Experimental demonstration of topological error correction

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Quantum computers exploit the laws of quantum mechanics and can solve many problems exponentially more efficiently than their classical counterparts1–4. However, in the laboratory the ubiquitous decoherence of quantum states makes it notoriously hard to achieve the required high degree of quantum control. To overcome this problem, quantum error correction has been invented4–6. The principal result in quantum error correction, the threshold theorem7–9, states that as long as the error rate, p, per gate in a quantum computer is smaller than a threshold value, p_c, arbitrarily long and accurate quantum computation is efficiently possible. However, most methods of fault-tolerant quantum computing with a high threshold error rate (10^{-10}) require strong and long-range interactions7–9, and are thus difficult to implement. Local architectures are normally associated with much lower thresholds. For traditional concatenated codes on a two-dimensional lattice of quantum bits (qubits) with nearest-neighbour interactions, the highest threshold known at present10 is 2.02 \times 10^{-5}.

In such lattices, it is advantageous to use topological error correction11–15 (TEC) in the framework of topological cluster-state quantum computing. This scheme makes use of the topological properties in three-dimensional (3D) cluster states, which form an inherently error-robust ‘fabric’ for computation. Local measurements drive the computation and, at the same time, implement the error correction. Active error correction and topological methods are combined, yielding a high error threshold12,13 of 0.7–1.1% and tolerating loss rates15 up to 24.9%. This allows for the unavoidable imperfections of physical devices, and makes our implementation of TEC close to the experimental state of the art. For practical quantum computation with TEC, a larger cluster state of more qubits would be needed. The 3D architecture can be further mapped onto a local setting in two spatial dimensions plus time14, also with nearest-neighbour interactions only. Two detailed architectures have already been proposed16,17. We note that a different topological scheme has been proposed in which quantum computation is driven by non-Abelian anyons18,19 and fault tolerance is achieved through passive stabilization afforded by a ground-state energy gap.

Some simple quantum error correction codes have been experimentally demonstrated in nuclear magnetic resonance22,23 and optical systems24,25. However, the experimental realization of topological quantum error correction methods remains challenging. At present, multipartite cluster states can be generated with up to six photons and work is under way to create non-Abelian anyons for topological quantum computing16,19. Here we develop an ultrabright entangled-photon source by using an interferometric Bell-type synthesizer. With this and a noise-reduction interferometer, we generate a polarization-encoded eight-photon cluster state, which is shown to possess the required topological properties for TEC. In accordance with the TEC scheme, we measure each photon (qubit) locally. Error syndromes are constructed from the measurement outcomes, and one topological quantum correlation is protected. We demonstrate that if only one physical qubit suffers an error, the faulty qubit can be located and corrected, and that if all qubits are simultaneously subjected to errors with equal probability, the effective error rate is significantly reduced by error correction. This constitutes a proof-of-principle experiment that demonstrates the viability of TEC, a central ingredient in topological cluster-state computing.

Cluster states and quantum computing

In cluster-state quantum computing26, projective one-qubit measurements replace unitary evolution as the elementary process driving a quantum computation. The computation begins with a highly entangled multi-qubit state, the ‘cluster state’ |G⟩ (ref. 27), which is specified by an interaction graph, G, and can be created from a product state through the pairwise Ising interaction over the edges in G. For each vertex i \in G, we define a stabilizer as K_i \equiv X_i \otimes e_i Z_i where the product is over all the interaction edges, e_{ij} connecting vertex i to its nearest-neighbouring vertices, j. The symbols X_i and Z_j denote the bit- and phase-flip Pauli operators, respectively, acting on qubits i and j. State |G⟩ is the unique joint eigenstate of a complete set of stabilizers K_i such that K_i |G⟩ = |G⟩ for all i \in G.

Cluster states in d \geq 3 dimensions are resources for universal fault-tolerant quantum computing27, in which the TEC capability—shared with Kitaev’s toric code28,29 and the colour code29—is combined with the capability to process quantum information.

