Josephson effect with superfluid fermions in the two-dimensional BCS-BEC crossover

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We investigate the macroscopic quantum tunneling of fermionic superfluids in the two-dimensional BCS-BEC crossover by using an effective tunneling energy which explicitly depends on the condensate fraction and the chemical potential of the system. We compare the mean-field effective tunneling energy with the beyond-mean-field one finding that the mean-field tunneling energy is not reliable in the BEC regime of the crossover. Then we solve the Josephson equations of the population imbalance and the relative phase calculating the frequency of tunneling oscillation both in the linear regime and in the nonlinear one. Our results show that the Josephson frequency is larger in the intermediate regime of the BCS-BEC crossover due to the peculiar behavior of the effective tunneling energy in the crossover.

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I. INTRODUCTION

Recent developments in confinement, cooling, and control of interaction with alkali-metal atoms have renewed the interest on the BCS-BEC crossover. This crossover from the Bardeen-Cooper-Schrieffer (BCS) state to the Bose-Einstein condensate (BEC) has been observed in two-hyperfine-components Fermi vапours of $^{40}$K atoms and $^{40}$Li atoms [5–7] with the use of Fano-Feshbach resonances [8]. Quite remarkably, the Josephson effect [9], i.e. the macroscopic quantum tunneling of Cooper pairs [10], has been experimentally investigated in the BEC-BCS crossover [11] with a three-dimensional (3D) configuration of neutral ultracold atoms. Moreover, also quasi two-dimensional (2D) ultracold Fermi gases have been realized [12] and studied in the 2D BCS-BEC crossover [13–15].

In this paper we investigate theoretically the Josephson effect in the 2D BCS-BEC crossover by taking into account the crucial role of the condensate fraction and the chemical potential on the effective tunneling energy [16]. To obtain these quantities we solve the extended-BCS gap and number equations at zero temperature in the full 2D crossover. We analyze the effects of quantum fluctuations, comparing the results obtained applying mean-field and beyond mean-field approximations to the pairing field. Having a reliable beyond-mean-field effective tunneling energy, we solve the atomic Josephson junction equations for the population imbalance and the relative phase reported in Ref. [18] and adapted to the 2D case. In this way we obtain immediately the critical current of the direct-current Josephson effect, but also the oscillation frequency of the alternate-current Josephson effect, as a function of the interaction strength of the 2D BCS-BEC crossover. We also study the oscillation frequency of macroscopic quantum tunneling in the nonlinear regime finding that is higher with respect to the one of the linear regime.

II. 2D BCS-BEC CROSSOVER

We consider a 2D attractive Fermi gas of ultracold and dilute two-spin component neutral atoms. The Mermin-Wagner theorem [19, 20] says that for a system with spatial dimension $d = 2$ there cannot be spontaneous symmetry breaking at finite temperature. In other words, there is a Bose-Einstein condensate only at zero temperature. Nonetheless, a 2D system can exhibit superfluidity both at zero and finite temperature: below the Berezinskii-Kosterlitz-Thouless (BKT) critical temperature $T_{BKT}$ [21–22], there is superfluidity. The BKT phase transition is a topological phase transition with a jump of the superfluid density at $T_{BKT}$. As we shall see, in our 2D fermionic superfluid quantum and thermal fluctuations play a crucial role to describe accurately various properties of the system [23].

The Hamiltonian density of a fermionic system made of ultracold and dilute alkali-metal atoms is given by

$$\hat{H} = \Psi_\sigma(\vec{r}, t) \left[ -\frac{\nabla^2}{2m} - \mu \right] \Psi_\sigma(\vec{r}, t) - g \Psi_\uparrow(\vec{r}, t) \Psi_\downarrow(\vec{r}, t) \Psi_\downarrow(\vec{r}, t) \Psi_\uparrow(\vec{r}, t) \right]$$

where the $\Psi_\sigma(\vec{r}, t)$ and $\Psi_\sigma(\vec{r}, t)$ are the complex Grassmann fields with spins $\sigma = \uparrow, \downarrow$. This Hamiltonian density described a system with single-channel interaction where $g > 0$ is the strength of the s-wave inter-atomic coupling. Contrary to the 3D case, in 2D attractive interatomic potentials exists a bound state for any value of the interaction strength $g$ [24–25]. In this way, it is possible to define the binding energy $\epsilon_B$, the energy that keeps the molecules together, in all the crossover. This latter can be written in terms of 2D fermionic scattering length $a_{2D}$ [26] as

