Dynamical conductivity of ungated suspended graphene

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Frequency dependent conductivity of Coulomb interacting massless Dirac fermions coupled to random scalar and random vector potentials is found to be a function of frequency in the regime controlled by a line of fixed points. Such model provides a low energy description of a weakly rippled suspended graphene. The main finding is that at the neutrality point the a.c. conductivity is not frequency independent and may either increase or decrease with decreasing \( \omega \), depending on the values of the disorder variances \( \Delta_\phi, \Delta_A \) and the Coulomb coupling \( \alpha = e^2/(\epsilon v_F) \). The low frequency behavior is characterized by the values of two dimensionless parameters \( \gamma = \Delta_\phi/\alpha^2 \) and \( \Delta_A \) which are RG invariants, and for small values of which the electron-hole "puddles" are effectively screened making the results asymptotically exact.

The physics of massless Dirac fermions in two spatial dimensions has received renewed attention since their discovery in single-layer graphene\[1,2,3\]. The great interest is not unrelated to the quantum critical\[4,5,6,7,8,9\] nature of the system near the neutrality point, where the Fermi level lies precisely at the (Dirac-like) band crossing. Indeed, absence of any intrinsic long distance lengthscale sets constraints on the (low) frequency or temperature dependence of any physical quantity. In this regard, electrical conductivity \( \sigma \) plays a special role since in two spatial dimensions it is expected to be proportional to \( e^2/\hbar \); the proportionality constant, which need not be finite, depends only on the nature of the renormalization group (RG) fixed point characterizing the low energy-long distance physics\[4,8,10,11,12\]. Electrical conductivity measurements at the neutrality point therefore constitute a direct probe of the non-trivial physics emerging at the end of the RG trajectory.

Recent experiments performed on monolayer graphene, both suspended\[13\] and on the substrate\[14,15\], have found that the optical conductivity near the neutrality point is \( \sigma(\omega) = \frac{\pi e^2}{\hbar} \), i.e. largely frequency independent and equal to \( \pi/2 \) in the natural units. What little frequency dependence there is in the regime where \( \hbar \omega \sim 1eV \) can be attributed to the curvature corrections to the electronic dispersion which deviates from the perfectly conical massless Dirac-like at such large energies\[13\]. While presently there is no conductivity data at the lower frequencies of interest (at sub meV scales), it is natural to ask whether such frequency independent \( \sigma(\omega) \) should persist down to \( \hbar \omega \sim k_B T \). Since the role of charged impurities located at the substrate is naturally eliminated in the suspended samples, the dominant source of scattering is most likely the random configuration of strain due to the graphene sheet rippling and possibly from the boundary effects imposed by the scaffolding necessary for the actual suspension. Such long wavelength strain fields are known to couple to the massless Dirac particles of graphene as a vector potential and a scalar potential\[16,17,18\], and their combined effect on the a.c. conductivity, together with the effects of the electron-electron (Coulomb) interactions, are analyzed below.

In the non-interacting model of massless Dirac particles coupled to the random scalar and random vector potentials with variances \( \Delta_\phi \) and \( \Delta_A \), it has been long known that within the perturbative RG, \( \Delta_\phi \) grows upon approaching low energies and the theory flows to a perturbatively inaccessible fixed point\[19,20\]. The effects of Coulomb interactions, parameterized by a dimensionless coupling \( \alpha = e^2/(\epsilon v_F) \), and random vector potential (without scalar potential randomness) on the a.c. conductivity has been studied in Ref.\[4\] where it was found that the a.c. conductivity is non-universal and dependent only on \( \Delta_A \) which is marginal in the RG sense. Such non-universality is directly tied to the appearance of the infra-red (IR) locally stable line of fixed points. The two loop effects on the RG flow diagram have been incorporated in Ref.\[21\] and extended to other types of disorder in Ref.\[22\].

