GEOMETRIC INEQUALITIES, STABILITY RESULTS AND KENDALL’S PROBLEM IN SPHERICAL SPACE

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ABSTRACT. In Euclidean space, the asymptotic shape of large cells in various types of Poisson driven random tessellations has been the subject of a famous conjecture due to David Kendall. Since shape is a geometric concept and large cells are identified by means of geometric size functionals, the resolution of the conjecture is inevitably connected with geometric inequalities of isoperimetric type and their improvements in the form of geometric stability results, relating geometric size functionals and hitting functionals. The latter are deterministic characteristics of the underlying random tessellation. The current work explores specific and typical cells of random tessellations in spherical space. A key ingredient of our approach are new geometric inequalities and quantitative strengthenings in terms of stability results for quite general and some specific size and hitting functionals of spherically convex bodies. As a consequence we obtain probabilistic deviation inequalities and asymptotic distributions of quite general size functionals. In contrast to the Euclidean setting, where the asymptotic regime concerns large size, in the spherical framework the asymptotic analysis is concerned with high intensities.

1. INTRODUCTION

Tessellations, or mosaics, have been a subject of interest for a very long time. Even ancient cultures, like the Sumerians or the Romans, used colored tiles to decorate floors and walls. Formal, mathematical definitions and deterministic tilings of the plane or higher dimensional spaces were considered much later. By a tessellation of \( \mathbb{R}^d \), we understand a system of convex polytopes in \( \mathbb{R}^d \) which cover the whole space and have pairwise no common interior points.

Random tessellations of Euclidean spaces are a classical topic in stochastic geometry. They are extensively studied in the literature, see e.g. [60, Chap. 10], [12, Chap. 9], [44] or [46] for an overview and results for general tessellations. There are also many articles discussing various properties of special models such as hyperplane tessellations or Voronoi tessellations, see e.g. [39, 42, 22], [57, Chap. 6] or [45, 47, 5, 7], and their applications [50, 66].

In the present work, we consider random tessellations of the unit sphere \( S^d \) in Euclidean space \( \mathbb{R}^{d+1} \). This setting is not as extensively studied in the literature as the Euclidean one is. The intersection of the unit sphere with a \( d \)-dimensional linear subspace is the unit sphere in the intersecting subspace and thus a great subsphere of \( S^d \) having unit radius. At the same time, \( d \)-dimensional linear subspaces partition the Euclidean space \( \mathbb{R}^{d+1} \) into polyhedral cones. This relation plays an important role in spherical geometry, see e.g. [1], [13], [59] (and the literature cited there). Tessellations of the sphere generated by intersecting the unit sphere with \( d \)-dimensional linear subspaces are called spherical hyperplane tessellations. Random spherical hyperplane tessellations, where the subspaces are selected randomly, are studied in [13, 40, Section 6], [2] and in the recent work [28] on conical tessellations. Voronoi tessellations in spherical space can be defined as in the Euclidean case, using the geodesic distance on \( S^d \). Random Voronoi tessellations on the sphere and their applications are investigated in [40 Sec. 7], [53], [62], [49 Sec. 3.7.6, Sec. 5.10] and [63]. A cell splitting scheme on \( S^d \) has been considered in [16], a systematic study of splitting tessellations in spherical space (in analogy to STIT tessellations in Euclidean space) is carried out in [29].

In the following, we focus on what became known as ‘Kendall’s Problem’ or ‘Kendall’s Conjecture’ and in particular on its geometric foundations (for surveys and background information, see [60] Note 9 for Sec. 10.4], [52], [11], [57], [20], [21], and the recent work [8], [9]). So far this line of investigation was only considered in the Euclidean setting. In our present work, we now formulate and investigate a spherical analogue. To recall the problem in Euclidean space, consider a stationary and isotropic Poisson line process in the Euclidean plane and

Date: August 30, 2017.
2010 Mathematics Subject Classification. Primary: 60D05, 52C22, 52A55; secondary: 52A40.
Key words and phrases. Spherical space, random tessellation, mosaic, Kendall’s problem, geometric inequality, stability result.
Authors supported in part by DFG grant FOR 1548.
denote the almost surely unique cell containing the origin by \( Z_0 \). This cell is called the zero cell or Crofton cell. In the foreword of the first edition of [12], David G. Kendall stated the following conjecture: The conditional law for the shape of \( Z_0 \), given the area \( A(Z_0) \), converges weakly, as \( A(Z_0) \to \infty \), to the degenerate law concentrated at the circular shape. This conjecture was strongly supported by heuristic arguments from R. Miles [45]. Two years later, a proof was given by Kovalenko in [31]. Kovalenko also provided a simplified proof in [33] and an extension to the typical cell of a Poisson–Voronoi tessellation in the plane in [32]. Further extensions to arbitrary dimensions and not necessarily isotropic Poisson hyperplane tessellations were made in [22], where the size of the Crofton cell was measured by the volume. In [23] the problem was extended and solved for typical cells of stationary Poisson–Voronoi tessellations in arbitrary dimensions and the size was measured by an intrinsic volume. In [25] a very general setting with a very general class of size functionals was considered, containing the aforementioned results as special cases. In [26], Kendall’s Problem was extended to the typical \( k \)-faces of a Poisson hyperplane tessellation \( (k \in \{2, \ldots, d-1\}) \) and in [27] to the typical \( k \)-faces of a Poisson–Voronoi tessellation. In [24] typical cells of Poisson–Delaunay tessellations were considered. In all these previous works, geometric inequalities, stability results and polytopal approximation have been crucial ingredients in the analysis.

We continue with some notation in order to present some of our results. On the unit sphere \( S^d \), there is no naturally distinguished point similar to the Euclidean origin in \( \mathbb{R}^{d+1} \), so we choose an arbitrary fixed point \( \mathbf{u} \in S^d \) as the spherical origin. Let \( d_s \) denote the geodesic metric on \( S^d \), and let \( B(x, r) \) denote a spherical ball with radius \( r \leq \pi \) and centre \( x \) (if \( r \leq \pi/2 \) we call \( B(x, r) \) a spherical cap). A proper (spherically) convex body in \( S^d \) is the intersection of the unit sphere with some line-free closed convex cone in \( \mathbb{R}^{d+1} \) which does not only consist of the Euclidean origin \( \{0\} \). We denote the set of proper convex bodies by \( K^d \). If we do not require the cone to be line-free but only require that the cone is not equal to some linear subspace of \( \mathbb{R}^{d+1} \), the resulting set will be denoted by \( \mathcal{C}^d \) and its elements are called spherically convex bodies. A spherical polytope is the intersection of \( S^d \) with a polyhedral cone (a finite intersection of half-spaces whose bounding hyperplanes contain the origin) which is also a spherically convex body. For more details on spherical geometry, we refer to [60] Section 6.5 and [19].

By a tessellation of \( S^d \) we understand a finite collection of spherical polytopes which have nonempty interiors, which cover \( S^d \) and have pairwise disjoint interiors, or the trivial tessellation consisting of \( S^d \) only. Prominent examples are spherical hyperplane and Voronoi tessellations which are based on a finite number of subspaces and point sets, respectively. Random tessellations then arise, for instance, by selecting the underlying subspaces and points randomly, thus giving rise to random spherical hyperplane and random Voronoi tessellations. In general, a random tessellation is said to be isotropic, if its distribution is invariant under any rotation in \( SO_{d+1} \). In this case, almost surely there exists a unique cell containing \( \mathbf{u} \) in its interior. We call this cell the spherical Crofton cell (or spherical zero cell) of the given random tessellation and denote it by \( Z_0 \).

We denote the spherical Lebesgue measure on \( S^d \) by \( \sigma_d \) and the surface area of the unit sphere by \( \omega_{d+1} := \sigma_d(S^d) \). Let \( X \) be an isotropic Poisson point process on \( S^d \). This means that \( X \) can be viewed as a random collection of finitely many points in \( S^d \) such that for any Borel set \( A \subset S^d \) the random number \( X(A) \) of points of \( X \) in \( A \) is Poisson distributed with Poisson parameter proportional to \( \sigma_d(A) \). Then the intensity measure of \( X \) equals \( \mathbb{E}[X(\cdot)] = \gamma_s \sigma_d(\cdot) \) for some \( \gamma_s \geq 0 \). Henceforth, we always assume that the intensity \( \gamma_s \) is finite and positive. From \( X \) we obtain a spherical Poisson hyperplane tessellation by tessellating the unit sphere with the random codimension one subspaces \( \{x^\perp : x \in X\} \), where \( x^\perp \) denotes the orthogonal complement of \( x \). We aim to show that the Crofton cell of such an isotropic spherical Poisson hyperplane tessellation, given a fixed lower bound for its spherical volume, converges to a spherical cap as \( \gamma_s \to \infty \). Therefore, we have to quantify the deviation of \( Z_0 \) from a spherical cap. A functional \( \vartheta : \mathcal{C}^d \to [0, \infty) \) is called a deviation functional for the class of spherical caps, if it is continuous and if \( \vartheta(K) = 0 \) holds, for some \( K \in \mathcal{C}^d \) with \( \sigma_d(K) > 0 \), if and only if \( K \) is a spherical cap. An example of such a deviation functional is the difference between spherical circumradius and spherical inradius of \( K \) (we minimize over all suitable common centre points). Another example, denoted by \( \Delta_2 \), measures the deviation of the spherical radial function of \( K \) with respect to a centre point) from its integral average in the \( L^2 \)-sense and minimizes over all suitable centre points. A crucial tool in the probabilistic analysis of the asymptotic shape of the spherical Crofton cell of a spherical Poisson hyperplane tessellation is the use of general inequalities of isoperimetric type and stability improvements thereof, for size and hitting functionals in spherical space. While isoperimetric results for a variety of geometric functionals in Euclidean space have been the subject of numerous investigations (see [53] for a detailed and profound exposition of geometric inequalities and [41-50] for some recent stability results, applications and further references), much less is known in the spherical setting.
The hitting functional associated with an isotropic spherical Poisson hyperplane tessellation is the functional $\Phi = 2U_1$ on $\overline{K}_s^d$, where $U_1$ is given by

$$U_1(K) = \frac{1}{2\omega_{d+1}} \int_{\mathbb{S}^d} 1\{x^\perp \cap K \neq \emptyset\} \sigma_d(dx), \quad K \in \overline{K}_s^d.$$ 

It can be interpreted as one half of the measure of all great subspheres hitting $K$. Thus, it is a spherical analogue of the Euclidean mean width functional $V_1$. To resolve Kendall’s problem in spherical space, with the spherical volume as the size functional, we need a stability improvement of a geometric inequality of isoperimetric type involving spherical mean width $U_1$ and volume $\sigma_d$. In [17], the following inequality (1) is shown. It can be interpreted as a spherical version of the classical Urysohn inequality. The latter provides a lower bound for the mean width functional (the first intrinsic volume $V_1$) in terms of the volume functional $V_d$ in Euclidean space $\mathbb{R}^d$, that is, $V_1(K) \geq (V_d(K)/V_d(B))^{1/d} V_1(B)$, where $B \subset \mathbb{R}^d$ is a Euclidean ball and $K \subset \mathbb{R}^d$ is a convex body (see [58, p. 382]). Equality holds if and only if $K$ is a ball. Stated in this form, the homogeneity of the involved functionals is crucial. However, an equivalent form of the inequality states that $V_1(K) \geq V_1(B)$, whenever $K$ is a convex body and $B$ is a ball of equal volume. In spherical space, a corresponding result can be stated in the following form.

Let $K \in \overline{K}_s^d$ and let $C$ be a spherical cap with $\sigma_d(K) = \sigma_d(C)$. Then

$$U_1(K) \geq U_1(C)$$

and equality holds if and only if $K$ is a spherical cap. Two proofs are provided in [17]. The second proof exhibits an interesting connection to the Euclidean Blaschke-Santaló inequality [58, Chap. 10.7]. Furthermore, this proof can be strengthened to yield the following more general stability estimate. For a spherical cap $C$, we denote its radius by $\alpha_C$.

**Theorem A.** Let $K \in \overline{K}_s^d$ and let $C \subset \mathbb{S}^d$ be a spherical cap with $\sigma_d(K) = \sigma_d(C) > 0$. Let $\alpha_0 \in (0, \pi/2)$ be such that $\alpha_0 \leq \alpha_C \leq \pi/2 - \alpha_0$. Then there is a constant $\beta^{(\ast)}_C > 0$ such that

$$U_1(K) \geq (1 + \beta^{(\ast)}_C \Delta_2(K)^2) U_1(C),$$

where $\beta^{(\ast)}_C$ depends on $\alpha_0, d$.

An improved version of this result is stated as Theorem 4.1 where instead of an upper bound on $\alpha_C$ the dependence of $\beta^{(\ast)}_C$ on $\alpha_0$ and $\alpha_C$ is made explicit.

In Section 6, we prove the following result, which is based on Theorem A. It provides not only the asymptotic shape (which is a spherical cap) of the spherical Crofton cell given a lower bound for its volume, but also gives deviation inequalities for fixed intensities.

**Theorem B.** Let $Z_0$ be the zero cell of an isotropic spherical Poisson hyperplane tessellation with intensity $\gamma_s$. Let $0 < a < \omega_{d+1}/2$ and $\varepsilon \in (0, 1]$. Then there are constants $\tilde{c}, \tilde{\beta} > 0$ such that

$$\mathbb{P}(\Delta_2(Z_0) > \varepsilon \mid \sigma_d(Z_0) \geq a) \leq c^* \exp \left(-\beta^* \varepsilon^{2(d+1)} \gamma_s\right),$$

where the constant $c^*$ depends on $a, \varepsilon, d$ and $\beta^*$ depends on $a, d$.

For both, Theorem A and Theorem B, similar results with different stability exponents, are obtained in Section 4 for the inradius as the size functional and a suitably chosen deviation functional. In the case of a general size functional and an associated deviation functional, an isoperimetric inequality and a corresponding stability result are obtained in Section 3. This finally leads to a general resolution of Kendall’s problem, but without an explicit bound on the stability order with respect to $\varepsilon$ as in Theorem B (see Section 6).

As a consequence of our approach, we also obtain the asymptotic distribution of the size functional of the zero cell of an isotropic spherical Poisson hyperplane tessellation as the intensity goes to infinity.

**Theorem C.** Let $Z_0$ be the zero cell of an isotropic spherical Poisson hyperplane tessellation with intensity $\gamma_s$. Let $\Sigma$ be a general increasing size functional and let $a > 0$ be such that $\Sigma^{-1}([a, \infty)) \neq \emptyset$. Then

$$\lim_{\gamma_s \to \infty} \gamma_s^{-1} \ln \mathbb{P}(\Sigma(Z_0) \geq a) = -\omega_{d+1} \tau(a),$$

where $\tau(a)$ is the isoperimetric constant (introduced in Section 3) associated with the hitting functional $\Phi = 2U_1$, the size functional $\Sigma$ and the threshold $a$. 

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In particular, this shows that the probability $\mathbb{P}(\Sigma(Z_0) \geq a)$ decays exponentially fast as $\gamma_s \to \infty$ (see Section 7).

Results similar to Theorem B and Theorem C (and to results stated in the following) can also be obtained for tessellations derived from a binomial process of size $N \geq d + 1$, where the deterministic number $N$ (of points or subspaces) replaces the intensity. Since the arguments are similar, and preliminary versions of such results are contained in [61], we do not provide further details.

After investigating Crofton cells, a natural next step is to look at typical cells. Thus, in Section 8 we consider typical objects in spherical space. Since $\mathbb{S}^d$ is a homogeneous $SO_{d+1}$-space (see [60, p. 584]), we use the framework of random measures on homogeneous spaces (see [33] and [54]). We start by recalling briefly the Euclidean framework which has been thoroughly studied. A process of compact convex particles in Euclidean space $\mathbb{R}^d$ is a point process on the space $\mathcal{K}^d$ of nonempty compact convex subsets of $\mathbb{R}^d$ (see [60, Chap. 4.1]). If $Y$ is a (simple) stationary particle process (that is its distribution is invariant under translations) with intensity $\gamma_Y$ and $c : \mathcal{K}^d \to \mathbb{R}^d$ is a translation covariant centre function, then a very intuitive representation for the distribution $\mathbb{Q}$ of the typical particle of $Y$ (see [60, p. 106]) is

$$\mathbb{Q}(\cdot) = \frac{1}{\gamma_Y} \mathbb{E} \sum_{K \in Y} 1\{K - c(K) \in \cdot\} 1\{c(K) \in [0, 1]^d\},$$

where $\mathbb{Q}$ is concentrated on sets having the Euclidean origin as their centre. Here we implicitly use that $K - c(K)$ is the unique translate of $K$ whose centre is the origin. In contrast, in spherical space there are infinitely many rotations $\varphi \in SO_{d+1}$ such that $\varphi \mathcal{O} = x$ for any fixed $x \in \mathbb{S}^d$. This is the reason why an additional randomization is used in the following definition of a typical particle of an isotropic particle process in spherical space. Let $X'$ be a (simple) isotropic spherical particle process with intensity $\gamma_{X'}$ (precise definitions are given in Section 8). Then the distribution of the associated typical particle $Z$ can be defined by

$$\mathbb{P}(Z \in \cdot) = \frac{1}{\gamma_{X'} \omega_{d+1}} \mathbb{E} \sum_{K \in X'} \int_{SO_{d+1}} 1\{\Phi^{-1}K \in \cdot\} \kappa(c_s(K), d\varphi),$$

where $c_s$ is a rotation covariant centre function, $\kappa$ is a probability kernel such that $\kappa(x, \cdot)$, for $x \in \mathbb{S}^d$, is a probability measure on $SO_{d+1}$ concentrated on the set $\{\varphi \in SO_{d+1} : \varphi \mathcal{O} = x\}$. (In a systematic context, we will obtain $\mathbb{P}(Z \in \cdot)$ as the (normalized) mark distribution of a certain marked random measure $\zeta'$ associated with $X'$, which is also equal to the (normalized) marginal measure of the Palm measure of $\zeta'$). Although $Z$ is not isotropic, its distribution $\mathbb{P}_Z$ is invariant under rotations fixing $\mathcal{O}$ and satisfies a disintegration result for isotropic particle processes on $\mathbb{S}^d$ (a Euclidean analogue can be found in [60, Theorem 4.1.1]). The disintegration result and the partial invariance together are characteristic for the distribution $\mathbb{P}_Z$, as stated in Theorem 8.2.

