Barbero–Immirzi parameter, manifold invariants and Euclidean path integrals

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Abstract

The Barbero–Immirzi parameter \( \gamma \) appears in the real connection formulation of gravity in terms of the Ashtekar variables, and gives rise to a one-parameter quantization ambiguity in loop quantum gravity. In this paper, we investigate the conditions under which \( \gamma \) will have physical effects in Euclidean quantum gravity. This is done by constructing a well-defined Euclidean path integral for the Holst action with a non-zero cosmological constant on a manifold with a boundary. We find that two general conditions must be satisfied by the spacetime manifold in order for the Holst action and its surface integral to be non-zero: (i) the metric has to be non-diagonalizable; (ii) the Pontryagin number of the manifold has to be non-zero. The latter is a strong topological condition and rules out many of the known solutions to the Einstein field equations. This result leads us to evaluate the on-shell first-order Holst action and corresponding Euclidean partition function on the Taub–NUT–ADS solution. We find that \( \gamma \) shows up as a finite rotation of the on-shell partition function which corresponds to shifts in the energy and entropy of the NUT charge. In an appendix, we also evaluate the Holst action on the Taub–NUT and Taub–bolt solutions in flat spacetime and find that in that case as well \( \gamma \) shows up in the energy and entropy of the NUT and bolt charges. We also present an example whereby the Euler characteristic of the manifold has a non-trivial effect on black hole mergers.

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1. Introduction

Since its discovery within the context of canonical gravity [1], the Barbero–Immirzi parameter, denoted \( \gamma \), has remained an elusive one-parameter quantization ambiguity in loop quantum gravity [2, 3]. For details, see [4–6]. This parameter can be fixed by comparing the
quantum theory to the semi-classical theory. This procedure for fixing $\gamma$ has been done by matching the Bekenstein–Hawking entropy of black holes with spacetime topology $\mathbb{R}^2 \times \mathbb{C}^2$, with $\mathbb{C}^2$ a compact 2-manifold, to the corresponding isolated horizon quantum geometry [7–10]. It turns out that $\gamma$ has the same value regardless of the topology of the black hole.

The Einstein field equations admit more solutions in four dimensions than just the black holes whose event horizons have topology $\mathbb{R} \times \mathbb{C}^2$; in particular, the NUT-charged spacetimes [11, 12] have topologies that cannot be foliated by a time function and contain Misner strings with a non-zero entropy [13]. A comparison of the semi-classical and quantum geometry descriptions of NUT-charged spacetimes would be of interest in order to provide an independent determination of $\gamma$ that can be compared to the previous results for black holes [7–10].

Motivated by this interest, we want to know what effects, if any, $\gamma$ may have on the Euclidean path integral. In this paper, we study the Euclidean Holst action [14] with non-zero cosmological constant, and derive the semi-classical energy and entropy of Taub–NUT–ADS spacetime in the presence of $\gamma$. This can only be done in the first-order formalism. Having said that, we need to make sure that the first-order Holst–ADS action satisfies two important conditions in order that we may be able to evaluate the on-shell Euclidean Holst–ADS action: (I) the action has to have a well-posed action principle on a manifold with boundary and (II) the action has to be finite.

Condition I leads us to consider the Dirichlet boundary value problem for a generic first-order action that includes curvature and torsion as functionals of the connection and co-frame, and we present a general prescription for determining the corresponding surface terms for which the first variation of the action vanishes identically. We then present, in an example, a derivation of the surface terms that are required for a generalized Hilbert–Palatini action in four dimensions to be functionally differentiable; the form of this functional is motivated by requiring consistency with the Hamiltonian theory coupled to fermions when $\gamma = i$. A special case of this action is the Holst action with non-zero cosmological constant—the main focus of study in this paper. Some general properties of the Holst surface term are discussed in section 3. Condition II is also addressed in section 3. The ADS and Holst–ADS actions are finite without the need of adding infinite counter terms to the boundary integral—a procedure done in the infinite subtraction method [15–21].

Once the two conditions are shown to be satisfied by the Holst–ADS action, we are able to evaluate the Euclidean Holst action on solutions to the field equations, and obtain the corresponding on-shell partition functions. We find that generically $\gamma$ shows up in the Euclidean path integral as a finite rotation of the on-shell partition function, and this rotation corresponds to shifts in the energy and entropy of the spacetime. This property of the Holst–ADS action is a result of a strong topological condition that we find: the Holst term and its surface term are non-zero if the Pontryagin number of the corresponding spacetime manifold is non-zero. We confirm these results by explicitly evaluating the Holst–ADS action and partition function on the Taub–NUT–ADS solution; $\gamma$ shows up as a finite shift in the energy and entropy of the NUT charge.

