Asymptotic Behaviours of $q$-orthogonal Polynomials from a $q$-Riemann Hilbert Problem

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Abstract
We describe a Riemann–Hilbert problem for a family of $q$-orthogonal polynomials, \( \{ P_n(x) \}_n \), and use it to deduce their asymptotic behaviours in the limit as the degree, \( n \), approaches infinity. We find that the $q$-orthogonal polynomials studied in this paper share certain universal behaviours in the limit $n \to \infty$. In particular, we observe that the asymptotic behaviour near the location of their smallest zeros, $x \sim q^{n/2}$, and norm, $\| P_n \|_2$, are independent of the weight function as $n \to \infty$.

Keywords Riemann–Hilbert problem · $q$-orthogonal polynomials and $q$-difference calculus

Mathematics Subject Classification 33C45 · 35Q15 · 39A13

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1 Introduction

The theory of orthogonal polynomials is a source of major developments in modern mathematical physics. But the spectacular outcomes of the classical theory of orthogonal polynomials with continuous measure are not yet matched by \( q \)-orthogonal polynomials, which are orthogonal with respect to a discrete measure supported on a lattice \( \{ \pm q^k \}_{k \in \mathbb{N}^+} \), for some \( 0 < q < 1 \). In this paper, we focus on a class of such polynomials and deduce their asymptotic behaviours as their degree grows by expressing them in terms of a Riemann–Hilbert Problem (RHP).

We denote monic \( q \)-orthogonal polynomials by \( \{ P_n(x) \}_{n=0}^{\infty} \). They satisfy the orthogonality relation

\[
\int_{-1}^{1} P_n(x) P_m(x) w(x) d_q x = \gamma_n \delta_{n,m},
\]

where \( d_q x \) refers to the (discrete) Jackson integral (see Eq. (1.4)). A fundamental consequence of Eq. (1.1) is the 3-term recurrence relation

\[
x P_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x),
\]

where the recurrence coefficients are given by

\[
a_n = \frac{\gamma_n}{\gamma_{n-1}}, \quad b_n = \frac{\int x P_n(x)^2 d\mu(x)}{\gamma_n}.
\]

A natural question, which motivated many studies through the past century [9, 15, 32], is to ask what the behaviours of \( P_n, \gamma_n, a_n \) and \( b_n \) are as \( n \to \infty \). Classical results focused on polynomials which satisfy the orthogonality relation

\[
\int P_n(x) P_m(x) w(x) d\mu(x) = \gamma_n \delta_{n,m},
\]

where \( d\mu(x) \) is a continuous measure on the real line and \( w(x) \) is a weight function whose rate of change satisfies certain conditions. Measures on the unit circle in the complex plane were also a focus of interest [31, 32]. Extensions to a wider class of weight functions, leading to so-called semi-classical orthogonal polynomials [14, 30], have attracted more attention in recent times, due to their appearance in Random Matrix Theory [7, 23] and relationship to the Painlevé equations [13].
More recently, discrete orthogonal polynomials have been of growing interest. For those on a multiplicative $q$-lattice, particular attention has been paid to cases such as little $q$-Jacobi and discrete $q$-Hermite I \cite[Chapter 18.27]{24} polynomials. However, very little appears to be known about the asymptotic behaviour of $q$-orthogonal polynomials outside these specific cases. More recent discoveries of $q$-orthogonal polynomials related to multiplicative discrete Painlevé equations \cite[12]{4} have reignited a need for further mathematical tools to answer questions about their asymptotic behaviours.

The aim of this paper is to consider such questions for a general class of $q$-orthogonal polynomials which includes, but is not limited to, a large subset of the $q$-Hahn class \cite[Chapter 18.27]{24}. Our main results are Theorems 1.5 and 1.6, where, under certain mild assumptions on the weight $w(x)$ in Eq. (1.1), we deduce the asymptotic behaviour of $P_n$, $\gamma_n$ and $a_n$ in the limit $n \to \infty$ and show that the error term is of size $O(q^n)$.

1.1 Background

In the past two decades, $q$-orthogonal polynomials have appeared in many areas of applied mathematics and physics \cite[17, 19, 28]{2,17,19,28}, particularly in quantum physics. However, little is known about the behaviour of $q$-orthogonal polynomials. Earlier work in the field focused on specific examples. In 2003, Postelmans and Van Assche introduced two kinds of multiple little $q$-Jacobi polynomials and described some asymptotic properties \cite[25]{25}. In 2005, Ismail described the asymptotic behaviour of discrete $q$-Hermite II polynomials using $q$-Airy functions. He then extended these results to $q$-orthogonal polynomials satisfying a certain $q$-difference equation \cite[15]{15}. In 2013, Driver and Jordaan studied the asymptotic behaviour of extreme zeros of $q$-orthogonal polynomials \cite[10]{10}. In 2017, Chen and Filipuk studied generalised $q$-Laguerre polynomials and determined resulting asymptotics for their recurrence coefficients \cite[6]{6}.

Recently, there has been improved understanding of the asymptotics of larger families of $q$-orthogonal polynomials. In 2020, Van Assche et al. \cite[33]{33} showed that the leading order of $\gamma_n$ is $q^{n^2}$ for $q$-orthogonal polynomials with weights satisfying

$$
\lim_{n \to \infty} \frac{1}{n^2} \log(w(x^n)) = 0,
$$

where $x \in (0, 1)$. They also describe the location of $\{x_i^{1/n}\}_{i=1}^n$ (where $\{x_i\}_{i=1}^n$ are the $n$ zeros of $P_n$) in the limit $n \to \infty$. However, it remained an open question to obtain a more precise asymptotic description of large classes of $q$-orthogonal polynomials.

RHPs have been extensively used to study the asymptotics of orthogonal polynomials \cite[21,22]{21,22} since the asymptotic behaviour of semi-classical Freudian polynomials was derived by Deift et al. using a model RHP \cite[9]{9}. Their work was based on earlier advancements by Deift and Zhou on the steepest descent method for oscillatory RHPs \cite[8]{8}. Expanding on the approach of Deift et al., Baik et al. \cite[3]{3} deduced the asymptotics of orthogonal polynomials on a discrete lattice using what they call an interpolation problem, which can be seen as the discrete analogue to a RHP. Although this work yielded interesting results for general discrete weights, we find that it misses some key details of the behaviour of $q$-orthogonal polynomials. In particular, the results do not
accurately describe the behaviour of \( a_n, \gamma_n \) and \( P_n \) as \( n \to \infty \). We note that in terms of Baik et al.’s notation, 0 is an accumulation point of the \( q \)-lattice.

Extending on our earlier theory [18], in this paper we will use a RHP to obtain detailed asymptotic results for a large class of \( q \)-orthogonal polynomials. We will observe an interesting intersection with \( q \)-RHP theory and provide explicit examples highlighting aspects of \( q \)-RHP theory discussed in the literature [1, 26, 29]. In particular, we will solve the model \( q \)-RHP by deducing an equivalent connection matrix between two solutions of a \( q \)-difference equation represented by a power series about 0 and \( \infty \).

The asymptotic results obtained in this paper also pertain to multiplicative discrete Painlevé equations. It has been shown that the recurrence coefficients of \( q \)-orthogonal polynomials can satisfy multiplicative-type discrete Painlevé equations [4], for example Eq. (1.3)

\[
a_n(a_{n+1} + q^{1-n}a_n + q^2a_{n-1} + q^{3-2n}a_{n+1}a_{n-1}) = q^{n-1}(1 - q^n),
\]

(1.3)

where the non-autonomous term in the equation is iterated on multiplicative lattices. (For the terminology distinguishing types of discrete Painlevé equations, we refer to Sakai [27].) Very little is known about the asymptotic behaviour of the solution to this equation. The results in this paper provide detailed asymptotics for the real positive solution to Eq. (1.3).

1.2 Notation and Previous Results in the Literature

For completeness, we recall some well known definitions and notations from the calculus of \( q \)-differences. These definitions can be found in [11]. Throughout the paper we will assume \( q \in \mathbb{R} \) and \( 0 < q < 1 \).

**Definition 1.1** We define the Pochhammer symbol \((z; q)_\infty\), and Jackson integrals as follows.

(1) The Pochhammer symbol \((z; q)_\infty\) is defined as

\[
(z; q)_\infty = \prod_{j=0}^{\infty} (1 - zq^j).
\]

Furthermore, we define \((z_1, z_2; q)_\infty\) as

\[
(z_1, z_2; q)_\infty = \prod_{j=0}^{\infty} (1 - z_1q^j)(1 - z_2q^j).
\]

(2) The unnormalised Jackson integral of \( f(z) \) from -1 to 1 is defined as

\[
\int_{-1}^{1} f(z) \delta_q z = \sum_{k=0}^{\infty} (f(q^k) + f(-q^k))q^k.
\]

(1.4)
We remark on an equivalence between two types of $q$-orthogonal polynomials seen in the literature.

