Higher Dimensional Cosmology: Relations among the radii of two homogeneous spaces.

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Abstract

We study a cosmological model in 1+D+d dimensions where D dimensions are associated with the usual Friedman-Robertson-Walker type metric with radio a(t) and d dimensions corresponds to an additional homogeneous space with radio b(t). We make a general analysis of the field equations and then we obtain solutions involving the two cosmological radii, a(t) and b(t). The particular case D=3, d=1 is studied in some detail.

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I. Introduction.

Extra-dimensional models have been subject of interest since its appearance in physics, mainly as a basis for unification models [1]-[4]. In fact, although there has been renewed interest in such models through the development of string theory, the very first works on the subject (Nordström, Kaluza and Klein)[1] contained already the essential techniques for building theories in higher dimensions, particularly the idea of compactification.

Different approaches to the problem eventually arose (two nice reviews are [1] and [2]), as well as considerations regarding the possibility of noncompact extra dimensions [5][6] or any type of dynamical compactification [7]-[11]. All possibilities studied had ultimately as common point the cosmological implications of the theory. This is because the cosmological scales may amplify any particular effect of a model with extra dimensions.

However, the extra-dimensional cosmological model appropriate to any particular approach relies on some assumptions by itself. This in turn permit us to observe that the study of cosmological models with extra dimensions deserves attention by itself. In this context, we review a cosmological model with extra dimensions, as it was presented in Ref. [12], such that $D$ dimensions have an evolving radius $a$ and $d$ dimensions have another evolving radius $b$.

Specifically, we consider a higher dimensional cosmological model with metric [12][13]

$$ds^2 = -dt^2 + a^2(t)\tilde{g}_{ij}(x)dx^idx^j + b^2(t)\hat{g}_{ab}(y)dy^ady^b.$$ (1.1)

Here, indices $(i, j, ...)$ run from 1 to $D$, while $(a, b, ...)$ run from $D+1$ to $D+d$. Also, $\tilde{g}_{ij}(x)$ and $\hat{g}_{ab}(y)$ are the metrics for two homogeneous spaces that depend only on the co-moving coordinates $(x^1, ..., x^D)$ and $(y^{D+1}, ..., y^{D+d})$, respectively. From here on, these spaces will be called $D$-space and $d$-space, with an obvious connotation.

With the prescriptions given above, Einstein equations in vacuum (cf. Appendix) are

$$D(D-1)\ddot{a} + D(D-1)\frac{k_1}{a^2} + d(d-1)\frac{\dot{b}^2}{b^2} + d(d-1)\frac{k_2}{b^2} + 2dD\frac{\dot{a}\dot{b}}{ab} = 0;$$ (1.2)

$$(D - 1)\left\{ \ddot{a} + \frac{(D-2)}{2} \left[ \frac{\dot{a}^2}{a^2} + \frac{k_1}{a^2} \right] \right\} + d\left\{ \ddot{b} + \frac{(d-1)}{2} \left[ \frac{\dot{b}^2}{b^2} + \frac{k_2}{b^2} \right] \right\}$$

$$+ d(D - 1)\frac{\dot{a}\dot{b}}{ab} = 0;$$ (1.3)

$$(d - 1)\left\{ \ddot{b} + \frac{(d-2)}{2} \left[ \frac{\dot{b}^2}{b^2} + \frac{k_2}{b^2} \right] \right\} + D\left\{ \ddot{a} + \frac{(D-1)}{2} \left[ \frac{\dot{a}^2}{a^2} + \frac{k_1}{a^2} \right] \right\}$$

$$+ D(d - 1)\frac{\dot{a}\dot{b}}{ab} = 0.$$ (1.4)
The structure of this article is as follows: In this section (I) we have presented the model, and shown the corresponding Einstein equations. Next two sections are focused in the particular case of $3 + 1$ spatial dimensions. In section II the vacuum case is presented briefly, while Section III deals with the inclusion of matter. We solve there $a$ as function of time, as well as $b$ as function of $a$. Section III ends with the general solutions when $\Lambda \neq 0$. Then, we present in section IV an interesting result where radii $a$ and $b$ are related in a form that resembles a conserved "angular momentum". Finally, we present a summary of the work and make some final comments in section V.

