MAXIMAL TIME EXISTENCE OF UNNORMALIZED CONICAL KÄHLER-RICCI FLOW

LIANGMING SHEN

Abstract. We generalize the maximal time existence of Kähler-Ricci flow in Tian-Zhang [25] and Song-Tian [22] to conical case. Furthermore, if the twisted canonical bundle $K_M + (1 - \beta)[D]$ is big or big and nef, we can expect more on the limit behaviors of such conical Kähler-Ricci flow. Moreover, the results still hold for simple normal crossing divisor.

1. INTRODUCTION

Kähler-Ricci flow has become a powerful tool in the study of Kähler geometry for many years. In [25], Tian-Zhang adapted Tsuji’s idea [26] to consider Kähler-Ricci flow in the level of cohomology class on projective manifolds. They proved that Kähler-Ricci flow can exist in maximal time interval, where the cohomology class of the Kähler form $\omega(t)$ stays nondegenerate. Based on this maximal time existence, they obtained existence results and limit behavior analysis in cases that the canonical line bundle is big or big and nef. Later, in the series of papers [20] [21] [22], Song-Tian extended this idea to construct an analytic method of minimal model program in algebraic geometry.

Recently, conic Kähler metric has become an attracting topic for geometers. More specifically, conic Kähler-Einstein metric plays a critical role in recent great progress of Kähler-Einstein problem, see [23] [5] [6] [7]. Besides this seminal work, there are a lot of papers considering the existence of conical Kähler-Einstein metric, for instance, see [1] [2] [3] [11] [13] [16]. On the other hand, conical Kähler-Ricci flow has been studied in [14] [8] [9] [18] [27]. As the study of Kähler-Ricci flow, conical Kähler-Ricci flow first was considered in hope of evolving conic Kähler metrics to conical Kähler-Einstein metrics. First, we recall that a Kähler current $\omega$ is a conical Kähler metric with angle $2\pi \beta (0 < \beta < 1)$ along the divisor $D$, if $\omega$ is smooth away from $D$ and asymptotically equivalent along $D$ to the model conic metric

$$\sqrt{-1} \left( \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^{n} dz_i \wedge d\bar{z}_i \right),$$

where $(z_1, z_2, \ldots, z_n)$ are local holomorphic coordinates and $D = \{z_1 = 0\}$ locally. We call $\omega$ a conic Kähler-Einstein metric with angle $2\pi \beta (0 < \beta < 1)$ along the divisor $D$ if it is a conic Kähler metric and satisfies the equation

$$Ric(\omega) = \mu \omega + 2\pi(1 - \beta)[D]$$
in the sense of currents, and a smooth Kähler-Einstein metric outside the divisor D. In [8] [9] [18], they set \( \mu = \beta \) and studied such normalized conical Kähler-Ricci flow

\[
\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) + \beta \omega + 2\pi(1 - \beta)[D]
\]

starting with a conical Kähler metric with angle \( 2\pi \beta \) along the anticanonical divisor D. In [18], Liu-Zhang used smooth approximation of conic metric, which was set up by Campana-Guenancia-Păun in [3], [11] to obtain a long time solution for the normalized normalized conical Kähler-Ricci flow. Moreover, they proved that when \( \beta \in (0, \frac{1}{2}] \), the flow converges to a conical Kähler-Einstein metric if it exists.

Now we can consider whether the conical Kähler-Ricci flow exists on general Kähler manifolds starting with arbitrary conic Kähler metric, and how long it will last, moreover, what will happen when the flow extincts. Actually, we can extend the work of Tian-Zhang [25] and Song-Tian [22] to conic case. And in the study of these problems, we can generalize the irreducible divisor D to be a simple normal crossing divisor, i.e,

\[
D = \sum_{i=1}^{l} (1 - \beta_i)D_i (l \leq n),
\]

where each \( D_i \) is irreducible and \( \beta_i \in (0, 1) \).

Locally near a point \( p \in \bigcap_i D_i \), \( D_i = \{ z_i = 0 \} \). Then a conic metric \( \omega \) along each \( D_i \) with cone angle \( 2\pi \beta_i \) is asymptotically equivalent to the model metric

\[
\omega_\beta = \sqrt{-1} \left( \sum_{i=1}^l \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_i)}} + \sum_{i=l+1}^n dz_i \wedge d\bar{z}_i \right).
\]

And the unnormalized conical Kähler-Ricci flow is

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) + 2\pi \sum_{i=1}^l (1 - \beta_i)[D_i] \\
\omega(0) = \omega^* = \omega_0 + \sum_{i=1}^l k\sqrt{-1}d\bar{\partial}\|S_i\|_{S_i}^{2\beta_i} 
\end{array} \right. \tag{1.1}
\]

where the initial metric is conic Kähler metric \( \omega^* \) with conic angles \( 2\pi \beta_i \) along \( D_i \). Suppose \( S_i \) is a holomorphic section of the bundle associated to the divisor \( D_i \) then this conic Kähler metric can be written as \( \omega^* = \omega_0 + \sum_{i=1}^l k\sqrt{-1}d\bar{\partial}\|S_i\|_{S_i}^{2\beta_i} \), where \( \omega_0 \) is a smooth Kähler metric on \( M \), \( k \) is a small positive constant and \( || \cdot ||_{S_i}^2 \) is a Hermitian metric associated to \([D_i]\). Similar to Tian-Zhang and Song-Tian, we can consider this flow equation (1.1) in cohomology lever and obtain the following maximal time existence theorem:

**Theorem 1.1.** Let

\[
T_0 := \sup\{ t \mid |\omega_0| - t(c_1(M) - \sum_{i=1}^l (1 - \beta_i)[D_i]) > 0 \},
\]

then starting with a conic Kähler metric \( \omega^* \) defined above, the unnormalized conical Kähler-Ricci flow (1.1) has a unique solution on \([0, T_0]\), which is smooth outside the divisors \( D_i \) and \( C^{2,\alpha} \) along the divisors \( D_i \).

we can prove this theorem by Tian-Zhang’s method [25]. However, as the conic metric is not smooth, we need to apply Liu-Zhang’s approach, making use of the approximation developed in
to obtain a family of solutions for the approximating equation, then we prove this family of solutions converge to the solution of the original flow equation. Finally, we prove the solution is smooth outside the divisors $D_i$ and $C^{2,\alpha}$ along the divisor in suitable sense.

We recall that in [25] [20] [21] [22], Kähler-Ricci flow is used to study minimal model program. More precisely, they analysed the behavior of Kähler-Ricci flow near the singular time or infinite time on different types of projective manifolds. As we mentioned before that conic Kähler metric has becoming an interesting topic, we hope that unnormalized conical Kähler-Ricci flow can play some similar role in the study of conic Kähler metrics on projective manifolds. We note that conical Kähler-Ricci flow always preserves conical singularities so it’s natrual to expect that we can have a good picture of the long time behavior besides conical singularities. Similarly, if we assume that so-called twisted canonical line bundle $K_M + \sum_{i=1}^{l}(1 - \beta)[D_i]$ is big, we have such a theorem, as [25],

**Theorem 1.2.** If the twisted canonical line bundle is big on the projective manifold $M$, starting with a conic Kähler metric $\omega^*$, as $t$ tends to $T_0 < \infty$ in (1.1), the solution converges to a Kähler current which is a smooth Kähler metric outside the divisor $D$ and the stable base locus set of $K_M + \sum_{i=1}^{l}(1 - \beta)[D_i]$, say $E$, and $C^{2,\alpha}$ with conic angle $2\pi\beta_i$ along $D_i \setminus E$. Moreover, on smooth part, the flow solution converges to the limiting metric in the local $C^\infty$-sense and $C^{2,\alpha}$ along the divisor $D \setminus E$.

Furthermore, if we assume the twisted canonical line bundle $K_M + \sum_{i=1}^{l}(1 - \beta)[D_i]$ is big and nef, by [1.1] we can see that $T_0 = \infty$. To get a more clear picture of the limiting metric, we can normalized such conical Kähler-Ricci flow equation (1.1) to

$$\frac{\partial}{\partial t} \omega = -\omega - \text{Ric}(\omega) + 2\pi \sum_{i=1}^{l}(1 - \beta)[D_i]).$$

We note that the solutions to these two flow equations are different only by scaling. Then similarly we have such a theorem

**Theorem 1.3.** All conditions follow 1.2, additionally, if the twisted canonical line bundle $K_M + \sum_{i=1}^{l}(1 - \beta)[D_i]$ is big and nef, then starting from a conic Kähler metric $\omega^*$, the normalized conical Kähler-Ricci flow (1.2) converges to a current as $t$ tends to infinity such that modulo the stable base locus $E$, the limiting current is a conical Kähler-Einstein metric which is $C^{2,\alpha}$ with conic angle $2\pi\beta_i$ along $D_i \setminus E$, and independent of the choice of the initial conic metric.

Later we will introduce the definitions of bigness and numerical effectiveness. For the proof, we will make use of Kodaira’s lemma and set up A priori estimates outside the base locus. Note that we still need Liu-Zhang’s approach to overcome the conic singularities. For simplicity, in this paper, we only study the conic metrics along only one irreducible divisor $D$ with cone angle $2\pi\beta$, and in the end we will briefly discuss how to generalize the proofs to simple normal crossing case.

