I. INTRODUCTION

General relativity (GR) considered as the best classical gauge theory to explore gravitational interaction locally and globally \(^1\). Several tests provided a trustable framework for GR. Because quantum mechanics (QM) also proved to be the best non relativistic framework to study tiny scales systems, it is naturally arised a question that whether QM can be written in a covariant form to include all quantum effects on curved space time manifolds or not?. Since GR has locally Lorentz invariance, as a result to build covariant QM we need first make QM relativistic and it is adequate to make it in canonical formalism. It was Stueckelberg who built a relativistic quantum mechanics in canonical formalism Stueckelberg\(^2\). Since relativistic wave equations have always many body interpretations, it was adequate to generalize Stueckelberg’s works to such many body cases. A generalization of the Stueckelberg’s relativistic canonical formalism to many body systems investigated by Horwitz and Piron\(^3\). This theory which we will refer to as SHP theory , recently revisited by Horwitz and generalized it to the general Lorentzian curved manifold \(^4\). In a recent paper I investigated exact solutions for SHP theory both in classical and quantum domains for a covariant simple harmonic oscillator (CSHO) in the vicinity of a black hole \(^5\). In continuation of my recent study and because SHP theory plays an important role to build covariant QM in curved space times, in this work I studied exact solutions for a charged CSHO in the background of magnetically Charged black hole. I allow that the CSHO have also a background dependet interaction with an external gauge field. The first application of the results in this letter will be an starting point to investigate covariant Aharonov and Bohm effect on curved backgrounds. It is well known that using the Schrödinger equation, the problem of the scattering of an electron in an external static magnetic field, in flat space showed an independent depth of penetration of the electrons into the region of non zero magnetic force lines. This effect discovered in 1959 by Aharonov and Bohm \(^6\) and named as Aharonov-Bohm (AB) effect. An direct interpretation of AB is that the external \(U(1)\) electromagnetic field interacts with the charged particles and penetrateto the region in which the field is localized and according to the QM , cannot be reached by the particles (see, the reviews of Refs. \(^7\) and \(^8\)). AB effect in curved space proposed in past in Ref. \(^9\) but in that interesting study the wave equation for QM didn’t preserve general covariance. Consequently still it is very important to investigate the effect of curvature of space on the Aharonov-Bohm effect. As an starting point, in this letter I considered covariant QM wave equation proposed in the SHP theory and I consider the situation in which there is an external magnetic force as well as an additional external static spherically symmetric gravitational field. I will solve the wave equation and will investigate the possible lower and higher modes scattering of this particle from the horizon of the black hole. Note that \(c = \hbar = 1\) in our units convention through whole analysis in this letter.

II. QUANTUM MECHANICS VIA SCHROEDINGER-STUECKELBERG-HORWITZ-PIRON WAVE EQUATION ON MAGNETICALLY CHARGED BLACK HOLES

According to the pionnering work of Horwitz, the covariant quantum mechnical wave equation for quantum particle on a curved general relativity background well formulated in \(^4\) and formulated as the below Schrödinger-Stueckelberg-Horwitz-Piron wave equation:

\[
i\hbar \frac{\partial}{\partial \tau} \Psi_\tau(x^\mu) = \hat{H} \Psi_\tau(x^\mu)
\]  \hspace{1cm} (2.1)

In the analogous to the QM, \(\hat{H}\) is quantum mechanical operator. The wave equation presented in Eq. (2.1) de-
fines a Hilbert space with scalar product as follows:

\[
(\psi, \chi) = \int d^4x \sqrt{-g} \Psi^*_\tau(x^\mu) \chi(x^\mu) \tag{2.2}
\]

In the above definition, the volume element is written as \(d^4x \sqrt{-g}\) and * denotes complex conjugate. The appropriate form for a quantum mechanical Hamiltonian \(\hat{H}\) by following convention of indices given in ref. 4 defined as:

\[
\hat{H} = \frac{1}{2M\sqrt{-g}}(p_\mu - a_\mu)p_\mu g^{\mu \nu}(p_\nu - a_\nu) + V(x) \tag{2.3}
\]

