Aspects of the $D = 6$, $(2,0)$ Tensor Multiplet

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Abstract

Some aspects of the $D = 6$, $(2,0)$ tensor multiplet are discussed. Its formulation as an analytic superfield on a suitably defined superspace and its superconformal properties are reviewed. Powers of the field strength superfield define a series of superconformal fields which correspond to the KK multiplets of $D = 11$ supergravity on an $AdS_7 \times S^4$ background. Correlation functions of these operators are briefly discussed.
1 Introduction

The $D = 6, (2,0)$ tensor multiplet plays an important rôle in modern string theory through the fact that it is the world-volume multiplet of the M-theory 5-brane. In the context of the Maldacena conjecture \[1\] the near-horizon geometry of a large number of such branes is $AdS_7 \times S^4$, and $D = 11$ supergravity on such a background is supposed to be related to a conformal field theory of the boundary of $AdS_7$.\[2\] In the $AdS_5 \times S^5$ case for IIB supergravity, the boundary conformal field theory is $N = 4$ super Yang-Mills theory, in the large $N_c$ limit (where the gauge group is $SU(N_c)$), but in the $D = 6$ case the interacting conformal field theory is not known. Nevertheless some progress can be made assuming that there is such a superconformal field theory.

The $D = 6, (2,0)$ tensor multiplet was first written down in \[3\] and reformulated in an appropriate harmonic superspace in \[4\] where it was also shown how the superconformal fields which correspond to the KK multiplets of $D = 11$ supergravity can be easily constructed as powers of the underlying field strength superfield. This construction is very similar to the construction of the KK multiplets which was used for $N = 4$ Yang-Mills theory in \[6, 7\] following the realisation that the field strength tensor and the supercurrent can be written in a simple way in superspace \[5\].

In this paper we give more details of the results outlined in \[4\]; after discussing the multiplet in ordinary superspace\[1\], we reformulate it in harmonic superspace. We then go on to discuss its conformal properties and finally we make some remarks on the properties of correlation functions that can be deduced from symmetry properties alone.

2 The tensor multiplet

The $(2,0)$ tensor multiplet consists of five scalars, a set of four chiral fermions transforming under the four-dimensional representation of the internal symmetry group $USp(4)$ (symplectic Majorana-Weyl fermions) and a two-form gauge potential $A$ whose three-from field strength is self-dual. The coordinates of $D = 6, (n, 0)$ Minkowski superspace are $(x^a, \theta^{\alpha i})$ where $a = 0, \ldots, 5$ is a Lorentz vector index, $\alpha = 1, \ldots, 4$ is a chiral spinor index and $i = 1, \ldots, 2n$ is an internal symmetry index for the group $USp(2n)$. The spinors satisfy a reality condition of the form

$$\psi_{\alpha i} \rightarrow \bar{\psi}_i^\alpha = \eta^{ij} \psi_{\alpha j}$$

(1)

where the conjugation denoted by the bar includes multiplication by a matrix $B$ which converts dotted to undotted spinor indices and satisfies $BB = -1$, so that they have altogether $4n$ real components. The antisymmetric matrix $\eta^{ij}$ is the symplectic invariant and we use the convention that $\eta_{ij}$ has the same components as $\eta^{ij}$. The superspace covariant derivatives are $\partial_a$ and $D_{\alpha i}$ where

\[1\]More recent discussions of the multiplet in ordinary superspace can be found in \[8, 9\].
\[ [D_{\alpha i}, D_{\beta j}] = i\eta_{ij}(\gamma^a)_{\alpha\beta} \partial_a \] (2)

The (2,0) tensor multiplet is described by a superfield \(W_{ij}\) which is real, antisymmetric and traceless with respect to \(\eta^{ij}\). It obeys the constraint

\[ D_{\alpha i}W_{jk} = 2\eta_{ij}\lambda_{ak} - 2\eta_{ik}\lambda_{aj} + \eta_{jk}\lambda_{\alpha i} \] (3)

from which it follows that

\[ D_{\alpha i}\lambda_{\beta j} = -\frac{i}{2}(\gamma^a)_{\alpha\beta} \partial_a W_{ij} + \frac{i}{24}(\gamma^{abc})_{\alpha\beta} F_{abc} \] (4)

where \(F_{abc}\) is self-dual. By applying further \(D\)'s to these relations one finds that the leading components of \(W_{ij}\), \(\lambda_{\alpha i}\) and \(F_{abc}\) are the only independent spacetime fields and that they obey the equations of motion

\[ \partial^a \partial_a W_{ij} = (\gamma^a)^{\alpha\beta} \partial_a \lambda_{\beta i} = 0 \]
\[ \partial^a F_{abc} = \partial_{[a} F_{bcd]} = 0 \] (5)

