The Relativistic Origins of Pseudospin Symmetry in Nuclei

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Abstract. Jerry Draayer has worked in approximate pseudospin and pseudo-U(3) symmetry in nuclei in the non-relativistic shell model. We show that pseudospin symmetry is a SU(2) relativistic symmetry of the Dirac Hamiltonian for which the sum of the vector and scalar potentials is a constant. We show that the Dirac Hamiltonian for which the sum of the vector harmonic oscillator potential and a scalar harmonic oscillator is a constant has a pseudo-U(3) symmetry and we derive the generators for this symmetry.

1. Introduction
In 1968 Akito Arima and collaborators [1] and Ted Hecht and collaborators [2] discovered quasi-degenerate states in single particle energy levels which were not spin doublets. In fact the doublets differed in orbital angular momentum by units of two and radial nodes by one. These states have single nucleon quantum numbers \([n, \ell, \ell + \frac{1}{2}]\) and \([n - 1, \ell + 2, \ell + \frac{3}{2}]\) where \(n, \ell\) are the radial and orbital quantum numbers, respectively, and the last quantum number in the brackets is the total angular momentum, \(j\). For example, in the usual notation, \((1s_\frac{1}{2}, 0d_\frac{3}{2}), (1p_\frac{3}{2}, 0f_\frac{5}{2})\), etc, are pseudospin doublets [3]. These quasi-degeneracies persist in recent measurements in nuclei far from stability [4]. They realized that, if they define the average of the orbital angular momenta as a pseudo-orbital angular momentum \((\tilde{\ell})\) and then couple a pseudospin \((\tilde{s} = \frac{1}{2})\) to the pseudo-orbital angular momentum they will get the total angular momenta \((j = \tilde{\ell} \pm \frac{1}{2})\). For example, for the \((1s_\frac{1}{2}, 0d_\frac{3}{2})\) orbits, \(\tilde{\ell} = 1\), which gives the total angular momenta \(j = \frac{1}{2}, \frac{3}{2}\). Subsequently pseudospin doublets in deformed nuclei were discovered [5]. Jerry and collaborators examined pseudospin symmetry in triaxial nuclei by calculating the triaxial orbits in the triaxial Nilsson model [6].

2. Pseudospin symmetry in the non relativistic shell model
Jerry and collaborators showed that the non relativistic spin-orbit potential derived from relativistic mean field theory approximately conserves pseudospin symmetry[7].The transformation from spin variables to pseudospin variables was then an open question. Jerry and his collaborators used translational, parity, and time reversal symmetry to suggest the helicity transformation of the form \(U_p = \tilde{\sigma} \cdot \tilde{p}\), where \(\tilde{p}\) is the unit momentum operator and the spin generators are \(\tilde{s} = \tilde{\sigma} \cdot 2\). With this transformation they were able to derive the conditions on the non-relativistic single-particle Hamiltonian for which pseudospin symmetry is a dynamical symmetry. If the strength of a constant single-particle spin orbit potential is \(C\) and the strength...
of constant single particle orbital angular momentum squared is $D$, then $C = 4D$ produced pseudospin symmetry [8].

The pseudospin generators are then $\tilde{\sigma} = U_p \tilde{\vec{\sigma}} U_p$. Since the helicity transformation is unitary this transformation preserves the spin commutation rules. The pseudospin doublets differ by two units of angular momentum and have different radial nodes. These generators intertwine momentum and spin. The momentum has the capability of changing the orbital angular momentum and the radial quantum number, and, when the details are worked out, these generators connect the pseudospin doublets in the correct way in the pseudospin limit.

3. Symmetries of the Dirac Hamiltonian

The Dirac Hamiltonian with a Lorentz scalar potential, $V_S(\vec{r})$, and a potential which is the fourth component of a Lorentz vector potential, $V_V(\vec{r})$, is

$$H = \vec{\alpha} \cdot \vec{p} + \beta (V_S(\vec{r}) + M) + V_V(\vec{r})$$

(1)

where $\vec{\alpha}$, $\beta$ are the Dirac matrices, $\vec{p}$ is the momentum, $M$ is the mass, $\vec{r}$ is the radial coordinate, and the velocity of light is set equal to unity, $c = 1$. The Dirac Hamiltonian has spin symmetry when the difference of the vector and scalar potentials in the Dirac Hamiltonian is a constant, $V_S(\vec{r}) - V_V(\vec{r}) = C_s$ [9]. Hadrons [10] and anti-nucleons in a nuclear environment have spin symmetry [11]. These are relativistic systems and normally, in such systems, we would expect large spin-orbit splittings, but, in this limit, spin doublets are degenerate.

