No-signaling from Gleason non-contextuality and the Tensor Product Structure

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The no-signaling principle in quantum mechanics is shown to be a consequence of Gleason non-contextuality and the tensor product structure.

INTRODUCTION

Gleason’s theorem [1] asserts that for a Hilbert space of dimension 3 or greater, the only possible probability measure $\mu(\alpha)$ associated with a particular linear subspace $\alpha$ of a Hilbert space will have the form $\text{Tr}(\Pi_{\alpha}\rho)$, where $\Pi_{\alpha}$ is the projector to the subspace, and $\rho$ is the density matrix for the system. The premise needed for proving the theorem, apart from certain continuity requirements, is the assumption of non-contextuality: the probability measure, $\sum_j \mu(e_j)$, associated with a Hilbert subspace, is independent of the choice of basis (context), $\{e_j\}$. The no-signaling theorem as applicable to a composite system, essentially asserts that in a bipartite composite system consisting of parts $A$ and $B$, the reduced density operator $\rho_A$ is unaffected by local operations in $B$ [2].

For our purpose, it will be useful to restate these two principles as follows. Given a system $S$, we say that it is tensor-product partitioned into sectors $J_i$ if the respective Hilbert spaces satisfy

$$\mathcal{H}_S = \bigotimes_i \mathcal{J}_i,$$

(1)

with $\text{dim}(\mathcal{H}_S) = \prod_i \text{dim}(\mathcal{J}_i)$. The no-signaling theorem asserts that the marginal probability distribution $p_i$ associated with sector $J_i$ is unaffected by local operations (in particular, measurement in some basis) in other sectors $J_j$ (i.e., operations of the form $I_{J_j} \otimes O_{\psi}$). It is customary to think of the sectors $J_i$ as being spatially separated in order to make the term ‘signaling’ meaningful, though spatiality is not essential to the formalism. No-signaling applies at the single-particle level also, as discussed below.

We say that it is tensor-sum partitioned into sectors $K_j$ if the respective Hilbert spaces satisfy

$$\mathcal{H}_S = \bigoplus_j \mathcal{K}_j,$$

(2)

with $\text{dim}(\mathcal{H}_S) = \sum_j \text{dim}(\mathcal{K}_j)$. The Gleason non-contextuality assumption asserts that the probability measure $q_j$ associated with sector $K_j$ is unaffected by local operations in that sector, i.e., rotations of the basis vectors such that $K_j$ is an invariant subspace of the operations. In other words, the choice of measurement basis in that sector, or by extension, the projectors used to complete the full basis in the other sectors, does not alter the probability measure associated with $K_j$.

Expressed thus, the result we wish to prove, which is that no-signaling in a multi-partite system is a manifestation of single system non-contextuality in a tensor product setting, seems tantalizingly plausible. Stated differently, we wish to show that the mutual independence of marginal probabilities under local transformations in distinct sectors across a tensor product cut reduce to the independence of probability measures across a tensor sum cut.

SINGLE-PARTICLE CASE

Suppose we are given the orthogonal states of a qutrit, whose state space $\mathcal{H}$ is spanned by the basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Alice and Bob are two spatially separated observers. Incoming qutrits prepared in the state $|\psi\rangle = \sum_j \beta_j |j\rangle$ ($j = 0, 1, 2$ and $\sum_j |\beta_j|^2 = 1$) are first subjected to a non-maximal test corresponding to the measurement of the degenerate observable $X = a|0\rangle\langle 0| + b(|1\rangle\langle 1| + |2\rangle\langle 2|) = a|0\rangle\langle 0| + b(|+\rangle\langle +| + |\rangle\langle -|)$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$ and $a, b$ are real numbers. Alice is located in station $A$ where she receives the qutrit if measurement of $X$ returns $a$. Otherwise the particle goes to Bob, who measures it either in the basis $Y_1 = \{|1\rangle, |2\rangle\}$ or in the basis $Y_2 = \{|\pm\rangle\}$.

To show that non-contextuality entails no-signaling at the single-particle level, assume that there is a nonlocal signal from Bob to Alice, depending on whether he measures in the basis $Y_1$ or $Y_2$. An instance of signaling implies that the probability $\text{Prob}(|0\rangle|Y_1) \neq \text{Prob}(|0\rangle|Y_2)$. 

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But this means that the probability measure associated with the subspace span\{ |1\rangle, |2\rangle \} depends on the choice of basis, and therefore provides a mechanism by which the probability \( \text{Prob}(|0\rangle) \) is contextual, and we obtain our required result. The converse is not true, however. In this example, if \( \mathcal{H} \) corresponded to an internal degree of freedom, so that the states \( |j\rangle \) \( (j = 0, 1, 2) \) are not spatially separated, then contextuality would not lead to signaling.

