Universal behavior of crossover scaling functions for continuous phase transitions

S. Lübeck

Weizmann Institute of Science, Department of Physics of Complex Systems, 76100 Rehovot, Israel,
Institut für Theoretische Physik, Universität Duisburg-Essen, 47048 Duisburg, Germany
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We consider two different systems exhibiting a continuous phase transition into an absorbing state. Both models belong to the same universality class, i.e., they are characterized by the same scaling functions and the same critical exponents. Varying the range of interactions we examine the crossover from the mean-field-like to the non-mean-field scaling behavior. A phenomenological scaling form is applied in order to describe the full crossover region which spans several decades. Our results strongly supports the hypotheses that the crossover function is universal.

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The critical behavior of a system exhibiting a second order phase transition with non-mean-like scaling behavior is strongly affected by the range of interactions. The longer the range of interactions the stronger will be the critical fluctuations reduced. In the limit of infinite interactions the system is characterized by the mean-field scaling behavior. But according to the well known Ginzburg criterion \cite{1}, mean-field-like behavior occurs even for finite interaction ranges sufficiently far away from the critical point. A crossover to the non-mean-field scaling behavior takes place if one approaches the transition point. Although crossover phenomena are well understood in terms of competing fixed points of the corresponding renormalization group approaches (see for instance \cite{2}), some aspects of crossover phenomena are still open. For instance it is an open question whether the so-called effective exponents fulfill certain scaling relations over the entire crossover region (see \cite{3,4,5,6} and references therein). A second open question is the main theme of this paper and addresses the universality of the crossover scaling functions. The range where the universal critical scaling behavior applies is usually restricted to a small vicinity around the critical point. Therefore it is questioned that the full crossover region, that spans several decades in temperature or conjugated field, can be described in terms of universal scaling functions. Renormalization group approaches predicted a non-universal behavior if one uses finite cutoff lengths whereas infinite cutoff lengths (which corresponds to an unphysical vanishing molecular size) lead to a universal crossover behavior (see for instance \cite{7,8}). On the other hand the experimental situation is also unclear since measurements over the whole crossover region are difficult and accurate results are rare (see \cite{9} for a short discussion). Thus several attempts were performed in order to address this question via numerical simulations. For instance the two- and three-dimensional Ising model with various interaction ranges is considered in a series of papers \cite{10,11,12,13}. Using a sophisticated cluster algorithm for long-range interactions it was possible to cover the full crossover region. In particular a collapse of the susceptibility for different values of the interaction range was observed. Thus the crossover can be described by a single scaling function in agreement with renormalization group approaches. But this result does not present an evidence that this scaling function is universal, since only one system of a given universality class was considered.

The purpose of this paper is to demonstrate via numerical simulations that the crossover from non-mean-field to mean-field-like scaling behavior can be represented by universal functions. We therefore consider two different systems exhibiting a continuous phase transition; both belong to the same universality class. The dynamics of the models is characterized by simple particle hopping processes, i.e., various interaction ranges can be easily implemented and highly accurate data are available. In this way it is possible to observe the full crossover region. Notice that we focus in our investigations on the particular universality class of absorbing phase transitions only for technical reasons. The demonstrated universality of crossover scaling functions can be applied to continuous phase transitions in general.

The first considered model is the so-called conserved lattice gas (CLG) which was introduced in \cite{11}. In the CLG lattice sites may be empty or occupied by one particle. In order to mimic a repulsive interaction a given particle is considered as active if at least one of its neighboring sites on the lattice is occupied by another particle. If all neighboring sites are empty the particle remains inactive. Active particles are moved in the next update step to one of their empty nearest neighbor sites, selected at random.

The second model is the so-called conserved transfer threshold process (CTTP) \cite{11}. Here, lattice sites may be empty, occupied by one particle, or occupied by two particles. Empty and single occupied sites are considered as inactive whereas double occupied lattice sites are considered as active. In the latter case one tries to transfer both particles of a given active site to randomly chosen empty or single occupied nearest neighbor sites.

