ALTERNATIVE DERIVATION OF THE HU-PAZ-ZHANG
MASTER EQUATION OF QUANTUM BROWNIAN MOTION

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Abstract

Hu, Paz and Zhang [B.L. Hu, J.P. Paz and Y. Zhang, Phys. Rev. D 45 (1992) 2843] have derived an exact master equation for quantum Brownian motion in a general environment via path integral techniques. Their master equation provides a very useful tool to study the decoherence of a quantum system due to the interaction with its environment. In this paper, we give an alternative and elementary derivation of the Hu-Paz-Zhang master equation, which involves tracing the evolution equation for the Wigner function. We also discuss the master equation in some special cases.

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I. INTRODUCTION

Quantum Brownian motion (QBM) models provide a paradigm of open quantum systems that has been very useful in quantum measurement theory [1], quantum optics [2] and decoherence [3-5]. One of the advantages of the QBM models is that they are reasonably simple, yet sufficiently complex to manifest many important features of realistic physical processes.

Central to the study of QBM is the master equation for the reduced density operator of the Brownian particle, derived by tracing out the environment in the evolution equation for the combined system plus environment. A variety of such derivation have been given [6-9]. The most general is that of Hu, Paz and Zhang [10,11], who used path integral techniques and in particular, the Feynman-Vernon influence functional.

The purpose of this paper is to provide an alternative and elementary derivation of the Hu-Paz-Zhang master equation for QBM, by tracing the evolution equation for the Wigner function of the whole system.

II. MASTER EQUATION FOR QUANTUM BROWNIAN MOTION

The system we considered is a harmonic oscillator with mass $M$ and bare frequency $\Omega$, in interaction with a thermal bath consisting of a set of harmonic oscillators with mass $m_n$ and natural frequency $\omega_n$. The Hamiltonian of the system plus environment is given by

$$H = \frac{p^2}{2M} + \frac{1}{2} M \Omega^2 q^2 + \sum_n \left( \frac{p_n^2}{2m_n} + \frac{1}{2} m_n \omega_n^2 q_n^2 \right) + q \sum_n C_n q_n ,$$

where $q, p$ and $q_n, p_n$ are the coordinates and momenta of the Brownian particle and oscillators, respectively, and $C_n$ are coupling constants.

The state of the combined system (1) is most completely described by a density matrix $\rho(q, q_i; q', q_i', t)$ where $q_i$ denotes $(q_1, ..., q_N)$, and $\rho$ evolves according to

$$\dot{\rho} = -\frac{i}{\hbar} [H, \rho] .$$
The state of the Brownian particle is described the reduced density matrix, defined by tracing over the environment,

\[ \rho_r(q, q', t) = \int \prod_n (dq_n dq'_n \delta(q_n - q'_n)) \rho(q, q; q', q', t). \]  

(3)

The equation of time evolution for the reduced density matrix is called the master equation. For a general environment, Hu, Paz, and Zhang [10] derived the following master equation by using path integral techniques:

\[
\begin{align*}
\frac{i\hbar}{\partial t} \rho_r & = -\frac{\hbar^2}{2M} \left( \frac{\partial^2 \rho_r}{\partial q^2} - \frac{\partial^2 \rho_r}{\partial q'^2} \right) + \frac{1}{2} M \Omega^2 (q^2 - q'^2) \rho_r \\
& + \frac{1}{2} M \delta \Omega^2(t) (q^2 - q'^2) \rho_r \\
& - i\hbar \Gamma(t) (q - q') \left( \frac{\partial \rho_r}{\partial q} - \frac{\partial \rho_r}{\partial q'} \right) \\
& - iM \Gamma(t) h(t) (q - q')^2 \rho_r \\
& + h \Gamma(t) f(t) (q - q') \left( \frac{\partial \rho_r}{\partial q} + \frac{\partial \rho_r}{\partial q'} \right). 
\end{align*}
\]

(4)

The explicit form of the coefficients of the above equation will be given later on. The coefficient \( \delta \Omega^2(t) \) is the frequency shift term, the coefficients \( \Gamma(t) \) is the “quantum dissipative” term, and the coefficients \( \Gamma(t) h(t), \Gamma(t) f(t) \) are “quantum diffusion” terms. Generally, these coefficients are time dependent

and of quite complicated behaviour.