$$\epsilon_B = \frac{4 \hbar^2}{\epsilon^2 m a^2_{2D}}$$

(2)
where $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant. The binding energy is conceptually more appealing than the scattering length, so in 2D case the crossover is mapped using the ratio $\epsilon_B/\epsilon_F$, where $\epsilon_F = \hbar^2\pi n/m$ is the 2D Fermi energy of two-spin-component non-interacting fermions with total number density $n$. It is possible to move from a BCS state of weakly-bound Cooper pairs to a BEC state of strongly-bound Cooper pairs by increasing $\epsilon_B$. The interaction strength $\epsilon_B$ by the expression

$$\frac{1}{g} = \frac{1}{L^2} \sum \frac{1}{\epsilon_k + \frac{\mu}{2}}$$  \hspace{1cm} (3)$$

where $\epsilon_k = \hbar^2 k^2 / 2m$ is the single-particle energy.

Through the Hubbard-Stratonovich transformation, the Hamiltonian density (1) can be rewritten introducing the bosonic complex field $\Delta(x)$ [27]. The Hamiltonian density then becomes

$$H = \Psi^\dagger \left[ -\frac{\nabla^2}{2m} - \mu \right] \Psi + \frac{|\Delta|^2}{g} - \nabla \Psi^\dagger \Psi - \nabla \Psi \Psi^\dagger$$  \hspace{1cm} (4)$$

Here we adopt a path integral approach [28] and we introduce the action $S$ and the partition function $Z$ at temperature $T$ of the system:

$$S = \int_0^\beta d\tau \int d\vec{r} \left[ \nabla \Psi \Psi^\dagger + H \right]$$

$$Z = \int \mathcal{D}[\Delta, \bar{\Delta}] / \mathcal{D}[\Psi, \bar{\Psi}] \exp(-S[\Delta, \bar{\Delta}, \Psi, \bar{\Psi}])$$  \hspace{1cm} (5)$$

where $\beta = 1/k_B T$ with $k_B$ Boltzmann constant. As we can see the density Hamiltonian (4) is quadratic with respect to $\Psi$, so it is possible to integrate over it obtaining the effective action $S_{eff}[\Delta(\vec{r},t), \bar{\Delta}(\vec{r},t)]$. We want to study the effects of Gaussian fluctuation of the gap field $\Delta(\vec{r},t)$ and so we set:

$$\Delta(x) = \Delta_0 + \eta(x)$$  \hspace{1cm} (6)$$

where $\eta(x)$ is the complex pairing field of bosonic fluctuations [11][29].

First of all we investigate the mean-field approximation imposing $\Delta(x) = \Delta_0$, where $\Delta_0$ is a real and spatial independent value. We replace it in (5), (6) and obtaining the mean-field effective action $S_{mf}$ and partition function

$$Z_{mf} = \exp \left[ -S_{mf} / \hbar \right] = \exp \left[ \beta \Omega_{mf} \right]$$  \hspace{1cm} (7)$$

where $\beta = 1/(k_B T)$ and $\Omega_{mf}$ is the mean-field grand potential. At zero temperature ($T = 0$, i.e. $\beta \rightarrow +\infty$) the 2D $\Omega_{mf}$ reads

$$\Omega_{mf} = -\sum_k (E_k + \epsilon_k + \mu) - L^2 \frac{\Delta_0^2}{g}$$  \hspace{1cm} (8)$$

where $E_k = \sqrt{(\epsilon_k - \mu)^2 + \Delta_0^2}$.

To describe the BCS-BEC crossover we solve the mean-field gap and number equation of the system. The gap equation is calculated by applying the saddle-point condition

$$\frac{\partial \Omega_{mf}}{\partial \Delta_0} = 0$$  \hspace{1cm} (9)$$

In this way, one obtains the familiar BCS gap equation

$$-\frac{1}{g} = \frac{1}{V} \sum k \frac{1}{2E_k}$$  \hspace{1cm} (10)$$

which can be combined with Eq. (3) to remove ultraviolet divergences and to get the energy gap $\Delta_0$ and the chemical potential $\mu$ as a function of the binding energy $\epsilon_B$.

