Constructive Graph Theory: Generation Methods, Structure and Dynamic Characterization of Closed Classes of Graphs - A survey

Mikhail Iordanski

Abstract

The processes of constructing some graphs from others using binary operations of union with intersection (gluing) are studied. For graph classes closed with respect to gluing operations the elemental and operational bases are introduced. The generating bases together with the system of restrictions on the gluing operation, that preserve the characteristic properties of graphs form a constructive descriptions of the closed classes of graphs. It is shown that each closed class of graphs has a unique elemental basis and at least one operational basis. For the closed class of all graphs and all basis precomplete closed subclasses of it the constructive descriptions are considered. For each of them its characteristic properties and a diagram of the inclusion of subclasses in superclasses is given. Constructive descriptions are obtained for some classes of graphs with classical properties. Some possible applications of constructive theory are discussed in conclusion.

1. Introduction

The work is a systematic review of the results on the constructive theory of graphs from Russian-language publications, mainly by the author, over the last 25 years.

The most complete description of the results is given in the monograph [1]. In the monograph and in the articles [2-5] different systems of operations on graphs are considered. In this work we based on the using binary set-theoretical operations of union with intersection (gluing operations), which is the most natural representation.

In the constructive theory, the processes of building some graphs from others are being studied. The generation methods, structure and dynamic characterization of graph classes closed with respect to gluing operations are considered. Restrictions on gluing operations are studied, under which various characteristic properties of graphs are saved. Knowing such restrictions allows you to build graphs that have the specified properties.

In the general case, loops and multiple edges are permissible in graphs. The following notation is used: $K_n$ – complete graph, $C_n$ – simple cycle, $L_n$ – simple chain, $O_n$ – empty graph, all of them contain $n$ vertices ($O_o$ – null graph without vertices).

Let $G_1$ and $G_2$ are disjoint graphs. The gluing operation consists in the identification of isomorphic subgraphs $G'_1 \subseteq G_1$ and $G'_2 \subseteq G_2$. Glueing operation is called trivial if $G'_1 = G_1$ and (or) $G'_2 = G_2$. For each of the graphs obtained as a result of the operation of gluing the graphs $\tilde{G}_1$ and $\tilde{G}_2$ on the subgraph $\tilde{G}$, isomorphic to $G'_1$ and $G'_2$ the notation $(G_1 \circ G_2) \tilde{G}$ is used. Operand graphs $G_1$ and $G_2$ are isomorphic to subgraphs of the resulting graph $G = (G_1 \circ G_2) \tilde{G}$ of gluing operation. The subgraph $\tilde{G}$ is called the subgraph of gluing. For fixed graphs $G_1$ and $G_2$ the resulting graph $G = (G_1 \circ G_2) \tilde{G}$ may depend on the type of subgraph gluing $\tilde{G}$ (Fig 1), the choice of identifiable subgraphs $G'_1$ and $G'_2$ in operand graphs (Fig 2) and the method of their identification (Fig 3).

*Lobachevsky State University, Minin Pedagogical State University of Nizhny Novgorod, Russia, iordanski@mail.ru
Let $\mathfrak{I}$ be a set of graphs. Graph $G$ is called superposition of graphs from $\mathfrak{I}$ if $G \in \mathfrak{I}$ or $G$ can be obtained by successive application of the operations of gluing to the graphs from $\mathfrak{I}$ or to the graphs obtained from $\mathfrak{I}$ with the operations of gluing. When performing each gluing operation, the type of the identified subgraphs their choice in operand graphs and the method of identification are determined independently. The process of construction graph $G$ from the graphs of the set $\mathfrak{I}$ determines the superposition operation of graphs from $\mathfrak{I}$. If in the superposition operation at least one of the graph-operands of each gluing operation belongs to the set of $\mathfrak{I}$, then the superposition operation is called canonical.

The set of graphs $\mathfrak{I}$, as well as graphs, derived from $\mathfrak{I}$ using superposition operations, denoted by $[\mathfrak{I}]$. If $[\mathfrak{I}] = \mathfrak{I}$, then the class $\mathfrak{I}$ is called closed. A closed class of graphs is
called trivial if used in superposition operations only trivial gluing operations. The set of $\mathfrak{S}' \subset \mathfrak{S}$ is a complete system of graphs in a closed class $\mathfrak{S}$ if $|\mathfrak{S}'| = \mathfrak{S}$. Minimal on inclusion the complete system of graphs $B_e$ is called elemental basis of a closed class $\mathfrak{S}$.

Operations with isomorphic gluing subgraphs $\tilde{G}$ refer to one type. The set of types gluing operations, use of which is enough to build from $B_e$ all graphs of a closed class $\mathfrak{S}$, forms a complete system of types gluing operations. Minimal on inclusion the complete system of types gluing operations $B_o$ is called operational basis of closed class $\mathfrak{S}$. The operational basis $B_o$ is described by set of graphs isomorphic to subgroups of gluing $\tilde{G}$. The elemental and operational bases are called generators bases. The generating bases define constructive description of the closed class $\mathfrak{S}$.

The numbering of all statements given in the text (theorems, lemmas and corollaries) is independent in each subsection. The section number is indicated first, then the subsection number.

Proofs of the fundamental statements of the first two subsections of the main part is given. Ends of the proofs are marked with $\square$.

2. The generation methods, structure and dynamic characterizations of closed classes of graphs

2.1. Generation Methods of closed classes of graphs

A number of theorems on the structure and methods of generating closed classes of graphs were announced without proofs in [6]. In subsequent works, all of them were proved. Some of the most important ones are listed below.

**Theorem 2.1.1** [7]. Every closed class of graphs $\mathfrak{S}$ has single elemental basis.

**Proof.** Consider an arbitrary closed class of graphs $\mathfrak{S}$. We associate with it an infinite oriented graph $G^\mathfrak{S}$. Its vertices correspond to the graphs from $\mathfrak{S}$. Arc $(v_i, v_j)$, where $v_i, v_j \in V(G^\mathfrak{S})$, $i \neq j$, is carried out then and only when the graph $G_i$ corresponding to the vertex $v_i$ is graph operand of at least one non-trivial operation binary gluing that implements the graph $G_j$, corresponding to vertex $v_j$. Since the arcs of the graph $G^\mathfrak{S}$ correspond only to non-trivial gluing operations, then $|V(G_j)| > |V(G_i)|$ or (and) $|E(G_j)| > |E(G_i)|$. From the finiteness of each graph in $\mathfrak{S}$ it follows that all paths leading to any vertex of the graph $G^\mathfrak{S}$, contain a finite number of different vertices. Graph $G^\mathfrak{S}$ cannot have oriented simple cycles since their vertices would correspond to isomorphic graphs. So all the paths leading to any vertex the graph $G^\mathfrak{S}$, have finite length. It follows that the set vertices of the graph $G^\mathfrak{S}$ with indegrees equal to zero is not empty. The graphs corresponding to such vertices form the elemental basis of a closed class of graphs $\mathfrak{S}$, since none of these graphs can be expressed as superposition of other graphs from $\mathfrak{S}$. The uniqueness of the elemental basis follows from its definition. $\square$

