DEFINITIONS OF $h$–LOGARITHMIC, $h$–GEOMETRIC AND $h$–MULTI CONVEX FUNCTIONS AND SOME INEQUALITIES RELATED TO THEM

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Abstract. In this paper, we put forward some new definitions and integral inequalities by using fairly elementary analysis.

1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality [1].

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq f(a) + \frac{f(b)}{2}$$

(1.1)

where $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$.

The following definitions is well known in the literature.

Definition 1. A function $f : I \rightarrow \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$, where $I$ is a convex set, is said to be convex on $I$ if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in (0, 1]$.

The concept of $h$-convexity was introduced by Varošanec [12] and was generalized by Házy [21].

Definition 2. [12] Let $h : J \rightarrow \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \rightarrow \mathbb{R}$ is an $h$-convex function, or that $f$ belongs to the class $SX(h, I)$, if $f$ is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

(1.3)

If inequality (1.3) is reversed, then $f$ is said to be $h$-concave, i.e. $f \in SV(h, I)$. Obviously, if $h(t) = t$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K^s$.

Definition 3. [4] A function $h : J \rightarrow \mathbb{R}$ is said to be a superadditive function if

$$h(x + y) \geq h(x) + h(y)$$

(1.4)

for all $x, y \in J$.

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Recently, in [2], the concept of geometrically and $s$-geometrically convex functions was introduced as follows.

**Definition 4.** A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a geometrically convex function if
\[ f \left( x^t y^{1-t} \right) \leq \left[ f(x)^t \right] \left[ f(y)^{1-t} \right] \]
for all $x, y \in I$ and $t \in [0, 1]$.  

**Definition 5.** A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an $s$-geometrically convex function if
\[ f \left( x^t y^{1-t} \right) \leq \left[ f(x)^t \right]^{s} \left[ f(y)^{1-t} \right]^{s} \]
for some $s \in (0, 1)$, $x, y \in I$ and $t \in [0, 1]$.  

If $s = 1$, the $s$-geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.  

In [3], Tunç and Akdemir introduced the class of $s$-logarithmically convex functions in the first and second sense as the following:

**Definition 6.** A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be $s$-logarithmically convex in the first sense if
\[ f \left( \alpha x + \beta y \right) \leq \left[ f(x)^\alpha \right] \left[ f(y)^\beta \right] \]
for some $s \in (0, 1]$, where $x, y \in I$ and $\alpha s + \beta s = 1$.  

**Definition 7.** A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be $s$-logarithmically convex in the second sense if
\[ f \left( t x + (1-t) y \right) \leq \left[ f(x)^t \right] \left[ f(y)^{1-t} \right] \]
for some $s \in (0, 1]$, where $x, y \in I$ and $t \in [0, 1]$.  

Clearly, when taking $s = 1$ in Definition 6 or Definition 7, then $f$ becomes the standard logarithmically convex function on $I$.

2. Results for $h$-log-Convex Functions

**Definition 8.** A positive function $f$ is called $h$-logarithmically convex on a real interval $I = [a, b]$, if for all $x, y \in I$ and $t \in [0, 1]$, 
\[ f \left( tx + (1-t) y \right) \leq \left[ f(x)^h(t) \right] \left[ f(y)^h(1-t) \right] \]
where $h(t)$ is a nonnegative function on $J$, with $h : J \subset \mathbb{R} \to \mathbb{R}$.

If $f$ is a positive $h$-logarithmically concave function, then inequality is reversed. On the other hand, a function $f$ is $h$-logarithmically convex on $I$ if $f$ is positive and log $f$ is $h$-convex on $I$.