Non-contextuality also implies the no-disturbance principle [4], which may be thought of as non-contextuality applied to whole observables rather than individual events or projectors. For, by construction, \( [X, Y_j] = 0; \ [Y_1^+, Y_2^+] \neq 0 \).

where \( Y_j = \{ |0\rangle \} \cup Y_j \). Thus, if there were disturbance, namely the probability measure associated with the subspace span \( \sum_i \{ |i\rangle \} \), this would constitute context dependence.

\[ \mu(\mathcal{K}_j | \mathcal{E}) \equiv \mu(\mathcal{K}_j | \mathcal{E}') \equiv \mu(\mathcal{K}_j). \]

Now,

\[ \mu(\mathcal{K}_j | \mathcal{E}) \equiv \sum_a \sum_{k \in \delta_j} \text{Prob}(A = a, B = k) = \text{Prob}_B(j | \mathcal{E}) \]

\[ \mu(\mathcal{K}_j | \mathcal{E}') \equiv \sum_{a'} \sum_{k \in \delta_j} \text{Prob}(A' = a', B = k) = \text{Prob}_B(j | \mathcal{E}'), \]

where \( \text{Prob}_B(j | \mathcal{E}) \) is the probability for Bob to obtain outcome \( j \) in the \( \xi \)-context (\( \xi = \mathcal{E}, \mathcal{E}' \)). If signaling were possible, it would mean that there is a \( j \) such that

\[ \text{Prob}_B(j | \mathcal{E}) \neq \text{Prob}_B(j | \mathcal{E}'), \]

which in view of Eq. (6), implies

\[ \mu(\mathcal{K}_j | \mathcal{E}) \neq \mu(\mathcal{K}_j | \mathcal{E}'). \]

Together with Eq. (5), this implies a violation of non-contextuality in sector \( \mathcal{K}_j \). This proves our stated result, which, as it happens, connects the Born rule to no-signaling via Gleason’s theorem.

An instance of contextuality, on the other hand, does not necessarily lead to signaling across spatially separated sectors (as noted earlier) or across a tensor product cut. As example of the latter: contextuality in a system with prime-numbered dimensionality cannot be represented as a signaling across two tensor product sectors of non-trivial dimensionality. We may regard signaling as the avatar of contextuality in a spatial situation where some events correspond to geographically separated locations. Non-contextuality then is a stronger condition than no-signaling assuming a tensor product structure.

A proof of no-signaling also obtains as a special case of no-disturbance, where \( X \) and either \( Y_j^+ \) in Eq. (5) are assumed to correspond to two different particles (tensor product sectors). However, when \( X \) and \( Y_j^+ \) belong to the same particle, as we saw, \( X \) must be non-maximal or degenerate (at least in the subspace where \( Y_1^+ \) and \( Y_2^+ \) fail to commute). On the other hand, no such restriction appears when \( X \) and \( Y_j^+ \) pertain to distinct particles or degrees of freedom or tensor-product sectors. This extra difference between the single-particle and multi-particle situation does not appear in our proof, where the more elementary events rather than observables are the primary objects, and thus the reduction is unconditional.
DISCUSSION AND CONCLUSIONS

We may regard non-contextuality as a more fundamental principle than no-signaling for several reasons. It pertains to a single system rather than a composite system. It enables unifying no-signaling into a single stronger no-go principle, as noted above. Moreover, no-signaling in our present sense arises in non-relativistic quantum mechanics. For it to be aligned with relativistic causality would be an odd conspiracy, that would need further explanation. No such difficulty arises when we regard non-contextuality as the more fundamental principle, with no-signaling an ‘innocent’ consequence of imposing it on a tensor-product structured space.

While no-signaling is no doubt a useful thumb-rule in deriving other results (e.g., as in Ref. [5, 6]), we believe that our observation would be of interest in axiomatic studies where no-signaling is treated as a primary postulate [7, 8]. It can also help clarify how, if potential violations of no-signaling arise in a more general theory than quantum mechanics, they may reduce or relate to single-particle effects. In Ref. [9], it is suggested that quantum optics is testably such a more general theory, with peculiarities introduced because of the lower-boundeness of energy imposed by the vacuum state.

Gleason non-contextuality is intimately related to, and yet quite distinct from, Kochen-Specker (KS) contextuality [10, 11], on which we report elsewhere [12]. It is the latter that makes the former surprising. To use existing terminology (as usually applied to quantum nonlocality in the multi-partite situation), Gleason non-contextuality refers to parameter independence [13] or signal locality, while KS contextuality to outcome dependence [13] or violation of Einstein locality. Parameter independence by itself would demand no more than ‘garden variety’ classicality: as for example, with vector components whose magnitude is given by the 2-norm, such as the length of a 3-D object or intensity of the electric field along a given direction. Outcome dependence by itself would demand a disturbance produced by measurement or (in a spatial setting) superluminal classicality. It is putting the two together that requires the subtle richness that is quantum contextuality or quantum nonlocality, based on the non-commutative structure of quantum mechanics, or any other generalized non-signaling probability distributions [14, 15].

It has recently been shown that local quantum mechanics and no-signaling imply quantum correlations [16]. Our result shows that assuming tensor product structure, no-signaling is a consequence of an aspect of local quantum mechanics, thus making the case for quantum correlations stronger. As one consequence, local quantum mechanics with super-quantum correlations entails superluminal signaling [17].