In resume, our main results of this study are: first, we obtain the behaviour of $a(t)$ and $b(a)$ for the two cases in which the cosmological constant is zero or nonzero, and secondly we show a general relation between the two radii in the presence of matter, reminiscent of the conservation of classical angular momentum. We start the analysis with a special case, where $D = 3$ and $d = 1$ [14][15].

II. $D=3$, $d=1$ dimensional model.

By setting $D = 3$ and $d = 1$, equations (1.2)-(1.4) are, respectively

\[
\frac{\dot{a}^2}{a^2} + \frac{k_1}{a^2} + \frac{\dot{a}\dot{b}}{ab} = 0, 
\]

\[
\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k_1}{a^2} + \frac{\ddot{b}}{b} + 2\frac{\dot{a}\dot{b}}{ab} = 0
\]

and

\[
\frac{\dddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k_1}{a^2} = 0.
\]

Equation (2.3) can be integrated respect to $a$ by mean of a change of variable [14], resulting

\[
\dot{a}^2 + k_1 = \frac{\Gamma}{a^2},
\]

where $\Gamma$ is a integration constant. Equation (2.4) serves two purposes. The first is to give the behaviour of $a$ in time. Second, it permit us to integrate (2.1) in order to obtain radius $b$ as a function of $a$, even with the inclusion of matter. We review both results more extensively in next section. By the moment, we verify that in this case -where $\rho = 0$-, we obtain $b$ as function of $a$ directly from (2.1) and (2.3), since both together imply $\frac{\dot{a}}{a} = \frac{\ddot{b}}{ab}$, and this in turn yields

\[
b = \beta \dot{a},
\]
where $\beta$ is a positive constant for an expanding Universe in 3-space. We mention that, by virtue of (2.3)-(2.5) we have that

$$\dot{b} = -\frac{\beta \Gamma}{a^3},$$  \hspace{0.5cm} (2.6)

i.e., with $\beta$ and $\Gamma > 0$ we have a decreasing radius $b$ in time for this vacuum case. As we will see, all this results are limit cases of a more general result, when matter is included.

III. $D=3$, $d=1$ model with matter.

In this section, we review a Kaluza-Klein cosmological model where $D = 3$ and $d = 1$, with a “five-dimensional dust”, as seen in [14]. In this case, (A.6) of the Appendix, where $T_{00}$ corresponds to $\rho$, the five-dimensional density, can be written as

$$\frac{\dot{a}^2}{a^2} + \frac{k_1}{a^2} + \frac{\dot{a}b}{ab} = \frac{8\pi \mu}{3a^3b}. \hspace{0.5cm} (3.1)$$

Here, Bianchi identities $G_{\mu\nu}^{\mu} = 0$ gave $\rho = \frac{4\pi \mu}{a^2}$, were $\mu$ is a constant.

First, from equation (2.4) we obtain the behaviour of radius $a$ for the cases of zero ($k_1 = 0$), positive ($k_1 = 1$) or negative curvature ($k_1 = -1$) in 3-space:

a) $k_1 = 0$. In this case $\Gamma$ is necessarily positive, and the integration yields

$$a = \sqrt{2\sqrt{\Gamma}t}. \hspace{0.5cm} (3.2)$$

This gives the evolution of the Hubble parameter as $H = \frac{1}{2t} = \frac{\sqrt{\Gamma}}{a^2}$. The result can be contrasted with that of the FRW model in a matter dominated Universe, where $a \propto t^{\frac{2}{3}}$, $H \propto a^{-\frac{2}{3}}$, and $H \propto \frac{1}{t}$ [16].