**Theorem 2.1.2** [7]. Power of the set of all closed classes graphs are continual.

**Proof.** The number of closed classes of graphs can be estimated from above the number of all subsets of the countable set of all graphs. For lower estimates it is enough to select an infinite sequence graphs, each of which cannot be represented by a superposition of others sequence graphs, for example, $C_n, n = 1, 2, \ldots$. Choosing all possible subsets
of this sequence as elementary bases of the corresponding closed classes, we obtain a continuous set of closed classes.

**Corollary 2.1.1** There are closed classes of graphs with $|B_e| = \infty$.

For operational bases, was obtained the following result.

**Theorem 2.1.3** [8]. Every closed class of graphs $\mathcal{Z}$ has an operational basis.

**Proof.** From the definition of a complete system of types of gluing operations it follows that each closed class of graphs $\mathcal{Z}$ has a nonempty set of such systems. This set contains, for example, a system including subgraphs of all graphs from $\mathcal{Z}$. If graphs from $\mathcal{Z}$ can be constructed using canonical superpositions then we have also the complete system including only subgraphs of graphs from $B_e$. Full can also be subgraph systems of various other subsets of graphs from $\mathcal{Z}$.

Each complete system of types of gluing operations is defined by the set of graphs. Put each graph $G$ in one-to-one match the positive integer $n(G)$ so that no graph with a higher number would not be isomorphic to a subgraph of a graph with a smaller number. This can always be done, for example, by numbering the graphs in the non-decreasing order of the sum of the number of their vertices and edges. Graph, corresponding to the number $n$, we denote by $G(n)$.

We can assign the characteristic binary fraction $0, r_1r_2...r_n...$ to each set of graphs $R$, in which $r_n = 0$ if $G(n)$ does not belong to the set $R$ and $r_n = 1$ if $G(n)$ belongs to the set $R$. According to this rule we can associate binary fraction as real number with each complete system of types of gluing operations.

The set of all complete systems of types of gluing operations corresponds to the set $A$ of real numbers. Since these numbers are positive, there exists a number $\inf A$. Show that the number $\inf A$ also belongs to the set $A$, that is, it corresponds to the complete system of types of gluing operations. Suppose that the system corresponding to the number $\inf A$ is not complete. Then there is a graph $G \in \mathcal{Z}$ that cannot be constructed from graphs of elemental basis $B_e$ using a system of types of gluing operations corresponding to the number $\inf A$.

From the definition of infimum, it follows that there is a complete system of types of gluing operations, which corresponds to the number $M < (\inf A + 2^{-n(G)})$. Since the graphs with numbers greater than $n(G)$ are not can be used in the construction of the graph $G$ (they are not isomorphic to own subgraphs of $G$), and the system corresponding to the number $M$ is complete, then the graph $G$ can be constructed using system of types of gluing operations corresponding to the number $\inf A$.

A complete system of types of gluing operations corresponding to the number $\inf A$ is minimal on inclusion, since any of its own subset corresponds to a smaller number, but the complete system of types of gluing operations corresponding to a number less than $\inf A$ does not exist. Thus, the system of types of operations corresponding to the number of $\inf A$, is the operational basis of the closed class of graphs $\mathcal{Z}$. □

The resulting graph of any gluing operation saves such properties of graphs-operands as the absence of isolated vertices, loops or edges.

It is not difficult to see that if all the graphs from elemental basis $B_e$ are connected, then to obtain disconnected graphs it is necessary to include a null graph $O_0$ in the operational basis $B_o$. 4
Theorem 2.1.4 [7]. Closed class of all graphs has an elemental basis $B_e = \{O_1, C_1, K_2\}$ and operational basis $B_o = \{O_0, O_1, O_2\}$.

Proof. As each gluing operation saves in graphs the lack of isolated vertices, loops and edges, we have the inclusion $\{O_1, C_1, K_2\} \subset B_e$. Since the elemental basis of each a closed class of graphs is unique, then for proof reverse inclusion enough to show that any graph $G$ we represent as a superposition of graphs from set $\{O_1, C_1, K_2\}$. It can be done, for example, like this:

1) construct an empty $|V(G)|$-vertex graph using $(|V(G)| - 1)$ gluing operations on $O_0$ implementing graphs of the form $(g \circ O_1)O_0$, where $g$ is the resulting graph of the previous one gluing operations ($g = O_1$ when performing the first operation);

2) supplement the empty graph with edges up to the graph $G$, using $|E(G)|$ gluing operations that implement graphs of the form $(g \circ K_2)O_2$ and (or) graphs of the form $(g \circ C_1)O_1$.

To complete the proof, it suffices to establish minimal on inclusion the numbers of graphs included in the set $B_o$. Without gluing operations on $O_0$ can not be implemented disconnected graphs since all graphs of elemental basis are connected. Graphs containing multiple loops and edges cannot be constructed without using gluing operations on $O_1$ and $O_2$ respectively. □

In the proof of Theorem 2.1.4, only canonical superpositions were used. This way of constructing graphs is always admissible in the following case.

Lemma 2.1.1 [9]. Graphs of a closed class $\mathcal{I}$ with generators bases $B_e$ and $B_o$ can be constructed using canonical superpositions if the operational basis $B_o = \{O_0, O_1, \ldots, O_n\}$, where $n = \max_{G \in B_e} |V(G)|$.

Proof. Each superposition operation that implements arbitrary graph from $\mathcal{I}$, you can match its coverage to the $B_e$ graphs. Consider the graph cover. Its vertices are subgraphs isomorphic to the graphs from $B_e$ and edges join vertices corresponding to intersecting subgraphs.

In a connected graph of a coverage containing at least two vertices, always there is a vertex whose removal preserves connectivity. Any vertex deletion process from the graph of the covering preserving the connectivity, the operation of canonical superposition corresponds to its reverse consideration.

Since when using gluing on empty subgraphs intersect, then for any order of graph assembling, all gluing subgraphs will be empty. Only the number of identified vertices in each specific operation can vary. When using canonical superpositions, it cannot exceed the number of vertices in the added graph $G \in B_e$, therefore, all operations satisfy the conditions of the lemma.

