ON CREATING MASS/MATTER BY EXTRA DIMENSIONS
IN THE EINSTEIN-GAUSS-BONNET GRAVITY

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Abstract

Kaluza-Klein (KK) black hole solutions in the Einstein-Gauss-Bonnet (EGB) gravity in $D$ dimensions obtained in the current series of the works by Maeda, Dadhich and Molina are examined. Interpreting their solutions, the authors claim that the mass/matter is created by the extra dimensions. To support this claim, one needs to show that such objects have classically defined masses. We calculate the mass and mass flux for 3D KK black holes in 6D EGB gravity whose properties are sufficiently physically interesting. Superpotentials for arbitrary types of perturbations on arbitrary curved backgrounds, recently obtained by the author, are used, and acceptable mass and mass flux are obtained. A possibility of considering the KK created matter as dark matter in the Universe is discussed.

1 Introduction

We study new exact solutions in the Einstein-Gauss-Bonnet (EGB) gravity in $D$ dimensions, which are $d$-dimensional Kaluza-Klein (KK) black holes (BHs) with $(D-d)$-dimensional submanifold, presented recently in [1] - [4] by Maeda, Dadhich and Molina. The authors treat them as a classical example of creating matter by curvature. The idea of such a kind is not new. Thus, to make inflation possible, a pioneer proposal was advanced by Starobinsky [5] that a high-energy density state was achieved by curved space corrections. Many other problems of modern cosmology may be solved in the framework of multidimensional gravity using high-order curvature invariants of KK type spacetimes, see, e.g., [6] and references there in.

To support the claim on creating ‘matter without matter’, it is necessary to calculate the mass and the mass flux by classical methods. It is the main goal of the present paper.
Here, we concentrate on 3D BHs in 6D EGB gravity [4]. These toy objects are rich enough in physical properties, e.g., they can have a radiative regime. For calculations we use the conservation laws developed by us in [7] - [9], where in the framework of EGB gravity, superpotentials (antisymmetric tensor densities) for arbitrary types of perturbations on arbitrary curved backgrounds have been constructed. Three important types of superpotentials [9] are used, those based on (i) Noether’s canonical theorem, (ii) Belinfante’s symmetrization rule and (iii) a field-theoretical derivation.

The paper is organized as follows. In section 2, we outline the solutions obtained in [1] - [4] and describe necessary properties of the 3D objects in 6D EGB gravity. In particular, in a natural way, we define a spacetime where a BH is placed. It can be considered as a possible background against which perturbations are studied. In section 3, in the preliminaries, the main notions and properties of the applied formalism are presented. Then we study the objects themselves: (a) as vacuum 6D solutions; (b) as 3D KK solutions with a ‘matter’ created by extra dimensions. Calculating the mass and the mass flux we support the second viewpoint. In section 4, we discuss (a) an ambiguity in the canonical approach related to a divergence in the Lagrangian; (b) a possibility of applying the KK BH solutions in cosmology. The Appendix presents explicit general expressions for all three types of superpotentials in EGB gravity.

2 Kaluza-Klein 3D black holes

We consider the action of the EGB gravity in the form:

$$S = -\frac{1}{2\kappa_D} \int d^D x \hat{L}_{EGB} = -\frac{1}{2\kappa_D} \int d^D x \sqrt{-g} \left[ R - 2\Lambda_0 + \alpha \left( R_{\mu\nu}^2 - 4R_{\mu\nu}^2 + R^2 \right) \right]_{L_{GB}}$$

(2.1)

where $\alpha > 0$. Here and below, curvature tensor $R_{\mu\nu}^\rho\sigma$, Ricci tensor $R_{\mu\nu}$ and scalar curvature $R$ are related to the dynamic metric $g_{\mu\nu}$; a ‘hat’ means densities of the +1, e.g., $\hat{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$; $(,\alpha) \equiv \partial_\alpha$ means ordinary derivatives; the subscripts ‘E’ and ‘GB’ are related to the Einstein and the Gauss-Bonnet parts in (2.1).

The main assumption in [1] - [4] is that the spacetime is locally homeomorphic to $\mathcal{M}^d \times \mathcal{K}^{D-d}$ with the metric $g_{\mu\nu} = \text{diag}(g_{AB}, r_0^2 \gamma_{ab})$, $A, B = 0, \cdots, d - 1$; $a, b = d, \cdots, D - 1$. Thus, $g_{AB}$ is an arbitrary Lorentzian metric on $\mathcal{M}^d$, $\gamma_{ab}$ is the unit metric on the $(D - d)$-dimensional space of constant curvature $\mathcal{K}^{D-d}$ with $k = 0, \pm 1$. Factor $r_0$ is a small scale of extra dimensions compactified by appropriate identifications. The gravitational equations
corresponding to the EGB gravity action (2.1) have the form:

\[ G^\mu_\nu \equiv G^\mu_\nu + \alpha H^\mu_\nu + \delta^\mu_\nu \Lambda_0 = 0, \tag{2.2} \]

where the Einstein tensor \( G^\mu_\nu \) and \( \delta^\mu_\nu \) correspond to the Einstein part and \( H^\mu_\nu \) corresponds to the GB part in (2.1). After all assumptions their decomposition is as follows:

\[
\begin{align*}
G^A_B & \equiv \left[ 1 + \frac{2k\alpha}{r^2_0} (D - d)(D - d - 1) \right] (d) G^A_B + \alpha (d) H^A_B \\
& + \left[ \Lambda_0 - \frac{k}{2r^2_0} (D - d)(D - d - 1) \left( 1 + \frac{k\alpha}{r^2_0} (D - d - 2)(D - d - 3) \right) \right] \delta^A_B = 0; \tag{2.3} \\
G^a_b & \equiv \delta^a_b \left\{ -\frac{(d) R}{2} + \Lambda_0 - \frac{k}{2r^2_0} (D - d - 1)(D - d - 2) - \alpha \left[ \frac{k}{r^2_0} (D - d - 1)(D - d - 2) \times \right. \right. \\
& \left. \left. \times \left( (d) R + \frac{k}{2r^2_0} (D - d - 3)(D - d - 4) \right) + \frac{(d) L_{GB}}{2} \right] \right\} = 0 \tag{2.4}
\end{align*}
\]

where the subscript ‘(d)’ means that a quantity is constructed with the use of \( g_{AB} \) only. As a result, one can see that (2.3) is a tensorial equation on \( M^d \), whereas (2.4) is a constraint for it. However to obtain more interesting solutions one has to consider a special case that the quantity \( G^A_B \) disappears identically. This is possible for \( d \leq 4 \) only because then \((d) H_{\mu\nu} \equiv 0\). Next, constants are chosen so as to suppress the coefficients in (2.3), which is possible if \( D \geq d + 2, k = -1 \) and \( \Lambda_0 < 0 \). Taking into account all the above, there remains a single governing equation, the scalar equation (2.4) on \( M^d \).

Here, we consider the solutions for \( D = 6 \) and \( d = 3 \) presented in [4]. A suitable set of constraints for the constants is \( r^2_0 = 12\alpha = -3/\Lambda_0 \). Then, the left hand side of (2.3) disappears identically. Keeping in mind that \((d)L_{GB} \equiv 0\), one simplifies (2.4) to obtain

\[(d) R = 2\Lambda_0, \tag{2.5}\]

to which the static solution \( g_{AB}(r) \) has been found:

\[ ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\phi, \quad f \equiv r^2/l^2 + q/r - \mu. \tag{2.6} \]

Here, \( \mu \) and \( q \) are integration constants, and \( l^2 \equiv -3/\Lambda_0 \). The Einstein tensor components for the solution (2.6) are

\[ G^0_0 = G^1_1 = 1/l^2 - q/2r^3, \quad G^2_2 = 1/l^2 + q/r^3. \tag{2.7} \]

As a space of a constant curvature, \((D - d = 3)\)-sector is completely presented by its scalar curvature:

\[(D-d) R = 6k/r^2_0 = 2\Lambda_0 = -1/2\alpha. \tag{2.8}\]
For comparison we consider the BTZ BH [10]. Its metric is presented in the form

\[ ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\phi , \quad f \equiv -r^2 \Lambda_0 - \mu , \]

(2.9)

which is a solution to the 3D pure Einstein equations. The horizon radius \( r_+ \) of the BH is defined as \( r_+^2 = -\mu / \Lambda_0 \), thus \( r_+ \) (and consequently a BH itself) disappears for vanishing \( \mu \). Therefore the integration constant \( \mu \) can be called the mass parameter. For \( \mu \to 0 \), the so-called real vacuum related to the BH (in another word, a spacetime where a BH is placed) is defined by (2.9) with \( \mathcal{F} = -r^2 \Lambda_0 \). However, such a spacetime is not maximally symmetric, unlike AdS one. The latter with \( \mathcal{F} = -r^2 \Lambda_0 + 1 \) is approached when \( \mu = -1 \). A difference between a real vacuum and a maximally symmetric vacuum is usual in BH solutions of modified metric theories (see, e.g., [11, 12]); the BTZ BH is the simplest illustration.

The solution (2.6) is more complicated than (2.9), although one has clear analogies with the BTZ case. Considering BH solutions for simulating dark matter (see a discussion in section 4) we are more interested in the cases with a horizon. In (2.6), the equation for the event horizon is \( l^2 q + r_+(r_+^2 - l^2 \mu) = 0 \). It is again natural to choose a mass parameter \( \tilde{\mu} \) in such a way that the BH horizon disappears under vanishing \( \tilde{\mu} \). This gives \( \tilde{\mu} = \mu - q/r_+ \) and \( r_+^2 = l^2 \tilde{\mu} \) (compare with the BTZ case), and consequently \( \tilde{\mu} > 0 \). Then a real vacuum is defined by (2.6) with \( \mathcal{F} \equiv r^2/l^2 + q/r - q/r_+ \), it is again not maximally symmetric. The maximally symmetric AdS vacuum is defined by (2.6) with \( \mathcal{F} \equiv r^2/l^2 + 1 \). For the latter, parameter \( q \) is considered entirely as a perturbation together with \( \mu + 1 \). For \( \tilde{\mu} \leq 0 \) a horizon does not exist, this takes place, when \( \mu > 0 \) with \( q > 2l (\mu/3)^{3/2} \) or \( \mu \leq 0 \) with \( q \geq 0 \).

