Comparative studies on the criteria for regularization parameter selection based on moving force identification

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ABSTRACT
The studies on inverse problems exist extensively in aerospace, mechanical, identification, detection, scanning imaging and other fields. Its ill-posed characteristics often lead to large oscillations in the solution of the inverse problem. In this study, the truncated generalized singular value decomposition (TGSVD) method is introduced to identify two kinds of moving forces, single and multi-axial forces. The truncating point is the most influential regularization parameter of TGSVD, which is initially selected by two classic regularization parameter selection criteria, namely, the L-curve criterion and the generalized cross-validation (GCV) criterion. Due to numerical non-uniqueness and noise disturbance in moving force identification (MFI), numerical simulation results show that neither of the two criteria can effectively help select the optimal truncating point of TGSVD. Hence, a relative percentage error (RPE) criterion is proposed for selecting the truncating point of TGSVD. Comparative studies show that the RPE criterion can be used to select the optimal truncating point of TGSVD more accurately against the GCV criterion and L-curve criterion. Moreover, the RPE criterion can be used to reflect the connections between certain properties and the ill-posedness problem existing in MFI, which should be adopted priority for the optimal truncating point selection of TGSVD.

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1. Introduction
The moving force identification (MFI) from dynamic responses of bridge structure has been studied for decades. The dynamic moving forces not only contribute to the living load on bridge structure but also can serve as moving exciter and dynamic responses receiver for the bridge structural health monitoring [1]. The dynamic interaction between moving forces and bridge structure can be described as two sets of second-order differential equations of motion, one for the vehicle and another for the bridge deck [2]. Based on the time-varying modal properties of the bridge-vehicle system, Tian et al. [3] presented a moving mass technique of identifying structural scaling factors from output-only data.

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Identification force from dynamic responses of a structure may suffer from measurement errors of responses and estimation error of the system, which belongs to the class of ill-posed inverse problems and meaning that the stable solution is not guaranteed [4]. Ill-posed problems existing in force reconstruction have attracted the attention of experts in the past two decades [5]. There were two common methods for solving ill-posed problems in force reconstruction. The first one was truncated singular value decomposition (TSVD) method by replacing small singular values with zeros to prevent the amplification of errors and the second one was the Tikhonov regularization method by imposing an additional constraint on the solution to be obtained [6].

With the development of computational theory and the emergence of innovative methods in recent years, many researchers have made great efforts on this issue to enhance the stability and accuracy of load identification. Abbasnia et al. [7] presented an Adjoint variable method to mitigate the ill-posed problem during the process of identification based on the sensitivity of structural response for identifying both the system parameters and input excitation force of a bridge structure. Xu and Ou [8] introduced the virtual work principle technique to transform the differential equation of motion into an integral equation, which can restrict the effect of random noise and accumulative error in the calculation. Jia et al. [9] proposed a weighted regularization method based on the proper orthogonal decomposition to address error propagation and ill-posed problem for random dynamic loads identification. Qiao et al. presented a series of novel methods for the large-scale ill-posed inverse problem of impact force reconstruction such as the regularized cubic B-spline collocation method [10] and the sparse reconstruction by separable approximation method [11]. Li et al. [12,13] presented a time–space domain decoupling approach based on Green’s function method and orthogonal polynomial fitting technique to separate identify the time history and spatial distribution function of dynamic load. Tran and Inoue [14] introduced the wavelet deconvolution technique to mitigate the ill-posed problem in reconstructing impact force. Jiang et al. [15] presented a fractional-order accumulative regularization filter to improve the ill-posedness of the transfer function of the reconstruction model. Liu et al. [16–18] proposed Gegenbauer polynomial approximation, time-domain Galerkin, and efficient interpolation-based approaches to reduce ill-posedness available and identify dynamic load more stable and accurate. He et al. [19,20] proposed an inverse pseudo excitation perturbation method (IPEPM) and an improved regularization method to identify random excitation for stochastic structures. Chang et al. [21] introduced an implicit Landweber method for single-source and multi-source force load reconstructions to avoid the aforementioned ill-posed phenomena. Chen et al. [22–25] developed a series of methods to overcome the ill-posed problem in MFI. Yu et al. [26–28] proposed a sparse self-estimated regularization method and a compressed sensing method to solve the ill-posed problem in MFI.

It is well-known that the ill-posed problems are usually sensitive to the regularization parameter and a priori regularization parameter is difficult to be obtained precisely in practice [29]. A common problem among these MFI methods is the absence of a properly defined criterion for the determination of the regularization parameter [30]. For a particular MFI method, the more reasonable regularization parameters are selected, the better identified results and more robust technique procedure can be achieved. Golub et al. [31] have presented the generalized cross-validation (GCV) function, which can be used in subset selection and singular value truncation methods for regression. Hansen [32,33] has
proposed the L-curve criterion for Tikhonov regularization and discrete ill-posed problems. In addition, the quasi-optimality criterion [34] and Morozov’s discrepancy principle [35] are also can be used to determine the optimal regularization parameter.

Among these regularization parameter selection criteria, the GCV criterion and the L-curve criterion are the most commonly used criteria. At present, the GCV criterion is widely applied to single out the optimal regularization parameter to deal with the ill-posedness and reduce the error amplification of force identification [10,12,15,19,36,37]. Similarly, the L-curve criterion is also a widespread application method for selecting the optimal regularization parameter of different regularization methods [11,13,14,17,38]. Comparative studies of the GCV criterion and the L-curve criterion also have been carried out [39]. Although many studies show that both of the criteria have good performance in regularization methods, the criteria often fail to determine the optimal regularization parameters when the noise contaminating the observed responses [40]. For instance, the GCV criterion has two typical difficulties in selecting regularization parameters. The first one is that the values of GCV function are too small to locate and the second one is that the GCV criterion sometimes mistakes interference noise for dynamic responses [41]. Vogel [42] indicated that the L-curve criterion yields regularized solutions that fail to converge for a certain class of problems. The L-curve has been carried out in a semi-discrete and a semi-stochastic setting, which shows that the L-curve criterion is not convergent under certain conditions. Moreover, he also strongly suggests that the L-curve method may fail for discrete ill-posed problems as well. Some relevant studies also show that the L-curve criterion can also give an accurate parameter when the noise level is high but it may fail when the noise level is low [43].

