Poisson superbialgebras

Imed Basdouri¹ *, Mohamed Fadous² †, Sami Mabrouk¹ ‡ and Abdenacer Makhlouf³ §

1. University of Gafsa, Faculty of Sciences, 2112 Gafsa, Tunisia
2. University of Sfax, Faculty of Sciences, BP 1171, 3038 Sfax, Tunisia
3. Université de Haute Alsace, IRIMAS - Département de Mathématiques,
   18, rue des frères Lumière, F-68093 Mulhouse, France

Abstract

We introduce the notion of Poisson superbialgebra as an analogue of Drinfeld’s Lie superbialgebras. We extend various known constructions dealing with representations on Lie superbialgebras to Poisson superbialgebras. We introduce the notions of Manin triple of Poisson superalgebras and Poisson superbialgebras and show the equivalence between them in terms of matched pairs of Poisson superalgebras. A combination of the classical Yang-Baxter equation and the associative Yang-Baxter equation is discussed in this framework. Moreover, we introduce notions of O-operator of weight λ ∈ K of a Poisson superalgebra and post-Poisson superalgebra and interpret the close relationships between them and Poisson superbialgebras.

Keywords: Poisson superbialgebra, O-operator, representation, post-Poisson superalgebra.

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Introduction

A Poisson algebra is both a Lie algebra and a commutative associative algebra which are compatible in a certain sense. Poisson algebras play important roles in many fields in mathematics and mathematical physics, such as the Poisson geometry, integrable systems, non-commutative (algebraic or differential) geometry, and so on (see [22] and the references therein).

As generalizations of Lie algebras, Lie superalgebras were introduced motivated by applications to physics and to deformations of Lie algebras, especially Lie algebras of vector fields. The notion of Lie superalgebras was firstly introduced in [12]. More precisely, Lie superalgebras are Z_2-graded version of a Lie algebras, where as the Jacobi identity is replaced by a superJacobi identity given by

\[ (-1)^{|x||y|}[x, [y, z]] + (-1)^{|y||z|}[z, [x, y]] + (-1)^{|x||z|}[y, [z, x]] = 0. \]

Superization of Poisson algebras was considered in classical dynamics of fermion fields and classical spin-1/2 particles, and also used in the BRST and Batalin-Vilkovisky formalism. It is also related to Poisson supermanifolds in differential geometry [15] [16] [17].

On the other hand, both Lie superalgebras and (commutative) associative superalgebras have a known theory of bialgebras which have been applied to a lot of fields. Lie bialgebras, which

*E-mail: basdourimed@yahoo.fr
†E-mail: mohamedfadous2011@gmail.com
‡E-mail: mabrouksami00@yahoo.fr
§E-mail: abdenacer.makhlouf@uha.fr (Corresponding author)
were introduced by Drinfeld in the early 1980s for studying the solutions of classical Yang-Baxter equation, are now well established as the infinitesimalisation of quantum groups. In the case of associative algebras, Joni and Rota introduced the notion of infinitesimal bialgebra in order to provide an algebraic framework for the calculus of divided differences [11]. With an additional supersymmetric solutions of the associative Yang-Baxter equation, it is called an associative D-bialgebra, see also balanced infinitesimal bialgebra or antisymmetric infinitesimal bialgebra. All these notions are analogues of a Lie bialgebra. Aguiar developed a systematic theory for infinitesimal bialgebras from this point of view [12][3].

The aim of this paper is to study Poisson superbialgebras from these perspectives and also dealing with matched pairs, manin triples and $O$-operators. The paper is organized as follows. In the first section we provide some basic definitions and in Section 2 we discuss a characterization of Poisson superalgebras with one operation and several constructions related to representations. In Section 3, we define various algebraic structures connected to Poisson superbialgebras. In Section 4, we introduce the notions of Manin triple of Poisson superalgebras and Poisson superbialgebras and then give the equivalence between them in terms of matched pairs of Poisson superalgebras. In Section 5, we consider the coboundary cases and discuss Poisson Yang-Baxter equation (PYBE) which is a combination of Classical Yang-Baxter equation (CYBE) and associative Yang-Baxter equation (AYBE). In Section 6, we introduce notions of $O$-operator of weight $\lambda \in \mathbb{K}$ of a Poisson superalgebra and post-Poisson superalgebra and interpret the close relationships between them and Poisson superbialgebras.

1 Definitions and Preliminaries

First, let us start by fixing some definitions and notations. Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a $\mathbb{Z}_2$-graded vector space over an arbitrary field $\mathbb{K}$ of characteristic 0. In the sequel, we will consider only elements which are $\mathbb{Z}_2$-homogeneous. For $x \in \mathcal{A}$, we denote by $|x| \in \mathbb{Z}_2$ its parity, i.e., $x \in \mathcal{A}_{|x|}$. We denote by $\tau$ the super-twist map of $\mathcal{A} \otimes \mathcal{A}$, namely $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$ for $x, y \in \mathcal{A}$. The super-cyclic map $\xi$ permutes the coordinates of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, it is defined as

$$\xi = (id \otimes \tau) \cdot (\tau \otimes id) : x \otimes y \otimes z \mapsto (-1)^{|x||y|+|z|} y \otimes z \otimes x,$$

for $x, y, z \in \mathcal{A}$, where $id$ is the identity map on $\mathcal{A}$. We denote by $\mathcal{A}^\circ$ =Hom($\mathcal{A}, \mathbb{K}$) the linear dual of $\mathcal{A}$. For $\phi \in \mathcal{A}^\circ$ and $x \in \mathcal{A}$, we often use the adjoint notation $\langle \phi, x \rangle$ for $\phi(x) \in \mathbb{K}$.

For a linear map $\Delta, \Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ (comultiplication), we use Sweedler’s notation $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$ for $x \in \mathcal{A}$. We will often omit the summation sign $\sum_{(x)}$ to make it simpler. The parity $|r|$ of $r \in \mathcal{A}^{\otimes 2}$ is defined as follows: since we assume $r$ homogenous, there exists $|r| \in \mathbb{Z}_2$, such that $r$ can be written as $r = \sum r_1 \otimes r_2 \in \mathcal{A}^{\otimes 2}$, $r_1, r_2$ are homogenous elements with $|r| = |r_1| + |r_2|$.

**Definition 1.1.** A Lie superalgebra is a pair $(\mathcal{A}, [\cdot, \cdot])$ consisting of a superspace $\mathcal{A}$ such that $[\mathcal{A}_r, \mathcal{A}_s] \subset \mathcal{A}_{r+s}$ equipped with a bilinear map $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ satisfying

$$[x, y] = (-1)^{|x||y|}[y, x], \tag{1.1}$$

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||z|}[z, [x, y]] + (-1)^{|y||x|}[y, [z, x]] = 0, \tag{1.2}$$

for all homogeneous elements $x, y, z \in \mathcal{A}$.

**Definition 1.2.** A commutative associative superalgebra is a pair $(\mathcal{A}, \mu)$ consisting of a superspace $\mathcal{A}$ an even bilinear map $\mu : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ satisfying

$$\mu(x, y) = (-1)^{|x||y|}\mu(y, x), \tag{1.3}$$

for $x, y \in \mathcal{A}$. 

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\[
\mu(x, \mu(y, z)) = \mu(\mu(x, y), z) \quad \text{(associativity)},
\]
for all \(x, y\) in \(A\).

**Definition 1.3.** A Poisson superalgebra is a triple \((A, \{\cdot\}, \mu)\) consisting of a superspace \(A\), an even bilinear map \(\{\cdot\} : A \times A \rightarrow A\) and \(\mu : A \times A \rightarrow A\), satisfying

1. \((A, \{\cdot\})\) is a Lie superalgebra,
2. \((A, \mu)\) is a commutative associative superalgebra,
3. for all \(x, y, z \in A\):

\[
\{x, \mu(y, z)\} = \mu(\{x, y\}, z) + (-1)^{|x||y|}\mu(y, \{x, z\}).
\]

The condition (1.5) expresses the compatibility between the Lie bracket \(\{\cdot,\}\) and the associative superalgebra product \(\mu\), it can be written equivalently as

\[
\{x, \mu(y, z)\} = \mu(x, \{y, z\}) + (-1)^{|x||y|}\mu(x, \{y, z\}).
\]

A homomorphism between two Poisson superalgebras is defined as a linear map between two Poisson superalgebras preserving the corresponding operations.

# 2 Modules and matched pairs of Poisson superalgebras

First, we recall how to construct a Lie superalgebra (or a commutative associative superalgebra) structure on the direct sum of two Lie superalgebra (or two commutative associative superalgebra) such that both of them are sub-superalgebra.

Let \(\cdot : (x, y) \rightarrow x \cdot y\) be an even bilinear map on the superspace \(A\). The associator as of \(\cdot\) is the trilinear map on \(A\) given by

\[
as(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z).
\]

**Proposition 2.1.** Let \((A, \cdot)\) be a \(K\)-superalgebra. Define the \(A\)-valued operations \(\{\cdot\}\) and \(\cdot\) on \(A \times A\) by

\[
x, y = \frac{1}{2}(x \cdot y - (-1)^{|x||y|} y \cdot x),
\]

\[
x \cdot y = \frac{1}{2}(x \cdot y + (-1)^{|x||y|} y \cdot x).
\]

Then \((A, \{\cdot\}, \cdot)\) is a Poisson superalgebra if and only if the operation \(x \cdot y\) satisfies the identity:

\[
3as(x, y, z) = (-1)^{|x||y|}(x \cdot z) \cdot y - (-1)^{|x||y|+|y||z|}(z \cdot x) \cdot y + (-1)^{|x||y|+|y||z|}(y \cdot z) \cdot x - (-1)^{|x||y|}(y \cdot x) \cdot z.
\]

**Proof.** For any \(x, y, z \in A\), we will check that the \((A, \{\cdot\})\) is a Lie superalgebra. We have

\[
(-1)^{|x||y|}\{x, \{y, z\}\} + (-1)^{|y||z|}\{y, \{z, x\}\} + (-1)^{|z||x|}\{z, \{x, y\}\}
\]

\[
= (-1)^{|x||y|}\{x, \frac{1}{2}(y \cdot z - (-1)^{|y||z|} z \cdot y)\} + (-1)^{|y||z|}\{y, \frac{1}{2}(z \cdot x - (-1)^{|z||x|} z \cdot x)\}
\]

\[
+ (-1)^{|z||x|}\{z, \frac{1}{2}(x \cdot y - (-1)^{|x||y|} y \cdot x)\}
\]

