Reconstruction of potentials of the hybrid inflation in the light of primordial black hole formation

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Abstract. The large enhancement of the primordial power spectrum of the curvature perturbation can seed the formation of primordial black hole, that can play as a dark matter component in the Universe. In multi-field inflation models, the curved trajectory of the scalar fields in the field space can generate a peak in the power spectrum on small scales due to the existence of the isocurvature perturbation. Here we show that a potential can be reconstructed from a given power spectrum, which is made of a scale-invariant one on large scales and the other function with a peak on small scales. In multi-field inflation models the reconstructed potential may not be unique and we can find different potentials from a given power spectrum.
1 Introduction

The observation of the large scale structures and the anisotropy of the cosmic microwave background (CMB) allowed the precise determination of the primordial power spectrum at large scales [1]. Cosmic inflation can naturally produce the required spectrum in addition to resolving the problems in the standard big bang model. However on small scales, the power spectrum is constrained weakly only by the formation of primordial black hole (PBH), ultracompact minihalo, or dark matter (DM) annihilation for some specific models [2, 3].

The PBHs can form through the collapse of large fluctuations of the density in the early Universe when a given scale enters the horizon [4–6]. The PBHs can evaporate via Hawking radiation, however those with the mass larger than $5 \times 10^{14}$g can survive until today. The present remnant of PBHs can contribute to the non-baryonic component of dark matter [7]. See the recent review [8] for PBH as a candidates for dark matter.

For the formation of PBH, typically the amplitude of the power spectrum need to be larger than around $10^{-2}$, that is $10^7$ bigger than that at the CMB scales, $P_{\text{CMB}} \sim 10^{-9}$. Several models were suggested to generate the large enhancement at small scales with a single scalar field, such as features in the potential, running mass of the inflaton, hilltop models, and inflection point [9–17]. However, for a single field inflation, the slow-roll must be violated at scales between the CMB scale and PBH mass scales. In this case, a numerical calculation is needed to evaluate the power spectrum properly.

In Ref. [18], Hertzberg et al. provided a method to reconstruct the inflaton potential from a given power spectrum within canonical singled field model. They applied this method to the formation of PBHs and confirmed again that the slow-roll conditions need to be violated in order to generate a significant spike in the spectrum.

The multi-field models which can generate PBHs include hybrid inflation, double inflation, and a curvaton field [19–25]. For non-canonical kinetic terms, the accurate analytic prediction was derived for the formation of PBHs in multfiled case [26].

In this paper, we propose a method to reconstruct a potential of multi-field scalar model from the given power spectrum. We consider a primordial power spectrum made of a scale-invariant one on large scales and the other with a peak on small scales in the light of generating PBHs. We use a hybrid-type potential and show that a few different examples of the reconstructed potential. We find that the reconstructed potential may not be unique.
Once we know the potential, it is possible to obtain the power spectrum numerically by solving the exact equations of motions. Lastly, we present these numerical results to support our analytical arguments.

In Sec. 2, we summarize the background evolution and the perturbations during inflation with two scalar fields, and in Sec. 3 we give the formulation of the $\delta N$ formalism approximation. In Sec. 4, we introduce a method to reconstruct the potential from a given power spectrum and show the analytical results with numerical calculation. We conclude in Sec. 5.

## 2 Background and perturbations of two-field model

In this section, we briefly revisit the evolutions of relevant background quantities and scalar perturbations (for more details, see [27, 28]). We shall focus on models with two real scalar fields, $\phi$ and $\psi$, described by the action

$$S[\phi_I] = -\int d^4x \sqrt{-g} \left[\frac{1}{2} \sum_{I=1}^{2} \partial_{\mu} \phi_I \partial^\mu \phi_I + V(\phi_I)\right], \quad (2.1)$$

where $\phi_I = \{\phi, \psi\}$. The equations of motion of the fields can be written as

$$\ddot{\phi}_I + 3H \dot{\phi}_I + V_{\phi_I} = 0, \quad (2.2)$$

where $H = \dot{a}/a$ is the Hubble parameter and $V_{\phi_I} = dV/d\phi_I$. Two Friedmann equations describing the evolution of the scale factor are given by

$$H^2 = \frac{1}{3M_P^2} \left[\frac{1}{2} \left(\dot{\phi}_I^2 + \dot{\psi}^2\right) + V(\phi, \psi)\right], \quad (2.3a)$$

$$\dot{H} = -\frac{1}{2M_P^2} \left(\dot{\phi}_I^2 + \dot{\psi}^2\right). \quad (2.3b)$$

