Kondo spin liquid in Kondo necklace model: Classical disordered phase versus symmetry-protected topological state

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We study possible topological features of Kondo spin liquid phase in terms of the one- and two-dimensional Kondo necklace models within the framework of quantum O(N) non-liner sigma model (NLSM). In the one-dimensional case, it is found that the bulk properties of the Kondo spin liquid phase are similar to the well-known Haldane phase at strong coupling fixed point. The difference between them mainly comes from their boundaries due to the effect of the topological term. In the two-dimensional case, the system can be mapped onto an O(4)-like NLSM with some O(3) anisotropy. Interestingly, we find that if hedgehog-like point defects are included together with the restoration of the full O(4) symmetry, our model is identical to a kind of SU(2) symmetry-protected topological (SPT) state. Additionally, if the system has the O(5) symmetry instead, the effective NLSM with Wess-Zumino-Witten term is just a description of the surface modes of a three-dimensional SPT state, though such O(5) NLSM could not be a proper description of Kondo spin liquid phase due to its gaplessness. We expect that the discussions might provide useful threads to identify certain microscopic bilayer antiferromagnet models (and related materials), which can support the desirable SPT states.

I. INTRODUCTION

It is still challenging to understand the emergent quantum phases and corresponding quantum criticality in heavy fermion systems. To tackle the problem, the Kondo lattice model has been introduced, which is believed to capture the nature of interplay between Kondo screening and the magnetic interaction, namely, the Ruderman-Kittel-Kasuya-Yosida (RKKY) exchange interaction, mediated by conduction electrons among localized spins. The former effect favors a non-magnetic spin singlet state in strong coupling limit while the latter tends to stabilize usual magnetic ordered states in weak coupling limit.

However, if we are only interested in the half-filled system with low-lying spin excitations, a further simplified theoretical model, namely, the Kondo necklace model originally introduced by Doniach, will be an alternative starting point. The Kondo necklace model is widely studied in many analytical and numerical methods, and most of them focus on the one-dimensional (1D) case. It is now believed that in 1D, the system is always gapped and the ground-state is in the Kondo spin singlet phase (also named Kondo spin liquid phase). The issues for higher dimension are still open.

Recently, topological properties of matter which are beyond the classic Landau symmetry-breaking theory have aroused great interest. Particularly, the idea of the symmetry-protected topological (SPT) state is proposed to classify many possible distinct insulating phases. The SPT states are bulk-gapped quantum phases with symmetries, and have gapless or degenerate boundary states as long as the symmetries are not broken. A simple example of the SPT state is the well-known Haldane phase in \( S = 1 \) antiferromagnetic spin chain, which is protected by its SO(3) spin rotation symmetry. Free fermion topological insulators are also SPT phases protected by their time reversal symmetry and U(1) charge conservation symmetry.

We note that the Haldane phase in \( S = 1 \) can be well described by two coupled antiferromagnetic \( S = 1/2 \) spin chains and the corresponding non-linear sigma model description correctly captures the non-trivial topological feature. It is noted that the Kondo necklace model may also be considered effectively as two coupled antiferromagnetic \( S = 1/2 \) chains (two-leg ladders) but with antiferromagnetic interaction between two chains. Since the Haldane phase is topologically nontrivial, it is interesting to study the Kondo spin liquid phase in the one-dimensional Kondo necklace model (also its higher-dimensional extensions) and to see whether it has hidden topological feature or not.

To uncover the possible topological properties, we will utilize the effective quantum O(N) non-linear sigma model (NLSM) field theory supported by topological \( \theta \) terms. It is found that at strong coupling fixed point, the Kondo spin liquid phase and the celebrated Haldane phase have similar bulk properties described by NLSM without topological term. The main distinction of these two phases comes from their boundary: on either end of the open chains, the Haldane phase has fractionalized \( S = 1/2 \) spins while the Kondo spin liquid does not support such degree of freedom. Therefore, we may consider the Kondo spin liquid in Kondo necklace model as a classical disordered phase.