The multiplet can also be described by a superspace three-form \(F = dA\) where \(A\) is a potential two-form. The non-vanishing components of \(F\) are

\[ F_{\alpha i\beta j c} = -i(\gamma_c)_{\alpha\beta} W_{ij} \]
\[ F_{ab\gamma k} = (\gamma_{ab})_{\gamma k} \] (6)

as well as \(F_{abc}\).

We note that there is also a (1,0) tensor multiplet described by a real scalar superfield \(W\) which is subject to a second-order constraint

\[ D_{[\alpha}^{[i} D_{\beta]}^{j]} W = 0 \] (7)

Its components are a scalar, a spinor \(\lambda_{\alpha i}\) and a self-dual field strength \(F_{abc}\). It can also be described by a (1,0) superspace three-form \(F\) whose components are the same as in (6) with \(W_{ij}\) replaced by \(\eta_{ij} W\). (In (1,0) \(\eta_{ij} = \epsilon_{ij}\).)

3 Harmonic superspace description

Harmonic superspaces are extensions of Minkowski superspace \(M\) of the form \(\hat{M} := M \times F\) where \(F\) is a coset space of the internal symmetry group which is usually chosen to be a
compact complex manifold [10]. Such superspaces can be thought of as applications of twistor geometrical techniques in a supersymmetric setting. The original idea of Penrose’s twistor theory was to encode information about spacetime in holomorphic data on projective twistor space [11]. However, in the case of Euclidean “spacetime” \( \mathbb{R}^4 \), twistor space can be thought of as an extension of spacetime by a two-sphere \( S^2 \sim \mathbb{C}P^1 \) [12]. The significance of this sphere is that it parametrises the self-dual complex structures on \( \mathbb{R}^4 \) (there is also an anti-self-dual set). One can then construct a natural complex structure on the total space by combining the complex structure on \( \mathbb{R}^4 \) parametrised by the given point of the sphere with the complex structure on the sphere itself. This idea can be generalised to superspace, but with complex structures replaced by (odd) CR structures which can be thought of as partial complex structures [13]. More formally, a CR structure is a subbundle of the complexified tangent bundle which has local bases of complex vector fields whose Lie brackets are fields of the same type. The prototype of an odd CR structure is provided by chirality in \( D = 4, N = 1 \) superspace and generalised chirality is referred to as Grassmann analyticity (G-analyticity). In the harmonic superspace context the space \( \mathcal{F} \) parametrises the odd, Lorentz covariant CR structures on Minkowski superspace, and on the total space \( \hat{M} \) there is a natural CR structure obtained by combining the odd CR structure specified by a given point of \( \mathcal{F} \) with the complex structure of \( \mathcal{F} \) itself.

In \( D = 6, (n, 0) \) superspace the problem is therefore to find subsets of the \( D \)’s which anticommute amongst themselves. If we demand that such subsets be Lorentz covariant then we are looking for derivatives \( D_{\alpha r} = u_{r i} D_{\alpha i} \) such that

\[
[D_{\alpha r}, D_{\beta s}] = 0
\]  

(8)

since in such a case it is consistent to look for superfields which are annihilated by \( D_{\alpha r} \). Clearly there will be a complementary set \( D_{\alpha r'} = u_{r' i} D_{\alpha i} \), and we should require that \( u_{t' i} = (u_{r' i}, u_{r'' i}) \) be an element of \( USp(2n) \). In order to have such odd CR structures we must have

\[
u_{r i} u_{s j} \eta_{ij} = 0
\]  

(9)

For \((1, 0)\) the only possibility is to choose \( r \) to have only one value, 1 say, so that the space of odd CR structures is again the two-sphere. In the \((2, 0)\) case a natural choice is to allow \( r \) to take two values, 1, 2, say, with \( r' = 3, 4 \). Then we can satisfy (8) above in this way if we choose the basis in which

\[
\eta_{rs} = \eta_{r's'} = 0
\]

\[
\eta_{rs'} = - \eta_{s'r} = \delta_{rs'}
\]  

(10)

The space of such odd CR structures is \( U(2)\backslash USp(4) \). Clearly we can generalise this particular type of structure to arbitrary \( n \) with associated coset space \( U(n)\backslash USp(2n) \).

