Random band matrices

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\[(H\Psi)(i) = \sum_{|i-j| \leq W} H_{ij} \Psi(j)\]
**Integrable models**: diagonal matrices \((W = 0)\) or the Gaussian Orthogonal Ensemble \((W = N/2, \text{Gaussian entries})\), which satisfies:

(i) Semicircle law as \(N \to \infty\) (Wigner):
\[
\rho(E) = \frac{1}{2\pi} \sqrt{(4 - E^2)_+}.
\]

(ii) Eigenvalues locally converge (Gaudin, Mehta, Dyson):
\[
\sum_i \delta_{N \rho(E)}(\lambda_k - E) \to \text{Sine}_1.
\]

(iii) Eigenvectors are uniform on the sphere, essentially supported on all sites (delocalization). The Lévy-Borel law holds: for any \(\|q\|_2 = 1\),
\[
\sqrt{N} \langle u_k, q \rangle_{N \to \infty} \to \mathcal{N}(0, 1).
\]
For $d = 1, 2, 3$, vertices are elements of $\Lambda = [1, N]^d$ and $H = (H_{ij})_{i,j \in \Lambda}$ are centered, real, independent up to the symmetry $H_{ij} = H_{ji}$. Band width $W$:

$$H_{ij} = 0 \text{ if } |i - j| > W.$$ 

Localization (eigenvector support length $\ell \ll N$) ?

**Conjecture**: transition at some critical band width $W_c(N)$. Localization for $W \ll W_c$, delocalization for $W \gg W_c$, where

$$W_c = \begin{cases} 
N^{1/2} & \text{for } d = 1 (\ell \sim \min(W^2, N)), \\
(\log N)^{1/2} & \text{for } d = 2 (\ell \sim \min(e^{W^2}, N)), \\
O(1) & \text{for } d = 3.
\end{cases}$$

Simultaneous transition of eigenvalues statistics, from Poisson to GOE.
Context
A spacially confined quantum mechanical system can only take certain discrete values of energy. Uranium-238:

These values are eigenvalues of a certain self-adjoint operator.

**Wigner’s universality idea (1956).** The local spectral statistics of highly correlated quantum systems are given by the random matrix statistics of the same symmetry type. Random matrix statistics are ”universal” probability laws for highly correlated systems.

(i) GOE : Hamiltonians of systems with time reversal invariance
(ii) GUE : no time reversal symmetry (e.g. application of a magnetic field)
(iii) GSE : time reversal but no rotational symmetry

Wigner’s vision is not proved for any realistic Hamiltonian.
Problem 1: GOE and delocalization with no randomness.

Bohigas-Giannoni-Schmit conjecture: if the billard is chaotic, GOE-type repulsion of eigenvalues.

Helmholtz equation: \(-\Delta \psi_n = \lambda_n \psi_n\).

Weyl law: \(|\{i : \lambda_i \leq \lambda\}| \sim \frac{\text{area}(D)}{4\pi} \lambda^\times\).

Spacings statistics:
\[
\chi^{(n)} = \frac{1}{n} \sum_{i \leq n} \delta \frac{4\pi}{\text{area}(D)} (\lambda_{i+1} - \lambda_i).
\]

(Numerics: A. Backer)

Delocalization on average (quantum ergodicity) is proved. GOE is not.
Problem 2: GOE and delocalization with non-trivial geometry.

Anderson’s model for metal-insulator transition:

\[ H = -\Delta + \lambda V_\omega \]

on \( \mathbb{Z}^d \cap [-L,L]^d \), random i.i.d. potential.

Depending on \( \lambda \) and \( d \), two distinct regimes:

(i) Localization for high disorder in any dimension by multiscale analysis (Frohlich-Spencer), fractional moment method (Aizenman-Molchanov). Poisson statistics (Minami).

(ii) Delocalization and GOE (e.g., small \( \lambda \) for \( d = 3 \))?

For Anderson, localization was the new phenomenon. Mathematically speaking, delocalization is harder: for all dimensions, nothing is proved. Minami showed that

Exponential decay of the resolvent implies Poisson statistics.

Mechanism for GOE statistics in the delocalized phase?
Delocalization, GOE, for at least one model beyond mean-field?

Content:

1. Mean-field models.
2. Results for band matrices.
3. Elements of proof for the delocalized regime, $d = 1, W \gg N^{3/4}$.
   Heuristics for transition exponents.
Mean field models
Wigner matrices: eigenvalues universality. $N \times N$ symmetric,

$$\mathbb{E}(H_{ij}) = 0, \mathbb{E}(H_{ij}^2) = \frac{1}{N},$$

higher moments are finite but arbitrary.

