Structure Entropy, Self-Organization, and Power Laws in Urban Street Networks: Evidence for Alexander’s Ideas

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The study of networking mechanism is of central importance for better understanding the broader properties and phenomena of networks. Here, we investigate the scale-free networking in urban street networks holistically within the framework of information physics and statistical physics. Although the number of times that a natural road (a substitute for “named” street) crosses an other one has been widely reported to follow a scale-free probability distribution among self-organized cities, the derivation of the statistics of urban street networks from fundamental principles has focused very little attention. We recover the discrete Pareto probability distribution for natural roads in self-organized cities, and foresee a nonstandard bell-shaped probability distribution with a Pareto tail for their junctions. Our approach explicitly emphasizes the road-junction hierarchy rather than implicitly inhibiting it as in most investigations. This holistic viewpoint reveals an underlying Galoisian algebraic structure. So that our approach fits with the mindset of information physics. This enables us to envisage urban street networks as evolving social systems subject to a Boltzmann-mesoscopic entropy conservation. The passage from the underlying Galoisian hierarchy to an underlying Paretoian coherence occurs by invoking Jaynes’s Maximum Entropy principle. Ultimately, to obtain the predicted statistics, we untangle the underlying discrete Pareto probability distribution with a binomial paired-agent social model taken at the asymptotic limit. The emerging paradigm may apply to systems with a more intricate hierarchy. Meanwhile, along your findings, it appears to reflect well Alexander’s ideas on cities. The established statistics can be useful to build realistic urban models and to discover underlying laws that govern our cities.

I. INTRODUCTION

Network theory has appeared as a powerful framework to investigate complex systems in the recent decades. It has become so powerful that nowadays we tend to identify networks here, there, and everywhere. Here we have essentially in mind real-world networks as emerging from effective data extracted from various concrete domains, such as social, biological, informational, technological, transportational, urban, or economical domains [1–4]. They present a rich multi-disciplinary-faceted zoology which is propitious to clashes of perspectives [1, 2]. Scale-freeness is one of the major facet, nonetheless its interpretation still conveys inter-disciplinary frictions [3]. New approaches, ideas, and mechanisms may contribute to converge views and efforts. In this paper, we expose a paradigm driven by structures — as modern physics is by symmetries — through an urban pre-network model that leads to scale-free networks effectively observed.

Two breakthroughs that occurred at the dusk of the last century were instrumental in the renewal of interest in network theory and real-world networks: the discoveries of (i) the small-world effect [5] and of (ii) the scale-free hierarchy [6] among real-world networks. The former exhibits high clustering coefficients and short characteristic path lengths compared to equivalent regular and random networks, respectively. The latter reveals valence distributions that essentially follow a power law, that is, nodes preferentially attach to already well-connected ones, known as hubs, essentially in a scale-free manner; in particular, no typical valence can be observed, unlike for regular or random networks. These two behaviours neither imply nor exclude each other [7, 8]; even so scale-free networks typically possess ultrashort characteristic path lengths, short otherwise [7]. In statistical physics, scale-invariance and its twin concept universality [9] had already emerged as crucial in hydrodynamics, phase transitions [9, 10], chaos, and dynamical systems with many degrees of freedom [9, 11]. Therefore, scale-invariance favours relevant features and bypasses fine details. Meanwhile fractal geometry had celebrated scale-invariance through self-similarity [12]. The small-world effect was coined after the so-called phenomenon in social science [5, 13]. Since then real-world networks have been intensively investigated: as a result, several novel features have emerged along with explanatory mechanisms [1, 2]. Nonetheless the field remains a vivid new frontier as seen in preliminaries.

Inter- and intra-urban environments are rich in concrete networks and they had been for network theory a source of case studies before the internet age [14–17]. The route network is certainly among the first global networks generated by humans: both country road networks...
and urban street networks appeared promptly to be subject to small-world and scale-free behaviours [17–23]. If the urban community has looked at urban street networks for additional common traits with tools developed for generic networks, it has also investigated for apparently more specific features with tools and approaches inherited from its own background [14–32]. Interestingly enough, the insightful thought of C. Alexander on cities [30–32] has appeared to resonate with scale-free invariance through the notion of “natural” city [22, 29, 30]. We must also mention the more mathematically oriented but not less insightful work of R. H. Atking on relation functions as pre-networking functions which led to Q-analysis [33].

Despite these efforts on observations and descriptions, few works have focused on deriving the statistics of urban street networks from fundamental principles. To this purpose, we explicitly emphasize the road-junction incidence relation of urban street networks rather than implicitly melting it into road-road and junction-junction bounds of two dual but isolated networks. Most investigations indeed seek to cast urban street networks into road-road (topological) networks and then to describe their valence probability distributions [20–23]. Here a natural road (or road) denotes an accepted substitute for a “named” street [22]. Our holistic preliminary is adopted from Q-analysis [33]. Q-analysis is subsequently applied in its paroxysmal but corrective variant due to Y.-S. Ho [34, 35], which is nothing but the Formal Concept Analysis (FCA) paradigm [36]. This holistic viewpoint also fits with the mindset of information physics [37–40], which is built upon partial-order relations [36]. Here the partial-order relation derives from the road-junction incidence relation using FCA [34, 36]. Then information physics enables us to envisage urban street networks as evolving social systems subject to an entropic equilibrium comparable to the Paretian entropic balance effectively observed among cities of a same cultural basin [41, 42]. Our approach recovers the discrete Pareto probability distribution widely observed for natural roads spreading in “natural” cities [20–23], and foresees a nonstandard bell-shaped distribution with a Paretian tail for their joining junctions found in agreement with observable data extracted from some typical “natural” urban street networks (see Fig. 3). Retrospectively, the cohering (or combining) part of our approach is the Paretian match for the Gaussian model in statistical physics, while the ordering (or structuring) part somehow reminisces C. Alexander’s ideas [30, 31] (see Fig. 2).

