General Relativistic Magnetoionic Theory

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Abstract. We have developed methods for tracing rays and performing radiative transfer through a magnetoactive plasma in a general relativistic environment. The two electromagnetic plasma modes propagate differently due to a combination of dispersive and gravitational effects. We have found that, when given an appropriate environment surrounding the central black hole, it is indeed possible to generate a significant degree of circular polarisation without an appreciable amount of linear polarisation due to these effects alone.

1. Introduction

Polarisation measurements now exist for many accreting compact objects (ostensibly black holes) at a number of frequencies. Typically, emission mechanisms are called upon to explain polarisation observations (see e.g. Bromley et al., 2001; Westfold, 1959). However, recent observations of Sgr A* and M81 (see e.g. Brunthaler et al., 2001; Bower et al., 1999; Sault and Macquart, 1999), as well as a number of blazars (see e.g. Komesaroff et al., 1984), have exhibited a significant amount of circular polarisation (CP) apparently unassociated with any linear polarisation (LP). This has proved difficult to explain with the standard set of polarised emission mechanisms alone, often requiring specialised magnetic field or disk structures. In addition to its anomalous size, the CP typically does not change in sign despite having a variability larger in frequency and magnitude than that of the LP (if present) and total intensity. Both of these suggest that the region responsible for the polarisation is compact, and perhaps the central compact object is playing a significant role, if only in moderating the local plasma and/or magnetic field structure. As a result, a substantial effort has been made to investigate the effects of the accretion environment upon polarisation.

These efforts have been primarily concentrated in two directions: (i) propagation effects due to a magnetised plasma (see e.g. Ruszkowski and Begelman, 2002; Macquart, 2002; Jones and Odell, 1977b & 1977a), and (ii) vacuum propagation effects due to general relativity, in particular near a rotating black hole (see e.g. Falcke et al., 2000; Agol, 1997; Laor et al., 1990; Connors et al., 1980). Most of these require an initial source of polarisation, presumably provided by the emission
mechanism. A notable exception is the scintillation mechanism proposed by Macquart and Melrose. However, for realistic conditions this has been unable to produce a polarisation of constant sign. The studies regarding (i) have thus far ignored general relativistic effects (and hence are inapplicable near the compact object), focusing upon non-dispersive plasma effects, e.g. Faraday rotation and conversion. The work considering (ii) has found general relativity to have a depolarising influence on LP due to frame dragging for photons passing near the black hole. However, the studies of general relativistic effects have ignored plasma effects completely, and hence are not always applicable in the case of a thick disk or when a dense and/or magnetised corona is present.

In contrast, magnetoionic effects, including dispersion, have been studied in detail in the context of radio waves in the upper atmosphere. This has, of course, been done in the absence of general relativity, where it has been found that for a specific range in frequency the dispersive effects can have a significant impact upon the propagation and polarisation of the radio waves (see e.g. Budden, 1964).

Here we present a fully general relativistic magnetoionic theory which takes into account general relativity as well as dispersive and non-dispersive plasma effects. This is a natural, albeit currently less well developed, extension of the previous investigations into the polarisation effects of accretion flows onto compact objects. The development of the theory can be succinctly separated into the problems of tracing rays and performing the radiative transfer. As such, these proceedings will be presented in five sections with §2 discussing ray tracing, §3 explaining the radiative transfer, §4 presenting results for Bondi flows, and §5 containing conclusions.

2. Ray Tracing

2.1. Formalism

The appropriate place to begin the study of photon propagation in a plasma are Maxwell’s equations,\[ \nabla_{\mu} F^{\nu\mu} = 4\pi J^{\nu} \quad \text{and} \quad \nabla_{\mu} * F^{\nu\mu} = 0, \]here expressed in covariant form in terms of the electromagnetic field tensor, \( F^{\nu\mu} \equiv \nabla^{\nu} A^{\mu} - \nabla^{\mu} A^{\nu} \), its dual \( * F^{\nu\mu} \), and the current fourvector, \( J^{\nu} \). A prescription is required to determine the current from the electromagnetic field. For small fields, this may be accomplished by a covariant extension of Ohm’s law,\[ J^{\nu} = \sigma^{\nu}_{\mu} E^{\mu} \quad \text{where} \quad E^{\mu} \equiv F^{\mu\nu} u_{\nu}, \]
is the fourvector coincident with the electric field vector in the locally flat comoving rest (LFCR) frame of the plasma ($u^{\nu}$ is the average plasma velocity fourvector). Inserting Ohm’s law into Maxwell’s equations and expressing the result in terms of $E^{\mu}$ and $B^{\mu} \equiv \ast F^{\mu\nu} u_{\nu}$, yields