In Section 8 we interpret an isotropic tessellation $X'$ of $\mathbb{S}^d$ as an isotropic particle process and use the aforementioned disintegration result to obtain that $\mathbb{E}[f(Z_0)] = \gamma_{X'} \mathbb{E}[f(Z) \sigma_d(Z)]$, for any measurable and rotation invariant function $f$ of the particles. Thus, we relate the distributions of the Crofton cell $Z_0$ and of the typical cell $Z$ of $X'$. Using this relation, we transfer Theorem B and Theorem C to the typical cell of a spherical Poisson hyperplane tessellation and the same can be done for the typical cell of the tessellation induced by a binomial hyperplane process of size $N \geq d + 1$.

Further, we investigate the typical cell of a spherical Poisson–Voronoi tessellation. In Euclidean space, the Theorem of Slivnyak characterizes a stationary Poisson process using its Palm distribution (see [60, Theorem 3.3.5]). We use a spherical version to show that the distribution of the typical cell of the Poisson–Voronoi tessellation, induced by an isotropic Poisson process $X$ on $\mathbb{S}^d$, is equal to the distribution of the Crofton cell associated with a special spherical Poisson hyperplane process $Y$. The spherical hyperplane process $Y$ is the set of all great sub-spheres having equal distance to the spherical origin $\mathcal{O}$ and a point in $X$ and thus clearly is not isotropic. This leads to a new functional $\tilde{U} := \tilde{U}_\mathcal{O}$ (for the ease of notation, we write $\tilde{U}$ instead of $\tilde{U}_\mathcal{O}$ if there is no danger of confusion) on $\mathcal{K}^d_\mathcal{O}$, the set of all spherically convex bodies $K \in \mathcal{K}^d_\mathcal{O}$ with $\mathcal{O} \in K \subset B(\mathcal{O}, \pi/2)$, defined by

$$\tilde{U}(K) := \int_{\mathbb{S}^d} 1\{(x - \mathcal{O})^\perp \cap K \neq \emptyset\} \sigma_d(dx), \quad K \in \mathcal{K}^d_\mathcal{O}.$$ 

In this setting, we measure the size with the centred spherical inball radius by

$$\Sigma_\mathcal{O}(K) := r_\mathcal{O}(K) := \max\{r \geq 0 : B(\mathcal{O}, r) \subset K\}, \quad K \in \mathcal{K}^d_\mathcal{O}.$$
Furthermore, let
\[ R_\mathbb{P}(K) := \min\{ r \geq 0 : K \subset B(\overline{0}, r) \}, \quad K \in \overline{\mathbb{K}_d}, \]
denote the centred spherical circumradius, and define a deviation functional \( \theta \) (for the class of spherical caps with centre \( \overline{0} \)) by
\[ \theta(K) := R_\mathbb{P}(K) - r_\mathbb{P}(K), \quad K \in \overline{\mathbb{K}_d}. \]

Section 4.3 is devoted to the following extremal and stability result for the size functional \( U \).

**Theorem D.** Let \( a \in (0, \pi/2) \), \( K \in \overline{\mathbb{K}_d} \) with \( r_\mathbb{P}(K) \geq a \), and let \( C := B(\overline{0}, a) \). Then
\[ U(K) \geq U(C) = \sigma_d(B(\overline{0}, 2a)) \]
with equality if and only if \( K = C \). Furthermore, if \( \theta(K) \geq \varepsilon \in (0, 1] \), then
\[ U(K) \geq \left( 1 + \tilde{\beta}_0 \varepsilon^{d+1} \right) U(C), \]
where the constant \( \tilde{\beta}_0 \) depends on \( a, d \).

Finally, in Section 10 we obtain an asymptotic result for the typical cell \( Z \) of a spherical Poisson–Voronoi tessellation, which is based on Theorem D.

**Theorem E.** Let \( a \in (0, \pi/2) \) and \( \varepsilon \in (0, 1] \). Let \( X \) be a Poisson process on \( \mathbb{S}^d \) with intensity \( \gamma_s \). Then
\[ \mathbb{P}(R_\mathbb{P}(Z) - r_\mathbb{P}(Z) \geq \varepsilon \mid r_\mathbb{P}(Z) > a) \leq \hat{c} \exp \left( -\tilde{\beta} \varepsilon^{d+1} \gamma_s \right), \]
where the constant \( \hat{c} > 0 \) depends on \( a, d, \varepsilon \) and the constant \( \tilde{\beta} > 0 \) depends on \( a, d \).

Asymptotic distributions of the spherical inradius or of general size functionals can be derived by similar arguments.

2. **Spherical hyperplane tessellations and the Crofton cell**

Let \( X \neq 0 \) be an isotropic Poisson process on \( \mathbb{S}^d \) in Euclidean space \( \mathbb{R}^{d+1} \) with scalar product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). We can view \( X \) as a simple random counting measure (a point process) on \( \mathbb{S}^d \) or as a finite random collection of points on \( \mathbb{S}^d \). Since the spherical Lebesgue measure \( \sigma_d \) is (up to a constant) the only rotation invariant measure on \( \mathbb{S}^d \), the intensity measure of \( X \) satisfies \( \Theta(\cdot) := \mathbb{E}[X(\cdot)] = \gamma_s \sigma_d(\cdot) \) for some constant \( \gamma_s \in (0, \infty) \). Here and in what follows we tacitly assume that \( \Theta \) is locally finite. The number \( \gamma_s \) is called the intensity of \( X \). The expected number of points on the sphere is \( \mathbb{E}[X(\mathbb{S}^d)] = \gamma_s \omega_{d+1} \).

Let \( G(d+1, k) \) denote the Grassmannian of \( k \)-dimensional linear subspaces of \( \mathbb{R}^{d+1} \). Applying the measurable mapping \( h : \mathbb{S}^d \rightarrow G(d+1, d) \cap \mathbb{S}^d, x \mapsto x^\perp \cap \mathbb{S}^d \), to every point in \( X \), we obtain the spherical hyperplane process (or great subsphere process) \( \tilde{X} := h(X) \), where \( h(X) \) denotes the image measure of \( X \) under \( h \). Clearly, \( h(X) \) is again an isotropic and simple Poisson process (see [60], [61] for an introduction to Poisson processes in general spaces). Given there is at least one spherical hyperplane in (the support of) \( \tilde{X} \), these spherical hyperplanes partition \( \mathbb{S}^d \) almost surely into a collection of spherical polytopes with pairwise disjoint interiors. Otherwise we obtain the empty tessellation of \( \mathbb{S}^d \) which is \( \mathbb{S}^d \) itself. In any case, such a partition is called a spherical hyperplane tessellation of \( \mathbb{S}^d \).

Let \( \overline{0} \) be an arbitrary fixed point of the unit sphere (which we call the spherical origin). The spherical Crofton cell or spherical zero cell is the (almost surely uniquely determined) cell which contains \( \overline{0} \) in its interior. We will denote it by \( Z_0 \). Clearly, \( Z_0 \) is almost surely a spherical polytope if \( X(\mathbb{S}^d) \neq 0 \), but \( Z_0 \subset \mathbb{K}_d \) only if \( X(\mathbb{S}^d) \geq d+1 \).

For \( K \subset \mathbb{S}^d \) we define \( H_K := \{ L \in G(d+1, d) \cap \mathbb{S}^d : L \cap K \neq \emptyset \} \). Then Campbell’s theorem (see [60], Theorem 3.1.2)) yields that \( \mathbb{E}[\tilde{X}(H_K)] = \gamma_s \omega_{d+1} 2U_1(K) \), for \( K \in \mathbb{K}_s^d \), and the hitting functional \( \Phi = 2U_1 \) associated with \( \tilde{X} \) satisfies \( \Phi(C_0) = 1 \) for any hemisphere \( C_0 \subset \mathbb{S}^d \). Furthermore,
\[ \mu(\cdot) := \frac{1}{\omega_{d+1}} \int_{\mathbb{S}^d} 1\{x^\perp \cap \mathbb{S}^d \in \cdot \} \sigma_d(dx), \]
is the rotation invariant probability measure on the space of great subspaces of \( \mathbb{S}^d \).

Our aim is to show that the Crofton cell \( Z_0 \), given a lower bound for its spherical volume, converges to a spherical cap for \( \gamma_s \rightarrow \infty \). This means that the conditional probability of \( Z_0 \) deviating from the shape of a
spherical cap, given \( Z_0 \) has spherical volume at least \( a \) for some \( a > 0 \), converges to 0 for \( \gamma_s \to \infty \). More generally, in the following we quantify the deviation of \( Z_0 \) from a spherical cap.

Sections [3-5] provide several geometric key results which are of interest in their own right and are needed for the investigation of Kendall’s problem in spherical space. The latter is treated in Sections [6-10].

3. A GENERAL FRAMEWORK FOR ISOPERIMETRY

The probabilistic deviation results outlined in the introduction are based on geometric inequalities and related stability results. Similarly as in the Euclidean setting, we can describe a very general setting for stability results of isoperimetric type, which are then applied to the solution of Kendall’s problem in spherical space.

The main ingredients in our analysis are a hitting functional \( \Phi \), a size functional \( \Sigma \), and a deviation measure \( \vartheta \). In general, by a hitting functional we mean a map \( \Phi : \mathcal{K}_s^d \to [0, \infty) \) which is continuous and such that \( \Phi(K) = 0 \), for some \( K \in \mathcal{K}_s^d \), if and only if \( K \) is a one-pointed set. Here and in the following, continuity on spherically convex bodies refers to the (spherical) Hausdorff metric \( \delta_s \). A first main example of a hitting functional is the hitting functional \( \Phi = 2U_1 \) of an isotropic spherical hyperplane process, which is proportional to the spherical mean width functional. In this case, \( \Phi \) is also increasing and rotation invariant. Another example arises in the study of Voronoi tessellations as the hitting functional of a non-isotropic spherical hyperplane tessellation and will be introduced later (on a restricted domain).

A size functional is a continuous map \( \Sigma : \mathcal{K}_s^d \to [0, \infty) \) which satisfies \( \Sigma \neq 0 \) and \( \Sigma(\{p\}) = 0 \) for all \( p \in S^d \). For the derivation of deviation inequalities and asymptotic distributions, we will also assume that \( \Sigma \) is increasing. Examples of size functionals are volume, surface area, any of the functionals \( s_i \) (for an introduction) are not increasing with respect to set inclusion in general [60, p. 262].

Under the assumptions of the proposition, we consider the nonempty set of extremal bodies associated with \( \vartheta \) and \( \Sigma \). The set
\[ E(\Phi, \Sigma, a) := \{ K \in \mathcal{K}_s^d : \Sigma(K) \geq a \text{ and } \Phi(K) = \tau(\Phi, \Sigma, a) \} \]
is the nonempty set of extremal bodies associated with \( \Phi, \Sigma \) and \( a \). Clearly, if \( \Phi \) and \( \Sigma \) are rotation invariant, then so is the class \( E(\Phi, \Sigma, a) \). In the following sections, we simply write \( \tau(a) \) and \( E(a) \) if \( \Phi, \Sigma \) are clear from the context.

Finally, a continuous functional \( \vartheta : \Sigma^{-1}([a, \infty)) \to [0, \infty) \) such that \( \vartheta(\Phi) = 0 \) if and only if \( K \in E(\Phi, \Sigma, a) \) is called a deviation functional for \( \Phi, \Sigma, a \). A general, canonical example is provided by
\begin{equation}
\vartheta(K) = \frac{\Phi(K)}{\tau(\Phi, \Sigma, a)} - 1, \quad K \in \Sigma^{-1}([a, \infty)).
\end{equation}

For specific choices of \( \Phi \) and \( \Sigma \), other choices of deviation functionals will be more natural. In particular, \( \vartheta \) as given in (2) is rotation invariant if this is true for \( \Phi \) and \( \Sigma \).

**Proposition 3.1.** Let \( \Phi \) be a hitting functional, \( \Sigma \) a size functional, and let \( a > 0 \) be such that \( \Sigma^{-1}([a, \infty)) \neq \emptyset \). Then there is a (stability) functional \( f_a = f_{\Phi, \Sigma, a} : [0, \infty) \to [0, 1] \) with \( f_a(0) = 0 \), \( f_a(t) > 0 \) for \( t > 0 \) and such that
\[ \Phi(K) \geq (1 + f_a(\varepsilon)) \tau(\Phi, \Sigma, a) \]
for all \( K \in \Sigma^{-1}([a, \infty)) \) with \( \vartheta(K) \geq \varepsilon \geq 0 \) and with \( \vartheta \) as in (2).

**Proof.** Under the assumptions of the proposition, we consider
\[ \mathcal{K}_s^d(\Phi, \Sigma, a, \varepsilon) := \{ K \in \Sigma^{-1}([a, \infty)) : \vartheta(K) \geq \varepsilon \} \].
Then $\overline{K}_x^d(\Phi, \Sigma, a, \varepsilon) \subset \overline{K}_x^d$ is compact. We can assume that this set is nonempty, since otherwise we simply define $f_a(\varepsilon) := 1$ for any such $\varepsilon > 0$. If $\overline{K}_x^d(\Phi, \Sigma, a, \varepsilon) \neq \emptyset$, we have

$$\tau(\Phi, \Sigma, a, \varepsilon) := \min \{\Phi(K) : K \in \overline{K}_x^d(\Phi, \Sigma, a, \varepsilon)\} \geq \tau(\Phi, \Sigma, a).$$

Assume that $\tau(\Phi, \Sigma, a, \varepsilon) = \tau(\Phi, \Sigma, a)$. Then there is some $K \in \overline{K}_x^d$ with $\Sigma(K) \geq a$, $\vartheta(K) \geq \varepsilon$ and $\Phi(K) = \tau(\Phi, \Sigma, a, \varepsilon) = \tau(\Phi, \Sigma, a)$, hence $K \in E(\Phi, \Sigma, a)$. By definition of $\vartheta$, we get $\vartheta(K) = 0$, a contradiction. Thus we obtain $\tau(\Phi, \Sigma, a, \varepsilon) > \tau(\Phi, \Sigma, a)$ and define

$$g_a(\varepsilon) := \frac{\tau(\Phi, \Sigma, a, \varepsilon)}{\tau(\Phi, \Sigma, a)} - 1 > 0$$

and $f_a(\varepsilon) := \min\{g_a(\varepsilon), 1\}$. Then, for $K \in \Sigma^{-1}([a, \infty))$ with $\vartheta(K) \geq \varepsilon > 0$, we have

$$\Phi(K) \geq \tau(\Phi, \Sigma, a, \varepsilon) = (1 + g_a(\varepsilon))\tau(\Phi, \Sigma, a) \geq (1 + f_a(\varepsilon))\tau(\Phi, \Sigma, a),$$

which proves the assertion. \hfill $\square$

In the following section, we will provide specific versions of stability results with the following choices of functionals. Explicit definitions will be given in Section [4]. In these specific situations, the domain of the functionals has to be adjusted. Instead of using axiomatic properties of functionals and deviation measures we will argue in a more direct way.

**Example 3.2.** Let $\Sigma = \sigma_d$, $\Phi = U_1$, $\vartheta = \Delta_2$ (an $L_2$ distance involving spherical radial functions). Continuity of $\sigma_d$ on $\overline{K}_x^d$ with respect to the Hausdorff metric is easy to see ([60, Theorem 6.5.2] and [19]), continuity of $U_1$ follows from [60] (6.63) and [60] Theorem 6.5.2. A direct argument as in the case of the functional $\tilde{U}$ (see Section 4) can also be given. The definition of the deviation measure $\Delta_2$ requires the restriction to $K_x^d$, the modification $\Delta_2^*$ works on the larger domain $\overline{K}_x^d$.