In this work, we study the combined effects of the unscreened Coulomb interactions, the quenched random vector and scalar potential disorder which arise naturally in the model of (randomly) strained sample, and analyze the frequency dependent conductivity within perturbative RG. This includes the effects of monolayer ripples and electron-hole "puddles". The perturbative RG adopted here is technically much simpler than the large \( N \) approximation adopted by Foster and Aleiner\[22\] to map out the phase diagram, and to leading order, leads to qualitatively similar results (for differences beyond the leading order, and the advantages of the former, see Ref.\[21\]). Moreover, the weak coupling RG can be easily extended to the calculation of \( \sigma(\omega) \), which was not calculated in\[22\]. The main finding is that \( \sigma(\omega) \) is not frequency independent and may either increase or decrease with decreasing \( \omega \), depending on the values of \( \Delta_\phi, \Delta_A \) and \( \alpha \). The low frequency behavior is characterized by the values of the dimensionless parameters \( \gamma = \Delta_\phi/\alpha^2 \) and \( \Delta_A \) which are RG invariants. The ultimate low fre-
only on the RG invariants $\Delta$ conductivity, in the collisionless limit of interest here, making the above result asymptotically exact. The ac parameter regime the weak coupling RG flow equations lead $<\gamma < \pi$ dependence of the conductivity, are presented below.

FIG. 1: The renormalization group flow diagram in the scalar disorder $\Delta_\phi$ - Coulomb interaction $\alpha = e^2/(\varepsilon v_F)$ plane. There are two marginal parameters (RG invariants): the variance of the random vector potential $\Delta_A$ and the ratio $\gamma = \Delta_\phi/\alpha^2$. For $\gamma < 32/(\pi^2 \Delta_A)$ each RG trajectory can cross three fixed points (the strong coupling one is not shown). The middle one, which is IR stable, merges with the IR unstable at a multicritical point (red circle) when $\gamma = 32/(\pi^2 \Delta_A)$. $\alpha_0 = \Delta_A/\pi$ lies along the fixed line discussed in [4, 21]. For $\gamma > 32/(\pi^2 \Delta_A)$, there are runaway flows with no perturbatively accessible fixed points. The phase diagram splits naturally into two regimes: Regime I (shaded) given by the locus of three fixed points which encompasses the parameter regime not included in Regime II (unshaded) with runaway flows. The $\omega \to 0$ limit of the (collisionless) a.c. conductivity along the IR stable fixed line is given in Eq. (4), and its $\omega$ dependence is discussed in the text.

frequency behavior depends on being in one of two regimes (see Fig 1).

Specifically, for the bare couplings in the Regime I, which is determined by the conditions $\alpha < 8 \Delta_A/\pi$ and $\gamma < \pi^2/(32 \Delta_A)$ or $\alpha > 8 \Delta_A/\pi$ and $\Delta_\phi < (\pi/2)\alpha - 2 \Delta$, then as $\omega \to 0$

$$\sigma \to \frac{e^2}{\hbar} \left[ \frac{\pi}{2} + \frac{\Delta_A}{6} + \frac{(23 - 6\pi)\pi^2}{96\gamma} \left(1 - \sqrt{1 - \frac{32}{\pi^2 \gamma \Delta_A}} \right) \right]$$

subject to the constraint $0 < \gamma < \pi^2/(32 \Delta_A)$. In this parameter regime the weak coupling RG flow equations lead to a perturbatively accessible IR stable line of fixed points, making the above result asymptotically exact. The ac conductivity, in the collisionless limit of interest here, therefore behaves as a universal amplitude, depending only on the RG invariants $\Delta_A$ and $\gamma$. In the Regime II, which encompasses the parameter regime not included in Regime I there are no perturbatively accessible fixed points, and the problem remains open. The details of the calculation leading to the above claims, as well as $\omega$ dependence of the conductivity, are presented below.

