Generalized gauge-invariance for gravitational waves

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Abstract

The aim of this paper is to show the gauge-invariance on the response of interferometers to gravitational waves (GWs). In this process, after a review of results on the Transverse-Traceless (TT) gauge, where, in general, the theoretical computations on GWs are performed, which is due for completeness, we analyse the gauge of the local observer, which represents the gauge of a laboratory environment on Earth. The gauge-invariance between the two gauges is shown in its full angular and frequency dependences. In previous works in the literature this gauge-invariance was shown only in the low frequencies approximation or in the simplest geometry of the interferometer in respect to the propagating GW (i.e. both of the arms of the interferometer are perpendicular to the propagating GW). As far as the computation of the response functions in the gauge of the local observer is concerned, a common misconception about interferometers is also clarified.

1 Introduction

The data analysis of interferometric GWs detectors has recently started (for the current status of GWs interferometers see [1, 2, 3, 4, 5, 6, 7, 8]) and the scientific community aims at a first direct detection of GWs in next years.

Detectors for GWs will be important for a better knowledge of the Universe and either to confirm or ruling out the physical consistency of General Relativity or any other theory of gravitation [9, 10, 11, 12, 13, 14]. In fact, in the context of Extended Theories of Gravity, some differences between General Relativity and the other theories can be pointed out starting from the linearized theory of gravity [9, 10, 12, 14].

In the framework of General Relativity, computations on GWs are usually performed in the so-called TT gauge [2, 15, 16]. This kind of gauge is historically
called Transverse-Traceless, because in these particular coordinates GWs have a transverse effect and are traceless, and the computations are, in general, simpler [15]. As interferometers work in a laboratory environment on Earth, the gauge in which the space-time is locally flat and the distance between any two points is given simply by the difference in their coordinates in the sense of Newtonian physics has to be used [12,13,15,16,17]. In this gauge, called the gauge of the local observer [15], GWs manifest themselves by exerting tidal forces on the masses (the mirrors and the beam-splitter in the case of an interferometer, see Figure 1). At this point, when approaching the first direct detection of GWs, it is very important, whatever the frequency and the direction of propagation of the GW will be, to demonstrate that the signal in the TT gauge, which has been computed in various theoretical approaches in lots of works in the literature, is equal to the one computed in the gauge of the local observer (which is the gauge where the detection will be observed on Earth).

In this paper, such a gauge-invariance on the response of interferometers to GWs between the two mentioned gauges is shown. In this process, after a review of results on the TT gauge following the lines of [2] and [3] (which is due for completeness and for a better understanding of the analysis), the response functions of interferometers are computed directly in the gauge of the local observer, obtaining the same result of the computation in the TT gauge. In this way, the gauge-invariance is shown in its full angular and frequency dependences. In previous works in the literature, this gauge-invariance was shown only in the low frequencies approximation (i.e. wavelength of the GW much large than the linear distance between test masses, see [18] for example) or in the simplest geometry of the interferometer in respect to the propagating GW (i.e. both of the arms of the interferometer are perpendicular to the propagating GW [16]). The presented results are consistent with previous approximations. As far as the computation of the response functions in the gauge of the local observer is concerned, a common misconception about interferometers is also clarified.

2 A review of the total response functions of interferometers in the TT gauge

Following [2,3], we work with \( G = 1, c = 1 \) and \( \hbar = 1 \) and we call \( h_+(t+z) \) and \( h_\times(t+z) \) the weak perturbations due to the + and the \( \times \) polarizations which are expressed in terms of synchronous coordinates in the TT gauge. In this way, the most general GW propagating in the \( z \) direction can be written in terms of plane monochromatic waves [15,16,17]

\[
\begin{align*}
    h_{\mu\nu}(t+z) &= h_+(t+z)\epsilon^{(+)}_{\mu\nu} + h_\times(t+z)\epsilon^{(\times)}_{\mu\nu} = \\
    &= h_{+0}\exp i\omega(t+z)\epsilon^{(+)}_{\mu\nu} + h_{\times0}\exp i\omega(t+z)\epsilon^{(\times)}_{\mu\nu},
\end{align*}
\]

and the correspondent line element will be
Photon
X
Y

Figure 1: photons can be launched from the beam-splitter to be bounced back by the mirror

\[ ds^2 = dt^2 - dz^2 - (1 + h_+)dx^2 - (1 - h_+)dy^2 - 2h_x dx dy. \] (2)

