Rethinking Renormalization for Quantum Phase Transitions

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This is a conceptual paper that re-examines the principles underlying the application of renormalization methods to quantum phase transitions in the light of quantum information theory. We start by describing the intuitive argument known as the Kadanoff “block-spin” construction for spins fixed on a lattice and then outline some subsequent ideas by Wilson and White. We then reconstruct these concepts for quantum phase transitions from first principles. This new perspective offers some very natural explanations for some features of renormalization theory that had previously seemed rather mysterious, even contrived. It also offers some suggestions as to how we might modify renormalization methods to make them more successful. We then discuss some possible order parameters and a class of functionals that are analogues of the correlation length in such systems.

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INTRODUCTION

It is an established principle in statistical mechanics that the state of a system undergoing a phase transition is invariant on all length scales: its microscopic structure looks the same under successive changes of scale, and reductions in the number of degrees of freedom, known as “coarse-grainings”. This is the starting point for much of modern statistical mechanics.

A distinction needs to be made between the two types of phase transition with quantum features. There are quantum-to-classical phase transitions and quantum-to-quantum transitions. The latter are often referred to simply as “quantum phase transitions” and are the main focus of this paper. These are a particular kind of phase transition that has critical points at zero temperature. These systems are therefore confined to their ground states, which are usually assumed to be non-degenerate. The quantum-to quantum transition behaviour usually persists a little way into the finite temperature regime. When the system becomes so hot that the low-temperature behaviour breaks down, a quantum-to-classical transition will occur.

This paper is by no means the first time quantum information theorists have thought about the role of entanglement in quantum phase transitions, among them [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. See also [13, 14, 15]. In [16], Aharonov demonstrated the existence of a quantum-to-classical phase transition in noisy quantum computers. Since then a number of papers have used methods from statistical mechanics to understand these systems...
I: RENORMALIZATION REVISITED

In order to explain what we’re doing and why, we will go right back to the origins of renormalization theory in statistical mechanics. This had its origin in some observations in classical statistical mechanics. Consider a system that undergoes a phase transition at a certain temperature, such as a chain of spins in a ferromagnetic material. If we coarse-grain a system that is below the critical temperature, we find that the coarse-grained looks hotter. What happens in the middle? This is where the phase transition occurs. For this special temperature, the coarse-grained system appears to have exactly the same temperature after this averaging procedure; it is invariant under rescaling. It is this entirely phenomenological observation that originally inspired the application of renormalization theory to statistical mechanics.

I.1: The Kadanoff construction

The first attempt to formulate a microphysical theory to account for this observation was by L.P. Kadanoff. His insight began with the realisation that the fact that a system at criticality looks the same at all length scales means that its dynamics must also be self-similar. How can a system made of spins on a lattice do this? Kadanoff’s proposal was that small collections or “blocks” of spins must behave collectively like single spins. He called these emergent spins “block-spins” and suggested that these would also team up to behave like a single emergent spin, and so on. This hierarchical structure should continue over distances less than the correlation length of the system. Likewise, the effective Hamiltonian coupling these emergent spins should have the same functional form as that between the original phys-

so it is well known that these systems fall within the larger class of phase transitions with quantum features. The aim of this paper is rather different. We will attempt to use ideas from quantum information theory to develop a more general theory of quantum phase transitions.

It has been suggested that entanglement should play an analogous role to the connected spin-spin correlation function in these systems as any correlations in a pure state that cannot be accounted for in terms of shared expectation values must be due to entanglement. Therefore a system at a quantum phase transition should exhibit entanglement at all length scales.

It is worth noting that this plausibility argument tacitly assumes that the ground state of the system is non-degenerate; otherwise the fact that the system is in its ground state would not immediately imply that the state is pure, and thus the fact that a reduced density matrix is mixed need not imply the presence of entanglement. Nevertheless, we will assume that this argument is indeed correct and proceed to examine its consequences in section III, even though this lacuna will come back to haunt us later on.

Renormalization methods have been enormously successful in statistical mechanics, but have had some difficulties when applied to quantum phase transitions. This paper will re-examine the conceptual foundations that underlie the application of renormalization methods to quantum phase transitions in condensed matter physics. We will only consider systems of spins on lattices, where each lattice site is permanently occupied by a single spin. While we hope that the ideas in this paper will be useful in other systems, they are beyond the remit of this paper.

We will start by outlining the history of this approach, beginning with the Kadanoff picture. We then describe two alternative ways of thinking about renormalization. The starting point for both is the Kadanoff picture, which we revisit using ideas from quantum information theory.

The first approach explores the consequences of the conjecture that entanglement plays an analogous role in quantum phase transitions to that of correlation functions in classical phase transitions. This implies that the state of a system undergoing a quantum phase transition should exhibit entanglement at all length scales. This idea is intuitively fruitful, but it does not seem to be possible to use this to obtain quantitative results using existing techniques.

The second approach starts by considering the interactions in the system, and leads to a new way of thinking about numerical renormalization methods which can explain both their previous successes and failures. This suggests both novel strategies for modifying these numerical renormalization techniques and new physical insights into the fundamental principles underlying the application of renormalization to condensed-matter systems. We finish by discussing what this implies about the order parameters of the system and define a class of functionals that are possible candidates to replace the correlation function in such systems.

The only initial assumptions we will make about the Hamiltonian is that it is dominated by short-range interactions which involve a small number of spins (which may be more than two). It can be shown that pairwise interactions can generate four-spin interactions in physically reasonable systems. We must also assume that the Hamiltonian’s structure repeats on a scale that remains a constant as the infinite lattice limit is taken, as short-rangedness on its own is not enough to guarantee that the problem will be tractable.

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FIG. 1: A Kadanoff “blocking” for a square lattice. The original lattice (black) is divided up into blocks (green/grey squares) and a single, collective “block spin” is defined for each block (blue/black rings). A rescaled lattice of block spins is obtained (thick blue/black lines) which has the same geometry as the original lattice.

ical spins. So he divided up the lattice into little blocks of $L$ spins and postulated a recursive procedure (called a \textit{“block-spin transformation”}) to map the spins in each block into their single, collective spin.

These block-spins would in turn give rise to emergent spins, coupled by their own effective Hamiltonian, and so on \textit{ad infinitum}. Figure 1 contains a heuristic picture of this construction for a square lattice. Unfortunately, while his intuitive argument seemed to account for the Widom scaling laws [28], it had a number of features that made little physical sense.

\section*{I.2: Wilson renormalization}

Wilson reworked this idea in two papers in 1971 [29, 30]. He began the first of these two papers by describing the “Kadanoff picture” and pointing out why it didn’t make any sense,

\textit{“The idea that blocks of spins act as a unit near the critical temperature does not stand up to close examination; in fact only near zero temperature [...] is it true \ldots \ldots In short the Kadanoff block picture, although absurd, will be the basis for generalizations which are not absurd, and it is helpful to understand the Kadanoff picture in differential form \ldots”}

He makes this point even more strongly in the second paper:

“No justification has been found for the Kadanoff picture. The difficulty is that Kadanoff assumes that the effective spin variable $s'_m$ has only two values, up or down. In the exact Ising model a block of spins of size $L$ and $z^L$ configurations (where $d$ is the dimensionality of the system) and it is hard to see how to reduce the $z^L$ different configurations to just two. Because there has been no justification of the Kadanoff picture, it has been impossible to calculate specific critical exponents within the Kadanoff picture \ldots ”

He then extensively modified Kadanoff’s block-spin rescaling idea. Having just explained why it doesn’t work for integer values of $L$, he argued that Kadanoff’s picture should be interpreted as an analyticity assumption, and that this \textit{was} correct. He then proceeded to assume that Kadanoff’s picture not only worked when averaging over blocks of integer size (to obtain a new block of size $nL$ in the original lattice spacing) but that it also worked for $L \rightarrow (1 + \delta)L$. This enabled Wilson to write the Kadanoff-Widom scaling laws in differential form and obtain a well-defined renormalization procedure. This led to a method to calculate the critical exponents for the transition using the behaviour of the system when the renormalization procedure is iterated. In his paper “II” [30] he successfully applied this methodology to the Kondo problem. Note also that Wilson’s renormalization implicitly assumes a Kadanoff block-like structure in the vicinity of the critical point, as well as at the critical point itself.

The so-called \textit{“standard real-space RG approach”} [31] consists of dividing a lattice into a set of blocks (assumed to be the same) and diagonalizing each block locally to find a set of eigenstates. This set is then truncated to leave $m$ states which are deemed to be the “most important” (where $m$ is a parameter that we are free to choose) and the reduced set is used as an eigenbasis to construct an approximate Hamiltonian $H_B$ for a new, larger block, made by merging two of the original blocks. All other operators are likewise “renormalized” by projection onto this $m$-dimensional subspace.

The original version of this idea completely neglected interactions between the blocks. When subsequent attempts to use this technique on other systems got into difficulties, Wilson suggested using degenerate perturbation theory to try and take account of the interactions between blocks, but this modification didn’t seem to work very well [31].

\section*{I.3: The density matrix renormalization group}

Wilson’s numerical approach worked well for impurity problems, but not much else [32]. In 1992 - 93, White (and Noack) defined a family of alternative renormalization procedures [31, 32, 33], of which the most successful
have come to be known collectively as density matrix renormalization group methods (DMRG). All of these various approaches were attempts to take better account of the interaction of the block with its surroundings. This is the reason for choosing the basis for the rescaled system using the state instead of the Hamiltonian, as explained by White in [33]:

"...in reality the block is not isolated, the density matrix is not \( \exp(-\beta H_B) \) [...] and eigenstates of the block Hamiltonian are not eigenstates of the block's density matrix. For a system which is strongly coupled to the outside universe, it is much more appropriate to use the eigenstates of the density matrix to describe the system rather than the eigenstates of the system's Hamiltonian."

