Optimal dynamic mechanisms with ex-post IR via bank accounts

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Abstract

Lately, the problem of designing multi-stage dynamic mechanisms has been shown to be both theoretically challenging and practically important. In this paper, we consider the problem of designing revenue optimal dynamic mechanism for a setting where an auctioneer sells a set of items to a buyer in multiple stages. At each stage, there could be multiple items for sale but each item can only appear in one stage. The type of the buyer at each stage is thus a multi-dimensional vector characterizing the buyer’s valuations of the items at that stage and is assumed to be stage-wise independent. Compared with the single-shot multi-dimensional revenue maximization, designing optimal dynamic mechanisms for this setting brings additional difficulties in that mechanisms can now branch on different realizations of previous random events and secure extra revenue.

In particular, we propose a novel class of mechanisms called bank account mechanisms. Roughly, a bank account mechanism is no different from any stage-wise individual mechanism except for an augmented structure called bank account, a real number for each node that summarizes the history so far. This structure allows the auctioneer to store deficits or surpluses of buyer utility from each individual mechanism and even them out on the long run. While such simplicity allows for easy implementation in practice, it turns out it is also sufficient to guarantee optimal revenue. We first establish that the optimal revenue from any dynamic mechanism in this setting can be achieved by a bank account mechanism, and we provide a simple characterization of the set of incentive compatible and ex-post individually rational bank account mechanisms. Based on these characterizations, we then investigate the problem of finding the (approximately) optimal bank account mechanisms. We prove that there exists a simple, randomized bank account mechanism that approximates optimal revenue up to a constant factor. Our result is general and can accommodate previous approximation results in single-shot multi-dimensional mechanism design. Based on the previous mechanism, we further show that there exists a deterministic bank account mechanism that achieves constant-factor approximation as well. Finally, we consider the problem of computing optimal mechanisms when the type space is discrete and provide an FPTAS via linear and dynamic programming.

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1 Introduction

A wide range of internet advertising applications including sponsored search and advertising exchanges employ a sequence of repeated auctions to sell their inventory of page-views or impressions. In a variety of settings, optimizing revenue for the seller (e.g., the search engine, publishers, or advertising exchange) boils down to designing auctions that allocate the most effective ads for users, and at the same time extract enough revenue and result in a stable market without incurring too much cost to enter the market.

In order to achieve this goal, the seller should provide good incentives for advertisers to reveal their true values. To cope with this goal, in many settings, advertising exchanges and search engines enforce incentive compatibility or truthfulness at the level of each individual auction, and as a result, end up running repeated 2nd-price auctions or variants of repeated Myerson optimal auctions [Ostrovsky and Schwarz, 2011].

In these cases, we say that the seller runs a static auction optimizing the social welfare and/or revenue of each individual auction. Advertisers, in turn, declare their bids for a sequence of such static auctions and not for one auction. On the other hand, more traditional online advertising campaigns employ long-term contracts between the seller and buyers which can result in higher revenue per impression for sellers.

One reason for lower per-impression revenue from the auction-based advertising is that the resulting individual auctions may become too thin [Celis et al., 2014], and therefore, they may lack enough competition to incur higher prices. More traditional contract-based advertising takes care of this problem by bundling their impressions together, and achieving higher average revenue per impression (while keeping the surplus of advertisers positive). Traditional contract-based advertising, on the other hand, does not have several advantages of the auction-based advertising, e.g., they incur a high cost for entering the market for smaller advertisers, and may not allow for real-time fine-tune targeting. A desirable goal is to design a solution that combines the advantages of auction-based and contract-based advertising. One way to achieve this goal is dynamic mechanisms which allow for combining several auctions, and optimize seller’s revenue across all auctions.

Motivated by the above discussion, the area of dynamic mechanism design has attracted a large body of research work in the last decade [Cavallo, 2008, Pai and Vohra, 2008, Pavan et al., 2008, Cavallo et al., 2009, Gershkov and Moldovanu, 2009, Bergemann and Välimäki, 2010, Gershkov and Moldovanu, 2010, Athey and Segal, 2013, Kakade et al., 2013, Papadimitriou et al., 2014, Pavan et al., 2014] (see [Bergemann and Saïd, 2011] for a survey). In these settings, unlike repeated auctions, the pricing and allocation of items in each auction may depend on the history of the bids, prices, and allocations in the previous auctions. While such dynamic auctions can produce more revenue for the sellers, a couple of issues has slowed down their adoption in the market. Main issues with a majority of results in dynamic mechanism design are two-fold: Complexity: The first issue is the high complexity and impracticality of implementing such auctions: If the seller is allowed to use the full power of dynamic mechanism design and the price and allocation of each auction depend on the outcome of all previous auctions, the complexity of computing the optimal mechanism or even describing a policy could be high. In fact, it has been recently shown that in some settings computing the optimal dynamic mechanism is computationally hard [Papadimitriou et al., 2014].

Future commitment and lack of individual rationality. The second issue is that they may require big initial commitments from buyers and sellers prior to the auction [Kakade et al., 2013]. Although there have been recent movements in adopting such new type of contracts [Mirrokni and Nazerzadeh, 2015], this issue makes it much harder for buyers and sellers to employ these mechanisms. In order to get around this issue, it would be desirable to design dynamic mechanisms
that satisfy a strong notion of individual rationality, i.e., ex-post individual rationality.

In this paper, we aim to design simple dynamic mechanisms that achieve ex-post individual rationality and at the same time, approximate the optimal revenue of the full-force dynamic mechanism. Toward this goal, we study a family of dynamic mechanisms by applying the idea of bank accounts, in which single-shot auctions are augmented with a structure we call bank account, a real number for each node that summarizes the history so far. This structure allows the seller to store deficits or surpluses of buyer utility from each individual auction and even them out on the long run. We prove a number of interesting properties for these simple mechanisms.

In this first part of the paper (Section 3), we prove a number of characterization results. These results ensure that to maximize revenue in dynamic mechanisms, one only needs to restrict attention to a subset of bank account mechanisms.

Theorem 3.5 states that for any dynamic mechanism, $M$, there is a constructive bank account mechanism (coined core BAM) that: (1) generates the same expected utility as $M$, (2) generates weakly higher expected revenue for the auctioneer than $M$, (3) and is deterministic if $M$ is deterministic. Theorem 3.11 provides sufficient and necessary conditions for such construction (Definition 3.9) to be an IC-IR bank account mechanism.

Based on the characterization results, in the second part of the paper (Sections 4-6), we investigate the problem of how to find the optimal bank account mechanisms.

Theorem 4.5 shows a simple bank account mechanism $B$ that guarantees a constant approximation to the optimal dynamic mechanism. Then Theorem 5.1 further derandomizes the construction to get a deterministic bank account mechanism $B^d$ that achieves a constant approximation as well. We remark that both of $B$ and $B^d$ are constructed based on a given (approximately) optimal (deterministic) static mechanism [1]. Particularly, for the one item per stage case, the optimal static mechanism is known and deterministic [Myerson, 1981], and $B$ and $B^d$ are 3— and 5—approximation of the optimal revenue, respectively. Finally, for the problem of computing optimal mechanisms when the type space is discrete, we provide an FPTAS in Theorem 6.1 via linear and dynamic programming.

In a parallel work [Mirrokni et al., 2016], we consider the bank account mechanisms under interim IR constraints. We also prove a characterization result concerning sufficiency of bank account mechanism in that setting and establish trade-offs between the maximum limit of bank account balance and revenue.

Ashlagi et al. [Ashlagi et al., 2016] independently study the same problem as ours except that, in their setting, they assume there is only one item for sale at each stage. They independently propose a dynamic program to compute the optimal mechanism, in the same spirit as ours in Section 6 but different in implementation details. They also propose a mechanism that 2-approximates the optimal revenue for the one-item-per-stage setting, in contrast to our 3-approximation in that specific setting. The approaches used to prove these results, however, are quite different. They finally extend their dynamic program algorithm to compute the optimal mechanism for the multi-bidder case and study the case with discounted infinite horizon. We note that our framework can naturally generalize to the multi-bidder case with one bank account for each of the bidders and we have performed stationary analysis for the infinite horizon case in our interim IR paper [Mirrokni et al., 2016].

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1. The gap between the optimal dynamic mechanism and static mechanism can be arbitrarily large (see Example 4.8).
2. Fully Polynomial Time Approximation Scheme.
2 Setting

Consider the problem of selling a sequence of items to a buyer who has additive valuations for the items. At each stage, there can be any number of items for sale, but each item only appears in one stage. The order in which the items arrive are fixed and commonly known, and the items cannot be sold in stages after they have arrived. At each stage $t \in [T]$, a type (valuation) $v_t \in \mathbb{V}_t = \mathbb{R}^{k_t}$ of the buyer ($k_t$ is the number of items for sale at stage $t$) is privately drawn from a public distribution $\mathcal{F}_t$, i.e., $v_t \sim \mathcal{F}_t$. We assume that the prior distributions $\mathcal{F}_t$’s are independent stage-wise, while within each stage, the distribution can be correlated over different items. Moreover, we assume that all valuations have finite expectations (\(E[v_t] < +\infty\)). This assumption is to rule out the cases where the buyer faces two different strategies that both lead to infinite utilities.

According to the definition above, $v_t$ is, in general, a multidimensional vector. None of our proofs, except the proof of Theorem 6.1, makes use of the assumption that $v_t$ is single dimensional. For the proof of Theorem 6.1, it still extends to the case where $v_t$ is multidimensional, by using the technique from Cai et al., 2012.

We also remark that all of our results extend to the setting with discounted utilities over time. The stage outcome at stage $t$ is specified by a pair $(x_t, p_t)$, where $x_t \in X_t = [0, 1]^{k_t}$ denotes the allocation vector at stage $t$ and $p_t \in \mathbb{R}$ denotes the stage payment. The buyer utility from this stage is

$$u_t(\cdot; v_t) = x_t(\cdot) \cdot v_t - p_t(\cdot).$$

The seller’s objective is to design a mechanism that maximizes the total revenue $\sum_{t=1}^{T} p_t$. For convenience, we use $v_{(\tau, \tau')} = v_{\tau}, \ldots, v_{\tau'}$ to denote the vector of buyer’s types from stage $\tau$ to $\tau'$. Similarly, $x_{(\tau, \tau')} = x_{\tau}, \ldots, x_{\tau'}$, $p_{(\tau, \tau')} = p_{\tau}, \ldots, p_{\tau'}$. We use $\mathcal{V}_{(\tau, \tau')} = \mathcal{V}_\tau \times \cdots \times \mathcal{V}_{\tau'}$ to denote the buyer’s type spaces from stage $\tau$ to $\tau'$.

2.1 Dynamic Mechanisms

In this paper, we consider direct mechanisms, where the agent reports its private type to the mechanism in each stage.

**Definition 2.1** (Direct Mechanism). A direct mechanism is a pair of allocation rule and payment rule. An allocation rule is a mapping $x_t : \mathcal{V}_{(1,t)} \rightarrow X_t$ that specifies at any given stage the probability of allocation of an item to the agent conditioned on the history, i.e., the sequence of types reported so far. A payment rule $p_t : \mathcal{V}_{(1,t)} \rightarrow \mathbb{R}$ specifies the payment the agent should make to the auctioneer at stage $t$, also conditioned on the sequence of types reported so far.

We are interested in mechanism satisfying two properties: incentive compatibility (IC) and individual rationality (IR) constraints. In dynamic mechanism design, there are various flavors of those notions. The flavors we will be interested in this work are as follows. For IC, we will consider incentive compatible in perfect Bayesian equilibrium, which means that at any given history, truth-telling is the best response for current and future stages in expectation over the randomness of future types. For IR, we will be interested in Ex-post individually rationality, which means that for any type realization, the buyer’s overall utility is non-negative. Notice that this is stronger than requiring individual rationality in expectation, where the buyer’s utility is non-negative in expectation over his type.

\[ \text{We use } 1 \text{ to denote a vector in which all entries are 1. Similarly for 0.} \]
To make those notions formal, let a bidding strategy \( b(t+1,T) = \langle b_t \rangle_{t+1,T} \) be a family of (possibly randomized) maps \( b_t : V_{(1,t)} \to V_t \) specifying which type to report at each stage given the history of types observed so far. For a fixed bidding strategy \( b(t+1,T) \), let \( U_t^{b(t+1,T)}(v_{(1,t)}) \) be the total utility that the buyer can obtain in expectation over his type from stages \( t+1 \) to \( T \) of the mechanism given that types \( v_{(1,t)} \) were reported in the stages \( 1 \) to \( t \) and the bidding strategy is employed from that point on. Formally:

\[
U_t^{b(t+1,T)}(v_{(1,t)}) = \mathbb{E}_{v_{(1,T)}} \left[ \sum_{\tau = t+1}^{T} u_{\tau}(v_{(1,t)}, b_{\tau+1}(v_{(1,t+1)}), \ldots, b_{\tau}(v_{(1,\tau)}); v_{\tau}) \right].
\]

(1)

We use \( U_t \) without superscript to denote the expected utility yielded by truthful bidding. This term is often called the promised utility \([\text{Li et al., 2013}]\), since it is the utility that the mechanism promises that the buyer will obtain in expectation if he/she behaves truthfully.

Now we can define IC and IR precisely. We say that the mechanism satisfies IC if at any given stage his present utility plus expected future utility is maximized by bidding truthfully in the current and future stages. In other words, for each time \( t \in [T] \), every type history \( v_{(1,T)} \in V_{(1,T)} \), any deviating report at this timestep \( v'_t \in V_t \) and any bidding strategy for the following timesteps \( b(t+1,T) \), we have:

\[
\text{IC: } u_t(v_{(1,t)}; v_t) + U_t(v_{(1,t)}) \geq u_t(v_{(1,t-1)}, v'_t; v_t) + U_t^{b(t+1,T)}(v_{(1,t-1)}, v'_t);
\]

(2)

We say that a mechanism satisfies individual rationality if for every history of types the utility is non-negative:

\[
\text{IR: } \sum_{\tau=1}^{T} u_{\tau}(v_{(1,\tau)}; v_{\tau}) \geq 0.
\]

(3)

It will be convenient to define a simplified notion of IC, called stage-wise IC, which we will show to be equivalent to the previous notion. We say that a mechanism is stage-wise IC if at any given stage, reporting truthfully at the current and future steps dominates deviating in the current step and reporting truthfully from that point on:

\[
\text{Stage-wise IC: } u_t(v_{(1,t)}; v_t) + U_t(v_{(1,t)}) \geq u_t(v_{(1,t-1)}, v'_t; v_t) + U_t^{b(t+1,T)}(v_{(1,t-1)}, v'_t);
\]

(4)

Lemma 2.2. IC (2) is equivalent to stage-wise IC (4).

The advantage to work with stage-wise IC is that we can focus on the current stage and ignore deviations from this point on. Using backwards induction, we will argue that IC and stage-wise IC are equivalent.

It is also convenient to define stage-wise IR, which means that at any given stage \( t \), the utility is non-negative.

\[
\text{Stage-wise IR: } u_t(v_{(1,t)}; v_t) \geq 0.
\]

(5)

Lemma 2.3. IR (3) is implied by stage-wise IR (5).

Clearly stage-wise IR implies IR, but not the other way round. What we will show in the next lemma, however, is that if there is a mechanism that is IC and IR, we can construct a mechanism that is IC and stage-wise IR with the same allocation rule and the same revenue. The idea is that if a mechanism is ex-post IR, one can anticipate the payments to earlier stages in such a way
the buyer is never required to pay more than his actual value. An extreme case of this reduction is the one where the buyer pays his bid in every round except the last, where the buyer is paid the difference between all the bids charged during the mechanism and his payment in the original mechanism.

**Lemma 2.4.** For any IR direct mechanism $M = \langle x_{(1,T)}, p_{(1,T)} \rangle$, there is another direct mechanism $M' = \langle x'_{(1,T)}, p'_{(1,T)} \rangle$, such that

1. $\forall t \in [T], \ x_t = x'_t$,
2. $\forall v_{(1,T)} \in V_{(1,T)}, \ \sum_{\tau=1}^{T} p_{\tau}(v_{(1,\tau)}) = \sum_{\tau=1}^{T} p'_{\tau}(v_{(1,\tau)})$,
3. $M'$ is stage-wise IR,
4. and $M'$ is IC, if and only if $M$ is IC.

In light of Lemma 2.2 to 2.4 we can use the stage-wise IC (4) as the equivalent definition of IC, and assuming that a direct mechanism satisfies stage-wise IR constraint (5) is without loss of generality.

For any mechanism $M$, denote the overall expected revenue by $\text{Rev}(M)$, overall expected buyer utility by $\text{Utl}(M)$, and overall expected social welfare by $\text{Eff}(M)$, i.e.,

$$\text{Rev}(M) = \mathbb{E}_{v_{(1,T)}} \left[ \sum_{\tau=1}^{T} p_{\tau}(v_{(1,\tau)}) \right],$$

$$\text{Utl}(M) = U_0 = \mathbb{E}_{v_{(1,T)}} \left[ \sum_{\tau=1}^{T} u_{\tau}(v_{(1,\tau)}; v_{\tau}) \right],$$

$$\text{Eff}(M) = \text{Utl}(M) + \text{Rev}(M).$$

Moreover, define the conditional expected utility with partially realized types $v_{(t,t')}$,

$$\text{Utl}(M|v_{(t,t')}) = \mathbb{E}_{V_{(1,T)}} \left[ \sum_{\tau=1}^{T} u_{\tau}(V_{(1,\tau)}; V_{\tau}) \bigg| V_{(t,t')} = v_{(t,t')} \right].$$

A mechanism is deterministic, if for each stage $t$, $x_t \in \{0,1\}^k_t$. A mechanism is history-independent, if both $x_t$ and $p_t$ only depend on $v_t$, but not $v_{(1,t-1)}$.

### 3 Bank account mechanisms

Designing dynamic mechanism is a hard task, mostly because the design space is so large. To remedy this situation, we propose a class of mechanisms, called bank account mechanisms, that are simpler and can be shown to contain the revenue optimal mechanism. Moreover, they are based on a simple principle that is easy to reason about: the mechanism keeps a bank balance for the buyer and the allocation rule depend on the bank balance and not on the full history of the mechanism.

Bank account mechanisms were introduced in [Mirrokni et al., 2016]. Here we show that this framework can be used to reduce the problem of designing dynamic mechanisms to the problem of designing a family of parametrized single-shot mechanism. To understand the spirit of this reduction, we bring the reader attention to the fact that the stage-wise IC constraint looks very much
like the IC constraint in single-shot mechanism design for period \( t \), when the quantity \( p_t(v_{1:t}) - U_t(v_{1:t}) \) is used instead of payments. To see that, rewrite the IC constraint (4) as follows:

\[
x_t(v_{1:t}) \cdot v_t - (p_t(v_{1:t}) - U_t(v_{1:t})) \geq x_t(v_{1:t-1}, v_t') \cdot v_t - (p_t(v_{1:t-1}, v_t') - U_t(v_{1:t-1}, v_t')),
\]

which can be considered as the IC condition for some one-shot stage mechanism with allocation rule \( \tilde{x}_{t|v_{1:t-1}}(\cdot) = x_t(v_{1:t-1}, \cdot) \) and payment rule \( \tilde{p}_{t|v_{1:t-1}}(\cdot) := (p_t(v_{1:t-1}, \cdot) - U_t(v_{1:t-1}, \cdot)) \).

Furthermore, rewrite the stage payment of a dynamic mechanism as \( p_t(v_{1:t}) = \tilde{p}_{t|v_{1:t-1}}(v_t) + U_t(v_{1:t}) \). This suggests us that the stage payment of any dynamic mechanism can be decomposed into the payment \( \tilde{p}_{t|v_{1:t-1}}(v_t) \) of some one-shot stage mechanism and a dynamic component \( U_t(v_{1:t}) \), i.e., the truthful utility from stages \( t + 1 \) to \( T \).

As a result, it is without loss of generality to think of any dynamic mechanism as a sequence of per-stage IC mechanisms, described by allocation rule \( \tilde{x}_{t|v_{1:t-1}}(\cdot) \) and payment rule \( \tilde{p}_{t|v_{1:t-1}}(\cdot) \), as well as an additional per-stage charge \( U_t(v_{1:t}) \) (a temporary utility deficit), which will eventually be compensated via future per-stage IC mechanisms.

By the incentive compatibility of the per-stage mechanism, according to the Envelope theorem [Rochet, 1985], the per-stage allocation rule is the sub-gradient of the per-stage utility,

\[
\frac{\partial (u_t(v_{1:t}) \cdot v_t + U_t(v_{1:t}))}{\partial v_t} = \frac{\partial (\tilde{x}_{t|v_{1:t-1}}(v_t) \cdot v_t - \tilde{p}_{t|v_{1:t-1}}(v_t))}{\partial v_t} = x_t(v_{1:t}),
\]

So, given the allocation rule of the per-stage mechanisms, according to our reinterpretation, the only flexibility of the seller is to design the per-stage charge \( U_t(v_{1:t}) \), the truthful utility for the buyer in future stages.

We are now ready to formally define a bank account mechanism (BAM). Imagine that the buyer has a bank account in his name. In each stage, before the buyer reports his type, the mechanism moves a certain amount of money from the account to the seller (we call it spend), then the buyer reveals his type and a single-shot IC mechanism is run. In the end of the mechanism, depending on the buyer’s reported type, a certain amount of money is deposited in the bank account. The amount deposited still belongs to the buyer, so it doesn’t correspond to a payment at the current step. The right way to think about a deposit is an amount of money that the buyer is setting aside so that the mechanism can spend.

Intuitively, the balance can be regarded as an explicit way to realize per-stage charge \( U_t \) — the change of balance of each stage specifies two things: (i) the amount of utility to compensate previously promised utilities; and (ii) the amount of newly promised utility starting from this stage.

First we define a generic bank account mechanism and then we discuss which properties it needs to have to satisfy IC and IR.

