Orthonormal Frame and SO(3) Kaluza-Klein Dyon

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Abstract

In previous paper, we present an SO(3) Wu-Yang-like Kaluza-Klein dyon solution satisfies the Einstein equation in the seven-dimensional spacetimes. In this note, we will show an alternative approach using an orthonormal frame, the Cartan’s structure equations, and calculating the affine spin connection one-form, curvature tensor and Ricci tensor. The results from these two different methods are coincident.

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1 Introduction

In previous paper,[1] we have shown that a new SO(3) Wu-Yang-like [2] seven-dimensional Kaluza-Klein (KK) dyon solution satisfies the Einstein equation, by calculating the Christoffel symbols and the Ricci tensor. In this note, we will show an alternative approach [3] using an orthonormal frame, the Cartan’s structure equations and calculating the affine spin connection one-form, curvature tensor and Ricci tensor.

Now consider the Kaluza-Klein theory [4] in a \((4 + N)\)-manifold \(M\) with a metric \(\bar{g}_{AB}(x)\) on \(M\) in local coordinates \(\bar{x}^A\). The line element is

\[
ds^2 = \bar{g}_{AB} d\bar{x}^A d\bar{x}^B
\]

where \(x\) parametrizes four-dimensional spacetimes, \(y\) parametrizes extra dimensions. We use \(A, B, C...\) indices to represent the total spacetimes; \(\mu, \nu, \rho...\) to represent the four-dimensional spacetimes; \(m, n, l...\) to represent the extra dimensions. \(g_{\mu\nu}\) is only a function of \(x\), and \(\gamma_{mn}\) is only a function of \(y\).

The ansatz [1] of the Kaluza-Klein dyonic metric admitting SO(3) Killing vectors is

\[
ds^2 = - e^{2\Psi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 + (dR + B^5_{\mu} dx^\mu)^2 + R^2 (d\Theta + B^6_{\mu} dx^\mu)^2 + R^2 \sin^2 \Theta (d\Phi + B^7_{\mu} dx^\mu)^2.
\]

Inserting \(L_a\) of (4) into equation (5), one gets the Killing’s equation

\[
\zeta^m_a \partial_m \zeta^n_b - \zeta^m_b \partial_m \zeta^n_a = - \epsilon_{abc} \zeta^n_c.
\]

With these Killing vectors, one can define

\[
B^m_{\mu} = \zeta^m_a A^a_{\mu},
\]
where $A^a_\mu$ is the true Yang-Mills field and $\zeta^m_a$ is only a function of $y$. Defining

$$\tilde{F}^m_{\mu\nu} \equiv \partial_\mu B^m_\nu + B^l_\mu \partial_\nu B^m_\mu - (\mu \leftrightarrow \nu) \quad (8)$$

$$= \zeta^m_a F^a_{\mu\nu}, \quad (9)$$

where $F^a_{\mu\nu}$ is the true field strength tensor of the Yang-Mills field,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon_{abc} A^b_\mu A^c_\nu. \quad (10)$$

It can be checked that the components of $\zeta^m_a$,

$$\zeta^5_a = 0, \quad \zeta^6_a = \hat{\Phi}_a, \quad \zeta^7_a = -\frac{1}{\sin \Theta} \hat{\Theta}_a, \quad (11)$$

satisfy the Killing equation (6). The gauge-field components of the Wu-Yang-like KK dyon are

$$A^6_0 = \frac{1}{r} \hat{r}^a, \quad A^a_1 = 0, \quad (14)$$

$$A^2_2 = -\hat{\phi}^a, \quad A^3_3 = \sin \theta \hat{\Theta}^a. \quad (15)$$

$A^a_1, A^a_2, A^a_3$ are just the spherical coordinate representation of the Wu-Yang monopole field in the ordinary gauge theory of four-dimensional spacetimes. The electric field of the KK dyon is

