Quantization of Lorentzian 3D gravity by partial gauge fixing

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Received 1 May 2012, in final form 8 June 2012
Published 10 July 2012
Online at stacks.iop.org/CQG/29/155011

Abstract

$D = 2 + 1$ gravity with a cosmological constant $\Lambda$ has been shown by Bonzom and Livine to present a Barbero–Immirzi-like ambiguity depending on a parameter $\gamma$. We make use of this fact to show that, for $\Lambda > 0$, the Lorentzian theory can be partially gauge fixed and reduced to an SU(2) Chern–Simons theory. We then review the already known quantization of the latter in the framework of loop quantization for the case of space being topologically a cylinder. We finally construct, in the same setting, a quantum observable which, although non-trivial at the quantum level, corresponds to a null classical quantity.

PACS numbers: 11.10.Kk, 11.15.Tk, 04.60.Ds, 04.60.Pp

1. Introduction

Since the discovery by Achucarro and Townsend [1] and the elaborated work of Witten [2], it is a sort of common sense to affirm that 3D gravity and Chern–Simons (CS) gauge theory are equivalent, up to a total derivative (boundary terms), with the Poincaré group as the underlying gauge group. However, the difference between the two theories is that in 3D gravity the triad is restricted to being invertible, whereas no such restriction exists in CS theory. Thus, we can think about the CS theory as an extension of 3D gravity including singular metrics or, alternatively, think about 3D gravity as a restricted version of the CS theory [2]. Questions regarding the role of a non-invertible triad is far from being trivial [3].

A richer structure emerges by enlarging the local symmetry to the (anti)-de Sitter group. In this case, besides the standard action for 3D gravity, it is possible to construct an ‘exotic’ action equivalent to the former at the level of classical field equations. This peculiarity of CS gravity
was not unnoticed in Witten’s original paper. This leads to an analogy with a well-established feature of loop quantum gravity (LQG) known as the Barbero–Immirzi parameter ambiguity [4, 5]. This analogy was studied in detail in [6] and represents the principal motivation of this work.

The introduction of the Ashtekar variables within the formalism of canonical gravity in 4D spacetime [7, 4, 8] was a big step into the simplification of the constraints. The simplicity was achieved thanks to the introduction of a complex phase space that turns out to be problematic at the moment of imposing reality conditions. A self-dual Lagrangian density corresponding to Ashtekar’s Hamiltonian was given independently by Samuel [9] and Jacobson and Smolin [10]. Then Barbero pointed out [5] the possibility of a real realization of canonical variables closely related to Ashtekar’s, but the price to be paid is the introduction of an arbitrary real/complex parameter \( \gamma \) known as the Barbero–Immirzi parameter. Also, the simplicity of Ashtekar’s Hamiltonian constraints is destroyed. Finally, Holst introduced [11] the most general action integrating all the previous cases, which contains a (on-shell) topological term coupled via the \( \gamma \) parameter. The theory is then interpreted as a class of actions for gravity classically indistinguishable but quantum mechanically inequivalent. Although the introduction of Ashtekar variables fits naturally in 2+1 gravity, it remained up to now unknown if there exists a corresponding Ashtekar–Barbero-like real connection defined on a compact Lie group, e.g. SU(2).

Bonzom and Livine [6] proposed a quantization of Euclidean 3D gravity with a cosmological constant based on the presence of two invariant actions, hence of two independent parameters: the gravitation constant and a Barbero–Immirzi-like parameter. In this paper we elaborate on this model, considering Lorentzian 3D gravity with a cosmological constant \( \Lambda \) as well. In the case of a positive \( \Lambda \), where the gauge symmetry is de Sitter SO(3,1), we succeed to quantize the theory in the LQG framework, thanks to a gauge fixing which reduces the gauge invariance group to the compact SO(3) or SU(2), for which a LQG-like quantization of SU(2) CS theory was developed in [12], in the case of a space manifold with the topology of a cylinder. We also construct a physical observable which has a certain analogy with the area operator of \( D = 3 + 1 \) gravity, although the former does not correspond to any classical geometrical quantity.

In the LQG scheme, the compactness of the gauge group—or of the residual gauge group after a partial gauge fixing—is known to be crucial, at least up to now. Apart from Riemannian gravity, this is achieved for Lorentzian \( D = 3 + 1 \) gravity in the time gauge [4], which holds only in four dimensions. An important development was brought by the authors of [13], who were able to reduce the gauge group of Lorentzian gravity in arbitrary dimension \( d + 1 \) to a compact group SO(d)—but this works only in the Hamiltonian formalism. Our case is different in three aspects: in three dimensions one does not need their simplicity constraints (see the second paper of [13]), we rely strongly on the existence of the Barbero–Immirzi-like parameter of Bonzom and Livine, a feature very peculiar to that dimension, and moreover we start from an existing Lagrangian formalism.

The plan of this paper is as follows. In the following section, which is mainly a review of results from [6], the basic tools to cast 3D gravity, with (or without) a cosmological constant, as a CS gauge theory will be presented, and the appearance of the Barbero–Immirzi-like parameter will be explained. In sections 3 and 4, the main argument of this paper will be presented, in which we consider the positive cosmological constant model and its gauge fixing. The detailed canonical analysis of the constrained theory will be developed. The construction of an observable and the computation of its spectrum are presented in section 5, where also its classical counterpart is discussed. The last section is reserved for some comments about the
gauge fixing reduction and quantization of the model in the spirit of LQG, as well as for the conclusions of the work.

2. Gravity from Chern–Simons theory with Barbero–Immirzi ambiguity

To begin with, let $\mathcal{M}$ be an orientable three-dimensional manifold. In addition, let $G$ be the gauge group and $g$ its Lie algebra equipped with a non-degenerate invariant quadratic form $\langle \cdot, \cdot \rangle$.

The CS action is defined as

$$S = -\frac{\kappa}{2} \int_{\mathcal{M}} \left( \Omega, d\Omega + \frac{2}{3} \Omega \right),$$

(2.1)

where $\Omega = \Omega_\mu dx^\mu$ is a $g$-valued 1-form connection and $\kappa$ is a dimensionless constant. The field equations read $F_\Omega = 0$, where $F_\Omega = d\Omega + \Omega \wedge \Omega$ is the field strength 2-form; this means that the CS theory is a topological theory with no truly propagating degrees of freedom. By construction, the CS action is diffeomorphism invariant.

Next we assume $\mathcal{M}$ to be of topology $\Sigma \times \mathbb{R}$, where $\Sigma$ is a two-dimensional manifold representing physical 2D space, and $\mathbb{R}$ represents time. The Hamiltonian formalism can be achieved by splitting the connection into its temporal and spatial components: $\Omega = \Omega_t dt + \Omega_\omega dx^\omega$. Replacing this into (2.1) the action can be written as

$$S = -\frac{\kappa}{2} \int_{\mathbb{R}} dt \int_{\Sigma} \langle \hat{\Omega}_t, \Omega_t \rangle + 2 \langle \Omega_\omega, F_\Omega \rangle,$$

(2.2)

where $\Omega = \Omega_\omega dx^\omega$ is the spatial connection and $F_\Omega = d\Omega + \Omega \wedge \Omega$ is the associated spatial field strength. The dot means time derivative.

