On the Hamiltonian Constraint of Loop Quantum Cosmology

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Abstract

In this paper we construct the Hamiltonian constraint operator of loop quantum cosmology using holonomies defined for arbitrary irreducible SU(2) representations labeled by spin $J$. We show that modifications to the effective semi-classical equations of motion arise both in the gravitational part of the constraint as well as matter terms. The modifications are important for phenomenological investigations of the cosmological imprints of loop quantum cosmology. We discuss the implications for the early universe evolution.

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I. INTRODUCTION

In the study of cosmology many questions remain unanswered despite the wealth of observational data at hand. In particular our understanding of the early universe is lacking. It is at these high energies that any classical cosmological models are no longer relevant and a quantum theory of gravity is needed. A deep quantum understanding of the regime is required to answer, for instance, questions related to the initial big-bang singularity and to understand what conditions preceded the onset of chaotic inflation.

A particularly exciting quantum theory of cosmology is known as loop quantum cosmology (LQC) (for a review and references see [1]). The progress made in LQC to date has been impressive. LQC has provided explanations to long standing questions in quantum cosmology. The reason for its success can be traced to the fact that it has as its basis a candidate full theory of quantum gravity known as loop quantum gravity (for reviews see [2, 3, 4, 5, 6]). Thus LQC is expected to handle on a firmer basis the extreme high energy regime of the early universe where non-perturbative effects are paramount. For instance it has been shown that the theory contains a robust mechanism for singularity avoidance [7]. The mechanism arises as a natural feature of the theory and does not rely on ad-hoc forms of matter. This is in contrast to earlier forms of quantum cosmology such as those based on Wheeler-DeWitt quantization where singularity avoidance does not arise naturally.

Yet, an even more exciting result is that LQC predicts modifications to the evolution of the universe for scales larger than the Planck scale. For instance it has been shown that LQC can modify the behavior of the inverse volume such that it becomes a bounded function [8]. This gives rise to modifications to the behavior of matter in the form of a scalar field which in turn predicts a period of super-inflation at energies near the Planck scale [9]. In this super-inflationary regime the inflaton is pushed up its potential due to the existence of an anti-friction term in the Klein-Gordon equation which can set the stage for the onset of slow-roll inflation [10, 11]. The modified dynamics in this regime can even lead to measurable signatures at large scales in the CMB [11].

Most of the phenomenological studies to date have exploited the modification of the inverse volume. For instance, in a collapsing universe with a scalar field the anti-frictional term in the Klein-Gordon equation becomes frictional and a bounce can occur [12]. This effect has been studied for a scalar field with a self-interaction potential in positive curvature
oscillatory universes where the effects of repeated expansion/contraction push the scalar field up its potential providing another mechanism to establish the initial conditions for slow-roll inflation \cite{13, 14}. In addition the modifications can lead to a bounce in the context of cyclic models with colliding branes, possibly removing singular behavior that has plagued these models \cite{15}.

These results are intriguing yet some open questions remain as to their validity. The modifications to the inverse volume arise by exploiting an ambiguity in defining the inverse volume operator which is required to quantize the matter part of the Hamiltonian constraint for a scalar field. In defining the inverse volume operator one traces over SU(2) valued holonomies and the ambiguity lies in choosing the irreducible representation in which to perform the trace. The ambiguity is labeled by a half-integer $J$ representing the spin of the representation. A neglected fact is that this same ambiguity arises in the gravitational part of the Hamiltonian constraint where one also traces over SU(2) valued holonomies. So far, the gravitational part of the constraint has been quantized using the $J = 1/2$ representation while the ambiguity has been freely exploited for the matter part.

It is therefore not clear if corrections arising in the gravitational part might change the dynamics and alter significantly the phenomenological consequences. From the full theory the gravitational part contains an inverse volume factor and the possibility remains that modifications here might cancel the effects in the matter part. It is thus of critical importance to determine the complete quantum corrections for arbitrary $J$ arising in both terms of the Hamiltonian constraint.

In this paper we aim to clear up the uncertainty with respect to this ambiguity. We explicitly construct the isotropic Hamiltonian constraint operator for arbitrary spin $J$ representations. We show that the resulting difference equation is higher order for larger $J$ and that geometric quantities in the gravitational part of the constraint obtain quantum modifications analogously to the inverse volume modifications for the matter part. We show that the higher $J$ constraint operator contains the same singularity resolution mechanism attesting to its robustness. We will derive and motivate effective semi-classical equations of motion from the constraint operator and comment on the phenomenological consequences and show that the qualitative results do not change. As an explicit example, the $J = 1$ constraint operator is constructed and analyzed with special attention payed to the semi-classical limit.
II. LOOP QUANTUM COSMOLOGY

We start with a brief overview of LQC and its kinematical Hilbert space. In Dirac quantization for systems with constraints (such as general relativity) one first starts by constructing a kinematical Hilbert space by quantizing the system ignoring the constraints. The constraints are then represented as self-adjoint operators on this space and physical solutions are wave functions annihilated by the constraint operators. In this paper we will restrict ourselves to homogeneous and isotropic models with zero spatial curvature. Upon the imposition of this symmetry reduction, the Hamiltonian of isotropic GR consists of a single constraint, the Hamiltonian constraint. The details of the construction of the kinematical Hilbert space and the constraint operator are given in [16]. We will follow the same conventions and notation here.

The first step toward quantization is a classical symmetry reduction of the action of general relativity written in connection form. In the full theory of loop quantum gravity the action is written in terms of a connection $A$ which contains the information about curvature, as well as a triad $E$ conjugate to $A$ which encodes the geometry. After reduction to homogeneity and isotropy the connection and triad are labeled by single quantities $c$ and $p$ respectively. These variables satisfy the commutation relation

$$\{c, p\} = \frac{1}{3} \kappa \gamma$$

where $\kappa = 8\pi G$ and $\gamma$ is the Barber-Immirzi parameter which represents a quantum ambiguity (black hole entropy calculations can fix its value to $\gamma \approx 0.2735$ [17, 18]). In terms of the standard metric variables given by the scale factor $a$, the new variables are related as

$$|p| = a^2$$  \hspace{1cm} (1a)

$$c = \gamma \dot{a}$$  \hspace{1cm} (1b)

Note the absence of the factor of 1/2 in the formula for the connection (1b) which appears in [15]. The correct factor (without the 1/2) can be derived using the relation from the full theory $A^i_\alpha = \Gamma^i_\alpha + \gamma K^i_\alpha$ where $A^i_\alpha$ is the full connection, $K^i_\alpha$ is the extrinsic curvature, and $\Gamma^i_\alpha$ is the spin connections which vanishes identically for isotropic flat models. The absolute value around the triad component $p$ indicates that we are extending the classical phase space of isotropic GR to include both orientations of the triad (positive and negative $p$). Note
that this does not mean that negative $p$ corresponds to a negative scalar factor. For both regions the scale factor, and hence the volume, is positive.

Dynamics is determined completely by the Hamiltonian constraint $H$ given by

$$H = -\frac{3}{\kappa^2} \text{sgn}(p) \sqrt{|p|} c^2 + H_m$$  \hspace{1cm} (2)$$

where $H_m$ is the matter Hamiltonian. Note again the factor of 3 which differs from previous works. With the correct factors here the resulting action $\mathcal{S}_{GR} = \int dt \frac{3}{\kappa} p \dot{c} - NH_{GR}$ is equivalent to the standard isotropic Einstein-Hilbert action $\mathcal{S}_{GR} = \frac{1}{2\kappa} \int \sqrt{|g|} R$, and the Hamiltonian equations $\dot{p} = \{p, H\}$ and $\dot{c} = \{c, H\}$ are equivalent to the equations of motion derived from the Einstein-Hilbert action.

