LOCAL STRUCTURE OF GENERALIZED COMPLEX MANIFOLDS

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Abstract. We study generalized complex manifolds from the point of view of symplectic and Poisson geometry. We start by showing that every generalized complex manifold admits a canonical Poisson structure. We use this fact, together with Weinstein’s classical result on the local normal form of Poisson manifolds, to prove a local structure theorem for generalized complex manifolds which extends the result Gualtieri has obtained in the “regular” case. Finally, we begin a study of the local structure of a generalized complex manifold in a neighborhood of a point where the associated Poisson tensor vanishes. In particular, we show that in such a neighborhood, a “first-order approximation” to the generalized complex structure is encoded in the data of a constant $B$-field and a complex Lie algebra.

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1. Introduction and main results

The main objects of study in this paper are irregular generalized complex (GC) structures on manifolds (the terminology is explained below). In this section we state and discuss our main results. The rest of the paper is devoted to their proofs.

1.1. Background on GC geometry. We begin by recalling the setup of generalized complex geometry. We use [Gua] as the main source for most basic results and definitions, a notable exception being the notion of a generalized complex submanifold of a GC manifold, which is taken from [BR].

The notion of a GC manifold was introduced by N. Hitchin (cf. [Hi1, Hi2, Hi3]) and developed by M. Gualtieri in [Gua]. If $M$ is a manifold (by which we mean a finite dimensional real $C^\infty$ manifold), specifying a GC structure on $M$ amounts to specifying either of the following two objects:

- an $\mathbb{R}$-linear bundle automorphism $J$ of $TM \oplus T^*M$ which preserves the standard symmetric bilinear pairing $\langle (X, \xi), (Y, \eta) \rangle = \xi(Y) + \eta(X)$ and satisfies $J^2 = -1$, or
- a complex vector subbundle $L \subset T\mathbb{C}M \oplus T^*\mathbb{C}M$ such that $T\mathbb{C}M \oplus T^*\mathbb{C}M = L \oplus \bar{L}$ and $L$ is isotropic with respect to the $\mathbb{C}$-bilinear extension of $\langle \cdot , \cdot \rangle$ to $T\mathbb{C}M \oplus T^*\mathbb{C}M$,

which are required to satisfy a certain integrability condition that is similar to the standard integrability condition for almost complex structures on real manifolds. A bijection between the two types of structure

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defined above is obtained by associating to an automorphism $\mathcal{J}$ its $+i$-eigenbundle. In terms of $L$, the integrability condition is that the sheaf of sections of $L$ is closed under the Courant bracket $\mathcal{C}$

$$[(X, \xi), (Y, \eta)]_{\mathcal{C}} = \left( [[X, Y], L_X \eta - L_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi) \right). \tag{1.1}$$

One can check that this condition is equivalent to the vanishing of the Courant-Nijenhuis tensor

$$\mathcal{N}_\mathcal{J}(A, B) = [\mathcal{J} A, \mathcal{J} B]_{\mathcal{C}} - \mathcal{J} [\mathcal{J} A, B]_{\mathcal{C}} - \mathcal{J} [A, \mathcal{J} B]_{\mathcal{C}} - [A, B]_{\mathcal{C}}, \tag{1.2}$$

where $A, B$ are sections of $TM \oplus T^*M$.

The two main examples of GC structures arise from complex and symplectic manifolds. If $M$ is a real manifold equipped with an integrable almost complex structure $J : TM \to T^*M$, it is easy to check that the automorphism $\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$ defines a GC structure on $M$; such a GC structure is said to be complex. Similarly, if $\omega$ is a symplectic form on $M$, we can view it as a skew-symmetric map $\omega : TM \to T^*M$, and then the automorphism $\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$ also defines a GC structure on $M$; such a GC structure is said to be symplectic.

A GC structure on a manifold $M$ induces a distribution $E \subseteq T_C M$ which is smooth in the sense of $\text{Sus}$. Namely, $E$ is the image of $L$ under the projection map $T_C M \oplus T_C^* M \to T_C M$. Note that $E$ may not have constant rank. The sheaf of sections of $E$ is closed under the Lie bracket (i.e., $E$ is involutive), as follows trivially from the definition of the Courant bracket. Moreover, there is a (complex) 2-form $\epsilon$ on $E$ defined as follows: if $X, Y$ are sections of $E$, choose a section $\xi$ of $T_C^* M$ such that $(X, \xi) \in L$, and set $\epsilon(X, Y) = \xi(Y)$. If $\eta$ is a section of $T_C^* M$ such that $(Y, \eta) \in L$, then $\xi(Y) = -\eta(X)$ because $L$ is isotropic with respect to the pairing $\langle \cdot, \cdot \rangle$, which implies that $\epsilon(X, Y)$ is independent of the choice of $\xi$; thus $\epsilon$ is well defined. Furthermore, one can define the tensor $d\epsilon \in \wedge^3(E^*)$ by the Cartan formula, which makes sense since $E$ is involutive.

**Proposition 1.1** (See $\text{Gua}$). The data $(E, \epsilon)$ determines the GC structure $L$ uniquely. Moreover, $d\epsilon = 0$.

A special type of operation defined for GC structures, which plays an important role in our discussion, is the transformation by a $B$-field. Specifically, if $B$ is a real closed 2-form on $M$, we define an orthogonal automorphism $\exp(B)$ of the bundle $TM \oplus T^*M$ via

$$\exp(B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix},$$

where we view $B$ as a skew-symmetric map $TM \to T^*M$. If $\mathcal{J}$ defines a GC structure on $M$, and the associated pair $(E, \epsilon)$ is constructed as above, then $\mathcal{J}' = \exp(B)\mathcal{J}\exp(-B)$ is another GC structure on $M$, which follows from the fact that $\exp(B)$ preserves the Courant bracket on $TM \oplus T^*M$, see $\text{Gua}$. Moreover, in this case, the $+i$-eigenbundle of $\mathcal{J}'$ is given by $L' = \exp(B)(L)$, and the associated pair $(E', \epsilon')$ is determined by $E' = E$, $\epsilon' = \epsilon + B|_{E'}$, where, by a slight abuse of notation, we also denote by $B$ the $C$-bilinear extension of $B$ to $T_C M$. In our paper, a $B$-field transformation will always mean a transformation of the form $\mathcal{J} \mapsto \exp(B)\mathcal{J}\exp(-B)$, where $B$ is a closed real 2-form. For a more detailed discussion and a more general notion of $B$-fields, see $\text{Gua}$ and references therein.

Another important construction is that of the canonical symplectic foliation on a GC manifold. Namely, let us consider $E \cap \bar{E}$; this is a distribution in $T_C M$ which is stable under complex conjugation, and hence has the form $\mathcal{I}_C = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{I}$ for some distribution $\mathcal{I} \subseteq TM$. Gualtieri proves in $\text{Gua}$ that $\mathcal{I}$ is a smooth distribution in the sense of $\text{Sus}$, and that the 2-form $\omega$ on $\mathcal{I}$ defined by $\omega = \text{Im}(\epsilon|_{\mathcal{I}})$ is (pointwise) nondegenerate. Moreover, it is now clear that sections of $\mathcal{I}$ is closed under the Lie bracket, and
that $\omega$ is a closed 2-form on $\mathcal{F}$, in the same sense as in Proposition \[\text{L1}\] It follows from the results of \[\text{Sus}\] that through every point of $M$ there is a maximal integral manifold of $\mathcal{F}$, which, by construction, inherits a natural symplectic structure.

For example, if the GC structure on $M$ is complex, then $\mathcal{F} = 0$, while if the GC structure on $M$ is symplectic, then $\mathcal{F} = TM$ and the canonical symplectic form on $\mathcal{F}$ coincides with the symplectic form defining the GC structure on $M$.

We now recall the notion of a generalized complex submanifold of a GC manifold. Let $L$ be a GC structure on a manifold $M$, and let $N \subset M$ be a (locally closed) submanifold. We define a (not necessarily smooth) distribution $L_N$ on $N$ as follows. Set

$$\tilde{L}_N = L_{|N} \cap \left(T_C N \oplus (T_C^* M_{|N})\right) \quad \text{and} \quad L_N = \text{pr}(\tilde{L}_N),$$

where $\text{pr} : T_C N \oplus (T_C^* M_{|N}) \to T_C N \oplus T_C^* N$ denotes the natural projection map, $(X, \xi) \mapsto (X, \xi|_{T_C N})$. It is proved in \[\text{BB}\] that $\dim_C L_{N, n} = \dim_R N$ for all $n \in N$. However, $L_N$ may not be a subbundle of $T_C N \oplus T_C^* N$. We say that $N$ is a generalized complex submanifold of $M$ provided $L_N$ is smooth, and defines a GC structure on $N$. It can be shown (cf. \[\text{BB}\]) that a necessary and sufficient condition for this is that $L_N$ is smooth and $L_N \cap \tilde{L}_N = 0$ (integrability is then automatic).

In conclusion, we would like to mention that there exists a way of describing GC structures on manifolds in terms of spinors. In fact, most of \[\text{Gua}\] is written in the language of spinors. However, in our paper we have made a conscious effort to state and prove all of our results in a spinor-free language. We hope that this approach helps illuminate the simple geometric ideas that underlie our main constructions.

1.2. The canonical Poisson structure on a GC manifold. From now on we fix a manifold $M$ equipped with a GC structure which, whenever convenient, we will think of in terms of either the automorphism $J$ or the subbundle $L \subset T_C M \oplus T_C^* M$. The starting point for our work is the observation that the canonical symplectic foliation $(\mathcal{F}, \omega)$ defined in \[\text{L1}\] is in fact the symplectic foliation associated to a certain Poisson structure on $M$. The existence of a canonical Poisson structure on a GC manifold was also independently noticed by M. Gualtieri \[\text{Gua2}\], and S. Lyakhovich and M. Zabzine \[\text{LZ}\].

