WALL CROSSING OF THE MODULI SPACES OF PERVERSE COHERENT SHEAVES ON A BLOW-UP.

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Abstract. We give a remark on the wall crossing behavior of perverse coherent sheaves on a blow-up \cite{Nakajima20} and stability condition \cite{Toda21}. 

0. Introduction

Let \( X \) be a smooth projective surface over an algebraically closed field \( k \) of characteristic 0. For \( (\beta, \omega) \in \text{NS}(X)_k \times \text{Amp}(X)_k \), Arcara and Bertram \cite{ArcaraBertram1} constructed stability conditions \( \sigma_{(\beta, \omega)} \) such that the structure sheaves of points \( k_x \ (x \in X) \) are stable of phase 1. In \cite{Toda22, Toda23}, Toda constructed new examples of stability conditions \( \sigma_{(\beta, \omega)} \) which is regarded to an extension of Arcara and Bertram’s examples to non-ample \( \omega \). Moreover Toda showed that new examples are related to stability conditions on blow-downs of \((-1)\)-curves on \( X \). Assume that \( \omega \) is close to ample in \( \text{NS}(X)_k \). Then \( \sigma_{(\beta, \omega)} \) is related to stability condition on a blow-down \( \pi: X \to Y \) of a \((-1)\)-curve \( C \) of \( X \). In this case, we can set \( \omega := \pi^*(L) + tC \), where \( L \in \text{Amp}(Y) \) and \( |t| \) is sufficiently small. If \( t < 0 \), then \( \sigma_{(\beta, \omega)} \) is an example of Arcara and Bertram \cite{ArcaraBertram1}. If \( t = 0 \), then instead of using a torsion pair of \( \text{Coh}(X) \), \( \sigma_{(0, \omega)} \) is constructed by using a similar torsion pair of a category of perverse coherent sheaves \( -1 \text{Per}(X/Y) \) \cite{Toda22}. Since \( k_x \ (x \in C) \) becomes reducible in \( -1 \text{Per}(X/Y) \), \( k_x \) is properly \( \sigma_{(0, \omega)} \)-semi-stable. If \( t > 0 \), then \( k_x \ (x \in C) \) is not \( \sigma_{(0, \omega)} \)-semi-stable. In this case, \( L \pi^*(k_y) \) is \( \sigma_{(0, \omega)} \)-stable and \( \sigma_{(0, \omega)} \)-semi-stable objects are parameterized by \( Y \).

In this note, we shall give examples of Bridgeland semi-stable objects on \( X \). In \cite{Nakajima20} and \cite{Toda21}, Nakajima and the author studied the relation of moduli spaces of Gieseker semi-stable sheaves on \( Y \) and \( X \) by looking at the wall crossing behavior in the category of perverse coherent sheaves. As expected from Toda’s papers and also the relation with the Gieseker semi-stability in the large volume limit, we shall show that Gieseker type semi-stability of perverse coherent sheaves corresponds to Bridgeland semi-stability in the large volume limit. Then we shall explain our wall crossing behavior of the moduli of perverse coherent sheaves in terms of Bridgeland stability condition (subsection 5.3).

For the proof of our results, we also need to study the large volume limit where \( \omega \) is ample. As is proved by Bridgeland \cite{Bridgeland6} and Lo and Qin \cite{LoQin14}, Bridgeland’s stability \( \sigma_{(\beta, \omega)} \) is related to (twisted) Gieseker stability if \( (\omega^2) \) is sufficiently large (Proposition 6.9). On the other hand if an object \( E \in \text{D}(X) \) satisfies \( (c_1(E) - \text{rk} E_{\beta_0, \omega_0}) = 0 \), then \( \sigma_{(\beta_0, \omega_0)} \)-stability of \( E \) is related to \( \mu \)-stability and the moduli space is related to the Uhlenbeck compactification of the moduli of \( \mu \)-stable locally free sheaves. By looking at the wall and chamber structure near \( \sigma_{(\beta_0, \omega_0)} \), we shall show that each adjacent chamber of \( (\beta_0, \omega_0) \) contains a point at the large volume limit, which implies that each chamber corresponds to Gieseker semi-stability. Thus we may say that Bridgeland stability unifies (twisted) Gieseker stabilities and the \( \mu \)-stability. We explain the usual wall crossing of moduli spaces of Gieseker semi-stable sheaves \cite{ArcaraBertram7, ArcaraBertram8, ArcaraBertram10} and also those of Uhlenbeck compactifications \cite{Uhlenbeck10} in terms of Bridgeland stability conditions. The blow-up case is a slight generalization of this consideration.

During preparation of this note, Bertram and Martinez informed us they also studied the wall crossing of twisted Gieseker stability by using Bridgeland stability, and get the same result of section 5.2 \cite{BertramMartinez}.

1. Background materials

1.1. Stability conditions on surfaces. Let \( X \) be a smooth projective surface over an algebraically closed field \( k \) of characteristic 0. As in Mukai lattice on abelian surfaces, let us introduce a bilinear form on \( H^*(X, \mathbb{Q})_{\text{alg}} := \mathbb{Q} \oplus \text{NS}(X)_{\mathbb{Q}} \oplus \mathbb{Q} \). For \( x = (x_0, x_1, x_2), y = (y_0, y_1, y_2) \in H^*(X, \mathbb{Q})_{\text{alg}} \), we set

\[
(x, y) := (x_1, y_1) - x_0 y_2 - x_2 y_0 \in \mathbb{Q}.
\]
Let $g_X := (0, 0, 1)$ be the fundamental class of $X$. We also use the notation $x = x_0 + x_1 + x_0g_X$ to denote $x = (x_0, x_1, x_2)$. For $x = (x_0, x_1, x_2)$, we set
\begin{align}
\text{rk } x &:= x_0, \\
c_1(x) &:= x_1.
\end{align}
For $E \in \mathcal{D}(X)$, we set $v(E) := \text{ch}(E) \in H^*(X, \mathbb{Q})$. Since
\begin{equation}
\Delta(E) := c_2(E) - \frac{\text{rk } E - 1}{2 \text{rk } E} (c_1(E)^2)
\end{equation}
the Bogomolov inequality is the following.

**Lemma 1.1.** Assume that $X$ is defined over a field of characteristic 0. Let $E$ be a $\mu$-semi-stable torsion free sheaf. Then $(v(E))^2 = 2(\text{rk } E)\Delta(E) \geq 0$.

Let $L$ be an ample divisor on $X$.
\[\mathcal{P}^+(X) := \{x \in \text{NS}(X) \mid (x^2) > 0, (x, L) > 0\}\]
denotes the positive cone of $X$ and $C^+(X)$ denotes $\mathcal{P}^+(X)/\mathbb{R}_{>0}$.

For a stability condition $\sigma$, $Z_\sigma : \mathcal{D}(X) \rightarrow \mathbb{C}$ is the central charge and $\mathcal{A}_\sigma$ is the abelian category generated by $\sigma$-stable objects $E$ with the phase $\phi_\sigma(E) \in (0, 1]$. Then $\sigma$ consists of the pair $(Z_\sigma, \mathcal{A}_\sigma)$ of a central charge $Z_\sigma : \mathcal{D}(X) \rightarrow \mathbb{C}$ and an abelian category $\mathcal{A}_\sigma$. Let $\text{Stab}(X)$ be the space of stability conditions. We have a map $\Pi : \text{Stab}(X) \rightarrow H^*(X, \mathbb{C})$ such that $(\Pi(\sigma), *) = (Z_\sigma(*))$. If $\sigma$ satisfies the support property, then $\Pi$ is locally isomorphic.

**1.2. Perverse coherent sheaves.** Let $\pi : X \rightarrow Y$ be a birational morphism of projective surfaces such that $X$ is smooth, $Y$ is normal and $R^1\pi_*\mathcal{O}_X = 0$. The notion of perverse coherent sheaves was introduced by Bridgeland [3]. Let us briefly recall a slightly different formulation of perverse coherent sheaves in [27]. Let $G$ be a locally free sheaf on $X$ which is a local projective generator of a category of perverse coherent sheaves $\mathcal{C}$ (27). Thus
\begin{align}
T := & \{ E \in \text{Coh}(X) \mid R^1\pi_* (G^\vee \otimes E) = 0\} \\
S := & \{ E \in \text{Coh}(X) \mid \pi_* (G^\vee \otimes E) = 0\},
\end{align}
is a torsion pair $(T, S)$ of $\text{Coh}(X)$, $G \in T$ and
\begin{equation}
\mathcal{C} = \{ E \in \mathcal{D}(X) \mid H^i(E) = 0, i \neq -1, 0, H^{-1}(E) \in S, \text{H}^0(E) \in T\}
\end{equation}
is the heart of a bounded t-structure of $\mathcal{D}(X)$. For $E \in \mathcal{D}(X)$, $p^H_i(E) \in \mathcal{C}$ denotes the $i$-th cohomology of $E$ with respect to the t-structure. We take a divisor $H$ on $X$ which is the pull-back of an ample divisor on $Y$. By using a twisted Hilbert polynomial $\chi(G, E(nH))$ of $E \in \mathcal{C}$, we define the dimension and the torsion freeness of $E$, which depend only on the category $\mathcal{C}$. We also define a $G$-twisted semi-stability of $E \in \mathcal{C}$ as in the Gieseker semi-stability [27 Prop. 1.4.3]. Then we have the following.

**Proposition 1.2 ([21], [27] Prop. 1.4.3).** There is a coarse moduli scheme $M^G_H(v)$ of $S$-equivalence classes of $\gamma$-semi-stable objects $E$ with $v(E) = v$.

**Definition 1.3.** For $\gamma \in \text{NS}(X) \cap \mathbb{Q}$ such that there is a local projective generator $G$ of $\mathcal{C}$ with $\gamma = c_1(G)/\text{rk } G$, we also define $\gamma$-twisted semi-stability as $G$-twisted semi-stability.

(1) We denote the moduli stack of $\gamma$-twisted semi-stable objects $E$ with $v(E) = v$ by $\mathcal{M}^G_H(v)$. It is a quotient stack of a scheme by a group action (see the construction in [27 Prop. 1.4.3]). $\mathcal{M}^G_H(v)^*$ denotes the open substack of $\mathcal{M}^G_H(v)$ consisting of $\gamma$-stable objects.

(2) $M^G_H(v)$ denotes the coarse moduli scheme of $S$-equivalence classes of $\gamma$-twisted semi-stable objects $E$ with $v(E) = v$.

**Remark 1.4.** If $\mathcal{C} = \text{Coh}(X)$, then every locally free sheaf is a local projective generator. Hence any $\gamma \in \text{NS}(X) \cap \mathbb{Q}$ is expressed as $\frac{c_1(G)}{\text{rk } G} = \gamma$ for a locally free sheaf $G$. $\gamma$-twisted semi-stability depends on the equivalence class $\gamma \mod \mathbb{Q}H$ and coincides with the twisted semi-stability of Matsuki and Wentworth [16].

**Definition 1.5.** By using the slope function $(c_1(E), H)/\text{rk } E$, we also define the $\mu$-semi-stability of a torsion free object $E$ of $\mathcal{C}$. $\mathcal{M}^H(v)^{\mu-ss}$ denotes the moduli stack of $\mu$-semi-stable objects $E$ with $v(E) = v$.

**Remark 1.6.** (1) $\mu$-semi-stability depends only on the category $\mathcal{C}$ and $H$.

(2) $\mathcal{M}^H(v)^{\mu-ss}$ is bounded.
Lemma 1.7. Assume that $G$ satisfies
\[
\frac{\chi(G)}{\text{rk } G} = e^{\frac{c_1(G)}{2}} + \left(\frac{1}{8}(K_X^2) - \chi(O_X)\right)g_X.
\]
Then
\[
\frac{\chi(G, E)}{\text{rk } G} = -(e^\beta, v(E)),
\]
where
\[
\beta - \frac{1}{2}K_X = \frac{c_1(G)}{\text{rk } G}.
\]

Proof. For $E \in \mathcal{D}(X)$, we have
\[
\chi(e^{\beta - \frac{1}{2}K_X}, E) = \int_X \left\{ e^{-\beta}e^{\frac{1}{2}K_X} \text{ch}(E) \left( 1 - \frac{1}{2}K_X + \chi(O_X)g_X \right) \right\}
\]
\[
= \int_X e^{-\beta} \text{ch}(E) \left( 1 + (\chi(O_X) - \frac{1}{8}(K_X^2))g_X \right)
\]
\[
= \int_X (e^{-\beta} \text{ch}(E)) - \text{rk } E \left( \frac{1}{8}(K_X^2) - \chi(O_X) \right)
\]
\[
= -(e^\beta, v(E)) - \text{rk } E \left( \frac{1}{8}(K_X^2) - \chi(O_X) \right).
\]
Hence the claim holds. \qed

1.3. Stability condition associated to $\mathcal{C}$. For the birational map $\pi : X \to Y$ in Lemma 1.2, let $H$ be the pull-back of an ample divisor on $Y$. For $\beta \in \text{NS}(X)_\mathbb{Q}$ in Lemma 1.7 and $\omega \in \mathbb{R}_{>0}H$, we set
\[
Z_{(\beta, \omega)}(E) := (e^{\beta + \omega\sqrt{-1}}, v(E)), \ E \in \mathcal{D}(X).
\]

For $E \in \mathcal{D}(X)$, we can write $v(E)$ as
\[
v(E) = e^\beta (r(E) + a_\beta(E)g_X + d_\beta(E)H + D_\beta(E)), \ D_\beta(E) \in H^1
\]
\[
= r(E)e^\beta + a_\beta(E)g_X + d_\beta(E)H + D_\beta(E) + (D_\beta(E)H + D_\beta(E), \beta)g_X,
\]
where $r(E) = \text{rk } E$ is the rank of $E$ and
\[
d_\beta(E) = \frac{(c_1(E) - r(E)\beta, H)}{(H^2)}, \ a_\beta(E) = -(e^\beta, v(E)).
\]

Then we have
\[
Z_{(\beta, \omega)}(E) = -a_\beta(E) + r(E)\frac{(\omega^2)}{2} + d_\beta(E)(H, \omega)\sqrt{-1}.
\]
In appendix, we shall prove the following inequality.

Lemma 1.8. For a $\mu$-semi-stable object $E$ of $\mathcal{C}$,
\[
(v(E)^2) - (D_\beta(E)^2) = d_\beta(E)^2(H^2) - 2\text{rk } E a_\beta(E) \geq 0.
\]

Definition 1.9. (1) $\mathcal{T}_\beta$ is the subcategory of $\mathcal{C}$ generated by torsion objects and $\mu$-stable objects $E$ with $d_\beta(E) > 0$.
(2) $\mathcal{F}_\beta$ is the subcategory of $\mathcal{C}$ generated by $\mu$-stable objects $E$ with $d_\beta(E) \leq 0$.

Then $(\mathcal{T}_\beta, \mathcal{F}_\beta)$ is a torsion pair of $\mathcal{C}$.

Definition 1.10. Let $\mathcal{A}_{(\beta, \omega)}$ be the tilting of the torsion pair $(\mathcal{T}_\beta, \mathcal{F}_\beta)$ of Coh$(X)$:
\[
\mathcal{A}_{(\beta, \omega)} = \{ E \in \mathcal{D}(X) \mid H^i(E) = 0, i \neq -1, 0, H^{-1}(E) \in \mathcal{F}_\beta, H^0(E) \in \mathcal{T}_\beta \}.
\]
By Lemma 1.8, $\sigma_{(\beta, \omega)} := (\mathcal{A}_{(\beta, \omega)}, Z_{(\beta, \omega)})$ is an example of stability condition. However we do not know whether $\sigma_{(\beta, \omega)}$ satisfies the support property in general. If $\pi$ is an isomorphism, a blow-up of a smooth point or the minimal resolution of rational double points, then the usual Bogomolov inequality holds and the support property holds (see subsection 2.3).

Remark 1.11. By Lemma 1.7 a torsion free object $E \in \mathcal{C}$ is $(\beta - \frac{1}{2}K_X)$-twisted semi-stable with respect to $\omega$ if and only if
\[
(i) \ \frac{d_\beta(F)}{\text{rk } F} < \frac{d_\beta(E)}{\text{rk } E} \text{ or } (ii) \ \frac{d_\beta(F)}{\text{rk } F} = \frac{d_\beta(E)}{\text{rk } E} \text{ and } \frac{a_\beta(F)}{\text{rk } F} \leq \frac{a_\beta(E)}{\text{rk } E}
\]
for all non-zero subobject $F$ of $E$.\]
Lemma 2.1. Some properties of perverse coherent sheaves.

