In this paper we discuss two approaches to anomaly-free quantization of a two-dimensional string. The first approach is based on the canonical Dirac prescription of quantization of degenerated systems. At the second approach we "weaken" the Dirac quantization conditions requiring the solving of first class constraints only in the sense of mean values. At both approaches there are no states with the indefinite metrics.

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1. Introduction

The quantum theory of gravity in four-dimensional space-time encounters fundamental difficulties, which now are not overcome. These difficulties can be divided on conceptual and computing. The main conceptual problem is that the Hamiltonian is a linear combination of a first class constrains. This fact makes unclear a role of time in gravitaty. The main computing problem consists in nonrenormalizability of the theory of gravity. The pointed difficulties are closely bound. For example, an anomaly (central charge) can be present or absent in algebra of first class constraints depending on computing procedure. In turn the presence or absence of anomaly in algebra of the first class constraints crucially influences on the resulting physical picture.

The listed fundamental problems are solved successfully for rather simple models of generally covariant theories in two-dimensional space-time. Both two-dimensional gravity and string theories belong to such models (see, for example [1-4] and reference there).

In the present work we develop ideology and technique of anomaly-free quantization of two-dimensional string. The Hamiltonian of a two-dimensional bosonic string actually coincides with Hamiltonian of two-dimensional pure gravity expressed in certain special variables [1,2]. The difference between two-dimensional string and two-dimensional gravity is that in the theory of two-dimensional string there is an additional external symmetry relative to abelian Lorentz group in two-dimensional space-time. However, the ideological basis of anomaly-free quantization in both models is identical. We explain this idea, using a material of works [1,2,4].

Let’s consider in two-dimensional space-time the following system of constraints:

\[ \mathcal{E} = -\mathcal{E}_0 + \mathcal{E}_1 \approx 0, \]

\[ \mathcal{E}_0 = \frac{1}{2} \left( (\pi_0)^2 + (r_0')^2 \right), \quad \mathcal{E}_1 = \frac{1}{2} \left( (\pi_1)^2 + (r_1')^2 \right), \]

\[ P = r^{a'} \pi_a = r^{0'} \pi_0 + r^{1'} \pi_1 \approx 0 \]  

(1.1a)

Here we use dimensionless values, \( r^a(x) \) and \( \pi_a(x) \), \( a = 0, 1 \) are canonically conjugated fields on a one-dimensional manyfold, so the nonzero commutational relations look like

\[ [r^a(x), \pi_b(y)] = i \delta^a_b \delta(x - y) \]  

(1.2)

The prime or overdot mean the derivatives \( \partial / \partial x \) or \( \partial / \partial t \), respectively.

Now we must determine the ground state of the theory. It allows to perform normal ordering of the operators in the constraints (1.1). The normal ordering in the constraints can result to anomalies in the commutators of the constrains. These anomalies partially destroy the weak equalities (1.1). To determine the ground state, the fields \( r^a \) and \( \pi_a \) are expanded in the modes which arise when solving the Heisenberg equations

\[ i \dot{r}^a = [r^a, \mathcal{H}], \quad i \dot{\pi}_a = [\pi_a, \mathcal{H}], \]

\[ \mathcal{H} = \int dx \mathcal{E} \]  

(1.3)

The solution of the equations (1.2), (1.3) can be written in the form

\[ r^a(t, x) = \int \frac{dk}{2 \pi} \frac{1}{\sqrt{2|k|}} \left\{ c_k^a e^{-i(|k|t-kx)} + c_k^{a+} e^{i(|k|t-kx)} \right\}, \]

\[ \pi^a(t, x) = -i \int \frac{dk}{2 \pi} \sqrt{\frac{2|k|}{k}} \left\{ c_k^a e^{-i(|k|t-kx)} - c_k^{a+} e^{i(|k|t-kx)} \right\}, \]

\[ [c_k^a, c_p^{b+}] = 2 \pi \eta^{ab} \delta(k-p) \quad [c_k^a, c_p^b] = 0 \]  

(1.4)

Here \( \eta^{ab} \) (below - \( \eta^{\mu\nu} \)) = diag\((-1, 1)\). We have also the following commutational relations:

\[ [\mathcal{H}, c_k^a] = -|k| c_k^a, \quad [\mathcal{H}, c_k^{a+}] = |k| c_k^{a+} \]  

(1.5)

Under traditional quantization the operators \( c_k^a \) are considered as the annihilation operators, and their hermitean conjugated \( c_k^{a+} \) are considered as creaton operators. The ground state \( |0\rangle \) satisfies the conditions:

\[ c_k^a |0\rangle = 0 \]  

(1.6)
Normal ordering of the operators \((c_k^+, c_k^a)\) in values (1.1) means an arrangement of the creation operators on the left of all annihilation operators.

Let’s consider the state
\[
|k, a\rangle = c_k^a |0\rangle
\]  
(1, 7)

From commutational relations (1.5) immediately follows, that
\[
\mathcal{H} |k, a\rangle = (|k| + E_0) |k, a\rangle,
\]
where \(E_0\) is the value of the operator \(\mathcal{H}\) for the ground state. The equality (1.8) implies the operator \(\mathcal{H}\) is positively defined.

In consequence of (1.4) and (1.6) we have for scalar product of vectors (1.7):
\[
\langle k, a | p, b \rangle = 2\pi \eta^{ab} \delta(k - p)
\]
(1.9)

Now we see that the metrics in complete space of states is indefinite.

Let’s calculate the commutator \([\mathcal{E}, \mathcal{P}]\), which can be represented according to (1.1) as a sum of two terms:
\[
[\mathcal{E}(x), \mathcal{P}(y)] = -[\mathcal{E}_0(x), r^0 \pi_0(y)] + [\mathcal{E}_1(x), r^0 \pi_1(x)]
\]
(1.10)

According to (1.2) the both commutators in the right hand side of Eqs.(1.10) coincide up to replacement of index ”\(a\)”. These commutators are proportional (up to the ordering) to quantities \(\mathcal{E}_0\) and \(\mathcal{E}_1\), respectively. The normal ordering of the operators in considered commutators, as is known, results in anomalies.

