QUADRATIC FORMS REPRESENTING ALL INTEGERS COPRIME TO 3

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Abstract. Following Bhargava and Hanke’s celebrated 290-theorem, we prove a universality theorem for all positive-definite integer-valued quadratic forms that represent all positive integers coprime to 3. In particular, if a positive-definite quadratic form represents all positive integers coprime to 3 and \( \leq 290 \), then it represents all positive integers coprime to 3. We use similar methods to those used by Rouse to prove (assuming GRH) that a positive-definite quadratic form representing every odd integer between 1 and 451 represents all positive odd integers.

1. Introduction and Statement of Results

The study of which integers are represented by certain quadratic forms dates back to the time of Diophantus in the 3rd century. Building on this work, Fermat classified the positive integers that can be written as a sum of two integer squares, i.e., the numbers represented by \( x^2 + y^2 \). In 1770, Lagrange proved that every positive integer can be written as a sum of four squares.

In 1916, Ramanujan [11] gave a list of 55 quadratic forms of the form \( ax^2 + by^2 + cz^2 + dw^2 \) and claimed that these quadratic forms are the only diagonal forms in four variables that represent all positive integers. Dickson [3] proved Ramanujan’s claim (modulo the error that Ramanujan had listed one form that fails to represent 15).

We say that a positive-definite quadratic form \( Q(\bar{x}) \) is an integer-matrix form if \( Q(\bar{x}) = \bar{x}^T A \bar{x} \), where \( A \) is a matrix with integer entries. We say that \( Q \) is integer-valued if \( Q(\bar{x}) = \frac{1}{2} \bar{x}^T A \bar{x} \) where \( A \) is a matrix with integer entries and even diagonal entries.

In her Ph.D. thesis, Willerding [16] classified universal integer-matrix quaternary forms, those that represent all positive integers. In 1993, Conway and Schneeberger proved the following theorem giving a nice classification of universal forms in any number of variables (see [14]).

Theorem (“The 15-Theorem”). A positive-definite integer-matrix quadratic form is universal if and only if it represents the numbers

\[ 1, 2, 3, 5, 6, 7, 10, 14, \text{ and } 15. \]

This theorem was elegantly reproven by Bhargava in 2000 (see [2]). Bhargava’s approach is to work with integral lattices, and to classify escalator lattices - lattices that must be inside any

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lattice whose corresponding quadratic form represents all positive integers. As a consequence, Bhargava was able to correct some errors in Willerding’s work.

Bhargava’s approach is quite general. Indeed, he has proven that for any infinite set $S$, there is a unique minimal finite subset $S_0$ of $S$ so that any positive-definite integral quadratic form represents all numbers in $S$ if it represents the numbers in $S_0$. Here the notion of integral quadratic form can mean either integer-matrix or integer-valued (and the set $S_0$ depends on which notion is used).

While working on the 15-Theorem, Conway and Schneeberger were led to conjecture that every integer-valued quadratic form that represents the positive integers between 1 and 290 must be universal. Bhargava and Hanke’s celebrated 290-Theorem proves this conjecture (see \[3\]). Their result is the following.

**Theorem ("The 290-Theorem").** If a positive-definite integer-valued quadratic form represents the 29 integers

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29,$$

$$30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, \text{and} \ 290,$$

then it represents all positive integers.

They also show that every one of the twenty-nine integers above is necessary. Indeed, for every integer $t$ on this list, there is a positive-definite integer-valued quadratic form that represents every positive integer except $t$. As a consequence of the 290-Theorem, they are able to prove that there are exactly 6436 universal integer-valued quaternion quadratic forms.

Bhargava has shown that if $Q$ is a positive-definite integer matrix form that represents the integers from 1 up to 33, then $Q$ represents all positive odd integers. In \[12\], the second author proved (assuming the generalized Riemann hypothesis) that an integer-valued form that represents the odd integers $\leq 451$ must represent all odd integers. Recently, Barowsky, Damron, Mejia, Saia, Schock and Thomson gave a classification of the possible sets $\{m, n\}$ of exceptions for an integer-matrix form with exactly two exceptions (see \[1\]). The goal of the present paper is to let $S$ be the set of positive integers coprime to 3, and to compute the minimal subset $S_0$ of integers that an integer-valued form $Q$ must represent in order to represent everything in $S$. Our main result is the following.

**Theorem 1 (The CCXC Theorem).** If a positive-definite integer-valued quadratic form represents the following 31 integers

$$1, 2, 5, 7, 10, 11, 13, 14, 17, 19, 22, 23, 26, 29, 31, 34, 35,$$

$$37, 38, 46, 47, 55, 58, 62, 70, 94, 110, 119, 145, 203, \text{and} \ 290,$$

then it represents all positive integers coprime to 3.

Here are two corollaries.
Corollary 2. For every single one of the positive integers \( t \) in the above list, there is a positive-definite integer-valued quadratic form \( Q \) that represents every positive integer coprime to 3 except \( t \).

Corollary 3. If a positive-definite integer-matrix quadratic form represents the following integers

\[ 1, 2, 5, 7, 10, 11, 14, 19, 22, 31, 35, \]

then it represents all positive integers coprime to 3.

