On A Bosonization Approach To Disordered Systems

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Abstract

In [1] a new bosonization procedure has been illustrated, which allows to express a fermionic gaussian system in terms of commuting variables at the price of introducing an extra dimension. The Fermi-Bose duality principle established in this way has many potential applications also outside the context of gauge field theories in which it has been developed. In this work we present an application to the problem of averaging the correlation functions with respect to random potentials in disordered systems and similar problems.

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1 Introduction

Both in statistical mechanics and in quantum mechanics there are several situations in which one has to average the correlations functions of a physical system with respect to disorder fields \( \mathcal{H} \). The averaging procedure is complicated by the fact that the dependence of the correlation functions on the disorder is not explicitly known. If we restrict ourselves to systems which admit a representation of the Green functions in terms of gaussian fields, one has two powerful tools at disposal to solve this problem, the replica method \( R \) and the supersymmetric method \( S \). The replica method can be applied to more general cases, but has the disadvantage that its mathematical consistency has not yet been proved, so that it has been subjected sometimes to some critiques concerning its validity in nonperturbative calculations \( R \). On the other side, the introduction of supersymmetric fields requires a fermionization of the system, in which the passage from bosonic to fermionic degrees of freedom is often obscure. For this reason, difficulties arise for instance in systems with spontaneous symmetry breaking, because it is not easy to interpret the symmetry breaking in terms of the resulting fermionic theory \( S \).

In this letter we propose an alternative to the above two methods. Our approach is very similar in spirit to the supersymmetric one, but it involves only bosonic fields. Instead of fermionization, we exploit the Fermi-Bose duality principle introduced by A. A. Slavnov in the context of lattice gauge field theories in order to provide an expression of fermion determinants which does not contain anticommuting fields \( F \). Successively, this principle has been also applied in \( F \) to rewrite the Faddeev-Popov determinants without anticommuting ghosts. Since in our case all operators are hermitian, it has been possible to simplify the original procedure, which otherwise would have lead to field theories with derivatives of the fourth order in the action. Another relevant change with respect to \( F \), \( F \) is the choice of boundary conditions of the auxiliary bosonic fields. Here boundary conditions are dictated by the compatibility requirement with the regularization needed in the path integral approach to quantum mechanics to guarantee convergence.

The material presented in this work is organized as follows. In Section 2 the problem of averaging over the disorder fields is briefly discussed. In Section 3 our alternative method based on bosonization is presented. The conclusions are drawn in Section 4.

2 The Averaging Problem In Disordered Systems

Let \( H \) be a local and hermitian Hamiltonian describing a system with \( D \) degrees of freedom \( \mathbf{q} = (q_1, \ldots, q_D) \). Further, we suppose that \( H \) depends on a set of random potentials \( \varphi(\mathbf{q}) = (\varphi_1(\mathbf{q}), \varphi_2(\mathbf{q}), \ldots) \) with a given distribution \( P(\varphi) \). Hamiltonians of this kind are widely applied in quantum mechanics \( H \). For instance, choosing \( \varphi = (\varphi_1) \), one obtains the energy operator of a disordered
system

\[ H = H_0 + \varphi_1 \]  

consisting of a fixed Hamiltonian \( H_0 \) and a random perturbation \( \varphi_1(q) \). If for example \( \varphi_1 \) is a source of Gaussian noise, the general form of \( P(\varphi_1) \) is given by:

\[
P(\varphi_1) = \exp \left\{ - \int d^D q d^D q' \varphi_1(q) K(q, q') \varphi_1(q') \right\}
\]

(2)

Analogously, with the same formalism it is possible to discuss the motion of \( n \)-dimensional particles immersed in an electromagnetic field with components \( A_i, i = 1, \ldots, n \), putting \( \vec{\varphi} = (A_1, \ldots, A_D) \) and taking as “distribution”

\[
P(A_1, \ldots, A_D) = \exp \{ -iS_{QED} \}
\]

(3)

where

\[
S_{QED} = \frac{1}{4g} \int d^D x F_{ij}^2
\]

is the usual action of quantum electrodynamics in \( n \) dimensions and \( F_{ij} = \partial_i A_j - \partial_j A_i \).