In terms of the e-folding number, $N_t$, defined as $N_t = \ln (a/a_i)$, where $a_i$ is the scale factor at a suitably chosen time, the field equations are

$$\frac{d^2\phi}{dN_t^2} + (3 - \epsilon_H) \frac{d\phi}{dN_t} + \frac{V_{\phi}}{H^2} = 0,$$

$$\frac{d^2\psi}{dN_t^2} + (3 - \epsilon_H) \frac{d\psi}{dN_t} + \frac{V_{\psi}}{H^2} = 0, \quad (2.4)$$

where $\epsilon_H = -\frac{1}{H} \frac{dH}{dN_t}$ is the Hubble slow roll parameter.

Since there are two fields involved, evidently, apart from the curvature perturbation, isocurvature perturbation also arises. In the spatially flat gauge, for instance, the Mukhanov-Sasaki variables associated with the curvature and the isocurvature perturbations $v_\sigma$ and $v_s$ are given by

$$v_\sigma = a \left(\cos \theta \, \delta \phi + \sin \theta \, \delta \psi\right), \quad (2.5)$$

$$v_s = a \left(-\sin \theta \, \delta \phi + \cos \theta \, \delta \psi\right), \quad (2.6)$$

where $\cos \theta = \dot{\phi}/\dot{\sigma}$, $\sin \theta = \dot{\psi}/\dot{\sigma}$ and $\dot{\sigma}^2 = \dot{\phi}^2 + \dot{\psi}^2$. The curvature and the isocurvature perturbations are defined as $R = v_\sigma/z$ and $S = v_s/z$, respectively, with $z = a \dot{\sigma}/H$ [28].
It is convenient to introduce the adiabatic and entropy vectors $E^I_\sigma$ and $E^I_s$ in the field space, defined as

$$E^I_\sigma = (\cos \theta, \sin \theta),$$

$$E^I_s = (-\sin \theta, \cos \theta),$$

where $I = \{\phi, \psi\}$. The equations governing the gauge invariant Mukhanov-Sasaki variables $v_\sigma$ and $v_s$ can be expressed as [28]

$$v''_\sigma + \left(k^2 - \frac{z''}{z}\right) v_\sigma = \frac{1}{z} \left(z \xi v_s\right)',$$

$$v''_s + \left(k^2 - \frac{a''}{a} + a^2 \mu^2_s\right) v_s = -z \xi \left(\frac{v_\sigma}{z}\right)' ,$$

where $\xi = -2 a V_s/\dot{\sigma}$ and the quantity $\mu^2_s$ is given by

$$\mu^2_s = V_{ss} - \left(\frac{V_s}{\dot{\sigma}}\right)^2 ,$$

with the subscript $\phi$ or $\psi$ indicating differentiation with respect to the fields. Also, the quantities $V_\sigma$, $V_s$ and $V_{ss}$ are given by $V_\sigma = E^I_\sigma V_I$, $V_s = E^I_s V_I$ and $V_{ss} = E^I_s E^J_s V_{IJ}$, with implicit summations assumed over the repeated indices $I$ and $J$.

