A simple model for citation curve

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ABSTRACT

There is considerable interest in the citation count for an author’s publications. This has led to many proposals for citation indices for characterizing citation distributions. However, there is so far no tractable model to facilitate the analysis of these distributions and the design of these indices. This paper presents a simple equation for such design analysis of these distributions, and the design of citation indices. This paper presents a simple equation for such design analysis of these distributions, and the design of citation indices. The equation has three parameters that are calibrated by three geometrical characteristics of a citation distribution. Its simple form makes it tractable. To demonstrate, the equation is used to derive closed-form expressions for various citation indices, analyze the effect of time and identify individual contribution to the Hirsch index for a group.

1. INTRODUCTION

Since the launch in 2004 of the web search engine Google Scholar\(^1\), it has become easy to look for the papers and publication record of a researcher. The information provided currently includes the citations for each publication, and a h-index for the citation count.

The h-index was proposed by Hirsch in 2005\(^9\). It gained much attention and triggered numerous proposals for alternative citation indices\(^4\), but there is controversy over characterizing an author’s research record by such indices\(^2\).

In this paper, we do not advocate one index or another, and propose none ourselves. Instead, we offer a simple equation for approximating the distribution of citation count. We claim that this equation can facilitate the analysis of these distributions, and the design of citation indices.

To demonstrate our claim, we use the equation to derive closed-form expressions for various indices in Sec. 3. In Sec. 4, we further apply the equation to examine the effect of time, and how individual h-indices contribute to the h-index of a group of researchers.

\(^1\)https://scholar.google.com/

2. THE PROPOSED MODEL

Let \(\Psi(n)\) denote the number of citations for an author’s \(n\)-th publication, where the publications are sorted based on their citation numbers so that \(\Psi(n) \geq \Psi(n')\) for \(n < n'\). Let \(M = \Psi(1)\), i.e. the maximum number of citations for the author’s most cited publication. Suppose the author has \(N\) cited publications, so \(\Psi(N) > 0\) but \(\Psi(N + 1)\) is either 0 or undefined. We seek a closed-form expression to define a function \(f\) that approximates \(\Psi\).

A frequently-used expression \(^8\) is the power law

\[
\frac{1}{f(x)} = \frac{C}{x^\lambda},
\]

where \(C\) and \(\lambda\) are parameters that vary among authors, \(C \geq 0\) and \(\lambda > 0\). This \(f: \mathbb{R}^+ \to \mathbb{R}^+\) has 3 issues:

First, the vertical asymptote at \(x = 0\) can make the approximation bad for authors who do not have hugely different citation counts for top-ranked papers. Second, the horizontal asymptote as \(x \to \infty\) can give a poor approximation for authors with a small number of papers \(N\). Third, two parameters do not suffice: We can think of \(M\) and \(N\) as anchoring \(f\), but they leave much ambiguity for the curvature in between.

(For the power law, if we require \(f(1) = M\), then \(C = M\) in Eqn. \(1\), and curvature is determined by \(\lambda\).

We therefore need at least 3 parameters to specify \(f\). We use the point where \(f\) cuts the diagonal line to fix the curvature for \(f\), i.e. if the real value \(h\) is defined by \(f(h) = h\), then \(f\) is determined by \(M\), \(N\) and \(h\).

What should we choose for \(f^n\)? For the power law, \(1/f(x)\) is proportional to \(x^\lambda\). Fig. 1 plots \(1/\Psi(n)\) for 5 researchers in engineering \((M = 718, N = 171, h = 50)\), mathematics \((M = 8763, N = 410, h = 77)\), medicine \((M = 2019, N = 345, h = 114)\), psychology \((M = 3740, N = 982, h = 243)\) and sociology \((M = 1057, N = 116, h = 24)\). The citation data is from a dataset of 226 authors that we sampled from Publish or Perish\(^2\), the histograms for \(M\), \(N\) and \(h\) are in the Appendix. We use this dataset throughout this paper.

In each case, the regression line shows that \(1/\Psi(n)\) is approximately linear in \(n\) for small \(n\), where most of an author’s citations are. This suggests \(f\) should have the form \(1/f(x) = \gamma_1 x + \gamma_0\) for some constants \(\gamma_0\) and \(\gamma_1\). Since \(f\) needs to have 3 parameters, we define it as

\[
f(x) = \frac{b}{x + c} - a.
\]

where \(a\), \(b\) and \(c\) are positive real values.

