WILL WE EVER CLASSIFY SIMPLY-CONNECTED SMOOTH 4-MANIFOLDS?

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Abstract. These notes are adapted from two talks given at the 2004 Clay Institute Summer School on Floer homology, gauge theory, and low dimensional topology at the Alfred Rényi Institute. We will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

1. Introduction

Despite spectacular advances in defining invariants for simply-connected smooth and symplectic 4-dimensional manifolds and the discovery of important qualitative features about these manifolds, we seem to be retreating from any hope to classify simply-connected smooth or symplectic 4-dimensional manifolds. The subject is rich in examples that demonstrate a wide variety of disparate phenomena. Yet it is precisely this richness which, at the time of these lectures, gives us little hope to even conjecture a classification scheme. In these notes, adapted from two talks given at the 2004 Clay Institute Summer School on Floer homology, gauge theory, and low dimensional topology at the Alfred Rényi Institute, we will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

Algebraic Topology. The critical algebraic topological information for a closed, simply-connected, smooth 4-manifold $X$ is encoded in its Euler characteristic $e(X)$, its signature $\sigma(X)$, and its type $t(X)$ (either 0 if the intersection form of $X$ is even and 1 if it is odd). These invariants completely classify the homeomorphism type of $X$ ($\mathbb{Z}/2\mathbb{Z}$). We recast these algebraic topological invariants by defining

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class is the Poincaré dual pair \((A, \psi)\) with \(\omega\) an orientation. Fix a characteristic homology class \(\beta\) with \(\text{SW}_X\) reduced mod 2 to \(w\), preserving diffeomorphism invariant of \(X\). For \(X\) of this type, the Seiberg-Witten invariant as an element of the integral group ring \(\mathbb{Z}\). Then the Seiberg-Witten invariant of \(X\) is an orientation-preserving diffeomorphism invariant of \(X\) (together with the choice of a homology orientation). We recast the Seiberg-Witten invariant as an element of the integral group ring \(\mathbb{Z}H_2(X)\), where for each \(\alpha \in H_2(X)\) we let \(t_{\alpha}\) denote the corresponding element in \(\mathbb{Z}H_2(X)\). Suppose that \(\{\pm \beta_1, \ldots, \pm \beta_n\}\) is the set of nonzero basic classes for \(X\). Then the Seiberg-Witten invariant of \(X\) is the Laurent polynomial

\[
\text{SW}_X = \text{SW}_X(0) + \sum_{j=1}^{n} \text{SW}_X(\beta_j) \cdot (t_{\beta_j} + (-1)^{(e+\text{sign}(X))/4} t_{\beta_j}^{-1}) \in \mathbb{Z}H_2(X).
\]

In the case \(\chi_2 = 1\), the Seiberg-Witten invariant depends on a given orientation of \(H_2^+(X; R)\), a given metric \(g\), and a self-dual 2-form as follows. There is a unique \(g\)-self-dual harmonic 2-form \(\omega_g \in H_2^+(X; R)\) with \(\omega_g^2 = 1\) and corresponding to the positive orientation. Fix a characteristic homology class \(k \in H_2(X; \mathbb{Z})\). Given a pair \((A, \psi)\), where \(A\) is a connection in the complex line bundle whose first Chern class is the Poincaré dual \(\hat{k} = \frac{i}{2\pi}[F_A]\) of \(k\) and \(\psi\) a section of the bundle \(W^+\) of self-dual spinors for the associated \(\text{spin}^c\) structure, the perturbed Seiberg-Witten equations are:

\[
D_A \psi = 0, \\
F_A^+ = q(\psi) + i\eta
\]

where \(F_A^+\) is the self-dual part of the curvature \(F_A\), \(D_A\) is the twisted Dirac operator, \(\eta\) is a self-dual 2-form on \(X\), and \(q\) is a quadratic function. Write \(\text{SW}_{X,g,\eta}(k)\) for the corresponding invariant. As the pair \((g, \eta)\) varies, \(\text{SW}_{X,g,\eta}(k)\) can change only at those pairs \((g, \eta)\) for which there are solutions with \(\psi = 0\). These solutions occur for pairs \((g, \eta)\) satisfying \((2\pi \hat{k} + \eta) \cdot \omega_g = 0\). This last equation defines a wall in \(H^2(X; \mathbb{R})\).
The point $\omega_g$ determines a component of the double cone consisting of elements of $H^2(X; \mathbb{R})$ of positive square. We prefer to work with $H_2(X; \mathbb{R})$. The dual component is determined by the Poincaré dual $H$ of $\omega_g$. (An element $H' \in H_2(X; \mathbb{R})$ of positive square lies in the same component as $H$ if $H' \cdot H > 0$.) If $(2\pi \hat{k} + \eta) \cdot \omega_g \neq 0$ for a generic $\eta$, $\text{SW}_{X,g,\eta}(k)$ is well-defined, and its value depends only on the sign of $(2\pi \hat{k} + \eta) \cdot \omega_g$. Write $\text{SW}^+_X,H(k)$ for $\text{SW}_{X,g,\eta}(k)$ if $(2\pi \hat{k} + \eta) \cdot \omega_g > 0$ and $\text{SW}^-_{X,H}(k)$ in the other case.