To prove the CCXC Theorem, we must determine exactly which positive, squarefree integers that are coprime to 3 are represented by a collection of 9611 quaternary quadratic forms. Any form that represents all positive integers coprime to 3 must represent either one of 11 regular ternary quadratic forms, or one of the quaternary forms on this list.

To analyze the quaternary forms, we use a combination of four methods. These methods are the same ones used in the proof of the 451-theorem of [13]. The first method checks to see if a given quaternary represents any of the 11 regular ternaries mentioned above. If so, it represents all positive integers coprime to 3. This method succeeds for 646 of the 9611 quaternary forms.

The second method attempts to find, given the integer lattice \( L \) corresponding to \( Q \), a regular ternary sublattice \( K \) so that \( K \oplus K^\perp \) locally represents everything coprime to 3. We make use of the classification of regular ternary quadratic forms due to Jagy, Kaplansky, and Schiemann [8]. This method is successful for 3631 of the remaining forms.

The last two methods rely on the theory of modular forms. For a positive-definite quaternary form \( Q \), we define \( \chi(n) = \left( \frac{\det(A)}{n} \right) \) to be the usual Kronecker character. Then the theta series

\[
\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n)q^n, \quad q = e^{2\pi iz}
\]

is a modular form of weight 2, level \( N \) and character \( \chi \). We can decompose \( \theta_Q(z) \) as

\[
\theta_Q(z) = E(z) + C(z) = \sum_{n=0}^{\infty} a_E(n)q^n + \sum_{n=1}^{\infty} a_C(n)q^n.
\]

A lower bound on \( a_E(n) \) is given in Theorem 5.7 of [6] and shows that

\[
a_E(n) \geq C_{E,n} \prod_{\substack{p|n \\chi(p)=-1}} \frac{p-1}{p+1}
\]

for some some constant \( C_E \), depending on \( Q \), provided \( n \) is squarefree and locally represented by \( Q \). We may decompose the form \( C(z) \) into a linear combination of newforms (and the images
of newforms under \( V(d) \). It is known that the \( n \)th Fourier coefficient of a newform of weight 2 is bounded by \( d(n)n^{1/2} \) by the Deligne bound. Thus, there is a constant \( C_Q \) so that

\[
|a_C(n)| \leq C_Q d(n)n^{1/2}.
\]

If we can compute or bound the constants \( C_E \) and \( C_Q \), we can determine the squarefree integers represented by \( Q \) via a finite computation.

Finally, the third method explicitly computes the constant \( C_Q \) by doing extensive exact linear algebra using Magma. This method handles 2267 forms.

The fourth method gives an upper bound on \( C_Q \) without explicitly computing it. This method may only be used when \( \det(A) \) is a fundamental discriminant. The Petersson inner product is defined for \( f, g \in S_2(\Gamma_0(N), \chi) \) by

\[
\langle f, g \rangle = \frac{3}{\pi|\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)|} \int_{\mathbb{H}/\Gamma_0(N)} f(x + iy) \overline{g}(x + iy) \, dx \, dy.
\]

If one can compute an upper bound on \( \langle C(z), C(z) \rangle \) and a lower bound on \( \langle g_i(z), g_i(z) \rangle \) for each newform \( g_i(z) \in S_2(\Gamma_0(N), \chi) \), one can derive an upper bound on \( C_Q \). We use the same machinery from [13] to accomplish these things. We use this method for the remaining 3067 forms.

One different feature of the present work is the use of the exact formulas due to Yang [17] for the local densities appearing in the formula for \( a_E(n) \). These provide for efficient computation.

An outline of the paper is as follows. In Section 2 we review background about quadratic forms and modular forms. In Section 3 we describe the theory of escalator lattices and our use of it. In Section 4 we describe in detail our first two methods relying on properties of ternary quadratic forms. In Section 5 we describe our first modular form based method (the only method used by Bhargava and Hanke in [3]), and in Section 6 we describe our second modular form method (pioneered in [13]). Finally, in Section 7 we prove Theorem 1, Corollary 2, and Corollary 3.

Acknowledgements. The authors used the computer software package Magma [4] version 2.21-1 extensively for the computations. Magma scripts and log files from the computations done are available at [http://users.wfu.edu/rouseja/CCXC/]. This work represents the master’s thesis of the first author completed at Wake Forest University in the spring of 2015.

2. Background

If \( Q \) is an integer-valued quadratic form in \( r \) variables, then \( Q(\vec{x}) = \frac{1}{2} \vec{x}^T A \vec{x} \) for some \( r \times r \) matrix \( A \). We say that the discriminant of \( Q \) is \( \det(A) \), and the level of \( Q \) is the smallest positive integer \( N \) so that \( NA^{-1} \) has integer entries and even diagonal entries.
If $N$ is a positive integer, define $\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) : N|c \right\}$. If $k$ is a positive even integer, let $M_k(\Gamma_0(N), \chi)$ denote the $\mathbb{C}$-vector space of modular forms $f$ so that

$$f \left( \frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^k f(z).$$

Let $S_k(\Gamma_0(N), \chi)$ denote the subspace of cusp forms. The operator $V(d)$ is defined by $\sum a(n)q^n \mapsto \sum a(n)q^{dn}$ and maps $S_k(\Gamma_0(N), \chi)$ to $S_k(\Gamma_0(dN), \chi)$. The old subspace of $S_k(\Gamma_0(N), \chi)$ is defined to be the span of the images of $V(e) : S_k(\Gamma_0(N/d), \chi) \to S_k(\Gamma_0(N), \chi)$ where $e$ runs over divisors of $d$, and $d$ runs over divisors of $N$. The new subspace of $S_k(\Gamma_0(N), \chi)$ is defined to be the orthogonal complement of the old subspace under the Petersson inner product. This new subspace is spanned by newforms - Hecke eigenforms lying in the new subspace that are normalized so the Fourier coefficient $q = e^{2\pi i z}$ is 1.