In our simulations (see \cite{12,13} for details) we have used square lattices of linear size $L \leq 2048$. Every simulation
starts from a random distribution of particles. After a transient regime both models reach a steady state characterized by the density of active sites \( \rho_a \). The density \( \rho_a \) is the order parameter and the particle density \( \rho \) is the control parameter of the absorbing phase transition, i.e., the order parameter vanishes at the critical density \( \rho_c \) according to, \( \rho_a \propto \delta \rho^\beta \), with the reduced control parameter \( \delta \rho = \rho/\rho_c - 1 \). Additionally to the order parameter we consider its fluctuations \( \Delta \rho_a \). Approaching the transition point from above (\( \delta \rho > 0 \)) the fluctuations diverge according to (see [12, 13]) \( \Delta \rho_a \propto \delta \rho^{-\gamma} \). Below the critical density (in the absorbing phase) the order parameter as well as its fluctuations are zero in the steady state.

It was shown recently that the order parameter as well as its fluctuations obey the scaling forms [14]

\[
\begin{align*}
\rho_a(\delta \rho, h) &\sim \lambda^\beta \tilde{R}(a_c \delta \rho, \lambda, a_h \lambda^\gamma), \\
a_\Delta \Delta \rho_a(\delta \rho, h) &\sim \lambda^{-\gamma} \tilde{D}(a_c \delta \rho, \lambda, a_h \lambda^\gamma),
\end{align*}
\]

where \( h \) denotes an external field which is conjugated to the order parameter \( x \). The universal scaling functions \( \tilde{R}(x, y) \) and \( \tilde{D}(x, y) \) are the same for all systems belonging to a given universality class whereas all non-universal system-dependent features (e.g., the lattice structure, the update scheme, etc.) are contained in the so-called non-universal metric factors \( a_c, a_h \), and \( a_\Delta \). The universal scaling functions are normed by the conditions \( \tilde{R}(1,0) = \tilde{R}(0,1) = \tilde{D}(0,1) = 1 \) and the non-universal metric factors can be determined from the amplitudes of

\[
\begin{align*}
\rho_a(\delta \rho, h = 0) &\sim (a_c \delta \rho)^\beta, \\
\rho_a(\delta \rho = 0, h) &\sim (a_h h)^{\beta/\sigma}, \\
a_\Delta \Delta \rho_a(\delta \rho = 0, h) &\sim (a_h h)^{-\gamma/\sigma}.
\end{align*}
\]

These equations are obtained by choosing in the scaling forms [Eqs. (1, 2)] \( a_c \delta \rho \lambda = 1 \) and \( a_h h \lambda^\sigma = 1 \), respectively.

Usually scaling functions are only known above the upper critical dimension \( D_c \) where the mean-field theory applies. In the case of the CLG model and CTTP the mean-field scaling functions are given by [14, 16] \( \tilde{R}_{MF}(x, y) = x/2 + (y + (x/2)^2)^{1/2} \) as well as \( \tilde{D}_{MF}(x, y) = R_{MF}(x, y)/(y + (x/2)^2)^{1/2} \) i.e., the mean-field exponents are \( \beta_{MF} = 1, \sigma_{MF} = 2 \), and \( \gamma_{MF} = 0 \) (corresponding to a finite jump of the fluctuations). Below the upper critical dimension the universal scaling functions depend on the dimension and are unknown due to a lack of analytical solutions.

In the original CLG model and the original CTTP particles of active sites are moved to nearest neighbors only, i.e., the range of interactions is \( R = 1 \). In the following we consider a modified CLG model and a modified CTTP where particles of active sites are moved (according to the rules of each model) to randomly selected sites within a radius \( R \). The order parameter is plotted in Fig. 1 for various ranges of interactions \( (R \in \{1, 2, 4, \ldots, 128\}) \). In the following we examine how the varying interaction range affects the scaling behavior in the vicinity of the absorbing phase transition which now takes place at the critical density \( \rho_{c,R} \).

