A MATHEMATICAL MODEL FOR MARINE DINOFLAGELLATES BLOOMS

M. DAMBRINE AND B. PUIG

Université de Pau et des Pays de l’Adour
E2S UPPA, CNRS, LMAP, Pau, France

G. VALLET

Université de Pau et des Pays de l’Adour
UPPA, CNRS, LMAP, Pau, France

Abstract. We present a model for the life cycle of a dinoflagellate in order to describe blooms. We prove the mathematical well-posedness of the model and the possibility of extinction in finite time of the alga form meaning that the full population is under the cysts from.

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1. Introduction. An algal bloom is a rapid increase or accumulation of the population of algae in freshwater or marine water systems. A harmful algal bloom is an algal bloom that causes negative impacts to other organisms via production of natural toxins. Harmful algal blooms caused by the dinoflagellates *Alexandrium catanella* occur every spring on the west coast of South Africa and less often of Chile. These blooms of organisms cause severe disruptions in fisheries of these waters as the toxins in the phytoplankton cause filter-feeding shellfish in affected waters to become poisonous for human consumption. The modelling of such event is therefore an important question in order to monitor such events. Existing models written by biologists ([4, 9]) focus on the bloom events and do not try to describe the full life-cycle of the dinoflagellate.

In this work, we propose a simplified model for the complete cycle of the alga taking into account the various forms of the alga. This model allows to describe the after bloom phenomenon: the algal form completely disappears and the whole population has evolved into a cyst form.

The interaction with the environment is assumed simple: the alga has no action on the environment. Of course, this is questionable with respect to the usual growth law for bacteria. Usually, one considers that it follows Monod’s law. As it would require to enrich the model with equations for diluted nitrates and phosphates that would increase the complexity of the model for the alga form and not modify the interactions between the life phase of the alga, we choose here to use a simple logistic term to model this law.

We present the model in Section 2. In Section 5, we state the results concerning the well-posedness of our model is well-posed. We also prove that the model recovers a characteristic feature of the life cycle of the alga: if the environmental conditions becomes bad, the alga form can completely disappear from the water and the species is only present under the form of cysts that will eventually germinate and give birth to a new generation. In Section 6, we prove a result of existence, uniqueness of solutions and their stability with respect to the environmental conditions.

One of the main characteristic of a bloom is the complete disappearance of the alga form between two events. If the environmental conditions becomes unfavorable, the alga cannot survive and its strategy is then to transform itself under the form of a resting cysts. Any model aiming at addressing a bloom phenomenon should be able to recover an extinction in finite time. We prove in Section 7 that the proposed model can recover such a phenomenon.

2. The model.

2.1. The compartments and the notations. Taking into account at the same time the full life cycle of the dinoflagellates, the hydrodynamic and the environmental model is too complex. Interested readers can refer for the vast biological literature (see for example [1, 10, 4, 9]). Hence, we make the following assumptions:

- the environmental conditions and the environment variables are given as functions of time and space.
- we consider only three different forms for the species: the algal form that is active and two distinct states for the inactive cysts: one is resting and the second one is mature and can reactivate itself when the environmental conditions are favorable. Notice that for technical reasons, we distinguish the resting cysts dropping in the water column from those laying at the bottom of the ocean, in the sediments.
Hence, we consider a four compartments model as follows:

\[
\begin{array}{c}
\text{Mature Cysts} \quad [C_{\text{mat}}] \quad \text{Alga} \quad [u] \quad \text{New Cysts} \quad [C_{\text{new}}] \quad \text{Resting Cysts} \quad [C_{\text{rest}}] \quad \ldots
\end{array}
\]

**Figure 1.** Sketch of the parts of *Alexandrium catenella* life we are interested in.

Our objective is to provide a model that explains the event from the germination of old cysts to the deposit of new formed cysts on the bottom of the ocean.

Let us start by defining the domain of interest. We should consider the evolution during a time period ranging from \( t = 0 \) to some \( T > 0 \). During that time, \( D \) denotes the domain occupied by the water and we assume that it is a bounded Lipschitz-domain of \( \mathbb{R}^3 \). As usually, \( Q \) denotes \((0,T) \times D\).

The boundary of \( D \) is split into three parts depending on its geographic nature. One corresponds to the upper surface corresponding to the water-air interface, we will denote it by \( \partial D^a \). A second one corresponds to the bottom of the ocean and a earth-water interface, we denote it by \( \partial D^e \). The last one corresponds to the interface with the rest of the ocean, we denote it by \( \partial D^w \). Notice that the vector \( \mathbf{n}(x) \) stands for the normal vector to \( D \) at the point \( x \) oriented to the exterior of \( D \). In the sequel, let us denote by \( C_{\text{mat}} \) the surface concentration of old cysts on the bottom of the ocean, by \( u \) the concentration of alga in the water and \( C_{\text{new}} \) the concentration of new cysts in the ocean.

2.2. **Deriving the equations for each compartment.**

2.2.1. *Equations for the mature cysts on the bottom of the ocean.* Let us first provide the equation satisfied by the mature cysts \( C_{\text{mat}} \). We neglect the near-bottom circulation and dispersion of sediment containing cysts and we assume that they are fixed to the sediments at the bottom of the ocean. The mature cysts can germinate to provide new algae that may lift to the surface. We therefore get the initial condition

\[
C_{\text{mat}}(0, \cdot) = C_{\text{0,mat}} \quad \text{in} \quad \partial D^e, \quad (1)
\]
and the equation:
\[
\frac{d}{dt} C_{\text{mat}} = -(\alpha + \mu_{\text{mat}}) C_{\text{mat}} + f \quad \text{in } (0,T) \times \partial D^e.
\]
(2)
Here, the function \(\alpha\) is the germination rate, depending on environmental conditions (temperature, salinity,...), it is measurable in time and space and there exists a positive constant \(\alpha_{\text{max}}\) such that \(\alpha_{\text{max}} \geq \alpha \geq 0\); \(\mu_{\text{mat}}\) stands for the natural mortality of cysts, it is assumed to be a positive constant. The function \(f\) stands for the arrival of resting cysts becoming mature, it will be described at the end of the model. Nevertheless, \(f\) is assumed to be a non-negative measurable function, bounded by a positive constant \(f_{\text{max}}\).

2.2.2. Equations for the alga. The alga, generated at the bottom then rose in the water column, is transported by the oceanic stream, grows and may blooms. We choose the same maturity \(u\) for all those stages and a logistic growth model is considered to yield the equations
\[
u(0,\cdot) = u_0 \quad \text{in } D, \tag{3}
\]
\[
\frac{\partial}{\partial t} u + \text{div} \left( u V - \kappa_a \nabla u \right) = ru \left( 1 - \frac{u}{u_m} \right) - \beta u \quad \text{in } Q. \tag{4}
\]
The parameters in the problem are:

- \(V\) is the sum \(V_w + V_a\) where \(V_w\) denotes the velocity of the water where the algae live and \(V_a\) stands for the average relative own speed of the alga with respect to the water. The velocity of the water is assumed to be known in this work, provided by a hydrodynamical model meaning that we consider that the presence of the alga does not modify the behavior of the water. It is divergence free. The vector \(V_a\) is \textit{a priori} assumed to be a vertical vector oriented to the top slowing down close to the upper surface. Note that this field is not divergence free since the alga cannot escape into the air.
- \(\kappa_a > 0\) is the diffusive coefficient of the algae in the water which corresponds to the self propulsion property of the alga. According to [10], it is such that the Peclet number is of order 10. This means that the transport is a dominant phenomenon.
- \(r \geq 0\) is the growth rate of the algae. It is a biological parameter determined experimentally, depending on the environment. In the case of interest, the environment influences the algae through the light intensity, the salinity, the concentration of nutrient and the temperature. Hence, it is a function depending on both time and space.
- \(u_m > 0\) is the natural capacity of the ocean with respect to the algae. It is also a biological parameter that depends on the environment. Hence, it is a function depending on both time and space.
- \(\beta\) denotes the rate of evolution of the alga into new cysts. This is also a space-time dependent parameter.
- \(u_0\) is the initial density of algae. It is therefore a nonnegative function, satisfying \(u_0(x) \leq u_m(0,x)\) in a natural situation: i.e. not a bloom situation.