Let us briefly explain why one could expect the existence of a natural Poisson structure on general grounds. Recall the definition of integrability as the vanishing of the Courant-Nijenhuis tensor \[\text{L2}\]. The condition $\mathcal{N}_J(A, B) = 0$ can be naturally rewritten as a collection of four equations corresponding to the possibilities of either $A$ or $B$ being a section of $TM$ or a section of $T^* M$. Let us also write $\mathcal{J}$ as a matrix

$$\mathcal{J} = \begin{pmatrix} J & \sigma \\ \pi & K \end{pmatrix},$$

where $J : TM \to TM$, $\pi : T^* M \to TM$, $\sigma : TM \to T^* M$ and $K : T^* M \to T^* M$ are bundle morphisms. The requirements that $\mathcal{J}^2 = -1$ and $\mathcal{J}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$ force $K = -J^*$, $\pi = -\pi^*$, $\sigma = -\sigma^*$; in particular, $\pi$ can be viewed as a bivector on $M$, i.e., a section of $\bigwedge^2 TM$. Moreover, it is a straightforward computation that in the case when $A = (0, \xi)$ and $B = (0, \eta)$, where $\xi, \eta$ are sections of $T^* M$, the $TM$-component of $\mathcal{N}_J(A, B)$ is the following expression:

$$[\pi \xi, \pi \eta] - \pi \left(\mathcal{L}_{\pi \xi} \eta - \frac{1}{2} d(\iota_{\pi \xi} \eta)\right) + \pi \left(\mathcal{L}_{\pi \eta} \xi - \frac{1}{2} d(\iota_{\pi \eta} \xi)\right).$$

Observe that this expression depends only on $\pi$ and not on the other components of the matrix defining $\mathcal{J}$. However, one can check that no other entry of the matrix can be separated from the rest in this way. This suggests that $\pi$ must play a special role in the theory. In fact, we prove

**Theorem 1.** The bivector $\pi$ defines a Poisson structure on $M$. Moreover, the canonical symplectic foliation associated to this Poisson structure coincides with $(\mathcal{F}, \omega)$. 

Given a real-valued \( f \in C^\infty(M) \), let us write
\[
\mathcal{J}(0, df) = (X_f, \xi_f).
\]
By construction, \( X_f = \pi(df) \) is the Hamiltonian vector field on \( M \) associated to \( f \). On the other hand, \( \xi_f \) is a certain differential 1-form on \( M \).

**Proposition 1.2.** The map \( f \mapsto \xi_f \) has the following properties.

1. For all \( f, g \in C^\infty(M) \), we have
   \[
   \xi_{f \cdot g} = f \cdot \xi_g + g \cdot \xi_f.
   \]
2. If \( \{ \cdot, \cdot \} \) is the Poisson bracket on \( C^\infty(M) \) defined by \( \pi \), then
   \[
   \xi_{\{f, g\}} = \mathcal{L}_{X_f}(\xi_g) - \mathcal{L}_{X_g}(d\xi_f).
   \]
3. If \((E, \epsilon)\) is associated to the GC structure \( \mathcal{J} \) as in \([L3]\), then for all \( f \in C^\infty(M) \), we have
   \[
   \mathcal{L}_{X_f}(\epsilon) = (d\xi_f)|_E.
   \]

The two results above are proved in Section 3. The properties of the map \( f \mapsto \xi_f \) turn out to be crucial in our proof of the local normal form for GC manifolds. Moreover, these result raise the question of whether one can give an explicit description of GC manifolds as Poisson manifolds equipped with additional structure. In other words, consider a GC structure on a manifold \( M \) defined by the matrix \([L3]\). By Theorem 1, the pair \((M, \pi)\) is a Poisson manifold. Then the problem is to describe, in the language of Poisson geometry, the extra data on \((M, \pi)\) that needed to recover all of \( \mathcal{J} \). Part (2) of Proposition 1.2 is a first step in this direction.

**1.3. The local structure theorem for GC manifolds.** We say that a GC structure on a manifold \( M \) is regular if the distribution \( \mathcal{J} \) (equivalently, \( E \)) has locally constant rank. The structure is said to be irregular otherwise. The original motivation for our work came from trying to extend the local structure theorem proved in [Gua] for regular GC structures to the irregular case. Gualtieri proved that if \( m \in M \) is a regular point of a given GC structure on \( M \) (i.e., the structure is regular in an open neighborhood of \( M \)), then there exists a neighborhood \( U \) of \( m \) in \( M \) such that the induced GC structure on \( U \) is a \( B \)-field transform of the product of a symplectic GC manifold and a complex GC manifold. However, it seems to be difficult to adapt the method of [Gua] to the irregular situation. In particular, it relies strongly on the “complex Frobenius theorem” [Nd], no irregular analogue of which is known to us. On the other hand, a powerful tool that is available to us in view of Theorem 1 is Weinstein’s local structure theorem for Poisson manifolds [Wei]. Our approach has the advantage that it uses neither the real nor the complex version of the Frobenius theorem; nor, indeed, any nontrivial result from the theory of partial differential equations.

Let us fix a GC manifold \( M \) and a point \( m_0 \in M \). We define the rank, \( \text{rk}_{m_0} M \), of \( M \) at \( m_0 \) to be the rank of the associated Poisson tensor \( \pi \) at \( m_0 \). The central result of our paper is the following

**Theorem 2.** There exists an open neighborhood \( U \) of \( m_0 \) in \( M \), a real closed 2-form \( B \) on \( U \), a symplectic GC manifold \( S \) and a GC manifold \( N \) with marked points \( s_0 \in S \), \( n_0 \in N \) such that \( \text{rk}_{n_0} N = 0 \), and a diffeomorphism \( S \times N \to U \) which takes \((s_0, n_0)\) to \( m_0 \) and induces an isomorphism between the product GC structure on \( S \times N \) and the transform of the induced GC structure on \( U \) via the 2-form \( B \).

This theorem is proved in Section 4. Note that it is different in nature from the recent results of Dufour and Wade [DW]. Due to the presence of \( B \)-fields, which have no analogue for Dirac structures, our work gives more complete information on the local structure of irregular GC manifolds than loc. cit. does for irregular Dirac structures. Our method of proof is also essentially different.

**Remark 1.3.** It is easy to recover the result of Gualtieri from Theorem 2. Namely, if, with the notation of the theorem, the GC structure on \( M \) is regular in a neighborhood of \( m_0 \), then the rank of \( N \) must be zero in a neighborhood of \( n_0 \). It then follows by linear algebra that the GC structure on \( N \) must be \( B \)-complex in a neighborhood of \( n_0 \), and the fact that this structure can be written as the transform of a complex structure by a closed real 2-form follows from the local vanishing of Dolbeault cohomology (cf. [Gua]).
1.4. Linear GC structures. The term "linear GC structure" should not be confused with the notion of a constant GC structure on a real vector space discussed in Section 2. Rather, it is used in the same way as the term "linear Poisson structure" is used to describe the canonical Poisson structure on the dual space of a real Lie algebra.

Recall that if \((M, \pi)\) is a Poisson manifold, and \(m \in M\) is a point at which the Poisson tensor \(\pi\) vanishes, then a "first-order approximation" to \(\pi\) at \(m\) defines a real Lie algebra of dimension \(\dim M\). Canoically, this Lie algebra can be identified with the quotient \(\mathfrak{g} = \mathfrak{m}_m / \mathfrak{m}^2_m\), where \(\mathfrak{m}_m\) denotes the ideal in the algebra of all real-valued \(C^\infty\) functions on \(M\) consisting of the functions that vanish at \(m\). Since \(\pi\) vanishes at \(m\), it is easy to check that \(\mathfrak{m}_m\) is stable under the Poisson bracket, and \(\mathfrak{m}^2_m\) is an ideal of \(\mathfrak{m}_m\) in the sense of Lie algebras, and hence we obtain an induced Lie algebra bracket on \(\mathfrak{g}\).

Therefore one expects that, near a point on a GC manifold where the associated Poisson tensor vanishes, the first-order approximation to the GC structure can be encoded in a real finite dimensional Lie algebra equipped with additional structure. Indeed, we prove the following

**Theorem 3.** In a neighborhood of a point on a GC manifold where the associated Poisson tensor vanishes, the first-order approximation to the GC structure is encoded in a complex Lie algebra of complex dimension \((\dim M)/2\), and a \(B\)-field which is constant in appropriate local coordinates (and hence, a fortiori, is closed).

The meaning of this statement is explained in Section 5.

A natural problem that arises is to give a local classification of GC manifolds near a point where the associated Poisson tensor vanishes. Together with our Theorem 2, a solution of this problem would yield a complete local classification of generalized complex manifolds.

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2. Linear algebra

2.1. In this section we present the auxiliary results on linear algebra that are used in the proofs of our main theorems. We begin by recalling that the notion of a GC structure has an analogue for vector spaces, which was studied in detail in \(\text{BB}\) and \(\text{Gua}\). Specifically, a constant generalized complex structure on a real vector space \(V\) is defined either as an \(\mathbb{R}\)-linear automorphism \(J\) of \(V \oplus V^*\) which preserves the standard symmetric bilinear pairing \(\langle \cdot, \cdot \rangle\) and satisfies \(J^2 = -1\), or as a complex subspace \(L \subset V_C \oplus V_C^*\) which is isotropic with respect to the \(\mathbb{C}\)-bilinear extension of \(\langle \cdot, \cdot \rangle\) and satisfies \(V_C \oplus V_C^* = L \oplus \overline{L}\). There is no integrability condition in this case. It is easy to see that constant GC structures on \(V\) correspond bijectively to GC structures on the underlying real manifold of \(V\) that are invariant under translations. Furthermore, it is obvious that if \(J\) is a GC structure on a manifold \(M\), then for every point \(m \in M\), the automorphism \(J_m\) of \(T_m M \oplus T_m^* M\) induced by \(J\) defines a constant GC structure on \(T_m M\). From now on, by a generalized complex vector space we will mean a real vector space equipped with a constant GC structure.