**Definition 3.1 (Bank Account Mechanism).** A bank account mechanism \( B \) is a tuple \((z_{1:T}, q_{1:T}, \text{bal}_{1:T}, d_{1:T}, s_{1:T})\), where for each \( t \),

- **allocation rule** \( z_t : \mathbb{R}_+ \times \mathcal{V}_t \rightarrow \mathcal{X}_t \), that maps balance and stage type to stage allocation,
- **payment rule** \( q_t : \mathbb{R}_+ \times \mathcal{V}_t \rightarrow \mathbb{R}_+ \), that maps balance and stage type to stage payment,
- **balance function** \( \text{bal}_t : \mathcal{V}_{1:t-1} \rightarrow \mathbb{R}_+ \) is defined recursively by the following equation,

\[
\forall v_t, \quad \text{bal}_{t+1} = \text{bal}_t - s_t(\text{bal}_t) + d_t(\text{bal}_t, v_t),
\]

where \( \text{bal}_1 = 0 \). Mathematically, \( \text{bal}_t \) is a function of history \( v_{1:t-1} \). We will often refer to \( \text{bal}_t(v_{1:t-1}) \) simply as \( \text{bal}_t \) and think of it as a variable that is updated in the course of the mechanism as the types are revealed.
Figure 1: Describing dynamic mechanisms with $U_t$’s (left) incurs a recursive structure over $U_t$. Introducing bank account balance (right) helps us to get rid of such recursions, i.e., the deposits $d_t$ and spends $s_t$ are rather independent with each other.

Figure 2: Transfer in bank account mechanisms. Revenue comes from both Spending (Step 1) and Payment (Step 2). In principle, the balance in the bank account belongs to the buyer, but the seller can spend it according to policy $s_t$.

- **deposit policy** $d_t : \mathbb{R}_+ \times \mathcal{V}_t \to \mathbb{R}_+$, that maps balance and stage type to a non-negative real that is to be added to the current balance,

- **spend policy** $s_t : \mathbb{R}_+ \to \mathbb{R}$, that maps balance to a real (no more than balance), that is to be subtracted from the current balance.

At stage $t$, a bank account mechanism works as follows, (see Figure 2)

1. The seller spends the balance by $s_t(bal_t)$, before learning the buyer’s stage type $v_t$. ($s_t(bal_t)$ directly transfers to the seller’s revenue.)

2. Upon receiving buyer’s report $v_t$, the seller allocates according to $z_t(bal_t, v_t)$ to the buyer and charges $q_t(bal_t, v_t)$.

3. The buyer deposits an amount of $d_t(bal_t, v_t)$ to the bank account.

It is useful to describe how a bank account mechanism maps to a direct revelation mechanism. Given a description of a BAM in terms of $z_{(1,T)}, q_{(1,T)}, bal_{(1,T)}, d_{(1,T)}, s_{(1,T)}$ we map it to the direct mechanism described by $x_{(1,T)}, p_{(1,R)}$.

For the allocation rule, we define:

$$x_t(v_{(1,t)}) = z_t(bal_t(v_{(1,t-1)}), v_t).$$

The payment to the seller is given by:

$$p_t(v_{(1,t)}) = s_t(bal_t(v_{(1,t-1)})) + q_t(bal_t(v_{(1,t-1)}), v_t)$$
While the deposit \( d_t(\text{bal}_t, v_t) \) is not present in the description above, it affects the mechanisms in the sense that it dictates how the balance is updated from one step to the other.

Consider the following example that illustrates the bank account mechanism.

**Example 3.2.** For the \( T = 2 \), one item per stage case. Let the valuation of each item be independently identically drawn from the equal revenue distribution, i.e.,

\[
\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}, \quad \mathcal{F}(v) = \begin{cases} 
0, & v \leq 1 \\
1 - 1/v, & 1 < v < v_{\max} \\
1, & v \geq v_{\max}
\end{cases}
\]

The following is a bank account mechanism for this setting.

- Starting with \( \text{bal}_1 = 0 \), spend 0 from the balance, i.e., \( s_1(0) = 0 \) (Step 1).
- The first item is sold at posted-price 1 (Step 2).
- The buyer deposits money of amount \( v_1 - 1 \) to the bank account, i.e., \( d_1(0, v_1) = v_1 - 1 \) (Step 3).
- All the balance so far are spent at the beginning of stage 2, i.e., \( s_2(\text{bal}_1) = \text{bal}_2 \) (Step 4).
- The second item is sold at posted-price \( v_{\max}/e^{v_1 - 1} \) (Step 2).
- The buyer deposit nothing at stage 2, i.e., \( d_1(\text{bal}_2, v_2) = 0 \) (Step 3).

First of all, one can verify that \( B \) is IC and IR: it is straightforward to verify that \( B \) is IR and IC for stage 2. For stage 1, (1) reporting \( v'_1 < 1 \) yields utility 0, since the buyer neither gets the first item (\( v'_1 \) is less than the reserve), nor gets the second item (the reserve is higher than \( v_{\max} \)); (2) reporting any \( v'_1 \geq 1 \) makes no difference to the buyer than truthfully reporting,

\[
v_1 - 1 - (v'_1 - 1) + \sum_{v_2 \geq r(v'_1)} (v_2 - r(v'_1)) = v_1 - v'_1 + \ln v_{\max} - \ln r(v'_1)
\]

\[
= v_1 - v'_1 + \ln v_{\max} - \ln v_{\max} + \ln e^{v'_1 - 1} = v_1 - 1.
\]

In fact, it is not hard to show that the bank account mechanism described above is revenue optimal among all dynamic mechanisms, with optimal revenue \( 2 + \ln \ln v_{\max} \), while any history independent mechanism can obtain revenue at most 2.

To sum up, the spend \( s_t(\text{bal}_t) \) in Step 1 can be considered as a prepayment for items to be sold in stage \( t \). Then, based on the prepayment, the seller provides a per-stage mechanism (allocation and payment) to the buyer for stage \( t \) in Step 2. The prepayment and mechanism are carefully chosen so as to guarantee stagewise IC. Finally, the buyer deposits part of his/her utility to the bank account balance in Step 3 to enable prepayments of the future stages.

### 3.1 Conditions for IC and IR bank account mechanisms

We discuss now sufficient conditions for a bank account mechanism to satisfy IC and IR constraints. At any given stage \( t \), the type \( v_t \) reported can affect the agent in two ways: it will influence the utility he/she gets in the single-shot mechanism defined by \( z_t(\text{bal}_t, \cdot), q(\text{bal}_t, \cdot) \) and it will affect how much is deposited in his bank account, which will in turn affect how much is spent in future rounds. If somehow we can design the mechanism such that the agent is indifferent about how
much we deposit in his/her account and spend from his/her account (i.e., indifferent about the actual balance he/she has), then we can focus on making the stage mechanism truthful.

Define the utility of the \( \langle z, q \rangle \)-mechanism at stage \( t \) as when the type is \( v_t \) and the reported type is \( v'_t \) as:

\[
\hat{u}_t(bal_t, v'_t; v_t) = z_t(bal_t, v'_t) \cdot v_t - q_t(bal_t, v'_t). \tag{8}
\]

The expected utility in stage \( t \) of the corresponding direct mechanism is:

\[
E_{v_t}[u_t(v_{(1,t)}; v_t)] = -s_t(bal_t) + E_{v_t}[\hat{u}_t(bal_t, v_t; v_t)]. \tag{9}
\]

Our next lemma shows that if the previous expression is independent of the bank balance \( bal_t \), it is enough for the \( \langle z, q \rangle \)-mechanism to be truthful as a single-shot mechanism for every bank balance \( bal_t \).

**Lemma 3.3.** Bank account mechanism \( B \) is IC, if the following two conditions are satisfied,

\[
\forall t \in [T], \quad \text{bal}_t, \text{bal}'_t \geq 0, \quad v_t, v'_t \in V_t,
\]

\[
\hat{u}_t(bal_t, v'_t; v_t) \geq \hat{u}_t(bal'_t, v'_t; v_t), \tag{10}
\]

\[
s_t(bal_t) - s_t(bal'_t) = E_{v_t}[\hat{u}_t(bal_t, v_t; v_t) - \hat{u}_t(bal'_t, v_t; v_t)]. \tag{11}
\]

**Lemma 3.4.** Bank account mechanism \( B \) is IR, if the following condition is satisfied,

\[
\hat{u}_t(bal_t, v'_t; v_t) \geq d_t(bal_t, v_t). \tag{12}
\]

In the remainder of this paper, we use BAMs to refer to bank account mechanisms satisfying \( \text{(10)-(12)} \). Given the previous lemmas, we refer \( \text{(10)} \) and \( \text{(11)} \) as IC constraint for BAMs, and \( \text{(12)} \) as IR constraint for BAMs.

### 3.2 It is without loss of generality to focus on Bank Account Mechanisms

The reason why we can focus on bank account mechanisms instead of general dynamic mechanisms (other than the fact that they are simple) is formally justified by the following theorem.

**Theorem 3.5.** For any direct mechanism \( M \), there is a BAM \( B \), such that,

\[
\text{Rev}(B) \geq \text{Rev}(M), \quad \text{Utl}(B) = \text{Utl}(M).
\]

In addition, if \( M \) is deterministic, \( B \) is deterministic.

Before conducting the proof of this theorem, we first need to understand that the set of BAMs is a strict subset of the set of direct mechanisms. In fact, for any BAM that defines the same balance for two different histories, \( \text{bal}_{t+1}(v_{(1,t)}) = \text{bal}_{t+1}(v'_{(1,t)}) \), the submechanisms after the two histories must be identical, since balance is the only history related information stored in a BAM. This is not the case for general direct mechanisms, which may be path-dependent.

Our idea is to conduct a two-step reduction from \( M \) to \( B \): firstly, from any direct mechanism \( M \) to a symmetric direct mechanism \( M' \). We say that a mechanism is symmetric if whenever conditioned on any two histories at time \( t \), if the expected utility is the same then the allocation and payment rules are also the same. Formally:
Definition 3.6 (Symmetric Mechanism). Given any direct mechanism $M$ define the equivalence relation $v_{(1,t-1)} \sim v'_{(1,t-1)}$ between histories as:

$$v_{(1,t)} \sim v'_{(1,t)} \iff \text{UTL}(M|v_{(1,t)}) = \text{UTL}(M|v'_{(1,t)}).$$

We say that mechanism $M$ is symmetric, if for any pair of equivalent histories, $v_{(1,t-1)} \sim v'_{(1,t-1)}$, the corresponding submechanisms are identical, i.e.,

$$\forall t \leq t', T, v_{(t,t')} \in V_{(t,t')}, \left\{ \begin{array}{l} x_{v}(v_{(1,t-1)},v_{(t,t')}) = x_{v'}(v'_{(1,t-1)},v_{(t,t')}) \\ p_{v}(v_{(1,t-1)},v_{(t,t')}) = p_{v'}(v'_{(1,t-1)},v_{(t,t')}) \end{array} \right..$$

Our first step is to show that for any mechanism, it is possible to construct a symmetric mechanism where the utility is the same and the revenue is at least as large:

Lemma 3.7. For any direct mechanism $M$, there is a symmetric direct mechanism $M'$ such that

$$\text{UTL}(M') = \text{UTL}(M), \text{ Rev}(M') \geq \text{Rev}(M).$$

In particular, if $M$ is deterministic, $M'$ is also deterministic.

The second step in the reduction is to show that any symmetric direct mechanism can be converted to a BAM.

Lemma 3.8. For any symmetric direct mechanism $M'$, there is a Core BAM $B$ such that

$$\text{UTL}(B) = \text{UTL}(M'), \text{ Rev}(B) = \text{Rev}(M').$$

In fact, we will show that any mechanism can be converted to a specific type of BAM, which we will denote Core BAM. The rest of this section is devoted to proving the previous lemmas, which taken together constitute a proof of Theorem 3.5. We will start by defining the concept of Core BAM which will be central to our analysis.

Let $H = H_0 \cup H_1 \cup \cdots \cup H_T$ be the set of all histories, where $H_t = V_{(1,t)}$ is the set of all histories of length $t$ ($0 \leq t \leq T$). In particular, $H_0 = \{\emptyset\}$.

For any $0 \leq t \leq T$, Let $g_t$ be a function that maps a history of length $t$ to a real number, and $y_t$ (for $t \geq 1$) be a function that maps a history of length $t$ to a stage allocation.

$$g_t : H_t \rightarrow \mathbb{R}, \forall 0 \leq t \leq T; \quad y_t : H_t \rightarrow X_t, \forall t \in [T].$$

Intuitively, we will construct a BAM, called core BAM, that yields the same conditional utility as $g$ and the same allocation as $y$, where $g$ and $y$ denote $g_0, \ldots, g_T$ and $y_1, \ldots, y_T$, respectively.

Definition 3.9 (Core BAM). Consider the following construction of bank account mechanism based on $g$ and $y$, denoted as $B^{g,y}$.

$$\text{bal}_{t+1}(v_{(1,t)}) = g_t(v_{(1,t)}) - 1$$

$$z_t(\text{bal}_t(v_{(1,t-1)}), v_t) = y_t(v_{(1,t)}),$$

$$q_t(\text{bal}_t, v_t) = z_t(\text{bal}_t, v_t) \cdot v_t - \int_0^{v_t} z_t(\text{bal}_t, v)dv,$$

$$d_t(\text{bal}_t, v_t) = \hat{u}_t(\text{bal}_t, v_t; v_t),$$

$$s_t(\text{bal}_t) = \text{bal}_t + d_t(\text{bal}_t, v_t) - \text{bal}_{t+1},$$

where for $t < T$, $\mu_t = \inf_{v_{(1,t)}} g_t(v_{(1,t)})$; $\mu_T = \min \{ 0, \inf_{v_{(1,T)}} g_T(v_{(1,T)}) \}$.

When the construction above is a valid BAM, we call it a core BAM.
For the construction in Definition 3.9 to be valid BAM we need certain conditions to be satisfied. For equation (15) to be well-defined, we need that 
\[ y_t(v(1,t)) = y_t(v_t'_{(1,t-1)}) \] whenever 
\[ g_t(v_{(1,t-1)}) = g_t(v'_t_{(1,t-1)}) \], otherwise equation (15) would imply different values of \( z_t \) for the same bank balance. Also for IC constraint (10) to be satisfied, we need the allocation function \( y_t(v_{(1,t)}) \) to be weakly increasing in \( v_t \). In Theorem 3.11 we will provide necessary and sufficient conditions on \( g_t \) and \( y_t \) for the mechanism to be a valid core BAM.

We also want to remind that reader that our theorems hold in the case where more than one item is auctioned in each stage. In the first time the reader reads our theorems, we recommend him to focus on the single-item per round case and interpret the integrals as one-variable integrals. However, they hold in general multi-dimensional case as well, by interpreting the integration in (16) as a multidimensional path-integral \( \int_{v_t}^v \) to \( v_t \). If \( z_t(bal_t, v) \) is the sub-gradient of some multi-dimensional valued input, then the value of the integral is independent of the integration path. As we will see in Theorem 3.11 a necessary condition for the mechanism \( B^{g,y} \) to be a Core BAM is that \( y_t \) is the sub-gradient of \( g_t \).

Core BAM is a key construction throughout this paper because it is without loss of generality to restrict to Core BAMs, as we will prove that Theorem 3.5 is still true even if we replace BAM by Core BAM in the statement of the theorem.

Now we are ready to prove the lemmas. Both of the proofs are constructive. The idea to prove Lemma 3.7 is sketched as follows:

1. Given a direct mechanism \( M \), we construct equivalence classes defined by (13).

2. For each equivalence class, select a representative history \( v^*_{(1,t-1)} \) such that the revenue from the submechanism at this history dominates the expected revenue from the submechanisms at the histories in this equivalence class.

3. At history \( v_{(1,t-1)} \) and stage type \( v_t \), \( M' \) simulates what \( M \) does at history \( v^*_{(1,t-1)} \) with stage type \( v_t \).

4. However, applying such simulation to histories at stage \( t \) will destroy the equivalence relation in stage \( t + 1 \) since it changes the utilities in stage \( t + 1 \).

5. Our idea is to replace each history with its representative history inductively, stage by stage (so that when the replacement procedure is applied up to stage \( t \), then equivalence relation in stage \( t + 1 \) is determined).

6. The so-constructed mechanism is symmetric and we verify that it satisfies the desirable properties.

The representative history in an equivalent class is a history in this class such that the seller’s revenue at this history is no less than the expected revenue in this equivalence class. Clearly, such a history always exists. Intuitively, by replacing all histories in an equivalent class with such a representative history, one can ensure that the seller’s expected revenue weakly increases while the buyer’s utility remains the same.

Instead of directly proving Lemma 3.8 we prove the following stronger version: for each symmetric direct mechanism, one can construct a Core BAM with the same overall outcome.

**Lemma 3.10.** For any symmetric direct mechanism \( M \), \( B^{g,y} \) is a BAM, if

\[ \forall 0 \leq t \leq T, h_t \in \mathcal{H}_t, g_t(h_t) = UTL(M|h_t), \forall t \in [T], h_t \in \mathcal{H}_t, y_t(h_t) = x_t(h_t). \quad (19) \]
Moreover, $B^{g,y}$ has the same overall allocation and payment, as a result,

$$Utl(B^{g,y}) = Utl(M) = g_0(\emptyset), \quad Rev(B^{g,y}) = Rev(M).$$

Then the proof of this lemma is to verify that so-constructed $B^{g,y}$ is indeed a BAM.

In light of Theorem 3.5, it is without loss of generality to consider core BAMs only for the purpose of this paper. Most of our results (approximation and computation) rely on the construction of core BAMs. However, as mentioned, the construction $B^{g,y}$ is not guaranteed to be a core BAM for an arbitrary pair of $g$ and $y$. At the end of this section, we characterize the set of pairs of $g$ and $y$ such that $B^{g,y}$ is a Core BAM (Theorem 3.11). The characterization will be used to construct feasibility constraints for finding the optimal Core BAMs in Section 6.

**Theorem 3.11.** $B^{g,y}$ is a core BAM, if and only if for any $t \in [T],$

- $y_t$ is the sub-gradient of $g_t$ with respect to $v_t$ with range being $\mathcal{X}_t$, i.e., $y_t(v_{(1,t)}) = \partial g_t(v_{(1,t)})/\partial v_t$ and $y_t(v_{(1,t)}) \in \mathcal{X}_t$.
- $g_t$ is consistent, symmetric, convex in $v_t$ and weakly increasing in $v_t$.

Consistency and symmetry are defined as follows,

$$\forall t \in [T], v_{(1,t)}, v'_{(1,t)} \in V_{(1,t)}$$

consistent: $g_{t-1}(v_{(1,t-1)}) - E_{v_t}[g_t(v_{(1,t)})] = c_t,$

symmetric: $g_{t-1}(v_{(1,t-1)}) = g_{t-1}(v'_{(1,t-1)}) \implies g_t(v_{(1,t)}) = g_t(v'_{(1,t-1)}, v_t),$ where $c_t$ is a constant.

As a result, it is without loss of generality to study BAMs and Core BAMs in the rest of the paper.

Figure 3 visualizes our findings so far. For more details and intuitions behind, we refer the readers to Appendix (along with the proof of Theorem 3.11).
4 Approximation of Optimal Mechanism

By Theorem 3.5 the revenue optimal bank account mechanism is optimal among any mechanism. However, the optimal bank account mechanism can still be complicated. This observation is formalized in a later section. Later in this section, we will use an example to informally illustrate this observation.

4.1 A simple BAM that 3-Approximates the optimal revenue

In this section we present a very simple BAM whose revenue is a 3-approximation of the revenue of the revenue optimal mechanism. By Theorem 3.5 we know that the optimal revenue is achieved by a BAM, so our strategy will be to first provide an upper bound on the maximum revenue that a BAM mechanism can extract. Then we will craft three individual mechanisms achieving one of the components of the upper bound as revenue. Then we will argue that their combination is a constant approximation to the optimal revenue.

We begin with an upper bound on the revenue of a BAM. The following theorem is a version of Theorem 3 in our previous paper where bank accounts were introduced [Mirrokni et al., 2016].

**Theorem 4.1.** For any BAM $B$,

$$\text{Rev}(B) \leq \text{Rev}(M^S) + \mathbb{E}_{v(1,T)} \left[ \sum_{\tau=1}^{T} s_{\tau}(\text{bal}_{\tau}) \right].$$

where $M^S$ denotes the optimal history-independent mechanism, i.e., within each stage, run the single-stage optimal mechanism.

There is a simple mechanism that achieves the first part of the upper bound, which is the optimal history-independent mechanism itself. In the case where a single item is sold per stage, this is achieved by running Myerson’s optimal auction in each stage. The main challenge is to come up with a mechanism whose revenue approximates the sum of spends of any BAM.

We start by providing an bound on the the sum of spends $\sum_{\tau=1}^{T} s_{\tau}(\text{bal}_{\tau})$ for any history $v(1,T)$. Denote

$$\text{Val}_t = \int_{V_t} 1 \cdot v d\mathcal{F}(v),$$

then

**Lemma 4.2.** For any BAM, the spend and deposit in any stage can be bounded as follows:

$$s_t(\text{bal}_t) \leq \min\{\text{bal}_t, \text{Val}_t\},$$

$$d_t(\text{bal}_t, v_t) \leq 1 \cdot v_t.$$

**Lemma 4.3.** For any BAM $B$, and any history $v(1,t) \in V(1,t)$ of length $t \in [T]$. The sum of spends of $B$ on history $v(1,t)$ can be bounded as follows,

$$\sum_{\tau=1}^{t} s_{\tau}(\text{bal}_{\tau}) \leq \sum_{\tau=1}^{t} s^*_\tau(\text{bal}^*_\tau),$$
where \( s^*_\tau \) and \( d^*_\tau \) are the spend and deposit policies of a bank account mechanism \( B^* \), defined as follows,

\[
\begin{align*}
    s^*_\tau(bal^*_\tau) &= \min\{bal^*_\tau, Val_\tau\}, \\
    d^*_\tau(bal^*_\tau, v_\tau) &= 1 \cdot v_\tau,
\end{align*}
\]

and \( bal^*_\tau \) is the balance of \( B^* \) updated accordingly.