The scalar equation (2.5) is also satisfied by the radiative Vaidya metric \( g_{AB}(v, r) \):

\[ ds^2 = -f dv^2 + 2 dv dr + r^2 d\phi , \quad f \equiv r^2/l^2 + q(v)/r - \mu(v) \]

(2.10)

where \( \mu(v) \) and \( q(v) \) now depend on the retarded/advanced time \( v \). Keeping in mind a possibility to form KK black holes [1] - [4], advanced time is more interesting. Then (2.10) can be connected with the solution of the form (2.6) by the transformation \( dt = dv - dr/f(v, r) \). After that, for every constant \( v_0 \), one can define its own horizon (if it exists) and a corresponding real vacuum analogously to the static case. The Einstein tensor components corresponding to (2.10) are

\[ G^0_0 = G^1_1 = 1/l^2 - q/2r^3 , \quad G^0_1 = (\dot{\mu} r - \dot{q})/2r^2 , \quad G^2_2 = 1/l^2 + q/r^3 , \]

(2.11)

where dot means \( \partial / \partial v \). The scalar curvature of \( (D - d = 3) \)-sector is expressed again by (2.8).

Considering (2.6) and (2.10) as solutions to the Einstein 3D equations on \( \mathcal{M}^3 \) (or, the same, EGB equations because in (2.2) one has \( \mathcal{H}_{\mu\nu} \equiv 0 \)), one concludes that they are not
vacuum equations with a redefined cosmological constant \( \Lambda = \Lambda_0 / 3 = -1/l^2 \). Indeed, both (2.7) and (2.11) show that a ‘matter’ source \( T_{AB} \) with zero trace \( T^{A}_A = 0 \) should exist, and the Einstein equations corresponding to (2.5) could be rewritten as

\[
(3) R_{AB} - \frac{1}{2} g_{AB}(3) R + g_{AB} \Lambda = \kappa_3 T_{AB}.
\] (2.12)

A natural treating in [1] - [4] is that \( T_{AB} \) is created by the compact extra dimensions.

3 The mass and the mass flux for 3D black holes

3.1 Preliminaries

Our calculation is based on differential conservation laws for perturbations in a given background spacetime in the form:

\[
\hat{T}^\alpha(\xi) = \partial_\beta \hat{T}^{\alpha\beta}(\xi)
\] (3.1)

where \( \xi^\alpha \) is a displacement vector, \( \hat{T}^\alpha \) is a vector density (current) and \( \hat{T}^{\alpha\beta} \) is an antisymmetric tensor density (superpotential). Thus, \( \partial_\alpha \hat{T}^{\alpha\beta} \equiv 0 \) and \( \partial_\beta \hat{T}^\alpha = 0 \). The current contains energy-momentum of both matter and metric perturbations, whereas the superpotential depends on metric perturbations only. Integrating \( \partial_\beta \hat{T}^\alpha = 0 \) and using the Gauss theorem one obtains the integral conserved charges in a generalized form:

\[
\mathcal{P}(\xi) = \int_\Sigma d^{D-1}x \hat{T}^0(\xi) = \oint_{\partial\Sigma} dS_i \hat{T}^{0i}(\xi)
\] (3.2)

where \( \Sigma \) is a \((D-1)\)-dimensional hypersurface \( x^0 = \text{const} \), \( \partial\Sigma \) is its \((D-2)\)-dimensional boundary, the zero indices correspond to time or lightlike coordinates, and small Latin indices correspond to space coordinates. Since we consider spherically symmetric systems, we need 01-components of the superpotentials in (3.2) only.

The formalism describes exact (not infinitesimal) perturbations in general. This is achieved if one solution (dynamical) is considered as a perturbed system with respect to another (background) solution of the same theory. Thus conserved quantities are defined with respect to a fixed (thought as known) spacetime, e.g., a mass of a perturbed system on a given background. A background can be both vacuum and non-vacuum, and usually is to be chosen to correspond with problems under consideration. The task of the present paper is calculating a global mass of the KK BHs presented above. It is more important the mass defined with respect to a spacetime, in which BH is placed because then with vanishing BH, one obtains a zero mass. Therefore, first of all a real vacuum described in previous section is chosen as a natural background. Although such backgrounds are curved
and nonsymmetric, the technique used is powerful. Besides, as interesting and important backgrounds we consider the AdS space. For such kinds of backgrounds, perturbations are not infinitesimal in general. However, we need in appropriate asymptotic of superpotentials in (3.2) only. As one can see below, the fall-off integrands in (3.2) both at spatial and at null infinity turns out to be sufficiently strong to allow surface integrals to converge and to give reasonable results.