**Theorem (2014)**

For any $E \in (-2, 2)$, \(\sum_i \delta_N \rho(E)(\lambda_k - E) \rightarrow \text{Sine}_1\).

Johansson (2000): Hermitian case with large Gaussian component, HCIZ formula

Erdős-Péché-Ramirez-Schlein-Yau (2009), Hermitian case, reverse heat flow

Tao-Vu (2009): Hermitian case, moment matching

Erdős-Schlein-Yau (2009): general case with averaging over $E$, relative entropy method

Erdős-Yau (2012): general case, gap universality, Hölder regularity

B.-Erdős-Yau-Yin (2014): general case, fixed energy, coupling method

Need a-priori estimates on the eigenvalues $\lambda_1 < \cdots < \lambda_N$.

**Theorem (Erdős, Yau, Yin, 2011)**

Define $\hat{k} = \min(k, N + 1 - k)$ and $\gamma_k$ implicitly by $\int_{-2}^{\gamma_k} d\rho = \frac{k}{N}$. Then

$$|\lambda_k - \gamma_k| < N^{-\frac{2}{3} + \varepsilon(\hat{k})} \hat{k}^{-\frac{1}{3}}.$$
Dyson Brownian Motion: $dH_t = \frac{dB_t}{\sqrt{N}}$, $H_0$ Wigner.

**Step 1: relaxation.** For $t \gg N^{-1+\epsilon}$, local equilibrium is reached. Relies on coupled dynamics. Let $x(0)$ be the spectrum of $H_0$, $y(0)$ of GOE.

$$
\frac{dx_i}{dy_i} = \sqrt{\frac{2}{N}} dB_i(t) + \frac{1}{N} \sum_{j \neq i} \frac{1}{x_i/y_i - x_j/y_j} dt.
$$

Then $\delta_\ell(t) = x_\ell(t) - y_\ell(t)$ satisfies the parabolic equation

$$
\partial_t \delta_\ell(t) = \sum_{k \neq \ell} B_{k\ell}(t) (\delta_k(t) - \delta_\ell(t)) ,
B_{k\ell} = \frac{1}{N(x_k - x_\ell)(y_k - y_\ell)}.
$$

Hölder regularity = universality. This coupling argument is local, it gives relaxation for any initial condition (Landon-Sosoe-Yau, Erdős-Schnelli).

**Step 2: density.** For $t \ll N^{-\frac{1}{2}-\epsilon}$, the local eigenvalues distribution has almost not changed. Relies on the matrix structure (Itô’s formula). This step fails for non mean field models.
Wigner matrices: eigenvectors universality. Let $u_1, \ldots, u_N$ be the eigenvectors associated to $\lambda_1 \leq \cdots \leq \lambda_N$.

By the density step, universality of bulk eigenvectors in the perturbative setting $E((\sqrt{N}H_{ij})^3) = 0$, $E((\sqrt{N}H_{ij})^4) = 3$ (Knowles-Yin, Tao-Vu, 2011).

**Theorem (B-Yau, 2013)**

For any deterministic $k, q \in \mathbb{R}^N$, $\sqrt{N}\langle q, u_k \rangle$ converges to a Gaussian.

Need a-priori estimates of (isotropic) delocalization.

**Theorem (Erdős, Knowles, Schlein, Yau, Yin, 2009-2011)**

For any $\varepsilon, D > 0$, for any deterministic $k, q \in \mathbb{R}^N$,

$$
\mathbb{P} \left( \sup_k \langle q, u_k \rangle \leq N^{-\frac{1}{2} + \varepsilon} \right) \geq 1 - N^{-D}.
$$

Relaxation step for eigenvectors?
Joint eigenvalues/eigenvectors dynamics (Bru, 1989)

\[ d\lambda_k = \frac{dB_{kk}}{\sqrt{N}} + \frac{1}{N} \sum_{\ell \neq k} \frac{1}{\lambda_k - \lambda_\ell} dt \]

\[ du_k = \frac{1}{\sqrt{N}} \sum_{\ell \neq k} \frac{dB_{k\ell}}{\lambda_k - \lambda_\ell} u_\ell - \frac{1}{2N} \sum_{\ell \neq k} \frac{dt}{(\lambda_k - \lambda_\ell)^2} u_k \]