We also include two appendices. In appendix A, we evaluate the Euclidean Holst action on Taub–NUT spacetimes with zero cosmological constant. We show that $\gamma$ shows up as a shift in the energy and entropy of the NUT and bolt charges, just as for the Taub–NUT–ADS solution. In appendix B, we present an example which shows how a topological invariant of the manifold can have non-trivial physical effects in gravity. In particular, it is shown that the presence of the Euler characteristic of the manifold can lead to violations of the second law during a black hole merging process.
2. First-order action principle: generalities

In the first-order formulation of general relativity (see e.g. [4]), the configuration space \( \mathcal{C} \) is the pair \([e, A]\), consisting of the co-frame \( e \) and a connection \( A \) valued in \( SO(D) \) or \( SO(D - 1, 1) \) depending on the signature of spacetime. The co-frame determines the spacetime metric \( g_{ab} = \eta_{IJ} e_a^I \otimes e_b^J \) and spacetime volume form \( e = e^0 \wedge \cdots \wedge e^{D-1} \), where \( e_b^J \) is the totally antisymmetric Levi–Civita tensor. In this paper, spacetime indices \( a, b, \ldots \in [0, \ldots, D - 1] \) are raised and lowered using the metric \( g_{ab} \), and internal indices \( I, J, \ldots \in [0, \ldots, D - 1] \) are raised and lowered using the flat metric \( \eta_{IJ} \). The connection determines the curvature 2-form \( \Omega^I_J = dA^I_J + A^I_K \wedge A^K_J = \frac{1}{2} R^I_{JKL} e^K \wedge e^L \), with \( R^I_{JKL} \) the Riemann tensor. The Lagrangian density is denoted \( \mathcal{L} \); this is a functional by which we mean a map from the space of functions \( (e, A) \) to \( \mathbb{R} \). The functional derivative of \( \mathcal{L} \) with respect to a function \( \varphi = \varphi(x^a) \) is denoted \( \delta \mathcal{L}/\delta \varphi \). The internal Hodge dual is denoted by \( \star \). In this paper, we write differential forms without indices.

2.1. Functional differentiability of the generic first-order action

Let us begin with the following.

**Proposition 1.** Let \( \Omega = dA + A \wedge A \) and \( T = de + A \wedge e \) be (resp.) the curvature and torsion of a D-dimensional manifold \( \mathcal{M} \) with a boundary \( \partial \mathcal{M} \), with \( A \) the connection and \( e \) the co-frame. Let \( \mathcal{L}[e, A, \Omega, T] \) be the Lagrangian density, a D-form in spacetime. The first-order action for an arbitrary diffeomorphism invariant theory of pure gravity on the configuration space \( \mathcal{C} = [e, A] \),

\[
I = \frac{1}{16\pi} \int_{\mathcal{M}} \mathcal{L}[e, A, \Omega, T] = \frac{(-1)^D}{16\pi} \oint_{\partial \mathcal{M}} \mathcal{Y}_\Omega \wedge A + \mathcal{Y}_T \wedge e, \tag{1}
\]

is functionally differentiable.

**Proof.** Take the variation of \( \mathcal{L} \):

\[
\delta \mathcal{L} = \mathcal{Y}_e \wedge \delta e + \mathcal{Y}_A \wedge \delta A + \mathcal{Y}_\Omega \wedge \delta \Omega + \mathcal{Y}_T \wedge \delta T
\]

\[
= \mathcal{Y}_e \wedge \delta e + \mathcal{Y}_A \wedge \delta A + \mathcal{Y}_\Omega \wedge (dA + 2A \wedge \delta A) + \mathcal{Y}_T \wedge (d\delta e - e \wedge \delta A + A \wedge \delta e)
\]

\[
= (-1)^D \mathcal{Y}_\Omega \wedge \delta A + (-1)^D d(\mathcal{Y}_T \wedge \delta e) + [\mathcal{Y}_e + \mathcal{Y}_T \wedge A - (-1)^D \mathcal{Y}_T \wedge \delta e]
\]

\[
+ \mathcal{Y}_A \wedge 2\mathcal{Y}_\Omega \wedge A - (-1)^D d\mathcal{Y}_\Omega - \mathcal{Y}_T \wedge e \wedge \delta A.
\]

The action will be functionally differentiable if the total derivative is canceled, and the equations of motion

\[
\mathcal{Y}_e + \mathcal{Y}_T \wedge A - (-1)^D d\mathcal{Y}_T = 0 \tag{2}
\]

\[
\mathcal{Y}_T \wedge e + (-1)^D d\mathcal{Y}_\Omega - 2\mathcal{Y}_\Omega \wedge A - \mathcal{Y}_A = 0 \tag{3}
\]

are satisfied. From the fundamental theorem of exterior calculus, the boundary term

\[
I_{\partial \mathcal{M}} = \frac{(-1)^D}{16\pi} \oint_{\partial \mathcal{M}} \mathcal{Y}_\Omega \wedge A + \mathcal{Y}_T \wedge e \tag{4}
\]

follows. This completes the proof. \( \square \)

From the identity \( d^2 = 0 \), terms involving higher derivatives in the configuration variables \( e \) and \( A \) do not show up in action (1). Action (1) is therefore generically first order in derivatives of \( e \) and \( A \).
2.2. Example: Generalized Hilbert–Palatini action in four dimensions

If we take the Wilsonian point of view on effective field theories, then we need to add all possible $D$-forms to the action for gravity that are diffeomorphism invariant. In four dimensions, this implies that in addition to the Hilbert–Palatini and cosmological terms, we should include in the action the Holst term $\theta_1 f e \wedge e \wedge \Omega$, as well as the characteristic classes: the Euler class $\theta_2 f \star \Omega \wedge \Omega$, the Pontryagin class $\theta_3 f \Omega \wedge \Omega$ and the Nieh–Yan class $\theta_4 f T \wedge T - e \wedge e \wedge \Omega$. See, e.g., [22, 23]. Here, $\theta_1 = 1/\gamma$ with $\gamma$ a non-zero real constant—the so-called Barbero–Immirzi parameter [1].

