HAMILTONIAN PSEUDO-ROTATIONS OF PROJECTIVE SPACES

VIKTOR L. GINZBURG AND BAŞAK Z. GÜREL

Abstract. The main theme of the paper is the dynamics of Hamiltonian diffeomorphisms of $\mathbb{CP}^n$ with the minimal possible number of periodic points (equal to $n+1$ by Arnold’s conjecture), called here Hamiltonian pseudo-rotations. We prove several results on the dynamics of pseudo-rotations going beyond periodic orbits, using Floer theoretical methods. One of these results is the existence of invariant sets in arbitrarily small punctured neighborhoods of the fixed points, partially extending a theorem of Le Calvez and Yoccoz and of Franks and Misiurewicz to higher dimensions. The other is a strong variant of the Lagrangian Poincaré recurrence conjecture for pseudo-rotations. We also prove the $C^0$-rigidity of pseudo-rotations with exponentially Liouville mean index vector. This is a higher-dimensional counterpart of a theorem of Bramham establishing such rigidity for pseudo-rotations of the disk.

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1. Introduction

The main theme of the paper is the dynamics of Hamiltonian diffeomorphisms of $\mathbb{CP}^n$, equipped with the standard symplectic structure, with exactly $n + 1$ periodic points, called here Hamiltonian pseudo-rotations. By Arnold’s conjecture established in this case by Fortune and Weinstein, [Fo, FW], and by Floer, [Fl], this is the minimal possible number of fixed points and, in particular, of periodic points. We prove several results on the dynamics of such maps. One is the existence of invariant sets in arbitrarily small punctured neighborhoods of the fixed points, partially extending the results of Le Calvez and Yoccoz and of Franks and Misiurewicz from [Fr99, FM, LCY] to higher dimensions. Another result is a strong variant of the Lagrangian Poincaré recurrence conjecture for pseudo-rotations. We also prove the $C^0$-rigidity of pseudo-rotations with exponentially Liouville mean index vector. This is a $2n$-dimensional analog of Bramham’s theorem from [Br15b].

Perhaps the most important general point of this paper is rather unexpectedly one can obtain a lot information about the dynamics of pseudo-rotations in dimensions greater than two, going far beyond periodic orbits, by purely Floer theoretical methods. Before turning to the precise statements of sample results we need to discuss the notion of a Hamiltonian pseudo-rotation – the key player in this work – in a greater generality and more detail.

1.1. Hamiltonian pseudo-rotations. In the framework of two-dimensional dynamics, pseudo-rotations are area preserving diffeomorphisms of $D^2$ or $S^2$ with exactly one or two periodic points, respectively, which are then automatically the fixed points. There are several ways of extending this notion to higher dimensions in the context of Hamiltonian dynamics and symplectic topology.

For instance, one can define a Hamiltonian pseudo-rotation of a closed symplectic manifold $(M^{2n}, \omega)$ as a Hamiltonian diffeomorphism $\varphi$ with finite and minimal possible number of periodic points, and such that the periodic points are the fixed points. This is the definition we prefer to use here even though the notion of the minimal possible number is ambiguous. However, when $M$ is sufficiently nice and Arnold’s conjecture is known to hold for $M$, this can be the sum of Betti numbers of $M$ when $\varphi$ is non-degenerate, and a Lusternik–Schnirelmann type lower bound (e.g., the category or the cup-length plus one) in general. For $\mathbb{CP}^n$ both lower bounds are $n + 1$; hence Definition 1.1.

One can also define a pseudo-rotation as a Hamiltonian diffeomorphism with finitely many periodic points. These are the so-called perfect Hamiltonian diffeomorphisms studied in the context of the Conley conjecture; see, e.g., [CGG, GG15] and references therein. As follows from the results in [Fr92, Fr96, FrHa, LeC], in dimension two this definition is equivalent to the one adopted here. We will further discuss the difference or a lack thereof between pseudo-rotations and perfect Hamiltonian diffeomorphisms below, but at the moment we only mention that in all known examples perfect Hamiltonian diffeomorphisms are non-degenerate pseudo-rotations.

It is believed that rather few manifolds admit pseudo-rotations. Namely, the Conley conjecture asserts that for a broad class of closed symplectic manifolds every Hamiltonian diffeomorphism has infinitely many simple (a.k.a. prime, i.e., not iterated) periodic orbits. At the time of writing, the conjecture has been established for all symplectic manifolds $(M^{2n}, \omega)$ such that there is no class $A \in \pi_2(M)$ with
$\omega(A) > 0$ and $\langle c_1(TM), A \rangle > 0$; see [GG17]. In particular, the conjecture holds for all symplectic CY manifolds, [GG09, He], and all negative monotone symplectic manifolds, [CGG, GG12]. (See also [FrHa, Gi10, Hi, LeC, SZ] for some milestone intermediate results and [GG15] for a general survey and further references.) One can also show that $N \leq 2n$ when $M$ admits a pseudo-rotation, where $N$ is the minimal Chern number of $M$, although it is not yet known if the Conley conjecture holds whenever $N > 2n$.

On the other hand, all symplectic manifolds $M$ which carry Hamiltonian circle (or torus) actions with isolated fixed points also admit pseudo-rotations. Indeed, then a generic element of the circle or torus induces a pseudo-rotation of $M$. In particular, $\mathbb{C}P^n$, the complex Grassmannians and flag manifolds, or more generally a majority of coadjoint orbits of compact Lie groups, and symplectic toric manifolds admit pseudo-rotations. For $\mathbb{C}P^n$ identified with the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, such a pseudo-rotation is a true rotation, by which we mean an element of $\text{SU}(n)$, and can be generated by a quadratic Hamiltonian $Q = \sum a_i |z_i|^2$ where $a_i - a_j \notin \mathbb{Q}$ for $i \neq j$. (The combinatorics of rotations is not entirely straightforward and we will examine it more closely in [GG18]; see also Examples 3.2 and 3.5.)

However, at least for $S^2$, not every pseudo-rotation is a true rotation. Indeed, diffeomorphisms arising as generic elements of $S^1$-actions have simple, essentially trivial dynamics. This is in general not the case for pseudo-rotations, and pseudo-rotations occupy a special place among low-dimensional dynamical systems. For instance, in [AK] Anosov and Katok constructed area preserving diffeomorphisms $\varphi$ of $S^2$ with exactly three ergodic measures, the area form and the two fixed points, by developing what is now known as the conjugation method; see also [FK] and references therein. Such a diffeomorphism $\varphi$ is automatically a pseudo-rotation. Indeed, $\varphi$ is area preserving and hence Hamiltonian, and $\varphi$ has exactly two periodic orbits, which are its fixed points. Furthermore, $\varphi$ is ergodic, necessarily has dense orbits, and thus is not conjugate to a true rotation. As a consequence, the products $(S^2)^n$ also admit pseudo-rotations which are not conjugate to true rotations.

The situation is more complicated for projective spaces. It is believed that $\mathbb{C}P^n$ and other toric manifolds admit dynamically interesting pseudo-rotations, e.g., ergodic or even with finite number of ergodic measures. However, in the Hamiltonian setting, the conjugation method encounters a conceptual difficulty along the lines of the symplectic packing or flexibility questions. This difficulty disappears for $\mathbb{C}P^2$ due to a theorem of McDuff asserting that any two symplectic embeddings of any fixed collection of closed balls into $\mathbb{C}P^2$ are Hamiltonian isotopic; see [McD98, McD09]. As a consequence, in this case the problem becomes more technical than conceptual, although still quite non-trivial due to the effect of the fixed points. A parallel question in the symplectomorphism case was studied in [HC] where a different variant of the conjugation method introduced in [FaHe] was used to construct minimal symplectomorphisms of symplectic manifolds admitting symplectic $S^1$-actions without fixed points.

We conclude this section by elaborating on the conjecture that every perfect Hamiltonian diffeomorphism, i.e., a Hamiltonian diffeomorphism with finitely many periodic points, is automatically a pseudo-rotation, i.e., it has the minimal possible number of periodic points, and all such points are fixed points. (Moreover, hypothetically every perfect Hamiltonian diffeomorphism is strongly non-degenerate.
and all its periodic orbits, which are then fixed points, are elliptic.) For instance, for $\mathbb{CP}^n$, according to this conjecture every Hamiltonian diffeomorphism with more than $n + 1$ fixed points must have infinitely many periodic orbits. As has been pointed out above, in dimension two this fact is essentially the combination of the celebrated theorem of Franks (see [Fr92, Fr96, LeC]) asserting that every area preserving diffeomorphism of $S^2$ with more than two fixed points has infinitely many periodic orbits and the Conley conjecture for surfaces proved in [FrHa]. (See also [CKRTZ] for a Floer theoretical proof of Franks’ theorem and [BH] for an approach utilizing finite energy foliations.) The earliest mentioning of this hypothetical generalization of Franks’ theorem known to us is on p. 263 in [HZ].

One can also make similar conjectures for symplectomorphisms and other types of “Hamiltonian dynamical systems”. Furthermore, looking at the question from a broader perspective one may expect that the presence of a periodic orbit which is homologically or geometrically unnecessary (e.g., degenerate, non-contractible, hyperbolic) forces the system to have infinitely many periodic orbits. We refer the reader to [Ba15a, Ba15b, GG14, GG16a, Gü13, Gü14, Or17a, Or17b] (and also to Theorem $4.2$) for some recent results in this direction in dimensions greater than two.

1.2. Sample results. In this section, we state without detailed discussion and in some instances in a simplified form the key results of the paper. This is a “sampler plate” and a much more thorough treatment is given in Sections 4 and 5. Here and throughout the paper, $\mathbb{CP}^n$ is equipped with the standard (sometimes up to a factor) Fubini–Study symplectic structure.

**Definition 1.1.** A pseudo-rotation of $\mathbb{CP}^n$ is a Hamiltonian diffeomorphism of $\mathbb{CP}^n$ with exactly $n + 1$ periodic points.

Our first theorem concerns invariant sets of pseudo-rotations. As has been mentioned above, the pseudo-rotations $\varphi$ of $S^2$ constructed by the conjugation method in [AK] have exactly three ergodic measures: the two fixed points and the area form. As a consequence, such pseudo-rotations are very close to being uniquely ergodic: $\varphi$ is uniquely ergodic on the complement to the two fixed points. Recall also that volume preserving uniquely ergodic maps of compact manifolds are necessarily minimal, i.e., all orbits are dense; see, e.g., [Wa]. These facts suggest that every orbit of $\varphi$ other than a fixed point should be dense. (Furthermore, minimal symplectomorphisms exist in abundance and can also be constructed by the conjugation method; [HCP].) However, this turns out to be false and $\varphi$ must have many proper, closed invariant subsets as proved in [FM, Prop. 5.5], which is in turn inspired by [LCY] and based on the approach developed in [Fr99]. Our first theorem is a partial generalization of these results to higher dimensions:

**Theorem 1.2.** No fixed point of a pseudo-rotation of $\mathbb{CP}^n$ is isolated as an invariant set.

This theorem is proved in Section 4. The proof hinges on a result of independent interest asserting that a Hamiltonian diffeomorphism of $\mathbb{CP}^n$ with a fixed point which is isolated as an invariant set and has non-vanishing local Floer homology must have infinitely many periodic orbits. This is Theorem $4.2$ extending [GG14, Thm. 1.1] to degenerate periodic orbits and proved in Section 6.

Our next result concerns the Lagrangian Poincaré recurrence conjecture put forth by the first author and independently by Viterbo around 2010. According
to this conjecture the images of a Lagrangian submanifold $L$ under the iterations of a Hamiltonian diffeomorphism with compact support cannot be all disjoint. In dimension two, this is an easy consequence of the standard Poincaré recurrence combined with the observation that when $L$ does not bound, it cannot be displaced by a Hamiltonian diffeomorphism. However, in higher dimensions the assertion is not obvious even for a particular Hamiltonian diffeomorphism unless, of course, it is periodic. We discuss Lagrangian Poincaré recurrence in more detail in Section 5.1, where we establish the conjecture for pseudo-rotations of $\mathbb{CP}^n$ and a sufficiently broad class of Lagrangians $L$. In particular, we prove the following.

**Theorem 1.3.** Let $\varphi$ be a pseudo-rotation of $\mathbb{CP}^n$ and let $L \subset \mathbb{CP}^n$ be a closed Lagrangian submanifold admitting a metric without contractible closed geodesics. Then $\varphi^k(L) \cap L \neq \emptyset$ for infinitely many $k \in \mathbb{N}$.

This theorem is a consequence of Corollary 5.1.1 ($\gamma$-norm convergence) asserting that $\gamma(\varphi^{k_i}) \to 0$ for some sequence $k_i \to \infty$, where $\gamma$ is the $\gamma$-norm. In fact, in Theorem 5.8 we establish a stronger version of the Lagrangian Poincaré recurrence conjecture relating the return rate, i.e., the frequency of intersections $\varphi^k(L) \cap L$, and the homological capacity $c_{hom}(L)$ of $L$. Moreover, the result holds for all compact subsets $L$ with $c_{hom}(L) > 0$. The proof is based on a quantitative version of the $\gamma$-norm convergence; see Section 5.1.1 and Theorem 5.1.

Finally, our third result is $C^0$-rigidity of pseudo-rotations of $\mathbb{CP}^n$. This is a higher-dimensional counterpart of the main theorem of [Br15b]. Let $\varphi$ be a Hamiltonian pseudo-rotation of $\mathbb{CP}^n$ with fixed points $x_0, \ldots, x_n$. These are also its periodic points. The mean index “vector” of $\varphi$ is by definition

$$\vec{\Delta} = (\hat{\mu}(x_0), \ldots, \hat{\mu}(x_n)) \in \mathbb{T}^{n+1} = \mathbb{R}^{n+1}/2(n+1)\mathbb{Z}^{n+1},$$

where we treat the mean indices $\hat{\mu}(x_i)$ of uncapped one-periodic orbits as elements of the circle $\mathbb{R}/2(n+1)\mathbb{Z}$; see Section 3 for more details. We say that $\vec{\Delta}$ is exponentially Liouville if the iterates $k\vec{\Delta}$ approximate $0 \in \mathbb{T}^{n+1}$ exponentially accurately: for any $c > 0$ there exists $k \in \mathbb{N}$ such that $\|k\vec{\Delta}\| < e^{-ck}$, where $\|\cdot\|$ is the distance to $0$; see Definition 5.14. In $\mathbb{T}^{n+1}$ or any closed subgroup of $\mathbb{T}^{n+1}$, exponentially Liouville elements form a zero measure, residual set (i.e., a countable intersection of open and dense sets).

**Theorem 1.4.** For every pseudo-rotation $\varphi$ of $\mathbb{CP}^n$ with exponentially Liouville mean index vector $\vec{\Delta}$, there exists a sequence $k_i \to \infty$ such that $\varphi^{k_i} \to id$. 

This theorem is proved in Section 5.2. Note that here we do not impose any non-degeneracy requirements on $\varphi$. For $\mathbb{CP}^1$, Theorem 1.4 is an easy consequence of [Br15b, Thm. 1] where a similar result is established for pseudo-rotations of $D^2$; see the end of Section 5.2 for further discussion.

Although the proofs of these theorems are formally independent of each other, the arguments share some common components. One of these, directly entering the proofs of Theorems 1.3 and 1.4, is a variant of Lusternik–Schnirelmann theory for pseudo-rotations going back to [GG09]. To state the key result, note that every point in the action spectrum $\mathcal{S}(H)$ of a pseudo-rotation $\varphi_H$ of $\mathbb{CP}^n$ is the value of an action selector (a spectral invariant) for some homology class in the quantum homology $HQ_*(\mathbb{CP}^n)$. Then, by Theorem 3.1, the ordering of $\mathcal{S}(H)$ by the degree or, equivalently, by the Conley–Zehnder index when $\varphi_H$ is non-degenerate agrees with
its natural ordering as a subset of $\mathbb{R}$. In particular, all action values are distinct. Furthermore, $S(H)$ coincides with the mean index spectrum up to scaling and a shift; see Theorem 3.3 and identities (3.5) and (3.7). We review these and other previously known results on symplectic topology of pseudo-rotations in Section 3. The notation, conventions and general preliminary facts are discussed in Section 2.

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2. Preliminaries

The goal of this section is to set notation and conventions, give a brief review of Hamiltonian Floer homology and several other notions from symplectic geometry used in the paper. We also state and prove several facts (e.g., Lemmas 2.2 and 2.4) which, although quite standard, seem to be unavailable elsewhere. The reader may consider consulting this section only as necessary.

2.1. Conventions and notation. Let $(M^{2n}, \omega)$ be a closed symplectic manifold. Throughout most of the paper, except parts of Sections 4 and 6, this manifold will be $\mathbb{CP}^n$ equipped with the standard symplectic form $\omega$. It will be convenient however to vary the normalization of $\omega$ and we set $\lambda = \langle [\omega], \mathbb{CP}^1 \rangle$. In other words, $\lambda$ is the positive generator of $\langle [\omega], \pi_2(M) \rangle \subset \mathbb{R}$, the rationality constant. For the standard Fubini–Study normalization $\lambda = \pi$. Recall also that the minimal Chern number $N$, i.e., the positive generator of $\langle c_1(TM), \pi_2(M) \rangle$, is $n + 1$ for $\mathbb{CP}^n$. On several occasions we will work with more general symplectic manifolds. Then $M$ is always assumed to be strictly monotone, i.e., $\langle [\omega], \pi_2(M) \rangle = \tau c_1(TM)|_{\pi_2(M)} \neq 0$ for some $\tau \in \mathbb{R}$, and hence $M$ is rational and $N < \infty$. In particular, $\tau = \lambda/N$.

