KARLIN–MCGREGOR POLYNOMIALS, GERONIMUS POLYNOMIALS
AND HAAR MEASURES OF HYPERGROUPS

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Abstract. Many symmetric orthogonal polynomial sequences \((P_n(x))_{n \in \mathbb{N}_0}\)
normalized by \(P_0(1) \equiv 1\), satisfy nonnegative linearization of products, i.e.,
the product of any two \(P_m(x), P_n(x)\) is a convex combination w.r.t. the basis \(\{P_k(x) : k \in \mathbb{N}_0\}\). Such
polynomials are accompanied by a hypergroup structure and a corresponding Haar measure.
The latter is the counting measure on \(\mathbb{N}_0\) weighted by the values of the function \(h : \mathbb{N}_0 \to [1, \infty)\), \(h(n) := 1/\int \mu_m^2(x) d\mu(x)\), where \(\mu\) denotes the orthogonalization measure and
the orthogonality implies that \(h(n) > 1\) for all \(n \in \mathbb{N}\). Our research was
motivated by the observation that many naturally occurring examples even satisfy the stronger property \(h(n) \geq 2\) \((n \in \mathbb{N})\). We give sufficient criteria which cover important
elements like ultraspherical or generalized Chebyshev polynomials. In particular, we
show that \(h(n) \geq 2\) \((n \in \mathbb{N})\) is always satisfied if \(\mathbb{N}_0\) equals the full interval \([-1, 1]\), where \(\mathbb{N}_0\)
corresponds to the Hermitian structure space (this is fulfilled by an abundance of examples).
Concerning our sufficient criteria, we particularly study the role of nonnegative
linearization of products (and of the resulting harmonic analysis). Moreover, we
construct two example types which do not satisfy the property \(h(n) \geq 2\) \((n \in \mathbb{N})\). To our
knowledge, these are the first such examples. The first type is based on Karlin–McGregor
polynomials and belongs to the class of Geronimus polynomials; we compute explicit for-
mulas for the corresponding orthogonalization measures, recurrence coefficients and Haar
measures. The second type is orthogonal w.r.t. discrete measures.

1. Introduction

1.1. Basic setting and observation. Let \((P_n(x))_{n \in \mathbb{N}_0} \subseteq \mathbb{R}[x]\) with \(\deg P_n(x) = n\)
be given by some recurrence relation \(P_0(x) = 1, P_1(x) = x,\)
\[ xP_n(x) = a_nP_{n+1}(x) + c_nP_{n-1}(x) \quad (n \in \mathbb{N}), \tag{1.1} \]
where \((c_n)_{n \in \mathbb{N}} \subseteq (0, 1)\) and \(a_n \equiv 1 - c_n\); to avoid case differentiations, we additionally define
\(a_0 := 1\). Obviously, the resulting polynomials are symmetric and normalized by \(P_n(1) \equiv 1\).
It is well-known from the theory of orthogonal polynomials\footnote{Standard results on orthogonal polynomials can be found in \cite{3}, for instance.} that \((P_n(x))_{n \in \mathbb{N}_0}\) is orthogonal
w.r.t. a unique probability (Borel) measure \(\mu\) on \(\mathbb{R}\) which satisfies \(\text{supp } \mu = \infty\) and
\(\text{supp } \mu \subseteq [-1, 1]\) (Favard’s theorem), which means
\[
\int_{\mathbb{R}} P_m(x)P_n(x) d\mu(x) \neq 0 \iff m = n.
\]
Moreover, it is well-known that the zeros of the polynomials are real, simple and located
in the interior of the convex hull of \(\text{supp } \mu\). In particular, all \(P_n\) are strictly positive at

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graphics were made with Maple.
If $f$ where $g$ Gibbs states on graphs [32]. We briefly recall some basics [15,17]. The nonnegativity of the product of any two polynomials $P_m(x), P_n(x)$ is a convex combination w.r.t. the basis $\{P_k(x) : k \in \mathbb{N}_0\}$. Due to orthogonality, one has $g(m, n; |m - n|), g(m, n; m + n) \neq 0$ and $g(m, n; k) = 0$ for $k < |m - n|$, so the summation in (1.2) starts with $k = |m - n|$ (and (1.2) can be regarded as an extension of the recurrence (1.1)) [17]. The nonnegativity of the linearization coefficients $g(m, n; k)$ gives rise to a commutative discrete hypergroup on $\mathbb{N}_0$, where the convolution $(m, n) \mapsto \sum_{k=|m-n|}^{m+n} g(m, n; k) \delta_k$ maps $\mathbb{N}_0 \times \mathbb{N}_0$ into the convex hull of the Dirac functions on $\mathbb{N}_0$, the identity on $\mathbb{N}_0$ serves as involution and 0 is the unit element $\delta_0^0$. Such hypergroups are called polynomial hypergroups, were introduced by Lasser in the 1980s and are generally very different from groups or semigroups [15]. There is an abundance of examples, and the individual behavior strongly depends on the underlying polynomials $(P_n(x))_{n \in \mathbb{N}_0}$. However, many concepts of harmonic analysis take a rather unified and concrete form. This makes these objects located at a fruitful and vivid cross-point between the theory of orthogonal polynomials and special functions, on the one hand, and functional and harmonic analysis, on the other hand. Recent publications deal with polynomial hypergroups vs. moment functions [5] vs. Ramsey theory [13] and vs. Gibbs states on graphs [62]. We briefly recall some basics [15,17]. The nonnegativity of the $g(m, n; k)$ implies that

$$\{\pm 1\} \cup \text{supp } \mu \subseteq \mathbb{N}_0 \subseteq [-1, 1],$$

where

$$\widehat{\mathbb{N}_0} := \left\{ x \in \mathbb{R} : \max_{n \in \mathbb{N}_0} |P_n(x)| = 1 \right\}.$$  

(1.3)

If $f : \mathbb{N}_0 \to \mathbb{C}$ is an arbitrary function, then, for every $n \in \mathbb{N}_0$, the translation $T_n f : \mathbb{N}_0 \to \mathbb{C}$ is given by

$$T_n f(m) = \sum_{k=|m-n|}^{m+n} g(m, n; k) f(k).$$

The corresponding Haar measure, normalized such that $\{0\}$ is mapped to 1, is the counting measure on $\mathbb{N}_0$ weighted by the values of the Haar function $h : \mathbb{N}_0 \to [1, \infty)$,

$$h(n) := \frac{1}{g(n, n; 0)} = \frac{1}{\int_{\mathbb{N}_0} P_n^2(x) d\mu(x)}.$$  

(1.4)

The orthonormal polynomials (with positive leading coefficients) $(p_n(x))_{n \in \mathbb{N}_0}$ correspond to $(P_n(x))_{n \in \mathbb{N}_0}$ satisfy $p_n(x) = \sqrt{h(n)} P_n(x)$ ($n \in \mathbb{N}_0$) and are given by the recurrence relation $p_0(x) = 1$, $p_1(x) = x/\sqrt{c_1}$,

$$xp_n(x) = \alpha_{n+1} p_{n+1}(x) + \alpha_n p_{n-1}(x) \quad (n \in \mathbb{N}),$$

where $\alpha_1 = \sqrt{c_1}$ and

$$\alpha_n = \sqrt{c_n \alpha_{n-1}} = \sqrt{c_n (1 - c_{n-1})}.$$
for \( n \geq 2 \). Moreover, the corresponding monic polynomials \((\sigma_n(x))_{n \in \mathbb{N}_0}\) fulfill \( \sigma_0(x) = 1 \), \( \sigma_1(x) = x \) and
\[
x \sigma_n(x) = \sigma_{n+1}(x) + \alpha_n^2 \sigma_{n-1}(x) \quad (n \in \mathbb{N}).
\]

If \( f \in \ell^1(h) := \{ f : \mathbb{N}_0 \to \mathbb{C} : \|f\|_1 < \infty \}, \|f\|_1 := \sum_{k=0}^\infty |f(k)|h(k) \), then \( T_n f \in \ell^1(h) \) and
\[
\sum_{k=0}^\infty T_n f(k)h(k) = \sum_{k=0}^\infty f(k)h(k)
\]
for every \( n \in \mathbb{N}_0 \). The norm \( \| \cdot \|_1 \), the convolution \((f, g) \mapsto f \ast g\), \( f \ast g(n) := \sum_{k=0}^\infty T_n f(k)g(k)h(k) \) and complex conjugation make \( \ell^1(h) \) a semisimple commutative Banach *-algebra with unit \( \delta_0 \), so the polynomials \((P_n(x))_{n \in \mathbb{N}_0}\) can be studied via methods coming from Gelfand’s theory. Polynomial hypergroups are accompanied by a sophisticated harmonic analysis and Fourier analysis; the orthogonalization measure \( \mu \) serves as Plancherel measure. \( \mathbb{N}_0 \) can be identified with the Hermitian structure space \( \Delta_\mu(\ell^1(h)) \) via the homeomorphism \( \mathbb{N}_0 \to \Delta_\mu(\ell^1(h)), x \mapsto \varphi_x \cdot \varphi_x(f) := \sum_{k=0}^\infty f(k)P_k(x)h(k) \), \( f \in \ell^1(h) \). If \( h \) is of subexponential growth, then \( \text{supp} \mu \) and \( \mathbb{N}_0 \) coincide \([26, 29, 30]\). It is obvious from (1.1) that \( h \) is also given by
\[
h(0) = 1, \quad h(n) = \prod_{k=1}^n \frac{\alpha_{k-1}}{c_k} \quad (n \in \mathbb{N}).
\]

Since \( g(n, n; 0) \) and \( g(n, n; 2n) \) are nonzero, nonnegative linearization of products always implies that \( h(n) = 1/g(n, n; 0) > 1 \) for all \( n \in \mathbb{N} \). Studying various examples, we observed that all of them satisfied the stronger property \( h(n) \geq 2 \) \((n \in \mathbb{N})\). The paper is devoted to questions concerning this eye-catching observation, as well as to corresponding criteria, to the role of nonnegative linearization of products and to (counter) examples.

