A CONJECTURED LOWER BOUND FOR THE COHOMOLOGICAL DIMENSION OF ELLIPTIC SPACES
RECENT RESULTS IN SOME SIMPLE CASES.
March 26, 2008

MOHAMED RACHID HILALI AND MY ISMAIL MAMOUNI

Abstract. Here we prove some special cases of the following conjecture: that the sum of the Betti numbers of a 1-connected elliptic space is greater than the total rank of its homotopy groups. Our main tool is Sullivan’s minimal model.

1. Introduction

The subject of our research is the following result:

Conjecture H (Topological version). If X is a 1-connected elliptic space then \( \dim H^*(X, \mathbb{Q}) \geq \dim (\pi_*(X) \otimes \mathbb{Q}) \).

For elliptic spaces X, it seems roughly that \( \dim H^*(X, \mathbb{Q}) \) becomes larger when \( \dim \pi_*(X) \otimes \mathbb{Q} \) does. That was the key idea for the first author to conjecture a lower bound for the cohomological dimension of an elliptic space in terms of the total rank of its homotopy groups. Let us recall that for any 1-connected space X of finite type, i.e., \( \dim H^*(X, \mathbb{Q}) < \infty \) for all \( k \geq 0 \), there exists a commutative differential graded algebra \((\Lambda V, d)\), called Sullivan model of X, which algebraically models the rational homotopy type of the space, more precisely \( H^*(\Lambda V, d) \cong H^*(X, \mathbb{Q}) \) as algebras, and \( V \cong \pi_*(X) \otimes \mathbb{Q} \) as vector spaces. By \( \Lambda V \) we mean the free commutative algebra generated by the graded vector space V, i.e., \( \Lambda V = TV / \langle v \otimes w - (-1)^{|v||w|} w \otimes v \rangle \), where \( TV \) denotes the tensor algebra over V. \( \Lambda^n V \) denotes the set of elements of \( \Lambda V \) of wordlength n and \( \Lambda^{\geq n} V := \bigcup_{k \geq n} \Lambda^k V \) denotes that of elements of \( \Lambda V \) of wordlength at least n. The differential d of any element of V is a “polynomial” in \( \Lambda V \) with no linear term, i.e., \( dV \subset \Lambda^{\geq 2} V \), we say that the model is minimal. A space X and its minimal Sullivan model are called elliptic if both \( \Lambda V \) and \( H^*(\Lambda V, d) \) are finite dimensional spaces. Because of this contravariant correspondence between spaces and their minimal models, the topological version of our conjecture admits the following algebraic interpretation:

Conjecture H (Algebraic version). If \( \Lambda V \) is a 1-connected elliptic Sullivan minimal model then \( \dim H^*(\Lambda V, d) \geq \dim V \).

We assume that the minimal model is simply connected, i.e., that the vector space V has no generators in degrees lower than 2. This assumption

2000 Mathematics Subject Classification. 55N34; 55P62; 57T99.
Key words and phrases. Rational Homotopy, Cohomology, Sullivan Minimal Model, Elliptic spaces, Nilmanifold, Two-stage model, homogeneous-length differential.
is necessary in order to translate our algebraic results into topological ones, although it is not necessary for the algebraic results themselves. For more details about minimal Sullivan models of spaces we refer the reader to [FHT01], (138-160).

Our main motivation to believe that the conjecture H holds in the elliptic case, is that it is for pure spaces [Hi90] and for $H$-spaces, symplectic manifolds,... [HM07]. Homogenous spaces are pure. Topological groups are $H$-spaces, and Kahler manifolds are symplectic.

2. Results and proofs

**Proposition 1.** If $X$ is a nilmanifold with $(\Lambda V, d)$ as a model, then $\dim H^*(\Lambda V, d) \geq \dim V$.

*Proof.* We know from [Dix55] that a nilmanifold $X$ of dimension $n$ has $b_i \geq 2$ for $1 \leq i \leq n - 1$ and hence $\dim H^*(\Lambda V, d) \geq 2f_d(X)$, where $f_d(X)$ called the formal dimension of $X$, denotes the largest integer $k$ such that $H^k(X, \mathbb{Q}) \neq 0$. We know also from ([FH79]-Proposition 2.6) that space of “type F” checks the inequality $f_d(X) \geq \dim V$. Nilmanifolds are of “type F”, since they are $K(G, 1)$ where $G$ is a nilpotent group. □

**Proposition 2.** If an elliptic minimal model $(\Lambda V, d)$ has a homogeneous-length differential and whose rational Hurewicz homorphism is non-zero in some odd degree. Then $\dim H^*(\Lambda V, d) \geq \dim V$.

