Non-Cartesian Thinking

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Abstract

As we typically teach in an introductory mechanics course, choosing a “good” reference frame with convenient axes may present a major simplification to a problem. Additionally, knowing some conserved quantities provides an extremely powerful problem-solving tool. While the former idea is typically discussed in the context of Newton’s Laws, the latter starts with introducing conservation of energy even later.

This work presents an elegant example of implementing both aforementioned ideas in the kinematical context, thus providing a “warm-up” introduction to the standard tools used later on in dynamics. Both the choice of the (non-orthogonal) reference frame and the conserved quantities are rather non-standard, yet at the same time quite intuitive to the problem at hand. Two such problems are discussed in detail with two alternative approaches. The first approach does not even require knowledge of calculus. In the appendix, I also present the brute-force solution involving a coupled system of differential equations. In addition, a few exercises and another similar problem for students’ “homework” are provided at the end.

Introduction

Chase problems were originally considered in the 17th century by Claude Perrault and later studied by Newton, Leibnitz, Huygens and others. In such a problem, there is typically a prey (leader) following a certain given trajectory and a pursuer whose trajectory (curve of pursuit) is to be found based on the specifics of the chasing algorithm. Chase problems have always received attention of physics teachers as an opportunity to illustrate some
concepts of kinematics. In general, the curve of pursuit follows from solving a system of differential equations describing the chase. However, there has been a number of papers (see e.g. [1] and [2]) that explain how, in some simple cases, the solutions can be found by use of just basic calculus, without resorting to differential equations. In this paper, we shall see that some non-trivial questions can even be answered without calculus at all!

In fact, the goal of this paper is mostly pedagogical. It is not about solving a chase problem *per se*, but rather using the rabbit chase as a playground for introducing two important ideas that go through the whole introductory physics sequence, as well as more advanced courses:

- *Choosing a good reference frame can simplify a solution dramatically.* From the very first course on (two- or higher-dimensional) kinematics, we are taught to use Cartesian coordinates as the reference frame. Later on we get familiar with some other coordinates, such as polar (cylindrical) or spherical, which also belong to the class of *orthogonal* coordinate frames. While such orthogonal frames are certainly motivated by the problem at hand, non-orthogonal frames hardly ever get mentioned. In this paper, a non-Cartesian frame arises rather naturally and leads to a major simplification.

- *If there are conserved quantities, it may be a very powerful problem-solving tool.*

Speaking of physics in general, there are two technically different approaches to solving problems: starting with equations of motion, compute the needed physical quantities or avoid solving equations of motion explicitly by using conservations laws. The latter method is based on that there are so-called *integrals of motion* - conserved quantities, like energy, momentum, angular momentum etc. - which can be used to relate provided data to the unknowns of the given problem.

Note that in a traditional introductory physics course, both ideas arise after 2D-Kinematics. In the light of this, the problem explained in the paper may facilitate an early students’ exposure to these fundamental concepts and enable them to think outside of the Cartesian box!

The paper is organized as follows. In the next two sections, two chase problems are solved using a non-trivial coordinate frame and non-trivial conserved quantity. Note that the solutions do not require knowledge of calculus, so any student familiar with concepts of uniform motion, vectors and
basics of relative motion should be able to follow. After the chase problems I provide a few control questions about the solutions, as well as a seemingly unrelated “homework” problem that can be solved using the same approach. The “brute-force” solutions, based on differential equations, are provided in the Appendix.

Problem 1. Rabbit Chase.

A rabbit is running along a straight road with constant speed $v$. A fox is hiding in the bushes at distance $R$ from the road and starts chasing the rabbit when it is passing the point of nearest approach (see Fig 1). The fox is always running at the rabbit with constant speed $u = v$.

