Large sum-free sets in abelian groups

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Abstract

Let $A$ be a subset of a finite abelian group $G$. We say that $A$ is sum-free if the equation $x + y = z$, has no solution $(x, y, z)$ with $x, y, z$ belonging to the set $A$. In this paper we shall characterise the largest possible sum-free subsets of $G$ in case the order of $G$ is only divisible by primes which are congruent to 1 modulo 3.

1 Introduction and statements of result

Throughout this paper $G$ will denote a finite abelian group of order $n$. If $A$ is a subset of $G$ then we say that $A$ is sum-free if the equation $x + y = z$ has no solution with $(x, y, z) \in A \times A \times A$. We say that $A$ is maximal sum-free if it is not a proper subset of any sum-free set. The present article is motivated by the following question:

**Question 1.1.** Find a “structure” of all “large” maximal sum-free subsets of $G$.

In this regard we prove Theorem 1.10, Theorem 1.12 and Theorem 1.13. The results are based on Theorem 1.14 which is a recent result of Ben Green and Imre Ruzsa [GR05]. Our results give complete characterisation of largest sum-free subsets of $G$ in case all the divisors of $n$ are congruent to 1 modulo 3. For all other groups a structure of all largest sum-free subsets was known before.

Before stating our results and previously known results related to the above question we shall explain what do we mean by “large” and what sort of ’structure” does one may expect?

To understand the meaning of large in the above question, we need to understand the following question.

**Question 1.2.** How big is the largest sum-free subset of $G$?
Definition 1.3. (i) Given any finite abelian group \( K \) and a set \( A \subset K \), we define the density of the set \( A \) to be \( \frac{|A|}{|K|} \) and denote it by \( \alpha(A) \).

(ii) We use \( \mu(G) \) to denote the density of a largest sum-free subset of \( G \).

We say a sum-free set \( A \subset G \) is large if the density of the set \( A \) is close to \( \mu(G) \); that is \( \mu(G) - \alpha(A) \) is “small”.

The value of \( \mu(G) \) is now known for any finite abelian group \( G \) [GR05].

Theorem 1.4. ([GR05], Theorem 2) Let \( G \) be a finite abelian group and \( m \) be its exponent. Then the value of \( \mu(G) \) is given by the following formula.

\[
\mu(G) = \max_{d|m} \left\{ \frac{\left\lfloor \frac{d-2}{3} \right\rfloor + 1}{d} \right\}
\]

The following facts are straightforward to check.

(i) For any positive integer \( d \), we have the natural projection \( p_d : \mathbb{Z} \rightarrow \mathbb{Z}/d\mathbb{Z} \). Let the set \( B_d \subset \mathbb{Z}/d\mathbb{Z} \) be the image of integers in the interval \( \left( \frac{d}{3}, \frac{2d}{3} \right] \) under the map \( p_d \). Then it is straightforward to check that \( B_d \) is sum-free and density of the set \( B_d \) is given by

\[
\alpha(B_d) = \frac{\left\lfloor \frac{d-2}{3} \right\rfloor + 1}{d}.
\]

(ii) For any positive integer \( d \), there is a surjective homomorphism \( f : G \rightarrow \mathbb{Z}/d\mathbb{Z} \) if and only if \( d \) divides the exponent of the group \( G \).

(iii) For any positive integer \( d \) and a surjective homomorphism \( f : G \rightarrow \mathbb{Z}/d\mathbb{Z} \), the set \( A = f^{-1}(B_d) \) is a sum-free subset of \( G \) and \( \alpha(A) = \alpha(B_d) \).

Therefore the following result follows.

Theorem 1.5. ([GR05]) Given any finite abelian group \( G \), there exists a sum-free set \( A \subset G \), a surjective homomorphism \( f : G \rightarrow \mathbb{Z}/d\mathbb{Z} \) (where \( d \) is some positive integer), a sum-free set \( B \subset \mathbb{Z}/d\mathbb{Z} \); such that the following hold.

(i) \( A = f^{-1}(B) \).

(ii) The density of \( A \) is \( \mu(G) \).

Now regarding the structure of large sum-free subsets of \( G \) we may ask the following question.

Question 1.6. Let \( A \) be a “large” sum-free subset of \( G \). Then given any such set \( A \), does there always exist a surjective homomorphism \( f : G \rightarrow \mathbb{Z}/d\mathbb{Z} \) (where \( d \) is some positive integer) and a sum-free set \( B \subset \mathbb{Z}/d\mathbb{Z} \), such that the set \( A \) is a subset of the set \( f^{-1}(B) \)?

Before discussing this question we describe the value of \( \mu(G) \) more explicitly by dividing the finite abelian groups in the following three classes.

Definition 1.7. Suppose that \( G \) is a finite abelian group of order \( n \).

(i) If \( n \) is divisible by any prime \( p \equiv 2 \text{ (mod 3)} \) then we say that \( G \) is type \( I \). We say that \( G \) is type \( I(p) \) if it is type \( I \) and if \( p \) is the least prime factor of \( n \) of the form \( 3l + 2 \). In this case the value of \( \mu(G) \) is equal to \( \frac{1}{3} + \frac{1}{3p} \).
(ii) If $n$ is not divisible by any prime $p \equiv 2$(mod 3), but $3|n$, then we say that $G$ is type $II$. In this case the value of $\mu(G)$ is equal to $\frac{1}{3}$.

(iii) The group $G$ is said to be of type III if all the divisor of $n$ (order of $G$) are congruent to 1 modulo 3. Let $m$ be the exponent of $G$. In this case the value of $\mu(G)$ is equal to $\frac{1}{3} - \frac{1}{3m}$. We also note the fact that if $G$ is a group of type III then any subgroup as well as quotient of $G$ is also a type III group.

**Remark 1.8.** We note the fact that if $G$ is a type III group and $d$ is any divisor of $m$, then $d$ is odd and congruent to 1 modulo 3. Therefore $d$ is congruent to 1 modulo 6 and $\frac{d-1}{6}$ is a non negative integer.

Theorem 1.10 was proven for type $I$ and type $II$ groups by Diananda and Yap [DY69]. For some special cases of type $III$ groups it was proven by various authors ( see [Yap72, Yap75, RN73]). For an arbitrary abelian groups of type $III$ the proof of Theorem 1.10 is due to Ben Green and Ruzsa [GR05].

Hamidoune and Plagne [oHP04] answered the question 3 affirmatively when $|A| \geq \frac{|G|}{3}$, in the case $|G|$ is odd. In case $|G|$ is even they answered the question 3 affirmatively if $|A| \geq \frac{|G|+1}{3}$. In case $G = (\mathbb{Z}/2\mathbb{Z})^r$ with $r \geq 4$ and $|A| \geq 5.2^{-r-4}$ then Davydov and Tombak [DT89] showed that answer of this question is affirmative. Recently Lev [Lev05] answered this question affirmatively in the case when $G = (\mathbb{Z}/3\mathbb{Z})^r$ (with an integer $r \geq 3$) and $|A| > \frac{5}{27}3^r$. Lev [Lev] has also characterised the sum-free subsets $A$ of $\mathbb{Z}/p\mathbb{Z}$ when $p$ is prime and $|A| > 0.33p$.

Notice that none of the above mentioned results tells us anything related to the question 3, in case $G$ is a finite abelian group of type $III$ and $G$ is not cyclic. In case $G$ is cyclic the answer of question 3 is obviously affirmative. In case $G$ is not cyclic and of type $III$ Theorem 1.10 shows that the answer of question 3 is negative.

For the rest of this paper unless specified differently, $G$ shall denote a finite abelian group of type $III$ and of order $n$. The symbol $m$ shall denote the exponent of $G$ and $k = \frac{m-1}{6}$.

**Remark 1.9.** If $G$ is an abelian group of type $III$ and $m$ is exponent of $G$, there exist $S \subset G$ and $C \subset G$ such that $S$ and $C$ are subgroups of $G$, $C$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ and $G = S \oplus C$. In case $G$ is not cyclic, $S$ will be a nontrivial subgroup of $G$.

Let $p : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be the natural projection from the group of integers to $\mathbb{Z}/m\mathbb{Z}$.

**Theorem 1.10.** Let $G$ be a finite abelian group of type III. Let $m$ denote the exponent of $G$ and $k = \frac{m-1}{6}$. Suppose $G$ is not a cyclic group and $S, C$ are non trivial subgroups of $G$ such that $G = S \oplus C$, and $C$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. Let $g : C \to \mathbb{Z}/m\mathbb{Z}$ be an isomorphism. Let $J$ be any proper subgroup of $S$ and $b$ is any element belonging to $S$. Then consider the following two examples:

(i) The set $A = ((J + b) \oplus g^{-1}p(\{2k\})) \cup (S \oplus g^{-1}p(\{2k + j : 1 \leq j \leq 2k - 1\}) \cup ((J + 2b) \oplus g^{-1}p(\{4k\})).$ (Here for any set $D \subset S$ the symbol $D^c$ denotes the set $S \setminus D$.)

(ii) The set $A = ((J + b) \oplus g^{-1}p(\{2k\})) \cup ((J - 2b) \oplus g^{-1}p(\{2k + 1\}) \cup (S \oplus g^{-1}p(\{2k + j : 2 \leq j \leq 2k - 1\}) \cup ((J + 2b) \oplus g^{-1}p(\{4k\})) \cup ((J - b) \oplus g^{-1}p(\{4k + 1\})).$

Let $A$ be any of the set as above. Then the following holds.
(i) The set $A$ is a sum-free subset of $G$ and $\alpha(A) = \mu(G)$.