Assume that $G$ is a locally free sheaf on $Y$. By [21, sect. 2.4]. We set

$$\chi(G, E) := \frac{\chi(O, E)}{\chi(G)} = -\langle e^\beta, v(E) \rangle.$$

Then

$$\chi(G, E) = \chi(O, E).$$

Assume that $(\beta, C) \notin \frac{1}{2} + \mathbb{Z}$. We take an integer $l$ satisfying $l - \frac{1}{2} < (\beta, C) < l + \frac{1}{2}$. Then there is a locally free sheaf $G$ satisfying (2.1) and

$$\text{Ext}^1(G, O_C(l)) = \text{Hom}(G, O_C(l - 1)) = 0$$

by [27, sect. 2.4]. We set

$$T := \{ F \in \text{Coh}(X) \mid R^1\pi_*(G^\vee \otimes F) = 0 \}$$

and

$$S := \{ F \in \text{Coh}(X) \mid \pi_*(G^\vee \otimes F) = 0 \}.$$ 

Then $(T, S)$ is a torsion pair of Coh$(X)$. Let $\mathcal{C}^\beta$ be a category of perverse coherent sheaves associated to $(T, S)$. Then $G$ is a local projective generator of $\mathcal{C}^\beta$. It is easy to see that $\mathcal{C}^\beta(lC)$ is the category of perverse coherent sheaves $^{-1}$ Per$(X/Y)$ defined by Bridgeland [7] and studied in [20], [21].

We set $G_1 := G(lC)$. By (2.4), $G_1|C \cong O_C^r \oplus O_C(-1)^{\oplus r-k}$, where $(c_1(G_1), C) = r-k$. Then we have an exact sequence

$$0 \to E \to G_1 \to O_C(-1)^{\oplus r-k} \to 0$$

such that $E|C \cong O_C^{r-k}$. Since $E$ is the pull-back of a locally free sheaf on $Y$, $\pi_*(G_1^\vee) \cong \pi_*(E)$ is a locally free sheaf on $Y$. By taking the dual of (2.5), we have an exact sequence

$$0 \to G_1 \to \pi^*(G_2) \to O_C^{\oplus r-k} \to 0,$$

where $\pi^*(G_2) = E^\vee$. Thus $G_1$ is an elementary transform of the pull-back of a locally free sheaf $G_2$ on $Y$. Conversely starting from a locally free sheaf $G_2$ on $Y$, we can construct a local projective generator $G_1$.

2. Some properties of perverse coherent sheaves. We recall some results on stable perverse coherent sheaves in [21].

Lemma 2.1. Assume that $l - \frac{1}{2} < (\beta, C) < l + \frac{1}{2}$, that is, $\mathcal{C}^\beta(lC) = -1$ Per$(X/Y)$. For a torsion free object $E$ of $\mathcal{C}^\beta$, there is an exact sequence

$$0 \to E \to E' \to T \to 0$$

in $\mathcal{C}^\beta$ such that $T$ is 0-dimensional, $E'(lC)$ is the pull-back of a locally free sheaf on $Y$.

Proof. By Lemma 6.1 in Appendix, there is an exact sequence

$$0 \to E \to E' \to T \to 0$$

such that $T$ is 0-dimensional, $E'$ is torsion free and $\text{Ext}^1(A, E') = 0$ for all 0-dimensional object $A$ of $\mathcal{C}^\beta$. Let $A$ be an irreducible object of $\mathcal{C}^\beta$. Then $A = O_C(l), A = O_C(l - 1)[1]$ or $k_x (x \in X \setminus C)$ (cf. [27, Lem. 1.2.16, Prop. 1.2.23]). We have an exact sequence

$$0 \to O_C(l - 1)^{\oplus 2} \to O_C(l) \to O_C(l - 2)[1] \to 0$$

in $\mathcal{C}^\beta(K_X)$. Hence $\text{Ext}^1(E', O_C(l)) = 0$. Since

$$\text{Ext}^1(E', O_C(l - 1)[1]) = \text{Hom}(O_C(l - 1)(-K_X), E')^\vee = \text{Hom}(O_C(l), E')^\vee$$

and $E'$ is torsion free, we also have $\text{Ext}^1(E', O_C(l - 1)[1]) = 0$. Therefore $E'$ is a local projective object of $\mathcal{C}^\beta$ ([27, Rem. 1.1.35]). In particular $E'$ is locally free. Then $\text{Ext}^1(E', O_C(l - 1)) = 0$ implies that $E'(lC) \cong O_C(l)^{\oplus r}$, which implies that $E'(lC)$ is the pull-back of a locally free sheaf on $Y$. \hfill \Box

Definition 2.2. Assume that $l - \frac{1}{2} < (\beta, C) < l + \frac{1}{2}$. Since $\mathcal{C}^\beta$ depends only on $l$, we denote this category by $\mathcal{C}$. We also denote the moduli stack of $\mu$-semi-stable objects by $\mathcal{M}_H^{-1/2K_X}(v)^{\mu-ss}$ or $\mathcal{M}_H^{-1/2}(v)^{\mu-ss}$ in order to indicate the category where the stability is defined (cf. Remark 1.0).
Remark 2.3. Assume that \( \gcd(r, (c_1(v), H)) = 1 \). Then \( E \in \mathcal{C} \) is \( \beta \)-twisted semi-stable if and only if \( \pi_*(E)^\vee \) is stable. In particular it is independent of the choice of \( \beta \) satisfying \( -\frac{1}{2} + l < (\beta, C) < \frac{1}{2} + l \).

We shall consider the relation with the \( \mu \)-semi-stability of torsion free sheaves.

Definition 2.4. A torsion free sheaf \( E \) is \( \mu \)-semi-stable with respect to \( H \), if

\[
\frac{\langle c_1(E_1), H \rangle}{\text{rk} E_1} \leq \frac{\langle c_1(E), H \rangle}{\text{rk} E}
\]

for any subsheaf \( E_1 \neq 0 \) of \( E \).

The set of \( \mu \)-semi-stable sheaves of a fixed Mukai vector is bounded (see [25 Lem. 2.2, sect. 2.3]). If a \( \mu \)-semi-stable perverse coherent sheaf \( E \) is torsion free in \( \text{Coh}(X) \), then it is a \( \mu \)-semi-stable sheaf. By the proof of [21 Prop. 3.37], we have the following lemma.

Lemma 2.5. For \( m \gg 0 \), \( \mathcal{M}_H^{\beta+mC-\frac{1}{2}K_X} (v)^{\mu\text{-ss}} \) consists of \( \mu \)-semi-stable torsion free sheaves with respect to \( H \).

Proposition 2.6. For \( m \gg 0 \), \( \mathcal{M}_H^{\beta+mC-\frac{1}{2}K_X} (v) = \mathcal{M}_H^{\beta-\frac{1}{2}K_X} (v) \), where \( q > 0 \) is sufficiently small.

Proof. We set \( \beta_m := \beta + mC \). Let \( \mathcal{M}_H(v)^{\mu\text{-ss}} \) be the moduli stack of \( \mu \)-semi-stable sheaves with respect to \( H \). By Lemma 2.5, we may assume that \( \mathcal{M}_H^{\beta-\frac{1}{2}K_X} (v) \subset \mathcal{M}_H(v)^{\mu\text{-ss}} \). We choose a sufficiently small \( q > 0 \) such that \( \mathcal{M}_H^{\beta-\frac{1}{2}K_X} (v) \subset \mathcal{M}_H(v)^{\mu\text{-ss}} \). We consider the set \( T \) of pairs \((E_1, E)\) such that

(i) \( E_1 \) is a subsheaf of a \( \mu \)-semi-stable sheaf \( E \) with respect to \( H \),
(ii) \( \frac{\langle c_1(E_1), H \rangle}{\text{rk} E_1} = \frac{\langle c_1(E), H \rangle}{\text{rk} E} \)

and

(iii) \( E/E_1 \) is torsion free.

Since \( (v(E_1)^2) \geq 0 \) and \( (v(E/E_1)^2) \geq 0 \), we see that \( \{v(E_1) \mid (E_1, E) \in T\} \) is a finite set. Since

\[
\frac{\chi(E(-\beta_m + \frac{1}{2}K_X))}{\text{rk} E} = \frac{\chi(E_1(-\beta_m + \frac{1}{2}K_X))}{\text{rk} E_1}
\]

we can take a sufficiently large \( m \) such that

\[
\frac{\chi(E_1(p(H - qC) - (\beta - \frac{1}{2}K_X)))}{\text{rk} E_1} \leq \frac{\chi(E(p(H - qC) - (\beta - \frac{1}{2}K_X)))}{\text{rk} E}, \quad p \gg 0
\]

if and only if

\[
\frac{\chi(E_1(-\beta_m + \frac{1}{2}K_X))}{\text{rk} E_1} \leq \frac{\chi(E(-\beta_m + \frac{1}{2}K_X))}{\text{rk} E}
\]

for \((E_1, E) \in T\). If \( E \) is a \((\beta_m - \frac{1}{2}K_X)\)-twisted semi-stable object of \( \mathcal{C}^{\beta_m} \), then for \((E_1, E) \in T\), we take a subsheaf \( E'_1 \) of \( E_1 \) such that \( E'_1 \in \mathcal{C}^{\beta_m} \) and \( E_1/E'_1 \in \mathcal{C}^{\beta_m}[-1] \). Then \( E'_1 \) is a subsheaf of \( E \) such that \( \chi(E'_1(-\beta_m + \frac{1}{2}K_X)) \geq \chi(E_1(-\beta_m + \frac{1}{2}K_X)) \). Hence

\[
\frac{\chi(E_1(-\beta_m + \frac{1}{2}K_X))}{\text{rk} E_1} \leq \frac{\chi(E'_1(-\beta_m + \frac{1}{2}K_X))}{\text{rk} E'_1} \leq \frac{\chi(E(-\beta_m + \frac{1}{2}K_X))}{\text{rk} E}
\]

which shows

\[
\frac{\chi(E_1(p(H - qC) - (\beta - \frac{1}{2}K_X)))}{\text{rk} E_1} \leq \frac{\chi(E(p(H - qC) - (\beta - \frac{1}{2}K_X)))}{\text{rk} E}, \quad p \gg 0.
\]

Moreover if the equality holds in \([2.15] \), then we see that \( E_1 = E'_1 \). Hence \( E \) is \((\beta - \frac{1}{2}K_X)\)-semi-stable with respect to \( H - qC \).

Conversely for \( E \in \mathcal{M}_H^{\beta-\frac{1}{2}K_X} (v) \), we may assume that \( \text{Hom}(E(-(m+l)C), \mathcal{O}_C(-1)) = 0 \). Hence \( E \) is an object of \( \mathcal{C}^{\beta_m} \). Then we see that \( E \) is a \((\beta_m - \frac{1}{2}K_X)\)-semi-stable object. \( \square \)

Bogomolov inequality for perverse coherent sheaves is a consequence of [21]. There is another proof in [22].

Lemma 2.7 (Bogomolov inequality). For a \( \mu \)-semi-stable object \( E \) of \( \mathcal{C}^{\beta_m} \), \( (v(E)^2) \geq 0 \).
Proof. Assume that $E \in \mathcal{M}_{H}^{\beta-\frac{1}{2}K_{X}}(v)^{\mu-ss}$. Take an integer such that $-m - \frac{1}{2} < (\beta, C) < -m + \frac{1}{2}$. By the proof of [21 Prop. 3.15], we have an exact sequence

\begin{equation}
0 \rightarrow \mathcal{O}_{C}(-m - 1)^{\oplus n} \rightarrow E \rightarrow E' \rightarrow 0
\end{equation}

such that $E' \in \mathcal{M}_{H}^{\beta-\frac{1}{2}K_{X}}(v')^{\mu-ss} \cap \mathcal{M}_{H}^{\beta-\frac{1}{2}K_{X} + C}(v')^{\mu-ss}$, $v' + nv(\mathcal{O}_{C}(-m - 1)) = v$ and

$$n \leq \dim \text{Ext}^{1}(E', \mathcal{O}_{C}(-m - 1)).$$

Since $\chi(E', \mathcal{O}_{C}(-m - 1)) = -\dim \text{Ext}^{1}(E', \mathcal{O}_{C}(-m - 1))$, we have $(c_{1}(E'), C) + rm \geq n$. Hence

\begin{equation}
(v^{2}) = (v'^{2}) + 2n((c_{1}(v'), C) + (\frac{1}{2} + m)r) - n^{2}
\end{equation}

$$\geq (v'^{2}) + n^{2} + nr > (v^{2}).$$

By the induction on $m$, the claim follows from Lemma 2.3. $\Box$

Lemma 2.8 ([21 Lem. 3.2]). Assume that $0 < (\beta - \frac{1}{2}K_{X}, C) < 1$. Then $-(c_{1}(v), C) \leq 0$ if $\mathcal{M}_{H}^{\beta-\frac{1}{2}K_{X}}(v)^{\mu-ss} \neq \emptyset$.

Proposition 2.9 (cf. [21 Prop. 3.3]). Assume that $(c_{1}(v), C) = 0$. Then $\mathcal{M}_{H}^{\beta-\frac{1}{2}K_{X}}(v) \cong \mathcal{M}_{H}^{\beta'-\frac{1}{2}K_{Y}}(v')$, where $\pi^{*}(v') = v$, $0 < (\beta - \frac{1}{2}K_{X}, C) < 1$, $H' \in \text{Amp}(Y)$ satisfies $\pi^{*}(H') = H$ and $\beta'$ is the NS($Y$)-component of $\beta$.

Proof. Under the assumption, $\pi^{*}(\pi_{*}(E)) \rightarrow E$ is an isomorphism by the proof of [21 Prop. 3.3]. Let $G$ be a local projective generator of $\mathfrak{c}^{0}$ with $c_{1}(G_{0}) = \beta - \frac{1}{2}K_{X}$. Then there is a locally free sheaf $G_{0}$ on $Y$ and an exact sequence

$$0 \rightarrow \pi^{*}(G_{0}) \rightarrow G' \rightarrow \mathcal{O}_{C}(-1)^{\oplus p} \rightarrow 0,$$

where $p = (c_{1}(G), C)$. Then $c_{1}(G_{0}) = \beta - \frac{1}{2}K_{Y}$ and $G' = \mathbb{R}\pi_{*}(G_{0}) = \mathbb{R}\pi_{*}(G')$. By the projection formula, $\pi_{1}(G' \otimes E) = G'_{0} \oplus \pi_{*}(E)$. Hence the stability coincides. $\Box$

2.3. Moduli spaces of perverse coherent sheaves on a categorical wall. We shall study the case where $\beta$ satisfies $(\beta, C) = l - \frac{1}{2}$. In this case, $(e^{\beta}, v(\mathcal{O}_{C}(l - 1))) = 0$ and we cannot consider a category of perverse coherent sheaves. We shall use the same idea in [21 sect. 3.7] to construct a moduli space of semi-stable perverse coherent sheaves.

We take a locally free sheaf $G$ such that $\chi(G, E)/\text{rk } G = -(e^{\beta}, v(E))$ and $\text{Hom}(G, \mathcal{O}_{C}(l - 1)[i]) = 0$ for all $i \in \mathbb{Z}$.

Definition 2.10. $E \in \mathfrak{c}^{l-1}$ with $\text{rk } E > 0$ is $(\beta - \frac{1}{2}K_{X})$-twisted semi-stable, if

\begin{equation}
\chi(G, E_{1}(nH)) \leq \text{rk } E_{1} \frac{\chi(G, E(nH))}{\text{rk } E}
\end{equation}

for all subobject $E_{1}$ of $E$ in $\mathfrak{c}^{l-1}$. If the inequality is strict for any non-trivial subobject $E_{1}$ of $E$, then $E$ is $(\beta - \frac{1}{2}K_{X})$-twisted stable.

Let $\mathcal{M}_{H}^{\beta-\frac{1}{2}K_{X}}(v)$ be the moduli stack of $\beta$-twisted semi-stable objects $E$ with $v(E) = v$.

Remark 2.11. If $E_{1} \in \mathfrak{c}^{l-1}$ is a subobject of $E$ with $\text{rk } E_{1} = 0$, then $\chi(G, E_{1}(nH)) = 0$ for all $n$. Then $E_{1} \cong \mathcal{O}_{C}(l - 1)^{\oplus k}$. In particular we see $E \in \text{Coh}(X)$ by $\chi(G, (H^{-1}(E)[1])(nH)) = 0$.

