ON LINEAR OPERATORS EXTENDING [PSEUDO]METRICS

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Abstract. For every closed subset $X$ of a stratifiable [resp. metrizable] space $Y$ we construct a positive linear extension operator $T : \mathbb{R}^{X \times X} \to \mathbb{R}^{Y \times Y}$ preserving constant functions, bounded functions, continuous functions, pseudometrics, metrics, [resp. dominating metrics, and admissible metrics]. This operator is continuous with respect to each of the three topologies: point-wise convergence, uniform, and compact-open.

An equivariant analog of the above statement is proved as well.

The problem of existing a linear operator extending [pseudo]metrics from a closed subset of a metric compactum $X$ over all of $X$ was posed by the second author in [4] and partly solved in [4], [5]. A complete solution of this problem appeared in [2] and [15] (see also [1] and [3]). M. Zarichnyi [16] presented a very simple construction of such extension operators.

In contrast to the mentioned results, the present paper, which is a simplified and generalized version of the preprint [1], allows to construct linear operators extending metrics which are continuous with respect to the point-wise convergence of functions.

For a space $Z$ we denote by $\mathbb{R}^Z$ the space of all, not necessarily continuous, real-valued functions on $Z$ with the Tychonoff product topology (which corresponds to the point-wise convergence of the functions).

Our first theorem is quite general and concerns stratifiable spaces, see [7] for their definitions and properties. Here we mention only that each metrizable space is stratifiable, each stratifiable space is perfectly paracompact, and every subspace of a stratifiable space is stratifiable too.

**Theorem 1.** Suppose $Y$ is a stratifiable space and $X$ is a closed subspace of $Y$ with $|X| \geq 2$. There exists a positive linear extension operator $T : \mathbb{R}^{X \times X} \to \mathbb{R}^{Y \times Y}$ preserving constant functions, bounded functions, continuous functions, pseudometrics, and metrics. This operator is continuous with respect to each of the three topologies: point-wise convergence, uniform, and compact-open.

Obviously the phrase "$T$ preserves bounded functions, etc." means that $T$ carries bounded functions, etc., on $X \times X$ into bounded functions, etc., on $Y \times Y$.

For metrizable spaces we are able to prove much more. It will be convenient to formulate the corresponding result in terms of uniform spaces (see Chapter 8 of [11] for the theory of uniform spaces). We remark that each metric space is automatically a uniform space. We call a uniform space metrizable if its uniformity is generated by a metric.

**Theorem 2.** Suppose $Y$ is a metrizable uniform space and $X$ is a closed subspace of $Y$ with $|X| \geq 2$. There exists a positive linear extension operator $T : \mathbb{R}^{X \times X} \to \mathbb{R}^{Y \times Y}$ preserving constant functions, bounded functions, continuous functions, pseudometrics, metrics, admissible metrics, dominating metrics, and uniformly dominating metrics. This operator is continuous with respect to each of the three topologies: point-wise convergence, uniform, and compact-open.

Moreover, if the uniform space $Y$ is complete, then $T$ preserves complete continuous uniformly dominating metrics. If $Y$ is totally bounded and $\dim(Y \setminus X) < \infty$, then $T$ preserves totally bounded pseudometrics.

A metric $d$ on a topological [resp. uniform] space $Z$ is called dominating [resp. uniformly dominating] if the formal identity map from the metric space $(Z, d)$ to $Z$ is [uniformly] continuous. A metric which is continuous and dominating is said to be admissible.

The proofs of the two theorems exploit Hartman-Mycielski space $HM(X)$ of all $X$-valued step functions defined on the interval $[0, 1)$ (in a similar way as Zarichnyi [16] applied the space of all $X$-valued measurable functions) and also Pikhurko’s [15] idea of constructing the required operator $T$ as sum of a series of operators “separating” points of $Y$.

Theorems 1 and 2 will be applied to construct linear operators extending invariant metrics. For a topological space $X$ by $C(X \times X)$ we denote the linear lattice of continuous functions on $X \times X$, equipped with the compact-open topology. If a compact topological group $G$ acts on $X$, let $C_{inv}(X \times X) = \{ f \in C(X \times X) : f(gx) = f(gy) \text{ for all } g \in G \text{ and } x, y \in X \}$ denote the subspace of $C(X \times X)$ consisting of all $G$-invariant functions.

**Theorem 3.** Suppose a compact topological group $G$ acts on a stratifiable space $Y$, and $X$ is a $G$-invariant subspace of $Y$ with $|X| \geq 2$. There exists a positive linear continuous (in the compact-open topology) extension operator

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functions, bounded functions, and \(p\) pseudo-metrics. From Proposition 5 of [9] and its proof it follows that if the space \(Y\) is metrizable, then additionally \(T\) preserves admissible metrics. If the group \(G\) is finite, then the operator \(T\) is continuous with respect to the point-wise convergence of functions.

The last theorem is an improvement obtained by the second author of a former result of [1] thanks to a discussion with C. Atkin. Another contribution of the second author is Section 5 containing a relatively simple construction of extension operators \(S, S_1, S_2\) having almost all properties of the operator \(T\) from Theorems 1 and 2 (except that \(S\) does not preserve metrics, \(S_1\) fails to preserve constants, and \(S_2\) is not positive).

1. Hartman-Mycielski Construction

This construction appeared in [14] in connection with some problems of topological algebra, see also [9]. For an \(n \in \mathbb{N}\) and a topological space \(X\) let \(HM_n(X)\) be the set of all functions \(f : [0, 1) \to X\) for which there exists a sequence \(0 = a_0 < a_1 < \cdots < a_n = 1\) such that \(f\) is constant on each interval \([a_{i−1}, a_i)\), \(1 \leq i \leq n\). Let \(HM(X) = \bigcup_{n \in \mathbb{N}} HM_n(X)\).

