Exact Operator Quantization of the Euclidean Black Hole CFT

Carsten Krüger

Institut für Theoretische Physik, Freie Universität Berlin, 14195 Berlin
E-mail: ckrueger@physik.fu-berlin.de

Abstract: We present an exact operator quantization of the Euclidean Black Hole CFT using a recently established free field parametrization of the fundamental fields of the classical theory [4, 5, 6, 7]. Quantizing the map to free fields, we show that the resulting quantum fields are causal and transform as covariant fields w.r.t. the Virasoro algebra. We construct the reflection operator of the quantum theory and demonstrate its unitarity. We furthermore discuss the W-algebra of the Euclidean Black Hole model. It turns out that unitarity of the reflection operator is a simple consequence of the fact that certain representations of the W-algebra are unitarily equivalent.
1. Introduction

The objective of the present article is to perform an exact quantization of the conformal field theory (CFT) classically defined by the action

\[ S = \frac{1}{4\pi\alpha'} \int_M d\sigma dt (\partial_\mu r \partial^\mu r + \tanh^2 r \partial_\mu \theta \partial^\mu \theta) \]  

(1.1)

which, given that \( M \) has cylindrical topology, \( M = \mathbb{R} \times S^1 \), is the action of a so-called non-linear sigma model describing a closed string propagating in a target space with metric

\[ ds^2 = dr^2 + \tanh^2 r d\theta^2 \]  

(1.2)

The target space has the shape of a semi-infinite cigar if \( \theta \) is an angular variable defined modulo \( 2\pi \).

It was realized in the early 90’s by Witten [1] that this model may be formulated as a gauged Wess-Zumino-Novikov-Witten (WZNW) theory [15, 16, 17] based on the noncompact group \( SL(2,\mathbb{R}) \), the so-called \( SL(2,\mathbb{R})/U(1) \) gauged WZNW model, and has since then attracted much attention since the target space geometry (1.2) may be seen to describe a 2d Euclidean black hole. It is thus an important toy model of string theory in a two-dimensional target space, and Witten’s discovery initiated quite some effort aiming at a better understanding of the Euclidean black hole CFT. Let us just mention the contribution of R. Dijkgraaf, E. Verlinde and H. Verlinde [2], where some interesting aspects of the dynamics of strings propagating in the curved geometry (1.2) were analyzed in the
point-particle limit, which reduces the problem to the quantization of the center of mass
dynamics of the string in an effective target space geometry. In this fashion, a scattering
amplitude could be obtained, relating incoming and outgoing plane waves on target space,
and the spectrum of the theory could be determined.

Recently, the 2d black hole model has become important for the study of little string
theories [18, 19, 20]. In this context exact results on the 2- and 3-point function in the 2d
black hole model have been obtained by coset construction from the $H^+_3$-WZNW model
[9, 10, 11]. In this article we will however follow an alternative approach. The classical
theory defined by the action (1.1) will be quantized in a fashion that preserves conformal
symmetry, and our quantization approach will give rise to a conformal quantum field theory
defined on the world-sheet $M$ of the closed string. We will use free field techniques very
similar to those already applied to Liouville theory [8]. Here, the classical Liouville field
is canonically mapped to a free field and then quantized by quantizing the map to free
fields. This method proves equivalently successful for the 2d black hole model, where the
parametrization of the classical Euclidean black hole fields in terms of free fields has been
established in [4, 5, 6, 7]. Taking into account that the free fields parametrizing the black
hole fields are asymptotic in/out fields makes it possible to determine an exact expression
for the scattering amplitude which in the semi-classical limit reduces to the amplitude of
2. Another central result of this article refers to the well known W-algebra [3] of the
Euclidean black hole model. It is demonstrated that representations of the W-algebra to
spins $j$ and $-j - 1$ where $j \in -\frac{1}{2} + i\mathbb{R}$ are unitarily equivalent. To establish this fact we
make use of the representation of the W-currents in terms of free fields. It turns out that
unitarity of the reflection operator (or S-matrix) of the model is closely linked to unitary
equivalence of these representations.

2. Structure of the classical solution

Classical Dynamics

Classically, the Euclidean Black Hole model may be defined by gauge invariant reduction
of the $SL(2, \mathbb{R})$ WZNW model. The action obtained in this fashion reads

$$S = \frac{k}{4\pi} \int_M d\sigma dt \ h^{\mu\nu} (\partial_{\mu} r \partial_{\nu} r + \tanh^2 r \partial_{\mu} \theta \partial_{\nu} \theta)$$

(2.1)

with flat Minkowskian world-sheet metric $h_{\mu\nu} = \text{diag}(+, -)$. The world-sheet $M$ has
cylindrical topology, i.e. $M = \mathbb{R} \times S^1$ corresponding to $\sigma \in [0, 2\pi], \ -\infty < t < \infty$. The
fields $r(\sigma, t), \theta(\sigma, t)$ map the world-sheet $M$ of a closed bosonic string into the curved
target-space (1.2). The fields $r, \theta$ obey the boundary conditions

$$r(\sigma + 2\pi, t) = r(\sigma, t), \quad \theta(\sigma + 2\pi, t) = \theta(\sigma, t) + 2\pi w$$

(2.2)

We note that $\theta$ is not strictly periodic w.r.t. the space coordinate. Instead, we choose $\theta$
to be an angular variable defined modulo $2\pi$ only in order to avoid a coordinate singularity
at \( r = 0 \). The target space then assumes the shape of a semi-infinite cigar. The integer \( w \) is called the winding number and counts how often the string winds round the cigar.

In \([4,5]\), the equations of motion derived from the action (2.1) have been shown to be completely integrable, and a parametrization of the on-shell physical fields \( r, \theta \) in terms of canonical free fields has been given. Let us briefly discuss the structure of the classical solution. Introducing complex Kruskal coordinates,

\[
\begin{align*}
\bar{u}(\sigma,t) &= \sinh(\sigma, t)e^{-i\theta(\sigma,t)}, \quad \bar{u}(\sigma,t) = \sinh(\sigma, t)e^{-i\theta(\sigma,t)}
\end{align*}
\]

the equations of motion assume the form

\[
\begin{align*}
\partial_{+}\partial_{-}u &= \frac{1}{1+uu}, & \partial_{+}\partial_{-}\bar{u} &= \frac{1}{1+uu}
\end{align*}
\]

Here we have introduced light-cone coordinates according to

\[
\begin{align*}
x_{+} &= t + \sigma, & x_{-} &= t - \sigma \\
\partial_{+} &= \frac{1}{2}(\partial_{t} + \partial_{\sigma}), & \partial_{-} &= \frac{1}{2}(\partial_{t} - \partial_{\sigma})
\end{align*}
\]

The on-shell physical fields \( u, \bar{u} \) may then be parametrized by canonical free fields

\[
\begin{align*}
\Phi_{1}(\sigma, t) &= \phi_{1}(x_{+}) + \bar{\phi}_{1}(x_{-}), & \Phi_{2}(\sigma, t) &= \phi_{2}(x_{+}) + \bar{\phi}_{2}(x_{-})
\end{align*}
\]

It will be useful to recall here the mode expansions for the free fields. For the (anti)chiral components of \( \Phi_{1} \) we have

\[
\begin{align*}
\phi_{1}(x_{+}) &= \frac{q_{1}}{2} + p_{1}x_{+} + i \sum_{n \neq 0} \frac{a_{n}^{(1)}}{n} e^{-inx_{+}} \\
\bar{\phi}_{1}(x_{-}) &= \frac{q_{1}}{2} + p_{1}x_{-} + i \sum_{n \neq 0} \frac{b_{n}^{(1)}}{n} e^{-inx_{-}}
\end{align*}
\]

Unlike \( \Phi_{1} \), the free boson \( \Phi_{2} \) is chosen to be compactified on a circle of radius \( R = \sqrt{k} \) which gives a slightly modified mode expansion for its (anti)chiral components,

\[
\begin{align*}
\phi_{2}(x_{+}) &= \frac{q_{2}}{2} + (p_{2} + \frac{\sqrt{k}}{2}w)x_{+} + i \sum_{n \neq 0} \frac{a_{n}^{(2)}}{n} e^{-inx_{+}} \\
\bar{\phi}_{2}(x_{-}) &= \frac{q_{2}}{2} + (p_{2} - \frac{\sqrt{k}}{2}w)x_{-} + i \sum_{n \neq 0} \frac{b_{n}^{(2)}}{n} e^{-inx_{-}}
\end{align*}
\]

with integer winding number \( w \). A canonical map from the free fields to the physical fields \( u, \bar{u} \) is then given by

\[
\begin{align*}
u[\Phi_{1}, \Phi_{2}] &= f_{1}(x_{+})\bar{f}_{1}(x_{-}) - f_{2}(x_{+})\bar{f}_{2}(x_{-}) \quad (2.11)
\end{align*}
\]

with building blocks

\[
\begin{align*}
f_{1}(x_{+}) &= e^{\frac{1}{\sqrt{k}}(\phi_{1}(x_{+}) + i\bar{\phi}_{2}(x_{+}))}, & f_{2}(x_{+}) &= A(x_{+})f_{1}(x_{+}) \\
f_{1}(x_{-}) &= e^{\frac{1}{\sqrt{k}}(\bar{\phi}_{1}(x_{-}) + i\phi_{2}(x_{-}))}, & f_{2}(x_{-}) &= \bar{A}(x_{-})\bar{f}_{1}(x_{-})
\end{align*}
\]

---
The screening charges $A(x_+), \bar{A}(x_-)$ are defined as follows. Introducing the quantities

$$V(x_+) = \frac{1}{\sqrt{k}} (\partial \phi_1(x_+) + i \partial \phi_2(x_+)) e^{-\frac{2}{\sqrt{k}} \phi_1(x_+)}$$

(2.13)

$$\bar{V}(x_-) = \frac{1}{\sqrt{k}} (\partial \bar{\phi}_1(x_-) + i \partial \bar{\phi}_2(x_-)) e^{-\frac{2}{\sqrt{k}} \bar{\phi}_1(x_-)}$$

(2.14)

the screening charges $A, \bar{A}$ are defined as solutions of the differential equations

$$\partial A(x_+) = V(x_+), \quad \partial \bar{A}(x_-) = \bar{V}(x_-)$$

(2.15)

The solutions to these equations are unique if we require the monodromy of the quantities $V, \bar{V}$ to be preserved under integration, i.e. with

$$V(x_+ + 2\pi) = e^{-\frac{4\pi}{\sqrt{k}} p_1} V(x_+)$$

(2.16)

we require the screening charges to have the same monodromy. One may check that the following expression for $A(x_+)$ solves this problem,

$$A(x_+) = \frac{2\pi}{2\sqrt{k} \sinh \frac{2\pi}{\sqrt{k} p_1}} \int_0^{2\pi} d\varphi \left( \partial \phi_1(x_+ + \varphi) + i \partial \phi_2(x_+ + \varphi) \right) e^{-\frac{2}{\sqrt{k}} \phi_1(x_+ + \varphi)}$$

(2.17)

The corresponding expression for $\bar{A}$ may be obtained by obvious replacements.