1.2. Motivation and outline of the paper. To start with, we give an additional and more detailed motivation for the problem: since the linearization coefficients \( g(m, n; k) \) are often not explicitly known or of cumbersome structure, it may be very hard to check whether a concrete sequence \((P_n(x))_{n \in \mathbb{N}_0}\) satisfies the crucial nonnegative linearization of products property. We are not aware of any convenient characterization (in terms of the recurrence coefficients \((a_n)_{n \in \mathbb{N}}\) and \((c_n)_{n \in \mathbb{N}}\) in terms of the orthogonalization measure \( \mu \) etc.). However, there are several sufficient criteria, starting with results of Askey [1] and continued by Szwarc et al. in a series of papers. One of these criteria [24, Theorem 1 p. 966] reads as follows:

**Theorem 1.1.** If \((c_n)_{n \in \mathbb{N}}\) is bounded from above by \(1/2\) and both \((c_{2n-1})_{n \in \mathbb{N}}\) and \((c_{2n})_{n \in \mathbb{N}}\) are nondecreasing, then nonnegative linearization of products is satisfied.

Now if \((c_n)_{n \in \mathbb{N}}\) is bounded from above by \(1/2\) (and thus \((a_n)_{n \in \mathbb{N}}\) is bounded from below by \(1/2\)) like in Theorem [1.1] then it is clear from [1.6] that indeed \( h(n) \geq 2 \) for all \( n \in \mathbb{N} \) (recall that \( a_0 = 1 \)). Therefore, it is at least not surprising that many examples satisfy this property because many examples are either constructed via Theorem [1.1] or satisfy \( c_n \leq 1/2 \) for other reasons. Recently, Kahler successfully applied Theorem [1.1] to the large class of associated symmetric Pollaczek polynomials (with monotonicity of the whole sequence \((c_n)_{n \in \mathbb{N}}\)), which is a two-parameter generalization of the well-known ultraspherical polynomials [11].

In [10], Kahler recently found the following example which, for certain choices of the parameters, satisfies nonnegative linearization of products without fulfilling the conditions of Theorem [1.1] for any \( \alpha, \beta > -1 \), let the sequence of generalized Chebyshev polynomials
The set $V \cap \{(\alpha, \beta) \in (-1, \infty)^2 : \alpha + \beta + 1 < 0\}$, cf. (1.7).

These polynomials are the quadratic transformations of the Jacobi polynomials and orthogonal w.r.t. $d\mu(x) = \Gamma(\alpha + \beta + 2)/(\Gamma(\alpha + 1)\Gamma(\beta + 1)) \cdot (1-x^2)^{\alpha/2}\chi(-1,1) dx$ (so $[-1,1] = \text{supp } \mu = \hat{\mathbb{N}}_0$) (Chapter V 2 (G) [15, 3 (f)]: for $\beta = -1/2$, one obtains the ultraspherical polynomials, including the Legendre polynomials and the Chebyshev polynomials of the first and second kind. The generalized Chebyshev polynomials are of particular interest concerning product formulas and duality structures [14,15,22]. In [10, Theorem 3.2], Kahler showed that $(T_n^{(\alpha,\beta)}(x))_{n \in \mathbb{N}_0}$ satisfies nonnegative linearization of products if and only if $(\alpha, \beta)$ is an element of the set $V \subseteq [-1/2, \infty) \times (-1, \infty)$ given by

$$V : = \{(\alpha, \beta) \in (-1, \infty)^2 : \alpha \geq \beta, a(a+5)(a+3)^2 \geq (a^2 - 7a - 24)b^2\}, \quad (1.7)$$

where $a := \alpha + \beta + 1$ and $b := \alpha - \beta$. However, the conditions of Theorem 1.1 are satisfied if and only if $\alpha \geq \beta$ and $\alpha + \beta + 1 \geq 0$. If $(\alpha, \beta) \in V$ but $\alpha + \beta + 1 < 0$, then $(c_{2n})_{n \in \mathbb{N}}$ is strictly decreasing and always greater than $1/2$.

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Figure 1. The set $V \cap \{(\alpha, \beta) \in (-1, \infty)^2 : \alpha + \beta + 1 < 0\}$, cf. (1.7).

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3This is the analogue to a well-known result of Gasper on the (nonsymmetric) class of Jacobi polynomials [8, Theorem 1].

4Such pairs $(\alpha, \beta)$ exist, cf. Figure 1 for instance, $(\alpha, \beta) = (-1/4, -5/6)$ has these properties.
Nevertheless, the explicit formulas
\[
h(2n - 1) = \frac{(\alpha + \beta + 2)_{n-1}}{(\beta + 1)_n}, \frac{(2n + \alpha + \beta)(\alpha + 1)_{n-1}}{(n-1)!} (1.8)
\]
\[
h(2n) = \frac{(\alpha + \beta + 2)_{n-1}}{(\beta + 1)_n}, \frac{(2n + \alpha + \beta + 1)(\alpha + 1)_n}{n!} (1.9)
\]
\[
\times \prod_{k=0}^{n-2} \left[ 1 + \frac{(2\alpha + 1)k + \alpha^2 + \alpha\beta + 3\alpha + 1}{(k + \beta + 1)(k + 1)} \right] (n \in \mathbb{N})
\]

[15, 3 (f)] and the estimation
\[
\alpha^2 + \alpha\beta + 3\alpha + 1 \geq 0 (1.10)
\]
show that \( h(n) \geq 2 \) for all \( n \in \mathbb{N} \). (1.10) can be seen as follows: if \( \alpha \geq 0 \), then
\[
\alpha^2 + \alpha\beta + 3\alpha + 1 \geq \alpha^2 - \alpha + 3\alpha + 1 = (\alpha + 1)^2 > 0,
\]
and if \( \alpha < 0 \), then
\[
\alpha^2 + \alpha\beta + 3\alpha + 1 \geq \alpha^2 + \alpha^2 + 3\alpha + 1 = (2\alpha + 1)(\alpha + 1) \geq 0.
\]

These observations yield the question whether \( h(n) \geq 2 \) \((n \in \mathbb{N})\) is true for every sequence \((P_n(x))_{n \in \mathbb{N}}\) which satisfies nonnegative linearization of products. In Section 2 we give sufficient criteria (Theorem 2.1, which is based on \( \mathbb{N}_0 \), and the subsequent corollaries) which cover many naturally occurring examples, including the generalized Chebyshev polynomials with \( \alpha + \beta + 1 < 0 \) considered above. Concerning these criteria, we will discuss the role of nonnegative linearization of products, and we will consider the examples of cosh-polynomials and Grinspun polynomials. Moreover, in Section 3 we show that there are also counterexamples (Theorem 3.1). In fact, for every \( \epsilon > 0 \) we will construct a polynomial hypergroup with \( h(1) < 1 + \epsilon \) (Theorem 3.2); our examples rely on Karlin–McGregor polynomials and belong to the class of Geronimus polynomials. To our knowledge, these are the first examples with \( h(1) < 2 \). We will compute explicit formulas for the corresponding orthogonalization measures, recurrence coefficients and Haar measures. The problem under consideration is also interesting for the following reason: for the well-known Chebyshev polynomials of the first kind, which play a fundamental role in asymptotics and optimization, \( h(n) \) equals 2 for all \( n \in \mathbb{N} \). Hence, our results show that under a large class of naturally occurring examples the Chebyshev polynomials of the first kind are optimal w.r.t. minimizing the Haar function—however, they are not optimal among all possible examples. In Theorem 3.3 we will obtain further examples with \( h(1) < 1 + \epsilon \); these do not rely on the Karlin–McGregor polynomials and are orthogonal w.r.t. discrete measures. Section 4 is devoted to some open problems.

2. Sufficient criteria and the Nevai class \( M(0, b) \)

In this section, we give some sufficient criteria for \( h(n) \geq 2 \) \((n \in \mathbb{N})\). They do not rely on boundedness properties of \((c_n)_{n \in \mathbb{N}}\) and they particularly cover examples where \((c_n)_{n \in \mathbb{N}}\) exceeds 1/2 as considered in Section 3. Our approach is based on the connection coefficients
to the Chebyshev polynomials of the first kind \((T_n(x))_{n \in \mathbb{N}_0}\): given an orthogonal polynomial sequence \((P_n(x))_{n \in \mathbb{N}_0}\) as in Section 1 let \(C_n(0), \ldots, C_n(n)\) be defined by the expansions

\[
P_n(x) = \sum_{k=0}^{n} C_n(k) T_k(x),
\]

where \(T_0(x) = 1, T_1(x) = x\) and

\[
xT_n(x) = \frac{1}{2} T_{n+1}(x) + \frac{1}{2} T_{n-1}(x) \quad (n \in \mathbb{N})
\]

or, equivalently, \(T_n(\cos(\varphi)) = \cos(n\varphi)\). It is clear that \(C_n(n) \neq 0\). Recall that the Chebyshev polynomials of the first kind satisfy nonnegative linearization of products and \(h(n) = 2\) \((n \in \mathbb{N})\); the orthogonalization measure \(\mu\) is absolutely continuous (w.r.t. the Lebesgue–Borel measure on \(\mathbb{R}\)) and satisfies \(d\mu(x) = 1/\pi \cdot (1 - x^2)^{-1/2} \chi_{(-1,1)}(x) dx\) [17 Sect. 6]. We need the following classical estimation result from Chebyshev theory [28 Theorem (3.1)]:

**Lemma 2.1.** Let \(P(x) \in \mathbb{R}[x]\) be a polynomial of degree \(n \in \mathbb{N}\) with leading coefficient 1. Then

\[
\max_{x \in [-1,1]} |P(x)| \geq \frac{1}{2^{n-1}},
\]

and equality holds if and only if \(P(x) = T_n(x)/2^{n-1}\).

In the following, we always assume that \((P_n(x))_{n \in \mathbb{N}_0}\) satisfies nonnegative linearization of products. The following theorem is the central result of this section.

