Information and cosmological physics

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Abstract. We have reviewed an information-theoretic approach to quantum cosmology, summarizing the key results obtained to date, including a suggestion that an accelerating universe will eventually turn around.

1. Preamble
There are several intriguing approaches to quantum spacetime, and quantum cosmology, in particular. Here, we have described the work of a number of people involved in one endeavour: An information-theoretic approach to quantum cosmology. Spacetime limit has made this review brief and qualitative.

2. A method of inference
The maximum entropy principle (MEP) [1] originated in statistical mechanics as an inference principle, which allowed one to obtain relevant probability distributions in a conceptually appealing manner. For example, consider a statistical system with unknown probability distribution \( p(x) \), but with specified mean energy \( E = \int \epsilon(x)p(x)dx \). In the MEP approach, one maximizes the Gibbs entropy, given by:

\[
I_{GS} = -\int p(x) \ln p(x)dx ,
\]  

under the given constraint to determine the form for the probability distribution: Introducing the Lagrange multiplier \( \beta \) and maximizing \( I_{GS} - \beta E \) with respect to variations in \( p(x) \) gives the well known canonical probability distribution \( p(x) \propto \exp(-\beta\epsilon(x)) \).

Although, in physics, the quantity in Eq. (1) is usually associated with problems in statistical mechanics, an identical expression was derived independently by Claude Shannon in his search for a measure that could be used to quantify the information content, or uncertainty, in a system. Indeed, the original MEP of physics is actually an example of a more general maximum uncertainty principle (MUP), a method of inference used in diverse fields of study [2]. The basic idea is that one should provide the most unbiased description of the state of the system, since maximizing the uncertainty measure acknowledges our ignorance of a more detailed structure.

In some cases, one may already have some a priori information about the system, encoded in the form of a reference probability distribution \( r(x) \). Then the relevant measure, that is maximized in a relative uncertainty measure, called the Kullback-Leibler (KL) information, is given by:

\[
I_{KL}(p, r) = -\int p(x) \ln \frac{p(x)}{r(x)}dx .
\]
If there is no useful \( a \ priori \) information, then \( r(x) \) can be taken to be a uniform distribution and the KL measure then reduces to the Gibbs-Shannon entropy.

Although the MUP is elegant, it does require some specific input: The precise form of the information measure is to be used. As Shannon has shown, the measure in Eq. (1) is the simplest measure that satisfies certain axioms appropriate for the context of some problems. Different situations may require one to relax those assumptions and thus, lead to measures such as in Eq. (2) or others like the Fisher measure to be discussed below.

### 3. Quantum physics

It is useful to re-examine the Schrödinger equation, which describes the evolution of probability amplitudes, from the perspective of the MUP [3]. In one dimension, the transformation \( \psi = \sqrt{p} e^{iS/\hbar} \) can be used to re-write the Schrödinger equation in terms of two real functions, \( p \) and \( S \) as:

\[
\frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V + Q = 0, \quad \text{and} \quad \frac{\partial p}{\partial t} + \frac{1}{m} \frac{\partial}{\partial x} \left( p \frac{\partial S}{\partial x} \right) = 0,
\]

with \( Q = -\frac{\hbar^2}{2m} \frac{1}{p} \frac{\partial^2 \sqrt{p}}{\partial x^2} \), the “quantum potential”. For \( Q = 0 \), the Eq. (3) is just the Hamilton-Jacobi equation for a classical ensemble of particles described by a probability distribution \( p(x, t) \), and with the function \( S(x, t) \) related to the velocity of a particle given by \( v = \frac{1}{m} \frac{\partial S}{\partial x} \). The Eq. (4) is just the expression for conservation of probability .

It was noted [3] that the classical \( Q = 0 \) limit of Eqs. (3) and (4) may be obtained by minimizing the action:

\[
\Phi_A \equiv \int p \left( \frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V \right) dx \, dt,
\]

through a variation of both \( p \) and \( S \). The full quantum equations follow if simultaneously the Fisher information (an inverse uncertainty measure) \( I_F = \int \frac{1}{p} (\frac{\partial p}{\partial x})^2 dx \, dt \) is also minimized: That is, \( \Phi_A + \xi I_F \) is minimized with respect to both \( p \) and \( S \). If the Lagrange multiplier \( \xi \) is set to \( \hbar/8m \), then one obtains the Eqs. (3) and (4), which are equivalent to the time-dependent Schrödinger equation.

Thus, the principle of minimum Fisher information is an example of the MUP, and the result above may be viewed as an extension of the method of inference to quantum mechanics. More specifically, one may argue that the method of inference allows one to supplement classical ensemble dynamics with additional fluctuations to arrive at quantum mechanics. The method is easily extended to many particles in higher dimensions.