In the following, we denote the quantum average and the average over disorder fields with the symbols \( \langle \cdot \rangle \) and \( (\cdot)_{\vec{\varphi}} \) respectively. With this notation, the advanced and retarded Green functions of the Hamiltonian \( H \) are given by:

\[
G^{\pm}_E(q, q'; \vec{\varphi}) = \lim_{\epsilon \to 0^+} \left\langle q \left| \frac{1}{E \pm i\epsilon - H} \right| q' \right\rangle
\]

(5)

where \( E \pm i\epsilon \) is a complex parameter with an arbitrary small imaginary part \( i\epsilon \). Relevant information about the system may be obtained computing averages over \( \vec{\varphi} \) of products of the above Green functions:

\[
(G^+_E(q, q'; \vec{\varphi})G^+_E(q, q'; \vec{\varphi}) \cdots) = \int D\vec{\varphi} P(\vec{\varphi}) G^+_E(q, q'; \vec{\varphi})G^+_E(q, q'; \vec{\varphi}) \cdots
\]

(6)

To this purpose, it is often convenient to use a representation of \( G^\pm_E(q, q'; V) \) in terms of complex scalar fields \( \phi, \bar{\phi} \):

\[
G^\pm_E(q, q'; \vec{\varphi}) = \frac{1}{iZ_\pm} \int D\phi D\bar{\phi} \phi(q) \bar{\phi}(q') \exp \left\{ \pm i \int d^D \phi \bar{\phi}(E \pm i\epsilon - H) \phi \right\}
\]

(7)

\( Z_\pm \) represents the partition function of the field theory:

\[
Z_\pm = \int D\phi D\bar{\phi} \exp \left\{ \pm i \int d^D \bar{\phi}(E \pm i\epsilon - H) \phi \right\}
\]

(8)

It is easy to realize that, even in the simple case of a single noise source with gaussian distribution as in Eq. (2), it is difficult to integrate over the random
potentials in the right hand side of Eq. (6) due to the presence of the factor $Z_{\pm}^{-1}$ in the definition of $G^{\pm}(q,q',E|\varphi)$ (see Eq.(7)). In fact, the partition function $Z_{\pm}$ is a functional depending on $\varphi$ in a complicated way. In the next Section, it will be shown how it is possible to perform the average with respect to the random potentials without introducing replica fields or fermionic degrees of freedom.

3 The Bosonization Method

First of all, we note that the class of problems under investigation has a gaussian nature, as it is shown by the field theory representation of Eq. (7), in which only gaussian fields are involved. Due to this fact, for our aims it will be sufficient to consider only the average of a single Green function. Let us study for instance the following average:

$$\langle G_E(q,q';\varphi) \rangle_{\varphi} = \int \mathcal{D}\varphi P(\varphi)G_E(q,q';\varphi)$$

(9)

Now it will be convenient to interpret the factor $Z_{\pm}^{-1}$ in Eq. (7) as the functional determinant of the operator $E - H$:

$$Z_{\pm}^{-1} = \det(E \pm i\epsilon - H)$$

(10)

To express the determinant appearing in the right hand side of Eq. (10), we apply the Fermi-Bose duality principle proposed in [1]. To this purpose, we introduce a fictitious time $\tau$ such that

$$-T \leq \tau \leq T$$

(11)

and two sets of auxiliary fields $c_n(q,\tau), \bar{c}_n(q,\tau)$ and $\chi_n(q)$, $n = 1, 2$. The field $\bar{c}_n$ is the hermitian conjugate of $c_n$

$$\bar{c}_n = (c_n)^\dagger$$

(12)

while $\chi_n$ is an hermitian scalar field, i.e. $(\chi_n)^\dagger = \chi_n$. The fields $c_n$ and $\bar{c}_n$ satisfy the boundary conditions:

$$c_n(q,-T) = B_n(q), \quad \bar{c}_n(q,T) = (B_n(q))^\dagger$$

(13)

where $B_n(q)$ is an arbitrary function of $q$.