\(^2\)https://harzing.com/resources/publish-or-perish
Figure 1: Regression lines show that $1/\Psi(n)$ is approximately linear for small $n$. (The research area for each author is indicated. Note the nonzero intercepts.)

Fig. 2 illustrates this $f$. The function has a horizontal asymptote at $y = -a$, vertical asymptote at $x = -c$, and intersects $y = x$ at $x = h$. These 3 parameters $(a, b, c)$ control the location and curvature of $f$. Their values are determined by

$$f(0) = M, \quad f(N) = 0 \quad \text{and} \quad f(h) = h. \quad (3)$$

It follows that $f(1) \approx f(0) = M = \Psi(1)$, and $f(N) = 0 \approx \Psi(N)$. Moreover, the $h$-index [9] is defined by solving $\Psi(n) = n$, so $f(h) = h \approx \Psi(h)$. We are only interested in $f(x)$ for $0 \leq x \leq N$.

Solving Eqn. (2) and Eqn. (3) gives

$$a = \frac{Mh^2}{MN - (M + N)h}$$

$$b = MN(M - h)(N - h)\left(\frac{h}{MN - (M + Nh)}\right)^2$$

$$c = \frac{Nh^2}{MN - (M + Nh)}$$

Fig. 3 shows how well $f(x)$ fits $\Psi(n)$ for 6 researchers from our dataset.

2.1 Approximations for the head and tail

Let $\Psi_{\text{head}}$ and $\Psi_{\text{tail}}$ denote the $\Psi$ for $n \leq h$ and $n > h$ respectively. One can simplify the expressions in Eqn. (4) by focusing on $\Psi_{\text{head}}$ and neglecting the fit for $\Psi_{\text{tail}}$. We do this by taking the limit $N \to \infty$ in Eqn. (4) and thus derive
Figure 3: Comparing number of citations $\Psi(n)$ and approximation $f(x)$ for 6 different authors whose research areas are in computer science, chemistry and medicine.
Equivalently, we use \( f \) if \( f(1) = M \) (instead of \( f(0) = M \)), but that would further complicate the expressions for \( a, b \) and \( c \) in Eqn. (4).

Let \( \Theta_{\text{head}} \) denote the total number of citations for an author’s first \( h \) publications. We can use \( f_{\text{head}} \) to approximate \( \Theta_{\text{head}} \) by

\[
F_{\text{head}} = \int_0^h \frac{b_{\text{head}}}{x + c_{\text{head}}} \, dx = b_{\text{head}} \ln \left( 1 + \frac{h}{c_{\text{head}}} \right)
= \frac{Mh^2}{M - h} \ln \left( 1 + \frac{M - h}{h} \right)
= \frac{Mh^2}{M - h} \ln \left( \frac{M}{h} \right)
= h \left( \frac{\ln \left( \frac{M}{h} \right) - \frac{M}{h}}{\frac{M}{h} - 1} \right)
\]

Similarly, if \( \Theta_{\text{tail}} \) denotes the total number of citations for the \( N - h \) publications in the tail, then we can use \( f_{\text{tail}} \) to approximate \( \Theta_{\text{tail}} \) by

\[
F_{\text{tail}} = \int_h^N \frac{b_{\text{tail}}}{x - a_{\text{tail}}} \, dx = b_{\text{tail}} \ln \left( \frac{N}{h} \right) - a_{\text{tail}}(N - h)
= \frac{Nh^2}{N - h} \left( \ln \frac{N}{h} \right) - h^2.
\]

3. Closed-Form Expressions for Indices

In this section, we relate \( f, f_{\text{head}} \) and \( f_{\text{tail}} \) to previous work. In particular, we use our equations to derive closed-form expressions for various indices.