The invariant $\text{SW}_{X,H}(k)$ is defined by $\text{SW}_{X,H}(k) = \text{SW}^+_X,H(k)$ if $(2\pi \hat{k}) \cdot \omega_g > 0$, or dually, if $k \cdot H > 0$, and $\text{SW}_{X,H}(k) = \text{SW}^-_{X,H}(k)$ if $H \cdot k < 0$. The wall-crossing formula \cite{12,15} states that if $H', H''$ are elements of positive square in $H_2(X; \mathbb{R})$ with $H' \cdot H > 0$ and $H'' \cdot H > 0$, then if $k \cdot H' < 0$ and $k \cdot H'' > 0$,

$$\text{SW}_{X,H'}(k) - \text{SW}_{X,H''}(k) = (-1)^{1+\frac{1}{2}d(k)}$$

where $d(k) = \frac{1}{4}(k^2 - (3 \\text{sign} + 2 \epsilon)(X))$ is the formal dimension of the Seiberg-Witten moduli spaces.

Furthermore, in case $b^- \leq 9$, the wall-crossing formula, together with the fact that $\text{SW}_{X,H}(k) = 0$ if $d(k) < 0$, implies that $\text{SW}_{X,H}(k) = \text{SW}_{X,H'}(k)$ for any $H'$ of positive square in $H_2(X; \mathbb{R})$ with $H \cdot H' > 0$. So in case $b^+_X = 1$ and $b^-_X \leq 9$, there is a well-defined Seiberg-Witten invariant, $\text{SW}_X(k)$.

**Possible Classification Schemes.** From this point forward and unless otherwise stated all manifolds will be closed and simply-connected. In order to avoid trivial constructions we consider *irreducible* manifolds, i.e. those that cannot be represented as the connected sum of two manifolds except if one factor is a homotopy 4-sphere. (We still do not know if there exist smooth homotopy 4-spheres not diffeomorphic to the standard 4-sphere $S^4$).

So the existence part of a classification scheme for irreducible smooth (symplectic) 4-manifolds could take the form of determining which $(\chi_h, c, t) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$ can occur as $(\chi_h(X), c(X), t(X))$ for some smooth (symplectic) 4-manifold $X$. This is referred to as the *geography problem*. The game plan would be to create techniques to realize all possible lattice points. The uniqueness part of the classification scheme would then be to determine all smooth (symplectic) 4-manifolds with a fixed $(\chi_h(X), c(X), t(X))$ and determine invariants that would distinguish them. Again, the game plan would be to create techniques that preserve the homeomorphism type yet change these invariants.

In the next two sections we will outline what is and is not known about the existence (geography) and uniqueness problems without detailing the techniques. Then we will list the techniques used, determine their interplay, and explore questions that may yield new insight. A companion approach, which we will also discuss towards the end of these lectures, is to start with a particular well-understood class of 4-manifolds and determine how all other smooth (symplectic) 4-manifolds can be constructed from these.
2. Existence

Our current understanding of the geography problem is given by Figure 1 where all known simply-connected smooth irreducible 4-manifolds are plotted as lattice points in the \((\chi_h, c)\)-plane. In particular, all known simply-connected irreducible smooth or symplectic 4-manifolds have \(0 \leq c < 9\chi_h\) and every lattice point in that region can be realized by a symplectic (hence smooth) 4-manifold.

![Figure 1](image-url)

**Problem 1.** Do there exist irreducible smooth 4-manifolds with \(\chi_h \notin \mathbb{Z}\)?

Here the work of Bauer and Furuta [2] on stable homotopy invariants derived from the Seiberg-Witten equations may be useful. To expose our ignorance, consider two copies of the elliptic surface \(E(2)\). Remove the neighborhood of a sphere with self-intersection \(-2\) from each and glue together the resulting manifolds along their
boundary $\mathbb{R}P^3$ using the orientation reversing diffeomorphism of $\mathbb{R}P^3$. The result has $\chi_h \notin \mathbb{Z}$ and it is unknown if it is irreducible.

All complex manifolds with $c = 9\chi_h > 9$ are non-simply-connected, in particular they are ball quotients. Thus obvious problems are:

**Problem 2.** Do there exist irreducible simply connected smooth or symplectic manifolds with $c = 9\chi_h > 9$?

**Problem 3.** Does there exist an irreducible non-complex smooth or symplectic manifold $X$ with $\chi_h > 1$, $c = 9\chi_h$ (with any fundamental group), $SW_X \neq 0$, and which is not a ball-quotient?

**Problem 4.** Do there exist irreducible smooth or symplectic manifolds with $c > 9\chi_h$?

The work of Taubes [28] on the relationship between Seiberg-Witten and Gromov-Witten invariants shows that $c \geq 0$ for an irreducible symplectic 4-manifold.

**Problem 5.** Do there exist simply connected irreducible smooth manifolds with $c < 0$?

There appears to be an interesting relationship between the number of Seiberg-Witten basic classes and the pair $(\chi_h, c)$. In particular, all known smooth 4-manifolds with $0 \leq c \leq \chi_h - 3$ have at least $\chi_h - c - 2$ Seiberg-Witten basic classes [4]. So

**Problem 6.** Does there exist an irreducible smooth manifold $X$ with $0 \leq c(X) \leq \chi_h(X) - 3$ and with fewer than $\chi_h(X) - c(X) - 2$ Seiberg-Witten basic classes? (There is a physics proof that there are no such examples [17].)