We find it convenient to use the Wigner function of the reduced density matrix,

\[ \tilde{W}(q, p, t) = \frac{1}{2\pi} \int du \ e^{iup/\hbar} \rho_r \left( q - \frac{u}{2}, q + \frac{u}{2}, t \right). \]  

(5)

Taking the Wigner transform of (4), we obtain

\[
\begin{align*}
\frac{\partial \tilde{W}}{\partial t} & = -\frac{1}{M^p} \frac{\partial \tilde{W}}{\partial q} + M[\Omega^2 + \delta \Omega^2(t)] q \frac{\partial \tilde{W}}{\partial p} \\
& + 2\Gamma(t) \frac{\partial (p \tilde{W})}{\partial p} + hM \Gamma(t) h(t) \frac{\partial^2 \tilde{W}}{\partial p^2} \\
& + h \Gamma(t) f(t) \frac{\partial^2 \tilde{W}}{\partial q \partial p}. 
\end{align*}
\]

(6)

\[ ^1 \text{We believe Eq. (2.48) in Ref. [10] contains some incorrect numerical factors.} \]
The inverse transformation of (5) is given by
\[
\rho_r(q, q', t) = \int dp \, e^{-ip(q-q')/\hbar} \tilde{W}\left(\frac{q + q'}{2}, p, t\right).
\]
(7)

Our strategy for deriving the master equation (4) is to derive the Fokker-Planck type equation (6) from the Wigner equation for the total system. The master equation can be obtained from the Wigner equation for the system by using the transformation (7).

We shall make the following two assumptions:

(1) The system and the environment are initially uncorrelated, \textit{i.e.} the initial Wigner function factors
\[
W_0(q, p; q_i, p_i) = W_s^0(q, p)W_b^0(q_i, p_i),
\]
where \(W_s^0\) and \(W_b^0\) are the Wigner functions of the system and the bath, respectively, at \(t = 0\).

(2) The heat bath is initially in a thermal equilibrium state at temperature \(T = (k_B \beta)^{-1}\).

This means that the initial Wigner function of bath is of Gaussian form,
\[
W_0^b = \prod_n W_n^b = \prod_n N_n \exp\{-\frac{2}{\omega_n \hbar} \tanh\left(\frac{1}{2} \hbar \omega_n \beta\right) H_n\},
\]
where \(H_n\) is the Hamiltonian of the \(n\)-th oscillator in the bath,
\[
H_n = \frac{p_n^2}{2m_n} + \frac{1}{2} m_n \omega_n^2 q_n^2.
\]
(10)

In addition, one easily see that the initial moments of the bath are
\[
\langle q_n(0) \rangle = \langle p_n(0) \rangle = 0,
\]
(11)
\[
\langle q_n(0) q_m(0) \rangle = 0 \text{ (if } m \neq n \text{)},
\]
(12)
\[
\langle p_n(0) p_m(0) \rangle = 0 \text{ (if } m \neq n \text{)},
\]
(13)
\[
\langle q_n(0) p_m(0) + p_m(0) q_n(0) \rangle = 0,
\]
(14)
and
\[ \langle q_n^2(0) \rangle = \frac{\hbar}{2m_n\omega_n} \coth\left(\frac{1}{2}\hbar\omega_n\beta\right), \]
\[ \langle p_n^2(0) \rangle = \frac{1}{2}\hbar m_n\omega_n \coth\left(\frac{1}{2}\hbar\omega_n\beta\right). \quad (15) \]

For the QBM problem described by (1) and (2), the Wigner function of the combined system plus environment satisfies

\[ \frac{\partial W}{\partial t} = -\frac{p}{M} \frac{\partial W}{\partial q} + M\Omega^2 q \frac{\partial W}{\partial p} + \sum_n \left( -\frac{p_n}{m_n} \frac{\partial W}{\partial q_n} + m_n\omega_n^2 q_n \frac{\partial W}{\partial p_n} \right) + \sum_n C_n \left( q_n \frac{\partial W}{\partial p} + q \frac{\partial W}{\partial p_n} \right). \quad (16) \]

By integrating over the bath variables on the both sides of the above equation, one obtains

\[ \frac{\partial \tilde{W}}{\partial t} = -\frac{p}{M} \frac{\partial \tilde{W}}{\partial q} + M\Omega^2 q \frac{\partial \tilde{W}}{\partial p} + \sum_n C_n \int \prod_i dq_i dp_i q_n \frac{\partial W}{\partial p}, \quad (17) \]

where \( \tilde{W}(q, p) \) is the reduced Wigner function and it follows from Eq. (3) that

\[ \tilde{W}(q, p) = \int_{-\infty}^{+\infty} \prod_i dq_i dp_i W(q, p; q_i, p_i). \quad (18) \]

This definition is equivalent to Eqs. (3) and (5). The first two terms on the right-hand side of the Eq. (17) give rise to the standard evolution equation of the system. The last term contains all the information about the behaviour of the system in the presence of interaction with environment.

