The Distribution of G.C.D.s of Shifted Primes and Lucas Sequences

Abhishek Jha and Ayan Nath

Abstract

Let \((u_n)_{n\geq0}\) be a nondegenerate Lucas sequence and \(g_u(n)\) be the arithmetic function defined by \(\gcd(n,u_n)\). Recent studies have investigated the distributional characteristics of \(g_u\). Numerous results have been proven based on the two extreme values 1 and \(n\) of \(g_u(n)\). Sanna investigated the average behaviour of \(g_u\) and found asymptotic formulas for the moments of \(\log g_u\). In a related direction, Jha and Sanna investigated properties of \(g_u\) at shifted primes.

In light of these results, we prove that for each positive integer \(\lambda\), we have

\[
\sum_{\substack{p\leq x \ \text{prime}}} (\log g_u(p-1))^\lambda \sim P_{u,\lambda} \pi(x),
\]

where \(P_{u,\lambda}\) is a constant depending on \(u\) and \(\lambda\) which is expressible as an infinite series. Additionally, we provide estimates for \(P_{u,\lambda}\) and \(M_{u,\lambda}\), where \(M_{u,\lambda}\) is the constant for an analogous sum obtained by Sanna [J. Number Theory 191 (2018), 305–315]. As an application of our results, we prove that for each positive integer \(\lambda\), we have

\[
\max\{g_u(n) : n \leq x\} \gg x^{0.4736}
\]

unconditionally, while \(\max\{g_u(p-1) : p \leq x\} \gg x^{1-o(1)}\) under the hypothesis of Montgomery’s or Chowla’s conjecture.

Contents

1 Introduction .................................................. 2
2 Lemmas and Preliminaries ................................. 6
3 Proofs of Theorems 1.4 and 1.5 ......................... 10
   Unconditional bounds ...................................... 10
   Bounds conditional on GRH ............................... 12
4 Proof of Theorem 1.6 ....................................... 12
   Lower bound on \(M_{u,\lambda}\) ............................ 13
   Upper bound on \(P_{\lambda,u}\) ............................... 13
5 Proof of Theorem 1.11 ..................................... 15
1 Introduction

Let \((u_n)_{n \geq 0}\) be an integral linear recurrence, that is, there exists \(a_1, a_2, \ldots, a_k \in \mathbb{Z}\) with \(a_k \neq 0\) such that

\[
u_n = a_1u_{n-1} + a_2u_{n-2} + \cdots + a_ku_{n-k}
\]

for all integers \(n \geq k\). The sequence is said to be nondegenerate if none of the ratios \(\alpha_i/\alpha_j\), \(i \neq j\), is a root of unity, where \(\alpha_1, \alpha_2, \ldots, \alpha_k\) are all the pairwise distinct zeroes of the characteristic polynomial

\[
\psi_u(x) = x^k - a_1x^{k-1} - a_2x^{k-2} - \cdots - a_k,
\]

The sequence \((u_n)_{n \geq 0}\) is said to be a Lucas sequence if \(u_0 = 0, u_1 = 1\), and \(k = 2\). Define \(\Delta_u := a_1^2 + 4a_2\) to be the discriminant of the characteristic polynomial of \((u_n)_{n \geq 0}\).

Let \(g_u(n)\) be the arithmetic function defined by \(\gcd(n, u_n)\). Several authors have studied the distributional properties of \(g_u\). For instance, the set of all positive integers \(n\) such that \(u_n\) divisible by \(n\) has been studied by Alba González, Luca, Pomerance, and Shparlinski in [AGLP+12] under the hypothesis that the characteristic polynomial of \((u_n)_{n \geq 0}\) has only simple roots. The same set was also studied by André-Jeannin [AJ91], Luca and Tron [LT15], Sanna [San17], and Somer [Som96], in the special case in which \((u_n)_{n \geq 0}\) is a Lucas sequence or the Fibonacci sequence.

On the other hand, Sanna and Tron [ST18; San19] have studied the fiber \(g_u^{-1}(y)\) where \(y = 1\) and \((u_n)_{n \geq 0}\) is nondegenerate, and in case \((u_n)_{n \geq 0}\) is the Fibonacci sequence with \(y\) being an arbitrary positive integer. The image \(g_u(\mathbb{N})\) has been analysed by Leonetti and Sanna [LS18] in case \((u_n)_{n \geq 0}\) is the Fibonacci sequence. Similar problems, with \((u_n)_{n \geq 0}\) replaced by an elliptic divisibility sequence or by the orbit of 0 under a polynomial map, were also studied [CGS17; GU20; Got12; Jha21; Kim20; SS11]. All recent developments in the study of \(g_u\) have been discussed in a recent survey by Trol [Tro20].

The previous results give rather convincing answers to the problem of determining extreme values of \(g_u(n)\), however, obtaining information about its average behaviour and distribution as arithmetic function has recently got particular interest. A natural question posed by Sanna in [San18] is–

**Question 1.1 ([San18])**. What is the average value of \(g_u\)? Or more generally, given a positive integer \(\lambda > 0\), is it possible to find an asymptotic for

\[
\sum_{n \leq x} g_u(n)^\lambda
\]

as \(x\) is large?

Hereafter, we assume that \((u_n)_{n \geq 0}\) is a nondegenerate Lucas sequence with \(a_1\) and \(a_2\) relatively prime integers. Given the oscillatory behaviour of \(g_u\), which makes it hard to investigate, the author succeeded in finding an asymptotic for the logarithms of \(g_u\).

**Theorem 1.2 ([San18, Theorem 1.1])**. Fix a positive integer \(\lambda\) and some \(\varepsilon > 0\). We have

\[
\sum_{n \leq x} (\log g_u(n))^{\lambda} = M_{u, \lambda} x + E_{u, \lambda}(x),
\]

where \(M_{u, \lambda} > 0\) is a constant depending on \(a_1, a_2,\) and \(\lambda\), and

\[
E_{u, \lambda}(x) \ll_{u, \lambda} x^{(3\lambda + 1)/(3\lambda + 2) + \varepsilon}.
\]
Furthermore, the author obtained an convergent infinite series for the constant \( M_{u,\lambda} \), but before stating it we need to introduce some notations. For each positive integer \( m \) relatively prime to \( a_2 \), let \( z_u(m) \) be the rank of appearance of \( m \) in the Lucas sequence \( (u_n)_{n \geq 0} \), that is, \( z_u(m) \) is the least positive integer \( n \) such that \( m \) divides \( u_n \). It is well known that the rank of appearance exists (see, e.g., [Ren13]). Also, define \( \ell_u(m) := \text{lcm}(m, z_u(m)) \).