There are other possibilities which do not halve the number of odd coordinates; for example, in \((2, 0)\) superspace we can consider the CR-structure specified by \( D_{\alpha 1} \) which again has isotropy
group $U(1) \times SU(2)$ (this is not the same group as that for $D_{\alpha_1}, D_{\alpha_2}$). Moreover, one can consider Lorentz covariant odd CR structures spanned by $p$ derivatives compatible with odd CR structures spanned by $q$ derivatives, $p < q$, and so on. The spaces associated with such sets of CR structures are clearly the spaces of nested subspaces of $\mathbb{C}^{2n}$ which are isotropic, i.e. for any two vectors $u, v$ in a given subspace one has $\eta_{ij} u^i v^j = 0$. These spaces are called isotropic flag manifolds and have been employed in the recent work of Ferrara and Sokatchev on conformal fields in six dimensions \cite{14}. In the following we shall only use spaces of the type $U(n) \setminus USp(2n)$.

The required harmonic superspaces are formed by adjoining these parameter spaces to Minkowski superspace, the idea being that constraints on Minkowski space superfields can be expressed more simply in terms of fields defined on the larger spaces. We therefore form the spaces $\hat{M} := M \times \mathbb{F}$ where $\mathbb{F} = U(n) \setminus USp(2n)$. Since $F$ is a homogeneous space it is often convenient to use the well-known group-theoretical technique of writing tensor fields on $F$ as equivariant maps from $USp(2n)$ to $V$ where $V$ is a representation space for the isotropy subgroup $U(n)$. Such a map $A(u)$ has the property that $A(hu) = R(h)A(u)$ where $h \in U(n)$ and $R(h)$ is the representation of $U(n)$ acting on $V$.

An element $u$ of $USp(2n)$ is both unitary and symplectic, so that

$$u_I^i u_J^j \eta_{ij} = \eta_{IJ}$$

and

$$\bar{u}_I^J = (u^{-1})_I^J := u_i^J$$

In our conventions the group $USp(2n)$ acts naturally to the right on $F$, and this right action is generated by the left-invariant vector fields. On the other hand, the left action of the group on itself is generated by the right-invariant vector fields which we denote by $\hat{D}_{IJ}$. These vector fields satisfy the following constraints

$$\hat{D}_{IJ} = \hat{D}_{JI}$$

$$\bar{\hat{D}}^{IJ} = \eta^{IK} \eta^{JL} \hat{D}_{KL}$$

The algebra of the right-invariant vector fields is

$$[\hat{D}_{IJ}, \hat{D}^{KL}] = -4 \delta_{(I}^{(K} \hat{D}_{J) L)}$$

and they act on $USp(2n)$ by

$$\hat{D}_{IJ} U_K^k = -\eta_{IK} u_J^k - \eta_{JL} u_I^k$$

The $\hat{D}_{IJ}$’s split into $\hat{D}_{rs}, \hat{D}_{rs'}$ and $\hat{D}_{rs''}$, the latter set corresponding to the isotropy algebra $u(n)$ while the former two sets correspond to the coset. We now define
\[ D_o = \eta^{rs'} \hat{D}_{rs'} \]
\[ D_{rs'} = \hat{D}_{rs'} - \frac{1}{n} \eta_{rs'} D_o \]

and drop the hats on \( D_{rs} \) and \( D_{r's'} \). \( D_o \) is the \( u(1) \) derivative while the operators \( D_{rs} \) span the \( su(n) \) part of the isotropy algebra. The operators \( D_{rs} \) transform as a symmetric second-rank tensor under \( SU(n) \) and can be regarded as the components of the \( \hat{D} \) operator on the coset, while the operators \( D_{r's'} \) are the conjugate set. With these conventions \( \nu_r^i \) and \( \nu_{r'}^i \) have \( U(1) \) charges +1 and −1 respectively.

The CR structure on the whole space \( \hat{M} = M \times U(n) \setminus USp(2n) \) is spanned by \( D_{ar} \) and \( D_{rs} \), with involutivity being guaranteed by virtue of the fact that each derivative of this set (anti)-commutes with any other in the set.