**Example 3.3.** Let $\Sigma = r$ (inradius), $\Phi = U_1$, $\vartheta = R - r$. Note that (in contrast to the Euclidean case) the inball is uniquely determined (if it is positive). Moreover, $r$ is a continuous functional on $\overline{K}_x^d$. For the sake of completeness, we provide an argument for the second assertion. Let $K, K_1 \in \overline{K}_x^d, i \in \mathbb{N}$, with $K_i \rightarrow K$ as $i \rightarrow \infty$. For the proof, we can assume that $r(K_1) \rightarrow r \geq 0$ and have to show that $r = r(K)$. For $i \in \mathbb{N}$ there is some $x_i \in K_i$ such that $B(x_i, r(K_i)) \subset K_i$. Then there are a subsequence $(x_{i_j})_{j \in \mathbb{N}}$ and some $x_0 \in S^d$ such that $x_{i_j} \rightarrow x_0$ as $j \rightarrow \infty$ and thus $B(x_0, r) \subset K$. This yields that $r(K) \geq r$. Aiming at a contradiction, we assume that $r(K) \in (r, \pi/2]$. Then there are $\bar{x}_0 \in S^d, \bar{r}_0 > r$ and $\varepsilon \in (0, \bar{r}_0)$ such that $\bar{r}_0 + 2\varepsilon \leq r(K) \leq \pi/2$, in particular, we have $B(x_0, r_0 + 2\varepsilon) \subset K$. There is some $i_0 \in \mathbb{N}$ such that $K \subset (K_i)_{e_i}$ for $i \geq i_0$. If $K_i \cap \text{int} B(x_0, \bar{r}_0) = \emptyset$ for some $i \geq i_0$, then a separation argument and $\bar{r}_0 > r$ show that $K \not\subset (K_i)_{e_i}$, a contradiction. Thus $K_i \cap \text{int} B(x_0, \bar{r}_0) \neq \emptyset$ for all $i \geq i_0$. Next we assume that $B(x_0, \bar{r}_0) \not\subset K_i$ for some $i \geq i_0$. Then there is some $x^* \in \partial K_i \cap B(x_0, \bar{r}_0)$. A separation argument and $B(x_0, \bar{r}_0 + 2\varepsilon) \subset K$ then imply that $K \not\subset (K_i)_{e_i}$, a contradiction. This yields that $B(x_0, \bar{r}_0) \subset K_i$ for all $i \geq i_0$. But then $r(K_i) \geq \bar{r}_0 > r$ for all $i \geq i_0$. This is a contradiction to $r(K_i) \rightarrow r$ as $i \rightarrow \infty$, which finally proves that $r(K) = r$.

**Example 3.4.** Let $\Sigma_r = r_\Theta, \Phi = \bar{U}_\Theta = \bar{U}$ (or a multiple thereof), $\vartheta_\Theta = R_\Theta - r_\Theta$, where $\bar{U} \subset S^d$ is a fixed point. Only spherically convex bodies are considered which contain $U$ and are contained in the closed hemisphere centred at $U$. As in the introduction, $r_\Theta, R_\Theta$ denote the centred inradius and circumradius, respectively. The functional $\bar{U}$ will be discussed further in Section [4]. Continuity of $r_\Theta$ follows as for $r$.

4. **Geometric inequalities and stability results**

In this section, we consider stability results which specify the general setting described in the preceding section.

4.1. **Framework of Example 3.2** In the following, we use the notation and some of the results from [17], specifically

$$D(x) := \int_0^x \sin^{d-1}(t) \, dt, \quad x \in [0, \pi/2],$$

and

$$h(y) := \tan^d(D^{-1}(y)), \quad y \in \text{im}(D).$$
For $e \in \mathbb{S}^d$, we put $S_e := \mathbb{S}^d \cap e^\perp$ and $T_e := e + e^\perp$. Further, we define the open halfspace $H^+(e) := \{ x \in \mathbb{R}^{d+1} : \langle x, e \rangle > 0 \}$, whose closure is denoted by $H^+(e)$. Then the mapping $R_e : \mathbb{S}^d \cap H^+(e) \to T_e$ with $R_e(u) := (e, u)^{-1}u$ is the radial projection to the tangent plane of $\mathbb{S}^d$ at $e$.

For $K \in \mathcal{K}_d^*$, the spherical polar $K^* := \{ u \in \mathbb{S}^d : \langle x, e \rangle \leq 0 \}$ for all $x \in K \}$ of $K$ is again in $\mathcal{K}_d^*$ and int$(K^*) \neq \emptyset$. If $e \in \operatorname{int}(K^*)$, then $K \subset H^+(e)$. The map $F_K : \operatorname{int}(K^*) \to (0, \infty)$ with

$$F_K(e) := \int_K \langle e, u \rangle^{-(d+1)} \sigma_d(du), \quad e \in \operatorname{int}(K^*),$$

assigns to $e$ the volume of $R_e(K)$ in $T_e$ (see [17] Section 3). The function $F_K$ on $\operatorname{int}(K^*)$ attains a minimum at some point $e \in \operatorname{int}(K^*)$ and any such point is the centroid of $R_e(K)$ in the hyperplane $T_e$.

For $e \in \operatorname{int}(K^*)$, the positive, continuous function $\alpha_{K,e} : S_e \to (0, \pi/2)$ defined by

$$\partial(R_e(K)) = \{ e + \tan(\alpha_{K,e}(u))u : u \in S_e \}$$

is the spherical radial function of $K$, whereas $\tan \circ \alpha_{K,e}$ is the radial function of $R_e(K)$ with respect to $e$ in $T_e$. If we only know that $K \subset H^+(e)$, then $\alpha_{K,e} : S_e \to [0, \pi/2]$. Using [60] Lemma 6.5.1, we can describe the volume of $K$ in the form

$$\sigma_d(K) = \int_{\mathbb{S}^d} \int_0^{\alpha_{K,e}(u)} \sin^{d-1}(t) dt \sigma_{d-1}(du),$$

and thus

$$\frac{\sigma_d(K)}{\kappa_d} = \int_{\mathbb{S}^d} D(\alpha_{K,e}(u)) \sigma_{d-1}^0(du),$$

where $\sigma_{d-1}^0 := \sigma_{d-1}(S^d)^{-1}\sigma_{d-1}$ on great spheres of $\mathbb{S}^d$. In particular, let $C \subset \mathbb{S}^d$ be a non-degenerate spherical cap contained in an open hemisphere. Then there is a constant $\alpha_C \in (0, \pi/2)$, independent of $e \in \mathbb{S}^d$, such that

$$\frac{\sigma_d(C)}{\kappa_d} = \int_0^{\alpha_C} \sin^{d-1}(t) dt = D(\alpha_C), \quad h \left( \frac{\sigma_d(C)}{\kappa_d} \right) = \tan^{d}(\alpha_C).$$

If $C^* \subset \mathbb{S}^d$ is the polar of $C$, then $\alpha_{C^*} + \alpha_C = \pi/2$.

For $K \in \mathcal{K}_d^*$, we define

$$\Delta_2(K) := \inf \left\{ \left\| D \circ \alpha_{K,e} - D \circ \alpha_{K,e} \right\|_{L^2(S_n, \sigma_{d-1}^0)} : e \in \operatorname{int}(K^*) \right\},$$

where $D \circ \alpha_{K,e}$ is the mean value of $D \circ \alpha_{K,e}$ with respect to $\sigma_{d-1}^0$ over $S_e$. Thus $\Delta_2(K)$ measures the deviation of the shape of $K$ from the shape of a spherical cap in the $L^2$ sense. Clearly, $\Delta_2(K) = 0$ if and only if $K$ is a spherical cap.

**Theorem 4.1.** Let $K \in \mathcal{K}_d^*$ and let $C \subset \mathbb{S}^d$ be a spherical cap with $\sigma_d(K) = \sigma_d(C) > 0$. Let $\alpha_0 \in (0, \pi/2)$ be such that $\alpha_0 \leq \alpha_C$. Then

$$U_1(K) \geq \left( 1 + \beta \Delta_2(K)^2 \right) U_1(C),$$

where

$$\beta := 2 \min \left\{ \frac{\sin^{d+1}(\alpha_0) \tan^{-2d}(\alpha_C)}{\pi^2} D \left( \frac{\pi}{2} - \alpha_C \right) : \frac{2}{\pi} d \left( \frac{d+1}{2} \right)^2 \right\}.$$

**Proof.** Let $K \in \mathcal{K}_d^*$ and let $e \in \operatorname{int}(K^*)$ be arbitrarily chosen. Throughout the proof, we put $\alpha := \alpha_{K,e}$ and restrict the domain of $D$ to $(0, \pi/2)$ so that $\operatorname{im}(D) = D((0, \pi/2))$. Since $U_1$ is rotation invariant, we can assume $C$ to be centred at $e$. We continue to use the notation from [17]. Then $x_0 := D \circ \alpha \in \operatorname{im}(D)$, since $\alpha(u) \in (0, \pi/2)$, for $u \in S_e$, $\sigma_{d-1}^0$ is a probability measure, and $D$ is strictly increasing and continuous. For any $z \in \operatorname{im}(D)$, we have

$$h(z) - h(x_0) = h'(x_0)(z - x_0) + \frac{1}{2} h''(x_0 + \theta(z - x_0))(z - x_0)^2,$$

for some $\theta = \theta(x_0, z) \in (0, 1)$. Since

$$h'(y) = \frac{d}{\cos^{d+1}(D^{-1}(y))} \geq d$$

and

$$h''(y) = \frac{d(d + 1)}{\cos^{d+2}(D^{-1}(y)) \sin^{-2}(D^{-1}(y))} \geq d(d + 1),$$
for \( y \in \text{im}(D) \), the functions \( h \) and \( h' \) are strictly increasing, and we deduce that

\[
(4) \quad h(z) - h(x_0) \geq h'(x_0)(z - x_0) + \frac{d + 1}{2}(z - x_0)^2.
\]

Substituting \( z = D(\alpha(u)), u \in \mathbb{S}_c \), in (4), and then integrating (4) with respect to \( \sigma_{d-1}^0 \) over \( \mathbb{S}_c \), we obtain

\[
\int_{\mathbb{S}_c} h(D(\alpha(u))) \sigma_{d-1}^0(du) - h\left(\frac{\sigma_d(K)}{d\kappa_d}\right) \geq 0 + \frac{d + 1}{2} \Delta_2(K)^2.
\]

Using that

\[
h\left(\frac{\sigma_d(K)}{d\kappa_d}\right) = h\left(\frac{\sigma_d(C)}{d\kappa_d}\right) = \tan^d(\alpha_C),
\]

the radial representation (5), and hence

\[
\lambda^d(R_e(K)) = \frac{1}{d} \int_{\mathbb{S}_c} \tan^d(\alpha(u)) \sigma_{d-1}^0(du) = \frac{1}{d} \int_{\mathbb{S}_c} h(D(\alpha(u))) \sigma_{d-1}^0(du),
\]

we conclude

\[
(5) \quad \frac{\lambda^d(R_e(K))}{\kappa_d} = \int_{\mathbb{S}_c} h(D(\alpha(u))) \sigma_{d-1}^0(du) \geq (1 + \beta_1 \Delta_2(K)^2) h\left(\frac{\sigma_d(K)}{d\kappa_d}\right),
\]

where \( \beta_1 := \frac{(d+1)}{2} \tan^{-d}(\alpha_C) \). Next, we recall some relations from [17]. The equality case of [17] (27) gives

\[
(6) \quad h\left(\frac{\sigma_d(K)}{d\kappa_d}\right) = h\left(\frac{\sigma_d(C)}{d\kappa_d}\right) = \frac{\lambda^d(R_e(C))}{\kappa_d},
\]

and the equality cases of [17] (26) and [17] (30) yield

\[
(7) \quad h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right) = \frac{\kappa_d}{\lambda^d(R_e(C))},
\]

Now we use (26) and (30) from [17] for the first inequality, (5) for the second inequality and the identities (6) and (7) to obtain

\[
h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right) \leq \frac{\kappa_d}{\lambda^d(R_e(K))} \leq \frac{1}{1 + \beta_1 \Delta_2(K)^2} h\left(\frac{\sigma_d(K)}{d\kappa_d}\right)
\]

\[
= \frac{1}{1 + \beta_1 \Delta_2(K)^2} \frac{\kappa_d}{\lambda^d(R_e(C))} = \frac{1}{1 + \beta_1 \Delta_2(K)^2} h\left(\frac{\sigma_d(C)}{d\kappa_d}\right)
\]

\[
= \left(1 - \frac{\beta_1}{1 + \beta_1 \Delta_2(K)^2} \Delta_2(K)^2\right) h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right)
\]

\[
\leq \left(1 - \beta_2 \Delta_2(K)^2\right) h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right),
\]

where

\[
\beta_2 := \frac{\beta_1}{1 + \beta_1 (\pi/2)^2} \leq \frac{\beta_1}{1 + \beta_1 \Delta_2(K)^2},
\]

since \( \Delta_2(K) \leq \pi/2 \). Next, we define

\[
\beta_3 := \min\left\{\beta_2 \tan^{-d}(\alpha_C) \sin^{d+1}(\alpha_0), \left(\frac{2}{\pi}\right)^2 \right\}
\]

\[
\leq \beta_3 \frac{\tan^d(\alpha_C)}{D(\alpha_C)} \frac{\sin^{d+1}(\alpha_C)}{d} = \beta_2 \frac{\sigma_d(C^*)}{d\kappa_d} h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right)^{-1},
\]

since \( \alpha_0 \leq \alpha_C \) and \( \alpha_{C^*} = \pi/2 - \alpha_C \). Note that the minimum in the definition of \( \beta_3 \) is taken in order to ensure that \( 1 - \beta_3 \Delta_2(K)^2 \geq 0 \). Then the mean value theorem and the fact that \( h \) and \( h' \) are increasing imply that

\[
-h\left(\left(1 - \beta_3 \Delta_2(K)^2\right) \frac{\sigma_d(C^*)}{d\kappa_d}\right) + h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right) \leq h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right) \beta_3 \Delta_2(K)^2 \frac{\sigma_d(C^*)}{d\kappa_d}
\]

\[
\leq \beta_2 \Delta_2(K)^2 h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right).
\]
Combining (3) and (9), we get
\[ h \left( \frac{\sigma_d(K^*)}{d\kappa_d} \right) \leq h \left( (1 - \beta_3 \Delta_2(K))^2 \frac{\sigma_d(C^*)}{d\kappa_d} \right), \]
and hence
\[ \frac{\sigma_d(K^*)}{\sigma_d(S^d)} \leq (1 - \beta_3 \Delta_2(K))^2 \frac{\sigma_d(C^*)}{\sigma_d(S^d)}. \]
Since
\[ \frac{1}{2} - U_1(K) = \frac{\sigma_d(K^*)}{\sigma_d(S^d)}, \quad \frac{1}{2} - U_1(C) = \frac{\sigma_d(C^*)}{\sigma_d(S^d)}, \]
we deduce that
\[ U_1(K) \geq \frac{1}{2} \beta_3 \Delta_2(K)^2 + (1 - \beta_3 \Delta_2(K))^2 U_1(C). \]
Finally, we use
\[ \frac{1}{2} - U_1(C) = \frac{\sigma_d(C^*)}{\sigma_d(S^d)} = D(\alpha_{C^*}) \geq 2D(\alpha_{C^*}) U_1(C), \]
and therefore
\[ \frac{1}{2} \geq \left( 1 + 2D \left( \frac{\pi}{2} - \alpha_C \right) \right) U_1(C), \]
to get
\[ U_1(K) \geq \left[ \beta_3 \Delta_2(K)^2 + 2D \left( \frac{\pi}{2} - \alpha_C \right) \beta_3 \Delta_2(K)^2 + 1 - \beta_3 \Delta_2(K)^2 \right] U_1(C), \]
and thus
\[ U_1(K) \geq \left( 1 + 2D \left( \frac{\pi}{2} - \alpha_C \right) \beta_3 \Delta_2(K)^2 \right) U_1(C), \]
which yields the assertion with \( \beta = 2D(\pi/2 - \alpha_C)\beta_3. \)

\textbf{Remark 4.2.} It is easy to see that
\[ \beta \geq \min \left\{ \frac{\sin^{d+1}(\alpha_0)}{\tan^{2d}(\alpha_C) + 2d \tan^{d}(\alpha_C)} \cdot \frac{0.4d}{\pi/2 - \alpha_C} \right\}. \]

\textbf{Remark 4.3.} For \( K \in \mathbb{K}_d^d, \) we define
\[ \Delta_2^*(K) := \inf \left\{ \| D \circ \alpha_{K,e} - D \circ \alpha_{e,e} \|_{L^2(S^d, q^{-1}_d)} : K \in H^*(e) \right\}. \]
Clearly, we have \( \Delta_2^*(K) \leq \Delta_2(K) \) for \( K \in \mathbb{K}_d^d. \) An approximation argument and Theorem 4.1 imply that
\[ U_1(K) \geq (1 + \beta \Delta_2^*(K)^2) U_1(C), \]
for \( K \in \mathbb{K}_d^d, \) where \( C \subset S^d \) is a spherical cap with \( \sigma_d(K) = \sigma_d(C) > 0 \) and \( \alpha_0 \in (0, \pi/2) \) is such that \( \alpha_0 \leq \alpha_C. \)

\textbf{Remark 4.4.} The deviation functional \( \Delta_2 \) can be compared to another natural deviation functional. For this we assume that the assumptions of the theorem are satisfied. If \( e \in -\text{int}(K^*), \) then \( \alpha = \alpha_{K,e} \) is well-defined and
\[ \underline{\alpha}_e(K) := \min \{ \alpha(u) : u \in S_e \}, \quad \bar{\alpha}_e(K) := \max \{ \alpha(u) : u \in S_e \}. \]
Then
\[ \Delta_0(K) := \inf \{ \bar{\alpha}_e(K) - \underline{\alpha}_e(K) : e \in -\text{int}(K^*) \} \]
measures the deviation of the shape of \( K \) from the shape of a spherical cap (centred at a point \( e \in -\text{int}(K^*) \)). For any \( e \in -\text{int}(K^*) \) we have
\[ \alpha(u) \in [\underline{\alpha}_e(K), \bar{\alpha}_e(K)], \quad u \in S_e, \]
and therefore
\[ \left| D(\alpha(u)) - \frac{\sigma_d(K^*)}{d\kappa_d} \right| \leq D(\bar{\alpha}_e(K)) - D(\underline{\alpha}_e(K)) \leq D'(\bar{\alpha}_e(K))(\bar{\alpha}_e(K) - \underline{\alpha}_e(K)) = \sin^{d-1}(\bar{\alpha}_e(K))(\bar{\alpha}_e(K) - \underline{\alpha}_e(K)) \leq \bar{\alpha}_e(K) - \underline{\alpha}_e(K). \]
Thus, we have
\[ \Delta_2(K) \leq \Delta_0(K). \]
Later, this relation will be used in the proof of Lemma 6.2. With the obvious definition of $\Delta_n^s$, we obtain the inequality $\Delta_2^s(K) \leq \Delta_0^s(K)$.

