Exact weak Majorana-type zero modes in a spin/fermion chain

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We study an exactly solvable one-dimensional spin-\(\frac{1}{2}\) model which can support weak zero modes in its ground state manifold. The spin chain has staggered XXZ-type and ZZ-type spin interaction on neighbouring bonds and is thus dubbed (XXZ, Z) chain. The model is equivalent to an interacting fermionic chain by Jordan-Wigner transformation. We study the phase diagram of the system and work out the conditions and properties of its weak zero modes. These weak zero modes are given by polynomials of Majorana fermion operators and are named “weak Majorana-type zero modes”. The fermionic chain representation contains only fermion hopping and interaction terms and may have potential realization in experiments.

The appearance of edge or defect zero modes is the key indication for topological phases. In free fermionic systems, a complete classification relating symmetries of the Hamiltonian and types of edge (or defect) zero modes that can exist has been given [1]. Among these zero modes the most fundamental is single Majorana zero modes [2, 3], for which a few superconducting toy models have been proposed, including the two-dimensional p\(x\) + ip\(y\) superconductor [4], the Kitaev p-wave superconducting chain [5] and some composite models [6]. In free fermionic systems, the zero modes are given by first-order fermionic operators that commute with the Hamiltonian itself. According to a classification by Alicea and Fendley [7], such operators are strong zero modes. For real systems in which interactions cannot be neglected, strong zero modes are difficult to find and write down explicitly [8]. In these systems, more common types of zero modes are weak zero modes, which are operators that commute with the Hamiltonian projected onto a subspace of the Hilbert space [7, 9]. From a broader perspective, a weak zero mode is degeneracy of some of the eigenstates of the Hamiltonian, instead of all the eigenstates as for strong zero modes. Among the first types of weak zero modes studied is the Majorana zero modes associated with vortices and edges in the Kitaev-type chiral spin liquids [10–13]. Weak zero modes have also been found and studied in one-dimensional systems including the interacting Kitaev chain [14, 15].

In this work we study an exactly solvable one-dimensional spin-\(\frac{1}{2}\) chain which has an equivalent interacting fermion-chain representation by Jordan-Wigner transformation [16, 17]. The model is solved exactly using another Jordan-Wigner transformation in a rotated basis [18, 19]. By examining its physical states, which are described by fermions coupled to static \(Z_2\) variables, we show that ground-state weak zero modes appear on the edges of the chain in some phases of the model. Such weak zero modes can be exactly written down in both the spin-chain and fermion-chain representations; moreover, the fermion-chain representation is a simple interacting model without superconducting terms, which makes it distinct from previously studied models for weak zero modes [14, 15]. The weak zero modes in the fermion chain are given by polynomials of Majorana operators and thus represent a particle-hole type degeneracy in certain subset of the physical Hilbert space. These weak zero modes can be seen as generalizations of the first-order Majorana strong zero modes in free systems and are thus named weak Majorana-type zero modes [19]. Depending on the order of the polynomials, these weak zero modes are not necessarily fermionic; in our case the weak Majorana-type zero modes are bosonic since the polynomials contain only even order terms, which is distinct from other models [19]. Despite the similarities, the weak zero modes in our spin chain model are also different from those of the two-dimensional Kitaev-type chiral spin liquids [11, 12] in that there exists a physical fermionic representation of the model without any constraint, in other words, the model has potential realization in experiments as a fermionic system. As can be seen later, the fermion-chain representation of our model is simple enough to facilitate such realizations.

THE MODEL AND ITS SOLUTION

To introduce the model, we start with a spin-\(\frac{1}{2}\) chain of finite length, the spins are labeled by integer \(n\) which goes from 1 to \(2N\) (\(N\) is a large integer). The spins interact with their nearest neighbours by a staggered XXZ and ZZ coupling, one unit cell of the model consists of two sites \((2n−1, 2n)\). The Hamiltonian of the spin chain is

\[
H = J \sum_{n=1}^{N} (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y) + K \sum_{n=1}^{N-1} (S_n^z S_{n+1}^z + S_n^x S_{n+1}^x + S_n^y S_{n+1}^y),
\]

where \(J\) and \(K\) are the coupling constants.

