More Efficient, Doubly Robust, Nonparametric Estimators of Treatment Effects in Multilevel Studies

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Abstract

When studying treatment effects in multilevel studies, investigators commonly use (semi-)parametric estimators, which make strong parametric assumptions about the outcome, the treatment, and/or the correlation between individuals. We propose two nonparametric, doubly robust, asymptotically Normal estimators of treatment effects that do not make such assumptions. The first estimator is an extension of the cross-fitting estimator applied to clustered settings. The second estimator is a new estimator that uses conditional propensity scores and an outcome covariance model to improve efficiency. We apply our estimators in simulation and empirical studies and find that they consistently obtain the smallest standard errors.

1 Introduction

When studying treatment effects in the social sciences, study units are often clustered together and form dependencies. For example, in educational assessment studies, such as the Trends in International Mathematics and Science Study, the Programme for International Student Assessment, or the Early Childhood Longitudinal Study (ECLS), students (i.e. the study units) are clustered at the classroom or school level. In the National Study of Learning Mindsets that was part of the workshop at the 2018 Atlantic Causal Inference Conference (Carvalho et al., 2019), students (i.e. the study units) were grouped into schools. In cross-classified educational studies, students belong to two, non-nested clusters, say school and neighborhood, and students’ data from the same school or the same neighborhood may be correlated due to shared school or neighborhood characteristics (Raudenbush and Bryk, 2002). Collectively, these studies are known as multilevel studies where data from study units may be correlated due to existing clustering or hierarchical relationships.

To account for clustering in multilevel studies, investigators often use parametric mixed effect models or semiparametric estimating equations (Liang and Zeger, 1986; Hong and Raudenbush, 2006; Gelman and Hill, 2006; Arpino and Mealli, 2011; Thoemmes and West, 2011; Li et al., 2013; Arpino and Cannas, 2016). Some well-known methods include generalized linear mixed effects models (GLMMs) and generalized estimating equations (GEEs). However, a key limitation with these methods is that they require a correctly specified parametric component in their estimators,
which, if mis-specified, could introduce bias. A bit more formally, let $O_{ij} = (Y_{ij}, A_{ij}, X_{ij})$ denotes the outcome, binary treatment, and pre-treatment covariates, respectively, of study unit $j$ in cluster $i$. GLMMs assume that the random effects, which model the correlation between study units in a cluster, are Normally distributed and the outcomes of a pair of units $ij$ and $ij'$ follow a conditional bivariate Normal distribution with a constant off-diagonal covariance matrix; see Fitzmaurice et al. (2012, Chapter 8) for a textbook discussion. GEEs and other semiparametric approaches do not make distributional assumptions, but the outcome model $E(Y_{ij} | A_{ij}, X_{ij})$ or the propensity score model $E(A_{ij} | X_{ij})$ (Rosenbaum and Rubin, 1983) is often assumed to be parametric (Robins, 1994; Gray and Brookmeyer, 1998; Robins et al., 1999; Gray and Brookmeyer, 2000; Robins et al., 2000; Goetgeluk and Vansteelandt, 2008; Stephens et al., 2012; Zetterqvist et al., 2016; Liu et al., 2018; Cafri et al., 2019; Fuentes et al., 2021; ?). van der Vaart (1996) proposed an extension of GEEs to allow for nonparametric mixtures of parametric models. But, all the aforementioned methods may lead to biased, inconsistent estimates of treatment effects when the parametric component of their estimators are mis-specified.

Recently, there has been growing number of works on using supervised machine learning (ML) algorithms to estimate treatment effects where both of the outcome and the treatment are modeled nonparametrically; see van der Laan et al. (2007a), Hill (2011), van der Laan and Rose (2011), Athey and Imbens (2016), Chernozhukov et al. (2018), Athey et al. (2019), Hahn et al. (2020), Nie and Wager (2020), and references therein. But, many of these estimators are designed for independent and identically distributed (i.i.d.) data. Also, while they have been applied to clustered settings (see Section 4.1 of Carvalho et al. (2019) for a summary), to the best of our knowledge, there is lack of theoretical justification on whether they have been appropriated applied and how to modify them for multilevel studies. Finally, there is some limited work on estimating nonparametric functionals in fixed effects settings (Henderson et al., 2008; Sun et al., 2009; Lin et al., 2014).

The goal of this paper is to study nonparametric, doubly robust estimation of treatment effects in multilevel studies where we allow both (i) the outcome regression $E(Y_{ij} | A_{ij}, X_{ij})$, (ii) the propensity score $E(A_{ij} | X_{ij})$ and (iii) the correlation structure between study units to be nonparametric; in other words, we allow both the “first-order” conditional moments (i.e. (i) and (ii)) and the “second-order” conditional moments (i.e. (iii)) to be nonparametric. Specific contributions include the following.

(a) We propose a nonparametric model for multilevel studies that places nonparametric, invariance restrictions on the outcome regression and the propensity score. Our nonparametric model encompasses GLMMs, GEEs, and even nonparametric latent variable models used to characterize dependency between study units. Our nonparametric model also encompasses two-level, three-level, cross-classified with non-nested clusters, or other complex/hierarchical structures.

(b) We present a naive, yet simple approach to modify existing ML algorithms initially designed
for i.i.d. data to achieve doubly robust, consistent, and asymptotically Normal estimates of treatment effects under the proposed nonparametric model. The proposed approach is a simple modification of sample splitting/cross-fitting (Schick, 1986; Chernozhukov et al., 2018) where the model training occurs at the cluster level. While the methodological and theoretical novelty underlying this method is minor, we believe the approach fills an important gap in the recent cross-fitting literature in causal inference, which has been typically used for i.i.d. data. Practically, this approach addresses a concern raised among practitioners (see Section 4.1 of Carvalho et al. (2019) for discussions) about how to properly use i.i.d.-based supervised ML algorithms in clustered settings. We also believe the approach would be useful as a simple, preliminary analysis of treatment effects in multilevel studies.

We present a new type of nonparametric, doubly robust, consistent, and asymptotically Normal estimators of treatment effects in multilevel studies. This new type of doubly robust estimators require nearly identical type of assumptions as existing ML-based estimators, but can be more efficient, especially compared to the simple estimator in (b). Our new estimator achieves these efficiency gains by estimating two new functionals that capture the correlation structure in the treatment and the outcome: (i) a conditional propensity score, which is the propensity score of a study unit given his/her peers’ treatment status and pre-treatment covariates, and (ii) an outcome covariance model, which measures the correlation in the residualized outcome between study units in the same cluster. More importantly, in real data examples, the estimator in (b) and other existing estimators have 8.8 to 32 percent larger variances than our new estimator; see Section 6 for details.

Despite the attractive properties mentioned above and the promising performance in real studies, the two proposed methods carry important limitations, which we detail throughout the text and Section 7 summarizes them. Also, in Section 5, we numerically assess the robustness of our methods when some of the underlying assumptions, specifically the bounded cluster size restriction in Section 2.3, are violated.

The paper is organized as follows. Section 2 introduce the notations, the assumptions, and our nonparametric model for multilevel data. Sections 3 and 4 present the two method and describe their properties. Sections 5 and 6 assess the performance of the proposed methods through a simulation study and real data analyses. We conclude by summarizing the strengths and weaknesses of the two methods in Section 7.

2 Setup

2.1 Review: Notation

Let $i = 1, \ldots, N$ index $N$ clusters and let $j = 1, \ldots, n_i$ be the number of individuals (i.e. study units) within cluster $i$. In cross-classified studies with non-nested clusters, $i$ would denote each
cluster defined by interacting non-nested clusters, say school and neighborhood, and \( j \) would denote individuals within the interacted cluster. More generally, we can expand the subscript \( i \) to better denote two, three, or \( k \)-level hierarchical structures in a multilevel study. But for simplicity, we suppress these double or triple subscripts.

For each study unit \( j \) in cluster \( i \), let \( \mathbf{X}_{ij} = (W_{ij}^\top, C_i^\top)^\top \) denote his/her pre-treatment covariates where \( W_{ij} \) denote unit-level pre-treatment covariates and \( C_i \) denote cluster-level pre-treatment covariates. In educational assessment studies like the ECLS, unit-level covariates may include student’s sex, age, prior test scores, and socioeconomic status. Cluster-level covariates may include type of school, say private or public schools, location of the school, say rural, suburban, or city, and cluster size \( n_i \). Often, cluster-level covariates drive the dependence between study units in a cluster. For example, school location may be one of the reasons why students’ test scores in the same school may be correlated. In Section 4, we exploit this feature about multilevel studies to help train one of the nonparametric functionals inside our new method.

Let \( A_{ij} \in \{0, 1\} \) denote treatment assignment where \( A_{ij} = 1 \) indicates that individual \( j \) in cluster \( i \) was treated and \( A_{ij} = 0 \) indicates that individual \( j \) in cluster \( i \) was untreated. Let \( Y_{ij} \) denote the observed outcome of individual \( j \) in cluster \( i \). We use the following notation for vectors of observed variable: \( \mathbf{Y}_i = (Y_{i1}, \ldots, Y_{in_i})^\top \), \( \mathbf{A}_i = (A_{i1}, \ldots, A_{in_i})^\top \), and \( \mathbf{X}_i = (X_{i1}^\top, \ldots, X_{in_i}^\top)^\top \). Also, let \( \mathbf{A}_{i(-j)} \) be the vector of treatment variables for all study units in cluster \( i \) except individual \( j \).

Overall, for each individual \( j \) in cluster \( i \), we observe \( O_{ij} = (Y_{ij}, A_{ij}, X_{ij}) \) and for each cluster \( i \), we observe \( \mathbf{O}_i = (O_{i1}, \ldots, O_{in_i}) \).

We adopt the following notations for convergence, norms of functions, and vectors. For a measurable function \( f \), let \( \|f\|_{P,2} \) denote the \( L_2(P) \)-norm of \( f \), i.e. \( \|f\|_{P,2} = \left\{ \int \|f(o)\|^2 dP(o) \right\}^{1/2} \). Let \( O(\cdot) \) and \( o(\cdot) \) be the usual big-o and small-o notation, respectively. We denote vectors of ones and zeros as \( \mathbf{1} \) and \( \mathbf{0} \), respectively.

### 2.2 Review: Causal Estimands and Causal Identification

We use the potential outcomes notation (Neyman, 1923; Rubin, 1974) to define causal effects in multilevel studies. Let \( Y_{ij}^{(1)} \) denote the potential outcome if individual \( i \) in cluster \( j \) were treated and let \( Y_{ij}^{(0)} \) denote the potential outcome if the individual were untreated; the notation implicitly assumes no interference between units (Cox, 1958; Rubin, 1986). The target estimand of interest is the (weighted) average treatment effect and is denoted as \( \tau^* \):

\[
\tau^* = E[w(C_i)\{Y_{ij}^{(1)} - Y_{ij}^{(0)}\}].
\]

The function \( w(C_i) \in \mathbb{R}, |w(C_i)| \leq C_w \), defines cluster-specific weights and is specified by the investigator based on the scientific question at hand. For example, if \( w(C_i) = 1 \), \( \tau^* \) becomes the usual (i.e. unweighted or equally-weighted) average treatment effect. But, as is commonly done in multilevel studies, if an investigator wishes to focus on certain clusters, say schools in rural areas are weighed more than schools in non-rural areas, \( w(C_i) \) can vary as a function of \( C_i \). For simplicity,
we will refer to both the unweighted/equally-weighted and weighted/unequally-weighted treatment effects as the average treatment effect in the paper.

To identify the average treatment effect \( \tau^* \) from the observed data, we assume the following.

(A1) (Stable Unit Treatment Value Assumption, Rubin (1986)) \( Y_{ij} = A_{ij}Y_{ij}(1) + (1 - A_{ij})Y_{ij}(0) \).

(A2) (Conditional Ignorability) \( Y_{ij}(1), Y_{ij}(0) \perp A_{ij} | X_{ij} \).

(A3) (Overlap) There exists a finite constant \( \delta > 0 \) such that \( \delta \leq P(A_i = a | X_i = x) \leq 1 - \delta \) for all \((a, x)\).

Briefly, Assumption (A1) and (A2) are standard identifying assumptions for causal effects (Imbens and Rubin, 2015; Hernán and Robins, 2020). Assumption (A3) is a slight generalization of the usual overlap assumption to allow for a vector of treatment indicators in cluster \( i \). Assumption (A3) implies the usual overlap assumption for the “individual-level” propensity score \( e^*(a | x) = P(A_{ij} = a | X_{ij} = x) \), i.e. \( \delta \leq P(A_{ij} = a | X_{ij} = x) \leq 1 - \delta \). In the paper, we refer to \( e^*(a | x) \) as the individual-level propensity score and \( P(A_i = a | X_i = x) \) as the cluster-level propensity score. Hong and Raudenbush (2006) and Hong and Raudenbush (2013) discuss the practical merits of Assumptions (A1)-(A3) when estimating treatment effects in multilevel studies.

Under Assumptions (A1)-(A3), the average treatment effect \( \tau^* \) can be written as a function of observed data via

\[
\tau^* = E[w(C_i)\{g^*(1, X_{ij}) - g^*(0, X_{ij})\}] \quad (1)
\]

where \( g^*(1, x) = E(Y_{ij} | A_{ij} = 1, X_{ij} = x) \) and \( g^*(0, x) = E(Y_{ij} | A_{ij} = 0, X_{ij} = x) \) are the true outcome regression models under treatment and control, respectively. The rest of the paper focuses on estimating the estimand \( \tau^* \), specifically the functional on the right-hand side of equation (1) written in terms of the observed data.

### 2.3 Nonparametric Model for Multilevel Studies

In addition to the causal identifying Assumptions (A1)-(A3), it is common to place additional parametric or nonparametric restrictions on the observed data, say in the form of modeling assumptions for the outcome variable or the propensity score, to arrive at consistent, asymptotically Normal estimator of the treatment effect. Here, we review two popular frameworks; for a more complete and recent review, see Fuentes et al. (2021) and Cafri et al. (2019).

First, generalized linear mixed effect models (GLMMs) and their variants usually place the following restrictions the observed data (Goldstein et al., 2002a; Hong and Raudenbush, 2006; Arpino and Mealli, 2011; Thoemmes and West, 2011; Arpino and Cannas, 2016; Shardell and Ferrucci, 2018).

\[
Y_{ij} = \alpha_0 + A_{ij}\alpha_1 + X_{ij}\beta + U_i + \epsilon_{ij}, \quad U_i \overset{i.i.d.}{\sim} N(0, \sigma_U^2), \quad \epsilon_{ij} \overset{i.i.d.}{\sim} N(0, \sigma_\epsilon^2), \quad (2a) \\
P(A_{ij} = 1 | X_i, V_i) = \expit(X_{ij}\gamma + V_i), \quad V_i \overset{i.i.d.}{\sim} N(0, \sigma_V^2), \quad (X_i, U_i, V_i) \perp \epsilon_{ij}, \quad X_i \perp (U_i, V_i). \quad (2b)
\]
The terms $U_i$ and $V_i$ are unobserved, Normally distributed random effect terms that govern the correlation between study units in a cluster. Estimation and inference of the model parameters are usually based on the likelihood principle; see Rencher and Schaalje (2008) for one textbook example.

Second, generalized estimating equations (GEEs) and their variants only place moment restrictions on the outcome and the treatment (Liang and Zeger, 1986; Gray and Brookmeyer, 1998; Robins et al., 1999; Gray and Brookmeyer, 2000; Goetgeluk and Vansteelandt, 2008; Stephens et al., 2012; Zetterqvist et al., 2016), say

$$E(Y_{ij} | A_{ij}, X_{ij}) = \alpha_0 + A_{ij}^\top \alpha_1 + X_{ij}^\top \beta,$$

$$E(A_{ij} | X_{ij}) = \expit(X_{ij}^\top \gamma), P(O_1, \ldots, O_N) = \prod_{i=1}^N P(O_i).$$

Unlike model (2), the within-cluster correlation between study units is not parametrized by Normal random variables $U_i$ and $V_i$. Instead, the correlation between units is left unspecified. For estimation and inference, investigators often use M-estimation with a working correlation matrix (Liang and Zeger, 1986; Diggle et al., 2002). Unfortunately, both GLMMs and GEEs in equations (2) and (3) have parametric components and if they are mis-specified or if there is insufficient justification for them, the corresponding estimates of the treatment effect may be biased.

Our proposal remedies the aforementioned deficiencies by only placing nonparametric constraints on the observed data. These constraints are stated as Assumptions (M1) and (M2).

(M1) (Independence and Invariance of Outcome and Propensity Score Models) The cluster-level observed data are independent and generated from a family of distributions $P$ that satisfies the following invariance restrictions on the outcome regression model and the propensity score:

$$P(O_1, \ldots, O_N) = \prod_{i=1}^N P(O_i),$$

$$P(O_i) \in \mathcal{P} = \left\{ P \left| E(P(Y_{ij} | A_{ij} = a, X_{ij} = x) = E(P(Y_{i'j'} | A_{i'j'} = a, X_{i'j'} = x),
\right.
\right.$$  

$$E(P(A_{ij} | X_{ij} = x) = E(P(A_{i'j'} | X_{i'j'} = x) \text{ for any } (a, x, i, j, i', j'), \text{ and } n_i \leq M \right\}. $$

(M2) (Bounded Moments) Let $g^*(A_i, X_i) = (g^*(A_{i1}, X_{i1}), \ldots, g^*(A_{in_i}, X_{in_i}))^\top$. For all $(A_i, X_i)$,

$$E\{\|Y_i - g^*(A_i, X_i)\|^4_2 \mid A_i, X_i\}$$

is bounded. Also, for any $X_i$, there exists $A_i$ such that the smallest eigenvalue of $E\{\{Y_i - g^*(A_i, X_i)\} \otimes^2 \mid A_i, X_i\}$ is positive.

In words, Assumption (M1) states that the cluster-level data are independent from each other. It also assumes that the outcome and the propensity score models are the same for every unit $ij$, and that the cluster size is bounded; see below for additional discussions on the boundedness assumption. Notably, Assumption (M1) does not assume that individual-level data must be i.i.d., i.e. $P(O_1, \ldots, O_N) = \prod_{i=1}^N \prod_{j=1}^{n_i} P(O_{ij})$. Quite surprisingly, we found that this cluster-level i.i.d.
assumption is widely used in recent works in causal inference under clustered settings to establish theoretical properties of various estimators (VanderWeele et al., 2014; Liu et al., 2016; Yang, 2018; Liu et al., 2019; Barkley et al., 2020; Smith et al., 2020; Kilpatrick and Hudgens, 2021). Assumption (M2) states that the residualized outcome \( Y_i - g^*(A, X_i) \) has finite fourth conditional moment and its conditional covariance matrix is nonsingular; see Chernozhukov et al. (2018) for a similar assumption. Overall, Assumptions (M1) and (M2) place nonparametric constraints on the observed data and GLMMs plus GEEs are special cases of (M1) and (M2) under mild conditions; see Section A.1 of the supplementary material for details.

Assumption (M1) also includes nonparametric latent models where the latent variables that govern the correlation between study units in a cluster; we denote such models as (M1-L).

(M1-L) (Nonparametric Latent Variable Model) There exists mutually independent latent variables \((U_i, V_i)\) that are i.i.d. and factorize the observed outcome and treatment variables, respectively, in cluster \(i\) as follows.

\[
P(Y_i | A_i, X_i, U_i, V_i) = \prod_{j=1}^{n_i} P(Y_{ij} | A_{ij}, X_{ij}, U_i), \quad P(A_i | X_i, U_i, V_i) = \prod_{j=1}^{n_i} P(A_{ij} | X_{ij}, V_i). \tag{4}
\]

See Figure 2.1 for a visual illustration of Assumption (M1-L) when there are two individuals in cluster \(i\). Assumption (M1-L) is also known as a “local independence” assumption (Lord, 1980; Holland and Rosenbaum, 1986) where the observed variables are conditionally independent given either an observed or unobserved variable. We remark that the latent variables \(U_i\) and \(V_i\) in (4) are not unmeasured confounders in that they do not simultaneously affect both the outcome and the treatment variables. Instead, the latent variables act similarly to random effects in GLMMs in that they create dependence in the outcome or the treatment variables among study units in the same cluster. Also, because the latent variables \(U_i\) and \(V_i\) are not confounders, we can still identify the average treatment effect \(\tau^*\) by equation (1).

![Graphical Illustration of (M1-L)](image)

Figure 2.1: Graphical Illustration of (M1-L) under a cluster of size \(n_i = 2\).

We conclude by highlighting two important limitation of Assumptions (M1) and (M2), notably (M1). First, Assumption (M1) does not encompass “covariate interference” models (Balzer et al., 2019) where individual \(j\)’s pre-treatment covariate affects the outcome of individual \(j' \neq j\) in cluster \(i\). However, models that encompass covariate interference often requires additional dimension-reducing assumptions on the pre-treatment covariates; see Section 3.2 and Appendices C and D of
Balzer et al. (2019) for details. Second, while Assumptions (M1) and (M2) include many parametric, semiparametric, and nonparametric models for multilevel studies (including latent-variable models), they are not all-encompassing and exclude a few type of models, say models where a study unit’s observed data \( O_{ij} \) follows a nonparametric Markov chain of the form \( P(O_{ij} | O_{i1}, \ldots, O_{im_i}) \) and the chain grows; these type of models would be able to simultaneously capture arbitrarily long-range, potentially latent, dependencies in a cluster. Third, Section 5 explores the robustness of our approaches when the bounded cluster size assumption in (M1) is not plausible. In particular, we show that our proposed methods show good performance even when the cluster size, say \( n_i \), is much larger than the number of clusters \( N \), say by five folds.

3 Method 1: A Simple Modification of Cross Fitting for Multilevel Studies

The first approach is a simple modification of the recent cross-fitting estimator by Chernozhukov et al. (2018). At a high level, we randomly split the observed data into non-overlapping folds, say two folds \( I_1 \) and \( I_2 \), at the cluster level instead of at the individual-level; we remark that the original proposal by Chernozhukov et al. (2018) randomly splits the data at the individual-level and assumes that the data was i.i.d. One fold, referred to as the auxiliary data and denoted as \( I^c_k \), is used to estimate the nuisance functions and the other fold, referred to as the main data and denoted as \( I_k \), evaluates the estimated nuisance functions obtained from the first fold. We then switch the roll of the two split samples to fully use the data for treatment effect estimation. Formally, let \( e(-k) \) and \( g(-k) \) be the estimated propensity score (i.e. \( E(A_{ij} | X_{ij}) \)) and the outcome regression (i.e. \( E(Y_{ij} | A_{ij}, X_{ij}) \)) based on the auxiliary data \( I^c_k \). Let \( \hat{e}(-k) \) and \( \hat{g}(-k) \) be the evaluation of these functions in the main data \( I_k \). Then, we use \( \hat{e}(-k) \) and \( \hat{g}(-k) \) as plug-ins to the augmented inverse probability weighted (AIPW) estimator (Robins et al., 1994; Scharfstein et al., 1999), denoted as \( \tau \) below.

\[
\tau = \tau(\hat{e}(-k), \hat{g}(-k)) = \frac{1}{2} \sum_{k=1}^{2} \sum_{i \in I_k} \varphi(O_i, \hat{e}(-k), \hat{g}(-k)), \tag{5}
\]

\[
\varphi(O_i, e, g) = \frac{w(C_i)}{n_i} \sum_{j=1}^{n_i} \left[ A_{ij} \left\{ Y_{ij} - g^*(1, X_{ij}) \right\} e(1 | X_{ij}) + g^*(1, X_{ij}) - \left( 1 - A_{ij} \right) \left\{ Y_{ij} - g^*(0, X_{ij}) \right\} e(0 | X_{ij}) - g^*(0, X_{ij}) \right].
\]

Similar to the original cross-fitting estimator in Chernozhukov et al. (2018), we make a parallel set of assumptions about the properties of the estimated nuisance functions.

(E1) (Estimated Nuisance Functions) For all \( k = 1, 2 \), and \( (A_{ij}, X_{ij}) \), we have \( \hat{e}(-k)(1 | X_{ij}) \in [c_e, 1 - c_e] \), and \( |\hat{g}(-k)(A_{ij}, X_{ij}) - g^*(A_{ij}, X_{ij})| \leq c_g \) for some positive constants \( c_e \) and \( c_g \).

(E2) (Convergence Rate of Estimated Nuisance Functions) For all \( k = 1, 2 \), consider one of the following three conditions.
We have \( \|\hat{c}^{(-k)}(1 \mid X_{ij}) - e^*(1 \mid X_{ij})\|_{P,2} = O_P(r_{e,N}) \) where \( r_{e,N} \) is \( o(1) \) and \( \|\hat{g}^{(-k)}(A_{ij}, X_{ij}) - g(A_{ij}, X_{ij})\|_{P,2} = O_P(r_{g,N}) \) where \( g \) is some function satisfying \( |g^*(A_{ij}, X_{ij}) - g(A_{ij}, X_{ij})| \leq c_g \) for all \( A_{ij}, X_{ij} \) and \( r_{g,N} \) is \( o(1) \).

We have \( \|\hat{g}^{(-k)}(A_{ij}, X_{ij}) - g^*(A_{ij}, X_{ij})\|_{P,2} = O_P(r_{g,N}) \) where \( r_{g,N} \) is \( o(1) \) and \( \|\hat{c}^{(-k)}(1 \mid X_{ij}) - e(1 \mid X_{ij})\|_{P,2} = O_P(r_{e,N}) \) where \( e \) is some function satisfying \( e(1 \mid X_{ij}) \in [c_e, 1 - c_e] \) for all \( X_{ij} \) and \( r_{e,N} \) is \( o(1) \).

We have \( \|\hat{c}^{(-k)}(1 \mid X_{ij}) - e^*(1 \mid X_{ij})\|_{P,2} = O_P(r_{e,N}) \) and \( \|\hat{g}^{(-k)}(A_{ij}, X_{ij}) - g^*(A_{ij}, X_{ij})\|_{P,2} = O_P(r_{g,N}) \) where \( r_{e,N} \) and \( r_{g,N} \) are \( o(1) \) and \( r_{e,N}r_{g,N} \) is \( o(N^{-1/2}) \).

In words, both Assumption (E1) and variants of (E2) state that the estimated nuisance functions \( \hat{c}^{(-k)} \) and \( \hat{g}^{(-k)} \) are well-behaved estimators. In particular, (E2.PS) states that \( \hat{c}^{(-k)} \) is a consistent estimator of the true propensity score \( e^* \), (E2.OR) states that \( \hat{g}^{(-k)} \) is a consistent estimator of the true outcome regression model \( g^* \), and (E2.Both) state that both \( \hat{c}^{(-k)} \) and \( \hat{g}^{(-k)} \) are consistent estimators of their true counterparts \( e^* \) and \( g^* \), respectively. Also, (E2.PS) and (E2.OR) do not require \( \sqrt{N} \) rates of convergence whereas (E2.Both) place restrictions on the convergence rates; these restrictions are satisfied if the estimated propensity score \( \hat{c}^{(-k)} \) and the estimated outcome regression \( \hat{g}^{(-k)} \) are estimated at rates faster than \( N^{-1/4} \).

Theorem 3.1 shows that the proposed adjustment to the original cross-fitting estimator leads to a consistent, doubly robust, and asymptotically Normal estimate of the average treatment effect in multilevel studies.

**Theorem 3.1.** Suppose Assumptions (A1)-(A3), (M1)-(M2) and (E1) hold. Then, the cross-fitting estimator \( \overline{\tau} \) in (5) satisfies the following properties.

1. **(Double Robustness)** If either (E2.PS) or (E2.OR) holds, the estimator \( \overline{\tau} \) is consistent for the true treatment effect, i.e. \( \overline{\tau} = \tau^* + o(1) \).

2. **(Asymptotic Normality)** If (E2.Both) holds, the estimator \( \overline{\tau} \) has an asymptotically Normal distribution centered at the true treatment effect, i.e. \( \sqrt{N}(\overline{\tau} - \tau^*) \overset{D}{\to} N(0, \text{Var}(\varphi(O_i, e^*, g^*))) \).

Also, the asymptotic variance of the estimator can be consistently estimated by the estimator \( \overline{\tau}^2 \),

\[
\varphi^2 = \frac{1}{N} \sum_{k=1}^{2} \sum_{i \in I_k} \left\{ \varphi(O_i, \hat{c}^{(-k)}, \hat{g}^{(-k)}) - \overline{\tau}_k \right\}^2, \quad \overline{\tau}_k = \frac{1}{N/2} \sum_{i \in I_k} \varphi(O_i, \hat{c}^{(-k)}, \hat{g}^{(-k)}) .
\]

In words, Theorem 3.1 states the original cross-fitting estimator, with a minor tweak on how the samples are split, retains its consistency, double robustness, and asymptotic Normality in multilevel studies where the data is no longer i.i.d. However, a critical difference between the traditional i.i.d. setting and the multilevel setting is that \( \overline{\tau} \) is not efficient when (E2.Both) holds. Roughly speaking, the efficiency loss is attributed to \( \overline{\tau} \) not accounting for the intra-cluster correlation of the outcomes and the treatment variables. For the outcome variable, even though there maybe intra-cluster
correlation among the residualized outcomes $Y_{ij} - g^{-k}(A_{ij}, X_{ij})$ in $\pi$, the estimator simply takes an equally-weighted average of these terms rather than an unequally weighted average where the weights depend on the correlation structure. For the treatment, the term $e^{-k}(a \mid X_{ij}) = P(A_{ij} = a \mid X_{ij})$ in $\pi$ integrates over peers’ treatment assignments and covariates in the same cluster, i.e. $(A_{i(-j)}, X_{i(-j)})$. But, if there was intra-cluster correlation among the treatment variables, the propensity score $P(A_{ij} = a \mid X_{ij})$ does not capture this correlation, leading to potential losses in efficiency. The next section presents a remedy for these two concerns by presenting a new estimator that is more efficient than $\pi$ in multilevel settings.

