TOWER POWER FOR $S$-ADICS

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Abstract. We explain and restate the results from our recent paper [2] in standard language for substitutions and $S$-adic systems in symbolic dynamics. We then produce as rather direct application an $S$-adic system (with finite set of substitutions $S$ on $d$ letters) that is minimal and has $d$ distinct ergodic probability measures.

As second application we exhibit a formula that allows an efficient practical computation of the cylinder measure $\mu([w])$, for any word $w \in A^*$ and any invariant measure $\mu$ on the subshift $X_\sigma$ defined by any everywhere growing but not necessarily primitive or irreducible substitution $\sigma : A^* \to A^*$. Several examples are considered in detail, and model computations are presented.

1. Introduction

Symbolic dynamics has undergone in the past 20-30 years a sequence of several fundamental changes in what people considered the most promising tool to investigate symbolic dynamical systems and their invariant measures. After first having developed the technology of Kakutani-Rokhlin towers, Bratteli diagrams with Vershik maps were recognized as (in many situations) more promising. More recently $S$-adic systems have moved into the limelight.

The authors of this note have very recently made public an exposition [2] of yet another technology, which we believe to be perhaps easier to learn and in some situations more efficient to apply than the previously existing ones. It also seems to have in many situations the potential for further reaching results.

While writing the paper [2] the authors also had in mind future applications to geometric group theory, most notably to currents on free groups $F_N$ and to the action of the automorphism group $\text{Out}(F_N)$ on current space. Since in this world it is natural to admit at all times inverses to the letters that generate the system, the paper [2] was written for graphs and graph maps rather than for monoids and substitutions, which makes our results less accessible for somebody working in the traditional settings of symbolic dynamics.

Hence a translation into the classical terminology of symbolic dynamics seemed to be called for, and it is provided here (see section 3). Instead of a proof (which is a direct consequence of the main result of [2], together with the translation dictionary given in section 3 of [2]), we proceed here to present two applications of our new technology. Both are new results (to our knowledge), but what is almost more important to the authors is to convince the reader about the relatively small effort by which they are derived from the results of [2]: We thus would like to point the symbolic dynamics community’s attention to this new technology, which we believe might be useful also for other people working in this area.

We now give a brief description of the results of this paper:

Let $A$ be a finite alphabet of cardinality $d \geq 1$, and let $X$ be a subshift over $A$. Assume that $X$ possesses an $S$-adic expansion over $A$ (see section 2), where $S$ is a possibly infinite set of substitutions $\sigma_n : A^* \to A^*$, then it is well known that $X$ supports at most $d$ distinct ergodic
probability measures. Our first result, presented in section 4, concerns the realization of this bound and can be stated as follows:

**Proposition 1.1.** (1) For any integer \( d \geq 1 \) there exists a directive sequence \( \sigma = \sigma_0 \circ \sigma_1 \circ \ldots \), with level alphabets \( A_\sigma \) all of cardinality \( d \), such that the associated subshift \( X_\sigma \) is minimal and supports \( d \) distinct invariant ergodic probability measures.  

(2) There is a finite set \( S \) (consisting of \( 4 \) substitutions) such that the above substitutions \( \sigma_0, \sigma_1, \ldots \) can all be chosen from \( S \).

This realization result contrasts in part (1) with the well-known upper bound \( \lfloor \frac{d^2}{2} \rfloor \) for the number of ergodic probability measures (see [9, 10]), in the case where \( X_\sigma \) is read off from an interval exchange transformation. Part (2) seems to contradict at first glance known results from Bratteli-Vershik theory, see Remark 4.6 below. We would also like to mention that Proposition 1.1 is preceded in the literature by several related results, see for instance [1, 4] and [8].

Our second result concerns the concrete calculation of the measure of any cylinder \( [w] \) with \( w \in A^* \), for an arbitrary subshift \( X \) over \( A = \{a_1, \ldots, a_d\} \). If \( X \) is uniquely ergodic, the question is unambiguous, but in general one needs to specify the measure \( \mu \) in question. The main result of \([2]\) as stated here in Theorem 3.2 presents a tool for such a specification in rather practical terms, so that a direct calculation (with controlled error term) is possible, once an everywhere growing \( S \)-adic expansion of \( X \) is chosen (see Corollary 3.6).

In section 5 we concentrate on the special case of a substitution subshift \( X = X_\sigma \) for any everywhere growing (but not necessarily primitive or irreducible) substitution \( \sigma : A^* \to A^* \). It is known (see Proposition 3.4) that the ergodic measures \( \mu_i \) on \( X_\sigma \) are in 1-1 correspondence with certain “distinguished” eigenvectors \( \vec{v}_i \geq 0 \) with eigenvalues \( \lambda_i > 1 \) of the incidence matrix \( M_\sigma \). Our determination of the cylinder measures is based on a natural extension of the incidence matrix \( M_\sigma \) to a \((d^2 + d) \times (d^2 + d)\)-matrix \( M_\sigma^+ \) (for \( d = \text{card} A \)), and on corresponding prolongations of the eigenvectors \( \vec{v}_i \) to eigenvectors \( \vec{v}_i^+ \), as well as on “occurrence vectors” \( \vec{v}_n(w) \) obtained from counting the occurrences of \( w \) as factor in the words \( \sigma^n(a_i) \) and \( \sigma^n(a_i,a_j) \). The precise terms of the following proposition are explained below in section 5; it should be noted though that both, \( \vec{v}_n(w) \) and the \( \vec{v}_i^+ \), can be readily computed from \( \sigma \) and \( w \), and also the lower bound “\((\sigma, w)\)-large” used below.

**Proposition 1.2.** Let \( \sigma : A^* \to A^* \) be an everywhere growing substitution, and let \( \mu = \sum c_i \mu_i \) be any invariant measure on the substitution subshift \( X_\sigma \), expressed as non-negative linear combination of ergodic measures \( \mu_i \).

Then for any \( w \in A^* \) and any \((\sigma, w)\)-large integer \( n \in \mathbb{N} \) the measure of the cylinder \( [w] \) is given by the scalar product

\[
\mu([w]) = \langle \vec{v}_n(w), \sum c_i \frac{1}{\lambda_i^n} \vec{v}_i^+ \rangle
\]

Several examples where the formula from Proposition 1.2 is applied to calculate the value of concretely given cylinders are given at the end of the paper in section 6.

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## 2. Preliminaries

In this section we only review standard definitions and facts, and we set up the notation used in this paper. We use [3] as standard reference; indeed, we try to use as much as possible their terminology and notations.
2.1. Subshifts.

Let \( A = \{a_1, \ldots, a_d\} \) be a finite set, called \textit{alphabet}. We denote by \( A^* \) the free monoid over \( A \). For any element \( w \in A^* \) we denote by \(|w|\) the length of \( w \) as word in the alphabet \( A \). For any any two words \( v, w \in A^* \) we write \(|w|_v \) for the number of occurrences of \( v \) as factor of \( w \).

We denote by

\[
\Sigma_A = \{ \ldots x_{-1}x_0x_1x_2 \ldots | \ x_i \in A \}
\]

be the set of biinfinite words in \( A \), called the \textit{full shift} over \( A \). For any word \( w = w_1 \ldots w_s \in A^* \) the \textit{cylinder}

\[
[w] \subseteq \Sigma_A
\]

is the set of all biinfinite words \( \ldots x_{-1}x_0x_1x_2 \ldots \) in \( A \) which satisfy \( x_1 = w_1, \ldots, x_s = w_s \). The full shift \( \Sigma_A \), being in bijection with the set \( A^\mathbb{Z} \), is naturally equipped with the product topology, where \( A \) is given the discrete topology. For any \( w \in A^* \) the cylinder \([w]\) is closed and open. The full shift \( \Sigma_A \) is compact, and indeed it is a Cantor set.

The shift map \( \sigma : \Sigma_A \to \Sigma_A \) is defined for \( x = \ldots x_{-1}x_0x_1x_2 \ldots \) by \( \sigma(x) = \ldots y_{-1}y_0y_1y_2 \ldots \), with \( y_n = x_{n+1} \) for all \( n \in \mathbb{Z} \). It is bijective and continuous with respect to the above product topology, and hence a homeomorphism.

A \textit{subshift} is a non-empty closed subset \( X \subset \Sigma_A \) which is invariant under the shift map \( \sigma \). Such a subshift \( X \) is called \textit{minimal} if it is the closure of the shift-orbit of any \( x \in X \).

Let \( \mu \) be a finite Borel measure supported on a subshift \( X \subseteq \Sigma_A \). The measure is called \textit{invariant} if for every measurable set \( A \subseteq X \) one has \( \mu(S^{-1}(A)) = \mu(A) \). Such a measure \( \mu \) is \textit{ergodic} if \( \mu \) can not be written in any non-trivial way as sum \( \mu_1 + \mu_2 \) of two invariant measures \( \mu_1 \) and \( \mu_2 \) (i.e. \( \mu_1 \neq 0 \neq \mu_2 \) and \( \mu_1 \neq \lambda \mu_2 \) for any \( \lambda \in \mathbb{R}_{>0} \)). An invariant measure is called a \textit{probability measure} if \( \mu(X) = 1 \), which is equivalent to \( \sum_{a_i \in A} \mu([a_i]) = 1 \). We denote by \( \mathcal{M}(X) \) the set of invariant measures on \( X \), and by \( \mathcal{M}_1(X) \subseteq \mathcal{M}(X) \) the subset of probability measures.

The set \( \mathcal{M}(X) \) is naturally equipped with an addition and an external multiplication with scalars \( \lambda \in \mathbb{R}_{\geq 0} \). It is well known (see [11]) that any invariant measure \( \mu \) is determined by the values \( \mu([w]) \) for all \( w \in A^* \). Hence the set \( \mathcal{M}(X) \) is a convex linear cone which through \( \mu \mapsto (\mu([w])_{w \in A^*} \) is naturally embedded into the non-negative cone of the infinite dimensional vector space \( \mathbb{R}^{A^*} \). The cone \( \mathcal{M}(X) \) is closed, and the extremal vectors of \( \mathcal{M}(X) \) are in 1-1 relation with the ergodic measures on \( X \). Furthermore, \( \mathcal{M}_1(X) \) is compact, and it is the closed convex hull of its extremal points. The following is well known (see [11]):

**Proposition 2.1.** For any subshift \( X \subseteq \Sigma_A \) any family of ergodic measures \( \mu_i \in \mathcal{M}(X) \), which are pairwise not scalar multiples of each other, is linearly independent.

In particular, if \( X \) admits (up to scalar multiples) only finitely many ergodic measures, then \( \mathcal{M}_1(X) \) is a finite simplex with vertices that are in 1-1 correspondence with the ergodic probability measures on \( X \). \( \Box \)

It is well known that for any subshift \( X \subseteq \Sigma_A \) the set \( \mathcal{M}(X) \) of invariant measures is not empty. If \( \mathcal{M}(X) \) consists of a single point (which then must be ergodic), then \( X \) is called \textit{uniquely ergodic}.

