ON SYMMETRIES OF KDV-LIKE EVOLUTION EQUATIONS

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The $x$-dependence of the symmetries of (1+1)-dimensional scalar translationally invariant evolution equations is described. The sufficient condition of (quasi)polynomiality in time $t$ of the symmetries of evolution equations with constant separant is found. The general form of time dependence of the symmetries of KdV-like non-linearizable evolution equations is presented.

1. Introduction

It is well known that provided scalar (1+1)-dimensional evolution equation (EE) with time-independent coefficients possesses the infinite-dimensional commutative Lie algebra of time-independent non-classical symmetries, it is either linearizable or integrable via inverse scattering transform [1, 2]. This algebra is usually constructed with usage of the recursion operator [2], but it may be also generated by the repeated commuting of mastersymmetry with few time-independent symmetries [3]. In its turn, to possess the mastersymmetry, EE in question must have (at least one) polynomial in time $t$ symmetry. This fact is one of the main reasons of growing interest to the study of whole algebra of time-dependent symmetries of EEs [4, 5, 6].

However, it is very difficult to describe this algebra even in the simplest case of scalar (1+1)-dimensional EE. To the best of author’s knowledge, in the class of scalar nonlinear EEs the complete algebras of time-dependent local symmetries were found only for KdV equation by Magadeev and Sokolov [7] and for KdV and Burgers equations by Vinogradov et al. [8]. In [8] there were also proved two no-go theorems, which show, when the symmetries of third order KdV-like and second order Burgers-like EEs are exhausted by Lie ones.

Surprisingly enough, using only the invariance of given EE under $\partial/\partial t$ and $\partial/\partial x$ and some simple observations on the explicit form of symmetries, we can show (vide Theorem 6 below) that any symmetry of KdV-like non-linearizable EE as a function of $t$ is a linear combination of quasipolynomials (i.e. of the products of exponents by polynomials). Moreover, for the class of EEs with constant separant our approach allows to find a simple sufficient condition for such a situation to take place (Theorem 3), and for the generic EE it enables us to describe the dependence of its symmetries on variable $x$ (Theorem 1).

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2. Some general properties of symmetries of evolution equations

Consider the scalar (1 + 1)-dimensional EE

$$\frac{\partial u}{\partial t} = F(t, u, u_1, \ldots, u_n), \quad n \geq 2,$$

(1)

where \( u_l = \partial^l u / \partial x^l, l \in \mathbb{N}, u_0 \equiv u, \) and its symmetries, i.e. the right hand sides \( G \) of EEs

$$\partial u / \partial \tau = G(x, t, u, u_1, \ldots, u_k),$$

(2)

compatible with equation (1). The biggest number \( k \) such that \( \partial G / \partial u_k \neq 0 \) is called the order of symmetry and is denoted as \( k = \text{ord} \, G \).

For any sufficiently smooth function \( h(x, t, u, u_1, \ldots, u_r) \) introduce the quantities \[1\]

$$h^* = \sum_{i=0}^r \partial h / \partial u^i D^i$$

and

$$\nabla h = \sum_{j=0}^\infty D^j(h) D \partial / \partial u^j,$$

where \( D = \partial / \partial x + \sum_{i=0}^\infty u_{i+1} \partial / \partial u^i \).

Now we can define the Lie bracket, which endows \( S \) by the structure of Lie algebra, as

$$\{ h, r \} = h^*(r) - r^*(h) = \nabla r(h) - \nabla h(r).$$

This definition differs from the conventional one \[1, 2, 9\] by the sign, but is more suitable for our purposes. Note that \( S^{(1)} \) is Lie subalgebra in \( S \).

Equation (2) is compatible with (1) if and only if

$$\frac{\partial G}{\partial t} = \{ F, G \}.$$

(3)

From (3) one may easily derive \[2\] that

$$\frac{\partial G_*}{\partial t} \equiv (\partial G / \partial t)_* = \nabla h(F) - \nabla F(h) + [F, G_*],$$

(4)

where \( \nabla h(F) \equiv \sum_{i=0}^\infty D^i(h) \partial^2 F / \partial u^i \partial u^l \) and similarly for \( \nabla F(h) \); \([\cdot, \cdot]\) stands for the usual commutator of linear differential operators.