If $Q$ is a positive-definite integer-valued quadratic form in $r$ variables, let $r_Q(n) = \{ \vec{x} \in \mathbb{Z}^r : Q(\vec{x}) = n \}$. The theta series of $Q$ is

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n)q^n.$$ 

If $N$ is the level of $Q$ and $r$ is even, then $\theta_Q \in M_{r/2}(\Gamma_0(N), \chi_D)$ (see Theorem 10.8 of [7]). Here $\chi_D(\cdot) = \left( \frac{-1}{r/2 \det A} \right)$ is the usual Kronecker symbol. As noted in Section 4 $\theta_Q(z)$ has a decomposition $E(z) + C(z) = \sum_{n=0}^{\infty} a_E(n)q^n + \sum_{n=1}^{\infty} a_C(n)q^n$, into an Eisenstein series and a cusp form.

We can associate a lattice $L$ to a positive-definite integer-valued quadratic form $Q$ by letting $L = \mathbb{Z}^r$ and defining an inner product on $L$ by setting

$$\langle \vec{x}, \vec{y} \rangle = \frac{1}{2} \left( Q(\vec{x} + \vec{y}) - Q(\vec{x}) - Q(\vec{y}) \right).$$

We have that $\langle \vec{x}, \vec{x} \rangle = Q(\vec{x})$ is integral, but arbitrary inner products $\langle \vec{x}, \vec{y} \rangle$ need not be integral. If $Q = \frac{1}{2} \vec{x}^T A \vec{x}$, we say that $A$ is the Gram matrix of $L$. We will move freely between a quadratic form and its corresponding lattice, and use adjectives that apply to quadratic forms to refer to lattices and vice versa.

For a prime $p$, let $\mathbb{Z}_p$ denote the ring of $p$-adic integers. We say that a positive-definite form $Q$ locally represents an integer $m$ if $m > 0$ and for all primes $p$, there is some $\vec{x} \in \mathbb{Z}_p^r$ so that $Q(\vec{x}) = m$. If $Q$ is fixed, we let $\text{Gen}(Q)$ denote the finite collection of positive-definite integral forms $R$ so that $R$ is equivalent to $Q$ over $\mathbb{Z}_p$ for all primes $p$. By work of Siegel [15], we have that

$$\sum_{n=0}^{\infty} a_E(n)q^n = \frac{\sum_{R \in \text{Gen}(Q)} \theta_R(z) / \# \text{Aut}(R)}{\sum_{R \in \text{Gen}(Q)} 1 / \# \text{Aut}(R)} = \sum_{n=0}^{\infty} \left( \prod_{p \leq \infty} \beta_p(n) \right) q^n,$$
where $\beta_p(n)$ is the local density associated to $p$, $Q$ and $n$. It follows from this formula that if $|\text{Gen}(Q)| = 1$, then $Q$ represents every integer $n$ that is locally represented by $Q$. If $Q$ is a quadratic form that represents all positive integers $n$ that are locally represented, we say that $Q$ is regular.

3. Escalators

Fix a set $S$ of positive integers. Given a quadratic form $Q$ with corresponding lattice $L$, we say that an element $t \in S$ is an exception for $Q$ if $Q$ does not represent $t$. We call the smallest exception the truant of $Q$. If $Q$ is a quadratic form with truant $t$ with corresponding lattice $L$, an escalation of $L$ is a lattice $L'$ generated by $L$ and a vector of norm $t$. We say that $L$ (or $Q$) is relatively universal if it represents everything in $S$ (or equivalently, if it has no truant). An escalator lattice is a lattice obtained by repeated escalation of the unique zero-dimensional lattice.

Write $S = \{t_1, t_2, t_3, \ldots\}$ with $t_i < t_j$ if $i < j$. If $L$ is a relatively universal lattice, then it contains a vector of norm $t_1$, and hence an escalator lattice $L_1$ generated by $t_1$. If $L_1$ is not relatively universal, then there is a vector in $L$ with norm equal to the truant of $L_1$. Then $L$ must contain some escalation $L_2$ of $L_1$. Continuing in this way, we get a sequence of escalator lattices $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots \subseteq L$ with the property that $L_i$ represents (at least) the first $i$ elements of $S$. Since $L \cong \mathbb{Z}^r$ is a noetherian $\mathbb{Z}$-module, this ascending chain of lattices stabilizes. Thus, there is some $L_n \subseteq L$ that is a relatively universal escalator lattice.