A rederivation of White’s procedure in terms of the Schmidt decomposition of density matrices and entanglement is given in [2]. The essential idea of the DMRG is to take two blocks and form a “superblock”, which is then diagonalized. If the superblock is in a pure state, this will simultaneously diagonalize the two subsystem (block) reduced density matrices. Then a basic result from linear algebra is used (by both White, and Osborne and Nielsen) to show that the optimal choice of states is to choose the \( m \) largest eigenvalues of the block density matrix.

Note that both these approaches think of the block-spin transformation used to rescale the system as an averaging process; Wilson treats it in terms of some kind of “mean” spin, whereas the DMRG is more like taking the “mode”. The leitmotif that unites all these approaches seems to be that the essential step in defining a workable renormalization procedure is to specify a block-spin transformation and be able to justify it.

**I.4: The renormalization group flow**

All of these methods require that the renormalization procedure is iterated. If we record the rescaled states and effective Hamiltonians at each step and plot these in their respective parameter spaces, we see some very distinctive patterns, such as those in figure 2. The motion of the state (or Hamiltonian) under iteration of the renormalization procedure is known as the renormalization group flow, or RNG flow.

A characteristic feature of the critical points under state renormalization is that they are associated with a set of states (often called the critical “manifold” or “surface”) which consists of a lower-dimensional set of neighbouring states which are drawn into the critical point under the renormalization group flow, in the limit as the number of iterations of the renormalization group tends to infinity. This manifold is represented by the thick line

![FIG. 2: A heuristic diagram of a renormalization group flow, which follows the arrows as shown. The points marked “S” are attractors, or “sink” points and can be thought of as local minima. Those marked “R” are repulsive, or completely unstable points, rather like maxima. The point marked “C” is a critical point. These are quasi-stable when approached in some special directions (thick line) and unstable from others, rather like saddle points, and are the focus of this paper. The thick line is the critical surface, or manifold.](image)

**I.5: Entanglement and quantum phase transitions**

In [2] the authors argued that the failures of existing renormalization schemes to find the fixed points corresponding to quantum phase transitions might be caused by the the fact that they either completely fail to take account of the role of entanglement, or do so only imperfectly. The authors then went on to propose a modification of the density matrix renormalization group (DMRG) to optimize the preservation of the entanglement of formation between successive steps of the renor-
malization procedure. They were unable to give a closed form for this procedure, because there is no known general closed form for the entanglement of formation. (Indeed, as the bipartite separability decision problem has since been proved to be NP-HARD [24], such a general closed form is unlikely to ever be found.)

This lack of closed forms has caused most previous papers that have examined the entanglement properties of specific systems to take at least one of two approaches. The first of these is to use the Wootters formula for the concurrence for two qubits [35] to calculate the entanglement of a single spin, as in [11, 12, 36]. The other approach is to look at the entanglement between blocks of spins, see for example [4]. The results in [11, 12] led the authors to suggest an alternative explanation for the difficulties encountered with the DMRG: they suggested that the cause of the convergence problems was that at criticality, there weren’t enough insignificant subspaces which could be safely discarded without disrupting the qualitative behaviour of the state.

II: THE QUANTUM KADANOFF CONSTRUCTION

We will now go right back to the first principles of renormalization and rework Kadanoff’s construction. We will begin by dividing the lattice up into blocks of $n$ spins each, and assume that the spins in the block are in some state that enables them to behave as a single, quantum spin. (This state might be entangled or it might not: this construction does not depend on entanglement.) For simplicity, we will assume that each block generates just a single spin; an assumption that we will justify later. We would like to find some procedure that would enable us to implement the transformation from the physical spins to the emergent block-spin.

Let $B$ be the transformation from the constituent spins to the block-spin, defined on a single block. The collective transformation for the entire lattice, $R$, is obtained by performing $B$ on all the blocks for some tiling over the entire lattice. I will use the superscript $(r)$ to denote the spins and block-spins of the system after this procedure has been iterated $r$ times.

The renormalization transformation is a way to change between the frame of reference of the physical spins, and that of the block spins, and its functional form should reflect that underlying dynamics. It may therefore be unwise to define a renormalization procedure a priori and then look for its fixed points. Thus we will not define an explicit $B, R$ at this stage, but instead attempt to find a natural way to determine what these transformations should be.

Let $\rho^{(r)}$ be the state of a single $r$th-order block, obtained by tracing out the rest of the lattice, and let

$$\chi_i^{(r)} = \text{tr}_{1,\ldots,i-1,i+1,\ldots,n}\rho^{(r)}$$

be the state of the $i$th single spin from that block. As $R$ is defined blockwise it will commute with the trace for subsystems consisting of intact blocks, so

$$\chi_i^{(r+1)} = B\rho^{(r)}$$

for some $i \in 1, \ldots, n$. The assumption that all the blocks look the same under the iteration of the procedure gives us that

$$\rho^{(r+1)} = \rho^{(r)}, \quad \chi^{(r+1)} = \chi^{(r)}$$

Where we allow that the local density matrix $\rho^{(r)}$ may be mixed for any finite $r$. The assumption that the state of the block spins after rescaling looks the same as the individual spins before rescaling can be written as

$$B\rho^{(r)} = \chi_i^{(r)} = \text{tr}_{1,\ldots,i-1,i+1,\ldots,n}\rho^{(r)},$$

where we have implicitly assumed a weak form of homogeneity: that the spins all have the same local density matrix.

It is common practice in statistical mechanics to include a projection operation, followed by multiplication by a scalar greater than 1 to “renormalize” the state as part of such coarse-graining procedures. This is rather disturbing to a quantum information theorist (see [25, 26], not least because there is a real risk that such “free” post-selection will alter the energy-density of the system, just as “collapse model” interpretations of quantum mechanics tend to have serious problems with energy conservation. I will take the position that decimation procedures should ultimately be interpreted passively: we are just choosing to ignore some degrees of freedom as being unimportant. I will therefore disallow free post-selection for the rest of this paper, as I consider it to be unnecessary and undesirable. I will assume that this map consists of some change of basis, followed by the tracing out of those degrees of freedom that we wish to ignore. It will sometimes prove to be convenient to use post-selection as a conceptual tool in some intermediate steps in this paper but it will ultimately be eliminated.

There are two defining features of the Kadanoff construction. One is the claim that each cluster of spins behaves like a single spin. In order to derive a fully quantum version of this idea we must incorporate a fundamental difference between a classical and a quantum spin. For a block to give rise to a classical spin, it need only support the eigenstates of that spin. A truly quantum spin requires more than this: the block must also support all linear combinations of those basis states as well. If the
The second feature is the assertion that the block-spin transformation is a recursive symmetry of the system. This assumption goes right back to the original phenomenological observation of scale invariance at a phase transition. Each spin, whether physical or emergent, will interact with its surroundings which it will perceive as its environment.

The two statements above together beg the question of whether the block spin is more or less strongly coupled to its environment than the constituent spins from which it is made? Clearly, the answer to this question must be, “sometimes more, sometimes less”. If the answer is “more”, then each block-spin will be subject to progressively more decoherence, until it ceases to be anything like a quantum spin, and the whole block structure is overwhelmed by disorder. This must be a sink point.

If the answer is “less”, something more interesting happens. Recall that we are discussing a fixed point of an iteration. So, if the first layer of block-spins is less strongly coupled to its environment, then the next layer must also be less strongly coupled in turn, by the same factor, or else we are not at a fixed point of the RNG flow. The only way adding a further layer of decoupling can leave the new block-spin’s interaction with its environment unchanged is if it was already completely decoupled from that environment.

Clearly, there must also be a boundary between these two basins of attraction, where the strength of coupling is exactly the same after each iteration. States exactly on this boundary can neither reach the perfectly decoupled point nor escape to the sink and might seem to be trapped in the boundary surface, but this midpoint must be unstable; any fluctuations will generally tip it into the basin of attraction for the sink point. Such fluctuations can either be thought of as a change in the state, or else as an additional term in the Hamiltonian, the effect will be the same. The fluctuations will act as an additional kick to the system, which it cannot tolerate. Thus the probability of finding the system in such a boundary state in the infinite lattice limit is zero. (Strictly speaking, it’s not impossible that such fluctuations could tip the system into the “perfectly decoupled” basin, it’s just very unlikely.) The continuum of possibilities available to finite sized systems has converged to just two in the infinite lattice limit. It can also be seen that if such fluctuations are sufficiently strong, they can overwhelm even the “perfectly decoupled” fixed point. Thus this fixed point can be only partially stable: it is a critical point.

III: ENTANGLEMENT ON ALL LENGTH SCALES?

In this section we will assume that the conjecture that systems undergoing quantum phase transitions exhibit entanglement “on all length scales” is correct and see where it leads us. For now, our goal is to build some intuition for the problem: we will make some of the ideas inspired by this more precise in a later section of this paper.

When trying to compare the role of entanglement in quantum phase transitions with that of classical correlation functions in more conventional phase transitions, it is important to note a few differences between the two. The quantitative study of entanglement arguably began with Bell’s theorem \( \text{[32]} \). This paper demonstrated a measurable difference between the predictions made by quantum mechanics and those made by a local hidden variables model. What is of interest for our purposes here is that these differences are only detectable when one compares the results from measurements made in more than one basis. If we restrict the allowed measurements to those in only one basis, a local hidden variables model can be constructed to account for what we see.

It has been noted elsewhere both that entanglement cannot be shared arbitrarily, unlike classical correlations \( \text{[40]} \) and that it is typically competitive. It should also be noted that there are many types of entanglement and in the case of multipartite entanglement, the number of different types grows exponentially in the number of parties \( \text{[41]} \). Furthermore, different types of entanglement cannot always be interconverted \( \text{[42]} \).

The only point we are trying to make at this stage is that we should not assume that the behaviour of the entanglement in a system will necessarily mimic the behaviour of the classical correlations in analogous-looking systems, or even the same system. In fact, a system has been found that exhibits a divergent “entanglement length” whilst having a finite correlation length \( \text{[5]} \). This should be a particular concern if we are using a renormalization procedure that includes a change of basis, such as the DMRG.