$$F^a_{01} = \frac{1}{r^2} \hat{r}^a, \quad (16)$$

while the magnetic field is

$$F^a_{23} = -\sin \theta \hat{r}^a. \quad (17)$$

The fields $B^m_\mu$ in (7) can be rewritten as

$$B^5_\mu = 0, \quad B^1_\mu = 0, \quad (18)$$

$$B^6_0 = \frac{1}{r} \hat{r} \cdot \hat{\Phi}, \quad B^6_0 = -\frac{1}{r \sin \Theta} \hat{r} \cdot \hat{\Theta}, \quad (19)$$

$$B^6_2 = -\hat{\phi} \cdot \hat{\Phi}, \quad B^7_2 = \frac{1}{\sin \Theta} \hat{\phi} \cdot \hat{\Theta}, \quad (20)$$

$$B^6_3 = \sin \theta \hat{\Theta} \cdot \hat{\Phi}, \quad B^7_3 = -\frac{\sin \theta}{\sin \Theta} \hat{\phi} \cdot \hat{\Theta}. \quad (21)$$

The nonzero components of $\tilde{F}^m_{\mu\nu}$ are

$$\tilde{F}^6_{01} = \frac{1}{r^2} \hat{r} \cdot \hat{\Phi}, \quad \tilde{F}^7_{01} = -\frac{1}{r^2 \sin \Theta} \hat{r} \cdot \hat{\Theta}, \quad (22)$$

$$\tilde{F}^6_{23} = -\sin \theta \hat{r} \cdot \hat{\Phi}, \quad \tilde{F}^7_{23} = \frac{\sin \theta}{\sin \Theta} \hat{r} \cdot \hat{\Theta}. \quad (23)$$
2 Orthonormal Frame

We now decompose the metric into vielbeins as

\[ \bar{g}_{AB} = \eta_{a\b} \bar{e}_A^a \bar{e}_B^b, \quad a, \b = 0, 1, 2, 3, 5, 6, 7, \]  

(24)

where the \( \eta_{a\b} \) is a seven-dimensional flat Minkowski space metric,

\[ \eta_{a\b} = \text{diag}(-1, 1, 1, 1, 1, 1, 1) \]  

(25)

The inverse of \( e_A^a \) is defined by

\[ E_A^a = \eta_{a\b} \bar{g}^{AB} \bar{e}_B^b \]  

(26)

which obeys

\[ E_A^a e_B^b = \delta_a^b \quad \eta_{a\b} E_A^a E_B^b = \bar{g}^{AB}. \]  

(27)

e_A^a is the matrix which transforms the coordinate basis \( dx^A \) of the cotangent space \( T^*_x(M) \) to an orthonormal basis of the same space \( T^*_x(M) \),

\[ e_A^a = e_A^a dx^A. \]  

(28)

The vielbein basis of the SO(3) KK dyonic metric can be written as

\[ e^0 = e^\Psi dt, \quad e^1 = e^\Lambda dr, \quad e^2 = r d\theta, \quad e^3 = r\sin\theta d\phi, \]  

\[ e^5 = dR, \quad e^6 = R(d\Theta + B_\mu^6 dx^\mu), \quad e^7 = R\sin\Theta(d\Phi + B_\mu^7 dx^\mu). \]  

(29)

The affine spin connection one-form \( \omega_{a\b}^b \) are introduced by

\[ de^a + \omega_{a\b}^b \land e^b = 0 \]  

(31)

and the metricity condition

\[ \omega_{a\b} = -\omega_{b\a}. \]  

(32)

The curvature 2-form is defined as

\[ R_{a\b}^d = d\omega_{a\b} + \omega_{c\b} \land \omega_{a}^c = \bar{R}_{d\b\a}^c \varepsilon^c \land \varepsilon^d. \]  

(33)

Equations (31) and (33) are called Cartan’s structure equations. The components of the curvature tensor have the relations,

\[ \bar{R}_{abcd} = -\bar{R}_{b\a\c\d} = -\bar{R}_{a\b\c\d} = \bar{R}_{c\d\a\b}. \]  

(34)

and satisfy the Bianchi identity,

\[ \bar{R}_{abcd} + \bar{R}_{a\c\d\b} + \bar{R}_{a\b\d\c} = 0. \]  