All Hamiltonians $H$ considered in this paper are assumed to be $k$-periodic in time, i.e., $H : S^1_k \times M \to \mathbb{R}$, where $S^1_k = \mathbb{R}/k\mathbb{Z}$ and $k \in \mathbb{N}$. When the period $k$ is not specified, it is equal to one and $S^1 = S^1_1 = \mathbb{R}/\mathbb{Z}$. We set $H_t = H(t, \cdot)$ for $t \in S^1_k$. The Hamiltonian vector field $X_H$ of $H$ is defined by $i_{X_H} \omega = -dH$. The (time-dependent) flow of $X_H$ is denoted by $\varphi_H^t$ and its time-one map by $\varphi_H$. Such time-one maps are referred to as Hamiltonian diffeomorphisms. A one-periodic Hamiltonian $H$ can always be treated as $k$-periodic, which we will then denote by $H^{2k}$ and, abusing terminology, call $H^{2k}$ the $k$th iterate of $H$.

Let $H$ and $K$ be one-periodic Hamiltonians such that $H_1 = K_0$ together with $t$-derivatives of all orders. We denote by $H^{2k}K$ the two-periodic Hamiltonian equal to $H_t$ for $t \in [0, 1]$ and $K_{t-1}$ for $t \in [1, 2]$. Thus $H^{2k} = H_2 \ldots 2H$ ($k$ times). More generally, when $H$ is $l$-periodic and $K$ is $k$-periodic, $H^{2k}K$ is $(l + k)$-periodic. (Strictly speaking, here we need to assume that $H_1 = K_0$ again together with all $t$-derivatives.)

Let $x : S^1_k \to M$ be a contractible loop. A capping of $x$ is an equivalence class of maps $A : D^2 \to M$ such that $A|_{S^1_k} = x$. Two cappings $A$ and $A'$ of $x$ are equivalent if the integrals of $\omega$ and $c_1(TM)$ over the sphere obtained by attaching $A$ to $A'$ are
equal to zero. A capped closed curve $\bar{x}$ is, by definition, a closed curve $x$ equipped with an equivalence class of cappings. In what follows, the presence of capping is always indicated by a bar.

The action of a Hamiltonian $H$ on a capped closed curve $\bar{x} = (x, A)$ is

$$A_H(\bar{x}) = -\int_A \omega + \int_{\mathbb{S}^1} H_t(x(t)) \, dt.$$ 

The space of capped closed curves is a covering space of the space of contractible loops, and the critical points of $A_H$ on this space are exactly the capped one-periodic orbits of $X_H$. The action spectrum $S(H)$ of $H$ is the set of critical values of $A_H$. This is a zero measure set; see, e.g., [HZ].

The $k$-periodic points of $\varphi_H$ are in one-to-one correspondence with the $k$-periodic orbits of $H$, i.e., of the time-dependent flow $\varphi^t_H$, which we denote by $\mathcal{P}_k(H)$. (Thus, for instance, $\mathcal{P}_1(H)$ can be identified with the fixed point set of $\varphi_H$.)

The collections of all periodic orbits and all capped periodic orbits of $H$ will be denoted by $\mathcal{P}(H)$ and, respectively, $\tilde{\mathcal{P}}(H)$. Recall also that a $k$-periodic orbit of $H$ is called simple or prime if it is not iterated. The definition of the action of $H$ extends to $k$-periodic orbits and Hamiltonians in an obvious way. Clearly, the action functional is homogeneous with respect to iteration:

$$A_{H^{\circ k}}(\bar{x}^k) = kA_H(\bar{x}),$$

where $\bar{x}^k$ is the $k$th iteration of the capped orbit $\bar{x}$. (The capping of $\bar{x}^k$ is obtained from the capping of $\bar{x}$ by taking its $k$-fold cover branched at the origin.)

A $k$-periodic orbit $x$ of $H$ is said to be non-degenerate if the linearized return map $d\varphi^k_H: T_{x(0)}M \to T_{x(0)}M$ has no eigenvalues equal to one. Following [SZ], we call $x$ weakly non-degenerate if at least one of the eigenvalues is different from one and totally degenerate if all eigenvalues are equal to one. A Hamiltonian $H$ is (weakly) non-degenerate if all its one-periodic orbits are (weakly) non-degenerate and $H$ is strongly non-degenerate if all periodic orbits of $H$ (of all periods) are non-degenerate.

Let $\bar{x} = (x, A)$ be a non-degenerate capped periodic orbit. The Conley–Zehnder index $\mu(\bar{x}) \in \mathbb{Z}$ is defined, up to a sign, as in [Sa, SZ]. In this paper, we normalize $\mu$ so that $\mu(\bar{x}) = n$ when $x$ is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian. The mean index $\bar{\mu}(\bar{x}) \in \mathbb{R}$ measures, roughly speaking, the total angle swept by certain unit eigenvalues of the linearized flow $d\varphi^k_H|_x$ with respect to the trivialization associated with the capping; see [Lo, SZ].

The mean index is defined even when $x$ is degenerate and depends continuously on $H$ and $\bar{x}$ in the obvious sense. Furthermore,

$$| \bar{\mu}(\bar{x}) - \mu(\bar{x}) | \leq n \tag{2.1}$$

and the inequality is strict when $x$ is weakly non-degenerate. The mean index is homogeneous with respect to iteration: $\bar{\mu}(\bar{x}^k) = k\bar{\mu}(\bar{x})$. For an uncapped orbit $x$, the mean index $\bar{\mu}(x)$ and the action $A_H(x)$ are well defined as elements of $S^1_{2N} = \mathbb{R}/2N\mathbb{Z}$ and, respectively, $S^1_{\lambda} = \mathbb{R}/\lambda\mathbb{Z}$; see (2.2) and (2.3). Likewise, when $x$ is non-degenerate, the Conley–Zehnder index $\mu(x)$ is well defined as an element of $\mathbb{Z}/2N\mathbb{Z}$.

2.2. Global and local Floer homology. In this subsection, we very briefly discuss, mainly to set notation, the constructions of filtered and local Floer homology.
We refer the reader to, e.g., \cite{GG09, MS, Sa, SZ} for detailed accounts and additional references.

2.2.1. Filtered Floer homology. Fix a ground field \( \mathbb{F} \). Let \( H \) be a non-degenerate Hamiltonian on \( M \). Denote by \( \text{CF}_{m(-\infty,b)}(H) \), with \( b \in (-\infty, \infty) \setminus \mathcal{S}(H) \), the vector space of formal finite linear combinations

\[
\sigma = \sum_{\bar{x} \in \mathcal{P}(H)} \sigma_{\bar{x}} \bar{x},
\]

where \( \sigma_{\bar{x}} \in \mathbb{F} \) and \( \mu(\bar{x}) = m \) and \( \mathcal{A}(\bar{x}) < b \). (Since \( M \) is strictly monotone, and thus \( N < \infty \), we do not need to take a completion here.) The graded \( \mathbb{F} \)-vector space \( \text{CF}_{-\infty}^{(-\infty,\infty)}(H) \) is endowed with the Floer differential counting solutions \( u: \mathbb{R} \times S^1 \to M \) of the Floer equation,

\[
\partial_s u + J_\# \partial_t u = -\nabla H_\lambda(u),
\]

where \((s,t)\) are the coordinates on \( \mathbb{R} \times S^1 \), i.e., the \( L^2 \)-anti-gradient trajectories of the action functional. Thus we obtain a filtration of the total Floer complex \( \text{CF}_{-\infty}^{(-\infty,\infty)}(H) \). Furthermore, set \( I = (a, b) \) and

\[
\text{CF}_{I}^f(H) := \text{CF}_{-\infty}^{(-\infty,\infty)}(H)/\text{CF}_{-\infty}^{(-\infty,a)}(H),
\]

where \(-\infty \leq a < b \leq \infty \) are not in \( \mathcal{S}(H) \). This is simply the complex generated by the capped periodic orbits of \( H \) with action in \( I \) and equipped with the Floer differential. The resulting homology, the \emph{filtered Floer homology} of \( H \), is denoted by \( \text{HF}_{I}^f(H) \) and by \( \text{HF}_{\ast}^f(H) \) when \( I = (-\infty, \infty) \). Working with filtered Floer homology, we will always assume that the end points of the action interval are \emph{not} in the action spectrum. The degree of a class \( \alpha \in \text{HF}_{I}^{f(a,b)}(H) \) is denoted by \( |\alpha| \).

Also recall that the energy \( E(u) \) of a solution \( u \) of the Floer equation is defined by

\[
E(u) = \int_{\mathbb{R} \times S^1} \|\partial_s u\|^2 \, ds \, dt = \int_{-\infty}^{\infty} \|\partial_s u(s)\|^2_{L^2} \, ds
\]

and that

\[
E(u) = \mathcal{A}_H(\bar{x}) - \mathcal{A}_H(\bar{y})
\]

when \( u \) is asymptotic to \( \bar{x} \) at \(-\infty\) and to \( \bar{y} \) at \( \infty \).

The total Floer complex and homology are modules over the \emph{Novikov ring} \( \Lambda \). In this paper, the latter is defined as follows. Set

\[
I_\omega(A) = -\omega(A) \quad \text{and} \quad I_{c_1}(A) = -2 \langle c_1(TM), A \rangle,
\]

where \( A \in \pi_2(M) \). Thus \( I_\omega = \tau I_{c_1}/2 \) since \( M \) is strictly monotone.

Let \( \Gamma \) be the quotient of \( \pi_2(M) \) by the equivalence relation where the two spheres \( A \) and \( A' \) are considered to be equivalent if \( I_\omega(A) = I_\omega(A') \) or, equivalently, since \( M \) is strictly monotone, \( I_{c_1}(A) = I_{c_1}(A') \). Clearly, \( \Gamma \simeq \mathbb{Z} \). The homomorphisms \( I_\omega \) and \( I_{c_1} \) descend to \( \Gamma \) and become isomorphisms onto the image.

The group \( \Gamma \) acts on \( \text{CF}_{\ast}^{f}(H) \) and \( \text{HF}_{\ast}^{f}(H) \) by recapping: an element \( A \in \Gamma \) acts on a capped one-periodic orbit \( \bar{x} \) of \( H \) by attaching the sphere \( A \) to the original capping. Denoting the resulting capped orbit by \( \bar{x} \# A \), we have

\[
\mu(\bar{x} \# A) = \mu(\bar{x}) + I_{c_1}(A)
\]

when \( x \) is non-degenerate. In a similar vein,

\[
\mathcal{A}_H(\bar{x} \# A) = \mathcal{A}_H(\bar{x}) + I_\omega(A) \quad \text{and} \quad \hat{\mu}(\bar{x} \# A) = \hat{\mu}(\bar{x}) + I_{c_1}(A)
\]
PSEUDO-ROTATIONS OF PROJECTIVE SPACES

2.1 (Projective Spaces) Let \( H \) be a Hamiltonian on \( \mathbb{CP}^n \). Then, by (2.4), \( \text{HF}_m(H) = \mathbb{F} \) when \( m \) has the same parity as \( n \) and \( \text{HF}_m(H) = 0 \) otherwise. To see this directly, one can, for instance, take a non-degenerate quadratic Hamiltonian as \( H \) and show by a calculation that all fixed points of \( \varphi_H \) are elliptic and hence their indices have the same parity as \( n \). (In particular, the Floer differential vanishes.) Recapping by the generator \( A_0 \) of \( \Gamma \cong \mathbb{Z} \) decreases the index by \( 2N = 2(n + 1) \).

2.2.2. Local Floer homology. The notion of local Floer homology goes back to the original work of Floer and it has been revisited a number of times since then. Here we only briefly recall the definition following mainly [Gi10, GG09, GG10] where the reader can find a much more thorough discussion and further references.

Let \( x \) be an isolated one-periodic orbit of a Hamiltonian \( H : S^1 \times M \to \mathbb{R} \). The local Floer homology \( \text{HF}(x) \) is the homology of the Floer complex \( \mathcal{CF}(\bar{H}, x) \) generated by the orbits \( x_i \) which \( x \) splits into under a \( C^2 \)-small non-degenerate perturbation \( \bar{H} \) of \( H \). This homology is well defined, i.e., independent of the perturbation. The homology \( \text{HF}(x) \) is only relatively graded and to fix an absolute grading one needs to pick a trivialization of \( TM \) along \( x \). This can be done by using, for instance, a capping of \( x \) and in this case we write \( \text{HF}_*(\bar{x}) \). Note that then the orbits \( x_i \) inherit a capping from \( \bar{x} \).

For example, if \( x \) is non-degenerate and \( \mu(\bar{x}) = m \), we have \( \text{HF}_l(\bar{x}) = \mathbb{F} \) when \( l = m \) and \( \text{HF}_l(\bar{x}) = 0 \) otherwise. This construction is local: it requires \( H \) to be defined only on a neighborhood of \( x \).

By definition, the support of \( \text{HF}_*(\bar{x}) \), denoted by \( \text{supp} \text{HF}_*(\bar{x}) \), is the collection of integers \( m \) such that \( \text{HF}_m(\bar{x}) \neq 0 \). By (2.1) and continuity of the mean index,

\[
\text{supp} \text{HF}_*(\bar{x}) \subset [\mu(\bar{x}) - n, \mu(\bar{x}) + n].
\] (2.5)

Moreover, when \( x \) is weakly non-degenerate, the closed interval can be replaced by the open interval.
The local Floer homology groups are building blocks for the filtered Floer homology. For instance, assume that \(c\) is the only point of \(S(H)\) in an interval \(I\) and that all one-periodic orbits of \(H\) with action \(c\) are isolated. Then, as is easy to see,\[
\HF^I_c(H) = \bigoplus_{\mathcal{A}_H(x) = c} \HF_*(x). \tag{2.6}
\]

To prove this and also with Lemma 2.2 in mind, first observe that, when the orbits are isolated, a finite energy solution of the Floer equation \(u\) is automatically asymptotic to unique one-periodic orbits as \(s \to \pm \infty\) and there is an \textit{a priori} lower bound \(\epsilon\) on the energy of \(u\); see, e.g., [Sa, Sect. 1.5]. (Hence, we have, in this case, the notion of a solution connecting two orbits.) Then one can shrink \(I\) to an interval with length below \(\epsilon\) and notice that for a sufficiently small non-degenerate perturbation \(\tilde{H}\) the Floer complex splits; see [GG09, Sect. 2.5].

We have the following simple result sharpening (2.6), which we state and prove in a slightly more general form than needed here. Denote by \(d(\cdot, \cdot)\) the distance between two subsets of \(\mathbb{R}\), i.e.,
\[
d(A, B) = \inf\{|a - b| \mid a \in A, b \in B\}.
\]

**Lemma 2.2.** Assume that all capped one-periodic orbits \(\bar{x}\) of \(H\) with action in \(I\) are isolated. Then there exists a spectral sequence with
\[
E^1 = \bigoplus_{\bar{x}} \HF_*(\bar{x}) \tag{2.7}
\]
converging to \(\HF^I_c(H)\). Furthermore, \(\HF_*(\bar{x})\) is a direct summand in \(\HF^I_c(H)\) when one of the following two conditions is met:

(i) \(\bar{x}\) is not connected to any other capped periodic orbit of \(H\) with action in \(I\) by a solution of the Floer equation;

(ii) for any \(\bar{y}\) with \(\mathcal{A}_H(\bar{y}) \in I\), we have
\[
d(\supp \HF_*(\bar{x}), \supp \HF_*(\bar{y})) > 1.
\]

Moreover, assume that (i) holds or the following condition is satisfied:

(ii’) for any \(\bar{y}\) with \(\mathcal{A}_H(\bar{y}) \in I\), we have
\[
|\mu(\bar{x}) - \mu(\bar{y})| > 2n + 1.
\]

Then, when \(\tilde{H}\) is sufficiently \(C^2\)-close to \(H\), the complex \(\CF_*(\tilde{H}, \bar{x})\) is a direct summand in \(\CF^I_c(\tilde{H})\).

Note that while requirement (ii’) is more restrictive than (ii), the third assertion of the lemma, that \(\CF_*(\tilde{H}, \bar{x})\) enters \(\CF^I_c(\tilde{H})\) as a direct summand, is stronger than the second concerning a similar fact on the level of homology. The proof of the lemma is routine and below we merely comment on the argument.

**Outline of the proof.** The required spectral sequence is associated with the action filtration on the Floer complex of a small non-degenerate perturbation of \(H\) and (2.7) is essentially a rephrasing of (2.6). (It is a variant of the Morse–Bott spectral sequence.) To prove the second assertion, it is enough to show that the parts of the differentials \(d_j\), \(j \geq 2\), connecting \(\HF_*(\bar{x})\) with the rest of the spectral sequence vanish when (i) or (ii) holds. When (i) is satisfied, this is a consequence of the third assertion proved below. When (ii) holds, the vanishing of the differentials follows from the fact that \(|\mu(\bar{x}_i) - \mu(\bar{y}_j)| > 1\) for all capped orbits \(\bar{x}_i\) and \(\bar{y}_j\) which
\( \bar{x} \) and, respectively, \( \bar{y} \) split into. This argument also shows that \( \text{CF}_*(\bar{H}, \bar{x}) \) is a direct summand when (ii') is satisfied.