For disconnected coverage graphs, firstly using canonical superposition of gluing operations a graph is constructed on $O_0$, each connected component of which is isomorphic to the graph $G \in B_e$, which is the original in the canonical superposition realizing this component. □

A canonical superposition is always possible if the gluing operation has the property of associativity. Restrictions under which the gluing operation is associative were considered in [10]. There, in particular, the associativity of operations over complete subgraphs of gluing was shown.
2.2. Structure of closed classes of graphs

In the theory of functional systems with operations the concept of studying the structure of closed classes of boolean functions by using precomplete classes is considered \[11\]. Closed class \(\mathcal{Z}_1 \subset \mathcal{Z}_2\) called precomplete in a closed superclass \(\mathcal{Z}_2\) if \(\mathcal{Z}_1 \neq \mathcal{Z}_2\), but adding to \(\mathcal{Z}_1\) of any element \(r \in \mathcal{Z}_2 \setminus \mathcal{Z}_1\) we get \(\mathcal{Z}_1 \cup r = \mathcal{Z}_2\). The precomplete class \(\mathcal{Z}_1\) is called trivial if the set \(\mathcal{Z}_2 \setminus \mathcal{Z}_1\) contains exactly one element. For closed classes of graphs the concept of a precomplete class is not informative.

**Theorem 2.2.1** [5]. All precomplete closed classes of graphs are trivial.

**Proof.** Assume that the subclass \(\mathcal{Z}_1\) does not contain two the graphs \(G_1\) and \(G_2\) from the superclass \(\mathcal{Z}_2\). If \(|V(G_1)| < |V(G_2)|\) and (or) \(|E(G_1)| < |E(G_2)|\), then the graph \(G_1\) cannot be built using the graph \(G_2\) because the graphs are operands gluing operations are isomorphic to subgraphs of the resulting graph. If \(|V(G_1)| = |V(G_2)|\) and \(|E(G_1)| = |E(G_2)|\), but \(G_1 \not\cong G_2\), then by the same reason, none of the graphs \(G_1\) and \(G_2\) cannot be constructed using another graph. Thus, all precomplete closed classes of graphs do not contain only one graph from their superclasses and are trivial. □

To describe the structure of closed classes of graphs, we introduce the concept of basis precompleteness \[4\]. Class \(\mathcal{Z}_1\) is precomplete in elemental basis in \(\mathcal{Z}_2\) if \(B_e\) of the class \(\mathcal{Z}_1\) does not contain only one of the graphs of elemental basis of the class \(\mathcal{Z}_2\) and the operational bases of both classes coincide. Similarly, the class \(\mathcal{Z}_1\) is precomplete in operational basis in \(\mathcal{Z}_2\) if \(B_o\) of the class \(\mathcal{Z}_1\) does not contain only one of the graphs of the operational basis of the class \(\mathcal{Z}_2\) and the elemental bases of both classes coincide.

**Theorem 2.2.2** [9]. The closed class of all graphs contains 35 nontrivial closed subclasses that are bases precomplete on elemental or operational basis in their supersclasses.

**Proof.** Consider the closed classes of graphs whose generators bases are subsets of the bases \(B_e\) or (and) \(B_o\) of closed class of all graphs. Constructive descriptions and the characteristic properties of these classes are listed in the table 1 for connected graphs and in table 2 for graphs that admit different number of connected components.

Number of vertices and edges in graphs (subgraphs) are denoted as \(N(n)\) and \(M(m)\), respectively. Characteristic properties are formulated on analysis of the generating bases used in the construction of graphs.

The union of the graphs \(G_1\) and \(G_2\) without intersections obtained with the help of gluing operations \((G_1 \circ G_2)O_0\). Adding edges and loops to the current graph \(G\) is implemented, respectively, by gluing operations \((G \circ K_2)O_2\) and \((G \circ C_1)O_1\). Adding edge with vertex to the current graph \(G\) can be done by one of the following gluing operations \((G \circ K_2)O_1\) or \(((G \circ O_1)O_0, \circ K_2)O_2\). Considering Lemma 2.1.1, the constructing of graphs can be restricted to canonical superpositions. As elemental bases various subsets of graphs from \(B_e\) are selected. As operational bases - the minimal subsets on inclusion of graphs from \(B_o\) specifying types of gluing operations, applicable to superpositions of graphs selected earlier from elemental bases. Minimal on inclusion means that a closed class with the same characteristic property cannot be obtained by using any of own subsets from \(B_o\). Subsets of graphs from \(B_o\), not satisfying the specified constraints for the graphs selected from \(B_e\) correspond to empty cells in the tables. The
cells in the table 1 also remain empty, if graphs selected from $B_o$ are not isomorphic to subgraphs of any graph selected from $B_e$ (cells with subsets of $B_e$ not containing $K_2$ and subset of $B_o$ containing only $O_2$). □

Table 1. The characteristic properties of the connected graphs

| $B_e \setminus B_o$ | $O_1, O_2$            | $O_2$                                      | $O_1$                                      |
|---------------------|-----------------------|-------------------------------------------|-------------------------------------------|
| $O_1, C_1, K_2$     | All connected graphs  | Graph $C_1$ or multigraphs with $N \leq 2$| Graphs without cycles $C_n, n \geq 2$     |
| $C_1, K_2$          | Graphs with $M \geq 1$| Graph $C_1$ or multigraphs with $N = 2$   | Graphs with $M \geq 1$ without cycles $C_n, n \geq 2$ |
| $O_1, K_2$          | Multigraphs           | Multigraphs with $N \leq 2$               | Trees                                     |
| $K_2$               | Multigraphs with $N \geq 2$ | Multigraphs with $N = 2$                   | Trees with $N \geq 2$                     |
| $O_1, C_1$          | —                     | —                                         | Graphs with $N = 1$                       |
| $C_1$               | —                     | —                                         | Graphs with $N = 1$ and $M \geq 1$         |