We are now ready to prove the following formula:

$$Z_{-}^{-1} = \det(E + i\epsilon - H) = \lim_{T \to +\infty} Z_{c,1} Z_{c,2}$$

(14)

where

$$Z_{c,n} = \int \mathcal{D}c_n \mathcal{D}\bar{c}_n \mathcal{D}\chi_n e^{iS_n}$$

(15)
and

$$S_n = \int_{-T}^{T} d\tau \int d^Dq \left[ -\left( \frac{i}{2} \frac{\partial \bar{c}_n}{\partial \tau} + (E - H)c_n \right) c_n \\
+ \left( \frac{i}{2} \frac{\partial c_n}{\partial \tau} - (E - H)c_n \right) \bar{c}_n + 2i\epsilon c_n \bar{c}_n + \chi(\bar{c}_n + c_n) \right]$$

(16)

The proof goes as follows. Since the operator $E - H$ is hermitian, it supports a complete system of orthonormal eigenfunctions $\psi_\alpha$ with eigenvalues $\lambda_\alpha$:

$$(E - H)\psi_\alpha(q) = \lambda_\alpha \psi_\alpha(q)$$

(17)

Thus, it is possible to expand the fields $c_n, \bar{c}_n$ and $\chi_n$ in terms of the $\psi_\alpha$:

$$c_n(q, \tau) = \sum_\alpha c^\alpha_n(\tau) \psi_\alpha(q)$$

(18)

$$\bar{c}_n(q, \tau) = \sum_\alpha \bar{c}^\alpha_n(\tau) \psi_\alpha(q)$$

(19)

$$\chi_n(q) = \sum_\alpha \chi^\alpha_n \psi_\alpha(q)$$

(20)

Here $c^\alpha_n(\tau)$, $\bar{c}^\alpha_n(\tau)$ depend only on the pseudo-time $\tau$, while the $\chi^\alpha_n$ are constant coefficients. To these equations, one should add the expansions of the fields $B_n$ and $B_n^\dagger$ which express the boundary conditions:

$$B_n(q) = \sum_\alpha B^\alpha_n \psi_\alpha(q)$$

$$B_n^\dagger(q) = \sum_\alpha \bar{B}^\alpha_n \psi_\alpha(q)$$

(21)

Substituting Eqs. (18–20) in the action (16) and remembering that:

$$\int d^Dq \psi_\alpha(q) \psi_\beta(q) = \delta_{\alpha\beta}$$

(22)

one obtains for $Z_{c,n}$:

$$Z_{c,n} = \prod_\alpha \int D\bar{c}^\alpha_n Dc^\alpha_n D\chi^\alpha_n e^{iS_{n,\alpha}}$$

(23)

with

$$S_{n,\alpha} = \int_{-T}^{T} d\tau \left[ -\left( \frac{i}{2} \frac{\partial \bar{c}^\alpha_n}{\partial \tau} + \lambda_\alpha c^\alpha_n \right) c^\alpha_n \\
+ \left( \frac{i}{2} \frac{\partial c^\alpha_n}{\partial \tau} - \lambda_\alpha c^\alpha_n \right) \bar{c}^\alpha_n + 2i\epsilon c^\alpha_n \bar{c}^\alpha_n + \chi^\alpha_n (\bar{c}^\alpha_n + c^\alpha_n) \right]$$

(24)

The gaussian path integral over the fields $\bar{c}^\alpha_n$ and $c^\alpha_n$ may be easily computed with the saddle point method. To this purpose, one has to solve the classical equations of motion of these fields:

$$\dot{c}^\alpha_n + \omega_n c^\alpha_n - i\chi^\alpha_n = 0$$

(25)

$$\dot{\bar{c}}^\alpha_n - \omega_n \bar{c}^\alpha_n + i\chi^\alpha_n = 0$$

(26)
In the above equation we have put \( \dot{c}_n^\alpha = dc_n^\alpha/d\tau, \dot{\bar{c}}_n^\alpha = d\bar{c}_n^\alpha/d\tau \) and
\[
\omega_\alpha = 2(i\lambda_\alpha + \epsilon)
\] (27)