Three-dimensional gravity meets CS theory when we choose as underlying gauge symmetry the Poincaré group $ISO(2,1)$. In this case, the connection is written as

$$\Omega = e^I P_I + \omega^I J_I,$$

$$e^I = e^I_\mu dx^\mu = e^I + e^I_\mu dt, \quad e^I = e^I_\mu dx^\mu,$$

$$\omega^I = \omega^I_\mu dx^\mu = \omega^I + \omega^I_\mu dt, \quad \omega^I = \omega^I_\mu dx^\mu,$$

(2.3)

where $e$ and $\omega$ are the spacetime co-frame field and spin connection, respectively, $e$ and $\omega$ are their spatial counterparts, whereas $P_I$ and $J_I$ correspond to the generators of translations and rotations of $ISO(2,1)$, respectively. We can go further and include a positive (negative) cosmological constant $\Lambda$ deforming the gauge symmetry to the (anti)-de Sitter $SO(3,1)$ ($SO(2,2)$) group. In any case, the generators will satisfy the general Lie algebra given by $[J_I, J_J] = \varepsilon_{IJK} J_K$, $[J_I, P_J] = \varepsilon_{IJK} P_K$ and $[P_I, P_J] = \sigma \Lambda \varepsilon_{IJK} J_K$.

A special feature of the $SO(3,1)$ group is the possibility of defining two non-degenerate quadratic forms in the algebra. They correspond to the two Casimir invariants

$$C_1 = \eta^{IJ} P_I J_J \quad \text{and} \quad C_2 = \eta^{IJ} \left( \frac{\sigma}{\Lambda} P_I P_J + J_I J_J \right).$$

(2.4)

We can associate with each Casimir a corresponding inner product:

$$\langle P_I, J_J \rangle_1 = \eta_{IJ}, \quad \langle P_I, P_J \rangle_1 = 0, \quad \langle J_I, J_J \rangle_1 = 0,$$

$$\langle P_I, J_J \rangle_2 = 0, \quad \langle P_I, P_J \rangle_2 = \sigma \Lambda \eta_{IJ}, \quad \langle J_I, J_J \rangle_2 = \eta_{IJ}.$$

2 We do not write explicitly the wedge symbol $\wedge$ for the external product of forms.

3 In what follows the Greek indices $\mu, \nu, \ldots$ run from 0 to 2 and the Latin indices from the beginning of the alphabets $a, b, \ldots$ (space indices) take values 1, 2 or, later on, $x, y$. Three-dimensional Lorentz frame indices are denoted by the Latin capital letters $I, J, \ldots$ running from 0 to 2. Our convention for the tangent spacetime metric is $\eta_{IJ} = \text{diag}(\sigma, 1, 1)$, where $\sigma = \pm 1$ allows us to switch between the Euclidean and Lorentzian cases, respectively. The indices $I, J, \ldots$ are raised and lowered with the metric $\eta_{IJ}$. 

3
The inner product defined by $C_1$ is non-degenerate for all $\Lambda$, whereas $C_2$ only for $\Lambda \neq 0$.

Starting with the general action (2.1), we can write one action for each inner product,

\[ S_1 = -\frac{\kappa}{2} \int \eta \left[ \epsilon^i \epsilon^j + \omega^i \epsilon^j + 2e_i^j (R + \sigma \Lambda e^2)^j + 2\omega^i T^j \right], \]

\[ S_2 = -\frac{\kappa}{2} \int \eta \left[ \epsilon^i \epsilon^j + \omega^i \omega^j + 2\sigma \Lambda e^i T^j + 2\omega^i (R + \sigma \Lambda e^2)^j \right], \]  

where $R^i = d\omega^i + \frac{1}{2} \epsilon^{ijk} \omega^j \omega^k$ is the spatial curvature and $T^i = d\epsilon^i + \epsilon^{ijk} \omega^j \omega^k$ the spatial torsion. The standard deviation for 3D gravity can be recognized in $S_1$. On the other hand, $S_2$ can be considered a kind of ‘exotic’ 3D gravity in the sense that, despite having a different action, it shares the same field equations. It makes perfect sense to add the exotic action, with an arbitrary coefficient $\gamma$, to the standard action, so the most general action is

\[ S = S_1 - \frac{1}{\gamma} S_2. \]  

(2.6)

It is worth noting that the equivalence can be established only at the level of the equations of motions, but this is not true at the level of the symplectic structure of the phase space, as can be seen from the canonical Poisson brackets deduced from the kinetic part of the action, which will be displayed in section 3 after some change of variables. This intriguing model was studied in detail within the context of LQG in [6], where the appearance of $\gamma$ is compared with the arbitrariness of the Barbero–Immirzi parameter [4].

This completes our brief review of the CS formulation of 3D gravity and the origin of the $\gamma$ parameter.4

### 3. Chern–Simons gravity with the positive cosmological constant

In what follows we will restrict the model to the $\Lambda > 0$ sector. (The case of the negative cosmological constant can be constructed analogously.) Also, we have kept open the possibility of switching between the Euclidean and Lorentzian theories by introducing the parameter $\sigma = \pm 1$, so the gauge group would be $SO(4)$ or $SO(3,1)$, respectively.

Let us start by writing the connection as $\Omega = A^i L_i + B^j K_j$, where $L_i$ are the generators of ‘rotations’ and $K_j$ are the generators of ‘boosts’ of the $SO(3,1)$ ($SO(4)$) group, i.e. in the $SO(3,1)$ case, of its compact subgroup $SO(3)$ and of its non-compact directions, respectively.

These generators satisfy the Lie algebra commutation rules

\[ [L_i, L_j] = \varepsilon_{ijk} L_k, \quad [L_i, K_j] = \varepsilon_{ijk} K_k, \quad [K_i, K_j] = \sigma \varepsilon_{ijk} L_k. \]

Here $i$ and $j$ take values 1, 2 and 3 and are raised or lowered with the delta Kronecker $\delta_{ij}$. Observe that $A^i$ are recognized as the components of an $SO(3)$ (or $SU(2)$) connection.

Relations between the old and new generators and variables read

\[ L = (P_i/\sqrt{\Lambda}, -P_i/\sqrt{\Lambda}, \sigma J_0), \quad (A^i, i = 1, 2, 3) = (\sqrt{\Lambda} e^3, -\sqrt{\Lambda} e^1, \sigma e^0); \]  

(3.1a)

\[ K = (J_2, -J_1, P_0/\sqrt{\Lambda}), \quad (B^i, i = 1, 2, 3) = (\omega^2, -\omega^1, \sqrt{\Lambda} e^0). \]  

(3.1b)

As before, the non-degenerate invariant quadratic forms in the algebra are given by the two Casimir of the group, but this time written in the new basis5:

\[ \{\Omega, \Omega\}'_1 = A \cdot B' + B \cdot A' \quad \text{and} \quad \{\Omega, \Omega\}'_2 = A \cdot A' + \sigma B \cdot B'. \]

4 Our notations differ slightly from those of [6]. One recovers the latter from ours by the substitutions $\sigma \rightarrow 1$, $\gamma \rightarrow -\gamma/\sqrt{\Lambda}$ and $\kappa \rightarrow -\kappa$, where $\Lambda = s|\Lambda|$ is the cosmological constant with $s = -1, 0, 1$.