The wordlines \( x, y, z = \text{const.} \) are timelike geodesics, representing the histories of free test masses \( [15, 17] \), that, in our case, are the beam-splitter and the mirrors of an interferometer, see Figure 1.

In order to obtain the response functions in the TT gauge, a generalization of the analysis in \( [16] \) will be used, the so called “bouncing photon method”: a photon can be launched from the beam-splitter to be bounced back by the mirror (Figure 1). This method has been generalized to scalar waves, angular dependences and massive modes of GWs in \( [2, 3, 12, 19] \). This includes the more general problem of finding the null geodesics of light in the presence of a weak GW \( [15, 20, 21, 22, 23] \).

We start by computing the variation of the proper distance that a photon covers to make a round-trip from the beam-splitter to the mirror of an interferometer (see \( [3] \) and Figure 1) with the gauge choice \( (2) \). With a treatment similar to the one of in \( [3] \), the analysis is translated in the frequency domain and the general response functions are obtained.

A special property of the TT gauge is that an inertial test mass initially at rest in these coordinates, remains at rest throughout the entire passage of the GW \( [3, 15, 16] \). Here, the use of words “at rest” has to be clarified: one wants to mean that the coordinates of the test mass do not change in the presence of the GW. The proper distance between the beam-splitter and the mirror of the interferometer changes even though their coordinates remain the same \( [3, 15, 16] \).

Let us start from the + polarization. The line element \( (2) \) becomes:

\[ ds^2 = -dt^2 + dz^2 + [1 + h_+(t - z)] dx^2 + [1 + h_+(t - z)] dy^2. \] (3)

But the arms of the interferometer are in the \( \overrightarrow{w} \) and \( \overrightarrow{v} \) directions, while the \( x, y, z \) frame is the proper frame of the propagating GW. Then, a spatial
rotation of the coordinate system has to be performed:

\[
\begin{align*}
u &= -x \cos \theta \cos \phi + y \sin \phi + z \sin \theta \cos \phi \\
v &= -x \cos \theta \sin \phi - y \cos \phi + z \sin \theta \sin \phi \\
w &= x \sin \theta + z \cos \theta,
\end{align*}
\]

(4)

or, in terms of the \( x, y, z \) frame:

\[
\begin{align*}
x &= -u \cos \theta \cos \phi - v \cos \theta \sin \phi + w \sin \theta \\
y &= u \sin \phi - v \cos \phi \\
z &= u \sin \theta \cos \phi + v \sin \theta \sin \phi + w \cos \theta.
\end{align*}
\]

(5)

In this way, the GW is propagating from an arbitrary direction \( \mathbf{P} \) to the interferometer (see Figure 2).

The coordinate transformation for the metric tensor is [2]:

\[
g^{ik} = \frac{\partial x^i}{\partial x'^l} \frac{\partial x^k}{\partial x'^m} g_{lm}'
\]

(6)

By using eq. (4), (5) and (6), in the new rotated frame the line element \( ds^2 \) in the \( \mathbf{u} \) direction becomes (note: the \( v \) and \( w \) directions can be neglected because bouncing photons will be used and the photon deflection into the \( v \) and \( w \) directions will be at most of order \( h_+ \). Then, to first order in \( h_+ \), the \( dv^2 \) and \( dw^2 \) terms can be neglected [3, 15, 16]):

\[
ds^2 = -dt^2 + [1 + (\cos^2 \theta \cos^2 \phi - \sin^2 \phi)h_+(t - u \sin \theta \cos \phi)]du^2.
\]

(7)
Unlike the line element of eq. 2 in [16], where there is a pure time dependence because of the simplest geometry, in the line element (7) both a spatial dependence in the $u$ direction and an angular dependence appear. Thus, the present analysis is more general than the analysis in [16], and similar to the one in [2, 3, 12] for the angular response functions.