**Remark 1.2** In general $q$-orthogonal polynomials can be orthogonal with respect to a weight supported on the Jackson integral from $[-1, 1]$ or from $(0, 1)$ [24, Chapter 18.27], where the latter is given by

$$\int_0^1 f(z) d_q z = \sum_{k=0}^{\infty} f(q^k)q^k.$$ 

Let $\{P_n\}_{n=0}^{\infty}$ be the normalised polynomials orthogonal with respect to the one-sided Jackson integral

$$\sum_{k=0}^{\infty} P_n(q^k)P_m(q^k)w(q^k)q^k = \delta_{n,m}.$$ 

Let $\rho = q^{1/2}$, hence

$$\sum_{k=0}^{\infty} P_n(\rho^{2k})P_m(\rho^{2k})w(\rho^{2k})\rho^{2k} = \delta_{n,m}. \quad (1.5)$$

Define $\omega(z) = w(z^2)|z|$, it follows that $\omega(z)$ is an even function. This implies that the corresponding set of orthogonal polynomials $\{Q_n(z)\}_{n=0}^{\infty}$ are even/odd for even/odd $n$ [16]. Thus, for positive integers $l$, $p$ the orthogonality condition for $\{Q_n(z)\}_{n=0}^{\infty}$ is given by

$$2\sum_{k=0}^{\infty} Q_{2l}(\rho^{2k})Q_{2p}(\rho^{2k})\omega(\rho^k)\rho^k = \delta_{l,p}, \quad (1.6)$$

which is equivalent to Eq. (1.5) (up to scaling of the normalisation factor by $\sqrt{2}$). Hence, we proved that the class of $q$-orthogonal polynomials with one-sided Jackson integrals are contained in the class of $q$-orthogonal polynomials with two-sided Jackson integrals. It follows that it is sufficient to study $q$-orthogonal polynomials with two-sided Jackson integrals.

We recall the definition of an *appropriate* Jordan curve and *admissible* weight function given in [18, Definition 1.2] (with slight modification).

**Definition 1.3** A positively oriented Jordan curve $\Gamma$ in $\mathbb{C}$ with interior $\mathcal{D}_- \subset \mathbb{C}$ and exterior $\mathcal{D}_+ \subset \mathbb{C}$ is called *appropriate* if

$$\pm q^k \in \begin{cases} \mathcal{D}_- & \text{if } k \geq 0, \\ \mathcal{D}_+ & \text{if } k < 0. \end{cases}$$
A weight function, $w(z)$, is called \textit{admissible} if there exists an appropriate Jordan curve, $\Gamma$, such that $w(z)$ is bounded on $\Gamma$, $w(z)$ is analytic in $D_-$, and, there exists constants $N_c$ and $c$ such that for $n > N_c$

$$|1 - w(\pm q^n/2)| < cq^n.$$ 

Furthermore, we require that

$$w(\pm q^k) \neq 0, \text{ for } k \in \mathbb{N}_0.$$ 

We define the function

$$h^\alpha(z) = \sum_{k=-\infty}^{\infty} \frac{2zq^{k(1+\alpha)}}{z^2 - q^{2k}} = \sum_{k=-\infty}^{\infty} \left( \frac{q^{k(1+\alpha)}}{z - q^k} + \frac{q^{k(1+\alpha)}}{z + q^k} \right), \quad (1.7)$$

which satisfies the $q$-difference equation

$$h^\alpha(qz) = q^\alpha h^\alpha(z). \quad (1.8)$$

Note that $h^0(z)$ is equivalent to the definition of $h(z)$ given in [18, Definition 1.3]. Consequently, we define the $q$-RHP:

\textbf{Definition 1.4} ($q$-RHP) Let $\Gamma$ be an appropriate curve (see Definition 1.3) with interior $D_-$ and exterior $D_+$, and $w(z)$ be a corresponding admissible weight. A $2 \times 2$ complex matrix function $Y_n(z)$, $z \in \mathbb{C}$, is a solution to the $q$-RHP if it satisfies the following conditions:

(i) $Y_n(z)$ is analytic on $\mathbb{C} \setminus \Gamma$.

(ii) $Y_n(z)$ has continuous boundary values $Y_n^-(s)$ and $Y_n^+(s)$ as $z$ approaches $s \in \Gamma$ from $D_-$ and $D_+$ respectively, where

$$Y_n^+(s) = Y_n^-(s) \begin{pmatrix} 1 & w(s)h^\alpha(s) \\ 0 & 1 \end{pmatrix}, \quad s \in \Gamma. \quad (1.9a)$$

(iii) $Y_n(z)$ satisfies

$$Y_n(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + O\left(\frac{1}{|z|}\right), \text{ as } |z| \to \infty. \quad (1.9b)$$

Following the arguments presented in [18, Section 2(a)] we deduce that the unique solution to the $q$-RHP given by Definition 1.4 is

$$Y_n(z) = \begin{bmatrix} P_n(z) & \int_{\Gamma} P_n(s)w(s)h^\alpha(s)\frac{1}{2\pi i(z-s)} ds \\ \gamma_{n-1} P_{n-1}(z) & \int_{\Gamma} P_{n-1}(s)w(s)h^\alpha(s)\frac{1}{2\pi i(z-s)\gamma_{n-1}} ds \end{bmatrix}, \quad (1.10)$$
where \( \{ P_n(z) \}_{n=0}^{\infty} \) is the family of monic \( n^{th} \)-degree orthogonal polynomials such that

\[
\int_{-1}^{1} P_n(z) P_m(z) w(z) |z|^\alpha dq z = \gamma_n \delta_{n,m}.
\]

### 1.3 Main Results

We are now in a position to state the main results of this paper, which are collected as Theorems 1.5 and 1.6. The first main result concerns the asymptotic behaviour of orthogonal polynomials as their degree approaches infinity.

**Theorem 1.5** Suppose that \( \{ P_n(z) \}_{n=0}^{\infty} \) is a family of monic \( q \)-orthogonal polynomials, orthogonal with respect to the weight \( |z|^\alpha w(z) dq z \), where \( \alpha \in (-1, \infty) \) and \( w(z) \) is an admissible weight function. Define \( t = zq^{-n/2} \). Then for any given positive integer \( m \) there exists an \( N_m \in \mathbb{N} \) and \( C(m) \) such that for even \( n > N_m \), we have

\[
|(-1)^{\frac{n}{2}} q^{-\frac{n}{2}(\frac{n}{2}-1)} P_n(z) - \psi(t)| < C(m)q^n, \quad \text{for } |t| \leq q^{-m-1/2},
\]

\[
|P_n(z) - z^n(\gamma^{-2}; q^2)_\infty| < C(m)q^n, \quad \text{for } 1 \geq |z| > q^{n/2-m-1/2},
\]

and

\[
|(-1)^{\frac{n}{2}} q^{-\frac{n}{2}(\frac{n}{2}-1+\alpha)} \gamma_{n-1}^{-1} P_{n-1}(z) - \varphi(t)| < C(m)q^n, \quad \text{for } |t| \leq q^{-m-1/2},
\]

\[
|P_{n-1}(z) - z^{n-1}(\gamma^{-2}; q^2)_\infty| < C(m)q^n, \quad \text{for } 1 \geq |z| > q^{n/2-m-1/2},
\]

where \( C(m) \) is a function of \( m \), independent of \( z, n, \) and \( \psi(t) \) and \( \varphi(t) \), given in Definition 2.3, are entire functions independent of \( n, m \).

Our second main result concerns the asymptotic behaviour of recurrence coefficients and \( L_2 \) norm of \( P_n \) as \( n \) approaches infinity.

**Theorem 1.6** Under the same hypotheses as Theorem 1.5 we have, for even \( n \in \mathbb{N} \), as \( n \to \infty \):

\[
\gamma_n = q^{\frac{n(n-1+\alpha)}{2}} \left( 2(q^2; q^2)_\infty^2 + O(q^n) \right),
\]

\[
\gamma_{n-1} = q^{-\frac{n-2}{2}(n-1+\alpha)} \left( 2(q^2; q^2)_\infty^2 + O(q^n) \right),
\]

\[
a_n = q^{n-1+\alpha}(1 + O(q^n)).
\]

where \( \gamma_n \) and \( a_n \) are defined in Eqs. (1.1) and (1.2) respectively.

**Remark 1.7** Theorem 1.6 gives information about \( a_n \) as \( n \to \infty \), but the methodology we present in this paper does not provide a similar level of information about the asymptotic behaviour of \( b_n \) as \( n \to \infty \). The reason lies in the fact that the model RHP (see Sect. 3) has a solution with an expansion of the form \( \mathcal{W}(z) = I + \mathcal{W}^{(1)}/z + O(1/z^2) \), as \( z \to \infty \), where \( \mathcal{W}^{(1)} \) has zero main diagonal. This diagonal is where \( b_n \) would typically appear and so our approach is only able to show that \( b_n = o(q^n) \) as \( n \to \infty \), without further information about the rate at which \( b_n \) vanishes.
Remark 1.8 Note that Theorems 1.5 and 1.6 do not require $w(z)$ to be positive in general.

Remark 1.9 The case of little $q$-Jacobi polynomials [20, Chapter 14.12] provides an illustration of Theorem 1.6. This case has the orthogonality weight

$$w(x) = |x|^\alpha (qx; q)_\infty / (bqx; q)_\infty,$$

and to leading order $\gamma_n$ is indeed independent of the parameter $b$ in the limit $n \to \infty$.

Remark 1.10 There is a number of generalisations one can make to the results in this paper that require a slight change in methodology and lead to slightly different final results.

(i) In Definition 1.3, the condition on admissible weights that there exists constants $N_c$ and $c$ such that

$$|1 - w(\pm q^n/2)| < cq^n,$$

for $n > N_c$, can be relaxed to $w(0) = 1$. However, this could result in a change of the asymptotic error terms in Theorems 1.5 and 1.6 (see the proof of Lemma 2.7). In general, $C(m)q^n$ is the smallest upper bound achievable on the error.

