Decoherence of quantum registers

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The dynamical evolution of a quantum register of arbitrary length coupled to an environment of arbitrary coherence length is predicted within a relevant model of decoherence. The results are reported for quantum bits (qubits) coupling individually to different environments (‘independent decoherence’) and qubits interacting collectively with the same reservoir (‘collective decoherence’). In both cases, explicit decoherence functions are derived for any number of qubits. The decay of the coherences of the register is shown to strongly depend on the input states: we show that this sensitivity is a characteristic of both types of coupling (collective and independent) and not only of the collective coupling, as has been reported previously. A non-trivial behaviour (‘recoherence’) is found in the decay of the off-diagonal elements of the reduced density matrix in the specific situation of independent decoherence. Our results lead to the identification of decoherence-free states in the collective decoherence limit. These states belong to subspaces of the system’s Hilbert space that do not get entangled with the environment, making them ideal elements for the engineering of ‘noiseless’ quantum codes. We also discuss the relations between decoherence of the quantum register and computational complexity based on the new dynamical results obtained for the register density matrix.

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I. INTRODUCTION

When any open quantum system, for example a quantum computer, interacts with an arbitrary surrounding environment, there are two main effects that have to be considered when examining the temporal evolution. First, there is an expected loss of energy of the initial system due to its relaxation, which happens at the rate $\tau_{\text{rel}}^{-1}$ where $\tau_{\text{rel}}$ is the relaxation time-scale of the system. Second, there is a process that spoils the unitarity of the evolution, the so-called decoherence $\tau$, where the continuous interaction between the quantum computer and the environment leads to unwanted correlations between them in such a way that the computer loses its ability to interfere, giving rise to wrong outcomes when executing conditional quantum dynamics. This phenomenon is characterized by a time that defines the loss of unitarity (i.e. the departure of coherence from unity), the decoherence time $\tau_{\text{dec}}$. Usually, the time-scale for this effect to take place is much smaller than the one for relaxation, hence quantum computers are more sensitive to decoherence processes than to relaxation ones. For practical applications in quantum computing there are several different systems that might provide a long enough $\tau_{\text{rel}}$: however, what really matters for useful quantum information processing tasks (e.g. quantum algorithms) to be performed reliably is the ratio $\tau_{\text{gating}}/\tau_{\text{dec}}$. $\tau_{\text{gating}}$, the time required to execute an elementary quantum logic gate, must be much smaller than $\tau_{\text{dec}}$. As a rough estimate, fault-tolerant quantum computation has been shown to be possible if the ratio $\tau_{\text{gating}}$ to $\tau_{\text{dec}}$ of a single qubit is of the order of $10^{-4}$ (a qubit is a two-state quantum system, the basic memory cell of any quantum information processor).

Decoherence and quantum theory are unavoidably connected. Indeed, the ubiquitous decoherence phenomenon has been ultimately associated with the “frontiers” between the quantum behaviour of microscopic systems and the emergence of the classical behaviour observed in macroscopic objects \textsuperscript{[1]}: roughly speaking, the decoherence time $\tau_{\text{dec}}$ determines the energy and length scales at which quantum behaviour is observed. It generally depends non-trivially on several different factors such as temperature, dimensionality, quantum vacuum fluctuations, disorder, and others whose origin is less well known (hardware characteristics). The time-scale for decoherence depends on the kind of coupling between the system under consideration and the environment, in a range that can go from pico—seconds in excitonic systems \textsuperscript{[2]} up to minutes in nuclear spin systems \textsuperscript{[3]}.

The discovery of algorithms for which a computer based on the principles of quantum mechanics \textsuperscript{[4]} can beat any traditional computer, has triggered intense research into realistic controllable quantum systems. Among the main areas involved in this active research field are ion traps \textsuperscript{[5]}, nuclear magnetic resonance (NMR) \textsuperscript{[6]}, quantum electrodynamics cavities \textsuperscript{[7]}, Josephson junctions \textsuperscript{[8]}, and semiconductor quantum dots \textsuperscript{[9]}. The main challenge that we face is to identify a physical system with appropriate internal dynamics and corresponding external driving forces which enables one to selectively manipulate quantum superpositions and entanglements. For this to be done, the candidate system should have a sufficiently high level of isolation from the environment: quantum information processing will be a reality when optimal control of quantum coherence in noisy environments can be achieved. The various com-
communities typically rely on different hardware methodologies, and so it is important to clarify the underlying physics and limits for each type of physical realization of quantum information processing systems. As mentioned above, these limits are mainly imposed by the decoherence time of each particular system. However, theoretical work has shown that, in addition to quantum error correcting codes \[ \text{[1,2]} \], there are two additional ways that may potentially lead to decoherence-free or decoherence-controlled quantum information processing: one of them is the so-called “error avoiding” approach where, for a given decoherence mechanism (e.g. collective decoherence discussed below), the existence of “decoherence-free” subspaces within the system’s Hilbert space can be exploited in order to obtain a quantum dynamics where the system is effectively decoupled from the environment \[ \text{[3,4]} \]. This approach requires the system-environment coupling to possess certain symmetry properties. The other approach is based on “noise suppression” schemes, where the effects of unwanted system-bath interactions are dynamically controlled using a sequence of ‘tailored external pulses’ \[ \text{[4,7]} \]. These strategies have motivated much theoretical and experimental work. In particular, there are some recent experimental results regarding the engineering of decoherence-free quantum memories \[ \text{[8,9]} \].

We devote this paper to the study of decoherence of an arbitrary quantum register (QR) of \( L \) qubits. In addition to providing a general theoretical framework for studying decoherence, we examine in detail the two limits where the qubits are assumed to couple (i) independently and (ii) collectively to an external (bosonic) reservoir. The reservoir is modelled by a continuum of harmonic modes. In Section II we show that the decoherence process is very sensitive to the input states of the register and give explicit expressions for the coherence decay functions. We have found that these functions possess a novel behaviour which we label “recoherence” (or coherence revivals) in the specific case of independent decoherence. This behaviour depends strongly on the temperature and the strength of the qubit-bath coupling. By contrast, for a certain choice of the QR input, the calculation of the reduced density matrix leads to the identification of decoherence-free quantum states \[ \text{[3,15]} \] when the qubits are coupled “collectively” to the environment, i.e. when the distance between qubits is much smaller than the bath coherence length and hence the environment couples in a permutation-invariant way to all qubits. The calculations in this paper were motivated by the results of Ref. \[ \text{[12,23]} \]. The present paper shows that some subtle but important details of these earlier results are incomplete. Particularly, the calculation of the \( L-\text{QR} \) density matrix reported here leads both to new qualitative results, when analyzing the behaviour of coherence decay, and new quantitative results, when estimating typical decoherence times: these novel results emerge for \( L > 1 \), as discussed in Section III. Concluding remarks are given in Section IV. We emphasize that the results of this paper are not restricted to a particular physical system; they are valid for any choice of the qubit system (e.g. photons, atoms, nuclei with spin 1/2, etc.) and any bosonic reservoir.

**II. INDEPENDENT VERSUS COLLECTIVE DECOHERENCE**

Let us consider the general case of a \( L-\text{QR} \) coupled to a quantized environment modelled as a continuum of field modes with corresponding creation (annihilation) bath operator \( b^\dagger (b) \). Throughout this work we will analyze pure dephasing mechanisms only. We will not consider relaxation mechanisms where the QR exchanges energy with the environment leading to bit-flip errors. Hence, the \( n^{th} \) qubit operator \( J^n_x \equiv J^n_x (J^n_z = J^n_y = 0) \text{[23]} \). As mentioned earlier, this is a reasonable approach since decoherence typically occurs on a much faster time scale than relaxation. The dynamics of the qubits and the environment can be described by a simplified version of the widely studied spin-boson Hamiltonian \[ \text{[22]} \]:

\[
H = \sum_{n=1}^{L} \epsilon_n J^n_x + \sum_{k} \epsilon_k b^\dagger_k b_k + \sum_{n,k} J^n_z (g^n_k b^\dagger_k + g^n^* b_k), \tag{1}
\]

where the first two terms describe the free evolution of the qubits and the environment, and the third term accounts for the interaction between them. Here \( g^n_k \) denotes the coupling between the \( n^{th} \) qubit and field modes, which in general depends on the physical system under consideration. The initial state of the whole system is assumed to be \( \rho^S (0) = \rho^B (0) \otimes \rho^T (0) \) (the superscripts stand for system, qubits, and bath), i.e. the QR and the bath are initially decoupled \[ \text{[24]} \]. We also assume that the environment is initially in thermal equilibrium, a condition that can be expressed as

\[
\rho^B (0) = \prod_k \rho_k (T) = \prod_k \frac{\exp(-\beta \hbar \omega_k b^\dagger_k b_k)}{1 + \langle N_{\omega_k} \rangle} , \tag{2}
\]