The solutions of (25–26) satisfying the desired boundary conditions are:
\[
c_{n,cl}(\tau) = e^{-\omega_\alpha(\tau+T)}B_n^\alpha + i\frac{\chi_n^\alpha}{\omega_\alpha} \left[ 1 - e^{-\omega_\alpha(\tau+T)} \right]
\] (28)
\[
\bar{c}_{n,cl}(\tau) = e^{\omega_\alpha(\tau-T)}\bar{B}_n^\alpha + i\frac{\chi_n^\alpha}{\omega_\alpha} \left[ 1 - e^{\omega_\alpha(\tau-T)} \right]
\] (29)

Let us note that \( c_{n,cl}^\alpha \) and \( \bar{c}_{n,cl}^\alpha \) do not diverge for large values of \( \tau \). Moreover, it is clear that the boundary conditions are irrelevant in the limit \( T \to +\infty \), because their contribution vanishes exponentially as \( e^{-2\epsilon T} \).

After the field transformation:
\[
c_n^\alpha(\tau) = c_{n,cl}^\alpha(\tau) + c_{n,q}^\alpha(\tau)
\] (30)
\[
\bar{c}_n^\alpha(\tau) = \bar{c}_{n,cl}^\alpha(\tau) + \bar{c}_{n,q}^\alpha(\tau)
\] (31)

\( Z_{c,n} \) becomes:
\[
Z_{c,n} = \lim_{T \to +\infty} \prod_\alpha \int \mathcal{D}c_n^\alpha \mathcal{D}c_{n,q}^\alpha e^{iS_{c,n}^\alpha}
\] (32)

Here we have put
\[
S_{c,n}^\alpha = \frac{1}{2} \int_{-T}^{T} d\tau \left[ \frac{2i(\chi_n^\alpha)^2}{\omega_\alpha} - i\frac{(\chi_n^\alpha)^2}{\omega_\alpha} e^{-\omega_\alpha T} \left( e^{-\omega_\alpha \tau} + e^{\omega_\alpha \tau} \right) + i\chi_n^\alpha e^{-\omega_\alpha T} \left( e^{-\omega_\alpha \tau} B_n^\alpha + e^{\omega_\alpha \tau} \bar{B}_n^\alpha \right) \right]
\] (33)

and
\[
\mathcal{N}_{n,\alpha} = \int \mathcal{D}c_n^\alpha \mathcal{D}c_{n,q}^\alpha \exp \left[ i \int_{-T}^{T} \left( -\frac{i}{2} \dot{c}_{n,q}^\alpha c_{n,q}^\alpha + i\frac{\chi_n^\alpha}{2} \dot{c}_{n,q}^\alpha \bar{c}_{n,q}^\alpha \right) \right]
\] (34)

It is easy to show that the constant factor \( \mathcal{N}_{n,\alpha} \) produced by the integration over the “quantum” fields \( c_{n,q}^\alpha, \bar{c}_{n,q}^\alpha \) is a just a constant, which is independent of \( \omega_\alpha \) and thus can be ignored. At this point it is possible to perform the integration over the pseudo-time \( \tau \) in the action \( S_{c,n}^\alpha \). The result is:
\[
S_{c,n}^\alpha = 2i\frac{(\chi_n^\alpha)^2 T}{\omega_\alpha} - i\frac{(\chi_n^\alpha)^2}{\omega_\alpha^2} (1 - e^{-2\omega_\alpha T}) + \frac{\chi_n^\alpha}{\omega_\alpha} (1 - e^{-2\omega_\alpha T}) \left( B_n^\alpha + \bar{B}_n^\alpha \right)
\] (35)

Finally, one has to integrate over the variables \( \chi_n^\alpha \) in \( Z_{c,n} \):
\[
Z_{c,n} = 
\int \mathcal{D}\chi_n^\alpha \exp \left\{ i \left[ 2i\frac{(\chi_n^\alpha)^2 T}{\omega_\alpha} - i\frac{(\chi_n^\alpha)^2}{\omega_\alpha^2} (1 - e^{-2\omega_\alpha T}) + \frac{\chi_n^\alpha}{\omega_\alpha} (1 - e^{-2\omega_\alpha T}) \left( B_n^\alpha + \bar{B}_n^\alpha \right) \right\}
\] (36)