FIG. 1: The lattice of the (XXZ, Z) spin-chain model. The black bonds indicate the ZZ spin coupling and the green bonds are the XY spin coupling. For the fermionic representation of the model in which the black bonds indicate the fermion interaction terms and the green bonds represent the fermion hopping terms.
FIG. 2: The model solved by a rotated Jordan-Wigner transformation. Majorana bilinears that commute with the Hamiltonian are denoted by green dashed lines, which in turn form the static $Z_2$ variables. The solution is a Majorana hopping model, connected by orange and black lines, coupled with static $Z_2$ variables.

given by
\[
\mathcal{H} = \sum_{n=1}^{N} \left[ J_n (\sigma_{2n-1}^x \sigma_{2n}^x + \sigma_{2n-1}^y \sigma_{2n}^y) + K_n \sigma_{2n-1}^z \right] - \sum_{m=1}^{N-1} \tilde{K}_m \sigma_{2m} \sigma_{2m+1}^z,
\]
in which $J_n$ and $K_n$ give the XXZ coupling strength on bonds $(2n-1, 2n)$ and $\tilde{K}_m$ gives ZZ coupling strength on bonds $(2m, 2m+1)$, as shown in Fig. 1. The model is thus referred to as $(XXZ, Z)$ spin chain.

The spin chain model (1) can be easily transformed into a fermionic chain by Jordan-Wigner transformation [16, 17]. To this end we introduce a fermion $c_i^\dagger$ for every spin $\sigma_i$ and require $\sigma_i^+ = c_i^\dagger \sum_{j<i} c_j^\dagger$, as well as $\sigma_i^\dagger = 2c_i^\dagger c_i - 1$. Under such transformation the original spin model (1) becomes
\[
\mathcal{H} = \sum_{n=1}^{N} \left[ 2J_n (c_{2n-1}^\dagger c_{2n} + c_{2n}^\dagger c_{2n-1}) 
+ K_n (c_{2n}^\dagger c_{2n-1} - 1) (c_{2n}^\dagger c_{2n-1} - 1) \right] 
+ \sum_{m=1}^{N-1} \tilde{K}_m (c_{2m}^\dagger c_{2m} - 1) (c_{2m+1}^\dagger c_{2m+1} - 1).
\]

This is an interacting fermionic Hamiltonian with no superconducting terms, it has a fine-tuned chemical potential and the kinetic hopping terms vanish for every second bond and thus break translational invariance (see Fig. 1). Experimentally the fermionic model can possibly be realized in Su-Schrieffer-Heeger systems [20] with Peierls instability, in which the translational symmetry is broken such that the fermion hopping can be neglected for every second bond although interaction between neighbouring sites cannot be neglected.

The spin chain Hamiltonian (1) and the fermionic Hamiltonian (2) form the two representations of the same model. The two representations enjoy a one-to-one correspondence between their Hilbert space and eigenstates, hence the physical properties such as degeneracies are identical. The model (1) can be solved by another Jordan-Wigner transformation in a rotated basis $(\tilde{x}, \tilde{y}, \tilde{z}) \rightarrow (y, z, x)$ [18, 19]; to this end we introduce another set of fermions $d_i$ and define $\sigma_i^\dagger = (d_i + d_i^\dagger) \sum_{j<i} d_j^\dagger d_j$, $\tilde{\sigma}_i = i(d_i - d_i^\dagger) \sum_{j<i} d_j^\dagger d_j$ as well as $\sigma_i^\dagger = 2d_i^\dagger d_i - 1$. It simplifies the problem to decouple the $d_i$ fermions into Majorana fermions $\eta_i^\alpha$ and $\eta_i^\beta$ by $d_i^\dagger = \frac{1}{2}(\eta_i^\alpha + i\eta_i^\beta)$. In terms of the Majorana fermions the new Jordan-Wigner transformation can be written as
\[
\begin{align*}
\tilde{\sigma}_i^\dagger &= -i\eta_i^\alpha \eta_i^\beta, \\
\sigma_i^\dagger &= \eta_i^\alpha \prod_{j=1}^{i-1} (i\eta_j^\beta \eta_j^\dagger), \\
\sigma_i^\dagger &= \eta_i^\beta \prod_{j=1}^{i-1} (i\eta_j^\alpha \eta_j^\dagger).
\end{align*}
\]

Using the transformation (3), the original spin model (1) can be written as
\[
\mathcal{H} = \sum_{n=1}^{N} \left[ J_n (\eta_{2n-1}^\alpha \eta_{2n}^\beta) (i\eta_{2n-1}^\beta \eta_{2n}^\alpha) + J_n \eta_{2n-1}^\alpha \eta_{2n}^\alpha 
- K_n \eta_{2n-1}^\beta \eta_{2n}^\beta \right] + \sum_{m=1}^{N-1} \tilde{K}_m (-i) \eta_{2m}^\alpha \eta_{2m+1}^\beta.
\]