where \( \langle N_{\omega_k} \rangle = \frac{1 + \exp(-\beta \hbar \omega_k)}{2} \) is the Bose-Einstein mean occupation number, \( k_B \) is the Boltzmann constant and \( \beta \equiv 1/k_B T \). For the model of decoherence presented here it is clear that we are in a situation where the qubit operator \( J^n_x \) commutes with the total Hamiltonian \( H \) and therefore there is no exchange of energy between qubits and environment, as expected. We will not attempt to perform a group-theoretic description for the study of quantum noise control \[ \text{[25]} \]. Instead, we study the specific real-time dynamics of the decay in the QR-density matrix coherences within the model given by the Hamiltonian Eq. (1) (i.e. Abelian noise processes, in the language of Refs. \[ \text{[26,27]} \]).
In the interaction picture, the quantum state of the combined (qubits + bath) system at time \( t \) is given by \( | \Psi(t) \rangle_i = U_i(t) | \Psi(0) \rangle_i \), where \( | \Psi(0) \rangle_i \) is the initial state of the system, \( U_i(t) \) is the time evolution operator, \( U_i(t) = T \exp \left[ -i \hbar \int_0^t H_i(t') dt' \right] \), and \( T \) is the time ordering operator. For the Hamiltonian (1) we introduce the notation \( H = H_0 + V \), where \( H_0 = \sum_{n,k} \varepsilon_n J^n_k \) and \( V = \sum_{n,k} J^n_k (g_n^r b_k^+ + g_n^a b_k) \) denotes the free evolution term and \( V = \sum_{n,k} J^n_k (g_n^r b_k^+ + g_n^a b_k) \) is the interaction term. Hence, the interaction picture Hamiltonian is given by \( H_i(t) = U_i^0(t)VU_0(t) \), with \( U_0 = \exp(-\frac{i}{\hbar}H_0 t) \). A simple calculation gives the result

\[
H_i(t) = \sum_{n,k} J^n_k \left( g_n^r e^{i\omega_n t} b_k^+ + g_n^a e^{-i\omega_n t} b_k \right),
\]

which allows us to calculate the time evolution operator \( U_i(t) \). The result is (see appendix B.1 for details)

\[
U_i(t) = \exp \left[ i \Phi_{\omega_k}(t) \right] \exp \left[ \sum_{n,k} J^n_k \left( \xi^n_k(t) b_k - \xi^n_k^*(t) b_k^+ \right) \right],
\]

with

\[
\Phi_{\omega_k}(t) = \sum_{n,k} \left| J^n_k \right|^2 \frac{\omega_n t - \sin(\omega_n t)}{(\hbar \omega_n)^2}, \quad \xi^n_k(t) = g_n^a e^{i\omega_n t} - \frac{1 - e^{i\omega_n t}}{\hbar \omega_n}. \]

This result differs from the one reported in Ref. 20 where the time ordering operation for \( U_i(t) \) was not performed. As will become clear later, this correction alters the resulting calculation of Ref. 20, and hence changes the results for the reduced density matrix of an arbitrary L-QR. Based on the time evolution operator of Eq. (4), we report here a different result for this density matrix and discuss its implications with respect to those of Ref. 20.

Unless we specify the contrary, we will assume that the coupling coefficients \( g_n^r \) \( (n = 1, \ldots, L) \) are position-dependent, i.e. that each qubit couples individually to a different heat bath, hence the QRR decoheres ‘independently’. Implications for the ‘collective’ decoherence case will be discussed later. Let us assume that the \( n \)-th qubit experiences a coupling to the reservoir characterized by

\[
g_n^r = g_n \exp(-ik \cdot r_n),
\]

where \( r_n \) denotes the position of the \( n \)-th qubit. It is easy to see that the unitary evolution operator given by Eq. (4) produces entanglement between register states and environment states \[20]. The degree of the entanglement produced will strongly depend on the qubit input states and also on the separation \( |r_m - r_n| \) between qubits because of the position dependent coupling. As we will show later, for some kind of input states no decoherence occurs at all despite the fact that all of the qubits are interacting with the environment.

We will also identify states with dynamics decoupled from thermal fluctuations; this fact may be relevant when designing experiments where the involved quantum states have dephasing times (mainly due to temperature dependent effects) on a very short time scale, as in the solid state example. We will also see that the above features are key issues when proposing schemes for maintaining coherence in quantum computers \[13].

Due to the pure dephasing (i.e. Abelian) characteristic of the noise model considered here, we can calculate analytically the functional dependence of the decay of the coherences of the QR by taking into account all the field modes of the quantized environment. We shall follow the notation of Ref. 20. Let us consider the reduced density matrix of the L-QR: the matrix elements of this reduced operator can be expressed as

\[
\rho^{Q}_{(i_n, j_n)}(t) = \langle i_L, i_{L-1}, \ldots, i_1 | T_{\rho} (\rho^S(t)) | j_L, j_{L-1}, \ldots, j_1 \rangle,
\]

where \( \{i_n, j_n\} \equiv \{i_1, j_1; j_2; \ldots; i_L, j_L\} \) refers to the qubit states of the QR and \( \rho^S(t) = U_i(t) \rho^S(0) \otimes \rho^{B}(0) U_i^T(t) \).

From this expression it becomes clear that the dynamics of the register is completely determined by the evolution operator \( U_i(t) \). In Eq. (8), the initial density matrix of the register can be expressed as \( \rho^{Q}(0) = \rho^{Q}_{1, 1}(0) \otimes \rho^{Q}_{2, 2}(0) \otimes \ldots \otimes \rho^{Q}_{L, L}(0) \), where \( \rho^{Q}_{i_n, j_n} = |i_n\rangle \langle j_n| \). In this expansion, \( |i_n\rangle = \pm \frac{1}{2} \) are the possible eigenstates of \( J_n^z \) and will be associated with the two level qubit states \( |1\rangle \) and \( |0\rangle \), respectively. The eigenvalues \( i_n = \pm \frac{1}{2} \) and \( j_n = \pm \frac{1}{2} \). In what follows, the subscripts of Eq. (7) will be renamed with the values 1 and 0 to indicate the actual values \( \frac{1}{2} \) and \( -\frac{1}{2} \), respectively. Hence we can rewrite Eq. (7) as

\[
\rho^{Q}_{(i_n, j_n)}(t) = T_{\rho} \left[ \rho^{B}(0) U_i^T(i_n) (t) U_i^{(i_n)}(t) \right] \rho^{Q}_{(i_n, j_n)}(0),
\]

(9)

\[
U_i^{(i_n)}(t) \equiv \exp \left[ i \sum_k g_k^r S(\omega_k, t) \sum_{m,n} e^{ik \cdot r_m} i_m i_n \right] \times \exp \left[ \sum_{n,k} \left( g_n^a \varphi_{\omega_n}(t) i_n e^{-ik \cdot r_n} \langle b_n^+ | g_n^a \varphi_{\omega_n}(t) i_n e^{ik \cdot r_n} b_n \rangle \right) \right],
\]

(10)

and calculate explicitly the decay of the coherences for the L-QR. The result is (see appendix B.2)

\[
\rho^{Q}_{(i_n, j_n)}(t) = \exp \left[ -\sum_{k,m,n} |g_k|^2 c(\omega_k, t) \times \right]
\]

(11)
pling. Hence, the functional dependence of the spectral
limit, Eq. (11) reads
\[
\exp \left[ -i \sum_{k,m,n} \left| g_{kmn} \right|^2 \frac{c(\omega_k, t) i m j m \sin k \cdot r_{mn}}{c(\omega_k, t) i m j m \sin k \cdot r_{mn}} \right] \rho_{(in,jn)}^Q(0),
\]
where \( r_{mn} = r_{m} - r_{n} \) is the relative distance between the
\( m^{th} \) and \( n^{th} \) qubits, \( s(\omega_j, t) = \left[ \omega_j t - \sin(\omega_j t) \right] / (\hbar \omega_j)^2 \),
and \( c(\omega_k, t) = [1 - \cos(\omega_j t)] / (\hbar \omega_j)^2 \). In the continuum
limit, Eq. (11) reads
\[
\rho_{(in,jn)}^Q(t) = \exp \left[ i \left\{ \Theta_d(t, t_s) - \Lambda_d(t, t_s) \right\} \right] \times \exp \left[ - \Gamma_d(t, t_s ; T) \right] \rho_{(in,jn)}^Q(0), \tag{12}
\]
where
\[
\Theta_d(t, t_s) = \int d\omega I_d(\omega) s(\omega, t) \times 2 \sum_{m=1 \atop m \neq n}^{L} (i m i n - j m j n) \cos(\omega t_s), \tag{13}
\]
\[
\Lambda_d(t, t_s) = 2 \int d\omega I_d(\omega) c(\omega, t) \sum_{m=1 \atop m \neq n}^{L} i m j n \sin(\omega t_s), \tag{14}
\]
and
\[
\Gamma_d(t, t_s ; T) = \int d\omega I_d(\omega) c(\omega, t) \frac{\omega}{2 \omega_p} \times \left\{ \sum_{m=1}^{L} (i m - j m)^2 + 2 \sum_{m=1 \atop m \neq n}^{L} \left( i m - j m i n - j n \right) \cos(\omega t_s) \right\}. \tag{15}
\]
Here \( \omega_p \equiv k_B T / \hbar \) is the thermal frequency (see discussion
below), \( \omega_p \equiv k_B \cdot r_{mn} \) sets the “transit time” \( t_s \), and
\( I_d(\omega) \equiv \sum \delta(\omega - \omega_j) |g_{kmn}|^2 \equiv \frac{d G(\omega)}{d\omega} |g(\omega)|^2 \) is the
spectral density of the bath. This function is charac-
terized by a cut-off frequency \( \omega_c \) that depends on the
particular physical system under consideration and sets
\( I_d(\omega) \rightarrow 0 \) for \( \omega >> \omega_c \). We see that an explicit
calculation for the coherences requires the knowledge of
the spectral density \( I_d(\omega) \). Here we assume that
\( I_d(\omega) = \alpha_d \omega^d e^{-\omega / \omega_c} \), where \( d \) is the dimension-
ality of the field and \( \alpha_d > 0 \) is a proportionality constant
that characterizes the strength of the system-bath cou-
pling. Hence, the functional dependence of the spectral
density relies on the dimensionality of the frequency depen-
dence of the density of states \( G(\omega) \) and of the coupling
\( g(\omega) \). In Eq. (12) the “transit time” \( t_s \) has been intro-
duced in order to express the QR-bath coupling in the
frequency domain. This transit time is of particular im-
portance when describing the specific “independent” de-
coherence mechanism, because of the position-dependent
coupling between qubits and bath modes. However, in
a scenario where the qubits are coupled “collectively”
to the same environment, \( t_s \) does not play any role (see
below).

