Abstract

This paper is a letter-type version of hep-th/9806236. We discuss properties of non-linear equations of motion which describe higher-spin gauge interactions for massive spin-0 and spin-1/2 matter fields in 2+1 dimensional anti-de Sitter space. The model is shown to have $\mathcal{N} = 2$ supersymmetry and to describe higher-spin interactions of $d3 \mathcal{N} = 2$ massive hypermultiplets. An integrating flow is found which reduces the full non-linear system to the free field equations via a non-local Bäcklund-Nicolai–type mapping.

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1 Introduction and Preliminaries

In this paper, we summarize some recent results in the study of the higher-spin (HS) interactions of massive matter fields in 2+1 dimensional space-time. A most important argument to consider theories of HS gauge fields is that they may open an alternative way towards a fundamental theory presently identified with M-theory. Although a route to M-theory passes through models in higher dimensions, particularly $d = 11$ and $d = 12$, it is useful to study a $d = 3$ model which has much simpler dynamics because $d3$ HS gauge fields do not propagate. As shown below, the simplicity of the model indeed allows one to study it in great detail, leading to some general conclusions on a structure of more complicated HS models. The main result consists of the constructive definition of a non-linear non-local mapping which links the non-linear problem with the free one. Due to the analogy of the $d = 3$ models with the self-dual $d = 4$ models [2], we speculate that the existence of the integrating flow may be an indication of integrability of the model.

It is convenient to describe HS gauge fields within “geometric approach” to gravity [3, 4, 5, 6, 7] with vielbein $h_\mu^a$ and Lorentz connection $\omega_{\mu}^{ab}$ identified with the connection 1-forms of an appropriate space-time symmetry algebra $g$. For example, one can use gauge fields $A_{\mu}^{BC} = -A_{\mu}^{CB}$ of the anti-de Sitter (AdS) algebra $g = o(d - 1, 2)$ to describe the geometry of the $d$-dimensional AdS space-time (the indices $B, C = 0, ..., d$ are raised and lowered by the flat metrics $\eta^{BC} = \text{diag}(+ - \cdots +)$) setting $\omega_{\mu}^{ab} = A_{\mu}^{ab}$ and $h_\mu^a = (\sqrt{2}\lambda)^{-1}A_\mu^a$ with the conventions $a, b = 0, ..., d - 1$, and $B = (b, \cdot)$. Here $\lambda \neq 0$ is some constant. The respective $o(d - 1, 2)$ gauge curvatures have the form

$$R_{\mu\nu}^{ab} = \partial_\mu\omega_{\nu}^{ab} + \omega_\nu^c\omega^{cb}_{\mu} - 2\lambda^2 h_\mu^a h_\nu^b - (\mu \leftrightarrow \nu),$$

$$R_{\mu\nu}^a = \partial_\mu h_\nu^a + \omega_\nu^c h^{cb}_\mu - (\mu \leftrightarrow \nu).$$

Lorentz connection $\omega_{\mu}^{ab}$ is expressed via vielbein $h_\mu^a$ with the aid of the constraint $R_{\mu\nu}^a = 0$ ($h_\mu^a$ is assumed to be non-degenerate). Substituting $\omega_{\mu}^{ab} = \omega_{\mu}^{ab}(h)$ into (1), the equation $R_{\mu\nu}^{ab} = 0$ becomes equivalent to $R_{\mu\nu}^{ab} = 2\lambda^2(h_\mu^a h_\nu^b - h_\nu^a h_\mu^b)$, where $R_{\mu\nu}^{ab} = \partial_\mu\omega_\nu^{ab}(h) + \omega_\nu^c(h)\omega_\mu^{cb}(h) - (\mu \leftrightarrow \nu)$ is the Riemann tensor. Therefore, the equations $R_{\mu\nu}^{ab} = 0$ and $R_{\mu\nu}^a = 0$ describe AdS space-time with radius $(\sqrt{2}\lambda)^{-1}$. This is how AdS space appears as a vacuum solution of the HS equations considered below. A role of the algebra $o(d - 1, 2)$ is twofold: its connection 1-forms are identified with the dynamical fields of the theory and it serves as the symmetry algebra of the most symmetric vacuum solution, AdS space-time symmetry.

In the case of $d = 2 + 1$, the AdS algebra is $o(2, 2)$, and the gravitational action is the Chern-Simons action, $S^W = \int_{M_3} \text{str}(w \wedge dw + \frac{2}{3}w \wedge w \wedge w)$, where $w$ is the $o(2, 2)$ connection 2-form [3].

The $d = 2 + 1$ HS superalgebras used below can be described as follows. Let $Aq(2; \nu)$ be an associative algebra [9] with a general element of the form

$$f(\hat{y}, k) = \sum_{\alpha_1, ... \alpha_n} \frac{1}{n!} f^{A_{\alpha_1}...A_{\alpha_n}}(k) \hat{y}_{\alpha_1} \cdots \hat{y}_{\alpha_n},$$

where $\nu = 0, 1$. The respective superalgebra is denoted $Aq(2; \nu)$. The simplicity of the $d = 2 + 1$ model indeed allows one to study it in great detail, leading to some general conclusions on a structure of more complicated HS models. The main result consists of the constructive definition of a non-linear non-local mapping which links the non-linear problem with the free one. Due to the analogy of the $d = 3$ models with the self-dual $d = 4$ models [2], we speculate that the existence of the integrating flow may be an indication of integrability of the model.
under condition that the coefficients $f^{A,a_1...a_n}$ are symmetric with respect to the indices $\alpha_j = 1, 2$, while the generating elements $\hat{g}_\alpha$, $k$ satisfy the relations