Based on the basic theory of the weigh-in-motion system [44] and the theory of MFI in the time domain [45], this study focuses on solving the ill-posed problem existing in dynamic force identification where a small noise in the measured responses can lead to large deviations in the identified results. Then the criterion for selecting the regularization parameter will be scrutinized, which is the heart of many regularization methods. Combining with the existing truncated generalized singular value decomposition (TGSVD) regularization method [46,47] regularization method, the characteristics of ill-posed problems in MFI are reflected by truncating small singular values. A regularization parameter selection criterion based on the relative percentage error (RPE) of identification results is proposed in this study. Meanwhile, comparative studies of the new criterion and two common criteria, namely, the GCV criterion and the L-curve criterion, have been carried out. The simulation results show that the RPE criterion can be used to reflect the influence of ill-posedness in MFI, which also can be applied to select the regularization parameter for the TGSVD method effectively.

2. Theory of moving force identification

2.1. Motion equation of bridge-vehicle system in time domain

The bridge-vehicle system is modeled as a Bernoulli-Euler simply-supported beam subject to a uniaxial time-varying force or biaxial time-varying forces as shown in Figure 1. The dynamic responses of the beam such as the bending moment responses and the acceleration responses can be measured when the vehicle passes over the bridge. Assuming the
Figure 1. MFI model with a Bernoulli-Euler simply-supported beam.

force $f(t)$ moving from left to right at a constant velocity $c$, the equation of motion in terms of the modal coordinate $q_n(t)$ can be written as

$$\ddot{q}_n(t) + 2\xi_n\omega_n\dot{q}_n(t) + \omega_n^2 q_n(t) = \frac{2}{\rho L} p_n(t) \quad (n = 1, 2, \ldots, \infty)$$

(1)

where $\xi_n = \frac{C}{2\rho\omega_n}$ is the $n$-th modal damping ratio; $C$ is the viscous damping; $\rho$ is the constant mass per unit length; $\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$ is the $n$-th modal frequency; $L$ is the span length of the beam; $E$ is Young’s modulus; $I$ is the moment of inertia of the beam cross-section and $p_n(t) = f(t)\sin\frac{n\pi ct}{L}$ is the modal force.

Combining the modal superposition and convolution integral, the dynamic deflection $v(x, t)$ of the beam at point $x$ and time $t$ can be obtained by solving Equation (1) in time domain [45]

$$v(x, t) = \sum_{n=1}^{\infty} \frac{2}{\rho L\omega_n} \sin\frac{n\pi x}{L} \int_0^t e^{-\xi_n\omega_n(t-\tau)}\sin\omega_n'(t-\tau)\sin\frac{n\pi c\tau}{L} f(\tau) d\tau$$

(2)

where $\omega_n' = \omega_n\sqrt{1 - \xi_n^2}$. Both the time-varying force $f(t)$ and the deflection $v(x, t)$ are step functions in a time interval $\Delta t$, and then the corresponding bending moment responses $M(t)$ and acceleration responses $\ddot{v}(t)$ at point $x$ and time $t$ can be derived from Equation (2) by mechanical derivation and calculus calculation as following.

At point $x$ and time $t$, the bending moment $M(x, t)$ of the simply-supported beam can be expressed as

$$M(x, t) = -EI \frac{\partial^2 v(x, t)}{\partial x^2}$$

$$= \sum_{n=1}^{\infty} \frac{2EI\pi^2}{\rho L^3} \frac{n^2}{\omega_n'} \sin\frac{n\pi x}{L} \int_0^t e^{-\xi_n\omega_n(t-\tau)}\sin\omega_n'(t-\tau)\sin\frac{n\pi c\tau}{L} f(\tau) d\tau$$

(3)

Similarly, at point $x$ and time $t$, the acceleration $\ddot{v}(x, t)$ of the simply-supported beam can be expressed as

$$\ddot{v}(x, t) = \sum_{n=1}^{\infty} \frac{2}{\rho L} \sin\frac{n\pi x}{L} \left[ f(t)\sin\frac{n\pi x}{L} + \int_0^t \ddot{h}_n(t-\tau) f(\tau)\sin\frac{n\pi c\tau}{L} d\tau \right]$$

(4)

where $\ddot{h}_n(t) = \frac{1}{\omega_n'\omega_n} e^{-\xi_n\omega_n t} \times \{ [(\xi_n\omega_n)^2 - \omega_n'^2] \sin\omega_n't - 2\xi_n\omega_n\omega_n'\cos\omega_n't \}$.

The relationship between the time-varying force $f(t)$ and the dynamic responses can be rewritten in discrete terms and rearranged into a set of linear algebraic equations,
which also can be modified for the identification of multi-forces in terms of the linear superposition principle.

2.2. Truncated singular value decomposition method (TSVD)

It is easy to find that the MFI in time domain eventually converts to a linear algebraic equation in the form $Ax = b$, where $A \in \mathbb{R}^{m \times n}$ is bridge-vehicle system matrix, $x \in \mathbb{R}^n$ is the unknown time-varying force vector and the vector $b \in \mathbb{R}^m$ is measured dynamic responses of bridge structure contaminated by unknown noise stemming from measurement error. The matrix $A$ can be decomposed as $A = U \Sigma V^T = \sum_{i=1}^n u_i \sigma_i v_i^T$ by singular value decomposition (SVD) algorithm, where $U = (u_1, u_2, \ldots, u_m)$ and $V = (v_1, v_2, \ldots, v_n)$ are orthonormal column matrices with $U^T U = I_m$ and $V^T V = I_n$, $u_i$ and $v_i$ are the left and right singular vectors of matrix $A$, respectively. $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ has non-negative diagonal elements with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. The $\sigma_n$ is the $n$-th singular value of $A$ and the condition number of $A$ is equal to the ratio $\sigma_1/\sigma_n$. Then the SVD solutions of $Ax = b$ can be expressed as $x = \sum_{i=1}^n \frac{u_i^T b}{\sigma_i} v_i$. With the ill-posed problem existing in MFI, bridge-vehicle system matrix $A$ is characterized by having one or more very small singular values, and hence the condition number of $A$ is very large. The best approach to reduce the abnormal large condition number of $A$ is to truncate very small singular values by using the TSVD algorithm.