b) $k_1 = 1$. Here $\Gamma$ is strictly positive too, and $a(t)$ is

$$a = \left[\Gamma - (\sqrt{\Gamma} - t)^2\right]^\frac{1}{2}. \hspace{0.5cm} (3.3)$$

c) $k_1 = -1$. Here $\dot{a} = \frac{\sqrt{\Gamma + a^2}}{a}$ and $\Gamma$ can be positive or negative. For $\Gamma > 0$, we have

$$a = \left[(t + \sqrt{\Gamma})^2 - \Gamma\right]^\frac{1}{2}. \hspace{0.5cm} (3.4)$$

Thus, the qualitative picture is a decelerating (expanding-) Universe.

If $\Gamma < 0$, the behaviour of $a$ in time is given by

$$a = \left[t^2 - \Gamma\right]^\frac{1}{2}, \hspace{0.5cm} (3.5)$$
where we set $t = 0$ as the time when $a$ reaches a minimum $a_{\text{min}} = \sqrt{\Gamma}$ (units where $c = 1$). We stress the fact that this solution doesn’t need the introduction of a cosmological constant for an accelerating radius $a$.

Fig. 1 shows the behaviour of $a(t)$ for the different cases just analysed.

![Figure 1: Evolution of radius $a(t)$ in time for the three cases of $k$ (-1, 0 and +1). The fourth case, with $\Gamma < 0$ is shown in blue.](image)

It is interesting to note that the mere introduction of the extra dimension—even in vacuum—gives the Friedmann type equation (2.4), and then we can consider an effective density $\rho_{\text{eff}} = \frac{3\Gamma}{8\pi a^3}$. In the classical FRW model this density corresponds to ultra-relativistic matter, related with pressure by $p = \rho/3$. Then in this model $\Gamma < 0$ corresponds to a solution with an effective negative pressure. We note that according to (2.1), we may also identify the term $\frac{\dot{a}b}{ab}$ as the effective density (more precisely $\frac{\dot{a}b}{ab} = -\frac{8\pi}{3} \rho_{\text{eff}}$).

By using (2.4) and (3.1), we obtain

$$\frac{\Gamma}{a^4} + \frac{\Gamma - k_1 a^2}{a^3 b} \frac{db}{da} = \frac{8\pi \mu}{3a^3 b}. \quad (3.6)$$

This can be integrated to obtain $b = b(a)$. We have three different cases:

$$k = 0 \rightarrow b = \frac{4\pi \mu a}{3\Gamma} + \frac{\alpha}{a}, \quad (3.7)$$
with $\alpha = \text{const}$. Observe also that making the identification $\alpha = \beta \sqrt{\Gamma}$ in (3.7) we obtain (2.5) as limit case when $\rho \to 0$ (contrast for instance with Ref. [3]).

$$k = 1 \to b = \frac{8\pi\mu}{3a} + \beta \dot{a},$$

where $\dot{a} = \frac{\sqrt{\Gamma - a^2}}{a}$ and $\beta$ is a constant consistent with (2.5). The third case is

$$k = -1 \to b = -\frac{8\pi\mu}{3a} + \beta \dot{a}.$$  

Here, $\dot{a} = \frac{\sqrt{a^2 + \Gamma}}{a}$ and $\Gamma$ can be positive or negative as we saw before.

We can see that the set of solutions for $b(a)$ given in (3.7)-(3.9) yield for $\mu \to 0$ the limit value $b \to \beta \dot{a}$, in agreement with (2.5). We show the schematic evolution of $b$ in time in Fig. 2 for the four cases considered.

![Schematic evolution of radius b(t).](image)

Figure 2: Schematic evolution of radius $b(t)$.