### 4.2. Framework of Example 4.3

For $K \in \mathbb{K}_e^d$ and $e \in K$, let $r_e(K)$ denote the (spherical) inradius and $R_e(K)$ the circumradius of $K$, with respect to $e$ as the centre of the insphere and the circumsphere, respectively. We consider the size functional $\Sigma_r(K) := \max \{ r_e(K) : e \in K \}$. If $e \in K$ is such that $r_e(K) = \Sigma_r(K)$, then the following lemma shows that $R_e(K) \leq \pi/2$.

**Lemma 4.5.** Let $K \in \mathbb{K}_e^d$, and let $e \in K$ be such that $r_e(K) = \Sigma_r(K)$. Then $K \subset H^*(e)$.

**Proof.** By assumption and a separation argument (applied to convex cones) it follows that $e \in \text{conv}(\partial K \cap B(e, r_e(K)))$. Hence there exist $k \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_k \in [0, 1]$ with $\sum_{i=1}^k \lambda_i = 1$, $\kappa > 1$, and $a_1, \ldots, a_k \in \text{conv}(\partial K \cap B(e, r_e(K))) \subset \mathbb{S}^d$ such that

\[
\sum_{i=1}^k \lambda_i (\kappa a_i - e) = o.
\]

Furthermore, there is some $\varrho > 1$ such that

\[
\langle a_i - \varrho e, x \rangle \leq 0, \quad x \in K, \quad i = 1, \ldots, k.
\]

From (12), we get

\[
\sum_{i=1}^k \lambda_i \kappa a_i \langle e - \varrho e, x \rangle \leq 0, \quad x \in K.
\]

Hence, (11) implies that $\langle e - \kappa \varrho e, x \rangle \leq 0$, that is, $\langle -e, x \rangle \leq 0$ for all $x \in K$. \hfill $\Box$

Put $\partial_r(K) := \min \{ R_e(K) - r_e(K) : e \in K, r_e(K) = \Sigma_r(K) \}$. Assume that $\Sigma_r(K) \geq a$ and $\partial_r(K) \geq \varepsilon \in (0, 1]$.

Let $e \in K$ be such that $r_e(K) = \Sigma_r(K)$ so that $a + \varepsilon \leq r_e(K) + \varepsilon \leq R_e(K) \leq \pi/2$. Then there is some $z_0 \in K$ such that $\text{conv}(B(e, a) \cup \{ z_0 \}) \subset K$ and $d_e(e, z_0) = a + \varepsilon > 0$. Hence, writing

\[
A := \{ x \in \mathbb{S}^d : x^\perp \cap B(e, a) = \emptyset, x^\perp \cap \text{conv}(B(e, a) \cup \{ z_0 \}) \neq \emptyset \},
\]

we have

\[
U_1(K) \geq U_1(\text{conv}(B(e, a) \cup \{ z_0 \})) = U_1(B(e, a)) + \frac{1}{2\omega_{d+1}} \int_{\mathbb{S}^d} 1\{ x \in A \} \sigma_d(dx).
\]

For fixed $e \in \mathbb{S}^d$, $u \in S_e$, $a > 0$, and $\varepsilon \in (0, 1]$, we define

\[
\delta(e, u, a, \varepsilon) := H^1([\text{conv}(B(e, a) \cup \{ x \}) \setminus B(e, a)] \cap \{ \lambda e + \mu u : \lambda \in \mathbb{R}, \mu \geq 0 \})
\]

and

\[
M(e, a, \varepsilon) := \{ u \in S_e : \delta(e, u, a, \varepsilon) \geq \varepsilon/2 \}.
\]

By symmetry, $M(e, a, \varepsilon)$ is a spherical cap $C(e, a, \varepsilon) \subset S_e$ with centre $(x - \langle x, e \rangle e)/\sqrt{1 - \langle x, e \rangle^2}$. We will show that the spherical radius of this cap is bounded from below by $\pi/2 - a \sqrt{\varepsilon}$. Once this is shown, we obtain

\[
\int_{\mathbb{S}^d} 1\{ x \in A \} \sigma_d(dx) \geq \int_{C(e,a,\varepsilon)} \int_{\frac{\varepsilon}{2} + a + \delta(e, u, a, \varepsilon)}^{\frac{\varepsilon}{2} + a} \sin^{d-1}(t) \, dt \, d\sigma_{d-1}(du)
\]

\[
\geq \int_{C(e,a,\varepsilon)} \int_{\frac{\varepsilon}{2} + a + \frac{\pi}{2}}^{\frac{\varepsilon}{2} + a} \sin^{d-1}(t) \, dt \, d\sigma_{d-1}(du)
\]

\[
= \int_{C(e,a,\varepsilon)} \int_{a + \frac{\pi}{2}}^{a + \frac{\pi}{2}} \cos^{d-1}(t) \, dt \, d\sigma_{d-1}(du).
\]

Using Hölder’s inequality, basic trigonometric identities and $\varepsilon \leq \pi/2 - a$, we get

\[
\int_{a}^{a + \frac{\pi}{2}} \cos^{d-1}(t) \, dt \geq \frac{\varepsilon}{2} \left( \frac{2}{\varepsilon} \int_{a}^{a + \frac{\pi}{2}} \cos t \, dt \right)^{d-1} = \frac{\varepsilon}{2} \left( \frac{2}{\varepsilon} \cos \left( a + \frac{\varepsilon}{2} \right) \sin \left( \frac{\varepsilon}{4} \right) \right)^{d-1}
\]

\[
\geq \frac{\varepsilon}{2} \left( \frac{2}{\varepsilon} \right)^{d-1} \sin^{d-1} \left( \frac{3}{4} \left( \frac{\pi}{2} - a \right) \right).
\]
we finally get
\[
\int_{\mathbb{R}^d} 1\{x \in A\} \sigma_d(dx) \geq \frac{1}{2} \omega_d \pi^{2(1-d)} a^{d-1} \sqrt{\varepsilon} d^{-1} \left(\frac{\pi}{2} - a\right)^{d-1}
\]
\[
= \frac{1}{4} 3^{d-1} \pi^{4(1-d)} \omega_d \varepsilon^{d+1} a^{d-1} \left(\frac{\pi}{2} - a\right)^{d-1}.
\]
Since
\[
U_1(B(e, a)) = 2 \frac{\omega_d}{\omega_{d+1}} \int_0^a \cos d^{-1}(t) dt \leq \frac{\omega_d}{\omega_{d+1}} a,
\]
we finally get
\[
U_1(K) \geq \left[ 1 + c(d) a^{d-2} \left(\frac{\pi}{2} - a\right)^{d-1} \varepsilon^{d+1} \right] U_1(B(e, a)),
\]
where \(c(d) \geq (1/8)(3\pi^{-4})^{d-1} \geq 4 \cdot 0.03^d\).

To complete the argument, we have to verify the asserted lower bound for the cap \(C(e, a, \varepsilon)\). By symmetry, it is sufficient to consider the case \(d = 3\). Then the boundary of \(\text{conv}(B(e, a) \cup \{z_0\})\) is the union of two geodesic segments, denoted by \([\bar{a}, \bar{b}]\) and \([\bar{b}, \bar{c}]\), and the arc of \(\partial B(e, a)\) connecting \(p\) and \(\bar{p}\) which does not meet the geodesic segment \([e, z_0]\). Let \(y \in [e, z_0]\) be such that \(d_y(y, e) = a + \varepsilon/2\). Further, let \(q\) and \(\varrho\) be the intersections of the geodesics through \(y\) orthogonal to \([e, z_0]\) with \([p, z_0]\) and \([\bar{p}, z_0]\). Finally, let \(\alpha := \angle(e, z_0, p) = \angle(y, z_0, q)\) and \(\omega := \angle(q, e, y)\).

By basic formulas of spherical trigonometry, we have
\[
\sin(\alpha) = \sin(a) \sin(a + \varepsilon), \quad \sin\left(\frac{\varepsilon}{2}\right) = \tan(d_y(y, q)) \cot(\alpha), \quad \sin\left(a + \frac{\varepsilon}{2}\right) = \tan(d_y(y, q)) \cot(\omega).
\]
Combining these relations, we deduce
\[
\tan(\omega) = \sin\left(\frac{\varepsilon}{2}\right) \sin(a) \sin\left(a + \frac{\varepsilon}{2}\right) \sqrt{\sin(a + \varepsilon) + \sin(a)} \sqrt{\sin(a + \varepsilon) - \sin(a)}
\]
\[
\geq \sqrt{2\pi^{-2}} a \sqrt{\varepsilon},
\]
and hence \(\omega \geq \pi^{-2} a \sqrt{\varepsilon}\) (here we distinguish the cases \(\omega \leq \pi/4\) and \(\omega > \pi/4\)).

Thus we have proved the following theorem.

**Theorem 4.6.** Let \(K \in \mathcal{K}_s^d\), \(a > 0\), and \(\varepsilon \in (0, 1]\). Assume that \(\Sigma_{\varepsilon}(K) \geq a\) and \(\vartheta_{\varepsilon}(K) \geq \varepsilon\). Then
\[
U_1(K) \geq \left[ 1 + c_\varepsilon(a, d) \varepsilon^{d+1} \right] U_1(B(e, a)),
\]
where \(c_\varepsilon(a, d) \geq (1/8)(3\pi^{-4})^{d-1} a^{-d+2} \left(\frac{\pi}{2} - a\right)^{d-1} \geq 4 \cdot 0.03^d a^{-d+2} \left(\frac{\pi}{2} - a\right)^{d-1}\).

4.3. **Framework of Example 3.4** Fix \(e \in \mathbb{S}^d\). (For notational reasons, we prefer to write \(e\) instead of \(\bar{e}\) for an arbitrary fixed point in \(\mathbb{S}^d\), in this section.) For \(K \in \mathcal{K}_s^d\) with \(e \in K \subset H^*(e)\), we define
\[
\tilde{U}(K) := \tilde{U}_e(K) := \int_{\mathbb{R}^d} 1\{(x - e) \perp K \neq \emptyset\} \sigma_d(dx).
\]
In analogy to the measure \(\mu\) on \(G(d + 1, d) \cap \mathbb{S}^d\), we also introduce
\[
\tilde{\mu}(\cdot) := \tilde{\mu}_e(\cdot) := \frac{1}{\omega_{d+1}} \int_{\mathbb{S}^d} 1\{(x - e) \perp \mathbb{S}^d \in \cdot\} \sigma_d(dx),
\]
so that \(\tilde{U}_e(K) = \omega_{d+1} \tilde{\mu}_e(H_K)\) for \(K \in \mathcal{K}_s^d\).

For \(a \in (0, \pi/2)\), we have \((x - e) \perp \mathbb{S}^d \in \cdot\) if and only if \(x \in B(e, 2a)\), and hence it follows that \(\tilde{U}(B(e, a)) = \sigma_d(B(e, 2a))\). Hölder’s inequality implies that
\[
\sigma_d(B(e, 2a)) = \omega_d \int_0^{2a} \sin d^{-1}(t) dt \geq \omega_d 2a \left(\frac{1 - \cos(2a)}{2a}\right)^{d-1} \geq 2 \omega_d \left(\frac{2}{\pi}\right)^{2(d-1)} a^d.
\]
For an upper bound, we distinguish the cases \( a \leq \pi/4 \) and \( a \in (\pi/4, \pi/2) \). Using \( \sin(t) \leq t, t \in [0, \pi/2] \), in the first case and the trivial bound \( \sigma_d(B(e, 2a)) \leq \omega_{d+1} \) in the second case, we obtain \( \sigma_d(B(e, 2a)) \leq 2^d \omega_{d+1} a^d \).

Thus we have

\[
2 \omega_d \left( \frac{2}{\pi} \right)^{2(d-1)} a^d \leq \tilde{U}(B(e, a)) = \sigma_d(B(e, 2a)) \leq 2^d \omega_{d+1} a^d.
\]

**Lemma 4.7.** The functional \( \tilde{U} \) is continuous on \( \overline{C} \), \( \tilde{\mu} \leq 2^d \mu \), and \( \tilde{U} \leq 2^{d+1} \omega_{d+1} U_1 \).

**Proof.** We first prove the second assertion in a slightly stronger form. A point \( x \in S^d \) can be parameterized in the form \( x = \cos(\varphi) e + \sin(\varphi) u \) with \( \varphi \in [0, \pi] \) and \( u \in S_x \). Then we obtain

\[
\int_{S^d} 1\{(x - e) \perp \cap S^d \in \cdot \} \sigma_d(dx)
\]

\[
= \int_{S^d} \int_0^\pi 1\{\cos \left( \frac{\varphi}{2} \right) e + \sin \left( \frac{\varphi}{2} \right) u, e^\perp \cap S^d \in \cdot \} \sin^{d-1}(\varphi) d\varphi \sigma_{d-1}(du)
\]

\[
= 2 \int_{S^d} \int_0^{\pi/2} 1\{\cos(s) e + \sin(s) u, e^\perp \cap S^d \in \cdot \} \sin^{d-1}(2s) ds \sigma_{d-1}(du)
\]

\[
= 2 \int_{S^x} \int_0^{\pi} 1\{\sin(t) e - \cos(t) u, e^\perp \cap S^d \in \cdot \} \sin^{d-1}(2t - \pi) dt \sigma_{d-1}(du).
\]

Since \( \sin(2t - \pi) \leq 2 \sin(t) \) for \( t \in [\pi/2, \pi] \), we obtain

\[
\int_{S^d} 1\{(x - e) \perp \cap S^d \in \cdot \} \sigma_d(dx)
\]

\[
\leq 2^d \int_{S^d} \int_0^{\pi/2} 1\{\sin(t) e - \cos(t) u, e^\perp \cap S^d \in \cdot \} \sin^{d-1}(t) dt \sigma_{d-1}(du)
\]

\[
= 2^d \int_{S^d} \int_0^{\pi/2} 1\{(\cos(t) e + \sin(t) u)^\perp \cap S^d \in \cdot \} \sin^{d-1}(t) dt \sigma_{d-1}(du)
\]

\[
\leq 2^d \int_{S^d} 1\{x^\perp \cap S^d \in \cdot \} \sigma_d(dx),
\]

which yields the second and the third assertion.

Now we prove the continuity assertion. For \( K \in \overline{C}_x \), we put \( A_K := \{z \in S^d : (z - e) \perp \cap K \neq \emptyset\} \).

Let \( K, K_i \in \overline{C}_x \), \( i \in \mathbb{N} \), with \( K_i \to K \) as \( i \to \infty \). For \( x \in S^d \), we distinguish the following cases.

If \( (x - e) \perp \cap \text{relint}(K) \neq \emptyset \) and \( K \not\subset (x - e) \perp \), then \( (x - e) \perp \cap K_i \neq \emptyset \) if \( i \) is sufficiently large, and hence \( \mathbf{1}_{A(K_i)}(x) \to \mathbf{1}_{A(K)}(x) \) as \( i \to \infty \).

If \( (x - e) \perp \cap K = \emptyset \), then \( (x - e) \perp \cap K_i = \emptyset \) if \( i \) is sufficiently large, and hence \( \mathbf{1}_{A(K_i)}(x) \to \mathbf{1}_{A(K)}(x) \) as \( i \to \infty \).

Finally, note that \( \{z \in S^d : K \subset (z - e) \perp\} \) has \( \sigma_d \)-measure zero and by (14) we have

\[
\sigma_d \left( \{z \in S^d : (z - e) \perp \cap \text{relint}(K) = \emptyset, (z - e) \perp \cap K \neq \emptyset\} \right)
\]

\[
\leq 2^d \sigma_d \left( \{x \in S^d : x^\perp \cap \text{relint}(K) = \emptyset, x^\perp \cap K \neq \emptyset\} \right)
\]

\[
\leq 2^d \omega_{d+1} \left( \{\rho \in SO_{d+1} : K \text{ and } \rho e_0^\perp \text{ touch each other}\} \right) = 0,
\]

where \( e_0 \in S^d \) is arbitrary and fixed and we used [19 Korollar 5.2.1] for the final equality.