Remark 2.12. If $E$ is $\beta$-twisted stable, then $\text{Hom}(\mathcal{O}_{C}(l - 1)) = \text{Hom}(\mathcal{O}_{C}(l - 1), E) = 0$. In particular $E \in \mathfrak{c}^{l}$. Moreover $E$ is a torsion free object of $\mathfrak{c}^{l}$ and $\mathfrak{c}^{l-1}$. We also have an exact sequence

\begin{equation}
0 \rightarrow \mathcal{O}_{C}(-1)^{\oplus N} \rightarrow \pi^{*}(\pi_{*}(E(lC))) \rightarrow E(lC) \rightarrow 0.
\end{equation}

Moreover $\text{Hom}(\mathcal{O}_{C}(-1), \pi^{*}(\pi_{*}(E(lC)))) \cong \mathbb{C}^{\oplus N}$. Hence $E$ is determined by $\pi_{*}(E(lC))$.

We take $\beta_{-} \in \text{NS}(X)_{\mathbb{Q}}$ which is sufficiently close to $\beta$ and $l - \frac{4}{2} < (\beta_{-}, C) < l - \frac{1}{2}$, and let $G_{-}$ be a local projective generator of $\mathfrak{c}^{l-1}$ with $\beta_{-} - \frac{1}{2}K_{X} = c_{1}(G_{-})/\text{rk } G_{-}$. For $E \in \mathcal{M}_{H}^{\beta-\frac{1}{2}K_{X}}(v)$, we have the Harder-Narasimhan filtration with respect to $G_{-}$:

\begin{equation}
0 \subset F_{0} \subset F_{1} \subset \cdots \subset F_{s} = E.
\end{equation}

Thus $E_{i} := F_{i}/F_{i-1}$ $(i \geq 0)$ satisfy the following.

(i) $F_{0} = \mathcal{O}_{C}(l - 1)^{\oplus k}$ and $E_{i}$ $(i \geq 1)$ are $(\beta_{-} - \frac{1}{2}K_{X})$-semi-stable objects,

(ii) $\frac{\chi(G_{-}, E_{1}(nH))}{\text{rk } E_{1}} > \frac{\chi(G_{-}, E_{2}(nH))}{\text{rk } E_{2}} > \cdots > \frac{\chi(G_{-}, E_{s}(nH))}{\text{rk } E_{s}}$, $n \gg 0$. and
Lemma 2.15. We shall introduce a scheme structure on this Brill-Noether locus as in [21, Prop. 3.31].

\[
\chi(G, E_1(nH)) = \chi(G, E_2(nH)) = \cdots = \chi(G, E_s(nH)), \quad n \gg 0.
\]

If \(E\) is \(\beta\)-twisted stable, then \(F_0 = 0\) and \(s = 1\). Moreover \(E\) is \((\beta - \frac{1}{2}K_X)\)-twisted stable.

We take \(\beta^+\) which is sufficiently close to \(\beta\) and \(\beta^+ + \beta = 2\beta\). Then \(l - \frac{1}{2} < (\beta^+, C) < l + \frac{1}{2}\). Let \(G_+\) be a local projective generator of \(\mathcal{E}^l\) with \(\beta^+ - \frac{1}{2}K_X = \frac{c_{\mathcal{O}_X}}{\text{rk}G_+}\). Then we have a Harder-Narasimhan type filtration with respect to \(G_+\):

\[
(2.21) \quad 0 \subset F_1 \subset F_2 \subset \cdots \subset F_t = E.
\]

Thus \(E_i := F_i/F_{i-1} (i = 1, 2, \ldots, t)\) satisfy

(i) \(E_i (i = 1, 2, \ldots, t - 1)\) are \((\beta^+ - \frac{1}{2}K_X)\)-twisted semi-stable objects of \(\mathcal{E}^l\) and \(E_t = \mathcal{O}_C(l - 1)^{\oplus k}\),

(ii) \(\frac{\chi(G_+, E_i(nH))}{\text{rk}E_i} > \frac{\chi(G_+, E_{i+1}(nH))}{\text{rk}E_{i+1}} > \cdots > \frac{\chi(G_+, E_{t-1}(nH))}{\text{rk}E_{t-1}}, \quad n \gg 0\)

and

(iii) \(\frac{\chi(G, E_1(nH))}{\text{rk}E_1} = \frac{\chi(G, E_2(nH))}{\text{rk}E_2} = \cdots = \frac{\chi(G, E_t(nH))}{\text{rk}E_t}, \quad n \gg 0\).

Remark 2.13. \(E_t = \text{im}(E \to \mathcal{O}_C(l - 1) \otimes \text{Hom}(E, \mathcal{O}_C(l - 1))^\vee)\).

If \(E\) is \((\beta - \frac{1}{2}K_X)\)-twisted stable, then \(F_t/F_{t-1} = 0\) and \(t = 2\). Moreover \(E\) is \((\beta^+ - \frac{1}{2}K_X)\)-twisted stable.

Lemma 2.14. \(\mathcal{M}^{\beta - \frac{1}{2}K_X} (v)^s = \mathcal{M}^{\beta - \frac{1}{2}K_X} (v) \cap \mathcal{M}^{\beta^+ - \frac{1}{2}K_X} (v)^s\).

Proof. We already know that

\[
\mathcal{M}^{\beta - \frac{1}{2}K_X} (v)^s \subset \mathcal{M}^{\beta - \frac{1}{2}K_X} (v) \cap \mathcal{M}^{\beta^+ - \frac{1}{2}K_X} (v)^s.
\]

Assume that \(E \in \mathcal{M}^{\beta - \frac{1}{2}K_X} (v)^s \cap \mathcal{M}^{\beta^+ - \frac{1}{2}K_X} (v)^s\). Then \(E \in \mathcal{E}^l\) and \(\text{Hom}(\mathcal{O}_C(l - 1), E) = 0\). If \(E\) is not \((\beta - \frac{1}{2}K_X)\)-twisted stable, then there is a subobject \(E_1\) of \(E\) in \(\mathcal{E}^{l-1}\) with

\[
(2.22) \quad \chi(G, E_1(nH)) \geq \text{rk}E_1 \frac{\chi(G, E(nH))}{\text{rk}E}.
\]

We have an exact sequence

\[
0 \to E' \to E_1 \to \mathcal{O}_C(l - 1)^\oplus q \to 0
\]

such that \(E_1' \in \mathcal{E}^l\). Then \(E_1'\) is a subobject of \(E\) in \(\mathcal{E}^l\) and \(\mathcal{E}^{l-1}\) with \(\text{rk}E' = \text{rk}E_1\) and \(\chi(G, E'_1(nH)) = \chi(G, E_1(nH))\). By \(\text{Hom}(\mathcal{O}_C(l - 1), E') = 0\), \(E_1' \neq 0\). Then the \(G_{\pm}\)-twisted stability implies

\[
(2.23) \quad \chi(G_{\pm}, E'_1(nH)) < \text{rk}E_1' \frac{\chi(G_{\pm}, E(nH))}{\text{rk}E},
\]

which shows

\[
(2.24) \quad \chi(G, E_1(nH)) < \text{rk}E_1 \frac{\chi(G, E(nH))}{\text{rk}E}.
\]

Therefore \(E\) is \((\beta - \frac{1}{2}K_X)\)-twisted stable.

We shall explain a construction of the moduli space. There is a locally free sheaf \(G_0\) on \(Y\) such that \(\pi^*(G_0)(-lC) \cong G\). Then \(E \in \mathcal{E}^{l-1}\) is \((\beta - \frac{1}{2}K_X)\)-twisted semi-stable if and only if \(\mathbf{R}^s\pi_*(E(lC))\) is \(G_0\)-twisted semi-stable. Moreover if \(E\) is \((\beta - \frac{1}{2}K_X)\)-twisted stable, then \(\pi_*(E(lC))\) is \((\beta - \frac{1}{2}K_Y)\)-twisted stable, where \(\beta' - \frac{1}{2}K_Y = \frac{c_{\mathcal{O}_Y}}{\text{rk}G_0}\). We have a morphism

\[
(2.25) \quad \phi : \mathcal{M}^{\beta - \frac{1}{2}K_X} (v) \to \mathcal{M}^{\beta' - \frac{1}{2}K_Y} (w),
\]

where \(w = v(\pi_*(E(lC)))\). If \(E\) is \(S\)-equivalent to \(\oplus_{i=0}^s E_i\) such that \(E_0 = \mathcal{O}_C(l - 1)^{\oplus p}\) and \(E_i (i > 0)\) are \((\beta - \frac{1}{2}K_X)\)-twisted stable objects with \(\text{rk}E_i > 0\). Then \(\pi_*(E(lC))\) is \(S\)-equivalent to \(\oplus_{i=0}^s \pi_*(E_i(lC))\) and \(\pi_*(E_i(lC))\) are \((\beta' - \frac{1}{2}K_Y)\)-twisted stable. By Remark 2.12, \(\oplus_{i=0}^s E_i\) is uniquely determined by \(\pi_*(E(lC))\).

Since \(pC = c_1(E) - \sum_{i=1}^s c_1(E_i), E_0\) is also uniquely determined.

We set \(k := r((\beta, C) - l)\) and

\[
(2.26) \quad \mathcal{M}^{\beta' - \frac{1}{2}K_Y} (w, k) := \{ F \in \mathcal{M}^{\beta' - \frac{1}{2}K_Y} (w) \mid \text{dim} \text{Hom}(\mathcal{O}_C(-1), \pi_* F) \geq k \}
\]

We shall introduce a scheme structure on this Brill-Noether locus as in [21, Prop. 3.31].

Lemma 2.15. The image of \(\phi\) is \(\mathcal{M}^{\beta' - \frac{1}{2}K_Y} (w, k)\).
\[ \text{Lemma 2.18} \]
For any \( E \in M_{H}^{\beta - \frac{1}{2}K_X} (v) \), we have an exact sequence
\[ (2.27) \quad 0 \to E'(IC) \to E(IC) \to \mathcal{O}_C(-1)^{\oplus q} \to 0 \]
where \( E' \in \mathcal{C} \). Then \( \pi_*(E(IC)) = \pi_*(E'(IC)) \) and
\[ \dim \text{Hom}(\mathcal{O}_C(-1), \pi^*(\pi_*(E'(IC)))) \geq \dim \text{Ext}^1(E'(IC), \mathcal{O}_C(-1)) \]
\[ = -\chi(E'(IC), \mathcal{O}_C(-1)) = q + k. \]
Hence \( \phi(E) \in M_{H}^{\beta - \frac{1}{2}K_Y} (w, k) \). Conversely for \( E' \in M_{H}^{\beta - \frac{1}{2}K_Y} (w, k) \), we take a \( k \)-dimensional subspace \( V \) of \( \text{Hom}(\mathcal{O}_C(-1), \pi^*(F)) \). Then \( E(IC) := \text{coker}(\mathcal{O}_C(-1) \otimes V \to \pi^*(F)) \in M_{H}^{\beta - \frac{1}{2}K_X} (v) \) and \( \phi(E) = F \). \( \square \)

\[ \text{Definition 2.16.} \]
We define the moduli scheme of the \( S \)-equivalence classes of \( (\beta - \frac{1}{2}K_X) \)-twisted semi-stable objects by \( M_{H}^{\beta - \frac{1}{2}K_X} (v) := M_{H}^{\beta - \frac{1}{2}K_Y} (w, k) \).

Since \( M_{H}^{\beta - \frac{1}{2}K_X} (v) \subset M_{H}^{\beta - \frac{1}{2}K_X} (v) \), we have the following.

\[ \text{Proposition 2.17.} \]
There are projective morphisms \( M_{H}^{\beta - \frac{1}{2}K_X} (v) \to M_{H}^{\beta - \frac{1}{2}K_X} (v) \).

\[ \text{2.4. Support property.} \]
Let us recall Bogomolov type inequality and the support property in \[22, 23\]. In these paper, Toda states results under the assumption \( \beta = 0 \). Obviously the same results hold for general cases.

\[ \text{Lemma 2.18 (}[22 \text{ Lem. 3.20}]\). \]
There is a constant \( C_\omega \) depending only on the class \( \omega \in C^+ (X) \) such that
\[ (2.29) \quad (c_1 (E)^2) (\omega^2) + C_\omega (c_1 (E), \omega)^2 \geq 0 \]
for all purely 1-dimensional objects \( E \in \mathcal{C} \).

\[ \text{Proof.} \]
If \( \omega \) is ample, then it is nothing but \[22 \text{ Lem. 3.20}]\). So we assume that \( \omega \in \pi^*(\text{Amp}(X)) \). If \( l - \frac{1}{2} < (\beta, C) < l + \frac{1}{2} \), then we have \( \mathcal{C} \beta (IC) = -1 \text{ Per}(X/Y) \) and \( c_1 (E) = c_1 (E(IC)) \). Hence the claim also follows from \[22 \text{ Lem. 3.20}]\). \( \square \)

Then the Bogomolov type inequality \[22 \text{ Cor. 3.24}]\) holds.

\[ \text{Lemma 2.19.} \]
For any \( \sigma_{(\beta, \omega)} \)-stable object \( E \in \mathcal{A}_{(\beta, \omega)} \),
\[ (v(E))^2 + C_\omega (c_1 (E(-\beta)), \omega)^2 \geq -1. \]

As in \[22 \text{ sect. 3.7}]\), there is a constant \( A \) depending on \( C_\omega, (\omega^2), \frac{1}{a_\beta (\mathcal{O}_C(l))} \) and \( \frac{1}{1 - a_\beta (\mathcal{O}_C(l))} \) such that
\[ \max \{ |r|, |a|, |d(\omega^2)|, \sqrt{(-\eta^2)} \} \leq A \]
for all \( \sigma_{(\beta, \omega)} \)-stable objects \( E \), where \( v(E) = e^\beta (v + d\omega + \eta + a \xi), \eta \in \omega^\perp \) and \( l - \frac{1}{2} < (\beta, C) < l + \frac{1}{2} \). More precisely, if \( E \notin \mathcal{O}_C (l), C_\mathcal{O}_C (l - 1) \{ 1 \}, \) then we have a bound \( A \) depending only on \( C_\omega \) and \( (\omega^2) \) as in \[22 \text{ sect. 3.7}]\). If \( E = \mathcal{O}_C (l), C_\mathcal{O}_C (l - 1) \{ 1 \}, \) then \( \frac{1}{a_\beta (\mathcal{O}_C(l))} \) and \( \frac{1}{1 - a_\beta (\mathcal{O}_C(l))} \) appear.

In particular, we have the following support property.

\[ \text{Lemma 2.20 (}[22 \text{ Prop. 3.13}]\). \]
Let \( B \) be a compact subset of \( \text{NS}(X) \times \text{Amp}(X) \) Then there is a constant \( C_B \) such that for \( (\beta, \omega) \in B \) satisfying \( \beta \in \text{NS}(X) \eta_2, \omega \in \mathbb{R}_{>0} H, H \in \text{Amp}(X) \eta_2 \) and \( \sigma_{(\beta, \omega)} \)-stable object \( E \),
\[ (2.32) \quad \frac{\| v(E) \|}{\| Z_{(\beta, \omega)} (E) \|} \leq C_B, \]
where \( \| v(E) \| \) is a fixed norm on \( H^* (X, \mathbb{R}) \text{alg.} \) Assume that \( B \) is a compact subset of \( \text{NS}(X) \times \pi^* (\text{Amp}(Y) \mathbb{R}) \) such that \( (\beta, \omega) \in B \) satisfies \( (\beta, C) \notin \frac{1}{2} + \mathbb{Z} \). Then \[23] also holds.

Let \( (\beta, \omega) \) satisfy the conditions in \[22 \text{ Lemma 2.20} \] By the support property, we have a wall/chamber structure in a neighborhood \( U \) of \( \sigma_{(\beta, \omega)} \). Thus
\[ (2.33) \quad \{ \sigma \in U \mid Z_{(\beta, \omega)} (E)/Z_{(\beta, \omega)} (E_1) \in \mathbb{R}_{>0}, E_1 \subset E \in \mathcal{A}, E, E_1 \text{ are } \sigma \text{-semi-stable}, (v(E) = v) \}
\]
is the set of walls for \( v \) and a connected component of the complement is a chamber in \( U \).

\[ \text{Remark 2.21.} \]
Assume that \( \text{NS}(X) = \mathbb{Z} H \). Then the Bogomolov inequality holds for any semi-stable object of \( \mathcal{A}_{(\beta, \omega)} \). As in \[13 \text{ Prop. 5.7}]\), if \( v_1 \) defines a wall, then we have \( (v_1^2) \geq 0 \), \( (v_2^2) \geq 0 \) and \( (v_1, v_2) > 0 \), where \( v_2 = v - v_1 \). Since we don’t know the sufficient condition for the existence of stable objects \( E_i \) with \( v(E_i) = v_1 \), the condition is only a necessary condition.
We set
\[ \mathcal{K}(X) := \{ (\beta, \omega) \mid \beta \in \text{NS}(X)_\mathbb{R}, \omega \in \text{Amp}(X)_\mathbb{R} \} \]
\[ \partial \mathcal{K}(X) := \{ (\beta, \omega) \mid \beta \in \text{NS}(X)_\mathbb{R}, \omega \in \pi^*(\text{Amp}(Y)_\mathbb{R}), (\beta, C) \notin \frac{1}{2} + \mathbb{Z} \} \]
\[ \overline{\mathcal{K}}(X) := \mathcal{K}(X) \cup \partial \mathcal{K}(X). \]

We have a natural embedding \( \overline{\mathcal{K}}(X) \to \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R} \).