5 Since all group indices are contracted with the three-dimensional metric $\delta_{ij}$, it is convenient to adopt a vector-like notation, e.g., $A'B_i = A \cdot B$, $\varepsilon_{ijk} A'B^k = (A \times B)_i$, etc.
With all this in mind, the general action (2.6) can be written as
\[ S = -\frac{\kappa}{2} \int_{\mathbb{R}} dt \left( \int_{\Sigma} \left( A \cdot \left( B - \frac{1}{\gamma} A \right) + B \cdot \left( A - \frac{\sigma}{\gamma} B \right) \right) - \mathcal{G}(A_t) - \mathcal{G}_0(B_t) \right), \tag{3.2} \]
where we are defining the smeared quantities
\[ \mathcal{G}(A_t) = \kappa \int_{\Sigma} A_t \cdot \left[ DB - \frac{1}{\gamma} (F_A + \frac{\sigma}{2} B \times B) \right], \tag{3.3a} \]
\[ \mathcal{G}_0(B_t) = \kappa \int_{\Sigma} B_t \cdot \left[ F_A + \frac{\sigma}{2} B \times B - \frac{\sigma}{\gamma} DB \right]. \tag{3.3b} \]

One readily sees that the theory defined by (3.2) is fully constrained. The conjugate momenta \( \Pi^i_A \) and \( \Pi^j_B \) of \( A^i_t \) and \( B^j_t \) are primary constraints, in Dirac’s terminology [14], whereas \( \mathcal{G}(A_t) \) and \( \mathcal{G}_0(B_t) \) are the secondary constraints, with \( A^i_t \) and \( B^j_t \) playing the role of Lagrange multipliers. Other primary constraints involve the conjugate momenta of the fields \( A^i_t \) and \( B^j_t \). They turn out to be of second class, whose solution according to the Dirac–Bergmann algorithm [14] gives rise to the Dirac–Poisson brackets

\[ \{A^i_t(x), A^j_t(x')\} = \frac{1}{\kappa} \epsilon_{a b} \delta^{i j} \frac{\sigma \gamma}{\sigma - \gamma^2} \delta^2(x - x'), \]
\[ \{B^i_t(x), A^j_t(x')\} = \frac{1}{\kappa} \epsilon_{a b} \delta^{i j} \frac{\gamma^2}{\sigma - \gamma^2} \delta^2(x - x'), \]
\[ \{B^i_t(x), B^j_t(x')\} = \frac{1}{\kappa} \epsilon_{a b} \delta^{i j} \frac{\gamma}{\sigma - \gamma^2} \delta^2(x - x'). \tag{3.4} \]

From these relations, we see that the 12 component of the fields \( A \) and \( B \) divide in 6 configuration fields and 6 momentum fields. It is important to mention that the gauge fields \( A \) and \( B \) are composed of mixed co-frame fields and spin connections (see (3.1)).

Performing the Legendre transformation, we obtain a fully constrained classical Hamiltonian, namely
\[ H = \mathcal{G}(A_t) + \mathcal{G}_0(B_t), \tag{3.5} \]
with \( A_t \) and \( B_t \) being Lagrange multipliers. The algebra of the constraints closes under Poisson brackets, and adopts the form
\[ \{\mathcal{G}(\epsilon), \mathcal{G}(\epsilon')\} = \mathcal{G}(\epsilon \times \epsilon'), \]
\[ \{\mathcal{G}_0(\epsilon), \mathcal{G}(\epsilon')\} = \mathcal{G}_0(\epsilon \times \epsilon'), \]
\[ \{\mathcal{G}_0(\epsilon), \mathcal{G}_0(\epsilon')\} = \sigma \mathcal{G}(\epsilon \times \epsilon'). \tag{3.6} \]

We can recognize here the structure of the \( \mathfrak{so}(3, 1) \) (\( \mathfrak{so}(4) \)) Lie algebra, in total agreement with the fact that in the Dirac–Bergmann formalism for constrained systems, first class constraints generate local gauge transformations. The infinitesimal gauge transformations generated by the constraints are
\[ \{\mathcal{G}(\epsilon), A\} = DB, \quad \{\mathcal{G}(\epsilon), B\} = B \times \epsilon; \]
\[ \{\mathcal{G}_0(\epsilon'), A\} = \sigma B \times \epsilon', \quad \{\mathcal{G}_0(\epsilon'), B\} = D \epsilon' \quad (D = d + A \times ). \tag{3.7} \]

These gauge transformations are related, on-shell, to local diffeomorphisms. This can be shown if we apply the Lie derivative to the gauge fields
\[ \mathcal{L}_A A = (d t_A A) + \sigma B \times (t_B A) + \text{field equations}, \]
\[ \mathcal{L}_B B = B \times (t_A A) + D(t_B A) + \text{field equations}. \tag{3.8} \]

6 We recall that, here as in equation (2.5), boldface letters represent space objects (2-forms, etc).
with $\ell = \delta \phi + t_\xi \delta$ being the Lie derivative. By comparison with (3.7) we identify in (3.8) infinitesimal gauge transformations with parameters $(\epsilon, \epsilon') = (t_\xi A, t_\xi B)$, up to field equations.

### 4. Axial gauge

We want to partially fix the gauge in such a way that the residual gauge symmetry group be compact, namely $\text{SO}(3)$ in the $\text{SO}(3,1)$ case to which we restrict hereafter. We shall verify that the gauge fixing condition, $B^i_j \approx 0$, is of first class and is compatible with the Dirac procedure (‘Dirac compatible’ according to the terminology of [15]). We shall check firstly that it does indeed involve only the phase-space variables and leads to the presence of second class constraints whose number is twice that of the gauge conditions, and secondly that it will reduce the gauge symmetry group in the desired way.