In the previous section, the bar meant a quantity related to a spacetime where a BH is ‘placed’; here and below, without contradictions the bar means a quantity related to a background spacetime as a structure of the formalism. As a natural choice, for the above described static and radiative solutions we use the background metric in the same forms (2.6) and (2.10), respectively, where \( \overline{f} = \overline{f}(r) \) can be arbitrary in general but should be static. For calculating the global mass \( M \) we use the timelike Killing vector

\[
\xi^\alpha = (-1, 0). \tag{3.3}
\]

It has this unique form for the above two generalized types of background metrics: the zero component in (3.3) can be both timelike and lightlike; 0 includes 5 or 2 space dimensions in a 6D or 3D derivation, respectively. The metrics of the real vacuum and AdS space just belong to the aforementioned two types of background metrics and consequently also have a timelike Killing vector of the unique form (3.3). Then, since (3.3) is used every time, we will not recall this frequently.

### 3.2 The BTZ solution

As an example, we calculate the mass of the BTZ BH [10] with the metric (2.9). We take the Einstein parts of each of the superpotentials (A.1), (A.5) and (A.7), and, keeping in mind a 3D consideration, calculate their 01-components

\[
E_{C}^{01} = \frac{\sqrt{-g_3}}{2\kappa_3 r} (f - \overline{f}) \left[ \frac{rf'}{2ff'} (f - \overline{f}) - 1 \right], \tag{3.4}
\]

\[
E_{B}^{01} = \frac{\sqrt{-g_3}}{2\kappa_3 r} (f - \overline{f}) \left[ \frac{rf'}{2ff'} (3f + \overline{f}) - \frac{rf''}{f'^2} (f + \overline{f}) - 1 \right], \tag{3.5}
\]

\[
E_{S}^{01} = -\frac{\sqrt{-g_3}}{2\kappa_3 r} (f - \overline{f}) \frac{\overline{f}}{f}, \tag{3.6}
\]

where the prime means \( \partial/\partial r \). Taking into account a background with \( \overline{f} = -r^2\Lambda_0 \), for which \( f - \overline{f} = -\mu \), and substituting (3.4) - (3.6) into (3.2), we obtain, as \( r \to \infty \), the unique result

\[
M = \oint_{r \to \infty} E^{01} d\phi = \frac{\pi \mu}{\kappa_3}, \tag{3.7}
\]
which is quite acceptable for the global mass of the BTZ BH (see, e.g., [13]). The canonical superpotential has already been checked for calculating (3.7) in [14], for the other superpotentials the result (3.7) could be considered as a nice test. Using the AdS background with \( \mathcal{F} = -r^2 \Lambda_0 + 1 \) one obtains 

\[ M = \pi (\mu + 1) / \kappa_3. \]

### 3.3 The static KK solution

Now let us turn to (2.6); since it is the solution of the EGB theory one should try to calculate the mass with using the full formulae (A.1), (A.5) and (A.7) for this theory. The full background metric is to be chosen as \( g_{\mu \nu} = g_{AB} \times r_0^2 \gamma_{ab} \). Many formulae below take place for arbitrary \( f \) in (2.6), although in specific calculations we choose \( f \equiv r^2 / l^2 + q/r - q/r_+ \).

Let us turn to the \((D-2)\)-dimensional surface integral (3.2). Really, the distant surface is considered in \((d = 3)\)-dimensional spacetime only, whereas the integral over the \((D-d = 3)\)-dimensional compact space could be interpreted as a constant, which 'normalizes' the 6D Einstein constant \( \kappa_6 \) to the 3D one \( \kappa_3 \). Indeed, one has for the global mass constructed by (3.2):

\[
M = \oint_{\partial \Sigma} d \Sigma^{D-2} \sqrt{-g_D} I_{01}^D = \int_{r_0}^{\infty} d \phi \sqrt{-g_D} I_{01}^D \int_{r_0}^{\infty} d x^{D-d} \sqrt{-g_D} I_{01}^D = V_{r_0} \int_{r_0}^{\infty} d \phi \sqrt{-g_D} I_{01}^D. 
\]

Thus, since \( I_{01}^D \sim 1/\kappa_6 \) one could set \( \kappa_3 = \kappa_6 / V_{r_0} \). At first we follow this prescription.

With our assumptions, we find out that the Einstein parts of the 01-components of the superpotentials (A.1), (A.5) and (A.7) for the solution (2.6) are described only by the \( d \)-sector. Therefore, to calculate the Einstein parts, it is sufficient to use Eqs. (3.4) - (3.6), but only with \( \sqrt{-g_3} / \kappa_3 \) replaced by \( \sqrt{-g_D} / \kappa_6 \). For all cases, in the natural background, the Einstein part in (3.8) gives a result corresponding to (3.7):

\[
M_E = \pi \tilde{\mu} V_{r_0} / \kappa_6. 
\]

We now construct the GB 01-components of the superpotentials (A.1), (A.5) and (A.7) for the solution (2.6). They consist of two parts. The first one is pure \((d = 3)\)-dimensional:

\[
(d) I_{01}^C \equiv 0, \quad (d) I_{01}^B \equiv \frac{\alpha \sqrt{-g_D} \mathcal{F}}{\kappa_6 l^2} \frac{f}{\mathcal{F}} (f - \mathcal{F}) (r f'' - f'), 
\]

\[
(d) I_{01}^S \equiv 0
\]