A good way to analyze variations in the proper distance (time) is by means of “bouncing photons” (see [2, 3, 12, 16, 19] and Figure 1). A photon can be launched from the beam-splitter to be bounced back by the mirror.

The condition for null geodesics ($ds^2 = 0$) in eq. (7) gives the coordinate velocity of the photon:

$$v^2 = \left(\frac{du}{dt}\right)^2 = \frac{1}{1 + (\cos^2 \theta \cos^2 \phi - \sin^2 \phi) h_+ (t - u \sin \theta \cos \phi)},$$

(8)

which is a convenient quantity for calculations of the photon propagation time between the beam-splitter and the mirror [2, 3, 12, 16, 19]. One assumes that the beam splitter is located in the origin of the coordinate system (i.e. $u_b = 0$, $v_b = 0$, $w_b = 0$). The coordinates of the beam-splitter $u_b = 0$ and of the mirror $u_m = L$ do not change under the influence of the GW, thus the duration of the forward trip can be written as

$$T_1(t) = \int_0^L \frac{du}{v(t' - u \sin \theta \cos \phi)},$$

(9)

with

$$t' = t - (L - u).$$

In the last equation $t'$ is the delay time (i.e. $t$ is the time at which the photon arrives in the position $L$, so $L - u = t - t'$).

At first order in $h_+$ this integral can be approximated with

$$T_1(t) = T + \frac{\cos^2 \theta \cos^2 \phi - \sin^2 \phi}{2} \int_0^L h_+ (t' - u \sin \theta \cos \phi) du,$$

(10)

where

$$T = L$$

is the transit time of the photon in absence of the GW. Similarly, the duration of the return trip will be

$$T_2(t) = T + \frac{\cos^2 \theta \cos^2 \phi - \sin^2 \phi}{2} \int_0^L h_+ (t' - u \sin \theta \cos \phi) (-du),$$

(11)

though now the delay time is

$$t' = t - (u - l).$$
The round-trip time will be the sum of $T_2(t)$ and $T_1[t - T_2(t)]$. The latter can be approximated by $T_1(t - T)$ because the difference between the exact and the approximate values is second order in $h_+$. Then, to first order in $h_+$, the duration of the round-trip will be

$$T_{\text{r.t.}}(t) = T_1(t - T) + T_2(t).$$

By using eqs. (10) and (11) one sees immediately that deviations of this round-trip time (i.e. proper distance) from its unperturbed value are given by

$$\delta T(t) = \frac{\cos^2 \theta \cos^2 \phi - \sin^2 \phi}{2} \int_0^T \left[ h_+ (t - 2T + u (1 - \sin \theta \cos \phi)) + 
+ h_+ (t - u (1 + \sin \theta \cos \phi)) \right] du. \quad (13)$$

Now, using the Fourier transform of the + polarization of the field, defined by

$$\tilde{h}_+ (\omega) = \int_{-\infty}^{\infty} dt h_+ (t) \exp(i\omega t), \quad (14)$$

the integral in (13) can be computed in the frequency domain, with the aid of the Fourier translation and derivation theorems:

$$\delta \tilde{T}(\omega) = \frac{1}{2} (\cos^2 \theta \cos^2 \phi - \sin^2 \phi) \tilde{H}_u(\omega, \theta, \phi) \tilde{h}_+ (\omega), \quad (15)$$

where

$$\tilde{H}_u(\omega, \theta, \phi) = \frac{-1 + \exp(2\omega L)}{2\omega (1 + \sin^2 \theta \cos^2 \phi)} + \frac{\sin \theta \cos \phi (1 + \exp(2\omega L) - 2 \exp \omega L (1 - \sin \theta \cos \phi))}{2\omega (1 + \sin^2 \cos^2 \phi)} \quad (16)$$

and one immediately sees that $\tilde{H}_u(\omega, \theta, \phi) \to L$ when $\omega \to 0$.

