Exploring the Jungle of Intuitionistic Temporal Logics

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Abstract
The importance of intuitionistic temporal logics in Computer Science and Artificial Intelligence
has become increasingly clear in the last few years. From the proof-theory point of view, intuitionistic
 temporal logics have made it possible to extend functional programming languages with new features via
 type theory, while from the semantics perspective several logics for reasoning about dynamical
 systems and several semantics for logic programming have their roots in this framework. We consider several
 axiomatic systems for intuitionistic linear temporal logic and show that each of these systems is sound
 for a class of structures based either on Kripke frames or on dynamic topological systems. We provide
two distinct interpretations of ‘henceforth’, both of which are natural intuitionistic variants of the classical
 one. We completely establish the order relation between the semantically-defined logics based on both
 interpretations of ‘henceforth’, and, using our soundness results, show that the axiomatically-defined logics
 enjoy the same order relations. Under consideration in Theory and Practice of Logic Programming (TPLP).

1 Introduction

Intuitionistic logic (IL) e.g. (Heyting 1930, Mints 2000) enjoys a myriad of interpretations based on
computation, information or topology, making it a natural framework to reason about dynamic processes
in which these phenomena play a crucial role. Thus, it should not be surprising that combinations of
intuitionistic logic and linear temporal

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logic (LTL) \cite{Pnueli1977} have been proposed for applications within several different contexts:

*Types for functional programming languages.* The Curry-Howard correspondence identifies intuitionistic proofs with the \( \lambda \)-terms of functional programming \cite{Howard1980}. Several extensions of the \( \lambda \)-calculus with operators from LTL have been proposed in order to introduce new features to functional programming languages: \cite{Davies1996, Davies2017} has suggested adding a ‘next’ (\( \odot \)) operator to IL in order to define the type system \( \lambda^\circ \), which allows extending functional programming languages with staged computation \cite{Ershov1977}. \cite{Davies2001} proposed the functional programming language Mini-ML\( ^\square \) which is supported by intuitionistic S4 and allows capturing complex forms of staged computation as well as runtime code generation. \cite{Yuse2006} later extended \( \lambda^\circ \) to \( \lambda^\Box \) by incorporating the ‘henceforth’ operator (\( \Box \)), useful for modelling persistent code that can be executed at any subsequent state.

*Semantics for dynamical processes.* Intuitionistic temporal logics have been proposed as a tool for modelling semantically-given processes. \cite{Maier2004} observed that an intuitionistic temporal logic with ‘henceforth’ and ‘eventually’ \( (\Diamond) \) could be used for reasoning about safety and liveness conditions in possibly-terminating reactive systems, and \cite{Fernandez-Duque2018} has suggested that a logic with ‘eventually’ can be used to provide a decidable framework in which to reason about topological dynamics.

*Temporal answer set programming.* In the areas of nonmonotonic reasoning, knowledge representation (KR), and artificial intelligence, intuitionistic and intermediate logics have played an important role within the successful answer set programming (ASP) \cite{Brewka2011} paradigm for KR, leading to several extensions of modal ASP \cite{Cabalar2007} that are supported by intuitionistic-based modal logics like temporal here and there \cite{Balbiani2016}.

Despite interest in the above applications, there is a large gap to be filled regarding our understanding of the computational behaviour of intuitionistic temporal logics. We have successfully employed semantical methods to show the decidability of the logic ITL\( ^e \) defined by a natural class of Kripke frames \cite{Boudou2017} and shown that these semantics correspond to a natural calculus over the \( \Box \)-free fragment \cite{Dieguez2018}. However, as we will see, in the presence of \( \Box \), new validities arise which may be undesirable from the point of view of an extended Curry-Howard isomorphism. Thus, our goal is to provide semantics for weaker axiomatically-defined intuitionistic temporal logics in order to provide tools for understanding their computational behaviour. We demonstrate the power of our semantics by separating several natural axiomatically-given calculi, which in particular answers in the negative a conjecture of \cite{Yuse2006} that the Gentzen-style and the Hilbert-style calculi presented there prove the same set of formulas.

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1. *Staged computation* is a technique that allows dividing the computation in order to exploit the early availability of some arguments.

2. In this paper, ‘eventually’ should be understood as ‘occurring at least once, either now or in the future,’ while ‘henceforth’ should be understood as ‘from now on.’
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There have already been some notable efforts towards a semantical study of intuitionistic temporal logics. (Kojima and Igarashi 2011) endowed Davies’s logic with Kripke semantics and provided a complete deductive system. Bounded-time versions of logics with henceforth were later studied by (Kamide and Wansing 2010). Both use semantics based on Simpson’s bi-relational models for intuitionistic modal logic (Simpson 1994). Since then, (Balbiani and Diéguez 2016) have shown that temporal here-and-there is continuous iff preimages of open sets are open. Similarly, setting $\llbracket \phi \rrbracket_S(x)$ to be interpreted as an open set $S(x) \subseteq X$, in order to interpret tenses, we equip $(X, T)$ with a continuous function $S: X \to X$. The classical semantics for next and eventually yield well-defined operations in this setting: for example, we define $\llbracket \Box \phi \rrbracket = S^{-1} \llbracket \phi \rrbracket$, which amounts to the standard definition where $x \in \llbracket \diamond \phi \rrbracket$ iff $S(x) \in \llbracket \phi \rrbracket$. The continuity of $S$ ensures that $\llbracket \diamond \phi \rrbracket$ is an open set whenever $\llbracket \phi \rrbracket$ is (recall that by definition, $S$ is continuous iff preimages of open sets are open). Similarly, setting $x \in \llbracket \diamond \phi \rrbracket$ iff there is $n \geq 0$ such that $S^n(x) \in \llbracket \phi \rrbracket$ ensures that $\llbracket \diamond \phi \rrbracket$ will always be open.

However, the classical definition of $\llbracket \square \phi \rrbracket$ would have that $x \in \llbracket \square \phi \rrbracket$ iff $S^n(x) \in \llbracket \phi \rrbracket$ for all $n \geq 0$ or, equivalently, $\llbracket \square \phi \rrbracket = \cap_{n \geq 0} \llbracket \phi \rrbracket$. The problem is that open sets need not be closed under infinite intersections, so an intuitionistic interpretation for $\square \phi$ must modify the classical semantics in a way that only open sets are produced. There are at
least two ways to achieve this. We call these the ‘weak’ and ‘strong’ interpretations of $\Box$. The first was originally proposed by (Kremer 2004) in an unpublished note, and is treated similarly to the universal quantifier in the context of intuitionistic semantics of first order logic. In order to distinguish it from the strong interpretation, we will denote it by $\Box^{\ast}$.  

As we will see, the operator $\Box^{\ast}$ does not satisfy some key LTL validities, namely $\Box p \rightarrow \Diamond \Box p$, $\Box \Diamond p \rightarrow \Diamond \Box p$, and $\Box p \rightarrow \Box \Box p$. Consequently, some of the standard LTL axioms are not sound for this interpretation. We thus propose a logic $\text{ITL}_{\Diamond \Box}$, where the axiom $\Box p \rightarrow \Diamond \Box p$ is replaced by the weaker $\Box p \rightarrow \Box \Diamond p$.

Nevertheless, $\Box p \rightarrow \Diamond \Box p$ is arguably one of the defining axioms for henceforth, so it is convenient to have semantics that validate it. In order to obtain semantics for $\text{ITL}_{\Diamond \Box}$, we propose a new interpretation for $\Box$. Our approach is natural from an algebraic perspective, as we define the interpretation of $\Box \varphi$ via a greatest fixed point in the Heyting algebra of open sets. We will show that dynamic topological systems provide semantics for the logics without the constant domain axiom, from which we conclude the independence of the latter. Moreover, we show that the Fischer Servi axioms are valid for the class of open dynamical topological systems, and that in this setting, the semantics for $\Box$ and $\Box$ coincide. The constant domain axiom shows that the $\{\Diamond, \Box\}$-logic of expanding posets is different from that of dynamic topological systems. We show via an alternative axiom that the $\{\Diamond, \Box\}$-logics are also different. We also consider the special case where topological semantics are based on Euclidean spaces. We show that this leads to logics strictly between that of all spaces and that of expanding posets. In the special case of the real line, we can prove that every formula falsified on a persistent poset is falsifiable on the real line.

**Layout.** Section 2 introduces the syntax and the axiomatic systems as well as its weak counterparts that we propose for intuitionistic temporal logic. Section 3 reviews dynamic topological systems, which are used in Section 4 to provide semantics for our formal language. Section 5 shows that four of our logics and their weak companions are each sound for a class of dynamical systems, and Section 6 shows that the remaining logics are sound for Euclidean spaces. In Section 7 we focus on $\text{ITL}_{\Diamond \Box}$ interpreted on persistent posets and its connection with the real line. In Section 8 we show that several of the logics we consider are pairwise distinct. Finally, Section 9 lists some open questions.

## 2 Syntax and axiomatics

In this section we will introduce several natural intuitionistic temporal logics. Most of the axioms we consider have appeared either in the intuitionistic logic, the temporal logic, or the intuitionistic modal logic literature. They will be based on the language of linear temporal logic, as defined next.

Fix a countably infinite set $P$ of propositional variables. The full language $\mathcal{L}_{\Diamond \Box \Box}$ of intuitionistic (linear) temporal logic ITL is given by the grammar in Backus-Naur form

$$ \varphi, \psi ::= \bot \mid p \mid \varphi \land \psi \mid \varphi \lor \psi \mid \varphi \rightarrow \psi \mid \Diamond \varphi \mid \Box \varphi \mid \Box \Diamond \varphi, $$

where $p \in P$. As usual, we use $\neg \varphi$ as a shorthand for $\varphi \rightarrow \bot$ and $\varphi \leftrightarrow \psi$ as a shorthand

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3. Kremer (2004) instead uses $\ast$.
4. We will not discuss Heyting algebras in this text, but see e.g. Heyting (1930) Mints (2000).
for \((\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)\). We read \(\circ\) as ‘next’, \(\Diamond\) as ‘eventually’, \(\Box\) as ‘strong henceforth’ and \(\lhd\) as ‘weak henceforth’. The intuition is that formulas are evaluated at moments of discrete time. The formula \(\circ \varphi\) indicates that \(\varphi\) will hold at the next moment, \(\Diamond \varphi\) that it will hold in some subsequent moment, and \(\Box \varphi\) and \(\lhd \varphi\) both indicate that \(\varphi\) will hold in every subsequent moment, including the current moment. However, as we will see, making sense of the latter notion in intuitionistic semantics is not straightforward, thus giving rise to two natural, but distinct, interpretations.

Given any formula \(\varphi\), we denote the set of subformulas of \(\varphi\) by \(\text{sub}(\varphi)\). For \(\Theta \subseteq \{\Diamond, \Box, \lhd\}\), the language \(L_\Theta\) is the sub-language of \(L_{\circ\Box\lhd}\) whose only tenses are \(\circ\) and those in \(\Theta\); we will not consider languages without \(\circ\). So, for example, \(L_\circ\) only has tenses \(\circ\) and \(\Diamond\). We will write \(L_\Xi\) instead of \(L_{\circ\Box\lhd}\).

The tenses \(\lhd, \Box\) represent two possible intuitionistic readings of ‘henceforth’ and thus we will rarely consider logics with both. In order to compare logics based on \(\lhd\) with logics based on \(\Box\), we introduce the translations \(t_\Xi\), where \(t_\Xi(\varphi) \in L_{\circ\Box\lhd}\) is the formula obtained by replacing every occurrence of \(\Box\) in \(\varphi\) by \(\lhd\), and similarly define \(t_\Xi\), which replaces every occurrence of \(\lhd\) by \(\Box\). The semantics for \(\lhd\) first appeared in the unpublished note (Kremer 2004), while those for \(\Box\) were first introduced in a preliminary version of this paper (Boudou et al. 2019).

We begin by establishing our basic axiomatization for logics over \(L_{\circ\Box}\). It is obtained by adapting the standard axioms and inference rules of LTL (Lichtenstein and Pnueli 2000) as well as their dual versions.

**Definition 2.1**

The logic \(\text{ITL}_{\circ\Box}\) is the least set of \(L_{\circ\Box}\)-formulas closed under the following axioms and rules.

(i) All intuitionistic tautologies;

(ii) \(\neg \circ \perp\);

(iii) \(\circ (\varphi \land \psi) \leftrightarrow (\circ \varphi \land \circ \psi)\);

(iv) \(\circ (\varphi \lor \psi) \leftrightarrow (\circ \varphi \lor \circ \psi)\);

(v) \(\circ (\varphi \rightarrow \psi) \rightarrow (\circ \varphi \rightarrow \circ \psi)\);

(vi) \(\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)\);

(vii) \(\Box (\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi)\);

(viii) \(\Box \varphi \rightarrow \Diamond \varphi\);

(ix) \(\Box \varphi \rightarrow \Diamond \varphi\);

(x) \(\varphi \rightarrow \Diamond \varphi\);

(xi) \(\circ \Diamond \varphi \rightarrow \Diamond \varphi\);

(xii) \(\Box (\varphi \rightarrow \Diamond \varphi) \rightarrow (\varphi \rightarrow \Box \varphi)\);

(xiii) \(\Box (\circ \varphi \rightarrow \varphi) \rightarrow (\Diamond \varphi \rightarrow \varphi)\);

(xiv) \(\varphi \rightarrow \psi \vdash \varphi \rightarrow \psi\);

(xv) \(\varphi \rightarrow \psi \vdash \Diamond \varphi \rightarrow \Diamond \psi\);

Axioms (V) and (VI) hold in any normal modal logic and (VIII) is a dual version of (VI). Such dual axioms are often needed in intuitionistic modal logic, since \(\Diamond\) and \(\Box\) are not typically inter-definable. The axioms (II) and (IV) have to do with the passage of time being deterministic in linear temporal logic, and are related to a functional modality, i.e. a modality that is interpreted using a function rather than a relation. The axioms (VIII) and (X) have to do with future tenses being interpreted reflexively, i.e. \(\varphi\) is considered to hold eventually if it holds now. The axiom (XI) states that if something will henceforth be the case, then in the next moment, it will still henceforth be the case, and (XIII) is successor induction, as time is interpreted over the natural numbers. Axioms (XI) and (XIII) are their duals. All rules are standard in any normal modal logic.

Each axiom is either included in the axiomatization of Goldblatt (Goldblatt 1992) page
or is a variant of one of them (e.g., a contrapositive); this is standard in intuitionistic modal logic, as such variants are needed to account for the independence of the basic connectives. We do not consider ‘until’ and ‘release’ in this paper, but these operators have previously studied within an intuitionistic context in Balbiani et al. (2019).