To proceed with the gauge fixing, let us rewrite the Hamiltonian (3.5), adding the gauge condition as a constraint

$$
H = G(A_t) + G_0(B_t) + \int_{\Sigma} d^2 x \mu(x) B^i_j(x).
$$

It turns out that the gauge fixing constraint together with the constraint $G_0$ given by (3.3b) is of second class: indeed the matrix of their Poisson brackets,

$$
C(x, y) = \begin{pmatrix}
G_0(x), G_0^{(1)}(x') & G_0(x), B^0_j(x') \\
B^0_j(x), G_0^{(1)}(x') & B^0_j(x), B^0_j(x')
\end{pmatrix}
$$

is (weakly) non-singular. Following the Dirac–Bergmann prescriptions, we introduce the Dirac bracket

$$
\{M, N\}_D = \{M, N\} - \{M, \chi_\alpha\} C^{-1} \beta^{(1)} \chi_\beta, N,
$$

where $M$ and $N$ are two phase-space functions, $\chi_\alpha, \alpha = 1, 2$ are the two second class constraints and $C^{-1}$ is the inverse—in the convolution sense—of the matrix (4.2). The Dirac brackets of the second class constraints with every phase-space function are zero by construction. Those of the fields $A^i_j$ are equal to their Poisson brackets (3.4):

$$
\{A^i_j(x), A^j_i(x')\}_D = \frac{1}{\kappa} \frac{\sigma \gamma}{\sigma - \gamma^2} \delta^2(x - x'),
$$

whereas those involving the remaining field $B^i_j$ are different. We shall however not write down the latter, since this field is not an independent variable. Indeed, the second class constraints are now considered as strong equalities: in particular, the constraint $G_0$ yields the equation

$$
\partial_\gamma A_x - D_\gamma \left( A_x - \frac{\sigma}{\gamma} B_x \right) = 0,
$$

which can be solved for $B^i_x$ as a functional of $A^i_x$ and $A^j_y$.

At this stage, we are left with one set of first class constraints, $G^{(1)}(x) \approx 0$. Let us now define the new variables

$$
A_x = A_x - \gamma B_x, \quad A_y = A_y.
$$
Using the Dirac brackets (4.3) we can check that they form a canonical pair of conjugate variables:

\[
\{A_i^a(x), A_j^b(x')\}_D = \frac{\gamma}{\kappa} \delta^{ij} \delta^2(x - x').
\]  

(4.5)

The Hamiltonian now reads

\[
H = -\frac{\kappa}{\gamma} \mathcal{G}(A_i),
\]  

(4.6)

with the first class constraint \(\mathcal{G}\) given by

\[
\mathcal{G}(\eta) = \int_{\Sigma} d^2 x \eta_i(x) \mathcal{F}^i \approx 0,
\]  

(4.7)

where we have introduced the curvature 2-form associated with \(A\):

\[
\mathcal{F}^i = \partial_x A_i^a - \partial_y A_i^a + \epsilon^{ijk} A_j^a A_k^b.
\]  

(4.8)

One recognizes in the Hamiltonian (4.6) of the CS theory for the connection \(A\), the latter transforming as an SO(3) or SU(2) connection:

\[
\{\mathcal{G}(\eta), A_i^a\}_D = \partial_a \eta_i + \epsilon^{ijk} \eta_j A_k^b,
\]  

(4.9)

under the infinitesimal gauge transformations generated by the constraint \(\mathcal{G}\).

Our gauge fixing has thus the effect of reducing the original gauge symmetry from SO(3,1) to SO(3) or SU(2), so we end up with the CS theory with a compact gauge group. This resembles the Ashtekar–Barbero variables formalism in 3+1 dimensions. In fact, the new variables defined here can be considered as the 2+1 analogous of the Ashtekar–Barbero variables. The role of the present axial gauge can be compared with the temporal gauge. In both cases the non-compact sector of the theory is frozen.

It is worth noting that the gauge fixing condition \(B_x = 0\) also fixes part of the spatial diffeomorphism invariance, leaving as residual invariance the diffeomorphisms generated by the vectors \(\xi = (\xi^i, \xi^r)\) with \(\xi^r\) being independent of \(y\). Indeed, if \(B_x = 0\), then \(\xi^r B_x = 0\), which shows that this gauge fixing is stable only if \(\partial_y \xi^r = 0\). However, the resulting CS theory in terms of the new connection \(A\) (4.4) is again fully diffeomorphism invariant as a consequence of its gauge invariance under the gauge transformations (4.9), as is well known [2].

From this point, one can exploit several known approaches for quantizing the CS theory with an SO(3) or SU(2) gauge group. For example, following the spirit of the canonical quantization, we can mention the work of Dunne, Jackiw and Trugenberger [16], or [12] for a LQG inspired treatment. The latter is summarized in the following section.

5. Quantization and observables

5.1. Quantum theory

In view of the result of the last section, we make here a quick review of the quantization of the CS theory with the gauge group \(G = \text{SU}(2)\) on a time-oriented 3-manifold, \(\mathcal{M} = \mathbb{R} \times \Sigma\). This subsection essentially follows [12].

The canonical variables are defined as operators satisfying, in correspondence with the Dirac brackets (4.5), the commutation rules

\[
[\hat{A}_i^a(x), \hat{A}_j^b(x')] = \frac{i\gamma}{\kappa} \delta^{ij} \delta^2(x - x'),
\]  

(5.1)

where \(i, j = 1, 2, 3\) are the gauge group indices.
We then choose a polarization such that \( \hat{A}_i \) is multiplicative and \( \hat{A}_f \) is a functional derivative\(^7\) acting on wave functionals \( \Psi[A_i] = \langle A_i|\Psi \rangle \):

\[
\hat{A}_i \Psi[A_i] = A_i \Psi[A_i]; \quad \hat{A}_f \Psi[A_i] = \frac{\gamma}{i \kappa} \frac{\delta}{\delta A_i} \Psi[A_i].
\] (5.2)

In this representation, the Gauss constraint is written as

\[
\left[ i \left( \frac{\delta}{\delta A_i} + f^{ij}_k A^j_k \frac{\delta}{\delta A^i_k} \right) + \frac{\kappa}{\gamma} \partial_i A_i \right] \Psi[A_i] = 0.
\] (5.3)

and a particular solution is given by \([16]\)

\[
\Psi_0[A_i] = \exp(2\pi i \alpha_0).
\] (5.4)

With

\[
\alpha_0 = \frac{\kappa}{6\pi \gamma} \int_{\Sigma} \epsilon^{\mu\nu\rho} \text{Tr}(h^{-1} \partial_\mu h h^{-1} \partial_\nu h h^{-1} \partial_\rho h) \, d^3x - \frac{\kappa}{2\pi \gamma} \int_{\Sigma} \text{Tr}(A_i h^{-1} \partial_i h),
\] (5.5)

The first term is the Wess–Zumino–Witten action, and it is an integer since the group is non-Abelian and compact, which requires that \( \kappa \) must be quantized, \( \kappa = \nu/4\pi, \nu \in \mathbb{Z} \), and \( h \in G \) is defined as a functional of \( A_i \) by

\[
A_i = h^{-1} \partial_i h.
\] (5.6)

It can be shown that taking into account the particular solution (5.4), the general wave functional solution of (5.3) can be written as

\[
\Psi[A_i] = \Psi_0[A_i] \psi^{im}[A_i],
\] (5.7)

where \( \psi^{im}[A_i] \) satisfies

\[
\left[ i \left( \frac{\delta}{\delta A_i} + f^{ij}_k A^j_k \frac{\delta}{\delta A^i_k} \right) \right] \psi^{im}[A_i] = 0.
\] (5.8)

The latter equation means that \( \psi^{im} \) is invariant under the infinitesimal \( x \)-gauge transformations

\[
\delta_{(x)} A^i_k = D_x \epsilon^i.
\] (5.9)

