GRAD-SHAFRANOV APPROACH TO AXISYMMETRIC STATIONARY FLOWS IN ASTROPHYSICS

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My lecture is devoted to the analytical results available for a large class of axisymmetric stationary flows in the vicinity of compact astrophysical objects. First, the most general case is formulated corresponding to the axisymmetric stationary MHD flow in the Kerr metric. Then, I discuss the hydrodynamical version of the Grad-Shafranov equation. Although not so well-known as the full MHD one, it allows us to clarify the nontrivial structure of the Grad-Shafranov approach as well as to discuss the simplest version of the 3 + 1-split language – the most convenient one for the description of ideal flows in the vicinity of rotating black holes. Finally, I consider several examples that demonstrate how this approach can be used to obtain the quantitative description of the real transonic flows in the vicinity of rotating and moving black holes.

Many astrophysical sources are axisymmetric and stationary to a good accuracy. These include both accreting neutron stars and black holes, axisymmetric stellar (solar) winds, jets from young stellar objects, and ejection of particles from magnetospheres of rotating neutron stars. It can not be ruled out that such magnetohydrodynamic flows also play an important role in other galactic sources, e.g., microquasars. The latter are regarded as candidates for black holes not to say about active galactic nuclei where the electrodynamical processes in the vicinity of the rotating supermassive black holes are considered as the most reasonable model of their central engine. So, it is not surprising that ideal magnetohydrodynamics, which allows sufficiently simple formalization of the problem, is actively applied when describing these flows.

The point is that due to axial symmetry and stationarity (as well as the ideal freezing-in condition), in the most general case it is possible to introduce five integral of motions which are constant at axisymmetric magnetic surfaces. This remarkable fact allows us to separate the problem of finding the poloidal field structure (the poloidal flow structure in the hydrodynam-
ics) from the problem of particle acceleration and the structure of electric currents. The solution of the latter task for a given poloidal field can be obtained in terms of quite simple algebraic relations. It is important that such an approach can be straightforwardly generalized to flows in the vicinity of the rotating black holes, as the Kerr metric is also axially symmetric and stationary.

On the other hand, it is much more difficult to find the two-dimensional poloidal magnetic field structure (the hydrodynamical flow structure). First of all, this is due to the complex structure of the equation describing axisymmetric stationary flows. In the general case, it is a nonlinear equation of the mixed type, which changes from elliptical to hyperbolical at singular surfaces and in addition contains integrals of motion in the form of free functions. Generally speaking, similar equations, which stem from the classical Tricomi equation, have been discussed since the beginning of the last century in connection with transonic hydrodynamic flows. Later on, the equations describing axially symmetric stationary flows were called Grad-Shafranov equations after the authors who formulated in the late 1950s an equation of such a type in connection with controlled thermonuclear fusion. This equation, however, was originally related to equilibrium static configurations only and required strong revision when it was generalized to the transonic case. The full version of such an equation was formulated by L.S. Soloviev in 1963 in the third volume of Problems of Plasma Theory and was well-known to physicists. However, as it often occurs, the full version of the Grad-Shafranov equation was little known in the astrophysical literature, so it was later ‘rediscovered’ scores of times.

As it turned out, the difficulty lay in the fact that the very formulation of the direct problem within the Grad-Shafranov approach proved to be nontrivial. For example, in the hydrodynamical limit, when there are only three integrals of motion, the problem requires four boundary conditions for the transonic flow regime. This implies that, for instance, two thermodynamic functions and two velocity components should be specified at some surface. However, to determine the Bernoulli integral, which naturally should be known in order to solve the Grad-Shafranov equation, all three components of the velocity must be specified, which is impossible since the third velocity component itself is to be obtained from the solution. This is in fact one of the main difficulties of the approach under consideration.

Nevertheless, several approaches exist that allow us to construct the analytical solution of direct problems within the framework of the Grad-Shafranov method. For example, this is possible when the exact solution
of this equation is known and we explore flows weakly diverging from the known one. Spherically symmetric accretion (ejection) of matter could be such an exact solution. As a result, the known structure of the flow in the zeroth approximation enables us to determine (with the required accuracy) both the location of singular surfaces and all the integrals of motion directly from boundary conditions, thus making it possible to solve the Grad-Shafranov equation within the direct formulation of a problem.