Next we define our base logic for weak henceforth. It is convenient to present it as a logic over $L_{\Box}$ and then translate to $L_{\Box^{\ast}}$. The main reason for this is that we view the weak and strong semantics of henceforth as two possible interpretations of intuitionistic temporal logic, rather than two independent tenses. From this point of view, the notation $\Box$ should be seen as an indication that weak semantics are being used. Moreover, we are interested in comparing logics based on $\Box$ with those based on $\square$, and uniform notation will be helpful for this. In particular, we will see that the weak semantics give rise to weaker logics, partially motivating the terminology. We will then use the translation $t_{\Box}$ (which, recall, replaces $\square$ by $\Box$) when we wish to indicate that we are working with weak semantics.

We define $ITL_{0}^{\Box}$ as $ITL_{\Box}$, but replacing axiom (IX) by

$$WH \overset{\text{def}}{=} \square \varphi \rightarrow \square \square \varphi.$$  

Here, WH stands for ‘weak henceforth’. This terminology is justified by the following.

**Lemma 2.2**

Every instance of WH is derivable in $ITL_{\Box}$.

**Proof**

From (VIII), (XV) and (VI) we obtain $\square \square \varphi \rightarrow \square \varphi$, which combined with axiom (IX) allows us to conclude $\vdash \square \varphi \rightarrow \square \varphi$. Thanks to (XV) and axiom (VII) we get

$$\vdash \square \square \varphi \rightarrow \square \varphi.$$  

(1)

From axiom (XV) necessitation and (XII) we obtain

$$\vdash \square (\square \varphi \rightarrow \square \square \varphi) \rightarrow (\square \varphi \rightarrow \square \square \varphi).$$  

(2)

From (1) and (2) we conclude $\vdash \square \varphi \rightarrow \square \square \varphi$.  

Thus $ITL_{0}^{\Box} \subseteq ITL_{\Box}$. We will later see that the inclusion is strict, since $\square \varphi \rightarrow \square \square \varphi$ is not valid in general for our semantics, but $\Box \varphi \rightarrow \Box \Box \varphi$ and $\Box \varphi \rightarrow \Box \Box \varphi$ are. We can then define $ITL_{\Box^{\ast}} = t_{\Box}(ITL_{0}^{\Box})$.

Modal intuitionistic logics often involve additional axioms, and in particular Fischer Servi (1984) includes the schema

$$FS_{\Box}(\varphi, \psi) \overset{\text{def}}{=} (\Box \varphi \rightarrow \Box \psi) \rightarrow \Box (\varphi \rightarrow \psi).$$

We also define

$$FS_{\Box}(\varphi, \psi) \overset{\text{def}}{=} (\Box \varphi \rightarrow \Box \psi) \rightarrow \Box (\varphi \rightarrow \psi).$$

This notation is justified in view of the fact that $\Box$ is self-dual in linear temporal logic: since time is modeled deterministically, ‘in some next moment’ is equivalent to ‘in every next moment,’ so $\Box$ may be regarded as a ‘box’ or a ‘diamond.’ It is further motivated by the following.
Proposition 2.3
The formula $FS_{\Diamond}(\Diamond p, \Box q) \rightarrow FS_{\Diamond}(p, q)$ is derivable in $\text{ITL}_{\Diamond\Box}$.

Proof
We reason within $\text{ITL}_{\Diamond\Box}$. Assume 1) $FS_{\Diamond}(\Diamond p, \Box q)$ and 2) $(\Diamond p \rightarrow \Box q)$. Notice that $\Diamond \Diamond p \rightarrow \Diamond p$ and $\Box p \rightarrow \Box \Box p$ are instances of axioms \text{[IX]} and \text{[XI]}.

From this and assumption \text{[2]} we conclude $\Diamond \Diamond p \rightarrow \Box \Box p$. Thanks to assumption \text{[1]} and Modus Ponens we conclude $\Diamond (\Diamond p \rightarrow \Box q)$. Therefore, $(\Diamond p \rightarrow \Box q) \rightarrow \Diamond (\Diamond p \rightarrow \Box q)$. By Rule \text{[XV]} we obtain $\Box ((\Diamond p \rightarrow \Box q) \rightarrow \Diamond (\Diamond p \rightarrow \Box q))$. By the induction axiom \text{[XII]} we derive $(\Diamond p \rightarrow \Box p) \rightarrow \Box (\Diamond p \rightarrow \Box q)$. From the assumption \text{[2]} and Modus Ponens it follows that $\Box (\Diamond p \rightarrow \Box q)$.

Note that $(\Diamond p \rightarrow \Box q) \rightarrow (p \rightarrow q)$ is derivable in $\text{ITL}_{\Diamond\Box}$. From rule \text{[XV]} and axiom \text{[VI]} we obtain $\Box (\Diamond p \rightarrow \Box q) \rightarrow \Box (p \rightarrow q)$. From this and $\Box (\Diamond p \rightarrow \Box q)$ it follows that $\Box (p \rightarrow q)$, as required. □

Later we will show that these schemas lead to logics strictly stronger than $\text{ITL}_{\Diamond\Box}$.

Next we consider additional axioms reminiscent of the constant domain axiom in first-order intuitionistic logic, namely $\forall x(\varphi(x) \lor \psi(x)) \rightarrow \exists x \varphi(x) \lor \forall x \psi(x)$; we maintain the terminology ‘constant domain’ due to this similarity, although they do not retain this meaning in our logics. As we will see, in the context of intuitionistic temporal logics, these axioms separate Kripke semantics from the more general topological semantics.

$$CD(\varphi, \psi) \overset{\text{def}}{=} \Box (\varphi \lor \psi) \rightarrow \Box \varphi \lor \Diamond \psi$$

$$BI(\varphi, \psi) \overset{\text{def}}{=} \Box (\varphi \lor \psi) \land \Box (\Diamond \psi \rightarrow \psi) \rightarrow \Box \varphi \lor \psi.$$  

Here, CD stands for ‘constant domain’ and BI for ‘backward induction’. We also define as a special case $CD^{-}(\varphi) = CD(\neg \varphi, \varphi)$.

The axiom CD might not be desirable from a constructive perspective, as from $\Box (\varphi \lor \psi)$ one cannot in general extract an upper bound for a witness for $\Diamond \psi$.

The axiom BI is a $\Diamond$-free version of CD, as witnessed by the following.

Proposition 2.4
The following formulas are derivable in $\text{ITL}_{\Diamond\Box}^{0}$:

1. CD$(p, q) \rightarrow BI(p, q)$;
2. BI$(p, \Diamond q) \rightarrow CD(p, q)$.

Proof
We reason within $\text{ITL}_{\Diamond\Box}^{0}$. For the first claim, assume that 1) CD$(p, q)$, 2) $\Box (\Box q \rightarrow q)$, and 3) $\Box (p \lor q)$.

From \text{[1]} and \text{[2]}, we obtain $\Box p \lor \Diamond q$, which together with \text{[2]} and axiom \text{[XIII]} gives us $\Box p \lor q$, as needed.

For the second, assume 1) BI$(p, \Diamond q)$ and 2) $\Box (p \lor q)$. From $\Box (p \lor q)$, axiom \text{[XI]} and some modal reasoning, $\Box (p \lor \Diamond q)$. Also from axiom \text{[XI]} and rule \text{[XV]} $\Box (\Diamond \Diamond q \rightarrow \Diamond q)$.

From BI$(p, \Diamond q)$ we obtain $\Box p \lor \Diamond q$, as needed. □

\footnote{For example, if $\varphi$ represents the ‘active’ states and $\psi$ the ‘halting’ states of a program, then CD would require us to decide whether the program halts, which is not possible to do constructively.}
Finally, we introduce the conditional excluded middle axiom

\[ \text{CEM}(p, q) \overset{\text{def}}{=} (\neg \lozenge p \land \lozenge \neg p) \rightarrow (\lozenge q \lor \neg \lozenge q). \]

This axiom states that a certain instance of excluded middle holds, provided some assumptions are satisfied. It is less familiar than others we have considered, but its role will become clear when we consider semantics based on the real line. With this, we define a handful of logics, listed in Table 2.1, along with definitions of the ‘optional’ axioms. The inclusions between these logics are summarized in Figure 2.4 as we will show in this paper, these are the only inclusions that hold between these logics.\(^6\)

| Symbolic Notation | Name |
|-------------------|------|
| \( \text{WH}(\varphi) \) | Weak Excluded Middle |
| \( \text{FS}_o(\varphi, \psi) \) | First-Order Excluded Middle |
| \( \text{FS}_\circ(\varphi, \psi) \) | First-Order Excluded Middle |
| \( \text{CD}(\varphi, \psi) \) | First-Order Excluded Middle |
| \( \text{CD}^{-}(\varphi) \) | First-Order Excluded Middle |
| \( \text{BI}(\varphi, \psi) \) | First-Order Excluded Middle |
| \( \text{CEM}(\varphi, \psi) \) | First-Order Excluded Middle |

\[ \text{ITL}_0 \circ = (\text{See Definition 2.1}) \]
\[ \text{ITL}^0 \circ = \text{ITL}_0 \circ - \{\text{IEX}\} + \text{WH} \]
\[ \text{ITL}^+ \circ = \text{ITL}^0 \circ + \text{CD}^{-} \]
\[ \text{ETL}_0 \circ = \text{ITL}_0 \circ + \text{CD}^{-} + \text{CEM} \]
\[ \text{RTL}_0 \circ = \text{ITL}_0 \circ + \text{CD}^{-} + \text{CEM} \]
\[ \text{CDTL}_0 \circ = \text{ITL}_0 \circ + \text{CD}^{-} + \text{CEM} \]
\[ \text{CDTL}^+ \circ = \text{ITL}_0 \circ + \text{FS}_o + \text{CD}^{-} \]
\[ \text{ETL}^+ \circ = \text{ITL}_0 \circ + \text{FS}_o + \text{CD}^{-} \]
\[ \text{CDTL}^+ \circ = \text{ITL}_0 \circ + \text{FS}_o + \text{CD}^{-} \]

Table 2.1: Axioms not listed in Definition 2.1 (above) and logics based on strong and weak henceforth (below). In the right-hand column, notice that only \( \text{ITL}^0 \circ \), \( \text{ETL}^0 \circ \) and \( \text{RTL}^0 \circ \) are based on \( \text{ITL}^0 \circ \).

Here, \( \text{RTL}_0 \circ \) stands for ‘real temporal logic’, \( \text{ETL}_0 \circ \) for ‘Euclidean temporal logic’ and \( \text{CDTL}_0 \circ \) for ‘constant domain temporal logic’. For a logic \( \Lambda \) in the above list, \( \Lambda^0 \) is defined analogously but replacing \( \text{ITL}_0 \circ \) by \( \text{ITL}^0 \circ \). Logics over \( \mathcal{L}_0 \circ \) are defined in Table 2.1. Note that logics with either CD or \( \text{FS}_o \) use the strong axiom, \( \boxast \varphi \rightarrow \lozenge \boxast \varphi \). This has to do with our semantics and will become clear later.

\(^6\) Note that our notation for logics has been modified from that in (Boudou et al. 2019), in order to accommodate the larger family we now consider. Specifically, \( \text{ITL}_0 \circ \) was denoted \( \text{ITL}^0 \circ \), \( \text{ITL}^+ \circ \) was denoted \( \text{ITL}^+ \circ \), \( \text{CDTL}_0 \circ \) was denoted \( \text{ITL}^{-} \circ \), and \( \text{CDTL}^+ \circ \) was denoted \( \text{ITL}^1 \circ \). Note that \( \text{ITL}^0 \circ \) is weaker than \( \text{ITL}^0 \circ \).
Fig. 2.1: Inclusions between the logics based on strong or weak henceforth we have defined; an arrow $\Lambda_1 \rightarrow \Lambda_2$ means that every theorem of $\Lambda_1$ is a theorem of $\Lambda_2$.

Our list is not meant to exhaust all combinations of axioms; rather, we only consider logics that arise from natural classes of models. Before discussing semantics, we establish the only non-trivial inclusion between these logics.

**Lemma 2.5**
Every instance of CEM is derivable in $\text{ITL}^+_\circ\Box$.

**Proof**
It is not hard to check that $\neg(\neg\Box p \land \Box \neg\neg p)$ is derivable in $\text{ITL}^+_\circ\Box$, hence so is CEM.

This immediately yields that $\text{RTL}_{\circ\Box} \subseteq \text{ETL}^+_{\circ\Box}$:

**Proposition 2.6**
Every formula derivable in $\text{RTL}_{\circ\Box}$ is derivable in $\text{ETL}^+_{\circ\Box}$.

We are also interested in logics over sublanguages of $\mathcal{L}_{\circ\Box}$ or $\mathcal{L}_{\circ\Box^*}$. For any logic $\Lambda$ defined above, let $\Lambda_{\Box}$ be defined by restricting similarly all rules and axioms to $\mathcal{L}_{\Box}$, except that when CD is an axiom of $\Lambda$, we add the axiom BI to $\Lambda_{\Box}$. In these cases, $\Lambda_{\Box^*}$ is similarly defined using $t_{\Box^*}(\text{BI})$. The logic $\text{ITL}_{\Box}$ is similar to a Hilbert calculus for the $\land, \lor$-free fragment considered by Yuse and Igarashi (Yuse and Igarashi 2006), although they do not include induction but include the axioms $\Box p \rightarrow \Box \Box p$ and $\Box \Box p \leftrightarrow \Box \Box p$. It is not difficult to check that the latter are derivable from our basic axioms, and hence their logic is contained in $\text{ITL}_{\Box}$.

We also define $\Lambda_{\Box}$ to be the logic obtained by restricting all rules and axioms to $\mathcal{L}_{\Box}$, and adding the rules $\frac{\psi \rightarrow \Box \varphi}{\varphi \rightarrow \Box \varphi}$ and $\frac{\Box \varphi \rightarrow \Box \psi}{\Box \varphi \rightarrow \Box \psi}$. Note that these rules correspond to axioms [(VII)](x) [(XIII)](x) respectively, but do not involve $\Box$. In this paper we are mostly concerned with logics including ‘henceforth’, but $\Box$-free logics are studied in detail by (Diéguez and Fernández-Duque 2018).

