Te-Comonoform Modules

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Abstract. In this paper, the concept of te-comonoform modules is introduced, where an R-module U is called te-comonoform if for each paper submodule V of U and for each f ≠ 0, f ∈ Hom(U,\frac{V}{W}), f(U) ≤_{es} U/V. Many properties related with this concept are given. Moreover many connections between this concept and certain modules are presented.

Keywords. monoform module, Small monoform module, essential submodules, te-essential submodules.

1.Introduction

Throughout this paper R be a ring with unity and U is a ring a right R-module. A nonzero module U is called monoform if for each nonzero submodule V of U and each nonzero homomorphism f ∈ Hom(V, U), f is monomorphism (i.e. kerf = 0). [1] [2] In 2014, Hadi and Mashaaon [2] introduced the concept of small monoform as a generalization of monomorphic module, where an R-module U is called small monoform if for each nonzero submodule V of U and for each 0 ≠ f ∈ Hom(V, U), kerf is small V, where a proper submodule V of U is called small and denoted by (V ≪ U), if V + W = U, W ≤ M, then W = U [4, p 4]. As a deal notion of small monoform. Muna Abbas in [5] introduce the concept co-small monoform, where an R-module U is called co-small monoform if for each V < U and for each nonzero f ∈ Hom(U, \frac{U}{W}), f(U) is essential in U/V(f(U) ≤_{es} U/V). Recall that a submodule V of U is called t-essential in U (briefly V ≤_{tes} U if whenever V ∩ W ≤ Z(U), W ⊆ U, then W ⊆ Z(U) where Z(U) the second singles submodule of U (or Z₂-torsion ) submodule and defined by \frac{Z(U)}{Z(U)} = \frac{Z(U)}{Z(U)} [7], where Z(U) is the sigules submodule of U.

Note that every essential submodules of U is t-essential, but not conversely. However they are concid in the class of nonsingular modules.

The module U is called Z₂-torsion if Z₂(U) = U [6]. As every essential submodule is t-essential. This motivates us to introduce (i.e section two) the concept te-comonoform module, where an R-module U is called te-comonoform if for each V < U and for each nonzero ∈ Hom(U, \frac{U}{V}), f(U) ≤_{es} U.

Hence every co-small module, but not conversely (see 2.6 (1), 5) and they are equivalent in the class of non-singular modules (see 2.6 (3)). Each class of singular or Z₂-torsion or modules whose all non zero submodules are t-essential or uniform, modules
contained in the class of te-comonoform, (see 2.6 (4), 2.6(8), 2.6 (9)). Moreover the direct sum of te-comonoform module may be not te-comonoform, (see 2.6.7). However under certain conditions hold, see theorem 2.17. Many other properties are introduced in S.2.

Section three is devoted for connections between te-comonoform and some classes of modules such as epiform modules, antihopfiam modules, quasi-Dedekined modules, co-quasi-dedekind module and t-semisimple modules.

2. t-essential-comonoform (te-comonoform)

In this section the concept of t-essential comonoform is introduced and many basic properties related with this concept are given.

First, some known properties which are useful for our work are listed.

Lemma 2.1 [7]:

The following are equivalent for a submodule \( V \) of an \( R \)-module \( U \):

\[
\frac{V \leq \text{tes} U}{Z_2(U)} \leq \frac{Z_2(U)}{Z_2(U)} \leq \frac{\text{ess} U}{Z_2(U)} \leq \frac{U}{Z_2(U)}
\]

\( \frac{U}{V} \) is \( Z_2 \)-torsion.

Note 2.2: Let \( U \) be an \( R \)-module, \( A \leq \text{tes} U \) if \( Z_2(U) \subseteq A \), then \( A \leq \text{ess} U \).

Proof: Since \( A \leq \text{tes} U \), then by lemma 2.1, \( A + Z_2(U) \leq \text{ess} U \). As \( Z_2(U) \subseteq A \), then \( A \leq \text{ess} U \).

Lemma 2.3: Let \( U \) be a singular (or \( Z_2 \)-torsion ). Then every submodule of \( U \) is t-essential.