We will argue that it is possible to bypass some of the problems caused by the lack of tractable entanglement measures by thinking about these systems in terms of their information dynamics. We will begin by assuming the existence of some kind of renormalization procedure and that it will take the form of an iterated rescaling.

If a state is entangled with some larger system, it cannot be in a pure local state. (Whether or not the converse of this statement is also true depends on your preferred interpretation of quantum mechanics. The reader may regard thermal mixtures as being entangled with external field degrees of freedom if they wish; it will make no difference to the arguments in this paper.)

We will look at the entanglement within (mixed) sub-
systems, rather than between the subsystem and the whole. This approach has some similarities to the conclusion reached at the end of [2], but with an important difference: we are thinking in terms of the entanglement within a block which we are assuming to be in a mixed state. By contrast, the modified DMRG suggested in [2] is formulated in terms of the entanglement between the block and the “superblock”. There are various reasons for this modification. One of the problems with only looking at the entanglement between a block and the rest of the system is that it implicitly requires you to assume that the system is globally pure, as there is no way that one can distinguish between “proper” and “improper” mixtures by looking at the local state alone. This is likely to cause problems when we try to extend such analyses to systems at finite temperature. It also makes it a little difficult to talk about the scales over which entanglement is present in a well-defined way: entanglement between the block and the rest of the lattice is implicitly global.

If instead we think in terms of the entanglement within a block at each iteration, we are not neglecting the entanglement between blocks, as at the next step in the iteration at least some of that will become “internal” entanglement. We shall see later (in the interaction formulation of renormalization) that the entanglement between the block and its environment will have a vital role to play. But for the purposes of defining the entanglement properties of critical points, we claim it makes more sense to think in terms of the entanglement within the block at each stage of the iteration.

We will attempt to make sense of the phrase “the state has entanglement on all length scales”, in this context. In particular, we will allow for the possibility that this entanglement may be irreducibly multipartite. The lack of closed forms for mixed state multipartite entanglement will mean that we cannot quantify that entanglement, but we may still be able to answer at least some questions of interest. This will, of necessity, result in a difference in emphasis in the role attributed to entanglement (compared with that in [2, 11, 12]).

III.1: Entanglement and renormalization

We will begin by clarifying what we mean when we say “entanglement on all length scales.” The Kadanoff construction asserts that both the state and the Hamiltonian have a self-similar structure, and we have concluded that the critical points correspond to the case when the block-spins are more decoupled from their environment than their constituent spins. We will look for entanglement in the context of this situation, i.e., states which have entanglement properties that cause them to behave in this way. The inequivalence of different kinds of entanglement mean that the choice of block structure may matter for lattices of quantum spins. In other words, the way in which those blocks generate their respective block spins is not just an averaging procedure that can be chosen arbitrarily, it represents an essential feature of the dynamics of these systems that gives rise to that emergent spin.

We will try to find states with the right entanglement properties and let these lead us to the renormalization procedure and the Hamiltonian.

Suppose that these emergent spins are themselves entangled in a way that has the same form of entanglement as that for the physical qubits, and that the rescaling procedure can be repeated on these emergent spins to obtain another entangled system and so on, ad infinitum. Then the system must have finite subsystems on all scales that are entangled, where the notion of “scale” refers to the associated renormalization procedure. Thus the block-spin must be a non-local structure and the entanglement of these states will have a branching, modular structure.

This feature is remarkably like one way of characterizing quantum error-correcting codes. In [43], the authors give a number of ways of defining codes. Suppose the “error” (for which read, the difference in the state \( \rho' \) from the code state, \( \rho_c \)) is of the form

\[
\rho' = \mathcal{S}(\rho_c) = \sum_a A_a \rho_c A_a^\dagger, \tag{5}
\]

and let \( \mathcal{A} = \{A_i\} \) be the set of Kraus operators for the error superoperator \( \mathcal{S} \).

**Theorem 1. (Knill and Laflamme)**

[Theorem 3.5 in [43]]

The subspace \( \mathcal{C} \) of \( \mathcal{H} \) is an \( \mathcal{A} \)-correcting code iff there is an isomorphism \( \sigma : \mathcal{H} \cong \mathcal{C} \otimes \mathcal{E} \oplus \mathcal{D} \) such that for all \( A_a \in \mathcal{A} \) and \( |\Psi\rangle \in \mathcal{C} \), \( A_a |\Psi\rangle = \sigma(|\Psi\rangle \otimes |\mathcal{E}(a)\rangle) \) for some vector \( |\mathcal{E}(a)\rangle \) depending on \( A_a \) alone. (For a perfect quantum code, \( \mathcal{D} \) is empty, and the vectors \( |\mathcal{E}(a)\rangle \) span \( \mathcal{E} \).)

Note that although the original formulation of the theory of quantum error correction codes was formulated actively in terms of projective measurements followed by correction and decoding, the authors realised that what really matters is the isomorphism in theorem II and that the whole structure could be understood in terms of passive transformations.

We can now write down some states with the requisite entanglement properties, provided we can solve a packing problem that depends only on the geometry of the lattice. In [12], the authors showed how to construct entangled states whose entanglement has precisely this branching property. These states are concatenated quantum error correction codes. A quantum error correction code is a highly entangled state that protects a quantum state from local noise. In a single encoding, the logical qubit that we wish to protect is encoded into several qubits, to form a codeblock. This encoding can then be concatenated: each constituent spin is itself encoded in its own
personal code block. This structure can then be repeated as many times as we like.

Now consider this construction in reverse, from the lowest level of encoding to the highest. Suppose that the lowest level of the encoding uses the physical qubits of the lattice. If we arrange the qubits of a codeblock into a tile, we can tile the lattice with them, and the encoded logical qubits of the error-correction code will be the block spins. When we repeat the block-spin construction (concatenate the encoding) the logical qubits of the next level of encoding will be the new block spins, and so on. If we specify that each level of encoding is performed using the same code, then we will obtain a state with precisely the kind of modular entanglement structure we were looking for. The corresponding renormalization procedure will just be the decoding operation for that code, which will leave the block in the state

$$B\rho^{(r)} = \chi^{(r+1)} \otimes |00\ldots0\rangle\langle 00\ldots0|$$

and the degrees of freedom that we discard will be the \(n-1\) “ancilla” spins.

Remark: It could be objected that equation (6) doesn’t tell us very much, as this statement will be true for almost any choice of unitary transformation if the state is locally pure. This is a quantum version of one of Wilson’s objections to the Kadanoff picture, and it would be particularly apt if we were to assume that the state of the block is locally pure. We are not making this assumption, but this is still an issue and we will return to it later in this paper. For now, we will call this the “arbitrary basis change problem” and leave it open, though as we shall see in section IV of this paper, it is solved “for free” in the interaction formulation in section IV.

III.2: A toy example

To see how this works, here is a toy example. It is a rather unphysical one, as the state is not translation invariant (amongst other things) so it should not be taken too seriously. However, it does have a particularly transparent renormalization procedure and so should serve to help develop our intuition. Figure 3 is an exact covering of the two-dimensional square lattice using the 5-qubit perfect code from [45, 46]. This is an example of an additive code. These codes are characterized in terms of their stabilizers; the actual choice of code-space is fundamentally arbitrary. Let us fix our choice of logical basis states for this code to be the one in [47] and then we can write:

$$|0_L\rangle = \frac{1}{4}[(00000) + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle - |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle]$$

The logical \(|1_L\rangle\) can be obtained from \(|0_L\rangle\) by interchanging the 1’s and 0’s. This stabilizer code uses five qubits to protect one logical qubit. The corresponding renormalization procedure is defined in terms of blocks of five spins, which behave like one spin.

Some remarks about tilings...

Not every code will work for every lattice system, because the packing problem is non-trivial for lattices of dimension two or higher. We can embed the code in tile(s) of any shape, so long as we use the same shape(s) for the whole lattice. We must also obtain a new lattice that differs from the old one by only a scale factor (and possibly a rotation) when we perform the renormalization. The 5-qubit code tiling in this example is also handed: there are two different ways of putting the tiles together, one of which generates a renormalized lattice that is rotated by + arctan(1/2) and the other by − arctan(1/2).

Whether or not a given tiling generates a rescaled lattice is a property of the tiling alone. Note that for our
5-qubit example, the choice of tiling is degenerate. This is one of the reasons why this example is probably unphysical: it is not translation invariant. In fact there are some codes which are translation invariant (under integer multiples of the lattice spacing) but their block-spin structure is not obvious, which is why I did not use them as a running example. We will look at these in a later section, and show that they can be described in this way. There is also no particular reason to believe that those translationally invariant codes are the most general class of states with these properties, because they are also exact codes. Exact codes have much stronger convergence properties than are required, and are therefore somewhat over-engineered as critical points.

A more realistic choice of code might be some kind of an approximate quantum code 48, 49, which are only required to protect the original state with high fidelity at each encoding, instead of exactly. Unfortunately, we do not yet know necessary and sufficient conditions for an encoding to be an approximate error-correction code; we only know what these conditions are for exact codes. This gap in our knowledge is not just a problem when we try to write down states: it also makes it difficult to discuss the renormalization group flow for these systems. Therefore we will return to our rather unphysical, exact concatenated code example for the rest of this section and return to this question in section IV.

This code-tiling construction is highly non-unique. The same code can also be used in to tile this lattice in a 5 × 1 brick tiling, as shown in figure 4. Note that this can be chosen to produce exactly the same rescaled lattice as the tiling in figure 3. It is also possible to tile the lattice in figure 3 using other codes, such as Shor’s 9-qubit code 50. The same is true for other lattices as well: a triangular lattice can be tiled using the Steane 7-qubit code 51, or other 7-qubit codes 52.

