Robust Learning of Trimmed Estimators via Manifold Sampling

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Abstract

We adapt a manifold sampling algorithm for the nonsmooth, nonconvex formulations of learning that arise when imposing robustness to outliers present in the training data. We demonstrate the approach on objectives based on trimmed loss. Empirical results show that the method has favorable scaling properties. Although savings in time come at the expense of not certifying optimality, the algorithm consistently returns high-quality solutions on the trimmed linear regression and multiclass classification problems tested.

1. Introduction

A frequent challenge in supervised learning is to perform training that is robust to outliers in a labeled training dataset $\mathcal{D} = \{(x^{i}, y^{i})\}_{i=1}^{N} \subset \mathbb{R}^{d} \times \mathbb{R}$. Here, we address this challenge by using trimmed estimators. Formally, for $l_{i}(F(x^{i}, w), y^{i})$, a scalar-valued loss function of the label $y^{i}$ and model prediction $F(x^{i}, w)$, we seek weights $w \in \mathbb{R}^{n}$ that solve

$$\min_{w} \frac{1}{q} \sum_{j=1}^{q} l_{(j)} \left( F(x^{(j)}, w), y^{(j)} \right),$$

(1)

where $(q)$ denotes the index associated with the $q$th-order statistic (i.e., $l_{(j-1)} (F(x^{(j-1)}, w), y^{(j-1)}) \leq l_{(j)} (F(x^{(j)}, w), y^{(j)})$ for $j = 2, \ldots, N$. Robustness properties of trimmed estimators have been explored in, for example, (Rousseeuw, 1984; Rousseeuw & Leroy, 1987). Such estimators have a high breakdown point; as a particular example in the context of least squares regression, when $q = \lfloor \frac{N}{2} \rfloor$, the estimator obtained by solving (1) can resist the adverse effects of up to 50% outliers present in the data. Despite such promising properties, however, a barrier to solving (1) has been its inherent nonconvexity and nonsmoothness (even if each $l_{i} (\cdot)$ is smooth and convex), rendering global solutions intractable for large values of $N$ or $d$. A majority of approaches to solving (1) in the context of least-squares regression (e.g., (Rousseeuw & Driessen, 2006)) have focused on heuristics that sample subsets of size $q$ from the dataset and then follow a sample average approximation-like approach of solving (1) restricted to this subset. Such an approach necessitates heuristics because complete enumeration would involve the prohibitive evaluation of $\binom{N}{q}$ many subsets in order to be assured of a set of smooth (possibly convex) problems. In the context of linear regression (i.e., $F(x, w) = w^{\top} x$), recent work (Bertsimas & Mazumder, 2014) has proposed solving the least quantile regression problem (of the form (1)) but with the sum starting at $j = q$ to global optimality through integer reformulations. Although that approach necessarily returns certificates of global optimality, integer optimization is not scalable, and least quantile regression is strictly more statistically inefficient than solving (1); see (Rousseeuw, 1984). Recently, a stochastic proximal gradient method targeting general problems of the form (1) by solving a relaxation of the problem in $n + N$ dimensions has been proposed (Aravkin & Davis, 2017).

In this paper we use the observation that the objective in (1) can be expressed as the composition of a piecewise linear function with the mapping $l(w) = [\ldots, l_{i}(F(x^{i}, w), y^{i}), \ldots]^{\top}$ in order to apply a manifold sampling algorithm. Let us make this piecewise linear structure explicit. Given a training dataset $\mathcal{D}$ and weights $w$, the values of $\{l_{i}(F(x^{i}, w), y^{i})\}_{i=1}^{N}$ are fixed, and the objective in (1) is defined by the set of $q$-tuples of active indices

$$I^{q,N}(l(w)) = \left\{(i_{1}, \ldots, i_{q}) : l_{i_{j}}(F(x^{i_{j}}, w), y^{i_{j}}), \ l_{(q)}(F(x^{(q)}, w), y^{(q)}) \leq l_{(q)}(F(x^{(q)}, w), y^{(q)}) \right\}.$$ 

Then, the objective in (1) is expressible as $(h_{q,N} \circ l) : \mathbb{R}^{n} \to \mathbb{R}$, where $h_{q,N}$ is the continuous selection (Scholtes, 2012)

$$h_{q,N}(l(w)) = \left\{g^{a}(l(w)) : a \in I^{q,N}(l(w)) \right\}$$

and the selection function $g^{a}$ for $a \in I^{q,N}(l(w))$ is defined componentwise by

$$g_{i}^{a}(l(w)) = \begin{cases} 1/q & i \in a \vspace{1ex} \\ 0 & \text{otherwise.} \end{cases}$$

(2)
Even though $I^{q,N}(l(w))$ need not be singleton, $h^{q,N}(l(w))$ is singleton for all $w$.