(ii) In Definition 1.3, the condition: $w(\pm q^k) \neq 0$, for $k \in \mathbb{N}_0$ can be removed. Suppose that $w(q^j) = 0$ for some $j \in \mathbb{N}_0$. Then, to compensate for this one has to change the function $f(z)$ defined in Eq. (2.1) to

$$f(z) = \prod_{j=0, j \neq i}^{\infty} (1 - z^{-2} q^{2j}).$$

(iii) The methodology presented in this paper can readily be extended to the Al-Salam Carlitz class of polynomials described in [16, Chapter 18]. In this case the discrete measure is supported on $\{q^k\} \cup \{-dq^k\}$ for $k \in \mathbb{N}_0$, where $d$ is a constant. To enable such an extension, we need new functions

$$h^{\alpha,d}(z) = \sum_{k=-\infty}^{\infty} \frac{q^k (1 + d) z}{z^2 + (d - 1) z q^k - dq^{2k}},$$

$$f_d(z) = (z^{-1}, q)_\infty (-d z^{-1}, q)_\infty,$$

that replace $h^\alpha(z)$ and $f(z)$ respectively. Furthermore, the $q$-difference equation

$$S_A(q t) = \left[ d^{-1} t q^{2-\alpha} q^{-\alpha} - d^{-1} t q^{2-\alpha} \right] S_A(t),$$

should be used instead of Eq. (3.1a). Repeating the methodology presented in this paper with these substitutions leads to similar asymptotic estimates.
1.4 Outline

The paper is structured as follows. In Sect. 2.1 we make a series of transformations to the the $q$-RHP given in Definition 1.4. This motivates the form of a model RHP by taking the limit $n \to \infty$ of the RHP defined by Eq. (2.9). Consequently in Sect. 2.2, we prove that the solution of the $q$-RHP approaches the solution to the model RHP and use this to prove Theorems 1.5 and 1.6. In Sect. 3, using $q$-difference calculus we show that there exists a unique solution to the model RHP and determine its form. In Appendix A we prove important properties about the solution to the model RHP. In Appendix B we motivate some of the arguments presented in this paper using discrete $q$-Hermite I polynomials as an example. In Appendix C we prove certain properties of $h^0(z)$ required in Sect. 3.

2 Proofs of Main Results

In this section, we provide the proofs of Theorems 1.5 and 1.6. To carry out the proofs, we rely on a sequence of transformations to the RHP described in Definition 1.4. This sequence ends with a limiting RHP, referred to as a model RHP, which is studied further in Sect. 2.2 to deduce our main results.

2.1 Deriving the Model RHP as $n \to \infty$, for Even $n$

We make a series of transformations to the RHP given by Definition 1.4. Recall $Y_n$ (see Eq. (1.10)) is the solution of this RHP. Inspired by a similar approach first described by Deift et al., we make the series of transformations

$$Y_n \to U_n \to V_n \to W_n,$$

which will enable us to deduce a model RHP governing $W$, such that $W_n \to W$ as $n \to \infty$.

We will use the functions $f$ and $g$ in the sequence of transformations, where

$$f(z) = (z^{-2}; q^2)_\infty,$$  \hspace{1cm} (2.1)

and

$$g(z) = (q^2z^2, z^{-2}; q^2)_\infty.$$  \hspace{1cm} (2.2)

It can verified by direct calculation that $f(z)$ satisfies the $q$-difference equation:

$$f(qz) = \left(1 - \frac{1}{q^2z^2}\right) f(z),$$  \hspace{1cm} (2.3)
and $g(z)$ satisfies the $q$-difference equation:

$$g(qz) = -q^{-2}z^{-2}g(z). \quad (2.4)$$

By induction, using Eq. (2.3), we find for even $n$

$$(q^{n/2}z)^n f(q^{n/2}z) = q^{\frac{n}{2}(\frac{q}{2}-1)} f(z) \prod_{j=1}^{n/2} \left(q^{2j}z^2 - 1\right).$$

To be concise, let us define $g_n(z)$ as

$$g_n(z) = f(z) \prod_{j=1}^{n/2} \left(1 - q^{2j}z^2\right). \quad (2.5)$$

Thus,

$$(q^{n/2}z)^n f(q^{n/2}z) = i^n q^{\frac{n}{2}(\frac{n}{2}-1)} g_n(z). \quad (2.6)$$

Furthermore,

$$\frac{g(z)}{g_n(z)} = (q^{n+2}z^2; q^2)_{\infty},$$

it follows that for a fixed $z$, $g_n(z) \to g(z)$ as $n \to \infty$.

The transformations consist of four steps.

I. We first define:

$$U_n(z) = \begin{cases} Y_n(z) \begin{bmatrix} 1 & 0 \\ 0 & f(z) \end{bmatrix}, & \text{for } z \in \text{ext}(\Gamma), \\ Y_n(z), & \text{for } z \in \text{int}(\Gamma). \end{cases}$$

Note that the zeros of $f(z)$ cancel with the simple poles of $h^a(z)$, at $\pm q^k$ for $k \in \mathbb{N}_0$. This allows us to deform the contour, $\Gamma$, so that the poles of $h(z)$ at $\pm q^k$ for $k \in \mathbb{N}_0$ can lie in $\text{ext}(\Gamma)$ without affecting analyticity of the solution (see [18, Section 2(a)] for a description of the holomorphicity of $Y_n$). We observe that $f(z)$ does not change the asymptotic condition; see Eq. (1.9b).

II. We now scale the contour $\Gamma$ so that the modulus of points on it are multiplied by $q^{n/2}$. (If $\Gamma$ were the unit circle, it would now be a circle with radius $q^{n/2}$.) Denote the new contour by $\Gamma_{q^{n/2}}$.

III. The next transformation is

$$V_n(z) = \begin{cases} U_n(z) \begin{bmatrix} f(z)^{-1} & 0 \\ 0 & 1 \end{bmatrix}, & \text{for } z \in \text{ext}(\Gamma_{q^{n/2}}), \\ U_n(z), & \text{for } z \in \text{int}(\Gamma_{q^{n/2}}). \end{cases}$$
Note we have now introduced simple poles at $\pm q^k$ for $k = 0, 1, 2, \ldots, n/2 - 1$.

IV. Our final transformation is

$$W_n(z) = \begin{cases} \begin{bmatrix} c_n^+ & 0 \\ 0 & c_n^- \end{bmatrix} V_n(z) \begin{bmatrix} (zc_n)^{-n} & 0 \\ 0 & (zc_n)^n \end{bmatrix} & \text{for } z \in \text{ext}(\Gamma_{qn/2}) \\ \begin{bmatrix} c_n^+ & 0 \\ 0 & c_n^- \end{bmatrix} V_n(z) & \text{for } z \in \text{int}(\Gamma_{qn/2}), \end{cases} \quad (2.7)$$

where $c_n$ is a constant, to be defined shortly. We observe that after these transformations $W_n$ has the asymptotic condition

$$W_n(z) = I + O \left( \frac{1}{|z|} \right).$$

Motivated by the form of Eq. (2.6) we set

$$c_n = -i q^{-\frac{1}{2}(\frac{n}{2} - 1)}. $$

After these transformations we are left with the following transformed RHP for which the $2 \times 2$ complex matrix function $W_n(z)$, defined in Eq. (2.7), is the solution:

(i) $W_n(z)$ is meromorphic in $\mathbb{C} \setminus \Gamma_{qn/2}$ with simple poles at $z = \pm q^k$ for $k = 0, 1, 2, \ldots, n/2 - 1$.

(ii) $W_n(z)$ has continuous boundary values $W_n^-(s)$ and $W_n^+(s)$ as $z$ approaches $s \in \Gamma_{qn/2}$ from $D_{-q^{n/2}}$ and $D_{+q^{n/2}}$ respectively, where

$$W_n^+(z) = W_n^-(z) \begin{bmatrix} g_n(sq^{-n/2})^{-1} q^{an/2} h^\alpha(sq^{-n/2})w(s)g_n(sq^{-n/2}) \\ 0 \end{bmatrix} \quad (2.8a)$$

for $s \in \Gamma_{qn/2}$. Note, we have used Eqs. (1.8) and (2.6) to evaluate the jump condition.

(iii) $W_n(z)$ satisfies

$$W_n(z) = I + O \left( \frac{1}{|z|} \right), \quad \text{as } |z| \to \infty. \quad (2.8b)$$

(iv) The residue at each pole $z = \pm q^k$ for $k = 0, 1, 2, \ldots, n/2 - 1$, is given by

$$\text{Res}(W_n(\pm q^k)) = \lim_{z \to \pm q^k} W_n(z) \begin{bmatrix} 0 \\ (z \mp q^k)g_n(zq^{-n/2})^{-2} h^\alpha(z)^{-1}w(z)^{-1} \end{bmatrix} \quad (2.8c)$$

We now make the change in variables $tq^{n/2} = z$. Furthermore, we scale the orthogonal weight, $|z|^n w(z)$, by $q^{-an/2}$. It follows from the RHP above for $W_n(z)$ that $W_{n,\text{scat}}(t) = W_n(z) = W_n(tq^{n/2})$ solves the following RHP:
(i) $W_{n,scal}(t)$ is meromorphic in $\mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}^{2 \times 2}$, with simple poles at $t = \pm q^{-k}$ for $k = 1, 2, ..., n/2$.