2.2. Substitutions.

**Definition 2.2.** (1) A \textit{substitution} \( \sigma \) is given by a map

\[
A \to A^*, \ a_i \mapsto \sigma(a_i).
\]

A substitution defines both, an endomorphism of \( A^* \), and a continuous map from \( \Sigma_A \) to itself which maps \([w] \) to \([\sigma(w)]\). Both of these maps are also denoted by \( \sigma \), and both are summarized under the name of “substitution”.

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(2) If \( \mathcal{A} \) and \( \mathcal{A}' \) are two possibly distinct alphabets, then any monoid homomorphism \( \sigma : \mathcal{A}^* \to \mathcal{A}'^* \) is also called a substitution, with the analogous convention for the induced map on \( \Sigma_\mathcal{A} \).

A substitution \( \sigma : \mathcal{A}^* \to \mathcal{A}^* \) is called everywhere growing if each \( a_i \in \mathcal{A} \) satisfies \( |\sigma^n(a_i)| \to \infty \) for \( n \to \infty \).

For any substitution \( \sigma \) we define the associated language \( L_\sigma \subseteq \mathcal{A}^* \) to be the set of factors of the words \( \sigma^n(a_i) \), with \( n \geq 1 \) and \( a_i \in \mathcal{A} \).

One defines the subshift \( X_\sigma \subseteq \Sigma_\mathcal{A} \) associated to the substitution \( \sigma \) as the set of all \( x = \ldots x_{k-1}x_kx_{k+1} \ldots \in \Sigma_\mathcal{A} \) with the property that for any integers \( m \geq n \) the word \( x_n \ldots x_m \) is an element of \( L_\sigma \).

For any substitution \( \sigma : \mathcal{A}^* \to \mathcal{A}'^* \) the non-negative matrix
\[
M_\sigma := (|\sigma(a)|_{a' \in \mathcal{A'}, a \in \mathcal{A}})
\]
is called the incidence matrix for the substitution \( \sigma \) (to be specific: \( a' \) gives the row index, while \( a \) gives the column index of \( M_\sigma \)). The substitution \( \sigma \) is called primitive if \( M_\sigma \) is primitive, i.e. there exists an integer \( k \geq 1 \) such that every coefficient of the power \( M_\sigma^k \) is positive.

2.3. \( S \)-adic sequences.

In \( S \)-adic theory (see for instance [3, 7]) one considers directive sequences of free monoids \( \mathcal{A}^*_n \) and of monoid morphisms \( \sigma_n : \mathcal{A}^*_n+1 \to \mathcal{A}^*_n \) (for \( n \geq 0 \)). The substitutions \( \sigma_n \) belong to a given set \( S \), which in many circumstances is assumed to be finite.

We sometimes call \( \mathcal{A}_n \) the level alphabets, and \( \mathcal{A}_0 \) the base alphabet of the directive sequence \( \sigma \). We also use the notation \( \sigma_{[m,n]} = \sigma_m \circ \sigma_{m+1} \circ \sigma_{m+2} \circ \cdots \circ \sigma_n \) for any \( n \geq m \geq 0 \). The directive sequence \( \sigma \) is often represented by writing:
\[
\sigma = \sigma_0 \circ \sigma_1 \circ \ldots
\]

To any such a directive sequence \( \sigma \) one associates the language \( L_\sigma \subseteq \mathcal{A}_0^* \), defined as the set of factors in \( \mathcal{A}_0^* \) of the words \( \sigma_0 \circ \sigma_1 \circ \ldots \circ \sigma_n(a_i) \), for any \( n \geq 0 \) and any \( a_i \in \mathcal{A}_{n+1} \). The subshift \( X_\sigma \subseteq \Sigma_{\mathcal{A}_0} \) associated to the directive sequence \( \sigma \) is the set of all \( x = \ldots x_{k-1}x_kx_{k+1} \ldots \in \Sigma_{\mathcal{A}_0} \) such that for any two integers \( m \geq n \) the word \( x_n \ldots x_m \) is an element of \( L_\sigma \). The directive sequence \( \sigma = \sigma_0 \circ \sigma_1 \circ \ldots \) is called an \( S \)-adic expansion of a subshift \( X \) if \( X = X_\sigma \) and if \( S \) is a set of substitutions which contains every \( \sigma_i \) that occurs in \( \sigma \).

The directive sequence \( \sigma \) is called everywhere growing if one has
\[
\min_{a_i \in \mathcal{A}_n} |\sigma_{[0,n]}(a_i)| \to \infty \quad \text{for} \quad n \to \infty .
\]

One says that \( \sigma \) is weakly primitive (or simply primitive by some authors) if for any \( m \geq 1 \) there is an integer \( n \geq m + 1 \) such the incidence matrix \( M_{\sigma_{[m,n]}} \) is positive. In this case it follows that \( \sigma \) is everywhere growing (unless all level alphabets have cardinality 1).

This terminology coincides with that for substitutions introduced in subsection 2.1: indeed, one recovers the latter as special case of a stationary \( S \)-adic sequence, i.e. all terms \( \sigma_n \) in the directive sequence \( \sigma \) are equal.

**Proposition 2.3** ([3]). **For any weakly primitive directive sequence \( \sigma \) the subshift \( X_\sigma \) is minimal. Furthermore, any minimal subshift \( X \) admits an \( S \)-adic expansion that is weakly primitive.** □

If the directive sequence \( \sigma \) in the last proposition is stationary (or “strongly minimal”, see Definition 5.1 of [3]), then one can deduce furthermore that \( X_\sigma \) is uniquely ergodic.

The hypothesis that our directive sequence \( \sigma \) is everywhere growing is crucial to everything done in [2]; it will always be assumed. Fortunately this is not really a restriction, as is shown by the following elementary fact (see Proposition 5.10 of [2]):
Lemma 2.4. Let \( \mathcal{A} \) be a finite alphabet, and let \( X \subseteq \Sigma_{\mathcal{A}} \) be an arbitrary subshift. Then there exists an everywhere growing directive sequence \( \sigma \) with base alphabet \( \mathcal{A}_0 = \mathcal{A} \) such that \( X = X_{\sigma} \). \( \square \)

The following seems to be well known (see [3], Remark 5 and [6]); a proof is provided through Corollary 2.11 of [2]:

Fact 2.5. (1) For any directive sequence \( \sigma \), where all level alphabets \( \mathcal{A}_n \) are equal to some fixed alphabet \( \mathcal{A} \) of cardinality \( d \geq 1 \), the number of distinct ergodic probability measures carried by the associated subshift \( X_{\sigma} \subseteq \Sigma_{\mathcal{A}} \) is bounded above by \( d \).

(2) In particular, the subset \( \mathcal{M}(X_{\sigma}) \) of probability measures on \( X_{\sigma} \) is a simplex of dimension \( \dim \mathcal{M}(X_{\sigma}) \leq d - 1 \). \( \square \)

3. Results from [2]

3.1. The general setting.

Throughout this section we assume that \( \sigma = \sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \cdots \) is a directive sequence of substitutions \( \sigma_n : \mathcal{A}^{*}_{n+1} \rightarrow \mathcal{A}^{*}_{n} \) as set up in the previous section. We also use the notation \( \sigma_{[m,n]} = \sigma_m \circ \sigma_{m+1} \circ \sigma_{m+2} \circ \cdots \circ \sigma_n \) for any \( n \geq m \geq 0 \), and \( M_n = M_{\sigma_n} \) and \( M_{[m,n]} = M_{\sigma_{[m,n]}} \) for the associated incidence matrices.

Let \( \mathcal{A}_1 = \{a_1, \ldots, a_d\} \) and \( \mathcal{A}_2 = \{a'_1, \ldots, a'_d\} \) be two alphabets, and let \( \sigma : \mathcal{A}_2^{*} \rightarrow \mathcal{A}_1^{*} \) be a substitution. We consider vectors \( \vec{v}_1 = (\vec{v}_1(a))_{a \in \mathcal{A}_1} \) and \( \vec{v}_2 = (\vec{v}_2(a'))_{a' \in \mathcal{A}_2} \) with real coordinates \( \vec{v}_1(a) \geq 0 \) and \( \vec{v}_2(a') \geq 0 \), and we say that \( \vec{v}_1 \) and \( \vec{v}_2 \) are \( \sigma \)-compatible if one has \( \vec{v}_1 = M_{\sigma} \vec{v}_2 \), where \( M_{\sigma} \) denotes the incidence matrix of \( \sigma \).

Definition 3.1. Let \( \sigma = \sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \cdots \) be a directive sequence, and let \( \vec{v} = (\vec{v}_n)_{n \in \mathbb{N} \cup \{0\}} \) be a family of non-negative vectors \( \vec{v}_n = (\vec{v}_n(a))_{a \in \mathcal{A}_n} \). We say that \( \vec{v} \) is a \( \sigma \)-compatible vector tower if for any \( n \geq 0 \) the vectors \( \vec{v}_n \) and \( \vec{v}_{n+1} \) are \( \sigma_n \)-compatible.

We notice that there is a natural addition for \( \sigma \)-compatible vector towers, and similarly an external multiplication with non-negative scalars \( \lambda \in \mathbb{R}_{\geq 0} \). We are now able to state the main result of of our previous paper, translated properly into \( S \)-adic terminology:

Theorem 3.2 ([2]). Let \( \sigma = \sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \cdots \) be an everywhere growing directive sequence with associated subshift \( X_{\sigma} \). Let \( \mathcal{M} := \mathcal{M}(X_{\sigma}) \) denote the set of invariant measures on \( X_{\sigma} \), and let \( \mathcal{V} = \mathcal{V}(\sigma) \) denote the set of \( \sigma \)-compatible vector towers.

(1) Every \( \sigma \)-compatible vector tower \( \vec{v} \) determines an invariant measure \( \mu_{\vec{v}} \) on \( X_{\sigma} \).

(2) Conversely, every invariant measure \( \mu \) on \( X_{\sigma} \) is given via \( \mu = \mu_{\vec{v}} \) by some \( \sigma \)-compatible vector tower \( \vec{v} \).

(3) The issuing map \( m : \mathcal{V} \rightarrow \mathcal{M}, \vec{v} \mapsto \mu_{\vec{v}} \) is linear (with respect to linear combinations with non-negative scalars).