3.1 Total number of citations

When introducing the \( h \)-index, Hirsch [10] postulated that the total number of citations

\[
\Theta = ah^2
\]
for some $\alpha$ that varies among authors and, empirically, $3 < \alpha < 5$. Eqn. (11) has the equivalent form

$$h = \frac{1}{\sqrt{\alpha} \Theta^{0.5}}.$$  

A regression analysis by van Raan [16] using data for chemistry research in Dutch universities also shows

$$h = 0.42\Theta^{0.45},$$

where the exponent 0.45 is close to 0.5. (See also the Yong’s “rule of thumb” [17].) Using our approximation $F$ for $\Theta$, Eqn. (8) shows that, in fact,

$$\alpha \approx \ln \left( \frac{MN}{eh^2} \right);$$

note that $\alpha$ itself depends on $h$.

### 3.2 A-index

To take into account $\Psi(n) - h$ for $n = 1, \ldots, h$, Jin et al. [11] defined an A-index

$$A = \Theta_{\text{head}} / h,$$  

i.e. the average citation count for the first $h$ papers. Using Eqn. (9), we can approximate this A-index as

$$A \approx F_{\text{head}} / h = \frac{Mh}{M - h} \ln \left( \frac{M}{h} \right).$$  

Fig. 6 compares the empirical value of the A-index for the 226 authors to the approximate value computed with Eqn. (13).

The regression line has a gradient of 0.86 and correlation coefficient $R^2 \approx 0.72$. This weak accuracy is expected, since the A-index is defined with $\Theta_{\text{head}}$, where there are huge differences among authors in the shape of $\Psi$ for their highly-cited papers. The outlier at (1705, 164) is from the computer scientist previously mentioned for Fig. 5. Another outlier at (455, 50) is from a chemist with just $N = 15$ publications, but has $M = 2259$.

Jin et al. proved that, using the power law model $f(x) = M/x^\lambda$, $A/h$ is a constant determined only by the curvature parameter $\lambda$. In contrast, Eqn. (13) shows that, using our model,

$$A / h \approx \frac{M}{M - h} \ln \left( \frac{M}{h} \right),$$

so $A/h$ also depends on the maximum citation $M$.

### 3.3 R-index

As an alternative to the A-index, Jin et al. [11] also defined an R-index

$$R = \sqrt{\Theta_{\text{head}}}. $$  

By Eqn. (9), we can approximate this as

$$R \approx \sqrt{F_{\text{head}}} = h \sqrt{ \frac{M}{M - h} \ln \left( \frac{M}{h} \right) }.$$  

Fig. (7) shows that, although there are outliers, Eqn. (14) provides a closed-form expression that, in general, gives an excellent approximation for the $R$-index (the regression line has gradient 1.00 and $R^2 \approx 0.97$).

Again, under the power law, $R/h$ is expressible in terms of $\lambda$, whereas Eqn. (14) shows that the ratio depends on $M$ as well.

### 3.4 g-index

An author’s $h$-index remains the same no matter how high $\Psi(n)$ is for $n = 1, 2, \ldots, h$. To overcome this issue, Egghe defined a $g$-index [8], which we approximate as

$$g \approx \int_0^g f(x) dx / g.$$  

(15)
Egghe has shown that $g > h$. We therefore use $f_{\text{head}}$ and $f_{\text{tail}}$ to further approximate $g$ by

\[
g^2 \approx \int_0^h f_{\text{head}}(x)\,dx + \int_h^g f_{\text{tail}}(x)\,dx
= \frac{Mh^2}{M-h} \ln \left( \frac{M}{h} \right) + b_{\text{tail}} \ln \left( \frac{g}{h} \right) - a_{\text{tail}}(g-h)
\]

by Eqn. (6) and Eqn. (9)

\[
\approx \frac{Mh^2}{M-h} \ln \left( \frac{M}{h} \right) + N h^2 \ln \left( \frac{g}{h} \right) \quad \text{for } a_{\text{tail}} \approx 0
\]

\[
\approx h^2 \ln \left( \frac{M}{h} \right) + h^2 \ln \left( \frac{g}{h} \right) \quad \text{for } M >> h \text{ and } N >> h.
\]

(16)

Consider some $\beta > g_h$. Then

\[
\ln \left( \frac{g}{h} \right) = \ln \beta + \ln \left( 1 - \left( 1 - \frac{g}{\beta h} \right) \right)
\]

\[
\approx \ln \beta - \left( 1 - \frac{g}{\beta h} \right)
\]

since $\ln(1-x) \approx -x$ for $0 < x < 1$.

