A tuning algorithm for a sliding mode controller of buildings with ATMD

Antonio Concha\textsuperscript{a}, Suresh Thenozhi\textsuperscript{b}, Ramón Octavio Jiménez Betancourt\textsuperscript{c}, S. K. Gadi\textsuperscript{d,∗}

\textsuperscript{a}Facultad de Ingeniería Mecánica y Eléctrica, Universidad de Colima, Coquimatlán, Colima 28400, México
\textsuperscript{b}Facultad de Ingeniería, Universidad Autónoma de Querétaro, Santiago de Querétaro, Querétaro 76010, México
\textsuperscript{c}Facultad de Ingeniería Electromecánica, Universidad de Colima, Manzanillo, Colima 28860, México
\textsuperscript{d}Facultad de Ingeniería Mecánica y Eléctrica, Universidad Autónoma de Coahuila, Torreón, Coahuila 27276, México

Abstract

This paper proposes an automatic tuning algorithm for a sliding mode controller (SMC) based on the Ackermann’s formula, that attenuates the structural vibrations of a seismically excited building equipped with an Active Tuned Mass Damper (ATMD) mounted on its top floor. The switching gain and sliding surface of the SMC are designed through the proposed tuning algorithm to suppress the structural vibrations by minimizing either the top floor displacement or the control force applied to the ATMD. Moreover, the tuning algorithm selects the SMC parameters to guarantee the following closed-loop characteristics: 1) the transient responses of the structure and the ATMD are sufficiently fast and damped; and 2) the control force, as well as the displacements and velocities of the building and ATMD are within acceptable limits under the frequency band of the earthquake excitation. The proposed SMC shows robustness against the unmodeled dynamics such as the friction of the damper. Experimental results on a reduced scale structure permits demonstrating the efficiency of the tuning algorithm for the SMC, which is compared with the traditional Linear Quadratic Regulator (LQR).

1. Introduction

Buildings can be subject to external forces such as earthquakes or winds, which can damage them. To protect the buildings against these natural events, they can be coupled to Mass Dampers (MD) to obtain a desired vibration attenuation. The passive version of the MD, known as Tuned Mass Dampers (TMD), are composed of a moving mass attached to a spring and a viscous damper. If the damper also has Coulomb-type dissipation force, then it can be denoted as FTMD \cite{1}. The active version, termed as Active Mass Damper (AMD), are constructed by coupling an actuator to the moving mass. By adding an actuator to the TMD results in a hybrid device, denoted as ATMD \cite{2}.

Active vibration control techniques using AMD or ATMD have been of great interest in recent years, due to their ability to provide higher vibration attenuation than the TMD. Linear controllers are by far the most widely applied techniques for active vibration control. They include the classical proportional-derivative (PD) or the proportional-integral-derivative (PID) controllers \cite{3,4}; acceleration feedback regulators \cite{5,6}; state-feedback with variable gain \cite{7}; full state feedback control using displacement, velocity, and acceleration of the structure \cite{8,11}; LQR using the knowledge of the seismic excitation \cite{12,13} or without it \cite{14}; Linear Quadratic Gaussian (LQG) \cite{15}; feedforward and feedback optimal tracking controller (FFOTC) \cite{16}; and robust controllers like $H_2$, $H_\infty$ \cite{17,18} or $H_2/H_\infty$ \cite{19}. On the other hand, intelligent techniques have also been applied for active control of structures. Yang et al. \cite{20} designed a neural-network for system identification and vibration control of a structure with AMD. Thenozhi and Yu \cite{21} designed Fuzzy

∗Corresponding author

Email addresses:aconcha@ucol.mx (Antonio Concha), suresh@uaq.mx (Suresh Thenozhi), rjimenez@ucol.mx (Ramón Octavio Jiménez Betancourt), Research@SKGadi.com (S. K. Gadi)
PD/PID controllers for structures with friction uncertainty. Genetic algorithms were applied in [22, 23] for optimization of structural active control laws.

An alternative to the aforementioned linear and intelligent control techniques is the SMC, that is widely accepted for structural control due to its robustness against disturbances and unmodeled dynamics. Yang et al. [24] presented a continuous SMC based on a saturation function for seismically excited structures, where the authors showed with numerical simulations that this controller avoids the undesirable chattering effect. Adhikari et al. [25] designed a SMC based on the theory of compensators to prevent a large response in the building due to its interaction with the ATMD. On the other hand, Wang et al. [26] developed a fuzzy SMC that uses a Mamdani inference method to determine the behavior of the closed-loop system in the sliding mode. Moreover, Li et. al [27] proposed a model reference SMC for a building with an ATMD at its top floor, where the reference model is the structure coupled to the TMD. Soleymani et al. [28] designed a SMC for a building modeled through a second-order reduced model, and they consider time delays in the control force applied to an AMD coupled to the structure. In addition, Mamat et al. [29] presented an adaptive nonsingular terminal SMC employed in a three-story building equipped with an ATMD, that was simulated in Matlab/Simulink.

This article proposes an algorithm to automatically tune the sliding variable and switching gain of a SMC based on the Ackermann’s formula, that is used for vibration control of a building containing an ATMD on its top floor. Considering that the first mode of the building response is dominant during the earthquake, the structure equipped with the ATMD is modeled as a fourth-order system. It is shown that the closed-loop system in the sliding mode is reduced to a third-order system, whose displacement of the dominant mode of the structure and that of the ATMD damper, as well as its control force behaves as the outputs of a dominant second-order filters, whose input is the seismic excitation. These filters have the advantage that are easier to design than the four-order filters presented in [5]. The aim of the proposed tuning algorithm is to design these dominant second-order filters to:

- minimize the displacement of the top floor of the structure as much as possible, or to minimize the control force applied to the ATMD while offering a great attenuation of this displacement.
- produce sufficiently fast and damped transient responses of the ATMD and building.
- guarantee that the Root Mean Square (RMS) values of the ATMD control force, displacements and velocities of the building and damper are within acceptable limits in the frequency band of the earthquake excitation.

Unlike the SMC techniques presented in [24–29], this article proves that the seismic excitation signal is not a coupled disturbance, whose effect on the controller and on the movements of the ATMD and building is analyzed. Moreover, in contrast to [25], that uses a compensator to filter out undesirable ATMD responses, the present work uses the dominant second-order filters to remove these undesirable responses, which are automatically designed using the proposed tuning algorithm. Thus, large responses in the building due to the movements of the ATMD are avoided.

The remaining of this paper is organized as follows. Section 2 introduces the mathematical model of a building equipped with an ATMD. The SMC designed with the Ackermann’s formula is presented in Section 3. The desired transient and frequency responses of the closed-loop structure are described in Section 4. The proposed algorithm for tuning the sliding variable and switching gain of the SMC is explained in Section 5. Section 6 demonstrates the effectiveness of the proposed tuning algorithm in both simulations and experiments. Finally, Section 7 gives the conclusions of this article.
2. Mathematical model of a building with an ATMD