Finally, let us prove that \( \text{CF}_*(\bar{H}, \bar{x}) \) is a direct summand in \( \text{CF}_*^I(\bar{H}) \) when (i) holds. Arguing by contradiction assume that there is a sequence of Floer trajectories \( u \) connecting \( \bar{x}_i \) and \( \bar{y}_j \) for some sequence of perturbations \( \bar{H} \to H \). (Say, \( u \) is asymptotic to \( \bar{x}_i \) at \( -\infty \) and \( \bar{y}_j \) at \( \infty \).) Fix a closed neighborhood \( U \) of \( x \) which contains no other one-periodic orbits of \( H \). Without loss of generality we may assume that \( u(0, 0) \in \partial U \) and that the half-cylinder \( (-\infty, 0] \times S^1 \) is mapped into \( U \). Applying the target-local compactness theorem from [Fi] to the restrictions of \( u \) to a finite cylinder \( [-L, L] \times S^1 \) and then the diagonal process as \( L \to \infty \), we obtain a solution \( v \) of the Floer equation for \( H \) mapping the half-cylinder \( (-\infty, 0] \times S^1 \) into \( U \). Hence, \( v \) is asymptotic to \( \bar{x} \) at \( -\infty \). It is easy to see that the capped orbit which \( v \) is asymptotic to at \( \infty \) has action in \( I \). This contradicts (i). \( \square \)

2.3. Cap product. The algebraic structure on the Floer homology, which is crucial for what follows, is that of a module over the (small) quantum homology of \( M \). The quantum homology \( \text{HQ}_*(M) \) is an algebra over the Novikov ring \( \Lambda \) defined above and \( \text{HQ}_*(M) = H_*(M)[-n] \otimes \Lambda \) as \( \Lambda \)-modules. We refer the reader to, e.g., [MS, Chap. 11] for the definition of the product \( * \) in \( \text{HQ}_*(M) \). The fundamental class \( [M] \) is the unit with respect to this product. The maps \( I_{c_i} \) and \( I_\omega \) naturally extend to \( \text{HQ}_*(M) \). For instance,

\[
I_{\omega}(\alpha) = \max \{ I_\omega(A) \mid \alpha_A \neq 0 \} = \max \{-\lambda_0 k \mid \alpha_k \neq 0\},
\]

where \( \alpha = \sum \alpha_A e^A = \sum \alpha_k q^k \in \text{HQ}_*(M) \).

Example 2.3. We will make an extensive use of the product structure on \( \text{HQ}_*(\mathbb{CP}^n) \). In this case \( N = n+1 \) and \( \text{HQ}_*(\mathbb{CP}^n) \) is generated by the hyperplane class \([\mathbb{CP}^{n-1}]^\ell\) when \( \ell \leq n \) and

\[
[\mathbb{CP}^{n-1}]^{n+1} = q[\mathbb{CP}^n],
\]

where \( |q| = -2(n+1) \); see, e.g., [MS, Sect. 11.3] and references therein. In other words, \([\mathbb{CP}^{n-1}]^\ell * \alpha_\ell = \alpha_{\ell-1} \), where \( \alpha_\ell \) is a generator of \( \text{HQ}_{2n-2\ell}(\mathbb{CP}^n) \). For instance, \([pt] *[\mathbb{CP}^{n-1}] = q[\mathbb{CP}^n] \), which reflects the fact that there is exactly one line passing through any two distinct points of \( \mathbb{CP}^n \) and when a point \( pt \) is fixed every line through \( pt \) intersects \( \mathbb{CP}^{n-1} \) exactly once.

Identifying the global Floer homology with \( \text{HQ}_*(M) \), we can view \( \text{HF}_*(H) \) as an \( \text{HQ}_*(M) \)-module. It is important for our purposes that this module structure extends in a certain way to the filtered Floer homology, and we refer to the resulting “HQ_*(M)-action”, (2.9), as the cap product. (Strictly speaking, the cap product is not a true algebra action due to a filtration shift; rather it should be thought of as an “action” of \( \text{HQ}_*(M) \) on the entire collection of the filtered Floer homology groups.) The definition of the cap product, recalled below, goes back to [Vi95] and [LO]. Here we closely follow [GG14, Sect. 2.3]; see also [MS, Rmk. 12.3.3].

On the level of cycles, the action of a pseudo-cycle \( \zeta \) with \([\zeta] \in H_*(M) \) on \( \bar{x} \) is given by counting the solutions \( u \) of the Floer equation with \( u(0, 0) \in \zeta \). More precisely, pick a generic almost complex structure and a generic \( \zeta \) and set

\[
\Phi_\zeta(\bar{x}) := \sum_y m(\bar{x}, \bar{y}; \zeta)\bar{y}.
\]
Here \( m(x, y; \zeta) \) is the number of the elements, taken with appropriate signs, in the moduli space \( \mathcal{M}(x, y; \zeta) \) of solutions \( u \) asymptotic to \( x \) at \( -\infty \) and \( y \) at \( \infty \) and such that \( u(0, 0) \in \zeta \). This moduli space has dimension \( |x| - |y| - \text{codim}(\zeta) \) and, by definition, \( m(x, y; \zeta) = 0 \) when \( \dim \mathcal{M}(x, y; \zeta) > 0 \).

The map \( \Phi_\zeta \) commutes with the Floer differential and gives rise to a well-defined map
\[
\Phi_\zeta: \text{HF}^{(a, b)}_*(H) \to \text{HF}^{(a, b)}_{* - \text{codim}(\zeta)}(H).
\]
The analytical details of this construction and complete proofs can be found in, e.g., [LO], in much greater generality than is needed here. Clearly,
\[
\Phi_{[M]} = \text{id}.
\]
The action of the class \( \alpha = q^\ell [\zeta] \in \text{HQ}_*(M) \) is induced by the map
\[
\Phi_{q^\ell \zeta}(x) := \sum_y m(q^\ell x, y; \zeta)y.
\]
Here, as in Section 2.2.1, \( q = e^{A_0} \) where \( A_0 \) is the generator of \( \Gamma \) with \( I_{c_1}(A_0) = -2N \). It is routine to check that \( \Phi_{q^\ell [\zeta]} = q^\ell \Phi_{[\zeta]} \). (This is a consequence of the fact that \( \mathcal{M}(q^\ell x, y; \zeta) = \mathcal{M}(x, q^{-\ell} y; \zeta) \).) The resulting map shifts the action interval by \( I_\omega (\alpha) \), i.e.,
\[
\Phi_\alpha: \text{HF}^{(a, b)}_*(H) \to \text{HF}^{(a, b) + I_\omega (\alpha)}_*(-2n)(H), \tag{2.9}
\]
where \( (a, b) + c \) stands for \( (a + c, b + c) \).

By linearity over \( \Lambda \), we extend \( \Phi_\alpha \) to all \( \alpha \in \text{HQ}_*(M) \) so that (2.9) still holds. The maps \( \Phi_\alpha \) are linear in \( \alpha \) once the shift of the action filtration is taken into account; see [GG14, Sect. 2.3]. These maps fit together to form an action of the quantum homology on the collection of the filtered Floer homology groups. The action is multiplicative. In other words, we have
\[
\Phi_\alpha \Phi_\beta = \Phi_{\alpha \ast \beta}, \tag{2.10}
\]
which can be thought of as a form of associativity of the quantum product. Strictly speaking, in (2.10), as in the case of additivity, the maps on the two sides of the identity have different target spaces, which can be accounted for by considering the shifts of the action filtration. We refer the reader to [GG14, Sect. 2.3] for the precise statement. The identity (2.10) was essentially established in [LO] and [PSS]; see also [MS, Rmk. 12.3.3].

The cap product action also extends to the local Floer homology. Namely, let \( \tilde{x} \) be an isolated one-periodic orbit of \( H \) and let \( \tilde{H} \) be a \( C^2 \)-small non-degenerate perturbation of \( H \). Applying the above construction word-for-word to the complex \( \text{CF}_*(\tilde{H}, x) \) from Section 2.2.2, we obtain an action (i.e., a module structure) of \( \text{HQ}_*(M) \) on \( \text{HF}_*(\tilde{x}) \). In other words, for every \( \alpha \in \text{HQ}_*(M) \) we have a map
\[
\Phi_\alpha: \text{HF}_*(\tilde{x}) \to \text{HF}_{* + |\alpha| - 2n}(\tilde{x}) \tag{2.10}
\]
and (2.10) is satisfied. However, the resulting action is trivial unless of course \( \alpha \) is a multiple of \([M]\):

**Lemma 2.4.** Assume that \( |\alpha| < 2n \). Then \( \Phi_\alpha = 0 \) on \( \text{HF}_*(\tilde{x}) \).

The proof of the lemma is standard and we just outline the argument.

*Proof.* We argue as in the proof of the fact that \( \text{CF}_*(\tilde{H}, \tilde{x}) \) is a subcomplex in \( \text{CF}_*(H) \); cf. [Gii10, GG09]. Let \( B \) be a small ball centered at the fixed point \( x(0) \) such that its closure \( \overline{B} \) contains no other fixed points, and let \( \tilde{H} \) be a \( C^2 \)-small, non-degenerate perturbation of \( H \). Denote by \( \tilde{x} \) the capped one-periodic orbits which
$\bar{x}$ splits into. Clearly, $x_i(0) \in B$. Every class $\alpha$ with $|\alpha| < 2n$ can be represented by a cycle avoiding $B$. Thus it is enough to show that for every solution $u$ of the Floer equation for $\tilde{H}$ asymptotic to some orbits $\bar{x}_i$ and $\bar{x}_j$, we have $u(s,0) \in B$ for all $s \in \mathbb{R}$ when $\tilde{H}$ is close to $H$. Arguing by contradiction assume that there is a sequence of $C^2$-small, non-degenerate perturbations $\tilde{H}_t \to H$ and solutions $u_t$ of the Floer equation for $\tilde{H}_t$ such that $u_t(s_0,0) \in \partial B = \bar{B} \setminus B$. Without loss of generality we can assume that $s_0 = 0$ by applying a shift in $s$. Clearly, $E(u_t) \to 0$ and, as a consequence, the loops $t \mapsto u_t(0,t)$ converge to an integral curve of $\varphi_H$; see, e.g., [Sa, Sect. 1.5]. Passing to a subsequence, we conclude that $p = \lim u_t(0,0)$ is a fixed point of $\varphi_H$ which contradicts our choice of $B$. □

Remark 2.5. Lemma 2.4 has a counterpart in the context of the algebra structure and the pair-of-pants product in Floer homology. This is the fact established in [Ci] that, unless $x$ is a so-called SDM, the local Floer homology algebra $\bigoplus_{k>0} HF_*(\tilde{x}^k)$ is non-uniformly nilpotent.

2.4. Spectral invariants and action carriers. In this section, we briefly discuss spectral invariants and action carriers to the extent necessary for our purposes. The theory of Hamiltonian spectral invariants was developed in its present Floer theoretical form in [Oh05b, Sc00], although the first versions of the theory go back to [HZ, Vi92]. Action carriers were introduced in [GG09] and then studied in [CGG, GG12].

Let $H$ be a Hamiltonian on a closed monotone (or even rational) symplectic manifold $M^{2n}$. The spectral invariant or action selector $c_\alpha$ associated with a class $\alpha \in HF_*(H) = HQ_*(M)$ is defined as

$$c_\alpha(H) = \inf \{ a \in \mathbb{R} \setminus S(H) \mid \alpha \in \text{im}(i^a) \} = \inf \{ a \in \mathbb{R} \setminus S(H) \mid j^a(\alpha) = 0 \},$$

where $i^a : HF(\mathbb{R},\mathbb{R}) \to HF_*(H)$ and $j^a : HF_*(H) \to HF(\mathbb{R},\mathbb{R})$ are the natural “inclusion” and “quotient” maps. It is easy to see that $c_\alpha(H) > -\infty$ when $\alpha \neq 0$ and one can show that $c_\alpha(H) \in S(H)$. In other words, there exists a capped one-periodic orbit $\bar{x}$ of $H$ such that $c_\alpha(H) = \mathcal{A}_H(\bar{x})$. As an immediate consequence of the definition,

$$c_\alpha(H + a(t)) = c_\alpha(H) + \int_{S^1} a(t) \, dt,$$

where $a : S^1 \to \mathbb{R}$.

Action selectors have several important properties. The function $c_\alpha$ is homotopy invariant: $c_\alpha(H) = c_\alpha(K)$ when $\varphi_H = \varphi_K$ in $\widehat{\text{Ham}}(M)$ and $H$ and $K$ have the same mean value. Furthermore, it is sub-additive:

$$c_{\alpha+\beta}(H \sharp K) \leq c_\alpha(H) + c_\beta(K).$$

In particular,

$$c_{[M]}(H \sharp K) \leq c_{[M]}(H) + c_{[M]}(K).$$

Finally, $c_\alpha$ is monotone and Lipschitz in the $C^0$-topology as a function of $H$.

When $H$ is non-degenerate, the action selector can also be evaluated as

$$c_\alpha(H) = \inf_{|\sigma|=\alpha} c_\sigma(H),$$

where we set

$$c_\sigma(H) = \max \{ \mathcal{A}_H(\bar{x}) \mid \sigma \bar{x} \neq 0 \}$$

for $\sigma = \sum \sigma_{\bar{x}} \bar{x} \in \text{CF}_*(H)$. (2.11)
The infimum here is attained, since $M$ is rational and thus $S(H)$ is closed. Hence there exists a cycle $\sigma = \sum \sigma_x \bar{x} \in \text{CF}_{|\alpha|}(H)$, representing the class $\alpha$, such that $c_\alpha(H) = A_H(\bar{x})$ for an orbit $\bar{x}$ entering $\sigma$. In other words, $\bar{x}$ maximizes the action on $\sigma$ and the cycle $\sigma$ minimizes the action over all cycles in the homology class $\alpha$. Such an orbit $\bar{x}$ is called a carrier of the action selector. This is a stronger requirement than just that $c_\alpha(H) = A_H(\bar{x})$ and $\mu(\bar{x}) = |\alpha|$. When $H$ is possibly degenerate, a capped one-periodic orbit $\bar{x}$ of $H$ is a carrier of the action selector if there exists a sequence of $C^2$-small, non-degenerate perturbations $\tilde{H}_l \to H$ such that one of the capped orbits $\bar{x}$ splits into is a carrier for $\tilde{H}_l$. An orbit (without capping) is said to be a carrier if it turns into one for a suitable choice of capping.

It is easy to see that a carrier necessarily exists, but, in general, is not unique. However, it becomes unique when all one-periodic orbits of $H$ have distinct action values.

As consequence of the definition of the carrier and continuity of the action and the mean index, we have

$$c_\alpha(H) = A_H(\bar{x}) \quad \text{and} \quad |\hat{\mu}(\bar{x}) - |\alpha|| \leq n,$$

and the inequality is strict when $x$ is weakly non-degenerate. Furthermore, a carrier $\bar{x}$ for $c_\alpha$ is in some sense homologically essential as the following result asserts.

**Lemma 2.6.** Let $\bar{x}$ be an action carrier for $c_\alpha$. Then $HF_{|\alpha|}(\bar{x}) \neq 0$ when $x$ is isolated.

This lemma is proved in [GG12, Lemma 3.2] for $c_{[M]}$, but the proof goes through word-for-word for any homology class. For the sake of brevity, we omit the argument.

### 3. Background results on pseudo-rotations

In this section, we assemble several known symplectic topological results on pseudo-rotations of projective spaces, crucial for what follows.

Let $\varphi = \varphi_H$ be a pseudo-rotation of $\mathbb{C}P^n$, which we do not assume to be non-degenerate. Denote by $\alpha_l$ the generator in $HF_{2l-n}(H) = \mathbb{F}$, $l \in \mathbb{Z}$, and let $\bar{x}_l \in \mathcal{P}_1(H)$ be an action carrier for $\alpha_l$. We will write $c_l := c_{\alpha_l}$. Then, in particular,

$$c_l(H) = A_H(\bar{x}_l) \quad \text{and} \quad HF_{2l-n}(\bar{x}_l) \neq 0.$$

Throughout the paper it is convenient to occasionally rescale the standard Fubini–Study symplectic structure $\omega$ on $\mathbb{C}P^n$. Hence, we state the results in a slightly more general form assuming that $\omega$ is proportional to the Fubini–Study form and $\lambda = \langle \omega, \mathbb{C}P^1 \rangle$ is the rationality constant. (For the Fubini–Study form, $\lambda = \pi$.)

**Theorem 3.1 ([GG09]).** For every $l \in \mathbb{Z}$ the action carrier $\bar{x}_l$ is unique and the resulting map

$$\mathbb{Z} \to \mathcal{P}_1(H), \quad l \mapsto \bar{x}_l$$

is a bijection. Furthermore, the map

$$\mathbb{Z} \to S(H), \quad l \mapsto c_l(H) = A_H(\bar{x}_l)$$

is strictly monotone, i.e., $l > l'$ if and only if $A_H(\bar{x}_l) > A_H(\bar{x}_{l'})$.

