A NOTE ON LOWER BOUNDS FOR THE FIRST EIGENVALUE OF THE WITTEN-LAPLACIAN

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Abstract. In this note, by extending the arguments of Ling (Illinois J. Math. 51, 853-860, 2007) to Bakry-Émery geometry, we shall give lower bounds for the first nonzero eigenvalue of the Witten-Laplacian on compact Bakry-Émery manifolds in the case that the Bakry-Émery Ricci curvature has some negative lower bounds and the manifold has the symmetry that the minimum of the first eigenfunction is the negative of the maximum. Our estimate is optimal among those obtained by a self-contained method.

1. Introduction

A Bakry-Émery manifold \((M, g, f)\) is a Riemannian manifold \((M, g)\) equipped with the weighted volume form \(d\mu := e^{-f} \, d\text{vol}_g\), where \(f \in C^2(M)\) is a real-valued \(C^2\)-function on \(M\). A Bakry-Émery manifold was introduced by Bakry-Émery [3] and has been received much attention in various areas of mathematics. Given a Bakry-Émery manifold \((M, g, f)\), we obtain a Bakry-Émery Ricci curvature and a Witten-Laplacian defined by

\[
\text{Ric}_f := \text{Ric} + \text{Hess} f, \quad \Delta_f := \Delta - \nabla f \cdot \nabla,
\]

respectively. Here, \(\Delta = g^{ij} \nabla_i \nabla_j\). As usual, for any functions \(u, v \in C_0^\infty(M)\) on \(M\) with a compact support, the following integration by parts formula holds:

\[
\int_M \langle \nabla u, \nabla v \rangle \, d\mu = - \int_M (\Delta_f u)v \, d\mu = - \int_M u(\Delta_f v) \, d\mu, \quad u, v \in C_0^\infty(M).
\]

Moreover, Bakry and Émery [3] proved that

\[
\frac{1}{2} \Delta_f |\nabla u|^2 = |\text{Hess} u|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \text{Ric}_f (\nabla u, \nabla u), \quad u \in C_0^\infty(M).
\]

This formula can be considered as a natural extension of the Bochner-Weitzenböck formula. The Bakry-Émery Ricci curvature and the Witten-Laplacian are good substitutes of the Ricci curvature and the Laplacian respectively extending many fundamental theorems in Riemannian geometry to Bakry-Émery geometry, for example, Myers type theorems [23, 21, 27, 18, 33], Cheeger-Gromoll splitting theorems [9], Gradient estimates [11, 28, 31], eigenvalue estimates [5, 11, 2, 10, 29], Liouville type theorems [14, 22, 24, 12, 10]. Moreover, it has been an important tool in Optimal transport theory [26] and in Perelman’s entropy formula for heat equations on complete Riemannian manifolds [15, 16].

The aim of the present paper is to give lower bounds for the first nonzero eigenvalue \(\lambda_1\) of the Witten-Laplacian \(\Delta_f\) on compact Bakry-Émery manifolds in the case of the Bakry-Émery Ricci curvature is bounded from below by some negative constants and the manifold has the symmetry that the minimum of the first eigenfunction is the negative of the maximum. Our main result is the following:

Date: May 3, 2014.

2010 Mathematics Subject Classification. Primary 58J50, 35P15. Secondary 53C21.

Key words and phrases. Witten-Laplacian, Eigenvalue, Lower bound, Bakry-Émery Ricci curvature.
Theorem 1.1. Let \((M, g, f)\) be an \(n\)-dimensional compact Bakry-Émery manifold. Suppose that the Bakry-Émery Ricci curvature has the lower bound
\[
\text{Ric}_f \geq K
\]
for some negative constants \(K < 0\) and the manifold has the symmetry that the minimum of the first eigenfunction of the Witten-Laplacian \(\Delta f\) is the negative of the maximum. Then the first nonzero eigenvalue \(\lambda_1\) of the Witten-Laplacian has the lower bound
\[
\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{1}{2} K,
\]
where \(d\) denotes the diameter of \((M, g)\).

Remark 1.2. If \(f\) is constant in (1.1), the Bakry-Émery Ricci curvature and the Witten-Laplacian become the ordinary Ricci curvature \(\text{Ric}\) and Laplacian \(\Delta\), respectively. The study of the first nonzero eigenvalue for the Laplacian has a long history. See [7, 25, 17, 8, 13, 34, 30, 32, 19, 20, 1] and references therein. Our estimate (1.3) can be considered as an extension of Ling [19] to the Witten-Laplacian via Bakry-Émery Ricci curvature.