Consider a $N$-story building that has an ATMD installed at its top floor, as shown in Figure 1. The behavior of this system is described by means of the following differential equations [30, 31]:

$$\begin{align*}
M(\ddot{x}(t) + L\ddot{\bar{x}}_g(t)) + C\dot{x}(t) + Kx(t) &= -\Gamma F(t) \\
m_d(\ddot{x}_a(t) + \ddot{\bar{x}}_g(t) + \ddot{x}_d(t)) &= F(t) \\
F(t) &= u(t) - k_d x_d(t) - c_d \dot{x}_d(t) - f(\dot{x}_d(t))
\end{align*}$$

where $M, C, K \in \mathbb{R}^{N \times N}$ are the mass, stiffness, and damping matrices, respectively. Term $M$ is a diagonal matrix composed by the floor masses $m_i, i = 1, 2, \ldots, N$. Moreover, $C$ and $K$ are tridiagonal matrices that contain the damping $c_i$ and stiffness $k_i$ coefficients between the $i$th and the $(i - 1)$th floors, as depicted in Figure 1. Furthermore, the term $\ddot{\bar{x}}_g$ is the earthquake acceleration, vector $x$ is given by $x = [x_1, x_2, \ldots, x_N]^T$, where $x_i$ represents the displacement of the $i$th floor relative to the ground. Variable $x_d$ is the relative displacement of the ATMD with respect to the top story, and the terms $m_d, k_d, c_d$, and $f(\dot{x}_d(t))$ are the mass, stiffness, damping, and non-linear friction of the ATMD, respectively. Variable $F$ is the net force acting upon the ATMD. In addition, signal $u(t)$ is the control force applied to the damper, term $I \in \mathbb{R}^{N \times 1}$ is an unity vector, and $\Gamma \in \mathbb{R}^{N \times 1}$ defines the localization of the ATMD and is represented as:

$$\Gamma = [0, 0, \ldots, 0, 1]^T$$

Since the first mode of vibration is dominant during an earthquake, the building model [1] can be approximated as [3]:

$$\begin{align*}
m_0\ddot{x}_0(t) + c_0\dot{x}_0(t) + k_0x_0(t) &= -\beta_0m_0\ddot{\bar{x}}_g(t) - F(t) \\
m_d(\ddot{x}_0(t) + \ddot{\bar{x}}_g(t) + \ddot{x}_d(t)) &= F(t)
\end{align*}$$

Signal $x_0$ represents the displacement of the dominant mode, which is given by

$$x_0(t) = \frac{\phi_0^T Mx(t)}{\phi_0^T M \phi_0}$$

where $\phi_0 \in \mathbb{R}^{N \times 1}$ is the the first mode, that is scaled to satisfy

$$\phi_0^T \Gamma = 1$$
This equality, in turn, produces the approximation

\[ x_N(t) \approx x_0(t) \]  
(8)

Parameters \( m_0, c_0, k_0, \) and \( \beta_0 \) are the mass, damping, stiffness, and participation factor of the dominant mode, respectively. They are defined as \[32\] :

\[ m_0 = \phi_0^T \mathbf{M}_0 \phi_0, \quad c_0 = \phi_0^T \mathbf{C}_0 \phi_0, \quad k_0 = \phi_0^T \mathbf{K}_0 \phi_0, \quad \beta_0 = \frac{\phi_0^T \mathbf{M}_l \phi_0}{\phi_0^T \mathbf{M}_0 \phi_0} \]  
(9)

Moreover, the natural frequency of the first mode is given by

\[ \omega_0 = \sqrt{\frac{k_0}{m_0}} \]  
(10)

Using the approximation (8) and substituting \( F(t) \) of (3) into (5) yields:

\[ \ddot{x}_N = -\frac{c_0}{m_0} \dot{x}_N - \frac{k_0}{m_0} x_N - \beta_0 \ddot{x}_g + \frac{c_d}{m_0} \dot{x}_d + \frac{k_d}{m_0} x_d - \frac{1}{m_0} u \]
\[ \ddot{x}_d = \left( \frac{m_0 + m_d}{m_0 m_d} \right) \left( u - c_d \dot{x}_d - f(\dot{x}_d) - k_d x_d \right) + \frac{k_0}{m_0} x_N + \frac{c_0}{m_0} \dot{x}_N + \ddot{x}_g (\beta_0 - 1) \]  
(11)

where argument \( t \) has been omitted in the time dependent signals.

Defining the following state variables

\[ z_1 = x_d, \quad z_2 = x_N, \quad z_3 = \dot{x}_d, \quad z_4 = \dot{x}_N \]  
(12)

allows rewriting system (11) in the following state-space representation:

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\dot{z}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
-k_d \left( \frac{m_0 + m_d}{m_0 m_d} \right) & 0 & 0 & 1 \\
-\frac{c_d (m_0 + m_d)}{m_0} & k_0 & -\frac{c_d (m_0 + m_d)}{m_0} & 0 \\
\frac{m_0 m_d}{k_0} & \frac{m_0 m_d}{k_0} & \frac{m_0 m_d}{m_0} & m_0
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0 \\
\frac{(m_0 + m_d)}{m_0 m_d} \\
1
\end{bmatrix} \begin{bmatrix}
u \\
f(z_3)
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0 \\
\beta_0 - 1 \\
-\beta_0
\end{bmatrix} \ddot{x}_g
\]

(13)

3. Sliding mode control of the structure

This section presents a SMC design based on the Ackermann’s formula for vibration attenuation of seismically excited buildings. For this purpose, let us first define a full-state feedback controller \( u_a \) for system (13), such that the closed-loop eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \) and \( \lambda_4 \) are placed in a desired location in the \( s \)-plane. The control law \( u_a \) is given by

\[ u_a = -k^T z \]  
(14)
According to the Ackermann’s formula \(33\), the feedback gain vector \(k_T \in \mathbb{R}^{1 \times 4}\) can be computed as
\[
k_T = e^T P(A)
\] (15)
with
\[
e^T = [0, 0, 0, 1] | B, AB, A^2B, A^3B |^{-1} \in \mathbb{R}^{1 \times 4}
\] (16)
\[
P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)
\] (17)

In order to design the SMC for the system \(13\), let us consider the following assumptions:

**Assumption 1.** The eigenvalue \(\lambda_4\) in \(17\) is real and negative.

**Assumption 2.** Bounds \(\delta\) and \(\varpi\), corresponding to the earthquake acceleration \(\ddot{x}_g\) and the non-linear friction \(f(z_3)\), respectively, are known and satisfy
\[
|\ddot{x}_g| \leq \delta
\] (18)
\[
|f(z_3)| \leq \varpi
\] (19)
where the bound \(\delta\) can be determined from the historical records of the ground acceleration in the region where the building is located, and the bound \(\varpi\) can be determined using friction estimation techniques.

Next theorem presents the SMC based on the Ackermann’s formula \(15\) and analyses the behavior of the resulting closed-loop system in the sliding mode at the plane \(\sigma = \eta^T z = 0\), where \(\eta^T \in \mathbb{R}^{1 \times 4}\) is a constant vector that will be automatically calculated through the proposed methodology outlined in Section 5.

**Theorem 3.1.** Let us consider the building structure in \(13\), that is equipped with an ATMD, whose control force \(u\) is provided by the following SMC
\[
u = -M_0 \text{sign}(\sigma)
\] (20)
where \(\sigma = \eta^T z\) and \(M_0 > 0\) are called sliding variable and switching gain, respectively. The vector \(\eta^T\) is given by
\[
\eta^T = [\eta_1, \eta_2, \eta_3, \eta_4] = e^T P_1(A), \quad P_1(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)
\] (21)
where \(\lambda_1, \lambda_2,\) and \(\lambda_3\) are the desired closed-loop eigenvalues; moreover, parameter \(M_0\) satisfies
\[
M_0 > \varpi + h_0
\] (22)
where \(h_0\) is a constant such that
\[
|u_a + \alpha_1 \ddot{x}_g| \leq h_0
\] (23)
with \(u_a\) given in \(14\) and \(\alpha_1\) defined as
\[
\alpha_1 = \beta_0(\eta_4 - \eta_3) + \eta_3
\] (24)
Then, the trajectories of closed-loop system reach the plane \(\sigma = \eta^T z = 0\) in a finite time \(t_\sigma\), and they are confined in this plane for \(t \geq t_\sigma\), where \(t_\sigma \leq \sigma(0)/(M_0 - [\varpi + h_0])\). Furthermore, when this plane is reached, the fourth-order dynamic system \(13\) is reduced to the following third-order system
\[
\dot{z}^* = A_1 z^* + B_1 \ddot{x}_g
\] (25)
where \(A_1 \in \mathbb{R}^{3 \times 3}\) is a matrix containing the eigenvalues \(\lambda_1, \lambda_2,\) and \(\lambda_3\); vector \(z^* \in \mathbb{R}^{3 \times 1}\) is given by
\[
z^* = [z_1, z_2, z_3]^T
\] (26)
and
\[
B_1 = \begin{pmatrix} 0 \\ 0 \\ \alpha_2 \end{pmatrix}
\] with \(\alpha_2 = (\beta_0 - 1) + \frac{\alpha_1(m_0 + m_d)}{m_0 m_d}
\] (27)
Proof. First, note that vector $\eta^T$ in (21) satisfies the following equalities (34):

$$\eta^T B = 1, \quad \eta^T A^* = \lambda_4 \eta^T$$

(28)

where $A^* = A - B k^T$.