Indeed, from the commutational relations (1.4) it follows that the correspondences \(c_k^0 \leftrightarrow c_k^{0+}\), \(c_k^{0+} \leftrightarrow c_k^1\) give the isomorphism of Heisenberg algebras \(H_0\) and \(H_1\) with generators \((c_k^0, c_k^{0+})\) and \((c_k^1, c_k^1)\) respectively. In this case the normal ordering of the operators in algebra \(H_1\) is mapped by specified isomorphism to antinormal ordering in algebra \(H_0\).

As is known, in such case the normal and antinormal orderings result in anomalies differing only in sign. Hence, the contribution of the first commutator in anomaly in the right hand side of Eq.(1.10) will be \((-A)\) and of the second commutator will be \((A)\). But as in front of the first commutator in (1.10) there is a sign ”minus”, the anomaly in (1.10) is equal \(-(-A) + A = 2A\).

Now we pass to another point of view.

In the work [1] Jackiw states that the condition (1.8) of positivity of operator \(\mathcal{H}\) is not necessary in the considered theory. The initial requirement of the theory is that the weak equalities (1.1) are valid. Therefore we can refuse the conditions of quantization (1.6) and replace them by the following:
\[
c_k^0 |0\rangle = 0, \quad c_k^1 |0\rangle = 0
\]
(1.11)

Under the quantization conditions (1.11) the bases of complete Fock space of the theory consists of the following vectors:
\[
c_k^0, c_k^1, \ldots c_{k_n}^0 c_{p_1}^{1+} \ldots c_{p_n}^{1+} |0\rangle
\]
(1.12)

From the commutational relations (1.4) it follows that the scalar product of states (1.12) is positively defined. Moreover, in the algebra of the operators (1.1) there is no anomaly.

Indeed, under conditions (1.11) the normal ordering consists in arrangement of the operators \((c_k^0, c_k^{1+})\) on the left of all operators \((c_k^{0+}, c_k^1)\). It means the normal ordering in both Heisenberg algebras \(H_0\) and \(H_1\). Now at the normal ordering the both commutators in (1.10) give in anomaly the same contribution, which is equal to \(A\). Since in front of the first commutator in the right hand side of Eq.(1.10) there is a sign ”minus” the complete anomaly appears equal to \(-A + A = 0\).

The absence of anomaly in algebra of the operators \((\mathcal{E}, \mathcal{P})\) enables to satisfy all weak equalities \(\mathcal{E} \approx 0, \mathcal{P} \approx 0\). In [1] two physical states annihilating all operators \(\mathcal{E}\) and \(\mathcal{P}\) are given:
\[
\Psi_{\text{gravity}}(r^a) = \exp \pm i \int dx \varepsilon_{ab} r^a r^{b'}
\]

In [2, 4] the continuous families of states parametrized by one real parameter which solve all constraints (1.1) are also considered.

In this work we reconsider the well known quantization conditions for relativistic bosonic string in two-dimensional space-time. As a result we formulate two another approaches to quantization of considered
model. The both approaches lead to the Virasoro algebra in which the central charge is absent. We use complete space of states with the positively defined scalar product. The complete space of states is decomposed to tensor product of physical states space and gauge states spaces. Such decomposition make possible a calculation of matrix elements for dynamic variables separately relative to physical states and states describing gauge degrees of freedom. This fact allows to change the Dirac quantization prescription for systems containing the first class constraints \cite{5}. The arising quantization rule can be useful for quantization of more complicated degenerated systems for example such as four-dimensional gravity.

2. Relativistic bosonic string in two-dimensional space-time

Let \( X^\mu = 0, 1 \) be the coordinates in two-dimensional Minkovski space. Consider the Nambu action for bosonic string:

\[
S = -\frac{1}{l^2} \int \sqrt{-g} \, d^2 \xi = \int d\tau \, \mathcal{L}
\]

Here \( \xi^a = (\tau, \phi) \) are the parameters of the world sheet of the string and

\[
g = \text{Det} \, g_{ab} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b}
\]

The parameter \( \tau \) is time like and \( \phi \) is spacial. Next the partial derivatives \( \partial / \partial \tau \) and \( \partial / \partial \phi \) will be denoted by an over dot and a prime, respectively. It is easy to show, that the generalized momenta \( \pi_\mu = \partial \mathcal{L} / \partial \dot{X}^\mu \) satisfy the conditions:

\[
\mathcal{E} = \frac{l^2}{2} \pi_\mu \pi^\mu + \frac{1}{2l^2} X^{\mu'} X'_\mu \approx 0, \\
\mathcal{P} = X^{\mu'} \pi_\mu \approx 0
\]

The quantities \( \mathcal{E}(\phi) \) and \( \mathcal{P}(\phi) \) exhaust the all first class constraints. The hamiltonian of the system

\[
\mathcal{H} = \int d\phi \, \pi_\mu \dot{\phi}^\mu - \mathcal{L} \approx 0
\]

is also equal to zero.

Therefore, following to Dirac, we must use the generalized hamiltonian which is an arbitrary linear combination of the first class constraints (2.2):

\[
\mathcal{H}_T = \int d\phi \left( v \mathcal{P} + w \mathcal{E} \right)
\]

The equations of motion can be obtained with the help of the variational principle

\[
\delta S = \delta \left\{ \int d\tau \left( \int d\phi \, \pi_\mu \dot{X}^\mu - \mathcal{H}_T \right) \right\} = 0
\]

In the case of an open string, when the parameter \( \pi \) changes from zero to \( \pi \), the variational principle (2.4) gives besides the equations of motion the boundary conditions

\[
(v \pi_\mu + \frac{1}{l^2} X'_\mu)|_{\phi=0, \pi} = 0
\]

Usually the boundary conditions (2.5) are replaced by conditions

\[
v|_{\phi=0, \pi} = 0, \quad X'_\mu|_{\phi=0, \pi} = 0
\]

