LINES HIGHLY TANGENT TO A HYPERSURFACE

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ABSTRACT. We study spaces of lines that meet a smooth hypersurface $X$ in $\mathbb{P}^n$ to high order. As an application, we give a polynomial upper bound on the number of planes contained in a smooth degree $d$ hypersurface in $\mathbb{P}^5$ and provide a proof of a result of Landsberg without using moving frames.

1. Introduction

We work in characteristic 0. Our intention in this paper is first to illustrate the simplifying role played by log tangent sheaves in the analysis of lines highly tangent to a hypersurface $X \subset \mathbb{P}^n$, and then to obtain an upper bound on the number of 2-planes contained in a smooth hypersurface in $\mathbb{P}^5$. The simple observation we exploit is: If $\ell \subset \mathbb{P}^n$ is a line, then sections of the restricted log-tangent sheaf $T_{\mathbb{P}^n}(-\log X)|_{\ell}$ parametrize those infinitesimal deformations of $\ell$ preserving the moduli of the point-configuration $(\ell, \ell \cap X)$.

This observation helps simplify arguments. To illustrate this, we provide a quick proof of the following theorem of Landsberg without using moving frames:

**Theorem 1.1.** [6, Theorem 2] Let $X \subset \mathbb{P}^n$ be a reduced hypersurface over a characteristic zero field, $p \in X$ a general point, and let $\Sigma_k \subset G(1, n)$ denote any irreducible component of the variety of lines meeting $X$ to order at least $k$ at $p$. If $\dim \Sigma_k$ exceeds its expected dimension $n - k$, then all lines parametrized by $\Sigma_k$ are entirely contained in $X$.

A special instance of Theorem 1.1 is the fact that the *flec-nodal* locus $F$ of a smooth, degree $d \geq 3$ surface $S$ in $\mathbb{P}^3$ is always one dimensional, a fact initially proved by Salmon [9]. This fact, coupled with the calculation of the degree of $F$, was used by Segre [10] to show that the number of lines contained in $S$ is at most $(d - 2)(11d - 6)$.

Using Theorem 1.1 together with a careful geometric analysis, we can show that for any smooth hypersurface $X$ of degree $d$ in $\mathbb{P}^5$, any component of the space of lines meeting $X$ to order 5 has dimension at most 2 (see Theorem 3.8), which generalizes the $n = 5$ case of the de Jong-Debarre Conjecture [1]. Related work on lines highly
tangent to hypersurfaces can also be found in [11]. Using an intersection theory calculation, we obtain an explicit upper bound on the space of 2-planes in $X$, which had for $d \geq 3$ been known to be finite [2, Corollary in Appendix].

**Theorem 1.2.** Let $X \subset \mathbb{P}^5$ be a smooth hypersurface of degree $d$. Then $X$ contains at most $120d^2 - 150d^3 + 35d^4$ 2-planes.

The Fermat hypersurface of degree $d$ in $\mathbb{P}^5$ is known to contain $15d^3$ 2-planes, which can be compared to the $O(d^4)$ bound that we obtain. See [4] for some recent results about the space of planes on cubic 4-folds using lattice theory. In characteristic $p$, many of these results do not hold. See [3] for some examples of the types of behavior that can occur.

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### 2. Deformations of lines with fixed contact order

The goal of this section is to prove [Theorem 1.1] using an analysis of the deformation theory of the space of lines meeting $X$ to high contact order. We begin by reviewing the deformation theory of these lines and its relation to the log tangent sheaf. Let $X$ be defined by the homogeneous polynomial $F(x_0, \ldots, x_n)$, and suppose that $X$ contains the point $p = [1 : 0 \ldots : 0]$. We let

$$ T_{\mathbb{P}^n}(-\log X) \subset T_{\mathbb{P}^n} $$

denote the logarithmic tangent sheaf, i.e. the sheaf of derivations $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}$ mapping the ideal sheaf $\mathcal{I}_X$ into itself. In general, the logarithmic tangent sheaf is reflexive if $X$ is reduced, and when $X$ is smooth (or more generally, simple normal crossing) the logarithmic tangent sheaf is a vector bundle. We fix homogeneous coordinates $[s : t]$ on $\mathbb{P}^1$.

Let $\iota : \mathbb{P}^1 \to \mathbb{P}^n$ be a linear embedding of $\mathbb{P}^1$.