At this point, one can choose to change the focus from the functionals of the connection \( A_i \), which does not transform homogeneously under the gauge transformations (as shown above), to functionals of the holonomies of \( A_i \) over a path \( c_y = [x_1, x_2] \) with constant \( y \) on \( \Sigma \), namely

\[
U(c_y, x_1, x_2) = P e^{\int_{c_y} A^i_k(x, y) \gamma^i dx},
\] (5.10)

whose transformations under the action of the gauge group are

\[
U(c_y, x_1, x_2) \mapsto g^{-1}(y, x_2) U(c_y, x_1, x_2) g(y, x_1).
\] (5.11)

Once these coordinates are themselves elements of the group, they turn out to be more suitable for constructing a Hilbert space with a well-defined scalar product. Taking into account the considerations above we construct the cylindrical space \( \text{Cyl} \) whose elements are functionals (see (5.7))

\[
\Psi_{\Gamma, f}[A_i] = \Psi_0[A_i] \psi^{im}_{\Gamma, f}[A_i],
\]

with

\[
\psi^{im}_{\Gamma, f}[A_i] = f(U(c_{y_1}, x_1, x'_1), \ldots, U(c_{y_k}, x_k, x'_k)),
\]

where \( \Gamma = \{ c_{y_1}, x_k, x'_k \}, k = 1, \ldots, K \) is a graph defined as a finite set of \( K \) \( y \)-constant paths on \( \Sigma \), and \( f \) is a function on \( SU(2)^N \) with complex values. Such a state is denoted by \( |\Gamma, f \rangle \).

\(^7\) We use either the wave functional representation or the abstract Dirac’s kets. The relation between both is given by \( \Psi_\alpha[A_i] = \langle A_i|\alpha \rangle \), where \( \alpha \) may represent the quantum numbers defining the state.
Since the wave functionals are written in terms of a finite number of holonomies, which are group elements, the Haar measure \( dU_k \) may be used to define the scalar product in Cyl:

\[
\langle \Gamma, f|\Gamma', f' \rangle = \int \prod_{k=1}^K dU_k f(U_1, \ldots, U_K) f'(U_1, \ldots, U_K). \tag{5.12}
\]

The kinematical Hilbert space \( \mathcal{H}_{\text{kin}} \) is then defined as the Cauchy completion of Cyl. Making use of the Peter–Weyl theorem, one finds a basis \( |\Gamma, j, \vec{\alpha}, \vec{\beta} \rangle \), with \( j = j_1, \ldots, j_K \), etc for \( \mathcal{H}_{\text{kin}} \):

\[
\Psi_{\Gamma, j, \vec{\alpha}, \vec{\beta}} = \Psi_0[A_{\Gamma}] \prod_{k=1}^K R^{h_k, h_k}_0(U(c_{j_k}, x_{j_k})), \tag{5.13}
\]

where \( R^{h_\alpha, h_\beta}_0(h) \) denote the \((\alpha, \beta)\) matrix element of the spin \( j \) representation of the holonomy. Note that one excludes the value \( j = 0 \) and completes the basis with the ‘null vector’ \(|0\rangle\), corresponding to the empty graph. These vectors form an orthogonal basis:

\[
\langle \Gamma, j, \vec{\alpha}, \vec{\beta}|\Gamma', j', \vec{\alpha}', \vec{\beta}' \rangle = \delta_{\Gamma, \Gamma'} \delta_{j, j'} \delta_{\vec{\alpha}, \vec{\alpha}'} \delta_{\vec{\beta}, \vec{\beta}'}. \tag{5.14}
\]

Thus, with every path \((c_{j_k}, x_{j_k}, x'_{j_k})\) of the graph \( \Gamma \) we associate a spin \( j_k \) representation of SU(2). Vectors associated with different graphs are orthogonal. Observe also that the kinematical Hilbert space is non-separable: it is the direct sum \( \bigoplus_{\Gamma} \mathcal{H}_{\Gamma} \), over all graphs \( \Gamma \), where \( \mathcal{H}_\Gamma \) is the separable Hilbert space associated with the graph \( \Gamma \). The expansion of a vector \(|\Psi\rangle \in \mathcal{H}_{\text{kin}} \) reads

\[
|\Psi\rangle = \sum_{\Gamma, j, \vec{\alpha}, \vec{\beta}} c_{\Gamma, j, \vec{\alpha}, \vec{\beta}} |\Gamma, j, \vec{\alpha}, \vec{\beta}\rangle ,
\]

where the sum over \( \Gamma \) covers a countable subset of graphs.

Before implementing the Gauss constraint (5.3) we will specify the spatial two-dimensional manifold \( \Sigma \) as being an infinite cylinder, and take \( x \) as the periodic coordinate. This choice allows us to impose the constraint in the form of the invariance of \( \psi^{\text{inv}} \) under all finite \( x \)-gauche transformations, implying, as can be easily realized from (5.11), that the functions should be reduced to the traces of the holonomies along closed paths (cycles, or Wilson loops—\( y \)-constant section of the cylinder) \( U_y \equiv \text{Tr}(U(c_y, x_1, x_1)) \), which depend on the \( y \) coordinate, but not on \( x \), after identifying the endpoints \( x_1 \) and \( x_2 \). Thus, each cycle is characterized by its ‘height’, and the graphs are now sets \( C \) of cycles. This defines the Hilbert space \( \mathcal{H}_{\text{Gauss}} \), whose basis is the orthonormal set of ‘spin network’ vectors \(|C, \vec{j}\rangle \), given by

\[
\Psi_{C, \vec{j}}[A_{\Gamma}] = \Psi_0[A_{\Gamma}] \prod_{k=1}^K \chi^{h_k}(U_{c_{j_k}}), \quad \text{with} \quad \chi^{h}(U_y) = \text{Tr} R^{h}(U_y), \tag{5.15}
\]

where \( \vec{j} \) stands for \((j_1, \ldots, j_K)\). These vectors are orthonormal, in the sense

\[
\langle C, \vec{j}|C', \vec{j}' \rangle = \delta_{C, C'} \delta_{\vec{j}, \vec{j}'}. \tag{5.16}
\]

Now we consider \( \Sigma \), the space of all finite linear combinations of spin networks, with \( \mathcal{H}_{\text{Gauss}} \) being its Cauchy completion. It is the direct sum \( \bigoplus_{C} \mathcal{H}^C_{\text{Gauss}} \), where \( \mathcal{H}^C_{\text{Gauss}} \) is the Hilbert space associated with a graph \( C \), which is separable. This is not the case for \( \mathcal{H}_{\text{Gauss}} \), since the graphs are indexed by finite arrays of real numbers.