We next analyze the topological properties of Kondo spin liquids in high dimensional Kondo necklace models. It seems that if hedgehog-like topological defect is ignored, the NLSM tells us that those Kondo spin liquids are still topologically trivial unless spin-orbit coupling-
like elements are introduced. On the other hand, and more interestingly, when hedgehogs-like point defects are included and the symmetry is expanded to $O(4)$ or $SU(2)$, our model is identical to a kind of $SU(2)$ SPT models. This indicates that certain microscopic bilayer antiferromagnet models may be found to support the desirable SPT state. Furthermore, if the symmetry is expanded to $O(5)$, the effective action just describes the surface state of a three-dimensional (3D) SPT state, though such $O(5)$ NLSM could not be a proper description of Kondo spin liquid phase due to its gaplessness.

The remainder of the paper is organized as follows. In Sec. II, two coupled spin-1/2 antiferromagnetic Heisenberg chains and the corresponding NLSM are introduced. Three limit cases are analyzed in the next section, which correspond to the decoupled case, Haldane phase, and featureless disordered state, respectively. In Sec. III, the Kondo necklace model is extended and studied in detail. In Sec. IV the discussion is generalized to higher dimension by employing an $O(4)$ and $O(5)$ NLSM with possible topological terms, and the issue of topological Kondo liquid with some marginal logarithmic corrections is included and the symmetry is expanded to $O(5)$, the effective action just describes the surface state of a unit sphere, the topological invariant is simply the number of times of the unit sphere being wrapped by this projection. For spin-half-integer like $S = 1/2$ systems, the topological term contributes to the partition function like $(-1)^{Q}$, while for the spin-integer (e.g., $S = 1$) situation, such topological term does not affect the partition function.

It is well-known that, the ground state and low-lying excitations of the spin-1/2 ($SU(2)$) antiferromagnetic Heisenberg chain can be described as a gapless Luttinger liquid with some marginal logarithmic corrections, which is given by the Hamiltonian

$$H = \frac{1}{2\pi} \int dx \left[ uK(\partial_x \theta)^2 + \frac{u}{K}(\partial_x \phi)^2 \right],$$

where $u$ denotes the non-universal velocity and $K = 1/2$ is the Luttinger parameter. The commutation relation of $\phi(x)$ and $\theta(x)$ fields reads $[\phi(x), \partial_y \theta(y)] = i\pi\delta(x - y)$. The right or left fermion operator is defined by $\psi_{x \rightarrow R,L}(x) \propto e^{-i(\phi(x) - \sigma \theta(x))}$. With these abelian bosonization formula in hand, the spin-spin correlation is readily found as $\langle S_i(x, 0) \cdot S_i(0, 0) \rangle \simeq C_1 + C_2 (-1)^i \frac{\log \sqrt{2}}{x}$ while the dimer-dimer correlation $\langle \phi_0(x) \phi_0(0) \rangle$ [where $\phi_0(x) \sim (-1)^x S_i(x) \cdot S_i(x + 1)$, denoting the dimer (valence bond) order parameter] has the same power-law-like decay. [Note that the alternative and more sophisticated $SU(2)_{k=2S}$ Wess-Zumino-Witten (WZW) model gives nice description of the gapless intermediate coupling fixed point for all half-integer antiferromagnetic chains. One may also notice that in Sec. IV the $SU(2)_{k=1}$ WZW model is used to describe the edge states of some $SU(2)$ SPT states.]

Importantly, it is widely believed that the NLSM with the topological term $i\pi Q$ provides the same low-energy description of the spin-1/2 antiferromagnetic Heisenberg chain as well, while the NLSM combined with $i2\pi Q$ leads to a correct description for the spin-1 Heisenberg chain.

III. THREE LIMIT SITUATIONS

In this section, let us inspect several simple and useful limits to get some insights. First, the simplest case is $V = 0$, where two spin-1/2 antiferromagnetic chains are decoupled and each of them is described by a gapless Luttinger liquid like Eq. (3) separately. Next we consider $V \rightarrow -\infty$ (ferromagnetic coupling between two chains)
FIG. 1. A cartoon for the edge state of the Haldane phase in $S = 1$ antiferromagnetic chain. One $S = 1$ spin is fractionalized into two $S = 1/2$ objects. Those $S = 1/2$ spins from neighboring sites form singlet states except for the ones on the boundary.

which fixes $\hat{n}_1 = \hat{n}_2 \equiv \hat{n}$, and in this case, Eq. (2) becomes

$$S = \int dt dx \frac{1}{2cg} (\partial_\mu \hat{n})^2 + i2\pi 2SQ,$$

$$Q = \frac{1}{4\pi} \int dtdx \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}).$$