$$[\hat{g}_\alpha, \hat{g}_\beta] = 2i\epsilon_{\alpha\beta}(1 + \nu k), \quad k\hat{g}_\alpha = -\hat{g}_\alpha k, \quad k^2 = 1, \quad (4)$$

where $\nu$ is an arbitrary number ($\alpha, \beta, \gamma = 1, 2$ are $d = 2 + 1$ spinor indices lowered and raised by the symplectic form $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \epsilon_{12} = \epsilon^{12} = 1, A^a = \epsilon^{a\beta}A_\beta, A_\alpha = A_\beta\epsilon_{\beta\alpha}$). Then, the $d3$ HS superalgebra $hs(2; \nu)$ is the Lie superalgebra canonically related to the associative algebra $Aq(2; \nu)$ (i.e. product law in $hs(2; \nu)$ is identified with (anti)commutators in $Aq(2; \nu)$ for the $Z_2$ grading counting a number of spinor indices, i.e. $\pi(\hat{g}_\gamma) = 1, \pi(k) = 0$).

To describe a doubling of the elementary algebras in $g = hs(2; \nu) \oplus hs(2; \nu)$ analogous to $o(2, 2) \sim sp(2) \oplus sp(2)$ it is convenient to introduce an additional central involutive generating element $\psi$,

$$[\psi, \hat{g}_\alpha] = 0, \quad [\psi, k] = 0, \quad \psi^2 = 1. \quad (5)$$

The two simple components of $g$ are singled out by the projection operators $P_\pm = \frac{1}{2}(1 \pm \psi)$. Field strengths for the gauge algebra $g$ are

$$R(\hat{g}, \psi, k|x) = d\omega(\hat{g}, \psi, k|x) - \omega(\hat{g}, \psi, k|x) \wedge \omega(\hat{g}, \psi, k|x), \quad (6)$$

where $d = dx^\nu \frac{\partial}{\partial x^\nu}$ and $\omega(\hat{g}, \psi, k|x)$ are the gauge fields for $g$ of the form (3) depending also on $\psi$ and the commuting space-time coordinates $x^\nu$, $\mu = 0, 1, 2$.

The $d3$ AdS space-time symmetry algebra $o(2, 2) \sim sp(2) \oplus sp(2)$ is the subalgebra of $g$ spanned by the bilinears

$$L_{\alpha\beta} = \frac{1}{4i}\{\hat{g}_\alpha, \hat{g}_\beta\}, \quad P_{\alpha\beta} = \frac{1}{4i}\{\hat{g}_\alpha, \hat{g}_\beta\}\psi. \quad (7)$$

The pure gauge HS action has the Chern-Simons form with the supertrace defined in [3]. It reduces to the Witten gravity action [8] in the spin 2 sector and to the Blencowe’s HS action [1] in the case $\nu = 0$.

An important question we address below at the level of equations of motion is how to introduce interactions of HS gauge fields with propagating matter fields. We will follow the “unfolded formulation” of [10], rewriting dynamical equations in a form of certain zero-curvature conditions and covariant constancy conditions

$$d\omega = \omega \wedge \omega, \quad dB^A = \omega^i(t_i)^A_B B_B^i, \quad (8)$$

supplemented with some gauge invariant constraints

$$\chi(B) = 0 \quad (9)$$

that do not contain space-time derivatives. Here $\omega(x) = dx^\nu \omega^i_\nu(x)T_i$ is a gauge field of some Lie superalgebra $l$ ($T_i \in l$), and $B^A(x)$ is a set of 0-forms that take values in a representation space of some representation $(t_i)^B_A$ of $l$.

An interesting property of this form of the equations is that their dynamical content is hidden in the constraints (3). Indeed, locally one can integrate (8) as $\omega = dg(x)g^{-1}(x)$, $B(x) = t_{g(x)}(B_0)$, where $g(x)$ is an arbitrary invertible element, while $B_0$ is an arbitrary
$x$ - independent representation element and $t(g(x))$ is the exponential of the representation $t$ of $l$. Since the constraints $\chi(B)$ are gauge invariant one is left with the only condition $\chi(B_0) = 0$. Let $g(x_0) = I$ for some point of space-time $x_0$. Then $B_0 = B(x_0)$. To understand how restrictions on values of some 0-forms at a fixed point of space-time can lead to non-trivial dynamics one should take into account that in the interesting examples the set of 0-forms $B$ is reach enough to describe all space-time derivatives of dynamical fields, while the constraints (9) just impose all restrictions on the space-time derivatives required by the dynamical equations under consideration. Given solution of (9) one knows all derivatives of the dynamical fields compatible with the field equations and can therefore reconstruct these fields by analyticity in some neighborhood of $x_0$.