We observe that (1) is generally a nonconvex, nonsmooth optimization problem. In particular, even if the loss function $l$ is smooth, the objective is potentially nondifferentiable at all values of $w$ where $I^{q,N}(l(w))$ is nonsingleton. Similarly, even if the loss function $l$ is convex, $h^{q,N}$ is generally a nonconvex piecewise function, and so (1) falls outside the scope of convex composite optimization; see (Duchi & Ruan, 2017) for details.

2. Manifold Sampling

Manifold sampling is an iterative method for minimizing compositions of a continuous selection $h^{q,N} : \mathbb{R}^N \rightarrow \mathbb{R}$ and a smooth function $l : \mathbb{R}^n \rightarrow \mathbb{R}^N$. The case where $h^{q,N}$ is piecewise linear, considered here, is analyzed in (Khan et al., 2017). The manifolds in the case of (1) can be thought of as the regions of $\mathbb{R}^d$ within which the objective is piecewise smooth. These regions can be catalogued by the set of active indices of $h^{q,N}(l(\cdot))$ (i.e., the set of indices $I^{q,N}(l(\cdot))$). At iteration $k$ of a manifold sampling method, there are a current iterate $w^k$ and a trust-region radius $\Delta_k$. The iteration involves a direction-finding routine and a manifold sampling loop which we now overview; see (Khan et al., 2017; Menickelly & Wild, 2017) for details. By the end of the loop, a generator set $k^k$ is determined, at least containing $I^{q,N}(l(w^k))$, and at most containing $\{I^{q,N}(l(y)) : y \in B(w^k, \Delta_k)\}$. Following the strategy in (Menickelly & Wild, 2017), the step $d^k$ is taken from the approximate solution to

$$\min_{\tau \in \mathbb{R}, d \in \mathbb{R}^n} \tau : ||d||_2 \leq \Delta_k, \quad \nabla^2 h^{q,N}(l(w^k)) + \nabla(g^a \circ l)(w^k)^T d \leq \tau, \quad \forall a \in k^k,$$

where $g^a$ is the selection function (see (2)) corresponding to $a \in k^k$. The iteration then proceeds similarly to a standard trust-region method.

We propose a stochastic variant by making two adjustments to the manifold sampling algorithm. First, given a sample $S \subseteq \{1, \ldots, N\}$, we consider a sampled smooth function $l_S(w) : \mathbb{R}^n \rightarrow \mathbb{R}^{|S|}$; intuitively, $l_S(w)$ is the projection of $l(w)$ onto the coordinates corresponding to indices in $S$. Then, in the $k$th iteration, the output of the manifold sampling loop can be regarded as a function of a sample $S^k$, a continuous selection $h$ (in particular, $h(l(q/N)(S^k)^{-1})$), the current iterate $w^k$, and the trust-region radius $\Delta_k$ (i.e., we write $(\tau_k, d^k) = M(S^k, h, w^k, \Delta_k)$). Second, we modify the step acceptance test so that given a second sample $S'$, continuous selection $h$ (here, $h[l(q/N)(S')^{-1}]$), and current iterate $w^k$,

$$\rho_k(S^k', h, w^k, \tau_k, d^k) = \frac{h(l_{g^a\circ l}(w^k)) - h(l_{g^a\circ l}(w^k + d^k))}{\tau_k}.$$

A statement of the algorithm is given in Algorithm 1.

Although some manifold sampling variants do not require gradients $\nabla l$, these gradients are readily available in the problems tested here, and hence we make direct use of them (but not second-order information $\nabla^2 l$). Consequently, we remark that because we are not using model/approximate Hessians, there is no need for the “acceptability test” (relative to a stationary measure) discussed in (Khan et al., 2017; Menickelly & Wild, 2017). We also remark that while the convergence analysis in (Khan et al., 2017; Menickelly & Wild, 2017) applies to the deterministic case wherein the sample $S^k = \{1, \ldots, N\}$ on every iteration $k$, a proper analysis of the stochastic variant where $|S^k| < N$ is beyond the scope of this paper. In future work, we will seek to establish convergence of the stochastic variant of Algorithm 1 to Clarke stationary points with probability 1 (given an appropriate sampling rate).