The citation data we have seen all show $g_h < 4$, so we choose $\beta = 4$. (A larger $\beta$ will increase $1 - \frac{g}{\beta h}$ and worsen the approximation.) Therefore,

\[
\ln \left( \frac{g}{h} \right) \approx \ln \left( \frac{4}{e} \right) + \frac{g}{4h}
\]

Substituting this into Eqn. (16), we get

\[
g^2 \approx \int_0^h f_{\text{head}}(x)\,dx + \int_h^g f_{\text{tail}}(x)\,dx
\]

\[
\approx \ln \left( \frac{M}{h} \right) + \ln \left( \frac{4}{e} \right) + \frac{g}{4h},
\]

so

\[
\left( \frac{g}{h} - \frac{1}{8} \right)^2 - \frac{1}{64} \approx \ln \left( \frac{4M}{eh} \right).
\]

Since $\frac{g}{h} > 1$, we can further simplify this as

\[
\left( \frac{g}{h} \right)^2 \approx \ln \left( \frac{4M}{eh} \right)
\]

\[
\text{i.e. } g \approx h \sqrt{\ln \left( \frac{4M}{eh} \right)}
\]

(17)

Egghe has proven that, for the power law model $f(x) = M/x^\lambda$, $g/h$ is a constant determined by the curvature parameter $\lambda$ only [8]. In contrast, Eqn. (17) shows that, for our model, $g/h$ also depends on $M$.

Fig. 8 plots the approximation (17) against actual $g$ values for the previously chosen 226 authors. The regression line shows that, on average, the approximation is accurate. The under-estimating outlier at (151, 63) is from the computer science previously mentioned for Fig. 5; the over-estimating outlier at (320, 573) is from an immunologist with unusually large $M$ and $N$ ($M = 12172$, $N = 995$, $h = 281$), for whom $f_{\text{head}}$ over-estimates $\Psi_{\text{head}}$.

### 3.5 $hg$-index

Alonso et al. [3] combined the $g$- and $h$-indices to get an $hg$-index $\sqrt{hg}$. By Eqn. (17),

\[
\sqrt{hg} \approx h \sqrt{\ln \left( \frac{4M}{eh} \right)},
\]

(18)

so the $hg$-index just reduces the square root in Eqn. (17) to a 4th root. Fig. 9 shows the closed-form accurately approximates the empirical value of the $hg$-index.
3.6 \textit{e-index}

Zhang [18] pointed out that the \(h\)-index holds no information for the head, and its integer value has coarse granularity. To address these issues, he proposed the \(e\)-index, defined by

\[
e^2 = \Theta_{\text{head}} - h^2.
\]

Using our \(F_{\text{head}}\) approximation (9), we have

\[
e^2 \approx Mh^2 - h \ln (\frac{M}{h}) - h^2,
\]

(19)

Fig. 10 shows the value of \(e\) computed from Eqn. (19) is a good approximation of the empirical value for our dataset: The regression line has gradient 1.00 and \(R^2 \approx 0.94\).

For the power law model \(f(x) = M/x^\lambda\), Zhang derived \(e^2 = M(1/2 \ln M - 1)\) if \(M = h^2\), in the case \(\lambda = 1\).

We can get this from Eqn. (19) for the case \(M = h^2\) and \(h^2 >> h\). In this sense, Zhang’s formula for \(e\) is a validation of Eqn. (9) for \(f(x) = M/x\) and \(M >> h\).

3.7 \textit{h’-index}

The \(h\)-index does not differentiate between authors whose \(\Theta_{\text{head}}\) and \(\Theta_{\text{tail}}\) are very different. To reflect such differences, Zhang defined another index \(h’\) [19], where

\[
h’ = \sqrt{\frac{\Theta_{\text{head}} - h^2}{\Theta_{\text{tail}}}} h.
\]

Using our approximations for \(\Theta_{\text{head}}\) and \(\Theta_{\text{tail}}\), we get

\[
h’ \approx \sqrt{\frac{F_{\text{head}} - h^2}{F_{\text{tail}}}} h
\]

\[
= \sqrt{\frac{M}{M+h} \left( \frac{\ln \frac{M}{h}}{\ln \frac{N}{h}} \right) - 1} h
\]

(20)
Figure 11: Comparing approximate value of $h'$ in Eqn. (20) to empirical value for the $h'$-index.