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2. Maximal Time Existence

2.1. Approximation, and $C^0$-Estimate. Now we restrict the situation to $l = 1$, i.e., there is only one irreducible divisor $D$. Then the unnormalized conical Kähler-Ricci flow is

$$\begin{align*}
\frac{\partial}{\partial t} \omega &= -Ric(\omega) + 2\pi(1 - \beta)|D| \\
\omega(0) &= \omega^* = \omega_0 + k\sqrt{-1}\partial\bar{\partial}|S|^2^\beta
\end{align*}$$

(2.1)

As [25] [22], first we want to transform the conical Kähler-Ricci flow equation (2.1) to Monge-Ampere flow equation. We set $T_\delta = T_0 - \delta$, where $T_0 := \sup\{t|\omega_0 - t(c_1(M) - (1 - \beta)|D|) > 0\}$, as we defined before. We know that when $t \in [0, T_\delta]$, the cohomology class $[\omega_0 - t(c_1(M) - (1 - \beta)|D|)$ is positive. Then when $t \in [0, T_\delta]$, there exist a hermitian metric $|| \cdot ||$ on the holomorphic line bundle associated to the divisor $D$ and a volume form $\Omega$ on $M$, such that

$$\omega_t = \omega_0 - t(Ric(\Omega) - (1 - \beta)R(|| \cdot ||)) > 0$$

(2.2)

where $R(|| \cdot ||) := -\sqrt{-1}\partial\bar{\partial}\log|| \cdot ||^2$ represents the curvature of the bundle $[D]$. Now we can write

$$\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t$$

where $k$ is so small that $\omega_t$ is positive as $t \in [0, T_\delta]$. Then if we set $\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi$, we can write the conical Kähler-Ricci flow (2.1) as following:

$$\frac{\partial}{\partial t}(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi) = \sqrt{-1}\partial\bar{\partial}\log(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n + (1 - \beta)\sqrt{-1}\partial\bar{\partial}\log|S|^2.$$ 

Plug the equation (2.2) into it, we obtain a Monge-Ampere flow equation:

$$\frac{\partial}{\partial t}\varphi = \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} + \log||S||^{2(1 - \beta)}$$

(2.3)

To approximate the solution to this equation, we first use the smoothing metric in [3] that

$$\omega_{t, \epsilon} = \omega_t + \epsilon\sqrt{-1}\partial\bar{\partial}\chi(\epsilon^2 + ||S||^2),$$

(2.4)

where

$$\chi(\epsilon^2 + t) = \beta \int_0^t \left(\frac{\epsilon^2 + r}{r}\right)^\beta - \epsilon^{2\beta}dr.$$ 

Before approximating the flow equation, we want to introduce some useful properties of this function $\chi(\epsilon^2 + t)$. First, we can find that for each $\epsilon > 0$, $\omega_{t, \epsilon}$ is a smooth Kähler metric, and as $\epsilon$ tends to $0$, $\omega_{t, \epsilon}$ converges to a conic metric $\omega_t$ in the sense of currents globally on $M$ and in $C^\infty_{loc}$ sense outside $D$. Moreover, for each $\epsilon > 0$, this function is smooth, and there exist constants $C > 0$ and $\gamma > 0$ independent of $\epsilon$ such that as $t$ is finite $0 \leq \chi(\epsilon^2 + t) \leq C$, and $\omega_{0, \epsilon} \geq \gamma\omega_0$. As for $t \in [0, T_\delta]$ $\omega_t$ is comparable to $\omega_0$ we can still use a constant $\gamma$ such that $\omega_{t, \epsilon} \geq \gamma\omega_0$.

Now we can introduce our approximation equation of (2.3) as following

$$\begin{align*}
\frac{\partial}{\partial t}\varphi_{t, \epsilon} &= \log\frac{(\omega_{t, \epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_{t, \epsilon})^n}{\varphi_{t, \epsilon}(\cdot, 0)} + \log(||S||^2 + \epsilon^2)^{1 - \beta} \\
\varphi_{t, \epsilon}(\cdot, 0) &= 0
\end{align*}$$

(2.5)

Equivalently, this equation can be written as the form of generalized Kähler-Ricci flow:

$$\begin{align*}
\frac{\partial}{\partial t}\omega_{t, \epsilon} &= -Ric(\omega_{t, \epsilon}) + (1 - \beta)\sqrt{-1}\partial\bar{\partial}\log\frac{||S||^{2+\epsilon^2}}{||S||^2} \\
\omega_{t, \epsilon}(\cdot, 0) &= \omega_{0, \epsilon}
\end{align*}$$

(2.6)
Note that the elliptic version of such approximation equations was set up in [23]. The main steps here are also similar to Tian’s approximation. We need to set up $C^0$-estimate, $C^2$-estimate and high order derivative estimate step by step and finally complete approximation argument. In the rest of this subsection we will complete $C^0$-estimate and leave other steps to the next subsection.

Now let’s consider the equation (2.5). We can rewrite this equation as following
\[
\frac{\partial}{\partial t} \varphi_\epsilon = \log \left( \frac{\omega_{t,\epsilon} + \sqrt{-1}\partial \bar{\partial} \varphi_\epsilon}{\Omega} \right)^n (||S||^2 + \epsilon^2)^{1-\beta}.
\]
To get an upper bound for $\varphi_\epsilon$, we can make use of maximal principle to get that
\[
\frac{\partial}{\partial t} \sup \varphi_\epsilon \leq \sup \log \frac{\omega_{t,\epsilon}^n (||S||^2 + \epsilon^2)^{1-\beta}}{\Omega}.
\]
Recall the definition of the function $\chi(\epsilon^2 + t)$, we know from [3],
\[
\sqrt{-1}\partial \bar{\partial} \chi(\epsilon^2 + ||S||^2) = \frac{\beta^2 \sqrt{-1}DS \wedge DS}{(\epsilon^2 + ||S||^2)^{1-\beta}} - \beta((\epsilon^2 + ||S||^2)^{\beta} - \epsilon^2)R(|| \cdot ||).
\]
From this computation, we observe that near the divisor $D$,
\[
\frac{\omega_{t,0}^n (||S||^2 + \epsilon^2)^{1-\beta}}{\Omega} \approx C,
\]
which is independent of $\epsilon$. So we can get a uniform upper bound that $\varphi_\epsilon \leq C_1$. Do the same argument for the lower bound, make use of maximal principle again, we can give a uniform $C^0$-estimate that
\[
|\varphi_\epsilon| \leq C_1.
\]

Next we hope to bound the time derivative for $\varphi_\epsilon$. We can apply the technique developed in [22]. For simplicity, we write $\rho = -\text{Ric}(\Omega) + (1 - \beta)R(|| \cdot ||) = \omega_{t,\epsilon}$. Take the derivative of (2.5), we have
\[
\frac{\partial}{\partial t} \dot{\varphi}_\epsilon = \Delta \dot{\varphi}_\epsilon + tr\omega \rho,
\]
where $\omega = \omega_{t,\epsilon} + \sqrt{-1}\partial \bar{\partial} \varphi_\epsilon$, and $\Delta$ is the Laplacian w.r.t $\omega$. First, we compute that
\[
(\frac{\partial}{\partial t} - \Delta)(t\dot{\varphi}_\epsilon - \varphi_\epsilon - nt) = tr\omega \rho + n - tr\omega \omega_{t,\epsilon} - n = -tr\omega \omega_{0,\epsilon} < 0.
\]
From this we have $\dot{\varphi}_\epsilon \leq n + \frac{C_1}{t}$. Combined with the equation (2.5) at time $t=0$, and local existence of parabolic equation, we obtain a uniform upper bound for $\dot{\varphi}_\epsilon$. Now we try to derive the lower bound for $\dot{\varphi}_\epsilon$,
\[
(\frac{\partial}{\partial t} - \Delta)(\dot{\varphi}_\epsilon + A\varphi_\epsilon - n \log t) = tr\omega (\rho + A\omega_{t,\epsilon}) + A \log \frac{\omega_t^n (||S||^2 + \epsilon^2)^{1-\beta}}{\Omega} - An - \frac{n}{t} \\
\geq C \left( \frac{\omega_{t,0}^n}{\omega_0^n} \right)^\frac{1}{n} + A \log \frac{\omega_t^n (||S||^2 + \epsilon^2)^{1-\beta}}{\Omega} - An - \frac{n}{t} \\
\geq C_1 \left( \frac{\omega_{0,0}^n}{\omega_0^n} \right)^\frac{1}{n} - \frac{C_2}{t}.
\]
Let’s explain these two inequalities. For the first, note that we can choose \( A \) sufficiently large such that \( \rho + A \omega \geq \omega_{0, \epsilon} \) in the time interval, then this inequality follows from Schwarz inequality.

For the second, note that in the proof of \( C^0 \)-estimate, we have got that \( \frac{\omega^n_{0, \epsilon} ||S||^2 + \epsilon^2}{\Omega} \approx C \), then this inequality follows from the behavior of logarithmic functions. Then by maximal principle, at the minimal point of the function \( \dot{\psi}_\epsilon + A \psi \geq n \log t \), we have that \( \omega \geq C t^n \omega_{0, \epsilon}^n \). Make use of this inequality and the relation of \( \omega_{0, \epsilon} \) and \( \Omega \) again, we get that

\[
\dot{\psi}_\epsilon + A \psi - n \log t = \log \frac{\omega^n(||S||^2 + \epsilon^2)^{1-\beta}}{\Omega} + A \psi - n \log t \\
\geq \log \frac{C t^n \omega_{0, \epsilon}^n (||S||^2 + \epsilon^2)^{1-\beta}}{\Omega} + A \psi - n \log t \geq C_5.
\]

As \( \psi \) is uniformly bounded, we get a uniform lower bound for \( \dot{\psi}_\epsilon \).