4 Method 2: A More Efficient, Nonparametric, Doubly Robust Estimator in Multilevel Studies

4.1 Overview of Key Ideas: Conditional Propensity Score and Outcome Covariance Model

To address the loss in efficiency, we propose a new nonparametric, doubly robust estimator for multilevel studies. The new estimator retains some of the favorable properties of the simple estimator $\pi$ above, notably double robustness, consistency, and asymptotic Normality. But, the new estimator can be more efficient when the data from study units in the same cluster are correlated. In this section, we first introduce the new estimator when the nuisance functions are known. The next section discusses how to estimate the nuisance functions.

The new estimator contains two key functions to improve efficiency: (i) the conditional propensity score, denoted as $\pi^*(a \mid \alpha', x, x') = P(A_{ij} = a \mid A_{i(-j)} = \alpha', X_{ij} = x, X_{i(-j)} = x')$, and (ii) the outcome covariance model $\beta(C_i)$. These two functions are used to obtain a more efficient, doubly robust estimator of the treatment effect and we denote the new estimator as $\hat{\tau}$ below.

$$\hat{\tau}(\pi, g, \beta) = \frac{1}{N} \sum_{i=1}^{N} \phi(O_i, \pi, g, \beta),$$

$$\phi(O_i, \pi, g, \beta) = \frac{w(C_i)}{n_i} \sum_{j=1}^{n_i} \left[ \frac{A_{ij} \{Y_{ij} - g(1, X_{ij})\} - \beta(C_i) \sum_{k \neq j} \{Y_{ik} - g(A_{ik}, X_{ik})\}}{\pi(1 \mid A_{i(-j)}, X_{ij}, X_{i(-j)})} + g(1, X_{ij}) - \frac{(1 - A_{ij}) \{Y_{ij} - g(0, X_{ij})\} - \beta(C_i) \sum_{k \neq j} \{Y_{ik} - g(A_{ik}, X_{ik})\}}{\pi(0 \mid A_{i(-j)}, X_{ij}, X_{i(-j)})} - g(0, X_{ij}) \right].$$

Compared to the simple estimator $\pi$, the new estimator of the average treatment effect $\hat{\tau}$ captures the correlation structure in the treatment and the outcome variables to increase efficiency. For the treatment variable, $\hat{\tau}$ uses the conditional propensity score $\pi$ to capture the (conditional) correlation between $ij$’s treatment $A_{ij}$ and his/her peers $A_{i(-j)}$ conditional on $ij$’s and his/her peers’ covariates, $X_{ij}$ and $X_{i(-j)}$, respectively. It then uses the estimated conditional correlation in the treatment variable as inverse probability weights, similar to the original AIPW estimator in i.i.d. settings. For the outcome variable, the new estimator $\hat{\tau}$ uses the outcome covariance model $\beta(C_i)$ to capture
the relationship between $ij$’s (residual) outcome relative to his/her peers’ (residual) outcomes. Specifically, $\beta(C_i)$ acts as a (functional) regression coefficient from regressing $ij$’s residual (i.e. $Y_{ij} - g(A_{ij}, X_{ij})$) onto the sum of his/her peers’ residuals (i.e. $\sum_{k \neq j} \{Y_{ik} - g(A_{ik}, X_{ik})\}$); here, the residual refers to the leftover variation in an individual’s outcome after conditioning on his/her own treatment and pre-treatment covariates.

In relation to the simple estimator $\tau$, if the $ij$’s treatment is independent of his/her peers’ treatment and pre-treatment covariates conditional on his/her own pre-treatment covariates, $\pi^{*}$ reduces the usual, individual-level propensity score $e^{*}$ and the inverse weights are identical between $\hat{\tau}$ and $\tau$. Also, if the residuals of everyone are independent of each other, or equivalently, everyone’s outcomes are independent of each other conditional on their own treatment and pre-treatment covariates, a hypothetical regression between $ij$’s residuals onto his/her peers residuals would lead to an estimated coefficient of $\beta(C_i) = 0$, matching the equivalent expression in equation (5) for $\bar{\tau}$. Finally, if both independence conditions hold, $\hat{\tau}$ equals $\tau$ and thus, $\tau$ is a special case of $\bar{\tau}$ where $\bar{\pi}$ does not account for correlation between study units in a multilevel study.

For a given conditional propensity score $\pi$ and outcome regression $g$, let $\beta^{*}$ be the minimizer of the variance of the estimator $\bar{\tau}(\pi, g, \beta)$ over some model space for the outcome covariance model $B$ that contains 0, i.e. $\beta^{*} = \arg\min_{\beta \in B} \text{Var}\{\bar{\tau}(\pi, g, \beta)\}$ where $0 \in B$; the next section discusses some potential models for the outcome covariance model. Lemma 4.1 highlights some simple, but useful characteristics of the proposed estimator of $\bar{\tau}$ when we assume, for a moment, that the nuisance functions are fixed, non-random functionals.

**Lemma 4.1.** Suppose Assumptions (A1)-(A3) and (M1)-(M2) hold. Then, the following properties hold for the estimator $\bar{\tau}$.

1. (Mean Double Robustness) For any fixed outcome covariance model $\beta$, we have $E\{\bar{\tau}(\pi, g, \beta)\} = \tau^{*}$ as long as the conditional propensity score or the outcome model are correct, i.e. $\pi = \pi^{*}$ or $g = g^{*}$.

2. (Efficiency Gain Under Conditionally Independent Treatment Assignment) Suppose the cluster-level propensity score is decomposable into the product of individual-level propensity scores, i.e. $P(A_i \mid X_i) = \prod_{j=1}^{n_i} P(A_{ij} \mid X_{ij})$. Then, we have $\text{Var}\{\bar{\tau}(\pi^{*}, g, \beta^{*})\} \leq \text{Var}\{\pi(e^{*}, g)\}$ for any outcome model $g$.

In words, part 1 of Lemma 4.1 states that the new estimator $\bar{\tau}$ is doubly robust (in expectation) for any outcome covariance model $\beta$; that is, whatever the investigator specifies for the outcome covariance model $\beta$, $\bar{\tau}$ will remain unbiased if either the conditional propensity score or the outcome regression, but not necessarily both, is correct. This property is similar to a well-known property of the GEE estimator where, irrespectively of the specification of the weighting/covariance matrix, the GEE estimator remains consistent. Indeed, as we show in Section 4.3, with a few mild assumptions, notably on how the nuisance functions are nonparametrically estimated, we can also achieve double
where \( B \) is the asymptotic expansion of \( \tau \) for a working, estimated outcome covariance model.

Part 2 of Lemma 4.1 states that if the individuals’ treatment assignment mechanism is independent conditional on \( X_i \) and is not dependent on others’ covariates, the new estimator \( \tau \) is at least as efficient than the simple estimator \( \tau \). The conditional independence assumption is satisfied in most randomized experiments and some observational studies where each individual is randomly assigned to treatment with equal probability or with probability that is a function of his/her own pre-treatment covariates. However, when the treatment assignment depends on peers’ covariates or treatment, the gain in relative efficiency becomes data-specific, in part because the variance term in the asymptotic expansion of \( \tau \) can start to behave like a first-order bias term in non-i.i.d. settings; Section C.2 of the supplementary materials provides further theoretical insights and Section A.2 of the supplementary material numerically illustrates this phenomena by comparing the relative efficiencies between \( \tau \) and \( \tau \). Regardless, even if the conditional independent assumption is not satisfied and the relative efficiency cannot be theoretically characterized, subsequent sections show that the proposed estimator \( \tau \) achieves consistency, asymptotic Normality, and is almost always the most efficient estimator among existing estimators in the numerical studies.

### 4.2 Nonparametric Estimation of Nuisance Functions \( \pi, g \) and \( \beta \) in \( \tau \)

In this section, we discuss nonparametric estimation of the nuisance functions \( \pi, g \) and \( \beta \) inside \( \tau \). For estimating \( \pi \) and \( g \), we use the cross-fitting procedure used in \( \tau \) where we estimate the functional forms from the auxiliary sample \( I_k^e \) and evaluate the estimated functions on the main sample \( I_k \), denoted as \( \pi(-k) \) and \( g(-k) \). In contrast, for estimating \( \beta \), we only use the main sample. Specifically, let the estimate of \( \beta \) be denoted as \( \hat{\beta}(k) \), which is a solution to the following optimization problem.

\[
\hat{\beta}(k) = \arg \min_{\beta \in \mathcal{B}} \frac{1}{N/2} \sum_{i \in I_k} w(C_i) \beta(A_i, X_i, \pi(-k))^T B(\beta(C_i)) S_i(-k) B(\beta(C_i)) \beta(A_i, X_i, \pi(-k))^T \ 
\]

where

\[
I(A_i, X_i, \pi(-k)) = \frac{1}{n_i} \left[ \begin{array}{cccc}
1(A_{i1} = 1) & 1(A_{im} = 1) & \cdots & 1(A_{in_i} = 1) \\
\pi(-k)(1 | A_{i(-1)}, X_{i1}, X_{i(-1)}) & \pi(-k)(1 | A_{i(-1)}, X_{im}, X_{i(-1)}) & \cdots & \pi(-k)(1 | A_{i(-1)}, X_{in_i}, X_{i(-1)}) \\
\pi(-k)(0 | A_{i(-1)}, X_{i1}, X_{i(-1)}) & \pi(-k)(0 | A_{i(-1)}, X_{im}, X_{i(-1)}) & \cdots & \pi(-k)(0 | A_{i(-1)}, X_{in_i}, X_{i(-1)})
\end{array} \right]
\]

\[
B(\beta(C_i)) = \begin{bmatrix}
1 & -\beta(C_i) & \cdots & -\beta(C_i) \\
-\beta(C_i) & 1 & \cdots & -\beta(C_i) \\
\vdots & \vdots & \ddots & \vdots \\
-\beta(C_i) & -\beta(C_i) & \cdots & 1
\end{bmatrix}, \quad S_i(-k) = \{Y_i - g(-k)(A_i, X_i)\} \otimes 2
\]

The motivation for equation (7) comes from the sandwich variance/Huber-White variance estimator (Huber, 1967; White, 1980) where \( B(\beta(C_i)) \) and \( S_i(-k) \) correspond to the outer and inner part,
respectively, of the sandwich estimator. Also, from a theoretical perspective, the model space for the outcome covariance model \( \mathcal{B} \) in (7) only needs to be complete and \( P \)-Donsker with a finite envelope; see Theorem 4.1 for details. Some examples of complete and \( P \)-Donsker classes include a collection of bounded functions indexed by a finite-dimensional parameter in a compact set, the \( \alpha \)-Hölder class with \( \alpha > \dim(C_i)/2 \), the Sobolev class, the collection of bounded monotone functions, and a collection of variationally bounded functions; see Chapter 2 of van der Vaart and Wellner (1996) and Chapter 19 of van der Vaart (1998) for textbook discussions about Donsker classes.

Practically speaking, we find that using a simple, parametric outcome covariance model already leads to good performance. For example, consider the following parametric (and \( P \)-Donsker) model space \( \mathcal{B} \), denoted as \( \mathcal{B}_\gamma \)

\[
\mathcal{B}_\gamma = \left\{ \beta(C_i) \right\} \quad \beta(C_i) = \sum_{\ell=1}^{J} \mathbb{1}(L(C_i) = \ell) \gamma_\ell, \\
\gamma_\ell \in [-B_0, B_0] \text{ for some } B_0 > 0, \; L : C_i \rightarrow \{1, \ldots, J\}.
\]

(8)

Functions in \( \mathcal{B}_\gamma \) are parametrized by a finite dimensional parameter \( \gamma = (\gamma_1, \ldots, \gamma_J)^T \in [-B_0, B_0]^J \) where \( \beta(C_i) = \gamma_\ell \) if cluster \( i \) belongs to stratum \( \ell \) based on a user-specified label function \( L \). The label function \( L \) can be a function about cluster size, say \( L(C_i) = \mathbb{1}(n_i \leq M_1) + 2 \cdot \mathbb{1}(M_1 < n_i \leq M_2) + 3 \cdot \mathbb{1}(M_2 < n_i) \) with thresholds \( M_1 \) and \( M_2 \), or \( L \) can be defined over a discrete covariate \( C_i^{(d)} \), say \( L(C_i) = \sum_{j=1}^{J} j \cdot \mathbb{1}(C_i^{(d)} = d_j) \). Under \( \mathcal{B}_\gamma \), estimation of \( \beta(C_i) \) is equivalent to solving the following estimating equation for \( \gamma \).

\[
0 = \sum_{i \in I_k} w(C_i)^2 \mathbb{1}(L(C_i) = \ell) \left[ 2I(A_i, X_i, \pi^{(-k)})^T (I - 11^T) \widetilde{S}_i^{(-k)} (I - 11^T) I(A_i, X_i, \pi^{(-k)}) \gamma_\ell \right. \\
- I(A_i, X_i, \pi^{(-k)})^T \left\{ 2\widetilde{S}_i^{(-k)} - 11^T \widetilde{S}_i^{(-k)} - \widetilde{S}_i^{(-k)} 11^T \right\} I(A_i, X_i, \pi^{(-k)}) \right], \quad \ell = 1, \ldots, J .
\]

(9)

Combining the estimates of \( \pi, g, \beta \) from above, we arrive at the estimator of \( \tau \) where the nuisance functions \( \pi, g, \beta \) are estimated nonparametrically

\[
\widehat{\tau} = \widehat{\tau}(\pi, g, \beta) = \frac{1}{2} \sum_{k=1}^{2} \frac{1}{N/2} \sum_{i \in I_k} \phi(O_i, \pi^{(-k)}, g^{(-k)}, \beta^{(k)}) .
\]

(10)

Finally, we conclude with a comment about the models for the outcome covariance model \( \beta \). It may be tempting to consider more complex models of \( \beta \) beyond \( P \)-Donsker, say by using flexible ML methods. Indeed, if this is desired, we can use additional steps in cross-fitting to estimate \( \beta \) without invoking the Donsker class assumption. However, in our experience, we find that this strategy is only useful in terms of reducing standard errors if the sample size is sufficiently large to estimate the underlying covariance relationships, which is often difficult to find in multilevel or, more generally, clustered studies.
4.3 Statistical Properties

To characterize the statistical properties of \( \hat{\tau} \), we make the following assumptions about the behavior of the estimated nuisance functions. These assumptions parallel Assumptions (E1) and (E2), which were used to describe the statistical properties of \( \bar{\tau} \).

(EN1) (Estimated Nuisance Functions) For all \( k = 1, 2, \) and \((A_i, X_i)\), we have \( \hat{\pi}^{(k)}(1 | A_{i(-j)}, X_{ij}, X_{i(-j)}) \in [c_\pi, 1 - c_\pi] \), and \( |\hat{g}^{(k)}(A_{ij}, X_{ij}) - g^*(A_{ij}, X_{ij})| \leq c_g \) for some positive constants \( c_\pi \) and \( c_g \).

(EN2) (Convergence Rate of Estimated Nuisance Functions) \( B \) is complete and \( P \)-Donsker with

\[
\| \hat{\beta}^{(k)}(C_i) - \beta(C_i) \|_{P,2} = O_P(r_{\beta,N}), \quad \beta \text{ is some fixed function, and } r_{\beta,N} = o(1) .
\]

Additionally, we consider one of the following three conditions.

(EN2.PS) We have \( \| \hat{\pi}^{(k)}(1 | A_{i(-j)}, X_{ij}, X_{i(-j)}) - \pi^*(1 | A_{i(-j)}, X_{ij}, X_{i(-j)}) \|_{P,2} = O_P(r_{\pi,N}) \) where \( r_{\pi,N} \) is \( o(1) \) and \( \| \hat{g}^{(k)}(A_{ij}, X_{ij}) - g(A_{ij}, X_{ij}) \|_{P,2} = O_P(r_{g,N}) \) where \( g \) is some function satisfying \( |g(A_{ij}, X_{ij}) - g(A_{ij}, X_{ij})| \leq c_g \) for all \((A_{ij}, X_{ij})\) and \( r_{g,N} \) is \( o(1) \).

(EN2.OR) We have \( \| \hat{\pi}^{(k)}(1 | A_{i(-j)}, X_{ij}, X_{i(-j)}) - \pi^*(1 | A_{i(-j)}, X_{ij}, X_{i(-j)}) \|_{P,2} = O_P(r_{\pi,N}) \) where \( r_{\pi,N} \) is \( o(1) \) and \( \| \hat{g}^{(k)}(A_{ij}, X_{ij}) - g(A_{ij}, X_{ij}) \|_{P,2} = O_P(r_{g,N}) \) where \( r_{g,N} \) is \( o(1) \).

(EN2.Both) We have \( \| \hat{\pi}^{(k)}(1 | A_{i(-j)}, X_{ij}, X_{i(-j)}) - \pi^*(1 | A_{i(-j)}, X_{ij}, X_{i(-j)}) \|_{P,2} = O_P(r_{\pi,N}) \) and \( \| \hat{g}^{(k)}(A_{ij}, X_{ij}) - g(A_{ij}, X_{ij}) \|_{P,2} = O_P(r_{g,N}) \) where \( r_{\pi,N} \) and \( r_{g,N} \) are \( o(1) \). Furthermore, \( r_{\pi,N} r_{g,N} \) and \( r_{g,N} r_{\beta,N} \) are \( o(N^{-1/2}) \).

Assumptions (EN1) and (EN2) are similar to Assumptions (E1) and (E2); the only difference is in (i) the type of propensity score (i.e. \( \pi \) versus \( \epsilon \)) and (ii) a condition about the estimated outcome covariance function \( \hat{\beta}^{(k)} \). In particular, Assumption (EN2) simply states that the estimated outcome covariance model \( \hat{\beta}^{(k)} \) converges to some limit \( \beta \). This limit does not have to equal to the “true” \( \beta \); in the extreme case, we can choose the estimator \( \hat{\beta}^{(k)} \) to be some constant value, say \( \hat{\beta}^{(k)} = 0 \), and the convergence condition in (EN2) will automatically hold. Also, if we only desire consistency of \( \hat{\tau} \) without the asymptotic Normality, we do not need an assumption about the convergence rate of \( \hat{\beta}^{(k)} \). However, if we desire asymptotic Normality of \( \hat{\tau} \), Assumption (EN2.Both) places additional restrictions on the convergence rate of \( \hat{\beta}^{(k)} \) where the product of the convergence rate of \( \hat{\beta}^{(k)} \) and the outcome regression function \( \hat{g}^{(k)} \) is of order \( N^{-1/2} \). Note that the naive estimator \( \hat{\beta}^{(k)} = 0 \) or the parametric estimator of \( \beta \) within \( B \), in equation (8) will satisfy this rate condition.

Theorem 4.1 formally characterizes the property of \( \hat{\tau} \).

**Theorem 4.1.** Suppose Assumptions (A1)-(A3), (M1)-(M3), and (EN1) hold. Then the estimator \( \hat{\tau} \) in equation (10) satisfies the following properties.

1. (Double Robustness) If either (EN2.PS) or (EN2.OR) holds, \( \hat{\tau} \) is consistent for the true treatment effect, i.e. \( \hat{\tau} = \tau^* + o_P(1) \)
(2) **(Asymptotic Normality)** If (EN2.Both) holds, \( \hat{\tau} \) has an asymptotically Normal distribution centered at the true treatment effect, i.e. \( \sqrt{N} ( \hat{\tau} - \tau^* ) \xrightarrow{D} N (0, \text{Var} \{ \phi (O, \pi^*, g^*, \beta) \} ) \). Also, a consistent estimator of the variance of \( \hat{\tau} \) is

\[
\hat{\sigma}^2 = \frac{1}{N} \sum_{k=1}^{2} \sum_{i \in I_k} \left\{ \phi (O, \hat{\tau}^{(-k)}, \hat{g}^{(-k)}, \hat{\beta}^{(k)}) - \hat{\tau}_k \right\}^2, \quad \hat{\tau}_k = \frac{1}{N/2} \sum_{i \in I_k} \phi (O, \hat{\tau}^{(-k)}, \hat{g}^{(-k)}, \hat{\beta}^{(k)}) .
\]

(3) **(Efficiency Gain Under Known Treatment Assignment Mechanism)** Suppose (E1) and (E2.Both) hold. If the same condition in part 2 of Lemma 4.1 concerning the treatment assignment mechanism and \( \mathcal{B} = \mathcal{B}_\gamma \), the asymptotic relative efficiency (ARE) of \( \hat{\tau} \) relative to \( \tilde{\tau} \) is not smaller than 1, i.e. \( \text{ARE}(\hat{\tau}, \tilde{\tau}) = \text{Var} \{ \varphi (O, e^*, g^*) \} / \text{Var} \{ \phi (O, \pi^*, g^*, \beta) \} \geq 1 \).

Theorem 4.1 shows that the estimator \( \hat{\tau} \) is consistent, doubly robust, and asymptotically Normal. Critically, we do not need our estimated outcome covariance model \( \hat{\beta} \) to converge to some underlying truth to guarantee double robustness, consistency, and asymptotic Normality of \( \hat{\tau} \). As long as \( \mathcal{B} \) is a complete \( P \)-Donsker class, \( \hat{\tau} \) is still doubly robust and if \( \hat{\beta} \) converges at a sufficiently fast rate (but not necessarily \( N^{-1/2} \)) to some function \( \beta \), \( \hat{\tau} \) is asymptotically Normal.

### 4.4 Practical Considerations

This section highlights some practical implementation details concerning the proposed estimators \( \tau \) and \( \hat{\tau} \), especially for investigators who are more familiar with using parametric or semiparametric methods like GLMM and GEE to estimate treatment effects in multilevel studies. Much of these tips already exist in the literature and the references below contain additional details.

First, because the estimators \( \tau \) and \( \hat{\tau} \) are constructed using cross-fitting, the finite sample performance of the estimators may depend on the particular random split. To mitigate the impact associated with the random split, Section 3.4 of Chernozhukov et al. (2018) suggests repeating the sample splitting \( S \) times and obtain the estimators \( \tau^{(s)} \) and \( \hat{\tau}^{(s)} \) \( (s = 1, \ldots, S) \) and the corresponding variance estimators \( \hat{\sigma}^{2,(s)} \) and \( \hat{\sigma}^{2,(s)} \). Afterwards, the medians of the estimators are computed, i.e.

\[
\tau^{\text{med}} = \text{median}_{s=1,...,S} \tau^{(s)}, \quad \hat{\tau}^{\text{med}} = \text{median}_{s=1,...,S} \hat{\tau}^{(s)} .
\]

Also, the following estimators for standard errors are used:

\[
\tau^{2,\text{med}} = \text{median}_{s=1,...,S} \left[ \{ \tau^{(s)} - \tau^{\text{med}} \}^2 + \sigma^{2,(s)} \right], \quad \hat{\tau}^{2,\text{med}} = \text{median}_{s=1,...,S} \left[ \{ \hat{\tau}^{(s)} - \hat{\tau}^{\text{med}} \}^2 + \hat{\sigma}^{2,(s)} \right].
\]

As shown in Corollary 3.3 of Chernozhukov et al. (2018), the median-adjusted estimators can replace the established results about \( \tau \) and \( \hat{\tau} \).

Second, for estimating \( e \) and \( g \), we can use existing nonparametric estimation methods. Specifically, for \( e \), we can treat \( A_{ij} \) as the dependent variable and \( X_{ij} \) as the independent variables. For \( g \), we can treat \( Y_{ij} \) as the dependent variable and \( (A_{ij}, X_{ij}) \) as the independent variables.

Third, to estimate the conditional propensity score \( \pi \), naively using the raw variables \( (A_{i(-j)}, X_{ij}, X_{i(-j)}) \) as independent variables in existing nonparametric regression estimator may not work because each
cluster may have different cluster size, leading to different number of independent variables. For example, without covariates, a cluster of size 2 will have 1 independent variable while a cluster of size 10 will have 9 independent variables when we use $A_{i(-j)}$ as independent variables in a nonparametric regression estimator. To resolve the issue, van der Laan (2014), Sofrygin and van der Laan (2017), and Ogburn et al. (2020) suggested making structural assumptions on $\pi$ where the investigator uses fixed, dimension-reducing summaries of peers’ data ($A_{i(-j)}$, $X_{i(-j)}$) as independent variables. Some popular examples of these dimension-reducing summaries include the proportion of treated peers, $A_{i(-j)} = \frac{\sum_{\ell \neq j} A_{i\ell}}{(n_i - 1)}$, and the average of peers’ covariates, $X_{i(-j)} = \frac{\sum_{\ell \neq j} X_{i\ell}}{(n_i - 1)}$. With the additional structural assumptions, nonparametric estimation of $\pi$ is similar to nonparametric estimation of $e$ where $A_{ij}$ is the dependent variable and $(A_{i(-j)}, X_{ij}, X_{i(-j)})$ are the independent variables. Note that investigators can use additional summaries of peers’ data as independent variables or use more flexible models of contagion in network settings (Salathé and Jones, 2010; Keeling and Rohani, 2011; Deeth and Deardon, 2013; Imai et al., 2015).

Fourth, in some multilevel studies, there may be few “outliers” clusters where the clusters are either very large (or very small) compared to other clusters in the study. In such cases, one may remove these outlying clusters or reweigh them by tweaking the weights $w(C_i)$ in the average treatment effect. Or, for large outlying clusters, one may randomly remove observations so that the number of individuals per cluster used in the estimation procedures are roughly similar to each other; this adjustment is often referred to as undersampling (Fernández et al., 2018, Chapter 5.2). One can also take the median of multiple conditional propensity score estimates that are obtained from repeated undersampling procedures to improve numerical stability.

5 Simulation

5.1 Finite-Sample Performance of $\hat{\tau}$ and $\tau$

We study the finite-sample performances of $\hat{\tau}$ and $\tau$ through a simulation study. For pre-treatment covariates $X_{ij}$, we use five variables $(W_{1ij}, W_{2ij}, W_{3ij}, C_{1i}, C_{2i}, n_i)^T$ where $W_{1ij}$, $W_{2ij}$, and $C_{1i}$ are from the standard Normal distribution and $W_{2ij}$ and $C_{2i}$ are from the Bernoulli distribution with mean parameter 0.3 and 0.7, respectively. The number of clusters varies from $N \in \{25, 50, 100, 250, 500\}$ and the cluster size takes on values between $2000/N$ and $3000/N$ so that the expected total number of individuals in the simulated dataset is 2500. The treatment follows a mixed effects model with non-linear terms, i.e.

$$P(A_{ij} = 1 | X_{ij}, V_i) = \exp\left\{-0.5 + 0.5Z_{1ij} - 1(W_{2ij} > 1) + 0.5W_{3ij} - 0.25C_{1i} + C_{2i} + V_i\right\},$$

(11)

where $V_i \sim N(0, \sigma^2_V)$. When $\sigma_V = 0$, the random effect $V_i$ in the propensity score model vanishes and each element of $A_i$ are conditionally independent from each other given $X_i$. But, when $\sigma_V \neq 0$, treatments in cluster $i$ are correlated with each other. Similarly, for the outcome, we use a mixed
effects model with the following non-linear terms.

\[ Y_{ij} \mid (A_{ij}, X_{ij}, U_i) \sim N \left( 3 + (2.1 + W_{2ij}^2 + 3W_{3ij})A_{ij} + 2W_{1ij} - C_{1i}^2 + W_{2ij}C_{2i} + U_i, 1 \right) \] (12)

where \( U_i \) follows \( N(0, \sigma_U^2) \). Similar to the propensity score model, individuals’ outcomes in cluster \( i \) are correlated via \( \sigma_U \). The target estimand is the average treatment effect with \( w(C_i) = 1 \), which equals 4 under the models above, i.e. \( \tau^* = 4 \).

We consider four different values of \( (\sigma_V, \sigma_U) \): (i) no treatment correlation and weak outcome correlation where \( (\sigma_V, \sigma_U) = (0, 0.5) \); (ii) no treatment correlation and strong outcome correlation where \( (\sigma_V, \sigma_U) = (0, 1.5) \); (iii) strong treatment correlation and weak outcome correlation where \( (\sigma_V, \sigma_U) = (1.5, 0.5) \); and (iv) strong treatment correlation and strong outcome correlation where \( (\sigma_V, \sigma_U) = (1.5, 1.5) \). To estimate the nuisance functions inside \( \hat{\tau} \) and \( \tau \), we use ensembles of multiple ML methods via the super learner algorithm (van der Laan et al., 2007b; Polley and van der Laan, 2010). We include the following methods and the corresponding R packages in our super learner library: linear regression via \texttt{glm} (Friedman et al., 2010), spline via \texttt{earth} (Friedman, 1991) and \texttt{polspline} (Kooperberg, 2020), generalized additive model via \texttt{gam} (Hastie and Tibshirani, 1986), boosting via \texttt{xgboost} (Chen and Guestrin, 2016) and \texttt{gbm} (Greenwell et al., 2019), random forest via \texttt{ranger} (Wright and Ziegler, 2017), and neural net via \texttt{RSNNS} (Bergmeir and Benítez, 2012). For estimating the conditional propensity score \( \pi \), we follow the details in Section 4.4 where we use \( A_{i(-j)} \) and \( X_{i(-j)} \) as the summary statistics of \( A_{i(-j)} \) and \( X_{i(-j)} \), respectively, perform undersampling five times by randomly choosing \( \min n_i \) (i.e. size of the smallest cluster) individuals from each cluster, and use the median adjustment for \( \hat{\tau} \) based on five random splits. For estimating the outcome covariance model \( \beta \), we choose the function space \( B_\gamma \) in equation (8) where we use 1, 2, 3, 5, and 3 strata parameters according to the cluster size under \( N = 25, 50, 100, 250, \) and 500, respectively. We repeat the simulation 200 times and report the empirical bias, empirical standard error, and the coverage rate of 95% confidence intervals.

Also, to make the terms \( \sigma_V \) and \( \sigma_U \) governing the correlation structures more interpretable, we compute intra-cluster correlation coefficients (ICC), a commonly used measure in multilevel studies to assess correlation between study units in the same cluster. For the outcome ICC, denoted as \( ICC_Y(\sigma_U) \), it varies from 0 to 0.8. For the treatment ICC, denoted as \( ICC_A(\sigma_V) \), it varies from 0.05 to 0.36. For the exact details on computing ICCs, especially for binary variables, see Section A.2 of the supplementary materials.