The intuition behind Lemma 4.3 is: (i) for any fixed history \( v(1,t) \), increasing any stage deposit \( d_\tau \) into the bank account never disables the seller to spend from the bank account at any stage as before, so we can set each of the stage deposit \( d_\tau \) to reach its upper bound \( 1 \cdot v_\tau \); (ii) after increasing the stage deposits, they become a sequence of constant incoming flows to the bank account, so taking money out from the bank account as much as possible at each stage is clearly the optimal way to maximize the cumulative spends until any stage.

The takeaway from Lemma 4.3 is that in order to maximize the sum of spends, we should try to keep the balance as large as possible by depositing as much as possible and at the same time we should try to spend as much as possible. Instead of trying to find a BAM with those two properties simultaneously, we define two mechanisms: the first one has the largest possible deposit in each stage. This is called the give-for-free mechanism where all items are given to the buyer for free at every stage. Here the utility of the buyer is maximized and all the utility is deposited in the bank account. The second mechanism seeks to spend as much as possible from the balance in each round. This is called posted-price-for-the-grand-bundle mechanism. This mechanism will post a price for the grand-bundle ensuring that the buyer will have enough utility in expectation so that he is comfortable with a large spend from his/her bank account.

Then, by the approach stated earlier, the uniform randomization of these three mechanisms (optimal history-independent mechanism, give-for-free mechanism, and posted-price-for-the-grand-bundle mechanism) 3-approximates the optimal revenue.

To formalize our reasoning so far, let \( M^S = \langle x^S_{(1,T)}, p^S_{(1,T)} \rangle \) denote the optimal history independent mechanism, i.e., within each stage, separately run the single-stage optimal mechanism 4.

Let \( \langle x^P_t(\theta, \cdot), p^P_t(\theta, \cdot) \rangle \) be the posted-price-for-the-grand-bundle mechanism with parameter \( \theta \) for stage \( t \), i.e.,

\[
    x^P_t(\theta, v_t) = 1 \cdot I[v_t \cdot 1 \geq r_t(\theta)], \quad p^P_t(\theta, v_t) = r_t(\theta) \cdot I[v_t \cdot 1 \geq r_t(\theta)], 
\]

where \( r_t(\theta) \) is the posted-price such that the expected buyer utility yielded by the single-stage posted-price-for-the-grand-bundle mechanism is exactly \( \theta \), formally,

\[
    r_t(\theta) = r, \text{ s.t., } \int_{V_t} (v \cdot 1 - r) \cdot I[v \cdot 1 \geq r]dv = \theta, \implies E_{v_t}[u^P_t(\theta, v_t; v_t)] = \theta. \tag{21}
\]

Since the function \( \int_{V_t} (v \cdot 1 - r) \cdot I[v \cdot 1 \geq r]dv \) is continuous in \( \theta \), a price \( r_t(\theta) \) exists for every \( \theta \) in the interval \([0, Val_t] \). Note that the integration above is the expected utility when the items are sold as a grand bundle at posted-price \( r \) in stage \( t \).
Mechanism 1 (3-Approximation BAM). Consider the following BAM, $B$ — the uniform randomization of the optimal history-independent mechanism $M^S$, the posted-price-for-the-grand-bundle mechanism (with parameter $3s_t(\text{bal}_t)$ for each stage $t$), and the give-for-free mechanism.

Formally defined as follows, where $\text{bal}_t$ is defined by $d_t$ and $s_t$ according to the balance update formula $[7]$.

\[
\begin{align*}
  z_t(\text{bal}_t, v_t) &= \frac{1}{3} \left( z_t^S(v_t) + 1 + z_t^P(3s_t(\text{bal}_t), v_t) \right), \\
  q_t(\text{bal}_t, v_t) &= \frac{1}{3} \left( p_t^S(v_t) + p_t^P(3s_t(\text{bal}_t), v_t) \right), \\
  d_t(\text{bal}_t, v_t) &= \frac{1}{3} \cdot v_t, \\
  s_t(\text{bal}_t) &= \min\{\text{bal}_t, \text{Val}_t/3\},
\end{align*}
\]

where $\text{Val}_t = \int_{\mathcal{V}_t} 1 \cdot v d\mathcal{F}_t(v)$.

Such a uniform randomization ensures that for any core BAM $B'$, the following properties hold simultaneously.

- The one third fraction on the optimal history-independent mechanism ensures enough revenue from per stage payment $q_t$:
  \[
  \forall t \in [T], \text{bal}_t, \text{bal}'_t \geq 0, v_t \in \mathcal{V}_t, \quad q_t(\text{bal}_t, v_t) \geq \frac{1}{3} q_t'(\text{bal}'_t, v_t).
  \]

- The one third fraction on the posted-price-for-the-grand-bundle mechanism with parameter $3s_t(\text{bal}_t)$ ensures enough spend $s_t$ from the bank account in each stage:
  \[
  \forall t \in [T], \text{bal}_t \geq 0, \quad s_t(\text{bal}_t) \geq \frac{1}{3} s_t'(\text{bal}_t).
  \]

- The one third fraction on the give-for-free mechanism ensures enough deposit $d_t$ into the bank account in each stage:
  \[
  \forall t \in [T], \text{bal}_t, \text{bal}'_t \geq 0, v_t \in \mathcal{V}_t, \quad d_t(\text{bal}_t, v_t) \geq \frac{1}{3} d_t'(\text{bal}'_t, v_t).
  \]

Lemma 4.4. Mechanism 1 is a BAM.

Theorem 4.5. Mechanism 1 3-approximates the optimal revenue.

In particular, if $M^S$ is given, then such BAM can be explicitly constructed.

Proof of Theorem 4.5. By Theorem 4.1, we have,

\[
\text{REV}(B^\text{OPT}) \leq \text{REV}(M^S) + \mathbb{E}_{(1,T)} \left[ \sum_{\tau=1}^{T} s_{\tau}^\text{OPT}(\text{bal}_\tau^\text{OPT}) \right].
\]

\(^5\)Since it is only one third fraction on the posted-price-for-the-grand-bundle mechanism, having parameter $3s_t(\text{bal}_t)$ ensures the buyer utility yielded from this one third fraction being exactly $s_t(\text{bal}_t)$.
By Lemma 4.3, we have,

$$\sum_{\tau = 1}^{T} s_{\tau}^{OPT}(bal_{\tau}^{OPT}) \leq \sum_{\tau = 1}^{T} s_{\tau}^{*}(bal_{\tau}^{*}). \tag{27}$$

By the definition of $B$ (see (24) and (25)), we have that $bal_t = bal_t^{*}/3$, $d_t = d_t^{*}/3$ and $s_t = s_t^{*}/3$. Hence by the definition of $B$ (see (23)),

$$\text{Rev}(B) = \frac{1}{3} \left( \text{Rev}(M^S) + \frac{1}{3} \sum_{\tau = 1}^{T} s_{\tau}^{*}(bal_{\tau}^{*}) \right). \tag{28}$$

Combining (26)(27)(28) above, we complete the proof,

$$\text{Rev}(B) \geq \frac{1}{3} \left( \text{Rev}(M^S) + \frac{1}{3} \sum_{\tau = 1}^{T} s_{\tau}^{*}(bal_{\tau}^{*}) \right) \geq \frac{1}{3} \text{Rev}(B^{OPT}).$$

In particular, for the case where only one item is sold in each stage, the optimal history-independent mechanism is the separate Myerson auction for each stage. So we can explicitly construct Mechanism 1 in this case.

Also note that, the optimal history-independent mechanism is unknown for general cases. However, there have been many recent results concerning approximately optimal mechanism for certain valuation assumptions [Hart and Nisan, 2012, Li and Yao, 2013, Babaioff et al., 2014, Cai et al., 2016]. The following corollary allows us to transform these approximately optimal mechanisms into an approximately optimal mechanism in our setting.

**Corollary 4.6. If there is a history-independent mechanism $M$ approximates $M^S$, i.e., $\text{Rev}(M) \geq \alpha \text{Rev}(M^S)$, one can explicitly construct a BAM $B$ that approximates the optimal dynamic mechanism $B^{OPT}$, i.e.,

$$\text{Rev}(B) \geq \frac{\alpha}{2\alpha + 1} \text{Rev}(B^{OPT}).$$

$B$ is a randomization of $M$, the give-for-free mechanism, and the posted-price-for-the-grand-bundle with $r_t((2+1/\alpha)s_t(bal_t))$ for each $t$, on weights $1/(2\alpha+1)$, $\alpha/(2\alpha+1)$, and $\alpha/(2\alpha+1)$, respectively.**

4.2 Global connectivity is necessary to achieve any constant approximations

We show by example that the bank account structure (or some other structure that summarizes the entire history) is necessary to achieve a constant approximation as the complementary to the 3-approximation results.
A local mechanism is a submechanism that only involves a subset of stages regardless of the valuation distribution, and is completely independent with what happens in other stages. For example, the stage mechanisms of a history-independent mechanism are local mechanisms, which operates independently with each other.

Observation 4.7. Any dynamic mechanism that can be decomposed into several independent local mechanisms cannot be optimal — in fact, it cannot guarantee any constant approximation.

In particular, it is also true for the mechanisms that are distributions over such mechanisms.

The observation above informally states the fact that a global structure such as the “bank account” introduced in this paper is essential to preserve the optimal revenue.

Example 4.8 (Necessity of global connectivity). Consider a dynamic mechanism M that can be decomposed into two independent local mechanisms, i.e., there exists T′, 1 < T′ < T, such that the outcomes of stages T′ + 1 to T are independent with what has happened in the first T′ stages.

We construct the following valuation distributions,

- the first and the (T′ + 1)-th items are i.i.d. equal-revenue distributions (see Example 3.2);
- all other items have 0 valuations.

By our assumption, M treats the two items with equal-revenue valuations independently, and can extract revenue at most 1 from each such stage without violating the ex-post IR constraint. (Because all other zero-valued items can be ignored.)

However, for example 3.2 we have shown a BAM with revenue \( 2 + \ln \ln v_{\max} \gg 2 \) (as \( v_{\max} \) tends to infinity) on a similar instance. To adjust the mechanism introduced in Example 3.2 to this example, let B be a BAM as follows,

\[
\begin{align*}
\text{bal}_1 = 0, z_1(\text{bal}_1, v_1) &= 1, q_1(\text{bal}_1, v_1) = 1, d_1(\text{bal}_1, v_1) = v_1 - 1, s_1(\text{bal}_1) = 0; \\
z_{T'+1}(\text{bal}_{T'+1}, v_{T'+1}) &= I[v_{T'+1} \geq v_{\max}/e^{\text{bal}_{T'+1}}], \\
q_{T'+1}(\text{bal}_{T'+1}, v_{T'+1}) &= z_{T'+1}(\text{bal}_{T'+1}, v_{T'+1}) \cdot v_{\max}/e^{\text{bal}_{T'+1}}, \\
d_{T'+1}(\text{bal}_{T'+1}, v_{T'+1}) &= 0, s_{T'+1}(\text{bal}_{T'+1}) = \min\{\ln v_{\max}, \text{bal}_{T'+1}\}; \\
\forall t \neq 1, T' + 1, z_t(\text{bal}_t, v_t) &= 0, q_t(\text{bal}_t, v_t) = 0, d_t(\text{bal}_t, v_t) = 0, s_t(\text{bal}_t) = 0. \\
\Rightarrow \text{REV}(B) &= \mathbb{E}_{v_1, v_{T'+1}}[q_1(v_1) + q_{T'+1}(v_{(1,T'+1)}) + s_1(\text{bal}_1) + s_{T'+1}(\text{bal}_{T'+1})] \\
&= 2 + \mathbb{E}_{v_1} \left\{ \min\{v_1 - 1, \ln v_{\max}\} \right\} = 2 + \ln \ln v_{\max}.
\end{align*}
\]

In other words, the gap between BAM (hence the optimal mechanism) and independent local mechanisms can be arbitrarily far on such instances.

5 Constant approximation via deterministic BAM

In this section, we prove the existence of deterministic BAM that guarantees constant approximation of the optimal revenue.

However, since there is an arbitrarily large gap between deterministic and optimal mechanisms for non-dynamic multidimensional case (even for 2-item, 1-buyer case [Hart and Nisan, 2013]), there is no general approximation guarantee of deterministic mechanisms. Instead, we prove that if there is a reasonably good (constant) approximation of the optimal history-independent mechanism, there is also a constant approximation of the optimal dynamic mechanism.
Theorem 5.1. If there is a deterministic (history-independent) mechanism \(M\) that is a constant approximation of the optimal history-independent mechanism \(M^S\), i.e., \(\text{Rev}(M) \geq \alpha \text{Rev}(M^S)\), then one can construct a deterministic BAM \(B\) based on \(M\), such that \(B\) is a constant approximation to the revenue of the optimal (possibly randomized) dynamic mechanism \(B^{\text{OPT}}\), i.e.,

\[
\text{Rev}(B) \geq \frac{\alpha}{4\alpha + 1} \text{Rev}(B^{\text{OPT}}).
\]

In particular, for the one item per stage case, \(\alpha = 1\) (since in this case, \(M^S\) is just a posted-price mechanism), and \(B\) guarantees a 5-approximation.

Proof of Theorem 5.1. First of all, if \(\text{Rev}(M^S) > \text{Rev}(B^{\text{OPT}})/(4\alpha + 1)\), then \(M\) is a good approximation, \(\text{Rev}(M) \geq \alpha \text{Rev}(M^S) \geq \alpha \text{Rev}(B^{\text{OPT}})/(4\alpha + 1)\).

Otherwise, suppose \(\text{Rev}(M^S) < \text{Rev}(B^{\text{OPT}})/(4\alpha + 1)\), then by Theorem 4.1

\[
E_{v(1,T)} \left[ \sum_{\tau=1}^{T} s^\text{OPT}_\tau(bal^\text{OPT}_\tau) \right] \geq \frac{4\alpha}{4\alpha + 1} \text{Rev}(B^{\text{OPT}}).
\]

Then we prove that there is a deterministic BAM \(B\) that approximates the expected overall spend of \(B^{\text{OPT}}\), i.e.,

\[
E_{v(1,T)} \left[ \sum_{\tau=1}^{T} s_\tau(bal_\tau) \right] \geq \frac{1}{4} E_{v(1,T)} \left[ \sum_{\tau=1}^{T} s^\text{OPT}_\tau(bal^\text{OPT}_\tau) \right].
\]

Recall that the randomization over the give-for-free mechanism and the posted-price-for-the-grand-bundle mechanism (each with probability 1/3) guarantees a 3-approximation to \(E_{v(1,T)} \left[ \sum_{\tau=1}^{T} s^\tau(bal^\tau) \right]\), (see inequality (28)).

Similarly the following construction of \(B^\dagger\), a uniform randomization of the give-for-free mechanism and the posted-price-for-the-grand-bundle mechanism (with posted-price at \(r_t^\dagger\) for each \(t\), always spends half of the maximum of overall spend.

\[
z_t^\dagger(bal_t^\dagger, v_t) = \frac{1}{2} \left( 1 + x_t^p(2s_t^\tau(bal_t^\tau), v_t) \right), \quad q_t^\dagger(bal_t^\dagger, v_t) = \frac{1}{2} p_t^p(2s_t^\tau(bal_t^\tau), v_t),
\]

\[
d_t^\dagger(bal_t^\dagger, v_t) = \frac{1}{2} v_t, \quad s_t^\dagger(bal_t^\dagger) = \min\{bal_t^\dagger, Val_t/2\},
\]

\[
\Rightarrow \forall v(1,T) \in V(1,T), \sum_{\tau=1}^{T} s^\dagger_\tau(bal^\dagger_\tau) = \frac{1}{2} \sum_{\tau=1}^{T} s^\tau_\tau(bal^\tau_\tau).
\]

Combining (27), \(B^\dagger\) also guarantees a 2-approximation to the expected overall spend of the \(B^{\text{OPT}}\).

\[
E_{v(1,T)} \left[ \sum_{\tau=1}^{T} s^\dagger_\tau(bal^\dagger_\tau) \right] \geq \frac{1}{2} E_{v(1,T)} \left[ \sum_{\tau=1}^{T} s^\tau_\tau(bal^\tau_\tau) \right] \geq \frac{1}{2} E_{v(1,T)} \left[ \sum_{\tau=1}^{T} s^\text{OPT}_\tau(bal^\text{OPT}_\tau) \right].
\]

Unfortunately, \(B^\dagger\) is not deterministic, since it is a mixture of two deterministic mechanism, i.e., \(z_t^\dagger(bal_t^\dagger, v_t) \in \{1/2, 1\}\). Then the rest of the proof focuses on the derandomization of \(B^\dagger\) to get a deterministic BAM that guarantees a constant approximation to the expected overall spend of \(B^\dagger\).
Consider BAM$^B$ defined by a binary string $\sigma \in \{0, 1\}^T$ as follows,

$$\begin{align*}
\text{if } \sigma_t = 0 : & \quad z^\sigma_t (\text{bal}_t^\sigma, v_t) = 1, \quad q^\sigma_t (\text{bal}_t^\sigma, v_t) = 0, \quad d^\sigma_t (\text{bal}_t^\sigma, v_t) = 1 \cdot v_t, \quad s^\sigma_t (\text{bal}_t^\sigma) = 0; \\
\text{if } \sigma_t = 1 : & \quad z^\sigma_t (\text{bal}_t^\sigma, v_t) = x^\sigma_t (s^\sigma_t (\text{bal}_t^\sigma), v_t), \quad q^\sigma_t (\text{bal}_t^\sigma, v_t) = p^\sigma_t (s^\sigma_t (\text{bal}_t^\sigma), v_t), \\
& \quad d^\sigma_t (\text{bal}_t^\sigma, v_t) = 0, \quad s^\sigma_t (\text{bal}_t^\sigma) = \min \{\text{bal}_t^\sigma, \text{Val}_t\}.
\end{align*}$$

Intuitively, if $\sigma_t = 0$, then $B^\sigma$ runs the give-for-free mechanism and deposits to the balance; if $\sigma_t = 1$, then $B^\sigma$ spends the balance and runs the posted-price-for-the-grand-bundle mechanism with parameter equal to the spend $s^\sigma_t (\text{bal}_t^\sigma)$.

We then argue that for binary string $\sigma$ chosen at uniformly random, $E_\sigma [\text{Rev} (B^\sigma)] \geq \Omega (\text{Rev} (B^{\text{OPT}}))$. Thus at least one out of them guarantees a constant approximation to the optimal revenue.

Consider the following lemma.

**Lemma 5.2.**

$$\forall t \in [T], v_{(1,T)} \in \mathcal{V}_{(1,T)}, \quad E_\sigma \left[ s^\sigma_t (\text{bal}_t^\sigma) \right] \geq \frac{1}{4} s^*_{t} (\text{bal}_t^*) . \tag{34}$$

Hence

$$E_\sigma [\text{Rev} (B^\sigma)] = E_{\sigma, v_{(1,T)}} \left[ \sum_{\tau=1}^{T} \left( q^\sigma_\tau (\text{bal}_\tau^\sigma, v_\tau) + s^\sigma_\tau (\text{bal}_\tau^\sigma) \right) \right] \geq E_{\sigma, v_{(1,T)}} \left[ \sum_{\tau=1}^{T} s^\sigma_\tau (\text{bal}_\tau^\sigma) \right] \geq \frac{1}{4} E_{v_{(1,T)}} \left[ \sum_{\tau=1}^{T} s^\text{OPT}_\tau (\text{bal}_\tau^{\text{OPT}}) \right],$$

where the last inequality is from [27].

Therefore there exists one $\sigma^* \in \{0, 1\}^T$, such that

$$\text{Rev} (B^{\sigma^*}) \geq E_\sigma [\text{Rev} (B^\sigma)] \geq \frac{1}{4} E_{v_{(1,T)}} \left[ \sum_{\tau=1}^{T} s^\text{OPT}_\tau (\text{bal}_\tau^{\text{OPT}}) \right],$$

and by assumption [30], we conclude that

$$\text{Rev} (B^{\sigma^*}) \geq \frac{1}{4} E_{v_{(1,T)}} \left[ \sum_{\tau=1}^{T} s^\text{OPT}_\tau (\text{bal}_\tau^{\text{OPT}}) \right] \geq \frac{\alpha}{4\alpha + 1} \text{Rev} (B^{\text{OPT}}).$$

Note that for any $\sigma$, $B^\sigma$ is a deterministic BAM by construction. So in summary, at least one out of these $2^T + 1$ deterministic mechanisms, $\{M\} \cup \{B^\sigma | \sigma \in \{0, 1\}^T \}$, guarantees the desired constant approximation to the optimal revenue.

\section*{6 Computing Optimal Bank Account Mechanism via Dynamic Programming}

In this section, we put forward a dynamic programing algorithm and FPTAS to compute the optimal mechanism for the discrete type case. We show a proof for the one item per stage case (see Appendix A.13), and argue that it can be directly generalized to the multi-item per stage case by using the method from [Cai et al., 2012].

\footnote{We omit the proof that $B^\sigma$ is a BAM, since it is almost the same with proving Mechanism 1 is a BAM in the proof of Theorem 4.5.}
Theorem 6.1. The optimal bank account mechanism can be computed through a dynamic programming algorithm.

Moreover, for any $\epsilon > 0$, there is an FPTAS to achieve an $\epsilon$-approximation (multiplicative) of the optimal revenue for the one item per stage case with discrete valuation distributions for each stage.

Corollary 6.2. For any $\epsilon > 0$, there is an FPTAS to achieve an $\epsilon$-approximation (multiplicative) of the optimal revenue for the general (multi-item per stage) case with discrete valuation distributions.

Proof sketch of Theorem 6.1. We outline the proof idea here and leave the complete proof to Appendix A.13.

By Theorem 3.5, we assume that the optimal mechanism is a core BAM $B^{g,y}$. By Theorem 3.11 and Lemma 3.10, it is without loss of generality to assume that $g_t(h_t) = U_t(B|h_t)$. Since $y_t$ is the sub-gradient of $g_t$ and $g_t$ is symmetric, we denote $z_t(bal_t, v_t)$ as $y_t(g_t(v(1,t−1)), v_t)$, by overloading the definition of $y_t$.

We claim that the optimal mechanism can be computed via a dynamic program with state function $\phi_t(\xi)$, which is the optimal revenue for the sub-problem (consists of stages $t + 1$ to $T$) but with additional constraint that the promised utility must equal to $\xi$ ($U_t = \xi$).