Adding and subtracting the term $B u_a$ to (13) yields

$$\dot{z} = (A - B k^T) z + B [u - u_a - f(z_3)] + D \ddot{x}_g$$

(29)

The system (29) is transformed into a new set of equations using the following state transformation

$$w = \begin{bmatrix} z^* \sigma \end{bmatrix} = \begin{bmatrix} \eta^T \\ I_{3 \times 3} \end{bmatrix} \begin{bmatrix} T \sigma \end{bmatrix} = Tz$$

(30)

where $T$ is an invertible matrix. The dynamics of the transformed system $\dot{w} = Tz$ is given by

$$\dot{w} = T \left( (A - B k^T) z + B [u - u_a - f(z_3)] + D \ddot{x}_g \right) = T (A - B k^T) T^{-1} w + TB [u - u_a - f(z_3)] + TD \ddot{x}_g$$

(31)

Using the equalities in (25) allows deducing the following structure of matrices $T (A - B k^T) T^{-1}$ and $TB$ in (31)

$$T (A - B k^T) T^{-1} = \begin{bmatrix} A_1 & a^* \\ 0_{1 \times 3} & \lambda_4 \end{bmatrix}, \quad TB = \begin{bmatrix} b^* \\ 1 \end{bmatrix} \text{ with } b^* = \begin{pmatrix} 0 \\ 0 \\ \frac{m_0 + m_d}{m_0 m_d} \end{pmatrix}$$

(32)

where $a^* \in \mathbb{R}^{3 \times 1}$, and $A_1$ is a matrix whose eigenvalues are equal to the closed-loop eigenvalues $\lambda_1$, $\lambda_2$ and $\lambda_3$. Finally, vector $TD$ in (31) is given by

$$TD = \begin{bmatrix} d_1 \\ -\alpha_1 \end{bmatrix} \text{ with } d_1 = \begin{pmatrix} 0 \\ 0 \\ \beta_0 - 1 \end{pmatrix}$$

(33)

Substituting equations (32) and (33) into (31) produces

$$\dot{z}^* = A_1 z^* + a^* \sigma + b^* [u - u_a - f(z_3)] + d_1 \ddot{x}_g$$

(34)

$$\dot{\sigma} = \lambda_4 \sigma + u - u_a - f(z_3) - \alpha_1 \ddot{x}_g$$

(35)

The control law $u$ is designed so that the trajectories of the structural system (13) converge to the plane $\sigma = 0$ in finite time, which is ensured when

$$\frac{1}{2} \frac{d}{dt} \sigma^2 = \sigma \dot{\sigma} < 0.$$ The product $\sigma \dot{\sigma}$ is given by

$$\sigma \dot{\sigma} = \sigma [\lambda_4 \sigma + u - u_a - f(z_3) - \alpha_1 \ddot{x}_g] = \lambda_4 \sigma^2 + \sigma [u - u_a - f(z_3) - \alpha_1 \ddot{x}_g]$$

(36)

Since $\lambda_4 < 0$, the last expression satisfies

$$\sigma \dot{\sigma} \leq \sigma [u - u_a - f(z_3) - \alpha_1 \ddot{x}_g]$$

(37)

Substituting the control law $u = -M_0 \text{sign}(\sigma)$ into (37), and using the inequalities (10), (22) and (23) yields

$$\sigma \dot{\sigma} \leq \sigma [-M_0 \text{sign}(\sigma) - u_a - f(z_3) - \alpha_1 \ddot{x}_g] \leq |\sigma| [-M_0 + \omega + h_0] \leq -|\sigma||M_0 - (\omega + h_0)| < 0$$

(38)
Therefore, the trajectories of system $\text{(13)}$ reach the surface $\sigma = \eta^T z = 0$ in a finite time $t_\sigma$, and remain there for $t \geq t_\sigma$, where $t_\sigma$ is computed by integrating $\text{(35)}$ and its value is given by

$$t_\sigma \leq \frac{\sigma(0)}{M_0 - (w + h_0)}$$

Finally, in order to determine the behavior of the closed-loop system in the surface $\sigma = 0$, the next solution of $\dot{\sigma} = 0$ with respect to $u$

$$u = u_n + f(z_3) + \alpha_1 \tilde{x}_g$$

is substituted into $\text{(41)}$ to produce the sliding motion equation $\text{(25)}$.

Remark Since the SMC $\text{(20)}$ and the dynamics of the closed-loop system $\text{(25)}$ in the sliding mode do not depend of $\lambda_4$, this parameter can take any negative value, and it is used only for the stability analysis of the SMC.

Remark Note that the building and the ATMD are at rest before an earthquake, therefore, the initial condition $z_i(0)$ of variables $z_i$, $i = 1, 2, 3, 4$ is zero, i.e., $z(0) = 0$, which implies that $\sigma(0) = \eta^T z(0) = 0$, and as a consequence $t_\sigma = 0$.

Remark Parameter $h_0$ in $\text{(23)}$ is required to design the SMC. In section 5.2, a methodology is proposed to compute this parameter using the frequency response of the closed-loop system in the sliding mode, and the knowledge of the bound $\delta$ in $\text{(18)}$ of the earthquake acceleration $\tilde{x}_g$.

4. Analysis of the closed-loop system at the sliding mode

The closed-loop system dynamics in the sliding mode is described in the Laplace domain as

$$(sI_{3 \times 3} - A_1)Z^*(s) = B_1L[\tilde{x}_g(t)]$$

where $L$ is the Laplace transform operator and $Z^*(s) = [Z_1(s), Z_2(s), Z_4(s)]^T$.

Using $\text{(41)}$, the transfer function $Z^*(s)/L[\tilde{x}_g(t)]$ can be expressed as

$$\frac{Z^*(s)}{L[\tilde{x}_g(t)]} = (sI_{3 \times 3} - A_1)^{-1}B_1$$

The characteristic polynomial $P_1(s)$ in $\text{(21)}$ corresponding to $\text{(12)}$ can be rewritten as

$$P_1(s) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3) = (s^2 + 2\zeta_\omega_n s + \omega_n^2)(s - \lambda_3)$$

where $\zeta$ and $\omega_n$ are positive constants, which are called damping ratio and undamped natural frequency, respectively.

Substituting $P_1(s)$ into $\text{(12)}$ leads to the following transfer functions $Z_1(s)/L[\tilde{x}_g(t)]$, $Z_2(s)/L[\tilde{x}_g(t)]$ and $Z_3(s)/L[\tilde{x}_g(t)]$:

$$G_1(s) = \frac{Z_1(s)}{L[\tilde{x}_g(t)]} = C_{21}(sI_{3 \times 3} - A_1)^{-1}B_1 = \frac{\alpha_2(s - \psi_1)}{(s - \lambda_3)(s^2 + 2\zeta_\omega_n s + \omega_n^2)}$$

$$G_2(s) = \frac{Z_2(s)}{L[\tilde{x}_g(t)]} = C_{22}(sI_{3 \times 3} - A_1)^{-1}B_1 = \frac{-\alpha_2 \eta_4 (s - \psi_2)}{\eta_4(s - \lambda_3)(s^2 + 2\zeta_\omega_n s + \omega_n^2)}$$

$$G_3(s) = \frac{Z_3(s)}{L[\tilde{x}_g(t)]} = C_{23}(sI_{3 \times 3} - A_1)^{-1}B_1 = \frac{\alpha_2(s - \psi_1)}{(s - \lambda_3)(s^2 + 2\zeta_\omega_n s + \omega_n^2)}$$

where

$$\psi_1 = \frac{\eta_4}{\eta_4}, \quad \psi_2 = \frac{\eta_4}{\eta_3}$$

$$\eta_3 = \frac{-\alpha_2 \eta_4 (s - \psi_2)}{\eta_4(s - \lambda_3)(s^2 + 2\zeta_\omega_n s + \omega_n^2)}$$
\[ C_{z_1} = [1, 0, 0], \quad C_{z_2} = [0, 1, 0], \quad C_{z_3} = [0, 0, 1] \] (48)

