is also very large, which may cause the effective four-dimensional Newtonian potential deviate the squared inverse law at very large distance. Therefore, we consider \(f(T)\) gravity and see what effects the torsion will have.

B. Model 2: \(f(T) = T + \alpha T^2\)

Secondly, we consider \(f(T) = T + \alpha T^2\), where the mass dimension of the parameter \(\alpha\) is \(-2\). For convenience, we define the dimensionless parameter \(\bar{\alpha} = \alpha k^2\). In this model, Eq. \((18)\) becomes

\[
\phi'^2 = \frac{3}{32} k^2 \text{sech}^4(k(y - b)) \text{sech}^4(k(b + y))
\]

\[
\times \left( \cosh(2k(y - b)) + \cosh(2k(b + y)) \right)
\]

\[
\times 2 \cosh(2k(y - b)) + 2 \cosh(2k(b + y))
\]

\[
+ (1 - 288\alpha k^2) \cosh(4ky) + \cosh(4bk)
\]

\[
+ 2 + 288\alpha k^2 \right).
\]

In order to ensure that the scalar field \(\phi\) is real, the parameter \(\alpha\) should satisfy \(\alpha \leq \frac{1}{288k^2}\).

We can solve Eq. \((51)\) numerically and show the plots of the scalar field in Figs. 5(a) and 5(b). It can be seen that, the shape of the scalar field depends on the values of \(\bar{\alpha}\) and \(b\). The scalar field has the configuration of a single kink for small \(|\bar{\alpha}|\) and \(b\). With the increase of \(|\bar{\alpha}|\), the single kink becomes a double one, and the asymptotic value of \(|\phi(\pm \infty)|\) increases accordingly. The influence of the parameter \(b\) on the scalar field is the same as the case of \(f(T) = T\).

Since the expression of the effective potential for \(f(T) = T + \alpha T^2\) is complicated and tedious, we only show its plots in Figs. 5(c) and 5(d). From Fig. 5(c) we can see that the depth of the effective potential well and the height of the potential barrier decrease with the parameter \(\bar{\alpha}\). With the decrease of \(\bar{\alpha}\), the potential splits from one well to two wells. From Fig. 5(d) we can see that the effective potential has four peaks. As \(-0.046 < \bar{\alpha} \leq 1/288 \approx 0.00347\) for the case of \(b = 4\), the four peaks will become two peaks. We label two of them as peak 1 and as peak 2 in Fig. 5(d). As \(\bar{\alpha}\) increasing, the height of peak 1 decreases, but the height of peak 2 slowly increases. When \(\bar{\alpha} = -0.212\), the heights of peak 1 and peak 2 are approximately the same. The relative probabilities \(P(m^2)\) of the gravitational resonances for \(b = 4, \bar{\alpha} = 0.00347, -0.212\), and \(-1\) are shown in Fig. 6.
In order to investigate the effects of the parameter $\tilde{b}$, we fix $\tilde{\alpha} = -1$. Plots of the relative probability for different values of $\tilde{b}$ are shown in Fig. 7. We find that the number of resonances increases with $\tilde{b}$, and the mass $m_1$ of the first resonance decreases with $\tilde{b}$. This result is the same as $f(T) = T$. The relation between the number of the resonances and the parameters $\tilde{\alpha}$ and $\tilde{b}$ is shown in Fig. 8. We can see that when $\tilde{\alpha} < -0.9$, even if the value of $\tilde{b}$ is very small, there is still a resonance. As mentioned earlier, when $|\tilde{\alpha}|$ is large, there are two wells for the effective potential, which can be seen from Fig. 5(c). It is because of the existence of these two potential wells that when $\tilde{b}$ is small, the massive KK gravitons can be quasilocalized on the brane. However, when the value of $\tilde{\alpha}$ increases from $-0.9$ to $0.00347$, a larger value of $\tilde{b}$ is required for the existence of resonances. The emergence of these properties is due to the difference between $f(T) = T + \alpha T^2$ and GR.