It was pointed out in [24, 25], however, that in the case when fermions are coupled to gravity, the Holst term needs to be replaced with the Nieh–Yan term. This extension of the Holst term is necessary in the presence of non-zero torsion so that the corresponding Hamiltonian with arbitrary $\gamma$ reduces to the Ashtekar–Romano–Tate Hamiltonian [26], when $\gamma$ is set equal to the imaginary unit. Therefore, we consider here the Nieh–Yan term in place of the Holst term in our action principle (with $\theta_4 = 1/\gamma$). When torsion is zero the Nieh–Yan term reduces to the Holst term. This makes the topological origin of the Holst term in the first-order action manifest!

From proposition 1, then, we have the following.

**Corollary 2.** The generalized Hilbert–Palatini action for general relativity with cosmological constant $\Lambda = 3\kappa^2 \epsilon^{-2} \in \mathbb{R} (\epsilon \in \{-1, 1\})$, Barbero–Immirzi parameter $\gamma \in \mathbb{R} \setminus 0$ and $\theta_2, \theta_3 \in \mathbb{R}$ on a four-dimensional manifold $(M, e, A)$ with boundary $\partial M$,

$$I = \frac{1}{16\pi} \int_M \star (e \wedge e) \wedge \Omega - 2\Lambda e + \frac{1}{\gamma} (T \wedge T - e \wedge e \wedge \Omega) + \theta_2 \star \Omega \wedge \Omega + \theta_3 \Omega \wedge \Omega$$

$$- \frac{1}{16\pi} \int_{\partial M} \star (e \wedge e) \wedge A - \frac{1}{\gamma} e \wedge de + 2\theta_2 \star \Omega \wedge A + \theta_3 \Omega \wedge A,$$

(5)

is functionally differentiable.

**Proof.** For the Nieh–Yan–Holst term, we find that $\Upsilon_T = T$ and $\Upsilon_\Omega = -e \wedge e$ so that the surface term has density $T \wedge e - e \wedge e \wedge A = de \wedge e + A \wedge e - A \wedge e \wedge e = de \wedge e$. For the Euler term, we find that $\Upsilon_\Omega = 2 \star \Omega$ so that the surface term has density $2 \star \Omega \wedge A$. For the Pontryagin term, we find that $\Upsilon_\Omega = \Omega$ so that the surface term has density $\Omega \wedge A$. The surface term in action (5) follows. $\square$

Action (6), with $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$, was previously studied within the context of black hole mechanics. In these works, the spacetimes under consideration include four-dimensional Einstein–Maxwell theory with zero cosmological constant [27, 28], non-trivial matter couplings [29–31], higher dimensional flat and ADS spacetimes [32, 33], Gauss–Bonnet gravity [34] and supergravity with $p$-form matter couplings [35–37].

In this formalism, the topology of the boundary in the action principle is taken to be $\partial M \cong M_1 \cup M_2 \cup \Delta \cup I$, with $\Delta$ a $(D - 1)$-dimensional null hypersurface equipped with a null normal and a degenerate metric with signature $(0 \, 0 \, \cdots \, +)$. $M_1$ and $M_2$ are partial Cauchy surfaces that extend from $\Delta$ to the boundary $I$ at infinity; $M_1$ and $M_2$ intersect $\Delta$ and $I$ in $(D - 2)$-surfaces $C^{D-2}$. Adding the surface term $\int_I \star (e \wedge e) \wedge A$ to the action at $I$ and fixing the geometry of $\Delta$ are sufficient conditions for the action to be functionally differentiable and for the zeroth and first laws of black hole mechanics to be satisfied. In particular, all the conserved charges are defined locally at the horizon $\Delta$; these include the non-monopolar (dipole) charge of the five-dimensional black ring solution [37, 38].

Subsequently, it was shown that the action is finite on asymptotically flat spacetimes [39, 40], and that a partition function can be given for Euclidean metrics [41] without having...
to add infinite counter-terms to the boundary [41]. It was then shown that the Holst term and its surface term together are finite [42]. In this paper, we further explore properties of the Holst action and corresponding Euclidean path integral on a manifold with boundary, in the presence of a non-zero cosmological constant.

