PERIODIC ORBITS IN VIRTUALLY CONTACT STRUCTURES

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Abstract. We prove that certain non-exact magnetic Hamiltonian systems on products of closed hyperbolic surfaces and with a potential function of large oscillation admit non-constant contractible periodic solutions of energy below the Mañé critical value. For that we develop a theory of holomorphic curves in symplectisations of non-compact contact manifolds that arise as the covering space of a virtually contact structure whose contact form is bounded with all derivatives up to order three.

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1. Introduction

One of the important questions in contact geometry is whether the Reeb vector field of a given contact form $\alpha$ on a contact manifold $(M', \xi = \ker \alpha)$ admits a periodic solution. The existence of closed Reeb orbits on closed contact manifolds $(M', \xi)$ was conjectured by Weinstein [59]. For compact contact type hypersurfaces in $\mathbb{R}^{2n}$ the conjecture was verified by Viterbo [58] and for closed 3-dimensional contact manifolds by Taubes [57]. Hofer [38] used finite energy holomorphic planes in symplectisations to verify the Weinstein conjecture in many instances of 3-dimensional contact manifolds. Based on Hofer’s finite energy holomorphic curves the Weinstein conjecture was shown to hold true for a large class of higher-dimensional closed contact manifolds, see [1, 6, 21, 22, 29, 31, 33, 47, 55].

Non-closed contact manifolds $(M', \xi)$ in general admit aperiodic Reeb vector fields. For example the Reeb flow of the Reeb vector field $\partial_z$ of the contact form $dz + xdy$ on $\mathbb{R}^{2n-1}$ is linear. Imposing additional topological conditions on the manifold $M'$ and asymptotic conditions on the contact form $\alpha$ existence of periodic Reeb orbits can be shown in many interesting cases, see [13, 14, 32, 54].

In this article we will consider open contact manifolds $(M', \xi)$ that are induced by so-called virtually contact structures. This means that $M'$ is the total space of a covering $\pi: M' \to M$ of a closed odd-dimensional symplectic manifold $(M, \omega)$ such that $\pi^*\omega = d\alpha$ and $\alpha$ admits uniform lower and upper bounds with respect to a lifted metric, cf. Section 2.1. We will consider non-trivial virtually contact structures exclusively whose odd-dimensional symplectic form $\omega$ itself is not the exterior differential of a contact form on $M$.

It was asked by Paternain whether the Reeb vector fields of virtually contact structures admit periodic orbits. Under an additional $C^3$-bound on the contact form $\alpha$ with respect to the Levi–Civita connection of the lifted metric we are able to make Hofer’s theory about finite energy planes work for virtually contact manifolds and answer Paternain’s question affirmatively in the following cases:

**Theorem 1.1.** Let $(M', \xi = \ker \alpha)$ be the total space of a virtually contact structure on a closed odd-dimensional symplectic manifold $(M, \omega)$. Assume that the contact form $\alpha$ is $C^3$-bounded. Then the Reeb vector field of $\alpha$ on $M'$ admits a contractible periodic orbit provided that one of the following conditions for the $(2n - 1)$-dimensional contact manifold $(M', \xi)$ is satisfied:

1. $n = 2$ and $\xi$ is overtwisted,
2. $n = 2$ and $\pi_2 M' \neq 0$,
3. $n \geq 3$ and $(M', \xi)$ contains a Legendrian open book with boundary,
4. $n \geq 3$ and $(M', \xi)$ contains the upper boundary of the standard symplectic handle of index $1 \leq k \leq n - 1$ whose belt sphere $S^{2n-1-k} \subset M'$ represents a non-trivial element in
(a) $\pi_{2n-2}M'$ if $k = 1$,
(b) $\pi_3M'$ if $n = 3$ and $k = 2$,
(c) $\pi_4M'$ if $n = 4$ and $k = 3$,
(d) the oriented bordism group $\Omega^{SO}_{2n-1-k}M'$ if $k \geq 2$,
(5) $n \geq 3$ and $(M', \xi)$ is obtained by a covering contact connected sum (see [60]) such that the underlying connected sum decomposition of $M$ is non-trivial and $\omega$ is not the exterior differential of a contact form on $M$.

We prove Theorem [1.1] in Section [6.1]. Before discussing the ingredients of the proof we mention applications to Hamiltonian dynamics: In classical mechanics one studies the motion of charged particles on a configuration space $Q$ under the influence of a magnetic field $\sigma$ and a conservative force field $V$. It is assumed that a Riemannian metric is chosen on $Q$ so that a Levi–Civita connection $\nabla$, the gradient of $V$, and the vector potential $B$ given by $\langle B., .\rangle = \sigma$ are defined. The equations of motion

$$\nabla_q \dot{q} = -\text{grad} V(q) + B(q) \dot{q}$$

are determined by the Hamiltonian function

$$H(q, p) = \frac{1}{2} |p|^2 + V(q)$$

given by the sum of kinetic and potential energy and the symplectic form

$$d(p \, dq) + \sigma_{ij}(q) \, dq^i \wedge dq^j$$

taking local coordinates $(q, p)$ on the cotangent bundle $T^*Q$. If the magnetic form is exact variational methods can be used to prove existence of periodic solutions as done for example by Contreras [19] using the Mañé critical value

$$\inf_{\theta} \sup_{Q} H(\theta),$$

where the infimum is taken over all primitive 1-forms $\theta$ of $\sigma$. We remark that the Mañé critical value is always bounded from below by $\max_Q V$. If the magnetic form is non-exact but lifts to an exact form to the universal cover of $Q$ it was shown by Merry [44] how to utilize the Mañé critical value of the lifted Hamiltonian on the universal cover instead. Finiteness of the Mañé critical value in that situation is equivalent to the existence of a bounded primitive with respect to the lifted metric.

Arnol'd [7] and Novikov [48, 49] initiated the study of magnetic flows in the context of symplectic topology. Motivated by their existence results obtained by Rabinowitz–Floer homology Cieliebak–Frauenfelder–Paternain [18] formulated a paradigm how a magnetic flow should behave in terms of the Mañé critical value of the magnetic system. In order to confirm the conjectured paradigm for energies below the Mañé critical value in parts we state the following theorem:

**Theorem 1.2.** Let $Q$ be a closed hyperbolic surface, i.e. a Riemannian surface of curvature $-1$. Let $V$ be a Morse function on $Q$ that has a unique local maximum which is required to be positive. Let $v_0 > 0$ be a positive real number such that $\{V \geq -v_0\}$ contains no critical point of $V$ other than the maximum. Let $\sigma$ be a 2-form on $Q$ that vanishes on the disc $\{V \geq -v_0\}$. If

$$v_0 > \inf_{\theta} \sup_{Q} \frac{1}{2} |\theta|^2,$$
where the infimum is taken over all $C^3$-bounded primitive 1-forms $\theta$ of the lift of $\sigma$ to the universal cover $\tilde{Q}$ that vanish on $\{V \geq -v_0\}$ for the lifted potential $\tilde{V}$, then the equations of motion of a charged particle on $Q$ under the influence of the magnetic field $\sigma$ and the presence of the potential $V$ have a non-constant periodic solution of energy $H = 0$ that is contractible in $\{V \leq 0\}$.

We will prove the theorem in Section 6.4. Observe that the 2-form $\sigma$ on the surface $Q$ is automatically closed but not necessarily exact. In the formulation of the theorem we explicitly allow $\sigma$ to be non-exact. As shown in Section 6.3 there always exist $C^3$-bounded primitive 1-forms $\theta$ on the hyperbolic upper half-plane with the vanishing condition as stated, so that the inequality in the theorem indeed can be satisfied. We remark that a change of the metric in the definition of the kinetic energy results in a change of $v_0$ and that the Mañé critical value of the described magnetic system is positive. In Example 6.4.1 we describe a class of magnetic Hamiltonian systems for which the trace of the solution found in Theorem 1.2 intersects the region where the magnetic field does not vanish.

Periodic low-energy solutions for Hamiltonian systems with a non-vanishing magnetic field were found by Schlenk [53]. If the magnetic form is symplectic the reader is referred to the work of Ginzburg–Gürel [35]. The case where $Q$ equals the 2-sphere is discussed by Benedetti–Zehmisch in [12]. If $Q$ is a 2-torus we refer to Schlenk [53] as well as to Frauenfelder–Schlenk [24], where periodic magnetic geodesics (i.e. Hamiltonians with vanishing potentials) are studied. For more existence results of periodic solutions in magnetic Hamiltonian mechanics inspired by the work of Ginzburg [34], Polterovich [51], and Taïmanov [56] confer [2, 3, 4, 5, 9, 8, 11, 12, 43, 53] and the citations therein.

A higher-dimensional analogue to Theorem 1.2 is:

**Theorem 1.3.** Let $Q$ be a product of closed hyperbolic surfaces. Assume that $Q$ admits a potential function $V$ together with a choice of regular value $-v_0 < 0$ and a closed magnetic 2-form $\sigma$ satisfying the conditions described in Theorem 1.2. In addition, assume that $\sigma$ is cohomologous to a $\mathbb{R}$-linear combination of the area forms corresponding to the factors of $Q$. Then the magnetic flow of the Hamiltonian $H$ carries a non-constant periodic solution of zero energy that is contractible in $\{V \leq 0\}$.

In fact, the proof given in Section 6.4 shows that Theorem 1.3 holds for any closed Riemannian manifold $(Q, h)$ that carries a 2-form $\sigma$ that is contained in a cohomology class that is a linear combination of classes represented by 2-forms admitting a $C^3$-bounded primitive on the universal cover of $Q$. For example one could take additional products with closed Riemannian manifolds in the formulation of Theorem 1.3.

In order to prove the theorems we consider the characteristics on the energy surface $M$ of the Hamiltonian, which is contained in $T^*Q$ equipped with the Liouville symplectic structure twisted by the pull back of the magnetic form, see [60 Section 2.3] and Section 6.2. The main observation due to Cieliebak–Frauenfelder–Paternain [18] is that the energy surface $M'$ of the lifted Hamiltonian on the universal cover of $T^*Q$ is of contact type uniformly. In fact, the energy surface $M$ is virtually contact in the sense of Cieliebak–Frauenfelder–Paternain [18], i.e. $M'$ admits a contact form $\alpha$ that is the restriction of a suitable primitive of the twisted symplectic form, cf. [60 Section 2.1] and Section 2.1. Due to the presence of a
non-exact magnetic form the energy surface $M'$ is typically non-compact, see [60, Theorem 1.2].

With the help of holomorphic curves we will find periodic orbits on $M'$. In the work of Cieliebak–Frauenfelder–Paternain [18] a variant of Rabinowitz–Floer homology on the universal cover of $T^*Q$ is used. In this article we will utilize the fundamental work of Hofer [38] and study holomorphic curves in the symplectization of the contact manifold $(M', \alpha)$. The Weinstein handle body structure of the sublevel set of the Hamiltonian is used in order to find germs of holomorphic disc fillings as in the work of Geiges–Zehmisch [31, 32] and Ghiggini–Niederkrüger–Wendl [33], see Section 6.1.

To handle the non-compactness of $M'$ local compactness properties of holomorphic curves have to be ensured. For holomorphic curves uniformly close to the zero section in the symplectization of $(M', \alpha)$ the required uniform lower and upper bounds on the contact form $\alpha$ suffice to guaranty a tame geometry, see Section 2. This leads to monotonicity type estimates in Section 5. In order to mimic Hofer’s asymptotic analysis we will use the deck transformation group of $\mathbb{R} \times M'$ which acts by isometries, see Section 5. The action on the tame structure in the $\mathbb{R}$-direction is controlled by the use of the Hofer energy, see [38]. The action on the contact form $\alpha$ in the direction of $M'$ is controlled by an Arzelà–Ascoli type argument that requires higher order bounds on the contact form, see Section 4. The upshot is that the holomorphic analysis developed in this paper allows a reduction in the search of periodic orbits to finding bounded primitives of higher order on non-compact covering spaces. Examples of how this principle works are formulated in Section 6.1. Furthermore, it gives a method of how to find periodic Reeb orbits on non-compact energy surfaces that is different from the method in the work of Suhr–Zehmisch [54].

2. A tame geometry

A virtually contact structure defines a bounded geometry on the corresponding covering space. In this section we describe the relation to the geometry which is defined by the contact form on the covering space and the induced complex structure on the contact plane distribution.

2.1. A virtually contact structure. Let $M$ be a closed connected manifold of dimension $2n - 1$ for $n \geq 2$. Let $\omega$ be an odd-dimensional symplectic form on $M$, i.e. a closed 2-form whose kernel is a 1-dimensional distribution. We assume that $(M, \omega)$ is virtually contact in the sense of [60]. This means that we can choose a virtually contact structure

$$(\pi: M' \to M, \alpha, \omega, g)$$

on $(M, \omega)$, where $\pi$ is a covering of $M$, $\alpha$ is a contact form on the covering space $M'$ such that $\pi^*\omega = d\alpha$, and $g$ is a Riemannian metric on $M$, whose lift $\pi^*g$ to $M'$ is denoted by $g'$. By definition of a virtually contact structure the primitive $\alpha$ is bounded with respect to the norm $|.|_{(g')}^*$ of the dual metric $(g')^*$ of $g'$. Moreover, there exists a constant $c > 0$ such that for all $v \in \ker d\alpha$ the following lower bound holds true:

$$(LB) \quad |\alpha(v)| \geq c|v|_{g'}. $$
In addition, we assume that the chosen virtually contact structure is **non-trivial**, i.e., that \( \omega \) is not the exterior differential of a contact form on \( M \).

Observe that the characteristic line bundle \( \ker \omega \) of the odd-symplectic manifold \( (M, \omega) \) is orientable precisely if \( M \) is orientable. In this situation we orient \( M \), resp., \( \ker \omega \) by the contact form \( \alpha \) on \( M' \) requiring that the covering map \( \pi \) preserves the orientation.

### 2.2. A characterization.

The virtually contact property of the odd-dimensional symplectic manifold \( (M, \omega) \) can be formulated in a slightly different language. We denote by \( \xi = \ker \alpha \) the contact structure on \( M' \) defined by the contact form \( \alpha \) of \( \pi \ast \omega \). The Reeb vector field of \( \alpha \) is denoted by \( R \) and defines a splitting \( \mathbb{R} R \oplus \xi \) of \( TM' \). Moreover, the metric \( g' \) defines an orthogonal splitting \( \xi^\perp \oplus \xi \) of \( TM' \) and the orthogonal projection onto \( \xi^\perp \) is denoted by \( \mathrm{proj}_\perp \).

With these notions understood we formulate the geometric requirements on \( \alpha \), which are given in terms of \( g' \), equivalently as follows: There exist constants \( c > 0 \) and \( C > 0 \) such that

\[
|\Reeb|_{g'} \leq \frac{1}{c} \quad \text{and} \quad \frac{1}{C} \leq |\mathrm{proj}_\perp \Reeb|_{g'}.
\]

The constant \( c \) corresponds to the one given in Section 2.1. The constant \( C \) can be taken to be \( \sup_{M'} |\alpha|(g')^{\flat} \), where \( (g')^{\flat} \) is the dual metric of \( g' \). We remark that \( |\alpha|(g')^{\flat} \) is equal to the operator norm \( \|\alpha\| \) of the linear form \( \alpha \) on \( T M' \) with respect to \( g' \), where the norm \( \|\alpha\| \) is taken pointwise.

### 2.3. The induced complex structure.

The equation

\[
d\alpha = g' \left( \Phi(\cdot, \cdot) \right) \quad \text{on} \quad \xi
\]

determines a skew adjoint vector bundle isomorphism \( \Phi : \xi \to \xi \) whose square multiplied by \(-1\) is self adjoint and positive definite. By [40, Proposition 1.2.4] the bundle endomorphism given by

\[
j := \Phi \circ (\sqrt{-\Phi^2})^{-1}
\]

is a complex structure on \( \xi \) that is compatible with \( d\alpha \), i.e.

\[
g_j := d\alpha(\cdot, j \cdot)
\]

defines a bundle metric on \( \xi \).

**Lemma 2.3.1.** The norm \( |\cdot|_j \) induced by \( g_j \) and the restriction of the norm \( |\cdot|_{g'} \) to \( \xi \) are uniformly equivalent, i.e., there exist constants \( c_1, c_2 > 0 \) such that

\[
\frac{1}{c_1} |\cdot|_{g'} \leq |\cdot|_j \leq c_2 |\cdot|_{g'}
\]
on \( \xi \).

**Proof.** Denote by \( \|\Phi^{-1}\| \) and \( \|\Phi\| \) pointwise operator norms with respect to \( g' \), which equal 1 over the smallest resp. the largest pointwise eigenvalue of \( \sqrt{-\Phi^2} \). Observe that

\[
g_j = g' \left( \sqrt{-\Phi^2}(\cdot, \cdot) \right) \quad \text{on} \quad \xi,
\]

so that

\[
\frac{1}{\sqrt{\|\Phi\|}} |\cdot|_{g'} \leq |\cdot|_j \leq \sqrt{\|\Phi\|} |\cdot|_{g'} \quad \text{on} \quad \xi.
\]
Therefore, we have to show that the eigenvalues of $-\Phi^2$ are uniformly positive and uniformly bounded.

Let $\Omega: TM \to T^*M$ be the linear bundle map given by $v \mapsto i_v \omega$. The characteristic line bundle of $(M, \omega)$ is equal to $\ker \Omega$. For a constant $c_0 > 0$ we define the convex subbundle $C$ of $TM$ to be the set of all tangent vectors $v \in TM$ whose angle with $\ker \Omega$ with respect to $g$ is greater or equal than $c_0$. In view of the alternative characterization of the virtually contact property given in Section 2.2, we can choose $c_0$ so small such that $T\pi(\xi) \subset C$.

We consider the map

$$TM \ni v \mapsto |\Omega(v)|_{g'} \in [0, \infty),$$

where $g'$ denotes the dual metric of $g$. This map is uniformly bounded from above and away from zero on the compact set $C \cap STM$, where $STM$ denotes the unit tangent bundle of $M$ with respect to $g$. We define a cone like subbundle $C'$ of $TM'$ to be the preimage of $C$ under $T\pi$ with respect to $g'$ that contains $\xi$ such that $C' \cap STM' \ni v \mapsto |\Omega'(v)|_{(g')^*} \in (0, \infty)$ has uniform upper and lower bounds, where $\Omega': TM' \to T^*M'$ is the corresponding map $v \mapsto i_v \omega_d\alpha$. Because of

$$|\Omega'(v)|_{(g')^*} = |\Phi(v)|_{g'}$$

for all $v \in \xi$ we obtain that the map

$$\xi \cap STM' \ni v \mapsto |\Phi(v)|_{g'} \in (0, \infty)$$

is uniformly bounded from above and away from zero. With

$$|\Phi^2(v)|_{g'} = |\Phi(v)|_{g'} \left| \Phi \left( \frac{\Phi(v)}{|\Phi(v)|_{g'}} \right) \right|_{g'}$$

we get similar bounds for

$$\xi \cap STM' \ni v \mapsto |\Phi^2(v)|_{g'} \in (0, \infty).$$

Inserting all possible eigenvectors $v \in \xi$ of unit length with respect to $g'$ we see that all eigenvalues of $-\Phi^2$ are uniformly positive and uniformly bounded. This proves the lemma.