(ii) For any positive integer $d$, there does not exist any surjective homomorphism $f : G \to \mathbb{Z}/d\mathbb{Z}$, a sum-free set $B \subset \mathbb{Z}/d\mathbb{Z}$, such that the set $A$ is a subset of the set $f^{-1}(B)$.

We got to know the above examples from certain remarks made in [GR05] about the group $(\mathbb{Z}/7\mathbb{Z})^*$.

We prove that if $G$ is a type $III$ group and $A$ is a sum-free subset $G$ of largest possible cardinality then either $A$ is an inverse image of a sum-free subset of a cyclic quotient of $G$ or $A$ is one of the set as given in Theorem 1.10.

**Definition 1.11.** Given a sum-free set $A \subset G$, a surjective homomorphism $h : G \to \mathbb{Z}/d\mathbb{Z}$ the following definition and notations are useful.

(i) For any $i \in \mathbb{Z}/d\mathbb{Z}$ the symbol $A(h, i)$ denote the set $A \cap h^{-1}\{i\}$.

(ii) For any $i \in \mathbb{Z}/d\mathbb{Z}$ we define $\alpha(h, i) = \frac{d}{n}|A(h, i)|$.

(iii) Let $l = \frac{4n-1}{6}$ and $p_d : \mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ be the natural projection. The sets $H_d, T_d, M_d, I_d \subset \mathbb{Z}/d\mathbb{Z}$ denote the following sets.

\[
H_d = p_d\{l + 1, l + 2, \cdots, 2l - 1, 2l\} \\
M_d = p_d\{2l + 1, 2l + 2, \cdots, 4l - 1, 4l\} \\
T_d = p_d\{4l + 1, 4l + 2, \cdots, 5l - 1, 5l\} \\
I_d = p_d\{l + 1, l + 2, \cdots, 5l - 1, 5l\}
\]

(iv) The symbol $H, T, M, I$ denote the sets $H_m, T_m, M_m, I_m$ respectively. The symbol $p$ denotes the map $p_m$.

**Theorem 1.12.** Let $G$ be a finite abelian group of type $III$. Let $A$ be a sum-free subset of an abelian group $G$. Let $n$ denotes the order of $G$ and $m$ denotes the exponent of $G$. Let $k = \frac{m-1}{6}$ and $\eta = 2^{-23}$. Suppose that $\alpha(A) > \mu(G) - \min(\eta, \frac{5}{42m})$. Then there exists a surjective homomorphism $f : G \to \mathbb{Z}/m\mathbb{Z}$ such that the following holds.

$A \subset f^{-1}p(\{2k + j : 0 \leq j \leq 2k + 1\})$

Further the following also holds.

(i) For all $i \in p(\{2k + j : 2 \leq j \leq 2k - 1\})$ the inequality $\alpha(f, i) \geq 1 - m (\mu(G) - \alpha(A))$ holds.

(ii) $\alpha(f, 2k) + \alpha(f, 4k) \geq 1 - m (\mu(G) - \alpha(A))$.

(iii) $\alpha(f, 4k + 1) + \alpha(f, 2k + 1) \geq 1 - m (\mu(G) - \alpha(A))$

(iv) If $A$ is maximal then $f^{-1}p(\{2k + j : 2 \leq j \leq 2k - 1\}) \subset A$
Using Theorem 1.12 the following theorem follows easily.

**Theorem 1.13.** Let $A$ be a sum-free subset of an abelian group $G$ of type $III$. Let the symbol $m$ denote the exponent of $G$ and $k = \frac{m-1}{6}$. Let the density of the set $A$ is equal to $\mu(G)$ and $f : G \to \mathbb{Z}/m\mathbb{Z}$ be a surjective homomorphism as given by Theorem 1.12.

Let the set $S$ denotes the kernel of $f$ and $C$ is a subgroup of $S$ such that $G = S \oplus C$. Let $g : C \to \mathbb{Z}/m\mathbb{Z}$ be an isomorphism obtained by restricting $f$ to the subgroup $C$. Then there exist $J$ a subgroup of $S$ and $b \in S$ such that one of the following holds:

(i) The set $A = f^{-1}p(\{2k + j : 1 \leq j \leq 2k\})$.

(ii) One of the set $A$ or $-A$ is equal to the following set

\[
((J+b) \oplus g^{-1}p(\{2k\})) \cup (f^{-1}p(\{2k + j : 1 \leq j \leq 2k - 1\})) \cup ((J+2b)^c \oplus g^{-1}\{4k\}).
\]

(Here and in the following for any set $D \subset S$ the symbol $D^c$ denotes the set $S \setminus D$.)

(iii) The set $A$ is union of the sets $f^{-1}p(\{2k+j : 2 \leq j \leq 2k-1\})$, $((J+b) \oplus g^{-1}p(\{2k\}))$, $((J+2b)^c \oplus g^{-1}p(\{4k\}))$, $((J-b) \oplus g^{-1}p(\{4k + 1\}))$

and $((J-2b)^c \oplus g^{-1}p(\{2k + 1\}))$.

The proof of Theorem 1.12 is based on the following result of Ben Green and Ruzsa [GR05].

**Theorem 1.14.** ([GR05], Proposition 7.2.) Let $A$ be a sum-free subset of an abelian group $G$ of type $III$. Let $\eta = 2^{-23}$. Suppose that $\alpha(A) \geq \mu(G) - \eta$, then there exists a surjective homomorphism $\gamma : G \to \mathbb{Z}/q\mathbb{Z}$ with $q \neq 1$ such that the following holds.

$$A \subset \gamma^{-1}I_q.$$ 

We require the following definitions and notations.

**Definition 1.15.** Let $A$ be a sum-free subset of $G$ and $\alpha(A) \geq \mu(G) - \eta$, then we choose a $\gamma$ a surjective homomorphism from $Z$ to $\mathbb{Z}/q\mathbb{Z}$ with $q \neq 1$ and $A \subset \gamma^{-1}I_q$. Following the terminology of [GR05] we call $\gamma$ the special direction of the set $A$ and $q$ the order of special direction. We use the symbols $\alpha_i$ and $A_i$ to denote the number $\alpha(\gamma,i)$ and the set $A(\gamma,i)$ respectively. We use symbol $S$ to denote the set $\ker(\gamma)$ and $S_i$ to denote $\gamma^{-1}\{i\}$.

**Remark 1.16.** Notice that in case $G = (\mathbb{Z}/7\mathbb{Z})^r$, the Theorem 1.14 is equivalent to Theorem 1.12 as in this special case $l + 1 = 2l$.

## 2 An outline of the paper

Let $H(A)$ is the largest subset of $G$ such that $H(A) + A = A$. The set $H(A)$ as defined is called period or stabiliser of the set $A$. For any set $A$ as as in Theorem 1.10 we prove that $H(A) = J$ where $J$ is as in Theorem 1.10. Using this Theorem 1.10 is easy to prove.

Let $A$ be as in Theorem 1.12 and $\gamma$ be the special direction of $A$. The proof of Theorem 1.12 is divided into following three parts.

(i) The order of the special direction of $A$ is $m$.

(ii) We define the set $L = \{i \in \mathbb{Z}/m\mathbb{Z} : \alpha_i > \frac{1}{4}\}$. We show that for any $i, j \in L$, $\alpha_{i+j} = 0$.

In particular the set $L$ is a sum-free subset of $\mathbb{Z}/m\mathbb{Z}$. Moreover the cardinality of $L$ is $2k = 2\frac{m-1}{6}$; that is $L$ is a largest sum-free subset of $\mathbb{Z}/m\mathbb{Z}$.
(iii) We describe all the largest sum-free subset of \( \mathbb{Z}/m\mathbb{Z} \) which are subsets of \( I \). Using Theorem 1.14 this characterises all the largest sum-free subsets of \( \mathbb{Z}/m\mathbb{Z} \).

There are two facts which we use repeatedly. One is that for any \( i, j \in \mathbb{Z}/q\mathbb{Z} \) the sets \( A(\gamma, i) + A(\gamma, j) \) and \( A(\gamma, i + j) \) are disjoint. Another is that for any divisor \( d \) of \( m \), the set \( I_d \) is divided into \( 2^{d-1}/6 \) disjoint pairs of the form \((i, 2i)\) with \( i \) belonging to the set \( H_d \cup M_d \).

3 Stabiliser of largest sum-free subset

In this section we shall give the proof of Theorem 1.10. Any abelian group acts on itself by translation. Given any set \( B \subset G \) we define the set \( H(B) \) to be those elements of the group \( G \) such that the set \( B \) is stable under the translation by the elements of the set \( H(B) \). In other words the set \( H(B) = \{ g \in G : g + B = B \} \). For any set \( B \) the set \( H(B) \) is a subgroup.

Let \( G \) be an abelian group of type \( III \) and let \( A \subset G \) be as in Theorem 1.10. To prove Theorem 1.10 we shall prove the following

**Lemma 3.1.** Let \( S \) and \( C \) are as in Theorem 1.10 and \( \pi_C : G \to G/S = C \) be the natural projection. Then the set \( \pi_C(H(A)) = \{0\} \).