Very recently we have investigated exact mode decomposition solutions for wave equation (2.1) in \([5]\). The aim in this letter is to use this covariant wave equation as:

\[
ds^2 = -B(r)dt^2 + B^{-1}(r)dr^2 + r^2 d\Omega^2. \tag{2.4}
\]

here \(B(r) = 1 - \frac{2MG}{r} + \frac{4\pi G}{r^2} \sqrt{Q_E^2 + Q_M^2}\). In the above metric function, \(Q_E, Q_M\) are electric and magnetic \(U(1)\) charges. The corresponding electric and magnetic fields due to the metric (2.6) are as follow:

\[
E_r = \frac{Q_E}{r^2}. \tag{2.5}
\]

\[
B_r = \frac{Q_M}{r^2}. \tag{2.6}
\]

Note that the geometrical structure of the metric (2.6) can be described by defining of an extremal mass parameter as follows:

\[
M_{ext} = \sqrt{4\pi (Q_E^2 + Q_M^2)} M_{pl}. \tag{2.7}
\]

here \(M_{pl} = G^{-1/2}\). There are three different cases for geometry of metric:

- if \(M > M_{ext}\), there is a pair of real zeros for algebraic equation \(B(r) = 0\), given by \(r_\pm\) (singularities). The region outside \(r > r_+\), is a region out of the horizon.
- if \(M = M_{ext}\), the radius \(r_+ = r_-\) coincides, we end up by an extremal Reissner-Nordstrom blackhole.
- if \(M < M_{ext}\), the spacetime has a naked singularity, it implies a no horizon solution. We obey the cosmic censorship consequently we aviod from the naked singularity.

Though this study we will consider \(M > M_{ext}\) as our possible black hole background.

It is adequate to mention here about the role of Dirac-string when the magnetic field eq. (2.6) arises from a monopole with magnetic charge \(Q_M\) gives rise to a radial \(\vec{B} = Br\). The magnetic field (2.6) yields to a vector potential \(\vec{A}_{\text{Dirac}} = \vec{A}\). Using the soloroid condition, \(\nabla \cdot \vec{B} = 0\), we have \(\vec{B} = \nabla \times \vec{A}\), in the coordinates \(x^\mu = (r, \theta, \varphi)\) we can opt the vector potential as follows:

\[
\vec{A} = (0, 0, Q_M(1 - \cos \theta)). \tag{2.8}
\]

The above vector potential eq. (2.8), has a singularity for \(z(= r \cos \theta) < 0\), called Dirac singularity. A suitable gauge transformation can move (but not remove) this singularity.

In flat space, the AB effect makes sense to this string unobservable. We can write \(Q_M\) using Dirac quantization as

\[
Q_M = \frac{QE}{e} \in \mathbb{Z} \quad \text{or} \quad \mathbb{Z} + \frac{1}{2}. \tag{2.9}
\]

Note that still we have a singularity at \(r = 0\). This singularity can be explained via a spontaneously broken gauge have non singular classical solutions,

\[
SU(2) \to U(1) \times \phi_{\text{Higgs}}. \tag{2.10}
\]

where the vacuum expectation value of the Higgs field \(<\phi_{\text{Higgs}}>) \neq 0\). In our study on curved spacetime which is based on the charged black hole solution (2.6), the aim is to split the Hamiltonian to \(\hat{H} = \hat{H}_0 + \Delta \hat{H}\), here \(\Delta \hat{H}\) is the perturbation term. I will compute the scattering cross section as well as exact solutions for unperturbated part \(\hat{H}_0\). We use an ansatz \(a_\mu = (a(r), \vec{A}(\theta))\), where \(a(r)\) is the scalar electric potential and \(\vec{A}\) is the Dirac vector potential in eq. (2.8). Following our former study in [5], we consider the potential \(V(r)\) as a covariant harmonic oscillator. In the background metric (2.6), the decomposition of equation (2.3) with the below form is possible:

\[
\Psi_\tau(x^\mu) = \exp \left[ - \frac{iE \tau}{\hbar} - i\omega t \right] \Phi(r, \theta, \varphi) \tag{2.11}
\]

In the Lorentz’s gauge,

\[
\partial_\mu (g^{\mu \nu} a_\nu) = 0. \tag{2.12}
\]