Using data from tables 1 and 2, we construct for a closed class of all graphs the diagram of the inclusions of all its closed subclasses, being basis precomplete in the relevant superclasses. Add for completeness the lower part of the diagram with four trivial closed classes (Fig. 4).
Table 2. The characteristic properties of the disconnected graphs

| $B_e \setminus B_o$ | $O_0, O_1, O_2$ | $O_0, O_2$ | $O_0, O_1$ | $O_0$ |
|---------------------|-----------------|------------|------------|-------|
| $O_1, C_1, K_2$     | All graphs      | No graphs with $N = 1$ and $M \geq 2$ | Graphs without cycles $C_n, n \geq 2$ | Connectivity components isomorphic to $O_1 \vee C_1 \vee K_2$ |
| $C_1, K_2$          | Graphs without isolated vertices | Graphs with perfect edge matching | Graphs without isolated vertices and cycles $C_n, n \geq 2$ | Connectivity components isomorphic to $C_1 \vee K_2$ |
| $O_1, K_2$          | —               | Multigraphs | Woods      | Connectivity components isomorphic to $O_1 \vee K_2$ |
| $K_2$               | Multigraphs     | Multigraphs with perfect edge matching | Woods without isolated vertices | Connectivity components isomorphic to $K_2$ |
| $O_1, C_1$          | —               | Connectivity components with $n = 1$ and $m \geq 0$, if $N = 1$ then $M \leq 1$ | Connectivity components with $n = 1$, $m \geq 0$ | Connectivity components isomorphic to $O_1 \vee C_1$ |
| $C_1$               | —               | Connectivity components with $n = 1$, $m \geq 1$, $M - N = 2k$, $k = 0, 1, \ldots$ | Connectivity components with $n = 1$, $m \geq 1$ | Connectivity components isomorphic to $C_1$ |
| $O_1$               | —               | —          | —          | Empty graphs |

*The parity of the sum of the number of loops in the components with $n \geq 2$ must coincide with the parity of the sum $\sum_{i=1}^{q}(m_i - 1)$, where $q$ is the number of components with $n = 1$, $m_i$ is the number of loops in the $i$-th component*
Denote the subclasses of the set $\mathcal{I}$ with generator bases $B'_e \subseteq B_e$ and $B'_o \subseteq B_o$ as $\mathfrak{T}(B'_e, B'_o)$. The generating bases are given below.

1. $\mathfrak{T}([O_1, C_1, K_2], \{O_0, O_1, O_2\})$. 2. $\mathfrak{T}([O_1, C_1, K_2], \{O_1, O_2\})$.
3. $\mathfrak{T}([C_1, K_2], \{O_0, O_1, O_2\})$. 4. $\mathfrak{T}([O_1, C_1, K_2], \{O_0, O_1\})$. 5. $\mathfrak{T}([O_1, C_1, K_2], \{O_0, O_2\})$.
6. $\mathfrak{T}([O_1, K_2], \{O_1, O_2\})$. 7. $\mathfrak{T}([C_1, K_2], \{O_1, O_2\})$. 8. $\mathfrak{T}([O_1, C_1, K_2], \{O_1\})$.
9. $\mathfrak{T}([O_1, C_1, K_2], \{O_2\})$. 10. $\mathfrak{T}([C_1, K_2], \{O_0, O_1\})$. 11. $\mathfrak{T}([C_1, K_2], \{O_0, O_2\})$.
12. $\mathfrak{T}([K_2], \{O_0, O_1, O_2\})$. 13. $\mathfrak{T}([O_1, K_2], \{O_0, O_1\})$. 14. $\mathfrak{T}([O_1, C_1], \{O_0, O_1\})$.
15. $\mathfrak{T}([O_1, C_1], \{O_0, O_2\})$. 16. $\mathfrak{T}([O_1, C_1, K_2], \{O_0\})$. 17. $\mathfrak{T}([O_1, K_2], \{O_0, O_2\})$.
18. $\mathfrak{T}([K_2], \{O_1, O_2\})$. 19. $\mathfrak{T}([C_1, K_2], \{O_1\})$. 20. $\mathfrak{T}([C_1, K_2], \{O_2\})$.
21. $\mathfrak{T}([O_1, C_1], \{O_1\})$. 22. $\mathfrak{T}([O_1, K_2], \{O_1\})$. 23. $\mathfrak{T}([O_1, K_2], \{O_2\})$.
24. $\mathfrak{T}([C_1, K_2], \{O_0\})$. 25. $\mathfrak{T}([K_2], \{O_0, O_1\})$. 26. $\mathfrak{T}([C_1], \{O_0, O_2\})$.
27. $\mathfrak{T}([C_1], \{O_0, O_1\})$. 28. $\mathfrak{T}([K_2], \{O_0, O_2\})$. 29. $\mathfrak{T}([O_1, C_1], \{O_0\})$.
30. $\mathfrak{T}([O_1, K_2], \{O_0\})$. 31. $\mathfrak{T}([K_2], \{O_1\})$. 32. $\mathfrak{T}([K_2], \{O_2\})$. 33. $\mathfrak{T}([C_1], \{O_1\})$.
34. $\mathfrak{T}([C_1], \{O_0\})$. 35. $\mathfrak{T}([K_2], \{O_0\})$. 36. $\mathfrak{T}([O_1], \{O_0\})$. 37. $\mathfrak{T}([K_2], \{K_2\})$.
38. $\mathfrak{T}([C_1], \{C_1\})$. 39. $\mathfrak{T}([O_1], \{O_1\})$. 40. $\mathfrak{T}([O_0], \{O_0\})$. 41. $\mathfrak{T}([O_1], \{O_1\})$.

Figure 4. The diagram of inclusions of all basis precomplete subclasses of the set of all graphs.
The graphs from the considered closed classes possess the most "strong" characteristic properties because for their constructive descriptions enough using finite elemental and operational bases.

Constructive descriptions of closed classes with more "weak" characteristic properties also include restrictions on the choice of identified subgraphs in the operand graphs and can be on the method of identification (situations corresponding to Fig.2 and Fig.3). Gluing operations satisfying such restrictions are denoted as $H$-gluing operations. A class of graph closed with respect to the operations of $H$-gluing is called for brevity an $H$-closed class.

These restrictions determine the dynamic characterization of classes of graphs with a given property. Together with generating bases, they give constructive descriptions of closed classes of graphs.

2.3. Constructive descriptions of closed classes of graphs.

We will consider closed classes of graphs with some classical properties.

2.3.1. Triangulated graphs.

A graph $G$ is called triangulated, or chordal if it does not contain a simple cycle $C_n, n \geq 4$ without a chord - edge connecting non-adjacent vertices of a cycle.

Dynamic characterization

From the definition of triangulated graphs it follows that the presence or absence in the graph of multiple edges does not affect the triangulation property, then we restrict ourselves to considering simple triangulated graphs.

Lemma 2.3.1.1 [7]. Operations over the complete subgraphs of gluing preserve the triangulation of graphs.

Generating bases

The operations preserve the triangulation of graphs are denoted as the $H_t$-gluing.