2.4. Bounding the geometry. On the covering space $M'$ we define a second Riemannian metric by setting

$$g_\alpha = \alpha \otimes \alpha + g_j$$

with respect to the splitting $\mathbb{R}R \oplus \xi$. We denote the projection of $TM'$ onto $\xi$ along the Reeb vector field by $\pi_\xi$.

**Lemma 2.4.1.** The norm $|.|_{\alpha}$ induced by $g_\alpha$ and the norm $|.|_{g'}$ are uniformly equivalent on $M'$, i.e. there exist constants $c_1, c_2 > 0$ such that

$$\frac{1}{c_1} |.|_{g'} \leq |.|_{\alpha} \leq c_2 |.|_{g'}.$$
Proof: We begin with the first inequality: Decompose any given tangent vector \( v \in TM' \) into \( v = v^1 R + Y \) with respect to \( \mathbb{R}R \oplus \xi \). Then

\[
|v|^2 = |\alpha(v^1 R)|^2 + |Y|^2.
\]

With \( \ref{13} \) and Lemma \( 2.3.1 \) we obtain the lower estimate

\[
|v|^2 \geq \min\left( c^2, \frac{1}{c^2} \right) \cdot \left( |v^1 R|^2 + |Y|^2 \right).
\]

Using the parallelogram identity \( 2(|x|^2 + |y|^2) = |x + y|^2 + |x - y|^2 \) this leads to

\[
|v|^2 \geq \frac{1}{2} \min\left( c^2, \frac{1}{c^2} \right) \cdot |v|^2_{g'}
\]

proving the first inequality.

In order to show the second inequality observe that

\[
|v|^2 = |\alpha(v)|^2 + |\pi_\xi v|^2.
\]

Using the boundedness of \( \alpha \) and Lemma \( 2.3.1 \) this gives

\[
|v|^2 \leq \max\left( c^2, \frac{1}{c^2} \right) \cdot |v|^2_{g'}.
\]

It remains to prove finiteness of the supremum of all pointwise operator norms \( \|\pi_\xi\| \) with respect to \( g' \), where the supremum is taken over all points of \( M' \). For \( p \in M' \) let \( u \) be a unit tangent vector with respect to \( g' \) that is either contained in \( \xi \) if \( R_p \in \xi^\perp \) or lies in the span of \( \mathbb{R}R \) and \( \xi^\perp \) at \( p \) being perpendicular to \( R_p \) and pointing in the same co-orientation direction of \( \xi \) as \( R_p \). The norm \( \|\pi_\xi|_p\| \) is the length \( |Y|_{g'} \) of the vector \( Y = \pi_\xi u \in T_p M' \). Consequently,

\[
\|\pi_\xi|_p\| = \frac{1}{\sin \angle_{g'}(R_p, \xi_p)},
\]

which is uniformly bounded as the angle \( \angle_{g'}(R_p, \xi_p) \) between \( R_p \) and \( \xi_p \) stays uniformly away from zero according to the alternative formulation of the virtually contact property given in Section \( 2.2.2 \). In other words, the supremum of all \( \|\pi_\xi|_p\|, p \in M' \), is finite proving the second inequality. \( \square \)

2.5. Length and area. The norms of \( g_\alpha \) and \( g' \) are uniformly equivalent by Lemma \( 2.4.1 \) so that we can formulate isoperimetric type inequalities with respect to either metric.

By definition the metric \( g' \) is locally isometric to \( g \) via the covering \( \pi: M' \to M \). The compactness of \( M \) implies that \( g' \) is of bounded geometry. In particular, the absolute values of the sectional curvature of \( g' \) are uniformly bounded and the injectivity radius of \( g' \) is uniformly bounded away from zero by, say, \( 2i_0 > 0 \). Further, the metric \( g' \) is complete. Moreover, for all \( p \in M' \) the exponential map \( \exp_p \) is defined for all tangent vectors \( v \in T_p M' \) of length \( |v|_{g'} < i_0 \) inducing a diffeomorphism

\[
T_p M' \supset B_{i_0}(0) \longrightarrow B_{i_0}(p) \subset M'
\]

onto the geodesic ball \( B_{i_0}(p) \) with respect to \( g' \). Denote by \( \mathcal{E}_p \) the restriction of \( \exp_p \) to \( B_{i_0}(0) \). By \( \ref{50} \) p. 318 the linearization of \( \mathcal{E}_p \) and its inverse \( \mathcal{E}_p^{-1} \) are uniformly bounded, i.e. there exists a constant \( C > 0 \) such that for all \( p \in M' \)

\[
\|T\mathcal{E}_p\|, \|T\mathcal{E}_p^{-1}\| < C,
\]

where the operator norm is taken pointwise with respect to \( g' \).
We will use this to formulate an isoperimetric inequality for smooth loops that are contained in a geodesic ball of radius \( r_0 \). Let \( c : \mathbb{R} \to M' \) be a \( 2\pi \)-periodic map with image in \( B_{r_0}(c(0)) \). Define a loop \( X \) of tangent vectors in \( T_{c(0)}M' \) via \( \exp_{c(0)}X(\theta) = c(\theta) \). This defines a disc map \( f_c : D^2 \to M' \) via

\[
f_c(re^{i\theta}) = \exp_{c(0)}(rX(\theta)),
\]

where we use polar coordinates \( z = re^{i\theta} \) on the closed unit disc \( D^2 \). In view of the above first order bounds on \( E \) and to the metric \( \tilde{g} \) we will use monotonicity type phenomena of holomorphic discs to obtain distance estimates in \( M' \)-distance bounds. Writing \( r \)

\[
\text{Area}_{g'}(f_c(D^2)) = \int_{(0,1) \times (0,2\pi)} \sqrt{\det(f_c^*g')}_{ij} \, dr \wedge d\theta
\]

of the disc \( f_c(D^2) \)

\[
\text{Area}_{g'}(f_c(D^2)) \leq C^3 \left( \text{length}_{g'}(c) \right)^2.
\]

Observe, that a similar inequality holds if area and length are measured with respect to the metric \( g_\alpha \).

3. Holomorphic discs

Holomorphic discs in symplectisations \( \mathbb{R} \times M' \) cannot only escape to \(-\infty\) in the \( \mathbb{R} \)-direction but they need \( C^0 \)-control in the \( M' \)-directions too. In this section we will use monotonicity type phenomena of holomorphic discs to obtain distance estimates in \( M' \)-directions in terms of symplectic energy and \( \mathbb{R} \)-distance bounds.

3.1. An almost complex structure. On \( \mathbb{R} \times M' \) we define an almost complex structure \( J \) that is translation invariant, restricts to \( j \) on \( \xi \), and sends \( \partial_t \) to \( R \) denoting by \( t \) the \( \mathbb{R} \)-coordinate. We consider a holomorphic disc, which is a smooth map \( u : \mathbb{D} \to \mathbb{R} \times M' \) such that \( Tu \circ i = J(u) \circ Tu \), where \( \mathbb{D} \) denotes the closed unit disc in \( \mathbb{C} \) equipped with the complex structure \( i \). We will assume that the following boundary condition is satisfied \( u(\partial \mathbb{D}) \subset \{0\} \times M' \). Writing \( u = (a, f) \) holomorphicity of \( u \) can be expressed as

\[
\begin{cases}
-\partial \alpha \circ i = f^*\alpha, \\
\pi \xi T f \circ i = j(f) \circ \pi \xi T f.
\end{cases}
\]

In particular, \( a : \mathbb{D} \to \mathbb{R} \) is a subharmonic function due to the choice of \( j \). The maximum principle implies that \( u(\mathbb{D}) \) is contained in \((-\infty, 0] \times M'\). Moreover,

\[
u^*(dt \wedge \alpha) = (a_x^2 + a_y^2) \, dx \wedge dy
\]

and

\[
f^*d\alpha = \frac{1}{2} \left(|f_x|^2 + |f_y|^2\right) \, dx \wedge dy.
\]
Observe that both expressions are non-negative.

3.2. Area growth. For \( p \in M' \) and \( t \in (0, i_0) \) we consider the solid cylinder \( \mathbb{R} \times B_t(p) \) over the open geodesic ball \( B_t(p) \) with respect to \( g' \) and denote the preimage of the intersection with the holomorphic disc by

\[
G_t = u^{-1}(\mathbb{R} \times B_t(p)) = f^{-1}(B_t(p)) .
\]

We assume that

\[
f(\partial \mathbb{D}) \subset M' \setminus B_{i_0}(p)
\]

so that \( G_t \) is disjoint from the boundary \( \partial \mathbb{D} \) of \( \mathbb{D} \). The radial distance function of \( g' \) at \( p \) is denoted by

\[
r : B_{i_0}(p) \rightarrow [0, i_0) ; \quad x \mapsto \text{dist}_{g'}(p, x) .
\]

Notice, that using the Gauß-lemma the pointwise operator norm of \( T r \) with respect to \( g' \) equals

\[
\| T r \| = 1 ,
\]

see \([50, \text{Lemma 6.12}]\). The restriction of the radial distance function to the holomorphic disc is denoted by

\[
F : G_{i_0} \rightarrow [0, i_0) ; \quad z \mapsto r(f(z)) .
\]

With this notation introduced we see that \( \partial G_t = F^{-1}(t) \). Denote by \( \text{Reg} \subset (0, i_0) \) the set of regular values of \( F \) that are not contained in the image \( r(\{ \pi_T T f = 0 \}) \), which is a finite set, see \([32, \text{Lemma 7}]\). Observe that \( f \) has no critical points on \( F^{-1}(\text{Reg}) \). Let \( h \) be a Riemannian metric on \( F^{-1}(\text{Reg}) \subset \mathbb{C} \) for which there exists a universal constant \( C_0 > 0 \) that is independent of the choice of \( p \in M' \) satisfying

\[
| \text{grad}_h F |_h \leq \frac{1}{C_0} .
\]

In order to find such a universal constant \( C_0 \) we remark that

\[
| \text{grad}_h F |^2_h = dF(\text{grad}_h F) .
\]

Since the Gauß lemma implies \( \| T r \| = 1 \) we see that for all \( v \in TG_t \)

\[
| dF(v) |_h \leq | Tf(v) |_{g'}
\]

applying the chain rule to \( F = r \circ f \).

With the co-area type arguments in the proof of the monotonicity lemma given in \([40, \text{p. 27/28}]\) we obtain:

**Lemma 3.2.1.** For all \( t \in \text{Reg} \) the \( t \)-derivative of the area of \( G_t \) exists and satisfies

\[
(\text{Area}_h(G_t))' \geq C_0 \text{length}_h(\partial G_t) .
\]

3.3. Symplectization. Denote by \( T \) the set of all smooth strictly increasing functions \( \tau : (-\infty, 0] \rightarrow [0, 1] \) with \( \tau(0) = 1 \). Any \( \tau \in T \) defines a symplectic form

\[
d(\tau \alpha) = \tau' dt \wedge \alpha + \tau d\alpha
\]

on \( \mathbb{R} \times M' \). The almost complex structure \( J \) is compatible with \( d(\tau \alpha) \) defining a metric

\[
g_\tau = d(\tau \alpha)(., ., J .) = \tau'(dt \otimes dt + \alpha \otimes \alpha) + \tau g_j .
\]

The holomorphicity of \( u \) implies that \( u^* g_\tau \) defines a conformal metric \( h \) on \( F^{-1}(\text{Reg}) \). Inserting

\[
v = \frac{\text{grad}_h F}{| \text{grad}_h F |_h}
\]
into [3] with that choice of \( h \) yields the universal constant \( C_0 \) required in Lemma 3.2.1 via

\[
|\text{grad}_h F|_h \leq c_1 \max_{a(\partial)} \left( \frac{1}{\tau'}, \frac{1}{\tau} \right),
\]

where \( c_1 \) is a constant from Lemma 2.4.1. Notice, that the right hand side depends on the holomorphic disc \( u = (a, f) \).

3.4. An isoperimetric inequality. We continue the discussion from Section 3.2. For \( \tau \in \mathcal{T} \) we will study the area

\[
A(t) = \int_{G_t} u^* d(\tau a),
\]

t \in [0, i_0), which is cut out by the solid cylinder \( \mathbb{R} \times B_i(p) \) about the \( g' \)-geodesic ball \( B_i(p) \) in \( M' \). As in Section 3.2 we only allow those geodesic balls that do not hit the boundary \( f(\partial D) \). The length of the maybe disconnected curves \( u(\partial G_t) \), to which we will compare the area \( A(t) \), is measured with respect to the metric

\[
g_0 = dt \otimes dt + g_a.
\]

**Lemma 3.4.1.** There exists a positive constant \( c_3 \), which only depends on the geometry of \( (M', g') \), such that

\[
A(t) \leq c_3 \left( 1 + \max_{G_t} (\tau'(a)) \right) \left( \text{length}_{g_0} (u(\partial G_t)) \right)^2,
\]

where \( \text{length}_{g_0} (u(\partial G_t)) \) is the sum of the lengths of the components.

**Proof.** Let \( N \) be the number of boundary components of \( G_t \). For the \( \ell \)-th component of \( \partial G_t \) define a disc map \( f_\ell: D^2 \to M' \) via \( g' \)-geodesics along the \( \ell \)-th component of the curve \( f(\partial G_t) \) as examined in Section 2.5. Similarly, disc maps \( a_\ell: D^2 \to \mathbb{R} \) are defined via convex interpolations along the components of \( a(\partial G_t) \). Choosing orientations appropriately Stokes’ theorem implies

\[
A(t) = \int_{\partial G_t} u^* (\tau a) = \sum_{\ell=1}^{N} \int_{D^2} (a_\ell, f_\ell)^* d(\tau a).
\]

The integrand decomposes into

\[
\tau'(a_\ell) da_\ell \wedge f_\ell^* \alpha + \tau(a_\ell) f_\ell^* da_{\alpha}.
\]

We will tread the summands separately. Beginning with the second, which is bounded by \( |f_\ell^* da| \) because of \( \tau \leq 1 \), we find

\[
\int_{D^2} |f_\ell^* da| \leq \frac{c_3^2 C^3}{2} \left( \text{length}_{g'} (f_\ell (\partial D^2)) \right)^2
\]

for all \( \ell = 1, \ldots, N \) using Lemma 2.4.1 and equations (1) and (2) from Section 2.5. Hence, the sum of the \( (a_\ell, f_\ell)^* (\tau d\alpha) \)-integrals is estimated by

\[
c_4 \left( \text{length}_{g'} (f(\partial G_t)) \right)^2 \leq c_7 c_4 \left( \text{length}_{g_0} (f(\partial G_t)) \right)^2
\]

setting \( c_4 = c_3^2 C^3/2 \) and using Lemma 2.4.1 again. Similarly, invoking the boundedness of \( \alpha \) and equations (1) and (2) again we obtain

\[
\int_{D^2} \tau'(a_\ell) da_\ell \wedge f_\ell^* \alpha \leq c_5 \max_{G_t} (\tau'(a)) \left( \text{length}_{g_0} (a_\ell, f_\ell (\partial D^2)) \right)^2
\]
for a positive constant $c_5$. Therefore, the sum of the $(a_\ell, f_\ell)^*(\tau' dt \wedge \alpha)$--integrals is bounded by
\[
c_5 \max_{G_t} (\tau'(a)) \left( \text{length}_{g_0}(u(\partial G_t)) \right)^2.
\]
Combining both estimates proves the claim. \qed

3.5. **Monotonicity.** For $h$ being induced by $u^*g_\tau$ as in Section 3.3, Lemma 3.2.1 implies that there exists a positive constant $c_6$ such that
\[
A'(t) \geq \frac{c_6}{(\max_{a(\mathbb{D})} \left( \frac{1}{\tau'}, \frac{1}{\tau} \right))^2} \text{length}_{g_0}(u(\partial G_t))
\]
for all $t \in \text{Reg}$. Combined with Lemma 3.4.1 which says that
\[
\text{length}_{g_0}(u(\partial G_t)) \geq \sqrt{\frac{c_3^{-1}}{1 + \max_{a(\mathbb{D})}(\tau')} \sqrt{A(t)}},
\]
we obtain
\[
A'(t) \geq 2 m \sqrt{A(t)},
\]
for all $t \in \text{Reg}$, where
\[
m = m(\tau(a)) := \frac{c_6}{2\sqrt{c_3}} \frac{1}{(\max_{a(\mathbb{D})} \left( \frac{1}{\tau'}, \frac{1}{\tau} \right))^2} \sqrt{\frac{1}{1 + \max_{a(\mathbb{D})}(\tau')}}.
\]

With the reasoning on [40, p. 28] equation (4) implies the following monotonicity lemma in symplectisations as described in the present context:

**Proposition 3.5.1.** Let $u = (a, f)$ be a holomorphic disc map that sends $(\mathbb{D}, \partial\mathbb{D})$ into $(\mathbb{R} \times M', \{0\} \times M')$. Let $p \in M'$ be a point on $f(\mathbb{D})$ such that the $g'$-geodesic ball $B_{i_0}(p)$ and the boundary curve $f(\partial \mathbb{D})$ are disjoint. Then the area functional
\[
A(t) = \int_{f^{-1}(B_{i_0}(p))} u^*(\tau \alpha)
\]
satisfies
\[
A(t) \geq m^2 t^2
\]
for all $t \in [0, i_0)$, where the constant $m = m(\tau(a))$ is positive and depends on the $\mathbb{R}$-coordinate of the holomorphic disc $u = (a, f)$.

3.6. **A distance estimate.** If we take $\tau(t) = e^t$ we obtain for the monotonicity constant $m$ in Proposition 3.5.1
\[
m(e^a) = c_7 e^{-2\max_0 |a|}
\]
for a positive constant $c_7$. We will use this in order to estimate the maximal distance
\[
\text{dist}_{g'}(L, f(\mathbb{D})) = \sup_{f(\mathbb{D})} \text{dist}_{g'}(L, .)
\]
between a maximally $J$-totally real submanifold $L \subset M'$ that has compact closure and the $M'$-part of a holomorphic disc $f(\mathbb{D})$ that has boundary $f(\partial \mathbb{D})$ on $L$. 
**Proposition 3.6.1.** There exists a positive constant $K$ that only depends on the geometry of $(M', g')$ such that for all holomorphic discs $u = (a, f) : (D, \partial D) \longrightarrow (\mathbb{R} \times M', \{0\} \times L)$, where $L \subset M'$ is a relatively compact maximally totally real submanifold with respect to $J$, the following estimate holds

$$\text{dist}_{g'}(L, f(D)) \leq K e^{4\max_0 |a|} E(u)$$

denoting by

$$E(u) = \int_D u^* d(e^t \alpha)$$

the symplectic energy of $u$ with respect to $d(e^t \alpha)$.

