Is $U_{e3}$ really related to the solar neutrino solutions?

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Abstract

It has been said that the measurements of $U_{e3}$ in the lepton flavor mixing matrix would help discriminate between the possible solar neutrino solutions under the natural conditions with the neutrino mass hierarchies of $m_1 \ll m_2 \ll m_3$ and $m_1 \sim m_2 \gg m_3$, where $m_i$ is the $i$-th generation neutrino absolute mass. However, it is not true, and the relation between $\sin^2 2\theta_{12}$ and $U_{e3}$ obtained by Akhmedov, Branco, and Rebelo is trivial in actual. We show in this paper that the value of $U_{e3}$ cannot predict the solar neutrino solutions without one additional nontrivial condition.
Recent neutrino oscillation experiments suggest the strong evidences of tiny neutrino masses and lepton flavor mixings[1, 2, 3, 4]. Studies of the lepton flavor mixing matrix, which is so-called Maki-Nakagawa-Sakata (MNS) matrix[5], will give us important cues of the physics beyond the standard model. The mixing angle between the second and the third generations is expected to be almost maximal[6], and the large mixing between the first and the second generations is also favored[7] as the large angle MSW (MSW-L) solution[8]. On the other hand, the mixing between the first and the third generations, which corresponds to $U_{e3}$ in the MNS matrix, is small as the present upper bound of CHOOZ experiments show $U_{e3} < 0.16[8]$. It is very interesting if value of $U_{e3}$ is related to the solar neutrino solutions. In Ref.[8], Akhmedov, Branco, and Rebelo said that the measurements of $U_{e3}$ would help discriminate between the possible solar neutrino solutions under the natural conditions with the neutrino mass hierarchies of $m_1 \ll m_2 \ll m_3$ and $m_1 \sim m_2 \gg m_3$, where $m_i$ is the $i$-th generation neutrino absolute mass. However, it is not true, and the relation between $\sin^2 2\theta_{12}$ and $U_{e3}$ obtained in Ref.[8] is trivial in actual. We will show in this paper that the value of $U_{e3}$ cannot predict the solar neutrino solutions without one additional nontrivial condition. This is because we know only four parameters, $\sin^2 2\theta_{12}, \sin^2 2\theta_{23}, \Delta m^2_{s\odot},$ and $\Delta m^2_{ATM}$, from experiments, although five parameters must be needed in order to obtain the MNS matrix, and its element $U_{e3}$.

Let us start our discussions with each type of neutrino mass hierarchy. Neutrino mass spectra can be classified in three types[9] as, Type A: $m_1 \ll m_2 \ll m_3$, Type B: $m_1 \sim m_2 \gg m_3$, and Type C: $m_1 \sim m_2 \sim m_3$. It is expected that $\Delta m^2_{ATM} \simeq |m_3^2 - m_2^2|$ and $\Delta m^2_{s\odot} \simeq |m_3^2 - m_1^2|$. By using $\theta_{23} = \pi/4$ and $U_{e3} = \epsilon(\ll 1)$ according to the data of the Super-Kamiokande and the CHOOZ experiments, respectively, the MNS matrix is given by [10]

$$U = \begin{pmatrix} c & s & \epsilon \\ -\sqrt{\frac{1}{2}}(s + c\epsilon) & \sqrt{\frac{1}{2}}(c - s\epsilon) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}(s - c\epsilon) & -\frac{1}{\sqrt{2}}(c + s\epsilon) & \frac{1}{\sqrt{2}} \end{pmatrix},$$  \hspace{1cm} (1)

where $c \equiv \cos \theta_{12}$ and $s \equiv \sin \theta_{12}$. The Majorana mass matrix of neutrino in the diagonal base of the charged lepton mass matrix is given by [10]

$$M_{\nu} = U \text{diag.}(m_1, m_2, m_3) U^T,$$  \hspace{1cm} (2)