1. Grad-Shafranov equation

Let us consider the axisymmetric stationary plasma flow in the vicinity of a rotating black hole, i.e., in the Kerr metric:

\[ ds^2 = -\alpha^2 dt^2 + g_{ik}(dx^i + \beta^i dt)(dx^k + \beta^k dt), \]

(1)

where

\[ \alpha = \frac{\rho}{\Sigma} \sqrt{\Delta}, \quad \beta^r = \beta^\theta = 0, \quad \beta^\varphi = -\omega = -\frac{2aMr}{\Sigma^2}, \]

\[ g_{rr} = \frac{\rho^2}{\Delta}, \quad g_{\theta\theta} = \rho^2, \quad g_{\varphi\varphi} = \varpi^2. \]

(2)

Here \( \alpha \) is the lapse function (gravitational red shift) vanishing on the horizon

\[ r_g = M + \sqrt{M^2 - a^2}, \]

(3)

\( \omega \) is the angular velocity of local nonrotating observers (the so-called Lense-Thirring angular velocity), and

\[ \Delta = r^2 + a^2 - 2Mr, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \]

\[ \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad \varpi = \frac{\Sigma}{\rho} \sin \theta. \]

(4)

As usual, \( M \) and \( a \) are the black hole mass and angular momentum per unit mass \((a = J/M)\) respectively. Here indices without hats denote components of vectors with respect to the coordinate basis \( \partial/\partial r, \partial/\partial \theta, \) and \( \partial/\partial \varphi, \) and indices with hats correspond to their physical components. Finally, below we shall use the system of units with \( c = G = 1. \)

In what follows we shall also use the 3 + 1 split language. Within this approach, the physical quantities are expressed in terms of three-dimensional vectors which would be measured by observers moving around the rotating black hole with the angular velocity \( \omega \) (so-called ZAMOs – zero angular momentum observers). The convenience of the 3 + 1 split language is connected with the fact that the representation of many expressions has
the same form as in the flat space. On the other hand, all thermodynamic quantities are determined in the comoving reference frame.

Now, we shall demonstrate how the five 'integrals of motions', which are constant at the magnetic surfaces, can be derived in the general case of axisymmetric stationary flows. It is convenient to introduce the scalar function \( \Psi(r, \theta) \) which has a meaning of magnetic flux. As a consequence, the magnetic field is defined in the following way:

\[
B = \frac{\nabla \Psi \times \hat{e}_\phi}{2\pi \alpha} - \frac{2I}{\alpha \omega} \hat{e}_\phi,
\]

where \( I(r, \theta) \) is the total electric current inside the region \( \Psi < \Psi(r, \theta) \).

As usual, we assume that the magnetosphere contains sufficient amount of plasma to satisfy the freezing-in condition which, using the 3 + 1 split language, preserves the form \( E + v \times B = 0 \). On the other hand, the stationarity (as well as the condition for zero longitudinal electric field) implies that the field \( E \) can be written as

\[
E = -\frac{\Omega_F - \omega}{2\pi \alpha} \nabla \Psi.
\]

By substituting relation (6) into the Maxwell equations, it is easy to verify that the condition \( B \cdot \nabla \Omega_F = 0 \) is satisfied, i.e. that \( \Omega_F \) must be constant at the magnetic surfaces (Ferraro’s isorotation law):

\[
\Omega_F = \Omega_F(\Psi).
\]

Next, the Maxwell equation \( \nabla \cdot B = 0 \), the continuity equation, and the freezing-in condition allow us to write the four-velocity of matter \( u \) in the form

\[
u = \frac{\eta}{\alpha n} B + \gamma (\Omega_F - \omega) \frac{\alpha}{\Omega} \hat{e}_\phi,
\]

where \( \gamma = 1/\sqrt{1 - v^2} \) is the Lorentz factor of matter (measured by ZAMOs), and the quantity \( \eta \) is the particle flux to magnetic flux ratio. Due to the relationship \( \nabla \cdot (\eta B_p) = 0 \), \( \eta \) must be constant at the magnetic surfaces \( \Psi(r, \theta) = \text{const} \) as well, i.e.,

\[
\eta = \eta(\Psi).
\]

The next two integrals of motions result from our assumption that the flow is axisymmetric and stationary. This yields the conservation law of the energy \( E \) and the \( z \)-component of angular momentum \( L_z \):

\[
E = E(\Psi) = \frac{\Omega_F I}{2\pi} + \mu \eta (\alpha \gamma + \omega u_\phi);
\]

\[
L = L(\Psi) = \frac{I}{2\pi} + \mu \eta \Omega \, u_\phi,
\]
where \( \mu = (\rho_m + P)/n \) is the relativistic enthalpy (\( \rho_m \) is the internal energy density, and \( P \) is the pressure). Finally, in the axially symmetric case the isentropy condition yields

\[
s = s(\Psi),
\]

so that the entropy per particle, \( s(\Psi) \), is the fifth integral of motion.