**Proof.** Since \( \pi_C \) is a homomorphism and \( H(A) \) is a subgroup of \( G \), the set \( \pi_C(H(A)) \) is a subgroup of \( M \). Since \( H(A) + A = A \) by the definition of \( H(A) \), it follows that \( \pi_C(H(A)) + \pi_C(A) = \pi_C(A) \). Therefore the set \( \pi_C(A) \) is a union of cosets of \( \pi_C(H(A)) \). Therefore the cardinality of the subgroup \( \pi_C(H(A)) \) divides the cardinality of the set \( \pi_C(A) \). Since the set \( \pi_C(H(A)) \) is a subgroup of \( C \), it is also true that \( \pi_C(H(A)) \) divides \( m \). This implies that \( |\pi_C(H(A))| \) divides \( \gcd(|\pi_C(A)|, m) \). Now if \( A \) is a set as in Theorem 1.10 (I) then the cardinality of the set \( \pi_C(A) \) is equal to \( 2k + 1 \) and if \( A \) is a set as in Theorem 1.10 (II) then the cardinality of the set \( \pi_C(A) \) is equal to \( 2k + 2 \). Since \( m = 6k + 1 \) it follows that in first case the number \( \gcd(|\pi_C(A)|, m) \) divides \( 2 \) and in the second case it divides \( 5 \). But as \( G \) is type \( III \) group, any divisor of \( m \) which is not equal to \( 1 \) is greater than or equal to \( 7 \). Hence \( \gcd(|\pi_C(A)|, m) = 1 \) for any of the set \( A \) as in Theorem 1.10. This forces \( |\pi_C(H(A))| = 1 \) and hence the lemma follows.

**Proposition 3.2.** Let \( A \) be any of the set as in Theorem 1.10 and \( S \) be a subgroup of \( G \), \( J \) be a proper subgroup of \( S \) as in Theorem 1.10. Then the stabiliser of the set \( A \) is equal to the set \( J \).

**Proof.** Using the previous lemma it follows that \( H(A) + ((J + b) \oplus g^{-1}\{2k\}) = (J + b) \oplus g^{-1}\{2k\} \). This implies that \( H(A) + J + b = J + b \). This implies that \( H(A) \subset J \). But it is straightforward to check that \( J + A = A \). Therefore it follows that \( J = H(A) \), proving the claim.

**Proof. of Theorem 1.10.**

(i) This is straightforward to check.

(ii) Suppose \( A \) is one of a set as in Theorem 1.10. Suppose the claim is not true for this set. Then there exist a positive integer \( q \), a surjective homomorphism \( f : G \to \mathbb{Z}/q\mathbb{Z} \), a sum-free set \( B \subset \mathbb{Z}/q\mathbb{Z} \) such that the set \( A \) is a subset of \( f^{-1}(B) \). Since from (I), the set \( A \) is a sum-free set of largest possible cardinality, it follows
that the set $A = f^{-1}(B)$. Therefore the kernel of $f$ is contained in the set $H(A)$.

Therefore we have the following inequality

$$|H(A)| \geq \frac{n}{q}. \quad (1)$$

But from Proposition 5.2 the stabiliser of the set $A$ is $J$ which is a proper subgroup of $S$. Since $m$ is the exponent of $G$ it follows that $q$ is less than or equal to $m$. Therefore we have the following inequality

$$|H(A)| < |S| = \frac{n}{m} \leq \frac{n}{q} \quad (2)$$

This contradiction proves the claim.

\[ \square \]

4 Order of the special direction

Let $A$ be as sum-free subset of $G$ and $\alpha(A) > \min(\eta, \frac{n}{4m})$. Let $\gamma$ be the special direction of the set $A$ as given by Theorem 1.14. In this section we shall show that the order of $\gamma$ is equal to $m$. The proof of this result is inherent in \cite{GR05}. We reproduce the proof here for the sake of completeness.

**Lemma 4.1.** \cite{GR05}, Lemma 7.3. (ii) Let $A$ be any surjective homomorphism $g : G \to \mathbb{Z}/d\mathbb{Z}$, where $d$ is a positive integer. Then for any $i \in \mathbb{Z}/d\mathbb{Z}$, the following inequality holds.

$$\alpha(g, i) + \alpha(g, 2i) \leq 1 \quad (3)$$

Here $\alpha(g, i)$ is a number as defined in section 1.

**Proof.** For any $i \in \mathbb{Z}/d\mathbb{Z}$, let the set $A(g, i)$ be as defined in section 1. The fact that $g$ is a homomorphism implies that the set $A(g, i) + A(g, i)$ is a subset of the set $g^{-1}\{2i\}$. The fact that the set $A$ is sum-free implies that the set $A(g, i) + A(g, i)$ is disjoint from the set $A(g, 2i)$. Therefore we have the following inequality.

$$|A(g, i) + A(g, i)| + |A(g, 2i)| \leq |g^{-1}\{2i\}| \quad (4)$$

The claim follows by observing that the set $A(g, i) + A(g, i)$ has cardinality at least $|A(g, i)|$.

\[ \square \]

The following lemma is straightforward to check, but is very useful.

**Lemma 4.2.** Let $d$ be a positive integer congruent to 1 modulo 6. Let $d = 6l + 1$. Let the set $I_d, H_d, M_d, T_d$ are subsets of the group $\mathbb{Z}/d\mathbb{Z}$ as defined in section 1. The set $I_d$ is divided into $2l$ disjoint pairs of the form $(i, 2i)$ where $i$ belong to the set $H_d \cup T_d$ and $2i$ belong to the set $M_d$.

**Proposition 4.3.** Let $A$ be a sum-free subset of an abelian group $G$, of type III. Let $m$ be the exponent of $G$ and $\alpha(A) > \min(\eta, \frac{n}{4m})$. Let $\gamma : G \to \mathbb{Z}/q\mathbb{Z}$ be the special direction of the set as given by Theorem 1.14. Then the order of $\gamma = q = m$.
Proof. Since $G$ is type III group, therefore any prime divisor of the order of $G$ is greater than or equal to 7. Therefore if $q$ is not equal to $m$, then $q \leq \frac{m}{7}$.

Using Theorem 1.14 we have the following equality for the density of the set $A$.

$$\alpha(A) = \frac{1}{q} \sum_{i \in \mathbb{Z}/q\mathbb{Z}} \alpha_i = \frac{1}{q} \sum_{i \in I_q} \alpha_i$$

Now from Lemma 4.2 it follows that

$$\alpha(A) = \frac{1}{q} \sum_{i \in H_q \cup M_q} (\alpha_i + \alpha_{2i}) \leq \frac{1}{q} \sum_{i \in H_q \cup M_q} 1.$$  

Since the cardinality of the set $H_q \cup M_q$ is equal to $q - 1$, it follows that

$$\alpha(A) \leq \frac{1}{3} - \frac{7}{3m}.$$  

But the last inequality above is contrary to assumption that

$$\alpha(A) > \mu(G) - \frac{5}{42m} > \frac{1}{3} - \frac{7}{3m}.$$  

Hence the lemma follows. \hfill \Box

5 Element with large fiber

Given a set $A \subset G$ such that $\alpha(A) \geq \mu(G) - \min(\eta, \frac{5}{12m})$, from Theorem 1.14 and Proposition 4.3 it follows that $\gamma$ is a surjective homomorphism from $G$ to $\mathbb{Z}/m\mathbb{Z}$ such that $A$ is a subset of the set $\gamma^{-1}(I)$, where $\gamma$ is the special direction of the set $A$ and $I$ is a subset of $\mathbb{Z}/m\mathbb{Z}$ as defined in section 4. Then we define $L \subset \mathbb{Z}/m\mathbb{Z}$ as follows.

$$L = \{i \in \mathbb{Z}/m\mathbb{Z} : \alpha(g, i) > \frac{1}{2} \}$$

We say that the fiber of an element $i \in \mathbb{Z}/m\mathbb{Z}$ is large if $i$ belong to the set $L(g, A)$. It is clear that $L$ is a subset of the set $I$. In this section we shall show that the set $L$ is a sum-free subset $\mathbb{Z}/m\mathbb{Z}$ and the cardinality of the set $L$ is $2k$, where $k$ is equal to $\frac{m-1}{6}$.

The fact that the set $L$ is sum-free is a consequence of the following folklore in additive number theory. We give a proof of the following lemma for the sake of completeness.

Lemma 5.1. (folklore) Let $C$ and $B$ are subsets of a finite abelian group $K$ such that $\min(|C|, |B|) > \frac{1}{2}|K|$. Then $C + B = K$.

Proof. Suppose there exist $x \in K$ such that $x$ does not belong to $C + B$. This is clearly equivalent to the fact that $C \cap (x-B) = \phi$. But this means that $|K| > |C| + |x-B| > |K|$. This is not possible. Hence the lemma is true. \hfill \Box

Lemma 5.2. For any two elements $i, j \in L$, we have $\alpha_{i+j} = \alpha_{i-j} = 0$. In particular the set $L$ is sum-free.
Proof. The fact that $\gamma$ is a homomorphism implies that the set $A_i + A_j$ is a subset of the set $S_{i+j}$. Take any $x \in S_i, y \in S_j$. Let $C = A_i - x$ and $B = A_j - y$ so that we have the sets $C$ and $B$ are subsets of group $S$. Then applying Lemma 5.1 it follows that $C + B = S$. Therefore we have $A_i + A_j = S_{i+j}$. The fact that $A$ is sum-free implies that $A_i + A_j \cap (A_i + A_j) = \phi$. Since we have shown that the set $A_i + A_j = S_{i+j}$, it follows that the set $A_i + j = \phi$. In other words it follows that $\alpha_{i+j} = 0$. From similar arguments it also follows that $\alpha_{i-j} = 0$.