Although our approach is specifically applied to urban street networks, it provides a generic paradigm for the study of complex networks underlying a more intricate partial-order. Within this broader perspective, urban street networks become an ideal toy model and C. Alexander’s ideas fall into the domain of network theory. The emerging paradigm is sketched as the first course (Sec. II). Then a brief survey of the state of the art in urban street networks is given before we proceed forwards (Sec. III). Once the paradigm is applied, we discuss further our results from the perspective of C. Alexander’s ideas (Sec. IV). Eventually, we point to future investigations.

II. PARADIGM

A. Structure before measure

‘Structure before measure (but without alteration)’ is the dominant leitmotif of the present work. It is borrowed from Q-analysis [33] but with a severe and fundamental constraint (in parenthesis) after a correction [34, 35] due to Y.-S. Ho [34]: ‘We should not include anything which is not given’. The Q-paradigm as revisited by Y.-S. Ho [34] leads to plain algebraic ordering structures known as Galois lattices [34, 36] instead to an insightful but in fine deficient [34, 35] simplistic geometrical interpretation [33]. As partially ordered structure, each Galois lattice is equipped with an order relation; as algebraic structure, with a join operator. Two elements are either comparable or not; an element is either join-irreducible or the join of two distinct elements.

In general, a Galois lattice organizes itself in layers with respect to its order relation to give rise to a Hasse diagram [36]. For finite distributive Galois lattices [36], which might be considered typical [36], the join-irreducible elements constitute the smallest nontrivial elements [36] from which the whole builds itself through the join operator, so that they form the lowest nontrivial layer of their Hasse diagrams. From now on, let us imagine this layer as a network of homogeneous elements that links each pair of them when they can join to generate a greater element. Along this line, each greater element itself belongs to an upper layer envisaged as another network of homogeneous elements arbitrarily bonded with respect to the order relation.

2. Measure

What about ‘measure’? As answer, let us invoke the formal statement that arises from the emerging theory of information physics [38]: ‘Measuring is the quantification of ordering’. More precisely, imposing natural algebraic consistency constraints permit us not only to evaluate Galois lattices but also to recover and generalise contemporary information measures (modulo two successive latticial exponentiations) [37–40] — information physics is to structures what Noether’s theorem [43] is to symmetries. For finite distributive Galois lattices [36], the evaluation reduces to the evaluations of their join-irreducible elements, the constraints determining the evaluations of the join-reducible elements. Latticial exponentiations generate distributive Galois lattices.
In other words, we have the freedom to evaluate each join-irreducible element as we wish. Nevertheless, while valuation functions associated to first exponentiations are recognized as probability distributions, further natural consistency constraints dictate linear combinations of the Shannon and Hartley entropies \([44]\) as valuation functions associated to second exponentiations \([38, 39]\). And, evidently, the valuation of the initial Galois lattice is governed by the underlying physics, viz., the evaluation of each initial join-irreducible element is meant to express its physical state. Meanwhile, the probability distribution might be as plausible as possible with respect to both our lack of comprehensive knowledge for each element on their concealed microscopic details and our macroscopic viewpoints. This is nothing other than Jaynes’s maximum entropy principle \([38, 45–48]\).

3. Principle of Maximum Entropy

Thence, the physical content of the paradigm shifts from an algebraic structure to a fluctuating environment, from Galois lattice partial-order to entropic coherence. Our initial ignorance \([48]\) yielding on the elements of the Galois lattice, the probability distribution is over their number of possible states.

Let \(\text{Pr}(\Omega)\) denote the probability of an element to count \(\Omega\) configurations and recap: the most plausible realization of \(\text{Pr}(\Omega)\) is the one that maximizes the entropy \(-\sum \text{Pr}(\Omega) \ln \text{Pr}(\Omega)\) \([49]\) with suitable characterizing moments as constraints \([45, 46]\). As characterizing moments, assuming among the elements no typical number of configurations but rather a typical scale, we must discard any classical moment and may consider logarithmic moments instead. Imposing the first logarithmic moment \(\sum \text{Pr}(\Omega) \ln \Omega\) as the unique characterizing moment appears to lead to the scale-free probability distribution \(\text{Pr}(\Omega) \propto \Omega^{-\lambda}\). Since \(\ln \Omega\) measures nothing but our complete ignorance on the state effectively occupied by any element having \(\Omega\) possible states, this constraint actually forces to preserve on average our complete ignorance on the elements of the Galois lattice — as an analogy, the Maxwell-Boltzmann statistics describing ideal gases can be deduced by solely enforcing a constant mean energy \([45, 47]\).