**Proposition 2.22** ([23]). We have a map \( s : \overline{\mathcal{K}}(X) \to \text{Stab}(X) \) such that \( Z_s(\beta, \omega) = e^{\beta + \sqrt{-1} \omega} \) and \( s(\beta, \omega) = \sigma(\beta, \omega)(\beta, \omega) \) if \( \beta \in \text{NS}(X)_\mathbb{Q} \) and \( \omega \in \mathbb{R} \) and \( H \in \text{NS}(X) \).

For the proof of this result, we also need the following. Then the same proof of [23] works.

**Lemma 2.23** ([22] Prop. 3.14). Assume that \( (\beta_0, \omega_0) \in \partial \mathcal{K}(X) \). Let \( U \) be an open neighborhood of \( \sigma_0 := \sigma(\beta_0, \omega_0) \) in \( \text{Stab}(X) \). We set \( V := \{ \sigma \mid \Pi(\sigma) = e^{\beta + \sqrt{-1} \omega}, (\omega, C) \geq 0 \} \). Then there is an open neighborhood \( U' \) of \( \sigma_0 \) such that \( M_\sigma(g_X) = \{ k_x \mid x \in X \} \) for all \( \sigma \in U' \cap V \).

**Proof.** For \( E \in M_{\sigma_0}(g_X), \phi_{\sigma_0}(E) = 1 \). We first classify \( \sigma_0 \)-stable objects \( E \) with \( \phi_{\sigma_0}(E) = 1 \). Since \( \phi_{\sigma_0}(pH^{-1}(E)[1]) \leq 1, \phi_{\sigma_0}(pH^0(E)) = 1 \). Hence \( pH^0(E) = 0 \) or \( pH^{-1}(E) = 0 \). In the first case, \( 0 = -r^k E = r^k pH^{-1}(E) > 0 \). Hence this case does not occur. For the second case, \( E = pH^0(E) \) satisfies \( r^k pH^0(E) = (1, E, \omega) = 0 \). Hence \( E \) is a 0-dimensional object of \( \mathcal{O}^0 \). If \( \text{Supp}(E) \subset X \setminus C \), then \( E \cong k_x, x \in X \setminus C \). Assume that \( \text{Supp}(E) \subset C \). We may assume \( \mathcal{O}^0(iC) = -1 \text{Per}(X/Y) \). By the classification of 0-dimensional objects, \( E \) is generated by \( \mathcal{O}_C(l) \) and \( \mathcal{O}_C(l-1)[1] \). Thus \( E = \mathcal{O}_C(l) \) or \( E = \mathcal{O}_C(l-1)[1] \). Then we see that \( g_X = v(\mathcal{O}_C(l)) + v(\mathcal{O}_C(l-1)[1]) \). \( M_{\sigma_0}(g_X) \) consists of \( E \) such that

(i) \( E \) is a non-trivial extensions
\[ 0 \to \mathcal{O}_C(l) \to E \to \mathcal{O}_C(l-1)[1] \to 0 \]

or

(ii) \( 0 \to \mathcal{O}_C(l-1)[1] \to E \to \mathcal{O}_C(l) \to 0 \)

or

(iii) \( E = \mathcal{O}_C(l) \oplus \mathcal{O}_C(l-1)[1] \).

We may assume that \( \mathcal{O}_C(l) \) and \( \mathcal{O}_C(l-1)[1] \) are \( \sigma \)-stable for all \( \sigma \in U' \). If \( \sigma \in V \), then \( \phi_\sigma(\mathcal{O}_C(l)) < 1 < \phi_\sigma(\mathcal{O}_C(l-1)[1]) \). Hence if \( E \) is \( \sigma \)-semi-stable for \( \sigma \in U' \cap V \), then \( E \) is an extension of the first type. Therefore \( E \cong k_x, x \in C \). \( \square \)

### 3. Structures of walls and chambers

Let \( X \) be a smooth projective surface. In this section, we shall study the structure of walls and chambers. Let \( P^+(v)_\mathbb{R} \subset v_\mathbb{R} \) be the positive cone in \( v_\mathbb{R} \) and set \( C^+(v) := P^+(v)_\mathbb{R}/\mathbb{R}_{>0} \). We shall construct a map
\[ \xi : \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R} \to C^+(v) \]
and study its property (Proposition 3.2, Corollary 3.3).

Let \( v = r + c_1(v) + a_{\mathcal{O}_X} \) be an element of \( v(K(X)) \subset H^*(X, \mathbb{Q})_\text{alg} \) with \( r > 0 \). From now on, we set
\[ \delta := \frac{c_1(v)}{r} \]

Then we have
\[ v = re^{\delta} - \frac{(v^2)}{2r^2} g_X. \]

As in [23], we set
\[ \xi(\beta, \omega) := \xi(\beta, \omega, 1)/r \]
\[ = \left( \frac{\omega^2}{2} - \frac{(\beta - \delta)^2}{2} + \frac{(v^2)}{2r^2} \right) (\omega + (\omega, \delta)g_X) \]
\[ + (\beta - \delta)(\beta - \delta + (\beta - \delta, \delta)g_X) + (\beta - \delta, \omega) \left( e^{\delta} + \frac{(v^2)}{2r^2} g_X \right) \in C^+(v) \]

for \( (\beta, \omega) \in \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R} \). We have
\[ \xi(\beta, \omega) = \text{Im} e^{\beta + \sqrt{-1} \omega}/Z(\beta, \omega)(v) \in C^+(v). \]

For \( v_1 \in H^*(X, \mathbb{Q})_\text{alg}, Z(\beta, \omega)(v_1) \in \mathbb{R}Z(\beta, \omega)(v) \) if and only if \( \xi(\beta, \omega) \in v_1^\perp \).
**Remark 3.1.** For \( u \in v^\perp \) with \((u^2) > 0\), \((\beta, \omega) \in \xi^{-1}(u)\) if and only if \( Z_{(\beta, \omega)}(x) \in \mathbb{R}Z_{(\beta, \omega)}(v)\) for all \( x \in u^\perp \).

The main result of this subsection is the following.

**Proposition 3.2.** \( \xi : \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R} \to C^+(v) \) is a regular map whose fibers are connected. In particular, \( \xi^{-1}(U) \) is connected if \( U \) is connected.

Before proving this proposition, we shall give consequences of the proposition. We consider the chamber structure in \( \text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R} \) and the map

\[
(3.3) \quad \xi : \text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R} \to C^+(v).
\]

By the identification \( \{X, \xi(\mathcal{X}(X))\} \), we have a map

\[
\tilde{\xi} : s(\mathcal{X}(X)) \to C^+(v).
\]

For \( E \in D(X) \) with \( v(E) = v \), assume that \( E \) is \( \sigma \)-semi-stable and is \( S \)-equivalent to \( \oplus_i E_i \), where \( \sigma \in D := \hat{\xi}^{-1}(u) \) and \( E_i \) are \( \sigma \)-stable with the same phase. Then there is an open neighborhood \( U \) of \( \sigma \) in \( D \) such that \( E_i \) are \( \sigma' \)-stable for \( \sigma' \in U \). Since \( \phi_{\sigma'}(E_i) \) are continuous over \( U \) and \( \phi_{\sigma'}(E_i) = \phi_{\sigma'}(E_j) \) mod \( \mathbb{Z} \) is constant over \( D \), \( \phi_{\sigma'}(E_i) = \phi_{\sigma'}(E_j) \) for all \( i, j \). Hence \( E \) is an \( \sigma' \)-semi-stable object which is \( S \)-equivalent to \( \oplus_i E_i \). Therefore semi-stability is an open condition. If \( E \) is not \( \sigma' \)-semi-stable for \( \sigma' \in D \), then in a neighborhood of \( \sigma' \), \( E \) is not semi-stable. Since \( D \) is connected, \( E \) is \( \sigma' \)-semi-stable for all \( \sigma' \in D \). Assume that \( E \) is \( \sigma \)-semi-stable and is \( S \)-equivalent to \( \oplus_i E_i \), where \( E_i \) are \( \sigma \)-stable with the same phase. Then we also see that \( E_i \) are \( \sigma' \)-stable for all \( \sigma' \in D \). Thus the \( S \)-equivalence class of \( E \) is independent of \( \sigma \in D \). In particular, we have the following.

**Lemma 3.3.** If \( \sigma_0 \) is on a wall, then every \( \sigma \in \hat{\xi}^{-1}(\tilde{\xi}(\sigma_0)) \) is on the same wall.

**Lemma 3.4.** The set of walls in \( P^+(v)_\mathbb{R} \) is locally finite.

**Proof.** Let \( V \) be a compact small neighborhood of a point of \( C^+(v) \). Then there is a compact subset \( \bar{V} \) of \( \text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R} \) such that \( \xi(\bar{V}) = V \) by Proposition 5.2. Then there are finitely many walls intersecting \( \bar{V} \). By Lemma 3.3 the set of walls in \( P^+(v)_\mathbb{R} \) is locally finite. \( \square \)

**Corollary 3.5.** Let \( W \) be the set of vectors defining walls with respect to \( v \). Let \( U \) be a connected component of \( C^+(v) \setminus \cup_{u \in W} u^\perp \). Then \( \xi^{-1}(U) \cap \text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R} \) is a chamber for \( v \).

**Proof.** A chamber is a connected component of \( \text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R} \setminus \cup_{u \in W} W_u \) and \((\beta, \omega) \in W_u \) if and only if \( \xi(\beta, \omega) \in u^\perp \). By Proposition 3.2 \( \xi^{-1}(U) \) is connected. Hence it is a chamber. \( \square \)

**Corollary 3.5** implies that we can study the wall crossing behavior by looking at linear walls in \( C^+(v) \).

**Proof of Proposition 3.2.**

We set

\[
(3.4) \quad u := \zeta + (\zeta, \delta)g_X + y \left( e^\delta + \frac{(u^2)}{2r^2} g_X \right).
\]

Then \( u \in C^+(v) \) if and only if \((u^2) > 0 \) and \((u, H, (\delta)g_X) = (\zeta, H) > 0 \). Assume that \( y \neq 0 \). For \( H \in P^+(X)_\mathbb{R} \), we have a decomposition

\[
(3.5) \quad \zeta = \frac{(\zeta, H)}{(H^2)} H + D, \quad D = \zeta - \frac{(\zeta, H)}{(H^2)} H \in H^\perp.
\]

Then

\[
(3.6) \quad \frac{\xi(\delta + sH + D', tH)}{(sH + D', tH)} = \frac{(H^2)}{2}(t^2 + s^2) + \frac{(u^2)}{2r^2} - \frac{(D'^2)}{2}(H + (\delta)g_X + D' + (D', \delta)g_X + \left( e^\delta + \frac{(u^2)}{2r^2} g_X \right),
\]

where \( D' \in H^\perp \). Hence \( \frac{\xi(\delta + sH + D', tH)}{(sH + D', tH)} = \frac{u}{y} \) if and only if

\[
(3.7) \quad D' = D \frac{H^2}{y}, \quad \frac{(H^2)}{2}(t^2 + s^2) + \frac{(u^2)}{2r^2} - \frac{(D'^2)}{2y} = \frac{(\zeta, H)}{(H^2)}.
\]
Since
\[
\frac{(H^2)}{2}(t^2 + s^2) + \frac{(v^2)}{2r^2} - \frac{(D^2)}{2y^2} - s\left(\frac{(\zeta, H)}{y}\right)
\]
\[
= \frac{(H^2)}{2}(t^2 + s^2) + \frac{(v^2)}{2r^2} - \frac{1}{2y^2}\left(\frac{(\zeta^2)}{(H^2)} - \frac{(\zeta, H)^2}{y}\right) - s\left(\frac{(\zeta, H)}{y}\right)
\]
\[
= \frac{(H^2)}{2}t^2 + \frac{(H^2)}{2}\left(s - \left(\frac{(\zeta, H)}{y(H^2)}\right)\right)^2 + \frac{(v^2)}{2r^2} - \frac{\zeta^2}{2y^2}
\]
\[
= \frac{(H^2)}{2}t^2 + \frac{(H^2)}{2}\left(s - \left(\frac{(\zeta, H)}{y(H^2)}\right)\right)^2 - \frac{(u^2)}{2y^2}.
\]
(3.8)
\[\xi^{-1}(u)\] is parameterized by \(H\) and the circle
\[
\frac{(H^2)}{2}t^2 + \frac{(H^2)}{2}\left(s - \left(\frac{(\zeta, H)}{y(H^2)}\right)\right)^2 = \frac{(u^2)}{2y^2}.
\]
(3.9)
For the general case, we apply Proposition [3.3] to isometries \((r, \xi, a) \mapsto e^\eta(r, \xi, a)\) and \((r, \xi, a) \mapsto (a, -\xi, r)\) in order to reduce to the case \(y \neq 0\). Thus \(\xi\) is a regular map as a \(C^\infty\)-map. Since we are restricted to \(\text{Amp}(X)_\mathbb{R}\), for the connectedness of the fiber of \(\xi\), we need to describe it directly. For \(u = H + (H, \delta)g_X\),
\[
\xi^{-1}(u) = \{(\beta, \omega) \mid \beta = \delta + D, \omega \in \mathbb{R}_{>0}H, D \in H^\perp\}.
\]
So it is connected.

\[\square\]

Remark 3.6. In order to clarify the dependence of \(\xi(\beta, \omega)\) on \(v\), we set
\[
\xi_v(\beta, \omega) := \text{Im} \frac{e^{\beta + \sqrt{-1}\omega}}{Z_{(\beta, \omega)}(v)}.
\]
For \(u \in v^\perp \cap w^\perp\),
\[
\xi_{v}^{-1}(u) = \xi_{w}^{-1}(u).
\]
Indeed for \(u \in v^\perp\) with \(\text{rk}\, u \neq 0\), by using the expression of \(u\) in (3.4), \(c_1(u) = \zeta + y\delta\) and (3.9), we see that \(\xi_v(\beta, \omega) = u\) in \(C^+(v)\) if and only if
\[
\frac{(\omega^2)}{2} + \frac{(y\beta - c_1(u), \omega)}{y(\omega^2)}^2 \frac{(\omega^2)}{2} = \frac{(u^2)}{2y^2}.
\]
If \(\text{rk}\, u = 0\), then \((c_1(v) - r\beta, \omega) = 0\) and \(\omega \in \mathbb{R}_{>0}c_1(u)\). Hence it does not depend on the choice of \(v\). The same claim also holds if \(r = 0\).

Remark 3.7. By [15], the walls form nested circles, if we fixed \(\mathbb{R}_{>0}\omega\). By our proof of Proposition [3.2] the circles are the fibers of \(\xi\).

We set
\[
x_0 := e^\delta H = H + (H, \delta)g_X,
\]
(3.10)
\[
x_1 := -e^\delta(D + (1 + \frac{(v^2)}{2r^2} g_X)) = -(D + (D, \delta)g_X + (e^\delta + \frac{(v^2)}{2r^2} g_X)).
\]
Let \(L\) be the line in \(\mathbb{P}(v^\perp)\) passing through \(x_0\) and \(x_1\). Then \(\xi(\delta + sH + D, tH) \in L\) for all \((s, t)\). The image of the unbounded chamber in \((s, t)\) with \(s < 0\) and \(t > 0\) is the interior of a segment connecting \(x_0\) and \(x_0 + c\epsilon x_1\) \((0 < c \ll 1)\).

Lemma 3.8. Let \(H\) be an ample divisor on \(X\).
\begin{enumerate}
\item \(\mathcal{M}_{(\delta + sH + D, tH)}(v) = \mathcal{M}_{H}^{D - \frac{D}{y}K_X}(v)\) if \(-1 < s < 0\).
\item If \((\beta, \omega)\) is general, then there is \(\beta' \in \text{NS}(X)_\mathbb{Q}\) with \(\mathcal{M}_{(\beta, \omega)}(v) \cong \mathcal{M}_{(\beta', tH)}(v)\). In particular, \(\mathcal{M}_{\omega - \frac{D}{y}K_X}(v) \cong \mathcal{M}_{(\beta', tH)}(v)\).
\end{enumerate}

Proof. (1) is a consequence of Proposition [3.9]. (2) For a chamber in \(C^+(v)\) containing \(\xi(\beta, \omega)\), we take a vector \(u\) and write it as in (3.4) with (5.5). Then there is \((s, t)\) such that \(\xi(\delta + sH + D/\gamma, tH) = u\). For \(\beta' := \delta + sH + D/\gamma, M_{(\beta, \omega)}(v) \cong \mathcal{M}_{(\beta', tH)}(v)\).