**Theorem 1.3 ([San18, Theorem 1.2])**. For all positive integers \( \lambda \), we have

\[
M_{u,\lambda} = \sum_{(m,a_2)=1} \frac{\rho_\lambda(m)}{\ell_u(m)},
\]

where the sum runs over all positive integers relatively prime to \( a_2 \).

Recently, Mastrostefano [Mas19] obtained a partial answer to the question posed by Sanna [San18] and proved a nontrivial upper bound on the moments of \( g_u \). Given the rich structure of \( g_u \), and in order to investigate the relationships between shifted primes and Lucas sequences; Jha and Sanna in [JS22] studied the set

\[
\mathcal{P}_k = \{ p \leq x : g_u(p-1) = k, \, p \text{ prime} \}
\]

for positive integers \( k \), and proved the existence of relative density of \( \mathcal{P}_k \) in the set of prime numbers and also obtained it as an absolutely convergent series. Furthermore, they proved bounds on the distribution of positive integers of the form \( g_u(p-1) \) for primes \( p \). Exploring further in this direction we investigate the average behaviour and distribution of \( g_u \) under shifted prime arguments. We prove the following theorem concerning the average behaviour of \( g_u(p-1) \).

**Theorem 1.4.** Fix a positive integer \( \lambda \) and some \( A, \epsilon > 0 \). We have

\[
\sum_{p \leq x} (\log g_u(p-1))^{\lambda} = P_{u,\lambda} \pi(x) + E_{u,\lambda}(x),
\]

where \( P_{u,\lambda} > 0 \) is a constant depending on \( a_1, a_2, \) and \( \lambda \), and

\[
E_{u,\lambda}(x) \ll_{u,\lambda} \frac{x}{(\log x)^A}.
\]

Conditional on the Generalized Riemann Hypothesis (GRH), we have

\[
E_{u,\lambda}(x) \ll_{u,\lambda} x^{(6\lambda+3)/(6\lambda+4)+\epsilon}.
\]

**Theorem 1.5.** For all positive integers \( \lambda \), we have

\[
P_{u,\lambda} = \sum_{(n,a_2)=1} \frac{\rho_\lambda(n)}{\varphi(\ell_u(n))},
\]

where the sum runs over all positive integers relatively prime to \( a_2 \).

As remarked by Sanna [San18] and Tron [Tro20], Theorems 1.2 and 1.3 bear a formal resemblance with work of Luca and Shparlinski [LS07, Theorem 2]. The authors studied sums of the form \( \sum_{n \leq x} f(u_n)^k \) for arbitrary arithmetic functions \( f \) satisfying some growth conditions, and obtained asymptotics of the form \( \sum_{n \leq x} f(u_n)^k \sim M_{f,k}x \). They also pointed out that \( \log M_{f,k} \ll k \log k \). Motivated by these results, we obtain estimates of the constants \( M_{u,\lambda} \) and \( P_{u,\lambda} \) of Theorems 1.3 and 1.5, respectively.
Theorem 1.6. For each positive integer $\lambda$, we have
\begin{enumerate}[(a)]
    \item $\log M_{u,\lambda} = \lambda \log 2 + O_u(\lambda)$, where $M_{u,\lambda}$ is defined in Theorem 1.3.
    \item $\log P_{u,\lambda} = \lambda \log 2 + O_u(\lambda)$, where $P_{u,\lambda}$ is defined in Theorem 1.5.
\end{enumerate}

Another important direction to investigate is estimating the distribution function of $g_u$. As an application of Sanna’s results in [San18], the author obtained an upper bound on the count $\#\{n \leq x : g_u(n) > y\}$ for all $x, y > 1$. Mastrostefano in [Mas19] improved these bounds for a specific range of $y$. As a corollary of our Theorem 1.4, we obtain upper bounds on the count $\#\{p \leq x : g_u(p-1) > y\}$ for all $x, y > 1$.

Corollary 1.7. For each positive integer $\lambda$, we have
$$\#\{p \leq x : g_u(p-1) > y\} \leq u, \lambda \frac{1}{(\log y)^\lambda} \log x,$$
for all $x, y > 1$.

Exploring further in this direction, we prove that for a fixed $y > 0$, there exist infinitely many runs of $m$ consecutive primes in short intervals such that $g_u(p-1) > y$. The result is essentially based on a recent remarkable framework of Zhang, Maynard, Tao, et al [May15] on small gaps between primes. In particular, we require a theorem of Freiberg [Fre15, Theorem 1] on consecutive primes in arithmetic progressions.

Proposition 1.8. Let $p_1 = 2 < p_2 = 3 < \cdots$ be the sequence of prime numbers. Let $a$ and $q$ be a relatively prime pair of integers, and $m \geq 2$ be an integer. For infinitely many $n$, we have
$$p_{n+1} \equiv \cdots \equiv p_{n+m} \equiv a \pmod{q} \text{ and } p_{n+m} - p_{n+1} \leq qB_m.$$ 

It has been shown in [May15, Theorem 1.1] that we can take $B_m = cm^3e^{4m}$ for an absolute and effective constant $c$.

The above proposition gives the following immediate corollary.

Corollary 1.9. For positive integers $m$ and $y$, we have infinitely many runs of $m$ consecutive primes such that $g_u(p_{n+i} - 1) > y$ for each $0 \leq i \leq m - 1$ and
$$\liminf_{n \to \infty} (p_{n+m-1} - p_n) < C y^2 m^3 e^{4m},$$
where $p_n$ denotes the $n$th prime and $C$ is an absolute constant. Moreover, if $\Delta_u \neq 1$, we have
$$\liminf_{n \to \infty} (p_{n+m-1} - p_n) \leq C y^2 m^3 e^{4m},$$
and in case $m = 2$, we have
$$\liminf_{n \to \infty} (p_{n+1} - p_n) \leq 246y.$$ 

Proof. Choose a positive integer $s \in (y, 2y]$ and consider the arithmetic progression $(1 + \ell(s)n)_{n=1}^\infty$. Noting that $\ell_u(n) \leq 2n^2$ (see Lemma 2.1-(d)) and applying Proposition 1.8, we get the first result. For the second part, as $\Delta_u \neq \pm 1$, we have a prime $p \mid \Delta_u$ such that $\ell_u(p') = p'$ for any positive integer $r$. Consider $g = p^{[\log p^y]}$ and the arithmetic progression $(1 + gn)_{n=1}^\infty$. Applying Proposition 1.8 again, we get that there exist infinitely many runs $p_n, p_{n+1}, \ldots, p_{n+m-1}$ of $m$ consecutive primes such that $p_{n+m-1} - p_n \leq C y^2 m^3 e^{4m}$ and $g_u(p_{n+i} - 1) > y$, $0 \leq i \leq m - 1$. In the case of $m = 2$, it has been obtained that $B_2 = 246$ works in [Pol14, Theorem 3.2], which gives us the last assertion. \qed
Having considered the distribution of \( g_a \), it is natural to consider the problem of analysing the growths of \( \max \{ g_a(n) : n \leq x \} \) and \( \max \{ g_a(p - 1) : n \leq x \} \). For the remainder of the paper, we assume that \( \Delta_u \neq 1 \).