(35)
The components of the affine spin connection one-form $\omega^a_b$ of the SO(3) KK dyon can be written more explicitly as

$$\omega^0_1 = \Psi' e^{-\Lambda} e^0 + \frac{R}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Phi}) e^0 - \frac{R}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Theta}) e^7,$$

$$\omega^0_2 = 0, \quad \omega^0_3 = 0, \quad \omega^0_5 = 0,$$

$$\omega^0_6 = \frac{R}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Phi}) e^1,$$

$$\omega^0_7 = -\frac{R}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Theta}) e^1,$$

$$\omega^1_2 = -\frac{1}{r} e^{-\Lambda} e^2,$$

$$\omega^1_3 = -\frac{1}{r} e^{-\Lambda} e^3, \quad \omega^1_5 = 0,$$

$$\omega^1_6 = \frac{R}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Phi}) e^0,$$

$$\omega^1_7 = -\frac{R}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Theta}) e^0,$$

$$\omega^2_3 = -\frac{1}{r} \cot \theta e^3 + \frac{R}{2r^2} (\hat{r} \cdot \hat{\Phi}) e^6 - \frac{R}{2r^2} (\hat{r} \cdot \hat{\Theta}) e^7, \quad \omega^2_5 = 0,$$

$$\omega^2_6 = \frac{R}{2r^2} (\hat{r} \cdot \hat{\Phi}) e^3,$$

$$\omega^2_7 = -\frac{R}{2r^2} (\hat{r} \cdot \hat{\Theta}) e^3, \quad \omega^3_5 = 0,$$

$$\omega^3_6 = -\frac{R}{2r^2} (\hat{r} \cdot \hat{\Phi}) e^2, \quad \omega^3_7 = \frac{R}{2r^2} (\hat{r} \cdot \hat{\Theta}) e^2,$$

$$\omega^5_6 = -\frac{1}{R} e^6, \quad \omega^5_7 = -\frac{1}{R} e^7,$$

$$\omega^6_7 = -\frac{1}{R} \cot \theta e^7 - \frac{1}{r} e^{-\Psi} \cot \theta (\hat{r} \cdot \hat{\Phi}) e^0 - \frac{1}{r} e^{-\Psi} (\hat{r} \cdot \hat{R}) e^0 + \frac{1}{r} \cot \theta (\hat{\Phi} \cdot \hat{\Theta}) e^2 + \frac{1}{r} (\hat{\Phi} \cdot \hat{R}) e^2 - \frac{1}{r} \cot \theta (\hat{\Theta} \cdot \hat{\Theta}) e^3 - \frac{1}{r} (\hat{\theta} \cdot \hat{R}) e^3.$$

The nonzero components of the curvature tensor are

$$\bar{R}_{0101} = e^{-2\Lambda} (\Psi'' - \Psi' \Lambda' + (\Psi')^2) - \frac{3}{4} R^2 e^{-2\Psi - 2\Lambda} \{(\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2\}$$

$$\bar{R}_{0116} = \frac{R}{2r^2} e^{-\Psi - 2\Lambda} (\Psi' + \Lambda' + \frac{2}{r})(\hat{r} \cdot \hat{\Phi}),$$

$$\bar{R}_{0117} = -\frac{R}{2r^2} e^{-\Psi - 2\Lambda} (\Psi' + \Lambda' + \frac{2}{r})(\hat{r} \cdot \hat{\Theta}),$$

$$\bar{R}_{0123} = \frac{R^2}{2r^4} e^{-\Psi - \Lambda} \{(\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2\}$$