(4)

Obviously, this action just represents a spin-2S antiferromagnetic chain with the topological term $i2\pi 2SQ$. The coupling strength and velocity are renormalized as $\frac{1}{cg} = \frac{1}{cg_{1g}} + \frac{1}{cg_{2g}}$, $c = \sqrt{cg_{1g}cg_{2g}}$. We can see that when $V$ is large negative, the coupled spin-1/2 chains are effectively described by a single spin-1 antiferromagnetic chain. And it is well established that the spin-1 antiferromagnetic Heisenberg chain always situates in its gapped phase (Haldane phase), and there exist two free spin-1/2 spins on the boundary if the open boundary condition is imposed (as shown in Fig. [1]). In other respect, the Haldane phase is a 1D symmetry-protected-topological phase since as long as the SO(3) spin rotation symmetry is preserved, both the bulk Haldane phase and its edge state (two free spins) are stable. Any perturbation preserving the SO(3) spin rotation symmetry can only affect the $S = 1$ object but cannot affect the fractionalized $S = 1/2$ free spins.

It is worth to mention that the two free spins on the boundary can be revealed from the derivation of the topological term in Eq. (2) as

$$iS \sum_m (-1)^m \Omega_{WZ}(\hat{n}) \simeq iS 2 \int_0^L dx \frac{\partial \Omega_{WZ}(\hat{n}(x))}{\partial x}$$

$$= iS \frac{1}{2} [\Omega_{WZ}(\hat{n}(x = L)) - \Omega_{WZ}(\hat{n}(x = 0))] + i2\pi SQ,$$

where $\Omega_{WZ}(\hat{n})$ denotes the Berry phase of unit vector $\hat{n}(m)$ on the $m$-th site. If one does not choose the periodic boundary condition, the first term will not be canceled and it is easy to see that they are indeed two free spin-S/2 spins on the boundary. (A spin-S spin has the Berry phase of $S\Omega_{WZ}(\hat{n})$.) In our case, we observe that the effective spin-1 chain has the non-trivial edge state (two free spins).

In the opposite limit with $V \to +\infty$ (antiferromagnetic coupling between two chains), one may set $\hat{n}_1 = -\hat{n}_2 \equiv \hat{n}$ and the resulting action simply reads

$$S = \int dtdx \frac{1}{2cg} (\partial_\mu \hat{n})^2,$$

(6)

where we note that the topological term totally vanishes in this situation. Thus, in contrast to the ferromagnetic case, no non-trivial edge state like two free spins appears when the coupling between the two chains is negative (antiferromagnetic). Since the one-dimensional NLSM without topological term is gapped and described by the strong coupling $g \to +\infty$ at fixed point, we may instead use the Hamiltonian formalism (quantum rotor) to get an intuitive understanding (setting $c = 1$ below)

$$H = \frac{g}{2} \sum_i \hat{L}_i^2 + \frac{1}{2g} \sum_i \hat{n}_i \cdot \hat{n}_{i+1},$$

(7)

where we have the commutation relations $[\hat{n}_{ia}, \hat{n}_{ib}] = i\delta_{ij} \epsilon_{\alpha\beta\gamma} \hat{n}_{i\gamma}$, $[\hat{L}_{ia}, \hat{L}_{ib}] = i\delta_{ij} \epsilon_{\alpha\beta\gamma} \hat{n}_{i\gamma}$ and $[\hat{L}_{ia}, \hat{n}_{ib}] = i\delta_{ij} \epsilon_{\alpha\beta\gamma} \hat{n}_{i\gamma}$. $\hat{n}$ acts like the canonical position and $\hat{L}$ corresponds to the angular momentum. The eigenstates of $\hat{L}^2$ read $|l, m\rangle$ with $-l \leq m \leq l$ and $l, m$ being integers. When $g$ is large, the ground state is the product-state $|\Psi\rangle \simeq \prod_i |l = 0\rangle$. The first excitation state is a triplet with $|l = 1, m = \pm 1\rangle$ on certain site, which corresponds to the physical $S = 1$ triplet excitation.

