GLOBAL LARGE SOLUTIONS AND OPTIMAL TIME-DECAY ESTIMATES TO THE KORTEweg SYSTEM

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Abstract. We prove the global solutions to the Korteweg system without smallness condition imposed on the vertical component of the incompressible part of the velocity. The weighted Chemin-Lerner-norm technique which is well-known for the incompressible Navier-Stokes equations is introduced to derive the a priori estimates. As a byproduct, we obtain the optimal time decay rates of the solutions by using the pure energy argument (independent of spectral analysis). In contrast to the compressible Navier-Stokes system, the time-decay estimates are more accurate owing to smoothing effect from the Korteweg tensor.

1. Introduction and the main result. We are concerned with the existence of global large solutions and the time-decay estimates for an isothermal model of capillary fluids derived by J.E Dunn and J. Serrin (see Arch. Rational Mech. Anal. 88 (2):95-133, 1985) in $\mathbb{R}^3$, which can be used as a phase transition model:

$$
\begin{cases}
\partial_t \rho + \text{div}(\rho U) = 0, & x \in \mathbb{R}^3, \ t \in \mathbb{R}^+, \\
\partial_t (\rho U) + \text{div}(\rho U \otimes U) - \text{div}(\mu(\rho)D(U)) - \nabla(\lambda(\rho)\text{div} U) + \nabla P(\rho) = \text{div} K, \\
(\rho, U)|_{t=0} = (\rho_0, U_0),
\end{cases}
$$

(1.1)

where the Korteweg tensor can be written as follows:

$$
\text{div} K = \nabla \left( \rho \kappa(\rho) \Delta \rho + \frac{\kappa(\rho)}{2} |\nabla \rho|^2 \right) - \text{div} \left( \kappa(\rho) \nabla \rho \otimes \nabla \rho \right). 
$$

(1.2)

Above, $\rho = \rho(t, x) \in \mathbb{R}^+$ is the density, $U = (U^1, U^2, U^3)$ is the velocity field. $\mu$ and $\lambda$ are two Lamé viscosity coefficients depending on the density $\rho$ and satisfying $\mu > 0$ and $\mu + 3\lambda \geq 0$. The notation $D(U)$ designates the deformation tensor defined by

$$
D(U) \equiv \frac{1}{2}(\nabla U + \nabla U^T) \quad \text{with} \quad \nabla U = \partial_i U^j \quad \text{and} \quad \nabla U^T = \partial_j U^i.
$$

The pressure $P$ is suitably smooth increasing function of the density $\rho$. $\kappa$ is the coefficient of capillarity and is a regular function. The Korteweg tensor $\text{div} K$ allows to describe the variation of density at the interfaces between two phases, generally a mixture liquid-vapor. Concerning derivation of (1.1), Van der Waals [48] observes
that a phase transition boundary can be regarded as a thin transition zone, i.e.,
diffuse interface caused by a steep gradient of the density. Based on this idea,
Korteweg [38] suggests the stress tensor including the term $\nabla \rho \otimes \nabla \rho$ of the Navier-
Stokes equation. Then Dunn and Serrin [22] generalize the Korteweg’s work and provide
the system (1.1) with (1.2). In recent works, Heida and Málek [35] derive
(1.1) by the entropy production method in difference from [22]. Gorban and Karlin
[27] derive the Korteweg tensor from the Boltzmann equation.

In the last decade, a large amount of literature has been devoted to the system
(1.1). Let us recall brief history of mathematical analysis for the compressible fluid
model of Korteweg type. In the case of constant capillary coefficient $\kappa(\rho)$, Hattori
and Li [33] proved the existence of the global strong solutions for (1.1) with small
initial data in $\mathbb{R}^d$ ($d \geq 2$). Danchin and Desjardins [17] improved this result by
working with initial data $(\rho_0 - 1, U_0)$ belonging to the homogeneous Besov spaces
$\dot{B}^{\frac{d}{2}+1}_{2,1} \times \dot{B}^{\frac{d}{2}-1}_{2,1}$. Haspot [31] further relaxed the initial data in [17] to $(\rho_0 - 1, U_0) \in
(\dot{B}^{\frac{d}{2}+1}_{2,2} \cap L^\infty) \times \dot{B}^{\frac{d}{2}-1}_{2,2}$. In [39], Kotschote showed the existence of strong solution for
the isothermal model in bounded domain by using Dore-Venni Theory and $\mathcal{H}^\infty$
calculus. In [49], Watanabe proved the existence of a unique global strong solutions
to the compressible Navier-Stokes-Korteweg equations in the $L^p$-in-time and $L^q$-in-
space framework. Readers interested by Korteweg type systems are referred to the
following articles [1, 2, 3, 4, 5, 6, 8, 9, 21, 29, 36, 47, 42, 53] and references cited
therein.

In the sequel, we are focusing on the case of shallow-water viscosity coefficients,
which means $\mu(\rho) = 2\mu\rho$ with $\mu > 0$ and $\lambda(\rho) = 0$. In addition we will deal
only with the case of the quantum compressible Navier-Stokes system studied in
particular in [36] which corresponds to the capillary coefficient $\kappa(\rho) = \frac{\kappa}{\rho^2}$. Following
the specific choice on the capillary coefficient $\kappa(\rho) = \frac{\kappa}{\rho}$ with $\kappa = \mu^2$ in [32]. A
direct computation implies $\kappa(\rho) + \rho\kappa'(\rho) = 0$, and $\text{div} \mathbb{K} = \mu^2 \text{div} (\rho \nabla \ln \rho)$. Thus,
as long as $\rho$ does not vanish, by introducing the effective velocity (which has been
introduced by Jiingel in [36] in order to prove the existence of global weak solution)
$u = U + \mu \nabla \ln (1 + a)$ with $a = \rho - 1$ such that we can reformulate the system (1.1)
with (1.2) equivalently as follows:

$$
\begin{alignat}{2}
\partial_t a - \mu \Delta a + \text{div} u &= -\text{div}(au), \\
\partial_t u - \mu \Delta u + \nabla a &= 2\mu \nabla (\ln (1 + a)) \nabla u - k(a) \nabla a - u \cdot \nabla u, \\
(a, u)|_{t=0} &= (a_0, u_0),
\end{alignat}
$$

where

$$
k(a) \overset{\text{def}}{=} \frac{P'(1 + a)}{1 + a} - 1, \quad \text{with} \quad P'(1) = 1,
$$

$$
G(a) \overset{\text{def}}{=} \int_0^a \left( \frac{P'(1 + t)}{1 + t} - 1 \right) dt \quad \text{such that} \quad \nabla G(a) = k(a) \nabla a.
$$

It’s obvious that the density equation verifies parabolic equation, this point is very
different from the compressible Navier-Stokes equations (see [10, 14, 15, 16, 18, 19,
23, 24, 25, 26, 30, 34, 52]). Indeed, the density regularity of the compressible Navier-
Stokes equations is separated by a frequency threshold (see [10, 14, 15, 16, 18, 30,
34, 52]): in low frequencies, the solution is subject to a heat-type smoothing, and
in the high frequencies, there is only a damping effect due to the term of pressure.
The solutions of the Korteweg system is more regular: for all frequencies, we have
a parabolic regularization on the density. Making full use of the smoothing effect of the density in (1.3), Haspot [32] proved the existence of strong solution in finite time for large initial data with a precise bound by below on the life span $T^*$. Moreover, the author extended this solution beyond $T^*$ with small initial data.

The aim of the present paper in the first part is to go beyond the smallness imposed on the initial data, especially on the incompressible part of the initial velocity. Before presenting our main result, we first separate the system (1.3) from incompressible part and compressible part with the aid of operators $P$ and $Q$ where

$$P \equiv \mathcal{I} - Q \equiv \mathcal{I} - \nabla \Delta^{-1} \text{div}:$$

(Incompressible part) \begin{align*}
\partial_t P u - \mu \Delta P u + P(u \cdot \nabla u) &= 2\mu P(\nabla(\ln(1 + a)) \nabla u), \\
P u|_{t=0} &= P u_0 = P U_0, \tag{1.4}
\end{align*}

and Compressible part

$$\begin{cases}
\partial_t a - \mu \Delta a + \text{div} Q u = -\text{div}(au), \\
\partial_t Q u - \mu \Delta Q u + \nabla a = Q(2\mu \nabla(\ln(1 + a)) \nabla u - k(a) \nabla a - u \cdot \nabla u), \\
(a, Q u)|_{t=0} = (a_0, Q u_0).
\end{cases} \tag{1.5}
$$

A main observation is that the incompressible part (1.4) is the same as the classical incompressible Navier-Stokes equations with the forcing term $2\mu P(\nabla(\ln(1 + a)) \nabla u)$. Thus, motivated by [12, 28, 44, 45, 51, 54] concerning the global well-posedness of 3-D incompressible Navier-Stokes (anisotropic) system with some large initial velocity field, we expect to relax the smallness condition imposed on the vertical component of the initial incompressible velocity. We remark that the algebraical structure of the equation on the vertical component of the incompressible velocity is a linear equation with coefficients depending on the horizontal components and $a$ plays the crucial role in our argument. The main mathematical tool we shall use here will be a weighted Chemin-Lerner type norm introduced in [44], [45]. As the system (1.3) contains the incompressible part (1.4) and compressible part (1.5), we need more technical lemmas and a careful study of the nonlinear terms compared to [44, 45, 54].

In the second part of the present paper, we are concerned with the large-time behavior of solutions to the compressible Korteweg system. More precisely, under the assumption that the system (1.3) has a global solutions with small initial data in the critical Besov spaces, we obtain the optimal time-decay estimates about this solutions. The study of the large-time behavior of solutions to the partial different equations is also an old subject. Here, we only set the compressible Navier-Stokes equations as an example. Starting with the pioneering work by Matsumura and Nishida [41], in which the authors proved that if the initial data are a small perturbation in $H^3(\mathbb{R}^3) \times L^1(\mathbb{R}^3)$ of $(\bar{\rho}, 0)$ then

$$\|\nabla^k (\rho - \bar{\rho}, U)(t)\|_{L^2} \lesssim (1 + t)^{-\frac{d}{2} - \frac{k}{2}} \quad \text{for} \quad k = 0, 1.$$ 

Subsequently, Ponce [46] obtained more general $L^p$ decay rates

$$\|\nabla^k (\rho - \bar{\rho}, U)(t)\|_{L^p} \lesssim (1 + t)^{-\frac{d}{2}(1 - \frac{1}{p}) - \frac{k}{2}}, \quad 2 \leq p \leq \infty, \quad 0 \leq k \leq 2, \quad d = 2, 3.$$ 

One can also refer to [20, 37, 40, 43] for the decay rate of the compressible Navier-Stokes equations by small perturbation in different Sobolev spaces or Besov spaces. Recently, Xin and Xu [50] obtained the decay rate of the compressible Navier-Stokes equations without the smallness of low frequencies of initial data. Let us go back to the compressible Korteweg system, there are few works on the large-time
behavior of solutions. Charve et al. [11] proved that the global solutions with critical regularity that have been constructed in [17] by the Danchin and Desjardins are Gevrey analytic. As a byproduct, they also obtained algebraic time-decay estimates for any derivatives of the solution. Motivated by Xin and Xu [50], we are continue to study the decay rate of the compressible Korteweg system. Compared with [11], the decay rate obtained here is more accurate and has no smallness condition imposed on the low frequencies of the initial data. Before presenting the main result, we give the following notations:

**Notations**

- Let \( z^h = (z^1, z^2) \) be the horizontal components and \( z^v = z^3 \) be the vertical component.
- Let \( \mathcal{S}(\mathbb{R}^3) \) be the space of rapidly decreasing functions over \( \mathbb{R}^3 \) and \( \mathcal{S}'(\mathbb{R}^3) \) its dual space. For any \( z \in \mathcal{S}'(\mathbb{R}^3) \), the lower and higher frequency parts are expressed as

\[
  z^\ell \overset{\text{def}}{=} \sum_{j \leq j_0} \Delta_j z \quad \text{and} \quad z^h \overset{\text{def}}{=} \sum_{j > j_0} \Delta_j z
\]

for a large integer \( j_0 \geq 1 \). The corresponding truncated semi-norms are defined as follows:

\[
  \| z \|_{B^s_{p,1}} \overset{\text{def}}{=} \| z^\ell \|_{B^s_{p,1}} \quad \text{and} \quad \| z \|_{B^s_{p,1}}^h \overset{\text{def}}{=} \| z^h \|_{B^s_{p,1}}.
\]

- Let \( \Lambda^s z \overset{\text{def}}{=} \mathcal{F}^{-1} (|\xi|^s \mathcal{F} z) \), \( s \in \mathbb{R} \).

We are now in the position to state the main result of the present paper.

**Theorem 1.1.** Let \( 0 < \alpha \leq 1 \) and \( 2 \leq p \leq \min(4, \frac{6}{1+2\alpha}) \). Assume \( (a^\ell, Qu^\ell) \in B^{\frac{3}{2}}_{2,1} (\mathbb{R}^3) \), \( a^h \in B^{\frac{3}{2}}_{p,1} (\mathbb{R}^3) \), \( (Pu_0, Qu^h) \in B^{\frac{3}{2} - 1}_{p,1} (\mathbb{R}^3) \). If there exist two positive constants \( c_0 \) and \( C_0 \) such that

\[
  \| (a^\ell, Qu^\ell) \|^\frac{1}{2}_{B^{\frac{3}{2}}_{2,1}} + \| (Pu_0, Qu^h) \|^\frac{1}{2}_{B^{\frac{3}{2} - 1}_{p,1}} + \| a^h \|^\frac{1}{2}_{B^{\frac{3}{2}}_{p,1}} \\
  \leq c_0 \exp \left( - C_0 \| (Pu_0)^v \|_{\frac{2}{p}} - 1 \right),
\]

then the system (1.3) admits a unique global solution \((a, u)\) satisfying,

\[
  (a^\ell, Qu^\ell) \in C(\mathbb{R}^+; B^{\frac{3}{2}}_{2,1}) \cap L^1(\mathbb{R}^+; B^{\frac{5}{2}}_{2,1}),
\]

\[
  a^h \in C(\mathbb{R}^+; B^{\frac{3}{2}}_{p,1}) \cap L^1(\mathbb{R}^+; B^{\frac{3}{2} + 1}_{p,1}), \quad (Pu, Qu^h) \in C(\mathbb{R}^+; B^{\frac{3}{2} + 1}_{p,1}) \cap L^1(\mathbb{R}^+; B^{\frac{5}{2} + 1}_{p,1}).
\]

Moreover, if we replace (1.7) by

\[
  \| (a^\ell, u^\ell) \|^\frac{1}{2}_{B^{\frac{3}{2}}_{2,1}} + \| u^\ell \|^\frac{1}{2 - \epsilon}_{B^{\frac{3}{2} - \epsilon}_{p,1}} + \| a^h \|^\frac{1}{2}_{B^{\frac{3}{2}}_{p,1}} \leq c_0,
\]

and assume further \((a_0, u_0) \in B^{\sigma}_{2,1}(\mathbb{R}^3)\) with \(-\frac{3}{2} < \sigma < \frac{1}{2}\). For any

\[
  p \leq q \leq \infty, \quad \frac{3}{q} - \frac{3}{p} + \sigma < \beta \leq \frac{3}{q} - 1,
\]

there holds

\[
  \| \Lambda^\beta (a, u) \|_{L^q} \leq C(1 + t)^{-\frac{3}{4} - \frac{(\beta - \sigma)q - 3}{2q}}.
\]
Remark 1.2. Compared with [32], on the one hand, we relax the smallness condition imposed on the initial data. More precisely, the global solutions constructed here allow the vertical component of the incompressible velocity could be arbitrarily large initially. Moreover, condition (1.7) allows us to consider the case $p > \frac{3}{2}$, so that the regularity exponent $3/p - 1$ for the velocity becomes negative, our result thus applies to large highly oscillating initial velocities (see [10, 14] for more explanation). On the other hand, we obtain the optimal time decay rates of the solutions.