To illustrate this point let us consider an example of a scalar field $\phi$ obeying the massless Klein-Gordon equation $\Box \phi = 0$ in a flat space-time of an arbitrary dimension $d$. Here $l$ is identified with the Poincaré algebra $iso(d - 1, 1)$ which gives rise to the gauge fields $\omega_{\nu} = (h_{\nu}^a, \omega_{\nu \ab})$ $(a, b = 0, \ldots, d - 1)$. The zero curvature conditions of $iso(d - 1, 1)$

\[ R_{\nu\mu}^a = 0, \quad R_{\nu\mu}^{ab} = 0 \quad (10) \]

imply that the vielbein $h_{\nu}^a$ and Lorentz connection $\omega_{\nu \ab}$ describe the flat geometry. Fixing the local Poincaré gauge transformations one can set

\[ h_{\nu}^a = \delta_{\nu}^a, \quad \omega_{\nu \ab} = 0 \quad (11) \]

To describe dynamics of a spin zero massless field $\phi(x)$ let us introduce an infinite collection of 0-forms $\phi_{a_1 \ldots a_n}(x)$ which are totally symmetric traceless tensors

\[ \eta^{bc} \phi_{bca_3 \ldots a_n} = 0 \quad (12) \]

where $\eta^{bc}$ is the flat Minkowski metrics. The “unfolded” version of the Klein-Gordon equation has a form of the following infinite chain of equations

\[ \partial_{\nu} \phi_{a_1 \ldots a_n}(x) = h_{\nu}^b \phi_{a_1 \ldots a_n b}(x) \quad (13) \]

where we have replaced the Lorentz covariant derivative by the ordinary flat derivative $\partial_{\nu}$ using the gauge condition (11). The tracelessness condition (12) is a specific realization of the constraints (9), while the system of equations (10), (13) is a particular example of the equations (8). It is easy to see that this system is formally consistent, i.e. $\partial_{\mu}$ differentiation of (13) does not lead to new conditions after antisymmetrization $\nu \leftrightarrow \mu$. This property is equivalent to the fact that the set of zero forms $\phi_{a_1 \ldots a_n}$ spans some representation of the Poincaré algebra.

To show that the system (13) is equivalent to the free massless field equation $\Box \phi(x) = 0$ let us identify the scalar field $\phi(x)$ with the $n = 0$ member of the tower of 0-forms $\phi_{a_1 \ldots a_n}$. Then the first two equations (13) read

\[ \partial_{\nu} \phi = \phi_{\nu} \quad (14) \]

and

\[ \partial_{\nu} \phi_{\mu} = \phi_{\mu \nu} \quad (15) \]
respectively. Eq. (14) tells us that $\phi_\nu$ is a first derivative of $\phi$. Eq. (15) implies that $\phi_{\nu\mu}$ is a second derivative of $\phi$. However, because of the tracelessness condition (12) it imposes the Klein-Gordon equation $\Box \phi = 0$. It is easy to see that all other equations in (13) express highest tensors in terms of the higher-order derivatives

$$\phi_{\nu_1...\nu_n} = \partial_{\nu_1} \ldots \partial_{\nu_n} \phi$$

and impose no additional conditions on $\phi$. The tracelessness conditions are all satisfied once the Klein-Gordon equation is true.

Let us note that the system (13) without the constraints (12) remains formally consistent but is dynamically empty just expressing all highest tensors in terms of derivatives of $\phi$ according to (16). This simple example illustrates how constraints can be equivalent to the dynamical equations. The specificity of the HS dynamics considered below that makes such an approach adequate is that HS symmetries mix all orders of derivatives which therefore are contained in a representation space of HS symmetries.

2 Nonlinear System

Now let us turn to the system for the massive matter fields interacting via HS gauge potentials in $d = 2 + 1$. The full nonlinear system of equations, which is a particular realization of the equations (8) and (9), is formulated in terms of the generating functions $W(z, y; \psi_{1, 2}, k, \rho|x)$, $B(z, y; \psi_{1, 2}, k, \rho|x)$, and $S_\alpha(z, y; \psi_{1, 2}, k, \rho|x)$ that depend on the space-time coordinates $x^\nu (\nu = 0, 1, 2)$, auxiliary commuting spinors $z_\alpha, y_\alpha (\alpha = 1, 2)$, $[y_\alpha, y_\beta] = 0$, a pair of Clifford elements $\{\psi_i, \psi_j\} = 2\delta_{ij} (i = 1, 2)$ that commute to all other generating elements, and another pair of Clifford-type elements $k$ and $\rho$ which have the following properties

$$k^2 = 1, \quad \rho^2 = 1, \quad \kappa \rho + \rho k = 0, \quad k y_\alpha = -y_\alpha k, \quad k z_\alpha = -z_\alpha k, \quad \rho y_\alpha = y_\alpha \rho, \quad \rho z_\alpha = z_\alpha \rho.$$ (17)

The space-time 1-form $W = dx^\nu W_\nu(z, y; \psi_{1, 2}, k, \rho|x)$,

$$W_\mu(z, y; \psi_{1, 2}, k, \rho|x) = \sum_{A, B, C, D = 0} \frac{1}{m!n!} W^{ABCD}_{\mu; \alpha_1...\alpha_m, \beta_1...\beta_n}(x) \times k^A \rho^B \psi_i^C \psi_i^D z^{\alpha_1} \ldots z^{\alpha_m} y^{\beta_1} \ldots y^{\beta_n}$$ (18)

is the generating function for HS gauge fields. $B = B(z, y; \psi_{1, 2}, k, \rho|x)$ is the generating function for the matter fields. The components of its expansion analogous to (18) are identified with the $d3$ matter fields and all their on-mass-shell non-trivial derivatives. $S_\alpha(z, y; \psi_{1, 2}, k, \rho|x)$ describes auxiliary and pure gauge degrees of freedom. The multi-spinorial coefficients in the expansions like (18) of the functions $W_\mu$, $B$, and $S_\gamma$ carry standard Grassmann parity in accordance with the number of spinor indices.