$$\begin{pmatrix} \mu & \frac{1}{\sqrt{2}}[\epsilon(m_3 - \mu) + m_- c\epsilon] & \frac{1}{\sqrt{2}}[\epsilon(m_3 - \mu) - m_- c\epsilon] \\ \frac{1}{\sqrt{2}}[\epsilon(m_3 - \mu) + m_- c\epsilon] & \frac{1}{2}(m_3 + m_1 - 2m_- c\epsilon) & \frac{1}{2}(m_3 - m_1) \\ \frac{1}{\sqrt{2}}[\epsilon(m_3 - \mu) - m_- c\epsilon] & \frac{1}{2}(m_3 - m_1) & \frac{1}{2}(m_3 + m_1 - 2m_- c\epsilon) \end{pmatrix}$$  \hspace{1cm} (3)

where

$$\mu \equiv m_1c^2 + m_2s^2, \quad \mu' \equiv m_1s^2 + m_2c^2, \quad m_- \equiv m_2 - m_1.$$  \hspace{1cm} (4)

In Type A with the mass hierarchy of $m_1 \ll m_2 \ll m_3$, the neutrino mass matrix of
Eq. (3) is written by

\[ M_\nu = m_0 \begin{pmatrix} \kappa & \alpha + \beta & \alpha - \beta \\ \alpha + \beta & 1 + \delta - \delta' & 1 - \delta \\ \alpha - \beta & 1 - \delta & 1 + \delta + \delta' \end{pmatrix}, \tag{5} \]

where we just normalize Eq. (3) by \( m_3 \) as \( m_0 \equiv m_3 / 2 \), \( \kappa \equiv \sqrt{2} \mu / m_3 \), \( \beta \equiv \sqrt{2} m_{cs} / m_3 \), \( \delta \equiv \mu' / m_3 \), and \( \delta' \equiv (2 m_{cs} / m_3) \epsilon \). The values of \( m_0, \kappa, \beta, \) and \( \delta \) are determined by the atmospheric and the solar neutrino solutions. Only \( \alpha \) and \( \delta' \) are unknown parameters, since they have the free parameter \( \epsilon \). Equation (5) induces the mixing angles of

\[ \tan 2\theta_{12} = \frac{\sqrt{2} \beta}{\delta - \frac{\alpha}{2}}, \quad \sin \theta_{13} = \frac{\alpha}{\sqrt{2} (1 - \frac{\alpha}{2})}. \tag{6} \]

By using the approximations of \( \Delta m^2_{ATM} \simeq m_3^2 \) and \( \Delta m^2_{sol} \simeq m_2^2 \), they become

\[ \tan 2\theta_{12} \simeq \frac{\sqrt{2} \beta}{\cos 2\theta_{12} \sqrt{\frac{\Delta m^2_{sol}}{\Delta m^2_{ATM}}}}, \quad \sin \theta_{13} \simeq \frac{\alpha}{\sqrt{2} \left(1 - \sin^2 \theta_{12} \sqrt{\frac{\Delta m^2_{sol}}{\Delta m^2_{ATM}}} \right)}. \tag{7} \]

Here we must notice that the value of \( \beta \) is determined by the atmospheric and the solar neutrino solutions. Only \( \epsilon = \sin \theta_{13} \) is the free parameter with \( U_{e3} (= \sin \theta_{13} ) < 0.16 \), which makes the value of \( \alpha \) be also free parameter. If \( O(\alpha) \approx O(\beta) \), which dose not have physical meaning, Eqs. (7) induce

\[ U_{e3} \simeq \frac{1}{\sqrt{2} \sin 2\theta_{12} \sqrt{\frac{\Delta m^2_{sol}}{\Delta m^2_{ATM}}}}. \tag{8} \]

The right-hand side of this equation gives the following values of \( U_{e3} \) corresponding to the solar neutrino solutions as

\[ U_{e3} \sim 10^{-1.5} \quad (\text{MSW} - L), \quad 10^{-3.5} \quad (\text{VO}), \quad 10^{-3} \quad (\text{MSW} - S). \tag{9} \]