**Definition 2.1.** A first order deformation of $\iota$ is a map

$$ \iota_\epsilon : \mathbb{P}^1 \times \text{Spec } \mathbb{K}[\epsilon]/(\epsilon^2) \to \mathbb{P}^n $$

which restricts to $\iota$ modulo $(\epsilon)$. We say a first order deformation $\iota_\epsilon$ preserves $\iota^{-1}(X)$ if the scheme $\iota_\epsilon^{-1}(X)$ is the preimage of $\iota^{-1}(X)$ under the projection to the first factor

$$ \mathbb{P}^1 \times \text{Spec } \mathbb{K}[\epsilon]/(\epsilon^2) \to \mathbb{P}^1. $$
The significance of logarithmic tangent sheaves in the present setting is captured by the following lemma.

**Lemma 2.2 (Lemma 2.1).** Global sections of $\iota^*T_{\mathbb{P}^n}(-\log X)$ are in one-to-one correspondence with first order deformations of $\iota$ preserving $\iota^{-1}(X)$.

We take a moment to concretely describe the correspondence in Lemma 2.2 using coordinates. Suppose $[s : t]$ are homogeneous coordinates on $\mathbb{P}^1$ and that the map $\iota$ is defined by $[a_0 : \ldots : a_n]$, where each $a_i$ is a linear form in $s, t$. Then, by the Euler exact sequence, the global sections of $\iota^*T_{\mathbb{P}^n}$ are identified with tuples $(b_0, \ldots, b_n)$ of linear forms in $s, t$, considered modulo the tuple $(a_0, \ldots, a_n)$.

A global section of $\iota^*T_{\mathbb{P}^n}$ represented by such a tuple $(b_0, \ldots, b_n)$ corresponds to the first-order deformation $\iota_\epsilon$ defined by $[s : t] \mapsto [a_0 + \epsilon b_0 : \ldots : a_n + \epsilon b_n]$.

Within the space of all deformations $H^0(\mathbb{P}^1, \iota^*T_{\mathbb{P}^n})$, the subspace $H^0(\mathbb{P}^1, \iota^*T_{\mathbb{P}^n}(-\log X))$ is represented by tuples $(b_0, \ldots, b_n)$ obeying the condition

$$\sum_{i=0}^{n} b_i \frac{\partial F}{\partial x_i}(a_0, \ldots, a_n) \equiv 0 \mod F(a_0, \ldots, a_n).$$

The next construction will effectively allow us to replace the degree $d$ hypersurface $X$ with a degree $k < d$ hypersurface. Let $y_j = x_j/x_0$, $j = 1, \ldots, n$ be affine coordinates on the corresponding affine chart $x_0 \neq 0$ around $p$, and let $f(y_1, \ldots, y_n)$ denote the dehomogenization of $F$ on this chart. We expand

$$f = f_1 + \ldots + f_k,$$

where $f_i$ is the degree $i$ part of $f$, and we let $F_k$ be the homogenization of $f_1 + \ldots + f_k$. Explicitly,

$$F_k(x_0, \ldots, x_n) = \sum_{j=1}^{k} x_0^{k-j} \cdot f_j(x_1, \ldots, x_n).$$

We let $X_k$ denote the hypersurface defined by $F_k(x_0, \ldots, x_n) = 0$, and we note that $p \in X_k$ is a smooth point. Hence, $X_k$ is generically smooth.

**Definition 2.3.** Suppose $\iota : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ is a linear map such that $\iota^*F \in \mathbb{K}[s, t]$ is non-zero and divisible by $s^k$. A first order deformation $\iota_\epsilon$ of $\iota$ preserves order $k$ contact at $[0 : 1]$ if $\iota_\epsilon^*F$ is divisible by $s^k$ in the ring $\mathbb{K}[\epsilon, s, t]/(\epsilon^2)$.

**Corollary 2.4.** Let $\iota : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ parametrize a line such that $\iota^*F$ is non-zero and divisible by $s^k$, and such that $\iota([0 : 1]) = p$. Then the global sections of $\iota^*T_{\mathbb{P}^n}(-\log X_k)$ are in one-to-one correspondence with first order deformations $\iota_\epsilon$ preserving order $k$ contact at $[0 : 1]$. 
Proof. Let \([s : t] \mapsto [a_0 : \ldots : a_n]\) define \(\iota\), and choose coordinates on \(\mathbb{P}^n\) so that \(a_0\) is not a scalar multiple of \(s\), but \(a_i, i \geq 1\) are multiples of \(s\). By assumption, \(F(a_0, \ldots, a_n)\) is a non-zero form divisible by \(s^k\). This, in turn, implies that \(F_k(a_0, \ldots, a_n)\) is a non-zero multiple of \(s^k\).