One important consequence of the theorem is that distinct capped one-periodic orbits of $\varphi_H$ necessarily have different actions. Another is that, by Lemma 2.6, $HF_{2l-n}(\bar{x}_l) \neq 0$. In particular, $HF(x) \neq 0$ for all $x \in \mathcal{P}(H)$.
When $H$ is non-degenerate we have $\mu(\bar{x}_l) = 2l - n$, and the proof of the theorem is rather straightforward. The general case of the theorem is established in [GG9, Sect. 6]. First, one shows that the map (3.2) is strictly monotone. This is a consequence of the facts that the orbits are isolated and $[\mathbb{C}P^n-1] * \alpha_l = \alpha_{l-1}$; see [GG9, Prop. 6.2]. Next, we have the relation

$$c_{l+n+1}(H) = c_l(H) + \lambda,$$

which follows from (2.8). Now the chain of the inequalities

$$\cdots < c_0(H) < \cdots < c_n(H) < c_{n+1}(H) = c_0(H) + \lambda < \cdots,$$

is used to infer the uniqueness of $\bar{x}_l$ and the assertion that the map (3.1) is a bijection. (The argument is similar to the proof of the degenerate case of Arnold’s conjecture for $\mathbb{C}P^n$; see [Fl, Fo, FW] and also [Oh05b, Se98] and a detailed account in [GG9, Sect. 6.2.3].) Note also that we automatically have the identity $\bar{x}_{l+n+1} = \bar{x}_l \# \mathbb{C}P^1$.

Example 3.2 (Rotations of $\mathbb{C}P^n$, I). In the context of this paper, a rotation of $\mathbb{C}P^n$ is a Hamiltonian diffeomorphism $\varphi_Q$ generated by a quadratic Hamiltonian $Q = \sum a_i |z_i|^2$, where we identified $\mathbb{C}P^n$ with the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ by the diagonal (Hopf) $S^1$-action. A calculation shows that $\varphi_Q$ is strongly non-degenerate if and only if it is a pseudo-rotation and if and only if $a_j - a_i \notin \mathbb{Q}$ for all pairs $j \neq i$. Then $P(Q) = P_{1}(Q) = \{x_0, \ldots, x_n\}$ is the set of the coordinate axes and, without loss of generality, we may assume that $a_0 < \ldots < a_n$. Furthermore, we can also require that $\sum a_i = 0$ or, equivalently, that the mean value of $Q$ is zero; for the Hamiltonian $\sum |z_i|^2$ is constant on $\mathbb{C}P^n$. We have

$$S(H) = \bigsqcup a_i + \lambda \mathbb{Z},$$

where $\lambda = \pi$ when $\omega$ is the standard Fubini–Study symplectic form. Denote by $\hat{x}_i$ the constant orbit $x_i$ equipped with the trivial capping. By a calculation, one can see that the eigenvalues of $d\varphi_Q$ at $x_i$ are $\exp(\pm 2\lambda \sqrt{-1}(a_j - a_i))$, where $j \neq i$, and $\hat{\mu}(\hat{x}_i) = 2(n+1)a_i/\lambda$. If $Q$ is $C^2$-small and non-degenerate, $\mu(\hat{x}_i) = 2i - n$. We will elaborate on this example in [GG18].

Theorem 3.1 enables us to extend the notion of the Conley–Zehnder index to capped one-periodic orbits $\bar{x}_l$ of degenerate pseudo-rotations by setting $\mu(\bar{x}) = 2l - n$ for $\bar{x} = \bar{x}_l$. We will call $\mu(\bar{x})$ the LS-index: for it ultimately comes from a version of Lusternik–Schnirelmann theory for action selectors which the proof of the theorem is based on. When $x$, the one-periodic orbit underlying $\bar{x}$, is non-degenerate this is just the ordinary Conley–Zehnder index. Without non-degeneracy, the LS-index is a global rather than local invariant. However, it has some of the expected properties of the Conley–Zehnder index, e.g., $\mu(\bar{x}#A) = \mu(\bar{x}) + \mu_1(A)$ and $HF_\mu(\bar{x}) \neq 0$. Moreover, when $n = 1$, $HF(\bar{x})$ is concentrated in only one degree which is $\mu(\bar{x})$; see [GG10]. With this notion in mind, Theorem 3.1 can be rephrased as that the ordering of $P_{1}(H)$ by the LS-index agrees with that by the action.

Next, recall that the augmented action of $H$ on $x \in P_{1}(H)$ is defined by

$$\tilde{A}_H(x) = A_H(\bar{x}) - \frac{\lambda}{2N} \hat{\mu}(\bar{x}),$$

where on the right we used an arbitrary capping of $x$. (Note that the augmented action is defined for every monotone manifold; see [GG9].) It is clear that the augmented action is well defined, i.e., independent of the capping, and homogeneous
with respect to iteration. Furthermore, $\tilde{A}_H(x)$ is completely determined, once $H$ is normalized to have zero mean, by the time-one map $\varphi_H$ viewed as an element of $\text{Ham}(\mathbb{C}P^n)$ rather than $\tilde{\text{Ham}}(\mathbb{C}P^n)$; [EP].

**Theorem 3.3** (Action-index Resonance Relations; Thm. 1.12 in [GG09]). Let $x_0, \ldots, x_n$ be the fixed points of a pseudo-rotation $\varphi_H$ of $\mathbb{C}P^n$. Then

$$\tilde{A}_H(x_0) = \ldots = \tilde{A}_H(x_n).$$

This theorem has been extended to some other symplectic manifolds and a broader class of Hamiltonian diffeomorphisms; see [CGG]. It also has an analog for Reeb flows; [GG16b, Sect. 6.1.2].

As a consequence of Theorem 3.3,

$$\tilde{A}_H(\bar{x}) = \frac{\lambda}{2(n+1)} \hat{\mu}(\bar{x}) + \text{const} \quad (3.5)$$

for every $\bar{x} \in \bar{P}_1(H)$, where $\text{const} = \tilde{A}_H(x)$ is independent of $x$. Surprisingly, very little is known about $\tilde{A}_H(x)$. Conjecturally, $\tilde{A}_H(x) = 0$ when $H$ is normalized to have zero mean. It is known however (see [GG17, Thm. 5.1]) that

$$\tilde{A}_H(x) = \lim_{k \to \infty} \frac{c[M] \langle H^{2k} \rangle}{k} \quad (3.6)$$

for pseudo-rotations of $M = \mathbb{C}P^n$. The limit on the right is the asymptotic spectral invariant, extensively studied in [EP] where it is shown to be equal to a Calabi quasimorphism for many symplectic manifolds including $\mathbb{C}P^n$ once $H$ is normalized to have zero mean.

The marked action spectrum $\hat{S}(H)$ is, by definition, the bijection

$$\hat{S} : \mathbb{Z} \xrightarrow{(3.1)} \bar{P}_1(H) \xrightarrow{(3.2)} S(H),$$

i.e., $\hat{S}(H)$ is simply the spectrum $S(H)$ with its points labelled by $\mathbb{Z}$ (essentially the indices) or, equivalently, by $\bar{P}_1(H)$. In a similar vein, the marked index spectrum $S_{\text{ind}}(\varphi_H)$ is the map

$$S_{\text{ind}} : \mathbb{Z} \xrightarrow{(3.1)} \bar{P}_1(H) \longrightarrow S_{\text{ind}}(\varphi),$$

where

$$S_{\text{ind}}(\varphi) = \{ \hat{\mu}(x) \mid x \in \bar{P}_1(H) \}$$

is the mean index spectrum of $H$ and the second arrow is the map $\bar{x} \mapsto \hat{\mu}(\bar{x})$. Then (3.5) can be rephrased as

$$\hat{S}(H) = \frac{\lambda}{2(n+1)} S_{\text{ind}}(\varphi_H) + \text{const},$$

i.e., the action spectrum and the index spectrum agree up to a factor and a shift. The factor can be made equal 1 by scaling $\omega$, and the shift can be made zero by adding a constant to $H$:

$$\hat{S}(H) = \hat{S}_{\text{ind}}(\varphi_H). \quad (3.7)$$

Finally, we have a different type of resonance relations involving only the indices. To state the result, recall that for an uncapped one-periodic orbit $x \in \bar{P}_1(H)$ the mean index is well defined modulo $2(n+1) = 2N$, i.e., $\hat{\mu}(x) \in S^1_{2N} = \mathbb{R}/2(n+1)\mathbb{Z}$. 

Theorem 3.4 (Mean Index Resonance Relations; Sect. 1.2 in [GK]). Let \( x_0, \ldots, x_n \) be the fixed points of a pseudo-rotation \( \varphi_H \) of \( \mathbb{CP}^n \). Then for some non-zero vector \( \vec{r} = (r_0, \ldots, r_n) \in \mathbb{Z}^{n+1} \), we have
\[
\sum r_i \hat{\mu}(x_i) = 0 \text{ in } \mathbb{R}/2(n+1)\mathbb{Z}.
\]

In other words, the closed subgroup \( \Gamma \subset \mathbb{T}^{n+1} \) topologically generated by the mean index vector
\[
\Delta = \Delta(\varphi_H) := (\hat{\mu}(x_0), \ldots, \hat{\mu}(x_n)) \in \mathbb{T}^{n+1} = \mathbb{R}^{n+1}/2(n+1)\mathbb{Z}^{n+1}
\]
has positive codimension. Moreover, the codimension is equal to the number of linearly independent resonances, i.e., the rank of the subgroup \( \mathcal{R} \subset \mathbb{Z}^{n+1} \) formed by all resonances \( \vec{r} \); see [GK].

Clearly, \( \mathcal{R} \) depends on \( \varphi_H \). However, conjecturally, the resonance relation
\[
\sum \hat{\mu}(x_i) = 0 \text{ in } \mathbb{R}/2(n+1)\mathbb{Z}. \tag{3.9}
\]
is universal, i.e., satisfied for all pseudo-rotations. (Up to a factor this is the only possible universal resonance relation; for, as is easy to see, any other relation breaks down for a suitably chosen rotation; [GG18].) For \( S^2 \), (3.9) asserts, roughly speaking, that \( D\varphi \) rotates the tangent spaces at the fixed points by the same angle but in opposite directions. This is a consequence of the Poincaré–Birkhoff theorem, [Br15a, Appendix A.2]; see also [CKRTZ] for a different approach based on Theorem 3.4. In [GG18] we will revisit this conjecture and establish it for non-degenerate pseudo-rotations of \( \mathbb{CP}^2 \).

Example 3.5 (Rotations of \( \mathbb{CP}^n \), II). In the setting of Example 3.2, we have \( \hat{A}_Q(x_i) = 0 \) as a direct calculation shows. As we have done throughout this section, let us denote by \( \bar{x}_i \) the orbit \( x_i \) capped so that \( \mu(\bar{x}_i) = 2i - n \). (By Theorem 3.1, such a capping exists and is unique.) When \( Q \) is \( C^2 \)-small, \( \bar{x}_i = \hat{x}_i \). One can show that
\[
\sum \hat{\mu}(\bar{x}_i) = 0,
\]
although this is not obvious. Thus, in particular, (3.9) holds. When \( Q \) is \( C^2 \)-small, this follows from that \( \sum a_i = 0 \) and \( \hat{\mu}(\hat{x}_i) = 2(n+1)a_i/\lambda \) by Example 3.2. The general case is proved in [GG18].

We conclude this discussion by the following sample illustrative application of the three theorems from this section to dynamics of pseudo-rotations.

Corollary 3.6. Every pseudo-rotation of \( S^2 \) is strongly non-degenerate and its fixed points are elliptic.

The corollary is a standard, although ultimately highly non-trivial, result in two-dimensional dynamics; see [Fr92].

Proof. Arguing by contradiction, assume that \( \varphi_H \) is a pseudo-rotation of \( S^2 \) and some iterate of \( \varphi_H \) is degenerate or that one of its fixed points is hyperbolic. In dimension two, a hyperbolic or degenerate fixed point necessarily has integer mean index. Hence, replacing \( \varphi_H \) by a sufficiently large iterate and using Theorem 3.4, we may assume that both fixed points \( x \) and \( y \) of \( \varphi_H \) have mean index equal to zero modulo 4. Let us scale the symplectic structure and adjust the Hamiltonian so that \( \tilde{S}(H) = \tilde{S}_\text{nd}(\varphi_H) \). Then, by (3.5), \( \tilde{A}_H(\tilde{x}) = \tilde{A}_H(\tilde{y}) \) for suitable cappings of \( x \) and \( y \), which is impossible by Theorem 3.1. \( \square \)
4. Invariant sets

Our goal in this section is to partially generalize some of the results of Le Calvez and Yoccoz, [LCY], and of Franks and Misiurewicz, [Fr99, FM], on invariant sets in dimension two to higher dimensions. To lay out the context, recall that the key feature of the examples of pseudo-rotations of \( S^2 \) constructed by Anosov and Katok in [AK] (see also, e.g., [FK]) is that these pseudo-rotations \( \varphi \) have exactly three ergodic measures: the two fixed points and the area form. Thus on the complement to the two fixed points \( \varphi \) is uniquely ergodic and even though \( \varphi \) is not uniquely ergodic on \( S^2 \) it is very close to being so. Volume preserving, uniquely ergodic maps of compact manifolds are necessarily minimal, i.e., all orbits are dense; see, e.g., [Wa]. Therefore, one might also expect every orbit of \( \varphi \) other than a fixed point to be dense. (In a similar vein, every symplectic manifold that admits a symplectic \( S^1 \)-action without fixed points also admits a minimal symplectomorphism; [HC].) This turns out to be false. As shown in [FM, Prop. 5.5] drawing from the aforementioned results, the set of periodic points of a topologically transitive homeomorphism of \( S^2 \) cannot be simultaneously finite and isolated as an invariant set. As a consequence, a pseudo-rotation of \( S^2 \) must have orbits other than the fixed points entirely contained in an arbitrarily small neighborhood of the fixed point set. While the full scope of [FM, Prop. 5.5] is certainly out of the reach of symplectic methods, its application to area-preserving smooth pseudo-rotations fits well in the symplectic framework. We have the following partial generalization of this result already mentioned in the introduction as Theorem 1.2.

**Theorem 4.1.** No fixed point of a pseudo-rotation of \( \mathbb{C}P^n \) is isolated as an invariant set.

The theorem is in turn a consequence of the following general result; cf. Example 3.2.

**Theorem 4.2.** Let \( M^{2n} \) be a strictly monotone symplectic manifold. Assume that the minimal Chern number \( N \geq n + 1 \) and

\[
\alpha \ast \beta = q[M]
\]

in \( \text{H}_{\ast}(M) \) for some homology classes \( \alpha \in \text{H}_{\ast}(M) \) and \( \beta \in \text{H}_{\ast}(M) \) with \( |\alpha| < n \) and \( |\beta| < n \). Let \( \varphi \) be a Hamiltonian diffeomorphism of \( M \) with a contractible periodic orbit \( x \) which is isolated as an invariant set and such that \( \text{HF}(x^k) \neq 0 \) for all \( k \in \mathbb{N} \). Then \( \varphi \) has infinitely many periodic points.

Theorem 4.2, proved in Section 6, significantly relaxes the dynamical requirements in [GG14, Thm. 1.1], where the orbit \( x \) is assumed to be hyperbolic, at the expense of imposing more restrictive conditions on \( M \) and, in particular, on \( N \). As in that theorem the assumption that the orbit \( x \) is contractible can be relaxed when \( M \) is toroidally monotone, but we have no examples of positive monotone manifolds with \( \pi_1(M) \neq 1 \) meeting the conditions of the theorem. (See however Remark 6.3.) In fact, the only positive monotone manifold with \( N \geq n + 1 \) known to us is \( \mathbb{C}P^n \). The requirements of the theorem are met by numerous negative monotone manifolds, but in this case the Conley conjecture holds and \( \varphi \) has infinitely many periodic orbits unconditionally; see [CGG, GG12]. However, the result we actually prove (Theorem 6.2) is slightly more precise than Theorem 4.2 and it gives additional information about the augmented actions of periodic orbits, even when \( M \) is negative monotone.
We also note that the assumption that a periodic orbit $x$ is isolated as an invariant set imposes a strong restriction on the dynamics. In particular, $x$ can easily be isolated as a periodic orbit for all iterations but not isolated as an invariant set.

**Proof of Theorem 4.1.** Let $\varphi$ be a Hamiltonian pseudo-rotation of $\mathbb{CP}^n$. As is pointed out in Section 3, $HF(x) \neq 0$ for every fixed point $x$ of $\varphi$ by Lemma 2.6. Since $\varphi^k$ is also a pseudo-rotation for every $k \in \mathbb{N}$, we have $HF(x^k) \neq 0$. Now it remains to apply Theorem 4.2 to $M = \mathbb{CP}^n$. Indeed, if $x$ were isolated as an invariant set, $\varphi$ would have infinitely many periodic orbits by Theorem 4.2, and hence would not be a pseudo-rotation. \qed

**Remark 4.3.** The condition that $HF(x^k) \neq 0$ is automatically satisfied when the Hopf index of $x^k$ is non-zero. In all examples known to us, $HF(x^k) \neq 0$ for all $k \in \mathbb{N}$ whenever $HF(x) \neq 0$ and $x^k$ is isolated. Furthermore, $HF(x^k) \neq 0$ when $HF(x) \neq 0$ and $k$ is admissible, i.e., none of the Floquet multipliers of $x$ is a root of unity of degree $k$ (see [GG10, Thm. 1.1]), and we conjecture that this is true for all $k$.

5. LAGRANGIAN POINCARÉ RECURRENCE AND $C^0$-RIGIDITY

In this section we prove two results which roughly speaking assert that under suitable extra conditions a sequence of iterates $\varphi^k$ of a pseudo-rotation converges to the identity in the appropriate metric. The first of these results (Theorem 5.1) is closely related to the Lagrangian Poincaré recurrence conjecture (see Theorem 5.8) while the second (Theorem 5.16) is a higher-dimensional analog of the $C^0$-rigidity theorem from [Br15b].