The TSVD algorithm can improve the ill-posed problem by replacing bridge-vehicle system matrix $A$ with $A_k$ ($k \leq n$). The matrix $A_k$ can be decomposed as $A_k = U \Sigma V^T = \sum_{i=1}^k u_i \sigma_i v_i^T$ by the TSVD algorithm. Then the $\min A x - b_2$ transforms into $\min A_k x - b_2$, the TSVD solutions of linear algebraic equation $Ax = b$ can be expressed as [48]

$$x_k = \sum_{i=1}^k \frac{u_i^T b}{\sigma_i} v_i$$ (5)

The 2-norm of $x_k$ satisfies $x_k^2 = \sum_{i=1}^k \sigma_i^{-2}(u_i^T b)^2$, and thus $x_2$ is increased with $k$. The truncating point $k$ is an important regularization parameter of the TSVD, which controls the amount of stabilization imposed on $x_k$ and the ill-posedness immunity of TSVD. Although the TSVD is well known as a useful method for model regularization, it still has some limitations such as the data over-fitting problem.

2.3. Truncated generalized singular value decomposition method (TGSVD)

The generalized singular values of matrix pair $(A, L)$ are the square roots of the matrix pair $(A^T A, L^T L)$, where $L \in \mathbb{R}^{p \times n}$ satisfies $m \geq n \geq p$. Then the generalized singular value decomposition of $A$ and $L$ can be expressed as

$$A = U \left( \begin{array}{cc} \Sigma & 0 \\ 0 & I_{n-p} \end{array} \right) X^{-1}, \quad L = V \left( \begin{array}{cc} M & 0 \\ 0 & I_{m-p} \end{array} \right) X^{-1}$$ (6)

where $I_{n-p} \in \mathbb{R}^{(n-p) \times (n-p)}$ is a unit matrix, $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p)$ has non-negative diagonal elements with $1 \geq \sigma_p \geq \cdots \geq \sigma_2 \geq \sigma_1 \geq 0$, $U$ and $V$ are orthonormal column matrices with $U^T U = I_m$ and $V^T V = I_p$, $X \in \mathbb{R}^{n \times n}$ is nonsingular, $M = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)$. 

$$\begin{align*}
\text{diag} & = \text{diagonal elements} \\
\text{diag}^{-1} & = \text{diagonal elements of inverse matrix} \\
\text{diag}(\cdot) & = \text{diagonal matrix}
\end{align*}$$

where $\text{diag}(\cdot)$ is a diagonal matrix.
$\mu_p$ are $p \times p$ non-negative diagonal elements with $1 \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_p \geq 0$ and $\sigma_i^2 + \mu_i^2 = 1$ ($i = 1, 2, \cdots, p$). The generalized singular values of $(A, L)$ are defined as the ratios $\gamma_i = \frac{\sigma_i}{\mu_i}$ ($i = 1, 2, \cdots, p$).

Define the $A$-weighted generalized inverse of $L$ as $L_A^+ = X (M^{-1}) V^T$ and the vector $x_0 = \sum_{i=p+1}^{n} (u_i^T b)x_i$. Then the standard form quantities of $\bar{A}, \bar{b}$ and $\bar{x}$ can be defined as $\bar{A} = AL_A^+, \bar{b} = b - Ax_0$ and $\bar{x} = Lx$, the least-squares problem $\min A_k x - b_2$ of TSVD can be transformed into $\min \bar{A}_k \bar{x} - \bar{b}_2$ and the TGSVD solutions of $Ax = b$ can be expressed as [46]

$$x_{k,L} = L_A^+ \bar{x}_k + x_0 = \sum_{i=p-k+1}^{p} \frac{u_i^T b}{\sigma_i} x_i + \sum_{i=p+1}^{n} (u_i^T b)x_i \quad (7)$$

where $L$ is the regularization matrix and $k$ is the truncating point, both of which affect the calculation accuracy and ill-posed immunity of TGSVD. The optimal regularization matrix $L$ has been selected by numerical simulation prudently in previous studies [47], which is shown as follow:

$$L = \begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & -2 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\
\end{pmatrix} \quad (n-2) \times n \quad (8)$$

In this study, the truncating point $k$ will be scrutinized by three different criteria for regularization parameter selection. Then the simulation results will be used to evaluate the effectiveness of the three criteria.

3. Theory of criteria for regularization parameter selection

3.1. Theory of L-curve criterion

The L-curve criterion is a common method for choosing the regularization parameter of regularization methods, which is a plot for all valid regularization parameters of the norm $Lx_{k,L}$ of the regularized solution versus the corresponding residual norm $Ax_{k,L} - b_2$. To provide a strict mathematical definition of the ‘corner’ of the L-curve, Hansen and O’leary [33] suggested using the point on the L-curve $(\hat{\rho}, \hat{\eta})$ with maximum curvature, where $\hat{\rho}$ and $\hat{\eta}$ measure the size of the corresponding residual and the regularized solution. Recall that $\hat{\rho}$ and $\hat{\eta}$ are a function of truncating point $k$, and let $\hat{\rho}', \hat{\eta}', \hat{\rho}''$, and $\hat{\eta}''$ denote the first and second derivatives of $\hat{\rho}$ and $\hat{\eta}$ with respect to $k$. Then the curvature $\mathcal{K}$ of the L-curve $(\hat{\rho}, \hat{\eta})$ can be expressed as a function of truncating point $k$:

$$\mathcal{K} = L(k) = \frac{\hat{\rho}' \hat{\eta}'' - \hat{\rho}'' \hat{\eta}'}{((\hat{\rho}')^2 + (\hat{\eta}')^2)^{3/2}} \quad (9)$$

Then the optimal truncating point $L(k_{\text{optimal}})$ of L-curve criterion can be defined as $L(k_{\text{optimal}}) = \max_{k > 0} \{ L(k) \}$.

For a continuous regularization parameter, the optimal regularization parameter can be selected by computing the maximum curvature of the L-curve, which is defined as the
The curvature of the L-curve can be computed and the optimal regularization parameter can be selected for the continuous regularization parameter. However, the truncating point of TGSVD is discrete. To select the optimal truncating point of TGSVD, the discrete L-curve in log–log scale can be drawn by a two-dimensional spline curve approximately. The corner of the discrete L-curve can be selected as the optimal truncating point of TGSVD, which is closest to the corner of the spline curve.