All notions and constructions discussed in \(\text{BB}\) have obvious analogues for GC vector spaces. In particular, for a real vector space \(V\), we let \(\rho : V \oplus V^* \to V\), \(\rho^* : V \oplus V^* \to V^*\) denote the natural projection maps. Given a GC structure on \(V\) defined by a subspace \(L \subset V_C \oplus V_C^*\), we let \(E = \rho(L) \subset V_C\). There is an induced \(\mathbb{C}\)-bilinear 2-form \(\epsilon\) on \(E\) defined in the same way as in \(\text{BB}\), and the pair \((E, \epsilon)\) determines the GC structure on \(V\) uniquely. Moreover, if \(S \subset V\) is the real subspace satisfying \(\mathbb{C} \otimes_{\mathbb{R}} S = E \cap \overline{E}\), then \(\omega = \text{Im}(\epsilon|_S)\) is
a symplectic form on $S$. Finally, the notion of a generalized complex subspace of a GC vector space $V$ is defined in the obvious way: if $W \subseteq V$ is a real subspace, set

$$\tilde{L}_W = L \cap (W_C \oplus V_C^*),$$

and $L_W = \text{pr}(\tilde{L}_W)$,

where $\text{pr} : W_C \oplus V_C^* \to W_C \oplus W_C^*$ is the projection map $(w, \lambda) \mapsto (w, \lambda|_{W_C})$. We say that $W$ is a generalized complex subspace of $V$ if $L_W \cap \tilde{L}_W = (0)$; it is shown in [BB] that in this case $L_W$ is automatically a GC structure on $W$, called the induced generalized complex structure.

The notion of a $B$-field transform is also defined in the obvious way. If $B \in \bigwedge^2 V^*$ is a skew-symmetric bilinear form on $V$, then the map

$$\exp(B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

is a linear automorphism of $V \oplus V^*$ which preserves the standard pairing $\langle \cdot, \cdot \rangle$, and hence acts on constant GC structures on $V$ via

$$L \mapsto \exp(B) \cdot L, \quad \text{or} \quad J \mapsto \exp(B) \cdot J \cdot \exp(-B).$$

It is easy to check that, in terms of the pairs $(E, \epsilon)$, the transformation above is given by

$$(E, \epsilon) \mapsto (E, \epsilon + B_C|_{E}),$$

where $B_C$ is the unique $\mathbb{C}$-bilinear extension of $B$ to $V_C$.

In what follows, we will occasionally need to consider GC structures on different vector spaces at the same time. Therefore, whenever a confusion may arise, we will use the notation $L_V \subseteq V_C \oplus V_C^*$, $J_V \in \text{Aut}_R(V \oplus V^*)$, $S_V \subseteq V$, $E_V \subseteq V_C$, etc., to denote the objects $L, J, S, E$, etc., that are associated to a given GC structure on a vector space $V$.

2.2. For future use, we make explicit the notions of an isomorphism and a product of GC structures. Given two real vector spaces, $P$ and $Q$, equipped with GC structures $L_P$ and $L_Q$, an isomorphism of GC vector spaces between $P$ and $Q$ is an $\mathbb{R}$-linear isomorphism $\phi : P \to Q$ such that the induced map

$$(\phi_C, (\phi_C)^{-1}) : P_C \oplus P_C^* \to Q_C \oplus Q_C^*$$

carries $L_P$ onto $L_Q$. The direct sum of the GC vector spaces $P$ and $Q$ is the vector space $P \oplus Q$ equipped with the GC structure $L_P \oplus L_Q$ (called the product GC structure), where we have made the natural identification

$$(P \oplus Q)_C \oplus (P \oplus Q)_C^* \cong P_C \oplus P_C^* \oplus Q_C \oplus Q_C^*.$$ 

Finally, if $V$ is a GC vector space and $P, Q \subseteq V$ are two subspaces, we say that $V$ is the direct sum of $P$ and $Q$ as GC vector spaces provided $P, Q$ are GC subspaces of $V$, and if we equip $P, Q$ with the induced GC structures and $P \oplus Q$ with the product GC structure, then the map $P \oplus Q \to V$ given by $(p, q) \mapsto p + q$ is an isomorphism of GC vector spaces.

The notions of an isomorphism and a product of GC structures have obvious extensions to GC manifolds, see [BB].

2.3. The main results of generalized complex linear algebra that we need are summarized in the following

**Theorem 2.1.** Let $V$ be any GC vector space, and let $(S, \omega)$ be defined as above.

(a) The notion of being a GC subspace is transitive; in fact, the following stronger statement holds: if $W_1 \subseteq V$ is a GC subspace and $W_2 \subseteq W_1$ is any real subspace, then $W_2$ is a GC subspace of $V$ if and only if it is a GC subspace of $W_1$ with respect to the induced GC structure on $W_1$. Moreover, if this is the case, then the induced GC structure on $W_2$ is the same in both cases.

(b) A subspace $W \subseteq V$ is a GC subspace if and only if $W \cap S$ is a symplectic subspace of $S$ (in the sense that $\omega|_{W \cap S}$ is nondegenerate) and $W_C = (W_C \cap E) + (W_C \cap E)$.

\footnote{In general, however, GC subspaces do not behave well with respect to taking sums and intersections.}
(c) In particular, $S$ itself is a GC subspace of $V$; the induced GC structure on $S$ is $B$-symplectic, and moreover, $S$ is the largest GC subspace of $V$ with this property. The underlying symplectic structure on $S$ is given by $\omega$.

(d) The notion of being a GC subspace is invariant under $B$-field transformations of the GC structure on $V$.

(e) If $W \subseteq V$ is a real subspace such that $W + S = V$ (the sum is not necessarily direct), then $W$ is a GC subspace of $V$ if and only if $W \cap S$ is a symplectic subspace of $S$. In particular, any subspace of $V$ that is complementary to $S$ in the sense of linear algebra is automatically a GC subspace of $V$.

(f) Let $W \subseteq V$ be a real subspace such that $W + S = V$, and let $S_0$ denote any real subspace of $S$ such that $S = S_0 \oplus (S \cap W)$, so that $V = S_0 \oplus W$. Then the following two conditions are equivalent:

(i) $S_0$ and $S \cap W$ are orthogonal with respect to $\omega$;

(ii) $W$ and $S_0$ are GC subspaces of $V$, and there exists a $B$-field $B \in \bigwedge^2 V^*$ which transforms the GC structure on $V$ into the direct sum of the induced GC structures on $S_0$ and $W$.

(g) If the equivalent conditions of part (f) hold, then the choice of $B$ is unique provided we insist that $B|_{S_0} = 0$ and $B|_W = 0$.

**Remark 2.2.** As a byproduct of our discussion, we obtain an alternate proof of the structure theorem for constant GC structures (see [BD] and [Gu]) which does not use spinors. Indeed, if $S \subseteq V$ is as above and $W \subseteq V$ is any complementary subspace to $S$, then parts (e) and (f) of the theorem imply that $W$ is a GC subspace of $V$ and the GC structure on $V$ is a $B$-field transform of the direct product GC structure on $S \oplus W$. It is then easy to check that the induced GC structure on $S$ (resp., $W$) is $B$-symplectic (resp., $B$-complex), see, e.g., [BD].

**Proof of Theorem 2.1.** (a) It is trivial to check that the two definitions of $L_{W_2}$ we obtain by viewing $W_2$ either as a subspace of $V$ or as a subspace of $W_1$ coincide, whence the claim.

(b) We first show the necessity of the two conditions. It follows from the results of [BD] that a subspace of $S$ is a GC subspace if and only if it is a symplectic subspace with respect to the form $\omega$. Now if $W$ is any GC subspace of $V$, then $W \cap S = S_W$, whence $W \cap S$ is a GC subspace of $W$ by the results of [BD]. By part (a), it follows that $W \cap S$ is also a GC subspace of $V$, and hence a GC subspace of $S$.

Suppose now that $W$ is a GC subspace of $V$, yet $(W \cap E) + (W \cap \overline{E}) \subseteq W$. Then there exists a nonzero real subspace $U \subset W$ with

$$U_{C} \oplus [(W_{C} \cap E) + (W_{C} \cap \overline{E})] = W_{C}.$$ 

This implies that

$$U_{C} \cap [E + (W_{C} \cap \overline{E})] = (0) \quad \text{and} \quad U_{C} \cap [E + (W_{C} \cap E)] = (0).$$

Hence we can find $\ell, \ell' \in V_{C}^{*}$ with $\ell|_{U_{C}} = \ell'|_{U_{C}} \neq 0$ and $\ell|_{E + (W_{C} \cap \overline{E})} = 0 \equiv \ell'|_{E + (W_{C} \cap \overline{E})}$. This forces $\ell \in L_{W_{2}}$, $\ell' \in T \cap V_{C}^{*}$ and $\ell|_{W_{C}} = \ell'|_{W_{C}} \neq 0$, which means that

$$(\rho(\ell), \rho^{*}(\ell)|_{W_{C}}) = (0, \ell|_{W_{C}}) = (0, \ell'|_{W_{C}}) = (\rho(\ell'), \rho^{*}(\ell')|_{W_{C}}) \neq 0,$$

contradicting the assumption that $W$ is a GC subspace of $V$.

Conversely, suppose that $W \subseteq V$ is a subspace such that $W_{C} = (W_{C} \cap E) + (W_{C} \cap \overline{E})$ and $W \cap S$ is a GC subspace (equivalently, a symplectic subspace) of $S$. We will prove that $W$ is a GC subspace of $V$. Assume that $\ell \in L$, $\ell' \in T$ and $\rho(\ell) = \rho(\ell') \in W_{C}$, $\rho^{*}(\ell)|_{W_{C}} = \rho^{*}(\ell')|_{W_{C}}$. Then, in particular, $\rho(\ell) = \rho(\ell') \in (W \cap S)^{c}$ and $\rho^{*}(\ell)|_{(W \cap S)^{c}} = (\rho^{*}(\ell')|_{(W \cap S)^{c}}$, so we deduce from the second assumption that $\rho(\ell) = \rho(\ell') = 0$ and $\rho^{*}(\ell)|_{(W \cap S)^{c}} = \rho^{*}(\ell')|_{(W \cap S)^{c}} = 0$. It remains to check that $\rho^{*}(\ell)|_{W_{C}} = \rho^{*}(\ell')|_{W_{C}} = 0$. But

$$\rho^{*}(\ell) = \ell \in L \cap V_{C}^{*} = \text{Ann}_{V_{C}^{*}}(E) \quad \text{and} \quad \rho^{*}(\ell') = \ell' \in T \cap V_{C}^{*} = \text{Ann}_{V_{C}^{*}}(\overline{E}),$$

whence $\ell|_{W_{C} \cap E} = 0 = \ell'|_{W_{C} \cap \overline{E}}$, and also, since $\ell|_{W_{C}} = \ell'|_{W_{C}}$, we find from our first assumption that $\ell|_{W_{C}} = \ell'|_{W_{C}} = 0$, completing the proof.
(c) This is easy. We omit the proof since the straightforward argument is presented in [BB].