**Proof.** There are unique $d_0 \in [0, 2i0)$ and $N \in \mathbb{Z}$ such that $\text{dist}_{g'}(L, f(D)) = 2Ni_0 + d_0$.

We assume that $N \neq 0$ because the claim holds by Proposition 3.5.1. Choose points $p_1, \ldots, p_N$ on $f(D)$ such that $\text{dist}_{g'}(L, .)$ maps $B_{i0}(p_\ell)$ into the shifted intervall $\text{dist}_{g'}(L, f(D)) - (2i0(N - \ell + 1), 2i0(N - \ell))$ for all $\ell = 1, \ldots, N$. By Proposition 3.5.1 the symplectic $d(e^t \alpha)$–energy of the intersection $u(D) \cap \mathbb{R} \times B_{i0}(p_\ell)$ is bounded from below as

$$\int_{f^{-1}(B_{i0}(p_\ell))} u^* d(e^t \alpha) \geq c_7^2 e^{-4\max_0 |a|} i_0^2$$

for all $\ell = 1, \ldots, N$. Because the preimages $f^{-1}(B_{i0}(p_\ell))$ are mutually disjoint taking the sum over all points $p_1, \ldots, p_N$ yields

$$E(u) \geq c_7^2 e^{-4\max_0 |a|} N i_0^2.$$ 

Combining this with $2Ni_0 + d_0 < (N + 1)2i0 \leq 4Ni_0$ yields

$$\text{dist}_{g'}(L, f(D)) \leq \frac{4}{c_7^2 i_0} e^{4\max_0 |a|} E(u)$$

proving the claim. \(\square\)

In particular, families of holomorphic discs with uniform energy bounds that lie above a certain slice $\{-R\} \times M'$, $R \gg 1$, have uniformly $C^0$-bounded projections into $M'$ with respect to $g'$. Therefore, Gromov's compactness theorem as formulated in \cite{25} applies:

**Corollary 3.6.2.** Let $u_\nu = (a_\nu, f_\nu) : (D, \partial D) \longrightarrow (\mathbb{R} \times M', \{0\} \times L)$ be a sequence of holomorphic discs, where $L \subset M'$ is a relatively compact maximally totally real submanifold with respect to $J$. Assume that there exists a compact subset of $L$ that contains all boundary curves $u_\nu(\partial D)$. If

$$\sup_{\nu \in \mathbb{N}} \sup_D |a_\nu| < \infty \quad \text{and} \quad \sup_{\nu \in \mathbb{N}} E(u_\nu) < \infty,$$

then $u_\nu$ has a subsequence that Gromov converges to a stable holomorphic disc.
4. Higher order bounds on primitives

The group of isometries acts on virtually contact structures via pull back. The induced action on the space of holomorphic discs in symplectisations \( R \times M' \) that have no uniform \( R \)-distance bounds will be part of the bubbling off analysis, see Section 5. In this section we will work out the required compactness properties on sequences of virtually contact structures obtained by the group action.

4.1. Higher order covariant derivative. We consider a connected Riemannian manifold \( (M', g') \). The covariant derivative of the Levi–Civita connection is denoted by \( \nabla \). Let \( \tau \) be a \((0, k)\)-tensor, \( k \in \mathbb{N} \), on \( M' \). Following [26, p. 73] and [41, p. 52] we define the covariant derivative \( \nabla_X \tau \) of \( \tau \) in direction of the vector field \( X \) on \( M' \) via the following formula:

\[
(\nabla_X \tau)(Y_1, \ldots, Y_k) := X(\tau(Y_1, \ldots, Y_k)) - \sum_{j=1}^k \tau(Y_1, \ldots, Y_{j-1}, \nabla_X Y_j, Y_{j+1}, \ldots, Y_k),
\]

where \( Y_1, \ldots, Y_k \) are test vector fields on \( M' \). Setting

\[
\nabla \tau(X, Y_1, \ldots, Y_k) := (\nabla_X \tau)(Y_1, \ldots, Y_k)
\]

defines a \((0, k + 1)\)-tensor \( \nabla \tau \) on \( M' \). For a 1-form \( \alpha \) on \( M' \) we inductively define the \( k \)-th covariant derivative by \( \nabla^0 \alpha = \alpha \) and

\[
\nabla^k \alpha = \nabla(\nabla^{k-1} \alpha)
\]

so that \( \nabla^k \alpha \) is a \((0, k + 1)\)-tensor \( \nabla \tau \) on \( M' \). The pointwise norm of \( \nabla^k \alpha \) at \( p \in M' \) is defined by

\[
|\nabla^k \alpha|_p := \sup\{|\nabla^k \alpha(v, w_1, \ldots, w_k)|: v, w_1, \ldots, w_k \in (M', g') \}
\]

where the supremum is taken over all tuples \((v, w_1, \ldots, w_k)\) of unit tangent vectors of \((M', g')\) at \( p \). We set

\[
\|\nabla^k \alpha\|_{C^0} := \sup_{M'} |\nabla^k \alpha|,
\]

which is the supremum of \( \nabla^k \alpha \) on the \((k + 1)\)-fold Whitney sum of the unit tangent bundle \( STM' \). The \( C^k \)-norm of \( \alpha \) on \((M', g')\) is defined via

\[
\|\alpha\|_{C^k} := \sup_{\ell=0,1,\ldots,k} \|\nabla^\ell \alpha\|_{C^0}.
\]

We provide the space of smooth 1-forms on \( M' \) that are bounded in all \( C^k \)-norms with the \( C^\infty \)-topology with respect to the sequence of \( C^k \)-norms as defined. Observe that convergence on the restrictions to relatively compact open subsets of \( M' \) is the same as the convergence induced by the compact open topology as introduced in [37, Section 2.1]. In this situation we will simply speak about convergence in the \( C^\infty_{\text{loc}} \)-topology.

Remark 4.1.1. Similarly, for a smooth function \( f \) on \( M' \) one defines \( \nabla^0 f = f \) and

\[
\nabla^k f = \nabla(\nabla^{k-1} f) = \nabla^{k-1} df
\]

as well as

\[
\|f\|_{C^k} := \sup_{\ell=0,1,\ldots,k} \|\nabla^\ell f\|_{C^0} = \sup \left\{ \|f\|_{C^0}, \|df\|_{C^{k-1}} \right\}.
\]
The space of smooth functions on $M'$ that have finite $C^k$-norm for all $k \in \mathbb{N}$ is provided with the $C^\infty$-topology induced by the $C^k$-norms. The $C^\infty_{loc}$-convergence is understood in the same manner as for 1-forms.

**Remark 4.1.2.** Let $\varphi$ be an isometry of $(M', g')$. The generalized *theorema egregium* can be phrased as

$$\varphi^*(\nabla_X Y) = \nabla_{\varphi_* X} \varphi_* Y$$

for all vector fields $X$ and $Y$ on $M'$, cf. [52, p. 183]. It follows that

$$\varphi^*(\nabla\tau) = \nabla(\varphi^*\tau)$$

on $(0, k)$-tensors $\tau$ on $M'$. Inductively, for all 1-forms $\alpha$ on $M'$ we get

$$\varphi^*(\nabla^k \alpha) = \nabla^k (\varphi^* \alpha) .$$

Because $\varphi$ induces a bundle isomorphism on $\bigoplus_{k=1}^{\infty} ST M'$ we obtain

$$\|\varphi^* \alpha\|_{C^k} = \|\alpha\|_{C^k} .$$

**4.2. Local computations.** We continue the discussions from Section 4.1. Let $x^1, \ldots, x^{2n-1}$ be local coordinates on $M'$. For a given Riemannian metric $g'$ and a given 1-form $\alpha$ we write $g' = g_{ij} dx^i \otimes dx^j$ and $\alpha = \alpha_j dx^j$. Denoting the Christoffel symbols of the Levi–Civita connection $\nabla$ of $g'$ by $\Gamma^{k}_{ij}$ we get $(\nabla \alpha)_{ij} = \alpha_{j,i} - \Gamma^{k}_{ij} \alpha_k$. Inductively,

$$(\nabla^{k+1} \alpha)_{i_1 \ldots j_{k+1}} = (\nabla^k \alpha)_{j_1 \ldots j_{k+1}, i} - \sum_{\ell=1}^{k+1} \Gamma^{m}_{i\ell j_k} (\nabla^k \alpha)_{j_1 \ldots m j_{k+1}}$$

where the $m$ ist placed at the $\ell$-th position.

In order to estimate the tensor $\nabla^k \alpha$ we notice that for the symmetric positive definite matrix $(g_{ij})_{ij}$ we find an orthogonal matrix $A$ such that $(g_{ij})_{ij} = ADA^T$, where $D$ denotes the diagonal matrix of all eigenvalues $\lambda_1 \leq \ldots \leq \lambda_{2n-1}$ of $(g_{ij})_{ij}$. Hence, writing $v = v^i \partial_i$ for a tangent vector we get $|v|^2 = \lambda_i |v^i|^2$ for all $i = 1, \ldots, 2n-1$. Taking unit tangent vectors $(v, w_1, \ldots, w_k)$ we finally get

$$|\nabla^k \alpha(v, w_1, \ldots, w_k)| \leq \left(\frac{2n-1}{\sqrt{\lambda_1}}\right)^{k+1} \max_{i_1 j_1 \ldots j_k} \left\{|(\nabla^k \alpha)_{i_1 \ldots j_k}|\right\} .$$

**4.3. Uniform $C^\infty$-bounds – An example.** Using the results from Section 4.2 we consider the following example: Denote by $H^+$ the open upper half-plane $\{y > 0\}$ provided with the standard hyperbolic metric. Let $M'$ be $\mathbb{R} \times H^+$ provided with the product metric

$$g' = dt \otimes dt + \frac{1}{y^2} (dx \otimes dx + dy \otimes dy) .$$

Labeling the coordinates $(t, x, y) \in \mathbb{R} \times H^+$ by $(x^1, x^2, x^3)$ the Christoffel symbols read as

$$\Gamma^k_{ij} = \frac{1}{y} \left(\delta_{i3} (\delta_{2j} \delta_{3j} - \delta_{3j} \delta_{ji}) - \delta_{i2} (\delta_{3j} \delta_{2j} + \delta_{3j} \delta_{3j})\right),$$

which can be brought in the form

$$\Gamma^k_{ij} = \frac{1}{y} \delta_{i3} \delta_{j2} .$$
for constants $\gamma_{ij}$ that vanish if at least one index equals 1. The covariant derivatives of the contact form
\[
\alpha = dt + \frac{1}{y} dx = \left( \delta_{i1} + \frac{1}{y} \delta_{i2} \right) dx^i
\]
with respect to the Levi–Civita connection $\nabla$ of $g'$ are given by
\[
(\nabla \alpha)_{ij} = \frac{1}{y^2} \delta_{i2} \delta_{j3} = \frac{1}{y^2} \Delta_{ij}
\]
and
\[
(\nabla^{k} \alpha)_{j_1...j_{k+1}} = \frac{1}{y^{k+1}} \Delta_{j_1...j_{k+1}}
\]
as an induction shows, where $\Delta_{ij}$ and $\Delta_{j_1...j_{k+1}}$ are constants that vanish provided that at least one index is equal to 1. In particular, in the non-vanishing case we find for all $k \in \mathbb{N}$ a positive constant $c_k$ such that for all index tuples $(j_1, \ldots, j_{k+1})$
\[
\left| (\nabla^{k} \alpha)_{j_1...j_{k+1}} \right| \leq \frac{c_k}{y^{k+1}}.
\]
Because the eigenvalues of the matrix obtained by the metric coefficients of the hyperbolic metric on $H^+$ are equal to $\lambda = \frac{1}{y^2}$ we obtain $|\alpha|_{(g')^\flat} \leq 2$ and
\[
|\nabla^{k} \alpha|_{(g')^\flat} \leq c_k \left( \frac{2}{\sqrt{\lambda}} \cdot \frac{1}{y} \right)^{k+1} = 2^{k+1} c_k =: C_k.
\]
In other words, for all $k \in \{0\} \cup \mathbb{N}$ we have that $\|\alpha\|_{C^k}$ is globally bounded.

**Remark 4.3.1.** Let $\Sigma$ be a closed hyperbolic surface. On $M = S^1 \times \Sigma$ an odd-symplectic form $\omega$ is given by the area form of $\Sigma$. The lift of $\omega$ to the universal covering has primitive $\alpha$ as described and defines a virtually contact structure that is $C^k$-bounded for all $k \in \mathbb{N}$. Replacing $\Sigma$ by a product of closed hyperbolic surfaces and $\omega$ by the direct sum of the corresponding area forms one obtains examples of $C^\infty$-bounded virtually contact structures in all odd dimensions in a similar manner.

### 4.4. An Arzelà–Ascoli argument

We consider a covering $\pi: M' \to M$. Let $g$ be a Riemannian metric and $\omega$ be an odd-symplectic form both on $M$. We set $g' = \pi^* g$ and denote the pull back form along the covering map $\pi$ by $\omega' = \pi^* \omega$. The group $G$ of deck transformations of the covering $\pi$ acts by isometries and by odd-symplectomorphisms on $(M, g', \omega')$, i.e. $\varphi^* g' = g'$ and $\varphi^* \omega' = \omega'$ for all $\varphi \in G$.

We assume that $\omega'$ has a primitive 1-form $\alpha$ on $M'$ so that
\[
\omega' = d\alpha.
\]
For a sequence $\varphi_{\nu} \in G$, $\nu \in \mathbb{N}$, of deck transformations we define a sequence $\alpha_{\nu} := \varphi_{\nu}^* \alpha$ of 1-forms on $M'$. We remark that because all $\varphi_{\nu}$ are isometries we get with Remark 4.1.2 for all $k \in \mathbb{N}$
\[
\|\alpha_{\nu}\|_{C^k} = \|\alpha\|_{C^k}
\]
and that because all $\varphi_{\nu}$ are odd-symplectic the 1-form $\alpha_{\nu} - \alpha$ is closed on $M'$.

**Proposition 4.4.1.** We assume that the base manifold $M$ is closed and that for all $k \in \mathbb{N}$ there exists $C_k > 0$ such that
\[
\|\alpha\|_{C^k} < C_k.
\]
Then $\alpha_{\nu}$ has a subsequence that converges in $C^\infty_{\text{loc}}(M')$. 


Proof. We first prove the statement under the additional assumption that the first de Rham cohomology group of the covering space $M'$ vanishes. Therefore, the 1-form $\alpha_\nu - \alpha$ is exact for all $\nu$. Denoting the base point of $M'$ by $o$ we get in fact that for all $\nu$ there exists a unique $f_\nu \in C^\infty(M')$ such that $f_\nu(o) = 0$ and $d f_\nu = \alpha_\nu - \alpha$.

In particular, we get
$$\|d f_\nu\|_{C^k} \leq \|\alpha_\nu\|_{C^k} + \|\alpha\|_{C^k} < 2C_k$$
providing a uniform bound
$$\sup_{\nu \in \mathbb{N}} \|d f_\nu\|_{C^k} \leq 2C_k.$$ 

In order to prove the proposition it suffices to show that for a subsequence of $f_\nu$ there exists $f \in C^\infty(M')$ such that $f_\nu \to f$ in $C^\infty_{loc}$ as $\nu$ tends to $\infty$. Indeed, this will imply $d f_\nu \to d f$ in $C^\infty_{loc}$ and, therefore, $\alpha_\nu = d f_\nu + \alpha \to d f + \alpha =: \alpha_0$ in $C^\infty_{loc}$ as $\nu$ tends to $\infty$.

Because the base manifold $M$ is closed by [26, 2.91 and 2.105] ($M'$, $g'$) is geodesically complete. By the Hopf–Rinow theorem (see [26, 2.103 and 2.105]) we find for any given point $p \in M'$ a minimal geodesic $c$ in $(M', g')$ of unit speed that connects $p$ with the base point $o$. By the mean value theorem we find for all $\nu \in \mathbb{N}$ a real number $t_\nu$ such that
$$|f_\nu(p)| \leq \text{dist}_{g'}(o, p) \left|T_{c(t_\nu)} f_\nu\right|_{g'}.$$ 

Denoting by $B_r$ the open $g'$-geodesic ball with center $o$ and radius $r > 0$ this implies
$$\|f_\nu\|_{C^0(B_r)} \leq r \|d f_\nu\|_{C^0(B_r)} \leq 2rC_0$$
for all $\nu \in \mathbb{N}$, which results in a uniform bound
$$\sup_{\nu \in \mathbb{N}} \|f_\nu\|_{C^0(B_r)} \leq 2rC_0.$$ 

This means that for all $r > 0$ the subset $\{f_\nu|_{B_r}\}$ of $C^0(B_r)$ is bounded. Similarly, replacing $o$ by any point $q \in B_r$ we get for all $p, q \in B_r$
$$|f_\nu(p) - f_\nu(q)| \leq \text{dist}_{g'}(p, q) 2C_0.$$ 

Therefore, the subset $\{f_\nu|_{B_r}\}$ of $C^0(B_r)$ is equicontinuous, i.e.
$$\sup_{\nu \in \mathbb{N}} |f_\nu(p) - f_\nu(q)| \to 0$$
as $\text{dist}_{g'}(p, q)$ tends to zero for points $p, q$ in $B_r$. Observe that by the Hopf–Rinow theorem the closure of $B_r$ is compact for all $r > 0$ and the union $\bigcup_{r > 0} B_r$ is equal to $M'$. Hence, using the Arzelà-Ascoli theorem (see [10]) there exists a continuous function $f$ on $M'$ such that a subsequence of $f_\nu$ converges to $f$ in $C^0_{loc}$. Furthermore, in view of the above estimate and the assumptions on the covariant derivatives of $\alpha$ we obtain
$$\sup_{\nu \in \mathbb{N}} \|f_\nu\|_{C^k(B_r)} \leq \max \{2rC_0, 2C_{k-1}\}$$
for all $k \in \mathbb{N}$ and for all $r > 0$. Therefore, as long as the closure $\overline{B}_r$ of $B_r$ is contained in a chart domain a diagonal sequence argument and [10, Theorem 8.6] imply that there exists a subsequence $f_{\nu_k}$ that converges in $C^\infty(\overline{B}_r)$ to the $a \text{ posteriori}$ smooth
function \( f|_{B_r} \). If \( B_r \) is not contained in a chart domain we can work with a finite covering of \( B_r \) by chart domains taking subsequences successively with respect to an ordering of the covering chart domains. Hence, for all \( r \in \mathbb{N} \) there exists a subsequence \( f_{\nu r} \) that converges in \( C^\infty(B_r) \) to \( f|_{B_r} \). A further diagonal sequence argument yields a in \( C^\infty_{\text{loc}} \) converging subsequence \( f_{\mu r} \to f \) so that \( \alpha_{\mu r} \to \alpha_0 \) in \( C^\infty_{\text{loc}} \).