**Proof.** Let’s rewrite $g = \log f(x)$. Since $g$ is $h-$convex function, for $x, y \in I$ and $t \in [0, 1]$ we get
\[ g \left( tx + (1-t) y \right) \leq h(t) g(x) + h(1-t) g(y) \]
\[ \log f \left( tx + (1-t) y \right) \leq h(t) \log f(x) + h(1-t) \log f(y) \]
\[ = \log f(x)^h(t) + \log f(y)^h(1-t) \]
So we have 
\[ f \left( tx + (1-t) y \right) \leq \left[ f(x)^h(t) \right] \left[ f(y)^h(1-t) \right] \]
\[ = \left[ f(x) \right]^{h(t)} \left[ f(y) \right]^{h(1-t)} \]
\[ \square \]
Remark 1. If we take $h(t) = t$ in Definition 3, $h$-logarithmically convex (concave) become ordinary log-convex (concave) function, and if we take $h(t) = t^s$ in Definition 3, $h$-logarithmically convex (concave) become $s$-log-convex (concave) function in the second sense.

Proposition 1. Let $f$ be an $h$-log-convex function. If the function $h$ satisfies the condition
\[ h(t) + h(1-t) = 1 \]
for all $t \in [0,1]$, then $f$ is also $h$-convex function.

Proof. As we choose $f$ is $h$-log-convex function we can write
\[ f(tx + (1-t)y) \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)}. \]
From a simple inequality
\[ x^\alpha y^{1-\alpha} \leq \alpha x + (1-\alpha) y \]
for $x, y > 0$ and by using the condition $h(t) + h(1-t) = 1$ we have
\[ f(tx + (1-t)y) \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)} \leq h(t) f(x) + h(1-t) f(y) \]
which shows that $f$ is $h$-convex function. \qed

Theorem 1. Let $f$ be an $h$-log-convex function. If $f$ is monotonically increasing or decreasing and $h$ is superadditive function on $[0,1]$, we have
\[ \frac{1}{b-a} \int_a^b f(x) f(a + b - x) \, dx \leq [f(a) f(b)]^{h(1)} \]  \hspace{1cm} (2.2)
and
\[ \frac{1}{(b-a)^2} \left( \int_a^b f(x) \, dx \right)^2 \leq [f(a) f(b)]^{h(1)}. \]  \hspace{1cm} (2.3)

Proof. Since $f$ is an $h$-log-convex function, for $a, b \in I$, $t \in [0,1]$ we have
\[ f(ta + (1-t)b) \leq [f(a)]^{h(t)} [f(b)]^{h(1-t)} \]
and
\[ f(tb + (1-t)a) \leq [f(b)]^{h(t)} [f(a)]^{h(1-t)} \]
If we multiply both sides we have
\[ f(ta + (1-t)b) f(tb + (1-t)a) \leq [f(a) f(b)]^{h(t)} [f(a) f(b)]^{h(1-t)} = [f(a) f(b)]^{h(t) + h(1-t)}. \]

As we choose $h$ is superadditive function we get
\[ f(ta + (1-t)b) f(tb + (1-t)a) \leq [f(a) f(b)]^{h(1)}. \]
By integrating the last inequality over $t$ from 0 to 1 we get
\[ \frac{1}{b-a} \int_a^b f(x) f(a + b - x) \, dx \leq [f(a) f(b)]^{h(1)}. \]
So the proof of (2.2) is completed.

On the other hand we use Chebyshev inequality on (2.2) we have
\[ \frac{1}{b-a} \int_a^b f(x) f(a + b - x) \, dx \geq \frac{1}{(b-a)^2} \int_a^b f(x) \, dx \int_a^b f(a + b - x) \, dx \]
\[ = \frac{1}{(b-a)^2} \left( \int_a^b f(x) \, dx \right)^2. \]
So we have
\[
\frac{1}{(b-a)^2} \left( \int_a^b f(x) \, dx \right)^2 \leq [f(a) f(b)]^{h(1)}.
\]

Then proof of (2.3) is completed. □

**Corollary 1.** If we choose \( h(t) = \frac{1}{t} \) at Theorem 2 as a superadditive function on \([0, 1]\) we have
\[
\frac{1}{b-a} \int_a^b f(x) f(a + b - x) \, dx \leq f(a) f(b)
\]
and
\[
\frac{1}{(b-a)^2} \left( \int_a^b f(x) \, dx \right)^2 \leq f(a) f(b).
\]