To anticipate students’ question as to why the fox would try catching the rabbit in a such inefficient way, you do not really have to mention that the fox never took a basic kinematics course. Foxes (as well as some other predators) really hunt like that, as they do not want to lose sight of their prey, especially in a forest environment. At this point, it could be pertinent to ask the students why this chase is inefficient, whether the fox will catch the rabbit and - once they say “no” - what will happen after a sufficiently long time. When they realize that in the “long run” the fox will end up on the road right behind the rabbit, running with the same speed hence at the same distance from its prey, you can ask the main question of the problem:

*After a long time, what will be the distance between the fox and the rabbit?*

As the fox’s trajectory is highly non-trivial (see Eq. (14) in the Appendix), you might want to discourage your students from trying to figure the shape of
its path out and look for an elegant solution. The solution itself is as follows. First of all, it is convenient to consider the chase in the reference frame associated with the rabbit with the $x$-axis pointing along the rabbit’s path. The second axis, $z$, passes through the fox and the rabbit and makes a variable angle $\theta$ with the $x$-axis (Fig. 2). In this frame, the rabbit is sitting at the origin, while the fox is following some trajectory that starts at $(0, -R)$ and ends somewhere on the $x$-axis. Denote the distance between the fox and the rabbit as $Z$ and their horizontal separation as $X$.

Let us now consider how these two quantities change over a short time interval $\Delta t$. The rabbit is running to increase $X$ at the speed $v$, while the fox is trying to decrease the distance, but not as efficiently so. The fox’s contribution to the decrease of $X$ is related to the horizontal component of its speed ($u \cos \theta$). Thus

\[ \Delta X = (v - u \cos \theta) \Delta t. \]  

(1)

Similarly, during $\Delta t$ the distance between the fox and the rabbit will change by

\[ \Delta Z = (-u + v \cos \theta) \Delta t. \]  

(2)
Since \( u = v \), the righthand sides of these equations are exact opposites of one another. Therefore, the sum \( \Delta X + \Delta Z = 0 \), which implies that the quantity

\[
Z + X = \text{const.}
\]

In the beginning, \( X = 0 \) whereas \( Z = R \), which fixes the constant to be \( R \). After a long time, when the fox is right behind the rabbit, the two axes coincide, hence \( X = Z \), which immediately yields

\[
X_f = Z_f = R/2.
\]  

(3)

**Problem 2. Missile vs Plane.**

No animals were hurt in the previous problem. Let us consider a similar situation:

* A plane is following a straight horizontal trajectory at altitude \( R \) with constant speed \( v \). A missile is launched from the ground when the plane is passing over it. The missile is always flying at the plane with constant speed \( u \) \((u > v)\) (see Fig 3). How long will it take for the missile to hit the plane?

The problem can be solved using the same \((x, z)\)-reference frame with the plane being at the origin. Although the speeds \( v \) and \( u \) are no longer equal. Nevertheless, Eqs. (1) and (2) will still do the trick. Multiplying Eq.(1) by \( v \) and adding Eq.(2) multiplied by \( u \) leads to

\[
\Delta (vX + uZ) = (v^2 - u^2) \Delta t.
\]

Importantly, the coefficient in front of \( \Delta t \) is a constant, so the quantity inside the parenthesis on the lefthand side is changing at a constant rate. Therefore, the *overall* time interval can be found by taking the *overall* change \((v\Delta X + u\Delta Z)\) divided by \((v^2 - u^2)\):

\[
\Delta t = \frac{Ru}{u^2 - v^2},
\]  

(4)

since \( \Delta X = 0 \) and \( \Delta Z = -R \). At this point your students could verify that the answer has the right units. Moreover in the “Fox & Rabbit” limit \( u \to v \), the amount of time becomes infinite, so that the fox never catches the rabbit. Finally, if the missile is very much faster than the plane \( u \gg v \), it is easy to see that the catch up time is simply \( \Delta t = R/u \), as if the plane was not moving at all.
Discussion

The solution did not require any knowledge of calculus. Students should know all the physical and mathematical ingredients of the method by the time they have covered vectors and relative motion. Having students work on such problems should help to facilitate the idea that a reference frame is a tool for describing and analyzing motion, rather than a fixed entity with the horizontal and vertical axes $x$ and $y$. And as a tool, it can be chosen differently and appropriately for each situation. We typically say that the reference frame should be convenient, which, with little to no experience, may sound a bit too abstract to students. In the case at hand, however, the choice is quite intuitive and appealing. Later on, when dealing with Newton’s Laws and especially with objects on inclines (or swinging objects), it should be much more obvious for students that a tilted reference frame would be handy.

