Rank 2 preservers on symmetric matrices with zero trace

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Abstract. Let $F$ be a field, $V_1$ and $V_2$ be vector spaces of matrices over $F$ and let $\rho$ be the rank function. If $T: V_1 \rightarrow V_2$ is a linear map, and $k$ a fixed positive integer, we say that $T$ is a rank $k$ preserver if for any matrix $A \in V_1$, $\rho(A) = k$ implies $\rho(T(A)) = k$. In this paper, we characterize those rank 2 preservers on symmetric matrices with zero trace under certain conditions.

1 Introduction

Let $F^{nn}$ be the algebra of all $n \times n$ matrices over a field $F$. Let $sl_n(F)$ denote the subspace of $F^{nn}$ consisting of all matrices with zero trace. In [1], Botta, Pierce and Watkins obtained a useful result concerning the structure of nonsingular linear mapping on $sl_n(F)$ that preserve nilpotent matrices where $F$ is infinite. In [2], Li and Pierce characterized linear mappings on $sl_n(F)$ that preserve nonzero nilpotent matrices with rank at most $k$ where $k$ is a fixed positive integer less than $n$ and $F$ is algebraically closed of characteristic zero. Then, Watkins characterized linear mappings from $sl_n(F)$ to $F^{nn}$ that preserve rank one matrices where $F$ is an algebraically closed field of characteristic not equal to 2. He applied this result to determine the structure of bilinear mappings on $F^{nn}$ that have certain rank-preserving properties in [3] and [4] respectively.

Let $S_n(F)$ be the vector space of all $n \times n$ symmetric matrices over $F$ and $Z_0(S_n(F))$ be its subspace consisting of all symmetric matrices with zero trace. Let $n \geq 4$ and $F$ be a field of characteristic greater than 3. Motivated by work of Lim [5] in the characterization of linear rank one preservers on matrices with zero trace, we characterize those rank 2 preservers on symmetric matrices with zero trace under certain conditions in this paper and will discuss some consequences of this characterization in our next paper.

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2 Some definitions and preliminary results

Let $U$ be a vector space over $F$. We use tensor language in our investigation. This provides us with a larger context. We denote by $U^{(2)}$ the second symmetric product space over $U$ and denoted by $x \cdot y$, $x, y \in U$, the decomposable elements of $U^{(2)}$. For each $u$ in $U$, let $u^2$ denote $u \cdot u$.

A scalar product on $U$ is a function which assigns a scalar $(x, y) \in F$ to each ordered pair of vectors $x, y \in U$ such that for any $x, y, z \in U$ and any $c \in F$

(i) $(x + y, z) = (x, z) + (y, z)$
(ii) $(cx, y) = c(x, y)$
(iii) $(x, y) = (y, x)$

We say $x$ is orthogonal to $y$ or $x$ and $y$ are orthogonal if $(x, y) = 0$. Let $S$ be a set of vectors in $U$. Then $S$ is called an orthogonal set if $(x, y) = 0$ for all $x, y \in S, x \neq y$. If in addition, $(x, x) = 1$ for every $x \in S$, then $S$ is called an orthonormal set.

Now we let $U$ be equipped with a scalar product $(,): U \times U \to F$ and $U$ has an orthonormal basis $E$. Let $Z_0(U^{(2)})$ be the subset of $U^{(2)}$ that consists of all vectors of the form

$$\sum_{1 \leq i \leq j \leq n} a_{ij} u_i \cdot u_j$$

where $\{u_1, \ldots, u_n\}$ is an arbitrary finite subset of $E$ and $a_{ij}$ ($1 \leq i \leq j \leq n$) are arbitrary scalars in $F$ such that $\sum_{i=1}^{n} a_{ii} = 0$. Clearly $Z_0(U^{(2)})$ is a subspace of $U^{(2)}$ and we call $Z_0(U^{(2)})$ the space of traceless 2nd order symmetric tensors over $U$.