Remark 1.3. Due to $-\frac{3}{p} \leq \frac{3}{2} - \frac{6}{p}$, for any $p \geq 2$, the decay estimate pointed out in Theorem 1.1 is much better than that of the usual compressible Navier-Stokes equations obtained in [50]. This reflects the parabolicity of the compressible Navier-Stokes-Korteweg system. In fact, if we relax the condition $(a_0, u_0) \in \dot{B}^{s}_{2,1}(\mathbb{R}^3)$ to $(a'_0, u'_0) \in \dot{B}^{s}_{2,\infty}(\mathbb{R}^3)$, then for any $\frac{3}{2} - \frac{6}{p} \leq \sigma < \frac{1}{2}$, following similarly to the argument in [50], we can also get the same decay rate (1.8).

The paper is organized as follows. In Section 2, we recall the Littlewood-Paley theory and give some useful lemmas about product laws, commutators estimates in Besov spaces. In Section 3, we derive the estimates for the compressible part of (1.3) in the low frequencies and high frequencies respectively. The estimates for incompressible parts of (1.3) which contain the horizontal components and vertical components will be presented in Section 4. We set the continuous argument in Section 5 to complete the proof of the global solutions. In the last section, we get the optimal decay of the solutions by a pure energy argument.

2. Preliminaries. The Littlewood-Paley decomposition plays a central role in our analysis.

Definition 2.1. Let us consider a smooth function $\varphi$ on $\mathbb{R}$, the support is included in $[3/4, 8/3]$ such that

$$\forall \tau > 0, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \tau) = 1 \quad \text{and} \quad \chi(\tau) \overset{\text{def}}{=} 1 - \sum_{j \geq 0} \varphi(2^{-j} \tau) \in \mathcal{D}([0, 4/3]).$$

Let us define

$$\hat{\Delta}_j u = \mathcal{F}^{-1}(\varphi(2^{-j} |\xi|) \hat{u}), \quad \text{and} \quad \hat{S}_j u = \mathcal{F}^{-1}(\chi(2^{-j} |\xi|) \hat{u}).$$

In order to ensure that

$$u = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \in \mathcal{S}'(\mathbb{R}^3),$$

we restrict our attention to those tempered distributions $u$ such that

$$\lim_{k \to -\infty} \|\hat{S}_k u\|_{L^\infty} = 0.$$

One can now define what an homogeneous Besov spaces $\dot{B}^{s}_{p,1}(\mathbb{R}^3)$ is.

Definition 2.2. Let $p$ be in $[1, +\infty]$ and $s$ in $\mathbb{R}$. We define the Besov norm by

$$\|u\|_{\dot{B}^{s}_{p,1}} \overset{\text{def}}{=} \left\| \left(2^{js}\|\hat{\Delta}_j u\|_{L^p} \right)_{j \in \mathbb{Z}} \right\|_{\ell^1(\mathbb{Z})}.$$
Lemma 2.3. (a) For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, for any $u \in \dot{B}^s_{p,1}(\mathbb{R}^3)$, then there holds the embedding

$$
\|u\|_{\dot{B}^s_{p,1}} \lesssim \|\nabla u\|_{\dot{B}^{s-1}_{p,1}} \lesssim \|u\|_{\dot{B}^s_{p,1}}.
$$

(2.1)

Here and in what follows, $f \lesssim g$ represents the inequality $f \leq Cg$ for a generic constant $C$.

(b) Let $1 \leq p \leq \infty$ and $s_1, s_2 \in \mathbb{R}$ with $s_1 > s_2$, for any $u \in \dot{B}^{s_1}_{p,1} \cap \dot{B}^{s_2}_{p,1}(\mathbb{R}^3)$, there holds

$$
\|u^f\|_{\dot{B}^{s_1}_{p,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{p,1}}, \quad \|u^h\|_{\dot{B}^{s_2}_{p,1}} \lesssim \|u\|_{\dot{B}^{s_2}_{p,1}}.
$$

(2.2)

(c) For $s \in \mathbb{R}$ and $1 \leq p < \infty$, then we have the embedding

$$
\dot{B}^s_{p,1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s-\frac{3}{2}-\frac{3}{q}}_{q,1}(\mathbb{R}^3).
$$

(2.3)

In this paper, we frequently use the so-called “time-space” Besov spaces or Chemin-Lerner space first introduced by Chemin and Lerner [13].

Definition 2.4. Let $s \in \mathbb{R}$ and $0 < T \leq +\infty$. We define

$$
\|u\|_{L^q_T(\dot{B}^s_{p,1})} \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} 2^{js} \left( \int_0^T \|\Delta_j u(t)\|_{L^p_T}^q \, dt \right)^{\frac{1}{q}}
$$

for $q, p \in [1, \infty)$ and with the standard modification for $p, q = \infty$.

By Minkowski’s inequality, we have the following relations between the Bochner space $L^q_T(\dot{B}^s_{p,1})$ and the Chemin-Lerner space $L^q_T(\dot{B}^s_{p,1})$:

$$
\|u\|_{L^q_T(\dot{B}^s_{p,1})} = \|u\|_{L^q_T(\dot{B}^s_{p,1})} \quad \text{and} \quad \|u\|_{L^q_T(\dot{B}^s_{p,1})} \leq \|u\|_{L^q_T(\dot{B}^s_{p,1})}.
$$

The following Bernstein’s lemma will be repeatedly used throughout this paper.

Lemma 2.5. Let $\mathfrak{B}$ be a ball and $\mathcal{C}$ a ring of $\mathbb{R}^3$. A constant $C$ exists so that for any positive real number $\lambda$, any non-negative integer $k$, any smooth homogeneous function $\sigma$ of degree $m$, and any couple of real numbers $(a, b)$ with $1 \leq a \leq b$, there hold

$$
\text{Supp} \, \hat{u} \subset \lambda \mathfrak{B} \Rightarrow \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^a} \leq C^{k+1} \lambda^{k+3(1/a-1/b)} \|u\|_{L^b},
$$

$$
\text{Supp} \, \hat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \lambda^k \|u\|_{L^b},
$$

$$
\text{Supp} \, \hat{u} \subset \lambda \mathcal{C} \Rightarrow \|\sigma(D) u\|_{L^b} \leq C_{\sigma, m} \lambda^{m+3(1/a-1/b)} \|u\|_{L^b}.
$$

Let us now recall a few nonlinear estimates in Besov spaces. Formally, any product of two distributions $u$ and $v$ may be decomposed into

$$
w = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v),
$$

(2.4)

with

$$
\dot{T}_u v \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{S}_j u \dot{\Delta}_j v, \quad \dot{R}(u, v) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \dot{\Delta}_j v, \quad \dot{\Delta}_j v \overset{\text{def}}{=} \sum_{|j-j'| \leq 1} \dot{\Delta}_{j'} v.
$$

The above operator $T$ is called the “paraproduct” whereas $R$ is called the “remainder”. Moreover, the paraproduct $\dot{T}$ and the remainder $\dot{R}$ operators satisfy the following continuous properties.
Lemma 2.6. ([7, Theorem 2.47, Theorem 2.52]) For all \( s \in \mathbb{R}, \sigma \geq 0, \) and \( 1 \leq p, p_1, p_2 \leq \infty, \) the paraproduct \( \mathcal{T} \) is a bilinear, continuous operator from \( \dot{B}^{-\sigma}_{p_1,1} \times \dot{B}^s_{p_2,1} \) to \( \dot{B}^{s-\sigma}_{p,1} \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \) The remainder \( \dot{R} \) is bilinear continuous from \( \dot{B}^{s_1}_{p_1,1} \times \dot{B}^{s_2}_{p_2,1} \) to \( \dot{B}^{s_1+s_2}_{p,1} \) with \( s_1 + s_2 > 0, \) and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \)

The following lemmas will be needed to estimate the nonlinear terms of (1.3). It is just a consequence of Bony decomposition and of continuity results for the paraproduct and remainder operators.

Lemma 2.7. ([7, Corollary 2.54]) For any \( p \in [1, \infty], \) let \( (u, v) \in L^\infty \cap \dot{B}^{s_1}_{p,1}(\mathbb{R}^3) \) for some \( s > 0. \) Then there exists a constant \( C \) depending only on \( p \) and \( s \) such that

\[
\|uv\|_{\dot{B}^{s_1}_{p,1}} \leq C(\|u\|_{u^\infty} \|v\|_{\dot{B}^{s_1}_{p,1}} + \|v\|_{V^\infty} \|u\|_{\dot{B}^{s_1}_{p,1}}).
\]

Lemma 2.8. ([20, Proposition A.1]) Let \( 1 \leq p, q \leq \infty, \) \( s_1 \leq \frac{3}{q}, \) \( s_2 \leq 3 \min\left\{ \frac{1}{p}, \frac{1}{q} \right\} \) and \( s_1 + s_2 > 3 \max\{0, \frac{1}{p} + \frac{1}{q} - 1\}. \) For \( \forall (a, b) \in \dot{B}^{s_1}_{q,1}(\mathbb{R}^3) \times \dot{B}^{s_2}_{q,1}(\mathbb{R}^3), \) we have

\[
\|ab\|_{\dot{B}^{s_1+s_2-\frac{3}{q}}_{p,1}} \leq C\|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{q,1}}.
\]

Lemma 2.9. Let \( 2 \leq p \leq 4. \) For any \( (u, v) \in \dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3) \cap \dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3), \) we have

\[
\|uv\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \leq C(\|u\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|v\|_{\dot{B}^{\frac{3}{p}}_{p,1}} + \|u\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|v\|_{\dot{B}^{\frac{3}{p}}_{p,1}}).
\]

Proof. We first deduce from Bony’s decompose that

\[ uv = \dot{T}_uv + \dot{T}_vu + \dot{R}(u, v). \]

By the Hölder inequality, the first term \( \dot{T}_uv \) can be estimated as follows:

\[
\left\| \dot{T}_uv \right\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \lesssim \sum_{j \in \mathbb{Z}} \sum_{|j-k| \leq 4} 2^{\frac{j}{2}} \left\| \Delta_j \left( \tilde{S}_{k+1} u \Delta_k v \right) \right\|_{L^2}
\]

\[
\lesssim \sum_{j \in \mathbb{Z}} \sum_{|j-k| \leq 4} 2^{\frac{j}{2}} \sum_{k' \leq k-2} \left\| \Delta_{k'} u \right\|_{L^{\frac{2p}{p-2}}} \left\| \Delta_k v \right\|_{L^p}
\]

\[
\lesssim \sum_{j \in \mathbb{Z}} \sum_{|j-k| \leq 4} 2^{\frac{j}{2}} \left( \sum_{k' \leq k-2} 2^{\frac{3}{p} - \frac{1}{2}(\frac{3}{p} - 1)k'} \left\| \Delta_{k'} u \right\|_{L^p} \right) \left\| \Delta_k v \right\|_{L^p}
\]

\[
\lesssim \left\| u \right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \left( \sum_{j \in \mathbb{Z}} \sum_{|j-k| \leq 4} 2^{\frac{j}{2} - \frac{3}{2k'}} \left\| \Delta_k v \right\|_{L^p} \right)
\]

\[
\lesssim \left\| u \right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \left\| v \right\|_{\dot{B}^{\frac{3}{p}}_{p,1}},
\]

where we have used \( p \leq 4 \) in the third step. The second term \( \left\| \dot{T}_vu \right\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \) can be obtained similarly. We now turn to the last term:

\[
\left\| \dot{R}(u, v) \right\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \lesssim \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{\frac{j}{2}} \left\| \Delta_j \left( \dot{S}_{k+1} u \dot{\Delta}_k v \right) \right\|_{L^2}
\]

\[
\lesssim \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{\frac{j}{2}} 2^{\frac{3}{2}(\frac{3}{p} - 1)k} \left\| \Delta_k u \right\|_{L^p} \sum_{|k' - k| \leq 1} 2^{-\frac{3}{2}k'} 2^{\frac{3}{2}k} \left\| \Delta_{k'} v \right\|_{L^p}
\]

\[
\lesssim \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{\frac{j}{2}} 2^{\frac{3}{2}(\frac{3}{p} - 1)k} \left\| \Delta_k u \right\|_{L^p} \left\| v \right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}.
\]
respectively. The main tool used here is the weighted-Chemin-Lerner norm. Let \( \nabla u \in \dot{B}^2_{p,1} \) and the extension from a local to a global solution. Hence, next, we only need to estimate to the solutions.

Proposition 2.11. \( \{7, \text{Lemma 2.100}\} \) Let \( 1 \leq p, q \leq \infty, -3 \min\{\frac{1}{p}, \frac{1}{q}\} < s \leq 1 + 3 \min\{\frac{1}{p}, \frac{1}{q}\} \). For any \( \nabla u \in \dot{B}^2_{p,1}(\mathbb{R}^3) \) and \( v \in \dot{B}^s_{q,1}(\mathbb{R}^3) \), there holds

\[
\| (\dot{\Delta}_j, u \cdot \nabla)v \|_{L^q} \lesssim d_j 2^{-js} \| \nabla u \|_{\dot{B}^\frac{s}{2}_{p,1}} \| v \|_{\dot{B}^s_{q,1}}, \quad \text{with} \quad \sum_{j \in \mathbb{Z}} d_j = 1.
\]

Finally, we recall a composition result.

Proposition 2.11. \( \{7\} \) Let \( G \) with \( G(0) = 0 \) be a smooth function defined on an open interval \( I \) of \( \mathbb{R} \) containing 0. Then the following estimates

\[
\| G(a) \|_{B^s_{p,1}} \lesssim \| a \|_{B^s_{p,1}} \quad \text{and} \quad \| G(a) \|_{\dot{B}^s_{p,1}(B^s_{p,1})} \lesssim \| a \|_{\dot{B}^s_{p,1}(B^s_{p,1})}
\]

hold true for \( s > 0, 1 \leq p, q \leq \infty \) and \( a \) valued in a bounded interval \( J \subset I \).