The renormalization flow for the state

Theorem 1 implies that any state that differs from the concatenated code state by the equivalent of a correctable error moves to the fixed point under the corresponding renormalization: the effects of the local physics on these states is confined to a subspace. The detect-and-correct parts of the protocol merely return the system to some standard form for our convenience when reading out the state. This is essentially a relabelling, and the physics of the system should not depend on our choice of basis labels for the block-spins. As such, we can arguably interpret the measurement and conditional operations passively, as something that we bring to the system as human beings trying to analyse it, rather than an active interpretation in terms of something that we would need to actually do to the system.

In a concatenated code, if a small number of lower-level codeblocks fail, then the larger encoding can still cope but there is a limit to what errors it can correct, even if we are allowed to use arbitrarily many levels of encoding. This “failure” can be interpreted as the boundary of the basin of attraction for this critical point under our renormalization procedure. States just inside the boundary will eventually reach the code state (critical point) under the error-correction procedure. States just outside that boundary will move off towards some other fixed point because the error-correction procedure is unable to move them towards the code state.

It can now be seen that the difference between a “correctable” and a “fatal” error in the concatenated error correcting code will be closely related to the difference between the “irrelevant” and “relevant” parameters under the renormalization group flow. (Errors that the code can only detect must count as failures.) Eventually, any state that differs from the fixed point state by such a relevant parameter will be driven away under the action of the renormalization group. States exactly on the boundary of the basin of attraction can neither escape nor reach the critical point and must therefore be trapped in the boundary surface, until some fluctuation enables them to escape. For as long as they are trapped, they may follow exotic orbits confined to that surface.

Remark: The best codes for this kind of construction are those that encode only a single spin, because multi-spin codes are generally poor concatenation candidates and would therefore correspond to critical points that are...
so unstable that we can ignore them. This is the reason why we can assume that each block generates a single spin without loss of generality.

**The effective Hamiltonian and the RNG flow**

The code-tiling intuition also suggests a class of candidate effective Hamiltonians for the entangled fixed points. The system is in the ground state of its Hamiltonian by assumption, so we are looking for an operator for which these critical point states are eigenstates. On its own, that cannot tell us very much about these Hamiltonians. However, if the system is going to exhibit the desired flow behaviour under renormalization, its excited states must have certain properties as well.

Every error-correction code has a set of error-detection “check” operators associated with it. These are a minimal generating set for the stabilizer of the code states. For example, here are the four stabilizer generators for the 5-qubit code we have used as our toy example in this paper:

\[
\begin{array}{c|ccccc}
1 & 2 & 3 & 4 & 5 \\
\hline
M_1 & \sigma_z \otimes \sigma_z \otimes \sigma_x \otimes \mathbb{I} \otimes \sigma_x \\
M_2 & \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x \otimes \mathbb{I} \\
M_3 & \mathbb{I} \otimes \sigma_z \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \\
M_4 & \sigma_x \otimes \mathbb{I} \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z
\end{array}
\]

The code space is a pair of common eigenstates of all of these operators, with eigenvalue +1 (by convention). These operators can detect single qubit errors by projection onto the error basis states. States for which a basis error has occurred are also eigenstates of these operators, but at least one of the stabilizer generators will return an eigenvalue with the opposite sign if such an error has occurred. The stabilizer operators all intercommute, so we can simply add the generators, as pointed out by Kitaev [17].

Suppose our system has a Hamiltonian $H'$, which is close to the (as yet incompletely specified) critical point Hamiltonian, but not identical to it. This Hamiltonian can be written as

\[ H' = H_C + \varepsilon H_E, \]

where $H_C$ is the critical point Hamiltonian, and that $\varepsilon H_E$ is some small perturbation. Our critical point Hamiltonian must have the property that at least some nearby Hamiltonians move towards it under the action of the renormalization procedure that we had previously obtained for the state.

If we revert to thinking about our renormalization procedure in active terms for a moment, then it is possible to convince oneself that the process of active error-correction is equivalent to thinking about a perturbation expansion in active terms: instead of just projecting onto the eigenbasis of the unperturbed Hamiltonian as a matter of mathematical convenience, we instead think of actually performing the corresponding projective measurement. So, the perturbative regime for this Hamiltonian will correspond to the regime where the “correction term” generates only correctable “errors”, provided that the eigenbasis of the Hamiltonian coincides with that of the check operators for the code.

For example, consider the operator

\[ T = -K_1 M_1 - K_2 M_2 - K_3 M_3 - K_4 M_4 \] (9)

where the $K_i$’s are real and positive coupling constants. (The minus sign is needed to ensure that the code space corresponds to the ground states.) We will call this operator a “tile-Hamiltonian”, because it only includes interactions between spins within the tile. Therefore it is not the Hamiltonian for the entire system as it does not include the interactions between tiles. These $M_i$’s in equation (9) are a minimal generating set, so they do not uniquely define the functional form of the tile-Hamiltonian for the 5-qubit code; any candidate tile-Hamiltonian which contained a complete generating set would do just as well.

We do not yet have an effective Hamiltonian for the system as a whole, because the tile-Hamiltonian obtained from the stabilizer generators is one which has the singly encoded state as (degenerate) ground state. The codewords of the full concatenated code will always appear as rank 2 mixtures in any finite subsystem consisting of intact tiles.

In fact we cannot naively write down a closed-system Hamiltonian for this system. The usual problems with infinite tensor products will prevent us from writing down a global Hamiltonian, even if we didn’t mind writing down an expression of that size. We also cannot just look at a subsystem in isolation because the dynamics of these systems isn’t closed; the system is always interacting with larger systems in long-range many-body interactions. Such open systems do not have a local closed-system Hamiltonian, they have an evolution superoperator. However, we may still be able to construct effective Hamiltonians for these systems, using techniques from the theory of the dynamics of open quantum systems [54].

There is a result due to Kitaev, in [17, 54], in which he shows that if the perturbation term is sufficiently weak and sufficiently more localized than the stabilizer generators, then when the state drops back into the ground state, it will do so into the “correct” ground state (critical point) instead of the “wrong” one (which would eventually cause the system to be overwhelmed by errors and therefore approach some noisy sink point). In other words, any state that differs from the code state by a correctable error will be in some linear combination of the ground (code) states, and some excited states. These eigenstates will themselves be eigenstates of some Hamiltonian, which will differ from the code Hamiltonian by...
a perturbation term. If the “error” is correctable, that perturbation must be weak (compared with the couplings that generate the code-block behaviour) and it must be only weakly nonlocal.

To obtain the evolution operator for the whole system, we would have to concatenate the tile-Hamiltonians and add correction terms to take account of the interactions between logical qubits of all orders. In order to construct these, we will need the encoded operations for this code. These have the effect of performing $\sigma_x$ and $\sigma_z$ on the encoded logical qubit, and are usually denoted $X$ and $Z$ respectively. (The bar notation is used to denote encoded qubits, and the Pauli operators are often abbreviated to “$X$” and “$Z$” to minimize notational clutter.) For the 5-qubit code, these are usually chosen to be

$$X = \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$$

(10)

$$Z = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z,$$

(11)

but the degeneracy of the concatenated code means that they could also be chosen to be

$$\overline{X} = -\sigma_z \otimes \sigma_x \otimes \sigma_x \otimes 1 \otimes 1 = M_1 M_2 X$$

(12)

$$\overline{Z} = -1 \otimes \sigma_z \otimes 1 \otimes \sigma_x \otimes \sigma_x = M_1 M_2 M_4 Z.$$  

(13)

The stabilizer generators for the encoded qubits and hence the correction terms must be constructed from these, to all orders ad infinitum, with no guarantee that we will ever obtain a closed form for a single block. All we can say is that by an extension of the earlier perturbation argument, it can be seen that longer-range terms must have correspondingly weaker coupling constants. That still leaves considerable ambiguity in the effective Hamiltonian, in both its functional form and the values of its coupling constants. This ambiguity extends to the tile-Hamiltonian and it can be seen that this is a consequence of the fact that the stabilizer for this code is abelian. (We will return to this point in section IV. For now, we will just add it to the list of problems with this example.)

**Remark:** There are two distinct meanings of the phrase “effective Hamiltonian” in common use. One term originates in statistical mechanics; this is the Hamiltonian of the rescaled system obtained by renormalization. The other meaning of this term is from the theory of open quantum systems, sometimes called “Quantum Operations theory” The above example does not really constitute a rigorous argument. By now, anyone reading this in a critical manner will have noticed a number of problems with our toy example. This is as far as our toy example can take us and these problems must now be addressed.

The first and most obvious problem is that there are a very large number of codes and tilings that would seem to work for a given lattice. Once a given tiling has been chosen, the fact that it has a period longer than a single lattice spacing means that the choice of position of the tiling is degenerate. There is also the fact that for any given code and tiling combination, there are typically several inequivalent ways of embedding the code by assigning the qubit labels to the various lattice sites in each tile. This choice can affect how non-local the associated tile-Hamiltonian will be. How do we choose which of these many possibilities is the right one for the system we are interested in?

The second problem is that the Hamiltonians associated with any particular choice of code, tiling and embedding seem to be intrinsically non-local. Certainly, this is the case for our 5-qubit code example, and this may well be the case whenever we use a code that is non-degenerate when singly encoded. (Concatenated stabilizer codes are always degenerate, even if the original code is non-degenerate.) There are irreducibly non-local terms in the effective Hamiltonian that do not seem to be decomposable into short-range, constant terms. This is not just an aesthetic problem that makes it difficult to write down the effective Hamiltonian in a tidy closed form. It implies the presence of interactions that are intrinsically long-range, which would seem to be unphysical.

There is also no apparent connection to the native Hamiltonian of the system. So far we have started with the state and then written down an associated Hamiltonian. We would like to be able to start with the Hamiltonian and try to find out about its ground state. This is also more natural from the point of view of quantum field theory, where we are accustomed to thinking in terms of renormalizing the coupling constants in the Hamiltonian, whereas this approach just seems to leave these as essentially free parameters. There is even some considerable ambiguity about the functional form of the Hamiltonian (the “abelian stabilizer ambiguity”).