(4) For any word \( w \in \mathcal{A}_0^* \) and any \( \sigma \)-compatible vector tower \( \vec{v} = (\vec{v}_n)_{n \in \mathbb{N} \cup \{0\}} \), with \( \vec{v}_n = (\vec{v}_n(a))_{a \in \mathcal{A}_n} \), the sequence of sums

\[
\sum_{a \in \mathcal{A}_n} \vec{v}_n(a) |\sigma_{[0,n]}(a)|_w
\]

is bounded above and increasing, and one has:

\[
\mu_{\vec{v}}([w]) = \lim_{n \rightarrow \infty} \sum_{a \in \mathcal{A}_n} \vec{v}_n(a) |\sigma_{[0,n]}(a)|_w
\]

\( \square \)

This is precisely the statement of Theorem 2.9 of [2], except that the “increasing” property from (4) has been shown in Remark 9.5 of [2]. The canonical translation from the more general language
we obtain a canonical linear map

\[ m_0 : \mathcal{V}(\sigma) \to \mathbb{R}^{A_0}, \quad \bar{v} = (\bar{v}_n)_{n \in \mathbb{N} \cup \{0\}} \mapsto \bar{v}_0, \]

and it follows (see Proposition 10.2 (2) of [2]) that its image is equal to the nested intersection

\[ C_{\infty} := \bigcap C_0^n \text{ of the cones } C_0^n. \]

This gives (see [2], Proposition 10.2 (1)):

**Lemma 3.3.** The map \( \zeta : \mathcal{M}(X_\sigma) \to \mathbb{R}^{A_0}, \mu \mapsto (\mu([a]))_{a \in A_0} \) satisfies \( m_0 = \zeta \circ m \) and thus \( C_{\infty} = \zeta(m(\mathcal{V}(\sigma))) \). In particular, \( \dim C_{\infty} \) is a lower bound to the number of distinct ergodic probability measures on \( X_\sigma \).

Of special interest are directive sequences where every level alphabet \( A_n \) has the same cardinality \( d \geq 1 \), so that we can postulate them to be equal to \( A_n := A = \{a_1, \ldots, a_d\} \). In this case we say that \( \sigma \) is a directive sequence over \( A \), and we say that \( \sigma \) is of tower dimension \( \dim \sigma = d \).

Examples are stationary sequences, or sequences derived through telescoping from directive sequences that have finite tower dimension \( d \): the number \( d \) is the inferior limit of the sequence of the card \( A_n \in \mathbb{N} \). (We believe that the notion of “finite tower dimension” is related or perhaps even equivalent to the condition “finite rank” as defined through the Bratteli-Vershik setting.)

### 3.2. Application to substitutions.

As pointed out in section 2, for any substitution \( \sigma : A^* \to A^* \) the stationary directive sequence \( \sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \ldots \), with \( \sigma_n = \sigma \) for all \( n \geq 0 \), has as associated subshift the substitution subshift \( X_\sigma \). This gives the possibility to interpret a compatible vector tower as infinite sequence of vectors in \( \mathbb{R}^{d_0} \) (with \( d = \text{card } A \)), obtained from each other through iteration of the linear map \( M_\sigma \). This observation has been used to derive in Theorem 10.8 of [2] the following result, which is a slight improvement of a result of Bezuglyi, Kwiatkowski, Medynets and Solomyak obtained in [5]:

**Proposition 3.4.** For any everywhere growing substitutions \( \sigma \) the set of ergodic measures on the substitution subshift \( X_\sigma \) is in 1-1 relation with the set of extremal vectors in the cone

\[ C_{\infty} = \bigcap \{M^n_{\sigma}(\mathbb{R}^{d_0}) \mid n \geq 1\}. \]

The latter are also the non-negative extremal eigenvectors of a suitable positive power \( M^k_{\sigma} \) (for example \( k = (\text{card } A)! \) would do, see Appendix 11.3 of [2]).

The determination of the extremal eigenvectors named in the above proposition is in practice for any given reducible matrix \( M \) quite convenient, once one has penetrated the slightly intricate logic of the two “conflicting” natural partial orders on the primitive diagonal blocks of the power \( M^k \).

A concise description of all ingredients needed is given in Appendix 11.3 of [2]; for the convenience of the reader we will now single out the most frequently occurring non-primitive case:

**Corollary 3.5.** Let \( \sigma \) be an everywhere growing substitution, and assume that the incidence matrix \( M_\sigma \) satisfies the following conditions:

(a) \( M_\sigma \) is a 2 \( \times \) 2 block lower triangular matrix.

(b) The two diagonal blocks \( M_{1,1} \) and \( M_{2,2} \) are primitive, with Perron-Frobenius eigenvalues \( \lambda_1 \geq 1 \) and \( \lambda_2 > 1 \) respectively.

(c) The lower left off-diagonal block \( M_{2,1} \) is non-zero.

(1) If \( \lambda_2 \geq \lambda_1 \), then there is (up to scalar multiples) only one non-negative eigenvector \( \bar{v} \) of \( M_\sigma \) (with eigenvalue \( \lambda_2 \)), which has zero-coordinates on the top block (corresponding to \( M_{1,1} \), and
non-zero coordinates on the bottom block (corresponding to \( M_{2,2} \)). In this case \( X_\sigma \) has only one ergodic probability measure, and its support is the sub-subshift of \( X_\sigma \) generated by the letters of \( \mathcal{A} \) that define the bottom block.

(2) If \( \lambda_1 > \lambda_2 \), then there are (up to scalar multiples) precisely two non-negative eigenvectors \( \vec{v}_1 \) and \( \vec{v}_2 \) of \( M_\sigma \). The vector \( \vec{v}_2 \) (with eigenvalue \( \lambda_2 \)) has the same properties as the eigenvector \( \vec{v} \) in case (1). On the other hand, the eigenvector \( \vec{v}_1 \) (with eigenvalue \( \lambda_1 \)) is positive in all coordinates.

In this case \( X_\sigma \) has precisely two ergodic probability measures \( \mu_1 \) and \( \mu_2 \): The support of \( \mu_2 \) is, as in the above case (1), only the sub-subshift of \( X_\sigma \) generated by the letters of \( \mathcal{A} \) that define the bottom block. The support of \( \mu_2 \) is all of \( X_\sigma \). \( \square \)

3.3. Cylinder measures. Let \( \sigma = \sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \cdots \) be a directive sequence of substitutions \( \sigma_n : \mathcal{A}_{n+1}^* \rightarrow \mathcal{A}_n^* \), and let \( \vec{v} = (\vec{v}_n)_{n \in \mathbb{N} \cup \{0\}} \) be a \( \sigma \)-compatible vector tower as in Definition 3.1. We now define, for any level \( n \geq 0 \) and any two letters \( a, a' \in \mathcal{A}_n \), a \emph{weight} \( \omega_{n,a,a'}^n \) through equality (3.2) below. The latter can be viewed as a particular case of equality (3.1), thus ensuring the existence of the limit on the right hand side:

\[
\omega_{n,a,a'}^n := \lim_{k \to \infty} \sum_{b \in \mathcal{A}_k} \vec{v}_k(b) |\sigma_{[n,k]}(b)|_{aa'}
\]

Here the pair \((a, a')\) has to be understood as the “transition” from \( a \) to \( a' \), and by its “\( \sigma_{n-1}\)-image” \((c, c')\) we understand correspondingly the transition of the last letter \( c \) of \( \sigma_{n-1}(a) \) to the first letter \( c' \) of \( \sigma_{n-1}(a') \). We write \((c, c') = \sigma_{n-1}^{-1}(a, a')\).

Of course, such transitions occur also inside \( \sigma_{n-1}(a) \) or \( \sigma_{n-1}(a') \), and indeed, we derive from (3.2) and from the \( \sigma \)-compatibility of \( \vec{v} \):

\[
\omega_{c,c'}^{n-1} = \sum_{\{(a,a')\} = \sigma_{n-1}^+(a,a')} \omega_{n,a,a'}^n + \sum_{a \in \mathcal{A}_n} \vec{v}_n(a) |\sigma_n(a)|_{cc'}
\]

The following has not been stated explicitly in [2]; we will though derive it quickly from the set-up studied there:

**Corollary 3.6.** Let \( \sigma \) and \( \vec{v} \) be as above. Then, for any word \( w \in \mathcal{A}_0^* \), and any integer \( n \geq 0 \) with \( |\sigma_{[0,n]}(a)| \geq |w| - 1 \) for all \( a \in \mathcal{A}_n \), one obtains:

\[
\mu_{\vec{v}}([w]) = \sum_{a \in \mathcal{A}_n} \vec{v}_n(a) \cdot |\sigma_{[0,n]}(a)|_w + \sum_{a,a' \in \mathcal{A}_n} \omega_{a,a'}^n (|\sigma_{[0,n]}(aa')|_w - |\sigma_{[0,n]}(aa')|_w - |\sigma_{[0,n]}(a')|_w)
\]

**Proof.** In [2] the vector towers \( \vec{v} \) from Theorem 3.2 are defined by means of weight function \( \omega_n \) on 1-vertex graphs \( \Gamma_n \) that topologically realize the monoid \( \mathcal{A}_n \). For any two letters \( a, a' \in \mathcal{A}_n \) there is a \emph{local edge} \( \varepsilon_{a,a'} \) defined in \( \Gamma_n \), and if \( \sigma \) is everywhere growing, then the vector tower \( \vec{v} \) in turn determines the weight functions \( \omega_n \). In particular, see Remark 9.5 of [2], the value of \( \omega_n(\varepsilon_{a,a'}) \) is given via \( \omega_n(\varepsilon_{a,a'}) := \omega_{a,a'}^n \) through equality (3.2). One obtains now the claimed statement as a direct translation of Propositions 6.9 and 7.4 from [2] into the \( S \)-adic language used here, following the instructions carefully laid out in section 3 of [2]. \( \square \)

4. Minimal subshifts with many ergodic measures

This section is devoted to the proof of Proposition 1.1. We give first in subsection 4.1 the proof of part (1) of this proposition, and improve this coarser approach in subsection 4.2 to a proof of part (2).
4.1. The general construction procedure.

In this subsection we present (in a purposefully concrete and “simplistic” way) in 7 steps a construction of a directive sequence \( \sigma \) with the desired properties:

(1) We first consider matrices of type \( M_\varepsilon = I_d + \varepsilon 1_{d \times d} \), where \( \varepsilon > 0 \), \( I_d \) denotes the \((d \times d)\)-unit matrix, and \( 1_{d \times d} \) denotes the \((d \times d)\)-matrix that has all coefficients equal to 1.