Now consider a nonvanishing cosmological constant; in this case, in the LHS of equations (2.3) and (3.1) results an extra term $-\frac{\Lambda}{3}$. The analogue of (2.4) turns out to be

$$\dot{a}^2 + k_1 = \frac{\Lambda}{6} a^2 + \frac{\Gamma}{a^2}.$$  

(3.10)
Solving for $a(t)$, we obtain

$$
a^2 = \frac{3k}{\Lambda} + \frac{M}{2} e^{\sqrt{2\Lambda} \Delta t} + \frac{1}{M} \left( \frac{9k^2}{2\Lambda^2} - \frac{3\Gamma}{\Lambda} \right) e^{-\sqrt{2\Lambda} \Delta t},
$$

(3.11)

where we have defined $M = \left( a_0^2 - \frac{3k}{\Lambda} \right) + \sqrt{\left( a_0^2 - \frac{3k}{\Lambda} \right)^2 + \frac{6\Gamma}{\Lambda} - \frac{9k^2}{\Lambda^2}}$; also $\Delta t = t - t_0$ and $a_0 = a(t_0)$. Fig. 3 shows the schematic behaviour for the evolution of $a(t)$.

![Figure 3: Evolution of radius $a(t)$ in time when $\Lambda \neq 0$.](image)

By using (3.10) and the first Einstein equation with cosmological constant, we have

$$
\frac{\Gamma}{a^4} + \frac{\dot{a}b}{ab} - \frac{\Lambda}{6} = \frac{8\pi\mu}{3a^3b}.
$$

(3.12)

Taking $b$ as function of $a$, the result is

$$
b = \frac{8\pi\mu (\Lambda a^2 - 3k)}{3 (2\Gamma\Lambda - 3k^2)} a + \beta \sqrt{\frac{\Gamma + \frac{\Lambda}{6} a^4 - ka^2}{a}}.
$$

(3.13)

We have used the same integration constant $\beta$ stressing the fact that $\Lambda = 0$ reduces to the previous cases.
IV. (1+D+d)-dimensional model.

Now we turn to the general case \( D + d + 1 \). In this case it cannot be defined
exact solutions for the model, except for some assumptions (e.g. see Ref.
[13]). Here, we remark a general relation between \( a(t) \) and \( b(t) \). We define
three functions of time \( F, G \) and \( H \), given by

\[
F \equiv \frac{\ddot{a}}{a} + (D - 1) \left( \frac{\dot{a}^2 + k_1}{a^2} \right) + d \frac{\dot{a} \dot{b}}{ab},
\]

\[
G \equiv \frac{\ddot{b}}{b} + (d - 1) \left( \frac{\dot{b}^2 + k_2}{b^2} \right) + D \frac{\dot{a} \dot{b}}{ab}
\]

and

\[
H \equiv D \left[ \frac{\ddot{a}}{a} + (D - 1) \left( \frac{\dot{a}^2 + k_1}{a^2} \right) \right] + d \left[ \frac{\ddot{b}}{b} + (d - 1) \left( \frac{\dot{b}^2 + k_2}{b^2} \right) \right] + dD \frac{\dot{a} \dot{b}}{ab}.
\]

In terms of them, (1.2) can be written as

\[
D \left[ F - \frac{\ddot{a}}{a} \right] + d \left[ G - \frac{\ddot{b}}{b} \right] = 0.
\]

Also, (1.3) and (1.4) can be written respectively as

\[
H - F = 0
\]

and

\[
H - G = 0.
\]

Both equations imply that

\[
F = G.
\]

Now we consider a perfect fluid in higher dimensions, such that the energy-
momentum tensor takes the form

\[
T_{\mu\nu} = (p + \rho)U_\mu U_\nu + pg_{\mu\nu};
\]

here \( p \) and \( \rho \) are the pressure and the density of the fluid respectively, and
\( U_\mu \) is the \( n \)-velocity vector, which has component \( U_0 = -1 \) in this comoving
coordinate frame defined by (1.1).
It is easy to show that the first Einstein equation, namely (A.6) of the Appendix, is

\[ D\left[F - \frac{\ddot{a}}{a}\right] + d\left[G - \frac{\ddot{b}}{b}\right] = 8\pi T_{00}. \]  

