Abstract

Since the 1970s, the LIBOR has served as a fundamental measure for floating term rates across multiple currencies and maturities. Loans and many derivative securities, including swaps, caps and swaptions, still rely on LIBOR as the reference forward-looking term rate. However, in 2017 the Financial Conduct Authority announced the discontinuation of LIBOR from the end of 2021 and the New York Fed declared the backward-looking SOFR as a candidate for a new reference rate for interest rate swaps denominated in U.S. dollars. We first outline the classical single-curve modelling framework before transitioning to the multi-curve framework where we examine arbitrage-free pricing and hedging of SOFR-linked swaps without and with collateral backing. As hedging instruments, we take liquidly traded SOFR futures and either common or idiosyncratic funding rates for the hedge and margin account. For concreteness, a one-factor model based on Vasicek’s equation is used to specify the joint dynamics of several overnight interest rates, including the SOFR, EFFR, and unsecured funding rate, although multi-factor term structure models could also be employed.

Keywords: LIBOR, SOFR, EFFR, OIS, futures, interest rate swap, cap, swaption
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Introduction

One of the most important financial instruments for institutions to manage cash flows are interest rate swaps, which provide an institution the ability to hedge their exposure to interest rate in a single or multiple currencies. The first swap agreements were negotiated in the 1970s to hedge against price movements by the British sterling. Since the contracts inception, the volume of swap contracts traded has grown exponentially and in 2019, 119,439 USD billion worth of swap contracts were traded for U.S. dollar alone ([35]). There are two key swap contracts, an interest rate swap and a currency swap. An interest rate swap is an agreement between two parties to exchange the interest on a notional principal in a single currency where, in the most common case of fixed-to-floating swap, one of parties pays a fixed rate, and the other party pays a reference floating rate.

One of the most important concepts associated with interest rate swaps has been the London Interbank Offered Rate (LIBOR), which aimed to reflect the forward-looking cost of interbank borrowing and lending. LIBOR has also served as the key benchmark rate for loans and swaps worldwide. It was formally established in 1984 and, historically, the British Bankers Association (BBA) published LIBOR for five currencies (USD, EUR, GBP, CHF and JPY), across seven tenors ranging from overnight to 12 months. This was continued after the administration of LIBOR was turned over by the Financial Conduct Authority (FCA) to the Intercontinental Exchange (ICE). The ICE LIBOR is based on a survey of a panel of 20 international banks as to what rate they could borrow funds for each currency and maturity. To help guard against extreme highs or lows that might skew LIBOR, the ICE Benchmark Administration strips out the four highest submissions and the four lowest submissions before calculating an average. The trimmed mean is published by every business day at 11:55 am London time. We refer to [34] for more details on LIBOR methodology and current panel composition.

Although financial instruments benchmarked on LIBOR have been traded in high volumes for over 35 years, in July 2017 the FCA announced the discontinuation of LIBOR from the end of the 2021. Since then, the FCA and other official sector bodies have strongly encouraged market participants to transition from LIBOR to alternative rates. More recently, it was announced by the FCA that 1-week and 2-month USD LIBOR will be last published on 31 December 2021 but 1-month, 3-month, 6-month, 12-month USD LIBOR settings will continue till 30 June 2023. In contrast, all sterling, euro, Swiss franc and Japanese yen LIBOR settings will cease immediately after 31 December 2021, although 1-month, 3-month and 6-month GBP and JPY LIBOR will be published under a new “synthetic” methodology for 2022 (see [22]).

The cessation of LIBOR is due to multiple factors, many of which came to light during the GFC of 2008. One of the main concerns surrounding LIBOR was market manipulating activities by some members of the panel (see, e.g., Taylor-Brill [57] and the references therein). Several panel banks (including Barclays, UBS, RBS, Deutsche Bank, Rabobank, Lloyds, Citigroup and JPMorgan) were alleged to be submitting LIBOR estimates that were not reflective of their recent transactions. There was also evidence of collusion between banks to distort rates to generate profits on the trading desks. Due to the discontinuation, existing contracts referencing LIBOR require a new reference rate; the transition from LIBOR to a new benchmark rate in outstanding contracts is referred to as the LIBOR fallback.

Global markets have pre-existing rates that can be used for reference in fixed income derivatives, such as: EONIA (Euro Overnight Indexed Average), SONIA (Sterling Overnight Indexed Average), SARON (Swiss Average Rate Overnight) and TONA (Tokyo Overnight Average Rate). EONIA is computed as a weighted average of all overnight unsecured interbank lending transactions undertaken in the European Union and European Free Trade Association (EFTA) countries. It will gradually be replaced by the Euro Short Term Rate (€STR), which is calculated using overnight unsecured fixed rate deposit transactions over €1 million and published by the European Central Bank since October 2019. SONIA and SONIA Compounded Index are administered by the Bank of England since April 2016 and are based on actual transactions in overnight indexed swaps for unsecured transactions in the sterling market. The above mentioned interest rates are commonly referred to as RFR (Risk-Free Rates) since they hardly reflect the credit risk of a bank when it raises funds.
In the United States, the Alternatives Reference Rate Committee (ARRC) announced the Treasury repo financing rate, formally referred to as the Secured Overnight Financing Rate (SOFR), would be the most effective rate to replace LIBOR. The SOFR is a backward-looking measure of the overnight repurchasing rate for Treasury securities and it has been published by the New York Federal Reserve since 2018. The SOFR is a broad measure of the cost of borrowing cash overnight collateralized by Treasury securities. Each business day, the New York Fed publishes the SOFR at approximately 8:00 a.m. ET. As an extension of the Secured Overnight Financing Rate (SOFR), the SOFR Averages are compounded averages of the SOFR over rolling 30-, 90-, and 180-calendar day periods. It is worth mentioning that SOFR was also recently chosen to replace EFFR as the Price Alignment Interest (PAI). The PAI is the overnight cost of funding collateral, which is debited from the receiver and transferred to the payer to cover the loss of interest on posted collateral (variation margin) in centrally cleared transactions. For convenience, the PAI for centrally cleared trades will be called the collateral rate, which is the term commonly used for collateralized OTC trades. We will provide a more in-depth definition of SOFR in Section 2.1.

The advantage of transitioning from LIBOR to SOFR is that the latter is determined by observing market transactions, rather than merely taking a survey. SOFR is therefore considerably less susceptible to banks attempting to manipulate the market. Additionally, while the LIBOR referenced unsecured borrowing and hence reflects also the interbank credit risk (see a study by Filipovič and Trolle [25]), the repurchase transactions are backed by the Treasury securities as collateral. The collateral backing protects the lender in the event of default, and thus SOFR can be interpreted as having negligible credit risk. Although SOFR has many advantages compared to LIBOR, there are several key issues regarding the transition. One of the primary challenges is SOFR has a single reference period, overnight, in contrast to LIBOR that has a range of maturities. Since in the case of SOFR there are no directly observable long-term maturities, the forward interest rate needs to be inferred from the overnight rate and derivative instruments. Another key issue, specifically in regard to swap contracts, is that the SOFR Average is a backward-looking compound overnight rate, while LIBOR is a forward-looking term rate.

Our research into this area is partially motivated by the recent papers by Mercurio [42] and Lyashenko and Mercurio [38, 39, 40]. Mercurio [42] provides an outline for the joint modelling of SOFR/OIS rates and considers the pricing of the respective SOFR futures contracts. His work is further expanded in Lyashenko and Mercurio [38, 39] where the classical LIBOR Market Model is generalized to cover also the backward-looking rates, such as the SOFR. We study replication of SOFR swaps and arbitrage-free pricing of some other SOFR derivatives. One of the key concepts in arbitrage-free pricing is the assumption that the payoff can be replicated by a self-financing trading strategy. Therefore, the paper attempts to provide greater context to [38, 39] by demonstrating that a SOFR swap and cap can be replicated by a dynamic strategy based on SOFR futures.

The structure of this work is as follows. In Section 1, we give a brief overview of the classical single-curve arbitrage pricing theory for fixed income instruments. In particular, we recall the single-curve definitions of forward LIBOR and swap rates (see also Mercurio [41] for a multi-curve extension) and we give some results on classical pricing of LIBOR derivatives, such as caps and swaptions. Section 2 describes an overview and the respective definitions for a multi-curve market. In contrast to the single-curve approach, there is no longer a single interest rate in the market but we deal instead with a family of interest rates corresponding to different segments of interest rates markets. We introduce five key rates for consideration; EFFR, SOFR, OIS rate, the unsecured funding rate and collateral rate. Having defined the overnight rates, we propose in Section 3 a framework for the modelling of the joint dynamics of these interest rates under Vasicek’s model. For simplicity, we focus here on Vasicek’s dynamics for the factor process but it is worth noting that Gellert and Schloegl [28] use an extended HJM approach, while Backwell et al. [3] choose an affine term structure model to represent the dynamics of SOFR. Under Assumption 3.1, we furnish a closed-form expression for the SOFR futures price and compute the respective hedge ratios for the replicating strategy for swaps referencing SOFR. We also give an explicit representation for a SOFR cap and make an attempt to price SOFR swaptions. The case of different funding rates available to firms for replication, in addition to factoring in the level of collateralization, is also considered.
1 Single-Curve Term Structure Models

We stress that all term structure models considered in this section belong to the class of single-curve term structure models, that is, models where the issues of credit risk, roll-over risk and funding costs are completely ignored. Although several alternative approaches to credit risk were developed during the last twenty years, for the sake of easy comparison between swaps and other derivatives referencing either LIBOR or SOFR, we first give a very brief overview of the classical single-curve term structure theory, which was mainly developed in the 1990s. In a single-curve approach to arbitrage-free pricing of fixed-income derivatives, we denote by $B$ and $B(t, T)$ the money market account and the price of a unit zero-coupon bond maturing at $T$, respectively. They are all assumed to be traded assets and they are not subject to credit risk or any other market imperfections such as taxes or transaction costs.

Recall that the spot martingale measure, denoted henceforth as $Q$, and the forward martingale measure for the date $T > 0$, denoted as $Q_T$, are formally defined as probability measures, which are equivalent to the statistical probability measure, and enjoy the property that under $Q$ and $Q_T$ the wealth of any self-financing trading strategy is a local martingale when divided by the process $B$ and $B(t, T)$, respectively. In other words, $Q$ and $Q_T$ are martingale measures obtained when either the money market account or the zero-coupon bond maturing at $T$ are chosen to play the role of a numeraire, respectively. We write $W^Q$ and $W^T$ to denote a Brownian motion under the spot martingale measure $Q$ and the forward martingale measure $Q_T$, respectively. For a detailed exposition of the classical theory (in particular, the HJM approach), we refer to monographs by Brigo and Mercurio [10] and Musiela and Rutkowski [45].

1.1 Vasicek’s Model

A single-factor Gaussian model of the term structure introduced in Vasicek [59] proposes a mean-reverting version of the Ornstein-Uhlenbeck diffusion process where the short term rate $r$ is given by the unique solution to the stochastic differential equation

$$dr_t = (a - br_t) dt + \sigma dW^Q_t$$

where $r_0, a, b$ and $\sigma$ are strictly positive constants. In a multi-curve setup introduced in Section 3, we will use equation (1.1) to describe the dynamics of the factor process $x$ (see equation (3.10)).

Proposition 1.1. The price at time $t$ of the zero-coupon bond maturing at time $T$ in Vasicek’s model equals

$$B(t, T) = v(r_t, t, T) = e^{m(t, T) - n(t, T)r_t}$$

where

$$m(t, T) = \frac{\sigma^2}{2} \int_t^T n^2(u, T) \, du - a \int_t^T n(u, T) \, du$$

and

$$n(t, T) = \frac{1}{b} \left( 1 - e^{-b(T-t)} \right).$$

The bond price volatility is a deterministic function $b(\cdot, T) :\to \mathbb{R}$ given by $b(t, T) = -\sigma n(t, T)$ and the dynamics of the bond price under $Q$ are given by

$$dB(t, T) = B(t, T)(r_t dt - \sigma n(t, T) dW^Q_t).$$

A large body of research was devoted to the class of affine term structure models, which are known to enjoy some fundamental properties of Vasicek’s model and hence are amenable to explicit computations (see, for instance, Cuchiero et al. [19], Duffie et al. [20], Duffie and Kan [21], Fontana et al. [26], Grbac et al. [31], Keller-Ressel et al. [37], and Papapantoleon and Wardenga [50]).
1.2 Single-Period LIBOR Swaps

We first consider the case of a single-period swap, before expanding to other derivative contracts. Let \( L_T(t, T + \delta) \) denote the LIBOR at time \( T \) for the accrual period \([T, T + \delta]\) where \( \delta > 0 \) is a constant. According to the single-curve paradigm, the no-arbitrage condition \( 1 + \delta L_T(t, T + \delta) = B^{-1}(t, T + \delta) \) holds since LIBOR is implicitly assumed to be a default-free.

**Definition 1.1.** A payer single-period LIBOR swap settled in arrears, with fixing date \( T \) and settlement date \( T + \delta \), is a fixed-for-floating swap contract where the long party makes a fixed payment \( X_1(t) = 1 + \delta \kappa(t) \) and receives a floating payment \( X_2(t) = 1 + \delta L_T(t, T + \delta) = B^{-1}(t, T + \delta) \) at time \( T + \delta \). Note that we assume here that the notional principal of a swap is \( P = 1 \).

Under the standard assumption that risk-free bonds are traded, simple replication arguments show that the arbitrage-free price of the fixed and floating legs of the swap satisfy \( \pi_t(X_1) = B(t, T) (1 + \delta \kappa(t)) \) and \( \pi_t(X_2) = B(t, T) \) for every \( t \in [0, T] \). By definition, forward LIBOR swap rate \( \kappa_t(T, T + \delta) \) associated with the single-period LIBOR swap results in the contract having no value at time \( t \). Notice that it is also called the forward LIBOR and henceforth denoted as either \( \kappa_t(T, T + \delta) \) or \( L_t(T, T + \delta) \).

**Lemma 1.1.** The forward LIBOR swap rate satisfies, for every \( t \in [0, T] \),

\[
\pi_t(X_1) - \pi_t(X_2) = B(t, T + \delta) \left(1 + \delta \kappa_t(T, T + \delta)\right) - B(t, T) = 0
\]

and thus

\[
\kappa_t(T, T + \delta) = \frac{1}{\delta} \left( \frac{B(t, T)}{B(t, T + \delta)} - 1 \right).
\]

1.3 Futures Contracts

A *futures contract* is constructed in a similar manner to that of forward contract but futures contracts are traded on organised exchanges whereas forward contracts are traded over-the-counter (OTC) between financial institutions and their customers. Similar to a forward contract, a futures contract is an agreement between two parties to buy or sell at a certain date in the future at a pre-determined price. The key distinction between the forward and futures contract is in the associated cash flows. In a forward contract, the cash flows are settled at maturity, however, the cash flows of a futures contract are settled daily on a marked-to-market basis. Intuitively, a futures contract can be viewed as a forward contract where profits/losses are realised continuously throughout the contract and thus the contract matures and is renewed every day. The following definition summarizes the basic features of a futures contract.

**Definition 1.2.** A futures contract written at time \( t \leq T \) on a contingent claim \( X \) is a financial asset satisfying:

(i) for every \( 0 \leq t \leq T \), there exists a futures price \( f(t, T, X) \) for \( X \) at \( t \),
(ii) at maturity \( T \), the holder of the long futures contract pays \( f(T, T, X) \) and receives the claim \( X \) and thus \( f(T, T, X) = X \),
(iii) during an arbitrary time interval \((t, t + 1)\), typically one day, the holder of the long futures contract receives \( f(t + 1, T, X) - f(t, T, X) \),
(iv) the cost \( S_t \) of entering into the futures contract is zero at any date \( t \in [0, T] \).

A formal definition of futures contract on a contingent claim \( X \) reads as follows.

**Definition 1.3.** The futures contract on a contingent claim \( X \) for settlement at \( T \) is a price dividend pair \((S, D)\) with a spot price process \( S \) and cumulative dividend process \( D \) such that:

(i) \( D_t = f(t, T, X) \),
(ii) \( D_T = f(T, T, X) = X \),
(iii) \( S_t = 0 \) for every \( t \in [0, T] \).
Although the next result is commonly known, it is given here since in Section 3 we will focus on hedging of SOFR derivatives using extended self-financing futures strategies with funding costs and collateralization and thus Proposition 1.2 will be extended (see Propositions 3.1 and 3.2).

**Proposition 1.2.** For a given contingent claim \( X \) integrable under a martingale measure \( Q \), the futures price for settlement at time \( T \) equals \( D_t = f(t, T, X) = E_Q(X | \mathcal{F}_t) \) for all \( t \in [0, T] \).

**Proof.** Consider a self-financing trading strategy for a dividend process such that we hold one futures contract and invest all dividends into the money market account \( B \). Its wealth is therefore given by \( V_t(\varphi) = \varphi_t B_t \) and \( \varphi \) is a self-financing futures strategy if, for every \( t \in [0, T] \),

\[
V_t(\varphi) = V_0(\varphi) + \int_0^t \varphi_u dB_u + D_t - D_0.
\]

Hence the discounted wealth \( \tilde{V}_t(\varphi) := B_t^{-1} V_t(\varphi) \) satisfies

\[
\tilde{V}_t(\varphi) = \tilde{V}_0(\varphi) + \int_0^t B_u^{-1} dD_u,
\]

which in turn implies that the process \( D \) is a \( Q \)-martingale and \( D_T = X \). It is thus clear that the asserted equality holds. \( \square \)

### 1.4 Forward, Spot and Futures LIBOR

Under the conventional LIBOR models, the short-term rate corresponding to LIBOR is the risk-free rate denoted as \( r \). Under the classical framework, term rates are formulated using \( r \) with the respective money market account \( B \) and the zero-coupon bond \( B(t, T) \). We begin by defining the forward LIBOR.

**Definition 1.4.** The forward LIBOR \( L_t(T, T + \delta) \) at time \( t \) prevailing at time \( T \) over the period \([T, T + \delta]\) equals for every \( t \in [0, T] \)

\[
1 + \delta L_t(T, T + \delta) = \frac{B(t, T)}{B(t, T + \delta)} = F_t^B(T, T + \delta)
\]

We may note the forward LIBOR directly corresponds to \( \kappa_t(T, T + \delta) \) from Lemma 1.1 and is a martingale under forward measure \( Q_{T, T + \delta} \) so that

\[
E_{Q_{T, T + \delta}}(L_t(T, T + \delta) | \mathcal{F}_s) = L_s(T, T + \delta)
\]

for all \( 0 < s < t \leq T \). The spot LIBOR can be seen as a specific case of the forward LIBOR given by \( L_t(t, t + \delta) \) for all \( t \in [0, T] \).