Let \( \bar{e} \in \mathbb{R}^d \) denote the column vector with all coefficients equal to 1, and let \( \bar{e} = \bar{e}_k \) be any one of the standard basis vectors of \( \mathbb{R}^d \). We compute:

\[
M_\varepsilon \cdot \bar{e} = \bar{e} + \varepsilon \bar{c} \quad \text{and} \quad M_\varepsilon \cdot \bar{c} = \bar{e} + d \varepsilon \bar{c} = (1 + d\varepsilon) \bar{c}
\]

As a consequence, for any \( \varepsilon_1, \ldots, \varepsilon_q > 0 \) we obtain:

\[
M_{\varepsilon_1} \cdot M_{\varepsilon_2} \cdots \cdot M_{\varepsilon_q} \cdot \bar{e} = \bar{e} + \left[ \varepsilon_1 + (1 + d\varepsilon_1) \varepsilon_2 + (1 + d\varepsilon_1)(1 + d\varepsilon_2) \varepsilon_3 + \cdots + (1 + d\varepsilon_1)(1 + d\varepsilon_2) \cdots (1 + d\varepsilon_{q-1}) \varepsilon_q \right] \bar{c}
\]

\[
= \bar{e} + \left[ L_1 \varepsilon_1 + L_2 \varepsilon_2 + L_3 \varepsilon_3 + \cdots + L_q \varepsilon_q \right] \bar{c}
\]

where we set \( L_1 := 1 \) and \( L_q := (1 + d\varepsilon_1)(1 + d\varepsilon_2) \cdots (1 + d\varepsilon_{q-1}) \) for any \( s \in \{2, \ldots, q\} \).

(2) We now choose for every \( n \in \mathbb{N} \) a value \( \varepsilon_n > 0 \) which is small enough so that it satisfies

(a) \( \log(1 + d\varepsilon_n) < 2^{-n} \log 2 \)

(b) \( \varepsilon_n < \frac{1}{2^{2n+1}} \).

From (a) we deduce \( L_n < 2 \), so that we compute:

\[
M_{\varepsilon_1} \cdot M_{\varepsilon_2} \cdots \cdot M_{\varepsilon_q} \cdot \bar{e} = \bar{e} + K_q \bar{c}
\]

with \( 0 < K_q \leq K_{q+1} < 1 \) for all \( q \in \mathbb{N} \).

We note that the same result stays valid if one further lowers the value of the \( \varepsilon_n \), so that we can assume that \( \varepsilon_n = \frac{1}{\ell(n)} \) for some integer \( \ell(n) \in \mathbb{N} \).

(3) For the family of \( \varepsilon_n \) as in (2) we consider any of the standard basis vectors \( \bar{e}_k \) and obtain

\[
\lim_{n \to \infty} M_{\varepsilon_1} \cdot M_{\varepsilon_2} \cdots \cdot M_{\varepsilon_n} \cdot \bar{e}_k = \bar{e}_k + K_{\infty} \bar{c}
\]

for some \( 0 < K_{\infty} \leq 1 \).

This shows that the nested intersection of the cones \( M_{\varepsilon_1} \cdot M_{\varepsilon_2} \cdots \cdot M_{\varepsilon_n}(\mathbb{R}^d_{\geq 0}) \) for all \( n \in \mathbb{N} \) is equal to the cone generated by the vectors \( \bar{e}_k + K_{\infty} \bar{c} \), which is simplicial of dimension \( d \).

(4) We now define for any \( \ell \in \mathbb{N} \) the integer matrix \( M_\ell' = \ell I_d + 1_{d \times d} \), and note that for \( \varepsilon = \frac{1}{\ell} \) we have

\[
M_\ell' = \ell M_\varepsilon
\]

It follows, since in (2) we chose \( \ell(n) \in \mathbb{N} \) such that \( \varepsilon_n = \frac{1}{\ell(n)} \), that the nested intersection of the cones \( M_{\ell(1)}' \cdot M_{\ell(2)}' \cdots \cdot M_{\ell(n)}'(\mathbb{R}^d_{\geq 0}) \) for all \( n \in \mathbb{N} \) is also equal to the cone generated by the vectors \( \bar{e}_k + K_{\infty} \bar{c} \) and thus simplicial of dimension \( d \).

(5) Next we consider an alphabet \( \mathcal{A} = \{a_1, \ldots, a_d\} \) and substitutions

\[
\sigma_n : \mathcal{A}^* \to \mathcal{A}^*, \quad a_k \mapsto a_{\ell(n)}^k a_1 \ldots a_d \quad (\text{for all } a_k \in \mathcal{A}),
\]

so that one has \( M_{\sigma_n} = M_{\ell(n)}' \) for all \( n \in \mathbb{N} \).

For formal reasons we add the substitution \( \sigma_0 := \text{id}_{\mathcal{A}^*} \) to our list.

(6) Since for \( n \geq 1 \) any of the incidence matrices \( M_{\sigma_n} \) is positive, it follows that the directive sequence \( \sigma = \sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \ldots \) is weakly primitive as defined in section 2, so that Proposition 2.3 shows that the associated subshift \( X_\sigma \) is minimal.
(7) We now apply Theorem 3.2, Lemma 3.3 and Fact 2.5 to deduce that \( M(X_\sigma) \) is a simplicial cone of dimension \( d \), so that \( X_\sigma \) supports \( d \) distinct ergodic probability measures.

**Remark 4.1.** Once the matrices \( M'_{\ell(n)} \) have been established in step (4) of the above construction, the choice of the substitutions \( \sigma_n \) with \( M_{\sigma_n} = M'_{\ell(n)} \) in step (5) is of course only one of many possible such choices.

4.2. **Reduction to finite \( S \).**

We will now explain how the method presented in the previous subsection can be refined to obtain the same result, but for a set \( S \) of substitutions that is finite. The latter are given by the substitutions \( \rho_j, \theta_j, \tau_j \), for any \( j \in \{1, \ldots, d\} \), which all fix any \( a_k \) with \( k \neq j \) and map \( a_j \) to \( \rho_j(a_j) = a_j^3 \), \( \theta_j(a_j) = a_j^2 \) and \( \tau_j(a_j) = a_j a_{j+1} \ldots a_d a_1 \ldots a_{j-1} \) respectively. We also use the abbreviation \( \theta_j' := \theta_d \ldots \theta_{j+1} \theta_{j-1} \ldots \theta_1 \).

We first need to recall the following well known fact (deduced easily from the non-commensurability of the logarithms of 2 and 3):

**Fact 4.2.** (1) The set of numbers \( \frac{m}{3^m} \), for any integers \( m, q \geq 0 \), is dense in \( \mathbb{R}_{\geq 0} \).

(2) More concretely, we need the following, which follows from (1):

For any \( \varepsilon > 0 \) there are integers \( m, q \geq 0 \) such that \( 1 \geq \frac{2^m}{3^m} \geq 1 - \frac{\varepsilon}{2} \). We then define the integer \( h \geq 1 \) through \( h := 3^q - 2^m \).

For any \( \varepsilon > 0 \) and \( m, q \) and \( h \) as in Fact 4.2 (2) we now define a substitution

\[
\sigma'_{\varepsilon,j} := \rho_j^q \theta_j^m \tau_j^h
\]

and observe that on the generators \( \sigma'_{\varepsilon,j} \) acts as follows:

\[
\begin{align*}
a_j &\mapsto a_j^{3^q} (a_{j+1} \ldots a_d a_1 \ldots a_{j-1})^h \\
a_k &\mapsto a_k^{2^m} \quad \text{for all} \quad k \neq j
\end{align*}
\]

From the above definition of \( h \) we quickly deduce that the incidence matrix \( M_{\sigma'_{\varepsilon,j}} =: M'_{\varepsilon,j} \) maps the center vector \( \vec{c} \) from the previous section to

\[
M'_{\varepsilon,j} \cdot \vec{c} = 3^q \vec{c},
\]

and similarly any basis vector \( \vec{e}_k \) with \( k \neq j \) to

\[
M'_{\varepsilon,j} \cdot \vec{e}_k = 2^m \vec{e}_k.
\]

For \( \vec{e}_j \) we compute:

\[
M'_{\varepsilon,j} \cdot \vec{e}_j = h \vec{c} + (3^q - h) \vec{e}_j = h \vec{c} + 2^m \vec{e}_j.
\]

**Lemma 4.3.** For any \( 1 \geq \varepsilon > 0 \) the matrices \( M'_{\varepsilon,j} \) above and \( M_\varepsilon \) from the previous subsection satisfy the following inclusion:

\[
M_\varepsilon(\mathbb{R}^d_{\geq 0}) \subseteq M'_{\varepsilon,j}(\mathbb{R}^d_{\geq 0})
\]

**Proof.** We recall from the previous subsection that \( M_\varepsilon \cdot \vec{c} = (1 + d \varepsilon) \vec{c} \) and \( M_\varepsilon \cdot \vec{e}_k = \vec{e}_k + \varepsilon \vec{c} \) for all \( k \in \{1, \ldots, d\} \). We note that both, \( M_\varepsilon \) and \( M'_{\varepsilon,j} \) fix the vector \( \vec{c} \) up to a scalar multiple and map the vector \( \vec{e}_k \), for any \( k \in \{1, \ldots, d\} \), to a linear combination of \( \vec{c} \) and \( \vec{e}_k \). Hence the claim follows if we show for any \( k \neq j \) that the projectivized line segment \( [\vec{c}, \vec{e}_k] \) is mapped by \( M'_{\varepsilon,j} \) to a larger segment than by \( M_\varepsilon \).

For \( k \neq j \) this is clear, since the projectivized line segment \( [\vec{c}, \vec{e}_k] \) is set-wise fixed by \( M'_{\varepsilon,j} \). For \( k = j \) we compute the coefficient quotient of \( M'_{\varepsilon,j} \cdot \vec{e}_j \) and apply the inequalities from Fact 4.2 (2) to obtain:

\[
\frac{h}{2^m} = \frac{3^q - 2^m}{2^m} = \frac{3^q}{2^m} - 1 \leq \frac{2}{2 - \varepsilon} - 1 = \frac{\varepsilon}{2 - \varepsilon}
\]
In comparison, the analogous coefficient quotient of $M_\varepsilon \cdot \vec{c}_j$ is equal to $\varepsilon$ and thus strictly bigger, since we can assume that $\varepsilon$ is small. This shows that $M_\varepsilon \cdot \vec{c}_j$ defines a point on the $M_{\varepsilon j}^\prime$-image of the projectivized line segment $[\vec{c}, \vec{c}_j]$. □

We now consider an infinite sequence of positive constants $\varepsilon_1, \varepsilon_2, \ldots$ as exhibited in the previous subsection, for which the limit intersection of the nested sequence of the cones $M_{\varepsilon_1} \cdot M_{\varepsilon_2} \cdot \ldots \cdot M_{\varepsilon_n}(\mathbb{R}^d_{\geq 0})$ is simplicial of dimension $d$.