5.1. Lagrangian Poincaré recurrence and the $\gamma$-norm convergence.

5.1.1. The $\gamma$-norm convergence. We start this section by briefly recalling the definition of the $\gamma$-norm and the related $\gamma$-metric on $Ham(M)$. This norm was originally introduced in [HZ, Vi92] for compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ using generating functions. The construction was then extended to closed symplectically aspherical manifolds in [Sc00] and to all rational weakly monotone manifolds in [Oh05a]. We refer the reader to these papers for the original definitions and a much more detailed treatment.

Let $M^{2n}$ be a closed, weakly monotone, rational symplectic manifold and let $\varphi$ be a Hamiltonian diffeomorphism of $M$ generated by a Hamiltonian $H$. It is convenient to assume that $H_t \equiv 0$ for $t$ close to 0. Set

$$H_t^{\text{inv}} = -H_t \circ \varphi_t.$$  

This is a periodic in time Hamiltonian generating the time-dependent flow $(\varphi^t)^{-1}$. (Alternatively, one can take the Hamiltonian generating the time-dependent flow $\varphi_H^{-t}$ as $H^{\text{inv}}$.) By definition,

$$\gamma(H) = c_{[M]}(H) + c_{[M]}(H^{\text{inv}}) \text{ and } \gamma(\varphi) = \inf_H \gamma(H),$$

where the infimum is taken over all $H$ generating the time-one map $\varphi$. (In fact, $\gamma(H)$ is independent of $H$ as long as the time-dependent flow $\varphi_H^{-t}$ remains in the same homotopy class with fixed end points. For a broad class of manifolds, but not for $\mathbb{CP}^n$, we have $\gamma(H) = \gamma(\varphi)$ for all Hamiltonians $H$ generating $\varphi$.) Furthermore, set

$$d_\gamma(\varphi, \psi) := \gamma(\varphi \psi^{-1}).$$
It is a standard fact that $d_*$ is a right-invariant distance on $Ham(M)$; see, e.g., [Oh05a].

As a consequence of the Poincaré duality in Floer homology (see, e.g., [EP, Lemma 2.2]), for some symplectic manifolds including $\mathbb{CP}^n$, we have

$$c_{[M]}(H^{\text{inv}}) = -c_{[\mu]}(H)$$

and thus

$$\gamma(H) = c_{[M]}(H) - c_{[\mu]}(H).$$

**Theorem 5.1.** Let $\varphi = \varphi_H$ be a pseudo-rotation of $\mathbb{CP}^n$. Then there exist a constant $C > 0$ and a non-negative integer $d \leq n$, both depending only on $\tilde{\Delta}(\varphi)$, such that for every $\epsilon$ such that $0 < \epsilon \leq \lambda$, we have

$$\liminf_{k \to \infty} \frac{|\{ \ell \leq k \mid \gamma(\varphi^\ell) < \epsilon \}|}{k} \geq C\epsilon^d.$$ \hfill (5.3)

In particular, the limit inferior is positive.

**Corollary 5.2** ($\gamma$-norm convergence). Let $\varphi$ be a pseudo-rotation of $\mathbb{CP}^n$. Then $\gamma(\varphi^k) \to 0$ for some sequence $k_i \to \infty$.

**Remark 5.3.** To the best of our knowledge, both the theorem and the corollary are new even when $n = 1$. It might be possible to relax the conditions of Theorem 5.1 by combining its proof with the proof of [CGG, Thm. 1.1] and extend the theorem, or at least Corollary 5.2, to perfect Hamiltonian diffeomorphisms of $\mathbb{CP}^n$.

Furthermore, recall that $\gamma(\varphi)$ is a priori bounded for all $\varphi \in Ham(\mathbb{CP}^n)$ and, in fact, $\gamma(\varphi) \leq \lambda$; see [EP] and also [McD10]. A simple way to see this in the context of the paper is to use (5.1) and (5.2) together with the fact that $c_{[\mu]}(H) \geq c_{[M]}(H) - \lambda$ by (3.3). This shows that the condition that $\epsilon \leq \lambda$ is not really restrictive: when $\epsilon > \lambda$ the density on the left-hand side of (5.3) is equal to one. In fact, the theorem is most interesting for small values of $\epsilon$.

The constant $d$ in Theorem 5.1 and also Theorem 5.8 below is the difference $d = n + 1 - r$, where $r$ is the number of linearly independent resonance relations the mean indices $\check{\mu}(\bar{x})$ satisfy; see [GK] or Section 3.

**Remark 5.4.** Few manifolds $M$ are expected to admit Hamiltonian diffeomorphisms $\varphi$ such that $\gamma(\varphi^k) \to 0$ for some sequence $k_i \to \infty$. For instance, hypothetically, this is never the case when $M$ is symplectically aspherical. (As far as we know, this question/conjecture is due to L. Polterovich; we learned of it from Seyfaddini.)

There is also a similar question for the $C^0$- or $C^1$-norm and in this instance some partial results are available. For example, it is not hard to see that one can never have $\varphi^k \to id$ in the $C^1$-sense when $M$ is symplectically aspherical; cf. [Po].

**Proof.** Throughout the proof, it will be convenient to rescale the symplectic structure on $\mathbb{CP}^n$ so that $[\omega] = 2c_1(T\mathbb{CP}^n)$ and hence $\lambda = 2(n + 1)$, and to normalize the Hamiltonian to ensure that all fixed points have zero augmented action. Then (3.7) holds, $\bar{S}(H) = \bar{S}_{nd}(\varphi)$, or, in other words,

$$A_H(\bar{x}) = \check{\mu}(\bar{x})$$

for every capped one-periodic orbit $\bar{x}$ of $\varphi$. Since the augmented action is homogeneous this is also true for all iterates $H^{2k}$. Furthermore, since $\gamma(\varphi^k) \leq \gamma(H^{2k})$, it suffices to prove the theorem for $\gamma(H^{2k})$ in place of $\gamma(\varphi^k)$.
Observe also that we only need to prove the theorem for small $\epsilon > 0$, e.g., for every $\epsilon < 4$. Then the result for the entire range of $\epsilon$ from 0 to $\lambda = 2(n + 1)$ will follow by adjusting the value of $C$.

Consider the mean index vector $\Delta$ defined by (3.8) and let $\Gamma \subset \mathbb{T}^{n+1}$ be the subgroup topologically generated by this vector. Thus $\Gamma$ is the closure of the positive semi-orbit $O = \{k\Delta \mid k \in \mathbb{N}\}$. The connected component of the identity in $\Gamma$ is a torus, and $\Gamma$ is isomorphic to the direct product of this torus and a cyclic group of order $k_0$. Replacing $\varphi$ by $\varphi^{k_0}$ we can assume that $\Gamma$ is connected, and hence isomorphic to a torus. Set $d = \dim \Gamma$. Since the components of $\Delta$ satisfy at least one resonance relation by Theorem 3.4, we have $d \leq n$.

The volume of the intersection of the $\epsilon/2$-neighborhood $B(\epsilon)$ of 0 in $\mathbb{T}^{n+1}$ with $\Gamma$ is bounded from below by $C\epsilon^d$, where $C$ is determined by the geometry of $\Gamma$:

$$\text{vol} (B(\epsilon) \cap \Gamma) \geq C\epsilon^d. \tag{5.4}$$

The semi-orbit $O$ is uniformly distributed in $\Gamma$ and hence to prove (5.3) it is enough to show that

$$\gamma(H^{2k}) < \epsilon \text{ whenever } k\Delta \in B(\epsilon). \tag{5.5}$$

To this end, it is convenient to equip $\mathbb{T}^{n+1}$ with the metric generated by the norm (the distance to the origin)

$$||\vec{\theta}|| = \max_i ||\theta_i||,$$

where $\vec{\theta} = (\theta_0, \ldots, \theta_n) \in \mathbb{T}^{n+1}$ and $||\cdot||$ on the right-hand side stands for the distance to zero in $\mathbb{R}/2(n+1)\mathbb{Z}$. Thus $B(\epsilon)$ is actually a cube with faces perpendicular to the “coordinate axes” and the diameter of $\mathbb{T}^{n+1}$ is $2(n + 1)$. (The choice of a norm on $\mathbb{T}^{n+1}$ effects only the value of the constant $C$ in (5.4) and (5.3).)

Assume that $k\Delta \in B(\epsilon)$. Then the spectrum $\mathcal{S}(H^{2k}) = \mathcal{S}_{\text{red}}(\varphi^k)$ is contained in the $\epsilon/2$-neighborhood of $2(n + 1)\mathbb{Z}$. We will call the part of this spectrum lying in the $\epsilon/2$-neighborhood of one point of $2(n + 1)\mathbb{Z}$ a cluster. To finish the proof we simply need to show that $c_{[M]}(H^{2k})$, where $M = \mathbb{C}P^n$, and $c_{[pt]}(H^{2k})$ are in the same cluster. Indeed, then

$$\gamma(H^{2k}) = c_{[M]}(H^{2k}) - c_{[pt]}(H^{2k}) < \epsilon.$$

We will prove that these action values are in fact in the cluster centered at 0. (Up to this point we could have worked directly with the “action vector” instead of $\Delta$, an element of $\mathbb{T}^{n+1}$ whose components are the actions of uncapped one-periodic orbits viewed as points in $\mathbb{R}/\lambda\mathbb{Z}$. However, in the next step, the role of $\Delta$ becomes essential because of (2.5).)

Focusing on $c_{[pt]}(H^{2k})$, denote by $2(n + 1)q$ the center of the cluster containing this point. Our goal is to show that $q = 0$. Let $\bar{x}$ be the action carrier for $[pt]$, i.e., $\bar{x}$ is the capped $k$-periodic orbit uniquely determined by the condition

$$c_{[pt]}(H^{2k}) = A_{H^{2k}}(\bar{x}).$$

This orbit has LS-index $-n$. (Recall that in the non-degenerate case this is simply the Conley–Zehnder index.) Since

$$\mu(\bar{x}) = -n \in \text{supp} \text{ HF}(\bar{x}) \subset [\hat{\mu}(\bar{x}) - n, \hat{\mu}(\bar{x}) + n]$$

by (2.5), we have $-2n \leq \hat{\mu}(\bar{x}) \leq 0$. On the other hand, $|\hat{\mu}(\bar{x}) - 2(n + 1)q| < \epsilon/2$, and thus $q = 0$ due to the assumption that $\epsilon < 4$. A similar argument shows that
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c_{[M]}(H^{2k}) is also in the cluster centered at 0. This proves (5.5) and completes the proof of the theorem. □

Remark 5.5. It is clear from the proof that we have also established the inequality

$$\gamma(\varphi^k) \leq \text{const} \|\Delta(\varphi^k)\|,$$

where \text{const} > 0 depends only on \(\Delta(\varphi)\). In particular, if \(\|\Delta(\varphi^k)\|\) converges to zero, the sequence \(\gamma(\varphi^k)\) converges to zero at least as fast.

5.1.2. Lagrangian Poincaré recurrence. Consider a compactly supported Hamiltonian diffeomorphism \(\varphi\) of a symplectic manifold \(M^{2n}\). The following conjecture was put forth by the first author and independently by Claude Viterbo around 2010.

Conjecture 5.6 (Lagrangian Poincaré Recurrence). For any closed Lagrangian submanifold \(L \subset M\) there exists a sequence of iterations \(k_i \to \infty\) such that

$$\varphi^{k_i}(L) \cap L \neq \emptyset.$$

Moreover, the density of the sequence \(k_i\) is related to a symplectic capacity of \(L\).

The requirement that \(\varphi\) is Hamiltonian is essential: the conjecture fails for symplectomorphisms of \(T^2\), e.g., for an irrational shift. Furthermore, the conjecture is most interesting when \(L\) is “small”. When it is not, e.g., if \(L\) is not displaceable, the assertion is often obvious. In dimension two (i.e., for \(n = 1\)), the conjecture readily follows from the standard Poincaré recurrence theorem when \(L\) bounds and the observation that otherwise \(L\) is not displaceable by a Hamiltonian diffeomorphism.

On the other hand, to the best of our knowledge, beyond \(n = 1\) the question has been completely open prior to now. Surprisingly, the conjecture is not straightforward to prove even for a given Hamiltonian diffeomorphism \(\varphi\) unless, of course, it is periodic, i.e., \(\varphi^k = id\) for some \(k\). The difficulty is present already when \(\varphi\) is as simple as an irrational rotation of \(\mathbb{CP}^n\), i.e., the Hamiltonian diffeomorphisms generated by a quadratic Hamiltonian.

It is also worth pointing out that it is sufficient to prove the existence of one iteration \(k = k(\varphi) > 1\), the first return time, such that \(\varphi^k(L) \cap L \neq 0\) for every \(\varphi\) or at least for the iterates of a fixed map. Then the existence of infinitely many such iterates will follow by replacing \(\varphi\) by \(\varphi^k\) and repeating the process.

To state our main result on Lagrangian Poincaré recurrence, we need to recall several definitions. Let \(U\) be an open subset of a closed, rational, weakly monotone symplectic manifold \(M\). The homological capacity of \(U\) is defined as

$$c_{\text{hom}}(U) = \sup_F \gamma(\varphi_F),$$

where \(F\) ranges through all Hamiltonians \(F\) supported in \(S^1 \times U\); see, e.g., [Sc00, Us10, Vi92] and references therein. This function of \(U\) has all the expected properties of a symplectic capacity and it is a standard fact that

$$c_{\text{hom}}(U) \leq \gamma(\varphi) \text{ when } \varphi(U) \cap U = \emptyset;$$

see, e.g., [Us10, Prop. 3.1]. (In applications, sometimes it is convenient to replace \(\gamma(\varphi_F)\) in (5.6) by \(c_{[M]}(F)\): the resulting capacity has the same properties as the one defined above; see [Gi05].) We extend \(c_{\text{hom}}\) to closed subsets \(L\) of \(M\) (for instance, to Lagrangian submanifolds) by setting

$$c_{\text{hom}}(L) = \inf_U c_{\text{hom}}(U),$$
where the infimum is taken over all open sets \( U \supset L \). Note that \( c_{\text{hom}}(L) \leq \lambda \); see Remark 5.3 and the references therein.

**Example 5.7.** Assume that \( L \subset M \) is a closed Lagrangian admitting a Riemannian metric without contractible closed geodesics. Then

\[
c_{\text{hom}}(L) > 0.
\]

The proof of this well-known fact, which we omit here, is quite standard and implicitly contained in, e.g., the proof of [Us11, Thm. 8.2]. Moreover, the same is true for certain classes of coisotropic manifolds (contact type, stable or with totally geodesic characteristic foliation); see [Gi07, Us11]. Conjecturally, (5.8) holds for all closed Lagrangians, but this, to the best of our understanding, is unknown.

Now we are in a position to state and prove the key result of this section.

**Theorem 5.8.** Let \( \varphi \) be a pseudo-rotation of \( \mathbb{C}P^n \) and let \( L \subset \mathbb{C}P^n \) be a closed subset (e.g., a Lagrangian submanifold). Then there exists a constant \( C > 0 \) and a non-negative integer \( d \leq n \), both depending only on \( \vec{\Delta}(\varphi) \) but not \( L \), such that

\[
\liminf_{k \to \infty} \frac{|\{\ell \leq k \mid \varphi^\ell(L) \cap L \neq \emptyset\}|}{k} \geq C \cdot c_{\text{hom}}(L)^d.
\]

In particular, the limit inferior is positive when \( c_{\text{hom}}(L) > 0 \).

As a consequence, we obtain Theorem 1.3 from the introduction:

**Corollary 5.9.** In the setting of Theorem 5.8, assume that \( c_{\text{hom}}(L) > 0 \) (e.g., \( L \) is as in Example 5.7). Then \( \varphi^{k_i}(L) \cap L \neq \emptyset \) for some sequence \( k_i \to \infty \).

**Remark 5.10 (Multiplicity of Intersections).** In the general framework of the Lagrangian Poincaré recurrence conjecture, we see no reason to expect a lower bound on the number of intersections of \( \varphi^k(L) \) and \( L \) – after all this is a dynamics rather than a symplectic topological question. However, in the setting considered here where the recurrence is a consequence of the \( \gamma \)-convergence, the situation is different. Namely, assume for the sake of simplicity that \( \varphi^k(L) \) and \( L \) are transverse for all \( k \), which is a generic condition on \( L \). Then, in Corollary 5.9, the number of intersections is bounded from below by \( \dim H(L) \), provided that one can replace the upper bound on the Hofer norm in Chekanov’s theorem, [Ch98], by an upper bound on the \( \gamma \)-norm. Some results in this direction have been recently announced in [KS].

**Proof of Theorem 5.8.** Without loss of generality we can assume that \( c_{\text{hom}}(L) > 0 \) – otherwise the assertion is void. Set \( \epsilon = c_{\text{hom}}(L) \); then \( \epsilon \leq \lambda \). Consider the set

\[
K = \{ k \mid \gamma(\varphi^k) < \epsilon \} \subset \mathbb{N}.
\]

Let \( U \) be a neighborhood of \( L \). Clearly, \( c_{\text{hom}}(U) \geq \epsilon \) and, by (5.7),

\[
\varphi^k(U) \cap U \neq \emptyset
\]

for every \( k \in K \). Since this holds for all \( U \supset L \), we have

\[
\varphi^k(L) \cap L \neq \emptyset
\]

for all \( k \in K \) and the result now follows from Theorem 5.1. \( \square \)
Remark 5.11 (Return Frequency for Small Balls). Let $U$ be a small ball of radius $\delta > 0$ in $\mathbb{C}P^n$. Then $\chi_{\text{com}}(U) \geq \text{const} \cdot \delta^2$. Arguing as in the proof of Theorem 5.8, it is easy to see that the return frequency for $\phi$ and $U$ is bounded from below by $C\delta^d$:

$$\lim_{k \to \infty} \left| \left\{ \ell \leq k \mid \phi^\ell(U) \cap U \neq \emptyset \right\} \right| \geq C\delta^d. \quad (5.9)$$

When $d = n$, (5.9) is exactly the lower bound guaranteed by the standard Poincaré recurrence theorem. In general, $d = n + 1 - r$, where $r$ is the number of linearly independent resonance relations which the mean indices $\hat{\mu}(x_i)$ satisfy. Thus (5.9) provides a stronger lower bound when $r \geq 2$.