We are concerned with the case that $S = \{n \geq 1 : \gcd(n, 3) = 1\}$. To prove Theorem 1, we begin by escalating the zero-dimensional lattice. We obtain the 1-dimensional lattice with Gram matrix $A = [2]$ corresponding to the quadratic form $Q(\vec{x}) = \frac{1}{2} \vec{x}^T A \vec{x} = x^2$. This has truant 2 and (by the Cauchy-Schwarz inequality), its escalations are those lattices with Gram matrices

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}.$$

The first two lattices have truant 5, and the third has truant 7. The escalation of these lattices result in 50 three-dimensional lattices. Of these, 11 have no truant below 1000. In fact, all of these are relatively universal.
**Theorem 4.** The following 11 ternary quadratic forms represent all positive integers coprime to 3.

\[
\begin{align*}
x^2 - xy + y^2 + z^2 \\
x^2 + xy + 2y^2 + yz + 4z^2 \\
x^2 + xy + 2y^2 - xz - 2yz + 4z^2 \\
x^2 + xy + 2y^2 - xz + 2z^2 \\
x^2 + xy + 2y^2 - xz + 5z^2 \\
x^2 + xy + 2y^2 + 3z^2 \\
x^2 + y^2 + xz - yz + 2z^2 \\
x^2 + y^2 + 3z^2 \\
x^2 + y^2 - xz + 4z^2 \\
x^2 + y^2 + 6z^2 \\
x^2 + y^2 - xz + 7z^2.
\end{align*}
\]

**Proof.** It is easy to verify that all 11 of these quadratic form locally represent all integers \( n \) coprime to 3. Of these 11 forms, 8 are in a genus of size 1, and this automatically implies that they are regular. The remaining three are also regular (since they occur on the list of forms proven regular in [8]). Since a regular form represents everything that is represented locally, and each form locally represents everything coprime to 3, each of these 11 forms represents all positive integers coprime to 3.

Escalating the remaining 39 ternary quadratic forms gives rise to 8894 quadratic forms \( Q \) in four variables. Of these 8884 locally represent everything coprime to 3. (We refer to these as the basic quaternary lattices.) The remaining 10 fail to locally represent all integers coprime to 3. These ten lattices all have either 14, 22 or 35 as their truant. We take a ternary sublattice of each, and escalate by the truant of the quaternary lattice - this gives rise to 727 extra quaternary lattices (all of which locally represent everything coprime to 3). Understanding the squarefree integers that are not represented by these \( 8894 + 727 - 10 = 9611 \) basic and extra quadratic forms suffices to prove Theorem [8]. The next several sections outline how we understand these 9611 quadratic forms.

### 4. Regular ternaries

Given a quaternary lattice \( L \), the first thing we check is if \( L \) has a ternary sublattice \( L' \) whose quadratic form is one of the 11 listed in Theorem [8]. If so, then for every number \( t \) coprime to 3, \( L \) has a vector with norm \( t \) and this means that the quadratic form corresponding to \( L \) represents \( t \). Therefore it is relatively universal.
Example. Form 8703 is $Q(x, y, z, w) = x^2 - xz + y^2 - yz + 2z^2 + 7w^2$. We have that $Q(-y - z, x, -y + z, 0) = x^2 + xy + 2y^2 - xz - 2yz + 4z^2$. This is the third universal ternary listed in Theorem 4. Since this ternary form represents everything coprime to 3, so does $Q$.

This first method applies to 644 of the 8884 basic quaternary lattices, and 2 of the 727 extra lattices.

The second method takes a quaternary lattice $L$ and searches for a ternary sublattice $K$ so that the quadratic form corresponding to $K$ is regular, and the quadratic form corresponding to $K \oplus K^\perp$ locally represents everything coprime to 3. We may write the quadratic form corresponding to $K \oplus K^\perp$ as $Q(x, y, z, w) = T(x, y, z) + dw^2$. Since $T$ is regular, whether an integer $m$ is represented by $T$ depends only on congruence conditions.

The paper [8] proves that there are at most 913 ternary quadratic forms, and 891 of those are definitely regular. In [10], another 8 ternary quadratic forms are proven regular (and in [9], the remaining 14 are proven regular assuming the Generalized Riemann Hypothesis). We use the list of 899 provably regular ternary quadratic forms.

We let $M$ be a positive integer divisible by all primes dividing the determinant of $T$ so that for every $a$, either $T$ represents every squarefree integer $\equiv a \pmod{M}$ or no integer $\equiv a \pmod{M}$.

We create a queue of residue classes to check, initially including all residue classes modulo $M$ that contain integers coprime to 3 that are not represented by $T$. If $a \pmod{M}$ is such a residue class, there is some integer $n = T(x, y, z) + dw^2 \equiv a \pmod{M}$ that is represented by $Q(x, y, z, w)$. If $T(x, y, z) = b \neq 0$, there is some arithmetic progression $b \pmod{M'}$ of positive integers represented by $T(x, y, z)$ and hence every integer $\equiv a \pmod{M'}$ and greater than or equal to $n$ is represented by $Q$.

If $M' = M$, then the only positive integers $\equiv a \pmod{M}$ not represented by $Q$ are those less than $n$. If $M' > M$, we add the residue classes $a + kM \pmod{M'}$ to the queue (for $0 < k < M'/M$). We proceed until the queue is empty.