### 3 Dynamic topological systems

The logics defined above are pairwise distinct. We will show this by introducing semantics for each of them. They will be based on dynamic topological systems (or dynamical systems for short), which, as was observed in (Fernández-Duque 2018), generalize their
Kripke semantics (Boudou et al. 2017). In this section, we review the basic notions of topological dynamics needed in the rest of the text. Let us first recall the definition of a topological space, as in e.g. (Dugundji 1975):

**Definition 3.1**
A topological space is a pair \((X, T)\), where \(X\) is a set and \(T\) a family of subsets of \(X\) satisfying a) \(\emptyset, X \in T\); b) if \(U, V \in T\) then \(U \cap V \in T\), and c) if \(O \subseteq T\) then \(\bigcup O \in T\). The elements of \(T\) are called open sets.

If \(x \in X\), a neighbourhood of \(x\) is an open set \(U \subseteq X\) such that \(x \in U\). Given a set \(A \subseteq X\), its interior, denoted \(A^\circ\), is the largest open set contained in \(A\). It is defined formally by

\[ A^\circ = \bigcup \{ U \in T : U \subseteq A \} \tag{3} \]

Dually, we define the closure \(\overline{A}\) as \(X \setminus (X \setminus A)^*\); this is the smallest closed set containing \(A\).

If \((X, T)\) is a topological space, a function \(S : X \to X\) is continuous if, whenever \(U \subseteq X\) is open, it follows that \(S^{-1}[U]\) is open. The function \(S\) is open if, whenever \(V \subseteq X\) is open, then so is \(S[V]\). An open, continuous function is an interior map, and a bijective interior map is a homeomorphism; equivalently, \(S\) is a homeomorphism if it is invertible and both \(S\) and \(S^{-1}\) are continuous.

A dynamical system is then a topological space equipped with a continuous function:

**Definition 3.2**
A dynamical (topological) system is a triple \(X = (X, T, S)\) such that \((X, T)\) is a topological space and \(S : X \to X\) is continuous. We say that \(X\) is open if \(S\) is an interior map and invertible if \(S\) is a homeomorphism.

Topological spaces generalize posets in the following way. Let \(\mathcal{F} = (W, \preceq)\) be a poset; that is, \(W\) is any set and \(\preceq\) is a transitive, reflexive, antisymmetric relation on \(W\). To see \(\mathcal{F}\) as a topological space, define \(\uparrow w = \{ v : w \preceq v \}\). Then consider the topology \(\mathcal{T}_\preceq\) on \(W\) given by setting \(U \subseteq W\) to be open if and only if, whenever \(w \in U\), we have \(\uparrow w \subseteq U\). A topology of this form is an up-set topology (Aleksandroff 1937). The interior operator on such a topological space is given by

\[ A^\circ = \{ w \in W : \uparrow w \subseteq A \} \tag{4} \]

i.e., \(w\) lies on the interior of \(A\) if whenever \(v \succeq w\), it follows that \(v \in A\).

Throughout this text we will often identify partial orders with their corresponding topologies, and many times do so tacitly. In particular, a dynamical system generated by a poset is called a dynamical or expanding poset, due to its relation to expanding products of modal logics (Gabelaia et al. 2006). It will be useful to characterize the continuous and open functions on posets (see Figure 3.1):

**Lemma 3.3**
Consider a poset \((W, \preceq)\) and a function \(S : W \to W\). Then,

1. the function \(S\) is continuous with respect to the up-set topology if and only if, whenever \(w \preceq w'\), it follows that \(S(w) \preceq S(w')\),
2. the function \(S\) is open with respect to the up-set topology if whenever \(S(w) \preceq v\), there is \(w' \in W\) such that \(w \preceq w'\) and \(S(w') = v\).
These are properties common in multi-modal logics and we refer to them as ‘confluence properties.’ A persistent function is an open, continuous map on a poset.

\[ w' \xrightarrow{S} w \]

\[ w' \xrightarrow{S} w \]

(a) Continuity

(b) Openness

Fig. 3.1: On a dynamic poset, the above diagrams can always be completed if $S$ is continuous or open, respectively.

4 Semantics

In this section we will see how dynamical systems can be used to provide a natural intuitionistic semantics for the language of linear temporal logic. Classically LTL may be interpreted over structures $(X,S)$ where $X$ is a set and $S: X \to X$. In this setting, $\Diamond \varphi$ is true on a point $x$ if $\varphi$ is true ‘at the next moment,’ i.e., on $S(x)$; $\varphi$ is true on a point $x$ if $\varphi$ is ‘eventually’ true, i.e. there is $n \geq 0$ such that $\varphi$ is true on $S^n(x)$, and $\Box \varphi$ is true on a point $x$ if $\varphi$ is ‘henceforth’ true, i.e. for all $n \geq 0$, $\varphi$ is true on $S^n(x)$. Meanwhile, topological spaces provide semantics for intuitionistic logic, where each formula is assigned an open set. Under this semantics, $\varphi \to \psi$ is true on $x$ if there is a neighborhood $U$ of $x$ (i.e., an open set $U$ with $x \in U$) such that every $y \in U$ satisfying $\varphi$ also satisfies $\psi$. Thus it is natural to interpret intuitionistic temporal logic on dynamical systems, which are endowed with both a topology and a transition function. In this setting, the classical definitions of $\Diamond$ and $\Diamond$ readily adapt to the topological setting without modification. On the other hand, the classical definition of $\Box$ does not necessarily yield open sets, and to this end we consider two variants of henceforth, the weak variant, $\Box$, and the strong variant, which we simply denote $\Box$.

Definition 4.1

Given a dynamical system $\mathcal{X} = (X,T,S)$, a valuation on $\mathcal{X}$ is a function $\llbracket \cdot \rrbracket: \mathcal{L}_{\Diamond\Box}\Box \to T$ such that

\[
\begin{align*}
\llbracket \top \rrbracket & = \emptyset \\
\llbracket \varphi \land \psi \rrbracket & = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket \\
\llbracket \varphi \lor \psi \rrbracket & = \llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket \\
\llbracket \varphi \to \psi \rrbracket & = ((X \setminus \llbracket \varphi \rrbracket) \cup \llbracket \psi \rrbracket)^\circ \\
\llbracket \Diamond \varphi \rrbracket & = \bigcup_{n \geq 0} S^{-n} \llbracket \varphi \rrbracket \\
\llbracket \Box \varphi \rrbracket & = \bigcup \left\{ U \in T : S[U] \subseteq U \subseteq \llbracket \varphi \rrbracket \right\} \\
\llbracket \Box \varphi \rrbracket & = \left( \bigcap_{n \geq 0} S^{-n} \llbracket \varphi \rrbracket \right)^\circ \\
\llbracket \Boxast \varphi \rrbracket & = \left( \bigcap_{n \geq 0} S^{-n} \llbracket \varphi \rrbracket \right)^\circ
\end{align*}
\]

A tuple $\mathcal{M} = (X,T,S,\llbracket \cdot \rrbracket)$ consisting of a dynamical system with a valuation is a dynamic topological model, and if $T$ is generated by a partial order, we will say that $\mathcal{M}$ is a dynamic poset model.

All of the semantic clauses are standard from either intuitionistic or temporal logic, with the exception of those for $\Boxast \varphi$ and $\Box \varphi$, which we discuss in greater detail in the
remainder of this section. It is not hard to check by structural induction on \( \varphi \) that \([\varphi]\) is uniquely defined given any assignment of the propositional variables to open sets, and that \([\varphi]\) is always open. We define validity in the standard way, and with this introduce additional semantically-defined logics, two of which were studied in (Boudou et al., 2017).

**Definition 4.2**
If \( \mathcal{M} = (X, \mathcal{T}, S, [\cdot]) \) is any dynamic topological model and \( \varphi \in \mathcal{L} \) is any formula, we write \( \mathcal{M} \models \varphi \) if \([\varphi]\) = \(X\). Similarly, if \( \mathcal{S} = (X, \mathcal{T}, S) \) is a dynamical system, we write \( \mathcal{S} \models \varphi \) if for any valuation \([\cdot]\) on \( \mathcal{X} \), we have that \((\mathcal{S}, [\cdot]) \models \varphi \); and, if \( \mathcal{X} = (X, \mathcal{T}) \) is any topological space, then \( \mathcal{X} \models \varphi \) if, for any continuous \( S: X \to X \), \((X, \mathcal{T}, S) \models \varphi \). Finally, if \( \Omega \) is a class of structures (either topological spaces, dynamical systems, or models), we write \( \Omega \models \varphi \) if for every \( \mathcal{A} \in \Omega \), \( \mathcal{A} \models \varphi \), in which case we say that \( \varphi \) is valid on \( \Omega \).

If \( \Omega \) is either a structure or a class of structures and \( \Theta \subseteq \{\bigcirc, \square, \Box\} \), we write \( \text{ITL}_\Theta^\Omega \) for the set of \( \mathcal{L}_\Theta \) formulas valid on \( \Omega \). We maintain the convention that \( \bigcirc \) is assumed to be in all languages, and we write \( \text{ITL}_\varnothing^\Omega \) when \( \Theta = \varnothing \).

The main classes of dynamical systems we are interested in are listed in Table 4.1.

For example, \( \text{ITL}_{\Box} \) denotes the set of \( \mathcal{L}_{\Box} \)-formulas valid over the class of all dynamical systems.

| c | all dynamical systems (with a continuous function) |
| e | expanding posets |
| p | persistent posets |
| o | open dynamical systems |
| \( \mathbb{R}^n \) | systems based on \( n \)-dimensional Euclidean space |

**Table 4.1:** The main classes of dynamical systems appearing in the text.

In practice, it is convenient to have a 'pointwise' characterization of the semantic clauses of Definition 4.1. For a model \( \mathcal{M} = (X, \mathcal{T}, S, [\cdot]) \), \( x \in X \) and \( \varphi \in \mathcal{L} \), we write \( \mathcal{M}, x \models \varphi \) if \( x \in [\varphi] \), and \( \mathcal{M} \models \varphi \) if \([\varphi]\) = \(X\). Then, in view of (4), given formulas \( \varphi \) and \( \psi \), we have that \( \mathcal{M}, x \models \varphi \rightarrow \psi \) if and only if there is a neighbourhood \( U \) of \( x \) such that for all \( y \in U \), if \( \mathcal{M}, y \models \varphi \) then \( \mathcal{M}, y \models \psi \); note that this is a special case of neighbourhood semantics (Pacuit, 2017). The following simple observation will be useful.

**Lemma 4.3**
If \( \mathcal{M} = (X, \mathcal{T}, S, [\cdot]) \) is any model and \( \varphi, \psi \in \mathcal{L}_{\Box} \), then \( \mathcal{M} \models \varphi \rightarrow \psi \) if and only if \( [\varphi] \subseteq [\psi] \).

**Proof**
If \( [\varphi] \subseteq [\psi] \) then \( (X \setminus [\varphi]) \cup [\psi] = X \), so \([\varphi \rightarrow \psi] = ((X \setminus [\varphi]) \cup [\psi])^\circ = X \). Otherwise, there is \( z \in [\varphi] \) such that \( z \notin [\psi] \), so that \( z \notin ((X \setminus [\varphi]) \cup [\psi])^\circ \), i.e. \( z \notin [\varphi \rightarrow \psi] \). \( \square \)

Using (4), this can be simplified somewhat in the case that \( \mathcal{T} \) is generated by a partial order \( \preceq \):
Proposition 4.4
If \((X, \preceq, S, [\cdot])\) is a dynamic poset model, \(x \in X\), and \(\varphi, \psi\) are formulas, then \(M, x \models \varphi \rightarrow \psi\) if and only if from \(y \geq x\) and \(M, y \models \varphi\), it follows that \(M, y \models \psi\).

This is the standard relational interpretation of implication, and thus topological semantics are a generalization of the usual Kripke semantics.

The semantics for \(\boxdot\) were originally introduced by [Kremer 2004] as an intuitionistic reading of ‘henceforth’. By analogy with \(\Diamond\), one might try to interpret \([\boxdot \varphi]\) as \(\bigcap_{n \geq 0} S^{-n} [\varphi]\). But this does not quite work since, on this interpretation, there would be no guarantee that \([\boxdot \varphi]\) is open. Instead, we consider interpreting \([\boxdot \varphi]\) as the interior of \(\bigcap_{n \geq 0} S^{-n} [\varphi]\). In other words, \(M, x \models \boxdot \varphi\) if and only if there is a neighbourhood \(U\) of \(x\) so that for every \(y \in U\) and every \(n \in \mathbb{N}\), one has that \(M, S^n(y) \models \varphi\).

This interpretation of ‘henceforth’ is analogous to the interpretation in [Rasiowa and Sikorski 1963], and going back to [Mostowski 1948], of \(\forall x\) in the topological semantics for quantified intuitionistic logic. We may interpret variables as ranging over some non-empty domain \(D\), and truth values as open sets in some topological space \((X, T)\). The semantic clauses for \(\exists x\) and \(\forall x\) are, essentially, the following:

\[
[\exists x \varphi] = \bigcup_{d \in D} [\varphi[d/x]]
\]
\[
[\forall x \varphi] = (\bigcap_{d \in D} [\varphi[d/x]])^\circ
\]

Note that if \(D\) is infinite then the intersection in the definition of \([\forall x \varphi]\) may also be infinite and hence the application of the interior operator is necessary in order to obtain an open truth value.

The semantics for \(\boxdot \varphi\) are also an intuitionistic interpretation of ‘henceforth’, but from a more algebraic perspective. In classical temporal logic, \([\boxdot \varphi]\) is the largest set contained in \([\varphi]\) which is closed under \(S\). In our semantics, \([\boxdot \varphi]\) is the greatest open set which is closed under \(S\). If \(M, x \models \Box \varphi\), this fact is witnessed by an open, \(S\)-invariant neighbourhood of \(x\), where \(U \subseteq X\) is \(S\)-invariant if \(S[U] \subseteq U\).

Proposition 4.5
If \((X, T, S, [\cdot])\) is a dynamic topological model, \(x \in X\), and \(\varphi\) is any formula, then \(M, x \models \Box \varphi\) if and only if there is an \(S\)-invariant neighbourhood \(U\) of \(x\) such that for all \(y \in U\), \(M, y \models \varphi\).

In fact, the open, \(S\)-invariant sets form a topology; that is, the family of \(S\)-invariant open sets is closed under finite intersections and arbitrary unions, and both the empty set and the full space are open and \(S\)-invariant (this follows readily from the fact that the topology \(T\) already has these properties, as does the family of \(S\)-invariant sets). This topology is coarser than \(T\), in the sense that every \(S\)-invariant open set is (tautologically) open. Thus \(\Box\) can itself be seen as an interior operator based on a coarsening of \(T\), and \([\Box \varphi]\) is always an \(S\)-invariant open set.