Let ord \( G = k \). Then equating the coefficients at \( D^l \) on both sides of (4) yields

$$\frac{\partial^2 G}{\partial u^l \partial t} = \sum_{m=0}^n D^m(G) \frac{\partial^2 F}{\partial u^m \partial u^l} - \sum_{r=0}^k D^r(F) \frac{\partial^2 G}{\partial u^r \partial u^l}$$

$$+ \sum_{j=\max(0, l+1-n)} \sum_{i=\max(l+1-j, 0)} \left[ C^{i+j-l}_{i} \frac{\partial F}{\partial u^i} D^{i+j-l} \left( \frac{\partial G}{\partial u^j} \right) \right]$$

$$- C^{i+j-l}_{j} \frac{\partial G}{\partial u^j} D^{i+j-l} \left( \frac{\partial F}{\partial u^i} \right), \quad l = 0, \ldots, n + k - 1,$$

(5)

where \( C^p_q = \frac{q!}{p!(q-p)!} \) and we assume that \( 1 / (-s)! = 0 \) for \( s \in \mathbb{N} \).
If \( k \geq 2 \), one easily obtains from (5) with \( l = n + k - 1, \ldots, n + 1 \) the formulas

\[
\frac{\partial G}{\partial u_k} = c_k(t)(\frac{\partial F}{\partial u_n})^{k/n},
\]

(6)

\[
\frac{\partial G}{\partial u_i} = c_i(t)(\frac{\partial F}{\partial u_n})^{i/n} + \sum_{p=i+1}^{k} \sum_{q=0}^{\lfloor \frac{p-i}{n} \rfloor} \chi_{pq}(t, x, u, \ldots, u_k) \frac{\partial^p c_p}{\partial t^q}, i = 2, \ldots, k-1,
\]

(7)

where \( c_j(t) \) are arbitrary functions of \( t \) (cf. [11]).

Furthermore, we see that by virtue of (11)

\[
[\nabla^p F - F, G] = 0 \pmod{D^p}, \quad p = \max(k, n)
\]

(8)

Equating the coefficients at powers of \( D \) in (8) yields the equations (5) with \( l = p + 1, \ldots, n + k - 1 \), from which we may find \( \partial G/\partial u_i, i = \max(k - n + 1, 2), \ldots, k \). By (7), the only arbitrary elements, which they may contain, are functions \( c_i(t) \), while their dependence on \( x, u, u \) is uniquely determined from (5).

On the other hand, since \( F \) is \( x \)-independent, the existence of the solution \( F_0 \) of (5) with \( p = n \) guarantees the solvability of the equations for \( \partial G/\partial u_i, i = \max(k - n + 2, 2), \ldots, k \), in terms of functions of \( u, \ldots, u_k \) and \( t \) only (cf. [10]). Therefore, \( \partial G/\partial u_i, i = \max(k - n + 2, 2), \ldots, k \), are \( x \)-independent. In particular, any \( G \in S^{(n)} \) has the form

\[
G = g(t, u, \ldots, u_k) + \Phi(t, x, u, u_1).
\]

(9)

Thus, for any symmetry \( G \in S \partial G/\partial x \in S \) and \( \text{ord} \partial G/\partial x \leq \max(1, \text{ord } G - n + 1) \). Applying this result to \( G = \partial G/\partial x \) and so on, we obtain that \( \partial^r G/\partial x^r \in S^{(n)} \) and hence is of the form (9), if \( r = r_{k,n,1} \), where for \( q = 0, 1 \)

\[
r_{k,n,q} = \begin{cases} \left\lfloor \frac{k}{n-1} \right\rfloor \text{ for } k \not\equiv 0, \ldots, q \pmod{n-1}, \\ \max(0, \left\lfloor \frac{k}{n-1} \right\rfloor - 1) \text{ for } k \equiv 0, \ldots, q \pmod{n-1}, \\ \end{cases} \quad \text{and } r_{k,n,-1} = \left\lfloor \frac{k}{n-1} \right\rfloor;
\]

\([s]\) denotes here the integer part of the number \( s \). The integration of \( \partial^r G/\partial x^r \) \( r \) times with respect to \( x \), taking into account the above, yields the following result:

**Theorem 1.** Any symmetry \( G \) of order \( k \) of (1) may be represented in the form

\[
G = \psi(t, x, u, u_1) + \sum_{j=0}^{s} x^j g_j(t, u, \ldots, u_{k-j(n-1)}), \quad s \leq r_{k,n,1}.
\]

(10)