In this Majorana Hamiltonian we notice that the Majorana bilinears $i\eta_{2n-1}^\alpha \eta_{2n}^\beta$ commute with the Hamiltonian; therefore a static $Z_2$ variable $\tau_n = \eta_{2n-1}^\alpha \eta_{2n}^\beta$, can be introduced for every unit cell. The Majorana Hamiltonian is illustrated in Fig. 2. With these definitions, the Hamiltonian is finally written as
\[
\mathcal{H} = \sum_{n=1}^{N} \left[ J_n (1 + \tau_n) i\eta_{2n-1}^\beta \eta_{2n}^\alpha 
- K_n \tau_n \right] 
+ \sum_{m=1}^{N-1} \tilde{K}_m (-i) \eta_{2m}^\alpha \eta_{2m+1}^\beta,
\]

which takes the form of a Majorana hopping model coupled with static $Z_2$ variables. To determine physical eigenstates of the model, one first take a set of $\{\tau_n\}$, under which the Hamiltonian becomes a Majorana hopping model; fermion eigenstates can then be worked out accordingly, and the physical eigenstates of the model take the form of $|\psi\rangle_E = |\tau_n\rangle \otimes |\eta_{2n-1}^\alpha, \eta_{2n-1}^\beta\rangle_{\tau}$. Every distribution of the $Z_2$ variables and its corresponding fermionic space can be understood as a sector of the Hilbert space. All physical eigenstates of the model can be worked out sector by sector. This solution of the model shares some similarities with the Jordan-Wigner transformation solution of the Kitaev honeycomb model [21, 22]; the model can also be solved using the SO(3) Majorana representation of spin [12, 23-25].

**PHASE DIAGRAM**

The model (4) is invariant under all the $J_n$ and $\tilde{K}_n$ change sign, accompanied by $\eta_{2m+1}^\beta \rightarrow -\eta_{2m+1}^\beta$ for all $m$. 
In light of this symmetry, to simplify further treatment of the Hamiltonian (5) we take all the parameters $K_m \equiv -1$ and assume that all $J_n$ and $K_n$ are constants, namely $J_n \equiv J$ and $K_n \equiv K$. Among all the physical eigenstates of the model, we are most interested in the ground state. Here we make the assumption that the ground state lies in the sectors in which all $\tau_n$ are equal. In other words, the ground state is searched for in two sectors $\{\tau_n \equiv 1\}$ and $\{\tau_n \equiv -1\}$. The sector that contains the ground state is then referred to as the ground state sector.

Under these assumptions the Hamiltonian (5) becomes

$$\mathcal{H} = \sum_{n=1}^{N} \left[ J(1+\tau)i\eta^\alpha_{2n-1}\eta^\beta_{2n} - K\tau \right] + \sum_{m=1}^{N-1} i\eta^\alpha_{2m}\eta^\beta_{2m+1}. \quad (6)$$

To find the ground state energy, we pair up $\eta^\beta_{2n-1}$ and $\eta^\alpha_{2n}$ in each unit cell to define a complex fermion $f^\dagger_n = \frac{1}{2}(\eta^\beta_{2n-1} + i\eta^\alpha_{2n})$. Due to the translational invariance of (6), a Fourier transformation can be performed, $f^\dagger_n = \frac{1}{\sqrt{N}} \sum_k f^\dagger_k e^{ikn}$. In the momentum space, the Hamiltonian (6) is given by

$$\mathcal{H} = -NK\tau + \sum_k \left( f^\dagger_k f^-_{-k} \right) \mathbf{H}_k \left( f^\dagger_k f^-_{-k} \right), \quad (7)$$

in which

$$\mathbf{H}_k = \begin{pmatrix} \cos k - J(1+\tau) & -i\sin k \\ i\sin k & J(1+\tau) - \cos k \end{pmatrix}. \quad (8)$$

The eigenvalues of the matrix $\mathbf{H}_k$ are $E_k = \pm \sqrt{1 + J^2(1+\tau)^2 - 2J(1+\tau)\cos k}$. As the number of unit cells $N \to \infty$, the ground state energy density is given by the integral

$$\frac{E_0}{N} = -\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \sqrt{1 + 4J^2 - 4J\cos k - K} \quad (9)$$

for the $\tau \equiv 1$ sector, and

$$\frac{E_0}{N} = K - 1 \quad (10)$$

for the $\tau \equiv -1$ sector. To determine which sector has the lowest ground state energy one needs to compare these two values for each $J$ and $K$.

