Irreversibility for All Bound Entangled States

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We derive a new inequality for entanglement for a mixed four-partite state. Employing this inequality, we present a one-shot lower bound for entanglement cost and prove that entanglement cost is strictly larger than zero for any entangled state. We demonstrate that irreversibility occurs in the process of formation for all non-distillable entangled states. In this way we solve a long standing problem, of how “real” is entanglement of bound entangled states. Using the new inequality we also prove the impossibility of local cloning of a known entangled state.

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Irreversibility in physical processes is one of the most fundamental phenomena both in classical mechanics and in quantum mechanics. In quantum information theory, entanglement plays a crucial role. As is well known, entanglement cannot be created from scratch and cannot be increased under local operations and classical communication (LOCC). Under LOCC operations, one can only change the form of entanglement. And there are two elementary processes for entanglement manipulation. One is formation of entanglement and the other is distillation of entanglement. Formation is the process to create a generic entangled state $\rho_{AB}$ from a singlet state. Entanglement cost $E_c(\rho)$ is the minimal number of singlets needed to prepare a bipartite state $\rho_{AB}$ by LOCC in asymptotic regime of many copies. Distillation is the process to obtain nearly perfect singlets from many identical copies of given state $\rho_{AB}$ by LOCC. Distillable entanglement $E_d(\rho)$ is defined as the maximal number of singlets that can be drawn from $\rho_{AB}$. Distillation is important because singlet can be used for quantum teleportation, which is the basic brick in quantum communication protocols. In some sense, formation and distillation are dual to each other. Now the problem arises of whether the two processes of formation and distillation are reversible or irreversible? Reversibility means $E_c(\rho_{AB}) = E_d(\rho_{AB})$ and irreversibility means $E_c(\rho_{AB}) > E_d(\rho_{AB})$ . Of course, we know $E_c(\rho_{AB}) \geq E_d(\rho_{AB})$ by the non-increasing axiom under LOCC. Here we want to separate ‘$\geq$’ into ‘$=$’ and ‘$>$’. It is proved that pure states can be converted reversibly . In contrast, it is commonly believed that irreversibility occurs for nontrivial mixed states. Perhaps this deep belief comes from the strongest indication that there exist so-called bound entangled states from which no entanglement can be distilled, but for which, in order to create a single copy, entanglement is required . This means that for bound entangled states $E_d = 0$ and so called entanglement of formation $E_f > 0$. Entanglement of formation is defined as $E_f(\rho_{AB}) = \min \sum_i p_i E(\rho_{AB}^i)$, where the minimum is over all pure ensembles $\{\rho_{AB}^i, p_i\}$ satisfying $\rho_{AB} = \sum_i p_i |\phi^i\rangle\langle\phi^i|_{AB}$, and entanglement for pure state $\phi_{AB}$ is $E(\phi_{AB}) = S(\text{tr}_A(\rho_{AB}|\phi\rangle\langle\phi|_{AB}))$, where $S(\rho)$ is von Neumann entropy defined as $S(\rho) = -\text{tr}\rho \log \rho$. $E_f$ is interpreted as entanglement needed to create a single copy of state. It was proved that the regularized form of entanglement of formation is equal to entanglement cost $E_c(\rho) = \lim_{n \to \infty} E_f(\rho_{AB}^\otimes n)/n$ . It is clear that if $E_f$ is additive, then immediately we get $E_c = E_f > 0$ and irreversibility for bound entangled states is quickly solved. Unfortunately, although additivity of $E_f$ is a very desirable property and is deeply believed, the proof escapes from us by far. Without additivity of $E_f$ , it is difficult to rule out the possibility that the amount of entanglement needed per copy vanishes in the asymptotic limit, that is $E_f > 0$ but $E_c = 0$. Although this seems unlikely, it has not been disproved. In it was shown that an additive quantity $S(A) - S(AB)$ is a lower bound for $E_f$. However if the quantity is positive then $E_d$ is also nonzero , hence the bound cannot be useful in the case of bound entangled states.

Irreversibility is proved for some special classes of mixed states . There are two approaches to prove irreversibility. One approach is to find a new entanglement measure that lies strictly between $E_d$ and $E_c$ for nontrivial mixed states. Note that any entanglement measure should satisfy $E_d \leq E \leq E_c$ . Here ’strictly’ means $E_d < E < E_c$ for nontrivial cases. There exist a few entanglement measures that are not ’strict’ measures. The other approach is to find a quantity that is a lower bound of $E_c$ and is nonzero for entangled state, but is unnecessarily a good measure. In this Letter, we follow the second approach. We find a new inequality of entanglement for a mixed four-partite state that can be employed to provide a one-shot lower bound for entanglement cost and the lower bound is strictly larger than zero for any entangled state. Irreversibility is immediately obtained for all non-distillable entangled states.