Formally,

$$\phi_t(\xi) = \max_{B: g_t(v(1,t)) = \xi} E_{v_t} \left[ \sum_{\tau=t+1}^{T} y_t(g_t(v(1,\tau−1)), v_\tau) \cdot v_\tau - bal_{T+1} \right].$$

It can be computed from $\phi_{t+1}(\xi)$ by the following program.

$$\begin{align*}
\text{maximize} & \quad E_{v_t} \left[ y_t(\xi, v_t) \cdot v_t + \phi_{t+1}(g(\xi, v_t)) \right] \\
\text{subject to} & \quad \hat{u}_t(\xi, v_t; v_t) \geq \hat{u}_t(\xi, v'_t; v_t), \forall v_t \in V_t \\
 & \quad g(\xi, v_t) = \xi + \hat{u}_t(\xi, v_t; v_t) - E_{v'_t} \left[ \hat{u}_t(\xi, v'_t; v'_t) \right] \geq 0, \forall v_t \in V_t
\end{align*}$$ (35)

Then $\max_{\xi \geq 0} \phi_0(\xi) = \max_B \text{REV}(B)$ is the optimal revenue, and function $\phi_0(\xi)$ can be computed via dynamic program starting from $\phi_T(\xi) = -\xi$. For any $t$, $\phi_t(\xi)$ is a concave function, and once the optimal revenue is determined, we can recover the entire mechanism by solving program (35) for each stage.

For discrete type space cases, program (35) can be written as an LP that can be efficiently solved by techniques introduced in [Cai et al., 2012]. Moreover, since $\phi_t(\xi)$ is concave, it can be bounded by two concave piece-wise linear functions within any constant $\delta > 0$. Since the number of pieces of the bounds is at most $O(N/\delta^7)$, this dynamic algorithm admits an FPTAS.

We conjecture that the computation of the exact solution is likely to be hard, because in general cases, the description of the BAM could be exponentially large (since there could be exponentially many different values of balance).

In addition, if one wants to further generalize the results for continuous valuation distributions by discretization of the valuation domain, one needs to relax the IC and IR constraints to $\epsilon$-IC and $\epsilon$-IR, which may be an interesting future direction.

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$^7$ $N$ is the input size.
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APPENDIX

A  Omitted proofs

A.1  Proof of Lemma 2.2

Proof of Lemma 2.2 IC \implies stage-wise IC:

Let \( b_{t+1,T} \) be the truthful bidding strategy from stage \( t + 1 \), we immediately have IC (2) \implies stage-wise IC (4).

IC \iff stage-wise IC:

Let \( b_t(v_{(1,t)}) = v'_t \) be some bidding strategy, and take expectation on the stage-wise IC constraint (4) over \( v_t \), we have

\[
E_{v_t} \left[ u_t(v_{(1,t)}; v_t) + U_t(v_{(1,t)}) \right] \geq E_{v_t} \left[ u_t(v_{(1,t-1)}, b_t(v_{(1,t)}); v_t) + U_t(v_{(1,t-1)}, b_t(v_{(1,t)})) \right],
\]

In other words,

\[
U_{t-1}(v_{(1,t-1)}) \geq U_{t-1}^{b_t}(v_{(1,t-1)}).
\]

Combining (4):

\[
u_t(v_{(1,t)}; v_t) + U_t(v_{(1,t)}) \geq u_t(v_{(1,t-1)}, v'_t; v_t) + U_t(v_{(1,t-1)}, v'_t)\]

we have

\[
u_t(v_{(1,t)}; v_t) + U_t(v_{(1,t)}) \geq u_t(v_{(1,t-1)}, v'_t; v_t) + U_t^{b_t}(v_{(1,t-1)}, v'_t).
\]

Take expectation on the last inequality above over \( v_t \), we have

\[
U_{t-1}(v_{(1,t-1)}) \geq U_{t-1}^{b_t(t+1)}(v_{(1,t-1)})
\]

Repeat the same argument above, we have

\[
u_t(v_{(1,t)}; v_t) + U_t(v_{(1,t)}) \geq u_t(v_{(1,t-1)}, v'_t; v_t) + U_t^{b_t(t+1,t+2)}(v_{(1,t-1)}, v'_t) \implies \cdots \implies (2).
\]

This proves the equivalence between IC and stage-wise IC.

A.2  Proof of Lemma 2.3

Proof of Lemma 2.3. To show that stage-wise IR implies IR, sum up the stage-wise IR constraints (5) from stage 1 to \( T \), we directly have the IR constraint (3).

A.3  Proof of Lemma 2.4

Proof of Lemma 2.4. Recall the construction of \( M' \) is as follows,

\[
\forall t \in [T], v_{(1,t)} \in V_{(1,t)}, x'_t(v_{(1,t)}) = x_t(v_{(1,t)});
\]
∀t < T, v(1,t) ∈ V(1,t), p'_t(v(1,t)) = x'_t(v(1,t)) · v_t;

∀v(1,T) ∈ V(1,T), p'_T(v(1,T)) = \sum_{\tau=1}^{T} p_\tau(v(1,\tau)) - \sum_{\tau=1}^{T-1} p'_\tau(v(1,\tau)).

We prove that the required properties in Lemma 2.4 are met. Properties 1 and 2 are straightforward from the construction. Property 3. For first T − 1 stages,

u'_t(v(1,t); v_t) = x'_t(v(1,t)) · v_t - p'_t(v(1,t)) = 0 \implies (5);

for the last stage,

u'_T(v(1,T); v_T) = x'_t(v(1,t)) · v_t - p'_t(v(1,t))
= x'_t(v(1,t)) · v_t - \left( \sum_{\tau=1}^{T} p_\tau(v(1,\tau)) - \sum_{\tau=1}^{T-1} p'_\tau(v(1,\tau)) \right)
= \sum_{\tau=1}^{T} x'_\tau(v(1,\tau)) · v_\tau - \sum_{\tau=1}^{T} p_\tau(v(1,\tau))
= \sum_{\tau=1}^{T} x_\tau(v(1,\tau)) · v_\tau - \sum_{\tau=1}^{T} p_\tau(v(1,\tau))
= \sum_{\tau=1}^{T} u_\tau(v(1,\tau); v_\tau)
\geq 0 \implies (5),

where the last inequality is from the IR condition (3) of M.

Property 4 is also straightforward. Firstly, rewrite the IC condition as follows by including all the utilities from past stages,

\sum_{\tau=1}^{t} u_\tau(v(1,\tau); v_\tau) + U_t(v(1,t)) \geq \sum_{\tau=1}^{t-1} u_\tau(v(1,\tau); v_\tau) + u_t(v(1,t-1), v'_t; v_t) + U'_t(v(1,t-1), v'_t).

Then by properties 1 and 2 the left-hand-side for M and M' are the same, and similar with the right-hand-side. \(\square\)

A.4 Proof of Lemma 3.3 & 3.4

Throughout the proof of Lemma 3.3 and 3.4 we use the following notations,

bal_\tau = bal_\tau(v(1,\tau-1)), bal_\tau^{(t)} = bal_\tau(v(1,\tau-1), v'_t, v_{(t+1,\tau)}), s_\tau = s_\tau(bal_\tau), s_\tau^{(t)} = s_\tau(bal_\tau^{(t)}),

where the superscript \(^{(t)}\) means the valuation of stage \(t\) is replaced by some \(v'_t\).

We also enhance both lemmas by generalizing the conclusions to a border set of direct mechanisms (beyond the direct mechanism we map the bank account mechanism \(B\) to).
For any given bank account mechanism $B$, let $M$ be any direct mechanism that has the same allocation rule and generates the same revenue with $B$. Formally,

$$\forall t \in [T], v_{(1,T)} \in \mathcal{V}_{(1,T)}, \quad \begin{cases} x_t(v_{(1,t)}) = z_t(\text{bal}_t, v_t) \\ \sum_{\tau=1}^T p_{\tau}(v_{(1,\tau)}) = \sum_{\tau=1}^T \left( q_{\tau}(\text{bal}_{\tau}, v_{\tau}) + s_{\tau}(\text{bal}_{\tau}) \right) \end{cases}$$ \quad (36)

$$\implies \sum_{\tau=1}^T u_{\tau}(v_{(1,\tau)}; v_{\tau}) = \sum_{\tau=1}^T \left( \hat{u}_{\tau}(\text{bal}_{\tau}, v_{\tau} ; v_{\tau}) - s_{\tau}(\text{bal}_{\tau}) \right).$$ \quad (37)

Then Lemma 3.3 and 3.4 work for $M$ accordingly.

**Proof of Lemma 3.3.** For any $v_{(1,t)}$, consider the following,

$$\sum_{\tau=1}^{t-1} u_{\tau}(v_{(1,\tau)}; v_{\tau}) + u_t(v_{(1,t-1)}, v'_t; v_t) + U_t(v_{(1,t-1)}, v'_t) = \sum_{\tau=1}^{t-1} u_{\tau}(v_{(1,\tau)}; v_{\tau}) + u_t(v_{(1,t-1)}, v'_t; v_t) + \mathbb{E}_{v_{(1,T)}} \left[ \sum_{\tau=t+1}^T u_{\tau}(v_{(1,\tau)}, v'_t(v_{\tau}) ; v_{\tau}) \right]$$

$$= \mathbb{E}_{v_{(1,T)}} \left[ \sum_{\tau=1}^{t-1} \left( \hat{u}_{\tau}(\text{bal}_{\tau}, v_{\tau} ; v_{\tau}) - s_{\tau} \right) + \hat{u}_t(\text{bal}_t, v'_t ; v_t) - s_t + \sum_{\tau=t+1}^T \left( \hat{u}_{\tau}(\text{bal}^{(t)}_{\tau}, v_{\tau} ; v_{\tau}) - s_{\tau}^{(t)} \right) \right],$$

where the last equation comes from replacing $\sum_{\tau=1}^T u_{\tau}$ according to (37).

Hence,

$$u_t(v_{(1,t)}; v_t) + U_t(v_{(1,t)}) - \left( u_t(v_{(1,t-1)}, v'_t; v_t) + U_t(v_{(1,t-1)}, v'_t) \right)$$

$$= \sum_{\tau=1}^{t-1} u_{\tau}(v_{(1,\tau)}; v_{\tau}) + u_t(v_{(1,t)}; v_t) + U_t(v_{(1,t)}) - \left( \sum_{\tau=1}^{t-1} u_{\tau}(v_{(1,\tau)}; v_{\tau}) + u_t(v_{(1,t-1)}, v'_t; v_t) + U_t(v_{(1,t-1)}, v'_t) \right)$$

$$= \mathbb{E}_{v_{(1,T)}} \left[ \sum_{\tau=1}^{t-1} \left( \hat{u}_{\tau}(\text{bal}_{\tau}, v_{\tau} ; v_{\tau}) - s_{\tau} \right) + \hat{u}_t(\text{bal}_t, v'_t ; v_t) - s_t + \sum_{\tau=t+1}^T \left( \hat{u}_{\tau}(\text{bal}^{(t)}_{\tau}, v_{\tau} ; v_{\tau}) - s_{\tau}^{(t)} \right) \right]$$

$$- \mathbb{E}_{v_{(1,T)}} \left[ \sum_{\tau=1}^{t-1} \left( \hat{u}_{\tau}(\text{bal}_{\tau}, v_{\tau} ; v_{\tau}) - s_{\tau} \right) + \hat{u}_t(\text{bal}_t, v'_t ; v_t) - s_t + \sum_{\tau=t+1}^T \left( \hat{u}_{\tau}(\text{bal}^{(t)}_{\tau}, v_{\tau} ; v_{\tau}) - s_{\tau}^{(t)} \right) \right]$$

$$= \mathbb{E}_{v_{(1,T)}} \left[ \hat{u}_t(\text{bal}_t, v'_t ; v_t) - \hat{u}_t(\text{bal}_t, v'_t ; v_t) + \sum_{\tau=t+1}^T \left( \hat{u}_{\tau}(\text{bal}_{\tau}, v_{\tau} ; v_{\tau}) - \hat{u}_{\tau}(\text{bal}^{(t)}_{\tau}, v_{\tau} ; v_{\tau}) - s_{\tau} + s_{\tau}^{(t)} \right) \right]$$

$$= \sum_{\tau=t+1}^T \mathbb{E}_{v_{(1,\tau-1)}} \left[ \mathbb{E}_{v_{\tau}} \left( \hat{u}_{\tau}(\text{bal}_{\tau}, v_{\tau} ; v_{\tau}) - \hat{u}_{\tau}(\text{bal}^{(t)}_{\tau}, v_{\tau} ; v_{\tau}) \right) - s_{\tau} + s_{\tau}^{(t)} \right] + \hat{u}_t(\text{bal}_t, v_t ; v_t) - \hat{u}_t(\text{bal}_t, v'_t ; v_t).$$

Note that by (11),

$$\mathbb{E}_{v_{\tau}} \left( \hat{u}_{\tau}(\text{bal}_{\tau}, v_{\tau} ; v_{\tau}) - \hat{u}_{\tau}(\text{bal}^{(t)}_{\tau}, v_{\tau} ; v_{\tau}) \right) - s_{\tau} + s_{\tau}^{(t)} = 0,$$

and by (10),

$$\hat{u}_t(\text{bal}_t, v_t ; v_t) - \hat{u}_t(\text{bal}_t, v'_t ; v_t) \geq 0.$$
Therefore we have
\[ u_t(v_{(1,t)}; v_t) + U_t(v_{(1,t)}) - \left( u_t(v_{(1,t-1)}, v'_t; v_t) + U_t(v_{(1,t-1)}, v'_t) \right) \geq 0, \]
which is exactly the IC constraint (4) for \( M \), namely, (10)(11) \( \Rightarrow \) (4).

\[ \square \]

**Proof of Lemma 3.7.** From the balance update formula (7), we have
\[ \sum_{\tau=1}^{T} d_\tau(\text{bal}_\tau, v_\tau) = \sum_{\tau=1}^{T} \left( \text{bal}_{\tau+1} - \text{bal}_\tau + s_\tau(\text{bal}_\tau) \right) = \text{bal}_{T+1} - \text{bal}_1 + \sum_{\tau=1}^{T} s_\tau(\text{bal}_\tau). \]

Then sum up (12) from 1 to \( T \),
\[ \sum_{\tau=1}^{T} u_\tau(\text{bal}_\tau, v_\tau; v_\tau) \geq \sum_{\tau=1}^{T} d_\tau(\text{bal}_\tau, v_\tau) = \text{bal}_{T+1} - \text{bal}_1 + \sum_{\tau=1}^{T} s_\tau(\text{bal}_\tau) \geq \sum_{\tau=1}^{T} s_\tau(\text{bal}_\tau), \]
where the last inequality is from the fact that bal_1 = 0 (by definition), and bal_{T+1} ≥ 0 (d_\tau is non-negative, and s_\tau is no more than bal_\tau). Then by (37),
\[ \forall v_{(1,T)} \in \mathcal{V}_{(1,T)}, \sum_{\tau=1}^{T} u_\tau(v_{(1,\tau)}; v_\tau) = \sum_{\tau=1}^{T} \hat{u}_\tau(\text{bal}_\tau, v_\tau; v_\tau) - \sum_{\tau=1}^{T} s_\tau(\text{bal}_\tau) \geq 0 \Rightarrow (3). \]

Thus \( M \) is IR. In other words, (12) \( \Rightarrow \) (3).

\[ \square \]

A.5 Proof of Lemma 3.7

**Proof of Lemma 3.7.** Without loss of generality\(^8\) suppose that
\[ \forall t < T, v_{(1,t)} \in \mathcal{V}_{(1,t)}, u_t(v_{(1,t)}; v_t) = 0. \]

Consider the following iterative procedure that constructs a sequence of direct mechanisms, \( M^{(1)}, \ldots, M^{(T)} \), where \( M^{(t)} = \langle x_{(1,T)}^{(t)}, p_{(1,T)}^{(t)} \rangle \). We start with \( M^{(1)} = M \), and end up with \( M^{(T)} = M' \) as the desirable symmetric direct mechanism.

Given that we have constructed \( M^{(1)}, \ldots, M^{(t)} \), we define the equivalence relation \( \sim_{(t)} \) between any two histories as follows,
\[ v_{(1,t')} \sim_{(t)} v'_{(1,t')} \iff \text{UTL}(M^{(t)}|v_{(1,t')}) = \text{UTL}(M^{(t)}|v'_{(1,t')}), \]
and denote the expected revenue of future stages as \( \pi^{(t)}_{v_{(1,t')}}(v_{(1,t')}) \). Formally,
\[ \pi^{(t)}_{v_{(1,t')}}(v_{(1,t')}) = \mathbb{E}_{v_{(t'+1,T)}} \left[ \sum_{\tau=t'+1}^{T} p^{(t)}_{v_{(1,t)}}(v_{(1,\tau)}) \right]. \]

If \( T = 1 \), \( M = M^{(1)} = M' \).
For \( T > 1 \), we construct \( M^{(t+1)} \) from \( M^{(t)} \) as follows.

\(^8\)For \( M \) which is not that case, construct another \( \overline{M} \) with the same allocation rules, but redefine payment rules as: for \( t < T, \overline{p}_{\tau}(v_{(1,t)}) = \pi_{\tau}(v_{(1,t)}); v_t \); for \( t = T, \overline{p}_{\tau}(v_{(1,T)}) = \pi_{\tau}(v_{(1,T)}); v_T - \text{UTL}(M|v_{(1,T)}) \). Note that \( \overline{M} \) is IC and IR, and has the same allocation, overall utility, and overall revenue for any \( v_{(1,T)}. \)
• For any \( \tau \leq t \), \( v_{(1,\tau)} \in V_{(1,\tau)} \), \( M^{(t+1)} \) is the same with \( M^{(t)} \),
\[
x_{\tau}^{(t+1)}(v_{(1,\tau)}) = x_{\tau}^{(t)}(v_{(1,\tau)}), \quad p_{\tau}^{(t+1)}(v_{(1,\tau)}) = p_{\tau}^{(t)}(v_{(1,\tau)}).
\] (40)

• For any \( \tau \geq t + 1 \), \( v_{(1,\tau)} \in V_{(1,\tau)} \), \( M^{(t+1)} \) simulates what \( M^{(t)} \) does at history \( v_{(1,t)}^{*} \),
\[
x_{\tau}^{(t+1)}(v_{(1,\tau)}) = x_{\tau}^{(t)}(v_{(1,t)}^{*}, v_{(t+1,\tau)}), \quad p_{\tau}^{(t+1)}(v_{(1,\tau)}) = p_{\tau}^{(t)}(v_{(1,t)}^{*}, v_{(t+1,\tau)}),
\] (41)
where \( v_{(1,t)}^{*} = h_{t}^{*}(v_{(1,t)}) \in V_{(1,t)} \) is the selected representative history for the equivalence class \([v_{(1,t)}]_{(t)}\) defined by the equivalence relation (39), in order to satisfy
– the representative history \( v_{(1,t)}^{*} \) is also in this equivalence class \([v_{(1,t)}]_{(t)}\),
\[
\text{UTL}(M^{(t)}|v_{(1,t)}^{*}) = \text{UTL}(M^{(t)}|v_{(1,t)});
\] (42)
– the selection of the representative history is unique within each equivalence class,
\[
v_{(1,t)} \sim (t) v_{(1,t)}' \implies h_{t}^{*}(v_{(1,t)}) = h_{t}^{*}(v_{(1,t)}');
\] (43)
– the seller’s revenue at the representative history \( v_{(1,t)}^{*} \) is no less than the expected revenue in this equivalence class,
\[
\pi_{t}^{(t)}(v_{(1,t)}^{*}) \geq E_{v_{(1,t)}'} \left[ \pi_{t}^{(t)}(v_{(1,t)}') \big| v_{(1,t)}' \sim (t) v_{(1,t)} \right].
\] (44)

We finally verify that the so constructed mechanism \( M' = M^{(T)} \) satisfies the desirable properties by induction: (i) \( M' \) is symmetric; (ii) \( M' \) guarantees the same buyer utility as \( M \) does, and generates revenue no less than \( M \) does.

For the inductive cases, we need to prove the following two properties, where [1] is identical with property (i) when \( t = T \), and property (ii) is implied if [2] holds for \( t = T \).

1. \( M^{(t)} \) is symmetric for any pair of equivalent histories of length at most \( t - 1 \), i.e.,
\[
\forall \tau \leq t - 1, v_{(1,\tau)}, v_{(1,\tau)}' \in V_{(1,\tau)}, \ s.t. \ v_{(1,\tau)} \sim (t) v_{(1,\tau)}',
\implies \forall \tau < t' \leq T, v_{(\tau+1,t')} \in V_{(\tau+1,t')},
\left\{ \begin{array}{c}
x_{\tau}^{(t)}(v_{(1,t')}) = x_{\tau}^{(t)}(v_{(1,\tau)}'), v_{(\tau+1,t')}
\end{array} \right.
\] (45)

2. \( \text{UTL}(M^{(t)}) = \text{UTL}(M^{(t-1)}), \ \text{REV}(M^{(t)}) \geq \text{REV}(M^{(t-1)}) \).

Base case: for \( t = 1 \), the properties hold trivially.

Assume by induction that the above two properties [1] and [2] hold for \( t = \overline{2} < T \), we prove that for \( t = \overline{3} + 1 \).