In MFI application, the dynamic responses are always contaminated by various types of errors, such as measurement noise, approximation errors, and rounding errors. Unfortunately, the dynamic system equation of MFI cannot has a unique solution and be stable with respect to noise in the measured data, i.e. the crucial property of MFI is ill-posedness. The dynamic responses \( \mathbf{b} \) of the bridge can be written as \( \mathbf{b} = \bar{\mathbf{b}} + \mathbf{e} \), where \( \mathbf{e} \) are the errors and \( \bar{\mathbf{b}} \) is the unperturbed dynamic response. The vertical part of the L-curve corresponds to solutions where \( \| \mathbf{Ax}_{k,L} - \mathbf{b} \|_2 \) is very sensitive to changes in the regularization parameter because of the perturbation error \( \mathbf{e} \). The horizontal part of the L-curve corresponds to solutions where it is the residual norm \( \| \mathbf{Lx}_{k,L_2} \|_2 \) that is most sensitive to the regularization parameter. A feature of the L-curve that has not previously been considered is that the 2-norm is not always the appropriate measure of the size of the identified forces and residual vectors.

### 3.2. Theory of generalized cross-validation criterion

The GCV function is another common approach for choosing the regularization parameter. The basic idea of cross-validation is as follows: if any measured responses \( \mathbf{b} \) are left out and a moving force \( \mathbf{x} \) is identified by the TGSVD method, then the estimate of \( \mathbf{b} \) computed from \( \mathbf{x} \) must be a good estimate. While ordinary cross-validation depends on the particular ordering of the measured responses, GCV function is invariant to the orthogonal transformation of the dynamic responses. The GCV function [31] can be expressed as:

\[
G(k) = \frac{\| \mathbf{Ax}_{k,L} - \mathbf{b} \|_2^2}{\left( \text{Trace}(\mathbf{I} - \mathbf{A}_k \mathbf{A}_k^T) \right)^2}
\]

where \( \mathbf{A}_k^T \) is any matrix that maps the dynamic responses \( \mathbf{b} \) onto the identified forces \( \mathbf{x}_{k,L} \), i.e. \( \mathbf{x}_{k,L} = \mathbf{A}_k^T \mathbf{b} \). Then the optimal truncating point \( G(k_{\text{optimal}}) \) of GCV function can be defined by minimizing the \( G(k) \) as \( G(k_{\text{optimal}}) = \min_{k>0} \{ G(k) \} \).

Although the GCV criterion works well for many regularization methods, there are still two difficulties that prevent it from selecting an exact regularization parameter. The first one is that the values of the GCV function are very small, which causes the tail of the curve to be too flat to locate the minimum \( G(k) \). The second one is that the GCV criterion sometimes mistakes interference noise for dynamic responses and then it is a failure for regularization methods.
3.3. Theory of relative percentage error criterion

To obtain a reasonably accurate approximation in MFI, the RPE criterion is proposed to overcome the deficiency of the ill-posedness of the system matrix.

The RPE criterion can be expressed as:

\[ RPE(k) = \frac{x_{k,L} - f_{true}^2}{f_{true}^2} \]  

(12)

where \( x_{k,L} \) is the force identified by TGSVD with truncating point \( k \) and \( f_{true} \) is the true force. Then the optimal truncating point \( RPE(k_{optimal}) \) of the RPE criterion can be defined by minimizing the \( RPE(k) \) as:

\[ RPE(k_{optimal}) = \min_{k > 0} \{ RPE(k) \} \]  

(13)

In Equation (12), an important prerequisite is needed, that is, the true force should be known in advance to construct the RPE criterion. However, in practical engineering, the true force is normally unknown. To improve this problem, a new equivalent evaluation standard is presented for the convenience of field application as:

\[ RPE(k) = \frac{R_{k, \text{reconstruction}} - R_{\text{measured}}^2}{R_{\text{measured}}^2} \]  

(14)

where \( R_{\text{measured}} \) are the measured dynamic responses of bridge and \( R_{k, \text{reconstruction}} \) are the dynamic responses reconstructed from the identified force with truncating point \( k \). Then the optimal truncating point \( RPE(k_{optimal}) \) of the RPE criterion can be obtained from Equation (13).

The GCV criterion and L-curve criterion use other indicators to evaluate the optimal regularization parameters, which is difficult to avoid causing additional errors. On the contrary, the RPE criterion intuitively expresses the deviation degree between identified force and true force, which can be used to select the optimal truncating point of TGSVD without mistake in theoretical and practical.

4. Numerical simulations

4.1. Simulation parameters of bridge-vehicle system

As shown in Figure 1, a simply-supported beam subjected to a moving force is taken as an example for numerical simulations. The parameters of the beam are as follows: \( L = 40 \) m, \( EI = 1.274916 \times 10^{11} \) N · m\(^2\), \( \rho A = 12000 \) kg · m\(^{-1}\) and the first three natural frequencies of the simply-supported beam are 3.2, 12.8, and 28.8 Hz. The analysis frequency is from 0 Hz to 40 Hz and the sampling frequency is 200 Hz. There are two types of moving forces with the moving speed is 40 m · s\(^{-1}\) are simulated as follow:

The first one is a uniaxial time-varying force, which is extracted from Pan et al. [49].

\[ f(t) = \begin{cases} 40[1 + 0.3 \sin(25 \pi t) + 0.2 \sin(60 \pi t)]kN & 0 \leq t < 0.6 \\ 40[1 + 0.3 \sin(25 \pi t) + 0.2 \sin(60 \pi t) + 3e^{-35(t-0.6)} \sin(125(t - 0.6))]kN & 0.6 \leq t \leq 1 \end{cases} \]  

(15)
The second one is biaxial time-varying forces, which are extracted from Law et al. [45]. The wheelbase of the vehicle is 8 m.

\[ f_1(t) = 20[1 + 0.1 \sin(10\pi t) + 0.05\sin(40\pi t)] \text{kN} \]

\[ f_2(t) = 20[1 - 0.1 \sin(10\pi t) + 0.05\sin(50\pi t)] \text{kN} \] (16)

Random noise is used to simulate the polluted responses as

\[ R_{\text{measured}} = R_{\text{calculated}} \cdot (1 + E_p \cdot N_{\text{noise}}) \] (17)

where \( E_p \) represents error level selecting as 1%, 5% and 10%, respectively; \( N_{\text{noise}} \) is a standard normal distribution vector.

The RPE values between the true moving force and the identified force are calculated by the following equation

\[ \text{RPE} = \frac{f_{\text{identified}} - f_{\text{true}}}{f_{\text{true}}} \times 100\% \] (18)

Both of the bending moment responses and acceleration responses or their combined responses can be used to identify the moving forces. The locations of the measuring points are arranged in 1/4, 1/2, and 3/4 bridge span, respectively.