By combining Proposition 3.8 in [11] and above Theorem 1.1, we immediately obtain:

Theorem 1.3. Let \((M, g, f)\) be an \(n\)-dimensional compact Bakry-Émery manifold. Suppose that the Bakry-Émery Ricci curvature has the lower bound (1.2) for some constants \(K \in \mathbb{R}\) and the manifold has the symmetry that the minimum of the first eigenfunction of the Witten-Laplacian \(\Delta f\) is the negative of the maximum. Then the first nonzero eigenvalue \(\lambda_1\) of the Witten-Laplacian has the lower bound (1.3).

Remark 1.4. Under only the assumption (1.2) for some positive constants \(K > 0\), by using the method of modulus of continuity, Andrews and Ni [2] showed that the first nonzero eigenvalue \(\lambda_1\) of the Witten-Laplacian \(\Delta f\) on compact Bakry-Émery manifolds satisfies (1.3).

Remark 1.5. Under only the assumption (1.2) for some constants \(K \in \mathbb{R}\), Futaki et al [10] showed that the first nonzero eigenvalue \(\lambda_1\) of the Witten-Laplacian \(\Delta f\) on compact Bakry-Émery manifolds satisfies
\[
\lambda_1 \geq \sup_{s \in (0,1)} \left\{ 4s(1-s)\frac{\pi^2}{d^2} + sK \right\} = \begin{cases} 0 & \text{if } Kd^2 < -4\pi^2, \\ \left(\frac{2}{n} + \frac{Kd}{4\pi}\right)^2 & \text{if } Kd^2 \in [-4\pi^2, 4\pi^2], \\ K & \text{if } Kd^2 \in (4\pi^2, (n-1)\pi^2). \end{cases}
\]
At the present time, the above estimate (1.4) is optimal. For \(K < 0\), taking \(s = \frac{1}{2}\) in the above, we recapture (1.3). Hence the lower bound (1.4) is stronger than that in Theorem 1.3. An advantage of our result is being self-contained as we see below, while they use a comparison theorem between eigenvalue problems. Note that our estimate in Theorem 1.3 is optimal among those obtained by a self-contained method.

Acknowledgements. The author would like to thank his supervisor Professor Toshiki Mabuchi. The author also wish to thank Professor Akito Futaki for his comments. This work was partly supported by Moriyasu graduate student fellowship.

2. Proof of Theorem 1.1

In this section, following [11] we extend the arguments in [19] to Bakry-Émery geometry in the case that the minimum of the first eigenfunction of the Witten-Laplacian \(\Delta f\) is the
negative of the maximum. Let \( u \) be an eigenfunction of the first nonzero eigenvalue \( \lambda_1 \) for the Witten-Laplacian, i.e., \( \Delta_f u + \lambda_1 u = 0 \). We may normalize the function \( u \) such that
\[
\max_M u = 1, \quad \min_M u = -1.
\]
Let \( b > 1 \) be an arbitrary constant. Define a function \( Z \) on \([- \sin^{-1}(1/b), \sin^{-1}(1/b)]\) and a constant \( \delta \) by
\[
Z(t) := \max_{x \in U(t)} \frac{|\nabla u|^2}{\lambda_1(b^2 - u^2)}, \quad \delta := \frac{K}{2 \lambda_1} < 0,
\]
where \( U(t) := \{ x \in M : \sin^{-1}(u(x)/b) = t \} \). Note that \( t \in [- \sin^{-1}(1/b), \sin^{-1}(1/b)] \).

Remark 2.1. In Proposition 3.1 and 3.4 of [11], estimates for an eigenvalue \( \lambda \) and a function \( Z(t) \) are obtained, respectively. On the other hand, the same way does not work in our case, since \( K < 0 \). However, the same argument as Proposition 3.5 in [11] still holds in our case. Note that (2.1) corresponds to the case of \( a = c = 0 \) in [11].