*Proof.* We say that $\Lambda V$ has differential of homogeneous-length $l$ if $d : V \longrightarrow \Lambda^l V$. We know from [Lu02] that under hypotheses above we have $\dim H^*(\Lambda V, d) \geq 2\text{cat}_0(\Lambda V)$ and from [FHT82] that $\dim V^{\text{even}} \leq \dim V^{\text{odd}} \leq \text{cat}_0(\Lambda V)$. We can then conclude that $\dim V \leq 2\text{cat}_0(\Lambda V) \leq \dim H^*(\Lambda V, d)$. □

**Proposition 3.** If an elliptic minimal model $(\Lambda V, d)$ has a differential, homogenous of length at least 3, then $\dim H^*(\Lambda V, d) \geq \dim V$.

*Proof.* The cohomology of such spaces admits a second grading $H^*(\Lambda V, d) = \bigoplus_{k \geq 1} H^*_k(\Lambda V, d)$, given by length of representative cocycle. We know from [Lu02]-Theorem 2.2 that $H^*_k(\Lambda V, d) \neq 0$ for each $k = 0, \cdots, e$ where $e = \dim V^{\text{odd}} + (l - 2) \dim V^{\text{even}}$. Then $\dim H^*(\Lambda V, d) \geq e \geq \dim V$, when $l \geq 3$.

**Proposition 4.** If an elliptic minimal model $(\Lambda V, d)$ has a differential, homogenous of length 2 (i.e : coformal) with odd degree generators only, i.e., $V^{\text{even}} = 0$, then $\dim H^*(\Lambda V, d) \geq \dim V$.

*Proof.* The proof is similar to that of that Proposition 3.

**Proposition 5.** If $X$ is an elliptic space wich has the homotopy type of the $r$-product of elliptic spaces satisfying the conjecture H, then it is also for $X$.

*Proof.* The argument is that : $\dim H^*(Y \times Z, d) = \dim H^*(Y, d) \cdot \dim H^*(Z, d)$ and that $\dim (\pi(Y \times Z) \otimes \mathbb{Q}) = \dim (\pi(Y) \otimes \mathbb{Q}) + \dim (\pi(Z) \otimes \mathbb{Q})$. □
Proposition 6. If \((\Lambda V, d)\) is a formal and hyperelliptic model, then \(\dim H^*(\Lambda V, d) \geq \dim V\).

Proof. The Sullivan model \((\Lambda V, d)\) is called hyperelliptic if \(d^{\text{even}} = 0\) and \(d^{\text{odd}} \subset \Lambda^* V^{\text{even}} \otimes \Lambda^* V^{\text{odd}}\). Set \(V^{\text{even}} = \Lambda\{x_1, \ldots, x_n\}\) and \(V^{\text{odd}} = \Lambda\{y_1, \ldots, y_{n+p}\}\) with \(p = \dim V^{\text{odd}} - \dim V^{\text{even}} \geq 0\) (cf. [FHT01]-Proposition 32.10 (444)). The first author (cf. [Hi90]) has showed that the conjecture \(H\) holds for pure spaces, then we assume that \((\Lambda V, d)\) is not pure. As it is formal, then \(\dim H^*(\Lambda V, d) \geq 2p\) when \(H^*(\Lambda V, d)\) is given by \(Q[x_1, \ldots, x_n] \otimes \Lambda\{y_1, \ldots, y_n\}\), \(dx_i = dy_i = 0\) for \(1 \leq i \leq n\). On the other hand it is proved (cf. [HM07]-Theorem C) for hyperelliptic models that \(\dim H^*(\Lambda V, d) \geq \dim V\) when \(\dim H^*(\Lambda V, d) \geq 2p - 1\), this simple remark completes the proof. □

Proposition 7. If \((\Lambda V = \Lambda(U, W), d)\) is a two-stage, elliptic minimal model with odd degree generators only and suppose that \(d : W \rightarrow \Lambda^2 U\) is an isomorphism, then \(\dim H^*(\Lambda V, d) \geq \dim V\).