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Next, we check that \((\mathcal{A}, \bullet)\) is a commutative associative superalgebra. Indeed, for any \(x, y, z \in \mathcal{A}\),

\[
(x \bullet y) \bullet z - x \bullet (y \bullet z) = \frac{1}{2} (x \cdot y + (-1)^{|y||z|} y \cdot x) \bullet z - x \bullet \frac{1}{2} (y \cdot z + (-1)^{|z||x|} y \cdot z)
\]

\[
= \frac{1}{4} (as(x, y, z) - (-1)^{|y||z|} as(y, x, z)) + (-1)^{|y||z|} as(y, z, x) + (-1)^{|z||x|} as(z, x, y)
\]

\[
+ (-1)^{|z||x|} as(x, y, z) - (-1)^{|y||z|} as(z, y, x) + (-1)^{|y||z|} as(y, z, x) + (-1)^{|z||x|} as(z, x, y) \bigg) = 0.
\]

Finally, we check the condition: \([x, y \bullet z] = [x, y] \bullet z + (-1)^{|y||z|} y \bullet [x, z]\). We have

\[
[x, y \bullet z] = [x, y] \bullet z - (-1)^{|y||z|} y \bullet [x, z]
\]

\[
= \frac{1}{4} (-2as(x, y, z) - 2(-1)^{|y||z|} as(y, x, z) - (-1)^{|y||z|} as(y, z, x) - (-1)^{|z||x|} as(z, x, y)
\]

\[
+ (-1)^{|z||x|} as(y, z, x) + (-1)^{|y||z|} as(x, y, z) + (-1)^{|z||x|} as(z, x, y) \bigg) = 0.
\]

The proof is finished. \(\square\)

**Definition 2.1.** Let \((\mathcal{A}, [\cdot, \cdot])\) be a Lie superalgebra and \(V = V_0 \oplus V_1\) an arbitrary vector superspace. A representation (or module) of the Lie superalgebra is given by a pair \((V, \rho)\), where \(\rho : \mathcal{A} \rightarrow \text{End}(V)\) is an even linear map such that \(\rho(\mathcal{A})(V_j) \subset V_{i+j}\) where \(i, j \in \mathbb{Z}_2\), and satisfying

\[
\rho([x, y]) = \rho(x) \circ \rho(y) - (-1)^{|y||z|} \rho(y) \circ \rho(x).
\]

(2.9)

for all homogeneous elements \(x, y \in \mathcal{A}\).

Let \((\mathcal{A}, [\cdot, \cdot], \mathcal{A})\) and \((\mathcal{A}', [\cdot, \cdot], \mathcal{A}')\) be two Lie superalgebras. Set \(\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1\) and \(\mathcal{A}' = \mathcal{A}'_0 \oplus \mathcal{A}'_1\). Let \(\rho : \mathcal{A} \rightarrow \text{End}(\mathcal{A}')\) and \(\rho' : \mathcal{A}' \rightarrow \text{End}(\mathcal{A})\) be two even linear maps. Define a skew-supersymmetric bracket \([\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \rightarrow \text{End}(\mathcal{A})\), where \(\mathcal{G}\) is given by \(\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1\), with \(\mathcal{G}_0 = \mathcal{A}_0 \oplus \mathcal{A}'_0\) and \(\mathcal{G}_1 = \mathcal{A}_1 \oplus \mathcal{A}'_1\). We set \(|x + x'| = |x| + |x'|\) and \(|y + y'| = |y| + |y'|\), for all homogeneous elements \(x, y \in \mathcal{A}\) and \(x', y' \in \mathcal{A}'\). On the direct sum of the underlying vector superspace \(\mathcal{A} \oplus \mathcal{A}'\), the bracket, on \(\mathcal{G}, [\cdot, \cdot] : \mathcal{A} \oplus \mathcal{A}' \times \mathcal{A} \oplus \mathcal{A}' \rightarrow \mathcal{A} \oplus \mathcal{A}'\) is defined by

\[
[x + x', y + y'] = [x, y] + \rho'(x')(y) - (-1)^{|y||z|} \rho'(y') \rho'(y) + [x', y']_{\mathcal{A}'} + \rho(x)(y') - (-1)^{|y||z|} \rho(y)(x').
\]

(2.10)

**Theorem 2.1.** [27] With the above notations, \((\mathcal{A} \oplus \mathcal{A}', [\cdot, \cdot])\) is a Lie superalgebra if and only if \(\rho\) and \(\rho'\) are representations of \(\mathcal{A}\) and \(\mathcal{A}'\) respectively and the following conditions are satisfied

\[
\rho'(z')(x, y)_{\mathcal{A}} = [\rho'(z')(x), y]_{\mathcal{A}} + (-1)^{|y||z|}[x, \rho'(z')(y)]_{\mathcal{A}}
\]

\[
+ (-1)^{|z||y| + |y||z|} \rho'(\rho(y)(z'))(x) - (-1)^{|z||y|} \rho'(\rho(y)(z'))(y),
\]

(2.11)

\[
\rho(z)(x', y')_{\mathcal{A}'} = [\rho(z)(x'), y']_{\mathcal{A}'} + (-1)^{|y'||z'|}[x', \rho(z)(y')]_{\mathcal{A}'}
\]

\[
+ (-1)^{|z'||y'| + |y'||z'|} \rho(\rho'(y')(z'))(x') - (-1)^{|z'||y'|} \rho(\rho'(y')(z'))(y'),
\]

(2.12)

for any \(x, y, z \in \mathcal{A}\) and \(x', y', z' \in \mathcal{A}'\).
This Lie superalgebra is denoted by $A \cong_{\phi} A'$ or simply $A \cong A'$. Moreover, every Lie superalgebra which is the direct sum of the underlying vector superspaces of two sub-superalgebras can be obtained from a matched pair of Lie superalgebras.

**Definition 2.2.** Let $(\mathcal{A}, \bullet)$ be a commutative associative superalgebra. Recall that a representation (or module) on a vector superspace $V = V_0 \oplus V_1$ with respect to $\mathcal{A}$ is an even linear map $\phi : \mathcal{A} \rightarrow \text{End}(V)$, such that for any $x, y \in \mathcal{A}$, the following equality is satisfied:

$$\phi(x \bullet y) = \phi(x) \circ \phi(y).$$  \hspace{1cm} (2.13)

We denote the representation by $(V, \phi)$.

Let $(\mathcal{A}_1, \bullet_1)$ and $(\mathcal{A}_2, \bullet_2)$ be two commutative associative superalgebras. Set $\mathcal{A}_1 = \mathcal{A}_{10} \oplus \mathcal{A}_{11}$ and $\mathcal{A}_2 = \mathcal{A}_{20} \oplus \mathcal{A}_{21}$. If there are linear maps $\phi_1 : \mathcal{A}_1 \rightarrow \text{End}(\mathcal{A}_2)$ and $\phi_2 : \mathcal{A}_2 \rightarrow \text{End}(\mathcal{A}_1)$ which are representations of $\mathcal{A}_1$ and $\mathcal{A}_2$. Define a bilinear map $\bullet : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$, where $\mathcal{G}$ is given by $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$, with $\mathcal{G}_0 = \mathcal{A}_{10} \oplus \mathcal{A}_{20}$ and $\mathcal{G}_1 = \mathcal{A}_{11} \oplus \mathcal{A}_{21}$. We set $|x + a| = |x| = |a|$ and $|y + b| = |y| = |b|$, for all $x, y \in \mathcal{A}_1$ and $a, b \in \mathcal{A}_2$. On the direct sum of the underlying vector superspace $\mathcal{A}_1 \oplus \mathcal{A}_2$, define a bilinear map $\bullet$ by

$$(x + a) \bullet (y + b) = (x \bullet_1 y + \phi_2(a)y) + (-1)^{|x||y|}\phi_2(b)x + a \bullet_2 b + \phi_1(b)x + (-1)^{|x||y|}\phi_1(y)a. \hspace{1cm} (2.14)$$

**Theorem 2.2.** With the above notations, $(\mathcal{A}_1 \oplus \mathcal{A}_2, \bullet)$ is a commutative associative superalgebra if and only if $\phi_1$ and $\phi_2$ are representations of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively and the following conditions are satisfied

$$\begin{align*}
\phi_1(x)(a \bullet_2 b) &= (\phi_1(x)x)(\bullet_2 b) + (-1)^{|x||y|}\phi_1(y)(\phi_2(a)x)b, \\
\phi_2(a)(x \bullet_1 y) &= (\phi_2(a)x)(\bullet_1 y) + (-1)^{|x||y|}\phi_2(y)(\phi_1(x)a)y,
\end{align*} \hspace{1cm} (2.15, 2.16)$$

for any $x, y \in \mathcal{A}_1$ and $a, b \in \mathcal{A}_2$.

**Proof.** If $(\mathcal{A}, \bullet)$ is a commutative associative superalgebra, with respect to $\bullet$ we have

$$(x + a) \bullet ((y + b) \bullet (z + c)) = ((x + a) \bullet (y + b)) \bullet (z + c), \hspace{1cm} (2.17)$$

for $x, y, z \in \mathcal{A}_1$ and $a, b, c \in \mathcal{A}_2$. Expanding (2.17), we obtain

$$\begin{align*}
\star) \quad (x + a) \bullet ((y + b) \bullet (z + c)) &= (x \bullet_1 (y \bullet_1 z) + x \bullet_1 (\phi_2(b)x) + (-1)^{|x||z|}x \bullet_1 (\phi_2(c)y) + \phi_2(a)(y \bullet_1 z) \\
&+ \phi_2(a)(\phi_2(b)z) + (-1)^{|x||y|}\phi_2(a)(\phi_2(c)y) + (-1)^{|x||y|+|x||z|}\phi_2(b \bullet_2 c)x + (-1)^{|x||y|+|x||z|}\phi_2(\phi_1(y)c)x \\
&+ (-1)^{|x||y|+|x||z|+|y||z|}\phi_2(\phi_2(b)z)x + a \bullet_2 (b \bullet_2 c) + a \bullet_2 (\phi_1(y)c) + (-1)^{|x||y|}a \bullet_2 (\phi_1(z)b) + \phi_1(x)(\phi_2(b)z)a \\
&+ \phi_1(x)(\phi_1(y)c) + (-1)^{|x||y|}\phi_1(x)(\phi_1(z)b) + (-1)^{|x||y|+|x||z|}\phi_1(y \bullet_1 z)a + (-1)^{|x||y|+|x||z|}\phi_1(\phi_2(b)z)a \\
&+ (-1)^{|x||y|+|x||z|+|y||z|}\phi_1(\phi_2(b)c)y.a.
\end{align*}$$

$$\begin{align*}
\star) \quad ((x + a) \bullet (y + b)) \bullet (z + c) &= (x \bullet_1 y) \bullet_1 z + \phi_2(a)y \bullet_1 z + (-1)^{|x||y|}\phi_2(b)x \bullet_1 z + \phi_2(a \bullet_2 b)z + \phi_2(a)(\phi_1(x)b)z \\
&+ (-1)^{|x||y|}\phi_2(\phi_1(y)a)z + (-1)^{|x||y|+|x||z|}\phi_2(c)x \bullet_1 y + (-1)^{|x||y|+|x||z|}\phi_2(b)\phi_2(a)y + (-1)^{|x||y|+|x||z|}\phi_1(\phi_2(b)x)c \\
&+ (-1)^{|x||y|+|x||z|+|y||z|}\phi_1(z)\phi_2(b) + (-1)^{|x||y|+|x||z|+|y||z|}\phi_1(y)\phi_2(c)\phi_2(b)x.
\end{align*}$$

\text{End}.
It implies (2.15), (2.16) and

\[ \varphi_1(x \cdot y)c = \varphi_1(x)\varphi_1(y)c, \]  
(2.18)

\[ \varphi_2(a \cdot b)z = \varphi_2(a)\varphi_2(b)z. \]  
(2.19)

By (2.18), we deduce that \( \varphi_1 \) is a representation of \( (\mathcal{A}_1, \cdot_1) \) on \( \mathcal{A}_2 \). By (2.19) we deduce that \( \varphi_2 \) is a representation of \( (\mathcal{A}_2, \cdot_2) \) on \( \mathcal{A}_1 \). This finishes the proof.