2.2. \( C^2 \)-estimate, and high order estimate outside the divisor. Now we continue our proof. We want to set up a uniform \( C^2 \)-estimate and high order estimate for \( \psi \). First, like \cite{[22], [3], [18]}, we prove a Laplacian estimate, which is essentially an application of generalized Schwarz lemma:

**Theorem 2.1.** Let \( \psi \) solve the equation (2.5). As we have

\[
|\psi| \leq C, \quad |\dot{\psi}| \leq C,
\]

on \([0, T_\delta]\), then on this time interval, there exists a uniform constant \( A \) which is independent of \( \epsilon \), such that

\[
A^{-1} \omega_{0, \epsilon} \leq \omega_{t, \epsilon} + \sqrt{-1} \partial \bar{\partial} \psi \leq A \omega_{0, \epsilon}.
\]

**Proof.** First we take an holomorphic orthonormal coordinates \((z_1, \cdots, z_n)\) at a point \( p \) for the metric \( \omega_{0, \epsilon} \), say, \( g_{0ij} = \delta_{ij} \), such that \( g_{ij}(\omega_{t, \epsilon}) = \lambda_i \delta_{ij}, \psi_{t, \epsilon} = \psi \), \( \partial \bar{\partial} \psi \). Write \( g_{ij} \) as the metric of \( \omega = \omega_{t, \epsilon} + \sqrt{-1} \partial \bar{\partial} \psi \) and assume that under such coordinates, \( \frac{\partial g_{ij}}{\partial z_k} = 0 \). From approximation
flow equations (2.5) (2.6), we compute that
\[
(\frac{\partial}{\partial t} - \Delta) \log \|S\|^2 + \epsilon^2 = \frac{1}{\|S\|^2 + \epsilon^2} \|\nabla \|S\|^2 + \epsilon^2 \| \|^2 
- g_{ij} g^{kk} \{ - R_{kkl} g^{li} + R_{kikk}(\omega_{0,\epsilon}) + \sum_j \frac{\partial g_{0lj}}{\partial z_k} \frac{\partial g_{0lj}}{\partial z_k} \} + \frac{1}{\|S\|^2 + \epsilon^2} \|\nabla \|S\|^2 + \epsilon^2 \| \|^2 
- \sum_{i \leq j} \hat{R}_{ij\hat{j}}(\omega_{0,\epsilon}) (\frac{\lambda_i + \varphi_{\epsilon \hat{i}}}{\lambda_j + \varphi_{\epsilon \hat{j}}} + \frac{\lambda_j + \varphi_{\epsilon \hat{j}}}{\lambda_i + \varphi_{\epsilon \hat{i}}} - 2).
\]

As [13], we choose a function \( \chi_\rho(||S||^2 + \epsilon^2) = \rho \int_0^{||S||^2 + \epsilon^2} \frac{(e^2 + ||S||^2)^{\rho - \epsilon^2 \rho}}{r} \). By the computation in [3] [11] we have that
\[
\hat{R}_{ij\hat{j}}(\omega_{0,\epsilon}) \geq -C_1 - \frac{C_2}{(||S||^2 + \epsilon^2)^{1-\beta}}.
\]

Meanwhile, we have
\[
\sqrt{-1} \partial \bar{\partial} \chi_\rho(\epsilon^2 + ||S||^2) = \rho^2 \sqrt{-1} DS \wedge DS \frac{(e^2 + ||S||^2)^{\rho - \epsilon^2 \rho}}{r} - \rho((e^2 + ||S||^2)^\rho - \epsilon^2 \rho) R(|| \cdot ||),
\]
so if we choose suitable constants \( \rho, C \) we can obtain that
\[
\hat{R}_{ij\hat{j}}(\omega_{0,\epsilon}) \geq -C - C' \chi_{\rho i\hat{i}},
\]
then we have such estimate
\[
\frac{1}{\|S\|^2 + \epsilon^2} \{ \sum_{i \leq j} \hat{R}_{ij\hat{j}}(\omega_{0,\epsilon}) \left( \frac{\lambda_i + \varphi_{\epsilon \hat{i}}}{\lambda_j + \varphi_{\epsilon \hat{j}}} + \frac{\lambda_j + \varphi_{\epsilon \hat{j}}}{\lambda_i + \varphi_{\epsilon \hat{i}}} - 2 \right) \}
\leq \sum_i \frac{1}{\lambda_i + \varphi_{\epsilon \hat{i}}} \sum_{i \leq j} \left( \frac{\lambda_i + \varphi_{\epsilon \hat{i}}}{\lambda_j + \varphi_{\epsilon \hat{j}}} (C + C' \chi_{\rho j\hat{j}}) + \frac{\lambda_j + \varphi_{\epsilon \hat{j}}}{\lambda_i + \varphi_{\epsilon \hat{i}}} (C + C' \chi_{\rho i\hat{i}}) \right)
\leq \sum_i \frac{C + C' \chi_{\rho i\hat{i}}}{\lambda_i + \varphi_{\epsilon \hat{i}}} = C \|S\|^2 + C' \Delta \chi_\rho.
\]
Now we want to estimate the other term. First we have that
\[(1 - \beta)\Delta_{\omega_0,\epsilon} \log \frac{||S||^2 + \epsilon^2}{|| \cdot ||^2} - R(\omega_{0,\epsilon})\]
\[= \Delta_{\omega_0,\epsilon} \log \frac{||S||^2 + \epsilon^2}{\Omega} - tr_{\omega_0,\epsilon} Ric(\Omega) + (1 - \beta)tr_{\omega_0,\epsilon} R(|| \cdot ||).\]

Now recall that \(\omega_{0,\epsilon} \geq \gamma \omega_0\), and \(Ric(\Omega), R(|| \cdot ||)\) are uniformly bounded independent of \(\epsilon\), moreover by \([3, 11]\),
\[
\sqrt{-1} \partial \bar{\partial} \log \frac{(||S||^2 + \epsilon^2)^{1-\beta} \omega_{0,\epsilon}^n}{\Omega} \leq C \omega_{0,\epsilon} + C \sqrt{-1} \partial \bar{\partial} \chi_\rho,
\]
we can bound this term. Considering that \(tr_{\omega_0,\epsilon} \omega \cdot tr_{\omega_0,\epsilon} \geq n\), we have
\[
\frac{\partial}{\partial t} - \Delta (\log tr_{\omega_0,\epsilon} \omega + C' \chi_\rho - B \varphi_\epsilon) \leq C tr_{\omega_0,\epsilon} + C C tr_{\omega_0,\epsilon} - B \varphi_\epsilon + B n - B tr_{\omega_0,\epsilon}.
\]

As \(\omega_{0,\epsilon}\) and \(\omega_{t,\epsilon}\) are equivalent, and \(\varphi_\epsilon\) are uniformly bounded, choose a suitable constant \(B\), we obtain that
\[
\frac{\partial}{\partial t} - \Delta (\log tr_{\omega_0,\epsilon} + C' \chi_\rho - B \varphi_\epsilon) \leq C - tr_{\omega_0,\epsilon}.
\]

By the maximal principle, at the maximal point \(p\) of \(\log tr_{\omega_0,\epsilon} + C' \chi_\rho - B \varphi_\epsilon\) we have \(tr_{\omega_0,\epsilon}(p) \leq C\). By the approximation flow equation \((2.5)\), we know that
\[
\frac{\omega_{\epsilon}}{\omega_{0,\epsilon}^n} = e^{\varphi_\epsilon} \frac{\Omega}{(||S||^2 + \epsilon^2)^{1-\beta} \omega_{0,\epsilon}^n}
\]
is uniformly bounded from above and away from \(0\), then we obtain that at that point \(p,\)
\[
tr_{\omega_0,\epsilon}(p) \leq \frac{\omega_{\epsilon}^n}{\omega_{0,\epsilon}^n} (tr_{\omega_0,\epsilon}(p)^{n-1} \leq C.
\]

As at that point \(p, \log tr_{\omega_0,\epsilon} + C' \chi_\rho - B \varphi_\epsilon\) attains the maximal, we conclude that on the whole manifold \(M, tr_{\omega_0,\epsilon} \omega \leq C\). Now make use of the inequality above again we have that
\[
tr_{\omega_0,\epsilon} \leq \frac{\omega_{\epsilon}^n}{\omega_{0,\epsilon}^n} (tr_{\omega_0,\epsilon})^{n-1} \leq C,
\]
which gives us a uniform constant \(A > 0\) such that
\[
A^{-1} \omega_{0,\epsilon} \leq \omega_{t,\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon \leq A \omega_{0,\epsilon}.
\]