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Topological error correction

Quantum error correction and fault-tolerant quantum computing are possible with cluster states whenever the underlying interaction graph can be embedded in a 3D cell structure known as a cell complex, which consists of volumes, faces, edges, and vertices. Qubits are encoded on the edges and faces of a cell complex. The associated interaction graph connects the qubit on each face to the qubits on its surrounding edges through the interaction edges. Consider the elementary cell complex in Fig. 1a, shown by the dashed lines: it has one cubic volume, six square faces, twelve edges and eight vertices. The interaction edges, represented by the solid lines, form an 18-qubit cell complex, G8. There are six face stabilizers, Kf (f = 1, 2, ..., 6). It follows that multiplication of these stabilizers cancels out all Z operators in Kf and thus yields a unit expectation value: \( \langle X_1X_2 \cdots X_6 \rangle = 1 \). This leads to the straightforward but important observation that despite the X measurement on each individual face qubit having the random outcome ±1, the product of all the outcomes on any closed surface, \( F \), is +1. That is, any closed surface has the topological quantum correlation \( C_F = \langle \chi_f \rangle = 1 \), where \( \chi_f \) is a face of \( F \).

A larger cell complex is displayed in Fig. 1b, which encodes and propagates a logical qubit. It consists of \( 5 \times 5 \times T \) cells, where \( T \) specifies a span of simulated time (t). A ‘defect’ along the t direction (Fig. 1b, line of green dots) is first produced by performing local Z measurements. Then the topological quantum correlation, \( C_{F_0} = 1 \), on a defect-enclosing closed surface (\( F_0 \)), combined with the boundary, is used to encode a logical qubit. The evolution of the logical state from \( t_1 \) to \( t_2 \) is achieved by local X measurements on all other physical qubits between \( t_1 \) and \( t_2 \) (see ref. 31 for details). Quantum computing requires a much larger cell complex and more defects, where quantum algorithms are realized by appropriate braiding-like manipulation of defects (a sketch of the logical controlled-NOT gate is shown in Supplementary Information).

The quantum computation is possible because the topological quantum correlation \( C_{F_0} = 1 \) holds on defect-enclosing closed surfaces. The TEC capability arises from the \( Z_2 \) homology, a topological feature, of a sufficiently large 3D cell complex (Supplementary Information). For a given \( F_0 \), there exist many homologically equivalent closed surfaces with the same topological correlation \( (C_{F_0} = 1) \). This redundancy leads to the topological protection of the correlation\(^3\).

Remarkably, in TEC it is sufficient to deal with Z errors, because an X error either has no effect, if it occurs immediately before an X measurement, or is equivalent to multiple Z errors. Finally, as TEC is implemented in topological cluster-state quantum computing—a measurement-based process—corrections suggested by TEC are not applied to the remaining cluster state but rather to the classical outcomes of X measurements.

Simpler topological cluster state

The cell complex in Fig. 1b encodes a propagating logical qubit in terms of one topological correlation, \( C_{F_0} = 1 \), and is robust against a local Z error. However, it contains 25 elementary cells and 180 physical qubits for each layer of complex over a unit time span, which is beyond the capacity of available experimental techniques. We design a simpler graph state, \( |G_8\rangle \) (Fig. 2a), to mimic the cell complex of Fig. 1b.

The topological feature of \( |G_8\rangle \) can be seen from its association with the 3D cell complex in Fig. 2b, which consists of four elementary volumes, \( \{v, w, y, z\} \); six faces, \( \{f_1, f_2, f_3, f_4, f_5, f_6\} \); two edges, \( \{e_5, e_6\} \); and two vertices, \( \{s, t\} \). All six faces have the same boundary, \( e_5 \cup e_6 \), and any two of them form a closed surface, \( F \). The centre volume is removed to resemble the defect in Fig. 1b, and the topological correlation to be protected, \( C_{F_0} \), reads

\[ C_{F_0} = \langle X_1X_2 \rangle = 1 \]  

In this simple cell complex, the topological correlation \( C_{F_0} = 1 \) is already multiply encoded: it is represented by any expectation \( \langle XX_i \rangle \) with \( i \in \{1, 2, 5\} \) and \( j \in \{3, 4, 6\} \). Moreover, there exist four other closed surfaces, corresponding to the respective boundaries of the volumes \( \{v, w, y, z\} \), that do not enclose the defect. The redundant topological correlations are

\[ \langle X_1X_2 \rangle = \langle X_3X_4 \rangle = \langle X_5X_6 \rangle = \langle X_1X_4 \rangle = 1 \]  

These can be used as error syndromes in TEC, which makes one or more of them equal to −1. As shown in Table 1, a single Z error on any physical qubit can be located and corrected.

Therefore, from the aspect of TEC capability, the cluster state \( |G_8\rangle \) is analogous to the cell complex in Fig. 1b. Each protects one topological correlation and is robust against a single Z error, despite the cell complex in Fig. 2b being too small to propagate a logical qubit (see Supplementary Information for details).