**Theorem 2.** Let \( f \) and \( g \) are \( h \)-log-convex functions on \( I \) and let \( h \) is symmetric about \( \frac{1}{2} \). For \( a, b \in I \) and \( t \in [0, 1] \) we have
\[
\frac{1}{b-a} \int_a^b (fg)(x) \, dx \leq \int_0^1 \left[ (f(a)(fg)(b))^{h(t)} \right] \, dt.
\] (2.4)

**Proof.** As we choose \( f \) and \( g \) are \( h \)-log-convex functions on \( I \) we have
\[
f(ta + (1-t)b) \leq [f(a)]^{h(t)} [f(b)]^{h(1-t)}
\]
\[
g(ta + (1-t)b) \leq [g(a)]^{h(t)} [g(b)]^{h(1-t)}.
\]

If we multiply both sides we get
\[
f(ta + (1-t)b) g(ta + (1-t)b) \leq [f(a) g(a)]^{h(t)} [f(b) g(b)]^{h(1-t)}.
\]

By integrating the inequality from 0 to 1 over \( t \), and change the variable \( x = ta + (1-t)b \) we have
\[
\frac{1}{b-a} \int_a^b (fg)(x) \, dx \leq \int_0^1 \left[ (f(a) g(a))^{h(t)} [f(b) g(b)]^{h(1-t)} \right] \, dt.
\]

Since \( h \) is symmetric about \( \frac{1}{2} \) we have \( h(t) = h(1-t) \). So we have
\[
\frac{1}{b-a} \int_a^b (fg)(x) \, dx \leq \int_0^1 \left[ (f(a) g(a))^{h(t)} [f(b) g(b)]^{h(1-t)} \right] \, dt.
\]

□

**Theorem 3.** Let \( f \) and \( g \) are \( h \)-log-convex functions. For \( \alpha, \beta > 0 \) and \( \alpha + \beta = 1 \) we have
\[
\frac{1}{b-a} \int_a^b (fg)(x) \, dx \leq \int_0^1 \left[ \alpha \left( [f(a)]^{h(t)} [f(b)]^{h(1-t)} \right)^{\frac{1}{\alpha}} + \beta \left( [g(a)]^{h(t)} [g(b)]^{h(1-t)} \right)^{\frac{1}{\beta}} \right] \, dt
\] (2.5)

and
\[
\frac{1}{b-a} \int_a^b (fg)(x) \, dx \leq \int_0^1 \left[ \alpha [f(a) g(a)]^{h(t)} + \beta [f(b) g(b)]^{h(1-t)} \right] \, dt.
\] (2.6)

**Proof.** Since \( f \) and \( g \) are \( h \)-log-convex functions we have
\[
f(ta + (1-t)b) \leq [f(a)]^{h(t)} [f(b)]^{h(1-t)}
\]
\[
g(ta + (1-t)b) \leq [g(a)]^{h(t)} [g(b)]^{h(1-t)}.
\]

If we multiply both sides and use the fact that \( cd \leq \alpha c^d + \beta d^d \) (for \( \alpha, \beta > 0, \alpha + \beta = 1 \)) we get
\[
(fg)(ta + (1-t)b) \leq \alpha \left( [f(a)]^{h(t)} [f(b)]^{h(1-t)} \right)^{\frac{1}{\alpha}} + \beta \left( [g(a)]^{h(t)} [g(b)]^{h(1-t)} \right)^{\frac{1}{\beta}}.
\]

By integrating the above inequality, we get the proof of (2.5).
On the other hand after multiplying both sides of (2.7) we can write
\[(fg)(ta + (1 - t)b) \leq \alpha [f(a)g(a)]^{\frac{h(t)}{\alpha}} + \beta [f(b)g(b)]^{\frac{h(1-t)}{\beta}}.\]
Then, by integrating the last inequality we get the proof of (2.6). \qed