In fact, the above argument works without making any assumption on the first de Rham cohomology of the covering space as follows: On each open ball \( B \subset M' \) whose closure \( \bar{B} \) is contained in a chart domain a unique primitive function \( f_{\nu B} \) of \( (\alpha_{\nu B} - \alpha_B)|_B \) that vanishes on the centre of \( B \) can be selected. Hence, \( \alpha_{\mu B} \to \alpha_B \) in \( C^\infty(B) \) as \( \mu \to \infty \) for a subsequence of \( \alpha_{\nu B} \) using the above argument. Taking finite coverings of the closure of \( B_r, r \in \mathbb{N} \), by ball-like chart domains \( B \) and using a diagonal sequence of \( \alpha_{\nu B} \) with respect to \( B_{11}, B_{k1}, B_{k^2}, \ldots, B_{k^{2j}} \) (covering \( B_2 \)), and so on, we find a subsequence \( \alpha_{\nu C} \) of \( \alpha_{\nu B} \) that converges in \( C^\infty_{\text{loc}} \) to \( \alpha_0 \), where for all \( r \in \mathbb{N} \) and for all \( j = 1, \ldots, k \) the restriction of \( \alpha_0 \) to \( B^{jr} \) equals \( \alpha_{B^{ir}} \). This proves the proposition in general.

We remark that a global \( C^0 \)-bound on \( \alpha \) is not sufficient in order to find a convergent subsequence of \( \alpha_{\nu B} \) in \( C^0_{\text{loc}} \). The above proof shows in fact:

**Corollary 4.4.2.** If there exists \( k \in \mathbb{N} \) such that \( \|\alpha\|_{C^k} < C \) for a positive constant \( C \), then the sequence \( \alpha_{\nu B} \) admits a subsequence \( \alpha_{\nu B} \to \alpha_0 \) that converges in \( C^{k-1}_{\text{loc}} \) to a 1-form \( \alpha_0 \) of class \( C^{k-1} \).

### 4.5. Induced convergence on complex structures.

We consider a covering \( \pi: M' \to M \) as in Section 4.4. Additionally, we assume that the primitive \( \alpha \) of \( \omega' \) is a contact form on \( M' \) so that all \( \alpha_{\nu B} \) are contact forms. We denote the induced contact structures by \( \xi_{\nu B} := \ker \alpha_{\nu B} \), which are provided with the symplectic form obtained by the restrictions of \( \omega' = d\alpha_{\nu B} \) to \( \xi_{\nu B} \). As shown in Section 4.4 for all \( \nu \in \mathbb{N} \) there exists a unique section \( \Phi_{\nu B} \) in the endomorphism bundle of \( \xi_{\nu B} \) such that

\[
\omega' = g'\left( \Phi_{\nu B}(\cdot, \cdot) \right) \quad \text{on} \quad \xi_{\nu B}.
\]

We obtain complex structures \( j_{\nu B} \) on \( (\xi_{\nu B}, \omega') \) by setting

\[
j_{\nu B} := \left( \Phi_{\nu B} (\cdot) \right)^{-1}
\]

so that

\[
g_{j_{\nu B}} := \omega'(\cdot, j_{\nu B} \cdot) \quad \text{on} \quad \xi_{\nu B}
\]

defines a bundle metric on \( \xi_{\nu B} \).

On the product \( \mathbb{R} \times M' \) we consider the non-degenerate 2-forms

\[
\eta_{\nu B} := dt \wedge \alpha_{\nu B} + \omega'
\]
denoting the \( \mathbb{R} \)-coordinate by \( t \). Notice that the exterior differentials are equal to

\[
d\eta_{\nu B} = -dt \wedge \omega'.
\]

For all \( \nu \in \mathbb{N} \) we define an endomorphism field \( \Psi_{\nu B} \) on \( \mathbb{R} \times M' \) by requiring \( \Psi_{\nu B} \) to be \( \mathbb{R} \)-translation invariant such that the restriction of \( \Psi_{\nu B} \) to \( \xi_{\nu B} \) equals \( \Phi_{\nu B} \) and such that

\[
\Psi_{\nu B}(\partial_t) = R_{\nu B} \quad \text{and} \quad \Psi_{\nu B}(R_{\nu B}) = -\partial_t,
\]

where \( R_{\nu B} \) denotes the Reeb vector field of \( \alpha_{\nu B} \). Taking the splitting

\[
T\left( \mathbb{R} \times M' \right) = \mathbb{R}\partial_t \oplus \mathbb{R}R_{\nu B} \oplus \xi_{\nu B}
\]
into account we get
\[ \Psi_\nu = i_\nu \oplus \Phi_\nu . \]

Therefore,
\[ J_\nu := \Psi_\nu \circ (\sqrt{-\Psi_\nu^2})^{-1} = i_\nu \oplus j_\nu \]
is the unique almost complex structure on \( \mathbb{R} \times M' \) that is \( \mathbb{R} \)-translation invariant, restricts to \( j_\nu \) on \( \xi_\nu \), and sends \( \partial_t \) to \( R_\nu \). Because of
\[ \alpha_\nu = -dt \circ J_\nu \]
the bilinear form
\[ \eta_\nu (\cdot, J_\nu \cdot) = dt \otimes dt + \alpha_\nu \otimes \alpha_\nu + g_{j_\nu} \]
is a metric on \( \mathbb{R} \times M' \).

Similarly, we define a sequence of metrics
\[ g_\nu := dt \otimes dt + \alpha_\nu \otimes \alpha_\nu + g'|_{\xi_\nu} \]
on \( \mathbb{R} \times M' \), where
\[ g'|_{\xi_\nu}(v, w) = g'(v - \alpha_\nu(v)R_\nu, w - \alpha_\nu(w)R_\nu) \]
for tangent vectors \( v, w \) parallel to \( M' \). We remark that
\[ \eta_\nu = g_\nu (\Psi_\nu (\cdot), \cdot) \]
on \( \mathbb{R} \times M' \) and that the equation characterizes \( \Psi_\nu \) uniquely.

**Lemma 4.5.1.** We assume that \( \alpha \) satisfies condition [LB]. If \( \alpha_\nu \to \alpha_0 \) converges in \( C^\infty_{\text{loc}} \), then \( \alpha_0 \) is a contact form and the uniquely associated sequence of almost complex structures \( J_\nu \to J_0 \) converges in \( C^\infty_{\text{loc}} \) to the almost complex structure \( J_0 \) that is defined by \( \alpha_0 \) via the above construction.

**Proof.** First of all observe that \( d\alpha_\nu = \omega' \) converges to \( d\alpha_0 \) in \( C^\infty_{\text{loc}} \). Therefore, \( d\alpha_0 = \omega' \) and \( \eta_\nu \) converges to \( \eta_0 := dt \wedge \alpha_0 + \omega' \) in \( C^\infty_{\text{loc}} \). Using condition [LB] we see that \( |\alpha_\nu(v)| \geq c|v|^n_{\nu} \) for all \( v \in \ker \omega' \) because all \( \varphi_\nu \in G \) are isometries. In other words the limiting 1-form \( \alpha_0 \) is a contact form.

We want to show that \( R_\nu \) converges to the Reeb vector field \( R_0 \) of \( \alpha_0 \). For that we observe \( R_\nu \) is the unique \( \mathbb{R} \)-invariant vector field on \( \mathbb{R} \times M' \) such that \( \iota_{R_\nu} \eta_\nu = -dt \). Choosing local coordinates \( x^1 = t, x^2, \ldots, x^m \) we get \( R^j = -\eta^j \) for the components of \( R_\nu \), where \( \eta^j \) is the inverse of the matrix of the \( \eta_\nu \)-coefficients. Hence, \( R_\nu \to R_0 \) in \( C^\infty_{\text{loc}} \).

This implies that \( g'|_{\xi_\nu} \) converges in \( C^\infty_{\text{loc}} \) to \( g'|_{\xi_0} \), which is defined with respect to the contact structure \( \xi_0 = \ker \alpha_0 \). In order to prove this we denote the projection of \( TM' \) onto \( \xi_\nu \) along the Reeb vector field \( R_\nu \) by \( \pi_{\xi_\nu} \). Setting \( P_\nu := 0 \oplus \pi_{\xi_\nu} \) on \( T(\mathbb{R} \times M') \) we obtain in local coordinates \( \gamma_{ij} = P^k_i P^j_l (g')_{kl} \) for the metric coefficients of \( g'|_{\xi_\nu} \) ignoring the subscript \( \nu \) in the notation for the coefficients of \( P_\nu \). Because of \( \pi_{\xi_\nu}(v) = v - \alpha_\nu(v)R_\nu \), which locally reads as \( P_\nu = \delta_i - \alpha_i R \), we get \( P_\nu \to P_0 \) in \( C^\infty_{\text{loc}} \), where \( P_0 \) is the projection defined with respect to \( \alpha_0 \). Therefore, \( g'|_{\xi_\nu} \to g'|_{\xi_0} \) in \( C^\infty_{\text{loc}} \), as claimed.

We claim that \( \Psi_\nu \) converges to \( \Psi_0 \) in \( C^\infty_{\text{loc}} \), where \( \Psi_0 \) is defined via the preliminary construction with respect to \( \alpha_0 \). Observe that \( g_\nu \) converges to \( dt \otimes dt + \alpha_0 \otimes \alpha_0 + g'|_{\xi_0} \) in \( C^\infty_{\text{loc}} \). Because of \( \eta_\nu = g_\nu (\Psi_\nu (\cdot), \cdot) \) we get \( \Psi^i_\nu = \eta^i_\nu g'^j \) for local representations of the (1,1)-tensors \( \Psi_\nu \). Here the right hand side is the product of the \( \eta_\nu \)-coefficients.
and the inverse of the metric coefficients of $g_\nu$. As both converge we get $\Psi_\nu \to \Psi_0$ in $C^\infty_{\text{loc}}$.

It remains to show that $J_\nu \to J_0$ converges in $C^\infty_{\text{loc}}$, where the almost complex structure $J_0$ is uniquely characterized by the preliminary construction. But this follows because

$$J_\nu = \Psi_\nu \circ (\sqrt{-\Psi_\nu^2})^{-1}$$

locally can be expanded in a power series in $\Psi_\nu$ with coefficients being independent of $\nu$. This proves $C^\infty_{\text{loc}}$-convergence of $J_\nu$ and hence the lemma. \qed

**Remark 4.5.2.** In order to conclude that $J_0$ can be constructed via $\alpha_0$ in the above proof, in which situation we will simply write $(\alpha_0, J_0)$, one needs convergence in $C^1_{\text{loc}}$ because we have to differentiate the limiting 1-form $\alpha_0$. Apart from that the above proof goes through in the case of $C^k_{\text{loc}}$-convergence for all $k \in \mathbb{N}$. Combined with Corollary 4.4.2 this implies that if $\alpha$ has a global $C^k$-bound for $k \in \mathbb{N}$ at least 2, then a subsequence of $(\alpha_\nu, J_\nu)$ can be selected that converges in $C^{k-1}_{\text{loc}}$ to $(\alpha_0, J_0)$. It follows that $\alpha_0$ admits a global $C^k$-bound and satisfies (11).

**Remark 4.5.3.** The almost complex structure $J$ introduced in Section 3.1 equals

$$J = \Psi \circ (\sqrt{-\Psi^2})^{-1},$$

where the endomorphism field $\Psi$ on $\mathbb{R} \times M'$ is uniquely determined by

$$\eta = (dt \otimes dt + \alpha \otimes \alpha + g_\nu'|\xi)(\Psi(\cdot), \cdot)$$

setting $\eta = dt \wedge \alpha + \omega'$. We consider the sequence of diffeomorphisms $F_\nu = a_\nu \times \varphi_\nu$ on $\mathbb{R} \times M'$, where $a_\nu$ denotes the translation by the real number $a_\nu$, and claim that

$$J_\nu = F_\nu^* J.$$

Indeed, $F_\nu^* \eta = \eta_\nu$ and

$$F_\nu^* (dt \otimes dt + \alpha \otimes \alpha + g_\nu'|\xi) = g_\nu$$

because of the following observations: Recall that $\pi_\xi(v) = v - \alpha(v)R$ and that $g_\nu'|\xi = \pi_\xi^* g_\nu^* g'$ so that because of $\pi_\xi \circ T_{\varphi_\nu} = T_{\varphi_\nu} \circ \pi_\xi$, we get

$$\varphi_\nu^* (g_\nu'|\xi) = \varphi_\nu^* \pi_\xi^* g_\nu^* \pi_\xi^* g' = \pi_\xi^* \varphi_\nu^* g' = \pi_\xi^* g'|\xi_\nu.$$

Therefore,

$$\eta_\nu = g_\nu \pi_\xi^* \Psi(\cdot), \cdot),$$

which characterizes $\Psi_\nu$ uniquely. In other words $\Psi_\nu = F_\nu^* \Psi$. This implies the claim because the eigenvalues of a matrix are invariant under conjugations.

5. **COMPACTNESS**

We consider a virtually contact structure $(\pi: M' \to M, \alpha, \omega, g)$ together with the associated almost complex structure $J$ on $\mathbb{R} \times M'$ constructed in Sections 2.3 and 3.1. We assume that the covering $\pi$ is regular, i.e. that the group of deck transformations $G$ acts transitively on the fibers of $\pi$. Furthermore, we assume that any sequence $\alpha_\nu = \varphi_\nu^* \alpha, \varphi_\nu \in G$, has a in $C^\infty_{\text{loc}}$ converging subsequence. In view of Lemma 4.5.1, the associated subsequence of $J_\nu$ converges in $C^\infty_{\text{loc}}$ as well. In the same manner we assume that for any accumulation point $\alpha_0$ of $\alpha_\nu$, the sequence $\varphi_\nu^* \alpha_0$ and the associated sequence of almost complex structures have converging subsequences, cf. Remark 4.5.2.
Let
\[ u_\nu = (a_\nu, f_\nu) : (\mathbb{D}, \partial \mathbb{D}) \to (\mathbb{R} \times M', \{0\} \times L) \]
be a sequence of \( J \)-holomorphic discs with boundary in an open relatively compact subset \( K_L \) of a maximally \( J \)-totally real submanifold \( L \subset M' \). We assume that the Hofer energy
\[ E_{\text{Hofer}}(u) := \sup_\tau \int_{\mathbb{D}} u^* d(\tau^1) \]
is uniformly bounded by \( E > 0 \) for all \( u = u_\nu \), where the supremum is taken over all smooth increasing functions \( \tau : \mathbb{R} \to [0, 1] \). By the maximum principle, the symplectic energy
\[ E(u) = \int_{\mathbb{D}} u^* d(e^1) \]
is uniformly bounded by \( E \) for all \( u = u_\nu \) too.

In this section we will carry out a bubbling off analysis which is largely analogous to [28, Section 6] and [30, p. 543-548]. But it is necessary to adapt the arguments to the present situation of a non-compact contact manifold \( M' \).

5.1. Bubbling off analysis. If the maximum of the \( |a_\nu| \) over \( \mathbb{D} \) is uniformly bounded, then by Corollary 3.6.2 the sequence \( u_\nu \) has a Gromov convergent subsequence that converges to a stable holomorphic disc whose underlying bubble tree consists of discs only.

If the sequence of maxima is not bounded, then we find a sequence \( \zeta_\nu \) in \( \mathbb{D} \) such that a subsequence of \( a_\nu(\zeta_\nu) \) tends to \(-\infty\). By the mean value theorem we find a point \( z_\nu \) on the line segment connecting \( \zeta_\nu \) with 1 in \( \mathbb{D} \) such that
\[ a_\nu(\zeta_\nu) = T_{z_\nu} a_\nu \cdot (\zeta_\nu - 1) \]
using \( a_\nu(1) = 0 \). Hence,
\[ |a_\nu(\zeta_\nu)| \leq 2 \left| T_{z_\nu} u_\nu \right|_{g'_0} \]
with respect to the complete metric \( g'_0 = dt \otimes dt + g' \), so that the sequence \( |Tu_\nu|_{g'_0} \) is not uniformly bounded. Passing to a further subsequence and writing
\[ \left| \nabla u_\nu \right|_{g'_0} = \sqrt{\left| \partial_x u_\nu \right|_{g'_0}^2 + \left| \partial_y u_\nu \right|_{g'_0}^2} \]
instead of \( |Tu_\nu|_{g'_0} \) we can assume that
\[ R_\nu := \max_{\mathbb{D}} \left| \nabla u_\nu \right|_{g'_0} = \left| \nabla u_\nu (z_\nu) \right|_{g'_0} \to \infty \]
for a sequence \( z_\nu \to z_0 \) in \( \mathbb{D} \). In the following we will distinguish the cases whether the limit point \( z_0 \) lies in the interior \( B_1(0) \subset \mathbb{C} \) or on the boundary \( \partial \mathbb{D} \) of \( \mathbb{D} \).

Case 1: \( z_0 \in B_1(0) \). We can assume that no \( z_\nu \) lies on the boundary \( \partial \mathbb{D} \). We choose \( \epsilon > 0 \) such that \( B_\epsilon(z_\nu) \subset B_1(0) \) for all \( \nu \in \mathbb{N} \). Let \( \mathcal{D} \subset M' \) be a fundamental domain of the covering \( \tau \) that contains the base point \( 0 \in M' \) and choose a sequence \( \varphi_\nu \in G \) of deck transformations such that \( \varphi_\nu^{-1} \) maps \( f_\nu(z_\nu) \) into the closure of \( \mathcal{D} \), see [16] p. 201. We consider the rescaled sequence \( v_\nu = (b_\nu, h_\nu) \) defined via
\[ b_\nu(z) := a_\nu(z_\nu + z/R_\nu) - a_\nu(z_\nu) \]
and
\[ h_\nu(z) := \varphi_\nu^{-1}(f_\nu(z_\nu + z/R_\nu)) \]
for all \( z \in B_{R_\nu(\epsilon)}(0) \) so that \( v_\nu(0) \in \{0\} \times \mathcal{D} \). Moreover, because \( F_\nu \) is an isometry with respect to the metric \( g_0' \) we have that \( |\nabla v_\nu(0)|_{g_0'} = 1 \) and that \( |\nabla v_\nu|_{g_0'} \leq 1 \).
uniformly on \( B_{R_{\nu}}(0) \). Observe that \( v_\nu \) is obtained from \( u_\nu \) by a reparametrisation with a Möbius transformation and the composition with the inverse of the diffeomorphism \( F_\nu = a_\nu(z_\nu) \times \varphi_\nu \). Therefore, \( v_\nu \) is \( J_\nu \)-holomorphic with respect to the almost complex structure \( J_\nu = F_\nu^*J \) on \( \mathbb{R} \times M' \) (see Remark 4.5.3), which is associated to \( \alpha_\nu = \varphi_\nu^* \alpha \), cf. Section 4.5. We can assume that a further subsequence of \( u_\nu \) is selected for which the sequence \((\alpha_\nu, J_\nu)\) converges in \( C_{\text{loc}}^\infty \) to \((\alpha_0, J_0)\). We finally remark that the Hofer energy

\[
\sup_\tau \int_{B_{R_{\nu}}(0)} v_\nu^* d(\tau \alpha_\nu) \leq E
\]

is uniformly bounded, where the supremum is taken over all smooth increasing functions \( \tau: \mathbb{R} \to [0, 1] \).

