Goldstone Modes in the Emergent Gauge Fields of a Frustrated Magnet

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We consider magnon excitations in the spin-glass phase of geometrically frustrated antiferromagnets with weak exchange disorder, focussing on the nearest-neighbour pyrochlore-lattice Heisenberg model at large spin. The low-energy degrees of freedom in this system are represented by three copies of a U(1) emergent gauge field, related by global spin-rotation symmetry. We show that the Goldstone modes associated with spin-glass order are excitations of these gauge fields, and that the standard theory of Goldstone modes in Heisenberg spin glasses (due to Halperin and Saslow) must be modified in this setting.

Gauge fields arise as low-energy degrees of freedom for frustrated magnets in a variety of contexts [1–4]. Their emergence is particularly transparent in the classical limit, where the systems of interest have macroscopically degenerate ground states and ground-state spin configurations can be mapped to configurations of a divergenceless vector field [2]. An important application of these ideas has been in research on spin-ice materials, represented by the Ising antiferromagnet on the pyrochlore lattice [5, 6]. In spin ice, the emergent gauge field is described by a U(1) theory, and magnetic monopole excitations act as its sources and sinks [7]. Extending the approach to n-component classical spins, a distinct flavour of U(1) field arises from each spin component: global spin rotations act as rotations between these flavours, and magnetisation density is a vector source for flux.

Such gauge fields may acquire dynamics by a number of different routes. Starting from a classical model, a natural step is to introduce quantum tunnelling between pairs of ground states that are related via rearrangement of small numbers of spins [8, 9]. In quantum versions of spin ice this leads to a theory of the standard form familiar from quantum electrodynamics [10, 11]. An alternative for Heisenberg models, however, is to build on the precessional dynamics of spins in the exchange fields arising from their neighbours. A treatment of the Heisenberg antiferromagnet on the pyrochlore lattice that combines precessional dynamics with a description in terms of emergent gauge fields is appropriate under two conditions: the system should be at low enough temperatures that it is close to its ground-state manifold, but not at such low temperature that quantum order-by-disorder [12] establishes the Néel state. Both conditions are satisfied in a window below the Curie-Weiss temperature that is wide at large spin. In this temperature range the gauge field dynamics arising from precession is overdamped, with a relaxation rate that is predicted [14] and observed [17] to be proportional to temperature.

Quenched exchange randomness offers a way to explore this physics further. It leaves the gauge fields as distinct degrees of freedom if its amplitude $\Delta$ is much less than the mean exchange $\bar{J}$, and it enables the system to evade order-by-disorder, instead stabilising the spin-glass state below a freezing temperature $T_F \sim \Delta$ [18–20]. The frozen state corresponds to a particular gauge field configuration (selected by the exchange randomness), which spontaneously breaks symmetry under global spin rotations. Consequently it offers a platform to understand the role of the global rotational symmetry in the gauge field dynamics. A key question is whether excitations in this state can be viewed both as excitations of the gauge degrees of freedom and also as Goldstone modes. In the following we establish a theoretical treatment of these modes, demonstrating how the two perspectives are consistent in the low-energy limit.

The theory of Goldstone modes in conventional spin glasses was established some time ago in work by Halperin and Saslow [21], and by Ginzburg [22]. Within their approach, the long-distance properties of the ordered state are characterised by the uniform magnetic susceptibility $\chi_0$ and the long-wavelength spin stiffness $\rho$. Modes of frequency $\omega$ and wavevector $k$ have a linear dispersion relation $\omega = ck$ with speed $c = \sqrt{\rho/\chi_0}$. For a spin glass having nearest-neighbour interactions with mean strength zero and variance $\bar{J}^2$, one has $\chi_0 \sim J^{-1}$ and $\rho \sim J a^2$, where $a$ is the lattice spacing. As a result, $c \sim J a$.

A direct attempt to extend the conventional theory to the spin-glass state in geometrically frustrated Heisenberg antiferromagnets with weak disorder suggests the result $c \sim a \sqrt{J \Delta}$. If applicable to the pyrochlore antiferromagnet, this form would imply that the modes mix gauge fields with high-energy degrees of freedom, since it combines $\Delta$ and $\bar{J}$. We show below that this is not in fact the behaviour. Instead the excitation speed is independent of the mean interaction strength $\bar{J}$, being $c \sim a \Delta$, as expected if the low-energy modes involve only the emergent gauge degrees of freedom.