**Theorem 2.1.** Let \(\tilde{\mathbb{N}}_0 = [-1,1]\). Then \(h(n) \geq 2\) for all \(n \in \mathbb{N}\).

**Proof.** Let \(n \in \mathbb{N}\setminus\{1\}\) and expand \(P_n(x) = \sum_{k=0}^{n} C_n(k) T_k(x)\). Since \(\tilde{\mathbb{N}}_0 = [-1,1]\), by Lemma 2.1 we have

\[
1 = \max_{x \in [-1,1]} |P_n(x)| = C_n(n) \max_{x \in [-1,1]} \left| \sum_{k=0}^{n} \frac{C_n(k)}{C_n(n)} T_k(x) \right| \geq C_n(n).
\]

Since the leading coefficient of \(P_n(x)\) is \(1/\prod_{k=1}^{n-1} a_k\) and the leading coefficient of \(T_n(x)\) is \(2^{n-1}\), we get

\[
\frac{1}{\prod_{k=1}^{n-1} a_k} = C_n(n) \cdot 2^{n-1} \leq 2^{n-1}
\]

and consequently

\[
4^{n-1} \prod_{k=1}^{n-1} a_k^2 \geq 1.
\]

Using (1.6), we have

\[
h(n) = \frac{1}{c_n} \prod_{k=2}^{n} \frac{a_{k-1}}{c_k} = \frac{1}{c_n} \prod_{k=1}^{n-1} \frac{a_k}{c_k} = \frac{1}{c_n} \prod_{k=1}^{n-1} \frac{a_k^2}{c_k(c_k - 1)}.
\]

Since \(c_k(1 - c_k) \leq 1/4\) for all \(k \in \{1, \ldots, n-1\}\), we now obtain

\[
h(n) \geq \frac{1}{c_n} \cdot 4^{n-1} \prod_{k=1}^{n-1} a_k^2 \geq \frac{1}{c_n}.
\]

Therefore, for every \(n \in \mathbb{N}\) we have both

\[
1 \leq c_n h(n)
\]
(with equality for \( n = 1 \)) and
\[
1 \leq c_{n+1}h(n+1) = a_nh(n),
\]
so
\[
2 \leq c_nh(n) + a_nh(n) = h(n). \quad \Box
\]

The essential condition \( \mathbb{N}_0 = [-1,1] \) in Theorem 2.1 is fulfilled by an abundance of examples (see \[2,15\] for instance). Some examples will be discussed later. We first give several corollaries.

**Corollary 2.1.** If all connection coefficients \( C_n(0), \ldots, C_n(n) \) are nonnegative, then \( h(n) \geq 2 \) for all \( n \in \mathbb{N} \).

**Proof.** As the connection coefficients \( C_n(0), \ldots, C_n(n) \) sum up to 1, the presumed nonnegativity allows to conclude in two ways: either obtain that \( \mathbb{N}_0 = [-1,1] \) as an immediate consequence and apply Theorem 2.1 or just use that the assumption particularly yields \( C_n(n) \leq 1 \) and proceed as in the proof of Theorem 2.1 above; the latter way avoids Lemma 2.1. \( \Box \)

**Corollary 2.2.** If there exists a function \( g : [-1,1] \to [0, \infty) \) such that \( |P_n(x)| \leq g(x) \) for all \( x \in [-1,1] \) and for all \( n \in \mathbb{N}_0 \), then \( h(n) \geq 2 \) for all \( n \in \mathbb{N} \).

**Proof.** It is a general result on polynomial hypergroups that the existence of such a function \( g \) implies that \( (P_n(x))_{n \in \mathbb{N}_0} \) is uniformly bounded on \([-1,1]\) by \( \pm 1 \); one always has
\[
\left\{ x \in \mathbb{R} : \sup_{n \in \mathbb{N}_0} |P_n(x)| < \infty \right\} = \hat{\mathbb{N}}_0
\]
[17]. Now Theorem 2.1 yields the assertion. \( \Box \)

**Corollary 2.3.** If \( \text{supp} \, \mu = [-a,a] \) for some \( a \in (0,1] \), then \( h(n) \geq 2 \) for all \( n \in \mathbb{N} \).

**Proof.** If \( \text{supp} \, \mu = [-a,a] \) for some \( a \in (0,1] \), then \([-a,a] \subseteq \hat{\mathbb{N}}_0 \). Since the zeros of the polynomials \( (P_n(x))_{n \in \mathbb{N}_0} \) are real, simple and located in \((-a,a)\), every \( P_n(x) \) is nondecreasing on \([a,1] \). This shows that also \((a,1] \subseteq \hat{\mathbb{N}}_0 \). Finally, by symmetry we can conclude that \( \hat{\mathbb{N}}_0 = [-1,1] \). Hence, the assertion follows from Theorem 2.1. \( \Box \)

**Corollary 2.4.** If \( (c_n)_{n \in \mathbb{N}} \) is convergent, then \( h(n) \geq 2 \) for all \( n \in \mathbb{N} \).

**Proof.** If \( (c_n)_{n \in \mathbb{N}} \) is convergent, then the limit \( c \) is an element of \((0,1/2]\) and
\[
\text{supp} \, \mu = [-2\sqrt{c(1-c)},2\sqrt{c(1-c)}]
\]
due to [16] Theorem (2.2), so the assertion follows from Corollary 2.3. Alternatively, one can obtain the result from Corollary 2.1 by [16] Theorem (2.6) or [18] Corollary 2.1, all connection coefficients \( C_n(0), \ldots, C_n(n) \) are nonnegative. \( \Box \)

**Example 2.1** (cosh-polynomials). Let \( a > 0 \), and let \((c_n)_{n \in \mathbb{N}} \) be given by
\[
c_n = \frac{\cosh(a(n-1))}{2 \cosh(2a) \cosh(a)}.
\]
\((c_n)_{n \in \mathbb{N}} \) is strictly decreasing, so Theorem 1.1 cannot be applied. Nevertheless, the corresponding sequence \( (P_n(x))_{n \in \mathbb{N}_0} \) satisfies nonnegative linearization of products (and therefore induces a polynomial hypergroup on \( \mathbb{N}_0 \)); the linearization [12] takes the very simple form
\[
P_m(x)P_n(x) = \frac{\cosh(a(n-m))}{2 \cosh(2a) \cosh(a)} P_{n-m}(x) + \frac{\cosh(a(m+n))}{2 \cosh(2a) \cosh(a)} P_{m+n}(x)
\]
for $n \geq m \geq 1$ [17 Sect. 6]. The orthogonalization measure $\mu$ is absolutely continuous and satisfies

$$d\mu(x) = \frac{1}{\pi} \left( \frac{1}{\cosh^2(a)} - x^2 \right)^{-\frac{1}{2}} \chi_{(-\infty,-\infty)}(x) \, dx$$

[2.4] Sect. 6. Since

$$\lim_{n\to\infty} c_n = \frac{1}{1 + x^2},$$

Corollary 2.4 can be applied. For this easy example, the Haar weights are also explicitly known and given by

$$h(n) = \begin{cases} 
1, & n = 0, \\
2\cosh^2(an), & n \in \mathbb{N},
\end{cases}$$

see [17 Sect. 6]. The desired estimation $h(n) \geq 2$ ($n \in \mathbb{N}$) also follows just from the boundedness of $(c_n)_{n \in \mathbb{N}}$ by $1/2$. Note that $h$ is of exponential growth. The limiting case $a = 0$ corresponds to the Chebyshev polynomials of the first kind. Moreover, it is easy to see from (2.3) that

$$T_n(x) = \frac{P_n \left( \frac{x}{\cosh(\alpha)} \right)}{P_n \left( \frac{1}{\cosh(\alpha)} \right)}$$

for every $n \in \mathbb{N}$; in other words: if one rescales the cosh-polynomials in such a way that the right endpoint of the support of the measure becomes 1, then one obtains the Chebyshev polynomials of the first kind (up to renormalization), independently from the parameter $a$.

A similar procedure for a less trivial example will occur and be crucial in Section 3 in order to construct examples with $h(1) < 2$.

Considering the orthonormal polynomials $(p_n(x))_{n \in \mathbb{N}_0}$ and following Nevai [21], one says $\mu \in M(0,b)$, $b \in (0,1]$, if $\lim_{n \to \infty} a_n = b/2$ or, equivalently\footnote{The implication $\lim_{n \to \infty} a_n = b/2 \Rightarrow \lim_{n \to \infty} c_n = (1 - \sqrt{1 - b^2})/2$ is not obvious and can be seen as follows: let $\lim_{n \to \infty} a_n = b/2$. Results on chain sequences [3 Theorem III-6.4] [25 Proposition 5] yield that $(c_n)_{n \in \mathbb{N}}$ converges to $(1 - \sqrt{1 - b^2})/2$ or $(1 + \sqrt{1 - b^2})/2$. Since $(c_n)_{n \in \mathbb{N}}$ cannot converge to a value which is strictly greater than $1/2$ [16 Theorem (2.2)], we have $\lim_{n \to \infty} c_n = (1 - \sqrt{1 - b^2})/2.$}

$$\lim_{n \to \infty} c_n = \frac{1}{2}(1 - \sqrt{1 - b^2}).$$

Many naturally occurring examples satisfy $\mu \in M(0,1)$. The cosh-polynomials considered in Example 2.2 satisfy $\mu \in M(0,b)$ with $b < 1$.

**Corollary 2.5.** If $\mu \in M(0,b)$ for some $b \in (0,1]$, then $h(n) \geq 2$ for all $n \in \mathbb{N}$.

**Proof.** This a reformulation of Corollary 2.4. \hfill $\square$

**Example 2.2** (generalized Chebyshev polynomials reconsidered). For all $(\alpha, \beta) \in V$, including those pairs where $\alpha + \beta + 1 < 0$ (cf. Figure 1) and $(c_n)_{n \in \mathbb{N}}$ exceeds $1/2$, the generalized Chebyshev polynomials $(T_n^{(\alpha,\beta)}(x))_{n \in \mathbb{N}_0}$ fulfill the assumptions of Theorem 2.1 and Corollary 2.1 to Corollary 2.5 (the latter with $\mu \in M(0,1)$).

**Remark 2.1.** The estimation $h(n) \geq 2$ ($n \in \mathbb{N}$), which is fulfilled with equality for the Chebyshev polynomials of the first kind, can be interpreted in the following sense: under the conditions of Theorem 2.1, the Chebyshev polynomials of the first kind are optimal w.r.t. minimizing the Haar function. Comparing this to the optimality assertion of Lemma 2.1 one might ask the question whether, under the conditions of Theorem 2.1 (or the subsequent corollaries), $h(n)$ is strictly greater than 2 for all $n \in \mathbb{N}$ as soon as $(T_n(x))_{n \in \mathbb{N}_0} \neq (T_n(x))_{n \in \mathbb{N}_0}$. 
Figure 2. The first Haar weights \( h(n) \) belonging to \( (T_n^{(\alpha,\alpha)}(x))_{n \in \mathbb{N}_0} \), cf. (1.8) and (1.9), for \( \alpha = -1/2 \) (box, corresponds to the Chebyshev polynomials of the first kind \( (T_n(x))_{n \in \mathbb{N}_0} \)), \( \alpha = 0 \) (diamond) and \( \alpha = 1/2 \) (circle). In all cases, one has \( h(1) = 2 \).