**Remark 2.** In complete analogy with the above, one may show that if

\[
\frac{\partial F}{\partial u_{n-i}} = \phi_i(t), \quad i = 0, \ldots, j,
\]

(11)

where \( \phi_i(t) \) are arbitrary functions of \( t \), then in (5) \( p = \max(k, n-1-j) \) and it is possible to find from (5) \( \partial G/\partial u_i, i = \max(k - n + 2, \max(1-j,0)), \ldots, k \), which again turn out to be \( x \)-independent, and hence (cf. [10]) in (10) \( s \leq r_{k,n,-\min(1,j)} \) and \( \psi \) satisfies

\[
\frac{\partial \psi}{\partial x_r} = 0, \quad r = \max(1-j,0), \ldots, 1.
\]

(12)

Note that if (11) holds true, (6), (7) hold for \( k \geq \max(1-j,0), i = \max(1-j,0), \ldots, k-1. \)
3. Symmetries of the equations with constant separant

Let us turn to the particular case, when \( \partial F/\partial t = 0 \) and \( F \) has the form

\[
F = u_n + f(u, \ldots, u_{n-1}), \tag{13}
\]
i.e. when \( F \) has a constant separant, equal to unity \([10]\). Note that any \( F \) with constant separant, different from unity, may be reduced to the form \( F \) by rescaling of time \( t \).

For the sake of brevity we shall refer to EE \([11]\) with \( F \) \([13]\) as to EE with constant separant. Let us also mention that if \( \partial F/\partial t = 0 \), then in \([1]\) \( \partial \chi_{\partial g}/\partial t = 0 \).

Assume that \( \text{ord } G \equiv k > n - 1 \). Then, by \([13]\) and \([6]\), \([5]\) with \( l = k \) reads

\[
nD(\partial G/\partial u_{k-n+1}) = \partial c_k(t)/\partial t + R, \tag{14}
\]
where \( R \) stands for the terms which depend only on \( F \) and its derivatives and on \( \partial G/\partial u_i, i = k - n + 2, \ldots, k \). Moreover, \( R = D(K) \) for some \( x \)-independent \( K \), as it follows from the fact that if \( F \) has a constant separant, \( F_k \) as \( p = n - 1 \). Really, if the term \( \partial G_*/\partial t \) in \([4]\) would be absent, \( G_* \) would satisfy \([8]\) with \( p = n - 1 \) and the equation for \( \partial G/\partial u_{k-n+1} \) would be solvable in terms of functions of \( t, u, u_1, \ldots \) (cf. the proof of Theorem 1 and \([10]\)). But the only term in \([14]\), generated by \( \partial G_*/\partial t \), is \( \partial c_k(t)/\partial t \), while \( R \) is the same as if it would be in absence of \( \partial G_*/\partial t \) in \([4]\). Hence, \( R = D(K), \partial K/\partial x = 0, \) and

\[
\partial G/\partial u_{k-n+1} = (x/n)\partial c_k(t)/\partial t + K + c_{k-n+1}(t). \tag{15}
\]

Since \( \partial G/\partial u_i, i = k - n + 2, \ldots, k \), are \( x \)-independent by Theorem 1, by \([15]\) \( \text{ord } \partial G/\partial x = k - n + 1 \) and \( \partial^p G/\partial u_{k-n+1} \partial x = (1/n)\partial c_k(t)/\partial t \).

Iterating this process shows that for \( r = r_{k,n,0} Q = \partial^r G/\partial x^r \in S^{n-1} \) and

\[
\partial Q/\partial u_q = (1/n^r)\partial^r c_k(t)/\partial t^r, \quad q \equiv \text{ord } Q. \tag{16}
\]

**Theorem 3.** If the symmetries from \( S^{n-1} \) of the equation \([1]\) with constant separant either are all polynomial in \( t \) or are all linear combinations of quasipolynomials\(^1\) in \( t \), then so does any symmetry of this equation.

**Proof.** If the conditions of theorem are fulfilled, then by \([10]\) for any symmetry \( G \), \( k \equiv \text{ord } G \geq 1 \), the function \( c_k(t) = \partial G/\partial u_k \) is either polynomial or linear combination of quasipolynomials in \( t \). Hence, there exists a differential operator \( \Omega = \sum_{l=0}^{m} a_l \partial^l/\partial t^l \), \( a_l \in \mathbb{C} \), such that \( \Omega(c_k(t)) = 0 \). In particular, if \( c_k(t) \) is polynomial in \( t \) of order \( p \), we may choose \( \Omega_0 = \partial^{p+1}/\partial t^{p+1} \) as \( \Omega \).