We use the notation \( \mathcal{A}_1 \trianglelefteq_{\psi_1} \mathcal{A}_2 \) or simply \( \mathcal{A}_1 \trianglelefteq \mathcal{A}_2 \). Moreover, every commutative associative superalgebra which is the direct sum of the underlying vector superspaces of two sub-superalgebras can be obtained from a matched pair of commutative associative superalgebras.

**Definition 2.3.** Let \( (\mathcal{P}, \{,\}, \bullet) \) be a Poisson superalgebra, \( V = V_0 \oplus V_1 \) an arbitrary vector superspace and \( \psi_{\{,\}}, \psi_{\bullet} : \mathcal{P} \rightarrow \text{End}(V) \) be two even linear maps. Then \( (V, \psi_{\{,\}}, \psi_{\bullet}) \) is called a representation (or module) of \( \mathcal{P} \) if \( (V, \psi_{\{,\}}) \) is a representation of \( (\mathcal{P}, \{,\}) \) and \( (V, \psi_{\bullet}) \) is a representation of \( (\mathcal{P}, \bullet) \) and they are compatible in the sense that for any \( x, y \in \mathcal{P} \) and \( v \in V \)

\[ \psi_{\{,\}}(x \cdot y)(v) = \psi_{\{,\}}(x)(\psi_{\{,\}}(y)(v) + (-1)^{|x||y|}\psi_{\bullet}(y))\psi_{\{,\}}(x)(v), \]  
(2.20)

\[ \psi_{\bullet}((x, y))(v) = \psi_{\bullet}(x)(\psi_{\bullet}(y)(v) + (-1)^{|x||y|}\psi_{\{,\}}(y)\psi_{\bullet}(x)(v)). \]  
(2.21)

In the case of Poisson superalgebras, we can construct semidirect product when given bimodules. Analogously, we have

**Proposition 2.2.** Let \( (\mathcal{P}, \{,\}, \bullet) \) be a Poisson superalgebra. Then \( (V, \psi_{\{,\}}, \psi_{\bullet}) \) is a representation of a Poisson superalgebra \( (\mathcal{P}, \{,\}, \bullet) \) if and only if the direct sum of vector superspaces \( V \oplus V \) is turned into a Poisson superalgebra by defining the operations by (we still denote the operations by \( \{,\} \) and \( \bullet \)):

\[ \{x_1 + v_1, x_2 + v_2\} = \{x_1, x_2\} + \psi_{\{,\}}(x_1)v_2 - (-1)^{|x_1||x_2|}\psi_{\{,\}}(x_2)v_1, \]  
(2.22)

\[ (x_1 + v_1) \cdot (x_2 + v_2) = x_1 \cdot x_2 + \psi_{\bullet}(x_1)v_2 + (-1)^{|x_1||x_2|}\psi_{\bullet}(x_2)v_1, \]  
(2.23)

for any \( x_1, x_2 \in \mathcal{P}, v_1, v_2 \in V \). We denote it by \( \mathcal{P} \triangleright_{\psi_{\{,\}}\psi_{\bullet}} V \) or simply \( \mathcal{P} \triangleright V \) [20].

**Proof.** The sufficient condition holds obviously. Here we just verify the necessary condition. For any \( x_1, x_2, x_3 \in \mathcal{P}, v_1, v_2, v_3 \in V \). From the construction above we have \( |x_1 + v_1| = |x_1| = |v_1| \) and \( |x_2 + v_2| = |x_2| = |v_2| \),

\[ \{x_1 + v_1, x_2 + v_2\} = \{x_1, x_2\} + \psi_{\{,\}}(x_1) - (-1)^{|x_1||x_2|}\psi_{\{,\}}(x_2)v_1 \]
\[ = (-1)^{|x_1||x_2|}((x_2, x_1) + \psi_{\{,\}}(x_2)v_1) = (-1)^{|x_1||x_2|}\psi_{\{,\}}(x_2)v_2 \]
\[ = (-1)^{|x_1||x_2|}\{x_2 + v_2, x_1 + v_1\} \]
\[ = (-1)^{|x_1||x_2|}\{x_2 + v_2, x_1 + v_1\}, \]

so the skew-supersymmetry holds. For the superjacobi identity, we have

\[ (-1)^{|x_1||v_1|}(|x_1 + v_1|)\{x_1 + v_1, x_2 + v_2, x_3 + v_3\} + (-1)^{|x_2||v_2|}(|x_2 + v_2|)\{x_2 + v_2, x_3 + v_3, x_1 + v_1\} \]
\[ + (-1)^{|x_3||v_3|}(|x_3 + v_3|)\{x_3 + v_3, x_1 + v_1, x_2 + v_2\} \]
\[ = (-1)^{|x_1||v_1|}\psi_{\{,\}}(x_1)|\psi_{\{,\}}(x_2)v_2 - (-1)^{|x_1||v_1|}\psi_{\{,\}}(x_2)v_2 \]
\[ = (-1)^{|x_1||v_1|}\psi_{\{,\}}(x_1)\psi_{\{,\}}(x_2)v_2 - (-1)^{|x_1||v_1|}\psi_{\{,\}}(x_2)v_2. \]
Thus \((\mathcal{P} \ltimes V, \cdot, \ast)\) is a Lie superalgebra. Next we will check that \((\mathcal{P} \ltimes V, \cdot, \ast)\) is a commutative associative superalgebra. In fact, for any \(x_1, x_2, x_3 \in \mathcal{P}, v_1, v_2, v_3 \in V\), also one may check directly that:

\[(x_1 + v_1) \cdot (x_2 + v_2) = (\ast)^{[x_1 + v_1], [x_2 + v_2]}(x_1 + v_1) \cdot (x_2 + v_2) \cdot (x_3 + v_3).\]

Now we check the following equality

\[(x_1 + v_1) \cdot ((x_2 + v_2) \cdot (x_3 + v_3)) = ((x_1 + v_1) \cdot (x_2 + v_2)) \cdot (x_3 + v_3).\]

For this, we calculate

\[\begin{align*}
(x_1 + v_1) \cdot ((x_2 + v_2) \cdot (x_3 + v_3)) &= (x_1 + v_1) \cdot (x_2 + v_2) \cdot (x_3 + v_3) + (\ast)^{[x_1 + v_1], [x_2 + v_2]}(x_1 + v_1) \cdot (x_2 + v_2) \\
&= (x_1 + v_1) \cdot (x_2 + v_2) \cdot (x_3 + v_3) + (\ast)^{[x_1 + v_1], [x_2 + v_2]}(x_1 + v_1) \cdot (x_2 + v_2) \\
&= \sum \left[ (x_1 + v_1) \cdot (x_2 + v_2) \cdot (x_3 + v_3) \right] + (\ast)^{[x_1 + v_1], [x_2 + v_2]}(x_1 + v_1) \cdot (x_2 + v_2).
\end{align*}\]

Finally, we check the condition

\[\{(x_1 + v_1), ((x_2 + v_2) \cdot (x_3 + v_3)) \} = \{(x_1 + v_1), x_2 \cdot x_3 + \psi_\ast(x_2) v_3 \} + (\ast)^{[x_1 + v_1], [x_2 + v_2]}(x_2 + v_2) \cdot \{x_1 + v_1, x_3 + v_3\}.\]

In fact, on the one hand, we have

\[\begin{align*}
\{(x_1 + v_1), ((x_2 + v_2) \cdot (x_3 + v_3)) \} &= \{(x_1 + v_1), (x_2 + v_2) \cdot (x_3 + v_3) + (\ast)^{[x_2 + v_2], [x_3 + v_3]}(x_2 + v_2) \cdot (x_3 + v_3) \} \\
&= \sum \left[ (x_1 + v_1) \cdot (x_2 + v_2) \cdot (x_3 + v_3) \right] + (\ast)^{[x_1 + v_1], [x_2 + v_2]}(x_2 + v_2) \cdot (x_3 + v_3).
\end{align*}\]

On the other hand, we have

\[\begin{align*}
\{(x_1 + v_1), ((x_2 + v_2) \cdot (x_3 + v_3)) \} &= \{(x_1 + v_1), (x_2 + v_2) \cdot (x_3 + v_3) + (\ast)^{[x_2 + v_2], [x_3 + v_3]}(x_2 + v_2) \cdot (x_3 + v_3) \} \\
&= \sum \left[ (x_1 + v_1) \cdot (x_2 + v_2) \cdot (x_3 + v_3) \right] + (\ast)^{[x_1 + v_1], [x_2 + v_2]}(x_2 + v_2) \cdot (x_3 + v_3).
\end{align*}\]

Thus \((\mathcal{P} \ltimes V, \cdot, \ast)\) is a Poisson superalgebra. The proof is completed. 