Table 5.1 summarizes the result. Broadly speaking, the estimator \( \hat{\tau} \) achieve the smallest standard errors compared to \( \tau \) across all simulation scenarios and \( \tau \) becomes less efficient under strong outcome correlation. Also, the coverage rates of confidence intervals based on \( \hat{\tau} \) are closer to the nominal coverage than those based on \( \tau \). In addition, even with small number of clusters, say \( N = 25 \) clusters with roughly 100 individuals per cluster and the bounded cluster condition in Assumption (M1) may be suspect, we see that both estimators have small bias and nominal coverage. Based on the result, we believe \( \tau \) and \( \hat{\tau} \) can be used in settings where each cluster has more study
units compared to the total number of clusters, say if households are the study units and U.S. states are clusters.

\[
\hat{\tau} (0.0, 0.5) (0.05, 0.20) 0.40 8.50 0.995 0.39 7.48 0.965 0.18 7.52 0.940 0.98 6.53 0.960 0.19 6.54 0.955 \\
\tau (0.0, 0.5) (0.05, 0.20) 1.27 27.10 0.995 -0.48 12.07 0.985 1.01 12.71 0.950 -0.17 10.69 0.965 1.00 11.00 0.950 \\
(1.5, 0.5) (0.27, 0.20) 7.37 88.19 0.990 3.12 31.34 0.980 1.35 16.92 0.975 0.69 8.59 0.975 0.48 7.11 0.970 \\
(1.5, 1.5) (0.27, 0.69) 7.64 222.33 0.990 7.31 85.20 0.965 1.00 19.79 0.970 0.75 13.42 0.950 \\
\hat{\hat{\tau}} (0.0, 0.5) (0.05, 0.20) -0.43 6.22 0.920 -0.12 6.13 0.965 0.18 6.65 0.920 0.55 6.25 0.935 0.20 6.20 0.950 \\
(0.0, 1.5) (0.05, 0.69) -0.08 6.58 0.945 0.01 6.21 0.955 0.07 6.19 0.960 0.27 6.59 0.970 1.05 7.58 0.950 \\
(1.5, 0.5) (0.27, 0.20) -0.38 6.63 0.930 -0.15 7.38 0.905 -0.29 7.15 0.925 -0.17 7.11 0.920 -0.06 6.14 0.980 \\
(1.5, 1.5) (0.27, 0.69) -0.51 7.09 0.940 -0.26 7.10 0.935 -0.08 7.00 0.955 -0.09 7.23 0.950 0.84 7.79 0.965 \\
\]

Table 5.1: Finite-sample performance of \( \hat{\tau} \) and \( \tau \) under different number of clusters. Each row represents the values of \( \sigma_V \) and \( \sigma_U \). Each column shows the empirical biases, empirical standard errors, and coverages of 95\% confidence intervals. \( N \) is number of clusters and the average number of individual per cluster (i.e. cluster size) is 2500/\( N \). Bias and SE columns are scaled by 100.

5.2 Comparison of \( \tau \) and \( \hat{\tau} \) to Existing Nonparametric Methods

Next, we compare the performance of \( \tau \) and \( \hat{\tau} \) to existing nonparametric estimators of treatment effects in the literature. Specifically, we use causal forests (Wager and Athey, 2018; Athey et al., 2019) implemented in the \texttt{grf} R-package (Tibshirani et al., 2021b), and R/U-learners (Nie and Wager, 2020) with gradient boosting and lasso via the \texttt{rlearner} R package. We remark that these competing ML methods were initially developed for i.i.d. data. But, \texttt{grf} provides some ways for users to accommodate clustering by using \texttt{weight.vector} and \texttt{clusters} options in \texttt{average\_treatment\_effect} function and we use these recommended tuning procedures in the simulation study; see Tibshirani et al. (2021a) for additional details. We use the same simulation models as in Section 5.1 except we fix \( N = 500 \).

Table 5.2 summarizes the result. Also, among competing ML methods initially developed for i.i.d. data, GRF with the recommended modifications for clustering seems to get close to the targeted nominal coverage, ranging from 91\% to 95.5\%. In contrast, R- and U-Learners’ coverage rates are often much lower, ranging from 70\% to 92\%. Finally, both of our estimators \( \tau \) and \( \hat{\tau} \) maintain nominal coverage, with \( \hat{\tau} \) having the smallest standard error among all estimators. Specifically, the other competing ML estimators have 5.4 to 93.9 percent larger variances than \( \hat{\tau} \).

Next, we study the robustness of \( \tau \) and \( \hat{\tau} \) against non-normal outcome regression random effect \( U_i \). Specifically, we use the same outcome regression as equation (12) but the distribution of \( U_i \) is generated from the mixture of the following four distributions: \( N(0, 0.5^2) \), \( 0.5 \cdot \text{t}(5) \), \( \text{Laplace}(0, 0.5) \), and \( \text{Unif}(−0.25, 0.25) \). The ANOVA-type estimate of the ICC of the outcome is 0.23. We keep the same propensity score model under \( \sigma_V \in \{0, 1.5\} \) and compare the estimators in Table 5.2. Table 5.3 summarizes the result. Under all settings, \( \hat{\tau} \) has the smallest standard error where
| Statistic | Bias   | SE    | Coverage | Bias   | SE    | Coverage | Bias   | SE    | Coverage |
|-----------|--------|-------|----------|--------|-------|----------|--------|-------|----------|
| R-Learner-B | -1.61  | 7.25  | 0.875    | -1.86  | 9.90  | 0.920    | -5.30  | 7.12  | 0.805    |
| R-Learner-L | -11.45 | 9.62  | 0.750    | -11.43 | 12.78 | 0.780    | -8.88  | 10.67 | 0.885    |
| U-Learner-B | -3.28  | 7.37  | 0.860    | -3.66  | 9.96  | 0.900    | -6.38  | 7.31  | 0.755    |
| U-Learner-L | -11.40 | 9.44  | 0.810    | -11.41 | 12.79 | 0.800    | -8.85  | 10.74 | 0.900    |
| GRF       | -0.10  | 7.58  | 0.930    | 0.81   | 11.18 | 0.915    | 0.07   | 7.61  | 0.955    |
| $\tau$    | 0.19   | 6.54  | 0.955    | 1.00   | 11.00 | 0.950    | 0.48   | 7.11  | 0.960    |
| $\hat{\tau}$ | 0.20   | 6.20  | 0.950    | 1.05   | 7.58  | 0.950    | -0.06  | 6.14  | 0.980    |

Table 5.2: Comparison of different estimators under varying correlation strength. Each row represents different estimators for the average treatment effect. Each column shows the empirical biases, empirical standard errors, and coverages of 95% confidence intervals under different values of $\sigma_V$ and $\sigma_U$. Bias and SE columns are scaled by 100. The values in boldface are the smallest standard errors under each data generating process.

the variances of the other competing ML estimators 1.7 to 69.6 percent larger than that of $\hat{\tau}$. Moreover, $\hat{\tau}$ achieves closer to nominal coverage compared to existing nonparametric estimators. The additional simulation supports the robustness and the efficiency gains of $\hat{\tau}$ under different correlation structures among individuals.

| Statistic | Bias   | SE    | Coverage |
|-----------|--------|-------|----------|
| R-Learner-B | -2.01  | 6.86  | 0.910    |
| R-Learner-L | -11.50 | 10.00 | 0.755    |
| U-Learner-B | -3.82  | 7.02  | 0.855    |
| U-Learner-L | -11.41 | 10.01 | 0.795    |
| GRF       | -0.41  | 7.80  | 0.910    |
| $\tau$    | 0.04   | 6.37  | 0.965    |
| $\hat{\tau}$ | -0.14  | 5.90  | 0.965    |

Table 5.3: Comparison of different estimators under non-normal $U_i$. Each row represents different estimators for the average treatment effect. Each column shows the empirical biases, empirical standard errors, and coverages of 95% confidence intervals under different values of $\sigma_V$ and $\sigma_U$. Bias and SE columns are scaled by 100. The values in boldface are the smallest standard errors under each data generating process.
6 Applications

6.1 Observational Study: Early Childhood Longitudinal Study

We apply our method to estimate the average treatment effect of center-based pre-school programs on children’s reading score in kindergarten from the Early Childhood Longitudinal Study’s Kindergarten Class of 1998-1999 (ECLS-K) dataset (Tourangeau et al., 2009); note that a similar question with respect to children’s math score was asked in Lee et al. (2021). Briefly, the dataset is a sample of children followed from kindergarten through eighth grade. Children define the individual study units, kindergarten classrooms define clusters, and the subscript $ij$ indicates the $j$th child in the $i$th kindergarten. The treatment takes on value $A_{ij} = 1$ if the child received center-based care before kindergarten and $A_{ij} = 0$ otherwise (i.e., parental care); note that the treatment was assigned at the individual-level. The outcome $Y_{ij} \in [22, 94]$ is the standardized reading score of each child, measured during the Fall semester and after the treatment assignment. For cluster-level pre-treatment covariates, we include cluster size, region (northeast/midwest/south/west), kindergarten location (central city/urban/rural), and kindergarten type (public/private). For individual-level pre-treatment covariates, we include children’s gender, age, race, motor skill, family type, parental education, and economic status. We restrict our analysis to children with complete data on the outcome, treatment, and pre-treatment covariates. This results in 15,980 children in 942 kindergartens, which corresponds to 16.96 children per kindergarten. Cluster size varies between 1 and 25 children; the first, second, and third quartiles of cluster size are 14, 19, and 21, respectively; see Section A.3 of the supplementary material for additional details.

To assess the outcome and the treatment correlations, we compute the ICCs as follows. For the outcome ICC, we fit a linear mixed effects regression model where $Y_{ij}$ is the independent variable, $(A_{ij}, X_{ij})$ are the dependent variables, and kindergarten classrooms serve as random effects. For the treatment ICC, we fit a logistic mixed effect regression model where $A_{ij}$ is the independent variable, $X_{ij}$ are the dependent variables, and kindergarten classrooms serve as random effects. Using these models, we find that the ICCs of the outcome and the treatment models are 0.099 and 0.062, respectively, suggesting a moderate amount of correlation/dependencies between children in the same kindergarten classroom.

The target estimand is the average treatment effect with $w(C_i) = 1$. We compute $\bar{\tau}$ and $\hat{\tau}$ using the same procedure in Section 5 except we repeat cross-fitting 100 times and use 24 strata parameters to estimate the outcome covariance model $\beta$ in $B_\gamma$. We remark that by Theorem 4.1, as long as either the outcome model or the conditional propensity score is correctly specified, $\hat{\tau}$ is consistent, irrespectively of the estimated outcome covariance model. Also, if the convergence rates for the outcome model, the conditional propensity score, and the outcome covariance model are sufficiently fast, $\hat{\tau}$ is asymptotically Normal. Similarly, by Theorem 3.1, as long as either the outcome model or the propensity score is correct, $\bar{\tau}$ is consistent for the average treatment effect and if the convergence rates are sufficiently fast, $\bar{\tau}$ should be asymptotically Normal.
For comparison, we estimate the average treatment effect using other methods as follows. For GLMM and GEEs, we use the parametric models used to compute ICCs to estimate the average treatment effect. For GRF and R/U-Learners, we use the same approach as in Section 5.2. Finally, we also compute the estimate from a generalized linear model (GLM) that does not consider correlation between study units in a cluster.

Table 6.1 summarizes the result. First, our estimates, GRF, R-Learner with gradient boosting (R-Learner-B) and U-Learner with gradient boosting (U-Learner-B) show similar effect estimates around 1.3. The effect estimates from parametric methods (i.e. GLM, GLMM, GEE) and the two lasso-based approaches (R-Learner-L, U-Learner-L) are larger, hovering around 1.5. The discrepancy may be due to potentially non-linear or higher-order terms in the outcome or the treatment models which the GLM, GLMM, GEE, and the two lasso-based approaches may fail to properly model and induce bias. In terms of standard error, our estimator \( \hat{\tau} \) shows the smallest standard error, where the other competing estimators have 17.3 to 32.2 percent larger variances than \( \hat{\tau} \).

|          | GLMM | GEE | GLM | R-Learner-B | R-Learner-L | U-Learner-B | U-Learner-L | GRF | \( \tau \) | \( \hat{\tau} \) |
|----------|------|-----|-----|-------------|-------------|-------------|-------------|-----|----------|-----------|
| Estimate | 1.519| 1.516| 1.505| 1.279       | 1.486       | 1.275       | 1.484       | 1.332| 1.333    | 1.323     |
| SE       | 0.229| 0.229| 0.232| 0.241       | 0.231       | 0.244       | 0.230       | 0.239| 0.238    | 0.212     |

Table 6.1: Summary of the data analysis from the ECLS-K dataset. Each column represents the estimators. Each row shows the point estimates, the standard errors, and the relative efficiency of each estimator compared to \( \hat{\tau} \). The value in boldface is the smallest standard error among all the standard errors.

### 6.2 Randomized Experiment: HIV Counseling in Kenya

We also apply our method to a randomized experiment by Duflo et al. (2015) and Duflo et al. (2019) on the effect of encouraging voluntary counseling and testing for HIV, abbreviated as VCT, on actually attending a VCT session. Briefly, the study took place in two districts of Western Kenya, Butere-Mumias and Bungoma during 2003. Students defined the individual study units, primary schools defined clusters, and the subscript \( ij \) indicates the \( j \)th individual who enrolled in the \( i \)th school. The treatment of individual \( j \) in school \( i \) takes on \( A_{ij} = 1 \) if the individual was encouraged to attend VCT and \( A_{ij} = 0 \) otherwise; note that the encouragement was randomized at individual level with \( P(A_{ij} = 1) = 0.5 \). The outcome is binary with \( Y_{ij} = 1 \) if individual \( j \) in school \( i \) attended VCT more than once and \( Y_{ij} = 0 \) otherwise. For cluster-level pre-treatment covariates, we include cluster size, location of school (urban/middle-rural/rural), treatment indicators from a prior experiment (two indicator variables corresponding to two prior interventions), and Kenyan Certificate of Primary Education exam score for each school. For individual-level pre-treatment covariates, we include individual’s gender, age, education, marriage status, and an indicator for whether the individual had any children. We restrict the sample to children with complete data, which results in 7,400 students in 325 schools. The cluster size varies between 1 and 77; the first,
second, and third quartiles of cluster size are 13, 21, and 32, respectively. The target estimand is the average treatment effect with \( w(C_i) = 1 \).

Unlike the observational study in Section 6.1, the treatment is randomized by the study design and there are no unmeasured confounders. By the same reason, the treatment is not correlated among students. But, within each school, students’ outcomes are likely correlated due to potentially, unmeasured, but shared characteristics among students in the same school, such as school funding, quality of teachers, and differences in how the HIV prevention curriculum is delivered. To assess this correlation between outcomes, we use a logistic mixed effect outcome model and find that the variance partition coefficient (Goldstein et al., 2002b; Merlo et al., 2005, 2016) is 0.057, showing mild outcome correlation structure.

We use the same estimators in Section 6.1 to estimate the average treatment effect, except we use the propensity score from the experiment design and we use 50 strata parameters according to the cluster size to estimate the outcome covariance model \( \beta \). In particular, because the treatment is not correlated due to the experimental design, the conditional independence assumption in Lemma 4.1 and Theorem 4.1 are satisfied and \( \hat{\tau} \) is asymptotically as efficient as \( \tau \), irrespectively of the outcome covariance model. Relatedly, because the outcome is bounded between 0 and 1, all of the technical conditions in Theorems 3.1 and 4.1 are satisfied, guaranteeing that \( \hat{\tau} \) and \( \tau \) are consistent and asymptotically Normal.

Table 6.2 summarizes the estimates of the average treatment effect from different methods. All estimators show similar effect estimates around 0.132, which corroborates the double robustness property of all the estimators when the propensity score is known. In terms of standard error, again, we find that our estimator \( \hat{\tau} \) shows the smallest standard error where the variances of the other competing estimators are 8.8 to 29.4 percent larger than that of \( \hat{\tau} \).

|        | GLM   | GLMM  | GEE   | R-Learner-B | R-Learner-L | U-Learner-B | U-Learner-L | GRF | \( \tau \) | \( \hat{\tau} \) |
|--------|-------|-------|-------|-------------|-------------|-------------|-------------|-----|-----------|-------------|
| Estimate| 0.1309 | 0.1308 | 0.1307 | 0.1341       | 0.1308       | 0.1339       | 0.1307       | 0.1320 | 0.1322    | 0.1311      |
| SE     | 0.0130 | 0.0130 | 0.0130 | 0.0120       | 0.0129       | 0.0122       | 0.0130       | 0.0131 | 0.0129    | 0.0115      |

Table 6.2: Summary of the data analysis from the VCT study. Each column represents the estimators. Each row shows the point estimates, the standard errors, and the relative efficiency of each estimator compared to \( \hat{\tau} \).

7 Conclusion

This paper presents two estimators \( \tau \) and \( \hat{\tau} \) to infer average treatment effects in multilevel studies. The estimator \( \tau \) is a simple extension of the cross-fitting estimator developed under i.i.d. settings to clustered settings. The estimator \( \hat{\tau} \) is a novel estimator that uses the conditional propensity score \( \pi \) and the outcome covariance model \( \beta \). Both estimators \( \tau \) and \( \hat{\tau} \) are doubly robust, asymptotically Normal, and do not require parametric specifications of the propensity score or the outcome model.
However, \( \hat{\tau} \) is often more efficient than \( \bar{\tau} \) and other existing nonparametric estimators for the average treatment effect. In particular, in the simulation study and the two empirical applications, we find that \( \hat{\tau} \) is about 15\% to 50\% more efficient than existing nonparametric estimators that do not account for the correlation structure.

We end by re-iterating some important limitations of our proposed methods and, in light of these limitations, offer some guidelines on how to use them in practice. First, while our proposed estimator \( \hat{\tau} \) performs well in the two empirical studies and all the simulation studies we considered, we do not show whether \( \hat{\tau} \) is optimal in terms of semiparametric efficiency. As hinted in Section 4.1, in nonparametric, multilevel settings, the semiparametric efficiency bound of \( \tau^* \) depends on both the smoothness of the conditional mean of the outcome as well as the covariance of the outcomes in a cluster as each part can have non-diminishing first-order terms in a locally linear, asymptotic expansion of the estimator. In light of this, we conjecture that (a) \( \hat{\tau} \) can achieve the semiparametric efficiency bound under additional assumptions about the exchangeability/invariance of the elements of the outcome covariance matrix, but (b) there does not exist a regular, asymptotically linear estimator that can achieve the bound with only Assumptions (M1) and (M2), which only make first-order invariance assumptions to allow arbitrary correlation structures between study units in a cluster. In short, our nonparametric model in (M1) and (M2) does not impose structure on the covariance structure, but it may come at a cost of not being able to construct an optimal estimator (among all regular, asymptotically linear estimators) that can achieve the semiparametric efficiency bound.

Second, while our theory currently relies on the bounded cluster size assumption, as demonstrated in Section 5.1, our methods seem to perform well even if the cluster size is larger than the number of clusters. Of course, we do not expect this phenomena to hold in settings where the cluster is very large and there are a couple number of clusters, say if study units consist of millions of residents in a handful of U.S. states. To the best of our knowledge, there aren’t any nonparametric methods that can accommodate this setting. If faced with such settings, we recommend investigators make some semiparametric or parametric assumptions to make inference tractable or to re-define the study unit so that our proposed methods can produce valid inference.

Third, whenever the number of clusters is larger or roughly comparable to the number of samples, say if clusters consist of households, villages, clinics, schools, or counties, and study units are individuals who belong to said clusters, we recommend investigators use our method, notably \( \hat{\tau} \), compared to existing nonparametric estimators in causal inference. Not only \( \hat{\tau} \) will produce doubly robust, consistent, asymptotically Normal estimates of the average treatment effect under the same regularity assumptions as those from cross-fitting estimators in the literature, our estimator constantly outperformed them in real data studies.
Supplementary Material

This document contains supplementary material for “More Efficient, Doubly Robust, Nonparametric Estimators of Treatment Effects in Multilevel Studies.” Section A presents additional results related to the main paper. Section B provides useful lemmas. Section C and D prove the theorems and lemmas introduced in the main paper and the supplementary material.

A Details of the Main Paper

A.1 Details of Section 2.3

In this section, we show that the generalized linear mixed effect model (GLMM) in (2) and the generalized estimating equation (GEE) model in (3) imply (M1) and (M2) under additional conditions.

A.1.1 GLMM

Let \( W_{ij} \) be the individual-level pre-treatment covariates (i.e. \( W_{ij} = X_{ij} \backslash C_i \)). Suppose that (i) \( W_{ij} \) are i.i.d. across \( ij \) given \( C_i \) and (ii) \( C_i \) are i.i.d. across \( i \) where \( n_i \) is included in \( C_i \) with a bounded support, i.e. \( n_i \leq M \). Then, the individual-level distribution given \( (U_i, V_i) \) is

\[
P(O_{ij} | U_i, V_i) = P(Y_{ij} | A_{ij}, W_{ij}, C_i, U_i, V_i) P(A_{ij} | W_{ij}, C_i, U_i, V_i) P(W_{ij} | C_i, U_i, V_i) P(C_i | U_i, V_i) = P(Y_{ij} | A_{ij}, X_{ij}, U_i) P(A_{ij} | X_{ij}, V_i) P(W_{ij} | C_i) P(C_i) .
\]

Note that the functional form of \( P(O_{ij} | U_i, V_i) \) is identical across \( ij \). The cluster-level distribution of \( O_i, P(O_i) \), is represented as follows.

\[
P(O_i) = \prod_{j=1}^{n_i} P(O_{ij} | U_i, V_i) P(U_i, V_i) .
\]

All distributions in the right hand side are identical across \( ijs \), so \( P(O_i) \) does not depend on \( i \).

To show (M1), we find that \( E(Y_{ij} | A_{ij} = a, X_{ij} = x) \) and \( E(A_{ij} | X_{ij} = x) \) do not depend on \( ij \):

\[
E(Y_{ij} | A_{ij} = a, X_{ij} = x) = \alpha_0 + a\alpha_1 + x^\top \beta , \quad E(A_{ij} | X_{ij} = x) = \int \expit(x^\top \gamma + v) f(v; \sigma_v) dv ,
\]

where \( f(v; \sigma_v) \) is the density of \( N(0, \sigma_v^2) \).

To show (M2), we find the following inequality holds for some constant \( c \):

\[
\|Y_i - g^*(A_i, X_i)\|_2^4 = \|1_{n_i} U_i + \epsilon_i\|_2^4 \leq c \left( n_i U_i^4 + \sum_{j=1}^{n_i} \epsilon_i^4 \right) .
\]

Therefore, the expectation of the above quantity is bounded because \( n_i \leq M \) and \( E(Z^4) = 3\sigma^4 < \infty \) for \( Z \sim N(0, \sigma^2) \). Lasty, we find

\[
E[\{Y_i - g^*(A_i, X_i)\}^2 \mid A_i, X_i] = E(1_{n_i} U_i^2 + 1_{n_i} U_i \epsilon_i^\top + \epsilon_i 1_{n_i} U_i + \epsilon_i \epsilon_i^\top \mid A_i, X_i) = \sigma^2 1_{n_i} 1_{n_i}^\top + \sigma^2 I .
\]
such that \( \operatorname{Var}(O) \) bounded by a positive constant \( c' \) for all \((A_i, X_i)\); and (iii) for any \( X_i \), there exists \( A_i \) and \( K > 0 \) such that \( \operatorname{Var}(c^T Y_i | A_i, X_i) \leq K \) for any non-zero \( c \in \mathbb{R}^n_i \).

To show (M1), we find that \( E(Y_{ij} | A_{ij} = a, X_{ij} = x) \) and \( E(A_{ij} | X_{ij} = x) \) do not depend on \( ij \):

\[
E(Y_{ij} | A_{ij} = a, X_{ij} = x) = \int E(Y_{ij} | A_{ij} = a, A_{i(-j)} = a', X_{ij} = x, X_{i(-j)} = x') P(A_{i(-j)} = a', X_{i(-j)} = x' | X_{ij} = x) d(a', x')
\]

\[
E(A_{ij} | X_{ij} = x) = \int E(A_{ij} | X_{ij} = x, X_{i(-j)} = x') P(X_{i(-j)} = x' | X_{ij} = x) dx'
\]

To show (M2), we find that

\[
\|g^*(A_i, X_i)\|_2^4 = \|E(Y_i | A_i, X_i)\|_2^4 \leq E\{\|Y_i\|_2^4 | A_i, X_i\} = \sum_{j,j'} E(Y_{ij}^2 Y_{ij'}^2 | A_i, X_i) \leq \left\{ \sum_{j=1}^{n_i} E(Y_{ij}^2 | A_i, X_i) \right\}^{1/2} \left\{ \sum_{j'=1}^{n_i} E(Y_{ij'}^2 | A_i, X_i) \right\}^{1/2} \leq \overline{M} c'.
\]

The first inequality holds from the Hölder’s inequality. The second inequality holds from \( E(XY | Z) \leq \{E(X^2 | Z)E(Y^2 | Z)\}^{1/2} \). The last inequality holds from \( n_i \leq \overline{M} \) and \( E(Y_{ij}^4 | A_i, X_i) \leq c' \). Lastly, we find

\[
\operatorname{Var}(c^T Y_i | A_i, X_i) = c^T E\{ (Y_i - g^*(A_i, X_i))^2 | A_i, X_i \} c \geq K \text{ for all } c.
\]

This implies the smallest eigenvalue of \( \{Y_i - g^*(A_i, X_i)\}^2 \) is strictly positive.

### A.2 Details of Section 5

#### A.2.1 Details of the Intra-cluster Correlation Coefficient

First, we presented the closed form of the ICC used in the main paper. The ICC of continuous \( Y \) is defined as \( \text{ICC}_Y(\sigma_U^2) = \sigma_U^2/(1 + \sigma_U^2) \). For a binary \( A_i \), we obtain the ANOVA-type estimate of
the ICC from Ridout et al. (1999) and Liljequist et al. (2019), i.e.

\[
\text{ICC}_A(\sigma_V) = \frac{MS_{A,b} - MS_{A,w}}{MS_{A,b} + (n_0 - 1)MS_{A,w}}, \quad n_0 = \frac{1}{N - 1} \left( \sum_{i=1}^{N} n_i - \sum_{i=1}^{N} n_i^2 \right)
\]

where \(MS_{A,b}\) and \(MS_{A,w}\) are the between-group and within-group mean squares of the residuals

\[
r_{ij} = A_{ij} - \epsilon^*(A_{ij} | X_{ij})
\]

\[
MS_{A,w} = \frac{1}{\sum_{i=1}^{N} n_i - N} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{n_i} r_{ij}^2 - \sum_{i=1}^{N} \frac{(\sum_{j=1}^{n_i} r_{ij})^2}{n_i} \right\},
\]

\[
MS_{A,b} = \frac{1}{N - 1} \left\{ \sum_{i=1}^{N} \frac{(\sum_{j=1}^{n_i} r_{ij})^2}{n_i} - \frac{(\sum_{i=1}^{N} \sum_{j=1}^{n_i} r_{ij})^2}{\sum_{i=1}^{N} n_i} \right\}.
\]

### A.2.2 Finite-Sample Relative Efficiency Between \(\hat{\tau}\) and \(\tau\)

We study the finite-sample efficiency characteristics of \(\hat{\tau}\) and \(\tau\) for different values of \((\sigma_V, \sigma_U)\). Specifically, we use the simulation models in Section 5.1 in the main paper when \(N = 500\). We vary both \((\sigma_V, \sigma_U)\) from a two-dimensional grid \(\{0, 0.1, \ldots, 2\} \otimes 2\). We then compare the efficiency between \(\tau(e^*, g^*)\) and \(\hat{\tau}(\pi^*, g^*, \hat{\beta})\) where \(\hat{\beta}\) is obtained from the estimating equation in equation (9) in the main paper. Here, we use the true nuisance functions \(\pi, g, \) and \(\epsilon\) to better highlight the differences in efficiency without the variability arising from estimating them. Specifically, for each \((\sigma_V, \sigma_U)\) combination, we generate the data 1,000 times and obtain the empirical relative efficiency between the two estimators, i.e.

\[
\rho(\sigma_V, \sigma_U) = \frac{\text{Var}\{\tau(e^*, g^*)\}}{\text{Var}\{\hat{\tau}(\pi^*, g^*, \hat{\beta})\}}.
\]

Figure A.1 shows the empirical relative efficiencies between the two estimators \(\hat{\tau}\) and \(\tau\) for different values of \((\sigma_V, \sigma_U)\). We see that \(\hat{\tau}\) is more efficient than \(\tau\) when the outcome correlation, as governed by \(\sigma_U\), is large. For example, if the outcome ICC is above 0.2, \(\hat{\tau}\) is always more efficient than \(\tau\), irrespective of the correlation between the treatment values. Also, when there is no correlation between treatments (i.e. when \(\sigma_V = 0\)), \(\hat{\tau}\) is always more efficient than \(\tau\), corroborating the result in Lemma 4.1 in the main paper. Interestingly, \(\tau\) is more efficient when the treatment correlation, as governed by \(\sigma_V\), is large, but not by a large margin compared to when the outcome correlation is large; \(\tau\) is no more than 20% efficient compared to \(\hat{\tau}\) when \(\sigma_V\) is large, but \(\hat{\tau}\) can be 300% more efficient compared to \(\tau\) when \(\sigma_U\) is large. In short, we expect that \(\hat{\tau}\) would perform better when (i) the units’ treatments in the same cluster are weakly correlated to each other and (ii) the outcomes of units are correlated.
A.3 Details of Section 6.1

We assess covariate balance and overlap in the ECLS-K dataset. We first obtain the median estimates of the propensity score and the conditional propensity score as follows.

\[
\hat{e}_{\text{med}}(a | X_{ij}) = \text{median}_{k=1, \ldots, 100} \hat{e}_k[a | X_{ij}],
\]

\[
\hat{\pi}_{\text{med}}(a | A_{i(-j)}, X_{ij}, X_{i(-j)}) = \text{median}_{k=1, \ldots, 100} \hat{\pi}_k[a | A_{i(-j)}, X_{ij}, X_{i(-j)}],
\]

where \(\hat{e}_k\) and \(\hat{\pi}_k\) are the estimated propensity score and conditional propensity score from \(k\)th cross-fitting procedure, respectively.