As we know, the perturbations considered are quantum in nature. We can quantise the perturbations by promoting the variables to quantum operators as [28]

$$\hat{v}_\sigma = f_\sigma \hat{a} + f^*_\sigma \hat{a}^\dagger + g_\sigma \hat{b} + g^*_\sigma \hat{b}^\dagger,$$

$$\hat{v}_s = f_s \hat{a} + f^*_s \hat{a}^\dagger + g_s \hat{b} + g^*_s \hat{b}^\dagger ,$$

where $f_{\sigma,s}$ and $g_{\sigma,s}$ are the solutions of Eq. (2.9), $(\hat{a}, \hat{b})$ and $(\hat{a}^\dagger, \hat{b}^\dagger)$ are the annihilation and creation operators. Vacuum states are defined as

$$\hat{a}|0\rangle = \hat{b}|0\rangle = 0 .$$

When the modes are very deep inside the Hubble radius, the equations of motion governing the set of variable $(f_\sigma, f_s)$ and $(g_\sigma, g_s)$ are decoupled and we shall set the initial conditions, as usual, by the Minkowski-like vacuum as

$$f_\sigma(\eta) = g_s(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}},$$

$$f_s(\eta) = g_\sigma(\eta) = 0 .$$

The two scalar power spectra can be expressed as [28, 29]

$$\mathcal{P}_R = \frac{k^3}{2\pi^2} \frac{|f_\sigma|^2 + |g_\sigma|^2}{z^2},$$

$$\mathcal{P}_S = \frac{k^3}{2\pi^2} \frac{|f_s|^2 + |g_s|^2}{z^2} .$$


3 Power spectrum with $\delta N$ formalism

Using the $\delta N$-formalism [30–33], we can evaluate the curvature perturbation on the hypersurfaces of constant energy density, $\zeta$, on super-horizon scales with the perturbation of the e-folding number $N$ defined as

$$N(t_e, t_*, x) \equiv \int_{t_*}^{t_e} H dt.$$  \hfill (3.1)

The integral is evaluated from an initial flat hyper-surface at $t = t_*$ to a final uniform density hyper-surface at $t = t_e$. The e-folding number $N(t_e, t_*, x)$ can be a function of the field at horizon exit at $t = t_*$ and its perturbation can be expanded in terms of the filed perturbations $\delta \phi(t_*, x)$ and $\delta \psi(t_*, x)$,

$$\zeta \simeq \delta N = \frac{\partial N}{\partial \phi_*} \delta \phi_* + \frac{\partial N}{\partial \psi_*} \delta \psi_*.$$  \hfill (3.2)

Here we assumed the slow-roll and ignored the dependence on the time derivative of the filed $\dot{\phi}$ and $\dot{\psi}$. The field perturbation satisfies the two-point correlation function

$$\langle \delta \phi_*(k_1) \delta \phi_*(k_2) \rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2) \frac{2\pi^3}{k_1^3} P_{\phi}(k_1), \quad P_{\phi}(k_1) \equiv \frac{H_*^2}{4\pi^2},$$  \hfill (3.3)

where $H_*$ is evaluated at Hubble exit $k = a_*H_*$. The similar relation is applied to $\delta \psi_*$. Then the power spectrum of the curvature perturbation, $P_{\zeta}$, is defined as

$$\langle \zeta(k_1) \zeta(k_2) \rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2) \frac{2\pi^3}{k_1^3} P_{\zeta}(k_1).$$  \hfill (3.4)

From Eq. (3.3) and Eq. (3.4), we obtain the power spectrum of the curvature perturbation as

$$P_{\zeta} = \frac{H_*^2}{4\pi^2} \left[ \left( \frac{\partial N}{\partial \phi_*} \right)^2 + \left( \frac{\partial N}{\partial \psi_*} \right)^2 \right].$$  \hfill (3.5)

In the slow-roll limit of $\phi$ field, the number of e-foldings can be written by [29]

$$N(\phi_*, \psi_*) = -\frac{1}{M_*^2} \int_{\phi_*}^{\phi_e} \frac{V}{V_\phi} d\phi,$$  \hfill (3.6)

where $\phi_e$ and $\psi_e$ are functions of $\phi_*$ and $\psi_*$. The partial derivatives of the e-folding number are

$$\frac{\partial N}{\partial \phi_*} = \frac{1}{M_*^2} \left( \frac{V}{V_\phi} \right)_* - \left( \frac{V}{V_\phi} \right)_e \frac{\partial \phi_e}{\partial \phi_*}, \quad \frac{\partial N}{\partial \psi_*} = -\left( \frac{V}{V_\phi} \right)_e \frac{\partial \psi_e}{\partial \psi_*}.$$  \hfill (3.7)