Now we shall show that the cardinality of the set $L$ is equal to $2k$. For this we require the following Lemma.

**Lemma 5.3.** Let $A, G$, be as in theorem 1.12. Let $m$ be the exponent of the group $G$ and $I, H, T, M$ are the subsets of $\mathbb{Z}/m\mathbb{Z}$ as defined in section 7. Let $m = 6k + 1$. Let $g$ be a surjective homomorphism $g : G \to \mathbb{Z}/m\mathbb{Z}$ such that the following holds.

$$A \subset g^{-1}(I).$$

Then we have the following inequality

$$\alpha(g, i) + \alpha(g, 2i) \geq 1 - m(\mu(G) - \alpha(A)), \quad \forall i \in H \cup T. \quad (5)$$

**Proof.** From the assumption on the homomorphism $g$ it follows that the density of the set $A$ satisfy the following equality

$$\alpha(A) = \frac{1}{m} \sum_{i \in I} \alpha(g, i).$$

Now from Lemma 4.2 it follows that the set $I$ is divided into $2k$ disjoint pairs of the form $(i, 2i)$ where $i$ belongs to the set $H \cup T$. Therefore it follows that

$$\alpha(A) = \frac{1}{m} \sum_{i \in I} \alpha(g, i) = \frac{1}{m} \sum_{i \in H \cup T} (\alpha(g, i) + \alpha(g, 2i)).$$

Now using Lemma 4.1 it follows that for any $i_0 \in H \cup T$ the following inequality holds

$$m\alpha(A) \leq 2k - 1 + \alpha(g, i_0) + \alpha(g, 2i_0).$$

From this the required inequality follows for any $i_0$ belonging to the set $H \cup T$ after observing that $\mu(G) = \frac{2k}{m}$.

We need the following well known theorem due to Kneser.

**Theorem 5.4.** (Kneser) Let $C, B$ are subsets of a finite abelian group $K$ such that $|C + B| < |C| + |B|$. Let $F = H(C + B) = \{g \in G : g + C + B = C + B\}$ be the stabiliser of the set $C + B$. Then the following holds

$$|C + B| = |C + F| + |B + F| - |F|.$$ 

In particular the set $F$ is a nontrivial subgroup of $K$ and

$$|F| \geq |C| + |B| - |C + B|. \quad (6)$$

For the proof of this theorem one may see [Nat91].
Lemma 5.5. Let $A, G, \gamma$ be as in theorem 1.12, $H, T, M$ be as defined earlier. Then the following holds

(i) For any $i \in H \cup T$ if $\alpha_i \leq \frac{1}{2}$ then $\alpha_{2i} > \frac{1}{2}$.

(ii) The cardinality of the set $L$ is equal to $2k$.

Proof. (i) Suppose the claim is not true. Then there exist $i_0 \in H \cup T$ such that

\[
\alpha_{i_0} \leq \frac{1}{2}, \quad \alpha_{2i_0} \leq \frac{1}{2}.
\]

Then from (5) it follows that

\[
\alpha_{i_0} > \frac{1}{2} - \frac{5}{42},
\]

\[
\alpha_{2i_0} > \frac{1}{2} - \frac{5}{42}.
\]

Take any $x \in S_{i_0}$ and consider the set $A_{i_0} - x$. Then invoking (4) and using (10), it follows that

\[
| (A_{i_0} - x) + (A_{i_0} - x) | = | A_{i_0} + A_{i_0} | \leq | S | - | A_{2i_0} | < \left( \frac{1}{2} + \frac{5}{42} \right) | S |
\]

Therefore using (4) it follows that

\[
2|A_{i_0} - x| = 2|A_{i_0}| > 2 \left( \frac{1}{2} - \frac{5}{42} \right) | S | > | (A_{i_0} - x) + (A_{i_0} - x) |
\]

Let $F$ denote the stabiliser of the set $(A_{i_0} - x) + (A_{i_0} - x)$. We can apply Theorem 5.4 with $C = B = A_{i_0} - x$ and using (3), (11), (12) we have the following inequality

\[
| F | > \left( \frac{1}{2} - \frac{15}{42} \right) | S | = \frac{1}{7} | S |.
\]

Therefore the cardinality of the group $S/F$ is strictly less than 7. But since $S$ is a group of type $III$, the group $S/F$ is also of type $III$. Hence it follows that $S = F$.

Therefore the stabiliser of the set $A_{i_0} - x$ is equal to the group $S$. This implies that the set $A_{i_0} = S_{i_0}$. This is in contradiction to the assumption that $\alpha_{i_0} \leq \frac{1}{2}$. Hence the claim follows.

(ii) The set $I$ is divided into $2k$ disjoint pairs of the form $(i, 2i)$ with $i \in H \cup T$. From (I) it follows that at least one element of any such pair belongs to the set $L$. The claim follows since we have shown that the set $L$ is sum-free and is a subset of the set $I$.

From Lemma 5.2 and Lemma 5.5 the following proposition follows

Proposition 5.6. (i) The set $L$ is a sum-free subset of $\mathbb{Z}/m\mathbb{Z}$ of cardinality $2k$. The set $L$ is a subset of the set $I$.

(ii) For any two elements $i, j \in L$, we have $\alpha_{i+j} = \alpha_{i-j} = 0$. 


6 Sum-free subset of cyclic group

Let the group $\mathbb{Z}/m\mathbb{Z}$ be of type $\text{III}$ group and $m = 6k + 1$. In this section we shall characterise all the sets $E \subset \mathbb{Z}/m\mathbb{Z}$ such that the set $E$ is sum-free and $|E| = 2k$. From Theorem 1.14 it is sufficient to characterise those sets $E$ which are subset of the set $I$.

Lemma 6.1. Let $E \subset \mathbb{Z}/m\mathbb{Z}$ be a sum-free set. Let the group $\mathbb{Z}/m\mathbb{Z}$ is of type $\text{III}$ and the cardinality of the set $E$ is $2k$, where $k$ is equal to $m - 1$. Let $H, T, M, I$ are subsets of $\mathbb{Z}/m\mathbb{Z}$ as defined in section 1 and the set $E$ is a subset of the set $I$. Then for any element $y$ belonging to the set $M$ exactly one of the element of the pair $(\frac{y}{2}, y)$ belongs to the set $E$.

Proof. This is straightforward from the fact that the set $I$ is divided into $2k$ disjoint pairs of the form $(\frac{y}{2}, y)$ with $y$ belonging to the set $M$ and the assumption that the set $E$ is a sum-free set and is a subset of the set $I$. □

We have the natural projection $p$ from the set of integers to $\mathbb{Z}/m\mathbb{Z}$.

$$p: \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$$

Since the map $p$ restricted to the set $\{0, 1, 2, \ldots, m - 2, m - 1\} \subset \mathbb{Z}$ is a bijection to the group $\mathbb{Z}/m\mathbb{Z}$ (as a map of the sets) we can define

$$p^{-1}: \mathbb{Z}/m\mathbb{Z} \to \{0, 1, 2, \ldots, m - 1\}$$

in an obvious way.

The following lemma is straightforward to check.

Lemma 6.2. (i) The set $H$ is equal to the set $-T$.

(ii) The set $M$ is equal to the set $-M$.

(iii) The set $I$ is equal to the set $-I$.

(iv) For any set $B \subset \mathbb{Z}/m\mathbb{Z}$ the set $B \cap T$ is same as the set $-(B \cap H)$. Also the set $p^{-1}(B \cap T) = m - p^{-1}((-B) \cap H)$.

(v) The set $H + H$ as well as the set $T + T$ are subsets of the set $M$.

(vi) Given any even element $y$ belonging to the set $p^{-1}(M)$ the element $\frac{p(y)}{2}$ belong to the set $H$. Also the element $p^{-1}(\frac{p(y)}{2})$ is equal to the element $\frac{y}{2}$.

(vii) Given any odd element $y$ belonging to the set $p^{-1}(M)$ the element $\frac{p(y)}{2}$ belong to the set $T$. Also the element $p^{-1}(\frac{p(y)}{2})$ is equal to the element $\frac{y + 6k + 1}{2}$.

(viii) Given any two elements $x, y$ belonging to the set $p^{-1}(H)$ which are of same parity, the element $p^{-1}(\frac{p(x+y)}{2}) = \frac{x+y}{2}$ and the element $\frac{x+y}{2}$ belongs to the set $p^{-1}(H)$. Moreover if $x$ is strictly less than $y$ then the following inequality holds.

$$x < \frac{x+y}{2} < y$$

Lemma 6.3. Let $E$ be a set as above. Then the following holds.

(i) Given any two elements $x, y$ which belong to the set $p^{-1}(E \cap H)$ and are of same parity, the element $\frac{x+y}{2}$ belong to the set $p^{-1}(E \cap H)$.
Proposition 6.4. Let $E$ be a set as above then the following holds.

(i) The set $p^{-1}(E \cap H)$ as well as the set $p^{-1}(E \cap T)$ is an arithmetic progression with an odd common difference.

(ii) Given any two elements $x, y$ which belong to the set $p^{-1}(E \cap H)$ and are of different parity, the element $\frac{x+y+6k+1}{2}$ belong to the set $p^{-1}(E \cap T)$.

(iii) Given an element $x$ belonging to the set $p^{-1}(E \cap H)$ and an element $y$ belonging to the set $p^{-1}(E \cap T)$, the element $\frac{p(x)+p(y)}{2}$ belong to the set $p^{-1}(E \cap (H \cup T))$.