However, this is not the case: for every \( \alpha > -1/2 \), the generalized Chebyshev polynomials \( (T_n^{(\alpha,\alpha)}(x))_{n \in \mathbb{N}_0} \neq (T_n(x))_{n \in \mathbb{N}_0} \) satisfy nonnegative linearization of products, as well as the conditions of Theorem 2.1 and of the subsequent corollaries (cf. Example 2.2), but \( h(1) = 2 \) (cf. (1.8) and Figure 2).

Note that the definitions of \( \hat{\mathbb{N}}_0 \) (1.3) and \( h \) (1.4) also make sense if nonnegative linearization of products is not satisfied (and hence without the underlying hypergroup structure). It is obvious that still \( \{\pm 1\} \subseteq \hat{\mathbb{N}}_0 \subseteq [-1,1] \). \( \text{supp } \mu \) does no longer have to be a subset of \( \hat{\mathbb{N}}_0 \), however. Also (1.6) remains true, but \( h \) can now map into the larger codomain \((0,\infty)\). The rest of the section is devoted to the question which of our results remain true under these more general conditions.

The proof of Theorem 2.1 remains fully true if the nonnegative linearization of products condition is dropped, as well as the proof of Corollary 2.1 which is based on nonnegativity of the connection coefficients \( C_n(0), \ldots, C_n(n) \). There are rather general sufficient criteria for the nonnegativity of all connection coefficients \( C_n(0), \ldots, C_n(n) \). In particular, the boundedness of \( (c_n)_{n \in \mathbb{N}} \) from above by 1/2 is sufficient [23] Corollary 1] (however, recall that the desired estimation \( h(n) \geq 2 \) \((n \in \mathbb{N}) \) is trivial in that case).

The following example shows that the further corollaries do not extend if the nonnegative linearization of products condition is dropped, however.
**Example 2.3** (Grinspun polynomials). Let $c_1 \in (0, 1)$ be arbitrary, and let $c_n = 1/2$ for every $n \geq 2$. The resulting polynomials $(P_n(x))_{n \in \mathbb{N}_0}$ are the Grinspun polynomials and orthogonal w.r.t. a measure $\mu$ with supp $\mu = [-1, 1]$ (Chapter VI 13 (C) (iv)). It is clear that $\mu \in M(0, 1)$. Via induction and (2.1), it is easy to see that 

$$P_n(x) = \frac{1}{2 - 2c_1} T_n(x) + \frac{1 - 2c_1}{2 - 2c_1} T_{n-2}(x) \quad (n \geq 2)$$

and therefore

$$P_n(x) = T_n(x) + \frac{2 - 2c_1}{2 - 2c_1} (T_n(x) - T_{n-2}(x)) \quad (n \geq 2)$$

(cf. also [3, VI-(13.9)] and [31, Section 3.2]). If $c_1 \leq 1/2$, then nonnegative linearization of products is satisfied (and hence a polynomial hypergroup on $\mathbb{N}_0$ is induced) [15, 3 (g) (ii)]. If $c_1 > 1/2$, however, then nonnegative linearization of products fails and the expansions (2.4) imply that $(P_n(x))_{n \in \mathbb{N}_0}$ is uniformly bounded on $[-1, 1]$ by $\pm c_1/(1 - c_1)$, but (1.6) yields

$$h(1) = \frac{1}{c_1} < 2$$

and

$$h(n) = 2 \frac{1 - c_1}{c_1} < 2 \quad (n \geq 2).$$

This shows that Corollary 2.4 is not valid without nonnegative linearization of products: if $c_1 > 2/3$, then $h$ does not even map to $[1, \infty)$. Reconsidering the proof of Corollary 2.2 we see that (2.4) made use of nonnegative linearization of products (and of the resulting harmonic analysis). Clearly, the example also shows that Corollary 2.3 and Corollary 2.4 and Corollary 2.5 are not valid without nonnegative linearization of products. It is already clear from the preceding considerations that neither Corollary 2.1 nor Theorem 2.1 can apply to the case $c_1 > 1/2$, and one can see from (2.3) and (2.5) at which stages an application exactly fails: let $c_1 > 1/2$. Then (2.4) yields that $C_n(n - 2) < 0$ for all $n \geq 2$. Moreover, one has $\mathbb{N}_0 = \{ \pm 1 \}$, which can be seen as follows: let $x \in (-1, 1)$ and $\varphi \in (0, \pi)$ with $x = \cos(\varphi)$. Then, by (2.5),

$$P_n(x) = T_n(\cos(\varphi)) + \frac{2c_1 - 1}{2 - 2c_1} (T_n(\cos(\varphi)) - T_{n-2}(\cos(\varphi)))$$

$$= \cos(n\varphi) + \frac{2c_1 - 1}{2 - 2c_1} (\cos(n\varphi) - \cos((n - 2)\varphi))$$

$$= \cos(n\varphi) + \frac{2c_1 - 1}{2 - 2c_1} (1 - \cos(2\varphi)) \cos(n\varphi) - \frac{2c_1 - 1}{2 - 2c_1} \sin(2\varphi) \sin(n\varphi)$$

for every $n \geq 2$. Now let $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N} \setminus \{1\}$ be a sequence with $\lim_{k \to \infty} \cos(n_k\varphi) = 1$ (and consequently $\lim_{k \to \infty} \sin(n_k\varphi) = 0$). Then

$$\lim_{k \to \infty} P_{n_k}(x) = 1 + \frac{2c_1 - 1}{2 - 2c_1} (1 - \cos(2\varphi)) > 1$$

and we can conclude that $x \notin \mathbb{N}_0$.

3. **Karlin–McGregor polynomials, Geronimus polynomials and examples with $h(1) < 2$**

Having seen criteria for $h(n) \geq 2 \quad (n \in \mathbb{N})$ in the previous section, we now construct examples where nonnegative linearization of products is satisfied but $h(1) < 2$. All such examples have to share the necessary condition

$$0 \notin \mathbb{N}_0,$$
which can be seen as follows: if \(0 \in \hat{\mathbb{N}}_0\), then
\[-\frac{c_1}{1 - c_1} = P_2(0) \geq -1\]
and therefore \(h(1) = 1/c_1 \geq 2\).

For \(\alpha, \beta \geq 2\), the Karlin–McGregor polynomials \((K^{(\alpha, \beta)}_n(x))_{n \in \mathbb{N}_0} \subseteq \mathbb{R}[x]\) are given by
\[c_{2n-1} = \frac{1}{\alpha}\]
and
\[c_{2n} = \frac{1}{\beta}\]
[17 Sect. 6]. For any choice of \(\alpha, \beta \geq 2\), \((K^{(\alpha, \beta)}_n(x))_{n \in \mathbb{N}_0}\) fulfills the conditions of Theorem [1] so nonnegative linearization of products is always satisfied and \(h(n) \geq 2\) \((n \in \mathbb{N})\). If \(\alpha = \beta = 2\), one obtains the Chebyshev polynomials of the first kind (which played a crucial role in Section 2). One has \(\hat{\mathbb{N}}_0 = [-1, 1]\) [17 Sect. 6]. Nevertheless, a modification of the Karlin–McGregor polynomials will yield examples which fulfill the necessary condition (3.1) and even the desired property \(h(1) < 2\) (see Theorem 3.1, the subsequent remarks and Theorem 3.2 below). The basic underlying idea is that there are choices of \(\alpha, \beta \geq 2\) such that \((K^{(\alpha, \beta)}_n(x))_{n \in \mathbb{N}_0}\) satisfies the condition
\[0 \notin \text{supp } \mu\] (3.2)
(cf. (3.6) below). As nonnegative linearization of products yields \(\text{supp } \mu \subseteq \hat{\mathbb{N}}_0\), (3.2) is a weaker condition than (3.1). In order to obtain the stronger condition (3.1), our strategy will be to modify the polynomials in such a way that (3.2) remains true and \(\text{supp } \mu\) coincides with \(\hat{\mathbb{N}}_0\).

We first recall some basics about the Karlin–McGregor polynomials. If \(\alpha \leq \beta\), then the orthogonalization measure \(\mu\) is absolutely continuous and satisfies
\[d\mu(x) = \begin{cases} \frac{\beta\sqrt{(\gamma_1^2 - x^2)(x^2 - \gamma_2^2)}}{2\pi|x|(1 - x^2)} \chi(-\gamma_1, -\gamma_2) \cup (\gamma_2, \gamma_1)(x) \; dx, & \alpha < \beta, \\ \frac{\alpha\sqrt{\gamma_1^2 - x^2}}{2\pi|x|(1 - x^2)} \chi(-\gamma_1, \gamma_1)(x) \; dx, & \alpha = \beta, \end{cases}\] (3.3)
where
\[\gamma_1 := \frac{1}{\sqrt{\alpha\beta}}(\sqrt{\alpha - 1} + \sqrt{\beta - 1})\]
and
\[\gamma_2 := \frac{1}{\sqrt{\alpha\beta}}|\sqrt{\alpha - 1} - \sqrt{\beta - 1}|.\]
If \(\alpha > \beta\), then \(\mu\) consists of an absolutely continuous part given by
\[\frac{\beta\sqrt{(\gamma_1^2 - x^2)(x^2 - \gamma_2^2)}}{2\pi|x|(1 - x^2)} \chi(-\gamma_1, -\gamma_2) \cup (\gamma_2, \gamma_1)(x) \; dx\] (3.4)
and a discrete part given by the point mass
\[\mu(\{0\}) = \frac{\alpha - \beta}{\alpha}.\] (3.5)

Essentially, the preceding formulas for \(\mu\) can be found in Karlin and McGregor’s paper [12], where a slightly different system of orthogonal polynomials was considered. Moreover, the first part of (3.3) can be found in [33, p. 49, p. 108] (without reference or explanation), and (3.5) can also be found in [6]. Up to a small mistake, the formulas for \(\mu\) and a proof via
Stieltjes transforms can also be found in [4, Theorem 4.2]. Since the deduction of (3.3) to (3.5) from the polynomials originally considered in Karlin and McGregor’s paper [12] is not obvious (and since the relevant passage of [12] contains a small mistake, too), we will give a proof sketch in the appendix. For the moment, we note that one particularly has

$$\text{supp } \mu = \left\{ [-\gamma_1, -\gamma_2] \cup [\gamma_2, \gamma_1], \quad \alpha \leq \beta, \right\} \cup \left\{ [-\gamma_1, -\gamma_2] \cup \{0\} \cup [\gamma_2, \gamma_1], \quad \alpha > \beta. \right\} \tag{3.6}$$