Since \(|C, \vec{j}\rangle \) depends on the \( y \) coordinate, \( \mathcal{H}_{\text{Gauss}} \) is not diffeomorphism invariant. The local invariance represented by the \( y \)-diffeomorphisms was not contemplated when we solved the Gauss constraint. Indeed, a diffeomorphism along the \( y \) coordinate generated by a vector field \( \xi = (0, \xi^y) \) acts on the configuration variable \( A_{\Gamma} \), as \( \xi^y A_{\Gamma} = D_y(\xi^y A_{\Gamma}) \), i.e. as a gauge transformation with parameter \( \xi^y A_{\Gamma} \) [12]. However, when applied to a wave functional, \( A_{\Gamma} \), must be replaced by the operator defined in (5.2). Such a ‘gauge transformation’ was not
contemplated when we solved the Gauss constraint. Therefore, we still have to implement this part of diffeomorphism invariance, namely invariance under the $y$-diffeomorphisms $y' = y'(x)$, $x' = x$. The invariance under the more general diffeomorphisms generated by vectors $\xi = (\xi^r, \xi^s)$ will then be obvious in view of the manifestly diffeomorphism invariant result. In order to show the $y$-diffeomorphism invariance, we use the group averaging method (see [4]), based on the Gel’fand triple $S_\circ \subset \mathcal{H}_{\text{Gauss}} \subset S'_\circ$, with $S'_\circ$ being the dual of the spin-network space $S_\circ$. The $y$-diffeomorphism invariant states are shown to be the elements of this dual space constructed from any spin-network state through the application of a functional ‘projector’ $P_{\text{diff}} : S_\circ \to S'_\circ$ defined by

$$
\langle P_{\text{diff}}^\dagger \Phi, \Phi \rangle = \sum_{\psi''} \langle \psi'' | \psi \rangle, \quad \forall | \psi \rangle \in S_\circ,
$$

where the sum is done over all vectors $| \psi \rangle$ obtained from $| \Psi \rangle$ by a $y$-diffeomorphism.

The linear forms $\Phi = P_{\text{diff}}^\dagger \Phi$ span the physical Hilbert space $\mathcal{H}_{\text{phys}}$, with an interior product induced from that of $\mathcal{H}_{\text{Gauss}}$ [4, 12]:

$$
\langle \Phi_1 | \Phi_2 \rangle = \langle \Phi_1, \Phi_2 \rangle = \langle P_{\text{diff}}^\dagger \Phi_1, \Phi_2 \rangle.
$$

The vectors of $\mathcal{H}_{\text{phys}}$ only depend on the equivalence classes of spin-network states under $y$-diffeomorphisms. In particular, a state defined as explained above from $| C, \vec{j} \rangle$ does not depend on the particular positions $y_\xi$ of the cycles, but only on the number of such cycles and on the spin value associated with each of them. We have then the s-knot states $| \vec{j} \rangle \equiv | j_1, \ldots, j_k \rangle = P_{\text{diff}}^\dagger | C, \vec{j} \rangle$, which form an orthonormal basis for the physical Hilbert space. They are the solutions of the Gauss constraint and are invariant under all diffeomorphisms. Once the set of s-knots is countable, the physical Hilbert space is separable.

5.2. Observable

Following the steps of what is done in LQG for defining the area operator, we look for an operator which will be diagonal in the spin $s$-knot basis of the physical Hilbert space $\mathcal{H}_{\text{phys}}$.

We begin with its construction in the Hilbert space $\mathcal{H}_{\text{Gauss}}$. We start by defining an operator $\hat{W}_j$ such that it acts on the wave functionals (5.7) in the following way:

$$
\hat{W}_j \psi[A] = \psi_0[A] \hat{A}_j^\dagger \psi | A \rangle.
$$

(5.19)

To find the explicit form of this operator, let us assume that we can split it into two terms which basically separate the canonical variable dependence from its conjugate momenta, i.e.

$$
\hat{W}_j = \hat{A}_j^\dagger + \hat{A}_j,
$$

(5.20)

with $\hat{\lambda}$ being a functional of the configuration variables to be defined. Combining this with (5.19), we obtain $\hat{\lambda} \psi_0 = -\hat{A}_j \psi_0$. One shows easily from the definition of $\psi_0$ that $\hat{A}_j \psi_0 = (h^{-1} \partial_j h) \psi_0$, where $h = h[A_A]$ is the nonlocal functional (5.6) [16]. With this result we can finally write

$$
\hat{W}_j = \hat{A}_j - h^{-1} \partial_j h.
$$

(5.21)

The advantage to work with $\hat{W}_j$ is that this operator transforms as an SU(2) vector because it is defined as the difference of two objects that transform like connections, so it is a good object with which we may construct gauge invariants observables.

In order to define $\hat{W}_j$, it is sufficient to define its action on the particular spin $j$ wave functional

$$
\psi^{(j)}(\alpha, A) = \psi_0[A] U^{(j)}(\alpha, A), \quad \text{with} \quad U^{(j)}(\alpha, A) \equiv R^{(j)}(U(\alpha, A)),
$$

(5.22)

We use Schwartz’s notation $\langle \Phi, \Psi \rangle$ for the value of the linear form $\Phi \in S'_\circ$ applied to the ‘test’ vector $\Psi \in S_\circ$. 

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corresponding to the holonomy along the \( y \)-constant path \( x = \alpha(s), s \in [0, 1] \) on \( \Sigma \), in the spin \( j \) representation. Once the result would be proportional to the \( \delta \)-distribution, it is more natural to consider the action of the integrated version of \( \hat{W}_c \) along a curve \( y = \beta(s) \) on \( \Sigma \) at \( x \) constant. The action of the latter on \( \Psi^{(j)} \) is then

\[
\hat{W}(\beta)\Psi^{(j)}(\alpha, A) = \int ds \beta(s)\hat{W}_c(x, \beta(s))\Psi^{(j)}(\alpha, A).
\]

This is non-zero if and only if \( \beta \) and \( \alpha \) intersect, with the result

\[
\hat{W}'(\beta)\Psi^{(j)}(\alpha, A) = \frac{\gamma}{iK} \Psi_0U^{(j)}(\alpha_1, A)T^{(j)}U^{(j)}(\alpha_2, A),
\]

where \( \alpha_1 \) and \( \alpha_2 \) are the parts of the curve \( \alpha \) before and after its intersection with \( \beta \), and \( T^{(j)} \) is the representation matrix of the generator \( T^i \). Observe that the action of \( \hat{W}_c \) results in the insertion of this matrix at the intersection point. The quadratic operator \( \hat{W}^2(\beta) \equiv \sum_{i=1}^{M} \hat{W}(\beta)\hat{W}'(\beta) \) acts on the same \( \Psi^{(j)} \) as

\[
\hat{W}^2(\beta)\Psi^{(j)} = \frac{\gamma^2}{K^2} j(j+1)\Psi^{(j)},
\]

where we have used the fact that the Casimir operator of SU(2) in the spin-\( j \) representation is given by \( \sum_{i=1}^{M} T^{(j)}T^{(j)} = -j(j+1) \times 1^{(j)} \).

In order to apply this operator to a general spin network vector, one needs to introduce a regularization. Similar to the case of the area operator in \( D = 3 + 1 \) LQG [4], a regularization is available for the square root \( \sqrt{W} \). The graph \( C \) of a spin network vector \( |C, j\rangle \) involves several cycles \( C_n, n = 1, \ldots, N \), endowed with spin \( j_n \) representations. These cycles cross the path \( \beta \) at different heights \( y_n \). Basically, the regularization scheme consists in subdividing the path \( \beta \) into \( K \) segments \( \beta_k, k = 1, \ldots, K \), such that each cycle crosses at most one of the segments \( \beta_k \). For each of these segments we can compute the action of \( \hat{L}_k \) defined as \( \hat{L}_k \equiv \sqrt{W}(\beta_k) \).