To motivate the discussion of the quantum mechanical reflection operator of the model to be discussed in subsection 3.9, it is essential to point out that the free fields parametrizing $u$ are in/out fields defined by

$$r(\sigma, t) \overset{t \rightarrow -\infty}{\sim} \frac{1}{\sqrt{k}} \Phi^{in}_1(\sigma, t), \quad \theta(\sigma, t) \overset{t \rightarrow -\infty}{\sim} \frac{1}{\sqrt{k}} \Phi^{in}_2(\sigma, t)$$

(2.18)

$$r(\sigma, t) \overset{t \rightarrow +\infty}{\sim} \frac{1}{\sqrt{k}} \Phi^{out}_1(\sigma, t), \quad \theta(\sigma, t) \overset{t \rightarrow +\infty}{\sim} \frac{1}{\sqrt{k}} \Phi^{out}_2(\sigma, t)$$

As the string propagates freely in the asymptotically flat region $r \rightarrow \infty$, the in/out fields defined in this fashion are indeed free fields as long as we consider solutions that satisfy $r_0(t) \rightarrow \infty$ for large positive and negative times, where $r_0(t)$ represents the zero mode of field $r(\sigma, t)$. Let us note that the free in-field $\Phi^{in}_1$ travelling towards the tip of the cigar at $r = 0$ has negative momentum zero-mode $p^{in}_1 < 0$, while the outgoing field $\Phi^{out}_1$ has momentum zero-mode $p^{out}_1 = - p^{in}_1 > 0$. It follows that the field $u$ has the asymptotic behaviour

$$u(\sigma, t) \overset{t \rightarrow -\infty}{\sim} e^{r + it} = e^{\frac{i}{\sqrt{k}} (\Phi^{in}_1 + \Phi^{in}_2)}$$

(2.19)

$$u(\sigma, t) \overset{t \rightarrow +\infty}{\sim} e^{r + it} = e^{\frac{i}{\sqrt{k}} (\Phi^{out}_1 + i \Phi^{out}_2)}$$

(2.20)

Now suppose we were given some solution $u(\sigma, t)$ of the dynamical equations that asymptotically behaves according to (2.19). The assertion is that the full interacting field $u(\sigma, t)$ for all times $t$ is given by

$$u(\sigma, t) = u[\Phi^{in}_1, \Phi^{in}_2] = u[\Phi^{out}_1, \Phi^{out}_2]$$

(2.21)
But this is easily proven by considering the asymptotic behaviour of \( u[\Phi_1^{in/out}, \Phi_2^{in/out}] \). Taking properly into account the dependence of the free-field exponentials on the momentum zero-mode of \( \Phi_1^{in/out} \) we find

\[
u[\Phi_1^{in}, \Phi_2^{in}] \overset{t \to -\infty}{\sim} e^{\frac{1}{\sqrt{k}}(\Phi_1^{in} + i\Phi_2^{in})}
\] (2.22)

and similarly

\[
u[\Phi_1^{out}, \Phi_2^{out}] \overset{t \to +\infty}{\sim} e^{\frac{1}{\sqrt{k}}(\Phi_1^{out} + i\Phi_2^{out})}
\] (2.23)

The term containing the screening charges is suppressed in both cases. Comparison with (2.19) leads us to conclude that the free fields parametrizing \( u \) are asymptotic \( in/out \) fields, depending on the sign of momentum zero-mode of \( \Phi_1 \). Finally, let us determine the classical “S-matrix” of our model, i.e. we would like to find some map between the \( in \) and the corresponding \( out \)-fields. The key to the construction of this map, which we call \( S \), is the fact that the \( in \) and the corresponding \( out \)-fields parametrize the same solution \( u(\sigma, t) \). For large positive times consider the asymptotics

\[
u[\Phi_1^{in}, \Phi_2^{in}] \overset{t \to +\infty}{\sim} -A(x_+)^\dagger A(x_-) e^{\frac{1}{\sqrt{k}}(\Phi_1^{in} + i\Phi_2^{in})}
\] (2.24)

Comparison with (2.23) yields the following relation between the \( in \) and the corresponding \( out \)-field:

\[
e^{\frac{1}{\sqrt{k}}(\Phi_1^{out} + i\Phi_2^{out})} = -A(x_+)^\dagger A(x_-) e^{\frac{1}{\sqrt{k}}(\Phi_1^{in} + i\Phi_2^{in})}
\] (2.25)

Similarly, we may derive the inverse relation

\[
e^{\frac{1}{\sqrt{k}}(\Phi_1^{in} + i\Phi_2^{in})} = -A(x_+)^\dagger A(x_-) e^{\frac{1}{\sqrt{k}}(\Phi_1^{out} + i\Phi_2^{out})}
\] (2.26)

What are the conclusions we can draw concerning the phase space of the Euclidean black hole model? To answer this question, introduce a splitting of the free field phase space \( \mathcal{P}^F \) according to \( \mathcal{P}^F = \mathcal{P}_+^F \oplus \mathcal{P}_-^F \), where \( \mathcal{P}_\pm^F \) corresponds to the subspace of free field configurations with positive resp. negative momentum zero-mode \( p_1 \). Then, \( S \) is a map \( S : \mathcal{P}_\pm^F \to \mathcal{P}_\mp^F \) by means of which an equivalence relation \( \sim \) may be introduced on \( \mathcal{P}^F \) identifying free field configurations which are images of each other under \( S \). As equivalent free field configurations parametrize the same solution \( u(\sigma, t) \) of the dynamical equations, the phase space of the Euclidean black hole model is isomorphic to \( \mathcal{P}^F / \sim \). In the following, our aim will be to carry over these considerations to the quantum mechanical model.

**Conserved classical currents**

As a classical field theory, the \( SL(2, \mathbb{R})/U(1) \) model is conformally invariant. Conformal transformations are generated by the (anti)chiral components of the energy momentum tensor

\[
T(x_+) = k \frac{\partial_+ \bar{u} \partial_+ u}{1 + uu}, \quad T(x_-) = k \frac{\partial_- \bar{u} \partial_- u}{1 + uu}
\] (2.27)
Interestingly, the Fourier modes \( l_n = \frac{1}{2\pi} \int_0^{2\pi} d\sigma T(\sigma)e^{in\sigma} \) of the energy momentum tensor satisfy a Poisson counterpart of the Virasoro algebra with vanishing central charge,

\[
\{l_n, l_m\} = i(n - m)l_{n+m} \tag{2.28}
\]

Substituting the free field parametrization into (2.27) transforms the components of the energy momentum tensor into that of a free theory,

\[
T^F(x_+) = (\partial \phi_1(x_+))^2 + (\partial \phi_2(x_+))^2, \quad \bar{T}^F(x_-) = (\partial - \bar{\phi}_1(x_-))^2 + (\partial - \bar{\phi}_2(x_-))^2 \tag{2.29}
\]

As the map to free fields preserves the symplectic structure of the theory, the modes of \( T^F \) satisfy the same Poisson Virasoro algebra as that of \( T \).

Besides the energy momentum tensor, another set of conserved currents can be found, as it is well known that upon reduction to the coset \( SL(2, \mathbb{R})/U(1) \) the classical Kac-Moody currents of the ungauged \( SL(2, \mathbb{R}) \) WZNW-model reduce to parafermionic coset currents. As argued for in [4], these parafermionic currents are given in terms of free fields as

\[
\psi^\pm(z) = i(\partial \phi_1(z) \pm i\partial \phi_2(z))e^{\mp 2\sqrt{\kappa} \phi_2(z)} \tag{2.30}
\]

It will turn out that a natural starting point for quantization is the quantization of these currents as it is known that the chiral algebra of the quantum mechanical \( SL(2, \mathbb{R})/U(1) \) model is generated by fusion of the quantum parafermions.

3. Quantization

**Quantum parafermions and energy momentum tensor**

Let us introduce (anti)holomorphic Euclidean free fields \( \phi_1(z), \phi_2(z), \bar{\phi}_1(z), \bar{\phi}_2(z) \) with short-distance behaviour

\[
\phi_k(z)\phi_l(w) = -\frac{1}{2} \delta_{kl} \ln(z - w), \quad \bar{\phi}_k(z)\bar{\phi}_l(w) = -\frac{1}{2} \delta_{kl} \ln(\bar{z} - \bar{w}) \tag{3.1}
\]

Then the quantum analoga of the classical parafermions turn out to be

\[
\psi^\pm(z) = i : \eta \partial \phi_1(z) \pm i\partial \phi_2(z) : e^{\mp 2\sqrt{\eta} \phi_2(z)} ; \tag{3.2}
\]

The corresponding antiholomorphic currents are obtained by obvious replacements, we will however restrict the discussion here to the holomorphic copy of the symmetry algebra. The deformation parameter \( \eta \) depends on the level \( k \) only and is given by

\[
\eta = \sqrt{\frac{k - 2}{k}} \tag{3.3}
\]

It turns out that the introduction of \( \eta \) in (3.2) is necessary for a consistent quantization and leads to the canonical form of the parafermion OPE as first introduced in [14]. Indeed, a standard calculation using Wick’s contraction theorem for free fields yields

\[
\psi^+(z)\psi^-(w) = (z - w)^{-2\Delta} \left\{ 1 + \frac{2\Delta}{c} (z - w)^2 T(w) + \mathcal{O}((z - w)^3) \right\} \tag{3.4}
\]
where $c$ and $\Delta_\psi$ are the central charge resp. the conformal weight of the parafermions, see below. The spin 2 current on the r.h.s. can be identified with the energy momentum tensor of the quantum theory. It has the free field parametrization

$$T(z) = -: (\partial \phi_1(z))^2 : -: (\partial \phi_2(z))^2 : - b \partial^2 \phi_1(z)$$

(3.5)

where we introduced the parameter

$$b = \frac{1}{\sqrt{k-2}}$$

(3.6)

A background charge $Q = -b$ that was absent classically, enters the quantum theory. Conformal symmetry in the quantum theory is generated by the Fourier-Laurent modes of $T(z)$ which satisfy a Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}$$

(3.7)

The central charge is found to be given in terms of the level $k$ as

$$c = \frac{3k}{k-2} - 1$$

(3.8)

This precisely coincides with what was found by Witten for the black hole coset model, see [1].