With the symmetry reduced action we now turn to quantization. The theory is quantized using ideas and techniques from the full theory of loop quantum gravity. The basic variables to be quantized are holonomies of the connection and the smeared triad integrated on a two-surface. Using this, the quantum configuration space inherits a discrete structure and an orthonormal basis is given by states labeled by a parameter $\mu$ given by

$$\langle c | \mu \rangle = \exp(i\mu c/2) \quad \mu \in \mathbb{R}.$$  

The basis states $|\mu\rangle$ are eigenstates of the triad operator $\hat{p}$ with eigenvalues

$$\hat{p} |\mu\rangle = \frac{\mu \gamma l_\rho^2}{6} |\mu\rangle.$$  \hspace{1cm} (3)$$

Geometric operators are built from the triad, which allows us to understand the physical meaning of the parameter $\mu$. In particular it is an eigenstate of the volume operator $\hat{V} = |\hat{p}|^{3/2}$ hence the physical volume is given by

$$V_\mu = \left( \left| \frac{\mu \gamma l_\rho^2}{6} \right| \right)^{3/2}.$$  \hspace{1cm} (4)$$

More complicated operators are built from the basic triad operator $\hat{p}$ and holonomy operators. Because of homogeneity and isotropy we only need consider holonomies along straight lines which can be parameterized as

$$h_i = \exp(\mu_0 c \tau_i)$$

where $\tau_i$ are the generators of the Lie algebra of SU(2) satisfying $[\tau_i, \tau_j] = \epsilon_{ij}^k \tau_k$, and $\mu_0$ is proportional to the length of the holonomy. In the fundamental representation the holonomies
are given by the simple formula

\[(\frac{1}{2}h_{i} = \cos(\mu_{0}c/2) + 2\tau_{i}\sin(\mu_{0}c/2))\]

where the notation \((\frac{1}{2})\) indicates that the holonomy is defined in the fundamental representation \((J = 1/2)\). In this paper we will use superscripts to the left of an object to indicate the irreducible representation of SU(2) used. The holonomy operators in turn behave as exponentiated position operators and thus act on the basis states by finite shifts. More precisely, the holonomy operators consist of the cosine and sine operators which we will for compactness notate \(\hat{\cos}\) and \(\hat{\sin}\) with actions

\[
\hat{\cos} |\mu\rangle \equiv \cos\left(\frac{\mu_{0}\hat{c}}{2}\right) |\mu\rangle = \frac{1}{2} \left[ |\mu + \mu_{0}\rangle + |\mu - \mu_{0}\rangle \right] \\
\hat{\sin} |\mu\rangle \equiv \sin\left(\frac{\mu_{0}\hat{c}}{2}\right) |\mu\rangle = -\frac{i}{2} \left[ |\mu + \mu_{0}\rangle - |\mu - \mu_{0}\rangle \right] 
\]

Knowing the action of the holonomy operators let us now turn to two essential operators: the inverse volume and the gravitational part of the Hamiltonian constraint. The inverse volume operator is needed to quantize certain forms of matter, for instance a scalar field. Since the eigenstates of the volume operator are normalizable, the naive inverse volume operator (with eigenvalues equal to the inverse of the volume eigenvalues) is not a densely designed self-adjoint operator. The solution is to recast the classical formula for the inverse volume using the Poisson bracket between the connection and the volume \cite{19}. When quantized the connection is represented with a holonomy operator, and the Poisson bracket is replaced with a commutator. The resulting operator is diagonal in the \(|\mu\rangle\) basis and is given by \cite{16}

\[(\frac{1}{2})\hat{\nu}^{-1} |\mu\rangle = \left[ \frac{4}{\gamma l_{p}^{2}} \left( V^{1/2}_{\mu+1} - V^{1/2}_{\mu-1} \right) \right]^{6} |\mu\rangle . \]

The eigenvalues are in fact bounded and approach zero near the classical singularity \(\mu = 0\). The fact that the inverse volume is cutoff for small volumes is the principal reason for the period of super-inflation for a scalar field in the early universe. We will discuss this more later.

There are two important points we wish to make regarding the inverse volume operator given here. First, as noted the formula is obtained using traces of holonomies in the spin 1/2 representation. The generalization to higher \(J\) has been performed \cite{20} and the effect is that the cutoff region gets pushed to larger volumes for larger \(J\) allowing for phenomenological
modifications for scales larger than the Planck length (the cutoff for $J = 1/2$ is deep in the Planckian regime where semi-classical equations of motion are no longer expected to be valid). Most of the phenomenological investigations are based on the modifications for larger values of $J$. The second point we make is that the holonomy operators involve the length parameter $\mu_0$ and an implicit value of $\mu_0 = 1$ has been used to arrive at equation (6). As stated in [16] one can use an arbitrary value $\mu_0$ in the regularization and in that reference the Hamiltonian constraint is constructed using a value of $\sqrt{3}/4$ determined on physical grounds from the smallest allowed area of the full theory of LQG. It is natural that the same value should be used both for the Hamiltonian constraint and the inverse volume operator since both are regularized using holonomies. We would like to update the formulas for the inverse volume operator to account for this ambiguity and in the next section we will do so.

Finally we wish to represent the Hamiltonian constraint operator, classically given in equation (2). There are two important non-trivialities that arise when attempting to quantize the constraint. The first is that the connection parameter $c$ is no longer one of the basic variables and must be represented using the holonomy operators. The other important non-triviality deals with how the $\text{sgn}(p)\sqrt{p}$ term of the constraint gets quantized from the full theory. While this term in the isotropic constraint could be simply quantized using the $p$ operator, this approach is not possible in the full theory. There the full constraint is given by

$$H = \frac{1}{\kappa} \int d^3 x \ N \ \epsilon^{ij}_{k} E^{a}_i E^{b}_j / \sqrt{|q|} F^{k}_{ab} + \ldots$$

where $N$ is the lapse, $F^{k}_{ab}$ is the curvature of the connection, $q$ is the determinant of the three-metric, and the dots indicate the Lorentzian part of the constraint in which we are not interested at this point (upon symmetry reduction it is proportional to the first half of the constraint). Most important is the presence of the $E^{a}_i E^{b}_k / \sqrt{|q|}$ term which leads to the $\text{sgn}(p)\sqrt{p}$ in the symmetry reduced action, but involves inverse triad operators in the full theory. Thus as for the inverse volume this term is represented using a commutator between the volume operator and holonomies. We will see that for higher $J$ this terms acquires quantum modifications for small volume analogously to the inverse volume operator.
The resulting operator is given by

$$\frac{1}{2} \hat{H}_{GR} = \frac{2i}{\kappa l_p^2 \gamma^2 \mu_0^3} \sum_{ijk} \epsilon^{ijk} \left( \hat{h}_i \hat{h}_j^{-1} \hat{h}_k^{-1} \hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right] \right).$$  \hspace{1cm} (7)$$

where we have explicitly indicated that the trace is performed in the fundamental representation. The curvature term of the classical constraint \((c^2)\) has been regulated with holonomies taken around a closed loop (the \(\hat{h}_i \hat{h}_j \hat{h}_k^{-1} \hat{h}_j^{-1}\) term). The \(\text{sgn}(p) \sqrt{p}\) term is now represented as \(\hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right]\) in the constraint operator. Several key remarks are to be made. First, the holonomy length parameter \(\mu_0\) has been introduced explicitly with the term in the denominator and in the holonomy operators. Second, a factor ordering has been chosen in that the \(\text{sgn}(p) \sqrt{p}\) term has been ordered to the right of the curvature term. This ordering is crucial in the singularity removal mechanism as we shall see. However, through this choice in ordering, the constraint is not self-adjoint. Appropriate self-adjoint constraints built from \((7)\) have been proposed \([21, 22]\) which have been shown to be non-singular. In this paper we will consider the non self-adjoint spin \(J\) constraint though we note that self-adjoint constraints can be constructed in an analogous way which does not effect the results presented here.