3. Assumptions on the parameters and comments. We make the following regularity assumptions on the biological parameters : \(u_m, r, \beta \in L^2(Q)\). Concerning the field \(V\), we assume
\[
V \in L^2(0,T,H^1(D))^3 \quad \text{with } V \cdot n \leq 0 \text{ on } (0,T) \times \partial D^e.
\]
(5)
The regularity $V \in L^2(0, T, H^1(D))$ is mainly to give a sense to the trace of $V \cdot n$ at the bottom of the ocean $\partial D^e$, it could have been a $H(\text{div})$ one. Usually, in hydrodynamical models, one assumes that $V_w \cdot n = 0$ on that part of the boundary. Therefore, the condition $V \cdot n \leq 0$ means that, when germinating, cysts enter $D$ via the fact that $V_a \cdot n < 0$ on $\partial D^e$, else, they stay in the earth and $V_a \cdot n = 0$ too.

Additional assumptions on the velocity will be used for the boundedness of $u$:

$$\text{div} V \in L^\infty(Q) \quad \text{and} \quad V \cdot n \leq 0 \text{ on } (0, T) \times \partial D^a. \quad (6)$$

The assumption on the divergence avoids local excessive concentrations of population in $D$. Since $V_w$ is divergence free, it concerns mainly the algae velocity. The boundary assumption forbids the escape of algae into the air. Again, one may have $V_w \cdot n = 0$ on $\partial D^w$ and $V_a \cdot n \leq 0$ means that the vertical rise of algae decreases and even cancels when approaching the surface.

We also make some boundedness assumptions for the parameters of the system: there are numbers $V_\infty, u_{\text{min}}, u_{\text{max}}, r_{\text{max}}, k_{\text{max}}, k_{\text{min}}, \beta_{\text{max}}, \beta_{\text{min}} > 0$ with

$$|V| \leq V_\infty, \quad u_{\text{min}} \leq u_m \leq u_{\text{max}}, \quad 0 \leq r \leq r_{\text{max}},$$

$$0 < k_{\text{min}} \leq k \leq k_{\text{max}} \quad \text{and} \quad 0 \leq \beta \leq \beta_{\text{max}}. \quad (7)$$

4. Boundary conditions for the alga. We need to complete these equations by boundary conditions. On the earth-water interface $\partial D^e$, the only phenomenon is the germination of mature cysts and no other aquatic organism crosses the air-water interface $\partial D^a$:

$$[uV - \kappa_a \nabla u] \cdot n = \alpha C^{\text{mat}} V \cdot n \quad \text{on } (0, T) \times \partial D^e, \quad (8)$$

$$[uV - \kappa_a \nabla u] \cdot n = 0 \quad \text{on } (0, T) \times \partial D^a. \quad (9)$$

To state the boundary condition of the interface $\partial D^w$ with the rest of the ocean is more difficult. We assume that the algae can cross this artificial boundary and that we know the concentration of algae there.

Hence, a Dirichlet condition depending on time and space is considered for $u$ on this interface via

$$u = g \quad \text{on } (0, T) \times \partial D^w. \quad (10)$$

As assumed, as for $u_0$, that $g$ is a measurable function, satisfying $0 \leq g \leq u_m$ in a natural situation: $i.e.$ not a bloom situation. One will consider in the sequel that if a bloom occurs, it will be produced by an abnormal rate of germination inside the studied coastal zone. Thus, one assumes that $D$ is big enough to be able to consider that $g \leq u_m$ on $(0, T) \times \partial D^w$. Of course, this is an arbitrary choice. However this boundary is artificial, only introduced to bound the computational domain. In practical situations, one can not expect to measure data on this boundary. Nevertheless, it can be provided by numerical simulations on a larger domain.

4.0.1. Equations for the new cysts falling to the ground. The newly created cysts are more dense than the water and fall due to gravity. Their own diffusion can be neglected and we assume that they reach immediately their terminal velocity and that their relative speed $V_c$ with respect to the water is downward and constant. Let us emphasize that the cysts do not enjoy the self propulsion property of the algae. Therefore, there is no diffusion term in the equation for the cysts and we get

$$\frac{\partial}{\partial t} C^{\text{new}} + \text{div} \left(C^{\text{new}}(V_w + V_c)\right) = \beta u \quad \text{in } (0, T) \times D. \quad (11)$$
Let us emphasize that this equation is not parabolic but hyperbolic. The notations of the parameters are similar to the ones of the algae: $V_c$ introduced above is the self motion capacity of the new cysts, it is a downward constant field, so it is divergence free; $\beta \geq 0$ already introduced denotes the rate of evolution of the alga into new cysts.

We need to complete the system by an initial condition and boundary conditions. Initially, there are $C_{\text{new}}^{\text{new}} \in L^\infty(D)$ new formed cysts so the initial condition is

$$C_{\text{new}}^{\text{new}}(0, \cdot) = C_{\text{new}}^{\text{new}} \geq 0 \text{ in } D. \quad (12)$$

On the air-water interface $\partial D^a$, there is a need of a boundary condition since the characteristics curves are entering due to $V_c$: since there are no cysts in the air, one just have

$$C_{\text{new}} = 0 \text{ on } (0, T) \times \partial D^a. \quad (13)$$

On the artificial water/water interface, depending on the sign of $V_w \cdot n$, the characteristics are either incoming or outgoing. Hence, the following Dirichlet boundary condition has to be understood in the hyperbolic sense: that is to say, considered only when $[V_w + V_c] \cdot n < 0$.

$$C_{\text{new}} = g_{C_{\text{new}}} \text{ on } (0, T) \times \partial D^w, \quad (14)$$

where $g_{C_{\text{new}}} \in L^\infty((0, T) \times D)$ is a non-negative function related to an a priori amount of known new cysts sedimenting on the artificial water boundary. By assumptions on $V_w$ and $V_c$, the characteristics are outgoing on $(0, T) \times \partial D^c$ and no boundary condition is needed.

4.0.2. Equations for the resting cysts. Once the cysts lay on the ground, they settle in the sediments and rest in order to become mature and to be able to transform themselves into algae again [1]. To take this phenomenon into account, we propose a age structured description of the cysts following McKendrick equation (see e.g. [7, §. 1.2.1 p.13]). Let $C_{\text{rest}}(t, a, x)$ be the population of resting cysts of age $a \in (0, A)$, at time $t$, and position $x$ at the bottom of the ocean $\partial D^r$. The generation comes by the sedimentation

$$C_{\text{rest}}(\cdot, 0, \cdot) = C_{\text{new}}V_c \cdot n \text{ in } (0, T) \times \partial D^c. \quad (15)$$

Then, the evolution is given by the transport equation:

$$\partial_t C_{\text{rest}} + \partial_a C_{\text{rest}} = -\mu_{\text{rest}} C_{\text{rest}} \text{ in } (0, T) \times (0, A) \times \partial D^c, \quad (16)$$

where $\mu_{\text{rest}}$ is a nonnegative mortality rate and the initial condition is given by

$$C_{\text{rest}}(0, \cdot) = C_{\text{rest}}^0 \text{ in } (0, A) \times \partial D^c \quad (17)$$

where $C_{\text{rest}}^0 \in L^\infty((0, A) \times \partial D^c)$ is assumed to be non-negative. The previous equation is nothing but a differential equation of the first order along its characteristic curves. When the resting cysts reach the age of $A$ it is mature again so that the source term $f$ in (2) is given by

$$f(t, x) = C_{\text{rest}}(t, A, x). \quad (18)$$

This relation closes the equations.
4.1. The final system of equations. To sum up, the system to solve is:

\[
\begin{align*}
\frac{d}{dt} C^{\text{mat}}(t) &= -(\alpha + \mu_{\text{mat}}) C^{\text{mat}}(t) + C^{\text{rest}}(t, A, \cdot) \text{ in } (0, T) \times \partial D^e, \\
\frac{\partial}{\partial t} u + \text{div} \left( u \mathbf{V} - \kappa_a \nabla u \right) &= ru \left( 1 - \frac{u}{u_m} \right) - \beta u \text{ in } (0, T) \times D, \\
\frac{\partial}{\partial t} C^{\text{new}} + \text{div} \left( C^{\text{new}}(\mathbf{V}_w + \mathbf{V}_c) \right) &= \beta u \text{ in } (0, T) \times D, \\
\frac{\partial}{\partial t} C^{\text{rest}} + \frac{\partial}{\partial a} C^{\text{rest}} &= -\mu_{\text{rest}} C^{\text{rest}} \text{ in } (0, T) \times (0, A) \times \partial D^e,
\end{align*}
\]  

for the boundary conditions

\[
\begin{align*}
[u \mathbf{V} - \kappa_a \nabla u] \cdot \mathbf{n} &= \alpha C^{\text{mat}} \mathbf{V} \cdot \mathbf{n} \text{ on } (0, T) \times \partial D^e, \\
[u \mathbf{V} - \kappa_a \nabla u] \cdot \mathbf{n} &= 0 \text{ on } (0, T) \times \partial D^a, \\
u &= g \text{ in } \partial D^w, \\
C^{\text{new}} &= 0 \text{ on } \partial D^a, \\
C^{\text{new}} &= g C^{\text{new}} \text{ on } \partial D^w, \\
C^{\text{rest}}(t, 0, \cdot) &= C^{\text{new}}(t, 0, \cdot) \mathbf{V}_c \cdot \mathbf{n} \text{ in } (0, T) \times \partial D^e,
\end{align*}
\]

and the initial conditions

\[
\begin{align*}
C^{\text{mat}}(0, \cdot) &= C^{\text{mat}}_0 \geq 0 \text{ on } \partial D^e, \\
u(0, \cdot) &= u_0 \geq 0 \text{ in } D, \\
C^{\text{new}}(0, \cdot) &= C^{\text{new}}_0 \geq 0 \text{ in } D, \\
C^{\text{rest}}(0, \cdot) &= C^{\text{rest}}_0 \geq 0 \text{ in } (0, A) \times \partial D^e.
\end{align*}
\]

5. Study of the system (19): the results.

5.1. Strategy for solving (19). For biological reasons, we are interested in time intervals of size less than \( A \). Indeed, we consider the evolution of the situation on a season while the maturation time \( A \) is much larger, typically a couple of years. This is justified by the variation of environmental conditions that allows alga formation only for short time periods during the year. In this section, we consider the system (19) on the time interval \([0, T]\) for \( T < A \).

This point will be crucial in the resolution since it allows to solve the system time-step by time-step. Indeed, the feature of that alga lies in a delay between the time a new cyst sediments and the time it becomes mature and is able to germinate. It has first to achieve a certain maturation, denoted \( A \) in our model, following the transport equation (16).

Therefore, our strategy to prove the result of existence and uniqueness of the solution to our system is the following program:

**Step 1:** there exists a unique \( C^{\text{mat}} \) based on the data \( C^{\text{rest}}_0 \) and \( C^{\text{mat}}_0 \);

**Step 2:** there exists a unique (bounded) solution \( u \) based on given data and the knowledge of \( C^{\text{mat}} \);

**Step 3:** there exists a unique solution \( C^{\text{new}} \) based on given data and the knowledge of \( u \);

**Step 4:** there exists a unique solution \( C^{\text{rest}} \) based on given data and the knowledge of \( C^{\text{new}} \).

We now state the results of the paper following each of these steps. The proofs are postponed in Section 6.
5.2. Main results.

5.2.1. Dealing with the resting, sedimenting and mature cysts - steps 1, 3 and 4. Equations concerning \( C^{\text{rest}} \) and \( C^{\text{new}} \) are transport equations with \( L^\infty \) data and with a regular and divergence-free velocity vector field. Therefore, the well-posedness theory is known. The equation concerning \( C^{\text{mat}} \) is an ordinary differential equation with source term \( f = C^{\text{rest}}(\cdot, A, \cdot) \), depending only on \( C^{\text{rest}}_0 \). The first lemma states that once the sedimentation of new cysts \( C^{\text{new}} \) is known, the equations for the resting cysts can be solved.

**Lemma 5.1** (Existence for the resting cysts). For any \( T < A \) and \( C^{\text{new}}_0 \) given, Problem (15)-(16)-(17) has a unique solution \( C^{\text{rest}} \), given by (22). This solution belongs to \( L^\infty((0,T) \times (0,A) \times \partial \mathcal{D}^e) \). Moreover, its trace \( C^{\text{rest}}(\cdot, A, \cdot) \) on \( \{a = A\} \) lays in \( L^\infty((0,T) \times \partial \mathcal{D}^e) \) with the bounds

\[
\|C^{\text{rest}}(\cdot, A, \cdot)\|_{L^\infty((0,T) \times \partial \mathcal{D}^e)} \leq \|C^{\text{rest}}_0\|_{L^\infty((0,A) \times \partial \mathcal{D}^e)}.
\]

Its trace \( C^{\text{rest}}(T, \cdot, \cdot) \) on \( \{t = T\} \) belongs to \( L^\infty((0,A) \times \partial \mathcal{D}^e) \) with the bound

\[
\|C^{\text{rest}}(T, \cdot, \cdot)\|_{L^\infty((0,A) \times \partial \mathcal{D}^e)} \leq \|C^{\text{rest}}_0\|_{L^\infty((0,A) \times \partial \mathcal{D}^e)} + c(V_c)\|C^{\text{new}}\|_{L^\infty((0,T) \times \partial \mathcal{D}^e)}.
\]

Using the fact that the source term \( f \) of the equation concerning \( C^{\text{mat}} \) depends only on \( C^{\text{rest}}_0 \), one can state the following lemma:

**Lemma 5.2** (Existence for the mature cysts). For any \( T < A \), \( C^{\text{mat}}_0 \in L^\infty(\partial \mathcal{D}^e) \) and \( C^{\text{rest}}_0 \) given in the above lemma, Problem (2)-(1)-(18) has a unique solution \( C^{\text{mat}} \) in \( W^{1,2}(0,T, L^2(\partial \mathcal{D}^e)) \) with the bounds

\[
\|C^{\text{mat}}\|_{L^\infty((0,T) \times \partial \mathcal{D}^e)} \leq \|C^{\text{mat}}_0\|_{L^\infty(\partial \mathcal{D}^e)} + T\|C^{\text{rest}}_0\|_{L^\infty((0,A) \times \partial \mathcal{D}^e)}
\]

and

\[
\frac{d}{dt}C^{\text{mat}} \|_{L^\infty((0,T) \times \partial \mathcal{D}^e)} \leq C(T, \mu_{\text{max}}, \rho_{\text{mat}}) \left( \|C^{\text{mat}}_0\|_{L^\infty(\partial \mathcal{D}^e)} + \|C^{\text{rest}}_0\|_{L^\infty((0,A) \times \partial \mathcal{D}^e)} \right).
\]

Concerning the sedimenting cysts \( C^{\text{new}} \), the transport field is more complex. However, it is regular and divergence free. We apply Boyer's results [2] to determine \( C^{\text{new}} \) if the alga density \( u \) is known.

**Lemma 5.3** (Existence for the sedimenting cysts). For any \( T < A \) and \( u \) given, there exists a unique couple \( (C^{\text{new}}, \gamma C^{\text{new}}) \), weak solution to (11), with boundary conditions (13)-(14) and initial condition (12). This weak solution \( C^{\text{new}} \) belongs to \( L^\infty((0,T) \times \mathcal{D}) \) and \( \gamma C^{\text{new}} \) its trace on \( (0,T) \times \partial \mathcal{D}^e \) lays in \( L^\infty((0,T) \times \partial \mathcal{D}, |(V_w + V_c) \cdot \mathbf{n}|^2 d\mathcal{H}^d) \) where, \( |(V_w + V_c) \cdot \mathbf{n}|^2 d\mathcal{H}^d \) denotes the \( d \)-dimensional Hausdorff measure with weight \( |(V_w + V_c) \cdot \mathbf{n}|^2 \).

Moreover, it satisfies the upper bound

\[
0 \leq C^{\text{new}} \leq \max \left[ \|C^{\text{new}}_0\|_{\infty} + \|\gamma C^{\text{new}}_0\|_{\infty} \right] + \beta_{\text{max}} \int_0^T \|u(t)\|_{\infty} dt.
\]

5.2.2. Dealing with the alga - step 2. Let us consider the equation for the alga. We first introduce some notations. Since a non-homogeneous Dirichlet boundary condition is considered on \( \partial \mathcal{D}^w \), one denotes by

\[
V = \{u \in H^1(\mathcal{D}), \ u = 0 \text{ on } \partial \mathcal{D}^w \} \text{ and } H = L^2(\mathcal{D})
\]
with Poincaré’s norm on $V$ and by $\bar{u}$ a suitable Dirichlet lift. The solution $u$ is a priori in $H^1(D)$ with the condition that $u - \bar{u} \in V$ and we are interested in the resolution of the following advection reaction diffusion problem:
\[ \frac{d}{dt} u(t) + A(t)u = F(u)(t) \]
where $A(t) : H^1(D) \to H^1(D)'$ denotes the operator defined for any $u, v \in H^1(D)$ by
\[ \langle A(t)u, v \rangle = \int_D \{ \kappa_a(t, \cdot) \nabla u \cdot \nabla v - u \mathbf{V}(t, \cdot) \cdot \nabla v + \beta(t, \cdot) uv \} \, dx \]
and $F$ is the Nemytskii-type operator associated with $f : Q \times \mathbb{R} \to \mathbb{R}$, the function defined by
\[ f(t, x, \lambda) = r(t,x)\lambda \left( 1 - \frac{\lambda}{u_m(t,x)} \right) . \]

Note that $A(t)$ is associated with a time-family of bilinear forms
\[ a(t, \cdot) : (u, v) \in (H^1(D))^2 \mapsto \langle A(t)u, v \rangle \]
satisfying, thanks to the assumptions on the parameters (cf. Lemma 6.2), the framework of [3, section 1.2 p.509]. As usual, one considers
\[ W(0, T) = \{ u \in L^2(0, T, V) \text{ with } \partial_t u \in L^2(0, T, V') \} \]
where Poincaré’s norm $\| \cdot \|_V : u \mapsto \| \nabla u \|_{L^2(D)^3}$ is considered on $V$ and the norm on $W(0, T)$ is the graph norm
\[ \| \cdot \|_W : u \mapsto [\| u \|^2_{L^2(0,T,V)} + \| \partial_t u \|^2_{L^2(0,T,V')} ]^{1/2} . \]

As mentioned previously, the problem is with non-homogeneous boundary conditions. So, one needs to assume that the initial condition and the boundary conditions are compatible in the sense that there exists at least
\[ \bar{u} \in L^2(0, T, H^1(D)) \cap C([0, T], L^2(D)) \text{ with } \partial_t \bar{u} \in L^2(0, T, V') \]
satisfying the trace conditions, that is to say
\[ \bar{u}(0,x) = u_0(x), \quad \text{in } D; \quad \bar{u}(t,x) = g(t,x), \quad \text{on } (0, T) \times \partial D^w, \]
with $0 \leq u_0 \leq u_m(0, \cdot) \text{ in } D \quad \text{and} \quad 0 \leq g \leq u_m \text{ in } (0, T) \times \partial D^w$. \tag{21}

Let us note that the regularity $\bar{u} \in C([0, T], L^2(D))$ may be implicit to $\bar{u} \in L^2(0, T, H^1(D))$ with $\partial_t \bar{u} \in L^2(0, T, V')$
even if $\bar{u}$ is not in the classical framework of $W(0, T)$ (see e.g. [6]); but we add it for convenience.

We are now in a position to give the definition of a solution.

**Definition 5.4.** A solution to the algae problem is any $u \in L^2(0, T, H^1(D))$ with $0 \leq u \text{ a.e. in } Q$ such that $u - \bar{u} \in W(0, T)$, $F(u) \in L^{3/2}(Q)$ and satisfying $t \text{ a.e. in } (0, T)$
\[ \forall w \in V, \quad \langle \partial_t u, w \rangle_{V', V} + a(t, u, w) = \int_D F(u) w \, dx - \int_{\partial D^w} \alpha C^{\text{mat}} \mathbf{V} \cdot \mathbf{n} \, w \, d\mathcal{H}^2(x), \]
and the initial condition $u(t = 0) = u_0$. 


Let us note that formally, the variational formulation is obtained by multiplying Equation (4) by a test function \( w \), integrating over the spatial domain and integrating by parts thanks to the boundary conditions. Note also that since \( u - \bar{u} \in W(0,T) \rightarrow C([0,T], L^2(D)) \), \( u \in C([0,T], L^2(D)) \) and the initial condition \( u(t=0) = u_0 \) makes sense in \( L^2(D) \).

Our main result of this sub-section is then.

**Theorem 5.5.** Under the assumptions (5), (7) and (21), there exists a unique solution to the algae problem (4) in the sense of Definition 5.4. Moreover, if \( u_1 \) and \( u_2 \) denote two given solutions associated to different data: \( u_0^i, \alpha_i \) and \( c_{\text{mat}}^i, i = 1, 2 \), satisfying the assumptions of the model, a positive constant \( C(T) \) exists such that

\[
\|u_1 - u_2\|^2_{C([0,T], L^2(D))} + \|u_1 - u_2\|^2_{L^2(0,T,H_1(D))} \\
\leq C(T) \left[ \|u_1^0 - u_2^0\|^2_{L^2(D)} + \|\alpha_1 c_{\text{mat}} - \alpha_2 c_{\text{mat}}\|^2_{L^2(0,T,L^2(\partial D^c))} \right].
\]

Assuming in addition (6) and \( c_{\text{mat}} \) bounded, the solution \( u \) is bounded by a constant \( M \) depending on \( T \) and \( \|c_{\text{mat}}\|_{\infty} + T\|c_{\text{mat}}^i\|_{\infty} \).

5.2.3. About global existence. It is possible to solve the problem over a long time \( T_1 \) by splitting the time interval \((0, T_1)\) in time intervals \((kA/2, (k + 1)A/2)\) and by considering as time initial condition on \((kA/2, (k + 1)A/2)\), the time ending condition on \((k - 1)A/2, kA/2)\).

6. Proofs.

6.1. Equations for the resting, sedimenting and mature cysts.

**Proof of Lemma 5.1:** The McKendrick (16) equation can be solved by the method of characteristics. The characteristics are all parallel lines of equation \( a(t) = t + a_0 \), \( a_0 \in (-T, A) \) (since \( T \) is assumed smaller than \( A \)) in the \((t, a)\)-plane. Thus, one has to solve the ordinary differential equation

\[
\frac{d}{dt} C_{\text{rest}}(t, t + a_0, x) = -\mu_{\text{rest}} C_{\text{rest}}(t, t + a_0, x).
\]

Therefore, the solution is given by

\[
\begin{align*}
C_{\text{rest}}(t, a, x) &= C_{\text{rest}}^0(a - t, x)e^{-\int_0^t \mu_{\text{rest}}(s, x)ds} & \text{ in } \{ a > t \}, \\
C_{\text{rest}}(t, a, x) &= (C_{\text{new}} V_e \cdot n)(t - a, x)e^{-\int_{t-a}^t \mu_{\text{rest}}(s, x)ds} & \text{ in } \{ a < t \}, \quad (22)
\end{align*}
\]

thanks to (15) with a possible shock along the line \( \{ t = a \} \). Since \( A > T \), one gets back the fact that the solution at age \( a = A \) is only given by the knowledge of \( C_{\text{rest}}^0 \).

Then, the solution at age \( a < t \) is given by the knowledge of \( C_{\text{new}} \) then one can check that \( C_{\text{rest}}(t, A, x) \geq 0 \). The estimates are direct consequences of (22).

**Proof of Lemma 5.2:** As a byproduct, the source term \( f = C_{\text{rest}}^0(\cdot, \cdot, \cdot) \) in equation for mature cysts (2) is nonnegative. For a fixed position \( x \), Equation (2) is an ordinary differential equation that can be solved, for any \( t \in [0, T] \) and almost any \( x \in \partial D^c \), by

\[
C_{\text{mat}}(t, x) = C_{\text{mat}}^0(x)e^{-\int_0^t \alpha(s, x)ds - \mu_{\text{mat}}t} + \int_0^t e^{-\int_s^t \alpha(s, x)ds - \mu_{\text{mat}}(t-s)} C_{\text{rest}}(\cdot, A, \cdot)(s, A, x)ds \geq 0. \quad (23)
\]
Assuming that the initial condition $C_0^{\text{mat}}$ is in $L^\infty(\partial D^c)$, the previous formula indicates that $C^{\text{mat}} \in W^{1,2}(0,T,L^2(\partial D^c))$ with the bounds
\[
\|C^{\text{mat}}\|_{L^\infty((0,T) \times \partial D^c)} \leq \|C_0^{\text{mat}}\|_{L^\infty(\partial D^c)} + T\|C^{\text{rest}}(\cdot, A, \cdot)\|_{L^\infty((0,T) \times \partial D^c)}
\]
and
\[
\frac{d}{dt} C^{\text{mat}} \|_{L^\infty((0,T) \times \partial D^c)} 
\leq C(T, \alpha_{\max}, \mu_{\text{mat}}) \left[ \|C_0^{\text{mat}}\|_{L^\infty(\partial D^c)} + \|C^{\text{rest}}(\cdot, A, \cdot)\|_{L^\infty((0,T) \times \partial D^c)} \right].
\]
Then to deal with the first non-linear problem with a linearized problem through Lions theorem [3, Th. 1 & 2, p.512-513].

The lemma holds by using the first estimate of Lemma 5.1.

**Proof of Lemma 5.3:** Indeed, the main part of the lemma is a consequence of [2, Th. 4.1,Th. 3.1]. The estimates are given by [2, Lem. 4.2] for the majoration and [2, Prop. 4.1] for the minorization since the data are non-negative and 0 is the unique solution corresponding to null data.

### 6.2. Dealing with the equations for the alga. Proof of theorem 5.5.

The problem will be solved by using a usual fixed point strategy: we first study the linearized problem through Lions theorem [3, Th. 1 & 2, p.512-513]. Then to deal with a first non-linear problem with $F^M$, a Lipschitz-continuous perturbation of function $F$ is considered and, finally, making use of a maximum principle and a priori estimates, one will recover the general case.

**Step 1: The linearized problem.**

Let us consider the new unknown $v = u - \bar{u} \in W(0,T)$ and set
\[
\psi : w \in V \mapsto -(\partial_t \bar{u} - A(t)\bar{u}, w)_{V',V} - \int_{\partial D} \alpha C^{\text{mat}} V \cdot n w \, d\mathcal{H}^2(x) \in L^2(0,T,V').
\]

Then, one has that

**Proposition 6.1.** $u$ is a weak solution to the algae problem in the sense of Definition 5.4 if and only if $v = u - \bar{u}$ is a solution to:

$v \in W(0,T)$, $F(v + \bar{u}) \in L^{3/2}(Q)$ with $v(t = 0) = 0$ such that, $t$ a.e. in $(0,T)$, $\forall w \in V$,
\[
\langle \partial_t v, w \rangle_{V' \times V} + a(t, v, w) = \int_D F(v + \bar{u})w \, dx + \langle \psi, w \rangle_{V' \times V}.
\]

**Lemma 6.2.** [Properties of $A$] Under Assumptions (5)-(7) of regularity and boundedness for the advection speed, the growth rate and the algal capacity, there are non-negative constants $C_1$, $C_2$ and $C_3$ such that for all $t \in [0,T]$ any $u, v \in H^1(D)$,
\[
|a(t, v, w)| \leq C_1\|v\|_{H^1} \|w\|_{H^1}, \quad (24)
\]
\[
|a(t, v, v)| \geq C_2\|\nabla v\|^2_{L^2} - C_3\|v\|^2_{L^2}. \quad (25)
\]

Moreover, the map $t \mapsto a(t, u, v)$ is measurable.

**Proof of Lemma 6.2:** The measurability and continuity properties of $a$ are clear. The coercivity (25) classically follows from Cauchy’s inequality. Indeed, we have the upper bounds
\[
\left| \int_D v V \cdot \nabla v \right| \leq V_\infty \|v\|_{L^2} \|\nabla v\|_{L^2} \leq V_\infty \left[ \frac{1}{\alpha} \|v\|^2_{L^2} + \alpha \|\nabla v\|^2_{L^2} \right].
\]
Hence, for $0 < \alpha < \frac{\kappa_n}{2V_\infty}$, we get
\[ a(t, v, v) \geq \frac{\kappa_n}{2} \| \nabla v \|_{L^2}^2 - \frac{V_\infty}{\alpha} \| v \|_{L^2}^2. \]

\[ \square \]

**Second step: Defining the fixed point operator.** By a direct application of Lions theorem (see \[3, Th. 1 & 2, p.512-513\]), we obtain a first existence result in $W(0, T)$ for a linear problem:

**Lemma 6.3.** The functions $\bar{u}$ and $\psi$ being fixed above, for any $s \in L^2(0, T, L^2(D))$, there is a unique $v \in W(0, T)$ such that $v(t = 0) = 0$ and $t \in (0, T)$ a.e., for all $w \in V$

\[ \langle \partial_t v, w \rangle_{V' \times V} + a(t, v, w) = \int_D swdx + \langle \psi, w \rangle_{V' \times V}. \] (26)

Moreover, the operator $S : L^2(0, T, L^2(D)) \to L^2(0, T, L^2(D))$ that maps $s$ to $Ss$ the unique solution of (26) is continuous.

**Third step: Dealing with a Lipschitz nonlinearity.** We define the Carathéodory function on $Q \times \mathbb{R}$, Lipschitz-continuous in $\lambda$, uniformly in $(t, x)$ with Lipschitz-constant less than $(2M + 1)\tau_{\text{max}}$ by

\[ f^M(t, x, \lambda) = r(t, x)\lambda^+ \text{ max } [-M, 1 - \frac{\lambda}{u_m(t, x)}]. \]

Therefore, \[8, Th. 2.1, p.22\], the associated Nemitskii operator $F^M : u \mapsto f^M(\cdot, u)$ is Lipschitz-continuous from $L^2(0, T, L^2(D))$ to $L^2(0, T, L^2(D))$ and one is able to tackle the problem: find $v \in W(0, T)$ such that $v(t = 0) = 0$ and, $t$ a.e. in $(0, T)$, for all $w \in V$

\[ \langle \partial_t v, w \rangle_{V' \times V} + a(t, v, w) = \int_D F^M(v + \bar{u})wdx + \langle \psi, w \rangle_{V' \times V}. \] (27)

To that end, we use a fixed point argument for the operator $T$ considering the recursion $v_{n+1} = Tv_n$ where $v_{n+1}$ is the unique solution introduced in the first step when $s = F^M(v_n + \bar{u})$. The fixed-point argument is based on the following contraction property of $T$:

**Lemma 6.4.** \([T \text{ is contractant!}]\) There exists a constant $C_4 \leq 0$ such that $T$ is a contraction mapping on $L^2(0, T, L^2(D))$ for the equivalent norm

\[ v \mapsto \left( \int_0^T e^{C_4 t} \| v(t) \|_{L^2}^2 dt \right)^{1/2}. \]

**Proof of Lemma 6.4:** Let $v_1$ and $v_2$ be in $L^2(0, T, L^2(D))$, we get for any $w \in V$

\[ \langle \partial_t v_i, w \rangle_{V' \times V} + a(t, v_i, w) = \int_D F^M(v_i + \bar{u})wdx + \langle \psi, w \rangle_{V' \times V} \quad i = 1, 2. \]

so that for any $w \in W(0, T)$ and $t$ a.e in $(0, T)$,

\[ \langle Tv_1 - Tv_2, w \rangle_{V' \times V} + a(t, Tv_1 - Tv_2, w) = \int_D (F^M(v_1 + \bar{u}) - F^M(v_2 + \bar{u})) wdx. \]
In particular for $w = Tv_1 - Tv_2$, we get thanks to Lemma 6.2
\[
\frac{1}{2} \partial_t \|Tv_1 - Tv_2\|_{L^2}^2 + C_2 \|Tv_1 - Tv_2\|_V^2 - C_3 (Tv_1 - Tv_2\|_{L^2}^2 \\
\leq (2M + 1) C(r_{\max}, u_m, M) \|Tv_1 - Tv_2\|_{L^2}^2 \|v_1 - v_2\|_{L^2}^2 \\
\leq \delta \|v_1 - v_2\|_{L^2}^2 + C_3 \|Tv_1 - Tv_2\|_{L^2}^2
\]
that yields, for a given $\delta \in (0, \frac{1}{2T})$,
\[\partial_t \|Tv_1 - Tv_2\|_{L^2}^2 + 2C_2 \|Tv_1 - Tv_2\|_V^2 - 2[C_3 + C_3] (Tv_1 - Tv_2\|_{L^2}^2 \leq 2\delta \|v_1 - v_2\|_{L^2}^2.
\]
Multiplying this relation by $e^{-2(C_3 + C_3)t} = e^{Ct}$, we get:
\[\partial_t [e^{Ct} \|Tv_1 - Tv_2\|_{L^2}^2] = e^{Ct} \partial_t \|Tv_1 - Tv_2\|_{L^2}^2 + C_4 e^{Ct} \|Tv_1 - Tv_2\|_{L^2}^2.
\]
Integrating in time over $[0, t]$, we get for all $t \in [0, T]$ since $Tv_1(0) = 0$
\[e^{Ct} \|Tv_1(t) - Tv_2(t)\|_{L^2}^2 + 2C_2 \int_0^t e^{C\tau} \|Tv_1(\tau) - Tv_2(\tau)\|_{L^2}^2 d\tau \\
\leq 2\delta \int_0^t e^{C\tau} \|v_1(\tau) - v_2(\tau)\|_{L^2}^2 d\tau,
\]
and, in particular,
\[\int_0^T e^{Ct} \|Tv_1(t) - Tv_2(t)\|_{L^2}^2 dt \leq 2\delta T \int_0^T e^{C\tau} \|v_1(\tau) - v_2(\tau)\|_{L^2}^2 d\tau,
\]
and the result holds as soon as $\delta < \frac{1}{2T}$. □

Then, by Banach’s fixed-point theorem, one comes to justify the following existence result:

**Lemma 6.5.** There exists a unique $v_M \in W(0, T)$ satisfying (27). In particular, $u_M = v_M + \bar{u}$ is the unique solution to the perturbed problem when $F$ is replaced by $F^M$, in the sense of Definition 5.4.

**Fourth step:** Proving $u_M \geq 0$. The function $u_M$ solves: for $t$ a.e. in $(0, T)$ and any $w$ in $V$,
\[\langle \partial_t u_M, w \rangle_{V', V} + a(t, u_M, w) = \int_D ru^+ \max \left(-M, 1 - \frac{u_M}{u_m}\right) w dx - \int_{\partial D^o} \alpha C^\text{mat} V \cdot n w dH^2(x).
\]

Since $-u_M = -(v_M + \bar{u})$ is an admissible test-function, one gets by Mignot-Bamberger-Alt-Luckhaus’s integration by part formula reminded in [5, p.31] or [6] that
\[\frac{1}{2} \frac{d}{dt} \|u_M\|_{L^2}^2 + a(t, u_M, u_M) = - \int_D F^M(u_M) u_M^- dx + \int_{\partial D^o} \alpha C^\text{mat} V \cdot nu_M^- dH^2(x) \\
= \int_{\partial D^o} \alpha C^\text{mat} V \cdot nu_M^- dH^2(x) \leq 0.
\]
Thus, one concludes that
\[ \frac{1}{2} \frac{d}{dt} \| u_M^i \|_{L^2}^2 \leq C_3 \| u_M^- \|_{L^2}^2 \]
and, since \( u(t = 0) = u_0 \geq 0 \), Gronwall’s Lemma yields \( u_M^- = 0 \) and \( u_M \geq 0 \) a.e. in \( D \), for any \( t \).

**Fifth step: A priori estimates uniform in \( M \) of \( u_M \).** Due to boundary conditions \( u_M \) is not in \( V \), but \( v_M = u_M - \bar{u} \) is. Hence, we consider the variational formulation (27) in \( v_M \):

\[ \langle \partial_t v_M, w \rangle_{V', V} + a(t, v_M, w) = \int_D F^M(u_M) wdx + \langle \psi, w \rangle_{V', V} \]

where
\[ \psi : w \in V \mapsto - \langle \partial_t \bar{u} - A(t) \bar{u}, w \rangle_{V', V} - \int_{\partial D^s} \alpha C_{\text{max}} V.n w d\mathcal{H}^2(x) \in L^2(0, T, V'). \]

Testing that problem by \( v_M \), for \( t \) a.e. in \((0, T)\),

\[ \frac{1}{2} \frac{d}{dt} \| v_M \|_{L^2}^2 + a(t, v_M, v_M) = \int_D r u_M (u_M + \bar{u}) \max \left( -M, 1 - \frac{u_M}{u_m} \right) dx + \langle \psi, v_M \rangle_{V', V}. \]

Let us note that

\[ r u_M (u_M + \bar{u}) \max \left( -M, 1 - \frac{u_M}{u_m} \right) \leq r_{\text{max}} u_{\text{max}}^2 + r u_M \bar{u} \max \left( -M, 1 - \frac{u_M}{u_m} \right) \]

so that

\[ \int_D F^M(u_M) v_M dx \leq C(r_{\text{max}} u_{\text{max}}) + C(r_{\text{max}} u_{\text{max}} u_{\text{min}}) \int_D u_M |\bar{u}| (1 + u_M) dx \]

\[ \leq C \left( 1 + \| u_M \|_{L^2}^2 + \| \bar{u} \|_{L^2}^2 + \int_D u_M^2 |\bar{u}| |dx| \right) \]

\[ \leq C \left( 1 + \| v_M \|_{L^2}^2 + \| \bar{u} \|_{L^2}^2 + \int_D v_M^2 |\bar{u}| |dx| + \int_D \| \bar{u} \|_{H^1}^3 |dx| \right) \]

\[ \leq C \left( 1 + \| v_M \|_{L^2}^2 + \| \bar{u} \|_{L^2}^2 + \| v_M \|_{L^2}^2 \| v_M \|_{H^1} \| \bar{u} \|_{H^1} + \| \bar{u} \|_{L^2}^2 \| \bar{u} \|_{H^1}^2 \right) \]

by using \( H^1(D) \hookrightarrow L^6(D) \) with the extended Hölder’s inequality.

Thus, one concludes that

\[ \int_D F^M(u_M(t)) v_M(t) dx \leq C \left( 1 + \| v_M(t) \|_{L^2}^2 + \| \bar{u}(t) \|_{L^2}^2 \right) + \epsilon \| v_M(t) \|_{V'}^2 \]

\[ + C_\epsilon \| v_M(t) \|_{L^2}^2 \| \bar{u}(t) \|_{H^1}^2 + C \| \bar{u} \|_{C([0, T], L^2)} \| \bar{u}(t) \|_{H^1}^2. \]

We gather this inequality with the usual estimate

\[ \| \langle \psi, v_M \rangle_{V', V} \|_{V} \leq \| \psi \|_{V'} \| v_M \|_V \leq \frac{1}{\epsilon} \| \psi \|_{V'}^2 + \epsilon \| v_M \|_{V'}^2, \]
to derive
\[
\frac{1}{2} \frac{d}{dt} \|v_M(t)\|_{L^2}^2 + C_2 \|\nabla v_M(t)\|_{L^2}^2 - C_3 \|v_M(t)\|_{L^2}^2 \\
\leq C \left(1 + \|v_M(t)\|_{L^2}^2 + \|\bar{u}(t)\|_{L^2}^2 + \|v_M(t)\|_{L^2}^2 \|\bar{u}(t)\|_{H^1} + \\
\|\bar{u}\|_{C([0,T],L^2)} \|\bar{u}(t)\|_{H^1} + \|\psi(t)\|_{V'}^2\right) + 2\epsilon \|v_M(t)\|_{V'}^2.
\]
Choosing \(\epsilon = C_2/8\) so that \(C_2 - 2\epsilon \geq C_2/2 > 0\), we get
\[
\frac{1}{2} \frac{d}{dt} \|v_M(t)\|_{L^2}^2 + \frac{C_2}{2} \|v_M(t)\|_{V'}^2 \\
\leq C \left(1 + \|\bar{u}(t)\|_{L^2}^2 + \|\bar{u}\|_{C([0,T],L^2)} \|\bar{u}(t)\|_{H^1} + \|\psi(t)\|_{V'}^2\right) \tag{28}

\]
Thus,
\[
\frac{d}{dt} \|v_M(t)\|_{L^2}^2 \leq h(t) + C \|v_M(t)\|_{L^2}^2 (1 + \|\bar{u}(t)\|_{H^1})
\]
for an integrable function \(h \) on \((0,T)\). Therefore, there is a constant \(C\) independent of \(M\) such that
\[
\|v_M(t)\|_{L^2}^2 \leq C \int_0^t (1 + \|\bar{u}(s)\|_{H^1}) \|v_M(s)\|_{L^2(D)}^2 ds + C
\]
and Gronwall’s lemma yields that
\[
\|v_M(t)\|_{L^2}^2 \leq C e^{\int_0^T (1 + \|\bar{u}(s)\|_{H^1}) ds}.
\]
In conclusion, \(v_M\) and \(u_M\) are bounded in \(C([0,T],L^2(D))\) independently of \(M\). Using a second time \((28)\), we obtain also that both \(v_M\) and \(u_M\) are bounded in \(L^2(0,T,H^1(D))\).

**Sixth step: A priori estimates uniform in \(M\) of \(\partial_t u_M\).** For any \(w \in V\), we check that
\[
\left| - \int_{\partial D^\epsilon} \alpha \mathbf{C}^{\text{mat}} \mathbf{V} \cdot \mathbf{n} w dH^2(x) \right| \leq C \|\mathbf{C}^{\text{mat}}\|_{L^2(\partial D^\epsilon)} \|w\|_V
\]
by Cauchy-Schwarz’s inequality on the boundary \(\partial D^\epsilon\) and the trace inequality. Then,
\[
\langle \partial_t u_M, w \rangle_{V', V}
\]
\[
\leq C_1 \|u_M\|_V \|w\|_V + \int_D r u_M \max \left(-M, 1 - \frac{u_M}{u_m}\right) w dx + C \|\mathbf{C}^{\text{mat}}\|_{L^2(\partial D^\epsilon)} \|w\|_V.
\]
Again, \(H^1(D) \hookrightarrow L^6(D)\) with the extended Hölder’s inequality yield \(t\) a.e. in \((0,T)\)
\[
\left| \int_D r u_M \max \left(-M, 1 - \frac{u_M}{u_m}\right) w dx \right|
\]
\[
\leq C(r) \int_D |u_M| \left|1 - \frac{u_M}{u_m}\right| |w| dx \leq C(r) \|u_M\|_1 - \frac{u_M}{u_m} \|w\|_{L^6} \leq C(r) \|u_M\|_{L^2} \|w\|_{L^6} \leq C(r) \|u_M\|_{L^2} \|u_M\|_{H^1} \|w\|_V
\]
\[
\leq C(r) \|u_M\|_{H^1} \|w\|_V \text{ since } \langle u_M \rangle_M \text{ is bounded in } C([0,T],L^2(D)).
\]
Therefore, \(\|\partial_t u_M\|_{V'} \leq C(1 + \|u_M\|_{H^1})\) and \(\langle \partial_t u_M \rangle\) is bounded in \(L^2(0,T,V')\).
By Gronwall’s Lemma, a constant 

Then, choosing the admissible test-function 

where 

is bounded in 

with the additional information that 

in \( D \) and \( v = 0 \) in \((0,T) \times \partial D^w\).

Back to the variational formulation satisfied by \( v_M \), one gets, for any \( w \in V \),

\[
\langle \partial_t v, w \rangle_{V^\prime \times V} + a(t,u,w) = \lim_{M \to +\infty} \int_D F^M(u_M)wdx + \langle \psi, w \rangle_{V^\prime \times V}.
\]

Note that, arguments similar to the previous ones lead to

\[
\int_Q |F^M(u_M)|^\frac{3}{2}dx \leq r_1^\frac{3}{2}\max_Q |u_M|^\frac{3}{2} \leq C(1 + \int_Q |u_M|^3dx)
\]

\[
\leq C(1 + \int_0^T ||u_M(t)||_{L^2(D)}||u_M(t)||^2_{L^1(D))}dt \leq C \left( 1 + \int_0^T ||u_M(t)||_{H^1}^2dt \right) \leq C.
\]

Therefore, Fatou’s Lemma, the a.e. convergence in \( Q \) and the above boundedness yield the existence of a solution \( v \) as presented in Proposition 6.1, thus a solution to the alga problem in the sense of Definition 5.4.

Eighth step: Uniqueness and stability of the solution.

Denote by \( u_1 \) and \( u_2 \) two given solutions associated to different data: \( u_0^i \) and 

\[ h_i = \alpha_i C^{\text{mat}} V \cdot n, \quad i = 1, 2, \]

satisfying the assumptions of the model. Then, for any \( w \) in \( V \),

\[
\langle \partial_t (u_1 - u_2), w \rangle_{V^\prime \times V} + a(t,u_1 - u_2, w) = \int_D r \left[ u_1 \left( 1 - \frac{u_1}{u_m} \right) - u_2 \left( 1 - \frac{u_2}{u_m} \right) \right] wdx - \int_{\partial D^w} [h_1 - h_2]wdH^2(x).
\]

Note that

\[
f(t,x,\lambda) = r\lambda^+ \left( 1 - \frac{\lambda}{u_m} \right) = r\lambda^+ \left( 1 - \frac{\lambda}{u_m} \right)^+ - r\lambda^+ \left( 1 - \frac{\lambda}{u_m} \right)^- = f_1(t,x,\lambda) + f_2(t,x,\lambda)
\]

where \( f_1 \) in a non-negative function bounded by \( \frac{1}{2}r_{\text{max}}u_{\text{max}} \), Lipschitz-continuous in \( \lambda \) with Lipschitz-constant less than \( r_{\text{max}} \): \( f_2 \) in a non-positive function, decreasing in \( \lambda \). Thus,

\[
\int_D r \left[ u_1 \left( 1 - \frac{u_1}{u_m} \right) - u_2 \left( 1 - \frac{u_2}{u_m} \right) \right] (u_1 - u_2)dx \leq r_{\text{max}} \|u_1 - u_2\|^2_{L^2}.
\]

Then, choosing the admissible test-function \( w = u_1 - u_2 \) leads us to

\[
\frac{1}{2} \frac{d}{dt} \|u_1 - u_2\|^2_{L^2} + C_1 \|u_1 - u_2\|_V^2 - C_3 \|u_1 - u_2\|^2_{L^2} \leq r_{\text{max}} \|u_1 - u_2\|^2_{L^2} + C \|h_1 - h_2\|^2_{L^2(\partial D^w)} + \frac{1}{2} C_1 \|u_1 - u_2\|^2_{V}.
\]