The generating functions are treated as elements of an associative algebra with the product law

$$(fg)(z, y; \psi_{1, 2}, k, \rho) = \frac{1}{(2\pi)^2} \int d^2ud^2v \exp(iu_\alpha v^\alpha) f(z+u, y+v; \psi_{1, 2}, k, \rho) g(z-v, y+v; \psi_{1, 2}, k, \rho),$$ (19)
where the integration variables \( u \) and \( v \) are required to satisfy the commutation relations similar to those of \( y \) and \( z \) in [17]. This product law yields a particular realization of Heisenberg-Weyl algebra, \([y_\alpha, y_\beta]_* = -z_\alpha z_\beta = 2i\epsilon_{\alpha\beta} = 2i\epsilon_{\alpha\beta}(1 + B*K), \quad [y_\alpha, z_\beta]_* = 0 ([a, b]_* = a*b - b*a). \]

The full system of equations is analogous to the d3 massless system of [11],

\[
dW = W * \wedge W, \quad dB = W * B - B * W, \quad dS_\alpha = W * S_\alpha - S_\alpha * W, \tag{20}
\]

\[
S_\alpha * S_\beta - S_\beta * S_\alpha = -2i\epsilon_{\alpha\beta}(1 + B*K), \quad S_\alpha * B = B * S_\alpha. \tag{21}
\]

Here \( d = dx^\nu \frac{\partial}{\partial x^\nu} \) and \( K = ke^{i(zy)} \), \((zy) = za^\alpha y^a\).

With the aid of the involutive automorphism \( \rho \to -\rho, \quad S_\alpha \to -S_\alpha \) one can truncate the system (20), (21) to the one with the fields \( W \) and \( B \) independent of \( \rho \) and \( S_\alpha \) linear in \( \rho \),

\[
W(z, y; \psi_{1,2}, k; \rho|x) = W(z, y; \psi_{1,2}, k|x), \quad B(z, y; \psi_{1,2}, k; \rho|x) = B(z, y; \psi_{1,2}, k|x),
\]

\[
S_\alpha(z, y; \psi_{1,2}, k; \rho|x) = \rho s_\alpha(z, y; \psi_{1,2}, k|x). \tag{22}
\]

From now on we consider this reduced system. Eqs. (20), (21) are invariant under the infinitesimal HS gauge transformations

\[
\delta W = d\varepsilon - W * \varepsilon + \varepsilon * W, \quad \delta B = \varepsilon * B - B * \varepsilon, \quad \delta S_\alpha = \varepsilon * S_\alpha - S_\alpha * \varepsilon, \tag{23}
\]

where \( \varepsilon = \varepsilon(z, y; \psi_{1,2}, k|x) \) is an arbitrary gauge parameter.

To elucidate the dynamical content of the system (20), (21), one first of all has to find an appropriate vacuum solution. There exists a class of vacuum solutions [12]. The simplest one is

\[
B_0 = \nu = \text{const}, \tag{25}
\]

\[
S_{0\alpha} = \rho \left(z_\alpha + \nu(z_\alpha + y_\alpha) \int_0^1 dt e^{it(zy)}k\right), \tag{26}
\]

\[
W_0(z, y; \psi_{1,2}, k|x) = W_0(y; \psi_{1,2}, k|x), \tag{27}
\]

where

\[
\tilde{y}_\alpha = y_\alpha + \nu(z_\alpha + y_\alpha) \int_0^1 dt(t - 1)e^{it(zy)}k \tag{28}
\]

are the elements with the defining property \([\tilde{y}_\alpha, S_{0\beta}]_* = 0\). An arbitrary “function” \( W_0(y; \psi_{1,2}, k|x) \) on the r.h.s. of (27) contains star-products of \( \tilde{y}_\alpha \). It is important that \( \tilde{y}_\alpha \) obey the deformed oscillator algebra commutation relations (18),

\[
[\tilde{y}_\alpha, \tilde{y}_\beta]_* = 2i\epsilon_{\alpha\beta}(1 + \nu k), \quad \tilde{y}_\alpha k = -k\tilde{y}_\alpha. \tag{29}
\]

Eqs. (27)-(27) solve all the equations (20), (21) except for

\[
dW_0 = W_0 * \wedge W_0, \tag{30}
\]

which requires a further specification of \( W_0 \). An appropriate ansatz is

\[
W_0 = \omega_0 + \lambda h_0 \psi_1, \quad \omega_0 = \frac{1}{\delta i} \omega_0^\beta(x)\{\tilde{y}_\alpha, \tilde{y}_\beta\}_*, \quad h_0 = \frac{1}{\delta i} h_0^\beta(x)\{\tilde{y}_\alpha, \tilde{y}_\beta\}_*, \tag{31}
\]

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where $\omega_0^{\alpha\beta}(x)$ and $h_0^{\alpha\beta}(x)$ are identified with Lorentz connection and dreibein of the background space and are required to solve (30). Here the properties of the deformed oscillators (29) play a crucial role, guaranteeing that the anticommutators $\{y_\alpha, y_\beta\}_* = y_\alpha y_\beta y_\beta y_\alpha$ satisfy the $sp(2)$ commutation relations for all $\nu$. As a result, the gauge fields (31) take values in the $d3$ AdS algebra $o(2,2) \sim sp(2) \oplus sp(2)$ and (30) describes AdS background.