The result of Eq. (12) differs in several respects from
the one reported in \cite{2}: the decoherence function
\( \Gamma_d(t, t_s ; T) \) contains additional information about
the characteristics of the independent decoherence and
the way in which the individual qubits couple to the en-
vironment through the position-dependent terms which
are proportional to \( \cos(\kappa \cdot r_{mn}) \). In essence this means
that the entanglement of the register with the noise
field depends on the qubit separation. Also the ex-
pression for \( \rho_{(in,jn)}^Q(t) \) reveals the new dynamical factor
\( N_d(t, t_s) \equiv \Theta_d(t, t_s) - \Lambda_d(t, t_s) \) which must be taken into
account when determining typical decoherence times for
the \( L-\QR \).

It is interesting that the decoherence effects aris-
ing from thermal noise can be separated from the
ones due purely to vacuum fluctuations. This is simply
because the average number of field excitations
at temperature \( T \) can be rewritten as
\( \langle N_\omega | T \rangle = \frac{1}{2} \exp(-\hbar \omega/2k_B T) \cosech(\hbar \omega/2k_B T) \),
and hence \( \coth(\hbar \omega/2k_B T) = 1 - 2 \langle N_\omega | T \rangle \) in Eq. (15).
The other term contributing to the decay of the coherences
in Eq. (12), \( N_d(t, t_s) \), is due purely to quantum vacuum
fluctuations. The separation made above allows us to
examine the different time-scales present in the (\( QR +
\)) bath system’s dynamics. The fastest time scale of the
environment is determined by the cut-off: \( \tau_c \sim \omega_c^{-1} \), i.e.
\( \tau_c \) sets the “memory” time of the environment. Hence,
the vacuum fluctuations will contribute to the dephasing
process only for times \( t > \tau_c \). Also note that the charac-
\[
teristic thermal frequency \( \omega_T \equiv k_B T / \hbar \) sets another
fundamental time scale \( \tau_T \sim \omega_T^{-1} \); thermal effects will affect
the qubit dynamics only for \( t > \tau_T \). Hence we see that
quantum vacuum fluctuations contribute to the dephasing
process only in the regime \( \tau_c < t < \tau_T \). From
this identification it becomes clear that the qubit dynamics
and hence the decoherence process of our open quantum
system will depend on the ratio of the temperature-to-
cut-off parameter \( \omega_p / \omega_c \) and the spectral function \( I_d(\omega) \).
Later we will analyze how different qualitative behaviours
are obtained for the decoherence depending on the rela-
tionship between the cut-off and the thermal frequency.
It is worth noticing that the analytical separation be-

tween the thermal and vacuum contributions to the over-
all decoherence process is a convenient property of the
pure dephasing (Abelian) model considered here. Fur-
ther generalizations of this model, e.g. by including rel-
axation mechanisms, should make this separation non-
solvable.

Next, we analyze the case of “collective decoherence”.
This situation can be thought of as a bath of “long” de-
coherence length (mean effective wave length \( \lambda \)) if com-
pared with the separation \( r_{mn} \) between qubits, in such a
way that $\lambda >> r_{mn}$ and hence the product of Eq. (6) has $\exp(-ik \cdot r_n) \approx 1$. Roughly speaking, we are in a situation where all the qubits "feel" the same environment, i.e. $g^n_k \equiv g_n$. A similar calculation to the one followed in Appendix B gives the following result for the decay of the coherences for the case of collective coupling to the reservoir:

$$\rho^{Q}_{i_j,n} (t) = \exp \left\{ i \Theta_d(t) \left[ \left( \sum_{m=1}^{L} i_m \right)^2 - \left( \sum_{m=1}^{L} j_m \right)^2 \right] \right\} \times$$

$$\exp \left\{ - \Gamma_d(t; T) \left[ \sum_{m=1}^{L} (i_m - j_m)^2 \right] \right\} \rho^{Q}_{i_j,n} (0)$$

(16)

where

$$\Theta_d(t) = \int d\omega I_d(\omega) s(\omega, t) ,$$

(17)

and

$$\Gamma_d(t; T) = \int d\omega I_d(\omega) c(\omega, t) \coth \left( \frac{\omega}{2\omega_r} \right) .$$

(18)

Expressions (12) and (16) are to be compared with those reported in Refs. [20] and [21]. Clearly, the new result for the evolution operator $U(t)$ of Eq. (4) induces a QR-environment dynamics different to the one reported in [24]. This fact has been pointed out in Ref. [21] for the situation of collective decoherence; in this particular case, our results coincide with theirs. However, we obtain different results when considering the situation of independent decoherence: the new dynamical factor $H_d(t, t_x)$ includes additional information about individual qubit dynamics that cannot be neglected. Indeed, from the results derived in Ref. [21], the authors argue that the damping of the independent decoherence is insensitive to the type of initial states and hence the sensitivity to the input states is only a property of the collective decoherence. As can be deduced from Eqs. (12) and (16), we find that this statement is not generally true and that the sensitivity to the initial states is a property of both collective and independent decoherence. This result is particularly illustrated for the case of a 2–QR in the next section. From the expressions (17) and (18) we see that for $\hbar \omega << k_B T$, the high-temperature environment (high-TE), the phase damping factor $\Gamma_d(t; T)$ is the main agent responsible for the qubits’ decoherence while the other dynamical damping factor $\Theta_d(t)$ plays a minor role. In this case $\omega_c$ is actually the only characteristic frequency accessible to the system ($\omega_c << \omega_r$) and thermal fluctuations always dominate over the vacuum ones. However, when we consider the situation $\omega_c >> \omega_r$, the low-temperature environment (low-TE), these damping factors compete with each other over the same time scale, and $\Theta_d(t)$ now plays a major role in eroding the qubits’ coherence. Here there is a much more interesting dynamics between thermal and vacuum contributions (see next section). This shows the difference and the importance of the additional terms of the reduced density matrix reported here when compared with those of Refs. [20, 21].

The above statements will be illustrated in the next section.

So far, the dynamics of the qubits and their coupling to the environment has been discussed in the interaction representation. To go back to the Schrödinger representation, recall $\rho_{q,r}(t) = U(t) \rho_{r} U^\dagger(t)$, with $\rho_{r}(t)$ as calculated before for the qubits decoherence (Eqs. (12) and (16)). Also note that in the Schrödinger representation (here denoted with subindex $Sch$) $U(t)$ introduces mixing but not decoherence. Next, we consider some particular cases which allow us to evaluate the expressions (12) and (16) and hence give a qualitative picture of the respective decoherence rates for both collective and independent decoherence situations.

### III. DIMENSIONALITY OF THE FIELD AND DECOHERENCE RATES FOR FEW QUBIT SYSTEMS

In this section we analyze the qualitative behaviour of the decay of the coherences given by Eqs. (12) and (16) for single and two qubit systems.