Lastly there is the arbitrary basis change problem, mentioned at the end of subsection III.1. We could argue that the requirement that short-range physics washes out under the renormalization group flow would force us to choose such an encoding but as this is what we are trying to show, such an argument would be circular.
III.4: Conclusions from this approach

So, what can we say for this approach? There would seem to be good reasons to suspect a link between the stabilizer formalism and renormalization. The formalism even seems to produce gauge-like symmetries naturally, because the choice of basis labelling for the encoded spin is arbitrary, but must nevertheless be made; if we were simply to average over all the possibilities, the effect would be equivalent to a fatal error syndrome and would destroy the very structures we were looking for.

However, all this is still contingent upon the unproven conjecture that systems undergoing quantum phase transitions exhibit entanglement on all length scales. The only plausibility argument for this explicitly invokes a pure and non-degenerate ground state, whereas all these code-tiling ansatzen have highly degenerate ground states. These would be locally (and could also be globally) mixed at zero temperature. So, it is still not unreasonable to disbelieve the initial assumption on which the entire argument in this section is based.

Even if the entanglement conjecture is correct, there are also many conceptual problems with the intuitive picture we have drawn from it, as we have just detailed. Fortunately, the quantum Kadanoff construction in the previous section (II) is logically independent of this conjecture. In order to resolve these problems, we must now return to this foundation and make a shift in our perspective once again.

IV: THE INTERACTION-BASED APPROACH

We now go right back to the beginning again, to the quantum Kadanoff construction. We will also recall White’s insight that we should think in terms of the interaction between the block and the surrounding lattice but we will make some important modifications. Instead of trying to estimate the block-spin and then correct for the boundary conditions, we will instead show that the Kadanoff block-spin construction can be understood entirely as a manifestation of the interaction between the block and the rest of the lattice, up to a (possible) gauge-like freedom in labelling the bases for the block-spins.

As it happens, the solution to all the problems in the previous section is a very simple one: let the system’s dynamics choose the code. When QECCs were first developed they were thought of in terms of active, external intervention in the system, designed to drive it in the direction we want it to go. But in this situation, there is no outside intervention by assumption, so it would make more sense to look at what the system is already doing on its own.

For any readers who are not quantum information theorists, this will mean using the theory of decoherence-free subspaces (or, more generally, decoherence-free subsystems). This alternative approach to fault-tolerance was invented by different groups of people to those who invented QECCs. These constructions were originally thought of in different terms, and so were given different names; the term “decoherence-free subspaces” was first proposed in [54]. The reason for the apparent difference is that while QECCs assume that the error syndrome consists of independent, single-spin errors, decoherence-free subspaces originally assumed a collective error syndrome that affected all the spins in the same way. It was only later that it was realized that the two approaches to fault-tolerance could both be understood in terms of the stabilizer formalism [57] and that decoherence-free subspaces were in fact highly degenerate QECCs in the sense of [58]. In between these two extremes we have the theory of decoherence-free sub-systems, which were first introduced in [59], and their connection to the stabilizer formalism in terms of non-abelian stabilizer groups was outlined in [59].

We will follow the standard practice of using the acronym “DFS” to denote both decoherence-free subspaces and subsystems, when no confusion will result. Otherwise, we will call these DFS-spaces and DFS-systems, as appropriate. There are a number of very good introductions to the theory of fault-tolerant quantum computing using decoherence-free subspaces in the literature (see, for example, [60]). For an in-depth introduction into DFS theory focussing on their uses as quantum memories (as distinct from fault-tolerant quantum computation) and a much more detailed set of references, see [61].

It should be noted that DFSs are a property of atypical Hamiltonians: there needs to be at least an approximate symmetry in the interaction between the system and its environment for them to occur. This is as it should be, as phase-transitions are atypical behaviour. (This counters the “arbitrary basis change” objection, above.) DFS-spaces are also robust against sufficiently small perturbations [52]. Recall that such robustness is an essential feature if we are to obtain the RNG flow behaviour characteristic of a critical point. (It is also worth noting that the fact that DFS-spaces are highly degenerate QECCs would be consistent with our earlier suspicion that one of the problems with our toy example was that we had used a non-degenerate code and therefore found ourselves dealing with a massively non-local effective Hamiltonian.)

Theorem 2. (Knill, Laflamme and Viola) (Theorem 5 from [58])

Let $A$ be a $\dagger$-closed algebra of operators on the state space $S$, including the identity. Recall that the commutant $Z(A)$ is the space of all operators commuting with
\[ \mathcal{S} \sim \sum_i C_i \otimes Z_i, \]  
(14)

in such a way that in the representation on the righthand-side,

\[ A = \sum_i \text{Mat}(C_i) \otimes I(Z_i) \]  
(15)

and the commutant of \( A \) is given by

\[ Z(A) = \sum_i I(C_i) \otimes \text{Mat}(Z_i) \]  
(16)

where \( \text{Mat}(\mathcal{H}) \) denotes the set of all linear operators from \( \mathcal{H} \) to itself.

Necessary and sufficient conditions for a DFS-space to be present in a system were found in [57] and for a DFS-system in [62]. These results draw heavily on the representation theory of operator algebras, see [63]. See also [58, 64, 65, 66].

\section*{IV.1: The role of the block-lattice interaction}

In the conventional picture of renormalization, the block-spin is regarded as some kind of average property of the block, with the interactions between the block and the lattice being accounted for as something separate; as boundary effects that force us to adjust the construction as we go along [58]. By contrast, the usual explanation of renormalization offered in the quantum information literature is in terms of the entanglement, as in [2].

The central claim of this paper is that we must incorporate both these perspectives, but go beyond them. We claim that theorem 2 is the algebraic principle underlying the Kadanoff block-spin transformation. The justification we are proposing for the Kadanoff construction is that it is the interaction between the block and the rest of the lattice which defines the block spin in the first place. This structure can only arise when the block-lattice interaction satisfies a suitable symmetry sufficiently closely to meet the conditions in [62]. For a critical point to occur, there must be a hierarchical tiling of the lattice into blocks such that the blocks meet this condition. For physically reasonable systems where the Hamiltonian consists of short-range interactions, these states will typically be highly entangled: to hide a spin from local interactions, secrete it in a non-local degree of freedom. Thus we may deduce that their entanglement will have the expected fractal structure, with entanglement on all length scales.

In this picture the block-lattice interaction chooses the tiling, the embedding, the code, the block-spin transformation and the equivalent of the value of \( m \), the number of states we can keep at each renormalization step. There will most likely be little or no freedom to choose these parameters, thus answering what we consider to be the most fundamental of Wilson’s objections to the Kadanoff construction: why that particular transformation? The only non-trivial freedom we will have left is in how we choose to label the basis states for the block-spin subsystem, which we will interpret as a type of gauge symmetry.

\section*{V: HOW MIGHT WE IMPLEMENT THIS IN PRACTICE?}

We will now outline how we might use these ideas in practice. For simplicity, consider a one-dimensional spin-chain, with your favourite local, periodic Hamiltonian. The procedure will have two main parts. First of all, we must determine our renormalization procedure; we must find the appropriate block-spin transformation for this system. We should then be able to use this in the conventional way to determine the critical exponents for the system.

\subsection*{V.1: Determining the block-spin transformation}

Begin by choosing a trial block. (As this is a 1-dimensional system, the tiling problem is trivial.) It is probably best to start with a three-spin block. The smallest-known DFS-space needs four spins [67] and the smallest-known DFS-system requires three spins [58]. Look for a DFS in this system that encodes a single spin.

There are at least two potential difficulties here. First of all, the conditions for an exact DFS are only satisfied on lower-dimensional sets in parameter space. Fortunately, these structures are very robust against perturbation [56, 62]. That still leaves the problem of how we should determine whether we are in “sufficiently close” to satisfying the DFS conditions to be in the stability regime for a DFS. Unlike conventional renormalization, we don’t start out knowing what our renormalization procedure should be: we have to find it. For the time being at least, we will need to know approximately where the critical point is located in parameter space.

This statement immediately implies that we will need to devise some form of search procedure. Such a search procedure implicitly relies on the existence of some kind of distance measure. How close would we need to get to the basin of attraction of a critical point before we could detect its presence? The structure of this basin of attraction suggests that this should be possible. The framework in [56] implies that we should be able to apply Kitaev’s result from [17, 54] to this problem. Together they imply that the basin of attraction will have a layered structure. This is because both conventional QECCs and...
DFSs can correct errors in two different ways. The first way is for trivial errors that consist of operations in the stabilizer. The second is the active correction performed by recovery operators [58, 59]. Conventional QECCs rely primarily on active correction, but can be thought of as DFSs in their passive regime. DFSs rely primarily on passive correction, but one can still define an active recover operator for these if required.

The innermost layer is where the DFS behaves as an “error-avoiding” code: the errors (differences in the Hamiltonian from the true fixed point) are trivial because they are in the stabilizer of the state and so have no effect. Then there is the “active correction” regime, where the DFS can be understood as behaving as an active QECC, so we can still define a recovery operation and hence a renormalization procedure. This single-encoding error structure could then be concatenated, as per Kitaev’s theorem. Error terms that might be fatal for a single encoding may end up in the stabilizer of the concatenated code or the active correction regime, and so on. The renormalization would still be able to find the fixed point, if we only knew what the renormalization was.

So, we are still left with something of a chicken-and-egg problem. This is one of the reasons why this outline does not constitute a precise description of how to implement this procedure. For the time being at least, we would need to know where the critical point is located in order to be able to use this procedure at all. The problem of determining how close is close enough will have to await further work; we anticipate that it will require a generalization of the results in [17, 54, 55, 58, 62].