Once a vacuum solution is known, one can study the system (20), (21) perturbatively expanding the fields as

$$ B = B_0 + B_1 + \ldots, \quad S_\alpha = S_{0\alpha} + S_{1\alpha} + \ldots, \quad W = W_0 + W_1 + \ldots. $$

Substitution of these expansions into (20), (21) gives in the lowest order

$$ D_0 W_1 = 0, $$

$$ D_0 C = 0, $$

$$ D_0 S_{1\alpha} = [W_1, S_{0\alpha}]_*, $$

$$ [S_{0\alpha}, S_{1\beta}]_* - [S_{0\beta}, S_{1\alpha}]_* = -2i\epsilon_{\alpha\beta} C*K, $$

$$ [S_{0\alpha}, C]_* = 0, $$

where we denote $C = B_1$ and $D_0$ is the background covariant derivative that acts on a $r$-form $P$ as $D_0 P = dP - W_0 \wedge P + (-)^r P \wedge W_0$.

To analyze the system (33)-(37) one proceeds as follows. From (37), one concludes that $C$ has a form similar to (27), i.e. $C = C(\tilde{y}; \psi_{1,2}, k|x)$. Expanding $C$ as $C = C^{aux}(\tilde{y}; \psi_{1}, k|x) + C^{dyn}(\tilde{y}; \psi_{1}, k|x)\psi_{2}$, one identifies $C^{aux}$ with some topological fields which carry no degrees of freedom, and $C^{dyn}$ with the generating function for the spin 0 and spin 1/2 matter fields. Namely, in accordance with the normal spin-statistics, $\tilde{y}$-even (odd) part of $C^{dyn}$ identifies with the generating function for spin 0 (1/2) matter fields, along with all their on-mass-shell non-trivial derivatives [13]. The equation (34) amounts to free field equations. Resolving the constraints (36), one reconstructs the auxiliary field $S_{1\alpha}$ as a linear functional of $C$, $S_{1\alpha} = S_{1\alpha}(C)$, up to a gauge ambiguity. Then, (35) allows one to express a part of degrees of freedom in $W_1$ via $C$, while the rest modes, which belong to the kernel of the mapping $[S_{0\alpha}, \ldots]_*$, remain free. These free modes are again arbitrary functions of $\tilde{y}_\alpha$, i.e.

$$ W_1 = \omega(\tilde{y}; \psi_{1,2}, k|x) + \Delta W_1(C), $$

where $\omega(\tilde{y}; \psi_{1,2}, k|x)$ corresponds to the HS gauge fields, and the dynamical equations for them are imposed by eq. (33) after (35) is solved. Eq. (33) describes the $C$-dependent first order corrections to the HS strengths for $\omega$, which are argued below to vanish. In principle, one can proceed similarly in the highest orders. However, the computation complicates enormously in the second order, and one needs some efficient methods to proceed.
3 Integrating Flow

A remarkable property of eqs. (20), (21) is that they admit a flow which allows one to express constructively solutions of the full system in terms of free fields. Since our perturbation expansion is just an expansion in powers of the physical fields which are identified with the deviation $C$ of $B$ from its vacuum value $\nu$, let us introduce a formal perturbation expansion parameter $\eta$ (i.e. the coupling constant) as follows

$$B(\eta) = \nu + \eta B(\eta).$$

Simultaneously, the rest of the fields acquire a formal dependence on $\eta$, $W = W(\eta)$ and $S_{\alpha} = S_{\alpha}(\eta)$. The system (20), (21) takes a form

$$dW = W \wedge W, \quad dB = W \wedge B - B \wedge W, \quad dS_{\alpha} = W \wedge S_{\alpha} - S_{\alpha} \wedge W,$$

$$S_{\alpha} \wedge S_{\alpha} = -2i(1 + \nu K + \eta B \wedge K), \quad S_{\alpha} * B = B * S_{\alpha}.$$

Now, one observes that for the limiting case $\eta = 0$ the system (40), (41) reduces to the free one. Indeed, setting $\nu = B_0$, $B(0) = C$, $W(0) = \omega$, $S_{\alpha}(0) = S_{0\alpha}$, we see that at $\eta = 0$, the system (40), (41) amounts to $S_{\alpha} = S_{0\alpha}$, $W = \omega(\tilde{y};\psi_{1,2},k|x)$, $B = C(\tilde{y};\psi_{1,2},k|x)$, $d\omega = \omega \wedge \omega$ and $dC = \omega \wedge C - C \wedge \omega$. The latter two equations describe vacuum background gauge fields and free matter field equations, respectively. This situation is similar to that with contractions of Lie algebras. For all values of $\eta \neq 0$, the systems of equations (40), (41) are pairwise equivalent since the field redefinition (39) is non-degenerate. On the other hand, although the field redefinition (39) degenerates at $\eta = 0$, eqs. (40), (41) still make sense for $\eta = 0$. This limiting system describes the free field dynamics.