**Definition 1.5.** Let \( L_t(t, t + \delta) \) denote the spot LIBOR prevailing at time \( t \) for the period \([t, t + \delta]\) and for all \( t > 0 \)

\[
1 + \delta L_t(t, t + \delta) = \frac{1}{B(t, t + \delta)} = F_t^B(t, t + \delta).
\]

Finally, we may consider the futures LIBOR. Intuitively, the futures LIBOR corresponds to a futures contract where the spot LIBOR is the underlying asset so that the terminal contingent claim at time \( T \) equals \( X = L_T(T, T + \delta) \) (or, more formally, \( \delta(1 - L_T(T, T + \delta)) \)). Let \( L_t^f(T, T + \delta) \) denote the futures LIBOR at time \( t \) for maturity \( T \) so that, in particular, \( L_T^f(T, T + \delta) = L_T(T, T + \delta) \).

**Definition 1.6.** The futures LIBOR \( L_t^f(T, T + \delta) \) satisfies, for every \( t \in [0, T] \),

\[
1 + \delta L_t^f(T, T + \delta) = E_Q(B^{-1}(T, T + \delta) | \mathcal{F}_t) = E_Q(L_T(T, T + \delta) | \mathcal{F}_t).
\]

We note that \( L_t^f(T, T + \delta) \) is an \( \mathbb{F} \)-martingale under the spot martingale measure \( Q \), which is consistent with Proposition 1.2 applied to \( X = L_T^f(T, T + \delta) = L_T(T, T + \delta) \). For more details on relationship between forward and futures LIBOR (in particular, the so-called convexity correction linking forward and futures LIBOR), we refer to [53].
1.5 Multi-Period LIBOR Swaps

The concept of a single-period swap may be extended over multiple time periods of varying lengths by considering a collection of fixing and payment dates \( 0 < T_0 < T_1 < \cdots < T_n \), referred to as the tenor structure. Let \( \delta_j = T_j - T_{j-1} \) denote the length of the \( j \)th accrual period for \( j = 1, 2, \ldots, n \).

**Definition 1.7.** At any date \( T_j \) where \( j = 1, 2, \ldots, n \), the net cash flow of a (payer) forward LIBOR swap equals \( FS_{T_j}(\kappa) = (L_{T_{j-1}}(T_{j-1}, T_j) - \kappa)\delta_j P \). The dates \( T_0, T_1, \ldots, T_{n-1} \) are the fixing dates and the dates \( T_1, T_2, \ldots, T_n \) are the payment dates. The symbols \( n, P \) and \( \kappa \) denote the number of payments (length) of the swap, the notional principal, and the preassigned fixed rate of interest, respectively. Without loss of generality, we may set \( P = 1 \).

**Lemma 1.2.** The arbitrage-free price at time \( t \in [0, T_0] \) of a forward LIBOR swap, denoted by \( FS_t(\kappa) \), equals

\[
B(t, T_0) - \sum_{j=1}^{n-1} \delta_j \kappa B(t, T_j) - (1 + \delta_n \kappa) B(t, T_n).
\]

If \( t \) belongs to the accrual period \( (T_{k-1}, T_k) \), then it suffices to replace \( T_0 \) by \( T_k \) and \( \sum_{j=1}^{n-1} \) by \( \sum_{j=k}^{n-1} \).

**Proof.** Using the classical change of a martingale measure to the forward measure, we obtain

\[
FS_t^\kappa = \mathbb{E}_{Q} \left( \sum_{j=1}^{n} \frac{B_t}{B_{T_j}} (L_{T_{j-1}}(T_{j-1}, T_j) - \kappa)\delta_j \middle| \mathcal{F}_t \right)
\]

\[
= \sum_{j=1}^{n} B(t, T_j) \mathbb{E}_{Q_{T_j}} \left( (L_{T_{j-1}}(T_{j-1}, T_j) - \kappa)\delta_j \middle| \mathcal{F}_t \right)
\]

\[
= \sum_{j=1}^{n} B(t, T_j)(L_i(T_{j-1}, T_j) - \kappa)\delta_j
\]

\[
= \sum_{j=1}^{n} \left( B(t, T_{j-1}) - B(t, T_j) - \delta_j \kappa B(t, T_j) \right)
\]

\[
= B(t, T_0) - \sum_{j=1}^{n-1} \delta_j \kappa B(t, T_j) - (1 + \delta_n \kappa) B(t, T_n),
\]

as was required to show. \( \square \)

One of the key features of a forward swap is that it has no value at initiation. Therefore, if a forward swap is initiated at \( t \leq T_0 \), the corresponding fixed rate of interest should be set such that \( FS_t^\kappa = 0 \). More generally, the forward LIBOR swap rate \( \kappa_i(T_0, n) \) at time \( t \) for a swap contract starting at \( T_0 \) and with length \( n \) is the value of the fixed rate \( \kappa \) such that price at time \( t \) of the swap equals zero.

**Lemma 1.3.** The forward LIBOR swap rate \( \kappa_i(T_0, n) \) equals

\[
\kappa_i(T_0, n) = \frac{B(t, T_0) - B(t, T_n)}{\sum_{j=1}^{n} \delta_j B(t, T_j)}.
\]

Finally, although the value of a swap at initiation is zero, once the swap has been written and the rate \( \kappa \) fixed, the marked-to-market value of the swap may change for \( t \in [0, T_0] \). Therefore, we may want to consider the swap value relative to the value of the swap written at \( t > 0 \) with rate \( \kappa_i(T_0, n) \).

**Lemma 1.4.** The following equality holds, for every \( t \in [0, T_0] \)

\[
FS_t^\kappa = \sum_{j=1}^{n} \delta_j B(t, T_j)(\kappa_i(T_0, n) - \kappa).
\]
1.6 LIBOR Caps

An interest rate cap is a derivative where the holder of the contract receives the cash difference between the benchmark interest rate and a pre-determined level of interest if and only if the designated interest rate exceeds a pre-determined strike level $\kappa > 0$. In the cap referencing LIBOR, the rate considered over the period $[T_{j-1}, T_j]$ is the forward LIBOR determined at time $T_{j-1}$.

**Definition 1.8.** A LIBOR cap with strike $\kappa > 0$ and maturity $T_0$, which is settled in arrears at dates $T_j$, $j = 1, 2, \ldots, n$ has cash flows at time $T_j$ equal to $FC_{T_j}(\kappa) = (LT_{j-1}(T_{j-1}, T_j) - \kappa)^+ \delta_j P$, where $FC_{T_j}(\kappa)$ denotes the $j$th caplet. In a similar manner, a LIBOR floor under the same tenor structure has cash flows at dates $T_j$ equal to $FF_{T_j}(\kappa) = (\kappa - LT_{j-1}(T_{j-1}, T_j))^+ \delta_j P$.

Without loss of generality, we may set $P = 1$ since the arbitrage-free pricing functional is a linear mapping in any frictionless market.

**Lemma 1.5.** The arbitrage-free price of a LIBOR cap with strike $\kappa > 0$ equals, for all $t \in [0, T_0]$,

$$FC_t^c = \sum_{j=1}^{n} B(t, T_j) \mathbb{E}_{T_j} \left( (LT_{j-1}(T_{j-1}, T_j) - \kappa)^+ | \mathcal{F}_t \right).$$

The LIBOR Market Model (LMM) was first developed in papers by Brace et al. [7], Jamshidian [36], Miltersen et al. [43] and Musiela and Rutkowski [44]. Their approach (as well as the HJM methodology) was subsequently modified and extended to multi-currency and multi-curve setups in numerous works, to mention just a few, Cuchiero et al. [18], Glasserman and Zhao [29], Grbac et al. [31], Mercurio [41], Rutkowski [54] and Schlögl [58]. However, most works on multi-curve extensions of HJM and LIBOR models were focusing on the joint modeling of the (default-free) OIS term structure and (defaultable) LIBORs, which is not the goal of the present work. It should be noted in this regard that in recent influential papers by Lyashenko and Mercurio [38, 39, 40] where the authors develop an extension of the LIBOR market model, called the Forward Market Model (FMM), which covers in a single-curve framework not only forward-looking (IBOR-like) term rates but also the backward-looking rates.

**Proposition 1.3.** Let the forward LIBORs $L(T_{j-1}, T_j)$ satisfy, for all $t \in [0, T_{j-1}]$,

$$dL_t(T_{j-1}, T_j) = L_t(T_{j-1}, T_j) \lambda(t, T_{j-1}, T_j) dW^T_{t},$$

where $W^T$ is a $d$-dimensional Brownian motion under the forward martingale measure $\mathbb{Q}_T$ and $\lambda(\cdot, T_{j-1}, T_j) : [0, T_{j-1}] \to \mathbb{R}^d$ is a deterministic function. Then the arbitrage-free price of a LIBOR cap with strike $\kappa > 0$ equals, for $t \in [0, T_0]$,

$$FC_t^c = \sum_{j=1}^{n} \delta_j B(t, T_j) \left( L_t(T_{j-1}, T_j) N(d_+(t, T_{j-1}, T_j)) - \kappa N(d_-(t, T_{j-1}, T_j)) \right)$$

where $N$ denotes the standard Gaussian cumulative distribution function and for all $j = 1, 2, \ldots, n$

$$d_\pm(t, T_{j-1}, T_j) = \frac{\ln (L_t(T_{j-1}, T_j)/\kappa) \pm \frac{1}{2} v^2(t, T_{j-1}, T_j)}{v(t, T_{j-1}, T_j)}$$

and

$$v^2(t, T_{j-1}, T_j) = \int_t^{T_{j-1}} |\lambda(u, T_{j-1}, T_j)|^2 du.$$
1.7 LIBOR Swaptions

Another widely used interest rate derivative is a swaption, which grants the holder of the contract the right, but not the obligation, to enter the underlying swap agreement at time $T_0$ for no additional cost. We may note the subtle differences between a swaption and a cap. The cap can be viewed as a basket of options where the holder pays a premium to receive the potentially positive payoffs at each $T_j$. In contrast, a swaption is an OTC option on the entire swap contract, granting the holder opportunity to enter the swap contract at time $T_0$ for the duration of the swap. We may use the following generic definition of a swaption on a payer swap with $n$ periods.

**Definition 1.9.** A *payer swaption* with strike rate $\kappa > 0$ and maturity $T_0$ yields the payoff $(\text{FS}^{\kappa}_{T_0})^+$ at time $T_0$ for the holder.

In view of Lemma 1.2 the arbitrage-free price of the swaption can be represented as follows

$$\text{PS}_t^\kappa = B_t \mathbb{E}_Q \left[ B_{T_0}^{-1} \left( B(t, T_0) - \sum_{j=1}^{n-1} \delta_j B(t, T_j) - (1 + \delta_n \kappa) B(t, T_n) \right) + \mid \mathcal{F}_t \right]$$

which is a useful representation for the price of a LIBOR swaption in any Markovian factor model of the short-term rate. We also deduce from Lemma 1.4 that the payoff of a LIBOR swaption can be conveniently represented as follows

$$(\text{FS}^{\kappa}_{T_0}(\kappa))^+ = \sum_{j=1}^{n} \delta_j B(T_0, T_j)(\kappa_{T_0}(T_0, n) - \kappa)^+ = N_{T_0}(\kappa_{T_0}(T_0, n) - \kappa)^+, \quad \text{where } N_t := \sum_{j=1}^{n} \delta_j B(t, T_j),$$

which leads to the following lemma.

**Lemma 1.6.** The arbitrage-free price of a LIBOR swaption equals, for all $t \in [0, T_0]$,

$$\text{PS}_t^\kappa = N_t \mathbb{E}_{Q^N} \left( (\kappa_{T_0}(T_0, n) - \kappa)^+ \mid \mathcal{F}_t \right)$$

where $Q^N$ is the martingale measure associated with the numeraire $N$.

The Swap Market Model (SMM) for a co-terminal family of forward swap rates referencing LIBOR was introduced in Jamshidian [36]. In the next result, we focus on arbitrage-free pricing of a swaption written on a swap with the longest duration, denoted by $n$, but an analogous pricing formula is valid for every swaption for a given family of underlying co-terminal swaps. Notice that the pricing formula of Proposition 1.4 agrees with the prevailing market convention for pricing of plain vanilla swaptions but, unfortunately, models assumed in Propositions 1.3 and 1.4 are incompatible with each other. For a detailed survey of LIBOR and swap market models, the reader may consult [54] or the monographs [10, 45].

**Proposition 1.4.** Assume that the dynamics of the forward LIBOR swap rate $\kappa(t_0, n)$ are, for $t \in [0, T_0]$,

$$d\kappa(t, T_0, n) = \kappa(t, T_0, n)\nu(t, T_0, n) dW^N_t$$

where $W^N$ is a $d$-dimensional Brownian motion under the forward swap measure $Q^N$. Then the arbitrage-free price of a LIBOR swaption with strike $\kappa > 0$ equals, for $t \in [0, T_0]$

$$\text{PS}_t^\kappa = \sum_{j=1}^{n} \delta_j B(t, T_j) \left( \kappa(t, T_0, n)N(d_+(t, T_0, n)) - \kappa N(d_-(t, T_0, n)) \right)$$

where $N$ denotes the standard Gaussian cumulative distribution function and

$$d_+(t, T_0, n) = \frac{\ln(\kappa(t, T_0, n)/\kappa) + \frac{1}{2} \nu^2(t, T_0, n)}{\nu(t, T_0, n)}$$

and

$$\nu^2(t, T_0, n) = \int_t^{T_0} |\nu(u, T_0, n)|^2 du.$$
2 Overnight Interest Rates

As explained in the previous section, single-curve term structure models hinge on the postulate of existence of a risk-free interest rate, which is available at all times to all market participants. In the classical single short rate modeling approach, it is common to denote by $r_t$ the theoretical instantaneous risk-free rate such that any entity is entitled to a short-term borrowing or lending of cash at rate $r_t$ at time $t$. Then there exists a unique money market account $B$ growing at the rate $r_t$ as well as a zero-coupon bond price $B(t, T)$ for any maturity $T > 0$. All market prices are then driven by the short term rate process $r_t$, which usually is assumed to enjoy the Markov property. It is also postulated that for every $T$, the bond maturing at time $T$ is a traded asset, which is default-free, and the bond market is arbitrage-free in the classical sense.

Under the multi-curve model, we depart from the classical framework where models are built around the assumption of a single instantaneous risk-free rate $r$, which is available to all market participants for borrowing and lending at any time $t \in [0, T^*)$. In the extended term structure model, the various interest rates represent different segments of financial markets and sources of funds. In contrast to the classical approach, access to the various markets may be restricted and every market is not always available to certain market participants. In addition, the issues of credit risk and roll-over risk needs to be examined.

The aim of this section is relatively modest since we only wish to outline the setup of the multi-curve framework, which consists of multiple overnight interest rates and related futures contracts. We assume that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where the filtration $\mathbb{F}$ satisfies the usual conditions of right-continuity and $\mathbb{P}$-completeness, and the initial $\sigma$-field $\mathcal{F}_0$ is trivial. All processes introduced in what follows are assumed to be $\mathbb{F}$-adapted.

We start by introducing the notation and assumptions about overnight rates in the extended market model, namely, the SOFR, EFFR, an unsecured funding rate and a collateral rate. We thus have a number of overnight rates, although none of them is assumed to be risk-free in the classical single-curve sense. Furthermore, we may formally associate each short term rate with a corresponding zero-coupon bond. However, in contrast to the classical approach, these bonds are not necessarily assumed to be traded assets. Rather, our goal is to focus on SOFR futures, SOFR-linked swaps and overnight indexed swaps, as opposed to the bond market and, unlike in the classical approach where the bond and swap markets are considered to be one market, we acknowledge that the bond and swap markets are autonomous (albeit related) sectors of the global fixed income market and thus each of them requires a distinct modeling approach.

2.1 Secured Overnight Financing Rate

As discussed in the introduction, due to concerns about the sustainability of the LIBOR, the Federal Reserve Board and the Federal Reserve Bank of New York initiated an effort to introduce an alternative reference rate in the United States. In 2017, the Alternative Reference Rates Committee (ARRC) convened by the Federal Reserve Board identified the Secured Overnight Financing Rate (SOFR) as its preferred alternative reference rate. The SOFR represents a broad measure for the cost of borrowing cash overnight collateralized by Treasury securities. The Federal Reserve Bank of New York began publishing the SOFR in April 2018 as part of an effort to replace LIBOR, the longstanding benchmark rate used around the world. The daily fixing is published by the New York Federal Reserve regarding the previous overnight transactions at approximately 8:00 am ET. Since its introduction, the SOFR has gradually became an influential interest rate that banks use to price U.S. dollar-denominated loans and fixed income derivatives. Historically, daily SOFR was a few basis points per annum lower than daily EFFR, which is due to the fact that the first rate is secured and the second one is unsecured so that it bears the credit risk.

The repo market for a given security refers to the sale of an asset between two parties, with the agreement to repurchase the asset at a later date (often overnight), typically for a higher price. The repo agreement may therefore be considered as a short term loan with the collateral backing of the underlying asset, where the corresponding rate of return upon repurchase is called the repo rate. In
the case of SOFR, the rate refers to the overnight repurchase market where the underlying assets are Treasury securities. Recently, the repo market has been very active with the Federal Reserve actively buying securities to generate liquidity. The repo market is critical for firms to be able to meet short term cash flow requirements by exchanging the Treasury securities (i.e., bonds, notes and bills) for cash.

The published daily SOFR fixing is calculated as a volume-weighted median, which is the rate associated with transactions at the 50th percentile of transaction volume with “specials” excluded. “Specials” are repos for specific-issue collateral, which can take place at much lower rates than general collateral trades since cash providers may be willing to accept a lesser return on their cash, or even at times accept a negative return, in order to secure a particular security. The volume-weighted median rate of general collateral repos is calculated by ordering the transactions from lowest to highest rate, taking the cumulative sum of volumes of these transactions, and identifying the rate associated with the trades at the 50th percentile of dollar volume. At publication, the volume-weighted median is rounded to the nearest basis point.