For any $\varepsilon_n > 0$ let $m_n, q_n$ and $h_n$ be integers as in Fact 4.2 (2), and for any choice of indices $j(n) \in \{1, \ldots, d\}$ let

$$\sigma'_{n,j(n)} := \rho^{q_n}_{j(n)} \cdot \tau^{h_n}_{j(n)} \cdot \theta^{m_n}_{j(n)}$$

be the corresponding substitution as introduced above. We denote its incidence matrix by $M'_{n} := M'_{\sigma'_{n,j(n)}}$, and obtain directly from Lemma 4.3 that for any $n \in \mathbb{N}$ one has:

$$M_{\varepsilon_1} \cdot M_{\varepsilon_2} \cdot \ldots \cdot M_{\varepsilon_n}(\mathbb{R}^d_{\geq 0}) \subseteq M'_{1} \cdot M'_{2} \cdot \ldots \cdot M'_{n}(\mathbb{R}^d_{\geq 0})$$

It follows that the same inclusion is true for the limit cones, so that the limit cone for the $\sigma'_{n,j(n)}$ must also have dimension $d$.

It remains to specify the indices $j(n)$: if they are chosen so that $j(n)$ varies cyclically through the set of all indices $1, 2, \ldots, d$, then the product of any subsequent $d$ incidence matrices is positive, which suffices to guaranty that the resulting directive sequence is weekly primitive.

It follows as in the previous subsection that the $S$-adic subshift defined by the directive sequence of $\sigma'_{n,j(n)}$ for the finite set $S$ given above, is minimal and has $d$ ergodic probability measures.

Remark 4.4. Similar to Remark 4.1 we note that there are many possible variations of the construction presented in this subsection. For examples the exponents of the powers $\rho_{j}(a_j)$ and $\theta_{j}(a_j)$ can be arbitrary relative prime integers, and $\tau_{j}(a_j)$ can be any word that involves every letter of $\mathbb{A}$ a fixed number of times.

Remark 4.5. We now observe that any two of the substitutions $\rho_j$ and $\rho_{j+1}$ are conjugate to each other, through a cyclic permutation $\pi : a_j \mapsto a_{j+1}$ (for $j$ understood modulo $d$). The same holds for the $\theta_j$ and for the $\tau_j$. It follows that the above constructed directive sequence $\sigma' := \sigma'_{0,j(0)} \circ \sigma'_{1,j(1)} \circ \ldots$ can be understood as obtained through telescoping from a directive sequence $\sigma'' := \sigma''_{0} \circ \sigma''_{1} \circ \ldots$ with same associated subshift $X_{\sigma''} = X_{\sigma''}$, where all $\sigma''_n$ belong to $S = \{\rho_1, \theta_1, \tau_1, \pi\}$. This proves the statement (2) of Proposition 1.1.

Remark 4.6. The relevance of statement (2) of Proposition 1.1 is emphasized by a comparison with F. Durand’s results in [6], where it is shown that a subshift with Bratteli-Vershik representation based on a finite set of positive incidence matrices is linearly recurrent and hence uniquely ergodic.

In our construction above it is crucial that the 4 generating substitutions $\rho_1, \theta_1, \tau_1, \pi$ are not positive (indeed, not even primitive), so that suitable products of them, as for example the ones pointed out in the above proof, allows one to compose an infinite set of positive matrices without (linear or else) bound on the resulting recurrence.

4.3. Some questions.

Inspired by Remark 4.5 we define for any set $S$ of substitutions of some free monoid $\mathbb{A}^*$ the ergodic size $\eta(S)$ as the maximal number of ergodic probability measures supported by any subshift $X \subseteq \Sigma_A$ which admits an $S$-adic expansion. One can specify this further to a min-ergodic size $\eta_{\text{min}}(S)$ by adding the additional requirement that $X$ be minimal. This gives

$$1 \leq \eta(S) \leq \text{card} \mathbb{A} \quad \text{and} \quad \eta_{\text{min}}(S) \leq \eta(S)$$

for any $S \neq \emptyset$, as well as

$$\eta_{\text{min}}(S') \leq \eta_{\text{min}}(S) \quad \text{and} \quad \eta(S') \leq \eta(S)$$
for any subset $S' \subseteq S$. We say that $S$ is \textit{ergodically rich} if $\eta_{\min}(S) \geq 2$; if $\eta(S) = 1$, we call $S$ \textit{uniquely ergodic}.

In Corollary 10.10 of [2] it has been shown that for any directive sequence $\sigma$ with stationary incidence matrix $M$ the number of ergodic probability measures on the associated subshift $X_\sigma$ depends only on $M$ and not on the particular choice of $\sigma$. It follows that for the finite set $S_M$ of substitutions $\sigma_i$ with $M_{\sigma_i} = M$ the ergodic size satisfies $\eta(S) = 1$, if $M$ is chosen suitably (for example primitive).

On the other hand, the min-ergodic size of the 4-elements set $S$ exhibited in Remark 4.5 is shown there to satisfy $\eta_{\min}(S) = d$, thus motivating the following:

\textbf{Question 4.7.} (1) Given integers $k \geq 1$ and $d \geq 1$, what are the “generic” values for $\eta(S)$ and $\eta_{\min}(S)$ of any set $S$ that consists of at most $k$ substitutions over an alphabet of $d$ letters?

(2) More specifically, does there exist an ergodically rich set $S$ that consists of 2 primitive substitutions?

5. \textbf{Determination of cylinder weights for substitutions}

Let $A = \{a_1, \ldots, a_d\}$ be an alphabet, and let $\sigma : A^* \to A^*$ be a substitution which is everywhere growing, but possibly reducible. As before we denote by $X_\sigma \subseteq \Sigma_A$ the subshift associated to $\sigma$, and by $\mu$ an invariant measure on $X_\sigma$. Recall that for any $w \in A^*$ the cylinder determined by $w$ is denoted by $[w]$.

\textbf{Question 5.1.} How can one determine the measure $\mu([w])$ for an arbitrary word $w \in A^*$?

We will now describe a (fairly practical) answer to this question, based on our previous results. Along the description of this algorithm, we will illustrate the main steps on the two (famous) examples of the Thue-Morse substitution $\sigma_{TM}$ and the Fibonacci substitution $\sigma_{Fib}$:

$$
\sigma_{TM} : a \mapsto ab \quad \sigma_{Fib} : a \mapsto ab \quad b \mapsto ba 
$$

The algorithm we present consists of three steps:

\textbf{Step 1:} We first compute from the given words $\sigma(a_i)$ the incidence matrix $M_\sigma = (m_{x,y})_{x,y \in A}$, where $m_{x,y} := |\sigma(y)|_x$ denotes the number of occurrences of $x$ in the word $\sigma(y)$.

We then pass to the \textit{augmented incidence matrix}

$$
M_{\sigma}^+ := (m_{x,y}^+)_{x,y \in A_2}
$$

which is defined as follows: Its rows and columns are indexed by the set $A_2$ of all words in $A^*$ of length 1 or 2, and it contains $M_\sigma$ as diagonal block: $m_{x,y}^+ := m_{x,y}$ for all $x, y \in A$. The complementary diagonal block is a “pre-permutation matrix” (i.e. every column contains precisely one non-zero entry, and the latter is equal to 1), defined by the rule that for any word $X = x_1x_2$ of length 2 the coefficient $m_{x,y}^+$ is equal to 1 if and only if for $Y = y_1y_2$ the letter $x_1$ is the last letter of $\sigma(y_1)$ and the letter $x_2$ is the first letter of $\sigma(y_2)$. Otherwise one sets $m_{x,y}^+ = 0$.

The off-diagonal blocks are defined by the rule that $m_{x,y}^+ = 0$ if $|X| = 1$ and $|Y| = 2$, while for $|X| = 2$ and $|Y| = 1$ one sets $m_{x,y}^+ := |\sigma(Y)|_X$.

\textbf{Example 5.2.} For the Thue-Morse substitution $\sigma_{TM}$ and the Fibonacci substitution $\sigma_{Fib}$ one has $M_{\sigma_{TM}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $M_{\sigma_{Fib}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and we obtain, for the row and column indexes given by
(a, b, aa, ab, ba, bb):

\[
M^+_{\sigma_{\text{TM}}} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}, \quad M^+_{\sigma_{\text{Fib}}} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}.
\]

Remark 5.3. It is not hard to verify that with this definition one obtains for any two substitutions \(\sigma, \sigma'\) the equality \(M^+_{\sigma} \cdot M^+_{\sigma'} = M^+_{\sigma \circ \sigma'}\). Thus we have in particular, for any integer \(n \in \mathbb{N}\):

\[
M^+_{\sigma^n} = (M^+_{\sigma})^n
\]

Remark 5.4. The augmented incidence matrix \(M^+_{\sigma}\) is lower triangular by blocks, with two diagonal blocks. The first diagonal block is given by the incidence matrix \(M_{\sigma}\). The second diagonal block can be described as a Kronecker product \(S_{\sigma} \otimes P_{\sigma}\). For this, we order lexicographically the elements of \(A_2 \setminus \mathcal{A}\) which serve as indices for the second diagonal block. Now \(P_{\sigma} = (p_{xy})_{x,y \in A}\) is the “prefix matrix” of \(\sigma\); we set \(p_{xy} = 1\) if \(x\) is the first letter of \(\sigma(y)\), and otherwise \(p_{xy} = 0\). Similarly, \(S_{\sigma} = (s_{xy})_{x,y \in A}\) is “suffix matrix” of \(\sigma\): \(s_{xy} = 1\) if \(x\) is the last letter of \(\sigma(y)\), and otherwise \(s_{xy} = 0\).

As a consequence, we notice that the spectrum of \(M^+_{\sigma}\) is given by the spectrum of \(M_{\sigma}\) plus possibly 0 and some \(d'\)-th roots of the unity, for \(d' \leq d\).

Example 5.5. For the Thue-Morse substitution \(\sigma_{\text{TM}}\) and the Fibonacci substitution \(\sigma_{\text{Fib}}\), we get:

\[
S_{\sigma_{\text{TM}}} = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}, \quad P_{\sigma_{\text{TM}}} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad S_{\sigma_{\text{Fib}}} = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}, \quad P_{\sigma_{\text{Fib}}} = \begin{bmatrix}
1 & 1 \\
0 & 0 \\
\end{bmatrix}.
\]

Step 2: For any word \(w \in \mathcal{A}^+\) we define an integer \(n \geq 0\) to be \((\sigma, w)\)-large if \(|\sigma^n(a_i)| \geq |w| - 1\) for all \(a_i \in \mathcal{A}\). For such \(n\) we now define an occurrence vector \(\vec{v}(w)_n = (v_X)_{X \in \mathcal{A}_2}\) with coefficients \(v_X\) defined by the rule \(v_X = |\sigma^n(X)|_w\) if \(|X| = 1\), and \(v_X = |\sigma^n(X)|_w - |\sigma^n(x_1)|_w - |\sigma^n(x_2)|_w\) if \(X = x_1x_2\) has length 2. Thus, in the latter case, the coefficient \(v_X\) counts the number of times that \(w\) occurs as factor of \(\sigma^n(x_1)\sigma^n(x_2)\), but ignores those occurrences that are either completely contained in the first or completely contained in the second factor of this product.

