Cramer’s Rule for Generalized Inverse Solutions of Some Matrices Equations

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Abstract

By a generalized inverse of a given matrix, we mean a matrix that exists for a larger class of matrices than the nonsingular matrices, that has some of the properties of the usual inverse, and that agrees with inverse when given matrix happens to be nonsingular. In theory, there are many different generalized inverses that exist. We shall consider the Moore Penrose, weighted Moore-Penrose, Drazin and weighted Drazin inverses.

New determinantal representations of these generalized inverse based on their limit representations are introduced in this paper. Application of this new method allows us to obtain analogues classical adjoint matrix. Using the obtained analogues of the adjoint matrix, we get Cramer’s rules for the least squares solution with the minimum norm and for the Drazin inverse solution of singular linear systems. Cramer’s rules for the minimum norm least squares solutions and the Drazin inverse solutions of the matrix equations $AX = D$, $XB = D$ and $AXB = D$ are also obtained, where $A$, $B$ can be singular matrices of appropriate size. Finally, we derive determinantal representations of solutions of the differential matrix equations, $X' + AX = B$ and $X' +XA = B$, where the matrix $A$ is singular.

1 Preface

It’s well-known in linear algebra, an $n$-by-$n$ square matrix $A$ is called invertible (also nonsingular or nondegenerate) if there exists an $n$-by-$n$ square matrix $X$ such that

$$AX =XA = I_n.$$ 

If this is the case, then the matrix $X$ is uniquely determined by $A$ and is called the inverse of $A$, denoted by $A^{-1}$.

By a generalized inverse of a given matrix, we mean a matrix that exists for a larger class of matrices than the nonsingular matrices, that has some of the properties of the usual inverse, and that agrees with inverse when given matrix happens to be nonsingular.

For any matrix $A \in \mathbb{C}^{m \times n}$ consider the following equations in $X$:

$$AXA = A; \tag{1.1}$$

$$XAX = X; \tag{1.2}$$

$$(AX)^* = AX; \tag{1.3}$$
and if \( m = n \), also
\[
\mathbf{AX} = \mathbf{AX}; \quad (1.5)
\]
\[
\mathbf{A}^{k+1} \mathbf{X} = \mathbf{A}^k. \quad (1.6)
\]

For a sequence \( \mathcal{G} \) of \( \{1, 2, 3, 4, 5\} \) the set of matrices obeying the equations represented in \( \mathcal{G} \) is denoted by \( \mathbf{A}\{\mathcal{G}\} \). A matrix from \( \mathbf{A}\{\mathcal{G}\} \) is called an \( \mathcal{G} \)-inverse of \( \mathbf{A} \) and denoted by \( \mathbf{A}^{(\mathcal{G})} \).

Consider some principal cases.

If \( \mathbf{X} \) satisfies all the equations (1.1)-(1.4) is said to be the Moore-Penrose inverse of \( \mathbf{A} \) and denote \( \mathbf{A}^+ = \mathbf{A}^{(1,2,3,4)} \). The Moore-Penrose inverse was independently described by E. H. Moore [1] in 1920, Arne Bjerhammar [2] in 1951 and Roger Penrose [3] in 1955. R. Penrose introduced the characteristic equations (1.1)-(1.4).

If \( \det \mathbf{A} \neq 0 \), then \( \mathbf{A}^+ = \mathbf{A}^{-1} \).

The group inverse \( \mathbf{A}^g \) is the unique \( \mathbf{A}^{(1,2,5)} \) inverse of \( \mathbf{A} \), and exists if and only if \( \text{Ind} \mathbf{A} = \min \{ k : \text{rank} \mathbf{A}^{k+1} = \text{rank} \mathbf{A}^k \} = 1 \).

A matrix \( \mathbf{X} = \mathbf{A}^D \) is said to be the Drazin inverse of \( \mathbf{A} \) if (1.6) (for some positive integer \( k \)), (1.2) and (1.5) are satisfied, where \( k = \text{Ind} \mathbf{A} \). It is named after Michael P. Drazin [4]. In particular, when \( \text{Ind} \mathbf{A} = 1 \), then the matrix \( \mathbf{X} \) is the group inverse, \( \mathbf{X} = \mathbf{A}^g \). If \( \text{Ind} \mathbf{A} = 0 \), then \( \mathbf{A} \) is nonsingular, and \( \mathbf{A}^D = \mathbf{A}^{-1} \).

Let Hermitian positive definite matrices \( \mathbf{M} \) and \( \mathbf{N} \) of order \( m \) and \( n \), respectively, be given. For any matrix \( \mathbf{A} \in \mathbb{C}^{m \times n} \), the weighted Moore-Penrose inverse of \( \mathbf{A} \) is the unique solution \( \mathbf{X} = \mathbf{A}^{+\mathbf{M},\mathbf{N}} \) of the matrix equations (1.1) and (1.2) and the following equations in \( \mathbf{X} \) [5]:
\[
(3\mathbf{M}) (\mathbf{MAX})^* = \mathbf{MAX}; \quad (4\mathbf{N}) (\mathbf{NXA})^* = \mathbf{NXA}.
\]

In particular, when \( \mathbf{M} = \mathbf{I}_m \) and \( \mathbf{N} = \mathbf{I}_n \), the matrix \( \mathbf{X} \) satisfying the equations (1.1), (1.2), (3\mathbf{M}), (4\mathbf{N}) is the Moore-Penrose inverse \( \mathbf{A}^+ \).

The weighted Drazin inverse is being considered as well.

To determine the inverse and to give its analytic solution, we calculate a matrix of cofactors, known as an adjugate matrix or a classical adjoint matrix. The classical adjoint of \( \mathbf{A} \), denote \( \text{Adj}[\mathbf{A}] \), is the transpose of the cofactor matrix, then \( \mathbf{A}^{-1} = \frac{\text{Adj}[\mathbf{A}]}{\|\mathbf{A}\|} \). Representation an inverse matrix by its classical adjoint matrix also plays a key role for Cramer’s rule of systems of linear equations or matrices equations.

Obviously, the important question is the following: what are the analogues for the adjoint matrix of generalized inverses and, consequently, for Cramer’s rule of generalized inverse solutions of matrix equations?

This is the main goal of the paper.

In this paper we shall adopt the following notation. Let \( \mathbb{C}^{m \times n} \) be the set of \( m \) by \( n \) matrices with complex entries, \( \mathbb{C}^{r \times r} \) be a subset of \( \mathbb{C}^{m \times n} \) in which any matrix has rank \( r \), \( \mathbf{I}_n \) be the identity matrix of order \( m \), and \( \|\cdot\| \) be the Frobenius norm of a matrix.
Denote by $a_{j}$ and $a_{i}$ the $j$th column and the $i$th row of $A \in \mathbb{C}^{m \times n}$, respectively. Then $a_{j}^{*}$ and $a_{i}^{*}$ denote the $j$th column and the $i$th row of a conjugate and transpose matrix $A^{*}$ as well. Let $A_{j}(b)$ denote the matrix obtained from $A$ by replacing its $j$th column with the vector $b$, and by $A_{i}(b)$ denote the matrix obtained from $A$ by replacing its $i$th row with $b$.

Let $\alpha := \{\alpha_{1}, \ldots, \alpha_{k}\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_{1}, \ldots, \beta_{k}\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. Then $[A_{\alpha}^{\beta}]$ denotes the minor of $A$ determined by the rows indexed by $\alpha$ and the columns indexed by $\beta$. Clearly, $|A_{\alpha}^{\alpha}|$ denotes a principal minor determined by the rows and columns indexed by $\alpha$. The cofactor of $a_{ij}$ in $A \in \mathbb{C}^{n \times n}$ is denoted by $\frac{\partial}{\partial a_{ij}} |A|$.

For $1 \leq k \leq n$, $L_{k,n} := \{\alpha : \alpha = (\alpha_{1}, \ldots, \alpha_{k}), 1 \leq \alpha_{1} \leq \cdots \leq \alpha_{k} \leq n\}$ denotes the collection of strictly increasing sequences of $k$ integers chosen from $\{1, \ldots, n\}$. Let $N_{k} := L_{k,m} \times L_{k,n}$. For fixed $\alpha \in L_{p,m}$, $\beta \in L_{p,n}$, $1 \leq p \leq k$, let

\[
I_{k,m}(\alpha) := \{I : I \in L_{k,m}, I \supseteq \alpha\},
J_{k,n}(\beta) := \{J : J \in L_{k,n}, J \supseteq \beta\},
N_{k}(\alpha, \beta) := I_{k,m}(\alpha) \times J_{k,n}(\beta)
\]

For case $i \in \alpha$ and $j \in \beta$, we denote

\[
I_{k,m}\{i\} := \{\alpha : \alpha \in L_{k,m}, i \in \alpha\}, J_{k,n}\{j\} := \{\beta : \beta \in L_{k,n}, j \in \beta\},
N_{k}\{i, j\} := I_{k,m}\{i\} \times J_{k,n}\{j\}.
\]

The paper is organized as follows. In Section 2 determinantal representations by analogues of the classical adjoint matrix for the Moore Penrose, weighted Moore-Penrose, Drazin and weighted Drazin inverses are obtained.

In Section 3 we show that the obtained analogues of the adjoint matrix for the generalized inverse matrices enable us to obtain natural analogues of Cramer’s rule for generalized inverse solutions of systems of linear equations and demonstrate it in two examples.

In Section 4, we obtain analogs of the Cramer rule for generalized inverse solutions of the matrix equations, $AX = B$, $XA = B$ and $AXB = D$, namely for the minimum norm least squares solutions and the Drazin inverse solutions. We show numerical examples to illustrate the main results as well.

In Section 5, we use the determinantal representations of the Drazin inverse solution to solutions of the following differential matrix equations, $X' + AX = B$ and $X' + AX = B$, where $A$ is singular. It is demonstrated in the example.

Facts set forth in Sections 2 and 3 were partly published in [6]. In Section 4 were published in [7, 8] and in Sections 5 were published in [8].

Note that we obtained some of the submitted results for matrices over the quaternion skew field within the framework of the theory of the column and row determinants (914).
2 Analogues of the Classical Adjoint Matrix for Generalized Inverse Matrices

For determinantal representations of the generalized inverse matrices as analogues of the classical adjoint matrix, we apply the method, which consists on the limit representation of the generalized inverse matrices, lemmas on rank of some matrices and on characteristic polynomial. We used this method at first in [6] and then in [8]. Liu et al. in [15] deduce the new determinantal representations of the outer inverse $A^{(2)}_{T,S}$ based on these principles as well. In this paper we obtain detailed determinantal representations by analogues of the classical adjoint matrix for the Moore Penrose, weighted Moore-Penrose, Drazin and weighted Drazin inverses.

2.1 Analogues of the Classical Adjoint Matrix for the Moore - Penrose Inverse

Determinantal representation of the Moore - Penrose inverse was studied in [1], [16–19]. The main result consists in the following theorem.

**Theorem 2.1** The Moore - Penrose inverse $A^+ = (a^+_{ij}) \in \mathbb{C}^{n \times m}$ of $A \in \mathbb{C}^{m \times n}$ has the following determinantal representation

$$a^+_{ij} = \frac{\sum_{(\alpha, \beta) \in N_r \{j, i\}} \left| \left( A^* \right)_\alpha^\beta \frac{\partial}{\partial a_{\alpha \beta}} A^\alpha_\beta \right|}{\sum_{(\gamma, \delta) \in N_r \{j, i\}} \left| \left( A^* \right)_\gamma^\delta \right| \left| A^\gamma_\delta \right|}, \ 1 \leq i, j \leq n.$$ 

This determinantal representation of the Moore - Penrose inverse is based on corresponding full-rank representation [16]: if $A = PQ$, where $P \in \mathbb{C}^{m \times r}$ and $Q \in \mathbb{C}^{r \times n}$, then

$$A^+ = Q^* (P^*AQ^*)^{-1} P^*.$$ 

For a better understanding of the structure of the Moore - Penrose inverse we consider it by singular value decomposition of $A$. Let

$$A A^* u_i = \sigma_i^2 u_i, \quad i = 1, m,$$

$$A^* A v_i = \sigma_i^2 v_i, \quad i = 1, n,$$

$$\sigma_1 \leq \sigma_2 \leq ... \sigma_r > 0 = \sigma_{r+1} = \sigma_{r+2} = ...$$

and the singular value decomposition (SVD) of $A$ is $A = U \Sigma V^*$, where

$$U = [u_1 u_2...u_m] \in \mathbb{C}^{m \times m}, \quad U^* U = I_m,$$

$$V = [v_1 v_2...v_n] \in \mathbb{C}^{n \times n}, \quad V^* V = I_n,$$

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_r) \in \mathbb{C}^{m \times n}.$$ 

Then [8], $A^+ = V \Sigma^+ U^*$, where $\Sigma^+ = \text{diag}(\sigma_1^{-1}, \sigma_2^{-1}, ..., \sigma_r^{-1})$.

We need the following limit representation of the Moore-Penrose inverse.
Lemma 2.2 [27] If $A \in \mathbb{C}^{m \times n}$, then
\[ A^+ = \lim_{\lambda \to 0} A^* (AA^* + \lambda I)^{-1} = \lim_{\lambda \to 0} (A^* A + \lambda I)^{-1} A^*, \]
where $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is the set of positive real numbers.

Corollary 2.3 [27] If $A \in \mathbb{C}^{m \times n}$, then the following statements are true.

i) If $\text{rank } A = n$, then $A^+ = (A^* A)^{-1} A^*$.

ii) If $\text{rank } A = m$, then $A^+ = A^* (AA^*)^{-1}$.

iii) If $\text{rank } A = n = m$, then $A^+ = A^{-1}$.

We need the following well-known theorem about the characteristic polynomial and lemmas on rank of some matrices.

Theorem 2.4 [22] Let $d_r$ be the sum of principal minors of order $r$ of $A \in \mathbb{C}^{n \times n}$. Then its characteristic polynomial $p_A(t)$ can be expressed as $p_A(t) = \det(tI - A) = t^n - d_1 t^{n-1} + d_2 t^{n-2} - \ldots + (-1)^n d_n$.

Lemma 2.5 If $A \in \mathbb{C}_r^{m \times n}$, then $\text{rank } (A^* A)_{ik} (a^*_j) \leq r$.

Proof. Let $P_{i,k}(-a_{j,k}) \in \mathbb{C}^{n \times n}$, $(k \neq i)$, be the matrix with $-a_{j,k}$ in the $(i,k)$ entry, 1 in all diagonal entries, and 0 in others. It is the matrix of an elementary transformation. It follows that
\[
(A^* A)_{ik} (a^*_j) \cdot \prod_{k \neq i} P_{i,k}(-a_{j,k}) = \begin{pmatrix}
\sum_{k \neq j} a^*_{1,k} a_{k,1} & \ldots & a^*_{1,j} & \ldots & \sum_{k \neq j} a^*_{1,k} a_{k,n} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\sum_{k \neq j} a^*_{n,k} a_{k,1} & \ldots & a^*_{n,j} & \ldots & \sum_{k \neq j} a^*_{n,k} a_{k,n}
\end{pmatrix}_{i-th}.
\]
The obtained above matrix has the following factorization.
\[
\begin{pmatrix}
\sum_{k \neq j} a^*_{1,k} a_{k,1} & \ldots & a^*_{1,j} & \ldots & \sum_{k \neq j} a^*_{1,k} a_{k,n} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\sum_{k \neq j} a^*_{n,k} a_{k,1} & \ldots & a^*_{n,j} & \ldots & \sum_{k \neq j} a^*_{n,k} a_{k,n}
\end{pmatrix}_{i-th} = \begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1m} \\
a_{21} & a_{22} & \ldots & a_{2m} \\
\ldots & \ldots & \ldots & \ldots \\
a_{n1} & a_{n2} & \ldots & a_{nm}
\end{pmatrix}_{j-th} \begin{pmatrix}
a_{11} & \ldots & 0 & \ldots & a_{n1} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
a_{m1} & 0 & \ldots & a_{mn}
\end{pmatrix}_{i-th}.
\]
Denote by $\tilde{A} := \begin{pmatrix} a_{11} & \ldots & 0 & \ldots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \ldots & 0 & \ldots & a_{mn} \end{pmatrix}$ the matrix obtained from $A$ by replacing all entries of the $j$th row and of the $i$th column with zeroes except that the $(j, i)$ entry equals 1. Elementary transformations of a matrix do not change its rank. It follows that rank $(A^*A)_{i,j} (a_{i,j}^*) \leq \min \{\text{rank} A^*, \text{rank} \tilde{A}\}$. Since rank $\tilde{A} \geq \text{rank} A = \text{rank} A^*$ and rank $A^*A = \text{rank} A$ the proof is completed. \[\blacksquare\]

The following lemma can be proved in the same way.

**Lemma 2.6** If $A \in \mathbb{C}^{m \times n}$, then rank $(AA^*)_{i,j} (a_{i,j}^*) \leq r$.

Analogues of the characteristic polynomial are considered in the following two lemmas.

**Lemma 2.7** If $A \in \mathbb{C}^{m \times n}$ and $\lambda \in \mathbb{R}$, then

$$
\det (\lambda I_n + A^*A)_{i,j} (a_{i,j}^*) = c_1^{(ij)} \lambda^{n-1} + c_2^{(ij)} \lambda^{n-2} + \ldots + c_n^{(ij)}, \tag{2.1}
$$

where $c_n^{(ij)} = \left| (A^*A)_{i,j} (a_{i,j}^*) \right|$ and $c_s^{(ij)} = \sum_{\beta \in J_{s,n}(i)} \left| ((A^*A)_{i,j} (a_{i,j}^*))_{\beta} \right|$ for all $s = 1, n-1$, $i = 1, n$, and $j = 1, m$.

**Proof.** Denote $A^*A = V = (v_{ij}) \in \mathbb{C}^{n \times n}$. Consider $(\lambda I_n + V)_{i,j} (v_{i,j}) \in \mathbb{C}^{n \times n}$. Taking into account Theorem 2.4 we obtain

$$
\left| (\lambda I_n + V)_{i,j} (v_{i,j}) \right| = d_1 \lambda^{n-1} + d_2 \lambda^{n-2} + \ldots + d_n, \tag{2.2}
$$

where $d_s = \sum_{\beta \in J_{s,n}(i)} \left| (V)_{\beta} \right|$ is the sum of all principal minors of order $s$ that contain the $i$-th column for all $s = 1, n-1$ and $d_n = \det V$. Since $v_{i,j} = \sum l a_{i,l}^* a_{i,j}$, where $a_{i,l}^*$ is the $l$th column-vector of $A^*$ for all $l = 1, n$, then we have on the one hand

$$
\left| (\lambda I_n + V)_{i,j} (v_{i,j}) \right| = \sum_{l} \left| (\lambda I_n + V)_{i,l} (a_{i,l}^*) a_{i,j} \right| = \sum_{l} \left| (\lambda I_n + V)_{i,l} (a_{i,l}^*) \right| a_{i,j}. \tag{2.3}
$$

Having changed the order of summation, we obtain on the other hand for all $s = 1, n-1$

$$
ds_s = \sum_{\beta \in J_{s,n}(i)} \left| (V)_{\beta} \right| = \sum_{\beta \in J_{s,n}(i)} \sum_{l} \left| (V)_{i,l} (a_{i,l}^*) a_{i,j} \right| = \sum_{l} \sum_{\beta \in J_{s,n}(i)} \left| (V)_{i,l} (a_{i,l}^*) \right| a_{i,j}, \tag{2.4}
$$

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By substituting (2.3) and (2.4) in (2.2), and equating factors at \( a_{ij} \) when \( l=j \), we obtain the equality (2.4). ■

By analogy can be proved the following lemma.

**Lemma 2.8** If \( A \in \mathbb{C}^{m \times n} \) and \( \lambda \in \mathbb{R} \), then

\[
\det((\lambda I_m + AA^*)_j . (a_i^*)) = r^{(ij)}_1 \lambda^{m-1} + r^{(ij)}_2 \lambda^{m-2} + \ldots + r^{(ij)}_m,
\]

where \( r^{(ij)}_m = |(AA^*)_j . (a_i^*)| \) and \( r^{(ij)}_s = \sum_{\alpha \in I, m \{ j \}} |((AA^*)_j . (a_i^*))^s_j| \) for all \( s = 1, n-1, i = 1, n, \) and \( j = 1, m \).

The following theorem and remarks introduce the determinantal representations of the Moore-Penrose by analogs of the classical adjoint matrix.

