PERTURBATIONS OF JORDAN MATRICES

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Abstract. We consider perturbations of a large Jordan matrix, either random and small in norm or of small rank. In both cases we show that most of the eigenvalues of the perturbed matrix are very close to a circle with centre at the origin. In the case of random perturbations we obtain an estimate of the number of eigenvalues that are well inside the circle in a certain asymptotic regime. In the case of finite rank perturbations we completely determine the spectral asymptotics as the size of the matrix increases. The paper provides an elementary illustration of some standard techniques of spectral theory.

1. Introduction

It is well known that the eigenvalues of large non-normal matrices can be highly unstable under very small perturbations. In this note we discuss a very simple example of this phenomenon. We show that in a wide variety of cases almost all of the eigenvalues of a slightly perturbed Jordan matrix lie near a circle with centre at the origin, with high probability in the random case. We also examine the exceptional eigenvalues, which remain well inside the circle.

A quantitative measure of spectral instability is provided by the notion of pseudospectra, which become interesting when the operator involved is far from being normal; see [10, 2] for detailed discussions and many references. If \( \delta > 0 \) the \( \delta \)-pseudospectra of an operator \( A \) are defined by

\[
\text{Spec}_\delta(A) = \text{Spec}(A) \cup \{ z \notin \text{Spec}(A) : \| (A - z)^{-1} \| > \delta^{-1} \} = \bigcup_{\{ K : \|K\| < 1 \}} \text{Spec}(A + \delta K),
\]

where \( \text{Spec} \) denotes the spectrum of a matrix. The second equality in (1) implies that a perturbation of \( A \) of size \( \delta \) can move the eigenvalues anywhere inside \( \text{Spec}_\delta(A) \). In particular the computed eigenvalues of a large matrix may be very inaccurate if \( \text{Spec}_\delta(A) \) is a large region, where \( \delta \) is the rounding error of the computations. In this note, we study this phenomenon in some detail for the Jordan block matrix, perturbed either by a matrix of small rank, in which case the analysis is much sharper, or by a random matrix with a small norm. The problem studied in this paper was proposed by Zworski, who showed how the general methods of Sjöstrand and Zworski ([3]) could be adapted to this particular setting. Our results go beyond the theory of Lidskii ([5, 3]) by allowing larger (but still extremely small) perturbations, for which the Puiseux series is not convergent and the eigenvalues are not where the first few terms of that series would predict.

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We define the standard $N \times N$ Jordan matrix $J$ by
\[
J_{r,s} := \begin{cases} 
1 & \text{if } s = r + 1, \\
0 & \text{otherwise},
\end{cases}
\]
where $r, s = 1, \ldots, N$, and we always assume $N > 2$ from now on. Figure 1 shows the results of a MATLAB computation of $\text{Spec}(J + \delta K)$, when $N = 100$, $\delta = 10^{-10}$ and $K$ is a complex gaussian random matrix. Our goal is to explain the form of the figure and others obtained by similar methods.

Most of the eigenvalues accumulate around a circle with centre at the origin, hence far away from $\text{Spec}(J) = \{0\}$, even though the perturbation is very small in norm (with a high probability). In Section 2 we explain the origin of this instability and prove that it is very likely to happen for this type of perturbation. The section illustrates the methods and ideas of [3], [4] in a very concrete setting.

If one adds a strictly upper triangular matrix to $J$ then the spectrum is not changed. In Section 3 we therefore concentrate on perturbations whose non-zero entries are all close to the bottom left-hand corner of the matrix; some generalizations are considered in Section 4. We give a complete asymptotic analysis of the spectrum for all such perturbations. The problem is described in detail in the following section. The equation to be solved is written down in Theorem 11. The asymptotic form of the solutions to this equation is described in Theorem 12, that will also be related to a Grushin problem as in [9], see also Section 2 and then in much more...
2. Random perturbations

Let us start by investigating the pseudospectra of the Jordan matrix. Let $D(w, r) = \{ z \in \mathbb{C} : |z - w| \leq r \}$.

**Lemma 1.** If $0 < |z| < 1$, then

$$|z|^{-N} \leq \|(J - z)^{-1}\| \leq N|z|^{-N},$$

which implies that for $\delta < 1/N$,

$$D(0, \delta^{1/N}) \subseteq \text{Spec}_\delta(J) \subset D(0, (\delta N)^{1/N}) .$$

**Proof.** Let $e(z) = (1, ..., z^{N-1})'$. If $0 < |z| < 1$, the identity

$$\|(J - z)e(z)\| = |z|^N \leq |z|^N\|e(z)\|$$

implies the lower bound in (2). We have the following expression for the resolvent for $z \neq 0$:

$$(J - z)^{-1}_{r,s} = \begin{cases} -z^{r-s-1} & \text{if } 1 \leq r \leq s \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that for $|z| < 1$,

$$\|(J - z)^{-1}u\|^2 \leq \left( \sum_{j=1}^{N} |z|^{-j}\right)^2\|u\|^2 \leq (N|z|^{-N})^2\|u\|^2 .$$

Using (1), we deduce that the eigenvalues of a perturbation of norm $\delta$ may be anywhere in $D(0, (\delta N)^{1/N})$. In Figure 1 most, but not all, of the eigenvalues are close to the boundary of this disc, and our goal is to understand why this is the case.

**Theorem 2.** Let $K$ be a $N \times N$ random matrix such that

$$P[\|K\| < 1] \geq 1 - p_1(N) ,$$

and

$$P[|K_{N,1}| < s] \leq p_2(s; N) .$$

Then for any $0 < \delta < \frac{1}{2N}$, $\alpha \geq \delta$, and $\sigma > 0$, with probability at least

$$1 - p_1(N) - p_2(5\alpha; N) ,$$

we have

$$\text{Spec}(J + \delta K) \subseteq D(0, (\delta N)^{1/N}) ,$$

and

$$\#(\text{Spec}(J + \delta K) \cap D(0, (\delta N)^{1/N}e^{-\sigma})) \leq \frac{1}{\sigma}(\ln N - \ln \alpha) .$$
The following theorem is obtained by putting $\alpha = N^{-3}$ and estimating $p_1(N)$ and $p_2(5\alpha, N)$. We see that for any fixed $\sigma > 0$, the proportion of eigenvalues that lie in the annulus
$$\{ z : (\delta N)^{1/N} e^{-\sigma} \leq |z| \leq (\delta N)^{1/N} \}$$
converges to one with probability one as $N \to \infty$.

**Theorem 3.** Let $\tilde{K}$ be a $N \times N$ random matrix with its entries independently and identically distributed according to a complex gaussian law centered at 0 and of variance 1. Let $K = \tilde{K}/N^2$. Then for any $0 < \delta \leq N^{-3}$ and $\sigma > 0$, with probability at least $1 - 26/N^2$, (8) is valid and
$$\# \left( \Spec(J + \delta K) \cap D(0, (\delta N)^{1/N} e^{-\sigma}) \right) \leq \frac{4}{\sigma} \ln N .$$

Choosing $\delta = \gamma^N$, we also obtain the following result.

**Corollary 4.** Let $K$ be as in Theorem 3. Then for any $0 < \gamma \leq N^{-3/N}$ and $\sigma > 0$, with probability at least $1 - 26N^2$,
$$\Spec(J + \gamma^N K) \subseteq D(0, \gamma N^{1/N})$$
and
$$\# \left( \Spec(J + \gamma^N K) \cap D(0, \gamma N^{1/N} e^{-\sigma}) \right) \leq \frac{4}{\sigma} \ln N .$$

2.1. The Grushin problem. For the proof of Theorem 2, we set up a Grushin problem as in [9, Sect. 2.2]. Let $A \in M_N(\mathbb{C})$, let $m \in \mathbb{N}$, and let $R_+$ and $R_-$ be $m \times N$ matrices. We put
$$A = \left( \begin{array}{cc} A & R_- \\ R_+ & 0 \end{array} \right) \in M_{N+m}(\mathbb{C}) .$$

**Lemma 5.** (Schur, Grushin) If $A$ is invertible, and
$$E = \left( \begin{array}{cc} E & E_+ \\ E_- & E_{-+} \end{array} \right)$$
is the matrix inverse of $A$, then $A$ is invertible if and only if $\det(E_{-+}) \neq 0$.

**Proof.** If $A$ is invertible and $\det(E_{-+}) \neq 0$, then
$$A(E - E_+ E_{-+}^{-1} E_-) = 1 ,$$
hence $A$ is invertible. The converse affirmation goes along the same path. \hfill \Box

**Corollary 6.** If
$$m = 1 , \quad R_- = e_N , \quad R_+ = e_1' ,$$
where $e_1, e_2, ..., e_N$ is the standard basis of column vectors in $\mathbb{C}^N$, then $A$ is invertible if and only if $\det(\tilde{A}) \neq 0$, where $\tilde{A}$ is obtained by deleting the first column and last row of $A$. In that case
$$E_{-+} = (-1)^N \frac{\det(A)}{\det(\tilde{A})} .$$
Example 7. If $A = J - zI$ and $R_\pm$ are as in (15), we write $\mathcal{J} = A$. By the previous corollary $\mathcal{J}$ is invertible, and

$$\mathcal{E}_{r,s} = \begin{cases} z^{r-s-1} & \text{if } s + 1 \leq r \leq N + 1, \\ z^{r-1} & \text{if } 1 \leq r \leq N + 1 \text{ and } s = N + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Assuming $|z| \leq 1$ we deduce that

\begin{align*}
\|\mathcal{E}(z)\| & \leq N + 1, \quad \|E(z)\| \leq N, \quad \|E_{\pm}(z)\| \leq N^{1/2}, \quad E_{-+}(z) = z^N. \\
\|\mathcal{E}(0)\| & \leq 1, \quad \|E(0)\| \leq 1, \quad \|E_{\pm}(0)\| \leq 1, \quad E_{-+}(0) = 0.
\end{align*} \hspace{1cm} (16)

Finally, using (14), we can also find the explicit expression for the resolvent.

2.2. Perturbation. Let us assume that $\|K\| < 1$. Then by (3), (8) holds. We analyze the part of the spectrum within $D(0, (\delta N)^{1/N})$ in more detail by using the Grushin problem for $J + \delta K$. We will show that the matrix $J + \delta K - zI$ in (13) may be inverted by using a Neumann series. Denoting the inverse by $E_{\delta}$, this implies that $\text{Spec}(J + \delta K) \cap D(0, R)$ coincides with the set of zeros of $E_{\delta} -+ (z)$ such that $|z| \leq R$.

Lemma 8. Let $\|K\| < 1$, and let $\delta < 1/2N$. Then if $N \geq 2$, for any $R < 1$,

$$\text{Spec}(J + \delta K) \cap D(0, R) = \{ z \in D(0, R); E_{-+}^\delta(z) = 0 \} ,$$ \hspace{1cm} (17)

where

$$E_{-+}^\delta(z) = z^N - \delta p_K(z) + q_{\delta K}(z)$$ \hspace{1cm} (18)

and

$$p_K(z) = \sum_{r,s=0}^{N-1} K_{(N-r),s+1} z^{r+s}.$$ \hspace{1cm} (19)

If $|z| < (\delta N)^{1/N} = R$, we have

$$\|E_{-+}^\delta\|_\infty \leq 3\delta N ,$$ \hspace{1cm} (20)

in $L^\infty(D(0, R))$, and

$$|E_{-+}^\delta(0)| \geq \delta(|K_{N,1}| - 2\delta) .$$ \hspace{1cm} (21)

Proof. The resolvent expansion implies invertibility provided $\|\delta K\|\|\mathcal{E}\| < 1$, which is proved by using (16) and $\delta\|K\|N < 1/2$. Moreover we have the following Neumann series expansion for the inverse:

$$\mathcal{E}^\delta = \mathcal{E}^0 + \left( \sum_{j \geq 1} E_{-}(\delta KE_{+})^j \right) \sum_{j \geq 1} \sum_{j \geq 1} E_{-}(\delta KE_{+})^j \left( \sum_{j \geq 1} (-\delta KE_{+})^j \right) .$$ \hspace{1cm} (22)

Evaluating the bottom right coefficient of each term yields (18) with

$$p_K(z) = E_KE_{+},$$

$$q_{\delta K}(z) = \delta^2 E_KEKE_{+} - \delta^3 E_KEKEKE_{+} + ... .$$
The bounds on the various quantities now follow by combining (16), $\|K\| < 1$, $\delta < 1/2N$, and $|z| < (\delta N)^{1/N}$:

\[
\begin{align*}
\|p_K\|_\infty & \leq \|K\|N \leq N , \\
\|q_\delta K\|_\infty & \leq 2\delta^2 \|K\|^2 N^2 \leq \delta N , \\
\|E_\delta^+\|_\infty & \leq \delta N + \delta N + \delta N \leq 3\delta N . 
\end{align*}
\] (23)

Moreover, using also the second line of (16),
\[
\begin{align*}
|p_K(0)| &= |K_{N,1}| , \\
|q_\delta K(0)| &\leq 2\delta^2 \|K\|^2 \leq 2\delta^2 , \\
|E_\delta^+(0)| &\geq \delta (|K_{N,1}| - 2\delta) .
\end{align*}
\] (24)

To estimate the number of zeros of the holomorphic function $E_\delta^+(z)$ we will need the next proposition.

2.3. Counting the zeros and proof of Theorem 2.

**Proposition 9** (The Poisson-Jensen formula). Let $f$ be a holomorphic function that does not vanish anywhere on the boundary of $D(0, R)$, where $0 < R < \infty$. Let $M$ be the number of zeros of $f$ in $D(0, Re^{-\sigma})$ for some positive constant $\sigma$. Then
\[
M \leq \frac{1}{\sigma} \left( - \ln \frac{|f(0)|}{\|f\|_{L^\infty(D(0,R))}} \right) .
\] (25)

This is a direct consequence of formula (1.2'), p.163 in [7]: if $f$ is a holomorphic function in $D(0, R)$ with zeros $a_\mu$ there, then
\[
\ln |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(Re^{i\theta})|d\theta - \sum_{|a_\mu|<R} \ln \frac{R}{|a_\mu|} \leq \ln \|f\|_\infty - \sum_{|a_\mu|<R} \ln \frac{R}{|a_\mu|} .
\] (26)

Hence
\[
- \ln |f(0)| + \ln \|f\|_\infty \geq \sum_{|a_\mu|<R} \ln \frac{R}{|a_\mu|} \geq \sum_{|a_\mu|<Re^{-\sigma}} \ln \frac{R}{Re^{-\sigma}} = M\sigma ,
\] (27)

which is our proposition.

**Proof of Theorem 2** With probability at least $1 - p_1(N)$, we know that $\|K\| < 1$, which we will assume from now on. Then Lemma 8 holds. Setting $f(z) = E_\delta^+(z)$ and $R = (\delta N)^{1/N}$, this implies that $\|f\|_{L^\infty(D(0,R))} \leq 3\delta N$. We also assume that $f$ does not vanish on $|z| = R$, since otherwise we may diminish $R$ slightly and obtain our result as a limit over increasing radii fulfilling this assumption.

If $\alpha > \delta$, with probability at least $1 - p_2(5\alpha; N)$, we have $|K_{N,1}| - 2\delta \geq 3\alpha$, hence again by Lemma 8 $|f(0)| \geq 3\delta \alpha$.

Thus we obtain that with probability at least $1 - p_1(N) - p_2(5\alpha; N)$,
\[
- \ln \frac{|f(0)|}{\|f\|_{L^\infty(D(0,R))}} \leq - \ln \frac{3\delta \alpha}{3\delta N} \leq \ln N - \ln \alpha .
\] (28)
Lemma 8 and Proposition 9 imply that with the same probability
\[
\# \left( \text{Spec}(J + \delta K) \cap D(0, (\delta N)^{1/N} e^{-\sigma}) \right) \leq \frac{1}{\sigma} (\ln N - \ln \alpha),
\]
which finishes the proof of Theorem 2. \(\square\)

### 2.4. Proof of Theorem 3
We end this section by showing that a complex Gaussian random perturbation fulfills the assumptions of Theorem 2.

**Proof.** Let \( K \) be as in Theorem 3. Let us first show that (5) is fulfilled with \( p_1(N) = 1/N^2 \).

If \( \tilde{K} \) is a random matrix with independent complex Gaussian normal distributed entries, and \( a > 0 \) then
\[
P[\|K\| > a] = P[\sum_{j,k=1}^N |\tilde{K}_{jk}|^2 > a^2] \leq E[a^{-2} \sum_{j,k=1}^N |\tilde{K}_{jk}|^2] = N^2/a^2.
\]
Hence
\[
P[\|K\| < 1] = P[\|\tilde{K}\| < N^2] \geq 1 - N^{-2}.
\]

Next, we have to estimate the following probability:
\[
P[|K_{N,1}| \leq s] = P[|\tilde{K}_{N,1}| \leq sN^2] = 1 - \exp\left( -\frac{(sN^2)^2}{2} \right) \leq (sN^2)^2 = p_2(s; N).
\]
Let us choose \( \alpha = N^{-3} \). Then \( p_2(5\alpha; N) = (5/N)^2 \), and Theorem 2 implies that with probability at least \( 1 - 1/N^2 - (5/N)^2 \), we have
\[
\# \left( \text{Spec}(J + \delta K) \cap D(0, (\delta N)^{1/N} e^{-\sigma}) \right) \leq \frac{4}{\sigma} \ln N,
\]
which finishes the proof of Theorem 3. \(\square\)

### 3. Small rank perturbations

#### 3.1. Description of the Example
Let \( C \) be a \( k \times k \) matrix with entries \( c_{r,s} \) and let \( \delta > 0 \) be the perturbation parameter. We consider the spectrum of \( A := J + \delta K \) where the \( N \times N \) matrix \( K \) has the block form
\[
K := \begin{pmatrix}
0 & 0 \\
0 & C
\end{pmatrix}
\]
and the (zero) top right hand entry is of size \((N-k) \times (N-k)\).

The asymptotic behaviour of Spec(\( A \)) depends on the choice of \( \delta \). From this point onwards we assume that \( \delta := \gamma N \) for some \( \gamma \in (0, \infty) \). If \( 0 < \gamma < 1 \) then \( \delta K \) is a very small perturbation of \( J \), but for \( \gamma > 1 \) the reverse holds. Surprisingly the same analysis applies in both cases; the choice \( \gamma := 1 \) is not special in any way.
Our results may have connections with the analysis of paraorthogonal polynomials on the unit circle in [8].

The spectral behaviour is different for random perturbations of the type considered in section [2]. The condition number of the diagonalizing matrix is much smaller in the random model, and the analysis is harder. For a fixed $\gamma \in (0, 1)$ the appearance of the spectrum is similar, but as $\gamma \to 1$ the number of eigenvalues inside the circle increases rapidly, and for $\gamma > 1$ the eigenvalues appear to be randomly distributed.

**Example 10.** Computations of the eigenvalues in our model tend to be numerically unstable because of the high condition numbers involved. We consider the example in which $N = 80$, $\gamma = 0.6$, $k = 3$ and

$$C := \begin{pmatrix} 8 & 0 & 0 \\ 2 & 5 & 0 \\ 1 & -2 & 3 \end{pmatrix}.$$  

The eigenvalues of $A$ inside the circle are close to $\pm i/4$, while the radius of the circle is close to $\gamma$. The condition number of the diagonalizing matrix is $4.5 \times 10^{17}$, and increases for smaller $\gamma$ or larger $N$.

**Theorem 11.** If $\delta := \gamma^N$ where $\gamma \in (0, \infty)$ then

$$\text{Spec}(A) = \{ \gamma z : z^N = f(z) \}_{8}$$
provided \(N\) is large enough, where

\[
f(z) := \sum_{r=0}^{2k-2} b_r z^r
\]

and

\[
b_r := \gamma^r \sum_{k-i+j=r+1} c_{i,j}.
\]

**Proof.** The spectrum of \(A\) is the set of solutions of \(g_N(\lambda) = 0\), where

\[
g_N(\lambda) := \det(\lambda I - J - \delta K).
\]

Let \(g_r(\lambda)\) be the determinant of the \(r \times r\) matrix obtained from \(\lambda I - J + \delta K\) by deleting its top \(N - r\) rows and the leftmost \(N - r\) columns. By expanding the determinant (31) down the leftmost column and assuming that \(N > 2(k+1)\) we obtain

\[
g_N(\lambda) = \lambda g_{N-1}(\lambda) - \delta c_{k,1} - \delta c_{k-1,1} \lambda - \delta c_{k-2,1} \lambda^2 - \cdots - \delta c_{1,1} \lambda^{k-1}.
\]

The formula for \(g_N(\lambda)\) follows inductively, and the proof is completed by making the change of variables \(\lambda := \gamma z\).

**Alternative proof using the Grushin problem.** Inserting the special form of \(K\) in the series expansion for \(E^\delta_+\) and using \(N > 2(k+1)\), we see that the series only contains terms up to first order in \(\delta\), so no condition on the smallness of \(\gamma\) is needed for convergence. Putting \(\delta = \gamma^N\) and \(\lambda = \gamma z\) yields

\[
E^\delta_+(\lambda) = \lambda^N - \delta E_+ K E_+ = \gamma^N (z^N - p_K(\gamma z))
\]

where \(p_K\) was defined in (19). We finally observe that \(p_K(\gamma z) = f(z)\) for all \(z\). Although the Grushin problem is equivalent to a direct analysis of the determinant in this particular case, it also permits estimates in cases where the determinant is quite hard to analyze directly.

**3.2. The Equation** \(z^N = f(z)\). Let \(U\) be a region in the complex plane that contains \(D(0,1+\delta)\) for some \(\delta > 0\). Let \(f\) be a bounded analytic function defined on \(U\). We assume that \(f(z) = 0\) has \(h\) distinct solutions \(z_i\) satisfying \(|z_i| < 1\), each with multiplicity \(m_i\). We put \(n := \sum_{i=1}^h m_i\). By reducing \(\delta > 0\) we may assume that \(|z_i| < 1 - \delta\) for all \(i\). We will determine the distribution of the solutions of \(z^N = f(z)\) asymptotically as \(N \to \infty\).

**Theorem 12.** For every \(\varepsilon \in (0,\delta)\) there exists \(N_\varepsilon\) such that if \(N \geq N_\varepsilon\) then \(z^N = f(z)\) has \(m_i\) solutions in the \(\varepsilon\)-neighbourhood of \(z_i\); for each \(i \in \{1,\ldots,h\}\), no other solutions in \(D(0,1-\varepsilon)\), no solutions in \(U \setminus D(0,1+\varepsilon)\) and \(N - n\) solutions in \(\{z : 1 - \varepsilon < |z| < 1 + \varepsilon\}\).

**Proof.** If \(N\) is large enough then \((1 + \varepsilon)^N > \max\{|f(z)| : z \in U\}\), so the equation has no solutions in \(U \setminus D(0,1+\varepsilon)\). By applying Rouche’s theorem to \(z^N - f(z)\) regarded as a small perturbation of \(z^N\), we see that for all large enough \(N\) the equation has \(N\) solutions inside \(D(0,1+\varepsilon)\). A similar argument but regarding \(f(z) - z^N\) as a small perturbation of \(f(z)\), implies that the equation has \(n\) solutions
inside \( D(0, 1-\varepsilon) \), provided \( N \) is large enough, and that these converge to the zeros of \( f(z) \) as \( N \to \infty \). The remaining \( N-n \) solutions must lie in the stated annulus.

In order to determine the asymptotic behaviour of the \( N-n \) solutions in the annulus as \( N \to \infty \), we assume for simplicity that \( f(z) \neq 0 \) whenever \( |z| = 1 \). We then put

\[
    f(e^{is}) := \rho(s)e^{i\phi(s)}
\]

where \( \rho(s) \) is positive and periodic on \([0, 2\pi]\) while \( \phi(2\pi) = \phi(0) + 2\pi n \). Both \( \rho(s) \) and \( \phi(s) \) are real analytic functions of \( s \). It is easy to see that for all large enough \( N \) the equation

\[
    \phi(s) = Ns \mod (2\pi)
\]

has \( N-n \) solutions in \([0, 2\pi]\). If these are labelled in increasing order then \( s_{r+1} - s_r = 2\pi/N + O(1/N^2) \). We will show that for large enough \( N \) the solutions of \( z_N = f(z) \) are very close to the points \( a_r := \rho(s_r)^{1/N} e^{is_r} \).

**Theorem 13.** Given \( \alpha \in (1, 2) \) there exists a constant \( b \) such that for all large enough \( N \) and every \( r \in \{0, ..., N-n-1\} \) the equation \( z_N = f(z) \) has a solution \( z_r \) satisfying

\[
    |z_r - a_r| \leq b N^{-\alpha}
\]

To leading order the \( N-n \) solutions of \( z_N = f(z) \) that are close to the unit circle are uniformly distributed around it.

**Proof.** An elementary calculation shows that finding the solution of \( z_N = f(z) \) closest to \( e^{is_r} \) is equivalent to finding the solution of \( z^N = f_r(z) \) closest to 1, where

\[
    f_r(z) := e^{-i\phi(s_r)} f(e^{is_r}z).
\]

We have

\[
    f_r(e^{is}) = \rho_r(s)e^{i\phi_r(s)}
\]

where

\[
    \phi_r(s) := \phi(s_r + s) - \phi(s_r),
    \rho_r(s) := \rho(s_r + s).
\]

Moreover \( \phi_r(0) = 0 \) and the equation \( \phi_r(s) = Ns \) is equivalent to \( \phi(s_r + s) = N(s_r + s) \). From this point onwards we drop the subscript \( r \), assume that \( \phi(0) = 0 \), and leave the reader to verify that the bounds obtained are uniform with respect to \( r \).

We define the sequence \( u_m := r_m e^{i\theta_m} \) for \( m = 1, 2, ... \) by \( u_1 := 1 \) and \( u_{m+1} := \{f(u_m)\}^{1/N} \), where we always take the \( N \)th root with the smallest argument. Note that \( \theta_1 = \theta_2 = 0 \), \( r_1 = 1 \) and \( r_2 = |f(1)|^{1/N} \). In the following arguments \( c_j \) denote positive constants that do not depend on \( N \) or \( m \) provided \( N \) is large enough.

We prove that if

\[
    S_N := \{re^{i\theta} : |r - r_2| \leq N^{-\alpha} \text{ and } |\theta| \leq N^{-\alpha}\}
\]
then for all large enough $N$, $u \in S_N$ implies $v := \{f(u)\}^{1/N} \in S_N$. Put $u := re^{i\theta}$ and $v := se^{i\phi}$. If $u \in S_N$ then

$$|s^N - r_2^N| = |\|f(u)\| - |f(1)||$$

$$\leq |f(u) - f(1)|,$$

$$\leq c_1|u - 1|$$

$$\leq c_1(|u - r_2| + |r_2 - 1|)$$

$$\leq c_2/N.$$

Therefore

$$s^N \geq r_2^N - c_2/N = |f(1)| - c_2/N \geq c_3 > 0$$

for some $c_3 > 0$. This also implies that $r_2^N \geq c_3$. Hence

$$\sigma := \sum_{i+j=N-1} s^i r^j_2 \geq Nc_3^{(N-1)/N} \geq Nc_4.$$

Combining the above estimates yields

$$|s - r_2| \leq c_2/N \sigma \leq c_5/N^2 \leq N^{-\alpha}$$

for all large enough $N$.

We next observe that

$$N s^N |\phi| \leq c_6 |s^N \sin(N\phi)| = c_6 |\Im (v^N - r_2^N)| \leq c_6 |v^N - r_2^N|$$

$$= c_6 |f(u) - f(1)| \leq c_7 |u - 1| \leq c_8/N.$$

Therefore

$$|\phi| \leq c_8/c_3 N^2 \leq N^{-\alpha}$$

for all large enough $N$.

Having established that $S_N$ is invariant under the map $u \rightarrow \{f(u)\}^{1/N}$ provided $N$ is large enough, we now apply a contraction mapping argument within $S_N$. Let $z_j \in S_N$ and put $w_j := s_j e^{i\phi_j} := \{f(z_j)\}^{1/N}$ for $j = 1, 2$. Then

$$|w_1^N - w_2^N| = |f(z_1) - f(z_2)| \leq c_9 |z_1 - z_2|.$$

Moreover

$$\left| \sum_{i+j=N-1} w_i^j w_2^j \right| \geq \sum_{i+j=N-1} \Re (w_i^j w_2^j)$$

$$= \sum_{i+j=N-1} s_i^j s_2^j \cos(\phi_1 + j\phi_2)$$

$$\geq Nc_{10}$$

where $c_{10} > 0$. Therefore

$$|w_1 - w_2| \leq c_9 |z_1 - z_2|/c_{10} N \leq |z_1 - z_2|/2$$

provided $N$ is large enough. Since $u_2 \in S_N$, the contraction mapping principle now implies that the sequence $u_m$ converges as $m \rightarrow \infty$ to a solution $u \in S_N$ of $u^N = f(u)$, again provided $N$ is large enough.
Figure 3. Solutions of the polynomial equation of Example 14.

Note Although we have proved that the eigenvalues of $A$ all lie on or inside the unit circle asymptotically, this does not imply that $|\det(A)| \leq 1$ asymptotically. Indeed $\det(A) = (-1)^{N-1} f(0)$ may be of any magnitude. If $|f(0)| > 1$ then the bound

$$|f(0)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{is}) \, ds \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(s) \, ds$$

implies that $\rho(s) > 1$ on average, so the eigenvalues close to the unit circle are actually slightly outside it, again on average.

Example 14. If there exists $z$ such that $|z| = 1$ and $f(z) = 0$ then Theorem 13 needs to be modified. The estimates in the theorem are local, so the conclusions are applicable to all the solutions of $z^N = f(z)$ that lie in

$$\{ z : 1 - \delta \leq |z| \leq 1 + \delta \text{ and } \alpha \leq \arg(z) \leq \beta \},$$

provided $f$ does not vanish in this set. Figure 3 shows the solutions of $z^N = 100(z - 1)$ when $N = 40$.

4. Some Generalizations

4.1. Other finite rank perturbations. In this section we allow the perturbation of the Jordan matrix $J$ to have non-zero entries in all corners of the matrix. We do not require the perturbation to be small, since we have already indicated that this possibility can be accommodated by introducing a scale factor.
Let $B, C, D, E$ be four $k \times k$ matrices and put

$$A := J + K$$

(33)

where $N \gg k$ and $K$ has the block form

$$K := \begin{pmatrix} B & 0 & C \\ 0 & 0 & 0 \\ D & 0 & E \end{pmatrix},$$

the central entry being of size $(N - 2k) \times (N - 2k)$. A direct calculation shows that

$$\det(\lambda I_N - A) = \lambda^{N-h}p(\lambda) - q(\lambda)$$

(34)

where $p, q$ are polynomials of degree (at most) $2k$ which depend on $B, C, D, E$ but not on $N$, and the $\lambda^h$ coefficient of $p$ is 1.

The following theorem describes the asymptotic distribution of the eigenvalues of $A$ as $N \to \infty$. The proof is an obvious adaptation of the proofs of Theorems 12 and 13.

**Theorem 15.** Let $p, q$ be two non-zero polynomials, let $m$ be a large enough natural number and let $0 < \delta < 1$. Suppose that $p(z) = 0$ and $q(z) = 0$ have no solutions satisfying $1 - \delta \leq |z| \leq 1 + \delta$. Then the solutions of

$$z^m p(z) = q(z)$$

satisfying $|z| \leq 1 - \delta$ converge to the zeros of $q$ in this region as $m \to \infty$. The solutions satisfying $|z| \geq 1 + \delta$ converge to the zeros of $p$ in this region. The solutions satisfying $1 - \delta \leq |z| \leq 1 + \delta$ converge to the unit circle and are given asymptotically by Theorem 13 where $f(z) := q(z)/p(z)$.

**Example 16.** We consider the above model with

$$B := \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad C := \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$

$$D := \begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}, \quad E := \begin{pmatrix} 1 & -3 \\ -2 & 0 \end{pmatrix}.$$ 

The determinant of $A$ equals $(-1)^N$. Its characteristic polynomial is $z^{N-4}p(z) - q(z)$, where

$$p(z) := z^4 - 4z^3 + 6z^2 + 15z - 33,$$

$$q(z) := -2z^2 - 2z - 1.$$

The zeros of $p$ are all outside the unit circle, at $2.0605 \pm 2.3672i$, $1.7709$ and $-1.8920$. The zeros of $q$ are both inside the unit circle, at $-1/2 \pm i/2$. The remaining $N - 6$ zeros of the characteristic polynomial are distributed almost uniformly around the unit circle. Figure 4 was obtained by putting $N = 50$.
4.2. **Matrix Pencils.** A direct calculation shows that the eigenvalues of the model operator (33) are the same as those of the $(2k + 1) \times (2k + 1)$ matrix pencil

$$A_n(z) := P_n(z) - J_{2k+1} - Q$$

where

$$P_n(z) := \begin{pmatrix} zI_k & 0 & 0 \\ 0 & z^{n-2k} & 0 \\ 0 & 0 & zI_k \end{pmatrix}, \quad Q := \begin{pmatrix} B & 0 & C \\ 0 & 0 & 0 \\ D & 0 & E \end{pmatrix},$$

the central entries of both block matrices being of size $1 \times 1$. By considering other finite rank perturbations of the Jordan matrix one is led to investigate the spectral asymptotics as $n \to \infty$ of the more general $m \times m$ matrix pencil

$$A_n(z) := B(z)z^n + C(z)$$

where $B(z)$ and $C(z)$ are analytic matrix-valued functions of $z$, defined for all $z$ in an open region $U$. In this more general problem, $n$ and $A_n$ are not necessarily the same as in (35).

The eigenvalues of this pencil are, by definition the values of $z \in U$ such that $\det(A_n(z)) = 0$. One sees that

$$\det(A_n(z)) = \sum_{r=0}^{m} p_r(z)z^{rn}$$

where $p_0(z) := \det(C(z))$ and $p_m(z) := \det(B(z))$. If $p_m$ does not vanish on $U$ then the zeros of the above expression are the same as those of

$$f_n(z) := z^{mn} + \sum_{r=0}^{m-1} q_r(z)z^{rn}$$

where $q_r := p_r/p_m$ are all analytic functions on $U$. 

**Figure 4.** Eigenvalues of the matrix $A$ of Example 16
Consider the asymptotic behaviour of the solutions of an equation of the form

\[ f(z) = z^n + q_1(z)z^n + q_2(z) = 0 \]  \hspace{1cm} (37)

as \( n \to \infty \). The proof is essentially the same as that of Theorem 12. There is also an analogue of Theorem 13, but we deal here only with the case \( m = 2 \). In other words we consider the asymptotic behaviour of the solutions of an equation of the form

\[ p_n(z) := z^{2n} + q_1(z)z^n + q_2(z) = 0 \]  \hspace{1cm} (37)

as \( n \to \infty \). Examples 20 and 21 illustrate the behaviour that we need to explain.

The following lemma sets up some notation that will be used in the following theorem.

**Lemma 18.** Let \( 0 < \delta < 1/2 \) and let \( q_1, q_2 \) be two bounded (uniformly in \( \delta \)) continuous functions on \( A := \{ z : 1 - \delta \leq |z| \leq 1 + \delta \} \) which are analytic in the interior of this annulus. Suppose that neither \( q_2 \) nor the discriminant \( v := q_1^2 - 4q_2 \) vanish anywhere in \( A \). Let \( h \), resp. \( k \), be the winding numbers of \( v(e^{i\theta}) \), resp. \( q_2(e^{i\theta}) \), around the origin, where \( \theta \in [0, 2\pi] \). Then the number of solutions of (37) in \( A \) is \( 2n - k \) for all large enough \( n \).

**Case 1** If \( h \) is even then there exist two non-vanishing analytic functions \( f_+ \) on \( A \) such that \( z \in A \) is a solution of (37) if and only if either \( z^n = f_+(z) \) or \( z^n = f_-(z) \). Moreover \( q_2(z) = f_+(z)f_-(z) \) for all \( z \in A \). If we define the real-valued analytic functions \( \rho_+ > 0 \) and \( \phi_+ \) on \( [0, 2\pi] \) by

\[ f_\pm(e^{i\theta}) := \rho_\pm(\theta)e^{i\phi_\pm(\theta)} \]

and put \( 2\pi k_\pm := \phi_\pm(2\pi) - \phi_\pm(0) \) then \( k := k_+ + k_- \).