In what follows, we shall demonstrate that the quantity

\[ G(q, p) = \sum_n C_n \int \prod_i dq_i dp_i q_n W \]

appearing (differentiated with respect to \( p \)) in (17) can be expressed in terms of \( \tilde{W} \) and its derivatives. To this end, we first perform Fourier transform of \( G(q, p) \)

\[ G(k, k') = \int dq dp e^{ikq+ik'p} G(q, p) = \sum_n C_n \int dq dp \prod_i dq_idp_i q_n e^{ik_q+ik'_p} W(q, p; q_i, p_i). \quad (20) \]
It is well known that $q(t), p(t)$ and $q_n(t), p_n(t)$ are related to the classical evolution of their initial values $q(0), p(0)$ and $q_n(0), p_n(0)$ through a canonical transformation:

$$z(t) = U(t)z(0),$$

where

$$z(t) = (q(t), q_1(t)...q_N(t); p(t), p_1(t)...p_N(t)).$$

Since the Hamiltonian (1) is quadratic, the Eq. (16) has the same form as the classical Liouville equation, so the solution of Eq. (16) is of the form,

$$W_t(z) = W_0(U^{-1}(t)z).$$

Changing the integration variables into their initial values by this canonical transformation, we obtain

$$G(k, k') = \int dq(0)dp(0) \prod_i dq_i(0)dp_i(0)$$

$$\times \left[ f q(0) + g p(0) + \sum_n (f_n q_n(0) + g_n p_n(0)) \right]$$

$$\times \exp \left[ ik \left( \alpha q(0) + \beta p(0) + \sum_n (a_n q_n(0) + b_n p_n(0)) \right) \right]$$

$$\times \exp \left[ ik' M \left( \dot{\alpha} q(0) + \dot{\beta} p(0) + \sum_n (\dot{a}_n q_n(0) + \dot{b}_n p_n(0)) \right) \right]$$

$$\times W_0^s(q(0), p(0)) W_0^b(q_i(0), p_i(0)).$$

Here the coefficients $f, g, f_n, g_n, \alpha, \beta, a_n, b_n$ are time dependent. Their explicit values are not required.

Similarly, the Fourier transform of the reduced Wigner function is

$$\tilde{W}(k, k') = \int dq dp e^{ikq + ik'p} \tilde{W}(q, p)$$

$$= \int dq(0)dp(0) \prod_i dq_i(0)dp_i(0)$$

$$\times \exp \left[ ik \left( \alpha q(0) + \beta p(0) + \sum_n (a_n q_n(0) + b_n p_n(0)) \right) \right]$$

$$\times \exp \left[ ik' M \left( \dot{\alpha} q(0) + \dot{\beta} p(0) + \sum_n (\dot{a}_n q_n(0) + \dot{b}_n p_n(0)) \right) \right]$$

$$\times W_0^s(q(0), p(0)) W_0^b(q_i(0), p_i(0)).$$

(23)
Now compare $G(k, k')$ and $\tilde{W}(k, k')$. They differ by the terms linear in $q(0), p(0), q_n(0), p_n(0)$ in the preexponential factor in $G(k, k')$. Consider the factors $f_n q_n(0)$ and $g_n p_n(0)$ in $G(k, k')$. Since they multiply $W^b_0(q_i(0), p_i(0))$, and since $W^b_0(q_i(0), p_i(0))$ is Gaussian in $q_n(0), p_n(0)$, the terms $f_n g_n(0) W^b_0$ and $g_n p_n(0) W^b_0$ may be replaced by terms of the form $\partial W^b_0 / \partial q_n(0), \partial W^b_0 / \partial p_n(0)$ up to time dependent factors. An integration by parts then may be performed, and these factors are then effectively replaced by multiplicative factors of $k, k'$.

Similarly, the factors $f q(0), g p(0)$ in the prefactor in $G(k, k')$ may be replaced by $\partial / \partial k, \partial / \partial k'$ (plus some more factors of $k$ and $k'$). Hence, it is readily seen that $G(k, k')$ is a linear combination of terms of the form $k, k', \partial / \partial k, \partial / \partial k'$ operating on $\tilde{W}(k, k')$, with time dependent coefficients.