First we recall a measure for classical correlation of
bipartite state $\rho_{AB}$ proposed in \cite{19},
\begin{align*}
C_+(\rho_{A:B}) & = \max_{A_1^i A_i} S(\rho_B) - \sum_i p_i S(\rho_B^i), \\
C_-(\rho_{A:B}) & = \max_{B_1^i B_i} S(\rho_A) - \sum_i p_i S(\rho_A^i),
\end{align*}
where $A_1^i A_i$ is a PVM performed on subsystem $A$, $\rho_B^i = tr_A(A_i \otimes I_\rho_{AB} A_i^\dagger \otimes I)/p_i$ is remaining state of $B$ after obtaining the outcome $i$ on $A$, and $p_i = tr_B(A_i \otimes I_\rho_{AB} A_i^\dagger \otimes I)$. In general, $C_-(\rho_{A:B}) \neq C_+(\rho_{A:B})$. We denote $C(\rho_{A:B}) = \max\{C_+(\rho_{A:B}), C_-(\rho_{A:B})\}$. It is proved that $C(\rho_{A:B}) = 0$ if and only if $\rho_{AB} = \rho_A \otimes \rho_B$ \cite{19}. Now we define a new quantity for $\rho_{AB}$
\begin{align*}
G_-(\rho_{A:B}) & = \inf_i \sum p_i C_-(\rho_{A:B}^i), \\
G_+(\rho_{A:B}) & = \inf_i \sum p_i C_+(\rho_{A:B}^i), \\
G_{HV}(\rho_{A:B}) & = \inf_i \sum p_i C(\rho_{A:B}^i),
\end{align*}
where infimum is taken over $\{\rho_{A:B}^i, p_i\}$, generally a mixed ensemble of realization of $\rho_{AB}$. The function $G_{HV}$ is not an entanglement measure, however it can be called "entanglement parameter", as it satisfies the following property:

**Theorem 1.** $G_{HV}(\rho_{A:B}) = 0$ if and only if $\rho_{AB}$ is separable.

**Proof.** It is easy to prove 'if' part. For the 'only if' part, it is sufficient to prove that if $G_-(\rho) = 0$, then $\rho_{AB}$ is separable.

In \cite{20} it was argued that $C_-(\rho_{A:B})$ is asymptotically continuous. The fastest argument comes from the duality relation between dual states. For a tripartite pure state $|\phi\rangle_{ABC}$, $\rho_{AB} = tr_C|\phi\rangle\langle\phi|$ is dual to $\sigma_{AC} = tr_B|\phi\rangle\langle\phi|$ and vice versa. The duality relation between dual states is \cite{21}
\begin{align*}
S(\rho_A) & = E_f(\rho_{A:C}) + C_-(\rho_{A:B}). \tag{1}
\end{align*}
Further notice the fact that if states $\rho_{AB}$ and $\sigma_{AB}$ are close to each other, then there exist purifications $\phi_{ABC}$ and $\psi_{ABC}$ such that the dual states $\rho_{AC}$ and $\sigma_{AC}$ are close \cite{22}. Since it is known that $E_f$ is continuous and entropy is continuous, then $C_-(\rho) = C_+(\rho)$ is continuous too. By Proposition 3 in Appendix, from continuity of $C_-$ it follows that there exists an optimal decomposition $\{\rho_{A:B}^i, p_i\}$ realizing $G_-$. and the decomposition contains $d^2 + 1$ elements at most where $d$ is the dimension of Hilbert space $H_{AB}$. If the state is entangled, there must be a non-product state in decomposition, and of course $C_-$ is nonzero on this state (because for a non-product state there always exists Alice’s measurement that is correlated with Bob's system \cite{10}). Thus $G_-$ is nonzero for every entangled state.

More generally, mixed convex roof from any continuous function that vanishes only on product states, gives a function that vanishes only on separable states. If, in addition, the function does not increase under conditioning upon local classical register, its convex roof is entanglement measure. (We prove this, and explore the consequences elsewhere). However $C_-(A : B)$ can increase under conditioning on Alice’s side (a counterexample can be found in \cite{23}).

**Lemma 1.** For any four-partite pure state $|\phi\rangle_{AA'BB'}$, the following inequality of entanglement is satisfied
\begin{align*}
E_f(\phi_{AA'BB'}) & \geq E_f(\rho_{A:B}) + C_-(\rho_{A':B'}), \tag{2}
\end{align*}
where $\rho_{AB} = tr_{A'B'}\phi_{AA'BB'}$ and $\rho_{A'B'} = tr_{AB}\phi_{AA'BB'}$.