To assess covariate balance after the (conditional) propensity score adjustment, we test the following hypotheses:

\[
H_{0,\text{no}} : E\left\{ \frac{A_{ij}X_i}{P(A_{ij} = 1)} - \frac{(1 - A_{ij})X_{ij}}{P(A_{ij} = 0)} \right\} = 0 ,
\]

\[
H_{0,e} : E\left\{ \frac{A_{ij}X_{ij}}{e(1 | X_{ij})} - \frac{(1 - A_{ij})X_{ij}}{e(0 | X_{ij})} \right\} = 0 ,
\]

\[
H_{0,\pi} : E\left\{ \frac{A_{ij}X_{ij}}{\pi(1 | A_{i(-j)}, X_{ij}, X_{i(-j)})} - \frac{(1 - A_{ij})X_{ij}}{\pi(0 | A_{i(-j)}, X_{ij}, X_{i(-j)})} \right\} = 0 .
\]

We randomly sample one individual from each cluster 10,000 times to guarantee the independence between observations used in the hypothesis testing. For each randomly chosen sample, we obtain three \(t\)-statistics associated with the above hypotheses by using \(\sum_{ij} A_{ij} / \sum_{i} n_i\), \(\hat{e}_{\text{med}}\), \(\hat{\pi}_{\text{med}}\) as the estimates of \(P(A_{ij} = 1)\), \(e\), and \(\pi\), respectively. To mitigate the effect of the particular random split, we obtain the median of 10,000 \(t\)-statistics.

Table A.1 shows the result. The \(t\)-statistics regarding \(H_{0,e}\) suggest that covariate balance is
achieved in all covariates. Similarly, the $t$-statistics regarding $H_{0,\pi}$ show that covariate balance is achieved in all covariates except socioeconomic status ($t$-statistic= 2.390). However, compared to the $t$-statistic regarding $H_{0,\text{no}}$ in socioeconomic status ($t$-statistic= 5.295), we find that adjustment with $\pi$ improved the balance of socioeconomic status. Moreover, given the number of hypotheses of interest is 34, one significant $t$-statistic may occur even though covariate balance is not violated. Therefore, we conclude that covariate balance is achieved for both $e$ and $\pi$.

| Covariate                      | Test statistic $H_{0,\text{no}}$ | $H_{0,e}$ | $H_{0,\pi}$ |
|--------------------------------|----------------------------------|-----------|-------------|
| Cluster size                   | -0.539                           | -0.162    | -0.320      |
| Census region=Northeast        | -0.759                           | -0.641    | -0.566      |
| Census region=South            | -0.766                           | -0.811    | -0.819      |
| Census region=West             | -1.985                           | 0.037     | -0.463      |
| Location=City                  | -0.910                           | -0.198    | -0.543      |
| Location=Rural                 | -1.353                           | -0.031    | -0.050      |
| Public kindergarten            | -2.522                           | -0.601    | -1.058      |
| Male                           | 0.206                            | 0.960     | 1.002       |
| Age                            | -1.060                           | -0.333    | -0.433      |
| Race=Asian                     | -0.200                           | 0.188     | -0.326      |
| Race=Black                     | 1.814                            | 0.384     | 0.665       |
| Race=Hispanic                  | -3.197                           | -1.060    | -1.513      |
| Race=White                     | 0.018                            | -0.154    | 0.073       |
| Motor skill                    | -0.403                           | -0.157    | -0.103      |
| Intact family                  | -1.050                           | -0.278    | -0.470      |
| Parental education $\geq$ College | 2.168                            | 0.254     | 0.561       |
| Socioeconomic status           | 5.925                            | 1.408     | 2.390       |

Table A.1: Covariate Balance Assessment: Each column shows $t$-statistics under $H_{0,\text{no}}$, $H_{0,e}$, and $H_{0,\pi}$, respectively. Each row shows pre-treatment covariates.

Next, we assess overlap for $e$ and $\pi$ via histograms and box plots. As shown in Figure A.2, we find the overlap is not violated for both $e$ and $\pi$.

![Propensity Score (e)](image1.png) ![Conditional Propensity Score (π)](image2.png)

Figure A.2: Overlap Assessment: histograms (top) and box plots (bottom) of $\hat{e}_{\text{med}}(1 \mid X_{ij})$ (left) and $\hat{\pi}_{\text{med}}(1 \mid A_{i(-j)}, X_{ij}, X_{i(-j)})$ (right).
B Lemma

In this section, we introduce four lemmas that are used in the proof in Section C.

Lemma B.1. Let $I(A_i, X_i; e)$ be

$$I(A_i, X_i; e) = \frac{1}{n_i} \begin{bmatrix} 1(A_{i1} = 1)/e(1 | X_{i1}) - 1(A_{i1} = 0)/e(0 | X_{i1}) \\ \vdots \\ 1(A_{in_i} = 1)/e(1 | X_{in_i}) - 1(A_{in_i} = 0)/e(0 | X_{in_i}) \end{bmatrix}.$$ 

Under conditions (A1)-(A3), (M1)-(M2), and (E1) of the main paper, the following conditions hold.

(a) The cluster size is bounded as $n_i \leq M$ for some integer $M$. The weight $w(C_i)$ satisfies $|w(C_i)| \leq C_w$ for all $C_i$ with some positive constant $C_w$.

(b) We have $E\{\|Y - g^*(A_i, X_i)\|^4 | A_i, X_i\} \leq C_4$ for all $A_i, X_i$, and for some some constant $C_4$. Furthermore, we have $\|\Sigma(A_i, X_i)\|_2 \leq C_2$. Moreover, for all $X_i$, the smallest eigenvalue of $\Sigma(A_i, X_i)$ is lower bounded by $C_2^{-1}$ for some $A_i$.

(c) For all $A_i, X_i$, we have $\|\hat{g}^{(-k)}(A_i, X_i) - g^*(A_i, X_i)\|_2 \leq C_g$, $\|I(A_i, X_i; e^*)\|_2 \leq C_e$, and $\|I(A_i, X_i; \hat{e}^{(-k)})\|_2 \leq C_e$ for some constants $C_g$ and $C_e$.

Additionally, the following conditions hold for all $k = 1, 2$ under (E2.PS), (E2.OR), and (E2.Both) of the main paper, respectively.

(d) PS $\|I(A_i, X_i; \hat{e}^{(-k)}) - I(A_i, X_i; e^*)\|_{P^2} = O_P(r_{e,N})$ with $r_{e,N} = o(1)$, and $\|\hat{g}^{(-k)}(A_i, X_i) - g'(A_i, X_i)\|_{P^2} = O_P(r_{g,N})$ for some function $g'$ satisfying $|g^*(A_i, X_i) - g'(A_i, X_i)| \leq C_g$ with $r_{g,N} = o(1)$.

(d) OR $\|I(A_i, X_i; \hat{e}^{(-k)}) - I(A_i, X_i; e')\|_{P^2} = O_P(r_{e,N})$ for some function $e'$ satisfying $\|I(A_i, X_i; e')\|_2 \leq C_e$ with $r_{e,N} = o(1)$, and $\|\hat{g}^{(-k)}(A_i, X_i) - g^*(A_i, X_i)\|_{P^2} = O_P(r_{g,N})$ with $r_{g,N} = o(1)$.

(d) Both $\|I(A_i, X_i; \hat{e}^{(-k)}) - I(A_i, X_i; e^*)\|_{P^2} = O_P(r_{e,N})$ with $r_{e,N} = o(1)$, and $\|\hat{g}^{(-k)}(A_i, X_i) - g^*(A_i, X_i)\|_{P^2} = O_P(r_{g,N})$ with $r_{g,N} = o(1)$ and $r_{e,N} r_{g,N} = o(N^{-1/2})$. 

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Lemma B.2. Let $I(A_i; X_i; \pi)$ be

$$I(A_i; X_i; \pi) = \frac{1}{n_i} \left[ \begin{array}{c} 1(A_{i1} = 1)/\pi(1 | A_{i(-1)}, X_{i1}, X_{i(-1)}) - 1(A_{i1} = 0)/\pi(0 | A_{i(-1)}, X_{i1}, X_{i(-1)}) \\ \vdots \\ 1(A_{ini} = 1)/\pi(1 | A_{i(-ni)}, X_{ini}, X_{i(-ni)}) - 1(A_{ini} = 0)/\pi(0 | A_{i(-ni)}, X_{ini}, X_{i(-ni)}) \end{array} \right] .$$

Under conditions (A1)-(A3), (M1)-(M2), and (EN1) of the main paper, the following conditions hold.

(a) The cluster size is bounded as $n_i \leq M$ for some integer $M$. The weight $w(C_i)$ satisfies $|w(C_i)| \leq C_w$ for all $C_i$ with some positive constant $C_w$.

(b) We have $E\{\|Y_i - g^*(A_i, X_i)\|^4 | A_i, X_i\} \leq C_4$ for all $A_i, X_i$, and for some constant $C_4$. Furthermore, we have $\|\Sigma(A_i, X_i)\|_2 \leq C_2$. Moreover, for all $X_i$, the smallest eigenvalue of $\Sigma(A_i, X_i)$ is lower bounded by $C_2^{-1}$ for some $A_i$.

(c) For all $A_i, X_i$, we have $\|\hat{g}^{(-k)}(A_i, X_i) - g^*(A_i, X_i)\|_2 \leq C_g$, $\|I(A_i, X_i; \pi^*)\|_2 \leq C_\pi$, and $\|\hat{I}(A_i, X_i; \hat{\pi}^{(-k)})\|_2 \leq C_\pi$ for some constants $C_g$ and $C_\pi$. Furthermore, we have $\|\hat{\beta}^{(k)}(C_i) - \beta_1^{(k)}(C_i)\|_2 = O_P(r_{\beta,N})$ where $r_{\beta,N} = o(1)$ with some function $\beta_1^{(k)}(C_i) \in \mathcal{B}$ that is fixed given $T_k^c$. Moreover, there exists a fixed function $\beta_2^{(k)}(C_i) \in \mathcal{B}$ so that $\|\beta_2^{(k)}(C_i) - \beta_1^{(k)}(C_i)\|_2 = O_P(r_{\pi,N}) + O_P(r_{g,N})$.

Additionally, the following conditions hold for all $k = 1, 2$ under (EN2.PS), (EN2.OR), and (EN2.Both) of the main paper, respectively.

(d.PS) $\|I(A_i, X_i; \hat{\pi}^{(-k)}) - I(A_i, X_i; \pi^*)\|_2 = O_P(r_{\pi,N})$ with $r_{\pi,N} = o(1)$, and $\|\hat{g}^{(-k)}(A_i, X_i) - g'(A_i, X_i)\|_2 = O_P(r_{g,N})$ for some function $g'$ satisfying $\|g^*(A_i, X_i) - g'(A_i, X_i)\|_2 \leq C_g$ with $r_{g,N} = o(1)$.

(d.OR) $\|I(A_i, X_i; \hat{\pi}^{(-k)}) - I(A_i, X_i; \pi')\|_2 = O_P(r_{\pi,N})$ for some function $\pi'$ satisfying $\|I(A_i, X_i; \pi')\|_2 \leq C_\pi$ with $r_{\pi,N} = o(1)$, and $\|\hat{g}^{(-k)}(A_i, X_i) - g^*(A_i, X_i)\|_2 = O_P(r_{g,N})$ with $r_{g,N} = o(1)$.

(d.Both) $\|I(A_i, X_i; \hat{\pi}^{(-k)}) - I(A_i, X_i; \pi^*)\|_2 = O_P(r_{\pi,N})$ with $r_{\pi,N} = o(1)$, and $\|\hat{g}^{(-k)}(A_i, X_i) - g^*(A_i, X_i)\|_2 = O_P(r_{g,N})$ with $r_{g,N} = o(1)$. Furthermore, $r_{\pi,N}r_{g,N} = o(N^{-1/2})$ and $r_{\beta,N}r_{g,N} = o(N^{-1/2})$. 

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Lemma B.3. Suppose that the conditions in Lemma B.2 hold. For given $\mathcal{B}$, let $\beta^{\dagger,(-k)}$ and $\beta^*$ be the solutions to the following minimization problems (13) and (14), respectively.

$$
\beta^{\dagger,(-k)} \in \arg\min_{\beta \in \mathcal{B}} E\left\{ w(C_i)^2 I_i^{(-k)} \mathcal{T} B(C_i; \beta) \hat{S}_i^{(-k)} B(C_i; \beta) \mathcal{I}_i^{(-k)} \bigg| \mathcal{I}_k^c \right\}, \quad (13)
$$

$$
\beta^* \in \arg\min_{\beta \in \mathcal{B}} E\left\{ w(C_i)^2 I_i^{*\dagger} B(C_i; \beta) S_i B(C_i; \beta) \mathcal{I}_i^* \right\} \quad (14)
$$

where $\mathcal{I}_i^* = I(A_i, X_i; \pi^*)$, $\hat{I}_i^{(-k)} = I(A_i, X_i; \hat{\pi}^{(-k)})$, $g_i^*(A_i) = g^*(A_i, X_i)$, $\hat{g}_i^{(-k)}(A_i) = \hat{g}^{(-k)}(A_i, X_i)$, $S_i = \{Y_i - g_i^*(A_i)\} \otimes 2$, and $\hat{S}_i^{(-k)} = \{Y_i - \hat{g}_i^{(-k)}(A_i)\} \otimes 2$. Then, we have $\|\beta^{\dagger,(-k)}(C_i) - \beta^*(C_i)\|_P^2 = O_P(r_{\pi,N}) + O_P(r_{g,N})$.

Lemma B.4. Suppose that the conditions in Lemma B.2 hold. Let $\beta^{\dagger,(-k)}(C_i) = \sum_{\ell=1}^J 1\{L(C_i) = \ell\} \gamma^{\dagger,(-k)}_\ell$ be the solution to the minimization problem (13) under $\mathcal{B}_\gamma$. Additionally, suppose that $\gamma^{\dagger,(-k)}$ belongs to the interior of $[-B_0, B_0] \otimes J$. Then, we have $\|\beta^{\dagger,(-k)}(C_i) - \beta^{\dagger,(-k)}(C_i)\|_P^2 = O_P(N^{-1/2})$.
C Proofs

C.1 Proof of Theorem 3.1

We prove the theorem in order of (2) Asymptotic Normality and (1) Double Robustness. Note that the assumptions in Theorem 3.1 implies the conditions in Lemma B.1.

C.1.1 Proof of (2) Asymptotic Normality

We use the following forms of \( \hat{\varphi}_k \) and \( \varphi^* \) in the proof.

\[
\hat{\varphi}_k(O_i) = w(C_i) \left[ I(A_i, X_i; \hat{g}^{(-k)}(A_i, X_i)) + \frac{1}{n_i} \left\{ \hat{g}^{(-k)}(1, X_i) - \hat{g}^{(-k)}(0, X_i) \right\} \right]
\]

\[
= w(C_i) \left[ I_k^{(-k), T} \{ Y_i - \hat{g}^{(-k)}(A_i) \} + \frac{1}{n_i} \left\{ \hat{g}^{(-k)}(1) - \hat{g}^{(-k)}(0) \right\} \right],
\]

\[
\varphi^*(O_i) = w(C_i) \left[ I(A_i, X_i; e^*) \{ Y_i - g^*(A_i, X_i) \} + \frac{1}{n_i} \left\{ g^*(1, X_i) - g^*(0, X_i) \right\} \right]
\]

\[
= w(C_i) \left[ I_k^{*, T} \{ Y_i - g_k^*(A_i) \} + \frac{1}{n_i} \left\{ g_k^*(1) - g_k^*(0) \right\} \right].
\]

We find \( \sqrt{N}(\bar{\tau} - \tau^*) \) is decomposed as follows

\[
\sqrt{N}(\bar{\tau} - \tau^*) = \frac{1}{\sqrt{2}} \sum_{k=1}^{2} \frac{1}{\sqrt{N/2}} \left( \hat{\varphi}_k - \varphi^* \right)
\]

\[
= \frac{1}{\sqrt{2}} \sum_{k=1}^{2} \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \hat{\varphi}_k(O_i) - \varphi^*(O_i) \right\}
\]

\[
= \frac{1}{\sqrt{2}} \sum_{k=1}^{2} \left[ \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \hat{\varphi}_k(O_i) - \varphi^*(O_i) \right\} + \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \varphi^*(O_i) - \tau^* \right\} \right]. \tag{15}
\]

The second term in the bracket is trivially satisfies the asymptotic Normality from the central limit theorem so it suffices to show the first term the bracket is \( o_P(1) \), which is decomposed into \([B]\) and \([C]\) as follows.

\[
\frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \hat{\varphi}_k(O_i) - \varphi^*(O_i) \right\}
\]

\[
= \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left[ \left\{ \hat{\varphi}_k(O_i) - \varphi^*(O_i) \right\} - \underbrace{E \left\{ \hat{\varphi}_k(O_i) - \varphi^*(O_i) \mid T_k \right\}}_{[B]} \right] + \sqrt{\frac{N}{2}} \underbrace{E \left\{ \hat{\varphi}_k(O_i) - \varphi^*(O_i) \mid T_k \right\}}_{[C]}.
\]

We show that \([B] = O_P(r_{e,N}^2) + O_P(r_{g,N}^2)\) and \([C] = O_P(r_{e,N}r_{g,N})\).

- **Rate of \([B]\):** The squared expectation of \([B]\) is

\[
E\left\{ [B]^2 \mid T_k \right\} = \frac{1}{N/2} \sum_{i \in I_k} E \left\{ \left\{ \hat{\varphi}_k(O_i) - \varphi^*(O_i) \right\}^2 \mid T_k \right\} = E \left\{ \left( \hat{\varphi}_k(O_i) - \varphi^*(O_i) \right)^2 \mid T_k \right\}. \tag{17}
\]
We find \( \hat{\varphi}_k(O_i) - \varphi^*(O_i) \) is represented as

\[
\hat{\varphi}_k(O_i) - \varphi^*(O_i) = w(C_i) \left[ \hat{I}_i^{(-k)} \{ Y_i - \hat{g}_i^{(-k)}(A_i) \} - I_i^{*T} \{ Y_i - g_i^*(A_i) \} + \frac{1}{n_i} \left\{ \hat{g}_i^{(-k)}(1) - g_i^*(0) - (g_i^*(1) + g_i^*(0)) \right\} \right]
\]

\[
= \frac{w(C_i)}{2} \left\{ \hat{I}_i^{(-k)} - I_i^* \right\} \{ 2Y_i - \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \} + \frac{w(C_i)}{2} \left\{ \hat{I}_i^{(-k)} + I_i^* \right\} \left\{ \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \right\}
\]

\[
+ \frac{w(C_i)1^T}{n_i} \left\{ \hat{g}_i^{(-k)}(1) - g_i^*(1) \right\} - \frac{w(C_i)1^T}{n_i} \left\{ \hat{g}_i^{(-k)}(0) - g_i^*(0) \right\} .
\]

We use the inequality \((a_1 + \ldots + a_d)^2 \leq 8(a_1^2 + \ldots + a_d^2)\) and \(\|AB\|_2 \leq \|A\|_2 \|B\|_2\) to obtain an upper bound of \(\{ \hat{\varphi}_k(O_i) - \varphi^*(O_i) \}^2\) as follows.

\[
\{ \hat{\varphi}_k(O_i) - \varphi^*(O_i) \}^2 \leq 2w(C_i)^2 \left[ \| \hat{I}_i^{(-k)} - I_i^* \|_2^2 \| 2Y_i - \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \|_2^2 \right.
\]

\[
+ \| \hat{I}_i^{(-k)} + I_i^* \|_2^2 \| g_i^{(-k)}(A_i) - g_i^*(A_i) \|_2^2
\]

\[
+ \frac{4}{n_i} \| \hat{g}_i^{(-k)}(1) - g_i^*(1) \|_2^2 + \frac{4}{n_i} \| \hat{g}_i^{(-k)}(0) - g_i^*(0) \|_2^2 .
\]

Since \(\|w(C_i)\| \leq C_w\), it is sufficient to study the asymptotic behavior of the following terms.

\[
E \left\{ \| \hat{I}_i^{(-k)} - I_i^* \|_2^2 \| 2Y_i - \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \|_2^2 \left| I_k^c \right\} \right.
\]

\[
E \left\{ \| \hat{I}_i^{(-k)} + I_i^* \|_2^2 \| g_i^{(-k)}(A_i) - g_i^*(A_i) \|_2^2 \left| I_k^c \right\} \right.
\]

\[
E \left\{ \| \hat{g}_i^{(-k)}(a) - g_i^*(a) \|_2^2 \left| I_k^c \right\} , a = 1, 0 .
\]

First, we find \(E \{ \| \hat{I}_i^{(-k)} - I_i^* \|_2^2 \left| I_k^c \right\} = O_P(v_i^2 \nu_e, N)\) and \(\| \hat{I}_i^{(-k)} + I_i^* \|_2^2 \leq 4C^2 \) from conditions (c) and (d) of Lemma B.1.

Second, we have \(E \{ \| \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \|_2^2 \left| I_k^c \right\} = O_P(v_i^2 \nu_{g_i, N})\) from condition (d) of Lemma B.1.

Moreover, \(E \{ \| 2Y_i - \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \|_2^2 \left| I_k^c \right\} \) is upper bounded by a constant as follows.

\[
E \{ \| 2Y_i - \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \|_2^2 \left| I_k^c \right\} \]

\[
= E \left[ E \{ \| 2Y_i - \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \|_2^2 \left| A_i, X_i, I_k^c \right\} \right] \left| I_k^c \right\}
\]

\[
= E \left[ 4\epsilon_i^T \epsilon_i + 4\epsilon_i^T \{ \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \} + \| \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \|_2^2 \right] \left| I_k^c \right\}
\]

\[
= E \left[ 4\tilde{I}^T \Sigma(A_i, X_i) 1 + \| \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \|_2^2 \right] \left| I_k^c \right\]
\]

\[
\leq MC' + C_g^2 .
\]

The three equalities are straightforward and the last inequality is from conditions (b) and (c) of Lemma B.1.
Finally, under condition (d), we find
\[
E\left\{ \left\| \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \right\|_2^2 \mid T_k^c \right\}
\]
\[
= \sum_{m=1}^M P(n_i = m) E\left\{ \left\| \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \right\|_2^2 \mid n_i = m, T_k^c \right\}
\]
\[
= \sum_{m=1}^M P(n_i = m) \sum_{a_i \in (0,1)^m} P(A_i = a_i \mid X_i, n_i = m)
\times E\left\{ \left\| \hat{g}_i^{(-k)}(a_i) - g_i^*(a_i) \right\|_2^2 \mid A_i = a_i, X_i, n_i = m, T_k^c \right\}.
\]

From the positivity assumption, \(P(A_i = a_i \mid X_i, n_i = m) \geq \delta\) for some constant \(\delta\), especially at \(a_i = 1\). Therefore, \(E\left\{ \left\| \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \right\|_2^2 \mid T_k^c \right\}\) is lower bounded by
\[
E\left\{ \left\| \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \right\|_2^2 \mid T_k^c \right\}
\]
\[
\geq \delta \sum_{m=1}^M P(n_i = m) E\left\{ \left\| \hat{g}_i^{(-k)}(1) - g_i^*(1) \right\|_2^2 \mid A_i = 1, X_i, n_i = m, T_k^c \mid n_i = m, T_k \right\}
\]
\[
= \delta \sum_{m=1}^M P(n_i = m) E\left\{ \left\| \hat{g}_i^{(-k)}(1) - g_i^*(1) \right\|_2^2 \mid n_i = m, T_k \right\}
\]
\[
= \delta E\left\{ \left\| \hat{g}_i^{(-k)}(1) - g_i^*(1) \right\|_2^2 \mid T_k \right\}.
\]

Since \(E\left\{ \left\| \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \right\|_2^2 \mid T_k^c \right\} = O_P(r_{g,N}^2)\), we obtain \(E\left\{ \left\| \hat{g}_i^{(-k)}(1) - g_i^*(1) \right\|_2^2 \mid T_k^c \right\} = O_P(r_{g,N}^2)\). Similarly, \(E\left\{ \left\| \hat{g}_i^{(-k)}(0) - g_i^*(0) \right\|_2^2 \mid T_k^c \right\} = O_P(r_{g,N}^2)\).

Using the established results, we find the convergence rates of (19)-(21). First, the rate of (19) is \(O_P(r_{e,N}^2)\).

\[
E\left\{ \left\| \tilde{I}_i^{(-k)} - I_i^* \right\|_2^2 \mid 2Y_i - \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \mid T_k^c \right\}
\]
\[
= E\left\{ \left\| \tilde{I}_i^{(-k)} - I_i^* \right\|_2^2 \mid \left\| 2Y_i - \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \right\|_2 \mid A_i, X_i, T_k \right\} \right\} T_k
\]
\[
\leq 4C_B^2(MC' + C_f^2) E\left\{ \left\| \tilde{I}_i^{(-k)} - I_i^* \right\|_2^2 \mid T_k \right\} = O_P(r_{e,N}^2) .
\]

Second, the rate of (20) is \(O_P(r_{g,N}^2)\).

\[
E\left\{ \left\| \tilde{I}_i^{(-k)} + I_i^* \right\|_2^2 \mid \left\| \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \right\|_2 \mid T_k \right\} \leq 4C_M^2 C_B^2 E\left\{ \left\| \hat{g}_i^{(-k)}(A_i) - g_i^*(A_i) \right\|_2 \mid T_k \right\} = O_P(r_{g,N}^2) .
\]

Lastly, (21) is \(O_P(r_{g,N}^2)\) from the established result. As a consequence, by plugging in the rate in (18), we have \(E\left\{ |B| \mid T_k \right\} = O_P(r_{e,N}^2) + O_P(r_{g,N}^2)\). Moreover, this implies \([B] = O_P(r_{e,N}^2) + O_P(r_{g,N}^2)\) from Lemma 6.1 of Chernozhukov et al. (2018).
• (Rate of $|C|$): $|C|$ is represented as
\[
|C| = E\left\{ \tilde{\varphi}_k(O_i) - \varphi^*(O_i) \right\} |\mathcal{T}_k^*| \\
= E\left[ w(C_i) \left( \tilde{I}_i^{(-k),T} \{ Y_i - \tilde{g}_i^{(-k)}(A_i) \} \right) \right] + \frac{1}{n_i} \left\{ \tilde{g}_i^{(-k)}(1) - \tilde{g}_i^{(-k)}(0) \right\} - \frac{1}{n_i} \left\{ g_i^*(1) - g_i^*(0) \right\} |\mathcal{T}_k^*|.
\]

The first term in the second line is decomposed into
\[
E\left[ w(C_i) \tilde{I}_i^{(-k),T} \{ Y_i - \tilde{g}_i^{(-k)}(A_i) \} \right] |\mathcal{T}_k^*| = E\left[ w(C_i) I_i^{*-T} \{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \} \right] |\mathcal{T}_k^*| + E\left[ w(C_i) \{ I_i^{(-k)} - I_i^{*} \}^{T} \{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \} \right] |\mathcal{T}_k^*|.
\]

From some algebra, the first term of (22) is equivalent as
\[
E\left[ w(C_i) I_i^{*-T} \{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \} \right] |\mathcal{T}_k^*| = E\left[ \frac{w(C_i)I_i^{*}}{n_i} \left\{ g_i^*(1) - g_i^*(0) - \tilde{g}_i^{(-k)}(1) + \tilde{g}_i^{(-k)}(0) \right\} \right] |\mathcal{T}_k^*|.
\]

The second term of (22) is
\[
\left| E\left[ w(C_i) \{ I_i^{(-k)} - I_i^{*} \}^{T} \{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \} \right] \right| \\
\leq C_{w} \| I_i^{(-k)} - I_i^{*} \|_{P,2} \| g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \|_{P,2} = O_{P}(r_{e,N},g,N). 
\]

The inequality holds from the Hölder’s inequality. This implies $|C| = O_{P}(r_{e,N},r_{g,N})$.

We use the established rates of $[B]$ and $[C]$ in (16) which leads to the following result.
\[
\frac{1}{\sqrt{N/2}} \sum_{i \in \mathcal{T}_k} \left\{ \tilde{\varphi}_k(O_i) - \varphi^*(O_i) \right\} = [B] + \frac{\sqrt{N}}{2} |C| \\
= O_{P}(r_{e,N}) + O_{P}(r_{g,N}) + \sqrt{N}O_{P}(r_{e,N},r_{g,N}) = o_{P}(1). \tag{23}
\]

From (15), we find
\[
\sqrt{N}(\tau - \tau^*) = \frac{1}{\sqrt{N}} \left\{ \varphi^*(O_i) - \tau^* \right\} + o_{P}(1) \xrightarrow{D} N(0, \text{Var}\{\varphi^*(O_i)\}). \tag{24}
\]

Lastly, we show that the variance estimator is consistent. Let $\sigma_k^2 = (N/2)^{-1} \sum_{i \in \mathcal{T}_k} \{ \tilde{\varphi}_k(O_i) - \tau_k \}^2$. We decompose $\sigma_k^2 - \text{Var}\{\varphi^*(O_i)\}$ as $\sigma_k^2 - S_k^2 + S_k^2 - \text{Var}\{\varphi^*(O_i)\}$ where $S_k^2 = (2/N) \sum_{i \in \mathcal{T}_k} \{ \varphi^*(O_i) - \tau^* \}^2$. From the law of large numbers, we have $S_k^2 - \text{Var}\{\varphi^*(O_i)\} = o_{P}(1)$ so it is sufficient to show
\[ \sigma_k^2 - S_k^2 = o_P(1) \text{ which is represented as follows.} \]

\[
\sigma_k^2 - S_k^2 = \frac{1}{N^2} \sum_{i \in I_k} \left( \hat{\varphi}_k(O_i) - \tau_k \right)^2 - \frac{1}{N^2} \sum_{i \in I_k} \left( \varphi^*(O_i) - \tau^* \right)^2
\]

\[
= \frac{1}{N^2} \sum_{i \in I_k} \left( \hat{\varphi}_k(O_i) - \tau_k - \varphi^*(O_i) + \tau^* \right) \left( \hat{\varphi}_k(O_i) - \tau_k - \varphi^*(O_i) + \tau^* + 2\varphi^*(O_i) - 2\tau^* \right)
\]

\[
= \frac{1}{N^2} \sum_{i \in I_k} \left( \hat{\varphi}_k(O_i) - \tau_k - \varphi^*(O_i) + \tau^* \right)^2 + \frac{2}{N^2} \sum_{i \in I_k} \left( \hat{\varphi}_k(O_i) - \tau_k - \varphi^*(O_i) + \tau^* \right) \left( \varphi^*(O_i) - \tau^* \right)
\]

\[
\leq V_N + 2\sqrt{V_N S_k^2}.
\]

The inequality holds from the Hölder’s inequality. Since \( S_k^2 = \sigma^2 + o_P(1) = O_P(1) \), it suffices to show that \( V_N = o_P(1) \).