Inverting the Fourier transform, it follows that

$$G = A(t) q \tilde{W} + B(t) p \tilde{W} + C(t) \frac{\partial \tilde{W}}{\partial q} + D(t) \frac{\partial \tilde{W}}{\partial p}. \quad (25)$$

for some coefficients $A(t), B(t), C(t), D(t)$ to be determined. This result immediately leads to the general form Wigner equation :

$$\frac{\partial \tilde{W}}{\partial t} = -\frac{p}{M} \frac{\partial \tilde{W}}{\partial q} + M \Omega^2 q \frac{\partial \tilde{W}}{\partial p} + A(t) q \frac{\partial \tilde{W}}{\partial p}$$

$$+ B(t) \frac{\partial (p \tilde{W})}{\partial p} + C(t) \frac{\partial^2 \tilde{W}}{\partial p \partial q} + D(t) \frac{\partial^2 \tilde{W}}{\partial p^2}. \quad (26)$$

### III. Determination of the Coefficients (General Case)

Having found the functional form of the Wigner equation (26) of the Brownian particle, the next step is to determine the coefficients in the equation. Undoubtedly, there is more than one way to do this. Here we shall choose a way which is both mathematically simple and physically heuristic. Towards this direction, let us consider the time evolution of the expectation values of the system variables: $q, p, q^2, p^2$ and $\frac{1}{2}(pq + qp)$.

By using Eq. (16), we have
\[ \frac{d}{dt} \langle q \rangle = \frac{1}{M} \langle p \rangle, \quad (27) \]
\[ \frac{d}{dt} \langle p \rangle = -M\Omega^2 \langle q \rangle - \sum_n C_n \langle q_n \rangle, \quad (28) \]
\[ \frac{d}{dt} \langle q^2 \rangle = \frac{1}{M} \langle pq + qp \rangle, \quad (29) \]
\[ \frac{d}{dt} \langle p^2 \rangle = -M\Omega^2 \langle pq + qp \rangle - 2 \sum_n C_n \langle pq_n \rangle, \quad (30) \]
\[ \frac{d}{dt} \langle pq + qp \rangle = \frac{2}{M} \langle p^2 \rangle - 2M\Omega^2 \langle q^2 \rangle - 2 \sum_n C_n \langle qq_n \rangle. \quad (31) \]

Similarly, using Eq. (26) yields
\[ \frac{d}{dt} \langle q \rangle = \frac{1}{M} \langle p \rangle, \quad (32) \]
\[ \frac{d}{dt} \langle p \rangle = -(M\Omega^2 + A) \langle q \rangle - B \langle p \rangle, \quad (33) \]
\[ \frac{d}{dt} \langle q^2 \rangle = \frac{1}{M} \langle pq + qp \rangle, \quad (34) \]
\[ \frac{d}{dt} \langle p^2 \rangle = -(M\Omega^2 + A) \langle pq + qp \rangle - 2B \langle p^2 \rangle + 2D, \quad (35) \]
\[ \frac{d}{dt} \langle pq + qp \rangle = \frac{2}{M} \langle p^2 \rangle - 2(M\Omega^2 + A) \langle q^2 \rangle - B \langle pq + qp \rangle + 2C. \quad (36) \]

Since the evolution equations of the expectation values are confined to the system variables, the above two sets of equations must be identical.

Now by comparing (28) with (33) we see that
\[ \sum_n C_n \langle q_n \rangle = A \langle q \rangle + B \langle p \rangle. \quad (37) \]

Similarly, by comparing (31) with (36), (30) with (35), respectively, we get
\[ \sum_n C_n \langle qq_n \rangle = A \langle q^2 \rangle + \frac{B}{2} \langle qp + pq \rangle - C, \quad (38) \]
\[ \sum_n C_n \langle pq_n \rangle = \frac{A}{2} \langle pq + qp \rangle + B \langle p^2 \rangle - D. \quad (39) \]