$$\nabla_{\mu} \left( u^{\nu} E_{\mu}^{\nu} - E_{\nu}^{\nu} u^{\mu} + \varepsilon^{\nu\mu\alpha\beta} u_{\alpha} B_{\beta} \right) = 4\pi \sigma^{\nu}_{\mu} E^{\mu}, \quad (3)$$

$$\nabla_{\mu} \left( u^{\nu} B_{\mu}^{\mu} - B_{\nu}^{\nu} u^{\mu} + \varepsilon^{\nu\mu\alpha\beta} u_{\alpha} E_{\beta} \right) = 0, \quad (4)$$

where $\varepsilon^{\nu\mu\alpha\beta}$ is the Levi-Civita pseudo tensor. These are eight partial differential equations which may be solved for $E^{\mu}$ and $B^{\mu}$ given a specific form for the conductivity tensor, $\sigma^{\nu}_{\mu}$.

Solving these equations can be greatly simplified by making use of a two length scale expansion (the so-called WKB or Eikonal approximations). This is permitted because the photon wavelengths of interest are much smaller than both, the typical general relativistic length scale (the size of the black hole), and the typical plasma scale length. In covariant form this expansion takes the form of assuming that $E^{\mu}$ and $B^{\mu}$ are proportional to a phase factor $\exp (i S)$ where the action, $S$, is related to the wave fourvector by $k^{\mu} = \nabla_{\mu} S$. Keeping only the lowest order terms and combining equations (3) and (4) gives

$$\left( \frac{k^{\delta} k_{\delta}}{k^{\mu} k_{\mu}} + k^{\mu} k_{\mu} + 4\pi i \omega \sigma^{\mu\nu} \right) E^{\nu} = 0, \quad (5)$$

where $\omega \equiv k^{\mu} u_{\mu}$ is the photon frequency in the LFCR frame. From this equation it is possible to determine the polarisation (for conductivities with non-degenerate polarisation eigenmodes) and a dispersion relation, $D(x^{\mu}, k^{\mu})$, a scalar function of the position and wave fourvectors that vanishes along a ray. From the latter it is possible to construct the rays directly using a covariant extension of the Hamilton-Weinberg equations (cf. Weinberg, 1962),

$$\frac{dx^{\mu}}{d\lambda} = \frac{\partial D}{\partial k^{\mu}} \text{ and } \frac{dk^{\mu}}{d\lambda} = -\frac{\partial D}{\partial x^{\mu}}, \quad (6)$$

where $\lambda$ is an affine parameter, the details of which depend upon the precise form of $D$ chosen.

### 2.2. Dispersion Relations

In order to investigate the implications of the ray equations (equations (6)) it is instructive to consider a number of particular dispersion relations. First, consider that corresponding to de Broglie waves, or particles,

$$D = k^{\mu} k_{\mu} + m^{2}. \quad (7)$$
When inserted into the ray equations this produces the geodesic equations, corresponding to test particles in general relativity.

Second, consider the dispersion relation associated with an isotropic plasma (derived from equation (5) with the appropriate conductivity, cf. Kulsrud and Loeb, 1992),

\[ D = k^\mu k_\mu + \omega_P^2, \]  

(8)

where \( \omega_P \equiv \sqrt{4\pi e^2 n_e/m_e} \) is the plasma frequency and \( n_e \) is the proper electron density. This bears a striking resemblance to equation (7), with the plasma frequency taking the place of a mass. Hence photons in a plasma act as if they have mass, with one significant difference: now this “mass” depends upon position through the plasma density. Therefore, in general photons in a plasma will not follow geodesics, and in particular will not follow the null geodesics that photons follow in vacuum.

Third, consider the dispersion relation associated with a magnetoactive plasma in the quasi-longitudinal approximation (again this is derived from equation (5)),

\[ D = k^\mu k_\mu + \frac{\omega \omega_P^2}{\omega \pm \omega_B}, \]  

(9)

where \( \omega_B \equiv e \sqrt{B^\mu B_\mu/m_e} \) is the cyclotron frequency associated with the externally imposed magnetic field, \( B^\mu \), and the \( \pm \) runs over the two different polarisation eigenmodes. Again, this is similar to equation (7) with the exception that now the “mass” depends upon polarisation as well as position. As a result, the different polarisation modes will propagate along different paths. This is simply an expression of the dispersive nature of magnetoactive plasmas.