Thus we have \( \mathbf{1}_{A(K_i)}(x) \to \mathbf{1}_{A(K)}(x) \) as \( i \to \infty \) for \( \sigma_d \text{-almost all } x \in S^d \), so that the assertion follows from the dominated convergence theorem.

For the given \( e \in S^d \) and for \( K \in \overline{C}_x \) with \( e \in K \subset H^+(e) \), we consider the size functional \( \Sigma_e(K) := r_e(K) \) and the deviation functional \( \tilde{\vartheta}_e(K) := R_e(K) - r_e(K) \). If \( \Sigma_e(K) \geq a \in (0, \pi/2) \) and \( \tilde{\vartheta}_e(K) \geq e \in (0, 1] \), then there is some \( z_0 \in K \subset B(e, \pi/2) \) with \( d_e(z_0) = a + e \leq \pi/2 \) and \( \text{conv}(B(e, a) \cup \{z_0\}) \subset K \). We put

\[
A_e := \{x \in S^d : (x - e) \perp \cap B(e, a) = \emptyset, (x - e) \perp \cap \text{conv}(B(e, a) \cup \{z_0\}) \neq \emptyset\}.
\]

Arguing as before, we get

\[
\tilde{U}(K) \geq \tilde{U}(B(e, a)) + \int_{S^d} \mathbf{1}_{A_e} \sigma_d(dx).
\]
Then we obtain
\[
\int_{S^d} \mathbf{1}\{x \in A_e\} \sigma_d(dx) \geq \int_{S^d} \int_0^\pi \mathbf{1}\{\cos \left(\frac{\varphi}{2}\right) e + \sin \left(\frac{\varphi}{2}\right) u \in \text{conv}(B(e, a) \cup \{z_0\}) \setminus B(e, a)\} \\
\times \sin^{d-1}(\varphi) \, d\varphi \, \sigma_{d-1}(du)
\]
\[
\geq \int_{C(e,a,\varepsilon)} \int_0^\pi \mathbf{1}\{\varphi/2 \in (a,a+\varepsilon/2)\} \sin^{d-1}(\varphi) \, d\varphi \, \sigma_{d-1}(du).
\]
For \(\varphi \in (2a, 2a + \varepsilon)\), we have
\[
\sin \varphi \geq \min\{\sin(2a), \sin(2a + \varepsilon)\} \geq \sin(a + \pi/2) \geq \frac{1}{2} \min\{a, \pi/2 - a\},
\]
and hence
\[
\int_{S^d} \mathbf{1}\{x \in A_e\} \sigma_d(dx) \geq 2^{1-d}(\min\{a, \pi/2 - a\})^{d-1}\sigma_{d-1}(C(e,a,\varepsilon)) \varepsilon \\
\geq 2^d\omega_{d+1} a^{d-1} \pi^{2(1-d)} \frac{\omega_d}{\omega_{d+1}} \min\{a, \pi/2 - a\}^{d-1} \varepsilon^{d+1} \\
\geq \tilde{U}(B(e, a))(2\pi)^{-2d} a^{-1} \min\{a, \pi/2 - a\}^{d-1} \varepsilon^{d+1},
\]
where we used (13) and the rough lower bound \(\omega_d/\omega_{d+1} \geq 1/\pi^2\).

Thus we have proved the following theorem.

**Theorem 4.8.** Let \(e \in S^d\). Let \(K \in \mathcal{K}_s^d\) with \(e \in K \subset H^*(e)\) and \(a \in (0, \pi/2)\). Then
\[
\tilde{U}(K) \geq \tilde{U}(B(e, a)) = \sigma_d(B(e, 2a)).
\]
If also \(\vartheta_e(K) \geq \varepsilon \in (0, 1)\), then
\[
\tilde{U}(K) \geq \left[1 + c(a, d) \varepsilon^{d+1}\right] \tilde{U}(B(e, a)),
\]
where \(c(a, d) \geq (2\pi)^{-2d} a^{-1} \min\{a, \pi/2 - a\}^{d-1}\).

5. APPROXIMATION RESULTS

We provide an approximation result for spherical polytopes and then derive a consequence for hitting functionals. Recall that \(\delta_s\) denotes the spherical Hausdorff distance.

**Lemma 5.1.** Let \(K \in \mathcal{K}_s^d\). Then there are constants \(k_1\) and \(b_1\), depending only on \(d\), such that for all \(k \geq k_1\) there is a spherical polytope \(Q\) with \(k\) vertices (which can be chosen on the boundary of \(K\) if \(K \in \mathcal{K}_s^d\)) satisfying
\[
\delta_s(K, Q) \leq b_1 k^{-2/(d-1)}.
\]

**Proof.** Let \(\varepsilon \in (0, \pi/2)\) and assume \(R(K) \leq \pi/2 - \varepsilon\). Below \(\varepsilon\) will be chosen as a function of \(d\). Let \(z(K) \in S^d\) denote the centre of the spherical circumball of \(K\) and \(B^s(z(K))\) the open half sphere with centre \(z(K)\). We consider the radial projection mapping
\[
P_K : B^s(z(K)) \to z(K) + z(K)^\perp, \quad x \mapsto \langle x, z(K) \rangle^{-1} x.
\]
The Euclidean circumradius of the image of \(K\) under \(P_K\) then satisfies the inequality
\[
R := R(P_K(K)) \leq \tan(\varepsilon)^{-1}.
\]
Applying the main result from [10], we get constants \(k_0 = k_0(d)\) and \(b_0 = b_0(d)\) such that the following is true. For \(k \geq k_0,\) there is a polytope \(Q_0 \subset z(K) + z(K)^\perp\) with \(k\) vertices, located on the boundary of \(P_K(K)\), satisfying
\[
\delta(R^{-1}P_K(K), R^{-1}Q_0) \leq b_0 k^{-2/(d-1)}.
\]
Here \(\delta\) denotes the Hausdorff-distance in \(\mathbb{R}^{d+1}\). The polytopes \(R^{-1}P_K(K)\) and \(R^{-1}Q_0\) lie in a common affine subspace parallel to \(z(K)^\perp\). Therefore \(P_K(K) \subset Q_0 + Rb_0 k^{-2/(d-1)} B_E\), where \(B_E\) is the unit ball in \(z(K)^\perp\), and thus
\[
\delta(P_K(K), Q_0) \leq \tan(\varepsilon)^{-1} b_0 k^{-2/(d-1)}.
\]
The mapping $\Pi_{S^d}: z(K) + z(K)^\perp \to S^d$, $x \mapsto \|x\|^{-1}x$, is Lipschitz continuous with Lipschitz constant at most 2. Since also \((\pi/2)\|x-y\| \geq 2\arcsin((\|x-y\|)/2) = d_s(x, y)\), for $x, y \in S^d$, the spherical polytope $Q := \Pi_{S^d}(Q_0)$ satisfies $Q \subset \Pi_{S^d}(F_K(K)) = K$ and
\[
\delta_s(Q, K) \leq \tan(\varepsilon)^{-1} b_0 \pi k^{2/(d-1)}, \quad k \geq k_0.
\]
For arbitrary $K \in \mathbb{K}_d$, we divide $K$ into $2^d$ pieces by intersecting with $d$ hyperplanes, where each hyperplane is the linear span of $0$ and $d - 1$ of the remaining $d$ standard basis vectors of $\mathbb{R}^{d+1}$ (or equivalently the orthogonal complement of one of those $d$ remaining standard basis vectors). Then each piece of $K$ is contained in a regular spherical $d$-simplex of edge-length $\pi/2$, which is the spherical convex hull of $d + 1$ unit vectors. Its circumradius is $\arccos(1/\sqrt{d + 1}) \in (\pi/4, \pi/2)$, see [14, Theorem 2]. Defining
\[
\varepsilon := \pi/2 - \arccos \left(\sqrt{d + 1}^{-1}\right),
\]
the individual pieces satisfy $R_s(K_i) \leq \pi - \varepsilon, i = 1, \ldots, 2^d$. Applying the reasoning above to every piece, we obtain spherical polytopes $Q_i, i = 1, \ldots, 2^d$, such that
\[
\delta_s(Q_i, K_i) \leq \tan(\varepsilon)^{-1} b_0 \pi k^{2/(d-1)}, \quad i = 1, \ldots, 2^d, \quad k \geq k_0.
\]
Defining $Q := \text{conv}_s(\bigcup_{i=1}^{2^d} Q_i)$, we obtain a polytope with (at most) $2^d k$ vertices and
\[
\delta_s(Q, K) \leq \tan(\varepsilon)^{-1} b_0 \pi k^{2/(d-1)}.
\]
Since $\tan(\varepsilon) = \tan \left(\pi/2 - \arccos(1/\sqrt{d + 1})\right) = \sqrt{d + 1}^{-1}$, the assertion follows with $k_1 = 2^d k_0$ and $b_1 = b_0 \pi \sqrt{d + 1}^{(d-1)/2}$.

The following lemma states that the values of $\Phi$ on spherical polytopes can be approximated by the values of $\Phi$ on polytopes with a controlled number of vertices (extreme points). We write $\text{ext}(P)$ for the set of extreme points of a spherical polytope $P$ and $f_0(P)$ for the number of its extreme points.

**Lemma 5.2.** Let $\Sigma$ be a size functional, $\Phi$ a hitting functional, and let $a, \alpha > 0$. Then there is an integer $\nu \in \mathbb{N}$, depending only on $\Phi, \Sigma, d, a, \alpha$, such that for every spherical polytope $P \in \mathbb{K}_d$ with $\Sigma(P) \geq a$ there is a spherical polytope $Q = Q(P)$ satisfying $\text{ext}(Q) \subset \text{ext}(P)$, $f_0(Q) \leq \nu$ and
\[
\Phi(Q) \geq (1 - \alpha)\Phi(P).
\]
Furthermore, the mapping $P \mapsto Q(P)$ can be chosen to be measurable.

**Proof.** The functional $\Phi$ is continuous on the compact space $\mathbb{K}_d$ (with respect to the spherical Hausdorff metric $\delta_s$). Thus $\Phi$ is uniformly continuous. Let $\Phi, \Sigma, \alpha, a$ be as in the statement of the lemma and define $\varepsilon := \alpha \tau(\Phi, \Sigma, a)$. From the uniform continuity of $\Phi$ it follows that there is some $\delta = \delta(\varepsilon) > 0$ such that $|\Phi(K) - \Phi(K')| \leq \varepsilon = \alpha \tau(\Phi, \Sigma, a)$ for all $K, K' \in \mathbb{K}_d$, with $\delta_s(K, K') \leq \delta(\varepsilon)$. Let $P$ be a spherical polytope with $\Sigma(P) \geq a$. From Lemma 5.1 we now obtain a spherical polytope $Q = Q(P)$ and a number $\nu = \nu(\Phi, \Sigma, d, a, \alpha)$ such that $\text{ext}(Q) \subset \text{ext}(P)$, $f_0(Q) \leq \nu$ and $\delta_s(Q, P) \leq \delta(\varepsilon)$. Since $\Sigma(P) \geq a$, we conclude that $\Phi(P) \geq \tau(\Phi, \Sigma, a)$, and therefore
\[
\Phi(P) - \Phi(Q) \leq |\Phi(P) - \Phi(Q)| \leq \varepsilon = \alpha \tau(\Phi, \Sigma, a) \leq \alpha \Phi(P),
\]
which yields the first assertion.

After identifying each spherical polytope with a Euclidean polytope which is the convex hull of the Euclidean origin and the vertices of the spherical polytope, the second assertion follows as in [22, Lemma 4.2].

## 6. Probabilistic Inequalities

After the geometric preparations of the preceding sections, we can proceed with estimating the conditional probabilities involved in resolving Kendall’s problem in spherical space. In each case, the conditional probability is the ratio of two probabilities. The probability in the denominator is easy to treat. In the following lemma, we provide an upper bound for the numerator, which is the main step in the probabilistic estimate.

In the following, we first consider the hitting functional $\Phi = 2U_1$ of an isotropic spherical Poisson hyperplane tessellation with Crofton cell $Z_0$, a general increasing size functional $\Sigma$, and an associated canonical deviation functional $\vartheta$. Then we write $\tau(a) = \tau(\Phi, \Sigma, a)$ for the isoperimetric constant based on the parameter $a > 0$ and $f_a$ for the stability function as in Proposition 3.1.
Lemma 6.1. Let \( a > 0, \varepsilon \in (0, 1) \) and \( \mathcal{K}_{a, \varepsilon} := \{ K \in \mathcal{K}_d^\alpha : \Sigma(K) \geq a, \vartheta(K) \geq \varepsilon \} \). Then

\[
\mathbb{P}(Z_0 \in \mathcal{K}_{a, \varepsilon}) \leq c_1 \max \{ \gamma_j a \nu d+1 \} d \nu \exp \left( -\gamma_j a \nu d+1 \left( 1 + \frac{1}{3} f_a(\varepsilon) \right) \tau(\alpha) \right),
\]

where the constants \( c_1 \) and \( \nu \) depend on \( a, d, \varepsilon \) and \( \Sigma, \vartheta \).

Proof. For the proof, we can assume that \( \Sigma^{-1}([a, \infty)) \neq \emptyset \). Let \( N \in \mathbb{N} \). For \( H_1, \ldots, H_N \in \mathcal{H}_d^\alpha \) such that \( \emptyset \notin H_i \), for \( i = 1, \ldots, N \), we define \( H_{(N)} := (H_1, \ldots, H_N) \) and let \( P(H_{(N)}) \) denote the spherical Crofton cell of the tessellation induced by \( H_1, \ldots, H_N \). In what follows, we consider \( H_1, \ldots, H_N \in \mathcal{H}_d^\alpha \) such that \( P(H_{(N)}) \in \mathcal{K}_{a, \varepsilon} \cap \mathcal{K}_d^\alpha \). This requires \( N \geq d+1 \). If \( N \geq d+1 \) and \( H_1, \ldots, H_N \) are i.i.d. with a distribution which has a density with respect to the invariant measure, then \( P(H_{(N)}) \in \mathcal{K}_{a, \varepsilon} \cap \mathcal{K}_d^\alpha \) is satisfied almost surely.

Define \( \alpha := f_a(\varepsilon)/(2 + f_a(\varepsilon)) \), hence \((1 - \alpha)(1 + f_a(\varepsilon)) = 1 + \alpha \). Since \( f_a(\varepsilon) \leq 1 \), we have \( \alpha \geq f_a(\varepsilon)/3 \).