It is obvious from (1.6) that the Haar weights are given by

$$h(2n - 1) = \alpha(\alpha - 1)^{n-1}(\beta - 1)^{n-1}$$ \tag{3.7}

and

$$h(2n) = \beta(\alpha - 1)^n(\beta - 1)^{n-1}$$ \tag{3.8}

for \(n \in \mathbb{N}\) [17, Sect. 6]. Moreover, it is easy to see via induction that

$$K_{2n}^{(\alpha, \beta)}(\gamma_1) = \frac{(\alpha - 2)\sqrt{\beta - 1} + (\beta - 2)\sqrt{\alpha - 1}}{\beta(\alpha - 1)^{2n+1}(\beta - 1)^{2n}} \cdot n + \frac{1}{(\alpha - 1)^2(\beta - 1)^2}$$ \tag{3.9}

and

$$K_{2n+1}^{(\alpha, \beta)}(\gamma_1) = \frac{(\alpha - 2)\sqrt{\beta - 1} + (\beta - 2)\sqrt{\alpha - 1}}{\sqrt{\alpha\beta(\alpha - 1)^{2n+1}(\beta - 1)^{2n+1}}} \cdot n + \frac{\sqrt{\alpha - 1} + \sqrt{\beta - 1}}{\sqrt{\alpha\beta(\alpha - 1)^{2n}(\beta - 1)^{2n}}}$$ \tag{3.10}

for all \(n \in \mathbb{N}_0\). Concerning the special case \(\alpha = 2\), it was shown in [4, p. 72] via sieved polynomials that

$$K_{2n-1}^{(2, \beta)}(x) = \frac{1}{(\beta - 1)^{2n}} \left( \sqrt{\beta - 1}U_{n-1} \left( \frac{(2x^2 - 1)\beta}{2\sqrt{\beta - 1}} \right) - U_{n-2} \left( \frac{(2x^2 - 1)\beta}{2\sqrt{\beta - 1}} \right) \right)$$ \tag{3.11}

and

$$K_{2n}^{(2, \beta)}(x) = \frac{1}{(\beta - 1)^{2n}} \left( \frac{\beta - 1}{\beta} U_n \left( \frac{(2x^2 - 1)\beta}{2\sqrt{\beta - 1}} \right) - \frac{1}{\beta} U_{n-2} \left( \frac{(2x^2 - 1)\beta}{2\sqrt{\beta - 1}} \right) \right)$$ \tag{3.12}

for every \(n \in \mathbb{N}\), where \((U_n(x))_{n \in \mathbb{N}_0}\) denotes the sequence of Chebyshev polynomials of the second kind (i.e., \(U_n(\cos(\varphi)) = \sin((n+1)\varphi)\)) with the convention \(U_{-1}(x) := 0\).

The following result provides our first example with \(h(1) < 2\).

**Theorem 3.1.** Let

$$c_{2n-1} = \frac{6n + 4}{9n + 9}$$

and

$$c_{2n} = \frac{n + 1}{3n + 5}$$

Then \((P_n(x))_{n \in \mathbb{N}_0}\) satisfies nonnegative linearization of products, i.e., \((P_n(x))_{n \in \mathbb{N}_0}\) induces a polynomial hypergroup on \(\mathbb{N}_0\). The Haar function \(h : \mathbb{N}_0 \to [1, \infty)\) satisfies

$$h(2n - 1) = \frac{9}{5} \left( \frac{n}{2} + \frac{1}{2} \right)^2$$

and

$$h(2n) = \frac{9}{5} \left( \frac{n}{2} + \frac{5}{6} \right)^2$$

for every \(n \in \mathbb{N}\). In particular, one has \(h(1) = 9/5 < 2\).
Proof. Let \((Q_n(x))_{n \in \mathbb{N}_0} \subseteq \mathbb{R}[x]\) be defined by

\[ Q_{2n}(x) = \frac{3n + 5}{2^{n - 1}} P_{2n} \left( \frac{\sqrt{10}}{3} x \right) \]

and

\[ Q_{2n+1}(x) = \frac{3n + 6}{2^{n+1} \sqrt{10}} P_{2n+1} \left( \frac{\sqrt{10}}{3} x \right). \]

We claim that \((Q_n(x))_{n \in \mathbb{N}_0}\) coincides with the Karlin-McGregor polynomials \((K_n(2,5)(x))_{n \in \mathbb{N}_0}\). Once the claim is established, we obtain the assertions as follows: since \((K_n(2,5)(x))_{n \in \mathbb{N}_0}\) satisfies nonnegative linearization of products, \((P_n(x))_{n \in \mathbb{N}_0}\) satisfies nonnegative linearization of products, too; the explicit formulas for \(h\) are easily verified by \((1.6)\) and induction.

\[ Q_0(x) = 1 = K_0(2,5)(x) \]

and \(Q_1(x) = x = K_1(2,5)(x)\) are immediate from the definitions. Let \(n \in \mathbb{N}_0\) be arbitrary but fixed, and assume that \(Q_{2n}(x) = K_{2n}(2,5)(x)\) and \(Q_{2n+1}(x) = K_{2n+1}(2,5)(x)\). Then

\[
\begin{align*}
K_{2n+2}(2,5)(x) &= \frac{x K_{2n+2}(2,5)(x)}{2} - \frac{1}{2} K_{2n+1}(2,5)(x) \\
&= \frac{x \cdot \frac{3n+6}{2^{n+1} \sqrt{10}} P_{2n+1} \left( \frac{\sqrt{10}}{3} x \right)}{2} - \frac{1}{2} \cdot \frac{3n+5}{2^{n-1}} P_{2n} \left( \frac{\sqrt{10}}{3} x \right) \\
&= \frac{3n+6}{2^{n+1} \sqrt{10}} \cdot \frac{\sqrt{10}}{3} \left( \frac{3n+6}{2^{n+1} \sqrt{10}} P_{2n+2} \left( \frac{\sqrt{10}}{3} x \right) \right) + \frac{6n+10}{2^{n+1} \sqrt{10}} P_{2n+1} \left( \frac{\sqrt{10}}{3} x \right) - \frac{1}{2} \cdot \frac{3n+5}{2^{n-1}} P_{2n} \left( \frac{\sqrt{10}}{3} x \right) \\
&= \frac{3n+8}{2^{n+1} \sqrt{10}} P_{2n+2} \left( \frac{\sqrt{10}}{3} x \right) \\
&= Q_{2n+2}(x)
\end{align*}
\]

and

\[
\begin{align*}
K_{2n+3}(2,5)(x) &= \frac{x K_{2n+3}(2,5)(x)}{2} - \frac{1}{2} K_{2n+2}(2,5)(x) \\
&= \frac{x \cdot \frac{3n+8}{2^{n+2} \sqrt{10}} P_{2n+3} \left( \frac{\sqrt{10}}{3} x \right)}{2} - \frac{1}{2} \cdot \frac{3n+6}{2^{n+1} \sqrt{10}} P_{2n+2} \left( \frac{\sqrt{10}}{3} x \right) \\
&= \frac{3n+8}{2^{n+2} \sqrt{10}} \cdot \frac{\sqrt{10}}{3} \left( \frac{3n+8}{2^{n+2} \sqrt{10}} P_{2n+3} \left( \frac{\sqrt{10}}{3} x \right) \right) + \frac{n+2}{2^{n+1} \sqrt{10}} P_{2n+2} \left( \frac{\sqrt{10}}{3} x \right) - \frac{1}{2} \cdot \frac{3n+6}{2^{n+1} \sqrt{10}} P_{2n+2} \left( \frac{\sqrt{10}}{3} x \right) \\
&= \frac{3n+9}{2^{n+2} \sqrt{10}} P_{2n+3} \left( \frac{\sqrt{10}}{3} x \right) \\
&= Q_{2n+3}(x).
\]

□
Reconsidering the proof of Theorem 3.1 and following the strategy outlined above, we can find further examples which satisfy $h(1) < 2$ or even $h(1) < 1 + \epsilon$ (and nonnegative linearization of products). We start with the Karlin–McGregor polynomials $(K_n^{(\alpha,\beta)}(x))_{n \in \mathbb{N}_0}$, $\alpha, \beta \geq 2$. Next, we rescale them in such a way that the right endpoint of the support of the measure becomes 1. Finally, we renormalize the resulting polynomials in such a way that $P_n(1) \equiv 1$ again. This procedure ends up in the sequence $(P_n(x))_{n \in \mathbb{N}_0} \subseteq \mathbb{R}[x]$ of modified Karlin–McGregor polynomials given by

$$P_n(x) = \frac{K_n^{(\alpha,\beta)}(\gamma_1 x)}{K_n^{(\alpha,\beta)}(\gamma_1)},$$

and $(P_n(x))_{n \in \mathbb{N}_0}$ still satisfies nonnegative linearization of products. The above-mentioned further examples will be obtained below for suitable choices of $\alpha$ and $\beta$. We first study the polynomials $(P_n(x))_{n \in \mathbb{N}_0}$ in detail and compute the associated orthogonalization measures, Haar measures and recurrence coefficients.

Using (3.3) to (3.5), we find that the Haar weights associated with $P_n(x)$ are linked to each other by multiplication with

$$\beta \gamma_1 \sqrt{(1-x^2)(\gamma_1^2 x^2 - \gamma_2^2)} 2\pi |x| (1 - \gamma_1^2 x^2) \chi(-1, -\frac{\gamma_1}{\gamma_1}) (x) dx, \quad \alpha < \beta,$$

and a discrete part given by

$$\mu(\{0\}) = \frac{\alpha - \beta}{\alpha}.$$  

Plots for different values of $(\alpha, \beta)$ can be found in Figure 3.

By construction, the Haar weights corresponding to the modified polynomials $(P_n(x))_{n \in \mathbb{N}_0}$ and the Haar weights corresponding to the original Karlin–McGregor polynomials $(K_n^{(\alpha,\beta)}(x))_{n \in \mathbb{N}_0}$ are linked to each other by multiplication with $(K_n^{(\alpha,\beta)}(\gamma_1))^2$. Using (3.7) to (3.10), we obtain that the Haar weights associated with $(P_n(x))_{n \in \mathbb{N}_0}$ satisfy $h(0) = 1$ and

$$h(2n-1) = 1 - \beta \left[ \frac{\alpha - 2}{\sqrt{\alpha - 1}} + \frac{\beta - 2}{\sqrt{\beta - 1}} \right] \cdot (n-1) + \sqrt{\alpha - 1} + \sqrt{\beta - 1}$$

and

$$h(2n) = 1 - \beta \left[ \frac{\alpha - 2}{\sqrt{\alpha - 1}} + \frac{\beta - 2}{\sqrt{\beta - 1}} \right] \cdot n + \frac{\beta}{\sqrt{\beta - 1}}$$

for $n \in \mathbb{N}$. Observe that $h$ is always of quadratic (and therefore subexponential) growth, which particularly implies that $\mathbb{N}_0 = \sup \mu$ now as desired. Plots can be found in Figure 4.