The five integrals of motions \( \Omega_F(\Psi), \eta(\Psi), s(\Psi), E(\Psi), \) and \( L(\Psi) \), as well as the poloidal magnetic field \( B_p \), allow us to find the toroidal magnetic field \( B_\phi \) and all other plasma parameters:

\[
\frac{I}{2\pi} = \frac{\alpha^2 L - (\Omega_F - \omega)\omega^2(E - \omega L)}{\alpha^2 - (\Omega_F - \omega)^2\omega^2 - M^2};
\]

\[
\gamma = \frac{1}{\alpha \mu \eta} \frac{\alpha^2(E - \Omega_F L) - M^2(E - \omega L)}{\alpha^2 - (\Omega_F - \omega)^2\omega^2 - M^2};
\]

\[
u_\phi = \frac{1}{\omega \mu \eta} \frac{(E - \Omega_F L)(\Omega_F - \omega)\omega^2 - LM^2}{\alpha^2 - (\Omega_F - \omega)^2\omega^2 - M^2},
\]

where

\[
M^2 = \frac{4\pi \eta^2 \mu}{n}.
\]

It is easy to see that \( M^2 \) is proportional (with the factor of \( \alpha^2 \)) to the Mach number squared of the poloidal velocity \( u_p \) with respect to the Alfvén velocity \( u_A = B_p/\sqrt{4\pi \eta \mu} \), i.e., \( M^2 = \alpha^2 u_p^2/u_A^2 \).

Since \( \mu = \mu(n, s) \), definition (16) allows us to express the concentration \( n \) (and hence the specific enthalpy \( \mu \)) as a function of \( \eta, s, \) and \( M^2 \). This means that along with the five integrals of motion, the expressions for \( I, \gamma, \) and \( u_\phi \) depend only on one additional quantity, namely the Mach number \( M \). To determine the Mach number \( M \), it is necessary to use the obvious relation \( \gamma^2 - u^2 = 1 \), which, owing to equations (14) and (15), can be rewritten in the form

\[
\frac{K}{\omega^2 A^2} = \frac{1}{64\pi^4} \frac{M^4(\nabla \Psi)^2}{\omega^2} + \alpha^2 \eta^2 \mu^2,
\]

where

\[
A = \alpha^2 - (\Omega_F - \omega)^2\omega^2 - M^2
\]

and

\[
K = \alpha^2 \omega^2 (E - \Omega_F L)^2 \left[ \alpha^2 - (\Omega_F - \omega)^2\omega^2 - 2M^2 \right] + M^4 \left[ \omega^2 (E - \omega L)^2 - \alpha^2 L^2 \right].
\]
As for the Grad-Shafranov equation itself, i.e., the equilibrium equation for magnetic field lines, it can be written in the form

\[
\frac{1}{\alpha} \nabla_k \left\{ \frac{1}{\alpha \omega^2} \left( \alpha^2 - (\Omega_F - \omega)^2 \omega^2 - M^2 \right) \nabla^k \Psi \right\} + \frac{\Omega_F - \omega}{\alpha^2} (\nabla \Psi)^2 d\Omega_F = \frac{1}{\alpha} \nabla \left( \frac{G}{A} \right) - \frac{1}{\alpha^2 \omega^2} \frac{1}{2 \partial \Psi} - 16\pi^3 \mu n \frac{1}{\eta d\Psi} - 16\pi^3 n T \frac{d s}{d\Psi} = 0,
\]

(20)

where

\[
G = \alpha^2 \omega^2 (E - \Omega_F L)^2 + \alpha^2 M^2 L^2 - M^2 \omega^2 (E - \omega L)^2,
\]

(21)

and the derivative \( \partial / \partial \Psi \) acts on the integrals of motion only. Finally, expressing in Eq. (20) the terms \( \nabla_k M^2 \) according to Eq. (19) we obtain

\[
A \frac{1}{\alpha} \nabla_k \left( \frac{1}{\alpha \omega^2} \nabla^k \Psi \right) + \frac{1}{\alpha^2 \omega^2 (\nabla \Psi)^2} \nabla_i \Psi \cdot \nabla^k \Psi \cdot \nabla_i \nabla_k \Psi 
\]

\[
+ \frac{\nabla^k A \nabla^k \Psi}{\alpha^2 \omega^2} - \frac{A}{\alpha^2 \omega^2 (\nabla \Psi)^2} \frac{1}{2 D} \nabla_i F \nabla^k \Psi + \frac{\Omega_F - \omega}{\alpha^2} (\nabla \Psi)^2 d\Omega_F = \frac{1}{\alpha} \nabla \left( \frac{G}{A} \right) - 16\pi^3 \mu n \frac{1}{\eta d\Psi} - 16\pi^3 n T \frac{d s}{d\Psi} = 0.
\]