**Proposition 2.1** If $\{e_i : i \in A\}$ where $A \supseteq \{1, 2\}$ is an orthonormal basis for $U$, then $B = \{e_i \cdot e_j, e_i^2 - e_j^2 : i \neq j, k \neq 1 \text{ and } 1, i, j, k \in A\}$ is a basis for $Z_0(U^{(2)})$.

**Proof.**
Clearly $B$ is a linearly independent set. Hence it is sufficient to show that $B$ spans $Z_0(U^{(2)})$.

Let $x \in Z_0(U^{(2)})$. Then $x = \sum_{1 \leq i \leq j \leq n} a_{ij} u_i \cdot u_j$ and $\sum_{i=1}^{n} a_{ii} = 0$ where $\{u_1, \ldots, u_n\}$ is a finite subset of $\{e_i : i \in A\}$ and $a_{ij}$ ($1 \leq i \leq j \leq n$) are scalars in $F$. It follows that

$$x = \sum_{1 \leq i \leq j \leq n} a_{ij} u_i \cdot u_j - \sum_{k=1}^{n} a_{kk} (e_i^2 - u_k^2).$$

Therefore, $B$ spans $Z_0(U^{(2)})$.  

Let $Z_0(S_n(F))$ denote the subspace of $S_n(F)$ such that for any $A \in Z_0(S_n(F))$, $tr(A) = 0$. If $U$ is a finite dimensional vector space with an orthonormal basis $\{e_i : i = 1, \ldots, n\}$, then $Z_0(U^{(2)})$ is isomorphic in a natural way to $Z_0(S_n(F))$ by the restricted
isomorphism $\varphi \big|_{Z_0(U^{(2)})}$ where $\varphi$ is the isomorphism from $U^{(2)}$ to $S_n(F)$ defined by $\varphi(e_i \cdot e_j) = E_{ij} + E_{ji}, 1 \leq i \leq j \leq n$.

**Remark.** If $U$ is a Euclidean space, then there does not exist any rank 1 vector in $Z_0(U^{(2)})$.

Let $J_k$ denote the set of vectors in $U^{(2)}$ of the form $\sum_{i=1}^k \lambda_i x_i^2$, where $x_1, \ldots, x_k$ are linearly independent vectors and $\lambda_1, \ldots, \lambda_k \in F \setminus \{0\}$. For each vector $u \in U$, let $u \cdot U = \{u \cdot v : v \in U\}$.

**Lemma 2.2** Let $M$ be a subspace of $U^{(2)}$ such that $M \subseteq \{0\} \cup J_1 \cup J_2$. Then either

(i) $M \subseteq W^{(2)}$ for some subspace $W$ of $U$ that is 2 dimensional or

(ii) $M \subseteq u \cdot U$ for some $u \in U \setminus \{0\}$.

**Proof.**

If $M \cap J_2 = \emptyset$, then clearly $M = \{x^2\}$ for some $x$ in $U$. Let $A_i = au_i^2 + u_i^2 \in J_2 \cap M$.

Assume that $M \subseteq V^{(2)}$ where $V = \langle u_1, u_2 \rangle$. Then it is clear that there exists $A_2 = bu_1^2 + v^2 \in J_2 \cap M$ where $u_1, u_2, u_3$ are linearly independent. Clearly $v \in \langle u_1, u_2, u_3 \rangle$, otherwise $A_1 + A_2 \in J_1$, a contradiction. Let $v = cu_1 + du_2 + eu_3$ where $c, d, e \in F$.

Since for any $\lambda \in F$,

$$\lambda A_1 + A_2 = (\lambda a + c^2)u_1^2 + (\lambda + d^2)u_2^2 + (b + e^2)u_3^2 + 2cdu_1 \cdot u_2 + 2ceu_1 \cdot u_3 + 2deu_2 \cdot u_3 \in J_1 \cup J_2,$$

it follows that

$$\begin{vmatrix}
\lambda a + c^2 & cd & ce \\
\lambda + d^2 & de \\
ce & de & b + e^2
\end{vmatrix}
= a(b + e^2)\lambda^2 + b(ad^2 + c^2)\lambda$$

for any $\lambda \in F$. Since $|F| \geq 3$, we have

$$b + e^2 = 0$$

(1)

and

$$ad^2 + c^2 = 0.$$  

(2)

From (1) we have

$$A_2 = (cu_1 + du_2) \cdot (cu_1 + du_2 + 2eu_3).$$

(3)

From (2) we have $ad^{-1} + d^{-1}c = 0$ and hence

$$A_1 = (cu_1 + du_2) \cdot (ac^{-1}u_1 + d^{-1}u_2).$$

(4)

From (3) and (4) we have $A_1 = u \cdot y_1$, $A_2 = u \cdot y_2$ where $u, y_1, y_2$ are linearly independent.