Example 4.6
As usual, the real number line is denoted by \(\mathbb{R}\) and we assume that it is equipped with the standard topology, where \(U \subseteq \mathbb{R}\) is open if and only if it is a union of intervals of the form \((a, b)\). Consider a dynamical system based on \(\mathbb{R}\) with \(S: \mathbb{R} \rightarrow \mathbb{R}\) given by \(S(x) = 2x\).
We claim that for any model $\mathcal{M}$ based on $(\mathbb{R}, S)$ and any formula $\varphi$, $\mathcal{M}, 0 \models \square \varphi$ if and only if $\mathcal{M} \models \varphi$.

To see this, note that one implication is obvious since $\mathbb{R}$ is open and $S$-invariant, so if $\llbracket \varphi \rrbracket = \mathbb{R}$ it follows that $\mathcal{M}, 0 \models \square \varphi$. For the other implication, assume that $\mathcal{M}, 0 \models \square \varphi$, so that there is an $S$-invariant, open $U \subseteq \llbracket \varphi \rrbracket$ with $0 \in U$. It follows from $U$ being open that for some $\varepsilon > 0$, $(-\varepsilon, \varepsilon) \subseteq U$. Now, let $x \in \mathbb{R}$, and let $n$ be large enough so that $|2^{-n}x| < \varepsilon$. Then, $2^{-n}x \in U$, and since $U$ is $S$-invariant, $x = S^n(2^{-n}x) \in U$. Since $x$ was arbitrary, $U = \mathbb{R}$, and it follows that $\mathcal{M} \models \varphi$.

On the other hand, suppose that $0 < a < x$ and $(a, \infty) \subseteq \llbracket \varphi \rrbracket$. Then, $(a, \infty)$ is open and $S$-invariant, so it follows that $x \in \llbracket \square \varphi \rrbracket$. Hence in this case we do not require that $\llbracket \varphi \rrbracket = \mathbb{R}$. Similarly, if $x < a < 0$ and $(-\infty, a) \subseteq \llbracket \varphi \rrbracket$, we readily obtain $x \in \llbracket \square \varphi \rrbracket$.

We will see more examples in Section 8 where we show, among other things, that the two interpretations of 'henceforth' are not equivalent. In general, we only obtain one implication.

Lemma 4.7
The formula $\square p \rightarrow \lozenge p$ is valid over the class of dynamical systems.

Proof
Let $\mathcal{M} = (X, \mathcal{T}, S, [\cdot])$ be any dynamical model. Suppose that $w \in \llbracket \square p \rrbracket$, and let $U$ be an $S$-invariant neighbourhood of $x$ such that $U \subseteq \llbracket p \rrbracket$. Then, using the $S$-invariance of $U$ we see by a routine induction on $n$ that $U \subseteq S^{-n} \llbracket p \rrbracket$, hence $U \subseteq \bigcap_{n \in \mathbb{N}} S^{-n} \llbracket p \rrbracket$. As $U$ is open, $U \subseteq \left( \bigcap_{n \in \mathbb{N}} S^{-n} \llbracket \varphi \rrbracket \right)^{\circ}$, and hence $w \in \llbracket \lozenge p \rrbracket$. Since $w$ was arbitrary, $\mathcal{M} \models \square p \rightarrow \lozenge p$. \(\square\)

However, when restricted to 'nice' dynamical systems, the two versions of 'henceforth' coincide.

Proposition 4.8
Let $\mathcal{M} = (W, \leq, S, [\cdot])$ be any dynamic poset model, $w \in W$ and $\varphi \in \mathcal{L}$. Then, the following are equivalent:

a) $w \in \llbracket \square \varphi \rrbracket$;  
b) $w \in \llbracket \lozenge \varphi \rrbracket$;  
c) $\forall n \in \mathbb{N} \ (S^n(w) \in \llbracket \varphi \rrbracket)$.

Proof
By Lemma 4.7 [a] implies [b]. That [b] implies [c] is immediate from $\left( \bigcap_{n \in \mathbb{N}} S^{-n} \llbracket \varphi \rrbracket \right)^{\circ} \subseteq \bigcap_{n \in \mathbb{N}} S^{-n} \llbracket \varphi \rrbracket$, so it remains to show that [c] implies [a]. Suppose that for all $n \in \mathbb{N}$, $\mathcal{M}, S^n(w) \models \varphi$, and let $U = \bigcup_{n \in \mathbb{N}} \uparrow S^n(w)$. That the set $U$ is open follows from each $\uparrow S^n(w)$ being open and unions of opens being open. If $v \in U$, then $v \triangleright S^n(w)$ for some $n \in \mathbb{N}$ and hence by upwards persistence, from $\mathcal{M}, S^n(w) \models \varphi$ we obtain $\mathcal{M}, v \models \varphi$; moreover, $S(v) \triangleright S^n+1(w)$ so $S(v) \in U$. Since $v \in U$ was arbitrary, we conclude that $U$ is $S$-invariant and $U \subseteq \llbracket \varphi \rrbracket$. Thus $U$ witnesses that $\mathcal{M}, w \models \square \varphi$. \(\square\)

As we will see later, Proposition 4.8 fails over the class of general dynamical systems, but a weaker version holds over the class of open dynamical systems.

Proposition 4.9
The formula $\lozenge p \leftrightarrow \square p$ is valid over the class of open dynamical systems.
Proof
One implication is Lemma 4.7, so we focus on the other. Let $(X, T, S, [\cdot])$ be a dynamical model. Assume that $w \in [\Box p]$, and let $U = \left( \bigcap_{n \in \mathbb{N}} S^{-n} [p] \right)^\circ$. Clearly $U$ is open; we claim that it is $S$-invariant. We have that
\[
S[U] = S \left( \left( \bigcap_{n \in \mathbb{N}} S^{-n} [p] \right)^\circ \right) \subseteq S \left( \bigcap_{n \in \mathbb{N}} S^{-n} [p] \right) \subseteq \bigcap_{n \in \mathbb{N}} S^{-n} [p],
\]
where the latter inclusion is obtained by distributing $S$ over the intersection. Moreover, $S[U]$ is open, since $S$ is assumed to be an open function. Thus
\[
S[U] \subseteq \left( \bigcap_{n \in \mathbb{N}} S^{-n} [p] \right)^\circ = U,
\]
witnessing that $w \in [\Box p]$. □

5 Soundness

In this section we will show that several of the logics we have considered are sound for their semantics based on different classes of dynamic topological systems. First we show that our basic logics (as given in Definition 2.1 and Table 2.1) are sound for the class of all dynamical systems. Below, recall that $c$ denotes the class of all dynamical systems (see Table 4.1).

Theorem 5.1
The logics $\operatorname{ITL}_{\Box}$ and $\operatorname{ITL}_{\Diamond}$ are sound for the class of dynamical systems; that is, $\operatorname{ITL}_{\Box} \subseteq \operatorname{ITL}_{\Box}$ and $\operatorname{ITL}_{\Diamond} \subseteq \operatorname{ITL}_{\Diamond}$.

Proof
Let $\mathcal{M} = (X, T, S, [\cdot])$ be any dynamical topological model; we must check that all the axioms (XIII) and the rules (XIV) (XV) preserve validity. Note that all intuitionistic tautologies are valid due to the soundness for topological semantics (Mints 2000). Many of the other axioms can be checked routinely, so we focus only on those axioms involving the continuity of $S$ or the semantics for $\Box$ or $\Diamond$.

Axiom (V): $\Diamond(\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi)$. Suppose that $x \in [\Diamond(\varphi \rightarrow \psi)]$. Then, $S(x) \in [\varphi \rightarrow \psi]$. Since $S$ is continuous and $[\varphi \rightarrow \psi]$ is open, $U = S^{-1} [\varphi \rightarrow \psi]$ is a neighbourhood of $x$. Then, for $y \in U$, if $y \in [\Diamond \varphi]$, it follows that $S(y) \in [\varphi] \cap [\varphi \rightarrow \psi]$, so that $S(y) \in [\psi]$ and $y \in [\Diamond \psi]$. Since $y \in U$ was arbitrary, $x \in [\Diamond \varphi \rightarrow \Diamond \psi]$, and by Lemma 4.3 (which we will henceforth use without mention), axiom (V) is valid on $\mathcal{M}$.

Axiom (VI): $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$. Observe that $[\Box(\varphi \rightarrow \psi)]$ is an $S$-invariant open subset of $[\varphi \rightarrow \psi]$. Similarly, $[\Box \varphi]$ is an $S$-invariant open subset of $[\varphi]$. Let $U = [\Box(\varphi \rightarrow \psi)] \cap [\Box \varphi]$. Since $U$ is open, it suffices to prove that $U \subseteq [\Box \varphi]$. Moreover, $U$ is $S$-invariant, therefore it suffices to prove that $U \subseteq [\varphi]$, which is direct because $U \subseteq [\varphi \rightarrow \psi] \cap [\varphi]$ and $[\varphi \rightarrow \psi] \subseteq (X \setminus [\varphi]) \cup [\psi]$.

Axiom (VII): $\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi)$. As before, suppose that $x \in [\Box(\varphi \rightarrow \psi)]$, and let $U$ be an $S$-invariant neighbourhood of $x$ such that $U \subseteq [\varphi \rightarrow \psi]$. If $y \in U \cap [\Diamond \varphi]$,
then $S^n(y) \in [\varphi]$ for some $n$; since $U$ is $S$-invariant, $S^n(y) \in U$, hence $S^n(y) \in [\psi]$ and $y \in [\Diamond \varphi \rightarrow \Diamond \psi]$. We conclude that $x \in [\Diamond \varphi \rightarrow \Diamond \psi]$.

**Axioms (VIII)** (IX): $\Box \varphi \rightarrow \varphi \land \Box \Box \varphi$. Suppose that $x \in [\Box \varphi]$, and let $U \subseteq [\varphi]$ be an $S$-invariant neighbourhood of $x$. Then, $x \in U$, so $x \in [\varphi]$. Moreover, $U$ is also an $S$-invariant neighbourhood of $S(x)$, so $S(x) \in [\Box \varphi]$ and thus $x \in [\Box \Box \varphi]$. We conclude that $x \in [\varphi] \cap [\Box \Box \varphi]$.

**Axiom (XIII)** $\Box (\Box \varphi \rightarrow \varphi) \rightarrow (\varphi \rightarrow \Box \varphi)$. Suppose that $x \in [\Box (\Box \varphi \rightarrow \varphi)]$, then $U = [\varphi] \cap [\Box (\Box \varphi \rightarrow \varphi)]$ is open (by the intuitionistic semantics) and $S$-invariant, since if $y \in U$, from $y \in [\varphi \rightarrow \varphi]$ we obtain $S(y) \in [\varphi]$. It follows that $U$ is an $S$-invariant neighbourhood of $x$, so $x \in [\Box \varphi]$.

**Axiom (XIII)** $\Box (\Box \varphi \rightarrow \varphi) \rightarrow (\Box \varphi \rightarrow \varphi)$. Suppose that $x \in [\Box (\Box \varphi \rightarrow \varphi)] \cap [\Box \varphi]$. Let $U \subseteq [\Box \varphi \rightarrow \varphi]$ be an $S$-invariant neighbourhood of $x$. Let $n$ be least so that $S^n(x) \in [\varphi]$; if $n > 0$, since $U$ is $S$-invariant we see that $S^{n-1}(x) \in U \subseteq [\Box \varphi \rightarrow \varphi]$, hence $S^{n-1}(x) \in [\varphi]$, contradicting the minimality of $n$. Thus $n = 0$ and $x \in [\varphi]$.

**Axiom $t_{\theta}(VI)$**: $\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$. Assume that $x \in [\Box (\varphi \rightarrow \psi)]$. We claim that if $y \in [\Box (\varphi \rightarrow \psi)] \cap [\Box \psi]$ then $y \in [\Box \psi]$, from which we obtain $x \in [\Box \varphi \rightarrow \Box \psi]$. If $y \in [\Box (\varphi \rightarrow \psi)] \cap [\Box \psi]$ then note that for all $z \in [\Box (\varphi \rightarrow \psi)] \cap [\Box \psi]$, by definition of the semantics, we have that for all $k \geq 0$, $S^k(z) \in [\varphi \rightarrow \psi]$ and $S^k(z) \in [\psi]$, so $S^k(z) \in [\varphi]$. Since $U := [\Box (\varphi \rightarrow \psi)] \cap [\Box \psi]$ is a neighbourhood of $y$ and $z \in U$ was arbitrary, $y \in [\Box \psi]$. Since $O := [\Box (\varphi \rightarrow \psi)]$ is a neighbourhood of $x$ and $y \in O$ was arbitrary, this witnesses that $x \in [\Box \varphi \rightarrow \Box \psi]$.

**Axiom $t_{\theta}(WH)$**: $\Box \varphi \rightarrow \Box \Box \varphi$. Suppose that $x \in [\Box \varphi]$. Let $U := [\Box \varphi]$ be a neighbourhood of $x$ such that, for all $y \in U$ and $n \in \mathbb{N}$, $S^n(y) \in [\varphi]$. Therefore, for all $y \in U$ and $n \in \mathbb{N}$, $S^{n+1}(y) \in [\varphi]$, i.e., $y \in [\Box \varphi]$, so that $U$ witnesses that $x \in [\Box \Box \varphi]$.

**Axiom $t_{\theta}(XIII)$**: $\Box (\varphi \rightarrow \Box \varphi) \rightarrow (\varphi \rightarrow \Box \varphi)$. Suppose that $x \in [\Box (\varphi \rightarrow \Box \varphi)]$, then $U = [\varphi] \cap [\Box (\varphi \rightarrow \Box \varphi)]$ is a neighbourhood of $x$. It can be proved by induction on $i$ that for all $i \geq 0$ and $y \in U$, $S^i(y) \in [\psi]$, so that $U$ witnesses that $x \in [\Box \psi]$.

**Axioms $t_{\theta}(VII), t_{\theta}(XIII)$**: These are treated as their analogues for $\Box$.

The additional axioms we have considered are valid over specific classes of dynamical systems. Specifically, the constant domain axiom is valid for the class of expanding posets, while the Fischer Servi axioms are valid for the class of open systems. Let us begin by discussing the former in more detail. In the next few results, recall that the relevant definitions are summarized in Tables 2.1 and 3.1.

**Theorem 5.2**

The logics CDTL$_{\varnothing}$ and CDTL$_{\Box}$ are sound for the class of expanding posets; that is, CDTL$_{\varnothing} \subseteq$ ITL$^e_{\varnothing}$ and CDTL$_{\Box} \subseteq$ ITL$^e_{\Box}$.