We observe that \( V_N \) is upper bounded by

\[
V_N \leq \frac{2}{N^2} \sum_{i \in I_k} \left( \hat{\varphi}_k(O_i) - \varphi^*(O_i) \right)^2 + \left( \tau_k - \tau^* \right)^2. \tag{25}
\]

From (23), the first term of (25) is \( o_P(1) \). The second term is also \( o_P(1) \) from (24). Therefore, 
\( S_k^2 - \text{Var}\{\varphi^*(O_i)\} = o_P(1) \) and \( \sigma_k^2 - \text{Var}\{\varphi^*(O_i)\} = o_P(1) \). This concludes the proof.

### C.1.2 Proof of (1) Double Robustness

The outline of the proof is given as follows. In [Case PS] and [Case OR], we assume (E2.PS) (i.e., \( e' = e^* \)) and (E2.OR) (i.e., \( g' = g^* \)), respectively. The proof is similar to the proof of Theorem 3.1-(b) in Section C.1.1 except we consider the empirical mean instead of \( \sqrt{N} \)-scaled empirical mean.

- **[Case PS]:** We assume \( e' = e^* \). Let \( \varphi'(O_i) \) be

\[
\varphi'(O_i) = w(C_i) \left[ I_i^* \{ Y_i - g_i'(A_i) \} + \frac{1}{n_i} \{ g_i'(1) - g_i'(0) \} \right].
\]

To study \( \mathbb{E}\{\varphi'(O_i)\} \), we first find that the expectation of \( w(C_i) I_i^* \{ Y_i - g_i'(A_i) \} \).

\[
\mathbb{E}\left[ w(C_i) I_i^* \{ Y_i - g_i'(A_i) \} \right] = \mathbb{E}\left[ \frac{w(C_i)}{n_i} \sum_{j=1}^{n_i} \left\{ \frac{1}{e^*(1|X_{ij})} - \frac{1}{e^*(0|X_{ij})} \right\} \left\{ g^*(A_{ij}, X_{ij}) - g'(A_{ij}, X_{ij}) \right\} \right]
\]

\[
= \mathbb{E}\left[ \frac{w(C_i)}{n_i} \sum_{j=1}^{n_i} \left\{ g^*(1, X_{ij}) - g^*(0, X_{ij}) \right\} - \left\{ g'(1, X_{ij}) - g'(0, X_{ij}) \right\} \right]
\]

\[
= \mathbb{E}\left[ \frac{w(C_i) 1^T}{n_i} \left\{ \{ g_i'(1) - g_i'(0) \} - \{ g_i'(1) - g_i'(0) \} \right\} \right]. \tag{26}
\]
Therefore, the expectation of \( \varphi'(O_i) \) is \( \tau^* \).

\[
E\{\varphi'(O_i)\} = E\left[w(C_i) \left[ I_i^{\star\top} \{Y_i - g_i'(A_i)\} + \frac{1}{n_i} \{g_i'(1) - g_i'(0)\} \right]\right] = E\left[\frac{w(C_i)}{n_i} \left\{ g_i'(1) - g_i'(0) \right\}\right] = \tau^* .
\]

We find that the proof in Section C.1.1 now involves with \( \varphi'(O_i) \) instead of \( \varphi^*(O_i) \). Specifically, (15) divided by \( \sqrt{N} \) becomes

\[
\tau - \tau^* = \frac{1}{2} \sum_{k=1}^{2} \frac{1}{N/2} \sum_{i \in I_k} \left\{ \hat{\varphi}_k(O_i) - \varphi'(O_i) \right\} + \frac{1}{2} \sum_{k=1}^{2} \frac{1}{N/2} \sum_{i \in I_k} \left\{ \varphi'(O_i) - \tau^* \right\}
\]

\[
= \frac{1}{2} \sum_{k=1}^{2} E\left\{ \hat{\varphi}_k(O_i) - \varphi'(O_i) \right\} I_k^n \] + \( o_P(1) \).
\]

The second equality is from the law of large numbers. The first term of the right hand side is represented as

\[
E\left\{ \hat{\varphi}_k(O_i) - \varphi'(O_i) \right\} I_k^n \]

\[
= E\left[w(C_i) \left[ \hat{I}_i^{(-k),\top} \{Y_i - \hat{g}_i^{(-k)}(A_i)\} + \frac{1}{n_i} \{\hat{g}_i^{(-k)}(1) - \hat{g}_i^{(-k)}(0)\} - \frac{1}{n_i} \{g_i'(1) - g_i'(0)\} \right] I_k^n \right].
\]

The first term in the second line is decomposed into

\[
E\left[w(C_i) \hat{I}_i^{(-k),\top} \{Y_i - \hat{g}_i^{(-k)}(A_i)\} \right] \quad E\left[w(C_i) I_i^{\star\top} \{g_i'(A_i) - \hat{g}_i^{(-k)}(A_i)\} \right] I_k^n \quad (27)
\]

\[
+ E\left[w(C_i) \left( \hat{I}_i^{(-k)} - I_i^{\star} \right)^\top \{g_i'(A_i) - \hat{g}_i^{(-k)}(A_i)\} \right] I_k^n .
\]

From (26), the first term of (27) is equivalent as

\[
E\left[w(C_i) I_i^{\star\top} \{g_i'(A_i) - \hat{g}_i^{(-k)}(A_i)\} \right] I_k^n = E\left[w(C_i) \frac{1}{n_i} \{g_i'(1) - g_i'(0) - \hat{g}_i^{(-k)}(1) + \hat{g}_i^{(-k)}(0)\} \right] I_k^n .
\]

The second term of (27) is

\[
E\left[w(C_i) \left( \hat{I}_i^{(-k)} - I_i^{\star} \right)^\top \{g_i'(A_i) - \hat{g}_i^{(-k)}(A_i)\} \right] I_k^n \leq C_w \left\| \hat{I}_i^{(-k)} - I_i^{\star}\right\|_{P,2} \left\| g_i'(A_i) - \hat{g}_i^{(-k)}(A_i) \right\|_{P,2}
\]

\[
\leq C_w C_g \left\| \hat{I}_i^{(-k)} - I_i^{\star}\right\|_{P,2} = O_P(r_{e,N}) .
\]

The inequality holds from the H"older's inequality. This implies \( E\{\hat{\varphi}_k(O_i) - \varphi'(O_i) \} = O_P(r_{e,N}) = o_P(1) \), so does \( \tau - \tau^* \). This concludes the proof.

- **[Case OR]**: We assume \( g' = g^* \). Let \( \varphi'(O_i) \) be

\[
\varphi'(O_i) = w(C_i) \left[ I_i^{\star\top} \{Y_i - g_i'(A_i)\} + \frac{1}{n_i} \{g_i'(1) - g_i'(0)\} \right] .
\]

It is trivial that \( E\{\varphi'(O_i)\} = \tau^* \).

We find that the proof in Section C.1.1 now involves with \( \varphi'(O_i) \) instead of \( \varphi^*(O_i) \). Specifically,
\[ \tau - \tau^* = \frac{1}{2} \sum_{k=1}^{2} \frac{1}{N/2} \sum_{i \in I_k} \left\{ \tilde{\varphi}_k(O_i) - \varphi'(O_i) \right\} + \frac{1}{2} \sum_{k=1}^{2} \frac{1}{N/2} \sum_{i \in I_k} \left\{ \varphi'(O_i) - \tau^* \right\} \]

\[ = \frac{1}{2} \sum_{k=1}^{2} \mathbb{E}\left\{ \tilde{\varphi}_k(O_i) - \varphi'(O_i) \mid I_k^c \right\} + o_P(1). \]

The second equality is from the law of large numbers. The first term of the right hand side is represented as

\[ \mathbb{E}\left\{ \tilde{\varphi}_k(O_i) - \varphi'(O_i) \mid I_k^c \right\} \]

\[ = \mathbb{E}\left[ w(C_i) \left[ \tilde{I}_i^{(-k),T} \{ Y_i - \tilde{g}_i^{(-k)}(A_i) \} + \frac{1}{n_i} \left\{ \tilde{g}_i^{(-k)}(1) - \tilde{g}_i^{(-k)}(0) \right\} - \frac{1}{n_i} \left\{ g_i^*(1) - g_i^*(0) \right\} \right] \mid I_k^c \right] . \]

The first term in the second line is decomposed into

\[ \mathbb{E}\left[ w(C_i)I_i^{*T}\{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \} \mid I_k^c \right] = \mathbb{E}\left[ w(C_i)I_i^{*T}\{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \} \mid I_k^c \right] \]

\[ + \mathbb{E}\left[ w(C_i)\{ \tilde{I}_i^{(-k)} - I_i^* \}^T\{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \} \mid I_k^c \right] . \]

The first term of (28) is equivalent as

\[ \mathbb{E}\left[ w(C_i)I_i^{*T}\{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \} \mid I_k^c \right] = \mathbb{E}\left[ \frac{w(C_i)}{n_i} \frac{1}{n_i} \left\{ \tilde{g}_i^*(1) - g_i^*(0) - \tilde{g}_i^{(-k)}(1) + \tilde{g}_i^{(-k)}(0) \right\} \mid I_k^c \right] . \]

The second term of (27) is

\[ \left| \mathbb{E}\left[ w(C_i)\{ \tilde{I}_i^{(-k)} - I_i^* \}^T\{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \} \mid I_k^c \right] \right| \leq C_w \| \tilde{I}_i^{(-k)} - I_i^* \|_{P,2} \| g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \|_{P,2} \]

\[ \leq 2C_wC_e \| g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \|_{P,2} = O_P(r_{g,N}) . \]

The inequality holds from the Hölder’s inequality. This implies \( \mathbb{E}\left\{ \tilde{\varphi}_k(O_i) - \varphi'(O_i) \mid I_k^c \right\} = O_P(r_{g,N}) = o_P(1), \) so does \( \tau - \tau^* \). This concludes the proof.

### C.2 Proof of Lemma 4.1

#### C.2.1 Proof of (1) Mean Double-Robustness

Let \( \phi(O_i; \pi, g, \beta) \) be

\[ \phi(O_i; \pi, g, \beta) = w(C_i) \left[ I(A_i, X_i; \pi)B(C_i; \beta) \{ Y_i - g(A_i, X_i) \} + \frac{1}{n_i} \left\{ g(1, X_i) - g(0, X_i) \right\} \right] . \]

We observe that \( I \) under the true \( \pi^* \) is expressed as

\[ I(A_i, X_i; \pi^*) = \frac{1}{P(A_i \mid X_i)} \times \frac{1}{n_i} \left[ \begin{array}{c} 1(A_{i1} = 1) - 1(A_{i1} = 0) \right\} P(A_{i(-1)} \mid X_i) \\
\vdots \\
1(A_{im_i} = 1) - 1(A_{im_i} = 0) \right\} P(A_{i(-m_i)} \mid X_i) \end{array} \right]. \]
We find the expectation of \( \phi \) under \( \pi^* \) and \( g' \) is

\[
E\{\phi(O_i; \pi^*, g', \beta)\} = E \left[ w(C_i) \left\{ I(A_i, X_i; \pi^*)^T B(C_i; \beta) \left\{ Y_i - g'(A_i, X_i) \right\} + \frac{1^T}{n_i} \left\{ g'(1, X_i) - g'(0, X_i) \right\} \right\} \mid X_i \right]
\]

\[
= E \left[ w(C_i) \left\{ I(A_i, X_i; \pi^*)^T B(C_i; \beta) \left\{ g^*(A_i, X_i) - g'(A_i, X_i) \right\} \right\} \mid X_i \right] 
+ E \left[ w(C_i) \frac{1^T}{n_i} \left\{ g'(1, X_i) - g'(0, X_i) \right\} \right]
\]

\[
= E \left[ w(C_i) \sum_{a_i \in \{0,1\}^{n_i}} v(a_i \mid X_i)^T B(C_i; \beta) \left\{ g^*(a_i, X_i) - g'(a_i, X_i) \right\} \right] 
+ E \left[ w(C_i) \frac{1^T}{n_i} \left\{ g'(1, X_i) - g'(0, X_i) \right\} \right]. 
\]

The terms in (29) is represented as

\[
w(C_i) \sum_{a_i \in \{0,1\}^{n_i}} v(a_i \mid X_i)^T \left\{ g^*(A_i, X_i) - g'(0, X_i) \right\} 
+ \sum_{a_i \in \{0,1\}^{n_i}} v(a_i \mid X_i)^T \left\{ B(C_i; \beta) - I \right\} \left\{ g'(1, X_i) - g'(0, X_i) \right\}
\]

\[
= \frac{w(C_i)}{n_i} \sum_{a_i \in \{0,1\}^{n_i}} \sum_{j=1}^{n_i} \left\{ 1(a_{ij} = 1) - 1(a_{ij} = 0) \right\} P(A_{i(-j)} = a_{i(-j)} \mid X_i) \left\{ g^*(a_{ij}, X_{ij}) - g'(a_{ij}, X_{ij}) \right\}
\]

\[
- \frac{w(C_i) \beta(C_i)}{n_i} \sum_{a_i \in \{0,1\}^{n_i}} \sum_{j=1}^{n_i} \left\{ 1(a_{ij} = 1) - 1(a_{ij} = 0) \right\} P(A_{i(-j)} = a_{i(-j)} \mid X_i) \sum_{\ell \neq j} \left\{ g^*(a_{i\ell}, X_{i\ell}) - g'(a_{i\ell}, X_{i\ell}) \right\}
\]

\[
= \frac{w(C_i)}{n_i} \sum_{j=1}^{n_i} \sum_{a_{ij} = 0}^{1} \left\{ g^*(a_{ij}, X_{ij}) - g'(a_{ij}, X_{ij}) \right\} \left\{ 1(a_{ij} = 1) - 1(a_{ij} = 0) \right\} \sum_{a_{i(-j)}} P(A_{i(-j)} = a_{i(-j)} \mid X_i)
\]

\[
- \frac{w(C_i) \beta(C_i)}{n_i} \sum_{j=1}^{n_i} \sum_{a_{ij} = 0}^{1} \left\{ 1(a_{ij} = 1) - 1(a_{ij} = 0) \right\} \sum_{a_{i(-j)}} P(A_{i(-j)} = a_{i(-j)} \mid X_i) \sum_{\ell \neq j} \left\{ g^*(a_{i\ell}, X_{i\ell}) - g'(a_{i\ell}, X_{i\ell}) \right\}
\]

\[
= \frac{w(C_i)}{n_i} \sum_{j=1}^{n_i} \left\{ g^*(1, X_{ij}) - g'(1, X_{ij}) \right\} - \left\{ g^*(0, X_{ij}) - g'(0, X_{ij}) \right\}
\]

\[
= w(C_i) \frac{1^T}{n_i} \left\{ g^*(1, X_i) - g^*(0, X_i) \right\} - \left\{ g'(1, X_i) - g'(0, X_i) \right\}
\]

Plugging in the result in \( E\{\phi(O_i; \pi^*, g', \beta)\} \), we find

\[
E\{\phi(O_i; \pi^*, g', \beta)\} = E \left[ w(C_i) \frac{1^T}{n_i} \left\{ g^*(1, X_i) - g^*(0, X_i) \right\} \right] = \pi^*.
\]
Next we find the expectation of $\phi$ under $g^*$ and $\tau^*$ as follows.

$$
E\{\phi(O_i; \pi', g^*, \beta)\} = E\left[ w(C_i) \left( I(A_i, X_i; \pi')^T B(C_i; \beta) \left\{ Y_i - g^*(A_i, X_i) \right\} + \frac{1}{n_i} \left\{ g^*(1, X_i) - g^*(0, X_i) \right\} \right) \right| X_i]
$$

$$
= E\left[w(C_i) \frac{1}{n_i} \left\{ g'(1, X_i) - g'(0, X_i) \right\} \right] = \tau^*.
$$

### C.2.2 Proof of (2) Efficiency Gain in Decomposable Propensity Score

When $P(A_i | X_i) = \prod_{i=1}^n P(A_{ij} | X_{ij}), I(A_i, X_i; \pi^*)$ reduces to

$$
I(A_i, X_i; \pi^*) = \frac{1}{P(A_i | X_i)} \frac{1}{\prod_{i=1}^n P(A_{ij} | X_{ij})} \left[ \frac{1(A_{i1} = 1) - 1(A_{i1} = 0)}{P(A_{i1} | X_{i1})} \right] \cdots \left[ \frac{1(A_{in_i} = 1) - 1(A_{in_i} = 0)}{P(A_{in_i} | X_{in_i})} \right]
$$

$$
= \frac{1}{n_i} \left[ \frac{1(A_{i1} = 1) - 1(A_{i1} = 0)}{P(A_{i1} | X_{i1})} \right] \cdots \left[ \frac{1(A_{in_i} = 1) - 1(A_{in_i} = 0)}{P(A_{in_i} | X_{in_i})} \right].
$$

Therefore, $\tau(e^*, g')$ is a special case of $\tau(\pi^*, g', \beta)$ when $\beta = 0$. From the definition of $\beta^*$, we have $\operatorname{Var}\{\tau(\pi^*, g', \beta^*)\} \leq \operatorname{Var}\{\tau(\pi^*, g', \beta = 0)\} = \operatorname{Var}\{\tau(e^*, g')\}$.

### C.3 Proof of Theorem 4.1

We prove the theorem in order of (2) Asymptotic Normality, (1) Double Robustness, and (3) Efficiency Gain Under Known Treatment Assignment Mechanism. Note that the assumptions in Theorem 4.1 implies the conditions in Lemma B.2.

#### C.3.1 Proof of (2) Asymptotic Normality

The outline of the proof is given as follows. In [Step 1], we show that $\sqrt{N}$-scaled empirical mean of $\widehat{\phi}_k(O_i; \delta^{(k)})$ is asymptotically identical to $\sqrt{N}$-scaled empirical mean of $\widehat{\phi}_k(O_i; \beta^{(-k)})$. In [Step 2], we show that $\sqrt{N}$-scaled empirical mean of $\widehat{\phi}_k(O_i; \beta^{(-k)})$ is asymptotically identical to $\sqrt{N}$-scaled empirical mean of $\phi*(O_i; \beta^{(-k)})$. In [Step 3], we show that $\beta^{(-k)} = \beta^*$ under $B = B_\gamma$.
• **Step 1:** \(\sqrt{N}\)-scaled empirical mean of \(\widehat{\phi}_k(O_i \mid \beta^{(k)})\) is asymptotically identical to \(\sqrt{N}\)-scaled empirical mean of \(\widehat{\phi}_k(O_i \mid \beta^{(k)})\).

To show the desired result, we use Example 19.7 and Lemma 19.24 of van der Vaart (1998). Let \(\Phi := \{\widehat{\phi}_k(O_i \mid \beta) \mid \beta \in B\}\) be the collection of influence function at function \(\beta\) where

\[
\widehat{\phi}_k(O_i \mid \beta) = w(C_i) \left[ \widehat{T}_i^{(k), \top} B(C_i \mid \beta) \left\{ Y_i - \widehat{g}_i^{(-k)}(A_i) \right\} + \frac{1}{n_i} \left\{ \widehat{g}_i^{(-k)}(1) - \widehat{g}_i^{(-k)}(0) \right\} \right]
\]

\[
= w(C_i) \left[ \widehat{T}_i^{(k), \top} \left\{ I + (I - 11^\top) B(C_i) \right\} \left\{ Y_i - \widehat{g}_i^{(-k)}(A_i) \right\} + \frac{1}{n_i} \left\{ \widehat{g}_i^{(-k)}(1) - \widehat{g}_i^{(-k)}(0) \right\} \right]
\]

\[
= w(C_i) \left[ \widehat{T}_i^{(k), \top} \left\{ Y_i - \widehat{g}_i^{(-k)}(A_i) \right\} + \frac{1}{n_i} \left\{ \widehat{g}_i^{(-k)}(1) - \widehat{g}_i^{(-k)}(0) \right\} \right]
\]

\[
\overset{\widehat{\psi}_{1,k}(O_i)}{\longrightarrow}
\]

\[
\overset{\widehat{\psi}_{2,k}(O_i)}{\longrightarrow}
\]

(30)

Therefore, the collection \(\Phi\) can be understood as the composition \(\rho \circ \{\widehat{\psi}_{1,k}\}, \{\widehat{\psi}_{2,k}\}, B\) where \(\rho : \mathbb{R}^3 \to \mathbb{R}\) is a fixed map with

\[
\rho(\widehat{\psi}_{1,k}, \widehat{\psi}_{2,k}, \beta) = \widehat{\psi}_{1,k}(O_i) + \widehat{\psi}_{2,k}(O_i) \beta(C_i) = \widehat{\phi}_k(O_i \mid \beta) .
\]

Note that the classes \(\{\widehat{\psi}_{1,k}\}\) and \(\{\widehat{\psi}_{2,k}\}\) are Donsker because they are singleton sets and square-integrable (van der Vaart, 1998, page 270). For two parameters \(\beta_1\) and \(\beta_2\), we find

\[
\left\{ \rho(\widehat{\psi}_{1,k}, \widehat{\psi}_{2,k}, \beta_1) - \rho(\widehat{\psi}_{1,k}, \widehat{\psi}_{2,k}, \beta_2) \right\}^2 = \left( \widehat{\phi}_k(O_i \mid \beta_1) - \widehat{\phi}_k(O_i \mid \beta_2) \right)^2 = \widehat{\psi}_{2,k}(O_i)^2 |\beta_1(C_i) - \beta_2(C_i)|^2 .
\]

Therefore, equation (2.10.19) of van der Vaart and Wellner (1996) is satisfied with \(L_{\alpha,1} = 0, L_{\alpha,2} = 0, L_{\alpha,3} = \widehat{\phi}_k, \alpha_1 = \alpha_2 = \alpha_3 = 1\). Moreover, \(\{\widehat{\psi}_{1,k}\}\), \(\{\widehat{\psi}_{2,k}\}\), and \(B\) have envelope functions as \(\widehat{\psi}_{1,k}, \widehat{\psi}_{2,k}\), and constant \(B_0\), respectively. Moreover, we find \((L_{\alpha} F_{\alpha})^2 = \widehat{\psi}_{2,k}(O_i)^2 B_0^2\) which is integrable as follows.

\[
\mathbb{E} \left\{ \widehat{\psi}_{2,k}(O_i)^2 \left| T_k \right. \right\} \leq \mathbb{E} \left\{ w(C_i)^2 \left\| \widehat{T}_i^{(k)} \right\|_2^2 \left| I - 11^\top \right\|_2^2 \left\| Y_i - \widehat{g}_i^{(-k)}(A_i) \right\|_2^2 \left| T_k \right. \right\}
\]

\[
\leq 2C_w^2 C_n^2 \mathbb{E} \left\{ \left\| Y_i - \widehat{g}_i^*(A_i) \left\|_2^2 + \left\| \widehat{g}_i^*(A_i) - \widehat{g}_i^{(-k)}(A_i) \right\|_2^2 \right| T_k \right\}
\]

\[
\leq 2C_w^2 C_n^2 \mathbb{E} \left\{ \left\| Y_i - \widehat{g}_i^*(A_i) \right\|_2^2 \left| T_k \right. \right\} + C_2^2
\]

\[
\leq C'' .
\]

(31)

This implies every function in \(\Phi\) is square integrable because \(\left\{ \widehat{\phi}_k(O_i \mid \beta) \right\}^2 \leq \widehat{\psi}_{2,k}(O_i)^2 B_0^2\).

Therefore, Theorem 2.10.20 of van der Vaart and Wellner (1996) can be applies and, as a consequence, \(\Phi = \rho \circ \{\widehat{\psi}_{1,k}\}, \{\widehat{\psi}_{2,k}\}, B\) is Donsker.
Therefore, we find

\[
\int \left\{ \hat{\phi}_k(o_i; \hat{\beta}^{(k)}) - \hat{\phi}_k(o_i; \beta^{t,(-k)}) \right\}^2 dP(o_i)
\]

\[
= \int \hat{\psi}_{2,k}(o_i)^2 \left\{ \hat{\beta}^{(k)}(c_i) - \beta^{t,(-k)}(c_i) \right\}^2 dP(o_i)
\]

\[
\leq \int w(c_i)^2 \left\| \hat{I}_i^{(-k)} \right\|^2_2 \left\| I - 11^T \right\|^2_2 \left\| y_i - \hat{g}_i^{(-k)}(a_i) \right\|^2_2 \left\{ \hat{\beta}^{(k)}(c_i) - \beta^{t,(-k)}(c_i) \right\}^2 dP(o_i)
\]

\[
\leq 2C_w^2 \hat{C}_w^2 \hat{M}^2 \int \left\{ \left\| y_i - g_i^*(a_i) \right\|^2_2 + \left\| g_i^*(a_i) - \hat{g}_i^{(-k)}(a_i) \right\|^2_2 \left\{ \hat{\beta}^{(k)}(c_i) - \beta^{t,(-k)}(c_i) \right\}^2 dP(o_i)
\]

\[
\leq 2C_w^2 \hat{C}_w^2 \hat{M}^2 \int \left\{ 1^T \Sigma (a_i, x_i) 1 + \left\| g_i^*(a_i) - \hat{g}_i^{(-k)}(a_i) \right\|^2_2 \left\{ \hat{\beta}^{(k)}(c_i) - \beta^{t,(-k)}(c_i) \right\}^2 dP(a_i, x_i)
\]

\[
\leq C' \int \left\{ \hat{\beta}^{(k)}(c_i) - \beta^{t,(-k)}(c_i) \right\}^2 dP(c_i)
\]

\[
= O_P(r_{\beta,N}^2) .
\]

Therefore, we can apply Lemma 19.24 of van der Vaart (1998) to show the empirical process of \( \hat{\phi}_k \) at \( \hat{\beta}^{(k)} \) is asymptotically the same as that at \( \beta^{t,(-k)} \).

\[
\frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left[ \hat{\phi}_k(O_i; \hat{\beta}^{(k)}) - m(\beta^{(k)}) \right] = \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left[ \hat{\phi}_k(O_i; \beta^{t,(-k)}) - m(\beta^{t,(-k)}) \right] + o_P(1)
\]

where

\[
m(\beta) = \int \hat{\phi}_k(o_i; \beta) dP(o_i) = \int \left\{ \hat{\phi}_{1,k}(o_i) + \hat{\phi}_{2,k}(o_i) \beta(c_i) \right\} dP(o_i) .
\]

Therefore, we find

\[
\frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \hat{\phi}_k(O_i; \hat{\beta}^{(k)}) - \tau^* \right\}
\]

\[
= \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \hat{\phi}_k(O_i; \beta^{t,(-k)}) - \tau^* \right\} + \sqrt{\frac{N}{2} \left\{ m(\beta^{(k)}) - m(\beta^{t,(-k)}) \right\}} + o_P(1) .
\]

From simple algebra, \([A] \) is \( o_P(1) \) from conditions of Lemma B.2.

\[
\left| [A] \right| = \sqrt{\frac{N}{2} \left\{ m(\beta^{(k)}) - m(\beta^{t,(-k)}) \right\}}
\]

\[
\leq \sqrt{\frac{N}{2} \left\{ \int \hat{\phi}_{2,k}(o_i) \left\{ \hat{\beta}^{(k)}(c_i) - \beta^{t,(-k)}(c_i) \right\} dP(o_i) \right\}^2}
\]

\[
= \sqrt{\frac{N}{2} \left\{ \int I(a_i, x_i; \hat{\pi}^{(-k)})^T (I - 11^T) \left\{ g^*(a_i, x_i) - \hat{g}^{(-k)}(a_i, x_i) \right\} \left\{ \hat{\beta}^{(k)}(c_i) - \beta^{t,(-k)}(c_i) \right\} dP(o_i) \right\}^2}
\]

\[
\leq \sqrt{\frac{N}{2} C_w^2 \hat{C}_w^2 \hat{M}^2 \left\| g^*(a_i, x_i) - \hat{g}^{(-k)}(a_i, x_i) \right\|_{P_2} \left\{ \int \left\{ \hat{\beta}^{(k)}(c_i) - \beta^{t,(-k)}(c_i) \right\}^2 dP(o_i) \right\}^{1/2}}
\]

\[
= O(N^{1/2}) O_P(r_{g,N}) O_P(r_{\beta,N}) = o_P(1) .
\]
As a consequence, (33) reduces to
\[
\frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \hat{\phi}_k(O_i; \hat{\beta}^{(k)}) - \tau^* \right\} = \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \hat{\phi}_k(O_i; \beta^{\dagger(-k)}) - \tau^* \right\} + o_P(1) \tag{34}
\]

• **[Step 2]**: $\sqrt{N}$-scaled empirical mean of $\hat{\phi}_k(O_i; \beta^{\dagger(-k)})$ is asymptotically identical to $\sqrt{N}$-scaled empirical mean of $\phi^*(O_i; \beta^{\dagger(-k)})$.