Fig. 11 shows a regression line of gradient 0.94 for approximate $h'$ values calculated with Eqn. (20) plotted against the empirical values. This is a good approximation for expected $h'$ value, but the data points are quite dispersed, giving a correlation coefficient of $R^2 \approx 0.86$.

For authors with $M >> h$ and $N >> h$, this simplifies to

$$h' \approx \sqrt{\frac{\ln M}{h} - 1}$$

(Note: It is not uncommon for authors to have $M < e^h$ or $N < e^h$, for whom the assumption $M >> h$ or $N >> h$ is violated and the approximation fails.)

To better understand the expression in Eqn. (21), consider a simplified geometry using two triangles, as shown in Fig. 12, with areas

$$\Delta_{\text{head}} = \frac{1}{2} (M - h)h \quad \text{and} \quad \Delta_{\text{tail}} = \frac{1}{2} (N - h)h.$$ 

For this geometry, we get

$$h' = \sqrt{\frac{\Delta_{\text{head}}}{\Delta_{\text{tail}}}} h = \sqrt{\frac{M - h}{N - h}} h.$$ 

Eqn. (21) shows how $h'$ for this simplified geometry is modified when we take into account the citation curvatures.

### 3.8 $h_2$-index

In analyzing the citations for an author’s publications, one must first filter out those by another author with a similar name. (E.g. The Computer Science bibliography website DBLP lists more than 300 authors named “Wei Wang”.)

To reduce the effort needed to disambiguate authorship, Kosmulski defined a $h_2$-index as the greatest integer such that the $h_2$ most-cited papers have at least $h^2$ citations each. He observed that

$$h_2 \propto \Theta^{\frac{1}{3}} \quad (22)$$

The focus of $h_2$ is in the head, so we can approximate $h_2$ by

$$h^2_2 \approx f_{\text{head}}(h_2) = \frac{b_{\text{head}}}{h_2 + c_{\text{head}}},$$

i.e.

$$h^2_2 = \frac{Mh^2}{h_2(M - h) + h^2}$$

by Eqn. (5).

Therefore,

$$h^3_2M = h^2(M - h^2)$$

so

$$h^3_2 \approx h^2 \quad \text{for} \quad M >> h^2_2. \quad (23)$$

This last approximation makes $h^2$ an over-estimate of $h^3_2$, as indicated by the regression line (gradient 1.27) in Fig. 13.

Eqn. (23) thus confirms Kosmulski’s observation (from a small dataset) that $h^3_2 = O(h^2)$. Using Eqn. (8), we get

$$h^3_2 \approx F \ln \left( \frac{MN}{e^h} \right) \quad \text{so} \quad h_2 \approx \left( \frac{1}{\ln \left( \frac{MN}{e^h} \right)} \right)^{\frac{1}{2}} \Theta^{\frac{1}{3}}$$

since $F$ is an approximation for $\Theta$. We thus see how Kosmulski’s approximation (22) depends on $M$, $N$, and $h$.

#### 3.9 $dc_i$ and $dc_o$: impact and potential

To measure the impact of an author’s publications, Silva and Grácio defined an index [14]

$$dc_i = \frac{1}{h} \sum_{n=1}^{h} (\psi(n) - h),$$

i.e. $dc_i = e^2/h$, using the $e$-index. From the approximation for $e$ in Sec. 3.6, we get

$$dc_i \approx h \left( \frac{M}{M - h} \ln M - 1 \right) \quad (24)$$
where, for the same $h$, $dc_o$ is higher for the author with a larger $N$. We can rewrite Eqn. (27) as

$$dc_o \approx h - \frac{h^2}{N} \ln\left(\frac{N}{\Theta h}\right),$$

so $dc_o$ is linear in $h$, with an additive noise term $\frac{h^2}{N} \ln\left(\frac{N}{\Theta h}\right)$ induced by $N$ and biased by $h$. Indeed, Silva and Grácio’s data shows a strong linear correlation (Pearson coefficient $\approx 0.96$) between $dc_o$ and $h$.

4. CONSIDERING TIME AND GROUP ACTIVITIES

We now apply our approximations to analyze the effect of time and the aggregation of citation counts.

4.1 Modeling the effect of time

In using $dc_o$ to measure the potential for increasing an author’s $h$ value, one can make a prediction. Supporting such a prediction requires some model of how citations and publications increase over time.