Proof: Let \( A \leq U \). Since \( U \) is singular (or \( Z_2 \)-torsion ), the \( Z_2(U) = U \) and hence \( A + Z_2(U) = U \leq \text{ess} U \). Thus by lemma 2.1, \( A \leq \text{tes} U \).

Lemma 2.4[6]: Let \( U \) be an \( R \)-module, \( V \leq U \). Then

\[
Z_2\left(Z_2(U)\right) = Z_2(U) \quad Z_2(V) = Z_2(U) \cap V.
\]

Def 2.5: Let \( U \) be an \( R \)-module. \( U \) is called t-essential comonoform (brifly, te-comonoform) if foe every nonzero homomorphism and every proper submodule \( V \) of \( U \), \( f: \frac{U}{V} \rightarrow \frac{U}{V} \), then \( f(U) \leq \text{tes} \frac{U}{V} \).

Remarks and Examples 2.6

1- It is clear that every co-small monoform module is te-comonoform, but the converse is not true in general, see (part (5)).
2- Every simple module is te-comonoform.
3- Let \( U \) be a nonsingular \( R \)-module. Then \( U \) is co-small monoform if and only if \( U \) is te-comonoform.
4- Every singular (or \( Z_2 \)-torsion ) module is te-comonoform.

Proof: let \( U \) be a singular (or \( Z_2 \)-torsion )

Let \( V < U \). Then \( \frac{U}{V} \) is singular (or \( Z_2 \)-torsion ).
Hence for any \( U \rightarrow \frac{u}{v}, f \neq 0 f(U) \leq_{\text{tes}} \frac{u}{v} \) by lemma 2.3.

5- The Z-module \( Z_6 \) is singular, hence by part (4), it is te-comonofrom. However, for \( h: Z_6 \rightarrow \frac{Z_6}{(0)} = Z_6 \) such that \( h(x) = 2x, \forall x \in Z_6 \), \( h(Z_6) = \langle \frac{Z}{Z_6} \rangle \) \( \approx_{\text{ess}} \frac{Z_6}{(0)} = Z_6 \). Hence \( Z_6 \) as Z-module is not co-small-monoform.

6- Let \( U \) be a te-comonoform \( R \)-module. The for \( \text{End}(U), f \neq 0, f(U) \leq U \).

Proof: let \( f \in \text{End}, f \neq 0 \). Hence \( h: U \rightarrow \frac{u}{v} \). Since \( U \) is te-comonoform, \( f(U) \leq_{\text{tes}} \frac{u}{v} \), that is \( f(U) \leq_{\text{tes}} U \).

7- As application of part (7): let \( U = Z_2 \oplus Z_2 \) as \( Z_2 \)-module. Let \( f: U \rightarrow V \) defined by \( f(\bar{x}, \bar{y}) = (\bar{x}, \bar{y}), \forall (\bar{x}, \bar{y}) \in Z_2 \oplus Z_2 \). It is clear that \( f(U) = Z_2 \oplus (\bar{0}) \). But \( (Z_2 \oplus (\bar{0})) \oplus Z_2(U) = (Z_2 \oplus (\bar{0})) \oplus (\bar{0}) = Z_2 \oplus (\bar{0}) \) which is not essential in \( Z_2 \oplus Z_2 \). Hence \( Z_2 \oplus (\bar{0}) \approx_{\text{tes}} U \) by lemma 2.1. Thus \( f(U) \leq_{\text{tes}} U \) and so by (7), \( U \) is not te-comono.

Also notice that \( U \) is semisimple module need not be te-comono.

8- Let \( U \) be an \( R \)-module such that for \( (0) \neq V \subseteq U, V \leq_{\text{tes}} U \). Then \( U \) is te-comonoform.

Proof: let \( V < U \). If \( V \neq (0), \) then \( V \leq_{\text{tes}} U \) and so by lemma (2.1), \( \frac{u}{v} \) is \( Z_2 \)-torsion. Now for every nonzero, \( f: U \rightarrow \frac{u}{v} \). Then by lemma (2.3), \( f(U) \leq_{\text{tes}} \frac{u}{v} \). If \( V = (0), \) then for any \( 0 \neq f: U \rightarrow \frac{u}{v} \) as \( \frac{u}{v} \) isomorphism theorem, \( (0) \neq f(U) \leq_{\text{tes}} \frac{u}{v} \). Hence by hypothesis \( f(U) \leq_{\text{tes}} U = \frac{u}{v} \).