Using this formula we can now determine the action of the constraint operator on the basis states \(|\mu\rangle\). Using the formulas for the holonomies in the fundamental representation \([5]\) we get

$$\frac{1}{2} \hat{H}_{GR} = \frac{48i}{\kappa l_p^2 \gamma^2 \mu_0^3} \hat{s}_n \hat{c}_s^2 \left[ \hat{s}_n \hat{V} \hat{c}_s - \hat{c}_s \hat{V} \hat{s}_n \right],$$

whose action on the basis states is

$$\frac{1}{2} \hat{H}_{GR} |\mu\rangle = \frac{3}{2\kappa l_p^2 \gamma^2 \mu_0^3} \left( V_{\mu+\mu_0} - V_{\mu-\mu_0} \right) \left[ |\mu + 4\mu_0\rangle - 2|\mu\rangle + |\mu - 4\mu_0\rangle \right].$$  \hspace{1cm} (8)$$

To understand the singularity removal mechanism we need to consider physical wave functions which are those annihilated by the constraint operator. Such states can be expanded using the basis states as \(|\psi\rangle = \sum_\mu \psi_\mu |\mu\rangle\). Imposing the constraint equation leads to a difference equation for the coefficients \(\psi_\mu\)

$$\frac{3}{2\kappa l_p^2 \gamma^2 \mu_0^3} \left[ (V_{\mu+5\mu_0} - V_{\mu+3\mu_0}) \psi_{\mu+4\mu_0} - 2(V_{\mu+\mu_0} - V_{\mu-\mu_0}) \psi_\mu + (V_{\mu-3\mu_0} - V_{\mu-5\mu_0}) \psi_{\mu-4\mu_0} \right] = -\hat{H}_m(\mu) \psi_\mu \hspace{1cm} (9)$$

where we have assumed that the matter constraint \(\hat{H}_m\) acts diagonally on the basis states. The key fact about singularity removal is that the wave function coefficient \(\psi_0\) corresponding
to the value of the wave functions at the classical singularity decouples from the difference
equation. This can be seen from the fact that the \((V_{\mu+\mu_0} - V_{\mu-\mu_0})\) factor that always
accompanies \(\psi_0\) on the lhs of the difference equation vanishes. The singularity is then absent
provided the matter side of the constraint annihilates the \(|\mu = 0\rangle\) state. This has been
shown to be true for various forms of matter including a scalar field.

Let us understand the singularity removal mechanism on a more general level. The key
observation is that the constraint operator \(\text{annihilates}\) the \(|\mu = 0\rangle\) state since \((V_{\mu+\mu_0} - V_{\mu-\mu_0})\)
vanishes identically for \(\mu = 0\) in equation (8). Let us assume in general that the constraint
operator acts by raising and lowering the basis states
\[
\hat{H} |\mu\rangle = \sum_{k=-\sigma}^{\sigma} \alpha_k^\mu |\mu + k\rangle
\]
where \(\sigma\) and \(\alpha_k^\mu\) are parameters depending on the details of the operator. Assuming this,
the difference equations for the components \(\psi_\mu\) will be
\[
\sum_{k=-\sigma}^{\sigma} \alpha_k^\mu \psi_{\mu-k} = 0
\]
for all values of \(\mu\). The statement that the constraint operator annihilates the \(|\mu = 0\rangle\) state
implies that \(\alpha_0^k = 0\). In the difference equation the \(\psi_0\) component is always accompanied
with the \(\alpha_0^k\) term and thus decouples. Had the \(\text{sgn}(p)\sqrt{p}\) been ordered to the left of the
curvature term, the constraint operator would not annihilate the \(|\mu = 0\rangle\) state and the
singularity would not decouple.

III. HOLONOMY REPRESENTATIONS

We would now like to generalize the calculation of the Hamiltonian constraint operator
(7) and the inverse volume operator to arbitrary irreducible representations labeled by spin
\(J\). That this ambiguity is possible in the full theory was first elaborated in [23]. Of par-
ticular interest is what effective semi-classical equations of motion can be inferred from the
constraint operator and what phenomenological differences can arise from the modifications.
After constructing the arbitrary spin \(J\) constraint operator and inverse volume operator, we
will give an explicit construction of the \(J = 1\) constraint operator and the resulting difference
equation will be investigated. Following that, we will propose effective classical equations of
motion and bounds on their validity.
A. Quantum Constraint Operator and Inverse Volume Operator

Let us propose the generalization of the formula of (7) and show that it is a proper representation of the classical Hamiltonian constraint (2). The formula is

\[ (J)H_{GR} = -\frac{3}{\kappa^2 \gamma^3 \mu_0^3 J(J+1)(2J+1)} \sum_{ijk} \hat{e}^{ijk} (J) \text{tr} \left( h_i h_j h_i^{-1} h_j^{-1} h_k \{ h_k^{-1}, V \} \right). \]  

(10)

We can restore the classical constraint by considering the limit as \( \mu_0 \) goes to zero. The holonomies are given by

\[ h_i = e^{\mu_0 c \tau_i} = 1 + \mu_0 c \tau_i + \frac{(\mu_0 c)^2}{2} \tau_i^2 + O(\mu_0^3) \]  

(11)

from which we find

\[ \sum_{ij} \epsilon^{ijk} h_i h_j h_i^{-1} h_j^{-1} = 2 (\mu_0 c)^2 \tau_k + O(\mu_0^2) \]

\[ h_k \{ h_k^{-1}, V \} = -\mu_0 \{ c, V \} \tau_k + O(\mu_0^2) = -\frac{1}{2} \kappa \gamma \mu_0 \text{sgn}(p) \sqrt{|p|} \tau_k + O(\mu_0^2). \]

Now using the formula \( (J)\text{tr}(\tau_i \tau_j) = -\frac{1}{3} J(J+1)(2J+1) \delta_{ij} \) we get

\[ \sum_{ijk} \epsilon^{ijk} (J) \text{tr} (h_i h_j h_i^{-1} h_j^{-1} h_k \{ h_k^{-1}, V \}) = \mu_0^3 \gamma \kappa J(J+1)(2J+1) \text{sgn}(p) \sqrt{|p|} c^2 + O(\mu_0^2) \]

from which we recover the classical expression given in (2).

As in the previous section the classical expression of the Hamiltonian constraint is promoted to a quantum operator using the holonomy operators and promoting the Poisson bracket to a commutator. The resulting expression is given by

\[ (J)\hat{H}_{GR} = \frac{3i}{\kappa \ell_p^2 \gamma^3 \mu_0^3 J(J+1)(2J+1)} \sum_{ijk} \epsilon^{ijk} (J) \text{tr} \left( \hat{h}_i \hat{h}_j \hat{h}_i^{-1} \hat{h}_j^{-1} \hat{h}_k \left[ \hat{h}_k^{-1}, \hat{V} \right] \right) \]

\[ = \frac{9i}{\kappa \ell_p^2 \gamma^3 \mu_0^3 J(J+1)(2J+1)} (J) \text{tr} \left\{ \left( \hat{h}_{i1} \hat{h}_{i2} \hat{h}_{i2}^{-1} \hat{h}_{i1}^{-1} - \hat{h}_{i2} \hat{h}_{i1} \hat{h}_{i2}^{-1} \hat{h}_{i1}^{-1} \right) \hat{h}_3 \left[ \hat{h}_3^{-1}, \hat{V} \right] \right\} \]  

(12)

where we have used the fact that the operator is gauge invariant to arrive at the second line.

To calculate the action of the operator we need expressions for the matrix elements of the holonomies in the spin \( J \) representations in order to perform the trace. Fortunately such
formulas do exist and a particular form we will use is

\[
(h_1)_{mn} = T_{mn} \sum_{s=|m-n|}^{2J-|m+n|} \frac{(-i)^s}{Y_{mn} s} \hat{c} s^{2J-s} \hat{s} n^s
\]

(13a)

\[
(h_2)_{mn} = T_{mn} \sum_{s=|m-n|}^{2J-|m+n|} \frac{(-i)^{n-m+s}}{Y_{mn} s} \hat{c} s^{2J-s} \hat{s} n^s
\]

(13b)

\[
(h_3)_{mn} = e^{im\mu_0 \hat{c}} \delta_{mn}
\]

(13c)

where the matrix elements \(m, n \in [-J, J]\) and \(T_{mn}\) and \(Y_{mn, s}\) are constant coefficients given by

\[
T_{mn} = \sqrt{(J+m)! (J-m)! (J+n)! (J-n)!}
\]

\[
Y_{mn, s} = [J + \frac{1}{2}(m + n - s)]! \frac{1}{2} (m + n + s)! \frac{1}{2} (m + s)! \frac{1}{2} (n + m + s)! (14)
\]

The index \(s\) in the sum increments by two which implies that if \((m - n)\) is an even number, the sum over \(s\) will comprise even values and vice versa. We can see that the holonomy operators are more complicated sums and products of the basic sine and cosine operators whose action is given in equations (5). Hence, the holonomies have a well-defined action on the basis states \(|\mu\rangle\).