We denote by \( \rho^s(t_j) \) the SOFR for overnight transactions negotiated on business day \( t_j \) and published on the next good business day after transaction day. The SOFR Averages are calculated through compounding of the overnight rates over a reference period. Let \( n_c \) (resp. \( n_u \)) denote the number of calendar days (resp. business days) during the period \([T, T+\delta]\). Then the (compound) SOFR Average for the period \([T, T+\delta]\) equals

\[
R^s(T, T + \delta) := \frac{360}{n_c} \prod_{j=1}^{n_b} \left( 1 + \frac{n_j \rho^s(t_j)}{360} \right) - 1
\]

where \( n_j \) is the number of calendar days to which \( \rho^s(t_j) \) applies. Typically, \( n_j = 1 \) for every business day, except for Friday where \( n_j = 3 \) and Saturday where \( n_j = 2 \). In accordance with broader U.S. dollar money market convention, interest is calculated using the actual number of calendar days, but assuming a 360-day year.

The simple SOFR Average for the period \([T, T + \delta]\) is given by the following expression

\[
\bar{R}^s(T, T + \delta) := \frac{360}{n_c} \sum_{j=1}^{n_b} \frac{n_j \rho^s(t_j)}{360}.
\]

Intuitively, the SOFR value on Saturday and Sunday is implicitly assumed to be equal to the SOFR fixing on preceding Friday and an analogous rule applies to public holidays. The SOFR Averages are published with tenors of 30, 90, and 180 calendar days, that is, for \( n_c \) equal to 30, 90 or 180.

For more details on calculations of averages for the SOFR and other overnight rates, see [23, 27].

**Definition 2.1.** We denote by \( r^s \) the instantaneous SOFR so that the continuously compounded SOFR account \( B^s \) over \([T, T + \delta]\) satisfies

\[
\frac{B^s_{T+\delta}}{B^s_T} = \exp \left( \int_T^{T+\delta} r^s_u \, du \right) \approx \prod_{j=1}^{n_b} \left( 1 + \frac{n_j \rho^s(t_j)}{360} \right). \tag{2.1}
\]

The compound SOFR over the period \([T, T + \delta]\) is an \( \mathcal{F}_{T+\delta} \)-measurable random variable given by

\[
R^s(T, T + \delta) := \frac{1}{\delta} \left( \exp \left( \int_T^{T+\delta} r^s_u \, du \right) - 1 \right) = \frac{1}{\delta} \left( \frac{B^s_{T+\delta}}{B^s_T} - 1 \right). \tag{2.2}
\]

It is clear that the compound SOFR for the period \([T, T + \delta]\) may only be observed at time \( T + \delta \), which again emphasizes the difference between a forward-looking LIBOR and a backward-looking SOFR. We henceforth assume the instantaneous SOFR is driven by the underlying factor process \( x \) supplemented by a basis \( \alpha^s \)

\[
r^s_t = x_t + \alpha^s_t, \quad \forall t \in [0, T^*]. \tag{2.3}
\]
2.2 Effective Federal Funds Rate

Federal funds are excess reserves that commercial banks and other financial institutions deposit at regional Federal Reserve banks. These funds can be lent, then, to other market participants with insufficient cash on hand to meet their lending and reserve needs. The Federal Funds Rate is the interest rate at which depository institutions (banks and credit unions) are expected to lend reserve balances to other depository institutions overnight on an uncollateralized basis. Since 2008, the Federal Reserve no longer defines a target rate but rather a target range (0 to 0.25%, as of November 2021). This change was in part due to the GFC and the lack of liquidity creating difficulty to maintain the target rate. Besides the Federal Funds Rate, the Federal Reserve also sets the Interest Rate on Reserve Balances (IORB), which is the rate of interest that the Federal Reserve pays on balances maintained by or on behalf of eligible institutions in master accounts at Federal Reserve Banks. The IORB is set by the Board of Governors (0.15%, as of November 2021) and it is an important tool of monetary policy.

In fact, the Federal Open Market Committee (FOMC) cannot force banks to charge the Federal Funds Rate but rather sets a target rate as a guidepost. Open market operations (OMOs) – the purchase and sale of securities in the open market by a central bank – are a key tool used by the Federal Reserve in the implementation of monetary policy. The actual interest rate a lending bank will charge is determined through negotiations between the two banks and the weighted average of interest rates across all transactions of this type is known as the Effective Federal Funds Rate (EFFR). The EFFR fixing is based on unsecured transactions between commercial banks, borrowing and lending excess reserves in their exchange account with Federal Reserve banks.

To provide a better insight into transaction-based information available to the public on broad overnight unsecured funding costs, the New York Fed publishes also the Overnight Bank Funding Rate (OBFR) for which the fixing is calculated using transactions in both federal funds and Eurodollars. The Eurodollar transactions included in the overnight bank funding rate represent borrowing booked at offshore branches that are managed by U.S.-based banking offices, and primarily reflect transactions executed in the United States. The OBFR is published daily based on trades executed on the previous day. For more details on calculation methodology for the EFFR and OBFR, see [23].

The EFFR fixing published on business day \( t_j \), which we denote as \( \rho^e(t_j) \), is calculated using the same methodology as used for the SOFR. The EFFR Average for the period \([T, T+\delta]\) becomes fully known at time \( T+\delta \) and is given by

\[
R^e(T, T+\delta) := \frac{360}{n_c} \prod_{j=1}^{n_b} \left( 1 + \frac{n_j \rho^e(t_j)}{360} \right) - 1
\]

where \( n_c \) (resp. \( n_b \)) is the number of calendar days (resp. business days) during the period \([T, T+\delta]\) and \( n_j = 1 \) for every business day, except for Friday where \( n_j = 3 \). As in the case of SOFR, the EFFR value on Saturday and Sunday is assumed to be equal to the EFFR fixing on preceding Friday.

**Definition 2.2.** Let \( r^e_t \) denote the instantaneous EFFR so that the continuously compounded EFFR account \( B^e \) over \([T, T+\delta]\) satisfies

\[
\frac{B^e_{T+\delta}}{B^e_T} = \exp \left( \int_T^{T+\delta} r^e_u \, du \right) \approx \prod_{j=1}^{n_b} \left( 1 + \frac{n_j \rho^e(t_j)}{360} \right).
\]  

(2.4)

The compound EFFR over the period \([T, T+\delta]\) is an \( \mathcal{F}_{T+\delta} \)-measurable random variable given by

\[
R^e(T, T+\delta) := \frac{1}{\delta} \left( \exp \left( \int_T^{T+\delta} r^e_u \, du \right) - 1 \right) = \frac{1}{\delta} \left( \frac{B^e_{T+\delta}}{B^e_T} - 1 \right).
\]  

(2.5)

Similarly to the SOFR, the EFFR will be modeled as a factor process \( x \) plus a basis \( \alpha^e \) so that

\[
r^e_t = x_t + \alpha^e_t, \quad \forall t \in [0, T^*].
\]  

(2.6)
2.3 Unsecured Funding Rate

Commercial banks in the United States have two primary ways to borrow money for their short-term operating needs. They can borrow and loan money to other banks without the need for any collateral using the market-driven interbank rate. The interbank lending system is short-term, typically overnight, and rarely more than a week. They also can borrow the money for their short-term operating requirements from the Federal Reserve Bank. The discount rate is the interest rate charged to commercial banks and other financial institutions for short-term loans they take from the Federal Reserve Bank.

We introduce the notion of an unsecured funding rate for loans with no collateral backing. As an unsecured rate is uncollateralized there exists a credit risk component in the event of default. In contrast to the previously defined overnight rates, an unsecured funding rate is not market wide, but rather firm specific and thus different banks will be able to borrow money at different funding rates. Banks with a high credit rating will be able to borrow funds at a lower rate than banks with a poor credit rating.

Definition 2.3. We denote by $r_u$ the unsecured funding rate and thus the respective continuously compounded account $B_u$ equals

$$B_u^t = \exp \left( \int_0^t r_u^s \, ds \right), \quad \forall t \in [0, T^*].$$  \hspace{1cm} (2.7)

As for other overnight rates, we assume that the unsecured funding rate $r_u$ is modeled as the factor process $x$ plus a basis $\alpha^u$

$$r_u^t = x_t + \alpha_u^t, \quad \forall t \in [0, T^*].$$  \hspace{1cm} (2.8)

2.4 Price Alignment Interest and Collateral Rate

Another key concept when considering the riskiness of a contract is the collateral backing. Collateralization (also known as the margining) refers to a party depositing capital (typically, Treasury notes or bonds) as backing to offset liabilities in case of default. The collateral rate is the interest rate paid on collateral (variation margin) account by the collateral receiver to the pledging party, that is, the collateral provider. Although cash is considered the safest form of collateral, many other assets may be deposited as collateral.

Often, when an entity borrows funds to purchase an asset, the asset may be deposited as collateral. For example, an options trader may post the underlying shares purchased for delta hedging against the borrowed funds. In the case of centrally cleared trades, the collateral rate is called the Price Alignment Interest (PAI) and the SOFR was recently chosen to replace the EFFR as standard PAI.

More generally, the collateral rate negotiated in Credit Support Annex (CSA) of an OTC trade does not have a specific market definition. Rather, the remuneration rate on collateral is agreed upon by the two parties at the inception of the trade. Therefore, we find it convenient to introduce a generic instantaneous interest rate, henceforth denoted as $r^c$, to represent the interest on margin account and to consider alternative specifications for the generic process $r^c$. For instance, we may set $r^c = r^e$, $r^c = r^u$ or $r^c = r^s$, depending on the trade’s CSA and prevailing market conventions.

Definition 2.4. Let $r^c = x + \alpha^c$ denote the instantaneous collateral rate, that is, the remuneration rate on the margin account representing the pledged or received collateral. The respective continuously compounded account $B^c$ satisfies

$$B^c_t = \exp \left( \int_0^t r^c_u \, du \right), \quad \forall t \in [0, T^*].$$  \hspace{1cm} (2.9)
2.5 Overnight Indexed Swaps

An Overnight Indexed Swap (OIS) is a swap in which one party pays a fixed rate of interest known as the OIS rate, which depends on the term of the swap and is known at trade inception, whereas the floating payment is based on a daily compound overnight interest rate and thus is not known until the end of the life of the OIS. The main use of OIS swaps is to allow banks to lock in the cost of unsecured overnight funding in advance. In the United States, futures referencing EFFR have traded for more than 30 years and overnight indexed swaps referencing EFFR have traded for almost 20 years. The terms to maturity for overnight indexed swaps are between one week to one year, with the bulk of the trading concentrated in relatively short maturities. An OIS carries very little credit risk as there is no exchange of principal; the only payment takes place at the maturity of the swap to reflect the net interest obligation of one party to the other. To fully define an OIS, the following attributes need to be specified: the notional amount, start and end dates, the reference overnight rate and the (fixed) OIS rate. Intuitively, to make the OIS swap have zero initial value at inception, the OIS rate must equal the market’s risk-neutral expectation of what the daily compounded index rate will be over the lifetime of the OIS.

An OIS may reference any overnight rate index, such as either SOFR or EEFR if the cash flows are in U.S. dollars (a similar role is played by SONIA and EONIA in sterling and euro markets, respectively). In the United States, the OIS rate was traditionally based on the EFFR as the reference overnight rate. However, the Financial Accounting Standards Board (FASB) issued on October 25, 2018 an Accounting Standards Update (ASU) to include the OIS rate based on the SOFR as a benchmark interest rate for hedge accounting purposes. OIS markets also use compound interest, and thus instruments that use compound interest will be easier to hedge. Because OIS is based on overnight lending, it has a lower credit risk than LIBOR and thus the OIS-LIBOR basis can be interpreted as a measure of interbank risk (see, e.g., Filipović and Trolle [25]).

As mentioned above, historically OIS contracts in the United States have referenced EFFR and hence high volume of contracts creates a very liquid asset for trading and a corresponding rate that accurately reflects market expectations. OIS contracts referencing SOFR are traded at a fraction of the volume since being introduced in 2018. The low volume creates illiquidity issues for market players and respective rate serves as a poor reflection of market expectations. Therefore, when referencing OIS as a traded asset, we shall focus on the OIS rate corresponding to EFFR.

**Definition 2.5.** An EFFR-linked overnight index swap is a fixed-to-floating swap contract settled in arrears where the floating leg is given by the compound EFFR Average over the period. The fixed leg of the OIS initiated at time $t$ is an $F_t$-measurable random variable, which is determined such that the OIS has zero value at time $t$.

For quotation purposes, the fixed leg is given by the OIS rate $o(t,T)$ over the period $[t,T]$. Specifically, the net payoff at time $T$ of the EFFR OIS over the period $[t,T]$ equals

$$X_T^c := R^c(t,T) - \text{OIS}^c(t,T)$$

(2.10)

where the floating leg $R^c(t,T)$ is the (compound) EFFR Average over the period $[t,T]$. The fixed leg OIS$^c(t,T)$ is given by the OIS rate $o(t,T)$ reset at time $t$ and compounded daily over the period $[t,T]$ so that

$$\text{OIS}^c(t,T) = \frac{360}{n_c} \left[ \prod_{j=1}^{n_b} \left( 1 + \frac{o_j(t,T)}{360} \right) - 1 \right]$$

(2.11)

where $n_c$ (resp. $n_b$) is the number of calendar days (resp. business days) during the reference period $[t,T]$. It is clear that the OIS rate $o(t,T)$ is a forward-looking term rate and thus, if the OIS floating leg is given by $R^c(t,T)$, rather that $R^o(t,T)$, then the forward-looking term SOFR could be, in principle, inferred from market data for OISs, provided that a satisfactory level of liquidity is achieved. Another option for identification of implied forward-looking term rates referencing SOFR is to use prices of SOFR futures contracts, which are presented and formalized in the next section.
2.6 SOFR Futures

In 2018, the Chicago Mercantile Exchange (CME) developed futures contracts referencing SOFR Averages. Since their release, SOFR futures have grown significantly in trading volume (see [15]) and thus the futures price reflects an accurate representation of the markets expectations for the overnight SOFR over the contract period. The CME applies compounding of daily SOFR values between quarterly IMM dates (the third Wednesday of every March, June, September, and December) to determine final settlement prices of expiring Three-Month SOFR futures. To be more specific, the final settlement price for an expiring Three-Month SOFR futures contract is 100 minus the SOFR rate, compounded over the contract’s reference quarter. The reference quarter for Three-Month SOFR futures is the interval from the third Wednesday (inclusive) of the month three months prior to the delivery month, to the third Wednesday (exclusive) of the delivery month. In the case of Three-Month SOFR futures, the method for calculation of compound SOFR is exactly the same as for the SOFR Average, which was presented in Section 2.1.

It is worth noting that compounding conventions are fully consistent with standard U.S. dollar OIS, for which a swap’s floating-rate leg is based on the business-day-compounded EFFR. Each Three-Month SOFR futures contract expires by cash settlement, by reference to the three-month term interest rate implied by compounded daily SOFR interest between the third Wednesday of the contract month and IMM Wednesday of the contract delivery month. Hence the Three-Month SOFR futures critical dates are aligned with the IMM calendar for Eurodollar futures. Like Eurodollar futures, Three-Month SOFR futures are sized at $25 per basis point per annum of contract interest.

The rules for One-Month SOFR futures contract traded on CME are slightly different. Final settlement for the One-Month SOFR futures contract is based on the average daily values of the SOFR benchmark over the given delivery month, that is, the simple arithmetic average of the daily SOFR rates of the calendar month (the sum of all rates in the month period divided by the number of calendar days in the month period) rounded to the nearest 1/10th of a basis point. The final contract settlement value is equal to 100 minus the rounded arithmetic average (e.g., the settlement value equals 100 – 1.041 = 98.959 if the rounded arithmetic average of SOFR equals 1.041%).

As mentioned above, Three-Month SOFR futures contracts listed by the CME group are defined as the average compounded daily SOFR over the period. However, for our purposes, we find it convenient to assume continuous compounding of the short term rate. Similar to the definition of futures LIBOR, we may provide a formal definition of SOFR futures rate and hence also SOFR futures price. Recall that the interpretation of a martingale measure \( Q^f \) will be given in Section 3.

**Definition 2.6.** A **SOFR futures** contract for the period \([T, T + \delta]\) is defined as a futures contract referencing SOFR Average and thus the **SOFR futures rate** is given by, for all \( t \in [0, T + \delta] \),

\[
R_t^*, f(T, T + \delta) := \mathbb{E}_{Q^f}(R^*(T, T + \delta) \mid \mathcal{F}_t). \tag{2.12}
\]

Using (2.2) and (2.12), it is easy to check that the process \( R^*, f(T, T + \delta) \) is a martingale under \( Q^f \) and the following equality holds, for every \( t \in [0, T] \),

\[
1 + \delta R_t^*, f(T, T + \delta) = \mathbb{E}_{Q^f}(e^{\int_t^{T+\delta} r_u^* \, du} \mid \mathcal{F}_t) \tag{2.13}
\]

and, for every \( t \in [T, T + \delta] \),

\[
1 + \delta R_t^*, f(T, T + \delta) = e^{\int_t^{T+\delta} r_u^* \, du} \mathbb{E}_{Q^f}(e^{\int_t^{T+\delta} r_u^* \, du} \mid \mathcal{F}_t). \tag{2.14}
\]

Notice that \( R_t^*, f(T, T + \delta) \) can be replaced by \( f_t^*(T, T + \delta) := 1 - R^*, f(T, T + \delta) \) in equation (2.12) but this would not have any nontrivial impact on mathematical modeling and properties of SOFR futures. If we denote by \( f_t^*(T, T + \delta) \) the SOFR futures price at time \( t \), then clearly \( f_t^*(T, T + \delta) = 1 - R_t^*, f(T, T + \delta) \) and thus the two processes have the same dynamics with respect to \( t \), up to the sign since \( df_t^*(T, T + \delta) = -dR_t^*, f(T, T + \delta) \). For that reason, we will focus in Section 3.2 on the dynamics of SOFR futures rate \( R_t^*, f(T, T + \delta) \).
3 Multi-Curve Pricing of SOFR Swaps

When considering the pricing of derivative claims, the price of the contract will vary based on the short rate used for discounting. Under the traditional LIBOR models the risk free rate $r$ was implemented for discounting. The rate $r$ was also the short rate corresponding to LIBOR in the case of bond prices. However, post GFC, it has been clear that this model is not reflective of the realities of financial markets. In addition, most fixed income derivatives, such as interest rate swaps, are now subject to collateralization and, more recently, they also started to use some overnight rate as a variable reference rate for a floating leg. Finally, the question of funding costs for hedging strategies has become much more important than in pre GFC framework where a unique short-term rate was identified with the funding rate for a hedge.