The second problem is caused by the fact that the existence proofs for DFSs in [57, 60] depend on the Kraus Representation Theorem (see [68], or alternatively, [69, 70]). This result is known to break down for the case of open system evolutions when there is prior entanglement between the system and its environment [71]. There is also no known way to make the Kraus representation theorem fully Lorentz covariant [72]. In a realistic system, there will typically be entanglement between the block and the rest of the lattice, indeed if we believe the conjecture that systems undergoing quantum phase transitions exhibit entanglement on all length scales, we should expect this.

A quick and dirty solution to this problem is simply to ignore this prior entanglement, proceed assuming that the Kraus representation theorem is still applicable, and treat any effects caused by the entanglement between the block and the rest of the lattice as correction terms.

Fortunately, a much more sophisticated approach is possible that bypasses the difficulties the Kraus representation theorem has with block-lattice entanglement. In fact, this was implicit in [58]. Working in terms of the error operators directly instead of the states removes the need to trace out the bath, and thus avoids the need to assume that the system and the bath are initially decoupled [73]. These ideas are developed further in [74]. Indeed, there is an explicit algorithm for computing the structure of the interaction algebra in [62].

Suppose we fail to find a DFS at the first attempt. If so, the next step should be to abandon this trial block and try again with a new, larger, block by including another spin. (If we were working with a higher-dimensional system we should also be prepared to search over different concatenated tilings.) If we started this procedure sufficiently close to a critical point, then this search method should eventually find a DFS. In fact, DFSs occur in families in systems which exhibit them, but in practice only one DFS can be used, as there is a loss of phase coherence between the different DFSs in the system. We can invoke the arguments in [58] and define the renormalization procedure to be the data-recovery superoperator as before, but in order to define that, we must choose which subspace (or subsystem) to be the codespace. With DFS-spaces, this is generally unproblematic; there is a natural, obvious choice. With DFS-systems however, this may not be the case; sometimes there is more than one reasonable candidate. If so, we might need to repeat the next step a few times with different choices for the encoded block-spin.

We now keep our newly decoded block-spin, and trace out the decoherence-full subsystems. This is the block-spin transformation needed for the Kadanoff construction. Note that the transformation is unique, up to a possible choice of basis-labelling of the block-spin (and apart from some mathematical ambiguities in codespace choice that should be resolved in time with further work). Note that we have made almost no choices here; the block-spin transformation is entirely dictated by the dynamics of the system.

We can now invoke the periodicity of the system dynamics to rescale the entire lattice in the same way; any DFS structures we found should be congruent with the symmetry properties of the whole-system dynamics. This decode and discard step completes the first iteration and leaves us with a rescaled state and lattice. We can then proceed to the next step.

Suppose for a moment that we had found our DFS by assuming there was no initial entanglement present, and using the test in [57] or [60]. The correction terms inherited from our assumption that there was no prior entanglement between the block and the rest of the lattice would then become the rescaled interaction terms. A similar rescaling should also be produced by the approach in [58, 72], although this is rather less obvious in the machinery of this construction. Note that we have also obtained a rescaled effective Hamiltonian by this procedure, complete with rescaled coupling constants. This is much more informative than the approach described in section III where these were left as free parameters.
We now search for a new DFS in the rescaled system. At each step we need to record the DFS, the effective Hamiltonian and the corresponding recovery operation. In the original formulation of the Kadanoff construction, the system's dynamics are exactly scale invariant and the fixed points of the transformation were assumed to be strictly pointlike. If we insist on that property, then there is no obvious guarantee that the DFS from the next iteration will have the same form, but other limit cycles are possible besides pointlike ones.

V.2: Moving away from the fixed point

If we are only interested in the structure in the immediate vicinity of the fixed point, then we would be done. If we want to use our renormalization procedure in a more conventional way, we need to move on to the next stage. The conditions for the existence of a DFS are only satisfied for atypical Hamiltonians that exhibit a particular symmetry property. How can we apply this construction to the generic case, which occurs precisely when these conditions are not satisfied? Recall that Wilson's renormalization method tacitly assumed a block-spin structure in the vicinity of the critical point, as well as at the fixed point itself. Under these conditions, the iteration flows away from the critical fixed point, into an attractor or sink point. Now that we have found the concatenated DFS for the critical point, we can invoke the unified stabilizer picture in [59] again and consider the renormalization as an explicit code recovery super-operator. The point of doing this is that we can always implement an explicit recovery super-operator on a state of the right size, even if that state is not within the stability region surrounding a concatenated-DFS critical point. We can still perform the recovery operator concatenation, and it will converge on something, but not the critical point. If we attempt to implement a concatenated coding scheme on a system that is outside the fault-tolerance threshold for that scheme, further layers of encoding will make the state more noisy, not less so. In terms of a quantum computer, this will correspond to a catastrophic failure of the code and the notional “computer” will undergo a runaway error syndrome, thus moving the system further and further away from the critical point, as required. This superoperator concatenation can be used to renormalize any observables in the obvious way.

V.3: A few remarks

This is not a complete specification by any means, but it could provide a useful framework for future investigations [72]. A fully worked out algorithm would need to contain a proper search routine, and that will require the development of an appropriate distance measure. This might need to be the entanglement fidelity [76] as this is the appropriate measure for quantifying the effectiveness of a quantum error correction scheme [77]. Having said that, provided the code performs sufficiently well for all pure states, the simple fidelity will do [43]. Likewise, a fully Lorentz-covariant version of the Kraus Representation Theorem would greatly improve this method.

There seems to be no particular reason to assume that only one DFS concatenation will be found. If these occur at slightly different positions in the parameter space, these would constitute a fine structure to the renormalization group flow. If they happen at the same values of the system parameters, they should reflect some kind of equivalence class structure among concatenated DFSs; either that, or the would seem to be some ambiguity in the RNG flow defined by this procedure.

The procedure outlined above employs a concatenated tiling renormalization. A more conventional “progressive addition” method would add a single new spin to the original block at each step, and the next step in the renormalization uses essentially the same block. The standard DMRG is an example of a progressive addition renormalization. We believe that this modification may not be benign, in the sense that they may not be equivalent to a concatenated tiling renormalization and so should not be used interchangeably. However, that is not to say that such addition renormalization schemes are incorrect, but rather that they correspond to a different encoding structure, which we discuss further in the Appendix.

Lastly, I would like to highlight an important qualitative difference between this renormalization scheme and the toy example in section III. One of the objections to this example made in subsection III.3 is that it seems to require irreducibly non-local interactions in its associated Hamiltonian and that these are unphysical. Contrast this with the way we obtained our rescaled Hamiltonian in the procedure we have just outlined above. Here the non-local terms arise entirely as residual correction terms from the blocking; the native Hamiltonian is actually local. Thus the effectively non-local interactions that emerge under iteration of this renormalization are entirely mediated by the discarded subsystems we have chosen to ignore in our renormalization. No intrinsically long-range interactions are required in the native Hamiltonian at all.

V.4: Relationship to the DMRG

Although the thinking behind this renormalization procedure looks rather different from that behind the DMRG the previous methodology is consistent with it, where it works. Those situations where it does seem to
make sense to think of the block-spin as an average of the physical spins can be understood in the coding picture; in this case, the code is what is known as a “naïve repetition” code, and the decoding operation is simply to take the majority vote. Thinking in terms of a code also makes the otherwise rather artificial rescaling of the spin variable seem more natural as well. (It is interesting to note that the direct quantum analogues of naïve repetition codes do not exist; they are forbidden by the No-Cloning Theorem.)

In fact the connection can be made even closer than that, as we can incorporate excited states as well. Note that in the 5-qubit code example in section III, the recovery operation is one of the transformations that will exactly locally diagonalize this state, and that in order for the recovery operator to work, the excited states of the system must also satisfy certain properties. A caveat is needed at this point: as the ground state is degenerate in this example, the exact diagonalization transformation is non-unique. Note that if we just chose one of the diagonalizing transformations at random each time, the effect of this would be to induce a random error in each encoded qubit on decoding, which is an error syndrome this coding scheme cannot withstand; the RNG flow would therefore not converge on the critical point but rather on some noisy sink point.

The relationship between the two approaches will not always be so close. In this paper we have argued that the principle underlying renormalization can be understood as a hidden, non-local structure in the local density matrix and that the goal of a renormalization step is to find that structure and pick out some factors in that tensor product, while discarding others. Compare this with the way the DMRG can be understood in terms of the ordered Schmidt decomposition. It is already known in quantum information theory that strange things can happen when the ordered Schmidt decomposition is applied to tensor products, albeit from a completely different context.

Sometimes the sorting operation preserves the tensor product structure, sometimes it doesn’t; it depends on relative sizes of the gaps between the eigenvalues in the various subsystems. So for example, if one subsystem was degenerate, with its eigenvalues taking the almost same values, while the rest of the system had a strongly non-degenerate eigenspectrum, then provided the system with the nearly degenerate eigenspectrum was the “correct” one to keep, then the DMRG should pick it out correctly. Systems like this should exhibit strongly stepped eigenspectra. But if the sizes of the gaps between the eigenvalues are roughly comparable, things can get very messy indeed. In general, there is no simple way to reconstruct the tensor product structure of a combined system from its ordered Schmidt decomposition alone and this is likely to be a particular problem with highly degenerate systems.

VI: ENTANGLEMENT.

VI.1: The trouble with entanglement measures

A number of authors have suggested using measures of multipartite entanglement for mixed states to try to understand these systems, in direct analogy to the spin-spin correlation function. This approach has been rather hamstrung by the lack of closed forms for these quantities, but even if we did have a full set of closed forms for multipartite entanglement measures, this approach would still have some serious problems. This is because it can be shown via a simple dimension-counting argument that the number of algebraically independent multipartite entanglement measures grows exponentially in the number of parties (assuming no one party controls a majority share). A similar argument can be used to show this number is doubly exponential in the number of parties for mixed states, although in this case that figure includes classical correlations.