Suppose now $A = x^2 \in J_1 \cap M$. Then $A + A_1, A + A_2 \in J_1 \cap J_2$ imply that $x \in \langle u, y_1 \rangle \cap \langle u, y_2 \rangle = \langle u \rangle$ and hence $A \in u \cdot U$.

Suppose that $B = \lambda_1 v_1^2 + \lambda_2 v_2^2 \in J_2 \cap M$. For each $\lambda \in F$, let $C_\lambda = u \cdot z_\lambda$, where $z_\lambda = y_1 + \lambda y_2$. Then $C_\lambda \in M$. Clearly there exists a subset $D$ of $F$ with 2 elements such
that \( \langle v_1, v_2 \rangle \neq \langle u, z_\lambda \rangle \) for all \( \lambda \in D \). Hence \( \dim \langle v_1, v_2, u, z_\lambda \rangle \geq 3 \) for all \( \lambda \in D \). By our previous argument, for any \( \lambda \in D \), there exists \( w_\lambda \in U \) such that
\[
B, C_\lambda \in w_\lambda \cdot U.
\]

Suppose \( B \neq u \cdot U \). Then
\[
B = \alpha z_i \cdot z_j
\]
for some \( \alpha \in F \setminus \{0\} \) where \( i \) and \( j \) are distinct element in \( D \). Let
\[
H = u \cdot z_k
\]
where \( k \in F \setminus \{i, j\} \). Since \( \dim \langle u, z_i, z_j, z_k \rangle = 3 \),
\[
B, H \in v \cdot U
\]
for some \( v \in U \) by previous argument. This yields a contradiction since \( B = \alpha z_i \cdot z_j \) and \( H = u \cdot z_k \) do not have a common factor. Therefore \( B = u \cdot U \).

\[
\Box
\]

3 Rank 2 preservers

Let \( U \) and \( W \) be vector spaces over \( F \). We always assume that \( U \) has an orthonormal basis, \( \{ e_i \in A \} \), with respect to a scalar product \( (,) : U \times U \rightarrow F \), where \( A \supseteq \{1,2,\ldots,n\} \) if \( A \) has at least \( n \) elements. For each vector \( u \in U \), let \( \langle u \rangle^\perp = \{ v \in U : (v, u) = 0 \} \).

**Lemma 3.1** Let \( T : Z_0(U^{(2)}) \rightarrow W^{(2)} \) be a rank 2 preserver. If \( V \) is a subspace of \( U \) such that \( V \subseteq \langle u \rangle^\perp \) for some \( u \in U \setminus V \), then \( \dim T(u \cdot V) = \dim u \cdot V \). Moreover, if \( \dim V \geq 3 \), then \( T(u \cdot V) \subseteq w \cdot W \) for some \( w \in W \setminus \{0\} \).

**Proof.**
Suppose \( T(u \cdot v_1) = T(u \cdot v_2) \) for some \( v_1, v_2 \in V \). Then \( T(u \cdot (v_1 - v_2)) = 0 \) and this implies that \( v_1 = v_2 \), since \( T \) is a rank 2 preserver. Hence \( \dim T(u \cdot V) = \dim u \cdot V \). If \( \dim V \geq 3 \), then \( \dim(T(u \cdot V)) \geq 3 \). Since \( T(u \cdot V) \) is a subspace of \( W^{(2)} \) contained in \( J_2 \cup \{0\} \), it follows by Lemma 2.2 that \( T(u \cdot V) \subseteq w \cdot W \) for some \( w \in W \setminus \{0\} \).