Thus, defining the signal in the $u$ arm as

$$\tilde{\delta T_u(\omega)} = \frac{\tilde{T_u(\omega)}}{T} \equiv \tilde{Y}_u^+(\omega) \tilde{h}_+ (\omega), \quad (17)$$

the total response function $Y_u^+(\omega)$ of the $u$ arm of the interferometer to the + component is:

$$Y_u^+(\omega) = \frac{(\cos^2 \theta \cos^2 \phi - \sin^2 \phi)}{2L} \tilde{H}_u(\omega, \theta, \phi), \quad (18)$$

where $2L = 2T$ is the round-trip time of the photon in absence of gravitational waves.

An analogous analysis can be used for the arm in the $v$ direction (see [2, 3] for details) obtaining the response function of the $v$ arm of the interferometer to the + polarization:

6
\[ \Upsilon^+ (\omega) = \frac{\cos^2 \theta \sin^2 \phi - \cos^2 \phi}{2L} \tilde{H}_v (\omega, \theta, \phi) \]  

where, now

\[ \tilde{H}_v (\omega, \theta, \phi) \equiv \frac{-1 + \exp(2i\omega L)}{2i\omega(1 + \sin^2 \theta \sin^2 \phi)} + \frac{-\sin \theta \sin(\omega L)(1 + \exp(2i\omega L))}{2i\omega(1 + \sin^2 \theta \sin^2 \phi)}, \]  

with \( \tilde{H}_v (\omega, \theta, \phi) \to L \) when \( \omega \to 0 \).

Thus, the total frequency and angular dependent response function (i.e. the detector pattern) of an interferometer to the + polarization of the GW is:

\[ \tilde{H}^+ (\omega) \equiv \Upsilon^+_u (\omega) - \Upsilon^+_v (\omega) = \frac{\cos^2 \theta \cos^2 \phi - \sin \phi}{2L} \tilde{H}_u (\omega, \theta, \phi) + \frac{-\cos^2 \theta \sin^2 \phi}{2L} \tilde{H}_v (\omega, \theta, \phi) \]  

that, in the low frequencies limit (\( \omega \to 0 \)) gives the well known low frequency response function of [24, 25] for the + polarization:

\[ \tilde{H}^+ (\omega \to 0) = \frac{1}{2}(1 + \cos^2 \theta) \cos 2\phi. \]  

The same analysis works for the \( \times \) polarization (see [2, 3] for details in this case too). One obtains that the total frequency and angular dependent response function of an interferometer to the \( \times \) polarization is:

\[ \tilde{H}^\times (\omega) = -\frac{\cos \theta \cos \phi \sin \phi}{L} [\tilde{H}_u (\omega, \theta, \phi) + \tilde{H}_v (\omega, \theta, \phi)], \]  

that, in the low frequencies limit (\( \omega \to 0 \)), gives the low frequency response function of [24, 25] for the \( \times \) polarization:

\[ \tilde{H}^\times (\omega \to 0) = -\cos \theta \sin 2\phi. \]  

The importance of the presented results is due to the fact that in this case the limit where the wavelength is shorter than the length between the splitter mirror and test masses is calculated. The signal drops off the regime, while the calculation agrees with previous calculations for longer wavelengths [24, 25]. The contribution is important especially in the high-frequency portion of the sensitivity band.

In fact, one can see the pronounced difference between the low-frequency approximation angular pattern [22] of the Virgo interferometer for the + polarization, which is shown in Figure 3, and the frequency-dependent angular pattern [21], which is shown in Figure 4 at a frequency of 8000 Hz, i.e. a frequency which falls in the high-frequency portion of the sensitivity band. The
same angular patterns are shown in Figures 5 and 6 for the LIGO interferometer. The difference between the low-frequency approximation angular patterns and the frequency-dependent ones is important for the $\times$ polarization too, as it is shown in Figures 7, 8 for Virgo and in Figures 9, 10 for LIGO.