We know that there are positive correlations between the lifetime of the resonance and width and height of the effective potential. For the case of $f(T) = T$, we adjust the value of $\tilde{b}$ to change the width of the potential well, but keep the height of the potential unchanged. For $f(T) = T + \alpha T^2$, the height of the potential and the lifetime of the first resonance increase with $|\tilde{\alpha}|$. When $|\tilde{\alpha}|$ is large enough, the height of the potential increases very slowly, and when $\tilde{\alpha} \rightarrow -\infty$, it approaches a finite value. In other words, the parameter $\tilde{b}$ plays a key role in the lifetime of the first resonance. This is similar to the case of $f(T) = T$. Therefore, for $f(T) = T + \alpha T^2$, such KK graviton resonances could be considered as one of the candidates of dark matter. But as the case of GR, there are two problems with $f(T) = T + \alpha T^2$. The first problem is that we cannot guarantee that fit function is still valid when the value of $\tilde{b}$ is very large. The second problem is that, with such large $\tilde{b}$, the effective four-dimensional gravitational potential may deviate from the squared inverse law at a large distance.
C. Model 3: $f(T) = -T_0 \left( e^{-\frac{T}{T_0}} - 1 \right)$

Lastly, we consider the model with

$$f(T) = -T_0 \left( e^{-\frac{T}{T_0}} - 1 \right),$$  \hspace{1cm} (52)

where the mass dimension of the parameter $T_0$ is 2. For convenience, we define the dimensionless parameter $\bar{T}_0 = T_0 / k^2$. Then $f(T)$ can be rewritten as

$$f(T) = -\frac{T_0}{k^2} \left( e^{-\frac{2kT}{T_0}} - 1 \right).$$  \hspace{1cm} (53)

The expression (52) can be expanded at $T = 0$ as

$$f(T) = T - \frac{T^2}{2T_0} + \frac{T^3}{6T_0^2} + \cdots.$$  \hspace{1cm} (54)

It can be seen that the smaller $|T_0|$, the more deviation from GR. The warp factor is also considered as (36). In this model, Eq. (18) becomes

$$\phi'^2 = \frac{3k^2 \sinh^4(2bk) \cosh^8(k(y - b)) \cosh^8(k(b + y))}{32T_0} \left( \tanh(k(y - b)) - \tanh(k(b + y)) \right)^4$$

$$\times \left[ 2T_0 \left( \cosh(2k(y - b)) + \cosh(2k(b + y)) + 1 \right) - 96k^2 + T_0 \cosh(4bk) + (96k^2 + T_0) \cosh(4k(b + y)) \right]$$

$$\times \left[ \cosh(2k(y - b)) + \cosh(2k(b + y)) + 2 \right]$$

$$\times e^{\frac{2b}{T_0} \left( \tanh(k(y - b)) + \tanh(k(b + y)) \right)^2}. \hspace{1cm} (55)$$

Same as before, from Eq. (55) we can find the numerical solution of the scalar field $\phi$. In order to ensure that the scalar field $\phi$ is real, the parameter $\bar{T}_0$ should satisfy $\bar{T}_0 > 0$ or $\bar{T}_0 \leq -96$. To compare the deviation of these two branches of $\bar{T}_0$ from GR, we rewrite expression (54) as

$$f(T) = T \left( 1 - \frac{T}{2T_0} + \frac{T^2}{6T_0^2} + \cdots \right).$$  \hspace{1cm} (56)

The $\frac{T}{2T_0}$ term in the above equation is the dominant term deviating from GR. From $T = -12A'(y)^2$ and (36) we get that the range of $\frac{T}{2T_0}$ is $-48 \leq \frac{T}{2T_0} \leq 0$. For $\bar{T}_0 \leq -96$, $0 \leq \frac{T}{2T_0} \leq \frac{1}{2}$; for $\bar{T}_0 > 0$, $-\infty < \frac{T}{2T_0} < 0$. Obviously, the effect of torsion for the case $\bar{T}_0 > 0$ can be more significant than the case $\bar{T}_0 \leq -96$, this is even more evident in the effective potential. Plots of the scalar field are shown in Figs. 9(a) and 9(b). It can be seen that the scalar field has the configuration of a double kink for a large $\bar{b}$. Similarly, we get the effective potential $U(z(y))$ through Eq. (32). Because the expression of the effective potential is complicated, we only show its plots. The width of the potential well increases with the parameter $\bar{b}$, and the height of the potential barrier decreases with the parameter $\bar{T}_0$, which can be seen from Figs. 9(c) and 9(d). The effective potential for $\bar{b} = 10$, $\bar{T}_0 = -96, -500$, and $\bar{T}_0 \to -\infty$ (corresponding to GR) are shown in Fig. 10. We can see that all shapes of the effective potential for all values of $\bar{T}_0$ are almost the same. That is to say, when $\bar{T}_0 \leq -96$, the torsion has an insignificant effect on the effective potential. Therefore, we only consider the case of $\bar{T}_0 > 0$.