Remarkably, the two inequivalent systems are still related to each other. To show this let us define a flow with respect to $\eta$ as follows:

$$\frac{\partial W}{\partial \eta} = (1 - \mu) B * \frac{\partial W}{\partial \nu} + \mu \frac{\partial W}{\partial \nu} * B,$$

$$\frac{\partial B}{\partial \eta} = (1 - \mu) B * \frac{\partial B}{\partial \nu} + \mu \frac{\partial B}{\partial \nu} * B,$$

$$\frac{\partial S_{\alpha}}{\partial \eta} = (1 - \mu) B * \frac{\partial S_{\alpha}}{\partial \nu} + \mu \frac{\partial S_{\alpha}}{\partial \nu} * B,$$

where $\mu$ is an arbitrary parameter. This flow is compatible with (40), (41). Therefore, solving the system (42)-(44) with the initial data $B(\eta = 0) = C$, $W(\eta = 0) = \omega$, $S_{\alpha}(\eta = 0) = S_{0\alpha}$, we can express solutions of the full nonlinear system at $\eta = 1$ via solutions of the free system at $\eta = 0$. The existence of the integrating flow (42)-(44) takes its origin in the fact that from the viewpoint of the system (20), (21), $B$ behaves like a constant: it commutes to $S_{\alpha}$ and satisfies covariant constancy condition. Knowledge of the vacuum solution with $B = \nu$ can be used to reconstruct the full dependence of $B$. Indeed, the
A qualitative meaning of (42)-(44) is that a derivative with respect to $\eta B$ is the same as that with respect to $\nu$. (All fields acquire a non-trivial dependence on $\nu$ via the vacuum solution as it follows e.g. from eqs. (28), (29).) The flows (42)-(44) at different $\mu$ are equivalent modulo gauge transformations $[12]$. This approach allows one to derive the relevant field redefinitions order by order since the r.h.s. of (42)-(44) contain one extra power of $B$. In particular, one can easily derive in the first order a field redefinition necessary to show that the HS gauge field strengths do not admit nontrivial sources linear in fields. In the first order this field redefinition is local. This result is expected since in the lowest order the non-trivial r.h.s. of the equations for HS gauge fields are the HS currents bilinear in the matter fields.

Remarkably, the method works in all higher orders, thus reducing the full non-linear problem to the free one. The point, however, is that beyond the first order one has to be careful in making statements on the locality of the mapping induced by the flow (12)-(14). Actually, although it does not contain explicitly space-time derivatives, it contains them implicitly via highest components $C_{\alpha_1...\alpha_n}$ of the generating function $C(\bar{y})$ which are identified $[13]$ with the highest derivatives of the matter fields due to the equations (34). For example, at $\mu = 0$ in the second order in $C$ one gets $\partial_{\eta} B_2(z, y) = C(\bar{y}) * \partial_{\nu} C(\bar{y})$. Because of the properties of the $*$-product, for each fixed rank multispinorial component of the l.h.s. of this formula, its r.h.s. is an infinite series involving bilinear combinations of the components $C_{\alpha_1...\alpha_n}$ with all $n$. Therefore, the r.h.s. of (12)-(14) effectively contain all orders of the space-time derivatives, i.e. the mapping resulting from (12)-(14) describes some non-local transformation. This means that one cannot treat the system (20), (21) as locally equivalent to the free system.

To illustrate this issue it is instructive to consider an example of some matter field $C$ interacting with the gravitational field fluctuating near the AdS vacuum solution. Schematically, the mechanism is as follows. Linearized Einstein equations have a form (with appropriate gauge fixings)

\[(L^C - \Lambda^2)h_{\mu\nu} = T_{\mu\nu}(C),\]  

where $h_{\mu\nu}$ is the fluctuational part of the metric tensor, $L^C$ is the linear operator corresponding to the l.h.s. of the free field equations of the matter fields $L^C C = 0$, while $\Lambda = \alpha \lambda$ with some numerical coefficient $\alpha \neq 0$. It is important that when the cosmological constant is non-vanishing, the term with $\Lambda^2$ turns out to be non-vanishing too. This property allows one to solve formally (15) by a field redefinition

\[h'_{\mu\nu} = h_{\mu\nu} - (L^C - \Lambda^2)^{-1}T_{\mu\nu}(C) = h_{\mu\nu} + \Lambda^{-2} \sum_{n=0}^{\infty} (\Lambda^{-2} L^C)^n T_{\mu\nu}(C).\]  

Clearly, a non-vanishing dimensionful constant, the cosmological constant, plays an important role in this analysis. It can be shown $[12]$ that this field redefinition admits a natural realization in terms of the generating function $C$ in agreement with the general analysis above.
4 $N = 2$ Supersymmetry and Truncations

The full system (20), (21) is explicitly invariant under the HS gauge transformations (24). Fixation of the vacuum solution (23), (26), (31) breaks this local symmetry down to some global symmetry, the symmetry of the vacuum. This global symmetry is generated by the parameter $\varepsilon_{gl}(x)$ obeying the conditions

$$
\begin{align*}
d\varepsilon_{gl} &= [W_0, \varepsilon_{gl}]_*, \\
[\varepsilon_{gl}, S_{0\alpha}]_* &= 0,
\end{align*}
$$

(47)

which follow from the requirement that $\delta W_0 = 0$, $\delta S_{0\alpha} = 0$ under the transformations (24) ($\delta B_0 = 0$ holds automatically for $B_0 = \text{const}$). The condition (48) implies that

$$
\varepsilon_{gl}(z, y; \psi_{1,2}, k|x) = \varepsilon_{gl}(\tilde{y}; \psi_{1,2}, k|x).
$$

(49)

The dependence of $\varepsilon_{gl}$ on the space-time coordinates $x_\mu$ is fixed by the differential equation (47) with arbitrary initial data $\varepsilon_{gl}(x_0) = \varepsilon_{gl}^0$ at any space-time point $x_0$. This infinite-dimensional global symmetry is also the symmetry of the linearized system (33)-(37).