#### A. Single qubit case

Here we consider the case of only one qubit in the presence of a thermal reservoir, as defined in Eq. (2). Hence, for both types of coupling (Eqs. (12) and (16)) we get

$$\rho^{Q}_{i,n} (t) = \exp[-\Gamma_d(t, T)] \rho^{Q}_{i,n} (0) .$$

(19)

By using Eq. (4), with $d = 1$, it is easy to show that the populations remain unaffected: $\rho^Q_{ii}(t) = \rho^Q_{ii}(0)$, $i = 0, 1$. In general ($i \neq j$), the decay of coherences is determined by Eq. (19). Here we can identify the main time regimes of decoherence for different dimensionalities of the field. In what follows, we consider the case of reservoirs with both one-dimensional density of states ("Ohmic") and three-dimensional density of states ("super-Ohmic"), i.e. $G(\omega) = constant$ and $G(\omega) = \omega^2$, respectively, where the frequency dependent coupling $g(\omega) \propto \sqrt{\omega}$, as considered in [24]. From Eq. (19) we see that a general solution for the case $d = 1$ requires numerical integration (see Appendix A). However, for the case where the interplay between thermal and vacuum effects is more complex, i.e. the low-TE ($\omega_r << \omega_c$), we can solve it analytically. Here we get

$$\Gamma_1(t, T) \approx c_1 \left[ 2\omega_r t \arctan(2\omega_r t) + \frac{1}{2} \ln \left( \frac{1 + \omega_r^2 t^2}{1 + 4\omega_r^2 t^2} \right) \right] ,$$

(20)
where \( c_1 \equiv \alpha_1 / h^2 \). On the other hand, an exact solution for the super-Ohmic case \( d = 3 \) (Eq. (19)) can be found for any temperature \( T \). The result is:

\[
\Gamma_3(t, T) = c_3 \left\{ 3 \left[ \zeta(2, \theta) + \zeta(2, 1 + \theta) - \zeta(2, \theta - i\omega_rt) \right] + \frac{1 - \omega_r^2 t^2}{1 + \omega_r^2 t^2} \right\},
\]

(21)

where \( \zeta(z, q) = [1/\Gamma(z)] \int_0^\infty dt \left[ t^{z-1}e^{-qt} / (1 - e^{-t}) \right] \) is the generalized Riemann zeta function, \( \Gamma(z) = \int_0^\infty dt t^{z-1}e^{-t} \) is the Gamma function, and \( c_3 = \alpha_3 \omega_r^2 / h^2 \).

For the purpose of this paper, we concentrate on the case \( L > 1 \) for which we have several novel results. We leave the analysis of the \( L = 1 \) case to Appendix A, where we discuss the process of identifying typical decoherence times for a single qubit, and the interplay between the different decoherence regimes as a function of the temperature.

### B. L = 2 qubit register

Let us analyze the case of two qubits in the presence of the bosonic reservoir discussed in the present paper. Using the same expressions for the density of states \( G(\omega) \) and for the frequency dependent coupling \( g(\omega) \) as above, we will analyze the coherence decay for several different input states. We set the qubits at positions \( r_a \) and \( r_b \) with coupling factors given by \( g^a = g_b = e^{-ik \cdot r} \), and \( n = a, b \). It is easy to see that the unitary evolution operator induces entanglement between qubit states and field states: \( U(t) \) acts as a conditional displacement operator for the field with a displacement amplitude depending on both qubits of the QR, as discussed in more detail in [23]. As we have pointed out previously, it is this entanglement that is responsible for the decoherence processes described in the present paper. In particular, the case of two qubits has \( \rho_{ia,ib}^Q(t) = \langle i_a, i_b | T_R | \langle \rho^S(t) \rangle | j_a, j_b \rangle \). Next we analyze the register dynamics for the limiting decoherence situations described above. First, let us study the case of **independent decoherence**:

(i) \( i_a = j_a, i_b \neq j_b \):

\[ \rho_{ia,ib}^Q(t) = \exp \left[ \Theta_d(t)f_{ia,i_b} - \Gamma_d(t; T) \right] \times \rho_{ia,ib}^Q(0), \]

where \( f_{ia,i_b} = 2ia(i_b - j_b) \cos k \cdot r_{ab} \). Hence \( f_{0b,01} = f_{11,10} = \cos k \cdot r_{ab} \), and \( f_{0a,01} = f_{11,01} = -\cos k \cdot r_{ab} \). \( \rho_{ia,ib}^Q(t) \) shows collective decay. This result is contrary to the one reported in Ref. [21].

(ii) \( i_a = j_a, i_b = j_b \):

\[ \rho_{ia,ib}^Q(t) = \rho_{ia,ia}^Q(0); \] as expected, the populations remain unaffected.

(iii) \( i_a \neq j_a, i_b \neq j_b \):

\[ \rho_{ia,ib}^Q(t) = \exp \left[ -\Gamma_d(t; T)h_{ia,j_a,i_b,j_b} \right] \rho_{ia,ib}^Q(0), \]

where \( h_{ia,j_a,i_b,j_b} = 2[1 + (a_i - j_a)(b_i - j_b) \cos k \cdot r_{ab}] \). Clearly, \( h_{10,10} = h_{01,01} = 2[1 + \cos k \cdot r_{ab}] \), and \( h_{10,01} = h_{01,10} = 2[1 - \cos k \cdot r_{ab}] \). Hence \( \rho_{ia,ib}^Q(t) \) also shows collective decay.

In the above cases, analytic expressions for the corresponding decoherence functions can be found. As in Section III (a), we shall consider two different surrounding environments. In the low-TE regime, the ‘Ohmic environment’ \( (d = 1) \) induces the following coherence decay (the high-TE requires numerical integration):

\[ \rho_{d=1}^\pm(t) = \exp \left[ -\Gamma(t; T) \mp i c_1 \left( \frac{1}{2} \arctan(\omega_c t_-) - \frac{1}{2} \arctan(\omega_c t_+) + \frac{\omega_c t_-}{1 + \omega_c^2 t_-^2} \right) \right] \rho_{d=1}^\pm(0) \]

(22)

for \( i_a = j_a \), and \( i_b \neq j_b \). For brevity, we have dropped the subindices of the reduced density matrix, and set \( t_+ = t_a + t, t_- = t_a - t \). For \( i_a \neq j_a \), and \( i_b \neq j_b \) we obtain the result

\[ \rho_{d=1}^\pm(t) = \exp \left[ -2\Gamma(t; T) + 2c_1 \left( \frac{1}{4} \left\{ 2 \ln \left( \frac{1 + 4\omega_c^2 t_-^2}{1 + \omega_c^2 t_-^2} \right) + \ln \left( \frac{1 + \omega_c^2 t_-^2}{1 + 4\omega_c^2 t_-^2} \right) \right\} - \omega_c^2 \left\{ 2t_a \arctan(2\omega_c t_a) - t_- \arctan(2\omega_c t_-) \right\} \right\} \right] \rho_{d=1}^\pm(0). \]

(23)

In Figs. 1 and 2 we have plotted the decay of the two qubit coherence due to the coupling to an environment of the Ohmic type (Eqs. (23)). Here, \( \Gamma(t; T) \) are defined from Eqs. (23) as \( \rho_{d=1}^\pm(t) = \exp[-\Gamma(t; T)] \rho_{d=1}^\pm(0) \). From these figures we can see the variations of the onset of decay when increasing the temperature. Figure (1) shows how the departure of coherence from unity changes with \( t_s \) (plot 1 (i)) whilst for high temperature (plot 1 (ii)) there is no variation with \( t_s \) at all. In the limit of large \( t_s \) (Table 1), we recover the onset of decay of Fig. 6 (Appendix A). Note the difference between the time scales of plots 1 (i) and (ii), and how an estimation of typical decoherence times strongly relies on the temperature of the environment. Figure (2) shows how the coherence decay shown in Fig. 1 disappears for small \( t_s \) values, i.e. for a given temperature, it is possible to find a \( t_s \) from which there is no decoherence of the two qubit system. This interesting behaviour occurs only for the density matrix elements \( \langle 10 | T_R | \rho^S(t) | 10 \rangle = 0 \) (Table 1).

In Table 1 we give some typical two qubit decoherence times \( \tau_{dec} \) for an Ohmic environment in terms of the temper-
perature, coupling constants $c_1$, and $t_s$. In all of the tables in this paper, we have taken the beginning of the decoherence process to occur when the reduced density matrix of the whole system exhibits a 2% deviation from the initial condition, i.e. when $\exp[-\Gamma_1(t = \tau_{dec}, T)] = 0.98$. The tables have been generated from the corresponding decoherence functions reported in this paper. Here, $t_f$ is defined from $\exp[-\Gamma_1(t = t_f, T)] = 0.01$, i.e. the difference between $t_f$ and $\tau_{dec}$ gives an estimate for the duration of the decoherence process, say $t_{\text{decay}}$, and $\tau_{\text{dec}}^\pm$ and $t_f^\pm$ are evaluated from the respective decoherence functions $\Gamma_1^\pm(t, T)$, with $i = 1, 3$. In order to gain insight into some characteristic time-scales, consider for example the case of the solid state, where in many situations the noise field can be identified with the phonon field. Here the cut-off $\omega_c$ can be immediately associated with the Debye frequency $\omega_D$. A typical Debye temperature $\Theta_D = 100$ K has $\omega_D \equiv \omega_c \approx 10^{13}$ s$^{-1}$, so $\theta \equiv \omega_c / \omega_D \approx 10^{-2}$ T. Hence we can see from Table 1 (Figs. 1 and 2) that for $c_1 = 0.25$, $\omega_c t_s = 0.5$, and $T = 0.1$ K, the decoherence process starts at $\tau_{\text{dec}}^+ \approx 23.5$ fs and $\tau_{\text{dec}}^- \approx 43.7$ fs, and it lasts for $t_{\text{decay}}^+ \approx 10.3$ ps (for $t_{\text{decay}}^-$, $\exp[-\Gamma_1^- (t \to \infty, 0.1)$ K] saturates above 0.01). If the strength of the coupling goes down to $c_1 = 0.01$ then $\tau_{\text{dec}}^+ \approx 0.2$ ps, and $t_{\text{decay}}^+ \approx 3.5$ ns. In this latter case, $\tau_{\text{dec}}^+$ and $t_{\text{dec}}^+$ are not reported since the coherence saturates to a value above 0.98. From the data reported in Table 1 is clear that the weaker the coupling between the qubits and the environment, the longer the decoherence times and the slower the decoherence process. In addition the higher the temperature, the faster the qubits decohere, as expected intuitively.