\section*{Theorem 2.3} Let \((\mathcal{P}, \{, \}, \cdot, \ast)\) be a Poisson superalgebra and \((V, \psi_{\{\cdot\}}, \psi_{\ast})\) be a representation of a Poisson superalgebra \((\mathcal{P}, \{, \}, \cdot, \ast)\). Define \(\psi_{\{\cdot\}}, \psi_{\ast} : \mathcal{P} \rightarrow \text{End}(V^\ast)\) by, for any \(x \in \mathcal{P}, \xi, \psi \in V^\ast, v \in V\)

\[\langle \psi_{\{\cdot\}}(x) \xi, v \rangle = -\langle \ast^{[x]}(\xi, \psi_{\{\cdot\}}(x)v) \rangle, \quad \langle \psi_{\ast}(x) \xi, v \rangle = -\langle \ast^{[\xi]}(\xi, \psi_{\ast}(x)v) \rangle.\]
If, in addition, for any \( x, y \in \mathcal{P}, \xi \in V^*, v \in V \)
\[
\psi_{(1)}(x \bullet y) = \psi_{(1)}(x)\psi_{*}(y) + (-1)^{|x||y|}\psi_{(1)}(y)\psi_{*}(x),
\]
\[
\psi_{*}([x, y]) = \psi_{(1)}(x)\psi_{*}(y) - (-1)^{|x||y|}\psi_{*}(y)\psi_{(1)}(x),
\]
(2.25)

then \((V^*, \psi^\dagger_{(1)}, -\psi^\dagger_{*})\) is a representation of Poisson superalgebra \((\mathcal{P}, \{\}, \bullet)\). Moreover, \((\mathcal{P} \ltimes V^*, \{\}, \bullet)\) is also a Poisson superalgebra.

**Proof.** We know that if \((V, \psi_{(1)})\) is a representation of a Lie superalgebra \((\mathcal{P}, \{\}, \bullet)\), then \((V^*, \psi^\dagger_{(1)}\) is a representation of the Lie superalgebra \((\mathcal{P}, \{\}, \bullet)\). Also, if \((V, \psi_{*})\) is a representation of a commutative associative superalgebra \((\mathcal{P}, \bullet)\), then \((V^*, -\psi_{*})\) is a representation of the commutative associative superalgebra \((\mathcal{P}, \bullet)\). It remains to show
\[
\psi^\dagger_{(1)}(x \bullet y) + \psi^\dagger_{*}(y)\psi^\dagger_{(1)}(x) + (-1)^{|x||y|}\psi^\dagger_{(1)}(x)\psi^\dagger_{*}(y) = 0,
\]
\[
-\psi^\dagger_{*}([x, y]) + \psi^\dagger_{(1)}(x)\psi^\dagger_{*}(y) - (-1)^{|x||y|}\psi^\dagger_{*}(y)\psi^\dagger_{(1)}(x) = 0.
\]
(2.26)

For this, we calculate
\[
\langle (\psi^\dagger_{(1)}(x \bullet y) + \psi^\dagger_{*}(y)\psi^\dagger_{(1)}(x) + (-1)^{|x||y|}\psi^\dagger_{(1)}(x)\psi^\dagger_{*}(y))\xi, v \rangle
\]
\[
= \langle \xi, (-1)^{|x||\xi|+|y||\xi|} - \psi_{(1)}(x \bullet y) + \psi_{(1)}(x)\psi_{*}(y) + (-1)^{|x||y|}\psi_{(1)}(y)\psi_{*}(x)) v \rangle = 0,
\]
\[
\langle (\psi^\dagger_{*}([x, y]) + \psi^\dagger_{(1)}(x)\psi^\dagger_{*}(y) - (-1)^{|x||y|}\psi^\dagger_{*}(y)\psi^\dagger_{(1)}(x))\xi, v \rangle
\]
\[
= \langle \xi, (-1)^{|x||\xi|+|y||\xi|} (\psi^\dagger_{*}([x, y]) - \psi_{(1)}(x)\psi_{*}(y) + (-1)^{|x||y|}\psi_{*}(y)\psi_{(1)}(x)) v \rangle = 0.
\]

So \((V^*, \psi^\dagger_{(1)}, -\psi^\dagger_{*})\) is a representation of Poisson superalgebra \((\mathcal{P}, \{\}, \bullet)\). The remaining results follow from Proposition 2.2 directly.

**Example 2.4.** Let \((\mathcal{P}, \{\}, \bullet)\) be a Poisson superalgebra. Then \((\mathcal{P}, \text{ad}_{(1)}, L_*)\) and \((\mathcal{P}^*, \text{ad}^\dagger_{(1)}, -L^*_*)\) are representation of Poisson superalgebra \((\mathcal{P}, \{\}, \bullet)\).

In the sequel, we will present a method to construct a Poisson superalgebra structure on a direct sum \(\mathcal{P}_1 \oplus \mathcal{P}_2\) of the underlying vector spaces of two Poisson superalgebras \(\mathcal{P}_1\) and \(\mathcal{P}_2\) such that \(\mathcal{P}_1\) and \(\mathcal{P}_2\) are Poisson sub-superalgebras.

**Theorem 2.5.** Let \((\mathcal{P}_1, \{\}, \bullet_1)\) and \((\mathcal{P}_2, \{\}, \bullet_2)\) be two Poisson superalgebras. Let \(\rho_{(1)}: \mathcal{P}_1 \rightarrow \text{End}(\mathcal{P}_2)\) and \(\rho_{(1)}: \mathcal{P}_2 \rightarrow \text{End}(\mathcal{P}_1)\) be four even linear maps such that \((\mathcal{P}_1, \mathcal{P}_2, \rho_{(1)}, \rho_{(1)})\) is a matched pair of Lie superalgebras and \((\mathcal{P}_1, \mathcal{P}_2, \rho_{*}, \rho_{*})\) is a matched pair of commutative associative superalgebras. If in addition \((\mathcal{P}_2, \rho_{(1)}, \rho_{*})\) and \((\mathcal{P}_1, \rho_{(1)}, \rho_{*})\) are representations of the Poisson superalgebras \((\mathcal{P}_1, \{\}, \bullet_1)\) and \((\mathcal{P}_2, \{\}, \bullet_2)\) respectively, and \(\rho_{(1)}, \rho_{(1)}, \rho_{*}, \rho_{*}\) are compatible in the following sense:
\[
\rho_{(1)}(x \bullet_1 y) = (\rho_{(1)}(x) y) \bullet_1 y + (-1)^{|x||y|} x \bullet_1 (\rho_{(1)}(y) y)
\]
\[
- (-1)^{|x||y|}\rho_{*}(\rho_{(1)}(x) y) y - (-1)^{|x||y|+|x||y|}\rho_{*}(\rho_{(1)}(y) y) x,
\]
(2.27)

\[
[x, \rho_{*}(y) y] = (-1)^{|x||y|+|y||x|}-\rho_{*}(\rho_{(1)}(x) y) y - (-1)^{|x||y|}(\rho_{(1)}(x) y) \bullet_1 y
\]
\[
- (-1)^{|x||y|}\rho_{*}(\rho_{(1)}(y) y)(x, y)_{1},
\]
(2.28)
3 Poisson superbialgebras

Definition 3.1. A Lie supercoalgebra is a pair \((\mathcal{A}, \delta)\) consisting of a superspace \(\mathcal{A}\) and a linear map \(\delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}\) satisfying the following conditions:

\[
\delta(\mathcal{A}) \subset \sum_{i=j+k} \mathcal{A}^i \otimes \mathcal{A}^k \text{ for } i \in \mathbb{Z}_2,
\]

\[
\text{Im} \delta \subset \text{Im}(\text{id} \otimes \text{id} - \tau) \text{ i.e. } \delta \text{ is skew-supersymmetric},
\]

\[
(id \otimes id \otimes \text{id} + \xi + \xi^2) \circ (id \otimes \delta) \circ \delta = 0 : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}.
\]
Definition 3.2. A coassociative supercoalgebra is a pair $\mathcal{A}, \Lambda$ consisting of a superspace $\mathcal{A}$ and a linear map $\Lambda: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ satisfying the following conditions:

$$\Lambda(\mathcal{A}) \subset \sum_{i+j=k} \mathcal{A}^i \otimes \mathcal{A}^k \text{ for } i \in \mathbb{Z}_2,$$

(3.39)

$$\text{(id} \otimes \Delta) \circ \Delta = (\Lambda \otimes \text{id}) \circ \Delta \quad \text{(coassociativity).}$$

(3.40)

Remark 3.1. 1) If $(\mathcal{A}, \langle, \rangle)$ is a Lie superalgebra, we let $ad_{1,1}(x) = ad(x)$ denote the adjoint operator, that is, $ad_{1,1}(x)y = ad(x)y = \{x, y\}$ for any $x, y \in \mathcal{A}$. For an element $x$ in a Lie superalgebra $(\mathcal{A}, \langle, \rangle)$ and $n \geq 2$, define the adjoint map $ad(x): \mathcal{A}^n \rightarrow \mathcal{A}^n$ by

$$ad(x)(y_1 \otimes \cdots \otimes y_n) = \sum_{i=1}^n (-1)^{\left|y_1\right| + \left|y_2\right| + \cdots + \left|y_{i-1}\right|} y_1 \otimes \cdots \otimes y_{i-1} \otimes [x, y_i] \otimes y_{i+1} \cdots \otimes y_n.$$

For $n = 2$, $ad(x)(y_1 \otimes y_2) = [x, y_1] \otimes y_2 + (-1)^{\left|y_1\right|} y_1 \otimes [x, y_2]$. Conversely, given $\gamma = y_1 \otimes \cdots \otimes y_n$, we define the map $ad(\gamma): \mathcal{A} \rightarrow \mathcal{A}^n$ by $ad(\gamma)(x) = ad(x)(y)$, for $x \in \mathcal{A}$.

2) Let $(\mathcal{A}, \bullet)$ be an associative superalgebra with a bilinear map $\langle, \rangle: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, $L_\bullet(x)$ and let $R_\bullet(x)$ denote the left and right multiplication operator respectively, that is, $L_\bullet(x)y = (-1)^{\left|y\right|} R_\bullet(y)x = x \bullet y$, for all homogeneous elements $x, y \in \mathcal{A}$.

We also simply denote them by $L(x)$ and $R(x)$ respectively when there is no confusion.

Definition 3.3. (10) A Lie superbialgebra is a triple $(\mathcal{A}, \langle, \rangle, \delta)$ such that

1. $(\mathcal{A}, \langle, \rangle)$ is a Lie superalgebra.
2. $(\mathcal{A}, \delta)$ is a Lie supercoalgebra.
3. The following compatibility condition holds for all $x, y \in \mathcal{A}$:

$$\delta([x, y]) = (ad(x) \otimes \text{id} + \text{id} \otimes ad(x))\delta(y) - (-1)^{\left|y\right|}(ad(y) \otimes \text{id} + \text{id} \otimes ad(y))\delta(x).$$

(3.41)

The map $f: \mathcal{A} \rightarrow \mathcal{A}'$ is called even (resp. odd) map if $f(\mathcal{A}) \subset \mathcal{A}'$ (resp. $f(\mathcal{A}) \subset \mathcal{A}'_{+1}$), for $i = 0, 1$. A morphism of Lie superbialgebras is an even linear map such that

$$f \circ \langle, \rangle = \langle, \rangle \circ f^{\otimes 2} \quad \text{and} \quad \delta \circ f = f^{\otimes 2} \circ \delta.$$

An isomorphism of Lie superbialgebras is an invertible morphism of Lie superbialgebras. Two Lie superbialgebras are said to be isomorphic if there exists an isomorphism between them.