(d) It follows from the remarks of [BB] that a \( B \)-field transform changes neither \( E \), nor \( S \), nor \( \omega = \text{Im}(\epsilon|_S) \).

Hence the claim follows from the characterization of GC subspaces given in part (b).

(e) We will show that if \( W \subseteq V \) is a subspace such that \( V = W + S \), then we automatically have \( W_C = (W_C \cap E) + (W_C \cap \overline{E}) \). The claim then follows from part (b). Let \( w \in W_C \), and write \( w = e_1 + \overline{e}_2 \), with \( e_j \in E \) for \( j = 1, 2 \). Further, we can write \( e_j = w_j + s_j \), where \( w_j \in W_C \) and \( s_j \in S_C \). A fortiori, \( s_j \in E \), so \( w_j \in E \cap W_C \). Hence

\[
    w = w_1 + \overline{w}_2 + (s_1 + \overline{s}_2),
\]

where \( w_1, w_2 \in W_C \cap E \). This forces \( s_1 + \overline{s}_2 \in W_C \), and since we also have \( s_j \in S_C \), it follows that \( s_1 + \overline{s}_2 \in W_C \cap S_C \subseteq W_C \cap E \). Finally, we conclude that

\[
    w = (w_1 + s_1 + \overline{s}_2) + \overline{w}_2,
\]

where \( w_1 + s_1 + \overline{s}_2 \in W_C \cap E \) and \( \overline{s}_2 \in W_C \cap \overline{E} \), as desired.

(f), (g) First, it is clear that (ii) implies (i), since \( B \)-field transforms cannot change the imaginary part of \( \epsilon \). Conversely, assume that \( S_0 \) and \( S \cap W \) are orthogonal with respect to \( \omega \). We will show that there exists exactly one \( B \)-field \( B \in \mathcal{A}^2 V^* \) such that \( B|_{S_0} = B|_W = 0 \) and \( B \) transforms the given GC structure on \( V \) into the direct sum of the induced GC structures on \( S_0 \) and \( W \).

Observe that \( E_V = E_{S_0} \oplus E_W \). Indeed, it is clear that \( E_{S_0} \oplus E_W \subseteq E_V \). Conversely, let \( e \in E_V \) and write \( e = e_1 + e_2 \), where \( e_1 \in (S_0)_C \) and \( e_2 \in W_C \). Then, a fortiori, \( e_1 \in E_V \), so we also have \( e_2 \in E_V \), whence \( e_1 \in E_V \cap (S_0)_C = E_{S_0} \) and \( e_2 \in E_V \cap W_C = E_W \), proving the claim.

Note now that if the original GC structure on \( V \) is determined by \( (E_V, \epsilon) \), then the product GC structure on \( S_0 \oplus W \) is determined by

\[
    (E_{S_0} \oplus E_W, \epsilon|_{E_{S_0}} + \epsilon|_{E_W}).
\]

To complete the proof, we must therefore show that there exists exactly one \( B \in \mathcal{A}^2 V^* \) such that \( B|_{S_0} = B|_W = 0 \) and the pairing between \( E_{S_0} \) and \( E_W \) induced by (the complexification of) \( B \) is the same as the one induced by \( \epsilon \).

Suppose that such a \( B \) exists. Let \( s \in S_0, \ w \in W \). Since \( w \) is real, we can write \( w = e + \overline{e} \), where \( e \in E \cap W_C \). Then we must have

\[
    B(s, w) = B(s, e) + \overline{B(s, e)} = 2 \cdot \text{Re} \epsilon(s, e),
\]

which proves that \( B \) is unique if it exists. Conversely, let us define \( B \) on \( S_0 \times W \) by this formula, and define \( B \) to be zero on \( S_0 \) and on \( W \). We claim that \( B \) is well defined. Indeed, consider a different representation \( w = e' + \overline{e}' \), where \( e' \in E \cap W_C \). Then

\[
    e - e' = e' - e \in (W \cap S)_C,
\]

which implies that \( e - e' = i \cdot t \) for some \( t \in W \cap S \), where \( i = \sqrt{-1} \). Hence

\[
    \text{Re} \epsilon(s, e - e') = - \text{Im} \epsilon(s, t) = - \omega(s, t) = 0 \quad \text{by assumption},
\]

which implies that \( B \) is well defined.

Finally, to show that \( B \) satisfies the required condition, it is enough to check (by linearity) that if \( s \in S_0 \) and \( e \in E_W = W_C \cap E \), then \( B(s, e) = \epsilon(s, e) \). We have

\[
    e = \frac{e + \overline{e}}{2} + i \cdot \frac{e - \overline{e}}{2i} \quad \text{and} \quad \frac{e + \overline{e}}{2}, \frac{e - \overline{e}}{2i} \in W.
\]

By construction,

\[
    B \left( s, \frac{e + \overline{e}}{2} \right) = 2 \text{Re} \epsilon \left( s, \frac{e}{2} \right) = \text{Re} \epsilon(s, e),
\]
and similarly $B\left(s, (e - \tau)/(2i)\right) = \text{Im}\, e(s, e)$, which completes the proof. \hfill \Box

Remark 2.3. The following comment will be used in our proof of the local structure theorem for GC manifolds. Consider a variation of generalized complex linear algebra where the vector space $V$ is replaced by a smooth real vector bundle $V$ over a base manifold $\mathcal{B}$, and a GC structure on $V$ is a subbundle $\mathcal{L} \subseteq \mathcal{V}_\mathbb{C} \oplus \mathcal{V}_\mathbb{C}$ such that for every point $b \in \mathcal{B}$, the subspace $\mathcal{L}_b \subseteq \mathcal{V}_b, \mathcal{C} \oplus \mathcal{V}_b, \mathcal{C}$ defines a constant GC structure on the real vector space $\mathcal{V}_b$. Then we have the subdistributions $\mathcal{E} \subseteq \mathcal{V}_\mathbb{C}$ and $\mathcal{S} \subseteq \mathcal{V}$ which are the global analogues of $E$ and $S$, respectively, which may have nonconstant rank, but are nevertheless smooth in the sense of $\text{Sus}$, by the argument given in $\text{Gum}$. We claim that, in fact, the proofs of parts (e), (f) and (g) of Theorem 2.1 go through in this setup with appropriate modifications that ensure smooth dependence on the point $b \in \mathcal{B}$.

First, consider the analogue of part (e), where $W$ is replaced by a smooth subbundle $\mathcal{V} \subseteq \mathcal{V}$ such that $\mathcal{V} = S + W$ pointwise. We assume also that there exists a smooth subbundle $\mathcal{S}_0 \subseteq \mathcal{V}$ which is contained in $\mathcal{S}$ and satisfies $\mathcal{V} = \mathcal{S}_0 \oplus W$. Then we claim that $\mathcal{W}_\mathbb{C} = (\mathcal{W}_\mathbb{C} \cap \mathcal{E}) + (\mathcal{W}_\mathbb{C} \cap \mathcal{S})$ in the sense that every smooth section $w$ of $\mathcal{W}_\mathbb{C}$ can be written as $w = w' + w''$, where $w'$ and $w''$ are smooth sections of $\mathcal{W}_\mathbb{C}$ that lie in $\mathcal{E}$ and $\mathcal{S}$, respectively. Indeed, let us go through the proof of part (e) given above. By assumption, $\mathcal{V}_\mathbb{C} \oplus \mathcal{V}_\mathbb{C}^* = \mathcal{L} \oplus \mathcal{L}^*$, so we can write $(w, 0) = (e_1, f_1) + (e_2, f_2)$ for smooth sections $(e_1, f_1)$ and $(e_2, f_2)$ of $\mathcal{L}$ and $\mathcal{L}^*$, respectively. A fortiori, $e_1$ and $e_2$ are smooth sections of $\mathcal{E}$ and $\mathcal{S}$ that satisfy $w = e_1 + e_2$. Further, we can write uniquely $e_j = w_j + s_j$, where $w_j$ are smooth sections of $\mathcal{V}_\mathbb{C}$ and $s_j$ are smooth sections of $\mathcal{S}_0, \mathcal{C}$. The rest of the proof of part (e) goes through without changes.

Next we consider the analogue of parts (f) and (g). We assume that $\mathcal{S}_0$, $\mathcal{W}$ are as in the previous paragraph, and that, in addition, $\mathcal{S}_0$ and $\mathcal{W}$ are orthogonal with respect to the canonical symplectic form on $\mathcal{S}$. If $\mathcal{B}$ is the 2-form on $V$ constructed pointwise as in the proof of parts (f) and (g) given above, we claim that $\mathcal{B}$ is in fact smooth. Clearly, it suffices to show that if $s$ and $w$ are smooth sections of $\mathcal{S}_0$ and $\mathcal{W}$, respectively, then $B(s, w)$ is a smooth function on $\mathcal{B}$. By the previous paragraph, we can write $w = e + e'$, where $e, e'$ are smooth sections of $\mathcal{W}_\mathbb{C}$ that lie in $\mathcal{E}$. Since $w$ itself is real, we have

$$w = \frac{1}{2} \cdot [(e + e') + (\overline{e + e'})] = \frac{1}{2} \cdot [(e + e') + e + e'],$$

which implies that we may assume that $e = e'$ without sacrificing smoothness. By the proof above, we then have $B(s, w) = 2 \cdot \text{Re}(s, e)$, which is a smooth function since $e$ is smooth.

2.4. As one can see from part (ii) of Theorem 2.4, if $W \subseteq V$ is a GC subspace such that $S + W = V$, it is important to know the orthogonal complement $(W \cap S)\perp$ of $W \cap S$ in $S$ with respect to $\omega$. The following useful result provides a construction of this orthogonal complement that does not involve $S$ or $\omega$ explicitly.