\[
\Box
\]

**Theorem 3.2** Let \( T \) be a rank 2 preserver from \( Z_0(U^{(2)}) \) to \( W^{(2)} \). If \( \dim U \geq 4 \), then one of the following holds:

(i) \( T = \lambda P_2(f) \big|_{Z_0(U^{(2)})} \) for some \( \lambda \in F \setminus \{0\} \) and some one-to-one linear mapping
\[
f : U \rightarrow W
\]
where \( P_2(f) \) is a second induced power of \( f \) such that
\[
P_2(f)(x \cdot y) = f(x) \cdot f(y);
\]
(ii) \( \text{Im} T \subseteq w \cdot W \) for some \( w \in W \setminus \{0\} \).
Proof.

Note that $\langle e_i \rangle \perp e_i \not\in \langle e_i \rangle$. In view of Lemma 3.1, $T(\langle e_i \cdot e_i \rangle) \subseteq w_i \cdot W$ for some $w_i \in W \setminus \{0\}$, $i \in A$. Now we have either $\{w_i : i \in A\}$ is a pairwise linearly independent set or $\langle w_i \rangle = \{w_i\}$ for some distinct $i, j$. We will consider these two cases separately.

**Case 1:** $\{w_i : i \in A\}$ is a pairwise linearly independent set.

Since $T(e_i \cdot e_2) \in w_1 \cdot W$, $T(e_2 \cdot e_i) \in w_2 \cdot W$ and $w_1, w_2$ are linearly independent, we have $T(e_1 \cdot e_2) = \alpha_{12} w_i \cdot w_j$ for some $\alpha_{12} \in F \setminus \{0\}$.

Likewise, $T(e_j \cdot e_i) = \alpha_{ij} w_i \cdot w_j$, (5)

where $\alpha_{ij} \in F \setminus \{0\}$ for all distinct $i, j$. Clearly

$\alpha_{ij} = \alpha_{ji}$. (6)

Now we claim that $\{w_i : i \in A\}$ is a linearly independent set. Suppose the contrary. Let $w_i = \sum_{i \in A \setminus \{1\}} a_i w_i$ for some $a_i \in F$. Then from (5), we have

$$T \left( e_2 \cdot \left( \frac{1}{\alpha_{12}} - \sum_{i \in A \setminus \{1\}} \frac{a_i}{\alpha_{12}} e_i \right) \right) = w_2 \cdot \left( w_i - \sum_{i \in A \setminus \{1\}} a_i w_i \right) = a_2 w_2^2$$

is of rank $\leq 1$, a contradiction. So, $\{w_i : i \in A\}$ is a linearly independent set. Let

$$M = \langle e_1 + e_2, e_3, e_4 \rangle.$$

Since $\langle e_1 + e_2, e_3, e_4 \rangle$ is a 3-dimensional subspace of $\langle e_i \rangle \perp$ and $e_1 + e_2 \not\in \langle e_1 + e_2, e_3, e_4 \rangle$, it follows by Lemma 3.1 that $\dim T(M) = 3$ and $T(M) \subseteq u \cdot W$ for some $u \in W \setminus \{0\}$.

Let

$$T((e_1 + e_2) \cdot (e_1 - e_2)) = u \cdot u_1, \quad T((e_1 + e_2) \cdot e_3) = u \cdot u_2, \quad T((e_1 + e_2) \cdot e_4) = u \cdot u_3,$$

where $\dim \langle u_1, u_2, u_3 \rangle = 3$. In view of (5), we have

$$T((e_1 + e_2) \cdot e_3) = T(e_1 \cdot e_3) + T(e_2 \cdot e_3) = (\alpha_{13} w_i + \alpha_{23} w_2) \cdot w_3,$$

$$T((e_1 + e_2) \cdot e_4) = T(e_1 \cdot e_4) + T(e_2 \cdot e_4) = (\alpha_{14} w_i + \alpha_{24} w_2) \cdot w_4.$$