It is easy to prove that

\[
x \ll \max \{ g_a(n) : n \leq x \} \leq x.
\]

To see this, we have the trivial upper bound \( \max \{ g_a(n) : n \leq x \} \leq x \). Since \( \Delta_u \neq \pm 1 \), we have a prime \( p \mid \Delta_u \) such that \( \ell_u(p^r) = p^r \) for any positive integer \( r \). Thus, we get that \( \max \{ g_a(n) : n \leq x \} \geq p^{[\log_p x]} \geq x/p \). This gives us desired conclusion.

One can observe that the problem of obtaining estimates for \( \max \{ g_u(n) : n \leq x \} \) is related to the study of positive integers \( n \) such that \( n \mid u_a \). If we were able to demonstrate that for any sufficiently large \( x \), there is an integer \( n \in (x - o(x), x) \) such that \( n \mid u_a \), we could then prove \( \max \{ g_a(n) : n \leq x \} \sim x \). If we consider the case of Fibonacci sequence \( (F_n)_{n \geq 0} \), then it is known that at least \( x^{1/4} \) positive integers \( n \) less than \( x \) satisfy \( n \mid F_n \) [AGLP+12, Theorem 1.3], which is similar to the lower bound of \( x^{1/3} \) obtained for Carmichael numbers less than \( x \) [Har08].

Lucas and Tron [LT15] pointed out that one should expect heuristics for self-Fibonacci divisors to be similar to those for Carmichael numbers. Larson [Lar21] recently proved that for all \( \delta > 0 \) and \( x \gg \delta \), there exist at least \( e^{\log x/(\log \log x)^{1/3}} \) Carmichael numbers between \( x \) and \( x + x/(\log x)^{1/2 + \delta} \). Thus, it is reasonable to expect similar results to hold for positive integers \( n \) dividing \( u_a \). Based on this observation, we make the following conjecture–

**Conjecture 1.10.** For Lucas sequences \( (u_n)_{n \geq 0} \) with \( \Delta_u \neq 1 \),

\[
\max \{ g_u(n) : n \leq x \} \sim x.
\]

Finding nontrivial lower bounds on the shifted prime analogue \( \max \{ g_u(p - 1) : p \leq x \} \) is notably more difficult. For the ease of notation, let us set

\[
\mathcal{G}(x) := \max \{ g_u(p - 1) : p \leq x \}.
\]

Our next theorem is in this direction.

**Theorem 1.11.** Let \( (u_n)_{n \geq 0} \) be a Lucas sequence such that \( \Delta_u \neq 1 \). Then we have that \( \mathcal{G}(x) \gg x^{0.4736} \) unconditionally, while under Montgomery’s conjecture (see Hypothesis 2.9) or Chowla’s conjecture (see [Cho34]), we obtain that \( \mathcal{G}(x) \gg x^{1-o(1)} \).

The paper is organized as follows. In Section 2, we state some well-known results and prove preliminary lemmas. In Section 3, we prove Theorems 1.4 and 1.5. Proofs of Theorems 1.6 and 1.11 are presented in Sections 4, and 5, respectively.

**Notations**

We employ the Landau-Bachman “Big Oh” and “little oh” notations \( O \) and \( o \), as well as the associated Vinogradov symbols \( \ll \) and \( \gg \), with their usual meanings. Any dependence of implied constants is explicitly stated or indicated with subscripts. Notations like \( O_a \) and \( o_u \) are shortcuts for \( O_{a_1, a_2} \) and \( o_{a_1, a_2} \), respectively. Throughout, the letters \( p \) and \( q \) reserved for prime numbers. We write \( (a, b) \) or \( \gcd(a, b) \) to denote the greatest common divisor of \( a \) and \( b \), and \( [a, b] \) or \( \lcm(a, b) \) to
denote the least common multiple of the same. As usual, denote by \( \tau(n) \), \( \omega(n) \), and \( P(n) \), for the number of divisors, the number of prime factors, and the greatest prime factor, of a positive integer \( n \), respectively.

For every \( x > 0 \) and for all integers \( a \) and \( b \), let \( \pi(x; b; a) \) be the number of primes \( p \leq x \) such that \( p \equiv a \pmod{b} \). Also denote the error in prime number theorem as

\[
\Delta(x; b, a) := \pi(x; b, a) - \frac{\pi(x)}{\varphi(b)}.
\]

Lastly, the incomplete gamma function \( \Gamma \) is defined as

\[
\Gamma(s, x) := \int_x^\infty e^{-t} t^{s-1} dt = \int_0^{\infty} \frac{(\log t)^{s+1}}{t^2} dt.
\]

2 Lemmas and Preliminaries

In what follows, let \((u_n)_{n \geq 0}\) be a nondegenerate Lucas sequence with \( \gcd(a_1, a_2) = 1 \). Note that the discriminant \( \Delta_u \neq 0 \) as \((u_n)_{n \geq 0}\) is nondegenerate.

**Lemma 2.1.** For all positive integers \( m, n, j \) and for all prime numbers \( p \nmid a_2 \), we have:

(a) \( m \mid g_n(m) \) if and only if \((m, a_2) = 1 \) and \( \ell_u(m) \mid n \).

(b) \( \text{lcm}(\ell_u(m), \ell_u(n)) = \ell_u(\text{lcm}(m, n)) \), whenever \((mn, a_2) = 1 \).

(c) \( \ell_u(p^j) = p^j \lambda_u(p) \) if \( p \nmid \Delta_u \), and \( \ell_u(p^j) = p^j \) if \( p \mid \Delta_u \).

(d) \( \ell_u(n) \leq 2n^2 \).

