The Anti-quark Distribution Function of the Baryon

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We derive the Deep Inelastic anti-quark distribution in a baryon at a low value of \( Q^2 \) using the variational principle of Quantum HadronDynamics, an alternative formulation of Quantum ChromoDynamics. It is determined by a variational approach generalizing the “valence” quark approximation of earlier papers. We find that the “primordial” anti-quarks carry less than a percent of the baryon momentum. In the limit of chiral symmetry and \( N_c \rightarrow \infty \), we show that the anti-quark content of the proton vanishes at low \( Q^2 \).

Keywords: Structure Functions; Parton Model; Deep Inelastic Scattering; Anti-quarks; Sea quarks; QCD; Skyrme model; Quantum HadronDynamics.

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In previous papers we have outlined a way of calculating the Deep Inelastic Structure functions of the baryon from Quantum ChromoDynamics (QCD). After some approximations, (i) Dimensional Reduction to two dimensions, (ii) Ignoring transverse gluon degrees of freedom the theory reduces to two dimensional QCD, which was transformed into a new form called Quantum HadronDynamics (QHD). In this form the basic dynamical variable is a color invariant quantity

\[
\hat{M}(x,y) = \frac{1}{N_c} [\chi_\alpha(x),\chi^{\dagger\alpha}(y)];
\]

(\( \chi, \chi^{\dagger} \) are the annihilation-creation operators of quarks) which can be thought of as the field operator of a meson. The main advantage of this new point of view is that the (iii) semi-classical approximation of QHD corresponds to the large \( N_c \) limit of QCD, and so is capable of describing non-perturbative phenomena such as the structure of hadrons.

The baryon is a topological soliton in this theory and its structure functions (within these approximations) can be determined by a variational principle. In previous papers we made yet another approximation, (iv) the assumption of a factorized ansatz for the classical meson variable

\[
M(x,y) = -2\psi(x)\psi^*(y),
\]

which corresponds to the valence quark approximation in the parton model.

We have already discussed the consequences of relaxing some of these simplifying assumptions. For example the effect of transverse momenta (departure from two dimensionality: relaxing (i)) can be studied within perturbative QCD; indeed as emphasized by Altarelli and Parisi, this is the physical meaning of the usual DGLAP evolution equations for the structure functions. The effect of reinstating transverse gluons (relaxing (ii)) is to produce a slightly more involved two dimensional field theory, which we will study in a separate paper. (This is important to derive the gluon distribution functions of the baryon.) The effect of finite \( N_c \) is (in the leading order) to

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restrict the range of values of the parton momentum. In this paper we will study the departure from the factorized ansatz for $M(x, y)$; in other words, we will study the departure from the valence parton model. This is the same as studying the anti-quark content of the baryon.

We will see that the probability of finding an anti-quark inside a proton is quite small ($< 1\%$) justifying the valence parton approximations made in previous papers. Using a variational ansatz we will obtain the anti-quark distribution functions. These can be used as initial data for evolution in $Q^2$ using the DGLAP equations, which take into account the perturbative corrections. That the initial anti-quark content is quite small, is consistent with the phenomenological model of Glück and Reya: we now have a theoretical derivation of this picture. However, we expect the initial gluon distribution to be non-zero. There are other approaches to studying parton distribution functions, see for example Ref. [7].

Let us begin by summarizing the large $N_c$ limit (which is the classical limit) of QHD. The dynamical variable is a complex valued function $M(x, y)$ of two space-time points $(x, y)$ lying along a null-line. This variable satisfies $M^*(x, y) = M(y, x)$ so that we can regard it as the integral kernel of a hermitean operator on $L^2(R)$. (For technical reasons we assume that this operator is Hilbert-Schmidt; i.e., $\int |M(x, y)|^2dxdy < \infty$.) Moreover, it satisfies the non-linear constraint $[\epsilon + M]^2 = 1$ where the operator $\epsilon$ is the celebrated Hilbert transform operator with $\epsilon^2 = 1$ and the integral kernel $\epsilon(x, y) = \int \text{sgn} (p)e^{ip(x-y)}dp = \frac{1}{\pi} P \frac{\epsilon}{x-y}$. This constraint can be understood as a consequence of the Pauli principle for fermions as explained in [6]. The static solutions of the theory are then the minima of the energy functional