Having identified the energy momentum tensor of the quantum theory, we find that the parafermionic currents obey the following commutation relations with the modes of the energy momentum tensor,

$$[L_n, \psi^\pm(z)] = z^n(z\partial + \Delta_\psi(n + 1))\psi(z)$$

(3.9)

which implies that they are primary with conformal dimension $\Delta_\psi = 1 + \frac{1}{k}$.\n
The $W$-algebra of the Euclidean black hole model

A remark on the $W$-algebra of the Euclidean black hole CFT is in order, as it will turn out that the quantum counterpart of the map $S : \mathcal{P}_+^E \rightarrow \mathcal{P}_+^E$ intertwines certain representations of this algebra. The currents of this algebra appear on the r.h.s. of the OPE (3.4). Indeed, there appears an infinite set of integer spin currents $W_s(z)$ which constitute a closed nonlinear operator algebra. They furthermore transform as primary fields of the Virasoro algebra,

$$[L_n, W_s(z)] = z^n(z\partial + s(n + 1))W_s(z)$$

(3.10)

For example, including terms of order $(z - w)^3$ inside the curly bracket on the r.h.s. of (3.4), we find the OPE

$$\psi^+(z)\psi^-(w) = (z-w)^{-2\Delta_\psi} \left\{ 1 + \frac{2\Delta_\psi}{c}(z-w)^2 \left( T(w) + \frac{1}{2}(z-w)\partial T(w) \right) - (z-w)^3 W_3(w) + O((z-w)^4) \right\}$$

(3.11)
where the spin 3 current $W_3(z)$ is given in terms of free fields as

$$W_3(z) = \alpha : (\partial \phi_2(z))^3 : + \beta \partial^2 \phi_2(z) + \gamma : (\partial \phi_1(z))^2 \partial \phi_2(z) :$$
$$+ \delta \partial^2 \phi_2(z) \partial \phi_1(z) + \varepsilon \partial^2 \phi_1(z) \partial \phi_2(z)$$

(3.12)

with coefficients given as functions of $k$ by

$$\alpha = \frac{2i^{3k-4}}{k^{k-1}} \quad \beta = \frac{i}{k} \quad \gamma = \frac{2i^{k-2}}{k^{k-1}}$$
$$\delta = i^{k-2} \sqrt{\frac{k-2}{k}} \quad \epsilon = -i^{k-2} \sqrt{\frac{k-2}{k}}$$

(3.13)

For more details on the W-algebra of the Euclidean black hole model, see for example [3], let us just add one important property of the currents of this algebra. It turns out that the structure of the OPE implies that the currents $W_s(z)$ transform under parity $\phi_2 \rightarrow -\phi_2$ as

$$W_s[\phi_1, -\phi_2] = (-)^s W_s[\phi_1, \phi_2]$$

(3.14)

We will see that using this relation one may show that certain representations of the W-algebra are unitarily equivalent, which implies unitarity of the quantum mechanical reflection operator.

**Free bosonic quantum fields in Minkowskian $\mathbb{R} \times S^1$**

As the quantization of the Euclidean black hole fields $u, \bar{u}$ will be done throughout in the Minkowskian formulation of CFT, we shall briefly discuss the quantization of the free bosons $\Phi_1, \Phi_2$ on the Minkowskian cylinder $\mathbb{R} \times S^1$. The chiral field $\phi_1(x_+)$ and its antichiral counterpart corresponding to the holomorphic resp. antiholomorphic Euclidean fields introduced in the previous subsection have the well-known mode expansion

$$\phi_1(x_+) = q_1 + p_1 x_+ + i \sum_{n \neq 0} \frac{a_n^{(1)}}{n} e^{-inx_+}$$

(3.15)

$$\bar{\phi}_1(x_-) = q_1 + p_1 x_- + i \sum_{n \neq 0} \frac{b_n^{(1)}}{n} e^{-inx_-}$$

(3.16)

For the chiral and antichiral components of compactified boson $\Phi_2$, we introduce commuting copies of the zero-mode algebra for the left and right-moving sectors. The mode expansion then reads

$$\phi_2(x_+) = q_2^L + p_2^L x_+ + i \sum_{n \neq 0} \frac{a_n^{(2)}}{n} e^{-inx_+}$$

(3.17)

$$\bar{\phi}_2(x_-) = q_2^R + p_2^R x_- + i \sum_{n \neq 0} \frac{b_n^{(1)}}{n} e^{-inx_-}$$

(3.18)

The nonvanishing commutators for the zero-modes are

$$[q_1, p_1] = \frac{i}{2} \quad [q_2^L, p_2^L] = \frac{i}{2} \quad [q_2^R, p_2^R] = \frac{i}{2}$$

(3.19)
and for the oscillators

\[ [a_k^{(i)}, a_l^{(j)}] = \frac{k}{2} \delta_{ij} \delta_{n+m,0} , \quad [b_k^{(i)}, b_l^{(j)}] = \frac{k}{2} \delta_{ij} \delta_{n+m,0} \quad (3.20) \]

The commutation relations have to be supplemented by hermiticity relations, which for the oscillators and zero modes read

\[
q_1^\dagger = q_1, \quad (q_{2L/R}^\dagger)^\dagger = q_{2L/R}^\dagger \\
p_1^\dagger = p_1, \quad (p_{2L/R}^\dagger)^\dagger = p_{2L/R}^\dagger \\
(a_n^{(k)})^\dagger = a_{-n}^{(k)}, \quad (b_n^{(k)})^\dagger = b_{-n}^{(k)}
\]  

(3.21)

The Hilbert space on which the oscillators and zero-modes take their action admits a decomposition into Fock modules which diagonalize the action of the zero modes \( p_1, p_{2L/R} \).

The compactification of free boson \( \Phi_2 \) implies that \( p_{2L/R} \) have discrete spectra

\[
\text{spec } p_{2L} = \left\{ \frac{1}{2\sqrt{k}} (m + nk) | m, n \in \mathbb{Z} \right\} \\
\text{spec } p_{2R} = \left\{ \frac{1}{2\sqrt{k}} (m - nk) | m, n \in \mathbb{Z} \right\}
\]

(3.22)

(3.23)

whereas the momentum zero mode \( p_1 \) of uncompactified free boson \( \Phi_1 \) has real continuous spectrum. We note that \( m \) resp. \( n \) correspond to the momentum resp. winding number in the compactified \( \theta \)-direction in asymptotic regions of the cigar and correspond to the eigenvalues of the operators

\[
m = \sqrt{k}(p_{2L}^2 + p_{2R}^2), \quad n = \frac{1}{\sqrt{k}}(p_{2L}^2 - p_{2R}^2)
\]

(3.24)

The decomposition of the Hilbert space \( \mathcal{H}^F \) into charged Fock modules then reads

\[
\mathcal{H}^F = \bigoplus_{m, n \in \mathbb{Z}} \int_{j=-1/2+i\rho}^\oplus dj \mathcal{F}^j_{mn} \otimes \bar{\mathcal{F}}^j_{mn}
\]

(3.25)

Here, \( j = -\frac{1}{2} + i\rho, \rho \in \mathbb{R} \) for \( k > 2 \) corresponds to the eigenvalue of the operator

\[
j = -\frac{1}{2} + ib^{-1}p_1
\]

and may be interpreted as the spin of the \( SL(2, \mathbb{R}) \) representation in the principal continuous series. We may furthermore observe that the operator \( a_0 = -ibj = p_1 + \frac{ib}{2} \) corresponds to the momentum zero mode of the Euclidean free boson \( \phi_1(z), \bar{\phi}_1(\bar{z}) \), where the shift is due to the presence of the background charge.

The Fock module \( \mathcal{F}^j_{mn} \otimes \bar{\mathcal{F}}^j_{mn} \) is spanned by acting with the \( a_{-n}^{(k)}, b_{-n}^{(k)} \) on the vacuum state \( |jmn\rangle \), which is annihilated by the oscillators \( a_n^{(k)}, b_n^{(k)} \) with \( n > 0 \). For Fock states \( f \) in the Fock module \( \mathcal{F}^j_{mn} \otimes \bar{\mathcal{F}}^j_{mn} \) we introduce the notation \( |jmn, f\rangle \), where we identify \( |jmn\rangle \equiv |jmn, \Omega\rangle \), with \( \Omega \) being the Fock vacuum, and \( a_{-n}^{(k)}|jmn\rangle = |jmn, a_{-n}^{(k)}\Omega\rangle \).
We conclude this subsection by considering the spectrum of the $L_0, \bar{L}_0$ generators which, taking into account the free field representation of the energy momentum tensor, may be written

$$L_0 = p_1^2 + \frac{b_1^2}{4} + (p_2^L)^2 + 2 \sum_{k>0} (a_{-k}^1 a_k^1 + a_{-k}^2 a_k^2)$$

$$\bar{L}_0 = p_1^2 + \frac{b_1^2}{4} + (p_2^R)^2 + 2 \sum_{k>0} (b_{-k}^1 b_k^1 + b_{-k}^2 b_k^2)$$

(3.26) (3.27)

It immediately follows that their spectra are given by

$$\text{spec } L_0 = \left\{ -j \left( \frac{j+1}{k} \right) + \frac{(m+nk)^2}{4k} + N \mid j \in -\frac{1}{2} + i\mathbb{R}; m, n \in \mathbb{Z}, N \in \mathbb{N} \cup \{0\} \right\}$$

$$\text{spec } \bar{L}_0 = \left\{ -j \left( \frac{j+1}{k} \right) + \frac{(m-nk)^2}{4k} + \bar{N} \mid j \in -\frac{1}{2} + i\mathbb{R}; m, n \in \mathbb{Z}, \bar{N} \in \mathbb{N} \cup \{0\} \right\}$$

(3.28) (3.29)

This coincides with the spectra of the $L_0, \bar{L}_0$ generators derived in [2] in the point-particle limit.