This gauge fixing condition gives us \(a(r) = \frac{Q_E}{e}\) as a coulomb’s scalar potential in the theory. We will end up by the following partial differential equation, eq. (2.11),
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi(r, \theta, \varphi)}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi(r, \theta, \varphi)}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi(r, \theta, \varphi)}{\partial \varphi^2} \\
+ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \delta B \frac{\partial \Phi(r, \theta, \varphi)}{\partial r} \right) - \left[ \frac{2 \omega a(r)}{B(r)} + \frac{2 \hbar \mu A(\theta)}{r^2 \sin^2 \theta} - E + V(r) \right] \Phi(r, \theta, \varphi) = 0. \quad (2.13)
\]

where we rewrite the metric function in the weak regime as follows,

\[
B(r) = 1 + \delta B(r), \quad |\delta B(r)| \ll 1. \quad (2.14)
\]
in our case, \[\delta B(r) = -\frac{2GM}{r} + \frac{GM_{\text{ext}}}{4\pi M_{\text{pl}}} \frac{1}{r}.\] We can rewrite eq. (2.13), in the following operator form in the representation theory,

\[
\hat{H}_0 \Phi + \epsilon (\Delta \hat{H}) \Phi = 0. \quad (2.15)
\]

Using the iteration technique, we need to substitute \(\Phi^0\) in \(\epsilon (\Delta \hat{H})\Phi\), and taking the first approximation, we obtain:

\[
\Phi^1 = -\hat{H}_0^{-1} (\Delta \hat{H}) \Phi^0. \quad (2.20)
\]

Now to solve the total Hamiltonian wave equation, i.e., Eq. (2.15) we use iteration method. We take the unperturbated solution (2.19) as zeroth order approximation and by inserting it to the full perturbed system Eq. (2.15), on the first level of the perturbation theory, we obtain:

\[
\Phi(\vec{r}) \approx \Phi^0(\vec{r}) + \int_{\Omega} G(\vec{r}, \vec{r}') \hat{H}_0^{-1}(\epsilon \Delta \hat{H}(\vec{r}')) \Phi^0(\vec{r}') d^3r'. \quad (2.21)
\]
bounded by \( r_+ \leq r < \infty \) obtained in terms of the harmonic functions of the Laplace operator \( \Delta = \hat{H}_0^{-1} \) as follows:

\[
G(\vec{r}, \vec{r}') = \sum_{l=0}^{\infty} \sum_{\mu=-l}^{+l} \frac{4\pi}{2l+1} Y_{l\mu}(\theta, \phi) \hat{Y}_{l\mu}^*(\theta', \phi') \left[ \frac{r_<}{r_+^{l+1}} - \frac{r_+^{2l+1}}{(r_< r_+)^{l+1}} \right] 
\]

(2.23)

where \( r_<, r_+ \) is the smaller (larger) of \( r \) and \( r' \). The first order solution for \( \phi^1 \) is obtained by

\[
\Phi^1(\vec{r}) = \int_{\Omega} G(\vec{r}, \vec{r}') \epsilon \hat{H}(\vec{r}') \Phi^0(\vec{r}') d\vec{r}' 
\]

(2.24)

The next task is to calculate closed form for (2.22) and specially study its lower order term, when \( l = 0 \), called s-wave. Comparing \( \Phi^1 \) with the zeroth order approximated solution, provides a way to compute the amplitude between reflected wave to the incident wave.

III. S-WAVE SCATTERING CROSS SECTION

Note that if we focus on s-wave, when \( \Phi^0(\vec{r}') = \frac{a}{r} \) where \( a \) can be computed via the inner product defined in Eq. (2.2) with metric (2.6), then \( \epsilon \hat{H}(\vec{r}') \Phi^0(\vec{r}') = s_1(r') + s_2(r') s_3(\theta') \) here the radial and azimutal functions represented as follow:

\[
s_1(r) = -\frac{a \partial_r (\delta B(r))}{r^2} - \frac{a}{r} \left( \frac{2\omega Q_E}{r B(r)} - E + \frac{1}{2} m \Omega^2 r^2 \right) 
\]

(3.1)

\[
s_2(r) = -\frac{2Q_M a \hbar \mu}{r^3} 
\]

(3.2)

\[
s_3(\theta) = (1 + \cos \theta)^{-1} 
\]

(3.3)