Let \( k \) be a natural number and choose \( \nu_0 \in \mathbb{N} \) such that the closure of \( B_k(0) \) is contained in \( B_{R_{\nu}}(0) \) for all \( \nu \geq \nu_0 \). For all \( z \in B_k(0) \) the \( g' \)-distance between \( v_\nu(0) \) and \( v_\nu(z) \) is bounded by \( k \) as the uniform gradient bound \( |\nabla v_\nu|_{g'} \leq 1 \) shows. By [18] p. 117/18 and p. 201] the fundamental domain \( \mathcal{D} \) can be chosen such that \( \pi(\mathcal{D}) \) equals the complement of the cut locus of \( \pi(o) \) in \((M, g)\), where \( o \) denotes the base point of \( M' \). Because the diameter \( d_0 \) of \((M, g)\) is finite (as \( M \) is compact) the \( g' \)-distance between \( o \) and \( h_\nu(0) \) is bounded by \( d_0 \). Therefore, for all \( \nu \geq \nu_0 \) the discs

\[ v_\nu(B_k(0)) \subset [-k, 0] \times \overline{B_{d_0+k}(o)} \]

are contained in the product of the interval \([-k, 0]\) and the closure of the \( g' \)-geodesic ball \( B_{d_0+k}(o) \), which by the Hopf–Rinow theorem is compact.

Because the restriction of \((\alpha_\nu, J_\nu)\) to \([-k, 0] \times \overline{B_{d_0+k}(o)}\) converges uniformly with all covariant derivatives to the restriction of \((\alpha_0, J_0)\), with elliptic regularity we see that \( v_\nu|_{B_k(0)} \) has a subsequence \( v_{\nu_k} \) that converges uniformly with all derivatives, see [16] Theorem B.4.2] and [28 p. 559]. Inductively, using a diagonal sequence argument we see that there exists a converging subsequence (again denoted by)

\[ v_\nu \to v \quad \text{in} \quad C_{\text{loc}}^\infty(\mathbb{C}) \]

that converges to a non-constant \( J_0 \)-holomorphic map \( v: \mathbb{C} \to \mathbb{R} \times M' \). By Fatou’s lemma we find for all \( k \in \mathbb{N} \) and all smooth increasing functions \( \tau: \mathbb{R} \to [0, 1] \)

\[
\int_{B_k(0)} v^* d(\tau \alpha_0) = \int_{B_k(0)} \lim_{\nu \to \infty} v_\nu^* d(\tau \alpha_\nu) \leq \liminf_{\nu \to \infty} \int_{B_k(0)} v_\nu^* d(\tau \alpha_\nu) = E
\]

so that \( v \) is a finite energy plane with Hofer energy

\[
\sup_\tau \int_{\mathbb{C}} v^* d(\tau \alpha_0) \leq E
\]

bounded by \( E \). This finishes our considerations for \( z_0 \in B_1(0) \).

**Case 2:** \( z_0 \in \partial \mathbb{D} \). We identify \( \mathbb{D} \setminus \{-z_0\} \) with the closed upper half-plane \( \mathbb{H} \) conformally such that \((z_0, 0, -z_0)\) corresponds to \((0, i, \infty)\). Under this identification we regard \( u_\nu \) as a \( J \)-holomorphic map

\[
u_{\nu}: (\mathbb{H}, \mathbb{R}) \to (\mathbb{R} \times M', \{0\} \times L).
\]

The sequence of the corresponding bubble points again denoted by \( z_\nu \rightarrow 0 \) can be assumed to be contained in

\[ \mathbb{D}^+ := \mathbb{D} \cap \mathbb{H} . \]
Moreover, we find a positive constant \( c \) such that for all \( z \in \mathbb{D}^+ \) and for all \( \nu \in \mathbb{N} \)

\[
\frac{1}{c}R_\nu \leq |\nabla u_\nu(z)|_{g_0'} \quad \text{and} \quad |\nabla u_\nu(z)|_{g_0'} \leq cR_\nu.
\]

By conformal equivalence the Hofer energy stays unchanged so that

\[
\sup_{\tau} \int_{\mathbb{D}^+} u_\nu^*(d\tau) \leq E
\]

for all \( \nu \in \mathbb{N} \), where the supremum is taken over all smooth increasing functions \( \tau : \mathbb{R} \to [0, 1] \). Passing to a suitable subsequence we can assume that there exists \( \varepsilon > 0 \) such that for all \( \nu \in \mathbb{N} \)

\[
B_\varepsilon^+(z_\nu) := B_\varepsilon(z_\nu) \cap \mathbb{H}
\]

is contained in \( \mathbb{D}^+ \) and that \( R_\nu y_\nu \to \check{\rho} \in [0, \infty] \) writing \( z_\nu = x_\nu + iy_\nu \).

**Case 2(a):** \( \check{\rho} = \infty \). Exactly as in Case 1 one defines a rescaled sequence \( v_\nu \) of \( J_0 \)-holomorphic maps on \( B_{R_\nu} \cdot (0) \cap \{ y \geq -R_\nu y_\nu \} \) which has the property that \( v_\nu(0) \) is contained in \( \{0\} \times \mathbb{D} \) and that \( |\nabla v_\nu(0)|_{g_0'} \) is bounded from below by \( 1/c \) independently of \( \nu \). Moreover, we have that \( |\nabla v_\nu|_{g_0'} \leq c \) uniformly on \( B_{R_\nu} \cdot (0) \cap \{ y \geq -R_\nu y_\nu \} \) and that the Hofer energy with respect to \( \alpha_0 \) satisfies

\[
\sup_{\nu \in \mathbb{N}} E_{\text{Hofer}}(v_\nu) \leq E.
\]

Arguing as in Case 1 one selects a \( C^\infty_{\text{loc}}(\mathbb{C}) \)-converging subsequence \( v_\nu \to v \) that converges to a non-constant \( J_0 \)-holomorphic finite energy plane \( v \) whose Hofer energy is taken with respect to \( \alpha_0 \).

**Case 2(b):** \( \check{\rho} < \infty \). This time we define a rescaled sequence \( v_\nu \) by

\[
v_\nu(z) := u_\nu(x_\nu + z/R_\nu)
\]

for all \( z \in B^+_{R_\nu}(iR_\nu y_\nu) \), where \( z_\nu = x_\nu + iy_\nu \). Observe that \( B^+_{R_\nu}(iR_\nu y_\nu) \) defines an exhausting sequence of open subsets of \( \mathbb{H} \). Furthermore, we have that \( |\nabla v_\nu(0)|_{g_0'} \) is greater than or equal to \( 1/c \) independently of \( \nu \), that \( |\nabla v_\nu|_{g_0'} \leq c \) uniformly on \( B^+_{R_\nu}(iR_\nu y_\nu) \), and that the Hofer energy with respect to \( \alpha \) satisfies

\[
\sup_{\nu \in \mathbb{N}} E_{\text{Hofer}}(v_\nu) \leq E.
\]

We remark that \( v_\nu(x) \) is contained in \( \{0\} \times K_L \) for all \( x \in \mathbb{R} \).

For given \( k \in \mathbb{N} \) and \( \nu \) such that \( B^+_k(0) \) is contained in \( B^+_{R_\nu}(iR_\nu y_\nu) \) we find that the \( g_0' \)-distance from \( v_\nu(z) \) to \( \{0\} \times K_L \) is bounded by \( ck \) because of the above uniform gradient bound. Hence, for all \( \nu \) sufficiently large \( v_\nu(B^+_k(0)) \) is a subset of the \( ck \)-neighbourhood of \( \{0\} \times K_L \) with respect to \( g_0' \), whose closure is compact. Invoking \[46\], Theorem B.4.2 and \[28\], p. 559 as in Case 1 we find a \( C^\infty_{\text{loc}}(\mathbb{H}) \)-converging subsequence \( v_\nu \to v \), where

\[
v : (\mathbb{H}, \mathbb{R}) \longrightarrow (\mathbb{R} \times M', \{0\} \times K_L)
\]

is a boundary condition preserving non-constant \( J \)-holomorphic finite energy half-plane of Hofer energy \( E_{\text{Hofer}}(v) \leq E \) with respect to \( \alpha \).

### 5.2. A finite energy cylinder

Let \( v \) be a non-constant \( J_0 \)-holomorphic finite energy plane with Hofer energy \( E_{\text{Hofer}}(v) \leq E \) with respect to \( \alpha_0 \) which can be obtained as in Section 5.1 Case 1 or Case 2(a).

Set

\[
T^1 := \mathbb{R}/2\pi\mathbb{Z}.
\]

Using the conformal map \( \mathbb{R} \times T^1 \to \mathbb{C} \setminus \{0\} \) given by \( (s, t) \mapsto e^{s+it} \) we regard \( v \) as a finite energy cylinder

\[
v : \mathbb{R} \times T^1 \longrightarrow \mathbb{R} \times M'.
\]

We claim that \( |\nabla v|_{g_0'} \) is bounded globally on \( \mathbb{R} \times T^1 \).
Arguing by contradiction as in [38, Proposition 30] we select sequences $\varepsilon_\nu \to 0$ in $(0, \infty)$, $z_\nu$ on the universal cover $\mathbb{R} \times \mathbb{R} \equiv \mathbb{C}$ with $|z_\nu| \to \infty$, and $R_\nu := |\nabla v(z_\nu)|_g^0$, such that for a subsequence we have $R_\nu \varepsilon_\nu \to \infty$ and $|\nabla v|_g^0 \leq 2R_\nu$ on $B_{R_\nu}(z_\nu)$ according to the Hofer lemma [38, Lemma 26]. Writing $v = (b, h)$ let $\varphi_\nu \in G$ be a sequence of deck transformations whose inverse sends $h(z_\nu)$ into the closure of the fundamental domain $D$. We define $u_\nu = (a_\nu, f_\nu)$ by

$$a_\nu(z) := b(z + z/R_\nu) - b(z)$$

and

$$f_\nu(z) := \varphi_\nu^{-1} \left( h(z + z/R_\nu) \right)$$

for all $z \in B_{R_\nu}(z_\nu)$. As observed in Section 5.1 Case 1 we have that $u_\nu(0) \in \{0\} \times D$, $|\nabla u_\nu(0)|_g^0 = 1$, and that $|\nabla u_\nu|_g^0 \leq 2$ uniformly on $B_{R_\nu}(0)$. Moreover, $u_\nu$ is $J_0^-$-holomorphic with respect to the almost complex structure $J_\nu = F_\nu^* J_0$ on $\mathbb{R} \times M'$, where $F_\nu = b(z_\nu) \times \varphi_\nu$, and the Hofer energy with respect to $\alpha_0' = \varphi_\nu^* \alpha_0$

$$\sup_{\tau} \int_{B_{R_\nu}(0)} u_\nu^* |d\alpha_0'| < E$$

is uniformly bounded, where the supremum is taken over all smooth increasing functions $\tau : \mathbb{R} \to [0, 1]$. Similarly,

$$\int_{B_{R_\nu}(0)} f_\nu^* |d\alpha_0| = \int_{B_{R_\nu}(0)} h^* |d\alpha_0| \to 0.$$ 

Additionally, we can assume that a subsequence of $(\alpha_\nu', J_\nu')$ converges in $C_\infty^0 \to (\alpha_\infty, J_\infty)$. Using the argumentation from Section 5.1 Case 1 we find a $C_\infty^0(\mathbb{C})$-converging subsequence $u_\nu \to u$, where $u$ is a non-constant $J_\infty$-holomorphic finite energy plane of Hofer energy

$$\sup_{\tau} \int_{\mathbb{C}} u^* |d\alpha_\infty| \leq E.$$ 

For the contact area we obtain

$$\int_{\mathbb{C}} f^* |d\alpha_\infty| = 0$$

writing $u = (a, f)$ because for all $k \in \mathbb{N}$

$$\int_{B_k(0)} f^* |d\alpha_\infty| = \lim_{\nu \to \infty} \int_{B_k(0)} f_\nu^* |d\alpha_\nu'| \leq \lim_{\nu \to \infty} \int_{B_{R_\nu}(0)} f_\nu^* |d\alpha_\nu'| = 0$$

by Fatou’s lemma. Using [38, Lemma 28], which holds for non-compact contact manifolds $(M', \alpha_\infty)$ as well, this is a contradiction to $u$ being non-constant. Therefore, the gradient $|\nabla v|_g^0$ of the finite energy cylinder $v = (b, h)$ is globally bounded.

Modifying the proof of [38, Theorem 31] we choose a sequence $\varphi_\nu \in G$ such that $\varphi_\nu^{-1}$ maps $b(\nu, 0)$ into the closure of $D$. We define $u_\nu = (a_\nu, f_\nu)$ via

$$a_\nu(s, t) := b(s + \nu, t) - b(\nu, 0)$$

and

$$f_\nu(s, t) := \varphi_\nu^{-1} \left( h(s + \nu, t) \right).$$

Observe that $u_\nu(0, 0) \in \{0\} \times D$ and that $u_\nu$ is $J_0^-$-holomorphic with respect to the almost complex structure $J_\nu = F_\nu^* J_0$ on $\mathbb{R} \times M'$ for $F_\nu = b(\nu, 0) \times \varphi_\nu$. The gradient
removal of boundary singularity. Because \( J_\nu^c \) is associated to \( \alpha_\nu^c = \varphi_\nu^c \alpha_0 \) we remark that the Hofer energy satisfies

\[
\sup \int_{\mathbb{R} \times T^1} u_\nu^* \, d(\tau \alpha_0^\nu) \leq E
\]

for all \( \nu \in \mathbb{N} \). Moreover, for given \( k \in \mathbb{N} \)

\[
\int_{[-k,k] \times T^1} f_\nu^* \, d\alpha_0^\nu = \int_{[-k,r_k+k+\nu] \times T^1} h^* \, d\alpha_0 \rightarrow 0
\]

as \( \nu \) tends to \( \infty \) because the contact area \( h^* \, d\alpha_0 \) is non-negative (see Section 3.1) and the total integral

\[
\int_h h^* \, d\alpha_0 \leq E
\]

is bounded by the Hofer energy. Similarly,

\[
\int_{(0) \times T^1} f_\nu^* \, d\alpha_0^\nu = \int_{(\nu) \times T^1} h^* \, d\alpha_0 = \int_{(-\infty,\nu) \times T^1} h^* \, d\alpha_0
\]

tends to the contact area \( \int_h h^* \, d\alpha_0 \) as \( \nu \) tends to \( \infty \). Assuming that a subsequence of \( (\alpha_0^\nu, J_\nu^c) \) converges to \( (\alpha_\infty, J_\infty) \) in \( C_\infty \) we find as in Section 5.1 Case 1 a subsequence \( u_\nu \rightarrow u \) that converges to a non-constant \( J_\infty \)-holomorphic finite energy cylinder \( u = (a, f): \mathbb{R} \times T^1 \rightarrow \mathbb{R} \times M' \) of Hofer energy

\[
\sup \int_{\mathbb{R} \times T^1} u^* \, d(\tau \alpha_\infty) \leq E
\]

vanishing contact area

\[
\int_{\mathbb{R} \times T^1} f^* \, \alpha_\infty = 0
\]

and action

\[
\int_{(0) \times T^1} f^* \, \alpha_\infty = \int_h h^* \, d\alpha_0
\]

which by [38, Lemma 28] does not vanish. Further, the gradient \( |\nabla u_\nu|_{g_0^\nu} \) is globally bounded on \( \mathbb{R} \times T^1 \). As in the last part of the proof of [38, Theorem 31] one shows that a subsequence of \( f_\nu(0, \cdot) \) converges in \( C_\infty \) to a contractible periodic \( \alpha_\infty \)-Reeb orbit whose \( \alpha_\infty \)-action is equal to the contact area \( \int_h h^* \, d\alpha_0 \) of \( v = (b, h) \).

5.3. Removal of boundary singularity. In Section 5.1 Case 2(b) we obtained a non-constant \( J \)-holomorphic finite energy half-plane \( v \) of Hofer energy \( E_{\text{Hofer}}(v) \leq E \) with respect to \( \alpha \). Let \( S \) be the strip \( \mathbb{R} \times [0, \pi] \), which is conformally equivalent to \( \mathbb{H} \setminus \{0\} \) via \( (s, t) \mapsto e^{s+it} \). Using this identification we regard \( v \) as finite energy strip

\[
v: (S, \partial S) \rightarrow (\mathbb{R} \times M', \{0\} \times K_L).
\]

We claim that \( |\nabla v|_{g_0} \) is bounded globally on \( S \).

Arguing by contradiction as in [38, Theorem 32] we obtain with help of the Hofer lemma [38, Lemma 26] \( \varepsilon_\nu \rightarrow 0 \) in \( (0, \infty) \), \( z_\nu = x_\nu + iy_\nu \) on the strip \( S \) with \( |x_\nu| \rightarrow \infty \), and \( R_\nu := |\nabla v(z_\nu)|_{g_0^\nu} \), such that a subsequence satisfies \( R_\nu \varepsilon_\nu \rightarrow \infty \) and \( |\nabla v|_{g_0^\nu} \leq 2R_\nu \) on \( B_{\varepsilon_\nu}(z_\nu) \). Furthermore, we can assume that \( R_\nu y_\nu \rightarrow g_0 \in [0, \infty] \) and \( R_\nu (\pi - y_\nu) \rightarrow g_\pi \in [0, \infty] \). If both \( g_0 \) and \( g_\pi \) equal \( \infty \) we can argue as in the first part of Section 5.2 to show that a rescaled sequence \( u_\nu \) converges in \( C_\infty \) to a non-constant \( J_\infty \)-holomorphic finite energy plane with respect to \( \alpha_\infty \) that has vanishing contact area. By [38, Lemma 28] this is a contradiction. If at least one
of the limits $\varrho_0$ or $\varrho_\pi$ is finite, where in the second case we precompose $v$ with $z \mapsto -(z - 1)$, we repeat the argumentation from Section 5.1 Case 2(b), i.e. the rescaled sequence $u_\nu$ converges in $C^\infty_{\text{loc}}(\mathbb{H})$ to a non-constant $J_\infty$-holomorphic finite energy half-plane with respect to $\alpha_\infty$ with vanishing contact area $\int_\mathbb{H} u^*d\alpha_\infty = 0$. This is a contradiction as shown in the first half of the proof of \cite[Theorem 32]{38}. Hence, the gradient $|\nabla v|_{g_0}$ of the finite energy strip $v$ is globally bounded.