With the formula for the holonomies (13) we now can determine the action of the constraint operator. The \(h_3 \left[ h_3^{-1}, \hat{V} \right]\) term is easiest to formulate given the simple expression (13c) for \(h_3\). We find

\[
\left( h_3 \left[ h_3^{-1}, \hat{V} \right]\right)_{mn} = \hat{V} \delta_{mn} - e^{im\mu_0 \hat{c}} \delta_{mo} \hat{V} e^{-im\mu_0 \hat{c}} \delta_{on}
\]

\[
= \left[ \hat{V} - e^{im\mu_0 \hat{c}} \hat{V} e^{-im\mu_0 \hat{c}} \right] \delta_{mn}.
\]

This operator acts diagonally on the basis states \(|\mu\rangle\) owing to the action of \(e^{-im\mu_0 \hat{c}}|\mu\rangle = |\mu - 2m\mu_0\rangle\). The eigenvalues are given by

\[
\left( h_3 \left[ h_3^{-1}, \hat{V} \right]\right)_{mn} |\mu\rangle = [V_\mu - V_{\mu - 2m\mu_0}] \delta_{mn} |\mu\rangle
\]

(15)

and so the SU(2) matrix elements of this operator are diagonal.

Since \(h_3 \left[ h_3^{-1}, \hat{V} \right]\) has diagonal matrix elements we need only consider the diagonal elements \(\left( h_1 h_2 h_1^{-1} h_2^{-1} - h_2 h_1 h_2^{-1} h_1^{-1} \right)_{mn}\) when performing the trace. Using the holonomy
formulas \( \text{13a, 13b} \) we get

\[
\left( \hat{h}_1 \hat{h}_2 \hat{h}_1^{-1} \hat{h}_2^{-1} - \hat{h}_2 \hat{h}_1 \hat{h}_2^{-1} \hat{h}_1^{-1} \right)_{mm} = \sum_{n,o,p=-J}^{J} T_{mn} T_{no} T_{op} T_{pm} \sum_{s1=|m-n|, s2=|n-o|, s3=|o-p|, s4=|p-m|} \sum_{2J-|m+n|, 2J-|o+p|} \times (-i)^{s} \left(-1\right)^{s+4} \left[ (-i)^{o-n+m-p} - (-i)^{n-m+p-o} \right] Y_{mn} s_1 Y_{no} s_2 Y_{op} s_3 Y_{pm} s_4 \]

\[
= \sum_{S=0}^{8J} Z_{m}^{S} \, c_{S}^{8J-S} \, \hat{s}_{n}^{S} \]

where the parameter \( S \) is simply \( S = s_1 + s_2 + s_3 + s_4 \) and we have hidden the complexity of the rhs of the first line in the parameters \( Z_{m}^{S} \) which are just constant coefficients. Taken together and performing the trace the constraint operator is given by

\[
\langle J \rangle \hat{H}_{GR} = \frac{9i}{k \ell_{p}^{2} \gamma \mu_{p}^{2} J (J+1)(2J+1)} \sum_{m=-J}^{J} \sum_{S=0}^{8J} Z_{m}^{S} \, c_{S}^{8J-S} \, \hat{s}_{n}^{S} \left[ \hat{V} - e^{im\mu_{c}t} \hat{V} e^{-im\mu_{c}t} \right] \quad (16)
\]

We can simplify this formula greatly by looking in detail at the coefficients \( Z_{m}^{S} \) many of which vanish. Let us again display the formula

\[
Z_{m}^{S} = \sum_{n,o,p=-J}^{J} T_{mn} T_{no} T_{op} T_{pm} \sum_{s1=|m-n|, s2=|n-o|, s3=|o-p|, s4=|p-m|} \sum_{2J-|m+n|, 2J-|o+p|} (-i)^{S} \left(-1\right)^{s+4} \left[ (-i)^{o-n+m-p} - (-i)^{n-m+p-o} \right] Y_{mn} s_1 Y_{no} s_2 Y_{op} s_3 Y_{pm} s_4 \]

\[
(17)
\]

with the restriction that \( s_1 + s_2 + s_3 + s_4 = S \). To simplify this we will use two important facts. First, note that when \( (o-n) + (m-p) \) in the sum is an even number, the numerator vanishes. Second, as we have stated since the \( s \) parameters increment by two, their evenness is determined by the parameters \( m, n, o, p \), thus for instance if \( m-n \) is even then \( s_1 \) is even.

We now show that the coefficients satisfy the following:

\[
Z_{m}^{S} = 0 \quad \text{for } S \text{ odd} \quad (18a)
\]

\[
Z_{m}^{0} = Z_{m}^{8J} = 0 \quad (18b)
\]

\[
Z_{m}^{S} = -Z_{-m}^{S} \quad (18c)
\]

To show \( 18a \) we note that if \( S \equiv s_1 + s_2 + s_3 + s_4 \) is odd then this implies that \( (m-n) + (n-o) + (o-p) + (p-m) \) is also odd. However, this quantity is equal to zero and hence
even. Thus $S$ must be even and $Z_m^S = 0$ for $S$ odd. (18b) can be shown by calculating $Z_m^S$ explicitly. If $S = 0$ then this implies that all $s_i$ are zero. This only occurs in the sum over $n, o, p$ when $n = o = p = m$. Hence, $o - n + m - p = 0$ which is even and thus the numerator vanishes. A similar calculation holds for $Z_m^{8J}$. To show (18c) one can make the substitution in (17) $m, n, o, p \rightarrow -m, -n, -o, -p$. Using the fact that $T_{mn} = T_{-m-n}$ and $Y_{mn} = Y_{-m-n}$ (which follow from the definitions (14)) it is trivial to show that $Z_m^S = -Z_m^{-S}$.

Using these properties we simplify the constraint operator to

$$(J) \hat{H}_{GR} = \frac{-9i}{\kappa l \alpha^3 \mu_0^3 J(J+1)(2J+1)} \sum_{m=-J}^{J} \sum_{S'=-J}^{J} Z_m^{2S'} \hat{c}_{S'}^8 \hat{c}_{S'} e^{i\mu_0 \hat{c}} \hat{V} e^{-i\mu_0 \hat{c}}$$

(19)

where the term in (16) involving only $\hat{V}$ vanishes due to (18c). The action of this operator on the basis states $|\mu\rangle$ can be understood as follows. We showed that the term $e^{i\mu_0 \hat{c}} \hat{V} e^{-i\mu_0 \hat{c}}$ acts diagonally on the basis states and in the next section we will show that it approximates the $\text{sgn}(p) \sqrt{p}$ term in the classical constraint. The $\hat{c}_{S}$ and $\hat{s}_{n}$ terms each raise and lower the basis states by discrete steps up to $\pm 8J$. Since the sum is over even powers of $\hat{c}_{S}$ and $\hat{s}_{n}$, $\hat{H}$ will raise and lower the basis states by even values of $\mu_0$. Thus in general we can say the action on the basis states is

$$(J) \hat{H}_{GR} |\mu\rangle = \sum_{k=-4J}^{4J} \alpha_k^\mu |\mu + 2k\mu_0\rangle$$

where $\alpha_k^\mu$ are coefficients determined by equation (19). The resulting difference equation will be of order $8J$. Furthermore, the fundamental step size of the difference equation is $\delta_\mu = 2\mu_0$. Note that for the $J = 1/2$ operator some further cancellation occurs and the difference equation given in equation (19) has step size equal to $4\mu_0$. While for arbitrary $J$ similar cancellations may occur, we will show that for $J = 1$ the step size is indeed $2\mu_0$ and the difference equation is of order $8J = 8$. We can state for certain that the smallest possible step size is $2\mu_0$.

With regards to the singularity behavior we have stated that singularity removal occurs if the constraint operator annihilates the $|\mu = 0\rangle$ state. To show that this occurs for arbitrary $J$ we will use the fact that $Z_m^S = -Z_m^{-S}$. The constraint will thus contain the following term

$$\sum_{m=1}^{J} Z_m^{2S'} \left( e^{i\mu_0 \hat{c}} \hat{V} e^{-i\mu_0 \hat{c}} - e^{-i\mu_0 \hat{c}} \hat{V} e^{i\mu_0 \hat{c}} \right)$$
whose action on the basis states contributes a $\sum_{m=1}^{2S'} (V_{\mu-2m\mu_0} - V_{\mu+2m\mu_0})$ term. It is easy to see that this vanishes for $\mu = 0$ and hence the singularity decouples from the difference equation.