Theorem 2.3.1.1 [7]. $H_t$-closed class of triangulated graphs has countable generating bases $B_e = \{O_1, K_2, K_3, \ldots\}$ and $B_o = \{O_1, K_2, K_3, \ldots\}$

2.3.2. Planar graphs.

A graph is called planar if it admits a geometric implementation on the plane, that is, the vertices of the graph can arrange on the plane so that none of its edges intersect and do not go through extraneous vertices.

Dynamic characterization

Two characterizations of planar graphs were considered: based on traditional geometric representations and using only the set-theoretic approach based on constructive descriptions of graphs.

Geometric representations

Lemma 2.3.2.1 [7]. Each planar graph $G$ can be flat packed into where all the vertices of an arbitrary face $f$ with a connected boundary are located on a circle inscribed in the face $f$ in the order of circular traversal of faces.

Suppose that all vertices from $V(G'_1)$ and $V(G'_2)$ belong to flat stacking of planar graphs $G_1$ and $G_2$ respectively to faces $f_1$ and $f_2$ with connected boundaries. Convert
flat styling graphs $G_1$ and $G_2$ so that all vertices of the faces $f_1$ and $f_2$ were located on a circle inscribed in the face $f$ in the order of circular traversal of faces. We identify the subgraphs $G'_1$ and $G'_2$, choosing pairs identified vertices in accordance with circular rounds of these circles. Gluing operations matching the specified restrictions on the choice and method of identifying the subgraphs $G'_1$ and $G'_2$ are denoted as operations of $H_p$-gluing.

Lemma 2.3.2.2 [7]. $H_p$-gluing operations preserve planar graphs.

Set-theoretic approach

Let $G' \subset G$. The subgraph of the graph $G$ generated by the edges of set of $E(G) \setminus E(G')$ is the shell of subgraph $G'$ in the graph $G$. A connected subgraph of a planar graph $G$, all the vertices and edges of which belong to the boundary of any connected face $f$, is denoted by $G_f$.

The graph $G_f$ is represented in the form of superposition of simple cycles and trees with gluing subgraphs $O_1$. The chain connecting two vertices of the same simple cycle is called chordal if its edges and interior vertices do not belong to the cycle. $G_f \subset G$ is the graph of maximal face if its shell in $G$ consists only from chordal chains.

Lemma 2.3.2.3 [12]. The subgraph $G'$ of the planar graph $G$ is graph of a maximal face in some plane packing of the connected graph $G$ if and only if:

1. $G'$ is realized by a superposition of simple cycles and trees with gluing subgraphs $O_1$;
2. $G'$ is selected in $G$ so that:
   a) the shell $G'$ consists of a set of chordal chains;
   b) each pair of chordal chains connecting the vertices, arranged along the cycle in alternating order, has a common inner vertex;
   c) no three vertices of one cycle connect to the two vertices from the shell of the subgraph $G'$ by disjoint chains.

Consider the depth first search procedure, in which the following restrictions are used:

1) do not select edges that are bridges in unfulfilled subgraph of the source graph $G'$, if there are other possibilities;
2) among the edges that are bridges, do not choose those that belong to the two connected components of the source graph $G'$, if there are other possibilities.

This procedure is called db-search. In [13] it was shown that the numbering of the vertices of a planar graph, realized by superposition of trees and simple cycles with subgraphs of gluing $O_1$, in accordance with a db-search allows single page flat lay.

Theorem 2.3.2.1 [14]. Each planar graph $G$ admits a flat packing in which all vertices of the face $f$ are located on the circle inscribed in the face $f$, in accordance with any db-search on the subgraph $G_f$.

Theorem 2.3.2.2 [1,15]. The graph $G = (G_1 \circ G_2)\bar{G}$ preserves the planarity of the operand graphs $G_1$ and $G_2$ if:

1) $G'_1 \subseteq G'_2 \subseteq G_1$ and $G'_2 \subseteq G'_1 \subseteq G_2$, $G'_1 \cong G'_2$ and the subgraphs $G'_1$ and $G'_2$ have the following properties:
   a) they are realized by superpositions of simple cycles and trees with gluing operations at $O_1$;
b) their shells consist of sets of chordal chains;
c) each pair of chordal chains of the shell connecting the vertices, arranged along the
cycle in alternating order, has a common inner vertex;
d) no three vertices of one cycle connect to the two vertices of the shell by disjoint
chains.

2. Pairs of identifiable vertices of the subgraphs \(G_1'\) and \(G_2'\) are selected according to
their order enumerations for arbitrary db-search for \(G_1'\) and \(G_2'\). Identifiable pairs edges
are selected from the set of multiple edges formed as a result of the identification of the
vertices.

So, the restrictions on operations of \(H_p\)-gluing preserving the planarity of graphs
can be formulated using only set-theoretic approach.

**Generating bases**

The generating bases of the closed class of all planar graphs coincide with the bases
of the class of all graphs. Only the above restrictions are introduced on the choice of
identifiable subgraphs in the operand graphs and on the identification method.

**Theorem 2.3.2.3** [7]. The \(H_p\)-closed class of planar graphs has an elemental basis
\(B_e = \{O_1, C_1, K_2\}\) and the operational basis \(B_o = \{O_0, O_1, O_2\}\).

We restrict ourselves further to the consideration of simple planar graphs. The gluing
operation preserves the absence of multiple edges, if each a pair of vertices non-adjacent
in \(\tilde{G}\) corresponds to a pair of non-adjacent vertices in at least one of the operand graphs
\(G_1\) or \(G_2\). Such gluing operations are referred to as \(\prec H \succ\)-gluing operations.

**Corollary 2.3.2.1.** \(\prec H_p \succ\)-closed class of simple planar graphs has elemental basis
\(B_e = \{O_1, K_2\}\) and operational basis \(B_o = \{O_0, O_2\}\).

All restrictions on \(H\)-gluing operation can be divided into internal ones, without
which gluing operations cannot preserve the required characteristic property of graphs
(note that all previously used restrictions were internal ones), and additional external
ones that affect the power of generating bases, order of graph assembly, the amount of
redundancy of the constructive description, etc.

Consider as an example gluing operations in which the set vertices of the subgraph
gluing \(V(\tilde{G})\) is separating set in the resulting graph. Denote such operations as \(H_s\)-gluing. Then, for the class of simple planar graphs we have \(\prec H_{ps} \succ\)-gluing operations.

**Theorem 2.3.2.4** [16]. \(\prec H_{ps} \succ\)-closed class of simple planar graphs has the generating
bases \(B_e = \{O_1, K_2, K_3, K_4\}\) and \(B_o = \{O_0, O_1, O_2, O_3, O_4, O_5\}\).