**Proof**

Let $M = (X, \leq, S, [\cdot])$ be a dynamic poset model; in view of Theorem 5.1 it only remains to check that CD and BI are valid on $M$. However, by Proposition 2.3 BI is a consequence of CD, so we only check the latter. Suppose that $x \in [\Box (\varphi \lor \psi)]$, but $x \not\in [\Box \varphi]$. Then, in view of Proposition 4.3 for some $n \geq 0$, $S^n(x) \not\in [\varphi]$. It follows that $S^n(x) \not\in [\psi]$, so that $x \in [\Diamond \varphi]$.
Note that the relational (rather than topological) semantics are used in an essential way in the above proof, since Proposition 4.8 is not available in the topological setting, and indeed we will show in Proposition 5.2 that these axioms are not topologically valid. But before that, let’s turn our attention to the Fischer Servi axioms (see Table 2.1).

Recall that $\lozenge$ denotes the class of dynamical systems with a continuous and open map.

**Theorem 5.3**

$\text{ITL}^+_{\lozenge} \subseteq \text{ITL}^+_{\lozenge}$, i.e., the logic $\text{ITL}^+_{\lozenge}$ is sound for the class of open dynamical systems.

**Proof**

Let $\mathcal{M} = (X, T, S, [\cdot])$ be a dynamical topological model where $S$ is an interior map. We check that axiom $\text{FS}_0$ is valid on $\mathcal{M}$. Suppose that $x \in [\Box \phi \rightarrow \Box \psi]$, and let $U = [\Box \phi \rightarrow \Box \psi]$. Since $S$ is open, $V = S[U]$ is a neighbourhood of $S(x)$. Let $y \in V \cap [\phi]$, so that $S(y) = y \in [\Box \phi]$. Since $y \in V$ was arbitrary, $S(x) \in [\phi \rightarrow \psi]$, and $x \in [\Box (\phi \rightarrow \psi)]$.

As an easy consequence, we mention the following combination of Theorems 5.2 and 5.3. Recall that dynamic posets with an interior map are also called persistent, and the class of persistent posets is denoted $p$.

**Corollary 5.4**

The logics $\text{CDTL}^+_{\lozenge}$ and $\text{CDTL}^+_{\Box}$ are sound for the class of persistent posets, that is, $\text{CDTL}^+_{\lozenge} \subseteq \text{ITL}^+_{\lozenge}$ and $\text{CDTL}^+_{\Box} \subseteq \text{ITL}^p$.

### 6 Euclidean spaces

The celebrated McKinsey-Tarski theorem states that intuitionistic propositional logic is complete for the real line, and more generally for a wide class of metric spaces which includes every Euclidean space $\mathbb{R}^n$ (Tarski 1938). Thus, it is natural to ask if a similar result holds for intuitionistic temporal logics, which could lead to applications in spatio-temporal reasoning. As we will see in this section, the answer to this question is negative; however, we identify some principles which could lead to an axiomatization for Euclidean systems.

Let us consider the real line. The conditional excluded middle axiom shows that even the $\Box$-logic of the real line is different from the logic of all dynamical systems.

**Lemma 6.1**

The formula

$$\text{CEM}(p, q) = (\neg \Box p \land \neg \neg \neg \neg p) \rightarrow (\Box q \lor \neg \neg \neg \neg p)$$

is valid on $\mathbb{R}$.

**Proof**

Suppose that $(\mathbb{R}, S, [\cdot])$ is a model based on $\mathbb{R}$ and that $x \in [\neg \Box p \land \neg \neg \neg \neg p]$. From $x \in [\neg \neg \neg \neg p]$ and the semantics of double negation (discussed in Fernández-Duque 2018) we see that there is a neighbourhood $V$ of $S(x)$ such that $V \subseteq [p]$. It follows from the intermediate value theorem that if $U$ is a neighbourhood of $x$ and $S[U]$ is not a singleton, then $S[U] \cap V$ contains an open set and hence $S[U] \cap [p] \neq \emptyset$. Meanwhile, from $x \in [\neg \neg \neg \neg p]$
Remark 6.2

It is tempting to conjecture that CEM axiomatizes the $\mathcal{L}_\mathcal{O}$-logic of the real line, but there is a possibility that additional axioms are required. CEM is an intuitionistic variant of similar formulas in (Kremer and Mints 2005; Slavnov 2003) showing that the dynamic topological logic (DTL) of the real line is different from the dynamic topological logic of arbitrary spaces. We will not review DTL here, but it is a classical cousin of intuitionistic temporal logic; see (Fernández-Duque 2018; Diéguez and Fernández-Duque 2018). The problem of axiomatizing DTL over the real line has long remained open, and (Nogin and Nogin 2008) give further examples of valid formulas not derivable from the classical analogue of CEM. We do not know if these formulas also have intuitionistic counterparts, and leave this line of inquiry open.

Since CEM is not valid for $\mathbb{R}^n$ in general, it cannot be used to show that our base logic is incomplete for Euclidean spaces. However, this can be shown using $\text{CD}^-$, which is valid on any locally connected space. Recall that a subset $C$ of a topological space $X$ is connected if, whenever $A, B$ are disjoint open sets such that $C \subseteq A \cup B$, it follows that $C \subseteq A$ or $C \subseteq B$. The space $X$ is locally connected if whenever $U$ is open and $x \in U$, there is a connected neighbourhood $V \subseteq U$ of $x$. It is well-known that $\mathbb{R}^n$ is locally connected for all $n$.

The following properties regarding connectedness will be useful below. Suppose that $X$ is a topological space and $S : X \to X$ is continuous. Then, for every $C \subseteq X$, if $C$ is connected, then so is $S[C]$. Moreover, any $A \subseteq X$ can be partitioned into a family of maximal connected sets called the connected components of $A$. If $X$ is locally connected and $A$ is open, then the connected components of $A$ are also open (see e.g. (Dugundji 1975)).

Lemma 6.3

The formulas $\text{CD}^-(p)$ and $t_{\Box}(\text{CD}^-(p))$ are valid on the class of locally connected spaces.

Proof

Let $(X, T, S, [\cdot])$ be a model based on a locally connected space, and $x \in X$. First we show that $x \in [t_{\Box}(\text{CD}^-(p))]$. Recall that $\text{CD}^-(p) = \Diamond(p \lor \neg p) \to \Box \neg p \lor \Diamond p$, so that $t_{\Box}(\text{CD}^-(p)) = \Box(p \lor \neg p) \to \Box \neg p \lor \Diamond p$. We may thus assume that $x \in [\Box(p \lor \neg p)]$, so that using the local connectedness of $X$, there is a connected neighbourhood $U$ of $x$ with $U \subseteq \bigcap_{n \in \mathbb{N}} S^{-n} [p \lor \neg p]$. This means that, for each $n \in \mathbb{N}$, $U \subseteq S^{-n} [p \lor \neg p] = S^{-n} [p] \cup S^{-n} [-p]$ and, from the connectedness of $U$, $U \subseteq S^{-n} [p]$ or $U \subseteq S^{-n} [-p]$, as these two sets are disjoint and open. If $x \in [\Box \neg p]$ there is nothing to prove, so we assume otherwise. This means that for all $n$, $S^n(x) \notin [p]$, which implies that $U \notin S^{-n} [p]$, and hence $U \subseteq S^{-n} [-p]$. But then, $U$ witnesses that $x \in [\Box \neg p]$.

To see that $x \in [\Box(\text{CD}^-(p))]$, suppose that $x \in [\Box(p \lor \neg p)]$; we must show that $x \in [\Box \neg p \lor \Diamond p]$. Let $U \subseteq [p \lor \neg p]$ be an $S$-invariant neighbourhood of $x$. For $n \in \mathbb{N}$, let $V_n$ be the connected component of $U$ containing $S^n(x)$, and set $V = \bigcup_{n \in \mathbb{N}} V_n$. Since each $V_n \subseteq U \subseteq [p] \cup [-p]$ and the latter are disjoint and open, it follows that either $V_n \subseteq [p]$
or \( V_n \subseteq \lbrack \neg p \rbrack \). If \( V_n \subseteq \lbrack p \rbrack \) for some \( n \), it immediately follows that \( x \in \lbrack \Diamond p \rbrack \), and we are done. Otherwise, \( V_n \subseteq \lbrack \neg p \rbrack \) for all \( n \). Clearly \( V \) is open; we claim that it is also \( S \)-invariant. To see this, note that \( S[V_n] \subseteq U \). Note that \( U \) can be written as the disjoint union of two open sets as \( U = V_{n+1} \cup (U \setminus V_{n+1}) \); since \( V_n \) is connected, so is \( S[V_n] \), hence \( S[V_n] \subseteq V_{n+1} \) or \( S[V_n] \subseteq U \setminus V_{n+1} \). However, \( S[V_n] \cap V_{n+1} \) is non-empty, so we must have \( S[V_n] \subseteq V_{n+1} \), and since \( n \) was arbitrary, \( S[V] \subseteq V \), as claimed. Hence \( V \) witnesses that \( x \in \lbrack \Box \neg p \rbrack \), as needed. \( \square \)

In conclusion, we obtain the following.

**Theorem 6.4**

1. \( \text{RTL}_\Box \) and \( \text{RTL}_\Diamond \) are sound for \( \mathbb{R} \).
2. \( \text{ETL}_\Box \) and \( \text{ETL}_\Diamond \) are sound for \( \{ \mathbb{R}^n : n > 0 \} \).
3. \( \text{ETL}_\Box^+ \) and \( \text{ETL}_\Diamond^+ \) are sound for the class of invertible systems based on \( \{ \mathbb{R}^n : n > 0 \} \).

In the remainder of this section we show that \( \text{ITL}_\Box^2 \subseteq \text{ITL}_\Box^+ \cap \text{ITL}_\Diamond^+ \). We show this using results from [Fernández-Duque 2007](#), originally developed for dynamic topological logic, but applicable to \( \text{ITL} \) as well. We begin with the notion of dynamic morphism.

**Definition 6.5**

Let \( X = (X, T_X, S_X) \) and \( Y = (Y, T_Y, S_Y) \) be dynamic topological systems. Let \( U \subseteq X \) be open and \( S_X \)-invariant. A **dynamic morphism** from \( X \) to \( Y \) is an interior map \( f : U \to Y \) such that for all \( x \in X \), \( fS_X(x) = S_Yf(x) \).

**Proposition 6.6**

Let \( X = (X, T_X, S_X) \) and \( Y = (Y, T_Y, S_Y) \) be dynamic topological systems. Let \( U \subseteq X \) be open and \( S_X \)-invariant and \( f : U \to Y \) be a dynamic morphism, and let \( \llbracket \cdot \rrbracket_X \) be any valuation on \( X \). Then, there is a valuation \( \llbracket \cdot \rrbracket_Y \) on \( Y \) such that for every formula \( \varphi \in \mathcal{L}_\Box \), \( \llbracket \varphi \rrbracket_X \cap U = f^{-1} \llbracket \varphi \rrbracket_Y \).

**Proof**

Let \( \llbracket \cdot \rrbracket_X \) be the unique valuation such that for any propositional variable \( p \), \( \llbracket p \rrbracket_X = f^{-1} \llbracket p \rrbracket_Y \). We prove by induction on \( \varphi \) that \( \llbracket \varphi \rrbracket_X \cap U = f^{-1} \llbracket \varphi \rrbracket_Y \). Most cases are standard, save the cases for \( \varphi = \Box \psi \) and \( \varphi = \Diamond \psi \), so we focus on those. First assume that \( x \in U \cap \llbracket \Box \psi \rrbracket_X \). Let \( V \subseteq U \) be a neighbourhood of \( x \) such that \( V \subseteq \bigcap_{n \geq 0} S_X^{-n} \llbracket \psi \rrbracket_X \). Then \( f[V] \) is open, and since \( fS_X = S_Yf \),

\[
\begin{align*}
\llbracket V \rrbracket_X &\subseteq \bigcap_{n \geq 0} S_X^{-n} \llbracket \psi \rrbracket_X \\
&\subseteq \bigcap_{n \geq 0} S_X^{-n} f \llbracket \psi \rrbracket_X &\subseteq \bigcap_{n \geq 0} S_Y^{-n} \llbracket \psi \rrbracket_Y.
\end{align*}
\]

So, \( f[V] \) witnesses that \( f(x) \in \llbracket \psi \rrbracket_Y \).

If \( f(x) \in \llbracket \Diamond \psi \rrbracket_Y \), we instead let \( V' \) be a neighbourhood of \( f(x) \) so that \( V' \subseteq \bigcap_{n \geq 0} S_Y^{-n} \llbracket \psi \rrbracket_Y \). Then,

\[
\begin{align*}
\llbracket V' \rrbracket_Y &\subseteq \bigcap_{n \geq 0} S_Y^{-n} \llbracket \psi \rrbracket_Y \\
&\subseteq \bigcap_{n \geq 0} S_Y^{-n} f^{-1} \llbracket \psi \rrbracket_Y &\subseteq \bigcap_{n \geq 0} S_X^{-n} \llbracket \psi \rrbracket_X.
\end{align*}
\]

Since \( f^{-1}[V'] \) is open, this witnesses that \( x \in \llbracket \Box \psi \rrbracket \).
The arguments for $\Box \psi$ are similar. As above, if $x \in [\Box \psi]_X$, we let $V \subseteq [\psi]_Y$ be an open, $S_X$-invariant neighbourhood of $x$. Since $U$ is open and $S_X$-invariant, $V \cap U$ is also open and $S_X$-invariant, so we may assume that $V \subseteq U$. Then, $f[V]$ is open and $f[V] \subseteq [\psi]_Y$ by the induction hypothesis. It remains to check that $f[V]$ is $S_Y$-invariant. But if $y \in f[V]$ then $y = f(z)$ for some $z \in V$, hence $S_Y(y) = S_Y(f(z)) = fS_X(z)$ and $S_X(z) \in V$ by $S_X$-invariance, so $S_Y(y) \in f[V]$, as needed.

Similarly, if $V'$ witnesses that $f(x) \in [\Box \psi]_Y$, then $f^{-1}[V']$ witnesses that $x \in [\Box \psi]_X$. \hfill \Box

From here, we easily obtain that the logics based on $\mathbb{R}^2$ are contained in those based on $\mathbb{R}$.

**Theorem 6.7**

Every formula of $\mathcal{L}_{\Box\Diamond \Box}$ valid on $\mathbb{R}^2$ is valid on $\mathbb{R}$; that is, $\text{ITL}_{\Box\Diamond \Box}^{\mathbb{R}^2} \subseteq \text{ITL}_{\Box\Diamond \Box}^{\mathbb{R}}$.