Here

\[
D = \frac{A}{M^2} + \frac{\alpha^2 B_\perp^2}{M^2 B_p^2} - \frac{1}{u_p^2} \frac{A}{M^2} \frac{c_s^2}{1 - c_s^2},
\]

(23)

\[
F = \frac{64\pi^4}{M^4} \frac{K}{A^2} - \frac{64\pi^4}{M^4} \alpha^2 \omega^2 \eta^2 \mu^2,
\]

(24)

and the gradient \( \nabla^k \) denotes the action of \( \nabla_k \) under the condition that \( M \) is fixed. Let us stress that in equation (22) the pressure \( P \), the temperature \( T \), sound velocity \( c_s \), and the specific enthalpy \( \mu \) are to be expressed via an equation of state in terms of the entropy \( s(\Psi) \) and the square of the Mach number \( M^2 \). In turn, the quantity \( M^2 \) is to be considered as a function of \( (\nabla \Psi)^2 \) and the integrals of motion,

\[
M^2 = M^2 \left[ (\nabla \Psi)^2, E(\Psi), L(\Psi), \eta(\Psi), \Omega_F(\Psi), s(\Psi) \right].
\]

(25)

The latter relation is the implicit form of Eq. (17). The stream equation (22) coupled with definitions (5) – (11) is the desired equation for the poloidal field which contains only the magnetic flux \( \Psi \) and the five integrals of motion \( \Omega_F(\Psi), \eta(\Psi), s(\Psi), E(\Psi), \) and \( L(\Psi) \) depending on it.
Equation (22) is a second-order equation linear with respect to the highest derivatives. It changes its type from elliptical to hyperbolical at singular surfaces where the poloidal velocity of matter becomes equal to either fast or slow magnetosonic velocity \( D = 0 \), or to the cusp velocity \( D = -1 \). Although at the Alfvénic surface, \( A = 0 \), the type of equation does not change, the Alfvénic surface does represent a singular surface of the Grad-Shafranov equation because a regularity condition must be satisfied there.

2. Examples

Bondi-Hoyle accretion. As a first example, we consider the hydrodynamic accretion onto a moving black hole (the Bondi-Hoyle accretion), which is one of the classical problems of modern astrophysics. First of all, let us formulate the hydrodynamical limit of the Grad-Shafranov equation, where we can neglect the electromagnetic field contribution. In this case, it is convenient to introduce a new potential \( \Phi(\Psi) \) satisfying the condition \( \eta(\Psi) = \frac{d\Phi}{d\Psi} \). Using definition (8) we obtain

\[
\alpha n = \frac{1}{2\pi \omega} (\nabla \Phi \times \mathbf{e}_\phi).
\]

Surfaces \( \Phi(r, \theta) = \text{const} \) define the streamlines of matter.

In the hydrodynamic limit, there are only three integrals of motion. These are the energy flux and the \( z \)-component of the angular momentum:

\[
E(\Phi) = \mu (\alpha \gamma + \omega \omega_\phi);
\]

\[
L(\Phi) = \mu \omega_\phi,
\]

as well as the entropy \( s = s(\Phi) \). Now the algebraic Bernoulli equation (17) takes the form

\[
(E - \omega L)^2 = \alpha^2 \mu^2 + \frac{\alpha^2}{\omega^2} L^2 + \frac{\hat{M}^4}{64 \pi^4 \omega^2} (\nabla \Phi)^2,
\]

where the ‘Mach number’ squared \( \hat{M}^2 \) is defined as \( \hat{M}^2 = 4\pi \mu/n \). Then the Grad-Shafranov equation (20) can be rewritten in the form

\[
-\frac{1}{\alpha} \nabla_k \left( \frac{\hat{M}^2}{\alpha \omega^2} \nabla^k \Phi \right) - 16\pi^3 n T \frac{ds}{d\Phi}
\]

\[
+ \frac{64 \pi^4}{\alpha^2 \omega^2 \hat{M}^2} \left[ \frac{\omega^2}{(E - \omega L)} \left( \frac{dE}{d\Phi} - \omega \frac{dL}{d\Phi} \right) - \alpha^2 L \frac{dL}{d\Phi} \right] = 0,
\]

where now

\[
D = -1 + \frac{1}{u_p^2} \frac{c_s^2}{1 - c_s^2}.
\]
As we see, equation (30) contains only one singular surface, i.e. the sonic surface, determined from the condition $D = 0$.