(ii) $W_{n,scal}(t)$ has continuous boundary values $W^{-}_{n,scal}(s)$ and $W^{+}_{n,scal}(s)$ as $t$ approaches $s \in \Gamma$ from $\mathcal{D}_{-}$ and $\mathcal{D}_{+}$ respectively, where

$$W^{+}_{n,scal}(t) = W^{-}_{n,scal}(t) \begin{bmatrix} g_{n}(s)^{-1} h^{\alpha}(s) w(sq^{n/2})g_{n}(s) & 0 \\ 0 & g_{n}(s)^{-1} \end{bmatrix}, \ s \in \Gamma,$$

(2.9a)

(iii) $W_{n,scal}(t)$ satisfies

$$W_{n,scal}(t) = I + O\left(\frac{1}{|t|}\right), \ \text{as} \ |t| \rightarrow \infty.$$ 

(2.9b)

(iv) The residue at each pole $t = \pm q^{-k}$ for $k = 1, 2, ..., n/2$, is given by

$$\text{Res}(W_{n,scal}(\pm q^{-k})) = \lim_{t \rightarrow \pm q^{-k}} W_{n,scal}(t) \begin{bmatrix} 0 & 0 \\ (t \mp q^{-k})g_{n}(t)^{-2}h^{\alpha}(t)^{-1}w(tq^{n/2})^{-1} & 0 \end{bmatrix}. \ (2.9c)$$

As seen in the statement of Theorems 1.5 and 1.6, we are interested in orthogonal weights which satisfy $w(zq^{n/2}) \rightarrow 1$ as $n \rightarrow \infty$. Taking the limit $n \rightarrow \infty$ of the RHP for $W_{n,scal}(t)$, motivates the following model RHP.

**Definition 2.1** (Model RHP) Assume that the contour $\Gamma$ and regions $\mathcal{D}_{\pm}$ satisfy the conditions of Definition 1.3.

(i) $\mathcal{W}(t)$ is meromorphic in $\mathbb{C} \setminus \Gamma$, with simple poles at $t = \pm q^{-k}$ for $k \in \mathbb{N}_{1}$.

(ii) $\mathcal{W}(t)$ has continuous boundary values $\mathcal{W}^{-}(s)$ and $\mathcal{W}^{+}(s)$ as $t$ approaches $s \in \Gamma$ from $\mathcal{D}_{-}$ and $\mathcal{D}_{+}$ respectively, where

$$\mathcal{W}^{+}(t) = \mathcal{W}^{-}(t) \begin{bmatrix} g(s)^{-1} & g(s)h^{\alpha}(s) & 0 \\ 0 & g(s)^{-1} \end{bmatrix}, \ s \in \Gamma,$$

(2.10a)

(iii) $\mathcal{W}(t)$ satisfies

$$\mathcal{W}(t) = I + O\left(\frac{1}{|t|}\right), \ \text{as} \ |t| \rightarrow \infty.$$ 

(2.10b)

However we also have that $\mathcal{W}(q^{-k})$ has poles in the LHS column for $k \in \mathbb{N}_{1}$. Thus, the decay condition does not hold near these poles. For example the decay condition holds for $t$ such that $|t \pm q^{-k}| > r$, for all $k \in \mathbb{N}_{1}$, for fixed $r > 0$.

(iv) The residue at the poles $t = \pm q^{-k}$ for $k \in \mathbb{N}_{1}$ is given by

$$\text{Res}(\mathcal{W}(\pm q^{-k})) = \lim_{t \rightarrow \pm q^{-k}} \mathcal{W}(t) \begin{bmatrix} 0 & 0 \\ (t \mp q^{-k})g(t)^{-2}h^{\alpha}(t)^{-1} & 0 \end{bmatrix}. \ (2.10c)
In Sect. 3 we prove that there exists a unique solution to the model RHP. We now show that $W_n(t) \rightarrow \mathcal{W}(t)$ in the limit $n \rightarrow \infty$.

**Remark 2.2** In Sect. 3 we show $\mathcal{W}(t)$ restricted to $t \in \mathcal{D}^-$ can be analytically extended for $t \in \mathcal{D}^+$ to a $2 \times 2$ matrix with entire entries. Let us denote this function as $\hat{\mathcal{W}}(t)$ (note that $\hat{\mathcal{W}}(t) = \mathcal{W}(t)$ for $t \in \mathcal{D}^-$). We also show that $\mathcal{W}(t)$, restricted to $t \in \mathcal{D}^+$, can be meromorphically extended for $t \in \mathcal{D}^- \setminus 0$ to a $2 \times 2$ matrix with simple poles for entries in the LHS column at $t = \pm q^k$, for $k \in \mathbb{N}$. Let us denote this function as $\hat{\mathcal{W}}(t)$ (note that $\hat{\mathcal{W}}(t) = \mathcal{W}(t)$ for $t \in \mathcal{D}^+$). For all $t \in \mathbb{C} \setminus 0$ the identity

$$\hat{\mathcal{W}}(t) = \hat{\mathcal{W}}(t) \begin{bmatrix} g(t)^{-1} h^\alpha(t)g(t) & 0 \\ 0 & g(t) \end{bmatrix},$$

holds. An analogous statement holds for $W_n(t)$ such that

$$\hat{W}_n(t) = \hat{W}_n(t) \begin{bmatrix} g_n(t)^{-1} h^\alpha(t)w(tq^n/2)g_n(t) & 0 \\ 0 & g_n(t) \end{bmatrix}.$$

**Definition 2.3** We define $\psi(t)$, $\phi(t)$, $\varphi(t)$ and $\varrho(t)$ as the $(1, 1)$, $(1, 2)$, $(2, 1)$ and $(2, 2)$ entries of $\hat{\mathcal{W}}$ respectively.

$$\hat{\mathcal{W}}(t) = \begin{bmatrix} \psi(t) & \phi(t) \\ \varphi(t) & \varrho(t) \end{bmatrix}.$$

In Sect. 3 we show that these four functions are entire and can be explicitly written in terms of a power series about 0.

### 2.2 Proofs of Theorems 1.5 and 1.6

We prove Theorems 1.5 and 1.6. First, Theorem 1.5 is proved by showing that $W_n(t) \rightarrow \mathcal{W}(t)$ as $n \rightarrow \infty$. To do this we will construct a RHP, given by Eq. (2.12), that has the unique solution $R(t)$ such that $R(t) = \hat{W}_n(t)/(\hat{\mathcal{W}}(t))^{-1}$ for $R(t)_{|1 \in \text{ext}(\Gamma_R)}$, where $\Gamma_R$ is defined shortly. We will show that $R(t) \rightarrow I$ as $n \rightarrow \infty$, Theorems 1.5 and 1.6 then follow immediately.

Before stating the RHP for $R(t)$ we define a number of identities.

**Definition 2.4** Define the piece-wise Jordan curve $\Gamma_R$ as $\Gamma_R = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where:

- $\Gamma_1 = \partial B(0, q^{-m-1/2})$, where $m \in \mathbb{N}$ is a free parameter, which will later be restricted in Theorem 2.9
- $\Gamma_2 = \bigcup_{k=m+1}^{m/2-1} \partial B(\pm q^{-k}, r)$, where there is a large degree of freedom in choosing $r$. It is sufficient to choose $r$ such that the contours do not intersect and the orthogonality weight $w(tq^{n/2})$ is analytic in int$(\Gamma_2)$.
- $\Gamma_3 = \bigcup_{k=n/2}^{\infty} \partial B(\pm q^{-k}, r)$, where there is a large degree of freedom in choosing $r$. It is sufficient to choose $r$ such that the contours do not intersect and the orthogonality weight $w(tq^{n/2})$ is analytic in int$(\Gamma_3)$. 
Fig. 1 RHP for $R$ in the $t$ plane. $\Gamma_R$ is the union of the solid black, dashed red and dotted blue circles. $\Gamma_1$ is delineated by the solid black circle, across which the jump $\tilde{R} = \hat{R}_1J_1$ holds. $\Gamma_2$ is delineated by the dashed red circles, across which the jump $\tilde{R} = \hat{R}_2J_2$ holds. $\Gamma_3$ is delineated by the dotted blue circles, across which the jump $\tilde{R} = \hat{R}_3J_3$ holds.

See Fig. 1 for an illustration of $\Gamma_R$. Note that $\Gamma_3$ is composed of an infinite union of circles whilst $\Gamma_2$ is composed of a finite union $(n/2 - m - 1)$.

**Definition 2.5** Define the three matrix functions:

\[
J_1(t) = \tilde{W}(\tilde{W}_n)^{-1}\tilde{W}_n(\tilde{W})^{-1}
\]
\[
J_2(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{g_n(t)}h^\alpha(t)^{-1}w(tq^{n/2})^{-1} & 1 \end{bmatrix}(\tilde{W})^{-1},
\]
\[
J_3(t) = (\tilde{W})^{-1}.
\]

In general these matrices have meromorphic entries with simple poles at $t = \pm q^k$, for $k \in \mathbb{N}$.

We now prove the following lemma.