Figure 1 | Topological cluster states. a, Elementary lattice cell. Dashed lines represent the edges of the associated cell complex and solid lines represent the edges of the interaction graph. Qubits (spheres) are encoded on the faces and edges of the elementary cell. b, Larger topological cluster state of \( 5 \times 5 \times T \) cells. Green dots represent local Z measurements, which effectively remove the measured qubits from the cluster state and thereby create a non-trivial topology capable of supporting a single correlation. Red dots represent Z errors. Red cells indicate the ends of error chains where \( C_{F_0} = -1 \). One axis of the cluster can be regarded as simulating the ‘circuit time’, t. The evolution of logical states from \( t_1 \) to \( t_2 \) is achieved by performing local X measurements on all physical qubits between \( t_1 \) and \( t_2 \).

Figure 2 | Cluster state \( |G_8\rangle \) and its cell complex. a, \( |G_8\rangle \), the interaction graph of \( |G_8\rangle \). b, The corresponding 3D cell complex, with volumes \( \{v, w, y, z\} \), faces \( \{f_1, f_2, f_3, f_4, f_5, f_6\} \), edges \( \{e_5, e_6\} \) and vertices \( \{s, t\} \). The exterior and the centre volume are not in the complex. For better illustration, the cell complex is cut open and the foreground quarter is removed (silhouette view from right is shown for clarity).
Preparation of the eight-photon cluster state

In our experiment, we create the desired eight-photon cluster state using spontaneous parametric down-conversion and linear optics. The first step is to develop an ultrabright, high-fidelity entangled-photon source. As shown in Fig. 3a, an ultraviolet mode-locked laser pulse (power, 915 mW) passes through a β-barium borate crystal, creating a pair of polarization-entangled photons in the state $|\psi\rangle = (|HH\rangle + |VV\rangle)/\sqrt{2}$. Using an interferometric Bell-state synthesizer\(^{32}\), we guide photons of different bandwidths (Fig. 3a, red and blue dots, respectively) along separate paths. This disentangles the temporal information from the polarization information. By contrast with the conventional narrowband filtering technique, this process does not result in photon loss and we thus achieve ultrahigh brightness. Four pairs of such entangled photons are prepared and labelled as 1–2, 3–4, 5–6 and 7–8 (Fig. 3b). Then we generate two graph states, each of four photons. The first is a Greenberger–Horne–Zeilinger state, $|\psi\rangle_{\text{GHZ}} = (|HH\rangle + |VV\rangle)/\sqrt{2}$, obtained by superposing photons 2 and 4 on a polarizing beam splitter (PBS\(_1\)), which transmits horizontal polarization (H) and reflects vertical polarization (V). At the same time, photons 6 and 8 are interfered on a polarization-dependent beam splitter (PDBS) and then separately pass through two other PDBSs. The first PDBS has transmitting probabilities $T_H = 1$ and $T_V = 1/3$, and the second and third have $T_H = 1/3$ and $T_V = 1$. The combination of these three PDBSs acts as a controlled-phase gate\(^{33,34}\). With a success probability of one-ninth, there is twofold coincidence in paths 6' and 8', yielding a four-photon cluster state\(^{35}\) $|\psi\rangle_{\text{CL}} = |HH\rangle_{\text{78}} |VV\rangle_{\text{78}} + |VV\rangle_{\text{78}} |HH\rangle_{\text{78}} - |VV\rangle_{\text{78}}|VV\rangle_{\text{78}}]/2$. Finally, photons 4' and 6' are superposed on PBS\(_2\). When eight photons come out of the output ports simultaneously, we obtain an entangled eight-photon cluster state:

$$|\psi\rangle = \frac{1}{2} \left[ |HH\rangle_{\text{78}} + |VV\rangle_{\text{78}} + |VV\rangle_{\text{78}} |HH\rangle_{\text{78}} - |VV\rangle_{\text{78}}|VV\rangle_{\text{78}}\right].$$

This is exactly the cluster state $|G_8\rangle$ shown in Fig. 2a under Hadamard operations $H^\otimes 8$ on all qubits. We note that the photons, which are interfered on the PBS\(_3\) or the PDBS\(_3\), have the same bandwidth, and that a star topology of the eight-photon interferometer\(^{32}\) leads to an effective noise reduction.