These results are the same as those of Ref. [8]. It seems that the measurements of \( U_{e3} \) can predict the solar neutrino solutions from Eq. (9). However, we must notice that the relation of Eq. (9) is satisfied just only in the case of \( O(\alpha) \approx O(\beta) \). This is the trivial condition,

\[ * \text{ Equation (8) is not the same as the result in Ref. [8]} \]

\[ \sin \theta_{13} \simeq \frac{1}{2} \frac{\tan 2\theta_{12}}{1 + \tan^2 2\theta_{12}} \left( \frac{\Delta m^2_{sol}}{\Delta m^2_{ATM}} \right)^{1/2}, \]

which can not apply to the large angle solutions. Our result of Eq. (8) can apply not only to the small angle solution but also to the large angle solutions.
since \( \alpha \) is the free parameter which has nothing to do with \( \beta \) at all. In Ref.\[8\], they have denoted \( \varepsilon \equiv \alpha + \beta \) and \( \varepsilon' \equiv \alpha - \beta \), and said that \( \varepsilon + \varepsilon' \) and \( \varepsilon - \varepsilon' \) are expected to be of the same order if there are no accidental cancellations. However, the condition \( \varepsilon + \varepsilon' \simeq \varepsilon - \varepsilon' \) means \( \alpha \simeq \beta \), which is not the natural condition but just the trivial assumption. Any relations between \( \alpha \) and \( \beta \) are considerable, and for example, if we take \( O(\alpha) \ll O(\beta) \), Eq.(8) becomes

\[
\sin \theta_{13} \ll \frac{1}{\sqrt{2}} \sin 2\theta_{12} \sqrt{\frac{\Delta m_{sol}^2}{\Delta m_{ATM}^2}}. \tag{10}
\]

We stress again that Eqs.(8) and (10) are just the trivial relations which are induced by the trivial assumptions of \( O(\alpha) \simeq O(\beta) \) and \( O(\alpha) \ll O(\beta) \), respectively.

We can similarly analyze the case of Type B with the mass hierarchy of \( m_1 \sim m_2 \gg m_3 \). In the case of (b1), which is \( M_{d} \simeq \text{diag}.(m_1, m_2, 0) \) in the first order, with the notation of \( m_2 = m_1 + d \) \(|d| \ll m_1\) the neutrino mass matrix \( M_{\nu} \) is given by

\[
M_{\nu} = m_1 \begin{pmatrix}
1 + \kappa & \alpha + \beta & \alpha - \beta \\
\alpha + \beta & \frac{1}{2} + \delta + \gamma + \delta' & -\frac{1}{2} + \delta - \gamma \\
\alpha - \beta & -\frac{1}{2} + \delta - \gamma & \frac{1}{2} + \delta + \gamma + \delta'
\end{pmatrix}, \tag{11}
\]

where \( \kappa \equiv ds^2/m_1 \), \( \alpha \equiv -(\varepsilon/\sqrt{2}) [1 + (m_3 - ds^2)/m_1] \), \( \beta \equiv dcs/\sqrt{2}m_1 \), \( \delta \equiv m_3/2m_1 \), \( \gamma \equiv dc^2/2m_1 \), and \( \delta' \equiv (dcs/m_1)\varepsilon \). The values of \( \kappa, \beta, \delta \), and \( \gamma \) are determined by the atmospheric and the solar neutrino solutions. Only \( \alpha \) and \( \delta' \) are free parameters, since they contain \( \varepsilon \). We can easily obtain mixing angles

\[
\tan 2\theta_{12} = \frac{\sqrt{2}\beta}{\gamma + \frac{\kappa}{2}}, \quad \sin \theta_{13} = -\frac{\sqrt{2}\alpha}{1 + \kappa - 2\delta}, \tag{12}
\]

from Eq.(11). By using the approximations of \( \Delta m_{ATM}^2 \simeq m_1^2 \), and \( \Delta m_{sol}^2 \simeq 4m_1^2(\gamma + \frac{\kappa}{2})(1 + \tan^2\theta_{12})^{1/2} \), we can obtain

\[
\alpha \simeq -\frac{1}{\sqrt{2}} \sin \theta_{13}, \quad \beta \simeq \frac{1}{4\sqrt{2}} \sin 2\theta_{12} \left( \frac{\Delta m_{sol}^2}{\Delta m_{ATM}^2} \right). \tag{13}
\]

