Space-filling curves of self-similar sets (II): edge-to-trail substitution rule

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Abstract
It is well known that the constructions of space-filling curves depend on certain substitution rules. For a given self-similar set, finding such rules is somehow mysterious, and it is the main concern of the present paper.

Our first idea is to introduce the notion of skeleton for a self-similar set. Then, from a skeleton, we construct several graphs, define edge-to-trail substitution rules, and explore conditions ensuring the rules lead to space-filling curves. Thirdly, we summarize the classical constructions of the space-filling curves into two classes: the traveling-trail class and the positive Euler-tour class. Finally, we propose a general Euler-tour method; using this method we show that if a self-similar set satisfies the open set condition and possesses a skeleton, then space-filling curves can be constructed. Especially, all connected self-similar sets of finite type fall into this class. Our study provides an algorithm to construct space-filling curves of self-similar sets.

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1. Introduction

Space-filling curves (SFC) have fascinated mathematicians for more than a century since Peano’s monumental work in 1890. In a series of three papers ([37], the present paper, and [38]), we develop a theory to construct SFCs of self-similar sets. For a given connected self-similar set, finding substitution rules leading to SFCs is a long-standing and difficult problem, and it is the main concern of the present paper.

1.1. A brief history of space-filling curves

SFCs of the first generation were constructed by Peano (1890), Hilbert (1891), Sierpiński (1912) and Pólya (1913) ([19, 32, 33, 41]), where the supporting sets of the SFCs are squares or triangles. Later on, people found many beautiful reptiles as well as their SFCs, where the boundaries of the reptiles are fractals; for instance, Heighway dragon curve ([11]), Gosper curve ([17]) (figure 1), etc. (A reptile is a compact set with non-empty interior such that it is a non-overlapping union of similarity copies of itself with the same size.) A survey of the early results can be found in Sagan [40]. In recent years, various interesting SFCs appear on the internet, see for example, Ventrella [48] and Teachout [45]. Figure 2 illustrates two of them. Some applications of SFC can be found in [9, 23, 29] (in analysis) and [6] (in computer science).

From the 1960’s to the 1980’s, two systematic methods were introduced to handle the SFCs. The first one is the L-system method, introduced by Lindenmayer [26], a biologist, is known to a very wide audience, see Bader [6]. The second method is the recurrent set method introduced by Dekking [12], which is an improvement of the L-system method. See [37] for the comparing of the two methods.

All the known constructions of SFC depend on certain ‘substitution rules’. Indeed, the L-system method and the recurrent set method provide exact meaning of ‘substitution rule’ and build a bridge from substitution rules to SFCs, but they do not tell us how to construct substitution rules.

There are few works on the construction of substitution rules. Hata [18] shows that if a self-similar set is generated by a linear IFS (see section 4 for the definition), then a substitution rule can be obtained. Another way is to consider the attractor of the so-called path-on-lattice IFS (this name is given by [37]). A path on a planar lattice defines a substitution rule as well as a self-similar set; if the self-similar set happens to satisfy the open set condition, then the substitution rule leads to a SFC. This method is widely used to find reptiles and SFCs by computer searching, see Fukuda et al [16], Arndt [3] and Ventrella [48]. But in this method, the self-similar sets are not given a priori. Other attempts of constructing substitution rules for special self-similar sets can be found in Remes [39] and Sirvent [42–44].

1.2. Basic definitions

Before stating our results, we give some definitions. We note that as we did in [37], the terminology ‘space-filling curve’ is used in a strong sense, that is, it is a kind of optimal parametrization. Let \(H^s\) denote the \(s\)-dimensional Hausdorff measure.
Definition 1.1. Let $K$ be a compact subset of $\mathbb{R}^d$ with $0 < \mathcal{H}^s(K) < \infty$. An onto mapping $\psi : [0, 1] \to K$ is called an optimal parametrization of $K$ if

(i) $\psi$ is almost one-to-one, that is, there exist $K' \subset K$ and $I' \subset [0, 1]$ with full measures such that $\psi : I' \to K'$ is a bijection;

(ii) $\psi$ is measure-preserving in the sense that $\mathcal{H}^s(\psi(F)) = c \mathcal{L}(F)$ and $\mathcal{L}(\psi^{-1}(B)) = c^{-1} \mathcal{H}^s(B)$, for any Borel set $F \subset [0, 1]$ and any Borel set $B \subset K$, where $c = \mathcal{H}^s(K)$.

(iii) $\psi$ is $1/s$-Hölder continuous, that is, there is a constant $c' > 0$ such that

$$|\psi(x) - \psi(y)| \leq c'|x - y|^{1/s} \quad \text{for all } x, y \in [0, 1].$$

Recall that a non-empty compact set $K \subset \mathbb{R}^d$ is called a self-similar set, if there exist contractive similitudes $S_1, \ldots, S_N : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$K = \bigcup_{j=1}^N S_j(K).$$

The family $\{S_1, \ldots, S_N\}$ is called an iterated function system, or IFS in short; $K$ is called the invariant set of the IFS. (See for instance, [2, 15, 20]). The IFS $\{S_1, \ldots, S_N\}$ is said to satisfy the open set condition (OSC), if there is an open set $U$ such that $\bigcup_{j=1}^N S_j(U) \subset U$ and the sets...
Si(U) are disjoint ([20]). If a self-similar set \( K \) satisfies the OSC and has non-empty interior, then we call \( K \) a self-similar tile; if in addition, the contraction ratios of \( S_i \) are all equal, then \( K \) is called a reptile ([21]).

In this paper, we introduce a notion of skeleton of a self-similar set, which is crucial in our theory. As soon as we have a skeleton, we construct SFCs along the following line:

\[
\text{Skeleton} \rightarrow \text{graphs and edge-to-trail substitution} \rightarrow \text{space-filling curve} \rightarrow \text{visualization.}
\]

To ‘see’ a SFC, we need to visualize or to approximate the curve, and this has been studied in [37]. In the following, we give a brief description of the first three steps.

1.3. Skeleton of a self-similar set

Let \( \{ S_j \}_{j=1}^N \) be an IFS with invariant set \( K \). Let \( A \) be a finite subset of \( K \). We call \( A \) a skeleton if it satisfies the following two conditions:

(i) It is stable under iteration, that is, \( A \subset \bigcup_{j=1}^N S_j(A) \);

(ii) It is a representative with respect to the connectedness, that is, the so-called Hata graph associated with \( A \) is connected. (See section 2 for the precise definition.)

From now on, we always assume that \( K \) is a connected self-similar set possessing a skeleton which we denote by

\[
A = \{ a_1, \ldots, a_m \}.
\]
1.4. Graphs and edge-to-trail substitution

First, we recall some terminologies of graph theory, see for instance, [7]. Let $H$ be a directed graph. We shall use $e_1 + \cdots + e_k$ to denote a walk consisting of the edges $e_1, \ldots, e_k$. We call the starting vertex and terminating vertex of a walk the origin and terminus, respectively. The walk is closed if the origin of $e_1$ and the terminus of $e_k$ coincide. A walk is called a trail, if all the edges appearing in the walk are distinct. A trail is called a path if all the vertices are distinct. A closed path is called a cycle.

A subgraph $H'$ of $H$ is called spanning, if $H'$ contains all the vertices of $H$. An Euler trail in $H$ is a spanning trail in $H$ that contains all the edges of $H$. An Euler tour of $H$ is a closed Euler trail of $H$.

Using the skeleton $A = \{a_1, \ldots, a_m\}$, we define

$$G_0 = \{\vec{a}_i a_j; 1 \leq i, j \leq m\}$$

which is the directed complete graph with vertex set $A$.

We select a spanning subgraph $\Lambda = (\Lambda, \mathcal{E})$ of $G_0$ as the initial graph of our construction, where $\Lambda$ is the vertex set and $\mathcal{E}$ is the edge set. Let $G$ be the union of affine images of $\Lambda$ under $S_j$, i.e.

$$G = \bigcup_{j=1}^{N} S_j(\Lambda)$$

We call $G$ the refined graph induced by $\Lambda$. (See section 3.)
If for each edge \( u \in \mathcal{E} \), we can find a trail \( P_u \) in \( G \) which shares the origin and terminus with \( u \), then we call the mapping \( u \mapsto P_u, \quad u \in \mathcal{E} \) an edge-to-trail substitution.

1.5. Edge-to-trail substitutions to SFCs

Now we investigate when a substitution \( \tau \) leads to a SFC, or equivalently, an optimal parametrization. Rao and Zhang [37] introduce linear graph-directed iterated function system (linear GIFS), which provides a criterion for optimal parameterizations (section 6 gives the exact definitions of GIFS, linear GIFS and a detailed descriptions of the related results).

**Theorem 1.1 ([37]).** Let \( \{ E_j \}_{j=1}^N \) be the invariant sets of a linear GIFS satisfying the open set condition and \( 0 < H^\delta(E_j) < \infty \) for \( 1 \leq j \leq N \), where \( \delta \) is the similarity dimension. Then \( E_j \) admits optimal parameterizations for every \( j = 1, \ldots, N \).

The SFC of \( E_j \) in the above theorem is obtained in the following way: for any \( n \geq 1 \), the cylinders of rank \( n \) of \( E_j \) have a linear ordered. Imagine a beetle is moving on \( E_j \). It runs over all the points in a cylinder of rank \( n \) before it moves to the next one, and the time the beetle staying in a cylinder is proportional to the Hausdorff measure of the cylinder.

However, for a self-similar set \( K \), usually the generating IFS is not linear. The main purpose of this paper is to build a new GIFS for \( K \) such that the new GIFS is linear and satisfies the OSC. To this end, we introduce the edge-to-trail substitution which induces a linear GIFS in a natural way (theorem 7.1). Now our task is to construct ‘nice’ edge-to-trial substitutions which do lead to SFCs.
First, using our point of view, we summarize the constructions of SFC in the literature into two classes, the traveling-trail method and the positive Euler-tour method, and provide rigorous treatments. Then we introduce an universal method, the general Euler-tour method, which can construct SFCs for all connected self-similar sets with skeleton.

(1) Traveling-trail method.
A trail of length $N$ in the refined graph $G = \bigcup_{j=1}^{N} S_j(\Lambda)$ is called a traveling trail, if for every $j \in \{1, \ldots, N\}$, the trail contains exactly one edge in $S_j(\Lambda)$. We show that

**Theorem 1.2.** If all the trails $P_\tau$ in an edge-to-trail substitution $\tau$ are traveling trails, then $\tau$ leads to a space-filling curve of $K$.

A special class of IFS, called self-similar zipper (see definition 4.1), is first introduced by Thurston [47] and plays a role in complex analysis ([5, 47]). Recently, there are some works on self-similar zippers on the fractal aspect [4, 34, 46]. Indeed, the self-similar zipper is a special case of linear GIFS with two states (theorem 6.2). As a consequence of theorems 1.1 and 6.2, we have

**Corollary 1.1.** If a self-similar zipper satisfies the OSC, then its invariant set admits optimal parameterizations.

(2) Positive Euler-tour method.
A natural selection of the initial graph is $\Lambda = a_1a_2 + \cdots + a_{m-1}a_m + \bar{a}_n\bar{a}_1$, the cycle passing all the elements of $A$. Next, we choose an Euler tour of the refined graph $G$ and a partition of this Euler tour. This partition can give us an edge-to-trail substitution if a consistency condition is fulfilled, and leads to a SFC of $K$ if it further satisfies the primitivity condition and pure-cell condition (theorem 8.2).

Does every self-similar set admit SFCs? To answer this question, we introduce a general Euler-tour method.

(3) (General) Euler-tour method.
Let $\Lambda^{-1}$ be the reverse cycle of the initial cycle $\Lambda$ in the positive Euler-tour method. Now, in the refined graph $G$, we allow negative orientation, that is, for some $j$, we replace $S_j(\Lambda)$ by $S_j(\Lambda^{-1})$. Then we have much more choices of refined graphs and Euler tours, which increase the possibility of finding an edge-to-trail substitution that can produce SFCs. Our main result is

**Theorem 1.3.** Let $\{S_j\}_{j=1}^{N}$ be an IFS possessing skeletons and satisfying the open set condition. Then the invariant set $K$ admits space-filling curves. More precisely, either an rearrangement of $\{S_j\}_{j=1}^{N}$ is a self-similar zipper, or $K$ admits space-filling curves constructed by the Euler-tour method.

The difficult part of theorem 1.3 is to prove the non self-similar zipper case, which we divide into two steps. First, we prove theorem 8.2, which transfers the SFC problem to a graph theory problem. Then, we solve the graph theory problem in sections 9 and 10, where we use bubbling process and orientation adjustment to produce the desired Euler tours.

Self-similar sets of finite type is an important class of fractals, see for instance [8, 27, 36]. In a subsequent paper [38], we show that if $K$ is a self-similar set of finite type, then $K$ possesses skeletons. Consequently,
Theorem 1.4. Let $K$ be a connected self-similar set of finite type and satisfying the open set condition. Then $K$ admits space-filling curves.

Remark 1.2. The following SFCs are constructed by traveling-trial method: Peano curve, Hilbert curve, Heighway dragon curve, Gosper curve, curves in Fukuda et al [16] and the websites [45, 48]. (See section 4.)