**Quantum counterparts of Euclidean black hole fields**

We now wish to construct the quantum counterparts of the classical building blocks $f_1(x_+), f_2(x_+)$. The basic building blocks $E_n^{(k)}(x_+) =: e^{2\phi_n(x_+)}$ of this construction are the normal ordered exponentials which are discussed in appendix A. As the quantum counterpart of $f_1$ we define

$$f_1(x_+) = E^{(1)}_{\frac{x_+}{2}} e^{\frac{2b}{x_+}}$$

(3.30)

Up to the chosen deformation of the coupling of $\phi_1$ in the exponential, the relation to the classical building block is quite obvious. The construction of the quantum counterpart of building block $f_2(x_+)$ is a little bit more involved. Let us first define the quantum counterpart of the classical screening charge $A(x_+)$. In analogy to the classical theory we require the quantized screening charge $A(x_+)$ to satisfy the differential equation

$$\partial A(x_+) = V(x_+)$$

(3.31)

where $V(x_+)$ is the screening current which in terms of chiral free fields is given by

$$V(x_+) = \frac{1}{\sqrt{k}} : (\eta \partial \phi_1(x_+) + i \partial \phi_2(x_+)) e^{-2b\phi_1(x_+)} :$$

(3.32)

The deformation of the coupling in the exponential as compared to the classical expression $\left(\frac{1}{\sqrt{k}} \rightarrow \frac{1}{\sqrt{k-2}}\right)$ is required to ensure vanishing of the conformal dimension of $A(x_+)$. The deformation parameter $\eta$ ensures that the screening current $V$ obeys a simple (parafermionic) exchange relation with itself.

The solution to the differential equation (3.31) is unique if we require monodromy to be preserved under integration. Taking into account the ordering of the zero-modes, we find the monodromy of $V(x_+)$ to be given by

$$V(x_+ + 2\pi) = e^{-4\pi b(b_1 - \frac{4}{2})} V(x_+)$$

(3.33)
The unique solution to (3.31) preserving monodromy is then given by

\[ A(x_+) = \frac{e^{2\pi b(p_1 - \frac{ib}{2})}}{2i \sin \pi b(2ip_1 + b)} Q(x_+) \]  

(3.34)

where

\[ Q(x_+) = \int_0^{2\pi} d\varphi V(x_+ + \varphi) \]  

(3.35)

The quantum counterpart of \( f_2 \) may then be defined as

\[ f_2(x_+) = A(x_+)f_1(x_+) \]  

(3.36)

However, we have to analyze carefully the arising short-distance singularities upon forming this product. To this purpose introduce the bilocal field

\[ f_2(x'_+, x_+) = A(x'_+)f_2(x_+) \]  

(3.37)

and analyze its behaviour as \( x'_+ \) approaches \( x_+ \). We first integrate the total derivative appearing in the integrand of \( A(x_+) \) and obtain

\[ f_2(x'_+, x_+) = -\frac{\eta^2}{2} E^{(1)}_{-b}(x'_+)f_1(x_+) + \frac{e^{2\pi b(p_1 - \frac{ib}{2})}}{2i \sin \pi b(2ip_1 + b)} \tilde{Q}(x_+)f_1(x_+) \]  

(3.38)

with

\[ \tilde{Q}(x_+) = \frac{i}{\sqrt{k}} \int_0^{2\pi} d\varphi : \partial \phi_2(x_+ + \varphi)e^{-2b\phi_2(x_+ + \varphi)} : \]  

(3.39)

The short-distance behaviour of the first summand on the r.h.s. is of special interest. According to eq. (A.5) it is given by

\[ E^{(1)}_{-b}(x'_+)f_1(x_+) = e^{\frac{i\pi b^2}{2}(x'_+-x_+)|1 - e^{-i(x'_+-x_+)|b^2} : E^{(1)}_{-b}(x'_+)f_1(x_+) : \]  

(3.40)

Here \( \epsilon(x) \) denotes the stairstep function defined by

\[ \epsilon(x) = 2n + 1, \quad 2\pi n < x < 2\pi(n + 1), \quad n \in \mathbb{Z} \]  

(3.41)

For \( k > 2 \) resp. real \( b \) from (3.40) we read off

\[ \lim_{x'_+ \to x_+} E^{(1)}_{-b}(x'_+)f_1(x_+) = 0 \]  

(3.42)

which leads us to conclude

\[ f_2(x_+) = \lim_{x'_+ \to x_+} A(x'_+)f_2(x_+) = \frac{e^{2\pi b(p_1 - \frac{ib}{2})}}{2i \sin \pi b(2ip_1 + b)} \tilde{Q}(x'_+)f_1(x_+) \]  

(3.43)
This may be seen as a kind of “gauge degree of freedom” in the sense that adding a total derivative of the dimensionless field \( e^{-2b\phi_1} \) to the integrand of the screening charge \( \tilde{Q} \) leaves physics unaltered as \( f_2 \) remains invariant under the “gauge transformation”, i.e. upon constructing the building block \( f_2 \), the screening charges \( Q, \tilde{Q} \) are equivalent. We will benefit from this kind of “gauge invariance” when we calculate the exchange algebra of the building blocks in the next subsection.

Note that our definition of \( f_2 \) makes sense if the integrand of the screening charge \( Q \) develops an integrable short-distance singularity with \( f_1 \). Our result is that \( f_2 \) is well-defined for \( k > 2 \), in which case the following identity holds:

\[
Q(x_+)f_1(x_+) = e^{rac{i\pi b_2}{2}} \int_0^{2\pi} d\varphi [1 - e^{-i\varphi}b^2] : V(x_+ + \varphi)f_1(x_+) : \quad (3.44)
\]

It remains to define the quantum counterparts of the antichiral building blocks. Quite obviously, the quantum counterpart of \( \tilde{f}_1(x_-) \) should be defined as

\[
\tilde{f}_1(x_-) = E(1)_{k}(x_-)E^{(2)}_{\lambda_0}(x_-) \quad (3.45)
\]

The quantum counterpart of the antichiral building block \( \tilde{f}_2(x_-) \) will then be defined as

\[
\tilde{f}_2(x_-) = \tilde{f}_1(x_-)\tilde{A}(x_-) \quad (3.46)
\]

where the antichiral screening charge satisfies the differential equation

\[
\partial \tilde{A}(x_-) = \tilde{V}(x_-) = \frac{1}{\sqrt{k}} : (\eta \partial \tilde{\phi}_1(x_-) + i \partial \tilde{\phi}_2(x_-))e^{-2b\phi_1(x_-)} : \quad (3.47)
\]

The monodromy preserving solution to the above equation is given by

\[
\tilde{A}(x_-) = \int_0^{2\pi} d\varphi \tilde{V}(x_- + \varphi) e^{\frac{2\pi b_2 (p_1 + \frac{ib}{2})}{2i \sin \pi b(2ip_1 - b)}} \quad (3.48)
\]

Again, the so defined building block \( \tilde{f}_2 \) will be well defined for \( k > 2 \), in case of which we have

\[
\tilde{f}_1(x_-) \int_0^{2\pi} d\varphi \tilde{V}(x_- + \varphi) = e^{-i\varphi^2} \int_0^{2\pi} d\varphi [1 - e^{i\varphi b^2}] : \tilde{f}_1(x_-)\tilde{V}(x_- + \varphi) : \quad (3.49)
\]

Note that the respective operator orderings chosen for the building blocks \( f_2, \tilde{f}_2 \) are fixed by requiring locality of the quantum counterpart of the fundamental field \( u(\sigma, t) \), which, having identified the quantum counterparts of our building blocks, may be written

\[
u(\sigma, t) = f_1(x_+)e^{-bq_1}\tilde{f}_1(x_-) - f_2(x_+)e^{bq_1}\tilde{f}_2(x_-) \quad (3.50)
\]

The quantum counterpart of the fundamental field \( \tilde{u}(\sigma, t) \) is then defined by \( \tilde{u}(\sigma, t) = u^\dagger(\sigma, t) \).
Conformal covariance

The fundamental quantum fields $u, \bar{u}$ will transform covariantly, as primary fields of the Virasoro algebra, given that the (anti)chiral building blocks $f_k, \bar{f}_k$ transform covariantly with equal conformal dimensions. This in turn holds if the screening charges $Q(x_+), Q(x_-)$ have vanishing conformal dimensions, which we will verify now. First of all one may note that the building block $f_1$ transforms due to (A.9) as

$$[L_n, f_1(x_+)] = e^{inx_+}(-i\partial_+ + n\Delta)f_1(x_+)$$

with conformal dimension

$$\Delta = -\frac{3}{4} + \frac{1}{4k}$$

The building block $f_2(x_+)$ will then transform covariantly with the same conformal dimension given that the screening charge $Q(x_+)$ has vanishing conformal dimension. Indeed, the screening current $V(x_+)$ is primary of conformal dimension equal to one, i.e. it obeys the commutation relation

$$[L_n, V(x_+)] = e^{inx_+}(-i\partial_+ + n)V(x_+)$$

which implies that the screening charge $Q$ transforms as a primary field of vanishing dimension according to

$$[L_n, Q(x_+)] = -ie^{inx_+}\partial_+ Q(x_+)$$

One may argue analogously for the antichiral building blocks. It immediately follows that $u$ is primary,

$$[L_n, u(\sigma,t)] = e^{inx_+}(-i\partial_+ + n\Delta)u(\sigma,t)$$

$$[\bar{L}_n, u(\sigma,t)] = e^{inx_-}(-i\partial_- + n\Delta)u(\sigma,t)$$

w.r.t. the chiral and antichiral copies of the Virasoro algebra, with conformal dimension $\Delta$ given by (3.52).