3. The estimates for the compressible parts. Given \( a_0, u_0 \) satisfy the conditions listed in Theorem 1.1, it follows by a similar argument as \( [31], [32] \) that there exists a positive time \( T \) so that the equations (1.3) have a unique solution \((a, u)\) with

\[
(a^t, Qu^t) \in C([0, T]; \dot{B}^\frac{1}{2}_{2,1}) \cap L^1((0, T); \dot{B}^\frac{5}{2}_{2,1}),
\]

\[
a^h \in C([0, T]; \dot{B}^\frac{3}{2}_{p,1}) \cap L^1((0, T); \dot{B}^\frac{5}{2}+2_{p,1}),
\]

\[
(\mathbb{P}u, Qu^h) \in C([0, T]; \dot{B}^{\frac{3}{2}-1}_{p,1}) \cap L^1((0, T); \dot{B}^{\frac{5}{2}+1}_{p,1}). \quad (3.1)
\]

The proof of global solution in Theorem 1.1 relies heavily on the a priori estimates and the extension from a local to a global solution. Hence, next, we only need to give the \textit{a priori} estimates to the solutions.

In the rest of this section, we are concerned with the estimates of compressible parts of (1.3) which will be split into low frequencies and high frequencies respectively. The main tool used here is the weighted-Chemin-Lerner norm. Let \( f(t) \in L^1_{\text{loc}}(\mathbb{R}^+), f(t) \geq 0 \). For some \( \lambda \geq 0 \) and \( s \in \mathbb{R} \), we define

\[
w_\lambda \equiv w \exp\left\{ -\lambda \int_0^t f(\tau) \, d\tau \right\}, \quad \| w_\lambda \|_{L^1_{\text{loc}}(B^s_{p,1})} \equiv \int_0^t f(\tau) \| w_\lambda(\tau) \|_{B^s_{p,1}} \, d\tau, \quad (3.2)
\]

and similar notations for \( a_\lambda, u_\lambda \).

In the whole paper, we assume the coefficient \( \mu \) appeared in (1.3) equal to one and choose the weight function

\[
f(t) \equiv \| (\mathbb{P}u)^{Y}(t) \|_{\dot{B}^\frac{s}{p}+1_{p,1}} + \| (\mathbb{P}u)^{Y}(t) \|_{\dot{B}^\frac{s}{p}+2_{p,1}} + \| (\mathbb{P}u)^{Y}(t) \|_{\dot{B}^\frac{s}{p}+1+a_{n}}, \quad (3.3)
\]

with \( a \) and \( p \) satisfy the restriction relations in Theorem 1.1.

Throughout we make the assumption that

\[
\sup_{t \in \mathbb{R}, x \in \mathbb{R}^3} |a(t, x)| \leq \frac{1}{2}, \quad (3.4)
\]
which will enable us to use freely the composition estimate stated in Proposition 2.11. Note that as $B^\frac{3}{2}_{p,1}(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$, condition (3.4) will be ensured by the fact that the constructed solution about $a$ has small norm.

3.1. The low frequencies estimates for the compressible parts. To cancel out the linear terms in the compressible equations, we shall set the estimates of the low frequencies in the $L^2$ critical Besov framework.

We begin to apply $\hat{\Delta}_j$ to the first equation of (1.5) and then multiply the resultant equation by $\exp \left\{ -\lambda \int_0^t f(\tau) \, d\tau \right\}$ that

$$\partial_t \hat{\Delta}_j a_\lambda + \lambda f(t) \hat{\Delta}_j a_\lambda - \hat{\Delta}_j a_\lambda + \hat{\Delta}_j \text{div} Q u_\lambda = -\hat{\Delta}_j (u \cdot \nabla a)_\lambda - \hat{\Delta}_j (\text{div} u)_\lambda. \quad (3.5)$$

Taking the $L^2$ inner product with $\hat{\Delta}_j a_\lambda$ and using the Bernstein’s inequality give

$$\frac{1}{2} \frac{d}{dt} \| \hat{\Delta}_j a_\lambda \|^2_{L^2} + \lambda f(t) \| \hat{\Delta}_j a_\lambda \|^2_{L^2} + 2^{2j} \| \hat{\Delta}_j a_\lambda \|^2_{L^2} + \int_{\mathbb{R}^3} \hat{\Delta}_j \text{div} Q u_\lambda \cdot \hat{\Delta}_j a_\lambda \, dx \lesssim (\| \hat{\Delta}_j (u \cdot \nabla a)_\lambda \|_{L^2} + \| \hat{\Delta}_j (\text{div} Q u)_\lambda \|_{L^2}) \| \hat{\Delta}_j a_\lambda \|_{L^2}. \quad (3.6)$$

By a similar argument as above, we deduce from the second equation of (1.5) that

$$\frac{1}{2} \frac{d}{dt} \| \hat{\Delta}_j Q u_\lambda \|^2_{L^2} + \lambda f(t) \| \hat{\Delta}_j Q u_\lambda \|^2_{L^2} + 2^{2j} \| \hat{\Delta}_j Q u_\lambda \|^2_{L^2} + \int_{\mathbb{R}^3} \hat{\Delta}_j u \cdot \hat{\Delta}_j Q u_\lambda \, dx \lesssim (\| \hat{\Delta}_j Q (\nabla (\ln(1 + a))) \|_{L^2} + \| \hat{\Delta}_j Q (u \cdot \nabla a)_\lambda \|_{L^2}) \| \hat{\Delta}_j Q u_\lambda \|_{L^2}. \quad (3.7)$$

Adding up (3.6), (3.7), using the Hölder inequality and integrating about time from 0 to $t$ yield

$$\| (\hat{\Delta}_j a_\lambda, \hat{\Delta}_j Q u_\lambda) \|_{L^2} + \int_0^t \lambda f(\tau) \| (\hat{\Delta}_j a_\lambda, \hat{\Delta}_j Q u_\lambda) \|_{L^2} \, d\tau$$

$$+ 2^{2j} \int_0^t \| (\hat{\Delta}_j a_\lambda, \hat{\Delta}_j Q u_\lambda) \|_{L^2} \, d\tau$$

$$\lesssim \| (\hat{\Delta}_j a_0, \hat{\Delta}_j Q u_0) \|_{L^2}$$

$$+ \int_0^t \| (\hat{\Delta}_j (\text{div} Q u)_\lambda) \|_{L^2} + \| \hat{\Delta}_j (u \cdot \nabla a)_\lambda \|_{L^2} + \| \hat{\Delta}_j Q (\nabla (\ln(1 + a))) \|_{L^2} \, d\tau$$

$$+ \int_0^t \| (\hat{\Delta}_j (u \cdot \nabla a)_\lambda) \|_{L^2} \, d\tau. \quad (3.8)$$

Multiplying the above inequality by $2^{2j}$ and summing up about $j \leq j_0$ give

$$\| (a^\ell_\lambda, Q u^\ell_\lambda) \|_{L^\infty_{\tau} (B^\frac{3}{2}_{2,1})} + \lambda \| (a^\ell_\lambda, Q u^\ell_\lambda) \|_{L^1_{\tau} (B^\frac{3}{2}_{2,1})} + \| (a^\ell_\lambda, Q u^\ell_\lambda) \|_{L^1_{\tau} (B^\frac{3}{2}_{2,1})}$$

$$\lesssim \| (a_0, Q u_0) \|_{B^\frac{3}{2}_{2,1}}$$

$$+ \int_0^t \| (\text{div} Q u) \|_{B^\frac{3}{2}_{2,1}} \, d\tau$$

$$+ \int_0^t \| (\nabla (\ln(1 + a))) \|_{B^\frac{3}{2}_{2,1}} \, d\tau$$

$$+ \int_0^t \| (u \cdot \nabla a) \|_{B^\frac{3}{2}_{2,1}} \, d\tau.$$

We are now turn to estimate each nonlinear term in the right hand side of (3.9).
In the further argument, we shall use repeatedly the following interpolation inequalities:
\[
\|w\|_{B^2_{\infty,1}} \lesssim \|w\|_{B^2_{p,1}}^{1/2} \|w\|_{B^{3/2+\varepsilon}_{p,1}}^{1/2}, \quad \|w\|_{B^{3/2+\varepsilon}_{p,1}} \lesssim \|w\|_{B^2_{p,1}}^{1/3} \|w\|_{B^{3/2+2\varepsilon}_{p,1}}^{1/3}.
\] (3.10)

By Lemma 2.9 and the above interpolation inequality (3.10), we have
\[
\|(\text{adiv } Q u)^\ell\|_{B^2_{\frac{3}{2}}}
\lesssim \|a\|_{B_{p,1}^3} \|Q u\|_{B_{p,1}^{3/2+\varepsilon}} + \|a\|_{B_{p,1}^3} \|Q u\|_{B_{p,1}^{3/2+\varepsilon}},
\]
\[
\lesssim \|a\|_{B_{p,1}^1} \|Q u\|_{B_{p,1}^{2}} + \|a\|_{B_{p,1}^3} \|Q u\|_{B_{p,1}^{3/2+\varepsilon}},
\]
\[
\lesssim \|a\|_{B_{p,1}^1} \|Q u\|_{B_{p,1}^{1/2}} + \|a\|_{B_{p,1}^1} \|Q u\|_{B_{p,1}^{2}} + \|Q u\|_{B_{p,1}^{3/2+\varepsilon}},
\]
\[
\lesssim (|a|^1_{B_{p,1}^1} + |a|_{B_{p,1}^3}^{3/2}) (\|Q u\|_{B_{p,1}^{1/2}} + \|Q u\|_{B_{p,1}^{2}} + \|Q u\|_{B_{p,1}^{3/2+\varepsilon}}).
\] (3.11)

From which
\[
\int_0^T \|(\text{adiv } Q u)^\ell\|_{B^2_{\frac{3}{2}}} \, d\tau \lesssim \int_0^T (|a|_{B_{p,1}^1}^{1/2} + |a|_{B_{p,1}^3}^{3/2}) (\|Q u\|_{B_{p,1}^{1/2}} + \|Q u\|_{B_{p,1}^{2}} + \|Q u\|_{B_{p,1}^{3/2+\varepsilon}}) \, d\tau
\]
\[+ \int_0^T (|a|_{B_{p,1}^1} + |a|_{B_{p,1}^3}) (\|Q u\|_{B_{p,1}^{1/2}} + \|Q u\|_{B_{p,1}^{2}} + \|Q u\|_{B_{p,1}^{3/2+\varepsilon}}) \, d\tau.
\] (3.12)

In order to bound \((Q(k(a)\nabla a))^\ell\), we first claim that, for any \(p\) given in Theorem 1.1, there holds
\[
\|(bc)^\ell\|_{B^2_{\frac{3}{2}}} \lesssim (\|b\|_{B_{p,1}^3}^{\frac{1}{2}} + \|b\|_{B_{p,1}^{2}}^{\frac{3}{2}}) \|c\|_{B_{p,1}^{1/2}}^{\frac{1}{2}}.
\] (3.13)

Indeed, one can first use Bony’s decomposition (2.4) to write
\[
(bc)^\ell = \hat{S}_{j_0+1} \left( \hat{T}_b c + \hat{R}(b, c) \right) + \hat{T}_c \hat{S}_{j_0+1} b + [\hat{S}_{j_0+1}, T_c] b.
\] (3.14)

By Lemma 2.6, we have
\[
\|\hat{T}_c \hat{S}_{j_0+1} b\|_{B^2_{\frac{3}{2}}} \lesssim \|c\|_{B^2_{p,1}} \|b\|_{B^2_{p,1}}^{\frac{1}{2}} \|c\|_{B^2_{p,1}}^{\frac{1}{2}},
\] (3.15)

and, for \(\frac{1}{p} + \frac{1}{p'} = 1\) that
\[
\|\hat{S}_{j_0+1} (T_b c + \hat{R}(b, c))\|_{B^2_{\frac{3}{2}}} \lesssim \|b\|_{B^2_{p,1}} \|c\|_{B^2_{p,1}} \|b\|_{B^2_{p,1}} \|c\|_{B^2_{p,1}}^{\frac{1}{2}}.
\] (3.16)

By Lemma 6.1 in [18], we obtain
\[
\|\|\hat{S}_{j_0+1}, \hat{T}_c \|b\|_{B^2_{\frac{3}{2}}} \lesssim \|\nabla c\|_{B^2_{p,1}} \|b\|_{B^2_{p,1}} \|b\|_{B^2_{p,1}} \|c\|_{B^2_{p,1}}^{\frac{1}{2}}.
\] (3.17)