Furthermore, when $V$ deviates from the above three limits, the standard perturbative renormalization group (RG) [see Fig. 2] tells us that even small perturbation around the fixed point $V = 0$ will drive the system to either strong coupling fixed point $V \to -\infty$ or $V \to +\infty$. In contrast, the latter two are rather stable for small perturbations and can be identified as two genuine phases. Therefore, we expect the qualitative analysis presented here may capture the basic physics. (The above features can also be obtained by standard abelian bosonization on two coupled spin-1/2 antiferromagnetic Heisenberg chains.)

IV. THE KONDO NECKLACE MODEL

Now, it is ready to study the Kondo necklace model, which is a highly simplified model only including spin degree of freedom for both local and conduction electrons.
at half-filling

\[ H_{KN} = t \sum_i (\tau^z_i \tau^z_{i+1} + \tau^x_i \tau^x_{i+1}) + J \sum_i \tilde{S}_i \cdot \tilde{S}_{i+1} \]  \hspace{1cm} (8)

where \( \tau^x_i, \tau^y_i \) represents the spin degrees of freedom coming from original conduction electrons and \( S_i \) denoting local spins. The first term is like a one-dimensional quantum XY model and can be easily solved by using the Jordan-Wigner transformation \( \tau^z_i = 2c_i^\dagger e^{i \pi \sum_{j<i} c_j^\dagger c_j}, \tau^x_i = 2c_i^\dagger c_i - 1 \). The resulting Hamiltonian reads \( H_{XY} = 2t \sum_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) \) with spin-spin correlation \( \langle \tau^x(x) \tau^x(0) \rangle \sim \frac{1}{2} \), \( \langle \tau^z(x) \tau^z(0) \rangle \sim 2e^{-t} \). The second part of \( H_{KN} \) describes the Kondo coupling between the local spins and spin degrees of freedom from conduction electrons, which is fundamental to form the Kondo spin singlet state (It is also called Kondo spin liquid when the system is insulating). Based on many analytical and numerical studies, it is well established that for the 1D Kondo necklace model, its ground-state is a gapped Kondo spin singlet state for all value of \( J/t \) except for \( J = 0.21^{24-26}, 32, 33 \). However, we should remind the reader that the Kondo necklace model cannot be derived from the original Kondo lattice model at half-filling but could only be considered as a phenomenological model devised for studying the low-lying spin excitations.

In the study of the Kondo necklace-like model, an analytical approach called bond-operator representation could be useful for such spin-only models.\textsuperscript{24-28} When the bond-operator mean-field approximation is used, the low-lying spin triplet excitation is gapped for all \( J/t > 0 \), which reproduces the expected results. The existence of the triplet quasiparticle gap denotes that there only exists a Kondo spin singlet phase in this model.\textsuperscript{24}

It is interesting to inspect the low energy properties of the Kondo necklace model in terms of the effective NLSM. However, since no exchange interaction exists among any local spins \( S_i \), the original Kondo necklace model is unable to give rise a useful formalism of NLSM. To resolve this drawback, we may add the Heisenberg exchange interaction for local spins. We note that this treatment is indeed widely used in literature, which could mimic the effect of RKKY interaction and is friendly to further approximation treatments.\textsuperscript{4,8,9} Moreover, one can also introduce exchange interaction between 2-component spins \( \tau^z \). We argue that the main feature of spin degrees of freedom of conduction electrons is its gaplessness and such feature does not rely on the specific XY-like formalism. We can also imagine that there exists strong interaction among original electrons and this interaction leads to the exchange interaction of \( \tau^z \). Then, we propose the extended Kondo necklace model as following

\[ H_{EKN} = t \sum_i \tilde{S}_i \cdot \tilde{S}_{i+1} + J \sum_i \tilde{S}_i \cdot S_i + W \sum_i S_i \cdot \tilde{S}_{i+1} \] \hspace{1cm} (9)

and its corresponding NLSM reads as

\[ S = \sum_{i=1,2} \int dx \mathcal{L}_i + i2\pi SQ_i + \int dx J(\tilde{n}_1 \cdot \tilde{n}_2), \]

\[ \mathcal{L}_i = \frac{1}{2c_i g_i} [(\partial_x \tilde{n}_i)^2 + c_i^2 (\partial_x \tilde{n}_i)^2], \hspace{1cm} (10) \]