In order to determine the transfer function \( U(s)/L[\ddot{x}_g(t)] \), consider that non-linear function \( f(z_3) \) in (40) is zero. Then, the control signal \( u \) in the sliding mode is given by:

\[
u = u_a + \alpha_1 \ddot{x}_g = -k^T z + \alpha_1 \ddot{x}_g = -e^T P(A)z + \alpha_1 \ddot{x}_g
\]
\[
= -e^T P_1(A)[A - I_{4 \times 4}]z + \alpha_1 \ddot{x}_g
\]
\[
= -\eta^T[A - I_{4 \times 4}]z + \alpha_1 \ddot{x}_g
\] (49)

Using the state transformation equation in (39), the vector \( z \) in (49) can be written as follows

\[
z = T^{-1} w = T^{-1} \begin{bmatrix} z^* \\ \sigma \end{bmatrix}
\] (50)

Since \( \sigma = 0 \) in the sliding mode, the above expression becomes

\[
z = T^{-1} \begin{bmatrix} z^* \\ 0 \end{bmatrix}
\] (51)

Substituting (51) into (49) yields

\[
u = -\eta^T[A - I_{4 \times 4}]T^{-1} \begin{bmatrix} z^* \\ 0 \end{bmatrix} + \alpha_1 \ddot{x}_g
\] (52)

Let

\[
\nu = -\eta^T[A - I_{4 \times 4}]T^{-1} = -\eta^T AT^{-1} + \lambda_4 \eta^T T^{-1} = -\eta^T AT^{-1} + [0, 0, 0, \lambda_4] = [\nu_1, \nu_2]
\] (53)

where \( \nu \in \mathbb{R}^{1 \times 4}, \nu_1 \in \mathbb{R}^{1 \times 3}, \) and \( \nu_2 \in \mathbb{R} \). Using this definition allows rewriting (52) as

\[
u = [\nu_1, \nu_2] \begin{bmatrix} z^* \\ 0 \end{bmatrix} + \alpha_1 \ddot{x}_g = \nu_1 z^* + \alpha_1 \ddot{x}_g
\] (54)

Note that \( \nu_1 \) is a constant vector and does not depend on \( \lambda_4 \). Applying the Laplace transform to (54) and using (42) produces:

\[
U(s) = \nu_1 (sI_{3 \times 3} - A_1)^{-1} B_1 L[\ddot{x}_g(t)] + \alpha_1 L[\ddot{x}_g(t)]
\] (55)

which can be rewritten as the following transfer function \( U(s)/L[\ddot{x}_g(t)] \)

\[
G_d(s) = \frac{U(s)}{L[\ddot{x}_g(t)\]} = \nu_1 (sI_{3 \times 3} - A_1)^{-1} B_1 + \alpha_1
\] (56)

4.1. Transient response for \( z_1(t) \) and \( z_2(t) \)

This section analyzes the transient responses of the damper and the top floor displacements \( z_1(t) \) and \( z_2(t) \), respectively. This analysis will permit tuning the SMC to produce sufficiently fast and damped transient responses of \( z_1(t) \) and \( z_2(t) \) under sudden changes in the input excitation \( \ddot{x}_g \). For this purpose, assume that this input is a step function, and consider the following assumption:

Assumption 3. The damping ratio \( \zeta \) of the characteristic polynomial \( P_1(s) \) in (43) satisfies \( \zeta < 1 \), which implies that \( P_1(s) \) has a real pole \( \lambda_3 \) and two complex conjugate poles \( \lambda_1 \) and \( \lambda_2 \), i.e.,

\[
P_1(s) = (s^2 + 2\zeta \omega_n s + \omega_n^2)(s - \lambda_3) = (s + \zeta \omega_n + j\omega_d)(s + \zeta \omega_n - j\omega_d)(s - \lambda_3)
\] (57)

\[
\lambda_1 = -\zeta \omega_n - j\omega_d, \quad \lambda_2 = -\zeta \omega_n + j\omega_d
\] (58)

where \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \) is the damped natural frequency.

8
Taking into account Assumption 3, the transient response of \( z_i(t), i = 1, 2 \) can be considered as the response of a standard second-order system, that is affected by the additional pole \( \lambda_3 \) and zero \( \psi_3 \) of the transfer function \( G_i(s) \). Thus, the response of \( z_i(t), i = 1, 2 \) can be specified by means of its rise time \( T_r \) and maximum overshoot \( M_p \). The response \( z_i(t) \) is affected by the pole \( \lambda_3 \) and zero \( \psi_3 \) \(^{35}\), as described below:

- **Effect of the additional zero**: The zero \( \psi_3 \) of \( G_i(s) \) can either be positive or negative, and its effect on \( z_i(t) \) is the following:
  1. If the zero \( \psi_3 \) is negative, then it has the effect of decreasing \( T_r \) and increasing \( M_p \) of the step response \( z_i(t) \), as shown in Figure 2 (a). The increase of \( M_p \) depends on the relation \( \gamma_i = |\psi_3|/(\zeta \omega_n) \). The smaller the relation \( \gamma_i \), the larger the increment of \( M_p \).
  2. If the zero \( \psi_3 \) is positive, then the transfer function \( G_i(s) \) is non-minimum phase. In this case, the zero \( \psi_3 \) slightly increases \( T_r \) and \( M_p \) of the step response \( z_i(t) \), but it produces an initial drop that appears at the beginning of the response, see Figure 2 (a). The peak of this drop depends on the relation \( \gamma_i = |\psi_3|/(\zeta \omega_n) \). The smaller the relation \( \gamma_i \), the larger this peak will be.

In conclusion, the zero \( \psi_3 \), either positive or negative, has little effect on the transient response of \( z_i(t) \) for \( \gamma_i \geq 3 \), but as \( \gamma_i \) decreases below 3, it has an increasing effect, especially when \( \gamma_i < 1 \).

- **Effect of the additional pole**: The additional pole \( \lambda_3 < 0 \) of \( G_i(s) \) tends to increase \( T_r \) and to decrease \( M_p \) of the step response \( z_i(t) \), see Figure 2 (b). The percentage of \( M_p \) is a function of \( \xi = |\lambda_3|/(\zeta \omega_n) \); the larger this relation, the smaller the percentage of \( M_p \). The pole \( \lambda_3 \) has little effect for \( \xi \geq 3 \), otherwise has an increasing effect.

![Figure 2: Effect of the zero \( \psi_3 \).](image1.png)

![Figure 2: Effect of the pole \( \lambda_3 \).](image2.png)

From the above analysis, it is clear that for certain values of the additional pole and zero, they have less effect on the transient response. In that case, the parameters \( \zeta \) and \( \omega_n \) of the dominant underdamped second-order system are tuned such that the transient responses of \( z_1(t) \) and \( z_2(t) \) are sufficiently fast and damped, and they do not have excessive overshoot. To this end, the tuning of the parameters \( \zeta, \omega_n, \lambda_3, \psi_1 \) and \( \psi_3 \) is carried out as follows.