Another important idea that students should take home is the concept of conserved quantities. While in this chase problem the constancy of $(v_X + u_Z)$ might look as a mere coincidence and did not have a transparent physical meaning, as a rule conserved quantities do have a physical interpretation. Thus the fox and rabbit problem may become a natural introduction to the later conservation laws for energy, momentum and angular momentum.

There is, of course, a brute-force approach to the problem. That requires solving a coupled system of two differential equations and can be found in the appendix. Remarkably, the equations can be solved completely analytically. The solution also yields the trajectory equation for the fox/missile, which is a highly non-trivial curve. Interestingly, the knowledge of the trajectory is not necessary to answer the questions, which makes the elegant solution
exist in the first place. It may also be curious to offer this problem not only to freshmen (pre-calculus students), but to more senior students who have taken an ODE course.

It would not be a pedagogical paper, if not for the “homework assignment”. In order to reinforce students’ understanding of the solution explained above, you can offer them control questions on the following page. In addition to the control questions, you can challenge your students with a different problem that can be cracked using the same two ideas: a non-Cartesian reference frame and the concept of conserved quantities.
Homework

Control Questions

1) In the “Fox & Rabbit” problem, how will the answer change if
   • the fox starts chasing the rabbit earlier/later: when the rabbit is at distance $R/2$ from the perpendicular?
   • when the fox takes off, the rabbit starts running at angle $\theta$ with respect to the $x$-axis?

2) In the Missile vs. Plane problem, how will the answer change if
   • the missile is launched earlier/later: when the plane is at distance $R/2$ from the perpendicular?
   • when the missile is launched, the plane starts flying at angle $\theta$ with respect to the $x$-axis?

Challenge Problem

A hockey puck on a rough wedge with a coefficient of friction $\mu = \tan \theta$ receives a horizontal kick (see Fig 4). Assume that the wedge is very large and answer the questions below.

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{figure4.png}
  \caption{A puck on an incline.}
\end{figure}

• What will happen after a long time?

• If the initial speed of the puck is $v$, what is its final speed?
Figure 5: Positions of the fox and rabbit at an arbitrary instant of time (big dots) and \( dt \) later (small dots). During the \( dt \), the fox’s trajectory can be considered straight, so that the fox covers \( udt \), while the rabbit moves by \( vdt \). The distance between the fox and the rabbit \( z \) and the angle between the axes \( \theta \) change by \( dz \) and \( d\theta (< 0) \) respectively.

Appendix

In this appendix, we consider the “brute-force” approach to the problem, which amounts to deriving and then solving differential equations. As before, it is convenient to begin consideration in the reference frame associated with the rabbit and introduce the \( x \)- and \( z \)-axes.

Fig. 5 depicts the chase at two infinitesimally close instants of time: \( t \) and \( t + dt \). The corresponding separations between the fox and the rabbit are denoted \( z(t) \equiv z \) and \( z(t + dt) \equiv z + dz \) respectively. Applying the law of sines to the triangle formed by \( z - udt \), \( z + dz \) and the segment \( vdt \) covered by the rabbit during \( dt \), up to higher order terms we obtain

\[
z d\theta = -v \sin \theta dt,
\]

which results in the differential equation for \( \theta \)

\[
\dot{\theta} = -\frac{v \sin \theta}{z}.
\]

(5)

Rewriting Eq. (2) as an explicit differential equation

\[
\dot{z} = v \cos \theta - u,
\]