A neighborhood sub-base of the topology of \(HM(X)\) at an \(f \in HM(X)\) consists of sets \(N(a, b, V, \varepsilon)\), where

1. \(0 \leq a < b \leq 1\), \(f\) is constant on \([a, b)\), \(V\) is a neighborhood of \(f(a)\) in \(X\), and \(\varepsilon > 0\);
2. \(g \in N(a, b, V, \varepsilon)\) means that \(\{t \in [a, b) : g(t) \notin V\} < \varepsilon\), where \(|\cdot|\) denotes the Lebesgue measure.

As noted in [9] Proposition 2 for every subspace \(A\) of \(X\), the space \(HM(A)\) can be considered as a subspace of \(HM(X)\). Also, the space \(X\) can be identified with the subspace \(HM_1(X)\) of \(HM(X)\).

For an element \(f \in HM(X)\) let \(\text{supp}(f)\) denote the smallest subset \(A \subset X\) such that \(f \in HM(A) \subset HM(X)\). Evidently that \(\text{supp}(f) = f([0, 1))\).

Recall that for a space \(Z\) the space \(\mathbb{R}^Z\) is endowed with the Tychonoff product topology (which corresponds to the point-wise convergence on \(\mathbb{R}^Z\) considered as a function space).

**Proposition 1.** The formula

\[
{\text{hm}}(d)(f, g) = \int_0^1 d(f(t), g(t))dt
\]

defines a positive linear continuous extension operator

\[
{\text{hm}} : \mathbb{R}^{X \times X} \to [0, \infty)^{HM(X) \times HM(X)}
\]

preserving constant functions, bounded functions, and bounded continuous functions, pseudometrics, metrics, dominating metrics, and bounded admissible metrics. Moreover, for any totally bounded pseudometric \(d\) on \(X\) the pseudometric \(\text{hm}(d)\) is totally bounded on each \(HM_n(X), n \in \mathbb{N}\).

**Proof.** It is an easy exercise to show that \(hm\) is a positive linear continuous extension operator preserving constant functions, bounded functions, and \([\text{pseudo}]\)metrics. From Proposition 5 of [9] and its proof it follows that \(hm\) preserves dominating metrics and bounded admissible metrics.

Let us show that \(hm\) preserves bounded continuous functions. For this fix a bounded continuous function \(d : X \times X \to \mathbb{R}, \varepsilon > 0\) and two elements \(f, g \in HM(X)\). Without loss of generality, \(|d(x, x')| \leq 1\) for every \(x, x' \in X\). Let \(0 = a_0 < a_1 < \cdots < a_n = 1\) be a sequence such that both \(f\) and \(g\) are constant on each interval \([a_{i−1}, a_i)\), \(1 \leq i \leq n\).

Using the continuity of \(d\), for every \(i \in \{1, \ldots, n\}\) pick neighborhoods \(U_i, V_i \subset X\) of \(f(a_i), g(a_i)\), respectively, such that for every \(x \in U_i, y \in V_i\) we have \(|d(x, y) − d(f(a_i), g(a_i))| < \varepsilon/2\). Then \(U = \bigcap_{i=1}^n N(a_{i−1}, a_i, U_i, \varepsilon/2n)\) and \(V = \bigcap_{i=1}^n N(a_{i−1}, a_i, V_i, \varepsilon/2n)\) are neighborhoods of \(f\) and \(g\), respectively, such that for every \(f' \in U, g' \in V\) we have \(|hm(d)(f', g') − hm(d)(f, g)| < \varepsilon\). That means the function \(hm(d) : HM(X) \times HM(X) \to \mathbb{R}\) is continuous.

Finally, we show that for every totally bounded pseudometric \(d\) on \(X\) the pseudometric \(hm(d)\) is totally bounded on each \(HM_n(X), n \in \mathbb{N}\). Fix \(n \in \mathbb{N}\) and \(a, b, V, \varepsilon > 0\) such that \(d(a, b) = \varepsilon\). Consider the equivalence relation \(\sim\) on \(X\), where \(x \sim y\) if \(d(x, y) = 0\). Then the pseudometric \(d\) induces a totally bounded metric \(\rho\) on the quotient space \(X/\sim\).

For a space \(X\) by \(\exp_{\omega} X\) we denote the set of all finite subsets of \(X\). A map \(u : X \to \exp_{\omega} X\) is called upper-continuous provided for every open set \(U \subset X\) the set \(\{y \in Y \mid u(y) \subset U\}\) is open in \(Y\).

Next, we prove that the spaces \(HM(X)\) over stratifiable spaces have an important extension property. Below for a metric \(d\) on a space \(X\) the open \(d\)-ball \(B(x, \varepsilon)\) of radius \(\varepsilon\) around a point \(x \in X\) is denoted by \(O_d(x, \varepsilon)\).

**Proposition 2.** For every closed subset \(X\) of a stratifiable space \(Y\) there exist

1. an upper semi-continuous map \(u : Y \to \exp_{\omega} X\) such that \(u(x) = \{x\}\) and
2) a continuous map \( h : Y \to \text{HM}(X) \) extending the identity embedding \( X \hookrightarrow \text{HM}(X) \) such that \( \text{supp}(h(y)) \subset u(y) \) for every \( y \in Y \).

Moreover, if \( \dim(Y \setminus X) < n \), then \( h(Y) \subset \text{HM}_n(X) \). If \( d \) is an admissible metric for \( Y \), then the map \( u \) can be chosen so that \( u(y) \subset O_d(y, 2d(y, X)) \) for every \( y \in Y \).