We now turn to discuss the possible zero modes in the Majorana hopping model in every sector. To do so, it helps to consider the chain as located on a closed circle; As can be seen from the Hamiltonian (5) the edge corresponds to a single broken $\tilde{K}$ bond. The ground state sector can only have edge zero modes since we have assumed that it does not break translational invariance. For the excited states sectors, the situation varies according to the distribution of $\tau_n$: for some of the distributions the chain can have broken $J - K$ bonds (when $\tau_n = -1$ on that bond), causing defects for the corresponding Majorana hopping model. In these excited-state-sectors, the possible Majorana zero modes are complicated. For the translationally invariant ground state sector, we have the following results on its edge zero modes.

(i) If the ground state of the model is in the sector $\tau_n \equiv -1$, then we have strictly localized Majorana zero modes $\eta^\alpha_n$ and $\eta^\beta_n$ on both edges of the chain. As can be seen in the Hamiltonian (6), this case is effectively equivalent to $J = 0$. We call this type of edge zero modes type $I$ zero modes and such phase is referred to as phase $A$. The ground state of the model is degenerate in phase $A$.

(ii) If the ground state of the model is in the sector $\tau_n \equiv 1$, the Majorana hopping model from the Hamiltonian (6) becomes a first order model as defined in Ref. [6], and we have two situations for its Majorana zero modes. (a) When $|J| < \frac{1}{2}$, there are localized edge Majorana zero modes on both ends whose wavefunction is exponentially decaying into the system [6]. We refer to this kind of edge zero modes as type $II$ zero modes and the phase as phase $B$. For excited states in other sectors without translational invariance, some of the bonds have $\tau_n = -1$, but no defect zero modes are associated with these bonds. (b) When $|J| > \frac{1}{2}$, there is no localized edge zero modes in the ground state sector. But localized defect zero modes exist around $\tau_n = -1$ bonds in other excited-state-sectors, provided that these defects are well-separated from each other. This phase is called phase $C$. The ground state is not degenerate in phase $C$.

Unlike type $I$ zero modes, whose existence implies exact degeneracy among the states within its sector, type $II$ zero modes and other defect zero modes whose wavefunctions are not strictly localized means that the states are only approximately degenerate. For all sectors, the split between energy levels, which exponentially decays with the distances between these zero modes, can only
be taken as vanishing when the $\tau_n = -1$ bonds are well-separated from each other. It is thus impossible to write down a universal operator that commute with the original Hamiltonian (5) which brings degeneracy in all sectors. Therefore both type I and type II edge zero modes are weak zero modes [7, 9] which only implies degeneracy in certain subspaces (or sectors) of the Hilbert space.

After numerically evaluating the ground state energies in the two translationally invariant sectors, we arrive at the phase diagram Fig. 3.

**WEAK MAJORANA-TYPE ZERO MODES**

To understand the nature of the two types of zero modes, we consider a simple XXZ model of a two-spin system, whose Hamiltonian is $H = J(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y) + K \sigma_1^z \sigma_2^z$. It has two degenerate eigenstates $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, with energy $K$; the other two eigenstates of the model are $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, with energy $2J - K$, and $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, with energy $-2J - K$. The pair of degenerate states with energy $K$ corresponds to type I zero mode in our original model (1), which is strictly localized in $d$-fermion language but strictly non-localized in spin language and $c$-fermion language. This leads us to guess that the degenerate ground states of the original spin model are $|\uparrow\uparrow\uparrow\cdots\uparrow\rangle$ and $|\downarrow\downarrow\downarrow\cdots\downarrow\rangle$ in phase A. This is indeed the case as we have translations from Majorana fermions back to the spin operators $\eta^{\alpha}_1 = \sigma_1^x \prod_{j=1}^{n-1} (\sigma_j^z)$ and $\eta^{\beta}_1 = \sigma_1^y \prod_{j=1}^{n-1} (\sigma_j^z)$; for type I Majorana zero modes in phase A we have $\eta^{\alpha}_1 \rightarrow i\sigma_1^\tau$ and $\eta^{\beta}_1 \rightarrow i\sigma_2^\tau \prod_{j=1}^{n}\sigma_j^\tau$, these are the zero-mode operators in spin language. In fact we have the product operator $\prod_{j} \sigma_j^\tau$ commutes with the spin Hamiltonian (1); but just like in the two-spin XXZ model, this does not lead to strong zero modes. As for type II zero modes in phase B, we focus on the two non-degenerate states in the two-spin model. Upon enlarging the length of the spin chain, these two states evolve into a spin-wave-like band of eigenstates and the type II zero modes exist in this band. The condition for the existence of the type II zero modes is exactly given by the topological condition of the Kitaev chain [5, 6, 26], such condition distinguishes phase B and phase C.