Suppose now that \((b_0, \ldots, b_n)\) is a tuple of linear forms representing a deformation \(\iota\) preserving order \(k\) contact at \([0 : 1]\). This condition is equivalent to the condition that the form

\[
\sum_{i=0}^{n} b_i \cdot \frac{\partial F}{\partial x_i}(a_0, \ldots, a_n)
\]

is also divisible by \(s^k\). Next, a direct computation shows that

\[
\frac{\partial F}{\partial x_i}(a_0, \ldots, a_n) \equiv a_0^{d-k} \frac{\partial F_k}{\partial x_i} \mod s^k
\]

Therefore, since \(a_0\) is not divisible by \(s\), \((2.4)\) is divisible by \(s^k\) if and only if

\[
\sum_{i=0}^{n} b_i \cdot \frac{\partial F_k}{\partial x_i}(a_0, \ldots, a_n)
\]

is also divisible by \(s^k\). In other words

\[
\sum_{i=0}^{n} b_i \cdot \frac{\partial F_k}{\partial x_i}(a_0, \ldots, a_n) \equiv 0 \mod F_k(a_0, \ldots, a_n)
\]

meaning that \((b_0, \ldots, b_n)\) represents a global section of \(\iota^* T_{\mathbb{P}^n}(-\log X_k)\), which is what we needed to prove. \(\square\)

2.1. Some incidence varieties. Before recovering the result of Landsberg, we need to study the locus swept out by a family of high-contact lines. The main goal is Theorem 2.8.

**Definition 2.5.** Define the incidence variety

\[\text{Inc} \subset \mathbb{G}(1, n) \times \mathbb{P}^n\]

to be the set of pairs \((\ell, p)\) such that \(p \in \ell\).

Let \(\lambda : \text{Inc} \rightarrow \mathbb{G}(1, n)\) and \(\pi : \text{Inc} \rightarrow \mathbb{P}^n\) denote the two projections.

The projection \(\lambda\) expresses \(\text{Inc}\) as the universal line over the Grassmannian \(\mathbb{G}(1, n)\).

**Definition 2.6.** Suppose \(V \subset \text{Inc}\) is any subset. Define

\[\text{Tot}(V) \subset \mathbb{G}(1, n) \times \mathbb{P}^n \times \mathbb{P}^n\]
to be the set of triples \((\ell, x, y)\) such that \((\ell, x) \in V\) and \(y \in \ell\). Define 
\[
\overline{\text{Tot}}(V) \subset \mathbb{P}^n
\]
to be the image of \(\text{Tot}(V)\) under the projection to the third factor, i.e. \(\overline{\text{Tot}}(V)\) is the union of all lines in \(\lambda(V)\).

Observe that for any subset \(V \subset \text{Inc}\),
\[
\pi(V) \subset \overline{\text{Tot}}(V).
\]

**Definition 2.7.** Let \(X \subset \mathbb{P}^n\) be a degree \(d\) hypersurface. For each \(k \leq d\) define the variety
\[
V_k(X) \subset \text{Inc}
\]
to be the locus of pairs \((\ell, x)\) with \(x \in X\), such that \(\ell\) meets \(X\) at the point \(x\) to order \(\geq k\).

If the hypersurface \(X\) is understood in context, we will sometimes write \(V_k\) for \(V_k(X)\).

**Theorem 2.8.** Let \(X \subset \mathbb{P}^n\) be a smooth hypersurface. If \(W \subset V_k\) is an irreducible component satisfying 
\[
\overline{\text{Tot}}(W) = \mathbb{P}^n,
\]
then \(W\) has the expected dimension \(2n - k - 1\).

**Proof.** That every irreducible component of \(V_k\) has dimension at least \(2n - k - 1\) is a simple dimension count which we omit. It is enough to show therefore that the tangent space \(T_W|_{(\ell,p)}\) of \(W\) at a general point \((\ell, p) \in W\) has dimension at most \(2n - k - 1\). Further, it is enough to prove the result only for components where the general line meets \(X\) at \(p\) with multiplicity exactly \(k\).