**Proof.** Suppose \( X \) is a closed subset of a stratifiable space \( Y \). By the proof of Theorem 4.3 of [7], there exists a locally finite open cover \( \mathcal{U} \) of \( Y \setminus X \) and a map \( \alpha : \mathcal{U} \to X \) such that the map \( u : Y \to \text{exp}_\mathcal{U}(X) \) defined by \( u(y) = \{ y \} \) for \( y \in X \) and \( u(y) = \{ \alpha(U) \mid y \in \text{cl}(U), \ U \in \mathcal{U} \} \) for \( y \in Y \setminus X \) is upper-semicontinuous. Let \( \mathcal{U} \) be any linear ordering of the set \( \mathcal{U} \) and let \( \{ \lambda_U : Y \setminus X \to [0, 1] \}_{U \in \mathcal{U}} \) be a partition of unity subordinate to the cover \( \mathcal{U} \). For a \( y \in Y \setminus X \) define a function \( h(y) \in \text{HM}(X) \) letting

\[
h(y)(t) = \alpha(U), \quad \text{if} \quad \sum_{V < U} \lambda_V(y) \leq t < \sum_{V \leq U} \lambda_V(y).
\]

Because only finitely many of \( \lambda_V(y) \)'s are distinct from zero, the function \( h(y) \) is well-defined.

For \( y \in X \) let \( h(y) = y \in X \subset \text{HM}(X) \).

We claim that the so-defined map \( h : Y \to \text{HM}(X) \) is continuous and satisfies the requirements of Proposition 2. The inclusion \( \text{supp}(h(y)) \subset u(y), \ y \in Y \), follows from the definitions of \( h(y) \) and \( u(y) \).

The continuity of \( h \) on the set \( Y \setminus X \) easily follows from the local finiteness of the cover \( \mathcal{U} \). Let us verify the continuity of \( h \) at a point \( x \in X \). Fix any neighborhood \( U \) of \( h(x) = x \) in \( \text{HM}(X) \). According to the definition of the topology of \( \text{HM}(X) \), there exists a neighborhood \( V \) of \( x \) in \( X \) such that \( \text{HM}(X) \subset U \). Since the map \( u : Y \to \text{exp}_\mathcal{U}(X) \) is upper-semicontinuous and \( u(x) = \{ x \} \), there is a neighborhood \( W \) of \( x \) in \( X \) such that \( u(y) \subset V \) for every \( y \in W \). Then for such \( y \) we have \( h(y) \in \text{HM}(u(y)) \subset \text{HM}(V) \subset U \), i.e. \( h \) is continuous at the point \( x \).

If \( \dim(Y \setminus X) < n \) then the cover \( \mathcal{U} \) can be chosen to be of order \( \leq n \). In this case, according to the construction, we get \( h(Y) \subset \text{HM}_n(X) \).

If \( Y \) is a metrizable space with an admissible metric \( d \), then using the classical technique of Dugundji [10] we may construct the map \( u \) so that \( u(y) \subset O_d(y, 2d(y, X)) \) for every \( y \in Y \).

**Question 1.** Is \( \text{HM}(X) \) an absolute extensor for stratifiable spaces? The answer is “yes” for separable metrizable \( X \). (This can be shown applying the arguments of [8] Ch.VI, §7)

2. CONSTRUCTION OF AN EXTENSION OPERATOR \( T \)

Suppose \( X \) is a closed subset of a stratifiable space \( Y \) and \( a, b \) be two distinct points of \( X \). An operator \( T \) satisfying the requirements of Theorems 1 and 2 will be constructed as the sum of a series \( \sum_{n=1}^{\infty} 2^{-n} T_n \), where the collection of extension operators \( \{ T_n : \mathbb{R}^X \times X \to \mathbb{R}^X \} \) is “separates” points of \( Y \).

It is known that every stratifiable space admits a bijective continuous map onto a metrizable space (combine [Bo, Lemma 8.2] with [Bo, property (A) on p.2]). Therefore, there is a continuous metric \( d \leq 1 \) on \( Y \). Moreover, applying Theorem 5.2 of [7], we may adjust the metric \( d \) so that \( d(y, X) > 0 \) for every \( y \in Y \setminus X \), where, as usual, \( d(y, X) = \inf \{d(y, x) : x \in X\} \). If \( Y \) is a metrizable uniform space, then \( d \) will be assumed to generate the uniformity of \( Y \).

Let \( h : Y \to \text{HM}(X) \) and \( u : Y \to \text{exp}_\mathcal{U}(X) \) be the maps from Proposition 2 (in case \( \dim(Y \setminus X) < \infty \) we assume that \( h(Y) \subset \text{HM}_k(X) \) for some \( k \in \mathbb{N} \)).

For every \( n \in \mathbb{N} \) we shall define an extension operator \( T_n : \mathbb{R}^X \times X \to \mathbb{R}^X \) as follows. Fix \( n \in \mathbb{N} \). Let \( \mathcal{U}_n \) be a locally finite (resp. finite, if the metric \( d \) is totally bounded) open cover of the space \( Y \) such that \( \text{diam}_d(U) < 2^{-n} \) for every \( U \in \mathcal{U}_n \), and let \( \{ \lambda_U^V : Y \to [0, 1] \}_{U \in \mathcal{U}_n} \) be a partition of unity, subordinate to the cover \( \mathcal{U}_n \). Further we consider \( \mathcal{U}_n \) as a discrete topological space. Let \( \leq \) be any linear ordering on \( \mathcal{U}_n \) and let \( h_n : Y \to \text{HM}(\mathcal{U}_n) \) be the map defined for a \( y \in Y \) by the formula

\[
h_n(y)(t) = U, \quad \text{if} \quad \sum_{V < U} \lambda_V(y) \leq t < \sum_{V \leq U} \lambda_V(y).
\]

As in the proof of Proposition 2 it can be shown that the map \( h_n \) is continuous.