The coefficients \( A, B, C, D \) may now be determined from (37)-(39) by regarding the expectation values \( \langle q \rangle, \langle q_n q \rangle \) etc. as expectation values of Heisenberg picture operators, and by solving the operator equation of motion. For simplicity, we still use ordinary notation to represent an operator without adding a hat on it.
The solution to the equation of motion may be written,

\[q_n(t) = \alpha_n q(t) + \beta_n p(t) + \sum_m (a_{nm} q_m(0) + b_{nm} p_m(0)) , \quad (40)\]
\[q(t) = \alpha q(0) + \beta p(0) + \sum_n (a_n q_n(0) + b_n p_n(0)) , \quad (41)\]

for some time-dependent coefficients \(\alpha_n, \beta_n, a_{nm}, b_{nm}, \alpha, \beta, a_n, b_n\). Note that \(q_n(t)\) has been expressed in terms of the final, not initial values of \(q, p\). By substituting Eq. (40) into Eq. (37), keeping (11) in mind, and comparing the two sides of the resulting equation, we have

\[A = \sum_n C_n \alpha_n , \quad B = \sum_n C_n \beta_n . \quad (42)\]

Similarly, substituting (40) and (41) into (38) and (39), respectively, we get

\[C = -\sum_{mn} C_n (a_{nm} a_m \langle q_m^2(0) \rangle + b_{nm} b_m \langle p_m^2(0) \rangle) , \quad (43)\]
\[D = -M \sum_{mn} C_n (a_{nm} \dot{a}_m \langle q_m^2(0) \rangle + b_{nm} \dot{b}_m \langle p_m^2(0) \rangle) . \quad (44)\]

Here we have made use of \(p = M\dot{q}\). The coefficients \(A, B, C, D\) are therefore completely determined by solving the equation of motion. We now do this explicitly.

We have

\[\ddot{q}(t) + \Omega^2 q(t) = \frac{1}{M} \sum_n C_n q_n(t) , \quad (45)\]
\[\ddot{q}_n(t) + \omega_n^2 q_n(t) = -\frac{C_n}{m_n} q(t) . \quad (46)\]

The solution to Eq. (46) is as follows:

\[q_n(t) = q_n(0) \cos(\omega_n t) + \frac{p_n(0) \sin(\omega_n t)}{m_n \omega_n} - C_n \int_0^t ds \frac{\sin[\omega_n(t-s)] q(s)}{m_n} . \quad (47)\]

Combining (45) and (47) gives

\[\ddot{q}(t) + \Omega^2 q(t) + \frac{2}{M} \int_0^t d\tau q(t-s) q(s) = \frac{f(t)}{M} , \quad (48)\]
where

\[ f(t) = -\sum_n C_n \left( q_n(0) \cos(\omega_n t) + \frac{p_n(0) \sin(\omega_n t)}{m_n \omega_n} \right). \tag{49} \]

The kernel \( \eta(s) \) is defined as

\[ \eta(s) = \frac{d}{ds} \gamma(s), \tag{50} \]

where

\[ \gamma(s) = \int_0^{+\infty} d\omega \frac{I(\omega)}{\omega} \cos(\omega s). \tag{51} \]

Here \( I(\omega) \) is the spectral density of the environment:

\[ I(\omega) = \sum_n \delta(\omega - \omega_n) \frac{C_n^2}{2m_n \omega_n}. \tag{52} \]

In order to get the expressions (40) and (41), we solve equation (48) with the following two different initial conditions:

\[ q(s = 0) = q(0), \quad \dot{q}(s = 0) = \frac{p(0)}{M}. \tag{53} \]

and

\[ q(s = t) = q(t), \quad \dot{q}(s = t) = \frac{p(t)}{M}. \tag{54} \]

where \( t \) is any given time point. In doing so, we consider the elementary functions \( u_i(s)(i = 1, 2) \) introduced by Hu, Paz and Zhang [10] which satisfy the following homogeneous integro-differential equation

\[ \ddot{\Sigma}(s) + \Omega^2 \Sigma(s) + \frac{2}{M} \int_0^s d\lambda \eta(s - \lambda) \Sigma(\lambda) = 0 \tag{55} \]

with the boundary conditions:

\[ u_1(s = 0) = 1, \quad u_1(s = t) = 0, \tag{56} \]

and
\[ u_2(s = 0) = 0, \quad u_2(s = t) = 1. \quad (57) \]

The solution to equation (55) with the initial condition (53) is obtained as the linear combination of \( u_1, u_2, \)

\[
w(s) = \left( u_1(s) - \frac{\dot{u}_1(0)}{\ddot{u}_2(0)} u_2(s) \right) q(0) + \frac{u_2(s)}{u_2(0)} \frac{p(0)}{M}. \quad (58)
\]

The solution to equation (48) with the homogeneous initial conditions can be formally written as

\[
\tilde{w}(s) = \frac{1}{M} \int_0^s d\tau G_1(s, \tau) f(\tau). \quad (59)
\]