To complete the picture we need the notion of a real structure. In the \((1, 0)\) case we define a conjugation by combining the antipodal map on the two-sphere with complex conjugation. For an equivariant function \( A \) on \( SU(2) \) we define

\[ A(u) \mapsto \tilde{A}(u) = \overline{A(\epsilon u)} \]  \hspace{1cm} (17)

where \( \epsilon \) is the usual \( \epsilon \)-matrix. Fields which have an even \( U(1) \) charge can be real under this operation, which simply means that such a field contains real representations of \( SU(2) \) in a harmonic expansion.

In the \((n, 0)\) case there is a very similar operation defined on equivariant fields by

\[ A(u) \mapsto \tilde{A}(u) = \overline{A(\eta u)} \]  \hspace{1cm} (18)

Again a field can be real under this conjugation if it has even \( U(1) \) charge. Explicitly, \( \tilde{u} := \overline{\eta u} \) is given by

\[
\begin{align*}
(\eta_{r'}^i) &= \tilde{u}_{r'}^i = (u^{-1})_{r'}^i = \eta_{s'}^j u_{si} \\
(\eta_{r'}^i) &= -\tilde{u}_{r'}^i = -(u^{-1})_{r'}^i = -\eta_{s'}^j u_{s'i}
\end{align*}
\]  \hspace{1cm} (19)

We shall now apply this formalism to the \((2, 0)\) tensor multiplet. We define

\[ W := \frac{1}{2} \epsilon^{rs} u_r^i u_s^j W_{ij} \]  \hspace{1cm} (20)

We claim that \( W \) is analytic. Since \( D_{rs} u_t^i = 0 \), \( W \) is clearly analytic on \( F \). Grassmann analyticity follows because

\[
D_{ar} W = \frac{1}{2} \epsilon^{st} u_s^j u_t^k u_r^i D_{ai} W_{jk} = \epsilon^{st} u_s^j u_t^k u_r^i \left( \eta_{ij} \lambda_{ak} - \eta_{ik} \lambda_{aj} - \frac{1}{2} \eta_{jk} \lambda_{ai} \right)
\]  \hspace{1cm} (21)
Since \( u^i u^j \eta_{ij} = 0 \) we deduce immediately that \( D_{\alpha i} W = 0 \). We note as well that the reality of \( W_{ij} \) ensures that \( W \) is real under the conjugation defined above.

The converse of this result is also true, namely, a real analytic field \( W \) of charge 2 defined on \( \hat{M} \) can be expressed in the form of (20) where the Minkowski superspace field \( W_{ij} \) obeys the constraints of the \((2,0)\) tensor multiplet discussed previously. The first part of this comes from the Bott-Borel-Weil theorem which, applied to this case, states that such an analytic field \( W \) must have a short expansion of the form of (20). Note that this implies that \( W_{ij} \) is symplectic traceless, since the trace is automatically absent in (20). Reality then implies that \( W_{ij} \) is real, and G-analyticity implies that it must satisfy the constraint (3). This is because \( D_{\alpha i} W_{jk} \) will in general decompose into a 4 and a 16 under \( USp(4) \), but the latter will not give zero if substituted into the right-hand side of (21).

It is obvious that we can use \( W \) to generate a family of superfields \( A_p := W^p \) which are analytic on \( \hat{M} \) and which have \( U(1) \) charge \( 2p \) [4]. The field \( A_2 \) is particularly important as it is the supercurrent, which in Minkowski superspace has the form \( W_{ij} W_{kl} |_{14} \) [3]. The fact that only the 14-dimensional representation of \( W \times W \) appears in the product is due to the properties of the \( u'\)s. Indeed, in \( SO(5) \) notation \( W \) transforms as a vector and the powers of \( W \) that appear in the series \( A_p \) fall into the symmetric traceless representations of \( SO(5) \). This is very similar to the \( N = 4 \) Yang-Mills case, where the leading components of the KK multiplets fall into symmetric traceless representations of \( SO(6) \). To show that these composite fields are superconformal we now turn to a discussion of superconformal transformations in six dimensions.