(iv) Any two consecutive element of the set $p^{-1}(E \cap H)$ (or of the set $p^{-1}(E \cap T)$) are of different parity.

Proof. Since the set $E$ is sum-free, it follows that given any two elements $x, y$ belonging to the set $p^{-1}(E)$, neither the element $p(x) + p(y)$ nor the element $p(x) - p(y)$ belong to the set $E$. Using this we prove all the claims.

(i) Under the assumption, the element $p(x) + p(y)$ belong to the set $M$. From Lemma 6.4 it follows that the element $\frac{p(x)+p(y)}{2}$ belong to the set $E$. Also the element $p^{-1}(p(x)+p(y))$ is equal to $x+y$ and is even. Therefore invoking Lemma 6.2 it follows that the element $p^{-1}\left(\frac{p(x)+p(y)}{2}\right)$ is equal to $\frac{x+y}{2}$ and belong to the set $p^{-1}(H)$. Hence the claim follows.

(ii) Under the assumption, the element $p(x) + p(y)$ belong to the set $M$. From Lemma 6.1 it follows that the element $\frac{p(x)+p(y)}{2}$ belong to the set $E$. In this case the element $p^{-1}(p(x+y))$ is equal to the element $x+y$ and is odd. Therefore invoking Lemma 6.2 it follows that the element $p^{-1}\left(\frac{p(x)+p(y)}{2}\right)$ is equal to the element $\frac{x+y+6k+1}{2}$ and belong to the set $p^{-1}(T)$. Hence the claim follows.

(iii) Under the assumption, the element $p(x) - p(y)$ belong to the set $M$. Therefore the claim follows invoking Lemma 6.1.

(iv) Let the set $p^{-1}(E \cap H) = \{x_1 < x_2 < \cdots < x_h\}$. Suppose there exist $1 \leq i_0 \leq h-1$ such that the element $x_{i_0}$ and the element $x_{i_0+1}$ have same parity. Then from (I) it follows that the element $\frac{x_{i_0} + x_{i_0+1}}{2}$ belong to the set $p^{-1}(E \cap H)$. From Lemma 6.2 the following inequality also follows.

$$x_{i_0} < \frac{x_{i_0} + x_{i_0+1}}{2} < x_{i_0+1}$$

But this contradicts the fact that the elements $x_{i_0}$ and $x_{i_0+1}$ are consecutive elements of the set $p^{-1}(E \cap H)$. Therefore the claim follows for the set $p^{-1}(E \cap H)$. Replacing the set $E$ by the set $-E$, it follows that the any two consecutive element of the set $p^{-1}((-E) \cap H)$ are also of different parity. Noticing that the set $p^{-1}(E \cap T) = m - p^{-1}((-E) \cap H)$, the claim follows for the set $p^{-1}(E \cap T)$ also.
Proof. Let the set $p^{-1}(E \cap H) = \{x_1 < x_2 < \cdots < x_h\}$.

(i) In case the cardinality of the set $E \cap H$ is less than or equal to 2, the claim is trivial for the set $p^{-1}(E \cap H)$. Otherwise for any $1 \leq i \leq h - 2$, consider the elements $x_i, x_{i+1}, x_{i+2}$, then from Lemma 6.3 it follows that the parity of elements $x_i$ and $x_{i+1}$ are different. For the same reason the the parity of elements $x_i$ and $x_{i+2}$ are different. Therefore the parity of elements $x_i$ and $x_{i+2}$ are same. Therefore from Lemma 6.3 it follows that the element $\frac{x_i + x_{i+2}}{2}$ belong to the set $p^{-1}(E \cap H)$. But from Lemma 6.2 the following inequality follows.

$$x_i < \frac{x_i + x_{i+2}}{2} < x_{i+1}$$

Hence it follows that for any $1 \leq i \leq h - 2$

$$\frac{x_i + x_{i+2}}{2} = x_{i+1}.$$

This is equivalent to the fact that the set $p^{-1}(E \cap H)$ is an arithmetic progression. It also follows that the common difference is odd. The claim for the set $p^{-1}(E \cap T)$ follows by replacing the set $E$ by the set $-E$.

(ii) From Lemma 6.3 it follows that the set

$$\left\{\frac{x_1 + x_2 + 6k + 1}{2} < \frac{x_2 + x_3 + 6k + 1}{2} < \cdots < \frac{x_{h-1} + x_h + 6k + 1}{2}\right\} \subset p^{-1}(E \cap T). \quad (15)$$

Therefore it follows that

$$|E \cap T| \geq |E \cap H| - 1.$$

Replacing the set $E$ by the set $-E$, it also follows that

$$|E \cap H| \geq |E \cap T| - 1.$$

Hence the claim follows.

6.1 $\max(|E \cap H|, |E \cap T|) \geq 2$

Proposition 6.5. Let $E$ be a set as above and $H, T, M$ as defined above.

(i) Suppose the inequality $\min(|E \cap H|, |E \cap T|) \geq 2$ is satisfied. Then the set $p^{-1}(E \cap H)$ and the set $E \cap T$ are arithmetic progression with same common difference $d(H, E) = d(T, E) = d(E)$ (say).

(ii) Suppose the inequality $\min(|E \cap H|, |E \cap T|) \geq 2$ is satisfied. The set $p^{-1}(E^c \cap M)$ is an arithmetic progression with common difference $d(E)$, where $d(E)$ is a positive integer given by (I).

(iii) Suppose the cardinality of the set $p^{-1}(E \cap H)$ (resp. $p^{-1}(E \cap T)$ ) is equal to 2 and the cardinality of the set $p^{-1}(E \cap H)$ (resp. $p^{-1}(E \cap T)$ ) is equal to 1, then the set $p^{-1}(E^c \cap M)$ is an arithmetic progression with common difference $d(H, E)$ (resp. $d(T, E)$ ) which is equal to 1.
(iv) Let the inequality \( \max(|E \cap H|, |E \cap T|) \geq 2 \) is satisfied. Then the set \( p^{-1}(E^c \cap M) \) is an arithmetic progression with common difference equal to 1.

**Proof.** (i) We discuss the two cases.

**Case 1:** \( \max((|E \cap H|, |E \cap T|)) \geq 3 \).

Under the assumption, either the inequality \( |E \cap H| \geq 3 \) holds or the inequality \( |E \cap T| \geq 3 \) holds (both the inequalities may also hold). Since the claim holds for the set \( E \) if and only if it holds for the set \( -E \), it is sufficient to prove the assertion under the assumption that the inequality \( |E \cap H| \geq 3 \) holds.

Now from Proposition 6.4 it follows that the sets \( p^{-1}(E \cap H) \) and \( E \cap T \) are arithmetic progression. Let the set \( p^{-1}(E \cap H) = \{x \in p^{-1}(E \cap H) : x \leq 2d(H, E)\} \).

Then from \( \text{Proposition 6.3} \) it follows that the following set

\[
\{x + \frac{d(H, E) + 6K + 1}{2}, x + \frac{d(H, E) + 6K + 1}{2} + d(H, E)\}
\]

is a subset of the set \( p^{-1}(E \cap T) \). From this the following inequality follows immediately

\[d(H, E) \leq d(T, E).\]

In the case the cardinality of the set \( E \cap T \) is equal to 2 it also follows that the set

\[
\{x + \frac{d(H, E) + 6K + 1}{2}, x + \frac{d(H, E) + 6K + 1}{2} + d(H, E)\}
\]

is equal to the set \( p^{-1}(E \cap T) \). Hence the claim follows in case the cardinality of the set \( E \cap T \) is equal to 2. Suppose the cardinality of the set \( E \cap T \) is also greater than or equal to 3, then replacing the set \( E \) by the set \( -E \), it follows that

\[d(H, -E) \leq d(T, -E).\]

Since the numbers \( d(H, -E) \) and \( d(T, -E) \) are equal to the numbers \( d(T, E) \) and \( d(H, E) \) respectively, the claim follows.

**Case 2:** \( |E \cap H| = |E \cap T| = 2 \).

Let the set \( p^{-1}(E \cap H) = \{x, y\} \) and the set \( p^{-1}(E \cap T) = \{z, w\} \). Then from Lemma 6.3 it follows that the parity of the elements \( x \) and \( y \) are different and the element \( \frac{x + y + 6k + 1}{2} \) belong to the set \( p^{-1}(E \cap T) \). Since we are not assuming that \( z < w \) we can assume without any loss of generality that the element

\[\frac{x + y + 6k + 1}{2} = z.\]

For the similar reason the element \( \frac{z + w - 6k + 1}{2} \) belong to the set \( p^{-1}(E \cap T) \) and we can assume without any loss of generality that the element

\[\frac{z + w - 6k + 1}{2} = x.\]

Therefore it follows that

\[w - z = x - y.\]

This proves the claim.
(ii) First we notice that the claim is true for the set $E$ if and only if it is true for the set $-E$. This is because the sets $(-E)^c$ and $M$ are equal to the sets $-(E)^c$ and $-M$ respectively. Therefore the set $p^{-1}(E^c \cap M)$ is same as the set $6k + 1 - p^{-1}((-E)^c \cap M)$. Therefore replacing the set $E$ by the set $-E$ if necessary we may assume that the following inequality holds.

$$|E \cap H| \geq |E \cap T|$$

Let the smallest member of $p^{-1}(E \cap H)$ be $x$ so that $p^{-1}(E \cap H) = \{x + jd(E) : 0 \leq j \leq h - 1\}$. Using (15) it follows that the set

$$\{a + jd(E) : 0 \leq j \leq h - 2\} \quad \text{where} \quad a = x + 3k + \frac{d(E) + 1}{2}, \quad (16)$$

is a subset of $p^{-1}(E \cap T)$ and its cardinality is $h - 1$. Since the cardinality of $p^{-1}(E \cap T)$ is at-most $h$ and since $p^{-1}(E \cap T)$ is in arithmetic progression with common difference $d(E)$, it follows that it is contained in \{a + jd(E) : -1 \leq j \leq h-1\}. On the other hand, $p^{-1}(E^c \cap M)$ is the disjoint union of $\{2x : x \in p^{-1}(E \cap H)\}$ and $\{2x - 6k - 1 : x \in p^{-1}(E \cap T)\}$. Therefore, we have $\{2x + jd(E) : 0 \leq j \leq 2h - 2\} \subset p^{-1}(E^c \cap M) \subset \{2x + jd(E) : -1 \leq j \leq 2h - 1\}$. From this the claim is immediate.