Proof. A minimal model \((\Lambda V, d)\) is said to be two-stage if \(V\) decomposes \(V = U \oplus W\) with \(dU = 0\) and \(dW \subset \Lambda U\), then the Mapping theorem ([FHT01], page 375) implies that \(W\) has generators of odd degree only. On the other hand, \(U\) may have generators of odd or even degree. We know from ([JL04]-Proposition 2.1), that under hypotheses here above we have \(\dim H^*(\Lambda(U, W), d) \geq 2^{\dim W}\). Set \(\dim U = n\), then \(\dim W = \frac{n(n+1)}{2} \geq n = \dim U\) and \(\dim W \geq \frac{\dim V}{2} = m\). Finally \(\dim H^*(\Lambda V, d) \geq 2^m \geq 2m = \dim V\). □

Open Questions:

- The author in [Lu02] believes that his Conjecture 3.4 holds at least in the coformal case. Has this been settled?
- We know from ([JL04]-Corollary 3.5) that if \(X\) has a two-stage model with odd degree generators only, then \(\dim H^*(X, \mathbb{Q}) \geq 2^{\dim G_*(X)}\), where \(G_*(X)\) denoted the subgroup of \(\pi_*(X)\) called the Gottlieb group. A question is: are there any cases where the equality \(\dim G_*(X) = \dim (\pi_*(X) \otimes \mathbb{Q})\) holds?
- If \(F \rightarrow E \rightarrow B\) is a fibration where \(F\) and \(B\) are elliptic and both verify the conjecture \(H\), what conditions on the fibration will guarantee that \(E\) will too?

Acknowledgements. It is for us a pleasure to thank Micheline Vigué (Univ. Paris 13, French) and Barry Jessup (Univ. Ottawa, Canada) for their interest and for their several readings and corrections.
References

[Dix55] J. Dixmier, *Cohomologie des algèbres de Lie nilpotentes*, Acta Sci. Math, Szeged 16 (1955), 246-250.

[FH79] J. Friedlander & S. Halperin, *An arithmetic characterization of the rational homotopy groups of certain spaces*, Invent. Math, 53 (1979), 117-133.

[FH82] J. Friedlander & S. Halperin, *Rational LS category and its applications*, Trans. Amer. Math. Soc., 273 (1982), no. 1, 1–38.

[FHT01] Y. Félix, S. Halperin & J-C Thomas, *Rational homotopy theory*, Graduate Texts in Math, Vol. 205, Springer-Verlag, New York, 2001.

[FH82] J. Friedlander & S. Halperin, *Rational LS category and its applications*, Trans. Amer. Math. Soc., 273 (1982), no. 1, 1–38.

[Hi90] M.R. Hilali, *Action du tore $T^n$ sur les espaces simplement connexes*, Thesis, Université catholique de Louvain, Belgique, 1990.

[HM07] M.R. Hilali & M.I. Mamouni, *The conjecture H : A lower bound of cohomologic dimension for an elliptic space*, Preprint submitted to Topology and its Applications, (2007).

[JL04] B. Jessup & G. Lupton, *Free torus actions and two-stage spaces*, Mathematical Proceedings of the Cambridge Philosophical Society, Cambridge University Press, Vol. 137 (2004), 191-207, arXiv : math/0309434.

[Lu02] G. Lupton, *The Rational Toomer Invariant and Certain Elliptic Spaces*, Contemporary Mathematics, Vol. 316 (2002), 135–146, arXiv : math/0309392v1.

Département de Mathématiques, Faculté des sciences Ain Chok, Université Hassan II, Route d’El Jadida, Casablanca, Maroc

E-mail address: rhilali@hotmail.com

Classes préparatoires aux grandes écoles d’ingénieurs, Lycée Med V, Avenue 2 Mars, Casablanca, Maroc

E-mail address: myismail1@menara.ma