The relative probability $P(m^2)$ of the gravitational resonances for $\bar{b} = 10$, $\bar{T}_0 = 2, 3, 4$ are shown in Fig. 11. We find that the number of the resonances decreases with $\bar{T}_0$. The relation between the number of the resonances and the parameters $\bar{T}_0$ and $\bar{b}$ is shown in Fig. 12. We can see that, when $\bar{T}_0 < 6$, even if the value of $\bar{b}$ is very small, there is still a resonance. This is similar to the case of $f(T) = T + \alpha T^2$.

The relations of the scaled mass $\bar{m}_1$ and the scaled lifetime $\bar{\tau}_1$ of the first resonance with the parameter $\bar{T}_0$ are shown in Fig. 13. It can be seen that, both the scaled mass $\bar{m}_1$ and the scaled lifetime $\bar{\tau}_1$ of the first resonance
parameter $\tilde{T}_0$, and the fit functions can be expressed as
\begin{equation}
\tilde{m}_1 = 0.693 + \frac{0.009}{\tilde{T}_0} - 0.027 \tilde{T}_0, \quad \text{(57)}
\end{equation}
\begin{equation}
\log(\tilde{\tau}_1) = -0.783 + \frac{8.738}{\tilde{T}_0}. \quad \text{(58)}
\end{equation}

On the other hand, since we have no analytical formulation for the integral expression (44) for $f(T) = -\tilde{T}_0 \left(e^{-\frac{\tilde{T}_0}{T_0}} - 1\right)$, we cannot obtain an analytical constraint for the five-dimensional fundamental scale $M_5$. In this case, we consider the constraint imposed by the test of the gravitational inverse-square law. According to the method of Ref. [41], for the brane world model considered here, the four-dimensional gravitational potential is
\begin{equation}
U(r) \sim G_N \frac{M}{r} \left[1 + \frac{C}{(kr)^2}\right], \quad \text{(59)}
\end{equation}
where $C$ is a dimensionless constant determined by the structure of the brane. The correction term is $G_N \frac{C M}{r^2}$. For the case of $f(T) = -\tilde{T}_0 \left(e^{-\frac{\tilde{T}_0}{T_0}} - 1\right)$, the correction occurs at the scale of $r \sim 1/k$. Experimentally, in the recent experiments of the test of the gravitational inverse-square law, the usual Newtonian potential still holds down to the scale of about $50 \mu$m. Therefore, the thickness of the brane, $1/k$, should be less than $50 \mu$m, this is equivalent to
\begin{equation}
k > 4 \times 10^{-3} \text{ eV}. \quad \text{(60)}
\end{equation}

On the other hand, the restriction of the lifetime of the first resonance on the parameter $k$ is given by
\begin{equation}
k \lesssim 1.5 \times 10^{\frac{5738}{70} - 33.783} \text{ eV}. \quad \text{(61)}
\end{equation}

By combining the fit function (57) and the two conditions (60), (61), the restricted expressions of the mass of the first resonance $m_1$ with the parameter $\tilde{T}_0$ can be obtained
\begin{equation}
m_1 > \left(2.8 - 0.1 \tilde{T}_0 + \frac{0.04}{\tilde{T}_0}\right) \times 10^{-3} \text{ eV}, \quad \text{(62)}
\end{equation}
\begin{equation}
m_1 \lesssim \left(1.0 - 0.04 \tilde{T}_0 + \frac{0.014}{\tilde{T}_0}\right) \times 10^{\frac{5738}{70} - 33.783} \text{ eV}. \quad \text{(63)}
\end{equation}