To ensure good spatial and temporal overlap, the photons are also spectrally filtered, with full-widths at half-maximum of $\Delta \lambda_{\text{FWHM}} = 8$ nm for photons 1, 3, 5 and 7 and $\Delta \lambda_{\text{FWHM}} = 2.8$ nm for photons 2, 4, 6 and 8, and are coupled by single-mode fibres. We obtain an average twofold coincidence count of $\sim 3.4 \times 10^5$ s\(^{-1}\) and a visibility of $\sim 94\%$ in the $|H\rangle$ basis as well as in the $|+\rangle$, $|-\rangle$ basis, where $|\pm\rangle = (|H\rangle \pm |V\rangle)/\sqrt{2}$. Fine adjustments of the delays between the different paths are tuned to ensure that all the interfering photons arrive at the PBS\(_3\) and the PDBS simultaneously.

Measurement of each photon is made using a polarization analyser, which contains a combination of a QWP, a HWP and a PBS together

| $G_8\rangle$ and the syndromes $\langle XX\rangle$ | $\langle XX\rangle$ | $\langle XX\rangle$ | $\langle XX\rangle$ | $\langle XX\rangle$
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | -1 | -1 |
| 4 | 1 | 1 | 1 | -1 |
| 5 | 1 | -1 | 1 | 1 |
| 6 | 1 | -1 | -1 | 1 |

$\text{Table 1 | } G_8\rangle$ and the syndromes $\langle XX\rangle$
with a single-mode, fibre-coupled single-photon detectors in each output of the PBS (Fig. 3c). The complete set of all 256 possible combinations of eight-photon coincidence events is registered by a home-made programmable coincidence logic unit based on a field-programmable gate array. We obtain an eightfold coincidence rate of 3.2 per hour. On the basis of the measurements for the 256 possible polarization combinations in the \(|HH, VV\rangle\) basis (Fig. 4a), we obtain a signal-to-noise ratio, defined as the ratio of the average number of desired components to that of non-desired components, of about 200:1. This indicates that we have been successful in preparing the desired eight-photon cluster state.

To characterize the cluster state more precisely, we use the entanglement witness method to determine its fidelity. For this purpose, we construct a witness that allows for the lower bound on the state fidelity (equation (3)), which differs from using a series of single-qubit measurements and classical correction. The experimental topological error correction (PEC) and its witness method to determine its fidelity. For this purpose, we construct a witness that allows for the lower bound on the state fidelity (equation (3)), which differs from using a series of single-qubit measurements and classical correction. The measured eightfold coincidence in the \(|HH, VV\rangle\) basis is shown in Fig. 4b. These yield the witness \(\mathcal{W}_8\) given by

\[
\mathcal{W}_8 = \frac{1}{2} - \left( \langle \psi^* | \psi \rangle - |\psi^*\rangle |\psi\rangle \right)
\]

\[
= \frac{1}{2} - \left[ \frac{1}{4} \langle \mathcal{H} \mathcal{H}^{\otimes 6} - |V\rangle \langle V| \mathcal{V} |\mathcal{V}\rangle \rangle^{\otimes 6} \right]_{-6} \otimes (X_7 X_8 - Y_7 Y_8)
\]

\[
+ \frac{1}{12} \sum_{k=0}^{5} \left( -1 \right)^k M_k^{\otimes 6} \otimes \left( \mathcal{H} \mathcal{H}^{\otimes 2} - |V\rangle \langle V| \mathcal{V} |\mathcal{V}\rangle \rangle^{\otimes 2} \right)_{78}
\]

Here \(\langle \psi^* | \psi \rangle = 0\) and \(M_k = \cos \left(\frac{k}{2} \pi / 6\right) X + \sin \left(\frac{k}{2} \pi / 6\right) Y\). The measured expectation value of each measurement setting in \(\mathcal{W}_8\) is shown in Fig. 4b. These yield the witness \(\langle \mathcal{W}_8 \rangle = -0.105 \pm 0.023\), which is negative by 4.5 s.d. The state fidelity is \(F > 0.5 - \langle \mathcal{W}_8 \rangle = 0.605 \pm 0.023\). This confirmed the presence of genuine eight-photon entanglement.

**Experimental topological error correction**

Given such a cluster state, topological error correction is implemented using a series of single-qubit measurements and classical correction operations. In the laboratory, operations are performed on state \(|\psi\rangle\) (equation (3)), which differs from \(|G_8\rangle\) in Fig. 2a by the Hadamard operation \(H^{\otimes 8}\). Therefore, the correlation to be protected in equation (2) corresponds to \(|Z_2 Z_8\rangle\) in the experiment; similarly, each \(X_i X_j\) in equation (2) corresponds to \((Z_i Z_j)\). Furthermore, \(X\) errors are simulated instead of \(Z\) errors.