3. Holst action with the cosmological constant

In this paper, we will focus particular attention on ADS spacetimes with $\Lambda < 0$, but the mathematical results obtained also apply to de Sitter spacetimes with $\Lambda > 0$. Therefore, we consider the Holst action in the presence of a non-zero cosmological constant $\Lambda = 3\varepsilon \ell^{-2} \in \mathbb{R} \setminus 0$, with $\varepsilon \in \{-1, 1\}$ and $\ell$ is the de Sitter radius. Specifically, we consider action (5) with $T = 0$ and with $\theta_2 = \theta_3 = 0$.

The first-order Holst action with cosmological constant $\Lambda = 3\varepsilon \ell^{-2} \in \mathbb{R} \setminus 0$ and Barbero–Immirzi parameter $\gamma \in \mathbb{R} \setminus 0$ on a four-dimensional manifold $(\mathcal{M}, e, A)$ with boundary $\partial \mathcal{M}$ is

$$I = \frac{1}{16\pi} \int_{\mathcal{M}} \left[ \left( 1 - \frac{1}{\gamma} \right) (e \wedge e) \right] \wedge \Omega + \frac{6e}{\ell^2} e - \frac{1}{16\pi} \int_{\partial \mathcal{M}} e (e \wedge e) \wedge A - \frac{1}{\gamma} e \wedge e \wedge A .$$

(6)

From corollary 2, this action is functionally differentiable. The surface term here is the same as the one that was previously found for the Holst action in flat spacetime [42]. This is because when $T = de + A \wedge e = 0$, we have $e \wedge de = e \wedge e \wedge A$.

An important property of the Holst surface term $e \wedge de$ is that it is identically zero for spacetimes with metrics that can be put into a diagonal form. To see this, consider the tetrad in components $e^i = e^i_a dx^a$. Differentiating the tetrad gives $de^i = (\partial e^i_a / \partial x^b) dx^b \wedge dx^a$. By direct substitution, we obtain

$$e \wedge de = \left[ e_0 \left( \frac{\partial e_0^0}{\partial x^b} \right) + e_1 \left( \frac{\partial e_1^0}{\partial x^b} \right) + e_2 \left( \frac{\partial e_2^0}{\partial x^b} \right) \right] \wedge dx^b \wedge dx^a .$$

(7)

If the metric is diagonal, then

$$e \wedge de = e_0 \left( \frac{\partial e_0^0}{\partial x^b} \right) \wedge dx^b \wedge dx^0 + e_1 \left( \frac{\partial e_1^1}{\partial x^b} \right) \wedge dx^b \wedge dx^1 + e_2 \left( \frac{\partial e_2^2}{\partial x^b} \right) \wedge dx^b \wedge dx^2 + e_3 \left( \frac{\partial e_3^3}{\partial x^b} \right) \wedge dx^b \wedge dx^3 = 0 .$$

(8)

Of particular interest in general relativity are spacetimes with a ‘$t$-$\phi$’ component in their corresponding metrics. For a general metric with $\gamma^0$-$\gamma^3$ cross term, $e \wedge de$ can be written in component form such that

$$e \wedge de = \left\{ e_{00} \left( \frac{\partial e_0^0}{\partial \theta} \right) - e_{03} \left( \frac{\partial e_0^3}{\partial \theta} \right) \right\} dx^0 \wedge dx^b \wedge dx^3 .$$

(9)

In spherical coordinates $x^a \in \{ \tau, r, \theta, \phi \}$, on a constant-$\tau$ hypersurface, the boundary term is

$$e \wedge de = \left\{ e_{00} \left( \frac{\partial e_0^0}{\partial \theta} \right) - e_{03} \left( \frac{\partial e_0^3}{\partial \theta} \right) \right\} d\tau \wedge d\theta \wedge d\phi .$$

(10)

This expression can be used to evaluate the Holst surface term on solutions with a $t$-$\phi$ component in the metric.

For spacetimes with non-zero cosmological constant, another condition can be found on the Holst surface term. Let us first substitute the equation of motion $de + A \wedge e = 0$ into action (6) to eliminate the tetrad derivative in the surface term:

$$I = \frac{1}{16\pi} \int_{\mathcal{M}} \left[ \left( 1 - \frac{1}{\gamma} \right) (e \wedge e) \right] \wedge \Omega + \frac{6e}{\ell^2} e - \frac{1}{16\pi} \int_{\partial \mathcal{M}} e (e \wedge e) \wedge A - \frac{1}{\gamma} e \wedge e \wedge A .$$

(11)
Then, with the equation of motion $\epsilon e \wedge e = (\epsilon \ell^2 / 6) \Omega$ the boundary term becomes (with $!$ denoting equality on-shell)

\[
- \frac{1}{16\pi} \oint_{\partial M} \star (e \wedge e) \wedge A - \frac{1}{\gamma} e \wedge e \wedge A = - \frac{\epsilon \ell^2}{6 \cdot 16\pi} \oint_{\partial M} \star \Omega \wedge A - \frac{1}{\gamma} \Omega \wedge A. \tag{12}
\]