Remark 1.3. The following curves are constructed by positive Euler-tour method: Sierpiński curve, Terdragon curve in Dekking [12] and the four-tile star in [45] (see figure 2(b)). We present two new examples: Sierpiński carpet (example 5.2) and the Rocket tile (example 5.3).

Example 1.1 (The Christmas tree). The Christmas tree is a fractal with 5 branches. Figure 13 provides a SFC of it, where the negative orientation is involved; the details are given in section 8.4. It can be parameterized with the positive Euler-tour method, but then the parametrization is not measure-preserving since the edge-to-trail substitution cannot be primitive.

Example 1.2 (Integral self-affine tiles). Let $A$ be an integral $n \times n$ expanding matrix, and $D \subset \mathbb{Z}^n$ be a set with $\#D = |\det A|$, where $\#D$ denotes the cardinality of $D$. Let $T$ be the unique compact set satisfying
\[
AT = \bigcup_{d \in D} (T + d).
\]
The set $T$ is called an integeral self-affine tile if it has positive Lebesgue measure. (See Lagarias and Wang [24] and the reference therein). By theorem 1.4, every integral self-affine tile admits SFCs. The Rocket tile is taken from Duvall et al [14]. It is an integral self-similar tile with 9 branches. A SFC is shown in figure 7. The details are given in example 5.3.

The paper is organized as follows. In section 2, we give a brief description of skeletons of self-similar sets. In section 3, we describe the general philosophy of constructing edge-to-trail substitutions by graphs. Sections 4 and 5 are devoted to the traveling-trail method and the positive Euler-tour method, respectively. In sections 6 and 7, we show an induced GIFS is always a linear GIFS, and theorem 1.2 is proved there. Sections 8–11 are devoted to the Euler-tour method, and theorem 1.3 is proved in section 11.
2. Skeleton of self-similar set

In this section, we give a brief introduction to skeletons of self-similar sets. A detailed study is carried out in Rao and Zhang [38].

Let $\mathcal{S} = \{S_j\}_{j=1}^N$ be an IFS with invariant set $K$. For any subset $A$ of $K$, we define a graph $H(A)$ as following: the vertex set is $\{S_1, S_2, \ldots, S_N\}$, and there is an edge between two vertices $S_i$ and $S_j$ if and only if $S_i(A) \cap S_j(A) \neq \emptyset$. We call $H(A)$ the Hata graph induced by $A$.

**Remark 2.1.** Such graphs are first studied by Hata [18], where he proved that a self-similar set $K$ is connected if and only if the graph $H(K)$ is connected.

**Definition 2.1.** Let $K$ be a connected self-similar set, and let $A$ be a finite subset of $K$. We call $A$ a skeleton of $\{S_j\}_{j=1}^N$ (or $K$), if $A \subset \bigcup_{j=1}^N S_j(A)$ and the Hata graph $H(A)$ is connected.

**Remark 2.2.** Kigami [22] and Morán [30] studied the ‘boundary’ (also called ‘vertices’ if it is finite) of a fractal. A skeleton is usually chosen to be a subset of the ‘boundary’ of a self-similar set; for a so-called p.c.f. self-similar set, the set of vertices is a skeleton. Indeed, the choices of skeletons are much more arbitrary.

2.1. Iteration

We denote $\Sigma = \{1, \ldots, N\}$ and call it an alphabet. Let $I = i_1i_2\ldots i_n \in \Sigma^n$, we call it a word of length $n$. We denote the length of a word $I$ by $|I|$. We define the $n$th iteration of $\mathcal{S}$ to be the IFS

$$\mathcal{S}^n = \{S_I; I \in \Sigma^n\},$$

where $S_I = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_n}$ if $I = i_1 \ldots i_n$.

It is well-known that the invariant set $K$ of the IFS $\mathcal{S}$ is also the invariant set of $\mathcal{S}^n$ (see Falconer [15]). Similarly, we have

**Proposition 2.3 ([38]).** If $A$ is a skeleton of $\mathcal{S}$, then $A$ is also a skeleton of $\mathcal{S}^n$.

Using neighbour graph of self-similar sets, it is shown that a connected self-similar set of finite type always possesses skeletons ([38]), and actually an algorithm of finding skeletons is given there. There do exist self-similar sets without skeletons (see figure 8).

3. Graphs and edge-to-trail substitutions: the general philosophy

Let $\mathcal{S} = \{S_j\}_{j=1}^N$ be an IFS possessing a skeleton and satisfying the OSC. Denote its invariant set by $K$, and let $A = \{a_1, a_2, \ldots, a_m\}$ be a skeleton. Recall that
$$G_0 = \{ \vec{a}_{ij}; 1 \leq i, j \leq m \},$$
is a directed complete graph with vertex set $A$. We note that the edges $\vec{a}_{ij}$ are abstract edges rather than oriented line segments; moreover, $i$ may be equal to $j$, see example 4.1.

Next, we choose a spanning subgraph $\Lambda = (A, \Gamma)$ of $G_0$ such that $\Lambda$ contains no isolated vertex. We call $\Lambda$ the initial graph. To continue our construction, we need to define the affine copy of a directed graph.

**Definition 3.1.** Let $G = (A, \Gamma)$ be a directed graph such that the vertex set $A \subset \mathbb{R}^d$. Let $S : \mathbb{R}^d \to \mathbb{R}^d$ be an affine mapping. We define a directed graph $S(G) = (S(A), S(\Gamma))$ as follows: there is an edge in $S(\Gamma)$ from $S(x)$ to $S(y)$ if and only if there is an edge $e \in \Gamma$ from vertex $x$ to $y$. Moreover, we denote this edge by $S(e)$ (Rigorously, we should understand $S(e)$ as a pair $(e, S)$).

**Remark 3.1.**
(i) If $(A_1, \Gamma_1)$ and $(A_2, \Gamma_2)$ are two graphs without common edges, then we define their union to be the graph $(A_1 \cup A_2, \Gamma_1 \cup \Gamma_2)$.
(ii) Even if $S_j(e_k)$ coincides with $S_j'(e_k)$ as oriented line segment, they should be regarded as different edges, since $(e_k, S_j) \neq (e_k, S_j')$.

### 3.1 Refined graph and edge-to-trail substitution

Let $G$ be the union of affine images of $\Lambda$ under $S_j$, that is,

$$G = \bigcup_{j=1}^{N} S_j(\Lambda),$$

and we call it the refined graph induced by $\Lambda$.

Let $\tau$ be a mapping from $\mathcal{E}$ to trails of $G$; we shall denote $\tau(u)$ by $P_u$ to emphasize that $\tau(u)$ is a trail. We call $\tau$ an edge-to-trail substitution, if for all $u \in \mathcal{E}$, $P_u$ has the same origin and terminus as $u$.

An edge-to-trail substitution $\tau$ can be thought as replacing each big edge $u$ by a trail $P_u$ consisting of small edges. Our goal is to construct edge-to-trail substitutions which lead to SFCs.

### 3.2 Iteration of edge-to-trail substitutions

We define $\tau^n(u)$ recurrently as follows. Let $n \geq 2$ and let

$$L = T_1(\gamma_1) + \cdots + T_k(\gamma_k), \quad T_j \in \mathcal{S}^{n-1} \text{ and } \gamma_j \in \mathcal{E}$$

be a trial in the refined graph $\bigcup_{|\mathcal{I}|=n-1} S_I(\Lambda)$. We define $\tau(L)$ to be the trial

$$\tau(L) = T_1(\tau(\gamma_1)) + \cdots + T_k(\tau(\gamma_k))$$

in the graph $\bigcup_{|\mathcal{I}|=n} S_I(\Lambda)$. Hence, we define $\tau^1(u) = \tau(u)$ and $\tau^n(u)$ is defined by using (3.2) repeatedly.

Geometrically, the action of $\tau$ is to replace an edge by a trail consisting of smaller edges, and finally $\tau^n(u)$ can be regarded as an oriented broken line, which provides an approximation of the corresponding SFC.
4. Traveling-trail method

Recall that a trail $P$ in $G = \bigcup_{j=1}^{N} S_j(\Lambda)$ is called a traveling trail, if for every $j$, $P$ contains exactly one edge in $S_j(\Lambda)$. Theorem 1.2 asserts that if all the trails $P_a$ in an edge-to-trail substitution $\tau$ are traveling trails, then $\tau$ leads to a SFC of $K$. We postpone the proof of theorem 1.2 to section 7. In this section, we summarize the SFCs constructed by this method.

4.1. Linear IFS

Let $K$ be a self-similar set. Assume that $K$ has a skeleton consisting of two points, say $A = \{a, b\}$. Denote $u = \overrightarrow{ab}$ and let $E = \{u\}$. If $S_1(u) + \cdots + S_N(u)$ is a trail from $a$ to $b$, where we use the symbol ‘+’ to connect the consecutive edges or sub-trails, then

$$\tau : u \mapsto S_1(u) + \cdots + S_N(u)$$

(4.1)

is an edge-to-trail substitution such that $\tau(u)$ is a traveling trail. Such IFS, called a linear IFS in [37], was first studied by De Rahm [13] and Hata [18], where they proved that a linear IFS leads to a SFC, which is a direct generalization of Peano’s original construction.

4.2. Self-similar zipper

**Definition 4.1.** Let $\{S_j\}_{j=1}^{N}$ be an IFS where the mappings are ordered. If there exists a set $\{x_0, \ldots, x_N\}$ of points and a sequence $(\beta_1, \ldots, \beta_N) \in \{-1, 1\}^N$ such that the mapping $S_j$ takes the pair $(x_0, x_{j-1})$ either into the pair $(x_j, x_{j-1})$ if $\beta_j = 1$ or into the pair $(x_j, x_{j-1})$ if $\beta_j = -1$, then we call $\{S_j\}_{j=1}^{N}$ a self-similar zipper. We call $\{x_0, \ldots, x_N\}$ the set of vertices and $(\beta_1, \ldots, \beta_N)$ the vector of signature.

Let $K$ be the invariant set of a self-similar zipper $\{S_j\}_{j=1}^{N}$. Then $K$ possesses the skeleton $A = \{x_0, x_N\}$. Denotes $u = \overrightarrow{x_0x_N}$ and set $E = \{u, u^{-1}\}$, where $u^{-1}$ denote the reverse edge of $u$. Clearly

$$\tau : \begin{cases} 
  u & \mapsto S_1(u^{\beta_1}) + \cdots + S_N(u^{\beta_N}) \\
  u^{-1} & \mapsto \text{reverse trail of } S_1(u^{\beta_1}) + \cdots + S_N(u^{\beta_N})
\end{cases}$$

is an edge-to-trail substitution such that both $\tau(u)$ and $\tau(u^{-1})$ are traveling trails.

The path-on-lattice IFS in [37] is a special case of the self-similar zipper; Hilbert curve, Heighway dragon curve and Gosper curve are obtained by this way, see section 5 of [37].

4.3. Space-filling curves of polygonal reptiles

In the web-site [45], there are many interesting SFCs of polygonal reptiles constructed by traveling-trail method. We take the Wedge tile as an example.

**Example 4.1 (The wedge tile).** The Wedge tile is a self-similar set generated by the IFS $\{S_j\}_{j=1}^{4}$, where the maps are indicated by figures 9(a) and (b), where we use an arrow to specify whether the linear part of $S_j$ contains reflection or not.
We choose $A = \{a_2, a_4\}$, two vertices of the wedge, to be the skeleton. Choose $E = \{\overrightarrow{a_4a_2}, \overrightarrow{a_2a_4}, \overrightarrow{a_4a_4}\} =: \{\alpha, \beta, \gamma\}$.

Then

$$
\tau : \begin{cases} 
\alpha & \mapsto S_1(\beta) + S_2(\beta) + S_3(\gamma) + S_4(\alpha), \\
\beta & \mapsto S_4(\beta) + S_3(\gamma) + S_2(\alpha) + S_1(\alpha), \\
\gamma & \mapsto S_3(\beta) + S_4(\gamma) + S_2(\alpha) + S_1(\alpha) 
\end{cases}
$$

is an edge-to-trail substitution. The trails $\tau(\alpha)$ and $\tau(\gamma)$ are illustrated by figures 9(c) and (d); the trail $\tau(\beta)$ is the reverse trail of $\tau(\alpha)$. A visualization of the SFC corresponding to $\tau$ is shown in figure 2(a).

### 4.4. Hexaflake

Hexaflake is a fractal constructed by iteratively exchanging each hexagon by a flake of seven hexagons, see figure 5(a). We choose $A = \{a_1, \ldots, a_6\}$, the vertex set of the original hexagon, to be our skeleton, and choose the initial graph

$$
\Lambda = \{\overrightarrow{a_i a_j}; i \neq j\}.
$$

Figures 5(d)–(f) show the trails for $\overrightarrow{a_4a_1}$, $\overrightarrow{a_4a_2}$, and $\overrightarrow{a_4a_6}$. For any $\overrightarrow{a_i a_j}$, there is a similitude which maps one of $\overrightarrow{a_4a_1}$, $\overrightarrow{a_4a_2}$, and $\overrightarrow{a_4a_6}$ to $\overrightarrow{a_i a_j}$, and hence this similitude induces a trail which we define to be $\tau(\overrightarrow{a_i a_j})$. The rule $\tau$ satisfies the condition of theorem 1.2. Figure 5(b) gives a visualization of the corresponding SFC.
5. Positive Euler-tour method

Another frequently used method of constructing SFC is the positive Euler-tour method. Let \( A = \{a_1, \ldots, a_m\} \) be a skeleton. Set

\[
\Lambda = \overrightarrow{a_1 a_2} + \cdots + \overrightarrow{a_{m-1} a_m} + \overrightarrow{a_m a_1}
\]

(5.1)
to by the cycle passing all the elements of \( A \). We set \( \Lambda \) to be our initial graph, and let \( G = \bigcup_{i=1}^{N} S_i(\Lambda) \) be the refined graph.