To construct the solution corresponding to the Bondi-Hoyle accretion, it is possible to seek the solution of the Grad-Shafranov equation for the flux function $\Phi(r, \theta)$ in the form of a small perturbation of the spherically symmetric flow in the reference frame moving with the black hole

$$\Phi(r, \theta) = \Phi_0[1 - \cos \theta + \varepsilon_1 f(r, \theta)].$$

(32)

Here we introduce a small parameter

$$\varepsilon_1 = \frac{v_\infty}{c_\infty}$$

(33)

which defines the ratio of the black hole velocity to the velocity of sound at infinity. For a nonmoving gravity center we return to the spherically symmetric flow.

As Grad-Shafranov equation (30) contains three invariants, it is necessary to specify four boundary conditions, say

(1) $v_{p, \infty} = \text{const}$,
(2) $v_\varphi = 0$ (and hence $L = 0$),
(3) $s_\infty = \text{const}$,
(4) $E_\infty = c_\infty^2/(\Gamma - 1)$.

In the last relation we neglect the terms $\sim \varepsilon_1^2$.

As a result, the Grad-Shafranov equation can be linearized:

$$-\varepsilon_1 \alpha^2 D \frac{\partial^2 f}{\partial r^2} - \frac{\varepsilon_1}{r^2} (D + 1) \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \theta} \right) + \varepsilon_1 \alpha^2 N_r \frac{\partial f}{\partial r} = 0,$$

(34)

where

$$N_r = \frac{2}{r} - \frac{\mu^2}{E^2 - \alpha^2 \mu^2} \frac{M}{r^2}.$$  

(35)

It is extremely important that according to (26) and (31),

$$D + 1 = \frac{\alpha^2 \mu^2}{E^2 - \alpha^2 \mu^2} \cdot \frac{c_s^2}{1 - c_s^2},$$

(36)

so the factor $\alpha^2$ enters every term of equation (34). Hence, equation (34) has no singularity at the horizon. In particular, it means that it is not necessary to specify any boundary conditions for $r = r_g$. It is not surprising because the horizon corresponds to the supersonic region which cannot affect the subsonic flow.
On the other hand, we see that all the terms contain the small value \( \varepsilon_1 \). Hence, the functions \( D, c_s, \) etc. can be taken from the zeroth solution. As for the spherically symmetric flow, the functions \( D, c_s, \) etc. do not depend on \( \theta \), and the solution of equation (34) can be expanded in eigen functions of the operator \( \sin \theta \partial/\partial \theta(1/\sin \theta \cdot \partial/\partial \theta) \). Thus, the solution can be presented in the form

\[
f(r, \theta) = \sum_{m=0}^{\infty} g_m(r) Q_m(\theta),
\]

the equations for the radial functions \( g_m(r) \) being

\[
r^2 D \frac{d^2 g_m}{dr^2} + r^2 N_r \frac{dg_m}{dr} + m(m+1) \frac{\mu^2}{E^2 - \alpha^2 \mu^2} \frac{c_s^2}{1 - c_s^2} g_m = 0.
\]

Here \( Q_0 = 1 - \cos \theta \), \( Q_1 = \sin^2 \theta \), \( Q_2 = \sin^2 \theta \cos \theta \), . . . are the eigen functions of the angular operator.

As to the boundary conditions, they can be formulated as follows:

1. No singularity on the sonic surface (where \( N_r = 0, D = 0 \)),
   \[
g_m(r^*) = 0.
\]

2. The homogeneous flow \( \Phi = \pi n_\infty v_\infty r^2 \sin^2 \theta \) at infinity which gives
   \[
g_1 \to \frac{1}{2} \frac{n_\infty c_\infty}{n_s c_s} \frac{r^2}{r_s^2}, \quad g_2, g_3, \cdots = 0.
\]

As a result, the complete solution can be presented in the form

\[
\Phi(r, \theta) = \Phi_0 [1 - \cos \theta + \varepsilon_1 g_1(r) \sin^2 \theta],
\]

where the radial function \( g_1(r) \) is the solution of the ordinary differential equation (38) for \( m = 1 \) with the boundary conditions (39) and (40).