**Proof**

Suppose that $\varphi$ is not valid on $\mathbb{R}$. Let $S: \mathbb{R} \to \mathbb{R}$ and $[\cdot]$ be such that $(\mathbb{R},S,[\cdot]) \not\models \varphi$. Define $f: \mathbb{R}^2 \to \mathbb{R}^2$ to be given by $f(x,y) = x$ and $S'(x,y) = (S(x),y)$. Then, it is not hard to see that $f$ is a surjective dynamic morphism from $(\mathbb{R}^2,S')$ onto $(\mathbb{R},S)$. It follows that $(\mathbb{R}^2,S',f^{-1}[\cdot]) \not\models \varphi$, and hence $\varphi \not\in \text{ITL}_{\Box\Diamond \Box}^{\mathbb{R}^2}$. \hfill \Box

In order to show that $\text{ITL}_{\Box\Diamond \Box}^{\mathbb{R}^2} \subseteq \text{ITL}_{\Box\Diamond \Box}^{\mathbb{R}}$, it would suffice to construct a dynamic morphism from $\mathbb{R}^2$ onto a given dynamic poset $(W,\preceq,S)$. We may assume that $W$ is finite in view of the following result, which is established by (Balbiani et al. 2019).

**Theorem 6.8**

Any formula satisfiable (falsifiable) on a dynamic poset is satisfiable (falsifiable) on a finite dynamic poset.

We may then use the following result, essentially proven in (Fernández-Duque 2007).

**Theorem 6.9**

If $(W,\preceq,S)$ is a finite dynamic poset and $w_0 \in W$, there exist continuous $T: \mathbb{R}^2 \to \mathbb{R}^2$, an open, $T$-invariant set $U \subseteq \mathbb{R}^2$, and a dynamic morphism $f: \mathbb{R}^2 \to W$, such that $0 \in U$ and $f(0) = w_0$.

**Proof sketch.**

(Fernández-Duque 2007) shows the result for any dynamic preorder with limits that commute with $S$. Recall that $\preceq$ is a preorder if it is transitive and reflexive (but not necessarily antisymmetric). We say that $(W,\preceq,S)$ is a dynamic preorder if $S: W \to W$ is $\preceq$-monotone, so that dynamic posets are a special case of dynamic preorders. We say that $(W,\preceq,S)$ is a dynamic preorder with limits if every monotone sequence $(w_i)_{i \in \mathbb{N}}$ (i.e., a sequence such that $w_i \preceq w_{i+1}$ for all $i$) is assigned a limit $\lim_{i \to \infty} w_i \in W$ with the property that $w_n \preceq \lim_{i \to \infty} w_i$ for all $n$ and $\lim_{i \to \infty} w_i \preceq w_n$ for $n$ large enough. The limits commute with $S$ if $S(\lim_{i \to \infty} w_i) = \lim_{i \to \infty} S(w_i)$.

However, if $W$ is finite and $\preceq$ is a partial order then every monotone sequence can trivially be assigned a limit, as in this case we have that $w_i$ is constant for $i$ large enough, and we can define $\lim_{i \to \infty} w_i$ to be this unique constant. It is easy to see that this is a limit assignment that commutes with $S$, hence the desired $U$, $T$ and $f$ exist by (Fernández-Duque 2007). \hfill \Box
Theorem 6.10
Every formula of $\mathcal{L}_{\Diamond \Box}$ valid on $\mathbb{R}^2$ is valid on the class of expanding posets; that is, $\mathit{ITL}_{\Diamond \Box}^2 \subseteq \mathit{ITL}_{\Diamond \Box}$.

Proof
We prove the claim by contrapositive. If $\varphi \notin \mathit{ITL}_{\Diamond \Box}^2$, then by Theorem 6.8 there are a finite dynamic poset model $M = (W, \preceq, T, [\cdot]_W)$ and $w_0 \in W \setminus [\varphi]_W$. By Theorem 6.9 there exist $T: \mathbb{R}^2 \to \mathbb{R}^2$, an open, $T$-invariant set $U \subseteq \mathbb{R}^2$, and a dynamic morphism $f: U \to W$, such that $0 \in U$ and $f(0) = w_0$. By Proposition 6.6 there is a valuation $[\cdot]_{\mathbb{R}^2}$ on $\mathbb{R}^2$ such that $[\varphi]_{\mathbb{R}^2} \cap U = f^{-1}[\varphi]_W$; in particular, since $w_0 \notin [\varphi]_W$, we have that $0 \notin [\varphi]_{\mathbb{R}^2}$, so that $\mathbb{R}^2 \not\models \varphi$ and hence $\varphi \notin \mathit{ITL}_{\Diamond \Box}^2$.

7 Persistent posets and the real line
We have seen that $\mathit{ITL}_{\Diamond \Box}^2 \subseteq \mathit{ITL}_{\Diamond \Box}$. The question naturally arises whether a similar result holds when restricting to open systems: is every formula falsifiable on a persistent poset (i.e., a poset equipped with a continuous, open map) falsifiable on some Euclidean space, also equipped with a continuous, open map? Surprisingly, not only is the answer affirmative, but in this case, any falsifiable formula is falsifiable on the real line. In this section, we will prove this fact, along with some properties of $\mathit{ITL}_{\Diamond \Box}$ which may be interesting on their own right. We remark that in this context $\Box$ and $\equiv$ coincide, so we restrict our attention to languages with the former.

One challenge is that we do not have the finite model property in this setting. This is already proven in (Balbiani et al. 2019) for $\mathcal{L}_{\Diamond}$, but in fact, the finite model property already fails over $\mathcal{L}_{\Box}$.

Proposition 7.1
The formula $\varphi = \square \neg p \to \neg \square p$ is valid over the class of all finite persistent posets, but not over the class of all persistent posets.

Proof
First we show that $\varphi$ is indeed valid over any finite persistent poset. Let $M = (W, \preceq, S, [\cdot])$ be any model based on a finite persistent poset, and let $w \in [\square \neg p]$. We show that $w \in [\neg \square p]$. It suffices to show that if $v \succ w$ is maximal, then $v \in [\square p]$. So, let $n \in \mathbb{N}$. Then, $S^n(w) \in [\neg p]$, and $S^n(v)$ is maximal (as order-preserving persistent functions preserve maximality), so $S^n(v) \in [p]$. It follows that $v \in [\square p]$.

To see that $\varphi$ is not valid over the class of persistent posets, let $W = \mathbb{Z}$, where $\preceq$ is the usual order and $S(x) = x - 1$. Let $[p] = \mathbb{N}$. Then, every point satisfies $\neg p$ (as every large-enough point satisfies $p$), so that in particular $0 \notin [\square \neg p]$. However, no point satisfies $\square p$, since $S^{x+1}(x) \notin [p]$. Hence, in particular, $0 \notin [\varphi]$. □

Thus, we cannot avoid working with infinite models. However, we can still work with models that have some ‘nice’ properties. For starters, $\mathit{ITL}_{\Diamond \Box}$ is complete for the class of product models (Kurucz et al. 2003). The products we consider will have a rather

\[\text{[In fact, the only property we use of } M \text{ is that for every } w \text{ there is a maximal } v \succ w, \text{ so } \varphi \text{ is valid over any persistent model with this property.]}\]
Lemma 7.3

Let \( f : X \to Y \) be a dynamical system. If \((x,n) \in X\) and suppose that \( f \) and \( S \) are continuous, then \( f^{-1} S^{-n} [U] \times \{n\} \) is a neighbourhood of \((x,n)\) contained in \( f^{-1}[U] \).

Define \( f : X \to Y \) is an interior map. Then, there exists a dynamic morphism \( f : X \to Y \) whose range is \( \bigcup_{n \in \mathbb{N}} S^f[Y][X] \).

Proof

Define \( f : X \times \mathbb{N} \to Y \) by \( f(x,n) = S_x^n \cdot f(x) \). First we must check that \( f \) is continuous and open. If \((x,n) \in X\) and \( U \) is a neighbourhood of \( f(x,n) \), then since both \( f \) and \( S_x^n \) are continuous, \( f^{-1} S_x^{-n}[U] \times \{n\} \) is a neighbourhood of \((x,n)\) contained in \( f^{-1}[U] \).

Since \((x,n)\) was arbitrary, \( f \) is continuous. Similarly, if \( O \) is a neighbourhood of \((x,n)\), then \( S_x^n [O \cap (X \times \{n\})] \) is a neighbourhood of \( f(x,n) \) contained in \( f[O] \), which since \((x,n)\) was arbitrary shows that \( f[O] \) is open. Hence, \( f \) is an open map.

Next we note \( S_Y f(x,n) = S_Y \circ S_x^n \cdot f(x) = S_x^{n+1} \cdot f(x) = f(x,n+1) = f S(x,n) \).

Finally, the range of \( f \) is

\[
f(X \times \mathbb{N}) = \bigcup_{n \in \mathbb{N}} f[X \times \{n\}] = \bigcup_{n \in \mathbb{N}} S_x^n f[X].
\]

As a corollary, we immediately obtain that any formula satisfiable (falsifiable) on a persistent poset is satisfiable (falsifiable) on a product poset, since we can take \( \mathcal{X} = \mathcal{Y} \) and \( f \) to be the identity. With this, we can easily check that we can restrict our attention to the class of countable models. Below, if \( \mathcal{M} = (W, \xi, S, [\cdot]) \) is a dynamical poset model and \( Z \subseteq W \), then \( \mathcal{M} \upharpoonright Z = (Z, \xi \upharpoonright Z, S \upharpoonright Z, [\cdot] \upharpoonright Z) \) is the sub-structure with domain \( W \) and such that each of \( \xi, S \) and \( [\cdot] \) are restricted to \( Z \). If \( Z \) is \( S \)-invariant but not necessarily open, then \( \mathcal{M} \upharpoonright Z \) is also based on a dynamic poset, although \( [\cdot] \upharpoonright Z \) may not be a valuation. However, this will indeed be the case if we choose \( Z \) appropriately.

Lemma 7.4

If \( \varphi \in \mathcal{L}_{\diamond} \) is falsifiable on a persistent poset, it is falsifiable on a countable model.
Proof
We proceed as in a standard proof of the downward Löwenheim-Skolem theorem. Let \( \mathcal{M} = (W, \leq, S, \llbracket \cdot \rrbracket) \) be a model based on a persistent poset such that \( \mathcal{M} \not\models \varphi \). In view of Lemma 7.3 we may assume that \( \mathcal{M} \) is a product model, so that \( W = U_\infty \) for some \( U \), and there is \( w_0 \in U \) such that \( (w_0, 0) \not\in \llbracket \varphi \rrbracket \). We define a sequence \( V_0 \subseteq V_1 \subseteq \ldots \subseteq U \) of countable sets, so that for \( V = \bigcup_{n \in \mathbb{N}} V_n \) we have that \( \mathcal{M} \vdash V \) falsifies \( \varphi \). The construction is straightforward: \( V_0 = \{ w_0 \} \times \mathbb{N} \), and if we are given \( V_n \), define
\[
V_{n+1} := V_n \cup \{(v_{w, k}^{\psi, \theta}, m) : (w, k) \in V_n, \psi, \theta \in L_{\Diamond/\Box}, \text{ and } m \in \mathbb{N}\},
\]
where \( v_{w, k}^{\psi, \theta} = w \) if \( (w, k) \in \llbracket \psi \rightarrow \theta \rrbracket \), and otherwise \( v = v_{w, k}^{\psi, \theta} \) is chosen to satisfy \( v \triangleright w \), \( (v, k) \in \llbracket \psi \rrbracket \) and \( (v, k) \not\in \llbracket \theta \rrbracket \). Note that at each stage we add \( (v_{w, k}^{\psi, \theta}, m) \) for all \( m \), which ensures that the resulting set is \( S \)-invariant and that \( S \) is open on each \( V_n \), hence on \( V \). One can then easily check that \( w_0 \) satisfies the same formulas on \( \mathcal{M} \vdash V \) as it did on \( \mathcal{M} \). \( \square \)

Given the lack of the finite model property for persistent posets we need to consider infinite posets, but in view of Lemma 7.4 it suffices to work with countable posets. Fortunately, we may work with a single, "universal" countable poset.

Definition 7.5
Define \( 2^{<\mathbb{N}} \) to be the set of (possibly empty) binary strings, where \( a \sqsubseteq b \) if and only if \( a \) is an initial segment of \( b \). We denote the empty string by \( \epsilon \).

The following is proven in (Goldblatt 1980) for finite structures, and as mentioned in (Kremer 2013), readily extends to countable structures.

Theorem 7.6
Given a countable poset \( (W, \leq) \) and \( w_0 \in W \), there is an interior map \( f : 2^{<\mathbb{N}} \to W \) such that \( w_0 = f(\epsilon) \).

In view of Lemma 7.3 one then immediately obtains the following.

Corollary 7.7
Given a countable persistent poset \( (W, \leq, S) \) and \( w_0 \in W \), there is a dynamic morphism \( f : 2^{\leq\mathbb{N}} \to W \) such that \( f(\epsilon) = w_0 \).

We extend the notation \( \sqsubseteq \) to elements of \( 2^{<\mathbb{N}} = 2^{<\mathbb{N}} \times \mathbb{N} \) by letting \( (a, n) \sqsubseteq (b, m) \) if \( a \sqsubseteq b \) and \( n = m \). It is not hard to check that the order \( \sqsubseteq \) generates the topology on \( 2^{\leq\mathbb{N}} \). It would remain to show that there is a dynamic morphism \( f : U \to 2^{\leq\mathbb{N}} \) for some suitable \( U \subseteq \mathbb{R} \) and some suitable \( S : \mathbb{R} \to \mathbb{R} \). Actually, as already observed in (Kremer 2013), this will not be possible. So, we must replace \( 2^{\leq\mathbb{N}} \) by a slightly different space.

Definition 7.8
Let \( 2^{\leq\mathbb{N}} \) be the set of all finite or infinite strings, and extend the notation \( \sqsubseteq \) to \( 2^{\leq\mathbb{N}} \) by also setting \( a \sqsubseteq b \) if \( a \) is an initial segment of \( b \). For finite \( b \), set \( \uparrow b = \{ a \in 2^{\leq\mathbb{N}} : b \sqsubseteq a \} \). Say that \( U \subseteq 2^{\leq\mathbb{N}} \) is open if whenever \( a \in U \), there is a finite \( b \sqsubseteq a \) such that \( \uparrow b \subseteq U \).