Remark 5.6. (1) If \(w\) has length 1 or 2, then independently of \(\sigma\) one has that \(n = 0\) is \((\sigma, w)\)-large. The vector \(\vec{v}(w)_0\) has only zero coefficients except for the coordinate \(X = w\), where it is equal to 1.

(2) It is not hard to verify (using Remark 5.3) that the above defined \(n\)-th occurrence vector \(\vec{v}(w)_n\) satisfies, for any two \((\sigma, w)\)-large integers \(n \geq m\) the equality

\[
\vec{v}(w)_n = \vec{v}(w)_m \cdot (M^+_{\sigma})^{n-m}.
\]

Example 5.7. (1) We now consider the substitutions \(\sigma_{\text{TM}}\) and \(\sigma_{\text{Fib}}\) from Example 5.2 and any word \(w\) of length \(|w| = 2\). We know from Remark 5.6 (1) that any \(n \geq 0\) is \((\sigma_i, w)\)-large, and as warm-up exercise we consider \(n = 1\) for \(\sigma_{\text{TM}}\) and \(n = 2\) for \(\sigma_{\text{Fib}}\). For the same ordering of coordinates as in Example 5.2 this gives for \(\sigma_{\text{TM}}\) the following occurrence vectors:

| \(\vec{v}(aa)_1\) | \(\vec{v}(ab)_1\) | \(\vec{v}(ba)_1\) | \(\vec{v}(bb)_1\) |
|------------------|------------------|------------------|------------------|
| 0, 0, 0, 0, 1, 0 | 1, 0, 0, 0, 0, 1 | 0, 1, 0, 0, 0, 0 | 0, 0, 0, 1, 0, 0 |

For \(\sigma_{\text{Fib}}\) we obtain the vectors:

| \(\vec{v}(aa)_2\) | \(\vec{v}(ab)_2\) | \(\vec{v}(ba)_2\) | \(\vec{v}(bb)_2\) |
|------------------|------------------|------------------|------------------|
| 0, 0, 1, 1, 0, 0 | 1, 1, 0, 0, 0, 0 | 1, 0, 0, 0, 1, 1 | 0, 0, 0, 0, 0, 0 |

For both sets of occurrence vectors one uses Remark 5.6 (1) to quickly verify the equation (5.1), for \(m = 0\) and \(n\) as above.
(2) We now want to illustrate how to deal with a more elaborate case, by picking “at random” a slightly longer word:

\[ w = baabab \]

One first iterates the given substitution on the generators, until their images are longer or equal to \(|w| - 1 = 5\). In our two cases this gives:

\[
\begin{align*}
\sigma_{TM}^3 : a &\mapsto abbabaab \\
b &\mapsto baababba \\
\sigma_{Fib}^4 : a &\mapsto abaababa \\
b &\mapsto ababa
\end{align*}
\]

We now compute for \(\sigma_{TM}^3\):

\[
\begin{align*}
v_a &:= |\sigma_{TM}^3(a)|_w = 0, \\
v_b &:= |\sigma_{TM}^3(b)|_w = 1, \\
v_{aa} &:= |\sigma_{TM}^3(aa)|_w = 1, \\
v_{ab} &:= |\sigma_{TM}^3(ab)|_w = 1, \\
v_{ba} &:= |\sigma_{TM}^3(ba)|_w = 1, \\
v_{bb} &:= |\sigma_{TM}^3(bb)|_w = 2
\end{align*}
\]

Similarly, we obtain for \(\sigma_{Fib}^4\):

\[
\begin{align*}
v_a &:= |\sigma_{Fib}^4(a)|_w = 1, \\
v_b &:= |\sigma_{Fib}^4(b)|_w = 0, \\
v_{aa} &:= |\sigma_{Fib}^4(aa)|_w = 2, \\
v_{ab} &:= |\sigma_{Fib}^4(ab)|_w = 1, \\
v_{ba} &:= |\sigma_{Fib}^4(ba)|_w = 2, \\
v_{bb} &:= |\sigma_{Fib}^4(bb)|_w = 1
\end{align*}
\]

We then use the general formula for the occurrence vector

\[ ^t\overline{\nu}(w)_n = (v_a, v_b, v_{aa} - 2v_a, v_{ab} - v_a - v_b, v_{ba} - v_b - v_a, v_{bb} - 2v_b) \]

to obtain for \(\sigma_{TM}^3\)

\[ ^t\overline{\nu}(aabab)_3 = (0, 1, 1, 0, 0, 0), \]

and for \(\sigma_{Fib}^4\)

\[ ^t\overline{\nu}(aabab)_4 = (1, 0, 0, 0, 1, 1). \]

**Step 3:** We know from Proposition 3.4 that every ergodic invariant measure \(\mu_i\) on the substitution subshift \(X_\sigma \subseteq \Sigma_A\) is given for some suitable exponent \(k \geq 1\) by a non-negative column eigenvector \(\overline{\nu}_i = (\nu_i(x))_{x \in A}\) of \(M_\sigma^k\) with eigenvalue \(\lambda_i > 1\), with the property \(\nu_x = \mu_i([x])\) for any \(x \in A\) (see Lemma 3.3). The converse statement also holds.

Since for the matrix \(M_\sigma^+\) all eigenvalues \(\lambda_j\) of the lower \(d^2 \times d^2\) diagonal block are of modulus \(|\lambda_j| \leq 1\), it follows from standard Perron-Frobenius theory for non-negative matrices (see Appendix 11.3 of [2]) that any eigenvector \(\overline{\nu}_i\) of \(M_\sigma^k\) as above determines uniquely a non-negative eigenvector \(\overline{\nu}_i^+\) of \((M_\sigma^+)^k\), which contains \(\overline{\nu}_i\) as “subvector” at the upper \(d\) coordinates (i.e. the coordinates for \(A \subseteq A_2\)), and which has the same eigenvalue \(\lambda_i\).

We have thus explained all terms used in Proposition 1.2, which we abbreviate here to:

**Proposition 5.8.** For any convex combination \(\mu = \sum c_i \mu_i\) of the above ergodic measures \(\mu_i\) the measure of the cylinder \([w]\) is given by the scalar product (written as matrix multiplication)

\[ \mu([w]) = ^t\overline{\nu}(w)_n \cdot \sum c_i \frac{1}{\lambda_i} \overline{\nu}_i^+. \]

**Proof.** It suffices to prove \(\mu_i([w]) = ^t\overline{\nu}(w)_n \cdot \frac{1}{\lambda_i} \overline{\nu}_i^+\) for a single ergodic measure \(\mu_i\). But this is precisely the formula from Corollary 3.6 for the measure \(\mu_i = \mu \overline{\nu}_i\) given by a vector tower \(\overline{\nu}_i\) that is compatible with the stationary directive sequence defined by the substitution \(\sigma\), where \(\overline{\nu}_i\) is defined by the above eigenvector \(\overline{\nu}_i\) through setting \(\overline{\nu}_i = (\frac{1}{\lambda_i} \overline{\nu}_i(x))_{x \in A \cup \{0\}}\).

**Remark 5.9.** Note that for any measure \(\mu_i\) as above the associated right eigenvector \(\overline{\nu}_i^+\) of the augmented matrix \(M_\sigma^+\) gives directly the measures of the cylinders of size 1 and 2. For the words of length 1 this is stated in Lemma 3.3, and for the words \(X\) of length 2 this follows from Proposition 5.8, since \(\overline{\nu}(X)_0\) is the unit vector for the coordinate \(X\), and 0 is \((\sigma, X)\)-large for any everywhere growing \(\sigma\) and \(X\) of length 2.
Example 5.10. The two substitutions $\sigma_{TM}$ and $\sigma_{Fib}$ from Example 5.2 are primitive and hence uniquely ergodic. For the Thue-Morse substitution $\sigma_{TM}$ on the full shift $\Sigma_{(a,b)}$, with incidence matrix $M_{\sigma_{TM}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the probability measure $\mu_{TM}$ is defined by the Perron-Frobenius eigenvector $\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda_{TM} = 2$. For the Fibonacci substitution $\sigma_{Fib}$ with incidence matrix $M_{\sigma_{Fib}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ we obtain analogously the measure $\mu_{Fib}$ through the eigenvector $\frac{1}{\varphi} \begin{bmatrix} \varphi \\ 1 \end{bmatrix}$ with eigenvalue $\lambda_{Fib} = \varphi$, for the golden mean $\varphi := \frac{1+\sqrt{5}}{2}$. For the augmented incidence matrices $M_{\sigma_{TM}}^+$ and $M_{\sigma_{Fib}}^+$ computed in Example 5.2 we thus calculate the augmented eigenvectors (written transposed) as $t^+v_{TM} = \frac{1}{2} t(1,1,\frac{1}{\varphi},\frac{2}{\varphi},\frac{1}{\varphi})$ and $t^+v_{Fib} = \frac{1}{\varphi^2}(\varphi,1,\varphi^{-1},1,1,0)$. Hence using Remark 5.6 (1) we can now evaluate the formula in Proposition 5.8 to obtain:

| $w$   | $a$ | $b$ | $aa$ | $ab$ | $ba$ | $bb$ |
|-------|-----|-----|------|------|------|------|
| $\mu_{TM}([w])$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

as well as

| $w$   | $a$ | $b$ | $aa$ | $ab$ | $ba$ | $bb$ |
|-------|-----|-----|------|------|------|------|
| $\mu_{Fib}([w])$ | $\frac{1}{\varphi}$ | $\frac{1}{\varphi^2}$ | $\frac{1}{\varphi^2}$ | $\frac{1}{\varphi^2}$ | $\frac{1}{\varphi^2}$ | $0$ |

It is not hard to check that Remark 5.9 gives the same answer, and is slightly more direct. But in order to compute the measures of cylinders of size bigger or equal to 3, Remark 5.9 does not apply, so that one needs to fall back onto Proposition 5.8. For example, for the word $w = baabab$ considered in Example 5.7 (3) we obtain

$$
\mu_{TM}([baabab]) = \frac{1}{8} t^+(baabab)_3 \cdot v_{TM}^+ = \frac{1}{12}
$$

and

$$
\mu_{Fib}([baabab]) = \frac{1}{\varphi^4} t^+(baabab)_4 \cdot v_{Fib}^+ = \frac{1}{\varphi^4}.
$$

For completeness we mention the fact that all the cylinders of length 3 have the same measure for Thue-Morse subshift, or else they have measure 0.

6. Examples

We conclude this paper with three slightly more challenging examples, concerning non-primitive everywhere growing substitutions.

6.1. An example with a periodic sequence in the substitution subshift.

The methods developed in [5] don’t seem to work for substitutions with periodic leaves in their associated subshift. Such a restriction doesn’t hold for the technology presented here, as is illustrated by the following.