If we go to gluing by the generated subgraphs \(\prec H \succ\)-gluing) and require that the
sets of vertices \(V(\tilde{G})\) are minimal separating in the resulting graphs then, for the class
of simple planar graphs, \(\prec H_{ps} \succ\)-gluing operations should be used.

**Theorem 2.3.2.5** [17]. The \(\prec H_{ps} \succ\)-closed class of simple planar graphs has
elemental basis \(B_e = \{O_1, K_2, K_3, K_4\}\). Operational basis \(B_o\) contains 16 types operations,
whose gluing subgraphs are isomorphic to graphs from the set
\[
\{ O_0, O_1, O_2, K_2, O_3, (O_1 \circ K_2)O_0, L_3, K_3, O_4, \\
(O_2 \circ K_2)O_0, (K_2 \circ K_2)O_0, (O_1 \circ L_3)O_0, L_4, C_4, L_5, C_5 \}.
\]
The operational basis will increase even more if you restrict using of gluing operations so that the number of edges in $E(\tilde{G})$ is also minimal. The corresponding operations denoted as operations of $< H_{\text{pee}} >$-gluing.

**Theorem 2.3.2.6** [7,18]. The $< H_{\text{pee}} >$-closed class of simple planar graphs has an elemental basis $B_e = \{O_1, K_2, K_3, K_4\}$. Operational basis $B_o$ contains 22 types operations, whose gluing subgraphs are isomorphic to graphs from the set

\[
\{ O_0, O_1, O_2, K_2, O_3, (O_1 \circ K_2)O_0, L_3, K_3, O_4, \\
(O_2 \circ K_2)O_0, (K_2 \circ K_2)O_0, (O_1 \circ L_3)O_0, L_4, C_4, O_5, (O_3 \circ K_2)O_0, \\
((K_2 \circ K_2)O_0 \circ O_1)O_0, (O_2 \circ L_3)O_0, (O_1 \circ L_4)O_0, (K_2 \circ L_3)O_0, L_5, C_5 \}.
\]

**Triangulated planar graphs**

**Theorem 2.3.2.7** [7]. $< H_p >$-closed class of triangulated simple planar graphs has such bases $B_e = \{O_1, K_2, K_3, K_4\}$ and $B_o = \{O_0, O_1, K_2, K_3\}$.

**Maximality planar graphs**

Simple planar graph $G$ is called **maximal**, if adding any edge to $G$ takes it out of class planar.

**Theorem 2.3.2.8** [7]. $< H_p >$-closed class of maximal planar graphs has countable elemental basis $B_e$ and operational basis $B_o = \{K_3\}$.

Constructive descriptions of closed classes of outerplanar as well as triangulated and maximal outerplanar graphs are given in [19].

### 2.3.3. Euler graphs.

A connected graph $G$ with even degrees of vertices is called **Euler**.

**Dynamic characterization**

**Lemma 2.3.3.1** [20]. If the graphs $G_1$ and $G_2$ are Euler, then the resulting graph $G = (G_1 \circ G_2)\tilde{G}$ will be Euler if and only if the degrees all vertices of the gluing subgraph $\tilde{G}$ are even.

In order to reduce the redundancy of constructive descriptions of Euler graphs we restrict ourselves to the use of glue operations on empty subgraphs. We denote them as operations of the $H^0$-gluing.

**Generating bases**

**Theorem 2.3.3.1** [21]. The $H^0$-closed class of Euler graphs has generating bases $B_e = \{C_1, C_2, ...\}$ and $B_o = \{O_1, O_2, ...\}$.

The infinity of the operational basis follows from the result of Alon [22].

### 2.3.4. Euler planar graphs.

Operations of gluing preserve the euler and planar properties of graphs are denoted as $H^0_{\emptyset}$-gluing. Based on studies performed in [8,20,21,23,24], the following result was obtained.
Theorem 2.3.4.1. $H^3_o$-closed class of Euler planar graphs has the elemental basis $B_e = \{C_1, C_2, \ldots\}$ and three operating bases $B^1_o = \{O_1, O_2, O_3\}$, $B^2_o = \{O_1, O_2, O_4\}$ and $B^3_o = \{O_1, O_2, O_5\}$.

Graphs that do not contain vertices of the second degree are called topological. If the gluing operations are carried out on the generated subgraphs, then the operational basis of the closed class of simple topological Euler planar graphs becomes infinite [25].

2.3.5. Hamiltonian graphs.

A graph $G$ is called Hamiltonian if it is possible to select in it a cycle containing all the vertices of the graph.

Dynamic characterization

Lemma 2.3.5.1 [26]. If $G_1$ and $G_2$ are Hamiltonian graphs, then the resulting graph $G = (G_1 \circ G_2) \tilde{G}$ will also be Hamiltonian under any of the following conditions:

1) the identified subgraph of at least one of the operand graphs contains all its vertices;
2) the identifiable subgraphs of the operand graphs consist of two vertices that are adjacent in their Hamiltonian cycles.

Gluing operations satisfying any of these restrictions are called $H_g$-gluing operations. Since the presence or absence in the graph of multiple edges does not affect the Hamiltonian property, then we restrict ourselves to using operations $\triangleleft H_g \triangleright$-glues excluding the appearance of multiple edges.

Generating bases

Here, as well as for Euler planar graphs, there are three operational bases.

Theorem 2.3.5.1 [26]. The $\triangleleft H_g \triangleright$-closed class of Hamiltonian graphs has elemental basis $B_e = \{C_1, C_2, \ldots\}$ and three following operational bases $B^1_o = \{O_1, K_2, C_4, C_5\ldots\}$, $B^2_o = \{O_1, K_2, L_3, L_4, \ldots\}$ and $B^3_o = \{O_1, K_2, (L_{n'} \circ L_{n''}) O_0\}, n', n'' \geq 2$.

If canonical superpositions are admissible when constructing graphs of some $H$-closed class, then such a class is briefly called as canonical $H$-closed class.

Corollary 2.3.5.1 Class of Hamiltonian graphs canonically $\triangleleft H_g \triangleright$-closed with elemental basis $B_e = \{C_1, C_2, \ldots\}$ and two operational bases $B^2_o = \{O_1, K_2, L_3, L_4, \ldots\}$ or $B^3_o = \{O_1, K_2, (L_{n'} \circ L_{n''}) O_0\}, n', n'' \geq 2$.

2.3.6. Bipartite graphs.

A graph $G$ is called bipartite if there exists a partition of the set of its vertices $V(G)$ into two subsets $V_1$ and $V_2$, each of which generates an empty graph. If the graph $G$ is not empty, then the ends each edge $e \in E(G)$ belongs to different parts.