Thus, the combination of (3.14)–(3.17) shows the validity of (3.13).
Using claim (3.13), Lemma 2.11 and the interpolation inequality (3.10),
\[
\| (Q(k(a) \nabla a)) \|_{B^{\frac{3}{2}}_{p,1}} \lesssim (\| \nabla a \|_{B^{\frac{3}{2}}_{p,1}} + \| \nabla \varepsilon \|_{B^{\frac{1}{2}}_{p,1}}) \| k(a) \|_{B^{\frac{3}{2}}_{p,1}}
\lesssim (\| a \|_{B^{\frac{3}{2}}_{p,1}} + \| a^\varepsilon \|_{B^{\frac{1}{2}}_{p,1}}) \| a \|_{B^{\frac{3}{2}}_{p,1}}
\lesssim \| a \|^2_{B^{\frac{3}{2}}_{p,1}} + \| a^\varepsilon \|^2_{B^{\frac{1}{2}}_{p,1}} + \| a^\varepsilon \|_{B^{\frac{1}{2}}_{p,1}} \| a_h \|_{B^{\frac{3}{2}}_{p,1}}
\lesssim \| a^\varepsilon \|_{B^{\frac{3}{2}}_{p,1}} \| a^\varepsilon \|_{B^{\frac{1}{2}}_{p,1}} + (\| a^\varepsilon \|_{B^{\frac{1}{2}}_{p,1}} + \| a_h \|_{B^{\frac{3}{2}}_{p,1}}) \| a^\varepsilon \|_{B^{\frac{3}{2}}_{p,1}} + 2, \quad (3.18)
\]
which implies
\[
\int_0^t \| (k(a) \nabla a) \|_{B^{\frac{3}{2}}_{p,1}} \, d\tau
\lesssim \int_0^t (\| a^\varepsilon \|_{B^{\frac{3}{2}}_{p,1}} \| a^\varepsilon \|_{B^{\frac{1}{2}}_{p,1}} + (\| a^\varepsilon \|_{B^{\frac{1}{2}}_{p,1}} + \| a_h \|_{B^{\frac{3}{2}}_{p,1}}) \| a^\varepsilon \|_{B^{\frac{1}{2}}_{p,1}} + 2) \, d\tau. \quad (3.19)
\]
From Lemma 2.9, one has
\[
\int_0^t \| (\nabla (\ln(1 + a)) \nabla u) \|_{B^{\frac{1}{2}}_{p,1}} \, d\tau
\lesssim \int_0^t \| a_\lambda \|_{B^{\frac{3}{2}}_{p,1}} \| u \|_{B^{\frac{5}{2}}_{p,1}} \, d\tau + \int_0^t \| a_\lambda \|_{B^{\frac{3}{2}}_{p,1}} \| a_\lambda \|_{B^{\frac{5}{2}}_{p,1}} \, d\tau
\]
\[
def J_1 + J_2. \quad (3.20)
\]
A direct computation from the embedding relation and the definition of weighted Chemin-Lerner norm gives
\[
J_1 \lesssim \int_0^t \| a_\lambda \|_{B^{\frac{3}{2}}_{p,1}} \| (( Pu )^\alpha, ( Pu )^\beta, Qu) \|_{B^{\frac{3}{2}}_{p,1}} \, d\tau
\lesssim \int_0^t (\| a_\lambda \|_{B^{\frac{3}{2}}_{p,1}} + \| a_h \|_{B^{\frac{3}{2}}_{p,1}}) \| (( Pu )^\alpha, ( Pu )^\beta, Qu) \|_{B^{\frac{3}{2}}_{p,1}} \, d\tau
\lesssim \| a_\lambda \|_{L^1_t (B^{\frac{3}{2}}_{p,1})} + \| a_h \|_{L^1_t (B^{\frac{3}{2}}_{p,1})}
\]
\[
+ \int_0^t (\| a_\lambda \|_{B^{\frac{3}{2}}_{p,1}} + \| a_h \|_{B^{\frac{3}{2}}_{p,1}}) (\| Qu \|_{B^{\frac{3}{2}}_{p,1}} + \| (( Pu )^\alpha, Qu) \|_{B^{\frac{3}{2}}_{p,1}} + \| (( Pu )^\beta, Qu) \|_{B^{\frac{3}{2}}_{p,1}}) \, d\tau. \quad (3.21)
\]
Due to appear representative term \| (( Pu )^\alpha) \|_{B^{\frac{3}{2}}_{p,1}}^2 \in J_2. \ We present their estimates in detail for the first time. Taking advantage of the interpolation inequality (3.10), we have
\[
J_2 \lesssim \int_0^t \| (( Pu )^\alpha, ( Pu )^\beta, Qu) \|_{B^{\frac{3}{2}}_{p,1}} \| a_\lambda \|_{B^{\frac{3}{2}}_{p,1}} \, d\tau
\lesssim \int_0^t (\| ( Pu )^\alpha \|_{B^{\frac{3}{2}}_{p,1}} \| a_\lambda \|_{B^{\frac{3}{2}}_{p,1}}) \| a_\lambda \|_{B^{\frac{3}{2}}_{p,1}} \, d\tau
\]
\[
+ \int_0^t (\| ( Pu )^\alpha, Qu ) \|_{B^{\frac{3}{2}}_{p,1}} \| (( Pu )^\alpha, Qu ) \|_{B^{\frac{3}{2}}_{p,1}} \| a_\lambda \|_{B^{\frac{3}{2}}_{p,1}} \| a_\lambda \|_{B^{\frac{3}{2}}_{p,1}} \, d\tau
\]
\[
def J_2^{(1)} + J_2^{(2)}. \quad (3.22)
\]
According to Young’s inequality and the definition of (3.2), the first term $J_2^{(1)}$ can be estimated as

$$J_2^{(1)} \lesssim \varepsilon \int_0^t \left( \|a_\lambda^a\|_{B^{\frac{3}{2}}_{2,1}} + \|a_\lambda^h\|_{B^{2+}_{p,1}} \right) \, d\tau + \int_0^t \| (Pu)^{a'} \|_{B^{\frac{3}{2}}_{p,1}} \varepsilon \lambda \|a_\lambda\|_{B^{\frac{3}{2}}_{p,1}} \, d\tau$$

$$\lesssim \varepsilon \int_0^t \left( \|a_\lambda^a\|_{B^{\frac{3}{2}}_{2,1}} + \|a_\lambda^h\|_{B^{2+}_{p,1}} \right) \, d\tau + \int_0^t \| (Pu)^{a'} \|_{B^{\frac{3}{2}}_{p,1}} \varepsilon \lambda \|a_\lambda\|_{B^{\frac{3}{2}}_{p,1}} + \|a_\lambda^h\|_{B^{\frac{3}{2}}_{p,1}} \, d\tau$$

$$\lesssim \varepsilon \int_0^t \left( \|a_\lambda^a\|_{B^{\frac{3}{2}}_{2,1}} + \|a_\lambda^h\|_{B^{2+}_{p,1}} \right) \, d\tau + \|a_\lambda^h\|_{L_{1,2}(B^{\frac{3}{2}}_{2,1})} + \|a_\lambda^h\|_{L_{1,2}(B^{\frac{3}{2}}_{2,1})}. \quad (3.23)$$

The second term $J_2^{(2)}$ can be dealt directly by using the Young inequality:

$$J_2^{(2)} \lesssim \int_0^t \left( \|Qa_\lambda\|_{B^{\frac{1}{2}}_{2,1}} + \|((Pu)^{a'}, Qa_\lambda)\|_{B^{\frac{3}{2}}_{2,1}} \right) \, d\tau$$

$$+ \int_0^t \left( \|a_\lambda\|_{B^{\frac{3}{2}}_{2,1}} + \|a_\lambda\|_{B^{2+}_{p,1}} \right) \left( \|Qa_\lambda^a\|_{B^{\frac{3}{2}}_{2,1}} + \|((Pu)^{a'}, Qa_\lambda^h)\|_{B^{\frac{3}{2}}_{2,1}} \right) \, d\tau. \quad (3.24)$$

Substituting (3.23), (3.24) into (3.22) and summing up (3.21), we deduce from (3.20) that

$$\int_0^t \left( \int_0^t (\nabla (1 + a) \cdot \nabla u)^2 \, dx \right) \, d\tau \lesssim \varepsilon \int_0^t \left( \int_0^t (\nabla (1 + a) \cdot \nabla u)^2 \, dx \right) \, d\tau$$

$$\lesssim \varepsilon \int_0^t \left( \|a_\lambda\|_{B^{\frac{3}{2}}_{2,1}} + \|a_\lambda^h\|_{B^{2+}_{p,1}} \right) \, d\tau + \|a_\lambda^h\|_{L_{1,2}(B^{\frac{3}{2}}_{2,1})} + \|a_\lambda^h\|_{L_{1,2}(B^{\frac{3}{2}}_{2,1})}. \quad (3.25)$$

Next, we estimate the term $(u \cdot \nabla a)_\lambda$. It follows from (3.13) and interpolation inequality and Young’s inequality that

$$\int_0^t \left( \int_0^t (u \cdot \nabla a)_\lambda \, dx \right) \, d\tau \lesssim \int_0^t \left( \int_0^t (u \cdot \nabla a)_\lambda \, dx \right) \, d\tau$$

$$\lesssim \int_0^t \left( \int_0^t (u \cdot \nabla a)_\lambda \, dx \right) \, d\tau \lesssim \int_0^t \left( \int_0^t (u \cdot \nabla a)_\lambda \, dx \right) \, d\tau$$

$$\lesssim \int_0^t \left( \int_0^t (u \cdot \nabla a)_\lambda \, dx \right) \, d\tau \lesssim \int_0^t \left( \int_0^t (u \cdot \nabla a)_\lambda \, dx \right) \, d\tau.$$
Finally, we have to deal with the last term \(Q(u \cdot \nabla u)\) in (3.9). A direct computation from \(u = Fu + Qu\) and \(\partial_x(Fu)^\nu = -\text{div}_h(Fu)^h\) gives

\[
\begin{align*}
u \cdot \nabla u &= (Fu + Qu) \cdot \nabla (Fu + Qu) \\
&= Qu \cdot \nabla Fu + Pu \cdot \nabla Qu + Pu \cdot \nabla Fu + Qu \cdot \nabla Qu \\
&= Qu \cdot \nabla (Fu)^h + Pu \cdot \nabla (Pu)^\nu + (Pu)^h \cdot \nabla_h Qu \\
&\quad + (Pu)^\nu \partial_u Qu + (Pu)^h : \nabla_h (Pu)^h + (Pu)^h \cdot \nabla_h (Pu)^\nu \\
&\quad + (Pu)^\nu \partial_u (Pu)^h - (Pu)^\nu \text{div}_h (Pu)^h + Qu \cdot \nabla Qu \\
&= Qu \cdot \nabla (Pu)^\nu + (Pu)^h \cdot \nabla_h (Pu)^\nu \\
&\quad + (Pu)^\nu \partial_u Qu + (Pu)^h \cdot \nabla_h Qu + (Pu)^h \cdot \nabla_h (Pu)^h + Qu \cdot \nabla Qu. \quad (3.27)
\end{align*}
\]

From the definition of the weighted function \(f(t)\) in (3.3), we must avoid appearing the norm of \(\|F(u)^\nu\|_{B_{p,1}^{\frac{3}{2}-1}}\), this point is very different from the construction of global small solutions. Thus, \(I_2\) is the most troublesome term among the above three terms.

Let us begin to deal with the first term \(I_1\). Thanks to Lemma 2.9 and (3.23), we infer that

\[
\int_0^t \| (I_1)_\lambda \|_{B_{p,1}^{\frac{3}{2}}} \, dt \leq \int_0^t \| ((Pu)_\lambda^h, Qu_x) \|_{B_{p,1}^{\frac{3}{2}-1}} \| \nabla (Pu)^\nu \|_{B_{p,1}^{\frac{3}{2}}} + \| \nabla (Pu)^\nu \|_{B_{p,1}^{\frac{3}{2}-1}} \| ((Pu)_\lambda^h, Qu_x) \|_{B_{p,1}^{\frac{3}{2}}} \, dt \\
\leq \int_0^t \| ((Pu)_\lambda^h, Qu_x) \|_{B_{p,1}^{\frac{3}{2}-1}} + \| (Pu)^\nu \|_{B_{p,1}^{\frac{3}{2}-1}} \| ((Pu)_\lambda^h, Qu_x) \|_{B_{p,1}^{\frac{3}{2}}} + \| (Pu)_\lambda^h \|_{L_{t,x}(B_{p,1}^{\frac{3}{2}-1})} + \| Qu_x \|_{L_{t,x}(B_{p,1}^{\frac{3}{2}})} + \| Qu_x \|_{L_{t,x}(B_{p,1}^{\frac{3}{2}-1})}. \quad (3.28)
\]

Similarly,

\[
\int_0^t \| (I_3)_\lambda \|_{B_{p,1}^{\frac{3}{2}}} \, dt \leq \int_0^t \| ((Pu)_\lambda^h, Qu_x) \|_{B_{p,1}^{\frac{3}{2}-1}} \| ((Pu)_\lambda^h, Qu_x) \|_{B_{p,1}^{\frac{3}{2}}} \, dt \\
\leq \int_0^t \| Qu_x \|_{B_{p,1}^{\frac{3}{2}}} + \| (Pu)_\lambda^h \|_{B_{p,1}^{\frac{3}{2}-1}} \times \| Qu_x \|_{B_{p,1}^{\frac{3}{2}}} + \| ((Pu)_\lambda^h, Qu_x) \|_{B_{p,1}^{\frac{3}{2}}} \, dt. \quad (3.29)
\]

The terms in \(I_2\) are more troublesome. We deal with the term \((Pu)^\nu \text{div}_h (Pu)^h\) for example, the rest two terms can be obtained analogously. With the aid of Bony
decomposition, we can rewrite
\[(\mathbb{P}u)^{\text{div}_h}(\mathbb{P}u)^{h} = \mathbb{T}_{(\mathbb{P}u)^{\text{div}_h}(\mathbb{P}u)^{h}} + \mathbb{T}_{\text{div}_h(\mathbb{P}u)^{h}}(\mathbb{P}u)^{\gamma} + \mathbb{R}(\mathbb{P}u)^{\gamma}, \text{div}_h(\mathbb{P}u)^{h}).\]
\[\tag{3.30}\]

Let \(p, \alpha\) satisfy the conditions stated in Theorem 1.1, by the Hölder inequality and Bernstein’s inequality, we have
\[
\begin{align*}
\left\| \hat{\Delta}_j \mathbb{T}_{(\mathbb{P}u)^{\gamma}}(\text{div}_h(\mathbb{P}u)^{h}) \right\|_{L^2} & \lesssim \sum_{|j-k| \leq 4} \left\| \hat{\Delta}_j (\mathbb{S}_{k-1}(\mathbb{P}u)^{\gamma}) \right\|_{L^p} \left\| \hat{\Delta}_k (\text{div}_h(\mathbb{P}u)^{h}) \right\|_{L^p} \\
& \lesssim \sum_{|j-k| \leq 4} \left( \sum_{|k'| \leq k-2} \left\| \hat{\Delta}_{k'} (\mathbb{P}u)^{\gamma} \right\|_{L^p} 2^{(\frac{p}{2} - \frac{1}{2})k'} \left\| \hat{\Delta}_k (\text{div}_h(\mathbb{P}u)^{h}) \right\|_{L^p} \right) \\
& \lesssim \sum_{|j-k| \leq 4} d_k \left\| (\mathbb{P}u)^{\gamma} \right\|_{B_{p,1}^{\frac{3}{2} - 1 - \alpha}} 2^{(\frac{3}{2} - \frac{1}{2})k} \left\| \hat{\Delta}_k (\text{div}_h(\mathbb{P}u)^{h}) \right\|_{L^p}
\end{align*}
\]
which implies
\[
\sum_{j \leq j_0} 2^{\frac{j}{2}} \left\| \hat{\Delta}_j \mathbb{T}_{(\mathbb{P}u)^{\gamma}}(\text{div}_h(\mathbb{P}u)^{h}) \right\|_{L^2} \lesssim \left\| (\mathbb{P}u)^{\gamma} \right\|_{B_{p,1}^{\frac{3}{2} - 1 - \alpha}} \left\| (\mathbb{P}u)^{h} \right\|_{B_{p,1}^{\frac{3}{2} + 1 - \alpha}}.
\]

Similarly, with the aid of \(\frac{3}{p} - \frac{1}{2} > 0\), we have
\[
\sum_{j \leq j_0} 2^{\frac{j}{2}} \left\| \hat{\Delta}_j \mathbb{T}_{\text{div}_h(\mathbb{P}u)^{h}}(\mathbb{P}u)^{\gamma} \right\|_{L^2} \lesssim \left\| (\mathbb{P}u)^{\gamma} \right\|_{B_{p,1}^{\frac{3}{2} - 1 - \alpha}} \left\| (\mathbb{P}u)^{h} \right\|_{B_{p,1}^{\frac{3}{2} + 1 - \alpha}}.
\]