The braid algebra and proof of locality

Locality of the quantum field $u$ will be established by means of an exchange algebra satisfied by the chiral building blocks $f_1, f_2$ and their antichiral companions. We claim that the following relations hold ($i \neq j$):

$$f_i(\sigma_1)f_j(\sigma_2) = f_j(\sigma_2)f_i(\sigma_1)C_{ji}^{ij}(\sigma_2 - \sigma_1, j) + f_i(\sigma_2)f_j(\sigma_1)C_{ij}^{ij}(\sigma_2 - \sigma_1, j)$$

and correspondingly for the antichiral building blocks

$$\bar{f}_i(-\sigma_1)\bar{f}_j(-\sigma_2) = \bar{f}_j(-\sigma_2)\bar{f}_i(-\sigma_1)\bar{C}_{ji}^{ij}(\sigma_1 - \sigma_2, j) + \bar{f}_i(-\sigma_2)\bar{f}_j(-\sigma_1)\bar{C}_{ij}^{ij}(\sigma_1 - \sigma_2, j)$$
The coefficients $C_{kl}^{ij}$ of the braiding matrix are given by

$$C_{21}^{12}(\sigma, j) = q^{(1/2)(1-\epsilon(\sigma))} \frac{[2j+2][2j]}{[2j+1]^2} \quad C_{21}^{21}(\sigma, j) = q^{(1/2)(1-\epsilon(\sigma))}$$

$$C_{12}^{12}(\sigma, j) = q^{1/2(\epsilon(\sigma))} \frac{q^{2\epsilon(\sigma)}}{[2j+1]^2} \quad C_{12}^{21}(\sigma, j) = -q^{1/2(\epsilon(\sigma))} \frac{q^{(-2j-2)\epsilon(\sigma)}}{[2j+1]}$$

Here we have introduced $q$-deformed numbers defined by $(q = e^{i\pi b^2})$

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$$

The coefficients of the antichiral exchange relations are related to the coefficients of the corresponding chiral exchange relations in a simple way:

$$C_{12}^{12} = C_{12}^{12}, \quad C_{21}^{21} = C_{21}^{21}, \quad C_{12}^{12} = C_{12}^{12}, \quad C_{12}^{21} = C_{21}^{12}$$

The exchange relations simplify for $i = j$ where we have

$$f_i(\sigma_1)f_i(\sigma_2) = q^{1/2(\epsilon(\sigma_2-\sigma_1))}f_i(\sigma_2)f_i(\sigma_1) \quad (3.62)$$

$$\bar{f}_i(-\sigma_1)\bar{f}_i(-\sigma_2) = q^{1/2(\epsilon(\sigma_1-\sigma_2))}\bar{f}_i(-\sigma_2)\bar{f}_i(-\sigma_1) \quad (3.63)$$

The details of the proof of the braid algebra can be found in appendix C. Having established the braid algebra, we may now prove locality of the Euclidean black hole fields. First of all note that application of the braid algebra on the r.h.s. of (3.57) leads to the following consistency conditions on the coefficients of the braid relations:

$$C_{ij}^{ij}(\sigma,j)C_{ji}^{ij}(-\sigma,j) + C_{ji}^{ij}(\sigma,j)C_{ij}^{ij}(-\sigma,j) = 1$$

$$C_{ji}^{ij}(\sigma,j)C_{ij}^{ij}(-\sigma,j) + C_{ij}^{ij}(\sigma,j)C_{ji}^{ij}(-\sigma,j) = 0$$

(3.64)

Corresponding relations hold for the coefficients $\bar{C}_{kl}^{ij}$ of the antichiral braid algebra. We will now see that the consistency conditions (3.64) and the relation (3.61) between the coefficients of the chiral and antichiral braid algebra suffice to establish locality of the quantum fields $u, \bar{u}$. Consider the operator product

$$u(\sigma_1, 0)u(\sigma_2, 0) = f_{11}(\sigma_1, \sigma_2) - f_{12}(\sigma_1, \sigma_2) - f_{21}(\sigma_1, \sigma_2) + f_{22}(\sigma_1, \sigma_2) \quad (3.65)$$

where we have introduced bilocal fields $f_{ij}$ according to

$$f_{11}(\sigma_1, \sigma_2) = f_1(\sigma_1)e^{-b\sigma_1}f_1(-\sigma_1)f_1(\sigma_2)e^{-b\sigma_2}f_1(-\sigma_2)$$

$$f_{22}(\sigma_1, \sigma_2) = f_2(\sigma_1)e^{b\sigma_1}\bar{f}_2(-\sigma_1)f_2(\sigma_2)e^{b\sigma_2}\bar{f}_2(-\sigma_2)$$

$$f_{12}(\sigma_1, \sigma_2) = f_1(\sigma_1)e^{-b\sigma_1}\bar{f}_1(-\sigma_1)f_2(\sigma_2)e^{b\sigma_2}\bar{f}_2(-\sigma_2)$$

$$f_{21}(\sigma_1, \sigma_2) = f_2(\sigma_1)e^{b\sigma_1}\bar{f}_2(-\sigma_1)f_1(\sigma_2)e^{-b\sigma_2}\bar{f}_1(-\sigma_2)$$

(3.66)

and we wish to show $u(\sigma_1, 0)u(\sigma_2, 0) = u(\sigma_2, 0)u(\sigma_1, 0)$. First of all let us note that by means of (3.62) it immediately follows that

$$f_{11}(\sigma_1, \sigma_2) = f_{11}(\sigma_2, \sigma_1)$$

$$f_{22}(\sigma_1, \sigma_2) = f_{22}(\sigma_2, \sigma_1)$$

(3.67)
Exchange relations for the remaining terms in (3.65) can be obtained by means of the exchange algebra (3.57) resp. (3.58), but it is first of all useful to note that the operator combinations $e^{-b q_1} f_1(-\sigma_1)$, $f_2(\sigma_2) e^{b q_1}$, $e^{b q_1} \tilde{f}_2(-\sigma_1)$ and $f_1(\sigma_2) e^{-b q_1}$ appearing in $f_{12}, f_{21}$ have no dependence on the zero-mode $q_1$. Therefore, the chiral objects commute with the corresponding antichiral objects. We may therefore change their ordering and rewrite the bilocal fields $f_{12}, f_{21}$ according to

$$
\begin{align*}
    f_{12}(\sigma_1, \sigma_2) &= f_1(\sigma_1) f_2(\sigma_2) \times \tilde{f}_1(-\sigma_1) \tilde{f}_2(-\sigma_2) \\
    f_{21}(\sigma_1, \sigma_2) &= f_2(\sigma_1) \tilde{f}_1(\sigma_2) \times f_1(-\sigma_1) \tilde{f}_2(-\sigma_2)
\end{align*}
$$

We then apply the exchange algebra (3.57) and (3.58) on the r.h.s. and find

$$
\begin{align*}
    f_{12}(\sigma_1, \sigma_2) &= \left( f_2(\sigma_2) f_1(\sigma_1) C_{21}^{12}(\sigma_2 - \sigma_1 j) + f_1(\sigma_2) f_2(\sigma_1) C_{12}^{12}(\sigma_2 - \sigma_1, j) \right) \\
    &\times \left( \tilde{f}_2(-\sigma_2) \tilde{f}_1(-\sigma_1) \bar{C}_{21}^{12}(\sigma_1 - \sigma_2, j) + \tilde{f}_1(-\sigma_2) \tilde{f}_2(-\sigma_1) \bar{C}_{12}^{12}(\sigma_1 - \sigma_2, j) \right) 
\end{align*}
$$

Now, since the $C_{kl}^{ij}(\sigma, j)$, $\bar{C}_{kl}^{ij}(\sigma, j)$ commute with the operator products $f_1 f_2$, $f_2 f_1$ and $\tilde{f}_1 \tilde{f}_2$, $\tilde{f}_2 \tilde{f}_1$ this may be rewritten as

$$
\begin{align*}
    f_{12}(\sigma_1, \sigma_2) &= f_{21}(\sigma_2, \sigma_1) C_{21}^{12}(\sigma_2 - \sigma_1 j) \bar{C}_{21}^{12}(\sigma_1 - \sigma_2, j) + f_{12}(\sigma_2, \sigma_1) C_{12}^{12}(\sigma_2 - \sigma_1, j) \bar{C}_{12}^{12}(\sigma_1 - \sigma_2, j) \\
    &+ f_1(\sigma_2) f_2(\sigma_1) \tilde{f}_2(-\sigma_2) \tilde{f}_1(-\sigma_1) C_{21}^{12}(\sigma_2 - \sigma_1, j) \bar{C}_{21}^{12}(\sigma_1 - \sigma_2, j) \\
    &+ f_2(\sigma_2) f_1(\sigma_1) \tilde{f}_1(-\sigma_2) \tilde{f}_2(-\sigma_1) C_{12}^{12}(\sigma_2 - \sigma_1 j) \bar{C}_{12}^{12}(\sigma_1 - \sigma_2, j) 
\end{align*}
$$

Correspondingly, we find the exchange relation

$$
\begin{align*}
    f_{21}(\sigma_1, \sigma_2) &= f_{12}(\sigma_2, \sigma_1) C_{12}^{21}(\sigma_2 - \sigma_1 j) \bar{C}_{12}^{21}(\sigma_1 - \sigma_2, j) + f_{21}(\sigma_2, \sigma_1) C_{21}^{21}(\sigma_2 - \sigma_1, j) \bar{C}_{21}^{21}(\sigma_1 - \sigma_2, j) \\
    &+ f_2(\sigma_2) f_1(\sigma_1) \tilde{f}_1(-\sigma_2) \tilde{f}_2(-\sigma_1) C_{21}^{21}(\sigma_2 - \sigma_1, j) \bar{C}_{21}^{21}(\sigma_1 - \sigma_2, j) \\
    &+ f_1(\sigma_2) f_2(\sigma_1) \tilde{f}_2(-\sigma_2) \tilde{f}_1(-\sigma_1) C_{12}^{21}(\sigma_2 - \sigma_1 j) \bar{C}_{12}^{21}(\sigma_1 - \sigma_2, j)
\end{align*}
$$
It follows that the operator product \( u(\sigma_1,0)u(\sigma_2,0) \) may be rewritten according to

\[
u(\sigma_1,0)u(\sigma_2,0) = f_{11}(\sigma_2,\sigma_1) + f_{22}(\sigma_2,\sigma_1) \\
- f_{12}(\sigma_2,\sigma_1) \left( C^{12}_{12}(\sigma_2 - \sigma_1,j)\bar{C}^{12}_{12}(\sigma_1 - \sigma_2,j) + C^{21}_{12}(\sigma_2 - \sigma_1)\bar{C}^{21}_{12}(\sigma_1 - \sigma_2) \right) \\
- f_{21}(\sigma_2,\sigma_1) \left( C^{12}_{21}(\sigma_2 - \sigma_1,j)\bar{C}^{12}_{21}(\sigma_1 - \sigma_2,j) + C^{21}_{21}(\sigma_2 - \sigma_1)\bar{C}^{21}_{21}(\sigma_1 - \sigma_2) \right) \\
- f_{1}(\sigma_2)f_{2}(\sigma_1)f_1(-\sigma_2)\bar{f}_1(-\sigma_1) \left( C^{12}_{12}(\sigma_2 - \sigma_1,j)\bar{C}^{12}_{12}(\sigma_1 - \sigma_2,j) + C^{21}_{12}(\sigma_2 - \sigma_1)\bar{C}^{21}_{12}(\sigma_1 - \sigma_2) \right) \\
- f_{2}(\sigma_2)f_1(\sigma_1)f_1(-\sigma_2)\bar{f}_1(-\sigma_1) \left( C^{12}_{21}(\sigma_2 - \sigma_1,j)\bar{C}^{12}_{21}(\sigma_1 - \sigma_2,j) + C^{21}_{21}(\sigma_2 - \sigma_1)\bar{C}^{21}_{21}(\sigma_1 - \sigma_2) \right)
\]