3 Computation in the gauge of the local observer.

A detailed analysis of the gauge of the local observer is given in Ref. [15], Sect. 13.6. Here, we only recall that the effect of GWs on test masses is described by the equation for geodesic deviation in this gauge

$$\ddot{x}^i = -\tilde{R}^i_{0k0}x^k,$$

where $\tilde{R}^i_{0k0}$ are the components of the linearized Riemann tensor [15].

In the computation of the response functions in this gauge, a common misconception about interferometers will be also clarified. This misconception purports that, because the wavelength of the laser light and the length of an interferometer’s arm are both stretched by a GW, no effect should be present, invoking an analogy with the cosmological redshift of the expanding Universe. This misconception has been recently clarified in a good way in [26], but only in
Figure 4: The angular dependence to the + polarization for the Virgo interferometer at 8000 Hz
the low frequency approximation. Here the misconception will be clarified in the full angular and frequency dependences of a GW, showing that the variation of proper time due to the photons redshift is different from the variation of proper time due to the motion of the arms.

We start with the + polarization. In the gauge of the local observer the equation of motion for the test masses are \[15\] [16]

\[
\ddot{x} = \frac{1}{2} \dot{h}_+ x, \quad (26)
\]

\[
\ddot{y} = -\frac{1}{2} \dot{h}_+ y, \quad (27)
\]

\[
\ddot{z} = 0, \quad (28)
\]

which can be solved using the perturbation method \[15\], obtaining

\[
x(t) = l_1 + \frac{1}{2} [l_1 h_+(t) - l_2 h_\times(t)]
\]

\[
y(t) = l_2 - \frac{1}{2} [l_2 h_+(t) + l_1 h_\times(t)] \quad (29)
\]

\[
z(t) = l_3,
\]
Figure 6: The angular dependence to the + polarization for the LIGO interferometer at 8000 Hz
Figure 7: The low-frequency angular dependence to the $\times$ polarization for the Virgo interferometer
where $l_1$, $l_2$ and $l_3$ are the coordinates of the mirror of the interferometer in absence of GWs, with the beam-splitter located in the origin of the coordinate system. The computation of the response function for an arbitrary propagating direction of the GW, can be achieved by performing the rotation (5). As a result, the $u$ coordinate of the mirror is

$$u = L + \frac{1}{2}LAh_+(t + u \sin \theta \cos \phi), \quad (30)$$

where

$$A \equiv \cos^2 \theta \cos^2 \phi - \sin^2 \phi, \quad (31)$$

and $L = \sqrt{l_1^2 + l_2^2 + l_3^2}$ is the length of the interferometer arms.

We consider a photon which propagates in the $u$ axis. The unperturbed (i.e. in absence of GWs) propagation time between the two masses is

$$T = L. \quad (32)$$

From eq. (30), the displacements of the two masses under the influence of the GW are

$$\delta u_{\pm}(t) = 0 \quad (33)$$

and

Figure 8: The angular dependence to the × polarization for the Virgo interferometer at 8000 Hz.
Figure 9: The low-frequency angular dependence to the × polarization for the LIGO interferometer
Figure 10: The angular dependence to the \( \times \) polarization for the LIGO interferometer at 8000 Hz

\[
\delta u_m(t) = \frac{1}{2} LAh_\times(t + L \sin \theta \cos \phi) \quad \text{(34)}
\]

In this way, the relative displacement, which is defined by

\[
\delta L(t) = \delta u_m(t) - \delta u_b(t) \quad \text{(35)}
\]
gives

\[
\frac{\delta T(t)}{T} = \frac{\delta L(t)}{L} = \frac{1}{2} LAh_\times(t + L \sin \theta \cos \phi) \quad \text{(36)}
\]