Example. Consider Form 238, $Q(x, y, z, w) = x^2 + y^2 + yz + 2z^2 - xw + 23w^2$ and let $L$ be the corresponding quaternary lattice. There is a lattice $L'$ of index two in $L$ of the form $K \oplus K'$. The quadratic form corresponding to $L'$ is $x^2 + y^2 + yz + 2z^2 + 91w^2$. The form $T(x, y, z) = x^2 + y^2 + yz + 2z^2$ is regular, and represents every positive integer except those of the form $7r^m$ where $r$ is odd and $\left(\frac{m}{7}\right) = -1$. Thus, the modulus of $M$ is 49, and initially our queue is set to include the classes 21 (mod 49), 35 (mod 49) and 42 (mod 49).

When we test the residue class 21 (mod 49), we find that neither 21 nor 70 is represented by $T(x, y, z) + 91w^2$. However, 119 = 91 + 28 is represented and since $Q$ represents everything $\equiv 28 \pmod{49}$, all $n > 70$ with $n \equiv 21 \pmod{49}$ are represented represented by $Q$.

When we test the residue class 35 (mod 49), we find that the smallest positive integer in this residue class represented is 378 = 91 · 2^2 + 14. Since $T$ represents everything $\equiv 14 \pmod{49}$, $Q$ represents everything $\equiv 35 \pmod{49}$ and greater than 378.
When we test the residue class $42 \pmod{49}$ we find that $42$ is not represented, that $91$ is represented but only as $91 \cdot 1^2 + 0$. We have $140 = 91 + 49$. Now, $T$ does not represent all numbers $\equiv 0 \pmod{49}$, but it does represent all those $\equiv 49 \pmod{343}$. This proves that $Q$ represents all numbers $\equiv 140 \pmod{343}$, but we add to the queue the six classes $42 \pmod{343}$, $91 \pmod{343}$, $189 \pmod{343}$, $238 \pmod{343}$, $287 \pmod{343}$, and $336 \pmod{343}$. These are easily checked.

We find in the end that $T(x, y, z) + 91w^2$ represents all positive integers except 21, 35, 42, 70, 84, 133, 182, 231, 280 and 329. Testing $Q$, we find that it represents all positive integers coprime to 3 except 70.

This method applies to 3465 of the basic quaternary lattices, and 166 of the extra lattices.

5. Modular forms

If $Q = \frac{1}{2}x^T Ax$ is a positive-definite quaternary quadratic form, then we use the theta series

$$\sum_{n=0}^{\infty} r_Q(n)q^n = \sum_{n=0}^{\infty} a_E(n)q^n + \sum_{n=1}^{\infty} a_C(n)q^n$$

and enumerate all squarefree $n$ so that $a_E(n) \leq |a_C(n)|$.

The Eisenstein coefficient $a_E(n) = \prod_{p \leq \infty} \beta_p(n)$. Formulas for local densities are known (see [6] and [17]) and imply that if $n$ is squarefree, then

$$\beta_p(n) = \begin{cases} \frac{\pi^2 n}{\sqrt{\det(A)}} & \text{if } p = \infty \\ \frac{1 - \chi_D(p)}{p} & \text{if } p \nmid N \text{ and } p \nmid n \\ \frac{1}{(p-1)(p^2 + (1 + \chi_D(p))p + 1)} & \text{if } p \mid N \text{ and } p | n. \end{cases}$$

Thus,

$$a_E(n) = \frac{\pi^2 n}{\sqrt{\det(A)}} \left( \prod_{p \mid N} \beta_p(n) \right) \prod_{p \mid n} \left( \frac{p-1}{p^3} + \frac{1}{(p^2 + (1 + \chi_D(p))p + 1)} \right) \prod_{p \mid N} \left( 1 - \frac{\chi_D(p)}{p^2} \right)$$

$$= \frac{\pi^2 n}{\sqrt{\det(A)}} \left( \prod_{p \mid N} \frac{\beta_p(n)}{1 - \chi_D(p)/p^2} \right) \left( \prod_{p \mid N} \left( \frac{p-1}{p^3} + \frac{1}{(p^2 + (1 + \chi_D(p))p + 1)} \right) \right) \prod_{p} \left( 1 - \frac{\chi_D(p)}{p^2} \right)$$

$$\geq \frac{\pi^2 n}{\sqrt{\det(A) \Lambda(2, \chi_D)}} \left( \prod_{p \mid N} \frac{\beta_p(n)}{1 - \chi_D(p)/p^2} \right) \prod_{p \mid N} \frac{1}{(p^2 + (1 + \chi_D(p))p + 1)} \prod_{p \mid N, \chi_D(p)=-1} \frac{p-1}{p+1}. $$
We compute $\beta_p(n)$ for all primes $p|2N$ and for all of the different $\mathbb{Z}_p$ square classes containing squarefree integers. Unlike the past work of Rouse [13], we do so by using the non-recursive formulas given in [17]. These are more efficient than the procedure given in [6]. In this way, we compute a constant $C_E$ so that

$$a_E(n) \geq C_E \prod_{p|n, p(N, \chi_D(p))=-1} \frac{p - 1}{p + 1}$$

for all squarefree positive integers $n$.