The above deus ex machina has been interpreted as some evolutionary based mechanism to maintain some opaque internal order \([41, 42]\). Imposing the second logarithmic moment as an extra characterizing moment leads to a statistics governed by the discrete lognormal probability distribution \(\text{Pr}(\Omega) \propto \Omega^{-\lambda} \exp(-((\ln \Omega - \eta)^2)/2\sigma^2_n)\); and so on. For now, let us restrict ourselves to our first attempt.

B. Overlying networks

1. The join-irreducible network

Now we shift our attention back to the network formed by the join-irreducible elements of the Galois lattice. As a working hypothesis, let us assume for each node that its number of configurations \(\Omega\) depends on its valence \(n\); we write \(\Omega(n)\). Therefrom, in this network, the probability distribution of node valences \(\text{Pr}(n)\) preserves the scale-free character when the number of configurations \(\Omega(n)\) grows powerly according to an exponent \(\nu_1\). Then we have \(\text{Pr}(n) \propto n^{-\lambda \nu_1}\) where the exponents \(\lambda\) and \(\nu_1\) characterize, respectively, the entropic coherence of the Galois lattice as a whole and the configurational growth of its join-irreducible elements as nodes of an homogeneous network. On the other hand, the number of configurations for every join-reducible element remains algebraically coerced by the Galois lattice, that is, it is obliged to algebraically depend on the number of configurations of its two joining elements through the valuation additive constraint \([37, 38]\). Now, let us envisage as a second network the layer that gathers the joins of two join-irreducible elements, two joins with a common generator being bonded.

2. The join-reducible networks

For the sake of argument, let us pretend that the nodes on the first and second network-layers undergo a powerly configurational growth with exponents \(\nu_1\) and \(\nu_2\), respectively. On our second network, we then have \(\text{Pr}(n) \propto C_n(\nu_1; n) n^{-\lambda \nu_2}\) where \(C_n(\nu_1; n)\) counts the occurrences of nodes of valence \(n\) with respect to the valuation additive constraint, so that it might be merely thought as a self-convolution operator acting on the valence probability distribution of our first network. Iterating this process gives for the \(k\)-th network-layer \(\text{Pr}(n) \propto C_n(\nu_1, \nu_2, \ldots, \nu_k; n) n^{-\lambda \nu_k}\) with obvious notations. Thence, under the rather favourable assumption that node configurations grow powerly with valences, the valence probability distribution for every reducible network-layer inherits a power tail from the underlying scale-free behaviour, whereas the irreducible network-layer plainly reveals it, and a mass function from the underlying Galois lattice algebraic structure. Notice that when the Galois lattice is flattened or ignored, the valence probability distribution sees its tail dominated by the strongest power tail and its mass function becoming a linear combinations of powerly weighted mass functions.

3. Misleading claim

So, within this scheme, we will observe no scale-free network if the underlying ordering structure is disregarded, if the involved network is unfortunately not the
irreducible one, or if the node configurations do not unlikely grow powerfully with valences. However, the claim that scale-free networks are rare would be misleading here since the system as a whole is effectively driven by a scale-free power law probability distribution, while the scale-free behaviour could possibly be observed only for the first network-layer.

III. URBAN STREET NETWORKS AS TOY MODEL

A. Geometry versus topology

1. Trivial versus nontrivial complexities

As pedestrians, cyclists, or drivers, we tend to envision at first glance the street junctions and segments of our cities as the natural nodes and edges, respectively, of urban street networks (see Fig. 1d). Their complexity is nonetheless trivial: three or four links for most street junctions [22, 25]. Indeed, in situ, any city-adventurer knows that at each street segment-end (or junction) they would have in most case only two alternatives: continue along or the other way. And this occurs independently of the city they explore or where they are in the city. Clearly, this first attempt to describe our urban environment — better known as the geometrical approach — appears to be too naive [17, 19, 22–26].

A second thought may lead us to realize that we rather reason in terms of streets than of street segments — and possibly in terms of junctions. Indeed, from townsmen we expect concise directional answers shaped as follows: “To go to Oasis office from Amethyst area: take Sunshine street, then Seaport street — at Jade junction — and, finally, Sunset street — at Jonquil junction.” Even though colourful, this typical directional answer implicitly reveals precious information: (i) neither position nor distance is awaited; (ii) each junction in itself plays a secondary role; (iii) each pair of successive streets critically shares a common junction — whichever it is. To wit, we expect topological responses. The topological approach reduces streets to nodes and links each pair of them that shares a common junction (see Fig. 1e). In contrast to geometrical networks, topological networks exhibit small-world and scale-free properties, that is, complex network behaviours [19, 22–27].

2. Data extraction overview

On the fly, we have neglected to ask ourselves how to define streets. This question should seem preposterous for most of us living in towns for which a cadaster has been scrupulously maintained over decades or centuries, but certainly not for the globetrotters among us. Even if perfect cadasters must exist, “named” streets essentially remain the result of intricate social processes where the underlying social physics likely interferes with local customs, past or present agency struggles between social groups, and so forth. Actually, the question “What is a street ?” has been addressed by introducing the notion of natural road.