Remark 5.12. The Lagrangian Poincaré recurrence can also be thought of as a consequence of a hypothetical obstruction to Lagrangian packing in the same way as the standard Poincaré recurrence can be viewed as coming from the volume obstruction to ball packing. For instance, one might conjecture that for a compact symplectic manifold $M$ (possibly with boundary) and a closed Lagrangian $L \subset M$, one can embed into $M$ only a finite number of disjoint Lagrangian submanifolds Hamiltonian isotopic to $L$. We do not have any counterexamples to this more general conjecture; nor are we aware of any results in this direction.

Remark 5.13. It is interesting to compare the Lagrangian Poincaré recurrence conjecture with Arnold’s Legendrian chord conjecture. While at first glance the two questions appear to be similar, there are some fundamental differences. For instance, in many cases the first return time in the chord conjecture is independent of the Legendrian submanifold and completely determined by the Reeb flow (see, e.g., [Mo]), but the first return time in the Lagrangian Poincaré recurrence must, clearly, depend on $L$. (Ultimately, the reason is that small, localized, Legendrian submanifolds have localized chords unrelated to global dynamics. In other words, the chord conjecture is a symplectic topological fact while Lagrangian Poincaré recurrence relates to dynamics.) On a more technical level, Legendrian chords can be treated in the framework of a Floer- or SFT-type homology theory (see, e.g., [Ch02, EGH]), but no such theory for the Lagrangian Poincaré recurrence is known.

5.2. $C^0$-rigidity. It turns out that under suitable additional conditions on the mean index vector $\vec{\Delta}$, $\gamma$-convergence in Corollary 5.2 can be replaced by $C^0$-convergence. For pseudo-rotations in dimension two this is the $C^0$-rigidity established in [Br15b], and our goal is to extend this result to higher dimensions. Let us begin with several preliminary observations.

Consider a compact abelian group $\Gamma$ equipped with some bi-invariant metric. As above we denote the norm of $\vec{\theta} \in \Gamma$, i.e., the distance to the origin, by $||\vec{\theta}||$.

Definition 5.14. A “vector” $\vec{\theta} \in \Gamma$ is exponentially Liouville if for every constant $c > 0$ there exists $k \in \mathbb{N}$ such that $||k\vec{\theta}|| < e^{-ck}$ or, equivalently, a sequence $k_i \to \infty$ satisfying the condition

$$||k_i\vec{\theta}|| < e^{-ck_i}. \quad (5.10)$$

It is clear that when $\Gamma$ is a subgroup of $\Gamma'$, a vector $\vec{\theta} \in \Gamma$ is exponentially Liouville in $\Gamma$ if and only if it is exponentially Liouville in $\Gamma'$. We will apply Definition 5.14 to the mean index vector $\vec{\Delta}$; see (3.8). But first we need to show that, even though exponentially Liouville vectors form a zero measure set, they are generic in the topological sense. The following observation is quite standard:
Proposition 5.15. Exponentially Liouville elements of a compact abelian group $\Gamma$ form a residual set $L$ in $\Gamma$, i.e., $L$ is a countable intersection of open and dense sets.

Proof. Note that it is enough to prove the proposition for the connected component of the identity in $\Gamma$. Thus we can assume that $\Gamma$ is a torus $T^m = \mathbb{R}^m/\mathbb{Z}^m$. (When $\Gamma$ is finite, the assertion is obvious.)

Following the proof of [Br15b, Lemma 10], consider the collection $G_k \subset T^m$ of points of the form $\vec{p}/k$, where $\vec{p}$ is an integer vector and $k \in \mathbb{N}$. This collection is $1/k$-dense in $T^m$ in the obvious sense. Set

$$U_{k,c} = \bigcup_{g \in G_k} \{ \vec{\theta} \in \Gamma \mid \| k\vec{\theta} - g \| < e^{-ck} \},$$

where $c$ and $k$ are positive integers. This is an open set, which is also at least $1/k$-dense. (For instance, $G_k \subset U_{k,c}$.) Thus the union

$$U_c = \bigcup_{k \in \mathbb{N}} U_{k,c}$$

is open and dense, and the intersection

$$L = \bigcap_{c \in \mathbb{N}} U_c$$

comprising the set of exponentially Liouville elements of $\Gamma$ is residual.

One reason we need to consider here a more general group than the torus $T^{n+1}$ is that the components of the mean index vector $\vec{\Delta}$ of a pseudo-rotation $\varphi$ must satisfy mean index resonance relations (see Theorem 3.4 and [GK]) and thus $\vec{\Delta}$ cannot be a generic vector in $T^{n+1}$, i.e., $\vec{\Delta}$ topologically generates a proper subgroup. (If (3.9) is indeed a universal resonance relation this subgroup is contained in the anti-diagonal subtorus $T^n$ in $T^{n+1}$.) Furthermore, one might be interested in further restricting the class of pseudo-rotations and hence of vectors $\vec{\Delta}$.

Theorem 5.16. Let $\varphi = \varphi_H$ be a pseudo-rotation of $\mathbb{C}P^n$ with exponentially Liouville mean index vector $\vec{\Delta}$. Then there exists a sequence $k_i \to \infty$ such that

$$\varphi^{k_i} \xrightarrow{C^0} id.$$

This is Theorem 1.4 from the introduction.

Remark 5.17. The sequence $k_i$ is, of course, exactly the sequence of iterations such that $\| k_i \vec{\Delta} \| < e^{-ck_i}$, where $c$ is completely determined by $\| H \|_{C^2}$. For instance, we can take any $c > 8\| H \|_{C^2}$. Furthermore, the convergence is exponential: $\| \varphi^{k_i} \|_{C^0} \leq e^{-a k_i}$ for some $a > 0$ which can be made arbitrarily large by choosing a large $c$. Finally, note that in this theorem we impose no non-degeneracy requirements on $H$.

Proof. The argument closely follows the proof from [Br15b], although we make several shortcuts and use Floer theory instead of pseudo-holomorphic curves.

Let us first assume that $\varphi$ is strongly non-degenerate, i.e., all its iterates are non-degenerate. It is easy to see that since $\varphi$ is a pseudo-rotation the Floer differential vanishes and hence we can identify the Floer homology of $\varphi$ with the Floer complex. Let $\bar{x}$ and $\bar{y}$ be the capped $k$-periodic orbits representing the fundamental class $[\mathbb{C}P^n]$ and the class of the point $[pt]$ in the Floer complex/homology of $\varphi^k$. Denote
by \( \mathcal{M}(\bar{x}, \bar{y}) \) the moduli space of Floer trajectories \( u: S^1_k \times \mathbb{R} \to \mathbb{CP}^n \), where \( S^1_k = \mathbb{R}/k\mathbb{Z} \), from \( \bar{x} \) to \( \bar{y} \). The image \( U \) of the evaluation map

\[
\mathcal{M}(\bar{x}, \bar{y}) \to \mathbb{CP}^n, \quad u \mapsto u(0, 0)
\]

contains an open and dense subset in \( \mathbb{CP}^n \). (This is true for any closed rational symplectic manifold \( M \) and any non-degenerate Hamiltonian when \( \bar{x} \) and \( \bar{y} \) are replaced by action carriers for \([M]\) and \([pt]\).) This is an immediate consequence of the standard fact that for a generic \( p \in \mathbb{CP}^n \), the number of \( u \in \mathcal{M}(\bar{x}, \bar{y}) \) with \( u(0, 0) = p \), taken with appropriate signs and viewed as an element of \( F \), represents the action of \( [p] = [pt] \in \mathcal{H}_* (\mathbb{CP}^n) \) on \( \mathcal{H}_* (\varphi^k) \) and \([pt] * [\mathbb{CP}^n] = [pt]\); see Section 2.3.

Next, recall that

\[
\| \partial_s u \|_{L^\infty} \leq O(E(u)^{1/4}), \tag{5.11}
\]

where \( s \) is the coordinate on \( \mathbb{R} \), and the energy \( E(u) \) is sufficiently small; see [Sa, Sect. 1.5] or [Br15b] for a simple self-contained proof. The upper bound on the right in (5.11) is uniform in \( k \), and in fact independent of \( k \), and completely determined by the \( C^2 \)-norm of \( H \) and the almost complex structure \( J \). (It is essential here that we view the iterated flow not as \( \varphi^k_H \) but as the flow \( \varphi_t^k \) with \( t \in [0, k] \).

Denote by \( d \) the distance in \( \mathbb{CP}^n \). We claim that

\[
d(p, \varphi^k(p)) \leq CkO(E(u)^{1/4}) \tag{5.12}
\]

for every \( p \in \mathbb{CP}^n \), where we can take any \( C > 2\|H\|_{C^2} \), provided that \( E(u) \) is small enough. To prove this, assume first that \( p \in U \). Set \( z(t) = u(0, t) \) for some \( u \) with \( p = u(0, 0) \) and \( \zeta(t) = (\varphi^t)^{-1}(z(t)) \). Then

\[
\dot{z}(t) = X_H(z(t)) + D\varphi^t(\dot{\zeta}(t)),
\]

and hence

\[
\dot{\zeta}(t) = (D\varphi^t)^{-1}(\dot{z}(t) - X_H(z(t))).
\]

It is easy to see that

\[
\| (D\varphi^t)^{-1} \| \leq C_0 t, \tag{5.13}
\]

where we can take \( C_0 = \|H\|_{C^2} \). Moreover, from the Floer equation and (5.11), we infer that

\[
\| X_H(z(t)) - \dot{z}(t) \| \leq O(E(u)^{1/4}),
\]

for all \( t \in [0, k] \). Therefore,

\[
\| \dot{\zeta}(t) \| \leq C_0 t O(E(u)^{1/4})
\]

and thus

\[
d(\zeta(0), \zeta(k)) \leq C_0 k O(E(u)^{1/4}).
\]

Clearly, \( \zeta(0) = p \) and, since \( z \) is a loop, \( \zeta(k) = \varphi^{-k}(p) \). Applying \( \varphi^k \) to \( \zeta(0) \) and \( \zeta(k) \) and using (5.13) again, we obtain (5.12) with \( C > 2C_0 \). Finally, the upper bound is uniform in \( p \) and \( U \) is dense in \( \mathbb{CP}^n \). Therefore, (5.12) holds on \( \mathbb{CP}^n \).

Next, let us find an upper bound on \( E(u) \). This is where the requirement that \( \hat{A} \) is exponentially Liouville enters the proof and the argument is quite similar to the end of the proof of Theorem 5.1. As in that proof, it is convenient to rescale the symplectic structure on \( \mathbb{CP}^n \) so that \( |\omega| = 2c_1(T\mathbb{CP}^n) \) and normalize \( H \) to ensure that \( S(H) = S_{ind}(\varphi) \). Then, in particular,

\[
\mathcal{A}_{\hat{H}^k}(\bar{x}) = \hat{\mu}(\bar{x}) \quad \text{and} \quad \mathcal{A}_{\hat{H}^k}(\bar{y}) = \hat{\mu}(\bar{y}).
\]
Assume that $k = k_i$ for the sequence $k_i$ from Definition 5.14 with $\tilde{\theta} = \bar{\Delta}$ and a sufficiently large $c$ to be specified later. Then, by (5.10), the spectrum $S(H^{2k}) = S_{ind}(\varphi^k)$ is located in a neighborhood of $2(n + 1)\mathbb{Z}$ of size $e^{-ck}$. Arguing exactly as in the proof of Theorem 5.1, it is easy to show that $\hat{\mu}(\bar{x})$ and $\hat{\mu}(\bar{y})$ both lie in the cluster centered at 0. Thus

$$|\hat{\mu}(x)| \leq e^{-ck} \text{ and } |\hat{\mu}(y)| \leq e^{-ck}$$

and

$$E(u) = A_{H^{2k}}(\bar{x}) - A_{H^{2k}}(\bar{y}) = \hat{\mu}(\bar{x}) - \hat{\mu}(\bar{y}) \leq 2e^{-ck}.$$ 

Set $c > 4C$. (For instance, $C = 2.1\|H\|_{C^2}$ and $c = 8.5\|H\|_{C^2}$.) Then, by (5.12) for $k = k_i$, we have

$$d(p, \varphi^{k_i}(p)) \leq O(e^{(C-c/4)k_i}) \to 0$$

as $k_i \to \infty$. Thus

$$\|\varphi^{k_i}\|_{C^0} \to 0,$$

which proves the theorem when $\varphi$ is strongly non-degenerate.

Dealing with the degenerate case, consider a $C^2$-small non-degenerate perturbation $F$ of $H^{2k}$. As above, let $\bar{x}$ and $\bar{y}$ be the action carriers for $[\mathbb{CP}^n]$ and, respectively, $[pt]$.

**Lemma 5.18.** Assume that $F$ is sufficiently $C^2$-close to $H^{2k}$. Then, for a generic point $p \in \mathbb{CP}^n$, there exists a solution $u$ of the Floer equation for $F$ with $u(0, 0) = p$ such that

$$E(u) \leq 2(A_{H^{2k}}(\bar{x}) - A_{H^{2k}}(\bar{y})). \tag{5.14}$$

**Proof.** We start with several general remarks. Fix $\epsilon > 0$. Then, when $F$ is sufficiently $C^2$-close to $H^{2k}$, every capped $k$-periodic orbit $\bar{z}$ of $H^{2k}$ splits under the perturbation $F$ into several capped orbits $\bar{z}_i$ located near $\bar{z}$ with actions and mean indices $\epsilon$-close to the action and the mean index of $\bar{z}$. All $k$-periodic orbits of $F$ arise in this way. (We view both $F$ and $H^{2k}$ as $k$-periodic Hamiltonians.) Moreover, for two $k$-periodic orbits $\bar{z}$ and $\bar{z}'$ we have

$$A_{H^{2k}}(\bar{z}) > A_{H^{2k}}(\bar{z}') \iff A_F(\bar{z}_i) > A_F(\bar{z}_j)$$

for all (or just one pair of) $i$ and $j$.

Let $\bar{z}$ be the action carrier for some class $\alpha \in HF(H^{2k})$. Then, in the notation from Section 2.4 and, in particular, (2.11), for every cycle $\sigma \in CF_*(F)$ representing $\alpha$,

$$c_{\sigma}(F) \geq c_{\alpha}(H^{2k}) - \epsilon = A_{H^{2k}}(\bar{z}) - \epsilon, \tag{5.15}$$

and there exists a representative $\sigma_{\min}$ such that

$$c_{\sigma_{\min}}(F) \leq c_{\alpha}(H^{2k}) + \epsilon = A_{H^{2k}}(\bar{z}) + \epsilon. \tag{5.16}$$

In particular, at least one of the orbits $\bar{z}_i$ enters $\sigma_{\min}$ with non-zero coefficient.

Let us pick such a cycle $\sigma_{\min}$ representing $[\mathbb{CP}^n]$ for $F$ and satisfying (5.16). Thus

$$c_{\sigma_{\min}}(F) \leq A_{H^{2k}}(\bar{x}) + \epsilon.$$

Next, we pick a generic $p \in \mathbb{CP}^n$ and view it as a cycle representing $[pt] \in HQ_*(\mathbb{CP}^n)$. Then acting by $p$ on $\sigma_{\min}$, we obtain a cycle $P \in CF_{-n}(F)$ also representing $[pt]$. By (5.15),

$$c_P(F) \geq A_{H^{2k}}(\bar{y}) - \epsilon.$$
A point \( \bar{y}_j \) with action \( c_p(F) \) enters the cycle \( P \), and hence there exists a solution \( u \) of the Floer equation for \( F \) connecting a point from \( \sigma_{\text{min}} \) to \( \bar{y}_j \). This solution has energy
\[
E(u) \leq c_{\sigma_{\text{min}}}(F) - c_p(F) \leq A_{H^{1k}}(\bar{x}) - A_{H^{1k}}(\bar{y}) + 2\epsilon
\]
and passes through \( p \), i.e., \( u(0,0) = p \). Making \( \epsilon > 0 \) small enough we obtain (5.14), which completes the proof of the lemma.

Now the proof of the theorem is finished essentially in the same way as in the non-degenerate case. First, note that \( \|F\|_{C^2} \) can be made arbitrarily close to \( \|H^{1k}\|_{C^2} = \|H\|_{C^2} \). (It is again essential here that \( H \) is periodic in time and \( H^{1k} \) is simply the Hamiltonian \( H_t \) with \( t \in S^1 \).) Then (5.11) still holds, where the upper bound on the right is uniform in \( k \) and completely determined by \( \|H\|_{C^2} \).

Next, the bound (5.12) turns into
\[
d(p,\phi_F(p)) \leq e^{Ck}O(E(u)^{1/4})
\]
for a generic \( p \), where we can again take any \( C > 2\|H\|_{C^2} \). By making \( F \) sufficiently close to \( H^{1k} \) we can make sure that \( d(\phi^k(p),\phi_F(p)) \) is arbitrarily small for all \( p \), and hence (5.12) holds in its original form for all \( p \) uniformly in \( k \), provided that \( E(u) \) is small enough (depending on \( \|H\|_{C^2} \)).