**Proof.** Apply the duality relation (1) to four-partite pure state $\phi_{AA'BB'}$ and regard $AA'$ as one part, then we get
\begin{align*}
S(\rho_{AA'}) & = E_f(\rho_{AA'}) + C_-(\rho_{A':B'}), \tag{3a} \\
& \geq E_f(\rho_{A:B}) + C_-(\rho_{A':B'}), \tag{3b}
\end{align*}
where $\geq$ comes from the fact that both $E_f$ and $C_-$ is non-increasing under local operations \cite{11,19}. Similarly we obtain
\begin{align*}
S(\rho_{BB'}) & \geq E_f(\rho_{A:B}) + C_-(\rho_{A':B'}). \tag{4}
\end{align*}
So we get inequality (2) that completes the proof.

**Proposition 1.** (Main inequality) For a mixed four-partite state $\rho_{AA'BB'}$,
\begin{align*}
E_f(\rho_{AA'BB'}) & \geq E_f(\rho_{A:B}) + G_{HV}(\rho_{A':B'}), \tag{5}
\end{align*}
where $\rho_{AB} = tr_{A'B'}\rho_{AA'BB'}$ and $\rho_{A'B'} = tr_{AB}\rho_{AA'BB'}$.

**Proof.** Now we consider a mixed four-partite state $\rho_{AA'BB'}$. Suppose the optimal realization of $E_f$ of $\rho_{AA'BB'}$ is $\{\phi_{AA'BB'}^i, p_i\}$, then we have
\begin{align*}
E_f(\rho_{AA'BB'}) & = \sum_i p_i S(\rho_{AA'}^i) \\
& \geq \sum_i p_i E_f(\rho_{A'B}^i) + \sum_i p_i C(\rho_{A':B'}^i) \tag{6a} \\
& \geq E_f(\rho_{A:B}) + \sum_i p_i C(\rho_{A':B'}^i) \tag{6b} \\
& \geq E_f(\rho_{A:B}) + G_{HV}(\rho_{A:B}), \tag{6c}
\end{align*}
where (6a) comes from (2) and (6c) from the convexity of $E_f$, and (6b) from the definition of $G_{HV}$. This ends the proof.

From Proposition 1, an immediate corollary is as follows:
Corollary 1. For a four-partite state $\rho_{AA'BB'}$, if the reduced state $\rho_{A'B'}$ is entangled, then $E_f(\rho_{AA'BB'}) > E_f(\rho_{A'B'B})$.

We will now use the above corollary to solve the open problem in [27]. In [27], impossibility of cloning a known entangled state under LOCC is reduced to whether $E(\rho_{AA'BB'}) > E(\rho_{AB})$ for some entanglement measure when $\rho_{AA'B'B}$ is entangled [28]. By Corollary 1, we obtain no-go theorem for LOCC cloning:

Proposition 2. It is impossible to clone a known entangled state by LOCC.

Let us now pass to the proof of the main result of the Letter.

Theorem 2. For any entangled state $\rho_{AB}$, $E_c(\rho_{AB}) \geq G_{HV}(\rho_{AB}) > 0$.

Proof. Entanglement cost is defined as the asymptotic cost of singlets to prepare a bipartite mixed state and proved to be $E_c(\rho) = \lim_{n \to \infty} E_f(\rho^\otimes n)/n$ [10]. Now consider $\rho^\otimes n$ and we have

$$E_f(\rho^\otimes n) = E_f(\rho^\otimes n-1 \otimes \rho) \geq E_f(\rho^\otimes n-1) + G_{HV}(\rho) \geq \cdots \geq E_f(\rho) + (n-1)G_{HV}(\rho),$$

(7)

all inequalities $\geq$ come from [10]. Then

$$\frac{E_f(\rho^\otimes n)}{n} \geq \frac{(n-1)}{n}G_{HV}(\rho).$$

(8)

Let $n \to \infty$ and we get $E_c(\rho) \geq G_{HV}(\rho) > 0$. This ends the proof.

It is notable that $G_{HV}$ is a one-shot lower bound for $E_c$, an asymptotic quantity. Recall that $\rho$ is irreducible if $E_c(\rho_{AB}) > E_d(\rho_{AB})$ in the processes of formation and distillation. Irreversibility is proved for some specific mixed states [13, 14, 15]. It is conjectured that irreversibility occurs for nontrivial mixed states, especially for non-distillable entangled states. It is well known that any PPT (positive partial transpose) entangled state is bound entangled [20] and it is conjectured that NPT (negative partial transpose) bound entangled states exist [20, 21]. Whatever the case is, we have the strict inequality $E_c \geq 0$ for entangled states. Therefore we conclude that irreversibility occurs for any non-distillable entangled state.