We start with the imaginary time partition function

$$Z = \int D\psi \bar{\psi} e^{-\left(S_0 + S_{\text{dis}} + S_{\text{int}}\right)} \tag{2}$$

where

$$S_0 = \int_0^\beta d\tau \int d^2 r \bar{\psi}(r, \tau) \left(\partial_\tau + v_F \sigma \cdot \mathbf{p}\right) \psi(r, \tau) \tag{3}$$

$$S_{\text{dis}} = \int_0^\beta d\tau \int d^2 r \bar{\psi}(r, \tau) \left(\phi(r) + v_F \sigma \cdot \mathbf{a}\right) \psi(r, \tau) \tag{4}$$

$$S_{\text{int}} = \frac{1}{2} \int_0^\beta d\tau \int d^2 r d^2 r' \bar{\psi}(r, \tau) V(|r - r'|) \psi(r', \tau) \tag{5}$$

The last term corresponds to the (Coulomb) electron-electron interaction $V(|r - r'|) = \frac{e^2}{\varepsilon |r - r'|}$, where $\varepsilon$ is the dielectric constant which may differ from 1. We assume that the disorder is uncorrelated with variances:

$$\langle \phi_k \phi_k' \rangle = (2\pi)^2 \delta(k - k') v_F^2 \Delta_\phi \tag{6}$$

$$\langle \phi_k^v \phi_k^v' \rangle = (2\pi)^2 \delta(k - k') \delta_{\mu\nu} \Delta_A. \tag{7}$$

As has been discussed extensively in the past, the scalar and vector potentials are naturally connected to the appearance of strain tensor $u_{ij}$ as $\phi = g(u_{xx} + u_{yy})$, $a_x = b(u_{yy} - u_{xx})$, $a_y = 2bu_{xx}$ [16, 17, 18, 23], with the estimates $g \approx 20 - 30 eV$ [16, 18] and $b \approx A^{-1} [16, 23]$.

We can perform the (quenched) average over the gaussian disorder fields $\phi(r)$ and $\mathbf{a}(r)$ using the standard replica trick of including $n$ copies of the fermion fields: $\psi \to \psi^n$, where $i = 1, 2, \ldots, n$. The resulting replica field theory is

$$\langle Z^n \rangle_{\text{dis}} = \int D\bar{\psi} \psi^n e^{-\left(S_0 + S_\phi + S_A + S_{\text{int}}\right)} \tag{8}$$

where

$$S_0 = \int_0^\beta d\tau \int d^2 r \bar{\psi} \left(\partial_\tau + v_F \sigma \cdot \mathbf{p}\right) \psi \tag{9}$$

$$S_\phi = -\frac{\varepsilon v_F^2 \Delta_\phi}{2} \int_0^\beta d\tau d\tau' \int d^2 r \bar{\psi} \psi \left(\partial_\tau + v_F \sigma \cdot \mathbf{p}\right) \psi \tag{10}$$

$$S_A = -\frac{\varepsilon v_F^2 \Delta_A}{2} \int_0^\beta d\tau d\tau' \int d^2 r \bar{\psi} \sigma^n \psi \left(\partial_\tau + v_F \sigma \cdot \mathbf{p}\right) \psi \tag{11}$$

$$S_{\text{int}} = \frac{1}{2} \int_0^\beta d\tau \int d^2 r d^2 r' \bar{\psi} \psi \left(\partial_\tau + v_F \sigma \cdot \mathbf{p}\right) \psi \tag{12}$$

As usual, we assume large momentum cutoff $\Lambda$ for the above fermion modes, and perform the renormalization of the bare coupling constants $e^2, v_F, \Delta_\phi, \Delta_A$. To first order we find that the imaginary time Greens function (Fig. 2) satisfies

$$G_{\omega^{-1}}(k) = -i\omega \left(1 + \frac{\Delta_\phi}{2\pi} \log \frac{\Lambda}{|\omega|} + \frac{\Delta_A}{\pi} \log \frac{\Lambda}{|\omega|}\right) + \left(v_F + \frac{e^2}{4} \log \frac{\Lambda}{|\omega|}\right) \sigma \cdot k \tag{9}$$
The renormalization condition demands that we absorb the dependence on the cut-off $\Lambda$ into a field rescaling constant $Z$ and the bare couplings. We do so at an arbitrary scale $\omega = k = \kappa$ where we demand that

$$G_{\omega}(k)|_{\kappa} = ZG_{\omega}^{R}(k)|_{\kappa} = Z(-i\kappa + v_{F}\sigma \cdot k)^{-1}$$