(iii) It is sufficient to prove the assertion in the case the cardinality of the sets $p^{-1}(E \cap H)$ and $p^{-1}(E \cap T)$ are equal to 2 and 1 respectively. In the other case then the assertion follows by replacing the set $E$ by the set $-E$. Suppose the set $p^{-1}(E \cap H)$ is $\{x, y\}$ and $p^{-1}(E \cap T)$ is $\{z\}$. Then it follows that the element $2x - 6k - 1$ is equal to the element $x + y$. It also follows that the set $p^{-1}(E^c \cap M)$ is $\{x, x + y, 2y\}$. Hence the assertion follows.

(iv) Replacing the set $E$ by the set $-E$ we may assume $|E \cap H| \geq 2$. From (I), (II) and (III) we know that the set $p^{-1}(E^c \cap M)$ is an arithmetic progression with common difference equal to $d(H,E)$ which is an odd positive integer. Suppose the assertion is not true. Then it means that $d(H,E) \geq 3$. Then the smallest element of the set $p^{-1}(E \cap H)$ (let say $x$) is less than or equal to $2k - 3$. This implies that $k$ is at least 3. It also follows that the cardinality of the set $\{2k+1, 2k+2, 2k+3\} \cap (E \cap M) \geq 2$. But since $x \leq 2k - 3$, the set $\{x + 2k + 1, x + 2k + 2, x + 2k + 3\}$ is a subset of the set $p^{-1}(M)$ and contains at least 2 elements of the set $p^{-1}(E^c \cap M)$. But this contradicts the fact that the set $p^{-1}(E^c \cap M)$ is in arithmetic progression of common difference $d(H,E)$ which is at least 3. Hence the claim follows.

\[\square\]

**Definition 6.6.** We shall say that a set $B \subset Z$ is an interval if either it is an arithmetic progression with common difference equal to 1 or the cardinality of the set $B$ is equal to 1.

**Proposition 6.7.** Let $E$ be a set as above. Suppose $\max(|E \cap H|, |E \cap T|) \geq 2$ then $E = H \cup T$.

**Proof.** From Lemma 6.1, the conclusion of proposition is equivalent of the assertion that $p^{-1}(E^c \cap M) = p^{-1}(M)$. Let $y$ be the smallest member of $p^{-1}(E^c \cap M)$ and it’s cardinality is $s$. Then using Proposition 6.5 it follows $p^{-1}(E^c \cap M)$ is equal to $\{y + j : 0 \leq j \leq s - 1\}$. Suppose the claim is not true. Therefore at least one of the element of the set $\{2k+1, 4k\}$
belong to the set $p^{-1}(E)$. The assumption of the proposition is satisfied for the set $-E$ as well and the conclusion is true for the set $E$ if and only if it is true for the set $-E$. Therefore replacing the set $E$ by the set $-E$ if necessary we may assume that the element $2k + 1$ belong to the set $p^{-1}(E)$. Using this we prove the following.

Claim: $p^{-1}(E \cap H) = p^{-1}(H)$.

Let $x$ be the smallest member of $p^{-1}(E^c \cap M)$ and it’s cardinality is $h$. Then using Proposition 6.5 it follows $p^{-1}(E \cap H)$ is equal to $\{x + j : 0 \leq j \leq h - 1\}$. Therefore it is sufficient to show that $x + h - 1$ is $2k$ and $x$ is $k + 1$.

Claim The largest element of $p^{-1}(E \cap H)$ is $2k$.

Suppose $x + h - 1$ is not equal to $2k$. Since the set $E$ is sum-free, it follows that $\{2x + j : 0 \leq j \leq 2h - 2\}$ is a subset of $p^{-1}(E^c \cap M)$. As $x + h - 1$ is not $2k$ therefore $2x + 2h$ belong to $p^{-1}(E \cap M)$. Therefore $y + s - 1 \leq 2x + 2h - 1$. Since $2k + 1$ belong to the set $p^{-1}(E)$ it also follows that $x + h - 1 + 2k + 1$ belong to the set $p^{-1}(E^c)$. Since $x + h - 1 < 2K$, we have $x + h - 1 + 2k + 1$ belong to the set $p^{-1}(E^c \cap M)$. Therefore it follows that $x + h - 1 + 2k + 1 \leq 2x + 2h - 1$ which implies $x + h - 1 \geq 2k$. Since $x + h - 1$ belong to the set $p^{-1}(H)$ it follows that $x + h - 1 = 2k$, contradictory to the assumption $x + h - 1 \neq 2k$. Therefore $x + h - 1$ is $2k$.

Claim The smallest element of $p^{-1}(E \cap H)$ is $k + 1$.

Now we will show that $x$ is $k + 1$. Since we have shown that $p^{-1}(E \cap H) = \{x + j : 0 \leq j \leq 2k - x\}$, using Lemma 6.3 it follows that $\{x + j : 3k + 1 \leq j \leq 5k - x\}$ is a subset of $p^{-1}(E \cap T)$ and it’s cardinality is $h - 1 = 2k - x$. The set $p^{-1}(E \cap T)$ is an interval and it’s cardinality is at most $h + 1$. Therefore we have that $p^{-1}(E \cap T)$ is a subset of $\{x + j : 3k - 1 \leq j \leq 5k - x\}$. But in case $\{x + 3k - 1, x + 3k + 1\}$ is a subset of $p^{-1}(E \cap T)$, from Lemma 6.4 it follows that $x - 1$ belong to $p^{-1}(E \cap T)$. This contradicts that $x$ is the smallest element of $p^{-1}(E \cap H)$. Therefore $p^{-1}(E \cap T)$ is a subset of $\{x + j : 3k \leq j \leq 5k - x\}$. Therefore the least element of $p^{-1}(E \cap T)$ is either $x + 3k$ or $x + 3k + 1$.

Case The least element of $p^{-1}(E \cap T)$ is $x + 3k$

In this case the element $x + 3k - 2k = x + k$ does not belong to the set $p^{-1}(E)$. Since $x$ belong to $p^{-1}(H)$, the element $x + k$ belong to $p^{-1}(M)$. Therefore in case $x + k$ is even, the element $\frac{x + k}{2}$ belong to the set $p^{-1}(E \cap H)$. But if $x + k$ is even then $\frac{x + k}{2} < x$. This contradicts the assumption that the element $x$ is the least element of $p^{-1}(E \cap H)$. In case $x + k$ is odd, then the element $\frac{x + k + 6k + 1}{2}$ belong to the set $p^{-1}(E \cap T)$. Now in case $x$ is not $k + 1$ then $x \geq k + 2$ and the inequality $\frac{x + k + 6k + 1}{2} < x + 3k$ is satisfied. This contradicts that $x + 3k$ is the least element of $p^{-1}(E \cap T)$. Therefore $x$ has to be $k + 1$ in this case.

Case The least element of $p^{-1}(E \cap T)$ is $x + 3k + 1$.

In this case the element $x + k + 1$ belong to the set $p^{-1}(E^c \cap M)$. Then again if $x$ is not $k + 1$, this leads to a contradiction.

Therefore $p^{-1}(E \cap H) = H$. The assumption of proposition can hold only in case $k \geq 2$. In that case $k + 1$ and the element $k + 2$ belong to $p^{-1}(H) = p^{-1}(E \cap H)$. Therefore
the element $2k + 3$ does not belong to $p^{-1}(E)$ and invoking Lemma 6.1 the element $p^{-1}p \left(\frac{2k+3}{2}\right) = 4k + 2$ belong to $p^{-1}(E \cap T)$. But since by assumption $2k + 1$ belong to the set $p^{-1}(E)$, this contradicts that the set $E$ is sum-free. Therefore finally it follows that the set $p^{-1}(E^c \cap M) = M$ and hence the proposition follows. \hfill \Box

6.2 \quad \text{max}(|E \cap H|, |E \cap T|) \leq 1

Replacing the set $E$ by the set $-E$ if necessary we may assume that the following inequality holds.

\[ |E \cap H| \geq |E \cap T| \]

Then we have the following three possible cases.

(i) The equality $|E \cap H| = |E \cap T| = 1$ holds.

(ii) The cardinality of the sets $E \cap H$ and $E \cap T$ are 1 and 0 respectively.

(iii) The equality $|E \cap H| = |E \cap T| = 0$.