9- Let \( U \) be a uniform \( R \)-module. Then \( U \) is te-comonoform.

Proof: let \( V < U \). If \( V \neq 0, V \leq_{\text{tes}} U \), hence \( V \leq_{\text{tes}} U \). Thus by a similar proof of (8), and \( 0 \neq f: U \rightarrow \frac{u}{v} \). \( f(U) \leq_{\text{tes}} \frac{u}{v} \). If \( V = (0), \) then for each \( 0 \neq f: U \rightarrow \frac{u}{v} \) we have \( f(U) \leq_{\text{tes}} \frac{u}{v} \).

**Proposition 2.7:** Let \( V < U \) and \( U \) be a te-comonoform module. Then \( \frac{u}{v} \) is a te-comonoform module.

**Proof:** Assume \( \frac{W}{V} < \frac{u}{v} \) and \( 0 \neq f: \frac{u}{v} \rightarrow \frac{u}{w} \) since \( \frac{v}{w} = \frac{u}{w} \) by 3rd isomorphism theorem, so there exists an isomorphism \( f: \frac{u}{v} \rightarrow \frac{u}{w} \). Consider \( U \rightarrow \frac{u}{v} \rightarrow \frac{u}{w} \). Then \( g/f: U \rightarrow \frac{u}{v} \) and \( (g/f)(U) = (g/f)\frac{u}{v} \neq 0 \). Since \( U \) is te-comonoform \((g/f)(U) \leq_{\text{tes}} \frac{u}{w} \). Hence \((g/f)(\frac{u}{v}) \leq_{\text{tes}} \frac{u}{w} \) i.e. \( g(f(\frac{u}{w})) \leq_{\text{tes}} \frac{u}{w} \). It follows that \( g^{-1}[g(f(\frac{u}{v}))] \leq_{\text{tes}} g^{-1}(\frac{u}{v}) = \frac{u}{v} = \frac{u}{w} \) and so \( f(\frac{u}{v}) \leq_{\text{tes}} \frac{u}{w} \). Therefore \( \frac{u}{v} \) is te-comonoform.

**Note (2.8):** let \( A \) be a proper nonzero submodule of an \( R \)-module s.t. \( \frac{V}{A} \) is te-comonoform. Then it is not neccessary that \( U \) is te-comonoform, for example consider \( Z_6 \) as \( Z_6 \)-module. Let
\(A = \langle \mathfrak{3} \rangle\), then \(Z_6/\langle \mathfrak{3} \rangle \cong Z_3\) as \(Z_6\)-module. Hence \(Z_6/\langle \mathfrak{3} \rangle\) is simple, so it is comonoform, but \(Z_6\) as \(Z_6\)-module is te-comonoform.

**Corollary 2.9:** the epimorphic image of te-comonoform is te-comonoform.

**Corollary 2.10:** A direct summand of te-comonoform module is te-comonoform.

Recall that a submodule of an \(R\)-module \(U\) is called t-closed (briefly \(A \leq_{tc} U\)) if \(A\) has no proper t-essential \([\ ]\), equivalently \(A \leq_{tc} U\) if \(\frac{U}{A}\) is nonsingular. However Goodearl in [7], called a submodule \(A\) of \(U\), \(Y\)-closed if \(\frac{U}{A}\) is nonsingular. Thus t-closed and \(Y\)-closed coincide.

**Proposition 2.11:** let \(U\) be a te-comonoform \(R\)-module and \(A \leq_{tc} U\). Then \(\frac{U}{A}\) is co-small monoform.

**Proof:** Since \(U\) is te-comonoform, then by proposition 2.7, \(\frac{U}{A}\) is te-comonoform. But \(A \leq_{tc} U\), so \(\frac{U}{A}\) is nonsingular. Then \(\frac{U}{A}\) is co-small- monoform by (Rem&Ex 2.6(3)).

**Remark 2.12:** The direct sum of te-comonoform modules need not te-comonoform, see (Rem. &Ex. 2.6 (7)).

Now we shall give conditions, which make the direct sum of te-comonoform is te-comonoform, but first, we need the following lemmas.