But, for a large separation between the test masses (in the case of Virgo the distance between the beam-splitter and the mirror is three kilometers, four in the case of LIGO), the definition (35) for relative displacements becomes unphysical because the two test masses are taken at the same time and therefore cannot be in a casual connection \[12, 16, 19\]. The correct definitions for the bouncing photon are

\[
\delta L_1(t) = \delta u_m(t) - \delta u_b(t - T_1) \quad \text{(37)}
\]

and

\[
\delta L_2(t) = \delta u_m(t - T_2) - \delta u_b(t) \quad \text{(38)}
\]
where $T_1$ and $T_2$ are the photon propagation times for the forward and return trip correspondingly. According to the new definitions, the displacement of one test mass is compared to the displacement of the other at a later time, in order to allow a finite delay for the light propagation. The propagation times $T_1$ and $T_2$ in eqs. (37) and (38) can be replaced with the nominal value $T$ because the test mass displacements are already first order in $h_{+}$. Thus, the total change in the distance between the beam splitter and the mirror, in one round-trip of the photon, is

$$
\delta L_{r.t.}(t) = \delta L_1(t - T) + \delta L_2(t) = 2\delta u_m(t - T) - \delta u_b(t) - \delta u_b(t - 2T), \quad (39)
$$

and, in terms of the amplitude of the $+ \text{polarization of the GW}$:

$$
\delta L_{r.t.}(t) = LA h_{+}(t + L \sin \theta \cos \phi - L). \quad (40)
$$

The change in distance (40) leads to changes in the round-trip time for photons propagating between the beam-splitter and the mirror:

$$
\frac{\delta_T T(t)}{T} = A h_{+}(t + L \sin \theta \cos \phi + L). \quad (41)
$$

In the last calculation (variations in the photon round-trip time which come from the motion of the test masses induced by the GW), it has been implicitly assumed that the propagation of the photon between the beam-splitter and the mirror of the interferometer is uniform as if it were moving in a flat space-time. But the presence of the tidal forces indicates that the space-time is curved. As a result, one more effect after the first discussed has to be considered, and it requires spacial separation (note: in (16) the effects considered were three, but the third effect vanishes putting the beam splitter in the origin of the coordinate system (12)). This is exactly the contribution of the photons redshift. If it results different from the contribution of the test masses motion in previous analysis (i.e. the sum of the two contributions is different from zero), it also clarifies the misconception purporting that, because the wavelength of the laser light and the length of an interferometer’s arm are both stretched by a GW, no effect should be present.

From equations (26), (27) and (5) the tidal acceleration of a test mass caused by the GW in the $u$ direction is

$$
\ddot{u}(t + u \sin \theta \cos \phi) = \frac{1}{2} LA \ddot{h}_{+}(t + u \sin \theta \cos \phi). \quad (42)
$$

Equivalently, one can say that there is a gravitational potential (12 [14] 15):

$$
V(u, t) = -\frac{1}{2} LA \int_{0}^{u} \ddot{h}_{+}(t + l \sin \theta \cos \phi)dl, \quad (43)
$$

which generates the tidal forces, and that the motion of the test mass is governed by the Newtonian equation.
In the framework of weak-field gravity, the interval in the gauge of the local observer is given by \[15\] \[17\]

\[ds^2 = g_{00}dt^2 + du^2 + dv^2 + dw^2.\] (45)

Equations like eq. (42) work for the \(v\) and \(w\) directions too. Thus, photon momentum in these directions is not conserved and photons launched in the \(u\) axis will deflect out of this axis. But here this effect can be neglected, because the photon deflections into the \(v\) and \(w\) directions will be at most of order \(h^+\) (see \[2\] \[3\] \[12\] \[16\] \[17\] \[19\]). Then, to first order in \(h^+\), the \(dv^2\) and \(dw^2\) terms can be neglected. Thus, the line element (45), for photons propagating along the \(u\) - axis, can be rewritten as

\[ds^2 = g_{00}dt^2 + du^2.\] (46)