Any cusp form $C(z)$ can be decomposed as

$$C(z) = \sum_{d|N} \sum_{i=1}^{s} \sum_{e|\text{cond}} g_i(\sqrt{d}z)$$

where $g_i(z)$ is a normalized Hecke eigenform living in the new subspace of $S_2(\Gamma_0(d), \chi)$. The $n$th Fourier coefficient of $g_i(z)$ has size at most $d(n)\sqrt{n}$. Thus, if we set

$$C_Q = \sum_{d|N} \sum_{i=1}^{s} \sum_{\text{cond}} \left| c_{d,i,e} \right| \sqrt{d}$$

we have $|a_C(n)| \leq C_Q d(n)\sqrt{n}$. This implies that there is a constant $F$ so that if

$$F_4(n) := \frac{\sqrt{n}}{d(n)} \prod_{p|N, p|n, \chi_D(p)=-1} \frac{p - 1}{p + 1} > F,$$

then $n$ is represented by $Q$.

In the third method, we will explicitly compute the newforms $g_i(z)$ (using the modular symbols algorithm provided in Magma [4]) and compute the constants $C_Q$ and $F$. This procedure is somewhat time consuming. Once we have computed $F$, we will enumerate all squarefree integers with $F_4(n) \leq F$, and see which of these are represented by $Q$, and which are not.

**Example.** Consider Form 8819, $Q(x, y, z, w) = x^2 + y^2 + 7z^2 - xw - yw + 7zw + 12w^2$, with corresponding lattice $L$. This form has level 546. The dimension of $S_2(\Gamma_0(546), \chi_{273})$ is 104. We compute that $C_E = 12/37$, and that $C_Q \approx 23.925$. This yields that $F = 74.507$. Any squarefree $n$ with $F(n) \leq F$ has at most 8 prime factors, all of which are $\leq 79939$. There are a total of 395007 squarefree $n$ coprime to 3 for which $F(n) \leq F$. The form $Q'(x, y, z, w) = w^2 + 2x^2 + 14y^2 + 78z^2$ is the quadratic form corresponding to a sublattice of $L$. We make an array of the values represented by $T(x, y, z) = 2x^2 + 14y^2 + 78z^2$ with $0 \leq x, y \leq 800$ and $z \leq 278$. For each of the 395007 squarefree integers $n$, we check to see if there is some integer $w$ so that $n - w^2$ is represented by $T(x, y, z)$. Of the 395007 squarefree $n$ coprime to 3 with $F(n) \leq F$, at most 9 are not represented by $Q'$, and $Q$ represents all positive integers coprime to 3 except 19, 22, 31, 35 and 133.
A discriminant $D$ is an integer $\equiv 0$ or $1 \pmod{4}$. A **fundamental discriminant** is a discriminant $D$ that is not a square multiple of another discriminant. The method above is used for all remaining quadratic forms $Q$ for which $\det(A)$ is not a fundamental discriminant. There are 1906 such basic quaternary lattices, and 361 extra lattices. A few of these cases are quite time consuming. For example, Form 3391 is handled with this method, and requires about 22 hours of computation time to compute the $g_i$ and the constants $c_{d,i,e}$. This motivates an additional method for computing an upper bound on $C_Q$, but not exactly computing it.

6. Petersson inner products

For the remaining cases, we use the method introduced in [13]. If $f, g \in S_2(\Gamma_0(N), \chi)$, the Petersson inner product of $f$ and $g$ is defined by

$$\langle f, g \rangle = \frac{3}{\pi [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\mathbb{H}/\Gamma_0(N)} f(x + iy) \overline{g(x + iy)} \, dx \, dy.$$ 

We will assume throughout this section that $\chi$ is a primitive Dirichlet character modulo $N$. This means that the new subspace of $S_2(\Gamma_0(N), \chi)$ is the entire space, and that any cusp form $C(z)$ in this space has a decomposition

$$C(z) = \sum_{i=1}^{s} c_i g_i(z), \quad s = \dim S_2(\Gamma_0(N), \chi)$$

where the $g_i$ are newforms. It is known that distinct newforms are orthogonal with respect to the Petersson inner product, and hence $\langle C, C \rangle = \sum_{i=1}^{s} |c_i|^2 \langle g_i, g_i \rangle$. Therefore, if $Q$ is a quadratic form and $\theta_Q(z) = E(z) + C(z)$, we may bound $C_Q$ by finding positive $A$ and $B$ so that $\langle C, C \rangle \leq A$ and $\langle g_i, g_i \rangle \geq B$ for all $i$. Then, the Cauchy-Schwarz inequality gives

$$C_Q = \sum_{i=1}^{s} |c_i| \leq \sqrt{s} \sqrt{\sum_{i=1}^{s} |c_i|^2} \leq \sqrt{As} / B.$$ 

A newform $g_i$ is said to have complex multiplication or CM if $g_i$ arises from a Hecke Grössencharacter, or equivalently if there is some negative integer $-D$ so that $\left(\frac{-D}{p}\right) = -1$ implies that the $p$th coefficient of $g_i$ is equal to zero. If $g_i$ does not have complex multiplication, then Proposition 11 of [13] proves that

$$\langle g_i, g_i \rangle \geq \frac{3}{208\pi^4 \prod_{p|N} (1 + 1/p) \log(N)}.$$ 

For a given $Q$, we can also explicitly enumerate all CM forms in $S_2(\Gamma_0(N), \chi)$ and verify that the same bound holds for those.