Now let’s consider high order estimates. We know that until now we don’t have a uniform \(C^3\)-estimate for conic Kähler-Einstein metrics. However, to get \(C^\infty\)-convergence away from the divisor, we only need to have local high order estimates away from the divisor. Note that from the two forms of approximation flow equations \((2.5) (2.6)\), compare with \([18]\), we find that our generalized Kähler-Ricci flow has the form that
\[
\frac{\partial}{\partial t} \omega = -Ric(\omega) + \theta,
\]
where $\theta = (1 - \beta)\sqrt{-1}\partial \bar{\partial} \log \frac{||S||^2 + \epsilon^2}{||\nu||^2}$ is uniformly bounded away from the divisor. And we can also define that
\[
S = |\nabla_0 g|_\omega^2 = g^{ji} g^{pq} \nabla_0 g_{kj} \nabla_j g_{pi},
\]
where $g$ is the metric of $\omega = \omega_t + \sqrt{-1}\partial \bar{\partial} \varphi = \omega_0 - t(Ric(\Omega) - (1 - \beta)R(|| \cdot ||)) + \sqrt{-1}\partial \bar{\partial} \varphi$, and $\nabla_0$ denotes the covariant derivatives with respect to $\omega_0$. Then here we can apply Proposition 2.2 in [18] directly:

**Theorem 2.2.** Let $\omega = \omega_t + \sqrt{-1}\partial \bar{\partial} \varphi$ solve the generalized Kähler-Ricci flow (2.8) and satisfy
\[
A^{-1} \omega_0 \leq \omega \leq A \omega_0 \quad \text{on} \quad B_r(p) \times [0, T].
\]
Then there exist constant $C', C''$ such that
\[
S \leq \frac{C'}{t^2}, \quad |Rm(\omega)|^2 \leq \frac{C''}{t^4}
\]
on $B_{\frac{r}{2}}(p) \times [0, T]$. Here $C'$ depends on $\omega_0, T, A, ||\varphi(., 0)||_{C^3(B_r(p))}, ||\theta||_{C^1(B_r(p))}$ and $C''$ depends on $\omega_0, T, A, ||\varphi(., 0)||_{C^4(B_r(p))}, ||\theta||_{C^2(B_r(p))}$.

Locally outside the divisor, metrics $\omega_t, \omega_{0, \epsilon}, \omega_{t, \epsilon}$ are equivalent, and derivatives of these metrics are also uniformly bounded on the time interval, then we have $C^3$-local estimates for $\varphi_t$ under the metric $\omega_t$. By this theorem, using bootstrap methods we can also get local uniform high order estimates outside the divisor $D$.

2.3. uniqueness and convergence. In this subsection we want to examine how the approximation solutions converge and what the limit likes. First, in the sections above we proved that on time interval $[0, T_\delta]$ we have a priori estimates for the solution of approximation flow equation (2.5). Now we hope to solve this equation on $[0, T)$. What remains to do is to prove the solution is independent of the choice of $\delta$. We can prove this as [25]. We recall that for each $\delta > 0$, we can find a volume form $\Omega$ such that $\omega_t = \omega_0 - t(Ric(\Omega) - (1 - \beta)R(|| \cdot ||)) > 0$ on $[0, T_\delta]$. Now we suppose that for another $\delta' > 0$, we have $\Omega'$ such that $\omega_t' = \omega_0 - t(Ric(\Omega') - (1 - \beta)R(|| \cdot ||)) > 0$.

Now we assume that $\Omega' = e^\varphi \Omega$ where $f$ is a smooth function on $\mathcal{M}$ and $\varphi'$ solves the equation
\[
\frac{\partial}{\partial t} \varphi' = \log \frac{(\omega_t' + \sqrt{-1}\partial \bar{\partial} \varphi')^n(||S||^2 + \epsilon^2)^{1 - \beta}}{\Omega'}, \quad \varphi'(\cdot, 0) = 0.
\]
Now put $\varphi_\epsilon = \varphi' + tf$, as we have $Ric(\Omega') = Ric(e^\varphi \Omega) = Ric(\Omega) - \sqrt{-1}\partial \bar{\partial} f$, we compute that
\[
\frac{\partial}{\partial t} \varphi_\epsilon = \frac{\partial}{\partial t} \varphi' + f = \log \frac{(\omega_t' + \sqrt{-1}\partial \bar{\partial} \varphi')^n(||S||^2 + \epsilon^2)^{1 - \beta}}{\Omega'} + f
\]
\[
= \log \frac{(\omega_t + t\sqrt{-1}\partial \bar{\partial} f + \sqrt{-1}\partial \bar{\partial} \varphi')^n(||S||^2 + \epsilon^2)^{1 - \beta}}{e^f \Omega} + f
\]
\[
= \log \frac{(\omega_t + \sqrt{-1}\partial \bar{\partial} \varphi)^n(||S||^2 + \epsilon^2)^{1 - \beta}}{\Omega},
\]
and $\varphi_\epsilon(\cdot, 0) = 0$. From this we find that $\varphi_\epsilon$ just solves the equation (2.5). By the uniqueness of the solution for (2.5) we know $\varphi_\epsilon = \varphi_\epsilon$, which means that for arbitrary $\delta, \delta'$, their corresponding solutions are the same essentially. So we can glue these solutions together to get a solution for (2.5) on the time interval $[0, T)$. 
Finally, we want to prove that as \( \epsilon \) tends to 0, the solution \( \varphi_\epsilon \) to (2.5) converges to a solution of unnormalized conical Kähler-Ricci flow in suitable sense. By the argument above, for simplicity we can prove this on some closed time interval \([0, T_\delta]\). We can prove this as [18]. Recall that we have uniform bound for \( \varphi_\epsilon, \dot{\varphi}_\epsilon \) and Laplacian estimate \( A^{-1}\omega_{0, \epsilon} \leq \omega_{0, \epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon \leq A\omega_{0, \epsilon} \). For any compact set \( K \subseteq M \setminus \{D\} \) we have a number \( N \) which depends on \( A \) and \( K \) such that \( N^{-1}\omega_0 \leq \omega \leq N\omega_0 \), and on this set \( K \log(||S||^2 + \epsilon^2) \) converges to \( \log||S||^2 \), \( \omega_{0, \epsilon} \) converges to \( \omega^* \) in \( C_{loc}^\infty \) sense. Let \( K \) approximate \( M \setminus \{D\} \) and \( \epsilon_i \) tend to 0, on time interval \([0, T_\delta]\), by diagonal rule we have a sequence which is denoted by \( \varphi_\epsilon(t), \dot{\varphi}_\epsilon(t) \), and converges to a function \( \varphi(t) \), which is smooth away from \( D \), in \( C_{loc}^\infty \) sense outside \( D \). By [23] we know that \( \omega_\varphi(t) = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi(t) \) are conic Kähler metrics with cone angle \( 2\pi \beta \) along the divisor \( D \) on the whole time interval.

Now let’s show that the limit potential \( \varphi(t) \) we defined above satisfies the conical Kähler-Ricci flow equation (2.3) globally on \( M \times [0, T_\delta] \) in the sense of currents. For any \((n - 1, n - 1)\)-form \( \eta \), by the argument above, as \( \epsilon_i \) tends to 0, we have

\[
\int_M \sqrt{-1} \partial \bar{\partial} \frac{\partial \varphi_\epsilon}{\partial t} \wedge \eta = \int_M \log \frac{(\omega_{\epsilon, \delta} + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon)^n(||S||^2 + \epsilon^2)^{1-\beta}}{\Omega} \wedge \sqrt{-1} \partial \bar{\partial} \eta \\
\to \int_M \log \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n||S||^{2(1-\beta)}}{\Omega} \wedge \sqrt{-1} \partial \bar{\partial} \eta \\
= \int_M \sqrt{-1} \partial \bar{\partial} \log \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n||S||^{2(1-\beta)}}{\Omega} \wedge \eta.
\]

On the other hand, let \( K \subset \subset M \setminus \{D\} \) and when \( K \) approximates \( M \setminus \{D\} \), we have \( \int_{M \setminus \{D\}} \sqrt{-1} \partial \bar{\partial} \eta \) tends to 0. As we have \( \dot{\varphi}_\epsilon, \dot{\varphi} \) are uniformly bounded independent of \( \epsilon \), we see that

\[
| \int_M (\dot{\varphi}_\epsilon - \dot{\varphi}) \wedge \sqrt{-1} \partial \bar{\partial} \eta | \leq | \int_K (\dot{\varphi}_\epsilon - \dot{\varphi}) \wedge \sqrt{-1} \partial \bar{\partial} \eta | + | \int_{M \setminus K} (\dot{\varphi}_\epsilon - \dot{\varphi}) \wedge \sqrt{-1} \partial \bar{\partial} \eta |.
\]

The first integral tends to 0 as \( \varphi_\epsilon \) converges to \( \varphi \) in the sense of \( C^\infty(K) \), and the second one tends to as \( \dot{\varphi}_\epsilon, \dot{\varphi} \) are uniformly bounded and \( K \) approximates \( M \setminus \{D\} \). From above we can see that \( \varphi(t) \) satisfies the conical Kähler-Ricci flow equation (2.3) globally on \( M \times [0, T) \) in the sense of currents.