Figure 1 contains no information about the geography of spin 4-manifolds, i.e. manifolds with $t = 0$. For a spin 4-manifold there is the relation $c = 8\chi_h \mod 16$. Almost every lattice point with $c = 8\chi_h \mod 16$ and $0 \leq c < 9\chi_h$ can be realized by an irreducible spin 4-manifold [21]. Surprisingly not all of the lattice points with $2\chi_h \leq 3(\chi_h - 5)$ can be realized by complex manifolds [24], so these provide several examples of smooth irreducible 4-manifolds with $2\chi_h - 6 \leq c < 9\chi_h$ that support no complex structure (cf. [29]). The open issues here are a better understanding of manifolds close to the $c = 9\chi_h$ line, in particular those with $9\chi_h > c \geq 8.76\chi_h$ and not on the lines $c = 9\chi_h - k$ with $k \leq 121$ (cf. [24]).

The techniques used in all these constructions are an artful application of the generalized fiber sum construction (cf. [13]) and the rational blowdown construction [6], which we will discuss later in this lecture.

3. **Uniqueness**

Here is where we begin to lose control of the classification of smooth 4-manifolds. If a topological 4-manifold admits an irreducible smooth (symplectic) structure that
Ronald J. Stern has a smoothly (symplectically) embedded torus with self-intersection zero and with simply-connected complement, then it also admits infinitely many distinct smooth (symplectic) structures and also admits infinitely many distinct smooth structures with no compatible symplectic structure. The basic technique here is the knot-surgery construction of Fintushel-Stern \[7\], i.e. remove a neighborhood \(T^2 \times D^2 = S^1 \times S^1 \times D^2\) of this torus and replace it with \(S^1 \times S^3 \setminus K\) where \(K\) is a knot in \(S^3\). In particular, there are no known examples of (simply-connected) smooth or symplectic 4-manifolds with \(\chi > 1\) that does not admit such a torus. Hence, there are no known smooth or symplectic 4-manifolds with \(\chi_h > 1\) that admit finitely many smooth or symplectic structures. Thus,

**Problem 7.** Do there exist irreducible smooth (symplectic) 4-manifolds with \(\chi_h > 1\) that do not admit a smoothly (symplectically) embedded torus with self-intersection 0 and simply-connected complement?

All of the constructions used for the geography problem with \(\chi_h > 1\) naturally contain such tori, so the only hope is to find manifolds where these constructions have yet to work, i.e. those with \(8.76 < c \leq 9\chi_h\), that do not contain such tori.

**Problem 8.** Do manifolds with \(c = 9\chi_h\) admit exotic smooth structures?

The situation for \(\chi_h = 1\) is potentially more interesting and may yield phenomena not shared by manifolds with \(\chi_h > 1\). For example, the complex projective plane \(\mathbb{C}P^2\) has \(c = 9\chi_h = 9\) and is simply-connected. It is also known that \(\mathbb{C}P^2\) as a smooth manifold has a unique symplectic structure \[27, 28\]. Thus, a fundamental question that still remains is

**Problem 9.** Does the complex projective plane \(\mathbb{C}P^2\) admit exotic smooth structures?

**Problem 10.** What is the smallest \(m\) for which \(\mathbb{C}P^2 \# m \overline{\mathbb{C}P^2}\) admits an exotic smooth structure?

The primary reason for our ignorance here is that for \(c > 1\) (i.e. \(m < 9\)), these manifolds do not contain homologically essential tori with zero self-intersection. Since the rational elliptic surface \(E(1) \cong \mathbb{C}P^2 \# 9 \overline{\mathbb{C}P^2}\) admits tori with self-intersection zero, it has infinitely many distinct smooth structures. In the late 1980’s Dieter Kotschick \[14\] proved that the Barlow surface, which was known to be homeomorphic to \(\mathbb{C}P^2 \# 8 \overline{\mathbb{C}P^2}\), is not diffeomorphic to it. In the following years the subject of simply connected smooth 4-manifolds with \(m < 8\) languished because of a lack of suitable examples. However, largely due to a beautiful paper of Jongil Park \[22\], who found the first examples of exotic simply connected 4-manifolds with \(m = 7\), interest was revived. Shortly after this conference ended, Peter Ozsváth and Zoltán Szabó proved that Park’s manifold is minimal \[20\] by computing its Seiberg-Witten invariants. Then András Stipsicz and Zoltán Szabó
used a technique similar to Park’s to construct an exotic manifold with $m = 6$ [25].

The underlying technique in these constructions is an artful use of the rational blowdown construction.

Since $\mathbb{CP}^2 \# m \overline{\mathbb{CP}}^2$ for $m < 9$ does not contain smoothly embedded tori with self-intersection zero, it has not been known whether it can have an infinite family of smooth structures. Most recently, Fintushel and Stern [11] introduced a new technique which was used to show that for $6 \leq m \leq 8$, $\mathbb{CP}^2 \# m \overline{\mathbb{CP}}^2$ does indeed have an infinite family of smooth structures, and, in addition, none of these smooth structures support a compatible symplectic structure. Park, Stipsicz, and Szabó [23] used this construction to show that $m = 5$ also has an infinite family of smooth structures none of which support a compatible symplectic structure (cf. [11]). The basic technique in these constructions is a prudent blend of the knot surgery and rational blowdown constructions.

As is pointed out in [25], the Seiberg-Witten invariants will never distinguish more than two distinct irreducible symplectic structures on $\mathbb{CP}^2 \# m \overline{\mathbb{CP}}^2$ for $m < 9$. Basically, this is due to the fact that if there is more than one pair of basic classes for a $\chi_h = 1$ manifold, then it is not minimal. So herein lies one of our best hopes for finiteness in dimension 4.