The mean-field number equation is instead derived by using the thermodynamic relation

$$n = -\frac{\partial \Omega_{mf}}{\partial \mu}$$  \hspace{1cm} (11)$$

from which one obtains an implicit formula between the chemical potential $\mu$ and the number density $n$.

By considering the fluctuation field $\eta(\vec{r},t)$ we find a new effective action, $S_{eff} = S_{mg}[\Delta_0] + S_{g}[\eta, \bar{\eta}]$, from which is possible to obtain the 2D grand potential $\Omega$ in the beyond mean-field approximation

$$\Omega = \Omega_{mf} + \Omega_\eta = \Omega_{mf} + \frac{1}{2\beta} \sum_q \ln det(\mathcal{M}(q))$$  \hspace{1cm} (12)$$

where $\mathcal{M}(q)$ is the inverse pair fluctuation propagator, reported in the supplement material of Ref. [23]. It’s important to note that the gap equation is independent on the approximation used. The relation (11) is valid also in the beyond mean-field case. Instead the number equation depends on the approximation. Replacing the relation (13) in the Eq.(12) one obtains the beyond mean-field number equation

$$n = -\frac{\partial \Omega}{\partial \mu}$$  \hspace{1cm} (14)$$

The procedure of the the numerical calculations at the Gaussian (one-loop) level is detailed in Refs. [2][30]. Solving the gap equation (11) and the beyond-mean-field number equation (14) it is then possible to obtain the behavior of the single-particle chemical potential $\mu$ and energy gap $\Delta_0$ along the crossover.

In the panel a) of Fig. 1 we report the scaled chemical potential $\tilde{\rho}_B = \mu_B / \epsilon_F$, where $\mu_B = 2\mu + \epsilon_B$ is the chemical potential of composite boson. In the mean-field case, $\mu_B$ is constant and equal to $2\epsilon_F$ in all the crossover. Gaussian fluctuations introduce a nontrivial crossover dependence. The effects are more relevant in the strong coupling limit (BEC regime) than in the weak coupling limit (BCS regime), like in the three-dimensional case.
From BCS to BEC limit the Cooper pair size decreases until it becomes smaller than the average distance between the particles; at short distance fluctuations cannot be neglected. We have to go beyond the mean-field approximation to correctly describe the BCS-BEC crossover phenomenon. Another useful interesting quantity to study is the condensate density \( n_0 \), i.e. the density of the Cooper pairs with a vanishing center-of-mass linear momentum. We use Eq. (21) reported in the work [33] where the mean-field number of condensates couples \( N_0 \) is evaluated as the largest eigenvalue of the two-body density matrix, written through the Bogoliubov representation of the field operator \( \psi \)

\[
N_0 = \int d^3r_1 d^3r_2 |\Psi_L(r_1)\Psi_R(r_2)|^2
\]

The condensate fraction \( \lambda_0 \) of the fermionic system is then given by

\[
\lambda_0 = \frac{n_0}{n} = \frac{1}{2} \frac{\mu}{\Delta_0} + \frac{\mu}{\Delta_0} + \frac{\Delta_0}{\mu + \Delta_0}
\]

where \( n_0 \) is the condensate density of Cooper pairs and \( n \) is the total fermionic density. Notice that here \( n_0 \) is defined such that \( n_0 \rightarrow n \) in the deep BEC regime. In Ref. [33] Fukushima et al. calculated the effects of gaussian fluctuations on the condensate fraction for a 3D BCS-BEC crossover concluding that the Gaussian correction is quite small and it can be neglected in the full crossover. Here we evaluate the beyond-mean-field condensate fraction replacing in (16) the mean-field chemical potential and gap energy with their beyond mean-field counterparts [3]. As we can see from the panel b) of Fig. 1 in the deep BCS and BEC regime the two approximations give the same result, that is however quite obvious because the condensate fraction always starts from 0 and goes to 1. The main difference is found near the crossover point \( \epsilon_B = \epsilon_F \).

We look for the \( \epsilon_B/\epsilon_F \) value over which the system is considered in the BEC regime. A Bose-Einstein condensate is characterized by a macroscopic number of particles in the ground-state energy. Following Ref. [34] we choose to consider the system in the strong-coupling BEC regime if the 80% of the Cooper pairs are condensed. In the beyond mean-field approximation, the condensate fraction reaches the value of 0.8 at the point \( \ln(\epsilon_B/\epsilon_F) = -0.11 \). We note that at \( \ln(\epsilon_B/\epsilon_F) = -0.16 \) the single-particle chemical potential \( \mu \) becomes negative. Considering the assumptions made to obtain the beyond mean-field \( n_0/n \), we can also say that the system enters the BEC regime when the single-particle chemical potential \( \mu \) becomes negative.