We can rewrite the source term, i.e., \( \epsilon \hat{H}(\vec{r}') \Phi^0(\vec{r}') \) in terms of the spherical harmonic functions according to the completeness condition:

\[
\epsilon \hat{H}(\vec{r}') \Phi^0(\vec{r}') = \sqrt{4\pi} s_1(r') Y_{00}(\theta', \phi') + s_2(r') \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{+l'} a_{l'm'} Y_{l'm'}(\theta', \phi') 
\]

(3.4)

Note that since

\[
a_{l'm'} = \int d\Omega' s_3(\theta') Y_{l'm'}^*(\theta', \phi') = \sqrt{\frac{2l'+1}{4\pi}} \sqrt{\frac{(l'-m')!}{(l'+m')!}} \delta_{m',0} \int_{-1}^{1} dx p_{l'}(x) \frac{1}{1+x} 
\]

(3.5)

consequently the source term simplifies to the follows:
\[ c \Delta \tilde{H}(r') \Phi^l (r') = \sqrt{4\pi} s_1(r') Y_{00}(\theta', \varphi') + s_2(r') \sum_{l' = 0}^{\infty} \frac{2l'+1}{4\pi} p_{l'}(\cos \theta') c_{l'} \] (3.6)

here we define \( c_{l'} \equiv \int_{-1}^{1} \frac{p_{l'}(x)dx}{1+x} \), consequently plugging \( c_{l'} \) in (2.23) we obtain:

\[ \Phi^1 = 4\pi \int_{r_+}^{\infty} r'^2 dr' s_1(r') \left[ \frac{1}{r_>} - r_+ \right] + \sum_{l=0}^{\infty} c_l p_l(\cos \theta) \sqrt{\frac{4\pi}{2l+1}} \int_{r_+}^{\infty} r'^2 dr' s_2(r') \left[ \frac{r_l}{r_>^{l+1}} - \frac{r_+^{2l+1}}{(r_<r_>)^{l+1}} \right] \] (3.7)

To calculate the integral we have to split the integral as follows:

\[ \int_{r_+}^{\infty} [...] dr' = \int_{r_+}^{r} [...] dr' + \int_{r}^{\infty} [...] dr' \] (3.8)

\[ \int_{r_+}^{\infty} r'^2 dr' s_1(r') \left[ \frac{1}{r_>} - r_+ \right] = \int_{r_+}^{r} r'^2 dr' s_1(r') + \sum_{l=0}^{\infty} c_l p_l(\cos \theta) \int_{r_+}^{\infty} r'^2 dr' s_2(r') \left[ \frac{r_l}{r_>^{l+1}} - \frac{r_+^{2l+1}}{(r_<r_>)^{l+1}} \right] \] (3.9)

\[ \int_{r_+}^{\infty} r'^2 dr' s_2(r') \left[ \frac{r_+^l}{r_>^{l+1}} - \frac{r_<^{2l+1}}{(r_<r_>)^{l+1}} \right] = \int_{r_+}^{r} r'^2 dr' s_2(r') \left[ \frac{r_l}{r_>^{l+1}} - \frac{r_+^{2l+1}}{(r_<r_>)^{l+1}} \right] + \int_{r}^{\infty} r'^2 dr' s_2(r') \left[ \frac{r_l}{r_>^{l+1}} - \frac{r_+^{2l+1}}{(r_<r_>)^{l+1}} \right] \] (3.10)

Finally the first order solution is obtained in the closed form as follows:

\[ \Phi^0 = \frac{a}{r} + 4\pi \left( \alpha_0^0(r_+, r) + \beta_0^0(r_+, r) \right) + \sum_{l=0}^{\infty} c_l p_l(\cos \theta) \sqrt{\frac{4\pi}{2l+1}} \left( \alpha_l^0(r_+, r) + \beta_l^0(r_+, r) \right) \] (3.11)

here we define a set of auxiliary functions:

\[ \alpha_l^{1,2}(r_+, r) = \int_{r_+}^{r} r'^2 dr' s_1,2(r') \left[ \frac{r_l}{r_>^{l+1}} - \frac{r_+^{2l+1}}{(r_<r_>)^{l+1}} \right] \] (3.12)

\[ \beta_l^{1,2}(r_+, r) = \int_{r}^{\infty} r'^2 dr' s_1,2(r') \left[ \frac{r_l}{r_>^{l+1}} - \frac{r_+^{2l+1}}{(r_<r_>)^{l+1}} \right] \] (3.13)

here \( l = 0...\infty \). Note that in the scattering regime, when the fields are considered only at the asymptotic limit \( r \rightarrow \)
\[ \alpha^{1,2}_l(r_+ \to \infty) \approx \frac{1}{r_l^{l+1}} \int_{r_+}^{\infty} r^2 dr' s_{1,2}(r') = \beta^{1,2}_l(\infty) \]  