We decompose the the right hand side of (34) as follows.
\[
\frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \hat{\phi}_k(O_i; \beta^{\dagger(-k)}) - \tau^* \right\} = \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \hat{\phi}_k(O_i; \beta^{\dagger(-k)}) - \phi^*(O_i; \beta^{\dagger(-k)}) \right\} + \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \phi^*(O_i; \beta^{\dagger(-k)}) - \tau^* \right\} \tag{35}
\]
where
\[
\phi^*(O_i; \beta) = w(C_i) \left[ I(A_i, X_i; \pi^*)^t B(C_i; \beta) \{ Y_i - g^*(A_i, X_i) \} + \frac{1}{n_i} \left\{ g^*(1, X_i) - g^*(0, X_i) \right\} \right].
\]
The second term is trivially satisfies the asymptotic Normality from the central limit theorem so it suffices to show the first term is $o_P(1)$, which is decomposed into $[B]$ and $[C]$ as follows.
\[
\frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \hat{\phi}_k(O_i; \beta^{\dagger(-k)}) - \phi^*(O_i; \beta^{\dagger(-k)}) \right\} = \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \left\{ \hat{\phi}_k(O_i; \beta^{\dagger(-k)}) - \phi^*(O_i; \beta^{\dagger(-k)}) \right\} - \mathbb{E} \left\{ \hat{\phi}_k(O_i; \beta^{\dagger(-k)}) - \phi^*(O_i; \beta^{\dagger(-k)}) \left| T_k^i \right. \right\} \right\} + \sqrt{\frac{N}{2}} \mathbb{E} \left\{ \hat{\phi}_k(O_i; \beta^{\dagger(-k)}) - \phi^*(O_i; \beta^{\dagger(-k)}) \left| T_k^i \right. \right\} \tag{36}
\]
We show that $[B] = O_P(r^2_{\pi,N}) + O_P(r^2_{g,N})$ and $[C] = O_P(r_{\pi,N}r_g,N)$.

- **(Rate of $[B]$)**

The squared expectation of $[B]$ is
\[
\mathbb{E} \{ |B|^2 \left| T_k^i \right. \} = \frac{1}{N/2} \sum_{i \in I_k} \mathbb{E} \left\{ \left( \hat{\phi}_k(O_i; \beta^{\dagger(-k)}) - \phi^*(O_i; \beta^{\dagger(-k)}) \right)^2 \left| T_k^i \right. \right\} \\
= \mathbb{E} \left\{ \left( \hat{\phi}_k(O_i; \beta^{\dagger(-k)}) - \phi^*(O_i; \beta^{\dagger(-k)}) \right)^2 \left| T_k^i \right. \right\} \tag{37}
\]
We find $\hat{\phi}_k(O; \beta^\dagger_1(-k)) - \phi^*(O; \beta')$ is represented as

\[
\hat{\phi}_k(O; \beta^\dagger_1(-k)) - \phi^*(O; \beta') = w(C_i) \left[ \tilde{I}_i^{(k), \top} B_i^{\dagger, (k)} R_i^{(-k)} - I_i^{*\top} B_i^{t, (k)} R_i^* + \frac{1}{n_i} \left\{ \tilde{g}_i^{(-k)}(1) - \tilde{g}_i^{(-k)}(0) - g_i^*(1) + g_i^*(0) \right\} \right]
\]

\[
= \frac{w(C_i)}{4} \left\{ \tilde{I}_i^{(-k)} - I_i^* \right\}^\top \left\{ B_i^{\dagger, (k)} + B_i^{t, (k)} \right\} \left\{ R_i^{(-k)} + R_i^* \right\} + \frac{w(C_i)}{4} \left\{ \tilde{I}_i^{(-k)} + I_i^* \right\}^\top \left\{ B_i^{\dagger, (k)} - B_i^{t, (k)} \right\} \left\{ R_i^{(-k)} + R_i^* \right\} + \frac{w(C_i)}{2} \left\{ \tilde{I}_i^{(-k), \top} B_i^{\dagger, (k)} + I_i^{*\top} B_i^{t, (k)} \right\} \left\{ R_i^{(-k)} - R_i^* \right\} + \frac{w(C_i)}{n_i} \left\{ \tilde{g}_i^{(-k)}(1) - g_i^*(1) \right\} - \frac{w(C_i)}{n_i} \left\{ \tilde{g}_i^{(-k)}(0) - g_i^*(0) \right\} .
\]

We use the inequality $(a_1 + \ldots + a_5) \leq 16(a_1^2 + \ldots + a_5^2)$ and $\|AB\|_2 \leq \|A\|_2 \|B\|_2$ to obtain an upper bound of $\left\{ \hat{\phi}_k(O; \beta^\dagger_1(-k)) - \phi^*(O; \beta^\tau_1(-k)) \right\}^2$ as follows.

\[
\left\{ \hat{\phi}_k(O; \beta^\dagger_1(-k)) - \phi^*(O; \beta^\tau_1(-k)) \right\}^2 \leq w(C_i)^2 \left\{ \|I_i^{(-k)} - I_i^*\|_2^2 \left\| B_i^{\dagger, (k)} + B_i^{t, (k)} \right\|_2 \left\| R_i^{(-k)} + R_i^* \right\|_2^2 + \|I_i^{(-k)} + I_i^*\|_2^2 \left\| B_i^{\dagger, (k)} - B_i^{t, (k)} \right\|_2 \left\| R_i^{(-k)} + R_i^* \right\|_2^2 + 4 \|I_i^{(-k), \top} B_i^{\dagger, (k)} + I_i^{*\top} B_i^{t, (k)}\|_2 \left\| R_i^{(-k)} - R_i^* \right\|_2^2 + \frac{16}{n_i} \|\tilde{g}_i^{(-k)}(1) - g_i^*(1)\|_2^2 + \frac{16}{n_i} \|\tilde{g}_i^{(-k)}(0) - g_i^*(0)\|_2^2 \right\} .
\]

Since $|w(C_i)| \leq C_w$, it is sufficient to study the asymptotic behavior of the following terms.

\[
E \left\{ \left\| I_i^{(-k)} - I_i^* \right\|_2^2 \left\| B_i^{\dagger, (k)} + B_i^{t, (k)} \right\|_2 \left\| R_i^{(-k)} + R_i^* \right\|_2 \right\}, \quad E \left\{ \left\| I_i^{(-k)} + I_i^* \right\|_2^2 \left\| B_i^{\dagger, (k)} - B_i^{t, (k)} \right\|_2 \left\| R_i^{(-k)} + R_i^* \right\|_2 \right\}, \quad E \left\{ \left\| I_i^{(-k), \top} B_i^{\dagger, (k)} + I_i^{*\top} B_i^{t, (k)} \right\|_2^2 \left\| R_i^{(-k)} - R_i^* \right\|_2 \right\}
\]

\[ E \left\{ \left\| \tilde{g}_i^{(-k)}(a) - g_i^*(a) \right\|_2 \right\}, a = 1, 0 .
\]

First, we find $E \left\{ \left\| \tilde{I}_i^{(-k)} - I_i^* \right\|_2 \right\} = O_P(r_{\pi, N}^2)$ and $\left\| \tilde{I}_i^{(-k)} + I_i^* \right\|_2 \leq 4C_n^2$ from conditions (c) and (d) of Lemma B.2.

Second, using $B_i = I - (1I^\top - I)\beta_m$ for $n_i = m$, we have

\[
E \left\{ \left\| B_i^{\dagger, (k)} - B_i^{t, (k)} \right\|_2 \right\} \leq \left\| 1I^\top - I \right\|_2 E \left\{ \left\| \beta^\dagger_1(-k) - \beta^\tau_1(-k) \right\|_2 \right\},
\]

\[
\leq \left\| 1I^\top - I \right\|_2 E \left\{ \left\| \beta^\dagger_1(-k) - \beta^\tau_1(-k) \right\|_2 \right\} = O_P(r_{\pi, N}^2) + O_P(r_{g, N}^2) .
\]
The last line holds from $\|11^T - I\|_F^2 \leq \overline{M}^2$ and condition (c) of Lemma B.2. We also find that $\|\hat{B}_i^\dagger(-k)\|_2^2 \leq C_B^2$ for some constant $C_B$ as follows.

$$\|\hat{B}_i^\dagger(-k)\|_2^2 \leq \|I + (I - 11^T)\hat{B}_i^\dagger(-k)\|_2^2 \leq 2\|I\|_2^2 + 2\|I - 11^T\|_2^2 \|\hat{B}_i^\dagger(-k)\|_2^2 \leq C_B^2 := 8\overline{M} + 4\overline{M}^2 B_0^2,$$

and $\|B_i^\dagger(-k)\|_2^2 \leq C_B^2$ from similar manner. Thus, $\|\hat{B}_i^\dagger(-k) + B_i^\dagger(-k)\|_2^2 \leq 4C_B^2$. Moreover, we have

$$\|\hat{I}_i(-k) - \hat{B}_i^\dagger(-k)\|_2^2 \leq \|\hat{I}_i(-k)\|_2^2 \|\hat{B}_i^\dagger(-k)\|_2^2 \leq C_B^2 C_B^2, \quad \|\hat{I}_i(-k)\|_2^2 \|\hat{B}_i^\dagger(-k)\|_2^2 \leq \|\hat{I}_i\|_2^2 \|\hat{B}_i^\dagger(-k)\|_2^2 \leq C_B^2 C_B^2 .$$

Third, we have $E\{\|\hat{R}_i(-k) - R_i^*\|_2^2 \|\hat{I}_k\} = E\{\|\hat{g}_i(-k)(A_i) - g_i^*(A_i)\|_2^2 \|\hat{I}_k\} = O_p(r_g^2, N)$ from condition (d) of Lemma B.2. Moreover, $E\{\|\hat{R}_i(-k) + R_i^*\|_2^2 \|\hat{I}_k\}$ is upper bounded by a constant as follows.

$$E\{\|\hat{R}_i(-k) + R_i^*\|_2^2 \|\hat{I}_k\} = E\left[E\{\|\hat{R}_i(-k) + R_i^*\|_2^2 \| A_i, X_i, \hat{I}_k \} \|\hat{I}_k\}\right]$$

$$= E\left[4e_i^2 + 4e_i^T \hat{g}_i(-k)(A_i) - g_i^*(A_i) + \|\hat{g}_i(-k)(A_i) - g_i^*(A_i)\|_2^2 \|\hat{I}_k\}\right]$$

$$= E\left[41^T \Sigma(A_i, X_i) + \|\hat{g}_i(-k)(A_i) - g_i^*(A_i)\|_2^2 \|\hat{I}_k\}\right]$$

$$\leq \overline{M} C' + C_g^2 .$$

The three equalities are straightforward and the last inequality is from conditions (b) and (c) of Lemma B.2.

Finally, under condition (d), we find

$$E\{\|\hat{g}_i(-k)(A_i) - g_i^*(A_i)\|_2^2 \|\hat{I}_k\}\}$$

$$= \sum_{m=1}^M P(n_i = m) E\{\|\hat{g}_i(-k)(A_i) - g_i^*(A_i)\|_2^2 \| n_i = m, \hat{I}_k \}\}$$

$$= \sum_{m=1}^M P(n_i = m) E\left[\sum_{a_i \in \{0, 1\}^m} P(A_i = a_i \mid X_i, n_i = m) \times E\{\|\hat{g}_i(-k)(a_i) - g_i^*(a_i)\|_2^2 \mid A_i = a_i, X_i, n_i = m, \hat{I}_k \} \mid n_i = m, \hat{I}_k \}\right] .$$

From the positivity assumption, $P(A_i = a_i \mid X_i, n_i = m) \geq \delta$ for some constant $\delta$, especially
at $\mathbf{a}_i = 1$. Therefore, $\mathbb{E}\{\|\hat{g}_i^{(-k)}(\mathbf{A}_i) - g_i^*(\mathbf{A}_i)\|^2 | \mathcal{I}_k^c\}$ is lower bounded by

$$
\mathbb{E}\left\{ \|\hat{g}_i^{(-k)}(\mathbf{A}_i) - g_i^*(\mathbf{A}_i)\|^2 | \mathcal{I}_k^c\right\}
\geq \delta \sum_{m=1}^{\mathcal{M}} \left[ \mathbb{P}(n_i = m) \mathbb{E}\left\{ \|\hat{g}_i^{(-k)}(1) - g_i^*(1)\|^2 | \mathbf{A}_i = 1, X_i, n_i = m, \mathcal{I}_k^c\right\} | n_i = m, \mathcal{I}_k^c\right]\right]
= \delta \sum_{m=1}^{\mathcal{M}} \left[ \mathbb{P}(n_i = m) \mathbb{E}\left\{ \|\hat{g}_i^{(-k)}(1) - g_i^*(1)\|^2 | n_i = m, \mathcal{I}_k^c\right\} \right]
= \delta \mathbb{E}\left\{ \|\hat{g}_i^{(-k)}(1) - g_i^*(1)\|^2 | \mathcal{I}_k^c\right\}.
$$

Since $\mathbb{E}\{\|\hat{g}_i^{(-k)}(\mathbf{A}_i) - g_i^*(\mathbf{A}_i)\|^2 | \mathcal{I}_k^c\} = O_P(r_{g,N}^2)$, we obtain $\mathbb{E}\{\|\hat{g}_i^{(-k)}(1) - g_i^*(1)\|^2 | \mathcal{I}_k^c\} = O_P(r_{g,N}^2)$. Similarly, $\mathbb{E}\{\|\hat{g}_i^{(-k)}(0) - g_i^*(0)\|^2 | \mathcal{I}_k^c\} = O_P(r_{g,N}^2)$.

Using the established results, we find the convergence rates of (39)-(42). First, the rate of (39) is $O_P(r_{\pi,N}^2)$.

$$
\mathbb{E}\left\{ \|\hat{r}_i^{(-k)} - \mathbf{I}_i\|^2 | \mathcal{I}_k^c\right\} 
\leq 4C_B^2(MC' + C_2^2) \mathbb{E}\{\|\hat{r}_i^{(-k)} - \mathbf{I}_i\|^2 | \mathcal{I}_k^c\}
= O_P(r_{\pi,N}^2).
$$

Second, the rate of (40) is $O_P(r_{\pi,N}^2) + O_P(r_{g,N}^2)$.

$$
\mathbb{E}\left\{ \|\hat{r}_i^{(-k)} + \mathbf{I}_i\|^2 | \mathcal{I}_k^c\right\} 
\leq 4C_B^2(MC' + C_2^2) \mathbb{E}\{\|\hat{r}_i^{(-k)} + \mathbf{I}_i\|^2 | \mathcal{I}_k^c\}
= O_P(r_{\pi,N}^2) + O_P(r_{g,N}^2).
$$

Third, the rate of (41) is $O_P(r_{g,N}^2)$.

$$
\mathbb{E}\left\{ \|\hat{r}_i^{(-k)} + \mathbf{I}_i\|^2 | \mathcal{I}_k^c\right\} 
\leq 4C_B^2(MC' + C_2^2) \mathbb{E}\{\|\hat{r}_i^{(-k)} + \mathbf{I}_i\|^2 | \mathcal{I}_k^c\}
= O_P(r_{g,N}^2).
$$

Lastly, (42) is $O_P(r_{g,N}^2)$ from the established result. As a consequence, by plugging in the rate in (38), we have $\mathbb{E}\{\|B\|^2 | \mathcal{I}_k^c\} = O_P(r_{\pi,N}^2) + O_P(r_{g,N}^2)$. Moreover, this implies $[B] = O_P(r_{\pi,N}^2) + O_P(r_{g,N}^2)$ from Lemma 6.1 of Chernozhukov et al. (2018).

- **Rate of** $[C]$.}
[C] is represented as
\[ [C] = E\left\{ \widehat{\phi}_k(O_i; \beta^{\text{k},(-k)}) - \phi^*(O_i; \beta^{\text{t},(-k)}) \mid I_k^c \right\} \]
\[ = E\left[ w(C_i) \left[ \tilde{I}_i^{(-k)\top} B_i^{(-k)} \tilde{R}_i^{(-k)} \right] + \frac{1}{n_i} \left\{ g_i^{(-k)}(1) - \tilde{g}_i^{(-k)}(0) \right\} - \frac{1}{n_i} \left\{ g_i^*(1) - g_i^*(0) \right\} \mid I_k^c \right]. \]

The first term in the second line is decomposed into
\[ E\left[ w(C_i) \tilde{I}_i^{(-k)\top} B_i^{(+k)} \tilde{R}_i^{(-k)} \mid I_k^c \right] = E\left[ w(C_i) I_i^{\top} B_i^{(+k)} \left\{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \right\} \mid I_k^c \right] \]
\[ + E\left[ w(C_i) \left\{ \tilde{I}_i^{(-k)} - I_i^\top \right\} B_i^{(+k)} \left\{ g_i^{(-k)}(A_i) - \tilde{g}_i^{(-k)}(A_i) \right\} \mid I_k^c \right]. \]

Taking \( g' = g^* - \tilde{g}^{(-k)} \) in Section C.2.1, the first term of (43) is equivalent as
\[ E\left[ w(C_i) I_i^{\top} B_i^{(+k)} \left\{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \right\} \mid I_k^c \right] = E\left[ \frac{1}{n_i} \left\{ g_i^*(1) - g_i^*(0) \right\} - \frac{1}{n_i} \left\{ g_i^{(-k)}(1) - \tilde{g}_i^{(-k)}(0) \right\} \mid I_k^c \right]. \]

The second term of (43) is
\[ \left| E\left[ \left\{ \tilde{I}_i^{(-k)} - I_i^\top \right\} B_i^{(+k)} \left\{ g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \right\} \mid I_k^c \right] \right| \]
\[ \leq E\left[ \| \tilde{I}_i^{(-k)} - I_i^\top \|_2 \| B_i^{(+k)} \left\| g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \right\|_2 \mid I_k^c \right] \]
\[ \leq C' \| \tilde{I}_i^{(-k)} - I_i^\top \|_{P,2} \| g_i^*(A_i) - \tilde{g}_i^{(-k)}(A_i) \|_{P,2} \]
\[ = O_P(r_{\pi,Nr_{g,N}}). \]

We use the established rates of \([B]\) and \([C]\) in (36) which leads to the following result.
\[ \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \widehat{\phi}_k(O_i; \beta^{\text{k},(-k)}) - \phi^*(O_i; \beta^{\text{t},(-k)}) \right\} = [B] + \sqrt{\frac{N}{2}} [C] \]
\[ = O_P(r_{\pi,N}) + O_P(r_{g,N}) + \sqrt{N} O_P(r_{\pi,Nr_{g,N}}) \]
\[ = o_P(1). \]

From (34) and (35), we find
\[ \sqrt{\frac{N}{2}} (\widehat{\tau}_k - \tau^*) = \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \widehat{\phi}_k(O_i; \beta^{(k)}) - \tau^* \right\} \]
\[ = \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \left\{ \phi^*(O_i; \beta^{(-k)}) - \tau^* \right\} + o_P(1) \]
\[ \overset{D}{\Rightarrow} N\left(0, \text{Var}\{ \phi^*(O_i; \beta^{(-k)}) \} \right) \quad (44) \]
and this implies
\[
\sqrt{N}(\hat{\tau} - \tau^*) = \frac{1}{\sqrt{2}} \left\{ \sqrt{\frac{N}{2}}(\hat{\tau}_1 - \tau^*) + \sqrt{\frac{N}{2}}(\hat{\tau}_2 - \tau^*) \right\}
\]
\[
= \frac{1}{\sqrt{2}} \sum_{k=1}^2 \frac{1}{\sqrt{N/2}} \sum_{i \in I_k} \{ \phi^*(O_i; \beta^{t(-k)}) - \tau^* \} + o_P(1)
\]
\[
\overset{D}{\sim} N \left( 0, \sigma^2(\beta^t) \right), \quad \sigma^2(\beta^t) = \frac{1}{2} \left[ \text{Var} \{ \phi(O_i; \beta^{t(-1)}) \} + \text{Var} \{ \phi(O_i; \beta^{t(-2)}) \} \right].
\]

**[Step 3]:** $\beta^{t(-k)} = \beta^*$ under $B = B_\gamma$.

In Lemma B.2, we take $\beta^{t(-k)}(C_i) = \beta^{t(-k)}(C_i) = \sum_{\ell=1}^J 1 \{ L(C_i) = \ell \} \gamma_{\ell}^{t(-k)}$ and $\beta^{t(-k)}(C_i) = \beta^*(C_i) = \sum_{\ell=1}^J 1 \{ L(C_i) = \ell \} \gamma_{\ell}^*$. Under these choices, we find
\[
\| \hat{\beta}^{(k)}(C_i) - \beta^{t(-k)}(C_i) \|_{P^2} = \| \hat{\beta}^{(k)}(C_i) - \beta^{t(-k)}(C_i) \|_{P^2} = O_P(N^{-1/2}),
\]
\[
\| \beta^{t(-k)}(C_i) - \beta^{t(-k)}(C_i) \|_{P^2} = \| \beta^{t(-k)}(C_i) - \beta^*(C_i) \|_{P^2} = O_P(r_{\pi,N}) + O_P(r_{g,N})
\]
where the results are from Lemma B.3 and B.4. Therefore, conditions (a)-(d) of Lemma B.2 are satisfied, so Theorem 4.1 holds with $\beta^{t(-k)} = \beta^*$.

Lastly, we show that the variance estimator is consistent. Let $\sigma_k^2$ be $\text{Var} \{ \phi^*(O_i; \beta^{t(-k)}) \}$ and $\hat{\sigma}_k^2 = (N/2)^{-1} \sum_{i \in I_k} \{ \hat{\phi}_k(O_i; \hat{\beta}^{(k)}) - \hat{\tau}_k \}^2$. We decompose $\hat{\sigma}_k^2 - \sigma_k^2$ as $\hat{\sigma}_k^2 - S_k^2 + S_k^2 - \sigma_k^2$ where
\[
S_k^2 = \frac{1}{N/2} \sum_{i \in I_k} \left\{ \phi^*(O_i; \beta^{t(-k)}) - \tau^* \right\}^2.
\]
From the law of large numbers, we have $S_k^2 - \sigma_k^2 = o_P(1)$ so it is sufficient to show $\hat{\sigma}_k^2 - S_k^2 = o_P(1)$ which is represented as follows.
\[
\hat{\sigma}_k^2 - S_k^2 = \frac{1}{N/2} \sum_{i \in I_k} \left\{ \hat{\phi}_k(O_i; \hat{\beta}^{(k)}) - \hat{\tau}_k \right\}^2 - \frac{1}{N/2} \sum_{i \in I_k} \left\{ \phi^*(O_i; \beta^{t(-k)}) - \tau^* \right\}^2
\]
\[
= \frac{1}{N/2} \sum_{i \in I_k} \left\{ \hat{\phi}_k(O_i; \hat{\beta}^{(k)}) - \hat{\tau}_k - \phi^*(O_i; \beta^t + \tau^*) \right\}^2
\]
\[
+ \frac{2}{N/2} \sum_{i \in I_k} \left\{ \hat{\phi}_k(O_i; \hat{\beta}^{(k)}) - \hat{\tau}_k - \phi^*(O_i; \beta^t + \tau^*) \right\} \left\{ \phi^*(O_i; \beta^{t(-k)}) - \tau^* \right\}
\]
\[
\leq V_N + 2\sqrt{V_N S_k^2}.
\]
The inequality holds from the Hölder’s inequality. Since $S_k^2 = \sigma_k^2 + o_P(1) = O_P(1)$, it suffices to show that $V_N = o_P(1)$.

We observe that $V_N$ is upper bounded by
\[
V_N \leq \frac{2}{N/2} \sum_{i \in I_k} \left\{ \hat{\phi}_k(O_i; \hat{\beta}^{(k)}) - \phi^*(O_i; \beta^{t(-k)}) \right\}^2 + \left( \hat{\tau}_k - \tau^* \right)^2.
\]
To bound the empirical mean in (45), we again empirical process methods. Let $S$ be the collection
of functions \( \{ \hat{\phi}_k(O; \beta) - \phi^*(O; \beta^{(k)}(x)) \}^2 \) for \( \beta \in B \), i.e.,

\[
\mathcal{S} := \left\{ \{ \hat{\phi}_k(O; \beta) - \phi^*(O; \beta^{(k)}(x)) \}^2 \bigg| \beta \in B \right\}.
\]

We find that \( \{ \hat{\phi}_k(O; \beta) - \phi^*(O; \beta^{(k)}(x)) \}^2 \) has a form

\[
\begin{align*}
&= \frac{w(C_i)^2 \tilde{I}^{(-k),T}_i (I - 11^T) \{ Y_i - \hat{g}_i^{(-k)}(A_i) \}}{\hat{\psi}_{3,k}(O_i)} \beta(C_i)^2 \\
&\quad - 2 \frac{w(C_i)^2 \tilde{I}^{(-k),T}_i (I - 11^T) \{ Y_i - \hat{g}_i^{(-k)}(A_i) \}}{\hat{\psi}_{4,k}(O_i)} \{ Y_i - g_i^*(A_i) \} (I - 11^T) I_1^* \beta^*(-k)(C_i) \beta(C_i) \\
&\quad + \frac{w(C_i)^2 \tilde{I}^{*T}_i (I - 11^T) \{ Y_i - g_i^*(A_i) \}}{\psi_{5,k}(O_i)} \{ Y_i - g_i^*(A_i) \} (I - 11^T) I_1^* \beta^*(-k)(C_i) \beta(C_i)^2
\end{align*}
\]

\[
= \lambda(\hat{\psi}_{3,k}, \hat{\psi}_{4,k}, \hat{\psi}_{5,k}, \beta)
\]

where \( \lambda : \mathbb{R}^4 \to \mathbb{R} \) is a continuous map with \( \lambda(x_1, x_2, x_3, x_4) = x_1 x_4^2 - 2x_2 x_4 + x_3 \). Using \( \lambda \), we find \( \mathcal{S} = \lambda(\hat{\psi}_{3,k}, \hat{\psi}_{4,k}, \hat{\psi}_{5,k}, B) \). We find that \( \{ \hat{\psi}_{3,k}, \hat{\psi}_{4,k}, \hat{\psi}_{5,k} \} \) are Glivenko-Cantelli because they are singleton sets and integrable (van der Vaart, 1998, page 270). Moreover, \( B \) is Glivenko-Cantelli because it is Donsker (van der Vaart and Wellner, 1996, page 82). Lastly, for any \( \beta \in B \), we find \( |\beta(C_i)| \leq B_0 \) and, as a consequence, we have \( \{ \hat{\phi}_k(O; \beta) - \phi^*(O; \beta^{(k)}(x)) \}^2 \leq S(O_i) := B_0^2 \hat{\psi}_{3,k}(O_i) + 2B_0 \hat{\psi}_{4,k}(O_i) + \hat{\psi}_{5,k}(O_i) \), i.e. \( S(O_i) \) is the envelope function of \( \mathcal{S} \) which is integrable as follows

\[
E \left\{ S(O_i) \right\} = B_0^2 E \left\{ \hat{\psi}_{3,k}(O_i) \right\} + 2B_0 E \left\{ \hat{\psi}_{4,k}(O_i) \right\} + B_0 E \left\{ \hat{\psi}_{5,k}(O_i) \right\} \\
\leq B_0^2 C_{w,M}^2 E \left\{ 1^T \Sigma(A_i, X_i) 1 + \| g_i^*(A_i) - \hat{g}_i^{(-k)}(A_i) \|^2_2 \right\} + 2B_0^2 C_{w,M}^2 E \left\{ 1^T \Sigma(A_i, X_i) 1 \right\} + B_0^2 C_{w,M}^2 E \left\{ 1^T \Sigma(A_i, X_i) 1 \right\} \\
\leq 4B_0^2 C_{w,M}^2 C_2 + B_0^2 C_{w,M}^2 C_9^2.
\]

Therefore, Theorem 3 of van der Vaart and Wellner (2000) can be applied so \( \mathcal{S} \) is Glivenko-Cantelli. As a consequence, we have the following result for the empirical mean in (45).

\[
\frac{2}{N/2} \sum_{k \in I_N} \left\{ \hat{\phi}_k(O_i; \beta^{(k)}) - \phi^*(O_i; \beta^{(k)}(x)) \right\}^2 = 2 \int \left\{ \hat{\phi}_k(O_i; \beta^{(k)}) - \phi^*(O_i; \beta^{(k)}(x)) \right\}^2 dP(O_i) + o_P(1).
\]

The integral in the right hand side has the following rate.

\[
\int \left\{ \hat{\phi}_k(O_i; \beta^{(k)}) - \phi^*(O_i; \beta^{(k)}(x)) \right\}^2 dP(O_i) \\
\leq 2 \int \left\{ \hat{\phi}_k(O_i; \beta^{(k)}) - \hat{\phi}_k(O_i; \beta^{(k)}(x)) \right\}^2 dP(O_i) + 2 \int \left\{ \hat{\phi}_k(O_i; \beta^{(k)}(x)) - \phi^*(O_i; \beta^{(k)}(x)) \right\}^2 dP(O_i) \\
= O_P(r_{3,N}^2) + O_P(r_{3,N}^2) + O_P(r_{3,N}^2).
\]

To establish the rate we observe that the first term of (46) is \( O_P(r_{3,N}^2) \) from (32) and that the second
term (46) is equal to $2\mathbb{E}\left[\{\hat{\phi}_k(O_i; \beta^t(-k)) - \phi^*(O_i; \beta^t(-k))\}^2 \mid I_k^c\right] = 2\mathbb{E}\left[|B|^2 \mid I_k^c\right] = O_P(r^2_{\pi, N} + O_P(r^2_{\beta, N})$ from (37). Therefore, the first term of (45) is $o_p(1)$. The second term of (45) is also $o_p(1)$ from (44). This implies $V_N = o_p(1)$. This concludes the proof.

**C.3.2 Proof of (1) Double Robustness**

The outline of the proof is given as follows. In [Case PS] and [Case OR], we assume (EN2.PS) (i.e., $\pi' = \pi^*$) and (EN2.OR) (i.e., $g' = g^*$), respectively. The proof is similar to the proof of Theorem 4.1-(b) in Section C.3.1 except we consider the empirical mean instead of $\sqrt{N}$-scaled empirical mean.