(10)

The renormalized Greens function $G_{\omega}^{R}(k)$, at any $\omega$ and $k$, is now independent of $\Lambda$. This leads to the RG equation for the Fermi velocity

$$\beta_{v_{F}} = \frac{\partial v_{F}}{\partial \log \Lambda} = v_{F} \left( \frac{\Delta_{\phi}}{2\pi} + \frac{\Delta_{A}}{\pi} - \frac{e^{2}}{4v_{F}} \right)$$

(11)

FIG. 3: Diagrams which contribute to the disorder renormalization.

To determine the RG scaling of the disorder variances we need to analyze the $\beta$ functions of the effective replica coupling constants. We do so by writing the equations for the irreducible four point vertex:

$$\Gamma^{(4)} = \frac{-1}{4!} \langle \psi^{\dagger} \psi^{\dagger} \psi^{\dagger} \psi \rangle_{\text{con.amp.}}.$$  

The renormalization prescription demands that at a scale $\omega = k = \kappa$, $\Gamma^{(4)} = \frac{1}{2\pi} \Gamma_{\omega}^{(4)}$. This means that at arbitrary $\omega, k$, the quantity $Z^{2} \Gamma^{(4)}$ can be made independent of $\Lambda$. To this order in coupling constants (Fig. 3) we find

$$\beta_{\Delta_{\phi}} = \frac{\partial \Delta_{\phi}}{\partial \log \Lambda} = -2\Delta_{\phi} \left( \frac{\Delta_{\phi}}{2\pi} + \frac{\Delta_{A}}{\pi} - \frac{e^{2}}{4v_{F}} \right)$$

(12)

$$\beta_{\Delta_{A}} = \frac{\partial \Delta_{A}}{\partial \log \Lambda} = 0$$

(13)

$$\beta_{e^{2}} = \frac{\partial e^{2}}{\partial \log \Lambda} = 0,$$

(14)

which agrees with Ref. 22. As argued in Ref. 4, the last equation is exact. The corresponding flow diagram is shown in Fig 4.

Defining the dimensionless Coulomb coupling constant $\alpha = e^{2}/\epsilon v_{F}$, the above equations imply the existence of two RG invariants $\gamma = \Delta_{\phi}/\alpha^{2}$ and $\Delta_{A}$, i.e.

$$\frac{\partial \gamma}{\partial \log \Lambda} = \frac{\partial \Delta_{A}}{\partial \log \Lambda} = 0.$$

(15)

FIG. 4: Diagrammatic contribution to the (a.c.) electrical conductivity within Kubo formula. From left to right: free Dirac fermion contribution, disorder vertex and self energy contribution.

Since conductivity does not acquire anomalous dimension we have

$$\left( \frac{\partial}{\partial \log \Lambda} + \hat{B} \right) \sigma(\omega; \Lambda, \Delta_{\phi}, \Delta_{A}, v_{F}, e^{2}) = 0,$$

(16)

where the differential operator

$$\hat{B} = \beta_{\Delta_{\phi}} \frac{\partial}{\partial \log \Lambda} + \beta_{\Delta_{A}} \frac{\partial}{\partial \Delta_{A}} + \beta_{v_{F}} \frac{\partial}{\partial v_{F}} + \beta_{e^{2}} \frac{\partial}{\partial e^{2}}$$

(17)

The solution of the above RG equation must satisfy the scaling law

$$\sigma(\omega; \rho, \Delta_{\phi}(\rho \Lambda), \Delta_{A}(\rho \Lambda), v_{F}(\rho \Lambda), e^{2}(\rho \Lambda)) =$$

$$\sigma(\omega; \Lambda, \Delta_{\phi}(\Lambda), \Delta_{A}(\Lambda), v_{F}(\Lambda), e^{2}(\Lambda))$$

(18)

where $\rho$ is a positive real number.