The space \( 2^{\leq\mathbb{N}} \) is thus endowed with a topology. This space is more convenient since, indeed, there is a surjective interior map from \( (0, 1) \) to this space. The following is shown in (Kremer 2013).
Moreover, we show that \( f, g \) open sets, where \( \cdot \in \{ \leq, < \} \) since \( h \) hence \( \exists f, g \) order-preserving maps \( \cdot \in \{ \leq, < \} \) N defined as above. As we did for \( 2^N \), we further extend the notation \( \subseteq \) to elements of \( 2^N \) by letting \( (a, n) \subseteq (b, m) \) if \( n = m \) and \( a \subseteq b \).

**Lemma 7.10**

There exist a function \( S: \mathbb{R} \to \mathbb{R} \), an open, \( S \)-invariant \( U \subseteq \mathbb{R} \), and a surjective dynamic morphism \( f: U \to 2^N \).

**Proof**

Let \( S \) denote the map on \( 2^N \) as given in Definition 7.2. By Theorem 7.9, there is a surjective interior map \( f': (0, 1) \to 2^N \), which can be viewed as an interior map \( f'': (0, 1) \to 2^N \times \mathbb{N} \) with range \( 2^N \times \{0\} \). Hence Lemma 7.9 tells us that there is a dynamic morphism \( f: (0, 1) \to 2^N \) with range \( \bigcup_{n \in \mathbb{N}} S^n[2^N \times \{0\}] = 2^N \times \mathbb{N} \). Define \( S: \mathbb{R} \to \mathbb{R} \) by \( S(x) = x + 1 \), and let \( U = (0, \infty) \setminus \mathbb{N} \) (i.e., the set of positive reals that are not integers). Then it is easily seen that \( \iota: (0, 1) \to U \) given by \( (x, n) \to x + n \) is an isomorphism. Thus \( f\iota^{-1}: U \to 2^N \) is also a surjective dynamic morphism. \( \square \)

**Remark 7.11**

In fact, [Kremer 2013] proves Theorem 7.9 for any complete metric space without isolated points, and shows how the completeness assumption can be dropped using algebraic semantics. Using Kremer’s result, we could generalize Lemma 7.10 to many more spaces, including the rational numbers and the Cantor space. We require only the existence of suitable \( U \) and \( S \), so as to be able to adapt the proof.

Even though the dynamical systems \( 2^N \) and \( 2^N \) are not isomorphic (the former is countable and the latter is not), they have essentially the same open sets. This observation will be key in proving the following lemma.

**Lemma 7.12**

A formula of \( L_{\Box} \) is satisfiable (falsifiable) on \( 2^N \) if and only if it is satisfiable (falsifiable) on \( 2^N \).

**Proof**

We follow an algebraic approach. Let \( 1^N = (2^N \times \mathbb{N}, T_\leq, S_\leq) \) and \( 2^N = (2^N \times \mathbb{N}, T_\leq, S_\leq) \). We show that \( T_\leq \) and \( T_\leq \) are isomorphic as Heyting algebras by providing order-preserving maps \( f: T_\leq \to T_\leq \) and \( g: T_\leq \to T_\leq \) so that \( g \) is the inverse of \( f \). Moreover, we show that \( f, g \) commute with preimages under \( S \) and preserve \( S \)-invariant open sets, where \( \cdot \in \{ \leq, \leq \} \). From this it will readily follow that if \( [\cdot]_\leq \) is a valuation on \( 2^N \) then \( f \circ [\cdot]_\leq \) is a valuation on \( 2^N \), and similarly if \( [\cdot]_\leq \) is a valuation on \( 2^N \), then \( g \circ [\cdot]_\leq \) is a valuation on \( 2^N \), and hence the two structures have the same valid formulas.

So, for \( U \subseteq T_\leq \), define \( f(U) = \{ a \in X : \exists b \subseteq a \ b \in U \} \), and for \( V \subseteq T_\leq \), define \( g(V) \) to be the set of finite elements of \( V \). We need the following properties of \( f, g \).

\( g \circ f \) is the identity on \( T_\leq \): If \( a \in U \subseteq T_\leq \) then from \( a \subseteq b \) we obtain \( a \in f(U) \), and since \( a \) is finite, \( a \in g \circ f(U) \). Conversely, if \( a \in g \circ f(U) \) then \( a \) is finite and \( a \in f(U) \), hence \( \exists b \subseteq a \) such that \( b \subseteq U \), but \( U \) is upward-closed, so \( a \in U \).
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\( f \circ g \) is the identity on \( T_\leq \): if \( a \in V \in T_\leq \) then since \( V \) is open there is finite \( b \subseteq a \) such that \( b \in V \), so that \( b \in g(V) \); but \( b \subseteq a \) implies that \( a \in f \circ g(V) \). Conversely, if \( a \in f \circ g(V) \) then there is \( b \subseteq a \) which is finite and so that \( b \in g(V) \), which implies \( b \in V \) and, since \( V \) is upwards-closed, \( a \in V \).

\( f \) is order-preserving: If \( U \subseteq U' \in T_\leq \) and \( a \in f(U) \) there is finite \( b \in U \) such that \( b \subseteq a \), but then \( b \in U' \) and \( a \in f(U') \).

\( g \) is order-preserving: If \( V \subseteq V' \in T_\leq \), then clearly every finite element of \( V \) is a finite element of \( V' \), so \( g(V) \subseteq g(V') \).

\( f \circ S_\leq^{-1} = S_\leq^{-1} \circ f \): Let \( U \in T_\leq \) and \((a,n) \in 2^{\leq N} \times N \). If \((a,n) \in f \circ S_\leq^{-1}[U] \), there is finite \( b \subseteq a \) such that \((b,n) \in S_\leq^{-1}[U] \), so that \( S_\leq(b,n) = (b,n+1) \in U \), witnessing that \( S_\leq(a,n) = (a,n+1) \in f(U) \), hence \( a \in S_\leq^{-1} \circ f(U) \). Conversely, if \((a,n) \in S_\leq^{-1} \circ f(U) \), then \((a,n+1) \in f(U) \), and so there is finite \( c \subseteq a \) with \((c,n+1) \in U \), so that \((c,n) \in S_\leq^{-1}[U] \) and \((a,n) \in f \circ S_\leq^{-1}[U] \).

\( g \circ S_\leq^{-1} = S_\leq^{-1} \circ g \): Take \( V \in T_\leq \) and \( a \in 2^{\leq N} \times N \). If \( a \in g \circ S_\leq^{-1}[V] \), then \( a \) is finite and \( S_\leq(a) \in V \); but \( S_\leq(a) \) is also finite so \( S_\leq(a) = S_\leq(a) \in g(V) \), hence \( a \in S_\leq^{-1} \circ g(V) \). Conversely, if \( a \in S_\leq^{-1} \circ g(V) \), then \( S_\leq(a) \in g(V) \), so that \( S_\leq(a) \subseteq V \) and \( a \in S_\leq^{-1}[V] \), which since \( a \) is finite implies \( a \in g \circ S_\leq^{-1}[V] \).

\( f \) preserves invariant sets: If \( U \in T_\leq \) is \( S_\leq^{-1} \)-invariant and \((a,n) \in f(U) \) then there is finite \( b \subseteq a \) so that \((b,n) \in U \), hence \((b,n+1) \in U \), witnessing that \( S_\leq(a,n) = (a,n+1) \in f(U) \), and \( f(U) \) is \( S_\leq^{-1} \)-invariant.

\( g \) preserves invariant sets: If \( V \in T_\leq \) is \( S_\leq^{-1} \)-invariant and \( a \in g(U) \), then \( a \) is finite and \( S_\leq(a) = S_\leq(a) \subseteq V \), since \( a \) is finite implies \( S_\leq(a) \) finite so that \( S_\leq(a) \in g(V) \).

If \( [\cdot]_\leq \) is a valuation on \( 2^{\leq N} \) and the above considerations show that \( f \circ [\cdot]_\leq \) is a valuation on \( 2^{\leq N} \), and conversely, if \( [\cdot]_\leq \) is a valuation on \( 2^{\leq N} \), then \( g \circ [\cdot]_\leq \) is a valuation on \( 2^{\leq N} \). For example, we show that \( f \circ [\cdot]_\leq \) satisfies the clause for \( \Box \). We have that \( [\Box \varphi]_\leq \) is \( S_\leq^{-1} \)-invariant, so that \( f([\Box \varphi]_\leq) \) is \( S_\leq^{-1} \)-invariant; that \( [\Box \varphi]_\leq \subseteq [\varphi]_\leq \), so that \( f([\Box \varphi]_\leq) \subseteq f([\varphi]_\leq) \); and finally, if \( U \in T_\leq \) is \( S_\leq^{-1} \)-invariant and \( U \subseteq f([\varphi]_\leq) \), then \( g(U) \) is \( S_\leq^{-1} \)-invariant and \( g(U) \subseteq [\varphi]_\leq \), hence \( U = f \circ g(U) \subseteq f([\varphi]_\leq) \). We conclude that \( f([\Box \varphi]_\leq) \) is the greatest \( S_\leq^{-1} \)-invariant open set contained in \( f([\varphi]_\leq) \), as needed.

Finally, we observe that if \( [\varphi]_\leq \neq 2^{\leq N} \times N \), then also \( f([\varphi]_\leq) \neq 2^{\leq N} \times N \), since \( f \) is injective and \( f(2^{\leq N} \times N) = 2^{\leq N} \times N \), so any formula falsifiable on \( 2^{\leq N} \) is falsifiable on \( 2^{\leq N} \); and, similarly, if \( [\varphi]_\leq \neq 2^{\leq N} \times N \) then also \( g([\varphi]_\leq) \neq 2^{\leq N} \times N \). We conclude that \( \varphi \) is valid on \( 2^{\leq N} \) if and only if it is valid on \( 2^{\leq N} \).

Putting our results together, we obtain the following.

**Theorem 7.13**

Every formula of \( \mathcal{L}_\Diamond \) valid over \( \mathbb{R} \) equipped with an interior map is valid over the class of persistent posets; that is, \( \mathbf{ITL}^{\mathbb{R}_{\Diamond}}_\Box \subseteq \mathbf{ITL}^p_\Box \).
Proof
If \( \varphi \) is falsifiable on some persistent poset, then by Corollary 7.7 it is falsifiable on \( 2^{\leq N} \), hence by Lemma 7.12 on \( 2^{\leq N} \). From this and Lemma 7.10 it follows that \( \varphi \) is falsifiable on \( \mathbb{R} \) with a homeomorphism. \( \square \)

Remark 7.14
Compare the situation to that of DTL, where (Fernández-Duque 2011) showed that there is a formula that is sound for the class of complete metric spaces with an open map, but not sound for the class of persistent posets. The latter is due to the Baire category theorem; in view of Theorem 7.13, the Baire category theorem does not seem to affect the intuitionistic temporal logic of \( \mathbb{R} \).

Figure 7.1 summarizes the inclusions between the semantically-defined logics we have defined. As we will see in the remainder of this paper, these are the only inclusions that hold between said logics.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{inclusions.png}
\caption{Inclusions between the semantically defined logics; arrows point from the smaller logic to the larger one. See Table 4.1 for the definitions of the relevant classes of dynamical systems.}
\end{figure}

8 Independence
In this section we will use our soundness results to show that many of the logics we have considered are pairwise distinct. We begin by showing that the weak logics (based on \( \boxast \)) are in fact weaker than their strong counterparts. Indeed, certain key LTL principles are not valid for logics with \( \boxast \). Recall that the semantics for \( \Box \) and \( \square \) are given in Definition 4.1.

Proposition 8.1
The following are not valid over \( \mathbb{R} \).

1. \( \square p \to \bigcirc \square p \),
2. \( \Diamond \bigcirc p \to \bigcirc \Diamond p \), and
3. \( \square p \to \bigcirc \Box p \).
Proof
Let $\mathcal{M} = (\mathbb{R}, S, [\cdot])$, where $S$ is defined as follows:

\[
S(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
2x, & \text{if } x > 0
\end{cases}
\]

and $[p] = (-\infty, 1)$. Note that $S^{-n}([p]) = (-\infty, 1/2^n)$, for $n \geq 0$. Thus,

\[
\bigcap_{n \geq 0} S^{-n} [p] = \bigcap_{n \geq 1} S^{-n} [p] = (-\infty, 0].
\]

So, $[\boxast p] = [\boxast \circ p] = (-\infty, 0)$. Hence, $[\circ \boxast p] = [\boxast \boxast p] = \emptyset$, and

\[
[\boxast p \to \circ \boxast p] = [\circ \circ p \to \circ \boxast p] = [\boxast p \to \boxast \boxast p] = (0, \infty) \neq \mathbb{R}.
\]

Next, we note that the formulas CD and BI separate Kripke semantics from the general topological semantics.

Proposition 8.2
The formulas CD($p, q$) and BI($p, q$) are not valid over the class of invertible dynamical systems based on $\mathbb{R}$, hence $\mathbf{ETL}^+ \nvDash \text{CD($p, q$)}$ and $\mathbf{ETL}^+ \nvDash \text{BI($p, q$)}$.

Proof
Define a model $\mathcal{M} = (\mathbb{R}, S, [\cdot])$ on $\mathbb{R}$, with $S(x) = 2x$, $[p] = (-\infty, 1)$ and $[q] = (0, \infty)$. Clearly $[p \lor q] = \mathbb{R}$, so that $[[\Box(p \lor q)] = \mathbb{R}$ as well.

Let us see that $\mathcal{M}, 0 \not\models \text{CD($p, q$)}$. Since $\mathcal{M}, 0 \models \Box(p \lor q)$, it suffices to show that $\mathcal{M}, 0 \not\models \Box p \lor \Diamond q$. It is clear that $\mathcal{M}, 0 \not\models \Diamond q$ simply because $S^n(0) = 0 \not\in [q]$ for all $n$. Meanwhile, by Example 4.6, $\mathcal{M}, 0 \models \Box p$ if and only if $[p] = \mathbb{R}$, which is not the case. We conclude that $\mathcal{M}, 0 \not\models \text{CD($p, q$)}$.

To see that $\mathcal{M}, 0 \not\models \text{BI($p, q$)}$ we proceed similarly, where the only new ingredient is the observation that $\mathcal{M}, 0 \not\models \Box q$, then $x > 0$ so that $\mathcal{M}, x \models q$, hence $[[\Diamond q] \Rightarrow q] = \mathbb{R}$.

Remark 8.3
Proposition 8.2 also holds for $t_\boxast(\text{CD($p, q$)})$ and $t_\boxast(\text{BI($p, q$)})$. However, by Proposition 4.9, these are equivalent to their counterparts with $\Box$ over the class of invertible systems, so we do not need to mention them separately. A similar comment holds when working over the class of dynamic posets.