Notice the fact that \(1 \leq \frac{p}{2} \leq 2\), we have
\[
\begin{align*}
\sum_{j \leq j_0} 2^{\frac{j}{2}} \left\| \hat{\Delta}_j \mathbb{R}(\mathbb{P}u)^{\gamma}, \text{div}_h(\mathbb{P}u)^{h}) \right\|_{L^2} & \lesssim \sum_{j \leq j_0} 2^{(\frac{p}{2} - 1)j} \left\| \hat{\Delta}_j \mathbb{R}(\mathbb{P}u)^{\gamma}, \text{div}_h(\mathbb{P}u)^{h}) \right\|_{L^2} \\
& \lesssim \sum_{j \leq j_0} 2^{(\frac{p}{2} - 1)j} \sum_{k \geq j-3} 2^k \left\| \hat{\Delta}_k (\mathbb{P}u)^{\gamma} \right\|_{L^p} \left\| \hat{\Delta}_k (\mathbb{P}u)^{h} \right\|_{L^p} \\
& \lesssim \left\| (\mathbb{P}u)^{\gamma} \right\|_{B_{p,1}^{\frac{3}{2} - 1 - \alpha}} \left\| (\mathbb{P}u)^{h} \right\|_{B_{p,1}^{\frac{3}{2} + 1 - \alpha}}.
\end{align*}
\]

Collecting above three estimates gives
\[
\left\| (\mathbb{P}u)^{\gamma} \text{div}_h(\mathbb{P}u)^{h} \right\|_{B_{p,1}^{\frac{3}{2} - 1 - \alpha}} \lesssim \left\| (\mathbb{P}u)^{\gamma} \right\|_{B_{p,1}^{\frac{3}{2} - 1 - \alpha}} \left\| (\mathbb{P}u)^{h} \right\|_{B_{p,1}^{\frac{3}{2} + 1 - \alpha}} + \left\| (\mathbb{P}u)^{\gamma} \right\|_{B_{p,1}^{\frac{3}{2} - 1 - \alpha}} \left\| (\mathbb{P}u)^{h} \right\|_{B_{p,1}^{\frac{3}{2} + 1 - \alpha}}.
\]

The other two terms in \(I_2\) can be estimated with the same processes, hence, we have
\[
\begin{align*}
\int_0^t \left\| I_2 \right\|_{B_{p,1}^{\frac{3}{2}}} d\tau & \leq \int_0^t \left\| (\mathbb{P}u)^{\gamma} \right\|_{B_{p,1}^{\frac{3}{2}}} \left\| (\mathbb{P}u)^{h}, (\mathbb{P}u)^{h} \right\|_{B_{p,1}^{\frac{3}{2}}} d\tau \\
& \quad + \int_0^t \left\| (\mathbb{P}u)^{\gamma} \right\|_{B_{p,1}^{\frac{3}{2} - 1 - \alpha}} \left\| (\mathbb{P}u)^{h}, (\mathbb{P}u)^{h} \right\|_{B_{p,1}^{\frac{3}{2} + 1 - \alpha}} d\tau \\
& \overset{\text{def}}{=} I_3 + I_4. \tag{3.31}
\end{align*}
\]
As the term \(\|(|Pu|^\lambda)\|_{B_{p,1}^2}\) contained in \(J_3\), we can get similarly to (3.23) that
\[
J_3 \lesssim \varepsilon \int_0^t \left( \|(|Pu|^\lambda, QU^h)\|_{B_{p,1}^2} \right. \\
+ \|(|Pu|^\lambda)\|_{L_{1,1}(B_{p,1}^2)} + \|QU^h\|_{H_{1,1}(B_{p,1}^2)} \left. \right) d\tau
\]
\[
\lesssim \varepsilon \int_0^t \left( \|(|Pu|^\lambda, QU^h)\|_{B_{p,1}^2} \right. \\
+ \|(|Pu|^\lambda)\|_{L_{1,1}(B_{p,1}^2)} + \|QU^h\|_{H_{1,1}(B_{p,1}^2)} \left. \right) d\tau
\]
In view of interpolation inequality and Young’s inequality, the term \(J_4\) can be bounded by
\[
J_4 \lesssim \varepsilon \int_0^t \left( \|(|Pu|^\lambda, QU^h)\|_{B_{p,1}^2} \right. \\
+ \|(|Pu|^\lambda)\|_{L_{1,1}(B_{p,1}^2)} + \|QU^h\|_{H_{1,1}(B_{p,1}^2)} \left. \right) d\tau
\]
Taking the estimates \(J_3, J_4\) back to (3.31), we arrive at
\[
\int_0^t \left( \|(|Pu|^\lambda, QU^h)\|_{B_{p,1}^2} \right. \\
+ \|(|Pu|^\lambda)\|_{L_{1,1}(B_{p,1}^2)} + \|QU^h\|_{H_{1,1}(B_{p,1}^2)} \left. \right) d\tau
\]
Inserting estimates (3.11), (3.12), (3.19), (3.25), (3.26), (3.28), (3.29), (3.32) into (3.9) yields
\[
\|(|a^\lambda, QU^h)\|_{L^{p}_{t,x}(B_{p,1}^2)} \lesssim \varepsilon \int_0^t \left( \|(|Pu|^\lambda, QU^h)\|_{B_{p,1}^2} \right. \\
+ \|(|Pu|^\lambda)\|_{L_{1,1}(B_{p,1}^2)} + \|QU^h\|_{H_{1,1}(B_{p,1}^2)} \left. \right) d\tau
\]
3.2. The high frequencies estimates for the compressible parts. Compared to the low frequencies parts, the linear terms appeared in the compressible equations are low order terms, we can set the high frequencies estimates for \((a, Qu)\) lie in the \(L^p\) framework, thus, the processes are easy in contrast to the low frequencies parts.

We first get by taking \(\partial_x\) to the first equation of (1.5) that
\[
\partial_t \nabla a - \Delta \nabla a + \nabla \text{div} Qu = -\nabla \text{div}(au) \quad \text{(3.34)}
\]
from which, we get similarly to (3.6) that
\[
\frac{d}{dt} \| \Delta_j \nabla a^\lambda \|_{L^p} + \lambda f(t) \| \Delta_j \nabla a^\lambda \|_{L^p} + 2^{\alpha_j} \| \Delta_j \nabla a^\lambda \|_{L^p}
\]
\[
\leq C_1 2^{\alpha_j} \| \Delta_j Qu^\lambda \|_{L^p} + C_1 \| \Delta_j (\nabla \text{div}(au)) \|_{L^p} \quad \text{(3.35)}
\]
Multiplying by $2^{(\frac{3}{2}-1)j}$ and summing up for any $j > j_0$ imply
\[
\|\nabla a^h \|_{L^\infty_t(B^{\frac{3}{2}-1}_{p,1})} + \lambda \|\nabla a^h \|_{L^1_t(B^{\frac{3}{2}-1}_{p,1})} + \|\nabla a^2 \|_{L^1_t(B^{\frac{3}{2}+1}_{p,1})} \\
\leq \|\nabla a^0 \|_{B^{\frac{3}{2}-1}_{p,1}} + C_1 \int_0^t \|\nabla a^h \|_{B^{\frac{3}{2}-1}_{p,1}} + \int_0^t \|\nabla \text{div}(au)\|_{B^{\frac{3}{2}-1}_{p,1}} \, dt. \quad (3.36)
\]
Carrying out the analogous line, one deduce from the second equation in (1.5) that
\[
\|\nabla u \|_{L^\infty_t(B^{\frac{3}{2}-1}_{p,1})} + \lambda \|\nabla u \|_{L^1_t(B^{\frac{3}{2}-1}_{p,1})} + \|\nabla u^2 \|_{L^1_t(B^{\frac{3}{2}+1}_{p,1})} \\
\leq \|\nabla u^0 \|_{B^{\frac{3}{2}-1}_{p,1}} + C_2 \int_0^t \|\nabla a^h \|_{B^{\frac{3}{2}-1}_{p,1}} + \int_0^t \|\nabla \text{div}(au)\|_{B^{\frac{3}{2}-1}_{p,1}} \, dt \\
+ C_2 \int_0^t \|\nabla \ln(1 + a)\|_{B^{\frac{3}{2}-1}_{p,1}} \, dt + C_2 \int_0^t \|\nabla \nabla \|_{B^{\frac{3}{2}-1}_{p,1}} \, dt. \quad (3.37)
\]
Multiplying by a suitable constant $C_3 > C_1$ on both hand side of (3.37) and then summing up the resultant equations with (3.36) gives
\[
\|\nabla a^h \|_{L^\infty_t(B^{\frac{3}{2}-1}_{p,1})} + \lambda \|\nabla a^h \|_{L^1_t(B^{\frac{3}{2}-1}_{p,1})} + \|\nabla a^2 \|_{L^1_t(B^{\frac{3}{2}+1}_{p,1})} \\
\lesssim \|\nabla a^0 \|_{B^{\frac{3}{2}-1}_{p,1}} + \int_0^t \|\nabla a^h \|_{B^{\frac{3}{2}-1}_{p,1}} \, dt \\
+ \int_0^t \|\nabla \text{div}(au)\|_{B^{\frac{3}{2}-1}_{p,1}} + \int_0^t \|\nabla \nabla \|_{B^{\frac{3}{2}-1}_{p,1}} \, dt. \quad (3.38)
\]
Due to
\[
\|\nabla a^h \|_{B^{\frac{3}{2}-1}_{p,1}} \lesssim 2^{-2j_0} \|\nabla a^h \|_{B^{\frac{3}{2}+1}_{p,1}}, \quad j > j_0,
\]
hence, absorbing the second term in the right hand side of (3.38) to the left gives
\[
\|\nabla a^h \|_{L^\infty_t(B^{\frac{3}{2}-1}_{p,1})} + \lambda \|\nabla a^h \|_{L^1_t(B^{\frac{3}{2}-1}_{p,1})} + \|\nabla a^2 \|_{L^1_t(B^{\frac{3}{2}+1}_{p,1})} \\
\lesssim \|\nabla a^0 \|_{B^{\frac{3}{2}-1}_{p,1}} + \int_0^t \|\nabla \text{div}(au)\|_{B^{\frac{3}{2}-1}_{p,1}} + \int_0^t \|\nabla \nabla \|_{B^{\frac{3}{2}-1}_{p,1}} \, dt. \quad (3.39)
\]
From Lemmas 2.7, 2.8, one has
\[
\int_0^t \|\nabla \text{div}(au)\|_{B^{\frac{3}{2}-1}_{p,1}} + \int_0^t \|\nabla \nabla \|_{B^{\frac{3}{2}-1}_{p,1}} \, dt \\
\lesssim \int_0^t \|\nabla a\|_{B^{\frac{3}{2}-1}_{p,1}} + \int_0^t \|\nabla u\|_{B^{\frac{3}{2}-1}_{p,1}} \, dt \\
\lesssim \int_0^t \|\nabla a\|_{B^{\frac{3}{2}-1}_{p,1}} + \|\nabla u\|_{B^{\frac{3}{2}-1}_{p,1}} \, dt. \quad (3.40)
\]
Now the terms on the right hand sided of (3.40) are the same as (3.20), thus, from (3.25) we can get

\[
\int_0^t \| (\nabla \text{div}(au))_t \|_{B^2_{p,1}} \, d\tau + \int_0^t \| (\nabla (\ln(1+a)) \nabla u)_t \|_{B^2_{p,1}} \, d\tau \\
\lesssim \varepsilon \int_0^t \left( \| a \chi \|_{B^3_{p,1}}^2 + \| a \chi \|_{\dot{B}^3_{p,1}}^2 \right) \, d\tau + \int_0^t \| a \chi \|_{L^1_{t,f}(\dot{B}^3_{p,1})} \, d\tau \\
+ \int_0^t \left( \| \mathcal{Q}u \|_{\dot{B}^3_{p,1}}^2 + \| (\mathcal{P}u)^h \|_{\dot{B}^3_{p,1}}^2 \right) \| a \chi \|_{\dot{B}^3_{p,1}}^2 + \| a \chi \|_{\dot{B}^3_{p,1}}^2 \, d\tau \\
+ \int_0^t \left( \| a \|_{\dot{B}^2_{p,1}}^2 + \| a \|_{\dot{B}^2_{p,1}}^2 \right) \left( \| \mathcal{Q}u \|_{\dot{B}^2_{p,1}}^2 + \| (\mathcal{P}u)^h \|_{\dot{B}^2_{p,1}}^2 \right) \, d\tau. \quad (3.41)
\]

With the aid of Lemmas 2.8, 2.11, we have

\[
\int_0^t \| (k(a) \nabla u)_t \chi \|_{B^3_{p,1}} \, d\tau \lesssim \int_0^t \| k(a) \|_{B^3_{p,1}} \| \nabla a \chi \|_{B^3_{p,1}} \, d\tau \\
\lesssim \int_0^t \| a \|_{B^3_{p,1}} \| a \chi \|_{B^3_{p,1}} \, d\tau \lesssim \int_0^t \| a \|_{\dot{B}^2_{p,1}} \| a \chi \|_{\dot{B}^2_{p,1}} \, d\tau \\
\lesssim \int_0^t \left( \| a \|_{\dot{B}^2_{p,1}} + \| a \|_{\dot{B}^2_{p,1}} \right) \left( \| a \chi \|_{\dot{B}^2_{p,1}} + \| a \chi \|_{\dot{B}^2_{p,1}} \right) \, d\tau. \quad (3.42)
\]

For the last term in (3.39), we can use the equality in (3.27) to get the same estimates as (3.28), (3.29), (3.31) that

\[
\int_0^t \| (u \cdot \nabla u)_t \chi \|_{\dot{B}^2_{p,1}} \, d\tau \lesssim \varepsilon \int_0^t \left( \| \mathcal{Q}u \|_{\dot{B}^2_{p,1}}^2 + \| \mathcal{Q}u \|_{\dot{B}^2_{p,1}}^2 + \| (\mathcal{P}u)^h \|_{\dot{B}^2_{p,1}}^2 \right) \, d\tau \\
+ \| \mathcal{Q}u \|_{L^1_{t,f}(\dot{B}^2_{p,1})}^2 + \| \mathcal{Q}u \|_{L^1_{t,f}(\dot{B}^2_{p,1})}^2 + \| (\mathcal{P}u)^h \|_{L^1_{t,f}(\dot{B}^2_{p,1})}^2 \\
+ \int_0^t \left( \| \mathcal{Q}u \|_{\dot{B}^2_{p,1}}^2 + \| (\mathcal{P}u)^h \|_{\dot{B}^2_{p,1}}^2 + \| (\mathcal{P}u)^h \|_{\dot{B}^2_{p,1}}^2 \right) \, d\tau. \quad (3.43)
\]

Thus, plugging the above estimates (3.41), (3.42), (3.43) into (3.40) implies

\[
\| (\nabla a \chi, \mathcal{Q}u \chi) \|_{L_t^\infty(L^2_{p,1})} + \| \nabla a \chi, \mathcal{Q}u \chi \|_{L^1_{t,f}(L^2_{p,1})} + \| (\nabla a \chi, \mathcal{Q}u \chi) \|_{L^1_{t,f}(L^2_{p,1})} \\
\lesssim (\| a \chi \|_{\dot{B}^2_{p,1}}^2 + \| a \chi \|_{\dot{B}^2_{p,1}}^2) \, d\tau \\
+ \varepsilon \int_0^t \left( \| a \chi \|_{\dot{B}^2_{p,1}}^2 + \| a \chi \|_{\dot{B}^2_{p,1}}^2 \right) \, d\tau + \| a \chi \|_{L^1_{t,f}(L^2_{p,1})}^2 \\
+ \| a \chi \|_{L^1_{t,f}(L^2_{p,1})}^2 + \| a \chi \|_{L^1_{t,f}(L^2_{p,1})}^2 + \| (\mathcal{P}u)^h \|_{L^1_{t,f}(L^2_{p,1})}^2 \\
+ \int_0^t \left( \| a \chi \|_{\dot{B}^2_{p,1}}^2 + \| a \chi \|_{\dot{B}^2_{p,1}}^2 + \| (\mathcal{P}u)^h \|_{\dot{B}^2_{p,1}}^2 \right) \, d\tau. \quad (3.44)
\]
4. The estimates for the incompressible parts. The estimates for the incompressible parts of (1.3) will be obtained in this section. As we have stated in the introduction, we shall separate the estimates into two subsections: the vertical components of the incompressible parts and the horizontal components of the incompressible parts. The structure coming from the equations plays the important role in our argument. We begin to estimate the horizontal components.