By Lemma 5.2 and Proposition 5.1 there are at most \( \nu = \nu(\Phi, \Sigma, d, a, \varepsilon) \) vertices of \( P(H_{(N)}) \in \mathcal{K}_{a, \varepsilon} \cap \mathcal{K}_d^\alpha \) such that the spherical convex hull \( Q(H_{(N)}) \) of these vertices satisfies

\[
1 \geq \Phi(Q(H_{(N)})) \geq (1 - \alpha) \Phi(P(H_{(N)})) \geq (1 - \alpha)(1 + f_a(\varepsilon)) \tau(\alpha) = (1 + \alpha) \tau(\alpha),
\]

where we used \( \Phi(\cdot) = 2U_1(\cdot) \leq 1 \). By Lemma 5.2, we can assume that the map \( (H_1, \ldots, H_N) \mapsto Q(H_{(N)}) \) is measurable. Since \( \mu \) is isotropic, every vertex of \( Q(H_{(N)}) \) lies \( \mu_N \)-almost surely in exactly \( d \) of these great sub-spheres. The remaining sub-spheres do not hit \( Q(H_{(N)}) \). Hence, the number of great sub-spheres hitting \( Q(H_{(N)}) \) is \( j \in \{d+1, \ldots, dN\} \). Without loss of generality we assume \( H_1 \cap Q(H_{(N)}) \neq \emptyset, \ldots, H_j \cap Q(H_{(N)}) \neq \emptyset \). Then there are subsets \( J_1, \ldots, J_{f_0(Q(H_{(N)}))} \) of \( \{1, \ldots, j\} \), each of cardinality \( d \), such that

\[
\bigcap_{i \in J_1} H_i, \quad i = 1, \ldots, f_0(Q(H_{(N)})) \leq \nu,
\]

give the vertices of \( Q(H_{(N)}) \). In the following, let \( \sum_{(J_1, \ldots, J_{f_0(Q(H_{(N)}))})} \) denote the sum over all \( \nu \)-tuples of subsets of \( \{1, \ldots, j\} \) with \( d \) elements.

\[
\int_{\mathcal{H}_d^\alpha} 1\{ H \cap K = \emptyset \} \mu(dH) = 1 - \Phi(K).
\]

Assuming \( N \geq d+1 \) and using that \( \Phi(S^d) = 1 \), we obtain

\[
\mathbb{P}(Z_0 \in \mathcal{K}_{a, \varepsilon} \mid \mathcal{X}(\mathcal{H}_d^\alpha) = N) = \int_{\mathcal{H}_d^\alpha} 1\{ P(H_{(N)}) \in \mathcal{K}_{a, \varepsilon} \cap \mathcal{K}_d^\alpha \} \mu^N(d(H_1, \ldots, H_N))
\]

\[
\leq \sum_{j=d+1}^{dN} \binom{N}{j} \int_{\mathcal{H}_d^\alpha} 1\{ P(H_{(N)}) \in \mathcal{K}_{a, \varepsilon} \cap \mathcal{K}_d^\alpha \} 1\{ H_i \cap Q(H_{(N)}) \neq \emptyset, \ i = 1, \ldots, j \}
\]

\[
\times 1\{ H_i \cap Q(H_{(N)}) = \emptyset, \ i = j+1, \ldots, N \} \mu^N(d(H_1, \ldots, H_N))
\]

\[
\leq \sum_{j=d+1}^{dN} \binom{N}{j} \sum_{(J_1, \ldots, J_{f_0(Q(H_{(N)}))})} \int_{\mathcal{H}_d^\alpha} \int_{\mathcal{H}_d^{N-j}} 1\{ \Phi(\text{conv}_s \bigcup_{r=1}^{\nu} \bigcap_{K_r \in J_r} H_r) \geq (1 + \alpha) \tau(\alpha) \}
\]

\[
\times 1\{ H_i \cap \text{conv}_s \bigcup_{r=1}^{\nu} \bigcap_{K_r \in J_r} H_r = \emptyset, \ l = j+1, \ldots, N \}
\]

\[
\times \mu^{N-j}(d(H_{j+1}, \ldots, H_N)) \mu^j(d(H_1, \ldots, H_j))
\]

\[
= \sum_{j=d+1}^{dN} \binom{N}{j} \sum_{(J_1, \ldots, J_{f_0(Q(H_{(N)}))})} \int_{\mathcal{H}_d^\alpha} 1\{ \Phi(\text{conv}_s \bigcup_{r=1}^{\nu} \bigcap_{K_r \in J_r} H_r) \geq (1 + \alpha) \tau(\alpha) \}
\]

\[
\times [1 - \Phi(\text{conv}_s \bigcup_{r=1}^{\nu} \bigcap_{K_r \in J_r})]^{N-j} \mu^j(d(H_1, \ldots, j))
\]

\[
\leq \sum_{j=d+1}^{dN} \binom{N}{j} \frac{1}{d} [1 - (1 + \alpha) \tau(\alpha)]^{N-j}
\]

(16)
Summation over \(N\) gives

\[
\mathbb{P}(Z_0 \in \mathcal{K}_{a,e}) \leq \sum_{N=0}^{d} \mathbb{P}(\bar{N}(\mathcal{H}) = N) + \sum_{N=d+1}^{\infty} \mathbb{P}(Z_0 \in \mathcal{K}_{a,e} \mid \bar{N}(\mathcal{H}) = N)\mathbb{P}(\bar{N}(\mathcal{H}) = N).
\]

For the second sum, we use (16) and \(\alpha \geq f_a(\varepsilon)/3\) to obtain

\[
\sum_{N=d+1}^{\infty} \mathbb{P}(Z_0 \in \mathcal{K}_{a,e} \mid \bar{N}(\mathcal{H}) = N)\mathbb{P}(\bar{N}(\mathcal{H}) = N) \\
\leq \sum_{N=d+1}^{\infty} \frac{d\nu}{j!} \exp(-\gamma_N\omega_{d+1}) \frac{(\gamma_N\omega_{d+1})^j}{j!} \mathbb{P}(\bar{N}(\mathcal{H}) = N) \\
= \sum_{N=d+1}^{\infty} \frac{d\nu}{j!} \exp(-\gamma_N\omega_{d+1}) \frac{(\gamma_N\omega_{d+1})^j}{j!} \mathbb{P}(\bar{N}(\mathcal{H}) = N) \\
= \exp(-\gamma_N\omega_{d+1}(1 + \alpha)\tau(a)) \frac{d\nu}{j!} \frac{(\gamma_N\omega_{d+1})^j}{j!} \exp[-\gamma_N\omega_{d+1}(1 + f_a(\varepsilon))\tau(a)].
\]

For the first sum, we get

\[
\sum_{N=0}^{d} \mathbb{P}(\bar{N}(\mathcal{H}) = N) \\
\leq \sum_{N=0}^{d} \frac{(\gamma_N\omega_{d+1})^d\nu}{N!} \exp[-\gamma_N\omega_{d+1}(1 + f_a(\varepsilon))\tau(a)],
\]

since \(1 \geq (1 + \alpha)\tau(a) \geq (1 + f_a(\varepsilon)/3)\tau(a)\). Combining both estimates, we obtain

\[
\mathbb{P}(Z_0 \in \mathcal{K}_{a,e}) \leq c_1 \max\{1, \gamma_N\omega_{d+1}\}^{d\nu} \exp[-\gamma_N\omega_{d+1}(1 + f_a(\varepsilon))\tau(a)],
\]

where

\[
c_1 = c_1(a, \varepsilon, d) := \sum_{j=d+1}^{d\nu} \frac{(j)!}{j!} \frac{1}{N!} + \sum_{N=0}^{d} \frac{1}{N!} \leq e + (d\nu)^{d\nu+1}.
\]

By choosing \(\Sigma = \sigma_d\) and \(\vartheta = \Delta_2\), and using Theorem 4.1 instead of Proposition 3.1, we arrive at the following more explicit result. Let \(B_\alpha\) denote a spherical cap of volume \(\alpha\).

**Lemma 6.2.** Let \(0 < \alpha < \omega_d/2 > 0, \varepsilon \in (0, 1]\) and \(\mathcal{K}_{a,e} = \{K \in \mathcal{K}_a : \sigma_d(K) \geq \alpha, \Delta_2(K) \geq \varepsilon\} \). Then

\[
\mathbb{P}(Z_0 \in \mathcal{K}_{a,e}) \leq c_2 \max\{1, \gamma_N\omega_{d+1}\}^{d\nu} \exp[-\gamma_N\omega_{d+1}(1 + f_a(\varepsilon))\tau(a)],
\]

where the constant \(c_2\) depends only on \(a, d, \varepsilon\) and the constant \(\beta\) depends only on \(a, d\).

**Proof.** Let \(Z_0 \in \tilde{K}_{a,e} \cap \mathcal{K}_a\). Let \(e \in \text{int}(Z_0)\). Suppose that all points of \(X\) are either in \(B^\varepsilon(e, \varepsilon)\) or \(B^\varepsilon(-e, \varepsilon)\), where \(B^\varepsilon(s, r)\) denotes the open spherical cap with centre \(e \in \mathbb{S}^d\) and radius \(r > 0\). Then we immediately get \(\Delta_0(Z_0) < \varepsilon\). By (10), it follows that \(\Delta_2(Z_0) < \varepsilon\), a contradiction to \(Z_0 \in \tilde{K}_{a,e}\).

Therefore, there is a point \(x \in X\) such that \(x \in B(e, \pi/2)\) and \(d_s(e, x) \geq \varepsilon\) or there is a point \(x \in B(-e, \pi/2)\) and \(d_s(e, x) \geq \varepsilon\). In either case, we obtain

\[
\sigma_d(Z_0) \leq \frac{\omega_d}{2} - \frac{\omega_{d+1}\varepsilon}{2}.\]

Now let \(C\) be a spherical cap with \(\sigma_d(C) \leq \omega_{d+1}/2 - (\varepsilon/\pi) \cdot (\omega_{d+1}/2)\) and denote its radius by \(\alpha_C\). Since \(\alpha_C \leq \pi/2\),

\[
\sigma_d(C) = \omega_d \int_0^{\alpha_C} \sin^{d-1}(t) dt \leq \frac{\omega_{d+1}}{2} - \frac{\varepsilon \omega_{d+1}}{\pi} = \omega_d \int_0^{\pi/2} \sin^{d-1}(t) dt - \frac{\omega_{d+1}\varepsilon}{2\pi}.
\]
and thus

\[
\frac{\varepsilon \omega_{d+1}}{2\pi \omega_d} \leq \int_{\alpha C}^{\pi/2} \sin^{d-1}(t) \, dt \leq \frac{\pi}{2} - \alpha C,
\]

which gives us

(17)

\[
\alpha C \leq \frac{\pi}{2} - \frac{\varepsilon \omega_{d+1}}{2\pi \omega_d}.
\]

Analogously to the proof of Lemma 6.1, we consider \( N \in \mathbb{N}, N \geq d + 1, \) and \( H_1, \ldots, H_N \in \mathcal{H}_d \) such that the Crofton cell \( P(H_{(N)}) \) of the tessellation satisfies \( P(H_{(N)}) \in \bar{K}_{a, \varepsilon} \cap K_d^a \). Let \( C \) be a spherical cap satisfying \( \sigma_d(C) = \sigma_d(P(H_{(N)})) \geq a \) and denote its radius by \( \alpha C \). Using Theorem 4.1 instead of Proposition 3.1 and the monotonicity of \( \Phi \), we get

(18)

\[
\Phi(P(H_{(N)})) \geq (1 + \beta \varepsilon^2) \Phi(C) \geq (1 + \beta \varepsilon^2) \Phi(B_a),
\]

where

\[
\beta = 2 \min \left\{ \frac{(d+1)^2 \sin^2(a) \tan^2(a \alpha)}{d + (d+1)^2 \frac{\pi}{2} \tan^2(a \alpha)}, \left( \frac{2}{\pi} \right)^2 D \left( \frac{\pi}{2} - \alpha C \right) \right\},
\]

with \( a_0 = \alpha_{B(a)} \) and \( a_0 \leq \alpha C < \pi/2 \). By (17), applied to \( P(H_{(N)}) \), we have

\[
\alpha C \leq \frac{\pi}{2} - \frac{\varepsilon \omega_{d+1}}{2\pi \omega_d}.
\]

Recalling

\[
D(x) = \int_0^x \sin^{d-1}(t) \, dt \geq \int_0^x \left( \frac{2}{\pi} \right)^{d-1} t^{d-1} \, dt = \frac{2^{d-1}}{d} x^d,
\]

and using the fact that \( \tan \) is increasing on \([0, \pi/2]\) and \( \tan(x) \geq x \) for \( x \in [0, \pi/2) \), we obtain

\[
\beta \geq 2 \min \left\{ \frac{(d+1)^2 \sin^2(a_0) \tan^2(a_0 \alpha)}{d + (d+1)^2 \frac{\pi}{2} \tan^2(a_0 \alpha)}, \left( \frac{2}{\pi} \right)^2 D \left( \frac{\varepsilon \omega_{d+1}}{2\pi \omega_d} \right) \right\}
\]

\[
\geq 2 \min \left\{ \frac{(d+1)^2 \sin^2(a_0) \omega_{d+1}^2}{d + (d+1)^2 \frac{\pi}{2} \omega_d^2}, \left( \frac{2}{\pi} \right)^2 \frac{\omega_{d+1}^2}{d} \right\} =: \tilde{\beta} \cdot \varepsilon^{2d},
\]

where we made use of \( \varepsilon \leq 1 \) in the second to last line. Note that \( \tilde{\beta} > 0 \) depends only on \( a \) and \( d \), since \( a_0 = \alpha_{B_a} \). Combining this with (18), we get

\[
\Phi(P(H_{(N)})) \geq (1 + \tilde{\beta} \varepsilon^{2d}) \Phi(B_a).
\]

Proceeding now as in the proof of Lemma 6.1 gives the result. \( \square \)

Now we are able to prove the following general theorem.

**Theorem 6.3.** Let \( Z_0 \) be the Crofton cell of an isotropic spherical Poisson hyperplane tessellation with intensity \( \gamma_s \) and hitting functional \( \Phi = 2U_1 \). Let \( \Sigma \) be an increasing size functional and \( \vartheta \) an associated deviation functional as in (2). Let \( a > 0 \) be such that \( \Sigma^{-1}([a, \infty)) \neq \emptyset \) and \( \varepsilon \in (0, 1] \). Then there are constants \( c_3, c_4 > 0 \) such that

\[
\mathbb{P}(\vartheta(Z_0) \geq \varepsilon \mid \Sigma(Z_0) \geq a) \leq c_3 \exp(-c_4 \gamma_s),
\]

where the constants \( c_3, c_4 \) depends on \( a, d, \varepsilon \) and \( \Sigma \).

**Proof.** First we note that

(19)

\[
\mathbb{P}(\vartheta(Z_0) \geq \varepsilon \mid \Sigma(Z_0) \geq a) = \frac{\mathbb{P}(\vartheta(Z_0) \geq \varepsilon, \Sigma(Z_0) \geq a)}{\mathbb{P}(\Sigma(Z_0) \geq a)} = \frac{\mathbb{P}(Z_0 \in K_{a, \varepsilon})}{\mathbb{P}(\Sigma(Z_0) \geq a)}.
\]

Let \( K_a \in K_d^a \) be an extremal body, hence \( \Sigma(K_a) \geq a \) and \( \Phi(K_a) = \tau(a) \). Let \( e_a \in K_a \) be an arbitrary fixed point. If \( \bar{X}(K_{e_a}) = 0 \), then \( K_a \subset Z_{e_a} \), the Crofton cell of the point \( e_a \). Since \( \Sigma \) is increasing, we have \( \Sigma(Z_{e_a}) \geq \Sigma(K_a) \geq a \). Using the isotropy of \( \bar{X} \), we get

(20)

\[
\mathbb{P}(\Sigma(Z_0) \geq a) = \mathbb{P}(\Sigma(Z_{e_a}) \geq a) \geq \mathbb{P}(\bar{X}(K_{e_a}) = 0) = \exp(-\gamma_s \omega_{d+1} \tau(a)).
\]
Lemma 6.4. \( t \) otic shape of Crofton cells having large spherical inradii.\footnote{\text{19}}\footnote{\text{20}} and Lemma 6.1 we obtain
\[
\mathbb{P}(\vartheta(Z_0) \geq \varepsilon \mid \Sigma(Z_0) \geq a) \leq \frac{c_1 \max\{1, \gamma_s \omega_{d+1}\}^{d\nu} \exp\left(-\gamma_s \omega_{d+1} \left(1 + \frac{1}{3} f_a(\varepsilon)\right) \tau(a)\right)}{\exp(-\gamma_s \omega_{d+1} \tau(a))} = c_1 \max\{1, \gamma_s \omega_{d+1}\}^{d\nu} \exp\left(-\gamma_s \omega_{d+1} \frac{1}{3} f_a(\varepsilon) \tau(a)\right) \leq c_3 \exp\left(-c_4 \gamma_s\right),
\]
where the constants \( c_3, c_4, \nu \) depend on \( a, \varepsilon, d \), and, of course, on \( \Sigma \).

Using Lemma 6.2 instead of Lemma 6.1 (and of course \( \vartheta = \Delta_2 \)), we obtain a similar result but with a more explicit constant in the exponent, which was stated as Theorem B in the introduction.

By similar arguments, we also obtain the following lemma and the subsequent theorem concerning the asymptotic shape of Crofton cells having large spherical inradii.

Lemma 6.4. Let \( a \in (0, \pi/2), \varepsilon \in (0, 1) \) and \( \mathcal{K}_{a,\varepsilon} := \{K \in \mathcal{K}_a : \Sigma_r(K) \geq a, \vartheta_r(K) \geq \varepsilon\} \). Then
\[
\mathbb{P}(Z_0 \in \mathcal{K}_{a,\varepsilon}) \leq c_5 \gamma_s \omega_{d+1}^{d\nu} \exp\left(-\gamma_s \omega_{d+1} \left(1 + c_6 \varepsilon^{d+1}\right) \Phi(B(\varnothing, a))\right),
\]
where the constants \( c_5, \nu \) depend on \( a, d, \varepsilon \) and \( c_6 \geq 1.3 \cdot 0.03d^{-d/2}(\pi/2-a)^{d-1} \).

Proof. Using Theorem 4.6 and \( B(\varnothing, a) \) instead of \( B_a \), the argument is exactly the same as in the proof of Lemma 6.1.

\( \square \)

Theorem 6.5. Let \( a \in (0, \pi/2) \) and \( \varepsilon \in (0, 1) \). Then there are constants \( c_7, c_8 > 0 \) such that
\[
\mathbb{P}(\vartheta_r(Z_0) \geq \varepsilon \mid \Sigma_r(Z_0) \geq a) \leq c_7 \exp\left(-c_8 \varepsilon^{d+1} \gamma_s\right),
\]
where \( c_7 \) depends on \( a, d, \varepsilon \) and \( c_8 \geq 0.06^{-d} \omega_{d+1} a^{-1}(\pi/2-a)^{d-1} \).

Proof. Combining
\[
\mathbb{P}(\Sigma_r(Z_0) \geq a) = \mathbb{P}(X(\mathcal{H}(\mathcal{P}(\varnothing, a)) = 0) = \exp[-\gamma_s \omega_{d+1} \Phi(B(\varnothing, a))] \]
with Lemma 6.4 the result follows as before by using \( \Phi(B(\varnothing, a)) \omega_{d+1} \geq 2d^{-d-1} \omega_{d+1} \).