(3.72)

The field \( u \) thus satisfies the canonical equal-time commutation relation given that the following equations are satisfied:

\[
C^{12}_{12}(\sigma_2 - \sigma_1,j)\bar{C}^{12}_{12}(\sigma_1 - \sigma_2,j) + C^{21}_{12}(\sigma_2 - \sigma_1)\bar{C}^{21}_{12}(\sigma_1 - \sigma_2) = 1 \\
C^{12}_{21}(\sigma_2 - \sigma_1,j)\bar{C}^{12}_{21}(\sigma_1 - \sigma_2,j) + C^{21}_{21}(\sigma_2 - \sigma_1)\bar{C}^{21}_{21}(\sigma_1 - \sigma_2) = 1 \\
C^{12}_{12}(\sigma_2 - \sigma_1,j)\bar{C}^{12}_{21}(\sigma_1 - \sigma_2,j) + C^{21}_{12}(\sigma_2 - \sigma_1)\bar{C}^{21}_{21}(\sigma_1 - \sigma_2) = 0 \\
C^{12}_{21}(\sigma_2 - \sigma_1,j)\bar{C}^{12}_{21}(\sigma_1 - \sigma_2,j) + C^{21}_{21}(\sigma_2 - \sigma_1)\bar{C}^{21}_{21}(\sigma_1 - \sigma_2) = 0
\]

(3.73)

Taking into account the identities (3.61) relating the exchange algebra coefficients of the anichiral building-blocks to those of the chiral building-blocks, we find that the set of equations (3.73) reduces to the consistency-conditions (3.64), and the proof is complete. The proof of the equal time commutator \([\bar{u}(\sigma_1,t),u(\sigma_2,t)] = 0\) is then trivial since \(\bar{u}(\sigma,t) = u^\dagger(\sigma,t)\).

### Matrix elements

In this subsection we will consider the matrix elements of the Euclidean black hole fields between vacua \(|jmn\rangle\) in the Fock module \(\mathcal{F}^j_{mn} \otimes \mathcal{F}^j_{mn}\). Knowledge of these vacuum matrix elements will be crucial for the derivation of the reflection operator of the Euclidean black hole model to be presented in the next subsection. Upon calculating the matrix elements, we have to carefully keep track of the zero mode exponentials which shift the values of \(j,m,n\) when applied to the vacuum state \(|jmn\rangle\). Then, keeping in mind formula (3.44) for \(f_2\) (and the corresponding expression for \(\bar{f}_2\)), it turns out that the vacuum matrix element of \(u\) for \(t = 0\) may be written

\[
\langle j'm'n'|u(\sigma,0)|jmn\rangle = e^{i\sigma} \delta_{m',m+1} \delta_{n'n'} \left( \delta(j'-j-\frac{1}{2}) + \frac{1}{k^2} \left( j - \frac{1}{2}(m+nk) \right) \left( j - \frac{1}{2}(m-nk) \right) B(j) \bar{B}(j) \delta(j'-j+\frac{1}{2}) \right)
\]

(3.74)
where $\mathcal{B}(j), \mathcal{B}(j)$ are given by

$$\mathcal{B}(j) = \frac{e^{-i\pi b^2(2j+1)}}{2i \sin \pi b^2(2j+1)} \int_0^{2\pi} d\varphi \ (1 - e^{-i\varphi})^b \ e^{i\varphi b^2(2j+1)} \quad (3.75)$$

$$\mathcal{B}(j) = \frac{e^{-2i\pi b^2 j}}{2i \sin 2\pi b^2 j} \int_0^{2\pi} d\varphi \ (1 - e^{i\varphi})^b \ e^{i\varphi b^2 2j} \quad (3.76)$$

By a simple contour deformation, the integrals in the above equations may be related to the integral representation of the Euler beta function, giving the result

$$\int_0^{2\pi} d\varphi \ (1 - e^{-i\varphi})^b \ e^{i\varphi b^2(2j+1)} = 2\pi \Gamma(1 + b^2) \frac{e^{i\pi b^2(2j+1)}}{\Gamma(1 - 2b^2 j) \Gamma(1 + b^2(2j+1))} \quad (3.77)$$

$$\int_0^{2\pi} d\varphi \ (1 - e^{i\varphi})^b \ e^{i\varphi b^2 2j} = 2\pi \Gamma(1 + b^2) \frac{e^{2i\pi b^2 j}}{\Gamma(1 - 2b^2 j) \Gamma(1 + b^2(2j+1))} \quad (3.78)$$

This finally yields for $\mathcal{B}(j), \mathcal{B}(j)$ the expressions

$$\mathcal{B}(j) = i \Gamma(1 + b^2) \frac{\Gamma(-b^2(2j+1))}{\Gamma(1 - 2b^2 j)} \ , \quad \mathcal{B}(j) = -i \Gamma(1 + b^2) \frac{\Gamma(2b^2 j)}{\Gamma(1 + b^2(2j+1))} \quad (3.79)$$

where we have used the reflection property of the Gamma function, $\Gamma(x)\Gamma(1 - x) = \pi / \sin \pi x$.

**The reflection operator**

The quantum counterpart of $S : \mathcal{P}^F_\pm \rightarrow \mathcal{P}^F_\pm$ should be a unitary operator that maps $p_1$ into $-p_1$ (resp. $j$ into $-j - 1$) but leaves $u$ invariant,

$$SuS^{-1} = u \quad (3.80)$$

Consistency with the symmetry algebra of the model implies that $S$ commutes with the modes of the W-algebra currents, a suitable ansatz for $S$ is therefore

$$S = \mathcal{P}_1 R \quad (3.81)$$

where $\mathcal{P}_1$ denotes the parity operator on the zero mode space of free boson $\Phi_1$, and $R$ is an intertwiner between representations of the W-algebra,

$$RW_{s,n}(j,m,n) = W_{s,n}(-j-1,m,n) R \quad (3.82)$$

The operator $R$ will be unitary if the following conditions are satisfied. First of all, since $R$ commutes with $L_0, \bar{L}_0$, the Fock vacuum $|jmn\rangle$ should be an eigenstate of $R$, i.e. we require

$$R|jmn\rangle = R(j,m,n)|jmn\rangle \quad (3.83)$$
The operator $R$ will then be unitary iff $|R(j, m, n)|^2 = 1$ and the norm of any state in the $W$-algebra module $\mathcal{W}_{mn}^j$ is invariant under $j \to -j - 1$, i.e. if we have

$$||W_{s_1, -n_1}(j, m, n) \cdots W_{s_k, -n_k}(j, m, n)\Omega|| =$$

$$||W_{s_1, -n_1}(-j - 1, m, n) \cdots W_{s_k, -n_k}(-j - 1, m, n)\Omega||$$  \hspace{1cm} (3.84)

where the norm is induced by the scalar product in $\mathcal{W}_{mn}^j$ which is defined by means of the hermiticity relations

$$W_{s,n}^j = W_{s,-n}^j$$  \hspace{1cm} (3.85)

and the fact that the Fock vacua $|jmn\rangle$ are highest weight states of the $W$-algebra, $W_{s,n}|jmn\rangle = 0$, $n > 0$. In order to prove (3.84), note that the statement is equivalent to invariance of the eigenvalues of the zero-modes $W_{s,0}$ under the reflection $j \to -j - 1$, which is proven in appendix A. Note that corresponding statements hold for the antihermitian $W$-algebra module $\bar{\mathcal{W}}_{mn}^j$ spanned by acting with the $W_{s,-n}$ on the Fock vacuum $|jmn\rangle$.

Having established unitarity of $R$, we wish to define its action upon the Fock module $\mathcal{F}_{mn}^j$ spanned by acting with the oscillators $a_n^{(k)}$ on $|jmn\rangle$. We will argue that the action of $R$ is fixed up to a phase if $\mathcal{W}_{mn}^j$ is isomorphic to $\mathcal{F}_{mn}^j$. Clearly, any state in $\mathcal{W}_{mn}^j$ is also a Fock state since the $W_{s,n}$ may be expressed by the oscillators. The reverse statement is certainly true if $\mathcal{W}_{mn}^j$ is irreducible, i.e. does not contain any null vectors. We assume that this holds at least for $j \in -\frac{1}{2} + i\mathbb{R}$. Then, any Fock state $f$ in $\mathcal{F}_{mn}^j$ may be written

$$f = \mathcal{P}_{mn}^j(L_{-n}(j, m, n), W_{3,-n}(j, m, n))\Omega$$  \hspace{1cm} (3.86)

where $\mathcal{P}_{mn}^j$ is some polynomial in the $L_{-n}(j, m, n), W_{3,-n}(j, m, n)$ with coefficients depending on the quantum numbers $j, m, n$. Here we make the additional assumption that $\mathcal{W}_{mn}^j$ is already covered by acting with the $L_{-n}, W_{3,-n}$ on the Fock vacuum. Acting with the reflection operator $R$ on the Fock state $f$, we find, using (3.82) and (3.83),

$$Rf = R(j, m, n)\mathcal{P}_{mn}^j(L_{-n}(-j - 1, m, n), W_{3,-n}(-j - 1, m, n))\Omega$$  \hspace{1cm} (3.87)

Completely analogous, we may define the action of $R$ on Fock states in the Fock module $\bar{\mathcal{F}}_{mn}^j$ spanned by acting with the $\bar{b}_n^{(k)}$ on $|jmn\rangle$.