**Lemma 2.13** [8] let \(U_1\) and \(U_2\) be \(R\)-modules and \(\text{ann}U_1 + \text{ann}U_2 = R\). Then \(\text{Hom}(U_1, U_2) = 0, \text{Hom}(U_2, U_1) = 0\).

**Lemma 2.14:** let \(U_1\) and \(U_2\) be \(R\)-module, let \(V_1 < U_1, V_2 < U_2\). If \(\text{ann}U_1 + \text{ann}U_2 = R\), then \(\text{ann}U_1 + \text{ann}\frac{U_1}{V_2} = R\) and \(\text{ann}U_2 + \text{ann}\frac{U_2}{V_1} = R\).

**Proof:** Since \(\text{ann}U_2 \subseteq [V_2, U_2]\) = \(\frac{U_2}{V_2}\), hence \(R = \text{ann}U_1 + \text{ann}U_2 \subseteq \text{ann}U_1\frac{U_2}{V_2}\). Thus \(\text{ann}U_1 + \text{ann}\frac{U_2}{V_2} = R\). Similarly, \(\text{ann}U_2 + \text{ann}\frac{U_1}{V_1} = R\).

**Lemma 2.15:** let \(U_1\) and \(U_2\) be \(R\)-module, let \(V_1 < U_1, V_2 < U_2\). If and only if \(\text{ann}U_1 + \text{ann}U_2 = R\). Then \(\text{Hom}(U_1, \frac{U_2}{V_2}) = 0, \text{Hom}(U_2, \frac{U_1}{V_1}) = 0\).

**Proof:** It follows by lemma’s, 2.14, 2.15.

**Lemma 2.16:** Let \(U_1\) and \(U_2\) be \(R\)-module and let \(V_1 < U_1\) and \(V_2 < U_2\), \(\text{ann}U_1 + \text{ann}U_2 = R\). Then \(\text{Hom}(U_1 \oplus U_2, \frac{U_1}{V_1} \oplus \frac{U_2}{V_2}) = \text{Hom}(U_1 + \frac{U_1}{V_1}) \oplus \text{Hom}(U_2 + \frac{U_2}{V_2})\).

**Proof:** \(\text{Hom}(U_1 \oplus U_2, \frac{U_1}{V_1} \oplus \frac{U_2}{V_2}) = \text{Hom}(U_1, \frac{U_1}{V_1}) \oplus \text{Hom}(U_1, \frac{U_2}{V_2}) \oplus \text{Hom}(U_2, \frac{U_1}{V_1}) \oplus \text{Hom}(U_2, \frac{U_2}{V_2})\) then by lemma 2.15, \(\text{Hom}(U_1 + \frac{U_2}{V_2}) = 0, \text{Hom}(U_2 + \frac{U_1}{V_1}) = 0\). Thus the result follows thus for each \(V_1 < U_1\) \& \(V_2 < U_2\) where \(U_1\) and are \(R\)-modules with \(\text{ann}U_1 + \text{ann}U_2 = R\) and for each \(f \in \text{Hom}(U_1 \oplus U_2, \frac{U_1}{V_1} \oplus \frac{U_2}{V_2})\) \(f = f_1 + f_2\) some \(f_i \in \text{Hom}(U_i \oplus U_i, \frac{U_i}{V_i} \oplus \frac{U_i}{V_i})\), \(i=1,2\). We shall assume that: if \(f \neq 0\), then there exists \(f_1 \neq 0\) and \(f_2 \neq 0\) such that \(f = f_1 + f_2\) and this condition is denoted by (*) are te-comonoform, the \(U = U_1 \oplus U_2\) is te-comonoform provided that condition (*)
Proof: let \( V < U = U_1 \oplus U_2 \). Since \( \text{ann} U_1 + \text{ann} U_2 = R \), then \( V = V_1 \oplus V_2 \) for some \( V_1 \leq U_1, V_2 \leq U_2 \).