**Theorem 2.4.** Let us identify $\mathcal{L}$ with $L^*$ using the standard symmetric bilinear pairing $\langle \cdot, \cdot \rangle$, and define $A_W$ to be the annihilator inside $\mathcal{L}$ of the subspace $\mathcal{L}_W = L \cap (W \oplus V_\mathbb{C}^*)$. If $C_W$ is the projection of $A_W$ on $\mathcal{V}_\mathbb{C}$, then $(W \cap S)\perp = C_W$.

The proof of this theorem provides an illustration of the techniques developed in this section. It consists of four steps.

**Step 1.** The statement of the theorem holds when the GC structure on $V$ is symplectic, given by a symplectic form $\omega \in \Lambda^2 V^\ast$.

**Proof.** In this case $V = S$, the canonical symplectic form on $S$ is also given by $\omega$, and the condition $W + S = V$ is vacuous. We have

$$L = L_V = \{ (v, -i \cdot \omega(v)) | v \in V_\mathbb{C} \},$$

whence

$$\mathcal{L}_W = \{ (w, -i \cdot \omega(w)) | w \in W_\mathbb{C} \}.$$

Also,

$$\mathcal{L} = \{ (u, i \cdot \omega(u)) | u \in V_\mathbb{C} \},$$
whence
\[
A_W = \left\{ (u, i\omega(u)) \mid u \in V, \ i\omega(u, w) - i\omega(w, u) = 0 \text{ for all } w \in W \right\}
\]
and so
\[
C_W = \left( W^\perp \right)_{C}^r,
\]
as required.

**Step 2.** The statement of the theorem holds under the following assumption: there exists a GC subspace \( U \subseteq V \) such that
- \( W = (W \cap S) \oplus U \);
- the induced GC structure on \( S \) (resp., \( U \)) is symplectic (resp., complex);
- \( V \) is the direct sum of \( S \) and \( U \) as GC vector spaces.

**Proof.** We make the obvious identifications
\[
V^* \cong S^* \oplus U^*, \quad V_{C}^* \cong S_{C}^* \oplus U_{C}^*, \quad V_{C} \oplus V_{C}^* \cong S_{C} \oplus S_{C}^* \oplus U_{C} \oplus U_{C}^*.
\]
Under these identifications, we have, by assumption,
\[
L = L_{V} = L_{S} \oplus L_{U} \quad \text{and} \quad \tilde{L}_W = \tilde{L}_{(W \cap S)} \oplus L_{U}.
\]
Thus it is clear that
\[
A_W = A_{W \cap S} \oplus (0) \subseteq \mathcal{T}_{S} \oplus \mathcal{T}_{U} = \mathcal{T}_{V},
\]
whence the claim follows from Step 1.

**Step 3.** The statement of the theorem is invariant under \( B \)-field transformations.

**Proof.** Let \( B \in \wedge^2 V^* \), then (with self-explanatory notation) we have
\[
L^{\text{new}} = \exp(B)(L^{\text{old}}), \quad \mathcal{T}^{\text{new}} = \exp(B)(\mathcal{T}^{\text{old}}),
\]
\[
\tilde{L}^{\text{new}}_W = \exp(B)(\tilde{L}^{\text{old}}_W) \quad (\text{because } \exp(B) \text{ preserves the subspace } W \oplus V^* \subseteq V \oplus V^*),
\]
\[
A^{\text{new}}_W = \exp(B)(A^{\text{old}}_W) \quad (\text{because } \exp(B) \text{ is orthogonal with respect to the standard pairing } \langle \cdot, \cdot \rangle),
\]
and finally
\[
C^{\text{new}}_W = C^{\text{old}}_W \quad (\text{because } \exp(B) \text{ commutes with the projection onto } V_{C}).
\]
On the other hand, we know that a \( B \)-field transformation changes neither \( S \) nor the canonical symplectic form on \( S \), proving the claim.

**Step 4.** We now complete the proof of the theorem as follows. Since \( W + S = V \), there exists a subspace \( U \subseteq W \) such that \( V = S \oplus U \). It follows from Theorem 2.1 that \( U \) is a GC subspace of \( V \), and the GC structure on \( V \) is a \( B \)-field transform of the direct sum of the induced GC structures on \( U \) and \( S \). Moreover, it follows from the results of [BB] that the induced GC structure on \( S \) (resp., \( U \)) is \( B \)-symplectic (resp., \( B \)-complex). Hence there exists a \( B \in \wedge^2 V^* \) that transforms the GC structure on \( V \) into the direct sum of the underlying symplectic structure on \( S \) and the underlying complex structure on \( U \). Now Steps 2 and 3 complete the argument.
3. Generalized complex manifolds are Poisson

In this section we prove Theorem 1 and Proposition 1.2. However, it is convenient for us to first restate the definition of the Poisson bracket on a generalized complex manifold in a different way.

Let $M$ be a manifold and $C^\infty(M)$ the algebra of all real-valued smooth functions on $M$. Consider a GC structure on $M$, where $J$ is the corresponding automorphism of $TM \oplus T^*M$ and $L$ is the $+i$-eigenbundle of $J$. If $f \in C^\infty(M)$, then $(0,df) \in TM \oplus T^*M$ can be written as $(0,df) = (X,\xi) + (\overline{X},\overline{\xi})$ for some sections $(X,\xi), (X',\xi')$ of $L$. Because of the uniqueness of this decomposition, and since $df = \overline{df}$, we have $(X',\xi') = (X,\xi)$. We set

$$X_f = 2iX = -2\text{Im}(X),$$

and call it the Hamiltonian vector field associated to the function $f$. Further, we put $\xi_f = -2\text{Im}(\xi)$. Since $L$ (resp., $\overline{L}$) is the $+i$-eigenbundle (resp., $-i$-eigenbundle) of $J$, it is clear that

$$J(0,df) = (X_f,\xi_f).$$

Recall that $(\mathcal{S},\omega)$ denotes the canonical symplectic foliation of $M$, as defined in [11].

**Lemma 3.1.** The vector field $X_f$ lies in $\mathcal{S}$. Moreover, for every section $Y$ of $\mathcal{S}$, we have

$$\omega(X_f,Y) = Y(f).$$

*Proof.* We use the notation of the previous paragraph. Since $X + \overline{X} = X + \overline{X} = 0$, we see that $X_f$ is a real vector field. Moreover, $X_f = 2iX$ lies in $E$ by construction, which forces $X_f$ to lie in $\mathcal{S}$. Now let $Y$ be a section of $\mathcal{S}$. By definition, we have

$$\omega(X_f,Y) = \text{Im}\epsilon(X_f,Y) = 2 \cdot \text{Re}\epsilon(X,Y) = 2 \cdot \text{Re}\xi(Y) = (df)(Y) = Y(f).$$

□

**Lemma 3.2.** With the notation above, the flow of $X_f$ preserves the subbundles $\mathcal{S} \subseteq TM$, $E \subseteq T_C M$, and also preserves the symplectic form $\omega$ on $\mathcal{S}$.

*Proof.* The fact that the flow of $X_f$ preserves $\mathcal{S}$ and $E$ follows from the fact that $\mathcal{S} \subset E$, that $X_f$ lies in $\mathcal{S}$, and that both $\mathcal{S}$ and $E$ are integrable. Then the fact that $\omega$ is preserved is a standard computation:

$$\mathcal{L}_{X_f}\omega = d(\iota_{X_f}\omega) + \iota_{X_f}(d\omega) = d(df) + 0 = 0.$$

□

**Lemma 3.3.** Let $f,g \in C^\infty(M)$, and define $\{f,g\} = X_f(g)$. Then

$$X_{\{f,g\}} = [X_f,X_g].$$

*Proof.* With the same notation as above, write

$$(0,df) = (X,\xi) + (\overline{X},\overline{\xi}), \quad (0, dg) = (Y,\eta) + (\overline{Y},\overline{\eta}),$$

so that

$$X_f = 2iX, \quad X_g = 2iY.$$

Recall from [Gu] that the restriction of the Courant bracket to sections of $L$ can be written as follows:

$$\left\{(X,\xi),(Y,\eta)\right\}_\text{cou} = \left([X,Y],\mathcal{L}_X\eta - \iota_Y(d\xi)\right).$$

Therefore the integrability condition for $L$ implies that

$$\ell = \left(\frac{1}{2i}[X_f,X_g],\mathcal{L}_{X_f}\eta - \iota_{X_g}(d\xi)\right) = 2i \cdot ([X,Y],\mathcal{L}_X\eta - \iota_Y(d\xi)) \quad (3.1)$$

is a section of $L$. Now the first component of $\ell + \overline{\ell}$ is zero, and the second one is

$$\mathcal{L}_{X_f}(\eta + \overline{\eta}) - \iota_{X_f}(d\xi + d\overline{\xi}) = \mathcal{L}_{X_f}(dg) - \iota_{X_g}(ddf) = \iota_{X_f}(ddg) + d(\iota_{X_f}(dg)) = dX_f(g) = d\{f,g\}.$$
Consequently,

\[ X_{\{f,g\}} = 2i \cdot \frac{1}{2i} [X_f, X_g] = [X_f, X_g]. \]

\[ \square \]

**Theorem 3.4.** The bracket \( \{ \cdot, \cdot \} \) defined in Lemma 3.3 is a Poisson bracket on \( M \). Moreover, the Hamiltonian vector fields \( X_f \) locally span the distribution \( \mathcal{J} \subseteq TM \). Consequently, \( (\mathcal{J}, \omega) \) coincides with the canonical symplectic foliation associated to the Poisson structure \( \{ \cdot, \cdot \} \).