We shall remark the behavior of \(\xi(\beta, tH)\) under an isometry of \(H^*(X, \mathbb{Q})\). Let \(\Phi : H^*(X, \mathbb{Q})_{\text{alg}} \rightarrow H^*(X, \mathbb{Q})_{\text{alg}}\).
be an isometry. Then $\Phi$ induces an isomorphism $\Phi^+: H^*(X, \mathbb{Q})^+ \rightarrow H^*(X, \mathbb{Q})^+$ of positive 2-plane $H^*(X, \mathbb{Q})^+_{alg} = \mathbb{Q}(1-\rho X) + QH$. Assume that $\Phi$ is an isometry such that $\Phi(r_1e^\gamma) = \rho X_1$, $\Phi(\rho X) = r_1e^\gamma$ and $\Phi^+$ preserves the orientation of $H^*(X, \mathbb{Q})^+_{alg}$. Then we can describe the action as

$$\Phi(re^\gamma + a\rho X + \xi + (\xi, \gamma)\rho X) = \frac{r}{r_1}\rho X + r_1ae^\gamma - \frac{r}{r_1}(\xi + (\xi, \gamma)\rho X),$$

where $\xi \in NS(X)_\mathbb{Q}$ and $\tilde{\xi} := \frac{r}{|r_1|}(\Phi(\xi + (\xi, \gamma)\rho X)) \in NS(X)_\mathbb{Q}$. We note that $\xi$ belongs to the positive cone if and only if $\tilde{\xi}$ belongs to the positive cone. For $(\beta, \omega) \in NS(X)_\mathbb{R} \times P^+(X)_\mathbb{R}$, we set

$$\tilde{\omega} := -\frac{1}{|r_1|} \frac{(\omega^2)}{((\beta-\gamma)^2 - (\omega^2))^2 + (\beta-\gamma, \omega)^2},$$

$$\tilde{\beta} := \gamma' - \frac{1}{|r_1|} \frac{(\omega^2)}{((\beta-\gamma)^2 - (\omega^2))^2 + (\beta-\gamma, \omega)^2}.$$  

(3.11)

Then $(\tilde{\beta}, \tilde{\omega}) \in NS(X)_\mathbb{R} \times P^+(X)_\mathbb{R}$.

**Proof.** By our assumption, $\tilde{\omega} \in P^+(X)_\mathbb{R}$. It is sufficient to prove $(\tilde{\omega}, \tilde{\omega}) > 0$ and $(\tilde{\omega}, \tilde{\omega}) > 0$, which follows from the following equations:

$$(\tilde{\omega})^2 = \frac{1}{|r_1|^2} \frac{(\omega^2)}{((\beta-\gamma)^2 - (\omega^2))^2 + (\beta-\gamma, \omega)^2},$$

$$(\tilde{\omega}, \tilde{\omega}) = \frac{1}{|r_1|^2} \frac{(\omega^2)}{((\beta-\gamma)^2 - (\omega^2))^2 + (\beta-\gamma, \omega)^2}. $$

(3.12)

Then $\beta - \gamma = \lambda \omega + D$ ($\lambda \in \mathbb{R}, D \in \omega^\perp$).

By [18] sect. A.1, we get the following commutative diagram:

$$
\begin{array}{ccc}
H^*(X, \mathbb{Q})^+_{alg} & \rightarrow & H^*(X, \mathbb{Q})^+_{alg} \\
|Z(\beta, \omega)| & \downarrow & \downarrow \zeta^{-1} \\
\mathbb{C} & \rightarrow & \mathbb{C} \\
\end{array}
$$

where

$$\zeta = r_1 \left( \frac{(\gamma - \beta)^2 - (\omega^2)}{2} + \sqrt{-1(\beta - \gamma, \omega)} \right).$$

**Proposition 3.9.** For $(\beta, \omega) \in NS(X)_\mathbb{R} \times P^+(X)_\mathbb{R}$, we have $\Phi(\xi(\beta, \omega)) = \xi(\tilde{\beta}, \tilde{\omega})$.

**Proof.** The proof is completely the same as of [28] Prop. 3.7].

4. **Stability conditions on a blow-up**

4.1. **Stability conditions for** $(\beta, \delta H)$. Let $\pi: X \rightarrow Y$ be the blow-up of a point as in section [2]. In this subsection, we shall study the map $\xi$ in a neighborhood of $H + (H, \delta)\rho X$, where $H \in \pi^*(\text{Amp}(Y)_\mathbb{Q})$. We start with the following easy fact.

**Lemma 4.1.** Assume that $\omega \in \pi^*(\text{Amp}(Y)_\mathbb{R})$.

1. $Z(\beta, \omega)(\mathcal{O}_C(-a)) = 0$ if and only if $(\beta, C) = -a + \frac{1}{2}$.
2. If $(\beta - \delta, \omega) \neq 0$, then $Z(\beta, \omega)(\mathcal{O}_C(-a)) = 0$ if and only if $\xi(\beta, \omega) \in v(\mathcal{O}_C(-a))^\perp$.

**Proof.** (1) is obvious. Since $Z(\beta, \omega)(v) \notin \mathbb{R}$ for $(\beta - \delta, \omega) \neq 0$, $\xi(\beta, \omega) \in v(\mathcal{O}_C(-a))^\perp$ if and only if $Z(\beta, \omega)(\mathcal{O}_C(-a)) = 0$. Thus (2) holds.

**Lemma 4.2.**

1. For $u = H + (H, \delta)\rho X$ with $H \in \pi^*(\text{Amp}(Y)_\mathbb{Q})$,

$$(4.1) \quad \xi^{-1}(u) \cap \overline{\mathcal{X}} = \{ (\beta, \omega) \mid \beta - \delta \in H^+, (\beta, C) \notin \frac{1}{2} + \mathbb{Z}, \omega \in \mathbb{R}_{\geq 0}H \}.$$

In particular, $\xi^{-1}(u) \cap \overline{\mathcal{X}}$ is not connected.

2. Assume that $\text{rk} u = -1$.

a. If $u \notin \bigcup_{a \in \mathbb{Z}} v(\mathcal{O}_C(-a))^\perp$, then

$$(4.2) \quad \xi^{-1}(u) \cap \overline{\mathcal{X}} = \{ (\beta, \omega) \mid (\beta, \omega) \in \xi^{-1}(u) \mid \omega \in \text{Amp}(X)_\mathbb{R} \cup \pi^*(\text{Amp}(Y)_\mathbb{R}) \}.$$
Remark 4.4. Proof. By Lemma 4.2 (2) and Lemma 3.3, the claim follows. We fix $H \in \pi^*(\text{Amp}(Y)_0)$. We have an inclusion

$$\iota : H^\perp \times \mathbb{R} \to \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R}$$

such that $\iota(D, s) = (\delta + sH + D, H)$. By (3.6),

$$\xi(\delta + sH + D, H) = e^\delta \left\{ \frac{r^2((H^2)^2(\delta^2 + 1) - (D^2)) + (\nu^2)H + s(H^2)}{2r^2} \right\} .$$

We also have

$$\iota^{-1}(\xi^{-1}(H + \delta)q_X)) = \{(D, 0) \mid D \in H^\perp\}.$$
Proof. Assume that \(s\) satisfies (4.8). We set
\[
\gamma := \delta + \frac{(\xi_0 - r_0\delta, H)}{r_0(H^2)} H
\]
(4.9)
\[
= \delta + \frac{\mu}{r_0(H^2)} H.
\]
Then \(r_0(\gamma, H) = (\xi_0, H)\) and \((c_1(E), H) - \text{rk} E(\gamma, H) \in \frac{\mu}{r_0} \mathbb{Z}\) for all \(E \in D(X)\). Since \(d_\gamma(v)(H^2) = \frac{\mu}{r_0}\), there is no wall intersecting \((sH + D, tH) | t > 0\). Hence the claim holds. \(\square\)

Toda [22], [23] constructed a stability condition \(\sigma_{(0, \pi^*(\omega') - qC)} = (Z_{(0, \pi^*(\omega') - qC)}, A_{\omega'}(X/Y))\), where \(q < 0, Z_{\pi^*(\omega') - qC}(E) = (e^{-\pi^*\omega'} - qC, v(E)), A_{\omega'}(X/Y) = (L, A_{\omega'}(C^0_{X/Y})\) and \(C^0_{X/Y}\) is spanned by \(O_{C}[-1]\). By the same proof in [23], we shall generalize \(A_{\omega'}(X/Y)\) as \(A_{(\gamma', \omega')}(X/Y) := (L, A_{(\gamma', \omega')}(C^0_{X/Y}), O_{C}[-1])\), where \((\gamma', \omega') \in \text{NS}(Y)_Q \times \text{NS}(Y)_Q\). Then
\[
\sigma_{(\pi^*(\delta'), \pi^*(\omega') - qC)} := (Z_{(\pi^*(\delta'), \pi^*(\omega') - qC)}, A_{(\gamma', \omega')}(X/Y))
\]
is a stability condition. Assume that \(v = \pi^*(v')\) with \(v' \in H^+(Y, \mathbb{Q})\). Then \(\mathcal{M}_{(\beta, H + D)}(v) \cong \mathcal{M}_{(\beta', H')}(v')\), where \(H = \pi^*(H')\).

4.2. A classification of walls. For \((s, q) \in \mathbb{R}^2\) with \(|s|, |q| \ll 1\), we consider
\[
(4.10)
\]
\[
(4.11)
\]
We set \(\epsilon := s(H^2) + pq\) and assume that \(\epsilon \leq 0\). We have
\[
(4.12)
\]
where
\[
(4.13)
\]
Then we have expansions
\[
(4.14)
\]
where \(O_n(\epsilon, q)\) is a power series of \(\epsilon\) and \(q\) contained in the ideal \((\epsilon, q)\). If \(q = 0\), then
\[
(4.15)
\]
If \(p >> 0\), then \((x, y)\) is close to \(y = 0\).
We take \((\beta_0, H)\) such that \(-\frac{l}{2} + t < (\beta_0, C) < \frac{l}{2} + l (l \in \mathbb{Z})\) and \((\beta_0 - \delta, H) = 0\). By (4.14), we can take a neighborhood \(U\) of \((\beta_0, H)\) in \(L\) such that \(\overline{U}\) is compact and \(\xi : U \to \xi(U)\) is isomorphic. In particular \(\xi(U)\) is a neighborhood of \(\xi(\beta_0, H) = e^H\). By shrinking \(U\), we may assume that there are finitely many Mukai
vectors defining walls in $U$ and all walls passes $(\beta_0, H)$. For each Mukai vector $v_1$ defining a wall, we may also assume that all walls in $U$ with respect to $v_1$ passes $(\beta_0, H)$. We set
\begin{align*}
U^{\leq 0} := & U \cap \{(\delta + sH + pC, H - qC) \mid \epsilon \leq 0\}, \\
U^{< 0} := & U \cap \{(\delta + sH + pC, H - qC) \mid \epsilon < 0\}.
\end{align*}

Let $E_1$ be a $\sigma_{(\beta, \omega)}$-stable object defining a wall in $U^{\leq 0}$. Since there is no wall with respect to $v(E_1)$ between $(\beta, \omega)$ and $(\beta_0, H)$, $E_1$ is $\sigma_{(\beta_0, H)}$-semi-stable. Since $(\beta_0 - \delta, H) = 0$, $E_1$ is generated by $\mu$-semi-stable objects and objects $O_C(-1), O_C(-1), k_x \in \mathcal{C}[-1], (x \in X, C)$. Since $g^{H} \subseteq \epsilon \cdot (\mathbb{R}H + \mathbb{R}C + \mathbb{R}(1 + (\epsilon^2 / 2) qX))$ is $y = 0$, if $|\epsilon| < q$, then $\sigma_{(\beta, \omega)}$-twisted stability coincides with Gieseker semi-stability with respect to $H - qC$, where $(\beta, \omega) = (\delta + sH + pC, H - qC)$.

**Lemma 4.48.** If $\text{rk} E_1 \neq 0$, then $E_1$ is a $\mu$-semi-stable object of $\mathcal{C}^i$.

**Proof.** For $(\beta, \omega) \in U^{< 0}$, $E_1$ is $\sigma_{(\beta, \omega)}$-stable with $\phi_{(\beta, \omega)}(E_1) \in (0, 1]$. Hence $H^i(E_1) = 0$ for $i \neq -1, 0$. Since $E_1$ is $\sigma_{(\beta_0, H)}$-semi-stable with $\phi_{(\beta_0, H)}(E_1) = 0$, $H^i(E_1) = 0$ for $i \neq 0, 1$. Therefore $E_1 \in \mathcal{C}^i$. Thus $E_1$ is a $\mu$-semi-stable object of $\mathcal{C}^i$. \qed

All walls in $\xi^{-1}(\xi(U))$ are defined by an object $E_1$ which defines a wall in $U$. For $p_0 > (\delta - \beta_0, C)$, by shrinking $U$, we may assume that
\begin{equation}
\{\xi((\beta_0 + sH, H - qC) \mid \epsilon \leq 0, q > 0\} \cap \{e^{\delta}(H + xC + y(1 + (\epsilon^2 / 2q) qX)) \in \xi(U) \mid x < p_0y, y \leq 0\},
\end{equation}
where we also use
\begin{equation}
x = -q + py + \mathcal{O}_2(\epsilon, q).
\end{equation}
For $a \in \mathbb{Z}$ with $p_0 < a + (\delta, C) - \frac{1}{2}$, we take $p \in \mathbb{R}$ such that $a + (\delta, C) - \frac{1}{2} < p < a + (\delta, C) - \frac{1}{2}$. We set $\beta_p := \beta + pC$. Let $V_a$ be the open set defined by $x < (a + (\delta, C) - \frac{1}{2}y)$ and $y < 0$. Since $\mathcal{O}_C(-a)[1]$ is an irreducible object of $\mathcal{C}^{a+1}$, $\mathcal{O}_C(-a)[1]$ is $\sigma_{(\beta_p, H)}$-stable. Hence there is an open neighborhood $U_p$ of $(\beta_p, H)$ such that $\mathcal{O}_C(-a)[1]$ is a $\sigma_{(\beta_p + sH, H - qC)}$-stable object for $(\beta_p + sH, H - qC) \in U_p$. By shrinking $U_p$, we may assume that
\begin{equation}
\ell_a := \{e^{\delta}(H + xC + y(1 + (\epsilon^2 / 2q) qX)) \in \xi(U_p) \mid x = (a + (\delta, C) - \frac{1}{2}y), y \leq 0\}
\end{equation}
is a subset of
\begin{equation}
\{\xi((\beta_0 + sH, H - qC) \mid \epsilon < 0, q > 0\} \cap \{e^{\delta}(H + xC + y(1 + (\epsilon^2 / 2q) qX)) \in \xi(U) \mid x < p_0y, y \leq 0\}.
\end{equation}
We note that $v(\mathcal{O}_C(-a))^\perp$ is the line $x = (a + (\delta, C) - \frac{1}{2}y)$. By Remark 3.3, $\xi^{-1}(\ell_a) = \xi^{-1}(\xi(v(\mathcal{O}_C(-a))))(\ell_a)$. Applying Lemma 3.3 to $\xi^{-1}(\ell_a) \cap U$, $\mathcal{O}_C(-a)$ is $\sigma_{(\beta_0 + sH, H - qC)}$-stable, if $(\beta_0 + sH, H - qC) \in U \ (q > 0)$ belongs to $\xi^{-1}(\ell_a)$. Thus we obtain the following.

**Lemma 4.49.** For any $a$ with $p_0 < a + (\delta, C) - \frac{1}{2}$, there is a neighborhood $U_a$ of $\xi^{-1}(\ell_a) \cap U$ such that $\mathcal{O}_C(-a)$ is $\sigma_{(\beta_0 + sH, H - qC)}$-stable. Moreover
\begin{equation}
\phi_{(\beta_0 + sH, H - qC)}(\mathcal{O}_C(-a)) = \phi_{(\beta_0 + sH, H - qC)}(E), \ E \in \mathcal{M}_{(\beta_0 + sH, H - qC)}(v)
\end{equation}
if $(\beta_0 + sH, H - qC) \in \xi^{-1}(\ell_a) \cap U$.