The condition for a null trajectory \((ds = 0)\) and the well known relation between Newtonian theory and linearized gravity \((g_{00} = 1 + 2V)\) give the coordinate velocity of the photons

\[v^2_c \equiv \left(\frac{du}{dt}\right)^2 = 1 + 2V(t, u),\] (47)

which, to first order in \(h^+\), is approximated by

\[v_c \approx \pm[1 + V(t, u)],\] (48)

with + and − for the forward and return trip respectively. Knowing the coordinate velocity of the photon, the propagation time for its travelling between the beam-splitter and the mirror can be defined:

\[T_1(t) = \int_{u_b(t-T_1)}^{u_m(t)} \frac{du}{v_c}\] (49)

and

\[T_2(t) = \int_{u_m(t-T_2)}^{u_b(t)} \frac{(-du)}{v_c}.\] (50)

The calculations of these integrals would be complicated because the \(u_m\) boundaries of them are changing with time:

\[u_b(t) = 0\] (51)

and

\[u_m(t) = L + \delta u_m(t).\] (52)
But, to first order in \(h_+\), these contributions can be approximated by \(\delta L_1(t)\) and \(\delta L_2(t)\) (see eqs. (37) and (38)). Thus, the combined effect of the varying boundaries is given by \(\delta_1 T(t)\) in eq. (11). Then, only the times for photon propagation between the fixed boundaries 0 and \(L\) have to be computed. Such a propagation times will be indicated with \(\Delta T_{1,2}\) to distinguish from \(T_{1,2}\). In the forward trip, the propagation time between the fixed limits is

\[
\Delta T_1(t) = \int_0^L \frac{du}{v_c(t', u)} \approx L - \int_0^L V(t', u)du, \tag{53}
\]

where \(t'\) represents the delay time which corresponds to the unperturbed photon trajectory (see Section 2):

\[
t' = t - (L - u).
\]

Similarly, the propagation time in the return trip is

\[
\Delta T_2(t) = L - \int_L^0 V(t', u)du, \tag{54}
\]

where now the delay time is given by

\[
t' = t - u.
\]

The sum of \(\Delta T_1(t - T)\) and \(\Delta T_2(t)\) gives the round-trip time for photons travelling between the fixed boundaries. Then, the deviation of this round-trip time (distance) from its unperturbed value \(2T\) is

\[
\delta_2 T(t) = -\int_0^L [V(t - 2L + u, u)du + \\
- \int_0^0 V(t - u, u)]du, \tag{55}
\]

and, using eq. (13),

\[
\delta_2 T(t) = \frac{1}{2}LA \int_0^L \left[\int_0^u \ddot{h}_+(t - 2T + l(1 + \sin \theta \cos \phi))dl + \\
- \int_0^u \ddot{h}_+(t - l(1 - \sin \theta \cos \phi))dl\right]du, \tag{56}
\]

Thus, the total round-trip proper time in presence of the GW is:

\[
T_t = 2T + \delta_1 T + \delta_2 T, \tag{57}
\]

and

\[
\delta T_u = T_t - 2T = \delta_1 T + \delta_2 T \tag{58}
\]

is the total variation of the proper time for the round-trip of the photon in presence of the GW in the \(u\) direction.

Using eqs. (11), (56) and the Fourier transform of \(h_+\), defined by
\[ \tilde{h}_+(\omega) = \int_{-\infty}^{\infty} dt \ h(t) \exp(i\omega t), \]  
(59)

the quantity \( \tilde{h}_u(\omega) \) can be computed in the frequency domain as

\[ \tilde{T}_u(\omega) = \tilde{\delta}_1 T(\omega) + \tilde{\delta}_2 T(\omega) \]  
(60)

where

\[ \tilde{\delta}_1 T(\omega) = \exp[i\omega L(1 - \sin \theta \cos \phi)]L \tilde{h}_+(\omega) \]  
(61)

\[ \tilde{\delta}_2 T(\omega) = - \frac{LA}{2} \left[ \frac{-1 + \exp[i\omega L(1 - \sin \theta \cos \phi)] - iL\omega(1 - \sin \theta \cos \phi)}{(1 - \sin \theta \cos \phi)^2} \right] \tilde{h}_+(\omega). \]  
(62)

In the above computation, derivative and translation Fourier transform theorems have been used.