**Lemma 2.6** The unique solution to the RHP:

1. $R(t)$ is analytic in $\mathbb{C} \setminus \Gamma_R$, where $\Gamma_R$ is described above and illustrated in Fig. 1,
2. $R(t)$ satisfies

\[
\lim_{t \to \Gamma_1^\pm} R(t) = \lim_{t \to \Gamma_2^\pm} R(t)J_{R}(s), \ s \in \Gamma_R,
\]

where $J_{R}|_{\Gamma_1} = J_1|_{\Gamma_1}$, $J_{R}|_{\Gamma_2} = J_2|_{\Gamma_2}$ and $J_{R}|_{\Gamma_3} = J_3|_{\Gamma_3}$,
3. $R(t) = I + O\left(\frac{1}{|t|}\right)$, as $|t| \to \infty$.
is given by,

\[
R(t) = \begin{cases} 
\tilde{R} = \tilde{W}_n(\tilde{\mathcal{V}})^{-1} & \text{for } R|_{\text{ext}(0_R)}, \\
\tilde{R}_1 = \tilde{W}_n(\tilde{\mathcal{V}})^{-1}(J_1)^{-1} = \tilde{W}_n(\tilde{\mathcal{V}})^{-1} & \text{for } R|_{\text{int}(0_1)}, \\
\tilde{R}_2 = \tilde{W}_n(\tilde{\mathcal{V}})^{-1}(J_2)^{-1} = \tilde{W}_n(\tilde{\mathcal{V}})^{-1} & \text{for } R|_{\text{int}(0_2)}, \\
\tilde{R}_3 = \tilde{W}_n(\tilde{\mathcal{V}})^{-1}(J_3)^{-1} = \tilde{W}_n & \text{for } R|_{\text{int}(0_3)}. 
\end{cases}
\quad (2.13)
\]

**Proof** Existence. By definition \(\tilde{R}(t)\) is analytic in \(\text{ext}(\Gamma_R)\) (as \(\tilde{W}_n\) and \(\tilde{\mathcal{V}}\) are analytic in \(\text{ext}(\Gamma_R)\)). Thus, \(R(t)\), as defined in Eq. (2.13), is analytic in \(\text{ext}(\Gamma_R)\). We are left to show that \(R(t)\) (given by Eq. (2.13)) is analytic in \(\text{int}(\Gamma_R)\).

First we look at the region \(|t| < q^{-m-1/2}\). The matrix \(J_1\) is defined in Eq. (2.11a) as

\[
J_1 = \hat{\mathcal{V}}(\hat{W}_n)^{-1}\hat{W}_n(\hat{\mathcal{V}})^{-1} = \hat{\mathcal{V}}\begin{bmatrix} 1/g_n & g_n h^\alpha w(tq^{n/2}) \\ 0 & g_n \end{bmatrix} \begin{bmatrix} g - g h^\alpha \\ 0 \end{bmatrix} (\hat{\mathcal{V}})^{-1}.
\]

By definition, \(\tilde{R}_1 = \tilde{W}_n(\tilde{\mathcal{V}})^{-1}(J_1)^{-1} = \tilde{W}_n(\tilde{\mathcal{V}})^{-1}\) which is analytic in \(\text{int}(\Gamma_R)\).

Next we look at \(q^{-m-1/2} < |t| < q^{-n/2}\). By definition

\[
\tilde{R}_2 = \tilde{W}_n(\tilde{\mathcal{V}})^{-1}(J_2)^{-1},
\]

\[
= \tilde{W}_n(\tilde{\mathcal{V}})^{-1} \begin{bmatrix} 1 \\ -g_n(t)^{-2}h^\alpha(t)^{-1}w(tq^{n/2})^{-1} \\ 0 \end{bmatrix}.
\]

From the residue condition for \(W_n\), given by Eq. (2.9c), we conclude that \(\tilde{R}_2\) is analytic in \(\text{int}(\Gamma_2)\).

Finally we consider \(|t| > q^{-n/2}\). By definition

\[
\tilde{R}_3 = \tilde{W}_n(\tilde{\mathcal{V}})^{-1}(J_3)^{-1},
\]

\[
= \tilde{W}_n(\tilde{\mathcal{V}})^{-1}\tilde{\mathcal{V}},
\]

\[
= \tilde{W}_n.
\]

From the residue condition for \(W_n\), given by Eq. (2.9c), we know that \(\tilde{W}_n(t)\) is analytic for \(|t| > q^{-n/2}\) and thus \(\tilde{R}_3\) is analytic in \(\text{int}(\Gamma_3)\).

**Uniqueness** We note that \(\det(W_n) = \det(\mathcal{V}) = 1\). It follows that \(\det(J_R) = 1\), and applying the same arguments as in Sect. 3.2 we conclude that if a solution exists to the RHP given by Eq. (2.12), then it is unique.

We now prove that under certain conditions the solution, \(R(t)\), to the RHP given by Eq. (2.12) approaches the identity. We first prove a lemma about the jump matrix \(J_R\).

**Lemma 2.7** There exists an \(M\) such that for any fixed integer \(m > M\), the jump conditions \(J_i\) defined in Eqs. (2.11a), (2.11b) and (2.11c) satisfy:

\[
\|J_1(t) - I\|_{\Gamma_1} = C(m)q^n,
\]

\(\square\)
$$\| J_2(t) - I \|_{\Gamma_2} < 1/2, \quad (2.15)$$

$$\| J_3(t) - I \|_{\Gamma_3} = O(q^{n/2}), \quad (2.16)$$

for large \( n > N_m \). (Where \( \| J_1(t) - I \|_{\Gamma_1} \) is the infinity norm of the matrix \( J_1(t) - I \) restricted to the curve \( \Gamma_1 \), and \( C(m) \) is a function of \( m \), independent of \( n, t \).)

**Proof** First we consider \( J_3 \). By the asymptotic condition, Eq. (2.10b), we know that

$$\| \tilde{W}(t) - I \|_{\partial B(\pm q^{-k}, r)} < C_1 q^k, \quad (2.17)$$

where the radius \( r \) can be ignored as we will be considering the case \( r \ll q^{-k} \). Applying Eq. (2.17) and Definition 2.4 we find

$$\| \tilde{W}(t) - I \|_{\Gamma_3} < C_1 q^{n/2}.\quad \text{(2.18)}$$

By definition \( J_3(t) = \tilde{W}(t) \), hence we conclude

$$\| J_3(t) - I \|_{\Gamma_3} < C_1 q^{n/2}. \quad \text{(2.19)}$$

Next we study \( J_2 \). Applying Eq. (2.17) we find for \( \Gamma_2 = \bigcup_{k=m+1}^{n/2-1} \partial B(\pm q^{-k}, r) \)

$$\| \tilde{W}(t) - I \|_{\Gamma_2} < C_1 q^m. \quad \text{(2.20)}$$

Furthermore, we observe that the matrix

$$\Psi(t) = \begin{bmatrix} 1 & 0 \\ g_n(t)^{-2} h^\alpha(t)^{-1} w(tq^{n/2})^{-1} & 1 \end{bmatrix} - I,$$ 

vanishes much faster than \( \| \Psi(t) - I \|_{\partial B(\pm q^{-k}, r)} < C/|t| \). To see this, recall from Eq. (2.5) that

$$g_n(t) = \prod_{j=1}^{n/2} \left( 1 - q^{2j} t^2 \right) \prod_{j=0}^\infty \left( 1 - q^{2j} t^2 \right).$$

Thus, \( g_n(t)^{-2} \) is vanishingly small for large \( |t| \). (To see this, expand out the first few terms of the product \( \prod_{j=1}^{n/2} \left( 1 - q^{2j} t^2 \right) \). Hence we conclude that

$$\| J_2(t) - I \|_{\Gamma_2} < C_2 q^m.$$ 

It follows there exists an \( M \) such that for \( m > M \),

$$\| J_2(t) - I \|_{\Gamma_2} < 1/2.$$

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Next we study $J_1$. Define the function

$$p(t) = \frac{g(t)}{g_n(t)} = \prod_{j=n/2+1}^{\infty} \left(1 - q^{2j}t^2\right),$$

(2.18)

where $g_n$ and $g$ are given in Eqs. (2.5) and (2.2) respectively. For large $n$, we can take a Taylor series expansion of $\log(p(t))$ to find

$$\log(p(t)) = \sum_{j=1}^{\infty} \log(1 - q^{n+2j}t^2),$$

$$\leq 2 \sum_{j=1}^{\infty} -q^{n+2j}t^2,$$

$$= q^n \frac{-2q^2t^2}{1 - q^2}. \tag{2.19}$$

Furthermore, define

$$H(t) = h(t) \left(p(t) - p(t)^{-1}w(tq^{n/2})\right).$$

Then, expanding Eq. (2.11a) we find

$$J_1(1, 1) = \psi \varphi p - \phi \varphi / p - \psi \varphi H,$$

$$J_1(1, 2) = \phi \psi (1/p - p) + \psi^2 H,$$

$$J_1(2, 1) = \varphi \varphi (p - 1/p) - \varphi^2 H,$$

$$J_1(2, 2) = \psi \varphi / p - \phi \varphi p + \psi \varphi H,$$

where $\psi, \phi, \varphi, \varrho$ are defined in Definition 2.3. We note that $\psi \varphi - \phi \varphi = \det(\hat{W}) = 1$. Applying Eq. (2.19) it is clear that for a fixed $|t| = q^{-m-1/2}$, there exists an $N_m$ such that for $n > N_m$

$$\|J_1(t) - I\|_{\Gamma_1} < C_3(m)q^n,$$

where $C_3(m)$ is a function of $m$. Note that we choose $|t| = q^{-m-1/2}$ because there exists poles of the jump function, $J_1$, at $t = \pm q^{-m}$ for integer values of $m$. □

Remark 2.8 Lemma 2.7 holds for $t$ lying on $\Gamma_R$. However, from Eqs. (2.11a), (2.11b) and (2.11c), the matrix functions $J_i(t) : \mathbb{C} \to \mathbb{C}^{2 \times 2}$ are well defined for all $t \in \mathbb{C}$. In general, the matrices $J_i(t)$ do not approach the identity everywhere, but, they will have simple poles of order $O(q^n)$ at $\pm q^k$, for $k \in \pm \mathbb{N}$. Furthermore,

$$\sum_{k=-\infty}^{\infty} |\text{Res}(J_i(q^k))| = O(q^n).$$
This follows by direct computation, observing that $g(t)^{-1}$ and $g_n(t)^{-1}$ are vanishingly small for large $t$, and applying Eq. (A.1) which demonstrates that $\psi(q^{-k})$ and $\varphi(q^{-k})$ also become vanishingly small for large positive integer values of $k$.