Finally, when \( k = k_i \), we have
\[
E(u) \leq 2(A_{H^{1k}}(\bar{x}) - A_{H^{1k}}(\bar{y})) = 2(\hat{\mu}(\bar{x}) - \hat{\mu}(\bar{y})) \leq 4e^{-ck}
\]
by Lemma 5.18 and again \( \|\phi^{k_i}\|_{C^0} \rightarrow 0 \) exponentially fast when \( c > 4C \). \( \square \)

Remark 5.19. As has been pointed out to us by Seyfaddini, one can also derive Theorem 5.16 from Theorem 5.1 and Remark 5.5 by employing a “H"older type” inequality relating the \( C^0 \)-norm, the \( \gamma \)-norm and the \( C^0 \)-norm of the derivative. However, the direct proof given here is of independent interest and may have other applications.

Just as in [Br15b] we have the following corollary of Theorem 5.16:

**Corollary 5.20.** Let \( \varphi \) be a pseudo-rotation of \( \mathbb{C}P^n \) with exponentially Liouville mean index vector. Then \( \varphi \) is not topologically mixing and, in particular, not mixing with respect to the Lebesgue measure.

We conclude this section with several remarks. First, note that in dimension two the \( \gamma \)-norm is continuous with respect to the \( C^0 \)-norm; [Se]. Furthermore, similar results in higher dimensions for symplectically aspherical manifolds and in some other settings have been recently obtained in [BHS]. Thus, for \( S^2 \) and more generally when such continuity is established, Theorem 5.16 implies Corollary 5.2 in the exponentially Liouville case. Finally, we conjecture that a variant of Theorem 5.16 holds without the assumption that \( \Delta \) is exponentially Liouville and in this case one may also have an analog of the frequency bound similar to that in Theorem 5.1.

It is also worth mentioning that strictly speaking the results in [Br15b] are established for exponentially Liouville pseudo-rotations of \( D^2 \) while, for \( n = 1 \), our Theorem 5.16 concerns exponentially Liouville pseudo-rotations of \( S^2 \). The difference in the domains is rather technical than conceptual, although it does effect what symplectic topological tools are better suited for the task (holomorphic curves vs. Floer homology). In any event, the results for \( S^2 \) can be derived from those for \( D^2 \) and vice versa. In one direction, from \( D^2 \) to \( S^2 \), this can be done
by simply applying an oriented real blow-up to one of the fixed points. In the opposite direction, the argument is considerably more subtle and requires more work. The difficulty lies in the fact that a pseudo-rotation of $S^2$ obtained from one of $D^2$ by, e.g., collapsing $\partial D^2$ to a point or doubling $D^2$, is not smooth. However, it is Lipschitz and the proof seems to go through with only minor modifications including extending the resonance relations to this setting. Finally, the requirement in [Br15b] that the pseudo-rotation is irrational is more of terminological than of mathematical nature: a pseudo-rotation of $D^2$ is automatically irrational, for otherwise it would have periodic orbits on $\partial D^2$. One can interpret this requirement as an implicit non-degeneracy condition. Then similarly, it is also automatically satisfied for pseudo-rotations of $S^2$; see Corollary 3.6.

6. Crossing energy and the proof of Theorem 4.2

The proof of Theorem 4.2 hinges on a technical result generalizing [GG14, Thm. 3.1] and asserting, roughly speaking, that the energy of a Floer trajectory asymptotic to $x^k$ and crossing a fixed isolating neighborhood of $x$ is bounded from below by a constant independent of $k$. With future applications in mind we start by proving this result in a form more general than needed for the proof of the theorem.

6.1. Crossing energy. Let $K \subset M$ be a compact invariant set of a Hamiltonian diffeomorphism $\varphi = \varphi_H$ of a symplectic manifold $M$. Recall that $K$ is said to be isolated (as an invariant set) if there exists a neighborhood $U \supset K$ such that for no initial condition $p \in U \setminus K$ the orbit through $p$ is contained in $U$, i.e., there exists $k \in \mathbb{Z}$, possibly depending on $p$, such that $\varphi^k(p) \notin U$. The neighborhood $U$ is called an isolating neighborhood of $K$. Then any neighborhood of $K$ contained in $U$ is also isolating, and hence such neighborhoods can be made arbitrarily small. (Note that a fixed point may be isolated for all iterations as a fixed point while not isolated as an invariant set.)

Consider solutions $u : \Sigma \to M$ of the Floer equation for $H^{2k}$, where $\Sigma \subset \mathbb{R} \times S^1_k$ is a closed domain, i.e., a closed subset with non-empty interior. Note that the period $k$ is not fixed, and the domain $\Sigma$ of $u$ need not be the entire cylinder $\mathbb{R} \times S^1_k$. By definition, the energy of $u$ is

$$E(u) = \int_{\Sigma} \|\partial_s u\|^2 dsdt.$$  

Here $\|\cdot\|$ stands for the norm with respect to $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$, and hence $\|\cdot\|$ also depends on the background almost complex structure $J$. We say that $u$ is asymptotic to $K$ as $s \to \infty$ (or as $s \to -\infty$) if for any neighborhood $V$ of $K$ the domain $\Sigma$ contains a cylinder $[s_V, \infty) \times S^1_k$ (or $(-\infty, s_V] \times S^1_k$) which is mapped into $V$ by $u$.

Finally, let $U$ be a fixed (sufficiently small) isolating neighborhood of $K$. Set $\partial U := U \setminus U$.

Theorem 6.1 (Crossing Energy Theorem). There exists a constant $c_\infty > 0$, independent of $k$ and $\Sigma$, such that for any solution $u$ of the Floer equation with $u(\partial \Sigma) \subset \partial U$ and $\partial \Sigma \neq \emptyset$, which is asymptotic to $K$ as $s \to \infty$ or $s \to -\infty$, we have

$$E(u) > c_\infty. \quad (6.1)$$

Moreover, the constant $c_\infty$ can be chosen to make (6.1) hold for all $k$-periodic almost complex structures (with varying $k$) $C^\infty$-close to $J$ uniformly on $\mathbb{R} \times U$. 

The key point of this result is that the lower bound $c_\infty > 0$ can be taken independent of $k$. The requirements of the theorem are met when $K$ is just one fixed point $x$ of $\varphi$ and $x$ is hyperbolic – this is the setting of [GG14, Thm. 3.1] or more generally when $K = \{x\}$ is isolated as an invariant set as in Theorem 4.2. The requirements are also satisfied when $K$ is a hyperbolic invariant subset (e.g., a Smale’s horseshoe). The condition that $K$ is isolated is essential and cannot be removed; see [GG14, Thm. 3.1]. Also note that periodic orbits of $\varphi'_H$ in $M$ corresponding to the fixed points in $K$ need not be contractible. The proof of Theorem 6.1 follows closely the line of reasoning establishing [GG14, Thm. 3.1]. However, Theorem 6.1 is considerably more general than that result and we feel that giving a detailed proof is justified.

Proof: We will focus on solutions $u$ asymptotic to $K$ as $s \to \infty$. The case of $s \to -\infty$ can be handled in a similar fashion. Before going into details of the proof, let us spell out the idea. Note first that since $\tilde{U}$ is an isolating neighborhood, there exists $T_0 > 1$, an escape time, such that every integral curve $\varphi'_H(p), t \in [-T_0, T_0]$, touching $\partial U$ at some moment $\tau \in [0, 1]$ cannot be entirely contained in $\tilde{U}$. Arguing by contradiction, assume that there exists a sequence of solutions $u_i$ of the Floer equation with $E(u_i) \to 0$ asymptotic to $K$ at $+\infty$ and defined on $\Sigma_i \subset \mathbb{R} \times S^1_{i,0}$. Then for every $T > T_0$ one can also find a sequence of solutions $v_i$ defined on a subset of a rectangle $[-a, a] \times [-T, T] \subset \mathbb{C}$ containing $[-a, a] \times [0, T]$, mapping this half-rectangle into $\tilde{U}$ and such that $v_i(0, \tau_i) \in \partial U$ for some $\tau_i \in [0, 1]$. As $i \to \infty$, these solutions converge to an integral curve $(s, t) \mapsto \varphi'_H(p)$ in the sense of the target-local compactness theorem from [Fi]. This curve is parametrized by $[-T', T']$ with $T > T' > T_0$, touches $\partial U$ at some moment $\tau \in [0, 1]$ and is entirely contained in $\tilde{U}$, which is impossible by the definition of $T_0$.

Throughout the proof, it will be convenient to work in $M \times S^1$ rather than in $M$. Thus let $\tilde{\varphi}'$ be the flow on $M \times S^1$ induced by the isotopy $\varphi'_H$, i.e., $\tilde{\varphi}'(p, \theta) = (\varphi'_H(p), \theta + t \mod 1)$. This is indeed a true flow since $H$ is one-periodic in time. In a similar vein, the map $u$ gives rise to the map

$$\tilde{u}: \Sigma_i \to M \times S^1, \quad (s, t) \mapsto (u(s, t), t \mod 1).$$

Set $\tilde{K} = K \times S^1$, and let $\tilde{B}$ and $\tilde{U}$ be closed isolating neighborhoods of $\tilde{K}$ such that $\tilde{B} \subset \text{int}(\tilde{U})$. (For instance, we can initially take $\tilde{U} = \tilde{U} \times S^1$.)

Arguing by contradiction, assume that there exists a sequence of iterations $k_i \to \infty$, a sequence of $k_i$-periodic almost complex structures $J_i$ on $M$, compatible with $\omega$ and $C^\infty$-converging to $J$ uniformly on $\mathbb{R} \times U$, and a sequence $u_i: \Sigma_i \to U$ of solutions of the Floer equation for $J_i$ and $H^{k_i}$, satisfying the hypotheses of Theorem 6.1 and such that $E(u_i) \to 0$.

To proceed, let us first make several simplifying assumptions. Namely, without loss of generality we can assume that the boundaries $\partial \tilde{U}$ and $\partial \Sigma_i$ are smooth. Indeed, to this end we can shrink $\tilde{U}$ slightly and simultaneously make sure that $\partial \tilde{U}$ is transverse to the maps $\tilde{u}_i$. At this stage we do not need $\tilde{U}$ to be a direct product.

Furthermore, we can assume that $[0, \infty) \times S^1_{k_i}$ is the largest half-cylinder in $\Sigma_i$ mapped by $\tilde{u}_i$ into $\tilde{B}$, i.e., $\tilde{u}_i([0, \infty) \times S^1_{k_i}) \subset \tilde{B}$ and $\tilde{u}_i(\{0\} \times S^1_{k_i})$ touches $\partial \tilde{B}$ at least one point $\tilde{u}_i(0, \tau_i)$ with $0 \leq \tau_i \leq 1$. Here the first assertion readily follows since $H$ is independent of $s$, and hence the Floer equation is translation invariant. As a consequence, we can change $u_i$ by applying a translation in $s$ without affecting the energy. To ensure that $0 \leq \tau_i \leq 1$, we apply an integer translation in $t$ to $u_i$ and
Since the almost complex structures $J_i$ $C^\infty$-converge to $J$ uniformly in $t \in \mathbb{R}$, the same is true for the translated almost complex structures. (This changes the almost complex structures $J_i$ and the solutions $u_i$ by translation, but again does not affect the energy of $u_i$.)

Finally, by passing if necessary to a subsequence, we may assume that the sequence $\tau_i$ converges.

As has been pointed out above, since $\tilde{B}$ is an isolating neighborhood, there exists a constant $T_0 > 1$ (an escape time), depending only on $\tilde{B}$ and $H$, such that no integral curve of $\tilde{\phi}^t$ passing through a point of $\partial\tilde{B}$ (or near $\partial\tilde{B}$) at a moment $\tau \in [0, 1]$ can stay in $\tilde{B}$ for all $t$ with $|t| < T_0$. This observation is the key to the proof and this is where we use the condition that $K$ is isolated.

Next, factoring the universal covering map $\mathbb{C} \to \mathbb{R} \times S^1$ as

$$\mathbb{C} \to \mathbb{R} \times S^1_{k_i} \to \mathbb{R} \times S^1,$$

we lift the domains $\Sigma_i$ to the domains $\hat{\Sigma}_i$ in $\mathbb{C}$ and view the maps $u_i$ as maps $\hat{\Sigma}_i \to M$, which are $k_i$-periodic in $t$. The graph $\Gamma_i$ of $u_i$ is an embedded $\hat{J}_i$-holomorphic curve in $M \times \mathbb{C}$ with respect to an almost complex structure $\hat{J}_i$ which incorporates both $J_i$ and $X_H$. The projection $\pi: M \times \mathbb{C} \to \mathbb{C}$ is holomorphic, and hence so is the projection of $\Gamma_i$ to $\mathbb{C}$. Let $\tau = \lim \tau_i \in [0, 1]$.

Pick arbitrary constants $T > T_0$ and $a > 0$ and set

$$\Pi = [-a, a] \times [-T, T] \subset \mathbb{C}.$$

From now on we will focus on the restrictions $v_i := u_i|_{\Pi}$ and $\hat{v}_i := \hat{u}_i|_{\Pi}$. Let $S_i$ be the graph of $v_i$. Denoting by $\hat{U}_T$ the part of the lift of $\hat{U}$ to the covering $M \times \mathbb{R} \to M \times S^1$ lying over $[-T, T] \subset \mathbb{R}$, we have

$$S_i = \Gamma_i \cap P,$$

where $P = \hat{U}_T \times [-a, a] \subset M \times \mathbb{C}$.

Clearly, $\partial S_i \subset \partial P$ and

$$\text{Area}(S_i) \leq \text{Area}(\Pi) + E(u_i) < \text{const},$$

where the constant on the right is independent of $i$. Let us now shrink $\Pi$ and $\hat{U}$ slightly. To be more precise, set

$$\Pi' = [-a', a'] \times [-T', T'] \subset \Pi,$$

where $0 < a' < a$ and $T_0 < T' < T$, and let $\hat{U}'$ be a closed neighborhood of $K \times S^1$ with boundary close to $\partial\hat{U}$. (More specifically, $\hat{B} \subset \text{int}(\hat{U}')$ and $\hat{U}' \subset \text{int}(\hat{U})$.) We denote the resulting subset of $M \times \mathbb{C}$ by $P'$.

By the target-local Gromov compactness theorem, [Fi, Thm. A], the intersections of the embedded $\hat{J}_i$-holomorphic curves $S_i \cap P'$ Gromov-converge, after passing if necessary to a subsequence, to a (cusp) $\hat{J}$-holomorphic curve $S'$ in $P'$ with boundary in $\partial P'$. This holomorphic curve is a union of multi-sections over subsets of $\Pi'$ and possibly some components contained in the fibers of the projection $\pi: P' \to \Pi'$ (the bubbles). The latter are points since $E(u_i) \to 0$.

Furthermore, $S'$ is in fact a unique section over some subset $D$ of $\Pi'$. One way to see this is to observe that the intersection index of $S'$ with the fiber over a regular point $(s, t)$ of its projection to $\Pi'$ is either one or zero – the intersection index of $S_i$ with the fiber. For instance, the index is one when $(s, t)$ is in the domain of each $v_i$ and the distance from $\hat{v}_i(s, t)$ to $\partial\hat{U}'$ stays bounded away from zero as $i \to \infty$. 


Note also that here we need the parameters \( a' \) and \( T' \) and the neighborhood \( \tilde{U}' \) to be “generic”.

To summarize, \( S' \) is the graph of a solution \( v \) of the Floer equation defined on some connected subset \( D \) of \( \Pi' \). Moreover, as is easy to see from [Fi], after making an arbitrarily small change to \( a' \) and \( T' \) we can ensure that the domain \( D \) of \( v \) has piece-wise smooth boundary. The maps \( u_i \) uniformly converge to \( v \) on compact subsets of int(\( D \)).

The domain \( D \) contains the half-rectangle \( \Pi'_+ = \{ s > 0 \} \cap \Pi' \). Indeed, \( \tilde{u}_i \) maps \( \Pi'_+ \), the \( \{ s \geq 0 \}\)-part of \( \Pi \), into \( B \subset \text{int}(\tilde{U}') \). Hence the projection of \( S'_j \) to \( \Pi \) contains \( \Pi'_+ \) or, in other words, \( \Pi'_+ \) is in the domain of \( v \). Furthermore, \( D \) also contains the closure of \( \Pi'_+ \) and, in particular, the point \((0, \tau)\). In fact, \((0, \tau) \in \text{int}(D)\) since the distance from the points \( \tilde{v}_i(0, \tau) \in B \) to \( \partial\tilde{U}' \) stays bounded away from zero. Let \( \tilde{v} \) be the natural lift of \( v \) to a map to \( \tilde{U}' \). Then

\[
\tilde{v}(0, \tau) = p := \lim \tilde{v}_i(0, \tau_i) \in \partial B.
\]

Since \( E(u_i) \to 0 \), we have \( E(v) = 0 \). Thus \( \partial_n v(s, t) = 0 \) identically on \( D \), and hence \( v(s, t) \) is an integral curve \( \gamma(t) \) of the flow \( \tilde{\varphi}^s \) on \( M \times S^1 \). This integral curve passes through the point \( p \in \partial B \) at the moment \( \tau \), and \( \gamma(t) \in B \) for all \( t \in [-T', T'] \), which is impossible due to our choice of \( T_0 \) and the fact that \( T' > T_0 \). This contradiction completes the proof of the theorem.

\[\square\]

6.2. Proof of Theorem 4.2. With the crossing energy lower bound established, we are now in a position to prove Theorem 4.2. In fact, we prove a slightly more precise result. To state it, recall that the normalized augmented action of a \( k \)-periodic orbit \( y^k \) is simply \( \hat{A}_{H|_{\mathbb{R}^k}}(y^k)/k \), where the augmented action is defined by (3.4). Thus all iterations of an orbit have the same normalized augmented action.