The effect of the transit time becomes more evident from Table 1: for large $t_s$ values there is no difference between $\tau_{\text{dec}}^+$ and $\tau_{\text{dec}}^-$ (also $t_{\text{decay}}^+ \approx t_{\text{decay}}^-$. This is because for large $t_s$, the contribution due to terms involving $t_s$ in Eq. (23) is negligible, hence the reduced density matrix $\rho_{\text{dec}}(t) \equiv \rho_{\text{dec}}(t) \approx \exp[-2\Gamma_1(t, T)]$ and hence has a similar behaviour to the single qubit case (Eq. (19)). Hence for large $t_s$ (e.g. $t_s = 10^4$), Figs. 1 and 2 resemble the onset of decay of the single qubit case (see later Fig. 6) where there is no dependence on $t_s$. We note that $\exp[-\Gamma_1^+ (t, T)] (\exp[-\Gamma_1^- (t, T)] is the corresponding decay of the coherences $\rho_{10,10} = \rho_{01,01}$ ($\rho_{10,01} = \rho_{01,10}$), hence $\rho_{10,10}^Q(t) = \rho_{01,01}^Q(t)$ for large $t_s$.

By contrast, the super-Ohmic $d = 3$ field leads to the following coherence decay

$$\rho_{\text{dec}}^{+3}(t) = \exp\left[ -\Gamma_3(t, T) \right] \pm i\epsilon_3 \left( \frac{\sin(2 \arctan(\omega_c t_s))}{2(1 + \omega_c^2 t_s^2)} - \frac{\sin(2 \arctan(\omega_c t_f))}{2(1 + \omega_c^2 t_f^2)} + \frac{2\omega_c t \cos(3 \arctan(\omega_c t_s))}{(1 + \omega_c^2 t_s^2)^{3/2}} \right) \rho_{\text{dec}}^{\pm}(0)$$

(24)

for $i_a = j_a$, and $i_b \neq j_b$, and

| $c_1$ | $k_B T / \hbar \omega_c$ | $\omega_c t_s$ | $\omega_c \tau_{\text{dec}}^+$ | $\omega_c t_f^+$ | $\omega_c \tau_{\text{dec}}^-$ | $\omega_c t_f^-$ |
|-------|----------------|-------------|----------------|----------------|----------------|----------------|
| 0.25  | $10^{-3}$      | 0.5         | 0.436919       | saturates      | 0.235466       | 103.507        |
| 0.25  | $10^0$         | 0.5         | 0.183755       | saturates      | 0.104119       | 2.05958        |
| 0.25  | $10^{-3}$      | 0.5         | 0.290113       | 127.93         | 0.290113       | 127.96         |
| 0.25  | $10^0$         | 0.5         | 0.127778       | 3.45901        | 0.127778       | 3.45901        |
| 0.1   | $10^{-3}$      | 0.5         | 0.913573       | saturates      | 0.37654        | 2025.75        |
| 0.1   | $10^0$         | 0.5         | 0.303135       | saturates      | 0.16504        | 4.28334        |
| 0.1   | $10^{-3}$      | 0.5         | 0.47316        | 5696.69        | 0.47315         | 5670.15        |
| 0.1   | $10^0$         | 0.5         | 0.203549       | 7.86596        | 0.203549       | 7.86596        |
| 0.01  | $10^{-3}$      | 0.5         | 0.583502       | saturates      | 1.45274        | 35004.7        |
| 0.01  | $10^0$         | 0.5         | 2.55738        | saturates      | 2.55738        | 40816.8        |
| 0.01  | $10^{-3}$      | 0.1         | 0.709492       | 73.8325        | 0.709492        | 73.8325        |

Table 1. Characteristic times for two-qubit independent decoherence $\tau_{\text{dec}}^{\pm}$ for $d = 1$ dimensional density of states of the field. Different temperature, transit time, and coupling strength values are considered. $i_a \neq j_a$, $i_b \neq j_b$ (see text).

for $i_a \neq j_a$, and $i_b \neq j_b$. The results of Eqs. (24) and (25) are exact: no approximations have been made in obtaining them. Therefore, they are valid for any temperature of the environment.

In Figs. 3 and 4 we have plotted the decay of two qubit coherence due to the coupling to a reservoir with three-dimensional density of states (Eqs. (25)). Some estimates for $\tau_c$ and $t_{\text{dec}}$ have been given in Table 2. The decoherence functions $\Gamma_3^+(t, T)$ are defined as $\rho_{\text{dec}}^{+3}(t) = \exp[-\Gamma_3^+(t, T)]\rho_{\text{dec}}^{+}(0)$. As we can see, there is a non-monotonic behaviour for the decay of coherence for low temperature values, as can be seen from Figs. 3 (i), and (ii). The decay given by the functions $\Gamma_3^+(t, T)$ (Fig. 3) and $\Gamma_3^-(t, T)$ (Fig. 4) saturates to a particular value, which is fixed by the strength of the coupling and the temperature of the reservoir: the lower the temperature, the slower the decay, and the higher the residual coherence. Some estimates for these saturation values $\epsilon^{-\Gamma_3^+(t_f^+, T)}$ are shown in Table 2. From Fig. 4 it is possible to find small $t_s$ values for which the onset of decay does not change in time and coherence remains unaffected. This result is very different from the case of Fig. 3, where coherence either vanishes (at high temperatures) or saturates to a residual coherence value (at low temperatures). Also note that whilst nothing happens to the onset of decay of Fig. 4, the
coherence decay is amplified in the case of Fig. 3, for small $t_s$ values. From Figs. 3 and 4 (Figs. (i), and (ii)) we see that there is saturation of the decay in the presence of a non-trivial coherence process. However, for high temperatures (Figs. 3 (iii) and 4 (iii)) there is a monotonic behaviour where no saturation occurs at all and the residual coherence vanishes.

We note that the non-monotonic behaviour reported here for the low temperature regime is not just a characteristic of high dimensionality fields: it also occurs for the $d = 1$ field, as can be seen from Figs. 1 (i) and 2 (i). We believe that this purely quantum-mechanical phenomenon is associated with an interplay between the quantum vacuum and thermal fluctuations of the system and the quantum character of the field. The system undergoes a dynamics where the environment manages to ‘hit back’ at the qubits in such a way that the coherences then exhibit an effective revival before vanishing (at high temperatures) or saturating to a residual value (at low temperatures). An essential feature of the model studied here is the fact that the QR and bath are assumed to be initially uncorrelated. In future work we will analyze the behaviour of these ‘recoherences’ with more general initial conditions, where the initial state of the combined system is allowed to contain some correlations between the bath and the QR (see also Ref. [30]). Such studies will help to clarify the origins of this ‘recoherence effect’ and also the effectiveness of decoherence as a function of the initial conditions. We note that there is a previous work by Hu et al. [31] where the study of quantum brownian motion in a general environment gives rise to a similar behaviour (although in a different context) to the one reported here for the ‘recoherences’. A more detailed analysis of the physical implications of this interesting behaviour of the coherence decay is intended to be addressed elsewhere [31].

It is interesting that the dynamics of coherences depends so strongly on the strength of the coupling. It can be seen from Table 2 and Fig. 5 that i) the coherences saturate (sat.) to a very high ‘residual value’ and show less than a 2% decay for weak coupling (e.g. $c_3 = 0.01$) and low temperatures ($T = 0.1$ K). This result is to be compared with the $d = 1$ case where the coherences always vanish for long $t_s$; ii) even at high temperatures ($T = 10^4$ K) where the coherences vanish, and weak coupling, we find the appearance of the ‘recoherence effect’ discussed above. This makes more evident the role of the QR-bath coupling: recoherences should appear only under certain conditions imposed by the strength $c_3$ (hence by the spectral density) and the temperature. These conditions can be directly obtained from the decoherence functions reported in this paper. Typical $\tau_{dec}$ times for this $d = 3$ dimensional environment are given in Table 2. From these we can conclude that the two QR decoherence time scales are shorter, and the decoherence process occurs faster, than in the single qubit case.

| $c_3$ | $\omega_0/\omega_c$ | $\omega_c t_s$ | $\omega_c t_s^+$ | $\omega_c t_s^-$ | $e^{-\tau_{dec}^+(T)}$ | $e^{-\tau_{dec}^-(T)}$ |
|------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.25 | $10^{-3}$ | 0.5 | 0.1292 | sat. | 0.477 | 0.1068 | sat. | 0.771 |
| 0.25 | $10^{-3}$ | 0.5 | 0.01338 | 0.20 | 0.01 | 0.01522 | 0.24 | 0.01 |
| 0.25 | $10^{-3}$ | 0.5 | 0.1278 | sat. | 0.6065 | 0.11738 | sat. | 0.6065 |
| 0.25 | $10^{-3}$ | 0.5 | 0.1278 | sat. | 0.6065 | 0.11738 | sat. | 0.6065 |
| 0.01 | $10^{-3}$ | 0.5 | 0.1292 | sat. | 0.477 | 0.1068 | sat. | 0.771 |
| 0.01 | $10^{-3}$ | 0.5 | 0.01338 | 0.20 | 0.01 | 0.01522 | 0.24 | 0.01 |
| 0.01 | $10^{-3}$ | 0.5 | 0.1278 | sat. | 0.6065 | 0.11738 | sat. | 0.6065 |

Table 2. Characteristic times for two qubit independent decoherence $\tau_{dec}^\pm$ for $d = 3$ dimensional density of states of field. $i_a \neq j_a$, $i_b \neq j_b$.