### III. JOSEPHSON EFFECT

The Josephson effect was introduced by Brian Josephson in 1962 [9]. He predicted that between two superconductors separated by an insulating layer a supercurrent, made of Cooper pairs, can flows. The same hypothesis can be applied also to neutral superfluid fermionic atoms replacing the insulating layer by a potential barrier created with a laser beam. Here we investigate the problem of a double-well potential and we consider the Fermi gas of \( N \) atoms with two equally populated spin components and attractive inter-atomic strength at zero temperature. The system can be schematized considering two reservoirs, a left (L) one and a right (R) one, which contain superfluid particles with \( A/2 \) the area of each 2D reservoirs. Following the procedure of Ref.[13], it is possible to describe the time evolution of the system by introducing the two-state phenomenological model

\[
i\hbar \frac{\partial}{\partial t} \Psi_L(t) = E_L \Psi_L(t) - K \Psi_R(t)
\]

(17)

\[
i\hbar \frac{\partial}{\partial t} \Psi_R(t) = E_R \Psi_R(t) - K \Psi_L(t)
\]

(18)

where \( \Psi_\alpha(t) \) is the time-dependent macroscopic wavefunction in the 2D reservoir \( \alpha = L, R \) and \( E_\alpha = \)
$2\mu(2|\Psi_n(t)|^2/A)$ is the onsite energy of that reservoir. The hopping term \( K \) describes the effective tunneling energy between the two regions. It is quite common to use a phenomenological tunneling energy to describe Josephson effect in superconductors because a microscopic analytic derivation of \( K \), valid in all the crossover, is not yet available. Under the assumption that the potential barrier keeps the particles between the two reservoirs weakly-linked, we can write

\[
\Psi_n(t) = \sqrt{N_n(t)/2} \exp(i\theta_n(t))
\]

where \( N_n(t) \) and \( \theta_n(t) \) are the number of fermions and the superfluid phase in the reservoir \( n \) at the time \( t \). To describe the Josephson effect we introduce the relative phase

\[
\Phi(t) = \theta_R(t) - \theta_L(t)
\]

and the adimensional population imbalance

\[
z(t) = \frac{N_L(t) - N_R(t)}{N}
\]

Replacing (19), (20) and (21) in (17), (18) one obtains

\[
\dot{z} = -\frac{2K}{\hbar} \sqrt{1 - z^2} \sin(\phi)
\]

\[
\dot{\phi} = \frac{2}{\hbar} \left[ \mu_1 + \mu_2 \right] + \frac{2K}{\hbar} \sqrt{1 - z^2} \cos(\phi)
\]

where \( \mu_1 \) and \( \mu_2 \) are the chemical potential corresponding to the densities \( n(1+z) \) and \( n(1-z) \) respectively. Eqs. (22) and (23) are the atomic Josephson junction equations for the two dynamical variable \( z(t) \) and \( \phi(t) \). They describe the oscillations of \( N \) fermionic atoms that tunnel between the left region \( L \) and the right region \( R \). A crucial role is played by the chemical potential \( \mu(n) \), that is obtained by solving self-consistently the gap equation and the number equation. The macroscopic tunneling can be described as a tunneling current defined in the following way

\[
I = -\dot{z}N/2 = \left( \frac{KN}{\hbar} \right) \sqrt{1 - z^2} \sin(\phi) = \sqrt{1 - z^2} \sin(\phi)
\]

where \( I_c = KN/\hbar \) is the critical current of the direct-current Josephson effect where \( \phi \) is time independent. Thus, we need a coupling-dependent formula for the effective tunneling energy \( K \) to get the critical current \( I_c \), and also to study the alternate-current Josephson effect which is obtained by solving Eqs. (22) and (23) without restrictions on the dynamics of the relative phase.