**Problem 11.** Does $\mathbb{CP}^2 \# m \overline{\mathbb{CP}}^2$ for $m < 9$ support more than two irreducible symplectic structures that are not deformation equivalent?

4. **The Techniques Used for the Construction of All Known Simply-Connected Smooth and Symplectic 4-Manifolds**

The construction of simply-connected smooth or symplectic 4-manifolds sometimes takes the form of art rather than science. This is exposed by the lack of success in proving structural theorems or uncovering any finite phenomena in dimension 4. In this lecture we will list all the constructions used in building the 4-manifolds necessary for the results of the first two sections and try to bring all the unusual phenomena in dimension 4 into a framework that will allow us to at least understand those surgical operations that one performs to go from one smooth structure on a given simply-connected 4-manifold to any other smooth structure. This will take the form of understanding a variety of cobordisms between 4-manifolds.

Here is the list of constructions used in the first two sections.

**generalized fiber sum:** Assume two 4-manifolds $X_1$ and $X_2$ each contain an embedded genus $g$ surface $F_j \subset X_j$ with self-intersection 0. Identify tubular neighborhoods $\nu F_j$ of $F_j$ with $F_j \times D^2$ and fix a diffeomorphism $f : F_1 \to F_2$. Then the fiber sum $X = X_1 \# f X_2$ of $(X_1, F_1)$ and $(X_2, F_2)$ is defined as $X_1 \setminus \nu F_1 \cup \phi \cdot X_2 \setminus \nu F_2$, where $\phi$ is $f \times (\text{complex conjugation})$ on the boundary $\partial(X_j \setminus \nu F_j) = F_j \times S^1$. 
**Generalized logarithmic transform:** Assume that \( X \) contains a homologically essential torus \( T \) with self-intersection zero. Let \( \nu T \) denote a tubular neighborhood of \( T \). Deleting the interior of \( \nu T \) and regluing \( T^2 \times D^2 \) via a diffeomorphism \( \phi : T^2 \times D^2 \to \partial(X - \text{int} \; \nu T) = \partial \nu T \) we obtain a new manifold \( X_\phi \), the generalized logarithmic transform of \( X \) along \( T \).

If \( p \) denotes the absolute value of the degree of the map \( \pi \circ \phi : \{pt\} \times S^1 \to \pi(\partial \nu T) = S^1 \), then \( X_\phi \) is called a generalized logarithmic transformation of multiplicity \( p \).

If the complement of \( T \) is simply-connected and \( t(X) = 1 \), then \( X_\phi \) is homeomorphic to \( X \). If the complement of \( T \) is simply-connected and \( t(X) = 0 \), then \( X_\phi \) has the same \( c \) and \( \chi \) but with \( t(X_\phi) = 1 \).

**Blowup:** Form \( X \# \mathbb{CP}^2 \).

**Rational blowdown:** Let \( C_p \) be the smooth 4-manifold obtained by plumbing \( (p - 1) \) disk bundles over the 2-sphere according to the diagram

\[-(p + 2) \quad -2 \quad \ldots \quad -2\]

Then the classes of the 0-sections have self-intersections \( u_0^i = -(p + 2) \) and \( u_i^i = -2 \), \( i = 1, \ldots, p - 2 \). The boundary of \( C_p \) is the lens space \( L(p^2, 1 - p) \) which bounds a rational ball \( B_p \) with \( \pi_1(B_p) = \mathbb{Z}_p \) and \( \pi_1(\partial B_p) \to \pi_1(B_p) \) surjective. If \( C_p \) is embedded in a 4-manifold \( X \) then the rational blowdown manifold \( X_{(p)} \) is obtained by replacing \( C_p \) with \( B_p \), i.e., \( X_{(p)} = (X \setminus C_p) \cup B_p \) (cf. [4]). If \( X \setminus C_p \) is simply connected, then so is \( X_{(p)} \).

**Knot surgery:** Let \( X \) be a 4-manifold which contains a homologically essential torus \( T \) of self-intersection 0, and let \( K \) be a knot in \( S^3 \). Let \( N(K) \) be a tubular neighborhood of \( K \) in \( S^3 \), and let \( T \times D^2 \) be a tubular neighborhood of \( T \) in \( X \). Then the knot surgery manifold \( X_K \) is defined by

\[X_K = (X \setminus (T \times D^2)) \cup (S^1 \times (S^3 \setminus N(K)))\]

The two pieces are glued together in such a way that the homology class \([pt \times \partial D^2]\) is identified with \([pt \times \lambda]\) where \( \lambda \) is the class of a longitude of \( K \). If the complement of \( T \) in \( X \) is simply-connected, then \( X_K \) is homeomorphic to \( X \).

The Seiberg-Witten invariants are sensitive to all of these operations.

**Generalized logarithmic transform:** If \( T \) is contained in a node neighborhood, then

\[SW_{X_\phi} = SW_X \cdot (s^{-(p-1)} + s^{-(p-3)} + \cdots + s^{(p-1)})\]
where \( s = \exp(T/p) \), \( p \) the order of the generalized logarithmic transform (cf. [4]).