Finally, for completeness the general dispersion relation for the magnetoactive, cold electron plasma is given by

\[
D = k^\mu k_\mu - \delta \omega^2 - \frac{\delta}{2(1 + \delta)} \left\{ \left( \frac{eB^\mu k_\mu}{m_e \omega} \right)^2 - (1 + 2\delta) \omega_B^2 \right\} \\
\pm \sqrt{\left( \frac{eB^\mu k_\mu}{m_e \omega} \right)^4 + 2\left( 2\omega^2 - \omega_B^2 - \omega_P^2 \right) \left( \frac{eB^\mu k_\mu}{m_e \omega} \right)^2 + \omega_B^4},
\]  

(10)

where \( \delta \equiv \frac{\omega_P^2}{\omega_B^2 - \omega^2} \).

This is the covariant extension of the Appleton-Hartree dispersion relation (see e.g. Budden, 1964 or Boyd and Sanderson, 1969). The extension to a pair plasma is straightforward. Note that now the “mass”
Figure 1. Photon capture cross sections in units of the vacuum capture cross section, $\sigma_\gamma = 27\pi M^2$, for (a) the quasi-longitudinal and (b) the quasi-transverse approximations as a function of plasma density for a number of magnetic field strengths. The solid, dotted, short dashed, long dashed, and dash-dotted lines correspond to $\omega_B/\omega_{obs} = 0, 0.7, 1.4, 2.1, \text{ and } 2.8$, respectively. The insets show the CP fraction, $m_c$, in terms of the effective emission area $A$ for the same set of magnetic field strengths.

depends upon the direction of propagation as well as polarisation and position.

2.3. PHOTON CAPTURE CROSS SECTIONS

Even without a method for performing the radiative transfer it is possible to investigate how the combination of dispersion and general relativity can produce polarisation. This occurs when one polarisation eigenmode (either the extraordinary or ordinary) is preferentially captured by the black hole due to dispersive plasma effects. This can be quantified by considering the photon capture cross section of the central black hole for the two different polarisation eigenmodes in the case of Bondi accretion ($\omega_F \propto r^{-3/4}$ and $\omega_B \propto r^{-5/4}$). In order to make this a one-dimensional problem it is necessary to choose an approximation in regard to the orientation of the wave fourvector relative to the external magnetic field. Here we consider the two extremes, the quasi-longitudinal ($k^\mu$ parallel to $B^\mu$) and the quasi-transverse ($k^\mu$ perpendicular to $B^\mu$).

As shown in Figure 1, in both cases the capture cross section associated with the extraordinary mode decreases more rapidly with increasing density than that of the ordinary mode. As a result, the
black hole will effectively cast a larger “shadow” on the ordinary mode, leading to a net excess of photons in the extraordinary mode. If the intervening material is optically thin, this will lead to an observable net polarisation. The magnitude of the polarisation will depend upon the details of the emission (different regions will have different emissivities) and the amount of diluting emission from locations far from the black hole (further than $\sim 5 - 10M$). Both of these will ultimately depend upon the details of the accretion flow. However, insight into the second can be obtained by parameterising the net polarisation in terms of an unknown effective emission area (the relation of which to the actually emitting area will still depend upon the details of the emissivity). The two extreme cases are shown in the insets, where the effective emission area is in units of the vacuum photon capture cross section. That this will produce much more CP than LP is a result of the fact that the polarisation eigenmodes become significantly elliptical only when the angle between the wave fourvector and the external magnetic field is within $\sim \omega_P^2 \omega_B/\omega^3$ of $\pi/2$, which is typically small.

3. Radiative Transfer

3.1. Length Scales & Radiative Transfer Regimes

In general the two polarisation eigenmodes will propagate in a coupled fashion. Because of the dispersive nature of the plasma the general case can be extremely difficult. Fortunately, it is possible to denote regimes in which the rays are dispersive and weakly coupled, and non-dispersive and strongly coupled.