4.2. Comparative studies of three criteria with uniaxial time-varying force

In this subsection, the uniaxial time-varying force will be identified by the TGSVD methods. The bending moment responses and acceleration responses are measured under moving forces and the location of the measurement points are arranged in 1/4, 1/2, and 3/4 span of the bridge, respectively. There are 12 cases including identification from a single type of response or combined responses, which is extracted from Law et al. [45]. The white noise is superposed into the bridge dynamic responses to simulate the polluted responses and the error level selecting as 1%, 5%, and 10%, respectively. In addition, the total number of samples of the uniaxial time-varying force is 198 and then the total number of single values is 198. That is, the truncating point of uniaxial time-varying force ranges from 1 to 198.

Table 1 tabulates the identification error and truncating points of uniaxial time-varying force by TGSVD with three criteria in all 12 cases. The sensor locations correspond to the simulation cases, which contain information such as responses type, number of responses, and location of the measurement points. For example, the sensor location ‘1/4m&1/2m&1/2a’ corresponds to the sixth case, which means that this case includes three dynamic responses, one is acceleration response and the other two are bending moment responses. In addition, the location of the accelerometers is arranged in the middle of the bridge and the location of the two strain gauges are arranged in the 1/4 span of the bridge and the middle of the bridge. The underlined values listed in Table 1 are for the L-curve criterion, italics values are for GCV criterion and the others are for the RPE criterion, respectively. For the consistency of the paper, this setting is consistent throughout the rest of the paper.
As shown in Table 1, the RPE values of RPE criterion are the smallest among the three criteria in all 12 cases with 1%, 5%, and 10% noise levels. It is easy to understand that the optimal truncating points selected by the RPE criterion are based on the minimum RPE values, which are the real optimal truncating points of TGSVD. On the other hand, it is very confusing that neither the L-curve criterion nor the GCV criterion can help select the real optimal regularization parameter in all cases, the RPE values of identification results by L-curve criterion and GCV criterion are more or less bigger than that come from the RPE criterion. Furthermore, the RPE values of all three criteria increase with the increase of noise level. That is, with the increase of noise level, both of the weighted residuals of the system equation and the identification error increase, which leads to the decrease of the identification accuracy.

Figure 2 shows the optimal truncating point of TGSVD selected by the L-curve criterion in case 1 with 1% noise level. The L-corner of discrete L-curve is drawn by a two-dimensional spline curve approximately, which is easy to mistake a fine-grained local

| Case | Response | Selection criterion | 1% noise | 5% noise | 10% noise |
|------|----------|---------------------|----------|----------|----------|
|      |          | RPE(%) | k | RPE(%) | k | RPE(%) | k |
| 1    | 1/4m&1/2m | RPE criterion | 5.2 | 85 | 12.1 | 65 | 20.9 | 62 |
|      |          | L-curve | 22.4 | 32 | 23.5 | 71 | 22.2 | 33 |
|      |          | GCV criterion | 5.5 | 77 | 13.5 | 66 | 25.1 | 52 |
| 2    | 1/4m&1/2m&3/4m | RPE criterion | 5.0 | 84 | 11.0 | 71 | 16.3 | 64 |
|      |          | L-curve | 22.8 | 35 | 28.5 | 24 | 132.0 | 162 |
|      |          | GCV criterion | 5.4 | 76 | 12.0 | 61 | 18.4 | 56 |
| 3    | 1/4a&1/2a | RPE criterion | 7.6 | 124 | 36.7 | 64 | 27.9 | 75 |
|      |          | L-curve | 8.1 | 174 | 37.2 | 84 | 73.5 | 69 |
|      |          | GCV criterion | 5.2 | 57 | 1.5 | 3 | 1.5 | 2 |
| 4    | 1/4a&1/2a&3/4a | RPE criterion | 3.8 | 192 | 14.7 | 82 | 27.9 | 75 |
|      |          | L-curve | 4.1 | 181 | 34.6 | 42 | 42.7 | 42 |
|      |          | GCV criterion | 4.0 | 179 | 15.2 | 83 | 28.5 | 65 |
| 5    | 1/2m&1/2a | RPE criterion | 4.6 | 122 | 16.0 | 72 | 26.7 | 34 |
|      |          | L-curve | 5.0 | 138 | 47.9 | 14 | 41.6 | 138 |
|      |          | GCV criterion | 4.7 | 121 | 18.2 | 76 | 31.7 | 62 |
| 6    | 1/4m&1/2m&1/2a | RPE criterion | 2.8 | 122 | 11.0 | 73 | 20.2 | 73 |
|      |          | L-curve | 3.0 | 124 | 21.2 | 32 | 38.6 | 156 |
|      |          | GCV criterion | 2.8 | 122 | 12.3 | 77 | 21.4 | 62 |
| 7    | 1/4m&1/2m&1/4a&1/2a | RPE criterion | 2.7 | 154 | 7.2 | 94 | 12.6 | 74 |
|      |          | L-curve | 3.0 | 172 | 7.7 | 106 | 22.0 | 33 |
|      |          | GCV criterion | 2.8 | 152 | 7.4 | 81 | 13.3 | 63 |
| 8    | 1/4m&1/4a | RPE criterion | 4.4 | 138 | 14.5 | 65 | 22.7 | 49 |
|      |          | L-curve | 6.6 | 103 | 18.4 | 72 | 29.2 | 32 |
|      |          | GCV criterion | 6.1 | 91 | 15.7 | 63 | 31.3 | 55 |
| 9    | 1/4m&1/4a&1/2a | RPE criterion | 3.0 | 158 | 9.0 | 74 | 13.7 | 64 |
|      |          | L-curve | 3.8 | 122 | 13.6 | 130 | 25.9 | 32 |
|      |          | GCV criterion | 3.2 | 152 | 10.4 | 78 | 13.7 | 68 |
| 10   | 1/2m&1/4a | RPE criterion | 3.8 | 133 | 9.5 | 87 | 17.0 | 87 |
|      |          | L-curve | 5.2 | 94 | 10.5 | 95 | 18.9 | 78 |
|      |          | GCV criterion | 5.0 | 92 | 12.8 | 64 | 19.5 | 64 |
| 11   | 1/4m&1/2m&1/4a | RPE criterion | 3.7 | 130 | 8.6 | 83 | 14.8 | 72 |
|      |          | L-curve | 8.7 | 184 | 43.6 | 184 | 87.6 | 184 |
|      |          | GCV criterion | 4.7 | 90 | 10.6 | 66 | 15.0 | 64 |
| 12   | 1/2m&1/4a&1/2a | RPE criterion | 2.6 | 153 | 7.7 | 94 | 13.6 | 81 |
|      |          | L-curve | 3.1 | 162 | 8.5 | 107 | 38.6 | 187 |
|      |          | GCV criterion | 2.7 | 152 | 8.1 | 82 | 14.3 | 63 |
Figure 2. Optimal truncating point of TGSVD selected by L-curve criterion with uniaxial force (case1 with 1% Noise).