**Proof.** See [San18, Lemma 2.1] for facts (1)-(3). The last fact follows directly from the well-known inequality \( \lambda_u(n) \leq 2n \) (see, e.g., [SK13]). \( \square \)

For each positive integer \( \lambda \) and for each positive integer \( n > 1 \) with prime factorisation \( n = q_1^{h_1} \cdots q_s^{h_s} \), where \( q_1 < \cdots < q_s \) are prime numbers and \( h_1, \ldots, h_s \) are positive integers, define

\[
\rho_\lambda(n) := \lambda! \sum_{\lambda_1 + \cdots + \lambda_s = \lambda} \frac{s!}{\lambda_1! \cdots \lambda_s!} \prod_{i=1}^{s} (h_i^{\lambda_i} - (h_i - 1)^{\lambda_i})(\log q_i)^{\lambda_i},
\]

where sum is extended over all the \( s \)-tuples \((s \geq 1)\) of positive integers \((\lambda_1, \ldots, \lambda_s)\) such that \( \lambda_1 + \cdots + \lambda_s = \lambda \). Note that \( \rho_\lambda(n) = 0 \) when \( s > \lambda \). For the sake of convenience, we set \( \rho_{\lambda}(1) = 0 \).

The next lemma is a slightly improved upper bound on the arithmetic function \( \rho_\lambda \) than the one proved in [San18, Lemma 2.4].

**Lemma 2.2.** For all positive integers \( \lambda \) and \( n \), we have \( \rho_\lambda(n) \leq (\log n)^\lambda \).

**Proof.** Let \( n = q_1^{h_1} \cdots q_s^{h_s} \) be the prime factorisation of \( n \), with prime numbers \( q_1 < \cdots < q_s \) and positive exponents \( h_1, \ldots, h_s \). Assume also that \( s \leq \lambda \), since otherwise \( \rho_\lambda(n) = 0 \). Therefore,

\[
\rho_\lambda(n) \leq \sum_{\lambda_1 + \cdots + \lambda_s = \lambda} \frac{\lambda!}{\lambda_1! \cdots \lambda_s!} \prod_{i=1}^{s} (h_i \log q_i)^{\lambda_i} \leq (\log n)^\lambda,
\]

by the multinomial theorem. \( \square \)
For $x, y > 0$ and positive integer $r$, define

$$\gamma(r) := \# \{ n \in \mathbb{N} : (n, a_2) = 1 \text{ and } \ell_u(n) = r \} \quad \text{and} \quad \Phi(x, y) := \# \{ n \leq x : P(n) < y \}.$$ 

**Lemma 2.3.** For all positive integers $r$, we have that $\gamma(r) \leq \tau(r)$ where $\tau(r)$ denotes the number of divisors of $r$.

**Proof.** As $n | \ell_u(n) = r$, there are at most $\tau(r)$ possible values of $n$ such that $\ell_u(n) = r$. \hfill \square

**Lemma 2.4.** Let $C > 0$ be a constant. For all sufficiently large $x$, we have $\Phi(x, C) \leq (2 \log x)^C$.

**Proof.** Each of the positive integers $n$ counted by $\Phi(x, C)$ can be written as $n = p_1^{a_1} \cdots p_{\pi(C)}^{a_{\pi(C)}}$ where $p_1, \ldots, p_{\pi(C)}$ are all prime numbers less than $C$, and $a_1, \ldots, a_{\pi(C)}$ are non-negative integers. Clearly there are at most $1 + \log x / \log 2$ choices for each $a_i$. Therefore,

$$\Phi(x, C) \leq \left(1 + \frac{\log x}{\log 2}\right)^C \leq (2 \log x)^C,$$

as desired. \hfill \square

The next three lemmas are upper bounds for certain sums involving $\ell_u$.

**Lemma 2.5.** We have

$$\sum_{\substack{P(n) \geq y \\ (n, a_2) = 1}} \frac{1}{\ell_u(n)} \ll u \frac{1}{y^{1/3 - \varepsilon}}$$

for all $\varepsilon \in (0, 1/4]$ and $y \gg_{u, \lambda} 1$.

**Proof.** See [San18, Lemma 2.5]. \hfill \square

**Lemma 2.6.** We have

$$\sum_{\substack{n \geq y \\ (n, a_2) = 1}} \frac{\rho_\lambda(n)}{\ell_u(n)} \ll u, \lambda \frac{1}{y^{1/(1 + 3\lambda) - \varepsilon}}$$

for all positive integers $\lambda, \varepsilon \in (0, 1/5]$, and $y \gg_{u, \lambda, \varepsilon} 1$.

**Proof.** See [San18, Lemma 2.6]. \hfill \square

**Lemma 2.7.** We have

$$\sum_{\substack{n \geq y \\ (n, a_2) = 1}} \frac{\rho_\lambda(n)}{\varphi(\ell_u(n))} \ll u, \lambda \frac{\log \log y}{y^{1/(1 + 3\lambda) - \varepsilon}},$$

for all positive integers $\lambda, \varepsilon \in (0, 1/5]$, and $y \gg_{u, \lambda, \varepsilon} 1$. 

PROOF. From Lemma 2.6, it follows that

\[ S(t) := \sum_{n \geq t} \frac{\rho_\lambda(n)}{\ell_u(n)} \ll_u \lambda \frac{1}{t^{1/(1+3\lambda)} - \varepsilon} \]

for all \( y \gg u, \lambda, \varepsilon \). By partial summation,

\[
\sum_{n \geq y} \frac{\rho_\lambda(n) \log n}{\ell_u(n)} = S(y) \log \log y + \int_y^{+\infty} \frac{S(t)}{t \log t} \, dt
\]

\[
\ll_u \lambda \frac{\log \log y}{y^{1/(1+3\lambda)} - \varepsilon} + \int_y^{+\infty} \frac{1}{t^{1+1/(1+3\lambda)} - \varepsilon \log t} \, dt
\]

\[
\ll \frac{\log \log y}{y^{1/(1+3\lambda)} - \varepsilon}.
\]

Since \( \phi(n) \gg n/\log \log n \) (see, e.g., [Ten15, Chapter I.5, Theorem 4]) and \( \ell_u(n) \leq 2n^2 \) (see Lemma 2.1-(d)) for all positive integers \( n \), it follows that

\[
\sum_{n \geq y} \frac{\rho_\lambda(n)}{\ell_u(n)} = \sum_{n \geq y} \frac{\rho_\lambda(n) \log n}{\ell_u(n)} \ll \frac{\log \log y}{y^{1/(1+3\lambda)} - \varepsilon}.
\]

LEMA 2.8 (SIEGEL-WALFISZ UNDER GRH). Under the Generalized Riemann Hypothesis (GRH), we have

\[ \Delta(x; b, a) \ll x^{1/2}(\log x), \]

for all \( x \gg 1 \).

PROOF. See [MV07, Corollary 13.8].