From the fundamental theorem of exterior calculus, this boundary integral can be written as a bulk integral:

\[
- \frac{\epsilon \ell^2}{6 \cdot 16\pi} \oint_{\partial M} \star (e \wedge e) \wedge A - \frac{1}{\gamma} e \wedge e \wedge A = - \frac{\epsilon \ell^2}{16\pi} \int_{M} \star \Omega \wedge \Omega - \frac{1}{\gamma} \Omega \wedge \Omega. \tag{13}
\]

Putting this in (11), we see that the Holst action can be written as a bulk integral:

\[
I = \frac{1}{16\pi} \int_{M} \left[ \left( \star - \frac{1}{\gamma} \right) (e \wedge e) \right] \wedge \Omega + \frac{6 \epsilon \ell^2}{\ell^6} \epsilon + \frac{\epsilon \ell^2}{6} \star \Omega \wedge \Omega + \frac{\epsilon \ell^2}{6\gamma} \Omega \wedge \Omega. \tag{14}
\]

Written this way, the surface terms appearing in action (6) are invariants of the manifold $M$. It follows from this form of the action that the Holst surface term is identically zero for manifolds that have zero Pontryagin number. In addition, the Holst term itself can be written purely in terms of the connection by using the equation of motion $e \wedge e = (\epsilon \ell^2 / 6) \Omega$ to eliminate the tetrad. We find that

\[
\frac{1}{\gamma} \int_{M} e \wedge e \wedge \Omega = \frac{\epsilon \ell^2}{6\gamma} \int_{M} \Omega \wedge \Omega. \tag{15}
\]

Therefore, we see that on-shell the Holst term will be identically zero if the Pontryagin number of the manifold is zero.

The first-order Holst action with negative cosmological constant is finite. To see this, consider first the Einstein–Hilbert and cosmological terms only. Then, action (14) with $\epsilon = -1$ is precisely the action that was shown to be finite for ADS spacetimes [43]. It was also shown in [42] that the Holst term and its surface term together are finite. Therefore, action (14) is finite.

Let us briefly summarize this section. The Holst–ADS action is functionally differentiable and finite. In order for the Holst term and its surface term, and therefore $\gamma$ to be present in the on-shell action, the corresponding solution has to have a non-diagonalizable metric and a non-zero Pontryagin number. These are very restrictive conditions and rule out many of the known spacetimes that are critical points of the action. The Taub–NUT spacetime is known to have Pontryagin number 2 [44]. Therefore, in section 4 and in appendix A, we evaluate the Euclidean on-shell Holst–ADS and Holst actions and partition functions (resp.) on the Taub–NUT–ADS spacetime with $\Lambda < 0$ and the Taub–NUT and Taub–bolt spacetimes with $\Lambda = 0$.

### 4. Euclidean path integrals and Taub–NUT–ADS spacetime

#### 4.1. Partition functions and thermodynamics

Let us consider the formal path integral

\[
Z = \int D[\Psi] \exp \left\{ -I[\Psi] \right\}. \tag{16}
\]

with $I[\Psi]$ being the Euclidean action and $\Psi$ a generic field variable. Here, the measure $D[\Psi]$ includes all fields and not just the classical fields $\tilde{\Psi}$ that satisfy the equations of motion $\delta I[\tilde{\Psi}] = 0$. However, if the dominant contributions to the partition function come from fields that are close to the classical fields, then the action can be expanded in a Taylor series:

\[
I[\tilde{\Psi} + \delta \Psi] = I[\tilde{\Psi}] + \delta I[\tilde{\Psi}, \delta \Psi] + \delta^2 I[\tilde{\Psi}, \delta \Psi] + \cdots. \tag{17}
\]
In order for the path integral \( Z \) to make sense, at least up to second order in the Taylor series, we require that the first term \( I[\Psi] \) be finite and that the linear term \( \delta I \) vanish identically. If these conditions are satisfied, then the on-shell partition function is approximated by

\[
\widetilde{Z} = \exp \left\{ -I[\Psi] \right\}
\]  

(18)

the average energy \( \langle E \rangle \) and entropy \( S \) are then given by

\[
\langle E \rangle = -\frac{\partial \ln \tilde{Z}}{\partial \beta} \quad \text{and} \quad S = \beta \langle E \rangle + \ln \tilde{Z}.
\]  

(19)

The physical meaning of the energy may differ based on the boundary conditions that are used, i.e. holding the pressure or volume constant.

From section 3, we know that the Holst–ADS action (6) is functionally differentiable and finite. Therefore, we may consider the Holst–ADS partition function

\[
\widetilde{Z} = \exp \left\{ -\frac{1}{16\pi} \int_M \left[ \ast - \frac{1}{\gamma} \right] (e \wedge e) \wedge \Omega + \frac{6\epsilon}{\ell^2} \epsilon - \frac{\epsilon\ell^2}{6} \ast \Omega \wedge \Omega + \frac{\epsilon\ell^2}{6\gamma} \Omega \wedge \Omega \right\}
\]  

(20)

for spacetimes with negative cosmological constant and non-zero Pontryagin number. Let us therefore proceed by evaluating the partition function (20) on the Taub–NUT–ADS spacetime.