(6)
we obtain a system of two ODE's for the two unknown functions \( z(t) \) and \( \theta(t) \). The trajectory equation, \( z(\theta) \), can be derived by dividing Eq.(6) by Eq.(5), which results in

\[
\frac{dz}{d\theta} = -z \frac{v \cos \theta - u}{v \sin \theta} \equiv \frac{z}{\sin \theta} (1 - \cos \theta + \delta),
\]

where \( \delta \equiv \frac{u}{v} - 1 \). Using the identity

\[
\frac{1 - \cos \theta}{\sin \theta} \equiv \frac{\sin \theta}{1 + \cos \theta},
\]

Eq.(7) can be easily integrated by separation of variables:

\[
\ln z = -\ln |1 + \cos \theta| + \delta \ln \left| \tan \frac{\theta}{2} \right| + \text{const}.
\]

The constant of integration is fixed by the initial conditions. Specifically at \( t = 0 \), \( z = R \) and \( \theta = \pi/2 \). Thus we arrive at the fox’s trajectory equation in the rabbit’s reference frame

\[
\frac{z}{R} = \left( \frac{\tan \frac{\theta}{2}}{\frac{\sin \theta}{1 + \cos \theta}} \right)^\delta.
\]

Notice that the parameter \( \delta \) shows by how much the chaser is faster than the prey. For instance, \( \delta = 0 \) corresponds to \( u = v \), i.e. the “Fox & Rabbit” problem. The “Missile vs Plane” situation is characterized by a \( \delta > 1 \).

It is now easy to see what kind of a trajectory the fox is following. Setting \( \delta = 0 \) in Eq.(8) yields

\[
z = \frac{R}{1 + \cos \theta},
\]

which is nothing else but a parabola equation written in polar coordinates. In the rabbit’s reference frame, the rabbit sits in the focus of the parabola at \((0, 0)\), while the fox starts off at \((0, -R)\) and runs toward the top of the parabola at \((-R/2, 0)\), as shown in Fig.6. Hence the final distance between the fox and the rabbit is indeed \( R/2 \), as was found earlier.

The chase, however, is an asymptotic process, and it takes an infinite amount of time for the fox to get to the top. It is therefore insightful to consider the time dependence of the relevant coordinates. Substituting \( z(\theta) \) from Eq.(8) back into Eq.(5) and separating the variables \((t and \theta)\) in the resulting equation leads to
Figure 6: In the reference frame associated with the rabbit, the fox is running along a parabolic trajectory from \((0, -R)\) to the top of the parabola at \((-R/2, 0)\). The rabbit is sitting at the focus of the parabola \((0, 0)\).

\[ t = \frac{R}{2v} \left( 1 - \frac{\beta^\delta}{\delta} + \frac{1 - \beta^{\delta+2}}{\delta + 2} \right), \tag{9} \]

where \(\beta = \tan(\theta/2)\). During the chase, \(\theta\) goes from \(\pi/2\) down to 0, which corresponds to \(\beta\) ranging from 1 to 0 respectively. When arriving at the equation above, the constant of integration was fixed by requiring \(t = 0\) at \(\beta = 1\). Note that in terms of \(\beta\) the separation between the fox and the rabbit \(z\) can be conveniently rewritten as

\[ z = \frac{R}{2} \beta^\delta \left( 1 + \beta^2 \right) \tag{10} \]

The last two equations thus describe the (parametric) time dependence of the fox-rabbit distance. Restricting Eq. (9) to \(\delta = 0\) amounts to applying L’Hôpital’s rule to the first term, resulting in

\[ t = \frac{R}{2v} \left( -\ln \beta + \frac{1}{2} \left( 1 - \beta^2 \right) \right), \]
in addition to
\[ z = \frac{R}{2} \left( 1 + \beta^2 \right). \]

It easy to see that after a long time (\( \beta \to 0 \)), the second term in the \( t \)-equation is negligible compared to the first one. This implies that \( z \) approaches its asymptotic value exponentially fast:
\[ \left( z - \frac{R}{2} \right) \propto \exp \left( -\frac{4vt}{R} \right). \]

The complete time dependence of the fox-rabbit distance is displayed in Fig. 7.