By \( X \sqcup \mathcal{U} \) denote the disjoint union of the spaces \( X \) and \( \mathcal{U}_n \), \( n \in \mathbb{N} \). According to [9] Proposition 2, we may identify \( \text{HM}(X) \) and \( \text{HM}(\mathcal{U}_n) \) with subspaces of \( \text{HM}(X \sqcup \mathcal{U}) \). Finally, define a map \( f_n : Y \to \text{HM}(X \sqcup \mathcal{U}) \) letting for a \( y \in Y \)

\[
f_n(y)(t) = \begin{cases} h_n(y)(t), & \text{if } 0 \leq t < \min\{1, n d(y, X)\}; \\
(h(y)(t), & \text{if } \min\{1, n d(y, X)\} \leq t < 1.
\end{cases}
\]

It is easily seen that \( f_n \) is a continuous map extending the natural embedding \( X \subset \text{HM}(X) \subset \text{HM}(X \sqcup \mathcal{U}) \).
Let us consider the linear operator \( E : \mathbb{R}^{X \times X} \to \mathbb{R}^{(X \times U) \times (X \times U)} \) defined for every \( p \in \mathbb{R}^{X \times X} \) by

\[
E(p)(x, y) = \begin{cases} 
  p(x, y), & \text{if } x, y \in X; \\
  \frac{p(x, a) + p(x, b)}{2}, & \text{if } x \in X, y \in U; \\
  \frac{p(a, y) + p(b, y)}{2}, & \text{if } x \in U, y \in X; \\
  p(a, b), & \text{if } x, y \in U \text{ and } x \neq y; \\
  0, & \text{if } x = y;
\end{cases}
\]

(recall that \( a, b \) are two fixed point in \( X \)). One can easily verify that \( E \) is a positive linear continuous extension operator preserving constants, bounded, bounded continuous functions and [pseudo]metrics.

The operator \( T_n : \mathbb{R}^{X \times X} \to \mathbb{R}^{Y \times Y} \) is defined as the composition

\[
\mathbb{R}^{X \times X} \xrightarrow{E} \mathbb{R}^{(X \times U) \times (X \times U)} \xrightarrow{hm} \mathbb{R}^{HM(X \times U) \times HM(X \times U)} \xrightarrow{(f_n \times f_n)} \mathbb{R}^{Y \times Y},
\]

where \((f_n \times f_n)(p) = p \circ (f_n \times f_n)\) for \( p \in \mathbb{R}^{HM(X \times U) \times HM(X \times U)}\), equivalently, by the explicit formula

\[
T_n(p)(y, y') = \int_0^1 E(p)(f_n(y)(t), f_n(y')(t)) \, dt \quad \text{for } p \in \mathbb{R}^{X \times X}, \ y, y' \in Y.
\]

Remark that \( T_n \) is a positive linear continuous extension operator preserving constants, bounded, bounded continuous functions and pseudometrics.

Finally, we define the required operator \( T : \mathbb{R}^{X \times X} \to \mathbb{R}^{Y \times Y} \) by the formula

\[
T = \sum_{n=1}^{\infty} \frac{1}{2^n} T_n.
\]

We shall verify the properties of the operator \( T \). First, observe that the definition of \( T \) is correct, i.e. for every function \( p : X \times X \to \mathbb{R} \) and every \( y, y' \in Y \) the series \( \sum_{n=1}^{\infty} 2^{-n} T_n(p)(y, y') \) is convergent. This is trivial, when \( y, y' \in X \) (all \( T_n \)'s are extension operators). If \( y \in X \) and \( y' \not\in X \) then for every \( n \in \mathbb{N} \) with \( d(y', X) \geq \frac{1}{n} \), by the construction of \( T_n \), we have \( T_n(p)(y, y') = \frac{1}{2} p(y, a) + \frac{1}{2} p(y, b) \). If \( y, y' \not\in X \) then, for every \( n \in \mathbb{N} \) with \( d(y, X), d(y', X) \geq \frac{1}{n} \), we have \( |T_n(p)(y, y')| \leq |p(a, b)| \). These remarks imply that the series \( \sum_{n=1}^{\infty} 2^{-n} T_n(y, y') \) converges for every \( y, y' \in Y \), i.e. the definition of \( T \) is correct.

Since \( T_n \)'s are positive linear extension operators preserving constants, bounded functions, bounded continuous functions and pseudometrics, so is the operator \( T \).

3. Proof of Theorem

In an obvious way Theorem 1 follows from the above-mentioned properties of the operator \( T \) and the subsequent four lemmas. The first of them can be easily derived from the construction of \( T \).

**Lemma 1.** Let \( y, y' \in Y \) and \( A = \text{supp}(h(y)) \cup \text{supp}(h(y')) \cup \{a, b\} \). If \( p, p' : X \times X \to \mathbb{R} \) satisfy \( p|A \times A \leq p'|A \times A \), then \( T(p)(y, y') \leq T(p')(y, y') \). Moreover, if \( p|A \times A \equiv c \), then \( T(p)(y, y') = c \).

**Lemma 2.** The operator \( T : \mathbb{R}^{X \times X} \to \mathbb{R}^{Y \times Y} \) is continuous with respect to the uniform, point-wise or compact-open topologies on the function spaces \( \mathbb{R}^{X \times X} \) and \( \mathbb{R}^{Y \times Y} \).

**Proof.** Because the operator \( T \) is positive and preserves constant functions, it is continuous with respect to the uniform convergence of functions.