3. Numerical Results

We now demonstrate the proposed approach on two standard problems. We implemented Algorithm 1 in MATLAB. Throughout, we initialize $w^0 = 0$ and $\Delta_0 = 10$ and choose $\gamma_{\text{inc}} = 1.01, \gamma_{\text{dec}} = 0.99, \eta = 10^{-3}$.

Regression Problems. For the first set of experiments, we follow (Rousseeuw & Driessen, 2006; Bertsimas & Mazumder, 2014) and consider a trimmed form of linear regression, where the loss function is absolute loss:

$$\min_w \frac{1}{q} \sum_{i=1}^q |w^T x^{(i)} - y^{(i)}|.$$

Although we focused on smooth functions $l(w)$ for ease of presentation, it is straightforward to consider the manifolds
We terminate the deterministic variant (“DMS”) of our method (i.e., $S^k = S^{k'} = [1, \ldots, N]$ in each iteration) based on the stopping criterion $\Delta_k < .01$ and we terminate the stochastic variant (“SMS”) of our method (where we draw random samples without replacement of size $|S^k|, |S^{k'}| = \max\{0.01p, (10^{-6}p)/\Delta_k^{\frac{1}{2}}\}$) after 100 epochs (that is, we budget $100p$ draws from the data). Both tested variants consistently achieve a TPR of 100% and a FPR of 0%. Consequently, in these experiments our method effectively returned the global minimum of (3): having identified the outliers correctly, the solution of a single linear program in postprocessing would return a global solution $w^*$.

Table 1 shows, for a variety of $d$ and $p$, the TPR, FPR, and the time (in seconds) to return a solution for $\hat{\text{lqs}}$, DMS, and SMS. We note that we examine larger datasets than the largest considered in (Rousseeuw & Driessen, 2006) and (Bertsimas & Mazumder, 2014), which are $(d = 5, p = 50000)$ and $(d = 20, p = 10000)$, respectively. We also note that our times for identifying what are tantamount to global minima are significantly less than what is reported for the integer formulations for least quantile regression considered in (Bertsimas & Mazumder, 2014); the disadvantage of our approach is that our method cannot provide a certificate of global optimality. We recall that (Bertsimas & Mazumder, 2014) focused on the least quantile formulation, as opposed to the trimmed estimator formulation; from a combinatorial perspective, trying to solve (3) to global optimality likely would further increase solution time.

**Multiclass Classification on MNIST.** Following (Aravkin & Davis, 2017), we apply the stochastic variant of Algorithm 1 to the well-known task of classifying digits 0 through 9 in the MNIST dataset ($N = 60000$) (LeCun et al., 1998). Each data point $(x^i, y^i) \in \mathbb{R}^{785} \times \mathbb{R}^{10}$, where $y^i$ is a “one-hot” vector indicating the class membership. The learning is performed with the multiclass softmax function $\ell_i : \mathbb{R}^{10 \times 785} \rightarrow \mathbb{R}$ defined by

$$\ell_i(W) = - \log \left( \frac{(y^i)^\top \exp(W x^i)}{e^\top \exp(W x^i)} \right),$$

\[
\begin{array}{c|cccc|cccc|cccc|cccc|cccc}
 d \times p & 5 \times 5 & 5 \times 5 & 5 \times 5 & 5 \times 5 & 10 \times 10 & 10 \times 10 & 10 \times 10 & 10 \times 10 & 20 \times 20 & 20 \times 20 & 20 \times 20 & 20 \times 20 & 100 \times 100 & 100 \times 100 & 100 \times 100 & 100 \times 100 \\
 2 \times 5 & 10 \times 50 & 10 \times 50 & 10 \times 50 & 10 \times 50 & 2 \times 5 & 10 \times 50 & 10 \times 50 & 10 \times 50 & 2 \times 5 & 10 \times 50 & 10 \times 50 & 10 \times 50 & 2 \times 5 & 10 \times 50 & 10 \times 50 & 10 \times 50 \\
\end{array}
\]