(for brevity we suppress the subscript 'GB'). For \( \mathcal{F} \equiv r^2 / l^2 + q/r - q/r_+ \), the behavior of (3.11) as \( r \to \infty \) is \( \sim 1/r^3 \), thus each of the variants (3.10) - (3.12) gives a zero contribution into...
the integral (3.8). The other part of the GB 01-components is determined by the intersecting
terms of the \((d = 3)\)-sector and the scalar curvature of the \((D - d = 3)\)-sector (2.8):

\[
(D - d)\hat{\mathcal{I}}_{01}^C = \frac{\sqrt{-g_D}}{4\kappa_6}\left[ (f - \bar{f})' - \frac{\bar{f}}{f f'}(f - \bar{f})^2 + \frac{2(f - \bar{f})}{r} \right],
\]

\[
(D - d)\hat{\mathcal{I}}_{01}^B = \frac{\sqrt{-g_D}}{2\kappa_6}\left[ \frac{(f - \bar{f})^2}{2ff'}(f + \bar{f})' + \frac{f^2}{f} - \frac{\bar{f}^2}{f} \right],
\]

\[
(D - d)\hat{\mathcal{I}}_{01}^S = \frac{\sqrt{-g_D}}{2\kappa_6 r} (f - \bar{f}) \frac{\bar{f}}{f},
\]

where the subscript ‘\((D - d)\)’ means that a quantity is without pure ‘\(d\)’-terms. We remark
that both (3.10) and (3.13) are unique for each of (A.3) and (A.4). The asymptotic of each
of (3.13) - (3.15) at spatial infinity in the natural background is \(\sim -\tilde{\mu}\), and their substitution
into (3.8) gives the unique result:

\[
M_{GB} = -\pi \tilde{\mu} V_{r_0}/\kappa_6.
\]

Thus, keeping in mind (3.9) one can see that the global mass defined in the natural back-
ground by the total integral (3.8) is zero in all the three approaches. The same result is valid
if the AdS background with \(\bar{f} = r^2/l^2 + 1\) is chosen.\(^{1}\)

At least, this result could be anticipated for the field-theoretical approach. Indeed, the
superpotential (A.7) can be connected directly with the linearized equations [7]. Contracting
the latter with \(\xi^\alpha\) in (3.3), one selects the \(d\)-sector only. However, under the present assump-
tions, the tensor in (2.3) is equal to zero identically, therefore its linearization is equal to zero
identically as well. This conclusion is supported by combining the expressions (3.6), with
the replacement \(\sqrt{-g_3}/\kappa_3 \rightarrow \sqrt{-g_D}/\kappa_6\), (3.12) and (3.15), which leads to zero identically.
At the same time, the canonical and Belinfante corrected approaches give a zero result only
asymptotically.

Of course, the zero result cannot be acceptable. Analyzing (2.6), one can find out that,
considering this system from the point of view of the Newtonian-like limit in 3 dimensions
(see, e.g., [13]), this system must have a total mass. Thus one should conclude that a \textit{vacuum}
6D interpretation (2.2) with (3.8) is not successful. By this argument, one should consider the
3D Einstein interpretation (2.12) with a \textit{created ‘matter’}. Calculating the global conserved
quantity basing on (3.2), we can use only the surface integral, whereas a source (maybe not
determined explicitly, as in (2.12)) is included into the current in the volume integral. Thus,

\(^{1}\)The zero result has been recently obtained for a similar situation by other methods as well by R.G. Cai,
L.M. Cao, and N. Ohta, “Black holes without mass and entropy in Lovelocj gravity”, \textit{Phys. Rev. D,} \textbf{81},
024018; (\textit{Preprint} arXiv:0911.0245 [hep-th]).
considering the solution (2.6), we can be restricted to only the Einstein parts of each of the superpotentials (A.1), (A.5) and (A.7) related to the non-vacuum equations (2.12). As a full background metric, one must again consider $\mathbf{g}_{AB}$ in (2.6) (without $r_0^2 \gamma_{ab}$); we choose $\mathcal{T} = r^2/l^2 + q/r - q/r_+$ again and use the Killing vector (3.3). Then, since the parameter $q$ describes a ‘created matter’ in (2.12), such a background is not vacuum in 3 dimensions now. Nevertheless, the meaning of the notion ‘real vacuum’ is not changed, although it could be called wider as a ‘real background’ now. Also, the applied formalism remains powerful in non-vacuum backgrounds, and the structure of the superpotentials remains the same. Then again we use (3.4) - (3.6) and obtain the acceptable result of the type (3.7):

$$M = \frac{\bar{\mu}}{\kappa_3}.$$  (3.17)

If AdS space with $\mathcal{T} = r^2/l^2 + 1$ is chosen as a background, the mass of the system is $M = \pi(\mu + 1)/\kappa_3$. Note that in both cases the parameter $q$ makes no contribution.

### 3.4 The radiative Vaidya KK solution

For the radiative solution (2.10) we have carried out calculations similar to those in Subsection 3.3. Though, in this case the lightlike $v$-coordinate is used instead of the time $t$-coordinate. We again calculate 01-components for the superpotentials, however, now $\Sigma$ in (3.2) is defined as $x^0 = v = \text{constant}$, and the mass calculation is related to null infinity. In Eqs. (3.18) - (3.23) below, an arbitrary $\mathcal{T} = \mathcal{F}(r)$ is considered. However, now there is no sense to connect a background (which must be static) with a horizon (which is changed in time). Therefore, in specific calculations we consider the AdS background with $\mathcal{T} = r^2/l^2 + 1$ only.