To compute an upper bound on $\langle C, C \rangle$, we use the following method. If $\epsilon \in \{\pm 1\}$, define

$$S_2^\epsilon(\Gamma_0(N), \chi) = \{ f \in S_2(\Gamma_0(N), \chi) : \text{if } f = \sum a(n)q^n, \text{ then } a(n) = 0 \text{ if } \chi(n) = -\epsilon \}.$$
Instead of working directly with $Q = \frac{1}{2}x^T A x$, we work with $Q^* = \frac{1}{2}x^T N A^{-1} x$. Let $\theta_{Q^*} = E^*(z) + C^*(z)$ be the decomposition into Eisenstein series and cusp forms. Proposition 15 of [13] shows that $\langle C, C \rangle = N \langle C^*, C^* \rangle$ and also that $C^* \in S_2^- (\Gamma_0(N), \chi)$. Finally, if $f(z) = \sum a(n) q^n \in S_2^- (\Gamma_0(N), \chi)$, Proposition 14 of [13] gives a formula for $\langle f, f \rangle$. Let $\psi(x) = -\frac{6}{\pi} K_1(4\pi x) + 24x^2 K_0(4\pi x)$, where $K_0$ and $K_1$ are the usual $K$-Bessel functions. Then

$$\langle f, f \rangle = \frac{1}{[SL_2(\mathbb{Z}) : \Gamma_0(N)]} \sum_{n=1}^{\infty} \frac{2^{\omega(\gcd(n,N))} a(n)^2}{n} \sum_{d=1}^{\infty} \psi \left( d \sqrt{\frac{n}{N}} \right).$$

We compute the first $15N$ coefficients of $C^*$ and let $C_1$ be the result of adding the terms above for $1 \leq n \leq 15N$. A bound on $\langle C^*, C^* \rangle$ translates into a bound on $a(n)$. Using this, we obtain the inequality

$$\langle C^*, C^* \rangle \leq C_1 + 1.71 \cdot 10^{-18} \frac{N^{7/4} s}{B} \left( 1 + \frac{1}{15N} \right)^{3/2} \langle C^*, C^* \rangle.$$

If the coefficient of $\langle C^*, C^* \rangle$ on the right hand side is less than 1, we can solve the inequality and obtain an upper bound on $\langle C^*, C^* \rangle$ and hence on $\langle C, C \rangle$. (The smallest $N$ for which the coefficient is larger than 1 is 44520, and all of the forms we work with have level $\leq 4292$.) Combining this with the lower bound on $\langle g_i, g_i \rangle$ for all $i$, this gives an upper bound on $C_Q$.

**Example.** Form 3995 is $Q(x, y, z, w) = x^2 + 2y^2 - xz + yz + 5z^2 - yw + 29w^2$ and has $\det(A) = N = 4273$, a fundamental discriminant. We have that $s = \dim S_2 (\Gamma_0(4273), \chi_{4273}) = 354$. There are no newforms with CM in this space, and $B = 2.66538 \cdot 10^{-5}$ is a lower bound for $\langle g_i, g_i \rangle$ for all $i$. Using the method described above, we find that $0.01297 \leq \langle C, C \rangle \leq 0.013026$ and this gives $C_Q \leq 415.9506$. It follows that if $n$ is squarefree and coprime to 3 and $F_4(n) > 996.385$, the $n$ is represented by $Q$. This calculation requires 77 seconds.

There are precisely 193766918 squarefree integers coprime to 3 for which $F_4(n) \leq 996.385$ and the only one of these that is not represented by $Q$ is 37. This calculation takes 177 seconds. Therefore, $Q$ represents every positive integer coprime to 3 except 37.

We use this method on all of the remaining forms. This accounts for 2869 basic quaternary forms, and 198 extra quaternary forms.

7. Proofs

**Proof of the CCXC Theorem.** If $Q$ is a positive-definite quadratic form that represents the numbers in the statement of Theorem [11] then the corresponding lattice $L$ contains either (i) a relatively universal ternary lattice, (ii) one of the 8884 basic quaternary lattices, or (iii) one of the 10 quaternary lattices that fails to locally represent everything coprime to 3.

In the first case, $Q$ is relatively universal.
In the second case, examining the squarefree positive integers not represented by the basic quaternary forms shows that the only integers coprime to 3 that might (i) not be represented by \( Q \), and (ii) not be in the statement of Theorem CCXC are 133 and 187. This is because form 6841 has truant 17 but fails to represent 187, and form 8819 has truant 19 but fails to represent 133. We compute all escalations of form 6821 (there are 22310) and none fails to represent 187. We compute all escalations of form 8819 (there are 20184) and find that each of them represents 133. Thus, \( Q \) is relatively universal.

In the third case, we must examine the extra lattices. Among the 727 extra lattices, we find forms that fail to represent 86, 91, 133, 139, 142, 154, 166, 182, 214, 238, 266, 287, 322, 329, 406 and 434. We escalate each of the 10 forms that fail to locally represent all integers coprime to 3 by their truant, and check each escalation to see if it fails to represent any of these numbers. All escalations of these 10 basic lattices represent all integers coprime to 3 except those in the statement of the theorem, and this shows that if \( L \) contains one of these 10 lattices, then \( Q \) must be relatively universal. This completes the proof. The Magma scripts used and log files are available at [http://users.wfu.edu/rouseja/CCXC/](http://users.wfu.edu/rouseja/CCXC/).