Now assume that the theorem is already proved for the symmetries from \( S^{k-1} \) (it is obviously true for \( k \leq n \)). For EE \([11]\) with \( F \) \([13]\) \( S^{k} \) is closed under \( \partial/\partial t \), and therefore \( \Omega(G) \in S^{k} \). Moreover, since \( \Omega(c_k(t)) = 0, \text{ord } \Omega(G) \leq k - 1 \) and hence, by our assumption, \( G \equiv \Omega(G) \) is either polynomial or linear combination of quasipolynomials in \( t \). Obviously, so does any solution \( R \) of the equation \( \Omega(R) = G \), including \( R = G \). If all the

\(^1\)We call quasipolynomials the products \( \exp(\lambda t)P(t) \), where \( \lambda \in \mathbb{C} \) and \( P \) is a polynomial.
elements of $S^{(n-1)}$ are polynomial in $t$, then so does $c_k(t)$ and hence the polynomiality of $G$ in $t$ is guaranteed, because we may take $\Omega = \Omega_0$ and because $\tilde{G}$ is polynomial in $t$ by our assumption. The induction by $k$, starting from $k = n$, completes the proof. □

Theorem 3 is a natural generalization of the result of [7] on polynomiality in $t$ of symmetries of KdV equation. It gives a very simple sufficient condition for all the symmetries of a given EE with constant separant to be polynomial in time $t$. Note that in such a situation all the time-dependent symmetries of EE in question may be constructed via the so-called generators of degree $s$ for different $s \in \mathbb{N}$, using the results of Fuchssteiner [3].

4. Symmetries of KdV-like equations

Now let us consider the equations with constant separant, whose $f$ satisfies
\[ \partial f/\partial u_{n-1} = \text{const}. \] (17)

We shall call the EEs (11) with $F$ (13), satisfying (17), KdV-like, since the famous Korteweg – de Vries equation has the form (11) with $F$ (13), where $n = 3$ and $f = 6uu_1$ obviously satisfies (17).

Let $G$ be the symmetry of KdV-like EE (11). Analyzing the leading term of $\partial^r G/\partial x^r$, $r = \left\lfloor \frac{\text{ord } G}{n-1} \right\rfloor$, like in the proof of Theorem 3, we obtain the following statement:

**Corollary 4.** If the symmetries from $S^{(n-2)}$ of KdV-like equation (11) either are all polynomial or are all linear combinations of quasipolynomials in $t$, then so does any symmetry of this equation.

**Example 5.** Consider third order formally integrable nonlinear KdV-like EEs (11):
\[
\begin{align*}
    u_t &= u_3 + uu_1, \\
    u_t &= u_3 + u_2^2 + c, \\
    u_t &= u_3 + u_2^2 u_1 + cu_1, \\
    u_t &= u_3 + u_2^2 + cu_1 + d, \\
    u_t &= u_3 - u_2^3/2 + (a \exp(2u) + b \exp(-2u) + d)u_1,
\end{align*}
\]
where $a, b, c, d \in \mathbb{C}$. All the symmetries of these EEs are polynomial in $t$ by Corollary 4, since so do their symmetries of orders 0 and 1.

Now let us analyze in more detail the general form of time dependence of symmetries of KdV-like EE (11). Assume that the EE in question may not be linearized by means of contact transformations (for the sake of brevity we shall call it non-linearizable). Then $\dim \Phi \leq n$ [9], where $\Phi = \{ \varphi(x,t)|\varphi(x,t) \in S \}$. Let us show that in such a case $\dim S^{(k)} < \infty$ for any $k = 0, 1, 2, \ldots$

Let $G \in S^{(k)}/S^{(k-1)}$, $k > n - 2$. Then, obviously, it is completely determined by its leading term $h_k(t) \equiv \partial G/\partial u_k$. Like the above, but taking into account Remark 2, we may show that for KdV-like EE (11) $Q = \partial^r G/\partial x^r \in S^{(n-2)}$, if $r = \left\lfloor \frac{k}{n-1} \right\rfloor$, and
\[
\partial Q/\partial u_q = c_q(t) = (1/n^r)\partial^r h_k(t)/\partial t^r, q \equiv \text{ord } Q. \tag{18}
\]
Since \( h_k \) satisfies (18), for \( k > n - 2 \)
\[
\dim S^{(k)}/S^{(k-1)} \leq \dim S^{(k_0)} + \left[ \frac{k}{n-1} \right], \quad k_0 = k - \left[ \frac{k}{n-1} \right](n-1),
\] (19)
whence \( \dim S^{(k)} < \infty \) for \( k = 0, \ldots, n - 2 \) implies the same result for any \( k \).