A natural road [22] is an exclusive sequence of successive street segments paired according to some behavioural based join principle (see Fig. 1c). Besides the de facto cadastral join principle, three geometrical join principles based on deflection angles [14, 15, 22, 26, 28] are...
mainly used. The every-best-fit join principle is a junction-centric one which only binds with respect to the deflection-angle-ordering of each junction, so that it is almost deterministic because of its local character. The self-best-fit and self-[random]-fit join principles are both path-centric ones which recursively append new segments, respectively, with respect to the deflection-angle-ordering of the end-segments and randomly. Unsurprisingly, the self join principles have appeared more realistic against relevant cadasters due to their global nature — the random variant being generally the best fit. Here we use the self-[random]-fit join principle, unless specified otherwise. Basically, our ‘raw material’ is geometrical networks extracted from map data fetched from well-known comprehensive archives (see Fig. 1a).

B. Galoisean hierarchy

1. Concealed Galois lattice

To knit a topological network we may first establish the incidence relation \( I \) that gathers for each natural road all junctions through which it passes, then infer its reciprocal \( I^{-1} \) that gathers for each junction all natural roads which it joins [22]: the composition of the former with the latter \( I \circ I^{-1} \) gives the road-road topological network encountered above, whereas the alternative composition \( I^{-1} \circ I \) leads to its dual the junction-junction topological network. Both networks are non-injective representation of \( I \), and so of the involved urban street network.

Let us now interpret any incidence relation \( I \) as an object/attribute relation where each natural road acts as an object and each junction as an attribute [33, 34, 36]. Thereby, relying on FCA, we can bijectively represent any incidence relation \( I \) as an ordered algebraic structure \( \mathcal{L} \) known as **Galois lattice** [34, 36]. As shown in the constructive proof provided by Y.-S. Ho [34], this paradigm combines objects and attributes into pairs of subsets of them to form without loss of information a Galois lattice — to achieve the emerging structure, the one-to-many relation \( I \) is naturally extended to a many-to-many relation.

Fortunately, for urban street networks, it turns out that incidence relations effortlessly reduce to Galois lattices with two nontrivial layers: the natural roads form the lower layer; the junctions compose the upper one; the ‘implicitly’ ordering relation is “passing through”. Still bearing in mind the triviality of geometrical networks, the reader has already noticed that when every junction joins only two natural roads the Galois lattice becomes distributive. They have also observed that any junction that joins more than two natural roads can be replaced by a roundabout so that it remains only junctions joining at most two natural roads. For these reasons, we will qualify as **canonical** any urban street network whose junctions effectively join only two natural roads. In short, for urban street networks, incidence relations bijectively reduce to essentially distributive Galois lattices with two nontrivial layers, while their canonicalization renders them plainly distributive.

Arguably this is nothing new, except that the complexity of urban street networks can now be holistically measured within the information physics framework. The detailed treatment of this subject is well outside the scope of this paper; thus, beyond the material formerly sketched (see Section II), we simply refer to the work of K. H. Knuth [37–40], and we will content ourselves with presenting the pertinent consequences for road/junction Galois lattices to elaborate further.

2. Complexity measurement

Without loss of generality, we may canonicalize urban street networks so that their Galois lattices are distributive. Henceforth, natural roads constitute their join-irreducible elements, viz., we have the freedom to evaluate each natural road as we desire while the Galois lattice algebraic structure dictates to evaluate each junction as the sum of the evaluation of their two joining natural roads. Thusly, for every junction \( j(r,s) \) joining the pair of natural roads \( (r,s) \), we are compelled to write

\[
Va(j(r,s)) = Va(r) + Va(s)
\]

where \( Va \) stands for the yet unknown valuation function. Further consistency requirements oblige to recognize any valuation function associated to the first exponentiation of each Galois lattice as a weight function \( w \); we read

\[
Pr = w \circ Va
\]

with \( Pr \) the probability distribution of the system. Meanwhile we may choose \( w \) as we want. Finally, same and further demanded consistency constraints force to identify the evaluation of the central element of the second exponentiation of the Galois lattice as the entropy \( H[VA,w] \) of the system which thusly expresses as a functional of the valuation and weight functions, \( Va \) and \( w \), respectively. For canonical urban street networks, the functional structure entropy \( H[VA,w] \) takes the form

\[
H[VA,w] = \sum_r (h \circ w)(Va(r)) + \sum_{j(r,s)} (h \circ w)(Va(r) + Va(s))
\]

(3)

where the first summation runs over the natural roads \( r \) and the second one over the junctions \( j(r,s) \) joining the pair of natural roads \( (r,s) \), while \( h: x \mapsto -x \ln x \) is the Shannon entropy function [49].

By reverting addition rule (1) in the right summation and then composing according to (2), the reader will readily recover the ‘flat’ expression of the functional entropy \( H[VA,w] \), namely \( H[Pr] = \sum_e (h \circ Pr)(e) \) where the summation occurs indiscriminately over all natural roads and junctions \( e \).
Therefore, in our context, the novelty brought by information physics theory sums up as follows: it enables us to measure the complexity of our heterogeneous system as a whole by taking its ordering hierarchy into account. In detail, it articulates as follows: locally, it reveals how the natural roads \( r \) impose their arbitrary valuations \( \nu_r \) to the junctions \( j \); globally, it unveils how an arbitrary weight function \( w \) cements the whole. Notice the slight abuse of language used in the article’s title: entropy (3) is qualified with structure to highlight this novelty.

C. Paretian coherence

1. Assumed complete ignorance

In any case, from their city, most dwellers do not perceive the underlying Galoisean hierarchy *per se* but rather the resulting emergent Paretian coherence. This passage from algebraic structure to organic arrangement appears to take place in our context as a consequence of Jaynes’s maximum entropy principle as outlined early (see Section II).