(4.9)

Also, (A.7) and (A.8) corresponds to

\[ H - F = -8\pi p \]  

(4.10)

and

\[ H - G = -8\pi p, \]  

(4.11)

respectively, implying that (4.7) still holds. If we assume (4.9) to be solvable by

\[ F - \frac{\dot{a}}{a} = \frac{8\pi}{D + d} \rho \]  

(4.12)

and

\[ G - \frac{\dot{b}}{b} = \frac{8\pi}{D + d} \rho, \]  

(4.13)

by combining (4.7), (4.12) and (4.13), then results

\[ \dot{a}b - b\dot{a} = \text{const.}, \]  

(4.14)

which is analogue to the definition of classical angular momentum.

V. Final comments.

In this work we studied a general model with \( D \) dimensions related by the same evolving radius \( a \), and \( d \) dimensions with evolving radius \( b \). From the introduction of the vacuum case with 3+1 spatial dimensions in section II, we have shown in section III the evolution in time of \( a \) and \( b \) for the case of a five-dimensional dust. In this part we have followed closely [9] and we have included the case with non-vanishing cosmological constant. We can mention that in the process we have obtained the correct expression (3.7) for \( b = b(a) \) in the flat case \( k = 0 \) -instead of the second expression in equation (2.6) of Reference [9]- that gives as limiting case (2.5). Also, we remark again the special case \( k < 0, \Gamma < 0 \) where the solution implies an accelerated rate of expansion for \( a(t) \), without the introduction of a cosmological constant, for instance. Further, we have obtained the general solutions when \( \Lambda \) is taken into account, both for \( a(t) \) and \( b(a) \). In section IV we have obtained an interesting relation in the form of eq. (4.14) for radii \( a \) and \( b \), valid for a special distribution.
of matter in the two homogeneous spaces. Our analysis can be complementary in the study of dynamical compactification [10]-[12] and in general for extra dimensional models [15][17].

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APPENDIX

Einstein equations

Here, we follow closely Ref. [5]. According to the notation in the text, from (1.1) we obtain the nonvanishing Christoffel symbols

$$\Gamma_{ij}^0 = a \dot{a} \tilde{g}_{ij}, \quad \Gamma_{0j}^i = \frac{\dot{a}}{a} \delta_j^i, \quad \Gamma_{jk}^i = \dot{\Gamma}_{jk}^i,$$

(A.1)

$$\Gamma_{ab}^0 = b \dot{b} \hat{g}_{ab}, \quad \Gamma_{0b}^a = \frac{\dot{b}}{b} \delta_a^b, \quad \Gamma_{bc}^a = \hat{\Gamma}_{bc}^a,$$

where \(\tilde{\Gamma}_{jk}^i\) and \(\hat{\Gamma}_{bc}^a\) refers to Christoffel symbols in the adequate dimensional reduction. We use the same notation as indicative of dimensional reduction; for instance, just as \(\tilde{\Gamma}_{jk}^i = \frac{1}{2} g_{il} \left( g_{jl,k} + g_{lk,j} - g_{jk,l} \right)\), we have \(\hat{\Gamma}_{ij}^a = \partial_i \hat{\Gamma}_{ab}^a - \partial_b \hat{\Gamma}_{ia}^a + \hat{\Gamma}_{ij}^c \hat{\Gamma}_{ab}^c - \hat{\Gamma}_{ic}^a \hat{\Gamma}_{ab}^c\) and \(\hat{R} = \tilde{g}^{ij} \tilde{\Gamma}_{ij}^a\). After calculating the needed components of the Riemann tensor \(R_{\alpha\beta\nu\lambda} = \partial_{\alpha} \Gamma_{\beta\nu}^\nu - \partial_{\beta} \Gamma_{\alpha\nu}^\nu + \Gamma_{\alpha\lambda}^\mu \Gamma_{\beta\nu}^\mu - \Gamma_{\beta\lambda}^\mu \Gamma_{\alpha\nu}^\mu\) and contracting it as \(R_{\mu\nu} = R_{\mu\nu}^\alpha\), we obtain three classes for the non-zero Ricci tensor components: for the time component,