By Theorem 1 and Remark 2 for KdV-like EE (1) (6) and (7) for \( k \leq n - 2 \) read
\[
\frac{\partial G}{\partial u_i} = c_i(t) + \sum_{p=i+1}^{k} \chi_p(u, \ldots, u_k) c_p(t), \quad i = 0, \ldots, k - 1,
\] (20)
\[
\frac{\partial G}{\partial u_k} = c_k(t),
\] (21)
and thus any symmetry \( G \) of order \( k \leq n - 2 \) has the form
\[
G = \psi(t, x) + g_0(t, u, \ldots, u_k).
\] (22)

Without loss of generality we can assume that the function \( g_0 \) is completely determined by \( \partial G/\partial u_i, \ i = 0, \ldots, k \). Since \( \partial \psi/\partial x \in \Phi \), we have
\[
\psi(t, x) = \gamma(t) + \sum_{p=1}^{\dim \Phi} a_p \int_0^x \varphi_p(y, t),
\] (23)
where \( a_p \in \mathbb{C}, \ \gamma(t) \) is arbitrary function of \( t \), and \( \varphi_p(x, t), \ p = 1, \ldots, \dim \Phi \), stand for some basis in \( \Phi \).

The substitution of \( G \) (22) with \( \psi \) (23) into equations (5) with \( l = 0, \ldots, n - 1 \) and into (3) yields in final account the system of first order linear ordinary differential equations in \( t \) (and, possibly, algebraic equations) for \( c_i(t), \ i = 0, \ldots, k \) and \( \gamma(t) \). Note that we must use (3) in order to obtain an ODE of the form \( \partial \gamma(t)/\partial t = \cdots \), allowing to find \( \gamma(t) \).

Obviously, the general solution of this system of ODEs for \( k \leq n - 2 \) may contain at most \( N_{k,n} = \dim \Phi + k + 2 \) arbitrary constants (including \( a_p, p = 1, \ldots, \dim \Phi \)).

Hence, \( \dim S^{(k)} \leq N_{k,n} < \infty \) for \( k \leq n - 2 \) and thus by (19) for any \( k = 0, 1, \ldots \)
\[
\dim S^{(k)} = \dim S^{(k_0)} + \sum_{j=k_0+1}^{k} \dim S^{(j)}/S^{(j-1)} < \infty.
\] (24)

Thus, for any \( k \) the space \( S^{(k)} \) is finite-dimensional. Since in addition this space is invariant under \( \partial/\partial t \), the dependence of its elements on \( t \) is completely described by Theorem 3.1 [12]. Namely, any symmetry of order \( k \) of KdV-like non-linearizable EE (1) is a linear combination of \( \dim S^{(k)} \) linearly independent symmetries of the form
\[
H = \exp(\lambda t) \sum_{j=0}^{m} t^j h_j(x, u, \ldots, u_k), \ \lambda \in \mathbb{C}, \ m \leq \dim S^{(k)} - 1.
\] (25)
Theorem 6. For any non-linearizable KdV-like EE \( (2) \) \( \dim S^{(k)} = k, k = 0, 1, 2, \ldots \) and any symmetry \( Q \) of order \( k \) is a linear combination of the symmetries \( (26) \). Thus, all the symmetries of non-linearizable KdV-like EEs are linear combinations of quasipolynomials in \( t \). This partially recovers the result of Corollary 4, but this corollary still remains of interest, providing the convenient sufficient condition of polynomiality of symmetries in time \( t \).

It is interesting to note that some general properties of time-dependent symmetries, which are linear combinations of the expressions \( (26) \), were studied by Ma [6]. However, while he considered this form as given a priori, we have proved that all the symmetries of non-linearizable KdV-like EE \( (1) \) indeed have this form.