A number of geometrically frustrated antiferromagnetic materials are well-described in a first approximation by the Heisenberg model. Many of them show spin freezing with a transition temperature much smaller than the dominant interaction scale (which is characterised by the Curie-Weiss constant) and this freezing is plausibly attributed to weak exchange disorder. A large magnetic heat capacity $C_M$ at low temperature is characteristic

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of these systems [23][26], suggesting soft, gapless modes. The Goldstone modes described by the theory we develop here give rise to a large value for $C_M$ because the excitation speed is small if exchange disorder is weak.

We study the classical Heisenberg model with Hamiltonian

$$\mathcal{H} = \sum_{(r,r')} J_{r,r'} S_r \cdot S_{r'} .$$

(1)

Here spins $S_r$ are three-component unit vectors, the sum is over nearest-neighbour pairs of sites $r, r'$ on the pyrochlore lattice, and $J_{r,r'} = J + \Delta \cdot R_r \cdot R_{r'}$, where $R_r \cdot R_{r'}$ is a Gaussian random variable with zero mean and unit variance. Our focus is on the weak-disorder limit $\Delta \ll J$.

In the absence of disorder ($\Delta = 0$) this model has a macroscopically degenerate ground state, with $N/2 + 3$ ground-state degrees of freedom for $N$ spins under periodic boundary conditions [14][15]. The ground states are ones in which each tetrahedron $\alpha$ of the pyrochlore lattice has total spin $L_\alpha \equiv \sum_{r \in \alpha} S_r = 0$, so that $\mathcal{H} = (J/2) \sum_{\alpha} |L_\alpha|^2 + \text{constant}$. These states can be represented as configurations of an emergent gauge field, as follows [2]. Noting that centres of tetrahedra lie on a lattice has total spin $L = \sum_{\alpha} L_\alpha \equiv \sum_{\alpha} \Theta(\ldots)$, where $\Theta$ is the step function. Our main results for behaviour in the regime $0 < \Delta \ll J$ are as follows. We show that: (i) $1/4$ of the Hessian eigenvalues $\lambda_n$ and $1/4$ of the magnon eigenfrequencies $\omega_n$ are $\mathcal{O}(\Delta)$, the remainder being $\mathcal{O}(J)$; (ii) The eigenvectors associated with low-lying Hessian eigenvalues and with low-frequency magnons are long-wavelength rotations of the minimum-energy spin configuration, involving only the emergent gauge-field degrees of freedom, with corrections that vanish as $\lambda_n$ or $\omega_n$ approach zero. (iii) In addition, we establish a continuum description of low-frequency magnons in terms of the local rotations of the spin configuration and local magnetisation density, which extends Halperin-Saslow theory to geometrically frustrated Heisenberg magnets with emergent gauge fields and weak exchange disorder. We first set out physical arguments leading to these conclusions and then present numerical evidence.

We begin by examining how accurately a smooth spin rotation can be represented by the gauge-field degrees of freedom. Using a continuum treatment, let the tensor field $B^{ai}(r)$ denote a ground state selected by disorder in the limit $\Delta/J \to 0$. Note that Eq. (2) implies $\partial_i B^{ai}(r) = 0$ and that divergenceful field configurations cost energy $\mathcal{O}(J)$. Rotations in spin-space can be described by an orthogonal matrix field $O^{ab}(r)$ that satisfies $O^{ab}(r)O^{bc}(r) = \delta^{ac}$ for all $r$. To ensure that a smoothly rotated configuration avoids an $\mathcal{O}(J)$ energy penalty, we write

$$B^{ai}(r) = O^{ab}(r)B^{bi}(r) + b^{ai}(r)$$

(5)

and choose $b^{ai}(r)$ so that $\partial_i B^{ai}(r) = 0$, implying

$$\partial_i b^{ai}(r) = -[\partial_i O^{ab}(r)]B^{bi}(r) \equiv \sigma^a(r).$$

(6)