**Lemma 5.1.** The refined graph \( G \) admits Euler tours.

**Proof.** First, we prove \( G \) is connected. Take two vertices \( S_i(a) \) and \( S_j(b) \) in \( G \), where \( a, b \in A \). Let \( S_i = S_{i_1}, S_{i_1}, \ldots, S_{i_k} = S_j \) be the vertices of a path connecting \( S_i \) and \( S_j \) in the Hata graph \( H(S, A) \). Then one can easily prove by induction that the graphs

\[
S_{i_\ell}(\Lambda) \cup \cdots \cup S_{i_k}(\Lambda), \quad 0 \leq \ell \leq k,
\]

are all connected. Set \( \ell = k \), we get that \( G \) is connected.

Finally, \( G \) admits Euler tour, since it is an edge-disjoint union of closed trails (see [10]).

Let \( P \) be an Euler tour of the refined graph \( G \). Let \( P = P_1 + \cdots + P_m \) be a partition of \( P \) such that for all \( j \), \( P_j \) is a sub-trail having the same origin and terminus as \( u_j \). Then

\[
\tau : u_j \mapsto P_j, \quad j = 1, 2, \ldots, m,
\]
gives us an edge-to-trail substitution. If \( P \) and the partition are carefully chosen, then \( \tau \) will lead to a SFC. The rigorous treatment is given in section 8.

**Example 5.1 (Terdragon).** Terdragon is generated by the IFS

\[
\{ S_1(z) = \lambda z + 1, \quad S_2(z) = \lambda z + \omega, \quad S_3(z) = \lambda z + \omega^2 \},
\]

where \( \lambda = \exp(\pi i/6)/\sqrt{3} \) and \( \omega = \exp(2\pi i/3) \). We choose the skeleton

\[
A = \{a_1, a_2, a_3\} = \{-\omega^2/\lambda, -1/\lambda, -\omega/\lambda\},
\]

Figure 10. A space-filling curve of Terdragon. (a) Initial pattern. (b) The second iteration. (c) Invariant sets of the GIFS.
which consists of the fixed points of \( S_j \), \( j = 1, 2, 3 \). Choose the initial graph to be

\[ \Lambda = \overrightarrow{a_1a_2} + \overrightarrow{a_2a_3} + \overrightarrow{a_3a_1} =: u_1 + u_2 + u_3 \]

(see figure 4(b)). The refined graph \( G = S_1(\Lambda) \cup S_2(\Lambda) \cup S_3(\Lambda) \) is the graph in figure 4(c).

According to the of the Euler-partition in figure 4(d), we obtain the following edge-to-trail substitution:

\[ \tau : \begin{align*}
    u_1 & \mapsto P_{u_1} = S_1(u_1) + S_2(u_3) + S_2(u_1), \\
    u_2 & \mapsto P_{u_2} = S_2(u_2) + S_1(u_1) + S_3(u_2), \\
    u_3 & \mapsto P_{u_3} = S_3(u_3) + S_1(u_2) + S_1(u_3).
\end{align*} \tag{5.2} \]

**Remark 5.2.** We shall see in section 8 that SFCs obtained by the Euler-tour method are all closed curves, and the invariant sets of the induced GIFS form a partition of the original self-similar set \( K \).

**Example 5.2 (Sierpiński carpet).** We choose the skeleton to be the middle points of the edges of the unit square, see figure 3 (right). An Euler tour of the refine graph is shown in figure 6(b), and a partition of this Euler tour is indicated by colors. The edge-to-trail substitution is indicated by figures 6(a) and (d). Figure 6(f) shows the invariants of GIFS associated with the edge-to-trail substitution, which form a partition of the carpet. (Dekking [12] gives a certain ‘parametrization’ of the carpet where there are infinitely many jumps.)

### 5.1. A stronger version of positive Euler-tour method

Instead of a cycle, we can choose the initial graph \( \Lambda \) to be a closed trail passing all the elements of \( A \), say,

\[ \Lambda = u_1 + \cdots + u_h. \]

Still we can define refined graphs, Euler tours, edge-to-trail substitutions and SFCs. In the following example, we make such a choice, where the advantage is that the visualizations of the resulted SFC are self-avoiding.

**Example 5.3 (The Rocket tile).** The Rocket tile is generated by the IFS \( \{S_j\}_{j=1}^9 = \{(x+d)/3; d \in D\} \), where \( D = \{-2, -1, -1 + i, 0, -i, -2i, 1 - i, 1 + i, 2 + 2i\} \). See figure 11(a). It is easy to check that \( A = \{a_1, a_2, a_3, a_4\} = \{0, -1, -i, 1 + i\} \) is a skeleton. (The \( a_j \)'s are fixed points of \( S_4, S_1, S_6 \) and \( S_9 \), respectively.)

We choose the initial graph to be the closed trail

\[ \Lambda = \overrightarrow{a_1a_2} + \overrightarrow{a_2a_4} + \overrightarrow{a_4a_3} + \overrightarrow{a_3a_1} + \overrightarrow{a_1a_4} + \overrightarrow{a_4a_1}, \]

see figure 11(b). The refined graph of \( \Lambda \) contains 54 edges. An Euler tour of the refined graph is indicated by figure 11(c), where a partition of this tour is indicated by 6 different colors. The edge-to-trail substitution is indicated in figures 11(b) and (e).
6. Linear GIFS

In this section, we recall the definition of the linear GIFS introduced in [37].

6.1. GIFS

Let \( G = (\mathcal{A}, \Gamma) \) be a directed graph with vertex set \( \mathcal{A} \) and edge set \( \Gamma \). Let

\[
\mathcal{G} = \left\{ g_\gamma : \mathbb{R}^d \to \mathbb{R}^d \right\}_{\gamma \in \Gamma}
\]  

be a family of similitudes. We call the triple \( (\mathcal{A}, \Gamma, \mathcal{G}) \), or simply \( \mathcal{G} \), a graph-directed iterated function system (GIFS). We call \( (\mathcal{A}, \Gamma) \) the base graph of the GIFS. In what follows, we shall call \( \mathcal{A} \) a state set instead of vertex set, to avoid confusion with other graphs. Very often but not always, we set \( \mathcal{A} \) to be \( \{1, \ldots, N\} \).

Let \( \Gamma_{ij} \) be the set of edges from state \( i \) to \( j \). It is well known that there exist unique non-empty compact sets \( \{E_i\}_{i=1}^N \) satisfying

\[
E_i = \bigcup_{j=1}^N \bigcup_{\gamma \in \Gamma_{ij}} g_\gamma(E_j), \quad 1 \leq i \leq N.
\]  

We call \( \{E_i\}_{i=1}^N \) the invariant sets of the GIFS [28]. The set equations (6.2) provide an alternative way to define a GIFS. We shall call (6.2) the set equation form of GIFS (6.1).

Figure 11. A space-filling curve of the Rocket tile. (a) Rocket. (b) Initial graph. (c) Euler tour. (d) Initial patterns. (e) The first iteration. (f) Invariant sets of the GIFS.
6.2. Symbolic space related to a graph G

A sequence of edges in \( G = (A, \Gamma) \), denoted by \( \omega = \omega_1 \omega_2 \ldots \omega_n \), is called a walk, if the terminus of \( \omega_i \) coincides with the origin of \( \omega_{i+1} \). (Here we do not use the notation \( \omega_1 + \cdots + \omega_n \) for simplicity.) For \( i \in A \), let

\[
\Gamma_i^k, \Gamma_i^1, \Gamma_i^\infty
\]

be the set of all walks with length \( k \), the set of all walks with finite length, and the set of all infinite walks, emanating from the state \( i \), respectively. Note that \( \Gamma_i^1 = \bigcup_{k \geq 1} \Gamma_i^k \).

For an infinite walk \( \omega = (\omega_n)_{n=1}^\infty \in \Gamma_i^\infty \), we set \( \omega|_k = \omega_1 \omega_2 \ldots \omega_k \) and call

\[
[\omega_1 \ldots \omega_n] := \{ \gamma \in \Gamma_i^\infty : \gamma|_n = \omega_1 \ldots \omega_n \}
\]
	he cylinder associated with \( \omega_1 \ldots \omega_n \). For a walk \( \gamma = \gamma_1 \ldots \gamma_n \), we denote

\[
E_\gamma := g_{\gamma_1} \circ \cdots \circ g_{\gamma_n}(E_{t(\gamma)}),
\]

where \( t(\gamma) \) denotes the terminus of the walk \( \gamma \) (and \( \gamma_0 \)). Iterating (6.2) \( k \)-times, we obtain

\[
E_i = \bigcup_{\gamma \in \Gamma_i^k} E_\gamma.
\]

We define a projection \( \pi : (\Gamma_i^\infty, \ldots, \Gamma_i^N) \rightarrow (\mathbb{R}^d, \ldots, \mathbb{R}^d) \), where \( \pi_i : \Gamma_i^\infty \rightarrow \mathbb{R}^d \) is defined by

\[
\{ \pi_i(\omega) \} : = \bigcap_{n \geq 1} E_{\omega|_n}.
\]

For \( x \in E_i \), we call \( \omega \) a coding of \( x \) if \( \pi_i(\omega) = x \). It is folklore that \( \pi_i(\Gamma_i^\infty) = E_i \).

6.3. Order GIFS and linear GIFS ([1, 37])

Let \( (A, \Gamma, G) \) be a GIFS. To study the ‘advanced’ connectivity property of the invariant sets, we equip a partial order on the edge set \( \Gamma \) enlightened by set equation (6.3). Let \( \Gamma_i = \Gamma_i^1 \) be the set of edges emanating from the state \( i \).

**Definition 6.1.** We call the quadruple \( (A, \Gamma, G, \prec) \) an ordered GIFS, if \( \prec \) is a partial order on \( \Gamma \) such that

(i) \( \prec \) is a linear order when restricted on \( \Gamma_j \) for every \( j \in A \);

(ii) elements in \( \Gamma_i \) and \( \Gamma_j \) are not comparable if \( i \neq j \).

The order \( \prec \) can be extend to \( \Gamma_i^n \) and \( \Gamma_i^\infty \) for every \( i \in A \). On \( \Gamma_i^n \), two elements \( \gamma_1 \gamma_2 \ldots \gamma_k \prec \omega_1 \omega_2 \ldots \omega_k \) if and only if \( \gamma_1 \ldots \gamma_{k-1} = \omega_1 \ldots \omega_{k-1} \) and \( \gamma_k \prec \omega_k \) for some \( 1 \leq k \leq k \). Observe that \( (\Gamma_i^n, \prec) \) is a linear order. In the same manner, we can obtain a linear order of \( \Gamma_i^\infty \), which we still denote by \( \prec \). In the following, we say \( \omega \) is lower than \( \gamma \) if \( \omega \prec \gamma \). Two walks \( \omega \prec \gamma \) in \( \Gamma_i^n \) are said to be adjacent if there is no walk \( \beta \) such that \( \omega \prec \beta \prec \gamma \).

**Definition 6.2.** Let \( (A, \Gamma, G, \prec) \) be an ordered GIFS with invariant sets \( \{E_i\}_{i=1}^N \). It is termed a linear GIFS, if for all \( i \in A \) and \( k \geq 1 \), \( E_\gamma \cap E_\omega \neq \emptyset \) provided \( \gamma, \omega \) are adjacent walks in \( \Gamma_i^k \).

6.4. Chain condition

Let \( (A, \Gamma, G, \prec) \) be an ordered GIFS. Fix \( i \in A \). Let \( \omega \) be an edge emanating from state \( i \). Let \( \delta(\omega) \) and \( \Delta(\omega) \)
be the lowest and highest walks in $\Gamma_i^{\infty}$ initialled by the edge $\omega$, respectively.

**Definition 6.3.** An ordered GIFS is said to satisfy the chain condition, if for any $i \in A$, and any two adjacent edges $\omega, \gamma \in \Gamma_i$ with $\omega \prec \gamma$, it holds that
\[
\pi_i(\Delta(\omega)) = \pi_i(\delta(\gamma)).
\]

The chain condition provides a simple criterion for linear GIFS.

**Theorem 6.1 ([37]).** An ordered GIFS is a linear GIFS if and only if it satisfies the chain condition.

6.5. Self-similar zippers and linear GIFS

For the definition of self-similar zippers, see definition 4.1.

**Theorem 6.2.** An IFS $\{S_i\}_{i=1}^N$ is a self-similar zipper with signature $(\beta_1, \ldots, \beta_N)$ if and only if the following ordered GIFS with two states
\[
\begin{align*}
E_1 &= S_1(E_{\beta_1}) + S_2(E_{\beta_2}) + \cdots + S_N(E_{\beta_N}), \\
E_{-1} &= S_N(E_{-\beta_N}) + S_{N-1}(E_{-\beta_{N-1}}) + \cdots + S_1(E_{-\beta_1})
\end{align*}
\]

(6.5)
is a linear GIFS. If the OSC holds in addition, then the invariant set $K$ of $\{S_i\}_{i=1}^N$ admits optimal parameterizations.