4 Superconformal transformations in Minkowski superspace

Superconformal transformations in \( D = 6, (n,0) \) Minkowski superspaces for \( n = 1,2 \) have been discussed in refs [15, 9]. The situation is rather similar to four dimensions (see [13] for a review). An infinitesimal superconformal transformation of Minkowski superspace can be defined to be an infinitesimal diffeomorphism which preserves the odd fermionic tangent bundle spanned by the vector fields \( D_{\alpha i} \). Dually, it can be defined to be an infinitesimal diffeomorphism which preserves the even cotangent bundle spanned by the one-forms \( E^a := dx^a - \frac{i}{2} d \theta^a (\gamma^a)_{\alpha \beta} \eta_{ij} \theta^j \).

Let \( V \) be the vector field which generates such a diffeomorphism, then we have

\[
< D_{\alpha i}, \mathcal{L}_V E^a > = 0 \tag{22}
\]

where \(<,>\) denotes the standard pairing between one-forms and vector fields. If

\[
V = F^a \partial_a + \varphi^{\alpha i} D_{\alpha i} \tag{23}
\]

we find that (22) implies that

\[
D_{\alpha i} F^a + i (\gamma^a)_{\alpha \beta} \varphi^\beta_i = 0 \tag{24}
\]
from which

$$\varphi^{\alpha i} = \frac{i}{6}(\gamma_a)^{\alpha \beta} D^b \phi^a$$

(25)

Thus a superconformal transformation is determined by a function $F^a$ obeying the constraint

$$D_{\alpha i} F^a = \frac{1}{6}(\gamma^a\gamma^b)_{\alpha \beta} D_{\beta i} F^b$$

(26)

It is not difficult to show that $F^a$ satisfying (26) contains precisely the parameters of the superconformal algebra. Indeed, differentiating (24) again one finds

$$D_{\alpha i} \varphi^{\beta j} = \delta_{\alpha}^{\beta} (f_{ij} + \frac{1}{4} \eta_{ij} \partial_a F^a) = \eta_{ij} (\gamma_a\gamma_b)_{\alpha \beta} L_{ab}$$

(27)

where $f_{ij} = f_{ji}$ and $L_{ab} = \partial_a F_b$. One also finds the conformal Killing equation

$$\partial_a F_b + \partial_b F_a = \frac{1}{3} \eta_{ab} \partial_c F_c$$

(28)

as well as the conformal Killing spinor equation

$$\gamma_a \partial_b \varphi_i + \gamma_b \partial_a \varphi_i = \frac{1}{3} \eta_{ab} \gamma^c \partial_c \varphi_i$$

(29)

If we define the components of $F$ in a $\theta$-expansion to be $\tilde{F}^a := F^a|, \tilde{\varphi}^{\alpha i} := \varphi^{\alpha i}|$ and $\tilde{f}_{ij} := f_{ij}|$, where the vertical bar denotes evaluation of a superfield at $\theta = 0$, then $\tilde{F}^a$ is a conformal Killing vector on ordinary Minkowski space, $\tilde{\varphi}^{\alpha i}$ is a conformal Killing spinor which contains the $Q$ and $S$ supersymmetry parameters and $\tilde{f}_{ij}$, which is constant in $x$, is the $USp(2n)$ parameter.

It is easy to see that the commutator of two vector fields satisfying (22) is a vector field of the same type so that the superconformal algebra can be expressed in terms of $F^a$ in the form

$$[F,F'] = F''$$

(30)

where

$$F''^a = F \cdot \partial F^a - F' \cdot \partial F^a - i \varphi^{\alpha i} (\gamma_a)^{\alpha \beta} \tilde{f}_{ij}$$

(31)

is easily checked to satisfy (24) provided that $F$ and $F'$ do.

5 Superconformal transformations on analytic fields

The superconformal algebra is represented on $M \times USp(2n)$ by vector fields $V$ of the form

$$V = V_0 + V_u$$

(32)
where $V_o$ is a superconformal Killing vector on $M$ and

$$V_u = -\frac{1}{2} f^{IJ} \hat{D}_{IJ}$$  \hspace{1cm} (33)$$

with

$$f^{IJ} = D^{(I}_{\alpha} \varphi^{\alpha J)}$$  \hspace{1cm} (34)$$

In addition we use the standard convention that $USp(2n)$ indices are converted from lower case to upper case by means of $u$.