By Gronwall’s Lemma, a constant \( C(T) \geq 0 \) exists such that

\[
\|u_1 - u_2\|^2_{C([0,T],L^2(D))} + \|u_1 - u_2\|^2_{L^2(0,T;H^1(D))} \leq C(T) \left[ \|u_0^1 - u_0^2\|^2_{L^2(D)} + \|h_1 - h_2\|^2_{L^2(0,T,L^2(\partial D^w))} \right].
\]

Ninth step: Assuming that \( C^{\text{mat}} \) is bounded. For \( a = \beta_{\text{max}} + ||\text{div}V||_{\infty} \), consider

\[
M(t) = (a_{\text{max}}(||C^{\text{mat}}_0\|_{\infty} + T||C^{\text{rest}}_0\|_{\infty}) + u_{\text{max}})e^{at},
\]
and test the problem with \((u - M(t))^+\), an element of \(V\):
\[
\frac{1}{2} \frac{d}{dt} \|(u(t) - M(t))^+\|^2_{L_2} + a(t, u(t) - M(t), (u(t) - M(t))^+) \\
+ a(t, M(t), (u(t) - M(t))^+) \\
= \int_D ru(1 - \frac{u(t)}{u_m})(u - M(t))^+ dx - \int_{\partial D^*} \alpha C^{mat} V \cdot n (u - M(t))^+ d\mathcal{H}^2(x) \\
- \int_D M'(t)(u - M(t))^+ dx \\
\leq \int_{\partial D^*} \alpha C^{mat} |V \cdot n|(u(t) - M(t))^+ d\mathcal{H}^2(x) - \int_D M'(t)(u - M(t))^+ dx
\]
since \(V \cdot n \leq 0\) on the boundary \(\partial D^*\) and \(ru(1 - \frac{u(t)}{u_m})(u(t) - M(t))^+ \leq 0\). Thus, we get
\[
\frac{1}{2} \frac{d}{dt} \|(u(t) - M(t))^+\|^2_{L_2} + C_1\|(u(t) - M(t))^+\|^2_V \\
\leq C_3\|(u(t) - M(t))^+\|^2_{L_2} - \int_D [M'(t) + \beta M(t)](u - M(t))^+ dx \\
+ \int_{\partial D^*} \alpha C^{mat} |V \cdot n|(u(t) - M(t))^+ d\mathcal{H}^2(x) + \int_D M(t)V \cdot \nabla(u(t) - M(t))^+ dx \\
= C_3\|(u(t) - M(t))^+\|^2_{L_2} - \int_D [M'(t) + (\beta + \text{div}V)M(t)](u(t) - M(t))^+ dx \\
+ \int_{\partial D^*} (M(t) - \alpha C^{mat}) |V \cdot n|(u(t) - M(t))^+ d\mathcal{H}^2(x) \\
+ \int_{\partial D^*} M(t)V \cdot n(u(t) - M(t))^+ d\mathcal{H}^2(x) \\
\leq C_3\|(u(t) - M(t))^+\|^2_{L_2} - \int_D [\alpha + \beta + \text{div}V]M(t)(u(t) - M(t))^+ dx \\
\leq C_3\|(u(t) - M(t))^+\|^2_{L_2}
\]
thanks to Green’s formula. Then, Gronwall’s Lemma and \(u_0 \leq u_{\text{max}}\) ensures that
\[
0 \leq u(t, x) \leq M(t) \leq M(T) \text{ a.e. in } D \text{ and for any } t.
\]

7. Testing the model: About the extinction of the alga form.

7.1. About extinction in finite time. We need to translate into mathematical assumptions the fact that the environmental conditions may become unfavorable around a given time \(T > 0\). These conditions are translated in the model into the following assumptions on the biological data thanks to the introduction of a function \(\gamma\). We assume that:

\(H_1\): there exists \(\gamma: [0, T] \to \mathbb{R}^+\), a regular increasing function with
\[
\lim_{t \to T} \gamma(t) = +\infty \text{ in such a way that } (H_\gamma) \int_0^T \gamma(t) dt = +\infty.
\]

\(H_2\): \(\beta(t, x) = \tilde{\beta}(x)\gamma(t)\) with \(0 < \beta_{\text{min}} \leq \tilde{\beta} \leq \beta_{\text{max}}\) and \(\tilde{\beta} \in L^2(D)\) meaning that the rate of evolution of the alga into new cysts increases substantially when \(t\) goes to \(T\);

\(H_3\): \(g = 0\) in order to assume that no algae are entering the domain from the ocean.
Theorem 7.1. Under the assumptions of Theorem 5.5, updated by (H1) – (H2) – (H3), there exists a unique solution \( u \) to the algae problem on \([0, T)\) and one has \( \lim_{t \to T} u(t) = 0 \) in \( L^2(D) \).

7.2. Proof of theorem 7.1. Since \( \gamma \) is uniformly bounded on \([0, \tilde{T})\) for \( \tilde{T} < T \), the result of wellposedness of Theorem 5.5 yields the existence and uniqueness of \( u \) in \([0, T)\). The proof of Thm 7.1 consists in showing the vanishing result when \( t \to T \).

Assuming (H2) and (H3), \( u \) is with values in \( V \) and testing the equation with \( u \) for \( t < T \) yields

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \frac{\kappa_2}{2} \|u(t)\|_V^2 - C(V) \|u(t)\|_{L^2}^2 + \beta_{\min} \gamma(t) \|u(t)\|_{L^2}^2
\]

\[
\leq \int_D ru^2(t)(1 - \frac{u(t)}{u_m(t)})dx - \int_{\partial D^0} \alpha C_{\text{mat}}(t) V \cdot n u(t) dH^2(x)
\]

\[
\leq C r_{\max} \|u\|_{H^1}^2 + C_{\epsilon} \|C_{\text{mat}}(t)\|_{L^2(\partial D^0)}^2 + \epsilon \|u(t)\|_{L^2}^2.
\]

Assume that \( t \in [t_0, T) \) such that \( \beta_{\min} \gamma(t) - C(V) \geq \frac{1}{2} \beta_{\min} \gamma(t) \), so that, for a suitable choice of \( \epsilon \), one gets that

\[
\frac{d}{dt} \|u(t)\|_{L^2}^2 + \beta_{\min} \gamma(t) \|u(t)\|_{L^2}^2 \leq C(1 + \|C_{\text{mat}}(t)\|_{L^2(\partial D^0)}) \leq C
\]

by assuming \( C_{\text{mat}} \) and \( C_{\text{rest}} \) bounded. Thus, if one denotes by \( \Gamma(t) = \int_{t_0}^t \gamma(s)ds \), we obtain

\[
\frac{d}{dt} e^{\beta_{\min} \Gamma(t)} \|u(t)\|_{L^2}^2 \leq C e^{\beta_{\min} \Gamma(t)}
\]

and

\[
\|u(t)\|_{L^2}^2 \leq e^{-\beta_{\min} \int_{t_0}^t \gamma(s)ds} \|u(t_0)\|_{L^2}^2 + C \int_{t_0}^t e^{-\beta_{\min} \int_{t_0}^\tau \gamma(s)ds} d\tau.
\]

By assumption (H1), the first term of the right hand side converges to 0 when \( t \) goes to \( T \), as well as \( e^{-\beta_{\min} \int_{t_0}^\tau \gamma(s)ds} 1_{(t_0, T)}(\tau) \) for any \( \tau \in (t_0, T) \). As the positive integrand is bounded by 1, the lemma holds thanks to Lebesgue’s theorem.

Remark 7.2. Note that thanks to equation (11), in spite of the extinction of the algae population in \( D \), it is stored as cysts. Then, after a maturity of age \( A \) and back to good environmental conditions, the population will reappear.

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E-mail address: marc.dambrine@univ-pau.fr
E-mail address: benedicte.puig@univ-pau.fr
E-mail address: guy.vallet@univ-pau.fr