Proposition 6.8. Let $E, H, T, M$ be as above. If $|E \cap H| = |E \cap T| = 1$, then the set $p^{-1}(E) = \{2k\} \cup \{2k + 2, 2k + 3, \ldots, 4k - 2, 4k - 1\} \cup \{4k + 1\}.$

Proof. Let the set $(E \cap H) = \{x\}$ and the set $(E \cap T) = \{y\}$. From Proposition 6.1 the set $E^c \cap M = \{2x, 2y\}$ and the set $p^{-1}(E^c \cap M) = \{p^{-1}(2x), p^{-1}(2y)\}.$ We claim that

\text{claim : } x = -y.

\text{proof of claim:} From Proposition 6.1 it follows that the element $\frac{-y + x}{2}$ belongs to the set $E \cap (H \cup T)$. Now if $p^{-1}(-y)$ and $p^{-1}(x)$ have same parity then the element $p^{-1}(\frac{-y + x}{2})$ belongs to the set $p^{-1}(E \cap H)$. But as the element $p^{-1}(x)$ is the only element belonging to the set $E \cap H$, in this case the claim follows. Otherwise the element $\frac{-y + x}{2}$ belongs to the set $E \cap T$ and hence is equal to $y$. Also then $p^{-1}(-x)$ and $p^{-1}(y)$ have different parity and the element $\frac{x + y}{2}$ is equal to $x$. But this implies after simple calculation that $x = 9x$. As $m$ is odd this is not possible. Hence the claim follows.

Next claim is

\text{claim: } p^{-1}(x) = 2k

\text{proof of claim: Suppose not, then } p^{-1}(y) \text{ is also not equal to } 4k + 1. \text{ Therefore the element } 2k + 1 \text{ belong to the set } p^{-1}(E) \text{. This implies that the element } p^{-1}(x) + 2k + 1 \text{ belongs to the set } p^{-1}(E^c \cap M). \text{ The element } p^{-1}(x) + 2k + 1 \text{ also satisfy the inequality } p^{-1}(x) + 2k + 1 > 2p^{-1}x. \text{ Therefore the element } p^{-1}(x) + 2k + 1 = p^{-1}(2y). \text{ Since } y = -x, \text{ it follows that } p^{-1}(2y) = 6k + 1 - 2p^{-1}(x). \text{ Therefore we have } 3p^{-1}(x) = 4k. \text{ This is possible only if } k \text{ is divisible by } 3. \text{ It is easy to check that case } k = 3 \text{ is not possible. So we may assume that } k \text{ is greater than } 3 \text{ and is divisible by } 3. \text{ As } k \text{ is strictly greater than } 3 \text{ therefore } p^{-1}(x) = \frac{4k}{3} \neq k + 1 \text{ and hence } 2k + 2 \text{ belong to the set } E. \text{ Also we have the inequality } \frac{4k}{3} \leq 2k - 2. \text{ Therefore the element } p^{-1}(x) + 2k + 2 \text{ belong to the set } E^c \cap M. \text{ Therefore the elements } p^{-1}(x) + 2k + 1 \text{ as well as } p^{-1}(x) + 2k + 2 \text{ belong to the set } p^{-1}(E^c \cap M) \text{ and neither of these elements are same as } 2p^{-1}(x). \text{ This implies that the cardinality of the set } p^{-1}(E^c \cap M) \text{ is greater than or equal to } 3. \text{ This is not possible. Hence the claim follows.}

Now the proposition follows immediately. \hfill \Box

Proposition 6.9. Let $E$ be as set as above, then following holds. If $|E \cap H| = 1$ and $|E \cap T| = 0$ then we have the set $p^{-1}(E) = \{2k, 2k + 1, \ldots, 4k - 2, 4k - 1\}.$
**Proof.** Suppose the set \( \{x\} \) is \( p^{-1}(E \cap H) \) The claim is immediate from the assertion \( x = \{2k\} \). Suppose the assertion is not true. Since we have assumed \( |E \cap T| = 0 \), using Lemma 6.4 it follows that \( 2k + 1 \) belong to the set \( p^{-1}(E) \). Therefore if \( x \neq \{2k\} \), then \( x + 2k + 1 \) belong to the set \( p^{-1}(M) \) and actually belong to \( p^{-1}(E^c \cap M) \). But trivially \( p^{-1}(E^c \cap M) = \{2x\} \) and the element \( x + 2k + 1 \) is not equal to \( 2x \). Hence there is a contradiction and the claim follows. \( \square \)

The following proposition is trivial.

**Proposition 6.10.** Let \( E \) be a set as above. In the case \( |E \cap H| = |E \cap M| = 0 \), then \( E = M \)

Therefore from Proposition 6.7 6.8 6.9 the proof of Theorem 1.13 in case \( G \) is cyclic follows. That is the following result follows.

**Theorem 6.11.** Let \( G \) be a cyclic abelian group of type III. That is \( G = \mathbb{Z}/m\mathbb{Z} \). Let \( k = \frac{m-1}{n} \). Let \( E \) be a sum-free subset of \( G \) of density \( \mu(G) \). Then there exist an automorphism \( f : G \to G \) such that one of the following holds.

(i) The set \( E = f^{-1}(M) \).

(ii) The set \( E \) is equal to \( f^{-1}(\{2k\} \cup \{2k + j : 2 \leq j \leq 2k - 1\} \cup \{4k + 1\}) \).

(iii) The set \( E \) or the set \( -E \) is equal to \( f^{-1}(\{2k\} \cup \{2k + j : 1 \leq j \leq 2k - 1\}) \).

**Proof.** From Theorem 1.13 and Proposition 1.3 it follows that there exist automorphism \( g : G \to G \) such that the set \( E \) is a subset of the set \( g^{-1}(I) \). Then in case \( \max(|g(E) \cap H|, |g(E) \cap T|) \geq 2 \) Proposition 6.7 we have \( g(E) = H \cup T \). Therefore taking \( f = 2g \) it follows \( E = f^{-1}(M) \). In the other cases taking \( f = g \) the claim follows from Proposition 6.8 6.9 6.10 \( \square \)

7 Sum-free set of general abelian group of type III

**Proof. of Theorem 6.12** Let \( \gamma \) be a special direction of the set \( A \). Then from Proposition 4.8 it follows that order of \( \gamma \) is equal to \( m \). We have also from Theorem 1.14 it follows that \( A \) is a subset of \( \gamma^{-1}(I) \). Let \( L \subset \mathbb{Z}/m\mathbb{Z} \) as defined in section 6. Then from Proposition 5.6 we have that \( L \) is a sum-free subset of \( \mathbb{Z}/m\mathbb{Z} \) and it’s cardinality is \( 2k \). Then from propositions 6.7 6.8 6.9 6.10 it follows \( p^{-1}(L) \) is one of the following set.

**Case 1:** \( p^{-1}(L) = p^{-1}(H \cup T) \).

In this case we easily check that given any element \( i \in \mathbb{Z}/m\mathbb{Z} \) such that \( i \) does not belong to the set \( L \), there exist \( x, y \in L \) such that either \( i = x + y \) or \( i = x - y \). Therefor invoking Proposition 5.6 it follows that for any \( i \in \mathbb{Z}/m\mathbb{Z} \) which does not belong to \( L \), the number \( a_i = 0 \). Therefore \( A \) is a subset of \( \gamma^{-1}(I) \). Hence \( f = 2\gamma \) is a surjective homomorphism from \( G \) to \( \mathbb{Z}/m\mathbb{Z} \) and moreover

\[
A \subset f^{-1}\{M\}.
\]

**Case 2:** \( p^{-1}(L) = \{2k\} \cup \{2k + j : 2 \leq j \leq 2k - 1\} \cup \{4k + 1\} \).

In this case again given any element \( i \in \mathbb{Z}/m\mathbb{Z} \) such that \( i \) does not belong to the set \( L \), there exist \( x, y \in L \) such that either \( i = x + y \) or \( i = x - y \). Therefore taking \( f = \gamma \) we have

\[
A \subset f^{-1}p(2k) \cup f^{-1}p(\{2k + j : 2 \leq j \leq 2k - 1\}) \cup f^{-1}p(4k + 1).
\]
Case 3: Either \( p^{-1}(L) \) or \( p^{-1}(-L) \) is \( \{2k\} \cup \{2k + j : 1 \leq j \leq 2k - 1\} \).

In both these cases, given any element \( i \in \mathbb{Z}/m\mathbb{Z} \) such that \( i \) does not belong to the set \( L \), there exist \( x, y \in L \) such that either \( i = x + y \) or \( i = x - y \). Therefore taking \( f = \gamma \) the claims follows.

\[
A \subset f^{-1}(p(2k) \cup f^{-1}(\{2k + j : 2 \leq j \leq 2k - 1\}) \cup f^{-1}(4k + 1).
\]

Case 4: \( p^{-1}(L) = p^{-1}(M) \).

In this case given any \( i \in \mathbb{Z}/m\mathbb{Z} \) which does not belong to \( \{2k + j : 0 \leq j \leq 2k + 1\} \) there exist \( x, y \in L \) such that either \( i = x + y \) or \( i = x - y \). Therefore taking \( f = \gamma \) we have

\[
A \subset f^{-1}(\{2k + j : 0 \leq j \leq 2k + 1\}).
\]

Therefore in all the cases there exist a homomorphism \( f : G \rightarrow \mathbb{Z}/m\mathbb{Z} \) such that

\[
A \subset f^{-1}(\{2k + j : 0 \leq j \leq 2k + 1\}).
\]

Now the claims (I), (II), (III) follows invoking Lemma 5.3. The claim (IV) follows observing that the set \( A \cup f^{-1}(\{2k + j : 0 \leq j \leq 2k + 1\}) \) is also a sum-free set. \( \blacksquare \)

Now we prove Theorem 1.18.