To prove the uniqueness we argue as Wang [27], which comes from Jeffery’s argument [12]. First we need to verify that \( \varphi(t) \) is Hölder continuous with respect to \( \omega_0 \). As [18] write \( \phi = \varphi + k||S||^{2\beta}, \) in any closed time interval, say \([0, T_\delta]\), we have uniform bound for \( \phi, \dot{\phi} \). Then we rewrite the equation of conical Kähler-Ricci flow (2.3) as

\[
(\omega_t + \sqrt{-1} \partial \bar{\partial} \phi)^n = e^{\phi} \frac{\Omega}{||S||^{2(1-\beta)}}
\]

away from \( D \). As \( \beta \in (0, 1) \) we find a constant \( \eta \) such that \( 2(1-\beta)(1+\eta) < 2 \) such that

\[
\int_M e^{(1+\eta)(\phi - \log||S||^{2(1-\beta)})} \Omega \leq C \int_M \frac{\Omega}{||S||^{2(1-\beta)(1+\eta)}} \leq C.
\]

By Kolodziej’s \( L^p \) estimate [15] we know that \( \varphi(t) \) is Hölder continuous with respect to \( \omega_0 \) on \([0, T_\delta]\), actually, on \([0, T)\). Now suppose there are two solutions \( \varphi_1(t), \varphi_2(t) \) which satisfy all the properties above and solve the conical Kähler-Ricci flow on the maximal time interval, then we
set \( \varphi_1 = \varphi_1(t) + a||S||^{2p} \) and write \( v = \varphi_1 - \varphi_2 \). Compare corresponding conical Kähler-Ricci flow equations, we have that

\[
\frac{\partial}{\partial t} v = \frac{\partial}{\partial t}(\varphi_1 - \varphi_2) = \log \left( \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi_1)^n}{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi_2)^n} \right) = \Delta v - a \Delta ||S||^{2p},
\]

(2.9)

where

\[
\Delta := \int_0^1 (\tilde{g}_t + \sqrt{-1} \partial \bar{\partial}((1 - s)\varphi_1 + s\varphi_2))^{i\bar{j}} \partial_i \bar{\partial}_j ds.
\]

Outside the divisor \( D \) we can compute that

\[
\sqrt{-1} \partial \bar{\partial}||S||^{2p} = p||S||^{2p} \sqrt{-1} \partial \bar{\partial}||S||^2 + \sqrt{-1} p^2 ||S||^{2p-2} \partial \bar{\partial}||S||^2 
\]

\[
\geq -p||S||^{2p} R(||.||).
\]

By the estimates above, we know that the metric \( \omega_t + \sqrt{-1} \partial \bar{\partial}((1 - s)\varphi_1 + s\varphi_2) \) is equivalent to the initial conic metric \( \omega^* \) and \( C^\alpha \) outside the divisor \( D \). Then from (2.9) we obtain that

\[
\frac{\partial}{\partial t} v \leq \Delta v + aC,
\]

by maximal principle, we have

\[
v(t) \leq \sup v(0) + aCt = a(Ct + ||S||^{2p}).
\]

Let a tend to 0, we have \( \varphi_1 \leq \varphi_2 \), and we can prove \( \varphi_2 \leq \varphi_1 \) by the same argument. Finally we obtain the uniqueness of the limit solution.

2.4. \( C^{2,\alpha} \)-estimate on the divisor \( D \). To complete the proof of Theorem 1.1 we need to give a \( C^{2,\alpha} \)-estimate for \( \varphi(t) \), as this estimate allows us to apply inverse function theorem to obtain the existence of the solution to conical Kähler-Ricci flow (2.3). In this part, we will only introduce the definition of these norms under conic setting and leave the proof to the next paper.

Let’s describe the basic construction. First we can fix a unit polydisk \( \mathbb{D}^n \subset \mathbb{C}^n \) with the origin as the center, and the divisor \( D = \{z_1 = 0\} \). Then the standard conic metric attached with \( (\mathbb{C}^n, D) \) is that

\[
\omega_\beta := \sqrt{-1} \left( \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^n dz_i \wedge d\bar{z}_i \right),
\]

which defines a Riemannian metric \( g_\beta \) and induces a distance \( d_\beta \). For simplicity, define \( \tilde{d}(p_1, p_2) := d_\beta(x_1, x_2) + |t_1 - t_2|^{\frac{\beta}{2}} \) as the spacetime distance, where \( p_i = (x_i, t_i) \). Now for a locally integrable function on \( \mathbb{D}^n \times [0, T] \) and \( \alpha \in (0, 1) \), we define the Hölder norm as [11]:

\[
[f]_\alpha := \sup_{\mathbb{D}^n \times [0, T]} |f| + \sup_{p_1 \neq p_2} \frac{|f(p_1) - f(p_2)|}{\tilde{d}(p_1, p_2)^\alpha}.
\]

We say the function \( f \) is \( C^\alpha \) if the norm \([f]_\alpha < +\infty\). Consider the vector fields \( \xi_1 = |z_1|^{1-\beta} \frac{\partial}{\partial z_1}, \xi_k = \frac{\partial}{\partial z_k} \), for \( k = 2, \cdots n \), then we say a \((1, 0)\)-form \( \tau \) is \( C^\alpha \) if \( \tau(\xi_i) \) is \( C^\alpha \) for any \( i = 1, \cdots n \), and say a \((1, 1)\)-form \( \sigma \) is \( C^\alpha \) if \( \sigma(\xi_i, \xi_j) \) is \( C^\alpha \) for any \( i, j = 1, \cdots n \). And we say \( f \) is \( C^{2,\alpha} \) if \( f, \partial f, \sqrt{-1} \partial \bar{\partial} f, \dot{f} \) are all \( C^\alpha \).

Consider the equation (2.3), as \( \omega_t \) is always smooth on the maximal interval, when \( r \) is sufficient small we can write \( \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi(t) = \sqrt{-1} \partial \bar{\partial} u \) locally, then we only need to show
that \( u \) is \( C^{2,\alpha} \). By computation before we know that \( \text{Ric}(\omega_{\beta \alpha}) = 2\pi(1 - \beta)[D] - \sqrt{-1}\partial\bar{\partial}F \) for some smooth function \( f \), so the equation for \( u \) can be written as

\[
\frac{\partial}{\partial t} u = \log \left( \frac{\sqrt{-1}\partial\bar{\partial}u}{\omega_\beta^n} \right) + F.
\]

Take the covariant derivatives with respect to \( \omega_\beta \), we have

\[
\frac{\partial}{\partial t} u_{k\bar{l}} = u^{\bar{j}}_{k} u_{\bar{j}k\bar{l}} - u^{\bar{q}}_{k} u_{\bar{q}j\bar{k}} u_{\bar{j}k\bar{l}} + F_{k\bar{l}}.
\]

Unfortunately, classic Evans-Krylov-Safanov estimate for nonlinear PDE can’t be used in conic setting. In conic Kähler-Einstein case, as the Ricci curvature of target metric has lower bound, by Cheeger-Colding-Tian theory \([4]\), the metric near the conic singularities is close to a standard flat conic metric, which indicates that perturbed Schauder estimate can be applied. Alternately, Tian extended his master thesis to conic case and gave a proof of \( C^{2,\alpha} \)-estimate in case of conic Kähler-Einstein metrics. In \([19]\), we will follow Tian’s approach to prove this theorem:

**Theorem 2.3.** \( u \) is \( C^{2,\alpha} \)-bounded on the divisor \( D \).

This theorem completes the proof for the first main theorem \([11]\).

3. LIMIT BEHAVIOR NEAR THE SINGULAR TIME WHEN THE TWISTED CANONICAL BUNDLE IS BIG

In the last section we discussed how long the unnormalized conical Kähler-Ricci flow would last. Now we want to understand what will happen when the flow runs to singular time. During the remaining part of this article we assume that \( M \) is an \( n \)-dimensional projective manifold. In \([25] \),\([22]\), to study minimal model program, they mainly considered the case that the canonical line bundle \( K_M \) is big, or big and nef (numerical effective). First let’s recall their definitions:

**Definition 3.1.** Suppose \( L \) is a holomorphic line bundle on a projective manifold \( M \), then we say \( L \) is big if \( |c_1(L)|^n = \int_M c_1(L)^n > 0 \). And we say \( L \) is nef if for any algebraic curve \( C \) on \( M \), \( c_1(L)(C) = \int_C c_1(L) \geq 0 \).

To study the limit behavior near the singular time, they mainly made use of so-called Kodaira’s lemma, which was proved by Kawamata and used in degenerate Monge-Ampère equations by Tsuji \([26]\) first:

**Lemma 3.2.** Let \( L \) be a line bundle on a projective manifold \( M \). If \( L \) is big, then there exists an effective line bundle \( E \) and two positive numbers \( a, b \) such that \( L - \delta E \) is positive for any \( \delta \in (a, b) \). Moreover, if \( L \) is big and nef, then the conclusion above is true for \( \delta \in (0, b) \).