(3.14)

\[ \beta^{1,2}_l(r_+, r) \approx 0. \]  

(3.15)

and finally we have:

\[ \Phi(\vec{r}) \approx \frac{a}{r} + \frac{4\pi \gamma_1^{\infty}}{r} + \sum_{l=1}^{\infty} c_l p_l(\cos \theta) \sqrt{\frac{4\pi}{2l + 1}} \frac{\gamma_1^{\infty}}{r^{l+1}} \]  

(3.16)

The cross section for the scattered waves are

\[ \sigma(\theta, \varphi) \sin \theta d\theta d\varphi = |f(\theta, \varphi)|^2 \sin \theta d\theta d\varphi. \]  

(3.17)

here we have

\[ f(\theta, \varphi) = \sqrt{\frac{\pi}{3}} 2 c_1 \gamma_1^{\infty} a^2 p_1(\cos \theta) \]  

(3.18)

here

\[ \gamma_1^{\infty} = 2 Q_M a h \mu \ln \frac{r_+}{\Lambda} \]  

(3.19)

\[ c_1 = 2 \tanh^{-1}(\Lambda^{-1}) \]  

(3.20)

and \( \Lambda \) is an ultraviolet cutoff parameter. The total cross section for the s-wave scattering is given as follows:

\[ \sigma_{\text{tot}} = \frac{256 \pi^2 \mu^2 \hbar^2 \text{Arctanh} \left( \frac{1}{\Lambda} \right)^2 Q_M^2 \log^2 \left( \frac{r_+}{\Lambda} \right)}{9 a^2} \]  

(3.21)

here \( r_+ = 1 + \sqrt{1 - \frac{2 M_{\text{ext}}}{GM_{\text{pl}}}} \). If we find the normalization factor \( a \) using the normalization integral (2.2) and by plugging the horizon radius \( r_+ \), finally we have:

\[ \sigma_{\text{tot}} = \frac{1024}{9} \pi^3 \mu^2 \hbar^2 Q_M^2 \text{Arctanh} \left( \frac{1}{\Lambda} \right)^2 \left( \Lambda - 1 - \sqrt{1 - \frac{2 M_{\text{ext}}}{GM_{\text{pl}}}} \right) \ln^2 \left( \frac{\sqrt{1 - \frac{2 M_{\text{ext}}}{GM_{\text{pl}}}} + 1}{\Lambda} \right) \]  

(3.22)

In limit \( \Lambda \to \infty \) it reduces to \( \sigma_{\text{tot}} \approx \frac{1024 \pi^3 \mu^2 \hbar^2 Q_M^2 \log^2(\Lambda)}{9 a^2} \approx 0 \) A possible explanation may be because black hole is magnetic as well, that might destroy the spherical symmetry and the S wave cross section could vanish \([11]\).

IV. GENERAL SOLUTION FOR \( \Phi \) FOR HIGHER ORDERS MOMENTS

In the previous section we focused on the lowest mode, when \( \Phi^0(\vec{r}) = ar^{-1} \). In this section we wanna find a more general solution by inserting the general solution \([2.19]\). The aim is to find \( \Phi^1(\vec{r}) \) by inserting \( \Phi^0(\vec{r}) \) as the zeroth order solution. Firstly it is more suitable to write the solution \( \Phi^0(\vec{r}) \) and Green’s function \([2.23]\) in the following form:

\[ \Phi^0(\vec{r}) = \sum_{l=0}^{+l} \sum_{\mu=-l}^{+l} j_{l\mu}(r) Y_{l\mu}(\theta, \varphi) \]  

(4.1)

\[ G(\vec{r}, \vec{r'}) = \sum_{l=0}^{+l} \sum_{\mu=-l}^{+l} g_{l\mu}(r, r') Y_{l\mu}(\theta, \varphi) Y_{l\mu}^*(\theta', \varphi') \]  