The crossover scaling function at zero field has to incorporate the range of interactions as an additional scaling field. We make the phenomenological ansatz

\[
\rho_a(\rho, R_{eff}) \sim \lambda^{-\beta_{MF}} \tilde{R}(a_c (\rho - \rho_{c,R}) \lambda, a_{R_{eff}}^{-1} R_{eff}^{-1} \lambda^\sigma),
\]

where the scaling function \( \tilde{R} \) is universal since we allow for the non-universal metric factors \( a_c \) and \( a_{R_{eff}} \). The Ginzburg criterion states that the mean-field picture is self-consistent in the active phase as long as the fluctuations within a correlation volume are small compared to the order parameter itself (see [17]). Thus the crossover exponent is given by \( \phi = (2\beta_{MF} - \nu_{MF} D)/D = (4 - D)/2D \), where \( \nu_{MF} = 1/2 \) denotes the critical exponents of the spatial correlation length. In order to avoid lattice effects we use the effective interaction range \( R_{eff} \)

\[
R_{eff}^2 = \frac{1}{z} \sum_{i \neq j} |x_i - x_j|^2, \quad |x_i - x_j| \leq R
\]

where \( z \) denotes the number of lattice sites within a radius \( R \) (see Table 1). The mean-field scaling behavior should be recovered for \( R \to \infty \), thus

\[
\tilde{R}(x, 0) = \tilde{R}_{MF}(x, 0) = x^{\beta_{MF}}
\]

which implies \( a_c = a_{R_{eff}}^{-1} \rho_{c,R_{eff}}^{-1} \). These factors were already determined in previous works where absorbing phase transitions with infinite particle hopping were investigated [16]. The non-universal metric factor \( a_R \) has
to be determined by a second condition. Several ways are possible (e.g. $\hat{R}(0, 1) = 1$) but for the sake of convenience we force that $\hat{R}$ scales as

$$\hat{R}(x, 1) \sim x^{-\beta_D}, \quad \text{for } x \to 0,$$

where $\beta_D$ denotes the non-mean-field order parameter exponent of the corresponding D-dimensional system. Setting $a^{-1}_r R^{-1}_\text{eff} \lambda^\phi = 1$ in Eq. (6) yields for zero field

$$\rho_s(\rho, R_\text{eff}) \sim (a_r R_\text{eff})^{-\beta_\text{MF}/\phi} \tilde{\Phi}(a_s(\rho - \rho_c) a^{-1}_r R^{-1}_\text{eff}, 1).$$

Taking into account that the D-dimensional scaling behavior is recovered for $R = 1$ we find

$$a_r = \left(\frac{\rho_c, R = 1, a_{p, R = \infty}}{\rho_c, R = 1, \rho_{c, R = \infty}}\right) \frac{\phi_D}{(\beta_\text{MF} - \beta_D)}.$$

According to the above scaling form we plot in Fig. 2 the rescaled order parameter $\rho_s (a_r R_\text{eff})^2$ as a function of the rescaled control parameter $a_s(\rho - \rho_c)(a_r R_\text{eff})^2$ for the two-dimensional (\(\phi = 1/2\)) CLG model and the two-dimensional CTTP. The values of the metric factors are listed in Table I and are determined from data of previous simulations (via a direct measurement of the amplitudes of the corresponding power-laws). Thus no parameter fitting is applied. We observe an excellent data collapse for the entire range of the crossover confirming the phenomenological ansatz. In the inset of Fig. 2 we plot the same data without metric factors. As can be seen each model is characterized by its own scaling function.

Since the entire crossover region covered several decades it could be difficult to observe small but systematic differences between the scaling functions of both models. It its therefore instructive to examine the crossover via the so-called effective exponent $\hat{R}$

$$\beta_\text{eff} = \frac{1}{\partial \ln x} \ln \hat{R}(x, 1).$$

The corresponding data are shown in Fig. 2. The excellent data collapse of $\beta_\text{eff}$ of both models over more than 7 decades strongly supports the hypothesis that the crossover function is a universal function.