**Proof.** Let $\{x_0, \ldots, x_N\}$ be the vertices of the self-similar zipper $\{S_i\}_{i=1}^N$. Clearly $E_1 = E_{-1} = K$ are the invariant sets of the ordered GIFS (6.5), where $K$ is the invariant set of the IFS $\{S_i\}_{i=1}^N$.

Let $\pi : \{1, \ldots, N\}^\infty \to K$ be the projection map associated with the IFS $\{S_i\}_{i=1}^N$. Let $\pi_1 : \Gamma_1^\infty \to E_1$ and $\pi_{-1} : \Gamma_1^{-\infty} \to E_{-1}$ be the projections associated with $E_1$ and $E_{-1}$ defined as (6.4).

Let $\delta$ and $\Delta$ be the lowest path and highest path emanating from the state $E_1$, respectively. The form of $\delta$ and $\Delta$ are completely determined by $\beta_1$ and $\beta_N$. We claim that
\[
\pi_1(\delta) = x_0, \quad \pi_1(\Delta) = x_N.
\]

If $\beta_1 = 1$, then the lowest edge emanating from $E_1$ is $\omega = (E_1, 1, S_1, E_1)$, where the components of $\omega$ means the initial state, the order, the contraction map, and the terminal state, respectively; hence $\delta = (E_1, 1, S_1, E_1)^\infty$ and the contraction maps associated with $\delta$ is $(S_1)^\infty$. On the other hand, the zipper condition implies that $S_1(x_0) = x_0$, so we obtain
\[
\pi_1(\delta) = \pi(1^\infty) = x_0.
\]

If $\beta_N = 1$, then the highest edge emanating from $E_1$ is $(E_1, N, S_N, E_1)$, hence $\Delta = (E_1, N, S_N, E_1)^\infty$ and the contraction maps associated with $\Delta$ is $(S_N)^\infty$. On the other hand, the zipper condition implies that $S_N(x_N) = x_N$, so we obtain
\[
\pi_1(\Delta) = \pi(N^\infty) = x_N.
\]

A similar calculation as above shows that both $(x_0, x_N)$ and $(\pi_1(\delta), \pi_1(\Delta))$ equal

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\[ (\pi(1^\infty), \pi(N^\infty)), \quad \text{if } (\beta_1, \beta_N) = (1, 1); \]
\[ (\pi(1N^\infty), \pi(N^\infty)), \quad \text{if } (\beta_1, \beta_N) = (-1, 1); \]
\[ (\pi(N^\infty), \pi(N1^\infty)), \quad \text{if } (\beta_1, \beta_N) = (1, -1); \]
\[ (\pi((1N)^\infty), \pi((N1)^\infty)), \quad \text{if } (\beta_1, \beta_N) = (-1, -1). \]

Our claim is proved.

Let \( \delta' \) and \( \Delta' \) be the lowest walk and highest walk emanating from the state \( E_{-1} \), respectively. Similarly, we have \( \pi_{-1}(\delta') = x_N, \pi_{-1}(\Delta') = x_0 \).

Finally, notice that the zipper condition is exactly the chain condition, so (6.5) is a linear GIFS.

The other side is obvious. The theorem is proved. \( \square \)

**Remark 6.1 (Criterion of self-similar zipper).** Let \( (S_i)_{i=1}^N \) be an ordered IFS. First, we calculate the four possibilities of the pair \( \{x_0, x_N\} \) by (6.6). Then \( (S_i)_{i=1}^N \) is a zipper if and only if there exists \( \{x_0, x_N\} \) and \( (\beta_1, \ldots, \beta_N) \in \{-1, 1\}^N \) such that \( S_1(x_0 x_N^{\beta_1}) + \cdots + S_N(x_0 x_N^{\beta_N}) \) is a trial.

## 7. Induced GIFS

In this section, we define the induced GIFS of edge-to-trail substitutions, especially we show that they are linear GIFS.

Let \( \tau : u \mapsto P_u, u \in \mathcal{E} \) be an edge-to-trail substitution defined in section 3. The trail \( P_u \) can be written as
\[
P_u = S_{u,1}(v_{u,1}) + \cdots + S_{u,k}(v_{u,k}),
\]
where \( S_{u,j} \in \mathcal{S} \) and \( v_{u,j} \in \mathcal{E} \) for \( j = 1, \ldots, k_u \).

According to \( \tau \) we can construct an ordered GIFS as follows. Replacing \( P_{u} \) by \( E_u \) on the left hand side, and replacing \( u \) by \( E_u \) on the right hand side of (7.1), we obtain an ordered GIFS:
\[
E_u = S_{u,1}(E_{v_{u,1}}) + \cdots + S_{u,k}(E_{v_{u,k}}), \quad u \in \mathcal{E},
\]
which we call the *induced GIFS* of \( \tau \). In an ordered GIFS, we use ‘+’ to replace the ‘∪’ in the set equation to emphasize the order structure.

**Example 7.1 (Induced GIFS of terdragon).** The edge-to-trail substitution \( \tau \) in example 5.1 induces the following GIFS:
\[
\begin{aligned}
E_1 &= S_1(E_1) + S_2(E_3) + S_2(E_1), \\
E_2 &= S_3(E_2) + S_3(E_1) + S_1(E_2), \\
E_3 &= S_3(E_3) + S_1(E_2) + S_1(E_3).
\end{aligned}
\]

The invariant sets \( E_j \) are illustrated in figure 10(c).

Let
\[
(\mathcal{E}, \Gamma, \mathcal{G}, \prec)
\]
be the basic-graph form of the ordered GIFS (7.2). The state set is \( \mathcal{E} \), which is the edges of the initial graph. The edge set \( \Gamma \) consists of quadruples \( (u, S_j, v, k) \), that is, if \( S_j(v) \) is the \( k \)th edge in the trail \( P_u \), then we add an edge to \( \Gamma \) and denote this edge by
The contraction associated with this edge is $S_j$.

**Theorem 7.1.** The induced GIFS (7.2) is a linear GIFS.

**Proof.** Let $u \in \mathcal{E}$. We denote by $a_u$ and $b_u$ the origin and the terminus of $u$ as an edge in the initial graph $\Lambda = (A, \mathcal{E})$. We claim that the lowest and highest elements in $\Gamma_u^\infty$ are codings of $a_u$ and $b_u$, respectively.

Let $S(v)$ be the first edge in $P_v$, then $\omega = (u, S, v, 1)$ is the lowest edge emanating from $u$ in the basic graph $\Gamma$. It follows that

$$a_u = S(a_v).$$

(7.5)

Therefore, if $(\omega_n)_{n=1}^\infty$ is a coding of $a_v$, then

$$\omega(\omega_n)_{n=1}^\infty$$

is a coding of $a_u$. Applying the same argument to $v$, we obtain a coding of $a_u$, such that the first two edges of this coding is the lowest walk in $\Gamma^2_u$. Continuing this argument, we conclude the lowest element in $\Gamma^\infty_u$ is a coding of $a_u$.

Similarly, the highest element in $\Gamma^\infty_u$ is a coding of $b_u$.

Now, let $\omega = (u, S, v, k)$ and $\gamma = (u, T, v', k + 1)$ be two consecutive edges in $\Gamma_u$. This means that $S(v)$ and $T(v')$ are two adjacent edges in $P_u$, so $S(b_u) = T(a_{v'})$.

On the other hand, write $\Delta(\omega) = \omega(\omega_n)_{n \geq 1}$, then $(\omega_n)_{n \geq 1}$ is the highest coding in $\Gamma_u$. So $\pi_v((\omega_n)_{n \geq 1}) = b_u$ by the claim above, and

$$\pi_u(\Delta(\omega)) = S \circ \pi_v((\omega_n)_{n \geq 1}) = S(b_u).$$

Similarly, we have $\pi_u(\delta(\gamma)) = T(a_{v'})$. This verifies the chain condition. Therefore, the ordered GIFS in consideration is linear. \qed

**Proof of theorem 1.2.** Let $\tau$ be an edge-to-trail substitution over $\mathcal{E}$ such that $\tau(u)$ are all traveling trails. By theorem 7.1, $\tau$ induces a linear GIFS and we denote it by $(\mathcal{E}, \Gamma, \mathcal{G}, \prec)$. Let $g_\omega$ be the contraction associated with the edge $\omega$ in $\Gamma$.

Fix an $u \in \mathcal{E}$. The traveling property of $\tau$ implies that the associated contractions with edges in $\Gamma^*_u$ are exactly the maps $S_l, l \in \{1, \ldots, N\}^*$, and each map appears only once. It follows that

$$\bigcup_{\omega \in \Gamma^*_u} g_\omega(\{0\}) = \bigcup_{l \in \{1, \ldots, N\}^*} S_l(\{0\}),$$

where $g_\omega = g_{\omega_1} \circ \cdots \circ g_{\omega_s}$ is the contraction associated to the trail $\omega$. Taking the limit in Hausdorff metric at both sides, we obtain $E_u = K$. This proves that the invariant sets of the GIFS are all equal to $K$.

Moreover, if we ignore the order structure, then the induced GIFS degenerates to the original IFS of $K$, so the induced GIFS satisfies the OSC. Hence, $K$ admits optimal parameterizations according to theorem 1.1. \qed
8. (General) Euler-tour method

In this section, we introduce the general Euler-tour method. Recall that $A = \{a_1, \ldots, a_m\}$ is a skeleton. We denote
\[ u_j = \overrightarrow{a_ja_{j+1}} \text{ and } u_j^{-1} = \overrightarrow{a_{j+1}a_j}, \quad j = 1, \ldots, m, \]
where we identify $a_{m+1}$ with $a_1$. Then
\[ \Lambda = u_1 + u_2 + \cdots + u_m, \]
is a cycle and $\Lambda^{-1} = u_m^{-1} + \cdots + u_1^{-1}$ is the reverse of $\Lambda$. Denote
\[ \mathcal{E}^+ = \{u_1, \ldots, u_m\}, \quad \mathcal{E}^- = \{u_1^{-1}, \ldots, u_m^{-1}\}. \]

Let $\beta = (\beta_1, \beta_2, \cdots, \beta_N) \in \{1, -1\}^N$, and we call it an orientation vector. Denote
\[ G(\mathcal{S}, A, \beta) = \bigcup_{j=1}^N S_j(\Lambda^\beta). \quad (8.1) \]

By the same argument as lemma 5.1, one can show that $G(\mathcal{S}, A, \beta)$ admits Euler tour.

**Definition 8.1.** Let $P$ be an Euler tour of $G(\mathcal{S}, A, \beta)$, and let
\[ P = P_1 + \cdots + P_m \]
be a partition of $P$. We call it an Euler-partition of $G(\mathcal{S}, A, \beta)$ if the initial points of $P_1, \ldots, P_m$, denoted by $a_{\tau(1)}, \ldots, a_{\tau(m)}$, is a permutation of $A$. We call $\tau$ the output permutation of $P_1 + \cdots + P_m$.

8.1. Consistency and induced edge-to-trail substitution

**Definition 8.2.** We say an Euler-partition $P_1 + \cdots + P_m$ of $G(\mathcal{S}, A, \beta)$ is consistent, if $P_i$ has the same origin and terminus as $u_i$ for each $i \in \{1, 2, \ldots, h\}$.

As soon as $P_1 + \cdots + P_m$ is consistent, we can define an edge-to-trail substitution over
\[ \mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^- = \{u_1, \ldots, u_m\} \cup \{u_1^{-1}, \ldots, u_m^{-1}\} \]
by
\[ \tau : u_j \mapsto P_j \text{ and } u_j^{-1} \mapsto P_j^{-1}, \quad j = 1, \ldots, m. \quad (8.2) \]
We call $\tau$ the induced edge-to-trail substitution of the Euler-partition $P_1 + \cdots + P_m$.

**Remark 8.1.** Set $G = \bigcup_{i=1}^N S_i(\Lambda \cup \Lambda^{-1}) = G(\mathcal{S}, A, \beta) \cup G(\mathcal{S}, A, -\beta)$, then it is seen that the generalized Euler-tour method is a special case of the construction in section 3.

Denote the length of $P_j$ by $\ell_j$, then $P_j$ has the form
\[ P_j = S_{j,1}(u_{j,1}) + S_{j,2}(u_{j,2}) + \cdots + S_{j,\ell_j}(u_{j,\ell_j}), \quad j = 1, \ldots, m \quad (8.3) \]
where $S_{j,1} \in \{S_1, \ldots, S_N\}$, and $u_{j,k} \in \mathcal{E}$. Accordingly,
\[ P_j^{-1} = S_{j,1}(u_{j,1}^{-1}) + \cdots + S_{j,2}(u_{j,2}^{-1}) + S_{j,\ell_j}(u_{j,\ell_j}^{-1}), \quad j = 1, \ldots, m. \quad (8.4) \]

The substitution (8.2) induces the following linear GIFS (by theorem 7.1):
\[
\begin{align*}
E_{uj} &= S_{j,1}(E_{uj}) + S_{j,2}(E_{uj}) + \cdots + S_{j,\ell_j}(E_{uj}), \\
E_{uj-1} &= S_{\ell_j}(E_{uj-1}) + \cdots + S_{j,2}(E_{uj}) + S_{j,1}(E_{uj}).
\end{align*}
\]

(8.5)

We call GIFS (8.5) the induced GIFS of the Euler-partition \( P_1 + \cdots + P_m \). We shall use \((\mathcal{E}, \Gamma, \mathcal{G}, \prec)\) to denote this GIFS, as we explained in section 7.

Now we give some basic properties of the induced GIFS.

**Lemma 8.2.**

(i) \( E_u = E_{u-1} \) for \( u \in \mathcal{E} \).

(ii) For each \( u \in \mathcal{E} \) and \( i \in \{1, \ldots, N\} \), \( S_i(E_u) \) appears exactly once in the right-hand side of (8.5).

(iii) \( K = \bigcup_{u \in \mathcal{E}^+} E_u \).

**Proof.**

(i) Set \( F_u = E_{u-1} \) for \( u \in \mathcal{E} \). Clearly \( \{F_u\}_{u \in \mathcal{E}} \) also satisfy the set equation (8.5) if we ignore the order structure. Hence (i) follows from the uniqueness of the invariant sets.

(ii) Since \( P \) is an Euler tour, for each \( u \in \mathcal{E} \) and \( i \in \{1, \ldots, N\} \), the edge \( S_i(u) \) appears exactly once in \( P \cup P^{-1} \), which implies (ii).

(iii) Taking the union of both sides of (8.5) and using the conclusion of (ii), we have

\[
\bigcup_{u \in \mathcal{E}} E_u = \bigcup_{j=1}^m \bigcup_{k=1}^l S_{j,k}(E_{uj} \cup E_{uj-1}) = \bigcup_{i=1}^N S_i(\bigcup_{u \in \mathcal{E}} E_u).
\]

(8.6)

Therefore, by the uniqueness of the invariant set of an IFS, we obtain

\[
K = \bigcup_{u \in \mathcal{E}^+} E_u = \bigcup_{u \in \mathcal{E}^+} E_u,
\]

where the last equality holds since \( E_u = E_{u-1} \).

**Lemma 8.2** (ii) means that for each \( u \in \mathcal{E} \), there are exactly \( N \) edges in \( \Gamma \) terminating at \( u \); moreover, the associated maps of these edges run over the maps \( S_1, \ldots, S_N \).

### 8.2. Primitivity and the open set condition

To ensure the induced GIFS satisfies the OSC, we need a primitivity condition.

Since \( E_u = E_{u-1} \), in the induced GIFS, we can identify \( E_u \) and \( E_{u-1} \) to simplify the discussion. By identifying \( u \) and \( u^{-1} \) in \( \tau \), and ignoring the maps \( S_i \) in \( \tau(u) \), we define a morphism over \( \{u_1, \ldots, u_m\} \) by

\[
\tau^* : u_j \rightarrow |u_{j,1}| |u_{j,2}| \ldots |u_{j,l_j}|, \quad j = 1, 2, \ldots, m,
\]

(8.7)

where \( |u^{-1}| = |u| \) for \( u \in \mathcal{E}^+ \). We call \( \tau^* \) the induced morphism. Recall that \( \tau^* \) is primitive if there exists an integer \( n \) such that for any \( i, j \in \{1, 2, \ldots, m\} \), \( u_i \) appears in \((\tau^*)^n(u_j)\); see for instance [35].

**Definition 8.3.** We say a consistent Euler-partition \( P_1 + \cdots + P_m \) of \( G(S,A,\beta) \) is primitive, if the induced morphism \( \tau^* \) is primitive.
Example 8.1. Consider the Sierpiński gasket. Let $\beta = (1, -1, -1)$. Figure 12(a) illustrates a consistent partition $P_1 + P_2 + P_3$, where the trails are indicated by colors. The induced edge-to-trail substitution is:

\[
\tau : \begin{cases} 
  u_1 \mapsto P_1 = S_1(u_1) + S_2(u_1^{-1}) + S_2(u_2^{-1}), \\
  u_2 \mapsto P_2 = S_2(u_1^{-1}) + S_1(u_2) + S_3(u_2^{-1}), \\
  u_3 \mapsto P_3 = S_3(u_2^{-1}) + S_2(u_1^{-1}) + S_1(u_3). 
\end{cases}
\] (8.8)

The induced morphism is

\[
\tau^* : \begin{cases} 
  u_1 \mapsto u_1u_2u_3, \\
  u_2 \mapsto u_1u_2u_3, \\
  u_3 \mapsto u_2u_1u_3 
\end{cases}
\]

and it is primitive. Figure 12(c) illustrates $\tau^2(\Lambda)$.

For a GIFS $(A, \Gamma, F)$, let $r_e$ be the contraction ratio of the mapping associated with the edge $e$. For $s > 0$, the Mauldin–Williams matrix $M(s)$ is defined to be ([28])

\[
\left( \sum_{e \in \Gamma \cup A} r_e^s \right)_{u,v \in A}.
\] (8.9)

The real number $s$ satisfying $\Phi(M(s)) = 1$ is called the similarity dimension of the GIFS, where $\Phi(\cdot)$ denotes the spectral radius of a matrix.

Let $H^s$ be the $s$-dimensional Hausdorff measure; a set $E$ is called an $s$-set, if $0 < H^s(E) < \infty$. A directed graph is said to be strongly connected, if for any pair of vertex $i$ and $j$, there is a trail from $i$ to $j$. The following criterion of the OSC is proved in [28] (the ‘only if’ part) and [25] (the ‘if’ part).

Lemma 8.3. Let $F$ be a GIFS with a strongly connected base graph. Denote its invariant sets by $\{E_j\}_{j=1}^N$, and the self-similar dimension by $\delta$. Then $F$ satisfies the open set condition if and only if

\[ 0 < H^\delta(E_j) < +\infty \]

for some $1 \leq j \leq N$ (or for all $1 \leq j \leq N$).
Theorem 8.1. Let $P_1 + \cdots + P_m$ be a consistent and primitive Euler-partition of $G(S, A, \beta)$ in (8.1). Then the induced GIFS (8.5) satisfies the OSC, and

$$0 < \mathcal{H}^s(E_u) < \infty, \quad \text{for all } u \in \mathcal{E},$$

where $s = \dim_H K$.

Proof. First, we show that the similarity dimension of the induced GIFS is $s = \dim_H K$. Let $c_j$ be the contraction ratio of $S_j$. Then $s$ fulfills the equation $\sum_{j=1}^N c_j^s = 1$ and $0 < \mathcal{H}^s(K) < \infty$, since $S$ satisfies the OSC. Recall that

$$M(s) = \left( \sum_{e \in \Gamma_{uw}} r_e^s \right)_{u,v \in \mathcal{E}}.$$  \hspace{1cm} (8.10)

By lemma 8.2(ii), we have (since $\cup_{u \in \mathcal{E}} \Gamma_{uw}$ is the set of edges with terminus $v$)

$$\sum_{w \in \mathcal{E}} \sum_{e \in \Gamma_{uw}} r_e^s = \sum_{i=1}^N c_i^s = 1, \quad v \in \mathcal{E},$$

namely, the sum of entries of each collum of $M(s)$ is 1. By Perron–Frobenius theorem (see lemma 8.4 below), the spectral radius of $M(s)$ equals 1, which implies that $s$ is the similarity dimension of the induced GIFS.

We identify $E_u$ and $E_{u-1}$ in the induced GIFS (8.5) and forget the order, then we obtain a simplified GIFS as follows:

$$E_u = S_{j,1}(E_{[u,1]}) \cup S_{j,2}(E_{[u,2]}) \cup \cdots \cup S_{j,\ell_j}(E_{[u,\ell_j]}), \quad j = 1, \ldots, m. \hspace{1cm} (8.11)$$

We have the following facts concerning the simplified GIFS:

(i) The base graph of the simplified GIFS is strongly connected, since the induced morphism $\tau^*$ is primitive.

(ii) The relation $\mathcal{H}^s(E_u) > 0$ holds for at least one $j$, since the union $K = \bigcup_{j=1}^m E_u$ has a finite and positive $s$-dimensional Hausdorff measure.

(iii) The simplified GIFS has the same similarity dimension as the induced GIFS, which is $s$.

This is true because the sum of each row of the associated matrix $\tilde{M}(s)$ of the simplified GIFS is still 1, where

$$\tilde{M}(s) = \left( \sum_{e \in \Gamma_{uw}} r_e^s + \sum_{e \in \Gamma_{uw-1}} r_e^s \right)_{u,v \in \mathcal{E}}.$$  \hspace{1cm} (8.10)

Item (i)–(iii) verify the conditions of lemma 8.3, hence the simplified GIFS satisfies the OSC, which implies $0 < \mathcal{H}^s(E_v) < \infty$ for all $v \in \mathcal{E}^+$ (again by lemma 8.3).

Finally, we observe that if the simplified GIFS satisfies the OSC with open sets $U_1, \ldots, U_m$, then the induced GIFS satisfies the OSC with the open sets $U_1, \ldots, U_m, U_1, \ldots, U_m$. \hfill $\Box$

The following is a part of the Perron–Frobenius theorem, see for example [31].

**Lemma 8.4 (Perron–Frobenius theorem).** Let $B = [a_{ij}]$ be a non-negative $k \times k$ matrix.

(i) There is a non-negative eigenvalue $\lambda$ such that it is the spectral radius $B$.

(ii) We have $\min_i \left( \sum_{j=1}^k a_{ij} \right) \leq \lambda \leq \max_i \left( \sum_{j=1}^k a_{ij} \right)$. 

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8.3. Pure-cell property and the disjointness of $K = \bigcup_{u \in E^+} E_u$

Finally, we investigate when $K = \bigcup_{u \in E^+} E_u$ is a disjoint union in Hausdorff measure. For $I \in \Sigma^n$, we call the set of edges

$$\{S_i(u); u \in E^+\} \text{ and } \{S_j(u); u \in E^{+1}\}$$

a positive and a negative $S_l$-cell, respectively.

**Definition 8.4.** Let $\tau$ be an edge-to-trail substitution induced by an Euler-partition $P_1 + \cdots + P_m$. Let $I \subset \Sigma^n$. The cell $S_I(\Lambda^{\pm1})$ is called a pure cell, if there exists $u \in E$ such that the $S_l$-cell is a subgraph of $\tau^n(u)$. In this case, we also say the partition $P_1 + \cdots + P_m$ potentially contains pure cells.

**Theorem 8.2.** Let $\mathcal{S} = \{S_i\}_{i \in \mathbb{N}}$ be an IFS satisfying the OSC, and let $A$ be a skeleton of $\mathcal{S}$. If there exist a vector $\beta \in \{-1, 1\}^N$ and an Euler-partition $P_1 + \cdots + P_m$ of $G(S, A, \beta)$ such that the partition is consistent, primitive and potentially contains pure cells, then

(i) $K = \bigcup_{u \in E^+} E_u$ is a disjoint union in the $s$-dimensional Hausdorff measure, where $s = \dim_H K$;

(ii) the edge-to-trail substitution $\tau$ in (8.2) leads to a a space-filling curve of $K$.

**Proof.**

(i) Suppose that an $S_l$-cell is a pure cell, i.e. all the edges of the $S_l$-cell belong to $\tau^n(u)$ for some $u \in E$, where $n = |I|$. Without loss of generality, we may assume that the orientation of the $S_l$-cell is positive. Then all $S_i(E_v), v \in E^+$ appear in the right hand side of the $n$th iteration of the set equation (8.5) corresponding to $E_v$. Hence, $\bigcup_{u \in E^+} S_l(E_v)$ is a disjoint union in the $n$-dimensional Hausdorff measure, since the induced GIFS satisfies the OSC (by theorem 8.1); it follows that its image under $S_l^{-1}$, $\bigcup_{u \in E^+} E_u$ is also a disjoint union in Hausdorff measure.

(ii) By theorem 7.1, the induced GIFS is a linear GIFS. By theorem 8.1, the induced GIFS satisfies the OSC, and all the invariant sets are s-sets.

Let $L_j = \mathcal{H}(E_v)$, and $L = \sum_{j=1}^{m} L_j$. By theorem 1.1, for each $j \in \{1, \ldots, m\}$, there exists an optimal parametrization $\varphi_j : [0, L_j] \to E_v$ such that $\varphi_j(0)$ and $\varphi_j(L_j)$ are the origins and terminus of $E_v$.

Let $\varphi : [0, L] \to E_v$ be the curve obtained by joining all the $\varphi_j$ one by one. Since $K = \bigcup_{u \in E^+} E_u$ is a disjoint union in measure, we conclude that $\varphi$ is an optimal parametrization of $K$.

**8.4. The Christmas tree**

The Christmas tree is the invariant set of the IFS $\{S_i\}_{i=1}^5 = \{(x + d)/3; d \in D\}$, where $D = \{0, 1, 2, 1 + i, 1 + 2i\}$. We choose $A = \{a_1, a_2, a_3, a_4\} = \{0, 1/2, 1/2 + i\}$, which are the fixed points of $S_1, S_2, S_3, S_4$. Let $u_l = \overrightarrow{a_ia_{l+1}}$ (assume $a_5 = a_0$) and let $\Lambda = u_1 + u_2 + u_3 + u_4$.