- The damping ratio \( \zeta \) of \( P_1(s) \) will be tuned between the next interval
  \[ \zeta_l < \zeta < \zeta_u \] \( (59) \)
  where the recommended values for the limits of the interval are \( \zeta_l = 0.5 \) and \( \zeta_u = 0.9 \).
The natural frequency $\omega_n$ will be tuned based on the dominant mode frequency of the structure $\omega_0$. If $\omega_n$ is near to $\omega_0$, it could provoke abrupt movements of the ATMD that could excite the structure instead of protect it. On the other hand, if $\omega_n$ is smaller than $0.5\omega_0$ could result in a slower transient response of the ATMD movements, hence an insignificant vibration attenuation. To avoid these problems, $\omega_n$ will be tuned between the following limits

$$\omega_{nl} < \omega_n < \omega_{nu}$$

where $\omega_{nl} = 0.5\omega_0$ and $\omega_{nu} = 0.8\omega_0$. Moreover, the above choice help to minimize the effect of the measurement noise and unmodeled dynamics in the closed-loop system.

To neglect the effect of the pole $\lambda_3$ on the transient responses of $z_1(t)$ and $z_2(t)$, it is fixed to:

$$\lambda_3 = -3\zeta\omega_n$$

The zeros $\psi_i$, $i = 1, 2$ of $G_i(s)$ are selected based on the following conditions, such that they have the least possible impact on the responses $z_1$ and $z_2$.

$$|\psi_1| \geq \gamma_1\zeta\omega_n \quad (62)$$

$$|\psi_2| \geq \gamma_2\zeta\omega_n \quad (63)$$

It is recommended to select $\gamma_1 = 5$ to guarantee that the effect of the zero $\psi_1$ on the ATMD displacement $z_1$ is insignificant. Moreover, the parameter $\gamma_2$ should be selected as large as possible, and we recommend $\gamma_2 \geq 1$.

### 4.2. Frequency responses of $z_1(t)$, $z_2(t)$, $z_3(t)$ and $u(t)$

The frequency responses of signals $z_1(t)$, $z_2(t)$, $z_3(t)$ and $u(t)$ will be computed to guarantee that these signals are between acceptable limits within the frequency band of $\tilde{x}_g(t)$. For that, assume that excitation signal $\tilde{x}_g$ is a sinusoidal input, denoted as $\tilde{x}_g = \delta \sin(\omega t)$, where the amplitude $\delta$ represents the upper bound of $\tilde{x}_g$, as indicated in (58). Moreover, the frequency $\omega$ corresponds to the earthquake frequency with a bandwidth $\omega \in [\omega_{BWl}, \omega_{BWu}]$. Then, the steady-state of responses $z_1(t)$, $z_2(t)$, $z_3(t)$ and $u(t)$ are also sinusoids. By varying the frequency $\omega$ of $\tilde{x}_g$ between its bandwidth permits to determine if the amplitudes of $z_1(t)$, $z_2(t)$, $z_3(t)$ and $u(t)$ are between tolerable limits within this band.

Let the transfer functions

$$H_i(s) = \delta G_i(s), \quad i = 1, 2, 3, u \quad (64)$$

where $G_1(s)$, $G_2(s)$, $G_3(s)$ and $G_u(s)$ are presented in section 4.3. The frequency responses of $z_1(t)$, $z_2(t)$, $z_3(t)$ and $u(t)$ are directly computed from the transfer functions $H_1(s)$, $H_2(s)$, $H_3(s)$ and $H_u(s)$ by substituting variable $s$ by $j\omega$, where $\omega \in [\omega_{BWl}, \omega_{BWu}]$. The root mean square (RMS) value $\kappa_i$ of $H_i(j\omega)$ is also calculated in the frequency band of $\tilde{x}_g(t)$. This value is defined as

$$\kappa_i = \text{RMS} \left( \left| H_i(j\omega) \right| \right), \quad i = 1, 2, 3, u \quad (65)$$

Since the predominant spectral content of earthquakes is between 1 to 20 Hz, the parameters $\omega_{BWl}$ and $\omega_{BWu}$ will be set as $\omega_{BWl} = 2\pi$ rad/s and $\omega_{BWu} = 40\pi$ rad/s.

### 5. Tuning algorithm for the SMC

This section presents the proposed tuning algorithm to compute: 1) the vector $\eta$ using the transient and frequency responses of the closed-loop system; and 2) the switching gain $M_0$ by analyzing the frequency response $H_u(j\omega)$.
5.1. Procedure to compute vector \( \eta \)

According to equation (21), vector \( \eta \) depends on the parameters \( \zeta \) and \( \omega_n \) of the characteristic polynomial \( P_1(s) \) in (13). Let us define \( \Upsilon \) as the possible set of vectors \( \eta_i \), with which the closed-loop system (25) in the sliding mode satisfies the following three conditions:

1. The limits for \( \zeta \) in (59) and for \( \omega_n \) in (60), the value \( \lambda_3 = -3\zeta \omega_n \) for the non-dominant pole, as well as the inequalities in (62) and (63) corresponding to the zeros \( \psi_1 \) and \( \psi_2 \), respectively.
2. The next upper limits \( \bar{\kappa}_i \) for \( \kappa_i \), \( i = 1, 2, 3 \) in (65) given by
   \[
   \kappa_i \leq \bar{\kappa}_i, \quad i = 1, 2, 3 \tag{66}
   \]
   where \( \bar{\kappa}_i \), \( i = 1, 2, 3 \) are positive constants, which constrain the maximum value of \( |H_u(j\omega)| \) under the bandwidth \( \omega \in [\omega_{BW_1}, \omega_{BW_u}] \), or in time-domain, the maximum permitted values of signals \( z_1, z_2 \) and \( z_3 \) in the bandwidth of \( \ddot{x}_g \).
3. The inequality
   \[
   \kappa_u + \sigma \leq \bar{\kappa}_u \tag{67}
   \]
deducted from (10), which indicates that the sum of the RMS value \( \kappa_u \) of frequency response \( |H_u(j\omega)| \) and the bound \( \sigma \) of the non-linear friction \( f(z_3) \) should be less than or equal to the specified limit \( \bar{\kappa}_u \).

Then, the vector \( \eta \in \Upsilon \) used by the SMC, denoted as \( \eta_\star \), is obtained by minimizing either of the following two performance indexes (PIs)

\[
J_{z2} = \min_{\eta \in \Upsilon} \kappa_2(\eta) \tag{68}
\]

\[
J_u = \min_{\eta \in \Upsilon} \kappa_u(\eta) \tag{69}
\]

where parameters \( \kappa_2 \) and \( \kappa_u \) show their dependency on \( \eta \). Note that \( J_{z2} \) and \( J_u \) are related to the ability of the SMC to minimize the top floor displacement \( z_2 \) and the control force \( u \), respectively. Hence the sliding variable \( \sigma = \eta^T_\star \chi \) guarantees a minimal of \( z_2 \) or \( u \), while ensuring that the RMS values of the closed-loop signals are within acceptable limits and their transient responses are sufficiently fast and damped.

5.2. Procedure to compute the switching gain \( M_0 \)

According to inequality (22), the value of \( M_0 \) should satisfy \( M_0 > \sigma + h_0 \). Among these two terms of the sum, only parameter \( \sigma \) is assumed to be known according to Assumption 2. This section presents a methodology for estimating parameter \( h_0 \), that is subsequently employed to compute \( M_0 \). Note that from (23), the parameter \( h_0 \) must satisfy the condition \( h_0 \geq |u_a + \alpha_1 \ddot{x}_g| \), where the frequency response of the signal \( u_a + \alpha_1 \ddot{x}_g \) is given by \( H_u(j\omega) \). Define \( \chi \) as the maximum magnitude of \( |H_u(j\omega)| \) between the interval \( \omega \in [\omega_{BW_1}, \omega_{BW_u}] \), where \( \chi \) is computed using the vector \( \eta_\star \) that minimizes either of the performance indexes (68) and (69), i.e.,

\[
\chi = \max_{\omega \in [\omega_{BW_1}, \omega_{BW_u}]} (|H_u(j\omega, \eta_\star)|) \tag{70}
\]

Since \( \chi \geq |u_a + \alpha_1 \ddot{x}_g| \) in the spectrum of \( \ddot{x}_g \), we will select \( h_0 = \chi \). Thus, the switching gain \( M_0 \) of the SMC is computed as:

\[
M_0 = \sigma + \chi + \zeta \tag{71}
\]

where \( \zeta \) is a small positive constant that guarantees the inequality (22), that in this work is fixed to 0.5.