For $(\beta, \omega) \in U^{\leq 0}$ with $q \geq 0$, we shall describe $\mathcal{M}_{(\beta, \omega)}(v)$ in terms of the moduli spaces of semi-stable perverse coherents sheaves $\mathcal{M}^{\delta}_{H}(v)$. We set
\begin{equation}
u := e^{\delta}(H + xC + y(1 + (\epsilon^2 / 2q) qX)).
\end{equation}
We first assume that $u \in \xi(U)$ satisfies $(u, v(\mathcal{O}_C(-a))) = -x - y(\frac{1}{2}a - a - (\delta, C)) \neq 0$ for all $a$. Under this condition, we shall describe $\mathcal{M}_{(\beta, \omega)}(v)$ with $\xi(\beta, \omega) = u$, where $(\beta, \omega) = (\delta + sH + pC, H - qC)$. There are $s', t' > 0$ such that $\xi(\delta + s'H + p'C, t'H) = u$, where
\begin{equation}
p' := \frac{x}{y} = \frac{q^3 + q(s^2 - 1)(H^2) + p^2q + 2ps(H^2) - (\frac{\epsilon^2}{2q})q}{2\epsilon}.
\end{equation}
Then $\{\xi(\delta + s'H + p'C, t'H) \mid s \leq 0, t > 0\}$ is a segment. Since there is no wall between $u$ and $e^{\delta}H$,
\begin{equation}
\mathcal{M}_{(\beta, \omega)}(v) = \mathcal{M}_{(\delta + s'H + p'C, t'_\infty H)}(v) = \mathcal{M}^{\delta + p'C - \frac{1}{2}K_X}_{H}(v),
\end{equation}
where $t'_\infty$ is sufficiently large. If $(\beta, \omega)$ is not general, then the wall is defined by
\begin{equation}
\frac{\chi(E(\xi(\delta + p'C - \frac{1}{2}K_X)))}{\text{rk} E} = \frac{\chi(E_1(\xi(\delta + p'C - \frac{1}{2}K_X)))}{\text{rk} E_1},
\end{equation}
where $E_1$ is a $(\delta + p'C - \frac{1}{2}K_X)$-twisted stable object with $(c_1(E_1) - \text{rk} E_1 \delta, H) = 0$.

We next consider the case where $(\beta, C) = l - \frac{1}{2}$. We set $\beta_{\pm} := \beta \mp \eta C$ for a sufficiently small $\eta > 0$. 


Definition 4.10. \( E \in \mathcal{A}_{(\beta,tH)} \) is \( \sigma(\beta,tH) \)-semi-stable, if
\[
Z(\beta,tH)(E_1)/Z(\beta,tH)(E) \in -\mathbb{H} \cup \mathbb{R}_{\geq 0}
\]
for all non-zero subobject \( E_1 \) of \( E \) in \( \mathcal{A}_{(\beta,tH)} \). Let \( \mathcal{M}(\beta,tH)(v) \) be the moduli stack of \( \sigma(\beta,tH) \)-semi-stable objects \( E \) with \( v(E) = v \).

Proposition 4.11. \( \mathcal{M}_{(\gamma,\omega)}(v) = \mathcal{M}(\beta,tH)(v) = \mathcal{M}_H^{\beta - \frac{1}{2} K_X}(v) \) for \( (\gamma,\omega) \in \xi^{-1}(\xi(\beta,tH)) \).

Proof. Let \( v = \sum_i v_i \) be a decomposition of \( v \) such that
\[
v_i = e^\beta (r_i + d_i H + d_i + a_i q_X), \quad d_i = r_i \frac{d}{r}, \quad a_i = r_i \frac{a}{r}.
\]
We note that \( v_i = e^\beta D_1 \), if \( r_i = 0 \). We first prove that
\[
\mathcal{M}_{(\gamma,\omega)}(v)^s = \mathcal{M}_H^{\beta - \frac{1}{2} K_X}(v)^s = \mathcal{M}_H^{\beta - \frac{1}{2} K_X}(v_i)^s
\]
for \( (\gamma,\omega) \in \xi^{-1}(\xi(\beta,tH)) \).

As in Remark 3.6 we set \( \xi_{v_i}(\gamma,\omega) := \text{Im}(Z(\gamma,\omega)(v_i)^{-1} e^{v + \sqrt{-1} \omega}) \). By our assumption, we see that \( \xi_{v_i}(\beta,tH) \in \mathbb{R}(\beta,tH) \). By Remark 3.6 \( \xi_{v_i}^{-1}(\xi_{v_i}(\beta,tH)) = \xi^{-1}(\xi(\beta,tH)) \). By Lemma 2.14 we have
\[
\mathcal{M}_H^{\beta - \frac{1}{2} K_X}(v_i)^s = \mathcal{M}_H^{\beta - \frac{1}{2} K_X}(v_i)^s \cap \mathcal{M}_H^{\beta - \frac{1}{2} K_X}(v_i)^s.
\]
We take \( (\gamma,\omega) \in \xi_{v_i}^{-1}(\xi_{v_i}(\beta,tH)) \) which are in a neighborhood of \( (\gamma,\omega) \). Then \( \mathcal{M}_H^{\beta - \frac{1}{2} K_X}(v_i)^s = \mathcal{M}_H^{\beta - \frac{1}{2} K_X}(v_i)^s \cap \mathcal{M}_H^{\beta - \frac{1}{2} K_X}(v_i)^s \), where \( t \) is sufficiently large. Since
\[
\mathcal{M}_{(\gamma,\omega)}(v)^s \cap \mathcal{M}_{(\gamma,\omega)}(v)^s = \mathcal{M}_{(\gamma,\omega)}(v)^s,
\]
we have \( \mathcal{M}_H^{\beta - \frac{1}{2} K_X}(v_i)^s = \mathcal{M}_{(\gamma,\omega)}(v)^s \).

Let \( E \in \mathcal{D}(X) \) be an object which is a successive extension of \( E_i \in \mathcal{D}(X) \) with \( v(E_i) = v_i \). Then \( E \) is \( \beta \)-twisted semi-stable if and only if \( E \) is \( \sigma(\gamma,\omega) \)-semi-stable. \( \square \)

5. Wall crossing for Gieseker semi-stability

5.1. Ample line bundles on the moduli spaces. Assume that there is a moduli scheme \( M(\beta,\omega)(v) \) of \( S \)-equivalence classes of semi-stable objects and consisting of stable objects. For simplicity, we assume that there is a universal family \( \mathcal{E}_v \) on \( M(\beta,\omega)(v) \times X \). For \( \alpha \in K(X)_Q \), we set
\[
\mathcal{L}(\alpha) := \text{det} p_{M(\beta,\omega)(v)}(\alpha^\vee \otimes \mathcal{E}_v).
\]
We set
\[
\mathcal{K}(X)_v := \{ \alpha \in K(X)_Q \mid \chi(\alpha^\vee \otimes v) = 0 \}.
\]
We have a morphism
\[
\kappa : \mathcal{K}(X)_v \rightarrow \text{Pic}(M(\beta,\omega)(v))_Q
\]
\[
\alpha \mapsto \mathcal{L}(\alpha).
\]
As in 18, we have a morphism \( a : M(\beta,\omega)(v) \rightarrow \text{Alb}(X) \times \text{Pic}^0(X) \). \( \kappa \) induces a homomorphism
\[
\theta_v : v^* \rightarrow \text{NS}(M(\beta,\omega)(v))_Q/a^*(\text{NS}(\text{Alb}(X) \times \text{Pic}^0(X))_Q),
\]
where
\[
v^* := \{ u \in H^*(X,\mathbb{Q})_{\text{alg}} \mid \chi(u(\text{td}^{-1}X)^\vee, v) = 0 \}
\]
\[
= \{ u \in H^*(X,\mathbb{Q})_{\text{alg}} \mid (u,v) = 0 \}.
\]
By works of Li [11], [13] (see also 18), if \( \text{rk} v = 1, 2 \) and \( M(\beta,\omega)(v) \) is the moduli of Gieseker semi-stable sheaves \( M_\omega^{\beta - \frac{1}{2} K_X}(v) \), then \( a \) is the Albanese morphism and \( \theta_v \) is isomorphic for all sufficiently large \( v^2 \) depending on \( \omega, \text{rk} v, c_1(v) \).

Remark 5.1. If there is a universal family as a twisted object, then \( \kappa \) is well-defined.

We take \( \xi_1, \xi_2 \in K(X)_Q \) such that
\[
\text{ch}(\xi_1) = (H + (H,\delta)q_X)(\text{td}^{-1}X)^\vee = H + (H,\delta - \frac{1}{2} K_X)q_X,
\]
\[
\text{ch}(\xi_2) = -\left( e^{\beta - \frac{1}{2} K_X} - \frac{\chi(e^{\beta - \frac{1}{2} K_X}, v)}{v}\right).
\]
By the construction of the moduli scheme, we have the following.

Lemma 5.2 (cf. 18 Lem. A.4)). \( L(n\xi_1 + \xi_2) \) is ample for \( n > 0 \).
We set
\[ u := \varepsilon^\delta((\zeta - 1 + \frac{(\zeta)}{2\pi^2})q_X). \]
Assume that there is no wall between \( H + (H, \delta)q_X \) and \( u \). Then there is a semi-circle in \((s, t)\)-plane such that \( \xi(\beta, tH) = u \), where \( \beta = \delta + sH - D \), \( D = \zeta - \frac{(\zeta)}{(H, H')}H \). Then Lemma 6.3 implies
\[ M_{(\delta + sH - D, tH)}(v) = M_{H}^{\beta - \frac{1}{2}K_X}(v). \]
Since \( \beta = \delta - \zeta + \left( s - \frac{(\zeta)}{(H, H')} \right)H \), we also have
\[ M_{(\delta + sH - D, tH)}(v) = M_{H}^{\beta - \zeta - \frac{1}{2}K_X}(v). \]

**Proposition 5.3.** Assume that there is no wall between \( H + (H, \delta)q_X \) and \( u := \zeta + (\zeta, \delta)q_X - (e^\delta + \frac{(\zeta)}{2\pi^2})q_X \).

Then for \( \alpha \in K(X)_Q \) with \( \text{ch}(\alpha) = u(t_{c_X}^{-1})^\vee \), \( \mathcal{L}(\alpha) \) is a \( Q \)-ample divisor. In particular, for an adjacent chamber \( C \) of \( g_X^1, \theta_v(C) \subseteq \text{Amp}(M_{H}^{\gamma}(v))_Q \), where \( \gamma = \delta - \zeta - \frac{1}{2}K_X \).

**Proof.** Let \( (\beta, tH) \) satisfy \( \xi(\beta, tH) = u \) as above, where \( \beta \equiv \delta - \zeta \mod RH \). Then we see that
\[ u = -e^\delta((1 - \zeta + \frac{(\zeta)}{2\pi^2})q_X) \]
\[ = -((1 + \beta + ((\beta, \delta) - \frac{\alpha}{\pi}))q_X) \mod RH \]
\[ = \left( \frac{e^\beta + (\beta, v)}{r}q_X \right) \mod RH \]
by (1.9), where \( v = r(1 + \delta + \frac{\alpha}{2\pi^2})q_X \). Hence
\[ u(t_{c_X}^{-1})^\vee = -\left( \frac{e^\beta - \frac{1}{2}K_X - \frac{\chi(e^\beta - \frac{1}{2}K_X, v)}{r}}{r} \right) \mod r \text{ch}(\xi_1) \]
\[ \equiv \text{ch}(\xi_2) \mod r \text{ch}(\xi_1). \]
We take \( u' \) such that \( u' = u - \epsilon(H + (H, \delta)q_X) (\epsilon > 0) \) belongs to the same chamber. We take \( \alpha' \) with \( \text{ch}(\alpha') = u'(t_{c_X}^{-1})^\vee \). Then
\[ c_1(\mathcal{L}(\alpha')) = c_1(pM_{H}^{\gamma}(v))_s(u'^\vee \text{ch}E_s). \]
By [3], \( \mathcal{L}(\alpha') \) is nef. We take an ample \( Q \)-divisor \( \mathcal{L}(\xi_1 + \frac{1}{2}\xi_2) \), \( n \gg 0 \). Since \( \mathcal{L}(\alpha) = \mathcal{L}((1 - t)\alpha' + t(\xi_1 + \frac{1}{2}\xi_2)) \) \((1 > t > 0)\), Lemma 5.2 implies \( \mathcal{L}(\alpha) \) is ample. \hfill \Box

**Remark 5.4.** If \( (\xi(\beta, \omega)) \) belongs to a wall \( v_2^+ \) of \( C \) and there are \( \sigma(\beta, \omega) \)-stable objects \( E_1 \) and \( E_2 \) with \( v(E_1) = v_1 \) and \( v(E_2) = v_2 \) such that \( \text{dim Ext}^1(E_2, E_1) \geq 2 \) and \( \phi(\xi(\beta, \omega))(E_1) < \phi(\xi(\beta, \omega))(E_2) \) for \( \xi(\beta, \omega) \) is in \( C \). Then \( \theta_v(\xi(\beta, \omega)) \) is not ample.

Let \( N_H(v) \) be the Uhlenbeck moduli space of \( \mu \)-semi-stable sheaves constructed by Li [12] (see also Remark 6.4). Thus a point of \( N_H(v) \) corresponds to a pair \((F, Z)\) of a poly-stable vector bundle \( F \) with respect to \( H \) and a 0-cycle \( Z = \sum_{i=1}^{p} n_i x_i \in S^n X \) such that \( v = v(F) - nq_X \), where \( n \geq 0 \). For \( (F, n_{\xi_1} x_{\xi}) \in N_H(v) \), we have \( F \oplus (\bigoplus_{i=1}^{p} k_{x_i}[-1]^{n_{\xi_i}}) = M_{(\delta, H)}(v) \). Hence \( N_H(v) \) parametrizes \( S \)-equivalence classes of \( \sigma(\delta, \omega) \)-semi-stable objects. By the construction of the moduli space, we have a morphism \( f_H : M_{H}^{\beta - \frac{1}{2}K_X}(v) \rightarrow N_H(v) \) and \( \mathcal{L}(\alpha) \in f_H^*(\text{Pic}(N_H(v))) \) if \( \text{ch}(\alpha) = (H + (H, \delta)q_X)(t_{c_X}^{-1})^\vee \).

**Remark 5.5.** We use a contraction of \( N_H(v) \) in [12]. \( N_H(v) \) may depend on the choice of \( H \). Assume that \( M_H(v)_{\mu, \omega} \) is irreducible, normal and \( M_H(v)_{\mu, \omega} \) is an open dense substack. Then we can construct \( N_H(v) \) as a normal projective variety and \( f_H : M_{H}^{\alpha}(v) \rightarrow N_H(v) \) is a birational map. If \( H' \) and \( H \) belongs to the same chamber, then we have a birational map \( f_H \circ f_H^{-1} : N_H(v) \rightarrow N_H(v) \). Since \( f_H \circ f_H^{-1}(z) \) is a point for all \( z \in N_H(v) \), it is an isomorphism. In particular, if \( (v^2) \gg \), then \( N_H(v) \) depends only on the chamber. By [9 1.4], \( \mathcal{L}(\alpha) \in f_H^*(\text{Pic}(N_H(v))) \) if \( \text{rk} \alpha = 0 \).

By the same argument, we have a contraction \( N_H(v) \rightarrow N_{H'}(v) \) if \( H \) belongs to a chamber and \( H' \) is on the boundary of the chamber.

**Remark 5.6.** We have a morphism \( f : M_H^{\alpha}(v) \rightarrow \text{Pic}^0(X) \) by sending \( E \) to \( \det(E - E_0) \), where \( E_0 \) is a fixed element of \( M_H^{\alpha}(v) \). In the construction of the Uhlenbeck moduli space [12], the determinant \( \det \) is fixed. In order to construct the Uhlenbeck moduli space for \( M_H^{\alpha}(v) \), we need to recall its construction.

We first note that \( f \) is étale locally trivial. Indeed we have an étale morphism \( f^{-1}(0) \times \text{Pic}^0(X) \rightarrow M_H^{\alpha}(v) \) by sending \( (E, L) \in f^{-1}(0) \times \text{Pic}^0(X) \) to \( E \otimes L \), and hence the composed morphism \( [E_0] \times \text{Pic}^0(X) \rightarrow M_H^{\alpha}(v) \rightarrow \text{Pic}^0(X) \) is the multiplication by \( \text{rk} E_0 \). Hence we have an identification \( i_x : f^{-1}(x) \cong f^{-1}(0) \).