where \( c_1 \sim t, c_2 \sim W \) and \( \tilde{n}_1, \tilde{n}_2 \) represent spins of conduction electrons and local moments, respectively. Clearly, the above NLSM is just the one studied in previous section (Eq. (2)). But it is important to note \( J > 0 \) in this case. Thus, based on the previous discussion of \( J > 0 (V > 0 \) in Eq. (2)), we expect the topological term will vanish and the system can be described by a trivial \( S = 1 \) chain (Eq. (9)), where the ground-state is a featureless disordered state and no free spin-1/2 spin exists on either side of the chains with open boundary condition. Such trivial disordered state should be the expected Kondo spin liquid state because \( \tilde{\tau} \) or \( \tilde{S}_i \) chain (both are effective \( S = 1/2 \) antiferromagnetic chains) alone cannot produce a gapped state due to their intrinsic \( i\pi Q \) topological term. Only when these two chains interact antiferromagnetically, their \( i\pi Q \) topological terms destructively interfere, which gives rise to the trivial disordered ground-state as shown in Fig. 3.\textsuperscript{8} We also know that the Kondo singlet state should be formed between \( \tilde{\tau} \) and \( \tilde{S}_i \) and in NLSM, it is described by \( \tilde{n}_1 = -\tilde{n}_2 \) in order to form a \( S = 0 \) object (\( S_{\text{total}} = S\tilde{n}_1 + S\tilde{n}_2 = 0 \)). We recall that \( \tilde{n}_1 = \tilde{n}_2 \) forces these two spins to form a \( S = 1 \) object (\( S_{\text{total}} = S\tilde{n}_1 + S\tilde{n}_2 = 1 \) with \( S = 1/2 \)) as what have been studied in Eq. (4) when the Haldane phase is concerned.\textsuperscript{8} Therefore, we may conclude that the Kondo spin liquid state is described by the featureless disordered state without noticeable non-trivial topological properties. In this respect, we may regard the Kondo spin liquid state as a classical disordered phase due to the vanished topological terms, which in fact represents the fundamental quantum features.

Comparing the Kondo spin liquid phase with the \( S = 1 \) Haldane phase, we may see that their bulk properties seem to be similar since the bulk excitation are both described by NLSM at strong coupling fixed point without the contribution form the topological term in closed manifold. The distinction mainly relies on their boundary, namely the Haldane phase has fractionalized \( S = 1/2 \) spins while the Kondo spin liquid does not support such degree of freedom on either side of the open chains.
V. KONDO SPIN LIQUID IN HIGHER SPATIAL DIMENSION AND POSSIBLE RELATION TO SPT STATES

Now, if we extend 1D Kondo necklace model to higher space dimension, besides the Kondo spin liquid phase, the usual antiferromagnetic insulating state appears (We only consider bipartite lattice as square and honeycomb lattices)\textsuperscript{24,25,30,31} It has been suggested that there may exist a second quantum phase transition between these two\textsuperscript{24,25} Based on previous discussion, we may considered the 2D Kondo necklace model as a bilayer antiferromagnet with antiferromagnetic inter-layer coupling. Each layer alone may form the antiferromagnetic Néel state or valence-bond solid (VBS) phase, while the inter-layer coupling can drive the system into the Kondo spin liquid phase. In the language of NLSM, we now have an effective two-dimensional O(3) NLSM but its usual topological term (Hopf term) vanishes in the strong coupled Kondo phase, similar to the 1D case.\textsuperscript{62} Therefore, we may conclude that the Kondo spin liquid phase are topologically trivial and cannot show any non-trivial edge (surface) states if no other novel elements are included.

However, the above argument is not the whole story. It has been realized that there are non-smooth solutions of the classical Euclidean equations of motion of (2+1)-D O(3) NLSM, known as hedgehogs, which have non-trivial topological (winding) numbers.\textsuperscript{62} Furthermore, if we introduce the VBS order parameter explicitly into the NLSM, the resulting model would be an O(4) or O(5) NLSM with appropriate topological terms. Although the inter-layer Kondo coupling might introduce possible O(3)\texttimes\texttimes Z\textsubscript{2} or O(3)\texttimes O(2) anisotropies, since the original conduction electron do not show spin-rotation or lattice-translation symmetry-breaking, we may neglect such anisotropies when the Kondo spin liquid phase is concerned. In the following subsections, we will focus on the O(4) and O(5) NLSM.