For the closed string instead of the boundary condition there is the periodicity condition. Further we consider an open string.
The first step of quantization consists in definition of commutational relations for generalized coordinates and momenta:

\[
[X^\mu(\phi), \pi^\nu(\phi')] = i \eta^{\mu\nu} \delta(\phi - \phi') \tag{2.7}
\]

The commutational relations (2.7) and the boundary conditions (2.6) are satisfied if

\[
X^\mu(\phi) = \frac{l}{\sqrt{\pi}} \left( x^\mu + i \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n \cos n\phi \right),
\]

\[
\pi^\mu(\phi) = \frac{1}{\sqrt{\pi} l} \sum_n \alpha^\mu_n \cos n\phi \tag{2.8}
\]

and the elements \((x^\mu, \alpha^\mu_n)\) satisfy the following commutational relations:

\[
[x^\mu, \alpha^\nu_n] = i \eta^{\mu\nu} \delta_n,
\]

\[
[x^\mu, x^\nu] = 0,
\]

\[
[\alpha^\mu_m, \alpha^\nu_n] = m \eta^{\mu\nu} \delta_{m+n} \tag{2.9}
\]

Since the quantities (2.8) are real then

\[
x^{\mu+} = x^\mu, \quad \alpha^{\mu+}_n = \alpha^{-\mu}_n \tag{2.10}
\]

The constraints (2.2) can be represented as follows:

\[
(\mathcal{E} \pm \mathcal{P})(\phi) = \frac{1}{2} (\xi^\mu_\pm(\phi))^2, \tag{2.11}
\]

where

\[
\xi^\mu_\pm(\phi) = \frac{1}{\sqrt{\pi}} \sum_n \alpha^\mu_n \exp \mp i n\phi \tag{2.12}
\]

From here it is seen, that \(\mathcal{E} - \mathcal{P}\) is obtained from \(\mathcal{E} + \mathcal{P}\) by replacement \(\phi \rightarrow -\phi\). This fact simplifies the further analysis, as the quantity \(\mathcal{E} + \mathcal{P}\) in interval \(-\pi \leq \phi \leq \pi\) contains the information about quantities \(\mathcal{E} \pm \mathcal{P}\) in interval \(0 \leq \phi \leq \pi\). Therefore the Fourier components

\[
L_n = \frac{1}{2} \int_\pi^\pi d\phi (\mathcal{E} + \mathcal{P}) \exp i n\phi \tag{2.13}
\]

are equivalent to the set of quantities (2.2) at \(0 \leq \phi \leq \pi\). According to (2.11) - (2.13) we have

\[
L_n = \frac{1}{2} : \sum_m \alpha^\mu_{n-m} \alpha_{\mu m} : \tag{2.14}
\]

The sense of the ordering operation in (2.14) is defined by a method of quantization.

Let’s write out also the expressions for momentum and angular momentum of the string:

\[
P^\mu = \int_0^\pi d\phi \pi^\mu, \quad J^{\mu\nu} = \int_0^\pi d\phi (X^\mu \pi^\nu - X^\nu \pi^\mu) \tag{2.15}
\]

With the help of (2.6) and (2.7) we can directly verify, that

\[
[P^\mu, \mathcal{H}_T] = 0, \quad [J^{\mu\nu}, \mathcal{H}_T] = 0
\]

It means, that the momentum and angular momentum of the string are conserved.

Under standard quantization the ground state \(|0\rangle\) satisfies the conditions

\[
\alpha^\mu_m |0\rangle = 0, \quad m \geq 0 \tag{2.16}
\]

The complete space of states has the orthogonal basis:

\[
\alpha^\mu_{m_1} \ldots \alpha^\mu_{m_s} |0\rangle, \quad m_i < 0 \tag{2.17}
\]
Thus, the all \( \alpha_m^m \) are linear operators in complete space of states. From (2.9) and (2.16) it follows, that the metrics in the space of states (2.17) is indefinite. The ordering in (2.14) means, that the operators \( \alpha_m^m \) with \( m < 0 \) are placed on the left of the all operators \( \alpha_n^n \) with \( n \geq 0 \). The such ordering results in the Virasoro algebra contains the anomalies:

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12}D(n^3 - n)
\] (2.18)

Here \( D \) is the dimension of \( x \)-space. In our case \( D = 2 \). Therefore the annihilation of the operators \( L_n \) with \( n \geq 0 \) is maximum, that can be reached. As a result the theory is self-consistent only for \( D = 26 \). The detailed study of problems, arising under the quantization (2.16), can be found in [4].

Now we shall state the way of quantization of two-dimensional string, which results in the self-consistent string theory [1-4]. This quantization method of the string is similar to the quantization method applied by Dirac to electromagnetic field (see. [7], and also Appendix).

Let’s introduce the designations

\[
x_\pm = x^0 \pm x^1, \quad \alpha_{m}^{(\pm)} = \alpha_{m}^0 \pm \alpha_{m}^1
\] (2.19)

From (2.9) we obtain:

\[
\begin{aligned}
[\alpha_{m}^{(+)} , \alpha_{n}^{(-)}] &= [\alpha_{m}^{(-)} , \alpha_{n}^{(+)}] = 0, \quad [\alpha_{m}^{(+)} , \alpha_{n}^{(-)}] = -2m \delta_{m+n} \\
x_+ x_- &= 0, \quad [x_+ , \alpha_{n}^{(+)}] = [x_- , \alpha_{n}^{(-)}] = 0 \\
x_+ \alpha_{n}^{(-)} &= -2i \delta_n, \quad [x_- , \alpha_{n}^{(+)}] = -2i \delta_n
\end{aligned}
\] (2.20)

Let’s write out the Virasoro operators in variables \( \alpha^{(\pm)} \):

\[
L_n = -\frac{1}{2} \sum_{m} \alpha_{n-m}^{(+)} \alpha_{m}^{(-)}
\] (2.21)

By definition the ordering in (2.21) means either the elements \( \alpha^{(+)} \) are arranged on the left of the all elements \( \alpha^{(-)} \) or the elements \( \alpha^{(-)} \) are arranged on the left of the all elements \( \alpha^{(+)} \). Both these orderings are equivalent. Indeed

\[
\sum_{m} \alpha_{n-m}^{(-)} \alpha_{m}^{(+)} = \sum_{m} \alpha_{n}^{(+)} \alpha_{m}^{(-)} + 2 \sum_{m} m,
\]

It is possible to consider the last sum as equal to zero, since it is represented in the form \( \zeta(-1) - \zeta(-1) \), where \( \zeta(s) \) is the Riemann zeta-function. As is known the zeta-function

\[
\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s}
\] (2.22)

has the unique analytical continuation to the point \( s = -1 \) and \( \zeta(-1) = -1/12 \).