When introducing the $h$-index, Hirsch gave a back-of-envelope derivation that shows

$$h = h_0t,$$

where $t$ is the time since the author’s first publication, and $h_0$ is a constant determined by publication rate and citation rate. There is some empirical validation of Eqn. (28) [10][11][12].

Burrell also provided numerical support using a stochastic model [6]. This model assumes that the number of publications for an author is Poisson distributed over time at a constant rate. One therefore expects

$$N = N_0t,$$

for some constant $N_0$. By a similar Poisson assumption, the number of citations for a particular publication is expected to be linear with respect to time. An author’s publications appear at different times and have different citation rates (that are gamma distributed in Burrell’s model). The publication with the highest citation — and the corresponding citation rate — may therefore change over time. Even so, we further assume

$$M = M_0t$$

for some constant $M_0$. In the following, we refer to Eqs. (28)–(30) as the linear model.

It follows from this model and Eqn. (4) that

$$a = \frac{M_0h_0^3}{M_0N_0 - (M_0 + N_0)h_0} t$$

$$b = M_0N_0(M_0 - h_0)(N_0 - h_0)\left(\frac{h_0}{M_0N_0 - (M_0 + N_0)h_0}\right)^2 t^2$$

$$c = \frac{N_0h_0^2}{M_0N_0 - (M_0 + N_0)h_0} t.$$
Burrell observed this linearity in two numerical examples for his stochastic model [7]. He pointed out that the correlation coefficient for $h_2$ vs $t$ is much smaller. In fact, we see from Eqn. (23) that

$$h_2 \approx h_0^2 t^2,$$

so $h_2$ is not linear in $t$.

Burrell’s numerical examples also showed that $\Theta_{\text{head}}$ and the $A$-index are approximately proportional to $t^2$ and $t$ respectively. Indeed, we see from Eqn. (9) and Eqn. (13) that

$$\Theta_{\text{head}} \approx F_{\text{head}} \approx \left( \frac{M_0 h_0^2}{M_0 - h_0} \ln \left( \frac{M_0}{h_0} \right) \right) t^2 \quad (31)$$

and

$$A \approx \left( \frac{M_0 h_0}{M_0 - h_0} \ln \left( \frac{M_0}{h_0} \right) \right) t,$$

so the multiplicative factors are constant if $M, N$ and $h$ are linear in $t$. Similarly, the factors

$$\ln \left( \frac{MN}{e h^2} \right) \quad \text{for } \Omega \quad \text{in Eqn. (11)},$$

$$\sqrt{\frac{M}{M - h}} \ln \left( \frac{M}{h} \right) \quad \text{for } R \quad \text{in Eqn. (14)},$$

$$\sqrt{4 \ln \left( \frac{4M}{eh} \right)} \quad \text{for } \sqrt{h g} \quad \text{in Eqn. (18)},$$

$$\frac{M}{M - h} \ln \left( \frac{M}{h} \right) - 1 \quad \text{for } e^2 \quad \text{in Eqn. (19)}$$

are constants in the linear model.

One issue with the $h$-index is that it provides no information for distinguishing two authors with the same integer value $h$. Even when they are different, authors may have larger $h$ values because they have been publishing for a longer time. Hirsch himself recommended using $h/t$, i.e. $h_0$ in the linear model, to compare authors with different seniority.

As mentioned above, Burrell used his probabilistic model to examine how $\Theta_{\text{head}}$ varies with time. If the linear model holds for our approximation, then

$$\frac{d\Theta_{\text{head}}}{dt} \approx 2 \left( \frac{M_0 h_0^2}{M_0 - h_0} \ln \left( \frac{M_0}{h_0} \right) \right) t \quad \text{from Eqn. (31)}$$

$$= 2 \left( \frac{M h^2}{M - h} \ln \left( \frac{M}{h} \right) \right)$$

$$\approx 2 \frac{\Theta_{\text{head}}}{t}.$$

We see that (like using $h/t$ to differentiate two authors with the same $h$) for two authors with the same $\Theta_{\text{head}}$, the senior author has a larger $t$ and thus a smaller growth rate $\frac{d\Theta_{\text{head}}}{dt}$.