Having proved Lemma 2.7 we are now in a position to show that the solution, $R(t)$, to the RHP given by Eq. (2.12) approaches the identity.

**Lemma 2.9** For a given integer $m > M$, where $M$ satisfies Lemma 2.7, the solution, $R(t)$, to the RHP defined in Eq. (2.12) is bounded for large $n$. Furthermore, $\hat{R}_1(t) = 1 + O(q^n)$ and $\tilde{R}(t) = 1 + O(q^n)$.

**Proof** Define

$$\Delta = J_R - I.$$ 

It immediately follows that for $s$ in $\Gamma_R$

$$\lim_{t \to \Gamma^+} R(t) = \lim_{t \to \Gamma^-} R(t)(I + \Delta(s)).$$

By the asymptotic condition, Eq. (2.12b), we conclude that

$$\tilde{R}(t) = I + \frac{1}{2\pi i} \oint_{\Gamma_R} \frac{R(s)\Delta(s)}{t - s} ds,$$

$$= I + \sum_{k=-\infty}^{\infty} \frac{\text{Res}(R(\pm q^k)\Delta(\pm q^k))}{t \pm q^k},$$

$$= I + \sum_{k=-m}^{\infty} \frac{\text{Res}(\hat{R}_1(\pm q^k)J_1(\pm q^k))}{t \pm q^k} + \sum_{k=-n/2}^{-m-1} \frac{\text{Res}(\hat{R}_2(\pm q^k)J_2(\pm q^k))}{t \pm q^k} + \sum_{k=-\infty}^{-n/2-1} \frac{\text{Res}(\hat{R}_3(\pm q^k)J_3(\pm q^k))}{t \pm q^k}. $$

(2.20)

Let $L$ be defined as $L = \sup_{t \in \text{int}(\Gamma_R)} |R(t)|$. As $R(t)$ is analytic in $\text{int}(\Gamma_R)$ it follows $|R(t)|$ achieves its maximum on the boundary (i.e. on $\Gamma_R$). Therefore,

$$L = \left| \left( I + \sum_{k=-\infty}^{\infty} \frac{\text{Res}(R(\pm q^k)\Delta(\pm q^k))}{s \pm q^k} \right)(I + \Delta(s))^{-1} \right|,$$

for some $s$ on $\Gamma_R$. Furthermore, $L > |R(\pm q^k)|$ for $k \in \pm \mathbb{N}$, as the points $t = \pm q^k$ lie in $\text{int}(\Gamma_R)$ for $k \in \pm \mathbb{N}$. As $\|\Delta\|_{\infty} < 1/2$, we can also determine $(I + \Delta(s))^{-1}$ using the Neumann series

$$(I + \Delta(s))^{-1} = \sum_{j=0}^{\infty} (-\Delta(s))^j.$$
Thus we find that,
\[
L < \left( I + L \sum_{k=-\infty}^{\infty} \left| \frac{\text{Res}(\Delta(\pm q^k))}{s \pm q^k} \right| \right) \sum_{j=0}^{\infty} \| \Delta(s) \|^j,
\]
\[
< 2 \left( I + L \sum_{k=-\infty}^{\infty} \left| \frac{\text{Res}(\Delta(\pm q^k))}{s \pm q^k} \right| \right).
\]
(2.21)

It follows from Remark 2.8 the sum on the RHS of Eq. (2.21) converges and
\[
\sum_{k=-\infty}^{\infty} \left| \frac{\text{Res}(\Delta(\pm q^k))}{s \pm q^k} \right| = O(q^n).
\]

Thus, Eq. (2.21) gives:
\[
L < 2 + 2L \times O(q^n). \tag{2.22}
\]

It follows there exists an $N$ such that for $n > N, L < 2$. Hence, we have just determined an upper bound for $|\hat{R}(t)|$ inside $\text{int}(\Gamma_R)$. Observing that $\text{Res}(\Delta(\pm q^k)) = O(q^n)$, Eq. (2.20) implies that for any fixed radius $r > 0$
\[
\hat{R}(t) = I + O(q^n)/r \text{ for } |t \pm q^\pm k| > r. \tag{2.23}
\]

By definition for $s$ lying on $\Gamma_1$,
\[
\hat{R}_1(s) = \hat{R}(s)(J_1)^{-1}. \tag{2.24}
\]

Recall that $|\hat{R}_1(t)|$ achieves its maximum on $\Gamma_1$. Applying Eqs. (2.23) and (2.24) and Lemma 2.7, where we showed $\|J_1 - I\| = C(m)O(q^n)$, we conclude $\hat{R}_1(t) = I + C(m)O(q^n)$. \hfill \Box

Having proved Lemma 2.9 we are now in a position to prove Theorem 1.5.

**Proof of Theorem 1.5** For $|t| \leq q^{-m-1/2}$ the result follows immediately from Lemma 2.9. Lemma 2.9 implies that
\[
\tilde{W}_n(z)\tilde{W}(t)^{-1} = I + C(m)O(q^n),
\]
for large $n$. Theorem 1.5 follows immediately after reversing the transformations $Y_n \rightarrow W_n$.

For $|t| > q^{-m-1/2}$, we observe that Eq. (2.23) implies that
\[
\tilde{R}(t) - I = O(q^n),
\]
for $|t \pm q^\pm k| > r$ (for some fixed $r$). We also observe that $\tilde{W} - I$ is bounded for $|t| > q^{-m-1/2}$ and $|t \pm q^\pm k| > r$, and goes to zero as $|t| \rightarrow \infty$. Furthermore, Eq.
(A.2) implies that the poles of $\mathcal{W}$ and $R(t)$ vanish for large $|t|$, much faster than the function $f(t)$ defined in Eq. (2.1) grows. As,

$$\tilde{W}_n(z)\tilde{W}(t)^{-1} = R(t),$$

this allows one to more accurately describe the behaviour of $P_n(q^k)$, $k \in \mathbb{N}$, as $n \to \infty$. \hfill \Box

We now prove Theorem 1.6.

**Proof of Theorem 1.6** Theorem 1.6 follows from Lemma 2.9. We note that in transforming from Eq. (2.8) to Eq. (2.9) the weight function was scaled by $q^{-n\alpha/2}$. Let $\zeta_n = q^{-n\alpha/2}\gamma_n$. Substituting in

$$Y_n(z) = \left[ \begin{array}{c} P_n(z) \\ \gamma_n^{-1} P_{n-1}(z) \end{array} \right] f_1^1 \frac{P_n(s)w(s)h_n(s)}{2\pi i(z-s)} ds$$

we can evaluate the expression

$$\begin{bmatrix} c_n^2 & 0 \\ 0 & c_n^{-2} \end{bmatrix} W_n(t) \begin{bmatrix} (c_n t q^{n/2})^{-n} & 0 \\ 0 & (c_n t q^{n/2})^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{t q^{n/2}} \begin{bmatrix} \beta_n c_n^{-1} \zeta_n c_n^2 \xi_n \xi_n^{-1} & \gamma_n c_n^{-2} \zeta_n \xi_n \end{bmatrix} + O(t^{-2}),$$

using the transformations detailed in Sect. 2.1. We note that the function $f(z)$, defined in Equation (2.3), is even and does not impact on the $z^{-1}$ coefficient during the transformations. Let,

$$\mathcal{W}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{t} \begin{bmatrix} B & 0 \\ C & 0 \end{bmatrix} + O(t^{-2}).$$

In Sect. 3 we show that this is a valid representation of the solution $\mathcal{W}(t)$ for large $t$. Applying Lemma 2.9 we observe that $\tilde{W}_n = \tilde{W} + O(q^n)$. Comparing coefficients of the $t^{-1}$ power in the top right term we find in the limit $n \to \infty$:

$$\zeta_n (-i q^{-\frac{1}{2}(\frac{\alpha}{2} - 1)} q^{2n} = B(1 + O(q^n)) q^{n/2},$$

$$\zeta_n = B(1 + O(q^n)) q^{n(\frac{\alpha}{2} - 1)} q^{n/2},$$

$$= B(1 + O(q^n)) q^{n(n-1)/2}.$$ 

Similarly in the bottom left term we find in the limit $n \to \infty$:

$$\zeta_n^{-1} = C(1 + O(q^n)) q^{-1} q^{(n-1)(n-2)/2},$$

The constants $B$ and $C$ can be evaluated by observing that generalised discrete $q$-Hermite I polynomials must satisfy these asymptotic conditions. Thus, $B = 2(q^2; q^2)_\infty$ and $C = q^{1+\alpha}/B$. Theorem 1.6 follows. \hfill \Box
3 On the Existence of a Unique Solution to the Model RHP

In this section we prove that there exists a unique solution to the model RHP given by Eq. (2.10).