**Proof of Theorem 1.18.** Let \( f \) be a surjective homomorphism from \( G \) to \( \mathbb{Z}/m\mathbb{Z} \) given by Theorem 1.12. Since \( m \) is the exponent of group \( G \) and \( f \) is a surjective homomorphism to \( \mathbb{Z}/m\mathbb{Z} \) there exist a subgroup \( C \) of \( G \) such that \( G = S \oplus C \) where \( S \) is a kernel of \( f \). Therefore \( f \) restricted to \( C \) is an isomorphism from \( C \) to \( \mathbb{Z}/m\mathbb{Z} \). We denote this restriction by \( g \). Since \( \alpha(A) = \mu(G) \) it follows from Theorem 1.12 we have

\[
f^{-1}(\{2k + j : 0 \leq j \leq 2k + 1\}) \subset A.
\]

Also the following equalities hold.

\[
|A(f, 2k)| + |A(f, 4k)| = \frac{n}{m} \quad (17)
\]

\[
|A(f, 4k + 1)| + |A(f, 2k + 1)| = \frac{n}{m} \quad (18)
\]

Now \( A(f, 2k) + A(f, 2k) \) is a subset of \( f^{-1}\{4k\} \) and it is disjoint from \( A(f, 4k) \). Therefore the following inequality follows.

\[
|A(f, 4k)| \leq \frac{n}{m} - |A(f, 2k) + A(f, 2k)| = |A(f, 4k)| + |A(f, 2k)| - |A(f, 2k) + A(f, 2k)|
\]

Hence we have

\[
|A(f, 2k) + A(f, 2k)| = |A(f, 2k)|. \quad (19)
\]

For any \( i \in \mathbb{Z}/m\mathbb{Z} \) there exist \( X_i \subset S \) such that \( A(f, i) = X_i \oplus g^{-1}\{i\} \). Then from 19 it follows that \( |X_{2k} + X_{2k}| = |X_{2k}| \). Therefore either \( X_{2k} = \phi \) or there exist \( J_1 \) a subgroup of \( S \) and an element \( b_1 \in S \) such that \( X_{2k} = J_1 + b_1 \). Similar arguments implies that either \( X_{4k+1} = \phi \) or there exist \( J_2 \) a subgroup of \( S \) and an element \( b_2 \in S \) such that \( X_{4k+1} = J_2 + b_1 \). Then there are three possibilities.

**Case 1:** Both the sets \( X_{2k} \) as well as the set \( X_{4k+1} \) are empty sets.

In this case from 17 and 18 it follows \( X_{2k+1} = S \) and \( X_{4k} = S \). Hence \( A \) is \( f^{-1}(M) \).
Then arguing as in case 2 it follows that $X$:

Case 2: Exactly one of the sets $X_{2k}$ and $X_{4k+1}$ is an empty set.
Replacing the set $A$ by $-A$ if necessary we may assume that $X_{4k+1}$ is an empty set. Since the set $A$ is sum-free it follows that $X_{4k}$ is a subset of $(J_1 + 2b_1)^c$. From (17) it follows trivially that $X_{4k}$ is $(J_1 + 2b_1)^c$.

Case 3: Both the sets $X_{2k}$ and $X_{4k+1}$ are not empty sets.
Then arguing as in case 2 it follows that $X_{4k}$ is $(J_1 + 2b_1)^c$ and $X_{4k+1}$ is $(J_2 + 2b_2)^c$. The assumption that $A$ is sum-free implies that $X_{4k+1}$ is a subset of $(X_{2k} + X_{2k+1})^c$. This means

$$J_2 + b_2 \subset (J_1 + b_1 + (J_2 + 2b_2)^c)^c.$$ 

This implies

$$(J_2 + b_2)^c = J_1 + b_1 + (J_2 + 2b_2)^c.$$ 

Therefore we have

$$J_1 + b_1 \subset J_2 - b_2. \quad (20)$$ 

Since $X_{2k}$ is a subset of $(X_{2k} + X_{2k+1})^c$, same arguments implies that

$$J_2 + b_2 \subset J_1 - b_1. \quad (21)$$ 

From (20) and (21) we have $J_1 + b_1 = J_2 - b_2$. Hence $J_1 = J_2$. This proves the theorem.

\[\Box\]

8 Remarks

In case $G$ is of type $I(p)$ group and $A$ is a maximal sum-free subset of $G$ such that $\alpha(A) > \frac{1}{3} + \frac{1}{2(3p^2)}$ then $\alpha(A) = \mu(G)$. For the proof of this one may see [GR05]. But in case $G$ is of type III then there exist $A$ such that $A$ is a maximal sum-free set of cardinality $\mu(G)n - 1$. For this consider the following example.

Example: $G = (\mathbb{Z}/7\mathbb{Z})^2$ and $A = \pi_2^{-1}\{3\} \cup (0, 2) \cup (1, 2) \cup (\pi_2^{-1}\{4\} \setminus \{(0, 4), (1, 4), (2, 4)\})$, where $\pi_2 : G \rightarrow \mathbb{Z}/7\mathbb{Z}$ is a natural projection to second co-ordinate.

Therefore Theorem 1.12 does not give complete characterisation of all large maximal sum-free subsets of $G$.

In general Hamidoune and Plagne [HPP04] have studied $(k, l)$ free subsets of finite abelian groups. For any positive integer $t$ we define $tA$ by

$$tA = \{x \in G : x = \sum_{i=1}^{t} a_i, \quad a_i \in A \quad \forall i\}.$$ 

Given any two positive integers we say $A$ is $(k, l)$ free if $kA \cap lA = \phi$. In case $k - l \equiv 0(\text{mod } m)$, where $m$ is the exponent of $G$ then it is easy to check that there is no set apart from empty set which is $(k, l)$ free. Therefore one assume that $gcd(|G|, k - l) = 1$. We denote the density of the largest $(k, l)$ free set by $\lambda_{k,l}$. Hamidoune and Plagne [HPP04] conjectured the following.

Conjecture. Let $G$ be a finite abelian group and $m$ is the exponent of $G$. Let $k, l$ be positive integers such that $gcd(|G|, k - l) = 1$. Then the density of largest $(k, l)$ free set...
is given by the following formula.

\[ \lambda_{k,l} = \max_{d \mid m} \frac{\left\lfloor \frac{d-2}{k+l} \right\rfloor + 1}{d} \]

They \[oHP04\] proved the above conjecture in case when there exist a divisor \(d_0\) of \(m\) such that \(d_0\) is not congruent to 1 modulo \(k + l\). In this situation they also showed that given any \((k, l)\) free set of density \(\lambda_{k,l}\), there exist a positive integer \(d\), a surjective homomorphism \(f : G \to \mathbb{Z}/d\mathbb{Z}\), a set \(B \subseteq \mathbb{Z}/d\mathbb{Z}\) such that \(B\) is \((k, l)\) free and \(A = f^{-1}(B)\). That is any \((k, l)\) free set of density \(\lambda_{k,l}\) is an inverse image of a \((k, l)\) free subset of a cyclic group. One may ask the following question:

**Question 8.1.** Let \(G\) be a finite abelian group and \(m\) is the exponent of \(G\). Suppose \(k, l\) are positive integers such that all the divisors of \(m\) are congruent to 1 modulo \(k + l\). Is it true that any \((k, l)\) free set of density \(\lambda_{k,l}\) is an inverse image of a \((k, l)\) free subset of a cyclic group?

We have already seen that the answer of the above question is negative in case \(k = 2\) and \(l = 1\). The following arguments show that the answer of the above question is negative for an arbitrary value of \(k\) provided \(l = 1\).

In case all the divisors of \(m\) are congruent to 1 modulo \(k + l\), it is easy to check that

\[ \max_{d \mid m} \frac{\left\lfloor \frac{d-2}{k+l} \right\rfloor + 1}{d} = \frac{\left\lfloor \frac{m-2}{k+l} \right\rfloor + 1}{m}. \]

Now consider the following example.

**Example:** Let \(G\) be a finite abelian group and \(m\) is the exponent of \(G\). We further assume that \(G\) is not cyclic. Let \(k\) be positive integers such that \(\gcd(|G|, k - 1) = 1\) and all the divisors of \(m\) are congruent to 1 modulo \(k + 1\). Then \(G = S \oplus \mathbb{Z}/m\mathbb{Z}\) with \(S \neq \{0\}\). Let \(q = \left\lceil \frac{m-2}{k+1} \right\rceil\). Let \(x \in \mathbb{Z}/m\mathbb{Z}\) such that \((k - 1)x \equiv 1 + q \pmod{m}\). Let \(J\) be any proper subgroup of \(S\) and

\[ A = (J \oplus \{x\}) \cup (S \oplus \{x + 1, x + 2, \cdots, x + q\}) \cup (J^c \oplus \{x + q + 1\}). \]

If \(A\) is in the example above then it is easy to check that \(A\) is \((k, 1)\) free and density of \(A\) is \(\frac{k}{m+1}\). Hamidoune and Plagne \[oHP04\] also proved the above conjecture for all cyclic groups. Using this and following the arguments as in section 3 it follows that stabiliser of \(A\) is \(J\). This shows that if \(A\) is the set as in above example then \(A\) is not an inverse image of any \((k, 1)\) free subset of a cyclic group.

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