Let us show that the operator \( T \) is continuous with respect to the point-wise convergence of functions. For this, fix points \( y, y' \in Y \) and notice that the set \( A = \{a, b\} \cup \text{supp}(h(y)) \cup \text{supp}(h(y')) \) is finite. By Lemma 1, for a function \( p : X \times X \to \mathbb{R} \) the inequality \( |p(x, x')| \leq 1 \) for every \( (x, x') \in A \times A \) implies \( |T(p)(y, y')| \leq 1 \). This means that the operator \( T \) is continuous with respect to the point-wise convergence of functions.

To show that \( T \) is continuous with respect to the compact-open topology fix a compactum \( C \subset Y \times Y \) and notice that the set \( K = \{u(y) \mid y \in \text{pr}_1(C) \cup \text{pr}_2(C)\} \subset X \) is compact because of the upper-semicontinuity of the map \( u : Y \to \text{exp}_\omega X \) (see [12, Theorem VII.7.10]) (by \( \text{pr}_1 : Y \times Y \to Y \) we denote the projection onto the corresponding factor). Consider the compact set \( K = K' \cup \{a, b\} \). Then \( \text{supp}(h(y)) \cup \text{supp}(h(y')) \subset u(y) \cup u(y') \subset K \) for every \( (y, y') \in C \). Now Lemma 1 yields that for a function \( p : X \times X \to \mathbb{R} \) if \( |p(x, x')| \leq 1 \) for every \( (x, x') \in K \times K \) then \( |T(p)(y, y')| \leq 1 \) for every \( (y, y') \in C \). But this means that the operator \( T \) is continuous in the compact-open topology.

**Lemma 3.** The operator \( T \) preserves continuous functions.
Lemma 5. Moreover, the map $E$ is upper-semicontinuous, there are neighborhoods $V, V' \subset Y$ of $y_0, y_0'$ respectively such that for every $y \in V$ and $y' \in V'$ we have $u(y) \cup u(y') \subset U$.

Now consider the bounded continuous function $\tilde{p} : X \times X \rightarrow \mathbb{R}$ defined by the formula

$$\tilde{p}(x, x') = \begin{cases} p(x, x'), & \text{if } -M - 1 \leq p(x, x') \leq M + 1 \\ M + 1, & \text{if } p(x, x') \geq M + 1 \\ -M - 1, & \text{if } p(x, x') \leq -M - 1. \end{cases}$$

Obviously that $\tilde{p}|U = p|U$. Moreover, since the operator $T$ preserves bounded continuous functions, the map $T(\tilde{p}) : Y \times Y \rightarrow \mathbb{R}$ is continuous. Now remark that for every $(y, y') \in V \times V'$ supp$(h(y)) \cup$ supp$(h(y')) \subset \{a, b\} \cup u(y) \cup u(y') \subset U$.

Since $\tilde{p}|U \times U = p|U \times U$, by Lemma 1, $T(p)(y, y') = T(\tilde{p})(y, y')$. Therefore, $T(p)|V \times V' = T(\tilde{p})|V \times V'$ and the function $T(p)$ is continuous.

Lemma 4. The operator $T$ preserves metrics.

Proof. Let $p$ be a metric on $X$. Since the operator $T$ preserves pseudometrics, it remains to prove that $T(p)(y, y') \neq 0$ for distinct $y, y' \in X$. So, fix $y, y' \in Y$ with $y \neq y'$.

If $y, y' \in X$ then $T(p)(y, y') = p(y, y') \neq 0$ because $p$ is a metric on $X$. Now assume that $y \in X$ and $y' \not\in X$. Then $d(y', X) > \frac{1}{n}$ for some $n \in \mathbb{N}$. Consequently, $f_n(y) = y \in X \subset HM(X \cup U)$ and $f_n(y') = h_n(y') \in HM(U_n) \subset HM(X \cup U)$. By the property of the operator $E$, we have $E(p)(y, h_n(y')(t)) = \frac{1}{2}(p(y, a) + p(y, b)) \geq \frac{1}{2}p(a, b)$ for every $t \in [0, 1]$ and thus

$$T_n(p)(y, y') = \int_0^1 E(p)(y, h_n(y')(t))dt \geq \frac{1}{2}p(a, b) > 0.$$ 

This yields $T(p)(y, y') \geq 2^{-n}T_n(p)(y, y') > 0$.

Now assume that $y, y' \in Y \setminus X$. Then there is an $n \in \mathbb{N}$ such that $d(y, X) > n^{-1}$, $d(y', X) > n^{-1}$ and $d(y, y') > 2^{-n+1}$. In this case, $f_n(y) = h_n(y)$ and $f_n(y') = h_n(y')$. Since diam$(U) < 2^{-n}$ for $U \in U_n$, there is no $U \in U_n$ with $\{y, y'\} \subset U$. Consequently, supp$(h_n(y)) \cap$ supp$(h_n(y')) = \emptyset$. By the definition of the metric $E(p)$, $E(p)(h_n(y)(t), h_n(y')(t)(t)) = p(a, b)$ for every $t \in [0, 1]$. Then

$$2^n T(p)(y, y') \geq T_n(p)(y, y') = \int_0^1 E(p)(h_n(y)(t), h_n(y')(t))dt \geq p(a, b) > 0.$$ 

Therefore, $T(p)$ is a metric on $Y$.

4. PROOF OF THEOREM 2

In this section we suppose that $Y$ is a metrizable uniform space and the metric $d$ generates the uniformity of $Y$. Moreover, the map $u$ constructed in Proposition 2 has the following property: $u(y) \subset O_d(y, 2d(y, X))$ for every $y \in Y$.