A. A possible topological term in O(4) NLSM and relation to 2D SU(2) SPT states

First, we consider the competition between the (\(\pi, \pi\)) Néel order and the \((\pi, 0)\) VBS order. In this case, the VBS order parameter is a one-component real scalar \(\phi_0\).\textsuperscript{61} The nontrivial winding number \(Q_H\) is expressed in terms of a four-component unit vector \(\phi = (\phi_0, \vec{n})\),

\[
Q_H = \frac{1}{12\pi^2} \int d\tau d^2x \epsilon_{\alpha \beta \gamma \delta} \epsilon_{\mu \nu \lambda} \phi_\alpha \partial_\mu \phi_\beta \partial_\nu \phi_\gamma \partial_\lambda \phi_\delta \tag{11}
\]

where \(\phi_0, \vec{n}\) denote the \((\pi, 0)\) VBS (dimer) order and the usual Néel order vector, respectively.\textsuperscript{61,63} [However, in the square lattice, the more general VBS order parameter is complex, which is used to describe both the columnar and the plaquette ordering patterns.\textsuperscript{61} This situation will be analyzed in next subsection as well.]

The resulting NLSM for two coupled antiferromagnetic layers with antiferromagnetic inter-layer coupling reads

\[
S = \sum_{i=1,2} \int d\tau d^2x L_i + i\pi Q_i + \int d\tau d^2x V(\vec{n}_1 \cdot \vec{n}_2),
\]

\[
L_i = \frac{1}{2c_3 g_i} (\partial_\mu \phi_i)^2, \quad Q_{Hi} = \frac{1}{12\pi^2} \int d\tau d^2x \epsilon_{\alpha \beta \gamma \delta} \epsilon_{\mu \nu \lambda} \phi_\alpha \partial_\mu \phi_\beta \partial_\nu \phi_\gamma \partial_\lambda \phi_\delta \tag{12}
\]

Due to the V-term which reflects the Kondo coupling between conduction and local spin degree of freedom, the above NLSM is not O(4) invariant but degenerates into global O(3)\texttimes Z\textsubscript{2} symmetry. The Z\textsubscript{2} refers to the VBS order parameter \(\phi_0\), while the O(3) invariant corresponds to the rotation symmetry of \(\vec{n}_1, \vec{n}_2\). Here, unfortunately, we do not know how to treat even the strong coupling limit \(V \rightarrow \pm \infty\). Instead, in order to restore the O(4) invariant, we may artificially add the term \(V \phi_0 \phi_2\). It seems that if certain multi-spin exchange interaction is added, this O(4) symmetry may be realized microscopically, as indicated from the example of four-spin interaction in the two coupled one-dimensional chains.\textsuperscript{24} Then, interestingly, one finds that the strong coupling limit \(V \rightarrow \pm \infty\) is described by the same action (setting \(c = 1\))

\[
S = \int d\tau d^2x \frac{1}{g} (\partial_\mu \phi)^2 + i2\pi Q_H. \tag{13}
\]

When \(g\) is large, this model disorders (the Kondo spin liquid phase) and its bulk excitation is gapped. The \(i2\pi Q_H\) term contributes a factor of unity to the partition function and has no noticeable contribution for the bulk spectrum. Interestingly, the boundary of this O(4) model is gapless because the edge state is described by the (1+1)-D SU(2)\(_{k=1}\) WZW model.\textsuperscript{64} This point can be made clear as follows. First, using the bulk-boundary correspondence, the edge action reads\textsuperscript{65}

\[
S = \int d\tau dx \frac{1}{g} (\partial_\mu \phi)^2 + i\frac{2\pi}{12\pi^2} \int d\tau dx \int_0^1 du \epsilon_{\alpha \beta \gamma \delta} \epsilon_{\mu \nu \lambda} \phi_\alpha \partial_\mu \phi_\beta \partial_\nu \phi_\gamma \partial_\lambda \phi_\delta \tag{14}
\]