Hence

$$u \cdot u_2 = (\alpha_{13} w_i + \alpha_{23} w_2) \cdot w_3,$$

$$u \cdot u_3 = (\alpha_{14} w_i + \alpha_{24} w_2) \cdot w_4.$$

Since $\dim \langle w_1, w_2, w_3, w_4 \rangle = 4$, we have $\langle u \rangle = \langle \alpha_{13} w_i + \alpha_{23} w_2 \rangle = \langle \alpha_{14} w_i + \alpha_{24} w_2 \rangle$. Hence

$$\frac{\alpha_{23}}{\alpha_{13}} = \frac{\alpha_{24}}{\alpha_{14}}.$$

Likewise
\[ \frac{\alpha_{ik}}{\alpha_{jk}} = \frac{\alpha_{il}}{\alpha_{jl}} \quad (7) \]

for all distinct \( i, j, k, l \). Now let

\[ N = \langle (e_1 - e_2) \cdot (e_1 + e_2), (e_1 - e_2) \cdot e_3, (e_1 - e_2) \cdot e_4 \rangle. \]

Since \( (e_1 + e_2, e_3, e_4) \) is a 3-dimensional subspace of \( (e_1 - e_2)^\perp \) and \( e_1 - e_2 \not\in (e_1 + e_2, e_3, e_4) \), it follows by Lemma 3.1 that \( \dim T(N) = 3 \) and \( T(N) \subseteq v \cdot W, v \in W \setminus \{0\} \). Let

\[ T((e_1 - e_2) \cdot (e_1 + e_2)) = v \cdot v_1, \]

\[ T((e_1 - e_2) \cdot e_3) = v \cdot v_2, \]

\[ T((e_1 - e_2) \cdot e_4) = v \cdot v_3. \]

where \( \dim \langle v_1, v_2, v_3 \rangle = 3 \). On the other hand, in view of (5)

\[ T((e_1 - e_2) \cdot e_3) = T(e_1 \cdot e_3) - T(e_2 \cdot e_3) = (\alpha_{13} w_1 - \alpha_{23} w_2) \cdot w_3, \]

\[ T((e_1 - e_2) \cdot e_4) = T(e_1 \cdot e_4) - T(e_2 \cdot e_4) = (\alpha_{14} w_1 - \alpha_{24} w_2) \cdot w_4. \]

Hence

\[ v \cdot v_2 = (\alpha_{13} w_1 - \alpha_{23} w_2) \cdot w_3, \]

\[ v \cdot v_3 = (\alpha_{14} w_1 - \alpha_{24} w_2) \cdot w_4. \]

Since \( \dim \langle w_1, w_2, w_3, w_4 \rangle = 4 \), we have

\[ \langle v \rangle = \langle \alpha_{13} w_1 - \alpha_{23} w_2 \rangle = \langle \alpha_{14} w_1 - \alpha_{24} w_2 \rangle. \]

Now we have

\[ T((e_1 + e_2) \cdot (e_1 - e_2)) \in (\alpha_{13} w_1 + \alpha_{23} w_2) \cdot W \]

and

\[ T((e_1 - e_2) \cdot (e_1 + e_2)) \in (\alpha_{13} w_1 - \alpha_{23} w_2) \cdot W. \]

Since \( \alpha_{13} w_1 + \alpha_{23} w_2 \) and \( \alpha_{13} w_1 - \alpha_{23} w_2 \) are linearly independent, we have

\[ T(e_1^2 - e_2^2) = T((e_1 + e_2) \cdot (e_1 - e_2)) = \lambda_{12}^{(3)} (\alpha_{13} w_1 + \alpha_{23} w_2) \cdot (\alpha_{13} w_1 - \alpha_{23} w_2) = \lambda_{12}^{(3)} (\alpha_{13}^2 w_1^2 - \alpha_{23}^2 w_2^2), \]

where \( \lambda_{12}^{(3)} \in F \setminus \{0\} \). Likewise

\[ T(e_1^2 - e_2^2) = \lambda_{ij}^{(k)} (\alpha_{ik}^2 w_i^2 - \alpha_{jk}^2 w_j^2) \quad (8) \]