The full global symmetry algebra contains elements $\varepsilon_{gl}$ linear in $\psi_2$. This part of the symmetry mixes the matter and topological modes in $C$ and does not allow unitary realization on quantum states. We therefore analyze the subalgebra $A^g$ of the full global symmetry algebra with the $\psi_2$-independent parameters [1].

The algebra $A^g$ is infinite-dimensional due to the dependence of $\varepsilon_{gl}(\tilde{y}; \psi_{1}, k|x)$ on $\tilde{y}_\alpha$. A maximal finite-dimensional subalgebra of $A^g$, $osp(2, 2) \oplus osp(2, 2)$, is spanned by the generators $\Pi \pm T^A$, where $\Pi \pm = \frac{1}{2}(1 \pm \psi_1)$ and $T^A = \{T_{\alpha\beta}, Q^{(1)}_{\alpha}, Q^{(2)}_{\alpha}, J\}$,

$$
T_{\alpha\beta} = \frac{1}{4i}(\tilde{y}_\alpha, \tilde{y}_\beta)_*, \quad Q^{(1)}_{\alpha} = \tilde{y}_\alpha, \quad Q^{(2)}_{\alpha} = \tilde{y}_\alpha k, \quad J = k + \nu.
$$

(50)

The fact that the generators $T^A$ close to $osp(2, 2)$ was shown in [14].

As shown in [13], the dynamical components $C_{\text{dyn}}(\tilde{y}; k, \psi_1|x)$ decompose into four bosonic and four fermionic infinite-dimensional representations of the AdS algebra $o(2, 2)$, each describing a single AdS particle. These free fields form altogether an irreducible $d3N = 2$ hypermultiplet constituted by 4 scalar and 4 spinor fields [1]:

$$
\{ C^0_+(x), C^0_-(x), C^1_+(x), C^1_-(x), C^0_+\alpha(x), C^0_{-\alpha}(x), C^1_+\alpha(x), C^1_{-\alpha}(x) \}.
$$

(51)

Here we use the following expansion of $C_{\text{dyn}}(\tilde{y}; k, \psi_1)$,

$$
C_{\text{dyn}}(\tilde{y}; k, \psi_1) = [C^0_+(\tilde{y}) + C^0_-(\tilde{y})] + [C^1_+(\tilde{y}) + C^1_-(\tilde{y})] \psi_1,
$$

(52)

1Also we factor out a trivial central element corresponding to a constant parameter $\varepsilon_{gl}(\tilde{y}; \psi_1, k|x) = \varepsilon_{gl}$.

2Note that this complexified multiplet is formally reducible with the irreducible subsets singled out by the projectors $\Pi \pm$. However, these subsets turn out to be complex conjugated to each other after imposing appropriate reality conditions (see [2] for details).
where $C_\pm = P_\pm C$, $P_\pm = \frac{1 \pm k}{2}$. Thus, the proposed field equations describe HS interactions of a $N = 2$ massive hypermultiplet. The values of mass are related to the parameter $\nu$ as follows,

$$M_\pm^2 = \lambda^2 \frac{\nu(\nu + 2)}{2}$$

for bosons, and

$$M_\pm^2 = \lambda^2 \frac{\nu^2}{2}$$

for fermions [12]. Here $\pm$ originates from $C_\pm$. The doubling of fields of the same mass is due to the operator $\psi_1$.

In the massless case $\nu = 0$ there exists [12] a truncation induced by some involutive symmetry of the system (21), (22), that preserves $N = 2$ SUSY. This symmetry is based on the automorphism $k \to -k$ and the antiautomorphism $\sigma$ [15, 16],

$$\sigma[A(z, y; \psi_{1,2}, k, \rho)] = A^{rev}(-iz, iy; \psi_{1,2}, k, \rho).$$

(Here the notation $A^{rev}(\ldots)$ means that an order of all product factors in the monomial expressions on the r.h.s. of (18) is reversed.) The reduced $N = 2$ massless supermultiplet consists of two bosonic and two fermionic fields [12],

$$\{ C_0^0(x), C_0^1(x), C_{0\alpha}^0(x), C_{1\alpha}^0(x) \}$$

(we use the convention $C(k) = C_0 + C_1 k$). This additional reduction compatible with $N = 2$ SUSY is a manifestation of the well-known shortening of massless supermultiplets.

There exists [12] also an alternative truncation of the system based on the antiautomorphism $\sigma$ (53), that breaks $N = 2$ SUSY down to $N = 1$ SUSY $osp(1,2) \oplus osp(1,2)$ with the generators

$$T_{\pm, \alpha\beta} = \frac{1}{4i} \Pi_{\pm} \{ \hat{y}_\alpha, \hat{y}_\beta \}, \quad Q_{\pm, \alpha} = \Pi_{\pm} \hat{y}_\alpha$$

and makes sense for arbitrary mass (i.e. $\nu$). The truncated $N = 1$ matter supermultiplet contains the following 2 scalars and 2 spinors [12],

$$\{ C_+^0(x), C_-^0(x), C_1^0(x), C_{0\alpha}^0(x) \}$$

with the masses (53) and (54). In the massless case $\nu = 0$, one can perform a further truncation and obtain a shortened $N = 1$ supermultiplet containing one scalar and one spinor massless field.