Via (1.6), (3.16) and (3.17), we can recursively compute the recurrence coefficients $(c_n)_{n \in \mathbb{N}}$ which correspond to the modified polynomials $(P_n(x))_{n \in \mathbb{N}_0}$. Alternatively, one can compute $(c_n)_{n \in \mathbb{N}}$ from (3.9) and (3.10) because the recurrence coefficients are linked to those belonging
Figure 3. The absolutely continuous part of $\mu$, cf. (3.13) and (3.14), for $(\alpha, \beta) = (2, 5)$ (solid line), $(\alpha, \beta) = (5, 5)$ (dashed line) and $(\alpha, \beta) = (8, 5)$ (dotted line). In the latter case, there is an additional discrete part with $\mu(\{0\}) = 3/8$, cf. (3.15).

to $(K_n^{(\alpha, \beta)}(x))_{n \in \mathbb{N}}$ by multiplication with $K_n^{(\alpha, \beta)}(\gamma_1)/(\gamma_1 K_n^{(\alpha, \beta)}(\gamma_1))$. We obtain

$$c_{2n-1} = \frac{\sqrt{\beta - 1}}{\sqrt{\alpha - 1} + \sqrt{\beta - 1}} \times \left[1 - \sqrt{\alpha - 1} \cdot \frac{\sqrt{\alpha - 1} \sqrt{\beta - 1} - 1}{((\alpha - 2)\sqrt{\beta - 1} + (\beta - 2)\sqrt{\alpha - 1}) \cdot n + \sqrt{\alpha - 1} + \sqrt{\beta - 1}}\right],$$

$$c_{2n} = \frac{\sqrt{\alpha - 1}}{\sqrt{\alpha - 1} + \sqrt{\beta - 1}} \left[1 - \sqrt{\beta - 1} \cdot \frac{\sqrt{\alpha - 1} \sqrt{\beta - 1} - 1}{((\alpha - 2)\sqrt{\beta - 1} + (\beta - 2)\sqrt{\alpha - 1}) \cdot n + \beta \sqrt{\alpha - 1}}\right],$$

and particularly

$$\lim_{n \to \infty} c_{2n-1} = \frac{\sqrt{\beta - 1}}{\sqrt{\alpha - 1} + \sqrt{\beta - 1}},$$

$$\lim_{n \to \infty} c_{2n} = \frac{\sqrt{\alpha - 1}}{\sqrt{\alpha - 1} + \sqrt{\beta - 1}}.$$
For every $n \in \mathbb{N}$, we compute
\[
\alpha_n = \begin{cases} 
\frac{\sqrt{\beta}}{\beta - 1}, & n = 1, \\
\frac{\sqrt{\alpha}}{\alpha - 1 + \sqrt{\beta - 1}}, & n \text{ even}, \\
\frac{\sqrt{\alpha}}{\alpha - 1 + \sqrt{\beta - 1}}, & \text{else}, 
\end{cases}
\]
so the coefficients in the orthonormal/monic normalization become periodic. This shows that $(P_n(x))_{n \in \mathbb{N}_0}$ belongs to the class of Geronimus polynomials \[19\] and that nonnegative linearization of products also follows directly from a general criterion in \[27\] (without using nonnegative linearization of products for the Karlin–McGregor polynomials): if $\alpha \leq \beta$, then \[27\] Theorem 3 (i) can be applied, and if $\alpha > \beta$, then \[27\] Theorem 3 (ii) combined with Remark 3 works. For the special case $\alpha = \beta$, nonnegative linearization of products also follows from \[15, 3 (g) (i)\] (via explicit formulas for the linearization coefficients of the corresponding Geronimus polynomials).

Moreover, our construction based on the Karlin–McGregor polynomials makes it possible to compute the linearization coefficients $g(m, n; k)$ explicitly because the linearization coefficients belonging to the Karlin–McGregor polynomials are explicitly known \[1\] Theorem 4.1] and just have to be multiplied by $K_k^{(\alpha, \beta)}(\gamma_1)/(K_m^{(\alpha, \beta)}(\gamma_1)K_n^{(\alpha, \beta)}(\gamma_1))$ (which is explicitly known due to \[3.9\] and \[3.10\]). We refrain from stating the corresponding formulas, however.

Coming back to our necessary condition \[3.1\] from the beginning of the section, we see from \[3.13\] to \[3.15\] that the modified polynomials $(P_n(x))_{n \in \mathbb{N}_0}$ satisfy this condition if and only if $\alpha < \beta$. We now also come back to the problem ”$h(1) < 2$” and first observe that if one is not interested in the full function $h$ but only in $h(1)$, then one can also argue in the following way which is based on the first recurrence coefficient $c_1$ and less computational than the proof of the full equations \[3.16\], \[3.17\]: $c_1$ satisfies
\[
x^2 = (1 - c_1)P_2(x) + c_1,
\]
so
\[
h(1) = \frac{1}{c_1} = 1 - \frac{1}{P_2(0)} = 1 - \frac{K_2^{(\alpha, \beta)}(\gamma_1)}{K_2^{(\alpha, \beta)}(0)}.
\]
Since
\[
K_2^{(\alpha, \beta)}(x) = \frac{\alpha x^2 - 1}{\alpha - 1},
\]
we obtain
\[
h(1) = \alpha \gamma_1^2 = \frac{1}{\beta} (\sqrt{\alpha - 1} + \sqrt{\beta - 1})^2
\]
(cf. \[3.16\]). In particular, one has $h(1) < 2$ if and only if $\alpha < 3\beta - 2\sqrt{2\beta^2 - 2\beta}$. Since $[2, 3\beta - 2\sqrt{2\beta^2 - 2\beta})$ is a (proper) subset of $[2, \beta)$, the corresponding measures are absolutely continuous with density given by the first case of \[3.13\]. Moreover, we obtain that the necessary condition \[3.1\] is not sufficient for $h(1) < 2$.

**Theorem 3.2.** Let $\alpha, \beta \geq 2$, and let $P_n(x) = K_n^{(\alpha, \beta)}(\gamma_1 x)/K_n^{(\alpha, \beta)}(\gamma_1)$ $(n \in \mathbb{N}_0)$.

(i) For every $\epsilon > 0$, there exists a polynomial hypergroup on $\mathbb{N}_0$ such that $h(1) < 1 + \epsilon$.

More precisely, for any choice of $\alpha$ the parameter $\beta$ can be chosen in such a way that the hypergroup induced by the sequence $(P_n(x))_{n \in \mathbb{N}_0}$ has the desired property.

(ii) For any choice of $\alpha, \beta$, the polynomial hypergroup induced by $(P_n(x))_{n \in \mathbb{N}_0}$ satisfies $h(n) \geq 2$ for all $n \geq 2$.
The first Haar weights \( h(n) \) belonging to \( (P_n(x))_{n \in \mathbb{N}_0} \), cf. (3.16) and (3.17), for \((\alpha, \beta) = (2, 5)\) (box), \((\alpha, \beta) = (5, 5)\) (diamond) and \((\alpha, \beta) = (8, 5)\) (circle). In the first case, one has \( h(1) < 2 \).

Proof. (i) Let \( \alpha \geq 2 \) be arbitrary. Then, by the preceding calculations,

\[
h(1) = \frac{1}{\beta} (\sqrt{\alpha - 1} + \sqrt{\beta - 1})^2 \rightarrow 1 \quad (\beta \to \infty).
\]

This yields the assertion.

(ii) For every \( n \in \mathbb{N} \), the explicit formulas (3.16) and (3.17) for \( h \) yield

\[
h(2n) \geq \frac{1}{\beta} \left[ \frac{\beta - 2}{\sqrt{\beta - 1}} + \frac{\beta}{\sqrt{\beta - 1}} \right]^2 = 4 \cdot \frac{\beta - 1}{\beta} \geq 2
\]

and

\[
h(2n + 1) \geq \frac{1}{\beta} \left[ \frac{\beta - 2}{\sqrt{\beta - 1}} + 1 + \sqrt{\beta - 1} \right]^2
\]

\[
= \left( \frac{\beta - 1 - 1)((2\beta - 1)\sqrt{\beta - 1} + 6\beta - 7) + 2}{(\beta - 1)\beta} \right)
\]

\[
\geq 2.
\]

\(\square\)

Corollary 3.1. The converse of Theorem 2.1 and the converses of Corollary 2.1 to Corollary 2.5 are not true.

Proof. Let \( \beta \geq 2 \) and \( \alpha \geq 3\beta - 2\sqrt{2\beta^2 - 2\beta} \) with \( \alpha \neq \beta \), and let \( P_n(x) = K_n^{(alpha, beta)}(x)/K_n^{(alpha, beta)}(1) \) \((n \in \mathbb{N}_0)\). Then, as consequence of the second part of Theorem 3.2 and the preceding notes, we have \( h(n) \geq 2 \) for all \( n \in \mathbb{N} \) but \( \mathbb{N}_0 \neq [-1, 1] \) (in fact, for
\( \alpha \in [3\beta - 2\sqrt{2\beta^2 - 2\beta}, \beta) \) we do not even have \( 0 \in \mathbb{N}_0 \). This shows that the converse of Theorem 2.1 is not true. The latter implies that the converses of Corollary 2.1 to Corollary 2.5 are also not true.