### 4.2 $h$-index for a group

The concept of $h$-index has been extended from an individual author to a group (department [15], journal [13], etc.). Here, we apply our equation to derive the $h$-index of...
Consider a group of \( r \) authors. Let \( M_i, N_i \) and \( h_i \) be the \( M, N \) and \( h \) values for the \( i \)-th author in the group, and \( M_* \), \( N_* \) and \( h_* \) the \( M, N \) and \( h \) values for the collection of publications from this group. Then,

\[
M_* = \max\{M_1, \ldots, M_r\}. \tag{32}
\]

For a first approximation, we assume no two authors in the group share a publication, so

\[
N_* = N_1 + \cdots + N_r. \tag{33}
\]

Suppose the \( i \)-th author has \( x_i \) publications with at least \( h_* \) citations each. By the definition of the \( h \)-index,

\[
h_* = x_1 + \cdots + x_r. \tag{34}
\]

Since \( x_i \leq h_i \) (see Fig. 13), we can use \( f_{\text{head}} \) to approximate the citation data for each author. Let \( a_i, b_i \) and \( c_i \) be the \( a_{\text{head}}, b_{\text{head}} \) and \( c_{\text{head}} \) values for the \( i \)-th author. By Eqn. (35),

\[
a_i = 0, \quad b_i = \frac{M_i h_i^2}{M_i - h_i}, \quad c_i = \frac{h_i^2}{M_i - h_i}
\]

and

\[
f_i(x) = \frac{b_i}{x + c_i},
\]

where \( f_i \) is \( f_{\text{head}} \) for the \( i \)-th author. Then

\[
h_* = \frac{b_i}{x_i + c_i} \quad \text{for } i = 1, \ldots, r,
\]

so

\[
\sum_{i=1}^{r} h_i (x_i + c_i) = \sum_{i=1}^{r} b_i.
\]

By Eqn. (34),

\[
h_*^2 + h_* \sum_{i=1}^{r} c_i - \sum_{i=1}^{r} b_i = 0.
\]

Thus

\[
h_* = \frac{-\sum_{i=1}^{r} c_i + \sqrt{\left(\sum_{i=1}^{r} c_i\right)^2 + 4\left(\sum_{i=1}^{r} b_i\right)}}{2} \tag{36}
\]

Note that \( b_i = M_i c_i \) and \( M_i \gg 1 \) for most authors. For \( 4\sum_{i=1}^{r} b_i \gg \left(\sum_{i=1}^{r} c_i\right)^2 \), we can use the following approximation:

\[
h_* \approx \frac{-\sum_{i=1}^{r} c_i + \sqrt{4\left(\sum_{i=1}^{r} b_i\right)}}{2}
\]

\[
\approx \sqrt{\sum_{i=1}^{r} b_i}
\]

\[
= \sqrt{\sum_{i=1}^{r} \frac{M_i h_i^2}{M_i - h_i}}. \tag{37}
\]

To validate this approximation, we selected 9 authors from our dataset. Table 1 lists, for author \( i \), the research area, \( M_i, N_i \) and \( h_i \). For \( r = 2, \ldots, 9 \), we grouped the first \( r \) authors’ publications to determine the empirical \( h_* \) value for the group.

Note that the approximation (37) takes into account the \( M_i \) value for each author in the collection when estimating the aggregate \( h_* \) value. The omission of \( \sum_{i=1}^{r} c_i \) leads to an under-estimation, but Fig. 16 shows that it nonetheless gives a good approximation for \( h_* \) (the regression line has gradient \( 0.91 \) and \( R^2 \approx 0.98 \)).

5. CONCLUSION

In this paper, we proposed a simple equation to approximate the citation count distribution of an author. The equation is based on the idea of using 3 geometrical characteristics \( (M, N, h) \) of the count distribution to calibrate 3 parameters \((a, b, c)\) for an equation to approximate the distribution.