**Theorem 2.9** If \( A \in \mathbb{C}^{m \times n} \) and \( r < \min\{m, n\} \), then the Moore-Penrose inverse \( A^+ = (a_{ij}^+) \in \mathbb{C}^{n \times m} \) possess the following determinantal representations:

\[
a_{ij}^+ = \sum_{\beta \in I, m \{ j \}} \left| (A^* A)^{\beta}_{\beta} \right| \sum_{\beta \in I, n \{ i \}} \left| (A^* A)^{\beta}_{\beta} \right|,
\]

or

\[
a_{ij}^+ = \frac{\sum_{\alpha \in I, m \{ j \}} \left| (AA^*)_j . (a_i^*) \right|^s_j}{\sum_{\alpha \in I, m} \left| (AA^*)_\alpha \right|^s_{\alpha}},
\]

for all \( i = 1, n, j = 1, m \).

**Proof.** At first we shall obtain the representation (2.5). If \( \lambda \in \mathbb{R}_+ \), then the matrix \( (\lambda I + A^* A) \in \mathbb{C}^{n \times n} \) is Hermitian and rank \( (\lambda I + A^* A) = n \). Hence, there exists its inverse

\[
(\lambda I + A^* A)^{-1} = \frac{1}{\det(\lambda I + A^* A)} \begin{pmatrix}
L_{11} & L_{21} & \ldots & L_{n1} \\
L_{12} & L_{22} & \ldots & L_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
L_{1n} & L_{2n} & \ldots & L_{nn}
\end{pmatrix},
\]

where \( L_{ij} \) \( (\forall i, j = 1, n) \) is a cofactor in \( \lambda I + A^* A \). By Lemma 2.2 \( A^+ = \lim_{\lambda \to 0} (\lambda I + A^* A)^{-1} A^* \), so that

\[
A^+ = \lim_{\lambda \to 0} \begin{pmatrix}
\frac{\det(\lambda I + A^* A)_1(a^*_1)}{\det(\lambda I + A^* A)} & \ldots & \frac{\det(\lambda I + A^* A)_1(a^*_n)}{\det(\lambda I + A^* A)} \\
\vdots & \ddots & \vdots \\
\frac{\det(\lambda I + A^* A)_n(a^*_1)}{\det(\lambda I + A^* A)} & \ldots & \frac{\det(\lambda I + A^* A)_n(a^*_n)}{\det(\lambda I + A^* A)}
\end{pmatrix}.
\]

From Theorem 2.4 we get

\[
\det(\lambda I + A^* A) = \lambda^n + d_1 \lambda^{n-1} + d_2 \lambda^{n-2} + \ldots + d_n,
\]
where \( d_r (\forall r = 1, n - 1) \) is a sum of principal minors of \( A^*A \) of order \( r \) and \( d_n = \det A^*A \). Since \( \text{rank } A^*A = \text{rank } A = r \), then \( d_n = d_{n-1} = \ldots = d_{r+1} = 0 \) and
\[
\det (\lambda I + A^*A) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n\lambda^{n-r} \quad (2.8)
\]
In the same way, we have for arbitrary \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \) from Lemma 2.7
\[
\det (\lambda I + A^*A)_{i,j} (a^*_j) = l_{ij}^{(i)}\lambda^{n-1} + l_{ij}^{(ii)}\lambda^{n-2} + \ldots + l_{ij}^{(ij)}\lambda^{n-r},
\]
where for an arbitrary \( 1 \leq k \leq n-1 \), \( l_{ij}^{(ij)} = \sum_{\beta \in J_{k,n}\{i\}} \left| ((A^*A)_{i,i}(a^*_j))_{\beta} \right| \)
and \( l_{ij}^{(ij)} = \det(A^*A)_{i,j}(a^*_j) \). By Lemma 2.5 \( \text{rank } (A^*A)_{i,j}(a^*_j) \leq r \) so that if \( k > r \), then \( \left| ((A^*A)_{i,i}(a^*_j))_{\beta} \right| = 0 \), \( (\forall \beta \in J_{k,n}\{i\}, \forall i = 1, n, \forall j = 1, m) \).
Therefore if \( r + 1 \leq k < n \), then \( l_{ij}^{(ij)} = \sum_{\beta \in J_{k,n}\{i\}} \left| ((A^*A)_{i,i}(a^*_j))_{\beta} \right| = 0 \) and \( l_{ij}^{(ij)} = \det(A^*A)_{i,j}(a^*_j) = 0 \), \( (\forall i = 1, n, \forall j = 1, m) \). Finally we obtain
\[
\det (\lambda I + A^*A)_{i,j} (a^*_j) = l_{ij}^{(i)}\lambda^{n-1} + l_{ij}^{(ii)}\lambda^{n-2} + \ldots + l_{ij}^{(ij)}\lambda^{n-r} \quad (2.9)
\]
By replacing the denominators and the numerators of the fractions in entries of matrix (2.7) with the expressions (2.8) and (2.9) respectively, we get
\[
A^+ = \lim_{\lambda \to 0} \frac{\begin{pmatrix} l_{11}^{(i)}\lambda^{n-1} + \ldots + l_{1m}^{(i)}\lambda^{n-r} \\ \vdots \\ l_{nm}^{(i)}\lambda^{n-1} + \ldots + l_{nm}^{(i)}\lambda^{n-r} \end{pmatrix}}{\lambda^n + d_1\lambda^{n-1} + \ldots + d_r\lambda^{n-r}}.
\]
From here it follows (2.5).
We can prove (2.6) in the same way. ■

**Corollary 2.10** If \( A \in \mathbb{C}^{m \times n} \) and \( r < \min \{m,n\} \) or \( r = m < n \), then the projection matrix \( P = A^+A \) can be represented as
\[
P = \left( p_{ij} \right)_{n \times n}^{d_r (A^*A)},
\]
where \( d_j \) denotes the \( j \)th column of \( (A^*A) \) and, for arbitrary \( 1 \leq i,j \leq n \),
\[
p_{ij} = \sum_{\beta \in J_{r,n}\{i\}} \left| ((A^*A)_{i,i}(d_j))_{\beta} \right|.
\]
**Proof.** Representing the Moore - Penrose inverse \( A^+ \) by (2.5), we obtain
\[
P = \frac{1}{d_r (A^*A)} \begin{pmatrix} l_{11} & l_{12} & \ldots & l_{1m} \\ l_{21} & l_{22} & \ldots & l_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ l_{nm} & l_{n2} & \ldots & l_{nm} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}.
\]
Therefore, for arbitrary $1 \leq i, j \leq n$ we get

$$p_{ij} = \sum_k \sum_{\beta \in J_{r,n}(i)} \left| (A^* A)_{i, (a^*_k)} \right|^\beta \cdot a_{kj} = \sum_{\beta \in J_{r,n}(i)} \sum_k \left| (A^* A)_{i, (a^*_k \cdot a_{kj})} \right|^\beta = \sum_{\beta \in J_{r,n}(i)} \left| (A^* A)_{i, (d^*_j)} \right|^\beta,$$

Using the representation (2.6) of the Moore - Penrose inverse the following corollary can be proved in the same way.

**Corollary 2.11** If $A \in \mathbb{C}^{m \times n}$, where $r < \min \{m, n\}$ or $r = n < m$, then a projection matrix $Q = AA^+$ can be represented as

$$Q = \left(\frac{q_{ij}}{d_r(AA^+)}\right)_{m \times m}.$$

where $q_{ij}$ denotes the $i$th row of $(AA^*)$ and, for arbitrary $1 \leq i, j \leq m$, $q_{ij} = \sum_{\alpha \in I_r(m)} \left| (AA^*)_{j, (g_{\alpha})} \right|^\alpha$.

**Remark 2.12** If rank $A = n$, then from Corollary 2.3 we get $A^+ = (A^* A)^{-1} A^*$. Representing $(A^* A)^{-1}$ by the classical adjoint matrix, we have

$$A^+ = \frac{1}{\det(A^* A)} \begin{pmatrix} \det(A^* A)_{1, (a^*_1)} & \ldots & \det(A^* A)_{1, (a^*_m)} \\ \ldots & \ldots & \ldots \\ \det(A^* A)_{n, (a^*_1)} & \ldots & \det(A^* A)_{n, (a^*_m)} \end{pmatrix}. \quad (2.10)$$

If $n < m$, then (2.10) is valid.

**Remark 2.13** As above, if rank $A = m$, then

$$A^+ = \frac{1}{\det(AA^*)} \begin{pmatrix} \det(AA^*)_{1, (a^*_1)} & \ldots & \det(AA^*)_{m, (a^*_1)} \\ \ldots & \ldots & \ldots \\ \det(AA^*)_{1, (a^*_n)} & \ldots & \det(AA^*)_{m, (a^*_n)} \end{pmatrix}. \quad (2.11)$$

If $n > m$, then (2.11) is valid as well.

**Remark 2.14** By definition of the classical adjoint $\text{Adj}(A)$ for an arbitrary invertible matrix $A \in \mathbb{C}^{n \times n}$ one may put, $\text{Adj}(A) \cdot A = \det A \cdot I_n$.

If $A \in \mathbb{C}^{m \times n}$ and rank $A = n$, then by Corollary 2.3 $A^+ = I_n$. Representing the matrix $A^+$ by (2.10) as $A^+ = \frac{b}{\det(A^* A)}$, we obtain $LA = \det(A^* A) \cdot I_n$.

This means that the matrix $L = \{l_{ij}\} \in \mathbb{C}^{m \times m}$ is a left analogue of $\text{Adj}(A)$, where $A \in \mathbb{C}^{m \times n}$, and $l_{ij} = \det(A^* A)_{i, (a^*_j)}$ for all $i = \overline{1, n}, j = \overline{1, m}$.

If rank $A = m$, then by Corollary 2.3 $AA^+ = I_m$. Representing the matrix $A^+$ by (2.11) as $A^+ = \frac{R}{\det(AA^*)}$, we obtain $AR = I_m \cdot \det(AA^*)$. This means that the matrix $R = \{r_{ij}\} \in \mathbb{C}^{m \times n}$ is a right analogue of $\text{Adj}(A)$, where $A \in \mathbb{C}^{m \times n}$, and $r_{ij} = \det(AA^*)_{j, (a^*_i)}$ for all $i = \overline{1, n}, j = \overline{1, m}$.  

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If \( A \in \mathbb{C}^{m \times n} \) and \( r < \min\{m, n\} \), then by (2.5) we have \( A^+ = \frac{1}{d_{r}(AA^*)} \), where \( L = (l_{ij}) \in \mathbb{C}^{n \times m} \) and \( l_{ij} = \sum_{\beta \in \mathcal{I}_{r,m}(j)} |(A^*A)_{i,j} (a^*_{\beta})| \) for all \( i = 1, n \), \( j = 1, m \). From Corollary 2.10 we get \( LA = d_{r}(A^*A) \cdot P \). The matrix \( P \) is idempotent. All eigenvalues of an idempotent matrix chose from 1 or 0 only. Thus, there exists an unitary matrix \( U \) such that

\[
LA = d_{r}(A^*A) U \text{diag}(1, \ldots, 1, 0, \ldots, 0) U^*,
\]

where \( \text{diag}(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^{n \times n} \) is a diagonal matrix. Therefore, the matrix \( L \) can be considered as a left analogue of \( \text{Adj}(A) \), where \( A \in \mathbb{C}^{m \times n} \).

In the same way, if \( A \in \mathbb{C}^{n \times m} \) and \( r < \min\{m, n\} \), then by (2.5) we have \( A^+ = \frac{R}{d_{r}(AA^*)} \), where \( R = (r_{ij}) \in \mathbb{C}^{n \times m}, r_{ij} = \sum_{\alpha \in \mathcal{I}_{r,m}(j)} |((AA^*)_{i,j} (a^*_{\alpha})| \) for all \( i = 1, n \), \( j = 1, m \). From Corollary 2.11 we get \( AR = d_{r}(AA^*) \cdot Q \). The matrix \( Q \) is idempotent. There exists an unitary matrix \( V \) such that

\[
AR = d_{r}(AA^*) V \text{diag}(1, \ldots, 1, 0, \ldots, 0) V^*,
\]

where \( \text{diag}(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^{m \times m} \). Therefore, the matrix \( R \) can be considered as a right analogue of \( \text{Adj}(A) \) in this case.

**Remark 2.15** To obtain an entry of \( A^+ \) by Theorem 2.14 one calculates \((C_n^r + C_{n-1}^r - 1)\) determinants of order \( r \). Whereas by the equation (2.5) we calculate as much as \((C_n^r + C_{n-1}^r - 1)\) determinants of order \( r \) or we calculate the total of \((C_n^r + C_{n-1}^r - 1)\) determinants by (2.6). Therefore the calculation of entries of \( A^+ \) by Theorem 2.14 is easier than by Theorem 2.14.

### 2.2 Analogues of the Classical Adjoint Matrix for the Weighted Moore-Penrose Inverse

Let Hermitian positive definite matrices \( M \) and \( N \) of order \( m \) and \( n \), respectively, be given. The weighted Moore-Penrose inverse \( X = A^+_{M,N} \) can be explicitly expressed from the weighted singular value decomposition due to Van Loan [23].

**Lemma 2.16** Let \( A \in \mathbb{C}^{m \times n} \). There exist \( U \in \mathbb{C}^{m \times m}, V \in \mathbb{C}^{n \times n} \) satisfying \( U^*MU = I_m \) and \( V^*N^{-1}V = I_n \) such that

\[
A = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^*.
\]

Then the weighted Moore-Penrose inverse \( A^+_{M,N} \) can be represented

\[
A^+_{M,N} = N^{-1}V \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*M,
\]

where \( D = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) \), \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0 \) and \( \sigma_1^2 \) is the nonzero eigenvalues of \( N^{-1}A^*MA \).
For the weighted Moore-Penrose inverse $X = A_{M,N}^+$, we have the following limit representation.

**Lemma 2.17** (\cite{24}, Corollary 3.4.) Let $A \in \mathbb{C}^{m \times n}$, $A^\sharp = N^{-1} A^* M$. Then

$$A_{M,N}^+ = \lim_{\lambda \to 0} (\lambda I + A^\sharp A)^{-1} A^\sharp.$$

By analogy to Lemma 2.17 can be proved the following lemma.

**Lemma 2.18** Let $A \in \mathbb{C}^{m \times n}$, $A^\sharp = N^{-1} A^* M$. Then

$$A_{M,N}^+ = \lim_{\lambda \to 0} A^\sharp (\lambda I + AA^\sharp)^{-1}.$$

Denote by $a_{j}^\sharp$ and $a_{i}^\sharp$ the $j$th column and the $i$th row of $A^\sharp$ respectively. By putting $A^\sharp$ instead $A^*$, we obtain the proofs of the following two lemmas and theorem similar to the proofs of Lemmas 2.5, 2.6, 2.7, 2.8 and Theorem 2.9, respectively.

**Lemma 2.19** If $A \in \mathbb{C}^{m \times n}$ and $A^\sharp$ is defined as above, then

$$\text{rank } (A^\sharp A)_{i,j}(a_{j}^\sharp) \leq \text{rank } (A^\sharp A),$$

$$\text{rank } (AA^\sharp)_{j,i}(a_{i}^\sharp) \leq \text{rank } (AA^\sharp),$$

for all $i = 1, n$ and $j = 1, m$.

Analogues of the characteristic polynomial are considered in the following lemma.

**Lemma 2.20** If $A \in \mathbb{C}^{m \times n}$ and $\lambda \in \mathbb{R}$, then

$$\det \left( (\lambda I_n + A^\sharp A)_{i,j}(a_{j}^\sharp) \right) = c_{1}^{(ij)} \lambda^{n-1} + c_{2}^{(ij)} \lambda^{n-2} + \ldots + c_{n}^{(ij)};$$

$$\det \left( (\lambda I_m + AA^\sharp)_{j,i}(a_{i}^\sharp) \right) = r_{1}^{(ij)} \lambda^{m-1} + r_{2}^{(ij)} \lambda^{m-2} + \ldots + r_{m}^{(ij)},$$

where $c_{n}^{(ij)} = \left| (A^\sharp A)_{i,j}(a_{j}^\sharp) \right|$, $r_{m}^{(ij)} = \left| (AA^\sharp)_{j,i}(a_{i}^\sharp) \right|$ and

$$c_{s}^{(ij)} = \sum_{\beta \in J_{s,n}(i)} \left| \left( (A^\sharp A)_{i,j}(a_{j}^\sharp) \right)_{\beta} \right| r_{t}^{(ij)} = \sum_{\alpha \in I_{t,m}(j)} \left| \left( (AA^\sharp)_{j,i}(a_{i}^\sharp) \right)_{\alpha} \right|$$

for all $s = 1, n-1$, $t = 1, m-1$, $i = 1, n$, and $j = 1, m$.

The following theorem introduce the determinantal representations of the weighted Moore-Penrose by analogs of the classical adjoint matrix.
Theorem 2.21 If $A \in \mathbb{C}^{m \times n}$ and $r < \min\{m, n\}$, then the weighted Moore-Penrose inverse $A_{M,N}^+ = (\tilde{a}_{ij}) \in \mathbb{C}^{n \times m}$ possess the following determinantal representation:

$$\tilde{a}_{ij} = \frac{\sum_{\beta \in J_{r,n}} \left| (A^\dagger A)_{ij} \right|}{\sum_{\beta \in J_{r,n}} \left| (A^\dagger A)_{\beta \beta} \right|},$$

or

$$\tilde{a}_{ij} = \frac{\sum_{\alpha \in I_{r,m}} \left| (A A^\dagger)_{ij} \right|}{\sum_{\alpha \in I_{r,m}} \left| (A A^\dagger)_{\alpha \alpha} \right|},$$

for all $i = 1, n$, $j = 1, m$.

2.3 Analogues of the Classical Adjoint Matrix for the Drazin Inverse

The Drazin inverse can be represented explicitly by the Jordan canonical form as follows.

Theorem 2.22 If $A \in \mathbb{C}^{n \times n}$ with $\text{Ind} A = k$ and

$$A = P \left( \begin{array}{cc} C & 0 \\ 0 & N \end{array} \right) P^{-1}$$

where $C$ is nonsingular and $\text{rank} C = \text{rank} A^k$, and $N$ is nilpotent of order $k$, then

$$A^D = P \left( \begin{array}{cc} C^{-1} & 0 \\ 0 & 0 \end{array} \right) P^{-1}.$$ 

(2.14)

Stanimirovic’ [26] introduced a determinantal representation of the Drazin inverse by the following theorem.

Theorem 2.23 The Drazin inverse $A^D = (a^D_{ij})$ of an arbitrary matrix $A \in \mathbb{C}^{n \times n}$ with $\text{Ind} A = k$ possesses the following determinantal representation

$$a^D_{ij} = \frac{\sum_{(\alpha, \beta) \in N_{rk} \{j, i\}} \left| (A^s)_{\alpha}^\beta \frac{\partial}{\partial a_{ij}} A^s_{\beta} \right|}{\sum_{(\gamma, \delta) \in N_{rk}} \left| (A^s)_{\gamma}^\delta \right|}, \quad 1 \leq i, j \leq n;$$

(2.15)

where $s \geq k$ and $r_k = \text{rank} A^s$.

This determinantal representations of the Drazin inverse is based on a full-rank representation.

We use the following limit representation of the Drazin inverse.
Lemma 2.24 If $A \in \mathbb{C}^{n \times n}$, then

$$A^D = \lim_{\lambda \to 0} (\lambda I_n + A^{k+1})^{-1} A^k,$$

where $k = \text{Ind} A$, $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.

Since the equation (1.6) can be replaced by follows

$$XA^{k+1} = A^k,$$

the following lemma can be obtained by analogy to Lemma 2.24.

Lemma 2.25 If $A \in \mathbb{C}^{n \times n}$, then

$$A^D = \lim_{\lambda \to 0} A^k (\lambda I_n + A^{k+1})^{-1},$$

where $k = \text{Ind} A$, $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.

Denote by $a^{(k)}_j$ and $a^{(k)}_i$ the $j$th column and the $i$th row of $A^k$ respectively.

We consider the following auxiliary lemma.