We now analyze how the above results are affected when we consider the situation of collective decoherence, as given by Eq. (16). The reduced density matrix for the two qubit system now reads: $\rho_{i_a j_a, i_b j_b}^Q(t) = \exp [i\Theta(t)f'_{i_a j_a, i_b j_b} - \Gamma(t; T)]\rho_{i_a j_a, i_b j_b}^Q(0)$, where $f'_{i_a j_a, i_b j_b} = 2i_a (i_b - j_b)$. Hence $f'_{00,00} = f'_{11,11} = 1$, and $f'_{00,11} = f'_{11,00} = -1$. The corresponding decoherence rates are

\[
\rho_{d_{x,z}}^+(t) \approx \exp \left[ -\Gamma_1(t; T) \pm ic_1(\omega_c t + \arctan(\omega_c t)) \right] \rho_{d_{x,z}}^+(0)
\]

\[
\rho_{d_{x,z}}^-(t) \approx \exp \left[ -\Gamma_3(t; T) \pm ic_3 \left(2\omega_c t - \frac{\sin(2\arctan(\omega_c t))}{1 + \omega_c^2 t^2} \right) \right] \rho_{d_{x,z}}^-(0) \tag{26}
\]

for the Ohmic environment and the $d = 3$-dimensional density of states, respectively.

(ii) $i_a = j_a$, $i_b = j_b$:

\[
\rho_{d_{x,z}}^Q(t) = \rho_{d_{x,z}}^Q(0)
\]

as expected, the populations remain unaffected.

(iii) $i_a \neq j_a$, $i_b \neq j_b$:

\[
\rho_{d_{x,z}}^Q(t) = \exp \left[ -\Gamma(t; T) h_{a,ja} h_{b,jb} \right] \rho_{d_{x,z}}^Q(0),
\]

where $h_{a,ja} h_{b,jb} = (i_a + i_b - j_a - j_b)^2$. Then, $h_{10,10} = h_{01,01} = 4$, and $h_{10,01} = h_{01,10} = 0$. Obviously, the corresponding decoherence rates are:

\[
\rho_{d_{x,z}}^+(t) = \rho_{d_{x,z}}^+(0),
\]

\[
\rho_{d_{x,z}}^-(t) \approx \exp \left[ -4\Gamma_1(t; T) \right] \rho_{d_{x,z}}^+(0) \tag{27}
\]

for the $d = 1$ dimensional field, and

\[
\rho_{d_{x,z}}^+(t) = \rho_{d_{x,z}}^+(0),
\]

\[
\rho_{d_{x,z}}^-(t) \approx \exp \left[ -4\Gamma_3(t; T) \right] \rho_{d_{x,z}}^+(0) \tag{28}
\]

for the $d = 3$ dimensional field. Hence, regarding the input states, the case of collective decoherence shows two
very well defined situations:

(a) A set of input states that shows no decoherence at all, despite the fact that the qubits are interacting with the environment. This is because under the specific situation of collective coupling there is a set of initial qubit states that does not entangle with the bosonic field and hence the states preserve their coherence. These states are the so-called “coherence-preserving” states and, for the case of a 2–QR, the corresponding reduced density matrix elements are $\langle 10 | \rho_{\text{QR}}(t) | 01 \rangle$, and $\langle 01 | \rho_{\text{QR}}(t) | 10 \rangle$. As studied in more detail in Ref. [13] (where relaxation effects are also included), this result can be used as an encoding strategy, where an arbitrary $L$–QR can be decoupled from its environment provided that every single qubit of the register can be encoded using 2 qubits: e.g. using the simple encoding (though not the most efficient one) $|0\rangle \mapsto |01\rangle$, and $|1\rangle \mapsto |10\rangle$. It turns out that this procedure is ensured even if relaxation effects are included in the Hamiltonian (1) [13]. Hence in the specific situation of collective decoherence one can find, for arbitrary $L$, a decoherence-free subspace (DFS) $C_{\perp} \subset \mathcal{H}^\otimes L$ (the Hilbert space $\mathcal{H} = \mathcal{H}_{\text{QR}} \otimes \mathcal{H}_B$) that does not get entangled with the environment, and hence the QR should evolve without decoherence. Besides quantum error correction codes, this is currently one of the most outstanding results in the battle against decoherence [13], particularly because of its relevance to maintaining a coherent qubit memory in quantum information processing.

(b) The other two input states have the decay of coherences $\langle 11 | \rho_{\text{QR}}(t) | 00 \rangle$ and $\langle 00 | \rho_{\text{QR}}(t) | 11 \rangle$; these give a situation of ‘superdecoherence’ [20], where the qubits are collectively entangled and hence these matrix elements give the fastest decay for the coherences. This means that whilst there is a subspace which is robust against decoherence as discussed above, the remaining part of the Hilbert space gets strongly entangled with the environment. This superdecoherence situation is illustrated in Fig. 6 for the case of reservoirs with one and three dimensional-density of states.

The above process of calculating explicit results for the decay of any coherence associated with the coupling of a $L$–QR to a bosonic reservoir, for both types of coupling (independent and collective), can be carried out for any $L > 2$ using the general formulas Eqs. (12) and (16). In so doing, we can obtain an estimate of typical decoherence times for a QR with an arbitrary number of qubits. We should point out that if no schemes for controlling the errors induced by the decoherence phenomenon are used [11, 13], $\tau_{\text{dec}}$ establishes an upper bound to the duration of any reliable quantum computing process.

IV. DISCUSSIONS AND CONCLUDING REMARKS

We have revisited a model of decoherence based upon the one previously studied by Leggett et al. [22] in connection with the tunneling problem in the presence of dissipation, and used later by Unruh and Palma et al. [22, 20] for describing the decoherence process of a quantum register. We have presented here a more complete description of this latter problem, which provides the following new results. The decoherence rates of the density matrix elements are correctly derived, leading to new quantitative results for both independent and collective decoherence situations. As discussed here, if no error correcting/preventing strategies are used, this has implications for an estimation of decoherence time-scales for which the quantum memory of a QR can be maintained in any reliable computation (we note, however, there has been a recent proposal by Beige et al. regarding the use of dissipation to remain and manipulate states within a DFS [23]). Our results agree with those reported in [21] for the case of collective decoherence but they are different to the ones reported there for the case of independent decoherence: in Ref. [21] it is argued that independent decoherence, as opposed to collective decoherence, is insensitive to the qubit input states. Here we have shown instead that both cases are very sensitive to the input states and that both of them show collective decay.

In the specific situation of independent decoherence we have found a non-trivial behaviour in the decay of the off-diagonal elements of the reduced density matrix. Here the coherences experience an effective revival before they either vanish or saturate to a residual value. This behaviour depends on the temperature of the environment and depends strongly on the strength of the coupling: the coherence dynamics are different in the weak and strong coupling regime. Also, there are important qualitative differences between the Ohmic $d = 1$ and the super-Ohmic environment $d = 3$, which is ultimately linked to the spectral density of the bath. In the former case the coherence is always lost, whilst in the latter the coherence generally saturates to a residual value which is fixed by the temperature and strength of the coupling and only vanishes in the high-TE. By contrast, in the case of collective decoherence we have identified QR input states that allow the system to evolve in a decoherence-free fashion, the so-called coherence-preserving states. We note that DFS’s do not exist in the specific situation of independent qubit decoherence. We also note that our attention has been drawn to a dynamical-algebraic description that unifies the currently known quantum errors stabilization procedures [27] (see also Refs. [10 (a), 26]). Within the framework of a system formed by a collection of $L$ uncorrelated clusters of subsystems where each cluster fulfills the requirements of collective decoherence, Zanardi has
shown that noiseless subsystems can be built [27].