where \( \lambda_{ij}^{(k)} \in F \setminus \{0\} \) for all distinct \( i, j, k \). As a consequence

\[ \lambda_{ij}^{(k)} \alpha_{ik}^2 = \lambda_{ij}^{(l)} \alpha_{jl}^2 \quad (9) \]

\[ \lambda_{ij}^{(k)} = \lambda_{ij}^{(l)} \quad (10) \]

for all distinct \( i, j, k, l \). Now let

\[ u_1 = e_1 + e_2 + e_3, \]

\[ u_2 = e_1 - e_2, \]

\[ u_3 = e_1 - e_3, \]

\[ u_4 = e_4. \]
Let $H = \langle u_1, u_2, u_3, u_4 \rangle$. Since $\langle u_2, u_3, u_4 \rangle$ is a 3-dimensional subspace of $\langle u_1 \rangle$ and $u_1 \not\in \langle u_2, u_3, u_4 \rangle$, it follows by Lemma 3.1 that $\dim(T(H)) = 3$ and $T(H) \subseteq v \cdot W$ for some \( v \in W \setminus \{0\} \). In view of (5), (7) and (8), we have

$$T(u_1 \cdot u_2) = T(e_1^2 - e_2^2 + e_1 \cdot e_3 - e_2 \cdot e_3)$$

$$= \lambda_{12}^{(3)} \left( \alpha_{13}^2 w_1^2 - \alpha_{23}^2 w_2^2 \right) + \alpha_{13} w_1 \cdot w_3 - \alpha_{23} w_2 \cdot w_3$$

$$= (\alpha_{13} w_1 - \alpha_{23} w_2) \cdot (\lambda_{12}^{(3)} \alpha_{13} w_1 + \lambda_{12}^{(3)} \alpha_{23} w_2 + w_3)$$

$$T(u_1 \cdot u_3) = T(e_1^2 - e_2^2 + e_1 \cdot e_3 - e_2 \cdot e_3)$$

$$= \lambda_{13}^{(3)} \left( \alpha_{13}^2 w_1^2 - \alpha_{23}^2 w_2^2 \right) + \alpha_{13} w_1 \cdot w_2 - \alpha_{23} w_2 \cdot w_3$$

$$= (\alpha_{13} w_1 - \alpha_{23} w_2) \cdot (\lambda_{13}^{(3)} \alpha_{13} w_1 + \lambda_{13}^{(3)} \alpha_{23} w_2 + w_3)$$

Since $(\alpha_{13} w_1 - \alpha_{23} w_2), (\alpha_{13} w_1 - \alpha_{23} w_2)$ and $(\lambda_{12}^{(3)} \alpha_{13} w_1 + \lambda_{13}^{(3)} \alpha_{23} w_2 + w_3)$ are pairwise linearly independent and $T(u_1 \cdot u_2), T(u_1 \cdot u_3)$ have a common factor, we have

$$\left\{ \lambda_{12}^{(3)} \alpha_{13} w_1 + \lambda_{12}^{(3)} \alpha_{23} w_2 + w_3 \right\} = \left\{ \lambda_{13}^{(3)} \alpha_{13} w_1 + \lambda_{13}^{(3)} \alpha_{23} w_2 + w_3 \right\}.$$

Hence

$$\frac{\lambda_{12}^{(3)} \alpha_{13}}{\lambda_{12}^{(3)} \alpha_{23}} = \frac{\lambda_{12}^{(3)} \alpha_{23}}{\alpha_{13} \alpha_{23}} = \frac{1}{\lambda_{12}^{(3)} \alpha_{23}}.$$