We note that the comoving curvature perturbation $R$ coincides with the perturbation on hypersurfaces of constant energy density $\zeta$ on scales far out side the horizon $k \ll aH$ [34]. In other words,

$$P_{\zeta} \approx P_R$$  \hfill (3.8)

on super horizon scales.
4 Reconstruction of a potential in two-field model

We are interested to reconstruct a potential that produces a power spectrum with a peak on small scales. For this we consider a power spectrum composed of two parts, almost-scale invariant one and the other with a peak given by

\[ P(\zeta(k)) = P_s(k) + P_p(k), \quad (4.1) \]

with

\[ P_s(k) = A_s \left( \frac{k}{k_p} \right)^{n_s - 1}. \quad (4.2) \]

Here \( k_p = 0.05 \text{ Mpc}^{-1} \) is the pivot scale used by Planck, and \( A_s \approx 2.0 \times 10^{-9} \) and \( n_s \approx 0.96 \) [1]. On large scales around CMB observation, \( P_s(k) \) is dominant and gives almost scale-invariant power spectrum for \( k \approx 10^{-4} - 1 \text{ Mpc}^{-1} \), however on small scales the peak spectrum is dominant \( P_p \ll P_s \). In the light of the PBH formation, we consider that \( P_p \gg 10^{7} P_s \) around scales of the peak.

It is known that, the power spectrum that peaks at a scale \( k_c = 10^{12} \text{ Mpc}^{-1} \) can generate stochastic background of gravitational waves which peaks in the frequency band targeted by the future interferometer LISA [35]. By following this, in our numerical calculations, we choose to work with the parameters such that the power spectrum peaks at the scale \( k_c = 10^{12} \text{ Mpc}^{-1} \).

In the two-field inflation models, we expect that the scale invariant power spectrum \( P_s \) comes from the \( \phi \) field, and the one with a peak from the \( \psi \) field, i.e. using \( \delta N \)-formalism,

\[ P_s = \frac{H_s^2}{4\pi^2} \left( \frac{\partial N}{\partial \phi} \right)^2, \quad P_p = \frac{H_p^2}{4\pi^2} \left( \frac{\partial N}{\partial \psi} \right)^2. \quad (4.3) \]

From these two equations, we will reconstruct the potential of two scalar fields. However, in multi-filed case, the reconstructed potential from the power spectrum may not be unique [36]. In the followings, we choose a potential of the type of hybrid inflation given by

\[ V = V_0 \left[ 1 + f(\phi) + g(\phi)h(\psi) \right], \quad (4.4) \]

where \( 1 \gg f(\phi) + g(\phi)h(\psi) \) during inflation to ensure the vacuum-domination.

The trajectories on the field space can be labelled by the integral of motion along the trajectory [29]

\[ C = \int \frac{g(\phi)}{f(\phi)} d\phi - \int \frac{1}{h(\psi)} d\psi = F(\phi) - H(\psi), \quad (4.5) \]

where we defined

\[ F(\phi) = \int \frac{g(\phi)}{f(\phi)} d\phi, \quad H(\psi) = \int \frac{1}{h(\psi)} d\psi. \quad (4.6) \]

Since the integral of motion connects the field values at the horizon exit and the end of inflation by

\[ F(\phi_e) - H(\psi_e) = F(\phi_e) - H(\psi_e), \quad (4.7) \]
we can find the relation of the partial derivatives

\[
\begin{align*}
\frac{g^*}{f_{\phi^*}} &= \frac{g_e}{f_{\phi_e}} \left( \frac{\partial \phi_e}{\partial \phi^*} \right) - \frac{1}{h_{\psi_e}} \left( \frac{\partial \psi_e}{\partial \phi^*} \right), \\
- \frac{1}{h_{\psi^*}} &= \frac{g_e}{f_{\phi_e}} \left( \frac{\partial \phi_e}{\partial \psi^*} \right) - \frac{1}{h_{\psi_e}} \left( \frac{\partial \psi_e}{\partial \psi^*} \right),
\end{align*}
\]