4.1. The estimates for the horizontal components. Let us first recall the equations for the horizontal components:
\[
\begin{align*}
\partial_t (Pu)^h - \Delta (Pu)^h + (Pu \cdot \nabla u)^h = 2(\nabla(\ln(1 + a))\nabla u)^h, \\
([Pu]^h)]_{t=0} = ([Pu]^h).
\end{align*}
\] (4.1)

For the above equations, we can get by a similar derivation of (3.9) that
\[
\begin{align*}
\|([Pu]^h)]_{B^{p,1}_{p,1}} + \|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} + \|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} & \\
& \leq \|([Pu]^h)]_{B^{p,1}_{p,1}} + C \int_0^t (\|u \cdot \nabla u\|_{B^{p,1}_{p,1}} + \|\nabla(\ln(1 + a))\nabla u\|_{B^{p,1}_{p,1}}) d\tau.
\end{align*}
\] (4.2)

The last term of the above inequality is the same as the last two terms in (3.39), thus, we get from (3.41), (3.43) that
\[
\begin{align*}
\|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} & \\
& \lesssim \|([Pu]^h)]_{B^{p,1}_{p,1}} + \|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} + \|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} \\
& + \|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} + \|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} + \|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} \\
& + \|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} + \|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} + \|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} \\
& + \int_0^t \left( \|u \cdot \nabla u\|_{L^p(B^{p,1}_{p,1})} + \|\nabla(\ln(1 + a))\nabla u\|_{L^p(B^{p,1}_{p,1})} \right) \\
& \times \left( \|u \cdot \nabla u\|_{L^p(B^{p,1}_{p,1})} + \|\nabla(\ln(1 + a))\nabla u\|_{L^p(B^{p,1}_{p,1})} \right) d\tau.
\end{align*}
\] (4.3)

4.2. The estimates for the vertical component. We pay attention to the equation of the vertical component:
\[
\begin{align*}
\partial_t ([Pu]^v - \Delta (Pu)^v + (Pu \cdot \nabla u)^v = 2(\nabla(\ln(1 + a))\nabla u)^v, \\
([Pu]^v)]_{t=0} = ([Pu]^v).
\end{align*}
\] (4.4)

By a standard energy argument as (3.9), we infer that
\[
\begin{align*}
\|([Pu]^h)]_{L^p(B^{p,1}_{p,1})} & \\
& \lesssim \|([Pu]^h)]_{B^{p,1}_{p,1}} + \int_0^t (\|u \cdot \nabla u\|_{B^{p,1}_{p,1}} + \|\nabla(\ln(1 + a))\nabla u\|_{B^{p,1}_{p,1}}) d\tau \\
& \lesssim \|([Pu]^h)]_{B^{p,1}_{p,1}} + \int_0^t (\|u \cdot \nabla u\|_{B^{p,1}_{p,1}} + \|\nabla(\ln(1 + a))\nabla u\|_{B^{p,1}_{p,1}}) d\tau.
\end{align*}
\] (4.5)
Thanks to the divergence free condition $\text{div } \mathbb{P}u = 0$, we can write
\[
   u \cdot \nabla u = (\mathbb{P}u + Qu) \cdot \nabla (\mathbb{P}u + Qu)
\]
\[
   = Qu \cdot \nabla Qu + Qu \cdot \nabla \mathbb{P}u + \mathbb{P}u \cdot \nabla Qu + \mathbb{P}u \cdot \nabla Qu + (\mathbb{P}u)^h \cdot \nabla_h (\mathbb{P}u)^h
\]
\[
   + (\mathbb{P}u)^h \cdot \nabla_h (\mathbb{P}u)^h + (\mathbb{P}u)^y \partial_x (\mathbb{P}u)^h - (\mathbb{P}u)^y \text{div}_h (\mathbb{P}u)^h.
\]

It follows from Lemma 2.8 that
\[
   \|Qu \cdot \nabla Qu\|_{B_{1,1}^{\frac{3}{2}}} \lesssim (\|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} - 1})(\|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} + 1}), \quad (4.6)
\]
\[
   \|Qu \cdot \nabla \mathbb{P}u\|_{B_{1,1}^\frac{3}{2}} \lesssim \|Qu\|_{B_{2,1}^\frac{3}{2} - 1}(\|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} + 1})
\]
\[
   \lesssim (\|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} - 1})(\|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} + 1}), \quad (4.7)
\]

Similarly, we have
\[
   \|\mathbb{P}u \cdot \nabla Qu\|_{B_{1,1}^{\frac{3}{2}}} \lesssim (\|(\mathbb{P}u)^h, (\mathbb{P}u)^y\|_{B_{2,1}^\frac{1}{2} - 1})(\|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} + 1}), \quad (4.8)
\]

and
\[
   \|(\mathbb{P}u)^h \cdot \nabla_h (\mathbb{P}u)^h\|_{B_{1,1}^{\frac{3}{2}}} + \|(\mathbb{P}u)^h \cdot \nabla_h (\mathbb{P}u)^y\|_{B_{1,1}^{\frac{3}{2} - 1}}
\]
\[
   + \|(\mathbb{P}u)^y \partial_x (\mathbb{P}u)^h\|_{B_{1,1}^{\frac{3}{2} - 1}} + \|(\mathbb{P}u)^y \text{div}_h (\mathbb{P}u)^h\|_{B_{1,1}^{\frac{3}{2} - 1}}
\]
\[
   \lesssim (\|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} - 1})(\|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} + 1} + \|(\mathbb{P}u)^y\|_{B_{2,1}^\frac{3}{2} - 1}), \quad (4.9)
\]

Thanks to Lemmas 2.8, 2.11 and the interpolation inequality, we get
\[
   \|\nabla (\ln (1 + a)) \nabla u\|_{B_{1,1}^{\frac{3}{2}}} \leq \|\nabla (\ln (1 + a))\|_{B_{1,1}^{\frac{3}{2}}} \|\nabla u\|_{B_{1,1}^{\frac{3}{2}}} \approx \|a\|_{B_{1,1}^{\frac{3}{2}}} \|u\|_{B_{1,1}^{\frac{3}{2} + 1}}
\]
\[
   \lesssim (\|Qu^h\|_{B_{2,1}^\frac{3}{2} - 1} + \|Qu^h\|_{B_{2,1}^\frac{1}{2}})(\|Qu^h\|_{B_{2,1}^\frac{3}{2} - 1} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} + 1} + \|(\mathbb{P}u)^h\|_{B_{2,1}^\frac{3}{2} + 1} + \|(\mathbb{P}u)^y\|_{B_{2,1}^{\frac{3}{2} + 1}}). \quad (4.10)
\]

Inserting estimates (4.6)–(4.10) into (4.5) gives
\[
   \|(\mathbb{P}u)^y\|_{L^\infty_t(B_{\frac{3}{2} - 1})} + \|(\mathbb{P}u)^y\|_{L^1_t(B_{\frac{3}{2} + 1})}
\]
\[
   \lesssim (\|Qu^h\|_{B_{2,1}^\frac{3}{2} - 1} + \|Qu^h\|_{B_{2,1}^\frac{1}{2}})(\|Qu^h\|_{B_{2,1}^\frac{3}{2} - 1} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} + 1} + \|(\mathbb{P}u)^h\|_{B_{2,1}^\frac{3}{2} + 1} + \|(\mathbb{P}u)^y\|_{B_{2,1}^{\frac{3}{2} + 1}}) \, d\tau
\]
\[
   + \int_0^t (\|Qu^h\|_{B_{2,1}^\frac{1}{2} - 1} + \|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} - 1})(\|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} + 1} + \|(\mathbb{P}u)^h\|_{B_{2,1}^\frac{3}{2} + 1} + \|(\mathbb{P}u)^y\|_{B_{2,1}^{\frac{3}{2} + 1}}) \, d\tau
\]
\[
   + \int_0^t (\|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} - 1})(\|Qu^h\|_{B_{2,1}^\frac{1}{2}} + \|Qu^h\|_{B_{2,1}^\frac{3}{2} + 1} + \|(\mathbb{P}u)^h\|_{B_{2,1}^\frac{3}{2} + 1} + \|(\mathbb{P}u)^y\|_{B_{2,1}^{\frac{3}{2} + 1}}) \, d\tau.
\]
5. Continuity argument. The goal of this section is to present the proof of global solutions in Theorem 1.1. We assume that $T^*$ is the lifespan of the solution $(u, v)$ defined in (3.1). Hence to prove Theorem 1.1, it amounts to prove that $T^* = \infty$. To complete the proof, we shall use the method of continuity. For this, let $0 < \tilde{c} \ll 1$ be a small enough positive constant, which will be determined later on, we define $T^{**}$ by

$$
T^{**} \overset{\text{def}}{=} \sup \left\{ t \in [0, T^*) : \| (a^\ell, Qu^\ell) \|_{L^\infty_t (B^{\frac{3}{2}}_{p,1})} + \| a^h \|_{L^\infty_t (B^{\frac{3}{2}}_{p,1})} + \| ((Pu)^h, Qu^h) \|_{L^\infty_t (B^{\frac{3}{2}}_{p,1})} + \| (a^\ell, Qu^\ell) \|_{L^1_t (B^{\frac{3}{2}}_{p,1})} + \| a^h \|_{L^1_t (B^{\frac{3}{2}}_{p,1})} + \| (Pu)^h, Qu^h \|_{L^1_t (B^{\frac{3}{2}}_{p,1})} \leq \tilde{c} \right\}.
$$

(5.1)

We shall prove $T^{**} = \infty$ under the assumption (1.7).

Combining with (3.33), (3.44), (4.3) and choosing $\varepsilon$ small enough and $\lambda$ sufficient large, we have

$$
\| (a^\ell, Qu^\ell) \|_{L^\infty_t (B^{\frac{3}{2}}_{p,1})} + \| ((Pu)^h, Qu^h) \|_{L^\infty_t (B^{\frac{3}{2}}_{p,1})} + \| a^h \|_{L^\infty_t (B^{\frac{3}{2}}_{p,1})} + \| a^\ell \|_{L^1_t (B^{\frac{3}{2}}_{p,1})} + \| (Pu)^h, Qu^h \|_{L^1_t (B^{\frac{3}{2}}_{p,1})}
$$

$$+ \left( \| (a^\ell, Qu^\ell) \|_{L^1_t (B^{\frac{3}{2}}_{p,1})} + \| a^h \|_{L^1_t (B^{\frac{3}{2}}_{p,1})} + \| (Pu)^h, Qu^h \|_{L^1_t (B^{\frac{3}{2}}_{p,1})} \right) dt.
$$

Using (5.1), we further get for any $t \leq T^{**}$,

$$
\| (a^\ell, Qu^\ell) \|_{L^\infty_t (B^{\frac{3}{2}}_{p,1})} + \| ((Pu)^h, Qu^h) \|_{L^\infty_t (B^{\frac{3}{2}}_{p,1})} + \| a^h \|_{L^\infty_t (B^{\frac{3}{2}}_{p,1})} + \| a^\ell \|_{L^1_t (B^{\frac{3}{2}}_{p,1})} + \| (Pu)^h, Qu^h \|_{L^1_t (B^{\frac{3}{2}}_{p,1})}
$$

$$\leq \| (a^\ell, Qu^\ell) \|_{B^{\frac{3}{2}}_{p,1}} + \| ((Pu)^h, Qu^h) \|_{B^{\frac{3}{2}}_{p,1}} + \| a^h \|_{B^{\frac{3}{2}}_{p,1}} + \| a^\ell \|_{B^{\frac{3}{2}}_{p,1}} + \exp(C \int_0^t f(\tau) d\tau). \quad (5.2)
$$

While thanks to (4.11) and (5.1), for any $t \leq T^{**}$, we arrive at by taking $\tilde{c}$ small enough that

$$
\| (Pu)^h \|_{L^\infty_t (B^{\frac{3}{2}}_{p,1})} + \| (Pu)^h \|_{L^1_t (B^{\frac{3}{2}}_{p,1})} + \| a^h \|_{B^{\frac{3}{2}}_{p,1}} \leq \| (Pu)^h \|_{B^{\frac{3}{2}}_{p,1}} + \tilde{c}.
$$

(5.3)
Thanks to interpolation inequality, one has
\[ \| (P u)^{\gamma} \|_{L^\alpha_t(B_{p,1}^\gamma)} \lesssim \| (P u)^{\gamma} \|_{L^\alpha_t(B_{p,1}^\gamma)} \| (P u)^{\gamma} \|_{L^\alpha_t(B_{p,1}^\gamma)}, \]
\[ \| (P u)^{\gamma} \|_{L^\alpha_t(B_{p,1}^\gamma)} \lesssim \| (P u)^{\gamma} \|_{L^\alpha_t(B_{p,1}^\gamma)} \| (P u)^{\gamma} \|_{L^\alpha_t(B_{p,1}^\gamma)} .\]