### 4.2. Taub–NUT–ADS spacetime

Here, we consider the Taub–NUT–ADS spacetime. The metric for four-dimensional Euclidean Taub–NUT spacetime, with a negative cosmological constant \( \Lambda = -3\epsilon^{-2} \), has the line element

\[
d\mathbf{s}^2 = V(r)[d\tau + 2N \cos \theta d\phi]^2 + \frac{dr^2}{V(r)} + (r^2 - N^2)(d\theta^2 + \sin^2 \theta d\phi^2),
\]

\[
V(r) = \frac{r^2 - 2Mr + N^2 + (r^4 - 6N^2r^2 - 3N^4)\epsilon^{-2}}{r^2 - N^2},
\]  

(21)

with \( N \) being the NUT parameter. Regularity of the metric requires that the Euclidean time \( \tau \) have a period \( \beta = 8\pi N \).

A suitable tetrad of co-frames for this spacetime is given by

\[
e^0 = \sqrt{V} d\tau + 2\sqrt{V} N \cos \theta d\phi, \quad e^1 = \frac{1}{\sqrt{V}} dr, \quad e^2 = \sqrt{r^2 - N^2} d\theta, \quad e^3 = \sqrt{r^2 - N^2 \sin^2 \theta} d\phi.
\]  

(22)

The Euclidean action for the Taub–NUT–ADS spacetime is then given by

\[
I = I_0 + \frac{64\pi^2N^2}{\gamma} \left( 1 - \frac{2N^2}{\ell^2} \right),
\]  

(23)

with \( I_0 \) being the on-shell action of the Taub–NUT–ADS solution without the Holst term (i.e. contributions from the Einstein–Hilbert and cosmological terms only) [20, 45]. Substituting this in (16) then gives the on-shell partition function

\[
\tilde{Z} = \exp \left\{ -I_0 - \frac{64\pi^2N^2}{\gamma} \left( 1 - \frac{2N^2}{\ell^2} \right) \right\}.
\]  

(24)

Whence the thermodynamic quantities are given by

\[
\langle E \rangle = \langle E \rangle_0 + \frac{N(\ell^2 - 4N^2)}{\gamma\ell^2} \quad \text{and} \quad S = S_0 + \frac{4\pi N^2}{\gamma} \left( 1 - \frac{N^2}{\ell^2} \right),
\]  

(25)

with \( \langle E \rangle_0 \) and \( S_0 \) denoting (resp.) the average energy and entropy of the Taub–NUT–ADS solution without the Holst term [20, 45].
We conclude that $\gamma$ appears in the thermodynamics of the Taub–NUT–ADS spacetime as a shift in the energy and entropy of the NUT charge (provided that $\gamma$ is finite and real). The quantities derived here are in agreement with previous results found by Mann [20] and Chamblin et al [45], but with a finite shift in the energy and entropy of the Taub–NUT–ADS spacetime; these shifts vanish in the limit when $\gamma$ is taken to infinity.

5. Summary and discussion

Let us briefly summarize the main results that are presented in this paper. We studied the properties of the first-order Holst action with non-zero cosmological constant. In particular, we showed that the spacetimes for which $\gamma$ will be present in the on-shell action (14), and hence in the partition function (20), are the ones that have non-diagonalizable metrics and non-zero Pontryagin number. This led us to evaluate (20) on the Taub–NUT–ADS spacetime. It was found that $\gamma$ shifts the energy and entropy of the NUT charge. The analogous results in the case where $\Lambda_1 = 0$ are presented in appendix A. Some results regarding the Euler characteristic and black hole mechanics are presented in appendix B.

The results found in this paper agree with recent results found by Durka and Kowalski-Glikman [46, 47]. Durka and Kowalski-Glikman derived the Noether–Wald charges for solutions of a constrained $BF$ theory with $SO(3, 2)$ symmetry, first introduced by Freidel and Starodubtsev [48]. They found that $\gamma$ shows up in the Noether–Wald charges of solutions that have $\partial_\theta g_{\phi\phi} \neq 0$. For the Taub–NUT–ADS solution, the energy and entropy are shifted by the same factor as found here, but with $1/\gamma \to \gamma$. This is because, in the action that they studied, the Holst and Nieh–Yan terms appear separately with different weight factors: the Holst term has coefficient $1/\gamma$ while the Nieh–Yan term has coefficient $(\gamma^2 + 1)/\gamma$, so their action is fundamentally different. In particular, the limit $\gamma \to \infty$ cannot be taken in this action to recover the Einstein–Hilbert action. Apart from this minor difference, the two approaches are equivalent: $\gamma$ contributes to the on-shell partition function and to the Noether–Wald charges through the Pontryagin number of the spacetime.