Formally, we assume our complete ignorance on what phenomena drive each natural road or junction; so that, the most we can state is that each one possesses a finite number of equally likely configurations. Thence, the system mean entropy \( \langle H \rangle \) writes

\[
\langle H \rangle = \sum_e \Pr (\Omega_e) \ln \Omega_e
\]  

whenever every natural road or junction \( e \) has reached an equilibrium; the summation happens indiscriminately over all natural roads and junctions \( e \), \( \Pr (\Omega_e) \) expresses the probability for the natural road or junction \( e \) to have \( \Omega_e \) states, and \( \ln \Omega_e \) its Boltzmann entropy. Then, using the same notation, Jaynes’s maximum entropy principle invoked with the first logarithmic moment as unique characterizing moment literally holds the Shannon Lagrangian expression

\[
\mathcal{L} (\{\Pr (\Omega_e)\}; \lambda, \nu) = - \sum_e \Pr (\Omega_e) \ln (\Pr (\Omega_e)) - \lambda \left[ \sum_e \Pr (\Omega_e) \ln \Omega_e - \langle H \rangle \right] - (\nu - 1) \left[ \sum_e \Pr (\Omega_e) - 1 \right]
\]

where the first and second constraints impose the conservation of the system mean entropy and the normalization condition satisfied by \( \Pr \), respectively, while \( \langle H \rangle \) stands for the constant mean entropy at which the system evolves. Resolving (5) readily gives the power law distribution

\[
\Pr (\Omega_e) = \frac{\Omega_e^{-\lambda}}{Z (\lambda)} \quad \text{with} \quad Z (\lambda) = \sum_e \Omega_e^{-\lambda}
\]

as Zustandssumme. Explicit computation of the mean entropy (4) yields the equation of state

\[
\langle H \rangle = - \frac{\partial}{\partial \lambda} \ln Z (\lambda)
\]

whose exploitation is deferred.

In this way, our complete ignorance helps us to discern a Paretian coherence, yet not plainly perceivable, among urban street networks.

2. Conceded partial knowledge

In fact we have feigned our complete ignorance, at least partially: we have blithely dismissed the underlying Galoisean hierarchy and that natural roads and junctions are likely driven by social interactions. It is time now to decompose accordingly the probability distribution (6) with respect to composition (2) and addition rule (1).
concerns each junction, the involved agents are merely the agents of the two joining natural roads combined together; hence the same crude maneuvers give

\[
\Omega_j(r,s) = \Omega_j (n_j = n_r + n_s) \simeq A^{2\nu_j} 2^{\nu_j} n_j^{2\nu_j} \tag{8b}
\]

along with some abuse of notation.

Therefrom, the valuation function \( V_a \) arises clearly as assigning to each natural road or junction the number of associated agents while the weight function \( w \) asymptotically counts the number of possible vital intraconnection layouts (modulo normalization) in the involved natural road or junction then envisioned as an intranetwork.

Returning to where we left off, we can now express the probability for natural roads and junctions in a more specific, perceivable fashion. Substituting (8a) into (6), we readily obtain for natural roads

\[
Pr (n_r) \propto n_r^{-2\lambda \nu_r} \tag{9a}
\]

which is a scale-free power law distribution. For the junction counterpart, inserting instead (8b) into (6), then gathering and counting with respect to the precedent probability distribution (9a) yields

\[
Pr (n_j) \propto \left( \sum_{j(r,s)} \frac{n_j = n_r + n_s}{(n_r n_s)^{2\lambda \nu_r}} \right) n_j^{-2\lambda \nu_j} \tag{9b}
\]

where Iverson bracket convention is used; the summation in parentheses is the self-convolution of the natural road probability distribution (9a). Given a natural road \( r \), its number of junctions \( n_r \) is nothing but essentially its degree in the involved road-road topological network: valence distribution (9a) has been empirically observed in self-organized cities [20–23]. The same argument dually applies for junctions: nonetheless, to the best of our knowledge, valence distribution (9b) can be neither confirmed nor refuted by the current literature.

In practical recognitions [51], we need to assume that the number of junctions per natural road spans from some minimal value \( n_\mu \geq 1 \). Then, the normalizing constants for probability distributions (9) can be effortlessly computed in terms of natural generalizations of known (very) special functions. While we readily have

\[
Pr (n_r) = \frac{n_r^{-2\lambda \nu_r}}{\zeta (2\lambda \nu_r; n_r)} \tag{10a}
\]

where \( \zeta (\alpha; n) = \sum_{n = n}^{\infty} n^{-\alpha} \) is the generalized (or Hurwitz) zeta function [51, 52], we find that

\[
Pr (n_j) = \frac{\sum_{n_j = n}^{\infty} \left[ n (n_j - n) \right]^{-2\lambda \nu_j} n_j^{-2\lambda \nu_j}}{W (2\lambda \nu_r, 2\lambda \nu_r; n_r)} \tag{10b}
\]

where \( W (\alpha, \beta; \gamma; n) = \sum_{m,n \geq 2} m^{-\alpha} n^{-\beta} (m + n)^{-\gamma} \) is the two-dimensional generalized (or Hurwitz-) Mordell-Tornheim-Witten zeta function [53]. The former probability distribution (10a) is known as the discrete Pareto distribution and is a shifted (or Hurwitz) version of the better known Zipf distribution [51, 54]; the latter (10b) is a nonstandard bell-shaped distribution with a Pareto-tail asymptotic to \( n_j^{-2\lambda (\nu_r + \nu_j)} \); as far as we can tell, and we have found it convenient to name it the Schwitten distribution [55].