This means that we have constructed the analytical solution of the problem, \( i.e. \), obtained the full description of the flow structure. For example, the sonic surface has now the nonspherical form

\[
r_*(\theta) = r_s \left[ 1 + \varepsilon_1 \left( \frac{\Gamma + 1}{5 - 3\Gamma} \right) k_2 \cos \theta \right],
\]

where the numerical coefficient \( k_2 = r_* g'_1(r_*) \) is expressed through the derivative of the radial function \( g_1(r) \) at the sonic point. As shown in Fig. 1, the analytical solution fully agrees with the numerical calculations in spite of the parameter \( \varepsilon_1 = 0.6 \) here being quite large. Here \( \Gamma = 4/3 \), and the numbers alongside the curves denote values \( \Phi/\Phi_0 \); the dashed lines show the streamlines and the form of the sonic surface obtained numerically.
As can be easily seen, outside the capture radius

$$R_c \approx \varepsilon^{-1/2} r_*$$

(43)

our main assumption, i.e., the smallness of the deviation from the spherically symmetric flow, is not valid. Nevertheless, the solution found remains correct. This remarkable property is due to the Grad-Shafranov equation becoming linear for constant concentration $n$. But as we learn from the spherically symmetric Bondi accretion, at large distances $r \gg r_*$ from the sonic surface the density of the accreting matter is virtually constant. Accordingly, the concentration is constant for a homogeneous flow as well. As a result, under the condition that $R_c \gg r_*$, which holds true for $\varepsilon_1 \ll 1$, near and beyond the capture radius (where the perturbation $\sim \varepsilon_1 g_1(r)$ becomes comparable to unity) equation (30) becomes linear. So that the sum of the two solutions, homogeneous and spherically symmetric ones, is also a solution.

**Thin transonic disk.** As a next example, we consider the internal two-dimensional structure of a thin accretion disk. Here, for simplicity we consider the case of a nonrotating (Schwarzschild) black hole. We recall that according to the standard model the accreting matter forms an equilibrium disk rotating around the gravitational center with the Keplerian velocity $v_K(r) = (GM/r)^{1/2}$. The disk will be thin provided that its temperature is sufficiently small ($c_s \ll v_K$) since the vertical balance of the gravity force and the pressure gradient implies that

$$H \approx r \frac{c_s}{v_K}$$

(44)
The General Relativity effects result in two important properties: the absence of stable circular orbits for \( r < r_0 = 3r_g \) and the transonic regime of accretion. The first point means that the accreting matter passing the marginally stable orbit approaches the black hole horizon sufficiently fast, namely, in the dynamical time \( \tau_d \sim \left[ \frac{v_r(r_0)}{c} \right]^{-1/3} \frac{r_g}{c} \). It is important that such a flow is realized in the absence of viscosity. The second statement results from the fact that up to the marginally stable orbit the flow is subsonic while at the horizon the flow is to be supersonic.

It is necessary to stress that the existence of the small parameter \( \varepsilon_2 = \frac{u_0}{c_0} \ll 1 \), where \( c_0 \) is the sound velocity and \( u_0 \) is the gas radial velocity on the marginally stable orbit, comes from the relation \( \frac{v_r}{v_K} \approx \frac{\alpha ss c_s^2}{v_K^2} \) for the radial velocity in the accretion disk 1. In the vicinity of the marginally stable orbit this estimate is apparently inapplicable. Nevertheless, below we shall consider the parameter \( \varepsilon_2 \) to be small because the presence of a small parameter allows us to investigate the flow structure analytically. In addition, the small parameter makes the effect under discussion more visible.

Up to now in the majority of works devoted to thin accretion disks the procedure of vertical averaging was used, where the vertical four-velocity \( u_\theta \) was assumed to be zero \( \text{13} \). As a result, the vertical component of the dynamic force \( nu^0 \nabla_b (\mu u_a) \) in the Euler equation was postulated to be inimportant up to horizon. For this reason the disk thickness was determined by the pressure gradient even in the supersonic region near the black hole \( \text{14} \). Here I am going to demonstrate that the assumption \( u_\theta = 0 \) is not correct.

As for the Bondi accretion, the dynamic force is to be important in the vicinity of the sonic surface.

Figure 2 shows the structure of a thin accretion disk after passing the marginally stable orbit \( r = 3r_g \) obtained by solving equation (30) numerically for \( c_0 = 10^{-2}, u_0 = 10^{-5} \), i.e., in the presence of a small parameter \( \varepsilon_2 = \frac{u_0}{c_0} \ll 1 \). The solid lines correspond to the range of parameters \( \frac{u_p^2}{c_s^2} < 0.2 \). The dashed lines indicate the extrapolation of the solution to the sonic-surface region. In the vicinity of the sonic surface the flow has the form of the standard nozzle.