Turning now to the inverse volume operator, the formula for arbitrary $J$ is

$$\hat{V}^{-1} = \left[ -\frac{l_p^2 \gamma \mu_0 J (J + 1)(2J + 1)}{4i} \sum_i \text{tr} \left( \tau_i \hat{h}_i \left[ \hat{h}_i^{-1}, \hat{V}^{1/2} \right] \right)^6 \right]$$

and can be approximated for large $J$ by the function

$$d_J(a) \approx \frac{1}{a^3} \left( \frac{a^2}{a_*^2} \right)^6 D(q) = 2q^{1/4} \left[ \frac{4}{7} [(q + 1)^{7/4} + \text{sgn}(q - 1)|q - 1|^{7/4}] - \frac{16}{77} [(q + 1)^{11/4} + |q - 1|^{11/4}] \right]$$

where $a_* = \sqrt{\frac{\gamma \mu_0 l_p}{3}}$ is the characteristic scale factor below which the quantum corrections are large. The eigenvalues are bounded with the maximum value occurring near $a_*$. For $a \gg a_*$ the function $d_J(a)$ approximates well the classical expression $a^{-3}$. Below $a_*$, $d_J(a)$ behaves polynomially and can be approximated by treating the sum as an integral to get

$$d_J(a) \approx \left( \frac{12}{7} \right)^6 \frac{a^{12}}{a_*^{15}}.$$

Our goal was to note explicitly the role played by the regularization length $\mu_0$. We note that the effect is simply to shift $a_*$ to larger or smaller volumes.

We have thus succeeded in formulating the constraint operator (19) and inverse volume operator (20) for arbitrary $J$. The key results for the Hamiltonian constraint are that the difference equation is higher order for larger $J$, yet the step size remains on the same order. The singularity removal mechanism remains valid for arbitrary $J$. For the inverse volume operator, the eigenvalues $d_J$ are bounded near a critical scale factor $a_*$ whose value scales
both with \( \mu_0 \) and \( J \). We do not explore any further the general properties of the quantum constraint for arbitrary \( J \) as we do not have a simple formula for the coefficients \( Z^S_m \) and the resulting difference equation would be quite complicated and none the enlightening.

B. Quantum Theory for \( J = 1 \)

We now wish to calculate an explicit example of the quantum operator given in equation (19). For simplicity we choose the \( J = 1 \) representation. We will derive the difference equation and show that under appropriate conditions the solutions approximately satisfy the Wheeler-DeWitt differential equation. We will also examine the local stability of the difference equation to determine the behavior of spurious solutions.

After a little labor the \( J = 1 \) operator is given by

\[
(1) \quad \hat{H}_{GR} = \frac{12}{\kappa l^2 p^3 \mu_0^3} \left[ \hat{c}s^6 \hat{s}^2 - 2\hat{c}s^4 \hat{s}^4 - \hat{c}s^2 \hat{s}^6 \right] e^{i\mu_0 \hat{c}} e^{-i\mu_0 \hat{c}} \hat{c} e^{i\mu_0 \hat{c}} e^{-i\mu_0 \hat{c}}
\]

whence the action on the basis states is

\[
(1) \quad \hat{H}_{GR} |\mu\rangle = -\frac{3}{32 \kappa^2 \gamma^2 \mu_0^2} s_1(\mu) \left\{ |\mu + 8 \mu_0\rangle - 4|\mu + 6 \mu_0\rangle - 4|\mu + 4 \mu_0\rangle + 4|\mu + 2 \mu_0\rangle + 6|\mu\rangle + 4|\mu - 2 \mu_0\rangle - 4|\mu - 4 \mu_0\rangle - 4|\mu - 6 \mu_0\rangle + |\mu - 8 \mu_0\rangle \right\}
\]

where we have defined

\[ s_1(\mu) \equiv \frac{V_{\mu+2\mu_0} - V_{\mu-2\mu_0}}{\gamma l^2 p \mu_0} \]

and it can easily be shown that for large volume \( s_1(p) \approx \sqrt{p} \). The resulting difference equation is

\[
-\frac{3}{32 \kappa^2 \gamma^2 \mu_0^2} \left\{ s_1(\mu + 8 \mu_0) \psi_{\mu+8\mu_0} - 4s_1(\mu + 6 \mu_0) \psi_{\mu+6\mu_0} - 4s_1(\mu + 4 \mu_0) \psi_{\mu+4\mu_0} + 4s_1(\mu + 2 \mu_0) \psi_{\mu+2\mu_0} + 6s_1(\mu) \psi_{\mu} + 4s_1(\mu - 2 \mu_0) \psi_{\mu-2\mu_0} - 4s_1(\mu - 4 \mu_0) \psi_{\mu-4\mu_0} - 4s_1(\mu - 6 \mu_0) \psi_{\mu-6\mu_0} + s_1(\mu - 8 \mu_0) \psi_{\mu-8\mu_0} \right\} = -\hat{H}_m(\mu) \psi_{\mu}.
\]

As predicted in the general case, the difference equation is of order \( 8J = 8 \) with step size \( \delta \mu = 2\mu_0 \).
To make contact with the Wheeler-DeWitt equation we will make the following assumptions. First is that we are in the large volume limit where \( \mu \gg \mu_0 \). Second is that the wave function coefficients \( \psi_\mu \) do not vary sufficiently fast on the order of \( \delta \mu = 2\mu_0 \) such that we can approximate the discrete coefficients with a continuous function \( \psi(\mu) \). Whether or not this assumption holds true for large volume is model dependent and an example where this continuum approximation fails is the isotropic flat model with a positive cosmological constant. The consequences for that model are discussed in depth in [24]. Yet, let us assume for the moment that we can make these assumptions, hence we can Taylor expand the function \( \psi(\mu + \delta) \approx \psi(\mu) + \frac{d\psi}{d\mu} \delta + \frac{1}{2} \frac{d^2\psi}{d\mu^2} \delta^2 \). Under these assumptions \( \psi(\mu) \) will approximately satisfy a differential equation.

The Wheeler-DeWitt equation is derived from the classical action (2) be performing the usual Schrodinger quantization where the operator \( \hat{c} \) is quantized to \(-\frac{i}{3} \hbar \kappa \gamma \frac{\partial}{\partial p} \). The resulting differential equation is

\[
\frac{\kappa \hbar^2}{3} \frac{\partial^2}{\partial p^2} \left( \sqrt{p} \psi(p) \right) + \hat{H}_m \psi(p) = 0.
\] (24)

Turning to the difference equation let us define \( t_\mu \equiv s_1(\mu) \psi_\mu \) and the difference equation simplifies to

\[
- \frac{3}{32 \kappa \gamma^2 \mu_0^2} \left\{ t_{\mu+8\mu_0} - 4t_{\mu+6\mu_0} - 4t_{\mu+4\mu_0} + 4t_{\mu+2\mu_0} + 4t_{\mu-2\mu_0} - 4t_{\mu-4\mu_0} - 4t_{\mu-6\mu_0} + t_{\mu-8\mu_0} \right\} = -\hat{H}_m(\mu) \psi_\mu.
\]

Using \( p = \frac{\mu \gamma l}{6} \) and Taylor expanding \( t(p) \) we find

\[
t_{\mu+8\mu_0} - 4t_{\mu+6\mu_0} - 4t_{\mu+4\mu_0} + 4t_{\mu+2\mu_0} + 6t_\mu + 4t_{\mu-2\mu_0} - 4t_{\mu-4\mu_0} - 4t_{\mu-6\mu_0} + t_{\mu-8\mu_0} \approx -\frac{32}{9} \mu_0^2 \gamma^2 t^4 \frac{\partial^2 t(p)}{\partial p^2}.
\]

For large volume \( t(p) \equiv s_1(p) \psi(p) \approx \sqrt{p} \psi(p) \) and plugging this in we find that the difference equation is approximated by the Wheeler-DeWitt equation.

Next we would like to examine the local stability of the difference equation. Since the difference equation is of higher order than the second order Wheeler-DeWitt equation, we must determine the behavior of the spurious solutions. As determined in [25], if the higher order difference equation admits solutions with amplitudes that grow (locally), then the difference equation is not locally stable and this might call into the question the validity
of the quantization as any semi-classical solutions would quickly become dominated by the expanding spurious solutions.