In view of (6), we have

$$\frac{1}{\lambda_{12}^{(3)}} = \lambda_{13}^{(2)} \alpha_{23} \alpha_{13} = \lambda_{13}^{(2)} \alpha_{13}^2,$$

$$\lambda_{13}^{(2)} = \frac{\alpha_{13}^2}{\alpha_{12} \alpha_{23}}.$$

Likewise,

$$\frac{1}{\lambda_{ji}^{(k)}} = \lambda_{ik}^{(j)} \alpha_{ij}^2$$

$$\lambda_{ij}^{(k)} = \frac{\alpha_{ij}}{\alpha_{ik} \alpha_{kj}}$$

for all distinct $i, j, k$. In view of (12) and (7), we have

$$\lambda_{ij}^{(k)} = \frac{1}{\alpha_{ik} \alpha_{kj}} \left( \frac{\alpha_{ij}}{\alpha_{kj}} \right) = \frac{1}{\alpha_{ik} \alpha_{kj}} \left( \frac{\alpha_{ik}}{\alpha_{ki}} \right) = \lambda_{ij}^{(k)}$$

for all distinct $i, j, k, l$. In view of (10) and (13), we find that

$$\lambda_{ij}^{(k)}$$

have a common value $\lambda^{(k)}$

for all $i, j \not\in \{k\}$. Now we define a linear mapping $f : U \to W$ by

$$f(e_i) = \begin{cases} \frac{1}{\lambda} w_3, & i = 3 \\ \alpha_{13} w_i, & i \neq 3 \end{cases}$$
where $\lambda = \lambda^{(3)}$. For any distinct elements $i, j, 3$ in $A$, in view of (5), (8), (9), (11), (12) and (14), we have

$$\lambda P_2(f)(e_i \cdot e_j) = \lambda^{(3)} f(e_i) \cdot f(e_j) = \lambda^{(3)} \alpha_i \alpha_j w_i \cdot w_j = \alpha_{ij} w_i \cdot w_j = T(e_i \cdot e_j),$$

$$\lambda P_2(f)(e_i \cdot e_j) = \lambda f(e_i) \cdot f(e_j) = \lambda (\alpha_i w_i) \cdot \left( \frac{1}{\lambda} w_j \right) = \alpha_{ij} w_i \cdot w_j = T(e_i \cdot e_j),$$

$$\lambda P_2(f)(e_i^2 - e_j^2) = \lambda^{(3)} f(e_i)^2 - \lambda^{(3)} f(e_j)^2 = \lambda^{(3)} \alpha_i^2 w_i^2 - \lambda^{(3)} \left( \frac{1}{\lambda^{(3)}} \right)^2 w_i^2 = \lambda^{(k)} \alpha_{ik}^2 w_i^2 - \frac{1}{\lambda^{(k)}} w_i^2 \text{ where } k \neq \{i, j, 3\}$$

$$= \lambda^{(k)} \alpha_{ik}^2 w_i^2 - \frac{1}{\lambda^{(k)}} w_i^2 = \lambda^{(k)} \alpha_{ik}^2 w_i^2 - \lambda^{(k)} \alpha_{ik}^2 w_i^2 = \lambda^{(k)} \alpha_{ik}^2 w_i^2 - \alpha_{ik}^2 w_i^2 = T(e_i^2 - e_j^2).$$

Therefore $T = \lambda P_2(f)|_{Z^{(3)}_{ij}(T^{(2)})}$ and $f$ is injective.

**Case 2:** $\langle w_i \rangle = \langle w_j \rangle$ for some distinct $i, j$.

Let $w = w_i$. Without loss of generality, we may assume that $w_i = w_2$. We first show that

$$T \left( e_i \cdot \langle e_i \rangle^{\perp} \right) \subseteq w \cdot W$$

for all $i \in A$.

If $w_3$ and $w$ are linearly independent, then

$$T(e_i \cdot e_i) = \alpha w \cdot w_3$$

for some $\alpha \in F \setminus \{0\}$. Similarly, $T(e_2 \cdot e_3) = \beta w \cdot w_3$ for some $\beta \in F \setminus \{0\}$. Hence

$$T(e_i \cdot (\beta e_i - \alpha e_i)) = 0.$$ 

This contradicts the hypothesis that $\rho(T(e_i \cdot (\beta e_i - \alpha e_i))) = 2$. Thus $\langle w_3 \rangle = \langle w \rangle$. Likewise, $\langle w_{ik} \rangle = \langle w \rangle$, for all $k \in A$. Therefore, we can conclude that
for all $i \in A$.

