The Five-Loop Four-Point Amplitude of $\mathcal{N} = 4$ Super-Yang-Mills Theory

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Using the method of maximal cuts, we construct the complete $D$-dimensional integrand of the five-loop four-point amplitude of $\mathcal{N} = 4$ super-Yang-Mills theory, including nonplanar contributions. In the critical dimension where this amplitude becomes ultraviolet divergent, we present a compact explicit expression for the nonvanishing ultraviolet divergence in terms of three vacuum integrals. This construction provides a crucial step towards obtaining the corresponding amplitude of $\mathcal{N} = 8$ supergravity useful for resolving the general ultraviolet behavior of supergravity theories.

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Recent years have seen remarkable progress in understanding and constructing scattering amplitudes in gauge and gravity theories, driven largely by the advent of on-shell techniques. The advances have had broad applications including computations in quantum chromodynamics of multijet processes at the Large Hadron Collider, resummations of planar $\mathcal{N} = 4$ super-Yang-Mills (sYM) amplitudes linking them to string theory via the S-matrix point of view, and studies of ultraviolet (UV) and infrared divergences in gauge and gravity theories. (See ref. [1] for recent reviews.)

These advances have been most striking for the maximally supersymmetric $\mathcal{N} = 4$ sYM amplitude, in the planar limit where the number of color charges is large. Significant progress has also been made for the less well understood nonplanar case, which is the subject of this Letter. Nonplanar contributions to amplitudes in $\mathcal{N} = 4$ sYM theory have been obtained previously through four loops [2–7], along with detailed studies of their UV properties in higher space-time dimensions. The planar part of the five-loop amplitude is found in ref. [8]. Here we carry out a similar study for the five-loop four-point amplitude and analyze the UV divergences in the lowest dimension where they occur. We anticipate that our results will become useful for detailed studies of the structure of the theory, including infrared singularities, anomalous dimensions and other observables related to amplitudes. Such studies can be a useful laboratory for quantum chromodynamics, for example, to help resolve the full structure of infrared singularities (see e.g. ref. [9]).

Beyond the intrinsic interest for understanding $\mathcal{N} = 4$ sYM theory, our construction of the five-loop four-point amplitude is a key step towards obtaining the corresponding amplitudes of $\mathcal{N} \geq 4$ supergravity, needed to help resolve the long-standing question on the possible UV finiteness these theories. In fact, whenever a representation of an $\mathcal{N} = 4$ sYM amplitude is constructed that exhibits a duality between color and kinematics [10, 11], a simple pathway exists for obtaining corresponding $\mathcal{N} \geq 4$ supergravity loop integrands [3, 7, 12, 13]. In particular, the $\mathcal{N} = 8$ supergravity integrand follows trivially when the duality is manifest, simply by replacing color factors with the kinematic numerators of the diagrams. Although the form of the five-loop four-point amplitude presented here does not manifest the required duality, it does offer an excellent starting point for finding such representations.

Explicit constructions of amplitudes have played a key role for determining the UV divergence structure of gauge and gravity theories as a function of dimension. $\mathcal{N} = 4$ sYM theory is known [3, 4, 14] to be UV finite in dimensions

$$D < 4 + \frac{6}{L}, \quad (L > 1) \quad (1)$$

where $L$ is the loop order. This exhibits the well-known UV finiteness in $D = 4$ [15]. A remaining open question is whether the bound (1) is saturated to all loop orders. From explicit computations, it is known to be saturated for $L \leq 4$ [3, 4, 15]. As commented on in ref. [6], the $L = 5$ planar amplitude [8] also saturates the bound (1). Below we give a simple expression for the divergence, including nonplanar parts.