**blowup:** The relationship between the Seiberg-Witten invariants of \( X \) and its blowup \( X \# \mathbb{CP}^2 \) is referred to as the blowup formula and was given in Witten’s original article [30] (cf. [5]). In particular, if \( e \) is the homology class of the exceptional curve and \( \{B_1, \ldots, B_n\} \) are the basic classes of \( X \), then the basic classes of \( X \# \mathbb{CP}^2 \) are \( \{B_1 \pm E, \ldots, B_n \pm E\} \) and 
\[
\text{SW}_{X \# \mathbb{CP}^2}(B_j \pm E) = \text{SW}_X(B_j).
\]

**rational blowdown:** The Seiberg-Witten invariants of \( X \) and \( X(p) \) can be compared as follows. The homology of \( X(p) \) can be identified with the orthogonal complement of the classes \( u_i, i = 0, \ldots, p - 2 \) in \( H_2(X; \mathbb{Z}) \), and then each characteristic element \( k \in H_2(X(p); \mathbb{Z}) \) has a lift \( \tilde{k} \in H_2(X; \mathbb{Z}) \) which is characteristic and for which the dimensions of moduli spaces agree, 
\[
d_{X(p)}(k) = d_X(\tilde{k}).
\]
It is proved in [3] that if \( b_X^+ > 1 \) then 
\[
\text{SW}_{X(p)}(k) = \text{SW}_X(\tilde{k}).
\]
In case \( b_X^+ = 1 \), if \( H \in H_2(X; \mathbb{R}) \) is orthogonal to all the \( u_i \) then it also can be viewed as an element of \( H_2^+(X(p); \mathbb{R}) \), and 
\[
\text{SW}_{X(p),H}(k) = \text{SW}_{X,H}(\tilde{k}).
\]

**knot surgery:** If, for example, \( T \) is contained in a a node neighborhood and \( \chi_h(X) > 1 \) then the Seiberg-Witten invariant of the knot surgery manifold \( X_K \) is given by
\[
\text{SW}_{X_K} = \text{SW}_X \cdot \Delta_K(t)
\]
where \( \Delta_K(t) \) is the symmetrized Alexander polynomial of \( K \) and 
\( t = \exp(2\pi i T) \). When \( \chi_h = 1 \), the Seiberg-Witten invariants of \( X_K \) are still completely determined by those of \( X \) and the Alexander polynomial \( \Delta_K(t) \) [7].

Here \( T \) contained in a node neighborhood means that an essential loop on \( \partial \nu T \) bounds a disk in the complement with relative self-intersection \(-1\). We sometimes refer to this disk as a vanishing cycle.

In many circumstances, there are formulas for determining the Seiberg-Witten invariants of a fiber sum in terms of the Seiberg-Witten invariants of \( X_1 \) and \( X_2 \) and how the basic classes intersect the surfaces \( F_1 \) and \( F_2 \).

**Interaction of the operations.** While knot surgery appears to be a new operation, the constructions in [2] point out that the knot surgery construction is actually a series of \( \pm 1 \) generalized logarithmic transformations on null-homologous tori. To see this, note that any knot can be unknotted via a sequence of crossing changes, which in turn can be realized as a sequence of \( \pm 1 \) surgeries on unknotted curves \( \{c_1, \ldots, c_n\} \) that link the knot algebraically zero times and geometrically twice. When crossed with \( S^1 \) this translates to the fact that \( X \) can be obtained from \( X_K \) via a sequence of \( \pm 1 \) generalized logarithmic transformations on the null-homologous tori \( \{S^1 \times c_1, \ldots, S^1 \times c_n\} \) in \( X_K \). So the hidden mechanism behind
the knot surgery construction is generalized logarithmic transformations on null-homologous tori. The calculation of the Seiberg-Witten invariants is then reduced to understanding how the Seiberg-Witten invariants change under a generalized logarithmic transformation on a null-homologous torus. This important formula is due to Morgan, Mrowka, and Szabó [18] (see also [29]). For this formula fix simple loops \( \alpha, \beta, \delta \) on \( \partial N(T) \) whose homology classes generate \( H_1(\partial N(T)) \). If \( \omega = p\alpha + q\beta + r\delta \) write \( X_T(p,q,r) \) instead of \( X_T(\omega) \). Given a class \( k \in H_2(X) \):

(1) \[ \sum_i SW_{X_T(p,q,r)}(k(p,q,r) + 2i[T]) = p \sum_i SW_{X_T(1,0,0)}(k(1,0,0) + 2i[T]) + q \sum_i SW_{X_T(0,1,0)}(k(0,1,0) + 2i[T]) + r \sum_i SW_{X_T(0,0,1)}(k(0,0,1) + 2i[T]) \]

In this formula, \( T \) denotes the torus which is the core \( T^2 \times 0 \subset T^2 \times D^2 \) in each specific manifold \( X_T(a,b,c) \) in the formula, and \( k(a,b,c) \in H_2(X_T(a,b,c)) \) is any class which agrees with the restriction of \( k \) in \( H_2(X \setminus T \times D^2, \partial) \) in the diagram:

\[
\begin{array}{ccc}
H_2(X_T(a,b,c)) & \longrightarrow & H_2(X \setminus T \times D^2, \partial) \\
\downarrow \cong & & \downarrow \cong \\
H_2(X) & \longrightarrow & H_2(X \setminus T \times D^2, \partial)
\end{array}
\]

Let \( \pi(a,b,c) : H_2(X_T(a,b,c)) \rightarrow H_2(X \setminus T \times D^2, \partial) \) be the composition of maps in the above diagram, and \( \pi(a,b,c)_* \) the induced map of integral group rings. Since we are often interested in invariants of the pair \( (X,T) \), it is sometimes useful to work with

\[ \overline{SW}_{X_T(a,b,c),T} = \pi(a,b,c)_*(SW_{X_T(a,b,c)}) \in \mathbb{Z}H_2(X \setminus T \times D^2, \partial). \]