**LQS**

| $d \times p$ | 5 \times 5 | 5 \times 5 | 5 \times 5 | 5 \times 5 | 10 \times 10 | 10 \times 10 | 10 \times 10 | 10 \times 10 | 20 \times 20 | 20 \times 20 | 20 \times 20 | 20 \times 20 | 100 \times 100 | 100 \times 100 | 100 \times 100 | 100 \times 100 |
|-------------|-----------|-----------|-----------|-----------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| TPR         | 100       | 100       | 100       | 100       | 100         | 100         | 100         | 100         | 65.6        | 67.1        | 66.8        | 66.0        | 43.8        | 43.5        | 43.9        | 43.9        |
| FPR         | 0         | 0         | 0         | 0         | 0           | 0           | 0           | 0           | 23.0        | 22.0        | 22.1        | 22.7        | 37.6        | 37.7        | 37.9        | 37.9        |
| time        | 0.1       | 0.3       | 0.6       | 2.6       | 0.2         | 0.4         | 0.8         | 4.0         | 0.2         | 0.6         | 1.0         | 4.9         | 2.4         | 4.4         | 9.0         | 42.0        |

**DMS**

| $d \times p$ | 5 \times 5 | 5 \times 5 | 5 \times 5 | 5 \times 5 | 10 \times 10 | 10 \times 10 | 10 \times 10 | 10 \times 10 | 20 \times 20 | 20 \times 20 | 20 \times 20 | 20 \times 20 | 100 \times 100 | 100 \times 100 | 100 \times 100 | 100 \times 100 |
|-------------|-----------|-----------|-----------|-----------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| TPR         | 100       | 100       | 100       | 100       | 100         | 100         | 100         | 100         | 65.6        | 67.1        | 66.8        | 66.0        | 43.8        | 43.5        | 43.9        | 43.9        |
| FPR         | 0         | 0         | 0         | 0         | 0           | 0           | 0           | 0           | 23.0        | 22.0        | 22.1        | 22.7        | 37.6        | 37.7        | 37.9        | 37.9        |
| time        | 8.5       | 9.9       | 9.9       | 31.2      | 10.6        | 10.1        | 12.6        | 38.2        | 15.1        | 13.7        | 18.4        | 45.7        | 37.9        | 47.4        | 91.9        | 436        |

**SMS**

| $d \times p$ | 5 \times 5 | 5 \times 5 | 5 \times 5 | 5 \times 5 | 10 \times 10 | 10 \times 10 | 10 \times 10 | 10 \times 10 | 20 \times 20 | 20 \times 20 | 20 \times 20 | 20 \times 20 | 100 \times 100 | 100 \times 100 | 100 \times 100 | 100 \times 100 |
|-------------|-----------|-----------|-----------|-----------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| TPR         | 100       | 100       | 100       | 100       | 100         | 100         | 100         | 100         | 100         | 100         | 100         | 100         | 100         | 100         | 100         | 100         |
| FPR         | 0         | 0         | 0         | 0         | 0           | 0           | 0           | 0           | 0           | 0           | 0           | 0           | 0           | 0           | 0           | 0           |
| time        | 0.9       | 1.0       | 1.1       | 2.2       | 1.2         | 1.4         | 1.6         | 2.9         | 1.7         | 2.0         | 2.3         | 4.6         | 2.5         | 4.6         | 5.3         | 15.0        |
Figure 1. 30 instances of stochastic implementation (SMS) of Algorithm 1 and standard SGD approach on the MNIST classifier problem. Top row: $q = \lfloor (1 - p - 0.1)N \rfloor$, bottom row: $q = \lfloor (1 - p - 0.05)N \rfloor$. Left: testing accuracy, center: TPR on training dataset, right: FPR on training dataset. Because of the experimental setup, the lower bound on FPR of any classifier is 10% or 5% for the first and second value of $q$, respectively. At the $p = 0$ level of contamination, TPR is undefined (there are no true outliers).

where $\exp$ is taken entrywise and $\epsilon$ is the vector of ones. Class predictions for any input $x \in \mathbb{R}^{785}$ are then made by evaluating the product $Wx \in \mathbb{R}^{10}$ and predicting the class as that corresponding to the index of the maximum entry. We consider the trimmed estimator resulting from (1), with $w$ denoting the vector form of $W$. To artificially produce outliers, we contaminate the labels of the data by choosing a fraction $p \in [0, 1]$ and changing the label $y^{'i}$ on a randomly selected subset of size $\lfloor pN \rfloor$ of the training data to $(y^{'i} + 1) \mod 10$.