Random walk in a dynamic random environment.
Configuration \( \eta \) of \( n \) points on \([1, N]\). Usual notations: \( \eta_x, \eta^{ij} \).
Define \( z_k(t) = \sqrt{N} \langle q, u_k(t) \rangle \). For a configuration \( \eta \) with \( j_k \) points at \( i_k \), let

\[ f(\eta) = f_{t,\lambda}(\eta) = \mathbb{E} \left( \prod_k z_{i_k}^{2j_k} \mid \lambda \right) / \mathbb{E} \left( \prod_k N_{i_k}^{2j_k} \right) . \]

Lemma

\[ \partial_t f(\eta) = \sum_{i \neq j} 2\eta_i (1 + 2\eta_j) \frac{f(\eta^{i,j}) - f(\eta)}{N(\lambda_i(t) - \lambda_j(t))^2} . \]
Numerics : L. Benigni
The dynamic approach applies beyond Wigner matrices. Examples:

(i) Matrices with general mean, variance profile, summable decay of correlations (Ajanki-Erdős-Krüger- Schröder, Che)

(ii) Sparse random graphs of type Erdős-Renyi (Erdős-Knowles-Yau-Yin, Lee-Schnelli, Huang-Landon-Yau) and \(d\)-regular models (Bauerschmidt-Huang-Knowles-Yau, B-Huang-Yau)

(iii) Convolution, free probability model \(D_1 + U^*D_2U\), \(U\) uniform on \(O(N)\) (Bao-Erdős-Schnelli, Che-Landon).

The above matrix models are all mean-field.
Random band matrices
We say that an eigenvector $\psi$ is subexponentially localized at scale $\ell$ if there exists $\epsilon > 0$, $I \subset [1, N]$, $|I| \leq \ell$, such that
$$\sum_{\alpha \notin I} |\psi(\alpha)|^2 < e^{-N\epsilon}.$$

**Conjecture**: $W_c = N^{1/2}$ for $d = 1$ for bulk eigenvectors?

1. Based on numerics (Casati-Molinari-Izrailev, 1990)
2. Theoretical evidence from supersymmetric method (Fyodorov-Mirlin, 1991)
   Corresponding localization length: $\ell \approx W^2(|4 - E^2| + N^{-2/3})$.

**Analogy with the Anderson model** $H = -\Delta + \lambda V_\omega$ : $\lambda \approx W^{-1}$.
Consistent with localization length $\lambda^{-2}$ for $d = 1$, $e^{\lambda^{-2}}$ for $d = 2$. 

$$H = \begin{pmatrix} 0 & 2W \\ W & 0 \end{pmatrix}$$
On large enough scales, the method of moments applies for convergence to
\[ \rho(E) = \frac{1}{2\pi \sqrt{4 - E^2}} \] (Bogachev, Molchanov, Pastur) and study linear statistics of
eigenvalues (Anderson-Zeitouni, Guionnet, Erdős-Knowles, Li-Soshnikov,…)

**Results on microscopic scale.**

1. Edge behavior
2. Gaussian models with specific variance profile
3. Localization for general models
4. Delocalization for general models

1. **Edge behavior.** The transition at \( W \approx N^{5/6} \) was made rigorous by a
subtle method of moments.

**Theorem (Sodin, 2010)**

If \( W \gg N^{5/6} \), then \( N^{2/3}(\lambda_N - 2) \) converges to the Tracy-Widom
distribution. If \( W \ll N^{5/6} \), this does not hold.
2. Gaussian models with specific variance profile, supersymmetry. Covariance structure $\mathbb{E}(H_{ij}H_{\ell k}) = \mathbb{1}_{i=k,j=\ell}J_{ij}$ where $J_{ij} = (-W^2\Delta + 1)_{ij}^{-1}$. Essentially a 1d band matrix of width $W$. Define

$$F_2(E_1, E_2) = \mathbb{E} (\det(E_1 - H) \det(E_2 - H)).$$

**Theorem (Shcherbina-Shcherbina, 2014-2017)**

$$(D_2)^{-1} F_2 \left( E + \frac{x}{N\rho(E)}, E - \frac{x}{N\rho(E)} \right) \to \begin{cases} 1 & \text{if } N^{\epsilon} < W < N^{1/2 - \epsilon} \\ \frac{\sin(2\pi x)}{2\pi x} & \text{if } N^{1/2 + \epsilon} < W < N \end{cases}$$

Proof idea: representation for the left hand side ($N = 2n + 1$)

$$\int e^{-\frac{W^2}{2} \sum_{-n+1}^{n} \text{Tr}(X_j - X_{j-1})^2 - \frac{1}{2} \sum_{-n}^{n} \text{Tr}(X_j + \frac{i\Delta E}{2} + i\frac{\Delta x}{N\rho(E)})^2 \prod_{-n}^{n} \det(X_j - i\frac{\Delta E}{2}) dX_j,$$

where $\Delta_E = \text{diag}(E, E)$, $\Delta_x = \text{diag}(x, -x)$. Then steepest descent.