We show below that $b^{ai}(r)$ is small for smooth rotations. Specifically, if $O^{ab}(r)$ varies on a scale $\ell$, then $|b^{ai}(r)|^2 \sim \ell^{-3}$ for large $\ell$ and corrections to a gradient expansion for the energy cost of the rotation are small:

$$\mathcal{H} - \mathcal{E} \sim \Delta \int d^3r \{ |\partial_i O^{ab}(r) |^2 + \mathcal{O}(\ell^{-3}) \} .$$

(7)

To determine the dependence of $|b^{ai}(r)|^2$ on $\ell$, we solve (6) and write

$$\int d^3r |b^{ai}(r)|^2 = \int d^3r_1 d^3r_2 \sigma^a(r_1)\sigma^a(r_2) \frac{4\pi |r_1 - r_2|}{r_5} .$$

(8)

Replacing the factor $B^{bi}(r_1)B^{ci}(r_2)$ appearing in this integrand by its average $\mathcal{E}$

$$\langle B^{bi}(0)B^{ci}(r) \rangle \propto \delta^{bc} \frac{3r_1^2 - r_2^2}{r_5^5}$$

(9)
over a Gaussian ensemble of divergenceless fields. The $t^{-3}$ scaling then follows from power-counting [27].

Low-frequency excitations involve an interplay between smooth rotations and the conserved magnetisation density. In a continuum description these are characterised by three-component vector fields $\theta(r)$ and $m(r)$, which are coarse-grained versions of their lattice counterparts. For a conventional spin glass the equations of motion proposed by Halperin and Saslow [21] are

$$\dot{\theta} = \lambda_0^{-1} m \quad \text{and} \quad \dot{m} = \rho \nabla^2 \theta,$$

(10)
giving the value $c = \sqrt{\rho/\chi_0}$ for the speed, as above.

To understand how this approach should be modified in the weakly disordered pyrochlore antiferromagnet, we start from the microscopic equation of motion [1], which can be recast in the two equivalent forms [22]

$$\dot{\theta}_r^a = [\chi^{-1}]^{ab}_{rr'} m_r^b \quad \text{and} \quad \dot{m}_r^a = -\tau^{ab}_{rr'} \dot{\theta}_r^b.$$  

(11)

It is useful to expand a fluctuation $m_r$ in the basis of Hessian eigenvectors. These span two subspaces, associated respectively with eigenvalues $\mathcal{O}(\Delta)$ and $\mathcal{O}(J)$. Separating the components of $m_r$ in each subspace, we write $m_r = m_{r,\Delta} + m_{r,J}$. Similarly, for $\theta_r$ in the basis of eigenvectors of $\tau$, we take $\theta_{r} = \theta_{r,\Delta} + \theta_{r,J}$. Since ground-state coordinates at $\Delta = 0$ form dynamically conjugate pairs, each $\theta_{r,\Delta}$ has a conjugate $m_{r,\Delta}$.

Under coarse graining, only the smooth parts of $m_r$ and $\theta_r$ survive. These are contained in $m_{r,J}$ and $\theta_{r,\Delta}$. To see this, note first that the ground-state condition $L_\alpha = 0$, which holds for $\Delta = 0$, implies that the average $\sum_{r \in \alpha} m_{r,\Delta}$ over a tetrahedron $\alpha$ is zero for $\Delta/J \rightarrow 0$; hence $m_{r,\Delta}$ is eliminated by coarse graining. Second, the expressions [3] and [7] for energy, in terms of $\theta_r$ and $\theta_r^2$ respectively, imply that smooth rotations are represented exclusively by $\theta_{r,\Delta}$. The relevant coarse-grained degrees of freedom in a continuum theory are therefore $m_{J}(r)$ and $\theta_{\Delta}(r)$. While $m_r$ and $\theta_r$ provide equivalent descriptions of the spin fluctuations on the microscopic level, this is not true after coarse-graining.