The available ranges of the parameters $k$ and $m_1$ are shown in Fig. 14. From Fig. 14(a), we can see that only if $\tilde{T}_0 < 0.28$, the two restricted conditions (60) and (61) of $k$ could be satisfied, which means that the parameter $\tilde{T}_0$ has an upper bound. From Fig. 14(b), we can see that the first resonance mass $m_1$ has a lower bound, i.e., $m_1 \gtrsim 2.8 \times 10^{-3} \text{ eV}$.

**IV. CONCLUSION AND DISCUSSION**

In this work, we investigated the gravitational resonances in various $f(T)$-brane models. The branes are
generated by a canonical scalar field. We first reviewed the tensor perturbations of \( f(T) \)-branes. Then we discussed the gravitational resonances in three \( f(T) \)-brane models with the same warp factor (36).

In model 1, we considered \( f(T) = T \), which is equivalent to GR. The influence of the parameter \( \tilde{b} \) on the gravitational resonances was analyzed. The result shows that, only when \( \tilde{b} > 1 \), the gravitational resonances could exist. The lifetime \( \tau_1 \) of the first resonance increases with the parameter \( \tilde{b} \). If \( \tilde{b} \) is large enough, the lifetime of the first resonance will be long enough as the age of our universe. This indicates that the first gravitational resonance could be one of the candidates of dark matter, which would lead to some constraints on the parameters \( k \) and \( \tilde{b} \), i.e., \( k \gtrsim 9.5 \times 10^{-7} \) eV, \( \tilde{b} > 7.9 \times 10^9 \). And the corresponding available range of the first resonance mass \( m_1 \gtrsim 4 \times 10^{-16} \) eV. But there are two problems. The first problem is that we cannot guarantee that the fit functions (42) and (43) are still valid when the value of \( \tilde{b} \) is very large. The second one is that, with such large \( \tilde{b} \), the effective four-dimensional gravitational potential may deviate from the squared inverse law at a large distance.

In model 2, we considered \( f(T) = T + \alpha T^2 \). The relation between the number of the resonances and the parameters \( \tilde{a} \) and \( \tilde{b} \) was obtained, which can be seen from Fig. 8. This can reflect the characteristics of the extra dimension. We found that for large \( |\tilde{a}| \), there is a gravitational resonance even for small \( \tilde{b} \), which is different from GR. Unfortunately, when considering the resonance as a candidate of dark matter, this model has the same problems as GR.

In model 3, \( f(T) = -T_0 \left( e^{-\frac{T}{T_0}} - 1 \right) \) was considered. We analyzed the effects of the parameters \( T_0 \) and \( \tilde{b} \) on the effective potential, and found that the parameter \( T_0 \) significantly changes the height of the potential barrier. Then, we got the relation between the number of the resonances and the parameters \( T_0 \) and \( \tilde{b} \) (see Fig. 12). The influence of the parameter \( T_0 \) on the gravitational resonances was analyzed by taking \( \tilde{b} = 10 \). The result shows that the lifetime \( \tau_1 \) of the first resonance decreases with the parameter \( T_0 \). If the lifetime of the first resonance exceeds the age of our universe, the parameters \( k, T_0 \), and mass of the first resonance must satisfy \( k \gtrsim 4 \times 10^{-3} \) eV, \( T_0 < 0.28 \), and \( m_1 \gtrsim 2.8 \times 10^{-3} \) eV. We can see that model 3 does not have the two problems in model 1 and model 2. This indicates that the first resonance could be a candidate for dark matter as \( T_0 < 0.28 \), for which the \( f(T) \) theory is very different from GR. Note that, if we choose a larger value of \( \tilde{b} \), the constraint on \( T_0 \) will be further relaxed.

In this paper, we only considered the KK gravitons in \( f(T) \) gravity as the candidate of dark matter and investigated the corresponding properties of such resonances. In fact, KK fermions and KK vector particles, etc., may also be candidates of dark matter, which deserves further study.

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