Property [2] holds. We first show that \( M^{(\overline{3}+1)} \) guarantees the same buyer utility as \( M^{(\overline{3})} \) does.
For any history \( v_{(1,\overline{3})} \in V_{(1,\overline{3})} \), by the definition of the conditional expected utility,
\[
\text{UTL}(M^{(\overline{3}+1)}|v_{(1,\overline{3})}) = E_{v_{(1,\overline{3})}} \left[ \sum_{\tau = 1}^{T} u_{\tau}^{(\overline{3}+1)}(v_{(1,\tau)}; v_{\tau}) \right]
\]
\[
= \sum_{\tau = 1}^{\overline{3}} u_{\tau}(v_{(1,\tau)}; v_{\tau}) + E_{v_{(1,\overline{3})}} \left[ \sum_{\tau = \overline{3}+1}^{T} u_{\tau}^{(\overline{3}+1)}(v_{(1,\tau)}; v_{\tau}) \right]
\]
\[
= \sum_{\tau = 1}^{\overline{3}} u_{\tau}^{(\overline{3}+1)}(v_{(1,\tau)}; v_{\tau}) + U_{\overline{3}}^{(\overline{3}+1)}(v_{(1,\overline{3})}).
\] (46)
Recall that by assumption (38), for $t < T$,

$$x_t(v_{(1,t)}) \cdot v_t - p_t(v_{(1,t)}) = u_t(v_{(1,t)}; v_t) = 0.$$  

According to construction (40) and (41), for $\tau \in \mathbb{Z}$, it is not hard to see that for $\tau \leq T$, there exists some $v'_{(1,\tau-1)} \in V_{(1,\tau-1)}$ such that,

$$u^*_\tau(v_{(1,\tau)}; v_\tau) = x^*_\tau(v_{(1,\tau)}; v_\tau).$$

Combining (47), we have

$$\text{ULTL}(M^{(\tau+1)}|v_{(1,\tau)}) = U^{(\tau+1)}_{\tau}(v_{(1,\tau)}).$$

By construction (41), for $\tau \geq T + 1$, $v'_{(1,\tau)} = h^*_\tau(v_{(1,\tau)})$ being the representative history,

$$u^*_\tau(v_{(1,\tau)}; v_\tau) = x^*_\tau(v_{(1,\tau)}; v_\tau) = x^*_\tau(v'_{(1,\tau)}; v_\tau) = x^*_\tau(v_{(1,\tau)}, v'_{(1,\tau)}; v_\tau) \cdot v_\tau - p^*_\tau(v'_{(1,\tau)}, v_{(1,\tau)}) = u^*_\tau(v_{(1,\tau)}, v_{(1,\tau)}; v_\tau).$$

Hence

$$U^*_\tau(v_{(1,\tau)}; v_\tau) = \left\{ \begin{array}{ll}
E_{v_{(1,\tau)}} \left[ \sum_{\tau=1}^{T} u^*_\tau(v_{(1,\tau)}; v_\tau) \right] \\
E_{v_{(1,\tau)}} \left[ \sum_{\tau=1}^{T} u^*_\tau(v_{(1,\tau)}, v'_{(1,\tau)}; v_\tau) \right] = U^*_{\tau}(v_{(1,\tau)}).
\end{array} \right.$$

Similar with (47), we have

$$\text{ULTL}(M^{(\tau)}|v^*_{(1,\tau)}) = U^*_{\tau}(v^*_{(1,\tau)}).$$

Combine (47) and the two equations above,

$$\text{ULTL}(M^{(\tau+1)}|v_{(1,\tau)}) = U^{(\tau+1)}_{\tau}(v_{(1,\tau)}) = U^*_{\tau}(v^*_{(1,\tau)}) = \text{ULTL}(M^{(\tau)}|v^*_{(1,\tau)}).$$

Furthermore, by construction (42), we conclude that

$$\forall v_{(1,\tau)} \in V_{(1,\tau)}, \text{ULTL}(M^{(\tau+1)}|v_{(1,\tau)}) = \text{ULTL}(M^{(\tau)}|v^*_{(1,\tau)}) = \text{ULTL}(M^{(\tau)}|v_{(1,\tau)}).$$

Then (48) implies $\text{ULTL}(M^{(\tau+1)}) = \text{ULTL}(M^{(\tau)})$, i.e.,

$$\text{ULTL}(M^{(\tau+1)}) = E_{v_{(1,\tau)}} \left[ \text{ULTL} \left( M^{(\tau+1)}|v_{(1,\tau)} \right) \right] = E_{v_{(1,\tau)}} \left[ \text{ULTL} \left( M^{(\tau)}|v_{(1,\tau)} \right) \right] = \text{ULTL}(M^{(\tau)}).$$

We then show that $M^{(\tau+1)}$ generates revenue no less than $M^{(\tau)}$ does.

Recall that by construction (40), for $\tau \leq T$,

$$p^*_\tau(v_{(1,\tau)}) = p^*_\tau(v_{(1,\tau)}),$$

and by construction (41), for $\tau \geq T + 1$,

$$p^*_\tau(v_{(1,\tau)}) = p^*_\tau(v^*_{(1,\tau)}, v_{(1,\tau)}).$$
Then according to the definition of $\pi$, we have

$$\pi^{(T+1)}_\pi(v_{(1,\mathbb{I})}) = \mathbb{E}_{v_{(T+1,\mathbb{T})}} \left[ \sum_{\tau = \mathbb{T}+1}^T p^{(T+1)}_{\tau}(v_{(1,\tau)}) \right]$$

$$= \mathbb{E}_{v_{(T+1,\mathbb{T})}} \left[ \sum_{\tau = \mathbb{T}+1}^T p^{(T)}_{\tau}(v^*_\pi, v_{(\mathbb{T}+1,\tau)}) \right] = \pi^{(T)}_\pi(v^*_\pi).$$

Moreover, by definition,

$$\text{REV}(M^{(T+1)}) = \mathbb{E}_{v_{(1,\mathbb{T})}} \left[ \sum_{\tau = 1}^T p^{(T+1)}_{\tau}(v_{(1,\tau)}) \right]$$

$$= \mathbb{E}_{v_{(1,\mathbb{T})}} \left[ \sum_{\tau = 1}^T p^{(T+1)}_{\tau}(v_{(1,\tau)}) + \mathbb{E}_{v_{(T+1,\mathbb{T})}} \left[ \sum_{\tau = \mathbb{T}+1}^T p^{(T+1)}_{\tau}(v_{(1,\tau)}) \right] \right]$$

$$= \mathbb{E}_{v_{(1,\mathbb{T})}} \left[ \sum_{\tau = 1}^T p^{(T)}_{\tau}(v_{(1,\tau)}) + \pi^{(T+1)}_\pi(v_{(1,\mathbb{T})}) \right]$$

$$= \mathbb{E}_{v_{(1,\mathbb{T})}} \left[ \sum_{\tau = 1}^T p^{(T)}_{\tau}(v_{(1,\tau)}) + \pi^{(T)}_\pi(h^*_\pi(v_{(1,\mathbb{T})})) \right].$$

(49)

For any fixed $v_{(1,\mathbb{T})} \in V_{(1,\mathbb{T})}$, by the property (44) of the representative history $v^*_\pi = h^*_\pi(v_{(1,\mathbb{T})})$, we have

$$\mathbb{E}_{v_{(1,\mathbb{T})}} \left[ \pi^{(T)}_\pi(h^*_\pi(v_{(1,\mathbb{T})})) \right] \geq \mathbb{E}_{v_{(1,\mathbb{T})}} \left[ \mathbb{E}_{v'_{(1,\mathbb{T})}} \left[ \pi^{(T)}_\pi(v'_{(1,\mathbb{T})}) \right] \right]$$

$$= \mathbb{E}_{v'_{(1,\mathbb{T})}} \left[ \mathbb{E}_{v_{(1,\mathbb{T})}} \left[ \pi^{(T)}_\pi(v_{(1,\mathbb{T})}) \right] \right]$$

$$= \mathbb{E}_{v_{(1,\mathbb{T})}} \left[ \pi^{(T)}_\pi(v_{(1,\mathbb{T})}) \right] = \pi^{(T)}_\pi(v_{(1,\mathbb{T})}).$$

(50)

By similar argument with (49),

$$\text{REV}(M^{(T)}) = \mathbb{E}_{v_{(1,\mathbb{T})}} \left[ \sum_{\tau = 1}^T p^{(T)}_{\tau}(v_{(1,\tau)}) + \pi^{(T)}_\pi(v_{(1,\mathbb{T})}) \right].$$

Then combining (49) and (50),

$$\text{REV}(M^{(T+1)}) - \text{REV}(M^{(T)}) = \mathbb{E}_{v_{(1,\mathbb{T})}} \left[ \pi^{(T)}_\pi(h^*_\pi(v_{(1,\mathbb{T})})) \right] - \mathbb{E}_{v_{(1,\mathbb{T})}} \left[ \pi^{(T)}_\pi(v_{(1,\mathbb{T})}) \right] \geq 0.$$

In other words, $\text{REV}(M^{(T+1)}) \geq \text{REV}(M^{(T)})$.

Property (II) holds.

Since the goal (45) has mathematically identical formulation on $x$ and $p$, we prove (45) for $x$ only.

Consider the following cases:
For $t' \leq \xi$, it follows from the construction (40) and the inductive assumption (45) as follows,

$$\forall \tau < t' \leq \xi, v(1,t') \in V(1,t'), v(1,\tau) \sim (\xi) v(1,\tau),$$

$$x'_T(\tau + 1) (v(1,t'), v'_T(1,\tau)) = x'_T(v(1,t'), v(1,\tau), v(\tau + 1, t')) = x'_T(\tau + 1) (v'_T(1,\tau), v(\tau + 1, t')).$$

Note that the second “=” comes from the inductive assumption (45) on $t = \xi$, where $\tau \leq t - 1 = \xi - 1$ is required.

For $t' > \xi$ and $\tau < \xi$, we show that $v(1,\tau) \sim (\xi + 1) v'_T(1,\tau)$ implies

$$v(1,\tau) \sim (\xi + 1) \left(v'_T(1,\tau), v(\tau + 1, \tau)\right),$$

and it reduces to the following $\tau = \xi$ case then.

Recall by (48), $\text{UTL}(M(\xi + 1)|v(1,\xi)) = \text{UTL}(M(\xi)|v(1,\xi))$. Hence by the definition of conditional expected utility,

$$\text{UTL}(M(\xi)|v(1,\tau)) = \mathbb{E}_{v(\tau + 1, \tau)} \left[\text{UTL}(M(\xi + 1)|v(1,\xi))\right]$$

$$= \mathbb{E}_{v(\tau + 1, \tau)} \left[\text{UTL}(M(\xi)|v(1,\xi))\right] = \text{UTL}(M(\xi)|v(1,\tau)).$$

Then $v(1,\tau) \sim (\xi + 1) v'_T(1,\tau)$ implies $v(1,\tau) \sim (\xi) v'_T(1,\tau)$, since

$$\text{UTL}(M(\xi)|v(1,\tau)) = \text{UTL}(M(\xi + 1)|v(1,\tau))$$

$$= \text{UTL}(M(\xi + 1)|v'_T(1,\tau)) = \text{UTL}(M(\xi)|v'_T(1,\tau)).$$

By inductive assumption (45) (on $t = \xi$), the submechanisms of $M(\xi)$ at history $v(1,\tau)$ and $v'_T(1,\tau)$ are identical. Therefore for any $v(\tau + 1, T) \in V(\tau + 1, T)$,

$$\text{UTL}(M(\xi)|v(1,T)) = \text{UTL}(M(\xi)|v'_T(1,T), v(\tau + 1, T)).$$

It then suggests that $v(1,\tau)$ and $(v'_T(1,\tau), v(\tau + 1, T))$ are equivalent with respect to the equivalence relation $\sim (\xi)$, i.e.,

$$\text{UTL}(M(\xi)|v(1,\tau)) = \mathbb{E}_{v(\tau + 1, T)} \left[\text{UTL}(M(\xi)|v(1,T))\right]$$

$$= \mathbb{E}_{v(\tau + 1, T)} \left[\text{UTL}(M(\xi)|v'_T(1,T), v(\tau + 1, T))\right] = \text{UTL}(M(\xi)|v'_T(1,T), v(\tau + 1, T)).$$

Again, by (48),

$$\text{UTL}(M(\xi + 1)|v(1,\xi)) = \text{UTL}(M(\xi)|v(1,\xi))$$

$$= \text{UTL}(M(\xi)|v'_T(1,\tau), v(\tau + 1, T)) = \text{UTL}(M(\xi + 1)|v'_T(1,\tau), v(\tau + 1, T)).$$

Hence we have proved (51), and it reduces to the following $\tau = \xi$ case.

---

9 Note that the demonstration based on the inductive assumption on $t = \xi$ only works for $\tau \leq t - 1 = \xi - 1$, which is certainly true when the integer $\tau < \xi$.

10 By assumption (38) and construction (40) and (41), $\text{UTL}(M(\xi)|v(1,\xi)) = u_T^M(v(1,\xi); v_T)$, which equals to $u_T^M(v(1,\tau), v(\tau + 1, T); v_T)$ because these two submechanisms are identical.
• For \( t' > \Xi \) and \( \tau = \Xi \), \( \forall v(1,\Xi) \sim (\Xi+1) v'(1,\Xi) \), by \((48)\), we have \( v(1,\Xi) \sim (\Xi) v'(1,\Xi) \).

Then by the uniqueness \((43)\) of the representative history, we have
\[
h^*_\Xi(v(1,\Xi)) = h^*_\Xi(v'(1,\Xi)) = v^*_\Xi.
\]

According to construction \((41)\), for any \( v(\Xi+1,t') \in V(\Xi+1,t') \),
\[
x^*(t') (v(1,\Xi),v(\Xi+1,t')) = x^*(t) (v(1,\Xi),v(\Xi+1,t')) = x^*(t+1) (v(1,\Xi),v(\Xi+1,t')).
\]

Therefore, we have finished the proof of the inductive properties \((1)\) and \((2)\) for \( t = \Xi + 1 \).

Hence \( M^{(\Xi+1)} \) is symmetric for any pair of equivalent histories of length at most \( \Xi \). Therefore, by induction, \( M' = M^{(T)} \) is symmetric and,
\[
\begin{align*}
\text{Utl}(M') &= \text{Utl}(M^{(T)}) = \cdots = \text{Utl}(M^{(1)}) = \text{Utl}(M) \\
\text{Rev}(M') &\geq \text{Rev}(M^{(T)}) \geq \cdots \geq \text{Rev}(M^{(1)}) \geq \text{Rev}(M).
\end{align*}
\]

In particular, by \((40)\) and \((41)\), if \( M \) is deterministic, \( M' \) is also deterministic. \( \square \)

A.6 Proof of Lemma 3.10

Proof of Lemma 3.10

To prove that \( B^{q,y} \) is a BAM, we must verify the following,

• \( B^{q,y} \) is a valid bank account mechanism:
  
  - \( B^{q,y} = (z(1,T), q(1,T), \text{bal}(1,T), d(1,T), s(1,T)) \) are defined on their domains.
  - \( q_t \) and \( d_t \) are always non-negative.
  - \( s_t(\text{bal}_t) \) is independent with \( v_t \), and no more than \( \text{bal}_t \).
  - The balance is updated according to formula \((7)\).

• \( B^{q,y} \) is a BAM: IC and IR constraints \((10)-(12)\) are satisfied.

We now verify these statements one by one.

• Since \( M \) is symmetric, then by definition,
\[
\forall t \in [T], v(1,t), v'(1,t) \in V(1,t), \text{Utl}(M|v(1,t-1)) = \text{Utl}(M|v'(1,t-1)) \implies x_t(v(1,t)) = x_t(v'(1,t-1), v_t).
\]

Combining with the construction of \( \text{bal}_t \) and \( z_t \) (see \((14)\) and \((15)\)), we have,
\[
\begin{align*}
\text{bal}_t(v(1,t-1)) &= \text{bal}_t(v(1,t-1)) \\
\implies \text{Utl}(M|v(1,t-1)) &= \text{bal}_t(v(1,t-1)) + \mu_t = \text{bal}_t(v'(1,t-1)) + \mu_t = \text{Utl}(M|v'(1,t-1)) \\
\implies z_t(\text{bal}_t(v(1,t-1)), v_t) &= x_t(v(1,t)) = x_t(v'(1,t-1), v_t) = z_t(\text{bal}_t(v(1,t-1), v_t),
\end{align*}
\]

which then suggests that \( z_t(\text{bal}_t, v_t) \) is a valid function, since for any input \( (\text{bal}_t, v_t) \), \( z_t \) is never assigned two different values.

Then it is also true for \( q_t \) and \( d_t \), since they are constructed based on \( z_t \) (see \((16)\) and \((17)\)).

Also note that by construction \((14)\), \( \text{bal}_t \) is always non-negative. Hence \( z_t, q_t, d_t, \) and \( s_t \) are defined on their domains (the first parameter is the balance — a non-negative real, and the second parameter is the value \( v_t \)).
• By the construction of \( z_t \), \( z_t(bal_t, v_t) = y_t(v(1, t)) \). Then by the statement \( 19 \) in Lemma 3.10, \( z_t(bal_t, v_t) = y_t(v(1, t)) = x_t(v(1, t)) \). Since the symmetric direct mechanism \( M \) satisfies IC condition \( 4 \), \( x_t \) must be (weakly) increasing in \( v_t \).

By the construction of \( q_t \), \( 16 \), we have

\[
q_t(bal_t, v_t) = z_t(bal_t, v_t) \cdot v_t - \int_0^{v_t} z_t(bal_t, v) dv \geq 0.
\]

By the construction of \( d_t \), \( 17 \), we have

\[
d_t(bal_t, v_t) = \hat{u}_t(bal_t, v_t; v_t) = \int_0^{v_t} z_t(bal_t, v) dv \geq 0.
\]

Hence both of \( q_t \) and \( d_t \) are always non-negative.

• We claim that according to the construction of \( B^{u,v} \), \( M \) is IC implies \( \partial s_t(bal_t)/\partial v_t = 0 \).

Formally, from the construction of \( s_t \), \( 18 \),

\[
s_t(bal_t) = bal_t + d_t(bal_t, v_t) - bal_{t+1}.
\]

By plugging the construction of \( d_t \) and \( bal_{t+1} \) into the formula above, we have

\[
s_t(bal_t) = bal_t + \hat{u}_t(bal_t, v_t; v_t) - g_t(v(1, t)) + \mu_t
\]

\[
= bal_t + \hat{u}_t(bal_t, v_t; v_t) - (UTL(M|v(1, t)) - \mu_t),
\]

where the second “=” is from the statement \( 19 \).

Then by taking partial derivative with respect to \( v_t \) on both sides, we have

\[
\frac{\partial s_t(bal_t)}{\partial v_t} = \frac{\partial bal_t}{\partial v_t} + \frac{\partial \hat{u}_t(bal_t, v_t; v_t)}{\partial v_t} - \frac{\partial (UTL(M|v(1, t)) - \mu_t)}{\partial v_t}.
\]

(52)

By the construction of \( bal_t(v(1, t-1)) \), \( 14 \), \( bal_t \) is independent with \( v_t \), hence \( \partial bal_t/\partial v_t = 0 \). Since \( \hat{u}_t(bal_t, v_t; v_t) = \int_0^{v_t} z_t(bal_t, v) dv, \partial \hat{u}_t(bal_t, v_t; v_t)/\partial v_t = z_t(bal_t, v_t) \).

Note that for \( \tau \leq t - 1, u_{\tau}(bal_{\tau}, v_{\tau}; v_{\tau}) \) is independent with \( v_t \), and \( \mu_t = \inf_{\nu(1, t)} g_t(v(1, t)) \) is also independent with \( v_t \). Hence

\[
\frac{\partial (UTL(M|v(1, t)) - \mu_t)}{\partial v_t} = \frac{\partial (\sum_{\tau=1}^t u_{\tau}(bal_{\tau}, v_{\tau}; v_{\tau}) + U_t(v(1, t)) - \mu_t)}{\partial v_t}
\]

\[
= \frac{\partial (u_t(bal_t, v; v_t) + U_t(v(1, t)))}{\partial v_t},
\]

which then by \( 6 \) equals to \( x_t(v(1, t)) \).

Now \( 52 \) reduces to the follows,

\[
\frac{\partial s_t(bal_t)}{\partial v_t} = \frac{\partial bal_t}{\partial v_t} + \frac{\partial \hat{u}_t(bal_t, v_t; v_t)}{\partial v_t} - \frac{\partial (UTL(M|v(1, t)) - \mu_t)}{\partial v_t}
\]

\[
= 0 + z_t(bal_t, v_t) - x_t(v(1, t)) = 0.
\]
Recall that by construction, \( z_t(\text{bal}_t, v_t) = x_t(v_{(1:t)}) \).

Hence \( s_t \) constructed as (18) is a function of \( \text{bal}_t \). By letting \( v_t = 0 \), we show that \( s_t(\text{bal}_t) \) is no more than \( \text{bal}_t \),

\[
 s_t(\text{bal}_t) = \text{bal}_t + d_t(\text{bal}_t, 0) - \text{bal}_{t+1}(\text{bal}_t, 0) \leq \text{bal}_t + d_t(\text{bal}_t, 0) = \text{bal}_t,
\]

where \( d_t(\text{bal}_t, 0) = \hat{u}_t(\text{bal}_t, 0; 0) = 0 \).

- The balance update formula (7) is directly implied by the construction of \( s_t \), (18).

- As we mentioned previously, \( z_t(\text{bal}_t, v_t) = x_t(v_{(1:t)}) \) is weakly increasing in \( v_t \). Then by Envelope theorem, the construction of \( q_t \), (16), implies the in-stage IC condition (10).

- According to the construction of \( B^{g,y} \), the fact that \( s_t(\text{bal}_t) \) is independent with \( v_t \) implies another IC condition (11). Formally, by adopting the construction of \( s_t, \text{bal}_t, \) and \( d_t \), we have

\[
 s_t(\text{bal}_t) = \text{bal}_t + d_t(\text{bal}_t, v_t) - \text{bal}_{t+1} \\
= \text{UTL}(M|v_{(1:t-1)}) - \mu_{t-1} + \hat{u}_t(\text{bal}_t, v_t; v_t) - (\text{UTL}(M|v_{(1:t)}) - \mu_t) \\
= \text{UTL}(M|v_{(1:t-1)}) - \text{UTL}(M|v_{(1:t)}) + \hat{u}_t(\text{bal}_t, v_t; v_t) - \mu_{t-1} + \mu_t.
\]

Taking expectation over \( v_t \) on both sides, we have

\[
 s_t(\text{bal}_t) = \mathbb{E}_{v_t}[s_t(\text{bal}_t)] \\
= \mathbb{E}_{v_t} \left[ \text{UTL}(M|v_{(1:t-1)}) - \text{UTL}(M|v_{(1:t)}) + \hat{u}_t(\text{bal}_t, v_t; v_t) - \mu_{t-1} + \mu_t \right] \\
= \mathbb{E}_{v_t} \left[ \hat{u}_t(\text{bal}_t, v_t; v_t) \right] - \mu_{t-1} + \mu_t,
\]

which implies the IC condition (11), because \( \mu_{t-1} \) and \( \mu_t \) are two constants with respect to \( \text{bal}_t \) and \( v_{(1:t)} \).