The related issue of idiosyncratic discounting in the risk-neutral valuation formula was examined by Piterbarg [51] who studied pricing of collateralized contracts on assets (e.g., collateralized stock options) under divergent interest rates, $r_R, r_F$ and $r_C$ for secured (repo) funding, unsecured funding, and collateral, respectively. One of his conclusions was that it suffices to focus on the collateral rate $r_C$ when pricing contracts with theoretical full collateral. Intuitively, the rate on collateral may be viewed as an equivalent of a funding rate when dealing with fully collateralized contracts. Piterbarg’s results on funding costs and hedging of collateralized contracts were extended to more general linear and nonlinear setups (for instance, with differing lending and borrowing rates) in subsequent works by several authors, including [4, 5, 8, 9, 11, 12, 16, 17, 30, 46, 47, 48, 49].

In the present work, we focus on the case of a linear trading model, which means that it is assumed throughout that any particular interest rate does not depend on the direction of a trade but, of course, the postulate of a single funding rate is not made. Such assumption is natural for SOFR and EFFR, which are indeed unique, but for unsecured funding it would be possible to introduce different lending and borrowing rates. However, the conditional expectation would then be replaced by a nonlinear backward stochastic differential equation (BSDE) and thus an explicit representation for solution would no longer be available (for some examples, we refer to, e.g., Nie and Rutkowski [46, 47, 48]). Our main goal is to examine multi-curve arbitrage-free pricing of collateralized derivatives referencing SOFR through self-financing hedging strategies based on idiosyncratic funding costs and liquidly traded contracts, such as SOFR futures and OIS based on SOFR. The main results furnish closed-form expressions for replicating strategies under the assumption that the factor process $x$ is governed by Vasicek’s dynamics and bases are deterministic. A possible extensions to more general dynamics of a factor process (e.g., a multi-factor affine term structure model; see Duffie and Kan [21], Christensen et al. [14] or Klingler and Syrstad [56]) is also briefly discussed in Remark 3.3. For modelling issues regarding the dynamics of overnight rates and related econometric studies, we refer to recent works by Alfeus et al. [1], Berndt et al. [6], Backwell et al. [3], Gellert and Schlögl [28], Haitfield and Park [33] and Klingler and Syrstad [56].

In a model-free study Haitfield and Park [33] show that SOFR futures-implied term rates accurately predict realized overnight rates during most periods and thus can be used as indicative forward-looking term rates derived from end-of-day SOFR futures prices. In that way, one can obtain synthetic term rates analogous to the term LIBOR rates, which were commonly used in loan contracts. Their approach was used till May 26, 2021 by the Federal Reserve to publish an indicative term SOFR rate derived from futures prices. More recently, the ARRC announced it selected CME Group as the administrator that it plans to recommend for a forward-looking SOFR term rate, once market indicators for the term rate are met. For more details on the time frame of transition plan from LIBOR to new forward-looking term rate, we refer to [2]. Klingler and Syrstad [56] conduct a detailed econometric analysis of overnight and term spreads and provide statistical tests of hypotheses regarding the origin of volatility and spikes observed in those spreads (see also Appendix C in [56] where a stylized model of interbank lending with asymmetric regulatory constraints is examined). The interested reader is also referred to Berndt et al. [6] who propose the introduction of the Across-the-Curve Credit Spread Index (AXI), which aims to represent the forward-looking credit-sensitive basis added to the SOFR to support the transition of loan products away from the LIBOR.
### 3.1 Futures Trading Strategies for Collateralized Contracts

Let a semimartingale \( f = (f^1, f^2, \ldots, f^m) \) represent a given family of futures prices. A futures trading strategy (or, simply, \( \varphi \)) is a process \( \varphi = (\varphi^0, \varphi^1, \ldots, \varphi^m) \) where the processes \( \varphi^k; k = 0, 1, \ldots, m \) are \( \mathbb{F} \)-adapted. Consider a collateral process \( C \) where \( C > 0 \) means that collateral is received by a bank and can be used for trading (that is, we work under the fairly common postulate of rehypothecation) whereas \( C < 0 \) means that a bank is the collateral pledger. By convention, we set \( C_{0-} = 0 \). We denote by \( r^c \) the interest rate paid on collateral amount (margin account) so that remuneration of collateral amount \( C_t \) over time period \([t, t + dt]\) is given by \(- r^c_t C_t dt\), as seen from the bank's perspective. We will sometimes assume that \( r^c = r^r \) but, of course, other conventions for remuneration of collateral are possible.

Finally, we henceforth denote by \( B^h \) the process satisfying \( dB^h_t = r^h_t B^h_t \, dt \) with \( B^h_0 = 1 \) where \( r^h \) represents the hedge funding rate. Hence \( B^h \) can be any of the processes \( B^r, B^e \) or \( B^u \), their combination or some other stochastic process, depending on a particular manner in which the cash component of a hedge for a given trade is funded by a bank.

**Definition 3.1.** The portfolio's value process \( V^p(\varphi) \) of a futures trading strategy \( \varphi \) with collateral \( C \) equals, for every \( t \in [0, T] \),

\[
V^p_t(\varphi) = \varphi^0_t B^h_t
\]

and we say that a futures portfolio \( \varphi \) is self-financing if, for every \( t \in [0, T] \),

\[
V^p_t(\varphi) = V^p_0(\varphi) + \int_0^t \varphi^0_u \, dB^h_u + \sum_{i=1}^m \int_0^t \varphi^i_u \, df^i_u + C_t - \int_0^t r^c_u \, dC_u.
\]

The hedger's wealth process is given by \( V_t(\varphi) = V^p_t(\varphi) - C_t \) for every \( t \in [0, T] \).

Let us denote \( \tilde{V}^h_t(\varphi) := (B^h)^{-1} V^p_t(\varphi), \tilde{V}^{p,h}_t(\varphi) := (B^h)^{-1} V^p_t(\varphi) \) and \( \tilde{C}^h_t := (B^h)^{-1} C_t \).

**Proposition 3.1.** The discounted wealth \( \tilde{V}^h_t(\varphi) := (B^h)^{-1} V^p_t(\varphi) \) of any self-financing futures portfolio \( \varphi \) with an arbitrary collateral \( C \) satisfies

\[
\tilde{V}^h_t(\varphi) = \tilde{V}^h_0(\varphi) + \sum_{i=1}^m \int_0^t \varphi^i_u (B^h_u)^{-1} \, df^i_u + \int_0^t (r^h_u - r^r_u) \tilde{C}^h_u \, du.
\]

**Proof.** The Itô integration by parts formula gives

\[
d\tilde{V}^h_t(\varphi) = (B^h)^{-1} \, dV_t(\varphi) + V_t(\varphi) \, dB^h_t - (B^h)^{-1} \, dV^p_t(\varphi) \, dC_t
\]

which shows that (3.3) is satisfied.

**Remark 3.1.** It is clear that if \( r^c = r^h \), then

\[
\tilde{V}^h_t(\varphi) = \tilde{V}^h_0(\varphi) + \sum_{i=1}^m \int_0^t \varphi^i_u (B^h_u)^{-1} \, df^i_u
\]
where we now have $\bar{V}^h(\varphi) = (B^h)^{-1}V(\varphi) = (B^c)^{-1}V(\varphi)$ and thus collateralization with any process $C$ does not affect the wealth dynamics of a self-financing futures strategy. Needless to say, the choice of collateral $C$ is still crucial for specification of the close-out payment, which is used to settle the trade in the occurrence of counterparty’s default (for related issues regarding credit risk, e.g., Bielecki et al. [4, 5], Brigo et al. [8, 9], Brigo and Pallavicini [11], Crépey [16, 17], and Pallavicini et al. [49]).

The level of collateralization is usually chosen to be proportional to the price of a contract, which is formally represented by the wealth of a hedging strategy. Therefore, we henceforth assume that $C = -\beta V(\varphi)$ where $\beta$ is an $\mathbb{F}$-adapted, nonnegative stochastic process. Notice that is not assumed that the process $\beta$ is fixed and applied to all trades between various counterparties.

On the contrary, we acknowledge that $\beta$ depends on the CSA complementing a particular trade and thus we allow for an arbitrary level of collateralization for every contract studied in our market model. In that sense, the level of collateralization can be seen as an additional parameter affecting the price of a contract, which entails that one gets a whole spectrum of prices for any particular contract within a common market model. The following proposition will allow us to effectively deal with proportional collateralization of an arbitrary derivative contract.

**Proposition 3.2.** The discounted wealth $\bar{V}^c(\varphi) := (B^c)^{-1}V(\varphi)$ of any self-financing futures portfolio $\varphi$ with proportional collateral $C = -\beta V(\varphi)$ satisfies

$$
\bar{V}_t^c(\varphi) = \bar{V}_0^c(\varphi) + \int_0^t (1 - \beta_u)(r^h_u - r^c_u)\bar{V}_u^c(\varphi) \, du + \sum_{i=1}^m \int_0^t \varphi^i_u(B^c_u)^{-1} \, df^i_u. \tag{3.5}
$$

**Proof.** We have

$$
d\bar{V}_t^c(\varphi) = (B^c_t)^{-1}dV_t(\varphi) + V_t(\varphi) \, d(B^c_t)^{-1}
$$

$$
= (B^c_t)^{-1}\left(\varphi^0_t dB^h_t + \sum_{i=1}^m \varphi^i_t df^i_t + \beta_t r^c_t V_t(\varphi) \, dt\right) - r^c_t (B^c_t)^{-1}V_t(\varphi) \, dt
$$

$$
= (B^c_t)^{-1}\left(r^h_t V_t^R(\varphi) \, dt + \sum_{i=1}^m \varphi^i_t df^i_t + \beta_t r^c_t V_t(\varphi) \, dt - r^c_t V_t(\varphi) \, dt\right)
$$

$$
= (B^c_t)^{-1}\left((1 - \beta_t)r^h_t V_t(\varphi) + \beta_t r^c_t V_t(\varphi) - r^c_t V_t(\varphi)\right) \, dt + \sum_{i=1}^m \varphi^i_t (B^c_t)^{-1} df^i_t
$$

$$
= (1 - \beta_t)(r^h_t - r^c_t)\bar{V}_t^c(\varphi) \, dt + \sum_{i=1}^m \varphi^i_t (B^c_t)^{-1} df^i_t
$$

and thus (3.5) holds.

**Remark 3.2.** The special case of full collateralization, which is a fairly common market practice, corresponds to $C_t = -V_t(\varphi)$ or, equivalently, $\beta_t = 1$ for all $t$. Then we obtain

$$
\bar{V}_t^c(\varphi) = \bar{V}_0^c(\varphi) + \sum_{i=1}^m \int_0^t \varphi^i_u(B^c_u)^{-1} \, df^i_u. \tag{3.6}
$$

The following immediate corollary to Proposition 3.2 is useful in a study of arbitrage-free property. Let us denote $r^h_t := (1 - \beta_t)r^h_t + \beta_t r^c_t$ and $dB^3_t = r^3_t B^3_t \, dt$ with $B^3_0 = 1$.

**Corollary 3.1.** The discounted wealth $\bar{V}^3(\varphi) := (B^3)^{-1}V(\varphi)$ of any self-financing futures portfolio $\varphi$ with proportional collateral $C = -\beta V(\varphi)$ satisfies

$$
\bar{V}_t^3(\varphi) = \bar{V}_0^3(\varphi) + \sum_{i=1}^m \int_0^t \varphi^i_u(B^3_u)^{-1} \, df^i_u. \tag{3.7}
$$
If Proposition 3.3. variation margin is assumed to be proportional to the wealth process.

\( \phi \tilde{\beta} \) and the discounted wealth is an \( Q \) as hence considered to be a unit, is, in principle, arbitrary and thus any strictly positive semimartingale can be used for this purpose, which depends not only on collateralization rate \( \beta \) but also on the choice of rates \( r^h \) and \( r^c \) used for hedge funding costs and remuneration of margin account.

We henceforth work under the assumption that \( C = -\beta V(\phi) \) and a martingale measure \( Q^f \) for \( f \) exists and thus we may formulate a minor extension of the classical result on arbitrage-free property of a market model.

\begin{definition}
Given the market model \( \mathcal{M}^f = \mathcal{M}^{f,s,e,u,c,h} = (f, B^s, B^e, B^u, B^c, B^h) \), we say that \( Q \) is an ELMM for any trade collateralized at some rate \( \beta \) in \( \mathcal{M}^f \) if \( Q \) is equivalent to \( P \) on \( (\Omega, \mathcal{F}_{T}) \) and the discounted wealth \( V^\beta(\phi) \) is a Q-local martingale with respect to \( P \) for any self-financing futures portfolio \( \phi \).
\end{definition}

In view of Proposition 3.2 and Corollary 3.1 we have the following result, which holds when the variation margin is assumed to be proportional to the wealth process.

**Proposition 3.3.** If \( C = -\beta V(\phi) \) for an arbitrary process \( \beta \), then \( Q \) is a martingale measure for a trade collateralized at the rate \( \beta \) in \( \mathcal{M}^f \) if and only if \( Q \) is a martingale measure for \( f \).

In view of Proposition 3.3, it is clear that we may consider trades with divergent collateralization rates and still employ the same martingale measure \( Q^f \in \mathcal{P}(f) \) combined with an idiosyncratic process \( B^\beta \) used for discounting of cash flows. Hence if we denote by \( \mathcal{P}(\mathcal{M}^f) \) the class of all martingale measures for collateralized contracts in the market model \( \mathcal{M}^f \), then the equality \( \mathcal{P}(f) = \mathcal{P}(\mathcal{M}^f) \) holds.

We henceforth work under the assumption that \( C = -\beta V(\phi) \) and a martingale measure \( Q^f \) for \( f \) exists and thus we may formulate a minor extension of the classical result on arbitrage-free property of a market model.

**Proposition 3.4.** If a martingale measure \( Q^f \) for \( f \) exists and we only allow for trading strategies with a nonnegative wealth, then the market model \( \mathcal{M}^f = \mathcal{M}^{f,s,e,u,c,h} \) is arbitrage-free with respect to any collateral rate \( \beta \) assuming that the margin account is proportional to the wealth process.

We have following immediate corollary to Proposition 3.4.

**Corollary 3.2.** If a contingent claim \( X \) maturing at time \( T \) can be replicated in \( \mathcal{M}^f \) through a collateralized strategy with rate \( \beta \), then its arbitrage-free price at any date \( t \in [0, T] \) equals

\[
\pi^f_{t,\beta}(X) = B^\beta_t \mathbb{E}_{Q^f}\left((B^\beta_{T+\delta})^{-1}X \mid \mathcal{F}_t\right). \tag{3.8}
\]

In particular, if \( \beta \equiv 0 \), then \( \pi^f_{t,\beta}(X) = \pi^f_{t,h}(X) \) where

\[
\pi^f_{t,h}(X) = B^h_t \mathbb{E}_{Q^f}\left((B^h_{T+\delta})^{-1}X \mid \mathcal{F}_t\right). \tag{3.9}
\]

### 3.2 Arbitrage-free Dynamics of SOFR Futures

The key traded instrument for hedging of contracts referencing SOFR are SOFR futures contracts. As already mentioned, SOFR futures are traded in high volumes, and therefore create a liquid
market for trading. Several papers, including Gellert and Schlägl [28] and Klingler and Syrstad [56] have focused on building the SOFR interest rate curve through stripping the discount factors from SOFR futures. In contrast to those papers, we are mainly concerned with SOFR futures strategies for the purpose of hedging swaps based on SOFR Average and other SOFR-linked derivatives in a multi-curve setup.

A preliminary step is to specify the dynamics for overnight rates \( r^s, r^e \) and \( r^n \), as well as the hedge funding rate \( r^h \) where the overnight rate \( r^h \) is specific to a bank and possibly also to a particular trade. For instance, it can be given as a (convex) combination of overnight rates \( r^s, r^e \) and \( r^n \), but such an assumption is by no means necessary for our further developments.

We henceforth work under the following standing assumption about the factor process \( x \). It should be noticed, however, that our results can be extended to multi-factor models with stochastic bases \( \alpha^s, \alpha^e, \alpha^h \) and \( \alpha^n \). For various applications of affine framework, we refer to Christensen et al. [14], Cuchiero et al. [19], Duffie and Kan [21], Duffie et al. [20] Fontana et al. [26], Grbac et al. [31] (see also Section 4.2 of Brigo and Mercurio [10]).

**Assumption 3.1.** The factor process \( x \) is given by Vasicek’s dynamics

\[
 dx_t = (a - bx_t) \, dt + \sigma \, dW^Q_t \quad (3.10)
\]

for all \( t \in [0, T^s] \), where \( a, b \) and \( \sigma \) are constants and \( W^Q_t \) is a Brownian motion under \( Q^f \). The basis terms \( \alpha^s, \alpha^e \) and \( \alpha^n \) introduced in (2.3), (2.6) and (2.8) are assumed to be deterministic. Furthermore, we postulate that \( r^h = x + \alpha^h \) where the basis \( \alpha^h \) is deterministic.

We first compute the SOFR futures rate, as given by Definition 2.6. Given our assumptions about the modelling of SOFR and the underlying rate process, we may derive an explicit solution for all \( t \in [0, T^s] \). For various applications of affine framework, we refer to Christensen et al. [14], Cuchiero et al. [19], Duffie and Kan [21], Duffie et al. [20] Fontana et al. [26], Grbac et al. [31] (see also Section 4.2 of Brigo and Mercurio [10]).

**Remark 3.3.** A much more flexible Gaussian affine model is examined and fitted to the market data in Skov and Skovmand [55] who focused on the dynamics of SOFR futures by postulating that \( r^s_t = c_0 + c_1^t X_t \) for some \( c_0 \in \mathbb{R} \) and \( c_1 \in \mathbb{R}^d \) where the Markov process \( X \) is governed under \( Q^f \) by the generalized Vasicek’s equation driven by \( dW^Q_t \)

\[
 dX_t = K(\theta - X_t) \, dt + \Sigma \, dW^Q_t
\]

for some \( d \)-dimensional matrices \( K \) and \( \Sigma \) and a vector \( \theta \in \mathbb{R}^d \). More specifically, they dealt with two- and three-factor Gaussian arbitrage-free Nelson-Siegel model (AFNS model) proposed in Christensen et al. [14]. Notice that their goal was different from ours since they aimed to infer the forward-looking term rates from SOFR futures data by identifying the convexity correction via an extended Kalman filter whereas our focus is on general properties of SOFR futures trading strategies under differential funding costs and either without and with collateral backing. In particular, Lemma 3.1 is a special case of results from Appendix A in [55] and our results on pricing and hedging of SOFR swaps, caps and swaptions can be, in principle, extended to the AFNS model, although one would need to address the issue of market incompleteness by introducing additional traded assets referencing SOFR.