Finally, Figure 3 shows the proposed tuning algorithm that computes the vector \( \eta_\star \) and gain \( M_0 \) of the SMC. This algorithm implements the procedure outlined in the present section, and it is programmed in the Matlab software. The terms \( \Delta_\zeta \) and \( \Delta_\omega_n \) in this algorithm represent the increments for \( \zeta \) and \( \omega_n \) from their lower to upper limits.
DEFINE

- System parameters: \( \omega_0 \) in (10) and \( A, B, D \) in (13);
- Bounds \( \delta \) in (18) and \( \eta \) in (19);
- Damping ratio limits \( \zeta_l \) and \( \zeta_u \) in (59), and increment \( \Delta \zeta \);
- Natural frequency limits \( \omega_{nl} \) and \( \omega_{nu} \) in (60), and increment \( \Delta \omega \);
- Lower bounds \( \gamma_1 \) in (62) and \( \gamma_2 \) in (63) for the zeros \( \psi_1 \) and \( \psi_2 \), respectively;
- Maximum signal limits: \( \bar{\kappa}_i \), \( i = 1, 2, 3, u \) in (66) and (67);
- The performance index \( J_z^2 \) or \( J_u \) to minimize.

CALCULATE

The vector \( \eta^* \) and the switching gain \( M_0 \) of the SMC, which satisfy the desired transient responses of \( z_i, i = 1, 2 \) and frequency responses of \( z_i, i = 1, 2, 3 \) and \( u \).

Figure 3: Proposed automatic tuning algorithm.
6. Results and discussion

The performance of the proposed tuning algorithm for the SMC, based on the Ackermann’s formula, is evaluated by means of numerical and experimental studies. In both studies the SMC is programmed in the Matlab/Simulink software and the data is sampled at 1 ms. The SMC in \[37\] causes the chattering effect, that is a discontinuous force at \(\sigma = 0\) that cannot be applied to the ATMD in practice. To overcome it, the sign function of the SMC is approximated by means of following continuous saturation function

\[
u = -M_0 \text{sign}(\sigma) = -M_0 \text{sat}(\sigma) = \begin{cases} -M_0 & \text{if } \sigma > \epsilon \\ \frac{\sigma}{\epsilon} M_0 & \text{if } -\epsilon \leq \sigma \leq \epsilon \\ M_0 & \text{if } \sigma < -\epsilon \end{cases} \tag{72}\]

where \(\epsilon\) is a positive constant that is fixed to \(\epsilon = 0.05\).

6.1. Numerical simulation

A reduced scale five-story building with an ATMD mounted on its top floor is simulated. The structure is excited through the North-South component of the El Centro (California, 1940) earthquake, whose magnitude has been scaled. The mass and stiffness of each floor are equal to 10 kg and 1.21 \(\times 10^4\) N/m, respectively. The natural frequency of this mode is \(\omega_0 = 9.9\) rad/s or \(f_0 = 1.58\) Hz. It is assumed that the structure has Rayleigh damping, where the damping ratios of the first and second modes are equal to 0.01. The mass \(m_d\) of the ATMD is 1.4 kg, that is 5% of the modal mass \(m_0\). Moreover, the tuned parameters \(c_d\) and \(k_d\) are selected as \(c_d = 3.54\) Ns/m and \(k_d = 121.66\) N/m, which are the optimum parameters according to \[38\]. Using these parameters, the natural frequency \(\omega_d = \sqrt{k_d/m_d}\) and damping ratio \(\zeta_d = c_d/(2m_d\omega_d)\) of the ATMD are equal to 0.94\(\omega_0\) and 0.135, respectively. Finally, the function \(f(z_i)\) in \[13\] is modeled as Coulomb friction acting between the ATMD and the floor on which it is attached. It is given by:

\[
f(z_i) = \mu_d |z_i| \tag{73}\]

where \(\mu_d\) is the Coulomb friction coefficient with a value of 0.35 N.

In order to compute the parameters \(\eta_u\) and \(M_0\) of the SMC, the proposed tuning algorithm is initialized as follows: \(\delta = 0.5\) m/s\(^2\), \(\tau = 0.5\) N, \(\gamma_1 = 5\) s, \(\gamma_2 = 1\) s, \(\zeta_1 = 0.5\), \(\zeta_2 = 0.9\), \(\Delta_1 = 0.01\), \(\omega_{nl} = 0.5\omega_0\), \(\omega_{nu} = \omega_0\), and \(\Delta_\omega = 0.01\). Moreover, the signals \(z_1\), \(z_2\), \(z_3\) and \(u\) are constrained using the bounds \(\bar{k}_1 = 20\) cm, \(\bar{k}_2 = 12\) mm, \(\bar{k}_3 = 70\) cm/s, and \(\bar{\kappa}_u = 12\) N, respectively. The proposed algorithm displays three-dimensional surfaces of \(\kappa_1\), \(\kappa_2\), \(\kappa_3\) and \(\kappa_u\) computed with the vectors \(\eta \in \mathbb{T}\). These plots are shown in Figure 4 and they indicate that the feasible sets of parameters \(\zeta_u\) and \(\omega_{nl}\) that satisfy the desired transient responses of \(z_i\), \(i = 1, 2, 3\) and \(u\), are the intervals \([0.5 - 0.58]\) and \([0.5\omega_0 - 0.78\omega_0]\), respectively. The red point over the surfaces of \(\kappa_2\) and \(\kappa_u\) indicates their minimum value. Table 1 presents the tuples \(\Gamma_u = \{\zeta_u, \omega_{nl}, \kappa_{u1}, \kappa_{u2}, \kappa_{u3}, \kappa_u, \lambda_1, \lambda_2, \lambda_3, \psi_1, \psi_2\}\) and the switching gains \(M_0\) returned by proposed algorithm by minimizing the performance indexes \(J_{z_2}\) and \(J_u\).

| PI  | \(\zeta\)   | \(\omega_n\) (rad/s) | \(\eta_u\) | \(\kappa_1\) (cm) | \(\kappa_2\) (mm) | \(\kappa_3\) (cm/s) | \(\kappa_u\) (N) | \(\lambda_1\) (rad/s) | \(\lambda_2\) (rad/s) | \(\psi_1\) (rad/s) | \(\psi_2\) (rad/s) | \(M_0\) (N) |
|-----|-------------|-----------------------|------------|---------------------|---------------------|---------------------|---------------------|----------------------|----------------------|----------------------|----------------------|-------------|
| \(J_{z_2}\) | 0.5 | 0.5\(\omega_0\) | [2.6, 289.2, 0.87, 9.76] | 4.0 | 1.0 | 33.8 | 5.76 | -2.48 | -2.48 | -7.43 | -29.64 | -2.99 | 24.13 |
| \(J_u\) | 0.5 | 0.62\(\omega_0\) | [5.312.3, 1.340.15] | 3.7 | 1.4 | 31.4 | 3.08 | -3.08 | -3.08 | -9.23 | 2114.6 | -3.72 | 20.35 |

According to Figure 4, the value of \(\kappa_2\) monotonically increases with the increment of either \(\zeta_u\) or \(\omega_{nl}\), i.e., the minimum of \(\kappa_2\) is located at \((\zeta_u, \omega_{nl})\). Moreover, the mesh plot shown in Figure 4(d) indicates that...
the minimum of $\kappa_u$ is located at $\zeta = \zeta_l$. Thus, using other initial parameters of $\zeta_l$ and $\omega_{nl}$ in the tuning algorithm changes the coordinates $(\zeta, \omega_n)$ of the minimums of $\kappa_2$ and $\kappa_u$.