Here we must notice that the value of \( \beta \) is determined by the atmospheric and the solar neutrino solutions, and \( \alpha \) (\( \sin \theta_{13} \)) is the free parameter. If \( O(\alpha) \simeq O(\beta) \), which does not have physical meaning, Eqs.(13) induce

\[
U_{e3} \simeq \frac{1}{4} \sin 2\theta_{12} \left( \frac{\Delta m_{sol}^2}{\Delta m_{ATM}^2} \right). \tag{14}
\]

Type B has two patterns of (b1) and (b2) according to the relative sign assignments of mass eigenvalues\[9\]. The stability of the mixing angles against the quantum corrections strongly depends on the relative assignments of mass eigenvalues\[11\].
This equation is the same as that of Ref.[5]. As we have shown in the case of Type A, Eq.(13) is the trivial equation which is obtained from the trivial assumption of \( O(\alpha) \approx O(\beta) \), since \( \alpha \) is the free parameter which has nothing to do with \( \beta \).

Similar discussions can be applied to the case of (b2), which is \( M^d_\nu \approx \text{diag.}(m_1,-m_2,0) \) in the first order. With the notation of \( m_2 = -m_1 + d \) (|d| \( \ll \) \( m_1 \)), the neutrino mass matrix \( M_\nu \) is given by

\[
M_\nu = m_1 \begin{pmatrix}
(c^2 - s^2) + \kappa & -\sqrt{2}cs + \alpha + \beta & \sqrt{2}cs + \alpha - \beta \\
-\sqrt{2}cs + \alpha + \beta & -\frac{1}{2}(c^2 - s^2) + \delta + \gamma - \delta' & \frac{1}{2}(c^2 - s^2) + \delta - \gamma \\
\sqrt{2}cs + \alpha - \beta & \frac{1}{2}(c^2 - s^2) + \delta - \gamma & -\frac{1}{2}(c^2 - s^2) + \delta + \gamma + \delta'
\end{pmatrix}
\]

(15)

where \( \kappa \equiv ds^2/m_1, \alpha \equiv -(\epsilon/\sqrt{2}) \ [(c^2 - s^2) - m_3/m_1 + ds^2/m_1], \beta \equiv dcs/\sqrt{2}m_1, \delta \equiv m_3/2m_1, \gamma \equiv dc^2/2m_1, \delta' \equiv [(d - 2m_1)cs/m_1]\epsilon \).

The values of \( \kappa, \beta, \delta, \) and \( \gamma \) are determined by the atmospheric and the solar neutrino solutions. The parameters \( \alpha \) and \( \delta' \) are free, which contain \( \epsilon \). The mixing angles are induced from Eq.(13) as

\[
\tan 2\theta_{12} = -\frac{\sqrt{2}(\beta - \sqrt{2}cs)}{(c^2 - s^2) - \gamma + \frac{\kappa}{2}}, \quad \sin \theta_{13} = -\frac{\sqrt{2}\alpha}{(c^2 - s^2) + \kappa - 2\delta}.
\]

(16)

By using the approximations of \( \Delta m_{\text{ATM}}^2 \approx m_1^2[(c^2 - s^2 - \gamma + \kappa/2)^2 + 2(\sqrt{2}cs - \beta)^2] \) and \( \Delta m_{\text{sol}}^2 \approx 4m_1^2[(\gamma + \kappa/2)[(c^2 - s^2 - \gamma + \kappa/2)^2 + 2(\sqrt{2}cs - \beta)^2]]^{1/2} \), the mixing angles in Eqs.(16) become

\[
\tan 2\theta_{12} \simeq -\sqrt{2}\beta, \quad \sin \theta_{13} \simeq -\sqrt{2}\alpha, \quad \text{(small mixing), (17)}
\]

\[
\sin 2\theta_{12} \simeq 1 - \sqrt{2}\beta, \quad \sin \theta_{13} \simeq -\sqrt{2}\alpha \left(\frac{d}{2m_1} - \frac{m_3}{m_1}\right)^{-1}, \quad \text{(large mixing). (18)}
\]

If \( O(\alpha) \approx O(\beta) \), Eqs.(17) and (18) induce

\[
\sin \theta_{13} \simeq \tan \theta_{12}, \quad \text{(small mixing), (19)}
\]

\[
\sin \theta_{13} \simeq (1 - \sin 2\theta_{12}) \left(\frac{m_3}{m_1} - \frac{d}{2m_1}\right)^{-1}, \quad \text{(large mixing). (20)}
\]

Similarly this is also the trivial relation.