The coarse-grained equations of motion can be inferred from [11]. The first follows from the observation that a spin fluctuation $m_{r,J}$ generates exchange fields of order $J$. This immediately yields

$$\dot{\theta}_{\Delta}(r) \sim J m_{J}(r).$$

(12)

The second uses a comparison of the right-hand sides of [3] and [7] to establish that the action of $\tau$ on a smooth rotation $\theta_{r,\Delta}$ can be represented in the continuum by $\tau \sim -\Delta \alpha^2 \nabla^2$. The prefactor $\Delta$ sets the magnitude of the microscopic exchange fields generated by $\theta_{r,\Delta}$, which in turn drive spin precession at frequency $\omega \sim \Delta$. As noted above, the canonically conjugate coordinate representing this precession can be written as $m_{r,\Delta}$ in the limit $\Delta \rightarrow 0$. At finite $\Delta/J$ it is accompanied by a correction $m_{r,J}$, with $|m_{r,J}| \sim (\Delta/J)|m_{r,\Delta}|$. Since these corrections alone survive coarse-graining, the second coarse-grained equation of motion is

$$\dot{m}_{J}(r) \sim a^2 \Delta^2 \nabla^2 \theta_{\Delta}(r).$$

(13)

In summary, a smooth magnetisation density, of magnitude $\mathcal{O}(\Delta/J)$ relative to the gauge field fluctuations, drives long-wavelength twists of the ground state configuration. From [12] and [13] we predict linearly dispersing Goldstone excitations with speed $c \sim a \Delta$ set only by exchange disorder. We next present numerical results that support this picture.

In Fig. 1, integrated densities $N(\lambda)$ of Hessian eigenvalues $\lambda$ ([a] and [b]) and $D(\omega)$ of magnon frequencies $\omega$ ([c] and [d]), with dependence on system size $L$ and disorder strength $\Delta/J$. Panels (a) and (c): for $2^{-10} \leq \Delta/J \leq 2^{-6}$ at $L = 3$; panels (b) and (d): for $3 \leq L \leq 7$ at $\Delta/J = 2^{-6}$. Dashed line in (b) represents $N(\lambda) \propto (\lambda/\Delta)^{3/2}$; dashed line in (d) represents $N(\omega) \propto (\omega/\Delta)^{3}$. Insets in (a) and (c) show full range of $N(\lambda)$ and $D(\omega)$.
spins. We average over 20 disorder realisations for \( L = 3 \) and over 10 realisations for \( 4 \leq L \leq 7 \). In the following we discard eigenvectors of the Hessian and dynamical matrix related to global rotations.

Results for Hessian eigenvalues \( \lambda \) are presented in Fig. 1(a) and (b), and those for the dynamical mode frequencies \( \omega \) in Fig. 1(c) and (d). The main panels of (a) and (c) show that, as \( \Delta/J \to 0 \), 1/4 of eigenvalues or frequencies are \( \mathcal{O}(\Delta) \); the insets to (a) and (c) show that the remainder are \( \mathcal{O}(J) \). Fig. 1(b) demonstrates that \( N(\lambda) \propto (\lambda/\Delta)^{3/2} \) for \( \lambda/\Delta \) and \( \Delta/J \) small. This form follows from Eq. (7). Fig. 1(d) shows \( D(\omega) \propto (\omega/\Delta)^{3/2} \) for \( \omega/\Delta \) and \( \Delta/J \) small. This form follows from Eqs. (12) and (13), which imply linearly dispersing excitations.

Next we test our picture of low-lying Hessian eigenvectors and dynamical modes as long-wavelength twists of the ground-state spin configuration. In both cases we consider the lowest-lying mode that is not simply a global rotation, and start from the coordinates \( \theta_r \). Since \( \theta_r \) is defined to have no component along the equilibrium spin direction \( \mathbf{S}_r \), it has spatial fluctuations even if it represents a global spin rotation. For this reason we redefine the coordinates to be \( \theta_r = \mathbf{S}_r \times \mathbf{m}_r + c_r \mathbf{S}_r \), where \( c_r \) is determined by minimisation of \( \sum_{(r,r')} (\theta_r - \theta_{r'})^2 \).

This scheme ensures that in the case \( \mathbf{m}_r = \theta_0 \times \mathbf{S}_r \) we recover \( \theta_r = \theta_0 \) for all \( r \). In Fig. 2(a) and (b) we present results for the correlator \( \langle \theta_r \cdot \theta_{r'} \rangle \) in the lowest non-trivial Hessian and dynamical modes, respectively. In both instances we find the scaling collapse \( \langle \theta_r \cdot \theta_{r'} \rangle = \langle \theta_0^2 \rangle f(|r-r'|/L) \) for data from system sizes \( 3 \leq L \leq 7 \). This demonstrates that these modes predominantly involve twists of the minimum-energy spin configuration on the scale of the system size.