**Proof.** By Lemma 3.3, we have \( \{ f, g \} = \omega(X_g, X_f) \), which shows that \( \{ \cdot, \cdot \} \) is skew-symmetric. The Leibniz rule is straightforward:

\[ \{ f, gh \} = X_f(gh) = gX_f(h) + hX_f(g) = g\{ f, h \} + h\{ f, g \}. \]

To check the Jacobi identity, we compute, using Lemma 3.3:

\[ \{ \{ f, g \}, h \} = X_{\{ f, g \}}(h) = [X_f, X_g](h) = X_f X_g(h) - X_g X_f(h) = \{ \{ f, g \}, h \} - \{ g, \{ f, h \} \}. \]

This proves the first statement of the theorem. The last statement follows immediately from the first two and the identity \( \{ f, g \} = \omega(X_g, X_f) \) proved in Lemma 3.3. Thus it remains to prove the second statement of the theorem. The question is a pointwise one; thus let us choose a point \( m \in M \), and let \( \{ f_j \} \) be any collection of functions in \( C^\infty(M) \) such that their differentials \( df_j \) span the cotangent space \( T^*_m M \). If \( Y \) is any local section of \( \mathcal{J} \) near \( m \) such that \( \omega(Y, X_{f_j}) = 0 \) at \( m \) for all \( j \), then Lemma 3.3 implies that \( \langle Y, df_j \rangle = 0 \) at \( m \) for all \( j \), whence \( Y \) vanishes at \( m \). This shows that the vector fields \( X_{f_j} \) span \( \mathcal{J} \) at \( m \). \( \square \)

Note that Theorem 3 in the introduction is identical to the result we have just proved. We can now give a

**Proof of Proposition 1.2.** Part (1) follows from the fact that \( \xi_f \) is defined by the equation \( \mathcal{J}(0, df) = (X_f, \xi_f) \), that \( (0, d(fg)) = f \cdot (0, dg) + g \cdot (0, df) \), and that \( \mathcal{J} \) is linear over \( C^\infty(M) \). We prove (2) using the notation of the proof of Lemma 3.3. If \( \ell \) is defined by (3.1), then, since \( \ell \) is a section of \( L \) and \( \ell + \bar{\ell} = (0, d\{ f, g \}) \), we have, by definition,

\[ \xi_{\{ f, g \}} = -2 \text{Im}(L_{X_f} \eta - \iota_{X_g}(d\xi_f)) = L_{X_f}(-2 \text{Im} \eta) - \iota_{X_g}(d(-2 \text{Im} \xi)) = L_{X_f}(\xi_g) - \iota_{X_g}(d\xi_f). \]

Finally, to prove (3), let us use the notation of the beginning of this section and write \( (0, df) = (X, \xi) + (\bar{X}, \bar{\xi}) \), where \( (X, \xi) \) is a section of \( L \). Then \( X_f = 2iX \), so \( (X_f, 2i\xi) \) is also a section of \( L \), which implies, by the definition of \( \epsilon \), that

\[ \iota_{X_f}(\epsilon) = 2i\xi|_E = 2i \cdot (\text{Re} \xi + i \text{Im} \xi)|_E = (i \cdot df + \xi)|_E. \]

Now we compute, using Cartan’s formula and the fact that \( de = 0 \) (cf. Proposition 1.1):

\[ L_{X_f}(\epsilon) = d(i_{X_f}(\epsilon)) + i_{X_f}(de) = d(i_{X_f}(\epsilon)) = d(i \cdot df + \xi)|_E = (d\xi_f)|_E. \]

This completes the proof. \( \square \)

### 4. Local Normal Form

#### 4.1. Strategy of the proof.

We begin by outlining the strategy of our proof of Theorem 2. Our argument is an extension of the inductive argument of [Wei]. If \( \text{rk}_{m_0} M = 0 \), then there is nothing to prove. Otherwise, following loc. cit., we can split \( M \), locally near \( m_0 \), as a product \( M = S \times N \) in the sense of Poisson manifolds, \( M = S \times N \), where \( S \) is an open neighborhood of \( 0 \) in \( \mathbb{R}^2 \) with the induced standard symplectic form \( \omega_0 \), and \( m_0 \in M \) corresponds to \( (0, m_0) \in S \times N \). By abuse of notation, we identify \( N \) with the submanifold \( \{ 0 \} \times N \) of \( M \). It is clear that each “horizontal leaf” \( S \times \{ n \} \) is a GC submanifold of \( M \).

**Lemma 4.1.** The “transverse slice” \( N \) is a GC submanifold of \( M \).
The proof of this lemma is given at the end of the section. We equip $N$ with the induced GC structure. It is clear that $\text{rk}_m N = \text{rk}_m M - 2$. Hence, by induction, it suffices to show that in a neighborhood of $m_0$, the GC structure on $M$ is a $B$-field transform of the product of the symplectic structure on $S$ and the induced GC structure on $N$. The proof of this fact consists of three steps, each involving a transformation by a closed 2-form and possibly replacing $M$ by a smaller open neighborhood of $m_0$. To save space, we will still use $M$ to denote any of these sufficiently small neighborhoods. The steps are listed below:

1. After a transformation by a closed 2-form $B''$ on $M$, the induced GC structure on each horizontal leaf $S \times \{n\}$ is the symplectic GC structure defined by $\omega_0$ via the obvious identification $S \cong S \times \{n\}$.

2. After a transformation by a closed 2-form $B'$ on $M$ that restricts to zero on the horizontal leaves $S \times \{n\}$ and on the transverse slice $\{0\} \times N$, we have that for each $n \in N$, the induced constant GC structure on $T_{(0,n)}M$ is the direct sum of the induced constant GC structures on $T_{(0,n)}(S \times \{n\})$ and on $T_n N$.

3. After a transformation by a closed 2-form $B$ on $M$ that vanishes along $N$, the GC structure on all of $M$ is the product of the symplectic GC structure on $S$ and the induced GC structure on $N$.

4.2. Step 1. We begin by introducing notation that will be used in the rest of the section. Let $(p, q)$ denote the standard coordinates on $S$, so that $\omega_0 = dp \wedge dq$; we will also view them as part of a coordinate system $(p, q, r_1, \ldots, r_d)$ on $M$, where $r_1, \ldots, r_d$ are local coordinates on $N$ centered at $n_0$. Note that for any such coordinate system on $M$, we have

$$X_p = -\frac{\partial}{\partial q} \quad \text{and} \quad X_q = \frac{\partial}{\partial p},$$

where $X_p$ and $X_q$ denote the Hamiltonian vector fields on $M$ associated to the functions $p$ and $q$, as in (4.1).

Without loss of generality, we may assume that $S$ is the open square on $\mathbb{R}^2$ defined by the inequalities $-1 < p < 1$, $-1 < q < 1$. A point $(s, n) \in S \times N = M$ will from now on be written as $(a, b, n)$, where $a = p(s), b = q(s) \in (-1, 1)$. We will denote by $\phi_s : M \to M$ and $\psi_t : M \to M$ the flows of the vector field $X_p$ and $X_q$, respectively. Of course, these flows are not everywhere defined. Explicitly, we have, from (4.1),

$$\phi_s(a, b, n) = (a, b - s, n) \quad \text{and} \quad \psi_t(a, b, n) = (a + t, b, n).$$

It is clear that the flows $\phi_s$ and $\psi_t$ commute with each other.

Furthermore, we define $\mathcal{X}_0$ (resp., $\mathcal{N}$) to be the distribution on $M$ which is tangent to the horizontal leaves $S \times \{n\}$ (resp., to the transverse slices $\{s\} \times N$); note that $\mathcal{X}_0$ is spanned by the vector fields $X_p, X_q$.

We now prove statement (1) of 4.1. Since $M = S \times N$ as Poisson manifolds, it follows that for each $n \in N$, the induced GC structure on $S \times \{n\}$ is $B$-symplectic, with the underlying symplectic structure being given by $\omega_0$. A general fact, proved in [Gua], is that on a $B$-symplectic GC manifold, both the underlying symplectic structure and the $B$-field are uniquely determined, and, moreover, depend smoothly on the original GC structure. In our situation, we obtain a family $\{B''_n\}$ of closed 2-forms on the leaves $S \times \{n\}$, depending smoothly on $n$, such that for every $n \in N$, the $B$-field $B''_n$ transforms the induced GC structure on $S \times \{n\}$ into the symplectic structure on $S \times \{n\}$ defined by $\omega_0$.

The usual proof of the Poincaré lemma shows that, after possibly shrinking $S$ and $N$, we can find a smooth family $\{\sigma_n\}_{n \in N}$ of 1-forms on the leaves $S \times \{n\}$ such that $d\sigma_n = B''_n$ for each $n \in N$. Now let $\sigma$ be an arbitrary 1-form on $M$ such that $\sigma|_{S \times \{n\}} = \sigma_n$ for each $n \in N$; such a $\sigma$ exists simply because $TM = \mathcal{X}_0 \oplus \mathcal{N}$ as vector bundles. By construction, the 2-form $B'' = d\sigma$ satisfies the requirement of statement (1) of 4.1.

4.3. Step 2. It follows now from parts (f) and (g) of Theorem 2.1 together with Remark 2.8 that for every point $n \in N$, there exists a unique 2-form $B'_n \in \bigwedge^2 T^*_{(0,0,n)} M$ with the following properties:

- $B'_n|_{T_n N} = 0$;
- $B'_n|_{T_{(0,0,n)}(S \times \{n\})} = 0$;
Lemma 4.2. For all differential forms on $\Omega^{p,0}(M)$ and $\Omega^{p,1}(M)$:

and, moreover, $B'_n$ depends smoothly on $n$. We must show that there exists a closed 2-form $B'$ on $M$ such that for each $n \in N$, we have $B'|_{S \times \{n\}} = 0$ and $B'|_{T(0,0,n)M} = B'_n$. In fact, we will define $B'$ by an explicit formula.

Let us choose a coordinate system $\{x_i\}$ on $S$ centered at $(0, 0)$ (one can take $\{x_i\} = \{p, q\}$, but this is not important in this step), and a coordinate system $\{y_j\}$ on $N$ centered at $n_0$, so that $\{x_i, y_j\}$ is a coordinate system on $M$ centered at $m_0$. We denote the corresponding coordinate vector fields by $s_i = \partial/\partial x_i, n_j = \partial/\partial y_j$.