(4.8)

where we used a notation \(g^* = g(\phi_e), \ g_e = g(\phi_e), \ f_{\phi^*} = \frac{df(\phi)}{d\phi} \bigg|_{\phi = \phi_e},\) and etc. In addition to this, if we know the condition of ending inflation

\[
E(\phi_e, \psi_e) = 0,
\]

(4.9)

then, in principle, we can obtain \(\phi_e\) and \(\psi_e\) in terms of \(\phi^*\) and \(\psi^*\) by solving Eq. (4.7) and Eq. (4.9) together.

From the potential in Eq. (4.4), the e-folding number in the slow-roll regime can be evaluated as

\[
N(\phi^*, \psi^*) \simeq - \frac{1}{M_P^2} \int_{\phi^*}^{\phi_e} \frac{d\phi}{f_{\phi}(\phi)},
\]

(4.10)

where we assumed \(f_{\phi}(\phi) \gg g_{\phi}(\phi)h(\psi)\). Then we obtain

\[
\begin{align*}
M_P^2 \frac{\partial N}{\partial \phi^*} &= - \left( \frac{\partial \phi_e}{\partial \phi^*} \right) \frac{1}{f_{\phi_e}} + \frac{1}{f_{\phi^*}}, \\
M_P^2 \frac{\partial N}{\partial \psi^*} &= - \left( \frac{\partial \phi_e}{\partial \psi^*} \right) \frac{1}{f_{\phi_e}}.
\end{align*}
\]

(4.11)

For simplicity we assume that the inflation ends by the condition given by only \(\psi_e\)

\[
E(\phi_e, \psi_e) = E(\psi_e) = 0,
\]

(4.12)

and \(\psi_e\) is independent of any \(\phi^*\) and \(\psi^*\). In this case, using Eq. (4.8), Eq. (4.11) becomes

\[
\begin{align*}
M_P^2 \frac{\partial N}{\partial \phi^*} &= \frac{1}{f_{\phi_e}} \left( 1 - \frac{g^*}{g_e} \right), \\
M_P^2 \frac{\partial N}{\partial \psi^*} &= \frac{1}{g_e h_{\psi^*}}.
\end{align*}
\]

(4.13)

Now, by matching this with the given power spectrum in Eq. (4.3), we obtain equations

\[
\begin{align*}
\mathcal{P}_s &= \frac{H_s^2}{4\pi^2 M_P^2} \left[ \frac{1}{f_{\phi^*}} \left( 1 - \frac{g^*}{g_e} \right) \right]^2, \\
\mathcal{P}_p &= \frac{H_s^2}{4\pi^2 M_P^2} \left[ \frac{1}{g_e h_{\psi^*}} \right]^2.
\end{align*}
\]

(4.14)

where \(N_s \equiv N(\phi^*, \psi^*).\) We solve these equations with the equations of motion of the fields Eq. (2.4) to reconstruct functions \(f(\phi), g(\phi), \) and \(h(\psi)\) in the potential. In the following subsections, we consider two cases for them and present the power spectra which are calculated using the equations Eq. (2.9) and Eq. (2.16) numerically.
4.1 Case 1: Gaussian peak

Here we consider a input power spectrum with a peak defined as:

$$P_p = \frac{H^2}{4\pi^2} \left[ \delta + \beta e^{-\alpha(N_t-N_{tc})} - e^{\lambda(N_t-N_{tc})} \right]^2.$$  \hspace{1cm} (4.15)

Note that $N$ is defined from the end of inflation. However, we also use the notation of $N_t$, which is defined from some initial time of inflation with a relation $N_t = N_{tot} - N$ with $N_{tot}$ the e-folding number between some initial time and the end. It is evident from the above expression that, the power spectrum is Gaussian near $N_t = N_{tc}$ and it decreases exponentially near to the end of inflation when $N_t > N_{tc}$.