$$E(M) = \frac{1}{N_c} \int M(p, p) \frac{dp}{2\pi} \left[ p + \frac{\mu^2}{p} \right] + \frac{\tilde{g}^2}{8} \int dxdy |M(x, y)|^2 \frac{1}{2} |x - y|$$

subject to the above constraints. The first term is just the kinetic energy in null co-ordinates; the second is the potential energy induced by the longitudinal gluon fields. (Recall that the linear potential $\frac{1}{\tilde{g}} |x - y|$ is the Fourier transform of the gluon propagator $\frac{1}{\tilde{g}^2}$ in two space time dimensions). The parameter $\tilde{g} \sim \lambda_{QCD}$ determines the strength of the strong interactions; also, $\mu^2 = m^2 - \frac{E^2}{\pi}$ is related to the current quark mass $m$ through a finite renormalization. We will be mainly interested in the case $m << \tilde{g}$ which corresponds to the limit of chiral symmetry. It has been shown elsewhere [7] that subject to the above constraints, the energy $E(M)$ is positive: the constraints being crucial for this conclusion. A Lorentz invariant form of the above variational principle is to minimize the invariant mass-squared $M^2$ rather than energy. Since the null momentum of a configuration is $p = -\frac{i}{2} \int p\tilde{M}(p, p) \frac{dp}{2\pi}$ we get

$$\frac{M^2}{N_c} = \left[ -\frac{1}{2} \int p\tilde{M}(p, p) \frac{dp}{2\pi} + \frac{\mu^2}{2p} \frac{2\pi}{2\pi} \right] - \frac{1}{2} \int p\tilde{M}(p, p) \frac{dp}{2\pi} + \frac{\tilde{g}^2}{8} \int dxdy |M(x, y)|^2 \frac{1}{2} |x - y|$$

The quantity $B = -\frac{1}{2} \text{tr} M = -\frac{1}{2} \int M(x, x)dx$ is an integer, a topological invariant of the configuration as shown in [6]. If we reexpress it in terms of the quark fields we can see that this is just the baryon number. Thus, a baryon is a topological soliton in this picture. We can get the structure functions of the baryon by minimizing the energy functional $E(M)$ subject to the above constraints. We have developed a method [8] to solve this problem: a variant of the steepest descent method that takes into account the non-linear constraint. However, this method is computationally intensive. A method based on a variational ansatz that builds on our previous results on the
The conditions 

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\[ \text{L vectors in configurations of higher rank.} \]

will find that in physically interesting situations, even this departure is very small, so we do not need to consider

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Thus the departure from the valence parton picture is determined by the dimensionless ratio \( \xi \) for this solution. Thus the departure from the valence parton picture is determined by the dimensionless ratio \( \xi \), respecting the constraint. In operator language this equation is \( \epsilon + M, \delta M^2 \frac{\delta M}{\delta \epsilon} = 0 \), and can be converted to a nonlinear integral equation. By direct computation, the separable ansatz with \( \tilde{\psi}(p) \sim e^{-\frac{p^2}{2}} \) can then be verified to be an exact solution of this equation, in the limit \( m = 0 \). (Details will be given in a longer paper.) Moreover, \( \mathcal{M}^2 \) is zero for this solution. Thus the departure from the valence parton picture is determined by the dimensionless ratio \( \frac{m^2}{g^2} \) which quantifies chiral symmetry breaking. (As we showed in Ref. 1 the leading effect of finite \( N_c \) is not to depart from the valence parton picture but rather to constrain the range of momenta of the partons.)

Thus we should expect the ‘primordial’ anti-quark distribution in the proton to be small: the up and down quarks have current quark masses small in comparison to \( \Lambda_{\text{QCD}} \), which means that \( \frac{m^2}{g^2} \ll 1 \) as well.