Then the total regularized operator is the sum of all pieces:

\[
\hat{L}(\beta) = \sum_{k=1}^{K} \hat{L}_k.
\]

Thus, when acting over a spin network vector, using the result (5.25) we obtain

\[
\hat{L}(\beta)|C, j\rangle = L_{C, j}(\beta)|C, j\rangle, \quad \text{with} \quad L_{C, j}(\beta) = \frac{\gamma}{K} \sum_{m=1}^{M} \sqrt{j_m(j_m+1)},
\]

where the summation runs over all intersections of the curve \( \beta \) with the graph \( C \). Note that the eigenvalue \( L_{C, j}(\beta) \) thus depends on the graph \( C \). The result is independent of \( K \), i.e. of the refinement of the regularization scheme: it only depends on the number \( M \) of cycles crossing the path \( \beta \) and on their associated SU(2) representations. This defines a ‘partial observable’ [4], i.e. a self-adjoint operator in \( \mathcal{H}_{\text{Gauss}} \).

An obvious question is about the possibility of extending the definition of this partial observable to an observable, i.e. a self-adjoint operator \( \hat{L}_{\text{phys}}(\beta) \) in \( \mathcal{H}_{\text{phys}} \). For any given finite curve \( \beta \), the answer is no, as we shall see now. A natural extension of \( \hat{L}(\beta) \) as an operator acting in \( S_c \), with domain in \( \mathcal{H}_{\text{phys}} \), is given by

\[
\langle \Phi, \hat{L}(\beta)| \Psi \rangle = \langle \Phi, \hat{L}(\beta)| \Psi \rangle, \quad \forall \Psi \in S_c,
\]

where \( \Phi \in \mathcal{H}_{\text{phys}} \) and \( \Psi \in S_c \) according to the definition (5.17). By linearity, it is sufficient to specialize to elements of the spin network basis of \( S_c \): \( \Phi \rightarrow \Phi_j = P_{\text{diff}}\Psi_{C, j} \) for some graph \( C \), and \( \Psi \rightarrow \Psi_{C, j} \). We have

\[
\langle \Phi_j, \Psi_{C, j} \rangle = \sum_{C \in [C]} \langle \Psi_{C, j} \Psi_{C, j} \rangle = \begin{cases} d_{ij} & \text{if } C' \in [C], \\ 0 & \text{if } C' \notin [C]. \end{cases}
\]

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where we have denoted by \([C]\) the \(y\)-diffeomorphism equivalence class of the graph \(C\). The last equality follows from the orthonormality of the spin network basis. Then, using (5.28):

\[
\hat{L}(\beta) \Phi_{\beta, \vec{j}} = \sum_{C \in [C]} \langle \Psi_{C, \vec{j}} | \hat{L}(\beta) \rangle \Psi_{C, \vec{j}} = L_{\vec{j}}(\beta) \sum_{C \in [C]} \langle \Psi_{C, \vec{j}} | \hat{L}(\beta) \rangle \Psi_{C, \vec{j}}
\]

where we have used the eigenvalue equation (5.27). We remark that, since \(L_{\vec{j}}(\beta)\) depends on the graph \(C\) of the spin network vector \(|\vec{j}\rangle\), argument of the linear form \(\Phi_{\beta, \vec{j}}\), the \(s\)-knot vector \(|\vec{j}\rangle\) is not an eigenvector of \(\hat{L}(\beta)\). Even more, the result, which should represent the matrix elements of \(\hat{L}(\beta)\) in the \(s\)-knot basis (see (5.18)), is not even \(y\)-diffeomorphism invariant.

Let us show however that if, instead of the finite curves \(B_{\infty} = \{x, y | x = \text{constant}, -\infty < y < +\infty\}\), we arrive at a well-defined observable, i.e. a self-adjoint operator \(\hat{L}_{\text{phys}}\) in \(H_{\text{phys}}\). Indeed, first, equation (5.27) becomes

\[
\hat{L}(B_{\infty}) |C, \vec{j}\rangle = L_{\vec{j}} |C, \vec{j}\rangle, \quad \text{with} \quad L_{\vec{j}} = \frac{\sqrt{N}}{\sqrt{\kappa}} \sum_{n=1}^{N} \sqrt{j_n(j_n + 1)},
\]

where the summation runs over all intersections of the curve \(B_{\infty}\) with the graph \(C\), thus leaving an eigenvalue \(L_{\vec{j}}\) independent of the graph \(C\)—provided that the latter consists of \(N\) cycles, with spin attributes \(\vec{j} = (j_1, \ldots, j_N)\). The eigenvalue is also independent of the position \(x\) of the level of the physical Hilbert space, we can rewrite (5.30) as a definition of the physical operator:

\[
\langle \hat{L}_{\text{phys}} \Phi_{\beta, \vec{j}} | \Psi_{C, \vec{j}} \rangle = \sum_{C \in [C]} \langle \Psi_{C, \vec{j}} | \hat{L}_{\text{phys}}(B_{\infty}) \rangle \Psi_{C, \vec{j}}
\]

where \(\hat{L}_{\text{phys}}\) is defined from \(\hat{L}(\beta)\) according to (5.28) with obvious substitutions. Comparing with (5.29), we obtain the eigenvalue equations

\[
\hat{L}_{\text{phys}} |\vec{j}\rangle = L_{\vec{j}} |\vec{j}\rangle,
\]

with \(L_{\vec{j}}\) given by (5.31), which shows that \(\hat{L}_{\text{phys}}\) is real diagonal in the \(s\)-knots basis of \(H_{\text{phys}}\), hence defines an observable as announced.

When we restore the full dimensional parameters of the model, we obtain

\[
\kappa = \frac{c^3}{16\pi G \sqrt{\Lambda}}.
\]

Therefore (5.33) reads

\[
\hat{L}_{\text{phys}} |\vec{j}\rangle = \gamma \sqrt{16\pi G \sqrt{\Lambda}} \sum_{n=1}^{N} \sqrt{j_n(j_n + 1)} |\vec{j}\rangle,
\]

where \(l_p = hG/c^3\) is the Planck length. We can see in these results an important difference which is the fact that the cosmological constant appears in the formula. Because of this, \(\hat{L}_{\text{phys}}(\beta)\) is dimensionless, but an operator with dimension of length can be defined as \(\hat{L}_{\text{phys}}/\sqrt{\Lambda}\) which has a spectrum very similar to that of the area operator in \(D = 3 + 1\) gravity.