The Fischer Servi axioms are also not valid in general, as already shown in (Balbiani et al. 2019). From this and the soundness of $\mathbf{ITL}^+ \diamondsuit$ (Theorem 5.3), we immediately obtain that they are not derivable in $\mathbf{ITL}^{+\diamondsuit}$.
Fig. 8.1: An expanding poset model falsifying both Fischer Servi axioms. Propositional variables that are true on a point are displayed; only one point satisfies $p$ and no point satisfies $q$. It can readily be checked that $FS_{\Diamond}(p, q)$ and $FS_{\Box}(p, q)$ fail on the highlighted point on the left. Note that $S$ is continuous but not open, as can easily be seen by comparing to Figure 3.1.

**Proposition 8.4**

1. $FS_{\Diamond}(p, q)$ is not valid over the class of expanding posets, hence $CDTL_{\Box} \not\vdash FS_{\Diamond}(p, q)$ and $CDTL_{\Box} \not\vdash FS_{\Box}(p, q)$.
2. $FS_{\Diamond}(p, q)$ and $t_{\Box}(FS_{\Diamond}(p, q))$ are not valid over $\mathbb{R}$, hence $RTL_{\Box} \not\vdash FS_{\Diamond}(p, q)$ and $RTL_{\Box} \not\vdash FS_{\Box}(p, q)$.

**Proof**

For the first claim, let us consider the model $\mathcal{M} = (W, \preceq, S, V)$ defined by 1) $W = \{w, v, u\}$; 2) $S(w) = v, S(v) = v$ and $S(u) = u$; 3) $v \preceq u$; 4) $[p] = \{u\}$, and 5) $[q] = \emptyset$ (see Figure S.1). Clearly, $\mathcal{M}, u \not\models p \rightarrow q$, so $\mathcal{M}, v \not\models p \rightarrow q$. By definition, $\mathcal{M}, w \not\models \Box(p \rightarrow q)$; however, $\mathcal{M}, w \models \Diamond p \rightarrow \Box q$, since the negation of each antecedent holds, so $\mathcal{M}, w \not\models (\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$.

For the second we let $(\mathbb{R}, S, [\cdot])$ be a model based on $\mathbb{R}$ with $S: \mathbb{R} \rightarrow \mathbb{R}$ given by $S(x) = 0$ for all $x$, $[p] = (0, \infty)$, and $[q] = \emptyset$. Then we have that $\Diamond p = (0, \infty)$, so that $-1 \in [\neg \Diamond p]$ and hence $-1 \in [\Diamond p \rightarrow \Box q]$. However, if $U$ is an $S$-invariant neighbourhood of $-1$ then $0 = S(-1) \in U$, but $0 \not\in [p \rightarrow q] = (-\infty, 0)$, hence $-1 \not\in [\Box(p \rightarrow q)]$. It follows that $-1 \notin [FS_{\Diamond}(p, q)]$. Similar reasoning shows that $-1 \notin [t_{\Box}(FS_{\Diamond}(p, q))]$. \qed

**Remark 8.5**

As mentioned previously, (Yuse and Igarashi 2006) present a Hilbert-calculus which yields a sub-logic of $ITL_{\Box}$. They also present a Gentzen-style calculus and conjecture that their two calculi prove the same set of formulas. However, (Kojima and Igarashi 2011) show that the formula $FS_{\Box}(p, q)$ is derivable in this Gentzen calculus, while Proposition 8.4 shows that it is not derivable in $ITL_{\Box}$. Hence, the two calculi are not equivalent.

Now we show that our axioms for Euclidean spaces are not valid in general. In particular, $CEM$ is valid for $\mathbb{R}$, but it is not valid for higher-dimensional spaces. In view of Theorem 6.10, it suffices to show that it is not valid over the class of expanding posets.

**Lemma 8.6**

The formula $CEM$ is not valid on the class of expanding posets, hence $CDTL_{\Box} \not\vdash CEM$. 

![Diagram](source_url)
Proof
Consider the model $\mathcal{M} = (W, \leq, S, [\cdot])$, where $W = \{w_0, w_1, v_0, v_1, v_2\}$, $[p] = \{v_2\}$ and $[q] = \{v_1, v_2\}$, and for $x, y \in W$, $x \leq y$ if and only if $x = y$ and $i \leq j$, and $S(x_i) = v_i$ (see Figure 8.2). Then, it is not hard to check that \(\text{CEM}(p, q) = (\neg \Diamond p \land \Diamond \neg p) \rightarrow \Diamond q \lor \neg \Diamond q\) fails at $w_0$. 

Similarly, $\text{CD}^{-}(p)$, which is valid on all Euclidean spaces, is not valid on all dynamical systems, even those based on $Q$.

Lemma 8.7
The formula $\text{CD}^{-}(p)$ is not valid on the class of invertible systems based on $Q$, hence $\text{ITL}^+_Q \not\models \text{CD}^{-}(p)$.

Proof
Recall that $\text{CD}^{-}(\varphi)\psi = \Box (p \lor \neg p) \rightarrow \Diamond p \lor \Diamond \neg p$. Let $S$ be given by $S(x) = x + 1$. Define a set

$$D = Q \cap \bigcup_{n \in \mathbb{N}} \left( n - \frac{1}{n + \pi}, n + \frac{1}{n + \pi} \right)$$

and let $[p] = Q \setminus D$. It is readily verified that $\frac{1}{n + \pi} \not\in Q$ for any $n \in \mathbb{N}$, and hence

$$Q \cap \left( n - \frac{1}{n + \pi}, n + \frac{1}{n + \pi} \right) = Q \cap \left[ n - \frac{1}{n + \pi}, n + \frac{1}{n + \pi} \right],$$

so that $D$ is both closed and open in $Q$. It follows that $[[\neg p]] = D$, and hence $[p \lor \neg p] = Q$; but $Q$ is open and $S$-invariant, so $[[\Box (p \lor \neg p)]] = Q$ as well. In particular, $0 \in [[\Box (p \lor \neg p)]]$.

Moreover, we claim that

(a) $0 \not\in [[\Diamond p]]$, but
(b) if $x \in (0, 1/2)$ and $n > 1/x$, then $S^n(x) \in [p]$.

Indeed, for (a) we see that any $n \in \mathbb{N}$, $S^n(0) = n \in D$, while for (b) if $x \in (0, 1/2)$ and $n > 1/x$, then

$$S^n(x) = n + x \in \left( n + \frac{1}{n + \pi}, n + \frac{1}{n + \pi} \right) \subseteq \left( n + \frac{1}{n + \pi}, (n + 1) - \frac{1}{n + \pi} \right),$$

so $S^n(x) \not\in D$. If $U$ is an $S$-closed neighbourhood of 0, $U$ contains some $x \in (0, 1/2)$. From (b) it follows that $S^n(x) \not\in [[\neg p]]$, hence $U \not\subseteq [[\neg p]]$; since $U$ was arbitrary, $0 \not\in [[\Box \neg \varphi]]$. 

The above independence results are sufficient to see that the only non-trivial inclusions between our axiomatic systems are given by Proposition 2.6.
Theorem 8.8
For each of the following families of axiomatically defined logics (see Table 2.1) or semantically defined logics (see Table 4.1) has pairwise distinct elements, and all subset relations are as indicated in Figures 2.1 or 7.1.

1. ITL κ, ITL + κ, CDTL κ, RTL κ, ETL κ, ETL + κ, and CDTL + κ;
2. ITL ⊗, ITL + ⊗, CDTL ⊗, RTL ⊗, ETL ⊗, ETL + ⊗, and CDTL + ⊗; and
3. ITL L, ITL + L, ITL R, ITL R 2, ITL ⊗ L, ITL ⊗ R, and ITL P.

The logics in the last item may be replaced by their fragments with only one of ⊗ or □.

Proof
For the first item, each arrow from Λ 1 to Λ 2 in Figure 8.3 is labelled by a formula which we have previously shown to belong to Λ 2 \ Λ 1. The same formulas may be used to separate the respective logics in the other two items. The non-trivial subset relations between the logics have been established in Propositions 2.6 and Theorems 6.7, 6.10, and 7.13.

We may also classify ◻-free logics.

Theorem 8.9
For each of the following families of logics, their elements are pairwise distinct, and all subset relations are as indicated in Figures 8.4.

1. ITL κ, ITL + κ, CDTL κ, RTL κ, and CDTL + κ;
2. ITL ⊗, ITL + ⊗, CDTL ⊗, RTL ⊗, and CDTL + ⊗; or
3. ITL L, ITL + L, ITL R, ITL R 2, ITL L L, ITL L R, and ITL P.

The logics in the last item may be replaced by their fragments with only one of ⊗ or □.

Proof
Similar to Theorem 8.8, Figure 8.4 displays formulas separating these logics, except that instances of CD should be replaced by BI.

Remark 8.10
Note that logics characterized by CD − are not included in the statement of Theorem 8.9. In particular, the formula BI(¬p, p) is already valid over the class of all dynamical systems. We do not know if the ◻-free logic of dynamical systems based on R 2 is different from that of all dynamical systems.

9 Concluding Remarks and Future Perspectives
We have proposed a natural ‘basic’ intuitionistic temporal logic, ITL κ, along with possible extensions including Fischer Servi or constant domain axioms, and weakened versions obtained by modifying the fixed-point axioms for ‘henceforth’. We have seen that relational semantics validate the constant domain axiom, leading us to consider a wider class of models based on topological spaces, with two possible interpretations for ‘henceforth’: the weak henceforth, ≺, and the strong henceforth, □. With this, we have shown that the logics ITL κ, CDTL κ, ITL + κ and CDTL + κ are sound for the class of all dynamical
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Fig. 8.3: Graph displaying the dependences among the different logics studied in this paper. Nodes correspond to different logic while edges mean two different kinds of relation. Edges of the form $Λ_1 - \varphi \rightarrow Λ_2$ mean that $Λ_1 \subseteq Λ_2$ and, moreover, $\varphi \in Λ_2 \setminus Λ_1$. Edges of the form $Λ_1 - \varphi \dashrightarrow Λ_2$ mean that $Λ_2 \not\subseteq Λ_1$ and $\varphi \in Λ_2 \setminus Λ_1$.

Fig. 8.4: Inclusions between $\Diamond$-free logics.
systems, of all dynamical posets, of all open dynamical systems, and of all persistent dynamical posets, respectively, which we have used in order to prove that the logics are pairwise distinct. We have also shown that the logics $\text{RTL}_\Box$, $\text{ETL}_\Box$, and $\text{ETL}_\Box^+$, based on Euclidean spaces, are distinct from any of the above-mentioned logics. We have performed a similar analysis for logics using $\boxast$ instead of $\Box$.

Of course this immediately raises the question of completeness, which we have not addressed. Specifically, the following are left open.

**Question 1**
Are the logics:

(a) $\text{ITL}_\Box$, $\text{ITL}_\Box^\p$, and $\text{ITL}_\Box^\p$ complete for the class of dynamical systems?
(b) $\text{CDTL}_\Box$ and $\text{CDTL}_\Box^\p$ complete for the class of expanding posets?
(c) $\text{ITL}_\Box^\p$, $\text{ITL}_\Box^\p$, and $\text{ITL}_\Box^\p$ complete for the class of open dynamical systems?
(d) $\text{ITL}_\Box^R$ complete for the class of systems based on $\mathbb{R}$?
(e) $\text{ETL}_\Box$, $\text{ETL}_\Box^\p$ complete for the class of systems based on Euclidean spaces?
(f) $\text{ETL}_\Box^\p$ complete for the class of systems based on Euclidean spaces with a homeomorphism?
(g) $\text{CDTL}_\Box^\p$, $\text{ITL}_\Box^\p$ and $\text{CDTL}_\Box^\p$ complete for the class of persistent posets?

We already know that $\text{ITL}_\Box$ is sound and complete for the class of expanding posets and for Euclidean spaces ([Diéguez and Fernández-Duque 2018]. However, the completeness of $\text{ITL}_\Box^\p$ and $\text{ITL}_\Box^\p$ is likely to be a more difficult problem than that of $\text{ITL}_\Box$, as in these cases it is not even known if the set of valid formulas is computably enumerable, let alone decidable.

**Question 2**
Are any of the logics $\Lambda$, $\Lambda_\Box$, or $\Lambda_\Box$ with $\Lambda \in \{\text{ITL}_\Box^\p, \text{ITL}_\Box^\p\}$ decidable and/or computably enumerable?

A negative answer is possible for any of these logics, since that is the case for their classical counterparts ([Konev et al. 2006]) and these logics do not have the finite model property ([Boudou et al. 2017]). Nevertheless, the proofs of non-axiomatizability in the classical case do not carry over to the intuitionistic setting in an obvious way, and these remain challenging open problems.

Note that the semantic counterpart for $\text{ETL}_\Box$ used in Theorem 8.8 is $\text{ITL}_\Box^\p \cap \mathbb{R}$. We could have used $\text{ITL}_\Box^\p \cap \mathbb{R}^{n \geq 1}$ instead, as $\text{ETL}_\Box^\p$ is also sound for this class. This raises the following.

**Question 3**
Is every formula falsifiable on some $\mathbb{R}^n$ with a homeomorphism also falsifiable on $\mathbb{R}$?

Note that we have not considered weak logics with CD or FS. However, this is only due to the fact that the topological semantics we have considered do not yield semantically-defined logics which satisfy the latter axioms without also satisfying $\text{ITL}_\Box$. It may yet be that semantics for such logics may be defined using other classes of dynamical systems. In particular, our techniques do not show whether the weak and standard logics coincide in these cases.
Question 4

Is the logic $\text{CDTL}^{+} + \Box + \Diamond$ distinct from $\text{ITL}^{0} + \Box + \Diamond$?

We conjecture that an affirmative answer could be given using more general algebraic semantics, but we leave this for future work. Finally, we remark that while we have not considered logics over the full language, it is possible to study logics which combine $\Box$ and $\Diamond$. Over dynamic posets or over open dynamical systems such an extension would be uninteresting since both operators are equivalent, but over the class of all dynamical systems, Lemma 4.7 suggests defining

$$\text{ITL}^{+ \Box \Diamond} \overset{\text{def}}{=} \text{ITL}^{+ \Box} + \text{ITL}^{+ \Diamond} + \Box p \rightarrow \Diamond p.$$ 

This leaves us with one final question.

Question 5

Is the logic $\text{ITL}^{5^+ \Box \Diamond}$ decidable, and does it enjoy a natural axiomatization?

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