We demonstrated the equation’s usefulness in the analysis of such distributions by deriving closed-form expressions for various citation indices, and using them to model the effect of time, and identify individual contribution to a group \( h \)-index.
### Table 1: Comparing approximate value of $h_*$ in Eqn. (37) to the empirical $h_*$.  

| r  | research area     | $M_i$ | $N_i$ | $h_i$ |
|----|-------------------|-------|-------|-------|
| 1  | physics           | 336   | 15    | 13    |
| 2  | city planning     | 423   | 90    | 27    |
| 3  | public health     | 2108  | 63    | 32    |
| 4  | physiology        | 1161  | 34    | 18    |
| 5  | computer science  | 262   | 396   | 44    |
| 6  | public policy     | 364   | 128   | 31    |
| 7  | sociology         | 901   | 64    | 24    |
| 8  | psychology        | 272   | 124   | 46    |
| 9  | artificial intelligence | 513 | 94    | 19    |

| r  | empirical value   | $h_*$ approximation (Eqn. (37)) |
|----|-------------------|---------------------------------|
| 2  | 32                | 30.9                            |
| 3  | 49                | 44.7                            |
| 4  | 57                | 48.2                            |
| 5  | 71                | 68.2                            |
| 6  | 77                | 75.5                            |
| 7  | 92                | 79.3                            |
| 8  | 103               | 91.0                            |
| 9  | 105               | 96.0                            |

### 6. REFERENCES

[1] H. A. Abt. A publication index that is independent of age. *Scientometrics*, 91(3):863–868, 2012.

[2] R. Adler, J. Ewing, and P. Taylor. Citation statistics. *Statistical Science*, 24(1):1–14, 2009.

[3] S. Alonso, F. J. Cabrerizo, E. Herrera-Viedma, and F. Herrera. $hg$-index: a new index to characterize the scientific output of researchers based on the $h$- and $g$-indices. *Scientometrics*, 82(2):391–400, 2010.

[4] A. Bihari, S. Tripathi, and A. Deepak. $h$-index and its alternative: A review. *CoRR*, abs/1811.03308, 2018.

[5] Q. L. Burrell. Hirsch index or Hirsch rate? some thoughts arising from Liang’s data. *Scientometrics*, 73(1):19–28, 2007.

[6] Q. L. Burrell. Hirsch’s $h$-index: A stochastic model. *J. Informetrics*, 1(1):16–25, 2007.

[7] Q. L. Burrell. On Hirsch’s $h$, Egghe’s $g$ and Kosmulkis’s $h(2)$. *Scientometrics*, 79(1):79–91, 2009.

[8] L. Egghe. Theory and practise of the $g$-index. *Scientometrics*, 69(1):131–152, 2006.

[9] J. E. Hirsch. An index to quantify an individual’s scientific research output. *Proc. Natl. Acad. Sci. USA*, 102(46):16569–16572, 2005.

[10] J. E. Hirsch. Does the $h$ index have predictive power? *Proc. Natl. Acad. Sci. USA*, 104(49):19193–19198, 2007.

[11] B. Jin, L. Liang, R. Rousseau, and L. Egghe. The R- and AR-indices: Complementing the $h$-index. *Chinese Sci Bull*, 52(6):855–863, 2007.

[12] L. Liang. $h$-index sequence and $h$-index matrix: Constructions and applications. *Scientometrics*, 69(1):153–159, 2006.

[13] J. Mingers. Measuring the research contribution of management academics using the Hirsch-index. *J. Oper. Res. Soc.*, 60(9):1143–1153, 2009.

[14] D. D. Silva and M. C. C. Grácio. Dispersion measures for $h$-index: a study of the Brazilian researchers in the field of mathematics. *Scientometrics*, 126(3):1983–2011, 2021.

[15] P. N. Tyrrell, A. R. Moody, J. O. C. Moody, and N. Ghiam. Departmental $h$-index: Evidence for publishing less? *Canadian Association of Radiologists Journal*, 68(1):10–15, 2017.

[16] A. F. J. van Raan. Comparison of the Hirsch-index with standard bibliometric indicators and with peer judgment for 147 chemistry research groups. *Scientometrics*, 67(3):491–502, 2006.

[17] A. Yong. Critique of Hirsch’s citation index: a combinatorial Fermi problem. *Notices of the AMS*, 61(9):1040–1050, 2014.

[18] C.-T. Zhang. The $e$-index, complementing the $h$-index for excess citations. *PLOS ONE*, 4(3):e5429, 2009.

[19] C.-T. Zhang. The $h’$-index, effectively improving the $h$-index based on the citation distributions. *PLOS ONE*, 8(4):e59012, 2013.

### Appendix

The following histograms describe the sample of 226 authors:
