Proximity Effects in Conical-Ferromagnet/Superconductor bilayers

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We present a study of various aspects of proximity effects in F/S (Ferromagnet/Superconductor) bilayers, where F has a spiral magnetic texture such as that found in Holmium, Erbium and other materials, and S is a conventional s-wave superconductor. We numerically solve the Bogoliubov-de Gennes (BdG) equations self-consistently and use the solutions to compute physical quantities relevant to the proximity effects in these bilayers. We obtain the relation between the superconducting transition temperature $T_c$ and the thicknesses $d_F$ of the magnetic layer by solving the linearized BdG equations. We find that the $T_c(d_F)$ curves include multiple oscillations. Moreover, the system may be reentrant not only with $d_F$, as is the case when the magnet is uniform, but also with temperature $T$: the superconductivity disappears in certain ranges of $d_F$ or $T$. The $T$ reentrance reported here occurs when $d_F$ is larger than the spatial period of the conical exchange field. We compute the condensation free energies and entropies from the full BdG equations and find the results are in agreement with $T_c$ values obtained by linearization. The inhomogeneous nature of the magnet makes it possible for all odd triplet pairing components to be induced. We have investigated their properties and found that, as compared to the singlet amplitude, both the $m = 0$ and $m = \pm 1$ triplet components exhibit long range penetration. For nanoscale bilayers, the proximity lengths for both layers are also obtained. These lengths oscillate with $d_F$ and they are found to be long range on both sides. These results are shown to be consistent with recent experiments. We also calculate the reverse proximity effect described by the three dimensional local magnetization, and the local DOS, which reveals important energy resolved signatures associated with the proximity effects.

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I. INTRODUCTION

The emerging field of spintronics has stimulated interest in fabricating solid state devices that make use of the intrinsic spins as a degree of freedom.1 Strides have been made recently towards exploiting the spin variable in hybrid ferromagnet/superconductor (F/S) systems. Such systems have shown promise for a number of practical applications, including nonvolatile information storage. The simplest of such potential devices usually involve layered F/S heterostructures. Owing to these potentially important spintronic applications, research on the fundamental physics of these systems has received great attention in the last decade.2,3 The most important basic physics elucidated by these studies is probably that of the superconducting proximity effects in F/S nanostructures,3 which describes the interplay between ferromagnetic and superconducting order parameters. Though these two order parameters are rarely found to coexist naturally in bulk materials, such coexistence can be and has been achieved near the interfaces of artificially created F/S composites. Thus, the subject has become important not only for its technological applications but also because of the underlying fundamental physics.

In elementary treatments, ferromagnetism is often deemed strictly incompatible with s-wave superconductivity due to their mutually exclusive order parameters. In ferromagnets, the exchange field tends to cause the electronic spins to align in the same direction, while in singlet s-wave superconductors, the Cooper pairs are composed of both spin-up and spin-down electrons. These two order parameters seem to naturally oppose each other. In fact, in F/S heterostructures this competition leads to a strong modification of the behavior of the superconducting Cooper pair amplitudes. When a Cooper pair in S encounters an F/S interface and enters the F region, the momenta of spin-up and spin-down electrons that make up the Cooper pair are changed, because of the exchange field in the F region. This leads to a nonzero center of mass momentum of the Cooper pair4,5 and an overall damped oscillating Cooper pair amplitude in the F side. It is because of these two competing order parameters that the oscillations decay over a relatively short length scale which decreases as the exchange field increases.

These oscillations of the superconducting wavefunctions are one of the most salient features governing proximity effects in F/S systems and form the basis for switching applications that require the manipulation of the superconducting transition temperature $T_c$ through variation of the experimental parameters. Due to the oscillatory nature of the Cooper pair amplitudes, the dependence of $T_c$ on the thickness of the ferromagnetic layer, $d_F$, in F/S bilayers is oscillatory too. Furthermore, the interference between the transmitted pairing wave function through the F/S interface and the reflected one from the boundary can become fully destructive: the superconductivity disappears for a certain $d_F$ range. This superconducting reentrant behavior with $d_F$ has been found experimentally in Nb/Cu$_{1-x}$Ni$_x$ and Fe/V/Fe.
trilayers and it is well understood theoretically.

Another important fact about $F/S$ proximity effects is the generation of induced triplet pairing correlations. These can be generated by the presence of spin active interfaces, or (and this is the case we will focus on here) in systems with clean interfaces and inhomogeneous $F$ structure. The simplest such cases are $F_1/S/F_2$ or $F_1/F_2/S$ layers in which the magnetizations of the two $F$ layers are misaligned. For $s$-wave superconductors, where the orbital part of the pair wave function is even, the Pauli principle requires the spin part to be odd and this would appear to forbid the existence of triplet correlations. However, triplet correlations that are odd in frequency (or equivalently in time) can be induced in $F/S$ systems, with $S$ being $s$-wave pairing, without violating the Pauli exclusion principle.

The importance of odd triplet correlations lies in their long range nature in the magnet, i.e., their proximity lengths can be in principle comparable to those found in the usual superconducting proximity effects involving nonmagnetic metals. Since the exchange fields tend to align the electronic spins of the Cooper pair electrons, the proximity length for singlet pairing is very short (and dependent on the magnitude of exchange field). However, the triplet pairing correlations can involve electron pairs with both spins aligned along the local magnetization direction, and thus be much less sensitive to the mechanism of exchange fields, penetrating much deeper in $F$ than their singlet counterparts. The possible appearance of both $m = 0$ and $m = \pm 1$ ($m$ denoting the usual spin quantum number) components of induced triplet correlations is controlled by the symmetry of the system and by conservation laws. In multilayer $F/S$ systems, when the $F$ layers are magnetically homogeneous (all exchange fields are along the same direction, the quantization axis) only the total spin projection corresponding to the $m = 0$ component can be induced. On the contrary, all three components ($m = 0$ and $m = \pm 1$) can arise if the direction of exchange fields differs in the ferromagnets, e.g., the exchange fields of $F_1$ and $F_2$ are not aligned in $F_1/S/F_2$ or $F_1/F_2/S$ types of trilayers. These long-range characteristics of triplet correlations have been experimentally detected in ferromagnetic multilayers by taking advantage of their magnetic inhomogeneity.

Besides ferromagnet misalignment, another possibility to generate long-range triplet correlations is to use a ferromagnet with an intrinsic inhomogeneous magnetic texture. Such structures are inherent to either known elements or chemical compounds. Examples of this kind of ferromagnets include most prominently Ho, which has a spiral magnetic structure at low temperatures. A similar spiral magnetic structure is found in metallic Erbium and MnSi thin films, and Fe(Se,Te) compounds. Indeed it has been experimentally confirmed that long-range triplet correlations are induced in Nb/Ho/Co multilayers with the periodicity of Ho playing an important role in triplet supercurrents. Superconducting phase-periodic conductance oscillations have also been observed in Al/Ho bilayers where the thickness of Ho is much larger than the penetration length of singlet amplitudes. This finding can be explained in the framework of the triplet proximity effects. Theoretically, the spin-polarized Josephson current in S/Ho/S junctions has been studied via quasi-classical Green function techniques. The triplet supercurrent in Ho/Co/Ho trilayers was also investigated in the diffusive and clean regimes. The long-range effects can however be limited by interface quality and impurities. These earlier works show that ferromagnets with an intrinsic spiral magnetic structure are of particular interest when studying superconducting proximity effects in $F/S$ nanostructures.

It was also recently predicted that superconductivity in conical-ferromagnet/superconductor bilayers can be reentrant with temperature, in addition to the standard reentrance with $d_F$ mentioned above. It was shown via numerical solution of the Bogoliubov-de Gennes (BdG) equations that in certain cases superconductivity can exist in a range $T_{c1} < T < T_{c2}$, where $T_{c1}$ is nonzero. This reentrance with temperature is quite different from that found in ternary rare earth compounds such as ErRh$_2$B$_4$ and HoMo$_6$S$_8$ where the disappearance of superconductivity below $T_{c1}$ results from the onset of long-range ferromagnetism. In the bilayers considered in Ref. the high $T$ phase and the low $T$ phase are the same. The physical reasons that account for the reentrance there are attributed to the proximity effects associated with the interference of Cooper pair amplitudes and the generation of triplet pairing correlations, resulting in nontrivial competition between the entropies and condensation energies.

In this paper, we present results for various properties of the proximity effects in $F/S$ bilayers, where the $F$ layer has a helical magnetic structure. We numerically find the self-consistent solutions to the BdG equations and use them to compute important physical quantities. By linearizing the BdG equations we calculate the critical temperature as a function of magnet thickness, exchange field strength and periodicity, and other parameters. We then discuss the effects of varying the superconductor thickness to coherence length ratio. We show that depending on the width of the superconductor, and for a broad range of magnetic strengths, reentrant behavior as a function of magnet thickness can arise. We find that under certain conditions, the superconductivity can also be reentrant with temperature, and for larger $d_F$ values than previously reported. To clarify these reentrant phenomena, we investigate the thermodynamic functions associated with the various ways reentrance can arise. We find that all components of the odd triplet correlations can be induced and we discuss their long range nature. We then characterize the important triplet long range behavior by introducing the corresponding proximity lengths. We find that these lengths oscillate as a function of $d_F$, and depend on details of the magnetic texture. Reverse proximity effects are also studied to determine the magnetic influence on the superconductor: we calculate the local
magnetization vector revealing greater penetration into S for weaker exchange fields. Lastly, the spectroscopic information is presented by means of the local density of states, to demonstrate consistency with the $T_c$ results.

II. METHODS

The procedures we employ to self-consistently solve the BdG equations and to extract the relevant quantities are very similar to those already described in the literature (see Refs. 17 and 38 and references therein). It is unnecessary to repeat details here. We consider $F/S$ bilayers that consist of one ferromagnetic layer with spiral exchange fields and a superconducting layer with $s$-wave pairing. The geometry is depicted in Fig. 1. Our systems are infinite in the $x-z$ plane and finite along $y$-axis. Their total thickness is denoted by $d$: the $F$ layer has width $d_F$ and the $S$ layer has width $d_S = d - d_F$. The left end of the bilayers is the $y = 0$ plane. We assume that the interface lies in the $x-z$ plane and the exchange field $\mathbf{h}$, which is present only in $F$, has a component that rotates in this plane plus a constant component in the $y$ direction, perpendicular to the interface:

$$\mathbf{h} = \hbar \left\{ \cos \alpha \hat{\mathbf{y}} + \sin \alpha \left[ \sin \left( \frac{\beta y}{a} \right) \hat{\mathbf{x}} + \cos \left( \frac{\beta y}{a} \right) \hat{\mathbf{z}} \right] \right\},$$

(1)

where the helical magnetic structure has a turning angle $\beta$, and opening angle $\alpha$. We will take $a$, the lattice constant, as our unit of length and vary the strength $\hbar$. The spatial period of the helix is $\lambda = 2\pi a/\beta$.

We first write down the effective Hamiltonian of our systems,

$$\mathcal{H}_{\text{eff}} = \int d^2 r \left\{ \sum_\rho \left[ \psi_\rho^\dagger(r) \left( -\frac{\nabla^2}{2m^*} - \mathcal{E}_F \right) \psi_\rho(r) \right. \right. \right.$$

$$+ \frac{1}{2} \sum_{\rho, \rho'} (i\sigma y)_{\rho\rho'} \Delta(r) \psi_\rho^\dagger(r) \psi_\rho(r) + h.c. \bigg]$$

$$\left. \left. - \sum_{\rho, \rho'} \psi_\rho^\dagger(r)(\mathbf{h} \cdot \sigma) \psi_\rho(r) \right\}, \right.$$

(2)

where $\rho$ and $\rho'$ are spin indices, $\psi_\rho(r)$ is the field operator, and $\sigma$ are the Pauli matrices. $\Delta(r)$ in the above equation is the pair potential. To apply the BdG formalism to spatially inhomogeneous systems, we first invoke the generalized Bogoliubov transformation,

$$\psi_\rho(r) = \sum_n \left[ u_{n\rho}(r) \gamma_n + v_{n\rho}^*(r) \gamma_n^\dagger \right],$$

(3)

where $u_{n\rho}(r)$ and $v_{n\rho}(r)$ can be interpreted as quasiparticle and quasi-hole wavefunctions and the creation operator $\gamma_n^\dagger$ and annihilation operator $\gamma_n$ obey the usual fermionic anti-commutation relations. To recast the effective Hamiltonian into a diagonalized form, $\mathcal{H}_{\text{eff}} = \sum_n \epsilon_n \gamma_n^\dagger \gamma_n$, one can use commutation relations between $\mathcal{H}_{\text{eff}}$ and field operators. By doing so, and making use of the quasi one dimensional nature of the problem one arrives at the BdG equations,

$$\begin{pmatrix}
\mathcal{H}_0 - h_x & -h_x + ih_y & 0 & 0 \\
-h_x - ih_y & \mathcal{H}_0 + h_x & -\Delta(y)^* & 0 \\
0 & -\Delta(y)^* & \mathcal{H}_0 + h_x & h_x - ih_y \\
\Delta(y)^* & 0 & h_x - ih_y & \mathcal{H}_0 - h_x \\
\end{pmatrix}
\begin{pmatrix}
\Delta(y) \\
0 \\
\Delta(y)^* \\
0 \\
\end{pmatrix}
= \epsilon_n
\begin{pmatrix}
u_{n\uparrow}(y) \\
v_{n\downarrow}(y) \\
u_{n\downarrow}(y) \\
u_{n\uparrow}(y) \\
\end{pmatrix},$$

(4)

where $\mathcal{H}_0 = -\frac{1}{2m^*} \frac{\partial^2}{\partial y^2} + \epsilon_\perp - \mathcal{E}_F$, is the usual single particle Hamiltonian for the quasi one-dimensional problem, with $\epsilon_\perp$ denoting the kinetic energy associated with the transverse direction. By using Eq. (3), the self consistency relation for the pair potential can be rewritten in the form,

$$\Delta(y) = \frac{g(y)}{2} \sum_n \left[ u_{n\uparrow}(y) v_{n\downarrow}^*(y) - u_{n\downarrow}(y) v_{n\uparrow}^*(y) \right] \tanh\left( \frac{\epsilon_n}{2T} \right),$$

(5)

where $g(y)$ is the usual BCS superconducting coupling constant in the $S$ region, and zero in the $F$ material. The
prime sign indicates summing over all eigenstates with
eigenenergies less than or equal to a cutoff “Debye”
frequency $\omega_D$. The solutions to the BdG equations must be
determined self-consistently. This self-consistency condition
is extremely important in studying proximity effects.

The singlet pair amplitude, i.e., the amplitude for finding
a Cooper pair, $F(y)$, is given by $F(y) = \Delta(y)/g(y)$.
One can determine the superconducting transition
temperatures $T_c$ by looking for the temperatures at which the
pair amplitudes becomes vanishingly small. However, it
is much more efficient to find $T_c$ by linearizing the self
consistency relation, Eq. (5) and using a perturbation expansion.
This technique has been discussed in other papers\textsuperscript{38,45} and the details are not repeated here.

Once a full set of self-consistent solutions is obtained,
all the additional quantities of interest can be computed.
For example, the triplet correlations, can in general be
defined\textsuperscript{21,22} as:

$$ f_0(r,t) \equiv \frac{1}{2}[\langle \psi^\dagger(r,t)\psi^\dagger(r,0) \rangle + \langle \psi^\dagger(r,t)\psi^\dagger(r,0) \rangle], \quad (6a) $$

$$ f_1(r,t) \equiv \frac{1}{2}[\langle \psi^\dagger(r,t)\psi^\dagger(r,0) \rangle - \langle \psi^\dagger(r,t)\psi^\dagger(r,0) \rangle], \quad (6b) $$

where $\langle \ldots \rangle$ represents ensemble averages. As discussed
in Sec. I, both $f_0$ and $f_1$ have to vanish at $t = 0$ to comply
with the Pauli principle. However, in general both of
them can be induced when $t \neq 0$ and the magnetic structure
is inhomogeneous. By using Eq. 3 and considering the
time evolution, one can rewrite the odd triplet correlations in terms of quasi-particle and quasi-hole wave-
functions in the form:

$$ f_0(y,t) = \frac{1}{2} \sum_n [u_{n \uparrow}(y)v_{n \downarrow}^\ast(y) + u_{n \downarrow}(y)v_{n \uparrow}^\ast(y)] \zeta_n(t), \quad (7a) $$

$$ f_1(y,t) = \frac{1}{2} \sum_n [u_{n \uparrow}(y)v_{n \downarrow}^\ast(y) - u_{n \downarrow}(y)v_{n \uparrow}^\ast(y)] \zeta_n(t), \quad (7b) $$

where $\zeta_n(t) \equiv \cos(\epsilon_n t) - i \sin(\epsilon_n t) \tanh(\epsilon_n/(2T))$.

Given the quasi-particle amplitudes and eigenvalues,
one is also able to evaluate the thermodynamic quantities
from the free energy $F(T)$. For an inhomogeneous system
it is most convenient to use the expression:\textsuperscript{46}

$$ F(T) = -2T \sum_n \ln \left[ 2 \cosh \left( \frac{\epsilon_n}{2T} \right) \right] + \left\langle \frac{\Delta^2(y)}{g(y)} \right\rangle_s, \quad (8) $$

where $\langle \ldots \rangle_s$ denotes spatial average. The condensa-
tion free energy $\Delta F$ is the difference between the free
energies of the superconducting state, $F_S$, and the normal
states $F_N$, i.e. $\Delta F = F_S - F_N$. $F_N$ can be calculated
by assuming that the pair potential is absent throughout
the system.

Another important physical quantity, which can be
determined experimentally by tunneling spectroscopy, is
the local density of states (LDOS). This quantity often
reveals important information about the superconducting
features of the sample studied. In our quasi-one-
dimensional model, the LDOS $N(y, \epsilon)$ depends spatially
only on $y$. $N(y, \epsilon)$ consists of both spin-up and spin-down
LDOS contributions, that is, $N(y, \epsilon) = N_\uparrow(y, \epsilon) +
N_\downarrow(y, \epsilon)$.

$$ N_\rho(y, \epsilon) = \sum_n \left[ |u_{n \rho}(y)|^2 \delta(\epsilon - \epsilon_n) + |v_{n \rho}(y)|^2 \delta(\epsilon + \epsilon_n) \right], \quad (9) $$

where $\rho = \uparrow, \downarrow$.

Just as the superconducting order parameter is changed by
the presence of ferromagnets, also, near the interface, the ferromagnetism can be modified by
the presence of the superconductor,\textsuperscript{5,47-50} a phenomenon
known as the reverse proximity effect. It is best described
by considering the local magnetization $\mathbf{m}$. The local
magnetization vector is defined as $\mathbf{m} = -\mu_B \langle m \sigma \Phi \rangle$
where $\Psi = (\psi^\dagger, \psi^\dagger)^T$, and, again, it depends on the
coordinate $y$ only. By using Eq. 3 we have,

$$ m_x(y) = -\mu_B \sum_n \left\{ (u_{n \uparrow}(y)v_{n \downarrow}^\ast(y) + u_{n \downarrow}(y)v_{n \uparrow}^\ast(y)) f_n \right\}, \quad (10a) $$

$$ m_y(y) = i\mu_B \sum_n \left\{ (u_{n \uparrow}(y)v_{n \downarrow}^\ast(y) - u_{n \downarrow}(y)v_{n \uparrow}^\ast(y))(1 - f_n) \right\}, \quad (10b) $$

$$ m_z(y) = -\mu_B \sum_n \left\{ \|u_{n \uparrow}(y)\|^2 - |u_{n \downarrow}(y)|^2 \right\} f_n \quad + |v_{n \uparrow}(y)\|^2 - |v_{n \downarrow}(y)|^2 (1 - f_n) \right\}, \quad (10c) $$

where $f_n$ is the Fermi function of $\epsilon_n$ and $\mu_B$ is the Bohr
magneton.

**III. RESULTS**

In the results shown here, capital letters will always
denote the dimensionless lengths denoted by the corre-
sponding small letter. For example, the dimensionless
thickness of the ferromagnet is written as $D_F \equiv d_F/a$
and that of superconductors is $D_S \equiv d_S/a$, where $a$ is
the lattice constant in Eq. (1). For the helical magnetic structure we take angular values (see Eq. (1)) $\alpha = 4\pi/9$
and $\beta = \pi/6$ which are\textsuperscript{29,51} appropriate to Ho, in which
case $D_F = 12$ contains one full period of the spiral ex-
change field. We will denote this dimensionless spatial
period by $\Lambda$ in the following subsections. For materi-
als other than Ho many of the results can be read off by
rescaling \( \lambda \) to the appropriate value. Throughout this paper, the dimensionless superconducting coherence length is fixed to be \( \Xi_0 = 100 \). In the same spirit, the dimensionless exchange field \( I \) is measured in terms of the Fermi energy: \( I \equiv h_0/E_F \). We choose the Fermi wavevector in \( S \) to equal \( 1/\alpha \). We take the “Debye” cutoff value to be \( \omega_D = 0.04E_F \). As usual, this value is irrelevant except for setting the overall transition temperature. Temperatures are given in dimensionless form in terms of \( T_0' \), the transition temperature of bulk \( S \) material. When discussing the triplet amplitudes, which are time dependent, we use the dimensionless time \( \tau \equiv t\omega_D \). Vertical dashed lines shown in figures, when present, denote the \( F/S \) interface.

### A. Transition Temperatures

To investigate the details of the predicted oscillatory nature of the \( d_F \) dependence of \( T_c \) as discussed in Sec. I, we calculated \( T_c \) as a function of \( D_F \) for several \( I \) and \( D_S \) values. These results are shown in Fig. 2 and Fig. 3. The \( D_F \) range in both figures includes three complete periods of the spiral magnetic order. This is reflected in the results shown: indeed, the presence of multiple oscillations in the included range of \( D_F \) is the most prominent feature in Figs. 2 and 3. The oscillations in \( T_c \) arise (as we discuss below) from a combination of the periodicity of the spiral magnetic structure and the usual \( T_c \) oscillations which arise, even when the magnet is uniform, from the difference in the wavevectors of the up and down spins. In Fig. 2, one can also see that with stronger exchange fields the oscillation amplitudes are larger. Despite this increase of the amplitudes with the exchange field (they are approximately proportional to \( I \)), the overall \( T_c \) decreases when the exchange field increases. This is consistent with expectations: a stronger exchange field destroys the superconductivity more efficiently. As mentioned in Sec. I, when the exchange field is strong enough, the systems can become normal in some range \( D_{F1} < D_F < D_{F2} \). Indeed, reentrance with \( d_F \) can be seen to occur in Fig. 2 near \( D_F = 4 \) at \( I = 0.2 \). Another feature seen in this figure is the decrease of the amplitude oscillations with increasing \( D_F \). This arises simply because the singlet Cooper pair amplitudes in \( S \) near the \( F/S \) boundary decay more strongly at larger \( d_F \) and therefore the effect of the pair amplitude oscillations in \( F \) is weaker.\(^{52}\)

In a \( F/S \) bilayer where the ferromagnet is homogeneous, the periodicity of the \( T_c \) oscillations is governed by the exchange field, or equivalently, by the magnetic coherence length\(^7 \) \( \Xi_F = 1/I \). Here, where a bilayer with a conical inhomogeneous ferromagnet is considered, the intrinsic spiral magnetic order with spatial period \( \Lambda \) plays an equally important and competing role in the \( T_c \) oscillations. In other words, both the strength and the periodicity of exchange fields influence the overall decay and the oscillatory nature of the superconducting transition temperatures. The existence of two different spatial periodicities leads to the obvious consequence that the \( T_c(D_F) \) curves are not describable in terms of one single period. However, when \( I \) is not very strong (\( I \lesssim 0.1 \)), the minima of \( T_c \) are near the locations where \( D_F = \lambda/2, \) \( 3\lambda/2, \) and \( 5\lambda/2 \) and similarly, the \( T_c \) maxima occur near \( D_F = \lambda, 2\lambda, \) and \( 3\lambda \). This indicates that the magnetic periodicity is dominant. Roughly speaking, the maxima and the minima are correlated with the strongest and weakest spatial average of the exchange field components in \( F \). As \( I \) increases and \( \Xi_F \) decreases deviations become obvious. Figure 2 shows that the distances between two successive maxima decrease when the exchange fields increase.

The existence of the multiple oscillations discussed above has been confirmed experimentally. In Ref. 53,
also demonstrates that not only a strong superconductive correlation with, but not simply described by, the spatial wavelength $\lambda$ of the Ho structure. Comparison with the theory discussed here was made, using $I$ as an adjustable parameter. Values near $I = 0.1$ were found to provide the best fit. The other parameters were extracted from other known properties of Ho and Nb or (e.g. $d_S$) from the experimental sample geometry. The results of the comparison were extremely satisfactory, showing clear agreement in all the features of the rather intricate $T_c(d_F)$ experimental curves. It was found also that one of the samples was close to being reentrant with $d_F$, near $D_F = 4$ and $T_c$ is depicted by the upper (green) squares and $T_c$ by the (blue) circles.

$T_c$ in Nb/Ho bilayers was measured as a function of $d_F$. The results exhibit an overall decay with Ho thickness, on which there are superimposed oscillations which are correlated with, but not simply described by, the spatial wavelength $\lambda$ of the Ho structure. Comparison with the theory discussed here was made, using $I$ as an adjustable parameter. Values near $I = 0.1$ were found to provide the best fit. The other parameters were extracted from other known properties of Ho and Nb or (e.g. $d_S$) from the experimental sample geometry. The results of the comparison were extremely satisfactory, showing clear agreement in all the features of the rather intricate $T_c(d_F)$ experimental curves. It was found also that one of the samples was close to being reentrant with $d_F$, near $D_F = 4$ and $T_c$ is depicted by the upper (green) squares and $T_c$ by the (blue) circles.

In previous work, we reported one specific case where superconductivity in Ho/S bilayers exhibits not only the usual reentrance with $d_F$ but also, at some fixed values of $d_S$, $h_0$, and $D_F$, reentrance with $T$, that is, superconductivity exists only in a temperature range $T_{c1} < T < T_{c2}$, where $T_{c1}$ is finite. In the example reported in Ref. temperature reentrance occurred near the first minimum of the $T_c(D_F)$ curve. We have investigated here whether this kind of reentrance can occur near some of the other minimum of $T_c(D_F)$. These locations appear favorable for such an occurrence since superconductivity is relatively weak near these minima. Also reentrance with $D_F$ is after all an extreme case of a minimum $T_c(D_F)$. We have found that other T-reentrant examples can indeed be found, although by no means universally. Here we report an example of reentrance occurring near the second minimum of $T_c(D_F)$. In Fig. 4, the main plot shows $T_c(D_F)$ for the parameter values specified in the caption. The first minimum of $T_c(D_F)$ drops to zero and is an example of $D_F$ reentrance. In the region near the second minimum ((green) symbols) reentrance with $T$ occurs. The region of interest is enlarged in the inset. There the upper (green) symbols represent $T_{c2}$ and the small dome of lower (blue) circles represent $T_{c1}$. In the dome region, but not outside it, the superconductivity is reentrant in $T$. When one lowers the temperature from the normal region, the $F/S$ bilayers become superconducting at $T_{c2}$. With further cooling, the bilayers return to normal state. Reentrance in this case occurs at the second minimum rather than the first because there is no upper transition associated with the first minimum: the system is normal. Near the second minimum the oscillatory effects are not as strong, and as as a result, the system becomes reentrant in $T$. This can be viewed as a “compromise”: near the second minimum, as opposed to the first, superconductivity is not completely destroyed but it becomes “fragile” and can disappear upon lowering $T$. The physics involved from the thermodynamic point of view will be discussed in the following subsection.

### Thermodynamics of Reentrance phenomena

To understand the reentrance phenomena in $T$ it is most useful to examine the thermodynamics of the two transitions, and in the region between them. From the condensation free energy $\Delta F$, which can be evaluated as explained in connection with Eq. (8), other quantities such as the condensation energy and entropy are easily obtained. For reentrance with $D_F$, it is sufficient to look at the free energy at constant low $T$.

Considering the reentrance with $T$, it is illuminating to consider first the $T$ dependence of the singlet pair amplitude $F(Y)$ well inside the S material. Thus, we focus on $F(Y)$ one coherence length from the interface: $Y = D_F + \Xi_0$. This quantity, normalized to its value in bulk $S$ material, is plotted in Fig. 5 as a function of $T$. 

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**FIG. 4.** (Color online) Calculated transition temperatures $T_c$ vs. $D_F$ for $D_S = 148$ and $I = 0.2$. The main plot shows ((red) symbols) the overall $T_c$ behavior from $D_F = 0$ to $D_F = 1.5A$. Reentrance with $D_F$ near $D_F = 4$ is seen. In this case there is also reentrance with temperature in the region indicated by (green) symbols near $D_F = 16$. The inset is a blow up of this region: superconductivity exists only in the region $T_{c1} < T < T_{c2}$, where $T_{c1}$ is depicted by the upper (green) squares and $T_{c2}$ by the (blue) circles.
for two contrasting values of $D_F$, one at $D_F = 16$ where reentrance occurs (see Fig. 4) and at a very nearby value, $D_F = 17$, which lies just outside the reentrance region and exhibits typical behavior. Fig. 5 ((green) circles) demonstrates that in the latter case the amplitude behaves qualitatively as the order parameter does in a conventional BCS superconductor: it decreases very slowly near $T = 0$ and eventually drops to zero very quickly but continuously near $T_c$, indicating the occurrence of a second order phase transition. This transition occurs at $T_c/T_c^0 \approx 0.32$ in agreement with Fig. 4. However, the behavior of the pair amplitude in the reentrant region ((red) squares in Fig. 5) is quite different. There are two transition temperatures: below a very low but finite temperature, $T_{c1}/T_c^0 \approx 0.02$, the singlet pair amplitude vanishes and the system is in its normal state. $F(Y)$ then begins to rise continuously, has a maximum at a temperature $T_m$, (where $T_m/T_c^0 \approx 0.1$) and eventually drops to zero, again continuously, at an upper transition $T_{c2}/T_c^0 \approx 0.22$. In the region $T_{c1} < T < T_{c2}$ the system is in the superconducting state. Both transitions are of second order. The values of $T_{c1}$ and $T_{c2}$ from the vanishing of the amplitude, seen in Fig. 5, agree with those calculated directly from linearization of the self consistent equation plotted in Fig. 4.

We now turn to the condensation free energy, $\Delta F$, and entropy, $\Delta S$, for the same $T$ reentrant case. $\Delta F$ is shown in the top panel of Fig. 6 as calculated from Eq. (8) and normalized by $2E_0$, where $E_0$ is the condensation energy of bulk $S$ material at $T = 0$. The lower panel shows the normalized condensation entropy, defined as the $\Delta S \equiv -d\Delta F/d(T/T_c^0)$. The meaning of the symbols in this figure is the same as in the previous one. When the system is near (but outside) the reentrant region the behavior of both quantities plotted is qualitatively the same as that found in textbooks for bulk BCS superconductors. Quantitatively, the magnitude of $\Delta F$ for our systems are much smaller than that for bulk $S$ where we would have $\Delta F = -0.5$ at $T = 0$ in our units. The value of $T_c$ in the non-reentrant case can also be identified from where the free energies of the normal and superconducting states are the same ($\Delta F(T) \equiv 0$), and it agrees with both Fig. 4 and 5. Moreover, the vanishing of the entropy difference at a finite $T_c$ confirms the occurrence of a second order phase transition. The value of this transition temperature is consistent with all above results.

The story for the reentrant case is quite different. There, although the values of $\Delta F$ are much smaller compared to those in the standard case, one can still find that the minimum of $\Delta F$ occurs at approximately the same value $T_m$ where the singlet pair amplitudes have a maximum. Thus the superconductivity is most robust at $T = T_m$. The two transition temperatures $T_{c1}$ and $T_{c2}$ can also be determined from the top panel of Fig. 6 and match with those found in Figs. 4 and 5. In the two $T$ ranges $T < T_{c1}$ and $T > T_{c2}$, the normal state is the
FIG. 7. (Color online) Reentrance with $D_F$. Top panel: normalized singlet pair amplitude, computed at a location one coherence length inside $S$ from the interface, as a function of $D_F$. Bottom panel: normalized condensation free energy, $\Delta F = F_S - F_N$, vs. $D_F$ at $T = 0$.

only self-consistent solution to the basic equations, as is evident from Fig. 5. The vanishing $\Delta F$ when $T < T_{c1}$ means that the electrons do not then condensate into Cooper pairs. This is exactly what happens for pure superconductors when $T > T_c$.

There are some remarkable facts about the behavior of $\Delta S$ in the reentrant case. First, the vanishing of $\Delta S$ (along with that of $\Delta F$) in Fig. 6 indicates that the system undergoes second order phase transitions at both $T_{c1}$ and $T_{c2}$. Also, $\Delta S$ is positive for $T_{c1} < T < T_m$ where $T_m$ is again the value of $T$ at which the singlet pair amplitude reaches its maximum and $\Delta F$ its minimum. That the entropy of the superconducting state is higher than that of the normal state indicates that the normal state at $T_{c1} < T < T_m$ is more ordered than the superconducting one. This truly unusual fact, which is the root cause of the reentrance, is due to the oscillating nature of both the Cooper pair condensates and of the exchange field, which leads to an uncommonly complicated structure for the pair amplitude. Above $T_m$, the superconducting state becomes more ordered than the normal state: $\Delta S$ is negative. From Fig. 5 and 6, we see that the singlet pair amplitudes, the condensation free energies, and entropy differences of reentrant case in the range $T_m < T < T_{c2}$ have a similar trend to those of the non-reentrant case in the range $0 < T < T_c$. We have found also examples of non-reentrant cases in which there is a finite temperature $T_m$ at which $\Delta F$ has a minimum but where on further lowering $T$, $\Delta F$ remains negative all the way to $T = 0$.

The situation in the more common $D_F$ reentrant region, where we find that the system does not become superconducting when it is heated from $T = 0$, is different from that of $T$ reentrance. A case where $T_r$ vanishes in the range $D_{F1} < D_F < D_{F2}$ for $I = 0.2$ was seen in Fig. 2. To further analyze this $D_F$ reentrance, we again calculated the singlet pair amplitudes inside $S$ at one coherence length from the interface, in the zero temperature limit. The top panel of Fig. 7 shows the normalized $F(D_F + \Xi_0)$ as a function of $D_F$, for the same parameters as the $I = 0.2$ case in Fig. 2. The singlet amplitudes drop to zero in the same range as where $T_r$ vanishes in Fig. 2: the normal state is the only self-consistent solution and the superconductivity is completely destroyed in this $D_F$ range. One can also see that the order parameter is continuous but its derivative discontinuous at $D_{F1}$ and $D_{F2}$. In the bottom panel, we plot the corresponding condensation free energies (at $T = 0$) as a function of $D_F$. The $D_F$ range and the temperature are the same as the top panel. The condensation free energies vanish in the same $D_F$ reentrance region. Unlike the derivatives of the singlet pair amplitudes, the derivatives of $\Delta F$ at $D_{F1}$ and $D_{F2}$ appear to be continuous.

The physical origins of these two kinds of reentrance are not identical. As mentioned in Sec. I, the interference effects of oscillating Cooper pair wavefunctions are responsible for the $D_F$ reentrance, provided that $I$ is strong and $D_F$ is not too thick. $D_F$ reentrance does not require a nonuniform magnet. The conical-ferromagnet structure introduces an additional nonuniform magnetic order which may coexist with nonuniform superconductivity as predicted in Ref. 54. This additional non-uniformity, with its concomitant introduction of triplet correlations and of a new periodicity, can produce, as we have shown, reentrant behavior in $T$, as opposed to the simpler behavior seen e.g. near the first minimum in the main plot of Fig. 4. Thermodynamically, the reentrance with $T$ is due to the competition between entropy and energy, and driven by the high entropy of the disordered superconducting state. When $T < T_m$, $\Delta S$ is positive and the roles of the normal and superconducting phases are exchanged: the high entropy phase is the superconducting one. Further lowering $T$ brings the system back to normal state. One can compare the instance of $T$ reentrance reported here with that reported in our previous work where it occurs near the first minimum of $T_r(D_F)$. In that work, $D_S = 150 = 1.55\Xi_0$ and $I = 0.15$. Here, not only is $D_S$ thinner but also $I$ is greater. The first minimum of $T_r(D_F)$ in the main plot of Fig. 4 drops to zero and becomes a $D_F$ reentrance region. Because stronger $I$ and thinner $D_S$ are unfavorable to superconductivity, the system can not sustain $T$ reentrance there. Thus, a delicate balance of geometrical and material parameters
is required.

C. Singlet to triplet conversion

In this subsection, we will discuss the general properties of the induced triplet pairing correlations in $F/S$ bilayers with $F$ being a conical ferromagnet. As mentioned in Sec. I, in the presence of inhomogeneous exchange fields in the $F$ layers both the $m = 0$ and the $m = \pm 1$ triplet pair amplitudes are allowed by conservation laws and the Pauli principle, but this says nothing about their size or shape, or indeed on whether they will exist at all. Thus detailed calculations are needed. The intrinsically inhomogeneous magnetic textures discussed here provide unique opportunities to study the triplet proximity effects in $F/S$ systems containing only a single $F$ layer. Triplet correlations in the ballistic regime for both $F_1/S/F_2$ and $F_1/F_2/S$ trilayers have been found in previous work\cite{21,22,24} to be long ranged and the expectation\cite{31} that they will also be in our case is fulfilled. We will here discuss and characterize this and other aspects (such as the effect of the strength of the exchange fields on the triplet pair amplitudes) of triplet pairing correlations in $F/S$ bilayers where the magnets maintain a spiral exchange field. Results presented in this subsection are all in the low $T$ limit.

To exhibit the long range nature of both types of triplet amplitudes, we show in Fig. 8 both the triplet and singlet pair amplitudes for a thick $F$ layer as a function of position, as given by the dimensionless coordinate $Y$. In this (and the next figure, Fig. 9), we will focus on the real parts of the in general complex (see Eqs. (7)) $f_0$ and $f_1$, since we have found that, for the cases shown, their imaginary parts are smaller by at least a factor of 2 to 5 and their behavior is similar to that of the real parts. To properly compare singlet and triplet quantities, both the singlet amplitude, $F(Y)$ and the triplet amplitudes are normalized the same way: to the value of the singlet amplitude in bulk $S$ material. For visibility, we have multiplied the triplet pair amplitudes by a factor of 10. In the left and right columns of Fig. 8,
we show in this way the real parts of both $f_0$ and $f_1$ when $I = 0.1$ and $I = 0.5$, respectively. The ferromagnet has a large thickness: $D_F = 3\Xi_0 = 2D_S$. The triplet correlations, which we recall vanish at equal times, are computed at a value of the dimensionless time $\tau = 9.6$. One sees right away that both the $f_0$ and $f_1$ components can be induced at the same time. This is always the case in our structures, as opposed to what occurs in $F_1/F_2/S$ and $F_1/S/F_2$ trilayers where the $f_0$ and $f_1$ components can be induced simultaneously only when the exchange fields in these $F$ layers are non-collinear. Secondly, the induced triplet correlations on the $F$ side are long ranged compared to the singlet amplitudes. The singlet amplitudes decay with a short\(^5\) proximity length $2\pi\Xi_F \approx 2\pi/I$ due to the pair-breaking effect of the exchange field. In contrast, the proximity length for the triplet amplitudes as seen in Fig. 8 is much longer: it is of the order of $\Xi_0$, and does not depend strongly on $I$. The triplet amplitudes spread over the $F$ side with an oscillatory behavior. This difference is more pronounced in the $I = 0.5$ case, where the decay length $\Xi_F$ is much shorter than $\Xi_0$ and the singlet amplitudes diminish much faster than in the $I = 0.1$ case. For both $I = 0.1$ and $I = 0.5$, one can also see that the singlet amplitudes begin to rise from the $F/S$ interface and saturate in the $S$ side about one superconducting coherence length from the interface. This agrees with our previous work\(^5\). Another interesting feature seen in the $I = 0.5$ case is that the peak height of the $f_1$ component near the interface is not much higher than that of its other peaks, as happens with its $f_0$ counterpart. In other words, the subsequent peak heights in the $F$ regions are comparable to that of the peak nearest to the interface.

In delineating the role of triplet correlations in other experimentally relevant quantities, it is necessary to understand their time dependence. Due to the self-consistent nature of the proximity effects and the fact that the triplet condensate amplitudes are odd in time, their time dependence is in general nontrivial. We illustrate this in Fig. 9, where we show the spatial dependence of both the $m = 0$ and $m = \pm 1$ components of the triplet amplitude for several $\tau$. The parameters used here are the same as in the right panels ($I = 0.5$) of Fig. 8. For easier comparison with Fig. 8, we have again multiplied the normalized triplet amplitudes by a factor of 10. Figure 9 shows that at small times triplet correlations are generated only near the interface. (We have of course verified that they always vanish when $\tau = 0$). One can extract information about the proximity length from the growing increase of peak heights in the $F$ regions. The peak heights grow faster when they are deeper inside the ferromagnet. Moreover, Fig. 9 clearly demonstrates that the triplet correlations penetrate into $F$ regions as $\tau$ increases, in the range studied. More remarkably, the peaks of the $f_1$ component that are not nearest to the interface grow very fast in time and have heights that are comparable to the one nearest to the interface, consistent with our remarks in our discussion of Fig. 8. In contradistinction with the oscillating behavior of the triplet amplitudes in the $F$ regions, one can see that both $f_0$ and $f_1$ decay monotonically into the $S$ side without any oscillations. However, the triplet correlations still spread over in the $S$ regions at larger values of $\tau$ just as they do in the $F$ layer.

In the above paragraphs, we have discussed the long range nature and other properties of the triplet amplitudes in our system when the conical ferromagnet is very thick. In the following paragraphs, we will consider the proximity effect of induced triplet pairing correlations for smaller scale conical-ferromagnets. To quantify the effect we introduce a set of proximity lengths $L_{i,M}$ defined as:

$$L_{i,M} = \frac{\int_M dY |f_i(Y, \tau)|}{\max_M |f_i(Y, \tau)|}, \quad i = 0, 1 \quad M = S, F.$$ \hspace{1cm} (11)

Here the first index denotes the spin component, and the second index $M$ denotes the region in which the given function is evaluated. If the decays were exponential, these lengths would coincide with the characteristic length in the exponent. Obviously, in the present situation the decays are more complicated but the $L_{i,M}$ can easily be extracted numerically. They depend on $D_S$, $D_F$, $I$, and $\tau$. The range of $D_F$ we will consider is from
Λ to 3Λ. In Fig. 10, we plot these proximity lengths on both the F and S sides for three different values of I, at \( \tau = 4.0 \). The left panels show the \( f_0 \) proximity lengths and the right panels that extracted from \( f_1 \). Recall that \( I = 1 \) corresponds to the half-metallic limit. We consider first the F side (top two panels) One can clearly see that both \( L_{0,F} \) and \( L_{1,F} \) are correlated to the strength of the exchange fields. Fig. 10 displays a period of near \( \Delta/2 \) for both \( L_{0,F} \) and \( L_{1,F} \) at \( I = 1.0 \). We also see that the peak heights increase only slowly with increasing \( D_F \). Also, the locations of the maxima or minima of \( L_{0,F} \) are locations of minima or maxima, respectively, of \( L_{1,F} \). This is as one might expect from the rotating character of the field. On the other hand, for \( I = 0.1 \) or \( I = 0.5 \) the periodicity is not clear since, for reasons already mentioned, the intermingling of periodicities becomes more complicated. Overall the proximity lengths are larger than those in the half-metallic limit. However, one can still say that both \( L_{0,F} \) and \( L_{1,F} \) gradually increase, although with fluctuations, with \( D_F \).

The superconductor on the S side is intrinsically s-wave but because of the F layer, triplet correlations can be induced in it, near the interface, as seen in Figs. 8 and 9. Their decay, which is now monotonic, can be equally characterized by the proximity lengths defined in Eq. (11). Results are plotted in the bottom panels of Fig. 10. The minimum of \( L_{0,S} \) is, for all three values of \( I \), at \( D_F = 23 \) which is near 2A. The maxima of \( L_{0,S} \) for \( I = 0.1 \) are at \( D_F = 16 \) and \( D_F = 28 \) which are not far from 1.5Λ and 2.5Λ respectively. On the other hand, for \( L_{1,S} \) the maxima for \( I = 0.1 \) are at \( D_F = 19 \) and \( D_F = 31 \) and there is a minimum at \( D_F = 26 \). The locations of these maxima are still near 1.5Λ and 2.5Λ and they are only slightly different than what they are for \( L_{0,S} \) case. If one recalls the above discussion of Fig. 3, maxima of \( T_c \) occur when \( D_F \) is close to an integer multiple of \( \Lambda \). Since a higher \( T_c \) is correlated with a higher singlet pair amplitude, this suggests again that there exists a conversion between singlet and triplet Cooper pairs. The dependence of \( L_{0,S} \) and \( L_{1,S} \), at \( I = 1.0 \), on \( D_F \) is harder to characterize. This is because the high value of \( I \) reduces the scale of the overall proximity effect in S (i.e. the depletion of the singlet amplitude). At \( I = 0.5 \), one still finds that the approximate periodicity of \( L_{0,S} \) and \( L_{1,S} \) is about \( \Delta/2 \). The proximity lengths \( L_{0,S} \) and \( L_{1,S} \) are again anti-correlated at \( I = 0.5 \): the maxima (minima) locations of \( L_{0,S} \) are near the minimum (maximum) locations of \( L_{1,S} \).

![FIG. 10. (Color online) The proximity lengths \( L_{i,M} \) (see Eq. (11)) of the induced triplet pair amplitudes vs. \( D_F \) for different \( I \), at \( \tau = 4.0 \) and \( D_S = 150 \). The left panels show the proximity lengths \( L_{0,F} \) and \( L_{0,S} \) (from \( f_0 \) in the F and S regions) and the right panels \( L_{1,F} \) and \( L_{1,S} \), similarly extracted from \( f_1 \). The lines are guides to the eye.](image)
Recent experiments in systems that consist of two superconducting Nb electrodes coupled via a Ho/Co/Ho trilayer have revealed that the long range effect of triplet supercurrents was much more prominent at particular thicknesses of the Ho layers. The magnetic coherence length in Ho in the experiment was \( \sim 5 \) nm which would correspond in our notation to \( I \sim 0.1 \). In the experiment, the Ho thickness was symmetrically varied and the critical current, \( I_c \), at \( T = 4.2K \) was measured. Peaks of \( I_c \) corresponding to \( D_F = 0.5\Lambda \) and \( D_F = 2.5\Lambda \) were found. These experimental findings are consistent with our theory. Here, we have shown (see Fig. 10) that \( L_{1,S} \) has maxima near \( 1.5\Lambda \) and \( 2.5\Lambda \) when \( I = 0.1 \) in the \( D_F \) range we have considered. We found another maximum at \( D_F \sim 0.5\Lambda \), not included in the range shown. The penetration lengths associated with \( S \) are as important as those associated with \( F \) when discussing the triplet proximity effect, because the system can open up the corresponding channels only when both of them are long ranged. We believe that no obvious peak near \( 1.5\Lambda \) was observed because of the layout of their symmetric system. Therefore, one can conclude that the spiral magnetic structures play an important role in the triplet proximity effects. Both experiment and theory confirm that the existence of the long range proximity effects depends on the relation between the thickness of the magnetic layers and the wavelength of their magnetic structure.

Having seen in the previous two figures that triplet amplitudes may substantially pervade even rather thick Ho layers at moderate values of \( \tau \), it is of interest to investigate the \( \tau \) dependence of the proximity lengths in these nanoscale F/S systems for times roughly up to \( 2\pi \), in our dimensionless units. We therefore present in Fig. 11, the triplet proximity lengths as a function of \( D_F \) for \( I = 0.5 \), and at different values of \( \tau \). The panel arrangement is as in the previous figure. Thus, in the top panels where we plot \( L_{0,F} \) and \( L_{1,F} \), we see that both of them depend only weakly on \( \tau \), in the range considered. This is in part because of the relatively thin \( F \) layers included in the plot. The triplet amplitudes vanish at \( \tau = 0 \) but can saturate quickly through the \( F \) region as soon as \( \tau \) increases. In contrast, on the much thicker \( (D_S = 150) \) \( S \) side (bottom panels) both \( L_{0,S} \) and \( L_{1,S} \) increase with \( \tau \), as is consistent with expectation and previous work involving F/S systems with misaligned exchange fields. Furthermore, the overall shape of the proximity lengths on the \( S \) side does not change with \( \tau \) and only the magnitude evolves. Quite remarkably, the minima of \( L_{0,S} \) and \( L_{1,S} \) are very deep, and the value of these lengths at their minima is almost \( \tau \) independent and nearly the same at all minima in
the range plotted. The minima are separated by $\Lambda/2$. If one compares the left and right panels one can see that the locations of maxima in one approximately coincide with the position of minima in the other: the left and right panels are again complementary to each other as was the case with the plots in Fig. 10.

### D. Local magnetization and LDOS

Next, we discuss other important physical quantities that are related to the proximity effects including the local magnetization, $m(y)$, and the the local DOS (LDOS).

Considering that the ferromagnetism can drastically alter the superconductivity, one might wonder about the opposite case: how the local magnetizations behave near the $F/S$ interface. These so-called reverse proximity effects have been studied\textsuperscript{5,17,24,47–50} for a number of multilayer $F/S$ configurations with uniform exchange fields in each magnetic layer. Here, the space-varying exchange fields in $F$ oscillate in the $x-z$ plane and are constant along the $y$ direction (see Fig. 1). We computed the local magnetizations using Eq. (10) for $D_S = 150$, $D_F = \Lambda$ and three different values of $I$. The results are normalized in the usual\textsuperscript{24,50} way so that for a putative bulk $F$ material with a uniform internal field characterized by the parameter $I$ the quantity plotted would have the value $[(1 + I)^{3/2} - (1 - I)^{3/2}] / [(1 + I)^{3/2} + (1 - I)^{3/2}]$. In Fig. 12, each component of $m$ is shown in a separate panel and their behavior plotted throughout the whole spiral magnet region and some distance into $S$ near the interface. Consider first the $x$ component: The corresponding component of the internal field (see Eq. 1) vanishes at the outer interface ($Y = 0$) and goes smoothly to zero at the $F/S$ interface which in this case is at $Y = D_F = \Lambda$. As a consequence, one can see that the $m_x$ component undergoes a full period of oscillation in the $F$ material. The maximum and minimum values as a function of $I$ are numerically what they should be, given our normalization. However, as Fig. 12 clearly shows, the self-consistently determined $m_x$ does not vanish at the $F/S$ interface and instead penetrates a short distance inside $S$. This is a manifestation of the reverse proximity effect. For the other transverse component, $m_y$, the situation is more complicated. The field component $h_x$, out of phase with $h_y$, does not vanish smoothly at $Y = 0$ nor at $Y = D_F$. Therefore the corresponding $m_y$ component in $F$ is squeezed, and in addition to the peak at $Y = \Lambda/2$, which has the expected location and value, there are two smaller peaks at intermediate values. At the interface between materials, penetration of this component is appreciably more considerable than for $m_x$. The longitudinal component $m_z$, which is induced by the uniform $h_y$ component, behaves qualitatively as transverse components do in uniform ferromagnet $F/S$ structures.\textsuperscript{14} Penetration into the $S$ layer occurs over a relatively short distance, except at the smallest value of $I$ where it is relatively larger although the overall scale is of course smaller. The value of $m_y$ in the $F$ layer is again the expected one, consistent with our normalization.

Finally, we wish to discuss the LDOS. Here we will present results for the LDOS, as given in Eq. (9), summed over spins, integrated over either the $F$ or the $S$ layer, and normalized, as usual, to its value in the normal state of bulk $S$ material. The results are given in Fig. 13 where the energy scale of the horizontal axis is in units of the superconducting gap of bulk $S$ material, $\Delta_0$. The left panels of Fig. 13 show the LDOS integrated over the $F$ (top) and $S$ regions (bottom) for $D_F = \Lambda$, $1.5\Lambda$, and $2\Lambda$. The superconductor has a thickness $D_S = 1.5\Xi_0$ and $F$ has a relatively weak exchange field, $I = 0.1$. For
$D_F = \Lambda$, one can clearly see, for the integrated DOS in the $S$ side, peaks near $\varepsilon/\Delta_0 = \pm 1$ as in the ordinary bulk spectrum. There is additional subgap structure including proximity induced bound states at smaller energies followed by a very deep dip-nearly a minigap. Overall, the DOS structure contains traces of the familiar DOS for a pure bulk superconductor. On the $F$ side, the integrated LDOS at this value of $D_F$ still exhibits BCS-like peaks at $\varepsilon/\Delta_0 = \pm 1$ and subgap dip, but the whole structure is much weaker and the depth of the dip much smaller. It is indicative of the superconducting correlations present in the $F$ region. In contrast, the subgap superconducting features in the integrated LDOS for larger $D_F$ values ($D_F = 1.5\Lambda$ and $2\Lambda$) are much less prominent, although the peaks near $\varepsilon/\Delta_0 = \pm 1$ remain. Nonetheless, there are still shallow and discernible signatures in the gap region, on both the $F$ and $S$ layers. For these two larger values of $D_F$, the results (as compared on the same side) are remarkably similar. This is surprising at first since we have already seen that $T_c(D_F)$ has in this range of $I$ maxima near $D_F = \Lambda$ and $D_F = 2\Lambda$ and a local minimum at $D_F = 1.5\Lambda$ as shown in Fig. 3. From that, one might naively guess that the integrated LDOS for $D_F = 2\Lambda$ should behave as that at $D_F = \Lambda$, with a different integrated LDOS for $D_F = 1.5\Lambda$. This expectation is incorrect because, as one can see on a closer inspection of Fig. 3, $T_c$ at $D_F = \Lambda$ is higher than that at $D_F = 2\Lambda$ although both are near local maxima. Furthermore, $T_c$ at $D_F = 1.5\Lambda$ is closer to the $T_c$ value at $D_F = 2\Lambda$ than to that at $D_F = \Lambda$. Since $T_c$ is associated with the magnitude of the singlet pair amplitudes, in which the LDOS is indirectly correlated to, one should conclude that the LDOS corresponding to $D_F = 2\Lambda$ should be similar to $D_F = 1.5\Lambda$ rather than $D_F = \Lambda$. Indeed, the results confirm this notion.

On the right panels of Fig. 13, we present the integrated LDOS on both the $F$ and the $S$ sides for different exchange fields, $I = 0.1$, $I = 0.5$, and $I = 1.0$, at $D_F = 12 = \Lambda$. We see that when $I$ is increased from $I = 0.1$, the integrated LDOS on the $F$ side becomes quite flat (at the value $1/2|\left| (1+I)^{1/2} + (1-I)^{1/2} \right|$ as per our normalization) and essentially devoid of a superconducting signature. On the $S$ side, the integrated LDOS at $I = 0.5$ and $I = 1.0$ still retains some vestiges of the structure seen in the $I = 0.1$ case. However, the integrated LDOS at $I = 0.5$ on the $S$ side is slightly different than that at $I = 1.0$. The dip in at $I = 1.0$ is wider than for $I = 0.5$, in a way more superconducting-

![Graph](image-url)
like. What happens is that at larger values of $I$ the mismatch between the Fermi wavevector in $S$ and the Fermi wavevectors in the up and down bands in $F$ increases. This diminishes the penetration of the Cooper pairs into $S$ and hence the overall scale of the proximity effects. We recall that the overall dimensionless scale of the proximity effect in $F$ is roughly $\Xi F = 1/I$. Consequently, superconductivity is impaired in $S$ over a smaller scale when it is in contact with a stronger ferromagnet. Having said that, one might argue that at $I = 0.1$ the integrated LDOS on the $S$ side should have a smaller dip than the other two curves for stronger $I$. However, we have to consider here also the overall behavior of the $T_F$ vs. $I$ curves at constant $D_F$. This behavior is once again oscillatory but with a superimposed decay. The overall decay results in $T_F$ being higher at $I = 0.1$ than at either $I = 0.5$ or $I = 1$, but the oscillations produce a higher value of $T_F$ at $I = 1$ than at $I = 0.5$. This explains the progression of the curves. All the above discussion and results indicate that the LDOS can provide, if properly analyzed, another perspective and additional information on the superconducting nature of our bilayers.

IV. CONCLUSIONS

We have studied several aspects of proximity effects in $F/S$ bilayers, where the ferromagnet has a spiral structure characteristic of rare earths such as Ho, by numerically solving the self consistent BdG equations. We have calculated $T_c(D_F)$, the critical temperature as a function of magnet thickness, for different parameter values. The $T_c(D_F)$ curves exhibit a fairly intricate oscillatory behavior which is found to be related to both the strength $I$ (as they would for a uniform magnet) and the periodicity $\Lambda$ of the spiral exchange fields inherent in the magnet. As is the case for $F/S$ structures in which $F$ is uniform, we observe reentrant behavior with $D_F$ when $I$ is strong enough. The physical reason behind this $D_F$ reentrance in our bilayers is similar to that in ordinary $F/S$ structures but the additional periodicity associated with the magnet, which in many cases dominates the oscillations, makes the behavior more complicated. As a function of $D_S$, we find that $T_c(D_F)$ can also exhibit $D_F$ reentrance even at small $I$ when $D_S$ is of the order of the superconducting coherence length. The additional oscillations produced by the magnetic structure lead also to pure reentrance with temperature: superconductivity occurs in a finite temperature range $T_{c1} < T < T_{c2}$. An example of this reentrance at a very small $D_F$ ($D_F \sim 0.5\Lambda$) was previously\textsuperscript{38} presented. Here we report that this reentrance can also occur when $D_F > \Lambda$, where it should be experimentally easier to realize. To elucidate the physics underlying these reentrant phenomena, we have evaluated the singlet pair amplitudes and thermodynamic functions. The competition between condensation energy and entropy is responsible for the $T$ reentrance: the superconducting state may be, under certain circumstances, the high entropy state, leading to recovery of the normal state as $T$ is lowered. The calculated thermodynamic quantities are fully consistent with the $T_c(D_F)$ phase diagrams and the singlet pair amplitudes.

When the magnet has a spiral structure both the $m = 0$ and $m = \pm 1$ odd triplet components can be induced simultaneously. This is not the case in uniform-magnet bilayers: at least two uniform misaligned $F$ layers are needed to generate the $m = \pm 1$ component. We studied the odd triplet pair amplitudes in our bilayers, and found them to be long-ranged in both the $S$ and $F$ layers. We have analyzed the time delay dependence of the odd triplet amplitudes. The results are consistent with our previous work on both $F_1/S/F_2$ and $F_1/F_2/S$ trilayers, but the additional $\Lambda$ periodicity leads to important differences. We characterized the triplet long range behavior by introducing the appropriately defined lengths. We found that the relevant proximity length oscillates with $D_F$ and these oscillations depend on the strength and periodicity of the exchange field. Our methods are likely appropriate for many experimental conditions, as evidenced by the consistency of our results with recent tunneling experiments.\textsuperscript{33}

We have also considered the reverse proximity effects: the influence of the superconductivity on the magnetism. We found all three components of the local magnetization penetrate in slightly different ways into the $S$ layer. At larger $I$ this is a short-ranged phenomenon, but it is otherwise for weak magnetism. Both $m_x$ and $m_z$ oscillate in the $F$ regions to reflect the spiral exchange field. Finally, the calculated LDOS reveals important information and discernible signatures linked to the proximity effects in these bilayers and are correlated to the superconducting transition temperatures.

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55455
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