Here the second term is usually called the Wess-Zumino term. The field \(\phi(x, \tau, u)\) is defined on a three-dimensional hemisphere with a boundary coinciding with the two-dimensional plane \((\tau, \gamma)\), where the original theory is defined. So we have \(\phi(x, \tau, u = 0) = \phi(x, \tau)\)\textsuperscript{24} Being a total derivative, the three-dimensional integral does not really depend on \(\phi(x, \tau, u)\).\textsuperscript{64} By introducing an SU(2) matrix \(\vec{U} = \phi_0 I + i\vec{n} \cdot \vec{\phi}\)\textsuperscript{61,63} the above action can be rewritten as the standard SU(2)\(_{k=1}\) WZW model

\[
S = \int d\tau dx \frac{1}{2g} Tr(\partial_\mu \vec{U}^{-1} \partial_\mu \vec{U}) + \frac{2\pi i}{24\pi^2} \int d\tau dx du \epsilon_{\mu \nu \lambda} Tr(\vec{U}^{-1} \partial_\mu \vec{U}^{-1} \partial_\nu \vec{U}^{-1} \partial_\lambda \vec{U}) \tag{15}
\]
And the ground-state is described by the following gapless fixed action (also called critical WZW model\textsuperscript{64})

\[
S = \int d\tau dx \frac{1}{8\pi} Tr(\hat{U}^{-1} \partial_\tau \hat{U} \hat{U}^{-1} \partial_\mu \hat{U}) +
\]

\[
\frac{i}{12\pi} \int d\tau dx du \partial_\mu \lambda \partial_\nu \lambda \partial_\rho \lambda \partial_\sigma \lambda \partial_\chi \phi \partial_\phi \partial_\psi \partial_\phi.
\]

For the WZW model Eq.(15) and Eq.(16), the symmetry is \( SU(2)_L \times SU(2)_R \), which states the action is invariant under transformation \( \hat{U} \rightarrow G_L \hat{U} G_R \) with \( G_L, G_R \) being \( SU(2) \) matrices. Such \( SU(2)_L \times SU(2)_R \) symmetry is in fact related to two decoupled edge excitations for the critical WZW model Eq.(18). One is the left mover \( J_+ = \frac{K}{2\pi} \partial_+ \hat{U} \hat{U}^{-1} \) and other is \( J_- = -\frac{K}{2\pi} \hat{U}^{-1} \partial_\tau \hat{U} \). (We have defined the light-cone or chiral coordinate as \( x^\pm = \frac{\sqrt{2}}{\pi}(\tau \pm i x) \) and \( \partial_{\pm} = \frac{\sqrt{2}}{2\pi}(\partial_\tau \mp i\partial_\tau) \).) Thus, the two movers satisfy the equation of motion \( \partial_\tau J_\pm = 0 \) and \( J_- \) is invariant under \( SU(2)_L \) transformation \( \hat{U} \rightarrow G_L \hat{U} \) while \( J_+ \) transforms as \( G_L J_+ G_L^{-1} \). Thus, the left mover \( J_+ \) carries the \( SU(2)_L \) charge and \( J_- \) is \( SU(2)_L \) neutral. Furthermore, if a suitable \( SU(2) \) external field is introduced, the left mover \( J_+ \) will respond to it and one may see a quantized Hall conductance\textsuperscript{64}.

A careful reader may notice that the Eq.(18) is just a form of a kind of \( (2+1) \)-D O(4) NSLM\textsuperscript{64} states and Eq.(16) corresponds to the expected symmetry-protected gapless edge state of the \( SU(2) \) SPT states\textsuperscript{64}. However, we should emphasize that the original model Eq.(12) does not have the O(4) invariant due to the V-term. In other words, the \( SU(2) \) symmetry (in fact the \( SU(2)_L \)), which leads to the symmetry-protected gapless edge state, will not be preserved. Therefore the disorder phase of our original model (Eq.(12)) is not an SPT state but seems a trivial one. However, we hope that the present study may help to identify certain microscopic bilayer antiferromagnets models which can support the desirable SPT state.

B. O(5) NLSM and relation to 3d SPT states

It is noted that in the square lattice, the more general VBS order parameter should be a complex field, in order to describe both the columnar and the plaquette ordering patterns\textsuperscript{66}. In this case, the complex VBS order parameter reads as \( \psi_{\text{VBS}} = \phi_{xx} + i \phi_{xy} \), with \( \phi_{xx}, \phi_{xy} \) being real fields. An O(5) vector field \( \phi = (\phi_{xx}, \phi_{xy}, \vec{n}) \) can be constructed accordingly, where the third component vector field \( \vec{n} \) is the usual Néel order parameter. Therefore, the effective O(4) NSLM (Eq.(18)) considered in the previous subsection has to be replaced by an O(5) version of NLSM\textsuperscript{66}.