These different entanglement types are not just algebraically independent, they are physically inequivalent as well. For example, in the 5-qubit example used in section III of this paper has no two or three qubit entanglement present whatsoever; the concurrence for any two qubits drawn from this state is identically zero. It can also be seen that there is no three-qubit entanglement present, as any 3-qubit reduced density matrix is proportional to the identity. In fact this example is an extreme case because the single encoding is non-degenerate and so coding existence bounds require that subsystems below a minimum size have density matrices proportional to the identity. Some of these bounds may not apply to DFSs, as they are highly degenerate codes.

Nevertheless, it would be unwise to assume that any one entanglement measure will be sufficient to detect the entanglement present in any given phase transition. This is because they have enlarged stabilizers, which has im-
plications for their entanglement properties:

**Theorem 3. (Carteret and Sudbery)**

*Theorem 2 in [82], generalized for mixed states*

Let $\mathcal{H}$ be the space of states, and let $G$ be the group of local unitary transformations of $\mathcal{H}$, and let the dimension of the stabilizer of a typical state be zero (which it is, for mixed states, $\frac{1}{2}$). Let $I_1, \ldots, I_k$ be a set of $k$ polynomial invariants which generate the algebra of local invariants in a neighbourhood of a state $\rho$. If the stabilizer of $\rho$ in $G$ has non-zero dimension, then there is a linear combination of $I_1 \ldots I_k$ which has a stationary value at $\rho$.

This theorem certainly applies to the abelian stabilizer groups of conventional QECCs and it should also be true of the stabilizers of DFS-systems if the non-local error syndromes these correct correspond to a simple merging of parties. (E.g., they can correct non-local error syndromes that act on particles 1 and 2 jointly, or on 3 and 4 jointly, but not on 2 and 3, say.)

Theorem 3 does not simply that states with enlarged stabilizers have maximal entanglement; they could also have minimal entanglement (such as the maximally mixed state) or, more typically, a combination of both maximal and minimal entanglement, or even saddle-points in the manifold of non-local invariants. The moral of the story is that states with enlarged stabilizers have highly specific entanglement properties. Note that the converse is not necessarily true: an enlarged stabilizer is not a necessary condition for extremal entanglement properties. The fact that the theorem is one way only may mean that DFS-systems whose stabilizers do not correspond to a simple merging of parties may also have extremal entanglement properties, but any proof of this will need a more sophisticated argument than that in theorem 3.

The practical implications of this are that if we were going to use these entanglement measures as our initial probe of a system that we know nothing else about, we would have to calculate almost all of these functions, even though there are exponentially many. If we omitted to check too many types of entanglement, we would risk missing the very structures we were looking for. To make matters worse, for each $n$-party measure, for which $n$ particles do you calculate it for? Even if we assume that it only makes sense to evaluate these functions for particles in contiguous clusters, that will still leave us with so many choices that this method would be impractical, even if we had a complete set of closed forms for multipartite entanglement measures, and could calculate each individual function with ease.

**VI.2: Order parameters**

Although this paper has concentrated on the critical fixed points of the renormalization group flow, at least some ideas from this picture can be applied to the completely repulsive fixed points mentioned in section II. The DFS renormalization picture suggests a very natural choice for an order parameter for these systems. This is the block-spin transformation symmetry we have invoked to define the entire theory and which is in turn directly linked to dynamical symmetries of the system. Recall that in section II we supposed the block-spin transformation was an exact symmetry of the system, but that as we moved away from the critical point, the symmetry would eventually break down, as in subsection V.2. In the infinite lattice limit there will be a sharp transition between the two basins of attraction, which produces the necessary singularity in the system’s behaviour.

It now remains to find a convenient way to quantify this symmetry, preferably as part and parcel of the renormalization procedure. As we have shown a close relationship between the theory of fault-tolerant quantum memory and quantum phase transitions via our quantum analogue of the Kadanoff construction, it is natural to look to the theory of fault-tolerant memory thresholds for inspiration. If we once again treat small changes to the system Hamiltonian as if they were noise syndromes, we can use some of these results. In [83, 84] a technique was developed for calculating exact quantum memory thresholds for concatenated codes, using the theory of quantum channel capacities (see also [85]). Given the error syndrome (correction to the Hamiltonian) and the encoding (which we will have found as our RNG) the net effect of the error and recovery superoperators can be represented as a quantum channel. One of the examples they calculate in detail is the case the 5-qubit under a simple error model.

In finite systems the quantum capacity of this channel can take a continuous range of values between 0 and 1 as the fault-tolerant quantum memory threshold is crossed, but in the infinite concatenation (i.e., infinite lattice) limit, the quantum channel capacity of the encoding must be either 1 or 0. This quantum channel capacity therefore has precisely the right properties to be an order parameter. Note that while there are many ways to define channels and hence channel capacities in these systems in terms of various fictional or actual measurements, we claim that the channel defined by the renormalization iteration is the only relevant one.

**VI.3: A natural correlation functional**

This does not yet give us an analogue of the correlation length. To see what this should be, note that for the case of a concatenated code that only just fails to withstand an error syndrome, each successive layer of concatenation actually makes the problem worse, not better. So, for such a failing code, how many layers of concatenation can we have before the quantum channel capacity falls...
away to less than some small threshold $\varepsilon < 1$? Note that it wouldn’t make much sense to specify a capacity of less than 1 here, as if the encoding achieves a capacity of 1 for infinitely many layers, it will achieve 1 for infinitely many and we must be at the critical point. We would like something that behaves in an analogous way to the correlation length. It should therefore be able to take a continuous range of values between 0 and infinity, so we should specify an $\varepsilon < 1$, strictly (though we may well find that we can take $\varepsilon \to 0$ in many cases).

Therefore it would make sense for us to define the **quantum $\varepsilon$-memory support** to be the maximum size of a region of the lattice that is able to support such an emergent spin, by analogy to Kadanoff’s definition of the correlation length. Suppose that it takes $r$ layers of encoding for the quantum channel capacity to fall below $\varepsilon$, and each Kadanoff block contains $k$ spins with a lattice spacing of $L$. If our system is one-dimensional, then the quantum $\varepsilon$ memory support will be $k^r L$. Likewise for a two-dimensional system with $L$ spins in each block, we will obtain an $\varepsilon$-memory support of $k^r L^2$, but this time with units of the lattice-spacing squared.

This function will typically scale with the dimension of the lattice, unless we choose a really strange tiling which is effectively lower-dimensional. Naturally the precise numerical value of $r$ will depend on our choice of $\varepsilon$, but its scaling characteristics for fixed $\varepsilon$ should not. In particular, this function will be infinite if we are sufficiently close to a critical point, as we will be within the quantum memory threshold for the DFS concatenation.

This is not (strictly speaking) an entanglement measure, but it will be closely related to the entanglement properties for any realistic system, because DFSs for these systems will be entangled states. This will also ensure that there is a close link to the yield of the corresponding (finite-batch) entanglement distillation protocol, particularly if we believe that the convergence of the renormalization procedure should be defined with respect to the entanglement fidelity in [76, 77].

## VII: TOWARDS A CONTINUUM LIMIT

We have still not managed to express this renormalization procedure in differential form (c.f. [28]) although the results in [83, 84] are a step in this direction. At the moment, we have an iteration that we would like to convert into a system of differential equations that we can solve exactly. This is not as easy to do rigorously as it may first appear. To see why this is non-trivial, consider the opposite problem, where we have a set of differential equations that we would like to approximate by an iteration, for numerical analysis.

It is well known that simply replacing the derivative with the corresponding finite-difference doesn’t always work, as the flow trajectories may be highly sensitive to small changes in the initial conditions. The fixed points of the system may also be affected. An example of this problem can be seen in the logistic equation. The continuous (derivative) version of this is

$$\frac{dN}{dt} = r \left(1 - \frac{N}{K}\right) N$$

(17)

where $N$ is the population, and $r$ and $K$ are parameters. The corresponding finite difference equation is

$$N_{n+1} = \mu \left(1 - \frac{N_n}{\kappa}\right) N_n,$$

(18)

obtained by substituting $dN/dt = (N_{n+1} - N_n)/\Delta t$ and then writing $\mu = 1 + r\Delta t$ and $\kappa = (1 + r\Delta t)K/(r\Delta t)$. These two equations share two fixed points (the extinction and steady-state points) but while the finite-difference equation exhibits period-doubling and even fully chaotic behaviour for sufficiently large values of $\mu$, the corresponding differential equation does not. This is admittedly a very simple example, but chaotic flows have been reported in some numerical renormalization schemes [84]. Note that the finite difference equation is not a single equation, but rather a family of equations whose parameters $\mu$ and $\kappa$ are functions of our choice of stepsize, $\Delta t$ and thus the point at which the onset of chaos occurs will be a function of the stepsize, if plotted as a function of the original system parameters $r$ and $K$.

There are some results in numerical analysis that tell us when an iteration is a reliable approximation to a differential equation. These results are known as “Shadowing Theorems”, and they all seem to rely on being able to take the step size to zero. We have the opposite problem that in our case, the iteration is the “real” system and the differential equation is the approximation. Thus our stepsize is fixed at 1 and we are not free to take it to zero. Having said that, there clearly are situations where this approximation is valid. A closer examination of the critical points involved in these cases reveals that they are insensitive to small changes in the initial conditions and indeed, they typically feature merging trajectories; multiple different starting points end up at the same fixed point and thus the system can be said to forget its initial conditions. In other words, the system’s flow is irreversible.

There has been considerable recent interest in the question of whether or not renormalization group flows are reversible or not [22, 35]. The problem above suggests a slightly different perspective on this question: it might just be that irreversibility is a necessary condition for a continuum limit to be trustworthy. Without some applicable converses to the shadowing theorems it is impossible to be more specific than that; this question will also have to await further work.