We then define $B'$ by the formulas

$$B'(s_i, s_k) = 0; \quad B'(s_i, n_j)(a, b, n) = B_n((s_i)(0,0,n), (n_j)(0,0,n))$$

(in particular, note that $B'(s_i, n_j)$ does not depend on the coordinates $x_k$);

$$B'(n_j, n_i) = \sum_i x_i \cdot (n_j B'(s_i, n_i) - n_i B'(s_i, n_j)).$$

By construction, $B'$ satisfies all the required pointwise conditions, so we only have to check that $B'$ is closed. Since $dB'$ is a differential 3-form on $M$, it suffices to prove the following identities:

$$0 = dB'(s_i, s_k, s_s) \overset{\text{def}}{=} s_k B'(s_s, s_s) - s_k B'(s_i, s_s) + s_s B'(s_i, s_k); (4.3)$$

$$0 = dB'(s_i, s_k, n_j) \overset{\text{def}}{=} s_l B'(s_k, n_j) - s_k B'(s_i, n_j) + s_j B'(s_i, s_k); (4.4)$$

$$0 = dB'(s_i, n_j, n_l) \overset{\text{def}}{=} s_i B'(n_j, n_l) - n_j B'(s_i, n_l) + n_l B'(s_i, n_l); (4.5)$$

$$0 = dB'(n_j, n_i, n_l) \overset{\text{def}}{=} n_l B'(n_i, n_l) - n_i B'(n_j, n_l) + n_j B'(n_i, n_l). (4.6)$$

The first two follow automatically from the definitions. The third one follows from the definition of $B'(n_j, n_l)$ and the fact that $s_i(x_k) = \delta_{ik}$. Finally, the fourth identity follows from a straightforward computation by substituting the definitions of $B'(n_j, n_l)$, $B'(n_j, n_i)$ and $B'(n_i, n_l)$ into (4.10) and using the fact that the vector fields $n_j$ commute with each other and annihilate $x_i$.

4.4. Step 3. We now complete the proof outlined in 4.1. Let us begin by exploring the consequence of the Fundamental Theorem of Calculus in the context of Lie derivatives. With the notation of 4.2, let $\tau$ be a differential form on $M$ of arbitrary degree.

Lemma 4.2. For all $(a, b, n) \in S \times N = M$, we have

$$\tau(a, b, n) = \phi_{a-s}^* \tau(a, 0, n) - \int_0^b \left( \phi_{a-s}^* (\mathcal{L}_{X_p} \tau) \right)(a, b, n) ds (4.7)$$

and

$$\tau(a, b, n) = \psi_{a-t}^* \tau(a, 0, n) + \int_0^a \left( \psi_{a-t}^* (\mathcal{L}_{Y_q} \tau) \right)(a, b, n) dt. (4.8)$$

The proof of this lemma is straightforward from the definition of Lie derivative and the Fundamental Theorem of Calculus. Combining (4.7) and (4.8), we deduce that

$$\tau(a, b, n) = \phi_{a}^* \psi_{a-t}^* \tau(a, 0, n) - \phi_{a}^* \int_0^a \left( \psi_{a-t}^* (\mathcal{L}_{X_p} \tau) \right)(a, b, n) dt - \int_0^b \left( \phi_{a-s}^* (\mathcal{L}_{X_p} \tau) \right)(a, b, n) ds (4.9)$$

$$= \psi_{a-t}^* \phi_{a}^* \tau(a, 0, n) - \psi_{a-t}^* \int_0^a \left( \phi_{a-s}^* (\mathcal{L}_{X_p} \tau) \right)(a, b, n) ds + \int_0^a \left( \psi_{a-t}^* (\mathcal{L}_{X_q} \tau) \right)(a, b, n) dt. (4.10)$$
In particular, Proposition 1.2(3) now implies that
\[
\epsilon_{(a,b,n)} = \phi_b^* \psi_{-a}^* (\epsilon_{(0,0,n)}) + \left\{ \phi_b^* \int_0^a (\psi_{t-a} (d\xi_q))_{(a,0,n)} dt - \int_0^b (\phi_{b-s}^* (d\xi_p))_{(a,b,n)} ds \right\}_{E_{(a,b,n)}}.
\] (4.11)

We now note that, due to the preparations of §4.2 and 1.3, the GC structure on \( S \times N \) defined as the product of the symplectic structure on \( S \) and the induced GC structure on \( N \) corresponds to the 2-form \( \epsilon' \) on \( E \) defined by
\[
\epsilon'_{(a,b,n)} = \phi_b^* \psi_{-a}^* (\epsilon_{(0,0,n)}).
\]
The proof will therefore be complete if we show that the (real) 2-form \( B \) on \( S \times N \) defined by
\[
B_{(a,b,n)} = \phi_b^* \int_0^a (\psi_{t-a} (d\xi_q))_{(a,0,n)} dt - \int_0^b (\phi_{b-s}^* (d\xi_p))_{(a,b,n)} ds
\]
is closed.

Recall first from Proposition 1.2 that \( \xi_{(f,g)} = \mathcal{L}_{X_f} (\xi_g) - \mathcal{L}_{X_g} (d\xi_f) \) for all \( C^\infty \) functions \( f, g \) on \( M \); on the other hand, the definition of the map \( f \mapsto \xi_f \) implies that if \( \{ f, g \} \) is a constant function on \( M \), then \( \xi_{(f,g)} = 0 \). We deduce that
\[
\begin{align*}
\mathcal{L}_{X_f} (\xi_g) &= \mathcal{L}_{X_f} (\xi_g), \\
\mathcal{L}_{X_f} (\xi_p) &= \mathcal{L}_{X_f} (d\xi_q), \quad \mathcal{L}_{X_f} (\xi_q) = \mathcal{L}_{X_q} (d\xi_q). \quad (4.12, 4.13)
\end{align*}
\]

We now compute
\[
\begin{align*}
(L_{X_f} B)_{(a,b,n)} &= \lim_{\gamma \to 0} \frac{1}{\gamma} \left[ \phi^*_\gamma (B_{(a,b-\gamma,n)}) - B_{(a,b,n)} \right],
\end{align*}
\]
and since the flows \( \phi_s \) and \( \psi_t \) commute, we obtain
\[
\phi^*_\gamma (B_{(a,b-\gamma,n)}) = \phi_b^* \int_0^a (\psi_{t-a} (d\xi_q))_{(a,0,n)} dt - \int_0^{b-\gamma} (\phi_{b-s}^* (d\xi_p))_{(a,b,n)} ds,
\]
whence
\[
\frac{1}{\gamma} \left[ \phi^*_\gamma (B_{(a,b-\gamma,n)}) - B_{(a,b,n)} \right] = \frac{1}{\gamma} \cdot \int_{b-\gamma}^b \phi_{b-s}^* (d\xi_p)_{(a,s,n)} ds.
\]
The limit of this expression as \( \gamma \to 0 \) is equal to \((d\xi_p)_{(a,b,n)}\). Thus
\[
\mathcal{L}_{X_f} B = d\xi_p. \quad (4.14)
\]

Similarly,
\[
\begin{align*}
(L_{X_q} B)_{(a,b,n)} &= \lim_{\gamma \to 0} \frac{1}{\gamma} \left[ \psi^*_\gamma (B_{(a+\gamma,b,n)}) - B_{(a,b,n)} \right],
\end{align*}
\]
and
\[
\begin{align*}
\psi^*_\gamma (B_{(a+\gamma,b,n)}) = \phi_b^* \int_0^{a+\gamma} (\psi_{t-a} (d\xi_q))_{(a,0,n)} dt - \int_0^b \psi_{b-s}^* (d\xi_p)_{(a+\gamma,s,n)} ds,
\end{align*}
\]
which leads to
\[
(L_{X_q} B)_{(a,b,n)} = \phi_b^* (d\xi_q)_{(a,0,n)} - \int_0^b \phi_{b-s}^* ((L_{X_q} (d\xi_p))_{(a,s,n)}) ds. \quad (4.15)
\]

However, we have, from 1.12 and Cartan’s formula for \( L_{X_q} \),
\[
\mathcal{L}_{X_q} (d\xi_p) = d\mathcal{L}_{X_q} (d\xi_p) = d\mathcal{L}_{X_p} (\xi_q) = \mathcal{L}_{X_q} (d\xi_q).
\]
Substituting this into 4.15 and combining with Lemma 1.2 we obtain
\[
\mathcal{L}_{X_q} B = d\xi_q. \quad (4.16)
\]
We now compute \( t_{X_p} B \). We use the fact that contraction commutes with integration of differential forms, and also that the vector field \( X_p \) is invariant under the flows \( \phi_s \) and \( \psi_t \):

\[
(t_{X_p} B)_{(a,b,n)} = \phi_b^* \int_0^a (\psi_{t-a}(t_{X_p}d\xi_q))(a,0,n)\, dt - \int_0^b (\phi_{b-s}(t_{X_p}d\xi_p))(a,b,n)\, ds
\]

\[
= \phi_b^* \int_0^a (\psi_{t-a}(\mathcal{L}_{X_q} \xi_q))(a,0,n)\, dt - \int_0^b (\phi_{b-s}(\mathcal{L}_{X_p} \xi_p))(a,b,n)\, ds
\]

\[
= \phi_b^* \left( (\xi_q)_{(a,0,n)} - \psi_{t-a}((\xi_q)(0,0,n)) \right) + (\xi_p)_{(a,b,n)} - \phi_b^*((\xi_p)(a,0,n))
\]

where we have used \( 4.12 \) in the second equality and Lemma \( 4.2 \) in the third equality. However, \( (\xi_p)(a,0,n) = 0 \). This follows from the fact that \( (\xi_q)(0,0,n) \) depends only on the value of \( dp \) at the point \( (0,0,n) \) and on the induced constant GC structure on \( T_{(0,0,n)} M \); on the other hand, after the preparations of \( 4.12 \) and \( 4.3 \) the constant GC structure on \( T_{(0,0,n)} M \) is the direct sum of the induced GC structure on \( T_n N \) and the symplectic GC structure on \( T_{(0,0,n)} (S \times \{ n \}) \).