We can repeat the same analysis for the “Missile vs Plane” problem. A set of parametric plots \( z(t) \) for several non-zero values of \( \delta \) is presented in Fig. 8a. The rightmost curve (\( \delta = 0.1 \)) corresponds to a missile flying barely faster than the plane, i.e. \( u = 1.1v \). The leftmost curve (\( \delta = 3 \)) describes a very fast missile, whose speed is four times that of the plane. Clearly, all these curves reach \( z = 0 \) in a finite amount of time that follows from Eq. 9.
Figure 8: Missile vs Plane graphs.

(a) Time dependence of the missile-plane distance for (right to left) \( \delta = 0.1, 0.2, 0.5, 1.0, 2.0, \) and 3.0. 

(b) Missile’s trajectory in the plane’s reference frame for (left to right) \( \delta = 0.1, 0.2, 0.5, 1.0, 2.0, \) and 3.0.

Setting \( \beta = 0, \) we obtain

\[
t = \frac{R}{v} \cdot \frac{\delta + 1}{\delta(\delta + 2)} \equiv \frac{Ru}{u^2 - v^2},
\]

which agrees with the earlier result (4).

Possible trajectories of the missile are also of interest. As before, it is most conveniently done in the reference frame associated with the plane. Importantly all such trajectories (for \( \delta > 0 \)) terminate at the origin, as shown in Fig 8b. The leftmost curve corresponds to the slowest missile (\( \delta = 0.1 \)), whereas the rightmost curve describes the fastest case (\( \delta = 3 \)). Remarkably, the trajectory equation (10) allows for an explicit expression of the chaser’s \((x, y)\)-coordinates one in term of the other. In the prey’s reference frame

\[
x = -z \cos \theta, \quad y = -z \sin \theta.
\]

By virtue of the standard trigonometric identities

\[
\cos \theta = \frac{1 - \beta^2}{1 + \beta^2} \quad \text{and} \quad \sin \theta = \frac{2\beta}{1 + \beta^2},
\]
the coordinates take the form
\[ x = -\frac{R}{2} \beta^\delta (1 - \beta^2), \quad y = -R\beta^{\delta+1}. \] (12)

Expressing $\beta$ in terms of $y$ from the second equation and substituting it into the first equation yields the trajectory equation
\begin{align*}
\left| \frac{x}{R} \right| &= \frac{1}{2} \left( \left| \frac{y}{R} \right|^{\frac{\delta}{\tau+1}} - \left| \frac{y}{R} \right|^{\frac{\delta+2}{\tau+1}} \right), \\
&\equiv \frac{1}{2} \left( \left| \frac{y}{R} \right|^{1-\frac{\nu}{u}} - \left| \frac{y}{R} \right|^{1+\frac{\nu}{u}} \right) \quad \text{for} \quad x, y < 0. \tag{13}
\end{align*}

In particular, setting $\delta = 0$ ($u = v$) recovers the parabolic path of the “Fox & Rabbit” problem. If you are looking for a way of discouraging your students to follow the brute force solution and Eq. (13) does not look scary enough, I would suggest to present the chaser’s trajectory equation in the laboratory reference frame (curve of pursuit) $x_{\text{Lab}}(y)$, with $x_{\text{Lab}} = x + vt$, where the time variable can be expressed in terms of $y$ using Eqs. (9) and (12). The curve-of-pursuit equation reads
\begin{align*}
x_{\text{Lab}} = \frac{uv}{u^2 - v^2} + \frac{1}{2} \left( \frac{u}{u+v} \left| \frac{y}{R} \right|^{1+\frac{\nu}{u}} - \frac{u}{u-v} \left| \frac{y}{R} \right|^{1-\frac{\nu}{u}} \right) \quad \text{for} \quad y < 0. \tag{14}
\end{align*}

References

[1] Carl E. Mungan, A classic chase problem solved from a physics perspective, Eur. J. Phys. 26 (2005) 985-990

[2] O. I. Chashchina and Z. K. Silagadze, The dog-and-rabbit chase problem as an exercise in introductory kinematics, Lat. Am. J. Phys. Educ. 3 (2009) 539-543