To that end, suppose \((\ell, p)\) is a general point of \(W\), and that
\[
\iota : \mathbb{P}^1 \rightarrow \mathbb{P}^n
\]
is a parametrization of \(\ell\) sending \([0 : 1]\) to \(p\), as in the setting of Corollary 2.4. Notice that we may assume that \(\ell\) meets \(X\) at \(p\) with multiplicity exactly equal to \(k\).

We invoke the auxiliary hypersurface \(X_k\) constructed in the last section. Since \(\ell\) meets \(X_k\) only at \(p\), and since \(p\) is a smooth point of \(X_k\), we deduce that \(\iota^* T_{\mathbb{P}^n}(-\log X_k)\) is a locally free on \(\mathbb{P}^1\), and its global sections parametrize first order deformations of the map \(\iota\) preserving order \(k\) contact at \([0 : 1]\) by Corollary 2.4.

The condition
\[
\overline{\text{Tot}}(W) = \mathbb{P}^n
\]
implies that the map which sends a deformation $\iota_\epsilon$ preserving order $k$ contact at $[0 : 1]$ to the deformation induced on the point $\iota(y) \in \mathbb{P}^n$ for a general $y \in \mathbb{P}^1$ is surjective. This means that the restriction map

$$H^0(\mathbb{P}^1, \iota_\epsilon^* T_{\mathbb{P}^n}(-\log X_k)) \rightarrow \iota_\epsilon^* T_{\mathbb{P}^n}(-\log X_k)|_y = \iota_\epsilon^* T_{\mathbb{P}^n}|_y$$

(2.8)

is surjective. We conclude that $\iota_\epsilon^* T_{\mathbb{P}^n}(-\log X_k)$ is a globally generated vector bundle. Since $\deg \iota_\epsilon^* T_{\mathbb{P}^n}(-\log X_k) = n + 1 - k$, and $\operatorname{rk} \iota_\epsilon^* T_{\mathbb{P}^n}(-\log X_k) = n$, it follows from global generation that $\dim H^0(\mathbb{P}^1, \iota_\epsilon^* T_{\mathbb{P}^n}(-\log X_k)) = 2n - k + 1$.

Finally, we note that there is a 2-dimensional space of deformations of the map $\iota$ which induce a trivial deformation on the pair $(\ell, p)$ (There is a 2-dimensional group of automorphisms of $\mathbb{P}^1$ fixing a point). Since every tangent vector of $W$ at the point $(\ell, p)$ can be lifted to a first order deformation of $\iota$ preserving order $k$ contact at $[0 : 1]$, we conclude that $\dim T_W|_{(\ell, p)} \leq (2n - k + 1) - 2$, as desired. $\square$

We are now in position to provide the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Assume the setting in the statement of Theorem 1.1. Suppose the dimension of $\Sigma_k$ exceeds $n - k$. Since $p \in X$ is a general point, this implies the existence of an irreducible component

$$W \subset V_k$$

containing $\Sigma_k$ having dimension exceeding $(n - k) + (n - 1) = 2n - k - 1$.

Theorem 2.8 now implies $\overline{\operatorname{Tot}(W)} \neq \mathbb{P}^n$. Since $X = \pi(W)$ is a subvariety of $\overline{\operatorname{Tot}(W)}$, it follows that $\overline{\operatorname{Tot}(W)} = X$, which is the theorem’s conclusion. $\square$

### 3. Fifth order tangent lines

This section culminates in the proof of Theorem 3.8, which states that the variety $V_5(X)$ has the expected dimension for any degree $d \geq n$ smooth hypersurface $X \subset \mathbb{P}^n$. We note here that similar statements for $V_k(X)$, $k \leq 4$ are also true – $k = 1$ is trivial, $k = 2$ follows because $X$ is smooth, and $k = 3$ follows from Theorem 3.1. Finally, the case $k = 4$ can be proved using a simpler version of the proof in Theorem 3.8 so we omit it.
Theorem 3.8 will then be used in §4 to provide a bound on the number of $m$-planes that a sufficiently large degree $2m$-dimensional hypersurface can contain.

After recalling some classical facts about the second fundamental form and the classification of varieties with many lines, we will prove a basic fact about cones on scrolls, and then prove the main theorem of the section, Theorem 3.8.