HYPOTHESIS 2.9 (MONTGOMERY’S CONJECTURE, [FG89, CONJECTURE 1(B)]). For any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon \) such that for all \( b \leq x \) we have

\[ \Delta(x; b, a) \leq C_\varepsilon x^{1/2+\varepsilon} b^{1/2}, \]

for all \( x \gg 1 \).

LEMA 2.10 (BOMBERI-VINOGRADOV). For any \( A > 0 \), there exists \( B = B(A) > 0 \) such that

\[
\sum_{d \leq \sqrt{x/((\log x)^B}}} \left| \Delta(x; d, a) \right| \ll \frac{x}{(\log x)^A},
\]

for all \( x \gg_A 1 \).

For our applications we need the following weighted version of the Bombieri-Vinogradov theorem.
LEMMA 2.11. For any $A > 0$, there exists $B = B(A) > 0$ such that
\[
\sum_{d \leq \sqrt{x}/(\log x)^A} \gamma(d)|\Delta(x; d, a)| \ll \frac{x}{(\log x)^A},
\]
for all $x \gg_A 1$.

PROOF. Replace $A$ with $2A + 4$ in Lemma 2.10 to get a constant $B = B(2A + 4)$. Since $\pi(x; d, a) \ll x/d$ for $d \leq x$, we have that
\[
\Delta(x; b, a) \ll \frac{x}{d},
\]
Therefore, by putting $z = x^{1/2}/(\log x)^B$, we get
\[
\sum_{d \leq z} \gamma(d)|\Delta(x; d, a)| \ll \sum_{d \leq z} \gamma(d) \left(\frac{x}{d}\right)^{1/2} |\Delta(x; d, a)|^{1/2}
\ll x^{1/2} \left( \sum_{d \leq z} \gamma(d)^2 \frac{1}{d} \right)^{1/2} \left( \sum_{d \leq z} |\Delta(x; d, a)| \right)^{1/2}
\]
by the Cauchy-Schwarz inequality. By Lemma 2.10 and the choice of $2A + 4$, we observe that
\[
\left( \sum_{d \leq z} |\Delta(x; d, a)| \right)^{1/2} \ll \frac{x^{1/2}}{(\log x)^{4+2}},
\]
and by Lemma 2.3, that
\[
\left( \sum_{d \leq z} \frac{\tau(d)^2}{d} \right)^{1/2} \ll \left( \sum_{d \leq z} \frac{\tau(d)^2}{d} \right)^{1/2} \ll (\log z)^2 \ll (\log x)^2,
\]
where the second inequality follows from the fact that
\[
\sum_{n \leq x} \tau(n)^2 = C_2 x (\log x)^3 + O(x(\log x)^2)
\]
for some constant $C_2 > 0$ (see, e.g., [LT17, Theorem 1]) and partial summation.

REM A R K 2.12. It is worth mentioning that the weighted Bombieri-Vinogradov theorem obtained above can be used in the proof of [JS22, Theorem 1.2] to improve the error term of the relative density estimate obtained. The authors originally obtained an error term of the form
\[
E(x) \ll \frac{x}{\log x} \cdot \frac{\log \log \log x}{\exp\left( \delta (\log \log x)^{1/2} (\log \log \log x)^{1/2} \right)},
\]
for some $\delta > 0$, while employing Lemma 2.11 instead leads to an effective upper bound on the error term of the form
\[
E(x) \ll \frac{x}{(\log x)^A}
\]
for any $A > 0$. 


3 Proofs of Theorems 1.4 and 1.5

Throughout the section, the letter $q$, with or without subscript, denotes a prime number not dividing $a_2$, while the letter $j$, with or without subscript, denotes a positive integer. We have

$$\log g_u(p-1) = \sum_{q^j \mid g_u(p-1)} j \log p = \sum_{q^j \mid g_u(p-1)} \log p = \sum_{p \equiv 1 (\mod \ell_u(q^j))} \log p$$

for all primes $p$ where the last equality follows from Lemma 2.1-(a). Consequently, for any prime $p$ and for all $x > 0$, it follows that

$$\sum_{p \leq x} (\log g_u(p-1))^\lambda = \sum_{p \leq x} \left( \sum_{p \equiv 1 (\mod \ell_u(q^j))} \log q \right)^\lambda$$

$$= \sum_{p \leq x} \sum_{\ell_u(q^j) \mid p-1, \ell_u(q^{j_2}) \mid p-1} \log q_1 \cdot \log q_\lambda$$

$$= \sum_{p \leq x} \sum_{p \equiv 1 (\mod \ell_u(q^j, ..., q^{j_\lambda}))} \log q_1 \cdot \log q_\lambda$$

$$= \sum_{q_1^{j_1}, ..., q_\lambda^{j_\lambda}} \log q_1 \cdot \log q_\lambda \pi(x; \ell_u(q_1^{j_1}, ..., q_\lambda^{j_\lambda}), 1)$$

$$= \sum_{n = (q_1^{j_1}, ..., q_\lambda^{j_\lambda})} \pi(x; \ell_u(n), 1) \sum_{n = [q_1^{j_1}, ..., q_\lambda^{j_\lambda}]} \log q_1 \cdot \log q_\lambda$$

where third equality follows from Lemma 2.1-(b). By the exact same reasoning as [San18, Section 3] we have

$$\sum_{n = [q_1^{j_1}, ..., q_\lambda^{j_\lambda}]} \log q_1 \cdot \log q_\lambda = \sum_{\lambda_1 + \cdots + \lambda_\lambda = \lambda} \frac{\lambda!}{\lambda_1! \cdots \lambda_\lambda!} \prod_{i=1}^\lambda (h_1^{\lambda_i} - (h_i - 1)^{\lambda_i})(\log q_i)^{\lambda_i}$$

$$= \rho_\lambda(n).$$

Therefore

$$\sum_{p \leq x} (\log g_u(p-1))^\lambda = \sum_{d \leq x} \rho_\lambda(d) \pi(x; \ell_u(d), 1)$$

for all $x > 0$.