$$R_{00} = -D \ddot{a} - d \ddot{b};$$

(A.2)

for the D dimensions \((x^i, x^j, ...)\),

$$R_{ij} = \tilde{R}_{ij} + \tilde{g}_{ij} \left[ a \ddot{a} + (D - 1) \dot{a}^2 + d \frac{\dot{a}}{b} \dot{b} \right];$$

(A.3)

and for the d dimensions \((x^a, x^b, ...)\)

$$R_{ab} = \hat{R}_{ab} + \hat{g}_{ab} \left[ b \ddot{b} + (d - 1) \dot{b}^2 + D \frac{\dot{b}}{a} \dot{a} \right].$$

(A.4)
With another contraction, the curvature scalar is

\[ R = 2D \dddot{a} + 2d \dddot{b} + \frac{1}{a^2} [\dddot{\hat{R}} + D(D-1)\dddot{a}^2 + dD \dddot{a} \dddot{b}] + \frac{1}{b^2} [\dddot{\hat{R}} + d(d-1)\dddot{b}^2 + dD \dddot{b} \dddot{a}]. \]  
(A.5)

Of course, all the results calculated above are symmetric respect to the D-dimensional and d-dimensional subspaces defined by the metric (1.1). More precisely, we obtain the same tensors if we perform the interchange \( a \leftrightarrow b \) and \( D \leftrightarrow d \), as well as the changes they induce, v.g. \( \dddot{\hat{R}} \leftrightarrow \dddot{\hat{R}} \). Furthermore, due to the homogeneity in the mentioned subspaces, we can write \( \dddot{\hat{R}}_{ij} = k_1(D-1)\dddot{g}_{ij} \) and \( \dddot{\hat{R}}_{ab} = k_2(d-1)\dddot{g}_{ab} \), were \( k_1 \) and \( k_2 \) can take values in \( \{-1, 0, 1\} \).

With all this at hand, -working in units where \( c = G = 1 \)- we can express the following sets of Einstein equations, \( G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 8\pi T_{\mu \nu} \):

\[ G_{00} = 8\pi T_{00} \rightarrow \]

\[ \frac{D(D-1)}{2} \left( \frac{\dddot{a}^2 + k_1}{a^2} \right) + \frac{d(d-1)}{2} \left( \frac{\dddot{b}^2 + k_2}{b^2} \right) + dD \dddot{a} \dddot{b} = 8\pi T_{00}; \]  
(A.6)

\[ G_{ij} = 8\pi T_{ij} \rightarrow \{(D-1) \left[ \frac{\dddot{a}}{a} + \frac{(D-2)}{2} \left( \frac{\dddot{a}^2 + k_1}{a^2} \right) \right] \}

\[ + d \left[ \frac{\dddot{b}}{b} + \frac{(d-1)}{2} \left( \frac{\dddot{b}^2 + k_2}{b^2} \right) \right] + d(D-1) \dddot{a}^2 \dddot{g}_{ij} = -8\pi T_{ij}; \]  
(A.7)

\[ G_{ab} = 8\pi T_{ab} \rightarrow \{(d-1) \left[ \frac{\dddot{b}}{b} + \frac{(d-2)}{2} \left( \frac{\dddot{b}^2 + k_2}{b^2} \right) \right] \]

\[ + D \left[ \frac{\dddot{a}}{a} + \frac{(D-1)}{2} \left( \frac{\dddot{a}^2 + k_1}{a^2} \right) \right] + D(d-1) \dddot{a} \dddot{b} \dddot{g}_{ab} = -8\pi T_{ab}. \]  
(A.8)
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