We briefly consider the special case \( \alpha = 2 \). As a consequence of (3.11) and (3.12), in this special case we have

\[
P_{2n-1}(x) = \frac{1}{\sqrt{\beta - 1}n - (n - 1)} x
\]

\[
\times \left( \sqrt{\beta - 1}U_{n-1}\left( \frac{(2\sqrt{\beta - 1} + \beta)x^2 - \beta}{2\sqrt{\beta - 1}} \right) - U_{n-2}\left( \frac{(2\sqrt{\beta - 1} + \beta)x^2 - \beta}{2\sqrt{\beta - 1}} \right) \right)
\]

and

\[
P_{2n}(x) = \frac{1}{(\beta - 2)n + \beta}
\]

\[
\times \left( (\beta - 1)U_n\left( \frac{(2\sqrt{\beta - 1} + \beta)x^2 - \beta}{2\sqrt{\beta - 1}} \right) - U_{n-2}\left( \frac{(2\sqrt{\beta - 1} + \beta)x^2 - \beta}{2\sqrt{\beta - 1}} \right) \right)
\]

for every \( n \in \mathbb{N} \). Choosing \((\alpha, \beta) = (2, 5)\) in Theorem 3.1 was motivated by getting "nice values" for \( \sqrt{\alpha - 1} \) and \( \sqrt{\beta - 1} \). In particular, this special choice allowed us to obtain very simple explicit expressions for the sequence \((c_n)_{n \in \mathbb{N}}\). Due to (3.13), the density takes the simple form

\[
d\mu(x) = \frac{15\sqrt{(1 - x^2)(9x^2 - 1)}}{2\pi|x|(10 - 9x^2)} \chi(-1, -\frac{1}{\sqrt{9}})\left(\frac{1}{\sqrt{9}}\right) dx
\]

in this case (cf. the solid line in Figure 3); furthermore, due to (3.18) and (3.19) the polynomials take the simple form

\[
P_{2n-1}(x) = \frac{1}{n + 1} x \left( 2U_{n-1}\left( \frac{9x^2 - 5}{4} \right) - U_{n-2}\left( \frac{9x^2 - 5}{4} \right) \right),
\]

\[
P_{2n}(x) = \frac{1}{3n + 5} \left( 4U_n\left( \frac{9x^2 - 5}{4} \right) - U_{n-2}\left( \frac{9x^2 - 5}{4} \right) \right)
\]

for all \( n \in \mathbb{N} \).

We finally construct another type of polynomial hypergroups with \( h(1) < 2 \) (and even \( h(1) < 1 + e \)). It does not rely on the Karlin–McGregor polynomials, and the corresponding measures are discrete.

**Theorem 3.3.** Let \((\lambda_n)_{n \in \mathbb{N}_0} \subseteq (0, 1)\) satisfy both \( \lambda_{2n-2} + \lambda_{2n-1} \leq \lambda_{2n} \) and \( \lambda_{2n-1} + \lambda_{2n} \leq \lambda_{2n+2} \) for every \( n \in \mathbb{N} \), and let \((Q_n(x))_{n \in \mathbb{N}_0} \subseteq \mathbb{R}[x]\) be given by the recurrence relation \( Q_0(x) = 1, Q_1(x) = x/\lambda_0 \).

\[
xQ_n(x) = \lambda_n Q_{n+1}(x) + \lambda_{n-1}Q_{n-1}(x) \quad (n \in \mathbb{N}).
\]

The following hold:

(i) The sequence \((Q_n(1))_{n \in \mathbb{N}_0}\) is strictly positive and strictly increasing.

(ii) The sequence \((P_n(x))_{n \in \mathbb{N}_0}\) defined by \(P_n(x) := Q_n(x)/Q_n(1)\) \((n \in \mathbb{N}_0)\) satisfies non-negative linearization of products, and \((Q_n(x))_{n \in \mathbb{N}_0}\) are the orthonormal polynomials which correspond to \((P_n(x))_{n \in \mathbb{N}_0}\). The corresponding measure \(\mu\) is discrete.

(iii) Let \( \lambda \in (0, 1) \) be the limit of the (strictly increasing and bounded, hence convergent) sequence \((\lambda_{2n})_{n \in \mathbb{N}_0}\). If \( \lambda = 1 \), then there exists a strictly increasing sequence \((x_n)_{n \in \mathbb{N}} \subseteq (0, 1)\) with \(\lim_{n \to \infty} x_n = 1\) and

\[
\mathbb{N}_0 = \text{supp } \mu = \{ \pm 1 \} \cup \{ \pm x_n : n \in \mathbb{N} \}.
\]
(iv) For every \( \epsilon > 0 \), the sequence \((\lambda_n)_{n \in \mathbb{N}_0}\) can be chosen with \( \lambda = 1 \) in such a way that the polynomial hypergroup induced by \((P_n(x))_{n \in \mathbb{N}_0}\) fulfills \( h(1) < 1 + \epsilon \).

(v) For any choice of \((\lambda_n)_{n \in \mathbb{N}_0}\), the polynomial hypergroup induced by \((P_n(x))_{n \in \mathbb{N}_0}\) satisfies \( h(n) > 4 \) for all \( n \geq 2 \).

**Proof.**

(i) For every \( n \in \mathbb{N} \), we compute

\[
Q_{n+1}(1) = Q_n(1) + \frac{1 - \lambda_{n-1} - \lambda_n}{\lambda_n} Q_n(1) + \frac{\lambda_{n-1}}{\lambda_n} (Q_n(1) - Q_{n-1}(1)).
\]

Since \( Q_0(1) = 1 \) and \( Q_1(1) = 1/\lambda_0 > 1 \), this yields the assertion.

(ii) As a consequence of (i), \((P_n(x))_{n \in \mathbb{N}_0}\) is well-defined. By [20, Corollary 2 (ii)], \((Q_n(x))_{n \in \mathbb{N}_0}\) satisfies nonnegative linearization of products. Hence, (i) implies that \((P_n(x))_{n \in \mathbb{N}_0}\) satisfies nonnegative linearization of products, too. It is clear from the recurrence relations that \((Q_n(x))_{n \in \mathbb{N}_0}\) are the orthonormal polynomials which correspond to \((P_n(x))_{n \in \mathbb{N}_0}\). Concerning the discreteness of \( \mu \), we refer to [20, Remark 2 p. 427].

(iii) Let \( \lambda = 1 \). Then, as a consequence of [20, Remark 2 p. 427], there exists a strictly increasing sequence \((x_n)_{n \in \mathbb{N}} \subseteq [0, 1)\) with \( \lim_{n \to \infty} x_n = 1 \) and

\[
\text{supp } \mu = \{ \pm 1 \} \cup \{ \pm x_n : n \in \mathbb{N}\}.
\]

It remains to show that \( \mathbb{N}_0 = \text{supp } \mu \) and \( 0 \notin \mathbb{N}_0 \). This can be seen as follows: let \((R_n(x))_{n \in \mathbb{N}_0} \subseteq \mathbb{R}[x] \) be defined by \( R_n(x^2) = P_{2n}(x) \) (this approach is motivated by [20, Section 6]). Then \((R_n(x))_{n \in \mathbb{N}_0}\) satisfies the recurrence relation \( R_0(x) = 1 \), \( R_1(x) = (x - c_1)/a_1 \),

\[
R_1(x) R_n(x) = a^R_n R_{n+1}(x) + b^R_n R_n(x) + c^R_n R_{n-1}(x) \quad (n \in \mathbb{N}),
\]

where

\[

a^R_n := \frac{a_{2n} a_{2n+1}}{a_1^2} \in (0, 1),

b^R_n := \frac{a_{2n} c_{2n+1} + c_{2n} a_{2n-1} - c_1}{a_1} \in (0, 1),

c^R_n := \frac{c_{2n} c_{2n-1}}{a_1} \in (0, 1)
\]

(as usual, \((c_n)_{n \in \mathbb{N}} \subseteq (0, 1)\) and \(a_n = 1 - c_n\) shall denote the recurrence coefficients which belong to the sequence \((P_n(x))_{n \in \mathbb{N}_0}\)). The estimation \( b^R_n > 0 \) follows from the monotonicity behavior of \((\lambda_{2n})_{n \in \mathbb{N}_0}\) because

\[
a_{2n} c_{2n+1} + c_{2n} a_{2n-1} = \lambda_{2n}^2 + \lambda_{2n-1}^2 > \lambda_{2n}^2 > \lambda_{2n-1}^2 = c_1
\]

for every \( n \in \mathbb{N} \); the remaining estimations are obvious from \( a^R_n + b^R_n + c^R_n = 1 \ (n \in \mathbb{N}) \). Since, for every \( n \geq 2 \),

\[
1 > c_{2n-1} = \frac{\lambda_{2n-2}^2}{a_{2n-2}} > \lambda_{2n-2}^2 \to 1 \ (n \to \infty),
\]

we have \( \lim_{n \to \infty} c_{2n-1} = 1 \) (hence \( \lim_{n \to \infty} a_{2n-1} = 0 \)) and consequently

\[
c_{2n} = 1 - \frac{\lambda_{2n}^2}{c_{2n+1}} \to 0 \ (n \to \infty)
\]
(hence \( \lim_{n \to \infty} a_{2n} = 1 \)). Therefore, we have
\[
\begin{align*}
\lim_{n \to \infty} a_n^R &= 0, \\
\lim_{n \to \infty} b_n^R &= 1, \\
\lim_{n \to \infty} c_n^R &= 0.
\end{align*}
\]

Let \( \mu_R \) denote the orthogonalization (probability) measure of \((R_n(x))_{n \in \mathbb{N}_0}\). It is clear from the construction that
\[
\text{supp } \mu_R = \{1\} \cup \{x_n^2 : n \in \mathbb{N}\}.
\]
By [7, Theorem 2, Proposition 4], the behavior of the sequences \((a_n^R)_{n \in \mathbb{N}_0}\), \((b_n^R)_{n \in \mathbb{N}_0}\) and \((c_n^R)_{n \in \mathbb{N}_0}\) as obtained above implies that
\[
\left\{ x \in \mathbb{R} : \max_{n \in \mathbb{N}_0} |R_n(x)| = 1 \right\} = \text{supp } \mu_R.
\]
Therefore, we obtain that
\[
\hat{N}_0 \subseteq \left\{ x \in \mathbb{R} : \max_{n \in \mathbb{N}_0} |P_{2n}(x)| = 1 \right\} = \left\{ x \in \mathbb{R} : \max_{n \in \mathbb{N}_0} |R_n(x)| = 1 \right\} = \{\pm 1\} \cup \{\pm x_n : n \in \mathbb{N}\}.
\]
Since, however,
\[
\{\pm 1\} \cup \{\pm x_n : n \in \mathbb{N}\} = \text{supp } \mu \subseteq \hat{N}_0,
\]
we obtain equality. Moreover, we have \(0 \not\in \hat{N}_0\) because
\[
|P_{2n}(0)| = \prod_{k=0}^{n-1} \frac{c_{2k+1}}{a_{2k+1}} \to \infty \quad (n \to \infty)
\]
by the limiting behavior of \((c_{2n-1})_{n \in \mathbb{N}}\) and \((a_{2n-1})_{n \in \mathbb{N}}\).