Lemma 2.26 If $A \in \mathbb{C}^{n \times n}$ with $\text{Ind} A = k$, then for all $i, j = 1, n$

$$\text{rank } A^{k+1}_{i.} (a^{(k)}_j) \leq \text{rank } A^{k+1}.$$ 

Proof. The matrix $A^{k+1}_{i.} (a^{(k)}_j)$ may be represent as follows

$$\begin{pmatrix} \sum_{s=1}^{n} a_{1s}a^{(k)}_{s1} & \cdots & \sum_{s=1}^{n} a_{1s}a^{(k)}_{sn} \\ \vdots & \ddots & \vdots \\ \sum_{s=1}^{n} a_{ns}a^{(k)}_{s1} & \cdots & \sum_{s=1}^{n} a_{ns}a^{(k)}_{sn} \end{pmatrix}.$$

Let $P_{ii} (-a_{ij}) \in \mathbb{C}^{n \times n}$, $(l \neq i)$, be a matrix with $-a_{ij}$ in the $(l, i)$ entry, 1 in all diagonal entries, and 0 in others. It is a matrix of an elementary transformation. It follows that

$$A^{k+1}_{i.} (a^{(k)}_j) \cdot \prod_{l \neq i} P_{ii} (-a_{ij}) = \begin{pmatrix} \sum_{s \neq j} a_{1s}a^{(k)}_{s1} & \cdots & \sum_{s \neq j} a_{1s}a^{(k)}_{sn} \\ \vdots & \ddots & \vdots \\ \sum_{s \neq j} a_{ns}a^{(k)}_{s1} & \cdots & \sum_{s \neq j} a_{ns}a^{(k)}_{sn} \end{pmatrix}.$$
The obtained above matrix has the following factorization.

\[
\begin{pmatrix}
\sum_{s \neq j} a_{1s} a_{s1} & \ldots & \sum_{s \neq j} a_{1s} a_{sn} \\
\ldots & \ldots & \ldots \\
\sum_{s \neq j} a_{ns} a_{s1} & \ldots & \sum_{s \neq j} a_{ns} a_{sn}
\end{pmatrix}
= \begin{pmatrix}
a_{11} & \ldots & 0 & \ldots & a_{1n} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
a_{n1} & \ldots & 0 & \ldots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
a_{11} & (k) & \ldots & a_{1n} \\
(k) & a_{21} & (k) & \ldots & a_{2n} \\
(k) & \ldots & \ldots & \ldots & \ldots \\
(k) & a_{n1} & \ldots & a_{nn}
\end{pmatrix}
\]

Denote the first matrix by

\[\tilde{A} := \begin{pmatrix}
a_{11} & \ldots & 0 & \ldots & a_{1n} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
a_{n1} & \ldots & 0 & \ldots & a_{nn}
\end{pmatrix}_{ith.}\]

The matrix \(\tilde{A}\) is obtained from \(A\) by replacing all entries of the \(ith\) row and the \(jth\) column with zeroes except for 1 in the \((i, j)\) entry. Elementary transformations of a matrix do not change its rank. It follows that \(\text{rank} \ A^{k+1} (a_{i,j}) \leq \min \{ \text{rank} \ A^k, \text{rank} \ \tilde{A} \}\). Since \(\text{rank} \ \tilde{A} \geq \text{rank} \ A^k\) the proof is completed. \(\blacksquare\)

The following lemma is proved similarly.

**Lemma 2.27** If \(A \in \mathbb{C}^{n \times n}\) with \(\text{Ind} \ A = k\), then for all \(i, j = 1, \ldots, n\)

\[\text{rank} \ A^{k+1} (a_{i,j}) \leq \text{rank} \ A^{k+1}.\]

**Lemma 2.28** If \(A \in \mathbb{C}^{n \times n}\) and \(\lambda \in \mathbb{R},\) then

\[\det \left((\lambda I_n + A^{k+1})_j (a_{i,}^{(k)})\right) = r_1^{(ij)} \lambda^{n-1} + r_2^{(ij)} \lambda^{n-2} + \ldots + r_n^{(ij)}, \quad (2.16)\]

where \(r_n^{(ij)} = \left|A_{j,}^{k+1} (a_{i,}^{(k)})\right|\) and \(r_s^{(ij)} = \sum_{\alpha \in I_{s,n}(j)} \left|\left(A_{j,}^{k+1} (a_{i,}^{(k)})\right)_\alpha\right|\) for all \(s = 1, n - 1\) and \(i, j = 1, n.\)

**Proof.** Consider the matrix \(\left((\lambda I_n + A^{k+1})_j (a_{i,}^{(k)})\right) \in \mathbb{C}^{n \times n}\). Taking into account Theorem 2.24 we obtain

\[\left|\left((\lambda I_n + A^{k+1})_j (a_{i,}^{(k)})\right)\right| = d_1 \lambda^{n-1} + d_2 \lambda^{n-2} + \ldots + d_n, \quad (2.17)\]
where \( d_s = \sum_{\alpha \in I_{r,n}(j)} |(A^{k+1})_{\alpha}^\alpha| \) is the sum of all principal minors of order \( s \) that contain the \( j \)-th row for all \( s = 1, n - 1 \) and \( d_n = \det A^{k+1} \). Since \( a_j^{(k+1)} = \sum_l a_{jl}a_l^{(k)} \), where \( a_l^{(k)} \) is the \( l \)-th row-vector of \( A^k \) for all \( l = 1, n \), then we have on the one hand

\[
\left| \left( (\lambda I_n + A^{k+1})_j \cdot (a_j^{(k)}) \right) \right| = \sum_l \left| (\lambda I_n + A^{k+1})_l \cdot (a_{jl}a_l^{(k)}) \right| = \sum_l a_{jl} \cdot \left| (\lambda I_n + A^{k+1})_l \cdot (a_l^{(k)}) \right| \tag{2.18}
\]

Having changed the order of summation, we obtain on the other hand for all \( s = 1, n - 1 \)

\[
d_s = \sum_{\alpha \in I_{r,n}(j)} |(A^{k+1})_{\alpha}^\alpha| = \sum_{\alpha \in I_{r,n}(j)} \sum_l \left| (A^{k+1}_j \cdot a_{jl}a_l^{(k)}) \right|^\alpha = \sum_{\alpha \in I_{r,n}(j)} \sum_l a_{jl} \cdot \left| (A^{k+1}_j \cdot a_l^{(k)}) \right|^\alpha \tag{2.19}
\]

By substituting (2.18) and (2.19) in (2.17), and equating factors at \( a_{jl} \) when \( l = i \), we obtain the equality (2.10).

**Theorem 2.29** If \( \text{Ind} A = k \) and \( \text{rank} A^{k+1} = \text{rank} A^k = r \leq n \) for \( A \in \mathbb{C}^{n \times n} \), then the Drazin inverse \( A^D = (a_j^{D}) \in \mathbb{C}^{n \times n} \) possess the following determinantal representations:

\[
a_j^{D} = \frac{\sum_{\alpha \in I_{r,n}(j)} \left| (A^{k+1}_j \cdot a_{jl}a_l^{(k)}) \right|^\alpha \sum_{\alpha \in I_{r,n}} |(A^{k+1})_{\alpha}^\alpha|}{\det (\lambda I + A^{k+1})} \tag{2.20}
\]

and

\[
a_j^{D} = \sum_{\beta \in I_{r,n}(i)} \left| (A^{k+1}_j \cdot a_{jl}a_l^{(k)}) \right| \frac{\sum_{\beta \in I_{r,n}} |(A^{k+1})_{\beta}^\beta|}{\det (\lambda I + A^{k+1})} \tag{2.21}
\]

for all \( i, j = 1, n \).

**Proof.** At first we shall prove the equation (2.20).

If \( \lambda \in \mathbb{R}_+ \), then \( \text{rank} (\lambda I + A^{k+1}) = n \). Hence, there exists the inverse matrix

\[
(\lambda I + A^{k+1})^{-1} = \frac{1}{\det (\lambda I + A^{k+1})} \begin{pmatrix}
R_{11} & R_{21} & \cdots & R_{n1} \\
R_{12} & R_{22} & \cdots & R_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
R_{1n} & R_{2n} & \cdots & R_{nn}
\end{pmatrix},
\]

where

\[
R_{ij} = \sum_{\alpha \in I_{r,n}(j)} \left| (A^{k+1}_j \cdot a_{jl}a_l^{(k)}) \right|^\alpha \sum_{\alpha \in I_{r,n}} |(A^{k+1})_{\alpha}^\alpha|.
\]
where $R_{ij}$ is a cofactor in $\lambda I + A^{k+1}$ for all $i, j = 1, n$. By Theorem 2.25
\[
A^D = \lim_{\lambda \to 0} (\lambda I + A^{k+1})^{-1},
\]
so that
\[
A^D = \lim_{\lambda \to 0} \frac{1}{\det(\lambda I + A^{k+1})} \begin{pmatrix}
\sum_{s=1}^{n} a_{1s}^{(k)} R_{1s} & \cdots & \sum_{s=1}^{n} a_{1s}^{(k)} R_{ns} \\
\vdots & \ddots & \vdots \\
\sum_{s=1}^{n} a_{ns}^{(k)} R_{1s} & \cdots & \sum_{s=1}^{n} a_{ns}^{(k)} R_{ns}
\end{pmatrix} = \lim_{\lambda \to 0} \begin{pmatrix}
det(\lambda I + A^{k+1}) \begin{pmatrix} a_{1i}^{(k)} \\ \vdots \\ a_{ni}^{(k)} \end{pmatrix} & \cdots & \det(\lambda I + A^{k+1}) \begin{pmatrix} a_{1i}^{(k)} \\ \vdots \\ a_{ni}^{(k)} \end{pmatrix} \\
\vdots & \ddots & \vdots \\
\det(\lambda I + A^{k+1}) \begin{pmatrix} a_{1i}^{(k)} \\ \vdots \\ a_{ni}^{(k)} \end{pmatrix} & \cdots & \det(\lambda I + A^{k+1}) \begin{pmatrix} a_{1i}^{(k)} \\ \vdots \\ a_{ni}^{(k)} \end{pmatrix}
\end{pmatrix}
\]  
(2.22)

Taking into account Theorem 2.24, we have
\[
det(\lambda I + A^{k+1}) = \lambda^n + d_1 \lambda^{n-1} + d_2 \lambda^{n-2} + \ldots + d_n,
\]
where $d_s = \sum_{\alpha \in I_s, \alpha} |(A^{k+1})_{\alpha}^{(s)}|$ is a sum of the principal minors of $A^{k+1}$ of order $s$, for all $s = 1, n - 1$, and $d_n = \det A^{k+1}$. Since rank $A^{k+1} = r$, then $d_n = d_{n-1} = \ldots = d_{r+1} = 0$ and
\[
det(\lambda I + A^{k+1}) = \lambda^n + d_1 \lambda^{n-1} + d_2 \lambda^{n-2} + \ldots + d_r \lambda^{n-r}.
\]
(2.23)

By Lemma 2.28 for all $i, j = 1, n$,
\[
det(\lambda I + A^{k+1})_{ij} \begin{pmatrix} a_{1i}^{(k)} \\ \vdots \\ a_{ni}^{(k)} \end{pmatrix} = l_{1}^{(ij)} \lambda^{n-1} + l_{2}^{(ij)} \lambda^{n-2} + \ldots + l_{n}^{(ij)},
\]
where for all $s = 1, n - 1$,
\[
l_{s}^{(ij)} = \sum_{\alpha \in I_{s,n}(j)} |(A_{j,\alpha}^{k+1})_{ij}^{(s)}|,
\]
and $l_{n}^{(ij)} = \det A_{j}^{k+1} \begin{pmatrix} a_{1i}^{(k)} \\ \vdots \\ a_{ni}^{(k)} \end{pmatrix}$.

By Lemma 2.26 rank $A_{j}^{k+1} \begin{pmatrix} a_{1i}^{(k)} \\ \vdots \\ a_{ni}^{(k)} \end{pmatrix} \leq r$, so that if $s > r$, then for all $\alpha \in I_{s,n}(i)$ and for all $i, j = 1, n$,
\[
|(A_{j,\alpha}^{k+1})_{ij}^{(s)}| = 0.
\]

Therefore if $r + 1 \leq s < n$, then for all $i, j = 1, n$,
\[
l_{s}^{(ij)} = \sum_{\alpha \in I_{s,n}(j)} |(A_{j,\alpha}^{k+1})_{ij}^{(s)}| = 0,
\]
and $l_{n}^{(ij)} = \det A_{j}^{k+1} \begin{pmatrix} a_{1i}^{(k)} \\ \vdots \\ a_{ni}^{(k)} \end{pmatrix} = 0$. Finally we obtain
\[
det(\lambda I + A^{k+1})_{ij} \begin{pmatrix} a_{1i}^{(k)} \\ \vdots \\ a_{ni}^{(k)} \end{pmatrix} = l_{1}^{(ij)} \lambda^{n-1} + l_{2}^{(ij)} \lambda^{n-2} + \ldots + l_{r}^{(ij)} \lambda^{n-r}.
\]
(2.24)
By replacing the denominators and the nominators of the fractions in the entries of the matrix (2.22) with the expressions (2.23) and (2.24) respectively, finally we obtain

\[
A^D = \lim_{\lambda \to 0} \begin{pmatrix}
\frac{f^{(11)}}{d_r} \lambda^{n-1} + \cdots + \frac{f^{(1)}(1)}{d_r} \lambda^{n-r} \\
\vdots \\
\frac{f^{(n)}}{d_r} \lambda^{n-1} + \cdots + \frac{f^{(n)}(1)}{d_r} \lambda^{n-r}
\end{pmatrix}
= \begin{pmatrix}
\frac{f^{(11)}}{d_r} \\
\vdots \\
\frac{f^{(n)}}{d_r}
\end{pmatrix},
\]

where for all \(i, j = 1, n\),

\[
I_{r}^{(ij)} = \sum_{\alpha \in I_{r,n}(j)} \left| \left( A_{j,\alpha}^{k+1} (a_{i,\alpha}^{(k)}) \right)^{\alpha} \right|, \\
d_r = \sum_{\alpha \in I_{r,n}} \left| (A_{\alpha}^{k+1})^{\alpha} \right|.
\]

The equation (2.24) can be proved similarly. This completes the proof. \(\blacksquare\)

Using Theorem 2.29 we evidently can obtain determinantal representations of the group inverse and the following determinantal representation of the identities \(A^D A\) and \(AA^D\) on \(R(A^k)\)

**Corollary 2.30** If \(\text{Ind} A = 1\) and \(\text{rank } A^2 = \text{rank } A = r \leq n\) for \(A \in \mathbb{C}^{n \times n}\), then the group inverse \(A^{\#} = (a_{ij}^\#) \in \mathbb{C}^{n \times n}\) possess the following determinantal representations:

\[
a_{ij}^\# = \frac{\sum_{\alpha \in I_{r,n}(j)} \left| (A_{j,\alpha}^{2})^{\alpha} \right|}{\sum_{\alpha \in I_{r,n}} \left| (A^{2})^{\alpha} \right|},
\]

\[
a_{ij}^\# = \frac{\sum_{\beta \in I_{r,n}(i)} \left| (A_{i,\beta}^{2})^{\beta} \right|}{\sum_{\beta \in I_{r,n}} \left| (A^{2})^{\beta} \right|},
\]

for all \(i, j = 1, n\).

**Corollary 2.31** If \(\text{Ind} A = k \) and \(\text{rank } A^{k+1} = \text{rank } A^k = r \leq n\) for \(A \in \mathbb{C}^{n \times n}\), then the matrix \(A A^D = (q_{ij}) \in \mathbb{C}^{n \times n}\) possess the following determinantal representation

\[
q_{ij} = \frac{\sum_{\alpha \in I_{r,n}(j)} \left| (A_{j,\alpha}^{k+1} (a_{i,\alpha}^{(k+1)}) \right|}{\sum_{\alpha \in I_{r,n}} \left| (A^{k+1})^{\beta} \right|},
\]

for all \(i, j = 1, n\).
Corollary 2.32 If \( \text{Ind} A = k \) and \( \text{rank} A^{k+1} = \text{rank} A^k = r \leq n \) for \( A \in \mathbb{C}^{n \times n} \), then the matrix \( A^D A = (p_{ij}) \in \mathbb{C}^{n \times n} \) possesses the following determinantal representation

\[
p_{ij} = \frac{\sum_{\beta \in J_{r,n}} \left| \left( A_{r,i}^{k+1} (a_{j}^{(k+1)}) \right)^{\beta} \right|}{\sum_{\beta \in J_{r,n}} \left| \left( A_{r,i}^{k+1} \right)^{\beta} \right|}, \tag{2.27}
\]

for all \( i, j = 1, n \).

2.4 Analogues of the Classical Adjoint Matrix for the W-Weighted Drazin Inverse

Cline and Greville [28] extended the Drazin inverse of square matrix to rectangular matrix and called it as the weighted Drazin inverse (WDI). The W-weighted Drazin inverse of \( A \in \mathbb{C}^{m \times n} \) with respect to \( W \in \mathbb{C}^{n \times m} \) is defined to be the unique solution \( X \in \mathbb{C}^{m \times n} \) of the following three matrix equations:

1) \( (AW)^{k+1}XW = (AW)^k \),
2) \( XWAWX = X \),
3) \( AXW = XWA \),

where \( k = \max\{\text{Ind}(AW), \text{Ind}(WA)\} \). It is denoted by \( X = A_{d,W} \). In particular, when \( A \in \mathbb{C}^{m \times m} \) and \( W = I_n \), then \( A_{d,W} \) reduce to \( A^D \). If \( A \in \mathbb{C}^{m \times m} \) is non-singular square matrix and \( W = I_m \), then \( \text{Ind}(A) = 0 \) and \( A_{d,W} = A^D = A^{-1} \).

The properties of WDI can be found in (e.g., [29–32]). We note the general algebraic structures of the W-weighted Drazin inverse [29]. Let for \( A \in \mathbb{C}^{m \times n} \) and \( W \in \mathbb{C}^{n \times m} \) exist \( L \in \mathbb{C}^{m \times m} \) and \( Q \in \mathbb{C}^{n \times n} \) such that

\[
A = L \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} Q^{-1}, \quad W = Q \begin{pmatrix} W_{11} & 0 \\ 0 & W_{22} \end{pmatrix} L^{-1}.
\]

Then

\[
A_{d,W} = L \begin{pmatrix} (W_{11}A_{11}W_{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1},
\]

where \( L, L, A_{11}, W_{11} \) are non-singular matrices, and \( A_{22}, W_{22} \) are nilpotent matrices. By [27] we have the following limit representations of the W-weighted Drazin inverse,

\[
A_{d,W} = \lim_{\lambda \to 0} \left( \lambda I + (AW)^{k+2} \right)^{-1} (AW)^k A \tag{2.29}
\]

and

\[
A_{d,W} = \lim_{\Lambda \to 0} A(WA)^k \left( \lambda I_n + (WA)^{k+2} \right)^{-1} \tag{2.30}
\]

where \( \lambda \in \mathbb{R}_+ \), and \( \mathbb{R}_+ \) is a set of the real positive numbers.