Let us be more precise. We will consider the large volume limit in a regime where the matter contribution is small. We assume that the volume is large enough such that the variations of $V_\mu$ and $H_m(\mu)$ are small. We can thus approximate the difference equation by one with constant coefficients

$$
\psi_{\mu+8\mu_0} - 4\psi_{\mu+6\mu_0} - 4\psi_{\mu+4\mu_0} + 4\psi_{\mu+2\mu_0} + \psi_{\mu} + 4\psi_{\mu-2\mu_0} - 4\psi_{\mu-4\mu_0} - 4\psi_{\mu-6\mu_0} + \psi_{\mu-8\mu_0} = \frac{32\kappa^2\mu_0^2 H_m(\mu)}{3s_1(\mu)} \psi_{\mu} = P \psi_{\mu}
$$

where we assume that on the order of $\delta \mu = 8\mu_0$ that $P \equiv \frac{32\kappa^2\mu_0^2 H_m(\mu)}{3s_1(\mu)}$ is constant. The difference equation with constant coefficients can be solved exactly by assuming solutions of the form $\psi_\mu = A z^{\mu/\mu_0}$ where $z \in \mathbb{C}$. The condition of local stability states that we should consider solutions of the homogeneous equation (with $P = 0$) and that the difference equation is locally stable if all solutions have norm equal to one, that is $|z|^2 = 1$ [25]. This will guarantee that spurious solutions which don’t approximate a semi-classical Wheeler-DeWitt solution will not come to dominate.

Plugging in our ansatz $\psi_\mu = A z^{\mu/\mu_0}$ to the homogeneous difference equation (25) we find the following condition on $z$

$$
z^8 - 4z^6 - 4z^4 + 4z^2 + 6 + 4z^{-2} - 4z^{-4} - 4z^{-6} + z^{-8} = 0.
$$

A numerical calculation of the roots shows that there are solutions with norms greater than one, thus the $J = 1$ difference equation is not locally stable. The result of this behavior is plotted in figure (1) for a massless scalar field with constant momentum where $H_m(\mu) = \frac{1}{2}d_J(\mu)P_0^2$. The figure shows a solution to the difference equation (25) and a solution to the Wheeler-DeWitt equation (24) for comparison. Initial conditions are specified on $\psi_\mu$ to match the Wheeler-DeWitt solution and the solution is evolved forward using the difference equation. It is clear that the solution of the difference equation follows the Wheeler-DeWitt solution briefly, after which the spurious non semi-classical solutions quickly dominate the wave function.

The presence of the ill behaved spurious solutions does not by itself represent a problem with the quantization. To truly determine the physical consequence requires a detailed understanding of the physical inner product. With a physical inner product, one could
FIG. 1: Log plot of the difference equation solution (dots) compared with Wheeler-DeWitt solution (solid line) for a massless scalar field at large volume. The presence of spurious solutions to the higher order difference equation eventually dominate to solution.

identify which solutions are physical based on the notion of probability, that is unphysical solutions would have either vanishing or infinite physical norm and would be modded out of the physical Hilbert space. An exampled where this happens is in the symmetry reduced Plebanski model of [24]. There, extra solutions have zero physical norm and the resulting physical Hilbert space is one dimensional for matter in the form of a cosmological constant. If the spurious solutions here do indeed have zero physical norm then the quantization presented would have the correct semi-classical limit. If not this would indicate the need for different quantizations or even the possibility that the higher spin quantizations do not represent consistent quantizations of isotropic LQC. This might indicate a preference of the quantum theory toward the use of the fundamental representation for defining the Hamiltonian constraint. There are possible indications that this behavior also occurs in the full theory where a large set of spurious solutions exist for the higher $J$ gravitational constraint operator [26].
C. Effective Classical Theory

We now turn to the question of what sort of semi-classical equations of motion can be inferred from the constraint operator given in equation (19). While in general the effective equations of motion should be derived from the wave function solutions themselves (for instance by finding semi-classical states), here we will take the simplest route. Our motivation comes from the path integral quantization of LQC where a discretized path integral involves integration over a classical action (for details on the path integral derivation see [24]). A detailed derivation of the path integral shows that the effective classical Hamiltonian constraint is given by the p-q symbol

\[
H_{\text{eff}} = \frac{\langle c | \hat{H} | p \rangle}{\langle c | p \rangle}.
\]

Since the Hamiltonian constraint of LQC consists of a self-adjoint part (the \( h_i h_j h_i^{-1} h_j^{-1} \) term) of which \( |c\rangle \) is an eigenstate, and a part \( \{ h_k^{-1}, V \} \) of which \( |p\rangle \) is an eigenstate we need only consider the eigenvalues of the two operators to get an effective classical constraint written in terms of a classical \( c \) and \( p \). We note that a more detailed consideration involving semi-classical states could lead to additional corrections. Equations of motion can then be derived from the effective Hamiltonian through the Hamiltonian equations

\[
\dot{p} = \{ p, H_{\text{eff}} \} \quad \text{and} \quad \dot{c} = \{ c, H_{\text{eff}} \}.
\]

We are interested in determining what modifications arise for instance to the Friedmann equations from the quantum effects.

We now show that two kinds of corrections arise in the effective constraint. The first occurs for small volumes and arises from the \( h_k \{ h_k^{-1}, V \} \) term which we will show behaves as a modification to the classical \( \text{sgn}(p) \sqrt{|p|} \) term in the constraint. The second modification is due to the curvature term \( h_i h_j h_i^{-1} h_j^{-1} \) and amounts to a modification of the classical \( c^2 \) term. This modification is evident when the connection \( c \) is large or equivalently (for flat models) when the extrinsic curvature is large. We will show that for large volume and small extrinsic curvature, the correct classical equations are recovered.

Let us now calculate the corrections explicitly. From equation (15) we get the formula for the \( h_3 \{ h_3^{-1}, V \} \) term as

\[
\left( \hat{h}_3 \left[ \hat{h}_3^{-1}, \hat{V} \right] \right)_{mn} |\mu\rangle = \left[ V_{\mu} - V_{\mu-2m\mu_0} \right] \delta_{mn}.
\]

Next the curvature term \( h_i h_j h_i^{-1} h_j^{-1} \) can be calculated by expanding the holonomies \( h_i = e^{\mu_0 c \tau_i} \) as done for equation (11). If taken to order \((\mu_0 c)^4\) we find that

\[
h_1 h_2 h_1^{-1} h_2^{-1} - h_2 h_1 h_2^{-1} h_1^{-1} = 2(\mu_0 c)^2 \tau_3 + (\mu_0 c)^3 (\tau_1 - \tau_2) - \frac{2}{3} (\mu_0 c)^4 \tau_3 + \mathcal{O}((\mu_0 c)^5).
\]
Now using \((\tau_3)_{mn} = \text{i}m\delta_{mn}\) and the fact that \((\tau_1)_{mn}\) and \((\tau_2)_{mn}\) are off-diagonal matrices we find after performing the trace

\[
H_{\text{eff}} = \frac{-9i}{\kappa \gamma^3 l_p^2 p_0^2} J(J+1)(2J+1) \left( 2\mu_0^2 c^2 - \frac{2}{3} \mu_0^4 c^4 + \mathcal{O}(\mu_0^5 \gamma^5) \right) \sum_{m=-J}^{J} \text{i}m[V_\mu - V_{\mu-2m\mu_0}]
\]

\[
= -\frac{3}{\kappa \gamma^2} s_J \left( c^2 - \frac{1}{3} \mu_0^2 c^4 \right) + \mathcal{O}\left( \frac{\mu_0^3 c^5}{\kappa \gamma^2} \right)
\]

(26)

and this is the formula we will use as our effective constraint. Notice the similarity to the classical constraint in (2) except for two modifications. The first is the presence of the function \(s_J\) which we have defined to be

\[
s_J \equiv -\frac{6}{\gamma l_p^2 p_0 J(J+1)(2J+1)} \sum_{m=-J}^{J} mV_{\mu-2m\mu_0}
\]