Let $\sigma$ be the (non-primitive) substitution given by:

$$
\begin{align*}
  a &\mapsto acbca \\
  b &\mapsto ba \\
  c &\mapsto cc
\end{align*}
$$
We compute the incidence matrix and both, the suffix and the prefix matrix for \( \sigma \):

\[
M_{\sigma} = \begin{bmatrix}
2 & 1 & 0 \\
1 & 1 & 0 \\
2 & 0 & 2
\end{bmatrix}, \quad S_{\sigma} = \begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad P_{\sigma} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

The occurrences of words of length 2 in the generator images \( \sigma(a), \sigma(b) \) and \( \sigma(c) \), which serve to determine the columns of the lower left off-diagonal block of the augmented incidence matrix, are given respectively (with indices in lexicographic order) by:

\[
\iota(0,0,1 : 0,0,1 : 1,1,0), \; \iota(0,0,0 : 1,0,0 : 0,0,0), \; \iota(0,0,0 : 0,0,0 : 0,0,1)
\]

We have now all ingredients to compute the augmented incidence matrix \( M_{\sigma}^+ \), which is of size 12 × 12. Recall that the upper diagonal block (of size 3 × 3) is equal to \( M_{\sigma} \), and the lower block (of size 9 × 9) is equal to the Kronecker product \( S_{\sigma} \otimes P_{\sigma} \).

\[
M_{\sigma}^+ = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

We now observe that the incidence matrix \( M_{\sigma} \) itself has two primitive diagonal blocks, with eigenvalue \( \lambda_2 = 2 \) for the bottom block of \( M_{\sigma} \), and \( \lambda_1 = 1 + \varphi \) for the top one (with \( \varphi = \frac{1 + \sqrt{5}}{2} \) as before). Since \( 1 + \varphi > 2 \) holds, the case (2) of Corollary 3.5 applies, giving rise to two non-negative eigenvectors \( \vec{v}_1 \) for \( \lambda_1 \) and \( \vec{v}_2 \) for \( \lambda_2 \), which determine invariant ergodic probability measures \( \mu_1 \) and \( \mu_2 \) respectively on the substitution subshift \( X_\sigma \). As explained in Step 3 of section 5, each \( \vec{v}_i \) gives rise to a non-negative eigenvector \( \vec{v}_i^+ \) of the augmented matrix \( M_{\sigma}^+ \), which we can use to determine the measure of specific cylinders \( [w] \) via Proposition 5.8. But first we consider the words of length 1 and 2, as for their cylinders the measure is read off directly from \( \vec{v}_1 \) and \( \vec{v}_2^+ \) respectively, using Remark 5.9:

- For the eigenvalue \( \lambda_2 = 2 \) we compute the eigenvector \( \vec{v}_2 \) and we obtain for the words of length 1:

\[
\mu_2([a]) = \mu_2([b]) = 0, \quad \mu_2([c]) = 1
\]

The computation of the augmented eigenvector \( \vec{v}_2^+ \) shows that every cylinder of size 2 has measure 0, except \([cc]\). Thus the measure \( \mu_2 \) is atomic: It only gives positive measure to the biinfinite periodic word \( \ldots ccc \ldots \).

- For the eigenvalue \( \lambda_1 = 1 + \varphi = \varphi^2 \) we compute the eigenvector

\[
\vec{v}_1^+ = \frac{1}{3} \iota(\varphi - 1, 2 - \varphi, 2 | 5\varphi - 8, 0, 7 - 4\varphi : 5 - 3\varphi, 0, 2\varphi - 3 : 2 - \varphi, 2 - \varphi, 2\varphi - 2)
\]

which defines the second ergodic measure \( \mu_1 \). We apply Remark 5.9 to obtain:
and now, according to Proposition 5.8, the matrix product
\[ \frac{1}{\varphi^2} \tilde{v}(bcacc) \cdot \tilde{v}_i^+ \]
For \( i = 2 \) this gives of course \( \mu_2([bdacc]) = 0 \), while for \( i = 1 \) we obtain:
\[ \mu_1([bdacc]) = \frac{1}{\varphi^4} \left( \varphi - 1 \right) + \frac{7 - 4\varphi}{3} \cdot \frac{2\varphi - 3}{3} = \frac{3 - \varphi}{\varphi^4} \]

6.2. Example from Bezuglyi, Kwiatkowski, Medynets and Solomyak.
This example has already been treated in [5], Example 5.8 (up to a permutation of the letters); we present here the alternative treatment by our methods.

\[ \sigma : \begin{align*}
    a & \mapsto baaad \\
    b & \mapsto baad \\
    c & \mapsto cb \\
    d & \mapsto de \\
    e & \mapsto ed
\end{align*} \]

The augmented incidence matrix \( M_\sigma^+ \) has size 30. Its upper diagonal block is equal to the incidence matrix \( M_\sigma \), while the lower one is given by \( S \otimes P \). We compute:
\[
M_\sigma = \begin{pmatrix}
3 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad P = \begin{pmatrix}
0 & 0 & 0 & 0 \end{pmatrix}
\]

Next we list the occurrences of words of length 2 in the images of the generators, which serve to determine the columns of the lower left off-diagonal block of \( M_\sigma^+ \). As before, these columns are written in transposed form, with \( A = \{a, b, c, d, e\} \) and indices ordered lexicographically:

\[
(|\sigma(a)|_{xy})_{y \in A} = (2, 0, 0, 1, 0; 1, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0)
\]
\[
(|\sigma(b)|_{xy})_{y \in A} = (0, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0)
\]
\[(|\sigma(c)|_{xy})_{x,y \in \mathcal{A}} = (0, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0)\]
\[(|\sigma(d)|_{xy})_{x,y \in \mathcal{A}} = (0, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0)\]
\[(|\sigma(e)|_{xy})_{x,y \in \mathcal{A}} = (0, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 1, 0)\]

We now observe that all three diagonal blocks of \(M_\sigma\) are distinguished, so that for each of them there is a non-negative eigenvector of \(M_\sigma^+\) with eigenvalue \(>1\), giving rise to 3 distinct ergodic probability measures \(\mu_{\text{full}}, \mu_{b,c}, \mu_{d,e}\) on the substitution subshift \(X_\sigma\).

- For each of the two lower diagonal blocks the substitution \(\sigma\) defines a “sub-substitution” \(\sigma_{b,c}\) and \(\sigma_{d,e}\), since both, the submonoids generated by \(b\) and \(c\) as well as that for \(d\) and \(e\), are \(\sigma\)-invariant. Both of these sub-substitutions are (up to renaming of the generators) equal to the Thue-Morse substitution \(\sigma_{\text{TM}}\) from section 5. They define minimal sub-shifts \(X_{b,c} \subseteq X_\sigma\) and \(X_{d,e} \subseteq X_\sigma\), which are precisely the support of the measures \(\mu_{b,c}\) and \(\mu_{d,e}\) respectively. Since the Thue-Morse substitutions has already been treated in detail in section 5, we will skip here the corresponding computations.

- The top diagonal block (of size 1 \(\times\) 1) of \(M_\sigma\) has eigenvalue 3 and determines a non-negative eigenvector \(\tilde{v}^+\) of \(M_\sigma^+\), which is chosen here as to have integer coefficients. In order to determine from \(\tilde{v}^+\) via Remark 5.9 the \(\mu_{\text{full}}\)-measures for the cylinders of size 1 and 2 we then rescale the values of the coordinates (by the factor \(\frac{1}{36}\)) to get total measure 1.

The first 5 coordinates of \(\tilde{v}^+\) define a “subvector” \(\tilde{v} = (v_x)_{x \in \mathcal{A}}\) which satisfies \(M_\sigma \tilde{v} = 3 \tilde{v}\). We compute \(\tilde{v} = t(12, 8, 4, 8, 4)\), which gives:

\[\mu_{\text{full}}([a]) = \frac{1}{3}, \mu_{\text{full}}([b]) = \frac{2}{9}, \mu_{\text{full}}([c]) = \frac{1}{9}, \mu_{\text{full}}([d]) = \frac{2}{9}, \mu_{\text{full}}([e]) = \frac{1}{9}\]

We then use the above described coefficients, for the lower diagonal and the lower left off-diagonal block of the matrix \(M_\sigma^+\), to compute the last 25 coordinates \(v_{xy}\) of \(\tilde{v}^+\), with \(x, y \in \mathcal{A}\):

\[
\begin{array}{cccccccc}
v_{aa} & v_{ab} & v_{ac} & v_{ad} & v_{ae} & v_{ba} & v_{bb} & v_{bc} & v_{bd} & v_{be} \\
8 & 0 & 0 & 4 & 0 & 4 & 1 & 3 & 0 & 0
\end{array}
\]

\[
\begin{array}{cccccccc}
v_{ca} & v_{cb} & v_{cc} & v_{cd} & v_{ce} & v_{da} & v_{db} & v_{dc} & v_{dd} & v_{de} \\
0 & 3 & 1 & 0 & 0 & 0 & 3 & 0 & 2 & 3
\end{array}
\]

\[
\begin{array}{cccc}
v_{ec} & v_{eb} & v_{ee} \\
0 & 1 & 0 & 2 & 1
\end{array}
\]

Hence all cylinders of size 2 have \(\mu_{\text{full}}\)-measure 0, except for:

\[\mu_{\text{full}}([aa]) = \frac{2}{9}, \mu_{\text{full}}([ad]) = \frac{1}{9}, \mu_{\text{full}}([ba]) = \frac{1}{9}, \mu_{\text{full}}([bb]) = \frac{1}{36}, \mu_{\text{full}}([bc]) = \frac{1}{12},\]

\[\mu_{\text{full}}([cb]) = \frac{1}{12}, \mu_{\text{full}}([cc]) = \frac{1}{36}, \mu_{\text{full}}([db]) = \frac{1}{12}, \mu_{\text{full}}([dd]) = \frac{1}{18},\]

\[\mu_{\text{full}}([de]) = \frac{1}{12}, \mu_{\text{full}}([eb]) = \frac{1}{36}, \mu_{\text{full}}([ed]) = \frac{1}{18}, \mu_{\text{full}}([ee]) = \frac{1}{36}\]

We observe that \(\mu_{\text{full}}\) has indeed full support on \(X_\sigma\), as follows in more generality from the fact that the eigenvector \(\tilde{v}\) of \(M_\sigma\) is positive.