As we see, the flow structure near the sonic surface is far from being radial. The appearance of the narrow waist has a simple physical meaning. Indeed, the density remains almost constant for subsonic flow while the radial velocity increases from \( u_0 \) to \( c_\ast \sim c_0 \), i.e., for \( \varepsilon_2 \ll 1 \) changes over several orders of magnitude. As a result, the disk thickness \( H \) should change
in the same proportion owing to the continuity equation as well,

\[ H(r_*) \approx \frac{u_0}{c_0} H(3r_*) \quad (45) \]

Here it is extremely important that both components of the dynamical force become comparable with the pressure gradient near the sonic surface:

\[ \frac{u_{\theta}}{r} \frac{\partial u_{\theta}}{\partial \theta} \approx u_r \frac{\partial u_{\theta}}{\partial r} \approx \frac{\nabla_{\theta}P}{\mu} \approx \frac{c_0^2 \theta}{u_0^2} r \quad (46) \]

where the angle \( \theta \) is counted off from the equatorial plane.

In other words, if there appears a nonzero vertical velocity component, the dynamical term \((\nabla \cdot v) v\) cannot be neglected in the vertical force balance near the sonic surface. It is clear that this property remains valid for arbitrary radial velocity of the flow, i.e., even when the transverse contraction of the disk is not so pronounced. Taking dynamical forces into account causes two additional degrees of freedom to appear, which relate
to the higher derivatives in the Grad-Shafranov equation. This also leads to extra conclusions independent of the value of $\varepsilon_2$. In thin accretion disks, the critical condition at the sonic surface does not fix the accretion rate any more; it determines the bending of the streamlines near the sonic surface.

Finally, the inclusion of the vertical velocity inevitably leads to the appearance of a small longitudinal scale $\delta r_{\parallel} \approx H_*$ in the vicinity of the sonic surface, which for a thin disk proves to be much smaller than the distance to the black hole for any value of the parameter $u_0/c_0$. In the standard one-dimensional approach, this scale does not emerge. As for the supersonic region (and, in particular, the region near the horizon), the disk thickness here will be determined not by the pressure gradient, but by the form of ballistic trajectories.

**The Blandford-Znajek process.** In conclusion, we discuss the energy loss of a rotating black hole embedded in an external magnetic field – the so-called Blandford-Znajek process $^{15}$. This process is considered to be the most preferential mechanism of energy release in active galactic nuclei, microquasars, and even cosmological gamma-ray bursters $^{16}$. Its main idea is based on the analogy with the energy transfer in the internal regions of radio pulsar magnetospheres. Indeed, let us suppose that there is a regular external magnetic field in the vicinity of a rotating black hole, and that the electric current $I$ flows along magnetic field lines. Then, the electric field $\mathbf{E}$, which is induced by plasma rotating with the angular velocity $\Omega_F$, and the toroidal magnetic field $\mathbf{B}_\phi$, which is due to the longitudinal current $I$, generate the electromagnetic energy flux (the Poynting vector flux) carrying the energy away along the magnetic field lines.

Of course, by definition, general relativity effects are important near the black hole. Consequently, it is not obvious that the pulsar analogy can be useful in all cases. For example, in radio pulsars, the braking of neutron stars results from the Ampère force acting on the star surface. This force results from the surface currents shorting the electric currents flowing in the pulsar magnetosphere $^{17}$. In the case of black holes such currents cannot lead to deceleration as the event horizon is not a physically preferred surface, though surface currents themselves can be formally introduced in the framework of the so-called membrane paradigm $^2$.

Indeed, let us consider the well-known condition at the horizon (the absence of infinite electromagnetic fields in the reference frame comoving with freely falling observer) $E'_\theta \to (E_\theta + B_\phi)/\alpha < \infty$. It means that...
\[ E_\theta + B_\varphi \rightarrow 0 \] which can be rewritten in the form

\[ 4\pi I(\Psi) = [\Omega_H - \Omega_F(\Psi)] \frac{r_s^2 + a^2}{r_g^2 + a^2 \cos^2 \theta} \sin \theta \left( \frac{d\Psi}{d\theta} \right)_{r_g}. \]  

(47)

Here \( a = J/M \) is the rotation parameter. This relation was used as the ‘boundary condition on the horizon’. But it is clear that the horizon is not causally connected with the outer space. For this reason the conclusion was made that there is no energy flux along magnetic field lines passing through the black hole horizon, and hence a black hole cannot work as a unipolar inductor extracting the rotation energy by electromagnetic stresses.