Where \( G_1(s_1, s_2) \) is the Green function which can be constructed in terms of \( u_i (i = 1, 2): \)

\[
G_1(s_1, s_2) = \begin{cases}
  \frac{u_1(s_2)u_2(s_1) - u_2(s_2)u_1(s_1)}{u_1(s_2)u_2(s_2) - u_1(s_2)u_2(s_2)} & \text{if } s_1 > s_2 \\
  0 & \text{otherwise.}
\end{cases}
\]

Note that \( G_1(s_1, s_2) \) as the function of \( s_1 \) satisfies equation (55) with

\[
G_1(s_1 = s_2, s_2) = 0, \quad \frac{d}{ds_1} G_1(s_1 = s_2, s_2) = 1. \quad (60)
\]

Then the solution to the equation (48) with initial condition (53) reads

\[
q(s) = w(s) + \tilde{w}(s), \quad (61)
\]

explicitly,

\[
q(s) = \left( u_1(s) - \frac{\dot{u}_1(0)}{\ddot{u}_2(0)} u_2(s) \right) q(0) + \frac{u_2(s)}{u_2(0)} \frac{p(0)}{M} \\
- \sum_n \frac{C_n}{M} \int_0^s d\tau G_1(s, \tau) \cos(\omega_n \tau) q_n(0) \\
- \sum_n \frac{C_n}{M} \int_0^s d\tau G_1(s, \tau) \frac{\sin(\omega_n \tau)}{\omega_n} \frac{p_n(0)}{m_n}. \quad (62)
\]

It can be shown that the solution to the homogeneous equation (55) with the initial conditions (54) is

\[
u(s) = \left( u_2(s) - \frac{\dot{u}_2(t)}{\ddot{u}_1(t)} u_1(s) \right) q(t) + \frac{u_1(s)}{\ddot{u}_1(t)} \frac{p(t)}{M}, \quad (63)
\]
and
\[ \ddot{u}(s) = \frac{1}{M} \int_t^s d\tau G_2(s, \tau) f(\tau), \quad (s \leq t) \]  \hspace{1cm} (64)

is the solution to the inhomogeneous equation (48) with the homogeneous initial conditions
\[ \ddot{u}(t) = 0, \quad \dot{u}(t) = 0, \]  \hspace{1cm} (65)

where Green function \( G_2(s_1, s_2) \) is
\[ G_2(s_1, s_2) = \begin{cases} \frac{u_1(s_1)u_2(s_2) - u_2(s_1)u_1(s_2)}{u_1(s_1)u_2(s_2) - u_2(s_1)u_1(s_2)} & \text{if } s_2 > s_1 \\ 0 & \text{otherwise.} \end{cases} \]

Hence, we get the solution to Eq. (48) with the initial conditions (54)
\[ q(s) = u(s) + \ddot{u}(s) \]
\[ = \left( u_2(s) - \frac{\dot{u}_2(t)}{u_1(t)} u_1(s) \right) q(t) + \frac{u_1(s)}{u_1(t)} \frac{p(t)}{M} \]
\[ + \sum_n \frac{C_n}{M} \int_s^t d\tau G_2(s, \tau) \cos(\omega_n \tau) q_n(0) \]
\[ + \sum_n \frac{C_n}{M} \int_s^t d\tau G_2(s, \tau) \frac{\sin(\omega_n \tau)}{\omega_n} \frac{p_n(0)}{m_n}. \]  \hspace{1cm} (66)

Substituting (66) into (47), one obtains
\[ q_n(t) = -\frac{C_n}{m_n \omega_n} \int_0^t ds \sin[\omega_n (t - s)] \left( u_2(s) - \frac{\dot{u}_2(t)}{u_1(t)} u_1(s) \right) q(t) \]
\[ - \frac{C_n}{m_n \omega_n} \int_0^t ds \sin[\omega_n (t - s)] \frac{u_1(s)}{u_1(t)} \frac{p(t)}{M} \]
\[ + q_n(0) \cos(\omega_n t) + \frac{p_n(0)}{m_n} \frac{\sin(\omega_n t)}{\omega_n} \]
\[ - \frac{1}{M} \sum_m \frac{C_n C_m}{m_n \omega_n} \int_0^t ds \int_s^t d\tau \sin[\omega_n (t - s)] G_2(s, \tau) \cos(\omega_m \tau) q_m(0) \]
\[ - \frac{1}{M} \sum_m \frac{C_n C_m}{m_n \omega_n} \int_0^t ds \int_s^t d\tau \sin[\omega_n (t - s)] G_2(s, \tau) \frac{\sin(\omega_m \tau)}{m_m \omega_m} p_m(0). \]  \hspace{1cm} (67)