Dynamic characterization

Lemma 2.3.6.1 [27]. If $G_1$ and $G_2$ are bipartite graphs, then the result of gluing them together graph $G = (G_1 \circ G_2) \tilde{G}, |V(\tilde{G})| \geq 2$ is bipartite if and only if when for any vertices $v_1, v_2 \in V(\tilde{G})$, connected by chains in $G(E_1 \setminus \tilde{E})$ and $G(E_2 \setminus \tilde{E})$, lengths these chains have the same parity.

This restriction on gluing operations is denoted by $H_0$. 

14
Generating bases

**Theorem 2.3.6.1** [27]. *The class of bipartite graphs $H_b$-closed with an elemental basis $B_e = \{O_1, K_2\}$ and the operational basis $B_o = \{O_0, O_2\}$.

3. Conclusion

Concluding the review, we note the following points.

1. The constructive descriptions of graphs show the efficiency of using gluing operations to uniformly formulate the conditions for preservation of various characteristic properties of graphs in terms of restrictions on the type of identified subgraphs, their choice in operand graphs and the identification method.

   The "payment" for this universality is the redundancy introduced by gluing operations in the information about the graph with labelled vertices. Estimates of the magnitude of this redundancy are obtained for the Eulerian graphs [28], some classes of triangulated graphs [29] and Hamiltonian graphs [30,31].

   When considering unlabelled graphs, knowledge of their construction processes can significantly reduce the length of the graph code and complexity of decoding algorithms by using the numbering of the vertices, reflecting graph assembly order [32,33].

2. The presence of several operational bases for some closed classes of graphs allows you to formulate tasks of optimal graph synthesis. For example, in the classical statement of minimizing the number of gluing operations needed to build a graph.

   Another class of optimal graph synthesis problems arises in supercomputer physical-mathematical modeling design of large graphs at the stage when it is necessary consistent return from the reduced graph to the original graph of large dimension with preserving the solution obtained on the previous steps.

   This process can be implemented using subgraphs duplication operations with full or partial preserving their neighborhoods in the current graph. Such operations named as cloning operations are discussed in [34,35].

   The task of optimal graph synthesis is put here as follows: based on the graph a small dimension it is necessary to construct a graph of large dimension with specified properties for minimum number of cloning operations. Such a task considered for trees and bipartite graphs in [36].

3. Constructive approach methodology to solving applied tasks on graphs will be successful if implemented the following principles:

   - choose as the source graphs a complete system of graphs (not necessarily elemental basis) for each of them the considered problem is solved most effectively
   - choose restrictions on the gluing operations so that the resulting graph can be built using canonical superposition ("brick by brick"). Its simplified structure analysis of graphs and therefore, finding the solution.

   Using the above methodology illustrated in [1] on examples of solving applied problems economical coding and optimal linear placement of graphs.

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References

[1] Iordanski M. A. Constructive graph theory and its applications. Nizhny Novgorod: Publishing house "Cyrillic 2016. 172 p.(Russian)

[2] Iordanski M. A. About operations on graphs // Discrete models in the theory of control systems // Proceedings of the IV International Conference, Krasnovidovo, June 19-25, 2000. / Edited by V.B. Alekseev, B. A. Zakharova. — M. : MAX Press, 2000. — P. 33–34.(Russian)

[3] Iordanski M. A. Constructive descriptions of graphs // Proceedings of the XI International Seminar-school "Synthesis and Complexity of Control Systems"(Nizhny Novgorod, November 20-25, 2000). Part I. — M.: Publishing House of the center for applied research on Department of the mechanical-mathematical Faculty of Moscow State University, 2001.— P. 80–84.(Russian)

[4] Iordanski M. A. The structure and methods of generating closed classes of graphs // Proceedings of the VII International Seminar "Discrete Mathematics and Its applications"(Moscow, January 29 - February 2, 2001). Part II. / Edited by O.B. Lupanov.— M.: Publishing House of the center for applied research on Department of the mechanical-mathematical Faculty of Moscow State University, 2001. — P. 218–221.(Russian)

[5] Iordanski M. A. The structure and methods of generating closed classes of graphs// Discrete Mathematics. 2003. Vol. 15, Iss. 3. — P. 105–116.(Russian)

[6] Iordanski M. A. Functional approach to the representation of graphs // Reports of the Russian Academy of Sciences. — 1997. — T. 353, No 3. — P. 303-305.(Russian)

[7] Iordanski M. A. Constructive graph descriptions // Discrete analysis and operation research. 1996. Vol. 3, Iss. 4. — P. 35–63.(Russian)

[8] Burkov E. V. Operational bases of closed classes of graphs // Proceedings of the IX International Seminar "Discrete Mathematics and Its applications Moscow, June 18-23, 2007. — M.: Publishing House of the Moscow State University M.V. Lomonosov, 2007. — P. 105–116.(Russian)

[9] Iordanski M. A. Constructive classification graphs // Modeling and analysis of information systems. 2012. Vol.19, Iss. 4. — P. 144–153.(Russian)

[10] Iordanski M. A. Some questions of analysis and synthesis of graphs // Proceedings of the First International Conference "Mathematical Algorithms"(Nizhny Novgorod, August 16-19, 1994). / Edited by M.A. Antonets, V. E. Alekseev, V.N. Shevchenko. — Nizhny Novgorod: Publishing House of the Nizhny Novgorod University, 1995. — P. 33–38.(Russian)

[11] Yablonsky S.V. Introduction to discrete mathematics. — M.: Publishing House "Science 2001. 384 p.(Russian)

[12] Iordanski M. A. Set-theoretic description of subgraphs of faces of planar graphs // Problems of Theoretical cybernetics. Materials of the XIV International Conference (Penza, May 23-28, 2005).— M.: Publishing House of the Department of the mechanical-mathematical Faculty of Moscow State University, 2005. — P. 56.(Russian)

[13] Iordanski M. A. Depth Search and single page graph stacking // Problems of Theoretical cybernetics. Materials of the XIII International Conference (Kazan, May 27-31, 2002). Part I / Edited by O.B. Lupanov. — M.: Publishing House of the center for applied research on Department of the mechanical-mathematical Faculty of Moscow State University, 2002. — P. 76.(Russian)
[14] Iordanski M. A. Properties of flat stackings of planar graphs // Proceedings of the VII International Seminar "Discrete Mathematics and Its applications" (Pokrovskoye, March 4-6, 2006). — M.: MAX Press, 2006. — P. 136–138. (Russian)