The indeterminacy due to the sum in (1) is caused by multiples of \( [T] \); so passing to \( \overline{SW} \) removes this indeterminacy, and the Morgan-Mrowka-Szabó formula becomes

(2) \[ \overline{SW}_{X_T(p,q,r),T} = p\overline{SW}_{X_T(1,0,0),T} + q\overline{SW}_{X_T(0,1,0),T} + r\overline{SW}_{X_T(0,0,1),T}. \]

So if we expand the notion of generalized logarithmic transformation to include both homologically essential and null-homologous tori, then we can eliminate the knot surgery construction from our list of essential surgery operations. Thus our list is of essential operations is reduced to

- generalized fiber sum
- generalized logarithmic transformations on a torus with trivial normal bundle
- blowup
- rational blowdown
There are further relationships between these operations. In [6], it is shown that if $T$ is contained in a node neighborhood, then a generalized logarithmic transformation can be obtained via a sequence of blowups and rational blowdowns. (This together with work of Margaret Symington [26] shows that logarithmic transformations ($\frac{p}{q} \neq 0$) on a symplectic torus results in a symplectic manifold. We do not know of any other proof that a generalized logarithmic transformation on a symplectic torus in a node neighborhood results in a symplectic manifold.) However, it is not clear that a rational blowdown is always the result of blowups and logarithmic transforms.

Rational blowdown changes the topology of the manifold $X$; while $\chi$ remains the same, $c$ is decreased by $p - 3$. So, an obvious problem would be

**Problem 12.** Are any two homeomorphic simply-connected smooth 4-manifolds related via a sequence of generalized logarithmic transforms on tori?

As already pointed out, there are two cases:

1. $T$ is essential in homology
2. $T$ is null-homologous

This leads to:

**Problem 13.** Can a generalized logarithmic transform on a homologically essential torus be obtained via a sequence of generalized logarithmic transforms on null-homologous tori?

For the rest of the lecture we will discuss these last two problems.

5. **Cobordisms between 4-manifolds**

Let $X_1$ and $X_2$ be two homeomorphic simply-connected smooth 4-manifolds. Early results of C.T.C. Wall show that there is an $h$-cobordism $W^5$ between $X_1$ and $X_2$ obtained from $X_1 \times I$ by attaching $n$ 2-handles and $n$ 3-handles. A long standing problem that still remains open is:

**Problem 14.** Can $W^5$ be chosen so that $n = 1$.

Let’s explore the consequences if we can assume $n = 1$. We can then describe the $h$-cobordism $W^5$ as follows. First, let $W_1$ be the cobordism from $X_1$ to $X_1 \# S^2 \times S^2$ given by attaching the 2-handle to $X_1$. To complete $W^5$ we then would add the 3-handle. Dually, this is equivalent to attaching a 2-handle to $X_2$. So let $W_2$ be the cobordism from $X_2$ to $X_2 \# S^2 \times S^2$ given by attaching this 2-handle to $X_2$. Then $W^5 = W_1 \cup f (\overline{W_2})$ for a suitable diffeomorphism $f: X_1 \# S^2 \times S^2 \to X_2 \# S^2 \times S^2$. Let $A$ be any of the standard spheres in $S^2 \times S^2$. Then the complexity of the $h$-cobordism can be measured by the type $k$, which is half the minimum of the number of intersection points between $A$ and $f(A)$ (as $A \cdot f(A) = 0$ there are
$k$ positive intersection points and $k$ negative intersection points). This complexity has been studied in [19]. A key observation is that if $k = 1$, then a neighborhood of $A \cup f(A)$ is diffeomorphic to an embedding of twin spheres in $S^4$ and that its boundary is the three-torus $T^3$. A further observation is that $X_2$ is then obtained from $X_1$ by removing a neighborhood of a null-homologous torus $T$ embedded in $X_1$ (with trivial normal bundle) and sewing it back in differently. Thus when $k = 1$, $X_2$ is obtained from $X_1$ by a generalized generalized logarithmic transform on a null-homologous torus.

Thus, the answers to Problems 12 and 13 are clearly related to the complexity $k$ of $h$-cobordisms. We expect that the answer to Problem 13 is NO and that ordinary generalized logarithmic transforms on homologically essential tori will provide examples of homeomorphic $X_1$ and $X_2$ that require $h$–cobordisms with arbitrarily large complexity.

Independent of this, an important next step is to study complexity $k > 2$ $h$-cobordisms. Here, new surgical techniques are suggested. In particular, the neighborhood of $A \cup f(A)$ above is diffeomorphic to the neighborhood $N'$ of two 2−spheres embedded in $S^4$ with $2n$ points of intersection. Let $N$ be obtained from $N'$ with one of the 2−spheres surgered out. Then it can be shown that $X_1$ is obtained from $X_2$ by removing an embedding of $N'$ and regluing along a diffeomorphism of its boundary. This could lead to a useful generalization of logarithmic transforms along null-homologous tori. It would then be important to compute its effect on the Seiberg-Witten invariants, and reinterpret generalized logarithmic transforms from this point of view.