$$\bar{R}_{0156} = -\frac{1}{r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Phi}), \quad \bar{R}_{0157} = \frac{1}{r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Theta}).$$
\begin{align}
R_{0167} &= -\frac{1}{r^2} e^{-\Psi} \Lambda (\hat{r} \cdot \hat{R}), \\
R_{0213} &= \frac{R^2}{4r^4} e^{-\Psi} \Lambda \{ (\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2 \}, \\
R_{0226} &= -\frac{R}{2r^3} e^{-\Psi - 2\Lambda} (\hat{r} \cdot \hat{\Phi}), \\
R_{0227} &= \frac{R}{2r^3} e^{-\Psi - 2\Lambda} (\hat{r} \cdot \hat{\Theta}), \\
R_{0303} &= -\frac{1}{r} \Psi e^{-2\Lambda}, \\
R_{0312} &= -\frac{R^2}{4r^4} e^{-\Psi - \Lambda} \{ (\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2 \}, \\
R_{0336} &= -\frac{R}{2r^3} e^{-\Psi - 2\Lambda} (\hat{r} \cdot \hat{\Phi}), \\
R_{0337} &= \frac{R}{2r^3} e^{-\Psi - 2\Lambda} (\hat{r} \cdot \hat{\Theta}), \\
R_{0516} &= -\frac{1}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Phi}), \\
R_{0517} &= \frac{1}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Theta}), \\
R_{0606} &= -\frac{R^2}{4r^4} e^{-2\Psi - 2\Lambda} (\hat{r} \cdot \hat{\Phi})^2, \\
R_{0607} &= -\frac{R^2}{4r^4} e^{-2\Psi - 2\Lambda} (\hat{r} \cdot \hat{\Theta})(\hat{r} \cdot \hat{\Phi}), \\
R_{0615} &= -\frac{1}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Phi}), \\
R_{0617} &= -\frac{1}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Theta}), \\
R_{0707} &= -\frac{R^2}{4r^4} e^{-2\Psi - 2\Lambda} (\hat{r} \cdot \hat{\Theta})^2, \\
R_{0715} &= -\frac{1}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{\Theta}), \\
R_{0716} &= -\frac{1}{2r^2} e^{-\Psi - \Lambda} (\hat{r} \cdot \hat{R}), \\
R_{1212} &= \frac{1}{r} \Lambda e^{-2\Lambda}, \\
R_{1236} &= \frac{R}{2r^3} e^{-\Lambda} (\hat{r} \cdot \hat{\Phi}), \\
R_{1237} &= -\frac{R}{2r^3} e^{-\Lambda} (\hat{r} \cdot \hat{\Theta}), \\
R_{1313} &= -\frac{1}{r} \Lambda e^{-2\Lambda}, \\
R_{1326} &= \frac{1}{2r^3} e^{-\Lambda} (\hat{r} \cdot \hat{\Phi}), \\
R_{1327} &= -\frac{1}{2r^3} e^{-\Lambda} (\hat{r} \cdot \hat{\Theta}), \\
R_{1616} &= -\frac{R^2}{4r^4} e^{-2\Psi - 2\Lambda} (\hat{r} \cdot \hat{\Phi})^2, \\
R_{1617} &= \frac{R^2}{4r^4} e^{-2\Psi - 2\Lambda} (\hat{r} \cdot \hat{\Theta})(\hat{r} \cdot \hat{\Phi}), \\
R_{1623} &= \frac{R}{r^3} e^{-\Lambda} (\hat{r} \cdot \hat{\Phi}), \\
R_{1717} &= -\frac{R^2}{4r^4} e^{-2\Psi - 2\Lambda} (\hat{r} \cdot \hat{\Theta})^2, \\
R_{1723} &= \frac{R}{r^3} e^{-\Lambda} (\hat{r} \cdot \hat{\Theta}), \\
R_{2323} &= \frac{1}{r^2} (1 - e^{-2\Lambda}) - \frac{3}{4} \frac{R^2}{r^4} \{ (\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2 \}, \\
R_{2356} &= \frac{1}{r^2} (\hat{r} \cdot \hat{\Phi}), \\
R_{2367} &= \frac{1}{r^2} (\hat{r} \cdot \hat{R}), \\
R_{2357} &= -\frac{1}{r^2} (\hat{r} \cdot \hat{\Theta}), \\
R_{2536} &= \frac{1}{2r^2} (\hat{r} \cdot \hat{\Phi}), \\
R_{2537} &= -\frac{1}{2r^2} (\hat{r} \cdot \hat{\Theta}), \\
R_{2627} &= \frac{R^2}{4r^4} (\hat{r} \cdot \hat{\Theta})(\hat{r} \cdot \hat{\Phi}), \\
R_{2635} &= \frac{1}{2r^2} (\hat{r} \cdot \hat{\Phi}) \end{align}
\[ \bar{R}_{2637} = \frac{1}{2r^2} (\hat{r} \cdot \hat{R}), \quad \bar{R}_{2727} = \frac{R^2}{4r^4} (\hat{r} \cdot \hat{\Theta})^2 \]  
\[ \bar{R}_{2735} = \frac{1}{2r^2} (\hat{r} \cdot \hat{\Theta}), \quad \bar{R}_{2736} = -\frac{1}{2r^2} (\hat{r} \cdot \hat{R}) \]  
\[ \bar{R}_{3636} = \frac{R^2}{4r^4} (\hat{r} \cdot \hat{\Phi})^2, \quad \bar{R}_{3637} = -\frac{R^2}{4r^4} (\hat{r} \cdot \hat{\Theta})(\hat{r} \cdot \hat{\Phi}) \]  
\[ \bar{R}_{3737} = \frac{R^2}{4r^4} (\hat{r} \cdot \hat{\Theta})^2. \]  