Finally, with the second half of the proof of \cite[Theorem 32]{38} (cf. \cite[Lemma 6.2]{30}) we see that $v$ extends to a holomorphic disc

$$v: (\mathbb{D}, \partial \mathbb{D}) \longrightarrow (\mathbb{R} \times M', \{0\} \times K_L)$$

after a suitable conformal change of the parametrization.

5.4. Aperiodicity and Gromov convergence. We summarize the results of the proceeding section. Let $(\pi: M' \rightarrow M, \alpha, \omega, g)$ be a virtually contact structure over a closed connected manifold $M$. Denote by $J$ the associated almost complex structure on $\mathbb{R} \times M'$ that is given by the construction in Section 2.3 and Section 3.1. Let $L \subset M'$ be a submanifold such that $\{0\} \times L$ maximally $J$-totally real in $\mathbb{R} \times M'$. Let

$$u_\nu = (u_\nu, f_\nu): (\mathbb{D}, \partial \mathbb{D}) \longrightarrow (\mathbb{R} \times M', \{0\} \times L)$$

be a sequence of $J$-holomorphic discs with boundary in an open relatively compact subset $K_L$ of $L$ such that the Hofer energy

$$E_{\text{Hofer}}(u_\nu) := \sup_\mathbb{D} \int_0^1 u_\nu^*d(\tau \alpha)$$

is uniformly bounded by $E > 0$ for all $\nu \in \mathbb{N}$, where the supremum is taken over all smooth increasing functions $\tau: \mathbb{R} \rightarrow [0, 1]$.

A closed characteristic on $(M, \omega)$ is a compact leaf of the 1-dimensional foliation $\text{ker}\omega$. Taking any orientation on a closed characteristic it represents a homology class. If a non-trivial multiple of a closed characteristic admits a contractible parametrization by a loop we will say that the closed characteristic is contractible.

Proposition 5.4.1. In the situation described above we assume that the covering $\pi$ is regular and that the $C^3$-norm of $\alpha$ defined in Section 4.1 is finite. If $(M, \omega)$ has no contractible closed characteristic, then $u_\nu$ has a Gromov converging subsequence that converges to a stable $J$-holomorphic disc with boundary on $K_L \subset L$ whose underlying bubble tree consists of discs only.

Proof. First we will prove the proposition under the additional assumption that all $C^k$-norms of $\alpha$ are bounded. By Proposition 4.3.1 this implies that for any sequence $\varphi_\nu \in G$ a in $C^\infty_{\text{loc}}$ converging subsequence of $\alpha_\nu = \varphi_\nu^*\alpha$ can be selected. The associated subsequence of almost complex structures $J_\nu$ on $\mathbb{R} \times M'$ converges in $C^\infty_{\text{loc}}$ too, see Lemma 4.5.1. The analogues convergence statement with $(\alpha, J)$ replaced by a resulting limit $(\alpha_0, J_0)$ of a subsequence of $(\alpha_\nu, J_\nu)$ holds true as well, see Remark 4.5.2. Observe that any periodic $\alpha_\infty$-Reeb orbit, which could be obtained as in Section 5.2 is a closed characteristic of $\omega = \omega'$ and, hence, projects to a closed characteristic of $\omega$ via $\pi$. Therefore, in the bubbling off analysis in Section 5.1 only Case 2(b) can occur. By Section 5.3 any resulting finite energy half-plane extends to a $J$-holomorphic disc and has Hofer energy less than or equal to $E$, which in the case of a disc equals the contact area. In fact, the contact areas of bubbled finite energy discs are uniformly bounded from below as a bubbling off argument as in \cite[Lemma 35]{38} and modified as in Sections 5.1, 5.2 and 5.3 shows.
Therefore, we can prove the proposition with the following arguments: As remarked at the beginning of Section 5.1 we have to rule out that there exists a sequence \( \zeta_\nu \) in \( D \) such that the minimum of \( a_\nu \) is attained for all \( \nu \in \mathbb{N} \) and \( a_\nu(\zeta_\nu) \to -\infty \) as \( \nu \to \infty \) for a subsequence. By the maximum principle we can assume that \( \zeta_\nu \) is contained in the interior \( B_1(0) \) of the unit disc \( D \). Because we are free to precompose \( u_\nu \) by a Möbius transformation we can assume that \( \zeta_\nu = 0 \) for all \( \nu \). By the preliminary remarks of the proof we have that all accumulation points \( z_0 \) of sequences \( z_\nu \) of bubbling points of \( u_\nu \) are contained in the boundary \( \partial D \).

In fact, there are only finitely many of them. Indeed, this is because the contact area \( \int_D f_\nu^*d\alpha \) of \( u_\nu \) is uniformly bounded by the Hofer energy of \( u_\nu \) and, hence, by \( E \). On the other hand the contact area of all possible bubbling discs is uniformly bounded from below. As the contact area is additive the arguments used to prove convergence modulo bubbling in [25, Section 2.5] carry over to the present context as in [38, pp. 542/43]. In other words, there are boundary points \( z_1^0, \ldots, z_N^0 \) of \( D \) such that for all \( \epsilon > 0 \) the restriction of \( |\nabla u_\nu|_{g_0} \) to \( D \setminus \bigcup_{j=1}^N B_\epsilon(z_j^0) \) is uniformly bounded. Using a mean value argument to obtain \( C^0 \)-bounds and [46, Theorem B.4.2] we find a subsequence of \( u_\nu \) that converges in \( C^\infty_{\text{loc}} \) on the deleted disc \( D \setminus \{z_1^0, \ldots, z_N^0\} \). This contradicts \( a_\nu(0) \to -\infty \). This proves the proposition under the assumption that \( \alpha \) is bounded with respect to all \( C^k \)-norms.

In order to obtain the proposition under the weaker assumption to have only \( C^3 \)-bounds on \( \alpha \) we observe that we have to select converging subsequences of \( (\alpha_\nu, J_\nu) \) precisely in Case 1 and Case 2(a) in Section 5.1 This involves the Arzelà–Ascoli argument and drops regularity by 1, see Corollary 4.4.2 and Remark 4.5.2. In order to obtain global gradient bounds for finite energy cylinders in Section 5.2 we have to select converging subsequences of \( (\alpha_\nu^0, J_\nu^0) \) via an Arzelà–Ascoli argument as discussed dropping regularity by 1 once more. The elliptic convergence holds for almost complex structures \( J_\nu \) that converge in \( C^1_{\text{loc}} \) resulting in \( C^1_{\text{loc}} \)-converging subsequences of \( u_\nu \), see [46, Remark B.4.3]. Because finite energy planes of class \( C^3 \) converge to periodic \( C^1 \)-orbits by the arguments in [38, Theorem 31] we see that we can work with an a priori \( C^3 \)-bound. \( \square \)

6. Contractible closed characteristics

In this section we will give the main applications of the compactness results of Section 5 in view of our periodicity questions of magnetic flows. The proofs of Theorem 1.2 and 1.3 are given.

6.1. Germs of holomorphic discs. We consider a non-trivial virtually contact structure \((\pi: M' \to M, \alpha, \omega, g)\) over a closed connected manifold \( M \) together with the associated almost complex structure \( J \) on \( \mathbb{R} \times M \) that is given by the construction in Section 2.3 and Section 3.1. The induced contact structure on \( M' \) defined by \( \alpha \) is denoted by \( \xi \).

**Theorem 6.1.1.** If the \( C^3 \)-norm of \( \alpha \) is finite and if \((M', \xi)\) is a 3-dimensional overtwisted contact manifold, then \((M, \omega)\) admits a contractible closed characteristic.
Proof. We can assume that the covering $\pi$ is regular as we always can pass to the universal covering. Denote by $D$ an overtwisted disc in $(M', \xi)$ such that the characteristic foliation $D_\xi$ has a unique singularity and a unique closed leaf given by the boundary $\partial D$. Denote by $e$ the singularity of $D_\xi$. Let $U$ be a open ball neighbourhood of $e$ whose closure in $M'$ is compact. Let $J_U$ be an almost complex structure on $\mathbb{R} \times M'$ that is translation invariant, sends $\partial_t$ to $R$, and restricts to a complex structure on $(\xi, \omega')$ such that $J_U$ equals $J$ in a neighbourhood of $\mathbb{R} \times (M' \setminus U)$ and allows a local $J_U$-holomorphic Bishop disc family emerging from $e$ in the sense of [38, Section 4.2] or [39, Section 3.1]. One considers the moduli space of all $J_U$-holomorphic discs with three marked points geometrically fixed by three mutually distinct leaves of $D_\xi$ not being $\partial D$ that are homologous relative $D \setminus \{e\}$ to one of the Bishop discs, see [38, 39]. According to automatic transversality, positivity of intersections, and the relative adjunction inequality as worked out in [25], the evaluation map from the moduli space to one of the distinguished leaves is a local diffeomorphism under which the Bishop discs have unique preimages. By E. Hopf’s boundary version of the maximum principle all holomorphic discs with boundary on the totally real punctured disc $D \setminus \{e\}$ are transverse to $D_\xi$, see [38, 39]. In particular, no holomorphic disc can touch the boundary $\partial D$. Therefore, the moduli space cannot be compact as this would imply surjectivity of the evaluation map.

Arguing indirectly we assume that $(M, \omega)$ has no contractible closed characteristic so that $(M', \alpha)$ does not have a contractible periodic Reeb orbit. We claim that under this assumption the above moduli space will be compact leading to the desired contradiction. A uniform Hofer energy bound is given by the $\frac{1}{2} |\omega'|$-area of $D$, see [38, Lemma 33]. Compactness is a consequence of Proposition 5.4.1 with the following modifications: In Proposition 3.6.1 one compares the distance to the universal covering. Denote by $e$ the singularity of $D_\xi$. Let $U$ be a open ball neighbourhood of $e$ whose closure in $M'$ is compact. Let $J_U$ be an almost complex structure on $\mathbb{R} \times M'$ that is translation invariant, sends $\partial_t$ to $R$, and restricts to a complex structure on $(\xi, \omega')$ such that $J_U$ equals $J$ in a neighbourhood of $\mathbb{R} \times (M' \setminus U)$ and allows a local $J_U$-holomorphic Bishop disc family emerging from $e$ in the sense of [38, Section 4.2] or [39, Section 3.1]. One considers the moduli space of all $J_U$-holomorphic discs with three marked points geometrically fixed by three mutually distinct leaves of $D_\xi$ not being $\partial D$ that are homologous relative $D \setminus \{e\}$ to one of the Bishop discs, see [38, 39]. According to automatic transversality, positivity of intersections, and the relative adjunction inequality as worked out in [25], the evaluation map from the moduli space to one of the distinguished leaves is a local diffeomorphism under which the Bishop discs have unique preimages. By E. Hopf’s boundary version of the maximum principle all holomorphic discs with boundary on the totally real punctured disc $D \setminus \{e\}$ are transverse to $D_\xi$, see [38, 39]. In particular, no holomorphic disc can touch the boundary $\partial D$. Therefore, the moduli space cannot be compact as this would imply surjectivity of the evaluation map.

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Theorem 6.1.2. If the $C^3$-norm of $\alpha$ is finite and if $M$ is a 3-dimensional manifold with non-trivial $\pi_2 M$, then $(M, \omega)$ admits a contractible closed characteristic.

Proof. Observe that $\pi_2 M'$ is non-trivial as well. By the sphere theorem there exists a non-contractible embedding of a two sphere into $M'$ whose image we denote by $S$. By Theorem 6.1.1 it suffices to consider the case of a tight contact structure $\xi$ so that we can assume that the characteristic foliation of $S$ has precisely two singular points $e^+$ and $e^-$, which are elliptic, but has no limit cycle, see [27, Section 4.6]. One uses a filling by holomorphic discs argument as in the proof of Theorem 6.1.1 based on two Bishop disc families emerging from $e^+$ and $e^-$, respectively. A contradiction to the non-existence of contractible closed characteristics can be obtained as compactness of the moduli space corresponding to the Bishop families would result in a 3-ball inside $M'$ that is bounded by $S$, see [38] and cf. [28], which is not possible. □
An embedded \((2n - 2)\)-sphere \(S\) in \((M', \xi)\) is called standard provided that the restriction of the contact form \(\alpha\) to \(TS\) equals the restriction of \(\frac{1}{2}(xdy - ydx)\) to \(TS\), where we identify \(S\) with the unit sphere \(S^{2n-2}\) in \(\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}\) equipped with coordinates \((w, x, y)\), see [31] p. 326/27.

**Theorem 6.1.3.** Let \(n \geq 3\). If the \(C^\infty\)-norm of \(\alpha\) is finite and if \((M', \xi)\) is a \((2n-1)\)-dimensional contact manifold that contains a standard sphere \(S\) whose class \([S]\) in \(\pi_{2n-2}M'\) is non-trivial, then \((M, \omega)\) admits a contractible closed characteristic.

**Proof.** We equip \(\mathbb{R} \times S^{2n-2}\) with the standard contact structure defined by the identification with the upper boundary of an index 1 handle, see [31] p. 329]. According to [20] Proposition 6.4 we find a contact embedding of \((-2,2) \times S^{2n-2}\) into \((M', \xi)\) mapping \([0] \times S^{2n-2}\) onto \(S\). We identify \((-2, 2) \times S^{2n-2}\) with its image \(U\) in \(M'\). Let \(\alpha_1\) be the \(\xi\)-defining contact form on \(M'\) that coincides with \(\alpha\) in a neighbourhood of \(M' \setminus U\) and with the \(\xi\)-defining contact form on the upper boundary of the 1-handle in a neighbourhood of \([-1, 1] \times S^{2n-2}\) by convex interpolation. Performing a reversed contact surgery along the belt sphere \(S\) we see that \((M', \xi)\) is the result of an index 1 surgery on a contact manifold \((N, \eta)\). The complement of the surgery region in \(N\) equals \(M' \setminus U\) and \(\eta\) admits a defining contact form that is given by \(\alpha\) on \(M' \setminus U\).

This places us in the situation of [31] Chapter 2. We assume that the index 1 surgery described in [31] Section 2.1 is realized in such a way that the contact form that corresponds to the thin handle equals \(\alpha_1\). This eventually requires a global contactomorphism with support in \(U\) along which \(\alpha\) is assumed to be pulled back. The contact form \(\alpha_R\) corresponding to the thick handle can be assumed to coincide with \(\alpha\) in a neighbourhood of \(M' \setminus U\). The scaling factor \(R\) is chosen according to [31] Lemma 6]. All this eventually results in a multiplication of \(\alpha\) by a large constant which we are free to ignore in the following.

We consider the manifold \(W\) obtained by gluing the region between the thin and the thick handle inside the surgery model (see [31] Section 2.1) to \((-\infty, 0] \times M'\) along the thin handle being contained in \([0] \times U\). Similarly to [31] Section 2.2 we equip \(W\) with the symplectic form \(d\lambda\), where \(\lambda\) on the model region is given by the dual of the Liouville vector field on [31] p. 330]. On \((-\infty, 0] \times M'\) we define \(\lambda\) as follows: Because \(\alpha\) and \(\alpha_1\) define the same contact structure we find a function \(h\) on \(M'\) that has a posteriori compact support in \(U\) and satisfies \(\alpha = e^h\alpha_1\). The Liouville form \(\lambda\) at \((t, p)\) on \((-\infty, 0] \times M'\) by definition is given by \(e^{t+b(t)}h\alpha_1\) for a smooth function \(b\) on \(\mathbb{R}\) that equals 0 on \([0, \infty)\), 1 on \((-\infty, t_0]\) for a suitable \(t_0 < 0\), and satisfies \(b'(t)h > -1\) on \(M'\) for all \(t\).

We remark that the interior of the complement of \((t_0, 0) \times U\) has two connected components. On the unbounded component \(\lambda\) is given by \(e^t\alpha_1\), on which the almost complex structure \(J\) is defined. We define a compatible almost complex structure \(J_U\) on \((W, d\lambda)\) that extends \(J\) such that \(J_U\) equals the complex structure on the model region, see [31] (J2) p. 331], is chosen to be generic on \((t_0, 0) \times U\) in the sense of [31] Section 3.5], and makes the boundary of \(W\), which is equal to a copy of \(M'\) with \(\lambda|_{\partial W} = \alpha_R\), \(J_U\)-convex. In other words we are in a situation as described in [31] Chapter 3] besides the fact that the entire construction is based on a non-compact contact manifold \((M', \xi)\).

We define a moduli space \(\mathcal{W}\) of holomorphic discs as in [31] Section 3.2]. All properties formulated in [31] Chapter 3] hold true with the following modification in the compactness argument: Assuming aperiodicity of the \(\alpha\)-Reeb flow on the
universal cover $M'$, to which we without loss of generality can pass, we can prove compactness as in [31 Section 3.4] for all $J_U$-holomorphic discs belonging to $W$ that have uniformly bounded projections to $M'$, cf. [30 Chapter 6]. Otherwise one argues as at the end of the proof of Theorem 6.1.1. In other words, $W$ is a $(2n-3)$-dimensional compact manifold with boundary.

If $W$ is not connected, we replace it by its connected component that contains the standard holomorphic discs, see [31 Section 3.1]. A deformation of the evaluation map as described in [31 Chapter 4 and Section 5.1] yields a continuous map

$$f: \left( W \times \mathbb{D}, \partial(W \times \mathbb{D}) \right) \to (M', S)$$

transverse to the belt sphere $S$ whose restriction to the boundary of $W \times \mathbb{D}$ has degree 1 and which maps $W \times \{1\}$ to a $(2n-3)$-dimensional cell contained in $S$. A first consequence is that the homology class represented by $S$ in $M'$ must vanish so that $S$ separates $M'$. Denote the closures of the connected components of $M' \setminus S$ by $M_1$ and $M_2$. Notice, that at least one of the $M_i$ has to be non-compact as $M'$ is not compact by our assumption on the non-triviality of the virtually contact structure. We denote by $V_i$ the preimage of $M_i$ under $f$ and observe that the restriction $f_i$ of $f$ to $V_i$ has a well-defined mapping degree. Counting the number of preimages with signs of generic points near $S$ of different components of $M' \setminus S$ we obtain

$$\deg f_1 - \deg f_2 = \pm 1$$

as in [31 Lemma 8]. Therefore, after eventually changing the notation we get that $M_1$ is compact and the degree of $f_1$ equals $\pm 1$. By [31 Proposition 11], which uses that $W \times \{1\}$ gets mapped to a cell, we infer that $M_1$ is simply connected. Moreover, with [31 Proposition 12] $M_1$ has the homology of a ball. Because $M_1$ is bounded by a $(2n-2)$-sphere for $n \geq 3$ the $h$-cobordism theorem implies that $M_1$ has to be a ball, see [45 Proposition A on p. 108]. Hence, the belt sphere $S$ is contractible in $M'$. This contradiction shows that the $\alpha$-Reeb flow on $M'$ cannot be aperiodic.