The following problem consists in the definition of vector space of states, in which the dynamic variables of the system act as linear operators.

In the beginning we apply the Dirac prescription for quantization of our model, marking the chosen way by the index \( D \). We represent the complete space of states \( H_{CD} \) as the tensor product of the gauge space of states \( H_G \) and physical space of states \( H_{PD} \):

\[
H_{CD} = H_G \otimes H_{PD}
\] (2.23)

The space \( H_G \) is generated by its vacuum vector \( |0; G\rangle \), which is determined by the following properties:

\[
\alpha_{-m}^{0} |0; G\rangle = 0, \quad \alpha_{m}^{1} |0; G\rangle = 0, \quad m > 0, \quad \langle 0; G |0; G\rangle = 1
\] (2.24)

The basis of the space \( H_G \) consists of vectors of the form

\[
\alpha_{m_1}^{0} \ldots \alpha_{m_r}^{0} \alpha_{-n_1}^{1} \ldots \alpha_{-n_r}^{1} |0; G\rangle, \quad m_i > 0, \quad n_i > 0
\] (2.25)

Thus, \( H_G \) is the Fock space with positively defined scalar product.
The basis in the Dirac physical space of states $H_{PD}$ consists of two series $|k\pm\rangle_D$ with the following properties ($\alpha_0^{(\pm)} = p_{\pm}$):

$$p_+ |k-\rangle_D = 2k |k-\rangle_D, \quad p_- |k+\rangle_D = 2k |k+\rangle_D,$$

(2.26)

$$a_m^{(-)} |k-\rangle_D = 0, \quad a_m^{(+)} |k+\rangle_D = 0, \quad m = 0, \pm 1, \ldots$$

(2.27)

and

$$\langle k \mp | k' \mp \rangle_D = k \delta(k - k'), \quad \langle k - | k' + \rangle_D = 0$$

(2.28)

Here $k$ is continuous real parameter, $0 < k < +\infty$. The quantization conditions (2.27) were applied in works [2,4]. Earlier the similar quantization conditions were applied by Dirac in electrodynamics [7].

From the given definitions it is seen, that in complete space of states the scalar product is positively defined.

Let’s emphasize, that variables $\alpha_n^\mu$, $n \neq 0$, being the linear operators in space $H_G$, are not operators in the space $H_{PD}$. Indeed, the action of the variables $\alpha_n^{(-)}$ on the vectors $|k+\rangle_D$, and also action of the variables $\alpha_n^{(+)}$ on the vectors $|k-\rangle_D$ is not defined, if $n \neq 0$. Nevertheless from the given definitions it follows, that the action of some combinations (generally speaking, nonlinear) of the variables $\alpha_n^\mu$ with $n \neq 0$ in the space $H_{PD}$ is determined correctly. For example, owing to (2.20) and (2.27), we have:

$$a_m^{(+)} a_m^{(-)} |k+\rangle_D = -2m |k+\rangle_D, \quad a_m^{(-)} a_m^{(+)} |k-\rangle_D = -2m |k-\rangle_D$$

(2.29)

Since

$$L_n = \frac{1}{2} \sum_m a_m^{(+)} a_m^{(-)} = \frac{1}{2} \sum_m a_{n-m}^{(-)} a_{m}^{(+)}$$

and as the consequence of (2.27) the equalities

$$L_n |p\rangle_D = 0, \quad |p\rangle_D \in H_{PD}$$

(2.30)

take place. Thus, all physical states are annihilated by the all Virasoro operators. The equalities (2.30) mean, that under quantization (2.24) - (2.28) the Virasoro algebra has no anomaly.

$$[L_n, L_m] = (n - m) L_{n+m}$$

(2.31)

Here we have a situation, which is common for all degenerated systems. In degenerated systems there is a set of first class constraints $\chi_m$. If evident resolution of these constraints is inexpedient, then according to Dirac prescription the following conditions

$$\chi_m |p\rangle_D = 0$$

(2.32)

are imposed on the physical states. It is obvious, that owing to (2.32) the physical states can not depend on gauge degrees of freedom, the changes of which are generated by the constraints $\chi_m$. Thus the variables, describing the gauge degrees of freedom can not be, generally speaking, the linear operators in the space of physical states. On the other hand, according to Dirac prescription it is necessary to consider only the physical states.

Let’s consider an example of investigated model the paradox which arise when the Dirac procedure is realized. We have a set of the oscillator variables $\alpha_m^\mu$, $m \neq 0$, and also the coordinate and momentum variables $(x^0, p^0)$. Under the Schrodinger quantization the complete space of states $H_C$ is decomposed to the tensor product of the spaces $H_G$ and $H_P$: $H_C = H_G \otimes H_P$. Here the space $H_G$ is as in (2.24) - (2.25).