**Round handlebody cobordisms.** Suppose that $X_1$ and $X_2$ are two manifolds with the same $c$ and $\chi_h$. It follows from early work of Asimov [1] that there is a round handlebody cobordism $W$ between $X_1$ and $X_2$. Thus $X_1$ can be obtained from $X_2$ by attaching a sequence of round 1−handles and round 2−handles. A round handle is just $S^1$ times a handle in one lower dimension. So for us, a round $r$−handle is a copy of $S^1 \times (D^r \times D^{4-r})$ attached along $S^1 \times (S^{r-1} \times D^{4-r})$ (see [1] for definitions).

**Problem 15.** Can $W$ be chosen so that there are no round 1−handles?

For a moment, suppose that the answer to Problem 15 is Yes. Then $W$ would consist of only round 2−handles. It then follows that $X_2$ would be obtained from $X_1$ via a sequence of generalized logarithmic transforms on tori. Thus the answer to Problem 15 is tightly related to Problem 12.

Note that if $X_1$ and $X_2$ are round handlebody cobordant, then the only invariant preventing them from being homeomorphic is whether $t(X_1) = t(X_2)$. So suppose $t(X_1) = 0$ and $t(X_2) = 1$. If the answer to Problem 15 were yes, then one could change the second Stiefel-Whitney class via a sequence of generalized logarithmic
transforms on tori. By necessity these tori cannot be null-homologous. So understanding new surgical operations that will change $t$ without changing $c$, $\chi_h$, and preserving the Seiberg-Witten invariants should provide new insights.

**Problem 16.** Suppose two simply-connected smooth 4-manifolds have the same $c$, $\chi_h$, number of Seiberg-Witten basic classes, and different $t$. Determine surgical operations that will transform one to the other.

There are explicit examples of this phenomena amongst complex surfaces, e.g. two Horikawa surfaces with the same $c$ and $\chi_h$, but different $t$.

**Modifying symplectic 4-manifolds**

To finish up this lecture, we point out that all known constructions of (simply-connected) non-symplectic 4-manifolds can be obtained from symplectic 4-manifolds by performing logarithmic transforms on null-homologous Lagrangian tori with non-vanishing framing defect (cf. [10]). Let’s look at a specific example of this phenomena. In particular, let’s consider $E(n)_K$.

The elliptic surface $E(n)$ is the double branched cover of $S^2 \times S^2$ with branch set equal to four disjoint copies of $S^2 \times \{\text{pt}\}$ together with $2n$ disjoint copies of $\{\text{pt}\} \times S^2$. The resultant branched cover has $8n$ singular points (corresponding to the double points in the branch set), whose neighborhoods are cones on $\mathbb{RP}^3$. These are desingularized in the usual way, replacing their neighborhoods with cotangent bundles of $S^2$. The result is $E(n)$. The horizontal and vertical fibrations of $S^2 \times S^2$ pull back to give fibrations of $E(n)$ over $\mathbb{CP}^1$. A generic fiber of the vertical fibration is the double cover of $S^2$, branched over 4 points — a torus. This describes an elliptic fibration of $E(n)$. The generic fiber of the horizontal fibration is the double cover of $S^2$, branched over 2$n$ points, and this gives a genus $n-1$ fibration on $E(n)$. This genus $n-1$ fibration has four singular fibers which are the preimages of the four $S^2 \times \{\text{pt}\}$’s in the branch set together with the spheres of self-intersection $-2$ arising from desingularization. The generic fiber $T$ of the elliptic fibration meets a generic fiber $\Sigma_{n-1}$ of the horizontal fibration in two points, $\Sigma_{n-1} \cdot T = 2$.

Now let $K$ be a fibered knot of genus $g$, and fix a generic elliptic fiber $T_0$ of $E(n)$. Then in the knot surgery manifold

$$E(n)_K = (E(n) \setminus (T_0 \times D^2)) \cup (S^1 \times (S^3 \setminus N(K))),$$

each normal 2-disk to $T_0$ is replaced by a fiber of the fibration of $S^3 \setminus N(K)$ over $S^1$. Since $T_0$ intersects each generic horizontal fiber twice, we obtain a ‘horizontal’ fibration

$$h : E(n)_K \to \mathbb{CP}^1$$
of genus $2g + n - 1$.

This fibration also has four singular fibers arising from the four copies of $S^2 \times \{\text{pt}\}$ in the branch set of the double cover of $S^2 \times S^2$. Each of these gets blown up at
2n points in \( E(n) \), and the singular fibers each consist of a genus \( g \) surface \( \Sigma_g \) of self-intersection \(-n\) and multiplicity 2 with 2n disjoint 2-spheres of self-intersection \(-2\), each meeting \( \Sigma_g \) transversely in one point. The monodromy around each singular fiber is (conjugate to) the diffeomorphism of \( \Sigma_{2g+n-1} \) which is the deck transformation \( \eta \) of the double cover of \( \Sigma_g \), branched over 2n points. Another way to describe \( \eta \) is to take the hyperelliptic involution \( \omega \) of \( \Sigma_{n-1} \) and to connect sum copies of \( \Sigma_g \) at the two points of a nontrivial orbit of \( \omega \). Then \( \omega \) extends to the involution \( \eta \) of \( \Sigma_{2g+n-1} \).

The fibration which we have described is not Lefschetz since the singularities are not simple nodes. However, it can be perturbed locally to be Lefschetz.

So in summary, if \( K \) is a fibered knot whose fiber has genus \( g \), then \( E(n)_K \) admits a locally holomorphic fibration (over \( \mathbb{CP}^1 \)) of genus \( 2g+n-1 \) which has exactly four singular fibers. Furthermore, this fibration can be deformed locally to be Lefschetz.