Therefore

\[
(t_{X_p} B)_{(a,b,n)} = \xi_p.
\]

Similarly,

\[
(t_{X_q} B)_{(a,b,n)} = \phi_b^* \int_0^a (\psi_{t-a}(t_{X_q}d\xi_q))(a,0,n)\, dt - \int_0^b (\phi_{b-s}(t_{X_q}d\xi_p))(a,b,n)\, ds
\]

\[
= \phi_b^* \int_0^a (\psi_{t-a}(\mathcal{L}_{X_q} \xi_q))(a,0,n)\, dt - \int_0^b (\phi_{b-s}(\mathcal{L}_{X_q} \xi_q))(a,b,n)\, ds
\]

\[
= (\xi_q)_{(a,0,n)} - \phi_b^*((\xi_q)(0,0,n))
\]

which forces

\[
(t_{X_q} B)_{(a,b,n)} = \xi_q.
\]

Let us compare \( 4.14 \) and \( 4.17 \). We can rewrite \( 4.14 \) as \( d(t_{X_p} B) + t_{X_p} (dB) = d\xi_p \), whence \( 4.17 \) implies that \( t_{X_p} (dB) = 0 \). Likewise, \( 4.16 \) and \( 4.18 \) force \( t_{X_q} (dB) = 0 \). But \( X_p, X_q \) span the tangent space to every horizontal leaf \( S \times \{ n \} \) at every point. Hence, to show that \( dB = 0 \), it remains to check that the restriction of \( dB \) to each transverse slice \( \{ s \} \times N \) is equal to zero. By construction, the restriction of \( B \) itself to \( \{ 0,0 \} \times N \) is zero. Let us pick three arbitrary sections \( Z_1, Z_2, Z_3 \) of \( \mathcal{N} \) which commute with \( X_p \) and \( X_q \).

Then \( (dB)(Z_1, Z_2, Z_3) = 0 \) along \( N = \{ (0,0) \} \times N \), and furthermore

\[
\mathcal{L}_{X_p}[(dB)(Z_1, Z_2, Z_3)] = (\mathcal{L}_{X_p} (dB))(Z_1, Z_2, Z_3) = (d\mathcal{L}_{X_p} B)(Z_1, Z_2, Z_3) = (dd\xi_p)(Z_1, Z_2, Z_3) = 0,
\]

where we have used \( 4.14 \) and the fact that \( X_p \) commutes with each \( Z_j \). Similarly, \( \mathcal{L}_{X_p} [(dB)(Z_1, Z_2, Z_3)] = 0 \). It follows that \( (dB)(Z_1, Z_2, Z_3) = 0 \) everywhere on \( M \) and completes the proof.

4.5. **Appendix: proof of Lemma 4.1** Let \( (\mathcal{S}, \omega) \) denote the canonical symplectic foliation associated to the GC structure on \( M \), and recall from \( 4.12 \) that \( \mathcal{S}_0 \subset \mathcal{S} \) denotes the foliation tangent to the leaves \( S \times \{ n \} \). Since \( M = S \times N \) as Poisson manifolds, it follows that at each point \( (s, n) \in N \), the tangent space \( T_{(s,n)} (S \times \{ n \}) \) is orthogonal to \( T_{s,n} ((s, n) \cap \mathcal{S}(s,n)) \) with respect to \( \omega \). In particular, by Theorem \( 2.1(e) \), the transverse slice \( N \) satisfies the pointwise condition for being a GC submanifold of \( M \), and hence we must only show that \( L_N \) is a subbundle of \( T_C N \oplus T_C^* N \).

Since \( L_N \) is the image of \( \tilde{L}_N = L_N \cap \left( T_C N \oplus (T_C^* M|_N) \right) \) under the projection map \( T_C N \oplus (T_C^* M|_N) \to T_C N \oplus T_C^* N \), it suffices to show that \( \tilde{L}_N \) is a subbundle of \( T_C N \oplus (T_C^* M|_N) \). Further, since \( \tilde{L}_N \) is defined as
the intersection of two subbundles of \((T_C M \oplus T^*_C M)|_N\), it suffices to show that \(\tilde{L}_N\) has constant rank on \(N\). Considering the projection of \(L_N\) onto \(T_C N\), we obtain a short exact sequence

\[0 \rightarrow \left( L \cap T^*_C M \right)|_N \rightarrow \tilde{L}_N \rightarrow \left( E \right)|_N \cap T_C N \rightarrow 0.\]

Now \(L \cap T^*_C M = \text{Ann}_{T^*_C M}(E)\), so

\[
\begin{align*}
\text{rk } \tilde{L}_N &= \text{rk}(\text{Ann}_{T^*_C M}(E)|_N) + \text{rk}(E|_N) - \text{rk}\left( (E|_N)/\left(E|_N \cap T_C N\right) \right) \\
&= \dim M - \text{rk}\left( (E|_N + T_C N)/T_C N \right) \\
&= \dim M - \text{rk}(T_C M|_N/T_C N) = \dim N;
\end{align*}
\]

we have used the fact that \(E|_N + T_C N = T_C M|_N\), which follows from \(\mathcal{J}|_N + TN = TM|_N\). Thus, in fact, not only is the rank of \(\tilde{L}_N\) constant, but the projection map \(\tilde{L}_N \rightarrow L_N\) is an isomorphism (since \(L_N\) has constant rank equal to \(\dim N\)).

5. Linearization of Generalized Complex Structures

In this section we consider a GC structure \(\mathcal{J}\) on a manifold \(M\) such that the associated Poisson tensor has rank zero at a certain point \(m \in M\). Our goal is to describe a “first-order approximation” to the GC structure in a neighborhood of \(m\). We will use the notation \(m^2_m \subset m_m \subset C^\infty(M)\) in the same sense as in [12]. Also, for \(f \in C^\infty(M)\), we will use the notation \((X_f, \xi_f)\) as defined in Section 3. Let us assume that \(\mathcal{J}\) is given by the matrix \(\mathbf{J}^m\). Thus, by assumption, \(\pi_m : T^*_m M \rightarrow T^*_m M\) is the zero map. Hence, if we consider the induced constant GC structure \(\mathcal{J}_m\) on \(T^*_m M\), its matrix has the form

\[
\mathcal{J}_m = \begin{pmatrix} J_m & 0 \\ \sigma_m & -J_m \end{pmatrix}.
\]

It is proved, for instance, in [12], that a constant GC structure of this form is a \(B\)-field transform of a complex GC structure on \(T^*_m M\). If \(B_m \in \Lambda^2(T^*_m M)\) is any two-form which transforms \(\mathcal{J}_m\) into a complex GC structure, we can extend \(B_m\) to a differential 2-form \(B\) on a neighborhood of \(m\) in \(M\) which is constant in the appropriate local coordinates, and hence, a fortiori, is closed. Applying the transformation defined by \(B\) to the structure \(\mathcal{J}\) reduces us to the situation where \(\sigma_m = 0\).

We now assume that \(\sigma_m = 0\) and explain what we mean by the first-order approximation to \(\mathcal{J}\) at the point \(m\), proving Theorem 3 at the same time. Let \(\mathfrak{g} = m_m/m^2_m\) be the real Lie algebra which encodes the first-order approximation to \(\pi\) at \(m\), as defined in [12]. Thus the Lie bracket on \(\mathfrak{g}\) is induced by the Poisson bracket on \(C^\infty(M)\) defined by \(\pi\). We can also think of \(\pi\) as a \(C^\infty(M)\)-linear map from \(\Gamma(M, T^*M)\) to \(m_m \cdot \Gamma(M, TM)\), which induces an \(\mathbb{R}\)-linear map

\[
\Gamma(M, T^*M)/m_m \cdot \Gamma(M, TM) \rightarrow m_m \cdot \Gamma(M, TM)/m^2_m \cdot \Gamma(M, TM).
\]

This map also encodes the first-order approximation to \(\pi\). It is then natural to define the first-order approximation to \(\mathcal{J}\) to be the \(\mathbb{R}\)-linear automorphism of

\[
\left( m_m \cdot \Gamma(M, TM)/m^2_m \cdot \Gamma(M, TM) \right) \oplus \left( \Gamma(M, T^*M)/m_m \cdot \Gamma(M, T^*M) \right)
\]

induced by \(\mathcal{J}\). Note, however, that the map

\[
m_m \cdot \Gamma(M, TM)\big/m^2_m \cdot \Gamma(M, TM) \rightarrow \Gamma(M, T^*M)/m_m \cdot \Gamma(M, T^*M)
\]

induced by \(\sigma\) clearly vanishes; moreover, since \(J\) and \(K = -J^*\) determine each other, we can concentrate our attention on the map

\[
\Gamma(M, T^*M)/m_m \cdot \Gamma(M, T^*M) \rightarrow \Gamma(M, T^*M)/m_m \cdot \Gamma(M, T^*M)
\]
induced by $-J^*$. Now the de Rham differential $d$ induces an $\mathbb{R}$-linear isomorphism
\[ \frac{m_m}{m_m^2} \cong \Gamma(M, T^*M) / \Gamma(M, T^*M), \]
and we have, by definition $\xi_f = -J^* (df)$ for any $f \in m_m$. By transport of structure, $-J^*$ induces an $\mathbb{R}$-linear automorphism of the Lie algebra $\mathfrak{g}$ which we will denote by $A$; by construction, $A^2 = -1$. To obtain further information on $A$, we will study it from the point of view of the map $f \mapsto \xi_f$.

Let $f, g \in m_m$. Part (2) of Proposition 1.2 yields
\[ \xi_{\{f, g\}} = \mathcal{L}_{X_f} (\xi_g) - \iota_{X_f} (d\xi_g) = [\iota_{X_f} (d\xi_g) - \iota_{X_g} (d\xi_f)] + d(\iota_{X_f} (\xi_g)). \]
But $X_f = \pi(df)$ and $X_g = \pi(dg)$, which implies that the first term vanishes modulo $m_m$. On the other hand, $\xi_g \equiv d(Ag) \mod m_m$, whence $\iota_{X_f} (\xi_g) \equiv \{f, Ag\} \mod m_m^2$. Thus we conclude that $A\{f, g\} \equiv \{f, Ag\}$ modulo $m_m^2$, which is precisely the condition for $A$ to make $\mathfrak{g}$ a complex Lie algebra. This completes the proof of Theorem 3.

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