In the experiment, the noisy quantum channels on polarization qubits are simulated by one HWP positioned between two QWPs, which are set at 90° relative to the horizontal. By randomly setting the HWP axis to be oriented at ± 0° with respect to the horizontal direction, the noisy quantum channel can be simulated with a bit-flip error probability of \(p = \sin^2(2\theta)\).

We first study the case in which only a single \(X\) error occurs on one of the six photons \([1, \ldots, 6]\). The syndrome correlations are measured (Fig. 5). For comparison, in Fig. 4c we plot the correlations without any simulated error. This comparison, together with Table 1, makes it possible to locate precisely the physical qubit undergoing an \(X\) error.

We then consider the case in which all six photons are simultaneously subject to a random \(X\) error with equal probability 0 < \(p < 1\) and study the rate of errors, \(\langle Z_2 Z_8 \rangle = -1\), for the topological quantum correlation \(|Z_2 Z_8\rangle\). Without error correction, the error rate of correlation \(|Z_2 Z_8\rangle\) is \(P = 1 - (1 - p)^2 - p^2\). With error correction, the residual error becomes

\[
P = 1 - \left[ (1 - p)^2 + p^2 \right] - 6p(1 - p)^5 + 6(1 - p)p^5
\]

\[
- 9p^2(1 - p)^4 + 9(1 - p)^2p^4
\]

For small \(p\), the residual error rate after error correction is significantly reduced relative to the unprotected case. As shown in Fig. 6, the experimental results are in good agreement with these theoretical predictions. Considerable improvement of the robustness of the correlation \(|Z_2 Z_8\rangle\) can be seen both in theory and in practice.

In the experiment, the whole measurement takes about 80 days. This requires our set-up to be extremely stable. The imperfections in the experiment are mainly due to the undesired components in the \(|HH, VV\rangle\) basis, which arise from higher-order emissions of entangled photons, and the imperfect photon overlapping at the PBSs and the PDBS. In spite of these issues, the viability of TEC is successfully demonstrated in the experiment.

**Discussion**

In this work, we have experimentally demonstrated TEC with an eight-photon cluster state. This state represents the current state of the art for preparation of cluster states in qubit systems and is of particular interest in studying multipartite entanglement and quantum information processing. The scalable construction of cluster...
Figure 5 | Experimental results of syndrome correlations for topological error correction. Only one qubit is subjected to an X error in each plot. The measurement for each error setting takes about 80 h. Error bars, 1 s.d., deduced from propagated Poissonian counting statistics of the raw detection events.

Figure 6 | Experimental results of topological error correction. All physical qubits are simultaneously subject to an X error with equal probability ranging from 0 to 1. The blue circles and blue lines represent the experimental and, respectively, theoretical values of the error rate for the protected correlation without TEC, and the red squares and red lines similarly represent the error rate with TEC. The agreement between the experimental and theoretical results demonstrates the viability of TEC. The measurement of each data point takes about 80 h. Error bars, 1 s.d., deduced from propagated Poissonian counting statistics of the raw detection events.
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**Supplementary Information** is linked to the online version of the paper at www.nature.com/nature.

**Acknowledgements** We acknowledge discussions with M. A. Martin-Delgado and O. Gühne. We are grateful to X.-H. Bao for his original idea of the ultrabright entanglement and to C.-Z. Peng for his idea of reducing high-order emission. We would also like to thank C. Liu and S. Fölling for their help in designing the figures. This work has been supported by the NNSF of China, the CAS, the National Fundamental Research Program (under grant no. 2011CB921300) and NSERC.

**Author Contributions** W.-B.G., A.G.F., R.R., Z.-B.C., Y.-J.D. and J.-W.P. had the idea for and initiated the experiment. A.G.F., R.R. and Y.-J.D. contributed to the general theoretical work. X.-C.Y., C.-Y.L., Y.-A.C. and J.-W.P. designed the experiment. X.-C.Y., T.-X.W. and H.-Z.C. carried out the experiment. X.-C.Y. and Y.-A.C. analysed the data. X.-C.Y., A.G.F., R.R., N.-L.L., C.-Y.L., Y.-J.D., Y.-A.C. and J.-W.P. wrote the manuscript. N.-L.L., Y.-A.C. and J.-W.P. supervised the whole project.

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