Therefore, due to symmetry considerations, the operator $R$ is uniquely characterized by the reflection amplitude $R(j, m, n)$. In order to determine $R(j, m, n)$ use the fact that $\mathcal{S}$ leaves $u$ invariant and consider the equation

$$(j'm'n'|\mathcal{S}u(\sigma, 0)\mathcal{S}^{-1}|jmn) = (j'm'n'|u(\sigma, 0)|jmn)$$  \hspace{1cm} (3.88)

Using equations (3.81),(3.83) and the expression (3.74) for the matrix elements of $u$, we find by comparison of the coefficients of the arising delta functions that the reflection amplitude

\footnote{It is possible to deduce the relevant result concerning the isomorphism of $\mathcal{F}_{mn}^j$ and $\mathcal{W}_{mn}^j$ from the determinant formula of $\mathcal{F}_{mn}^j$.}
$R(j, m, n)$ solves the difference equations

$$R(-j - \frac{1}{2}, m + 1, n)R(j, m, n) =$$

$$\frac{1}{k^2} \left( j - \frac{1}{2}(m + nk) \right) \left( j - \frac{1}{2}(m - nk) \right) \Gamma^2(1 + b^2) \frac{\Gamma(-b^2(2j + 1))\Gamma(2b^2j)}{\Gamma(1 - 2b^2j)\Gamma(1 + b^2(2j + 1))}$$ (3.89)

$$R(-j - \frac{3}{2}, m + 1, n)R(j, m, n) = k^2 \left( -j - 1 - \frac{1}{2}(m + nk) \right)^{-1} \left( -j - 1 - \frac{1}{2}(m - nk) \right)^{-1}$$

$$\times \frac{1}{\Gamma^2(1 + b^2)} \frac{\Gamma(1 + 2b^2(j + 1))\Gamma(1 - b^2(2j + 1))}{\Gamma(b^2(2j + 1))\Gamma(-2b^2(j + 1))}$$ (3.90)

It is easy to see that these equations are solved by

$$R(j, m, n) =$$

$$\left( \nu(k) \right)^{2j+1} \frac{\Gamma(j + 1 + \frac{1}{2}(m + nk))\Gamma(j + 1 + \frac{1}{2}(m - nk))\Gamma(-2j - 1)\Gamma(1 - \frac{2j+1}{k-2})}{\Gamma(-j + \frac{1}{2}(m + nk))\Gamma(-j + \frac{1}{2}(m - nk))\Gamma(2j + 1)\Gamma(1 + \frac{2j+1}{k-2})}$$ (3.91)

with

$$\nu(k) = \frac{1}{k^2} \Gamma^2\left( \frac{1}{k - 2} \right)$$ (3.92)

It remains to verify $|R(j, m, n)|^2 = 1$. Since we consider only representations where $j \in -\frac{1}{2} + i\mathbb{R}$, it follows that $j^* = -j - 1$. From this it can easily be inferred that $R(j, m, n)$ is unitary.

To conclude this subsection, let us consider the semiclassical limit $k \rightarrow \infty$ of our scattering amplitude. In this limit we have

$$R(j, m, n) \rightarrow \infty \frac{\Gamma(j + 1 + \frac{1}{2}(m + nk))\Gamma(j + 1 + \frac{1}{2}(m - nk))\Gamma(-2j - 1)\Gamma(1 - \frac{2j+1}{k-2})}{\Gamma(-j + \frac{1}{2}(m + nk))\Gamma(-j + \frac{1}{2}(m - nk))\Gamma(2j + 1)\Gamma(1 + \frac{2j+1}{k-2})}$$ (3.93)

This precisely coincides with the semiclassical results of [3].

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A. Normal ordered exponentials

The main ingredient of our construction of the quantum counterparts of the Euclidean black hole fields are (anti)chiral normal ordered exponentials. See for example [13] for a more detailed explanation of the techniques used here. The normal ordered exponentials are defined as

$$E_{\phi^+}(x^+) = \sum_{n=0}^{\infty} \frac{(2\alpha)^n}{n!} : \phi_k(x^+)^n :,$$

$$\bar{E}_{\phi^+}(x^-) = \sum_{n=0}^{\infty} \frac{(2\alpha)^n}{n!} : \bar{\phi}_k(x^-)^n :$$ (A.1)
where we impose the usual Wick-ordering for the oscillators and symmetric normal-ordering for the zero-modes, i.e.

$$: q^n f(p) := \frac{d^n}{d\alpha^n}(e^{i\Phi q f(p)e^{i\Phi q}})|_{\alpha=0} \quad (A.2)$$

Note that the definition (A.1) is equivalent to

$$E^{(1)}_{\alpha}(x_+) = e^{\alpha q_1} \exp \left\{ 2i\alpha \sum_{n<0} \frac{a_n^{(1)}(1)}{n} e^{-inx} \right\} e^{2\alpha q_1 x_+} \exp \left\{ 2i\alpha \sum_{n>0} \frac{a_n^{(1)}(1)}{n} e^{-inx} \right\} e^{\alpha q_1} \quad (A.3)$$

and

$$E^{(2)}_{\alpha}(x_+) = e^{\alpha q_2^\perp} \exp \left\{ 2i\alpha \sum_{n<0} \frac{a_n^{(2)}}{n} e^{-inx} \right\} e^{2\alpha q_2^\perp x_+} \exp \left\{ 2i\alpha \sum_{n>0} \frac{a_n^{(2)}}{n} e^{-inx} \right\} e^{\alpha q_2^\perp} \quad (A.4)$$

Corresponding expressions exist for the antichiral exponentials. From now on we will suppress the discussion of the antichiral exponentials since all results for the chiral exponentials can directly be applied to the antichiral counterparts. Using Wick’s theorem, one may show that the normal ordered exponentials have short-distance behaviour

$$E^{(k)}_{\alpha}(x_+) E^{(k)}_{\beta}(y_+) = e^{-i\pi \alpha \beta} e^{(x_+ - y_+)} [1 - e^{-i(x_+ - y_+)}]^{-\alpha \beta} : E^{(k)}_{\alpha}(x_+) E^{(k)}_{\beta}(y_+) : \quad (A.5)$$

Here, as starting point for the application of Wick’s theorem, we use the fact the the chiral quantum field $\phi_k$ satisfies

$$\phi_k(x_+) \phi_k(y_+) = : \phi_k(x_+) \phi_k(y_+) : - \frac{i\pi}{2} \delta_k \epsilon^+(x_+ - y_+) \quad (A.6)$$

where the distribution $\epsilon^+$ is given by

$$\epsilon^+(x) = \frac{1}{2} \epsilon(x) - \frac{i}{\pi} \ln |1 - e^{-ix}| = \frac{x}{2\pi} - \frac{i}{\pi} \ln (1 - e^{-ix}) \quad (A.7)$$

Here, $\ln$ denotes the principal value of the logarithm. From the above equation one easily derives that the normal-ordered exponentials obey the exchange-relations

$$E^{(k)}_{\alpha}(x_+) E^{(k)}_{\beta}(y_+) = e^{-2i\alpha \beta(x_+ - y_+)} E^{(k)}_{\beta}(y_+) E^{(k)}_{\alpha}(x_+) \quad (A.8)$$

The normal ordered exponentials are primary fields of the Virasoro algebra and satisfy commutation relations with the generators $L_n$ of the form

$$[L_n, E^{(k)}_{\alpha}(x_+)] = e^{inx} (-i\partial_+ + n\Delta^{(k)}_{\alpha}) E^{(k)}_{\alpha}(x_+) \quad (A.9)$$

The conformal weights are given by

$$\Delta^{(k)}_{\alpha} = \alpha (Q\delta_{k1} - \alpha) \quad (A.10)$$

where $Q = -b$ is the background charge of the theory.

Causal bosonic quantum fields are obtained by composing chiral and antichiral exponentials according to

$$: e^{2i\alpha \Phi_1(\sigma, t)} := E^{(1)}_{\alpha}(x_+) e^{-2\alpha q_1} E^{(1)}_{\alpha}(x_-) \quad (A.11)$$

$$: e^{2i\beta \Phi_2(\sigma, t)} := E^{(2)}_{\beta}(x_+) E^{(2)}_{\beta}(x_-) \quad (A.12)$$

where we impose the restriction $\beta \in \mathbb{R}$ since $\Phi_2$ is compactified.
B. Proof of unitary equivalence

In this appendix it will be demonstrated that representations of the W-algebra on the modules $W_{mn}^j$ and $W_{mn}^{-j-1}$ are unitarily equivalent. We restrict the discussion to the holomorphic copy of the algebra.

The Fock vacua $|jmn⟩$ diagonalize the action of the zero modes $W_{s,0}$. Consider the eigenvalue equation

$$W_{s,0}|jmn⟩ = W_s(j, m, n)|jmn⟩ \quad \text{(B.1)}$$

To prove unitary equivalence of representations to spin $j$ and spin $-j-1$ it suffices to show

$$W_s(j, m, n) = W_s(-j - 1, m, n) \quad \text{(B.2)}$$

A clarifying remark is necessary here. The norm of an arbitrary state in the module $W_{mn}^j$ is given by

$$||W_{s,-n_k}(j, m, n) \cdots W_{s,-n_1}(j, m, n)\Omega||^2 = \langle \Omega, W_{s,n_k}(j, m, n) \cdots W_{s,n_1}(j, m, n)W_{s,-n_k}(j, m, n) \cdots W_{s,-n_1}(j, m, n)\Omega \rangle \quad \text{(B.3)}$$