**Theorem 6.2.** Let \( M^{2n} \) be a strictly monotone symplectic manifold. Assume that \( N \geq n + 1 \) and

\[
\alpha \ast \beta = q[M]
\]

in \( \text{HQ}_*(M) \) for some homology classes \( \alpha \in \text{H}_*(M) \) and \( \beta \in \text{H}_*(M) \) with \( |\alpha| < n \) and \( |\beta| < n \). Let \( \varphi_H \) be a Hamiltonian diffeomorphism of \( M \) with a contractible periodic orbit \( x \) which has a neighborhood not intersecting any other periodic orbit of any period, is isolated as an invariant set and such that \( \text{HF}(x^k) \neq 0 \) for all \( k \in \mathbb{N} \). Furthermore, let \( I \) be an arbitrary interval containing \( \hat{A}_H(x) \). Then \( \varphi \) has infinitely many periodic orbits with normalized augmented action in \( \hat{I} \).

Theorem 4.2 readily follows from this result. Some remarks are due before the proof of the theorem.

**Remark 6.3.** The key new point of Theorem 6.2 when compared to Theorem 4.2 is the control of the augmented action. A similar refinement can also be made in [GG14, Thm. 1.1]. Theorem 6.2 and, as a consequence, Theorem 4.2 have analogs when the orbit \( x \) is not contractible. In this case, we need to require \( M \) to be toroidally monotone with \( N \) being the “toroidal” minimal Chern number; cf. [GG14, Rmk. 1.2 and 4.4]. Then the periodic orbits detected in the theorem lie in the free homotopy classes of the iterated orbits \( x^k \), \( k \in \mathbb{N} \).

**Remark 6.4.** As has been pointed out in Section 4, the only monotone manifold with \( N \geq n + 1 \) known to the authors is \( \mathbb{C}P^n \). There are numerous negative monotone
manifolds, simply connected and not, meeting the conditions of the theorem. Although in this case the Conley conjecture holds and \( \varphi \) has infinitely many periodic orbits unconditionally (see [CGG, GG12]), Theorem 6.2 gives additional information about the augmented actions of the orbits or their location and free homotopy classes; cf. [Ba15b, GG16a, Gii13, Or17a, Or17b].

**Proof of Theorem 6.2.** Before giving a detailed argument let us lay out the logic of the proof and explain the main idea. To this end, we assume that \( H \) is strongly non-degenerate and, arguing by contradiction, that it has finitely many periodic orbits; cf. the proof of [GG14, Thm. 1.1]. Then, when \( U \) is small enough, none of these orbits other than \( x \) enter \( U \). Fix an arbitrary capping of \( x \). By adding a constant to \( H \), we can ensure that \( \mathcal{A}_H(\bar{x}) = 0 \). Furthermore, set \( I = (-c, c) \), where \( c > c_\infty \), and pick \( \epsilon < c_\infty \).

One can then show that there exists an arbitrarily large \( k \) such that \( \bar{x}^k \) is not connected to any \( k \)-periodic orbit with action in \( I \) by a solution of the Floer equation of relative index one and that all capped \( k \)-periodic orbits have action in the \( \epsilon \)-neighborhood of \( \lambda \mathbb{Z} \). Acting by \( \alpha \) and \( \beta \) on \( [\bar{x}^k] \in \text{HF}^I_\ast(H^{2k}) \) we find a capped \( k \)-periodic orbit \( \bar{y} \) and two solutions, \( u \) and \( v \), of the Floer equation for \( H^{2k} \), the first of which is connecting \( \bar{x}^k \) to \( \bar{y} \) and the second is from \( \bar{y} \) to \( \bar{x}^k \# A_0 \). Here, as in Section 2.2.1, \( A_0 \) is the generator of \( \Gamma = \pi_2(M)/\ker I_\omega \), and thus \( \langle \omega, A_0 \rangle = \lambda \) and \( \langle c_1(TM), A_0 \rangle = 2N \). The action of \( \bar{y} \) is either in \((-\epsilon, \epsilon)\) or in \((-\epsilon, \epsilon) - \lambda \), and hence either \( E(u) < \epsilon \) or \( E(v) < \epsilon \), which is imposibly by Theorem 6.1 due to the assumption that \( \epsilon < c_\infty \).

Let us turn now to a rigorous proof in complete generality. As above, arguing by contradiction, assume that \( H \) has finitely many periodic orbits with normalized augmented action in \( \bar{I} \). Then, replacing \( H \) by an iterate and making a change of time, we can also assume that \( x \) and all simple periodic orbits of \( H \) with action in \( \bar{I} \) are one-periodic. We denote these orbits by \( x = x_0, x_1, \ldots, x_r \). As above, let us pick an arbitrary capping of \( x \), denote by \( \bar{x} \) the resulting capped orbit and adjust \( H \) so that \( \mathcal{A}_H(\bar{x}) = 0 \).

To further fix notation, set \( a_i = \mathcal{A}_H(x_i) \in S_i^1 \) and \( \Delta_i = \bar{\mu}(x_i) \in S_i^{2N} \). Here again we view the actions and mean indices of the orbits without capping as elements of the circles \( S_i^1 = \mathbb{R}/\mathbb{Z} \) and, respectively, \( S_i^{2N} = \mathbb{R}/2N\mathbb{Z} \). We have \( a_0 = 0 \) due to the normalization of \( H \). Also set

\[ \tilde{a}_i = \bar{\mathcal{A}}_H(x_i) \in \bar{I} \subset \mathbb{R}. \]

Clearly, \( \mathcal{A}_{H^{2k}}(x_0^k) = ka_i \) and \( \bar{\mu}(x_0^k) = k\Delta_i \) and \( \bar{\mathcal{A}}_{H^{2k}}(x_0^k) = k\tilde{a}_i \). Finally, note that without loss of generality we can assume that \( \bar{I} = (\tilde{a}_0 - \eta, \tilde{a}_0 + \eta) \) for some \( \eta > 0 \).

Fix a one-periodic in time almost complex structure \( J^0 \). Let \( U \) be an isolating neighborhood of \( x \) such that, in addition, no periodic orbit of \( \varphi_H \) with normalized augmented action in \( \bar{I} \) other than \( x \) enters \( U \). By Theorem 6.1 applied to \( K = \{x(0)\} \), there exists a constant \( c_\infty > 0 \) such that, for all \( k \in \mathbb{N} \), every non-trivial \( k \)-periodic solution of the Floer equation for the pair \( (H, J) \) asymptotic to \( x^k \) as \( s \to \pm\infty \) has energy greater than \( c_\infty \), where \( J \) is \( k \)-periodic and sufficiently \( C^\infty \)-close to \( J^0 \). (In what follows, such an almost complex structure \( J \) is chosen to be generic and is suppressed in the notation.)

Let \( c > 0 \) be outside the union of the action spectra \( \mathcal{S}(H^{2k}), k \in \mathbb{N} \), and set \( I = (-c, c) \). Next we pick a large constant \( C > 0 \) and a small constant \( \epsilon > 0 \) depending on \( c \), to be specified later. As is easy to show using the Kronecker
theorem, there exists an arbitrarily large \( k \in \mathbb{N} \) such that for all \( i \)

\[
\|ka_i\|_\lambda < \epsilon \tag{6.3}
\]

and

\[
\text{either } a_i = 0 \text{ or } k|a_i - a_0| > C. \tag{6.4}
\]

Here \( \|a\|_b \in [0, b/2] \) stands for the distance from \( a \in S^1_b = \mathbb{R}/b\mathbb{Z} \) to 0. We will also require that

\[
k > C/\eta. \tag{6.5}
\]

**Lemma 6.5.** Assume that \( C > 0 \) is sufficiently large and \( \epsilon > 0 \) is sufficiently small and that \( k \in \mathbb{N} \) satisfies the requirements (6.3), (6.4) and (6.5), and let \( F \) be a \( k \)-periodic, non-degenerate \( C^2 \)-small perturbation of \( H^{\ast k} \). Then, in the notation from Section 2.2.2, \( \text{CF}^* (F, \bar{x}^\#(\ell A_0)) \text{ is a direct summand in } \text{CF}^* (F) \) whenever \( \mathcal{A}_{H^{\ast k}} (\bar{x}^\#(\ell A_0)) \in I \). In particular,

\[
\bigoplus \text{HF}^* (\bar{x}^\#(\ell A_0))
\]

is a direct summand in \( \text{HF}^* (H^{\ast k}) \)

The specific bounds on \( C > 0 \) and \( \epsilon > 0 \) are (6.8), (6.9) and (6.10) below.

**Proof.** Set \( \bar{x}_k = \bar{x}^\#(\ell A_0) \), where \( |\ell| < c/\lambda \) as in the assertion of the lemma, and hence \( \mathcal{A}_{H^{\ast k}} (\bar{x}_k) \in I \). Let \( \bar{y}_k \) be any other capped \( k \)-periodic orbit with action in \( I \). We claim that \( \bar{y}_k \) and \( \bar{x}_k \) can be connected by a solution of the Floer equation only when

\[
|\hat{\mu}(\bar{x}_k) - \hat{\mu}(\bar{y}_k)| > 2n + 1. \tag{6.6}
\]

Once this is established, Lemma 6.5 will follow from Lemma 2.2.

Assume first that \( y_k \) is one of the orbits \( x_k \), e.g., \( y_k = \bar{x}_1 \). Then, by (6.4), there are two cases to consider.

If \( k|\bar{a}_1 - \bar{a}_0| > C \), we have

\[
|\mathcal{A}_{H^{\ast k}} (\bar{x}_k) - \mathcal{A}_{H^{\ast k}} (\bar{y}_k) | + \frac{\lambda}{2N} |\hat{\mu}(\bar{x}_k) - \hat{\mu}(\bar{y}_k) | > C.
\]

Here the first term is bounded from above by \( 2c = |I| \) since both orbits have action in \( I \). Hence, when \( C \) is sufficiently large, we infer that

\[
|\hat{\mu}(\bar{x}_k) - \hat{\mu}(\bar{y}_k) | > 2(n + 1) \tag{6.7}
\]

and (6.6) follows. Note that here it suffices to take

\[
C > \frac{4N}{\lambda} (c + n + 1). \tag{6.8}
\]

In fact, we can make the right-hand side in (6.6) arbitrarily large by taking a sufficiently large \( C \).

The second case is when \( \bar{a}_1 = \bar{a}_0 \). Then

\[
|\mathcal{A}_{H^{\ast k}} (\bar{x}_k) - \mathcal{A}_{H^{\ast k}} (\bar{y}_k) | = \frac{\lambda}{2N} |\hat{\mu}(\bar{x}_k) - \hat{\mu}(\bar{y}_k) |.
\]

Assume that

\[
\epsilon < c_{\infty}. \tag{6.9}
\]

Then, by Theorem 6.1, \( \bar{x}_k \) and \( \bar{y}_k \) can be connected by a solution of the Floer equation only when

\[
|\mathcal{A}_{H^{\ast k}} (\bar{x}_k) - \mathcal{A}_{H^{\ast k}} (\bar{y}_k) | > \epsilon.
\]
This, by the first inequality of (6.3), implies that
\[ |A_{H^2k}(\bar{x}_k) - A_{H^2k}(\bar{y}_k)| > \lambda - \epsilon, \]
and hence
\[ |\hat{\mu}(\bar{x}_k) - \hat{\mu}(\bar{y}_k)| > 2N\frac{\lambda - \epsilon}{\lambda}. \]
Recall that \( N \geq n + 1 \). Therefore, (6.6) holds when \( \epsilon > 0 \) is sufficiently small, e.g., such that
\[ \epsilon < \lambda/2(n + 1). \] (6.10)

Finally, if \( y_k \) is not one of the orbits \( x_t^k \), i.e., its normalized augmented action is not in \( \bar{I} \), we have
\[ |\tilde{A}_{H^2k}(y_k) - \tilde{A}_{H^2k}(x^k)| > \kappa \eta > C \]
by (6.5). Then, exactly as in the first case, we have (6.7) and hence (6.6). \( \square \)

Note also that the last argument shows that for any \( k \)-periodic orbit \( \bar{y}_k \) with action in \( I \), but normalized augmented action outside \( \bar{I} \), we have
\[ |\hat{\mu}(\bar{x}_k) - \hat{\mu}(\bar{y}_k)| \geq O(k) \] (6.11)
where the lower bound on the right is independent of the orbit.

Throughout the rest of the proof, we assume that \( \epsilon > \lambda \), and hence both \( \bar{x}_k \) and \( \bar{x}_k \# A_0 \) have action in \( I \); that \( C > 0 \) is sufficiently large and \( \epsilon > 0 \) is sufficiently small and, in particular, (6.9) holds; and \( k \) satisfying (6.3) and (6.4) is also sufficiently large. As a consequence, the requirements of Lemma 6.5 are met.

Let \( F \) be \( C^2 \)-close to \( H^{2k} \). Then the orbits which \( x^k \) splits into under the perturbation \( F \) lie in the isolating neighborhood \( U \) of \( x \).

Pick a non-zero class
\[ \gamma \in HF_*(\bar{x}_k) \subset HF_*(I)(H^{2k}). \]
Then the class
\[ q\gamma = \Phi_\alpha(\Phi_\beta(\gamma)) \in HF_*(\bar{x}_k \# A_0) \subset HF_*(\tilde{H}^{2k}) \]
is also non-zero. (Here we use (2.10) and (6.2).) Moreover, by Lemma 2.4, the intermediate class \( \Phi_\beta(\gamma) \) is not in \( HF_*(\bar{x}_k) \) or \( HF_*(\bar{x}_k \# A_0) \).

Now, it is a formal algebraic consequence of Lemmas 6.5 and 2.4 that there exists a capped \( k \)-periodic orbit \( \bar{z} \) of \( F \), one of the orbits that \( \bar{x}_k \) splits into, connected by a solution \( u_F \) of the Floer equation to a capped \( k \)-periodic orbit \( \bar{z}_* \), which, in turn, is connected to a closed orbit \( \bar{z}' \# A_0 \) by a solution \( v_F \), where again \( \bar{z}' \) is one of the orbits which \( \bar{x}_k \) splits into. Furthermore, \( z_* \) is not among the orbits arising from \( x \), and \( z_* \) does not enter the neighborhood \( U \) of \( x \).

Passing to the limit as \( F \to H^{2k} \) for a suitably chosen sequence of perturbations and using the target-local compactness theorem from [F] exactly as in the proof of Lemma 2.2, it is easy to show that \( \bar{x}_k \) is connected by a solution of the Floer equation for \( H^{2k} \) to some \( k \)-periodic orbit \( \bar{y}_* \) lying entirely outside \( U \) and such that, by Theorem 6.1,
\[ A_{H^{2k}}(x^k) - c_\infty > A_{H^{2k}}(y_*) \]
By passing if necessary to a subsequence of perturbations \( F \to H^{2k} \), we can also assume that the capped orbits \( \bar{z}_* \) converge to a capped \( k \)-periodic orbit \( \bar{y}_* \) of \( H \).

We claim that, when \( k \) is sufficiently large, the orbit \( y_* \) is necessarily one of the orbits \( x^k_t \). (In other words, \( z_* \) is among the orbits which \( y_* = x^k_t \) splits into
under the perturbation \( F \). Indeed, this follows from (6.11) and the fact that \( \tilde{y}_s \) has action in \( I \); for \( \mu(\tilde{z}_s) = \mu(\tilde{z}) - |\beta| \) and hence
\[
|\hat{\mu}(\tilde{y}_s) - \hat{\mu}(\tilde{x}_k)| \leq 2n + |\beta|.
\]

The orbit \( \tilde{y}_s \) might be different from \( \tilde{y}_+ \), but
\[
A_{H^1}(\tilde{y}_s) \geq A_{H^1}(\tilde{x}_k) = A_{H^1}(\tilde{x}_k^k - A_0) = A_{H^1}(\tilde{x}_k) - \lambda.
\]

If \( c_\infty > \lambda \), we have arrived at a contradiction and the proof is finished. Thus we can assume that \( c_\infty \leq \lambda \). Then, by (6.3), we have
\[
|A_{H^1}(\tilde{y}_s) - A_{H^1}(\tilde{x}_k^k - A_0)| < \epsilon.
\]

However, applying the same argument to the Floer trajectories \( v_F \), we find a \( k \)-periodic orbit \( \tilde{y}_- \) of \( H^1 \) which does not enter \( U \), is connected to \( \tilde{x}_k^k - A_0 \) by a solution of the Floer equation, and such that, again by Theorem 6.1,
\[
A_{H^1}(\tilde{y}_-) > A_{H^1}(\tilde{x}_k^k - A_0) + c_\infty
\]
and
\[
A_{H^1}(\tilde{x}_k^k - A_0) + \epsilon > A_{H^1}(\tilde{y}_s) \geq A_{H^1}(\tilde{y}_-)
\]
which is impossible by (6.9).

This contradiction concludes the proof of the theorem.

\[\square\]

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BG: Department of Mathematics, UCF, Orlando, FL 32816, USA
E-mail address: basak.gurel@ucf.edu

VG: Department of Mathematics, UC Santa Cruz, Santa Cruz, CA 95064, USA
E-mail address: ginzburg@ucsc.edu