![Mesh plot of $(\zeta, \omega_n, \kappa_1)$.](image1)

![Mesh plot of $(\zeta, \omega_n, \kappa_2)$.](image2)

![Mesh plot of $(\zeta, \omega_n, \kappa_3)$.](image3)

![Mesh plot of $(\zeta, \omega_n, \kappa_u)$.](image4)

Figure 4: Three-dimensional surfaces of $\kappa_1, \kappa_2, \kappa_3$ and $\kappa_u$.

Figures 5 (a) and 5 (b) depict, respectively, the earthquake excitation signal $\ddot{x}_g$ and the top floor displacement $x_5(t) = z_2(t)$ obtained using the TMD, i.e., with $u = 0$. Furthermore, Figures 6 (a), 6 (b), 7 (a), and 7 (b) presents, respectively, the signals $z_1$, $z_2$, $z_3$ and $u$, which are produced with the SMC using the parameters $\eta_*$ and $M_0$ that minimize the PIs $J_{z_2}$ and $J_u$. By comparing Figures 5 (b) and 6 (b), we can conclude that the ATMD, based on the proposed SMC and either of the PIs $J_{z_2}$ and $J_u$, provides greater reduction of the building displacements when compared to the TMD.

![Ground acceleration $\ddot{x}_g(t)$](image5)

![Top floor displacement $z_2(t)$](image6)

Figure 5: Signals $\ddot{x}_g(t)$ and $z_2(t)$. 

14
Now, let us define $z_{\text{rms}}$, $u_{\text{rms}}$ and $z_{\text{peak}}$, $u_{\text{peak}}$ as the RMS and maximum peak values of signals $z_i$ and $u$, respectively. Table 2 presents these values corresponding to the TMD and ATMD during the period of $t = 0$ to $t = 30$ s. This table indicates that the ATMD allows reducing more than three times the peak value of the floor displacement $z_2 = x_5$ when compared to the TMD. Note that the SMC that minimize the PI $J_{z_2}$ (respectively $J_u$) produce the smallest $z_{\text{rms}}$, as expected. Finally, by comparing Tables 1 and 2, it is possible to observe that the values $z_{\text{rms}}$, $i = 1, 2, 3, u$ can be considered as a scaled version of the parameters $\kappa_i$, since the relation $\kappa_i/z_{\text{rms}}$ is between 1.5 to 2.65.

| Damper | $z_{\text{rms}}$ | $z_{\text{peak}}$ | $z_{\text{rms}}$ | $z_{\text{peak}}$ | $z_{\text{rms}}$ | $z_{\text{peak}}$ | $u_{\text{rms}}$ | $u_{\text{peak}}$ |
|--------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| TMD    | 0.33           | 1.33          | 2.67           | 9.33           | 3.19           | 13.38          | 25.8           | 93.8           |
| ATMD $J_{z_2}$ | 2.61           | 9.84          | 0.48           | 2.0            | 14.0           | 66.9           | 3.93           | 17.61          |
| ATMD $J_u$      | 1.86           | 6.74          | 0.58           | 2.58           | 11.81          | 55.04          | 5.11           | 26.49          |

6.2. Experimental results

Figure 8 shows the experimental structure, which was developed by the Quanser company. It consists of a single-story building controlled by an AMD, where the structure is mounted over a shake table, model STI-40. The maximum travel or stroke of the AMD is ±9.5 cm. Moreover, the structure has an accelerometer that measures the floor acceleration, and two optical encoders to detect the AMD and the shake table displacements. The displacement $z_2$ and velocity $z_3$ of the story, as well as the velocity $z_4$ of the AMD are

Figure 6: Displacements $z_1(t)$ and $z_2(t)$.  

Figure 7: Signals $z_3(t)$ and $u(t)$.
estimated by means of a Luenberger state observer \cite{39}, that uses the measurements from the encoders and accelerometer. The SMC is programmed using the QUARC real-time control software of Matlab/Simulink. During the experiments, the structure is excited with the 1994 Northridge earthquake, which is scaled in time and amplitude, as shown in Figure 9. The model of the structure is given in \cite{39}, where $f(z_3)$ is given in (73). The parameters of the experimental structure are listed in Table 3, where $c_d$ represents the viscous friction between the damper and the floor.

![Experimental structure with an AMD.](image)

**Figure 8: Experimental structure with an AMD.**

![Shake table acceleration $\ddot{x}_g(t)$.](image)

**Figure 9: Shake table acceleration $\ddot{x}_g(t)$.**

| Parameter | Value | Units |
|-----------|-------|-------|
| $m_0$     | 1.84  | kg    |
| $m_d$     | 0.79  | kg    |
| $k_0$     | 226.23| N/m   |
| $k_d$     | 0     | N/m   |
| $c_0$     | 0.16  | N·s/m |
| $c_d$     | 6.85  | N·s/m |
| $\mu_d$  | 0.43  | N     |
| $\beta_0$| 1     | –     |
| $\omega_0$| 11.08 | rad/s |

**Table 3: Parameters of the experimental structure.**
In order to compute the parameters \( M_0 \) and \( \eta_* \) of the SMC, the tuning algorithm is initialized with the parameters employed in the numerical simulation described in section 6.1 with the exception of parameter \( \delta \), that is set as \( \delta = 3 \text{ m/s}^2 \), and the maximum allowed values of signals \( z_1, z_2, z_3 \) and \( u \), which are specified by means of the upper bounds \( \kappa_1 = 5 \text{ cm}, \kappa_2 = 10 \text{ mm}, \kappa_3 = 32 \text{ cm/s}, \) and \( \kappa_u = 10 \text{ N}, \) respectively. The tuning algorithm returns the mesh plots of \( \kappa \) parameters employed in the numerical simulation described in section 6.1, with the exception of parameter \( \delta \). The feasible intervals for \( \zeta \) returned by the proposed tuning algorithm are presented in Table 4.

![Damping factor ζ and Frequency [rad/s] (a) Mesh plot of (ζ,ωn,κ1).](image1)

![Damping factor ζ and Frequency [rad/s] (b) Mesh plot of (ζ,ωn,κ2).](image2)

![Damping factor ζ and Frequency [rad/s] (c) Mesh plot of (ζ,ωn,κ3).](image3)

![Damping factor ζ and Frequency [rad/s] (d) Mesh plot of (ζ,ωn,κu).](image4)

Figure 10: Three-dimensional surfaces of \( \kappa_1, \kappa_2, \kappa_3 \) and \( \kappa_u \).

| Table 4: Tuples \( \Gamma_* \) returned by the proposed algorithm for the experimental structure. |
| --- |
| \( \text{PI} \) | \( \zeta \) | \( \omega_n \) | \( \eta_* \) | \( \kappa_1 \) | \( \kappa_2 \) | \( \kappa_3 \) | \( \kappa_u \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \psi_1 \) | \( \psi_2 \) | \( M_0 \) |
| \( J_{x_1} \) | 0.5 | 0.55\( \omega_n \) | 1.64 - 19.94, 0.49 - 0.20 | 3.65 | 8.0 | 30.33 | 2.71 | -2.77 | -2.77 | -8.31 | -99.15 | -3.33 | 13.52 |
| \( J_u \) | 0.5 | 0.55\( \omega_n \) | 2.16 - 20.64, 0.59 - 0.13 | 3.5 | 9.2 | 29.61 | 2.54 | -3.04 | -3.04 | -9.11 | -99.15 | 162.45 | 3.65 | 12.05 |

Figure 11 presents the frequency responses of the filters \( \dot{H}_i(j\omega, \eta_*) \), \( i = 1, 2, 3 \), \( u \) corresponding to the vectors \( \eta_* \) that minimize the PIs \( J_{x_1} \) and \( J_u \). This figure also indicates the natural frequency of the structure \( f_0 = \omega_0 / (2\pi) \) in Hz. It shows that at low frequencies, the magnitude of \( \dot{H}_i(j\omega, \eta_*) \) is larger than \( \dot{\kappa}_i \); however, the RMS value \( \kappa_i \) of \( \dot{H}_i(j\omega, \eta_*) \) is smaller than \( \dot{\kappa}_i \) in the bandwidth of the earthquake \( \ddot{x}_g \).