To get a primitive substitution, we choose an orientation vector $\beta = (-1, 1, -1, 1, 1)$. Using the Euler-partition in figure 13(e), we get the following edge-to-trail substitution.
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\[
\begin{align*}
    u_1 &\mapsto S_1 (u_4^{-1}) + S_1 (u_3^{-1}) + S_2 (u_1), \\
    u_2 &\mapsto S_2 (u_2) + S_3 (u_4^{-1}) + S_3 (u_2^{-1}), \\
    u_3 &\mapsto S_3 (u_2^{-1}) + S_3 (u_3^{-1}) + S_2 (u_3) + S_4 (u_2) + S_5 (u_3) + S_5 (u_3), \\
    u_4 &\mapsto S_5 (u_4) + S_5 (u_1) + S_4 (u_4) + S_4 (u_1) + S_2 (u_4) + S_2 (u_4^{-1}) + S_1 (u_4^{-1}).
\end{align*}
\]

(8.12)

and the rules for \(u_1^{-1}, u_2^{-1}, u_3^{-1}, u_4^{-1}\) can be obtained accordingly.

9. Consistency

In this section and the next section, we will confirm the conditions in theorem 8.2. In this section, we study the existence of consistent Euler-partitions.
Let $S = \{S_j\}_{j=1}^N$ be an IFS possessing a skeleton $A = \{a_1, a_2, \ldots, a_m\}$. Let $K$ be the invariant set of $S$. Recall that $\Sigma = \{1, \ldots, N\}$.

Let $\tau$ be a permutation of $\{1, \ldots, m\}$. Denote by $\Lambda_\tau$ the cycle passing the elements of $A_\tau = \{a_\tau(1), \ldots, a_\tau(m)\}$ one by one. As before, set

$$G(S, A_\tau, \beta) = \bigcup_{j \in \Sigma} S_j(\Lambda_\beta^j).$$

(9.1)

For simplicity, we abbreviate $G(S, A_\tau, \beta)$ as $G(S, A_\tau)$ when $\beta = (1, 1, \ldots, 1)$ is totally positive. The main idea in this section is to generate an Euler tour of $G(S, A_\tau)$ by a bubbling process.

9.1. Bubbling property

First, we give some definitions. We say two subgraphs $H_1$ and $H_2$ are edge-disjoint, denoted by $H_1 \cap H_2 = \emptyset$, if they do not have a common edge.

Let $P$ be an Euler tour of a directed graph $G$, and let $C = e_1 + \cdots + e_h \subset G$ be a cycle. We call $C$ a remaining cycle of $P$, if $e_j$ appears earlier than $e_{j+1}$ in $P$ for all $1 \leq j < h$ when we regard $e_1$ as the starting edge of $P$. (See figure 14) In this case, $P$ can be uniquely written as

$$P = e_1 + L_1 + \cdots + e_h + L_h,$$

(9.2)

where some $L_j$ may be empty trails. We call $L_j$ out-trails w.r.t. $C$, and call (9.2) the out-trail decomposition of $P$ w.r.t. $C$.

**Definition 9.1 (Cell-exact subgraph).** A subgraph $H$ of $G(S, A_\tau)$ is said to be cell-exact, if for all cell $S_k(\Lambda_\tau)$, either $S_k(\Lambda_\tau) \subset H$ or $S_k(\Lambda_\tau) \cap H = \emptyset$.

**Definition 9.2 (Bubbling property).** Let $H \subset G(S, A_\tau, \beta)$ be a cell-exact subgraph, and let $P$ be an Euler tour of $H$. We say $P$ satisfies the bubbling property, if for all $S_j(\Lambda) \subset H$, the cell $S_j(\Lambda)$ is a remaining cycle of $P$, and all the outpaths w.r.t. $S_j(\Lambda)$ are cell-exact.

The goal of this section is to prove the following theorem.

**Theorem 9.1.** There exist an integer $n$ and a permutation $\tau$ such that $G(S^n, A_\tau)$ has an Euler-partition $P = P_1 + \cdots + P_m$ which is consistent and satisfies the bubbling property.

**Definition 9.3.** Let $P$ be an Euler tour of $G(S, A_\tau, \beta)$. We say a cell $S_j(\Lambda_\tau) = e_1 + \cdots + e_m$ is unbroken, if $e_k + \cdots + e_m$ is a subtrail of $P$ for some $k \in \{1, 2, \ldots, m\}$. 

![Figure 14. A remaining cycle C = e1 + e2 + e3 + e4 and its out-trials.](image)
9.2. Bubbling process

Recall that $H(S, A)$ is the Hata graph induced by $A$, see definition 2.1. It is a connected undirected graph.

A subgraph $R$ of $H(S, A)$ is called a spanning tree, if $R$ is a tree with vertex set $A$. (Recall that an undirected connected graph is called a tree, if it does not contain cycles.) A vertex $v$ of a tree $R$ is called a top, if $v$ is a vertex of degree one.

Now, we choose a spanning tree $R$ of $H(S, A)$ and fix it. We choose an order of the edges of $R$, say, $r_1, \ldots, r_{N-1}$, \( (9.3) \) such that for any $1 \leq j \leq N-1$, the subgraph consisting of the edges $r_1, \ldots, r_j$ is connected.

This means that starting from a certain cell, we can add the cells one by one in the order \( (9.3) \) to obtain $G(S, A)$, and all the subgraphs in the process are connected. By changing the names of $S_j$, we may assume without loss of generality that the cells are added in the order $S_1(\Lambda_\tau), \ldots, S_N(\Lambda_\tau)$. Then one vertex of $r_j$ is $S_j + 1$, and the other vertex of $r_j$ belongs to $\{S_1, S_2, \ldots, S_j\}$, which we denote by $S_{\psi(j)}$. We note that if $S_k$ is a top of $R$, then $\psi(j)$ cannot take the value $k$ for any $j$. See figure 15.

Now, we construct inductively a sequence $(\Delta_k)_{k=1}^N$ such that $\Delta_k$ is an Euler tour of $\bigcup_{j=1}^k S_j(\Lambda_\tau)$ satisfying the bubbling property (see definition 9.2).

Let $\Delta_1 = S_1(\Lambda_\tau)$ be the first Euler tour, which clearly satisfies the bubbling property.

Suppose $\Delta_k$ has been constructed, $1 \leq k \leq N-1$. Now we add the cell $S_{k+1}(\Lambda_\tau)$ to $\Delta_k$. In the tree $R$, $S_{k+1}$ is connected to $S_{\psi(k)} \in \{S_1, \ldots, S_k\}$, so

$$S_{k+1}(\Lambda_\tau) \cap S_{\psi(k)}(\Lambda_\tau) \neq \emptyset.$$  

Take any point $z^*$ from the above intersection. Write

$$S_{\psi(k)}(\Lambda_\tau) = e_1 + \cdots + e_m \quad \text{and} \quad S_{k+1}(\Lambda_\tau) = e'_1 + \cdots + e'_m,$$

where both $e_1$ and $e'_1$ have origin $z^*$. The outpath decomposition of $\Delta_k$ w.r.t. $S_{\psi(k)}(\Lambda_\tau)$ can be written as $\Delta_k = e_1 + L_1 + \cdots + e_m + L_m$. Set

$$\Delta_{k+1} = e_1 + L_1 + \cdots + e_m + L_m + (e'_1 + \cdots + e'_m),$$

then $\Delta_{k+1}$ is an Euler tour of the graph $\bigcup_{j=1}^{k+1} S_j(\Lambda_\tau)$.

Now we show that $\Delta_{k+1}$ satisfies the bubbling property. For a cell $S_j(\Lambda_\tau) \subset \Delta_{k+1}$, no matter it is $S_{k+1}(\Lambda_\tau)$, or is $S_{\psi(j)}(\Lambda_\tau)$, or belongs to a certain $L_i (1 \leq i \leq k)$, one can easily prove

Figure 15. A spanning tree and bubbling process.
that it is a remaining cycle of $\Delta_{k+1}$ and all its out-trials are cell-exact. The bubbling property is proved.

Set $k = N$, we obtain the following:

**Lemma 9.1.** For any permutation $\tau$ of $A$, there exists an Euler tour $\Delta_\tau$ of $G(S, A_\tau)$ satisfying the bubbling property. Moreover, if $S_i$ is a top of the spanning tree $R$, then $S_i(\Delta_\tau)$ is an unbroken cell of $\Delta_\tau$.

### 9.3. An Euler-partition of $\Delta_\tau$

Now we introduce a natural partition of $\Delta_\tau$. We start from the vertex $a_1$, and walk along $\Delta_\tau$. We record the elements of $A$ appearing on the Euler tour but without repetition, then we get a rearrangement of $A$, which we denote by $a_\tau^*(1), \ldots, a_\tau^*(m)$.

Let $P_j$ be the subtrail of $\Delta_\tau$ starting from the first visiting of $a_\tau^*(j)$ and ending at the first visiting of $a_\tau^*(j+1)$, then

$$P_\tau = P_1 + \cdots + P_m$$

(9.4)

is an Euler-partition of $G(S, A_\tau)$ with output permutation $\tau^*$.

### 9.4. A cycle of permutations of $A$

By sections 9.2 and 9.3, starting from a permutation $\tau$ of $A$, we can construct an Euler-partition $P_\tau$ of $G(S, A_\tau)$. Since there are only finite many permutations of $A$, there exist a sequence of permutations $\tau_1, \ldots, \tau_{h+1}$ such that $\tau_{j+1}$ is the output permutation of $P_{\tau_j}$, $1 \leq j \leq h$.

### 9.5. Composition of Euler-partitions

Let $P = P_1 + \cdots + P_m$ be an Euler-partition of $G(S, A_\tau)$ with output permutation $\tau_2$, and let $Q = Q_1 + \cdots + Q_m$ be an Euler-partition of $G(S^n, A_\tau)$ with output permutation $\tau_3$. Since $Q$ is an Euler tour of the refined graph $\bigcup_{I \in \Sigma_n} S_I(\Lambda\tau_2)$, each edge in $Q$ can be written as

$$S_I(a_{\tau_2(k)}a_{\tau_2(k+1)})$$

(9.5)

Notice that $S_I(P_k)$ is a trail having the same origin and terminus as the edge in (9.5). So, we can define an Euler tour of $G(S^{n+1}, A_{\tau_3})$ by replacing every edge $S_I(a_{\tau_2(k)}a_{\tau_2(k+1)})$ in $Q$ by the trail $S_I(P_k)$. We denote this replacement rule by $\sigma$, that is,

$$\sigma(S_I(a_{\tau_2(k)}a_{\tau_2(k+1)})) = S_I(P_k), \quad I \in \{1, \ldots, N\}^n, \ k \in \{1, \ldots, m\}$$

(9.6)

and set $\sigma(\omega) = \sigma(\omega_1) + \cdots + \sigma(\omega_\ell)$ for any trail $\omega = \omega_1 + \cdots + \omega_\ell$. We define the composition of $P$ and $Q$ to be

$$Q \circ P = \sigma(Q_1) + \cdots + \sigma(Q_m),$$

which is an Euler-partition of $G(S^{n+1}, A_{\tau_3})$. 
Lemma 9.2. Let \( P = P_1 + \cdots + P_m \) be an Euler-partition of \( G(S, A_\tau) \) with output permutation \( \tau_2 \), and let \( Q = Q_1 + \cdots + Q_n \) be an Euler-partition of \( G(S^m, A_{\tau_3}) \) with output permutation \( \tau_3 \). Then

(i) If both \( P \) and \( Q \) have the bubbling property, then \( Q \circ P \) also does.
(ii) The output permutation of \( Q \circ P = \sigma(Q_1) + \cdots + \sigma(Q_m) \) is \( \tau_3 \).
(iii) If the numbers of unbroken cells in \( P \) and \( Q \) are \( u \) and \( v \), respectively, then the number of unbroken cells in \( Q \circ P \) is no less than \((u - 1)v\).

Proof.

(i) Take \( I \in \Sigma^n \) and \( j \in \Sigma \). Since \( Q \) has the bubbling property, the outpath decomposition w.r.t. \( S_j(A_{\tau_3}) \) can be written as

\[
Q = e_1 + L_1 + \cdots + e_m + L_m,
\]

where \( e_j = S_j(a_{\tau_3(j)}a_{\tau_3(j+1)}) \), and the outpaths \( L_j \) are cell-exact. Then \( Q \circ P \) can be written as

\[
Q \circ P = \sigma(e_1) + \sigma(L_1) + \cdots + \sigma(e_m) + \sigma(L_m),
\]

where \( \sigma \) is the replacement rule defined in (9.6). Clearly \( (L_j) \) are cell-exact in \( G(S^{n+1}, A_{\tau_3}) \) since \( L_j \) are cell-exact in \( G(S^n, A_{\tau_3}) \). Since all \( (L_j) \) are edge-disjoint with \( S_j(A_{\tau_3}) \), to prove \( Q \circ P \) is bubbling w.r.t. \( S_j(A_{\tau_3}) \), we need only show that

\[
\sigma(e_1) + \cdots + \sigma(e_m) = S_j(P)
\]

is bubbling w.r.t. \( S_j(A_{\tau_3}) \). This is clearly true since \( P \) is bubbling w.r.t. \( S_j(A_{\tau_3}) \) by our assumption, and the bubbling property does not change under a linear transformation. This finishes the proof of (i).