(27)

which we now show approximates \(\text{sgn}(p) \sqrt{|p|}\). This can be seen by noting that for large volume \(V_{\mu-2m\mu_0} \approx V_\mu - m\gamma \mu_0 l_p^2/2 \text{sgn}(p) \sqrt{|p|} + \mathcal{O}(\frac{\mu_0^2 |p|^{3/2}}{\sqrt{p}})\). Thus we get

\[
s_J \approx -\frac{6}{\gamma l_p^2 p_0 J(J+1)(2J+1)} \sum_{m=-J}^{J} m \left( V_\mu - \frac{m\gamma \mu_0 l_p^2}{2} \text{sgn}(p) \sqrt{|p|} + \mathcal{O}(\frac{\mu_0^2 \gamma l_p^4}{\sqrt{p}}) \right)
\]

\[
\approx \text{sgn}(p) \sqrt{|p|} + \mathcal{O}\left( \frac{\mu_0 \gamma l_p^4}{\sqrt{p}} \right)
\]

where we have used \(\sum_{m=-J}^{J} m^2 = 1/3 J(J+1)(2J+1)\) to arrive at the second line. As for the formula for \(d_J(\mu)\) we can approximate \(s_J(\mu)\) for large values of \(J\) by treating the sum as an integral. The resulting formula is

\[
s_J(\mu) \approx \sqrt{\frac{\gamma l_p}{6}} \frac{1}{2\mu_0^3 J^3} \left\{ \frac{J \mu_0}{5} \left[ (\mu + 2J\mu_0)^{5/2} + \text{sgn}(\mu - 2J\mu_0) |\mu - 2J\mu_0|^{5/2} \right] - \frac{1}{35} \left[ (\mu + 2J\mu_0)^{7/2} - |\mu - 2J\mu_0|^{7/2} \right] \right\}
\]

which implies that written in terms of the scale factor \(a\)

\[
s_J(a) = a S(a^2/a_*^2)
\]

\[
S(q) = \frac{4}{\sqrt{q}} \left\{ \frac{1}{10} \left[ (q + 1)^{5/2} + \text{sgn}(q-1) |q - 1|^{5/2} \right] - \frac{1}{35} \left[ (q + 1)^{7/2} - |q - 1|^{7/2} \right] \right\}
\]

(28)

where as for \(d_J(a)\) we have the same critical scale factor \(a_* = \sqrt{\frac{\gamma l_p}{4}} l_p\) below which quantum corrections occur. The function \(S(q)\) for \(q > 1\) is approximately equal to one which implies
that for $a > a_*$, $s_J(a) \approx a = \sqrt{|p|}$. For small volume $S(q)$ behaves as $\frac{6}{5}\sqrt{q}$ hence $s_J(a)$ behaves quadratically with a

$$s_J(a) \approx \frac{6}{5} \frac{a^2}{a_*} \quad a \ll a_*.$$ 

(29)

A plot of the function $s_J(a)$ compared with $\sqrt{|p|} = a$ is shown in figure (2). It is clear that $s_J$ behaves quadratically for small $a$ and changes behavior near $a_*$ after which it approximates the classical expression $\sqrt{|p|}$.

FIG. 2: LQC modified function $s_J(a)$ compared with the classical expression for the function $\text{sgn}(p)\sqrt{|p|} = a$. The modified $s_J(a)$ function behaves quadratically for $a < a_* = \sqrt{\frac{\gamma J \mu_0}{3} l_p}$. The values of $a_*$ in the graph are 1.290, 2.89, 4.08 for $J = 10, 50, 100$ respectively.

Thus we have determined the effective Hamiltonian constraint given in equation (26) which is valid in the regime where the connection component $c$ is small or more precisely when $\mu_0 c \ll 1$. The function $s_J(\mu)$ in turn approximates the classical $\text{sgn}(p)\sqrt{|p|}$ term for large volume when $\mu \gg 2J \mu_0$ or equivalently when $a \gg a_*$. Let us now make contact with the standard formulation of isotropic cosmology written in terms of the scale factor $a$. We assume that the orientation of the triad is positive $(p, \mu > 0)$ so we can drop the absolute value and sign terms in the formulas. According to
equations (1a, 3) we have the relation between $\mu$ and the scale factor as $a^2 = \frac{2}{9} \mu \mu_m^2$. From the Hamiltonian equations and the vanishing of the constraint we can derive the Friedmann equation. We first have

$$\dot{p} = \{p, H_{eff}\} = -\frac{1}{3} \kappa \gamma \frac{\partial H_{eff}}{\partial c}$$

$$= \frac{1}{\gamma} \left(2c - \frac{4}{3} \mu_0^2 c^3\right) s_J.$$  

Using $p = a^2$ we can use this equation to get the Friedmann equation in terms of the connection as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{s_J^2 c^2}{\gamma^2 a^4} \left(1 - \frac{4}{3} \mu_0^2 c^2 + \frac{4}{9} \mu_0^4 c^4\right).$$

From this it is clear that the criteria $\mu_0 c \ll 1$ corresponds to a criteria on the scale factor velocity being small $\mu_0 \gamma \frac{a}{s_J} \dot{a} \ll 1$. To get this equation in standard form we need to solve for the connection in terms of the matter Hamiltonian. We can get this from the vanishing of the full Hamiltonian constraint $H = H_{eff} + H_m = 0$. This gives

$$c^2 = \frac{\kappa \gamma^2}{3 s_J} H_m + \frac{\kappa^2 \gamma^4 \mu_0^2}{27 s_J^2} H_m^2$$

where we have only kept the terms to second order in $H_m$. Putting these two together we get the modified Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3} \frac{s_J}{a^4} H_m - \frac{\kappa^2 \gamma^2 \mu_0^2}{9 a^4} H_m^2$$

$$= \frac{\kappa}{3} S(a) \rho_m - \frac{1}{9} \kappa^2 \gamma^2 \mu_0^2 \rho_m^2$$

(30)

where the matter density is simply $\rho_m \equiv a^{-3} H_m$ and as before $s_J(a) = a S(a)$. The modified Friedmann equation remains valid provided $\mu_0 \gamma \frac{a}{s_J} \dot{a} \ll 1$. This in turn places a bound on the matter density $\rho_m \ll \frac{3 S(a)}{\kappa \mu_0^2 \gamma^2 a}$ below which the effective equations remain valid. From the modified Friedmann equation it is clear that in the large volume regime (where $S(a) \approx 1$) and small curvature (where we neglect the $\rho_m^2$ term) we recover the standard Friedmann equation $(\frac{\dot{a}}{a})^2 = \frac{\kappa}{3} \rho_m$.

Let us reiterate the main results of the section. The modified effective Hamiltonian constraint is given in equation (26) and we rewrite it here

$$H_{eff} = -\frac{3}{\kappa \gamma^2 s_J} \left(c^2 - \frac{1}{3} \mu_0^2 c^4\right)$$

(31)
where the function $s_J$ is given in equation (27) and approximates $\text{sgn}(p)\sqrt{|p|}$ for large volume. The modified Friedmann dynamics given in equation (30) was calculated through the Hamiltonian equations derived from the effective constraint. The effective constraint is valid when $\mu_0 c \ll 1$ which in turn places a bound of validity determined by both the extrinsic curvature and on the matter density.

IV. DISCUSSION

In this paper we have sought to clear up certain issues in LQC, namely the complete consequences of quantizing the Hamiltonian constraint using higher spin representations for the holonomies. In addition we have fixed the constants necessary to make contact on a consistent basis with the standard action of GR. We have found that the use of higher spin holonomies to regulate the gravitational part of the constraint operator leads to modifications of the $\det([-q])$ term. The modifications are qualitatively similar to those of the inverse scale factor which lead to the $d_J(a)$ function. This is not unexpected since we are quantizing the constraint operator to take into account factors involving the inverse triad (the $1/det(|q|)$ term). This modification, leading to the $s_J(a)$ function, is another important consequence of a key mantra of LQC: to remain as close to the full theory of loop quantum gravity as possible. We could have exploited the symmetry of the isotropic model and quantized the $\text{sgn}(p)\sqrt{|p|}$ term avoiding any inverse triad operators. Because this is not possible in the full theory it is not done in LQC and the non-perturbative corrections are imported from the full theory.

We have been guided by certain simplicity considerations. Previous studies have used the $J = 1/2$ gravitational part of the constraint (and the resulting difference equation) while freely choosing an arbitrary representation to define the inverse volume operator in the matter part of the constraint. While there is certain freedom to specify the representations differently and quantize the two parts in a different manner, for instance to define a $J_G$ for the gravitational part and a $J_m$ for the matter part, the simplest choice is to use the same representation. This ambiguity also exists in the parameter $\mu_0$ appearing in the matter and gravitational part.