D. Case studies

We checked the statistical pertinence of the foreseen junction valence distribution (10b) for five urban street
networks for which the predicted natural road valence distribution (10a) is a plausible hypothesis with respect to the state-of-the-art statistical methods for power law distributions [51] which is based on Maximum Likelihood Estimations (MLE). A sixth urban street network which is recognized as planned was taken as counter-case study. A validation of the junction valence distribution (10b) along the lines of the state of the art [51] could not be managed because fast evaluation of the normalizing function $W$ has yet to be found; meanwhile a crude data analysis prevents us from grossly rejecting the foreseen junction valence distribution (10b).

The cities were chosen to have distinct cultural backgrounds and to feature an identifiable unremodeled historical urban street network; we picked: (a) London (United Kingdom), (b) Ahmedabad (India), (c) Xi’an (China), (d) Harar (Ethiopia), (e) Taroudant (Morocco), and (f) Levittown (Pennsylvania, United States). The boundary is either the innermost ring road (London), the city wall (Ahmedabad, Xi’an, Harar, Taroudant), or a consistent encircling series of connected roads (Levittown). The natural roads (see Fig. 1c) were joined with a consistent encircling series of connected roads (Levittown). For each skeleton, we generated one hundred natural road setups, and then we selected, among the setups with a relatively smooth RACFD for the valence of their junction-junction topological network, the one with the highest goodness-of-fit quantifier. Observed that for the first five urban street networks (a-e) the predicted natural road valence distribution (10a) is effectively a plausible hypothesis, since their goodness-of-fit quantifiers $p_r$ are greater than 0.1, while for the sixth one (f) it must be clearly rejected [51]. So, as expected, the first five are “natural” while the sixth is “artificial”.

**FIG. 3.** Relative Anti-Cumulative Frequency Distributions (RACFD) for five “natural” urban street networks (a-e) of cities with distinct cultural backgrounds and for an “artificial” urban street network (f) of a planned city: circles represent relative anti-cumulative frequencies for the valences of their respective road-road topological networks (see Fig. 1e); crosses represent relative anti-cumulative frequencies for the valences of their respective junction-junction topological networks (see Fig. 1g). The red fitted curves for the natural road statistics describe the Maximum Likelihood Estimates (MLE) for the discrete Pareto probability distribution (10a) estimated according to the state of the art [51, 56] (250 000 samples). The green fitted curves for the junction statistics show the Nonlinear Least-Squares Fittings (NLSF) for the nonstandard bell-shaped discrete probability distribution (10b) with $\nu_r$ and $2\lambda_r$ fixed to their respective MLE values; no MLE approach can be computationally envisaged for the time being. The MLE goodness-of-fit quantifier $p_r$ allows us to qualify the urban street networks as “natural” when it is greater than 0.1, otherwise as “artificial” [19–22, 51]; therefore, our choice of urban street networks is justified a posteriori. On the other hand, for now, the ad hoc NLSF data analysis prevents us from grossly rejecting the foreseen junction valence distribution (10b).
Our ad hoc crude data analysis appears promising in the sense that it forbids one from grossly rebutting the foreseen junction valence distribution (10b). Interestingly, the case studies reveal that the number of vital connections $v_j$ is negative, to wit that the associated generalized binomial combination number is smaller than one. We interpret this result as follows: the number of agent intraconnections for junctions is relatively much smaller than the one for natural roads.

IV. ALEXANDER’S IDEAS AS GUIDE

A. Retro-recapitulation

In summary, we can take for granted that our partial ignorance permits us to recognize a hierarchical Pareto-coherence among urban street networks. More precisely, within the framework of information physics [37–40], the emerging Pareto-coherence that characterizes self-organized (or “natural”) urban street networks [19–22] has not only been predicted but also shown to reveal the underlying Galoisian hierarchy that describes any of them, either planned or self-organized. The passage to the Pareto-coherence — organic by nature — from the Galoisian hierarchy — in essence algorithmic — occurs by imposing a logarithmic maximum-entropy constraint with complete ignorance as the initial knowledge condition [45–48].

Our partial knowledge hangs on the “passing through” partial-ordering that ties natural roads with junctions and on the “pairing” that typifies any social system. The former bijectively transforms urban street networks into Galois lattices whose algebraic structure, in turn, leads (modulo some natural algebraic constraints [37–40]) to a set of functional relations and equations meant to measure complexity; the latter furnishes a hint to figure out the two involved functional unknowns, namely the weight and the evaluation functions.

In the words of C. Alexander [29–31], the pre-passage part is “mechanical”; we have used Galoisian instead. The Formal Concept Analysis (FCA) algorithmic transformation [34, 36] is simply a prerequisite to apply information physics [37–40]. The hint was translated to a crude asymptotic binomial paired-agent model, which is compatible with the social machinery taking place “in Berkeley at the corner of Hearst and Euclid” in Ref. 30.