We first derive out the Einstein parts of all superpotentials:

$$E^{\hat{I}_C}_{01} = E^{\hat{I}_B}_{01} = E^{\hat{I}_S}_{01} = -\sqrt{-g_D} \frac{D}{2\kappa_6 r^2} (\mathcal{F} - \mathcal{T})$$  (3.18)

where $\mathcal{F} = f(v, r)$, which looks surprisingly simple, see, e.g., (3.4) - (3.6). The GB 01-components of the superpotentials (A.1), (A.5) and (A.7) for the solution (2.10) consist of two parts again. The pure ($d = 3$)-dimensional part is

$$\left. \frac{\partial}{\partial v} \hat{I}_S \right|_{(A.3)} = \frac{\alpha \sqrt{-g_D}}{\kappa_6 r^2} (f - \mathcal{T})(f' - rf''),$$  (3.19)

$$\left. \frac{\partial}{\partial v} \hat{I}_C \right|_{(A.4)} \equiv 0,$$  (3.20)

$$\left. \frac{\partial}{\partial v} \hat{I}_B \right|_{(A.7)} = \frac{\alpha \sqrt{-g_D}}{\kappa_6 r^2} \left[ (f - \mathcal{T})(rf'' - f') + 2 r \mathcal{T}' + 2 r \frac{\partial}{\partial v} (r(f - \mathcal{T})), \right.$$

$$\left. \left. \left. + \left( r \mathcal{T}' + 2 r \frac{\partial}{\partial v} \right) (r(f - \mathcal{T})), \right] \right. \left. \right. (3.21)$$

$$\left. \frac{\partial}{\partial v} \right|_{(A.7)} \equiv 0.$$  (3.22)
For the AdS background one has as $r \to \infty$: for (3.19) $\sim 1/r^3$ and for (3.21) $\sim 1/r^2$, thus all (3.19) - (3.22) again give a zero contribution into the integral (3.8). As in the static case, the other part of the GB 01-components is determined by the intersection terms of the $(d = 3)$-sector and the scalar curvature of the $(D - d = 3)$-sector (2.8):

$$\hat{T}^{01}_C = (D - d) \hat{T}^{01}_B = (D - d) \hat{T}^{01}_S = \frac{\sqrt{-g_D}}{2\kappa_6 r} (f - \mathcal{F}).$$

(3.23)

One can see that these components precisely compensate the components (3.18). Thus, as in the previous subsection, the global mass defined in 6 dimensions is zero. Then one should follow the interpretation of the static case and reject the vacuum 6D derivation (2.2) with (3.8) as unacceptable one.

We again consider Eq. (2.12) as a governing one. Restricting ourselves to the $d$-sector only and repeating the steps of Subsection 3.3, we obtain in the AdS background $M = \pi (\mu(v) + 1)/\kappa_3$. This is in a correspondence with the static case. The mass flux for the radiating metric (2.10) is obtained simply by differentiating with respect to $v$: $\dot{M} = \pi \dot{\mu}(v)/\kappa_3$. Comparing with the known BMS flux derivation [15], this looks acceptable.

Concluding the section we assert that since the KK BH objects have classically defined global mass and flux, they bring ‘matter’ created by extra dimensions and a special structure of the objects themselves. If we set $q = 0$ and $q(v) = 0$, then, at least in the static case, $\mathcal{T}_{AB} = 0$ in (2.2). However, this does not influence on our assertion because in all the cases $q$ and $q(v)$ do not contribute into the global mass. Thus, mass/matter is created in a more wide sense than creating $\mathcal{T}_{AB}$ in (2.2).

4 Concluding remarks

We will first discuss a well-known ambiguity in the canonical approach related to a choice of a divergence in the Lagrangian. We consider this problem in [9] and do not make a definite choice between [14] (or (A.3)) and [16] (or (A.4)). Indeed, both choices give an acceptable mass for the Schwarzschild-AdS BH tested in [9]. Here, the study of KK objects also does not give an answer because in all cases we have a unique result. However, in [16] arguments in favor (A.4) are given. In multtimendional GR, the Katz and Livshits superpotential [16] turns out uniquely the KBL superpotential [17]; in EGB gravity, their superpotential naturally transfers into the KBL superpotential for $D = 4$. This is in a correspondence with the Olea arguments [18] where GB terms in the Lagrangian regularize conserved quantities even if $D \leq 4$. Lastly, the choice (A.4) looks more preferable because (a) it is more ‘symmetric’ than (A.3), (b) the canonical superpotential with (3.20) gives a zero global integral in 6 dimensions identically, as in the field-theoretical approach.
Now we turn to cosmological problems. As well known, the properties of dark energy and of dark matter are very weakly constrained by the cosmological observable data, therefore their derivation remains very uncertain. Thus a search for acceptable models describing the cosmic ingredients is very important, it is carried out very intensively, and even dramatically, see, e.g., the recent papers, reviews [19] - [27] and references there in.