From this input power spectrum in Eq. (4.15), we try to reconstruct a potential by solving the relations in Eq. (4.14) from $\delta N$ formalism. By comparing both equations, we can obtain $h_{\psi}(N_t)$ from $P_p$ as

$$h_{\psi}(N_t) = \frac{1}{M_p^2 g_e} \left[ \delta + \beta e^{-\alpha(N_t-N_{tc})} - e^{\lambda(N_t-N_{tc})} \right].$$  \hspace{1cm} (4.16)

In order to reconstruct a potential $h(\psi)$ which can produce the above power spectrum, we consider that the function $h_{\psi}$ in Eq. (4.4) is proportional to $\psi^{n-1}$ with integer $n$ larger or equal to 2. In this case, $h(\psi)$ is simply

$$h(\psi) = \frac{1}{n} \left( \frac{\psi}{\kappa} \right)^n,$$  \hspace{1cm} (4.17)

where $\kappa$ is a constant with the same dimension as $\psi$. Then, Eq. (4.17) directly gives the relation $\psi(N_t)$ as

$$\psi(N_t) = \psi_c \left( \frac{h_{\psi}(N_t)}{h_{\psi}(N_{tc})} \right)^{\frac{1}{n-1}},$$  \hspace{1cm} (4.18)

where $\psi_c \equiv \psi(N_{tc})$. In the above expression, for convenience, we have rewritten $\kappa$ as $\kappa = \frac{\psi_{tc}}{(h_{\psi}(N_{tc})\psi_c)^{1/n}}$. From above expression we expect that, the $\psi$ is nearly constant at initial position and slowly evolves towards a minimum and then increases. For this evolution, we find that, the second derivative of $\psi$ cannot be neglected for a short duration near the point where $\psi$ begins to roll down towards the minimum and also near the point where the first derivative of $\psi$ is zero which happens at $\psi = \psi_c$. We check this deviation of slow roll condition of $\psi$ field in our numerical calculations as well. Then, from the equation of motion, Eq. (2.4), the function $g(\phi)$ is obtained in terms of $N_t$, as

$$g(\phi) = -\frac{H^2}{V_0 h_{\psi}(N_t)} \left[ \frac{d^2 \psi}{dN_t^2} + (3 - \epsilon_H) \frac{d\psi}{dN_t} \right],$$  \hspace{1cm} (4.19)

where in the second line we ignored the subdominant $\epsilon_H$, however we included the second derivative $d^2 \psi/dN_t^2$. Using the explicit form of $\psi(N_t)$, we find

$$g(\phi) = -\frac{ng_c^2 h(\phi)}{3(n-1)^2} \left\{ n [\lambda E_{\phi}(\phi) + 2 N_0(\phi) G(\phi)]^2 + (n-1) [\lambda(3 + \lambda) E_{\phi}(\phi) + 2\alpha(1 + N_0(\phi)(3 - 2\alpha N_0(\phi))) G(\phi)] N_0(\phi) \right\}$$  \hspace{1cm} (4.20)
where

\[
N_d[\phi] \equiv N_t(\phi) - N_{tc} \\
G(\phi) \equiv \beta e^{-\alpha(N_t(\phi) - N_{tc})^2} \\
Ex(\phi) \equiv e^{\lambda(N_t(\phi) - N_{ti})}
\]

The e-folding number \( N_t \) can be replaced as a function of \( \phi \) from the equation of motion in Eq. (2.4) once we know the evolution of \( \phi \) in terms of \( N_t \). As a specific choice, let us use a small field inflation with a function \( f(\phi) \),

\[
f(\phi) = -\left(\frac{\phi}{\mu}\right)^2.
\]

By solving the slow-roll equation of motion, we find the evolution of the field \( \phi \),

\[
\phi = \phi_i e^{\frac{2M_p^2}{\mu^2}(N_t - N_i)} = \phi_i e^{\frac{2M_p^2}{\mu^2}(N_t - N_i)},
\]

and the function \( g(\phi) \) is obtained by replacing \( N_t \) with \( \phi(N_t) \).