**Lemma 3.1.** Under Assumption 3.1, the SOFR futures rate satisfies, for every \( t \in [0, T] \),

\[
 1 + \delta R^s_{t,T,T} = A^e(T,T) \exp\left(\mu(t,T,T) \cdot \delta + \frac{1}{2} v^2(t,T,T) \cdot \delta^2\right)
\]

where

\[
 \mu(t,T,T) = (n(t,T) - n(t,T)) x_t + \int_t^T a u n(u,T) \, du - \int_t^T a u n(u,T) \, du
\]

and

\[
 v^2(t,T,T) = \int_t^T \sigma^2(n(u,T) - n(u,T))^2 \, du + \int_t^T \sigma^2 n^2(u,T) \, du
\]
Therefore, under the martingale measure $Q$, so that

$$1 + \delta R^s_{t}(T, T + \delta) = A^s(T, T + \delta) e^{\int_{t}^{T} r_u^s \, du} e^{\mu(t, t, T + \delta) + \frac{1}{2} \sigma^2(t, t, T + \delta)}$$  (3.12)

where

$$\mu(t, T + \delta) = \mu(t, t, T + \delta) = \int_{t}^{T + \delta} a_n(u, T + \delta) \, du$$

and

$$v^2(t, t, T + \delta) = \int_{t}^{T + \delta} \sigma^2(u, T + \delta) \, du.$$  

The dynamics of the SOFR futures rate are, for $t \in [0, T]$

$$dR^s_{t}(T, T + \delta) = \delta^{-1} (1 + \delta R^s_{t}(T, T + \delta)) (n(t, T + \delta) - n(t, T)) \sigma dW^Q_t$$  (3.13)

and, for every $t \in [T, T + \delta]$,

$$dR^s_{t}(T, T + \delta) = \delta^{-1} (1 + \delta R^s_{t}(T, T + \delta)) n(t, T + \delta) \sigma dW^Q_t.$$  (3.14)

Proof. Let us denote $A^s(T, T + \delta) := e^{\int_{T + \delta}^{T + \delta} \alpha^s_u \, du}$ where $\alpha^s$ is deterministic. The process $R^s T, T + \delta) is a martingale under $Q_f$ and satisfies, for every $t \in [0, T]$,

$$1 + \delta R^s_{t}(T, T + \delta) = E_{Q_f} \left( e^{\int_{t}^{T} r_u^s \, du} | F_t \right) = A^s(T, T + \delta) E_{Q_f} \left( e^{\int_{t}^{T} x_u \, du} | F_t \right).$$  (3.15)

and, for every $t \in [T, T + \delta]$,

$$1 + \delta R^s_{t}(T, T + \delta) = A^s(T, T + \delta) e^{\int_{t}^{T} r_u^s \, du} E_{Q_f} \left( e^{\int_{t}^{T} x_u \, du} | F_t \right).$$  (3.16)

Hence it is clear that it suffices to compute the conditional expectation, for $t \leq T < U$,

$$E_{Q_f} \left( e^{\int_{t}^{U} x_u \, du} | F_t \right) = E_{Q_f} \left( e^{X_{T, U}} | F_t \right)$$

where $X_{T, U} := \int_{T}^{U} x_u \, du$. It can be checked directly that for all $t < T$

$$X_{t, T} := \int_{t}^{T} x_u \, du = n(t, T)x_t + \int_{t}^{T} a_n(u, T) \, du + \int_{t}^{T} \sigma n(u, T) \, dW^Q_u$$

so that

$$X_{T, U} = \int_{T}^{U} x_u \, du - \int_{t}^{T} x_u \, du = X_{t, U} - X_{t, T}$$

$$= \mu(t, T, U) + \int_{t}^{U} \sigma n(u, U) \, dW^Q_u - \int_{t}^{T} \sigma n(u, T) \, dW^Q_u$$

where

$$\mu(t, T, U) := (n(t, U) - n(t, T))x_t + \int_{t}^{U} a_n(u, U) \, du - \int_{t}^{T} a_n(u, T) \, du.$$  

Therefore, under the martingale measure $Q_f$, the conditional distribution with respect to $F_t$ of the random variable $X_{T, U}$ is Gaussian with the conditional expectation $\mu(t, T, U)$ and the conditional variance

$$\text{Var}_{Q_f}(X_{T, U} | F_t) = \int_{t}^{T} \sigma^2(n(u, U) - n(u, T))^2 \, du + \int_{U}^{T} \sigma^2 n^2(u, U) \, du =: v^2(t, T, U).$$

Consequently, for all $t \leq T < U$,

$$E_{Q_f} \left( e^{X_{T, U}} | F_t \right) = e^{\mu(t, T, U) + \frac{1}{2} v^2(t, T, U)}.$$  

In view of (3.15) and (3.16), we conclude that equalities (3.11) and (3.12) are valid. It is now easy to check that the SOFR futures rate is governed by (3.13) and (3.14).
3.3 Uncollateralized and Collateralized SOFR Swaps

We are now in a position to introduce the crucial distinctions between the classical single-curve and multi-curve term structure models. Recall that according to the current market convention for U.S. dollar denominated interest rate swaps, the floating leg payoffs are based on SOFR Average over each accrual period $[T_i, T_i + \delta_i]$ and thus they become fully known at the end of each period. Furthermore, they become known gradually during that period, which is a new feature of a swap. In the classical case of a LIBOR-linked swap, the floating leg is known at the beginning of each accrual period and thus a dynamic hedging strategy for the cash flow occurring at time $T_j$ ceases at the beginning of each period and becomes static between the dates $T_{j-1}$ and $T_j$. Let us consider the case of a multi-period swap contract with the tenor structure $0 \leq T_0 < T_1 < \cdots < T_n$. As before, we denote $\delta_j = T_j - T_{j-1}$ for $j = 1, 2, \ldots, n$. Although the basic structure of an interest rate swap referencing SOFR Average is similar to the classical LIBOR-linked swap described in Section 1.5, it should be noticed that for each accrual period $[T_{j-1}, T_j]$ the fixing date $T_j$ of a SOFR swap coincides with the payment date $T_j$ due to the fact that the SOFR Average is backward-looking (in practice, the payment usually occurs one or two good-business days after the last day of the accrual period).

This should be contrasted with the standard LIBOR-in-advance swap where, for each accrual period, $T_{j-1}$ is the fixing date and $T_j$ is the payment date and thus the cash flow occurring at time $T_j$ is already known at time $T_{j-1}$. Notice also that in a Forward Rate Agreement (FRA), as well as LIBOR-in-arrears swap, the fixing and payment date coincide (once again, in practice the payment date may be slightly delayed).

**Definition 3.3.** At every payment date $T_j, j = 1, 2, \ldots, n$, the net cash flow associated with a payer forward SOFR swap equals $FS_{T_j}^j(\kappa) = (R^s(T_{j-1}, T_j) - \kappa) \delta_j P$. The dates $T_0, T_1, \ldots, T_{n-1}$ and $T_1, T_2, \ldots, T_n$ represent the respective start dates and the payment dates for the accrual period. The values $n, P$ and $\kappa$ denote the number of payments (length) of the swap, the notional principal, and the preassigned fixed rate of interest.

It is clear that the price of a SOFR swap will depend on the funding rate of the firm and the level of collateralization of a trade. We will first assume that a swap is uncollateralized and $r^h$ is used by a bank as a funding rate for a replicating strategy where the hedging instrument is the reference SOFR futures. Therefore, we will need to deal with a possible mismatch between the floating leg in a swap, which is based on SOFR Average, and the overnight funding rate $r^h$ used for hedging. In the next step, we will assume that a swap is either fully or partially collateralized and the margin account is remunerated at rate $r^c$.

3.4 Pricing of Uncollateralized SOFR Swaps

Let us assume that a swap is uncollateralized and the short term rate $r^h$ is used by a bank as a hedge funding rate for the swap. We will consider the process of replication in Section 3.5 and thus Proposition 3.5 will be fully justified through hedging arguments. For the reader’s convenience, we first examine a single-period SOFR swap referencing the period $[T, T + \delta]$.

**Definition 3.4.** A payer single-period SOFR swap over $[T, T + \delta]$ and settled in arrears at $T + \delta$ is a fixed-for-floating swap where at time $T + \delta$ and per one unit of nominal value, the long party makes a fixed payment $X_1 = \delta \kappa$ and receives a floating payment $X_2 = \delta R^s(T, T + \delta)$ linked to the SOFR Average $R^s(T, T + \delta)$ over $[T, T + \delta]$. Hence the net cash flow at time $T + \delta$ equals $FS_{T+\delta}^s(\kappa) = (R^s(T, T + \delta) - \kappa) \delta P$ where $P$ is the nominal value of a swap.

Unlike in the classical framework, as the floating leg is not known until time $T + \delta$, the net cash flow of the SOFR swap is not fixed at time $T$, as was the case for the LIBOR swap. Therefore, we need to examine the pricing and hedging of the forward swap rate during the accrual period $[T, T + \delta]$. Using the martingale measure corresponding to the numeraire $B^h$, under the temporary assumption that replication is feasible, we may compute the forward SOFR swap rate for an uncollateralized
single-period SOFR swap. Pricing results from this section are supported by replication results, which are established in Section 3.5 (see, in particular, Proposition 3.8).

We denote by $FS^{s,h,\kappa}(T, T+\delta)$ the price at time $t \in [0, T+\delta]$ of an uncollateralized SOFR swap with fixed rate $\kappa$ so that $FS^{s,h,\kappa}_t(T, T+\delta) = \pi^{f,h}_t(FS^{s}_T + \delta \kappa)$ for all $t \in [0, T+\delta]$ where $\pi^{f,h}$ is the pricing functional in the market model $\mathcal{M}^f$ from Section 3.1 with $\beta \equiv 0$.

**Definition 3.5.** The single-period uncollateralized forward SOFR swap rate $F^{s,h}_t(T, T+\delta)$ at time $t \in [0, T+\delta]$ is a unique $\mathcal{F}_t$-measurable random variable representing a fixed rate $\kappa$ such that $FS^{s,h,\kappa}_t(T, T+\delta) = 0$.

Similarly to Lyashenko and Mercurio [38] we define, for all $t \in [0, T]$, $B^h(t,T) := E_{Q^f}\left(e^{-\int_t^T r^h_u du} \middle| \mathcal{F}_t \right)$ and we set $B^h(t,T) := e^{\int_t^T r^h_u du}$ for all $t \in [T,T+\delta]$. Furthermore, for all $t \in [0, T+\delta]$ $B^h(t,T+\delta) := E_{Q^f}\left(e^{-\int_t^{T+\delta} r^h_u du} \middle| \mathcal{F}_t \right)$.

Notice that $B^h(t,T)$ and $B^h(t,T+\delta)$ can be interpreted as prices of fictitious zero-coupon bonds (or, in some instances, the existent corporate bonds). We will use the following auxiliary lemma, which is a minor extension of Proposition 1.1.

**Lemma 3.2.** For every $T > 0$, the process $B^h(t, T)$ equals, for all $t \in [0, T]$, $B^h(t,T) = \exp\left(m(t,T) - n(t,T)x_t - \int_t^T \alpha^h_u du \right)$ and the dynamics of $B^h(t, T)$ are, for all $t \in [0, T]$, $dB^h(t, T) = B^h(t, T) \left(r^h_t dt - \sigma n(t,T) dW^Q_t \right)$.

We first derive the arbitrage-free price of a single-period uncollateralized SOFR swap in terms of the fictitious bond $B^h(t,T)$ or, equivalently, in terms of the factor process $x$ (hence also in terms of $r^h$). Without loss of generality, we may assume that the notional principal is $P = 1$.

**Proposition 3.5.** Assume that a single-period uncollateralized SOFR swap can be replicated in $\mathcal{M}^f$. Then its arbitrage-free price equals, for all $t \in [0, T+\delta]$, $FS^{s,h,\kappa}_t(T, T+\delta) = B^h(t, T)A^{s,h}(T, T+\delta) - (1+\delta \kappa)B^h(t, T+\delta)$ (3.17)

and thus the single-period uncollateralized forward SOFR swap rate $F^{s,h}_t(T, T+\delta)$ equals, for all $t \in [0, T+\delta]$, $1 + \delta F^{s,h}_t(T, T+\delta) = A^{s,h}(T, T+\delta) \frac{B^h(t, T)}{B^h(t, T+\delta)}$ (3.18)

where $A^{s,h}(T, T+\delta) := e^{\int_t^{T+\delta}(\alpha^x_u - \alpha^h_u) du}$. 
Proof. From equation (3.9) in Corollary 3.2, we deduce that the price of the SOFR swap satisfies

\[ \pi_{t}^{h} (FS_{T+\delta}^{*}(\kappa)) = B_{t}^{h} \mathbb{E}_{Q^{f}} \left( (B_{T+\delta}^{h})^{-1} (\delta R^{s}(T, T+\delta) - \delta \kappa) | \mathcal{F}_{t} \right) \]

\[ = B_{t}^{h} \mathbb{E}_{Q^{f}} \left( (B_{T+\delta}^{h})^{-1} (B_{T+\delta}^{s} (B_{T+\delta}^{s})^{-1} - (1 + \delta \kappa)) | \mathcal{F}_{t} \right) \]

\[ = \mathbb{E}_{Q^{f}} \left( e^{-\int_{t}^{T+\delta} r_{u}^{f} du} e^{\int_{T+\delta}^{T} r_{u}^{f} du} | \mathcal{F}_{t} \right) - (1 + \delta \kappa) B^{h}(t, T+\delta) \]

\[ = \mathbb{E}_{Q^{f}} \left( e^{-\int_{t}^{T+\delta} r_{u}^{f} du} e^{\int_{T+\delta}^{T} (r_{u}^{s} - r_{u}^{f}) du} | \mathcal{F}_{t} \right) - (1 + \delta \kappa) B^{h}(t, T+\delta) \]

\[ = B^{h}(t, T) e^{\int_{T+\delta}^{T} (\alpha_{u}^{f} - \alpha_{u}^{s}) du} - (1 + \delta \kappa) B^{h}(t, T+\delta) = FS_{T+\delta}^{*}(h, \kappa)(T, T+\delta). \]

Since the forward SOFR swap rate is defined as an \( \mathcal{F}_{t} \)-measurable level of the fixed rate \( \kappa \) for which the single-period SOFR swap is worthless at time \( t \) we obtain equality (3.18) for all \( t \in [0, T+\delta] \). \( \square \)

Before stating an alternative pricing result for a SOFR, we need to introduce some auxiliary notation. We already know from (3.13) and (3.14) that the process \( Y := 1 + \delta R^{s,f}(T, T+\delta) \) satisfies the linear SDE

\[ dY_{t} = Y_{t} \sigma_{Y}^{f} dW_{t}^{Q^{f}} \quad (3.19) \]

where \( \sigma_{Y}^{f} = \sigma(n(t, T+\delta) - n(t, T)) \) for all \( t \in [0, T] \) and \( \sigma_{Y}^{s} = \sigma n(t, T+\delta) \) for all \( t \in [T, T+\delta] \). Hence we have that, for all \( t \in [0, T+\delta] \),

\[ Y_{T+\delta} = Y_{t} \exp \left( \int_{t}^{T+\delta} \sigma_{Y}^{f} dW_{u}^{Q^{f}} - \frac{1}{2} \int_{t}^{T+\delta} (\sigma_{Y}^{f})^{2} du \right) = Y_{t} e^{\xi(t, T+\delta) - \frac{1}{2} v_{Y}^{2}(t, T+\delta)} \]

where

\[ \xi(t, T+\delta) := \int_{t}^{T+\delta} \sigma_{Y}^{f} dW_{u}^{Q^{f}}, \quad v_{Y}^{2}(t, T+\delta) := \int_{t}^{T+\delta} (\sigma_{Y}^{f})^{2} du. \quad (3.20) \]

The next result gives a representation for the arbitrage-free price of a SOFR swap in terms of the SOFR futures \( R^{s,f}(T, T+\delta) \) and the factor process \( x \).