3.1. Preliminaries on the second fundamental form. Let $x$ be a point on a smooth hypersurface $X \subset \mathbb{P}^n$ of degree $d \geq 2$. Letting $\mathfrak{m}_x$ denote the maximal ideal of $x \in X$, since $X$ is smooth, the vector space $H^0(X, \mathfrak{m}_x^2 \otimes \mathcal{O}_X(1))$ is 1-dimensional, generated by an element $s_x$. Its residue in the $\mathbb{K}$-vector space $(\mathfrak{m}_x^2/\mathfrak{m}_x^3)(1)$ defines a quadratic form (well-defined up to scaling, and possibly zero) denoted

$$\Pi_x$$

on the Zariski tangent space $T_xX = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$. The quadratic form $\Pi_x$ is called the second fundamental form of $X$ at $x$.

We will need the following fundamental fact about the behavior of $\Pi_x$:

**Theorem 3.1.** (\cite{5} Theorem 7.11) For each $0 \leq i \leq n - 1$, the closed subset $R_i \subset X$

consisting of points $x \in X$ where $$\mathrm{rk} \Pi_x \leq (n - 1) - i$$

has codimension $\geq i$.

3.2. Fourfolds with many lines. First, we record a well-known proposition about sweeping families of lines on hypersurfaces. For any hypersurface $X$, we let $F_1(X)$ denote the Fano scheme of lines contained in $X$. For any subscheme $A \subset F_1(X)$, we let $\mathrm{Tot}(A) \to A$ denote the universal line over $A$; $\mathrm{Tot}(A)$ is a $\mathbb{P}^1$-bundle over $A$. Furthermore, we denote by $\overline{\mathrm{Tot}}(A)$ the union of all lines parametrized by $A$.

**Proposition 3.2.** Suppose $X \subset \mathbb{P}^n$ is a smooth hypersurface, and suppose an irreducible component $B \subset F_1(X)$ sweeps out $X$, i.e. $\overline{\mathrm{Tot}}(B) = X$. The normal bundle $N_{\ell/X}$ is globally generated for a general line $\ell \in B$.

**Proof.** Let $(\ell, p) \in \overline{\mathrm{Tot}}(B)$ denote a general point. Then the derivative of the natural projection $\pi : \overline{\mathrm{Tot}}(B) \to X$ is a map on Zariski tangent spaces:

$$\psi : T_{(\ell,p)} \overline{\mathrm{Tot}}(B) \to T_pX.$$ (3.1)
The sweeping hypothesis implies that (3.1) is surjective. Furthermore, (3.1) sends
the “vertical tangent subspace” space

\[ T_p \ell \subset T_{(\ell,p)} \text{Tot}(B) \]

isomorphically to \( T_p \ell \subset T_p X \), and therefore we get an induced surjection on quotient
spaces:

\[ T_{(\ell,p)} \text{Tot}(B)/T_p \ell \twoheadrightarrow N_{\ell/X}|_p. \tag{3.2} \]

The domain vector space in (3.2) is canonically identified with the Zariski tangent
space \( T_\ell B \), which by standard deformation theory is canonically isomorphic to
\( H^0(\ell, N_{\ell/X}) \). Under these identifications, (3.2) becomes the restriction map

\[ H^0(\ell, N_{\ell/X}) \twoheadrightarrow N_{\ell/X}|_p, \]

and its surjectivity is precisely what we needed to prove. □

**Corollary 3.3.** If \( X \subset \mathbb{P}^n \) is a smooth hypersurface of degree \( d \geq n \) then \( X \) is not
swept out by lines.

**Proof.** The normal bundle of any line \( \ell \subset X \) has degree \( n - d - 1 < 0 \), and therefore
cannot be globally generated. The corollary now follows from Proposition 3.2. □

We will also need the following theorem of Segre classifying four dimensional
projective varieties containing a five dimensional family of lines:

**Theorem 3.4** (Theorem 1 of [8]). If \( X \subset \mathbb{P}^N \) is a variety which contains a
\((2 \dim X - 3)\)-dimensional family of lines, then \( X \) is either a linear space, a quadric,
or a scroll.

### 3.3. Cones in scrolls.

In this section, we record a basic result on cones contained
in scrolls, which we were unable to find a reference for.