### 4. Non-linear Schrödinger equation

The pioneering approach in [3] did not provide a derivation of the Fisher measure in the same way that the Gibbs-Shannon measure was derived: From axioms appropriate for the information-theoretic context. Such a derivation was provided in [4], and it has clarified that the Fisher measure was the simplest to satisfy suitable assumptions, such as locality and separability. One of the assumptions [4] was that the number of derivatives in the information measure be at a minimum. Clearly then, a generalization would lead to higher derivative information measures and a generalized Schrödinger equation.
However, a direct expansion in derivatives would likely lead to highly singular expressions in the generalized Schrödinger equation, and so a different approach would be to construct a suitable information measure, which may be viewed as a “sum of higher derivative” terms. It turns out that there already exists such a measure! If in Eq. (2) one chooses the reference distribution $r(x)$ to be the same as $p(x)$, but with infinitesimally shifted arguments, that is, $r(x) = p(x + \Delta x)$, then to lowest order:

$$I_{KL}(p(x), p(x + \Delta x)) = -\frac{\Delta x^2}{2} I_F(p(x)) + O(\Delta x)^3.$$  (5)

So to the lowest order, minimizing the Fisher information is the same as maximizing the relative entropy for two probability distributions that are close to each other, $r(x) = p(x + \Delta x)$.

An interpretation of $\Delta x$ is readily available. For example, many heuristic arguments combining quantum theory and gravity suggest the existence of a minimal position uncertainty, thus, $\Delta x$ in Eq. (5) may be taken to be the scale at which the coordinates become distinguishable. Beyond leading order, the expansion in Eq. (5) involves higher derivatives. These too have a natural interpretation as describing situations, where fluctuations at increasingly shorter scales become important, as one might expect when quantum physics affects the spacetime.

This motivates one to study a generalized Schrödinger equation that results from using the KL measure, instead of the Fisher measure in the derivation of the last Section. It leads to a non-linear Schrödinger equation [5], whose consequences have been studied in a number of papers. Though non-linear, the equation still shares some important properties of the linear Schrödinger equation: The continuity Eq. (4) is unchanged, so that $p = \psi^* \psi$ is still conserved, and has a sensible interpretation as probability density; also, the equation does not depend on the normalization of the wave function.

A conservative attitude would be to view the information-theoretic non-linear Schrödinger equation as an effective equation, which models, perhaps in an approximate manner, the unknown. In passing, we have noted that non-linear generalizations of the Dirac equation using information-theoretic measures have been discussed in [6].

5. Quantum cosmology of de Sitter space

In quantum cosmology, one studies the universe as a single quantum entity, which in the Wheeler-DeWitt (WDW) approach is described by a wave function of the universe. Usually, situations of restricted symmetry are studied in the mini-superspace scheme, which reduces the functional differential equation to a manageable Schrödinger-like equation with a few degrees of freedom.

In the spirit of the MUP, it was suggested in [7] that an information-theoretic extension of the usual WDW equation be used to model the unknown structure of quantum spacetime. This was motivated in part by the fact that the usual linear WDW equation could not always resolve cosmological singularities.

The non-linear WDW equation for the de Sitter space studied in [7] was just the non-linear Schrödinger equation of [5], but using the potential appropriate for quantum cosmology. As the equation was complicated, being a non-linear difference-differential equation, the first attempt involved studying a truncated and linearized version, with an effective potential, which represented approximately the new ingredients. Encouraging results have been obtained. It was shown that the original Big Bang of the linear WDW equation was replaced by creation of the universe through tunnelling.

Recently [8], we have managed to transform the full non-linear difference-differential equation into a purely difference equation for the probability density by solving first, the current conservation constraint. The difference equation could then be studied easily, though still mostly numerically. The full treatment has shown some new features not seen in the previous
approximate study, such as the existence of a minimum and maximum allowable size to the quantum de Sitter universe.

At the quantum mechanical level, \( a_{\text{min}} \) and \( a_{\text{max}} \) corresponded to the location where the probability density \( p \) vanished. Unlike the case for normal quantum mechanical systems, the vanishing of \( p \) in the non-linear WDW equation with a de Sitter potential was shown to imply a termination of the evolution of the difference equation.

Since, for the de Sitter case, there is no other variable in the WDW equation to play the role of an internal clock, a semi-classical analysis was performed to interpret \( a_{\text{min}} \) and \( a_{\text{max}} \). It was shown that in the effective classical dynamics, the location of nodes in \( p \) corresponded to the position of barriers, which caused bounces at short and large distances.

The de Sitter model is reasonable at early times of our Universe. It will also be a reasonable model at late times if the current acceleration continues. However, our results, taken at face values, suggest that eventually the acceleration will stop and the universe collapse. The above results hold even when the cosmological constant varies slowly [9].

6. The quantum FRW - \( \phi \) universe
A massless scalar field may be used as an internal clock. In our first approximate study [7], using the truncated version of the non-linear WDW equation, we have found that the zero size singularity of the classical FRW - \( \phi \) model was resolved by a bounce in the effective classical dynamics.

Recently, we [9] have extended the study of the FRW - \( \phi \) model by using the full non-linear difference-differential equation in a similar manner mentioned in [8]. By using a suitable ansatz, wave packets with appropriate physical properties were constructed. It was observed that the quantum evolution of the wave packets displayed bounces at short and large distances leading to cyclic evolution, though the cycles observed so far were neither periodic nor everlasting.

7. Outlook
The results from our studies of toy quantum universe, within the information-theoretic approach, have been encouraging. Some natural extensions would involve adding other matter fields, studying less symmetrical situations, and examining the robustness of the results to deformations of the information measure used.

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