(iv) The following example is motivated by [20, Remark 1 p. 427]: choose \(\lambda_{2n} = 1 - q^{n+1}\) and \(\lambda_{2n+1} = (1-q)q^{n+2}\), where \(q \in (0,1)\). Then
\[
\lambda_{2n} - \lambda_{2n-2} - \lambda_{2n-1} = (1-q)^2q^n > 0
\]
and
\[
\lambda_{2n-1} + \lambda_{2n} = \lambda_{2n+2}
\]
for all \(n \in \mathbb{N}\), so \((\lambda_n)_{n \in \mathbb{N}_0}\) is as required. Moreover, as a consequence of (ii) we have
\[
b(1) = Q^2(1) = \frac{1}{\lambda_0^2} = \frac{1}{(1-q)^2} \to 1 \quad (q \to 0),
\]
which yields the assertion.

(v) As a consequence of (i) and (ii), the sequence \((h(n))_{n \in \mathbb{N}_0}\) coincides with \((Q^2(1))_{n \in \mathbb{N}_0}\) and is strictly increasing. Therefore, it suffices to establish that \(Q^2(1) > 2\). The latter can be seen from the estimation
\[
Q^2(1) = \frac{1 - \lambda_0^2}{\lambda_0\lambda_1} > \frac{1 - \lambda_0^2}{\lambda_0(1 - \lambda_0)} = 1 + \frac{1}{\lambda_0} > 2.
\]
\[\square\]
Remark 3.1. Reconsider the example studied in the proof of Theorem 3.3 (iv) for \( q \geq 1 - \sqrt{2}/2 \). In this case, one has \( h(1) \geq 2 \) (and consequently \( h(n) \geq 2 \) for all \( n \in \mathbb{N} \) by Theorem 3.3 (v)). However, \( \mathbb{N}_0 \) is a discrete subset of \([-1, 1]\) by Theorem 3.3 (iii). Therefore, the example provides an alternative proof of Corollary 3.1.

4. Open problems

We finish our paper with a collection of some open problems:

(i) Is \( h(2) \geq 2 \) always true?

(ii) Is \( \lim \inf_{n \to \infty} h(n) \geq 2 \) always true?

(iii) Is \( h(n) \geq 2 \) (\( n \in \mathbb{N} \setminus \{1\} \)) always true?

(iv) Is \( h(1) \geq 2 \) sufficient for \( h(2) \geq 2, \lim \inf_{n \to \infty} h(n) \geq 2 \) or \( h(n) \geq 2 \) (\( n \in \mathbb{N} \))?

(v) Is \( 0 \in \mathbb{N}_0 \) sufficient for \( h(2) \geq 2, \lim \inf_{n \to \infty} h(n) \geq 2 \) or \( h(n) \geq 2 \) (\( n \in \mathbb{N} \))?

The questions (i), (ii) and (iii) are motivated by our observations made in Theorem 3.2 (ii) and Theorem 3.3 (v). Concerning (iv) and (v), recall that \( 0 \in \mathbb{N}_0 \) implies at least \( h(1) \geq 2 \), see the beginning of Section 3.

Appendix A. On the orthogonalization measure of the Karlin–McGregor polynomials

As announced in Section 3, we sketch a proof of equations (3.3) to (3.5), which also corrects a small mistake in Karlin and McGregor’s paper [12]. We first note the following simple observation: let \((P_n(x))_{n \in \mathbb{N}_0}\) be an (arbitrary) orthogonal polynomial sequence as in Section 1 and let \((P^*_n(x))_{n \in \mathbb{N}_0} \subseteq \mathbb{R}[x]\) be the orthogonal polynomial sequence which corresponds to the probability measure

\[
d\mu^*(x) := \frac{1}{a_1} (1 - x^2) \, d\mu(x).
\]

Moreover, let \((\sigma_n(x))_{n \in \mathbb{N}_0}, (\sigma^*_n(x))_{n \in \mathbb{N}_0} \subseteq \mathbb{R}[x]\) denote the corresponding monic versions. Then

\[
(1 - x^2)\sigma^*_n(x) = -\sigma_{n+2}(x) + \frac{\sigma_{n+2}(1)}{\sigma_n(1)} \sigma_n(x)
\]

(A.1)

for all \( n \in \mathbb{N}_0 \). This is immediate from the relation

\[
\int_{\mathbb{R}} (1 - x^2)\sigma^*_n(x)\sigma_k(x) \, d\mu(x) = a_1 \int_{\mathbb{R}} \sigma^*_n(x)\sigma_k(x) \, d\mu^*(x) \quad (n, k \in \mathbb{N}_0)
\]

due to orthogonality and symmetry, yields that \( (1 - x^2)\sigma^*_n(x) \) has to be a linear combination of \( \sigma_{n+2}(x) \) and \( \sigma_n(x) \). Concerning the division in (A.1), we note that it is clear from Section 3 that \( \sigma_n(1) > 0 \) for all \( n \in \mathbb{N}_0 \).

From now on, let \( \alpha, \beta \geq 2 \) and \((P_n(x))_{n \in \mathbb{N}_0}\) be the sequence of Karlin–McGregor polynomials \((\tilde{K}^{(\alpha, \beta)}_n(x))_{n \in \mathbb{N}_0}\). Moreover, let \((\tilde{P}_n(x))_{n \in \mathbb{N}_0}\) be the sequence given by the modified recurrence relation \( \tilde{P}_0(x) = 1, \tilde{P}_1(x) = \beta x/((\beta - 1)\),

\[
x\tilde{P}_n(x) = a_n \tilde{P}_{n+1}(x) + c_n \tilde{P}_{n-1}(x) \quad (n \in \mathbb{N}),
\]

(A.2)

where \((c_n)_{n \in \mathbb{N}} \subseteq (0, 1) \) and \( a_n \equiv 1 - c_n \) are the Karlin–McGregor coefficients as in Section 3 (so \( c_{2n} = 1/\alpha, \ c_{2n+1} = 1/\beta \)). These are the polynomials studied in [12], and up to the aforementioned mistake, it was shown there that \((\tilde{P}_n(x))_{n \in \mathbb{N}_0}\) is orthogonal w.r.t. the probability

\[
\psi(x) = \sqrt{4p_1q x^2 - (x^2 - p_1q)(x^2 - p_1q)/2(p_1q)^2}.
\]

The formula for \( \psi'(x) \) on [12] p. 74] has to be corrected; the right formula reads \( \psi'(x) = \sqrt{4p_1q x^2 - (x^2 - p_1q)(x^2 - p_1q)/2(p_1q)^2} \).
measure $\tilde{\mu}$ satisfying the following properties: if $\alpha \leq \beta$, then $\tilde{\mu}$ is absolutely continuous with
\begin{equation}
\nonumber d\tilde{\mu}(x) = \begin{cases}
\frac{\alpha \beta \sqrt{(\gamma_2^2 - x^2)(2\gamma_2^2 - x^2)}}{2\pi (\alpha - 1) |x|} \chi_{(-\gamma_1, -\gamma_2) \cup (\gamma_2, \gamma_1)}(x) dx, & \alpha < \beta, \\
\frac{\alpha \beta}{\sqrt{1 - x^2}} \chi_{(-\gamma_1, \gamma_1)}(x) dx, & \alpha = \beta,
\end{cases}
\end{equation}
(A.3)
where $\gamma_1$ and $\gamma_2$ are as in Section 3. If $\alpha > \beta$, however, then $\tilde{\mu}$ consists of the absolutely continuous part
\begin{equation}
\nonumber \tilde{\mu}(\{0\}) = \frac{\alpha - \beta}{\alpha - 1}.
\end{equation}
(A.5)

It is now left to deduce (3.3) to (3.5) from (A.3) to (A.5), which can be done as follows: first, show that the monic versions $(\sigma_n(x))_{n \in \mathbb{N}_0}$ and $(\tilde{\sigma}_n(x))_{n \in \mathbb{N}_0}$ satisfy the recurrence relations $\sigma_0(x) = \sigma_0(x) = 1$, $\sigma_1(x) = \sigma_1(x) = x$,
$$x \sigma_n(x) = \sigma_{n+1}(x) + \lambda_n \sigma_{n-1}(x) (n \in \mathbb{N}),$$
$$x \tilde{\sigma}_n(x) = \tilde{\sigma}_{n+1}(x) + \tilde{\lambda}_n \tilde{\sigma}_{n-1}(x) (n \in \mathbb{N}),$$
where
$$\lambda_n := \begin{cases}
\frac{1}{\alpha}, & n = 1, \\
\frac{\alpha - 1}{\alpha^2}, & n \text{ even}, \\
\frac{\alpha - 1}{\alpha^2}, & n \text{ else},
\end{cases}$$
and
$$\tilde{\lambda}_n = \begin{cases}
\frac{\beta - 1}{\alpha \beta}, & n = 1, \\
\lambda_n, & n \text{ else}.
\end{cases}$$
Concerning $(\sigma_n(x))_{n \in \mathbb{N}_0}$, this follows from (1.5); concerning $(\tilde{\sigma}_n(x))_{n \in \mathbb{N}_0}$, just calculate the leading coefficients of the polynomials $P_n(x)$ from (A.2) and the initial conditions. Next, use these recurrence relations to show that
$$(1 - x^2) \tilde{\sigma}_0(x) = -\sigma_2(x) + \frac{\alpha - 1}{\alpha} \sigma_0(x)$$
and
$$\nonumber (1 - x^2) \tilde{\sigma}_n(x) = -\sigma_{n+2}(x) + \frac{\alpha - 1)(\beta - 1)}{\alpha \beta} \sigma_n(x) (n \in \mathbb{N})$$
via induction. Finally, use induction to show that
$$\frac{\sigma_{n+2}(1)}{\sigma_n(1)} = \frac{\sigma_{n+1}(1)}{\sigma_n(1)} - \lambda_{n+1} = \begin{cases}
\frac{\alpha - 1}{\alpha}, & n = 0, \\
\frac{(\alpha - 1)(\beta - 1)}{\alpha \beta}, & \text{else},
\end{cases}$$
Putting all together, we see that $(\tilde{\sigma}_n(x))_{n \in \mathbb{N}_0}$ coincides with the sequence $(\sigma^*_n(x))_{n \in \mathbb{N}_0}$. It is well-known that if the support of an orthogonalization measure is compact, then it is the unique orthogonalization measure of its orthogonal polynomial sequence $[3]$. Therefore, we can conclude that $\tilde{\mu} = \mu^*$, which yields the desired formulas for $\mu$. 


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