So, does our reinterpretation of the Kadanoff picture give rise to an irreversible RNG flow? This question can
be addressed through the thermodynamics of quantum error-correction \[87\]. This is because the error-correction process produces classical information about the error syndrome as a by-product. Once we have undone the error, the results of the syndrome measurement must be erased. Landauer’s principle states that erasure has a thermodynamic cost of \(k \log(2)\) per bit. We can minimise this cost by using efficient reversible data compression, but we cannot eliminate it. This result goes through for the case of the “passive” error correction of DFS theory, as the error syndrome will be discarded with the decoherence-full subsystem. It would seem likely that this condition will be met for the critical points described in this paper. Whether or not it would be met by the completely repulsive fixed points will most likely depend on whether we can find fixed-step-size converses to the shadowing theorems.

VIII: CONCLUSIONS AND FURTHER WORK

In this paper we have tried to understand renormalization theory in terms of quantum information theory. In the process we found ourselves going right back to the Kadanoff picture in order to build a bridge between these two conceptual systems. The Kadanoff picture has traditionally been regarded as not much more than a “Just So” story, with only intuitive value that cannot be made rigorous. We believe this evaluation should be reconsidered. We then incorporated White’s insights into the role of the interaction between the block and the rest of the lattice \(a\)\(\text{fortiori}\). This led us to conclude that the Kadanoff block-spin structure can be explained as a consequence of the block’s interaction with the rest of the lattice. The block-spin is created by the rest of the lattice. Every pearl in Indra’s net not only reflects every other, it consists entirely of those reflections.

Strangely, despite the fact that the conceptual development of this paper began with a conjecture about the entanglement properties of these systems, we have ended up concluding that entanglement measures may give little insight into these systems if used in \textit{isolation}. Rather we claim that the key to understanding these systems lies in realizing that the Kadanoff block-spin is induced by the interaction between the block and the rest of the lattice. It should be possible to interlace the entanglement properties with the renormalization procedure in a natural way, thus mitigating the problem in theorem \[\text{40}\] to some extent and enabling us to probe these in greater detail. I believe the papers \[\text{58, 59, 60}\] contain important steps in this direction, even though they do not mention renormalization \textit{per se}.

The symmetry properties of the system’s dynamics feature prominently in our model of renormalization. The condition for the existence of such a DFS structure is a symmetry condition and thus leads naturally to an order parameter. We can also define an analogue of the correlation length. While we have not proved that quantum phase transitions must necessarily exhibit entanglement on all length scales, we have shown that this is to be expected under the physically reasonable conditions of short-range interactions.

We are still quite some way from giving an explicit alternative to existing numerical renormalization procedures. There are many unresolved problems in the detection of DFS structures, the generalization of the Kraus representation theorem, and possible converses to the Shadowing theorems. We hope that the details required will become clearer in further work \[\text{72}\].

There is also the intriguing possibility of some kind of gauge symmetry in the choice of basis-labelling for the encoded qubits. It is not yet clear to me whether this freedom is congruent to known gauge symmetries or not. (Note that the choice of gauge may need to be time-dependent; some DFSs exhibit Lamb shifts \[\text{31}\].)

The reformulation of renormalization theory given in this paper is entirely in terms of local density matrices and open-system dynamics. As such, it should be straightforward to apply this to finite-temperature systems. This is because the mathematics we have used sees errors only as undesired admixtures; it does not care whether these are caused by unwanted entanglement with an external environment, “proper” thermal mixing, or any combination of the two. Therefore it should be possible to extend this framework to incorporate quantum-to-classical transitions, as these will be the completely repulsive fixed points (or rather, surfaces) introduced in section II. However, we anticipate that the problems with converse shadowing theorems will be much more severe for these cases, not least because they are unlikely to be pointlike structures in the parameter spaces.

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I am also grateful to Daniel Lidar for drawing to my attention a subtlety in reference \[\text{58}\] (and also \[\text{72}\]) which should provide a rather elegant way to fix one of the problems with the renormalization procedure I proposed in version 1 of this paper.

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FIG. 5: Minimal generators for a toric code. Top left: an X
"site" generator drawn on the dual lattice. Top right: the
same generator represented on the physical lattice. Bottom
left: a Z "plaquette" generator on the dual lattice. Bottom
right: Z-plaquette on the physical lattice.

APPENDIX: SOME OTHER EXAMPLES

Here we discuss some other examples which may be of
interest. They also serve to further illustrate some points
which were only mentioned in passing in the main text.

Some translationally invariant solutions

Another class of solutions which are also a type of sta-
bilizer code are the toric codes [17]. To show this, we need
to establish that these codes admit a recursive renor-
malization scheme. Toric codes are usually defined on a dual
lattice, where the spins live on the edges rather than
the vertices. Figure 5 illustrates the generators for these
codes, on both the dual and physical lattices.

Unlike the 5-qubit example in section III, these solu-
tions are degenerate codes, and they are translation in-
variant. They satisfy the Kadanoff construction, because
it is possible to choose a minimal generating set that is
scale invariant, using a construction which is illustrated
in figure 6 [92]. (This seems to be related to the “ex-
panding diamonds” construction [93].)

The stabilizer generators for these codes are usually
chosen to have interactions between no more than four
spins each. Minimal generating sets are not unique, so
we are free to choose a different one with a more obvi-
ously hierarchical structure. (Here the abelian Hamil-
tonian ambiguity is useful, rather than being a problem.)
The large X-site generator is the product of the 9 smaller
ones (one is hidden under the middle of the larger gen-
erator). If a minimal generating set is desired, we can
then omit one of the smaller generators and replace with
the large one. Likewise, the larger Z-plaquette generator
is the product of the 5 smaller ones it contains, and we
can choose to omit the small plaquette in the middle, as
the new, longer-range plaquette has made it redundant.
Note that there is more than one way of doing this. The
construction can be displaced by integer multiples of the
lattice spacing, and so is translation invariant. We could
also choose to make the rescaled generators larger, so
each cycle would scaled by a factor larger than 3. Thus
the corresponding effective Hamiltonians will have the
required self-similar structure.

Note that every spin is in only one unit cell. The tiling
does not overlap, but the tiles do interact with each other
directly, as they share some stabilizer generators. The
long-range interactions in these systems can claim to be
genuinely emergent, because all the longer-range interac-
tions arise as products of four-spin interactions. The cor-
responding renormalization procedure is a progressive,
fault-tolerant decoding operation. This is a special case
of the procedure is described in [19]. We can either mod-
ify the triangulation of the surface one move at a time,
or choose to collect the moves into batches that follow
our choice of blocking for the lattice.

Toric codes certainly possess short-range entanglement
and they must have global entanglement in order to func-
tion as error-correcting codes. However, proving that
they have entanglement on all length scales requires that
we take some care with our definitions.

Suppose we try to ascertain whether these codes have
n-party entanglement by looking for blocks of n spins,
and checking to see if they are entangled. If we find a
block that is, we then look at blocks of size n + 1 that
include our entangled block of n spins and repeat. This
calculation was done a few years ago by D. Aharonov and
Gottesman [11], using a reasonable measure of entanglement that can be defined on stabilizer states. They found entanglement for small \( n \), but for blocks above a certain size, they found nothing until they reached the global entanglement scale. Let us call the largest sub-global value of \( n \) for which they found entanglement, \( n_T \).

By contrast, if we perform one iteration of a renormalization scheme such as that described above, we will obtain another toric code on a smaller, rescaled lattice. This will also have short-range entanglement, by the result in [94], and the iteration can be repeated until the logical qubit is decoded. Thus in this sense these codes can be said to have entanglement on all length scales. It would seem to be essential to define this entanglement in terms of a renormalization procedure.

However, both of these measures make sense and the number \( n_T \) does capture an important feature of this code. Therefore, let us define the cardinality of an entanglement type to be the number of parties it involves, and the characteristic cardinality to be the largest cardinality present in whatever system we are currently looking at; in this case \( n_T \). Let us incorporate the rescaling symmetry of the renormalization by defining the scaling cardinality of entanglement of these codes to be \( n_T \), the characteristic cardinality of the local density matrix at each iteration. These definitions may be extended to other cases in the obvious way.

Note that while toric codes are usually defined on surfaces with non-trivial topology, there is a method for constructing them in planes while keeping the stabilizer generators local [95, 96]. Alternatively, one can define the code on a plane and have it encode no qubits. These solutions would still have global entanglement, even though they don’t encode any qubits. They would correspond to globally pure systems with non-degenerate ground states. It is an open question whether the definition of the quantum \( \epsilon \)-memory support given in subsection VI.3 can be adapted to this case. However, I suspect that the definition of such an empty “toric” code on the plane in terms of entirely local interactions would require a truly infinite plane with no boundaries, because these codes are defined topologically. Truly infinite planes do not occur in nature and so I suspect we would only ever see the “unwrapped” planar toric codes, as these can be defined for finite sized systems with boundaries in a sensible, local way. The problem of defining the quantum \( \epsilon \)-memory support for empty toric codes would therefore be moot.

**Solutions to addition RNGs**

The original Kadanoff construction is often modified so that after a block has been mapped to one or more spins a single new spin is added to the block from the rest of the lattice; the renormalization procedure is then repeated with what is essentially the original block instead of using a concatenated block structure. This seems to have been done to try and keep the problem computationally feasible. This seems to have become an almost universal practice, and indeed White’s DMRG is a case in point.

As hinted at above, this modification of the original Kadanoff construction may not be benign; there are classes of codes for which addition renormalization schemes may not capture the scaling behaviour of the entanglement.

However, there are families of codes for which such an addition renormalization would be natural. These are the quantum convolutional codes [97, 98, 99]. For an excellent introduction to these codes, see [100]. This features a modified version of the 5-qubit code, adapted for use on a chain of qubits, such as a spin chain. Note also that we could use the convolutional 5-qubit code to make the stripes in the alternative “brick” tiling of the square lattice mentioned earlier, and shown in figure 4.

Convolutional codes were developed to protect transmitted data (such as a radio broadcast) where the qubits pass through the device in a stream, and once they’ve gone, they are no longer accessible to processing. It is also possible to use DFS-spaces in this situation, as shown in [101].

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