We now consider the order parameter fluctuations. Analogous to the order parameter we make the scaling ansatz ($\gamma_\text{MF} = 0$)

$$a_\Delta \Delta \rho_s(\rho, R_\text{eff}) \sim \tilde{D}(a_s(\rho - \rho_c) a^{-1}_r R^{-1}_\text{eff} \lambda^\phi).$$

Again the mean-field behavior should be recovered for $R \to \infty$, implying $\tilde{D}(x, 0) = \tilde{D}_\text{MF}(x, 0) = 2$ as well as $a_\Delta = a_\Delta, R = \infty$. Setting $a^{-1}_r R^{-1}_\text{eff} \lambda^\phi = 1$ yields

$$a_\Delta \Delta \rho_s(\rho, R_\text{eff}) \sim \tilde{D}(a_s(\rho - \rho_c) a^{-1}_r R^{-1}_\text{eff}, 1).$$

For finite $R$, the fluctuations diverge at the critical point, i.e., the universal function $\tilde{D}$ scales as

$$\tilde{D}(x, 1) \sim m_{\Delta, \rho} x^{-\gamma_D}, \quad \text{for } x \to 0.$$ (15)

The universal amplitude $m_{\Delta, \rho}$ can be determined in the following way: The scaling form Eq. (13) has to equal for $R = 1$ the D-dimensional scaling behavior [see Eq. (14)]

$$\Delta \rho_s \sim a^{-1}_{\Delta, R = 1} \tilde{D}(1, 0) \left(\frac{\rho_c, R = 1}{\rho_c, R = 1} \right)^{-\gamma_D'/D}.$$

Thus we find

$$m_{\Delta, \rho} = \tilde{D}(1, 0) \frac{a_{\Delta, R = \infty}}{a_{\Delta, R = 1}} \left(\frac{\rho_c, R = \infty}{\rho_c, R = 1} \right)^{\gamma_D'/\gamma_\text{MF}}.$$ (17)
The inset of Fig. 1 shows the corresponding data for the CTTP. As can be seen, the data of various interaction ranges tend to the same power-law behavior if one approaches the transition point.

In conclusion, the crossover from mean-field to non-mean-field scaling behavior is numerically investigated for two different models exhibiting an absorbing phase transition. Increasing the range of interactions we are able to cover the full crossover region which spans several decades of the control parameter. The excellent collapse of the effective exponents of both models strongly supports the interpretation of the crossover scaling functions in terms of universality, i.e., the crossover function is universal.

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where the value of the universal scaling function \( D(1, 0) = 1.87 \pm 0.11 \) is obtained via direct measurements of the corresponding two-dimensional systems. According to the scaling form Eq. (13) we plot in Fig. 3 the rescaled fluctuations as a function of the rescaled control parameter for the two-dimensional CLG model as well as for the CTTP. We observe again a good collapse of the data over the entire region of the crossover. Furthermore, both asymptotic behaviors are recovered, confirming the scaling ansatz Eq. (13).

The corresponding effective exponent \( \gamma_{\delta \Delta} \) is \( \partial \ln D(x, 1) / \partial \ln x \) displayed in the inset of Fig. 3. Although the data of the effective exponent are suffering from statistical fluctuations one can see that both models are characterized by the same universal behavior.

At the end we consider the critical amplitudes of the scaling functions. Using the above discussed scaling forms it easy to show that these amplitudes display a singular dependence on the range of interactions. For instance the order parameter scales sufficiently close to the transition point \( (x \to 0) \) as [see Eqs. (1) (13)]

\[
\rho_c(\rho, R_{\text{eff}}) \sim R_c^{(\beta_D - \beta_{\text{MF}})} \left( \frac{\rho - \rho_c, R_{\text{eff}}}{\rho_c, R_{\text{eff}}} \right)^{\beta_D}
\]

(18)

Thus this scaling law and the corresponding scaling law for the fluctuations are only valid for finite interaction ranges whereas they become useless for infinite \( R \), signaling the change in the universality class for \( R \to \infty \). This amplitude scaling can be observed in simulations.

The following table shows the non-universal metric factors determined from previous simulations via direct measurements of the corresponding power-laws.

| Model | \( \rho_c, R_{\text{eff}} \) | \( a_{\Delta, R_{\text{eff}}} \) | \( \Delta c, R_{\text{eff}} \) | \( \rho_c, R_{\infty} \) | \( a_{\Delta, R_{\infty}} \) | \( \Delta c, R_{\infty} \) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|
| CLG   | 0.34494 | 0.5089 | 15.50 | 0.1244 | 0.1635 | 12.02 |
| CTTP  | 0.69392 | 0.3410 | 50.18 | 1/2 | 0.3345 | 24.85 |