We compare the stochastic variant of Algorithm 1 (“SMS”) with a stochastic gradient descent (SGD) method, which has well-tuned step-size and batch-size parameters but solves the untrimmed formulation (i.e., (1) with $q = N = 60000$). For various levels of $p$, when applying SMS, we consider an overestimation of the number of outliers in the data (i.e., we select either $q = \lfloor (1 - p - 0.1)N \rfloor$ or $q = \lfloor (1 - p - 0.05)N \rfloor$). This simulates a realistic application of an outlier detection method, since the proportion of outliers $p$ generally cannot be known exactly a priori but can, one hopes, be upper bounded. For SMS, we used a sample on each iteration of size $|S^k| = |S^{k^*}| = \max\{40, (10^{-6}/\Delta_k^*)\}$. We stop both methods after 40 epochs (i.e., $40N$ draws are made from the training data).

Figure 1 displays the testing accuracy - the accuracy of predictions made by learned classifiers $W^*$ on the uncontaminated testing MNIST set of size $N = 10000$ and the TPR and FPR of $W^*$ applied to the contaminated training dataset. Recall that “positive” here refers to a data point being an outlier; for the output $W^*$ returned by either method, we label all data points corresponding to $l_{(q+1)}(W^*) \leq \cdots \leq l_N(W^*)$ as outliers in postprocessing. Since both methods are stochastic, we performed 30 trials of the experiment with different random seeds. Because these problems are nonconvex and likely have many local minima, we show the 25th, 50th, and 75th percentiles of the results in boxes and the outliers in whiskers. The results illustrate how, even with the agnostic initialization $w^0 = 0$, SMS has a tendency to identify a robust solution in terms of the natural performance metrics. Comparisons with a well-tuned stochastic (sub)gradient method may be found in the Supplementary Material.

The median results of SMS are comparable to the results of the SMART algorithm reported in (Aravkin & Davis, 2017), but we remark that SMS works in $\mathbb{R}^{7850}$, while the SMART approach works in $\mathbb{R}^{7850+60000}$.
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A. Supplementary Material

It is of interest to compare our method to a simple stochastic subgradient descent (SSGD) method. Given \( w^k \in \mathbb{R}^n \), we can compute a particular subgradient of (1) by choosing a fixed element of \( I_{[q]}(I(w^k)) \) by using a lexicographic ordering; we label this element (which is a \( q \)-tuple) \( I^k \). We remark that in practice, and because of the almost-everywhere differentiability of \( h_{[q]}(S^k_t) \circ I(\cdot) \), it is often the case that \( I_{[q]}(S^k_t) \circ l(\cdot) \) consists of a single \( q \)-tuple; in this case \( I^k \) is well defined with no need for a lexicographic ordering.

The \( q \)-tuple \( I^k \) defines a particular subgradient of \( h_{[q]}(S^k_t) \circ l \) at \( w^k \), namely,

\[
g_{S^k}(w^k) = \frac{1}{|q|} \sum_{i_j \in I^k} \nabla l_{i_j}(F(x^{i_j}, w^k), y^{i_j}). \tag{4}\]

With a (sub)gradient defined, we can employ the SSGD method described in Algorithm 2.

A deterministic (i.e., \( |S^k| = N \) on every iteration \( k \)) variant of such a method was used in (Bertsimas & Mazumder, 2014) as a means of warmstarting their integer programming model. Under modest assumptions, deterministic methods of the form in Algorithm 2 can be shown to converge to minimizers of convex (almost-everywhere differentiable) functions; see, for instance, (Shor, 1985). However, most convergence results for methods of nonconvex nonsmooth minimization require some form of “stabilized descent” direction, like those generated in bundle or gradient-sampling methods (see Makelä, 2002 and (Burke et al., 2018) for respective literature reviews). Manifold sampling methods more closely resemble these latter two classes of methods.

In light of this lack of convergence guarantees for Algorithm 2, we performed a tuning of the three parameters, \((\alpha, \delta, s)\), of Algorithm 2. An illustration of results from our tuning experiments is shown in Figure 2. Here, we repeat the previously described MNIST experiment with \( p = 0.3 \) and \( q = \frac{(1-p-0.05)N}{N} \), but we apply the SSGD method to the randomly contaminated problems instead, varying the parameters \( \alpha \in \{0.01, 0.001, 0.01, 10^{-2}, 0.0^{-3}\} \) and \( s \in \{N, N/10, N^2/3, N/100, N^1/3, N^1/4\} = \{60000, 6000, 1533, 600, 40, 16\} \). In our experience, setting \( \delta > 0 \) generally deteriorates performance for any fixed level of \( \alpha, s \). Hence, in all the results shown, we set \( \delta = 0; \) that is, we effectively have a fixed step size.