3.1. Pseudospin Symmetry: A Symmetry of the Dirac Hamiltonian

In Ref [9], another SU(2) symmetry of the Dirac Hamiltonian was discovered to occur when the sum of the vector and scalar potentials in the Dirac Hamiltonian is a constant, $V_S(\vec{r}) + V_V(\vec{r}) = C_p$. This symmetry was shown to be pseudospin symmetry [12]. The potentials can be spherical [1, 2], deformed [5], or triaxial, [6]. The generators for this SU(2) algebra $\tilde{\vec{\sigma}}$, which commute with the Dirac Hamiltonian with any potentials that satisfy these conditions, $[H, \tilde{\vec{\sigma}}] = 0$, are given by [13]

$$\tilde{\vec{\sigma}} = \begin{pmatrix} U_p \vec{s} U_p & 0 \\ 0 & \vec{s} \end{pmatrix}$$

(2)

The eigenfunctions of the Dirac Hamiltonian in this limit will have degenerate doublets of states, one of which has pseudospin aligned and the other with pseudospin unaligned. The non-relativistic part of these relativistic pseudospin generators (upper quadrant of the matrix) are the generators proposed in [8].

These conditions hold for spherical, deformed, and triaxial potentials. If, in addition, the potentials are spherical, $V_S(\vec{r}) = V_S(r)$ and $V_V(\vec{r}) = V_V(r)$, then the pseudo-orbital angular momentum is conserved as well [13]. Its generators are given by

$$\tilde{\vec{L}} = \begin{pmatrix} U_p \vec{\ell} U_p & 0 \\ 0 & \vec{\ell} \end{pmatrix}$$

(3)

where $\vec{\ell} = \frac{(\vec{r} \times \vec{p})}{\hbar}$.

The “upper” components of the degenerate doublets have the same quantum numbers of the spherical pseudospin doublets and the deformed pseudospin doublets. The “upper” matrix of the pseudospin generators in Eqs (2, 3), $U_p \vec{s} U_p$ and $U_p \vec{\ell} U_p$, have the spin intertwined with the momentum which enables the generators to connect the states in the doublet, which differ by two units of angular momentum and one unit of radial quantum number.

Hence pseudospin symmetry is a relativistic symmetry. The approximate equality in magnitude of the vector and scalar fields in nuclei and their opposite sign have been confirmed in relativistic mean field theories [3] and in QCD sum rules [14, 3].
4. Consequences of Relativistic Pseudospin Symmetry

One immediate consequence of pseudospin symmetry as a relativistic symmetry is that the “lower” matrix of the pseudospin generators in Eqs (2, 3), $\vec{s}$ and $\vec{\ell}$ do not change the radial wavefunction of the “lower” component of the Dirac eigenfunctions. Hence this symmetry predicts that the radial wavefunctions of the “lower” component is the same for the two states in the doublet. Previous to this discovery many relativistic mean field calculations of nuclear properties have been made. Hence this prediction was tested with existing calculations and, indeed, these wavefunctions are very similar for both spherical [15, 16] and deformed nuclei [17, 18]. Because of the momentum dependence of the “upper” matrix of the generators the relationship between the “upper” components involves a differential equation but these have also been tested in spherical [19] and deformed nuclei [18] with success. There are also predictions about magnetic dipole transitions and Gamow-Teller transitions [20, 21], quadrupole transitions [3], and scattering matrices [22, 23]. These and other theoretical and experimental tests of pseudospin symmetry are not discussed in this paper because of limitation of space [3].

5. The Dirac Hamiltonian with Pseudo-U(3) Symmetry

It is well known that the Schrödinger equation with a non-relativistic spherical harmonic oscillator potential has a U(3) symmetry [24]. The energies depend linearly on the total number of harmonic oscillator quanta, $N = 2n + \ell$, and the generators of the U(3) symmetry are the angular momentum operator, a quadrupole operator, and a monopole operator which commute with the non-relativistic Hamiltonian. We know that the Dirac Hamiltonian for which the difference of the scalar and vector potentials is a constant has spin symmetry. But if, in addition, the potentials are spherical harmonic oscillators, the Dirac Hamiltonian can be solved analytically [25] and has an U(3) symmetry [26, 27]. The energies depend on the same harmonic quantum number $N$ as the non-relativistic harmonic oscillator, and, therefore has the same degeneracies. However, the energy dependence on $N$ depends on the ratio of the $M$ to the strength of the potential. For $M$ large, the dependence is linear just as in the non-relativistic oscillator. For $M$ small, the energy goes like $N^{\frac{3}{2}}$, which means that, in the relativistic limit, the oscillator is not harmonic. Hence even a relativistic system can have U(3) symmetry.