The space $H_P$ in the Schrodinger representation is, for example, the space of functions of two variables $(x^0, x^1)$, on which there is a positively determined Lorentz - invariant scalar product. By this definition in the space $H_P$ there are no states which are annihilated by the operators $\alpha_m^{(\pm)}$, $m \neq 0$. Moreover, also in complete space $H_C$ there are no states which are annihilated by the operators $\alpha_m^{(\pm)}$, $m \neq 0$. One can add to the space $H_G$ the states $|\pm\rangle$, on which the operators $\alpha_m^{(\pm)}$ are equal to zero. In the extended space we shall define the scalar product as follows:

$$\langle \pm | \pm \rangle = 1, \quad \langle + | - \rangle = 0,$$
\[ \langle \pm | G \rangle = 0, \quad |G \rangle \in H_G \]  

Further we define
\[ |k\pm\rangle_{D} = |\pm\rangle \otimes |k\pm\rangle, \]

where
\[ |k\pm\rangle \in H_P, \quad p_{\pm} |k\mp\rangle = 2k |k\mp\rangle \]  

Then the set of vectors \( \{ |k\pm\rangle_{D} \} \) form the basis of the space \( H_{PD} \).

Let’s pay attention to the reason by which the parameter \( k \) in (2.27), (2.28) and (2.34) is positive. This follows from the requirement of a positivity of scalar product in the space \( H_{P} \). Indeed, the space \( H_{P} \) is the space of states of a massless bose-particle. As is well known (see \[8\], Chapter 3, \$2) in such space the positively defined and Lorentz-invariant scalar product exists only for positive (or negative) energies.

From above consideration it is seen, that any vector from the space \( H_{PD} \) does not belong to initial normalizable space \( H_{C} \). But just in the space \( H_{C} \) the initial variables \( \alpha_{m}^{\mu} \), \( m \neq 0 \) are the well defined bose creation and annihilation operators.

Thus, the Dirac prescription imply the variables \( \alpha_{m}^{\mu} \) with \( m \neq 0 \) are not the operators in the physical space \( H_{PD} \). The variables \( \{ \alpha_{m}^{\mu}, x^{\mu} \} \) are generators of the associative noncommutative involutive algebra \( A \) with unit over the complex numbers (see \[4\]). The generators \( \{ \alpha_{m}^{\mu}, x^{\mu} \} \) satisfy only the relations (2.9). The generators \( \{ x^{\mu}, p^{\mu} \} \) are the operators in the space of physical states \( H_{PD} \). In the space \( H_{PD} \) there is the basis \( \{ |k\pm\rangle_{D} \} \), for which the relations (2.27) take place.

The variables \( \alpha_{m}^{\mu} \) with \( m \neq 0 \) are not the operators in the space of physical states. Nevertheless, the observable quantities can depend on these variables so, that the matrix elements of observable quantities relative to vectors from the space of physical states are determined correctly. Just such situation has a place in quantum electrodynamics for Dirac quantization \[7\] (see also Appendix).

Let’s continue the study of our model.

From the definitions (2.15) we obtain the following formulae:

\[
\begin{align*}
\exp i \omega J^{01} \alpha_{m}^{(\pm)} \exp -i \omega J^{01} &= \exp (\pm \omega) \alpha_{m}^{(\pm)}, \\
\exp i \omega J^{01} x_{\pm} \exp -i \omega J^{01} &= \exp (\pm \omega) x_{\pm} \\
\exp i a_{\mu} P^{\mu} x_{\pm} \exp -i a_{\mu} P^{\mu} &= x_{\pm} + \frac{\sqrt{\pi}}{l} a_{\pm} \\
\exp i a_{\mu} P^{\mu} \alpha_{m}^{(\pm)} \exp -i a_{\mu} P^{\mu} &= \alpha_{m}^{(\pm)}
\end{align*}
\]  

(2.35)

Here \( \omega \) and \( a^{\mu} \) are arbitrary real numbers. It is evident from Eqs. (2.29) and (2.30) that translations and Lorentz transformations conserve the condition (2.23).

According to Eq. (2.35)

\[ p_{\pm} \exp -i \omega J^{01} = \exp (\omega) \exp -i \omega J^{01} p_{\pm} \]  

(2.37)

Let us formally act with the equalities (2.37) on the states \( |k\mp\rangle \), accordingly. As a result of Eq. (2.26) we obtain

\[ p_{\pm} \exp -i \omega J^{01} |k\mp\rangle = 2k e^{\pm \omega} \exp -i \omega J^{01} |k\mp\rangle \]  

(2.39)

The last equality makes it possible to determine the action of the operators \( \exp -i \omega J^{01} \) on the physical states as follows:

\[ \exp -i \omega J^{01} |k\mp\rangle = f_{\mp}^{\mp} |\exp -\omega J^{01} |k\mp\rangle \]

Here \( f_{\mp}^{\mp} \) are some complex numbers different from zero. Since the scalar product on physical state vectors is defined in a Lorentz-invariant manner according to (2.28) so \( |f_{\mp}^{\mp}| = 1 \). From Eq. (2.39) it is evident that one can assume

\[ k > 0 \]  

(2.40)

According to Eqs. (2.8) and (2.15)

\[ P^{\mu} = \frac{\sqrt{\pi}}{l} a_{0}^{\mu} = \frac{\sqrt{\pi}}{2l} \{ (\delta_{0}^{\mu} + \delta_{1}^{\mu}) p_{+} + (\delta_{0}^{\mu} - \delta_{1}^{\mu}) p_{-} \} \]
Therefore from (2.26) and (2.27) we obtain:

\[ P^\mu |k\pm\rangle_D = \sqrt{\frac{\pi}{l}} k^\mu_\pm |k\mp\rangle_D , \quad k^\mu_\pm = (k, \pm k) \tag{2.41} \]

Thus, as a result of the above-described quantization procedure of two-dimensional string there arises a system similar to a massless quantum particle in two-dimensional space-time.

3. **The other way of anomaly-free quantization of a two-dimensional string.**

In the previous section we have emphasized the difficulties which arise under Dirac quantization of degenerated systems. These difficulties in general are reduced to the following problems:

1) The problem of normalizability of physical state vectors.

2) The impossibility of interpretation of some initial dynamic variables as linear operators in the space of physical states.

These problems are successfully solved in relatively simple models such as electrodynamics and two-dimensional string. However, the complete solution of these problems in more complicated models such as four dimensional gravity is extremely difficult.