As Proposition 3.6 (b) in [11], we have:

Proposition 2.2. Suppose that the function \( z : [- \sin^{-1}(1/b), \sin^{-1}(1/b)] \to \mathbb{R} \) satisfies the following conditions:
\begin{enumerate}
  \item \( z(t) \geq Z(t) \) for all \( t \in [- \sin^{-1}(1/b), \sin^{-1}(1/b)] \),
  \item there exists some \( x_0 \in M \) such that \( z(t_0) = Z(t_0) \) at \( t_0 = \sin^{-1}(u(x_0)/b) \),
  \item \( z(t_0) \geq 1 \), and
  \item \( \dot{z}(t_0) \sin t_0 \leq 0 \).
\end{enumerate}

Then we have the following:
\[
z(t_0) \leq \frac{1}{2} \ddot{z}(t_0) \cos^2 t_0 - \dot{z}(t_0) \cos t_0 \sin t_0 + 1 - 2 \delta \cos^2 t_0.
\]

Sketch of the Proof. By the same arguments as Proposition 3.5 in [11], we have
\[
0 \leq \frac{1}{2} \ddot{z}(t_0) \cos^2 t_0 - \dot{z}(t_0) \cos t_0 \sin t_0 - z(t_0) + 1 - 2 \delta \cos^2 t_0
\]
\[
- \frac{\dot{z}(t_0)}{4z(t_0)} \cos t_0 \{ \dot{z}(t_0) \cos t_0 - 2z(t_0) \sin t_0 + 2 \sin t_0 \}.
\]

By the assumption (3) and (4), the last term of the above is nonpositive. \( \square \)

Now, we give a proof of Theorem [11].

Proof of Theorem [11] The proof is the same as Theorem 3.1 in [19]. Then, we just give an outline of the proof. Let
\[
z(t) := 1 + \delta \xi(t),
\]
where \( \xi(t) \) is a function on \([-\pi/2, \pi/2]\] defined by
\[
\xi(t) = \frac{\cos^2 t + 2t \sin t \cos t + t^2 - \pi^2}{\cos^2 t}
\]
and \( \delta < 0 \) is the negative constant as in [22]. This function \( \xi(t) \) is introduced by Ling and needed properties are studied in [19] and [20]. By using such properties, we have
\[
\frac{1}{2} \ddot{z}(t) \cos^2 t - \dot{z} \cos t \sin t - z = -1 + 2 \delta \cos^2 t,
\]
\[
z(t) \geq 1, \quad \text{and}
\]
\[
\dot{z}(t) \sin t \leq 0.
\]
By using the above three conditions and (2.3), we can show that
\[ z(t) \geq Z(t), \quad t \in [-\sin^{-1}(1/b), \sin^{-1}(1/b)]. \]
The above implies
\[ (2.4) \quad \sqrt{\lambda_1} \geq \frac{|\nabla t|}{\sqrt{z(t)}}, \quad t \in [-\sin^{-1}(1/b), \sin^{-1}(1/b)]. \]
Let \( q_1 \) and \( q_2 \) be the two points such that \( u(q_1) = 1 \) and \( u(q_2) = -1 \), respectively. Let \( L \) be the minimizing geodesic between \( q_1 \) and \( q_2 \). We integrate the both sides of (2.4) along \( L \) and change variable. Letting \( b \to 1 \), we have
\[
d \sqrt{\lambda_1} \geq \int_L \frac{|\nabla t|}{\sqrt{z(t)}} \, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{z(t)}} \, dt \geq \frac{\left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{z(t)} \right)^{3/2}}{\left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} z(t) \, dt \right)^{1/2}} \geq \left( \frac{\pi^3}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} z(t) \, dt} \right)^{1/2}.
\]
By the definition of \( z(t) \) and the properties of \( \xi(t) \), we have
\[
\lambda_1 \geq \frac{\pi^3}{d^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} z(t) \, dt} = \frac{\pi^2}{d^2 (1-\delta)} \quad \text{and} \quad \lambda_1 \geq \pi^2 \frac{1}{d^2} + \frac{1}{2} K.
\]
The proof of Theorem 1.1 is completed. \( \Box \)

Remark 2.3. In [11], under only the assumption (1.2) for some constants \( K > 0 \), it is proved that the first nonzero eigenvalue \( \lambda_1 \) of the Witten-Laplacian \( \Delta_f \) on compact Bakry-Émery manifolds has the lower bound \( \lambda_1 \geq \frac{\pi^2}{d^2} + \frac{31}{100} K \). However, in the case of \( K < 0 \) same argument as in [11] does not hold, since there is no substitute for the Lichnerowicz type estimate. See the Case (B-a-2) in [11] and the Case (II-a) in [20].

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