**Case 1:** if \( V_1 < U_1 \) and \( V_2 \leq U_2 \). let \( f \neq 0 \), and \( f \in \text{Hom}(U_1 \oplus U_2, \frac{U_1 \oplus U_2}{V_1 \oplus V_2}) \cong \text{Hom}(U_1 \oplus U_2, \frac{U_1}{V_1} \oplus \frac{U_2}{V_2}) \). Hence by lemma 2.16

\[
\text{Hom}(U_1 \oplus U_2, \frac{U_1}{V_1} \oplus \frac{U_2}{V_2}) \cong \text{Hom}(U_1 + \frac{U_1}{V_1}) \oplus \text{Hom}(U_2 + \frac{U_2}{V_2})
\]

So by condition (*), \( f = f_1 + f_2 \), where \( 0 \neq f_1 \in \text{Hom}(U_1 \oplus U_2, \frac{U_1}{V_1} \oplus \frac{U_1}{V_1范式}), \ 0 \neq f_2 \in \text{Hom}(U_2 \oplus U_2, \frac{U_2}{V_2} \oplus \frac{U_2}{V_2}) \). But \( U_1 \) and \( U_2 \) are te-comonoform, so that \( f_1(U_1) \leq \text{tes} \frac{U_1}{V_1} \), \( f_2(U_2) \leq \text{tes} \frac{U_2}{V_2} \). It follows that \( f(U_1 \oplus U_2) = f_1(U_1) \oplus f_2(U_2) \leq \text{tes} \frac{U_1}{V_1} \oplus \frac{U_2}{V_2} \), since \( \frac{U_1}{V_1} \oplus \frac{U_2}{V_2} \simeq \frac{U_1 \oplus U_2}{V_1 \oplus V_2} \).

**Case 2:** \( V_1 < U_1 \) and \( V_2 = U_2 \) let \( 0 \neq f : U_1 \oplus U_2 \to \frac{U_1 \oplus U_2}{V_1 \oplus V_2} \simeq \frac{U_1}{V_1} \).

Now consider the following \( U_1 \xrightarrow{i} U_1 \oplus U_2 \xrightarrow{f} \frac{U_1}{V_1} \) where \( i \) is the induction mapping. Hence \( f \circ i : U_1 \to \frac{U_1}{V_1} \) and so \( f_1(U_1) \leq \text{tes} \frac{U_1}{V_1} \). But \( f_1(U_1) \subseteq f(U_1 \oplus U_1) \) which implies that \( f(U_1 \oplus U_1) \leq \text{tes} \frac{U_1}{V_1} \).

Case 3: \( V_1 = U_1, V_2 < U_2 \). The proof is similar to case (2) therefore \( U = U_1 \oplus U_2 \) is te-comonoform.

**Note 2.18** Consider \( Z_{15} \) as \( Z_{15} \)-module \( Z_{15} = (3) \oplus (5) \cong Z_5 \oplus Z_3 \), \( \text{ann} Z_{15} \). \( Z_5 \cong Z_3 \), \( \text{ann} Z_{15} \cong Z_5 \). Thus \( \text{ann} Z_{15} \) is a \( Z_{15} \)-module defined by \( f(z, z) = (0, 0) \), \( f \neq 0 \), \( \text{Hom}(Z_{15}, Z_5) = 0 \). Hence there is no \( f_1 \neq 0 \), so for any \( f_2 \in (Z_3, \frac{Z_3}{Z_3}) \), \( f \neq f_1 \oplus f_2 \).

### 3. Te-comonoform Modules and Other Related Concepts

In this section, some connections between te-comonoform modules and other classes of modules are introduced.

Recall that on \( R \)-modules \( U \) is called epimorphic if for each paper submodule \( V \) of \( U \) and for every nonzero, \( f \in \text{Hom}(U, \frac{U}{V}) \), \( f \) is an epimorphism.

**Remark 3.1:**
It is clear that every epimorphic module is co-small-monoform. However te-comonoform need not epimorphic, for example \( Z \) as \( Z \)-module is te-comonoform but it is not epimorphic.

**Proposition 3.2:** Let \( R \) be a semisimple ring and \( U \) be a te-comonoform \( R \)-module. Then \( U \) is epimorphic.
Proof: Let $V < U$ and let $0 \neq f: U \to \frac{u}{v}$ then $f(U) \leq_{tes} \frac{u}{v}$, since $U$ is te-comonoform. But $R$ is semisimple implies every $R$-module is nonsingular so that $\frac{u}{v}$ is nonsingular and hence $f(U) \leq_{tes} \frac{u}{v}$. On the other hand, $R$ is semisimple implies $\frac{u}{v}$ is semisimple, so $f(U) = \frac{u}{v}$. Thus $U$ is epiform.