\[
S = \int d\tau dx \frac{1}{g} (\partial_\mu \phi)^2 + i 2\pi \bar{Q},
\]

where the winding number \( \bar{Q} \) is defined as

\[
\bar{Q} = \frac{1}{64\pi^2} \int d\tau dx d\mu \frac{1}{g} (\partial_\mu \phi)^2
\]

Here, the \( i2\pi \bar{Q} \) is also known as the WZW term, and \( \phi(\tau, x, y, u) \) is an extension of the space-time configuration of \( \phi(\tau, x, y) \), which satisfies \( \phi(\tau, x, y, 0) = (0, 0, 0, 1) \) and \( \phi(\tau, x, y, 1) = \phi(\tau, x, y) \). Interestingly, such \((2+1)\)-D O(5) NLSM describes one of the gapless surface states of 3D SPT phase, whose effective action is\textsuperscript{66}.

\[
S = \int d\tau dx d\mu \frac{1}{g} (\partial_\mu \phi)^2
\]

\[
+i 2\pi \int d\tau dx d\mu d\nu d\rho d\sigma d\delta d\phi d\lambda \partial_\mu \phi d\nu \phi d\rho \phi d\sigma \phi d\delta \phi.
\]

Since it has been known that the O(5) NLSM with the WZW term is gapless, it seems that it can not describe the gaped Kondo spin liquid phase. On the other hand, if the WZW term is irrelevant in this situation, we might suggest that an O(5) NLSM without the WZW term may reproduce the correct physics when the Kondo spin liquid state is concerned. Again, it is consistent with the previous speculation that the state belongs to a trivial disorder phase.

C. An alternative route to the description of high-dimensional Kondo spin liquid

An alternative route to describe the Kondo necklace model in 2D is by using (coupled multi-chains) multi-leg ladders for each effective antiferromagnetic layer\textsuperscript{65}. However, the multi-leg-ladder construction itself suffers from the serious even-odd effect, where the physics heavily depends on whether the number of legs (chains) is even or odd\textsuperscript{65}. For a odd-leg ladder, it behaves like the single spin-1/2 Heisenberg chain, while an even-leg ladder has similar properties as the \( S = 1 \) Heisenberg chain\textsuperscript{65}. Based on the consideration that the realistic 2D system should not have such strong even-odd effect, we may not use this formalism in our discussion. But it is interesting to point out that there exist some interesting studies of multi-leg spin ladders. For example, see Ref.\textsuperscript{55} and references therein.

D. Comparison with topological Kondo insulator

In addition, we note that in literature, the so-called topological Kondo insulator has been proposed as an extension of the original topological (band) insulators\textsuperscript{68}. In contrast to usual topological insulators where the spin-orbit coupling is encoded in a spin-dependent hopping amplitudes between different unit cells, the topological Kondo insulator is produced by the spin-orbit coupling associated with the hybridization between conduction
and local electrons. Different from our case, these insulators are obviously topological non-trivial due to their intrinsic spin-orbit couplings, and their gapless surface states are protected by the topology of the bulk (time-reversal symmetry).

VI. CONCLUSION

In summary, we have studied the Kondo necklace model in terms of effective non-linear sigma model field theory and found that the bulk properties of the Kondo spin liquid phase are similar to the celebrated Haldane. The distinction of these two phases mainly relies on their boundary, where on either side of the open chains, the Haldane phase has a fractionalized $S = 1/2$ spin while the Kondo spin liquid does not support such degree of freedom. In addition, we analyze the topological properties of Kondo spin liquids in high dimensional Kondo necklace models. It seems that those Kondo spin liquids are still topologically trivial compared to the topological Kondo insulator unless some spin-orbit coupling-like elements are introduced. Yet, interestingly, if the system’s symmetry is extended to O(4), and the hedgehog-like point defects are taken into account, our model can be identified to a kind of SU(2) SPT models. This indicates that certain microscopic bilayer antiferromagnets models could support the desirable SPT state and even some real-world bilayer materials might be devised accordingly in the near future. Furthermore, if the symmetry is expanded to O(5), the effective action just describes the surface state of a 3D SPT state though such O(5) NLSM could not be a proper description of Kondo spin liquid phase due to its gaplessness. We expect the present work may be useful for further study on states with nontrivial topological properties.

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