Denote \( WA =: U \) and \( AW =: V \). Denote by \( \mathbf{v}_j^{(k)} \) and \( \mathbf{v}_i^{(k)} \) the jth column and the ith row of \( V^k \) respectively. Denote by \( \mathbf{V}^k := (AW)^k A \in \mathbb{C}^{m \times n} \) and \( W = WAW \in \mathbb{C}^{n \times m} \).
Lemma 2.33 If $AW = V = (v_{ij}) \in \mathbb{C}^{m \times m}$ with $\text{Ind} V = k$, then
\[
\text{rank } (V^{k+2})_{-i} (\vec{v}^{(k)}_j) \leq \text{rank } (V^{k+2})_{-i}.
\] (2.31)

Proof. We have $V^{k+2} = \tilde{W}^k$W. Let $P_{is} (-\vec{w}_{js}) \in \mathbb{C}^{m \times m}$, $(s \neq i)$, be a matrix with $-\vec{w}_{js}$ in the $(i, s)$ entry, 1 in all diagonal entries, and 0 in others. The matrix $P_{is} (-\vec{w}_{js})$, $(s \neq i)$, is a matrix of an elementary transformation. It follows that
\[
(V^{k+2})_{-i} (\vec{v}^{(k)}_j) \cdot \prod_{s \neq i} P_{is} (-\vec{w}_{js}) = \left( \begin{array}{ccc}
\sum_{s \neq j} \vec{v}^{(k)}_{1s} \vec{w}_{s1} & \cdots & \vec{v}^{(k)}_{1j} \\
\vdots & \ddots & \vdots \\
\sum_{s \neq j} \vec{v}^{(k)}_{ms} \vec{w}_{s1} & \cdots & \vec{v}^{(k)}_{mj}
\end{array} \right)_{i-th}.
\]

We have the next factorization of the obtained matrix.
\[
\left( \begin{array}{ccc}
\sum_{s \neq j} \vec{v}^{(k)}_{1s} \vec{w}_{s1} & \cdots & \vec{v}^{(k)}_{1j} \\
\vdots & \ddots & \vdots \\
\sum_{s \neq j} \vec{v}^{(k)}_{ms} \vec{w}_{s1} & \cdots & \vec{v}^{(k)}_{mj}
\end{array} \right) = \left( \begin{array}{ccc}
\vec{v}^{(k)}_{11} & \vec{v}^{(k)}_{12} & \cdots \\
\vec{v}^{(k)}_{21} & \vec{v}^{(k)}_{22} & \cdots \\
\vdots & \vdots & \ddots \\
\vec{v}^{(k)}_{m1} & \vec{v}^{(k)}_{m2} & \cdots \\
\end{array} \right) \left( \begin{array}{ccc}
\vec{w}_{11} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \\
\end{array} \right)_{j-th}.
\]

Denote $\tilde{W} := \left( \begin{array}{ccc}
\vec{w}_{11} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \\
\vec{w}_{n1} & \cdots & 0 \\
\end{array} \right)_{i-th}$. The matrix $\tilde{W}$ is obtained from $W = WAW$ by replacing all entries of the $j$th row and the $i$th column with zeroes except for 1 in the $(i, j)$ entry. Since elementary transformations of a matrix do not change a rank, then $\text{rank } V_{-i} = \min \{ \text{rank } \tilde{V}^k, \text{rank } \tilde{W} \}$. It is obvious that
\[
\text{rank } \tilde{V}^k = \text{rank } (AW)^k A \geq \text{rank } (AW)^{k+2},
\]
\[
\text{rank } \tilde{W} \geq \text{rank } WAW \geq \text{rank } (AW)^{k+2}.
\]

From this the inequality (2.31) follows immediately. □

The next lemma is proved similarly.
Lemma 2.34 If $WA = U = (u_{ij}) \in \mathbb{C}^{n \times n}$ with $\text{Ind} U = k$, then
\[ \text{rank} \left( U^{k+2} \right)_i \left( u^{(k)}_{ij} \right) \leq \text{rank} \left( U^{k+2} \right), \]
where $U^k := A(WA)^k \in \mathbb{C}^{m \times n}$.

Analogues of the characteristic polynomial are considered in the following two lemmas.

Lemma 2.35 If $AW = V = (v_{ij}) \in \mathbb{C}^{m \times m}$ with $\text{Ind} V = k$ and $\lambda \in \mathbb{R}$, then
\[ \left| (\lambda I_m + V^{k+2})_i \left( v^{(k)}_{ij} \right) \right| = c_1^{(ij)} \lambda^{m-1} + c_2^{(ij)} \lambda^{m-2} + \ldots + c_m^{(ij)}, \quad (2.32) \]
where $c_m^{(ij)} = \det (V^{k+2})_i \left( v^{(k)}_{ij} \right)$ and $c_s^{(ij)} = \sum_{\beta \in J_s, m \{i \}} \det \left( (V^{k+2})_i \left( v^{(k)}_{ij} \right) \right)_{\beta}$ for all $s = 1, m - 1, i = 1, m, and j = 1, n$.

Proof. Consider the matrix $(\lambda I + V^{k+2})_i \left( v^{(k+2)}_{ij} \right) \in \mathbb{C}^{m \times m}$. Taking into account Theorem 2.24 we obtain
\[ \left| (\lambda I + V^{k+2})_i \left( v^{(k+2)}_{ij} \right) \right| = d_1 \lambda^{m-1} + d_2 \lambda^{m-2} + \ldots + d_m, \quad (2.33) \]
where $d_s = \sum_{\beta \in J_s, m \{i \}} \left| (V^{k+2})_i \left( v^{(k+2)}_{ij} \right) \right|_{\beta}$ is the sum of all principal minors of order $s$ that contain the $i$-th column for all $s = 1, m - 1$ and $d_m = \det (V^{k+2})$. Since
\[ v^{(k+2)}_{ij} = \begin{pmatrix} \sum_l v^{(k)}_{il} \bar{w}_{li} \\ \sum_l v^{(k)}_{2l} \bar{w}_{li} \\ \vdots \\ \sum_l v^{(k)}_{ml} \bar{w}_{li} \end{pmatrix} = \sum_l v^{(k)}_{li} \bar{w}_{li}, \]
where $v^{(k)}_{ij}$ is the $l$th column-vector of $V^k = (AW)^k A$ and $WAW = \bar{W} = (\bar{w}_{li})$ for all $l = 1, n$, then we have on the one hand
\[ \left| (\lambda I + V^{k+2})_i \left( v^{(k+2)}_{ij} \right) \right| = \sum_l \left| (\lambda I + V^{k+2})_i \left( v^{(k)}_{li} \bar{w}_{li} \right) \right| = \sum_l \left| (\lambda I + V^{k+2})_i \left( v^{(k)}_{li} \bar{w}_{li} \right) \right| \cdot \bar{w}_{li} \quad (2.34) \]
Having changed the order of summation, we obtain on the other hand for all $s = 1, m - 1$
\[ d_s = \sum_{\beta \in J_s, m \{i \}} \left| (V^{k+2})_i \left( v^{(k+2)}_{ij} \right) \right|_{\beta} = \sum_{\beta \in J_s, m \{i \}} \sum_l \left| (V^{k+2})_i \left( v^{(k)}_{li} \bar{w}_{li} \right) \right|_{\beta} = \sum_l \sum_{\beta \in J_s, m \{i \}} \left| (V^{k+2})_i \left( v^{(k)}_{li} \bar{w}_{li} \right) \right|_{\beta} \cdot \bar{w}_{li}. \quad (2.35) \]
By substituting (2.34) and (2.35) in (2.33), and equating factors at \( \bar{w}_t \), when \( l = j \), we obtain the equality (2.32). ■

By analogy can be proved the following lemma.

**Lemma 2.36** If \( WA = U = (u_{ij}) \in C^{n \times n} \) with \( \text{Ind} U = k \) and \( \lambda \in \mathbb{R} \), then

\[
(\lambda I + U^{k+2})_{j}^{(k)} \cdot (\bar{u}_{i}^{(k)}) = r_{1}^{(ij)} \lambda^{n-1} + r_{2}^{(ij)} \lambda^{n-2} + \ldots + r_{n}^{(ij)},
\]

where \( r_{n}^{(ij)} = |(U^{k+2})_{j}^{(k)} \cdot (\bar{u}_{i}^{(k)}) | \) and \( r_{s}^{(ij)} = \sum_{\alpha \in L_{s,n} \{ j \}} |(U^{k+2})_{j}^{(k)} \cdot (\bar{u}_{i}^{(k)}) | \alpha | \) for all \( s = 1, \ldots, n-1 \), \( i = 1, \ldots, m \), and \( j = 1, \ldots, n \).

**Theorem 2.37** If \( A \in C^{m \times n} \), \( W \in C^{n \times m} \) with \( k = \max \{ \text{Ind}(AW), \text{Ind}(WA) \} \) and \( \text{rank}(AW)^{k} = r \), then the \( W \)-weighted Drazin inverse \( A_{d,W} = (a_{ij}^{d,W}) \in C^{m \times n} \) with respect to \( W \) possess the following determinantal representations:

\[
a_{ij}^{d,W} = \frac{\sum_{\beta \in J_{r,m} \{ i \}} |(AW)_{j}^{k+2} \cdot (\bar{u}_{i}^{(k)}) | \beta |}{\sum_{\beta \in J_{r,m} \{ i \}} |(AW)_{j}^{k+2} | \beta |}, \tag{2.36}
\]

or

\[
a_{ij}^{d,W} = \frac{\sum_{\alpha \in L_{r,n} \{ j \}} |(WA)_{j}^{k+2} \cdot (\bar{u}_{i}^{(k)}) | \alpha |}{\sum_{\alpha \in L_{r,n} \{ j \}} |(WA)_{j}^{k+2} | \alpha |}. \tag{2.37}
\]

where \( \bar{u}_{i}^{(k)} \) is the \( i \)-th row of \( \bar{U}^{k} = (AW)^{k} A \) for all \( j = 1, \ldots, m \) and \( \bar{u}_{i}^{(k)} \) is the \( i \)-th row of \( \bar{U}^{k} = A(WA)^{k} \) for all \( i = 1, \ldots, n \).

**Proof.** At first we shall prove (2.36). By (2.29),

\[
A_{d,W} = \lim_{\lambda \to 0} \left( \lambda I_{m} + (AW)^{k+2} \right)^{-1} (AW)^{k} A.
\]

Let

\[
\left( \lambda I_{m} + (AW)^{k+2} \right)^{-1} = \frac{1}{\det(\lambda I_{m} + (AW)^{k+2})} \begin{pmatrix}
L_{11} & L_{21} & \ldots & L_{m1} \\
L_{12} & L_{22} & \ldots & L_{m2} \\
\vdots & \vdots & \ddots & \vdots \\
L_{1m} & L_{2m} & \ldots & L_{mm}
\end{pmatrix},
\]

where \( L_{ij} \) is a left \( ij \)-th cofactor of a matrix \( \lambda I_{m} + (AW)^{k+2} \). Then we have

\[
\left( \lambda I_{m} + (AW)^{k+2} \right)^{-1} (AW)^{k} A =
\]

\[
= \frac{1}{\det(\lambda I_{m} + (AW)^{k+2})} \begin{pmatrix}
\sum_{s=1}^{m} L_{s1} \bar{v}_{s1}^{(k)} & \sum_{s=1}^{m} L_{s1} \bar{v}_{s2}^{(k)} & \ldots & \sum_{s=1}^{m} L_{s1} \bar{v}_{sn}^{(k)} \\
\sum_{s=1}^{m} L_{s2} \bar{v}_{s1}^{(k)} & \sum_{s=1}^{m} L_{s2} \bar{v}_{s2}^{(k)} & \ldots & \sum_{s=1}^{m} L_{s2} \bar{v}_{sn}^{(k)} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{s=1}^{m} L_{sm} \bar{v}_{s1}^{(k)} & \sum_{s=1}^{m} L_{sm} \bar{v}_{s2}^{(k)} & \ldots & \sum_{s=1}^{m} L_{sm} \bar{v}_{sn}^{(k)}
\end{pmatrix},
\]

where \( \bar{v}_{s}^{(k)} \) is the \( s \)-th row of \( \bar{V}^{k} = (AW)^{k} \) for all \( s = 1, \ldots, m \).
By (2.29), we obtain
\[
A_{d,W} = \lim_{\lambda \to 0} \begin{pmatrix}
\frac{(\lambda I_m + (AW)^{k+2})_{ij}}{\lambda I_m + (AW)^{k+2}} & \cdots & \frac{(\lambda I_m + (AW)^{k+2})_{is}}{\lambda I_m + (AW)^{k+2}} \\
\vdots & \ddots & \vdots \\
\frac{(\lambda I_m + (AW)^{k+2})_{js}}{\lambda I_m + (AW)^{k+2}} & \cdots & \frac{(\lambda I_m + (AW)^{k+2})_{sn}}{\lambda I_m + (AW)^{k+2}}
\end{pmatrix}.
\]

By Theorem 2.34, we have
\[
| (\lambda I_m + (AW)^{k+2}) | = \lambda^m + d_1 \lambda^{m-1} + d_2 \lambda^{m-2} + \ldots + d_m,
\]
where \( d_s = \sum_{\beta \in J_{s,m}} | (\lambda I_m + (AW)^{k+2})_{\beta} | \) is a sum of principal minors of \((AW)^{k+2}\) of order \( s \) for all \( s = 1, m-1 \) and \( d_m = |(AW)^k| \).

Consider \((AW)^k\) and \((AW)^{k+2}\). \( (AW)^{k+2} \) has no more \( r \) linearly independent columns.

We shall prove that det \( (\lambda I_m + (AW)^{k+2}) \) = \( \lambda^m + d_1 \lambda^{m-1} + d_2 \lambda^{m-2} + \ldots + d_r \lambda^{m-r} \).

By Lemma 2.33,
\[
| (\lambda I_m + (AW)^{k+2})_{ij} | = c_1^{(ij)} \lambda^{m-1} + c_2^{(ij)} \lambda^{m-2} + \ldots + c_m^{(ij)}
\]
for \( i = 1, m \) and \( j = 1, n \), where \( c_s^{(ij)} = \sum_{\beta \in J_{s,m}} | (AW)^{k+2}_{\beta} | \) for all \( s = 1, m-1 \) and \( c_m^{(ij)} = |(AW)^{k+2}_{\beta} | \).

We shall prove that \( c_s^{(ij)} = 0 \), when \( k \geq r + 1 \) for \( i = 1, m \). By Lemma 2.33, \( (AW)^{k+2}_{\beta} \) is a principal submatrix of \((AW)^{k+2}_{\beta} \) of order \( s \geq r + 1 \). Deleting both its \( i \)-th row and column, we obtain a principal submatrix of order \( s - 1 \) of \((AW)^{k+2}\). We denote it by \( M \). The following cases are possible.

- Let \( s = r + 1 \) and \( \det M \neq 0 \). In this case all columns of \( M \) are right-linear independent. The addition of all of them on one coordinate to columns of \((AW)^{k+2}_{\beta} \) keeps their right-linear independence.

Hence, they are basis in a matrix \((AW)^{k+2}_{\beta} \) and the \( i \)-th column is the right linear combination of its basis columns. From this,
\[
| (AW)^{k+2}_{\beta} \lambda | = 0, \text{ when } \beta \in J_{s,n} \text{ and } s = r + 1.
\]
Thus in all cases we have $ \left| \left( (AW)_{i,j} \right)^{k+2} \left( \bar{v}^{(k)} \right) \right|_{\beta}^{\beta} = 0$ as well.

Hence, $\left| \left( (AW)_{i,j} \right)^{k+2} \left( \bar{v}^{(k)} \right) \right|_{\beta}^{\beta} = 0$ for $i = \overline{1,m}$ and $j = \overline{1,n}$.

By substituting these values in the matrix from (2.38), we obtain

$$A_{d,W} = \lim_{\lambda \to 0} \begin{pmatrix}
    \frac{c_{1}^{(1)} \lambda^{m-1} + \ldots + c_{1}^{(1)} \lambda^{m-r}}{A + d_1 \lambda + \ldots + d_r \lambda}
    & \ldots
    & \frac{c_{n}^{(1)} \lambda^{m-1} + \ldots + c_{n}^{(1)} \lambda^{m-r}}{A + d_1 \lambda + \ldots + d_r \lambda}
    \\
    \ldots
    & \ldots
    & \ldots
    \\
    \frac{c_{1}^{(m)} \lambda^{m-1} + \ldots + c_{1}^{(m)} \lambda^{m-r}}{A + d_1 \lambda + \ldots + d_r \lambda}
    & \ldots
    & \frac{c_{n}^{(m)} \lambda^{m-1} + \ldots + c_{n}^{(m)} \lambda^{m-r}}{A + d_1 \lambda + \ldots + d_r \lambda}
    \\
    \frac{c_{1}^{(1)} \lambda^{1} \lambda^{m-1} + \ldots + c_{1}^{(1)} \lambda^{m-r}}{A + d_1 \lambda + \ldots + d_r \lambda}
    & \ldots
    & \frac{c_{n}^{(1)} \lambda^{1} \lambda^{m-1} + \ldots + c_{n}^{(1)} \lambda^{m-r}}{A + d_1 \lambda + \ldots + d_r \lambda}
    \\
    \ldots
    & \ldots
    & \ldots
    \\
    \frac{c_{1}^{(m)} \lambda^{m-1} + \ldots + c_{1}^{(m)} \lambda^{m-r}}{A + d_1 \lambda + \ldots + d_r \lambda}
    & \ldots
    & \frac{c_{n}^{(m)} \lambda^{m-1} + \ldots + c_{n}^{(m)} \lambda^{m-r}}{A + d_1 \lambda + \ldots + d_r \lambda}
    \\
\end{pmatrix}$$

where $c_{\beta}^{(i)} = \sum_{\beta \in J_{r,m} \{i\}} \left| \left( A_{i,j} \right)^{k+1} \right|_{\beta}^{\beta}$ and $d_{r} = \sum_{\beta \in J_{r,m}} \left| \left( A_{i,j} \right)^{k+1} \right|_{\beta}^{\beta}$.

Thus, we have obtained the determinantal representation of $A_{d,W}$ by (2.30).

By analogy can be proved (2.37). \(\blacksquare\)

### 3 Cramer’s Rules for Generalized Inverse Solutions of Systems of Linear Equations

An obvious consequence of a determinantal representation of the inverse matrix by the classical adjoint matrix is the Cramer rule. As we know, Cramer’s rule gives an explicit expression for the solution of nonsingular linear equations. In [33], Robinson gave an elegant proof of Cramer’s rule which aroused great interest in finding determinantal formulas for solutions of some restricted linear equations both consistent and nonconsistent. It has been widely discussed by Robinson [33], Ben-Israel [34], Verghese [35], Werner [36], Chen [37], Ji [38], Wang [39], Wei [31].

In this section we demonstrate that the obtained analogues of the adjoint matrix for the generalized inverse matrices enable us to obtain natural analogues of Cramer’s rule for generalized inverse solutions of systems of linear equations.
### 3.1 Cramer’s Rule for the Least Squares Solution with the Minimum Norm

**Definition 3.1** Suppose in a complex system of linear equations:

\[ A \cdot x = y \] (3.1)

the coefficient matrix \( A \in \mathbb{C}^{m \times n} \) and a column of constants \( y = (y_1, \ldots, y_m)^T \in \mathbb{C}^m \). The least squares solution with the minimum norm of (3.1) is the vector \( x^0 \in \mathbb{C}^n \) satisfying

\[
\| x^0 \| = \min_{\tilde{x} \in \mathbb{C}^n} \left\{ \| \tilde{x} \| \mid \| A \cdot \tilde{x} - y \| = \min_{x \in \mathbb{C}^n} \| A \cdot x - y \| \right\},
\]

where \( \mathbb{C}^n \) is an \( n \)-dimension complex vector space.

If the equation (3.1) has no precision solutions, then \( x^0 \) is its optimal approximation.

The following important proposition is well-known.

**Theorem 3.2** [21] The vector \( x = A^+ y \) is the least squares solution with the minimum norm of the system (3.1).

**Theorem 3.3** The following statements are true for the system of linear equations (3.1).

i) If \( \text{rank } A = n \), then the components of the least squares solution with the minimum norm \( x^0 = (x^0_1, \ldots, x^0_n)^T \) are obtained by the formula

\[
x^0_j = \frac{\det(A^+ A)_{j,j}(f)}{\det A^+ A}, \quad (\forall j = 1, n), \tag{3.2}
\]

where \( f = A^+ y \).

ii) If \( \text{rank } A = r \leq m < n \), then

\[
x^0_j = \sum_{\beta \in J_r \setminus \{j\}} \frac{\left| ((A^+ A)_{j,j}(f))_{\beta}^* \right|}{d_r(A^+ A)}, \quad (\forall j = 1, n). \tag{3.3}
\]

**Proof.** i) If \( \text{rank } A = n \), then we can represent \( A^+ \) by (2.10). By multiplying \( A^+ \) into \( y \) we get (3.2).

ii) If \( \text{rank } A = k \leq m < n \), then \( A^+ \) can be represented by (2.5). By multiplying \( A^+ \) into \( y \) the least squares solution with the minimum norm of the linear system (3.1) is given by components as in (3.3).

Using (2.4) and (2.11), we can obtain another representation of the Cramer rule for the least squares solution with the minimum norm of a linear system.

**Theorem 3.4** The following statements are true for a system of linear equations written in the form \( x \cdot A = y \).
i) If \( \text{rank} \, A = m \), then the components of the least squares solution \( x^0 = yA^* \) are obtained by the formula

\[
x^0_i = \frac{\det(AA^*)_i (g)}{\det AA^*}, \quad (\forall i = 1, m),
\]

where \( g = yA^* \).

ii) If \( \text{rank} \, A = r \leq n < m \), then

\[
x^0_i = \sum_{\alpha \in L_{r,m}(i)} \frac{|((AA^*)_i (g))_{i\alpha}|}{d_r(AA^*)}, \quad (\forall i = 1, m).
\]

**Proof.** The proof of this theorem is analogous to that of Theorem 3.3.

**Remark 3.5** The obtained formulas of the Cramer rule for the least squares solution differ from similar formulas in [34,36–39]. They give a closer analogue to usual Cramer’s rule for consistent nonsingular systems of linear equations.

### 3.2 Cramer’s Rule for the Drazin Inverse Solution

In some situations, however, people pay more attention to the Drazin inverse solution of singular linear systems [40–43].