[15] Iordanski M. A. Inductive description of the class of planar graphs // Proceedings of the XVI International Seminar-school "Synthesis and Complexity of Control Systems" (St. Petersburg, June 26-30, 2006). — M.: Publishing House of the Department of the mechanical-mathematical Faculty of Moscow State University, 2006. — P. 41–43. (Russian)

[16] Iordanski M. A. The complexity of constructive descriptions of planar graphs // Proceedings of the IX International Seminar-school "Synthesis and Complexity of Control Systems" (Nizhny Novgorod, December 16-19, 1998). — M.: Publishing House of the Department of the mechanical-mathematical Faculty of Moscow State University, 1999. — P. 20–24. (Russian)

[17] Iordanski M. A. Bases of planar graphs // Discrete models in the theory of control systems: V International Conference: (Ratmino, May 26-29, 2003). — M.: Publishing Department of the Faculty of Computational Mathematics and Cybernetics Moscow State University M.V. Lomonosov, 2003. — P. 36–38. (Russian)

[18] Iordanski M. A. Constructive descriptions of planar graphs // Problems of Theoretical cybernetics. Abstracts of the XI International Conference (Ulyanovsk, June 10-14, 1996). — M.: Publishing House of the Russian State University for the Humanities, 1996. — P. 76–77. (Russian)

[19] Iordanski M. A. Algorithmic descriptions of outerplanar graphs // Proceedings of the Second International Conference "Mathematical Algorithms" (Nizhny Novgorod, June 26 - July 1, 1995). / Edited by M.A. Antonets, V. E. Alekseev, V.N. Shevchenko. — Nizhny Novgorod: Publishing House of the Nizhny Novgorod University, 1997. — P. 78–82. (Russian)

[20] Iordanski M. A., Burkov E. V. Constructive descriptions of Eulerian planar graphs // Discrete models in the theory of control systems: VI International Conference: Moscow (December 7-11, 2004) — M.: Publishing Department of the Faculty of Computational Mathematics and Cybernetics Moscow State University M.V. Lomonosov. 2004. — P. 167–169. (Russian)

[21] Burkov E. V. Constructive descriptions of planar and Eulerian Counts // Bulletin of Nizhny Novgorod State university N.I. Lobachevsky. Mathematics. 2010. Iss. 5 (1). P. 165–170. (Russian)

[22] Alon N. Tough Ramsey Graphs Without Short Cycles // Journal of Algebraic Combinatorics. 1995. Vol. 4, Iss. 3. — P. 189–195.

[23] Burkov E. V. Another operational basis for the class of Euler planar graphs // Problems of Theoretical cybernetics. Abstracts of the XV International Conference (Kazan, June 2-7, 2008). — Kazan: Publishing house "Fatherland". — 2008. — P. 13. (Russian)

[24] Burkov E. V. Short cycles in planar graphs with a minimum degree of four // Bulletin of Nizhny Novgorod State University. Math modeling and optimal control. — 2009. — Iss. 4. — P. 146–148. (Russian)

[25] Iordanski M. A. Countable operational basis of topological Euler planar graphs // Discrete models in the theory of control systems: VIII International Conference:
Moscow (April 6-9, 2009): Proceedings. — M.: MAX Press, 2009. — P. 127–129. (Russian)

[26] Iordanski M. A. Constructive descriptions of Hamiltonian graphs // Bulletin of Nizhny Novgorod State University N.I. Lobachevsky. Mathematics. 2012. Iss. 3 (1). — P. 137–140. (Russian)

[27] Iordanski M. A. Constructive descriptions of bipartite graphs // Problems of Theoretical cybernetics. Abstracts of the XV International Conference (Kazan, June 2-7, 2008). - Kazan: Publishing house "Fatherland". — 2008. — P. 44. (Russian)

[28] Iordanski M. A. Redundancy of constructive descriptions of Eulerian graphs // Problems of Theoretical cybernetics. Materials of the XVII International Conference (Kazan, June 16-20, 2014). Edited by Yu.I. Zhuravlev. — Kazan: Fatherland, 2014. — P. 115-116. (Russian)

[29] Iordanski M. A. Redundancy of constructive descriptions of (r,s)-trees // Discrete models in the theory of control systems: IX International Conference, Moscow and Moscow Region, May 20-22, 2015: Proceedings / Edited by V.B. Alekseev, D. S. Romanov, B. R. Danilov. — M.: MAX Press, 2015. — P. 90–91. (Russian)

[30] Iordanski M. A. Redundancy of constructive descriptions of Hamiltonian graphs // Proceedings of the XII International Seminar "Discrete Mathematics and Its applications" named after academician O.B. Lupanov, Moscow, June 20-25, 2016. — M.: Publishing House of the Department of the mechanical-mathematical Faculty of Moscow State University, 2016. — P. 290–293. (Russian)

[31] Iordanski M. A. Redundancy of constructive descriptions of Hamiltonian planar graphs // Proceedings of the XI International Seminar "Discrete Mathematics and Its applications" (Moscow, June 18-22, 2012). — M.: Publishing House of the Department of the mechanical-mathematical Faculty of Moscow State University, 2012. — P. 285–288. (Russian)

[32] Iordanski M. A. Constructive descriptions and economical coding of graphs // Problems of Theoretical cybernetics. Materials of the XII International Conference (Nizhny Novgorod, May 17-22, 1999). — M.: Publishing House of the Department of the mechanical-mathematical Faculty of Moscow State University, 1999. — P. 87. (Russian)

[33] Iordanski M. A. Constructive descriptions and economical coding of graphs // Bulletin of Nizhny Novgorod State University. Math modeling and optimal control. — 2000. — Iss. 1 (22). — P. 88–93. (Russian)

[34] Iordanski M. A. Cloning graphs // Problems of Theoretical cybernetics. Abstracts of the XVIII International Conference (Penza, June 19-23, 2017). — M: MAX Press, 2017. — P. 108–110. (Russian)

[35] Iordanski M. A. On a class of graph transformations // Discrete models in the theory of control systems: X International Conference: Moscow and Moscow Region (May 23-25, 2018) Proceedings / Edited by V.B. Alekseev, D. S. Romanov, B. R. Danilov. — M.: MAX Press, 2018. — P. 139–142. (Russian)

[36] Iordanski M. A. On the complexity of graph synthesis by cloning operations // Proceedings of the XIII International Seminar "Discrete Mathematics and Its applications" named after academician O.B. Lupanov (Moscow, June 17-22, 2019) — M.: Publishing Department of the Faculty of Mechanics and Mathematics Moscow State University M.V. Lomonosov. 2019. — P. 220–223. (Russian)