IV. COUPLING-DEPENDENT TUNNELING ENERGY

In this Section we consider a time-independent phase difference \( \phi(t) = \bar{\phi} \) and a small population imbalance \( |z(t)| \ll 1 \). In this case the Josephson current \( I \) reads

\[
I(\bar{\phi}) = I_c \sin(\bar{\phi})
\]

This is the familiar formula of the direct-current Josephson effect [9,10]. In a recent work, Zaccanti and Zwerger [17] developed a model to describe 3D Josephson tunneling between two reservoirs of ultracold superfluid atoms which account the dependence of the critical current on the coupling. In particular, they found the critical current density \( j_c \), such that \( I_c = Aj_c \) where \( A \) is the transverse area of the Josephson junction, is given by

\[
\frac{j_c}{|t|(\mu_B)} = \frac{j_c}{|t|^2(\mu_F)} = \frac{nv_F}{8} \frac{\lambda_0}{\sqrt{\mu}}
\]

where \( \lambda_0 = n_0/n \) is the condensate fraction, \( \tilde{\mu} = \mu_B/2\epsilon_F \) is the renormalized bosonic chemical potential and \( v_F = \sqrt{2\epsilon_F/m} \) is the Fermi velocity of the non-interacting Fermi gas. The \( |t|(\mu_B) \) and \( |t|^2(\mu_F) \), where \( \mu_F = \mu_B/2 \), are the associated single-boson transmission amplitude and the transmission probability of a single fermion respectively. Eq. (26) provides a very useful formula valid for the BCS-BEC crossover both in 3D. This result has been obtained evaluating the tunneling amplitude like the problem is 1D, since the potential barrier extends only in one direction [35]. We are considering the same geometry for the potential barrier and for this reason we think that the results obtained by Zaccanti and Zwerger can be extended also in the 2D systems.

Under the condition of a high potential barrier both \( |t|(\mu_B) \) and \( |t|^2(\mu_F) \) are practically constant along the crossover and the dependence on the coupling strength is mainly in the product \( \lambda_0\sqrt{\mu} \). Its behavior along the 2D BCS-BEC crossover is shown in Fig. 2 both with mean-field and beyond-mean-field schemes. The figure shows that the mean-field curve grows monotonically. Instead, the beyond mean-field curve grows from the BCS regime to a maximum at the value \( \ln(\epsilon_F/E_F) = 0.26 \) and then it decreases in the BEC regime. Thus, also in this case the quantum fluctuations cannot be neglected in the strong-coupling regime. To better understand the meaning of this product we can write Eq. (26) in the form of the critical current \( hI_c = KN \) where \( K \) is the tunneling energy and \( N \) the total number of fermions.

\[
hI_c = hL_1j_c = hL_1|t|^2(\mu_F)\frac{nv_F}{8} \frac{\lambda_0}{\sqrt{\mu}}
\]

Replacing \( n = N/(2L_1L_2) \), where \( L_1 \) and \( L_2 \) are the transverse and longitudinal lengths of the 2D system, one obtains

\[
hI_c = h|t|^2(\mu_F)\frac{nv_F}{16L_2} \lambda_0 \sqrt{\mu}
\]

Comparing with \( hI_c = KN \), the tunneling energy \( K \) reads

\[
K = h|t|^2(\mu_F)\frac{v_F}{16L_2} \lambda_0 \sqrt{\mu} = K_0 \lambda_0 \sqrt{\mu}
\]

where the factor \( K_0 = h|t|^2(\mu_F)v_F/16L_2 \) encloses all the coupling independent factors. Once the system setup is
fixed, $K_0$ is a constant along the crossover. So the product $\lambda_0 \sqrt{\tilde{\mu}}$ provides useful informations about the behavior of the tunneling energy and the critical current along the crossover.

Figure 2. The mean-field (red dashed curve) and beyond mean-field (blue solid curve) $\lambda_0 \sqrt{\tilde{\mu}}$ as a function of the scaled binding energy $\epsilon_B/\epsilon_F$, where $\lambda_0 = n_0/n$ is the ratio between the condensate density and the total density, $\tilde{\mu} = \mu_B/2\epsilon_F$ is the ratio between the bosonic chemical potential and two times the Fermi energy $\epsilon_F$.