Moreover, acting on any symmetry \( (25) \) by \( (\partial/\partial t - \lambda)^m \) for \( \lambda \neq 0 \) or by \( \partial^{m-1}/\partial t^{m-1} \) for \( \lambda = 0 \), we obtain the symmetry which is either linear or exponential in \( t \). Hence, there is a very simple test of existence of any time-dependent symmetries for given non-linearizable KdV-like EE \( (1) \). Namely, it suffices to check whether there exist the symmetries of the form

\[
G = \exp(\lambda t)Q_0, \lambda \in \mathbb{C}, \lambda \neq 0
\]  

or of the form

\[
G = G_0 + tG_1, G_1 \neq 0,
\]  

where \( Q_0, G_0 \) and \( G_1 \) are time-independent. If non-linearizable KdV-like EE \( (1) \) (with time-independent coefficients!) has no time-dependent symmetries of the form \( (26) \) or \( (27) \), then it has no time-dependent symmetries at all (but of course it may have time-independent symmetries).

The substitution of \( (26) \) and \( (27) \) into \( (3) \) yields

\[
\{ F, Q_0 \} = \lambda Q_0, \quad \{ F, G_0 \} = G_1, \quad \{ F, G_1 \} = 0.
\]  

In the first case \( F \) is called scaling symmetry (or conformal invariance [13]) of \( Q_0 \). However, known scaling symmetries \( F \) of integrable hierarchies, such as KdV, depend usually only on \( x, u, u_1 \) but not on \( u_2 \) and higher derivatives [13] and hence do not generate EEs of the form \( (1) \), which we consider here. We guess that if KdV-like EE \( (1) \) is non-linearizable and integrable, there exist no functions \( Q_0 \), which satisfy \( (28) \) with \( \lambda \neq 0 \). Moreover, it is believed [4] that in such a case the only polynomial in \( t \) symmetries \( (2) \) that EE \( (1) \) may possess are those linear in \( t \).

Now let us consider the second case. Assume that there exists some commutative algebra \( Alg \) of time-independent symmetries of KdV-like non-linearizable EE \( (1) \), such that for any \( K \in Alg \) the Lie bracket \( \{ G_0, K \} \in Alg. \) Then \( G_0 \) is mastersymmetry of \( (1) \), and hence \( (1) \) possesses (under some extra conditions, vide [3]) the infinite set of time-independent symmetries and is probable to be integrable via inverse scattering transform. Let us mention that the condition of commutativity of \( Alg \) may be rejected if \( G_0 \) is scaling symmetry of \( F \), i.e. \( \{ F, G_0 \} = \mu F \) for some \( \mu \in \mathbb{C}, \mu \neq 0 \) [13].

Conjecture 7. For any KdV-like non-linearizable evolution equation \( (1) \) either all its symmetries are polynomial in \( t \) or all they are linear combinations of exponents in \( t \).
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REFERENCES
[1] V. V. Sokolov: Russian Math. Surveys 43, no. 5 (1988), 165.
[2] P. Olver: Applications of Lie Groups to Differential Equations, Springer, New York 1986.
[3] B. Fuchssteiner: Progr. Theor. Phys. 70 (1983), 1508.
[4] W. X. Ma, P. K. Bullough, P. J. Caudrey and W. I. Fushchych: J. Phys. A: Math. Gen. 30 (1997), 5141; preprint solv-int/9705014 (arXiv.org)
[5] B. Fuchssteiner: J. Math. Phys. 34 (1993), 5140; DOI: 10.1063/1.530295
[6] W. X. Ma: Science in China A 34 (1991), 769.
[7] B. A. Magadeev, V. V. Sokolov: Dinamika Sploshnoj Sredy, 52 (1981) 48 (in Russian).
[8] A. M. Vinogradov, I. S. Krasil’schik, V. V. Lychagin: Introduction to Geometry of Non-linear Differential Equations, Nauka, Moscow 1986 (in Russian).
[9] B. A. Magadeev: St. Petersburg Math. J. 5, no. 2 (1994), 345.
[10] N. H. Ibragimov: Transformation Groups Applied to Mathematical Physics, Reidel, Dordrecht 1985.
[11] A. V. Mikhailov, A. B. Shabat and V. V. Sokolov: The Symmetry Approach to Classification of Integrable Equations in What is Integrability?, V. E. Zakharov ed., Springer, New York 1991.
[12] A. V. Shapovalov, I. V. Shirokov: Theor. Math. Phys. 92 (1992), 697.
[13] W. Oevel: A Geometrical Approach to Integrable Systems Admitting Time-dependent Invariants in Proc. Conf. On Nonlinear Evolution Equations, Solitons and Inverse Scattering Transform, M. Ablowitz et al. eds., Singapore, World Scientific 1987.