Figure 3. Influence of noise level on L-curve criterion with uniaxial force (case1).

‘corner’ for the desired corner due to the discrete property of truncating points. The optimal truncating point selected by the L-curve criterion is corresponding to the L-corner at $k = 32$ while the real optimal truncating point at $k = 85$ is missed.

Figure 3 shows the influence of the noise level on the L-curve criterion with uniaxial force in case 1. Table 1 shows that the optimal truncating points of TGSVD selected by the L-curve criterion have no specific rule. As shown in Figure 3, the maximum truncating point comes from the 5% noise level and the minimum truncating point comes from the 1% noise level. With the increase of noise level, the dynamic responses are more contaminated by noise. Hence, more small singular values should be truncated to overcome the ill-posed problem caused by small singular values. Unfortunately, the L-curve criterion can neither be used to select the real optimal truncating points of TGSVD nor be used to reflect the variation rule of the optimal truncating points with the noise level.

Figure 4 shows the optimal truncating point of TGSVD selected by GCV criterion and RPE criterion in case 1 with 1% noise level, respectively. Figure 5 shows the optimal truncating point of TGSVD selected by the RPE criterion and RPE criterion in case 1 with three kinds of noise levels, respectively. The principle of the GCV criterion is very simple, it is to choose the minimum $G(k)$ as the optimal truncating point. Although the GCV criterion works well for many regularization methods, there are some cases in which the GCV criterion has difficulty in selecting the optimal regularization parameter. One difficulty is that the GCV function has many very close small values hence the minimum value may be difficult to localize numerically [41]. For example, the optimal truncating point of GCV criterion is selected at $k = 77$ corresponding to the minimum $G(k) = 4.769 \times 10^{-10}$ in
case 1 with 1% noise level. Similarly, the $G(k)$ equals to $4.776 \times 10^{-10}$ at $k = 78$ and the $G(k)$ equals to $4.795 \times 10^{-10}$ at $k = 79$ in this case. The $G(k)$ values of GCV criterion are too small to select accurately, which cannot be used to select optimal truncating point conveniently. As shown in Figure 5(a), the $G(k)$ values are too close to select in all three noise levels. In addition, the optimal truncating points decrease with the increase of noise level as shown in Figure 5(a) and Table 1. Although the GCV criterion has difficulties in selecting the accurate optimal truncating points of TGSVD, it can be used to reflect the variation rule of the optimal truncating points with the noise level.

Both of the RPE criterion and the GCV criterion have the same principle, i.e. the minimum value is chosen as the optimal truncating point. Although the objective of both criteria is to choose the minimum value to determine the optical truncating point, there are still obvious differences between the two criteria due to the different criterion parameters. For example, the optimal truncating point of the RPE criterion is selected at $k = 85$ corresponding to the minimum $RPE(k) = 5.23$ in case 1 with 1% noise level. Similarly, the $RPE(k)$ equals to 5.30 at $k = 84$ and the $RPE(k)$ equals to 5.32 at $k = 86$ in this case. Compared with the GCV criterion, the minimum value of the RPE criterion is easier to select as shown in Figure 4(b).

More importantly than all of that, the RPE criterion can be used to reveal the influence of small singular values on the identification error directly. As shown in Figure 4(b), with the increase of noise level, the tail of the curve rises rapidly, i.e. the identification error explodes with the very small singular values. Moreover, the big error values not only appear at the tail.
of the curve corresponding to the small singular values but also in some special intervals, such as between 10 and 20. Different from the big error values at the tail of the curve which can be eliminated by truncating the small singular values, the big error values at special intervals can be shown in Figure 5(b) but cannot be eliminated with a similar method. In order to avoid tremendous ill-posedness of MFI, the range of these big error values needs to pay special attention as same as the tail of the curve of the RPE criterion. The illustration results show that the RPE criterion can be used to select the real optimal truncating point accurately, which also can be used to reflect the variation rule of the optimal truncating points with the noise level.

Compared with the RPE criterion, the $G(k)$ values of GCV criterion are too small to select, which cannot be accurately selected even with local magnification coordinates as shown in Figure 6.

Figure 7 and Figure 8 show the power spectral density (PSD) curve of identified uniaxial force and the identified uniaxial time-varying moving force with three criteria in case 1 with 1% noise level, respectively. The identified results show that the RPE criterion and the GCV criterion are more suitable for TGSVD than the L-curve criterion in this type of moving force. The data in Table 1 also shows that the identified results with the RPE criterion and the GCV criterion are more accurate than the L-curve criterion in most of the cases. In addition, the optical truncating points of the RPE criterion and GCV criterion decrease with the increase of noise level. That is, with the increase of noise level, more contaminated responses are contained to identify moving force and more small singular values need to be truncated to remove noise disturbance signals. The optical truncating points of the L-curve criterion do not agree with this rule, which are selected particularly by L-corner of discrete L-curve. Compared with the GCV criterion, the L-curve criterion has lower accuracy in selecting the optical truncating points of TGSVD with uniaxial force. Miao et al. [50] also indicated that the L-curve generated by the experiment is unobservable and the optimal point on the L-curve is difficult to localize with accuracy.

### 4.3. Comparative studies of three criteria with biaxial time-varying force

In this subsection, the biaxial time-varying forces will be identified and the simulation cases are the same as the previous subsection. Table 2 tabulates the identification error and truncating points of biaxial time-varying forces by TGSVD with three criteria in all 12 cases. The total number of samples of biaxial time-varying forces is 396 and then the total
number of single values is 396. That is, the truncating point of biaxial time-varying forces ranges from 1 to 396.