In this paper, we considered initially the Holst action with a generic cosmological constant, and in particular focused on the case when $\Lambda < 0$. However, in light of cosmological data, it would be of interest to also study in detail the Holst action with $\Lambda > 0$. Expression (20) is mathematically valid for any sign $\varepsilon = \text{sign}(\Lambda)$ of the cosmological constant. An important step toward determining how $\gamma$ affects, e.g., the NUT-charged DS spacetimes [49, 50], is to first prove the finiteness of the Holst-DS action with $\varepsilon = 1$. Then, (20) will be a well-defined partition function for DS spacetimes with non-vanishing Pontryagin number. This approach may also reveal new insights regarding the Kodama wavefunction with arbitrary real values of $\gamma$ [51, 52].

In this paper, we looked at the torsion-free case. This field will be non-zero in the presence of fermion couplings, and therefore should be included in the action. Torsion-squared Lagrangian densities in the first-order action have recently been studied in [53, 54]; these terms are all consistent with corollary 2. Note that in the presence of fermion couplings, $\gamma$ will appear in the chiral anomaly [55]. Of particular interest would be to extend the formalism here to supergravity. To study the effects of $\gamma$ in supergravity, the supersymmetric Holst actions found by Kaul [56] have to be generalized to a manifold with boundary. In practice, however, finding supersymmetric boundary terms without imposing any boundary conditions on the fields is difficult. See [57–59] for details. Ideally, we would like to have an action principle for supergravity that is invariant under the off-shell supersymmetry algebra; this suggests that we extend the supergravity action to a manifold with boundary in superspace; such an action
without boundary has been recently found by Gates Jr et al [60] where it was found that $\gamma$ shows up in superspace as the complex component of the gravitational constant.

It would also be of interest to determine the effects of $\gamma$ in quantum gravity by studying more general Euclidean path integrals. Because any topology may occur in classical and quantum gravity, one may have in general a partition function that sums over all possible inequivalent topologies [61–63]. In section 3, we found that in order for $\gamma$ to be present in the partition function, the Pontryagin number of the manifolds have to be non-zero, so only those manifolds will contribute to the formal sum. In the case of supergravity with fermion couplings, the existence of a spinor structure on $M$ requires that the second Stiefel–Whitney class of the manifold be non-zero [64]. This condition places further restrictions on the formal sum.

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Appendix A. Taub–NUT and Taub–bolt spacetimes

Here, we consider the Taub–NUT spacetimes. The metric for four-dimensional Euclidean Taub–NUT spacetime, with zero cosmological constant, has line element

$$\text{d}s^2 = V(r) \left[ \text{d}r + 2N \cos \theta \text{d}\phi \right]^2 + \frac{\text{d}r^2}{V(r)} + (r^2 - N^2) (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2),$$

where $V(r) = \frac{r^2 - 2Mr + N^2}{r^2 - N^2}$. (A.1)

with $N$ being the NUT parameter. Regularity of the metric requires that $\tau$ have a period $\beta = \frac{8\pi}{N}$.

A suitable tetrad of co-frames for this spacetime is given by

$$e^0 = \sqrt{V} \text{d}\tau + 2\sqrt{V} N \cos \theta \text{d}\phi, \quad e^1 = \frac{1}{\sqrt{V}} \text{d}r, \quad e^2 = \sqrt{r^2 - N^2} \text{d}\theta,$$

$$e^3 = \sqrt{r^2 - N^2} \sin \theta \text{d}\phi.$$ (A.2)

Using (10), we find that the Euclidean action for the Taub–NUT spacetime is given by

$$I = 4\pi MN = \frac{2\pi N^2}{\gamma}.$$ (A.3)

Substituting this in (18) then gives the on-shell partition function

$$\tilde{Z} = \exp \left( -4\pi MN + \frac{2\pi N^2}{\gamma} \right).$$ (A.4)

The thermodynamic quantities can now be calculated. In particular, $M = N$ for the NUT charge and substituting this into (A.4) gives the average energy and entropy

$$\langle E \rangle = N \left( 1 - \frac{1}{2\gamma} \right) \quad \text{and} \quad S = 4\pi N^2 \left( 1 - \frac{1}{2\gamma} \right).$$ (A.5)
while \( M = 5N/4 \) for the bolt charge and substituting this into (A.4) gives the average energy and entropy

\[
\langle E \rangle = \frac{5N}{4} \left( 1 - \frac{2}{5\gamma} \right) \quad \text{and} \quad S = 5\pi N^2 \left( 1 - \frac{2}{5\gamma} \right).
\]  

(A.6)

Therefore, \( \gamma \) appears in the thermodynamics of Taub–NUT and Taub–bolt solutions with zero cosmological constant as shifts in the energies and entropies of the NUT and bolt charges, just as we found for the NUT charge in the Taub–NUT–ADS solution.

Appendix B. Euler characteristic and the second law

In this appendix, let us present an example that illustrates how a topological invariant of the spacetime manifold can have non-trivial physical effects on black hole thermodynamics. In particular, we will derive an upper bound on \( \theta_2 \) that must be satisfied in order for the second law to hold when two black holes merge.

The bound presented here is general and holds for solutions to the Gauss–Bonnet field equations in arbitrary dimensions. We will consider four-dimensional asymptotically flat black holes as a special case; in this case, the black holes must satisfy certain theorems and these can be used to put a tight upper bound on \( \theta_2 \) for the area theorem to hold when two Schwarzschild black holes merge [74].