For the numerical calculation, we choose the power spectrum has a peak at \( k_c = 10^{12} \text{ Mpc}^{-1} \) corresponding to \( N_{tc} = 40 \). We also assume that the pivot scale exit the Hubble radius at \( N_{tp} = 10 \). Using these information and also assuming slow roll condition, we fix,

\[
\mu = 10, \\
V_0 = 24\pi^2 2\phi(N_p)^2 A_s/\mu^4.
\]

In addition, we choose the smallest scale relevant for the perturbations in CMB is \( k_{p_2} = 1 \text{ Mpc}^{-1} \) and this scale exit the Hubble radius at \( N_{tp_2} = N_{tp} + \log \frac{k_{p_2}}{k_p} \). As we have mentioned earlier, on small scales the peak spectrum is dominant, \( i.e. P_s \ll P_p \) and also \( P_p \gtrsim 10^7 P_s \) around scales of the peak. These information leads to the constraints

\[
\delta < \frac{\mu^2}{2\phi(N_{tp_2})}, \\
\beta > \frac{\mu^2\sqrt{10^7}}{2\phi(N_{tp})}.
\]
The power spectrums calculated numerically from the potential in Case 1 with $n = 2$ (blue), $n = 3$ (green), and the input power spectrum (orange). We used $\alpha = 1/75, \beta = 4 \times 10^5, \delta = 12, N_{tc} = 40$ and $N_{t1} = 65$, with $\lambda = 5$.

For the numerical calculations, we choose to work with

$$\alpha = 1/75, \quad \delta = 12, \quad \beta = 4 \times 10^5. \quad (4.26)$$

The remaining parameters are the initial field values used at $N_t = 0$ which are fixed as

$$\phi_i = \phi(N_{tp})e^{-2N_{tp}/\mu^2}, \quad \psi_i = 10^{-3}.$$ 

In Fig. 1 (Left), we show the function $g(\phi)$ for the function $f(\phi)$ used in Eq. (4.22). The explicit form of the potential is obtained substituting Eq. (4.17) and Eq. (4.20) in Eq. (4.4), which is complicated and not shown here. Instead, in Fig. 1 (Right), we show the potential and the trajectory of the fields in the plane of $(\phi/\phi_i, \psi/\psi_i)$. It is interesting to note that, during the initial stages of inflation, the function $g(\phi)$ is very small. This is expected since the potential is dominated by the field $\phi$ alone. In the case of $n = 2$ the square of the mass of $\psi$ is proportional to the function $g(\phi)$. Around $N_t = N_{tc}$, $g(\phi)$ is linear in $N_t$ and changes its sign from positive to negative. This means that the field $\psi$ becomes tachyonic. Finally, the square of mass decreases exponentially and this leads inflation to end when the $\epsilon_H = 1$.

In Fig. 2, we show the power spectrums for this case: the input power spectrum (orange dashed) and the power spectrums from the reconstructed potential with $n = 2$ (blue solid), $n = 3$ (green solid). We can see that the input power spectrum matches the power spectrum obtained numerically quite well within an error of 10%.

4.2 Case 2: Hyperbolic peak

In the previous case, we have been able to reconstruct the potential by considering the Gaussian power spectrum and the function $h_\psi \propto \psi^n$. Let us now try to reconstruct the potential by assuming the same $h_\psi$ but a different type of peak power spectrum which is
Figure 3: Evolution of \( g(\phi) \) (left) and the trajectory of the fields on the potential \( V(\phi, \psi) \) (right) for case 2 with \( n = 2 \). We used the same parameters as in Fig. 2 except \( \gamma = 1/2, N_{t1} = 57, \) and \( \lambda = 3 \).