This vector field has the following properties: it defines a representation of the superconformal algebra; it commutes with the right-invariant vector fields on $USp(2n)$, and it preserves CR-analyticity modulo isotropy group derivatives. Thus Grassmann analyticity is preserved for H-analytic fields because

$$[D_{ar}, V] \sim D_{ar}, D_{rs}$$  \hspace{1cm} (35)$$

modulo isotropy group derivatives. These isotropy group terms have the effect that a transformation of the above type, acting on a G-analytic superfield $A$, say, with $U(1)$ charge $p$, does not preserve G-analyticity so that a correction term is required. The only possibility is to include a term proportional to $\partial \cdot F$, since this function is independent of $u$ and hence harmonic analytic. The required transformation law is

$$\delta A = VA + \frac{p}{3n} (\partial \cdot F) A$$  \hspace{1cm} (36)$$

It can be checked that this rule does preserve G-analyticity and harmonic analyticity and that it does define a representation because

$$V_F(\partial \cdot F') - V_{F'}(\partial \cdot F) = \partial \cdot [F, F']$$  \hspace{1cm} (37)$$

Equivalently we can write

$$\delta A = VA + \frac{p}{n(3 - n)} \Delta A$$  \hspace{1cm} (38)$$

where

$$V = F \cdot \partial + \varphi^{\alpha r'} D_{\alpha r'} - \frac{1}{2} f^{r's'} D_{r's'}$$  \hspace{1cm} (39)$$

and

8
\[ \Delta = \partial \cdot F - D_{\alpha r} \varphi^{\alpha r} - \frac{1}{2} D_{r's'} f^{r's'} \]
\[ = (3 - n) \left( \frac{1}{3} \partial \cdot F + \eta^{r's'} f_{r's'} \right) \]  

(40)

The vector field \( \mathcal{V} \) also gives a representation of the superconformal algebra and has the property that it preserves G-analyticity and CR analyticity up to isotropy group terms. The function \( \Delta \) is the divergence of \( \mathcal{V} \). Note that there is no singularity for \( n = 3 \) as the two \((3 - n)\) factors cancel from \( \Delta \) and the transformation rule.

For the \((2,0)\) tensor multiplet field strength \( W \) the \( U(1) \) charge is 2 and hence we expect that

\[ \delta W = \mathcal{V} W + \frac{1}{3} \partial \cdot F W = \mathcal{V} W + \Delta W \]  

(41)

To verify that this is indeed the case one starts from the transformation rule of the three-form \( F \) which is

\[ \delta F = \mathcal{L}_\mathcal{V} F \]  

(42)

Since

\[ F_{\alpha r \beta s c} = -i (\gamma_c)_{\alpha \beta} \epsilon_{r s} W \]  

(43)

we can find the transformation rule for \( W \) by examining this component of (42). It is straightforward to verify (41) by this means.

6 Correlation functions

In this section we consider the constraints which correlation functions of analytic operators in \( D = 6, (2,0) \) should satisfy. Even though we do not know the interacting theory it is still possible to examine the Ward Identities of the theory. The technique to be used here is the same one that has been employed to study the correlation functions of analytic operators in \( D = 4 \) superconformal field theories in a series of papers [6, 17, 18, 19, 20], although we are not able, in this case, to perform checks in harmonic superspace perturbation theory. Indeed, even though one can present the \((2,0)\) tensor multiplet in \((1,0)\) superspace as a tensor multiplet together with a hypermultiplet, the \((1,0)\) tensor multiplet does not admit an off-shell superfield formulation due to self-duality so that there there is no \((1,0)\) superspace action even for the free theory. There have been several studies of 3-point functions in the literature, mainly for scalars \([21]\) or energy-momentum tensors \([22]\), including some discussion of anomalies.

We denote an \( N \)-point correlation function of analytic operators with charges \( p_1, \ldots p_n \) by \(< p_1 \ldots p_N >\). The Ward identity for such a correlation function reads
\[
\sum_{i=1}^{N} \left( \mathcal{V}_i + \frac{1}{3} p_i \Delta_i \right) < p_1 \ldots p_N > = 0 \quad (44)
\]

where we assume that all the points are separated.