**Definition 3.5.** An \( n \)-dimensional variety \( P \subset \mathbb{P}^N \) is a scroll if there exists a
smooth, irreducible, projective curve \( B \), a \( \mathbb{P}^{n-1} \)-bundle \( \pi : \Lambda \to B \) and a morphism
\( \varphi : \Lambda \to \mathbb{P}^N \) satisfying:

1. \( \varphi \) restricts to a linear embedding on each fiber of \( \pi \), and
2. \( \varphi(\Lambda) = P \), and \( \varphi \) is birational onto its image.

If \( P \) is a scroll, and if \( \pi : \Lambda \to B \) is understood, we call the linear spaces \( \varphi(\Lambda_b) \subset \mathbb{P}^N \) the rulings of \( P \). To ease notation, we will denote \( \varphi(\Lambda_b) \) by just \( \Lambda_b \) as long as
no confusion arises.
Definition 3.6. A cone in $\mathbb{P}^N$ is a pair $(C, c)$ where $C \subset \mathbb{P}^N$ is a projective variety and $c \in C$ satisfying: For any $q \in C \setminus \{c\}$, the line joining $q$ and $c$ is entirely contained in $C$. If $(C, c)$ is a cone, we say a line $\ell \subset C$ is a cone line if $c \in \ell$.

Note that if $(C, c)$ is a cone, and if $H \subset \mathbb{P}^N$ is any hyperplane containing $c$, then $(C \cap H, c)$ is again a cone. Furthermore, if a cone $(C, c)$ is irreducible and if $\dim C \geq 3$, then for a general hyperplane $H$ containing $c$, the cone $(C \cap H, c)$ is again irreducible.

The next proposition constrains large cones in scrolls.

Proposition 3.7. Suppose $P \subset \mathbb{P}^N$ is an $n$-dimensional scroll which is not a linear space, with rulings $\Lambda_b, b \in B$. Let $p \in P$, and suppose $(C, p)$ is a $(n-1)$-dimensional irreducible cone contained in $P$. Assume only finitely many of the rulings contain $p$.

Then $C$ is a linear space.

Proof. Assume, for sake of contradiction, that $\deg C > 1$. Then, as $\dim C = n - 1$, it follows that $C$ is not contained in any ruling $\Lambda_b$. Therefore, a general cone line $\ell \subset C$ meets a general $\Lambda_b$ at a single point. Since this is true for a general $\ell$, it follows that every $\ell$ meets a general $\Lambda_b$—furthermore, this meeting is a single point, because $p$ is only contained in finitely many rulings.

Now pick any hyperplane $H$ avoiding $p$, and consider projection from $p$

$$\pi_p : P \setminus \{p\} \longrightarrow H.$$  

The image $W := \pi_p(C \setminus \{p\})$ is the $n - 2$-dimensional variety $C \cap H$, which has degree $> 1$ by assumption. Furthermore, the image $\pi_p(\Lambda_b)$ for $b \in B$ general, is a $(n-1)$-dimensional linear space in $H$, and

$$W \subset \pi_p(\Lambda_b).$$

However, as $\deg W > 1$, this implies that $\pi_p(\Lambda_b)$ is independent of $b$. Thus, the image of $\pi_p : P \setminus \{p\}$ is the $(n-1)$-dimensional linear space $\pi_p(\Lambda_b)$, forcing $P$ itself to be a linear space, contrary to assumption. The argument by contradiction is complete. \hfill \square

We can now prove the main theorem of this section.

Theorem 3.8. If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d \geq n \geq 4$, then $V_5(X)$ has the expected dimension $2n - 6$.

Proof. We proceed by contradiction, and suppose here onward that

$$W \subset V_5$$
is an irreducible subvariety satisfying
\[ \dim W = 2n - 5. \]
We let \( \pi : W \to X \) denote the map sending a pair \((\ell, x)\) to the point \(x \in X\). Observe that for every \(x \in X\), the preimage
\[ \pi^{-1}(x) \subset W \]
is naturally a subvariety of the \((n - 2)\)-dimensional projective space parametrizing lines in \(\mathbb{P}^n\) which are tangent to \(X\) at \(x\). Thus, if we set
\[ Y := \pi(W) \subset X, \]
there are three possibilities:

1. \(Y = X\), or
2. \(Y \subset X\) has codimension 2, or
3. \(Y \subset X\) has codimension 1.

The proof proceeds by eliminating each of these possibilities.