(ii) Is obvious, since \( \sigma(Q_k) \) has the same origin and terminus as \( Q_k \).

(iii) Let \( S_j(A_{\tau_3}) = e_1 + \cdots + e_m \) be an unbroken cell of \( Q_k \). Then

\[
\sigma(e_1 + \cdots + e_m) = S_j(P_1) + \cdots + S_j(P_m)
\]

is the affine image of \( P \) under \( S_j \), and it is a subtrail of \( Q \circ P \) by the unbroken property. This subtrail contains at least \( u - 1 \) unbroken cells of \( Q \circ P \), since under \( S_j \), all unbroken cells in \( P \) map to unbroken cells of \( Q \circ P \), except at most one of them may be no longer unbroken after gluing to the trail \( Q \circ P \setminus S_j(P) \). So \( Q \circ P \) contains at least \((u - 1)v\) unbroken cells.

\[\square\]

Proof of Theorem 9.1. We have shown that there exists a sequence of permutations \( \tau_1, \ldots, \tau_n \) and a sequence of Euler-partitions \( Q^1, \ldots, Q^h \) such that for \( k = 1, \ldots, h \),

(i) \( Q^k \) is an Euler-partition of \( G(S, A_{\tau_k}) \) with output permutation \( \tau_{k+1} \), where we identify \( \tau_1 \) and \( \tau_{h+1} \);

(ii) \( Q^k \) has the bubbling property.

By Lemma 9.2,

\[
P = Q^h \circ \cdots \circ Q^1
\]

(9.7)
is an Euler-partition of $G(S^h, A_r)$ with output permutation $\tau_1$ and satisfying the bubbling property. This $P$ is the desired Euler-partition.

**Remark 9.3.** If $H(S, A)$ has a spanning tree with 3 or more tops, then we can require the Euler-partition $P$ in theorem 9.1 containing at least $3 \cdot 2^{h-1}$ unbroken cells.

10. Primitivity

Let $Q = Q_1 + \cdots + Q_m$ be an Euler-partition of $G(S, A_r)$ which is consistent and satisfies the bubbling property. In this section, we show that we can always change the orientation of some cells in $G(S, A_r)$ and transfer $Q$ to a primitive Euler-partition.

We shall use $A$ and $\Lambda$ instead of $A_\tau$ and $\Lambda_\tau$, since the permutation $\tau$ is fixed in this section.

Our goal is to prove the following:

**Theorem 10.1.** Let $Q = Q_1 + \cdots + Q_m$ be a consistent Euler-partition of $G(S, A)$ satisfying the bubbling property. If

$$\min_{1 \leq j \leq m} |Q_j| \geq m^4 + 5m^2,$$

then there exist an orientation vector $\beta$ such that we can construct a consistent and primitive Euler-partition $P = P_1 + \cdots + P_m$ of $G(S, A, \beta)$. Moreover, $P$ contains pure cells whenever $Q$ does.

10.1. Classification of cells

We call the initial edges and terminate edges of the trails $Q_1, \ldots, Q_m$ special edges; there are $2m$ special edges. We say a trail visits a cell, if they have common edges.

Let $i \in \Sigma = \{1, \ldots, N\}$. We call $S_i(\Lambda)$ a special cell, if it contains special edges. We call $S_i(\Lambda)$ a pure cell, a bi-partition cell, or a poly-partition cell if it is visited by one, two or more than two members of $\{Q_k; 1 \leq k \leq m\}$.

Take a cell $S_i(\Lambda)$, let

$$Q = e_1 + L_1 + e_2 + L_2 + \cdots + e_m + L_m$$

be the outpath decomposition of $Q$ w.r.t. $S_i(\Lambda)$.

**Lemma 10.1.** Let $S_i(\Lambda)$ be a non-special cell. Then

$$\# \{Q_j; Q_j \text{ visits } S_i(\Lambda)\} = \# \{L_j; L_j \text{ contains special edges}\}.$$  

**Proof.** If $S_i(\Lambda)$ is pure, that is, it is a subset of one $Q_j$, then both sides of (10.3) equal 1.

Now suppose $S_i(\Lambda)$ is not pure. If $Q_j$ visits $S_i(\Lambda)$, then $Q_j$ visits exactly two $L_j$ which contain special edges. On the other hand, if an outpath $L_j$ contains special edges, then it intersects exactly two of $Q_j$ visiting $S_i(\Lambda)$. The lemma is proved. (If one boy shakes hands exactly with two girls and one girl shakes hands exactly with two boys, then the number of boys and the number of girls are the same.)

Now we estimate the number of poly-partition cells.
Lemma 10.2. The number of poly-partition cells is less than \( m(m-1)(m-2)/6 \).

Proof. For a cell \( S_i(\Lambda) \), we define \( \kappa(i) = \{ k : Q_k \text{ visits } S_i(\Lambda) \} \).

Let \( S_i(\Lambda) \) and \( S_j(\Lambda) \) be two poly-partition cells, that is, \( \# \kappa(i), \# \kappa(j) \geq 3 \).

Let \( L \) be the outpath of \( S_i(\Lambda) \) containing the cell \( S_j(\Lambda) \). Then there exist two edges \( e_h \) and \( e_{h+1} \) in the cell \( S_i(\Lambda) \) such that

\[
e_h + L + e_{h+1}
\]

is a subtrail of \( Q \). Hence, if \( Q_k \) visits both the \( S_i \)-cell and the closed trail \( L \), then it contains at least one of \( e_h \) and \( e_{h+1} \). This implies that at most two elements of \( \{ Q_k : k \in \kappa(i) \} \), visit \( L \). Therefore \( \kappa(i) \) and \( \kappa(j) \) share at most two elements. It follows that the smallest three elements of \( \kappa(i) \) and that of \( \kappa(j) \) cannot be the same. The lemma is proved.

10.2. Kingdom of a non-special bi-partition cell

Let \( S_i(\Lambda) \) be a non-special bi-partition cell, then \( S_i(\Lambda) \) is visited by two elements in \( \{ Q_j ; 1 \leq j \leq m \} \), say, \( Q_{k} \) and \( Q_{k'} \). By lemma 10.1, exactly two outpaths of \( S_i(\Lambda) \), denoted by \( L \) and \( L' \), contain special edges. Then the Euler tour \( Q \) has the following decomposition:

\[
Q = L + D + L' + D'.
\]

We call \( D \cup D' \) the \( S_i(\Lambda) \)-kingdom, and call \( D + D' \) the partition of the \( S_i(\Lambda) \)-kingdom. (See figure 16.) The following lemma is obvious.

Lemma 10.3. An \( S_i(\Lambda) \)-kingdom is a cell-exact subgraph; moreover, all cells in it are pure except \( S_i(\Lambda) \).

The trails \( Q_k \), also \( Q_{k'} \), are cut into three parts by \( L \) and \( L' \), and we denote the three parts by

\[
Q_k = C + D + E, \quad Q_{k'} = C' + D' + E'.
\] (10.4)

where \( C, E' \subset L \) and \( E, C' \subset L' \). (See figure 17.)

Lemma 10.4 (Disjointness of kingdoms) Let \( S_i(\Lambda) \) and \( S_j(\Lambda) \) be two non-special bi-partition cells. Then the \( S_i \)-kingdom and the \( S_j \)-kingdom are disjoint.
\textbf{Proof.} Since \(S_j(\Lambda)\) is a bi-partition cell, \(Q\) can be written as
\[
Q = L + D + L' + D'
\]
where \(D + D'\) is the \(S_j\)-kingdom. The cell \(S_j(\Lambda)\) does not belong to the \(S_j\)-kingdom since it is not pure (see lemma 10.3), so \(S_j(\Lambda)\) must belong to one of \(L\) and \(L'\). Let us assume that \(S_j(\Lambda) \subseteq L\) without loss of generality. Then
\[
Q \setminus L = L' + (S_j\text{-kingdom})
\]
is a closed subtrail of \(Q\) not intersecting \(S_j(\Lambda)\). So \(Q \setminus L\) must belong to an outpath \(L^*\) w.r.t. \(S_j(\Lambda)\). Now \(L^*\) contains special edges since \(L' \subseteq L^*\), so it does not belong to the \(S_j\)-kingdom, and \(Q\) can be written as \(Q = L^* \cup (S_j\text{-kingdom}) \cup L^*\). It follows that the \(S_j\)-kingdom, as a subset of \(L^*\), does not intersect the \(S_j\)-kingdom. The lemma is proved.

\section*{10.3. Operation cells}

First, let us reformulate the notion of primitivity. Let \(Q\) be an Euler-partition. We construct a graph \(\Gamma_Q\) as follows: the vertex set is \(\{1, \ldots, m\}\). If \(Q_k\) contains a cell \(S_j(\overline{a_1, a_0, a_u, a', u})\) or \(S_j(\overline{a_1, a_0, a_u, a', u})^{-1}\) for some \(j \in \Sigma\), then there is an edge from \(k\) to \(k'\). By definition 8.3, the following lemma is obvious.

\textbf{Lemma 10.5.} \(Q = Q_1 + \cdots + Q_m\) is primitive if and only if there is an integer \(n_0 \geq 1\) such that, for any \(i,j \in \{1, \ldots, m\}\), there is a trail in \(\Gamma_Q\) of length \(n_0\) from \(i\) to \(j\).

Set
\[
V = \{k; Q_k \text{ contains pure cells}\}, \quad V' = \{1, \ldots, m\}/V.
\]
If \(V' = \emptyset\), then \(T(Q)\) is a complete graph and clearly \(Q\) is primitive. So we assume that \(V' \neq \emptyset\). Let \(k_0\) be the minimal element of \(V'\).

For each \(k \in V\), we select a pure cell contained in \(Q_k\), and call it the \textit{protected cell}, or a \textit{black cell}, of \(Q_k\). It is possible that this black cell is contained in a \(S_j\)-kingdom. If this happens, then we call \(S_j(\Lambda)\) an \textit{indirectly protected cell}, or \textit{grey cell}, associated with \(Q_k\). (In this case \(S_j(\Lambda)\) is a non-special bi-partition cell.) By the disjointness of kingdoms, there is at most one grey cell associated with \(Q_k\). So there are at most \#\(V\) grey cells.

For \(k \in V', Q_k\) only visits bi-partition cells and poly-partition cells; we shall choose operation cells among the bi-partition cells which are non-special and not grey. The lemma below estimates the number of such cells. Let \(|Q_j|\) be the length of \(Q_j\), i.e. the number of edges in \(Q_j\).

\textbf{Lemma 10.6.} \[\min_{1 \leq i \leq m} |Q_i| \geq m^4 + 5m^2,\] (10.5) then for any \(k \in V'\), the number of bi-partition cells which are visited by \(Q_k\) non-special and not grey, is no less than \(2m\).

\textbf{Proof.} The total number of special cells, non-special poly-partition cells, and the grey cells are no more than \(2m, \frac{m(n-1)(m-2)}{6}\), and \(m\), respectively (see lemma 10.2). Let \(k \in V'\). The number of cells visited by \(Q_k\) is no less than \(|Q_k|/m\). So the number of the desired cells is no less than \(|Q_k|/m - (2m + \frac{m(n-1)(m-2)}{6} + m) \geq 2m.\]
From now on, we assume that (10.5) holds. Denote
\[ r = m + (\# V') - 1. \]
By the above lemma, we can choose \( r \) non-special, not grey, bi-partition cells
\[ T_1, \ldots, T_m, T_{m+1}, \ldots, T_r \] (10.6)
such that \( Q_k \) visits the first \( m \) of them, and for the other \( k \in V' \), each \( Q_k \) visits a different cell in \( T_{m+1}, \ldots, T_r \). We call the cells in (10.6) operation cells. In other words, we associate with \( k_0 \) the first \( m \) operation cells, and for each \( k \in V' \setminus \{k_0\} \), we associate with one operation cell.

10.4. \((v, k)\)-operation
Let \( v \) be an edge of \( \Lambda \) and \( k \in V' \). Let \( S_\ell(\Lambda) \) be an operation cell associated with \( k \), then \( Q_k \) visits \( S_\ell(\Lambda) \); let \( Q_{k^*} \) be the other element in \( \{Q_j; 1 \leq j \leq m\} \) visiting the cell \( S_\ell(\Lambda) \). We do a \((v, k)\)-operation as follows. Let \( D + D^* \) be the partition of the \( S_\ell\)-kingdom. By (10.4), \( Q_k \) and \( Q_{k^*} \) can be written as
\[ Q_k = C + D + E, \quad Q_{k^*} = C^* + D^* + E^*. \]
If \( S_\ell(v) \) belongs to \( Q_k \), the \( v \)-operation is a null operation, that is, we do nothing; otherwise, the \( v \)-operation is to construct a new Euler-partition \( Q' = Q'_1 + \cdots + Q'_m \) by exchanging \( D \) and \( D^* \), and reverse them. Precisely, we set
\[
\begin{align*}
Q'_k &= C + (D^*)^{-1} + E, \\
Q'_{k^*} &= C^* + D^{-1} + E^*, \\
Q'_j &= Q_j, \text{ if } j \notin \{k, k^*\}.
\end{align*}
\]
Then the following statements (i)–(iv) hold:

(i) \( Q' \) is an Euler tour in \( G(S, A, \beta') \) where \( \beta' \) is obtained by changing the orientation of all the cells in the \( S_\ell\)-kingdom. (Since an edge is reversed if and only it belongs to \( S_\ell\)-kingdom.)
(ii) \( Q' \) contains pure cells if \( Q \) does; (The protected cell of \( Q_k \), which is pure, still belongs to \( Q'_k \) since the operation does not change any protected cell.)
(iii) \( Q' \) is still consistent since \( Q'_k \) and \( Q_k \) have the same initial and terminate points for all \( 1 \leq k \leq m \).
(iv) Either \( S_\ell(v) \) or \( S_\ell(v^{-1}) \) belongs to \( Q'_k \). (If \( S_\ell(v) \) does not belong to \( Q_k \), then it must appear in \( D^* \). After operation, \( S_\ell(v^{-1}) \) belongs to \( Q_k \).