If, on the other hand, the sum of the vector and scalar potentials is a constant and are spherical harmonic oscillators, the Dirac Hamiltonian, $\tilde{H}$, has a pseudospin symmetry and a pseudo-U(3) symmetry. The energy depends on the total number of pseudo-quanta, $\tilde{N} = 2\tilde{n} + \tilde{\ell}$. The quantum numbers, $\tilde{N}, \tilde{n}, \tilde{\ell}$, refer to the “lower” component of the Dirac eigenfunction. In terms of the non-relativistic quantum numbers, $\tilde{n} = n$ of the spin unaligned partner. The eigenvalue equation in the pseudospin limit is [3, 27]

$$\sqrt{E_{\tilde{N}} - M} \ (E_{\tilde{N}} + M) = \sqrt{2 \hbar^2 \omega^2 M} \ (\tilde{N} + \frac{3}{2}).$$

(4)

This eigenvalue equation is solved on Mathematica,

$$E_{\tilde{N}} = \frac{M}{3} \left[ 3B(A_{\tilde{N}}) - 1 + \frac{4}{3} B(A_{\tilde{N}}) \right],$$

(5)

where $B(A_{\tilde{N}}) = \left[ \frac{A_{\tilde{N}} + \sqrt{A_{\tilde{N}}^2 + \frac{4}{27}}}{} \right]^{\frac{3}{2}}$, and $A_{\tilde{N}} = \frac{\sqrt{2} \hbar \omega}{M} (\tilde{N} + \frac{3}{2})$. The spectrum goes like $\tilde{N}^{\frac{3}{2}}$ for $M$ small and $\tilde{N}^{\frac{3}{2}}$ for $M$ large; i.e., the relativistic harmonic oscillator is not harmonic.
The pseudo-U(3) generators which commute with the Dirac Hamiltonian, are the pseudo-orbital angular momentum given in Eq (3), and the

$$\hat{Q}_m = \sqrt{\frac{3}{\sqrt{M^2} + (H - M)}} \left( \frac{M\omega^2 [pp]^{(2)}_m}{\hbar^2} \hat{\sigma} \cdot \hat{p} \right) + \frac{M\omega^2 [rr]^{(2)}_m}{\hbar^2} \hat{M}^2 (M\omega^2 r^2 - 2M) [rr]^{(2)}_m + [pp]^{(2)}_m, $$

along with the pseudo-quantum number operator

$$\hat{N} = \sqrt{\frac{H - M}{\hbar^2} \frac{\hat{H} + M}{\hbar^2}} - \frac{3}{2}. $$

These generators satisfy the commutation relations of a U(3) algebra,

$$[\hat{L}, \hat{L}]^{(t)}_1 = -\sqrt{2} \hat{L} \hat{\delta}_{t,1}, [\hat{L}, \hat{Q}]^{(t)} = -\sqrt{6} \hat{Q} \hat{\delta}_{t,2}, [\hat{Q}, \hat{Q}]^{(t)} = 3\sqrt{10} \sqrt{H} \hat{L} \hat{\delta}_{t,1}, $$

$$[\hat{N}, \hat{L}] = [\hat{N}, \hat{Q}_m] = 0. $$

In the above we use the coupled commutation relation between two tenors, $T^{(t_1)}_1, T^{(t_2)}_2$ of rank $t_1, t_2$ which is $[T^{(t_1)}_1, T^{(t_2)}_2]^{(t)} = [T^{(t_1)}_1 T^{(t_2)}_2]^{(t)} - (-1)^{t_1 + t_2 - t}[T^{(t_2)}_2 T^{(t_1)}_1]^{(t)}$ [28]. Theses generators, along with the pseudospin, commute with the Hamiltonian

$$[\hat{H}, \hat{S}] = [\hat{H}, \hat{L}] = [\hat{H}, \hat{Q}_m] = [\hat{H}, \hat{N}] = 0. $$

The pseudospin generators commute with the pseudo-U(3) generators as well as the Dirac Hamiltonian, and so the invariance group is pseudo-U(3)×pseudo-SU(2), where the pseudo-SU(2) is generated by the pseudospin generators. Hence pseudo-U(3) is a relativistic symmetry. However, the eigenfunctions of $\hat{H}$ are not valence Dirac eigenfunctions, but hole Dirac eigenfunctions. Or in common parlance, "negative" energy states. The energies are not negative, though, since the harmonic oscillator is not finite. But the nodal structure of the eigenfunctions is that of hole states [3]. There are no bound valence states of $\hat{H}$ and hence these solutions do not correspond to realistic nuclei. Of course, for real nuclei, the scalar and vector potentials are not exactly equal in magnitude, and so pseudospin is approximately conserved since the ratio of the sum of the potentials to their difference is approximately .03. For this reason we are attempting to solve analytically the Dirac Hamiltonian for general scalar and vector harmonic oscillator potentials.

Because, in general, the pseudospin symmetry limit has no bound valence Dirac eigenfunctions, it is not possible to do perturbation theory for approximate pseudospin symmetry around the exact pseudospin symmetry limit. Since this feature is unique to pseudospin symmetry, pseudospin symmetry is an interesting topic to understand better.

6. Summary

We have shown the pseudospin symmetry and pseudo-U(3) for which Jerry has made many contributions are relativistic symmetries. This is surprising since nuclei have been described well for most properties by non relativistic quantum mechanics. Perhaps there is a more fundamental rationale for pseudospin symmetry at the QCD level.

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