V. JOSEPHSON DYNAMICS OF 2D FERMIONIC SUPERFLUIDS

We replace the coupling dependent tunneling energy (29) in the Josephson equations (22) and (23) to study the time evolution of the population imbalance $z(t)$ and the relative phase $\phi(t)$. The Josephson junction equations become

\[ \dot{z} = -\frac{2K_0\lambda_0\sqrt{\tilde{\mu}}}{\hbar} \sqrt{1 - z^2} \sin(\phi) \]  
\[ \dot{\phi} = \frac{2}{\hbar} \left[ \mu n (1 + z) + \mu n (1 - z) \right] + \frac{2K_0\lambda_0\sqrt{\tilde{\mu}}}{\hbar} \frac{z}{\sqrt{1 - z^2}} \cos(\phi) \]  

Since in the previous sections we show in detail the crucial effect of the quantum fluctuations, in this part we treat the system with the beyond mean-field approximation directly.

We choose to study a fermionic superfluid of $^{40}K$ atoms with the total density $n = 0.01$ atoms/$\mu$m$^2$ and $K_0/\epsilon_B = 2 \times 10^{-8}$ Kelvin. In Fig. 3 we show the time evolution of $z(t)$ and $\phi(t)$ for three different crossover points: $\ln(\epsilon_B/\epsilon_F) = -10$, $\ln(\epsilon_B/\epsilon_F) = 10$ and $\ln(\epsilon_B/\epsilon_F) = 0.26$.

The oscillation frequency of both quantities changes along the crossover. We note anyway that $z(t)$ and $\phi(t)$ oscillate with the same frequency $\omega$ at a fixed scaled binding energy $\epsilon_B/\epsilon_F$.

We determine the frequency $\omega$ by evaluating the time interval between an oscillation peak and the next one.

\[ \omega_0 = \frac{K_0\lambda_0\sqrt{\tilde{\mu}}}{\pi \hbar} \sqrt{1 + \frac{2mc_s^2}{K_0\lambda_0\sqrt{\tilde{\mu}}}} \]  

where $c_s = \sqrt{(\partial \mu/\partial n)}$ is the speed of sound. This frequency $\omega_0$ is called zero-mode. Notice that also here we have replaced the coupling dependent relation (29).
in the tunneling energy $K$. Fig. 4 shows that $\omega$ is always larger than $\omega_0$ and their behavior as a function of $\ln(\epsilon_B/\epsilon_F)$ is very similar to the product $\lambda_0 \sqrt{\mu}$ both in mean-field approximation (red dashed curves) and in beyond mean-field one (blue solid curves).

**Figure 4.** (a) Josephson oscillation frequency $\omega$ in the nonlinear regime as a function of the scaled binding energy $\epsilon_B/\epsilon_F$. The frequency $\omega$ is obtained by solving Eqs. (30) and (31) with initial conditions: $z(0) = 0.5$ and $\phi(0) = 0$. (b) Josephson oscillation frequency $\omega_0$ in the linear regime as a function of the scaled binding energy $\epsilon_B/\epsilon_F$. The frequency $\omega_0$ is obtained from Eq. (32). In the tunneling energy $K$ of Eq. (29) we use $K_0 = 2 \cdot 10^{-8} k_B$ Kelvin. Red dashed curves: mean-field approximation; blue solid curves: beyond-mean-field (Gaussian) results. Here $\epsilon_F$ is the Fermi energy.

VI. CONCLUSIONS

The phenomenological Josephson equations for the relative phase and the population imbalance depend on two effective coupling parameters: the tunneling energy and the onsite energy. As recently shown by Zaccanti and Zwerger [17], in general these parameters are a function of the chemical potential and the condensate fraction of the system. For a two-dimensional fermionic superfluid we have derived all these quantities at zero temperature from a microscopic approach based on functional integration, finding that the Gaussian quantum fluctuations of the pairing field strongly modify the results in the BEC regime of the BCS-BEC crossover. By using the beyond-mean-field chemical potential and tunneling energy we have then numerically and analytically studied the Josephson equations. We have found that the critical current of the direct-current Josephson effect and the frequency of the alternate-current Josephson effect are larger in the intermediate regime of the BCS-BEC crossover because they are both proportional the effective tunneling energy which has the same behavior. The Josephson effect has been observed with fermionic superfluids made of alkali-metal atoms in a three-dimensional configuration of the BCS-BEC crossover [11] and very recently also with two-dimensional bosonic atoms [37]. We believe that our theoretical results can be a useful benchmark for the next future experiments on the macroscopic quantum tunneling with ultracold atoms in the two-dimensional BCS-BEC crossover.

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