Compared with data in Table 1, some similar rules can be obtained from Table 2 as follows. The RPE values of RPE criterion are the smallest among three criteria and neither the L-curve criterion nor the GCV criterion can help select the real optimal truncating points in all cases. The RPE values of all three criteria increase with the increase of noise level. The optical truncating points of the RPE criterion and GCV criterion decrease with the increase of noise level while the optical truncating points of the L-curve criterion do not agree with this rule. On the other hand, there are some differences between the uniaxial time-varying moving force. For instance, the identified biaxial results with the GCV criterion are much worse than the uniaxial force results. In some cases, the RPE values of the GCV criterion are the biggest in three criteria. With biaxial time-varying force, neither the L-curve criterion nor the GCV criterion can effectively help select the optimal truncating point of the TGSVD regularization method.

Figure 9 shows the optimal truncating point of TGSVD selected by the L-curve criterion in case 1 with 5% noise level. Figure 10 shows the influence of the noise level on the L-curve criterion with biaxial forces in case 1. Both of the illustration results and the Table 2 show that the selected optimal truncating points of L-curve criterion are far from the
real optimal truncating points in most of the 12 cases. Similar to the uniaxial time-varying moving force, the simulation results of biaxial time-varying moving forces show that the L-curve criterion cannot be used to reflect the variation rule of the optimal truncating points with noise level either.

Figure 11 shows the optimal truncating point of TGSVD selected by GCV criterion and RPE criterion in case 1 with 5% noise level, respectively. Figure 12 shows the optimal truncating point of TGSVD selected by the RPE criterion and RPE criterion in case 1 with three kinds of noise levels, respectively. As shown in Figure 11(b) and Figure 12(b), the tail of the RPE criterion curve rises obviously, indicating that the ill-posedness caused by small singular values increases significantly.

Similar to uniaxial time-varying moving force, the simulation results of biaxial time-varying moving forces show that the RPE criterion can not only be used to select the real optimal truncating point accurately but also can be used to reveal the influence of small singular values on the ill-posedness problem existing in MFI.

| Case | Response | Selection criterion | 1% noise | 5% noise | 10% noise |
|------|-----------|---------------------|----------|----------|----------|
|      |           | RPE(%)              | front    | rear     | k        | front    | rear     | k        | front    | rear     | k        |
| 1    | 1/4m&1/2m | RPE criterion      | 5.0      | 4.3      | 76       | 8.2      | 8.0      | 49       | 8.4      | 8.0      | 34       |
|      |           | L-curve             | 47.7     | 39.8     | 337      | 434.2    | 343.0    | 381      | 870.0    | 688.4    | 381      |
|      |           | GCV criterion       | 97.9     | 130.9    | 392      | 456.9    | 568.2    | 390      | 912.4    | 1115.3   | 390      |
| 2    | 1/4m&1/2m&3/4m | RPE criterion | 3.6      | 2.7      | 122      | 8.3      | 8.0      | 42       | 8.4      | 8.0      | 40       |
|      |           | L-curve             | 4.9      | 4.0      | 73       | 84.2     | 92.9     | 304      | 168.4    | 186.7    | 304      |
|      |           | GCV criterion       | 3.9      | 3.0      | 118      | 13.6     | 8.2      | 45       | 17.2     | 18.0     | 35       |
| 3    | 1/4a&1/2a | RPE criterion      | 1.5      | 1.0      | 357      | 2.8      | 3.4      | 242      | 5.7      | 4.8      | 233      |
|      |           | L-curve             | 2.3      | 1.0      | 379      | 3.8      | 3.4      | 289      | 6.1      | 5.7      | 291      |
|      |           | GCV criterion       | 1.5      | 2.0      | 309      | 3.4      | 3.5      | 191      | 6.1      | 5.3      | 173      |
| 4    | 1/4a&1/2a&3/4a | RPE criterion | 0.6      | 0.8      | 393      | 2.2      | 1.5      | 242      | 3.7      | 2.6      | 233      |
|      |           | L-curve             | 0.8      | 1.0      | 381      | 2.5      | 1.7      | 289      | 4.0      | 2.9      | 291      |
|      |           | GCV criterion       | 1.1      | 2.0      | 312      | 2.6      | 2.2      | 191      | 3.7      | 4.4      | 173      |
| 5    | 1/2m&1/2a | RPE criterion      | 3.3      | 2.9      | 159      | 8.8      | 6.1      | 122      | 12.6     | 9.7      | 85       |
|      |           | L-curve             | 3.6      | 9.5      | 267      | 245.2    | 195.9    | 348      | 18.2     | 12.8     | 95       |
|      |           | GCV criterion       | 87.3     | 98.5     | 393      | 431.5    | 476.6    | 393      | 862.7    | 950.0    | 393      |
| 6    | 1/4m&1/2m&1/2a | RPE criterion | 2.6      | 2.6      | 276      | 7.5      | 6.7      | 127      | 12.4     | 11.6     | 65       |
|      |           | L-curve             | 3.6      | 3.9      | 199      | 7.6      | 8.2      | 130      | 17.7     | 11.7     | 105      |
|      |           | GCV criterion       | 3.6      | 2.9      | 128      | 8.3      | 6.8      | 115      | 18.5     | 11.9     | 100      |
| 7    | 1/4m&1/2m&1/4a&1/2a | RPE criterion | 1.8      | 1.4      | 370      | 2.9      | 3.4      | 256      | 4.2      | 4.6      | 116      |
|      |           | L-curve             | 1.8      | 2.2      | 312      | 3.6      | 4.2      | 274      | 5.7      | 7.0      | 224      |
|      |           | GCV criterion       | 1.8      | 2.3      | 309      | 3.1      | 4.0      | 155      | 7.7      | 6.2      | 141      |
| 8    | 1/4m&1/4a | RPE criterion      | 3.9      | 4.2      | 254      | 9.5      | 7.4      | 95       | 9.1      | 7.9      | 94       |
|      |           | L-curve             | 4.1      | 4.5      | 241      | 16.0     | 11.4     | 150      | 16.9     | 12.6     | 82       |
|      |           | GCV criterion       | 444.0    | 299.1    | 390      | 2261.1   | 1523.9   | 390      | 4537.4   | 3060.8   | 390      |
| 9    | 1/4m&1/4a&1/2a | RPE criterion | 1.6      | 1.5      | 353      | 2.3      | 3.3      | 256      | 3.7      | 3.9      | 117      |
|      |           | L-curve             | 1.6      | 2.3      | 311      | 3.0      | 4.1      | 274      | 5.9      | 13.2     | 297      |
|      |           | GCV criterion       | 1.6      | 2.4      | 309      | 2.6      | 4.1      | 155      | 6.8      | 6.2      | 141      |
| 10   | 1/2m&1/4a | RPE criterion      | 5.5      | 4.8      | 183      | 9.1      | 9.7      | 62       | 9.1      | 10.2     | 61       |
|      |           | L-curve             | 41.5     | 29.3     | 372      | 205.9    | 142.4    | 372      | 411.5    | 284.1    | 372      |
|      |           | GCV criterion       | 45.7     | 35.1     | 387      | 229.0    | 170.8    | 388      | 458.8    | 341.2    | 388      |
| 11   | 1/4m&1/2m&1/4a | RPE criterion | 2.7      | 4.1      | 248      | 9.0      | 8.3      | 168      | 12.5     | 11.5     | 79       |
|      |           | L-curve             | 7.7      | 4.1      | 143      | 10.0     | 8.4      | 154      | 41.9     | 39.5     | 272      |
|      |           | GCV criterion       | 2.9      | 4.1      | 182      | 12.6     | 8.9      | 102      | 19.8     | 13.4     | 102      |
| 12   | 1/2m&1/4a&1/2a | RPE criterion | 1.3      | 1.3      | 362      | 3.9      | 3.1      | 283      | 5.9      | 6.0      | 117      |
|      |           | L-curve             | 1.8      | 1.3      | 339      | 4.0      | 3.2      | 273      | 6.9      | 6.2      | 274      |
|      |           | GCV criterion       | 1.5      | 2.0      | 309      | 3.9      | 3.4      | 155      | 9.2      | 6.3      | 141      |