- **[Case PS]:** We assume $\pi' = \pi^*$. By following [Step 1] and [Step 2] in the proof of Theorem 4.1, we find that $|A|/\sqrt{N}/2$ in (33) is upper bounded by

$$\left|A\right|/\sqrt{N}/2 = \left|m(\hat{\beta}(k)) - m(\beta^t(-k))\right|$$

$$= \int \hat{\phi}_{2,k}(O_i)\left\{\hat{\beta}(k)(c_i) - \beta^t(-k)(c_i)\right\}dP(o_i)$$

$$= \int I(a_i, x_i; \hat{\pi}(-k))^T(I - 11^T)\left\{g^*(a_i, x_i) - \hat{g}^(-k)(a_i, x_i)\right\}\left\{\hat{\beta}(k)(c_i) - \beta^t(-k)(c_i)\right\}dP(o_i)$$

$$= C_{\pi M} \left\|g^*(a_i, x_i) - \hat{g}^(-k)(a_i, x_i)\right\|_{p,2} \left[\int \left\{\hat{\beta}(k)(c_i) - \beta^t(-k)(c_i)\right\}^2 dP(o_i)\right]^{1/2}$$

$$\leq C_g.$$

Therefore, (33) divided by $\sqrt{N}/2$ is expressed as

$$\frac{1}{N/2} \sum_{i \in I_k} \left\{\hat{\phi}_k(O_i; \beta^t(k)) - \tau^*\right\} = \frac{1}{N/2} \sum_{i \in I_k} \left\{\hat{\phi}_k(O_i; \beta^t(-k)) - \tau^*\right\} + \frac{|A|}{\sqrt{N}/2} + o_p(N^{-1/2})$$

$$= \frac{1}{N/2} \sum_{i \in I_k} \left\{\hat{\phi}_k(O_i; \beta^t(-k)) - \tau^*\right\} + o_p(1).$$

We find that [Step 3] in the proof of Theorem 4.1 now involves with $\phi'(O_i; \beta^t(-k))$ instead of $\phi^*(O_i; \beta^t(-k))$ where

$$\phi'(O_i; \beta^t(-k)) = w(C_i) \left[I(A_i, X_i; \pi^*)^TB(C_i; \beta^t(-k))\left\{Y_i - g'(A_i, X_i)\right\} + \frac{1}{n_i} \left\{g'(1, X_i) - g'(0, X_i)\right\}\right].$$

Specifically, (35) divided by $\sqrt{N}/2$ becomes

$$\frac{1}{N/2} \sum_{i \in I_k} \left\{\hat{\phi}_k(O_i; \beta^t(-k)) - \tau^*\right\}$$

$$= \frac{1}{N/2} \sum_{i \in I_k} \left\{\hat{\phi}_k(O_i; \beta^t(-k)) - \phi'(O_i; \beta^t(-k))\right\} + \frac{1}{N/2} \sum_{i \in I_k} \left\{\phi'(O_i; \beta^t(-k)) - \tau^*\right\}. $$

50
Since $\text{E}\{\phi'(O_i; \beta_t^{(-k)})\} = \tau^*$, the second term in the right hand side is $o_P(1)$ from the law of large numbers. Consequently, it suffices to show that the first term in the right hand side is $o_P(1)$, which is expressed as follows from the law of large numbers.

$$
\begin{align*}
\frac{1}{N/2} \sum_{i \in \mathcal{I}_k} \left\{ \bar{\phi}_k(O_i; \beta_t^{(-k)}) - \phi'(O_i; \beta_t^{(-k)}) \right\} \\
= \text{E}\left[ \bar{\phi}_k(O_i; \beta_t^{(-k)}) - \phi'(O_i; \beta_t^{(-k)}) \Bigg| \mathcal{I}_k^c \right] + o_P(1) \\
= \text{E}\left[ w(C_i) \left( \bar{I}_i^{(-k),T} B_i^{t(-k)} \bar{R}_i^{(-k)} + \frac{\mathbf{1}^T}{n_i} \left\{ g_i^{(-k)}(1) - g_i^{(-k)}(0) \right\} - \frac{\mathbf{1}^T}{n_i} \left\{ g_i^*(1) - g_i^*(0) \right\} \right) \right| \mathcal{I}_k^c \\
= \text{E}\left[ \left\{ \bar{\phi}_k(O_i; \beta_t^{(-k)}) - \phi'(O_i; \beta_t^{(-k)}) \left| \mathcal{I}_k^c \right. \right\} \right].
\end{align*}
$$

The last inequality holds because $\text{E}\{\phi'(O_i; \beta_t^{(-k)})\} = \tau^* = \text{E}[w(C_i)1^T\{g_i^*(1) - g_i^*(0)\}/n_i]$.

The first term in the last line is equivalent to

$$
\text{E}\left[ \left\{ \bar{\phi}_k(O_i; \beta_t^{(-k)}) - \phi'(O_i; \beta_t^{(-k)}) \left| \mathcal{I}_k^c \right. \right\} \right].
$$

where the second inequality holds from Theorem 4.1. The last term has a following rate.

$$
\text{E}\left[ \left\{ \bar{\phi}_k(O_i; \beta_t^{(-k)}) - \phi'(O_i; \beta_t^{(-k)}) \left| \mathcal{I}_k^c \right. \right\} \right] \\
\leq \text{E}\left[ \left\| \bar{I}_i^{(-k)} - I_i^* \right\|_2 B_i^{t(-k)} \left\| g_i^*(A_i) - \bar{g}_i^{(-k)}(A_i) \right\|_2 \left| \mathcal{I}_k^c \right. \right] \\
\leq C' \left\| \bar{I}_i^{(-k)} - I_i^* \right\|_{P,2} \\
= o_P(r_{\pi,N}).
$$

The second inequality holds from Assumption in the theorem. Therefore, we find

$$
\frac{1}{N/2} \sum_{i \in \mathcal{I}_k} \left\{ \bar{\phi}_k(O_i; \beta_t^{(-k)}) - \phi'(O_i; \beta_t^{(-k)}) \right\} = o_P(r_{\pi,N}) + o_P(1) = o_P(1).
$$

This concludes the proof.

- **[Case OR]**: We assume $g' = g^*$. By following [Step 1] and [Step 2] in the proof of Theorem 4.1, we find that $[A]$ in (33) has the same rate $[A] = o_P(1)$. Therefore, (33) divided by $\sqrt{N/2}$ is expressed as

$$
\frac{1}{N/2} \sum_{i \in \mathcal{I}_k} \left\{ \bar{\phi}_k(O_i; \beta_t^{(-k)}) - \tau^* \right\} = \frac{1}{N/2} \sum_{i \in \mathcal{I}_k} \left\{ \bar{\phi}_k(O_i; \beta_t^{(-k)}) - \tau^* \right\} + \frac{[A]}{\sqrt{N/2}} + o_P(N^{-1/2}) \\
= \frac{1}{N/2} \sum_{i \in \mathcal{I}_k} \left\{ \bar{\phi}_k(O_i; \beta_t^{(-k)}) - \tau^* \right\} + o_P(1).
$$

We find that [Step 3] in the proof of Theorem 4.1 now involves with $\phi'(O_i; \beta_t^{(-k)})$ instead of
φ*(O_i; β(t-k)) where

φ'(O_i; β(t-k)) = w(C_i) \left[ I(A_i, X_i; \pi') B(C_i; \beta'(t-k)) \{Y_i - g^*(A_i, X_i)\} + \frac{1}{n_i} \left\{ g^*(1, X_i) - g^*(0, X_i) \right\} \right].

Specifically, (35) divided by √N/2 becomes

\[
\frac{1}{N/2} \sum_{i \in \mathcal{I}_k} \left\{ \hat{\phi}_k(O_i; \beta'(t-k)) - \tau^* \right\} 
= \frac{1}{N/2} \sum_{i \in \mathcal{I}_k} \left\{ \hat{\phi}_k(O_i; \beta'(t-k)) - \phi'(O_i; \beta'(t-k)) \right\} + \frac{1}{N/2} \sum_{i \in \mathcal{I}_k} \left\{ \phi'(O_i; \beta'(t-k)) - \tau^* \right\}.
\]

Since E{φ'(O_i; β'(t-k))} = τ*, the second term in the right hand side is o_P(1) from the law of large numbers. Consequently, it suffices to show that the first term in the right hand side is o_P(1), which is expressed as follows from the law of large numbers.

\[
\frac{1}{N/2} \sum_{i \in \mathcal{I}_k} \left\{ \hat{\phi}_k(O_i; \beta'(t-k)) - \phi'(O_i; \beta'(t-k)) \right\} 
= E\left\{ \hat{\phi}_k(O_i; \beta'(t-k)) - \phi'(O_i; \beta'(t-k)) \left| T_k^i \right. \right\} + o_P(1)
= E\left[ w(C_i) \left( I_i^{(k),*} B_i^{(t-k)} \hat{R}_i^{(k)} + \frac{1}{n_i} \left\{ \hat{g}_i^{(k)}(1) - \hat{g}_i^{(k)}(0) \right\} - \frac{1}{n_i} \left\{ g_i^*(1) - g_i^*(0) \right\} \right] \left| T_k^i \right. \right] + o_P(1).
\]

The last inequality holds because E{φ'(O_i; β'(t-k))} = τ* = E[ w(C_i)1^\top \{ g_i^*(1) - g_i^*(0) \} / n_i].

The first term in the last line is equivalent to

\[
E\left[ w(C_i) \left( I_i^{(k),*} B_i^{(t-k)} \hat{R}_i^{(k)} \right) \left| T_k^i \right. \right] = E\left[ w(C_i) I_i^{(k),*} B_i^{(t-k)} \left\{ g_i^*(A_i) - \hat{g}_i^{(k)}(A_i) \right\} \left| T_k^i \right. \right]
+ E\left[ w(C_i) \left\{ \hat{I}_i^{(k)} - I_i^{(k)} \right\} B_i^{(t-k)} \left\{ g_i^*(A_i) - \hat{g}_i^{(k)}(A_i) \right\} \left| T_k^i \right. \right]
= E\left[ \frac{1}{n_i} \left\{ g_i^*(1) - g_i^*(0) \right\} - \frac{1}{n_i} \left\{ \hat{g}_i^{(k)}(1) - \hat{g}_i^{(k)}(0) \right\} \left| T_k^i \right. \right]
+ E\left[ w(C_i) \left\{ \hat{I}_i^{(k)} - I_i^{(k)} \right\} B_i^{(t-k)} \left\{ g_i^*(A_i) - \hat{g}_i^{(k)}(A_i) \right\} \left| T_k^i \right. \right]
\]

where the second inequality holds from Theorem 4.1. The last term has a following rate.

\[
\left| E\left[ \left\{ \hat{I}_i^{(k)} - I_i^{(k)} \right\} B_i^{(t-k)} \left\{ g_i^*(A_i) - \hat{g}_i^{(k)}(A_i) \right\} \left| T_k^i \right. \right] \right|
\leq E\left[ \left\| \hat{I}_i^{(k)} - I_i^{(k)} \right\|_2 \left\| B_i^{(t-k)} \right\|_2 \left\| g_i^*(A_i) - \hat{g}_i^{(k)}(A_i) \right\|_2 \left| T_k^i \right. \right]
\leq C \left\| g_i^*(A_i) - \hat{g}_i^{(k)}(A_i) \right\|_{p,2}
= O_P(r_{g,N}).
\]

The second inequality holds from Assumption in the theorem. Therefore, we find

\[
\frac{1}{N/2} \sum_{i \in \mathcal{I}_k} \left\{ \hat{\phi}_k(O_i; \beta'(t-k)) - \phi'(O_i; \beta'(t-k)) \right\} = O_P(r_{g,N}) + o_P(1) = o_P(1).
\]

This concludes the proof.
C.3.3 Proof of (3) Efficiency Gain Under Known Treatment Assignment Mechanism

Under the assumptions, we find

\[
\text{ARE}(\hat{\tau}, \tau) = \frac{a \text{Var}(\tau)}{a \text{Var}(\hat{\tau})} = \frac{\text{Var}\{\phi(O_i, e^*, g^*)\}}{\text{Var}\{\phi(O_i, \pi^*, g^*, \beta^*)\}}.
\]

From Lemma 4.1 (b), we find the quantity is not smaller than 1. This concluded the proof.
D Proof of Lemmas in the Supplementary Material

D.1 Proof of Lemma B.1

Since condition (a) is trivially satisfied, we only need to show that conditions (b), (c), and (d) hold. First, we show condition (M2) of the main paper implies condition (b) of Lemma B.1. It suffices to show \( \| \Sigma(A_i, X_i) \|_2 \leq C_2 \), which is given as

\[
\| \Sigma(A_i, X_i) \|_2 = \| \mathbb{E}\{ Y_i - g^*(A_i, X_i) \} \|_2 \leq E\{ \| Y_i - g^*(A_i, X_i) \|_2^2 | A_i, X_i \} \leq \left[ E\{ \| Y_i - g^*(A_i, X_i) \|_4^4 | A_i, X_i \} \right]^{1/2} \leq \sqrt{C_4}.
\]

The first inequality holds from the Jensen’s inequality and the second inequality holds from the Hölder’s inequality.

Second, we show condition (E1) of the main paper implies condition (c) of Lemma B.1. First, we find \( e^*(A_{ij} | X_{ij}) \) obeys the positivity assumption with some constant \( c_e \).

\[
e^*(A_{ij} | X_{ij}) = \int P(A_{ij} | X_{ij}) dP(X_{i(-j)}) = \int \sum_{A_{i(-j)}} P(A_i | X_i) dP(X_{i(-j)}) \geq \int \sum_{A_{i(-j)}} \delta dP(X_{i(-j)}) \geq c_e.
\]

This implies \( e^*(A_{ij} | X_{ij}) \in [\delta_e, 1 - \delta_e] \). Therefore, we find the result from the following algebra

\[
\| \hat{g}(-k)(A_i, X_i) - g^*(A_i, X_i) \|^2 = \sum_{j=1}^{n_i} \left\{ \hat{g}(-k)(A_{ij}, X_{ij}) - g^*(A_{ij}, X_{ij}) \right\}^2 \leq M c_e^2 \equiv C_g^2
\]

\[
\| I(A_i, X_i; e^*) \|^2 = \sum_{j=1}^{n_i} \frac{1}{e^*(A_{ij} | X_{ij})^2} \leq \frac{M^2 c_e^2}{c_e} \equiv C_e^2
\]

\[
\| I(A_i, X_i; \hat{e}(-k)) \|^2 = \sum_{j=1}^{n_i} \frac{1}{\hat{e}(-k)(A_{ij} | X_{ij})^2} \leq \frac{M^2 c_e^2}{c_e} \equiv C_e^2
\]

Lastly, we only show condition (E2.PS) of the main paper implies condition (d.PS) of Lemma B.1 but other cases can be shown in a similar manner. Let \( \text{supp}(n_i) \subset \{1, \ldots, M\} \) be the support of \( n_i \). We find

\[
\| \hat{e}(-k)(1 | X_{ij}) - e^*(1 | X_{ij}) \|^2 \geq \mathbb{P}(n_i = m) \int \left\{ \hat{e}(-k)(1 | X_{ij}, n_i = m) - e^*(1 | X_{ij}, n_i = m) \right\}^2 dP(X_{ij} | n_i = m) \geq \mathbb{P}(n_i = m) \int \left\{ \hat{e}(-k)(1 | X_{ij}, n_i = m) - e^*(1 | X_{ij}, n_i = m) \right\}^2 dP(X_{ij} | n_i = m) \equiv R_{m(ij)}
\]

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Similarly, by choosing \( g' = g \), we find
\[
\left\| \hat{g}^{(k)}(A_{ij}, X_{ij}) - g'(A_{ij}, X_{ij}) \right\|_{P,2}^2 = \int \left\{ \hat{g}^{(k)}(A_{ij}, X_{ij}) - g'(A_{ij}, X_{ij}) \right\}^2 dP(A_{ij}, X_{ij})
\]
\[
= \sum_{m \in \text{supp}(n_i)} P(n_i = m) \int \left\{ \hat{g}^{(k)}(A_{ij}, X_{ij}, n_i = m) - g'(A_{ij}, X_{ij}, n_i = m) \right\}^2 dP(A_{ij}, X_{ij} | n_i = m)
\]
\[
\geq P(n_i = m) \int \left\{ \hat{g}^{(k)}(A_{ij}, X_{ij}, n_i) = m) - g'(A_{ij}, X_{ij}, n_i = m) \right\}^2 dP(A_{ij}, X_{ij} | n_i = m)
\]
\[
\geq T_m(ij)
\]
Since \( P(n_i = m) \) is a positive number, \( R_m(ij) = O_P(r^2_{e,N}) \) and \( T_m(ij) = O_P(r^2_{g,N}) \) for any \( j = 1, \ldots, m \) and \( m \in \text{supp}(n_i) \). As a consequence, \( \sum_{j=1}^m R_m(ij) = O_P(r^2_{e,N}) \) and \( \sum_{j=1}^m T_m(ij) = O_P(r^2_{g,N}) \) for \( m \in \text{supp}(n_i) \). Therefore,
\[
\left\| I(A_i, X_i ; \hat{e}^{(k)}) - I(A_i, X_i ; e^*) \right\|_{P,2}^2 = \int \sum_{j=1}^{n_i} \left\{ \hat{e}^{(k)}(A_{ij} | X_{ij}) - e^*(A_{ij} | X_{ij}) \right\}^2 dP(A_{ij}, X_i)
\]
\[
\leq \frac{1}{c^4} \int \sum_{j=1}^{n_i} \left\{ \hat{e}^{(k)}(A_{ij} | X_{ij}) - e^*(A_{ij} | X_{ij}) \right\}^2 dP(A_{ij}, X_i)
\]
\[
= \frac{1}{c^4} \sum_{m \in \text{supp}(n_i)} P(n_i = m) \sum_{j=1}^m R_m(ij)
\]
\[
= O_P(r^2_{e,N}) .
\]
Similarly, we have
\[
\left\| \hat{g}^{(k)}(A_{i}, X_{i}) - g'(A_{i}, X_{i}) \right\|_{P,2}^2 = \int \sum_{j=1}^{n_i} \left\{ \hat{g}^{(k)}(A_{ij}, X_{ij}) - g'(A_{ij}, X_{ij}) \right\}^2 dP(A_{i}, X_{i})
\]
\[
= \sum_{m \in \text{supp}(n_i)} P(n_i = m) \sum_{j=1}^m T_m(ij)
\]
\[
= O_P(r^2_{g,N}) .
\]
This concludes the proof.

**D.2 Proof of Lemma B.2**

The proof of condition (a) and (b) are the same as Lemma B.1.

We show condition (EN1) of the main paper implies condition (c) of Lemma B.2. From the
following algebra, we find the result.
\[
\| \tilde{g}^{(-k)}(A_i, X_i) - g^*(A_i, X_i) \|_2^2 = \sum_{j=1}^{n_i} \left\{ \tilde{g}^{(-k)}(A_{ij}, X_{ij}) - g^*(A_{ij}, X_{ij}) \right\}^2 \leq M c_g^2 \equiv C_g^2
\]
\[
\| I(A_i, X_i; \pi^*) \|_2^2 = \sum_{j=1}^{n_i} \frac{1}{\pi^*(A_{ij} | A_i(j), X_{ij}, X_i(j))^2} = \sum_{j=1}^{n_i} \frac{P(A_{i(j)} | X_i)^2}{P(A_i | X_i)^2} \leq \frac{M(1 - \delta)^2}{\delta^2}
\]
\[
\| I(A_i, X_i; \tilde{\pi}^{(-k)}) \|_2^2 = \sum_{j=1}^{n_i} \frac{1}{\tilde{\pi}^{(-k)}(A_{ij} | A_i(j), X_{ij}, X_i(j))^2} \leq \frac{M}{c_\pi^2}.
\]
We take \( C_g^2 = \max \{ M(1 - \delta)^2/\delta^2, M/c_\pi^2 \} \).

We take \( \beta^{\ell,(-k)} = \beta^{\ell,(-k)} = \beta \). Then, the claim about \( \beta \) is satisfied.
\[
\| \tilde{\beta}^{(k)}(C_i) - \beta^{\ell,(-k)}(C_i) \|_{p,2} = \| \tilde{\beta}^{(k)}(C_i) - \beta(C_i) \|_{p,2} = O_P(r_{\beta,N}) ,
\]
\[
\| \beta^{\ell,(-k)}(C_i) - \beta^{\ell,(-k)}(C_i) \|_{p,2} = \| \beta(C_i) - \beta(C_i) \|_{p,2} = 0 = O_P(r_{\pi,N}) + O_P(r_{g,N}) .
\]
Lastly, we only show condition (EN2.PS) of the main paper implies condition (d.PS) of Lemma B.1 but other cases can be shown in a similar manner. Let \( \text{supp}(n_i) \subset \{1, \ldots, M\} \) be the support of \( n_i \). We find
\[
\left\| \pi^{(-k)}(1 | A_i(j), X_{ij}, X_i(j)) - \pi^*(1 | A_i(j), X_{ij}, X_i(j)) \right\|_{p,2}^2 = \int \left\{ \pi^{(-k)}(1 | A_i(j), X_{ij}, X_i(j)) - \pi^*(1 | A_i(j), X_{ij}, X_i(j)) \right\}^2 dP(A_i(j), X_i)
\]
\[
\geq P(n_i = m) \int \left\{ \pi^{(-k)}(1 | A_i(j), X_{ij}, X_i(j), n_i = m) - \pi^*(1 | A_i(j), X_{ij}, X_i(j), n_i = m) \right\}^2 dP(A_i(j), X_i | n_i = m) \equiv R_{n}(i)
\]
Similarly, by choosing \( g' = g \), we find
\[
\left\| \tilde{g}^{(-k)}(A_{ij}, X_{ij}) - g'(A_{ij}, X_{ij}) \right\|_{p,2}^2 = \int \left\{ \tilde{g}^{(-k)}(A_{ij}, X_{ij}) - g'(A_{ij}, X_{ij}) \right\}^2 dP(A_{ij}, X_{ij})
\]
\[
\geq P(n_i = m) \int \left\{ \tilde{g}^{(-k)}(A_{ij}, X_{ij}, n_i = m) - g'(A_{ij}, X_{ij}, n_i = m) \right\}^2 dP(A_{ij}, X_{ij} | n_i = m) \equiv T_{n}(i)
\]
Since \( P(n_i = m) \) is a positive number, \( R_{m}(i) = O_P(r_{\pi,N}^2) \) and \( T_{m}(i) = O_P(r_{g,N}^2) \) for any \( j = 1, \ldots, m \) and \( m \in \text{supp}(n_i) \). As a consequence, \( \sum_{j=1}^{m} R_{m}(i) = O_P(r_{\pi,N}^2) \) and \( \sum_{j=1}^{m} T_{m}(i) = O_P(r_{g,N}^2) \)
Similarly, we have
\[
\|I(A_i, X_i ; \hat{\pi}^{(-k)}) - I(A_i, X_i ; \pi^*)\|_{P,2}^2
\]
\[
= \sum_{j=1}^{n_i} \left\{ \frac{\hat{\pi}^{(-k)}(A_{ij})}{\pi^*(A_{ij})} - \pi^*(A_{ij}) \right\}^2 dP(A_i, X_i)
\]
\[
\leq \frac{(1 - \delta)^2}{\delta^2 c_\pi^2} \sum_{j=1}^{n_i} \left\{ \frac{\hat{\pi}^{(-k)}(1 | A_{ij})}{\pi^*(1 | A_{ij})} - \pi^*(1 | A_{ij}) \right\}^2 dP(A_i, X_i)
\]
\[
= \frac{(1 - \delta)^2}{\delta^2 c_\pi^2} \sum_{m \in \text{supp}(n_i)} P(n_i = m) \sum_{j=1}^{m} R_m(ij)
\]
\[
= O_P(r_{\pi,N}^2).
\]

Similarly, we have
\[
\|g^{(-k)}(A_i, X_i) - g'(A_i, X_i)\|_{P,2}^2 = \sum_{j=1}^{n_i} \left\{ \frac{\hat{g}^{(-k)}(A_{ij}, X_{ij})}{g'(A_{ij}, X_{ij})} - g'(A_{ij}, X_{ij}) \right\}^2 dP(A_i, X_i)
\]
\[
= \sum_{m \in \text{supp}(n_i)} P(n_i = m) \sum_{j=1}^{m} T_m(ij)
\]
\[
= O_P(r_{\hat{g},N}^2).
\]

This concludes the proof.

**D.3 Proof of Lemma B.3**

To construct the estimator of $\beta^t\hat{,}(-k)$, we find an alternative representation of $\beta^t\hat{,}(-k)$ from the law of total expectation and the first order condition of (13) and (14).

\[
0 = 2E \left\{ w(C_i) \hat{I}_i^{(k)}(I - 11^T) \hat{S}_i^{(k)}(I - 11^T) \hat{I}_i^{(-k)} \mid \sigma(B), \hat{T}_i^{(k)} \right\} \beta^t\hat{,}(-k)(C_i)
\]
\[
- E \left\{ w(C_i) \hat{I}_i^{(-k)} \hat{S}_i^{(k)}(I - 11^T) \hat{I}_i^{(k)} \mid \sigma(B), \hat{T}_i^{(k)} \right\} \hat{\beta}_i^{(-k)}(C_i)
\]
\[
0 = 2E \left\{ w(C_i) ^2 I_i^{+T} (I - 11^T) S_i (I - 11^T) I_i^{*} \mid \sigma(B) \right\} \beta^{*}(C_i)
\]
\[
- E \left\{ w(C_i) ^2 I_i^{+T} \left( 2S_i - 11^T S_i - S_i 11^T \right) I_i^{*} \mid \sigma(B) \right\} .
\]

We define $\Psi(O_i ; \beta)$ and $\hat{\Psi}^{(-k)}(O_i ; \beta)$ as follows.

\[
\Psi(O_i ; \beta) = w(C_i) \beta(C_i) \left( I_i^{+T} S_i^* I_i^{*} - I_i^{+T} T_i^{*} I_i^{*} \right),
\]
\[
\hat{\Psi}^{(-k)}(O_i ; \beta) = w(C_i) \beta(C_i) \left( \hat{I}_i^{(-k),T} \hat{S}_i^{(-k)} \hat{I}_i^{(-k)} - \hat{I}_i^{(-k),T} \hat{T}_i^{(-k)} \hat{I}_i^{(-k)} \right).
\]

Here $S_i = (I - 11^T) S_i (I - 11^T)$, $T_i = 2S_i - 11^T S_i - S_i 11^T$, and $S_i = e_i e_i^T = \{ Y_i - g^*(A_i, X_i) \} \otimes 2$. Similarly, $\hat{S}_i^{(-k)} = (I - 11^T) \hat{S}_i^{(-k)} (I - 11^T)$, $\hat{T}_i^{(-k)} = 2\hat{S}_i^{(-k)} - 11^T \hat{S}_i^{(-k)} - \hat{S}_i^{(-k)} 11^T$, and $\hat{S}_i^{(-k)} = \{ Y_i - \hat{g}^{(-k)}(A_i, X_i) \} \otimes 2$. Then, we find $\beta^*$ defined in (14) and $\beta^t\hat{,}(-k)$ defined in (13) solve the
following estimating equations, respectively.

\[
E\{\Psi(O_i; \beta^*) \mid \sigma(B)\} = 0 \ , \ E\{\hat{\Psi}^{(-k)}(O_i; \beta^{\dagger}(-k)) \mid \sigma(B), I_k^c\} = 0 \ . \tag{49}
\]

As a consequence, if \(w(C_i) > 0\) with positive probability given \(\sigma(B)\), we have

\[
\begin{align*}
\beta^*(C_i) &= \frac{1}{2} E\{w(C_i)^2 I_1^c I_k^c \mid \sigma(B)\} \\
\beta^{\dagger}(-k)(C_i) &= \frac{1}{2} E\{w(C_i)^2 \hat{I}_1^{(-k)} \hat{I}_k^{(-k)} \mid \sigma(B), I_k^c\}
\end{align*}
\]  \tag{50}

If \(w(C_i) = 0\) for any \(C_i\) given \(\sigma(B)\), there is nothing to prove because \(\beta^*(C_i) = \beta^{\dagger}(-k)(C_i) = 0\) from the definition of \(\beta^*\).

From the definition of \(\beta^*\) and \(\beta^{\dagger}(-k)\) in (50), we find

\[
\begin{align*}
&\left|\beta^{\dagger}(-k)(C_i) - \beta^*(C_i)\right| \\
&= \frac{1}{2} \left| E\{w(C_i)^2 \hat{I}_1^{(-k)} \hat{I}_k^{(-k)} \mid \sigma(B), I_k^c\} - E\{w(C_i)^2 I_1^c I_k^c \mid \sigma(B)\} \right| \\
&= \frac{1}{2} \left| E\{w(C_i)^2 \hat{I}_1^{(-k)} \hat{I}_k^{(-k)} \mid \sigma(B), I_k^c\} - E\{w(C_i)^2 \hat{I}_1^{(-k)} \hat{I}_k^{(-k)} \mid \sigma(B)\} \right| \\
&= \frac{1}{2} \left| E\{w(C_i)^2 \hat{I}_1^{(-k)} \hat{I}_k^{(-k)} \mid \sigma(B), I_k^c\} - E\{w(C_i)^2 \hat{I}_1^{(-k)} \hat{I}_k^{(-k)} \mid \sigma(B)\} \right| \\
&= \frac{1}{2} E\{w(C_i)^2 \hat{I}_1^{(-k)} \hat{I}_k^{(-k)} \mid \sigma(B), I_k^c\} \\
&= \frac{1}{2} E\{w(C_i)^2 \hat{I}_1^{(-k)} \hat{I}_k^{(-k)} \mid \sigma(B), I_k^c\} \\
&= \frac{1}{2} E\{w(C_i)^2 \hat{I}_1^{(-k)} \hat{I}_k^{(-k)} \mid \sigma(B), I_k^c\}
\end{align*}
\]  \tag{51}

We find the first term of the denominator of (51) is bounded below by

\[
E\{w(C_i)^2 \hat{I}_1^{(-k)} \hat{I}_k^{(-k)} \mid \sigma(B)\} \\
= E\{w(C_i)^2 I_1^c I_k^c (I - 11^T) \Sigma(A_i, X_i) (I - 11^T) I_k^c \mid \sigma(B)\} \\
= E\{w(C_i)^2 \left( \sum_{A_i} P(A_i \mid X_i) I_1^{(-T)} (I - 11^T) \Sigma(A_i, X_i) (I - 11^T) I_k^c \right) \mid A_i, X_i, \sigma(B) \} \mid \sigma(B)\} \\
\geq \frac{\delta}{C_2} E\left\{w(C_i)^2 \left\| I_1^{(-T)} (I - 11^T) \right\|_2^2 \mid A_i = a_i, X_i, \sigma(B) \right\} \mid \sigma(B)\} \ . \tag{52}
\]

The inequality holds for some \(a_i\) from condition (b) of Lemma B.2 and the positivity assumption, i.e. \(\delta \leq P(A_i = 1 \mid X_i) \leq 1 - \delta\). Moreover, the positivity assumption implies that

\[
P(A_{ij} = a_{ij} \mid A_{i(-j)} = a_{i(-j)}, X_i) = a_{ij} \mid X_i) \geq \frac{\delta}{1 - \delta} \ .
\]

Therefore, at \(A_i = a_i\), we find

\[
\left\| I(A_i = a_i, X_i )^{\dagger} (I - 11^T) \right\|_2 = \frac{1}{n_i} \left\| \sum_{j \neq 1} \left\{ P(A_{ij} = a_{ij} \mid A_{i(-j)} = a_{i(-j)}, X_i) \right\}^{-1} \right\|_2 \geq \frac{\delta}{M(1 - \delta)}.
\]

Therefore, equation (52) is bounded below by

\[
E\{w(C_i)^2 I_1^{(-k)} \hat{I}_k^{(-k)} \mid \sigma(B)\} \geq \frac{\delta^2}{C_2 M(1 - \delta)} E\{w(C_i)^2 \mid \sigma(B)\} > 0 \ ,
\]

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and we find $E\{I_i^* S_i \mid \sigma(B)\}^{-1} \leq C'_1$ for some constant $C'_1 > 0$.