**Proposition 3.6.** If a single-period uncollateralized SOFR swap can be replicated in \( \mathcal{M}^{f} \), then its arbitrage-free price equals, for all \( t \in [0, T+\delta] \),

\[ FS_{t}^{*}(h, \kappa)(T, T+\delta) = A^{h}(t, T+\delta) e^{\rho(x_{t})} \quad (3.21) \]

where \( A^{h}(t, T+\delta) := e^{-\int_{T}^{T+\delta} \alpha_{h}^{f} du} \)

\[ \rho(x_{t}) := -n(t, T+\delta) x_{t} - \int_{t}^{T+\delta} an(u, T+\delta) du \]

and

\[ w^{2}(t, T) := \int_{t}^{T+\delta} \sigma^{2} n^{2}(u, T) du, \quad w^{2}(t, T+\delta) := \int_{t}^{T+\delta} \sigma^{2} n^{2}(u, T+\delta) du. \]

**Proof.** As in the proof of Proposition 3.5, we observe that, for every \( t \in [0, T+\delta] \),

\[ FS_{t}^{*}(h, \kappa)(T, T+\delta) = B_{t}^{h} \mathbb{E}_{Q^{f}} \left( (B_{T+\delta}^{h})^{-1} (R^{s}(T, T+\delta) - \kappa) | \mathcal{F}_{t} \right) \]

\[ = \mathbb{E}_{Q^{f}} \left( R^{h}(t, T+\delta) (R^{s,f}_{T+\delta}(T, T+\delta) - \kappa) | \mathcal{F}_{t} \right) \]

\[ = A^{h}(t, T+\delta) \mathbb{E}_{Q^{f}} \left( A^{f}(t, T+\delta) (Y_{T+\delta} - (1 + \delta \kappa)) | \mathcal{F}_{t} \right) \]
where \( Y_t := 1 + \delta R^{s,f}_t(T, T + \delta) \) for all \( t \in [0, T + \delta] \) and
\[
R^h(t, T + \delta) = B^{h}_t(B^{h}_{T + \delta})^{-1} = e^{-\int_t^{T+\delta} r^h du} = A^h(t, T + \delta)A^x(t, T + \delta)
\]
where \( A^x(t, T + \delta) := e^{-\int_t^{T+\delta} r^x du} \) and \( A^h(t, T + \delta) := e^{-\int_t^{T+\delta} \alpha^h du} \) so that \( A^h(t, T + \delta) \) is deterministic. Furthermore, the dynamics of \( x \) are given by (3.10) and thus
\[
\int_t^{T+\delta} x_u du = n(t, T + \delta)x_t + \int_t^{T+\delta} an(u, T + \delta) du + \int_t^{T+\delta} \sigma n(u, T + \delta) dW^Q_u
\]
and thus
\[
A^x(t, T + \delta) = e^{-\int_t^{T+\delta} r^x du} = e^{\rho(x_t)} e^{-\zeta(t, T + \delta)}
\]
where
\[
\rho(x_t) := -n(t, T + \delta)x_t - \int_t^{T+\delta} an(u, T + \delta) du, \quad \zeta(t, T + \delta) := \int_t^{T+\delta} \sigma n(u, T + \delta) dW^Q_u.
\]
Since \( \rho(x_t) \) and \( Y_t \) are \( \mathcal{F}_t \)-measurable, we obtain
\[
\mathbb{FS}^{s,h,\kappa}_t(T, T + \delta) = A^h(t, T + \delta)\mathbb{E}_Q^F \left( Y_{T+\delta} A^x(t, T + \delta) - (1 + \delta \kappa)A^x(t, T + \delta) \bigg| \mathcal{F}_t \right)
\]
\[
= A^h(t, T + \delta)e^{\rho(x_t)} \mathbb{E}_Q^F \left( Y_t e^{\xi(t, T + \delta) - \zeta(t, T + \delta)} - \delta \kappa^2(t, T + \delta) - (1 + \delta \kappa) e^{-\zeta(t, T + \delta)} \bigg| \mathcal{F}_t \right)
\]
\[
= A^h(t, T + \delta)e^{\rho(x_t)} \left( Y_t e^{\frac{1}{2}\sigma^2(t, T + \delta)} - \delta \kappa^2(t, T + \delta) - (1 + \delta \kappa) e^{-\frac{1}{2}\sigma^2(t, T + \delta)} \right)
\]
where we denote \( \eta_1 := \xi(t, T + \delta) - \zeta(t, T + \delta), \eta_2 := -\zeta(t, T + \delta) \) so that \( \eta_1 - \eta_2 = \xi(t, T + \delta) \), and we write \( \tilde{\eta}_i = \eta_i - \frac{1}{2} \text{Var}(\eta_i) \) for \( i = 1, 2 \). It is clear that \( Y_t \) is \( \mathcal{F}_t \)-measurable and the random variable \( \eta_1, \eta_2 \) is independent of \( \mathcal{F}_t \) and has the Gaussian distribution with mean zero and \( \text{Var}(\eta_1) \) and \( \text{Var}(\eta_1) \) given by
\[
\text{Var}(\eta_1) = w^2(t, T) = \int_t^{T+\delta} \sigma^2 n^2(u, T) du, \quad \text{Var}(\eta_2) = w^2(t, T + \delta) = \int_t^{T+\delta} \sigma^2 n^2(u, T + \delta) du,
\]
and
\[
\text{Var}(\eta_1 - \eta_2) = \text{Var}(\xi(t, T + \delta)) = \int_t^{T+\delta} (\sigma^2_u)^2 du = \sigma^2(t, T + \delta).
\]
Hence it is clear that the asserted pricing formula is valid. \( \square \)

We can formulate an immediate consequence of (3.17) and (3.18).

**Corollary 3.3.** The price of a single-period SOFR swap with the fixed rate \( \kappa \) satisfies, for all \( t \in [0, T + \delta] \),
\[
\mathbb{FS}^{s,h,\kappa}_t(T, T + \delta) = \delta(F^{s,h}_t(T, T + \delta) - \kappa)B^h(t, T + \delta).
\]
In particular, if the SOFR swap was initiated at time 0 with the fixed rate \( \kappa \) equal to the forward SOFR swap rate \( F^{s,h}_0(T, T + \delta) \), then its price at time \( t \) equals
\[
\mathbb{FS}^{s,h,\kappa}_t(T, T + \delta) = \delta(F^{s,h}_t(T, T + \delta) - F^{s,h}_0(T, T + \delta))B^h(t, T + \delta).
\]

**Remark 3.4.** If the SOFR swap is financed via the short term rate \( r^s \) so that \( r^h = r^s \), then \( A^{s,h}(T, T + \delta) = 1 \) and \( B^h(t, T) = B^s(t, T) \). Hence (3.18) yields the classical representation of the LIBOR single-period swap for all \( t \in [0, T] \). Furthermore, the realised backward-looking SOFR return on \([T, T + \delta] \) satisfies \( R^s(T, T + \delta) = F^{s}_T(T, T + \delta) \).
Since the pricing functional is linear, Propositions 3.5 and 3.6 can be easily extended to that of multi-period SOFR swap given by Definition 3.3. Let $FS_t^{s,h,\kappa}(T_0,n)$ denote the price at time $t \in [0,T_n]$ of a multi-period uncollateralized forward SOFR swap.

**Definition 3.6.** The multi-period uncollateralized forward SOFR swap rate $\kappa_t^{s,h}(T_0,n)$ at time $t \in [0,T_n]$ is a unique $\mathcal{F}_t$-measurable random variable, which represents a unique level of a fixed rate $\kappa$ for which $FS_t^{s,h,\kappa}(T_0,n) = 0$.

**Proposition 3.7.** The arbitrage-free price of a multi-period uncollateralized forward SOFR swap equals, for all $t \in [0,T_0]$,

$$FS_t^{s,h,\kappa}(T_0,n) = \sum_{j=1}^{n} \left( A_s^{s,h}(T_{j-1},T_j) B^h(t,T_{j-1}) - (1 + \delta_j \kappa) B^h(t,T_j) \right)$$

(3.23)

and the multi-period uncollateralized forward SOFR swap rate $\kappa_t^{s,h}(T_0,n)$ equals, for all $t \in [0,T_0]$,

$$\kappa_t^{s,h}(T_0,n) = \frac{\sum_{j=1}^{n} \left( A_s^{s,h}(T_{j-1},T_j) B^h(t,T_{j-1}) - B^h(t,T_j) \right)}{\sum_{j=1}^{n} \delta_j B^h(t,T_j)}$$

(3.24)

where

$$A_s^{s,h}(T_{j-1},T_j) := e^{\int_{T_{j-1}}^{T_j} (\alpha^s_t - \alpha^h_t) \, du}.$$ 

Furthermore, for all $t \in [0,T_0]$,

$$FS_t^{s,h,\kappa}(T_0,n) = \sum_{j=1}^{n} \delta_j B^h(t,T_j) (\kappa_t^{s,h}(T_0,n) - \kappa).$$

(3.25)

For $t \in [T_{k-1},T_k]$, where $k = 1, 2, \ldots, n-1$, the price of a SOFR swap $FS_t^{s,h,\kappa}(T_0,n)$ and SOFR swap rate $\kappa_t^{s,h}(T_k,n-k)$ are given by expressions analogous to (3.23) and (3.24) with $j = k, k+1, \ldots, n$ and $B^h(t,T_{k-1}) = \exp \left( \int_{T_{k-1}}^{t} r^h_u \, du \right)$.

**Proof.** The value of the cash flows of a multi-period SOFR swap can be derived in a similar way to the previous calculations

$$FS_t^{s,h,\kappa}(T_0,n) = B_t^h \mathbb{E}_Q \left( \sum_{j=1}^{n} (B_{T_j}^h)^{-1} (R^s(T_{j-1},T_j) - \kappa) \delta_j \mid \mathcal{F}_t \right)$$

$$= B_t^h \mathbb{E}_Q \left( \sum_{j=1}^{n} (B_{T_j}^h)^{-1} (B_{T_j}^h B_{T_{j-1}}^{-1})^{-1} - (1 + \delta_j \kappa) \mid \mathcal{F}_t \right)$$

$$= \sum_{j=1}^{n} \left( \mathbb{E}_Q \left( e^{-\int_{T_{j-1}}^{T_j} r^h_u \, du} e^{\int_{T_{j-1}}^{T_j} r^h_u \, du} \mid \mathcal{F}_t \right) - (1 + \delta_j \kappa) B^h(t,T_j) \right)$$

$$= \sum_{j=1}^{n} \left( \mathbb{E}_Q \left( e^{-\int_{T_{j-1}}^{T_j} (\alpha^s_t - \alpha^h_t) \, du} \mid \mathcal{F}_t \right) - (1 + \delta_j \kappa) B^h(t,T_j) \right)$$

$$= \sum_{j=1}^{n} \left( A_s^{s,h}(T_{j-1},T_j) B^h(t,T_{j-1}) - (1 + \delta_j \kappa) B^h(t,T_j) \right)$$

$$= \sum_{j=1}^{n} \left( A_s^{s,h}(T_{j-1},T_j) B^h(t,T_{j-1}) - (1 + \delta_j \kappa) B^h(t,T_j) \right),$$

which also yields the desired expression (3.24) for $\kappa_t^{s,h}(T_0,n)$ when $t \in [0,T_0]$. For any $k = 1, 2, \ldots, n$ and every $t \in [T_{k-1},T_k]$, there are $n-k$ cash flows remaining and thus we have the same representation as before but the summation starts at $j = k$ and $B^h(t,T_{k-1})$ is given by the expression given in the statement of the proposition. \(\square\)
Similar to the case of the single-period swap, if we take \( r^h = r^s \), then for all \( t \in [0, T_0] \) the forward swap rate has exactly the same representation as in the classical single-curve case from Lemma 1.3 but with a traded bond \( B(t, T) \) replaced by a theoretical bond \( B^s(t, T) \). Of course, this observation is no longer valid if we consider any date \( t \) after the start date \( T_0 \) of a swap.

### 3.5 Hedging of Uncollateralized SOFR Swaps

In the first case, we will consider the hedging of an uncollateralized SOFR swap where the bank uses the rate \( r^h \) for funding of the hedge. The dynamics of the wealth process are therefore given by Proposition 3.1 with \( C = 0 \) and \( r^h \) equal to either \( r^s, r^c \) or \( r^u \). A single-period SOFR swap referencing the period \([T, T + \delta]\) has the payoff at \( T + \delta \) given by

\[
X = \delta R^s(T, T + \delta) - \delta \kappa = B^h_{T + \delta}(B^s_T)^{-1} - (1 + \delta \kappa).
\]

To identify a replicating strategy for an uncollateralized SOFR swap, it suffices to use Proposition 3.1 with \( C = 0 \). Recall that the wealth process \( V(\varphi) \) of a futures portfolio \( \varphi \) equals, for every \( t \in [0, T] \),

\[
V_t(\varphi) = \varphi^0_t B^h_t
\]

and \( \varphi \) is self-financing if, for every \( t \in [0, T] \),

\[
dV_t(\varphi) = \varphi^0_t dB^h_t + \varphi^1_t df^s_t(T, T + \delta).
\]

In contrast to hedging methods where the replicating strategy takes a position in assets with a non-zero spot price, such as stocks or commodities, in the case of a futures trading strategy there is no cost of investing in the futures contracts. Rather, \( \varphi^1_t \) denotes the number of SOFR futures contracts held at time \( t \), which are required for replication of an uncollateralized SOFR swap.

We are in a position to give a result on replication of an uncollateralized SOFR swap by a futures trading strategy funded at the rate \( r^h \).

**Proposition 3.8.** An uncollateralized SOFR swap with a fixed rate \( \kappa \) can be replicated by a futures trading strategy \( \varphi = (\varphi^0, \varphi^1) \) where, for all \( t \in [0, T + \delta] \),

\[
\varphi^0_t B^h_t = A^{s,h}(T, T + \delta) B^h(t, T) - (1 + \kappa \delta) B^h(t, T + \delta).
\]

Furthermore, for all \( t \in [0, T] \),

\[
\varphi^1_t = -\frac{(1 + \delta \kappa) B^h(t, T + \delta) n(t, T + \delta) - A^{s,h}(T, T + \delta) B^h(t, T)n(t, T)}{\delta^{-1}(1 + \delta R^{s,f}(T, T + \delta))(n(t, T + \delta) - n(t, T))}
\]

and, for all \( t \in [T, T + \delta] \),

\[
\varphi^1_t = \frac{(1 + \delta \kappa) B^h(t, T + \delta)}{\delta^{-1}(1 + \delta R^{s,f}(T, T + \delta))}.
\]

**Proof.** We wish to find the trading strategy \( \varphi \) replicating the swap value given by Corollary 3.3 so that

\[
dV_t(\varphi) = \varphi^0_t dB^h_t + \varphi^1_t df^s_t(T, T + \delta) = dFS^{s,h,\kappa}(T, T + \delta).
\]

The floating leg of the SOFR swap and the SOFR futures contract are driven by same Brownian motion and thus we can compute \( \varphi^1_t \) by equating the respective diffusion terms. On the one hand, we deduce from equation (3.13), that the dynamics of the futures price are, for all \( t \in [0, T] \),

\[
df^s_t(T, T + \delta) = -\delta^{-1}(1 + \delta R^{s,f}(T, T + \delta))(n(t, T + \delta) - n(t, T)) \sigma dW^f_t.
\]
Therefore, the differential of the wealth of a trading strategy equals
\[
dV_t(\varphi) = \varphi_t^0 dB_t^h - \varphi_t^1 \delta^{-1} \left( 1 + \delta R_t^{s,f}(T, T + \delta) \right) \left( n(t, T + \delta) - n(t, T) \right) \sigma dW_t^{Q_f}.
\]

On the other hand, in the case of the SOFR swap, the dynamics of the price can be derived by applying Itô’s formula to equation (3.21), for all \( t \in [0, T] \),
\[
dFS_t^{s,h,\kappa}(T, T + \delta) = A^{s,h}(T, T + \delta) dB^h(t, T) - (1 + \delta \kappa) dB^h(t, T + \delta)
\]
\[
= A^{s,h}(T, T + \delta) \left( r^h_t \ dt - \sigma n(t, T) dW_t^{Q_f} \right)
\]
\[
- (1 + \delta \kappa) B^h(t, T + \delta) \left( r^h_t \ dt - \sigma n(t, T + \delta) dW_t^{Q_f} \right)
\]
\[
= r^h_t \left( A^{s,h}(T, T + \delta) B^h(t, T) - (1 + \delta \kappa) B^h(t, T + \delta) \right) dt
\]
\[
+ \sigma \left( (1 + \delta \kappa) B^h(t, T + \delta) n(t, T + \delta) - A^{s,h}(T, T + \delta) B^h(t, T)n(t, T) \right) dW_t^{Q_f}.
\]

By equating the drift and diffusion terms on \([0, T]\), we obtain (3.27) and (3.28), respectively. Notice that the component \( \varphi_t^0 \) can also be computed from the equalities \( V_t(\varphi) = FS_t^{s,h,\kappa}(T, T + \delta) = \varphi_t^0 B_t^h \) for all \( t \in [0, T + \delta] \).

For the case of \( t \in [T, T + \delta] \), the dynamics of the SOFR futures price follow from equation (3.14)
\[
df_t^s(T, T + \delta) = -\delta^{-1} \left( 1 + \delta R_t^{s,f}(T, T + \delta) \right) n(t, T + \delta) \sigma dW_t^{Q_f}
\]
and thus
\[
dV_t(\varphi) = \varphi_t^0 dB_t^h - \varphi_t^1 \delta^{-1} \left( 1 + \delta R_t^{s,f}(T, T + \delta) \right) \left( n(t, T + \delta) - n(t, T) \right) \sigma dW_t^{Q_f}.
\]

The price of SOFR swap satisfies, for all \( t \in [T, T + \delta] \),
\[
dFS_t^{s,h,\kappa}(T, T + \delta) = A^{s,h}(T, T + \delta) dB^h(t, T) - (1 + \delta \kappa) dB^h(t, T + \delta)
\]
\[
= A^{s,h}(T, T + \delta) B^h(t, T) r^h_t \ dt - (1 + \delta \kappa) B^h(t, T + \delta) \left( r^h_t \ dt - \sigma n(t, T + \delta) dW_t^{Q_f} \right).
\]

As in the previous step, we can obtain the hedge ratios \( \varphi_t^0 \) and \( \varphi_t^1 \) by equating the drift and diffusion terms of the respective processes. As expected, we obtain here the same expression for \( \varphi_t^0 \) as for \( t \in [0, T] \) (see (3.26)) but a different one for \( \varphi_t^1 \) (see (3.28)). The latter feature was expected since the dynamics of SOFR swap price and SOFR futures change when the reference period is reached.

As mentioned earlier, in contrast to the classical single-period LIBOR swaps, the value of the SOFR swap payoff continues to evolve for \( t \in [T, T + \delta] \). Therefore, the replicating strategy for the replicating strategy are no longer static during the accrual period \([T, T + \delta]\). As the replicating strategy is expected to be a continuous process, it is important to check that \( \varphi_{T-} = \varphi_{T+} \), which is indeed the case. In view of Lemma 3.2 and Proposition 3.5, the price at time \( t \) of a SOFR swap is a function of \( x_t \) (hence also \( r^h_t \)) but the hedge ratios are expressed in terms of the funding rate \( r^f_t \) (e.g., \( r^f_t \) or \( r^f_t \)) and the futures price \( R_t^{s,f}(T, T + \delta) \), which are observed in the market. We will see the same feature when examining hedging strategies for caps and swaptions.