The basic building blocks in the analysis of the Ward identities are the propagator, which is the two-point function of two free \( W \)'s, and the invariants. The propagator can be written

\[
<W(1)W(2)> \sim g_{12} = \frac{(12)}{x_{12}} \quad (45)
\]

where

\[
(12) := u^{ij}(1)u_{ij}(2) \quad (46)
\]

with

\[
u^{ij} := \frac{1}{2} \epsilon^{rs} u_r^i u_s^j \quad (47)
\]

The spacetime variable \( x \) is modified in order to satisfy Grassmann analyticity, and \( x_{12} = x_1 - x_2 \) denotes the difference as usual. The analytic \( x \) is related to the standard spacetime coordinate \( x_S \) by

\[
x^a = x_s^a - \frac{i}{2} \theta^{\alpha \beta} (\gamma^a)_{\alpha \beta} \theta^{\beta s'} \eta_{rs'} \quad (48)
\]

Given the propagator it is straightforward to write down the two- and three-point functions. They are

\[
<p_1p_2> \sim \delta_{p_1p_2} (g_{12})^{p_1} \quad (49)
\]

and

\[
<p_1p_2p_3> \sim (g_{12})^{p_{12}} (g_{23})^{p_{23}} (g_{12})^{p_{23}} \quad (50)
\]

where

\[
p_{ij} := \frac{1}{2} p_i + p_j - p_k, \quad k \neq i, j \quad (51)
\]

This formula for the three-point functions is rather similar to the one for \( D = 4, N = 4 \) Yang-Mills theory [17]. Note that it gives the functional form of all of the three-point functions for all of the component operators which are related by AdS/CFT to the KK multiplets of \( D = 11 \) supergravity on \( AdS_7 \times S^4 \).
For more than three points, using standard conformal field theory techniques, we can write the correlation functions, or at least those which do not vanish at leading order, as prefactors times functions of invariants, where the prefactors, which are constructed from the propagators, are solutions in the free theory. For a given correlator the prefactor will satisfy the full Ward identity (44), but in general will not be unique. We thus have, schematically,

\[ < p_1 \ldots p_N > \sim P \times F \]  

where \( F \) is a function of invariants.

It would be interesting to study, for example, four-point functions using similar techniques to those of [19] to see whether there can be non-trivial correlation functions even though one does not know the interacting theory. We can make a few speculations about certain special correlators, however, for which the charges are chosen in a special way.

We recall that in four dimensional SCFT there are certain extremal and next-to-extremal correlators whose functional form is free and which obey non-renormalisation theorems [23, 24, 25, 26, 27, 28, 20]. It has recently been proposed that similar simplifications should occur in the present context [29]. As in four dimensions the prefactors are uniquely determined for extremal correlators where the charge at point 1, say, is equal to the sum of the charges at the other points, \( p_1 = \sum_{i=2}^{N} p_i \). The prefactor, \( P \), is

\[ P = \prod_{i=2}^{N} (g_{1i})^{p_i} \]  

In the analysis for such correlators in four dimensions in analytic superspace [18, 20] it was found to be helpful to use the reduction formula [30] which relates \( N \)-point functions, differentiated with respect to the coupling, to \((N + 1)\)-point functions with one integrated insertion of the supercurrent. This trick is not available here, but we can still attempt to make a straightforward analysis for four points. There are no nilpotent invariants for four points, essentially because one has enough supersymmetries, 32, to transform away all of the odd coordinates, so that the invariants are the supersymmetric completions of the zeroth-order invariants. There are two independent spacetime cross-ratios and two independent internal cross-ratios and the supersymmetric completions of all of these have singularities in the internal space which occur in, say, \( y_{13} \) and \( y_{24} \) together, where we use \( y \) to denote the complex coordinates of the internal space.\footnote{The internal space can be thought of as complexified compactified 3-dimensional Minkowski space.} It is difficult to see how such singularities can be cancelled by the prefactors for extremal correlators in which each factor involves point 1. Although this is not a rigorous argument, it is very suggestive, and lends support to the conjecture recently made in [29].

Another way of looking at this problem would be to work in \((1, 0)\) harmonic superspace and study the simpler problem of extremal four-point functions of hypermultiplet composites. The internal space in this case is the same as in \( D = 4, N = 2 \) harmonic superspace, i.e. \( \mathbb{C}P^1 \), so that the analysis resembles that given for extremal correlators in \( D = 4, N = 2 \) given in [18].
For more than four points there will be nilpotent invariants exactly as in the four-dimensional
case. These invariants are rather complicated, but they are all singular. It is therefore very
plausible that $n$-point extremal correlators are also free [29]. It should be possible to make these
arguments using invariants rigorous by studying the singularities explicitly; one might also hope
to find strong constraints on next-to-extremal correlators as in four dimensions [18, 20]. This
has also been proposed in ref [29].

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