Consider a general system of linear equations (3.1), where \( A \in \mathbb{C}^{n \times n} \) and \( x, y \) are vectors in \( \mathbb{C}^n \). \( R(A) \) denotes the range of \( A \) and \( N(A) \) denotes the null space of \( A \).

The characteristic of the Drazin inverse solution \( A^Dy \) is given in [24] by the following theorem.

**Theorem 3.6** Let \( A \in \mathbb{C}^{n \times n} \) with \( \text{Ind}(A) = k \). Then \( A^Dy \) is both the unique solution in \( R(A^k) \) of

\[
A^{k+1}x = A^k y,
\]

and the unique minimal \( P \)-norm least squares solution of (3.1).

**Remark 3.7** The \( P \)-norm is defined as \( \|x\|_P = \|P^{-1}x\| \) for \( x \in \mathbb{C}^n \), where \( P \) is a nonsingular matrix that transforms \( A \) into its Jordan canonical form (2.14).

In other words, the Drazin inverse solution \( x = A^Dy \) is the unique solution of the problem: for a given \( A \) and a given vector \( y \in R(A^k) \), find a vector \( x \in R(A^k) \) satisfying \( Ax = y \) with \( \text{Ind} \, A = k \).

In general, unlike \( A^+y \), the Drazin inverse solution \( A^Dy \) is not a true solution of a singular system (3.1), even if the system is consistent. However, Theorem 3.6 means that \( A^Dy \) is the unique minimal \( P \)-norm least squares solution of (3.1).

A determinantal representation of the \( P \)-norm least squares solution of a system of linear equations (3.1) by the determinantal representation (2.15) of the Drazin inverse has been obtained in [24].
We give Cramer’s rule for the $\mathbf{P}$-norm least squares solution (the Drazin inverse solution) of (3.1) in the following theorem.

**Theorem 3.8** Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ with $\text{Ind} (\mathbf{A}) = k$ and rank $\mathbf{A}^{k+1} = \text{rank} \mathbf{A}^k = r$. Then the unique minimal $\mathbf{P}$-norm least squares solution $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^T$ of the system (3.1) is given by

$$\hat{x}_i = \frac{\sum_{\beta \in J_{r,n}(i)} \left| (\mathbf{A}^{k+1}_i)^\beta \right| f^\beta_{\beta}}{\sum_{\beta \in J_{r,n}} \left| (\mathbf{A}^{k+1})^\beta \right|} \forall i = 1, n, \quad (3.5)$$

where $f = \mathbf{A}^k y$.

**Proof.** Representing the Drazin inverse by (2.21) and by virtue of Theorem 3.6, we have

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_n \end{pmatrix} = \mathbf{A}^D y = \frac{1}{d_r (\mathbf{A}^{k+1})} \begin{pmatrix} \sum_{s=1}^{n} d_{1s} y_s \\ \vdots \\ \sum_{s=1}^{n} d_{ns} y_s \end{pmatrix}.$$

Therefore,

$$\hat{x}_i = \frac{1}{d_r (\mathbf{A}^{k+1})} \sum_{s=1}^{n} \sum_{\beta \in J_{r,n}(i)} \left| (\mathbf{A}^{k+1}_i)^\beta \right| f^\beta_{\beta} y_s = \frac{1}{d_r (\mathbf{A}^{k+1})} \sum_{\beta \in J_{r,n}(i)} \left| (\mathbf{A}^{k+1}_i)^\beta \right| y_s = \frac{1}{d_r (\mathbf{A}^{k+1})} \sum_{\beta \in J_{r,n}(i)} \sum_{s=1}^{n} \left| (\mathbf{A}^{k+1}_i)^\beta \right| f^\beta_{\beta} y_s.$$

From this (3.5) follows immediately. ■

If we shall present a system of linear equations as,

$$\mathbf{xA} = \mathbf{y}, \quad (3.6)$$

where $\mathbf{A} \in \mathbb{C}^{n \times n}$ with $\text{Ind} (\mathbf{A}) = k$ and rank $\mathbf{A}^{k+1} = \text{rank} \mathbf{A}^k = r$, then by using the Drazin inverse determinantal representation (2.20) we have the following analog of Cramer’s rule for the Drazin inverse solution of (3.6):

$$\hat{x}_i = \frac{\sum_{\alpha \in J_{r,n}(i)} \left| (\mathbf{A}^{k+1}_i)^\alpha \right| g^\alpha_{\alpha}}{\sum_{\alpha \in J_{r,n}} \left| (\mathbf{A}^{k+1})^\alpha \right|}, \quad \forall i = 1, n,$$

where $g = \mathbf{yA}^k$. 

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3.3 Cramer’s Rule for the W-Weighted Drazin Inverse Solution

Consider restricted linear equations
\[ W A W x = y, \quad (3.7) \]
where \( A \in \mathbb{C}^{m \times n} \), \( W \in \mathbb{C}^{n \times m} \), \( k_1 = \text{Ind}(AW) \), \( k_2 = \text{Ind}(WA) \) with \( y \in R((WA)^{k_2}) \) and \( \text{rank}(WA)^{k_2} = \text{rank}(AW)^{k_1} = r \).

In [31], Wei has showed that there exists an unique solution \( A_{d,W} y \) of the linear equations (3.7) and given a Cramer rule for the W-weighted Drazin inverse solution of (3.7) by the following theorem.

**Theorem 3.9** Let \( A, W \) be the same as in (3.7). Suppose that \( U \in \mathbb{C}^{n \times (n-r)} \) and \( V^* \in \mathbb{C}^{m \times (m-r)} \) be matrices whose columns form bases for \( N((WA)^{k_2}) \) and \( N((AW)^{k_1}) \), respectively. Then the unique W-weighted Drazin inverse solution \( x = (x_1, ..., x_m) \) of (3.7) satisfies
\[ x_i = \frac{\det \left( \begin{array}{cc} W A W (i \to y) & U \\ V (i \to 0) & 0 \end{array} \right)}{\det \left( \begin{array}{cc} W A W & U \\ V & 0 \end{array} \right)} \]
where \( i = 1, m \).

Let \( k = \max\{k_1, k_2\} \). Denote \( f = (AW)^{k} A \cdot y \). Then by Theorem 2.37 using the determinantal representation (2.36) of the W-weighted Drazin inverse \( A_{d,W} \), we evidently obtain the following Cramer’s rule of the W-weighted Drazin inverse solution of (3.7),
\[ x_i = \frac{\sum_{\beta \in J_{r,m} \setminus \{i\}} \left| \begin{array}{c} (AW)^{k+2} (f) \\ \beta \end{array} \right| \beta}{\sum_{\beta \in J_{r,m}} \left| (AW)^{k+2} \beta \right|}, \quad (3.8) \]
where \( i = 1, m \).

**Remark 3.10** Note that for \((3.8)\) unlike Theorem 3.9, we do not need auxiliary matrices \( U \) and \( V \).

3.4 Examples

1. Let us consider the system of linear equations.
\[
\begin{align*}
2x_1 - 5x_3 + 4x_4 &= 1, \\
7x_1 - 4x_2 - 9x_3 + 1.5x_4 &= 2, \\
3x_1 - 4x_2 + 7x_3 - 6.5x_4 &= 3, \\
x_1 - 4x_2 + 12x_3 - 10.5x_4 &= 1.
\end{align*}
\quad (3.9)
\]
The coefficient matrix of the system is \( A = \begin{pmatrix} 2 & 0 & -5 & 4 \\ 7 & -4 & -9 & 1.5 \\ 3 & -4 & 7 & -6.5 \\ 1 & -4 & 12 & -10.5 \end{pmatrix} \). The rank of \( A \) is equal to 3. We have
\[
A^+ = \begin{pmatrix} 2 & 7 & 3 & 1 \\ 0 & -4 & -4 & -4 \\ -5 & -9 & 7 & 12 \\ 4 & 1.5 & -6.5 & -10.5 \end{pmatrix}, \quad A^+ A = \begin{pmatrix} 63 & -44 & -40 & -11.5 \\ -44 & 48 & -40 & 62 \\ -40 & -40 & 299 & -205 \\ -11.5 & 62 & -205 & 170.75 \end{pmatrix}.
\]

At first we obtain entries of \( A^+ \) by (2.10):
\[
d_3(A^+ A) = \begin{vmatrix} 63 & -44 & -40 & + & 63 & -44 & -11.5 \\ -44 & 48 & -40 & + & -44 & 48 & 62 \\ -40 & -40 & 299 & + & -11.5 & 62 & 170.75 \\ -11.5 & -205 & 170.75 & + & -40 & 299 & -205 \\ 102060, \end{vmatrix} = 25779,
\]
and so forth. Continuing in the same way, we get
\[
A^+ = \frac{1}{102060} \begin{pmatrix} 25779 & -4905 & 20742 & -5037 \\ -3840 & -2880 & -4800 & -960 \\ 28350 & -17010 & 22680 & -5670 \\ 39558 & -18810 & 26484 & -13074 \end{pmatrix}.
\]

Now we obtain the least squares solution of the system (3.9) by the matrix method.
\[
x^0 = \frac{1}{102060} \begin{pmatrix} 25779 & -4905 & 20742 & -5037 \\ -3840 & -2880 & -4800 & -960 \\ 28350 & -17010 & 22680 & -5670 \\ 39558 & -18810 & 26484 & -13074 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 12193 \\ 17010 \\ 56700 \\ 68316 \end{pmatrix}.
\]

Next we get the least squares solution with minimum norm of the system (3.9) by the Cramer rule (3.3), where
\[
f = \begin{pmatrix} 2 & 7 & 3 & 1 \\ 0 & -4 & -4 & -4 \\ -5 & -9 & 7 & 12 \\ 4 & 1.5 & -6.5 & -10.5 \end{pmatrix}, \quad \begin{pmatrix} 26 \\ -24 \\ 10 \\ -23 \end{pmatrix}.
\]
Thus we have

$$x_1^0 = \frac{1}{102060} \begin{pmatrix}
26 & -44 & -40 \\
-24 & 48 & -40 \\
10 & -40 & 299 \\
\end{pmatrix} + \frac{1}{102060} \begin{pmatrix}
26 & -44 & -11.5 \\
-24 & 48 & 62 \\
10 & -40 & 299 \\
\end{pmatrix} + \frac{1}{102060} \begin{pmatrix}
26 & -40 & -11.5 \\
10 & 299 & -205 \\
23 & -205 & 170.75 \\
\end{pmatrix} = \frac{73158}{102060} = 0.717010;$$

$$x_2^0 = \frac{1}{102060} \begin{pmatrix}
63 & 26 & -40 \\
-44 & -24 & -40 \\
-40 & 10 & 299 \\
\end{pmatrix} + \frac{1}{102060} \begin{pmatrix}
63 & 26 & -11.5 \\
-44 & -24 & 62 \\
-40 & 10 & 299 \\
\end{pmatrix} + \frac{1}{102060} \begin{pmatrix}
-24 & -40 & 62 \\
10 & 299 & -205 \\
-23 & -205 & 170.75 \\
\end{pmatrix} = \frac{-24960}{102060} = -0.243107;$$

$$x_3^0 = \frac{1}{102060} \begin{pmatrix}
63 & -44 & 26 \\
-44 & 48 & -24 \\
-40 & -40 & 10 \\
\end{pmatrix} + \frac{1}{102060} \begin{pmatrix}
63 & 26 & -11.5 \\
-44 & -24 & 62 \\
-40 & 10 & 299 \\
\end{pmatrix} + \frac{1}{102060} \begin{pmatrix}
48 & -24 & 62 \\
-40 & -40 & 10 \\
-23 & -205 & 170.75 \\
\end{pmatrix} = \frac{56700}{102060} = 0.553106;$$

$$x_4^0 = \frac{1}{102060} \begin{pmatrix}
63 & -44 & 26 \\
-44 & 48 & -24 \\
-11.5 & 62 & -23 \\
\end{pmatrix} + \frac{1}{102060} \begin{pmatrix}
63 & 26 & -11.5 \\
-44 & -24 & 62 \\
-40 & 10 & 299 \\
\end{pmatrix} + \frac{1}{102060} \begin{pmatrix}
48 & -40 & -24 \\
-40 & 299 & 10 \\
62 & -205 & -23 \\
\end{pmatrix} = \frac{68316}{102060} = 0.668316.$$

2. Let us consider the following system of linear equations.

$$\begin{align*}
x_1 - x_2 + x_3 + x_4 &= 1, \\
x_2 - x_3 + x_4 &= 2, \\
x_1 - x_2 + x_3 + 2x_4 &= 3, \\
x_1 - x_2 + x_3 + x_4 &= 1. \\
\end{align*} \tag{3.10}$$

The coefficient matrix of the system is the matrix $A = \begin{pmatrix}
1 & -1 & 1 & 1 \\
0 & 1 & -1 & 1 \\
1 & -1 & 1 & 2 \\
1 & -1 & 1 & 1 \\
\end{pmatrix}$. It is easy to verify the following:

$$A^2 = \begin{pmatrix}
3 & -4 & 4 & 3 \\
0 & 1 & -1 & 0 \\
4 & -5 & 5 & 4 \\
3 & -4 & 4 & 3 \\
\end{pmatrix}, \quad A^3 = \begin{pmatrix}
10 & -14 & 14 & 10 \\
-1 & 2 & -2 & -1 \\
13 & -18 & 18 & 13 \\
10 & -14 & 14 & 10 \\
\end{pmatrix},$$
and rank $A = 3$, rank $A^2 = rank A^3 = 2$. This implies $k = \text{Ind}(A) = 2$. We obtain entries of $A^D$ by (2.21).

$$d_2(A^3) = \begin{vmatrix} 10 & -14 \\ -1 & 2 \\ 13 & 18 \\ 10 & 10 \end{vmatrix} + \begin{vmatrix} 10 & 14 \\ 2 & -1 \\ 18 & 13 \\ 14 & 10 \end{vmatrix} + \begin{vmatrix} 10 & 10 \\ -18 & 18 \\ -14 & 10 \\ 14 & 10 \end{vmatrix} = 8,$$

$$d_{11} = \begin{vmatrix} 3 & -14 \\ 0 & 2 \\ 3 & 14 \\ 3 & 10 \end{vmatrix} = 4,$$

and so forth.

Continuing in the same way, we get $A^D = \begin{pmatrix} 0.5 & 0.5 & -0.5 & 0.5 \\ 1.75 & 2.5 & -2.5 & 1.75 \\ 1.25 & 1.5 & -1.5 & 1.25 \\ 0.5 & 0.5 & -0.5 & 0.5 \end{pmatrix}$. Now we obtain the Drazin inverse solution $\hat{x}$ of the system (3.10) by the Cramer rule (4.3), where

$$g = A^2y = \begin{pmatrix} 3 & -4 & 4 & 3 \\ 0 & 1 & -1 & 0 \\ -5 & 5 & 4 \\ 3 & -4 & 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 13 \\ 10 \end{pmatrix}.$$

Thus we have

$$\hat{x}_1 = \frac{1}{8} \left( \begin{vmatrix} 10 & -14 \\ -1 & 2 \\ 13 & 18 \\ 10 & 10 \end{vmatrix} \right) = \frac{1}{2},$$

$$\hat{x}_2 = \frac{1}{8} \left( \begin{vmatrix} 10 & 10 \\ -1 & -1 \\ 13 & 18 \\ 10 & 10 \end{vmatrix} \right) = 1,$$

$$\hat{x}_3 = \frac{1}{8} \left( \begin{vmatrix} 10 & 10 \\ 13 & 13 \\ 2 & -1 \\ -18 & 18 \end{vmatrix} \right) = 1,$$

$$\hat{x}_4 = \frac{1}{8} \left( \begin{vmatrix} 10 & 10 \\ 10 & 10 \\ -10 & 10 \\ 14 & 10 \end{vmatrix} \right) = \frac{1}{2}.$$

4 Cramer’s Rule of the Generalized Inverse Solutions of Some Matrix Equations

Matrix equation is one of the important study fields of linear algebra. Linear matrix equations, such as

$$AX = C, \quad (4.1)$$

$$XB = D, \quad (4.2)$$

and

$$AXB = D, \quad (4.3)$$
play an important role in linear system theory therefore a large number of papers have presented several methods for solving these matrix equations \([45–49]\). In \([50]\), Khatri and Mitra studied the Hermitian solutions to the matrix equations \((4.1)\) and \((4.3)\) over the complex field and the system of the equations \((4.1)\) and \((4.2)\). Wang, in \([51, 52]\), and Li and Wu, in \([53]\) studied the bisymmetric, symmetric and skew-antisymmetric least squares solution to this system over the quaternion skew field. Extreme ranks of real matrices in least squares solution of the equation \((4.3)\) was investigated in \([54]\) over the complex field and in \([55]\) over the quaternion skew field.

As we know, the Cramer rule gives an explicit expression for the solution of nonsingular linear equations. Robinson’s result \((\[33]\)) aroused great interest in finding determinantal representations of a least squares solution as some analogs of Cramer’s rule for the matrix equations (for example, \([56–58]\)). Cramer’s rule for solutions of the restricted matrix equations \((4.1), (4.2)\) and \((4.3)\) was established in \([59–61]\).

In this section, we obtain analogs of the Cramer rule for generalized inverse solutions of the aforementioned equations without any restriction.

We shall show numerical examples to illustrate the main results as well.

### 4.1 Cramer’s Rule for the Minimum Norm Least Squares Solution of Some Matrix Equations

**Definition 4.1** Consider a matrix equation

\[
AX = B,
\]

where \(A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{m \times s}\) are given, \(X \in \mathbb{C}^{n \times s}\) is unknown. Suppose

\[
S_1 = \{X | X \in \mathbb{C}^{n \times s}, \|AX - B\| = \text{min}\}.
\]

Then matrices \(X \in \mathbb{C}^{n \times s}\) such that \(X \in S_1\) are called least squares solutions of the matrix equation \((4.4)\). If \(X_{LS} = \min_{X \in S_1} \|X\|\), then \(X_{LS}\) is called the minimum norm least squares solution of \((4.4)\).

If the equation \((4.4)\) has no precision solutions, then \(X_{LS}\) is its optimal approximation.

The following important proposition is well-known.

**Lemma 4.2** (\([38]\)) The least squares solutions of \((4.4)\) are

\[
X = A^+B + (I_n - A^+A)C,
\]

where \(A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{m \times s}\) are given, and \(C \in \mathbb{C}^{n \times s}\) is an arbitrary matrix. The least squares minimum norm solution is \(X_{LS} = A^+B\).

We denote \(A^+B =: \hat{B} = (\hat{b}_{ij}) \in \mathbb{C}^{n \times s}\).
Theorem 4.3  
(i) If \( \text{rank } A = r \leq m < n \), then we have for the minimum norm least squares solution \( X_{LS} = (x_{ij}) \in \mathbb{C}^{n \times s} \) of (4.4) for all \( i = 1, n \), \( j = 1, s \)
\[
x_{ij} = \frac{\sum_{\beta \in J_{r, n}} \left| \left( (A^* A)_{i} \left( \hat{b}_j \right) \right)_{\beta} \right|}{\sum_{\beta \in J_{r, n}} \left| (A^* A)_{\beta} \right|}.
\]

(ii) If \( \text{rank } A = n \), then for all \( i = 1, n \), \( j = 1, s \) we have
\[
x_{ij} = \frac{\det(A^* A)_{i} \left( \hat{b}_j \right)}{\det(A^* A)},
\]
where \( \hat{b}_j \) is the \( j \)-th column of \( \hat{B} \) for all \( j = 1, s \).

Proof. i) If \( \text{rank } A = r \leq m < n \), then by Theorem 2.9 we can represent \( A^+ \) by (2.5). Therefore, we obtain for all \( i = 1, n \), \( j = 1, s \)
\[
x_{ij} = \frac{\sum_{k=1}^{m} a_{ik}^* b_{kj} = \sum_{k=1}^{m} \frac{\sum_{\beta \in J_{r, n} \setminus \{i\}} \left| \left( (A^* A)_{i} \left( a_{k}^* \right) \right)_{\beta} \right|}{\sum_{\beta \in J_{r, n}} \left| (A^* A)_{\beta} \right|} \cdot b_{kj} =}{\sum_{\beta \in J_{r, n}} \left| (A^* A)_{\beta} \right|}.
\]
Since \( \sum_{k} a_{k}^* b_{kj} = \begin{pmatrix} \sum_{k} a_{1k}^* b_{kj} \\ \sum_{k} a_{2k}^* b_{kj} \\ \vdots \\ \sum_{k} a_{nk}^* b_{kj} \end{pmatrix} = \hat{b}_j \), then it follows (4.5).