From the point of view of complexity analysis (and without including any strategy for the stabilizing of quantum information), we should ask how the results reported in the present paper affect those of Ref. [20]. We must identify the coherences that are destroyed more rapidly: from Eqs. (12) and (16) it is easy to see that the coherences with the fastest decay are given by the matrix elements $\rho_{(0n,1a)}$ and $\rho_{(1a,0n)}$. These off-diagonal elements decay as $\exp[-\Gamma_d(t;T)f(L)]$, with

$$f(L) = L + 2 \sum_{m,n=2}^{L} (i_m - j_m)(i_n - j_n) \cos(\omega t_s),$$

for the limit of independent qubit decoherence, and as $\exp[-L^2\Gamma_d(t;T)]$ for the collective decoherence case. Hence, it is clear that for both cases, the longer the QR coherence length, the faster the coherence decay. Despite the fact that the results of Palma et al. [20] are not the same as ours, it turns out that both sets of results lead to the same unwelcome exponential increase of the error rate. We note that the result of Eq. (29) is in general different to the one reported in [20]. We also note that the coherence decay for the case of collective decoherence coincides with that of [20] only for the fastest off-diagonal element decay: if we consider different density matrix elements, the results of Ref. [20] no longer coincide with ours. If the information reported in our work is used for an estimation of the actual decoherence time associated with any given off-diagonal density matrix element (coherence), the results are in general quite different from the ones reported in [20].

We have shown how a bosonic environment destroys the coherences of an arbitrary quantum register. In doing so, we have identified DFS states that are invariant under the coupling to such an environment. This result could be of crucial importance for improving the efficiency of quantum algorithms, for example. We believe that the engineering of DFS’s will become intrinsic to the designs of future quantum computation architectures. There was a recent experimental demonstration of decoherence-free quantum memories [18,19]. This has been achieved for one qubit, by encoding it into the DFS of a pair of trapped $^9$Be$^+$ ions: in this way, Kielpinski et al. have demonstrated the immunity of a DFS of two atoms to collective dephasing [18]. Prior to this experiment with trapped ions, Kwiat et al. demonstrated the robustness of a DFS for two photons to collective noise [19]. Robust quantum memories seem therefore to be well on their way, both theoretically and experimentally, to overcoming the main obstacle to quantum information processing - namely, decoherence.

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APPENDIX A: SINGLE QUBIT DECOHERENCE

The decoherence rates for a single qubit coupled to a reservoir with $d = 1$, and $d = 3$ density of states are (Eq. (19))

$$\Gamma_1(t, T) = \frac{\alpha_1}{\hbar} \int d\omega e^{-\omega/\omega_c} \frac{1 - \cos(\omega t)}{\omega} \coth\left(\frac{\omega}{2\omega_c}\right)$$

(A1)

$$\Gamma_3(t, T) = \frac{\alpha_3}{\hbar} \int d\omega \omega e^{-\omega/\omega_c} [1 - \cos(\omega t)] \coth\left(\frac{\omega}{2\omega_c}\right).$$

(A2)

In Section III A we gave the analytic solutions to these integrals. However, we did not perform a full analysis of those results. We start here by recalling that the solution found for the integral (A1) was an approximate one, valid only for the low-TE ($\omega_c < < \omega_0$): a general solution to this integral requires numerical integration. The second integral was solved analytically for any temperature value $T$ (without making any approximation). In these calculations note that the constant coupling $\alpha_d$ changes its units with the dimensionality of the field: $[\alpha_1] = [(eV)^2 s^2]$, $[\alpha_3] = [(eV)^2 s^4]$, etc.

We first analyze the case $d = 1$. In the low-TE, Eq. (20) leads to the identification of three main regimes for the decay of the coherences: (a) a “quiet” regime, for which $t < \tau_c$, and $\Gamma_1(t, T) \approx c_1 \omega_c^2 t^2/2$; (b) a “quantum” regime, where $\tau_c < t < \tau_\rho$, and $\Gamma_1(t, T) \approx c_1 \ln(\omega_c t)$; and (c) a “thermal” regime, for which $t >> \tau_\rho$, and $\Gamma_1(t, T) \approx 2c_1 \omega_c t$. These regimes have also been discussed in Refs. [4,12] and can be easily identified in Fig. 7 for several different temperatures. In Fig. 7 (i) we have plotted Eq. (19) as a function of $\omega_c t$ for several different temperatures and for $d = 1$. Since the decoherence effects arising from thermal noise can be separated from the ones due to quantum vacuum fluctuations, we have also plotted these partial contributions in order to see their effects over the time-scales involved in the decoherence of the single qubit (Eq. (A1)). It can be seen that for a given value of the temperature parameter $\theta$, a characteristic time for which we start observing deviation of coherence from unity is determined by the shortest of the two time-scales $\tau_c$ and $\tau_\rho$, and that this value is increased when decreasing the temperature $T$, as expected.
From Fig. 7 (i.c) it can clearly be seen that at low temperatures the quantum vacuum fluctuations play the major role in eroding the qubit coherence whilst the contribution due to thermal fluctuations plays a minor role. From this plot we can see the three main regimes indicated above: a quiet ($t < \tau_c$), a quantum ($\tau_c < t < \tau_r$) and a thermal ($t > \tau_r$) regime. In this limit of low-TE, $\tau_r$ is the characteristic time that signals the departure of coherence from unity. Here the qubit dynamics shows a decay: once the end of the ‘quiet’ regime has been reached, the coherences decay to either zero, as in the case of Fig. 7 (ii), and the start of the decoherence process is ruled by $\tau_{dec}$. Similar conclusions can be obtained from Fig. 7 (i.c) and the extent of these residual values.

In the case of the high-TE, the decay due to thermal noise (see dashed line in Fig. 7 (i.a)) becomes more important than the vacuum fluctuation contribution and the start of the decoherence process is ruled by $\tau_r$.

From Fig. 7 (i.c) it can clearly be seen that at low temperatures and coupling strengths $c_3$. For $c_3 = 0.1$, and 0.25 the temperature value for which no saturation occurs at all and the expected decoherence takes place. In Table 3 we give some saturation values for different temperatures and coupling strengths $c_3$. For $c_3 = 0.1$, and 0.25 the temperature value for which any residual coherence vanishes falls in the interval $10 < \theta < 100$ (high-TE). Apparently these residual values shown in Fig. 8 (i) and Table 3 vanish when additional frequency modes associated with the three-dimensionality of the field are taken into account. Even if this is not the case, surely the effects of relaxation mechanisms would mark the extent of these residual values.

### APPENDIX B: TIME EVOLUTION AND THE REDUCED DENSITY MATRIX

In this Appendix we give details of the main steps followed in the calculation of the reduced density matrix given by Eq. (15).

#### 1. The time evolution operator $U_i(t)$

Since Eq. (3) gives

$$H_i(t) = \sum_{n,k} J_n^k (g_n^0 e^{i\omega_t t} b_k^\dagger + g_k^* e^{-i\omega_t t} b_k),$$

we have that

$$U_i(t) = T \exp \left[ -\frac{i}{\hbar} \int_0^t \sum_k g_k e^{i\omega_t t'} J_k b_k^\dagger + e^{-i\omega_t t'} J_k^* b_k dt' \right],$$  

(B1)

where we have introduced the shorthand notation $J_k = \sum_n e^{-i\mathbf{k} \cdot \mathbf{r}} J_n^k$. Hence, we can rewrite Eq. (B1) as

$$e^{-i\mathbf{k} \cdot \mathbf{r}} J_n^k b_k dt' \right].$$  

(B1)
\[ U_i(t) = \exp \left[ \sum_k g_k \varphi_{k_i}(t) J_{z_k}^i b_k \right] \times \]
\[ T \exp \left[ -\frac{i}{\hbar} \int_o^{t'} dt' \sum_k e^{-i\omega_k t'} \times \exp \left( -\sum_{k'} g_{k'k} \varphi_{k_i}(t') J_{z_k'}^i b_{k'} \right) g_k J_{z_k}^i b_k \right. \]
\[ \left. \exp \left( \sum_{k'} g_{k'k} \varphi_{k_i}(t') J_{z_k'}^i b_{k'} \right) \right] . \] (B2)

It is easy to show that the calculation of the product given by the last two lines Eq. (B2) gives the result
\[ g_k J_{z_k}^k \left[ b_k + g_k \varphi_{k_i}(t) J_{z_k}^i \right] . \] (B3)

Hence, the following expression for \( U_i(t) \) arises
\[ U_i(t) = e^{\sum_k g_k \varphi_{k_i}(t) J_{z_k}^i} \times \sum_k e^{-i\omega_k t} \times \exp \left\{ \frac{i}{\hbar} \sum_k g_k J_{z_k}^i \right\} \times \exp \left\{ \sum_k \varphi_{k_i}(t) J_{z_k}^i b_k \right\} \times \exp \left\{ \frac{i}{\hbar^2 \omega_k} - \frac{\varphi_{k_i}(t)}{\hbar} + \frac{|\varphi_{k_i}(t)|^2}{2} \right\} \] (B4)

where we have used the result \( e^{A+B} = e^A e^B e^{-[A,B]/2} \), which holds for any pair of operators \( A, B \) that satisfy \([A, [A, B]] = 0 = [B, [A, B]] \) (as in the case of Eq. (B4)). It is straightforward to see that Eq. (B4) gives the final result
\[ U_i(t) = \exp \left[ i \sum_k g_k \frac{2 \omega_k t - \sin(\omega_k t)}{(\hbar \omega_k)^2} J_{z_k}^i J_{z_k}^i \right] \times \exp \left[ \sum_k \left( A_k(t) - A_k(t) \right) \right] , \] (B5)

where \( A_k(t) = g_k^* \varphi_{k_i}(t) J_{z_k}^i b_k \).