Now we can begin the proof of theorem\([12]\). As the last section, we can do similar constructions and consider the approximation flow equation \((2.5)\) again that

\[
\frac{\partial}{\partial t} \varphi_\epsilon = \log \left( \frac{(\omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon)^n}{\Omega} \right) + \log(||S||^2 + \epsilon^2)^{1-\beta}.
\]

Although \( \omega_{t,\epsilon} \) may not be a Kähler metric near the singular time \( T \), as it’s controlled from above, we can still make use of maximal principle to get a uniformly upper bound for \( \varphi_\epsilon \). To get a lower bound, we need to apply Kodaira’s lemma to get a family of regular background metrics. Recall that we define \( \omega_t = \omega_0 - t(Ric(\Omega) - (1 - \beta)R(||\cdot||)) \), and we assume that the
twisted canonical bundle $K_M + (1 - \beta)[D]$ is big, then we can easily see that the current $[\omega_t]$ is big on the time interval $[0, T_0]$. By Kodaira's lemma, we have a $\delta \in (a, b)$ and an effective divisor $E$ with a Hermitian metric $|| \cdot ||'$ such that $\omega_t + \delta \sqrt{-1} \partial \overline{\partial} \log || \cdot ||'^2 > 0$. If we define $S'$ as a local holomorphic section for $E$, we will have

$$\omega_t + \delta \sqrt{-1} \partial \overline{\partial} \log || S'||^2 > 0.$$  

Set $\omega'_{t, \epsilon} = \omega_t + \delta \sqrt{-1} \partial \overline{\partial} \log || S'||^2$, we can rewrite the equation (2.5) as

$$\frac{\partial}{\partial t} (\varphi - \delta \log || S'||^2) = \log \left( \frac{\omega'_{t, \epsilon} + \sqrt{-1} \partial \overline{\partial} (\varphi - \delta \log || S'||^2))}{\Omega} \right)^n (||S||^2 + \epsilon^2)^{1-\beta}. \tag{3.1}$$

Note that in this equation, $\omega'_{t, \epsilon}$ is a smooth metric, by the argument in the last section, we know that

$$\log \frac{\omega'_{t, \epsilon}^n (||S||^2 + \epsilon^2)^{1-\beta}}{\Omega}$$

is bounded from below by a constant depending on $\delta$. By maximal principle, we know that $\varphi - \delta \log || S'||^2$ is bounded from below by $C_\delta$. Now we get a $C^0$-estimate for $\varphi$ that

$$C_\delta + \delta \log || S'||^2 \leq \varphi \leq C.$$

Now let's begin to estimate $\dot{\varphi}$. For simplicity, we denote $\varphi_{\epsilon, \delta} = \varphi - \delta \log || S'||^2$, and $\omega = \omega'_{t, \epsilon} + \sqrt{-1} \partial \overline{\partial} \varphi_{\epsilon, \delta}$. Then the equation (3.1) can be written as

$$\frac{\partial}{\partial t} \varphi_{\epsilon, \delta} = \log \frac{\omega'_{t, \epsilon}^n (||S||^2 + \epsilon^2)^{1-\beta}}{\Omega}.$$

Take the derivative of this equation, we obtain that

$$\frac{\partial}{\partial t} \dot{\varphi}_{\epsilon, \delta} = \Delta \dot{\varphi}_{\epsilon, \delta} + tr_\omega \rho,$$

which is the same with (2.7). Note that $\sqrt{-1} \partial \overline{\partial} \log || S'||^2 = -R(|| \cdot ||')$ outside the base divisor $E$, and $C_\delta \leq \varphi_{\epsilon, \delta} \leq C - \delta \log || S'||^2$, we can compute as [22],

$$\left( \frac{\partial}{\partial t} - \Delta \right) (\dot{\varphi}_{\epsilon, \delta} - A^2 \varphi_{\epsilon, \delta} + A \log || S'||^2)$$

$$= tr_\omega ( - A^2 \dot{\varphi}_{\epsilon, \delta} + A^2 (n - tr_\omega \omega'_{t, \epsilon}) + A tr_\omega R(|| \cdot ||`)$$

$$= tr_\omega ( - A^2 \omega'_{t, \epsilon} + A R(|| \cdot ||') - A^2 (\dot{\varphi}_{\epsilon, \delta} - A^2 \varphi_{\epsilon, \delta} + A \log || S'||^2)$$

$$+ (n A^2 - A^4 \varphi_{\epsilon, \delta} + A^3 \log || S'||^2).$$

Choose suitable large constant $A$ such that the first term is negative, make use of $C^0$-estimate for $\varphi_{\epsilon, \delta}$ and $\log || S'||^2$ is bounded from above, we have that

$$\left( \frac{\partial}{\partial t} - \Delta \right) (\dot{\varphi}_{\epsilon, \delta} - A^2 \varphi_{\epsilon, \delta} + A \log || S'||^2) \leq -A^2 (\dot{\varphi}_{\epsilon, \delta} - A^2 \varphi_{\epsilon, \delta} + A \log || S'||^2)$$

$$+ C - C' \log || S'||^2.$$

Now make use of maximal principle, note that the maximal of $\dot{\varphi}_{\epsilon, \delta} - A^2 \varphi_{\epsilon, \delta} + A \log || S'||^2$ can only be obtained outside $E$, we can finally get that

$$\dot{\varphi}_{\epsilon, \delta} \leq C - C' \log || S'||^2,$$
where the constants $C, C'$ may depend on $\delta$. Similarly, to get a lower bound, we can compute that

\[
\begin{align*}
&\left(\frac{\partial}{\partial t} - \Delta\right)(\dot{\varphi}_{\epsilon, \delta} + A^2 \varphi_{\epsilon, \delta} - A \log ||S'||^2) \\
= &\text{tr}_\omega \rho + A^2 \log \frac{\omega^m(\Omega)}{\omega^m_{t, \epsilon}} + nA^2 + A^2 \text{tr}_\omega \omega^m_{t, \epsilon} - A\text{tr}_\rho R(\| \cdot ||') \\
= &\text{tr}_\omega (\rho + A^2 \omega^m_{t, \epsilon} - AR(\| \cdot ||') - nA^2 \\
+ &A^2 (\log \frac{\omega^m_{t, \epsilon}}{\omega^m} + \frac{\text{tr}_\omega \omega^m_{t, \epsilon}}{2} + \log \frac{\omega^m_{t, \epsilon}(\|S||^2 + \epsilon^2)^{1-\beta}}{\Omega})).
\end{align*}
\]

Choose $A$ large enough such that the first term becomes positive, then make use of Schwarz inequality and the property of logarithmic functions, we obtain that

\[
\begin{align*}
&\left(\frac{\partial}{\partial t} - \Delta\right)(\dot{\varphi}_{\epsilon, \delta} + A^2 \varphi_{\epsilon, \delta} - A \log ||S'||^2) \\
\geq &\ -nA^2 + A^2 (\log \frac{\omega^m_{t, \epsilon}}{\omega^m} + \frac{1}{2} (\frac{\omega^m_{t, \epsilon}}{\omega^m})^2 + \log \frac{\omega^m_{t, \epsilon}(\|S||^2 + \epsilon^2)^{1-\beta}}{\Omega}) \\
\geq &A^2 (-\log \frac{\omega^m_{t, \epsilon}}{\omega^m} + \log \frac{\omega^m_{t, \epsilon}(\|S||^2 + \epsilon^2)^{1-\beta}}{\Omega}) - C_1 \\
= &\ -A^2 (\dot{\varphi}_{\epsilon, \delta} + A^2 \varphi_{\epsilon, \delta} - A \log ||S'||^2) + 2A^2 \log \frac{\omega^m_{t, \epsilon}(\|S||^2 + \epsilon^2)^{1-\beta}}{\Omega} \\
&\ + A^4 \varphi_{\epsilon, \delta} - A^3 \log ||S'||^2 - C_1 \\
\geq &\ -A^2 (\dot{\varphi}_{\epsilon, \delta} + A^2 \varphi_{\epsilon, \delta} - A \log ||S'||^2) - C_2,
\end{align*}
\]

By maximal principle and argue as above, we can get a lower bound for $\dot{\varphi}_{\epsilon, \delta}$ that

\[
\dot{\varphi}_{\epsilon, \delta} \leq -C + C' \log ||S'||^2.
\]

Now we can summarize these estimates and obtain such a theorem:

**Theorem 3.3.** Suppose the twisted canonical bundle $K_M + (1 - \beta)[D]$ is big on a projective manifold $M$, and $E$ is an effective divisor such that $[\omega] - \delta[E] = [\omega_0] - t[c_1(M) - (1 - \beta)c_1(D)] - \delta[E] > 0$ on the time interval $[0, T_0]$ for a positive $\delta \in (a, b)$. Let $\varphi_\epsilon$ solves the approximation flow equation (2.5), then we have that

\[
C_\delta + \delta \log ||S'||^2 \leq \varphi_\epsilon \leq C, \quad |\dot{\varphi}_\epsilon| \leq C - C' \log ||S'||^2.
\]

Now we can derive a Laplacian estimate as before, which is also an application of generalized Schwarz Lemma.

**Theorem 3.4.** On the time interval $[0, T_0]$ there exist constants $C, \alpha > 0$ such that $\text{tr}_{\omega_0, \cdot} \omega \leq C ||S'||^{\alpha - 2\alpha}$.

**Proof.** As theorem 2.1 we first compute that

\[
\left(\frac{\partial}{\partial t} - \Delta\right) (\log \text{tr}_{\omega_0, \cdot} \omega + C' \chi_\rho) \leq C\text{tr}_\omega \omega_0, \epsilon.
\]
As in the proof of theorem 3.3 we define

\[ H = \log \ tr_{\omega_{0,\epsilon}}(\omega + C'\chi_p - A^2\varphi_{\epsilon,\delta} + A\log ||S'||^2), \]

then for sufficiently large \( A > 0 \), we have that

\[
(\frac{\partial}{\partial t} - \Delta)H \\
\leq tr_{\omega}(C\omega_{0,\epsilon} - A^2\omega_{t,\epsilon} + AR(||\cdot||')) - A^2\hat{\varphi}_{\epsilon,\delta} + nA^2 \\
\leq - tr_{\omega}\omega_{0,\epsilon} + C - C' \log ||S'||^2.
\]

By maximal principle, we know that the maximal of \( H \) can only be obtained outside the divisor \( E \), and when this maximal is obtained, we have

\[ tr_{\omega}\omega_{0,\epsilon} \leq C - C' \log ||S'||^2, \]

which indicates that at the maximal of \( H \),

\[ tr_{\omega}n\omega \leq \omega^n_{\omega_{0,\epsilon}}(tr_{\omega}\omega_{0,\epsilon})^{n-1} = e^{\hat{\varphi}_{\epsilon}} \frac{\Omega}{(||S'||^2 + \epsilon^2)^{1-\beta}\omega^n_{0,\epsilon}} \leq C||S'||^{{1-2\alpha'}}. \]

Then the theorem follows. \( \square \)

As in the last section, we can give local high order estimate for \( \varphi_{\epsilon} \) outside the divisors \( D \) and \( E \).