By using (42) we immediately arrive at
\[ A(t) = -\sum_n \frac{C_n^2}{m_n \omega_n} \int_0^t ds \sin[\omega_n (t - s)] \left( u_2(s) - \frac{\dot{u}_2(t)}{u_1(t)} u_1(s) \right), \]  \hspace{1cm} (68)
\[ B(t) = -\frac{1}{M} \sum_n \frac{C_n^2}{m_n \omega_n} \int_0^t ds \sin[\omega_n (t - s)] \frac{u_1(s)}{u_1(t)}. \]  \hspace{1cm} (69)
Furthermore, $A, B$ can be written as

\[
A(t) = 2 \int_0^t ds \eta(t-s)u_2(s) - 2 \frac{\dot{u}_2(t)}{\dot{u}_1(t)} \int_0^t ds \eta(t-s)u_1(s),
\]

\[
B(t) = \frac{2}{M\dot{u}_1(t)} \int_0^t ds \eta(t-s)u_1(s).
\]

From (62), the momentum of the Brownian particle is then

\[
p(s) = M\dot{q}(s)
= \left( \dot{u}_1(s) - \frac{\dot{u}_1(0)}{\dot{u}_2(0)} \dot{u}_2(s) \right) M q(0) + \frac{\dot{u}_2(s)}{\dot{u}_2(0)} p(0)
- \sum_n \int_0^s d\tau G'_1(s, \tau) \cos(\omega_n \tau) q_n(0)
- \sum_n \int_0^s d\tau G'_1(s, \tau) \frac{\sin(\omega_n \tau)}{\omega_n} \frac{p_n(0)}{m_n}.
\]

Here “prime” stands for derivative with respect to the first variable of $G_1(s, \tau)$.

With these results, it can be easily shown that

\[
C(t) = \frac{\hbar}{M} \int_0^t d\lambda G_1(t, \lambda) \nu(t - \lambda)
- \frac{2\hbar}{M^2} \int_0^t ds \int_0^t d\tau \int_0^t d\lambda \eta(t-s)G_1(t, \lambda)G_2(s, \tau)\nu(\tau - \lambda),
\]

and

\[
D(t) = \hbar \int_0^t d\lambda G'_1(t, \lambda) \nu(t - \lambda)
- \frac{2\hbar}{M} \int_0^t ds \int_0^t d\tau \int_0^t d\lambda \eta(t-s)G'_1(t, \lambda)G_2(s, \tau)\nu(\tau - \lambda).
\]

where $\nu(s)$ is defined as

\[
\nu(s) = \int_0^{+\infty} d\omega \mathcal{I}(\omega) \coth(\frac{1}{2}\hbar\omega\beta) \cos(\omega s).
\]

It is seen that the coefficients $A(t), B(t), C(t), D(t)$ are dependent only on the kernels $\eta(s)$ and $\nu(s)$ and the initial state of the bath, not dependent on the initial state of the system.

Once the spectral density of the environment is given, the elementary functions $u_i(i = 1, 2)$ can be solved from equation (55), the Green functions $G_i(i = 1, 2)$ are then obtained. Thus the coefficients of master equation can be determined.
IV. PARTICULAR CASES

In this section, we will consider some special cases. Let us at first treat a special case in which we assume that the interaction between the system and environment is weak, so the $C_n$ are small. In this case, the coefficients are of simple forms, and the determination of these coefficients is very simple and straightforward. We shall work out these coefficients directly using the method in the last section, rather than the general formulae.