B. Alexander’s conjecture

Convinced that nature does not like trees, C. Alexander informally introduced the notion of “semilattice” [30]: whoever has seen their hand-representations is stuck by the resemblance between their line renderings and Hasse diagrams before they realize that the round ones swimmingly illustrate addition rule (1) (see Fig. 2). We believe that he intuitively grasped the idea of the partial-ordering relation reduction to Galois lattices — plainly apprehended and rigorously established earlier by Ø. Ore [58, 59] — along the concomitant algebraic structure [60].

Even so C. Alexander did not attempt to put numbers on “semilattices”, he nonetheless claimed that for “natural” cities their elements holistically arrange according to a “living” coherence: it is his legacy as urban architect. In the literature, it takes the form of straight lines on log-log plots of the natural road valence distribution; here, for urban street networks, it has been shown to emerge from Jaynes’s maximum entropy principle invoked with the first logarithmic moment as sole characterizing moment. Thus, in this work, we have established the statistical physics foundation for the “living” coherence occurring among “natural” cities, at least for their urban street networks; instead of “living” we have used Paretoian.

Adopting, as C. Alexander might have done, the more intuitive approach that interprets entropy as the average amount of surprisal [61], the Alexander’s conjecture becomes: “natural” cities evolve by maintaining their amount of surprisal constant on average. This conjecture applies to cities as a whole, from habitations to transportation.

C. Surprise

Besides giving an intuitive macroscopic physical content, stating Alexander’s conjecture in terms of surprisal implicitly gives to C. Alexander’s ideas a microscopic physical content. Surprisal (or surprise) $S_u = -\ln \Pr$ was introduced by M. Tribus as a measure that quantifies our astonishment and indecision when we face an arbitrary event [49, 61]. Along this line, Alexander’s conjecture expresses nothing but the conservation on average of the astonishment and indecision of dwellers when they perceive their own city. To draw an analogy from statistical physics, particles of an ideal gas conserve on average their motion, which is quantified in terms of kinematic momentum [45, 47]. So, from a statistical physics perspective, astonishment and indecision of dwellers of an Alexander city appears then to be for natural roads and junctions — and any other similar urban items — what motion is for particles of an ideal gas.

Carrying on the analogy between our system and an ideal gas as a parallel between a Paretoian system and a Gaussian system is relevant as well. The distribution of number of states would be a discrete Gaussian distribution instead of a discrete Paretoian distribution, for the elements of the Galois lattice, if Jaynes’s maximum entropy principle was invoked with the first and second moments rather than with the first logarithmic moment as characteristic moments. Then the nature of the underlying discrete Gaussian distribution might be almost preserved for both the natural road and the junction distributions provided that the numbers of vital connections are both equal to 1/2. We used the fact that the convolution of two discrete Gaussian distributions is almost a discrete
Gaussian distribution [62]. The noteworthy point is that the junction valence distribution would then appear similar to the natural road valence distribution. That is, a Gaussian physics would mainly dissolve the underlying Galois lattice of our system, while the Paretian physics typified by a binomial paired-agent model taken at the asymptotic limit [41]. We have recovered the discrete Pareto probability distribution widely observed for natural roads evolving in self-organized cities [20–23]. What is more interesting, however, is that we have also been able to foresee a nonstandard bell-shaped distribution with a Paretian tail for their junctions. Our statistical model for urban street networks appears fine enough to study urban macro behaviours.

Beyond urban street networks, we have argued that our paradigm reflects C. Alexander’s ideas on cities [30, 31]. From the viewpoint of statistical physics, the passage from Galoisian hierarchy to Paretian coherence looks like a missing piece of his ideas. This passage has given place to a concise eponymous conjecture expressed in terms of surprisal [61]. Surpirsal quantifies the astonishment and indecision of city-dwellers, which are for Paretian statistical physics of “natural” cities what motion is for Gaussian statistical physics of ideal gases [45, 47]. Ultimately we are facing a Galoisian Paretian statistical physics that challenges our “mechanical” and Gaussian ways of thinking [29–31, 63].

We have also shed a new light on how power law phenomena can emerge from complex systems that underlie a Galoisian hierarchy. Here urban street networks constitute an ideal toy model as they reduce to intuitive two-layer Galois lattices. In this regard we believe that Paretian networks are omnipresent in nature but also that neither their underlying partial-order and neither the logarithmic character of their statistics have been plainly taken into account.
[24] A. P. Masucci, D. Smith, A. Crooks, and M. Batty, Eur. Phys. J. B 71, 259 (2009), arXiv:0903.5440 [physics.data-an].

[25] B. Jiang and C. Liu, Int. J. Geogr. Inf. Sci. 23, 1119 (2009), arXiv:0709.1981 [physics.data-an].

[26] M. Rosvall, A. Trusina, P. Minnhagen, and K. Sneppen, Phys. Rev. Lett. 94, 028701 (2005), arXiv:cond-mat/0407054 [cond-mat.dis-nn].

[27] A. P. Masucci and C. Molinero, Eur. Phys. J. B 89, 53 (2016), arXiv:1509.01940 [physics.soc-ph].

[28] C. Molinero, R. Murcio, and E. Arcaut, Sci. Rep. 7, 4312 (2017), arXiv:1512.05659 [physics.soc-ph].

[29] B. Jiang, Physica A 463, 475 (2016), arXiv:1602.08939 [nlin.AO].