\footnote{With the difference that in \(D = 3 + 1\) gravity the area operator is only a partial observable, not even invariant under the space diffeomorphisms [4].}
We mentioned before that in the CS theory the level of the theory (i.e. the constant in front of the CS action) is quantized by topological arguments when the underlying gauge group is compact and simply connected, then we have \( \kappa / \gamma = \nu / (4\pi \hbar) \), with \( \nu \in \mathbb{Z} \). Therefore, the last equation can be rewritten as

\[
\hat{L}_{\text{phys}} | \vec{j} \rangle = \frac{4\pi}{\nu} \sum_{n=1}^{N} \sqrt{j_n(j_n + 1)} | \vec{j} \rangle.
\]

(5.36)

This quantization rule can also be interpreted as a quantization rule of the three fundamental constants of the theory:

\[
4\gamma \sqrt{\Lambda} l = \nu, \quad \nu \in \mathbb{Z}, \quad \nu \neq 0.
\]

(5.37)

5.3. Classical limit of the observable \( \hat{L}_{\text{phys}} \)

The obvious question is which classical object corresponds to \( \hat{L}(\beta) \) or \( \hat{L}(\beta_\infty) \). It is clear that the limit \( K \to \infty \) of the classical counterpart of the sum (5.26) is the Riemannian sum of the integral

\[
L(\beta) = \int d\hat{\beta} \sum_{i=1}^{3} W_i^j W_i^j = \int d\hat{\beta} \sqrt{-2\text{Tr}(A_y - h^{-1}\partial_y h)^2},
\]

(5.38)

which reinforces the analogy with the \( D = 3 + 1 \) area operator.

We are going now to show that, for a finite curve \( \beta \), and hence for the curve \( \beta_\infty \) in the limit, the classical gauge invariant quantity \( \hat{L}(\beta) \) is vanishing. In order to show this, let us first rewrite the CS connection \( A = A_\alpha dx^\alpha \) as

\[
A = \Lambda^{-1} d\Lambda + (A_y - \Lambda^{-1}\partial_y \Lambda) dx^\alpha
\]

where \( \Lambda \) is defined as a functional of \( A_\alpha \) by the equation \( \Lambda^{-1}\partial_y \Lambda = A_y \). This shows that the connection is gauge equivalent to \( A = E dx^\alpha \) where \( E = \Lambda(A_y - \Lambda^{-1}\partial_y \Lambda) \Lambda^{-1} \). Applying now the classical Gauss constraint (4.7), we obtain that the function \( E \) is independent of \( y \), and thus the connection is gauge equivalent to

\[
A = E(x) dx
\]

(5.39)

As a corollary, in the gauge where the latter equation holds, we have \( A_y = 0 \). From its definition (5.6) as a functional of \( A_\alpha \), we infer that \( h[A_y] \) is independent of \( y \) in this gauge. Then \( W_j \), as defined by the classical version of (5.21), is vanishing. Since the classical quantity \( L(\beta) \) defined by (5.38) is gauge invariant, we finally conclude that \( L(\beta) = 0 \); hence \( L(\beta_\infty) = 0 \) in the limit, as announced. We are thus led to the conclusion that the non-triviality of the quantum observable \( \hat{L}_{\text{phys}}(\beta) \) is a purely quantum effect.

**Remark 1.** That the gauge invariant local object \( L(\beta) = 0 \) be vanishing should be expected, since a topological theory is characterized by the absence of local invariant observables. In the limit of the curve \( \beta \) going to infinity, follows the vanishing of \( L(\beta_\infty) \), although the latter is in fact global.\(^{10}\)

\(^{10}\) It could a priori depend on the position \( x \) of the curve \( \beta_\infty \), but the vanishing of the curvature \( F_{ab} \) and the non-Abelian Stoke theorem [17] allows us to show easily that \( L(\beta_\infty) \) is independent of \( x \).
6. Concluding remarks

The origin of the Barbero–Immirzi ambiguity \( \gamma \) in Chern–Simons (CS) formulation of 3D Gravity lies in the fact that it is possible to define two non-equivalent inner products in the algebra of the SO\((3, 1)\) group. Let us note, however, that, in 4D gravity the gauge group is usually taken as that of local Lorentz transformations, whereas in this case of 3D gravity, it is that of local Lorentz transformations and translations (Poincaré, de Sitter or anti-de Sitter group). Therefore, it is expected a qualitative departure in the interpretation of \( \gamma \) in both contexts—although both gauge groups have the same dimension, namely 6.

We have elaborated a detailed analysis of the \( \Lambda > 0 \) sector of the theory, where we have succeeded to reduce the de Sitter SO\((3, 1)\) gauge group to its compact subgroup SO\((3)\) thanks to a suitable axial gauge fixing. We note that our results also apply to Riemannian gravity with a negative cosmological constant, where the gauge group is de Sitter SO\((3, 1)\), too. The case of Lorentzian gravity with \( \Lambda < 0 \) is not quite different, but technical difficulties to quantize the model are expected because the SO\((2, 2)\) group would only be reduced, by the same procedure, to a non-compact subgroup SO\((2, 1)\).

We have thus obtained a notable reduction of the model. Specifically, in the positive \( \Lambda \) case, we have gotten a CS theory with an SU\((2)\) gauge group. This result, in particular the compactness of the residual gauge group, opened the way to the LQG quantization of a \( D = 2 + 1 \) Lorentzian gravity theory.

Specializing to the particular case of a 2D space manifold with the topology of a cylinder, we have applied a recent LQG quantization scheme of the resulting CS theory [12] which yields a physical separable Hilbert space with an s-knot basis labeled by spin arrays \( j = \{ j_1, \ldots, j_N \} \). We have constructed a global observable \( \hat{L}_{\text{phys}} \), diagonal in that basis, with a spectrum very similar to that of the partial observable ‘area’ in \( D = 3 + 1 \) gravity.

Finally, despite the non-triviality of the quantum observable \( \hat{L}_{\text{phys}} \), we have found that its classical counterpart \( L(\beta_{\infty}) \) is trivial, indeed null. We have related this result to the fact that a non-zero \( L(\beta_{\infty}) \) would be the limit of a local gauge invariant quantity, hence of a classical local observable, \( L(\beta) \), and the existence of the latter would be in contradiction with the topological nature of the theory. Thus the observable \( \hat{L}_{\text{phys}} \) together with its spectrum appear as a purely quantum effect. To the best of our knowledge, there is no previous example in the literature of such a quantum observable whose classical counterpart is gauge equivalent to zero. We may note that zero is the eigenvalue of \( \hat{L}_{\text{phys}} \) in the null state \( |0\rangle \) corresponding to the empty graph\(^{11}\).

Acknowledgments

RMSB, CPC, ZO and OP were supported in part by the Conselho Nacional de Desenvolvimento Científico e Tecnológico—CNPq (Brazil) and by the PRONEX project no. 35885149/2006 from FAPES CNPq (Brazil). Work of RMSB was also supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico—CNPq (Brazil) grant no. 130253/2009-0. Work of ZO was also supported in part by the Centro Latino-Americano de Física—CLAF and the Conselho Nacional de Desenvolvimento Científico e Tecnológico—CNPq (Brazil) grant no. 141579/2008-0.

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\(^{11}\) We thank Alejandro Perez for this last remark, as well as one of the referees for asking the question.
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