Then, using eqs. (31), (61), (62) and the definition (16) the signal can be defined in the \( u \) arm in this gauge too:

\[ \tilde{T}_u(\omega) \equiv \Upsilon^+_u(\omega) \tilde{h}_+(\omega), \]  
(63)

where

\[ \Upsilon^+_u(\omega) \equiv \frac{(\cos^2 \theta \cos^2 \phi - \sin^2 \phi)H_u(\omega, \theta, \phi)}{2L} \]  
(64)

is the response function of the \( u \) arm of the interferometer to the GW. This is exactly the response function that has been obtained in eq. (18) for the TT gauge.

Note: the fact that this response function is, in general, different from zero implies that the contribution (61) to the total signal, due to the motion of the test masses, will be, in general, different from the contribution (62) due to the gravitational redshift of the GW. In this way the misconception on interferometers is clarified.

The same analysis works the \( v \) arm. One gets the total response function in the \( v \) direction for the GWs in the gauge of the local observer, which is

\[ \Upsilon^+_v(\omega) \equiv \frac{(\cos^2 \theta \sin^2 \phi - \cos^2 \phi)H_v(\omega, \theta, \phi)}{2L}, \]  
(65)

which is the same results of the TT gauge in this case too (see eq. (19)).

Thus, in the gauge of the local observer, the total frequency-dependent response function (i.e. the detector pattern) of an interferometer to the + polarization of the GW is given by:
\[ \mathcal{H}^+(\omega) \equiv \Upsilon_u^+(\omega) - \Upsilon_v^+(\omega) = \]
\[ = \frac{(\cos^2 \theta \cos^2 \phi - \sin^2 \phi)}{2L} \tilde{H}_u(\omega, \theta, \phi) + \]
\[ - \frac{(\cos^2 \theta \sin^2 \phi - \cos^2 \phi)}{2L} \tilde{H}_v(\omega, \theta, \phi), \tag{66} \]

which is the same result of the TT gauge (eq. (21)). This gauge-invariance agrees with lots of results which are well known in the literature, where the analysis has been made in the low frequency approximation, i.e. in the case in which the wavelength of the GW is much larger than the length of the interferometer’s arms (see [18] for example). Note that the gauge-invariance obtained with the equality between equation (21) and equation (66) is more general than the one in [16], where the computation was performed in the simplest geometry of the interferometer in respect to the propagating gravitational wave, i.e. in the case in which the arms of the interferometer are perpendicular to the propagating GW, an only for one arm. Putting \( \theta = \phi = 0 \) and \( v = 0 \) in equations (21) and (66), the result of [16] for one arm of an interferometer is recovered.

A similar analysis works for the \( \times \) polarization. One obtains the same result of eq. (23) in the TT gauge.

Then, the total response functions of interferometers for the \( + \) and \( \times \) polarization of GWs, in their full angular and frequency dependences, are equal in the TT gauge and in the gauge of a local observer. In this way, the gauge-invariance has been totally generalized.

4 Conclusions

In this paper, the gauge-invariance on the response of interferometers to GWs has been shown. In this process, after a review of results on the TT gauge, where, in general, the theoretical computations on GWs are performed, which is due for completeness, the gauge of the local observer has been analysed. The gauge-invariance between the two gauges has been shown in its full angular and frequency dependences while in previous works in the literature this gauge-invariance was shown only in the low frequencies approximation or in the simplest geometry of the interferometer in respect to the propagating gravitational wave. As far as the computation of the response functions in the gauge of the local observer has been concerned, a common misconception about interferometers has been also clarified.

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