The proposed SMC is compared with the LQR \(^{33} \), which minimizes the performance index

\[
J_{\text{LQR}} = \int_0^\infty (x^TQx + ru^2)dt
\]  

(74)
where $Q \in \mathbb{R}^{4 \times 4}$ is a symmetric matrix such that $Q \geq 0$; moreover $r$ is a positive constant. Both $Q$ and $r$ are designed according to the literature \cite{35, 40}, where the entries of $Q = \text{diag}(q_{11}, q_{22}, q_{33}, q_{44})$ satisfy

$$q_{ii} = \frac{1}{\text{maximum acceptable value of } z_i^2}, \quad i = 1, 2, 3, 4$$  \hspace{1cm} (75)$$

and the parameter $r$ is given by

$$r = \frac{1}{\text{maximum acceptable value of } u^2}$$  \hspace{1cm} (76)$$

The maximum acceptable values for $z_i$, $i = 1, 2, 3$ and $u$ are, respectively, specified by the constants $\bar{\kappa}_j$, $j = 1, 2, 3, u$, which were mentioned above. The remaining maximum acceptable value for $z_4$ is chosen as 0.1 m/s. Thus, coefficients $q_{ii}$, $i = 1, 2, 3, 4$ in (75) and $r$ in (76) are given by:

$$q_{11} = 400, \quad q_{22} = 10,000, \quad q_{33} = 9.77, \quad q_{44} = 100, \quad r = 0.01$$  \hspace{1cm} (77)$$

The control law $u$ that minimizes the performance index $J_{\text{LQR}}$ in (74) is computed as

$$u = -k_J z$$  \hspace{1cm} (78)$$

where $k_J = \frac{1}{r} B^T P$ and $P = P^T > 0$ is the solution of the Riccati equation

$$A^T P + PA - \frac{1}{r} PBB^T P = -Q$$  \hspace{1cm} (79)$$

Solving (79) yields $k_J = [200, -1276.5, 49, 17.32]$, and the closed-loop eigenvalues $\lambda_1 = -3.52 + j6.82$, $\lambda_2 = -3.52 - j6.82$, $\lambda_3 = -6.77$ and $\lambda_4 = -77.98$. Using these dominant poles $\lambda_1$ and $\lambda_2$ permits computing the parameters $\zeta$ and $\omega_n$ of the closed-loop system, which are $\zeta = 0.46$ and $0.69\omega_0$. Note that these LQR parameters do not belong to the feasible intervals of $\zeta$ and $\omega_n$ corresponding to the SMC.
Figure 12 depicts the uncontrolled response $z_2(t)$ of the structure under the earthquake excitation, which indicates that its maximum magnitude is approximately 3.5 cm. On the other hand, Figures 13 (a), 13 (b), 14 (a), 14 (b) and 15 show, respectively, the signals $z_1(t)$, $z_2(t)$, $z_3(t)$, $z_4(t)$ and $u(t)$ produced by the LQR and the SMC. From these plots, it is possible to see that the maximum displacement $z_2(t)$ corresponding to the SMC, based on either $\text{PI } J_{z_2}$ or $\text{J}_u$, is smaller than that produced by the LQR controller. Moreover, the maximum peak of signals $u(t)$ and $z_1(t)$ corresponding to the SMC is slightly larger than the one generated by the LQR controller, but these signals are within the specified limits.

In addition, Table 5 presents the RMS and maximum peak values of the signals $u(t)$ and $z_i$, $i = 1, 2, 3, 4$ corresponding to the SMC and LQR, during the period of $t = 2$ to $t = 6$ s. Note that the relations $\kappa_i/z_i^{\text{rms}}$, $i = 1, 2, 3, u$, are between 2.15 to 4.54. This table indicates that the SMC and LQR allow attenuating more than three times the peak of the uncontrolled response of $z_2$. Moreover, the SMC that minimizes the PI $J_{z_2}$ produces the smaller RMS value of $z_2$, as expected. Note also, that the value $z_2^{\text{rms}}$ corresponding to the SMC based on the PI $\text{J}_u$ is smaller than that produced by the LQR, and their corresponding control efforts $u^{\text{rms}}$ are similar. These results show that the SMC based on the Ackermann’s formula and either of the PIs $J_{z_2}$ or $\text{J}_u$ has great potential for active structural control.

![Figure 13: Displacements of the AMD and floor.](image)

![Figure 14: Velocities of the AMD and floor.](image)
Figure 15: Control force generated by SMC and LQR.

Table 5: RMS and maximum peak values of $z_i$, $i = 1, 2, 3, 4$ and $u(t)$.

| Controller | PI | $z_{rms}^1$ (cm) | $z_{peak}^1$ (mm) | $z_{rms}^2$ (cm/s) | $z_{peak}^2$ (cm/s) | $z_{rms}^3$ (cm/s) | $z_{peak}^3$ (cm/s) | $u_{rms}$ (N) | $u_{peak}$ (N) |
|------------|----|------------------|-------------------|-------------------|-------------------|-------------------|-------------------|----------------|----------------|
| Control OFF |     | 20.14            | 35.31             |       |       |       |       | 17.90          | 34.54          |
| SMC       | $J_{z2}$ | 0.84            | 2.56              | 2.62          | 7.39          | 9.59        | 29.06          | 2.52           | 8.49           | 1.25          | 3.36          |
| SMC       | $J_u$   | 0.77            | 2.32              | 3.21          | 9.73          | 9.39        | 29.95          | 2.89           | 10.71          | 1.18          | 3.43          |
| LQR       | $J_{LQR}$ | 0.74           | 2.17              | 4.21          | 11.80         | 8.75        | 26.62          | 3.50           | 12.68          | 1.05          | 2.92          |

7. Conclusions

This paper proposed a tuning algorithm for a SMC designed for attenuating the structural vibration of a seismically excited building equipped with an ATMD on its top floor. The SMC is derived based on the Ackermann’s formula and its robustness against friction uncertainty was demonstrated. It was proved that the seismic excitation signal is not a coupled disturbance, and as a consequence it cannot be eliminated by the SMC; however, its effect can be minimized by designing the SMC with the proposed tuning algorithm.

It was also shown that the responses of the ATMD and building under the SMC can be described by means of dominant second-order filters, whose input is the seismic excitation. Their parameters were automatically tuned by the proposed algorithm in order to minimize the PI $J_{z2}$ or $J_u$. The PI $J_{z2}$ is related to the minimization of the top floor displacement, while the PI $J_u$ is related to the minimization of the control force applied the ATMD while offering a great attenuation of this displacement. The effectiveness of the designed SMC was demonstrated in a numerical and an experimental structure. Moreover, the experiments showed that the designed SMC has a better vibration attenuation performance than the LQR, while ensuring that the structure and the ATMD signals are within the specified limits.

References

[1] S.-Y. Kim, C.-H. Lee, Peak response of frictional tuned mass dampers optimally designed to white noise base acceleration, Mechanical Systems and Signal Processing 117 (2019) 319–332.
[2] S. Chesne, G. Inquieté, P. Cranga, F. Legrand, B. Petitjean, Innovative hybrid mass damper for dual-loop controller, Mechanical Systems and Signal Processing 115 (2019) 514–523.
[3] S. Paul, W. Yu, A method for bidirectional active control of structures, Journal of Vibration and Control 24 (15) (2018) 3400–3417.
[4] A. E. Kayabekir, G. Bekdaş, S. M. Nigdeli, Z. W. Geem, Optimum design of pid controlled active tuned mass damper via modified harmony search, Applied Sciences 10 (8) (2020) 2976.
[38] R. Rana, T. Soong, Parametric study and simplified design of tuned mass dampers, Engineering structures 20 (3) (1998) 193–204.

[39] K. Ogata, Modern control engineering, Prentice Hall, Upper Saddle River, NJ, 2002.

[40] A. E. Bryson, Y. C. Ho, Applied Optimal Control, Taylor & Francis, Madison Avenue, NY, 1975.