Definition 3.4. An infinitesimal superbialgebra is a triple $(\mathcal{A}, \bullet, \Lambda)$ such that

1. $(\mathcal{A}, \bullet)$ is a associative superalgebra,
2. $(\mathcal{A}, \Lambda)$ is a coassociative supercoalgebra,
3. The following compatibility condition holds for all $a, b \in \mathcal{A}$:

$$\Lambda(a \bullet b) = (L_\bullet(a) \otimes \text{id})\Lambda(b) + (\text{id} \otimes R_\bullet(b))\Lambda(a).$$

(3.42)

A morphism of infinitesimal superbialgebras is a linear map that commutes the multiplications and the comultiplications.
The multiplication \( \cdot \) in this context is not explicitly defined in the text. However, we can infer that it is the usual multiplication of elements in a superalgebra.

Example 3.1. Let \( \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \) be a 2-dimensional superspace where \( \mathcal{A}_0 \) is generated by \( e_1 \) and \( \mathcal{A}_1 \) is generated by \( e_2 \). Then \( (\mathcal{A}, \cdot, \partial, \Delta) \) is a Poisson superbialgebra in the following situation:

**The pair** \((\mathcal{A}, \cdot)\) **is a Lie superalgebra** when

\[
\{e_1, e_2\} = de_2, \quad \{e_2, e_2\} = ce_1, \quad e_1, e_1 = 0, \quad \text{with} \quad bc = 0.
\]

**The multiplication** \( \cdot \) and **\( \Delta \) are of the form**:

\[
e_1 \cdot e_2 = de_2, \quad e_1 \cdot e_1 = ke_1, \quad e_2 \cdot e_2 = fe_1 \quad \text{and} \quad \Delta(e_1) = \Delta(e_2) = 0.
\]

**The cobracket** \( \partial \) **is a commutative associative superalgebra** when \( d = k \).

**The triple** \((\mathcal{A}, \cdot, \partial)\) **is a Lie superbialgebra** if \( c_1c_2 = 0 \) and \( cc_1 = -4bc_2 \).

**Also** \( \partial \) and **\( \Delta \)** are **compatible** if \( ke_1 = 2dc_1 \) and \( fc_2 = 0 \), where \( b, c, d, k, f, c_1, c_2 \) are parameters in \( \mathbb{K} \).

Therefore, we have a Poisson superbialgebra for \( c = d = k = c_2 = 0 \) or \( b = f = c_1 = 0 \) and \( d = k \).

### 4 Manin triples of Poisson superalgebras and Poisson superbialgebras

We introduce first a notion of Manin triple of Poisson superalgebras which is an analogue of the notion of Manin triple for Lie superalgebras [21](7).

**Definition 4.1.** A Manin triple of Poisson superalgebras \( (\mathcal{P}, \mathcal{P}^+, \mathcal{P}^-) \) is a triple of Poisson superalgebras \( \mathcal{P}, \mathcal{P}^+, \) and \( \mathcal{P}^- \) together with a nondegenerate supersymmetric bilinear form \( B \) on \( \mathcal{P} \) which is invariant in the sense that

\[
B([x, y], z) = B(x, [y, z]), \quad B(x \cdot y, z) = B(x, y \cdot z),
\]

for any \( x, y, z \in \mathcal{P} \), satisfying the following conditions:
1. \( P^+ \) and \( P^- \) are Poisson sub-superalgebra of \( P \),

2. \( P = P^+ \oplus P^- \) as \( \mathbb{K} \)-vector superspace,

3. \( P^+ \) and \( P^- \) are isotropic with respect to \( B \).

A homomorphism between two Manin triples of Poisson superalgebras \((P_1, P_1^+, P_1^-)\) and \((P_2, P_2^+, P_2^-)\) with two nondegenerate supersymmetric invariant bilinear forms \( B_1 \) and \( B_2 \) respectively is a homomorphism of Poisson superalgebras \( \phi : P_1 \to P_2 \) such that

\[
\phi(P_1^+) \subset P_2^+, \quad \phi(P_1^-) \subset P_2^-,
\]

\[
B_1(x, y) = \phi^* B_2(x, y) = B_1(\phi(x), \phi(y)),
\]

for any \( x, y \in P_1 \).

Obviously, a Manin triple of Poisson superalgebras is just a triple of Poisson superalgebras such that they are both a Manin triple of Lie superalgebra and a commutative associative version of Manin triple with the same nondegenerate supersymmetric bilinear form (and share the same isotropic sub-superalgebra). Moreover, it is easy to see that \( P^+ \) and \( P^- \) are both Lagrangian sub-superalgebras of \( P \).

In particular, there is a special (standard) Manin triple of Poisson superbialgebra as follows. Let \((P, \{\cdot, \cdot\}, \bullet)\) be a Poisson superalgebra. If there is a Poisson superalgebra structure on the direct sum of the underlying vector superspace of \( P \) and its dual superspace \( P^* \) such that \((P \oplus P^*, P, P^*)\) is a Manin triple of Poisson superalgebras with the invariant supersymmetric bilinear form on \( P \oplus P^* \) given by

\[
B_p(x + a, y + b) = (x, b) + (a, y), \quad \text{for any } x, y \in P, \ a, b \in P^*,
\]

then \((P \oplus P^*, P, P^*)\) is called a standard Manin triple of Poisson superalgebras.

Obviously, a standard Manin triple of Poisson superalgebras is a Manin triple of Poisson superalgebras. Conversely, it is easy to show that every Manin triple of Poisson superalgebras is isomorphic to a standard one. Furthermore, it is straightforward to get the following structure Theorem ([7] and [5]).

In the following, we concentrate on the case that \( P_1 \) is \( P \) and \( P_2 \) is \( P^* \), where \( P^* \) is the dual space of \( P \), and \( p_{\mid l, i} = ad_{\mid l, i}^\ast, \phi_\ast = -L_\ast, \) (i = 1, 2).

**Theorem 4.1.** Let \((P, \{\cdot, \cdot\}, \bullet)\) and \((P^*, \{\cdot, \cdot\}, \bullet)\) be two Poisson superalgebras. Then \((P \oplus P^*, P, P^*)\) is a standard Manin triple of Poisson superalgebras if and only if \((P, P^*, ad_{\mid l, 1}^\ast, -L_\ast, ad_{\mid l, 2}^\ast, -L_\ast)\) is a matched pair of Poisson superalgebras.

Like the Manin triples for Lie superalgebra corresponding to Lie superbialgebras and the commutative associative versions of Manin triples corresponding to commutative and cocommutative infinitesimal superbialgebras, [3], [25] there is also a bialgebra structure which corresponds to a Manin triple of Poisson superalgebras following from the equivalent conditions of matched pairs of Poisson superalgebras.

**Theorem 4.2.** Let \((P, \{\cdot, \cdot\}, \bullet)\) and \((P^*, \{\cdot, \cdot\}, \bullet)\) be two Poisson superalgebras equipped with two comultiplications \( \delta, \Lambda : P \to P \otimes P \). Suppose that \( \delta^*, \Lambda^* : P^* \otimes P^* \subset (P \otimes P)^* \to P^* \) induce a Poisson superalgebra structure on \( P^* \), where \( \delta^* \) and \( \Lambda^* \) correspond to the Lie bracket and the product of the commutative associative superalgebra respectively. Set \( \{\cdot\}_i = \delta^*, \bullet_i = \Lambda^* (i = 1, 2) \). Then the following conditions are equivalent:
1. \((\mathcal{P}, \{\}, \bullet, \Delta)\) and \((\mathcal{P}', \{\}, \bullet, \delta, \Delta)\) are Poisson superbialgebras.

2. \((\mathcal{P}, \mathcal{P}', \text{ad}^\prime_{t_{1/2}}, -L^*_{\cdot}, \text{ad}^\prime_{t_{1/2}}, -L^*_{\cdot})\) is a matched pair of Poisson superalgebras.

3. \((\mathcal{P} \oplus \mathcal{P}', \mathcal{P}, \mathcal{P}')\) is a standard Manin triple of Poisson superalgebras with the bilinear form defined by Eq. (4.43) and the isotropic subsuperalgebras are \(\mathcal{P}\) and \(\mathcal{P}'\).

Proof. We only need to prove that the fact (1) holds if and only if the fact (2) holds. In fact, it is known that \((\mathcal{P}, \{\}, \delta)\) is a Lie superbialgebra if and only if \((\mathcal{P}, \mathcal{P}', \text{ad}^\prime_{t_{1/2}}, \text{ad}^\prime_{t_{1/2}}, -L^*_{\cdot}, -L^*_{\cdot})\) is a matched pair of Lie superalgebra and \((\mathcal{P}, \bullet, \Delta)\) is a commutative and cocommutative infinitesimal superbialgebra if and only if \((\mathcal{P}, \mathcal{P}', -L^*_{\cdot}, -L^*_{\cdot})\) is a matched pair of associative superalgebras. Then by Theorem 2.5 we only need to prove Eqs. (2.27)–(2.30) are equivalent to Eqs. (3.44) and (3.45) in the case that \(p_{\{\}} = \text{ad}^\prime_{t_{1/2}}, \varphi^* = -L^*_{\cdot} (i = 1, 2)\). As an example, we give an explicit proof of the case that (for any \(x, y \in \mathcal{P}, a \in \mathcal{P}'\))

\[
-ad^*_{t_{1/2}}(a)(x \cdot 1) + (ad^*_{t_{1/2}}(a)x) \cdot 1 + (-1)^{|a||x|} x \cdot 1 (ad^*_{t_{1/2}}(a)y) + (-1)^{|a||x|} L^*_\cdot (ad^*_{t_{1/2}}(a)x)y = 0
\]

is equivalent to Eq. (3.44). The proof of other cases is similar. In fact, let the left hand side of the above equation acts on an arbitrary element \(b \in \mathcal{P}'\), we have

\[
0 = \langle -ad^*_{t_{1/2}}(a)(x \cdot 1) + (ad^*_{t_{1/2}}(a)x) \cdot 1 + (-1)^{|a||x|} x \cdot 1 (ad^*_{t_{1/2}}(a)y) + (-1)^{|a||x|} L^*_\cdot (ad^*_{t_{1/2}}(a)x)y, b \rangle
\]

Then the conclusion follows. \(\square\)

Furthermore, it is easy to show that two Manin triples of Poisson superalgebras are isomorphic if and only if their corresponding Poisson superbialgebras are isomorphic.