**Corollary 3.3:** Let $U$ be a module over semisimple ring. Then the following statements are equivalent:

1. $U$ is co-smal monoform,
2. $U$ is te-comonoform,
3. $U$ is epiform.

Recall that an $R$-module $U$ is called co-equi-retractable. If for each $V < U$, there exists $f: \frac{u}{v} \to U$ such that $f$ is monomorphism, [10]. Note that some author called it mono-coretractable module.

**Theorem 3.4:** Let $U$ be a co-equi-retractable $R$-module. Then $M$ is te-comonoform if and only if for each $f \in \text{End}(U)$, $f \neq 0$, $f(U) \leq U$.

**Proof:** $\Rightarrow$ it follows by Rem&Ex 2.6(6).

$\Leftarrow$ let $V < U$ and $g: U \to \frac{u}{v}, g \neq 0$.

As $U$ is co-equi-retractable, $\exists h: \frac{u}{v} \to U$ s.t. $h$ is monomorphism. Hence $h \circ g \in \text{End}(U)$ and $h \circ g \neq 0$. By hypothesis $(h \circ g)(U) \leq_{tes} U$, that is $h(g(U)) \leq_{tes} U$ and so that $h^{-1}[h(g(U))] \leq_{tes} h^{-1}(U)$. But we can show that $h^{-1}[h(g(U))] \leq_{tes} g(U)$ as follows: let $x \in h^{-1}[h(g(U))]$, so that $h(x) \in h(g(u))$, and hence $h(x) = h(g(u_1))$ for some $u_1 \in U$. Therefore $x = g(u_1)$, since $h$ is one-to-one. Thus $g(U) \leq_{tes} \frac{u}{v}$ and $U$ is te-comonform.

Recall that an $R$-module $U$ is called antihafian if $U = \frac{u}{v}$ for $V < U$ [11], [12].

It is clear that every antihafian implies co-equi-retractable module.

**Corollary 3.5:** Let $U$ be an antihafain $R$-module. $U$ is te-comonoform if and only if for each $0 \neq f \in \text{End}(U)$, $f(U) \leq_{tes} U$. Recall that an $R$-module $U$ is called quasi-Dedkind if for each $0 \neq f \in \text{End}(U)$ $f$ is monomorphism [13].

**Lemma 3.6:** Let $U$ be a quasi-dedkind $R$-module. Then for each $f \neq 0$, $f \in \text{End}(U)$, $f(U) \leq_{tes} U$.

**Proof:** let $0 \neq f \in \text{End}(U)$. Since $U$ is quasi-dedkind, $f$ is monomorphism. As $U \leq_{tes} U$ and $f$ is monomorphism, $f(U) \leq_{tes} U$.

**Corollary 3.7:** Let $U$ be a quasi-dedkind and co-equi-retractable. Then $U$ is te-comonoform.

**Proof:** it follows by lemma 3.6 and theorem 3.4.

Recall that an $R$-module is called regular (or simply regular) if every submodule is pure, [14].

A submodule $V$ of $U$ is pure if for each called $A$ of $R$, $A^2 \cap V = A^2 V$, [15].
**Theorem 3.8:** Let $U$ be a regular $R$-module. Then $U$ is te-comonoform if and only if $U$ is co-small monoform.

**Proof:** Since $u_1$ is regular, $R/\text{ann}(x)$ is a regular ring [14, th. 1.10], where $\text{ann}(x) = \{r \in R : xr = 0 \}$. It follows that $Z(U) = 0$ because if $x \in Z(U)$, then $\text{ann}(x) \leq_{ess} R$ and so $R/\text{ann}(x)$ is singular; that $Z \left( \frac{R}{\text{ann}(x)} \right) = R(\text{ann}(x))$. But $R/\text{ann}(x)$ is regular implies $Z \left( \frac{R}{\text{ann}(x)} \right) = 0$. Thus $\frac{R}{\text{ann}(x)} = 0$ and so $R = \text{ann}(x)$ which implies $x = 0$ and so $Z(U) = 0$, i.e. $U$ is nonsingular. Thus the result follows by Rem&Ex. 2.6(3).