[30] C. Alexander, Arch. Forum 122, 58 (1965).

[31] C. Alexander, The Nature of Order: an Essay on the Art of Building and The Nature of the Universe (Center for Environmental Structure, Berkeley, 2002).

[32] N. A. Salingaros, Principles of Urban Structure, Design Science Planning, Vol. 4 (Techne Press, Amsterdam, 2005).

[33] R. H. Atkin, Mathematical Structure in Human Affairs (Heinemann Educational Books, London, 1974); Combinatorial Connectivities in Social Systems: An Application of Simplicial Complex Structures to the Study of Large Organizations, Interdisciplinary Systems Research, Vol. 34 (Springer-Verlag, Basel, 1977).

[34] Y.-S. Ho, Environ. Plan. B 9, 397 (1982).

[35] S. M. Macgill and T. Springer, Environ. Plan. B 14, 39 (1987).

[36] B. A. Davey and H. A. Priestley, Introduction to Lattices and Order; 2nd ed. (Cambridge University Press, Cambridge, 2002); G. Grätzer, Lattice Theory: Foundation, Birkhäuser Mathematics (Springer-Verlag, Basel, 2011).

[37] K. H. Knuth, AIP Conf. Proc. 1305, 3 (2011), arXiv:1009.5161 [math-ph].

[38] K. H. Knuth, AIP Conf. Proc. 1073, 35 (2008).

[39] K. H. Knuth, Neurocomputing 67, 245 (2005).

[40] K. H. Knuth, Contemp. Phys. 55, 12 (2014), arXiv:1310.1667 [quant-ph].

[41] Y. Dover, Physica A 334, 591 (2004), arXiv:cond-mat/0309383 [cond-mat.stat-mech].

[42] M. Milaković, A Statistical Equilibrium Model of Wealth Distribution, Computing in Economics and Finance 2001 214 (Society for Computational Economics, 2001).

[43] E. A. Noether, Nachr. v. d. Ges. d. Wiss. Göttingen 2, 235 (1918); E. A. Noether and M. A. Tavel, Transport Theory Statist. Phys. 1, 186 (1971), arXiv:physics/0503066 [physics.hist-ph].

[44] J. Aczél, B. Forte, and C. T. Ng, Adv. in Appl. Probab. 6, 131 (1974).

[45] E. T. Jaynes, Phys. Rev. 106, 620 (1957); Phys. Rev. 108, 171 (1957).

[46] H. K. Kesavan, “Jaynes’ maximum entropy principle,” in Encyclopedia of Optimization, edited by C. A. Floudas and P. M. Pardalos (Springer-Verlag, Boston, 2009) pp. 1779–1782, 2nd ed.

[47] J. N. Kapur and H. K. Kesavan, “Entropy optimization principles and their applications,” in Entropy and Energy Dissipation in Water Resources, Water Science and Technology Library, Vol. 9, edited by V. P. Singh and M. Fiorentino (Springer-Verlag, Dordrecht, 1992) pp. 3–20.

[48] E. T. Jaynes, “Where do we stand on maximum entropy?” in E. T. Jaynes: Papers on Probability, Statistics and Statistical Physics, Synthese Library, Vol. 158, edited by R. D. Rosenkrantz (Kluwer Academic Publishers, Dordrecht, 1989) pp. 211–314.

[49] Entropies and surprisals are expressed in nat units.

[50] H. V. D. Parunak, S. Braeckner, and R. Savit, in Proceedings of the Third International Joint Conference on Autonomous Agents and Multiagent Systems, AAMAS 2004, Vol. 2 (IEEE Computer Society, Washington, 2004) pp. 930–937.

[51] A. Clauset, C. R. Shalizi, and M. E. J. Newman, SIAM Rev. 51, 661 (2009), arXiv:0706.1062 [physics.data-an].

[52] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., NIST Hand Book of Mathematical Functions (Cambridge University Press, 2010).

[53] J. M. Borwein and K. Dilcher, Ramanujan J. 45, 413 (2018).

[54] M. E. J. Newman, Contemp. Phys. 46, 323 (2005), arXiv:cond-mat/0412004 [cond-mat.stat-mech].

[55] Schwitten stands for Self-Convoluted Hurwitz Witten.

[56] C. S. Gillespie, J. Stat. Softw. 64, 1 (2015), arXiv:1407.3492 [stat.CO].

[57] OpenStreetMap contributors, openstreetmap.org.

[58] O. Ore, Bull. Amer. Math. Soc. 48, 173 (1942).

[59] O. Ore, Trans. Amer. Math. Soc. 55, 493 (1944); Trans. Amer. Math. Soc. 56, 570 (1944), erratum.

[60] Alexander’s semilattices are not Galois lattices: the notion of semilattice illustrated in diagram B [30, col. 5] is not a Galois lattice since the elements 2 and 3 imply the two incomparable elements 123 and 234 — eliminating either 123 or 234 causes B to become a Galois lattice.

[61] M. Tribus, Thermostatics and Thermodynamics, University Series in Basic Engineering (Van Nostrand, Princeton, 1961).

[62] P. J. Szabowski, Statist. Probab. Lett. 52, 289 (2001).

[63] F. Sattin, Phys. Rev. E 68, 032102 (2003), arXiv:cond-mat/0212077 [cond-mat.stat-mech].