Remark 6.1.4. Let $(M, \omega)$ be a closed connected odd-symplectic manifold that admits a non-trivial virtually contact structure $(\pi: M' \to M, \alpha, \omega, g)$. Assume that $(\pi: M' \to M, \alpha, \omega, g)$ is obtained by a covering connected sum (as defined in [60 Section 2.2]) up to a multiplication of $\alpha$ by a positive function that is different from 1 only on a compact subset of $M'$. We assume that $\alpha$ is $C^3$-bounded, cf. Remark 6.3.2. If the induced connected sum of the underlying manifold $M$ is non-trivial meaning that none of the summands is a homotopy sphere, then there exists a contractible closed characteristic on $(M, \omega)$.

Indeed, the case $n = 2$ follows with Theorem 6.1.2 denoting the dimension of $M$ by $2n - 1$, cf. [31 Theorem 1]. For the case $n \geq 3$ we can assume the situation of the proof of Theorem 6.1.3. Under the assumption of aperiodicity one shows that any belt sphere of the covering connected sum bounds a $(2n-1)$-dimensional disc. With the arguments of [30 Proposition 1.6 and Proposition 3.10] the existence of a closed characteristic on $(M, \omega)$ follows.

We say that the standard contact handle of index $k$ admits a contact embedding into $(M', \xi)$ if $(M', \xi)$ contains the image of the upper boundary $D^k \times S^{2n-1-k}$ of a symplectic handle $D^k \times D^{2n-k}$ under a suitable contact embedding as described in [27 Section 6.2] in the context of contact surgery. We call $\{0\} \times S^{2n-1-k}$ or its image the belt sphere of the handle. The index $k$ of the handle is subcritical provided that $k \leq n - 1$. 
Theorem 6.1.5. Let \( n \geq 3 \). If the \( C^3 \)-norm of \( \alpha \) is finite and if \((M', \xi)\) is a \((2n-1)\)-dimensional contact manifold that admits an embedding of the standard contact handle of subcritical index \( k \) with belt sphere \( S \) whose class \([S]\) in the oriented bordism group \( \Omega^{SO}_{2n-1-k}M' \) is non-trivial, then \((M, \omega)\) admits a contractible closed characteristic.

Proof. In view of [33, Theorem 1.4] one can argue as in the proof of Theorem 6.1.3. Under the assumption of aperiodicity of the \( \alpha \)-Reeb flow on the universal cover \( \tilde{M}' \) one shows with [33, Sections 3 and 7] that \( S \) is null-bordant via a deformation of the evaluation map defined on a surgered moduli space of holomorphic discs. \( \square \)

Remark 6.1.6. With the same reasoning and [33, Sections 5] it follows that in the situation of Theorem 6.1.5 \((M, \omega)\) admits a contractible closed characteristic provided that the class \( \{S\} \) of the belt sphere \( S \) is non-trivial in \( \pi_3M' \) if \( n = 3 \) and \( k = 2 \), resp., in \( \pi_4M' \) if \( n = 4 \) and \( k = 3 \).

Remark 6.1.7. The result of Theorem 6.1.5 is based on moduli spaces of holomorphic discs with boundary on a family of Legendrian open books in \((M', \xi)\). The involved arguments for the holomorphic curves in symplectisations of \( M' \) of the preceding section work equally well for Legendrian open books with boundary. Similarly to Theorem 6.1.1 and [6, 33, 42, 47] one shows: If the \( C^3 \)-norm of \( \alpha \) is finite and if \((M', \xi)\) admits a Legendrian open book with boundary, then \((M, \omega)\) admits a contractible closed characteristic. Such examples can be obtained with [60, Proposition 2.6.1 or Proposition 2.6.2].

6.2. Magnetic energy surfaces. Let \((Q, h)\) be a closed \( n \)-dimensional Riemannian manifold and denote by \( \tau: T^*Q \to Q \) the cotangent bundle of \( Q \). Using the Levi-Civita connection \( D^Q \) of \( h \) and the dual metric \( h^\ast \) we split the tangent bundle \( TT^*Q = H \oplus V \) of \( T^*Q \) into the horizontal and vertical distribution. A metric \( m \) on \( T^*Q \) is defined via

\[
m((v, a), (w, b)) := h(T\tau(v), T\tau(w)) + h^\ast(a, b)
\]

for all \((v, a), (w, b)\) in \( H \oplus V \). The Levi-Civita connection of \( m \) is denoted by \( D \). This turns \( \tau \) into a Riemannian submersion with totally geodesic fibers as is readily verified in Riemannian normal coordinates. Indeed, straight vector space lines contained in a cotangent fibre \( T^*_q Q \) minimize the length in the class of curves connecting two given sufficiently close points on \( T^*_q Q \).

On \( T^*Q \) we consider the twisted symplectic form \( \omega_\sigma = d\lambda + \tau^*\sigma \), where \( \lambda \) denotes the Liouville 1-form of \( \tau \) and \( \sigma \) is a magnetic 2-form on \( Q \), and the Hamiltonian function

\[
H = \frac{1}{2} |h^\ast|_{h^\ast}^2 + V \circ \tau,
\]

where \( V: Q \to \mathbb{R} \) is a potential function on \( Q \). For energies \( e > \max_Q V \) we denote the regular level set \( \{H = e\} \subset T^*Q \) by \( M \), which is equipped with the odd-symplectic form \( \omega = \omega_\sigma |_{TM} \) and the metric \( g = m|_{TM} \). The Levi-Civita connection on \((M, g)\) is denoted by \( \nabla \) and the second fundamental form of \((M, g)\) in \((T^*Q, m)\) by \( II \).

We assume that there exists a 1-form \( \theta \) on the universal cover \( \tilde{Q} \) such that the lift \( \mu^*\sigma = d\theta \) of the magnetic form along the universal covering map \( \mu: \tilde{Q} \to Q \) has primitive \( \theta \). To \( \tilde{Q} \) we lift the metric \( h \) via pull back \( \tilde{h} = \mu^*h \) so that \( \mu \) is turned into a local isometry. The induced Levi-Civita connection is denoted by \( \tilde{D}^Q \). In
the same way the Hamiltonian $H$ lifts to $\tilde{H}$ on $T^*\tilde{Q}$ defining $M'$ via $\{\tilde{H} = \epsilon\}$. Along the induced universal covering map $T^*\mu: T^*\tilde{Q} \to T^*Q$ we lift the geometry of $(T^*Q, m, D)$ to $(T^*\tilde{Q}, \tilde{m}, \tilde{D})$ turning $T^*\mu$ into a local isometry. This induces a metric $g' = \tilde{m}|_{TM'}$ on $M'$, which is the lifted metric, such that $\pi = T^*\mu|_{M'}$ is a local isometry. The Levi-Civita connection of $g'$ is denoted by $\nabla'$ and the second fundamental form of $(M', g')$ in $(T^*\tilde{Q}, \tilde{m})$ is denoted by $\Pi'$.

The restriction
\[ \alpha = (\tilde{\lambda} + \tilde{\tau}^*\theta)|_{TM'} , \]
where $\tilde{\lambda}$ denotes the Liouville 1-form of the cotangent bundle $\tilde{\lambda}$, serves as a primitive of $\omega' = \pi^*\omega$. By Proposition 2.4.1 the tuple
\[ (\pi: M' \to M, \alpha, \omega, g) \]
is a virtually contact structure for all energies $\epsilon > \sup_{\tilde{Q}} \tilde{H}(\theta)$ provided that $\|\theta\|_{C^0}$ is finite.

**Proposition 6.2.1.** Let $k \in \mathbb{N}$ be a natural number. If the $C^k$-norm of $\theta$ with respect to $(Q, \tilde{h}, D^Q)$ is finite, then the $C^k$-norm of $\alpha$ with respect to $(M', \tilde{g}', \nabla')$ is finite too.

**Proof.** We abbreviate
\[ \beta = \tilde{\lambda} + \tilde{\tau}^*\theta . \]
Because of the Gauß formula (cf. [15, Theorem 1.72]) we obtain for vector fields $X, Y$ on $M'$ suitably extended to $T^*\tilde{Q}$ that
\[ (\nabla_X\alpha)(Y) = (D_X\beta)(Y)|_{M'} + \beta(\Pi'(X,Y))|_{M'} . \]
For the extension we used a small tubular neighbourhood $\{\epsilon - \epsilon < \tilde{H} < \epsilon + \epsilon\}$ of $M'$ and the vector fields are required to be tangent to each level set of $\tilde{H}$. Moreover, $\Pi'$ stands for the $(1,2)$-tensor that is defined as in [15, (9.17)] and coincides with the second fundamental form restricted to each energy surface. Similarly,
\[ (\nabla'_X(\nabla'_\alpha))(Y_1, Y_2) \]
is in view of Section 4.1 the sum of
\[ (D_X(D\beta))(Y_1, Y_2)|_{M'} \]
and
\[ (D\beta)(\Pi'(X,Y_1), Y_2)|_{M'} + (D\beta)(Y_1, \Pi'(X,Y_2))|_{M'} \]
and
\[ \beta \circ \Pi' \left( \Pi'(X,Y_1), Y_2 \right)|_{M'} + \beta \circ \Pi' \left( Y_1, \Pi'(X,Y_2) \right)|_{M'} \]
as well as
\[ (D_X(\beta \circ \Pi'))(Y_1, Y_2)|_{M'} = (D_X\beta)(\Pi'(Y_1, Y_2))|_{M'} + \beta \left( (D_X \Pi')(Y_1, Y_2) \right)|_{M'} . \]
Inductively one finds that \((\nabla'^k\alpha)(X, Y_1, \ldots, Y_k)\) admits an analogous expression that involves $D$-covariant derivatives of $\beta$ up to order $k$ and $D$-covariant derivatives of $\Pi'$ up to order $k - 1$, cf. Appendix A for the third derivative. Invoking the local isometry $T^*\mu$, the $(1, \ell + 2)$-tensor $D\Pi'$ along $M'$ can be estimated along the compact manifold $M$ by $D^Q\Pi$, which is uniformly bounded, for all $\ell = 0, 1, \ldots, k - 1$.

Therefore, it is enough to estimate $D^\ell\beta$ on a disc subbundle $DT^*\tilde{Q}$ of $T^*\tilde{Q}$ that contains $M'$. Observe that the local isometry $T^*\mu$ from $(T^*\tilde{Q}, \tilde{m}, \tilde{D})$ to $(T^*Q, m, D)$
pulls the Liouville form $\lambda$ of $\tau$ back to the Liouville form $\tilde{\lambda} = (T^*\mu)^*\lambda$ of $\tilde{\tau}$. Hence, using Remark 4.1.2 we get $D\tilde{\lambda} = (T^*\mu)^*D\lambda$ for all $\ell$. This implies that the $C^k$-norm of $\tilde{\lambda}$ restricted to $DT^*\tilde{Q}$ is equal to the $C^k$-norm of $\lambda$ restricted to $DT^*Q$ relative to the respective geometries. As the latter is finite by the compactness of the disc bundle it remains to be shown finiteness of $\|\tilde{\tau}^*\theta\|_{C^k}$ on $(DT^*\tilde{Q}, \tilde{m}, \tilde{D})$.

We will estimate
\[ \gamma = \tilde{\tau}^*\theta \]
assuming $C^k$-bounds on $\theta$. According to the splitting into horizontal and vertical spaces the first covariant derivative $D\tilde{\gamma}$ on tangent vectors $(v, a), (w, b)$ in $\mathcal{H} \oplus \mathcal{V}$ at $u \in T^*\tilde{Q}$ equals to
\[ (D\tilde{\gamma})(v, w) + (D\tilde{\gamma})(v, b) + (D\tilde{\gamma})(a, w) + (D\tilde{\gamma})(a, b). \]
We will estimate each term of this expression separately.

Denote by $X, Y$ horizontal vector fields that are lifts along $T\tilde{\tau}$ of tangent vector fields $X\tilde{Q}, Y\tilde{Q}$ on $\tilde{Q}$. We obtain
\[ (D\tilde{\gamma})(X, Y)|_u = (D\tilde{\gamma})(X\tilde{Q}, Y\tilde{Q})|_{\tilde{\tau}(u)} \]
because $\gamma(Y) = \theta(Y\tilde{Q})$ is constant along the cotangent fibres and the horizontal part of $D\tilde{X}Y$ is projected to $D\tilde{X}\tilde{Q}Y\tilde{Q}$ via $T\tilde{\tau}$, see [15, p. 240]. Using the assumption of $\theta$ being $C^1$-bounded we infer that $D\tilde{\gamma}$ is finite on unit horizontal vectors $v, w$.

For a horizontal vector field $X$ and a vertical vector field $V$ we get
\[ (D\tilde{\gamma})(X, V) = -\theta(T\tilde{\tau}(A_XV)) \]
because $T\tilde{\tau}(V)$ vanishes identically. Here $A$ is the $(1, 2)$-tensor field on $T^*\tilde{Q}$ defined by decomposing $D\tilde{X}V$ orthogonally into the respective horizontal and vertical parts as in [15, Definition 9.20] measuring the non-integrability of the horizontal distribution. Because $\theta$ is a bounded 1-form on $\tilde{Q}$, $T\tilde{\tau}$ is an orthogonal projection, and $A$ is a geometric tensor of the lifted geometry of $(T^*\tilde{Q}, \tilde{m}, \tilde{D})$, the tensor field $D\gamma$ restricted to the chosen disc bundle $DT^*\tilde{Q}$ is bounded on mixed unit tangent vectors $v \in \mathcal{H}_u$ and $b \in \mathcal{V}_u$.

Similarly, for a vertical vector field $U$ and a lifted horizontal vector field $Y$
\[ (D\tilde{\gamma})(U, Y) = -\theta(T\tilde{\tau}(A_YU)) \]
because $\gamma(Y_u) = \theta(Y\tilde{Q})$ is constant along the cotangent fibres, so that an application of the vertical vector field $U$ vanishes, and by [15, Definition 9.21b] combined with the fact that $[Y, U]$ is vertical, see [15, p. 240]. Again, finiteness of the $C^0$-norm of $\theta$ implies that $\tilde{D}\gamma$ is bounded on mixed unit tangent vectors $a \in \mathcal{V}_u$ and $w \in \mathcal{H}_u$ for all $u \in DT^*\tilde{Q}$.

Finally, because $\gamma(V)$ vanishes identically and because the fibres of $\tilde{\tau}$ are totally geodesic we obtain with [15, 9.25a and 9.26] that
\[ (D\tilde{\gamma})(U, V) = 0 \]
for vertical vector fields $U, V$. Summarizing, we obtain that $\|\gamma\|_{C^1}$ is bounded by $2\|\theta\|_{C^1}(\|A\|_{C^0} + 1)$ on $DT^*\tilde{Q}$. Moreover, we get that
\[ (D\tilde{\gamma})(X \oplus U, Y \oplus V) \]
decomposes as
\[
(D^Q \theta)(T \tilde{\tau}(X), T \tilde{\tau}(Y)) = \theta(T \tilde{\tau}(A_X V)) - \theta(T \tilde{\tau}(A_Y U))
\]
for all lifted horizontal vector fields \(X, Y\) and vertical vector fields \(U, V\) on \(T^* \tilde{Q}\) with respect to \(H \oplus V\). In other words, the tensor \(\tilde{D}\gamma\) is equal to
\[
\tilde{D}\gamma = \tilde{D}^Q \theta \circ T \tilde{\tau} - \theta \circ T \tilde{\tau} \circ 2(A)
\]
by [15] 9.21 denoting the symmetrization of the tensor \(A\) by \(A\).

The second covariant derivative \(D^2\gamma\) can be computed in a similar fashion. For any tangent vector field \(X\) on \(T^* \tilde{Q}\) we consider \(\tilde{D}_X \tilde{D}\gamma\) read as \(\tilde{D}_X\) applied to the difference expression of \(\tilde{D}\gamma\). The minuend
\[
\tilde{D}_X \left( \tilde{D}^Q \theta \circ T \tilde{\tau} \right)
\]
can be treated as the first covariant derivative of \(\gamma\) analogously to the above considerations. The subtrahend can be computed according to the product rule for the composition \(\circ\) of tensors as
\[
\tilde{D}_X \left( \theta \circ T \tilde{\tau} \circ 2(A) \right) = \tilde{D}_X \left( \theta \circ T \tilde{\tau} \right) \circ 2(A) + \left( \theta \circ T \tilde{\tau} \right) \circ \tilde{D}_X \left( 2(A) \right).
\]
Because of \(\gamma = \tilde{\tau}^* \theta\) this sum can be discussed with the arguments for \(\gamma\) as above with the additional phenomenon that a covariant derivative of \(A\), which still is a geometric tensor, has to be taken into account. Inductively, one shows finiteness of \(\|\tilde{\tau}^* \theta\|_{C^k}\) on \((DT^* \tilde{Q}, \hat{m}, \hat{D})\) as claimed. \(\square\)

6.3. **Truncating the magnetic field.** We will continue the considerations and the use of the notation from Section 6.2. In the following we will recall the truncation construction from [60] Section 2.5: Let \(U\) be an embedded closed \(n\)-disc in \(Q\). The image of the origin is denoted by \(q \in Q\). We choose \(U\) so small such that the decomposition of \(\mu^{-1}(U)\) into the connected components \(U^p, p \in \mu^{-1}(q)\), results into a family of diffeomorphisms \(\mu|_{U^p}\) onto \(U\), which are in fact isometries. Let \(\chi\) be a cut off function on \(Q\) that is identically 1 on \(Q \setminus U\) and vanishes on a disc neighbourhood \(W\) of \(q\). We define a closed 2-form \(\sigma\) on \(Q\) that equals \(\sigma\) on \(Q \setminus U\) and \(d(\chi \sigma_U)\) on \(U\), where \(\sigma_U\) is a primitive on of \(\sigma|_{U}\). Notice, that \(\sigma\) and \(\sigma\) are cohomologous. In fact, \((\chi - 1)\sigma_U\) is a primitive of the difference \(\sigma - \sigma\). Therefore,
\[
\theta' := \theta + (\tilde{\chi} - 1)\mu^* \theta_U
\]
is a primitive of \(\mu^* \sigma\), where \(\tilde{\chi} := \mu^* \chi\). The restriction \(\theta'|_{W^p}\) of \(\theta'\) to all connected components \(W^p\) of \(\mu^{-1}(W)\) is closed, and, hence, exact. As in the proof of [60] Proposition 2.5.1 we select primitive functions \(f^p\) of \(\theta'|_{W^p} = df^p\) via the Poincaré Lemma. Define a primitive \(\hat{\theta}\) of \(\sigma\) to be the 1-form on \(\tilde{Q}\) that coincides with \(\theta'\) on \(\tilde{Q} \setminus \mu^{-1}(W)\) and with \(d(\tilde{\chi} f^p)\) on \(W^p\) for all \(p \in \mu^{-1}(q)\), where \(\chi_W\) is the \(\mu\)-pull back of a cut off function \(\chi_W\) that is equal to 1 on \(Q \setminus W\) and vanishes on \(W\). We call the pair \((\hat{\theta}, \tilde{\sigma})\) a truncation of \((\theta, \sigma)\).