Thus, we deduce from (5.3) that
\[ \int_0^t f(\tau) \, d\tau = \int_0^t \| (P u)^{\gamma} \|_{B_{p,1}^\gamma} + \| (P u)^{\gamma} \|_{B_{p,1}^\gamma} + \| (P u)^{\gamma} \|_{B_{p,1}^\gamma} \, d\tau \]
\[ \leq C \left( \| (P u)^{\gamma} \|_{B_{p,1}^\gamma} + \bar{c} + \| (P u)^{\gamma} \|_{B_{p,1}^\gamma} + \bar{c} \right) \]
\[ \leq C \| (P u)^{\gamma} \|_{B_{p,1}^\gamma} + 1. \] (5.4)

Combining (5.2) with (5.4), we reach for \( t \leq T^{**} \) that
\[ \| (a^f, Qu^f) \|_{L^\alpha_t(B_{1/2}^2)} + \| (P u)^{\gamma} \|_{L^\alpha_t(B_{1/2}^2)} + \| a^h \|_{L^\alpha_t(B_{1/2}^2)} + \| a^h \|_{L^\alpha_t(B_{1/2}^2)} \]
\[ + \| (a^f, Qu^f) \|_{L^t_t(B_{1/2}^2)} + \| (P u)^{\gamma} \|_{L^\alpha_t(B_{1/2}^2)} + \| a^h \|_{L^\alpha_t(B_{1/2}^2)} \]
\[ \leq (\| a^f_0, Qu^f_0 \|_{L^\alpha_t(B_{1/2}^2)} + \| (P u)^{\gamma} \|_{L^\alpha_t(B_{1/2}^2)} + \| a^h_0 \|_{L^\alpha_t(B_{1/2}^2)} + \| a^h_0 \|_{L^\alpha_t(B_{1/2}^2)}) \exp(C \| (P u)^{\gamma} \|_{B_{p,1}^\gamma} + 1). \] (5.5)

In particular, (5.5) implies that if we take \( C_0 \) large enough and \( c_0 \) sufficiently small in (1.7), there holds
\[ \| (a^f, Qu^f) \|_{L^\alpha_t(B_{1/2}^2)} + \| a^h \|_{L^\alpha_t(B_{1/2}^2)} + \| (P u)^{\gamma} \|_{L^\alpha_t(B_{1/2}^2)} \]
\[ + \| (a^f, Qu^f) \|_{L^t_t(B_{1/2}^2)} + \| a^h \|_{L^\alpha_t(B_{1/2}^2)} + \| (P u)^{\gamma} \|_{L^\alpha_t(B_{1/2}^2)} \leq \bar{c} \]
\[ \forall t \leq T^{**}, \]
which contradicts with (5.1). Whence we conclude that \( T^* = \infty \).

Consequently, we complete the proof of the global solutions in Theorem 1.1. \( \square \)

6. The decay of the constructed solutions. In this section, we are concerned with the long-time behavior of the global small solutions. By a similar argument from Section 3–Section 5, we can get the following theorem (strong smallness condition):

**Theorem 6.1.** Let \( 2 \leq p \leq 4 \). For any \( (a^0_0, u^0_0) \in \dot{B}_{2,1}^{3/2}(\mathbb{R}^3) \), \( a^0_0 \in \dot{B}_{p,1}^{3/2}(\mathbb{R}^3) \), \( u^0_0 \in \dot{B}_{p,1}^{3/2}(\mathbb{R}^3) \). If there exists a positive constant \( c_0 \) such that,
\[ \mathcal{X}_0 \overset{\text{def}}{=} \| (a^f_0, u^0_0) \|_{\dot{B}_{2,1}^{3/2}} + \| u^0_0 \|_{\dot{B}_{p,1}^{3/2}} + \| a^0_0 \|_{\dot{B}_{p,1}^{3/2}} \leq c_0, \]
then the system (1.3) has a unique global solution \((a, u)\) so that for any \( T > 0 \)
\[ (a^f, a^u) \in C(\mathbb{R}^+; \dot{B}_{2,1}^{3/2}) \cap \dot{L}^1(\mathbb{R}^+; \dot{B}_{2,1}^{3/2}), \]
\[ a^h \in C(\mathbb{R}^+; \dot{B}_{p,1}^{3/2}) \cap \dot{L}^1(\mathbb{R}^+; \dot{B}_{p,1}^{3/2}), \]
\[ u^h \in C(\mathbb{R}^+; \dot{B}_{p,1}^{3/2}) \cap \dot{L}^1(\mathbb{R}^+; \dot{B}_{p,1}^{3/2}). \] (6.1)

Moreover, there exists some constant \( C = C(p, d) \) such that
\[ \mathcal{X}(t) \leq C \mathcal{X}_0, \]
with \( X(t) \overset{\text{def}}{=} \| (a^t, u^t) \|_{\tilde{L}^\infty(B^{1/2}_{2,1})} + \| u^h \|_{\tilde{L}^\infty(B^{0}_{p,1})} + \| a^h \|_{\tilde{L}^\infty(B^{3}_{p,1})}
\]
+ \( \| (a^t, u^t) \|_{L^1_t(B^{3/2}_{2,1})} + \| u^h \|_{L^1_t(B^{1}_{p,1})} + \| a^h \|_{L^1_t(B^{1+2}_{p,1})} \).

By a similar argument in Section 3–Section 5, we can get the following inequality:
\[
\begin{align*}
\frac{d}{dt} \left(\| (a, u) \|_{\tilde{L}^\infty(B^{1/2}_{2,1})} + \| u \|_{B^{5/2}_{p,1}} + \| a \|_{B^{5}_{p,1}} \right) + C \left(\| (a, u) \|_{B^{2}_{2,1}} + \| u \|_{B^{2+1}_{p,1}} + \| a \|_{B^{2}_{p,1}} \right) \\
\leq C \left(\| (a, u) \|_{B^{1/2}_{2,1}} + \| u \|_{B^{3/2}_{p,1}} + \| a \|_{B^{3}_{p,1}} \right). 
\end{align*}
\]  
(6.2)

In view of Theorem 6.1,
\[
\| (a^t, u^t) \|_{\tilde{L}^\infty(B^{1/2}_{2,1})} + \| u^h \|_{\tilde{L}^\infty(B^{3/2}_{p,1})} + \| a^h \|_{\tilde{L}^\infty(B^{3}_{p,1})} \leq C \lambda_0 \ll 1, \quad \text{for all } t \geq 0.
\]

thus, we can get from (6.2) that
\[
\begin{align*}
\frac{d}{dt} \left(\| (a, u) \|_{B^{1/2}_{2,1}} + \| u \|_{B^{3/2}_{p,1}} + \| a \|_{B^{3}_{p,1}} \right) \\
+ C \left(\| (a, u) \|_{B^{1/2}_{2,1}} + \| u \|_{B^{3/2}_{p,1}} + \| a \|_{B^{3}_{p,1}} \right) \leq 0. 
\end{align*}
\]  
(6.3)

With (6.3) in hand, we want to use the interpolation inequality to get the Lyapunov-type inequality for the above energy norms. The core of the argument is to control the low frequencies of \( \| (a, u) \|_{B^{\sigma}_{2,1}} \) for some \( \sigma < \frac{1}{2} \). To do this, we first get by applying \( \hat{\Delta}_j \) to the equations in (1.3) that
\[
\begin{align*}
\partial_t \hat{\Delta}_j a - \Delta \hat{\Delta}_j a + \hat{\Delta}_j \text{div} u &= -\Delta_j \text{div}(au), \\
\partial_t \hat{\Delta}_j u - \Delta \hat{\Delta}_j u + \hat{\Delta}_j \nabla a &= 2\Delta_j (\nabla (\ln (1+a))) \nabla u - \Delta_j (k(a) \nabla a) - \Delta_j (u \cdot \nabla u).
\end{align*}
\]  
(6.4)

Taking \( L^2 \) inner product with \( \hat{\Delta}_j a, \hat{\Delta}_j u \) respectively, we can get by a standard energy argument that
\[
\begin{align*}
\| (a, u) \|_{B^{\sigma}_{2,1}} + \int_0^t \| (a, u) \|_{B^{\sigma+2}_{2,1}} \, d\tau \\
\leq \| (a_0, u_0) \|_{B^{\sigma}_{2,1}} + \int_0^t \| u \cdot \nabla u \|_{B^{\sigma+2}_{2,1}} \, d\tau + \int_0^t \| \text{div}(au) \|_{B^{\sigma+2}_{2,1}} \, d\tau \\
+ \int_0^t \| k(a) \nabla a \|_{B^{\sigma+2}_{2,1}} \, d\tau + \int_0^t \| \nabla (\ln (1+a)) \nabla u \|_{B^{\sigma+2}_{2,1}} \, d\tau.
\end{align*}
\]  
(6.5)

For any \( -\frac{3}{p} < \sigma \leq \frac{3}{p} \), we deduce from Lemma 2.8 that
\[
\begin{align*}
\| u \cdot \nabla u \|_{B^{\sigma}_{2,1}} + \| \text{div}(au) \|_{B^{\sigma}_{2,1}} \leq &\left(\| \nabla a, \nabla u \|_{B^{\sigma}_{2,1}} \right) \| (a, u) \|_{B^{\sigma}_{2,1}} \\
\leq &\left(\| (a, u) \|_{B^{3}_{p,1}} + \| u \|_{B^{3}_{p,1}} \right) \| (a, u) \|_{B^{\sigma}_{2,1}} \\
\leq &\left(\| (a, u) \|_{B^{3}_{p,1}} + \| a \|_{B^{3+2}_{p,1}} + \| u \|_{B^{3+2}_{p,1}} \right) \| (a, u) \|_{B^{\sigma}_{2,1}}.
\end{align*}
\]  
(6.6)
and
\[ \|k(a)\nabla a\|_{\dot{B}^\sigma_{2,1}} + \|\nabla(\ln(1 + a))\nabla u\|_{\dot{B}^\sigma_{2,1}} \lesssim \|k(a)\|_{\dot{B}^\sigma_{p,1}}^\frac{1}{2} \|\nabla a\|_{\dot{B}^\sigma_{2,1}} + \|\nabla(\ln(1 + a))\|_{\dot{B}^\sigma_{p,1}}^{\frac{1}{2}} \|\nabla u\|_{\dot{B}^\sigma_{2,1}} \]
\[ \lesssim (\|a\|_{\dot{B}^\sigma_{p,1}}^\frac{1}{2} + \|a\|_{\dot{B}^\sigma_{p,1}^{\frac{3}{2}}}) \|(a, u)\|_{\dot{B}^\sigma_{2,1}^{\frac{5}{2}}} \]
\[ \lesssim \frac{1}{2} (\|a, u\|_{\dot{B}^\sigma_{2,1}} + (\|a\|_{\dot{B}^\sigma_{p,1}}^2 + \|a\|_{\dot{B}^\sigma_{p,1}^{3}}}^\frac{1}{2} + \|a\|_{\dot{B}^\sigma_{p,1}^{\frac{3}{2}}}) \|(a, u)\|_{\dot{B}^\sigma_{2,1}} \]
\[ \lesssim \frac{1}{2} (\|a, u\|_{\dot{B}^\sigma_{2,1}} + (\|a\|_{\dot{B}^\sigma_{p,1}}^\frac{1}{2} + \|a\|_{\dot{B}^\sigma_{p,1}^{\frac{3}{2}}}) \|(a, u)\|_{\dot{B}^\sigma_{2,1}}.) \quad (6.7) \]

Plugging the above two estimates into (6.5) yields
\[ \|(a, u)\|_{\dot{B}^\sigma_{2,1}} + \frac{1}{2} \int_0^t \|(a, u)\|_{\dot{B}^\sigma_{2,1}^{\frac{5}{2}}} d\tau \lesssim \|(a_0, u_0)\|_{\dot{B}^\sigma_{2,1}} + \int_0^t G(\tau) \|(a, u)\|_{\dot{B}^\sigma_{2,1}} d\tau, \quad (6.8) \]
with
\[ G(\tau) \overset{\text{def}}{=} \|(a, u)\|_{\dot{B}^\sigma_{2,1}} + \|a\|_{\dot{B}^\sigma_{p,1}^{\frac{3}{2}}}^\frac{1}{2} + \|u\|_{\dot{B}^\sigma_{p,1}^{3}}}^\frac{1}{2} + (\|a\|_{\dot{B}^\sigma_{p,1}}^\frac{1}{2} + \|a\|_{\dot{B}^\sigma_{p,1}^{\frac{3}{2}}}) \|(a, u)\|_{\dot{B}^\sigma_{2,1}}.) \]

By virtue of Theorem 6.1, one deduces that
\[ \int_0^t G(\tau) d\tau \leq CA_0^2, \]
from which and the Gronwall inequality applied to (6.8), we have
\[ \|(a, u)\|_{\dot{B}^\sigma_{2,1}} \leq C, \quad -\frac{3}{p} < \sigma \leq \frac{3}{p}, \quad (6.9) \]
with C is a constant depending on p, \( \|(a_0, u_0)\|_{\dot{B}^\sigma_{2,1}} \) and \( \Lambda_0, \)

For any \( \sigma < \frac{1}{2}, \) it follows from interpolation inequality that
\[ \|(a, u)\|_{\dot{B}^\sigma_{2,1}} \leq C(\|(a, u)\|_{\dot{B}^\sigma_{2,1}}^\frac{1}{2} \|(a, u)\|_{\dot{B}^\sigma_{2,1}}^\frac{1}{2})^{1-\theta_1} \]
\[ \leq C(\|(a, u)\|_{\dot{B}^\sigma_{2,1}})^{\theta_1} (\|(a, u)\|_{\dot{B}^\sigma_{2,1}})\}
\[ \theta_1 = \frac{4}{5} - \frac{2}{\sigma} \in (0, 1), \]

this together with (6.9) implies that
\[ \|(a, u)\|_{\dot{B}^\sigma_{2,1}} \geq C(\|(a, u)\|_{\dot{B}^\sigma_{2,1}})\}
\[ (6.10) \]
Due to
\[ \|u^h\|_{L^\infty(B_{p,1}^{\frac{3}{2}-1})} + \|a^h\|_{L^\infty(B_{p,1}^{\frac{3}{2}})} \leq C \Lambda_0 \ll 1, \quad \text{for all} \ t \geq 0, \]
hence,
\[ (\|u^h\|_{B_{p,1}^{\frac{3}{2}-1}})^\frac{1}{\sigma - 1} \leq (\|u^h\|_{B_{p,1}^{\frac{3}{2}}})^\frac{1}{\sigma - 1} \leq \|u^h\|_{B_{p,1}^{\frac{3}{2}}} \]
\[ \leq (\|u^h\|_{B_{p,1}^{\frac{3}{2}}}^\frac{1}{\sigma - 1} \|u^h\|_{B_{p,1}^{\frac{3}{2}}}^\frac{1}{\sigma - 1} \leq \|u^h\|_{B_{p,1}^{\frac{3}{2}+1}}, \]
\[ (\|a^h\|_{B_{p,1}^{\frac{3}{2}}}^\frac{1}{\sigma - 1} \leq (\|a^h\|_{B_{p,1}^{\frac{3}{2}}}^\frac{1}{\sigma - 1} \|a^h\|_{B_{p,1}^{\frac{3}{2}}}^\frac{1}{\sigma - 1} \leq (\|a^h\|_{B_{p,1}^{\frac{3}{2}}}^\frac{1}{\sigma - 1} \|a^h\|_{B_{p,1}^{\frac{3}{2}+2}} \leq \|a^h\|_{B_{p,1}^{\frac{3}{2}+2}}. \quad (6.11) \]
Thus, inserting (6.10) and (6.11) into (6.3) yields
\[
\frac{d}{dt}\left\| (a, u) \right\|_{B_{2,1}^\gamma} + \left\| u \right\|_{\dot{B}_{2,1}^\gamma} + \left\| a \right\|_{\dot{B}_{2,1}^\gamma} + \tilde{c}\left(\left\| (a, u) \right\|_{B_{2,1}^\gamma} + \left\| u \right\|_{\dot{B}_{6,1}^\gamma} + \left\| a \right\|_{\dot{B}_{6,1}^\gamma}\right)^{\frac{5-2\sigma}{1-2\sigma}} \leq 0.
\]