A pedestrian perturbation theory calculation to the leading order in coupling constants (Fig 3) gives

$$\sigma_{pt}(\omega) = 4 e^{2} \frac{1}{h} \left[ \frac{\pi}{8} - \frac{\Delta_{\phi}}{24} - \frac{\Delta_{A}}{24} + \frac{\pi \alpha(\frac{25}{6} - \pi)}{16} \right].$$

(19)

This extends the result found in [4] to include the scalar-disorder potential contribution. Note that the above expression does not satisfy the scaling law (18), since according to Eqs. (11-14) the coupling constant $\alpha$ and the scalar disorder variance $\Delta_{\phi}$ do have non-trivial dependence on $\Lambda$. Nevertheless, to the same order in the coupling constants, both (18) and (19) can be satisfied if

$$\sigma(\omega) = 4 e^{2} \frac{1}{h} \left[ \frac{\pi}{8} - \gamma \alpha^{2}(\frac{25}{6} - \pi) \right]$$

(20)

where we used $\Delta_{\phi}(\rho) = \gamma \alpha^{2}(\rho)$. The dimensionless Coulomb coupling constant $\alpha(\rho)$ is defined as the solution of

$$\frac{\partial \alpha}{\partial \log \rho} = -\alpha \left( \frac{\gamma}{2\pi} \alpha^{2} - \frac{\alpha}{4} + \frac{\Delta_{A}}{\pi} \right)$$

(21)

with the initial condition $\alpha(1) = \alpha$. The functional dependence can be found implicitly

$$\frac{4\pi \alpha(\frac{25}{6} - \pi)}{\pi \alpha(\frac{25}{6} - \pi) - (1 - A)} \left( \frac{\alpha^{2}(\frac{25}{6} - \pi)}{(\frac{25}{6} - \pi \alpha(\frac{25}{6} - \pi) - 1) - A^{2}} \right)^{A} =$$

$$\frac{4\pi \alpha - (1 + A)}{4\pi \alpha - (1 - A)} \left( \frac{\alpha^{2}}{(\frac{25}{6} \alpha - 1) - A^{2}} \right)^{A} \left( \frac{\omega}{\Lambda} \right)^{4\pi \alpha^{2} \Lambda(A^{2} - 1)}$$

(22)
where $A = \sqrt{1 - \frac{32}{\pi^2} \gamma \Delta_A}$. The above equation is easily inverted numerically. As an illustration, the resulting conductivity as a function of frequency for $\Delta_A = 1$ and in the vicinity of the multicritical trajectory is plotted in Fig. 5.

The explicit dependence on $\omega$ can be found in some limiting cases. In the vicinity of the IR stable fixed line, but away from the multicritical point (see Fig. 1), we find

$$\alpha \left( \frac{\omega}{\Lambda} \right) \approx \frac{\pi}{4\gamma} \left[ 1 - \frac{1 - \sqrt{1 - \frac{32}{\pi^2} \gamma \Delta_A}}{1 - \sqrt{1 - \frac{32}{\pi^2} \gamma \Delta_A}} \right] \left( \frac{\omega}{\Lambda} \right)^\theta$$

where the (crossover) exponent

$$\theta = \frac{\pi}{16\gamma} \sqrt{1 - \frac{32}{\pi^2} \gamma \Delta_A} \left( 1 - \sqrt{1 - \frac{32}{\pi^2} \gamma \Delta_A} \right)$$

Note also that

$$\frac{\partial \alpha(\omega)}{\partial \log \omega} = 4 \frac{e^2}{h} \left[ \frac{\pi}{16} \left( \frac{25}{6} - \pi \right) - \frac{\gamma}{12} \alpha(\omega) \right] \frac{\partial \alpha}{\partial \log \omega}.$$
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