We observe from Figure 2 that parameter tuning has a significant impact on the empirical performance of the SSGD method. Arguably, only three of the \((\alpha, s)\) pairs (corresponding to \((0.1, 40), (0.1, 16), (0.01, 16)\)) produce a reasonable range of testing accuracy values across the randomly contaminated instances. Using the best median value as a selection criteria suggests that \((\alpha = 0.1, s = 40)\) is good for these instances.

In contrast, the empirical performance of SMS is less sensitive to a selection of its parameter values. SMS has more parameters to tune, but we observed that the most variance in the performance metrics in these MNIST experiments come from the selection of the “batch size” \( |S^k| = |S^k| \), and the choice of the trust-region expansion and contraction parameters, \( \gamma_{\text{inc}} \) and \( \gamma_{\text{dec}} \). In Figure 3 we perform a similar tuning experiment as done for SSGD, except we tune two parameters \( \beta \) and \( |S^k| = |S^k| \). For a single value of \( \beta \), we define \( \gamma_{\text{inc}} = \beta \) and \( \gamma_{\text{dec}} = 1/\beta \). We show results for all possible combinations of \( \beta \in \{2.0, 1.5, 1.33, 1.1, 101\} \), constant batch sizes in \{60000, 1533, 600, 40, 16\}, and \( \Delta_0 \in \{10, 1\} \). We remark that we did not test a batch size of 60000 only because it is typical for SMS to run for only one or two iterations of Algorithm 1 in this setting, since we only set a budget of 40 epochs.

For completeness, we show in Figure 4 the same plots as in Figure 1, but we simply replace SGD with the well-tuned SSGD (i.e., with \( \alpha = 0.1 \) and \( s = 40 \)). Interestingly, for \( p \in \{0, 0.1, 0.2, 0.3\} \), SSGD performs mildly better than does SMS in the \( q = \frac{(1-p-0.05)N}{N} \) setting in test accuracy, while SMS performs mildly better in identifying outliers in the training data. For the highest levels of contamination (i.e., \( p = 0.4 \)), however, SSGD achieves an undesirably wide range of values in all metrics. In the \( q = \frac{(1-p-0.05)N}{N} \) setting, SMS consistently outperforms SSGD in all metrics.

Although we do not illustrate this result here, we remark that a well-tuned version of SSGD can perform as well as SMS on the previously discussed regression problems, both in terms of solution time and identification of outliers.

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**Algorithm 2** Stochastic Subgradient Descent (SSGD) Method

**Input:** algorithmic parameters initial point \( w^0 \in \mathbb{R}^n \), initial step size \( \alpha > 0 \), decay parameter \( \delta \geq 0 \), and batch-size parameter \( s \geq 1 \).

\( k \leftarrow 0 \)

repeat

Randomly sample \( S^k \subset \{1, \ldots, N\} \) so that \( |S^k| = s \).

Compute \( g_{S^k}(w^k) \) as in (4).

\[
w^{k+1} \leftarrow w^k - \frac{\alpha}{k} g_{S^k}(w^k).
\]

until budget exhausted
Figure 2. Testing accuracy of classifiers learned by SSGD for $p = 0.3$ and $q = \lfloor (1 - p - 0.05)N \rfloor$ in the MNIST experiments over a variety of $(\alpha, s)$ parameter pairs.

Figure 3. Testing accuracy of classifiers learned by SMS for $p = 0.3$ and $q = \lfloor (1 - p - 0.05)N \rfloor$ in the MNIST experiments over a variety of $(\beta, |S^k| = |S^{k'}|)$ parameter pairs. **Left:** $\Delta_0 = 10$, **right:** $\Delta_0 = 1$
Figure 4. 30 instances of SMS and SSGD on the MNIST classifier problem. Top row: $q = \lfloor (1 - p - 0.1)N \rfloor$, bottom row: $q = \lfloor (1 - p - 0.05)N \rfloor$. Left: testing accuracy, center: TPR on training dataset, right: FPR on training dataset.