A related open question is whether maximally supersymmetric $\mathcal{N} = 8$ supergravity has the same finiteness bound as $\mathcal{N} = 4$ sYM theory, implying it is UV finite in $D = 4$, or if it has a worse behavior. (For a recent optimistic opinion see ref. [16]; for a recent pessimistic one see ref. [17].) Explicit calculations of the divergences [3, 4, 7, 18, 19] and symmetry and other arguments [20] show that through four loops, the bound (1) holds in $\mathcal{N} = 8$ supergravity. Beyond this, the arguments suggest that $\mathcal{N} = 8$ supergravity will have a worse behavior, leading to a seven-loop divergence in $D = 4$. However, when similar symmetry arguments are applied to $\mathcal{N} = 4$ supergravity, they imply the existence of a valid...
three-loop counterterm \([21]\); the coefficient of this counterterm has recently been explicitly shown to vanish \([13]\). (See ref. \([22]\) for a string-based argument.) This exhibits better behavior than implied by known symmetry considerations and is in line with cancellations suggested by unitarity arguments \([23]\). In particular, it emphasizes the importance of directly checking the amplitudes whether eq. \((4)\) holds for \(N = 8\) supergravity at \(L = 5\).

Our construction of the five-loop four-point amplitude of \(N = 4\) sYM theory organizes it in the form,

\[
A_4^{(5)} = i g^{12} s t A_3^{\text{tree}} \sum_{S_4} \frac{1}{S_4} \sum_{i=1}^{416} \prod_{j=3}^{9} d^D l_j \frac{1}{S_i} \prod_{m=3}^{20} t_m^{m-5},
\]

where the second sum runs over a set of 416 distinct (non-isomorphic) graphs with only cubic (trivalent) vertices. Some sample graphs are shown in fig. \(1\). The first sum runs over all 24 permutations of external leg labels indicated by \(S_4\). The symmetry factors \(S_i\) remove overcounts, including those arising from internal automorphism symmetries with external legs fixed. Here we absorb all contact terms (i.e. terms with fewer than the maximum number of propagators) into graphs with only cubic vertices, by multiplying and dividing by appropriate propagators. We denote external momenta by \(l_i\) for \(i = 1, \ldots, 4\) and the five independent loop momenta by \(l_j\) for \(j = 5, \ldots, 9\). The remaining \(l_j\) are linear combinations of these. The color factors \(C_i\) of all graphs are obtained by dressing every three-vertex in the graph with a factor of \(f^{abc} = \text{Tr}([T^a, T^b]T^c)\), where the gauge group generators \(T^a\) are normalized as \(\text{Tr}(T^a T^b) = \delta^{ab}\). The gauge coupling is \(g\) and the crossing symmetric prefactor \(s t A_3^{\text{tree}}\) is in terms of the color-ordered \(D\)-dimensional tree amplitude \(A_3^{\text{tree}} \equiv A_3^{\text{tree}}(1, 2, 3, 4)\) and \(s = (k_1 + k_2)^2\) and \(t = (k_2 + k_3)^2\).

To construct the numerators \(N_i\), we use the method of maximal cuts \([8]\), based on the unitarity method \([25]\). Application of this method and various strategies for greatly streamlining the construction of the numerators has been described in considerable detail in ref. \([6]\), so here we give only a brief summary. The method works in \(D\) dimensions and can be used to obtain local expressions, from which UV divergences can be straightforwardly extracted.

We start with an ansatz for the diagram numerators containing free parameters to be determined by matching against generalized unitarity cuts. Our ansatz is a polynomial of degree four in the kinematic invariants, subject to the power-counting constraint that no term has more than six powers of loop momentum. We also demand that each numerator respects the automorphism symmetries of the graph. Once a solution is found satisfying a complete set of cut conditions, we have the integrand. If an inconsistency is encountered, the ansatz must be enlarged. We note that the solutions for numerators are not unique and different choices can be mapped into each other by generalized gauge transformations \([11, 11, 20]\).