3.1 Existence

We first show that there exists a solution to the model RHP. This is achieved by determining the connection matrix \([5]\) between three solutions of a \(q\)-difference equation (Eq. (3.1b)), \(S_A(t)h^a(t)\), \(S_B(t)\) and \(S_C(t)/g(t)\) (defined shortly). We prove that the connection matrix is equivalent to the jump condition, Eq. (2.10a), of the model RHP. Consequently, we show that the model RHP is satisfied by \(S_A(t)\), \(S_B(t)\) and \(S_C(t)\), after appropriate transformations. To begin, let us consider the two \(q\)-difference equations

\[
S_A(qt) = \begin{bmatrix} 1 \\ qt^{2-\alpha} q^{-\alpha} \end{bmatrix} S_A(t),
\]

\[
S_B(qt) = \begin{bmatrix} q^\alpha \\ qt^2 \end{bmatrix} S_B(t),
\]

where \(S_A(t)\) and \(S_B(t)\) are vectors with complex entries and \(\alpha \in (-1, \infty)\) is a real parameter. Note that: \(S_B(qt)/S_B(t) = q^\alpha S_A(qt)/S_A(t)\). We motivate the form of Eq. (3.1a) in Appendix B. Writing the entries of \(S_A(t) = [S^1_A(t), S^2_A(t)]^T\) as a power series in \(t\), we find by direct substitution into Eq. (3.1a) that \(S^2_A(t)\) can be written in terms of the odd power series

\[
S^2_A(t) = A_{2,1}t + A_{2,3}t^3 + A_{2,5}t^5 + \cdots + A_{2,j}t^{2j+1} + \cdots,
\]

where

\[
A_{2,2j+1} = \frac{-q^{3-\alpha+2j} A_{2,2j-1}}{(q^{-\alpha} - q^{2j+1})(1 - q^{2j})}.
\]

Likewise, \(S^1_A(t)\) can be written as an even power series

\[
S^1_A(t) = A_{1,0} + A_{1,2}t^2 + A_{1,4}t^4 + \cdots + A_{1,l}t^{2l} + \cdots,
\]

where

\[
A_{1,2l} = \frac{A_{2,2l-1}}{1 - q^{2l}}.
\]

From the recurrence relations we can deduce that both entries of \(S_A\) are entire. To see this, observe that for \(0 < q < 1\)

\[
\lim_{j \to \infty} \frac{A_{2,2j+1}}{A_{2,2j-1}} = 0.
\]
Similarly, $S_B(t) = [S^1_B(t), S^2_B(t)]^T$ can be written in terms of power series which converge everywhere. However, in this case $S^1_B(t)$ is odd and $S^2_B(t)$ is even.

Now consider the $q$-difference equation

$$S_C(qt) = -\frac{1}{q^2t^2} \left[ q^\alpha - t q^\alpha \right] S_C(t).$$  \hfill (3.1c)

Note the similarity to Eq. (3.1b). One can readily show that there exists a solution to Eq. (3.1c), $S_C(t) = [S^1_C(t), S^2_C(t)]^T$ which can be represented by a power series at infinity

$$S^1_C = \sum_{j=0}^\infty C_{1,2j+1} t^{-(2j+1)}, \quad C_{1,1} \neq 0,$$

$$S^2_C = \sum_{l=0}^\infty C_{2,2l} t^{-2l}, \quad C_{2,0} \neq 0,$$ \hfill (3.2)

which converges everywhere (except obviously at 0). Earlier, in Eq. (2.2) we defined the even function $g(t)$, which satisfies

$$g(t)/g(qt) = -q^2t^2.$$  

We also earlier defined the function $h^\alpha(t)$ in Equation (1.7), which satisfies the $q$-difference equation

$$h^\alpha(qt)/h^\alpha(t) = q^\alpha.$$  

As both $S_C(t)/g(t)$ and $S_A(t)h^\alpha$ satisfy the $q$-difference equation (3.1b) we conclude that

$$S_C(t)/g(t) = P_1(t)h^\alpha(t)S_A(t) + P_2(t)S_B(t),$$  \hfill (3.3)

where $P_1(qt) = P_1(t)$ and $P_2(qt) = P_2(t)$. This is the equivalent to a column of the connection matrix in [5]. As $S_C(t)/g(t)$ is a meromorphic function with simple poles at $t = \pm q^k$ for $k \in \mathbb{Z}$ we conclude that $P_1(t)$ must be a constant and $P_2(t)$ must be either be a constant or a meromorphic function with simple poles at $t = \pm q^k$. Thus, by Corollary C.2 we conclude

$$P_2(t) = c_1 h^0(z) + c_0,$$

where $c_1$ and $c_0$ are constants, and $h^0(z)$ is defined in Eq. (1.7). Comparing odd and even terms in Equation (3.3) we conclude that $c_1 = 0$ and $P_2(t) = c_0$. Thus, after absorbing constants into the power series of $S_A$ and $S_B$, we have shown,

$$S_C(t) = g(t)h^\alpha(t)S_A(t) + g(t)S_B(t),$$  \hfill (3.4)
and

\[ S_C = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + O \left( \frac{1}{|t|} \right), \quad \text{as } t \to \infty. \]

Furthermore, in Section A we show that \( S_A(t)/g(t) \) satisfies the asymptotic condition,

\[ S_A(t)/g(t) = \begin{bmatrix} C_0 \\ 0 \end{bmatrix} + O \left( \frac{1}{|t|} \right), \quad \text{as } t \to \infty, \]

where \( C_0 \) is a constant. Hence, in summary we have proved

\[ \left[ S_A(t)/g(t), S_C(t) \right] = \left[ S_A(t), S_B(t) \right] \begin{bmatrix} g(t)^{-1} h^g(t)g(t) \\ 0 \\ g(t) \end{bmatrix}, \quad \text{(3.5)} \]

where \( S_A(t) \) and \( S_B(t) \) are analytic everywhere and \( S_C(t) \) is analytic everywhere except \( t = 0 \). Furthermore, we have proved the asymptotic behaviour

\[ \left[ C_0^{-1} 0 \\ 0 1 \right] \left[ S_A(t)/g(t), S_C(t) \right] = I + O \left( \frac{1}{|t|} \right), \quad \text{as } t \to \infty. \quad \text{(3.6)} \]

Thus, after appropriate scaling we have found a solution which satisfies condition (i) of the model RHP: by the holomorphicity of \( S_A(t), S_B(t) \) and \( S_C(t) \), condition (ii): by Equation (3.5), condition (iii): by Eq. (3.6), and condition (iv): by Eq. (3.5).

3.2 Uniqueness

Uniqueness follows by considering the determinant of a solution, \( \mathcal{W} \), to the model RHP, Eq. (2.10). By the residue condition, Eq. (2.10c), we can deduce that \( \det(\mathcal{W}) \) is analytic in \( \mathbb{C} \setminus \Gamma \). Furthermore, by the jump condition, Eq. (2.10a), we deduce that \( \det(\mathcal{W}) \) is entire. Applying Louiville’s theorem we conclude that the asymptotic condition, Equation (2.10b), implies that \( \det(\mathcal{W}) = 1 \) everywhere.

Suppose that there exists another solution, \( \mathcal{W}_2 \) to the model RHP. By the residue condition, Eq. (2.10c), \( \mathcal{W}_2 \mathcal{W}^{-1} \) is analytic everywhere. Furthermore, the jump conditions cancel and we can conclude \( \mathcal{W}_2 \mathcal{W}^{-1} \) is entire. Applying Louisille’s theorem we conclude that the asymptotic condition, Eq. (2.10b), implies that \( \mathcal{W}_2 \mathcal{W}^{-1} = I \).

Thus, \( \mathcal{W}_2 = \mathcal{W} \).

4 Conclusion

In this paper, we determined the asymptotic behaviour of a general class of \( q \)-orthogonal polynomials by using the \( q \)-RHP setting [18]. The work is motivated by the methods developed by Deift et al. [9], which used the RHP setting to determine the asymptotic behaviour of semi-classical orthogonal polynomials. The main results
are Theorems 1.5 and 1.6 which provide more detailed asymptotic results for a large class of $q$-orthogonal polynomials than we could find in the literature.

There are a number of observations we can make from the results of this paper. In particular we proved that $\lim_{n \to \infty} \gamma_n$ is only dependent on the $\lim_{z \to 0} w(z)$, where $w(z) d_q z$ is the orthogonality measure. Furthermore, we note that the results in this paper hold even if $w(q^k) < 0$ for some $k \in \mathbb{N}_0$. That is we do not require positivity of the weight function. When determining a solution to the model RHP we observed some interesting examples of $q$-RHP theory. For example we demonstrated how to explicitly determine a connection matrix between two solutions of a $q$-difference equation and, in Appendix A, also saw the relationship between $q$-Borel transforms and divergent power series arising in $q$-difference equations.

An interesting avenue for future exploration would be to extend the results of the current paper to a larger class of $q$-orthogonal polynomials. Another possible direction could be determining if the theory presented in this paper can be applied to other settings not just $q$-discrete weights and orthogonal polynomials.

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Appendix A. Properties of the Solution to the Model RHP

In this section we prove some important properties of the solution to the RHP given by Eq. (2.10). These results are used in Sects. 2 and 3. The section concludes with a remark which highlights an interesting connection between the present work and $q$-Stoke’s phenomena. Note this is a side observation and not necessary for the proofs of the main results of this paper. As shown in Sect. 3 studying the solution to the RHP given by Eq. (2.10) is equivalent to studying the solutions $S_A$, $S_B$ and $S_C$ of the $q$-difference equations given in Sect. 3.