**Theorem 4.** Let \( f \) be an \( h \)-log-convex function on \([a, b]\). For \( \alpha, \beta > 0 \), \( \alpha + \beta = 1 \) we have
\[ f \left( \frac{a + b}{2} \right) \leq \alpha \frac{1}{b-a} \int_a^b f(x) \frac{h(\frac{1}{2})}{\alpha} \, dx + \beta \frac{1}{b-a} \int_a^b f(x) \frac{h(\frac{1}{2})}{\beta} \, dx. \]
Proof. If we choose \( t = \frac{1}{2} \) on Definition 8 we have
\[ f \left( \frac{x + y}{2} \right) \leq |f(x)f(y)|^{\frac{h(\frac{1}{2})}{2}} \]
If we change the variable \( x = ta + (1 - t)b \) and \( y = (1 - t)a + tb \) we get
\[ f \left( \frac{a + b}{2} \right) \leq f(ta + (1 - t)b)^{h(\frac{1}{2})} f((1 - t)a + tb)^{h(\frac{1}{2})} \]
If we use the inequality \( cd \leq \alpha e^{\frac{c}{\alpha}} + \beta d^{\frac{1}{\beta}} \) (for \( \alpha, \beta > 0 \), \( \alpha + \beta = 1 \)) we get
\[ f \left( \frac{a + b}{2} \right) \leq \alpha f(ta + (1 - t)b)^{\frac{h(\frac{1}{2})}{\alpha}} + \beta f((1 - t)a + tb)^{\frac{h(\frac{1}{2})}{\beta}}. \]
By integrating the last inequality over \( t \) on \([0, 1]\) we have
\[ f \left( \frac{a + b}{2} \right) \leq \alpha \int_0^1 f(ta + (1 - t)b)^{\frac{h(\frac{1}{2})}{\alpha}} \, dt + \beta \int_0^1 f((1 - t)a + tb)^{\frac{h(\frac{1}{2})}{\beta}} \, dt. \]
By rewriting the inequality by using suitable variable changeings we get the desired result. \qed

3. Results for \( h \)-geometrically Convex Functions

**Definition 9.** A positive function \( f \) is called \( h \)-geometrically convex on a real interval \( I = [a, b] \), if for all \( x, y \in I \) and \( t \in [0, 1] \),
\[ f \left( x^t y^{(1-t)} \right) \leq |f(x)|^{h(t)} |f(y)|^{h(1-t)} \] (3.1)
where \( h(t) \) is a nonnegative function on \( J \), with \( h : J \subseteq \mathbb{R} \to \mathbb{R} \).

If \( f \) is a positive \( h \)-geometrically concave function, then inequality is reversed.

**Remark 2.** It is clear that when \( h(t) = t \) in Definition \( 9 \) \( h \)-geometrically convex (concave) become ordinary geometrically convex (concave) function, and if we take \( h(t) = t^s \) in Definition \( 9 \) \( h \)-geometrically convex (concave) become \( s \)-geometrically convex (concave) function.

**Remark 3.** As we can write
\[ x^t y^{(1-t)} \leq tx + (1 - t)y \]
for \( t \in [0, 1] \) and \( x, y > 0 \), we get all Theorems and Corollaries given at Section 2 for decreasing \( h \)-geometrically convex functions.