(ii) The proof of this case is similarly to that of (i) by using Corollary 2.3. □

Definition 4.4  Consider a matrix equation
\[
XA = B,
\]
where \( A \in \mathbb{C}^{m \times n} \), \( B \in \mathbb{C}^{s \times n} \) are given, \( X \in \mathbb{C}^{s \times m} \) is unknown. Suppose
\[
S_2 = \{ X | X \in \mathbb{C}^{s \times m}, \| XA - B \| = \min \}.
\]
Then matrices \( X \in \mathbb{C}^{s \times m} \) such that \( X \in S_2 \) are called least squares solutions of the matrix equation (4.7). If \( X_{LS} = \min_{X \in S_2} \| X \| \), then \( X_{LS} \) is called the minimum norm least squares solution of (4.7).
The following lemma can be obtained by analogy to Lemma 4.2.

**Lemma 4.5** The least squares solutions of (4.7) are

\[ X = BA^+ + C(I_m - AA^+), \]

where \( A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{s \times n} \) are given, and \( C \in \mathbb{C}^{s \times m} \) is an arbitrary matrix. The minimum norm least squares solution is \( X_{LS} = BA^+ \).

We denote \( BA^* =: \tilde{B} = (\tilde{b}_{ij}) \in \mathbb{C}^{s \times m} \).

**Theorem 4.6**

(i) If \( \text{rank } A = r \leq n < m \), then we have for the minimum norm least squares solution \( X_{LS} = (x_{ij}) \in \mathbb{C}^{s \times m} \) of (4.7) for all \( i = \overline{1, s}, j = \overline{1, m} \)

\[ x_{ij} = \frac{\sum_{\alpha \in I_{r,m}(j)} \left| \left( (AA^*)_{j, (\tilde{b}_{i,j})} \right) \alpha \right|}{\sum_{\alpha \in I_{r,m}} \left| (AA^*)_{\alpha} \right|}. \quad (4.8) \]

(ii) If \( \text{rank } A = m \), then for all \( i = \overline{1, s}, j = \overline{1, m} \) we have

\[ x_{ij} = \frac{\det(AA^*)_{j, i} \cdot (\tilde{b}_{i,j})}{\det(AA^*)}, \quad (4.9) \]

where \( \tilde{b}_{i,j} \) is the \( i \)th row of \( \tilde{B} \) for all \( i = \overline{1, s} \).

**Proof.** (i) If \( \text{rank } A = r \leq n < m \), then by Theorem 2.9 we can represent \( A^+ \) by (2.6). Therefore, for all \( i = \overline{1, s}, j = \overline{1, m} \) we obtain

\[ x_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}^* = \sum_{k=1}^{n} b_{ik} \cdot \sum_{\alpha \in I_{r,m}(j)} \left| \left( (AA^*)_{j, (a_{i,k}^*)} \right) \alpha \right| = \frac{\sum_{k=1}^{n} b_{ik} \sum_{\alpha \in I_{r,m}(j)} \left| (AA^*)_{\alpha} \right|}{\sum_{\alpha \in I_{r,m}} \left| (AA^*)_{\alpha} \right|}. \]

Since for all \( i = \overline{1, s} \)

\[ \sum_{k} b_{ik} a_{k}^* = \left( \sum_{k} b_{ik} a_{k1}^* \sum_{k} b_{ik} a_{k2}^* \cdots \sum_{k} b_{ik} a_{km}^* \right) = \tilde{b}_{i,j}, \]

then it follows (4.8).

(ii) The proof of this case is similarly to that of (i) by using Corollary 2.3. ■

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Definition 4.7 Consider a matrix equation

\[ AXB = D, \]  

(4.10)

where \( A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{p \times q}, D \in \mathbb{C}^{m \times q} \) are given, \( X \in \mathbb{C}^{n \times p} \) is unknown. Suppose

\[ S_3 = \{ X | X \in \mathbb{C}^{n \times p}, \|AXB - D\| = \min \}. \]

Then matrices \( X \in \mathbb{C}^{n \times p} \) such that \( X \in S_3 \) are called least squares solutions of the matrix equation (4.10). If \( X_{LS} = \min_{X \in S_3} \| X \| \), then \( X_{LS} \) is called the minimum norm least squares solution of (4.10).

The following important proposition is well-known.

Lemma 4.8 (36) The least squares solutions of (4.10) are

\[ X = A^+DB^+ + (I_n - A^+A)V + W(I_p - BB^+), \]

where \( A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{p \times q}, D \in \mathbb{C}^{m \times q} \) are given, and \( \{ V, W \} \subset \mathbb{C}^{n \times p} \) are arbitrary quaternion matrices. The minimum norm least squares solution is

\[ X_{LS} = A^+DB^+. \]

We denote \( \tilde{D} = A^*DB^* \).

Theorem 4.9 (i) If rank \( A = r_1 < n \) and rank \( B = r_2 < p \), then for the minimum norm least squares solution \( X_{LS} = (x_{ij}) \in \mathbb{C}^{n \times p} \) of (4.10) we have

\[ x_{ij} = \frac{\sum_{\beta \in I_{r_1,n} \setminus \{ i \}} |(A^*A)_{i\beta} (d^B_{\alpha \beta})^\beta|}{\sum_{\beta \in I_{r_1,n}} |(A^*A)_{\beta\beta}| \sum_{\alpha \in I_{r_2,p}} |(BB^*)_{\alpha\alpha}|}, \]

(4.11)

or

\[ x_{ij} = \frac{\sum_{\alpha \in I_{r_2,p} \setminus \{ j \}} |(BB^*)_{j\alpha} (d^A_{\alpha \alpha})^\alpha|}{\sum_{\beta \in I_{r_1,n}} |(A^*A)_{\beta\beta}| \sum_{\alpha \in I_{r_2,p}} |(BB^*)_{\alpha\alpha}|}, \]

(4.12)

where

\[ d^B_{\alpha \beta} = \left[ \sum_{\alpha \in I_{r_2,p} \setminus \{ j \}} |(BB^*)_{j \alpha} (d^B_{\alpha \alpha})^\alpha|, ..., \sum_{\alpha \in I_{r_2,p} \setminus \{ j \}} |(BB^*)_{j \alpha} (d^B_{\alpha \alpha})^\alpha| \right]^T, \]

(4.13)

\[ d^A_{\alpha \beta} = \left[ \sum_{\beta \in I_{r_1,n} \setminus \{ i \}} |(A^*A)_{i \beta} (d^A_{\beta \beta})^\beta|, ..., \sum_{\beta \in I_{r_1,n} \setminus \{ i \}} |(A^*A)_{i \beta} (d^A_{\beta \beta})^\beta| \right], \]

(4.14)

are the column-vector and the row-vector, respectively. \( \tilde{d}_i \) is the \( i \)-th row of \( \tilde{D} \) for all \( i = 1, n \), and \( d_j \) is the \( j \)-th column of \( \tilde{D} \) for all \( j = 1, p \).
(ii) If rank $A = n$ and rank $B = p$, then for the least squares solution $X_{LS} = (x_{ij}) \in \mathbb{C}^{n \times p}$ of (4.10) we have for all $i = 1, n$, $j = 1, p$,

$$x_{ij} = \frac{\det((A^*A)_{ij}(d^B))}{\det(A^*A) \cdot \det(BB^*)},$$

or

$$x_{ij} = \frac{\det((BB^*)_{ij}(d^A))}{\det(A^*A) \cdot \det(BB^*)},$$

where

$$d^B_j := [\det((BB^*)_{j1}(d_{11})), \ldots, \det((BB^*)_{jn}(d_{n1}))]^T,$$

$$d^A_i := [\det((A^*A)_{i1}(d_{11})), \ldots, \det((A^*A)_{in}(d_{n1}))].$$

are respectively the column-vector and the row-vector.

(iii) If rank $A = n$ and rank $B = r_2 < p$, then for the least squares solution $X_{LS} = (x_{ij}) \in \mathbb{C}^{n \times p}$ of (4.10) we have

$$x_{ij} = \frac{\det((A^*A)_{ij}(d^B))}{\det(A^*A) \sum_{\alpha \in I_{r_2,p}} |(BB^*)^\alpha_{ij}|},$$

or

$$x_{ij} = \frac{\sum_{\alpha \in I_{r_2,p}} |(BB^*)_{ij}(d^A)_{\alpha}|}{\det(A^*A) \sum_{\alpha \in I_{r_2,p}} |(BB^*)^\alpha_{ij}|},$$

where $d^B_j$ is (4.13) and $d^A_i$ is (4.18).

(iii) If rank $A = r_1 < m$ and rank $B = p$, then for the least squares solution $X_{LS} = (x_{ij}) \in \mathbb{C}^{n \times p}$ of (4.10) we have

$$x_{ij} = \frac{\det((BB^*)_{ij}(d^B))}{\sum_{\beta \in J_{r_1,n}} |(A^*A)^\beta_{ij}| \cdot \det(BB^*)},$$

or

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1,n}} |(A^*A)_{ij}(d^B)^{\beta}|}{\sum_{\beta \in J_{r_1,n}} |(A^*A)^\beta_{ij}| \cdot \det(BB^*)},$$

where $d^B_j$ is (4.17) and $d^A_i$ is (4.14).
Proof. (i) If \( A \in \mathbb{C}^{m \times n} \), \( B \in \mathbb{C}^{p \times q} \) and \( r_1 < n \), \( r_2 < p \), then by Theorem 4.8 the Moore-Penrose inverses \( A^+ = (a_{ij}^+) \in \mathbb{C}^{n \times m} \) and \( B^+ = (b_{ij}^+) \in \mathbb{C}^{p \times q} \) possess the following determinantal representations respectively,

\[
a_{ij}^+ = \frac{\sum_{\beta \in J_{r_1, n} \setminus \{i\}} |(A^*A)_{i, (a_{ij}^+)}\|_\beta^\beta |(A^*A)_{j, \beta}\|_\beta^\beta}{\sum_{\beta \in J_{r_1, n}} |(A^*A)_{j, \beta}\|_\beta^\beta},
\]

\[
b_{ij}^+ = \frac{\sum_{\alpha \in J_{r_2, p} \setminus \{j\}} |(BB^*)_{j, (b_{ij}^+)}\|_\alpha^\alpha |(BB^*)_{\alpha, \alpha}\|_\alpha^\alpha}{\sum_{\alpha \in J_{r_2, p}} |(BB^*)_{\alpha, \alpha}\|_\alpha^\alpha}.
\]

(4.23)

Since by Theorem 4.8 \( X_{LS} = A^+DB^+ \), then an entry of \( X_{LS} = (x_{ij}) \) is

\[
x_{ij} = \sum_{s=1}^q \left( \sum_{k=1}^m a_{ik}^+d_{ks} \right) b_{sj}^+.
\]

(4.24)

Denote by \( \hat{d}_s \) the \( s \)th column of \( A^*D =: \hat{D} = (\hat{d}_{ij}) \in \mathbb{C}^{n \times q} \) for all \( s = 1, q \). It follows from \( \sum_k a_{ik}^+d_{ks} = \hat{d}_s \) that

\[
\sum_{k=1}^m a_{ik}^+d_{ks} = \sum_{k=1}^m \frac{\sum_{\beta \in J_{r_1, n} \setminus \{i\}} |(A^*A)_{i, (a_{ik}^+)}\|_\beta^\beta |(A^*A)_{j, \beta}\|_\beta^\beta}{\sum_{\beta \in J_{r_1, n}} |(A^*A)_{j, \beta}\|_\beta^\beta} \cdot d_{ks} = \sum_{\beta \in J_{r_1, n}} \sum_{\{k=1 \}}^m \frac{|(A^*A)_{i, (a_{ik}^+)}\|_\beta^\beta d_{ks} |(A^*A)_{j, \beta}\|_\beta^\beta}{\sum_{\beta \in J_{r_1, n}} |(A^*A)_{j, \beta}\|_\beta^\beta}.
\]

(4.25)

Suppose \( e_s \) and \( e_s \) are respectively the unit row-vector and the unit column-vector whose components are 0, except the \( s \)th components, which are 1. Substituting (4.25) and (4.23) in (4.24), we obtain

\[
x_{ij} = \sum_{s=1}^q \frac{\sum_{\beta \in J_{r_1, n} \setminus \{i\}} |(A^*A)_{i, (d_{s})}\|_\beta^\beta |(BB^*)_{j, (b_{ij}^+)}\|_\alpha^\alpha}{\sum_{\beta \in J_{r_1, n}} |(A^*A)_{j, \beta}\|_\beta^\beta} \cdot \sum_{\alpha \in J_{r_2, p}} \frac{|(BB^*)_{j, \alpha}\|_\alpha^\alpha}{\sum_{\alpha \in J_{r_2, p}} |(BB^*)_{\alpha, \alpha}\|_\alpha^\alpha}.
\]

(4.26)

Since

\[
\hat{d}_s = \sum_{l=1}^n e_le_{ls}, \quad \hat{b}_s = \sum_{t=1}^p b_{st}e_{lt}, \quad \sum_{s=1}^q \hat{d}_{ls}b_{st} = \hat{d}_{lt},
\]

(4.26)

then we have

\[
x_{ij} = \sum_{s=1}^q \frac{\sum_{\beta \in J_{r_1, n} \setminus \{i\}} |(A^*A)_{i, (d_{s})}\|_\beta^\beta |(BB^*)_{j, (b_{ij}^+)}\|_\alpha^\alpha}{\sum_{\beta \in J_{r_1, n}} |(A^*A)_{j, \beta}\|_\beta^\beta} \cdot \sum_{\alpha \in J_{r_2, p}} \frac{|(BB^*)_{j, \alpha}\|_\alpha^\alpha}{\sum_{\alpha \in J_{r_2, p}} |(BB^*)_{\alpha, \alpha}\|_\alpha^\alpha}.
\]

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\[
\begin{align*}
&\sum_{a=1}^{q} \sum_{t=1}^{p} \sum_{l=1}^{n} \sum_{\beta \in J_{r_1,n}(i)} \left| (A^*A)_{i,t} \left( e_i \right)_{j,\beta} \right| \sum_{\alpha \in I_{r_2,p}(j)} \left| (BB^*)_{j,\alpha} \right| \\
&\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_{i,t} \left( e_i \right)_{j,\beta} \right| \sum_{\alpha \in I_{r_2,p}} \left| (BB^*)_{j,\alpha} \right| \\
&\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_{i,t} \left( e_i \right)_{j,\beta} \right| \sum_{\alpha \in I_{r_2,p}} \left| (BB^*)_{j,\alpha} \right|.
\end{align*}
\]

(4.27)

Denote by

\[
d_{it}^A := \sum_{\beta \in J_{r_1,n}(i)} \left| (A^*A)_{i,t} \left( \tilde{d}_t \right)_{\beta,\beta} \right| = \sum_{l=1}^{n} \sum_{\beta \in J_{r_1,n}(i)} \left| (A^*A)_{i,t} \left( e_i \right)_{j,\beta} \right| \tilde{d}_t
\]

the \( t \)-th component of a row-vector \( d_t^A = (d_{i1},...,d_{ip}) \) for all \( t = 1, p \). Substituting it in (4.27), we have

\[
x_{ij} = \frac{\sum_{t=1}^{p} d_{it}^A \sum_{\alpha \in I_{r_2,p}(j)} \left| (BB^*)_{j,\alpha} \right|}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_{i,t} \right| \sum_{\alpha \in I_{r_2,p}} \left| (BB^*)_{j,\alpha} \right|}.
\]

Since \( \sum_{t=1}^{p} d_{it}^A e_t = d_i^A \), then it follows (4.12).

If we denote by

\[
d_{lj}^B := \sum_{t=1}^{p} \tilde{d}_t \sum_{\alpha \in I_{r_2,p}(j)} \left| (BB^*)_{j,\alpha} \right| = \sum_{\alpha \in I_{r_2,p}(j)} \left| (BB^*)_{j,\alpha} \right|
\]

(4.28)

the \( l \)-th component of a column-vector \( d_l^B = (d_{1j},...,d_{nj})^T \) for all \( l = 1, n \) and substitute it in (4.27), we obtain

\[
x_{ij} = \frac{\sum_{l=1}^{n} \sum_{\beta \in J_{r_1,n}(i)} \left| (A^*A)_{i,t} \left( e_i \right)_{j,\beta} \right| d_{lj}^B \sum_{\alpha \in I_{r_2,p}} \left| (BB^*)_{j,\alpha} \right|}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_{i,t} \right| \sum_{\alpha \in I_{r_2,p}} \left| (BB^*)_{j,\alpha} \right|}.
\]

Since \( \sum_{l=1}^{n} e_i d_{lj}^B = d_j^B \), then it follows (4.11).

(ii) If rank \( A = n \) and rank \( B = p \), then by Corollary 2.3, \( A^+ = (A^*A)^{-1} A^* \)
and \( B^+ = B^* (BB^*)^{-1} \). Therefore, we obtain

\[
X_{LS} = (A^*A)^{-1}A^*DB^* (BB^*)^{-1} = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1p} \\
 x_{21} & x_{22} & \ldots & x_{2p} \\
 \vdots & \vdots & \ddots & \vdots \\
 x_{n1} & x_{n2} & \ldots & x_{np} \\
 \end{pmatrix} \begin{pmatrix} L_{A1}^A & L_{A2}^A & \ldots & L_{Am}^A \\
 L_{A1}^A & L_{A2}^A & \ldots & L_{Am}^A \\
 \vdots & \vdots & \ddots & \vdots \\
 L_{An}^A & L_{An}^A & \ldots & L_{An}^A \\
 \end{pmatrix}^{-1} \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1m} \\
 d_{21} & d_{22} & \ldots & d_{2m} \\
 \vdots & \vdots & \ddots & \vdots \\
 d_{n1} & d_{n2} & \ldots & d_{nm} \\
 \end{pmatrix}^{-1} \begin{pmatrix} R_{B1} & R_{B2} & \ldots & R_{Bp} \\
 R_{B1} & R_{B2} & \ldots & R_{Bp} \\
 \vdots & \vdots & \ddots & \vdots \\
 R_{B1} & R_{B2} & \ldots & R_{Bp} \\
 \end{pmatrix}^{-1},
\]

where \( \hat{d}_{ij} \) is the \( ij \)-th entry of the matrix \( \hat{D} \), \( L_{ij}^A \) is the \( ij \)-th cofactor of \((A^*A)\) for all \( i, j = 1, n \) and \( R_{ij}^B \) is the \( ij \)-th cofactor of \((BB^*)\) for all \( i, j = 1, p \). This implies

\[
x_{ij} = \frac{\sum_{k=1}^{n} L_{ki}^A \left( \sum_{s=1}^{p} \hat{d}_{ks} R_{js}^B \right)}{\det(A^*A) \cdot \det(BB^*)}. \tag{4.29}
\]

for all \( i = 1, n, j = 1, p \). We obtain the sum in parentheses and denote it as follows

\[
\sum_{s=1}^{p} \hat{d}_{ks} R_{js}^B = \det(BB^*)_j \left( \hat{d}_{k} \right) := d_{k,j}^B,
\]

where \( \hat{d}_{k} \) is the \( k \)-th row-vector of \( \hat{D} \) for all \( k = 1, n \). Suppose \( d_{j}^B := (d_{1,j}, \ldots, d_{n,j})^T \) is the column-vector for all \( j = 1, p \). Reducing the sum \( \sum_{k=1}^{n} L_{ki}^A \hat{d}_{k,j} \), we obtain an analog of Cramer’s rule for \((4.10)\) by \((4.15)\).

Interchanging the order of summation in \((4.29)\), we have

\[
x_{ij} = \frac{\sum_{s=1}^{p} \left( \sum_{k=1}^{n} L_{ki}^A \hat{d}_{ks} \right) R_{js}^B}{\det(A^*A) \cdot \det(BB^*)}.
\]

We obtain the sum in parentheses and denote it as follows

\[
\sum_{k=1}^{n} L_{ki}^A \hat{d}_{ks} = \det(A^*A)_i \left( \hat{d}_{s} \right) := d_{i,s}^A,
\]

where \( \hat{d}_{s} \) is the \( s \)-th column-vector of \( \hat{D} \) for all \( s = 1, p \). Suppose \( d_{i}^A := (d_{i,1}, \ldots, d_{i,n})^T \) is the row-vector for all \( i = 1, n \). Reducing the sum \( \sum_{s=1}^{n} d_{i,s}^A R_{js}^B \), we obtain another analog of Cramer’s rule for the least squares solutions of \((4.10)\) by \((4.16)\).