The mathematical advantage of the separable ansatz is that it ‘solves’ the nonlinear constraint on \( M \): more precisely, it replaces it with the condition that \( \psi \) is of norm one. In the same spirit, consider the configuration \( M = \sum_{a,b=1}^r \xi_a^b \psi_a \otimes \psi_b^\dagger \). Here we choose \( \psi_a \) to be a set of \( r \) orthonormal eigenvectors of the operator \( \epsilon \); i.e., \( \epsilon \psi_a = \epsilon_a \psi_a, \epsilon_a = \pm 1 \). This implies that the operator \( M \) is of rank \( r \): the special case of rank one is just the separable ansatz above. This ansatz will satisfy the constraint on \( M \) if the \( r \times r \) matrix \( \xi \) is hermitian and satisfies the constraint \( \xi_a^b \xi_b^c + [\epsilon_a + \epsilon_b] \xi_a^c = 0 \); a ‘mini’ version of the constraint on \( M \). Moreover, the baryon number is \( B = -\frac{1}{2} \text{tr} M = -\frac{1}{2} \text{tr} \xi \).

In the special case of rank one, we have simply \( \xi = -2 \). By choosing a large enough value of \( r \) this ansatz can produce as general a configuration in the phase space as needed: such configurations form a dense subset of the phase space.

The simplest configuration of baryon number one that departs from the separable ansatz is of rank three. We will find that in physically interesting situations, even this departure is very small, so we do not need to consider configurations of higher rank.

By a choice of basis among the \( \psi_a \), we can always bring a rank three configuration of baryon number one to the form \( M = -2 \psi \otimes \psi^\dagger + 2 \zeta_- \left\{ \zeta_- [\psi_- \otimes \psi_+^\dagger - \psi_+ \otimes \psi_-^\dagger] + \sqrt{1 - \zeta_-^2} [\psi_- \otimes \psi_+^\dagger + \psi_+ \otimes \psi_-^\dagger] \right\} \) where \( \psi_-, \psi, \psi_+ \) are three vectors in \( L^2(R) \) satisfying \( \epsilon \psi_- = -\psi_-, \epsilon \psi = \psi, \epsilon \psi_+ = \psi_+, ||\psi_-||^2 = ||\psi_+||^2 = 1, < \psi_-, \psi_+ >= 0 \). The conditions \( < \psi_-, \psi >= < \psi_-, \psi_+ >= 0 \) are then automatic. The parameter \( 0 \leq \zeta_- \leq 1 \) measures the deviation from the rank one ansatz and hence, the anti-quark content of the baryon. For example, baryon number is given by \( B = \int_0^\infty \left\{ ||\psi(p)||^2 + \zeta_-^2 \left[ ||\psi_+(p)||^2 - ||\psi_-(p)||^2 \right] \right\} \frac{dp}{2 \sqrt{p}} \). The wavefunctions \( \psi, \psi_+ \) both describe quarks and their orthogonality can be interpreted as a consequence of the Pauli principle. \( \psi \) describes “valence” quarks while \( \psi_+ \) is the wave function of the “sea” quarks. Since \( \psi_- \) contributes with a negative sign to the baryon number, it describes anti-quarks. From our previous result we expect \( \zeta_- \) to vanish as \( \frac{m^2}{g^2} \to 0 \).

We can substitute this ansatz into the energy \( E(M) \) or \( \mathcal{M}^2 \) and derive integral equations for the minimization. However, in keeping with the spirit of the variational ansatz, we can simplify the problem by assuming first some simple functional forms for the functions \( \tilde{\psi}, \tilde{\psi}_\pm \). The form of the exact solution suggests the choice
\[ \psi(p) = C \left( \frac{p^a}{g} \right) e^{-b \frac{p^2}{2g}}, \psi_+(p) = C_+ \left( \frac{p^a}{g} - C_1 \right) e^{-b \frac{p^2}{2g}} \text{ for } p > 0 \text{ and } \psi_-(p) = \psi(-p) \text{ for } p < 0. \] (For other ranges of \( p \) these functions must vanish.) The parameter \( C_1 \) is determined by the orthogonality condition while \( C, C_+ \) are fixed by the normalization conditions. The variational parameter \( b \) determines the reference frame. The Lorentz invariant quantity \( M^2 \) is independent of \( b \). Thus the variational principle will determine \( a \) and \( \zeta^- \) and hence the wavefunctions.