For black holes of Gauss–Bonnet gravity with the generic cosmological constant \( \Lambda \), the first law of black hole mechanics holds with an entropy that is given by [34, 65, 66]

\[
S = \frac{1}{4\pi} \oint_C \tilde{\epsilon}(1 + 2\theta_2 \mathcal{R});
\]  

(B.1)

here, \( \mathcal{R} \) is the Ricci scalar of the horizon cross section \( C \) and \( \tilde{\epsilon} \) is the area \((D-2)\)-form on \( C \).

Let us consider the merging of two black holes — one with mass \( m_1 \) and the other with mass \( m_2 \). The entropies of these black holes are (resp.)

\[
S_1 = \frac{1}{4\pi} \oint_C \tilde{\epsilon}(1 + 2\theta_2 \mathcal{R}(C_1)) \quad \text{and} \quad S_2 = \frac{1}{4\pi} \oint_C \tilde{\epsilon}(1 + 2\theta_2 \mathcal{R}(C_2));
\]  

(B.2)

here, we have defined the surface area \( \mathcal{A} \) and correction term \( X(C) \) via

\[
\mathcal{A} = \oint_C \tilde{\epsilon} \quad \text{and} \quad X(C) = \oint_C \tilde{\epsilon} \mathcal{R}.
\]  

(B.3)

Before the black holes merge, the total entropy is

\[
\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 = \frac{1}{4\pi} \left[ \mathcal{A}_1 + \mathcal{A}_2 + 2\theta_2 (X(C_1) + X(C_2)) \right].
\]  

(B.4)

After the black holes merge, the total entropy of the resulting black hole is

\[
\mathcal{S}' = \frac{1}{4\pi} \left[ \mathcal{A}' + 2\theta_2 X(C') \right].
\]  

(B.5)

The area theorem will hold if and only if \( \mathcal{S}' > \mathcal{S} \). Thus, we have the following bound:

\[
\theta_2 < \left( \frac{\mathcal{A}_1 + \mathcal{A}_2 - \mathcal{A}'}{2[X(C_1) + X(C_2) - X(C')]} \right).
\]  

(B.6)

This bound for \( \theta_2 \) is general, and holds for spacetimes in all dimensions with generic values of \( \Lambda \).

Without knowing more about the details of the black holes that are merging, nothing further can be said about bound (B.6) because the topological structure of black holes is much richer in \( D > 4 \) dimensions than in four dimensions. When \( \Lambda \geq 0 \), the topology of black holes in \( D \geq 5 \) dimensions can be any product manifold \( \mathbb{R}^2 \times \mathbb{C}^{D-2} \), with \( \mathbb{C}^{D-2} \) being a space of positive Yamabe type. For example, in five dimensions the topology of the event horizon has to be (a connected sum of) a 3-sphere \( \mathbb{C}^3 \cong S^3 \) or three-ring \( \mathbb{C}^3 \cong S^2 \times S^1 \). A complete
discussion of black holes in higher dimensions is presented in [67]; topological properties are presented in [68–73].

For concreteness, then, let us consider the merging of two non-rotating black holes in four dimensions, with \( \Lambda = 0 \). First, the Gauss–Bonnet theorem says that

\[
X(\mathbb{C}) = \oint_{\mathbb{C}} \tilde{e} R = 4\pi \chi(\mathbb{C}),
\]

with \( \chi(\mathbb{C}) \) being the Euler characteristic of \( \mathbb{C} \). Then, the Hawking topology theorem says that the horizon cross sections can only be 2-spheres so that \( \mathbb{C} \cong S^2 \), and then \( \chi(S^2) = 2 \). It follows that

\[
X(C_1) = X(C_2) = X(C') = 8\pi.
\]

Finally, the Birkhoff theorem says that the only static asymptotically flat solution to the field equations is the Schwarzschild solution. Since the surface area of a Schwarzschild black hole is related to its mass \( m \) via \( \mathcal{A} = 16\pi m^2 \), the surface areas of the initial and final black hole states are

\[
\mathcal{A}_1 = 16\pi m_1^2, \quad \mathcal{A}_2 = 16\pi m_2^2 \quad \text{and} \quad \mathcal{A}' = 16\pi (m_1 + m_2 - \alpha)^2.
\]

In the above definition for \( \mathcal{A}' \), the parameter \( \alpha \geq 0 \) has been added which corresponds to any mass that may be carried away by gravitational radiation during merging. Whence the bound on \( \theta_2 \):

\[
\theta_2 < 2m_1 m_2 - \alpha[2(m_1 + m_2) - \alpha].
\]

Therefore, in four-dimensional asymptotically flat spacetimes, the second law will be violated if \( \theta_2 \) is greater than twice the product of the masses of two Schwarzschild black holes before merging minus a correction due to gravitational radiation. This is an important property because it shows that a non-zero Euler characteristic of the manifold with boundary can have physical effects in four dimensions.

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