As an example, in the recent paper [28], recalling the ’t Hooft ideas of 1985, so-called ‘quantum black holes’ are discussed as elementary particles playing the role of the dark matter particles. The latter are assumed as weak interacting matter particles (WIMPs), which can have desirable TeV energies (see the aforementioned reviews). ‘Quantum black holes’ can be presented just like WIMPs, they can be stable and do not radiate in the Hawking-Bekenstein regime, unlike usual black holes.

Our main results show that the solutions (2.6) and (2.10) have a classically defined mass and mass flux. This just presents a possibility for the KK BHs to be presented in the regime of ‘quantum black holes’. Thus, the topic of the present paper, as we think, could be related to the dark matter problems. Concerning this, we remark the following. First, since the parameter $q$ can describe additional (to gravity) interactions, its presence can suppress the WIMP idea. Then one needs to set $q = 0$, which is permissible, as has been remarked above. Second, it is desirable to have a positive mass for WIMP objects. We support this because, if a BH exists, one has $\tilde{\mu} > 0$ that leads to $M > 0$. Third, basing on the radiating regime, in [1] - [4] a scenario of forming KK BHs in EGB gravity was suggested. One could try to develop this scenario for various epochs. Keeping in mind all that, in future studies we plan an examination of more realistic models presented in [1] - [4]: they are 4D KK objects in 6 and more dimensions of EGB gravity.

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A Superpotentials in the EGB gravity

In this Appendix, we represent an explicit form of the three types of superpotentials for perturbations in the EGB gravity [9]. The background quantities: Christoffel symbols $\Gamma^\sigma_{\tau\rho}$,
covariant derivatives \( \mathcal{D}_\alpha \), the Riemannian tensor \( \mathcal{R}^{\alpha}_{\beta\rho\sigma} \) and its contractions are constructed on the basis of a background \( D \)-dimensional spacetime metric \( \mathcal{g}_{\mu\nu} \). It is a known (fixed) solution of EGB gravity; the bar means that a quantity is a background one. One can find a detail derivation in [9]. We first present the superpotential in the canonical prescription:

\[
\hat{T}_{\alpha\beta} = E\hat{T}_{\alpha\beta} + GB\hat{T}_{\alpha\beta} = \kappa^{-1} \left( \tilde{g}^{\rho[\alpha} \mathcal{D}_{\rho} \xi^{\beta]} + \tilde{g}^{\rho[\alpha} \Delta^{\beta]}_{\rho\sigma} \xi^{\sigma} - \mathcal{D}[\alpha \xi^{\beta]} + \xi^{[\alpha} \mathcal{D}^{\beta]} \right) + GB\hat{T}_{\alpha\beta} - GB\hat{T}_{\alpha\beta} \tag{A.1}
\]

where \( \Delta^{\alpha}_{\mu\nu} \equiv \Gamma_{\mu\nu} - \nabla^\alpha \), and2

\[
g_{\alpha\beta} = \frac{2}{\sqrt{-\mathcal{g}}} \left( \mathcal{D}_{\mu} \mathcal{g}_{\rho\mu} + \mathcal{D}_{\nu} \mathcal{g}_{\rho\nu} - \mathcal{D}_{\rho} \mathcal{g}_{\mu\nu} \right) \text{ and } \sqrt{-\mathcal{g}} \left( \hat{R}_{\alpha\beta} - 4 \hat{R}_{[\alpha} \hat{g}^{[\beta]} + \delta^{[\alpha}_{\beta} \hat{g}^\lambda R \right) \Delta^{\alpha}_{\mu\nu} \tag{A.2}
\]

The vector density \( \hat{d}^\lambda = E\hat{d}^\lambda + GB\hat{d}^\lambda \) could be defined as in [14] or following the prescription of [16]:

\[
d_1 = 2\sqrt{-\mathcal{g}} \Delta^{\alpha}_{\beta[\alpha} \hat{g}^{\lambda]} + 4\sqrt{-\mathcal{g}} \left( \hat{R}_{\alpha\beta} - 2 \hat{R}_{[\alpha} \hat{g}^{[\beta]} + \delta^{[\alpha}_{\beta} \hat{g}^\lambda R \right) \Delta^{\alpha}_{\beta} \tag{A.3}
\]

The Einstein part in (A.1) is precisely the KBL superpotential [14, 17], which in 4D general relativity (GR) for the Minkowski background in the Cartesian coordinates and with the translation Killing vectors \( \xi^{\alpha} = \delta^{\alpha}_{[\beta] \gamma} \) is just the well-known Freud superpotential [29].