5 Coboundary Poisson superbialgebras

We recall first some relevant results of coboundary Lie superbialgebras and coboundary infinitesimal superbialgebras. For example, they lead to the famous classical Yang-Baxter equation (CYBE) [7], [9] and associative Yang-Baxter equation (AYBE) [11], respectively.

Definition 5.1. Denote by \(U(\mathcal{A})\), the universal enveloping algebra of a superspace \(\mathcal{A}\). If \(r = \sum r_1 \otimes r_2 \in \mathcal{A} \otimes \mathcal{A}\), then (here we also use \(\otimes\) to denote the identity element of \(U(\mathcal{A})\))

\[
r_{12} = \sum r_1 \otimes r_2 \otimes \text{id} = r \otimes \text{id}, \quad (5.49)
\]

\[
r_{13} = \sum r_1 \otimes \text{id} \otimes r_2 = (\text{id} \otimes \tau)(r \otimes \text{id}) = (\text{id} \otimes \tau)(\text{id} \otimes r),
\]

\[
r_{23} = \sum \text{id} \otimes r_1 \otimes r_2 = \text{id} \otimes r
\]

are elements of \(U(\mathcal{A}) \otimes U(\mathcal{A}) \otimes U(\mathcal{A})\).
Definition 5.2. The classical Yang-Baxter equation (CYBE) in a Lie superalgebra \((\mathcal{A}, [\cdot, \cdot])\) is
\[
C(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{5.50}
\]
for \(r \in \mathcal{A} \otimes \mathcal{A}\). The three brackets in (5.50) are defined as
\[
[r_{12}, r'_{13}] = \sum (-1)^{|r_2||r'|_2} [r_1, r'_1] \otimes r_2 \otimes r'_2, \tag{5.51}
\]
\[
[r_{12}, r'_{23}] = \sum r_1 \otimes [r_2, r'_1] \otimes r'_2,
\]
\[
[r_{13}, r'_{23}] = \sum (-1)^{|r_1||r'|_1} r'_1 \otimes [r_2, r'_2],
\]
where \(r = \sum r_1 \otimes r_2\) and \(r' = \sum r'_1 \otimes r'_2 \in \mathcal{A} \otimes \mathcal{A}\).

Definition 5.3. The associative Yang-Baxter equation (AYBE) (it is also called AYB in Ref. [2]) in a associative superalgebra \((\mathcal{A}, \cdot)\) is
\[
A(r) = r_{13} \cdot r_{12} - r_{12} \cdot r_{23} + r_{23} \cdot r_{13} = 0, \text{ for } r \in \mathcal{A} \otimes \mathcal{A}. \tag{5.52}
\]
The three elements in (5.52) are defined as
\[
r_{13} \cdot r'_{12} = \sum r_1 \cdot r'_1 \otimes r'_2 \otimes r_2, \tag{5.53}
\]
\[
r_{12} \cdot r'_{23} = \sum r_1 \otimes r_2 \cdot r'_1 \otimes r'_2,
\]
\[
r_{23} \cdot r'_{13} = \sum r'_1 \otimes r_1 \otimes r_2 \cdot r'_2,
\]
where \(r = \sum r_1 \otimes r_2\) and \(r' = \sum r'_1 \otimes r'_2 \in \mathcal{A} \otimes \mathcal{A}\).

Definition 5.4. A coboundary Lie superbialgebra \((\mathcal{A}, [\cdot, \cdot], \delta, r)\) consists of a Lie superbialgebra \((\mathcal{A}, [\cdot, \cdot], \delta)\) and an element \(r = \sum r_1 \otimes r_2 \in \mathcal{A} \otimes \mathcal{A}\) such that for all \(x \in \mathcal{A}\)
\[
\delta(x) = (ad(x) \otimes id + (-1)^{|x||r_1|} id \otimes ad(x))r = [x, r_1] \otimes r_2 + (-1)^{|x||r_1|} r_1 \otimes [x, r_2], \tag{5.54}
\]
Definition 5.5. A coboundary infinitesimal superbialgebra \((\mathcal{A}, \cdot, \Delta, r)\) consists of an infinitesimal superbialgebra \((\mathcal{A}, \cdot, \Delta)\) and an element \(r = \sum r_1 \otimes r_2 \in \mathcal{A} \otimes \mathcal{A}\) such that \(|r_1| = |r_2|\) and for all \(a \in \mathcal{A}\)
\[
\Delta(a) = (L_{\cdot}(a) \otimes id - (-1)^{|a||r_1|} id \otimes R_{\cdot}(a))r = a \cdot r_1 \otimes r_2 - r_1 \otimes r_2 \cdot a. \tag{5.55}
\]

Remark 5.1.
\[
\delta(x) = (-1)^{|x||r|} \sum [x, r_1] \otimes r_2 + (-1)^{|x||r_1|} r_1 \otimes [x, r_2]) \tag{5.56}
\]
for \(x \in \mathcal{A}\), where the parity \(|r|\) of \(r\) is defined as follows : since we assume \(r\) is homogenous, there exists \(|r| \in \mathbb{Z}_2\), such that \(r\) can be written as \(r = \sum r_1 \otimes r_2 \in \mathcal{A}^\otimes 2\), \(r_1, r_2\) are homogenous elements with \(|r| = |r_1| + |r_2|\). (Note that equations (5.56) and (5.55) show that we have \(|r| = 0\), namely \(|r_1| = |r_2|\). So we get (5.54).

Let \((\mathcal{A}, [\cdot, \cdot])\) be a Lie superalgebra and \(r \in \mathcal{A} \otimes \mathcal{A}\). The linear map \(\delta\) defined by Eq. (5.54) and \(|r| = 0\) makes \((\mathcal{A}, \delta)\) into a Lie supercoalgebra if and only if the following conditions are satisfied (for any \(x \in \mathcal{A}\):
1. \(ad(x) \otimes id + (-1)^{|a||r_1|} id \otimes ad(x))(r + \tau(r)) = 0 \iff \delta(x) + \tau(\delta(x)) = 0,
2. \(ad(x)([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]) = 0.\)
Let \((\mathcal{A}, \bullet)\) be a commutative associative superalgebra and \(r \in \mathcal{A} \otimes \mathcal{A}\). The linear map \(\Delta\) defined by Eq. (5.55) and \(|r| = 0\) makes \((\mathcal{A}, \Delta)\) into a cocommutative and coassociative superalgebra if and only if the following conditions are satisfied (for any \(x \in \mathcal{A}\)):

1. \(L(x) \otimes id - (-1)^{|x||r|}id \otimes L(x))(r + \tau(r)) = 0 \iff \Delta(x) - \tau(\Delta(x)) = 0,
2. \(L(x) \otimes id \otimes id - id \otimes id \otimes T(x))(r_{13} \bullet r_{12} - r_{12} \bullet r_{23} + r_{23} \bullet r_{13}) = 0,

where \(T(x)y = y \bullet x\).

Next we introduce the notion of coboundary Poisson superbialgebra.

**Definition 5.6.** A coboundary Poisson superbialgebra \((\mathcal{A}, \{,\}, \bullet, \delta, \Delta, r)\) consists of a Poisson superbialgebra \((\mathcal{A}, \{,\}, \bullet, \delta, \Delta)\) and an element \(r = \sum r_1 \otimes r_2 \in \mathcal{A} \otimes \mathcal{A}\) and \(|r| = 0\) such that

\[
\delta(x) = (ad_{|x|}(x) \otimes id + (-1)^{|x||r|}id \otimes ad_{|x|}(x))r,
\]

\[
\Delta(x) = (L(x) \otimes id - (-1)^{|x||r|}id \otimes L(x))r, \quad \text{for all } x \in \mathcal{A}.
\]

Obviously, a coboundary Poisson superbialgebra \((\mathcal{A}, \{,\}, \bullet, \delta, \Delta, r)\) is equivalent to the fact that both \((\mathcal{A}, \{,\}, \delta, r)\) (as a Lie superbialgebra) and \((\mathcal{A}, \bullet, \Delta, r)\) (as an infinitesimal superbialgebra) are coboundary.

Now we consider when \((\mathcal{A}, \{,\}, \bullet, \delta, \Delta, a)\) becomes a Poisson supercoalgebra, where \(\delta\) and \(\Delta\) are defined by Eqs. (5.57) and (5.58) for some \(r \in \mathcal{A} \otimes \mathcal{A}\), respectively. Let \(\mathcal{A}\) be a \(\mathbb{K}\)-vector space equipped with two comultiplications \(\delta, \Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}\). Then \((\mathcal{A}, \delta, \Delta)\) becomes a Poisson supercoalgebra for \(\delta\) corresponding to the Lie cobracket and \(\Delta\) corresponding to the coproduct of the cocommutative superbialgebra if and only if \((\mathcal{A}, \delta)\) is a Lie supercoalgebra and \((\mathcal{A}, \Delta)\) is a cocommutative superbialgebra and

\[
W(x) = (id \otimes \Delta)\delta(x) - (\delta \otimes id)\Delta(x) - (\tau \otimes id)(id \otimes \delta)\Delta(x) = 0, \quad \text{for all } x \in \mathcal{A}.
\]

Let \((\mathcal{A}, \{,\}, \bullet)\) be a Poisson superalgebra. Define two comultiplications \(\delta, \Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}\) by Eqs. (5.57) and (5.58) with \(r = \sum r_1 \otimes r_2 \in \mathcal{A} \otimes \mathcal{A}\) and \(|r| = 0\), respectively. Then in this case (for any \(x \in \mathcal{A}\))

\[
W(x) = -(ad_{|x|}(x) \otimes id \otimes id)A(r) + (-1)^{|x||r|}(id \otimes L_{\bullet}(x) \otimes id - id \otimes id \otimes L_{\bullet}(x))C(r)
\]

\[
- \sum (((-1)^{|r||x|}ad_{|x|}(r_1) \otimes id)(L_{\bullet}(x) \otimes id - id \otimes L_{\bullet}(x))(r + \tau(r))) \otimes r_2.
\]