Now we can proof the main theorem of this section.
**Proof of theorem 10.1.** Write \( A = v_1 + \cdots + v_m \). For \( j = 1, \ldots, m \), we do \((v_j, k_0)\)-operation to the \( j \)th cell in (10.6), which are operation cells associated with the minimal element \( k_0 \) in \( V' \). For the operation cells associated to the indices \( k \in V' \setminus \{k_0\} \), we do the \((v_k, k)\)-operation. We can do these \( r \) operations consecutively, since the operation cells belonging to the kingdoms are disjoint.

Let \( Q' = Q'_1 + \cdots + Q'_n \) be the resulted Euler-partition after doing all the operations.

First, \( Q' \) is an Euler-partition of \( G(S, A, \beta') \), where \( \beta' \) is obtained by changing the orientations of the cells in the kingdoms which are involved in the operations.

Secondly, \( Q' \) is consistent, since all the operations do not change the origins and terminates of the partition.

Thirdly, \( Q' \) is primitive, since \( v_{k_0} \) appears in every \( Q_k \) for \( k \in \{1, \ldots, m\} \), and all \( v_j \), \( j = 1, \ldots, m \), appear in \( Q_{k_0} \).

Finally, the pure cell property is clearly preserved. The theorem is proved. \( \square \)

### 11. Proof of theorem 1.3

In this section, we prove theorem 1.3. We consider two cases according to the behaviour of the Hata graphs \( H(S^n, A) \), \( n \geq 1 \): if \( H(S^n, A) \) is not a chain for some \( n \geq 1 \), then we show that \( K \) admits SFCs constructed by the Euler-tour method; otherwise, \( S \) is a self-similar zipper.

A graph with \( n \) vertices is a chain if there exists an order \( x_1, \ldots, x_n \) of the vertices such that there is exactly one edge between \( x_i \) and \( x_j \) if \( |i - j| = 1 \), and there is no edge otherwise.

In this case, we use \( (x_1, \ldots, x_n) \) to indicate the chain, and we regard the chain as a trail with origin \( x_1 \) and terminus \( x_n \).

**Theorem 11.1.** Let \( S = \{S_i\}_{i=1}^{\infty} \) be an IFS satisfying the OSC. Suppose that \( S \) has skeleton \( A \) and the Hata graph \( H(S^n, A) \) \((k \geq 1)\) is not always a chain. Then there exist an integer \( n \), a permutation \( \tau \) of \( \{1, \ldots, m\} \), a vector \( \beta \in \{-1, 1\}^n \), and an Euler-partition

\[
P = P_1 + \cdots + P_m
\]

of \( G(S^n, A_\tau, \beta) \) such that \( P \) is consistent, primitive and contains pure cells.

**Proof.** Let \( n_1 \) be an integer such that the Hata graph \( H(S^{n_1}, A) \) is not a chain. By theorem 9.1, there exist an integer \( n_2 \) and a permutation \( \tau \) on \( \{1, \ldots, m\} \), and an Euler-partition

\[
Q = Q_1 + \cdots + Q_m
\]

of \( G(S^{n_2}, A_\tau) \) such that \( Q \) is consistent and satisfies the bubbling property. Moreover, \( Q \) contains more than 3 unbroken cells by remark 9.3.

We choose \( \ell \) large enough such that \( 2^\ell > 2m \) and \( Q^\ell \), the \( \ell \)th iteration of \( Q \), satisfies the inequality (10.1) holds. Notice that \( Q^\ell \) is still consistent and satisfies the bubbling property. So \( Q^\ell \) satisfies all conditions of theorem 10.1. Therefore, there exist an orientation vector \( \beta \), and an Euler-partition \( P = P_1 + \cdots + P_m \) of \( G(S^{n_3}, A_\tau, \beta) \) such that \( P \) is consistent and primitive.

Finally, we show that \( P \) contains pure cells. Since \( 2^\ell > 2m \), by lemma 9.2 (iii), \( Q^\ell \) contains more than 2m unbroken cells. Since there are only \( 2m \) special edges in the Euler-partition \( Q^\ell \), there exists an unbroken cell which contains no special edge and it must be a pure cell of \( Q^\ell \).

Again by theorem 10.1, \( P \) also contains pure cells. The theorem is proved. \( \square \)
**Theorem 11.2.** Let \( S = \{ S_i \}_{i=1}^N \) be an IFS with connected invariant set \( K \) satisfying the OSC. Suppose that \( A \) is a skeleton of \( K \) such that the Hata graph \( H(S^n, A) \) is a chain for every \( n \geq 1 \), then an arrangement of \( \{ S_i \}_{i=1}^N \) is a self-similar zipper.

**Proof.** We will abbreviate \( H(S^n, A) \) as \( H_n \). Since \( H_1 \) is a chain, without lose of generality, we assume that \( H_1 \) passes the vertices in the order \( S_1, S_2, \ldots, S_N \). To simplify the notation, we denote \( H_1 \) by

\[
H_1 = (1, 2, \ldots, N)
\]

instead of \((S_1, S_2, \ldots, S_N)\). For \( 1 \leq j \leq N \), by self-similarity,

\[
(j_1, j_2, \ldots, jN) \text{ or } (jN, \ldots, j2, j1)
\]

is a sub-chain of \( H_2 \). The origin of \( H_2 \) is either \( S_{i_1} \) or \( S_{i_N} \), and the terminus is \( S_{i_N} \) or \( S_{i_1} \). We define a sequence \( (\beta_j)_{j=1}^N \) as follows: \( \beta_j = 1 \) if \((j_1, j_2, \ldots, jN) \) is a sub-chain of \( H_2 \), and \( \beta_j = -1 \) if \((jN, \ldots, j2, j1) \) is.

We use \( j(a_1, \ldots, a_k) \) to denote the chain \((ja_1, \ldots, ja_k)\), then

\[
H_2 = (1H_1)^{\beta_1} + (2H_1)^{\beta_2} + \cdots + (NH_1)^{\beta_N},
\]

where we use ‘+’ to denote the concatenation of two sequences, and denote \((a_1, \ldots, a_k)^{-1} = (a_k, \ldots, a_1)\) to be the reverse of a chain.

First, we prove by induction that

\[
H_n = (1H_{n-1})^{\beta_1} + \cdots + (NH_{n-1})^{\beta_N}.
\]

By self-similarity, for any \( j \in \{1, \ldots, N\} \), either \( jH_{n-1} \) or \((jH_{n-1})^{-1} \) is a sub-chain of \( H_n \), so there exist \( \epsilon_1, \ldots, \epsilon_N \in \{-1, 1\} \) such that

\[
H_n = (1H_{n-1})^{\epsilon_1} + \cdots + (NH_{n-1})^{\epsilon_N}.
\]

By \((jH_{n-1})^{-1} = jH_{n-1}^{-1}\) and using the induction hypothesis, we have

\[
H_n = 1(H_{n-1})^{\epsilon_1} + \cdots + N(H_{n-1})^{\epsilon_N}
\]

\[
= \cdots + j \left( H_{n-2}^{\beta_1} + \cdots + NH_{n-2}^{\beta_N} \right)^{\epsilon_j} + (j + 1) \left( H_{n-2}^{\beta_1} + \cdots + NH_{n-2}^{\beta_N} \right)^{\epsilon_{j+1}} + \cdots
\]

Notice that if \(i_1 \cdots i_n\) and \(j_1 \cdots j_n\) are adjacent in \( H_n \), then \(i_1 \cdots i_k\) and \(j_1 \cdots j_k\) are adjacent in \( H_k \) for any \( k < n \). The above decomposition of \( H_n \) forces that

\[
j(1, \ldots, N)^{\epsilon_j} + (j + 1)(1, \ldots, N)^{\epsilon_{j+1}}
\]

is a sub-chain of \( H_2 \), so \( \epsilon_j = \beta_j \) and \( \epsilon_{j+1} = \beta_{j+1} \), which proves (11.2).

Using \((\beta_1, \ldots, \beta_N)\), we define the following ordered GIFS:

\[
E_1 = S_1(E_{\beta_1}) + S_2(E_{\beta_2}) + \cdots + S_N(E_{\beta_N})
\]

\[
E_{-1} = S_N(E_{-\beta_N}) + \cdots + S_2(E_{-\beta_2}) + S_1(E_{-\beta_1}).
\]

Clearly, \( K = E_1 = E_{-1} \) by the uniqueness of invariant set of \( S \).
Let \((\mathcal{V}, \Gamma)\) be the basic graph of the above GIFS, then \(\mathcal{V} = \{E_1, E_{-1}\}\); the \(j\)th edge from \(E_1\) is an edge from \(E_1\) to \(E_{j_1}\) with similitude \(S_j\), which we denote by \((E_1, j, E_{j_1})\); the \(j\)th edge starting from \(E_{-1}\) goes to \(E_{-j_1}\) and has similitude \(S_{N-j+1}\), which we denote by \((E_{-1}, S_{N-j+1}, j, E_{-j_1})\).

By theorem 6.2, to prove the theorem, we need only prove (11.3) is a linear GIFS.

Let \(G_n\) be the sequence of trails with length \(n\) in the graph \(\Gamma\) emanating from the state \(E_1\), arranged in the increasing order. Notice that for each trails in \(G_n\), the associated contraction has the form \(S_j\) with \(|I| = n\), and for different trails, the associated contractions are distinct. So, we replace each trail in \(G_n\) by the associated contraction and further more replace \(S_j\) by the word \(I\), then we obtain a sequence which is a permutation of \(\Sigma^n\). For simplicity, we still denote this sequence by \(G_n\). Similarly, we denote by \(G_n^\prime\) the sequence of trails in the graph emanating from the state \(E_{-1}\), arranged in the increasing order. We apply the same simplification to \(G_n^\prime\).

We shall prove that \(G_n = H_n\) and \(G_n^\prime = G_n^\prime\).

Clearly, \(G_1 = \{1, 2, \ldots, N\}\), \(G_1^\prime = \{N, \ldots, 1\}\). Moreover, by (11.3),

\[
G_2 = (1G_1)^{\beta_1} + \cdots + (NG_1)^{\beta_N}.
\]

We claim that, for every \(n \geq 1\), it holds that

\[
G_n = (1G_{n-1})^{\beta_1} + \cdots + (NG_{n-1})^{\beta_N} \quad (11.4)
\]

and \(G_n^\prime = G_n^\prime\).

We prove by induction on \(n\). Let us consider the words in \(G_n\) initialed by \(j\). If \(\beta_j = 1\), then after one step, we are still in the state \(E_1\), hence the arrangement of such words in increasing order is \(jG_n-1\); if \(\beta_j = -1\), we switch to the state \(E_{-1}\) after one step, and the corresponding arrangement is \(jG_n-1 = j(G_n-1)^{-1}\). Therefore,

\[
G_n = (1G_{n-1})^{\beta_1} + (2G_{n-1})^{\beta_2} + \cdots + (NG_{n-1})^{\beta_N}.
\]

The same argument shows that

\[
G_n^\prime = (NG_{n-1})^{\beta_N} + \cdots + (2G_{n-1})^{\beta_2} + (1G_{n-1})^{\beta_1}
\]

\[
= (NG_{n-1})^{-\beta_N} + \cdots + (2G_{n-1})^{-\beta_2} + (1G_{n-1})^{-\beta_1}
\]

\[
= G_n^\prime.
\]

The claim is proved.

Comparing (11.2) and (11.4), we obtain that \(G_n = H_n\) for all \(n \geq 1\). Hence, if \(I\) and \(I'\) are two adjacent words in \(G_n\) (or in \(G_n^\prime\)), then they are also adjacent in \(H_n\), so

\[
S_I(K) \cap S_{I'}(K) \supset S_I(A) \cap S_{I'}(A) \neq \emptyset,
\]

which implies that (11.3) is a linear GIFS. \hfill \Box

**Proof of theorem 1.3.** Let \(\{S_i\}_{i=1}^N\) be an IFS satisfying the OSC. Let \(A\) be a skeleton of \(\mathcal{S}\).

If the Hata graph \(H(S^n, A)\) is always a chain, then \(K\) can be generated by a self-similar zipper by theorem 11.2, and hence admits a SFC.

If \(H(S^n, A)\) is not always a chain, by theorem 11.1, there exists an integer \(k\) such that an Euler-partition of \(G(S^k, A, \beta)\) satisfying the conditions in theorem 8.2. Hence the invariant set of \(S^k\), which is also \(K\), admits a SFC constructed by the Euler-tour method. \hfill \Box
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