In order to evaluate this expression, we have to move the annihilation operators $W_{s,n}(j, m, n)$ with $n > 0$ to the right until they hit the vacuum. Using the commutation relations of the W-algebra, this produces zero-modes $W_{s,0}(j, m, n)$ which upon application of the zero-modes to the vacuum yields the corresponding eigenvalues $W_s(j, m, n)$. The norm is thus invariant under $j \to -j - 1$ if $W_s(j, m, n) = W_s(-j - 1, m, n)$. In order to prove the invariance of the eigenvalues under $j \to -j - 1$ we need to remember that the W-algebra currents $W_s(z)$ are polynomials in $\partial\phi_1(z), \partial\phi_2(z)$ and higher derivatives thereof. We furthermore have to keep in mind the mode expansions

$$\partial\phi_1(z) = -i \sum_n a_n^{(1)} z^{-n-1}, \quad \partial\phi_2(z) = -i \sum_n a_n^{(2)} z^{-n-1} \quad \text{(B.4)}$$

where the Euclidean momentum zero-modes $a_0^{(k)}$ are related to their Minkowskian counterparts through

$$a_0^{(1)} = p_1 + \frac{ib}{2}, \quad a_0^{(2)} = p_2^L \quad \text{(B.5)}$$

From the structure of the OPE it follows that we can always normalize the currents $W_s(z)$ such that their modes match the requirement of hermiticity,

$$W_{s,n}^j = W_{s,-n} \quad \text{(B.6)}$$

This implies that the numerical coefficients of the above mentioned polynomials are real for $s$ even and imaginary for $s$ odd. Furthermore, with the mode expansions (B.4) and the behaviour of the currents under parity $\phi_2 \to -\phi_2$,

$$W_s[\phi_1, -\phi_2] = (-)^s W_s[\phi_1, \phi_2] \quad \text{(B.7)}$$
we claim that it follows that the zero-mode eigenvalues $W_s(j, m, n)$ may be represented as polynomials in $j$ according to

$$W_s(j, m, n) = \sum_{k=0}^{s} C_k^s(m, n) j^k$$

(B.8)

with real coefficients $C_k^s(m, n)$. In order to prove this statement consider a monomial of the form

$$\mathfrak{M}(z) = C : (\partial^{m_1} \phi_1)^{k_1} (\partial^{m_2} \phi_1)^{k_2} \times \cdots \times (\partial^{m_1} \phi_2)^{l_1} (\partial^{m_2} \phi_2)^{l_2} \cdots :$$

(B.9)

which is a typical building block of the current $W_s(z)$ given that $\sum(n_l k_l + m_l l_l) = s$. The constant $C$ is real for $s$ even and imaginary for $s$ odd, as argued above. Now consider the action of the zero-mode $\mathfrak{M}_0$ of $\mathfrak{M}(z)$ on the highest weight state $|jmn\rangle$. Taking into account the mode expansions (B.4) we find

$$\mathfrak{M}_0|jmn\rangle = C (n_1 - 1)! (-)^{n_1} a_0^{(1)} (n_2 - 1)! (-)^{n_2} a_0^{(1)} k_1 \times \cdots \times (m_1 - 1)! (-)^{m_1} a_0^{(2)} (m_2 - 1)! (-)^{m_2} a_0^{(2)} l_1 \times \cdots |jmn\rangle$$

(B.10)

which can be rewritten according to, where $\text{const.} \in \mathbb{R}$,

$$\mathfrak{M}_0|jmn\rangle = C \times \text{const.} i^{\sum k_i + \sum l_i} (a_0^{(1)})^{\sum k_i} (a_0^{(2)})^{\sum l_i} |jmn\rangle$$

(B.11)

Taking properly into account that the action of the zero-mode $a_0^{(1)}$ on the state $|jmn\rangle$ is given by $a_0^{(1)}|jmn\rangle = -ib j|jmn\rangle$, we finally find

$$\mathfrak{M}_0|jmn\rangle = C \times \text{const.'} i^{\sum l_i} (a_0^{(2)})^{\sum l_i} j^{\sum k_i} |jmn\rangle$$

(B.12)

where $\text{const.'}$ is real at least for real $b$ resp. $k > 2$. Now, $\sum l_i$ counts the number of occurrences of the field $\phi_2$ in $\mathfrak{M}$. Therefore, due to (B.7), $\sum l_i$ is even for $s$ even and odd for $s$ odd which implies that $i^{\sum l_i}$ is real for $s$ even and imaginary for $s$ odd. However, the constant $C$ that appears in (B.9) has the same property, as demonstrated above. This means, since $a_0^{(2)}$ is hermitean and has real eigenvalues, that the coefficient of $j^{\sum k_i}$ in $\mathfrak{M}_0|jmn\rangle$ is real which proves the asserted statement.

In order to conclude our proof of unitary equivalence of representations we note that hermiticity of $W_{s, 0}$ implies

$$(W_s(j, m, n))^* = W_s(j, m, n)$$

(B.13)

However, in the principal continuous series we have $j \in -\frac{1}{2} + i\mathbb{R}$ and $j^* = -j - 1$. It therefore follows from the reality of the $W_s(j, m, n)$ and the reality of the coefficients $C_k^s(m, n)$ that

$$W_s(j, m, n) = W_s(-j - 1, m, n)$$

(B.14)

and the proof is complete.
C. Proof of braid relations

Let us sketch only the basic ideas that enter the proof of (3.57). It is first of all useful to restrict the range of \( \sigma_1 \) and \( \sigma_2 \) to the fundamental interval \( |\sigma_1 - \sigma_2| < 2\pi, \sigma_1 < \sigma_2 \). Furthermore, we find it convenient to introduce a splitting of the screening charge \( Q(\sigma) = \int_0^{2\pi} d\varphi V(\sigma + \varphi) \) according to

\[
Q(\sigma_1) = Q_{I_c} + Q_{I_1} \\
Q(\sigma_2) = Q_{I_c} + Q_{I_2}
\]

where

\[
Q_{I_c} = \int_{I_c} d\varphi V(\varphi), \quad Q_{I_1} = \int_{I_1} d\varphi V(\varphi), \quad Q_{I_2} = \int_{I_2} d\varphi V(\varphi)
\]

with \( I_c = [\sigma_2, \sigma_1 + 2\pi], I_1 = [\sigma_1, \sigma_2], I_2 = [\sigma_1 + 2\pi, \sigma_2 + 2\pi] \). We note that \( Q_{I_1} \) and \( Q_{I_2} \) are linearly related according to

\[
Q_{I_2} = q^{4i+4}Q_{I_1}
\]

which is a simple consequence of the monodromy property (3.33) of the screening current. One may then use the fact that the screening current \( V \) obeys a simple exchange relation with the building block \( f_1 \) to derive that in the fundamental interval

\[
Q_{I_c}f_1(\sigma_1) = qf_1(\sigma_1)Q_{I_c} \\
Q_{I_1}f_1(\sigma_1) = qf_1(\sigma_1)Q_{I_1} \\
Q_{I_2}f_1(\sigma_1) = q^3f_1(\sigma_1)Q_{I_2}
\]

\[
Q_{I_c}f_1(\sigma_2) = qf_1(\sigma_2)f_{I_c} \\
Q_{I_1}f_1(\sigma_2) = q^{-1}f_1(\sigma_2)Q_{I_1} \\
Q_{I_2}f_1(\sigma_2) = qf_1(\sigma_2)Q_{I_2}
\]

(C.5)

We may use these relations to demonstrate that there exist functions \( C_{kl}^{ij}(j) \) such that the following identities hold:

\[
f_1(\sigma_1)f_2(\sigma_2) = f_2(\sigma_2)f_1(\sigma_1)C_{21}^{12}(j) + f_1(\sigma_2)f_2(\sigma_1)C_{12}^{12}(j) \quad \text{(C.6)}
\]

\[
f_2(\sigma_1)f_1(\sigma_2) = f_1(\sigma_2)f_2(\sigma_1)C_{12}^{21}(j) + f_2(\sigma_2)f_1(\sigma_1)C_{21}^{21}(j) \quad \text{(C.7)}
\]

Finally, one may use the monodromy of the building blocks to generalize these relations to arbitrary \( \sigma_1, \sigma_2 \), with the result (3.57). I will sketch how this can be done. Given arbitrary \( \sigma_1, \sigma_2 \), we introduce the variable \( \tilde{\sigma}_2 \) according to

\[
\tilde{\sigma}_2 = \sigma_2 - \pi(\epsilon\sigma_2 - \sigma_1) - 1
\]

(C.8)

One may convince oneself that this definition implies \( |\sigma_1 - \tilde{\sigma}_2| < 2\pi \) and \( \sigma_1 < \tilde{\sigma}_2 \). Given that we want to derive an exchange relation for \( f_1(\sigma_1)f_2(\sigma_2) \), we may reduce the calculation of the exchange relation for \( f_1(\sigma_1)f_2(\sigma_2) \) to the calculation of the exchange relation for
\( f_1(\sigma_1)f_2(\tilde{\sigma}_2) \). This is done by simply setting \( \sigma_2 = \tilde{\sigma}_2 + \pi(\epsilon(\sigma_2 - \sigma_1) - 1) \) and using the monodromy of the building blocks which is given by

\[
\begin{align*}
    f_2(x_+ + 2\pi) &= e^{2\pi b(-p_1 + ip_2 + i\frac{\theta}{2\pi})} f_2(x_+) \\
    f_1(x_+ + 2\pi) &= e^{2\pi b(p_1 + ip_2 + i\frac{\theta}{2\pi})} f_1(x_+) 
\end{align*}
\] (C.9) (C.10)

Now, since \(|\sigma_1 - \tilde{\sigma}_2| < 2\pi\) and \(\sigma_1 < \tilde{\sigma}_2\), we may apply the exchange relation (C.6) and finally by expressing \(\tilde{\sigma}_2\) in terms of \(\sigma_2\) and again using the monodromy relations of the building blocks, we arrive at an exchange relation for \(\sigma_1, \sigma_2\) arbitrary.

It remains to consider to exchange relations satisfied by \(f_i(\sigma_1)f_j(\sigma_2)\) for \(i = j\). The exchange relations of the building block \(f_1\) resp. \(\tilde{f}_1\) with itself is a simple consequence of (A.8) resp. the corresponding relation for the antichiral building blocks. To prove the relation \(3.62\) for the building block \(f_2\) (resp. \(\tilde{f}_2\)) is more delicate. Here we may use the fact that the screening current \(V(x_+)\) obeys a simple exchange relation with itself according to

\[
V(\sigma_1)V(\sigma_2) = e^{-2\pi ib^2(\sigma_1 - \sigma_2)}V(\sigma_2)V(\sigma_1)
\] (C.11)

This implies for the screening charges (C.3), restricting again to the fundamental interval \(|\sigma_1 - \sigma_2| < 2\pi\), the following set of exchange relations:

\[
Q_{I_1}Q_{I_2} = q^4 Q_{I_2}Q_{I_1} \quad Q_{I_c}Q_{I_1} = q^{-2} Q_{I_1}Q_{I_c} \quad Q_{I_c}Q_{I_2} = q^2 Q_{I_2}Q_{I_c}
\] (C.12)

One may use these relations and (C.5) to show that in the fundamental interval

\[
f_2(\sigma_1)f_2(\sigma_2) = q^\frac{1}{4} f_2(\sigma_2)f_2(\sigma_1)
\] (C.13)

This can be generalized to arbitrary values of \(\sigma_1, \sigma_2\) by again using the monodromy properties of the building blocks.

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