Having determined the replicating strategy for a single-period swap, a multi-period swap can simply be viewed as a specific portfolio of single-period swaps. The differential of the wealth of a futures portfolio is now given by
\[
V_t(\varphi) = V_0(\varphi) + \int_0^t \varphi_u^0 dB_u^h + \sum_{j=1}^n \int_0^t \varphi_u^j dJ_u^s(T_{j-1}, T_j)
\]
where \( f^s(T_{j-1}, T_j) \) is the SOFR futures price referencing the period \([T_{j-1}, T_j]\) for \( j = 1, 2, \ldots, n \).
Corollary 3.4. A multi-period SOFR swap can be replicated for all \( t \in [0, T_n] \) by the futures strategy \( \varphi = (\varphi^0, \varphi^1, \ldots, \varphi^n) \) such that

\[
\varphi_t^0 B_t^h = \sum_{j=1}^{n} (A^{s,h}(T_{j-1}, T_j) B^h(t, T_{j-1}) - (1 + \delta_j \kappa) B^h(t, T_j)) \mathbb{1}_{t \leq T_j}
\]

and for every \( j = 1, 2, \ldots, n \) we have, for all \( t \in [0, T_{j-1}] \),

\[
\varphi_t^j = -\frac{(1 + \delta_j \kappa) B^h(t, T_j) - A^{s,h}(T_{j-1}, T_j) B^h(t, T_{j-1}) n(t, T_j)}{\delta_j^{-1} (1 + \delta_j R^{s,j}(T_{j-1}, T_j)) (n(t, T_j) - n(t, T_{j-1}))}
\]

and, for all \( t \in [T_{j-1}, T_j] \),

\[
\varphi_t^j = -\frac{1 + \delta_j \kappa) B^h(t, T_j)}{\delta_j^{-1} (1 + \delta_j R^{s,j}(T_{j-1}, T_j))}.
\]

Proof. As mentioned previously, the multi-period SOFR swap with \( n \) accrual periods can be interpreted as portfolio of \( n \) individual single-period SOFR swaps where the \( j \)th accrual period of the swap corresponds to the reference period of the \( j \)th SOFR futures contract.

3.6 Pricing and Hedging of Collateralized SOFR Swaps

In the next step, we study a collateralized SOFR swap when \( r^h \) is used as a funding rate for the hedging strategy. Hence the dynamics of the wealth process are now given by Proposition 3.1 with \( C \neq 0 \) and \( r^h \). We henceforth assume that \( C = -\mathcal{V}(\varphi) \) where the process \( \beta \) takes values in \([0, 1]\). We will use Corollary 3.1 where discounting of the wealth process of a self-financing futures strategy is performed using the effective funding rate \( r^\beta \).

Before determining the value of the collateralized SOFR swap contract, we define the auxiliary process \( B^\beta(t, T) \). Recall that we denote \( r_t^\beta := (1 - \beta_t) r_t^h + \beta_t r_t^c \) and \( dB_t^\beta = r_t^\beta B_t^\beta dt \) with \( B_0^\beta = 1 \). We define the bond price (of course, \( B^\beta(t, T) \) does not represent the price of a traded bond)

\[
B^\beta(t, T) := \mathbb{E}_{Q^\beta} \left( e^{-\int_t^T r_u^\beta du} \bigg| \mathcal{F}_t \right).
\]

For \( t > T \), we define

\[
B^\beta(t, T) := \exp \left( \int_T^t r_u^\beta du \right)
\]

so that \( dB^\beta(t, T) = r_t^\beta B^\beta(t, T) dt \) for all \( t > T \).

Lemma 3.3. Assume that \( \beta \) is deterministic. Then \( B^\beta(t, T) \) equals, for all \( t \in [0, T] \),

\[
B^\beta(t, T) = \exp \left( m(t, T) - n(t, T)x_t - \int_t^T \alpha_u^\beta du \right)
\]

where \( \alpha^\beta := (1 - \beta) \alpha^h + \beta \alpha^c \) for all \( t \in [0, T] \) and thus the dynamics of \( B^\beta(t, T) \) are

\[
dB^\beta(t, T) = B^\beta(t, T) \exp \left( r_t^\beta dt - \sigma n(t, T) dW_{Q^\beta}^c \right).
\]

Proof. It is clear that

\[
r_t^\beta = (1 - \beta) r_t^h + \beta r_t^c = x_t + (1 - \beta) \alpha_t^h + \beta \alpha_t^c = x_t + \alpha_t^\beta
\]

where the basis \( \alpha^\beta := (1 - \beta) \alpha^h + \beta \alpha^c \) is deterministic. Therefore,

\[
B^\beta(t, T) = \mathbb{E}_{Q^\beta} \left( e^{-\int_t^T r_u^\beta du} \bigg| \mathcal{F}_t \right) = \mathbb{E}_{Q^\beta} \left( e^{-\int_t^T (x_u + \alpha_u^\beta) du} \bigg| \mathcal{F}_t \right) = e^{-\int_t^T \alpha_u^\beta du} \mathbb{E}_{Q^\beta} \left( e^{-\int_t^T x_u du} \bigg| \mathcal{F}_t \right)
\]

and thus it suffices to use equation (1.2). The dynamics of \( B^\beta(t, T) \) is now easy to obtain.

\[\square\]
We denote by $\mathbf{FS}^{s,h,κ,β}_t(T, T + \delta)$ the price at time $t \in [0, T + \delta]$ of a single-period collateralized SOFR swap with fixed rate $κ$, that is, $\mathbf{FS}^{s,h,κ,β}_t(T, T + \delta) = \pi^{f,h,β}_t(\mathbf{FS}^{*}_T + \delta(κ))$ for all $t \in [0, T + \delta]$ where $\pi^{f,h,β}$ is the pricing functional in the market model $\mathcal{M}_t$ from Section 3.1 with a collateralization rate equal to $β$. The collateralized forward SOFR swap rate $F^{s,β}_t(T, T + \delta)$ is the fair rate in a collateralized forward SOFR swap with the collateralization rate $β$. The following result extends Proposition 3.5 to the case of collateralized SOFR swaps.

**Proposition 3.9.** The arbitrage-free price of a collateralized SOFR swap equals, for all $t \in [0, T + \delta]$, 

$$\mathbf{FS}^{s,h,κ,β}_t(T, T + \delta) = B^h(t, T)A^{s,β}(T, T + \delta) - (1 + δκ)B^β(t, T + \delta) \tag{3.29}$$

and thus the collateralized forward SOFR swap rate $F^{s,β}_t(T, T + \delta)$ satisfies, for all $t \in [0, T]$, 

$$1 + δF^{s,β}_t(T, T + \delta) = A^{s,β}(T, T + \delta) \frac{B^β(t, T)}{B^β(t, T + \delta)} \tag{3.30}$$

where

$$A^{s,β}(T, T + \delta) = e^{\int_{T}^{T + \delta}(α^s_u - α^β_u) du}$$

and, for all $t \in [T, T + \delta]$,

$$1 + δF^{s,β}_t(T, T + \delta) = e^{\int_{T}^{T + \delta} r_u^β du} A^{s,β}(T, T + \delta) \frac{B^β(t, T)}{B^β(t, T + \delta)}. \tag{3.31}$$

**Proof.** The computations are similar to those from the proof of Proposition 3.5. We now apply (3.8) to the SOFR swap’s payoffs to obtain 

$$\pi^{f,β}_t(\mathbf{FS}^{*}_T + \delta(κ)) = B^β_t E_{Q^t} \left( (B^β_{T + \delta})^{-1} δR^s(T, T + \delta) - δκ \right) | F_t \right)$$

$$= B^β_t E_{Q^t} \left( (B^β_{T + \delta})^{-1} \left[ (B^β_T + (B^β_T)^{-1} - (1 + δκ) \right) | F_t \right)$$

$$= E_{Q^t} \left( e^{-\int_{T}^{T + \delta} r_u^β du} e^{\int_{T}^{T + \delta}(r_u^s - r_u^β) du} | F_t \right) - (1 + δκ)B^β(t, T + \delta)$$

$$= A^{s,β}(T, T + \delta)B^β(t, T) - (1 + δκ)B^β(t, T + \delta) = \mathbf{FS}^{s,h,κ,β}_t(T, T + \delta),$$

which yields the desired results.

We are now in a position to consider the replication of a SOFR swap with collateral backing. We continue working under the postulate that collateral is proportional to the wealth of the trading strategy, that is, $C = -βV(φ)$ for some predetermined process $β$. We obtain from Definition 3.1

$$φ^0_t B^h_t = V^P_t(φ) = V_t(φ) + C_t = (1 - β_t)V_t(φ)$$

and thus the wealth process of futures trading strategy under collateralization satisfies

$$V_t(φ) = V_0(φ) + \int_0^t φ^0_u B^h_u r_u^h du + \sum_{i=1}^m \int_0^t φ^i_u f^i_u + \int_0^t r_u^β B^β_u V_u(φ) du$$

$$= V_0(φ) + \int_0^t φ^0_u B^h_u (1 - β_u)^{-1} r_u^β du + \sum_{i=1}^m \int_0^t φ^i_u f^i_u.$$
Proposition 3.10. A collateralized SOFR swap with $C = -\beta V(\varphi)$ where $\beta$ is deterministic can be replicated by a SOFR futures strategy $\varphi = (\varphi^0, \varphi^1)$ where, for all $t \in [0, T + \delta]$,

$$\varphi^0_t B^h_t = (1 - \beta_1) \left( A^s,\beta(T, T + \delta) B^\beta (t, T) - (1 + \delta \kappa) B^\beta (t, T + \delta) \right).$$

Furthermore, for all $t \in [0, T]$,

$$\varphi^1_t = -\frac{(1 + \delta \kappa) B^\beta (t, T + \delta) - A^s,\beta(T, T + \delta) B^\beta (t, T)n(t, T)}{\delta^{-1} (1 + \delta R^s,F (T, T + \delta))(n(t, T + \delta) - n(t, T))}.$$ 

and, for all $t \in [T, T + \delta]$,

$$\varphi^1_t = -\frac{(1 + \delta \kappa) B^\beta (t, T + \delta)}{\delta^{-1} (1 + \delta R^s,F (T, T + \delta))}.$$ 

In particular, for a fully collateralized SOFR swap we have $\beta = 1$ and thus $\varphi^0_t = 0$ for all $t \in [0, T + \delta]$.

Proof. The computations are analogous to the proof of Proposition 3.8. We have, for all $t \in [0, T]$,

$$dV_t(\varphi) = \varphi^0_t B^h_t (1 - \beta_1)^{-1} r^\beta_t dt - \varphi^1_t \delta^{-1} (1 + \delta R^s,F (T, T + \delta))(n(t, T + \delta) - n(t, T)) \sigma dW^Q_t.$$ 

and, for all $t \in [0, T + \delta]$,

$$dV_t(\varphi) = \varphi^0_t B^h_t (1 - \beta_1)^{-1} r^\beta_t dt - \varphi^1_t \delta^{-1} (1 + \delta R^s,F (T, T + \delta))(n(t, T + \delta) - n(t, T)) \sigma dW^Q_t.$$ 

The price $FS^{s,h,\kappa}(T, T + \delta)$ of a collateralized SOFR swap is given by (3.29) and thus we obtain, for all $t \in [0, T]$,

$$dFS_t^{s,h,\kappa}(T, T + \delta) = \delta^{-1} \left( A^s,\beta(T, T + \delta) dB^\beta (t, T) - (1 + \kappa \delta) dB^\beta (t, T + \delta) \right) dt$$

$$= r^\beta_t \delta^{-1} \left( e^{\int_t^{T+\delta} (\alpha_t^\kappa - \alpha_t^\delta) du} B^\beta (t, T) - (1 + \delta \kappa) B^\beta (t, T + \delta) \right) dt$$

$$+ \sigma \delta^{-1} \left( 1 + \delta \kappa \right) B^\beta (t, T + \delta)n(t, T + \delta) - e^{\int_t^{T+\delta} (\alpha_t^\kappa - (1 - \beta) \alpha_t^\kappa - \beta \alpha_t^\kappa) du} B^\beta (t, T)n(t, T) \right) dW^Q_t.$$ 

and, for all $t \in [T, T + \delta]$,

$$dFS_t^{s,h,\kappa}(T, T + \delta) = r^\beta_t \delta^{-1} \left( A^s,\beta(T, T + \delta) dB^\beta (t, T) - (1 + \delta \kappa) B^\beta (t, T + \delta) \right) dt$$

$$+ \sigma \delta^{-1} \left( 1 + \delta \kappa \right) B^\beta (t, T + \delta)n(t, T + \delta) \right) dW^Q_t.$$ 

By equating the drift and diffusion terms, we obtain the hedge ratios $\varphi^0_t$ and $\varphi^1_t$ given in the statement of the proposition. \hfill $\square$

### 3.7 Pricing and Hedging of SOFR Caps

A SOFR *cap* is a portfolio of European style options where the holder of the receives the cash difference between the SOFR rate and a pre-determined level of interest if and only if the designated interest rate exceeds a pre-determined strike level. Let $\kappa > 0$ and $P$ denote the pre-determined strike and the notional principal, respectively. In the *cap* referencing LIBOR, the rate considered over the period $[T_{j-1}, T_j]$ is the forward LIBOR determined at time $T_{j-1}$ and thus the payoff of the $j$th caplet is known in advance since it equals $(L_{T_{j-1}}(T_j) - \kappa)^+ \delta_j P$. In contrast, the payoff from the $j$th SOFR caplet settled in arrears becomes fully known at time $T_j$ and equals $(R^s(T_{j-1}, T_j) - \kappa)^+ \delta_j P$ though, of course, a good approximation of the payoff is available for dates close to $T_j$.

**Definition 3.7.** A SOFR *cap* with strike $\kappa > 0$ and maturity $T_0$, which is settled in arrears at dates $T_j$, $j = 1, 2, \ldots, n$ has the cash flow at each date $T_j$ equal to $\text{FC}^{s}_T (\kappa) = (R^s(T_{j-1}, T_j) - \kappa)^+ \delta_j P$ where $\text{FC}^{s}_T (\kappa)$ denotes the payoff of the $j$th SOFR caplet. Similarly, a SOFR *floor* with strike $\kappa > 0$ and the same tenor structure has cash flows at dates $T_j$ equal to $\text{FF}^{s}_T (\kappa) = (\kappa - R^s(T_{j-1}, T_j))^+ \delta_j P$. Without loss of generality, we may set $P = 1$. 


Using equality (3.9), we obtain

\[
\mathbf{FC}^{s,h,\kappa}_t = B_t^h \mathbb{E}_{Q^f} \left( \sum_{j=1}^{n} (B_{T_j}^h)^{-1} \mathbf{FC}^{s}_t(\kappa) \mid \mathcal{F}_t \right) = B_t^h \mathbb{E}_{Q^f} \left( \sum_{j=1}^{n} (B_{T_j}^h)^{-1} (R^s(T_{j-1}, T_j) - \kappa)^{\dagger} \delta_j \mid \mathcal{F}_t \right)
\]

and thus it suffices to compute the arbitrage-free price of the \( j \)th caplet related to the accrual period \([T, T + \delta]\) where \( T = T_{j-1} \) and \( T + \delta = T_{j-1} + \delta_j = T_j \). The next result gives a closed-form expression for the arbitrage-free price of a SOFR caplet. Recall that the dynamics of the process \( Y \) and \( v_Y(t, T + \delta) \) are given by (3.19) and (3.20), respectively. We use here the same notation as in Proposition 3.6, in particular, \( \rho(x_t), w^2(t, T) \) and \( w^2(t, T + \delta) \) are defined in the statement of Proposition 3.6.

**Proposition 3.11.** The arbitrage-free price of a SOFR caplet with strike \( \kappa > 0 \) and related to the accrual period \([T, T + \delta]\) equals, for all \( t \in [0, T + \delta] \),

\[
\mathbf{FC}^{s,h,\kappa}_t = A^h(t, T + \delta) e^{\rho(x_t)} \left( e^{\frac{1}{2}(w^2(t, T) - v_Y(t, T + \delta))} N(h_+) - (1 + \delta\kappa) e^{\frac{1}{2}(w^2(t, T + \delta) - v_Y(t, T + \delta))} N(h_-) \right)
\]

where \( A^h(t, T + \delta) := e^{-\int_t^{T+\delta} \alpha_h^b du} \). Furthermore, for every \( t \in [T, T + \delta] \),

\[
h_+ = h_+(t, T + \delta) = \frac{\ln \left( 1 + \delta R_t^{s,f}(T, T + \delta) \right)}{1 + \delta_h^s} + \frac{1}{2} \left( w^2(t, T) - w^2(t, T + \delta) - v_Y(t, T + \delta) \right)
\]

(3.32)

and

\[
h_- = h_-(t, T + \delta) = \frac{\ln \left( 1 + \delta R_t^{s,f}(T, T + \delta) \right)}{1 + \delta_h^s} - \frac{1}{2} \left( w^2(t, T) - w^2(t, T + \delta) - v_Y(t, T + \delta) \right)
\]

(3.33)

Finally, for all \( t \in [T, T + \delta] \),

\[
h_+ = h_+(t, T + \delta) = \frac{\ln \left( 1 + \delta R_t^{s,f}(T, T + \delta) \right)}{1 + \delta_h^s} - \frac{1}{2} v_Y^2(t, T + \delta)
\]

(3.34)

and

\[
h_- = h_-(t, T + \delta) = \frac{\ln \left( 1 + \delta R_t^{s,f}(T, T + \delta) \right)}{1 + \delta_h^s} - \frac{1}{2} v_Y^2(t, T + \delta)
\]

(3.35)

**Proof.** We use similar arguments as in the proof of Proposition 3.6. To find the arbitrage-free price of a caplet, it suffices to evaluate the conditional expectation, for every \( t \in [0, T + \delta] \),

\[
\mathbf{FC}^{s,h,\kappa}_t = B_t^h \mathbb{E}_{Q^f} \left( (B_{T+\delta}^h)^{-1} (R^s(T, T + \delta) - \kappa)^{\dagger} \delta \mid \mathcal{F}_t \right)
\]

\[
= \mathbb{E}_{Q^f} \left( R^h(t, T + \delta) (R_t^{s,f}(T, T + \delta) - \kappa)^{\dagger} \delta \mid \mathcal{F}_t \right)
\]

\[
= A^h(t, T + \delta) \mathbb{E}_{Q^f} \left( A^x(t, T + \delta) (Y_{T+\delta} - (1 + \delta\kappa))^\dagger \mid \mathcal{F}_t \right)
\]

where \( Y_t := 1 + \delta R_t^{s,f}(T, T + \delta) \) for all \( t \in [0, T + \delta] \) and

\[
R^h(t, T + \delta) = \frac{B_t^h}{B_{T+\delta}^h} = e^{-\int_t^{T+\delta} \alpha_h^b du} A^h(t, T + \delta) A^x(t, T + \delta)
\]

where \( A^x(t, T + \delta) := e^{-\int_t^{T+\delta} \alpha_h^b du} \) and \( A^h(t, T + \delta) := e^{-\int_t^{T+\delta} \alpha_h^b du} \) so that \( A^h(t, T + \delta) \) is deterministic. Furthermore, the dynamics of \( x \) are given by (3.10) and thus

\[
\int_t^{T+\delta} x_u du = n(t, T + \delta)x_t + \int_t^{T+\delta} an(u, T + \delta) du + \int_t^{T+\delta} \sigma n(u, T + \delta) dW^{Q^f}_u
\]
and thus
\[ A^\delta(t, T + \delta) = e^{-\int_t^{T+\delta} x_u \, du} = e^{\rho(x_t)} e^{-\zeta(t, T + \delta)} \]
where
\[ \rho(x_t) := -n(t, T + \delta)x_t - \int_t^{T+\delta} an(u, T + \delta) \, du, \quad \zeta(t, T + \delta) := \int_t^{T+\delta} \sigma_n(u, T + \delta) \, dW^Q_u. \]