**Theorem 5.** Let \( f \) be an \( h \)-geometrically convex function on \( I \). For every \( x, y \in I \) with \( x < y \) we get
\[ \frac{1}{\ln y - \ln x} \int_x^y f(\gamma) f \left( \frac{x\gamma}{y} \right) \frac{d\gamma}{\gamma} \leq \int_0^1 [f(x)f(y)]^{h(t)+h(1-t)} \, dt. \]
Proof. Since we choose \( f \) is an \( h \)-geometrically convex function on \( I \), we can write
\[
\begin{align*}
  f (x^1 y^{1-t}) & \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)} \\
  f (x^{1-t} y^t) & \leq [f(x)]^{h(1-t)} [f(y)]^{h(t)}.
\end{align*}
\]
If we multiply both sides of inequalities we get
\[
 f (x^1 y^{1-t}) f (x^{1-t} y^t) \leq [f(x) f(y)]^{h(t)+h(1-t)}
\]
By integrating both sides respect to \( t \) over \([0, 1]\) we have
\[
\int_0^1 f (x^1 y^{1-t}) f (x^{1-t} y^t) \, dt \leq \int_0^1 [f(x) f(y)]^{h(t)+h(1-t)} \, dt
\]
If we change the variable \( \gamma = x^1 y^{1-t} \), we get the desired result. \( \square \)

**Theorem 6.** Let \( f \) and \( g \) are \( h \)-geometrically convex functions on \( I \). For \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \) we get
\[
\int_0^1 f (x^1 y^{1-t}) g (x^{1-t} y^t) \, dt \leq \left( \int_0^1 f (x)^{q h(t)} \, dt \right)^{\frac{1}{q}} \left( \int_0^1 g (y)^{p h(t)} \, dt \right)^{\frac{1}{p}}
\]
\[
\times \left( \int_0^1 f (y)^{p h(1-t)} \, dt \right)^{\frac{1}{q}} \left( \int_0^1 g (x)^{q h(1-t)} \, dt \right)^{\frac{1}{p}}
\]
for every \( x, y \in I \) with \( x < y \) and \( t \in [0, 1] \).

Proof. As we choose \( f \) and \( g \) are \( h \)-geometrically convex functions on \( I \) we can write
\[
\begin{align*}
  f (x^1 y^{1-t}) & \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)} \\
  g (x^{1-t} y^t) & \leq [g(x)]^{h(1-t)} [g(y)]^{h(t)}.
\end{align*}
\]
By multiplying both sides and integrate respect to \( t \) over \([0, 1]\) we have
\[
\int_0^1 f (x^1 y^{1-t}) g (x^{1-t} y^t) \, dt \leq \int_0^1 [f(x) g(y)]^{h(t)} [f(y) g(x)]^{h(1-t)} \, dt.
\]
If we apply Hölder’s inequality for \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \) we get
\[
\int_0^1 f (x^1 y^{1-t}) g (x^{1-t} y^t) \, dt \leq \left( \int_0^1 [f(x) g(y)]^{p h(t)} \, dt \right)^{\frac{1}{p}} \left( \int_0^1 [f(y) g(x)]^{q h(1-t)} \, dt \right)^{\frac{1}{q}}.
\]
Then by applying Hölder’s inequality again, we get
\[
\int_0^1 f (x^1 y^{1-t}) g (x^{1-t} y^t) \, dt \leq \left[ \left( \int_0^1 f(x)^{q h(t)} \, dt \right)^{\frac{1}{q}} \left( \int_0^1 g(y)^{p h(t)} \, dt \right)^{\frac{1}{p}} \right]^{\frac{1}{q}} \left[ \left( \int_0^1 f(y)^{p h(1-t)} \, dt \right)^{\frac{1}{p}} \left( \int_0^1 g(x)^{q h(1-t)} \, dt \right)^{\frac{1}{q}} \right]^{\frac{1}{p}}.
\]
By rearranging the inequality, we get the desired result. \( \square \)

4. **h-multi Convex Functions**

**Definition 10.** A positive function \( f \) is called \( h \)-multi convex on a real interval \( I = [a, b] \), if for all \( x, y \in I \) and \( t, \lambda \in [0, 1] \),
\[
\lambda f (x^1 y^{1-t}) + (1 - \lambda) f ((tx + (1-t)y) \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)}
\]
where \( h(t) \) is a nonnegative function on \( J \), with \( h : J \subseteq \mathbb{R} \rightarrow \mathbb{R} \).