## 5 Inner symmetries and $N$-extended SUSY

An important fact about HS dynamical systems is that they admit a natural extension to the case with non-Abelian internal (Yang-Mills) symmetries, as was first discovered in the $d4$ case in [15, 16] and analyzed in detail for the $d3$ case in [12]. The key observation is that the system (20)-(21) remains consistent if components of all fields take their values in an arbitrary associative algebra $M$ with a unit element $I_M$, i.e. the fields $W$, $B$, and
$S_\alpha$ take values in $A_{ext} = A \otimes M$, where $A$ is the associative algebra with the general element \[13\]. The gravitational sector is associated with $A \sim A \otimes I_M$ and commutes with $M \sim I_A \otimes M$, where $I_A$ is the unit element of $A$. Therefore, $M$ describes internal symmetries in the model. For the case of semisimple finite-dimensional inner symmetries, $M$ is identified with some matrix algebra $M = Mat_n$, i.e.

$$W(z, y; \psi_{1,2}, k) \rightarrow W^i(z, y; \psi_{1,2}, k), \quad B(z, y; \psi_{1,2}, k) \rightarrow B^i(z, y; \psi_{1,2}, k),$$

$$S_\alpha(z, y; \psi_{1,2}, k; \rho) \rightarrow S_{\alpha, i}^j(z, y; \psi_{1,2}, k; \rho).$$

The extended systems describe \[12\] HS interactions of the $N = 2$ hypermultiplets in the representations $n \otimes \bar{m} \oplus m \otimes \bar{n}$ of $u(n) \oplus u(m)$ Yang-Mills symmetries. There exist consistent truncations of these extended systems based on the antiautomorphism $\sigma$ \[55\] and appropriate antiautomorphisms of $Mat_n$, which break $N = 2$ SUSY down to $N = 1$ SUSY. The reduced systems describe $N = 1$ matter supermultiplets in the representations $n \otimes m$ of $o(n) \oplus o(m)$ or $usp(n) \oplus usp(m)$ internal symmetries (for more detail see \[12\]).

Finally let us discuss extended supersymmetries with $N > 2$. In \[17\], it was shown that there is a simple way to incorporate $N$-extended superalgebras $osp(N, 2m)$ ($m = 1$ for the $d = 3$ case under consideration) by supplementing the bosonic generating elements of the Heisenberg algebra $y_a$ with the Clifford elements $\phi^i (i = 1, \ldots, N)$. In this approach the Clifford algebra $C_N$ is a particular case of the matrix algebra $Mat_{2N}$ (for $N$ even), while the generators of the $osp(N, 2m)$ are realized in terms of the bilinears

$$T_{\alpha \beta} = \{ y_\alpha, y_\beta \}, \quad Q^i_\alpha = y_\alpha \phi^i, \quad M^{ij} = [\phi^i, \phi^j],$$

provided that

$$[y_\alpha, y_\beta] = 2i \epsilon_{\alpha \beta}, \quad \{ \phi^i, \phi^j \} = 2 \delta^{ij}.$$  \hspace{1cm} (59)

Now we observe that this construction is not working for the deformed oscillators because the generators \[59\] do not form a closed algebra if $y_a$ is replaced by $\hat{y}_a$ \[4\] with $\nu \neq 0$. This result can be explained as follows. For $\nu \neq 0$ the mass of the matter supermultiplet is non-vanishing and the spin range within a supermultiplet increases with $N$. Since massive fields of spins greater than 1/2 are not included in our model, $N > 2$ extended supersymmetry cannot be realized. In the massless case, however, one can realize higher supersymmetries within only scalar and spinor fields \[18\] due to trivialization of the notion of spin for the $d = 3$ massless case (i.e. trivialization of the $d = 3$ massless little group).

Thus we conclude that the model under consideration admits $N > 2$ extended supersymmetry only for the massless vacuum $\nu = 0$.

6 Conclusions

The main conclusion is that dynamical systems based on infinite sets of HS gauge fields admit interesting structures which allow their constructive perturbative solvability. The fact that some system can be integrated order by order with the help of a non-local

$^3$We use the convention that fields in $k \otimes \bar{l}$ are complex conjugated to those in $l \otimes \bar{k}$.
field redefinition is not surprising. What is special about the HS systems is that such a field redefinition is described in a systematic way by some flow with respect to an additional evolution parameter. As a result, one can reconstruct solutions of the non-linear HS equations in terms of those of the free system by integrating ordinary differential equations. We believe that this fact can be interpreted as some sort of integrability of the $d = 3$ HS equations, although a rigorous proof of this statement remains to be elaborated. Moreover, the very concept of integrability may need to be modified in application to HS models. Indeed, as mentioned at the end of sect. 2, the part of the equations that contain space-time derivatives has a form of zero-curvature conditions and can be integrated explicitly. As a result, one is to solve only the constraint part of the system (in some analogy with Hamiltonian reduction). The latter problem is not of the evolution type however, because the equations (21) are some integral equations in the auxiliary spinor space in which the star product acts. The main result reported in this paper is how the problem of solving these constraints is reduced to solving ordinary differential equations with respect to the coupling constant $\eta$ provided that a particular solution with $B = \nu = \text{const}$ is found.

We argue that the resulting field redefinition is essentially non-local in the space-time sense. A non-local character of the transformation manifests itself in the appearance of infinite series in the inverse cosmological constant. Taking into account that HS gauge interactions are known [13] to require non-analyticity in the cosmological constant, we conjecture that HS gauge theories are indeed non-local in a certain sense. This conjecture agrees with the light-cone analysis in [20] and fits the ideology of the modern string theory.

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