**Remark.** There are only two quadratic forms in our list that fail to represent 119: form 379 and form 8891 (one of those that fails to locally represent all integers coprime to 3). Form 379 has truant 70 and all 264341 escalations of this form by 70 represent 119. Form 8891 has a unique escalation by its truant 35 that fails to represent 119, namely \( x^2 + y^2 + 7z^2 + 7zw + 14w^2 + 35v^2 \).

**Proof of Corollary 2.** Let \( t \) be one of the positive integers in the statement of the CCXC Theorem. Then there is a form \( Q(\vec{x}) \) that has truant \( t \) (see Appendix A). The form

\[
Q(\vec{x}) + (t + 1)(a^2 + b^2 + c^2 + d^2) + \sum_{i=1}^{m-1} (t + 1 + i)x_i^2
\]

can be easily seen (by Lagrange’s four square theorem) to represent every positive integer larger than \( t \), and every positive integer coprime to 3 less than \( t \). This proves the desired claim.

**Proof of Corollary 3.** We escalate the one-dimensional lattice with Gram matrix [2] repeatedly, taking only integer-matrix escalations. We use the CCXC Theorem to determine if our escalator lattices are relatively universal, and proceed until we have found relatively universal escalator lattices. There are two binary escalator lattices corresponding to \( x^2 + y^2 \) and \( x^2 + 2y^2 \) with truants 7 and 5, respectively. Escalatings these two gives 12 ternary lattices of which two are relatively universal.

Escalating the other 10 gives rise to 261 quadratic lattices, of which all but 15 are relatively universal. All but two of the escalations of these 15 quaternary lattices are universal. These correspond to \( x^2 + y^2 + 4z^2 + 4v + 7w^2 + 13v^2 \) and \( x^2 + y^2 + 7z^2 + 13w^2 + 12wv + 13v^2 \). Of the integers relatively prime to 3 and less than 290, these fail to represent only 35. Hence, all escalations will represent all integers coprime to 3 less than 290 and so will be relatively universal.
Appendix A. Table of quadratic forms with given truants

| Form | Truant |
|------|--------|
| $2x^2$ | 1 |
| $x^2$ | 2 |
| $x^2 + 2y^2$ | 5 |
| $x^2 + y^2$ | 7 |
| $x^2 - xy + y^2 + 2z^2 + 2zw + 4w^2$ | 10 |
| $x^2 + y^2 + 5z^2$ | 11 |
| $x^2 + 2y^2 - xz + 2yz + 5z^2 - xw - 2zw + 4w^2$ | 13 |
| $x^2 + y^2 + 2z^2 - yw + 3w^2$ | 14 |
| $x^2 + xy + 2y^2 - xz - yz + 4z^2 + zw + 5w^2$ | 17 |
| $x^2 + xy + 2y^2 - xz - yz + 3z^2 - xw - 2zw + 6w^2$ | 19 |
| $x^2 + y^2 - xz + yz + 7z^2 - 2zw + 10w^2$ | 22 |
| $x^2 + 2y^2 + yz + 5z^2 - xw + 7w^2$ | 23 |
| $x^2 + 2y^2 + 3z^2 - 2yw + 3zw + 9w^2$ | 26 |
| $x^2 + 2y^2 - xz - yz + 4z^2$ | 29 |
| $x^2 + xy + 2y^2 - xz + yz + 5z^2 + 2zw + 10w^2$ | 31 |
| $x^2 + xy + 2y^2 - xz + yz + 3z^2 + 17w^2$ | 34 |
| $x^2 + 2y^2 + 5z^2 - xw - 2yw + zw + 10w^2$ | 35 |
| $x^2 + 2y^2 - xz + yz + 5z^2 - xw - 2zw + 12w^2$ | 37 |
| $x^2 + y^2 - xz + 5z^2 - xw - yw - 3zw + 11w^2$ | 38 |
| $x^2 + y^2 - xz + yz + 5z^2 - xw + 5zw + 11w^2$ | 46 |
| $x^2 + y^2 + yz + 6z^2 + 9w^2$ | 47 |
| $x^2 + y^2 + 7z^2 - xw + 7zw + 8w^2$ | 55 |
| $x^2 + 2y^2 + 3z^2 - xw - yw + 3zw + 8w^2$ | 58 |
| $x^2 + xy + 2y^2 + 5z^2 - wz + 5zw + 6w^2$ | 62 |
| $x^2 + y^2 + yz + 2z^2 - xw + 23w^2$ | 70 |
| $x^2 + y^2 - xz + yz + 5z^2 - xw - yw + 12w^2$ | 94 |
| $x^2 + y^2 + yz + 3z^2 + 22w^2$ | 110 |
| $x^2 + y^2 + 7z^2 + 7zw + 14w^2 + 35w^2$ | 119 |
| $x^2 + 2y^2 - xz - yz + 4z^2 + 29w^2$ | 145 |
| $x^2 + 2y^2 - xz - yz + 4z^2 + 29w^2 + 29zw + 58wv + 145v^2$ | 203 |
| $x^2 + 2y^2 - xz - yz + 4z^2 + 29w^2 + 29zw + 87wv + 145v^2$ | 290 |

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