Figure 4: The power spectrums calculated numerically from the reconstructed potential in Case 2 with \( n = 2 \) (red), \( n = 3 \) (purple) and the input power spectrum (orange). We used the same parameters as in Fig. 2 except \( \gamma = 1/2, N_{t1} = 57, \) and \( \lambda = 3 \).

given by

\[
\mathcal{P}_P = \frac{H_*^2}{4\pi^2} \left[ \delta + \beta \text{sech} \left[ \gamma(N_t - N_{tc}) \right] - e^{\lambda(N_t - N_{t1})} \right]^2.
\] (4.27)

The task is to reconstruct \( g(\phi) \) by using the Eq. (4.19). Following the same method used in case 1, it is straight forward to obtain \( g(\phi) \) as

\[
g(\phi) = -\frac{ng_e^2 h(\phi)}{3(n-1)^2} \left\{ n\lambda^2 E x(\phi)^2 + \lambda E x(\phi) [(n-1)(3+\lambda)N_{\psi}(\phi) + 2n\beta\gamma S[\phi]T[\phi]] \\
+ \gamma \beta S(\phi) [n\beta\gamma S(\phi)T(\phi)^2 + (n-1)N_{\psi}(\phi)(2\gamma S(\phi)^2 - \gamma + 3T(\phi))] \right\}
\] (4.28)
where

\begin{align}
S(\phi) &= \frac{2}{(\phi/\phi_c)^{\gamma^2/2} + (\phi/\phi_c)^{-\gamma^2/2}}, \\
T(\phi) &= \frac{(\phi/\phi_c)^{\gamma^2} - (\phi/\phi_c)^{-\gamma^2}}{(\phi/\phi_c)^{\gamma^2} + (\phi/\phi_c)^{-\gamma^2}}.
\end{align}
(4.29)

In Fig. 3, we show the evolution of \( g(\phi) \) and the trajectory of the fields on the potential \( V(\phi, \psi) \) for case 2 with \( n = 2 \). As one can see from this figure, the overall behavior of the function \( g(\phi) \) and the evolution of the inflationary trajectory are similar as in the case 1. In Fig.4, we show the power spectrum for the reconstructed potential in Case 2 with \( n = 2 \) (red), \( n = 3 \) (purple) with the input power spectrum (orange). Here we used \( \gamma = 1/2, N_{11} = 57, \) and \( \lambda = 3 \). It is clear from the figure that the given spectrum match the numerical results very well.

5 Discussion

The primordial black hole can be produced from the enhanced primordial power spectrum of the curvature perturbation on small scales, and may play as dark matter. In the literature, there have been several trials to calculate an enhanced power spectrum from a given potential in single field or multi-field inflation models. In this paper, however, we suggest new method to reconstruct a potential from a given power spectrum. With this way, we could find potentials that have a peak on small scales, which is large enough to generate primordial black holes.

In this work, we used the input power spectrum composed of a nearly scale-invariant one on large scales and the other with a peak on small scales. To reconstruct a potential with two canonical scalar fields with \( \phi \) and \( \psi \), we have used a hybrid type potential. Guided by the \( \delta N \) formalism, we matched each power spectrum to the contribution from each field and solved them. We have been able to solve them numerically and could reconstruct a potential. We evaluated the scalar power spectra in reconstructed models and confirmed that the resulting spectra are quite compatible with the input spectrum.

In the reconstructed models, the scale-invariant power spectrum is generated when the field \( \phi \) is dominant while the field \( \psi \) is nearly constant. After this period, the field \( \psi \) starts moving towards its minimum and then bounces at a critical value to increase. During this bounce, the power spectrum is dominated by the \( \psi \) field and the enhance peak is generated.

We should also mention that, though the correct form of the power spectrum for small scales is yet to be understood, for illustration, we have chosen two types of power spectrum, Gaussian and hyperbolic. It is also important to note that, in this work, we have focussed on constructing small-field hybrid type of potentials. We believe that, using our methods discussed in this work, one can explore more complex models beyond the models we have constructed here.

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