Thus it is important to reexamine the phenomenological consequences in light of the modifications to the Friedmann equation (30) that arise for arbitrary values of $J$. Let us
first concentrate on the modifications arising from $s_J(a)$. In the instance of a massless scalar field we have for the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa s_J(a) d_J(a)P_\phi^2}{3 a^4}$$

where we have used $H_m = \frac{1}{2}d_J(a)P_\phi^2$ with $P_\phi$ being constant. To understand the behavior of the scalar field let us de-parameterize the equations of motion to remove the notion of time. This is in accord with notions of quantum gravity where coordinate time does not have any physical meaning. Thus we are interested in how the scalar field evolves with the scale factor as opposed to coordinate time. From the Hamiltonian equations we find that $\dot{\phi} = d_J(a)P_\phi$ and we de-parameterize as $d\phi/da = \dot{\phi}/\dot{a}$ to get

$$\frac{d\phi}{da} = \sqrt{\frac{6a d_J(a)}{\kappa S(a)}}$$

which implies that the field is pushed to higher values in the region below $a_*$ since the $S(a)$ appearing in the denominator is suppressed in that region. The addition of the $s_J$ corrections can therefore improve the viability of LQC to push the inflaton up its potential to set the initial conditions for slow-roll inflation.

The consequences of the modifications to the Friedmann equation quadratic in matter have not been explored. Important though is that we have given a precise bound $\mu_0^2 \frac{\dot{a}}{s_J} \ll 1$ on the validity of the first order effective constraint. This is in contrast with the criteria for validity cited in other phenomenological investigations $a/\dot{a} \gg \sqrt{\gamma l_p}$ based on the notion that the Hubble length should not be smaller than the fundamental length of discreteness $\sqrt{\gamma l_p}$. The criteria given here can be understood heuristically as follows. The discreteness of the difference equation plays a role precisely where the Wheeler-DeWitt approximation breaks down, that is when the wave function changes significantly on the order of the fundamental step size $\delta \mu = 2\mu_0$. If we consider a Wheeler-DeWitt solution that is locally oscillatory $\psi(\mu) = \exp(i2\pi \mu/\lambda)$ where $\lambda$ is the wavelength, then the discreteness becomes important when $\lambda \approx 4\mu_0$. Given that in the Wheeler-DeWitt equation $\dot{c} = -\frac{i\hbar^2}{\sqrt{3}} \partial/\partial p$ we find that the connection $c$ is related to $\lambda$ as $c = 4\pi/\lambda$. Using this we recover the criteria that the higher order corrections occur when $\mu_0 c \approx 1$. We can therefore understand that these corrections are a direct result of discreteness effects of the difference equation.

The term quadratic in the matter density is negative definite in the Friedmann equation and hence can act as an effective compactness even in the non-compact model considered.
Whether or not this would indeed imply a re-collapse depends on the inclusion of the full corrections to arbitrary order of $\mu_0c$. Furthermore, classical descriptions in this regime are not expected to be valid since, as stated in the previous paragraph, this corresponds to the regime where the discreteness is dominant. A model where this occurs is deSitter space which is discussed in detail in [24]. In that model classically the extrinsic curvature grows with time and the wave functions become non semi-classical around a critical volume given by $V_c = \left(\frac{6}{\Lambda \gamma^2 \mu_0^2}\right)^{3/2}$ which is precisely when $\mu_0c = \pi$. Above the critical volume, the wave function decays rapidly indicating a classically forbidden region and thus an effective compactness. Thus this correction can have drastic consequences for the evolution of the universe. For ordinary matter such as dust or radiation, the matter density drops off sufficiently fast for these corrections not to be relevant at large volumes. Yet the fact that observations suggest that the universe might be in an asymptotically deSitter space call into question the validity of LQC for large volumes and a more detailed derivation of LQC from the full theory would likely solve this problem.

We are in the process of examining numerically the full phenomenological consequences of the modified Friedmann equation (30) for more complicated forms of matter. The fact that the $s_J$ corrections help push the inflaton up its potential is a benefit to this scenario and would help enlarge the parameter space that leads to successful inflation.

The key parameter involved in the quantum corrections is the regulating length $\mu_0$. A deep understand of the origin of this parameter is essential before exact phenomenological predictions can be made. In the full theory one takes the limit as $\mu_0$ goes to zero and a well-defined operator remains which due to diffeomorphism invariance is independent of the regulator [19]. In LQC the corrections for $s_J, d_J$ as well as for large extrinsic curvature depend explicitly on the value of $\mu_0$. If we were to take the limit of small $\mu_0$, the phenomenological consequences of the quantum corrections would be pushed into the deep Planckian regime. This might wash out any possible observational signals from LQC. Yet, a smaller value of $\mu_0$ would also push to larger volumes the validity of LQC for the deSitter example discussed above. In [16] it is argued heuristically that $\mu_0$ is fixed to a value of $\sqrt{3}/4$ in the Hamiltonian constraint based on the smallest allowed area of the full theory. A more thorough understand of the parameter $\mu_0$ and its derivation from the full theory is required to settle this issue.

A key issue with which we have not dealt is the need to derive the effective equations of
motion from solutions of the difference equation itself. Several proposals have been put forth to date on this matter. In one [21], kinematical semi-classical states are constructed and the effective constraint is calculated from the expectation value of the constraint operator. The validity of this is not entirely clear since the semi-classical states are not physical states in that they are not annihilated by the constraint operator. The corrections to the constraint do include a $\mu_0^2 c^4$ term as we showed in this paper (in that paper the corrections are calculated for large volume for the $J = 1/2$ constraint so would not include $s_J$ corrections). The exact factors for these corrections do not agree however. In addition corrections arise in that model due to the spread of the wave function, something of which we have not considered here in deriving the effective equations of motion.

In another technique [28], WKB type solutions of the difference equation are calculated and from that an effective constraint is extracted. In this paper the modified constraint includes a $-\frac{3}{2\gamma^2\kappa} \left[ s_{1/2}(\mu + 4\mu_0) + s_{1/2}(\mu - 4\mu_0) \right] c^2$ term which to leading order is equivalent to the effective constraint of equation (26) for $J = 1/2$. More detailed calculations in the WKB context [29] show the higher order corrections in $\mu_0 c$ appearing in our effective constraint (26). In that paper the effective constraint for large volume is given by $H_{eff} = -\frac{3}{\kappa \gamma^2 \mu_0^2} \sqrt{\mu} \sin^2(\mu_0 c)$ (where we have adjusted the multiplicative factors to agree with ours) which is precisely the effective constraint presented in this paper had we calculated all the higher order corrections for the $J = 1/2$ constraint (the $c^2 - \frac{1}{2} \mu_0^2 c^4$ term can be seen as the small $\mu_0 c$ expansion of $\sin^2(\mu_0 c)/\mu_0^2$). In addition the WKB effective constraint contains a potential term which contributes for small volume. Again the interpretation of dynamics in the WKB setting is not entirely clear especially in the setting of the discrete difference equation of LQC. In addition the approximation used in [28] relies on correlating wave function solutions of the difference equation on scales smaller than the fundamental step size $\delta \mu = 4\mu_0$. It can be shown in LQC that quantum interference only occurs for quantum states defined at values of $\mu$ differing by an integer times the fundamental step size $\delta \mu$. Thus, quantum mechanically LQC does not correlate the quantum wave function at a given volume $\mu$ and a neighboring value $\mu + \epsilon$ where $\epsilon < \delta \mu$.

A further method to determine semi-classical dynamics involves introducing a coordinate time parameter with which to consider evolution of Gaussian wave packets [22]. In that paper the wave packets are shown to follow the classical trajectory even to small volumes. The relevance of the solutions to physical solutions (annihilated by the constraint operator
and independent of any time parameterization) has yet to be determined.

Each of the techniques to derive effective equations of motion are performed with models with zero degrees of freedom, and thus none consider physical semi-classical states. In quantum gravity where coordinate time is not a physical degree of freedom, one needs to include higher degrees of freedom and choose one of them to play the role of a clock. The model of a massless scalar field provides a testing ground for these ideas. In this model, the volume is a monotonically increasing function (classically) and can play the role of a clock. A semi-classical state peaked around some value of the scalar field at a given volume can be evolved forward in "time" with the difference equation and the resulting trajectory compared with the effective equations of motion. This would have the benefit that the wave functions considered would be physical states annihilated by the constraint operator. Early results for the spin 1/2 difference equation indicate that the first order corrections for the effective equations match well with the results from the difference equations to volumes near the Planck scale. While this issue would be trickier to study for the higher order spin $J$ difference equation, the agreement for the spin 1/2 constraint is indicative of the validity of the effective equations.

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