The rest of the calculation is a straightforward evaluation of the energy integrals and then their minimization. (We use the symbolic package Mathematica for some of the computations, most of which can be done analytically. Some details are provided in [5]). We have done the calculation and shown that for physically interesting values of \( \frac{m_2^2}{g^2} \) \((\sim \frac{m_u d}{\Lambda_{QCD}})^2 \sim .001)\), the parameter \( \zeta^- \) is quite small. We present the results in the figures which show the small effects of deviations from the separable ansatz (i.e. the effects of anti-quarks) and from chiral symmetry. Finally, the effect of finite \( N_c \) is (in the leading order) to restrict the maximum value of parton momenta. We have already studied this correction in the case of the separable ansatz and find it to be small [3,8]. In the case of anti-quarks, we establish that they carry less than a percent of baryon momentum in the \( N_c \to \infty \) limit and therefore, corrections due to finite \( N_c \) are less relevant. They will be addressed in a longer paper.

Thus, we have derived the "primordial" anti-quark distribution function of the proton by a series of approximations from QCD. We have an explanation of why it is small in comparison to the valence quark distribution at the low initial value of \( Q_0^2 \sim 0.4GeV^2 \) [8]. The anti-quark distribution is in fact zero in the limit of chiral symmetry and when \( N_c \to \infty \), while deviations are small. This justifies the valence parton approximations made in earlier papers [4,3,8]. It is possible to compare our prediction with experimental data: there is a specific combination of deep inelastic structure functions that describes anti-quarks [3]. To make a comparison, we need to evolve our distribution from \( Q_0^2 \), according to the DGLAP equations. However, it is necessary to know the initial gluon distribution in order to solve the evolution equations. We will study the gluon distribution in a later paper and subsequently return to this issue.

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Figure 1: Variational estimates for (a) the exponent ‘\( a \)’ and (b) \( \zeta^- \) as a percent. They are plotted as functions of \( nu = 1000 * \frac{m_2^2}{g^2} \). The exponent \( a \) and the anti-quark content \( \zeta^- \) go to zero for small current quark masses.

Figure 2: (a) Variational estimate for the fraction of fermion momentum of the baryon carried by anti-quarks. It is plotted as a percent as a function of \( nu \). The “primordial” anti-quarks carry less than a percent of the portion
of baryon momentum shared between quarks and anti-quarks. (b) Variational upper-bound on the invariant \(M^2\), of the baryon in the two-dimensional approximation, plotted as a function of \(nu\). In the limit of chiral symmetry, we recover the exact exponential solution with \(\frac{M^2}{g^2 N_c} = 0\), \(\zeta_- = 0\) and \(a = 0\).

Figure 3: (a) The “valence” quark \((x_B \psi |^2)\), (b) anti-quark \((x_B |\psi^-|^2)\) and (c) “sea” quark \((x_B |\psi^+|^2)\) distributions plotted as a function of momentum fraction \(x_B = \frac{p}{P}\) at low \(Q^2\) (\(\sim 0.4 GeV^2\), see Ref. [8]). The exponential tails beyond \(x_B = 1\) are an artifact of the large-\(N_c\) limit. They are plotted for a small value of current quark mass \((m_u, d) \Lambda_{QCD} \sim 0.001\) in the reference frame in which the mean baryon momentum, \(P\) is 1. The fermions are assumed to carry \(f = \frac{1}{2}\) the mean baryon momentum. The rest is carried by gluons [8]. The node in the “sea” quark wavefunction is because it is required to be orthogonal to the “valence” quark wavefunction by the Pauli principle. The valence quark distribution shown above \((x_B*Valence = xF3)\), though calculated in the limit \(N_c \to \infty\) agrees well with the distribution obtained after taking into account the leading \(\frac{1}{N_c}\) corrections and also with experimental measurements of the neutrino structure function \(xF3(x_B, Q^2)\) when evolved to higher values of \(Q^2\) [8].

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