We propose in this section the alternative way of quantization of a two-dimensional string. The idea of this quantization consists in some weakening of the Dirac conditions (2.32) by replacement them with conditions:

\[ \langle P | \chi_m | P \rangle_G = 0 \tag{3.1} \]

Here the index \( P \) number any physical state. The index \( G \) indicates that an averaging in (3.1) goes only in gauge degrees of freedom. The quantization conditions (3.1) are similar in some sense a) to the Gupta-Bleuler conditions in electrodynamics, when the equality \( \partial_\mu A^\mu = 0 \) takes place only in the sense of mean value; b) to the quantization conditions in the usual string theory, when the generators of Virasoro algebra satisfy conditions \( L_n = 0 \) also only in the sense of mean value.

It is important that calculation of the mean value in (3.1) is performed relative to physical states.

The difference of offered here quantization from Gupta-Bleuler and standard string quantizations is that at our approach the scalar product in complete space of states is positively defined. It is shown below, that this fact allows to carry out the anomaly-free quantization of a two-dimensional string.

Insted of the Dirac selfconsistency conditions \( [\chi_m, \chi_n] = c_{mn}^{\prime} \chi_l \), now we have:

\[ \langle P | [\chi_m, \chi_n] | P \rangle_G = 0 \tag{3.2} \]

Explain the physical sense of the conditions (3.2). As is known Hamiltonian of a generally covariant system has the form \( \mathcal{H}_T = \sum v_m \chi_m \). Assume, that at the moment of time \( t \) the conditions (3.1) take place. In the infinitely close moment of time \( t + \delta t \) the constraint \( \chi_n \) is equal to

\[ \chi_n(t + \delta t) = \chi_n(t) + i \delta t \sum_m v_m [\chi_m, \chi_n](t) \]

Thus the selfconsistency conditions (3.2) provide the equalities (3.1) at any moment of time.

We believe, that the complete space of states, in which the initial dynamic variables (2.9) - (2.10) act, is the space \( H_C \), determined in the previous section:

\[ H_C = H_G \otimes H_P \tag{3.3} \]

Here the spaces \( H_G \) and \( H_P \) are defined according to (2.24), (2.25) and (2.34). Let in the space \( H_P \) the basis \( |k\rangle = |k^0, k^1\rangle \) be such, that

\[ p^\mu |k\rangle = k^\mu |k\rangle \tag{3.4} \]

For the further calculations it is necessary to define a suitable ordering of the operators. Next we shall use the ordering

\[ L_0 = \frac{1}{2} p^\mu p_\mu - \sum_{m>0} (\alpha^0_m \alpha^0_{-m} - \alpha^1_{-m} \alpha^1_m) \tag{3.5} \]
which is equivalent to the ordering (2.21).

We think that in the examined model the most convenient physical states satisfying conditions (3.1), are the states, which are coherent relative to gauge degrees of freedom. Let’s consider in the space \( H_G \) the coherent state

\[
|z; G\rangle = \prod_{m>0} \exp \left\{ -\frac{|z^0_m|^2 + |z^1_m|^2}{2m} + \frac{1}{m} \left( z^0_m \alpha^0_m + z^1_m \alpha^1_m \right) \right\} |0; G\rangle
\]  

(3.6)

Here \([z^\mu_m, m \neq 0] \) are complex numbers. Below we assume by definition that \( z^\mu_0 = k^\mu \) and \( z^\mu_m = z^\mu_{-m} \).

The bar above means the complex conjugation. As a consequence of (2.9) and (2.24) we have:

\[
\langle z; G | z; G \rangle = 1,
\]

\[
\alpha^0_{-m} | z; G \rangle = z^0_{-m} | z; G \rangle, \quad \alpha^1_{-m} | z; G \rangle = z^1_{-m} | z; G \rangle, \quad m > 0
\]  

(3.7)

Let’s introduce a designation \( |z\rangle \) for the state

\[
|z\rangle = |z; G\rangle \otimes |k\rangle \in H_G
\]  

(3.8)

From (3.4), (3.7) and (3.8) it follows, that

\[
\alpha^0_{-m} | z \rangle = z^0_{-m} | z \rangle, \quad \alpha^1_{-m} | z \rangle = z^1_{-m} | z \rangle, \quad m \geq 0
\]  

(3.9)

We call the set of complex numbers \( \{z^\mu_m\} \) as parameters of the state \( |z\rangle \). A state is called physical and designated by \( |z\rangle_P \), if its parameters satisfy the equations:

\[
L_n(z) = \frac{1}{2} \sum_m z^\mu_{n-m} z^{\mu m} = 0
\]  

(3.10)

We put \( z^{(\pm)}_n = z^0_n \pm z^1_n \) and

\[
z^{(\pm)}(\phi) = \frac{1}{\sqrt{2\pi}} \sum_n e^{-in\phi} z^{(\pm)}_n,
\]

\[
L(\phi) = \frac{1}{2\pi} \sum_n e^{-in\phi} L_n = -\frac{1}{2} z^{(+)}(\phi) z^{(-)}(\phi)
\]

The functions \( z^{(\pm)}(\phi) \) are real. Equations (3.10) are equivalent to the following one:

\[
L(\phi) = -\frac{1}{2} z^{(+)}(\phi) z^{(-)}(\phi) = 0
\]  

(3.10’)

Impose also the gauge invariant conditions

\[
-z^{(+)}_0 z^{(-)}_0 = k^\mu k_\mu = 0
\]  

(3.11)

Further it is supposed, that all solutions of the equations (3.10) satisfy condition (3.11). Therefore the basis in the space \( H_P \) can be given according to (2.34), and the scalar product According to (2.28).

From the formulas (3.5), (3.9), (3.10) it follows immediately, that in considered case the conditions (3.1) take place:

\[
\langle z | L_n | z \rangle_P = 0
\]  

(3.12)

Let’s verify the selfconsistency condition (3.2). In our case for this purpose it is enough to check up, that

\[
\langle z | (L_n L_{-n} - L_{-n} L_n) | z \rangle_P = 0
\]  

(3.13)

The simple calculation shows:

\[
L_n L_{-n} = \frac{1}{2} \sum_{m=1}^{n} m(n-m) + n (a^0_0)^2 + 2n \sum_{m=1}^{n} \alpha^1_{-m} \alpha^1_m +
\]
tum operator. Let for any variable $O$

Similarly:

The scalar product of physical states is conserved under the Lorentz transformations (3.21):

From here we see that the action of the angular momentum operator on physical states is well defined.