Note that the first “\( = \)” comes from the fact that \( s_t(\text{bal}_t) \) is independent with \( v_t \). For the second “\( = \)”, we use the fact that

\[
 \mathbb{E}_{v_t} \left[ \text{UTL}(M|v_{(1:t)}) \right] = \mathbb{E}_{v_{(t+1:T)}} \left[ \mathbb{E}_{v_{(t,T)}} \left[ \text{UTL}(M|v_{(1:T)}) \right] \right] \\
= \mathbb{E}_{v_{(t,T)}} \left[ \text{UTL}(M|v_{(1:T)}) \right] \\
= \text{UTL}(M|v_{(1:t-1)}).
\]

- Finally, the construction of \( d_t \), (17), directly implies the IR condition (12).

Note that \( M \) and \( B^{g,y} \) are equivalent in allocation rule, \( x_t(v_{(1:t)}) = z_t(\text{bal}_t, v_t) \). Hence \( \text{Eff}(B^{g,y}) = \text{Eff}(M) \). Then we show that they are also equivalent on buyer utility to conclude that they are also equivalent on revenue.

Recall that by definition,

\[
 \text{UTL}(B^{g,y}|v_{(1:T)}) = \sum_{\tau=1}^{T} z_\tau(\text{bal}_\tau, v_\tau) \cdot v_\tau - \sum_{\tau=1}^{T} q_\tau(\text{bal}_\tau, v_\tau) - \sum_{\tau=1}^{T} s_\tau(\text{bal}_\tau) \\
= \sum_{\tau=1}^{T} \left( (z_\tau(\text{bal}_\tau, v_\tau) \cdot v_\tau - q_\tau(\text{bal}_\tau, v_\tau)) - s_\tau(\text{bal}_\tau) \right) \\
= \sum_{\tau=1}^{T} \left( \hat{u}_\tau(\text{bal}_\tau, v_\tau; v_\tau) - s_\tau(\text{bal}_\tau) \right).
\]
Furthermore, by the construction of \( d_t \) and \( s_t \) (see (17) and (18)), we have

\[
U_{t|v(1,T)}(B_{g,y}|v(1,T)) = \sum_{\tau=1}^{T} \left( \hat{u}_{\tau}(\text{bal}_{\tau}, v_{\tau}) - s_{\tau}(\text{bal}_{\tau}) \right)
= \sum_{\tau=1}^{T} \left( d_{\tau}(\text{bal}_{\tau}, v_{\tau}) - (\text{bal}_{\tau} + d_{\tau}(\text{bal}_{\tau}, v_{\tau}) - \text{bal}_{\tau+1}) \right)
= \sum_{\tau=1}^{T} (\text{bal}_{\tau+1} - \text{bal}_{\tau})
= \text{bal}_{T+1} - \text{bal}_1.
\]

Note that by definition, \( \text{bal}_1 = 0 \), and

\[
\text{bal}_{T+1} = U_{t|v(1,T)}(M|v(1,T)) - \mu_T = U_{t|v(1,T)}(M|v(1,T)) - \min \left\{ 0, \inf_{v'(1,T)} U_{t|v'(1,T)}(M|v'(1,T)) \right\} = U_{t|v(1,T)}(M|v(1,T)),
\]

where \( \mu_T = 0 \), because \( U_{t|v'(1,T)}(M|v'(1,T)) \geq 0 \) by the IR condition (3).

Therefore,

\[
U_{t|v(1,T)}(B_{g,y}|v(1,T)) = \text{bal}_{T+1} - \text{bal}_1 = U_{t|v(1,T)}(M|v(1,T)),
\]

which then implies

\[
U_{t|v(1,T)}(B_{g,y}) = U_{t|v(1,T)}(M),
\]

\[
\text{REV}(B_{g,y}) = \text{REV}(M).
\]

A.7 Proof of Theorem 3.11

Before conducting the proof, we first give some intuitions behind the theorem as follows,

- \( g_t \) is convex in \( v_t \) and weakly increasing in \( v_t \): then \( g_t(v_{(1,t-1)}, v_t) \) is the utility function of some IC and IR stage mechanism for stage \( v_t \), by the Envelope theorem.

- \( g_t \) is consistent: then \( g_t(v_{(1,t)}) \) can be considered as the conditional utility \( U_{t|v(1,t)}(M|v(1,t)) \) plus a constant for some direct mechanism \( M \), i.e.,

\[
g_t(v_{(1,t)}) = U_{t|v(1,t)}(M|v(1,t)) + \text{constant}.
\]

- If \( g_t \) is further symmetric: then the direct mechanism \( M \) above is a symmetric direct mechanism.

- Finally, if \( y_t \) is the sub-gradient of \( g_t \) with respect to \( v_t \) and its range is \( X_t \): then \( y_t \) is exactly the allocation function of the \( M \).

In other words, \( B_{g,y} \) is a core BAM, if and only if there is a symmetric direct mechanism \( M \) such that \( g \) equals to its conditional utility plus some constants (one for each stage), and \( y \) is its allocation function.

The proof of Theorem 3.11 is similar to the proof of Lemma 3.10. The following corollary summarizes the important conclusions we have until the end of Section 3.
Corollary A.1. Let \( M \) be a revenue-optimal direct mechanism, subject to IC, IR, and with fixed buyer utility. One can construct a symmetric (Lemma 3.7), stage-wise IC and stage-wise IR (Lemma 2.2 to 2.4) mechanism \( M' \) that yields the same revenue and guarantees the same buyer utility. Furthermore, \( M' \) can be induced by a Core BAM (Theorem 3.5, Lemma 3.3 and 3.4).

Proof of Theorem 3.11. “\( \Rightarrow \)”: \( y \) is the sub-gradient of \( g \) and \( g \) is consistent, symmetric, and convex and increasing in \( v_t \), \( \Rightarrow B^{y,y} \) is a Core BAM.

By Definition 3.9 we need to show that \( B^{y,y} \) is a valid BAM. The following proof is almost the same with the proof of Lemma 3.10 so we only outline the differences here.

- Since \( g \) is symmetric,
  \[ g_{t-1}(v_{(1,t-1)}) = g_{t-1}(v'_{(1,t-1)}) \implies g_t(v_{(1,t)}) = g_t(v'_{(1,t-1)}, v_t), \]
  and \( y_t \) is the sub-gradient of \( g_t \) with respect to \( v_t \), so by the construction (14) of \( \text{bal}_t \)
  \[ \text{bal}_t(v_{(1,t-1)}) = \text{bal}_t(v'_{(1,t-1)}) \implies g_{t-1}(v_{(1,t-1)}) = g_{t-1}(v'_{(1,t-1)}) \]
  \[ \implies y_t(v_{(1,t)}) = \frac{\partial g_t(v_{(1,t)})}{\partial v_t} = \frac{\partial g_t(v'_{(1,t-1)}, v_t)}{\partial v_t} = y_t(v'_{(1,t-1)}, v_t), \]
  which then implies \( z_t(\text{bal}_t, v_t) = y_t(v_{(1,t)}) \) is a valid function on input \( (\text{bal}_t, v_t) \). Also note that the range of \( y_t \) (hence the range of \( z_t(\text{bal}_t, v_t) \)) is \( X_t \) by definition.

- \( z_t, q_t, d_t, \) and \( s_t \) are defined on their domains; \( q_t \) and \( d_t \) are always non-negative: the same with the proof of Lemma 3.10.

- By the construction of \( s_t, \text{bal}_t, d_t, \) and \( z_t \) (see (18), (14), (17) and (15)),
  \[ s_t(\text{bal}_t) = \text{bal}_t + d_t(\text{bal}_t, v_t) - \text{bal}_{t+1} \]
  \[ = g_{t-1}(v_{(1,t-1)}) - \mu_{t-1} + \int_0^{v_t} z_t(\text{bal}_t, v) dv - g_t(v_{(1,t)}) + \mu_t \]
  \[ = g_{t-1}(v_{(1,t-1)}) + \int_0^{v_t} y_t(v_{(1,t-1)}, v) dv - g_t(v_{(1,t)}) - \mu_{t-1} + \mu_t. \]

Taking partial derivative with respect to \( v_t \), we have
  \[ \frac{\partial s_t(\text{bal}_t)}{v_t} = 0 + y_t(v_{(1,t)}) - \frac{\partial g_t(v_{(1,t)})}{\partial v_t} - \frac{\partial g_t(v_{(1,t)})}{\partial v_t} = 0. \]
  Since \( y_t(v_{(1,t)}) = \frac{\partial g_t(v_{(1,t)})}{\partial v_t} \), we conclude that \( \partial s_t(\text{bal}_t)/\partial v_t = 0 \). Hence \( s_t \) constructed as (18) is a function of \( \text{bal}_t \).

- \( s_t(\text{bal}_t) \leq \text{bal}_t \); the balance update formula holds: the same with the proof of Lemma 3.10.

- Since \( z_t(\text{bal}_t, v_t) = y_t(v_{(1,t)}) \) is the sub-gradient of \( g_t(v_{(1,t)}) \) with respect to \( v_t \), which is also convex and weakly increasing in \( v_t \), then by the Envelope theorem, the construction (16) of \( q_t \) implies the in-stage IC condition (10).
• \( g_t \) is consistent implies that \( s_t(bal_t) - E_{\nu_t}[\hat{u}_t(bal_t, \nu_t; \nu_t)] \) is a constant, i.e.,

\[
s_t(bal_t) - E_{\nu_t}[\hat{u}_t(bal_t, \nu_t; \nu_t)] = E_{\nu_t}[s_t(bal_t) - d_t(bal_t, \nu_t)]
\]

\[
= E_{\nu_t}[(bal_t + d_t(bal_t, \nu_t) - bal_{t+1}) - d_t(bal_t, \nu_t)]
\]

\[
= E_{\nu_t}[bal_t - bal_{t+1}]
\]

\[
= E_{\nu_t}[g_{t-1}(v_{(1,t-1)}) - \mu_{t-1} - g_t(v_{(1,t)}) + \mu_t]
\]

\[
= g_{t-1}(v_{(1,t-1)}) - E_{\nu_t}[g_t(v_{(1,t)})] - \mu_{t-1} + \mu_t
\]

= constant,

where the last “=” is from the consistency of \( g_t \) and the fact that \( \mu_{t-1} \) and \( \mu_t \) are constants. It then implies the IC condition [11].

• The IR condition [12] holds: the same with the proof of Lemma 3.10.

“\( \Leftarrow \)” : \( B^{g,y} \) is a core BAM \( \implies y \) is the sub-gradient of \( g \) and \( g \) is consistent, symmetric, and convex and increasing in \( \nu_t \).

By definition, the Core BAM \( B^{g,y} \) is a valid BAM. Most of the arguments for the sufficient direction also apply to the necessary direction.

• By the construction [14]-[18],

\[
\frac{\partial s_t(bal_t)}{\nu_t} = y_t(v_{(1,t)}) - \frac{\partial g(v_{(1,t)})}{\nu_t},
\]

which must be 0, since \( s_t(bal_t) \) is invariant with respect to \( \nu_t \). Hence \( y_t \) is the sub-gradient of \( g_t \) with respect to \( \nu_t \), i.e.,

\[
y_t(v_{(1,t)}) = \frac{\partial g(v_{(1,t)})}{\nu_t}.
\]

• Since \( s_t(bal_t) = bal_t + d_t(bal_t, \nu_t) - bal_{t+1} \) is a valid function on input \( (bal_t) \), hence \( \forall v_{(1,t-1)}, v'_{(1,t-1)} \in \mathcal{V}_{(1,t-1)}, \nu_t \in \mathcal{V}_t, \)

\[
bal_t(v_{(1,t-1)}) = bal_t(v'_{(1,t-1)}) \implies s_t(bal_t(v_{(1,t-1)})) = s_t(bal_t(v'_{(1,t-1)})).
\]

By substituting \( s_t(bal_t) \), we have

\[
bal_t(v_{(1,t-1)}) + d_t(bal_t, \nu_t) - bal_{t+1}(v_{(1,t)}) = bal_t(v'_{(1,t-1)}) + d_t(bal_t, \nu_t) - bal_{t+1}(v'_{(1,t-1)}, \nu_t),
\]

which then implies

\[
bal_{t+1}(v_{(1,t)}) = bal_{t+1}(v'_{(1,t-1)}, \nu_t),
\]

since \( bal_t(v_{(1,t-1)}) = bal_t(v'_{(1,t-1)}) \).

Note that by the construction of \( bal_t \),

\[
bal_t(v_{(1,t-1)}) = bal_t(v'_{(1,t-1)}) \iff g_{t-1}(v_{(1,t-1)}) = g_{t-1}(v'_{(1,t-1)}),
\]

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and
\[
\text{bal}_{t+1}(v_{(1,t)}) = \text{bal}_{t+1}(v'_{(1,t-1)}, v_t) \iff g_t(v_{(1,t)}) = g_t(v'_{(1,t-1)}, v_t).
\]
Hence \(g_t\) must be symmetric, i.e., \(\forall v_{(1,t-1)}, v'_{(1,t-1)} \in V_{(1,t-1)}, v_t \in V_t\),
\[
g_{t-1}(v_{(1,t-1)}) = g_{t-1}(v'_{(1,t-1)}) \implies \text{bal}_t(v_{(1,t-1)}) = \text{bal}_t(v'_{(1,t-1)})
\implies \text{bal}_{t+1}(v_{(1,t)}) = \text{bal}_{t+1}(v'_{(1,t-1)}, v_t)
\implies g_t(v_{(1,t)}) = g_t(v'_{(1,t-1)}, v_t).
\]

- Since the range of \(z_t(\text{bal}_t, v_t)\) must be \(X_t\), so is the range of \(y_t(v_{(1,t)})\).
- By the in-stage IC condition \([10]\) and the Envelope theorem, \(y_t(v_{(1,t)})\) must be weakly increasing in \(v_t\), hence \(g_t\), being the integration of \(y_t\), must be convex and increasing in \(v_t\).
- Finally, the another IC condition \([11]\) implies that \(s_t(\text{bal}_t) - E_{v_t} [\hat{u}_t(\text{bal}_t, v_t; v_t)]\) must be a constant (with respect to \(\text{bal}_t\)). Hence,
\[
g_{t-1}(v_{(1,t-1)}) - E_{v_t} [g_t(v_{(1,t)})] = \text{bal}_t + \mu_{t-1} - E_{v_t} [\text{bal}_{t+1} + \mu_t]
= s_t(\text{bal}_t) - E_{v_t} [d_t(\text{bal}_t, v_t; v_t)] + \mu_{t-1} - \mu_t
= s_t(\text{bal}_t) - E_{v_t} [\hat{u}_t(\text{bal}_t, v_t; v_t)] + \mu_{t-1} - \mu_t
= \text{constant}.
\]

Therefore \(g_t\) is consistent.

A.8 Proof of Theorem 4.1

**Proof of Theorem 4.1** Assume by contradiction that there exists a core BAM, \(B\), such that,
\[
\text{Rev}(B) > \text{Rev}(M^S) + E_{v_{(1,T)}} \left[ \sum_{\tau=1}^{T} s_{\tau}(\text{bal}_\tau) \right].
\]
(53)

Then we construct a mechanism \(M\) that operates independently on each stage but generates strictly higher revenue than the optimal separate mechanism in at least one stage.

To ensure that the constructed mechanism is history-independent and IC-IR within each single stage. We create a fake bank account to mimic the behavior of the real bank account in \(B\). One can think of the balance \(\text{bal}_t^f\) of the fake bank account as simply a number, satisfying that the distribution of \(\text{bal}_t^f\) is the same as the distribution of the real balance (as a random-valued function of previous types).

Consider the following construction of \(M\) based on \(B\).
\[
x_t(v_t) = z_t(\text{bal}_t^f, v_t), \quad p_t(v_t) = q_t(\text{bal}_t^f, v_t),
\]
where in each stage \(t\), \(\text{bal}_t^f\) is a random variable defined by the following equations for \(\tau\) from 1 to \(t - 1\).
\[
\text{bal}_{\tau+1}^f = \text{bal}_{\tau}^f - s_{\tau}(\text{bal}_\tau^f) + d_{\tau}(\text{bal}_\tau^f, v_\tau^f), \quad v_\tau^f \sim F_\tau.
\]

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Thus $\text{bal}^*_t$ is independent of the buyer’s real types, and has the same distribution as $\text{bal}_t$. Note that $v_t$ is independent of the history, hence independent of $\text{bal}_t$ as well. Then

$$
\mathbb{E}_{v(t,i-1)}[q_t(\text{bal}_t, v_t)] = \mathbb{E}_{\text{bal}^*_t}[q_t(\text{bal}^*_t, v_t)] = \mathbb{E}_{\text{bal}^*_t}[p_t(v_t)],
$$

$$
\implies \text{REV}(M) = \sum_{\tau=1}^{T} \mathbb{E}_{\text{bal}^*_t,v_{\tau}}[p_{\tau}(v_{\tau})] = \text{REV}(B) - \mathbb{E}_{v(t,1)}[s_{\tau}(\text{bal}_\tau)] > \text{REV}(M^S)
$$

$$
\implies \exists \tau, s.t. \mathbb{E}_{\text{bal}^*_t,v_{\tau}}[p_{\tau}(v_{\tau})] > \rho_{\tau}, \quad (54)
$$

namely, at least in one stage $\tau$, $M$ generates strictly higher revenue than the optimal revenue for stage $\tau$, $\rho_{\tau}$.

By construction, $M$ is history-independent, and by (10)(12) and $q_t \geq 0$ (by the construction of core BAM), $M$ is also IC and IR within each single stage. Thus (54) contradicts the optimality of $M^S$.

### A.9 Proof of Lemma 4.2

**Proof of Lemma 4.2.** From previous definition and analysis, we have two upper bounds on $s_t(\text{bal}_t)$.

1. By Definition 3.1, $s_t(\text{bal}_t) \leq \text{bal}_t$.
2. By IC constraint (11) of BAM, we have

$$
\text{Val}_t = \mathbb{E}_{v_t}[\hat{u}_t(\text{bal}_t, v_t; v_t)] + s_t(0) \leq \mathbb{E}_{v_t}[\hat{u}_t(\text{bal}_t, v_t; v_t)] = \text{Val}_t.
$$

For any $\text{bal}_t$, the minimum of the above two bounds can be achieved if

$$
\mathbb{E}_{v_t}[\hat{u}(\text{bal}_t, v_t; v_t)] = \min \{ \text{bal}_t, \text{Val}_t \}
$$

$$
\hat{u}(0, v_t; v_t) = s_t(0) = 0.
$$

Furthermore, by IR constraint (12), we have,

$$
d_t(\text{bal}_t, v_t) \leq \hat{u}_t(\text{bal}_t, v_t; v_t) \leq \text{Val}_t.
$$

### A.10 Proof of Lemma 4.3

We prove the following stronger version of Lemma 4.3. Lemma 4.3 is then directly implied by Lemma 4.2 and Lemma A.2.

**Lemma A.2.** For deposit sequence $d(1,t)$ and spend sequence $s(1,t)$ with given bounds on each of the deposits and spends,

$$\forall \tau \in [t], \ 0 \leq d_{\tau} \leq 1 \cdot v_{\tau}, \ s_{\tau} \leq \text{Val}_{\tau},$$

and the corresponding balance defined with respect to the balance update formula,

$$\text{bal}_1 = 0, \ \text{bal}_{\tau+1} = \text{bal}_\tau + d_{\tau} - s_{\tau}, \ s_{\tau} \leq \text{bal}_\tau, \forall \tau \in [t].$$

We have the following upper bound on the cumulated spends,

$$\forall v(1,t) \in \mathcal{V}(1,t), \ \sum_{\tau=1}^{t} s_{\tau} \leq \sum_{\tau=1}^{t} s^*_t(\text{bal}^*_t). \quad (55)$$

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Proof of Lemma A.2. Let \( t' \leq t \) be the largest element such that \( s^*_t(\text{bal}^*_r) = \text{bal}^*_r \). Then according to the balance update formula,

\[
\sum_{\tau = 1}^{t} s^*_\tau = s^*_t + \sum_{\tau = t'+1}^{t} s^*_\tau + \sum_{\tau = 1}^{t-1} (\text{bal}^*_\tau + d^*_\tau - \text{bal}^*_r + 1)
\]

\[
= s^*_t + \sum_{\tau = t'+1}^{t} s^*_\tau + \text{bal}^*_1 - \text{bal}^*_t + \sum_{\tau = 1}^{t-1} d^*_\tau.
\]

Remember that by definition, \( \text{bal}^*_1 = \text{bal}^*_1 = 0 \), and the upper bounds on \( d^*_\tau \) and \( s^*_\tau \), \( \forall \tau \in [t] \), \( d^*_\tau \leq 1 \cdot v^*_\tau \), \( s^*_\tau \leq \text{bal}^*_\tau \).