If \( f \) is a positive \( h \)-multi concave function, then inequality is reversed.
Remark 4. It is clear that when \( \lambda = 0 \) in Definition 10, \( h \)-multi convex (concave) become \( h \)-logarithmically convex (concave) function, and if we take \( \lambda = 1 \) in Definition 10, \( h \)-multi convex (concave) become \( s \)-geometrically convex (concave) function.

Theorem 7. Let \( f \) be an \( h \)-multi convex function on \( I \). Then we get
\[
\frac{1}{2} \left[ \frac{1}{\ln y - \ln x} \int_x^y f'(\gamma) d\gamma + \frac{1}{y - x} \int_x^y f(\gamma) d\gamma \right] \leq \int_0^1 [f(x)]^{h(t)} [f(y)]^{h(1-t)} dt
\]
for all \( x, y \in I \) and \( t, \lambda \in [0, 1] \).

Proof. From Definition 10 we have
\[
\lambda f \left( x^t, y^{(1-t)} \right) + (1 - \lambda) f \left( tx + (1 - t)y \right) \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)}.
\]
If we integrate the inequality respect to \( \lambda \) over \( [0, 1] \) we have
\[
\int_0^1 \left( \lambda f \left( x^t, y^{(1-t)} \right) + (1 - \lambda) f \left( tx + (1 - t)y \right) \right) d\lambda \leq \int_0^1 [f(x)]^{h(t)} [f(y)]^{h(1-t)} d\lambda.
\]
So we have
\[
f \left( x^t, y^{(1-t)} \right) + \frac{f(tx + (1 - t)y)}{2} \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)}.
\]
Then by integrating the inequality respect to \( t \) over \( [0, 1] \) we get the desired result. \( \square \)

References
[1] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d’une fonction considérée par Riemann, J. Math Pures Appl., 58, (1893) 171–215.
[2] T.-Y. Zhang, A.-P. Ji and F. Qi, On Integral inequalities of Hermite-Hadamard Type for \( s \)-convex functions, J. Math. Inequal. 2 (3) (2008) 335–341.
[3] M. Bombardelli and S. Varošanec, Properties of \( h \)-convex functions related to the Hermite–Hadamard–Fejér inequalities, Comput. Math. Appl. 58 (2009) 1869–1877.
[4] M.Z. Sarıkaya, A. Sağlam and H. Yıldırım, On some Hadamard-type inequalities involving \( h \)-convex functions, Acta Math. Univ. Comenian. LXIX (2) (2010) 265–272.
[5] M.E. Özdemir, M. Gürbüz and A.O. Akdemir, Inequalities for \( h \)-Convex Functions via Further Properties, RGMIA Research Report Collection Volume 14, article 22, 2011.
[6] M. Bessenyei, The Hermite–Hadamard inequality on simplices, Amer. Math. Monthly 115 (2008), no. 4, 339–345. MR 2009h:52023.
[7] P. Burai and A. Házy, On Orlicz-convex functions, Proc. 12th Sympt. Math. Appl. (November 5-7, 2009) (University of Timisoara), Editura Politehnica, 2010, pp. 73–79.
[8] P. Burai, A. Házy and T. Juhász, Bernstein–Doetsch type results for \( s \)-convex functions, Publ. Math. Debrecen 75 (2009), no. 1-2, 23–31.
[20] M. Bessenyei and Zs. Páles, *Higher-order generalizations of Hadamard’s inequality*, Publ. Math. Debrecen 61 (2002), no. 3-4, 623–643. MR 2003k:26021
[21] A. Házy, *Bernstein–Doetsch type results for h-convex functions*, Math. Inequal. Appl. 14 (2011), no. 3, 499–508.
[22] B. G. Pachpatte, *Mathematical Inequalities*, North-Holland Mathematical Library, Elsevier Science B.V. Amsterdam, 2005.

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