Thus, the mean values of Heisenberg dynamic variables satisfy the classical equations of motion.

For the states (3.21). It means, that the states

Since:

Therefore, the mean values of Heisenberg variables change in time. The Heisenberg equations look like

Since:

Now consider the question about the Lorentz-invariance of the theory. Let’s write the operator of the angular momentum in the form:

\[
\alpha^m_n = \sum_{m=0}^{\infty} \alpha^m_m + \sum_{m=1}^{\infty} \alpha^m_{-m} + \sum_{m=0}^{\infty} \alpha^m_{-m} + \sum_{m=1}^{\infty} \alpha^m_m + :L_n L_{-n}:
\]

Similarly:

\[
L_{-n} L_n = \frac{1}{2} \sum_{m=1}^{n-1} m(n-m) + n(\alpha^0_0)^2 + 2n \sum_{m=1}^{n} \alpha^0_m \alpha^0_{-m} +
\]

\[
+ \sum_{m=n+1}^{\infty} (n+m) \alpha^1_m \alpha^1_m + \sum_{m=n+1}^{\infty} (m-n) \alpha^1_{-m} \alpha^1_{-m} + :L_n L_{-n}:
\]

Since:\n
\[
L_n L_n - L_{-n} L_n = 2n L_0,
\]

The ordering in the right hand side of Eq.(3.16) is given according to (3.5). From (3.16) it is seen, that the equalities (3.12) take place, so here the selfconsistency conditions (3.2) are valid.

Note that, generally speaking,

\[
\langle z | L_n L_{-n} | z \rangle_p \neq 0
\]

Let Hamiltonian of the system be $H_T = \sum_n v_n L_n$, where $\tilde{v}_n(\tau) = v_n(\tau)$. Give the answer for the question how the mean values of Heisenberg variables change in time. The Heisenberg equations look like

Define by following equations the system of complex time-dependent numbers:

\[
z^\mu_m(\tau) [k \delta(k' - k)] = \langle z' | \alpha^\mu_m(\tau) | z \rangle_p,
\]

where $\alpha^\mu_m(\tau)$ are the Heisenberg variables. In Eq.(3.18) we assume that the collections of parameters \{z\} and \{z'\} are physical, that are they satisfy the equations (3.10). Moreover $z^\mu_0 = z^\mu_0$ if $z^\mu_0 = z^\mu_0$. Obviously in this case $\langle z' | z \rangle = k \delta(k' - k)$. With the help of (3.17) and (3.18) we obtain the equations for change in time of the parameters (3.18):

\[
i \frac{d}{d\tau} z^\mu_m(\tau) = m \sum_n v_n(\tau) z^\mu_{m+n}(\tau)
\]

Thus, the mean values of Heisenberg dynamic variables satisfy the classical equations of motion.

Now consider the question about the Lorentz-invariance of the theory. Let’s write the operator of the angular momentum in the form:

\[
J^{01} = (x^0 p^1 - x^1 p^0) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} (\alpha^0_n \alpha^1_{-n} - \alpha^1_n \alpha^0_{-n})
\]

From here we see that the action of the angular momentum operator on physical states is well defined.

Let

\[
| z' \rangle_p = \{ \exp -i \omega J^{01} \} | z \rangle_p
\]

Since $|L_m, J^{01}| = 0$ and operator $J^{01}$ is hermitian, the equations (3.12) and (3.13) take place as well for the states (3.21). It means, that the states $| z' \rangle_p$ determined according to (3.21) are physical states. Owing to (2.35) the both states $| z \rangle_p$ and $| z' \rangle_p$ are the eigenstates for the momentum operator $p^\mu$.

The scalar product of physical states is conserved under the Lorentz transformations (3.21):

\[
\langle z' | z \rangle = \langle z_1 | z \rangle
\]

We shall calculate the mean values of dynamic variables variations generated by the angular momentum operator. Let for any variable $O$

\[
\delta O(L) = -i \delta \omega \{ O, J^{01} \}
\]
Then for fundamental variables we have:

$$\delta x_{(L)}^\mu = \delta \omega (\eta^{\mu 1} x^0_1 - \eta^{\mu 0} x^1_1), \quad \delta \alpha_{m(L)}^\mu = \delta \omega (\eta^{\mu 1} \alpha^0_m - \eta^{\mu 0} \alpha^1_m)$$  \hspace{1cm} (3.22)

For variations of mean values (3.18) we find:

$$\delta z_{m(L)}^\mu = \delta \omega (\eta^{\mu 1} z^0_m - \eta^{\mu 0} z^1_m)$$ \hspace{1cm} (3.23)

We see, that under Lorentz transformation the corresponding transformation of observable variables \(\{x^\mu, p^\mu\}\) occurs and also the gauge transformation of gauge degrees of freedom take place.

Let’s discuss shortly the superposition principle under the second quantization method.

Assume, that the states \(|z\rangle_p\) and \(|z'\rangle_p\) are physical. Is the state

$$|z\rangle_p + |z'\rangle_p$$

physical?

We think, that it is not necessary to extend the superposition principle to unphysical, gauge degrees of freedom. Therefore, if under the second quantization method the superposition principle will appear limited in the space \(H_G\), in our opinion, it does not depreciate the method. However, in physical space the superposition principle is kept completely.

4. Conclusion.

The quantization methods which was applied to the two dimensional string in the Sections 2, 3 next we shall call as first and second methods, accordingly. The results of this paper are reduced to the following theses:

1) Both methods lead to absence of the central charge in Virasoro algebra. In both cases the basis for such result is that in complete space of states the scalar product is positively defined.