Other results, 3d density of states (Disertori-Pinson-Spencer), 1d pair correlation if $W \gg cN$ (Shcherbina), 1d delocalization if $W \gg N^{6/7}$ (Bao-Erdős).
3. Localization for general models. The following implies localization for $W \ll N^{1/8}$.

**Theorem (Schenker (2009))**

Let $\mu > 8$. There exists $\tau > 0$ such that for large enough $N$, for any $\alpha, \beta \in [1, N]$ one has

$$
\mathbb{E} \left( \sup_{1 \leq k \leq N} |u_k(\alpha)u_k(\beta)| \right) \leq W^{\tau} e^{-\frac{\left| \alpha - \beta \right|}{W^{\mu}}}. 
$$

This was improved to $W \ll N^{1/7}$ for some specific Gaussian model (Peled-Schenker-Shamis-Sodin).

The Poisson eigenvalues statistics are still unknown.
4. Delocalization for general models. An advanced analysis of the resolvent of band matrices gives an average form of delocalization:

For $W \gg N^{7/9}$ and $\ell \ll N$, the fraction of eigenvectors localized on scale $\ell$ vanishes as $N \to \infty$ (Erdős-Knowles-Yau-Yin 2012, He-Marcozzi 2018).

**Theorem (B-Yau-Yin, 2018)**

Assume $W \gg N^{3/4}$.

(a) Universality. For any $E \in (-2, 2)$, $\sum_i \delta N \rho(E)(\lambda_k - E) \to \text{Sine}_1$.

(b) Delocalization. For any unit bulk eigenvector $\psi$, $\|\psi\|_\infty < N^{-\frac{1}{2}} + \varepsilon$.

(c) Quantum unique ergodicity. There exists $c > 0$ such that for any interval $I \subset [1, N]$, $|I| > W$, $|\sum_{\alpha \in I} (\psi(\alpha)^2 - \frac{1}{N})| < N^{-c} \frac{|I|}{N}$.

Previously proved for $W \geq cN$ in (B-Erdős-Yau-Yin 2016), where a mean-field reduction method was introduced.
Quantum unique ergodicity (QUE) is the main mechanism for GOE.

**QUE conjecture (Rudnick, Sarnak).** If $\mathcal{M}$ is compact and negatively curved, and $(\psi_k)_{k \geq 1}$ the Laplacian eigenfunctions,

$$\int_A |\psi_k(x)|^2 \mu(dx) \xrightarrow{k \to \infty} \int_A \mu(dx).$$

**Known:** arithmetic QUE (Lindenstrauss), more general cases in an average sense, quantum ergodicity (QE) (Shnirelman, Zelditch, Colin de Verdière), quantum ergodicity for random regular graphs (Anantharaman-Le Masson).

(i) QUE implies GOE

(ii) Heuristics: QUE and GFF give the threshold $W_c$.

(iii) QE and DBM imply QUE
Probabilistic QUE
From QUE to GOE: mean-field reduction.

Let $H_e = \begin{pmatrix} A & B^* \\ B & D \end{pmatrix}$ with $A$ of size $W \times W$, and $\psi_j$ the eigenvector associated with $\lambda_j : H\psi_j = \lambda_j \psi_j$.

Denote $\psi_j = \begin{pmatrix} w_j \\ p_j \end{pmatrix}$. Then

$$Q_{\lambda_j} w_j = \lambda_j w_j$$

where $Q_e = A - B^* (D - e)^{-1} B$. Let $\xi^e_k$'s be the eigenvalues of $Q_e$

$$\frac{d\xi^e_k}{de} = -\frac{\|p_k\|_2^2}{\|w_k\|_2^2}.$$

Knowing QUE, GOE for band follows from GOE for $Q_e$ by parallel projection.
**QUE and GFF : heuristics.** QUE for an eigenvector $\psi$ of $H$ is obtained from QUE for $Q_e$ and a simple patching:

1. If QUE holds for all $Q_e$, $\|\pi_{[1,W/2]}\psi\|_2 \approx \|\pi_{[W/2,W]}\psi\|_2$.
2. Patching this equality over $\frac{N}{W}$ pairs of intervals gives a flat $\psi$.
3. Central limit for sum of errors : $\sqrt{\frac{N}{W}} \cdot \frac{1}{\sqrt{W}} \ll 1$ when $W \gg N^{1/2}$.