Since the \(\sigma\)-image of every generator \(a_i\) has length \(|a_i| \geq 2\), we can use the above values also for a direct evaluation via Proposition 5.8 of any cylinder \([w]\) of size \(|w| = 3\), through a quick calculation.
of the occurrence vector $\vec{v}(w)_0$. For example, this occurrence vector for $w = edb$ has coefficients equal to 1 only in the coordinates $ea$ and $eb$, giving
\[
\mu_{\text{full}}([edb]) = \mu_{\text{full}}([ea]) + \mu_{\text{full}}([eb]) = \frac{1}{36},
\]
while for $w = cba$ we obtain
\[
\mu_{\text{full}}([cba]) = \mu_{\text{full}}([ba]) = \frac{1}{9}.
\]

6.3. A family of examples with varying number of ergodic measures.

For any integer $k \geq 1$ we consider the substitution $\sigma_k$ defined by:
\[
\tau : \begin{cases} 
  c \mapsto cd \\
  d \mapsto c 
\end{cases}, \quad \sigma_k : \begin{cases} 
  a \mapsto bacaab \\
  b \mapsto aba \\
  c \mapsto \tau^k(c) \\
  d \mapsto \tau^k(d) 
\end{cases}
\]

We compute the incidence matrix for $\sigma_k$:
\[
M_{\sigma_k} = \begin{bmatrix} 
  3 & 2 & 0 & 0 \\
  2 & 1 & 0 & 0 \\
  1 & 0 & a_k & b_k \\
  0 & 0 & c_k & d_k 
\end{bmatrix}
\]
with $\begin{bmatrix} a_k & b_k \\
  c_k & d_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\
  1 & 0 \end{bmatrix}^k = M_{\tau}^k$.

We note that for any $k \geq 1$ the Corollary 3.5 applies, and since $\tau$ is up to a change of generators equal to the Fibonacci substitution $\sigma_{\text{Fib}}$ treated extensively in section 5, we know already that the primitive bottom diagonal block $M_{\tau}^k$ of $M_{\sigma_k}$ has PF-eigenvalue $\varphi^k$. Its corresponding eigenvector determines an invariant probability measure which is supported only on the bottom stratum of $\sigma_k$, defined by $c$ and $d$, and the cylinder values are precisely those computed in section 5 for the Fibonacci substitution.

The top diagonal block of $M_{\sigma_k}$ is independent of $k$ and turns out to be actually equal to $M_{\tau}^3$: it thus has PF-eigenvalue $\varphi^3$. Hence for $k \geq 3$ case (1) of Corollary 3.5 holds, so that the above described invariant “Fibonacci” measure supported on the bottom stratum is the only invariant measure on the substitution subshift $X_{\sigma_k}$.

For $k = 1$ and $k = 2$, however, we find ourselves in case (2) of Corollary 3.5, and hence in both cases there is a second invariant probability measure $\mu_k$ with full support, which we will now consider in detail:

The case $k = 1$: We first compute an integer PF-eigenvector $^t(v_a, v_b, v_c, v_d)$ for the eigenvalue $\varphi^3$ of the above given (non-augmented) incidence matrix $M_{\sigma_1}$, where we note that $\varphi^3 = 2\varphi + 1$:
\[
\begin{array}{cccc}
  & v_a & v_b & v_c & v_d \\
 3\varphi^3 & 3\varphi^2 & \varphi^3 & 1 \\
\end{array}
\]

This gives directly the measure of the cylinders of size 1, through the normalization $\mu_1([x]) = \frac{1}{\Pi_{x\in A} v_x}$ for any $x \in A = \{a, b, c, d\}$.

For the cylinders of size 2 we compute (as in the previous examples) the augmented incidence matrix $M_{\sigma_1}^+$ of $\sigma_1$, which has size 20. We note:
\[
S_{\sigma_1} = \begin{bmatrix} 
  0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0 
\end{bmatrix}, \quad P_{\sigma_1} = \begin{bmatrix} 
  0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 1 & 1 \\
  0 & 0 & 0 & 0 
\end{bmatrix}.
\]
Next we list the occurrences of words of length 2 in the images of the generators, which serve to determine the columns of the lower left off-diagonal block of $M^+_{\sigma}$. These columns are written as in the previous examples in transposed form, with indices ordered lexicographically:

\[
(|\sigma_1(a)|_{xy})_{x,y \in A} = (1, 1, 1, 0 : 1, 0, 0, 0 : 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
\]
\[
(|\sigma_1(b)|_{xy})_{x,y \in A} = (0, 1, 0, 0 : 1, 0, 0, 0 : 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
\]
\[
(|\sigma_1(c)|_{xy})_{x,y \in A} = (0, 0, 0, 0 : 0, 0, 0 : 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
\]
\[
(|\sigma_1(d)|_{xy})_{x,y \in A} = (0, 0, 0, 0 : 0, 0, 0 : 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
\]

We now compute a right PF-eigenvector of $M^+_{\sigma}$ associated to $\varphi^3$, with coefficients $v_w$ for any $|w| \leq 2$ that agree for $|w| = 1$ with the above coefficients $v_x$ for the eigenvector of $M_{\sigma}$. From the above specified coefficients of $M^+_{\sigma}$ we derive through a minor computational effort:

\[
\begin{align*}
 v_a + v_bb &= \varphi^3 v_{aa} \\
 v_a + v_b + v_{ba} &= \varphi^3 v_{ab} \\
 v_a + v_{bc} + v_{bd} &= \varphi^3 v_{ac} \\
 0 &= \varphi^3 v_{ad} \\
 v_a + v_{db} &= \varphi^3 v_{ca} \\
 v_{da} &= \varphi^3 v_{cb} \\
 v_{dc} + v_{dd} &= \varphi^3 v_{cc} \\
 v_c &= \varphi^3 v_{cd}
\end{align*}
\]

This gives:

\[
\begin{array}{cccccccc}
 v_{aa} & v_{ab} & v_{ac} & v_{ad} & v_{ba} & v_{bb} & v_{bc} & v_{bd} \\
 \frac{\varphi^3 v_{aa}}{\varphi^3 - 1} & \frac{\varphi^3 v_{ab}}{\varphi^3 - 1} & 0 & \frac{\varphi^3 v_{ac}}{\varphi^3 - 1} & \frac{\varphi^3 v_{ad}}{\varphi^3 - 1} & \frac{\varphi^3 v_{ba}}{\varphi^3 - 1} & \frac{\varphi^3 v_{bb}}{\varphi^3 - 1} & \frac{\varphi^3 v_{bc}}{\varphi^3 - 1} & \frac{\varphi^3 v_{bd}}{\varphi^3 - 1} & 0
\end{array}
\]

As above, we obtain the measure of the cylinders of size $|w| = 2$ through the normalization $\mu_1([w]) = \frac{1}{11\varphi + 8} v_w$.

**The case $k = 2$:** We proceed precisely as in the case $k = 1$ to compute from $M_{\sigma}$ an integer eigenvector for the eigenvalue $\varphi^3$:

\[
\begin{array}{cccc}
 v_a & v_b & v_c & v_d \\
 5\varphi + 2 & 2\varphi + 3 & 2\varphi + 2 & \varphi
\end{array}
\]

We next determine

\[
S_{\sigma_2} = \begin{bmatrix}
 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
P_{\sigma_2} = \begin{bmatrix}
 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & 0 \\
 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0
\end{bmatrix}.
\]

as well as:

\[
(|\sigma_2(a)|_{xy})_{x,y \in A} = (1, 1, 1, 0 : 1, 0, 0, 0 : 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
\]
\[
(|\sigma_2(b)|_{xy})_{x,y \in A} = (0, 1, 0, 0 : 1, 0, 0, 0 : 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
\]
We thus compute
\[
\begin{align*}
\left( |\sigma^2(c)|_{xy} \right)_{x,y \in A} &= (0, 0, 0, 0 : 0, 0, 0, 1 : 0, 1, 0) \\
\left( |\sigma^2(d)|_{xy} \right)_{x,y \in A} &= (0, 0, 0, 0 : 0, 0, 0, 0, 1 : 0, 0, 0, 0, 0)
\end{align*}
\]

and obtain:
\[
\begin{align*}
&\begin{align*}
\phi^3 v_{\phi^3} &= v_a + v_b \\
v_a + v_b + v_{ba} &= \phi^3 v_{ab} \\
v_a + v_{bc} + v_{bd} &= \phi^3 v_{ac} \\
0 &= \phi^3 v_{ad}
\end{align*} \\
&\begin{align*}
v_a + v_{cb} &= \phi^3 v_{ca} \\
v_{ca} &= \phi^3 v_{cb} \\
v_{cc} + v_{cd} &= \phi^3 v_{cc} \\
v_c + v_{dd} &= \phi^3 v_{cd}
\end{align*}
\]

and obtain:
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
v_{a} & v_{b} & v_{ac} & v_{ad} & v_{ba} & v_{bb} & v_{bc} & v_{bd} \\
\hline
\phi^3 v_{\phi^3} & v_a + v_b & \phi^3 v_{\phi^3} & 0 & v_a + v_b & \phi^3 v_{\phi^3} & 0 & 0 \\
& v_a + v_b & 0 & v_a + v_b & v_a & v_a & v_a & v_a \\
\hline
\end{array}
\]

As for \( k = 1 \) we obtain the measure of any cylinder of size \( |w| \leq 2 \) from the coefficients \( v_w \) through a normalization given by:
\[
\mu_2([w]) = \frac{v_w}{10\phi + 7}
\]

References

[1] M. Adamska, S. Bezuglyi, O. Karpel, and J. Kwiatkowski, Subdiagrams and invariant measures on Bratteli diagrams. Ergodic Theory Dynam. Systems 37 (2017), 2417–2452

[2] N. Bédaride, A. Hilion and M. Lustig, Graph towers, laminations and their invariant measures. J. London Math. Soc. 95 (2020), 1–61

[3] V Berthé and V. Delecroix, Beyond substitutive dynamical systems: S-adic expansions. RIMS Kôkyûroku Bessatsu B46 (2014), 81–123

[4] S. Bezuglyi, O. Karpel and J. Kwiatkowski, Exact number of ergodic invariant measures for Bratteli diagrams. J. Math. Anal. Appl. 480 (2019), Article 123431

[5] S. Bezuglyi, J. Kwiatkowski, K. Medynets and B. Solomyak, Invariant measures on stationary Bratteli diagrams. Ergodic Theory Dynam. Systems 30 (2010), 973–1007

[6] F. Durand, Combinatorics on Bratteli diagrams and dynamical systems. In “Combinatorics, automata and number theory”. Encyclopedia Math. Appl. 135, 324–372, Cambridge Univ. Press, Cambridge 2010

[7] F. Durand, J. Leroy and G. Richomme, Do the properties of an S-adic representation determine factor complexity ? J. Integer Sequences 16 (2013), Article 13-2-6

[8] S. Ferenczi, A. M. Fisher and M. Talet, Minimality and unique ergodicity for adic transformations. J. Anal. Math. 109 (2009), 1–31

[9] A. B. Katok, Invariant measures of flows on orientable surfaces. Dokl. Akad. Nauk SSSR 211 (1973), 775–778

[10] W. A. Veech, Moduli spaces of quadratic differentials. J. Analyse Math. 55 (1990), 117–171

[11] P. Walters, Ergodic theory – introductory lectures, Lecture Notes in Mathematics 458, Springer-Verlag, Berlin-New York, 1975.