2. The reduced density matrix of a \( L \)-qubit register

We start by using the result for \( U_i(t) \) in order to calculate the decay of the coherences, i.e., \( T_{\rho} \left[ \rho^{B}(0) U_{i}^{(j_m)}(t) U_{j_n}^{(i_i)}(t) \right] \), with \( U_{i}^{(j_m)}(t) \) as defined in Eq. (10). In so doing, we first compute the operator algebra for the product \( U_{i}^{(j_m)}(t) U_{j_n}^{(i_i)}(t) \) by taking into account the expression (B5). The result gives
\[ U_{i}^{(j_m)}(t) U_{j_n}^{(i_i)}(t) = \exp \left[ i \sum_k g_k \frac{2 \omega_k t - \sin(\omega_k t)}{(\hbar \omega_k)^2} J_{z_k}^i J_{z_k}^i \right] \times \sum_{m,n} (i_m i_n - j_m j_n) \cos \mathbf{k} \cdot \mathbf{r}_{mn} \exp \left[ i \sum_k g_k \varphi_{k_i}(t) \right] \times \sum_{m,n} i_m j_n \sin \mathbf{k} \cdot \mathbf{r}_{mn} \exp \left[ \sum_k (\sigma_k b_k - \sigma_k^* b_k) \right] , \] (B6)

where we have set \( \sigma_k = g_k \varphi_{k_i}(t) \sum_m (i_m - j_m) e^{-i\mathbf{k} \cdot \mathbf{r}_m} \) as defined. From the above equation note that the first two exponential terms commute, hence we only have to take the trace over the third term. By doing this (see e.g. Ref. \[24\]), we obtain the result
\[ T_{\rho} \left[ \rho^{B}(0) \exp \left\{ \sum_k (\sigma_k b_k - \sigma_k^* b_k) \right\} \right] = \Pi \exp \left[ -|g_k|^2 \frac{1 - \cos(\omega_k t)}{(\hbar \omega_k)^2} \frac{\coth \left( \frac{\hbar \omega_k}{2k_B T} \right)}{2} \right] \times \sum_{m,n} (i_m - j_m)(i_n - j_n) \cos \mathbf{k} \cdot \mathbf{r}_{mn} , \] (B7)

from where Eq. (11) arises immediately.

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[1] W.H. Zurek, Phys. Rev. Lett. 85, R2265 (2000); E. Bio-
Formally, noise processes can be characterized in terms of factorizable conditions. This factorizable condition is the one that has been extensively studied so far. However, more general situations have been analyzed in connection with the influence functional path-integral method of Feynman and Vernon. See, e.g., Ref. 25.

If relaxation mechanisms are also considered, in general $J_x, J_y \neq 0$. The presence of distinct error generators along $x, y,$ and $z$ make the noise processes non-Abelian and the Hamiltonian can no longer be solved analytically.

This factorizable condition is the one that has been extensively studied so far. However, more general situations have been analyzed in connection with the influence functional path-integral method of Feynman and Vernon. See, e.g., Ref. 25.

Normally, noise processes can be characterized in terms of the interaction algebra that they generate, see e.g. Refs. 26 and 27.

Reflecting the proton's magnetic properties, the proton's mass is not affected by the magnetic field. The proton's mass is independent of the magnetic field. The proton's mass is independent of the magnetic field.

Figure 1. Two qubit “independent decoherence” due to the coupling to a reservoir of the Ohmic type $(d = 1)$ as a function of time $t$ and the transit time $t_o$, for the input states associated with $\Gamma_\alpha^\pm (t, T)$ ($i_a \neq j_a, i_b \neq j_b$). Here $c_1 = 0.25$, and (i) $\theta = 10^{-3}$, (ii) $10^0$, and (iii) $10^2$. $\Gamma_\alpha^\pm (t, T)$, with $i = 1, 3$ are defined in the text.

Figure 2. Two qubit “independent decoherence” caused by the coupling to an ‘Ohmic environment’ as a function of times $t$ and $t_o$, for the input states associated with $\Gamma_\alpha^\pm (t, T)$ ($i_a \neq j_a, i_b \neq j_b$). Here $c_1 = 0.25$, and (i) $\theta = 10^{-3}$, (ii) $10^0$, and (iii) $10^2$.

Figure 3. Two qubit “independent decoherence” due to the super-Ohmic environment $d = 3$ (Eq. (25)) as a function of times $t$ and $t_o$, for the input states associated with $\Gamma_\alpha^\pm (t, T)$. $c_3 = 0.25$, and (i) $\theta = 10^{-3}$, (ii) $10^0$, and (iii) $10^2$.

Figure 4. Two qubit “independent decoherence” due to the super-Ohmic environment $d = 3$ as a function of time $t$ and $t_o$, for the input states associated with $\Gamma_\alpha^\pm (t, T)$. $c_3 = 0.25$, and (i) $\theta = 10^{-3}$, (ii) $10^0$, and (iii) $10^2$.

Figure 5. Two qubit “independent decoherence”. $d = 3$, coupling strength $c_3 = 0.01$, and (i), (iii) $\theta = 10^{-3}$, and (ii), (iv) $\theta = 10^2$.

Figure 6. Two qubit “collective decoherence” for (i) $d = 1$ ‘Ohmic environment’ (Eq. (27)), and (ii) $d = 3$ ‘super-Ohmic environment’ (Eq. (28)), as a function of time and the temperature $\Theta \equiv \omega_r / \omega_c$. $c_1 = c_3 = 0.25$, and $i_a \neq j_a$, and $i_b \neq j_b$. $\Gamma_\alpha^\pm (t, T)$ ($d = 1, 3$) is defined using Eqs. (27) and (28).

Figure 7. (i) Decoherence of a single qubit for an ‘Ohmic environment’ as a function of $t$ (in units of $\omega_c$). The contributions arising from the separate integration of thermal $[\exp(-\Gamma_\alpha^\pm (t))]$ and vacuum $[\exp(-\Gamma_\alpha^\pm (t))]$ fluctuations are shown as dotted curves. $c_1 = 0.25$, (a) $\theta \equiv \omega_r / \omega_c = 0.001$, (b) $10^{-2}$, (c) $10^{-5}$. If $\omega_c$ is the Debye cutoff, $\Theta \approx 10^{-2} T$ (see text): the decoherence shown corresponds to $T = 100 K$, $T = 1 K$, and $T = 1$ mK, respectively. (ii) Coherence decay for (a) $\theta = 10^{-5}$, (b) $10^{-2}$, (c) $10^{-5}$. $c_1 = 0.25$. Here time is given in units of the thermal frequency $\omega_r \equiv k_b T / \hbar$.

Figure 8. Decoherence of a single qubit for (i) $d = 1$, and (ii) $d = 3$, as a function of time (in units of the cut-off $\omega_c$) and the temperature parameter $\Theta \equiv \omega_r / \omega_c$. $c_1 = c_3 = 0.25$. If $\omega_c$ is the Debye cutoff, the range of coherence decay goes from a few mK up to a $10^4 K$ (plot (i)) and (b) $1.5 \times 10^3 K$ (plot (ii)).

Figure Captions

[1] P.W. Shor Phys. Rev. A 52, 2493 (1995); A.M. Steane, Phys. Rev. Lett. 77, 793 (1996).
[2] E. Knill and R. Laflamme, Phys. Rev. A 55, 900 (1997).
[3] P. Zanardi and M. Rasetti, Phys. Rev. Lett. 79, 3306 (1997); L.-M. Duan and G.-C. Guo, *ibid*. 79, 1593 (1997); D.A. Lidar et al., *ibid*. 81, 2594 (1998).
[4] A. Beige et al., Phys. Rev. Lett. 85, 1762 (2000).
[5] J. Kempe et al., Phys. Rev. A 63, 042307 (2001), and references therein.
[6] L. Viola, E. Knill and S. Lloyd, Phys. Rev. Lett. 85, 3520 (2000); *ibid*. 82, 2417 (1999); L. Viola and S. Lloyd, Phys. Rev. A 58, 2733 (1998).
[7] G. S. Agarwal et al., Phys. Rev. Lett. 86, 4271 (2001), and references therein; G. S. Agarwal, Phys. Rev. A 61, 013809 (2000); C. Search and P.R. Berman, Phys. Rev. Lett. 85, 2272 (2000).
[8] D. Kielbiski et al., Science 291, 1013 (2001).
[9] P.G. Kwiat et al., Science 290, 498 (2000).
[10] G.M. Palma, K.-A. Suominen, and A.K. Ekert, Proc. Roy. Soc. London, Ser. A 452, 567 (1996).
[11] L.-M. Duan and G.-C. Guo, Phys. Rev. A 57, 737 (1998).
[12] A. J. Leggett et al., Rev. Mod. Phys. 59, 1 (1987).
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