In an obvious way Theorem 2 follows from Theorem 1 and the subsequent four lemmas.

Lemma 5. The operator $T$ preserves the class of dominating metrics.

Proof. Let $p$ be a dominating metric for $X$. To show that the metric $T(p)$ dominates the topology of $Y$, it suffices for every $y \in Y$ and every $\varepsilon \in (0, 1]$ to find $\delta > 0$ such that $T(p)(y, y') \geq \delta$ for every $y' \in Y$ with $d(y, y') > \varepsilon$.

First we consider the case $y \notin X$. Then we can find $n \in \mathbb{N}$ such that $d(y, X) > \frac{1}{n}$ and $2^{-n+1} < \varepsilon$. Let $\delta = 2^{-n-1}p(a, b)$ and $y' \in Y$ be any point with $d(y, y') > \varepsilon$. Then $d(y, y') > 2^{-n+1}$ and by the choice of the cover $U_n$ and the map $h_n$, we have supp$(h_n(y)) \cap$ supp$(h_n(y')) = \emptyset$. As we have observed in the proof of Lemma 4, this implies $E(p)(h_n(y)(t), h_n(y')(t)) = p(a, b)$ for every $t \in [0, 1]$. Besides, it follows that $E(p)(h_n(y)(t), h_n(y')(t)) \geq \frac{1}{2}p(a, b)$. Then

$$2^n T(p)(y, y') \geq T_n(p)(y, y') = \int_0^1 E(p)(h_n(y)(t), f_n(y')(t))dt \geq \frac{1}{2}p(a, b) = 2^n \delta.$$ 

Now assume that $y \in X$. Let $n \in \mathbb{N}$ be such that $2^{-n+1} < \varepsilon$. Since the metric $p$ is dominating for $X$, there is $\eta > 0$ such that $p(x, y) > \eta$ for every $x \in X$ with $d(x, y) > \varepsilon$. Let $\delta = \min\{2^{-n-1}p(a, b), n\varepsilon/2, 3\eta/8\}$ and fix any point $y' \in Y$ with $d(y, y') > \varepsilon$. To verify that $T(p)(y, y') \geq \delta$, consider two cases:

1) $d(y', X) \geq \frac{\eta}{4}$. By the property of the metric $E(p)$, we have $E(p)(y, h_n(y')(t)) \geq \frac{1}{2}p(a, b)$ for every $t \in [0, 1]$. Then

$$2^n T(p)(y, y') \geq T_n(p)(y, y') = \int_0^1 E(p)(y, f_n(y')(t))dt \geq \int_0^{\min\{1, d(y', X)\}} E(p)(y, h_n(y')(t))dt \geq \min\{1, n d(y', X)\} \frac{1}{2}p(a, b) > 2^{-n-1} \min\{1, n\varepsilon/4\}p(a, b) \geq 2^n \delta.$$
Now pass to the other case:

2) \(d(y', X) < \frac{\varepsilon}{4} \). Then \( \text{supp}(h(y')) \subset u(y') \subset O_d(y', \frac{\varepsilon}{4}) \) and because \( d(y, y') > \varepsilon \), we get \( d(y, h(y')(t)) > \frac{\varepsilon}{4} \) for every \( t \in [0, 1] \). By the choice of \( \eta \), this implies \( p(y, h(y')(t)) > \eta \) for every \( t \in [0, 1] \). Then

\[
2T(p)(y, y') \geq T_1(p)(y, y') = \int_0^1 E(p)(y, f_1(y')(t))dt \geq \int_{d(y', X)}^1 E(p)(y, h(y')(t))dt \geq \int_{\varepsilon/4}^1 p(y, h(y')(t))dt \geq (1 - \frac{\varepsilon}{4})\eta \geq \frac{3}{4} \cdot \eta \geq 2\delta.
\]

\[ \square \]

**Lemma 6.** The operator \( T \) preserves uniformly dominating metrics.

**Proof.** Let \( p \) be a uniformly dominating metric for the uniform space \( X \). To show that the metric \( T(p) \) is uniformly dominating for \( Y \), it suffices to verify that the formal identity map \( (Y, T(p)) \to (Y, d) \) between the respective metric spaces is uniformly continuous.

Fix any \( \varepsilon > 0 \). We have to find \( \delta > 0 \) such that for every \( y_1, y_2 \in Y \) the inequality \( T(p)(y_1, y_2) < \delta \) implies \( d(y_1, y_2) < \varepsilon \); equivalently, \( d(y_1, y_2) \geq \varepsilon \) implies \( T(p)(y_1, y_2) \geq \delta \). To find such \( \delta \), select \( n \in \mathbb{N} \) so that \( 2^{-n} < \frac{\varepsilon}{4} \) and \( 2^{-n+1} < \frac{\varepsilon}{4} \). Since the metric \( p \) is uniformly dominating for \( X \), there exists \( \delta > 0 \) such that \( d(x, x') < \frac{\varepsilon}{4} \) for all \( x, x' \in X \) with \( p(x, x') < 2\varepsilon + 1 \). Moreover, we may take \( \delta \) so small that \( 2^n\delta < \frac{\varepsilon}{4} \).

Observe that in a complete metrizable uniform space every continuous uniformly dominating metric is complete.

We claim that the so-chosen number \( \delta \) satisfies our requirements. To show this, fix any points \( y_1, y_2 \in Y \) with \( d(y_1, y_2) \geq \varepsilon \). Because \( T(p)(y_1, y_2) \geq 2^{-n}T_n(y_1, y_2) \), it suffices to verify the inequality \( T_n(y_1, y_2) \geq 2^n\delta \). Two cases will be considered separately:

1) \( \max\{d(y_1, X), d(y_2, X)\} \geq 2^{-n-2} \). Without loss of generality, \( d(y_1, X) \leq d(y_2, X) \). Since \( d(y_1, y_2) \geq \varepsilon \) and \( \sup_{U \in \mathcal{U}_n} \text{diam}(U) < 2^{-n} < \frac{\varepsilon}{4} \), we get \( \text{supp}(h_n(y_1)) \cap \text{supp}(h_n(y_2)) = \emptyset \). It follows that \( E(p)(h_n(y_1)(t), h_n(y_2)(t)) \geq p(a, b) \) and \( E(p)(h_n(y_2), h(y_1)) \geq \frac{\varepsilon}{4} p(a, b) \) for every \( t \in [0, 1] \).

Then

\[
T_n(p)(y_1, y_2) = \int_0^1 E(p)(f_n(y_1)(t), f_n(y')(y_2)(t))dt \geq \frac{1}{2} \min\{1, nd(y_2, X)\} p(a, b) \geq \frac{1}{2} p(a, b) \frac{n}{2^{n+2}} > 2^n\delta.
\]

Next, we consider the case:

2) \( \max\{d(y_1, X), d(y_2, X)\} < 2^{-n-2} \). By the definition of the map \( u \), we have \( \text{supp}(h(y_i)) \subset u(y_i) \subset O_d(y_i, 2d(y_i, X)) \subset O_d(y_i, \frac{\varepsilon}{4}) \) for \( i = 1, 2 \). Consequently,

\[
d(h(y_1)(t), h(y_2)(t)) > \frac{\varepsilon}{2} \quad \text{for every} \quad t \in [0, 1)
\]

and by the choice of \( \delta \), we get \( p(h(y_1)(t), h(y_2)(t)) > 2^{n+1}\delta \). Finally, for the pseudometric \( T_n(p) \) we obtain

\[
T_n(p)(y_1, y_2) = \int_0^1 E(p)(f_n(y_1)(t), f_n(y')(y_2)(t))dt \geq \int_{1-n\max\{d(y_1, X), d(y_2, X)\}}^1 p(h(y_1)(t), h(y_2)(t))dt \geq (1 - n\max\{d(y_1, X), d(y_2, X)\})2^{n+1}\delta \geq (1 - 2^{-n-2})2^{n+1}\delta \geq \frac{1}{2} 2^{n+1}\delta = 2^n\delta.
\]

\[ \square \]

**Lemma 7.** If the uniform space \( Y \) is complete, then the operator \( T \) preserves complete continuous uniformly dominating metrics.

**Proof.** Observe that in a complete metrizable uniform space every continuous uniformly dominating metric is complete and apply Lemmas 3 and 6.

\[ \square \]

**Lemma 8.** If the uniform space \( Y \) is totally bounded and \( \dim Y \setminus X < \infty \) then the operator \( T \) preserves totally bounded pseudometrics.

**Proof.** Fix a totally bounded pseudometric \( p \) on \( X \). It is enough to show that each pseudometric \( T_n(p) \) is totally bounded. Fix any \( n \in \mathbb{N} \). Since the metric \( d \) on \( Y \) is totally bounded, by the construction, the cover \( \mathcal{U}_n \) is finite. Then the metric \( E(p) \) on \( X \cup \mathcal{U}_n \) is totally bounded. Let \( k > |\mathcal{U}_n| \) be such that \( h(Y) \subset \text{HM}_{k}(X) \). Then \( f_n(Y) \subset \text{HM}_{2k}(X \cup \mathcal{U}_n) \) and \( T_n(p)(y, y') = \text{hm}(E(p))(f_n(y), f_n(y')) \) for every \( y, y' \in Y \). By Proposition 1, the pseudometric \( \text{hm}(E(p)) \) is totally bounded on \( \text{HM}_{2k}(X \cup \mathcal{U}_n) \). Hence, the pseudometric \( T_n(p) \) is totally bounded on \( Y \).

\[ \square \]
5. Proof of Theorem 3

Assume that \(G\) is a compact group, \(\mu\) the Haar measure on \(G\), \(Y\) is a (left) \(G\)-space and \(X\) is a closed subspace of \(Y\) consisting of at least two points and invariant under the action of \(G\). For \(Z \in \{X,Y\}\) by \(C(Z \times Z)\) we denote the linear lattice of continuous functions on \(Z \times Z\), equipped with the compact-open topology and by \(C_{\text{inv}}(Z \times Z)\) its linear subspace consisting of continuous invariant functions, i.e., such that \(f(gx,gy) = f(x,y)\) for every \(g \in G\) and \(x, y \in X\).

**Proposition 3.** The averaging operator \(A : C(Y \times Y) \to C_{\text{inv}}(Y \times Y)\) defined by

\[
Af(y, y') = \int_Y f(gy, gy') \, d\mu \quad \text{for } f \in C(Y \times Y), \quad y, y' \in Y
\]

is a continuous retraction of \(C(Y \times Y)\) onto \(C_{\text{inv}}(Y \times Y)\). The operator \(A\) takes constants, pseudometrics, metrics, admissible metrics into constants, invariant pseudometrics, invariant metrics, invariant admissible metrics, respectively.

**Proof.** Let \(d \in C(Y \times Y)\) be a metric on \(Y\) and \(d' = Ad\). Let \(a, b \in Y\), \(a \neq b\). There is a neighborhood \(U\) of the neutral element of the group \(G\) such that \(d(ga, gb) \geq 2^{-1}d(a, b)\) for \(g \in U\). Therefore \(d'(a, b) \geq 2^{-1}\mu(U)d(a, b) > 0\), i.e., \(d' = Ad\) is a metric.