Solving this differential inequality directly, we obtain
\[
\left\| (a, u) \right\|_{B_{2,1}^\gamma} + \left\| u \right\|_{\dot{B}_{2,1}^\gamma} + \left\| a \right\|_{\dot{B}_{2,1}^\gamma} \leq C \left( \lambda_0^{-\frac{4}{5-2\sigma}} + \frac{4\tilde{c}}{1-2\sigma} t \right)^{-\frac{1-2\sigma}{2}}.
\]

Moreover, from Lemma 2.3, we further get
\[
\left\| (a, u) \right\|_{B_{p,1}^{\frac{\gamma}{p}}} \leq C \left\| (a, u) \right\|_{B_{p,1}^{\frac{\gamma}{p}}}^{\frac{2}{3} - \frac{\gamma}{p}} \leq C(1 + t)^{-\frac{1-2\sigma}{2}}.
\]

For any \(\frac{3}{p} - \frac{3}{2} + \sigma < \gamma < \frac{3}{p} - 1\), by the interpolation inequality we have
\[
\left\| (a, u) \right\|_{B_{p,1}^{\gamma}} \leq C \left\| (a, u) \right\|_{B_{p,1}^{\gamma}}^{\frac{2}{3} - \frac{\gamma}{p}} \left\| (a, u) \right\|_{B_{p,1}^{\gamma}}^{1-\theta_2}, \quad \theta_2 = \frac{3}{p} - 1 - \gamma \in (0, 1),
\]

which combines (6.9) with (6.12) gives
\[
\left\| (a, u) \right\|_{B_{p,1}^{\gamma}} \leq C(1 + t)^{-\frac{1-2\sigma}{2}} = C(1 + t)^{-\frac{2}{3} (\frac{2}{3} - \frac{1}{p}) - \frac{2\sigma}{3}}.
\]

In the light of \(\frac{3}{p} - \frac{3}{2} + \sigma < \gamma < \frac{3}{p} - 1\), we see that
\[
\left\| (a^h, u^h) \right\|_{B_{p,1}^{\gamma}} \leq C \left\| u^h \right\|_{\dot{B}_{p,1}^{\gamma}} + \left\| a^h \right\|_{\dot{B}_{p,1}^{\gamma}} \leq C(1 + t)^{-\frac{1-2\sigma}{2}},
\]

from which and (6.13) gives
\[
\left\| (a, u) \right\|_{B_{p,1}^{\gamma}} \leq C \left\| (a, u) \right\|_{B_{p,1}^{\gamma}} + \left\| (a, u) \right\|_{B_{p,1}^{\gamma}} \leq C(1 + t)^{-\frac{1-2\sigma}{2}}.
\]

Thanks to the embedding relation \(B_{p,1}^{\theta_3}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)\), one infer that
\[
\left\| \Lambda_\gamma(a, u) \right\|_{L^p} \leq C(1 + t)^{-\frac{2}{3} (\frac{2}{3} - \frac{1}{p}) - \frac{2\sigma}{3}}.
\]

For any \(p \leq q \leq \infty\) and \(\frac{3}{q} - \frac{3}{2} + \sigma < \beta \leq \frac{3}{q} - 1\), by the Gagliardo-Nirenberg type interpolation inequality, which can be found in the Chap. 2 of [7], taking
\[
k\theta_3 + m(1 - \theta_3) = \beta + 3 \left( \frac{1}{p} - \frac{1}{q} \right), \quad m = \frac{3}{p} - 1,
\]

we get
\[
\left\| \Lambda^\beta(a, u) \right\|_{L^q} \leq C \left\| \Lambda^m(a, u) \right\|_{L^p}^{\frac{1-\theta_3}{2}} \left\| \Lambda^k(a, u) \right\|_{L^p}^{\theta_3} \leq C \left\{ (1 + t)^{-\frac{2}{3} (\frac{2}{3} - \frac{1}{p}) - \frac{m\sigma}{3}} \right\}^{1-\theta_3} \left\{ (1 + t)^{-\frac{2}{3} (\frac{2}{3} - \frac{1}{p}) - \frac{m\sigma}{3}} \right\}^{\theta_3} = C(1 + t)^{-\frac{2}{3} (\frac{2}{3} - \frac{1}{p}) - \frac{2\sigma}{3}}.
\]
Consequently, we have completed the proof of our theorem.

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REFERENCES

[1] P. Antonelli, L. E. Hientzsch and S. Spirito, Global existence of finite energy weak solutions to the Quantum Navier-Stokes equations with non-trivial far-field behavior, arXiv:2001.01652.
[2] P. Antonelli and P. Marcati, On the finite energy weak solutions to a system in quantum fluid dynamics, Commun. Math. Phys., 287 (2009), 657–686.
[3] P. Antonelli and P. Marcati, The quantum hydrodynamics system in two space dimensions, Arch. Rational Mech. Anal., 203 (2012), 499–527.
[4] P. Antonelli and S. Spirito, Global existence of finite energy weak solutions of the quantum Navier-Stokes equations, Arch. Rational Mech. Anal., 255 (2017), 1161–1199.
[5] P. Antonelli and S. Spirito, On the compactness of weak solutions to the Navier-Stokes-Korteweg equations for capillary fluids, Nonlinear Anal., 187 (2019), 110–124.
[6] C. Audiard and B. Haspot, Global well-posedness of the Euler Korteweg system for small irrotational data, Commun. Math. Phys., 351 (2017), 201–247.
[7] H. Bahouri, J.-Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehren Math. Wiss., vol. 343, Springer-Verlag, Berlin, Heidelberg, 2011.
[8] D. Bresch, B. Desjardins and C.-K. Lin, On some compressible fluid models: Korteweg, lubrication, and shallow water systems, Commun. Part. Diffe. Equ., 28 (2003), 843–868.
[9] D. Bresch, M. Gisclon and I. Lacroix-Violet, On Navier-Stokes-Korteweg and Euler-Korteweg systems: Application to quantum fluids models, Arch. Rational Mech. Anal., 233 (2019), 975–1025.
[10] F. Charve and R. Danchin, A global existence result for the compressible Navier-Stokes equations in the critical $L^p$ framework, Arch. Rational Mech. Anal., 198 (2010), 233–271.
[11] F. Charve, R. Danchin and J. Xu, Gevrey analyticity and decay for the compressible Navier-Stokes system with capillarity, arXiv:1805.01764.
[12] J.-Y. Chemin and I. Gallagher, Wellposedness and stability results for the Navier-Stokes equations in $R^3$, Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), 599–624.
[13] J.-Y. Chemin and N. Lerner, Flot de champs de vecteurs no lipschitziens et équations de Navier-Stokes, J. Differential Equations, 121 (1995), 314–328.
[14] Q. Chen, C. Miao and Z. Zhang, Global well-posedness for compressible Navier-Stokes equations with highly oscillating initial velocity, Comm. Pure Appl. Math., 63 (2010), 1173–1224.
[15] Z.-M. Chen and X. Zhai, Global large solutions and incompressible limit for the compressible Navier-Stokes equations, J. Math. Fluid Mech., 21 (2019), Art. 26, 23 pp.
[16] R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations, Invent. Math., 141 (2000), 579–614.
[17] R. Danchin and B. Desjardins, Existence of solutions for compressible fluid models of Korteweg type, Annales de l'IHP, Analyse nonlinéaire, 18 (2001), 97–133.
[18] R. Danchin and L. He, The incompressible limit in $L^p$ type critical spaces, Math. Ann., 366 (2016), 1365–1402.
[19] R. Danchin and P. B. Mucha, Compressible Navier-Stokes system: large solutions and incompressible limit, Adv. Math., 320 (2017), 904–925.
[20] R. Danchin and J. Xu, Optimal time-decay estimates for the compressible Navier-Stokes equations in the critical $L^p$ framework, Arch. Rational Mech. Anal., 224 (2017), 53–90.
[21] D. Donatelli, E. Feireisl and P. Marcati, Well/ill posedness for the Euler-Korteweg-Poisson system and related problems, Comm. Part. Diffe. Equa., 40 (2015), 1314–1335.
[22] J. E. Dunn and J. Serrin, On the thermomechanics of interstitial working, Arch. Rational Mech. Anal., 88 (1985), 95–133.
[23] E. Feireisl, Dynamics of Viscous Compressible Fluids. Oxford Univ. Press, Oxford, 2004.
[24] E. Feireisl and A. Novotný, H. Petzeltová, On the global existence of globally defined weak solutions to the Navier-Stokes equations of isentropic compressible fluids, J. Math. Fluid Mech., 3 (2001), 358–392.

[25] E. Feireisl, Compressible Navier-Stokes equations with a non-monotone pressure law, J. Differential Equations, 184 (2002), 97–108.

[26] E. Feireisl, A. Novotný and Y. Sun, Suitable weak solutions to the Navier-Stokes equations of compressible viscous fluids, Indiana Univ. Math. J., 60 (2011), 611–631.

[27] A. N. Gorban and I. V. Karlin, Beyond Navier-Stokes equations: Capillarity of ideal gas, Contemporary physics, 58 (2017), 70–90.

[28] G. Gui and P. Zhang, Stability to the global solutions of 3-D Navier-Stokes equations, Adv. Math., 225 (2010), 1248–1284.

[29] B. Haspot, Existence of global weak solution for compressible fluid models of Korteweg type, J. Math. Fluid Mech., 13 (2011), 223–249.

[30] B. Haspot, Well-posedness in critical spaces for the system of compressible Navier-Stokes in larger spaces, J. Differential Equations, 251 (2011), 2262–2295.

[31] B. Haspot, Existence of global strong solution for Korteweg system with large infinite energy initial data, J. Math. Anal. Appl., 438 (2016), 395–443.

[32] B. Haspot, Global strong solution for the Korteweg system with quantum pressure in dimension $N \geq 2$, Math. Ann., 367 (2017), 667–700.

[33] H. Hattori and D. Li, The existence of global solutions to a fluid dynamic model for materials for Korteweg type, J. Partial Differential Equations, 9 (1996), 323–342.

[34] L. He, J. Huang and C. Wang, Global stability of large solutions to the 3D compressible Navier-Stokes equations, Arch. Rational Mech. Anal., 234 (2016), 1167–1222.

[35] M. Heida and J. Málek, On compressible Korteweg fluid-like materials, Internat. J. Engrg. Sci., 48 (2010), 1313–1324.

[36] A. Jüngel, Global weak solutions to compressible Navier-Stokes equations for quantum fluids, SIAM J. Math. Anal., 42 (2010), 1025–1045.

[37] M. Kawashita, On global solution of Cauchy problems for compressible Navier-Stokes equations, Nonlinear Anal., 48 (2002), 1087–1105.

[38] D. J. Korteweg, Sur la forme que prennent les équations du mouvement des fluides si l’on tient compte des forces capillaires par des variations de densité, Arch. Néer. Sci. Exactes Sér., 6 (1901), 1–24.

[39] M. Kotschote, Strong solutions for a compressible fluid model of Korteweg type, Annales de l'IHP, Analyse non linéaire, 25 (2008), 679–696.

[40] H.-K. Li and T. Zhang, Large time behavior of isentropic compressible Navier-Stokes system in $\mathbb{R}^3$, Math. Methods Appl. Sci., 34 (2011), 670–682.

[41] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ., 20 (1980), 67–104.

[42] M. Murata and Y. Shibata, The global well-posedness for the compressible fluid model of Korteweg type, arXiv:1908.07224.

[43] M. Okita, Optimal decay rate for strong solutions in critical spaces to the compressible Navier-Stokes equations, J. Differential Equations, 257 (2014), 3850–3867.

[44] M. Paicu and P. Zhang, Global solutions to the 3-D incompressible anisotropic Navier-Stokes system in the critical spaces, Comm. Math. Phys., 307 (2011), 713–759.

[45] M. Paicu and P. Zhang, Global solutions to the 3-D incompressible inhomogeneous Navier-Stokes system, J. Funct. Anal., 262 (2012), 3556–3584.

[46] G. Ponce, Global existence of small solution to a class of nonlinear evolution equations, Nonlinear Anal. TMA., 9 (1985), 339–418.

[47] K. Takayuki and T. Kazuyuki, Global existence and time decay estimate of solutions to the compressible Navier-Stokes-Korteweg system under critical condition, Asympt. Anal., (2020), Publishing.

[48] J. F. Van der Waals, Thermodynamische Theorie der Kapillarität unter Voraussetzung stetiger Dichteänderung, Phys. Chem., 13 (1894), 657–725.

[49] K. Watanabe, Global existence of the Navier-Stokes-Korteweg equations with a non-decreasing pressure in $L^p$-framework, arXiv:1907.07752.

[50] Z. Xin and J. Xu, Optimal decay for the compressible Navier-Stokes equations without additional smallness assumptions, arXiv:1812.11714v2.

[51] H. Xu, Y. Li and F. Chen, Global solution to the incompressible inhomogeneous Navier-Stokes equations with some large initial data, J. Math. Fluid Mech., 19 (2017), 315–328.
[52] X. Zhai, Y. Li and F. Zhou, Global large solutions to the three dimensional compressible Navier-Stokes equations, *SIAM J. Math. Anal.*, **52** (2020), 1806–1843.

[53] S. Zhang, A class of global large solutions to the compressible Navier-Stokes-Korteweg system in critical Besov spaces, *J. Evol. Equ.* (2020).

[54] T. Zhang, Global wellposedness problem for the 3-D incompressible anisotropic Navier-Stokes equations in an anisotropic space, *Comm. Math. Phys.*, **287** (2009), 211–224.

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