Finally, we present evidence in Fig. 2(c) that the dynamics of \( \theta_{r,\Delta} \) for low-lying modes is, as argued in justification of (13), driven by the smooth part of \( \mathbf{m}_{r,j} \), which we denote by \( \mathbf{m}_{j..} \). Our expectation that \( \omega \propto |\mathbf{m}_{j..}| \) is vindicated by excellent scaling collapse of \( (J/\Delta)(\mathbf{m}_{j..}^2) \) vs. \( \omega/\Delta \) for a range of \( \omega/\Delta \) and \( \Delta/J \). In this computation \( \mathbf{m}_{j..} \) is isolated by projecting \( \mathbf{m}_{r,j} \) for a normalised dynamical eigenvector onto the subspace spanned by the Hessian eigenvectors associated with the highest quarter of \( \lambda_\Lambda \). A simple indication that this subspace includes the smooth part of \( \mathbf{m}_{r,j} \) is given in the inset to Fig. 2(c), which shows the correlator \( \langle \mathbf{m}_\alpha \cdot \mathbf{m}_{r} \rangle \) of magnetisations \( \mathbf{m}_\alpha \equiv \sum_{r \in \alpha} \mathbf{m}_r \) of neighbouring tetrahedra \( \alpha \) and \( \alpha' \) in Hessian eigenvectors. The correlator is positive in the subspace used to construct \( \mathbf{m}_{j..} \), as required if \( \mathbf{m}_r \) is smooth.

In conclusion, the data shown in Figs. 1 and 2 provide extensive support for our main results, listed above as (i) - (iii), and for the physical arguments used to derive them. Most importantly, the data and physical arguments together establish the description of Goldstone modes in these systems as excitations of the emergent gauge fields.

An experimental signature of these modes is their large magnetic contribution to the heat capacity, scaling as \( (T/\Delta)^\alpha \) at \( T \ll T_F \) with \( \alpha = 3 \). Several examples of pyrochlore antiferromagnets show spin freezing at a temperature much lower than the main interaction scale, which is attributed to weak exchange disorder induced by random strains. Power-law heat magnetic heat capacity (but with \( \alpha \approx 2 \)) is reported for: NaCaNi\(_2\)F\(_7\) (in which there is intrinsic disorder in the locations of non-magnetic cations)\(^{20}\); Y\(_2\)Mo\(_2\)O\(_7\)\(^{21}\) (in which local lattice distortions have been detected\(^{20}\)); and Lu\(_2\)Mo\(_2\)O\(_2\)\(^{22}\). It is also found in SrCr\(_2\)Ga\(_3\)O\(_19\)\(^{23}\); since this is a quasi-two dimensional material, the value \( \alpha = 2 \) is expected here. We note that, since \( D(\omega) \) [Fig. 1(d)] is convex, there is an obvious reason for measured values of \( \alpha \) to decrease as \( T \) increases towards \( T_F \).

Inelastic neutron scattering would potentially provide more detailed information, although the small energy scales involved (smaller than \( T_F \)) present a challenge. Specifically, we expect that the energy-dependence of scattering from emergent gauge degrees of freedom...
should evolve with temperature, from a Lorentzian above $T_F$ to a triple-peaked (elastic as well as gain and loss inelastic peaks) below $T_F$. The wavevector dependence should be the same in all cases, with pinch-points and a suppression of scattering for small momentum transfer.

The ideas we have developed are specific to Heisenberg antiferromagnets that have emergent gauge fields as semiclassical, low-energy degrees of freedom. Other frustrated magnets require different treatments of weak disorder, an example being the jammed spin-liquid systems studied recently [30, 31]. Equally, in some contexts the dependence $c \sim \sqrt{\rho/\chi_0}$ may hold with distinct energy scales $\chi_0^{-1}$ and $a^{-2} \rho$, as suggested [32] for NiGa$_2$S$_4$ [33].

In summary, we have developed a theory of the Goldstone modes in a frustrated Heisenberg magnet with weak exchange randomness, illustrating how the standard hydrodynamic theory must be modified to understand their propagation. We find gapless excitations with energies depending only on the magnitude of the exchange randomness.

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