However, a recent more accurate analysis (in which, in fact, the first solution of the Grad-Shafranov equation for nonzero particle mass in the Kerr metric was obtained) indicates that the braking torque acts in the plasma generation region above the black hole horizon. Such a torque appears due to the action of long-range gravitomagnetic forces which penetrate into the regions causally connected with the outer space. As a result, it was shown that for a finite mass of particles in the very vicinity of the horizon, there is a hyperbolic region of the Grad-Shafranov equation which is altogether absent in the force-free approximation. Hence, the full version of the Grad-Shafranov equation needs no boundary condition on the horizon. Actually, this property was already demonstrated above by the example of the Bondi-Hoyle accretion. Thus, in this case it is impossible to consider relation (47) as a boundary condition. Relation (47) is automatically true for any solution of the Grad-Shafranov equation which can be extended up to the horizon.

On the other hand, in the force-free approximation, when the Grad-Shafranov equation remains elliptical up to the black hole horizon, extra condition (47) is to be included into consideration. But as was demonstrated, this condition is actually the manifestation of the critical condition on the fast magnetosonic surface. Hence, in reality this condition is specified on the surface which does not coincide with the event horizon, and, hence, is in the causal connection with the outer magnetosphere.

Thus, the Blandford-Znajek mechanism of electromagnetic energy extraction from rotating black holes faces no causality problem. As in the pulsar magnetosphere, if there is enough secondary plasma to screen the longitudinal electric field, its charge density and electric currents produce the flux of electromagnetic energy propagating from the central star to infinity. For the same reason, a rotating black hole embedded into an external magnetic field works as a unipolar inductor extracting its energy of rotation.
by the flux of the electromagnetic energy. As a result, the energy loss can be evaluated as $W_{\text{tot}} \approx W_{\text{BZ}}$, where

$$W_{\text{BZ}} = \frac{\Omega_F (\Omega_H - \Omega_F)}{\Omega_H} \left( \frac{a}{M} \right)^2 B_0^2 r_s c$$

$$\approx 10^{45} \left( \frac{a}{M} \right)^2 \left( \frac{B_0}{10^4 \text{G}} \right)^2 \left( \frac{M}{10^9 M_\odot} \right)^2 \text{erg/s.}$$  \quad (48)$$

It is easy to check that for the extreme rotation of a black hole ($a = 1$) and for $B = B_{\text{Edd}} \approx 10^4 (M/10^9 M_\odot)^{-1/2}$ G, the energy loss $W_{\text{BZ}}$ (48) coincides with the Eddington luminosity.

![Figure 3. Magnetospheres of rotating black hole (a) and radio pulsar (b).](image)

It should be emphasized, however, that as follows from equation (48), the rate of the energy release needed to explain the luminosity of active galactic nuclei can be achieved only for the extreme black hole mass $\sim 10^9 M_\odot$, the extreme magnetic field $B \sim B_{\text{Edd}}$, and the extreme angular velocity $a \sim M$. Therefore, some papers have recently appeared in which the efficiency of the Blandford-Znajek process in real astrophysical conditions was questioned. In particular, it was pointed out that for rapid rotation the Wald solution for the vacuum magnetosphere leads to the magnetic field being pushed out from the horizon into the ergosphere, which causes the appearance of the additional factor $(1 - a^2/M^2) \to 0$ in expression (48). But as shown in Fig. 3a, in the black hole magnetosphere filled with plasma, all magnetic field lines crossing the surface of the internal light
cylinder $\alpha^2 = (\Omega_F - \omega)^2 \varpi^2 + M^2$ are to cross the black hole horizon as well, that is why to an order of magnitude the energy loss for extremely rotating black holes coincides with the loss given by equation (48). Indeed, here the situation is to be fully analogous to the pulsar magnetosphere where the field lines passing the external light cylinder do not intersect the equatorial plane (Fig. 3b). The magnetic field structure shown in Fig. 3a was recently obtained numerically.

3. Conclusion

In some simple cases the Grad-Shafranov equation allows us to construct the exact solution to the problem. In particular, this approach is very useful in studying the analytical properties of transonic flows and in determining the required number of boundary conditions. On the other hand, in the general case no consistent procedure exists regarding the construction of the solution within the Grad-Shafranov approach. The point is that the location of singular surfaces, at which critical conditions should be formulated, is not known beforehand and itself must be found from the solution to the problem. Moreover, it is impossible to generalize this approach to the case of nonideal, non-axially symmetric and nonsteady flows. So it is not surprising that most investigators, who are in the first place interested in astrophysical applications, have focused on a totally different class of equations, namely on time relaxation problems, that can only be solved numerically. Nevertheless, it is clear that the key physical results obtained using the Grad-Shafranov approach are independent of the computing method. For this reason one can hope that the results presented above can be useful for everyone working in this field.

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