**Lemma 3.9.** \(Y \neq X\).

*Proof.* Direct application of [Theorem 1.1] □

**Lemma 3.10.** \(Y \subset X\) cannot have codimension 2.

*Proof.* By contradiction. Assume \(\text{codim } Y = 2\). In this case, for a general (and hence every) point \(y \in Y\), the fiber \(\pi^{-1}(y) \subset W\) has dimension \((n - 2)\). Since, for every \((\ell, x) \in W\), the line \(\ell\) meets \(X\) at \(x\) with contact order \(> 2\), it follows that the second fundamental form \(\Pi_x(X)\) vanishes identically (has rank 0) for all points \(y \in Y\). This violates [Theorem 3.1] providing the desired contradiction. □

Thus, we may and shall assume \(Y \subset X\) has codimension 1 from here onward. First, we determine the dimension of \(\overline{\text{Tot}}(W)\). [Theorem 2.8] implies that \(\overline{\text{Tot}}(W) \neq \mathbb{P}^n\). Secondly, \(\overline{\text{Tot}}(W) \neq Y\): Otherwise, \(Y\) would be a \((n - 2)\)-dimensional variety which contains at least a \((2n - 6)\)-dimensional family of lines. Thus, \(Y\) would be a linear space. As \(n \geq 4\), \(X\) could not contain such a space.

We conclude that \(\overline{\text{Tot}}(W) \subset \mathbb{P}^n\) is a hypersurface. Since \(d \geq n\), [Corollary 3.3] implies that \(\overline{\text{Tot}}(W) \neq X\). Observe that, by Segre’s classification theorem, \(\overline{\text{Tot}}(W)\) is either a linear space, a quadric hypersurface, or a scroll.

Now pick a general point \(y \in Y\). Let \((C, y) \subset \mathbb{P}^n\) denote the \((n - 2)\)-dimensional cone swept out by the family of lines \(\pi^{-1}(y)\). Observe, by construction, that \(C \subset\)
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Furthermore, if $T_yX$ denotes the projective tangent space (an $(n - 1)$-dimensional linear space) of $X$ at $y$, then $C \subset T_yX$.

By Theorem 3.1, $\Gamma_y$ defines an irreducible quadric, and therefore for dimension reasons, $(C, y)$ must be the cone over this quadric. This in turn implies that $C$ spans the hyperplane $T_yX$. Furthermore, this also eliminates the possibility that $\text{Tot}(W)$ is a scroll: the hypotheses of Proposition 3.7 would be met, so $C$ could not be a quadric.

Thus, $\text{Tot}(W)$ is either a linear space or a quadric. Assume $\text{Tot}(W)$ is a linear space. At a general point $y \in Y$, the quadric cone $(C, y)$ is simultaneously contained in the linear space $\text{Tot}(W)$ and also spans $T_yX$. Thus all $T_yX$ are equal for $y \in Y$. This violates the finiteness of the Gauss map

$$\Gamma : X \rightarrow \mathbb{P}^n,$$

which sends $x \in X$ to the hyperplane $T_xX$.

Finally, assume $\text{Tot}(W)$ is a quadric hypersurface. Then, this quadric hypersurface cannot be singular at the general point $y \in Y$, because $\text{Tot}(W)$ is irreducible. As the cone $(C, y)$ is contained in $\text{Tot}(W)$, it follows that

$$T_y\text{Tot}(W) = T_yX$$

for $y \in Y$ general. Now, pick a general complete curve $D \subset Y$. On the one hand, the Gauss map $\Gamma$ induces a regular map of degree $\deg(D)(d - 1)$. On the other hand, this same map agrees with the (rational) Gauss map for $\text{Tot}(W)$, which is a map defined by linear forms on $\mathbb{P}^n$. Thus, $\deg(D)(d - 1) \leq \deg(D)$, which is absurd given $d \geq n$.

Thus, $Y \subset X$ cannot be a divisor, and we have exhausted all possibilities. The theorem follows.

\[\square\]

4. INTERSECTION THEORY COMPUTATION

**Theorem 4.1.** For $d \geq 5$, the number of 2-planes in a smooth 4-fold is at most $120d^2 - 150d^3 + 35d^4$.

**Proof.** Let $S$ be the tautological subbundle of the Grassmannian $G(1, 5)$ and $Z_{5,1} \subset \mathbb{P}(S) \times G(1, 5) \mathbb{P}(S)$ consist of the locus $(\ell, p_1, p_2)$ where $\ell$ is a line in $\mathbb{P}^5$ meeting $X$ at points $p_1$ and $p_2$ with orders 5 and 1, respectively.