The parameters of the ansatz are determined from generalized unitarity cuts that decompose a loop integrand into products of on-shell tree amplitudes summed over all intermediate states, \(\sum_{\text{states}} A_4^{\text{tree}} A_3^{\text{tree}} \cdots A_1^{\text{tree}}\). These cuts are organized according to the number of cut propagators that are replaced with on-shell conditions. We start from the maximal cuts (MCs) where all 16 internal propagators cut. After obtaining the MCs, we then construct all next-to-maximal cuts (NMCs), with 15 cut propagators. We continue this process, systematically constructing analytic expressions for \((\text{next-to})^k\)-maximal cuts (\(N^k\)MCs) with fewer and fewer imposed cut conditions. For the five-loop four-point \(N = 4\) sYM amplitude this process terminates at \(k = 3\), since the power counting of the theory prevents numerator factors from can-
celing more than 3 propagators. Representative cuts for
$k = 0, 1, 2, 3$ are shown in fig. 2. The number of nonzero
(color-stripped) cuts of type $N^k MC$ are 410, 2473, 7917,
15156 for $k = 0, 1, 2, 3$, respectively. This count does
not include the different independent color orderings of
each cut. In addition to the nonvanishing cuts, there is
a large class of $N^k \leq 3MCs$ that evaluate to zero because
they contain nontrivial ($n \leq 3$)-point subamplitudes.

Each cut can be reduced to a relatively simple analy-
tic expression. All $N^k MCs$ used in the construction
are evaluated in $D$ dimensions by embedding them in
auxiliary cuts that can be directly expressed in terms of
simplified analytic forms. As discussed in some detail in
ref. [6], two particularly useful cuts for this purpose are
two-particle cuts and box cuts. Whenever a two-particle
reducible cut can be factorized into two four-point am-
plitudes, as illustrated in fig. 3(a), all contributions to
the cut can be written down immediately using lower-
loop results [2]. Similarly, all cut contributions that pos-
sess a four-point loop or box subdiagram, illustrated in
fig. 3(b), are simple to evaluate [6]. A third type of
auxiliary cut, illustrated in fig. 3(c), allows us to map
known $D$-dimensional planar cuts to nonplanar ones via
the new tree-amplitude relations uncovered in ref. [10].
Alternatively, one can construct numerator representa-
tions that obey the color-kinematics duality for each cut
separately [6], giving local numerator relations between
planar and nonplanar diagrams, up to terms that vanish
on the cut. This technique is especially useful whenever
the cut contains massless bubble or tadpole subdiagrams
(as sometimes occurs for $N^3 MCs$), since the local nu-
merators are automatically free of spurious singularities
that can appear with other methods. A fourth type of
auxiliary cut valid in $D$ dimensions and used in our con-
struction is one that maps five-loop nonplanar cuts to
to six-loop planar cuts [27].

We have found a choice of parameters in the starting
ansatz whose cuts correctly reproduce the $N^k MCs$ at the
level of the integrand. We thus have a complete integral
representation of the five-loop amplitude. As a few sim-
ple examples, the numerators of graphs 1, 12 and 284 are

\[
\begin{align*}
N_1 &= s^4, \\
N_{12} &= 2s^3k_3 \cdot l_5, \\
N_{284} &= 2s^2((l_{10} \cdot l_{20})^2 + (l_{13} \cdot l_{18})^2),
\end{align*}
\]

(3)
corresponding to the graphs in fig. 1 labeled as (1), (12)
and (284) and matching the labeling in the ancillary
file [24]. The lines with arrows in fig. 1 give the momen-
tum labels and directions. The symmetry factors $1/S_i$
for these graphs are, respectively, 1/4, 1 and 1/4.

The complete set of 416 nonvanishing graphs with their
associated symmetry factors, numerators and color fac-
tors are included in the ancillary file [24]. We note that
graphs 61, 67, 133, 137, 263, 382, 412 have vanishing color
factors for a general gauge group (due to symmetry prop-
e rties of the graph), and hence do not contribute to the
amplitude. However, we include them in our represen-
tation because they are needed for constructing gravity amplitudes [6].