**Proposition 3.5.** For any \( K \subset M \setminus (D \cup E) \) and \( k > 0 \), there exists \( C_{k,K} \) such that

\[ ||\varphi_{\epsilon}||_{C^k(K \times [0,T_0])} \leq C_{k,K}. \]

By this proposition, we can argue as the last section. We put a sequence of compact set to approximate the regular part \( M \setminus (D \cup E) \). On each compact set \( K \), as we have uniform estimates for high order derivatives, we can choose a sequence \( \epsilon_i \) such that \( \varphi_{\epsilon_i} \) converges on \( K \times [0,T_0] \) in \( C^\infty_{\text{loc}} \) sense. By diagonal argument, we can take a sequence, say, \( \varphi_{\epsilon_i} \), converges to a function \( \varphi \) on \((M \setminus (D \cup E)) \times [0,T_0]) C^\infty_{\text{loc}} \) sense, which is smooth on \( M \setminus (D \cup E) \). By theorem 3.3 we know that

\[ \omega_\beta = \omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi \]

is a conic Kähler metric with angle \( 2\pi\beta \) along the part of divisor \( D \) outside the divisor \( E \) on \([0,T_0]\). Now as above, we want to prove that \( \varphi_{\epsilon_i} \) converges to \( \varphi \) globally in the sense of currents. Other arguments are the same except for the part near the divisor \( E \). Note that when we consider the integral

\[ \int_M \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n||S'||^{2(1-\beta)}}{\Omega} \wedge \sqrt{-1}\partial\bar{\partial}\eta, \]

near the divisor \( E \), by theorem 3.3 the integral function grows as \( \log ||S'||' \), so this integral exists. On the other hand, by theorem 3.3 as \( \varphi, \varphi_{\epsilon_i} \) grows like \( \log ||S'||' \), then it’s also true that

\[ |\int_M (\varphi_{\epsilon_i} - \varphi) \wedge \sqrt{-1}\partial\bar{\partial}\eta| \]

tends to \( 0 \). So we know that the limit \( \varphi(t) \) outside the divisor \( E \) satisfies conical Kähler-Ricci flow equation on \([0,T_0]\) in the sense of currents, and as \( t \) tends to \( T_0 \), \( \omega_\varphi = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi(t) \) converges smoothly to a Kähler metric on \( M \setminus (D \cup E) \), and conic along \( D \setminus E \) with cone angle \( 2\pi\beta \). Now we want to claim that the limit is unique, i.e. for any sequence \( \{\epsilon_i\} \) which tends to
0, the sequence $\varphi_{\epsilon_i}$ will converge to the same limit in our sense. Actually this follows from the uniqueness of the solution on the time interval $[0, T_0)$. Then consider that in any compact set in $M \setminus (D \cup E)$, as $\varphi_{\epsilon_i}$ is uniformly bounded, we can conclude that the limit of any sequence $\varphi_{\epsilon_i}$ converges to the same limit, say, $\varphi(t)$ at $T_0$, which completes our claim. Finally, note that the base locus $E$ may not be unique, however, we can consider the intersection of these $E$’s and call it the stable base locus, and still denote it as $E$. By the argument above, we can obtain a unique limit current $\omega_\varphi = \omega_{t, \epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi$ outside $E$ at time $T_0$, and this current is a smooth Kähler metric outside $D \cup E$, and conic along $D \setminus E$ with cone angle $2\pi \beta$. Finally, $C^{2, \alpha}$-estimate in the proof of theorem 1.1 still holds here. Now we complete the proof of theorem 1.2.

4. LIMIT IN THE CASE OF BIG AND NEF TWISTED CANONICAL BUNDLE

In this section we want to understand the limit behavior when the twisted canonical bundle is big and nef. By theorem 1.1 we know that this conical Kähler-Ricci flow exists forever. In this case we find that $\omega_t = \omega_0 - t(Ric(\Omega) - (1 - \beta)R(|| \cdot ||))$ will blow up at infinite time. To capture the limit behavior more precisely, we can consider normalized conical Kähler-Ricci flow

$$\frac{\partial}{\partial t} \omega = -\omega - Ric(\omega) + 2\pi(1 - \beta)[D].$$

(4.1)

Note that if we define $\tilde{\omega}(t) := (1 + t)\omega(\log(1 + t))$, then we can compute that

$$\frac{\partial}{\partial t} \tilde{\omega}(t) = \omega(\log(1 + t)) + (-Ric(\omega(\log(1 + t)) - \omega(\log(1 + t)) + 2\pi(1 - \beta)[D]))$$

$$= -Ric(\tilde{\omega}(t)) + 2\pi(1 - \beta)[D],$$

which means that the solution of unnormalized conical Kähler-Ricci flow is actually the same with normalized conical Kähler-Ricci flow only moduli a scaling. As we did in the proof of theorem 1.1 we can consider this equation in the level of cohomology class. Then we can write

$$\omega_t = e^{-t}\omega_0 + (1 - e^{-t})(-Ric(\Omega) + (1 - \beta)R(|| \cdot ||)),$$

and set $\overline{\omega_t} = \omega_t + k\sqrt{-1} \partial \bar{\partial} ||S||^{2\beta}$ and $\omega = \overline{\omega_t} + \sqrt{-1} \partial \bar{\partial} \varphi$. Now we can write the equation (4.1) in scalar form:

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \varphi = \log \left( \frac{(\overline{\omega_t} + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \right) + \log ||S||^{2(1 - \beta)} - \varphi - k||S||^{2\beta} \\
\varphi(\cdot, 0) = 0
\end{array} \right.$$  

(4.2)

As before, we can consider an approximation flow equation of (4.2)

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \varphi_\epsilon = \log \left( \frac{(\omega_{t, \epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon)^n}{\Omega} \right) ||S||^{2(1 - \beta)} - \varphi_\epsilon - k\chi(\epsilon^2 + ||S||^2) \\
\varphi_\epsilon(\cdot, 0) = 0
\end{array} \right.$$  

(4.3)

where

$$\omega_{t, \epsilon} := e^{-t}\omega_0 + (1 - e^{-t})(-Ric(\Omega) + (1 - \beta)R(|| \cdot ||)) + k\sqrt{-1} \partial \bar{\partial} \chi(\epsilon^2 + ||S||^2)$$

as before.

First consider $C^0$-estimate for $\varphi_\epsilon$. By maximal principle, as

$$\log \left( \frac{\omega_{t, \epsilon}^n}{\Omega} \right) ||S||^{2(1 - \beta)} - k\chi(\epsilon^2 + ||S||^2)$$
is bounded from above, it’s easy to see that $\varphi_\epsilon \leq C$. On the other hand, we recall lemma 3.2: is the twisted canonical bundle $K_M + (1 - \beta)[D]$ is big and nef, there exists an effective divisor $E$ such that $\omega_{t, \epsilon} + \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2 > 0$ for all $t \in [0, \infty)$ and any $\delta \in (0, a)$, where $S'$ is a local defining section of $E$. Then the equation (4.3) can be written as

$$\frac{\partial}{\partial t} (\varphi_\epsilon - \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2) = \log \left( \frac{(\omega_{t, \epsilon} + \sqrt{-1} \partial \bar{\partial} (\varphi_\epsilon - \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2))^n}{\Omega||S'||^{2\delta}} \right)$$

$$+ \log(||S'||^2 + \epsilon^2)^{1-\beta} - (\varphi_\epsilon - \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2) - k\chi(\epsilon^2 + ||S'||^2),$$

where $\omega_{t, \epsilon} = \omega_{t, \epsilon} + \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2$. By maximal principle again, as

$$\log \left( \frac{(\omega_{t, \epsilon} + \sqrt{-1} \partial \bar{\partial} (\varphi_\epsilon - \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2))^n}{\Omega||S'||^{2\delta}} \right)$$

is uniformly bounded from below, we have that $\varphi_\epsilon \geq -C_\delta + \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2$. Now let’s consider the estimate of $\dot{\varphi}_\epsilon$. Denote $\rho = -Ric(\Omega) + (1 - \beta)R(|| \cdot ||)$ again and recall in [25], we have that

$$\left( \frac{\partial}{\partial t} - \Delta \right) (e^{t}\dot{\varphi}_\epsilon - \dot{\varphi}_\epsilon - \varphi_\epsilon - n t)$$

$$= - \text{tr}_\omega (\omega_{0} - \rho) + e^{-t} \text{tr}_\omega (\omega_{0} - \rho) + \varphi_\epsilon - \dot{\varphi}_\epsilon + n + \text{tr}_\omega (e^{-t} \omega_{0} + (1 - e^{-t})\rho$$