Unconditional bounds

Choose a large constant $A > 0$ and $B = B(A) > 0$ in the statement of the weighted Bombieri-Vinogradov theorem (Lemma 2.11). Set $y := x^{1/4}/\sqrt{2}(|\log x|)^B/2$ for brevity. We split the right
hand side of (1) as
\[
\sum_{p \leq x} (\log g_u(p - 1))^\lambda = \sum_{d \leq y} \left( \frac{\pi(x) \rho_y(d)}{\varphi(\ell_u(d))} + \sum_{\ell_u(d) = 1} \rho_y(d) \Delta(x; \ell_u(d), 1) \right)
\]
\[= E_1 + E_2.\]

First we estimate $E_1$.

\[
E_1 = \sum_{d \leq y} \frac{\pi(x) \rho_y(d)}{\varphi(\ell_u(d))} + \sum_{d \leq y} \rho_y(d) \Delta(x; \ell_u(d), 1) \]
\[= \sum_{d \leq y} \pi(x) \cdot \frac{\rho_y(d)}{\varphi(\ell_u(d))} + \sum_{d \leq y} \rho_y(d) \Delta(x; \ell_u(d), 1). \tag{2}\]

As $y \to \infty$, the left sum in (2) converges to $P_{u, \lambda} \pi(x)$ where

\[
P_{u, \lambda} := \sum_{n, a_2 = 1} \frac{\rho_y(n)}{\varphi(\ell_u(n))}.
\]

This follows by convergence of the sum in Lemma 2.7. For the right sum in (2), if $d \leq y$, then $\ell_u(d) \leq x^{1/2}/(\log x)^B$ by Lemma 2.1-(d). Therefore, by Lemmas 2.2 and 2.11,

\[
\sum_{d \leq y} \rho_y(d) \Delta(x; \ell_u(d), 1) \leq (\log x)^\lambda \sum_{d \leq y} |\Delta(x; \ell_u(d), 1)|
\]
\[\leq (\log x)^\lambda \sum_{d \leq y} \gamma(d) |\Delta(x; d, 1)|
\]
\[\leq \frac{x}{(\log x)^{A - \lambda}}.\]

For $E_2$, we employ the trivial bound $\pi(x; b, a) \leq x/b$ and Lemma 2.6 as follows–

\[
E_2 = \sum_{d \geq y} \rho_y(d) \pi(x; \ell_u(d), 1) \leq x \sum_{d \geq y} \frac{\rho_y(d)}{\ell_u(d)} \leq u_{u, \lambda} \frac{x}{y^{1/(1 + 3\lambda) - \epsilon}}
\]

where $\epsilon \in (0, 1/5]$. Thus, finally we obtain that

\[
\sum_{p \leq x} (\log g_u(p - 1))^\lambda = P_{u, \lambda} \pi(x) + E_{u, \lambda}(x)
\]

where

\[
E_{u, \lambda}(x) \leq \frac{x}{(\log x)^A}
\]

for any $A > 0$. 

Bounds conditional on GRH

Let $z$ be a parameter to be chosen later. Just as before, we split (1) as

$$\sum_{p \leq x} (\log g_u(p-1))^\lambda = \sum_{d \leq z \atop (d, a_2) = 1} \rho_\lambda (d) \pi(x; \ell_u(d), 1) + \sum_{d \geq z \atop (d, a_2) = 1} \rho_\lambda (d) \pi(x; \ell_u(d), 1)$$

$$= E_1' + E_2'.$$

First we look at $E_1'$. Proceeding as before we just need to look at the sum

$$\sum_{d \leq z \atop (d, a_2) = 1} \rho_\lambda (d) \Delta(x; \ell_u(d), 1).$$

Again, we obtain that

$$\sum_{d \leq z \atop (d, a_2) = 1} \rho_\lambda (d) \Delta(x; \ell_u(d), 1) \ll z \cdot (\log x)^\lambda \cdot x^{1/2} (\log x)$$

by Lemmas 2.2 and 2.8. Similarly, for $E_2'$, we get that

$$E_2' \ll u, \lambda \frac{x}{z^{1/(1+\lambda)} - \varepsilon}.$$

Thus,

$$\sum_{p \leq x} (\log g_u(p-1))^\lambda = P_{u, \lambda} (x) + E_{u, \lambda} (x),$$

where

$$E_{u, \lambda} (x) \ll u, \lambda \frac{x}{z^{1/(1+\lambda)} - \varepsilon} + z (\log x)^\lambda \cdot x^{1/2} (\log x).$$

It is routine to check that the optimal value of $z$, while satisfying all restrictions, is $z = \frac{x^{(3\lambda + 1)} / (6\lambda + 1)}{(3\lambda + 1)}$.

In conclusion,

$$E_{u, \lambda} (x) \ll u, \lambda x^{(6\lambda + 3) / (6\lambda + 4) + \varepsilon}$$

for all sufficiently large $x$ depending on $a_1, a_2, \lambda$ and $\varepsilon$.

4 Proof of Theorem 1.6

It is obvious that $M_{\lambda, u} \leq P_{\lambda, u}$ from Theorems 1.3 and 1.5. So it is sufficient to prove that $\log M_{\lambda, u} \geq \lambda \log \lambda + O(\lambda)$ and $\log P_{\lambda, u} \leq \lambda \log \lambda + O(\lambda)$. We shall use the following basic lemma on the incomplete gamma function to estimate some integrals.

LEMMA 4.1. For positive integer $n$ and $x \geq 0$, we have

$$\Gamma(n, x) = (n - 1)! e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}$$

PROOF. See [Jam16, Theorem 3].

We split the proof into two parts– the lower bound and the upper bound.
Lower bound on $M_{u,\lambda}$

Note that $\rho_\lambda(m) = (\log m)^\lambda$ whenever $m$ is prime. Using Lemma 2.1–(d), it follows that

$$M_{\lambda, u} = \sum_{(a_2, 1)} \rho_\lambda(n) \ell_u(n) \geq \frac{1}{2} \sum_{p > u^2} \frac{(\log p)^\lambda}{p^2}. $$

By partial summation and the prime number theorem,

$$\sum_{p > x} \frac{(\log p)^\lambda}{p^2} = - \frac{\pi(x)(\log x)^\lambda}{x^2} + \int_x^\infty \pi(t) \left( \frac{2(\log t)^\lambda}{t^3} - \frac{\lambda(\log t)^{\lambda-1}}{t^3} \right) dt$$

$$= - \frac{(\log x)^{\lambda-1}}{x} + O\left( \frac{(\log x)^{\lambda-2}}{x} \right) - \int_x^\infty \left( \frac{2(\log t)^{\lambda-1}}{t^2} \right) dt$$

$$\quad + \frac{2(\log x)^{\lambda-2}}{x} + O\left( \frac{2(\log t)^{\lambda-3}}{t^2} \right) dt$$

$$= - \frac{(\log x)^{\lambda-1}}{x} + O\left( \frac{(\log x)^{\lambda-2}}{x} \right) + 2\Gamma(\lambda, \log x)$$

$$\quad - (\lambda - 2)\Gamma(\lambda - 1, \log x) + O(\lambda \Gamma(\lambda - 2, \log x))$$

$$= 2\Gamma(\lambda, \log x) - (\lambda - 2)\Gamma(\lambda - 1, \log x) + O(\lambda (\lambda - 3)!).$$

The last step follows from Lemma 4.1. Note that $\Gamma(\lambda, \log x) \geq (1 - o_\lambda(1))(\lambda - 1)!$ and $\Gamma(\lambda - 1, \log x) \leq (\lambda - 2)!$. Therefore, $2\Gamma(\lambda, \log x) - (\lambda - 2)\Gamma(\lambda - 1, \log x) \geq (1 - o_\lambda(1))\lambda (\lambda - 2)!$ and we conclude that $M_{\lambda, u} \geq (0.5 - o_\lambda(1))\lambda (\lambda - 2)!$, completing the proof of the lower bound.