(iii) If \( A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{p \times q} \) and \( r_1 = n, r_2 < p \), then by Remark \(2.12\) and Theorem \(2.9\) the Moore-Penrose inverses \( A^+ = (a_{ij}^+) \in \mathbb{C}^{n \times m} \) and \( B^+ = (b_{ij}^+) \in \mathbb{C}^{m \times p} \).
$(b^+_ij) \in \mathbb{C}^{q \times p}$ possess the following determinantal representations respectively,

$$a^+_ij = \frac{\det(A^*A)_{i,j} (a^*_i)}{\det(A^*A)}.$$  

$$b^+_ij = \frac{\sum_{\alpha \in I_2, p} |(BB^*)_j \cdot (b^*_i)_\alpha|}{\sum_{\alpha \in I_2, p} |(BB^*)_\alpha|}. \quad (4.30)$$

Since by Theorem 4.8 $X_{LS} = A^+ DB^+$, then an entry of $X_{LS} = (x_{ij})$ is (4.24). Denote by $\hat{d}_s$ the $s$-th column of $A^* D =: \hat{D} = (\hat{d}^*_{ij}) \in \mathbb{C}^{n \times q}$ for all $s = 1, q$. It follows from $\sum_k a^*_k d_{ks} = \hat{d}^*_s$ that

$$\sum_{k=1}^m a^*_{ik} d_{ks} = \sum_{k=1}^m \frac{\det(A^*A)_{i,k} (a^*_k)}{\det(A^*A)} \cdot d_{ks} = \frac{\det(A^*A)_{i,k} (\hat{d}^*_s)}{\det(A^*A)} \quad (4.31)$$

Substituting (4.31) and (4.30) in (4.24), and using (4.26) we have

$$x_{ij} = \sum_{s=1}^q \frac{\det(A^*A)_{i,k} (\hat{d}^*_s) \sum_{\alpha \in I_2, p} |(BB^*)_j \cdot (b^*_i)_\alpha|}{\det(A^*A) \sum_{\alpha \in I_2, p} |(BB^*)_\alpha|} =$$

$$\sum_{s=1}^q \sum_{t=1}^p \sum_{l=1}^n \frac{\det(A^*A)_{i,k} (\tilde{d}^*_{ls} b^*_{st}) \sum_{\alpha \in I_2, p} |(BB^*)_j \cdot (e^*_t)_\alpha|}{\det(A^*A) \sum_{\alpha \in I_2, p} |(BB^*)_\alpha|} =$$

$$\sum_{t=1}^p \sum_{l=1}^n \frac{\det(A^*A)_{i,k} (\tilde{d}^*_{lt} b^*_{st}) \sum_{\alpha \in I_2, p} |(BB^*)_j \cdot (e^*_t)_\alpha|}{\det(A^*A) \sum_{\alpha \in I_2, p} |(BB^*)_\alpha|}. \quad (4.32)$$

If we substitute (4.28) in (4.32), then we get

$$x_{ij} = \sum_{t=1}^n \frac{\det(A^*A)_{i,k} (e^*_t) d^B_{jt}}{\det(A^*A) \sum_{\alpha \in I_2, p} |(BB^*)_\alpha|}.$$  

Since again $\sum_{l=1}^n e^*_t d^B_{jt} = d^B_{ij}$, then it follows (4.19), where $d^B_{ij}$ is (4.13). If we denote by

$$d^A_{it} := \sum_{l=1}^n \det(A^*A)_{i,l} (\tilde{d}^*_{lt}) = \sum_{l=1}^n \det(A^*A)_{i,l} (e^*_t) \tilde{d}^*_{lt}$$

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the $t$-th component of a row-vector $d^A_\mathbf{i} = (d^A_{i1}, ..., d^A_{ip})$ for all $t = 1, p$ and substitute it in (4.32), we obtain

$$x_{ij} = \sum_{t=1}^p d^A_{it} \frac{\sum_{\alpha \in I_{r2,p}(j)} \left| (BB^*)_j \right|}{\det (A^*A) \sum_{\alpha \in I_{r2,p}} \left| (BB^*)_\alpha \right|}.$$  

Since again $\sum_{t=1}^p d^A_\mathbf{te}_t = d^A_\mathbf{i}$, then it follows (4.20), where $d^A_\mathbf{i}$ is (4.18).

(iii) The proof is similar to the proof of (iii).

4.2 Cramer’s Rule of the Drazin Inverse Solutions of Some Matrix Equations

Consider a matrix equation

$$AX = B,$$  

(4.33)

where $A \in \mathbb{C}^{n \times n}$ with $\text{Ind} A = k$, $B \in \mathbb{C}^{n \times m}$ are given and $X \in \mathbb{C}^{n \times m}$ is unknown.

**Theorem 4.10** ([62], Theorem 1) If the range space $R(B) \subset R(A^k)$, then the matrix equation (4.33) with constraint $R(X) \subset R(A^k)$ has a unique solution

$$X = A^D B.$$  

We denote $A^k B =: \hat{B} = (\hat{b}_{ij}) \in \mathbb{C}^{n \times m}$.

**Theorem 4.11** If rank $A^{k+1} = \text{rank} A^k = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then for the Drazin inverse solution $X = A^D B = (x_{ij}) \in \mathbb{C}^{n \times m}$ of (4.33) we have for all $i = 1, n$, $j = 1, m$,

$$x_{ij} = \frac{\sum_{\beta \in J_{r,n}(i)} \left| (A^{k+1} \mathbf{a}^{(k)} \hat{b}_j) \right|}{\sum_{\beta \in J_{r,n}} \left| (A^{k+1} \mathbf{a})_\beta \right|}.$$  

(4.34)

**Proof.** By Theorem 2.29 we can represent $A^D$ by (2.21). Therefore, we obtain for all $i = 1, n$, $j = 1, m$,

$$x_{ij} = \sum_{s=1}^n a^D_{is} b_{sj} = \sum_{s=1}^n \sum_{\beta \in J_{r,n}(i)} \left| (A^{k+1} \mathbf{a}^{(k)} \mathbf{a}^{(k)}_s) \beta \right| \sum_{\beta \in J_{r,n}} \left| (A^{k+1} \mathbf{a})_\beta \right| \cdot b_{sj} = \frac{\sum_{\beta \in J_{r,n}(i)} \sum_{s=1}^n \left| (A^{k+1} \mathbf{a}^{(k)} \mathbf{a}^{(k)}_s) \beta \right| \cdot b_{sj}}{\sum_{\beta \in J_{r,n}} \left| (A^{k+1} \mathbf{a})_\beta \right|}.$$  

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Since \( \sum_s a_s^{(k)} b_{sj} \) = \( \hat{b}_j \), then it follows (4.34).

Consider a matrix equation

\[ XA = B, \quad (4.35) \]

where \( A \in \mathbb{C}^{n \times m} \) with \( \text{Ind} \, A = k \), \( B \in \mathbb{C}^{n \times m} \) are given and \( X \in \mathbb{C}^{n \times m} \) is unknown.

**Theorem 4.12** ([62], Theorem 2) If the null space \( N(B) \supset N(A^k) \), then the matrix equation (4.35) with constraint \( |N(X) \supset N(A^k) | \) has a unique solution

\[ X = BA^D. \]

We denote \( BA^k =: \hat{B} = (\hat{b}_{ij}) \in \mathbb{C}^{n \times m} \).

**Theorem 4.13** If \( \text{rank} \, A^{k+1} = \text{rank} \, A^k = r \leq m \) for \( A \in \mathbb{C}^{n \times m} \), then for the Drazin inverse solution \( X = BA^D = (x_{ij}) \in \mathbb{C}^{n \times m} \) of (4.35), we have for all \( i = 1, n, j = 1, m \),

\[ x_{ij} = \frac{\sum_{a \in I_{r,m}} \left| \left( A^{k+1}_{j} \cdot (\hat{b}_{s}) \right)_{a} \right|}{\sum_{a \in I_{r,m}} \left| (A^{k+1})_{a} \right|}. \quad (4.36) \]

**Proof.** By Theorem 2.29 we can represent \( A^D \) by (2.20). Therefore, we obtain for all \( i = 1, n, j = 1, m \),

\[ x_{ij} = \sum_{s=1}^{m} b_{is} a_{sj} = \sum_{s=1}^{m} b_{is} \cdot \frac{\sum_{a \in I_{r,m}} \left| \left( A^{k+1}_{j} \cdot (a_s^{(k)}) \right)_{a} \right|}{\sum_{a \in I_{r,m}} \left| (A^{k+1})_{a} \right|} = \frac{\sum_{s=1}^{m} b_{ik} \sum_{a \in I_{r,m}} \left| \left( A^{k+1}_{j} \cdot (a_s^{(k)}) \right)_{a} \right|}{\sum_{a \in I_{r,m}} \left| (A^{k+1})_{a} \right|}. \]

Since for all \( i = 1, n \)

\[ \sum_s b_{is} a_s^{(k)} = \left( \sum_s b_{is} a_{s1}^{(k)} \sum_s b_{is} a_{s2}^{(k)} \ldots \sum_s b_{is} a_{sm}^{(k)} \right) = \hat{b}_{i}, \]

then it follows (4.36).

Consider a matrix equation

\[ AXB = D, \quad (4.37) \]

where \( A \in \mathbb{C}^{n \times n} \) with \( \text{Ind} \, A = k_1 \), \( B \in \mathbb{C}^{m \times m} \) with \( \text{Ind} \, B = k_2 \) and \( D \in \mathbb{C}^{n \times m} \) are given, and \( X \in \mathbb{C}^{n \times m} \) is unknown.
Theorem 4.14 (\cite{62}, Theorem 3) If $R(D) \subset R(A^{k_1})$ and $N(D) \supset N(B^{k_2})$, $k = \max\{k_1, k_2\}$, then the matrix equation (4.37) with constrain $R(X) \subset R(A^k)$ and $N(X) \supset N(B^k)$ has a unique solution

$$X = A^DDB^D.$$ 

We denote $A^{k_1}DB^{k_2} =: \tilde{D} = (\tilde{d}_{ij}) \in \mathbb{C}^{n \times m}$.

Theorem 4.15 If $\text{rank} A^{k_1+1} = \text{rank} A^{k_1} = r_1 \leq n$ for $A \in \mathbb{C}^{n \times n}$, and $\text{rank} B^{k_2+1} = \text{rank} B^{k_2} = r_2 \leq m$ for $B \in \mathbb{C}^{m \times m}$, then for the Drazin inverse solution $X = A^DDB^D =: (x_{ij}) \in \mathbb{C}^{n \times m}$ of (4.37) we have

$$x_{ij} = \frac{\sum_{\beta \in J_{1,n}(i)} |A^{k_1+1}_{ij} (d_B^,)_{\beta} |}{\sum_{\beta \in J_{1,n}} |(A^{k_1+1})_{ij} \beta \sum_{\alpha \in I_{2,m}} |(B^{k_2+1})_{\alpha}|},$$

or

$$x_{ij} = \frac{\sum_{\alpha \in I_{2,m}(j)} |B^{k_2+1}_{ij} (d_A^,)_{\alpha} |}{\sum_{\beta \in J_{1,n}} |(A^{k_1+1})_{ij} \beta \sum_{\alpha \in I_{2,m}} |(B^{k_2+1})_{\alpha}|},$$

where

$$d_{B}^j = \left[ \sum_{\alpha \in I_{2,m}(j)} B^{k_2+1}_{j,1} (\tilde{d}_1)_{\alpha}, \ldots, \sum_{\alpha \in I_{2,m}(j)} B^{k_2+1}_{j,n} (\tilde{d}_n)_{\alpha} \right]^T,$$

$$d_{A}^i = \left[ \sum_{\beta \in J_{1,n}(i)} A^{k_1+1}_{i,1} (\tilde{d}_1)_{\beta}, \ldots, \sum_{\beta \in J_{1,n}(i)} A^{k_1+1}_{i,n} (\tilde{d}_n)_{\beta} \right]$$

are the column-vector and the row-vector, $\tilde{d}_i$ and $\tilde{d}_j$ are respectively the $i$-th row and the $j$-th column of $\tilde{D}$ for all $i = 1, n, j = 1, m$.

**Proof.** By (2.21) and (2.20) the Drazin inverses $A^D = (a^D_{ij}) \in \mathbb{C}^{n \times n}$ and $B^D = (b^D_{ij}) \in \mathbb{C}^{m \times m}$ possess the following determinantal representations, respectively,

$$a^D_{ij} = \frac{\sum_{\beta \in J_{1,n}(i)} |A^{k_1+1}_{ij} (a^{k_1})_{\beta} |}{\sum_{\beta \in J_{1,n}} |(A^{k_1+1})_{ij} \beta \sum_{\alpha \in I_{2,m}} |(B^{k_2+1})_{\alpha}|},$$

$$b^D_{ij} = \frac{\sum_{\alpha \in I_{2,m}(j)} |B^{k_2+1}_{ij} (b^{k_2})_{\alpha} |}{\sum_{\alpha \in I_{2,m}} |(B^{k_2+1})_{ij} \alpha |}.$$

(4.41)
Then an entry of the Drazin inverse solution \( \mathbf{X} = \mathbf{A}^D \mathbf{D}\mathbf{B}^D =: (x_{ij}) \in \mathbb{C}^{n \times m} \) is

\[
x_{ij} = \sum_{s=1}^{m} \left( \sum_{t=1}^{n} a_{it}^D d_{ts} \right) b_{sj}^D. \tag{4.42}
\]

Denote by \( \mathbf{d}_{s} \) the \( s \)-th column of \( \mathbf{A}^D \mathbf{D} =: \mathbf{D} = (d_{ij}) \in \mathbb{C}^{n \times m} \) for all \( s = 1, m \). It follows from \( \sum_t a_{it}^D d_{ts} = \mathbf{d}_{s} \) that

\[
\sum_{k=1}^{n} a_{it}^D d_{ts} = \sum_{k=1}^{n} \sum_{\beta \in J_{1,n}(i)} \sum_{\beta \in J_{1,n}} |\mathbf{A}_{k,i}^{k+1}(a_{it}^{(k)})_{\beta}| \cdot d_{ts} = 
\]

\[
\sum_{\beta \in J_{1,n}(i)} \sum_{\beta \in J_{1,n}} |\mathbf{A}_{k,i}^{k+1}(a_{it}^{(k)})_{\beta}| \cdot d_{ts} = \sum_{\beta \in J_{1,n}(i)} \sum_{\beta \in J_{1,n}} |\mathbf{A}_{l,j}^{l+1}(\mathbf{d}_{s})_{\beta}| \tag{4.43}
\]

Substituting (4.43) and (4.41) in (4.42), we obtain

\[
x_{ij} = \sum_{s=1}^{m} \sum_{\beta \in J_{1,n}(i)} \sum_{\beta \in J_{2,m}} |\mathbf{A}_{k,i}^{k+1}(\mathbf{d}_{s})_{\beta}| \cdot \sum_{\alpha \in J_{2,m}} |\mathbf{B}_{j}^{l+1}(b_{s}^{(k)})_{\alpha}| = 
\]

\[
\sum_{\beta \in J_{1,n}(i)} \sum_{\beta \in J_{2,m}} |\mathbf{A}_{k,i}^{k+1}(\mathbf{d}_{s})_{\beta}| \cdot \sum_{\alpha \in J_{2,m}} |\mathbf{B}_{j}^{l+1}(b_{s}^{(k)})_{\alpha}|. \tag{4.44}
\]

Suppose \( \mathbf{e}_s \) and \( \mathbf{e}_t \) are respectively the unit row-vector and the unit column-vector whose components are 0, except the \( s \)-th components, which are 1. Since

\[
\mathbf{d}_{s} = \sum_{l=1}^{m} \mathbf{e}_l \mathbf{d}_{st}, \quad \mathbf{b}_{s}^{(k)} = \sum_{t=1}^{m} b_{st}^{(k)} \mathbf{e}_l, \quad \sum_{s=1}^{m} \mathbf{d}_{st}^{(k)} = \mathbf{d}_{it},
\]

then we have

\[
x_{ij} = 
\]

\[
\sum_{s=1}^{m} \sum_{t=1}^{n} \sum_{\beta \in J_{1,n}(i)} \sum_{\beta \in J_{2,m}} |\mathbf{A}_{k,i}^{k+1}(\mathbf{e}_l)^{\beta}_{\beta}| \cdot \sum_{\alpha \in J_{2,m}} |\mathbf{B}_{j}^{l+1}(\mathbf{e}_l)^{\alpha}_{\alpha}| = 
\]

\[
\sum_{\beta \in J_{1,n}(i)} \sum_{\beta \in J_{2,m}} |\mathbf{A}_{k,i}^{k+1}(\mathbf{d}_{s})_{\beta}| \cdot \sum_{\alpha \in J_{2,m}} |\mathbf{B}_{j}^{l+1}(b_{s}^{(k)})_{\alpha}|. \tag{4.44}
\]

Denote by

\[
a_{it}^A :=
\]
\[
\sum_{\beta \in J_{r,1,n}(i)} \left| A_{k_1+1,i}^{(\beta)} \tilde{d}_t \right| = \sum_{l=1}^{n} \sum_{\beta \in J_{r,1,n}(i)} \left| A_{k_1+1,i}^{(\beta)} (e,l) \beta \tilde{d}_t \right|
\]

the \( t \)-th component of a row-vector \( \tilde{d}_t = (d_{\tilde{t}1}, \ldots, d_{\tilde{t}n}) \) for all \( t = 1, m \). Substituting it in (4.44), we obtain

\[
\begin{aligned}
x_{ij} &= \frac{\sum_{l=1}^{m} d_{ij}^{A} \sum_{\alpha \in I_{r_2,m}(j)} \left| B_{k_2+1,j}^{(\alpha)} (e,l) \alpha \right|}{\sum_{\beta \in J_{r,1,n}(i)} \left| (A_{k_1+1,i}^{(\beta)}) \beta \sum_{\alpha \in I_{r_2,m}(j)} \left| (B_{k_2+1,j}^{(\alpha)}) \alpha \right|}
\end{aligned}
\]

Since \( \sum_{l=1}^{m} d_{il}^{A} e_{l} = d_{i}^{A} \), then it follows (4.39).

If we denote by

\[
\begin{aligned}
d_{ij}^{B} &= \sum_{l=1}^{m} \tilde{d}_t \sum_{\alpha \in I_{r_2,m}(j)} \left| B_{k_2+1,j}^{(\alpha)} (e,l) \alpha \right| = \sum_{\alpha \in I_{r_2,m}(j)} \left| B_{k_2+1,j}^{(\alpha)} (\tilde{d}_t) \alpha \right|
\end{aligned}
\]

the \( l \)-th component of a column-vector \( d_{lj}^{B} = (d_{lj}^{B}, \ldots, d_{jn}^{B})^{T} \) for all \( l = 1, n \) and substitute it in (4.44), we obtain

\[
\begin{aligned}
x_{ij} &= \frac{\sum_{l=1}^{n} \sum_{\beta \in J_{r,1,n}(i)} \left| A_{k_1+1,i}^{(\beta)} (e,l) \beta \tilde{d}_{lj}^{B} \right|}{\sum_{\beta \in J_{r,1,n}(i)} \left| (A_{k_1+1,i}^{(\beta)}) \beta \sum_{\alpha \in I_{r_2,m}(j)} \left| (B_{k_2+1,j}^{(\alpha)}) \alpha \right|}
\end{aligned}
\]

Since \( \sum_{l=1}^{n} e_{l} d_{lj}^{B} = d_{lj}^{B} \), then it follows (4.38). ■

### 4.3 Examples

In this subsection, we give an example to illustrate results obtained in the section.