Hence

\[
\sum_{\tau = 1}^{t} s^*_\tau = s^*_t + \sum_{\tau = t'+1}^{t} s^*_\tau + \text{bal}^*_1 - \text{bal}^*_t + \sum_{\tau = 1}^{t-1} d^*_\tau
\]

\[
\leq \text{bal}^*_t + \sum_{\tau = t'+1}^{t} \text{Val}_r + \text{bal}^*_1 - \text{bal}^*_t + \sum_{\tau = 1}^{t-1} 1 \cdot v^*_\tau
\]

\[
= \sum_{\tau = t'+1}^{t} \text{Val}_r + \sum_{\tau = 1}^{t-1} 1 \cdot v^*_\tau.
\]

(56)

Recall the construction of \( d^*_\tau \) and \( s^*_\tau \), \( \forall \tau \in [t] \), 
\[ d^*_\tau(\text{bal}^*_r, v^*_\tau) = 1 \cdot v^*_\tau, \quad s^*_\tau(\text{bal}^*_r) = \min\{\text{bal}^*_r, \text{Val}_r\}. \]

Combining the definition of \( t' \), we have

\[
\forall \tau > t', \quad s^*_t(\text{bal}^*_r) = \text{Val}_r.
\]

Therefore, apply the balance update formula again,

\[
\sum_{\tau = 1}^{t} s^*_\tau(\text{bal}^*_r) = s^*_t(\text{bal}^*_r) + \sum_{\tau = t'+1}^{t} s^*_\tau(\text{bal}^*_r) + \sum_{\tau = 1}^{t-1} (\text{bal}^*_\tau + d^*_\tau(\text{bal}^*_r, v^*_\tau) - \text{bal}^*_r + 1)
\]

\[
= \text{bal}^*_t + \sum_{\tau = t'+1}^{t} \text{Val}_r + \text{bal}^*_1 - \text{bal}^*_t + \sum_{\tau = 1}^{t-1} 1 \cdot v^*_\tau
\]

\[
= \sum_{\tau = t'+1}^{t} \text{Val}_r + \sum_{\tau = 1}^{t-1} 1 \cdot v^*_\tau,
\]

which, by (56), is the upper bound on \( \sum_{\tau = 1}^{t} s^*_\tau \).

Thus we conclude that

\[
\sum_{\tau = 1}^{t} s^*_\tau(\text{bal}^*_r) \geq \sum_{\tau = 1}^{t} s^*_\tau.
\]

\( \square \)
A.11 Proof of Lemma 4.4

Proof of Lemma 4.4. We prove that the constructed $B$ (by Mechanism 1) is a BAM.

With parameter $\theta = 3s_t(bal_t) = 3\min\{bal_t, Val_t/3\} \leq Val_t$, $(x^P_t(\theta, \cdot), p^P_t(\theta, \cdot))$ is the posted-price-for-the-grand-bundle mechanism for stage $t$ with non-negative posted-price by definition. Hence $z_t, q_t, d_t, s_t$, are defined on the domains required by the definition of bank account mechanism.

Within each single stage and for any balance $bal_t$, stage mechanism $(z_t(bal_t, \cdot), q_t(bal_t, \cdot))$ is truthful, because it is a randomization of three truthful stage mechanisms: the optimal history-independent mechanism, the give-for-free mechanism, and the posted-price-for-the-grand-bundle mechanism with a parameter independent with the reported type on current stage, $v_t$. Hence $B$ satisfies the IC constraint (10).

Another IC constraint (11) is verified as follows,

$$
\begin{align*}
\mathbb{E}_{v_t} \left[ \hat{u}_t(bal_t, v_t; v_t) - \hat{u}_t(bal'_t, v_t; v_t) \right] \\
= \frac{1}{3} \mathbb{E}_{v_t} \left[ (z_t(bal_t, v_t) \cdot v_t - q_t(bal_t, v_t)) - (z_t(bal'_t, v_t) \cdot v_t - q_t(bal'_t, v_t)) \right] \\
= \frac{1}{3} \mathbb{E}_{v_t} \left[ \left( x^S_t(v_t) + 1 + x^P_t(3s_t(bal_t), v_t) \right) \cdot v_t - \left( p^S_t(v_t) + p^P_t(3s_t(bal_t), v_t) \right) \right] \\
- \frac{1}{3} \mathbb{E}_{v_t} \left[ \left( x^S_t(v_t) + 1 + x^P_t(3s_t(bal'_t), v_t) \right) \cdot v_t - \left( p^S_t(v_t) + p^P_t(3s_t(bal'_t), v_t) \right) \right] \\
= \frac{1}{3} \left( \mathbb{E}_{v_t} [u^P_t(3s_t(bal_t), v_t; v_t)] - \mathbb{E}_{v_t} [u^P_t(3s_t(bal'_t), v_t; v_t)] \right).
\end{align*}
$$

By the definition of the posted-price-for-the-grand-bundle mechanism with parameter $\theta$, the buyer utility $u^P_t(\theta, v_t; v_t) = \theta$ (see (21)), hence

$$
\begin{align*}
\mathbb{E}_{v_t} \left[ \hat{u}_t(bal_t, v_t; v_t) - \hat{u}_t(bal'_t, v_t; v_t) \right] & = \frac{1}{3} \left( \mathbb{E}_{v_t} [u^P_t(3s_t(bal_t), v_t; v_t)] - \mathbb{E}_{v_t} [u^P_t(3s_t(bal'_t), v_t; v_t)] \right) \\
& = \frac{1}{3} (3s_t(bal_t) - 3s_t(bal'_t)) = s_t(bal_t) - s_t(bal'_t) \implies (11).
\end{align*}
$$

By Lemma 3.3 conditions (10) and (11) are sufficient conditions of stage-wise IC, so we have proved the IC property of $B$.

Finally, by the construction of $z_t$, $q_t$, and $d_t$, (see (22), (23), and (24) respectively), we can prove the IR constraint (12) as follows. Hence $B$ is a BAM.

$$
\hat{u}_t(bal_t, v_t; v_t) = \frac{1}{3} (u^S_t(v_t) + 1 \cdot v_t + u^P_t(3s_t(bal_t), v_t; v_t)) \geq \frac{1}{3} v_t \cdot 1 = d_t(bal_t, v_t) \implies (12).
$$

A.12 Proof of Lemma 5.2

Proof of Lemma 5.2. Since $s^\sigma_t(bal'^\sigma_t) = 0$ if $\sigma_t = 0$, we cannot prove the lemma by arguing $s^\sigma_t(bal'^\sigma_t) \geq s^\sigma_t(bal'_t)/4$. Even if when $\sigma_t = 1$, it is still nearly impossible to argue for similar lower bounds of $s^\sigma_t(bal'^\sigma_t)$, because $bal'^\sigma_t$ — the upper bound of $s^\sigma_t(bal'^\sigma_t)$ — depends on the first $t - 1$ bits of $\sigma$, which have $2^{t-1}$ different cases.

The trick here is to consider a pair of $\sigma$’s, say $\sigma$ and $\sigma(t)$, and to argue that if $\sigma_t = 1$, then $s^\sigma_t(bal'^\sigma_t) + s^\sigma(t)_t(bal'^\sigma(t)_t) \geq s^\sigma_t(bal'_t)$.
In particular, for any \( \sigma \in \{0, 1\}^T \) and \( t \in [T] \), let \( \sigma(t) = \bar{\sigma}_1 \cdots \bar{\sigma}_{t-1} \sigma_t \cdots \bar{\sigma}_T \), which is different from \( \sigma \) in the first \( t - 1 \) bits, and the same with \( \sigma \) in the rest of the bits.

In order to use Lemma A.2, consider the following deposit sequence, \( d_{(1, t-1)} \), and spend sequence, \( s_{(1, t-1)} \),
\[
\forall \tau < t, \; d_{\tau} = d_{\tau}^p(\text{bal}_{\tau}^p, v_{\tau}) + d_{\tau}^{s(t)}(\text{bal}^{s(t)}_{\tau}, v_{\tau}), \; s_{\tau} = s_{\tau}^p(\text{bal}_{\tau}^p) + s_{\tau}^{s(t)}(\text{bal}^{s(t)}_{\tau}). \quad (57)
\]
Recall the definitions of \( d_{\tau}^p \) and \( s_{\tau}^p \), i.e.,
\[
\begin{align*}
&\text{if } \sigma_{\tau} = 0: \; d_{\tau}^p(\text{bal}_{\tau}^p, v_{\tau}) = 1 \cdot v_{\tau}, \; s_{\tau}^p(\text{bal}_{\tau}^p) = 0; \\
&\text{if } \sigma_{\tau} = 1: \; d_{\tau}^p(\text{bal}_{\tau}^p, v_{\tau}) = 0, \; s_{\tau}^p(\text{bal}_{\tau}^p) = \min\{\text{bal}_{\tau}^p, \text{Val}_{\tau}\}.
\end{align*}
\]
Hence
\[
\forall \tau < t, \; d_{\tau} = 1 \cdot v_{\tau}, \; s_{\tau} = \begin{cases} 
\min\{\text{bal}_{\tau}^p, \text{Val}_{\tau}\}, & \text{if } \sigma_{\tau} = 1 \\
\min\{\text{bal}_{\tau}^{s(t)}, \text{Val}_{\tau}\}, & \text{if } \sigma_{\tau} = 0.
\end{cases} \quad (58)
\]
Consider the following balance \( \text{bal}_{(1, t)} \) defined by \( d_{(1, t-1)} \) and \( s_{(1, t-1)} \),
\[
\text{bal}_{1} = 0, \; \text{bal}_{\tau+1} = \text{bal}_{\tau} + d_{\tau} - s_{\tau}, \forall \tau \in [t-1].
\]
By (57), it is not hard to demonstrate the fact below through induction from \( \tau = 1 \) to \( \tau = t \),
\[
\forall \tau \in [t], \; \text{bal}_{\tau} = \text{bal}_{\tau}^p + \text{bal}_{\tau}^{s(t)}. \quad (59)
\]
Also note that from (58), we have
\[
\forall \tau < t, \; d_{\tau} = 1 \cdot v_{\tau}, \; s_{\tau} \leq \text{Val}_{\tau}.
\]
Hence the following inequality is from Lemma A.2
\[
\sum_{\tau=1}^{t-1} s_{\tau} \leq \sum_{\tau=1}^{t-1} s_{\tau}^*(\text{bal}_\tau^*).
\]
Apply the balance update formula (7) recursively,
\[
\text{bal}_{t} = \text{bal}_{1} + \sum_{\tau=1}^{t-1} d_{\tau} - \sum_{\tau=1}^{t-1} s_{\tau}.
\]
Combining \( \text{bal}_{1} = 0 \) (by definition), (58), (59), and (60), we have
\[
\text{bal}_{t}^p + \text{bal}_{t}^{s(t)} = \text{bal}_{t} \geq 0 + \sum_{\tau=1}^{t-1} 1 \cdot v_{\tau} - \sum_{\tau=1}^{t-1} s_{\tau}^*(\text{bal}_{\tau}^*),
\]
where by the definition of \( d_{(1, T)}^p \) and \( s_{(1, T)}^* \),
\[
0 + \sum_{\tau=1}^{t-1} 1 \cdot v_{\tau} - \sum_{\tau=1}^{t-1} s_{\tau}^*(\text{bal}_{\tau}^*) = \text{bal}_{1}^p + \sum_{\tau=1}^{t-1} d_{\tau}^p(\text{bal}_{\tau}^p, v_{\tau}) - \sum_{\tau=1}^{t-1} s_{\tau}^*(\text{bal}_{\tau}^*) = \text{bal}_{t}^p.
\]

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starting with $φ$ where $\max$ where a core BAM $B$

By Theorem 3.5, it is WLOG to assume that the optimal mechanism is $A$.

### A.13 Proof of Theorem 6.1

where the last inequality is from (62).

Now we are ready to finish the proof. $∀t \in [T], v_{(1,t)} \in \mathcal{V}_{(1,t)}$,

\[
E \left[ s_0^\sigma (\text{bal}_0^\sigma) \right] = E \left[ s_0^\sigma (\text{bal}_0^\sigma) \bigg| \sigma_t = 1 \right] \cdot Pr[\sigma_t = 1] + E \left[ s_0^\sigma (\text{bal}_0^\sigma) \bigg| \sigma_t = 0 \right] \cdot Pr[\sigma_t = 0]
\]

\[
= \frac{1}{2} E \left[ s_0^\sigma (\text{bal}_0^\sigma) \bigg| \sigma_t = 1 \right] + 0
\]

\[
= \frac{1}{4} E \left[ s_0^\sigma (\text{bal}_0^\sigma) + s_t^\sigma (\text{bal}_t^\sigma) \bigg| \sigma_t = 1 \right]
\]

\[
\geq \frac{1}{4} s_t^\sigma (\text{bal}_t^\sigma),
\]

where the last inequality is from (62).

### A.13 Proof of Theorem 6.1

**Proof of Theorem 6.1** By Theorem 3.5, it is WLOG to assume that the optimal mechanism is a core BAM $B^\sigma$. Moreover, by Theorem 3.11 and Lemma 3.10, it is WLOG to assume that $g(h) = Utl(B|h)$. Since $y$ is the sub-gradient of $g$ and $g$ is symmetric, we denote $z_t(\text{bal}_t, v_t)$ by $y_t(g(\xi, v_t))$ throughout the proof.

Let $φ_t(\xi)$ be defined as follows,

\[
φ_t(\xi) = \max_{B: g(\xi, v_t) \in \mathcal{V}_{(1,t)}} E \left[ \sum_{t+1}^T y_t(g(\xi, v_{t+1}), v_{t+1}) \cdot v_{t+1} - \text{bal}_{t+1} \right].
\]  

Consider the following program.

\[
\text{maximize } E \left[ y_t(\xi, v_t) \cdot v_t + φ_t(g(\xi, v_t)) \right]
\]

subject to

\[
\hat{u}_t(\xi, v_t; v_t) \geq \hat{u}_t(\xi, v'_t; v'_t), \forall v_t \in \mathcal{V}_t
\]

\[
g(\xi, v_t) = \xi + \hat{u}_t(\xi, v_t; v_t) - E_{v_t} \left[ \hat{u}_t(\xi, v'_t; v'_t) \right]
\]

\[
g(\xi, v_t) \geq 0, \forall v_t \in \mathcal{V}_t
\]

where

**Lemma A.3.** $\max_{\xi \geq 0} φ_0(\xi)$ is the maximum expected revenue, and $φ_{t-1}(\xi)$ is computed based on $φ_t$ according to the above program (64), where the entire algorithm is essentially a dynamic program starting with $φ_T(\xi) = -\text{bal}_{T+1} = -\xi^{11}$

\[11\text{Recall that } μ_T = 0, \text{ hence } \text{bal}_{T+1} = Utl(B|v_{(1,T)}).\]
Proof of Lemma A.3. According to the construction of Lemma 3.10,

\[
\max_{\xi \geq 0} \phi_0(\xi) = \max_{B: \text{UTL}(B) \geq 0} \mathbb{E} \left[ \sum_{\tau=1}^{T} y_t \left( \text{UTL}(B|v_{1,\tau-1}), v_{\tau} \right) \cdot v_{\tau} - \text{bal}_{T+1} \right]
\]

\[
= \max_{B} \mathbb{E}_{v_{1,T}} \left[ \sum_{\tau=1}^{T} \left( \hat{u}_{\tau}(\text{bal}_{\tau}, v_{\tau}) + q_{\tau}(\text{bal}_{\tau}, v_{\tau}) \right) - \text{bal}_{T+1} \right]
\]

\[
= \max_{B} \mathbb{E}_{v_{1,T}} \left[ \sum_{\tau=1}^{T} d_{\tau}(\text{bal}_{\tau}, v_{\tau}) - \text{bal}_{T+1} + \sum_{\tau=1}^{T} q_{\tau}(\text{bal}_{\tau}, v_{\tau}) \right]
\]

\[
= \max_{B} \mathbb{E}_{v_{1,T}} \left[ \sum_{\tau=1}^{T} \left( s_{\tau}(\text{bal}_{\tau}) + q_{\tau}(\text{bal}_{\tau}, v_{\tau}) \right) - \text{bal}_1 \right]
\]

\[
= \max_{B} \text{Rev}(B).
\]

Hence \( \max_{\xi \geq 0} \phi_0(\xi) \) is the maximum expected revenue.

In the meanwhile,

- \( \text{(65)} \iff g \) is convex in \( v_t \);
- \( \text{(66)} \iff g \) is consistent;
- the definition of \( \hat{u}_t \) and \( \text{(66)} \iff y_t \) is the sub-gradient of \( g(\xi,v_t) \);
- finally, by the assumption that \( g(h) = \text{UTL}(B|h) \), \( g \) must be symmetric, and non-negative, while \( \text{(67)} \) guarantees that \( \phi_t(g(\xi,v_t)) \) is valid, since \( \phi_t \) is not defined on negative inputs.

In summary, by Theorem 3.11, \( B^{g,y} \) is a core BAM (with assuming \( g(h) = \text{UTL}(B|h) \)), if and only if \( \forall t \in [T], (g,y) \) is feasible in program \( \text{(64)} \).

Particularly, for one item per stage case, \( \forall t \in [T], \forall v_t = \mathbb{R}_+ \), the buyer has discrete stage type distribution for each stage, i.e.,

\[
\forall t \in [T], \exists v^1_t < \cdots < v^{k_t}_t \in \mathcal{V}_t, \text{s.t. } \sum_{i=1}^{k_t} \Pr_{v \sim \mathcal{F}_t}[v = v^i_t] = 1.
\]

Then program \( \text{(64)} \) reduces to the following forms, i.e.,

maximize \( \mathbb{E}_{v_t} \left[ y_t(\xi,v_t) \cdot v_t + \phi_t(g(\xi,v_t)) \right] \)

subject to \( y_t(\xi,v_t) = \int_0^{v_t} w_t(\xi,v)dv \)

\[
\int_{\mathcal{V}_t} w_t(\xi,v)dv = 1
\]

\[
g(\xi,v_t) = \xi + \hat{u}_t(\xi,v_t) - \mathbb{E}_{v'_t}[\hat{u}_t(\xi,v'_t;v_t)]
\]

\[
g(\xi,v_t) \geq 0, w_t(\xi,v_t) \geq 0, \forall v_t \in \mathcal{V}_t
\]

Lemma A.4. For one item per stage case, the allocation rule must be a randomization of posted-price mechanisms.

Particularly, for discrete valuation distribution cases, it is WLOG to restrict to the convex combinations of posted-price mechanisms with the prices only from the set of discrete types, \( v^1_t, \ldots, v^{k_t}_t \).
While for discrete valuation distributions, WLOG, suppose that \( v_t^1 = 0 \), then the program further reduces to the following form, where \( \Pr_t^i \) denotes the probability of type \( v_t^i \) on stage \( t \), i.e., \( \Pr_t^i = \Pr_{v^i \in \mathcal{F}_t} [v = v_t^i] \).

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{k_t} \Pr_t^i \cdot \left( y_t^i \cdot v_t^i + \phi_t(g_t^i) \right) \\
\text{subject to} & \quad y_t^i = \sum_{j=1}^{i} w_t^j \\
& \quad \sum_{i=1}^{k_t} w_t^i = 1 \\
& \quad \hat{u}_t^i = \sum_{j=1}^{i} w_t^j (v_t^i - v_t^j) \\
& \quad g_t^i = \xi + \hat{u}_t^i - \sum_{j=1}^{k_t} \Pr_t^i \cdot \hat{u}_t^j \\
& \quad g_t^i \geq 0, w_t^i \geq 0, \forall i = 1, \ldots, k_t.
\end{align*}
\]

Denote the value of the program by \( \text{value}(\xi, \phi_t, w_t) \), and the optimal value by \( \text{Prog}(\xi, \phi_t) = \max_{w_t} \text{value}(\xi, \phi_t, w_t) \), where \( \xi \) and \( \phi_t \) are parameters (\( \phi_t \) is a function) of the program, and \( w_t = (w_t^1, \ldots, w_t^{k_t}) \) are the variables (others are all determined by \( w_t \)). Note that \( w_t \) takes different values for different parameters.

This program is a linear program, if \( \phi_t \) is a concave and piece-wise linear function. Moreover, the LP is of polynomial size, if \( \phi_t \) has at most polynomially many pieces.

Then the following lemma completes our proof that for any \( \epsilon > 0 \), the LP-based algorithm achieves an \( \epsilon \)-approximation by solving at most \( O(TN/\epsilon) \) many LPs, and each LP of size at most \( O(k_tN/\epsilon) \).

**Lemma A.5.** \( \phi_t(\xi) \) is concave, and can be \( \kappa \)-approximated by two concave piece-wise linear functions, \( \underline{\phi}_t(\xi) \) and \( \overline{\phi}_t(\xi) \). Formally,

\[
\forall \xi, \underline{\phi}_t(\xi) \leq \phi_t(\xi) \leq \overline{\phi}_t(\xi), \quad \overline{\phi}_t(\xi) - \underline{\phi}_t(\xi) \leq \kappa \max_\xi \phi_t(\xi),
\]

where the number of pieces of \( \underline{\phi}_t \) and \( \overline{\phi}_t \) is bounded by \( O(N + 1/\kappa) \).

**Proof of Lemma A.5.** We first show that \( \phi_t(\xi) \) is concave. Let \( B \) and \( B' \) be the sub-BAMs from stage \( t + 1 \) to \( T \) that reach \( \phi_t(\xi) \) and \( \phi_t(\xi') \) respectively. Note that \( \text{E}_{v(t+1,T)}[\text{bal}_{T+1}] = \text{E}_{v(t+1,T)}[\text{UTL}(B|v(1,T))] = \text{UTL}(B|v(1,T)) = \xi \). Then for any \( \eta \in [0,1], \xi'' = \eta \xi + (1-\eta)\xi' \),

\[
\phi_t(\xi'') \geq \text{E}_{v(t+1,T)} \left[ \sum_{\tau=t+1}^{T} y_{\tau}'' \cdot v_{\tau} - \text{bal}_{T+1}'' \right] \\
= \text{E}_{v(t+1,T)} \left[ \sum_{\tau=t+1}^{T} (\eta y_{\tau} + (1-\eta)y_{\tau}') \cdot v_{\tau} \right] - \xi'' \\
= \eta \phi_t(\xi) + (1-\eta)\phi_t(\xi'),
\]

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where \( y''_t \) is a convex combination of \( y_t \) and \( y'_t \). Note that in general, the convex combination of two BAMs (\( B \) and \( B' \)) is not a BAM, because it might not be symmetric. However, by Theorem 3.5, there is a BAM that admits the same expected utility, \( \xi'' \), and better expected efficiency. In other words, \( y''_t \) might indicate a mechanism that is not a BAM, but there is a BAM that generates higher revenue than this mechanism. Hence \( \phi_t(\xi'') \) is even larger.

Then by the following lemma, we could efficiently compute \( \Phi_{t-1} \) and \( \Phi_t \) from given \( \phi_t \) and \( \Phi_t \). Hence we get an LP-based algorithm to achieve an arbitrarily close approximation to the optimal mechanism.