Now we show that

$$T\left(e_i \cdot \{e_i\}^\perp\right) \subseteq w \cdot W$$

for all distinct $i, j$.

Let

$$T(e_i \cdot e_i) = w \cdot z_i,$$
$$T(e_j \cdot e_j) = w \cdot z_j,$$
$$T(e_i \cdot e_j) = w \cdot z_3,$$
$$T(e_2 \cdot e_4) = w \cdot z_4,$$

where $z_1, z_2, z_3, z_4 \in W$. Let

$$M = \langle (e_1 + e_2) \cdot (e_1 - e_2), (e_1 + e_2) \cdot e_3, (e_1 + e_2) \cdot e_4 \rangle.$$ Then $\langle e_1 - e_2, e_3, e_4 \rangle$ is a three dimensional subspace of $\langle e_1 + e_2 \rangle^\perp$ and in view of Lemma 3.1, $\dim T(M) = 3$ and $T(M) \subseteq v \cdot W$ for some $v \in W$.

Let

$$T((e_1 + e_2) \cdot (e_1 - e_2)) = v \cdot v_1,$$
$$T((e_1 + e_2) \cdot e_3) = v \cdot v_2,$$
$$T((e_1 + e_2) \cdot e_4) = v \cdot v_3,$$

where $\dim \langle v_1, v_2, v_3 \rangle = 3$. On the other hand,

$$T((e_1 + e_2) \cdot e_3) = T(e_1 \cdot e_3) + T(e_2 \cdot e_3)$$
$$= (z_1 + z_2) \cdot w.$$
$$T((e_1 + e_2) \cdot e_4) = T(e_1 \cdot e_4) + T(e_2 \cdot e_4)$$
$$= (z_3 + z_4) \cdot w.$$

Since $\dim T(M) = 3$, $z_1 + z_2$ and $z_3 + z_4$ are linearly independent. Then by comparing the two expressions for $T((e_1 + e_2) \cdot e_3)$ and $T((e_1 + e_2) \cdot e_4)$, clearly $\langle w \rangle = \langle v \rangle$ and we obtain

$$T(e_i^2 - e_j^2) \in w \cdot W.$$

Likewise,

$$T(e_i^2 - e_j^2) \in w \cdot W$$

for all distinct $i, j$. Now we have proved that

$$T\left(e_i \cdot \{e_i\}^\perp\right) \subseteq w \cdot W$$

for all $i \in A$, and

$$T(e_i^2 - e_j^2) \in w \cdot W$$

for all distinct $i, j$. Therefore $\text{Im} T \subseteq w \cdot W$. 

**Remark.** We conjecture that Theorem 3.2 is also true when $\dim U = 3$. If $\dim U = 2$, Theorem 3.2 is no longer true.
**Example**  Let $U$ be a 2-dimensional Euclidean space with an orthonormal basis $\{e_1, e_2\}$. Let $T : Z_0(U^{(2)}) \to U^{(2)}$ be a linear mapping such that $T(e_1^2 - e_2^2) = e_1 \cdot e_2$ and $T(e_1 \cdot e_2) = e_1^2 - e_2^2$. Then clearly $T$ is a rank 2 preserver. However $T$ is neither of the form (i) nor the form (ii) in Theorem 3.2.

Now we state Theorem 3.2 in matrix language when the dimension of $U$ and $W$ are finite.

**Corollary 3.3** Let $n$ and $m$ be positive integers and let $L$ be a rank 2 preserver from $Z_0(S_n(F))$ to $S_m(F)$. If $n \geq 4$, then one of the following holds:

(i) There exists a rank $n \times n$ matrix $P$ such that $L(A) = \lambda PAP'$ for all $A \in Z_0(S_n(F))$ where $\lambda \in F \setminus \{0\}$;

(ii) There exists a nonsingular $m$-square matrix $Q$ such that

$$\text{Im } L \subseteq \left\{ Q \begin{pmatrix} c_1 & c_2 & \cdots & c_m \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ c_m & \cdots & 0 \end{pmatrix} Q^t : c_j \in F, j = 1, 2, \ldots, m \right\}.$$

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