(4.2)
For our next purposes we mention here that

\[ \partial_r \Phi^0(r') = - \sum_{l=0}^{\infty} \sum_{\mu=-l}^{+l} (l+1) \frac{f_{l\mu}(r)}{r} Y_{l\mu}(\theta, \varphi) \] (4.3)

Using the above expressions we can find the following source term:

\[ \epsilon \Delta H(r') \Phi^0(r') = \sum_{l=0}^{\infty} \sum_{\mu=-l}^{+l} \left( E - \frac{2Q_{E\omega}}{r(1 + \delta B(r))} \right) f_{l\mu}(r) Y_{l\mu}(\theta, \varphi) \] (4.4)

By plugging them in the \( \Phi^1(r') \), Eq. (2.24) we obtain:

\[ s_3(\theta') = \sum_{l=0}^{\infty} \sum_{\mu=-l}^{+l} c_{l\mu} Y_{l\mu}(\theta, \varphi) \delta \mu, 0 \] (4.5)

\[ \Phi^1(r') = \sum_{l=0}^{\infty} \sum_{\mu=-l}^{+l} \sum_{l'=0}^{\infty} \sum_{\mu'=-l'}^{+l'} \gamma_{\mu\mu'}(r) Y_{l\mu}(\theta, \varphi) \Phi_{\mu'}(\theta', \varphi') \Phi_{l'}(\theta', \varphi') \] (4.6)

\[ + \sum_{l,l',l''=0}^{\infty} \sum_{m,m',m''=0}^{\infty} e_{l',m'} \delta_{\mu',0} \xi_{l' \mu' \mu''}(r) Y_{l\mu}(\theta, \varphi) \Phi_{\mu''}(\theta', \varphi') \Phi_{l''}(\theta', \varphi') \] (4.7)

here

\[ \gamma_{\mu\mu'}(r) = \int_{r_+} r'^2 r'^2 g_{\mu'}(r_-, r') f_{l\mu}(r') \left( E - \frac{1}{2} m \Omega^2 r'^2 - \frac{2Q_{E\omega}}{r'^2(1 + \delta B(r'))} \right) \] (4.8)

\[ \xi_{l' \mu' \mu''}(r) = \int_{r_+} r'^2 r'^2 g_{l\mu'}(r_-, r') f_{l\mu}(r') s_2(r') \delta_{\mu'', 0} \] (4.9)

remembering these identities for harmonic functions \[12\],

\[ \Phi_{\mu'}(\theta', \varphi') = \delta_{\mu', \mu} \Phi_{\mu'}(\theta', \varphi') \] (4.10)

\[ \Phi_{l''}(\theta', \varphi') = \sqrt{\frac{(2l' + 1)(2l'' + 1)}{4\pi(2L + 1)}} C(l, l', l''|\mu, \mu', \mu'') C(l, l', l''|\mu, \mu', \mu''). \] (4.11)

Here \( L = l + l' + l'' \) and \( C(l, l', l''|\mu, \mu', \mu'') \) is Clebsch-Gordan coefficients. Such solution can be used to make
a better approximation to the cross section obtained in previous section.

V. SUMMARY

SHP theory provides a covariant quantum mechanical wave equation to study quantum mechanics on curved space times. A direct application can be a realization of the role of the QM in the magnetic fields. In this letter I applied SHP theory in a general magnetic-electric charged background for a covariant harmonic oscillator as our quantum mechanical toy model. I solved the wave equation in lower and higher modes using perturbation (iteration) technique. I showed that in the s-mode, the total scattering cross section can be computed analytically by introducing a suitable ultraviolet cutoff parameter $\Lambda$. This UV cutoff can be imagined as the UV analogous to an infrared (near horizon) cutoff $\epsilon$. A remarkable observation was the total cross section vanishes at large values of cutoff parameter. We can interpret it as an isolation of the magnetic monopole by the dual electric charge in this charged black hole background. Furthermore I derived exact higher mode solution. In a forthcoming paper I will study Aharonov-Bohm effect using these covariance quantum mechanical wave solutions.

VI. ACKNOWLEDGMENT

I thank Prof. Lawrence P. Horwitz for carefully reading my first draft, very useful comments, corrections and discussions.

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