**Lemma A.6.** For any concave function \( \varphi \) defined on interval \([a, b]\), if

- \( \varphi(a) \) and \( \varphi(b) \) are given;
- there exist \( |\beta_a| \leq +\infty \) and \( |\beta_b| \leq +\infty \) such that

\[
\varphi(\xi) \leq \beta_a(\xi - a) + \varphi(a), \quad \varphi(\xi) \leq \beta_b(\xi - b) + \varphi(b);
\]

then for any \( \kappa > 0 \), a pair of lower and upper bounds, \( \varphi \) and \( \overline{\varphi} \) can be computed via \( O(n) \) queries to the evaluation oracle of \( \varphi \), such that

- let \( \beta = (\varphi(b) - \varphi(a))/(b - a) \), then

\[
n \leq \frac{4}{\kappa} + \log \left( \frac{(\beta_a - \beta_b)^2}{(\beta_a - \beta)(\beta_b - \beta)} \right);
\]

- both of \( \varphi \) and \( \overline{\varphi} \) are concave and piece-wise linear and have at most \( O(n) \) pieces;

- the gap between \( \varphi \) and \( \overline{\varphi} \) is \( \delta = (\max_{\xi} \varphi(\xi) - \min\{\varphi(a), \varphi(b)\}) \kappa \).

Suppose we have lower and upper bounds for \( \phi_t \), i.e., \( \phi_t \leq \phi_t \leq \Phi_t \). Then the functions \( \phi_{t-1}^*(\xi) = \text{Prog}(\xi, \phi_t) \) and \( \Phi_{t-1}^*(\xi) = \text{Prog}(\xi, \Phi_t) \) are lower and upper bounds for \( \phi_{t-1} \), respectively, i.e.,

\[
\text{Prog}(\xi, \phi_t) = \text{value}(\xi, \phi_t, w_t^*) \leq \text{value}(\xi, \phi_t, w_t^*) \leq \text{value}(\xi, \phi_t, w_t^*) \leq \text{Prog}(\xi, \phi_t) = \phi_t(\xi),
\]

\[
\text{Prog}(\xi, \Phi_t) = \text{value}(\xi, \Phi_t, w_t^*) \geq \text{value}(\xi, \phi_t, w_t^*) \geq \text{value}(\xi, \phi_t, w_t^*) \geq \text{Prog}(\xi, \phi_t) = \phi_t(\xi).
\]

Let \( \phi_{t-1}^* \) be the lower bound of \( \phi_{t-1}^* \) and \( \Phi_{t-1}^* \) be the upper bound of \( \Phi_{t-1}^* \). So \( \phi_{t-1}^* \leq \phi_{t-1} \leq \Phi_{t-1}^* \), and the gap is bounded as follows,

\[
\Phi_{t-1}^* - \phi_{t-1}^* = (\Phi_{t-1} - \phi_{t-1}) + (\phi_{t-1}^* - \phi_{t-1}) + (\phi_{t-1}^* - \phi_{t-1}) \\
\leq \kappa + 2(T - t)\kappa + \kappa = 2(T - t + 1)\kappa,
\]

where

\[
\Phi_{t-1}^* - \phi_{t-1}^* = \text{value}(\xi, \phi_{t-1}, w_{t-1}^*) - \text{value}(\xi, \phi_{t-1}, w_{t-1}^*) \\
\leq \text{value}(\xi, \phi_{t-1}, w_{t-1}^*) - \text{value}(\xi, \phi_{t-1}, w_{t-1}^*) \\
= \max_{\xi \geq 0} \left( \phi_{t-1}(\xi) - \phi_{t-1}(\xi) \right) \leq 2(T - t)\kappa.
\]

Hence the gap between \( \phi_t \) and \( \Phi_t \) is bounded by \( T\kappa \).

Let \( N \) be the input size of the problem. Then we prove that
Lemma \(\ref{lem:phi} \) can be used to approximate \(\tilde{\phi}_{i-1}^u\) and \(\overline{\phi}_{i-1}^u\) with \(\phi_i\) and \(\overline{\phi}_i\) being given;

- the multiplicative \(\epsilon\)-approximation can be achieved by letting \(\kappa = 4\epsilon / T\);
- the entire algorithm needs to solve \(O((N + T/\epsilon)T)\) many LPs, each of size at most \(O((T/\epsilon + N)k_i)\).

We verify that \(\phi_t(\xi)\) satisfies the requirements for Lemma \(\ref{lem:phi} \). Consider the boundary estimation of \(\phi_t(\xi)\). Note that by assumption, \(v_t^1 = 0\), hence \(\hat{u}_t^1 = 0\). Then

\[
g_t^1 \geq 0 \implies \sum_{j=1}^{k_t} \Pr_t^j \cdot \hat{u}_t^j \leq \xi.
\]

Meanwhile, for \(i < k_t\),

\[
y_t^i = \sum_{j=1}^{i} w_t^j \leq \max_{i' \leq i} \frac{1}{v_t^{i+1} - v_t^i} \cdot \sum_{j=1}^{i} (v_t^{i+1} - v_t^j) w_t^j \leq \frac{1}{v_t^{i+1} - v_t^i} \cdot \hat{u}_t^{i+1}
\]

\[
\implies \sum_{i=1}^{k_t} \Pr_t^i \cdot y_t^i \cdot v_t^i \leq \Pr_t^{k_t} \cdot y_t^{k_t} \cdot v_t^{k_t} + \sum_{i=1}^{k_t-1} \Pr_t^i \cdot \hat{u}_t^{i+1} \cdot \frac{v_t^{i+1}}{v_t^{i+1} - v_t^i} \cdot \sum_{i=1}^{k_t} \Pr_t^i \cdot \hat{u}_t^i \leq \Pr_t^{k_t} \cdot v_t^{k_t} + c_t \cdot \xi,
\]

where \(c_t = \max_{i<k_t} (\Pr_t v_t^i)/((v_t^i - v_t^{i-1})\Pr_t^{i+1}) < +\infty\). Combining with the concavity of \(\phi_{t+1}\), we conclude that

\[
\phi_t(\xi) \leq \Pr_t^{k_t} \cdot v_t^{k_t} + c_t \cdot \xi + \phi_{t+1}(\xi) \leq \cdots \leq \xi \sum_{t=1}^{T} c_t + \sum_{t=1}^{T} \Pr_t^{k_t} \cdot v_t^{k_t}.
\]

(68)

Meanwhile, when \(\xi = 0\), \(\hat{u}_t^1 \leq 0 \implies w_t^1 = \cdots = w_t^{k_t-1} = 0\), \(w_t^{k_t} = 1\). Hence

\[
\phi_t(0) = \Pr_t^{k_t} \cdot v_t^{k_t} + \phi_{t+1}(0) = \cdots = \sum_{t=1}^{T} \Pr_t^{k_t} \cdot v_t^{k_t},
\]

(69)

which meets the upper bound \(\ref{lem:phi} \) of \(\phi_t\) at \(\xi = 0\).

By letting \(y_t^1 = \cdots = y_t^{k_t} = 1\), we know that when \(\xi \geq \sum_{t=1}^{T} \text{Val}_t\), \(g_t^i \geq \sum_{t=1}^{T} \text{Val}_t\). Then by backward induction from \(T\), for large enough \(\xi \geq \sum_{t=1}^{T} \text{Val}_t\), \(\phi_t(\xi) = \sum_{t=1}^{T} \text{Val}_t - \xi\). Since \(\phi_t\) is concave, it is further upper bounded as follows,

\[
\phi_t(\xi) \leq \sum_{t=1}^{T} \text{Val}_t - \xi.
\]

(70)

In summary, \(\phi_t\) is a concave function on interval \([0, \sum_{t=1}^{T} \text{Val}_t]\), and the function values on two ends are \(\sum_{t=1}^{T} \Pr_t^{k_t} \cdot v_t^{k_t}\) and \(0\), respectively. Moreover, it also has two upper bounds, \(\ref{lem:phi} \), \(\ref{lem:phi} \), that go through the two ends respectively. Hence, \(\beta_a = \sum_{t=1}^{T} \text{Val}_t\), and \(\beta_b = -1\). So

\[
\log \frac{(\beta_a - \beta_b)^2}{(\beta_a - \beta)(\beta_b - \beta)} = O(N).
\]

Then by Lemma \(\ref{lem:phi} \), \(\phi_t(\xi)\) can be \(2(T-t)\kappa\)-approximated by \(\phi_t^*\) and \(\overline{\phi}_t\) with at most \(O(\frac{1}{\kappa} + N)\) many queries to \(\phi_{t+1}(\xi)\) and \(\overline{\phi}_{t+1}(\xi)\). \(\square\)
Therefore, there is an LP-based FPTAS via dynamic program to achieve a multiplicative \( \epsilon \)-approximation.

### A.14 Proof of Lemma [A.4]

**Proof of Lemma [A.4]** Note that for one item per stage case, the mechanism is truthful within any stage \( t \), if and only if \( \hat{u}_t(\xi, v_t) \) is increasing and convex in \( v_t \), and \( z_t(\xi, v_t) \) is the sub-gradient of \( \hat{u}_t \) with respect to \( v_t \). It is equivalent to \( z_t(\xi, v_t) \) being non-negative and increasing in \( v_t \), and \( \hat{u}_t \) being the integration of \( z_t(\xi, v_t) \) over \( v_t \).

\( z_t(\xi, v_t) \) is non-negative and increasing in \( v_t \), if and only if \( z_t(\xi, v_t) \) is a convex combination of step functions, i.e.,

\[
z_t(\xi, v_t) = \int_0^{+\infty} w_t(\xi, v) \cdot I[v_t \geq v] dv = \int_0^{v_t} w_t(\xi, v) dv,
\]

where \( \int_0^{+\infty} w_t(\xi, v) dv = 1 \) and \( w_t(\xi, v) \geq 0 \).

Particularly, for discrete types, to compute the optimal mechanism, we could assume that \( w_t(\xi, v) \) is non-zero only at points \( v_1^k, \ldots, v_k^{i_k} \).

To prove this, for any \( v^* \) such that \( v_i^k < v^* < v_{i+1}^k \), we show that if \( w_t(\xi, v^*) > 0 \), we can weakly improve the objective value without either breaking any constraint, or changing the values of \( g(\xi, v_i^k) \).

Let \( w'_t(\xi, v) = w_t(\xi, v) \) except for three points, i.e.,

\[
w'_t(\xi, v_i^j) = w_t(\xi, v_i^j) + \frac{v_i^{j+1} - v^*}{v_i^{j+1} - v_i^j} w_t(\xi, v^*),
\]

\[
w'_t(\xi, v_i^{j+1}) = w_t(\xi, v_i^{j+1}) + \frac{v^* - v_i^j}{v_i^{j+1} - v_i^j} w_t(\xi, v^*),
\]

\[
w'_t(\xi, v^*) = 0.
\]

Then it is straightforward to verify the following facts.

\[
w'_t(\xi, v) \geq 0,
\]

\[
\int_0^{+\infty} w'_t(\xi, v) dv = \int_0^{+\infty} w_t(\xi, v) dv = 1,
\]

\[
y'_t(\xi, v_i^j) = \int_0^{v_i^j} w'_t(\xi, v) dv = \int_0^{v_i^j} w_t(\xi, v) dv = y_t(\xi, v_i^j), \forall j \neq i
\]

\[
y'_t(\xi, v_i^j) = \int_0^{v_i^j} w'_t(\xi, v) dv = \int_0^{v_i^j} w_t(\xi, v) dv + \frac{v_i^{j+1} - v^*}{v_i^{j+1} - v_i^j} w_t(\xi, v^*) > y_t(\xi, v_i^j),
\]

\[
\hat{u}'_t(\xi, v_i^j; v_i^j) = \int_0^{v_i^j} y'_t(\xi, v) dv = \int_0^{v_i^j} y_t(\xi, v) dv = \hat{u}_t(\xi, v_i^j; v_i^j), \forall j \leq i
\]

\[
\hat{u}'_t(\xi, v_i^j; v_i^{j+1}) = \int_0^{v_i^{j+1}} y'_t(\xi, v) dv = \int_0^{v_i^{j+1}} y_t(\xi, v) dv + \int_{v_i^j}^{v_i^{j+1}} \left( y'_t(\xi, v) - y_t(\xi, v) \right) dv
\]

\[
= \hat{u}_t(\xi, v_i^j; v_i^{j+1}), \forall j \geq i + 1
\]

\[
\Rightarrow g'_t(\xi, v_i^j) = g(\xi, v_i^j), \forall j
\]

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where
\[
\int_{v_i}^{v_{i+1}} (y_i(\xi, v) - y_l(\xi, v)) \, dv = \int_{v_i}^{v_{i+1}} \int_0^v \left( w_i(\xi, \nu) - w_l(\xi, \nu) \right) \, d\nu \, dv
\]
\[
= \int_{v_i}^{v_{i+1}} \int_{v_i}^v \left( w_i(\xi, \nu) - w_l(\xi, \nu) \right) \, d\nu \, dv = \int_{v_i}^{v_{i+1}} \int_{v_i}^{v_{i+1}} \left( w_i(\xi, \nu) - w_l(\xi, \nu) \right) \, d\nu \, dv
\]
\[
= (v_{i+1} - v_i) \cdot (w_i(\xi, v_i) - w_l(\xi, v_i)) + (v_{i+1} - v^*) \cdot (w_i(\xi, v^*) - w_l(\xi, v_i)) + (v_{i+1} - v_i) \cdot (w_l(\xi, v_i^*) - w_l(\xi, v_i^+))
\]
\[
= (v_{i+1} - v_i) \cdot \frac{v_{i+1} - v^*}{v_{i+1}^* - v_i} w_l(\xi, v_i^*) + (v_{i+1} - v^*) \cdot ( - w_l(\xi, v_i^*) ) + 0 = 0.
\]

As desired, no constraint gets violated, but objective value gets improved. \qed

\section*{A.15 Proof of Lemma A.6}

\textbf{Proof of Lemma A.6} Since \( \varphi \) is concave, hence \( \beta_a \geq \beta_b \). If they coincide, i.e., \( \beta_a = \beta_b \), then we are done, and \( \varphi \) must be the line segment that connects \((a, \varphi(a))\) and \((b, \varphi(b))\).

Otherwise, \( \beta_a - \beta_b > 0 \), and since \( |\beta_a|, |\beta_b| < +\infty \), \( \beta_a - \beta_b < +\infty \).

By adding a linear offset to \( \varphi \), we assume WLOG that \( \varphi(a) = \varphi(b) = 0 \). Hence \( \beta_a > 0 > \beta_b \).

Note that \( \max \varphi(\xi) \min \{\varphi(a), \varphi(b)\} \) is not increased after adding the linear offset.

Then for any \( \delta > 0 \), we make \( n \) queries to the evaluation oracle of \( \varphi \) at \( l_1, \ldots, l_m, l_{m+1}, \ldots, l_n \), where \( l_m \) is the last point such that \( \varphi(l_{m-1}) \leq \varphi(l_m) > \varphi(l_{m+1}) \). Since \( \varphi \) is concave, such \( m \) is unique.

Then points \( l_1, \ldots, l_m \) are adaptively chosen according to the following equations, until \( \varphi \) starts to decrease.

\[
\begin{align*}
l_0 &= a, l_i = \min \{ l_{i-1} + \frac{\delta}{\beta_{l_{i-1}}}, b \}, \\
\beta_0 &= \beta_a, \beta_i &= \frac{\varphi(l_i) - \varphi(l_{i-1})}{l_i - l_{i-1}} - \frac{\varphi(l_i)}{l_i - b}.
\end{align*}
\]

Since \( \varphi \) is concave, \( \beta_i \geq 0 \), and \( \beta_i \leq \beta_a - \beta_b < +\infty \). Hence \( l_i > l_{i-1} \), as long as \( l_{i-1} < b \). Thus \( m < +\infty \).

For the last half of points, \( l_{m+1}, \ldots, l_n \), are chosen similarly but in the reverse order, starting from \( l_{n+1} = b \) until the next point is less than \( l_m \).

In the remainder of the proof, we show that \( m \leq 2 \max \varphi(\xi)/\delta + \log(\beta_a - \beta_b)/(-\beta_b) \), which then implies that \( n \leq 4 \max \varphi(\xi)/\delta + \log(\beta_a - \beta_b)^2/(-\beta_a \beta_b) \).

Note that to achieve a given multiplicative approximation ratio \( \kappa \), we can let \( \delta = \kappa \cdot \varphi(\frac{a+b}{2}) \), because \( 2\varphi(\frac{a+b}{2}) \geq \max \varphi(\xi) \geq \varphi(\frac{a+b}{2}) \).

Then by connecting points \((a, \varphi(a)), (l_1, \varphi(l_1)), \ldots, (l_n, \varphi(l_n)), (b, \varphi(b))\), we get a lower bound \( \varphi \) for \( \varphi \), which is concave and piece-wise linear, and has at most \( n + 1 \) pieces. The upper bound is simply \( \varphi(\xi) = \varphi(\xi) + \delta \).

From now on, we only consider \( 0 \leq i \leq m \). For any \( l_i \leq l \leq l_{i+1} \), \( \varphi(l) \) is \( \delta \)-approximated by the line segment that connects \((l_i, \varphi(l_i))\) and \((l_{i+1}, \varphi(l_{i+1}))\), i.e., by the concavity of \( \varphi \),

\[
\varphi(l) \geq \frac{\varphi(l_i)(b - l)}{b - l_i}, \quad \varphi(l) \leq \frac{\varphi(l_{i-1})(l - l_i)}{l_{i-1} - l_i} + \frac{\varphi(l_i)(l_{i+1} - l)}{l_i - l}.
\]

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The gap between the upper and lower bounds of \( \varphi(l) \) is,

\[
\frac{\varphi(l_i - 1)(l - l_i)}{l_i - l_i - 1} + \frac{\varphi(l_i)(l_i - 1 - l)}{l_i - l_i} - \frac{\varphi(l_i)(b - l)}{b - l_i}
\]

\[
= \left( \frac{\varphi(l_i) - \varphi(l_i - 1)}{l_i - l_i - 1} - \frac{\varphi(l_i)}{l_i - b} \right) l + \left( \frac{\varphi(l_i)(l_i - 1) - \varphi(l_i - 1)}{l_i - l_i - 1} - \frac{\varphi(l_i)b}{b - l_i} \right)
\]

\[
= \beta_i l + \left( \frac{\varphi(l_i)l_i - \varphi(l_i - 1)i}{l_i - l_i} - \frac{\varphi(l_i)b}{b - l_i} \right)
\]

\[
\leq \beta_i(l_i + \delta/\beta_i) + \left( \frac{\varphi(l_i)l_i - \varphi(l_i - 1)i}{l_i - l_i} - \frac{\varphi(l_i)b}{b - l_i} \right)
\]

\[
= \delta + \varphi(l_i) - \varphi(l_i) = \delta.
\]

Then, we show that \( m \) is small, so the \( \delta \)-approximation is achieved by a few of queries. Let \( \beta_i^* \) be defined as follows,

\[
\beta_i^* = \beta_i + \frac{\varphi(l_i)}{l_i - b} - \beta_b \geq \beta_i + \frac{\beta_b(l_i - b)}{l_i - b} - \beta_b = \beta_i.
\]

Note that \( \varphi(l_i) > 0 = \varphi(b) \), hence \( l_m < b \), then for \( i \leq m - 1 \),

\[
\beta_i = \frac{\varphi(l_i) - \varphi(l_i - 1)}{l_i - l_i - 1} - \frac{\varphi(l_i)}{l_i - b} \geq \frac{\varphi(l_i) - \varphi(l_i - 1)}{l_i - l_i - 1} = \beta_i^* - \beta_i^* + 1,
\]

\[
\beta_i^*(l_i + 1 - l_i) \geq \beta_i(l_i + 1 - l_i) = \beta_i \cdot \frac{\delta}{\beta_i} = \delta,
\]

\[
\implies \delta - (\beta_i^* - \beta_i^* + 1 + \beta_b)(l_i + 1 - l_i) \leq (\beta_i^* - \beta_i^* + 1 + \beta_b)(l_i + 1 - l_i) = \varphi(l_i) - \varphi(l_i),
\]

\[
\implies \delta \leq (\beta_i^* - \beta_i^* + 1 + \beta_b)(l_i + 1 - l_i) + \varphi(l_i) - \varphi(l_i)
\]

\[
= \frac{(\beta_i^* - \beta_i^* + 1 + \beta_b)}{\beta_i} + \varphi(l_i) - \varphi(l_i),
\]

\[
\implies m \delta \leq \sum_{i=0}^{m-1} \left( \frac{(\beta_i^* - \beta_i^* + 1 + \beta_b)}{\beta_i} + \varphi(l_i) - \varphi(l_i) \right)
\]

\[
= \varphi(l_m) + \delta \sum_{i=0}^{m-1} \frac{\beta_i^* - \beta_i^* + 1 + \beta_b}{\beta_i}
\]

\[
\leq \max_{\xi} \varphi(\xi) + \frac{(m + k)\delta}{2}
\]

\[
\implies m \leq \frac{2 \max_{\xi} \varphi(\xi)}{\delta} + k,
\]

where there are at most \( k \) different \( i \)'s such that \( (\beta_i^* - \beta_i^* + 1 + \beta_b)/\beta_i \geq 1/2 \). Because

\[
\frac{\beta_i^* - \beta_i^* + 1 + \beta_b}{\beta_i} \geq \frac{1}{2} \implies \beta_i^* \geq (\beta_i - \beta_i^*) + 2\beta_i^* + 2\beta_b \geq 2\beta_i^*.
\]

Since for any \( i \leq m - 1 \), \( \beta_i \leq \beta_a - \beta_b \), \( \beta_{i+1} \geq -\beta_b \), and \( \beta_i^* \geq \beta_{i+1}^* \), \( k \) must be no more than \( \log(\beta_a - \beta_b)/(-\beta_b) \).

Combining with the last half of points, we have

\[
n \leq \frac{4}{\delta} \max_{\xi} \varphi(\xi) + \log \frac{(\beta_a - \beta_b)^2}{-\beta_a \beta_b}.
\]