Since \( \rho(x_t) \) is \( \mathcal{F}_t \)-measurable and positive, we have
\[
\mathbf{FC}^{s,h,\kappa}_t = A^h(t, T + \delta) e^{\rho(x_t)} \mathbb{E}_{Q^t} \left( \left( Y_{t,T+\delta} A^\delta(t, T + \delta) - (1 + \delta\kappa) A^\delta(t, T + \delta) \right) \bigg| \mathcal{F}_t \right) = A^h(t, T + \delta) e^{\rho(x_t)} \mathbb{E}_{Q^t} \left( Y_{t,T+\delta} e^{\frac{1}{2} \sigma^2_n(t,T+\delta)} \, d\tilde{\eta}_t - (1 + \delta\kappa) e^{\frac{1}{2} \sigma^2_n(t,T+\delta)} \right) \bigg| \mathcal{F}_t \right)
\]
where we denote \( \eta_1 := \xi(t, T + \delta) - \zeta(t, T + \delta), \eta_2 := -\zeta(t, T + \delta) \) so that \( \eta_1 - \eta_2 = \xi(t, T + \delta) \), and we write \( \tilde{\eta}_t := \eta_t - \frac{1}{2} \sigma_n(t, T + \delta) \) for \( i = 1, 2 \). The random variable \( \eta_1, \eta_2 \) is independent of \( \mathcal{F}_t \) and has the Gaussian distribution with mean zero and \( \text{Var}(\eta_1) \) and \( \text{Var}(\eta_1) \) given by
\[
\text{Var}(\eta_1) = w^2(t, T) = \int_t^{T+\delta} \sigma^2_n^2(u, T + \delta) \, du, \quad \text{Var}(\eta_2) = w^2(t, T + \delta) = \int_t^{T+\delta} \sigma^2_n^2(u, T + \delta) \, du,
\]
and
\[
\text{Var}(\eta_1 - \eta_2) = \text{Var}(\xi(t, T + \delta)) = \int_t^{T+\delta} (\sigma_n^2 u^2) \, du = v^2(t, T + \delta).
\]
To complete the computations, we use Lemma 3.4 to obtain
\[
\mathbf{FC}^{s,h,\kappa}_t = A^h(t, T + \delta) e^{\rho(x_t)} \left( Y_{t,T+\delta} e^{\frac{1}{2} \sigma^2_n(t,T+\delta)} N(h_+) - (1 + \delta\kappa) e^{\frac{1}{2} \sigma^2_n(t,T+\delta)} N(h_-) \right)
\]
where \( h_\pm = h_\pm(t, T + \delta) = \frac{1}{\kappa} \ln(c_1 / c_2) \pm \frac{1}{\kappa} k, k = \sqrt{\text{Var}(\eta_1 - \eta_2)} = v(t, T + \delta) \) and
\[
\frac{c_1}{c_2} = \frac{Y_t}{1 + \delta\kappa} e^{\frac{1}{2} \sigma^2_n(t, T + \delta) - \frac{1}{2} \sigma^2_n(\eta_2)}.
\]

It is now easy to check that \( h_+(t, T + \delta) \) and \( h_-(t, T + \delta) \) are given by equalities (3.32)–(3.35).

**Lemma 3.4.** Let \( (\eta_1, \eta_2) \) be a zero-mean, jointly Gaussian (non-degenerate), two-dimensional random variable on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Then for arbitrary positive real numbers \( c_1 \) and \( c_2 \) we have
\[
\mathbb{E}_{\mathbb{P}} \left( c_1 e^{\eta_1 - \frac{1}{2} \text{Var}(\eta_1)} - c_2 e^{\eta_2 - \frac{1}{2} \text{Var}(\eta_2)} \right) = c_1 N(h_+) - c_2 N(h_-)
\]
where
\[
h_\pm = \frac{1}{k} \ln \frac{c_1}{c_2} \pm \frac{1}{2} k, \quad k = \sqrt{\text{Var}(\eta_1 - \eta_2)}.
\]

Using Proposition 3.11, we may find the hedging strategy for a SOFR caplet (and hence for a SOFR cap) by differentiating the pricing formula and identifying the hedge ratios \( \varphi^0_t \) and \( \varphi^1_t \) as in the proof of Proposition 3.8. Since the market model \( \mathcal{M}^f \) is complete, the existence of a replicating strategy for a SOFR caplet (or a SOFR swaption) is obvious. To make the expressions for the hedging strategy more concise, we denote
\[
\Gamma_t := A^h(t, T + \delta) e^{\rho(x_t)} e^{\frac{1}{2} \sigma^2_n(t, T + \delta) - \frac{1}{2} \sigma^2_n(\eta_2)}
\]
and

\[ \Lambda_t := A^h(t, T + \delta)e^{p(x_t)}(1 + \delta \kappa)e^{\frac{1}{2}w^2(t, T + \delta)} \]

so that the pricing formula for a SOFR caplet from Proposition 3.8 becomes

\[ \text{FC}^{s,h,\kappa}_t = \Gamma_t N(h_+(t, T + \delta)) - \Lambda_t N(h_-(t, T + \delta)). \]

We also write

\[ \hat{n}(t, T, T + \delta) := \frac{n(t, T + \delta)}{n(t, T + \delta) - n(t, T)}. \]

**Proposition 3.12.** A SOFR caplet with a fixed rate \( \kappa \) can be replicated by a futures trading strategy \( \varphi = (\varphi^0, \varphi^1) \) where, for all \( t \in [0, T + \delta] \),

\[ \varphi^0_t B^h_t = A^{s,h}(T, T + \delta)B^h(t, T) - (1 + \delta \kappa)B^h(t, T + \delta). \tag{3.36} \]

Furthermore, for all \( t \in [0, T] \),

\[ \varphi^1_t = -\frac{\Gamma_t n(h_+(t, T + \delta)) - \Lambda_t n(h_-(t, T + \delta)) - \nu_Y(t, T + \delta)\hat{n}(t, T, T + \delta)\text{FC}^{s,h,\kappa}_t}{\delta^{-1}v_Y(t, T + \delta)(1 + \delta R^{s,f}_t)(T, T + \delta)} \tag{3.37} \]

and, for all \( t \in [T, T + \delta] \),

\[ \varphi^1_t = -\frac{\Gamma_t n(h_+(t, T + \delta)) - \Lambda_t n(h_-(t, T + \delta)) - \nu_Y(t, T + \delta)\text{FC}^{s,h,\kappa}_t}{\delta^{-1}v_Y(t, T + \delta)(1 + \delta R^{s,f}_t)(T, T + \delta)}. \tag{3.38} \]

**Proof.** We already know from the proof of Proposition 3.8 that the dynamics of the wealth of a futures trading strategy \( \varphi \) are, for all \( t \in [0, T] \),

\[ dV_t(\varphi) = \varphi^0_t dB^h_t - \varphi^1_t \delta^{-1}(1 + \delta R^{s,f}_t)(T, T + \delta)(n(t, T + \delta) - n(t, T))\sigma dW^Q_t \]

and, for all \( t \in [T, T + \delta] \),

\[ dV_t(\varphi) = \varphi^0_t dB^h_t - \varphi^1_t \delta^{-1}(1 + \delta R^{s,f}_t)(T, T + \delta)n(t, T + \delta)\sigma dW^Q_t. \]

Replication of a caplet means that \( V_t(\varphi) = \varphi^0_t B^h_t = \text{FC}^{s,h,\kappa}_t \) for all \( t \in [0, T + \delta] \) and thus it is clear that \( \varphi^0_t = (B^h_t)^{-1}\text{FC}^{s,h,\kappa}_t \) for all \( t \in [0, T + \delta] \). To compute the hedge ratio \( \varphi^1_t \) it suffices to identify the martingale part (that is, the Itô integral term) in the dynamics of the price process \( \text{FC}^{s,h,\kappa}_t \). We obtain, for all \( t \in [0, T + \delta] \),

\[ d\text{FC}^{s,h,\kappa}_t = \Gamma_t dN(h_+(t, T + \delta)) - \Lambda_t dN(h_-(t, T + \delta)) - \text{FC}^{s,h,\kappa}_t n(t, T + \delta)\sigma dW^Q_t \]

where

\[ dN(h_\pm(t, T + \delta)) = n(h_\pm(t, T + \delta)) \frac{d(1 + \delta R^{s,f}_t)}{v_Y(t, T + \delta)(1 + \delta R^{s,f}_t)}. \]

Furthermore, from (3.13) and (3.14) we obtain, for all \( t \in [0, T] \),

\[ dN(h_\pm(t, T + \delta)) = n(h_\pm(t, T + \delta))(v_Y(t, T + \delta))^{-1}(n(t, T + \delta) - n(t, T))\sigma dW^Q_t \]

and, for all \( t \in [T, T + \delta] \),

\[ dN(h_\pm(t, T + \delta)) = n(h_\pm(t, T + \delta))(v_Y(t, T + \delta))^{-1}n(t, T + \delta)\sigma dW^Q_t. \]

It is now easy to obtain equalities (3.37) and (3.38).
3.8 Pricing of SOFR Swaptions

As opposed to caps, which are simple portfolios of caplets (that is, options on the value of a single-period swap settled in arrears), the pricing of swaptions on a multi-period SOFR swap is more computationally demanding. More importantly, options on multi-period swaps cannot be settled in arrears, that is, at time $T_n$. This means that the decision to exercise a swaption by its holder should be based on the arbitrage-free price of the underlying SOFR swap at the swaption’s maturity date $T_0$. Recall that the arbitrage-free pricing and hedging of SOFR swaps was examined in Sections 3.4–3.6 for various choices of the hedge funding rate and with different levels of proportional collateralization. In order to resolve whether to exercise or abandon the swaption at its expiry $T_0$, one needs to compute the arbitrage-free price of the underlying SOFR swap using, in particular, the funding rate $r^h$, which is not a universal quantity. Recall that the price of the SOFR swap depends on a bank’s ability to raise funds and that salient feature is reflected through the hedge funding rate $r^h$.

In view of Definition 3.3 the arbitrage-free price of the SOFR swaption equals

$$\text{PS}^{s,h,\kappa}_t(T_0, n) = B_t^h \mathbb{E}_{Q^f} \left[ \left( B_{T_0}^h \right)^{-1} \left( \text{FS}^{s,h,\kappa}_T(T_0, n) \right) \right] | \mathcal{F}_t$$

where in turn (see Proposition 3.7)

$$\text{FS}^{s,h,\kappa}_t(T_0, n) = \sum_{j=1}^n \left( A^{s,h}(T_{j-1}, T_j) B^h(T_0, T_{j-1}) - (1 + \delta_j \kappa) B^h(T_0, T_j) \right)$$

We thus obtain the following pricing formula, for all $t \in [0, T_0]$,

$$\text{PS}^{s,h,\kappa}_t(T_0, n) = B_t^h \mathbb{E}_{Q^f} \left[ \left( B_{T_0}^h \right)^{-1} \left( \sum_{j=1}^n \left( A^{s,h}(T_{j-1}, T_j) B^h(T_0, T_{j-1}) - (1 + \delta_j \kappa) B^h(T_0, T_j) \right) \right) \right] | \mathcal{F}_t$$

which can also be rewritten as follows

$$\text{PS}^{s,h,\kappa}_t(T_0, n) = B_t^h \mathbb{E}_{Q^f} \left[ \left( B_{T_0}^h \right)^{-1} \left( \sum_{j=1}^n a_j B^h(T_0, T_{j-1}) - c_j B^h(T_0, T_j) \right) \right] | \mathcal{F}_t.$$

Then, assuming that $r^h = x + \alpha^h$ where the function $\alpha^h$ is differentiable so that the process $r^h$ satisfies Vasicek’s SDE, one can apply standard result for pricing of swaptions in a single-curve setup to obtain the pricing formula in terms of $B^h(t, T_0), B^h(t, T_1), \ldots, B^h(t, T_n)$ (hence as a function of $r^h$) and involving integration of the Gaussian density functions (see Propositions 3.13 and 13.2.1 in [45]).

To formulate an alternative pricing result for swaptions, we denote $Y^j := 1 + \delta_j R^{s,j}(T_{j-1}, T_j)$ so that the dynamics of the process $Y^j$ under $Q^f$ are given by Lemma 3.1, that is,

$$dY^j_t = Y^j_t \sigma^j_t dW_t^{Q^f}$$

where $\sigma^j_t = \sigma(n(t, T_j) - n(t, T_{j-1}))$ for all $t \in [0, T_{j-1}]$ and $\sigma^j_t = \sigma(n(t, T_j)$ for all $t \in [T_{j-1}, T_j]$. As before, we denote $A^h(t, T + \delta) := e^{-\int_0^{T_0} s^h du} A^x(t, T + \delta)$ and $R^{s,j}(T, T + \delta)$ and to identify the hedging strategy for a SOFR swaption in $\mathcal{M}^f$. However, since such an expression would inevitably lengthy, it is not reported here.

**Proposition 3.13.** The arbitrage-free price of a SOFR swaption with strike $\kappa > 0$ equals, for all $t \in [0, T_0]$,

$$\text{PS}^{s,h,\kappa}_t(T_0, n) = \mathbb{E}_{Q^f} \left[ \left( \mathbb{E}_{Q^f} \left( \sum_{j=1}^n A^h(t, T_j) A^x(t, T_j) (Y^j_t - (1 + \delta_j \kappa)) \right) \right) \right] | \mathcal{F}_t.$$

(3.39)
Proof. The arbitrage-free price of the SOFR swaption can be represented as (see Proposition 3.7)

\[ \mathbf{FS}^{s,h,\kappa}_{T_0}(T_0, n) = B_{T_0}^h \mathbb{E}_{Q^f}\left( \sum_{j=1}^{n} (B_{T_j}^h)^{-1} \left( R^s(T_{j-1}, T_j) - \kappa \right) \delta_j \mid \mathcal{F}_{T_0} \right) \]

which gives, for all \( t \in [0, T_0] \),

\[
\mathbf{PS}^{s,h,\kappa}_{t}(T_0, n) = B_{T_0}^h \mathbb{E}_{Q^f}\left[ \left( B_{T_0}^h \right)^{-1} \left( \sum_{j=1}^{n} (B_{T_j}^h)^{-1} \left( R^s(T_{j-1}, T_j) - \kappa \right) \delta_j \mid \mathcal{F}_{T_0} \right) \right]^{+} \mid \mathcal{F}_t \]

\[
= \mathbb{E}_{Q^f}\left[ \mathbb{E}_{Q^f}\left( \sum_{j=1}^{n} R^s(t, T_j) \left( R^s(T_{j-1}, T_j) - \kappa \right) \delta_j \mid \mathcal{F}_{T_0} \right) \right]^{+} \mid \mathcal{F}_t \]

\[
= \mathbb{E}_{Q^f}\left[ \mathbb{E}_{Q^f}\left( \sum_{j=1}^{n} A^s(t, T_j) A^s(t, T_j) (Y_j^s - (1 + \delta_j \kappa)) \mid \mathcal{F}_{T_0} \right) \right]^{+} \mid \mathcal{F}_t \]

where for every \( j = 1, 2, \ldots, n \) we denote \( R^s(t, T_j) = B_{T_j}^h (B_{T_0}^h)^{-1} \) and the process \( Y_j^s \) is given by the equality \( Y_j^s = 1 + \delta_j R_{T_j, T_j}^s(T_{j-1}, T_j) \) for all \( t \in [0, T_j] \). \( \square \)

If either \( r^h = r^s \) or \( r^h = r^e \), then one may argue that the price becomes independent of particular bank’s circumstances. In particular, if \( r^h = r^e \) then we may use the pricing formula of Proposition 3.7 to obtain

\[ \mathbf{FS}^{s,h,\kappa}_{T_0}(T_0, n) = \sum_{j=1}^{n} \left( B^s(T_0, T_{j-1}) - (1 + \delta_j \kappa) B^s(T_0, T_j) \right) \]

which leads to the following swaption’s pricing formula

\[ \mathbf{PS}^{s,h,\kappa}_{t}(T_0, n) = B_{T_0}^s \mathbb{E}_{Q^f}\left[ \left( B_{T_0}^s \right)^{-1} \left( \sum_{j=1}^{n} \left( B^s(T_0, T_{j-1}) - (1 + \delta_j \kappa) B^s(T_0, T_j) \right) \right) \right]^{+} \mid \mathcal{F}_t \]

which coincides with the single-curve pricing formula (1.6) upon denoting \( r^s = r \). Similarly, if we take \( r^h = r^e \), then we get

\[ \mathbf{FS}^{s,h,\kappa}_{T_0}(T_0, n) = \sum_{j=1}^{n} \left( A^e(T_{j-1}, T_j) B^e(T_0, T_{j-1}) - (1 + \delta_j \kappa) B^e(T_0, T_j) \right) \]

which in turn yields the following expression for the SOFR swaption’s price

\[ \mathbf{PS}^{s,h,\kappa}_{t}(T_0, n) = B_{T_0}^e \mathbb{E}_{Q^f}\left[ \left( B_{T_0}^e \right)^{-1} \left( \sum_{j=1}^{n} \left( A^e(T_{j-1}, T_j) B^e(T_0, T_{j-1}) - (1 + \delta_j \kappa) B^e(T_0, T_j) \right) \right) \right]^{+} \mid \mathcal{F}_t \]

Let us conclude this work by acknowledging that the arbitrage-free pricing and hedging of SOFR caps, swaptions and other derivatives in a multi-curve framework is an important practical issue, which requires a lot of further theoretical and numerical studies in multi-factor models, also with allowance for stochastic spreads between overnight rates.

For instance, it would be natural to postulate that the basis \( r^e - r^s \) is a stochastic process driven by another Brownian motion. Then, in order to ensure that the model \( \mathcal{M}^f \) is complete, one could assume that the traded instruments are \( B^s, B^e \) and \( R^{s,f} \). However, it would be important to ensure that the trading conditions preclude simple arbitrage opportunities between \( B^s \) and \( B^e \), which rely on borrowing cash at a lower and simultaneously lending at a higher rate.
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