These radiative transfer regimes depend upon two length scales, the plasma scale length and the coherence length. The plasma scale length is the characteristic length scale over which the plasma changes appreciably,

$$\Lambda_S = \left| \frac{dx^\mu}{d\lambda} \nabla_\mu \ln n_e \right|^{-1}, \quad (11)$$

written here covariantly, in terms of the affine parameter. In general this should also include a measure of the length scales over which the external magnetic field magnitude and direction change appreciably. The coherence or Faraday length,

$$\Lambda_F = \left| \frac{dx^\mu}{d\lambda} (k_\Omega - k_\chi) \right|^{-1}, \quad (12)$$

is the length over which the two modes will maintain coherence. The three radiative transfer regimes are then denoted as follows:
\[ \Lambda_F \ll \Lambda_S \quad \text{Adiabatic (weakly coupled & highly dispersive)} \]
\[ \Lambda_F \approx \Lambda_S \quad \text{Transitional} \]
\[ \Lambda_F \gg \Lambda_S \quad \text{Nonadiabatic (strongly coupled & weakly dispersive)} \]

Fortunately, the transitional case occurs only for a very small spatial region, and can usually be safely ignored. In these proceedings we have simply transferred from the adiabatic to the nonadiabatic regimes, skipping the transitional regime altogether.

The differences between these regimes can be illustrated in context of Faraday rotation in the interstellar medium. If in this case the propagation were adiabatic, and hence the polarisation eigenmodes propagated independently, the rotation measure would be proportional to \( \int n_e B \, dl \). This is because in the adiabatic regime the extraordinary mode never evolves into the ordinary mode as a result of changes in the magnetic field, including field reversals. However, because in the case of the interstellar medium the propagation is in reality nonadiabatic, and hence the modes are strongly coupled, the extraordinary mode can evolve into the ordinary mode, leading to the familiar rotation measure, proportional to \( \int n_e B \cdot dl \).

### 3.2. Covariance

Because of the relativistic nature of the problem it is necessary to recast the radiative transfer in a covariant fashion. Because the emission and absorption are local processes, they are most easily dealt with in the LFCR frame. It is then necessary to transform from the differential distance in the LFCR frame, \( dl \), into a differential change in the affine parameter, \( d\lambda \). This is accomplished using

\[
dl = \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - \left( u_\mu \frac{dx^\mu}{d\lambda} \right)^2} \, d\lambda.
\]

In the adiabatic regime the polarisation propagates adiabatically, being defined by the local plasma conditions, hence only the total intensity need be transferred. It is then a simple matter to integrate the occupancy number instead of the intensity to maintain covariance.

The nonadiabatic regime creates more difficulties as it is now necessary to propagate a covariant form of the Stokes parameters. Because each of the Stokes parameters are defined in terms of intensities and a fiducial direction, it is possible to define analogous covariant quantities in terms of the occupancy numbers and fiducial directions defined by an orthonormal tetrad propagated along the ray. Because the ray is no longer strictly a geodesic this propagation should be done via...
Fermi-Walker transport (see e.g. Misner et al., 1973),
\[ v^\nu \nabla_\nu e^\mu = (v^\mu a^\nu - v^\nu a^\mu) e_\nu \] where \( a^\mu \equiv v^\nu \nabla_\nu v^\mu \). (14)

However, since in the nonadiabatic regime the rays are only weakly dispersive, using parallel transport \( (a^\mu = 0) \) introduces a negligible error.

### 3.3. Emission Models

We have considered two emission models. Both are low harmonic synchrotron emission arising from a power law tail of hot electrons. The densities of these electrons were made proportional to the plasma density. The first model was unpolarised, splitting the emitted power equally between the polarisation modes. This was done to better illustrate the creation of polarisation by the dispersion near the black hole. The second model splits the synchrotron flux among the two polarisations appropriately. This is not necessarily more realistic because in the accretion flows considered the magnetic field is uniform over the entire space. While this is not necessary for the dispersive polarisation mechanism to operate, it will lead to the production of a substantial amount of polarisation from the synchrotron emission that would not be present otherwise.

### 4. Bondi Flow

In order to obtain results which can ultimately be compared to observations, it is necessary to specify the density and velocity of the plasma in the accretion flow as well as the magnetic field geometry. In general this should be done in a self consistent manner. However, for simplicity in implementation and clarity of exposition, we have chosen instead to impose a Bondi accretion flow with a split monopolar magnetic field geometry, the strength of which is given by a fixed fraction of the equipartition value. The overall magnitudes of these parameters are scalable by the observation frequency. Here, they have been chosen so that interesting effects occur near 10 GHz, as is the case for spectra of Sgr A*.