Now assume that \(d \in C(Y \times Y)\) is an admissible metric, and \((y_n)\) is a sequence of points of \(Y\) such that \(\lim_n d'(y_n, y) = 0\) for some \(y \in Y\). Hence the sequence of real functions \(\varphi_n(y) = d(y_n, y)\) tends to zero in the \(L^1\)-norm, and since \(\mu(G) = 1 < \infty\), there is a subsequence \(\varphi_{n_k}\) which tends to zero almost everywhere, in particular, \(\lim_n d(g_0 y_{n_k}, y_0 y) = 0\) for some \(g_0 \in G\). “Multiplying the last relation from the left” by \(g_0^{-1}\) we get \(\lim_n d(y_{n_k}, y) = 0\).

The same arguments yield that every subsequence of the sequence \((y_n)\) contains a subsequence convergent (in the admissible metric) to \(y\). That means that the whole sequence \((y_n)\) tends to \(y\). We have proved that the \(d' = Ad\) is dominating, and (being continuous) is admissible. The other assertions of the proposition are evident.

**Proof of Theorem 3.** Let \(T\) be the operator appearing in Theorem 1 (Theorem 2 in case of metrizable \(Y\)). The required operator \(I\) is the composition \(I = A \circ T|C_{\text{inv}}(X \times X)\).

6. The Extension Operators \(S, S_1, S_2 : \mathbb{R}^{X \times X} \to \mathbb{R}^{Y \times Y}\)

In this section we present a simple construction of extension operators \(S, S_1, S_2\) having almost all properties of the operator \(T\).

Suppose \(Y\) is a stratifiable space, \(X\) is a closed subset of \(Y\) and \(a, b\) are two distinct points in \(X\). As we said, the space \(Y\) admits a continuous metric \(d \leq 1\) such that \(d(y, X) > 0\) for all \(y \in \Gamma \setminus X\). If \(Y\) is metrizable, we assume that \(d\) is an admissible metric for \(Y\).

For \(y, y' \in Y\) let

\[
d^*(y, y') = \min[d(y, y'), d(y, X) + d(y', X)],
\]

Clearly, \(d^*\) is a continuous pseudometric on \(Y\) (moreover, the restriction of \(d^*\) on \(Y \setminus X\) is a metric). Let \(h : Y \to \text{HM}(X)\) be the map appearing in Proposition 2 and define

\[
S(p)(y, y') = hm(p)(h(y), h(y')) = \int_0^1 p(h(y)(t), h(y')(t)) \, dt,
\]

\[
S_1(p) = S(p) + p(a, b)d^*, \quad S_2(p) = S(p) + (p(a, b) - p(a, a))d^*
\]

for \(p \in \mathbb{R}^{X \times X}, \ y, y' \in Y\). Thus we have defined three extension operators \(S, S_1, S_2 : \mathbb{R}^{X \times X} \to \mathbb{R}^{Y \times Y}\).

**Theorem 4.** The operators \(S, S_1\) and \(S_2\) satisfy the requirements of Theorem 1 except that \(S\) does not preserve metrics, \(S_1\) fails to preserve constants and \(S_2\) is not positive. Moreover, if the space \(Y\) is metrizable, then the operators \(S_1\) and \(S_2\) preserve dominating and admissible metrics.

**Proof.** The first statement of the theorem easily follows from Propositions 1 and 2 (to prove that these operators preserve continuous functions one should apply the arguments from Lemma 3). The fact that in the metric case, \(S_1\) and \(S_2\) preserve dominating metrics is an immediate consequence of the next two easy lemmas.

**Lemma 9.** For every dominating metric \(p\) on \(X\) the pseudometric \(\rho = S(p)\) has the following property:

\((*)\) Let \(y_n \in Y\) for \(n \in \mathbb{N}\) and \(x \in X\). Then \(\lim_n \rho(y_n, x) = 0\) and \(\lim_n d(y_n, X) = 0\) imply \(\lim_n d(y_n, x) = 0\).

**Proof.** (cf. proof of Lemma 3). Recall that \(d\) is a fixed admissible metric for \(Y\). According to the last assertion of Proposition 2 and the definition of the operator \(S\), for every \(y \in Y\) there is an \(y' \in u(y) \subset X\) such that

\[
d(y, y') \leq 2d(y, x) \quad \text{and} \quad p(y', x) \leq \rho(y, x).
\]

We have \(d(y_n, x) \leq d(y_n, y'_n) + d(y'_n, x) \leq 2d(y_n, X) + d(y'_n, x)\).

But \(0 \leq p(y'_n, x) \leq \rho(y_n, x) \to 0\) as \(n \to \infty\), and since \(p\) is a dominating metric for \(X\), we get \(\lim_n d(y'_n, x) = 0\), and by the assumption of \((*)\), \(\lim_n d(y_n, x) = 0\)
Lemma 10. For every pseudometric $\rho$ in $Y$ and every constant $c > 0$ the sum $\rho + cd^*$ is a dominating metric on $Y$, provided $\rho$ has the property $(\ast)$.

Proof. By $(\ast)$, the sum $\rho + cd^*$ is dominating “at each point” $x \in X$. In order to show the domination at the remaining points it is enough to examine the second term $d^*$ which is a metric, when restricted to $Y \setminus X$. □

Finally, we pose an open problem suggested by Theorem 2 and a known result of J.S. Isbell [13] according to which for every subspace $X$ of a uniform space $Y$, every bounded uniformly continuous pseudometric on $X$ extends to a bounded uniformly continuous pseudometric on $Y$.

Problem 1. Suppose $X$ is a subspace of a metrizable uniform space $Y$. Does there exist a “nice” operator extending bounded uniformly continuous pseudometrics from $X$ over $Y$.

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