By Theorem 3.8, $Z_{5,1}$ is 4-dimensional. Let $H$ be the $O(1)$ class of $\mathbb{P}(S)$, which is also the hyperplane class of $\mathbb{P}^5$ pulled back under $\mathbb{P}(S) \rightarrow \mathbb{P}^5$. Let $H_1$ and $H_2$ be $H$
pulled back to the two factors of \( \mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S) \) respectively under the projections \( \pi_1, \pi_2 : \mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S) \to \mathbb{P}(S) \).

To compute \( Z_{5,1} \subset \mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S) \), we note that it is the zero locus of a section of the sheaf of relative principal parts \( \pi_1^* P^4_{(d)}(G(1,5)) \times \pi_2^* \mathcal{O}(d) \) as defined in [5, Section 11.1.1]. The relative canonical divisor of \( \mathbb{P}(S) \to G(1,5) \) is \(-2H + \sigma_1\). Therefore, the class of \( Z_{5,1} \) is

\[
dH_1((d-2)H_1 + \sigma_1)((d-4)H_1 + 2\sigma_1)((d-6)H_1 + 3\sigma_1)((d-8)H_1 + 4\sigma_1)dH_2
\]

inside \( A^*(\mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S)) \) by [5, Theorem 11.2].

We will intersect \([Z_{5,1}] \in A^*(\mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S))\) with the pullback of \( G(1,4) \cong \sigma_{1,1} \subset G(1,5) \) and the product \( H_1H_2 \). This must be nonnegative on each component of \( Z_{5,1} \) by Kleiman’s transversality [5, Theorem 1.7] and yields

\[
\begin{align*}
\int_{\mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S)} & \sigma_{1,1}H_1H_2(dH_1((d-2)H_1 + \sigma_1) \cdots ((d-8)H_1 + 4\sigma_1)dH_2 = \\
\int_{(\mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S))|_{G(1,4)}} & H_1H_2(dH_1((d-2)H_1 + \sigma_1) \cdots ((d-8)H_1 + 4\sigma_1)dH_2. \quad (4.1)
\end{align*}
\]

In the final integral, the image of \( H_1 \) in \( A^*((\mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S))|_{G(1,4)}) \) can be identified as the image of \( H_1 \) under the map \( A^*(\mathbb{P}^5) \to A^*((\mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S))|_{G(1,4)}) \) in the commutative diagram of pullback maps of Chow rings

\[
\begin{array}{ccc}
A^*(\mathbb{P}^5) & \to & A^*((\mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S))|_{G(1,4)}) \\
\downarrow & & \downarrow \\
A^*(\mathbb{P}^4) & \to & A^*((\mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S))|_{G(1,4)}).
\end{array}
\]

Since \( H_1^5 = 0 \) in \( A^*(\mathbb{P}^4) \), the same is true in \( A^*((\mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S))|_{G(1,4)}) \). Therefore, the numerical answer in (4.1) is \( O(d^4) \). To get the exact numerical value for (4.1), we expand and evaluate with the help of Mathematica and the presentation of the Chow ring of a projective bundle [5, Theorem 9.6] to obtain

\[
120d^2 - 150d^3 + 35d^4.
\]

To finish, we note that every 2-plane \( \Lambda \) in \( X \) gives rise to a component of \( Z_{5,1} \) whose class is the pullback of \( \sigma_{2,2} \), the lines in \( \Lambda \). Since \( \sigma_{2,2}\sigma_{1,1} \) is a point on \( G(1,5) \), \( \sigma_{2,2}\sigma_{1,1}H_1H_2 \) evaluates to 1 in \( A^*((\mathbb{P}(S) \times_{G(1,5)} \mathbb{P}(S)) \). \( \square \)

**Remark 4.2.** If one could show that the space \( Z_6 \) consisting of pairs \((p, \ell)\) where \( \ell \) meets \( X \) to order 6 at \( p \) is of expected dimension 3, then the argument in the proof
of Theorem 4.1 would imply the number of 2-planes in $X$ is at most
\[ 1800d - 1370d^2 + 225d^3, \]
which one can compare with the $15d^3$ 2-planes in $X$ in the case $X$ is a Fermat hypersurface.

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