In fact, it follows from

\[
W(x) = \sum [x, r_1 \otimes r_2 \bullet r_1 \otimes r_2 - (-1)^{|x||r_1||r_2|}[x, r_1 \otimes r_2 \bullet r_2 - x \bullet r_1 \otimes r_2 \otimes r_2]
\]

\[
+ [x \bullet r_2, r_1 \otimes r_2 \bullet r_1 \otimes r_2 + (-1)^{|x||r_1||r_2|}[x, r_1 \otimes r_2 \bullet r_1 \otimes r_2 - (-1)^{|x||r_1||r_2|}r_1 \bullet r_2 \otimes r_2]
\]

\[
- [r_2, r_1 \otimes x \bullet r_1 \otimes r_2 - (-1)^{|x||r_1||r_2|}r_2 \otimes x \bullet r_2 \otimes r_2 - (-1)^{|x||r_1||r_2|}r_2 \bullet r_2 \otimes r_2]
\]

\[
+ (-1)^{|x||r_1|}[r_1 \otimes r_2 \otimes x \bullet r_2 + (-1)^{|x||r_1||r_2|}[r_2 \otimes r_1 \otimes x \bullet r_2 \otimes r_2 + r_2 \otimes r_1 \otimes x \bullet r_2 \otimes r_2]],
\]

in which the sum of the first four terms is

\[
-(ad_{|x|}(x) \otimes id \otimes id)A(r) - \sum (((-1)^{|r||x|}ad_{|x|}(r_1) \otimes id)(L_{\bullet}(x) \otimes id)(r + \tau(r))) \otimes r_2.
\]
induce a coboundary Poisson superbialgebra structure on $PD \oplus A$ given in Theorem 5.2, where $(\delta, \Delta)$ comultiplications that if $x \in A$ are satisfied (for any $x \in A$ supercoalgebra such that $\delta$ and $\Delta$ defined by Eqs. (5.57) and (5.58), respectively, make $(\mathcal{A}, \delta, \Delta, \alpha)$ is a skew-supersymmetric solution of PYBE in a Poisson super algebra $(\mathcal{A}, \{\}, \bullet, \delta, \Delta)$ into a Poisson superbialgebra if and only if the following conditions are satisfied (for any $x \in \mathcal{A}$):

1. $(ad_{L_1}(x) \otimes id + (\ad_{L_1} \otimes ad_{L_1})(x))(r + \tau(r)) = (L_1(x) \otimes id - (\ad_{L_1} \otimes L_1)(x))(r + \tau(r)) = 0$,

2. $(L_1(x) \otimes id \otimes id - id \otimes id \otimes T_1(x))A(r) = 0$,

3. $ad(x)(C(r)) = 0$,

4. $W(x) = 0$, where $W(x)$ is given by Eq. (5.60) and $r = \sum r_1 \otimes r_2$.

Let $(\mathcal{A}, \{\}, \bullet)$ be a Poisson superalgebra and $r \in \mathcal{A} \otimes \mathcal{A}$ such that $\rho = 0$. We say $r$ satisfies Poisson Yang-Baxter equation (PYBE) if $r$ satisfies both CYBE and AYBE. Therefore a direct consequence is that if $r$ is a skew-supersymmetric solution of PYBE in a Poisson superalgebra $(\mathcal{A}, \{\}, \bullet, \delta, \Delta)$, then the comultiplications $\delta$ and $\Delta$ defined by Eqs. (5.57) and (5.58), respectively, make $(\mathcal{A}, \{\}, \bullet, \delta, \Delta)$ into a Poisson superbialgebra.

Another important consequence of Theorem 5.1 is the following Poisson superalgebra analogue of the Drinfeld double construction.

**Theorem 5.2.** Let $(\mathcal{A}, \{\}, \bullet_1, \delta, \Delta)$ be a Poisson superbialgebra. Then there is a canonical coboundary Poisson superalgebra structure on $\mathcal{A} \oplus \mathcal{A}^r$.

**Proof.** Let $r \in \mathcal{A} \otimes \mathcal{A}^r \subset (\mathcal{A} \oplus \mathcal{A}^r) \otimes (\mathcal{A} \oplus \mathcal{A}^r)$ correspond to the identity map $id : \mathcal{A} \rightarrow \mathcal{A}$. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathcal{A}$ and $\{e_1^r, \ldots, e_r^r\}$ be its dual basis. Then $r = \sum_i e_i \otimes e_i^r$ such that $|\rho| = 0$. Suppose that the Poisson superalgebra structure $(\{\}, \bullet)$ on $\mathcal{A} \oplus \mathcal{A}^r$ is given by $PD(\mathcal{A}) = \mathcal{A} \triangleright \Delta_{L_1} - L_1^* \triangleright \mathcal{A}^r$, where $(\{\}, \bullet_1)$ is the Poisson superalgebra structure on $\mathcal{A}^r$ induced by $(\delta, \Delta)$. Then we have, for any $x, y \in \mathcal{A}$, $a, b \in \mathcal{A}^r$,

$$\{x, y\} = \{x, y\}_1, \quad x \bullet y = x \bullet_1 y, \quad \{a, b\}_1 = \{a, b\}_2, \quad a \bullet b = a \bullet_2 b, \quad [x, a] = \sum \rho_{|\rho|} [a, x] = \sum \rho_{|\rho|} \ad_{L_1}(a)a - \sum \rho_{|\rho|} \ad_{L_1^*}(a), \quad x \bullet a = \sum \rho_{|\rho|} x \bullet_1 a - \sum \rho_{|\rho|} L_1^*(a).$$

It is straightforward to prove that $r$ satisfies CYBE and AYBE such that $\rho = 0$ and for any $u \in PD(\mathcal{A})$

$$(\ad_{L_1}(u) \otimes id + (\ad_{L_1} \otimes ad)(u))(r + \tau(r)) = (L_1(u) \otimes id - (\ad_{L_1} \otimes L_1)(u))(r + \tau(r)) = 0.$$

So $r$ satisfies the conditions in Theorem 5.1. Thus, for any $u \in PD(\mathcal{A})$

$$\delta_{PD}(u) = (\ad_{L_1}(u) \otimes id + (\ad_{L_1} \otimes ad)(u)r, \quad \Delta_{PD}(u)(L_1(u) \otimes id - (\ad_{L_1} \otimes L_1)(u)r),$$

induce a coboundary Poisson superbialgebra structure on $PD(\mathcal{A})$. $\square$

Let $(\mathcal{A}, \{\}, \bullet_1, \delta, \Delta_1)$ be a Poisson superbialgebra. With the Poisson superbialgebra structure given in Theorem 5.2, $\mathcal{A} \oplus \mathcal{A}^r$ is called the Drinfeld classical double of $\mathcal{A}$. As in the proof of Theorem 5.2 we denote it by $PD(\mathcal{A})$. 16
6 O-operators of Poisson superalgebras, Post-Poisson superalgebras, and quasitriangular Poisson superbialgebras

In this section we introduce the notions of O-operator of weight \( \lambda \in \mathbb{K} \) on a Poisson superalgebra, post-Poisson superalgebra, and quasitriangular Poisson superbialgebras. We use O-operators on Poisson superalgebras to construct post-Poisson superalgebras. We show that a quasitriangular Poisson superbialgebra naturally gives a post-Poisson superalgebra.

**Definition 6.1.** Let \((\mathcal{A}, \{ \cdot, \cdot \}_\mathcal{A})\) and \((\mathcal{A}', \{ \cdot, \cdot \}_{\mathcal{A}'})\) be two Lie superalgebras. Suppose that \( \omega \) is a Lie superalgebra homomorphism from \( \mathcal{A} \) to \( \text{Der}_\mathbb{K}(\mathcal{A}') \) the Lie superalgebra consisting of all the derivations of \( \mathcal{A}' \). Then \((\mathcal{A}', \{ \cdot, \cdot \}_{\mathcal{A}'}, \omega)\) is called a \( \mathcal{A} \)-Lie superalgebra.

**Definition 6.2.** Let \((\mathcal{A}, \cdot)\) and \((R, \cdot)\) be two commutative associative superalgebras. Let \( \varphi : \mathcal{A} \rightarrow \text{End}(R) \) be an even linear map. Then the triple \((R, \cdot, \varphi)\) is called an \( \mathcal{A} \)-module superalgebra if

\[
\varphi(x \cdot y) = \varphi(x)\varphi(y), \quad \varphi(x)(u \cdot v) = (\varphi(x)u) \cdot v, \quad (6.61)
\]

for \( x, y \in \mathcal{A}, u, v \in R \).

**Definition 6.3.** Let \((\mathcal{P}, \{ \cdot, \cdot \}), (V, \{ \cdot, \cdot \}_1, \cdot_1)\) be two Poisson superalgebras. Let \( \psi_{\cdot_1}, \psi_\cdot : \mathcal{P} \rightarrow \text{End}(R) \) be two even linear maps such that

1. \((V, \{ \cdot, \cdot \}_1, \psi_{\cdot_1})\) is a \( \mathcal{P} \)-Lie superalgebra, where \( \mathcal{P} \) is seen as a Lie superalgebra with respect to the Lie bracket \( \{ \cdot, \cdot \} \).
2. \((V, \cdot_1, \psi_\cdot)\) is a \( \mathcal{P} \)-module superalgebra, where \( \mathcal{P} \) is seen as a commutative associative superalgebra with respect to the commutative associative product \( \cdot_1 \).
3. \((\psi_{\cdot_1}, \psi_\cdot, V)\) is a module of \( \mathcal{P} \).
4. The following equations hold:

\[
\psi_{\cdot_1}(x)(u \cdot_1 v) = (\psi_{\cdot_1}(x)u) \cdot_1 v + (-1)^{|x||u|} u \cdot_1 (\psi_{\cdot_1}(x)v), \quad \forall \ x \in \mathcal{P}, \ u, v \in V, (6.62)
\]

\[
\psi_\cdot(x)(u, v)_1 = (\psi_{\cdot_1}(x)u) \cdot_1 v + (-1)^{|x||u|} u, \psi_\cdot(x)v)_1, \quad \forall \ x \in \mathcal{P}, \ u, v \in V. (6.63)
\]

Then \((V, \{ \cdot, \cdot \}_1, \cdot_1, \psi_{\cdot_1}, \psi_\cdot)\) is called a \( \mathcal{P} \)-module Poisson superalgebra.

**Proposition 6.1.** With above notations, \((V, \{ \cdot, \cdot \}_1, \cdot_1, \psi_{\cdot_1}, \psi_\cdot)\) is a \( \mathcal{P} \)-module Poisson superalgebra if and only if the direct sum of vector spaces \( \mathcal{P} \oplus V \) is turned to a Poisson superalgebra with the operations defined as

\[
(x, u), (y, v) = ([x, y], \psi_{\cdot_1}(x)v - (-1)^{|x||u|} \psi_{\cdot_1}(y)u + [u, v]_1), \quad (6.64)
\]

\[
(x, u) \cdot (y, v) = (x \cdot_1 y, \psi_\cdot(x)v + (-1)^{|x||y|} \psi_\cdot(y)u + u \cdot_1 v), \quad (6.65)
\]

for any \( x, y \in \mathcal{P}, \ u, v \in V \).