The solution to Eq. (46) may be written as

$$ q_n(t) = q_n(0) \cos(\omega_n t) + \frac{p_n(0)}{m_n} \frac{\sin(\omega_n t)}{\omega_n} $$

$$ - \frac{C_n}{m_n} \int_0^t dt' \frac{\sin[\omega_n(t - t')] \cos[\Omega(t' - t)]}{\omega_n} q(t) $$

$$ - \frac{C_n}{m_n} \int_0^t dt' \frac{\sin[\omega_n(t - t')] \sin[\Omega(t' - t)]}{\Omega} p(t) \frac{\sin(\omega_n t)}{m_n\omega_n} \frac{\sin(\omega_n t)}{m_n\omega_n} + O(C_n^2) . \quad (76) $$

Using Eq. (37) and ignoring terms with higher than the second order of $C_n$, we get

$$ \sum_n C_n \langle q_n(t) \rangle = \{ \sum_n - \frac{C_n^2}{m_n} \int_0^t dt' \frac{\sin[\omega_n(t - t')] \cos[\Omega(t' - t)]}{\omega_n} \} \langle q(t) \rangle $$

$$ + \{ \frac{1}{M} \sum_n - \frac{C_n^2}{m_n} \int_0^t dt' \frac{\sin[\omega(t - t')] \sin[\Omega(t' - t)]}{\Omega} \} \langle p(t) \rangle . \quad (77) $$

Then we immediately get

$$ A(t) = 2 \int_0^t ds \eta(s) \cos(\Omega s) , \quad (78) $$

$$ B(t) = - \frac{2}{M\Omega} \int_0^t ds \eta(s) \sin(\Omega s) . \quad (79) $$

We next evaluate $\sum_n C_n \langle q(t)q_n(t) \rangle$ and $\sum_n C_n \langle p(t)q_n(t) \rangle$. After a few manipulations, we arrive at the expressions

$$ C(t) = - \sum_n C_n \left\{ \langle q(t)q_n(0) \rangle \cos(\omega_n t) + \langle q(t)p_n(0) \rangle \frac{\sin(\omega_n t)}{m_n\omega_n} \right\} , \quad (80) $$

and

$$ D(t) = - \sum_n C_n \left\{ \langle p(t)q_n(0) \rangle \cos(\omega_n t) + \langle p(t)p_n(0) \rangle \frac{\sin(\omega_n t)}{m_n\omega_n} \right\} . \quad (81) $$
To calculate $C(t)$ and $D(t)$, we need to expand $q(t)$ up to the second order of $C_n$:

\[
q(t) = q(0) \cos(Ωt) + \frac{p(0) \sin(Ωt)}{Ω} - \sum_n \frac{C_n}{M} \int_0^t ds \frac{\sin[Ω(t - s)]}{Ω} \cos(ω_ns) q_n(0)
- \sum_n \frac{C_n}{M} \int_0^t ds \frac{\sin[Ω(t - s)]}{Ω} \sin(ω_ns) \frac{p_n(0)}{ω_n} m_n
+ O(C_n^2).
\] (82)

The expansion of $p(t)$ is easily obtained from that of $q(t)$,

\[
p(t) = M\dot{q}(t).
\] (83)

With these results it is easy to compute $C(t)$ and $D(t)$:

\[
C(t) = \frac{\hbar}{MΩ} \int_0^t ds ν(s) \sin(Ωs),
\] (84)

\[
D(t) = \hbar \int_0^t ds ν(s) \cos(Ωs).
\] (85)

This simple example exhibits the time dependency of the coefficients of the master equation in a general environment. Eqs. (78), (79), (84), (85) are in agreement with Hu, Paz and Zhang [10].

As another example, we briefly discuss the purly Ohmic case in the Fokker-Planck limit (a particular high temperature limit), which has been extensively discussed in the literature [6,10]. In this case one has

\[
η(s - s') = Mγδ'(s - s')
\] (86)

\[
ν(s - s') = \frac{2Mγk_BT}{\hbar}δ(s - s')
\] (87)

Then the equation (55) reduces to

\[
\ddot{u}(s) + Ω_{ren}^2 u(s) + γ\dot{u}(s) = -2γδ(s)u(0)
\] (88)

where $Ω_{ren}^2 = Ω^2 - 2γδ(0)$. After solving this equation, a few calculations give
\[ A(t) = -2M\gamma\delta(0), \] (89)
\[ B(t) = 2\gamma, \] (90)
\[ C(t) = 0, \] (91)
\[ D(t) = 2M\gamma k_B T, \] (92)

then the Wigner equation reads:

\[
\frac{\partial}{\partial t} \tilde{W} = -\frac{p}{M} \frac{\partial \tilde{W}}{\partial q} + M\Omega_{\text{res}}^2 \frac{\partial \tilde{W}}{\partial p} + 2\gamma \frac{\partial \tilde{W}}{\partial p} + 2M\gamma k_B T \frac{\partial^2 \tilde{W}}{\partial p^2}.
\] (93)

In this regime, the coefficients of this Wigner equation are constants.

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