Upper bound on $P_{\lambda, u}$

We show bounds on tail sums of $P_{\lambda, u}$:

$$\sum_{n \geq y} \frac{\rho_\lambda(n)}{\varphi(\ell_u(n))}$$

Observe that $\rho_\lambda(n) = 0$ when $\omega(n) > \lambda$. Consider the sum

$$\sum_{\omega(n) \leq \lambda} \frac{\rho_\lambda(n)}{\varphi(\ell_u(n))}$$

and set $y = w^{12}$. By choosing $\epsilon = 1/4$ in Lemma 2.5 we have that

$$S(w) := \sum_{\omega(n) \leq \lambda, P(n) > y} \frac{1}{\ell_u(n)} \leq \frac{C_u}{w}.$$
for all $w \geq w_u$ where $C_u$ and $w_u$ are constants depending on $u$. Since $\varphi(n) \gg n/\log \log n$ (see, e.g., [Ten15, Chapter I.5, Theorem 4]) and $\ell_u(n) \leq 2n^2$ (see Lemma 2.1-(d)) for all positive integers $n$, it follows that

$$
\sum_{(n, a_2) = 1 \atop \omega(n) \leq \lambda} \frac{\rho_\lambda(n)}{\varphi(\ell_u(n))} \leq \sum_{(n, a_2) = 1 \atop \omega(n) \leq \lambda} \frac{(\log n)^{\lambda+1}}{\ell_u(n)} = S(w)(\log w)^{\lambda+1} + (\lambda + 1) \int_{w}^{\infty} \frac{(\log t)^{\lambda}}{t} dt
$$

$$
\leq C_u \left( \frac{(\log w)^{\lambda+1}}{w} + (\lambda + 1) \int_{w}^{\infty} \frac{(\log t)^{\lambda}}{t^2} dt \right)
$$

$$
\leq C_u \left( \frac{(\log w)^{\lambda+1}}{w} + (\lambda + 1) \Gamma(\lambda + 1, \log w) \right)
$$

for all $w \geq w_u$. Hence, by substituting $w = w_u$ and $y = y_u := w_u^{12}$, we obtain that

$$
\sum_{(n, a_2) = 1 \atop \omega(n) \leq \lambda} \frac{\rho_\lambda(n)}{\varphi(\ell_u(n))} \leq C_u \left( \frac{(\log w_u)^{\lambda+1}}{w_u} + (\lambda + 1) \Gamma(\lambda + 1, \log w_u) \right),
$$

which implies that for all sufficiently large $\lambda$, it follows that

$$
\sum_{(n, a_2) = 1 \atop \omega(n) \leq \lambda} \frac{\rho_\lambda(n)}{\varphi(\ell_u(n))} \leq 2C_u(\lambda + 1) \Gamma(\lambda + 1, \log w_u) \leq 2C_u(\lambda + 1)!.
$$

(3)

Now we focus on the sum

$$
\sum_{(n, a_2) = 1 \atop \omega(n) \leq \lambda} \frac{\rho_\lambda(n)}{\varphi(\ell_u(n))}
$$

Using Lemma 2.4, for all sufficiently large $\lambda$, we have that

$$
\sum_{(n, a_2) = 1 \atop \omega(n) \leq \lambda} \frac{\rho_\lambda(n)}{\varphi(\ell_u(n))} \leq \sum_{(n, a_2) = 1 \atop \omega(n) \leq \lambda} \frac{\rho_\lambda(n)}{n} \leq \frac{\Phi(w_u, y_u)(\log w_u)^{\lambda+1}}{w_u} + \int_{w_u}^{\infty} \frac{\Phi(t, y_u)((\log t)^{\lambda+1} - (\lambda + 1)(\log t)^{\lambda})}{t^2} dt
$$

$$
\leq \int_{w_u}^{\infty} \frac{(\log t)^{\lambda+1} + y_u}{t^2} dt
$$

$$
= \Gamma(\lambda + 1 + y_u, \log w_u)
$$

$$
\leq (\lambda + \lceil y_u \rceil)!
$$

(4)

Combining (3) and (4), we obtain the desired upper bound.
5 Proof of Theorem 1.11

In what follows, for relatively prime positive integers \(a\) and \(d\), \(p(a, d)\) denotes the least prime in the arithmetic progression \((a + dn)_{n=1}^{\infty}\). The remainder of this section depends heavily on different variants of Linnik’s theorem on least primes in arithmetic progressions. If \(n\) divides \(\ell_u(n)\), we have \(p \equiv 1 \mod \ell_u(n)\) (see Lemma 2.1-(a)). Henceforth, we need to look for the largest \(n\) such that \(p(1, \ell_u(n)) \leq x\). Since \(|\Delta_u| \neq 1\), it follows from Lemma 2.1-(c) that there exists a prime \(p | \Delta_u\) such that \(\ell_u(p^r) = p^r\) for any positive integer \(r\). Unconditionally, the best known bound on \(p(a, d)\) is of the form \(p(a, d) \ll d^{2.1115}\) when \(d\) varies over powers of a fixed prime number \(q\) (see [BS19, Theorem 3.6]). We solve for the maximal \(n\) such that

\[
p(1, \ell_u(q^n)) \ll (\ell_u(q^n))^{2.1115} = q^{2.1115n} \leq x,
\]

which gives us the lower bound

\[
\mathcal{G}(x) \gg x^{0.4736}.
\]

By an application of Hypothesis 2.9, one gets the folklore conjecture of Chowla [Cho34] that \(p(a, d) \ll d^{1+\varepsilon}\) for any \(\varepsilon > 0\). Thus, applying the same technique as before, conditionally, we solve for maximal \(n\) satisfying

\[
p(1, \ell_u(q^n)) \ll q^{n(1+\varepsilon)} \leq x
\]

giving us the lower bound

\[
\mathcal{G}(x) \gg x^{1-o(1)}.
\]

Remark 5.1. If we consider the case of Lucas sequence \((u_n)_{n \geq 0}\) with \(\Delta_u = 1\), then combining the best known upper bound on \(p(a, d)\) of the form \(p(a, d) \ll d^5\) (see [Xyl, Theorem 2.1]) with the fact \(\ell_u(n) \leq 2n^2\) (see Lemma 2.1-(d)) we get that \(\mathcal{G}(x) \gg x^{0.1}\) unconditionally. Conditional on Montgomery’s conjecture (Hypothesis 2.9), we can improve this bound to \(\mathcal{G}(x) \gg x^{0.5-o(1)}\).