\( \square \)

7. Asymptotic distribution of the size of the Crofton cell

Similar to [25] Theorem 2, we determine the asymptotic distribution function of a general increasing size functional \( \Sigma \) of the Crofton cell \( Z_0 \) associated with an isotropic spherical Poisson hyperplane tessellation with hitting functional \( \Phi = 2U_1 \), as the intensity \( \gamma_s \) tends to infinity. We use the techniques developed in the proof of Lemma 6.1 to obtain the following theorem. We write \( \tau(a) = \tau(\Phi, \Sigma, a) \) for the isoperimetric constant of the given parameters.

Theorem 7.1. Let \( \Sigma \) be an increasing size functional. Let \( a > 0 \) be such that \( \Sigma^{-1}([a, \infty)) \neq \emptyset \). Then
\[
\lim_{\gamma_s \to \infty} \gamma_s^{-1} \ln \mathbb{P}(\Sigma(Z_0) \geq a) = -\omega_{d+1} \tau(\Phi, \Sigma, a).
\]

Proof. Since we are only interested in \( \gamma_s \to \infty \), we can assume that \( \gamma_s \omega_{d+1} \geq 1 \). Let \( k \in (0, 1) \) and \( \mathcal{K}_{a,0} = \{K \in \mathcal{K}_a : \Sigma(K) \geq a\} \). Let \( N \in \mathbb{N}, N \geq d+1 \), and let \( H_1, \ldots, H_N \in \mathcal{H}_{d+1} \) be such that \( P(H_{(N)}) \in \mathcal{K}_{a,0} \cap \mathcal{K}_a \), in particular \( \Phi(P(H_{(N)})) \geq \tau(a) \). By Lemma 6.2 we obtain a number \( \nu = \nu(d, a, k) \) and a spherical polytope \( Q(P(H_{(N)})) = Q \) with at most \( \nu \) vertices and \( \text{ext}(Q) \subset \text{ext}(P(H_{(N)})) \) such that
\[
\Phi(Q) \geq \left(1 - \frac{k}{2}\right) \Phi(P(H_{(N)})) \geq \left(1 - \frac{k}{2}\right) \tau(a).
\]
Proceeding as in the proof of Lemma 6.1 we obtain
\[
\mathbb{P}(Z_0 \in \mathcal{K}_{a,0} \mid \mathcal{H}_{d+1} = N) \leq \sum_{j=d+1}^{d\nu} \binom{N}{j} \left[ \frac{j}{d} \right] \left(1 - \left(1 - \frac{k}{2}\right) \tau(a)\right)^{N-j}
\]
for \( N \geq d+1 \). After summation over \( N \), where we deal with the cases \( N \in \{0, \ldots, d\} \) as in the proof of Lemma 6.1 and since \( \gamma_s \omega_{d+1} \geq 1 \), we get
\[
\mathbb{P}(Z_0 \in \mathcal{K}_{a,0}) \leq c_9 \gamma_s \omega_{d+1}^{d\nu} \exp\left(-\left(1 - \frac{k}{2}\right) \tau(a) \gamma_s \omega_{d+1}\right).
\]
for suitable constants $c_9, c_{10} > 0$ which depend only on $a, d$ and $κ$. For the last inequality, we used that $x \mapsto x^{dν} \exp(-κ/2x)$ is bounded.

Combining (21) with (20), we get
\[
\exp(-κ\omega_d + 1τ(a)) \leq \mathbb{P}(Σ(Z_0) \geq a) \leq c_{10} \exp(-κ\omega_d + 1τ(a))
\]
This yields
\[
\lim_{γ_s \to ∞} γ_s^{-1} \ln \mathbb{P}(Σ(Z_0) \geq a) \geq -ω_d + 1τ(a)
\]
and
\[
\lim_{γ_s \to ∞} γ_s^{-1} \ln \mathbb{P}(Σ_d(Z_0) \geq a) \leq (1 - κ)ω_d + 1τ(a).
\]
The left-hand side of the second estimate is independent of $κ$ and therefore
\[
\lim_{γ_s \to ∞} γ_s^{-1} \ln \mathbb{P}(Σ_d(Z_0) \geq a) = -ω_d + 1τ(a),
\]
which completes this proof. □

8. Typical cells of particle processes in spherical space

After having studied the Crofton cell in the previous section, we now turn to typical cells. In Euclidean space, there is a very intuitive representation for the distribution of the typical grain of a stationary particle process. In the special case of a particle process derived from a tessellation, this leads to the notion of the typical cell of the tessellation. In spherical space, some modifications are required. We will specialize the general framework of [34, 35], where random measures are studied in a general homogeneous space, and start by giving some additional definitions and specific explanations.

We consider the compact group $SO_d + 1$ of proper rotations and denote the unique, rotation invariant probability measure on $SO_d + 1$ by $ν$. The group $SO_d + 1$ operates continuously and transitively on $Σ^d$ ([60, Theorem 13.2.2]). Defining $σ^0_d := ν ◦ π^{-1}$, we obtain the unique rotation invariant probability measure $σ^0_d = ω_d + 1^{-1}σ_d$ on $Σ^d$. Next we consider the stabilizer (isotropy group) of $[0$, which is given by $SO^0_d + 1 := \{ ϕ ∈ SO_d + 1 : ϕ[0] = [0]\}$ and denote by $κ([0, ·)$ the $SO^0_d + 1$-invariant probability measure on this (compact) subgroup. Putting $κ([0, SO_d + 1 \backslash SO^0_d + 1) := 0$, we extend the measure to a probability measure on $SO_d + 1$. For $x ∈ Σ^d$, we define $SO^x_d + 1 := \{ ϕ ∈ SO_d + 1 : ϕ[0] = x\}$, and for $ϕ_x ∈ SO^x_d + 1$ arbitrary but fixed let
\[
κ(x, B) := \int 1\{ ϕ_x ◦ ϕ ∈ B\} κ([0, dϕ), \quad B ∈ B(SO_d + 1).
\]
This definition is independent of the choice of $ϕ_x$ (see [34, (2.7)]), in particular we have
\[
\int 1\{ ϕ ∈ ·\} κ(x, dϕ) = \int 1\{ ϕ ◦ ψ ∈ ·\} κ(x, dϕ), \quad ψ ∈ SO^x_d + 1, x ∈ Σ^d,
\]
and
\[
\int 1\{ ϕ ∈ ·\} κ(ψ, x) = \int 1\{ ϕ ◦ ψ ∈ ·\} κ(x, dϕ), \quad ψ ∈ SO_d + 1, x ∈ Σ^d.
\]
In the following sections, we assume (without restriction of generality, as explained in [34, Rem. 3.6], by a suitable/canonical choice of a state space) all random elements to be defined on a common probability space $(Ω, F, P)$, equipped with a measurable flow $\{ θ_ϕ : ϕ ∈ SO_d + 1\}$. This is a family of measurable maps $Ω × SO_d + 1 → Ω$, $(ω, ϕ) → θ_ϕ(ω)$, such that $θ_0s_0Σ_d + 1 = id_Ω$ and $θ_ϕ ◦ θ_ψ = θ_ϕ ◦ ψ$. We further assume $P$ to be invariant, i.e.
\[
P ◦ θ_ϕ = P, \quad ϕ ∈ SO_d + 1.
\]
Then a random measure $ξ$ on $Σ^d$ is called adapted or isotropic, if
\[
ξ(θ_ϕω, φB) = ξ(ω, B), \quad ω ∈ Ω, φ ∈ SO_d + 1, B ∈ B(Σ^d),
\]
where $φB = \{ φx : x ∈ B\}$. For measurable $f : Σ^d → [0, ∞)$ and isotropic $ξ$, we have
\[
\int f(x) ξ(θ_ϕω, dx) = \int f(φx) ξ(ω, dx).
\]
For each locally finite measure \( \eta \) on \( \mathbb{S}^d \) and for each \( \varphi \in SO_{d+1} \), let the rotated measure \( \varphi \eta \) be given by \( (\varphi \eta)(\cdot) := \varphi(\varphi^{-1}(\cdot)) \) (the image measure of \( \eta \) under \( \varphi \)), hence \( (\varphi \eta)(\varphi B) = \eta(B) \), \( B \in \mathcal{B}(\mathbb{S}^d) \). Thus we can rewrite (24) as
\[
\mathbb{P}(\xi \mid \theta_z \omega, \cdot) = \varphi(\xi(\omega, \cdot)). 
\]
The Palm measure of an isotropic random measure \( \xi \) is a finite measure on \( \Omega \) defined by
\[
\mathbb{P}_\xi(A) := \int_{\Omega} \int_{\mathbb{S}^d} \int_{SO_{d+1}} 1\{\theta^{-1}_\varphi \omega \in A\} \kappa(x, d\varphi) \xi(\omega, dx) \mathbb{P}(d\omega), \quad A \in \mathcal{F};
\]
compare (34), (3.8). In what follows, \( \mathbb{E}_\xi \) denotes integration with respect to \( \mathbb{P}_\xi \). Then the refined Campbell Theorem ([34, Theorem 3.7], [54, Theorem 1]) states that, for an isotropic random measure \( \xi \) and any measurable map \( f : \Omega \times SO_{d+1} \to [0, \infty) \),
\[
\mathbb{E} \int \int \int 1\{\theta^{-1}_\varphi \omega \in A\} \kappa(x, d\varphi) \xi(\omega, dx) = \mathbb{E}_\xi \int \int \kappa(\theta d\varphi, \varphi) \nu(d\varphi).
\]
(This can be easily proved by starting with the right-hand side and using successively the definition of \( \mathbb{P}_\xi \), the right invariance of \( \nu \), Fubini’s theorem, the invariance of \( \mathbb{P} \), (24), (23), the flow property and the fact that \( \nu \) is a probability measure.) Note that \( \mathbb{P}_\xi(\Omega) = \mathbb{E} [\xi(\mathbb{S}^d)] \).

Let \( X' \) be an isotropic particle process on \( \mathbb{S}^d \) with particles in \( \widehat{\mathcal{K}}^d_s := \overline{\mathcal{K}}^d_s \cup \{\mathbb{S}^d\} \). This is a point process in \( \widehat{\mathcal{K}}^d_s \) satisfying
\[
X'(\theta_z \omega) = \varphi X'(\omega), \quad \omega \in \Omega, \quad \varphi \in SO_{d+1}.
\]
The intensity \( \gamma_{X'} \) of \( X' \) is defined as the expected number of particles, normalized by the surface area of \( \mathbb{S}^d \) so that \( \gamma_{X' \omega_{d+1}} = \mathbb{E}[X'(\widehat{\mathcal{K}}^d_s)] \), which we assume to be positive and finite. A rotation covariant spherical centre function is a mapping \( c_s : \widehat{\mathcal{K}}^d_s \to \mathbb{S}^d \subseteq \mathbb{S}^d \cup \{o\} \) (we write \( o \) for the Euclidean origin) satisfying \( c_s(\mathbb{S}^d) = o, c_s(K) \in \mathbb{S}^d \) for \( K \in \overline{\mathcal{K}}^d_s \) and
\[
c_s(\varphi K) = \varphi c_s(K), \quad K \in \overline{\mathcal{K}}^d_s, \quad \varphi \in SO_{d+1}.
\]
An example for such a centre function is the spherical circumcentre together with \( c_s(\mathbb{S}^d) = o \).

We complement the definition of the probability kernel \( \kappa \) by \( \kappa(o, \cdot) := \nu \). Then \( \kappa \) is a probability kernel from \( \widehat{\mathbb{S}}^d \) to \( SO_{d+1} \) and the crucial properties (22) and (23) are satisfied. In order to define the typical particle of \( X' \) with respect to a given center function, we consider the marked random measure on \( \widehat{\mathbb{S}}^d \times \widehat{\mathcal{K}}^d_s \) given by
\[
\kappa' (\omega) = \int_{\widehat{\mathcal{K}}^d_s} \int_{SO_{d+1}} 1\{(c_s(K), \varphi^{-1} K) \in \cdot\} \kappa(c_s(K), d\varphi) X'(\omega, dK).
\]
Here and in what follows, we will also use the notation \( \kappa' (\omega, \cdot) := \kappa'(\omega)(\cdot) \). This random measure is invariant (see [34, Remark 3.9]) in the sense that for \( \psi \in SO_{d+1} \) and measurable sets \( B \subseteq \mathbb{S}^d \), \( A \subset \widehat{\mathcal{K}}^d_s \) we have \( \kappa' (\theta_z \omega, \psi B \times A) = \kappa'(\omega, B \times A) \), which follows by using successively (27), (28) and (23). The Palm measure of \( \kappa' \) is a measure on \( \Omega \times \widehat{\mathcal{K}}^d_s \) defined by
\[
\mathbb{P}_{\kappa'}(\cdot) := \int_{\Omega} \int_{\widehat{\mathbb{S}}^d} \int_{\widehat{\mathcal{K}}^d_s} 1\{(\theta^{-1}_\varphi \omega, \cdot) \in \cdot\} \kappa(x, d\varphi) \kappa'(\omega, d(x, K)) \mathbb{P}(d\omega).
\]
The distribution of the typical particle \( Z \) of \( X' \) is defined as the normalized mark distribution of the random measure \( \kappa' \) with the mark space \( \mathcal{K}_\kappa := \{K \in \widehat{\mathcal{K}}^d_s : c_s(K) = \overline{\Xi} \} \cup \{\mathbb{S}^d\} \). Then we have
\[
\mathbb{P}_Z(\cdot) := \frac{1}{\gamma_{X' \omega_{d+1}}} \mathbb{E} \int_{\widehat{\mathcal{K}}^d_s} \int_{SO_{d+1}} 1\{\varphi^{-1} K \in \cdot\} \kappa(c_s(K), d\varphi) X'(dK).
\]
Note that \( \mathbb{P}_Z(\cdot) = (\gamma_{X' \omega_{d+1}})^{-1} \mathbb{P}_{\kappa'}(\Omega \times \cdot) \).

In the Euclidean case, the typical particle of an isotropic and stationary particle process is still isotropic, but not stationary. In the spherical setting, the typical particle cannot be isotropic, since its centre is almost surely \( \overline{\Xi} \), but its distribution still has some symmetry.

**Lemma 8.1.** Let \( X' \) be an isotropic particle process on \( \mathbb{S}^d \), and let \( Z \) denote the typical particle of \( X' \). Then the distribution of \( Z \) is invariant under rotations fixing \( \overline{\Xi} \), that is,
\[
\mathbb{P}(\varphi Z \in \cdot) = \mathbb{P}(Z \in \cdot), \quad \varphi \in SO_{d+1}^\Xi.
\]

**Proof.** This immediately follows from (22). \( \square \)
We also have a disintegration result for the intensity measure of $X'$, similar to the Euclidean case (see [60, Theorem 4.1.1]). Moreover, the following theorem expresses a natural characterization of the distribution of the typical grain. Although this will not be used in the following, it demonstrates that $\mathbb{P}_Z$ is defined in the right way.

**Theorem 8.2.** Let $X'$ be an isotropic particle process on $\mathbb{S}^d$ with intensity $\gamma_{X'} \in (0, \infty)$, and let $Z$ be the typical particle of $X'$. If $f : \hat{K}_a^d \to [0, \infty)$ is measurable, then

$$
\mathbb{E} \int_{\hat{K}_a^d} f(K) X'(dK) = \gamma_{X'} \omega_{d+1} \int_{\hat{K}_a^d} \int_{SO_{d+1}} f(\psi K) \nu(d\psi) \mathbb{P}_Z(dK).
$$

The probability measure $\mathbb{P}_Z$ is uniquely determined by (31) and its invariance under rotations fixing $\hat{0}$.

**Proof.** Using the definition of $\mathbb{P}_Z$, we obtain

$$
\gamma_{X'} \omega_{d+1} \int_{\hat{K}_a^d} \int_{SO_{d+1}} f(\psi K) \nu(d\psi) \mathbb{P}_Z(dK) = \int_{\hat{K}_a^d} \int_{SO_{d+1}} f(\psi K) \nu(d\psi) \mathbb{P}_Z(dK),
$$

where we successively also used the right invariance of $\nu$, the fact that $\kappa(x, \cdot)$ is a probability measure for each $x \in \hat{S}^d$, the isotropy of $X'$ and the fact that $\nu$ is a probability measure.