Assume that \( \text{ch}(\alpha) = ((H + (H, \delta)q_X)(t_{c_X}^{-1})^\vee \). Then \( \mathcal{L}(\alpha)(f_{-1}(x)) \) is independent of \( x \in \text{Pic}^0(X) \) under the identification \( i_x \). By the base change theorem, \( f_s(\mathcal{L}(\alpha)) \rightarrow (n \gg 0) \) is locally free and \( f_s(\mathcal{L}(\alpha)) \rightarrow (n \gg 0) \) is locally free and \( f_s(\mathcal{L}(\alpha)) \rightarrow (n \gg 0) \) is locally free.

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Since $L(na)_{s}$ is base point free, we have a surjective homomorphism $f^*(f_*(L(na))) \to L(na)$, which implies we have a morphism $M_n^H(v) \to \text{Pic}^0(X)$ for $n \gg 0$. Then the image is the Uhlenbeck moduli space.

5.2. Wall crossing for Gieseker and Uhlenbeck moduli spaces. Assume that $H \in \text{Amp}(X)$. Since the set of walls is locally finite (Lemma 5.1), in a small neighborhood $U$ of $H + (H, \delta)g_X$, we may assume that all walls pass through $H + (H, \delta)g_X$. Then $M_{(\beta, tH)}(v) = M_{H, \delta}^{e \frac{1}{2}K_X}(v)$ if $\xi(\beta, tH) \in U$ and $(\beta, H) < (\delta, H)$. We classify such walls. For $H + (H, \delta)g_X$, $v_1 := r_1 + \xi_1 + a_{\delta}g_X \in \text{U}(H + (H, \delta)g_X)$ if and only if $(\xi_1/r_1, H) = (\delta, H)$. We set $v_2 := v - v_1$. Assume that $\xi(\beta, tH) \in v_1^\perp$. We may assume that $M_{(\beta, tH)}(v_1) \neq 0$ and $M_{(\beta, tH)}(v_2) \neq 0$. By shrinking a neighborhood $U$, we may also assume that there is no wall for $v_1, v_2$ between $\xi(\beta, tH)$ and $H + (H, \delta)g_X$. Then $M_{(\beta, tH)}(v_1) = M_{H, \delta}^{e \frac{1}{2}K_X}(v_1)$ for $i = 1, 2$. In particular, $(v_1^2), (v_2^2) \geq 0. \ u = \xi(\beta, tH) \in v_1^\perp$ means that

$$\chi(e^{\beta - \frac{1}{2}K_X} - \chi(e^{\beta - \frac{1}{2}K_X}, v)\ g_X, v_1)$$

$$= - (e^{\beta}, v_1) + \frac{r_1}{r}(e^{\beta}, v) = 0$$

by (5.9), (5.8) and $(H + (H, \delta)g_X, v_1) = 0$. Thus

$$\frac{\chi(e^{\beta - \frac{1}{2}K_X}, v_1)}{r_1} = \frac{\chi(e^{\beta - \frac{1}{2}K_X}, v)}{r}.$$ 

Hence if $u$ crosses a wall $v_1^\perp$, then twisted semi-stability changes.

Therefore the wall crossing of Gieseker semi-stability is naturally understood by the wall crossing of Bridgeland's stability: For a family of vectors

$$u_H := n(H + (H, \delta)g_X) - \left(e^\delta + \frac{(v^2)}{2r^2}\ g_X\right).$$

$H \in U \subset \text{Amp}(X)$, we have a family of moduli spaces, where $U$ is a compact subset and $n$ is a sufficiently large integer depending on $U$. If $H$ crosses a wall for $\mu$-semi-stability, then $u_H$ crosses all walls for twisted semi-stability. If $H_0$ is on a wall and let $H_{\pm}$ be ample divisors in adjacent chambers, then we have morphisms $N_{H_{\pm}}(v) \to N_{H_0}(v)$ constructed by Hu and Li [10] (see also Remark 5.5). On the other hand, for the Gieseker moduli spaces, we have a sequence of flops as was constructed by Ellingsrud-Göttsche [7], Friedman-Qin [8], Matsuki- Wentworth [10].

5.3. The case of a blow-up. Assume that $k := (c_1(v), C)$ is normalized as $0 \leq k < 1$. We take $\beta_0 := \delta + p_0C$ such that $(\beta_0, C) = 0$, that is, $\beta_0 = \pi^*(\beta')$ ($\beta' \in \text{NS}(Y)_{\mathbb{Q}}$). Let $U$ be the open neighborhood of $(\beta_0, H)$ in subsection 1.2. We take $p_1$ with $p_1 \gg p_1 > p_0$. Then we may assume that

$$\{\xi(\beta_0 + sH, H - qC) \mid \epsilon \leq 0, q > 0\} \supset \{\epsilon(\delta(H + xC + y(1 + \frac{(v^2)}{2r^2})g_X)) \in \text{U}(U) \mid x < p_1y, y \leq 0\},$$

where $\epsilon := s(H^2) + p_0q$. Then $M_{(\beta_0 + sH, H - qC)}(v) = M_{(\beta_0, H)}(v)$ for $q \geq 0$ and $\epsilon \leq 0$, where $\xi(\beta, tH) = \xi(\beta_0 + sH, H - qC)$. In particular the wall crossing in $U$ covers the wall crossing for the moduli of perverse coherent sheaves in [21].

We first look at the wall crossing along $g_X^\perp$. We note that $\sigma(\beta_0, H)$ is on the wall defined by $g_X$ and also on walls defined by $v(O_C(-n))$. We define $v' \in H^*(Y, C)$, $\delta' \in \text{NS}(Y)_{\mathbb{Q}}$ and $H'$ by

$$\pi^*(v') = v + kv$O_C, \quad \delta' := c_1(v')/r, \quad H = \pi^*(H').$$

Lemma 5.7. (1) $M_{(\beta_0, H)}(v)$ consists of $E$ which is $S$-equivalence to $\pi^*(F \oplus A)$ where $F$ is a locally free poly-stable sheaf $F$ on $Y$ and

$$A = \bigoplus_{i=1}^m k_{x_i[-1]} \oplus O_C[-1] \oplus O_C[-1] \oplus O_C[-1] \oplus O_C[-1], \ (x_i \in X \setminus C).$$

(2) The $S$-equivalence class of $E$ in (1) is uniquely determined by the $S$-equivalence class of $R\pi_*(E) \in M_{(\delta', H')}(v')$, that is, it is determined by $R\pi_*(F)$ and

$$A' := R\pi_*(A) = \bigoplus_{i=1}^m k_{x_i[-1]} \oplus k_{p[-1]} \oplus k_{p[-1]} \oplus k_{p[-1]} \oplus k_{p[-1]}, \ (x_i \in X \setminus C).$$

Proof. (2) By $A'$, $\{x_1, ..., x_m\}$ and $n_1$ are determined. Since $c_1(E) = c_1(\pi^*(F)) + (n_2 - n_1)C$, $n_2$ is also determined. Hence the $S$-equivalence class of $E$ is uniquely determined by the $S$-equivalence class of $R\pi_*(E)$. For $\sigma(\beta_0, H - qC)$ and $\sigma(\beta_0, H)$ on $g_X^\perp$, we have a contraction $N_{H - qC}(v) \to N_{H'}(v')$. 


Lemma 5.8. For $E \in \mathcal{C}^0$ with $\text{Hom}(\mathcal{O}_C, E) = 0$, there is an exact sequence
\begin{equation}
0 \to E \to \mathbf{L}\pi^*(F) \to \text{Ext}^1(\mathcal{O}_C, E) \otimes \mathcal{O}_C \to 0.
\end{equation}

Proof. We note that $\text{Ext}^2(\mathcal{O}_C, E) = \text{Hom}(\mathcal{O}_C, \mathcal{O}_C(-1))^\vee = 0$. Hence
\[
\dim \text{Ext}^1(\mathcal{O}_C, E) = -\chi(\mathcal{O}_C, E) = (c_1(E), C).
\]
Then the universal extension $E'$ is an object of $\mathcal{C}'$ such that $\text{Hom}(\mathcal{O}_C, E') = 0$ and $(c_1(E'), C) = 0$. Since $F := \pi_*(E')$ is torsion free, $\mathbf{L}\pi^*(F) = \pi^*(F)$ and $\phi : \pi^*(F) \to E'$ is surjective in $\mathcal{C}'$. By comparing the first Chern classes, we see that $\phi$ is isomorphic. \qed

Remark 5.9. Since $R\pi_*(\mathcal{O}_X(C)) = \mathcal{O}_Y$, $F = R\pi_*(\mathbf{L}\pi^*(F)(C)) = \pi_*(E(C))$.

In $A_{(\beta', sH', H')}(X/Y)$, (5.15) gives an exact sequence
\begin{equation}
0 \to \text{Hom}(\mathcal{O}_C[-1], E) \otimes \mathcal{O}_C \to E \to \mathbf{L}\pi^*(F) \to 0.
\end{equation}

Therefore we get the following.

Proposition 5.10. (1) If $k \neq 0$, then $\mathcal{M}_{(\beta_0, H-qC)}(v) = \emptyset$ for $q < 0$.

(2) If $k = 0$, then $\mathcal{M}_{(\beta_0, H-qC)}(v) = \mathcal{M}_{(\beta', H')}(v')$ for $q < 0$.

We next consider the wall crossing for the Gieseker semi-stability. For simplicity, we assume that $\gcd(r, (c_1(v), H)) = 1$. In this case, the hyperplanes $v(\mathcal{O}_C(-n))^\perp$ ($n > 0$) in $C^+(v)$ are the candidates of walls. Let $C_n$ be a chamber between $v(\mathcal{O}_C(-n))^\perp$ and $v(\mathcal{O}_C(-n-1))^\perp$. Then $n - \frac{1}{2} < -(\beta, C) < n + \frac{1}{2}$ for $\xi(\beta, \omega) \in C_n$ with $\omega \in \mathbb{R}_{>, 0} H$ and $M_{C_n}(v) \cong M_{H, KX}^{\beta - KX}(v)$. By Lemma 5.9 or the finiteness of walls in $U$, there is an integer $N(v)$ such that $\mathcal{M}_{C_n}(v) = \mathcal{M}_{H-qC}^{\beta - KX}(v)$ for $n \geq N(v)$. If $(c_1(v), C) = 0$, then $\mathcal{M}_{C_n}(v)$ is the moduli space of semi-stable sheaves on $X$. Thus the wall-crossing in [21] is the wall crossing in the space of stability conditions. More generally, by (5.16), we have a morphism
\[
\mathcal{M}_{C_0}(v) \to \mathcal{M}_{(\beta', sH', H')}(v')
\]
whose general fibers are $Gr(r,k)$-bundles. If $\xi(\beta_0 + sH, H - qC) \in C_0$ crosses the wall $v(\mathcal{O}_C)^\perp$, then $\mathcal{M}_{(\beta_0 + sH, H - qC)}(v) = \emptyset$ for $q > 0$.

The relation between $\mathcal{M}_{C_0}(v)$ and $\mathcal{M}_{C_{N(v)}}(v)$ is described as Mumford-Thaddeus type flips:
\begin{equation}
\begin{array}{cccc}
M_{C_0}(v) & \to & M_{C_1}(v) & \to \cdots \to \mathcal{M}_{C_{N(v)}}(v),
\end{array}
\end{equation}

where $M_{C_{n+1}}(v)$ parameterizes $S$-equivalence classes of $\sigma(\beta, \omega)$-twisted semi-stable objects with $\xi(\beta, \omega) \in v(\mathcal{O}_C(-n-1))^\perp$.

Remark 5.11. Although we considered the wall-crossing behavior in a neighborhood of $(\beta_0, H)$, we may move the parameter $p$ as in [21]. In $C_n$, $n \geq 0$, the wall-crossing behaviour is the same. However for the wall $v(\mathcal{O}_C)^\perp$, the behavior is different. For $E \in \mathcal{M}_{C_0}(v)$, we have an exact sequence
\begin{equation}
0 \to E' \to E \to \text{Hom}(E, \mathcal{O}_C)^\vee \otimes \mathcal{O}_C \to 0
\end{equation}
with $E' \in \mathcal{C}^1 \cap \mathcal{C}^0$. If $k > 0$, then we have a morphism $M_{C_0}(v) \to M_{(\beta', H')}(w')$ by sending $E$ to $\pi_*(E(C))$, where $\pi^*(w') = v - (r-k)\omega(\mathcal{O}_C)$.

6. Appendix

6.1. Some properties of perverse coherent sheaves. Let $\pi : X \to Y$ be a resolution of a rational singularity. Let $\mathcal{C}$ be a category of perverse coherent sheaves and $G$ be a local projective generator of $\mathcal{C}$ which is a locally free sheaf on $X$ ([21] Defn. 1.13)).

Lemma 6.1. For a torsion free object $E$ of $\mathcal{C}$, there is an exact sequence
\begin{equation}
0 \to E \to \mathbf{E} \to T \to 0
\end{equation}
such that $T$ is 0-dimensional, $E'$ is torsion free and $\text{Ext}^1(A, E') = 0$ for all 0-dimensional objects $A$ of $\mathcal{C}$.

Proof. If $\text{Ext}^1(A, E) \neq 0$ for an irreducible object $A$ of $\mathcal{C}$, then a non-trivial extension is torsion free. So we inductively construct torsion free objects $E_n$
\[
0 \subset E = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n
\]
such that $E_i/E_{i-1}$ are irreducible objects of $\mathcal{C}^2$. We set $T_n := E_n/E_0$. We have an exact sequence
\begin{equation}
0 \to \pi_*(G^\vee \otimes E) \to \pi_*(G^\vee \otimes E_n) \to R\pi_*(G^\vee \otimes T_n) \to 0.
\end{equation}
By the torsion freeness of $E, E_n, \pi_*(G^\vee \otimes E)$ and $\pi_*(G^\vee \otimes E_n)$ are torsion free sheaves on $Y$ by
\begin{equation}
(6.3) \quad \text{Hom}(C_{y}, \pi_*(G^\vee \otimes E_n)) = \text{Hom}(G \otimes O_{\pi^{-1}(y)}, E_n) = 0, y \in Y.
\end{equation}
Hence $\pi_*(G^\vee \otimes E_n)$ is regarded as a subsheaf of $\pi_*(G^\vee \otimes E)^{\vee \vee}$ and
\[
\chi(\pi_*(G^\vee \otimes E)/\pi_*(G^\vee \otimes E)) \geq \chi(R\pi_*(G^\vee \otimes T_n)) = \chi(G, T_n).
\]
On the other hand, $\chi(G, T_i/T_{i-1}) > 0$ for all $i$ imply that $\chi(G, T_n) \geq n$. Therefore $n$ is bounded above. Hence there is $n$ such that $E_n$ satisfies the desired property. \hfill \blacksquare

**Lemma 6.2.** For a $\mu$-semi-stable object $E$ of $\mathcal{C}$, $d_\beta(E^2)(H^2) - 2 \text{rk } E\alpha_\beta(E) \geq 0$.

**Proof.** We take $E'$ in Lemma 6.4. Then $E'$ is $\mu$-semi-stable locally free sheaf with respect to $H$, since $\text{Ext}^1(k_x, E') = 0$ for all $x \in X$ (cf. [27, Lem. 1.1.31]). Since $a\alpha_\beta(T) \geq 0$, we have
\[d_\beta(E^2)(H^2) - 2 \text{rk } E\alpha_\beta(E) \geq d_\beta(E')^2(H^2) - 2 \text{rk } E'a\alpha_\beta(E').\]
Then the claim follows from the ordinary Bogomolov-Gieseker inequality. \hfill \blacksquare

The following weak form of Bogomolov inequality is an easy consequence of Lemma 6.2 and the proof of Lemma 2.19.

**Proposition 6.3.** For a $\sigma(\beta, \omega)$-semi-stable object $E$ of $A(\beta, \omega)$,
\begin{equation}
(6.4) \quad d_\beta(E^2)(H^2) - 2 \text{rk } E\alpha_\beta(E) \geq 0.
\end{equation}

We shall postpone the proof of this claim, since we need a claim in Proposition 6.3. So we shall prove (6.4) in the course of the proof of Proposition 6.3.

Let $\mathcal{C} = \mathcal{O}^D(-K_X)$ be a category of perverse coherent sheaves such that $G^\vee(-K_X)$ is a local projective generator (see [27, Lem. 1.14]). Since $(G^\vee(-K_X))^\vee(-K_X) = G$, we have $(\mathcal{C})^* = \mathcal{C}$. By [27, Lem. 1.14 (4)], the following claim holds.

**Lemma 6.4.** $A \in D(X)$ is a 0-dimensional object of $\mathcal{C}$ if and only if $A^\vee[2]$ is a 0-dimensional object of $\mathcal{C}^*$.