We have tacitly omitted nearly one half of the nonzero components of the curvature tensor, because of the relations in (34), for shorthand.

3 Einstein Equation

Contracting the curvature tensor, one will get the Ricci tensor

\[ \bar{R}_{ab} = \eta^{cd} \bar{R}_{cda\bar{b}}. \]  

Substituting the components of the dyonic metric,

\[ e^{2\Psi} = 1 - \frac{r_s}{r}, \quad e^{2\Lambda} = (1 - \frac{r_s}{r})^{-1}, \]

where \( r_s \) is the Schwarzschild radius, then we can obtain almost \( \bar{R}_{ab} \) are zero except

\[ \bar{R}_{00} = -\frac{1}{2} \frac{R^2}{r^4} \{(\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2 \}, \]  
\[ \bar{R}_{11} = +\frac{1}{2} \frac{R^2}{r^4} \{(\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2 \}, \]  
\[ \bar{R}_{22} = -\frac{1}{2} \frac{R^2}{r^4} \{(\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2 \}, \]  
\[ \bar{R}_{33} = -\frac{1}{2} \frac{R^2}{r^4} \{(\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2 \}. \]

Then one has the Ricci scalar curvature,

\[ \bar{R} = \eta^{ab} \bar{R}_{ab} \]  
\[ = -\bar{R}_{00} + \bar{R}_{11} + \bar{R}_{22} + \bar{R}_{33} \]  
\[ = 0. \]

From the fields \( \tilde{\mathcal{F}}_\mu^m \) in (22) and (23), the components of the Ricci tensor, (84) ~ (87), can be recast into the form,

\[ \bar{R}_{ab} = -\frac{1}{2} \bar{F}_a^\mu \bar{F}_b^\nu \bar{g}^{\alpha\beta} \gamma_{\alpha\beta\gamma\delta} \tilde{\mathcal{F}}_\mu^m \tilde{\mathcal{F}}_\nu^m. \]
or
\[ \bar{R}_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \gamma_{mn} \tilde{F}_m^{\mu\alpha} \tilde{F}_n^{\nu\beta}. \] (92)

Since the identity, \( \gamma_{mn} \tilde{F}_m^{\mu\nu} \tilde{F}_n^{\nu\mu} = 0 \), holds, the right-hand side of the equation (92) can be identified as \( 8\pi \) times the stress-energy tensor of the Yang-Mills field,

\[ \bar{R}_{\mu\nu} = 8\pi \bar{T}_{\mu\nu}, \quad \bar{T}_{\mu\nu} = -\frac{1}{16\pi} g^{\alpha\beta} \gamma_{mn} \tilde{F}_m^{\mu\alpha} \tilde{F}_n^{\nu\beta}. \] (93)

Then the Einstein equation, \( \bar{R}_{AB} - \frac{1}{2} g_{AB} \bar{R} = 8\pi \bar{T}_{AB} \), is satisfied, where some components of \( \bar{T}_{AB} \) are zero, \( \bar{T}_{\mu\mu} = 0 \) and \( \bar{T}_{mn} = 0 \).

The \( SO(3) \) KK dyonic metric satisfies the Einstein equation in the seven-dimensional. The results from two different methods are coincident.

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