The Belinfante corrected superpotential in EGB gravity is

\[
\hat{T}_{\alpha\beta} = E\hat{T}_{\alpha\beta} + GB\hat{T}_{\alpha\beta} = \kappa^{-1} \left( \tilde{g}^{\rho[\alpha} \mathcal{D}_{\rho} \hat{\xi}^{\beta]} - \mathcal{D}[\alpha \hat{g}^{\beta]} + \hat{g}^{\lambda[\alpha} \mathcal{D}_{\lambda} \xi^{\beta]} \right) + GB\hat{T}_{\alpha\beta} - GB\hat{T}_{\alpha\beta} \tag{A.5}
\]

where \( \hat{\xi}^{\alpha} = \hat{g}^{\alpha} - \overline{\mathcal{g}}^{\alpha} \) and

\[
g_{\alpha\beta} = \frac{2}{\sqrt{-\mathcal{g}}} \left( \mathcal{D}_{\mu} \mathcal{g}_{\rho\mu} + \mathcal{D}_{\nu} \mathcal{g}_{\rho\nu} - \mathcal{D}_{\rho} \mathcal{g}_{\mu\nu} \right) \sqrt{-\mathcal{g}} \left( \hat{R}_{\alpha\beta} - 4 \hat{R}_{[\alpha} \hat{g}^{[\beta]} + \delta^{[\alpha}_{\beta} \hat{g}^\lambda R \right) \Delta^{\alpha}_{\mu\nu} \tag{A.6}
\]

The Einstein part, \( E\hat{T}_{\alpha\beta} \), being constructed in arbitrary \( D \) dimensions, has precisely the form of the Belinfante corrected superpotential in 4D GR [30]. In the Minkowski background in

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2 The expression (A.2) is differed from the correspondent one in [9], where the mistake has been found. Nevertheless, the main results and conclusions in [9] are not changed; see Corrigendum: Class. Quantum Grav. 27 (2010) 069801 (2pp); Preprint arXiv:0905.3622 [gr-qc].
the Cartesian coordinates and with the translation Killing vectors \( \hat{E}^{\alpha \beta}_B \), it transforms to the well-known Papapetrou superpotential [31].

Lastly, the superpotential in the field-theoretical derivation in EGB gravity is

\[
\hat{T}_S^{\alpha \beta} = \hat{T}_S^{\alpha \beta} + G_B \hat{T}_S^{\alpha \beta} = \kappa^{-1} \left( \hat{\xi}_{\nu} \hat{D}^{[\alpha} \hat{h}^{\beta]}_{\nu} - \hat{\xi}^{[\alpha} \hat{D}_{\nu} \hat{h}^{\beta]}_{\nu} + \hat{\xi}^{[\alpha} \hat{D}^{\beta]}_{\nu} \hat{h} - \hat{h}^{[\alpha} \hat{D}_{\nu} \hat{\xi}^{\beta]} + \frac{1}{2} \hat{h}^{[\alpha} \hat{D}^{\beta]}_{\nu} \hat{\xi} \right) + \frac{4}{3} \left( 2 \hat{\xi}_{\sigma} \hat{D}_{\lambda} \hat{N}_{\sigma \beta} N^{[\alpha}_{GB} - \hat{N}_{\sigma \beta} \hat{D}_{\lambda} \hat{\xi}_{\sigma} \right),
\]

(A.7)

where \( \hat{h}_{\alpha \beta} = \sqrt{-\hat{g}} (g_{\alpha \beta} - \hat{g}_{\alpha \beta}) \) and

\[
\hat{N}_{\alpha \beta}^{[\alpha \beta]_{GB} \nu} = \frac{3 \alpha \sqrt{-\hat{g}}}{4 \kappa} \left\{ \hat{h}_{\sigma}^{\rho} \left[ \hat{g}^{[\rho \sigma \lambda \nu \rho \sigma]} \hat{T} + 2 \hat{g}^{[\rho \sigma \lambda \nu]_{\nu \sigma]} - 2 \hat{g}^{[\rho \sigma \lambda \nu]_{\nu \lambda]} \hat{T} - \hat{T}^{[\rho \sigma \lambda \nu]} \right] + \left( \hat{g}^{[\rho \sigma \lambda \nu]_{\nu \lambda]} - \hat{g}^{[\rho \sigma \lambda \nu]} \right) \hat{T} \right\} + 2 \left( \hat{h}_{\sigma}^{[\rho \sigma \lambda \nu]} - \hat{h}_{\sigma}^{[\rho \sigma \lambda \nu]_{\nu \lambda]} \right) + 2 \left( \hat{h}_{\sigma}^{[\rho \sigma \lambda \nu]} - \hat{h}_{\sigma}^{[\rho \sigma \lambda \nu]_{\nu \lambda]} \right) + 2 \left( \hat{h}_{\sigma}^{[\rho \sigma \lambda \nu]} - \hat{h}_{\sigma}^{[\rho \sigma \lambda \nu]_{\nu \lambda]} \right) + 2 \hat{h}_{\sigma}^{[\rho \sigma \lambda \nu]} + 4 \left( \hat{R}_{[\rho \sigma \lambda \nu]}^{[\rho \sigma \lambda \nu]} + \hat{R}_{[\rho \sigma \lambda \nu]}^{[\rho \sigma \lambda \nu]} \right) + 2 \hat{h}_{\sigma}^{[\rho \sigma \lambda \nu]} \hat{T} \}
\]

One obtains from (A.7) the Deser-Tekin superpotential [32] if one chooses the AdS background. Again, doing simplifications in 4 dimensions as above, one obtains the Papapetrou superpotential [31] (note, see [8], that in 4D GR the Belinfante and field-theoretical approaches give the same result). Under weaker restrictions, say, to AdS/dS backgrounds in 4D GR, the superpotential (A.7) goes to the Abbott-Deser expression [33].

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