It is now possible to explicitly see the dispersion mechanism by tracing rays, as shown in Figure 2. Note that the capture cross section for the ordinary mode is in fact larger than that of the extraordinary mode, as predicted.

The Stokes parameters (Figures 3 and 4) also confirm the predictions made in §2.3. While quantitative differences exist, qualitatively the two
emission models produce the similar results. For both models, \( I \approx 3 \) Jy, \( Q/I \approx -10^{-3} \), and \( U/I \approx -10^{-6} \). The disparity between \( Q \) and \( U \) is a result of the field reversal in the split monopolar magnetic field geometry occurring in the equatorial plane. For the unpolarised model, \( V/I \approx 0.2 \), and for the polarised model, \( V/I \approx 0.5 \). All of these numbers are the integrated values within the regions shown, and therefore do not include dilution from further out in the accretion flow. For the unpolarised model this makes little difference. For the polarised model, this means that the LP can be seriously underestimated. However, as mentioned in §3.3, this may be an artifact of the artificiality of the accretion flow geometry at large radii.

Plotted as a function of frequency (Figure 5), the sizable contribution of the dispersive effects to the total CP is clearly evident. At the maximum CP, nearly 75% of the total polarisation is due to the dispersive effects alone, as demonstrated by comparing the curves for the polarised and unpolarised emission models. Furthermore, the dispersive effects are capable of creating polarisation over more than a decade in frequency, and hence may be an important source of polarisation for a
significant portion of the spectrum. This large magnitude is necessary in order to maintain a significant residual polarisation after the inclusion of diluting unpolarised emission from the rest of the accretion flow.

5. Conclusions

Dispersive effects coupled with general relativistic effects will produce considerable amounts of CP when the plasma and/or cyclotron frequencies are commensurate with those being observed. This method of producing CP is unique in that it does not require a polarised emission mechanism — even unpolarised emission will become polarised after passing near a black hole. Unlike the non-dispersive processing mechanisms, e.g. Faraday conversion, this does not require uniform large scale magnetic fields over the entire disk. Rather, only uniformity near the black hole horizon is necessary, where the black hole’s influence can in principle moderate the magnetic field geometry. This is neither
Figure 4. Stokes parameters at 10 GHz as observed at infinity as a function of displacement in the two perpendicular directions ($\xi$ is perpendicular to the azimuthal axis) for the polarised emission model. Note that the scales listed in the titles.

dependent upon the details of the emission mechanism being employed nor contaminated by large degrees of LP.

The requirements of the dispersion mechanism place some constraints upon the emission mechanisms. The first is that the mechanism must be able to operate near $\omega_p, \omega_B \sim \omega$. This can be relaxed somewhat by having the black hole being backlit, eliminating the necessity for an emission mechanism that is capable of operating near the hole. A second constraint upon the emission mechanism is that it needs to have a large brightness temperature. This is equivalent to the fact that the fraction of the total intensity propagating through the inner $\sim 5 - 10M$ must be larger than the CP fraction. For blazars, this all but rules out this mechanism (however jets and/or plasma distributions which use dispersion to magnify the emitting regions may yet make a difference). For Sgr A* and M81, brightness temperatures on the order of $10^{12}$ K are necessary, pushing the upper bounds given by the paucity of the X-ray fluxes. Still, this remains a tenable source for the CP in low luminosity AGN and may be at work in Sgr A* and/or M81. Recent LP results
Figure 5. The Stokes V parameter as a function of frequency. The filled triangles (open squares) show V/I for the unpolarised (polarised) emission model.

(which confirm the earlier observations of Aitken et al., 2000) suggest that the gas density close to the black hole in Sgr A* is far lower than expected for a conservative accretion flow (Bower et al., 2002). Therefore, the Sgr A* environment is conducive to seeing relativistic magnetoionic effects at high frequencies close to the black hole.

In addition to applications to low luminosity AGN, this mechanism can have implications in stellar mass black hole systems as well. The degree of CP depends upon the relative sizes of the black hole and the accretion flow. Hence, high mass X-ray binaries can be expected to exhibit a significant amount of CP if (i) the accretion flow is optically thin at radio frequencies (e.g. if the black hole is being seen through the corona) and (ii) magnetic fields are present, presumably generated via the magnetorotational instability and ordered by the black hole.
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