**Lemma 6.3.1.** Let \(k \in \mathbb{N}\) be a natural number. If the \(C^k\)-norm of \(\theta\) with respect to \((\tilde{Q}, \hat{h}, \hat{D}^Q)\) is finite, then the \(C^k\)-norm of the truncation \(\hat{\theta}\) with respect to \((\tilde{Q}, \hat{h}, \hat{D}^Q)\) is finite too.
Proof. First of all observe that $\theta'$ is bounded in the $C^k$-norm as it is obtained from $\theta$ by adding the pull back of the compactly supported 1-form $(\chi - 1)\theta_U$ on $Q$ along the local isometry $\mu$. That $\hat{\theta}$ is bounded in $C^0$ was verified in [60, Proposition 2.5.1]. In order to estimate the $\ell$-th covariant derivative of $\hat{\theta}$ for $\ell \leq k$ it suffices to do so for the restrictions $d(\tilde{\chi}_W f^p)$ of $\hat{\theta}$ to $W^p$ for all $p \in \mu^{-1}(W)$. In view of Remark 4.1.1 we point out that the $(\ell + 1)$-th covariant derivative of the products $\tilde{\chi}_W f^p$ equals the sum of tensor products

$$\sum_{j=0}^{\ell+1} (\ell+1-j) (\tilde{D}^Q)^j \tilde{\chi}_W \otimes (\tilde{D}^Q)^{\ell+1-j} f^p$$

according to the Leibniz rule. All covariant derivatives of $\tilde{\chi}_W$ are bounded in view of Remark 4.1.2. Moreover, a $C^0$-bound for $f^p$ can be obtained as in [60, Proposition 2.5.1]. For $\ell + 1 - j \geq 1$ we have by Remark 4.1.1

$$(\tilde{D}^Q)^{\ell+1-j} f^p = (\tilde{D}^Q)^{\ell-j} \theta'|_{W^p},$$

which is bounded by the $C^k$-norm of $\theta'$ and, hence, by the $C^k$-norm of $\theta$. □

Remark 6.3.2. If $||\theta||_{C^k}$ is finite, then the primitive $\hat{\alpha} = (\hat{\lambda} + \check{\tau}^* \hat{\theta})|_{TM'}$ of $\pi^* \hat{\omega}$ is $C^k$-bounded, where $\hat{\omega} := \omega\hat{\sigma}|_{TM}$, and defines a virtually contact structure $$(\pi: M' \to M, \hat{\alpha}, \hat{\omega}, g)$$

for all energies $e > \sup_{\tilde{Q}} \tilde{H}(\hat{\theta})$ by [60] Proposition 2.4.1| that is somewhere contact in the sense of [60] Section 2.5. With the help of the covering connected sum described in [60] Section 2.2| a connected sum of somewhere contact virtually contact structures is defined. Given two $C^k$-bounded somewhere contact virtually contact structures the resulting virtually contact structure on the connected sum will be $C^k$-bounded too.

6.4. Classical Hamiltonians and magnetic fields. In view of Section 6.2 let $(Q, h)$ be a $n$-dimensional closed Riemannian manifold that splits off a product of closed hyperbolic surfaces. An $\mathbb{R}$-linear combination of the corresponding area forms coming from the surface factors defines a magnetic form $\sigma$ on $Q$. The lift of $\sigma$ to the universal cover $\tilde{Q}$ of $Q$ has a primitive 1-form $\tilde{\theta}$ that is the corresponding linear combination of $\frac{1}{2} dx$ for $(x, y) \in H^+$, which is $C^k$-bounded for all $k \in \mathbb{N}$ by the computations in Section 4.3. If we replace $\sigma$ by a 2-form on $Q$ whose cohomology class is contained in the subspace of $H^2_{dR} Q$ spanned by the classes of the area forms coming from the surface factors, we obtain a magnetic form whose lift admits a $C^\infty$-bounded primitive $\theta$. This is because adding an exact 2-form on $Q$ keeps the property of having a bounded primitive on $\tilde{Q}$. As discussed in Section 6.2 this leads to a rich class of examples of virtually contact type energy surfaces in classical mechanics with magnetic fields.

We would like to describe a particular class of Hamiltonian systems on $(Q, h)$. Let $\sigma$ and $\theta$ as above and consider a Morse function $V$ on the manifold $Q$. By composing $V$ with a strictly increasing function we can arrange that:

1. $V$ has a unique local maximum which is assumed to be positive.
All critical values of $V$ that correspond to critical points of index $\leq n - 1$ are strictly smaller than
\[ \frac{1}{2}t_0^2, \]
where $t_0 := \max_{\tilde{Q}} |\theta|_{(\tilde{h})}$.  

Denote by $c_{n-1}$ the largest critical value of $V$ not equal to the maximum of $V$. Let $-v_0 < 0$ be a regular value of $V$ such that
\[ -v_0 \in \left( c_{n-1}, -\frac{1}{2}t_0^2 \right). \]

We require that $\sigma$ vanishes on the $n$-disc
\[ \{ V \geq -v_0 \} \subset Q \]
and that $\theta$ vanishes on
\[ \{ \tilde{V} \geq -v_0 \} \subset \tilde{Q}, \]
denoting by $\tilde{V}$ the lift of $V$ to $\tilde{Q}$.

Notice that item (3) can be achieved with truncation from Section 6.3. It follows from [60, Section 3.3] that $M = \{ H = 0 \}$ is of virtually contact type and somewhere contact. We remark that the Mañé critical value of the magnetic system, which is greater or equal than the maximum of $V$, is positive.

**Proof of Theorem 1.2.** We can assume the situation of the proceeding section for $n = 2$. By [60] Theorem 1.2 the 3-dimensional energy surface $M$ has non-vanishing $\pi_2 M$. The existence of a periodic solution of the equations of motion that is contractible in $\{ V \leq 0 \}$ follows from Theorem 6.1.2 by projecting the obtained contractible periodic solution of $X_H$ on $M$ to $Q$ via $\tau$. The solution is non-constant according to the equations of motion. Indeed, a constant zero energy solution would be contained in the regular level set $\{ V = 0 \}$ in $Q$, on which the magnetic form $\sigma$ vanishes, cf. [54, p. 135].

**Example 6.4.1.** We will describe magnetic Hamiltonian systems to which Theorem 1.2 applies such that the obtained solution inside $\{ V \leq 0 \}$ cannot stay entirely in the annulus $\{ -v_0 \leq V \leq 0 \}$, on which the magnetic form vanishes: Integrating gradient flow lines as in [37], p. 153] we find a diffeomorphism of $[-v_0, 0] \times S^1$ onto $\{ -v_0 \leq V \leq 0 \}$ such that the metric tensor $h_{ij}$ is given by a diagonal matrix with respect to the $(r, \theta)$-coordinates and such that $V(r, \theta) = r$. In fact, we can conformally change the metric $h$ by multiplying with a function that restricted to the annulus equals the norm square of the gradient of $V$ so that $h_{11} = 1$. On $Q$ we choose magnetic fields subject to the requirements of Theorem 1.2. We claim that the trace of $\gamma$ cannot be contained in the annulus $\{ -v_0 \leq V \leq 0 \}$. Otherwise we will reach a contradiction as follows: Via the gradient flow the solution $\gamma$ can be homotoped into the boundary of $\{ V \leq 0 \}$, which is a circle. The induced mapping degree of the homotoped $\gamma$ must vanish as otherwise a multiple of the boundary circle $\{ V = 0 \}$ would be contractible in $\{ V \leq 0 \}$. This implies that $\gamma$ is contractible inside the annulus. Therefore, we can consider the lift $(r(t), \theta(t))$ of $\gamma$ to the universal cover $[-v_0, 0] \times \mathbb{R}$ that is equipped with $(r, \theta)$-coordinates for which the metric tensor of $h$ is diagonal such that $h_{11} = 1$ and grad $V$ equals $\partial_r$. As $\theta$ has
to have extremal points we find $t_0$ such that $(\dot{r}(t_0), \dot{\theta}(t_0)) = (\sqrt{-2r(t_0)}, 0)$. On the other hand, the Christoffel symbols $\Gamma_{11}^1$ and $\Gamma_{11}^2$ of the Levi–Civita connection $D^Q$ of $h$ vanish such that the curve $\beta(t) = (b(t), \theta(t_0))$ for a quadratic polynomial $b(t)$ with leading term $-\frac{1}{2}t^2$, and with $b(t_0) = r(t_0)$ and $\dot{b}(t_0) = \sqrt{-2r(t_0)}$ is a solution of the equation of motion $D^Q_\beta \beta = -(1, 0)$ of zero energy. Using uniqueness of solutions of second order ordinary differential equations for given initial data and an open–closed argument we see that the periodic solution $\gamma : \mathbb{R} \to [-v_0, 0] \times \mathbb{R}$ coincides with $\beta$ where defined. In particular, the trace of $\gamma$ connects the boundary components of the annulus along a gradient flow line of $V$ so that $\gamma$ has vanishing velocity at the intersection with the lower boundary corresponding to $-v_0$. This is a contradiction.

**Proof of Theorem 1.3.** We assume that $n \geq 3$. The proof is based on the fact that the energy surface $M = \{H = 0\}$ is of virtually contact type such that the upper boundary of the standard $(n - 1)$-handle $D^{n-1} \times D^{n+1}$ admits a contact embedding into $(M', \ker \alpha)$ such that the image of the belt sphere represents a non-trivial homology class of degree $n$ in $M'$.

We give a direct verification of the contact type property of $M'$. For this denote by $X$ the gradient vector field of $\tau^* V$ with respect to the metric $m$ on $T^* Q$ and define a function $F$ on $T^* Q$ by setting $F = \lambda(X)$. We remark that $F(u) = h^\flat(u, dV)$ for all covectors $u$ on $Q$. Lifting to the universal cover $T^* \tilde{Q}$ we obtain for all $\varepsilon > 0$ and covectors $\tilde{u}$ on $\tilde{Q}$ that

$$\left(\tilde{\lambda} + \tilde{\tau}^* \theta - \varepsilon d\tilde{F}\right)(X_{\tilde{H}}) (\tilde{u})$$

is equal to the sum of

$$|\tilde{u}|^2_{(\tilde{h})^\flat} + (\tilde{h})^\flat(\tilde{u}, \theta)$$

and $\varepsilon$ times

$$-\left(\text{Hess}_{\tilde{h}} \tilde{V}\right)(\tilde{u}^\#, \tilde{u}^\#) + |\text{grad}_{\tilde{h}} \tilde{V}|^2_{\tilde{h}} + (\tau^* \mu^* \sigma) \left(\tilde{u}^\#, \text{grad}_{\tilde{h}} \tilde{V}\right),$$

where $X_{\tilde{H}}$ denotes the Hamiltonian vector field of the system $(\tilde{\omega}_{d\theta}, \tilde{H})$, $\tilde{u}^\#$ is the $\tilde{h}$-dual vector of $\tilde{u}$, and $\text{Hess}_{\tilde{h}} \tilde{V} = D^Q d\tilde{V}$ is the Hessian of $\tilde{V}$. Under the assumption of the theorem we can assume the situation described at the beginning of this section. Distinguishing the cases $\frac{1}{2} |\tilde{u}|^2_{(\tilde{h})^\flat} \geq v_0$ and $\frac{1}{2} |\tilde{u}|^2_{(\tilde{h})^\flat} < v_0$ one shows that the above sum is uniformly positive along

$$\left\{ \frac{1}{2} |\tilde{u}|^2_{(\tilde{h})^\flat} = -\tilde{V}(\tilde{\tau}(\tilde{u})) \right\}$$

for some small $\varepsilon > 0$. It follows that $M'$ is of virtually contact type.

In fact, taking the family $t\theta$ of 1-forms for $t \in [0, 1]$, which corresponds to the family $t\sigma$ of magnetic forms, it follows that

$$\alpha_t = \left(\tilde{\lambda} + t \cdot \tilde{\tau}^* \theta - \varepsilon d\tilde{F}\right)|_{TM'}$$

is a family of contact forms on $M'$ that connects $\alpha = \alpha_1$ with a contact form $\alpha_0$, which descends to the contact form $\alpha_{W'} := (\lambda - \varepsilon dF)|_{TM}$ on $M$. The induced contact structures are contactomorphic along a global flow as the Gray stability argument in [27] p. 60/61] shows. Indeed, the Moser integrating time-dependent vector field on $M'$ is bounded with respect to the complete metric $g'$ because $\alpha_t$ is bounded in $C^1$ (see Sections 6.2 and 6.3) as local arguments used in Section 4.5.
show. By [17] Example 11.12 (2) \((M, \ker \alpha_W)\) is the contact type boundary of the
Weinstein handle body \((H \leq 0)\) in \((T^*Q, d\lambda)\) with Morse function \(H\) and Liouville
vector field \(p\partial_p + \varepsilon X_F\) because
\[
\partial H(p\partial_p + \varepsilon X_F) = (\lambda - \varepsilon dF)(X_H)
\]
on \((T^*Q, d\lambda)\). By [60] Section 3.1 and the assumptions on the potential \(V\) the
Weinstein handle body \(W\) is subcritical.

Up to Weinstein handle moves according to the Morse–Smale theory for Wein-
stein structures in [17] Chapters 10, 12, and 14 we can assume that the largest
critical value is taken at precisely one critical point \(p_0\) and the Morse index of \(p_0\)
equals \(n - 1\). In view of Gray stability this changes the contact structure on the
boundary by a contact isotopy. With [17] Proposition 12.12 we can assume after a
Weinstein isotopy supported in a neighbourhood of \(p_0\) that the Weinstein structure
near \(p_0\) is given by the standard model handle as it appears in contact surgery, cf.
[27]. Replacing \(\partial W\) by a regular level set slightly above the largest critical value,
what results in a further contact isotopy of \((M, \ker \alpha_W)\) by following the Liouville
flow, the upper boundary of the model handle can be assumed to be contained in
the boundary of the Weinstein handle body. After all we obtain a contact embed-
ding of the upper boundary \(D^{n-1} \times S^n\) of the standard handle into \((M, \ker \alpha_W)\)
whose belt sphere represents a non-zero homology class by [60] Section 3.2. Hence,
\(D^{n-1} \times S^n\) lifts to a contact handle in \((M', \ker \alpha)\) so that the image of \(S^n\) is non-
zero in the homology of \(M'\). In view of Theorem [54, p. 135] this proves
the theorem.

**Appendix A. The third covariant derivative**

In view of the proof of Proposition [61, 3.5] we remark that the third covariant
derivative \(\nabla_X((\nabla')^2 \alpha)\) evaluated on the triple of vector fields \((Y_1, Y_2, Y_3)\) is the
sum of the following terms after restricting to \(M'\):
\[
\left(\overline{D}_X(\overline{D}^2 \beta)\right)(Y_1, Y_2, Y_3)
\]
and \(\overline{D}^2 \beta\) evaluated on the triples
\[
(\Pi'(X, Y_1), Y_2, Y_3) \quad \text{and} \quad (Y_1, \Pi'(X, Y_2), Y_3) \quad \text{and} \quad (Y_1, Y_2, \Pi'(X, Y_3))
\]
and \(\overline{D}_X(\overline{D} \beta)\) evaluated on the tuples
\[
(\Pi'(Y_1, Y_2), Y_3) \quad \text{and} \quad (Y_1, \Pi'(Y_2, Y_3)) \quad \text{and} \quad (Y_2, \Pi'(Y_1, Y_3))
\]
and \(\overline{D} \beta\) evaluated on the tuples
\[
(\Pi'(Y_1, Y_2), \Pi'(X, Y_3)) \quad , \quad (\Pi'(X, Y_2), \Pi'(Y_1, Y_3)) \quad , \quad (\Pi'(X, Y_1), \Pi'(Y_2, Y_3))
\]
and \(\beta \circ \Pi'\) evaluated on the tuples
\[
\left(\Pi'(X, \Pi'(Y_1, Y_2)), Y_3\right) \quad , \quad \left(\Pi'(X, \Pi'(Y_1, Y_3))\right) \quad , \quad \left(Y_1, \Pi'(X, \Pi'(Y_2, Y_3))\right)
\]
and \(\beta \circ \Pi'\) evaluated on the tuples
\[
\left(\Pi'(Y_1, Y_2), \Pi'(X, Y_3)\right) \quad \text{and} \quad \left(\Pi'(X, Y_2), \Pi'(Y_1, Y_3)\right)
\]
and \(\beta \circ \Pi'\) evaluated on the tuples
\[
\left(\Pi'(X, \Pi'(Y_1, Y_2)), Y_3\right) \quad \text{and} \quad \left(Y_2, \Pi'(X, \Pi'(Y_1, Y_3))\right)
\]
and
\[
(\tilde{D}_X\beta)\left((\tilde{D}_{Y_1}\Pi')(Y_2,Y_3)\right) + \beta\left(\Pi'(X,(\tilde{D}_{Y_1}\Pi')(Y_2,Y_3))\right)
\]
as well as
\[
\left(\tilde{D}_X(\beta \circ \Pi')\right)(\Pi'(Y_1,Y_2),Y_3),
\]
which equals,
\[
(\tilde{D}_X\beta)(\Pi'(\Pi'(Y_1,Y_2),Y_3)) + \beta\left((\tilde{D}_X\Pi')(\Pi'(Y_1,Y_2),Y_3)\right)
\]
and
\[
\left(\tilde{D}_X(\beta \circ \Pi')\right)(Y_2,\Pi'(Y_1,Y_3)),
\]
which equals,
\[
(\tilde{D}_X\beta)(\Pi'(Y_2,\Pi'(Y_1,Y_3))) + \beta\left((\tilde{D}_X\Pi')(Y_2,\Pi'(Y_1,Y_3))\right).
\]

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