For a 0-dimensional object $A$ of $\mathcal{C}$,
\begin{equation}
(5.5) \quad \text{Hom}(A, E) = \text{Hom}(E^\vee[1], (A^\vee[2])[−1]),
\end{equation}
\begin{equation}
\text{Hom}(E, A[−1]) = \text{Hom}(A^\vee[2], E^\vee[1]).
\end{equation}

**Lemma 6.5.** Let $E \in D(X)$ satisfy $pH^i(E) = 0$ (i ≠ −1, 0) and $\text{Hom}(C_x, E) = 0$ for all $x \in X$. Then $H^i(E^\vee[1]) = 0$ for $i = 0, 1, 0 \text{ and } H^i(B, E^\vee[1]) = 0$ for all 0-dimensional object $B$ of $\mathcal{C}^*$. Moreover if $\text{Hom}(A, E) = 0$ for all 0-dimensional object $A$ of $\mathcal{C}$, then $pH^i(E^\vee[1]) = 0$ for $i = 0, 1, 0$.

**Proof.** Let $G$ be a local projective generator of $\mathcal{C}$. Since $\text{Ext}^2(G(−n), pH^{-1}(E)) = 0$ for $n > 0$, we have a morphism $G(−n)^{\oplus N} \rightarrow E$ such that $H^0(G(−n)^{\oplus N}) \rightarrow H^0(E)$ is surjective in $\mathcal{C}$. We set $V_0 := G(−n)^{\oplus N}$ and $V_{−1} := \text{Cone}(V_0 \rightarrow E)[-1]$. Then we see that $V_{−1} \in \mathcal{C}$. Since $V_0$ is a locally free sheaf, $\text{Ext}^2(C_x, V_0) = 0$ for all $x \in X$. By $\text{Ext}^2(E, C_x) = \text{Hom}(C_x, E^\vee) = 0$, $\text{Ext}^1(V_{−1}, C_x) = 0$ for all $x \in X$. Hence $V_{−1}$ is a locally free sheaf. Then $E^\vee[1]$ is represented by a two term complex of locally free sheaves. For a 0-dimensional object $B$ of $\mathcal{C}^*$, we set $A := B^\vee[2]$. Then $A^\vee[2] = B$ and Lemma 6.4 implies that $A$ is a 0-dimensional object of $\mathcal{C}$. Then 6.5 implies $\text{Hom}(B, E^\vee[1]) = \text{Hom}(E, A[−1]) = 0$.

Assume that $\text{Hom}(A, E) = 0$ for all 0-dimensional objects. Then by Lemma 6.3 and 6.4, we see that $\text{Ext}^0(E^\vee[1]) = 0$. For $F := H^{-1}(E^\vee[1])$, we have an exact triangle
\begin{equation}
(6.6) \quad pH^0(F) \rightarrow F \rightarrow pH^{-1}(F)[-1] \rightarrow pH^0(F)[1].
\end{equation}
Therefore $pH^i(E^\vee[1]) = 0$ for $i = −1, 0, pH^{-1}(E^\vee[1]) = pH^0(F)$ and we have an exact sequence
\begin{equation}
0 \rightarrow pH^{-1}(F) \rightarrow pH^0(E^\vee[1]) \rightarrow H^0(B^\vee[1]) \rightarrow 0
\end{equation}
in $\mathcal{C}^*$.

\hfill \blacksquare

**Remark 6.6.** $V_i^\vee$ are local projective objects of $\mathcal{C}^D$. However we do not know whether $V_i^\vee \in \mathcal{C}^*$ or not.

**Corollary 6.7.** For a complex $E \in D(X)$, $pH^i(E) = 0$ (i ≠ −1, 0) and $\text{Hom}(A, E) = 0$ for all 0-dimensional object $A$ of $\mathcal{C}$ if and only if $pH^i(E^\vee[1]) = 0$ for $i = −1, 0$ and $\text{Hom}(B, E^\vee[1]) = 0$ for all 0-dimensional object $B$ of $\mathcal{C}^*$.

**Remark 6.8.**
1. Let $E$ be a $\sigma$-stable object with $0 < \phi(E) < 1$. Then $\text{Hom}(A, E) = 0$ for all 0-dimensional object $A$ of $\mathcal{C}$.
2. For the object $E$ in Lemma 6.3 if $\text{rk } pH^0(E) = 0$, then $E^\vee[1] \in \text{Coh}(X)$. Moreover if $\text{Hom}(A, E) = 0$ for all 0-dimensional object $A$ of $\mathcal{C}$, then $E^\vee[1] \in \mathcal{C}^*$ and $E^\vee[1]$ does not contain any 0-dimensional object.
3. Let $V_{−1} \rightarrow V_0$ be a two term complex of locally free sheaves. Then $pH^i(E) = 0$ for $i ≠ 0, −1$ if $\text{Hom}(E, A[−1]) = 0$ for all 0-dimensional object $A$ of $\mathcal{C}$. 

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6.2. Large volume limit. We consider the stability condition $\sigma_{(\beta, \omega)}$ in [13]. This stability condition is very similar to that for an abelian surface. Thus almost same results in [17, 18] and [24] hold. In particular results in [24] sect. 4.1] hold.

We set

$$v := e^\beta (r + a_\beta q_X + d_\beta H + D_\beta), D_\beta \in H^\perp.$$  

By Lemma [13, Lem. 2.1.6] holds, where

$$\begin{align*}
(\omega^2) > & \begin{cases}
\frac{d_d - d_{\min, \beta}((v^2) - (D_\beta^2))}{r_d}, & r > 0 \\
\frac{d_d - d_{\min, \beta}((v^2) - (D_\beta^2)) + 2d_\omega|a_\omega|}{d_d}, & r = 0, \\
\frac{-d_d + d_{\min, \beta}((v^2) - (D_\beta^2))}{r_d}, & r < 0,
\end{cases}
\end{align*}$$

$$d_{\min, \beta} := \min\{d_\beta(E) > 0 \mid E \in K(X)\}$$ and $d_\omega = \frac{1}{(r_\beta)} \min\{(D, H) > 0 \mid D \in \Pic(X)\}$.

We set $\beta = \xi/r', \xi \in NS(X)$. Then

$$(c_1(E) - r\beta, H) = \frac{(r'c_1(E) - r\xi, H)}{r'} \in \mathbb{Z}\frac{d_0(H^2)}{r'}.$$  

If $\gcd\left(\frac{c_1(E, H)}{d_0(H^2)}, r\right) = 1$, that is, $r'(c_1(E, H) - ra) = 1$ $(r', a \in \mathbb{Z})$, then for $\xi \in NS(X)$ with $(\xi, H) = ad_0(H^2)$, we have

$$(c_1(E) - r\xi/r', H) = \frac{(r'c_1(E) - r\xi, H)}{r'} = \frac{d_0(H^2)}{r'} = d_{\min, \xi/r'}(H^2).$$

By the same proof of [17] Prop. 3.2.1, Prop. 3.2.7, we have the following results.

**Proposition 6.9.** Assume that $\omega$ satisfies (6.7) with respect to $v$. Then $E \in D(X)$ is $\sigma_{(\beta, \omega)}$-semi-stable with $\phi(E) \in (0, 1)$ if and only if

(i) $r \geq 0, E \in \mathcal{C}$ and is $(-\frac{1}{2}K_X)$-twisted semi-stable or

(ii) $r < 0, E^\vee[1] \in \mathcal{C}^*$ and is $(-\frac{1}{2}K_X)$-twisted semi-stable.

Thus we have

$$\mathcal{M}_{(\beta, \omega)}(v) \cong \begin{cases}
\mathcal{M}_\omega^{\beta - \frac{1}{2}K_X}(v), & r \geq 0, \\
\mathcal{M}_\omega^{-\beta - \frac{1}{2}K_X}(-v^\vee), & r < 0.
\end{cases}$$

**Proof.** Let $E$ be a $\sigma_{(\beta, \omega)}$-semi-stable object for $(\omega^2) \gg 0$. By using Lemma [6.2] for $(\beta - \frac{1}{2}K_X)$-semi-stable objects of $\mathcal{C}$ and $-(\beta + \frac{1}{2}K_X)$-semi-stable object of $\mathcal{C}^*$, we see that $E$ is $(-\frac{1}{2}K_X)$-semi-stable or $E^\vee[1]$ is $-(\beta + \frac{1}{2}K_X)$-semi-stable ([17] Prop. 3.2.1, Lem. 3.2.3, Prop. 3.2.7]). In particular, $E$ satisfies Proposition 6.3. For a general $\omega$, we use

$$\frac{(v_1^2 + v_2^2)^2 - (D_1 + D_2)^2}{(d_1 + d_2)^2} = \frac{(v_1^2) - (D_1^2)}{d_1^2} + \frac{(v_2^2) - (D_2^2)}{d_2^2} + \left(\frac{a_1}{d_1} - \frac{a_2}{d_2}\right)\left(\frac{r_1}{d_1} - \frac{r_2}{d_2}\right)$$

for

$$v_i := e^\beta (r_i + d_i H + D_i + a_i q_X), D_i \in H^\perp.$$  

to study the wall crossing behavior. Then Proposition 6.3 follows by the induction on $d_\beta$.

Thanks to Proposition 6.3, the converse direction also holds. \hfill \Box

**Remark 6.10.** If $\mathcal{C}^\beta = \text{Coh}(X)$, then the claim is a refinement of results of Lo and Qin [14] or Maciocia [15].

The following claim is due to Maciocia [15] Thm. 3.13] if $\mathcal{C}^\beta = \text{Coh}(X)$

**Lemma 6.11 ([17] sect. 3).** Fix $\beta \in NS(X)_{Q}$. Assume that $d_\beta(v) > 0$ and

$$v_1 \in \mathbb{Q}v \iff (r(v_1), d_\beta(v_1), a_\beta(v_1)) \in \mathbb{Q}(r(v), d_\beta(v), a_\beta(v))$$

for all $v_1 \in H^\vee(X, \mathbb{Q})_{\text{alg}}$ with $0 < d_\beta(v_1) < d_\beta(v)$. Then the set of walls intersecting $\{(\beta, tH) \mid t > 0\}$ is finite.

**Proof.** Let $B \in \mathbb{Z}_{\geq 0}$ be the denominator of $\beta$. Assume that we have a decomposition $v = v_1 + v_2$ such that there are $\sigma_{(\beta, \omega)}$-semi-stable objects $E_i$ with $v(E_i) = v_i$ $(i = 1, 2)$ and $Z(\beta, \omega)(v_1), Z(\beta, \omega)(v_2) \in \mathbb{R}_{>0} Z(\beta, \omega)(v)$. We set

$$v := e^\beta (r + dH + D + aq_X), D \in H^\perp$$

$$v_i := e^\beta (r_i + d_i H + D_i + a_i q_X), D_i \in H^\perp.$$  

Then

$$(\omega^2) = \frac{1}{2}(dr_i - dr) = (d_\beta - d_\beta a).$$
By Lemma 6.2, $(v_1^2) - (D_2^2) = d_1^2 (H^2) - 2r_i a_i \geq 0$. Since

\[
(v_1, v_2) - (D_1, D_2) = \frac{(v_1^2) - (D_1^2)}{2d_1^2} + \frac{(v_2^2) - (D_2^2)}{2d_2^2} + \left( \frac{r_1}{d_1} - \frac{r_2}{d_2} \right) \left( \frac{a_1}{d_1} - \frac{a_2}{d_2} \right),
\]

$\geq 0$. Hence $0 \leq (v_1^2) - (D_2^2) < (v_2^2) - (D_2^2)$ and $(v_1, v - v_2) - (D_1, D - D_2) > 0$. Since the choice of $d_i$ are finite, $r_i a_i$ are bounded. $(v^2) - (D^2) \geq 2((v, v) - (D, D)) > 0$ implies $r a_i + r a_i$ is also bounded. Therefore $r_i a_i$ are bounded. Since $r_i \in \mathbb{Z}$ and $2B^2 a_i \in \mathbb{Z}$, the choice of $r_i$ and $a_i$ are finite. Therefore the set of walls is finite. \[\square\]

REFERENCES

[1] Arcara, D., Bertram, A., Bridgeland-stable moduli spaces for $K$-trivial surfaces, arXiv:0708.2247 J. Eur. Math. Soc. 15 (2013), 1-38.
[2] Bayer, A., Macri, E., The space of stability conditions on the local projective plane, Duke Math. J. 160 (2011), no. 2, 263–322.
[3] Bayer, A., Macri, E., Projectivity and Birational Geometry of Bridgeland moduli spaces, arXiv:1203.4613.
[4] Bertram, A., Martinez C., Change of polarization for moduli of sheaves on surfaces as Bridgeland wall-crossing, a first draft.
[5] Bridgeland, T., Flops and derived categories, Invent. Math. 147 (2002), 613-632.
[6] Bridgeland, T., Stability conditions on K3 surfaces, math.AG/0307164 Duke Math. J. 141 (2008), 241–291.
[7] Ellingsrud, G., Göttsche, L., Variation of moduli spaces and Donaldson invariants under change of polarization, J. Reine Angew. Math. 467 (1995), 1–49.
[8] Friedman, R., Qin, Z., Flips of moduli spaces and transition formulas for Donaldson polynomial invariants of rational surfaces, Comm. Anal. Geom. 3 (1995), no. 1-2, 11–83.
[9] Göttsche, L., Nakajima, H., Yoshioka, K., K-theoretic Donaldson invariants via Instanton counting, Pure and Applied Mathematics Quarterly 5 (2009), 1029–1111.
[10] Hu, Y., Li, W.P., Variation of the Gieseker and Uhlenbeck compactifications, Internat. J. Math. 6 (1995), no. 3, 397–418.
[11] Li, J., The first two Betti numbers of the moduli spaces of vector bundles on surfaces, Comm. Anal. Geom. 5 (1997), no. 4, 625–684.
[12] Li, J., Picard groups of the moduli spaces of vector bundles over algebraic surfaces, Moduli of vector bundles (Sanda, 1994; Kyoto, 1994), 129–146, Lecture Notes in Pure and Appl. Math., 179, Dekker, New York, 1996.
[13] Li, J., Compactification of moduli of vector bundles over algebraic surfaces, Collection of papers on geometry, analysis and mathematical physics, 98–113, World Sci. Publ., River Edge, NJ, 1997.
[14] Lo, J., Qin, Z., Min-a-walls for Bridgeland stability conditions on the derived category of sheaves over surfaces, arXiv:1103.4552.
[15] Maciocia, A., Computing the Walls Associated to Bridgeland Stability Conditions on Projective Surfaces, arXiv:1202.4587.
[16] Matsuki, K., Wentworth R., Mumford-Thaddeus principle on the moduli space of vector bundles on an algebraic surface, Internat. J. Math. 8 (1997), no. 1, 97–148.
[17] Minamide, H., Yanagida, S., Yoshioka, K., Fourier-Mukai transforms and the wall-crossing behavior for Bridgeland’s stability conditions, arXiv:1105.5217.
[18] Minamide, H., Yanagida, S., Yoshioka, K., Some moduli spaces of Bridgeland’s stability conditions, arXiv:1111.6187 Int. Math. Res. Not. IMRN 2014, No.19, 5264–5327, doi:10.1093/imrn/rnt126.
[19] Nakajima, H., Yoshioka, K., Lecture on instanton counting, Proceeding of “Workshop on algebraic structure and moduli spaces”, July 14–20, 2003, CRM Proc. Lecture note 38, Amer. Math. Soc., Providence, RI, 2004 31–101.
[20] Nakajima, H., Yoshioka, K., Perverse coherent sheaves on blow-up. I. A quiver description, preprint, arXiv:0802.3120.
[21] Nakajima, H., Yoshioka, K., Perverse coherent sheaves on blow-up. II. wall-crossing and Betti numbers formula, Adv. Stud. Pure Math. 61 (2011), 349–386.
[22] Toda, Y., Stability conditions and extremal contractions, Math. Ann. 357 (2013), 631–685.
[23] Toda, Y., Stability conditions and birational geometry of projective surfaces, arXiv:1205.3602.
[24] Yanagida, S., Yoshioka, K., Bridgeland’s stabilities on abelian surfaces, arXiv:1203.0884 Math. Z. 276 (2014), 571–610.
[25] Yoshioka, K., Chamber structure of polarizations and the moduli of stable sheaves on a ruled surface, Int. J. Math. 7 (1996), 411–431.
[26] Yoshioka, K., A note on the universal family of moduli of stable sheaves, J. reine angew. math. 496 (1998), 149–161.
[27] Yoshioka, K., Perverse coherent sheaves and Fourier-Mukai transforms on surfaces I, Kyoto J. Math. 53 (2013), 261–344.
[28] Yoshioka, K., Bridgeland’s stability and the positive cone of the moduli spaces of stable objects on an abelian surface, arXiv:1206.8336 Adv. Stud. Pure Math. to appear.