Similarly, the second term of the denominator of (51) is bounded below by

$$E\{w(C_i)^2 I_i^{(-k), \top} S_i^{(-k)} I_i^{(-k)} \mid \sigma(B), I_k^c\} = E\left[ w(C_i)^2 I_i^{(-k), \top} (I - 11^\top)\left(\{g_i^*(A_i) - \hat{g}_i^{(-k)}(A_i)\} \otimes \Sigma(A_i, X_i)\right)(I - 11^\top) I_i^{(-k)} \mid \sigma(B), I_k^c\right]$$

$$= E\left[ w(C_i)^2 E\left\{ \sum_{A_i} P(A_i \mid X_i) I_i^{(-k), \top} (I - 11^\top) \Sigma(A_i, X_i)(I - 11^\top) I_i^{(-k)} \mid A_i, X_i, \sigma(B)\right\} \mid \sigma(B)\right]$$

$$\geq \frac{\delta}{C_2} E\left[ w(C_i)^2 \left\| I_i^{(-k), \top}(I - 11^\top)\right\|_2^2 \mid A_i = a_i, X_i, \sigma(B)\right] \mid \sigma(B)\right].$$

The inequality holds for some $a_i$ from condition (b) of Lemma B.2 and the positivity assumption, i.e. $\delta \leq P(A_i = 1 \mid X_i) \leq 1 - \delta$. We find that $\left\| I_i(A_i, X_i; \hat{\pi}^{(-k)})\right\|_2 \leq C_\pi$ implies $C_\pi^{-1} \leq \hat{\pi}^{(-k)}(A_{ij} \mid A_{i(-j), X_i})$ and, as a consequence, $C_\pi^{-1} \leq \hat{\pi}^{(-k)}(A_{ij} \mid A_{i(-j), X_i}) \leq 1 - C_\pi^{-1}$ for all $(A_i, X_i)$. Moreover, at $A_i = a_i$, we find

$$\left\| I_i(a_i, X_i; \hat{\pi}^{(-k)})\right\|_2 = \frac{1}{n_i} \left\| \sum_{j \neq 1} \left\{ \hat{\pi}^{(-k)}(A_{ij} = a_{ij} \mid A_{i(-j), X_i})\right\}^{-1} \right\|_2 \leq \frac{1}{M(C_\pi - 1)}.$$

Therefore, equation (53) is bounded below by

$$E\left[ w(C_i)^2 I_i^{(-k), \top} S_i^{(-k)} I_i^{(-k)} \mid \sigma(B), I_k^c\right] \leq \frac{\delta}{C_2 M(C_\pi - 1)} E\left[ w(C_i)^2 \mid \sigma(B)\right] > 0 \quad (54)$$

and we find $E\{w(C_i)^2 I_i^{(-k), \top} S_i^{(-k)} I_i^{(-k)} \mid \sigma(B), I_k^c\}^{-1} \leq C'_2$ for some constant $C'_2 > 0$. Therefore, (51) is upper bounded by

$$\beta^{(-k)}(C_i) - \beta^*(C_i)$$

$$\leq \frac{C'_1 C'_2^4}{2} \left[ \beta(C_i) E\left\{ w(C_i)^2 I_i^{*, \top} S_i I_i^* \mid \sigma(B)\right\} E\left\{ w(C_i)^2 I_i^{(-k), \top} S_i^{(-k)} I_i^{(-k)} \mid \sigma(B), I_k^c\right\} - E\left\{ w(C_i)^2 I_i^{(-k), \top} S_i^{(-k)} I_i^{(-k)} \mid \sigma(B), I_k^c\right\} \right]$$

$$= \frac{C'_1 C'_2^4}{2} \left[ E\left\{ I_i^{*, \top} S_i I_i^* \mid \sigma(B)\right\} E\left\{ I_i^{(-k), \top} S_i^{(-k)} I_i^{(-k)} - I_i^{*, \top} S_i I_i^* \mid \sigma(B), I_k^c\right\} - E\left\{ I_i^{(-k), \top} S_i^{(-k)} I_i^{(-k)} - I_i^{*, \top} S_i I_i^* \mid \sigma(B), I_k^c\right\} \right]$$

$$\leq \frac{C'_1 C'_2^4}{2} \left[ \left[ E\left\{ I_i^{*, \top} S_i I_i^* \mid \sigma(B)\right\} \right] E\left\{ I_i^{(-k), \top} S_i^{(-k)} I_i^{(-k)} - I_i^{*, \top} S_i I_i^* \mid \sigma(B), I_k^c\right\} + E\left\{ I_i^{*, \top} S_i I_i^* \mid \sigma(B)\right\} E\left\{ I_i^{(-k), \top} S_i^{(-k)} I_i^{(-k)} - I_i^{*, \top} S_i I_i^* \mid \sigma(B), I_k^c\right\} \right]$$

$$\leq \frac{C'_1 C'_2^4}{2} \left[ C_\pi^2 M^2 C_2 E\left\{ I_i^{(-k), \top} S_i^{(-k)} I_i^{(-k)} - I_i^{*, \top} S_i I_i^* \mid \sigma(B), I_k^c\right\} + 2C_\pi^2 M C_2 E\left\{ I_i^{(-k), \top} S_i^{(-k)} I_i^{(-k)} - I_i^{*, \top} S_i I_i^* \mid \sigma(B), I_k^c\right\} \right]. \quad (55)$$
In the last inequality of (55), we used

\[
E\{I^*_i \Sigma_i I^*_i | \sigma(B)\} = E\{I^*_i (I - 11^T) \Sigma(A_i, X_i)(I - 11^T) I^*_i | \sigma(B)\} \\
\leq E\left\{\|I^*_i\|_2^2 \|I - 11^T\|_2^2 \|\Sigma(A_i, X_i)\|_2 \right\} \sigma(B) \}
\leq C^2 \beta \\bar{\tilde{M}} C_2 ,
\]

\[
E\{I^*_i \Sigma_i I^*_i | \sigma(B)\} = E\{I^*_i \{2 \Sigma(A_i, X_i) - 11^T \Sigma(A_i, X_i) - \Sigma(A_i, X_i)11^T\} I^*_i | \sigma(B)\} \\
\leq 2E\left\{\|I^*_i\|_2^2 \|I - 11^T\|_2^2 \|\Sigma(A_i, X_i)\|_2 \right\} \sigma(B) \}
\leq 2C^2 \beta \\bar{\tilde{M}} C_2 .
\]

By taking square, we have the following result for some constants c_1 and c_2.

\[
\left\{\beta^{\bot,(-k)}(C_1) - \beta^*(C_1)\right\}^2 
\leq c_1 E\left\{\left\|I_i^{(-k),T} S_i^{(-k)} I_i^{(-k)} - I_i^{*,T} S_i^* I_i^*\right\|^2 \sigma(B), T_k^\circ \right\}^2 + c_2 E\left\{\left\|I_i^{(-k),T} \tilde{T}_i^{(-k)} \tilde{I}_i^{(-k)} - I_i^{*,T} \tilde{T}_i^* I_i^*\right\|^2 \sigma(B), T_k^\circ \right\}^2
\leq c_1 E\left\{\left\|I_i^{(-k),T} S_i^{(-k)} I_i^{(-k)} - I_i^{*,T} S_i^* I_i^*\right\|^2 \sigma(B), T_k^\circ \right\} + c_2 E\left\{\left\|I_i^{(-k),T} \tilde{T}_i^{(-k)} \tilde{I}_i^{(-k)} - I_i^{*,T} \tilde{T}_i^* I_i^*\right\|^2 \sigma(B), T_k^\circ \right\} .
\]

We study upper bounds of two terms in (56). Since the two terms in the conditional expectations have the same form, so we only present the upper bound of the second term which is represented as

\[
E\left\{\left\|I_i^{(-k),T} S_i^{(-k)} I_i^{(-k)} - I_i^{*,T} S_i^* I_i^*\right\|^2 \sigma(B), T_k^\circ \right\}
= E\left\{\left\|(I_i^* + R_I)^T (S_i^* + R_S) (I_i^* + R_I) - I_i^{*,T} S_i^* I_i^*\right\|^2 \sigma(B), T_k^\circ \right\}
= E\left\{\left\|I_i^{*,T} R_S I_i^* + 2R_I^T S_i^* I_i^* + 2R_I^T R_S I_i^* + R_I^T S_i^* R_I + R_I^T R_S R_I\right\|^2 \sigma(B), T_k^\circ \right\}
\leq 16E\left\{\left\|I_i^{*,T} R_S I_i^*\right\|^2 + 4\left\|R_I^T S_i^* I_i^*\right\|^2 + 4\left\|R_I^T R_S I_i^*\right\|^2 + \left\|R_I^T S_i^* R_I\right\|^2 + \left\|R_I^T R_S R_I\right\|^2 \right\} \sigma(B), T_k^\circ \right\} .
\]

Therefore, it is sufficient to obtain the convergence rate of R_S and R_T; note that \(\|R_I\|_{P,2} = O_P(\rho, N)\) from condition (d) of Lemma B.2.

We first derive the convergence rate of \(\tilde{S}_i^{(-k)} - e_i e_i^T = \tilde{S}_i^{(-k)} - S_i\). We have the following upper bound with some constant c_1 and c_2.

\[
\|\tilde{S}_i^{(-k)} - S_i\|_2^2 \leq c_1 \{g_i^*(A_i) - \hat{g}_i^{(-k)}(A_i)\}^T e_i^{\otimes 2}\{g_i^*(A_i) - \hat{g}_i^{(-k)}(A_i)\} + c_2 \|g_i^*(A_i) - \hat{g}_i^{(-k)}(A_i)\|^2_2
\leq c_1 \{g_i^*(A_i) - \hat{g}_i^{(-k)}(A_i)\}^T e_i^{\otimes 2}\{g_i^*(A_i) - \hat{g}_i^{(-k)}(A_i)\} + c_2 C^2_2 \|g_i^*(A_i) - \hat{g}_i^{(-k)}(A_i)\|^2_2 .
\]

The second inequality is from condition (c) of Lemma B.2. Given \(T_k^\circ\), we find

\[
E\left\{\|\tilde{S}_i^{(-k)} - S_i\|^2_2 \left| T_k^\circ \right\right\}
\leq c_1 E\left\{\|g_i^*(A_i) - \hat{g}_i^{(-k)}(A_i)\|^2 \Sigma(A_i, X_i) \{g_i^*(A_i) - \hat{g}_i^{(-k)}(A_i)\} \right\} T_k^\circ
\leq (c_1 C_2 + c_2 C^2_2) E\left\{\|g_i^*(A_i) - \hat{g}_i^{(-k)}(A_i)\|^2_2 \left| T_k^\circ \right\right\}
= O_P(\rho_{g_i}, N) .
\]

The first inequality is from \(\Sigma(A_i, X_i) = E(e_i^{\otimes 2} | A_i, X_i)\) and the second inequality is from condition
Similarly, we find $\hat{E}_i$ and $(c)$ of Lemma B.2. The last equality holds from condition (d) of Lemma B.2. Moreover, there exists a constant $O_P(r_{g,N})$ so that

$$\|S_i^* - \hat{S}_i\|^2_{P_2} = O_P(r_{g,N}^2) \quad \text{and} \quad \|T_i^* - \hat{T}_i\|^2_{P_2} = O_P(r_{g,N}^2).$$

(58)

Moreover, for any $a_i = a(A_i, X_i)$ and $b_i = b(A_i, X_i)$ be $n_i$-dimensional random and bounded vectors, there exists a constant $C'$ so that

$$E(a_i^T S_i b_i) \leq C E\{\|a_i\|_2 \|b_i\|_2\} \leq C' \|a_i\|_{P_2} \|b_i\|_{P_2}$$

and $E(a_i^T T_i b_i) \leq C' \|a_i\|_{P_2} \|b_i\|_{P_2}$ from similar manner.

The five terms in the last line of (57) have the following convergence rate by using conditions (c) and (d) of Lemma B.2, (58), (59), and the properties of matrix norms.

$$E\{\|I_k\|^2 R_k\|^2 | T_k\} \leq C^2 \|R_k\|^2_{P_2}$$

$$E\{\|R_k S_i I_i\|^2 | T_k\} \leq C^2 \|R_k\|^2_{P_2}$$

$$E\{\|R_k^* R_k I_i\|^2 | T_k\} \leq C \|R_k\|^4_{P_2}$$

$$E\{\|R_k S_i R_k I_i\|^2 | T_k\} \leq C \|R_k\|^4_{P_2}$$

Therefore, the expectation of (57) with respect to $\beta(C_i)$ is $O_P(r_{g,N}) + O_P(r_{g,N}^2)$.

$$E\{\|I_k\|^2 R_k\|^2 | T_k\} \leq C \|R_k\|^2_{P_2}$$

For $E\{\|R_k S_i I_i\|^2 | T_k\} \leq C \|R_k\|^2_{P_2}$, we find the desired result.

$$\|\hat{\beta}_i^{(-k)} - \beta^{(-k)}\|^2_{P_2} = O_P(r_{g,N}^2) + O_P(r_{g,N}^2).$$
D.4 Proof of Lemma B.4

The outline of the proof is given as follows. In [Step 1], we show that $\gamma_{1,-k}$ is identifiable. In [Step 2], we show that the estimating equation in (60) is uniformly consistent. In [Step 3], we show that $\tilde{\gamma}_{1}$ is consistent for $\gamma_{1,-k}$. In [Step 4], we find the rate of $\|\tilde{\beta}^{(k)}(\mathbf{C}_{i}) - \beta_{1,-k}^{(k)}(\mathbf{C}_{i})\|_{p/2}$.

• [Step 1]: Identifiability of $\gamma_{1,-k}$.

If $E\{w(\mathbf{C}_{i})^{2} \mid L(\mathbf{C}_{i}) = \ell\} = 0$ for some $\ell$, the average treatment effect does not use the clusters in $\ell$th stratum so $\beta(\mathbf{C}_{i})$ is not defined in $\ell$th stratum. Therefore, $E\{w(\mathbf{C}_{i})^{2} \mid L(\mathbf{C}_{i}) = \ell\} > 0$ for all $\ell = 1, \ldots, J$. We re-write (49) in terms of $\gamma$ as follows.

$$
E\{\hat{\Psi}^{(-k)}(\mathbf{O}_{i} ; \gamma_{1,-k}) \mid \sigma(\mathbf{B}_{i}, \mathcal{I}_{k})\} = \begin{bmatrix}
E\{\hat{\Psi}_{1}^{(-k)}(\mathbf{O}_{i} ; \gamma_{1,1,-k}) \mid \sigma(\mathbf{B}_{i}, \mathcal{I}_{k}^{c})\} \\
\vdots \\
E\{\hat{\Psi}_{J}^{(-k)}(\mathbf{O}_{i} ; \gamma_{1,J,-k}) \mid \sigma(\mathbf{B}_{i}, \mathcal{I}_{k}^{c})\}
\end{bmatrix} = 0 ,
$$

$\hat{\Psi}_{\ell}^{(-k)}(\mathbf{O}_{i} ; \gamma_{\ell}) = 1\{L(\mathbf{C}_{i}) = \ell\}w(\mathbf{C}_{i})^{2}\left[2\gamma_{\ell}\{\hat{T}_{1}^{(-k)}, \sigma_{\hat{T}_{1}^{(-k)}}^{(-k)}\hat{T}_{1}^{(-k)}\} - \hat{T}_{1}^{(-k)}\sigma_{\hat{T}_{1}^{(-k)}}^{(-k)}\hat{T}_{1}^{(-k)}\right]$. Accordingly, $\gamma_{1}$ is the solution to the following estimating equation.

$$
\frac{1}{N/2} \sum_{i \in \mathcal{I}_{k}} \hat{\Psi}_{i}^{(-k)}(\mathbf{O}_{i} ; \gamma_{1}) = 0 .
$$

And we find that the $\ell$th component of $\gamma_{1,-k}$ is

$$
\gamma_{1,-k}^{(\ell)} = \frac{1\{L(\mathbf{C}_{i}) = \ell\}}{2} \frac{E\{w(\mathbf{C}_{i})^{2} \hat{T}_{1}^{(-k)}\sigma_{\hat{T}_{1}^{(-k)}}^{(-k)}\hat{T}_{1}^{(-k)} \mid L(\mathbf{C}_{i}) = \ell, \mathcal{I}_{k}\}}{E\{w(\mathbf{C}_{i})^{2} \hat{T}_{1}^{(-k)}\sigma_{\hat{T}_{1}^{(-k)}}^{(-k)}\hat{T}_{1}^{(-k)} \mid L(\mathbf{C}_{i}) = \ell, \mathcal{I}_{k}\}} .
$$

From similar steps in (53), we can show that $E\{w(\mathbf{C}_{i})^{2} \hat{T}_{1}^{(-k)}\sigma_{\hat{T}_{1}^{(-k)}}^{(-k)}\hat{T}_{1}^{(-k)} \mid L(\mathbf{C}_{i}) = \ell, \mathcal{I}_{k}\}$ is strictly bounded away from zero. Therefore, $\gamma_{1,-k}$ is uniquely defined.

• [Step 2]: Uniform consistency.

To bound the first term, we use Lemma 2.4 of Newey and McFadden (1994) which is restated as follows in our context. Suppose that (i) the data is i.i.d., (ii) $\Omega = [-B_{0}, B_{0}]^{\otimes J}$ is compact, (iii) $\hat{\Psi}^{(-k)}(\mathbf{O}_{i} ; \gamma)$ is continuous at each $\gamma \in \Omega$ with probability 1, and (iv) there exists $F(\mathbf{O}_{i})$ with $\|\hat{\Psi}^{(-k)}(\mathbf{O}_{i} ; \gamma)\|_{2} \leq F(\mathbf{O}_{i})$ for all $\gamma \in \Omega$ and $E\{F(\mathbf{O}_{i})\} < \infty$. Then,

$$
\sup_{\gamma \in \Omega} \left\| \frac{1}{N/2} \sum_{i \in \mathcal{I}_{k}} \hat{\Psi}_{i}^{(-k)}(\mathbf{O}_{i} ; \gamma) - E\{\hat{\Psi}_{i}^{(-k)}(\mathbf{O}_{i} ; \gamma) \mid \mathcal{I}_{k}\} \right\|_{2} = o_{p}(1). \tag{61}
$$

Conditions (i), (ii), and (iii) are trivially satisfied, so it suffices to show condition (iv). From some algebra, we find

$$
\hat{S}_{i}^{(-k)} = \epsilon_{i} \otimes^{2} + \{g^{*}(\mathbf{A}_{i}, \mathbf{X}_{i}) - \hat{g}^{(-k)}(\mathbf{A}_{i}, \mathbf{X}_{i})\} \epsilon_{i}^\top + \epsilon_{i}\{g^{*}(\mathbf{A}_{i}, \mathbf{X}_{i}) - \hat{g}^{(-k)}(\mathbf{A}_{i}, \mathbf{X}_{i})\}^\top + \{g^{*}(\mathbf{A}_{i}, \mathbf{X}_{i}) - \hat{g}^{(-k)}(\mathbf{A}_{i}, \mathbf{X}_{i})\} \otimes^{2}
$$

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which leads to
\[
\left\| \hat{S}_i^{(-k)} \right\|_2 \leq \left\| I - 11^T \right\|_2 \left\| \hat{S}_i^{(-k)} \right\|_2 \\
\leq \left\| I - 11^T \right\|^2_F \left\{ \| \epsilon_i \|_2^2 + 2 \| \epsilon_i \|_2 \| g^*(A_i, X_i) - \hat{g}^{(-k)}(A_i, X_i) \|_2 + \| g^*(A_i, X_i) - \hat{g}^{(-k)}(A_i, X_i) \|_2^2 \right\} \\
\leq \bar{M}^2 \left\{ \| \epsilon_i \|_2^2 + 2C_g \| \epsilon_i \|_2 + C_g \right\} .
\]  
(63)

The first and second inequalities hold from the properties of matrix norms and the last inequality holds from condition (c) of Lemma B.2. Similarly,
\[
\left\| \hat{I}_i^{(-k)} \right\|_2 \leq 2 \left\| I - 11^T \right\|_2 \left\| \hat{I}_i^{(-k)} \right\|_2 \\
\leq 2 \left\| I - 11^T \right\|_F \left\{ \| \epsilon_i \|_2^2 + 2 \| \epsilon_i \|_2 \| g^*(A_i, X_i) - \hat{g}(A_i, X_i) \|_2 + \| g^*(A_i, X_i) - \hat{g}(A_i, X_i) \|_2^2 \right\} \\
\leq 2 \bar{M} \left\{ \| \epsilon_i \|_2^2 + 2C_g \| \epsilon_i \|_2 + C_g \right\} .
\]

As a consequence, we find the following inequality holds for any \( \beta \).
\[
\| \tilde{w}^{(-k)}(O_i; \beta) \|_2 \leq C_w \left\{ 2 \| \beta \|_2 \| I^{(-k)} \|_2 \left\| \hat{S}_i^{(-k)} \right\|_2 \right\} + \| \hat{I}_i^{(-k)} \|_2 \left\| \hat{I}_i^{(-k)} \|_2 \right\} \\
\leq C_w^2 C^2_\pi \left\{ 2 \| \epsilon_i \|_2^2 + 2C_g \| \epsilon_i \|_2 + C_g \right\} + \bar{M} \left\{ \| \epsilon_i \|_2^2 + 2C_g \| \epsilon_i \|_2 + C_g \right\} \\
\leq c_2 \| \epsilon_i \|_2^2 + c_1 \| \epsilon_i \|_2 + c_0
\]  
(64)

where \( c_0, c_1, \) and \( c_2 \) are generic constants. Since \( \text{E}\{ \| \epsilon_i \|_2^2 \mid \mathcal{I}_k^c \} \) and \( \text{E}\{ \| \epsilon_i \|_2 \mid \mathcal{I}_k^c \} \) are integrable, \( F(O_i) = c_2 \| \epsilon_i \|_2^2 + c_1 \| \epsilon_i \|_2 + c_0 \) satisfies condition (iv) of Lemma 2.4 of Newey and McFadden (1994). As a consequence, (61) holds.

- **[Step 3]**: Consistency of \( \hat{\gamma}^{(k)} \)

Conditioning on \( \mathcal{I}_k^c \), \( \hat{w}^{(-k)} \) and \( \hat{g}^{(-k)} \) can be understood as fixed quantities. Therefore, we find \( \hat{\gamma}^{(k)} = \gamma^{1,-(k)} + o_P(1) \) from Theorem 5.9 of van der Vaart (1998).

- **[Step 4]**: Rate of \( \| \tilde{\beta}^{(k)}(C_i) - \beta^{1,-(k)}(C_i) \|_{P_2}^2 = \int \| \tilde{\beta}^{(k)}(c_i) - \beta^{1,-(k)}(c_i) \|_2^2 dP(c_i) \)

We check conditions for Theorem 5.21 of van der Vaart (1998). First, for any \( \gamma_1 \) and \( \gamma_2 \) in \( \Omega \equiv [-B_0, B_0]^{\otimes J} \), we find
\[
\hat{\Psi}^{(-k)}(O_i; \gamma_1) - \hat{\Psi}^{(-k)}(O_i; \gamma_2) = 2w(C_i) \left\{ \tilde{I}_i^{(-k)} \right\} \gamma_1 \gamma_2 \right\} \left[ \left\{ \left\{ L(C_i) = 1 \right\} \gamma_1, 1 - \gamma_1, 2 \right\} \right] : \\
\leq \left\{ \left\{ L(C_i) = 1 \right\} \gamma_1, 1 - \gamma_1, 2 \right\} \\
\left\{ \left\{ L(C_i) = J \right\} \gamma_1, 1 - \gamma_1, 2 \right\}
\]

and this implies
\[
\left\| \hat{\Psi}^{(-k)}(O_i; \gamma_1) - \hat{\Psi}^{(-k)}(O_i; \gamma_2) \right\|_2 \leq 2C_w^2 C^2_\pi \left\| \hat{S}_i^{(-k)} \right\|_2 \left\| \right\|_\gamma_1 - \gamma_2 \right\|_2 \\
\leq \left\{ c_2 \| \epsilon_i \|_2^2 + c_1 \| \epsilon_i \|_2 + c_0 \right\} \left\| \gamma_1 - \gamma_2 \right\|_2 .
\]

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The second inequality holds from (63) with some constants $c_0$, $c_1$, and $c_2$. Therefore, we find the function on the right hand side is squared integrable as follows.

$$E \left[ \left( c_2 \| \epsilon_i \|_2^2 + c_1 \| \epsilon_i \|_2 + c_0 \right)^2 \right] = E \left\{ c_4' \| \epsilon_i \|_2^4 + c_3' \| \epsilon_i \|_2^3 + c_2' \| \epsilon_i \|_2^2 + c_1' \| \epsilon_i \|_2 + c_0' \right\} < \infty$$

where $c_0', \ldots, c_4'$ are some constants.

Second, $E\left\{ \| \hat{\Psi}^{(-k)}(O_i; \gamma^*) \|_2^2 | \mathcal{I}_k^c \right\} < \infty$ because (64) holds with $\gamma = \gamma^*$.

Third, $E\{ \hat{\Psi}^{(-k)}(O_i; \gamma) \mid \mathcal{I}_k\} \times \hat{\gamma}(k) = 2w(C_i)^2 \left\{ \tilde{I}_i^{(-k), \top} \tilde{S}_i^{(-k)} \tilde{I}_i^{(-k)} \right\} \text{diag} \left[ 1 \{ L(C_i) = 1 \} , \ldots , 1 \{ L(C_i) = J \} \right]$. The expectation of the $\ell$th diagonal of the above term is lower bounded by a positive constant $C_L$ as follows.

$$E \left[ 2w(C_i)^2 1 \{ L(C_i) = \ell \} \left\{ \tilde{I}_i^{(-k), \top} \tilde{S}_i^{(-k)} \tilde{I}_i^{(-k)} \right\} \mid \mathcal{I}_k\right] = 2P\{ L(C_i) = \ell \} E \left[ w(C_i)^2 \left\{ \tilde{I}_i^{(-k), \top} \tilde{S}_i^{(-k)} \tilde{I}_i^{(-k)} \right\} \mid L(C_i) = \ell, \mathcal{I}_k\right] \geq C_L \equiv \frac{2P\{ L(C_i) = \ell \} \delta}{C_2\overline{M}(C_2 - 1)} E\{ w(C_i)^2 \mid L(C_i) = \ell, \mathcal{I}_k\} > 0$$

(65)

The inequality holds from (54) and $E\{ w(C_i)^2 \mid L(C_i) = \ell, \mathcal{I}_k\} > 0$ (if this quantity is zero, the investigator should not define a parameter of statum $\ell$). This implies the derivative of $E\{ \hat{\Psi}^{(-k)}(O_i; \gamma) \mid \mathcal{I}_k\}$ is nonsingular at $\gamma = \gamma^*$.

Lastly, we have $\sum_{i \in \mathcal{I}_k} \hat{\Psi}^{(-k)}(O_i; \hat{\gamma}(k)) = 0$ from the definition of $\hat{\gamma}(k)$ and $\hat{\gamma}(k) = \gamma^{\dagger,-(k)} + o_P(1)$ from [Step 3]. Therefore, all conditions of Theorem 5.21 of van der Vaart (1998) are satisfied. Thus, we have the following asymptotic normality for some variance matrix $V_0$.

$$\sqrt{N/2} \left\{ \hat{\gamma}(k) - \gamma^{\dagger,-(k)} \right\} \overset{D}{\to} N(0, V_0) .$$

This implies $\hat{\gamma}(k) - \gamma^{\dagger,-(k)} = O_P(N^{-1/2})$.

•  [Step 4]: Rate of $\| \hat{\beta}^{(k)}(C_i) - \beta^{\dagger,-(k)}(C_i) \|_{P, 2}$

Since $\beta(C_i) = \sum_{\ell=1}^{J} 1 \{ L(C_i) = \ell \} \gamma_\ell$, we find

$$\int \| \hat{\beta}^{(k)}(C_i) - \beta^{\dagger,-(k)}(C_i) \|_2^2 dP(c_i) = \int \sum_{\ell=1}^{J} 1 \{ L(C_i) = \ell \} \left\{ \hat{\gamma}_\ell^{(k)} - \gamma_\ell^{\dagger,-(k)} \right\}^2 dP(c_i) \leq J \sum_{\ell=1}^{J} \left\{ \hat{\gamma}_\ell^{(k)} - \gamma_\ell^{\dagger,-(k)} \right\}^2 \leq \overline{M} \left\| \hat{\gamma}(k) - \gamma^{\dagger,-(k)} \right\|_2^2 = O_P(N^{-1}) .$$

As a consequence, $\| \hat{\beta}^{(k)}(C_i) - \beta^{\dagger,-(k)}(C_i) \|_{P, 2} = O_P(N^{-1/2})$. This concludes the proof.
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