**Case 2** If \( h \) is odd then there exists a non-vanishing analytic function \( f \) on the double covering \( \tilde{A} \) of \( A \) such that \( z \in \tilde{A} \) is a solution of (37) if and only if \( z^n = f(z_1) \) or \( z^n = f(z_2) \), where \( z_1 \) and \( z_2 \) are the two points in \( \tilde{A} \) above \( z \). Moreover \( q_2(z) = f(z_1)f(z_2) \) for all \( z \in A \). If we define the real-valued analytic functions \( \rho > 0 \) and \( \phi \) on \( [0, 4\pi] \) by

\[ f(e^{i\theta}) := \rho(\theta)e^{i\phi(\theta)} \]

then \( 2\pi k = \phi(4\pi) - \phi(0) \).

**Proof.** If \( |z| \geq 1 + \delta \), then using the uniform boundedness of \( q_1, q_2 \) we see that for \( n \) big enough the last two terms in (37) are small compared to \( |z|^{2n} \gg 1 \), hence \( p \) cannot vanish there and its \( 2n \) zeros are confined to a disc:

\[ \frac{1}{2\pi i} \int_{|z| = 1 + \delta} \frac{p_n'(z)}{p_n(z)} \, dz = 2n \, . \]

On the other hand, if \( |z| = 1 - \delta \), then for \( n \) large enough the first two terms in (37) are so small that adding them to \( q_2 \) will just have the effect of moving
its zeros slightly inside of \(|z| < 1 - \delta\) (recall that these cannot lie on \(|z| = 1 - \delta\) by assumption). Hence in this case they are equal to the zeros of \(p_n\) there asymptotically:

\[
\frac{1}{2\pi i} \int_{|z|=1-\delta} \frac{p_n'(z)}{p_n(z)} \, dz = k,
\]

for all large enough \(n\). If \(\gamma\) is the difference of these contours then the number of zeros of \(p_n\) in \(A\) is given by

\[
\frac{1}{2\pi i} \int_\gamma \frac{p_n'(z)}{p_n(z)} \, dz,
\]

which equals \(2n - k\).

To prove the statements in Cases 1 and 2, one only has to observe that in the formula

\[
2f_\pm(z) = -q_1(z) \pm \sqrt{v(z)}
\]
the square root has a single-valued branch on \(A\) if and only if \(h\) is even. The formulae for \(k\) follows directly from \(q_2(z) = f_+(z)f_-(z)\) or \(q_2(z) = f(z_1)f(z_2)\).

\[\square\]

**Theorem 19.** Under the assumptions of Lemma 18 the \(2n-k\) solutions of (37) in \(A\) converge to the unit circle. More precisely there exists a constant \(c > 0\) such that for all large enough \(n\) every zero \(z \in A\) satisfies

\[
1 - c/n \leq |z| \leq 1 + c/n.
\]

Moreover the zeros are asymptotically uniformly distributed around the circle in the sense that

\[
\lim_{n \to \infty} (2n)^{-1} \# \{z : \theta < \arg(z) < \phi\} = \phi - \theta
\]
provided \(0 \leq \theta, \phi \leq 2\pi\).

**Proof.** Case 1 follows directly from Theorem 13, while Case 2 involves slight modifications of the proof of that theorem.

\[\square\]

We conclude with two examples exhibiting the behaviour described in the two cases.

**Example 20.** Consider the equation

\[
p_n(z) := z^{2n} + q_1(z)z^n + q_2(z) = 0,
\]
where

\[
q_1(z) := -z^2 - z + 9/2,
q_2(z) := z^3 - z^2/2 - 4z + 2.
\]

The auxiliary equation

\[
w^2 + q_1(z)w + q_2(z) = 0
\]
with \(z\) replaced by \(e^{i\theta}\) has the two distinct solutions

\[
f_+(\theta) := e^{i\theta} - 1/2,
f_-(\theta) := e^{2i\theta} - 4,
\]
for all \(\theta \in [0, 2\pi]\). We deduce that \(1/2 \leq |f_+(\theta)| \leq 3/2\) and \(3 \leq |f_-(\theta)| \leq 5\) for all \(\theta \in [0, 2\pi]\). The winding numbers of these curves around the origin are \(m_+ := 1\)
and $m_- := 0$. The solutions of $q_2(z) = 0$ are $\pm 2$ and $1/2$. We should therefore anticipate that for large $n$ the equation (38) has one solution near $z = 1/2$ and two distinct rings of solutions, both close to the unit circle. One of these rings has $n - 1$ points on it while the other has $n$ points.

This example is particularly simple because one can factorize (38) in closed form, the left hand side being the product of $z^n - z + 1/2$ and $z^n - z^2 + 4$. Because the discriminant of the quadratic equation (39) is a perfect square its roots come in pairs, so there must be an even number inside the unit circle. If one replaces the coefficient $9/2$ of $q_1$ by $9$ one obtains a more typical example in which the roots form two distinct rings. The zeros of this modified polynomial $\tilde{p}_n$ are shown in Figure 3, for $n := 40$.

![Figure 3. Eigenvalues of the polynomial $\tilde{p}_{40}$ of Example 20.](image)

**Example 21.** Consider the equation

$$p_n(z) := z^{2n} - 4z^n - 8z + 3 = 0.$$ 

(40)

The solutions of the auxiliary equation

$$w^2 - 4w - 8e^{i\theta} + 3 = 0$$

(41)

are

$$w := 2 \pm \sqrt{8e^{i\theta} + 1}$$

and combine into a single closed curve winding twice around the origin and crossing itself on the negative real axis. Figure 4 shows the set of zeros of the polynomial equation (40) for $n := 20$. 

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