In Type C with the mass hierarchy of \( m_1 \approx m_2 \approx m_3 \), we show the case of (c4)[11], for example, which is \( M^d_\nu \approx \text{diag.}(m_1,m_2,m_3) \) in the first order. With the notation of
\[ m_2 = m_1 + d, \ m_3 = m_1 + D, \text{ and } |d| \ll |D| \ll m_1, \] the neutrino mass matrix \( M_\nu \) is given by

\[
M_\nu = m_1 \begin{pmatrix}
1 + \kappa & \alpha + \beta & \alpha - \beta \\
\alpha + \beta & 1 + \delta + \gamma - \delta' & \delta - \gamma \\
\alpha - \beta & \delta - \gamma & 1 + \delta + \gamma + \delta'
\end{pmatrix}
\]

(21)

where \( \kappa \equiv d s^2 / m_1 \), \( \alpha \equiv (\epsilon / \sqrt{2}) [D / m_1 - d s^2 / m_1] \), \( \beta \equiv d c s / \sqrt{2} m_1 \), \( \delta \equiv D / 2 m_1 \), \( \gamma \equiv d c^2 / 2 m_1 \), and \( \delta' \equiv (d c s / m_1) \epsilon \). Equation (21) induces the mixing angles as

\[
\tan \theta_{12} = \frac{\sqrt{2} \beta}{\gamma - \kappa / 2}, \quad \sin \theta_{13} = \frac{\sqrt{2} \alpha}{2 \delta - \kappa}.
\]

(22)

By using the approximations of \( \Delta m^2_{\text{sol}} \simeq 4 m_1^2 (1 + \gamma + \kappa / 2) [(\gamma - \kappa / 2)^2 + 2 \beta^2]^{1/2} \) and \( \Delta m^2_{\text{ATM}} \simeq 4 \delta m_1^2 \), we can obtain

\[
\frac{\beta}{\delta} \simeq \frac{1}{\sqrt{2}} \sin 2 \theta_{12} \left( \frac{\Delta m^2_{\text{sol}}}{\Delta m^2_{\text{ATM}}} \right), \quad \frac{\alpha}{\delta} \simeq \sqrt{2} \sin \theta_{13}.
\]

(23)

Under the trivial assumption of \( O(\alpha) \simeq O(\beta) \), Eqs.(23) induce the trivial relation

\[
\sin \theta_{13} \simeq \frac{1}{2} \sin 2 \theta_{12} \frac{\Delta m^2_{\text{sol}}}{\Delta m^2_{\text{ATM}}}.
\]

(24)

It has been said that the measurements of \( U_{e3} \) in the lepton flavor mixing matrix would help discriminate between the possible solar neutrino solutions under the natural conditions with the neutrino mass hierarchies of \( m_1 \ll m_2 \ll m_3 \) and \( m_1 \sim m_2 \gg m_3 \) [8]. However, it is not true, and the relation between \( \sin^2 2 \theta_{12} \) and \( U_{e3} \) obtained in Ref.[8] is trivial in actual.

Why can not we obtain the relations between the value of \( U_{e3} \) and the solar neutrino solutions? This is easily understood as follows. Neglecting the \( CP \) phases in the lepton sector, the number of independent parameters in the Majorana mass matrix of neutrino are six. Five parameters are enough to determine the MNS matrix, since overall factor in the neutrino mass matrix does not contribute to the MNS matrix. Thus we need five input parameters in order to determine the MNS matrix, and its element \( U_{e3} \). Since the neutrino oscillation experiments except for the CHOOZ give us only four input parameters \( \Delta m^2_{\text{ATM}}, \Delta m^2_{\text{sol}}, \sin^2 2 \theta_{12}, \) and \( \sin^2 2 \theta_{23} \), the value of \( U_{e3} \) remains as an unknown parameter, which we only know the upper bound from CHOOZ experiments as \( U_{e3} < 0.16 \) [4]. Therefore the value of \( U_{e3} \) cannot predict the solar neutrino solutions without one additional nontrivial condition.
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