Let $\mathbb{P}^*$ be another probability measure on $\hat{K}_a^d$ which satisfies (31) and is invariant under rotations fixing $\hat{0}$. Observe that $\mathbb{S}^d \to SO_{d+1}$, $x \mapsto \varphi_x \in SO_{d+1}^\infty$, can be chosen as a measurable map and the disintegration

$$
\nu(\cdot) = \int_{\mathbb{S}^d} \kappa(x, \cdot) \sigma_0^\nu(dx)
$$

holds (see [34, (2.9)]). Since (31) holds for $\mathbb{P}_Z$ and $\mathbb{P}^*$, we obtain

$$
\int_{\hat{K}_a^d} \int_{SO_{d+1}} f(\psi K) \nu(d\psi) \mathbb{P}_Z(dK) = \int_{\hat{K}_a^d} \int_{SO_{d+1}} f(\psi K) \nu(d\psi) \mathbb{P}^*(dK)
$$

for any measurable function $f : \hat{K}_a^d \to [0, \infty)$. Using (32), the definition of the kernel $\kappa$ and Fubini’s theorem, we get

$$
\mathbb{E} \int_{\mathbb{S}^d} \int_{SO_{d+1}} \int_{\hat{K}_a^d} f(\varphi_x \circ \varphi K) \mathbb{P}_Z(dK) \kappa(\hat{0}, d\varphi) \sigma_0^\nu(dx) = \int_{\mathbb{S}^d} \int_{SO_{d+1}} \int_{\hat{K}_a^d} f(\varphi_x \circ \varphi K) \mathbb{P}^*(dK) \kappa(\hat{0}, d\varphi) \sigma_0^\nu(dx).
$$

The invariance property of $\mathbb{P}_Z$ and $\mathbb{P}^*$ implies that

$$
\int_{\mathbb{S}^d} \int_{\hat{K}_a^d} f(\varphi_x K) \mathbb{P}_Z(dK) \sigma_0^\nu(dx) = \int_{\mathbb{S}^d} \int_{\hat{K}_a^d} f(\varphi_x K) \mathbb{P}^*(dK) \sigma_0^\nu(dx).
$$

Let $h : \hat{K}_a^d \to [0, \infty)$ be measurable and define $f(K) := h(\varphi_{-1}(K))$, for $K \in \hat{K}_a^d$, and $f(\mathbb{S}^d) := h(\mathbb{S}^d)$. Then $f$ is measurable and $f(\varphi_x K) = h(K)$ for all $K \in \hat{K}_a^d$. Therefore

$$
\int_{\mathbb{S}^d} \int_{\hat{K}_a^d} h(K) \mathbb{P}_Z(dK) \sigma_0^\nu(dx) = \int_{\mathbb{S}^d} \int_{\hat{K}_a^d} h(K) \mathbb{P}^*(dK) \sigma_0^\nu(dx),
$$

which yields the uniqueness assertion. \qed
9. The typical cell of spherical Poisson hyperplane tessellations

As in [2], we can interpret the tessellation generated by the isotropic spherical hyperplane process $\widetilde{X}$ as an isotropic particle process $X'$. The distribution of the typical cell $Z$ of $X'$ is given by (30). The following relation between the typical cell and the Crofton cell of an isotropic tessellation on $\mathbb{S}^d$ is a special case of a well-known relationship valid for all homogeneous tessellations, see e.g. [34, Corollary 8.4]. Its Euclidean counterpart can be found in [60, Theorem 10.4.1]. We include the simple proof for convenience and add an explicit expression for the intensity of $X'$ if the tessellation is induced by a spherical Poisson hyperplane process. In advance, we point out some properties of the functions

$$h_m : [0, \infty) \to \mathbb{R}, \quad t \mapsto (-1)^{m+1}e^{-t} + 2 \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{t^{m-2i}}{(m-2i)!},$$

for $m \in \mathbb{N}_0$, which will occur in the explicit expression of the intensity of $X'$.

**Lemma 9.1.** The functions $h_m$, $m \in \mathbb{N}_0$, have the following properties:

1. $h'_m = h_{m-1}$, $m \geq 1$;
2. $h_0(t) = 2 - e^{-t} \geq 1$, $h_1(t) = e^{-t} + 2t \geq 1$;
3. $h_m(0) = 1$, $m \geq 0$;
4. $h_m$ is strictly increasing and $h_m \geq 1$, $m \geq 0$;
5. $h_m$ is convex for $m \geq 1$;
6. $0 \leq h_m(t) - \left(1 + \frac{t}{m} + \frac{t^2}{m^2} + \ldots + \frac{t^m}{m!}\right) \leq \frac{t^m}{m!}$, $m \in \mathbb{N}_0$.

**Theorem 9.2.** Let $f : \mathbb{R}^d \to [0, \infty)$ be measurable and rotation invariant. Let $X'$ be an isotropic tessellation of $\mathbb{S}^d$ with intensity $\gamma_{X'} \in (0, \infty)$, let $Z_0$ denote the spherical Crofton cell and $Z$ the typical cell of $X'$. Then

$$E[f(Z)] = \gamma_{X'}E[f(Z) \sigma_d(Z)].$$

If $X'$ is the spherical hyperplane tessellation induced by a spherical Poisson hyperplane process $\widetilde{X}$ with intensity $\gamma_s$, then $\gamma_{X'}\omega_d+1 = h_d(\gamma_s\omega_d+1)$.

**Proof.** From (31) and the rotation invariance of $f$ we get

$$E[f(Z_0)] = E \sum_{K \in X'} f(K) 1 \{ \Omega \in \text{int}(K) \}$$

$$= \gamma_{X'}\omega_d+1 \int_{\mathbb{K}_d} \int_{SO_d+1} f(\varphi K) 1 \{ \Omega \in \text{int}(\varphi K) \} \nu(d\varphi) \mathbb{P}_Z(dK)$$

$$= \gamma_{X'}\omega_d+1 \int_{\mathbb{K}_d} f(K) \int_{SO_d+1} 1 \{ \varphi\Omega \in \text{int}(K) \} \nu(d\varphi) \mathbb{P}_Z(dK)$$

$$= \gamma_{X'}\omega_d+1 \int_{\mathbb{K}_d} f(K) \frac{\sigma_d(K)}{\omega_d+1} \mathbb{P}_Z(dK),$$

since the inner integral in the second to last line defines a rotation invariant probability measure on $\mathbb{S}^d$. This completes the first part of the proof.

For the second part, we use Schläflí’s theorem (see [56, p. 209–212] or [13] (1.1)) in modern language, which provides an explicit formula for the number of cells $N(k)$ generated by $k \geq 1$ great subspheres in general position,

$$N(k) = 2 \sum_{i=0}^{d-1} \binom{k-1}{i}.$$

Recall that the spherical hyperplane process $\widetilde{X}$ is defined by $\widetilde{X} = h(X)$, where $X$ is a spherical Poisson point process and $h(x) = x^+ \cap \mathbb{S}^d$, $c \in \mathbb{S}^d$, with $\mathbb{E}[X(\mathbb{S}^d)] = \gamma_s\omega_d+1$. If $X$ contains no points, we consider the whole of $\mathbb{S}^d$ as one cell and thus define $N(0) := 1$. Then

$$\gamma_{X'} = \frac{1}{\omega_d+1} \mathbb{E}[N(X(\mathbb{S}^d))]$$

$$= \frac{1}{\omega_d+1} \sum_{k=1}^{\infty} 2 \sum_{i=0}^{d-1} \binom{k-1}{i} \mathbb{P}(X(\mathbb{S}^d) = k) + \frac{1}{\omega_d+1} \mathbb{P}(X(\mathbb{S}^d) = 0)$$
Defining
\[ f_i(x) := \sum_{k=i+1}^{\infty} \frac{1}{k} \frac{x^k}{(k-i-1)!}, \]
we have
\[ f_i(0) = 0 \quad \text{and} \quad f_i'(x) = \sum_{k=i+1}^{\infty} \frac{x^{k-1}}{(k-i-1)!} = x^i \cdot e^x. \]

Applying [65, p. 174, Formula 419] iteratively yields
\[ f_i(\gamma_s \omega_d + 1) = \int_0^{\gamma_s \omega_d + 1} x^i e^x \, dx = \sum_{i=0}^{d} \left( e^{\gamma_s \omega_d + 1} (-1)^k (\gamma_s \omega_d + 1)^{i-k} \frac{i!}{(i-k)!} \right) - (-1)^i \cdot i!. \]

Combining this with (34) and using \( \sum_{i=0}^{d} (-1)^i = -\frac{1}{2} (1 + (-1)^d) \), we obtain
\[ \gamma X' = \frac{(-1)^d + 1}{\omega_d + 1} e^{-\gamma_s \omega_d + 1} + \frac{2}{\omega_d + 1} \sum_{i=0}^{d} \sum_{k=0}^{i} \left( (-1)^k \frac{(\gamma_s \omega_d + 1)^{i-k}}{(i-k)!} \right). \]

The proof can now be easily completed by interchanging the order of summation (twice) in the remaining double sum.

Now we are able to extend our results for the asymptotic shape of the spherical Crofton cell to typical cells. We state an abstract version of such a result, for specific size functionals, the argument can be adjusted as before. As before, we require that the size functional \( \Sigma \) be increasing, but now we also need that \( \Sigma \) is rotation invariant and simple. The latter means that \( \Sigma(K) = 0 \) whenever \( K \) is not \( d \)-dimensional. This condition is clearly satisfied by the volume and the inradius. Otherwise, \( \Phi \) and \( \vartheta \) are chosen as before. A simple compactness and continuity argument shows that if \( a > 0 \) with \( \Sigma^{-1}([a, \infty)) \neq \emptyset \) then the condition implies that there is a positive constant \( \gamma(a) > 0 \) such that \( \sigma_d(K) \geq \gamma(a) \) whenever \( \Sigma(K) \geq a \).

**Theorem 9.3.** Let \( \Sigma \) be an increasing, rotation invariant, simple size functional. Let \( a > 0 \) with \( \Sigma^{-1}([a, \infty)) \neq \emptyset \) and let \( \varepsilon \in (0, 1] \). Let \( X \) be an isotropic, spherical Poisson hyperplane process on \( S^d \) with intensity \( \gamma_s \). Let \( Z \) be the typical cell of the induced tessellation. Then there are constants \( c_{11}, c_{12} > 0 \), depending on \( a, d, \varepsilon \) and of course on \( \Sigma \), such that
\[ \mathbb{P}(\vartheta(Z) \geq \varepsilon \mid \Sigma(Z) \geq a) \leq \frac{c_{11}}{\gamma_s \omega_d + 1} \exp \left( -\frac{c_{12}}{\gamma_s} \right). \]

**Proof.** We first note the trivial upper bound \( \sigma_d(Z) \leq \omega_d + 1 \). In order to estimate the denominator, we use (33) and (20) to obtain
\[ \mathbb{P}(\Sigma(Z) \geq a) = \mathbb{E} \left[ \mathbf{1}\{\Sigma(Z) \geq a\} \right] = \frac{1}{\gamma X'} \mathbb{E} \left[ \mathbf{1}\{\Sigma(Z) \geq a\} \left\{ \frac{1}{\sigma_d(Z)} \right\} \mathbf{1}\{\Sigma(Z) \geq a\} \right] \]
\[ = \frac{1}{\gamma X'} \mathbb{E} \left[ \mathbf{1}\{\Sigma(Z) \geq a\} \right] \frac{1}{\sigma_d(Z)} \]
\[ \geq \frac{1}{\gamma X'} \mathbb{E} \left[ \mathbf{1}\{\Sigma(Z) \geq a\} \right] \frac{1}{\omega_d + 1} \]
\[ \geq \frac{1}{\gamma X' \omega_d + 1} \exp \left( -\gamma_s \omega_d + 1 \tau(a) \right), \]
where \( \tau(a) = \tau(\Phi, \Sigma, a) \). For the numerator, we use (33), (15), the fact that with \( \Phi \) and \( \Sigma \) also \( \vartheta \) is rotation invariant, and proceed as above to get
\[ \mathbb{P}(\Sigma(Z) \geq a, \vartheta(Z) \geq \varepsilon) = \frac{1}{\gamma X'} \mathbb{E} \left[ \mathbf{1}\{\Sigma(Z) \geq a, \vartheta(Z) \geq \varepsilon\} \right] \frac{1}{\sigma_d(Z)} \]
\[ \leq \frac{1}{\gamma X'} \mathbb{E} \left[ \mathbf{1}\{\Sigma(Z) \geq a, \vartheta(Z) \geq \varepsilon\} \right] \frac{1}{\gamma(a)} \]
\[ \leq \frac{c_{11}}{c(a)} \mathbb{E} \left[ \mathbf{1}\{\Sigma(Z) \geq a, \vartheta(Z) \geq \varepsilon\} \right] \]
\[ \leq \frac{c_{11}}{c(a)} \frac{\exp \left( -\gamma_s \omega_d + 1 \tau(a) \right)}{\exp \left( -\gamma_s \omega_d + 1 \tau(a) \right)} \]
\[ \leq \frac{c_{11}}{c(a)} \frac{\exp \left( 1 + \frac{1}{2} f_a(\varepsilon) \right)}{\gamma X' \omega_d + 1} \exp \left( -\gamma_s \omega_d + 1 \tau(a) \right). \]
Combining these two estimates, we obtain the result. □

A result similar to Theorem 7.1 holds for the asymptotic distribution of the typical cell under the assumptions of Theorem 9.3.

**Theorem 9.4.** Let \( a > 0 \) be such that \( \Sigma^{-1}([a, \infty)) \neq \emptyset \). Then

\[
\lim_{\gamma_s \to \infty} \gamma_s^{-1} \ln P(\Sigma(Z) \geq a) = -\omega_{d+1} \tau(\Phi, \Sigma, a).
\]

## 10. Spherical Poisson–Voronoi Tessellations

After the consideration of the spherical Crofton cell and the typical cell of Poisson hyperplane tessellations on the sphere, it is a natural step to take a look at the Poisson–Voronoi tessellations in spherical space, since in Euclidean space this tessellation is one of the classical models in stochastic geometry. Voronoi tessellations of a underlying Poisson point process and its Palm distribution. In order to define the distribution of the typical cell in this setting, we consider the random measure \( \zeta \) which is defined on \( S^d \times \hat{K}_d \) by

\[
\zeta(\omega) := \int_{S^d} \int_{SO_{d+1}} \delta_{(x, \varphi^{-1} C(x, X(\omega)))} \kappa(x, d\varphi) X(\omega, dx).
\]

This definition is very similar to definition (29), and \( \zeta \) is also invariant in the sense (see [34] Remark 3.9)) that for \( \psi \in SO_{d+1} \) and measurable sets \( B \subset S^d \) and \( A \subset \hat{K}_d \), we have \( \zeta(\theta \psi \omega, (\psi B) \times A) = \zeta(\omega, B \times A) \), where we made use of (24), (35) and (23). Recalling that \( X \) is an isotropic Poisson process with intensity measure.
Lemma 10.2. \( \nu, c \) where the constants \( \zeta \) depend on \( a, d, \varepsilon \) and \( c_{14} \geq 3^{-1}(2\pi)^{-2}a^{-1}\min\{a, \frac{\pi}{2} - a\}\)^{d-1}.

**Proof.** We proceed as in the proof of Lemma 6.1. Let \( \tilde{B}(\cdot) := (\omega_{d}+1)^{-1}\tilde{U}(\cdot) \) and \( N \in \mathbb{N} \).

For \( H_1, \ldots, H_N \in H_d \), we define \( H(d) := (H_1, \ldots, H_N) \) and let \( P(H(d)) \) denote the spherical Crofton cell of the tessellation induced by \( H_1, \ldots, H_N \). For \( N \geq d + 1 \), let \( H_1, \ldots, H_N \) be such that \( P(H(d)) \in \mathcal{K}_{d} \cap \mathbb{K}_{d} \) and define \( \alpha := c(a, d)\varepsilon^{d+1}/(2 + c(a, d)\varepsilon^{d+1}) \), where the constant \( c(a, d) \) is taken from Theorem 4.8. Then \( \alpha \geq 1/3c(a, d)\varepsilon^{d+1} \) and therefore \( (1 - \alpha)(1 + c(a, d)\varepsilon^{d+1}) = 1 + \alpha \). By Lemma 5.2 and Theorem 4.8 there are at most \( \nu = \nu(a, d, \varepsilon) \) vertices of \( P(H(d)) \) such that the spherical convex hull \( Q(H(d)) \) of these vertices satisfies

\[
\bar{V}(Q(H(d))) \geq (1 - \alpha)\bar{V}(P(H(d))) \geq (1 - \alpha)(1 + c(a, d)\varepsilon^{d+1})\bar{V}(B(\overline{U}, a)) = (1 + \alpha)\bar{V}(B(\overline{U}, a)).
\]

It follows from (14) that \( \bar{V}^{N} \)-almost surely any \( N \) great subspheres are in general position and therefore \( P(H(d)) \in \mathcal{K}_{d} \) holds almost surely for \( N \geq d + 1 \).

Thus, proceeding as in Lemma 6.1, we obtain

\[
\mathbb{P}(Z \in \mathcal{K}_{d} \cap \overline{U}) \leq c_{14}(a, \varepsilon, d)\max\{1, \zeta_{d}+1\}^{d}\nu \exp \left(-\zeta_{d}\zeta_{d+1}\left(1 + \frac{1}{3}c(a, d)\varepsilon^{d+1}\right)\bar{V}(B(\overline{U}, a))\right).
\]

Combining this result with Lemma 10.2 and using again (13), we obtain the following theorem for the asymptotic shape of the typical cell of a spherical Poisson–Voronoi tessellation.
Theorem 10.3. Let $0 < a < \pi/2$ and $\varepsilon \in (0, 1]$. Let $X$ be a Poisson process on $S^d$ with intensity $\gamma_s \in (0, \infty)$. Then the typical cell $Z$ of its spherical Poisson–Voronoi tessellation satisfies

$$P\left(R_a(Z) - r_a(Z) \geq \varepsilon \mid r_a(Z) \geq a\right) \leq c_{15} \exp\left(-c_{16} \varepsilon \frac{d+1}{2} \gamma_s\right),$$

where the constant $c_{15} > 0$ depends on $a, d, \varepsilon$ and $c_{16} \geq \pi^{-4d} \omega_d a^{d-1} \left(\min \left\{a, \frac{\pi}{2} - a\right\}\right)^{d-1}$.

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