Definition A.1 For conciseness, we will adopt the notation $f(qz) = \tilde{f}(z)$ throughout the appendix.

Let

$$[S_A, S_B] = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $S_A(t)$ and $S_B(t)$ satisfy Eqs. (3.1a) and (3.1b) respectively. From Eq. (3.1a) it can be deduced that $a$ satisfies the second order $q$-difference equation

$$\tilde{a} + a(t^2 q^{3-\alpha} - (1 + q^{1-\alpha})) + q^{1-\alpha} a = 0.$$ 

Let us consider $a$ evaluated at $t = q^{-k}$, for large integer $k$, note that these locations coincide with the poles of $h^a$ and the zeros of $g$ defined in Eqs. (1.7) and (2.2)
respectively. If
\[ \bar{a} \ll \hat{a} t^2 q^{3-\alpha}, \]
then this implies that \( a \) grows as \( a/\bar{a} = -t^2 q^2 (1 + o(1)) \). Now consider the jump condition given in Eq. (3.4)
\[ agh^\alpha + bg = S_C^1. \]
At the points of interest, \( t = q^{-k} \), it can readily be verified by direct calculation from Eqs. (2.4) and (1.8) that \( gh^\alpha \) satisfies the \( q \)-difference equation:
\[ \frac{gh^\alpha}{gh^\alpha} = -q^{-1+\alpha}t^{-2}. \]
Note that \( g(t) = 0 \) and \( h(t)^\alpha \) has a simple pole at \( t = q^{-k} \) for \( k \in \mathbb{N} \). If \( a \) grows as \( a/\bar{a} = -t^2 q^2 \) then \( agh^\alpha \) grows as \( agh^\alpha/agh\bar{a} = t^4 q^{3-\alpha} (1 + o(1)) \), but the first entry in \( S_C, S_C^1 \), decays as \( 1/t \) for large \( t \) which is a contradiction (remembering that \( bg = 0 \) as \( g = 0 \) at our points of interest). Thus, \( a \) shrinks like \( \bar{a}/\bar{a} = -t^2 q^{3-\alpha} (1 + o(1)) \). Hence,
\[ \bar{a}/a = -t^2 q^{1-\alpha} (1 + o(1)), \]
at \( t = q^{-k} \). Note that this indicates that the residue of the poles of \( a/g \) are rapidly shrinking, such that
\[ \text{Res}(\bar{a}/\bar{g})/\text{Res}(a/g) = t^4 q^{4-\alpha} (1 + o(1)). \]
Furthermore, as \( a \) is an entire function by Liouville’s theorem there must be a \( t \in \mathbb{C} \) such that
\[ \bar{a} \ll \hat{a} t^2 q^{3-\alpha}. \]
Thus, it follows that along this ray (given by iterating \( t \)) in the complex plane,
\[ a/\bar{a} = -t^2 q^2 (1 + o(1)), \]
which importantly means that
\[ a\bar{g}/\bar{a}g = 1 + o(1) \]
(A.3)
Note that \( a/g \) is a meromorphic function, of the form
\[ a/g = \sum_{j=0}^{\infty} c_j z^{-j} + \sum_{k=-\infty}^{\infty} \frac{d_k}{t^2 - q^{2k}} \]
with vanishingly small poles (Eq. (A.2)). Hence, Equation (A.3) implies that $c_0$ is non-zero and $a/g$ approaches a non-zero constant for large $t$. A similar argument for the second entry of $S_a$ shows that

$$S_a/g = \begin{bmatrix} c_0 \\ 0 \end{bmatrix} + O\left(\frac{1}{|t|}\right), \text{ as } t \to \infty,$$

for $|t \pm q^{-k}| > r, k \in \mathbb{N}$ (with fixed $r > 0$).

**Remark A.2** By solving the $q$-discrete equation satisfied by $a/g$ one can determine a divergent power series representation for $a/g$ at infinity. Taking a $q$-Borel transformation [26] we expect this series to represent the presence of a theta function ‘switching’, analogous to Stokes phenomena. This is reflected in the vanishing poles found in Eq. (A.2).

### Appendix B. $q$-Hermite $q$-difference Equation

In this section we motivate the form of Eq. (3.1a) by studying the $q$-difference equation satisfied by discrete $q$-Hermite I polynomials. Discrete $q$-Hermite I polynomials satisfy the $q$-difference equation:

$$\overline{Y}_n(z) = \begin{bmatrix} 1 \\ -q^{2-n} \\ z(q^n - 1) \\ 1 - z^2 q^{2-n} \end{bmatrix} Y_n(z),$$

where

$$Y_n(z) = \begin{bmatrix} P_n \\ P_{n-1} \end{bmatrix}.$$ 

After making the transformation $tq^{n/2} = z$ we find

$$\overline{Y}_n(t) = \begin{bmatrix} 1 \\ t q^{2-n/2} \\ t q^{n/2} (q^n - 1) \\ 1 - t^2 q^2 \end{bmatrix} Y_n(t).$$

After taking the linear transformation

$$S_n(t) = \begin{bmatrix} 1 & 0 \\ 0 & q^{n/2} \end{bmatrix} Y_n(t).$$

We find that $S_n$ satisfies the $q$-difference equation

$$\overline{S}_n(t) = \begin{bmatrix} 1 \\ t q^2 \\ t(q^n - 1) \\ 1 - t^2 q^2 \end{bmatrix} S_n(t). \quad (B.1)$$
Taking the limit $n \to \infty$ the $q$-difference equation for $S_n$ becomes,

$$\overline{S}_\infty(t) = \begin{bmatrix} 1 & -t \\ tq^2 & 1 - t^2 q^2 \end{bmatrix} S_\infty(t).$$  \hspace{1cm} (B.2)

We would expect that the solution to this difference equation solves the model RHP for the case $\alpha = 0$, and indeed that is what we find.

### Appendix C. Functions Invariant Under $z \mapsto qz$

In this section we prove some properties about meromorphic functions with simple poles which are invariant under the transformation $z \mapsto qz$, i.e. $C(qz) = C(z)$.

**Lemma C.1** Let $C(z)$ be a function defined on $\mathbb{C} \setminus 0$, which is analytic everywhere except for simple poles at $q^k$ for $k \in \mathbb{Z}$. Then, $C(qz) \neq C(z)$.

**Proof** We prove the result by contradiction. Assume $C(qz) = C(z)$. Define

$$G(z) = (-z, -qz^{-1}; q)_\infty.$$

By direct calculation one can show $G(qz) = z^{-1}G(z)$. Furthermore, by definition, $G(z)$ is zero on the $q$-lattice $q^k$, $k \in \mathbb{Z}$. Let

$$F(z) = C(z)G(z),$$

then it follows $F(z)$ is analytic in $\mathbb{C} \setminus 0$ and satisfies the difference equation

$$F(qz) = z^{-1}F(z).$$  \hspace{1cm} (C.1)

As $F(z)$ is analytic in $\mathbb{C} \setminus 0$ we can write $F(z)$ as the Laurent series

$$F(z) = \sum_{k=-\infty}^{\infty} F_k z^k.$$

Comparing the coefficients of $z$ in Eq. (C.1), one can readily determine

$$F_k = c_0 q^{k(k-1)/2}. \hspace{1cm} (C.2)$$

However, there is only one solution with $z$ coefficients given by Eq. (C.2) (up to scaling by a constant) and it follows that $F(z) = c_0 G(z)$. Thus, if $C(qz) = C(z)$, then $C(z) = c_0$, and $C(z)$ has no poles. \qed

**Corollary C.2** Let $C(z)$ be a function defined on $\mathbb{C} \setminus 0$, which is analytic everywhere except for simple poles at $\pm q^k$ for $k \in \mathbb{Z}$. Furthermore, suppose $C(z)$ satisfies $C(qz) = C(z)$. Then, $C(z) = c_1 h^0(z) + c_0$, where $c_0$ and $c_1$ are constants and $h^0(z)$ is as defined in Eq. 1.7.
**Proof** As both $C(z)$ and $h^0(z)$ have simple poles at $z = -1$ we conclude that there exists a $c_1 \neq 0$ such that

$$\text{Res}(C(-1)) = c_1 \text{Res}(h^0(-1)).$$

Furthermore, both $C(z)$ and $h^0(z)$ are invariant under the transformation $z \to qz$, hence for all $k \in \mathbb{Z}$

$$\text{Res}(C(-qk)) = c_1 \text{Res}(h^0(-qk)).$$

Thus, the function

$$D(z) = C(z) - c_1 h^0(z),$$

is meromorphic in $\mathbb{C} \setminus 0$, with possible simple poles at $q^k$ for $k \in \mathbb{Z}$, and satisfies $D(qz) = D(z)$. However, by Lemma C.1, $D(z)$ can not have simple poles at $q^k$ for $k \in \mathbb{Z}$. Hence, $D(z)$ is analytic in $\mathbb{C} \setminus 0$ and it follows that $D(z)$ can be written as a convergent Laurent series. Thus,

$$D(z) = \sum_{j=-\infty}^{\infty} d_j z^j.$$

Substituting this into the $q$-difference equation $D(qz) = D(z)$, we conclude $D(z) = d_0 (= c_0)$ and Corollary C.2 follows immediately. □

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