For any dimension, decompose $[1, N]^d$ into $(\frac{N}{W})^d$ cubes $C_v$ of side $W$. Let

$$X_v = \sum_{i \in C_v} |\psi(i)|^2.$$ 

Good model for $(X_v)_v$ : independent $\mathcal{N}(0, \frac{W^d}{N^{2d}})$ increments over adjacent cubes, conditioned to $\sum(X_{v_{i+1}} - X_{v_i}) = 0$ for any closed path.

This model is simply the Gaussian free field on $[1, N/W]^d$, with density

$$e^{-\frac{N^2}{2Wd} \sum_{v \sim w} (x_v - x_w)^2}.$$ 

QUE holds when $\text{Var}(X_v)^{1/2} \ll \mathbb{E}(X_v)$. For the GFF model, this means

$$W \gg (\log N)^{1/2} \ (d = 2), \quad W \gg 1 \ (d = 3).$$
From QE to QUE : dynamics. Consider the Dyson Brownian Motion: 
\[ \text{d}Q(t) = \frac{\text{d}B_t}{\sqrt{n}}. \]
Assume \( Q = Q_0 \) satisfies QE in the sense that
\[
\left| \frac{1}{|I|} \sum_{i \in I} ((Q - z)^{-1})_{ii} - \frac{1}{n} \text{Tr}(Q - z)^{-1} \right| \leq n^{-\alpha}, \quad \text{Im}(z) > n^{-\beta}.
\]

Lemma
Quantum unique ergodicity holds for \( t \gg n^{-\beta} \): if \( \psi_t \) is any eigenvector of \( Q(t) \), with overwhelming probability
\[
\left| \sum_{\alpha \in I} (\psi_t(\alpha)^2 - \frac{1}{n}) \right| < n^{-\alpha}.
\]

The initial QE estimate is hard to obtain for \( A - B^* (D - e)^{-1} B \) (singularity). Delicate estimates on a generalized Green’s function (B-Yang-Yau-Yin, Yang-Yin).
(a) A configuration $\eta$ with $N(\eta) = 6$, $\eta_{i_1} = 2$, $\eta_{i_2} = 3$, $\eta_{i_3} = 1$.

(b) A perfect matching $G \in \mathcal{G}_\eta$. Here, $P(G) = p_{i_1 i_1} p_{i_1 i_2} p_{i_2 i_2} p_{i_2 i_3} p_{i_3 i_1}$.

$$p_{ij} = \sum_{\alpha \in I} u_i(\alpha)u_j(\alpha) \quad (i \neq j), \quad p_{ii} = \sum_{\alpha \in I} u_i(\alpha)^2 - \frac{|I|}{n},$$

For any given configuration $\eta$, double the set of particles and consider the set of perfect matchings $(\mathcal{G}_\eta)$, each graph $G$ with edges $\mathcal{E}(G)$. Define

$$\tilde{f}_{\lambda,t}(\eta) = \frac{1}{\mathcal{M}(\eta)} \mathbb{E}\left( \sum_{G \in \mathcal{G}_\eta} P(G) \mid \lambda \right), \quad \mathcal{M}(\eta) = \prod_{i=1}^{n} (2\eta_i)!!,$$

**Lemma**

$$\partial_t \tilde{f}(\eta) = \sum_{i \neq j} 2\eta_i (1 + 2\eta_j) \frac{\tilde{f}(\eta^{i,j}) - \tilde{f}(\eta)}{N(\lambda_i(t) - \lambda_j(t))^2}.$$
For many mean field models, universality for eigenvalues and eigenvectors is understood.

Beyond mean field, one model with proved GOE and delocalization: random band matrices for $W \gg N^{3/4}$ ($d = 1$) and $W \gg N^{1-\kappa}$ for $d = 2, 3$. The exponents are not optimal.

Quantum unique ergodicity gives GOE (localization giving Poisson statistics).

For the proof of quantum unique ergodicity, new integrable dynamics connect to random walks in time-dependent random environments.

One major question is random matrix statistics in a deterministic setting.