Recall that: An $R$-module $U$ is called coquasi-dedkind if $\text{Hom}(U, V) = 0, \forall V < U$. Equivalently $U$ coquasi dedkind if for each $f \in \text{End}(U)$, $f \neq 0$, then $f$ is onto.

It is clear that every coquasi-dedkind module is indecomposable.

An $R$-module $U$ is called $t$-semisimple, if for each $U \leq U$, there is a direct summand with $K$ is $t$-essential in $V$ [16].

**Proposition 3.9:** Let $U$ be a coquasi-dedkind and $t$-semisimple module. Then $U$ is te-comonoform.

**Proof:** Since $U$ is couasi-dedkind, $U$ is indecomposable. But $U$ is $t$-semisimple so that $U = Z_2(U) \oplus \hat{U}$ for some semisimple nonsingular $\hat{U}$. [16]. But $U$ is indecomposable, so either $Z_2(U) = 0$ or $\hat{U} = 0$. If $Z_2(U) = 0$, then $U = \hat{U}$, then $U = \hat{U}$ and so $U$ is semisimple.

Again $U$ is indecomposable, so $U$ is simple and hence $U$ is te-comonoform. If $\hat{U} = 0$, then $U = Z_2(U)$; i.e. $U$ is $Z_2$-torsion thus $U$ is te-comonoform by Rem&Ex. 2.6 (4).

**Corollary 3.10** Let $R$ be a $t$-semisimple $R$-module. Then every coquasi-dedkind $R$-module is te-comonoform.

Proof: Since $R$ is $t$-semisimple, then every $R$-module is $t$-semisimple [prop 1.1.53.7]. Hence the result follows by proposition 3.9.

4. References

[1] Sahical M. Y, Coquasi-Dedekind Modules, (2003) Ph. D thesis, University of Baghdad.
[2] J. M. Zelmanowitze, Representations of Rings with Faithful Monoform Modules, J. London Math, Soci. (2)29, (1984), 237-248.
[3] I. M. Hadi, H. K. Marhoon, Smallmonoform modules, Ibn-AL-Haitham, J of pure and applied Sci. 27(2014), 229-240.
[4] Kasch, Modules and Rings, (1982), Acad, Press.
[5] Muna A.A, Co-small monoform . Italiar Journal of pure and applied mathematics, 2019, No. 42(230-241).
[6] Asgari, Sh, Haghany A., Density Cohopfion Modules, Journal of Algebra and its application, 9(2010), 989-1000.
[7] Codearl K. R, Ring Theory, Non Singular Ring and Modules, (1976), Dekker, Inc. New York and Basel.
[8] Frahan D. S, A Study of Modules Related with t-Semisimple Modules, (2018), Ph.D Thesis, University of Baghdad-Iraq.
[9] B. A. AL-Hashim, M. A. Ahmed, Some results on epiform modules, J. AL-Anbar for pure Sci, 4(2010)54-56.
[10] G. Oman, A. Salminen, Modules which are isomorphic to their factor Modules in, Algebra, 41(2013), 1300-1315.
[11] Y-Hirano, I. Mogami, On Restricted antihapfian Modules, Math. J. Okayama, Univ., 28(1986), 119-131.
[12] Asagari, Sh. Haghany, A. t-extending Modules and t-Bear Modules, Comm, Algebra, 39(2011), 1605-1623.
[13] Sahirah M. Y., F-Regular Modules, (1993), Ms.C Thesis, University of Baghdad-Iraq.
[14] Mijbess A. S, Quasi-dedekind Modules, Ph. D. University of Baghdad, 1997.
[15] Anderson, Fiw. Fuller. K.R., Rings and Categories of Modules, 2nd end. Graduats Texts in Math, (1974) Berlin-Heidelberg, New York: Springs-Verlag.
[16] Asgari, Sh. Haghany, Al-Tolonei Y, T-Semisimple Module and T-Semisimple Ring, Comm Algebra, 4(5), (2013), 1882-1902.
[17] A. Ghorbani, Co-Epi-Retracted Modules and co-pri-Rings, comm, Algebra 38, No. 10 (2010), 3589-3596.