To go to the $c$-fermion language for the fermionic representation of the model (2), we first decouple the $c$-fermion into Majorana fermions, $c_i = \frac{1}{2}(\gamma_i^\alpha - i\gamma_i^\beta)$. The transformation between the two types of Majoranas can be obtained by noting that $\sigma_j^\tau = i\gamma_j^\alpha \gamma_j^\beta$, as well as $\eta^{\alpha}_n = \gamma_1^\alpha \gamma_2^\beta \prod_{j=1}^{n-1} (\sigma_j^\tau)$ and $\eta^{\beta}_n = i\gamma_1^\alpha \gamma_2^\beta \prod_{j=1}^{n-1} (\sigma_j^\tau)$. In general the transformation between $\eta$ and $\gamma$ Majorana fermions can be worked out. To this end we have $\eta^{\alpha}_n$ is a product of $n$ $\gamma$ Majorana fermions, namely

$$\eta^{\alpha}_{2m} = -(i)^m \gamma_1^\beta \gamma_2^\alpha \cdots \gamma_{2m-1}^\beta \gamma_{2m}^\alpha,$$

$$\eta^{\beta}_{2m+1} = (i)^m \gamma_1^\alpha \gamma_2^\beta \cdots \gamma_{2m}^\alpha \gamma_{2m+1}^\beta. \quad (11)$$

And $\eta^{\alpha}_n$ is a product of $n + 1$ $\gamma$ Majorana fermions,

$$\eta^{\beta}_{2m} = -(i)^m \gamma_1^\beta \gamma_2^\alpha \cdots \gamma_{2m-2}^\alpha \gamma_{2m-1}^\beta \gamma_{2m}^\alpha,$$

$$\eta^{\beta}_{2m+1} = (i)^{m+1} \gamma_1^\beta \gamma_2^\alpha \cdots \gamma_{2m}^\alpha \gamma_{2m+1}^\beta + 1. \quad (12)$$

Therefore, the type I zero mode, which is a product of $\gamma$ Majorana fermions from all sites, is strictly non-localized. The type-II zero modes whose wavefunction is exponentially decaying into the system are given by $\zeta^{\beta}_L = \sum_{n=1}^{N} \lambda_{2n-1} \gamma_2^\tau \gamma_n^\alpha$, which is localized on the left end of the chain, and $\zeta^{\beta}_R = \sum_{n=N}^{1} \lambda_{2n-1} \gamma_2^\tau \gamma_n^\alpha$, which is localized on the right end of the chain. They are both translated into polynomials of $\gamma$ Majorana fermions, which we call “Majorana-type” zero modes [19]. The Majorana-type zero modes are a generalization of the Majorana zero modes since they represent a particle-hole type symmetry in the projected subset of the Hilbert space. In our model the Majorana-type zero modes on both ends involve terms that are even order of $\gamma$ Majorana fermion, indicating that these zero modes are bosonic in the $c$-fermion language. This is understandable because the static $Z_2$ variables $\tau_n$ are given by $\tau_n = -(1)^{m-n^2/2}$ in the $c$-fermion language, with $n_i$ being the occupation number on site $i$; these static $Z_2$ variables classify the subset of the Hilbert space and thus the weak zero modes in each subset must commute with the $\tau_n$ operators. Also notice that the polynomials involve terms that are macroscopic order of $\gamma$ Majorana fermions so that the locality of the weak zero modes has some subtleties [27].

**CONCLUSION**

To summarize, we have studied an exactly solvable spin chain which has a dual description of interacting fermions in one dimension. The model possesses weak Majorana-type zero modes in some sectors of the Hilbert space. Future studies can generalize the discussion beyond the translational invariance assumption and consider possible defect induced zero modes. The duality transformation between Majorana and Majorana-type zero modes is possible only for one-dimensional systems thanks to the equivalence between bosonic and fermionic degrees of freedom. Nevertheless, the results unveil some properties of weak Majorana-type zero modes in real systems in which interactions are included.

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