Table 2. Comparison of three criteria of TGSVD with biaxial time-varying moving forces.
Figure 9. Optimal truncating point of TGSVD selected by L-curve criterion with biaxial forces (case 1 with 5% Noise).

Figure 10. Influence of noise level on L-curve criterion with biaxial forces (case 1).

Figure 11. Optimal truncating point of TGSVD with biaxial forces (case 1 with 5% Noise). (a) GCV criterion; (b) RPE criterion.

Figure 12. Influence of noise level on selection criteria with biaxial forces (case 1). (a) GCV criterion; (b) RPE criterion.
As mentioned above, the GCV criterion has two typical difficulties in selecting regularization parameters. The first one is that the values of GCV function are too small to locate, which has been presented in the previous subsection with uniaxial force simulation. The second one is that the GCV criterion sometimes mistakes interference noise for dynamic responses. In this special case, the GCV criterion is not only difficult to select regularization parameters but also may select the wrong regularization parameters.

As shown in Figure 11(a) and Figure 12(a), due to mistaking interference noise for dynamic responses, the tail of the GCV criterion curve drops abnormal, which should be rising to reveal the influence of small singular values. In this special case, the selected truncating point corresponding to the minimum $G(k)$ is far from the real optimal truncating point. Obviously, the GCV criterion is a failure for TGSVD in this case.

Figure 13 and Figure 14 show the identified biaxial time-varying moving forces and the PSD curve of identified biaxial forces with three criteria (case1 with 5% Noise), respectively. The identified results show that the L-curve criterion and GCV criterion are failures in identifying biaxial forces in this case. On the contrary, the identified forces with the RPE criterion are very close to the true forces in both the front axle and rear axle. Evaluation results show that the RPE criterion remains at a high level with different kinds of moving forces, which is very beneficial for the application of the regularization method in MFI.

As mentioned above, the true force is normally unknown and the equivalent evaluation standard as shown in Equation (14) can be applied in the field. In the time domain method, there is a linear relationship between the force and the bridge response. In practical engineering, the measured response is fixed while the reconstruction response varies with the identified force. In other words, with different regularization parameters, the reconstruction responses are different. When the reconstruction response is closest to the measured response, the corresponding regularization parameter is the optimal one. Figure 15 shows
the reconstruction bending moment responses and acceleration responses of the bridge at 3/4 span with three criteria. As shown in Figure 15, the reconstruction bending moment response and acceleration response of the RPE criterion are very close to the measured responses. On the contrary, the reconstruction responses of the L-curve criterion and GCV criterion deviate greatly from the measured responses. Numerical simulations show that the equivalent evaluation standard in Equation (14) is consistent with the evaluation standard in Equation (12), which proves equivalence between the original and equivalent evaluation criterion.

5. Conclusions

In this paper, an RPE criterion is proposed to select the optimal truncating point of TGSVD. To compare the effectiveness of three criteria, two types of time-varying forces have been identified by TGSVD approach in this work, then some conclusions can be made as follows:

(1) To increase the identification accuracy and overcome the ill-posedness in MFI with a higher noise level, the small singular values need to be truncated to remove noise disturbance signals. The optical truncating points of the RPE criterion and GCV criterion decrease with the increase of noise level. That is, with the increase of noise level, more contaminated responses are contained to identify moving force and more small singular values need to be truncated. On the contrary, the optical truncating points of the L-curve criterion do not agree with this rule, which are selected particularly by L-corner of discrete L-curve.

(2) The L-corner of discrete L-curve is drawn by a two-dimensional spline curve approximately, which is easy to mistake a fine-grained local ‘corner’ for the desired corner. For this reason, the identified results with the L-curve criterion are not accurate enough for both uniaxial time-varying force and biaxial time-varying forces.

(3) The identified uniaxial force with the GCV criterion agrees well with the true force in most cases even though the values of GCV function are too small to locate. However, the identified biaxial forces with GCV criterion are far from the true forces in many cases. The GCV criterion sometimes mistakes interference noise for dynamic responses and then it is a failure for TGSVD.

(4) The RPE values of the RPE criterion are the smallest among the three criteria. It is the only one of the three criteria that can successfully select the real optimal truncating
points for TGSVD. More importantly, the RPE criterion can be used to reveal the influence of small singular values on the identification error directly, which also can be used to reflect the ill-posedness problem existing in MFI.

The RPE criterion has obvious advantages such as clearer concept, easier selection regularization parameters, and higher accuracy compared with the other two criteria, which is effective in determining the optimal truncating point of TGSVD and should be adopted priority.

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