2) The first method successively realizes the Dirac quantization prescription for degenerated systems. In this case in the theory there are such dynamic variables, which are not the operators in physical space of states. In simple models (two-dimensional string and electrodynamics) this fact is not important since the matrix elements of observables relative to physical states are well defined and calculated obviously. However, in the complicated theories such as four-dimensional gravity the specified difficulty can extremely complicate the decision of a problem, by ceasing to be in result only technical.

3) The second quantization method is based on some weakening of Dirac quantization conditions for degenerated systems. It enables to treat the all initial dynamic variables as the operators in physical space of states. The physical space of states is a vector subspace of complete space of states with positively defined scalar product. This statement is incorrect for the first quantization method. The scalar product in physical space of states is induced by scalar product in complete space of states.

4) The both quantizations result in isomorphic spaces of physical states. However, it is unknown, whether this result will be kept in more complicated models.

5) The both quantizations result in the Lorentz-invariant theories. Note that in the conventional quantization there exists a state which is invariant under Lorentz transformations. This state is the ground state. In this respect the conventional string theory is similar to the standard quantum field theory of point objects. In such field theories the ground state usually is Lorentz-invariant. Conversely, in our approach there does not exist a state that is invariant under Lorentz transformations. For this reason, the quantum string theory proposed above is analogous to a quantum theory of a single relativistic particle. In the letter case there does not exist a Lorentz-invariant quantum state of a single relativistic particle. For existence of Lorentz-invariant state in our theory we would have to introduce a second-quantized string field. In such theory the ground state or vacuum would be Lorentz-invariant.

It seems to us, that the second quantization method can give interesting results in application to more complicated models.

Appendix

Here we describe the Dirac quantization [7] of free electromagnetic Field. This quantization method is applied in Section 2.
The quantization of an electromagnetic field is presented in the form

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \{a_\mu(\vec{k}) e^{ikx} + a_\mu^+(\vec{k}) e^{-ikx}\} \quad (A1)$$

Here \(\mu, \nu, \ldots = 0, 1, 2, 3\), \(kx \equiv k_\mu x^\mu = -k^0 x^0 + \vec{k} \cdot \vec{x}, \ k^0 = |\vec{k}|\) and \(\{a_\mu(\vec{k}), a_\mu^+(\vec{k})\}\) are some generators of an associative involutive algebra \(\mathcal{A}\) with an identity element (see Sec. 2). The nonzero commutation relations between these generators have the form

$$[a_\mu(\vec{k}), a_{\bar{\nu}}^+(\vec{p})] = (2\pi)^3 \eta_{\mu\nu} \delta^{(3)}(\vec{k} - \vec{p}) \quad (A2)$$

One can see from the expansion (A1) that the set of elements \(\partial_\mu A^\mu(x)\) is linearly equivalent to the set of elements \(k^\mu a_\chi(\vec{k})\) and \(k^\mu a_\chi^+(\vec{k})\) from the algebra \(\mathcal{A}\). Let \(a_\chi(\vec{k})\) be two independent elements (for fixed \(\vec{k}\)) satisfying the conditions

$$\sum_{i=1}^3 k_i a_i^T(\vec{k}) = 0, \quad [a_i^T(\vec{k}), a_j^T(\vec{p})] = (2\pi)^3 \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \delta^{(3)}(\vec{k} - \vec{p}) \quad (A3)$$

Eqs.(A1) and (A2) imply the following commutation relations \(\{F_{\mu\nu}, \partial_\mu A_\nu - \partial_\nu A_\mu\}\):

$$[F_{\mu\nu}(x), k^\chi a_\chi(\vec{k})] = [F_{\mu\nu}(x), k^\chi a_\chi^+(\vec{k})] = 0, \quad (A4)$$

$$[k^\mu a_\chi(\vec{k}), p^\nu a_\nu^+(\vec{p})] = 0 \quad (A5)$$

We have also

$$[a_\chi^T(\vec{k}), k^\mu a_\chi(\vec{k})] = [a_\chi^T(\vec{k}), k^\mu a_\chi^+(\vec{k})] = 0 \quad (A6)$$

Dirac quantization presupposes that the conditions

$$a_\chi^T(\vec{k}) |0\rangle = 0 \quad (A7)$$

are imposed on the ground state and the conditions

$$k^\mu a_\chi(\vec{k}) |\rangle = 0, \quad k^\mu a_\chi^+(\vec{k}) |\rangle = 0 \quad (A8)$$

are imposed on all physical states. As a result of Eqs. (A5) and (A6) the conditions (A7) and (A8) are algebraically consistent. The states satisfying the conditions (A8) are called physical. The Fock space of all physical states is constructed with the help of the creation operators \(a_1^T(\vec{k})\) from the ground state satisfying the conditions (A7) and (A8). As a result of Eq. (A6) any state of the Fock space constructed satisfies the conditions (A8).

Let \(k_\mu^\nu = (-k^0, \vec{k})\). We find from Eq. (A2)

$$[k^\mu a_\chi(\vec{k}), p^\nu a_\nu^+(\vec{p})] = 2 k^2 (2\pi)^2 \delta^{(3)}(\vec{k} - \vec{p}) \quad (A9)$$

The relations (A4) and (A9) mean that the observables \(F_{\mu\nu}\) do not depend on the generators \(\{k^\mu a_\chi(\vec{k}), k^\mu a_\chi^+(\vec{k})\}\). Therefore all matrix elements of the form \(\langle \Lambda | F_{\mu\nu} | \Sigma \rangle\), where \(|\Lambda\rangle, |\Sigma\rangle\) are physical states, are determined.

We note that as a result of Eqs. (A3) and (A7) the scalar product in the space \(V\) is positive-defined provided that \(|0 \rangle 0\rangle = 1\). We call attention to the fact that the action of the generators \(k_\mu^\nu a_\chi(\vec{k})\) and \(k_\mu^\nu a_\chi^+(\vec{k})\) on the physical states is not determined in Dirac quantization, and therefore these generators of the algebra \(\mathcal{A}\) are not linear operators in the space of physical states.

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