$$+ k\sqrt{-1} \partial \bar{\partial} \chi) - n = - \text{tr}_\omega (\omega_{0} + k\sqrt{-1} \partial \bar{\partial} \chi) = - \text{tr}_\omega \omega_{0, \epsilon} < 0,$$

then by maximal principle, for $t > t_0 > 0$, we have $\dot{\varphi}_\epsilon \leq C e^{-t}$. On the other hand, consider

$$H = \varphi_\epsilon + (1 - e^{-t}) \dot{\varphi}_\epsilon - \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2,$$

for $t < T$ and compute that

$$\left( \frac{\partial}{\partial t} - \Delta \right) H = -n + \text{tr}_\omega (\rho + e^{-T} (\omega_{0} - \rho) + k\sqrt{-1} \partial \bar{\partial} \chi) = -n + \text{tr}_\omega \omega_{T, \epsilon}.$$ We know that for $t \to \infty$, $H$ tends to positive infinity near $E$, so the minimal point of $H$ is attained away from $E$, where $-n + \text{tr}_\omega \omega_{T, \epsilon} \leq 0$, by maximal principle. Then we have $\omega \geq \omega_{T, \epsilon} / n$. At the minimal point of $H$, we have that

$$H = \varphi_\epsilon + (1 - e^{-T}) (\log \frac{\omega^n ||S'||^2 + \epsilon^2)^{1-\beta}}{\Omega} - \varphi_\epsilon - k\chi) - \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2$$

$$\geq e^{-T} \varphi_\epsilon + (1 - e^{-T}) (\log \frac{\omega^n ||S'||^2 + \epsilon^2)^{1-\beta}}{n^n \Omega} - k\chi) - \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2$$

$$\geq -C_\delta,$$

then we have that

$$(1 - e^{-T}) \dot{\varphi}_\epsilon \geq -\varphi_\epsilon + \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2 - C_\delta \geq -C_\delta.$$ Now let $T$ tend to infinity, we have that $\dot{\varphi}_\epsilon \geq -C_\delta$. When we begin to study Laplacian estimate for $\varphi_\epsilon$, we note that the evolution equation for $\log tr\omega_{0, \epsilon} \omega$ is different from before only by a constant. So do the same computation we can derive that

$$tr\omega_{0, \epsilon} \omega \leq \frac{C}{||S'||^{2\delta}}.$$
again. By the argument in [18], we can derive local high order estimate for \( \varphi_\epsilon \) outside the base locus \( E \) and the divisor \( D \) that

\[
||\varphi_\epsilon||_{C^k([0,\infty) \times K)} \leq C(k, K)
\]

for any \( K \subset M \setminus (D \cup E) \).

Similar to last two sections, we can choose a sequence of compact sets to approximate \( M \setminus (D \cup E) \), and a sequence \( \varphi_{\epsilon_i} \), by diagonal method, such that \( \varphi_{\epsilon_i} \) converges to a function \( \varphi(t) \) on \( [0, \infty) \times M \setminus (D \cup E) \) in \( C^\infty_{loc} \) sense, which is smooth on \( M \setminus (D \cup E) \). By the same argument as the last section we know that this convergence is also in the sense of currents globally. Similarly, as time tends to infinity, \( \omega_\varphi = \omega_k + \sqrt{-1} \partial \bar{\partial} \varphi(t) \) is smooth on \( M \setminus (D \cup E) \) where \( E \) is the intersection of all base divisors, i.e. stable base locus of the twisted canonical bundle, and conic along \( D \setminus E \) with cone angle \( 2\pi \beta \).

As the flow exists forever, we hope to analyse the limit behavior of \( \varphi(t) \) and \( \omega_\varphi \). As \( \dot{\varphi}_\epsilon \leq C t e^{-4k} \) we can see that near the infinite time, \( \varphi_\epsilon \) is nonincreasing. Note that we also have \( \varphi_\epsilon \geq -C_\delta + \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2 \), which indicates that \( \varphi_\epsilon \) converges uniformly on arbitrary compact set \( K \subset M \setminus (D \cup E) \).

Similarly, as time tends to infinity, \( \frac{\partial \omega_\varphi}{\partial t} \) converges to a function \( \mathcal{F}(t) \) on \( M \setminus (D \cup E) \). The same as the last section we know that this convergence is also in the sense of currents globally. Moreover, \( C^{2,\alpha}\)-estimate still holds. Finally, as [23], we can prove that the limit current is independent of the choice of the initial conic metric. First, by the same argument as in the proof of theorem 1.1 we can see that the solution is independent of the choice of volume form \( \Omega \). Now fix \( \Omega \) and see whether different initial conic Kähler metrics matter. Recall that for any \( \delta \in (0, a) \), we have the estimates

\[
-C_\delta + \delta \sqrt{-1} \partial \bar{\partial} \log ||S'||^2 \leq \varphi_\epsilon \leq C,
\]

which indicates that \( \varphi_\epsilon + \varphi_\epsilon \leq C \). And we also know that \( \dot{\varphi}_\epsilon \) tends to 0 as \( t \) tends to infinity.

Then we obtain that

\[
\int_M e^{\varphi_{\epsilon_i} + k\lambda} \Omega = \lim_{t \to \infty} \int_M e^{\varphi_{\epsilon_i} + \varphi_\epsilon + k\lambda} \Omega
\]

\[
= \lim_{t \to \infty} \int_M (\omega_\epsilon + \sqrt{-1} \partial \bar{\partial} (\varphi_\epsilon + k\lambda))^n = \lim_{t \to \infty} \int_M \omega^n.
\]

Now suppose we have two initial conic metrics \( \omega^* = \omega_0 + k \sqrt{-1} \partial \bar{\partial} ||S||^{2\beta} \) and \( \omega'^* = \omega'_0 + k' \sqrt{-1} \partial \bar{\partial} ||S||^{2\beta} \). Without loss of generality, we can suppose that \( \omega_0 < \omega'_0 \) and \( k < k' \), otherwise
we can replace \( \omega^k \) by \( \omega^* + \omega^k \), then we can write the following flow equations:

\[
\frac{\partial}{\partial t} (\varphi \epsilon + k \chi) = \log \left( \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} (\varphi \epsilon + k \chi))}{\Omega} \right)^n - \left( \varphi \epsilon + k \chi \right)
\]

\[
\frac{\partial}{\partial t} (\varphi^' \epsilon + k^' \chi) = \log \left( \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} (\varphi^' \epsilon + k^' \chi))}{\Omega} \right)^n - \left( \varphi^' \epsilon + k^' \chi \right),
\]

\[\varphi \epsilon (\cdot, 0) = \varphi^' \epsilon (\cdot, 0) = 0,\]

where \( \omega_t = e^{-t} \omega^0 + (1 - e^{-t})(-\text{Ric}(\Omega) + (1 - \beta) R(||\cdot||)) = \omega_t + e^{-t}(\omega^0 - \omega_0) \). Take the difference, and denote \( \varphi \epsilon, k = \varphi \epsilon + k \chi \) and \( \varphi^' \epsilon, k = \varphi^' \epsilon + k^' \chi \), we have that

\[
\frac{\partial}{\partial t} (\varphi \epsilon, k - \varphi^' \epsilon, k) = \log \left( \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi \epsilon, k)}{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi^' \epsilon, k) + e^{-t} (\omega^0 - \omega_0) - \sqrt{-1} \partial \bar{\partial} (\varphi \epsilon, k - \varphi^' \epsilon, k)} \right)^n
\]

and \( (\varphi \epsilon, k - \varphi^' \epsilon, k)(\cdot, 0) = (k - k^' \chi) \leq 0 \). By maximal principle, we have \( \varphi \epsilon, k - \varphi^' \epsilon, k \leq 0 \) at any time, so at infinity, we have that \( \varphi \epsilon, \infty + k \chi \leq \varphi^' \epsilon, \infty + k^' \chi \). However, as \( \int_M (\frac{\varphi \epsilon, \infty + k \chi}{e^{\varphi^' \epsilon, \infty + k^' \chi}}) \Omega = \lim_{t \to \infty} \int_M \omega_t^n = \int_M \omega_0 \Omega \), we can obtain that \( \varphi \epsilon, \infty + k \chi = \varphi^' \epsilon, \infty + k^' \chi \). Take the limit as \( \epsilon \) tends to 0, we complete the proof for the uniqueness of the limit conical Kähler-Einstein current, which also completes the proof for theorem 1.3.

5. Further remarks

Until now we proved three main theorems under the assumption of one irreducible divisor. Now we discuss how to generalize our proofs to the case of simple normal crossing divisors briefly. By [11], we can add up all the approximation metrics to all divisors, and all the properties are preserved. In Laplacian estimate, we note that in [11], they proved that the approximation metric has a lower bound for bisectional curvature, which allows us to obtain the Laplacian estimate. The proof of \( C^{2, \alpha} \)-estimate is essentially the same, see [19].

Note that in [22], unnormalized Kähler-Ricci flow can smooth initial Kähler metrics with rough and degenerate datas. So naturally, we hope unnormalized Kähler-Ricci flow can play similar roles in the study of conic Kähler metrics and we will discuss this phenomenon in the future.

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544, USA.
E-mail address: liangmin@math.princeton.edu.