Summarizing, we present a new inequality of entanglement for a mixed four-partite state. Based on this inequality, the asymptotic quantity, entanglement cost is lower bounded by a one-shot quantity which is strictly larger than zero for entangled state. So irreversibility occurs in asymptotic manipulations of entanglement for all non-distillable entangled states that solves the problem announced in the original paper on bound entanglement [20]. Also the new inequality is employed to prove no-go theorem, saying that it is impossible to clone a known entangled state by LOCC operations.

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APPENDIX

Definition. For any continuous function $f$ of state on Hilbert space $C^d$ one defines a mixed convex roof $\hat{f}$

$$\hat{f}(\rho) = \inf \sum_i p_i f(\rho_i),$$

(9)

where the infimum is taken over all finite decompositions $\sum_i p_i \rho_i = \rho$.

Proposition 3. The infimum is attained, and the optimal ensemble can be chosen to have $d^2 + 1$ elements.

Proof. We use standard techniques from information theory [31] (see also [32]). First, let us show that for any finite decomposition $\rho = \sum_{i=1}^n p_i \rho_i$, we can provide a decomposition $\rho = \sum_{i=1}^{d^2+1} q_i \rho_i$ with $d^2 + 1$ elements, such that

$$\sum_{i=1}^n p_i f(\rho_i) = \sum_{i=1}^{d^2+1} q_i f(\rho_i).$$

(10)

To this, consider convex hull $A$ of the set $\{(\rho_i, f(\rho_i))\}_{i=1}^n$. The point $x = (\sum_{i=1}^n p_i \rho_i, \sum_{i=1}^n p_i f(\rho_i))$ belongs $A$. The set $A$ is a compact convex set, actually a polyhedron, in $d^2$-dimensional real affine space (this comes from the fact that states belongs to the real $d^2$ dimensional space of Hermitian operators and have unit trace). The set of extremal points is included in the set $\{\rho_i, f(\rho_i)\}_{i=1}^n$. Then from Caratheodory theorem it follows that $x$ can be written as a convex combination of at most $d^2 + 1$ extremal points, i.e. $x = \sum_{i=1}^n q_i (\rho_i, f(\rho_i))$ where $j = 1, \ldots, d^2 + 1$. Writing $q_i = q_j, \rho_i = \sigma_j$ we get $\sum_{i=1}^n p_i \rho_i = \sum_{j=1}^{d^2+1} q_j \sigma_j$ and $\sum_{i=1}^n p_i f(\rho_i) = \sum_{j=1}^{d^2+1} q_j f(\sigma_j)$. Thus we have a found a decomposition that has $d^2 + 1$ elements, and returns the same value of average, so that the infimum can be taken solely over such decompositions. Then from continuity of the function and compactness of the set of states it follows that the infimum is attained.
It should be emphasized that irreversibility of distillation-formation process is connected with the fact that the class of allowed operations is restricted to LOCC which plays the basic role in quantum communication theory. There are investigations towards lifting irreversibility by enlarging class of operations [4,5].

1 - F(\rho, \sigma) \leq \frac{\text{tr}|\rho - \sigma|}{2} \leq \sqrt{1 - F^2(\rho, \sigma)}.

Applying these formulas to \rho_{AB} and \sigma_{AB}, we know that there exist purifications \phi_{ABC} and \psi_{ABC} such that

\text{tr}|\rho_{AC} - \sigma_{AC}| \leq \sqrt{\epsilon(4 - \epsilon)} \leq 2\sqrt{\epsilon}.

So \rho_{AC} is close to \sigma_{AC} when \epsilon \to 0.

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1 - F(\rho, \sigma) \leq \frac{\text{tr}|\rho - \sigma|}{2} \leq \sqrt{1 - F^2(\rho, \sigma)}.

Applying these formulas to \rho_{AB} and \sigma_{AB}, we know that there exist purifications \phi_{ABC} and \psi_{ABC} such that

\text{tr}|\rho_{AC} - \sigma_{AC}| \leq \sqrt{\epsilon(4 - \epsilon)} \leq 2\sqrt{\epsilon}.

Then we get

F(\rho_{AC}, \sigma_{AC}) \geq |\langle \phi_{ABC} | \psi_{ABC} \rangle| \geq 1 - \frac{\epsilon}{2},

So \rho_{AC} is close to \sigma_{AC} when \epsilon \to 0.

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