References

[AGLP+12] J. J. Alba González, F. Luca, C. Pomerance, and I. E. Shparlinski. On numbers \(n\) dividing the \(n\)th term of a linear recurrence. Proc. Edinb. Math. Soc. (2), 55(2):271–289, 2012.

[AJ91] R. André-Jeannin. Divisibility of generalized Fibonacci and Lucas numbers by their subscripts. Fibonacci Quart., 29(4):364–366, 1991.

[BS19] W. D. Banks and I. E. Shparlinski. Bounds on short character sums and \(L\)-functions with characters to a powerful modulus. J. Anal. Math., 139(1):239–263, 2019.

[CGS17] A. S. Chen, T. A. Gassert, and K. E. Stange. Index divisibility in dynamical sequences and cyclic orbits modulo \(p\). New York J. Math., 23:1045–1063, 2017.

[Cho34] S. Chowla. On the least prime in an arithmetical progression. J. Indian Math. Soc., 1(2):1–3, 1934.

[Fre15] T. Freiberg. A note on the theorem of Maynard and Tao. In Advances in the theory of numbers. Volume 77, Fields Inst. Commun. Pages 87–103. Fields Inst. Res. Math. Sci., Toronto, ON, 2015.
[FG89] J. Friedlander and A. Granville. Limitations to the equi-distribution of primes. I. Ann. of Math. (2), 129(2):363–382, 1989.

[GU20] T. A. Gassert and M. T. Urbanski. Index divisibility in the orbit of 0 for integral polynomials. Integers, 20:Paper No. A16, 15, 2020.

[Got12] A. Gottschlich. On positive integers n dividing the nth term of an elliptic divisibility sequence. New York J. Math., 18:409–420, 2012.

[Har08] G. Harman. Watt’s mean value theorem and Carmichael numbers. Int. J. Number Theory, 4(2):241–248, 2008.

[Jam16] G. J. O. Jameson. The incomplete gamma functions. The Mathematical Gazette, 100(548):298–306, 2016.

[Jha21] A. Jha. On terms in a dynamical divisibility sequence having a fixed G.C.D. with their index, 2021. Preprint: https://arxiv.org/abs/2105.06190v4.

[JS22] A. Jha and C. Sanna. Greatest common divisors of shifted primes and Fibonacci numbers, 2022. Preprint: https://arxiv.org/abs/2204.05161v2.

[Kim20] S. Kim. The density of the terms in an elliptic divisibility sequence having a fixed G.C.D. with their indices. J. Number Theory, 207:22–41, 2020. With an appendix by M. Ram Murty.

[Lar21] D. Larsen. Bertrand’s postulate for carmichael numbers, 2021. Preprint: https://arxiv.org/abs/2111.06963v1.

[LS18] P. Leonetti and C. Sanna. On the greatest common divisor of n and the nth Fibonacci number. Rocky Mountain J. Math., 48(4):1191–1199, 2018.

[LT17] F. Luca and L. Tóth. The rth moment of the divisor function: an elementary approach. Journal of Integer Sequences, 20(2):3, 2017.

[LT15] F. Luca and E. Tron. The distribution of self-Fibonacci divisors. Fields Inst. Commun. 77:149–158, 2015.

[Mas19] D. Mastrostefano. An upper bound for the moments of a gcd related to Lucas sequences. Rocky Mountain J. Math., 49(3):887–902, 2019.

[May15] J. Maynard. Small gaps between primes. Ann. of Math. (2), 181(1):383–413, 2015.

[MV07] H. L. Montgomery and R. C. Vaughan. Multiplicative number theory. I. Classical theory, volume 97 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2007, pages xviii+552.

[Pol14] D. H. J. Polymath. Variants of the Selberg sieve, and bounded intervals containing many primes. Res. Math. Sci., 1:Art. 12, 83, 2014.

[Ren13] M. Renault. The period, rank, and order of the (a,b)-Fibonacci sequence modm. Math. Mag., 86(5):372–380, 2013.

[San19] C. Sanna. On numbers n relatively prime to the nth term of a linear recurrence. Bull. Malays. Math. Sci. Soc., 42(2):827–833, 2019.

[San17] C. Sanna. On numbers n dividing the nth term of a Lucas sequence. Int. J. Number Theory, 13(3):725–734, 2017.

[San18] C. Sanna. The moments of the logarithm of a G.C.D. related to Lucas sequences. J. Number Theory, 191:305–315, 2018.

[ST18] C. Sanna and E. Tron. The density of numbers n having a prescribed G.C.D. with the nth Fibonacci number. Indag. Math. (N.S.), 29(3):972–980, 2018.
[SS11] J. H. Silverman and K. E. Stange. Terms in elliptic divisibility sequences divisible by their indices. *Acta Arith.*, 146(4):355–378, 2011.

[Som96] L. Somer. Divisibility of terms in Lucas sequences of the second kind by their subscripts:473–486, 1996.

[SK13] L. Somer and M. Křížek. Fixed points and upper bounds for the rank of appearance in Lucas sequences. *Fibonacci Quart.*, 51(4):291–306, 2013.

[Ten15] G. Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 163 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, third edition, 2015.

[Tro20] E. Tron. The greatest common divisor of linear recurrences. *Rend. Semin. Mat. Univ. Politec. Torino*, 78(1):103–124, 2020.

[Xyl] T. Xylouris. Über die Nullstellen der Dirichletschen L-Funktionen und die kleinste Primzahl in einer arithmetischen Progression. Bonner Mathematische Schriften [Bonn Mathematical Publications], 404.

Abhishek Jha,
Indraprastha Institute of Information Technology, New Delhi, India
Email: abhishek20553@iiitd.ac.in

Ayan Nath,
Chennai Mathematical Institute, Siruseri, Tamil Nadu, India
Email: ayannath@cmi.ac.in