\footnote{Let us note that this factor is also independent on \( T \).}
Only the first term in the right hand side of the above equation becomes relevant when \( T \) becomes very large, as it is in our case. Since

\[
\frac{2iT}{\omega_\alpha} = T\frac{\lambda_\alpha + i\epsilon}{\lambda_\alpha^2 + \epsilon^2}
\]  

(37)

it turns out that, thanks to the presence of the \( \epsilon \)-term in the action \( S_n \) of Eq. (13), the integrals in \( D\chi_n^\alpha \) are convergent. Upon renormalizing the fields \( \chi_n^\alpha \) in Eq. (36) as follows: \( \chi_n^{\alpha'} = \chi_n^{\alpha T^{1/2}} \), one finds:

\[
Z_{c,n} = \prod_\alpha \sqrt{\lambda_\alpha + i\epsilon} = \sqrt{\text{det}(E - H + i\epsilon)}
\]  

(38)

This proves Eq. (14) as desired. An analogous formula can be derived for \( Z_{-1} \).

Coming back to the original averaging problem, we rewrite Eq. (39) in the form:

\[
\langle G_E(q, q'; \vec{\varphi}) \rangle_{\vec{\varphi}} = \lim_{T \to +\infty} \int D\vec{\varphi} P(\vec{\varphi}) Z_{c,1} Z_{c,2} \times \int D\phi D\bar{\phi}(q) \bar{\phi}(q') \exp \left\{-i \int_{-T}^{T} \frac{d\tau}{2T} \int d^Dq \bar{\phi}(E - i\epsilon - H)\phi \right\}
\]  

(39)

The dependence of \( G_E(q, q'; \vec{\varphi}) \) on the disorder fields \( \vec{\varphi} \) is now explicit and it is given by Eq. (16), which expresses the inverse of the partition function \( Z_\pm \) as a path integral over gaussian fields \( c_n, \bar{c}_n, \chi_n, n = 1, 2 \). At this point it is possible to perform the averaging over the disorder fields at least perturbatively. In the case of Hamiltonians like that of Eq. (1), the random potential \( \varphi_1 \) may be integrated out from the partition function with the help of a gaussian integral.

Let us note that in Eq. (39) the limit for \( T \to +\infty \) has been permuted with the integration over the disorder fields \( \vec{\varphi} \). In a similar way, in the method of replicas one needs to permute the limit of vanishing replica’s and the average with respect to the disorder. The difference between the two approaches is that in the present case the limit \( T \to +\infty \) does not require a complex analytical continuation as in the method of replicas and it is mathematically under control. In fact, the presence of the variable \( T \) is only limited to the partition functions \( Z_{c,n} \) of the bosonic fields \( c_n, \bar{c}_n, \chi_n \). These partition functions may always be rewritten as in Eq. (36), i.e. in terms of standard integrals over the real variables \( \chi_n^\alpha \), which are convergent in the limit \( T \to +\infty \) due to the presence of the \( \epsilon \) term. As a consequence, the permutation of the symbol \( \lim_{T \to +\infty} \) with the integrals which are necessary to compute \( Z_{c,n} \) is allowed.

4 Conclusions

Concluding, in this work it has been presented a bosonization approach to disordered systems based on the Fermi–Bose principle of [1], which is alternative to the supersymmetric method and to the method of replicas. The bosonization approach does not contain fermionic degrees of freedom because it involves only
bosonic fields. As a consequence, it may be useful in the investigations of systems with spontaneous symmetry breaking, which are sometimes complicated by the presence of anticommuting variables.

Let us note that in the case of quantum chromodynamics the Fermi-Bose principle used to express the Faddeev–Popov determinant without the help of ghost fields leads to a new symmetry, which replaces the usual BRST symmetry. It would be thus interesting to check if also the disordered path integral of Eq. (39) enjoys an analogous new symmetry, which would replace the fermion-boson symmetry of the supersymmetric method.

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