There is another way to view these constructions. Consider the branched double cover of \( \Sigma_g \times S^2 \) whose branch set consists of two disjoint copies of \( \Sigma_g \times \{pt\} \) and 2n disjoint copies of \( \{pt\} \times S^2 \). After desingularizing as above, one obtains a complex surface denoted \( M(n,g) \). Once again, this manifold carries a pair of fibrations. There is a genus \( 2g+n-1 \) fibration over \( S^2 \) and an \( S^2 \) fibration over \( \Sigma_g \).

Consider first the \( S^2 \) fibration. This has 2n singular fibers, each of which consists of a smooth 2-sphere \( E_i \), \( i = 1, \ldots, 2n \), of self-intersection \(-1\) and multiplicity 2, together with a pair of disjoint spheres of self-intersection \(-2\), each intersecting \( E_i \) once transversely. If we blow down \( E_i \) we obtain again an \( S^2 \) fibration over \( \Sigma_g \), but the \( i \)th singular fiber now consists of a pair of 2-spheres of self-intersection \(-1\) meeting once, transversely. Blowing down one of these gives another \( S^2 \) fibration over \( \Sigma_g \), with one less singular fiber. Thus blowing down \( M(n,g) \) \( 4n \) times results in a manifold which is an \( S^2 \) bundle over \( \Sigma_g \). This shows that (if \( n > 0 \)) \( M(n,g) \) is diffeomorphic to \((S^2 \times \Sigma_g) \# 4n \mathbb{CP}^2\).

The genus \( 2g+n-1 \) fibration on \( M(n,g) \) has 2 singular fibers. As above, these fibers consist of a genus \( g \) surface \( \Sigma_g \) of self-intersection \(-n\) and multiplicity 2 with 2n disjoint 2-spheres of self-intersection \(-2\), each meeting \( \Sigma_g \) transversely in one point. The monodromy of the fibration around each of these fibers is the deck transformation of the double branched cover of \( \Sigma_g \). This is just the map \( \eta \) described above.

Let \( \varphi \) be a diffeomorphism of \( \Sigma_g \setminus D^2 \) which is the identity on the boundary. For instance, \( \varphi \) could be the monodromy of a fibered knot of genus \( g \). There is an induced diffeomorphism \( \Phi \) of \( \Sigma_{2g+n-1} = \Sigma_g \# \Sigma_{n-1} \# \Sigma_g \) which is given by \( \varphi \) on the first \( \Sigma_g \) summand and by the identity on the other summands. Consider the twisted fiber sum

\[
M(n,g) \# \Phi M(n,g) = \{ M(n,g) \setminus (D^2 \times \Sigma_{2g+n-1}) \} \cup_{\text{id} \times \Phi} \{ M(n,g) \setminus (D^2 \times \Sigma_{2g+n-1}) \}
\]
where fibered neighborhoods of generic fibers $\Sigma_{2g+n-1}$ have been removed from the two copies of $M(n,g)$, and they have been glued by the diffeomorphism $\text{id} \times \Phi$ of $S^1 \times \Sigma_{2g+n-1}$.

In the case that $\varphi$ is the monodromy of a fibered knot $K$, it can be shown that $M(n,g)\#\varphi M(n,g)$ is the manifold $E(n)_K$ with the genus $2g + n - 1$ fibration described above. To view this, we view $S^2$ as the base of the horizontal fibration. Then it suffices to check that the total monodromy map $\pi_1(S^2 \setminus 4 \text{ points}) \rightarrow \text{Diff}(\Sigma_{2g+n-1})$ is the same for each. It is not difficult to see that if we write the generators of $\pi_1(S^2 \setminus 4 \text{ points})$ as $\alpha, \beta, \gamma$ with $\alpha$ and $\beta$ representing loops around the singular points of, say, the image of the first copy of $M(n,g)$ and basepoint in this image, and $\gamma$ a loop around a singular point in the image of the second $M(n,g)$ then the monodromy map $\mu$ satisfies $\mu(\alpha) = \eta, \mu(\beta) = \eta$ and $\mu(\gamma)$ is $\varphi \oplus \omega \oplus \varphi^{-1}$, expressed as a diffeomorphism of $\Sigma_g \# \Sigma_{n-1} \# \Sigma_g$. That this is also the monodromy of $E(n)_K$ follows directly from its construction.

Now let $E(n)_g$ denote $E(n)$ fiber summed with $T^2 \times \Sigma_g$ along an elliptic fiber. The penultimate observation is that $E(n)_K$, viewed as $M(n,g)\#\varphi M(n,g)$, is then the result of a sequence of generalized logarithmic transforms on null-homologous Lagrangian tori in $E(n)_g$. The effect of these surgeries is to change the monodromy of the genus $n + 2g - 1$ Lefschetz fibration (over $\mathbb{CP}^1$) on $E(n)_g$. This is accomplished by doing a $1/n$, with respect to the natural Lagrangian framing, generalized logarithmic transform on these Lagrangian tori (cf. [9,10]). The final observation is that if the Lagrangian framing of these tori differs from the null-homologous framing (cf. [10]), then a $1/n$ log transformations on $T$ with respect to the null-homologous framing can be shown, by computing Seiberg-Witten invariants, to result in non-symplectic 4-manifolds. Careful choices of these tori and framings will result in manifolds homotopy equivalent to $M(n,g)\#\varphi M(n,g)$ (cf. [9]).

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