We have carried out extensive cross checks on our re-
result. The cut construction automatically cross checks
the vast majority of contributions because they are de-
tected in multiple independent channels. As an addi-
tional rather nontrivial check, in four dimensions we con-
firmed numerically that all the analytically-obtained cuts
are correct; to carry out this check we used the simple
algorithms of ref. [28] for carrying out the supersums ap-
pearing in the cuts. We have also carried out systematic
cross checks using generalized cuts with up to six col-
collapsed propagators. Furthermore, we have evaluated a
set of cuts that suffices to detect all “snail” contribu-
tions, equivalent to bubbles on external legs (see sections
2D and 3C of ref. [3]), showing that such contributions
do not appear in our representation.

Starting with the constructed integrand, we obtain the
potential logarithmic divergence in the five-loop critical
dimension, $D = 26/5$, following the same strategy as at
lower loops [6, 7, 18]: We expand the amplitudes
at small external momentum and keep the leading term.
The result of this expansion is a sum of about 185 vac-
uum diagrams; a few of which are displayed in fig. 4.

As discussed in refs. [7, 19], the vacuum integrals in this
expansion are not all independent (so the precise num-
ber appearing initially can vary). We derive consistency
relations between the vacuum integrals by considering
auxiliary linearly divergent integrals of similar propaga-
tor structure, expanding them around zero external mo-
menta and requiring that the results of the expansion
be independent of different integrand parametrizations.
This also directly checks the procedure for inte-
that no further hidden cancellations remain at vacuum integrals in eq. (5) are all positive definite, proving with the chosen normalization, the Wick-rotated vacuum integrals giving, two examples are,

\[
V^{(j)} = 2V^{(c)} + 35V^{(i)} + \frac{365}{6}V^{(d)} - \frac{4175}{162}V^{(c)} - \frac{1045}{18}V^{(f)} - \frac{8965}{81}V^{(i)} + \frac{2405}{3}V^{(b)},
\]

where the labels correspond to the ones in fig. 4.

After using the consistency relations, the leading UV divergence is remarkably simple and given by only three vacuum integrals. For SU(N) color contributions, using FIESTA [29] we have numerically evaluated the integrals giving,

\[
\begin{align*}
V^{(a)} &= \frac{0.331K}{\epsilon}, & V^{(b)} &= \frac{0.310K}{\epsilon}, & V^{(c)} &= \frac{0.291K}{\epsilon},
\end{align*}
\]

where the dimensional regularization parameter is \( \epsilon = (26/5 - D)/2, K = 1/(4\pi)^{13} \) and numerical integration uncertainties are below the displayed digits. It is interesting that the ratio between the subleading and leading contributions 45.0/N^2 is rather close to the three- and four-loop ratios, 43.3/N^2 and 44.4/N^2. A striking feature of the result is that the divergence does not contain terms beyond \( \mathcal{O}(1/N^2) \) suppression, nor does it contain double-trace contributions when converted to an SU(N) color-trace representation, in line with expectations from lower loops [2]. The second of these features has already been discussed in refs. [6, 30]. Furthermore, the three integrals and their relative coefficients have a remarkable similarity with the corresponding ones at four loops, as seen by comparing to eq. (5.33) of ref. [6]. At lower loops, exactly the same combination of integrals appearing in the subleading-color contributions to the \( N = 4 \) sYM divergences appears in the corresponding ones of \( N = 8 \) supergravity [6]. A natural conjecture is that the same holds at five loops, so that the two theories share the same critical dimension, \( D = 26/5 \).

In summary, the five-loop amplitude we have constructed here offers detailed information on the structure of the nonplanar sector of \( N = 4 \) sYM theory. As a first application, we have shown that simple patterns for divergences in the dimension where they first appear continue to hold through five loops; this hints that the divergences are controlled by a deep structure of the theory. Our construction of the five-loop four-point amplitude is an excellent starting point to try to find a representation exhibiting the duality between color and kinematics. We expect that the results presented here will be crucial input for obtaining corresponding supergravity amplitudes and for studying their UV behavior.

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