1. Let us consider the matrix equation

\[
AXB = D,
\]

where

\[
A = \begin{pmatrix}
1 & i & i \\
i & -1 & -1 \\
0 & 1 & 0 \\
-1 & 0 & -i
\end{pmatrix}, \quad B = \begin{pmatrix}
i & 1 & -i \\
-1 & 1 & i \\
1 & i & 0 \\
1 & 0 & 1
\end{pmatrix}, \quad D = \begin{pmatrix}
1 & i & 1 \\
i & 0 & 1 \\
i & 1 & 0 \\
0 & 1 & i
\end{pmatrix}.
\]

Since rank \( A = 2 \) and rank \( B = 1 \), then we have the case (ii) of Theorem 4.9. We shall find the least squares solution of (4.45) by (4.11). Then we have
\[ A^*A = \begin{pmatrix} 3 & 2i & 3i \\ -2i & 3 & 2 \\ -3i & 2 & 3 \end{pmatrix}, \quad BB^* = \begin{pmatrix} 3 & -3i \\ 3i & 3 \end{pmatrix}, \quad \tilde{D} = A^*DB^* = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \]

and \[ \sum_{\alpha \in I_{1,2}} |(BB^*)_{\alpha}| = 3 + 3 = 6, \]

\[ \sum_{\beta \in J_{2,3}} \left| (A^*A)_{\beta} \right| = \det \left( \begin{array}{cc} 3 & 2i \\ -2i & 3 \end{array} \right) + \det \left( \begin{array}{cc} 2i & 3 \\ 2 & 3 \end{array} \right) + \det \left( \begin{array}{cc} 3i & 3 \\ -3i & 3 \end{array} \right) = 10. \]

By (4.17), we can get

\[ d^B_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad d^B_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}. \]

Since \[ (A^*A)_{.1} \left( d^B_1 \right), \]

\[ x_{11} = \frac{\sum_{\beta \in J_{2,3}} \left| (A^*A)_{\beta} \right| \left( \sum_{\alpha \in I_{1,2}} |(BB^*)_{\alpha}| \right) \left| (A^*A)_{\beta} \right|}{\sum_{\beta \in J_{2,3}} \left| (A^*A)_{\beta} \right| \sum_{\alpha \in I_{1,2}} |(BB^*)_{\alpha}|} = \frac{\det \left( \begin{array}{cc} 1 & 2i \\ -i & 3 \end{array} \right) + \det \left( \begin{array}{cc} 1 & 3i \\ -i & 3 \end{array} \right)}{60} = \frac{1}{60}. \]

Similarly,

\[ x_{12} = \frac{\det \left( \begin{array}{cc} -i & 2i \\ -1 & 3 \end{array} \right) + \det \left( \begin{array}{cc} -i & 3i \\ -1 & 3 \end{array} \right)}{60} = \frac{-i}{60}, \]

\[ x_{21} = \frac{\det \left( \begin{array}{cc} 3 & 1 \\ -2i & -i \end{array} \right) + \det \left( \begin{array}{cc} 3 & 2 \\ -i & 3 \end{array} \right)}{60} = \frac{-i}{60}, \]

\[ x_{22} = \frac{\det \left( \begin{array}{cc} 3 & -i \\ -2i & -1 \end{array} \right) + \det \left( \begin{array}{cc} -1 & 2 \\ -1 & 3 \end{array} \right)}{60} = \frac{-1}{60}, \]

\[ x_{31} = \frac{\det \left( \begin{array}{cc} 3 & 1 \\ -3i & -i \end{array} \right) + \det \left( \begin{array}{cc} 3 & -i \\ 2 & -i \end{array} \right)}{60} = \frac{-i}{60}, \]

\[ x_{32} = \frac{\det \left( \begin{array}{cc} 3 & -i \\ -3i & -1 \end{array} \right) + \det \left( \begin{array}{cc} 3 & 1 \\ 2 & -1 \end{array} \right)}{60} = \frac{-1}{60}. \]

2. Let us consider the matrix equation (4.45), where

\[ A = \begin{pmatrix} 2 & 0 & 0 \\ -i & i & i \\ -i & -i & -i \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 1 \\ i & -i & i \\ -1 & 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & i & 1 \\ i & 0 & 1 \\ 1 & i & 0 \end{pmatrix}. \]
We shall find the Drazin inverse solution of (4.45) by (4.11). We obtain

\[
A^2 = \begin{pmatrix}
4 & 0 & 0 \\
2 - 2i & 0 & 0 \\
-2 - 2i & 0 & 0 \\
\end{pmatrix},
A^3 = \begin{pmatrix}
8 & 0 & 0 \\
4 - 4i & 0 & 0 \\
-4 - 4i & 0 & 0 \\
\end{pmatrix},
\]

\[
B^2 = \begin{pmatrix}
-i & i & 3 - i \\
1 & -1 & 1 + 3i \\
-3 + i & 3 - i & 3 + i \\
\end{pmatrix}.
\]

Since \(\text{rank } A = 2\) and \(\text{rank } A^2 = \text{rank } A^2 = 1\), then \(k_1 = \text{Ind } A = 2\) and \(r_1 = 1\).

Since \(\text{rank } B = \text{rank } B^2 = 2\), then \(k_2 = \text{Ind } B = 1\) and \(r_2 = 2\). Then we have

\[
\tilde{D} = A^2DB = \begin{pmatrix}
-4 & 4 & 8 \\
-2 + 2i & 2 - 2i & 4 - 4i \\
2 + 2i & -2 - 2i & -4 - 4i \\
\end{pmatrix},
\]

and \(\sum_{\beta \in J, \alpha \in I} \left| (A^3)_{\beta} \right| = 8 + 0 + 0 = 8\),

\[
\sum_{\alpha \in I_2} \left| (B^2)_{\alpha} \right| = \det \begin{pmatrix}
-i & i \\
1 & -1 \\
\end{pmatrix} + \det \begin{pmatrix}
-1 & 1 + 3i \\
3 - i & 3 + i \\
\end{pmatrix} + \det \begin{pmatrix}
-i & 3 - i \\
-3 + i & 3 + i \\
\end{pmatrix} = 0 + (-9 - 9i) + (9 - 9i) = -18i.
\]

By (4.13), we can get

\[
d_{B,1} = \begin{pmatrix} 12 - 12i \\ -12i \\ -12 \end{pmatrix},
d_{B,2} = \begin{pmatrix} -12 + 12i \\ 12i \\ 12 \end{pmatrix},
d_{B,3} = \begin{pmatrix} 8 \\ -12 - 12i \\ -12 + 12i \end{pmatrix}.
\]

Since \(A_{3,1} \left( d_{B,1} \right) = \begin{pmatrix} 12 - 12i & 0 & 0 \\
-12i & 0 & 0 \\
-12 & 0 & 0 \end{pmatrix}\), then finally we obtain

\[
x_{11} = \frac{\sum_{\beta \in J, \alpha \in I} \left| (A^3)_{\beta} \right| (d_{B,1})_{\beta} \left| (B^2)_{\alpha} \right|}{\sum_{\beta \in J, \alpha \in I} \left| (A^3)_{\beta} \right| \sum_{\alpha \in I_2} \left| (B^2)_{\alpha} \right|} = \frac{12 - 12i}{8 \cdot (-18i)} = \frac{1 + i}{12},
\]

Similarly,

\[
x_{12} = \frac{-12 + 12i}{8 \cdot (-18i)} = \frac{-1 - i}{12},
x_{13} = \frac{8}{8 \cdot (-18i)} = \frac{i}{18},
x_{21} = \frac{-12i}{8 \cdot (-18i)} = \frac{1}{12},
x_{22} = \frac{12i}{8 \cdot (-18i)} = \frac{-1}{12},
x_{23} = \frac{-12 - 12i}{8 \cdot (-18i)} = \frac{1 - i}{12}.
\]
\[ x_{31} = \frac{12}{8 \cdot (-18i)} = -\frac{i}{12}, \quad x_{32} = \frac{-12}{8 \cdot (-18i)} = \frac{i}{12}, \quad x_{33} = \frac{-12 + 12i}{8 \cdot (-18i)} = -\frac{1 - i}{12}. \]

Then
\[
X = \begin{pmatrix}
\frac{1+i}{12} & \frac{1-i}{12} & \frac{i}{18} \\
\frac{1}{12} & \frac{1}{12} & \frac{1-i}{12} \\
-\frac{1}{12} & \frac{1}{12} & \frac{-1-i}{12}
\end{pmatrix}
\]
is the Drazin inverse solution of (4.45).

5 An Application of the Determinantal Representations of the Drazin Inverse to Some Differential Matrix Equations

In this section we demonstrate an application of the determinantal representations (2.20) and (2.21) of the Drazin inverse to solutions of the following differential matrix equations, \(X' + AX = B\) and \(X' +XA = B\), where the matrix \(A\) is singular.

Consider the matrix differential equation
\[
X' + AX = B 
\tag{5.1}
\]
where \(A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times n}\) are given, \(X \in \mathbb{C}^{n \times n}\) is unknown. It’s well-known that the general solution of (5.1) is found to be
\[
X(t) = \exp^{-At} \left( \int \exp^{At} dt \right) B
\]
If \(A\) is invertible, then
\[
\int \exp^{At} dt = A^{-1} \exp^{At} + G,
\]
where \(G\) is an arbitrary \(n \times n\) matrix. If \(A\) is singular, then the following theorem gives an answer.

**Theorem 5.1** ([63], Theorem 1) If \(A\) has index \(k\), then
\[
\int \exp^{At} dt = A^D \exp^{At} + (I - AA^D) t \left[ I + \frac{A}{2} t + \frac{A^2}{3!} t^2 + ... + \frac{A^{k-1}}{k!} t^{k-1} \right] + G.
\]

Using Theorem 5.1 and the power series expansion of \(\exp^{-At}\), we get an explicit form for a general solution of (5.1)
\[
X(t) = \left\{ A^D + (I - AA^D) t \left[ I - \frac{A}{2} t + \frac{A^2}{3!} t^2 - ... (-1)^{k-1} \frac{A^{k-1}}{k!} t^{k-1} \right] + G \right\} B.
\]
If we put $G = 0$, then we obtain the following partial solution of \((5.1)\),

$$
X(t) = A^DB + (B - A^DAB)t - \frac{1}{2} (AB - A^DAB^2)t^2 + \ldots
$$

$$
\frac{(-1)^{k-1}}{k!} (A^{k-1}B - A^DAB)^k.
$$

Denote $A'B =: \tilde{B}(i) = (\tilde{b}_{ij}) \in \mathbb{C}^{n \times n}$ for all $l = 1, 2k$.

**Theorem 5.2** The partial solution \((5.2)\), $X(t) = (x_{ij})$, possess the following determinantal representation,

$$
x_{ij} = \frac{\sum_{\beta \in J_{r,n}(1)} \left| \left( A^{k+1}\beta \right) \right|}{\sum_{\beta \in J_{r,n}} \left| \left( A^{k+1}\beta \right) \right|} + \left( b_{ij} - \frac{\sum_{\beta \in J_{r,n}(1)} \left| \left( A^{k+1}\beta \right) \right|}{\sum_{\beta \in J_{r,n}} \left| \left( A^{k+1}\beta \right) \right|} \right) t
$$

$$
-\frac{1}{2} \left( \tilde{b}_{ij}^{(1)} - \frac{\sum_{\beta \in J_{r,n}(1)} \left| \left( A^{k+1}\beta \right) \right|}{\sum_{\beta \in J_{r,n}} \left| \left( A^{k+1}\beta \right) \right|} \right) t^2 + \ldots
$$

$$
\frac{(-1)^{k}}{k!} \left( \tilde{b}_{ij}^{(k)} - \frac{\sum_{\beta \in J_{r,n}(1)} \left| \left( A^{k+1}\beta \right) \right|}{\sum_{\beta \in J_{r,n}} \left| \left( A^{k+1}\beta \right) \right|} \right) t^k
$$

for all $i, j = 1, n$.

**Proof.** Using the determinantal representation of the identity $A'^D A \ (2.27)$, we obtain the following determinantal representation of the matrix $A^D A^m B := (y_{ij})$,

$$
y_{ij} = \sum_{s=1}^{n} p_s \sum_{t=1}^{n} a_{st}^{(m-1)} b_{ij} = \sum_{s=1}^{n} \left| \left( A^{k+1} \left( a_{s, \beta}^{(k+1)} \right) \right) \beta \right| \frac{n}{\sum_{t=1}^{n} a_{st}^{(m-1)} b_{ij}} = \sum_{\beta \in J_{r,n}(1)} \left( A^{k+1} \left( a_{s, \beta}^{(k+1)} \right) \right) \beta \cdot b_{ij} = \sum_{\beta \in J_{r,n}(1)} \left( A^{k+1} \left( \tilde{a}_{st}^{(k+m)} \right) \right) \beta \cdot b_{ij} = \sum_{\beta \in J_{r,n}(1)} \left( A^{k+1} \left( \tilde{a}_{st}^{(k+m)} \right) \right) \beta \cdot b_{ij}
$$

for all $i, j = 1, n$ and $m = 1, k$. From this and the determinantal representation of the Drazin inverse solution (1.33) and the identity (2.27) it follows (5.3). 

**Corollary 5.3** If $\text{Ind} A = 1$, then the partial solution of \((5.7)\),

$$
X(t) = (x_{ij}) = A^g B + (B - A^g A B)t,
$$

possess the following determinantal representation

$$
x_{ij} = \frac{\sum_{\beta \in J_{r,n}(1)} \left| \left( A^{2}\beta \right) \right|}{\sum_{\beta \in J_{r,n}} \left| \left( A^{2}\beta \right) \right|} + \left( b_{ij} - \frac{\sum_{\beta \in J_{r,n}(1)} \left| \left( A^{2}\beta \right) \right|}{\sum_{\beta \in J_{r,n}} \left| \left( A^{2}\beta \right) \right|} \right) t,
$$

for all $i, j = 1, n$. 

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Consider the matrix differential equation

$$X' + XA = B$$  \hspace{1cm} (5.5)

where $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times n}$ are given, $X \in \mathbb{C}^{n \times n}$ is unknown. The general solution of (5.5) is found to be

$$X(t) = B \exp(-At) \left( \int \exp(At) \, dt \right)$$

If $A$ is singular, then an explicit form for a general solution of (5.5) is

$$X(t) = B \left\{ A^D + (I - AA^D)t \left( I - \frac{A}{2}t + \frac{A^2}{3!}t^2 + \ldots (-1)^{k-1} \frac{A^{k-1}}{k!}t^{k-1} \right) + G \right\}.$$  

If we put $G = 0$, then we obtain the following partial solution of (5.5),

$$X(t) = BA^D + (B - BAA^D)t - \frac{1}{2}(BA - BA^2A^D)t^2 + \ldots \frac{(-1)^k}{k!}(BA^{k-1} - BA^kA^D)t^k.$$  \hspace{1cm} (5.6)

Denote $BA^l = \tilde{B}^{(l)} = (\tilde{b}^{(l)}_{ij}) \in \mathbb{C}^{n \times n}$ for all $l = 1, 2k$. Using the determinantal representation of the Drazin inverse solution (2.36), the group inverse (2.25) and the identity (2.26) we evidently obtain the following theorem.

**Theorem 5.4** The partial solution (5.6), $X(t) = (x_{ij})$, possess the following determinantal representation,

$$x_{ij} = \sum_{\alpha \in I_{r,n}(j)} \frac{\left| \left( A^{k+1}_{ij} \left( \tilde{b}^{(k)}_{ij} \right) \right)^{\alpha} \right|}{\alpha!} \cdot t \left( b_{ij} - \sum_{\alpha \in I_{r,n}(j)} \frac{\left| \left( A^{k+1}_{ij} \left( \tilde{b}^{(k+1)}_{ij} \right) \right)^{\alpha} \right|}{\alpha!} \right) + \frac{1}{2} \sum_{\alpha \in I_{r,n}(j)} \frac{\left| \left( A^{k+1}_{ij} \left( \tilde{b}^{(k+2)}_{ij} \right) \right)^{\alpha} \right|}{\alpha!} t^2 + \ldots + \frac{(-1)^k}{k!} \sum_{\alpha \in I_{r,n}(j)} \frac{\left| \left( A^{k+1}_{ij} \left( \tilde{b}^{(2k)}_{ij} \right) \right)^{\alpha} \right|}{\alpha!} t^k$$

for all $i, j = 1, n$.

**Corollary 5.5** If $\text{Ind}A = 1$, then the partial solution of (5.5),

$$X(t) = (x_{ij}) = BA^g + (B - BAA^g)t,$$

possess the following determinantal representation

$$x_{ij} = \sum_{\alpha \in I_{r,n}(j)} \frac{\left| \left( A^{g}_{ij} \left( \tilde{b}^{(1)}_{ij} \right) \right)^{\alpha} \right|}{\alpha!} \cdot t \left( b_{ij} - \sum_{\alpha \in I_{r,n}(j)} \frac{\left| \left( A^{g}_{ij} \left( \tilde{b}^{(2)}_{ij} \right) \right)^{\alpha} \right|}{\alpha!} \right)$$

for all $i, j = 1, n$.  

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5.1 Example

1. Let us consider the differential matrix equation

\[ X' + AX = B, \quad (5.7) \]

where

\[ A = \begin{pmatrix} 1 & -1 & 1 \\ i & -i & i \\ -1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & i & 1 \\ i & 0 & 1 \\ 1 & i & 0 \end{pmatrix}. \]

Since \( \text{rank } A = \text{rank } A^2 = 2 \), then \( k = \text{Ind } A = 1 \) and \( r = 2 \). The matrix \( A \) is the group inverse. We shall find the partial solution of (5.7) by (5.4). We have

\[ A^2 = \begin{pmatrix} -i & i & 3 - i \\ 1 & -1 & 1 + 3i \\ -3 + i & 3 - i & 3 + i \end{pmatrix}, \quad \tilde{B}^{(1)} = AB = \begin{pmatrix} 2 - i & 2i & 0 \\ 1 + 2i & -2 & 0 \\ 1 + i & i & 0 \end{pmatrix}, \]

\[ \tilde{B}^{(2)} = A^2B = \begin{pmatrix} 2 - 2i & 2 + 3i & 0 \\ 2 + 2i & -3 + 2i & 0 \\ 1 + 5i & -2 & 0 \end{pmatrix}. \]

and

\[ \sum_{\alpha \in J_{2,3}} \left| (A^2)^\alpha \right| = 0 + (-9 - 9i) + (9 - 9i) = -18i. \]

Since \( (A^2)^1 \left( \tilde{b}^{(1)}_1 \right) = \begin{pmatrix} 2 - i & i & 3 - i \\ 1 + 2i & -1 & 1 + 3i \\ 1 + i & 3 - i & 3 + i \end{pmatrix} \) and

\[ (A^2)^1 \left( \tilde{b}^{(2)}_1 \right) = \begin{pmatrix} 2 - 2i & i & 3 - i \\ 2 + 2i & -1 & 1 + 3i \\ 1 + 5i & 3 - i & 3 + i \end{pmatrix}, \]

then finally we obtain

\[ x_{11} = \frac{\sum_{\alpha_1 \in J_{2,3}(1)} \left| (A^2)^1(\tilde{b}^{(1)}_1) \right|^\beta}{\sum_{\alpha \in J_{2,3}} |(A^2)^\beta|} + \left( b_{11} - \frac{\sum_{\alpha_1 \in J_{2,3}(1)} \left| (A^2)^1(\tilde{b}^{(2)}_1) \right|^\beta}{\sum_{\alpha \in J_{2,3}} |(A^2)^\beta|} \right) t = \frac{3 - 3i}{-18i} + \left( 1 - \frac{18i}{18t} \right) t = \frac{1 + i}{6}. \]

Similarly,

\[ x_{12} = \frac{-3 + 3i}{-18i} + \left( i - 9 + 9i \right) t = \frac{-1 - i}{6} + \frac{1 + i}{2} t, \quad x_{13} = 0 + (1 - 0) t = t, \]

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\[ x_{21} = \frac{3+3i}{-18i} + \left( i - \frac{-18}{-18i} \right) t = \frac{-1+i}{6}, \]
\[ x_{22} = \frac{-3-3i}{-18i} + \left( 0 - \frac{-9+9i}{-18i} \right) t = \frac{1-i}{6} + \frac{1+i}{2} t, \]
\[ x_{23} = 0 + (1-0) t = t, \]
\[ x_{31} = \frac{-12i}{-18i} + \left( 1 - \frac{-18i}{-18i} \right) t = \frac{2}{3}, \]
\[ x_{32} = \frac{9+3i}{-18i} + \left( i - \frac{-18}{-18i} \right) t = \frac{-1+3i}{6}, x_{33} = 0 + (0-0) t = 0. \]

Then
\[ X = \frac{1}{6} \begin{pmatrix} 1+i & -1-i+(3+3i)t & t \\ -1+i & 1-i+(3+3i)t & t \\ 4 & -1+3i & 0 \end{pmatrix} \]
is the partial solution of (5.7).

6 Conclusions

From student years it is well known that Cramer’s rule may only be used when the system is square and the coefficient matrix is invertible. In this paper we are considered various cases of Cramer’s rule for generalized inverse solutions of systems of linear equations and matrix equations when the coefficient matrix is not square or non-invertible. The results of this paper have practical and theoretical importance because they give an explicit representation of an individual component of solutions independently of all other components. Also the results of this paper can be extended to matrices over rings (and now this is done in the quaternion skew field), to polynomial matrices, etc.

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