**Proof.** Straightly from Proposition[2,2] \(\square\)

If \((\mathcal{P}, \{ \cdot, \cdot \}, \cdot_1, ad_{\cdot_1}, L_\cdot)\) is a \( \mathcal{P} \)-module Poisson superalgebra.
Definition 6.4. A (left) post-Lie superalgebra is a superspace $\mathcal{A}$ with two even bilinear operations $([\cdot,\cdot],\circ)$ satisfying the following equations:

\[
[x,y] = -(1)^{|x||y|}[y,x],
\]

\[
(1)^{|x||z|}[x,[y,z]] + (1)^{|x||y|}[z,[x,y]] + (1)^{|y||z|}[y,[z,x]] = 0,
\]

\[
(1)^{|y||z|} \circ (y \circ x) - y \circ (z \circ x) + (y \circ z) \circ x - (1)^{|z||y|}(z \circ y) \circ x + [y,z] \circ x = 0,
\]

\[
z \circ [x,y] - [z \circ x,y] - (1)^{|z||x|}[x,z \circ y] = 0,
\]

for any elements $x,y,z$ in $\mathcal{A}$.

Definition 6.5. A (left) commutative dendriform supertrialgebra is a superspace $\mathcal{A}$ equipped with two even bilinear operations $(\cdot,\cdot) : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ satisfying the following equations:

\[
x \cdot y = (1)^{|x||y|}y \cdot x,
\]

\[
(x \cdot y) \cdot z = x \cdot (y \cdot z),
\]

\[
x > (y > z) = (x > y + (1)^{|x||y|}y > x + x \cdot y) > z,
\]

\[
(x > y) \cdot z = x > (y \cdot z),
\]

for any elements $x,y,z$ in $\mathcal{A}$.

Definition 6.6. A (left) post-Poisson superalgebra is a superspace $\mathcal{A}$ equipped with four even bilinear operations $([\cdot,\cdot],\circ,\cdot,\cdot)$ such that $(\mathcal{A},[\cdot,\cdot],\circ)$ is a (left) post-Lie superalgebra, $(\mathcal{A},\cdot,\cdot)$ is a (left) commutative dendriform supertrialgebra, and they are compatible in the sense that, for any elements $x,y,z$ in $\mathcal{A}$,

\[
[x,y \cdot z] = [x,y] \cdot z + (1)^{|x||y|}y \cdot [x,z],
\]

\[
[x,z > y] = (1)^{|x||z|}z > [x,y] - (1)^{|y||z|+[y][z]}y \cdot (z \circ x),
\]

\[
x \circ (y \cdot z) = (x \circ y) \cdot z + (1)^{|y||z|}y \cdot (x \circ z),
\]

\[
(y > z + (1)^{|y||z|}z > y + y \cdot z) \circ x = (1)^{|y||z|}z > (y \circ x) + y > (z \circ x),
\]

\[
x \circ (z > y) = (1)^{|x||z|}z > (x \circ y) + (x \circ z - (1)^{|y||z|}z \circ x + [x,z] > y).
\]

Theorem 6.1. Let $(\mathcal{A},[\cdot,\cdot],\circ,\cdot,\cdot)$ be a post-Poisson superalgebra. Define two new even bilinear map $(\cdot,\cdot) : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ on $\mathcal{A}$ by

\[
[x,y] = x \circ y - (1)^{|x||y|}y \circ x + [x,y], \quad x \cdot y = x > y + (1)^{|x||y|}y > x + x \cdot y, \quad \forall x,y \in \mathcal{A}.
\]

Then $(\mathcal{A},\cdot,\cdot,\cdot)$ becomes a Poisson superalgebra.

It is called the associated Poisson superalgebra of $(\mathcal{A},[\cdot,\cdot],\circ,\cdot,\cdot)$ and is denoted by $(P(\mathcal{A}),\cdot,\cdot,\cdot)$. Moreover, $(\mathcal{A},[\cdot,\cdot],\cdot,L_1,L_\circ)$ is a $P$-module Poisson superalgebra of $(P(\mathcal{A}),\cdot,\cdot,\cdot)$.
Proof. For any \(x, y, z \in \mathcal{A}\), we will check that \((\mathcal{A}, \cdot)\) is a Lie superalgebra. In fact, we have

\[
[x, y] = x \cdot y - (-1)^{|x||y|} y \cdot x + [x, y]
\]

\[
= -(-1)^{|x||y|} y \cdot x + x \cdot y - (-1)^{|x||y|}[y, x]
\]

\[
= -(-1)^{|x||y|}(y \cdot x - (-1)^{|x||y|} x \cdot y + [y, x])
\]

\[
= -(-1)^{|x||y|}[y, x].
\]

So the skew-supersymmetry holds. For the superJacobi identity, we have

\[
-(-1)^{|x||y|}[x, y, z] + (-1)^{|y||z|}[y, z, x] + (-1)^{|z||x|}[z, x, y] \\
= -(-1)^{|x||y|}[x, y \circ z - (-1)^{|y||z|} y \circ z + [y, z]] + (-1)^{|y||z|}[y, z \circ x - (-1)^{|z||x|} z \circ x + [x, z]] \\
+ (-1)^{|z||x|}[z, x \circ y - (-1)^{|x||y|} x \circ y + [x, y]]
\]

\[
= -(-1)^{|x||y|} x \circ (y \circ z) - (-1)^{|x||y|+|y||z|} x \circ (z \circ y) + (-1)^{|x||y|} x \circ [y, z] - (-1)^{|x||y|}(y \circ z) \circ x \\
+ (-1)^{|y||z|+|y||z|}(z \circ y) \circ x - (-1)^{|x||y|}[y, z] \circ x - (-1)^{|z||x|}[x, y \circ z] - (-1)^{|z||x|+|y||z|}[x, z \circ y] \\
+ (-1)^{|x||y|}[y, [y, z]] + (-1)^{|x||y|} y \circ (z \circ x) - (-1)^{|x||y|+|x||z|} y \circ (x \circ z) + (-1)^{|x||y|} y \circ [x, z] \\
- (-1)^{|y||z|} (z \circ x) \circ y + (-1)^{|y||z|+|z||y|} (x \circ z) \circ y - (-1)^{|y||z|}[z, x] \circ y + (-1)^{|y||z|}[y, z \circ x]
\]

\[
- (-1)^{|x||y|+|y||z|}[y, x \circ z] + (-1)^{|y||z|}[y, [y, z]] + (-1)^{|y||z|} z \circ (x \circ y) - (-1)^{|y||z|+|y||z|} z \circ (y \circ x) \\
+ (-1)^{|y||z|} z \circ [y, x] - (-1)^{|x||y|+|x||z|}[z, x \circ y] + (-1)^{|y||z|}[z, [y, x]] = 0.
\]

Next, we check that \((\mathcal{A}, \bullet)\) is a commutative associative superalgebra. In fact, for any \(x, y, z \in \mathcal{A}\), also one may check directly that:

\[
x \bullet (y \circ z) = x \cdot y = x \cdot y + (-1)^{|x||y|} y \cdot x = (-1)^{|x||y|}(y \cdot x + (-1)^{|x||y|} x \cdot y) = (-1)^{|x||y|} y \bullet x.
\]

Now we check the following equality for any \(x, y, z \in \mathcal{A}\), \(x \bullet (y \circ z) = (x \bullet y) \circ z\). In fact, we have

\[
x \circ (y \circ z) = x \circ (y \circ z) + (-1)^{|y||z|} y \circ z + y \circ z \\
= x \circ (y \circ z) + (-1)^{|y||z|} y \circ z + y \circ z + [x, y \circ z] + (-1)^{|y||z|} y \circ z + y \circ z \\
+ (-1)^{|y||z|} y \circ z + (-1)^{|y||z|} y \circ z + x \cdot (y \cdot z) \\
= x \circ (y \circ z) + (-1)^{|y||z|} y \circ z + x \cdot (y \cdot z) + (-1)^{|y||z|} y \circ z + x \cdot (y \cdot z) \\
+ (-1)^{|y||z|} y \circ z + (-1)^{|y||z|} y \circ z + x \cdot (y \cdot z) + (-1)^{|y||z|} y \circ z + x \cdot (y \cdot z)
\]

\[
= (x \bullet y) \bullet z.
\]

Finally, we check the condition: \([x, y \bullet z] = [x, y] \bullet z + (-1)^{|x||y|} y \bullet [x, z]\). In fact, we have

\[
[x, y \bullet z] = [x, y \circ z + (-1)^{|y||z|} y \circ z + y \circ z] = x \circ (y \circ z + (-1)^{|y||z|} y \circ z + y \circ z) \\
- (-1)^{|x||y|+|x||z|}[y, z \circ x + (-1)^{|y||z|} y \circ z + y \circ z] \\
= x \circ (y \circ z + (-1)^{|y||z|} y \circ z + x \circ (y \circ z) - (-1)^{|x||y|+|x||z|} z \circ (y \circ z) - (-1)^{|x||y|+|x||z|} z \circ (y \circ z) \\
+ [x, y \circ z] + (-1)^{|y||z|} [x, z \circ y] + [x, y \circ z] \\
= (x \circ y) \circ z - (-1)^{|y||z|} (y \circ x) \circ z + [x, y] \circ z + (-1)^{|x||z|+|y||z|} z \circ (x \circ y) - (-1)^{|x||z|+|y||z|} z \circ (y \circ x)
\]

\[
= (x \circ y) \circ z + [x, y] \circ z + (-1)^{|x||y|+|x||z|} z \circ (x \circ y) - (-1)^{|x||z|+|y||z|} z \circ (y \circ x) = (x \circ y) \circ z.
\]

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In fact, we have
\[ (x \circ y) \cdot z = (x \circ y) \cdot z - (1)^{|x||y|} (y \circ x) \cdot z + [x, y] \cdot z + (1)^{|x||y|} y \cdot (x \circ z) \\
- (1)^{|x||y|} (y \circ z) > [x, z] + (1)^{|x||y|} y \cdot (x \circ z) > y - (1)^{|x||y|} (y \circ z) > [x, z] > y \\
+ (1)^{|x||y|} [x, z] > y + (1)^{|x||y|} y \cdot (x \circ z) - (1)^{|x||y|} [x, z] > (1)^{|x||y|} y \cdot [x, z] \\
= [x, y] \cdot z + (1)^{|x||y|} y \cdot [x, z], \]
as desired. Thus \((\mathcal{A},\{,\},\bullet)\) is a Poisson superalgebra.

\[\square\]

**Definition 6.7.** Let \((\mathcal{A},\{,\},\bullet)\) be a Poisson superalgebra and \((V,\{,\}_1,\bullet_1,\psi_1,\psi_\bullet)\) be an \(\mathcal{A}\)-module Poisson superalgebra. An even linear map \(T : V \rightarrow \mathcal{A}\) is called an \(O\)-operator of weight \(\lambda \in K\) associated to \((V,\{,\}_1,\bullet_1,\psi_1,\psi_\bullet)\) if for any \(u,v \in V\)

\[
T(u), T(v) = T((\psi_1)(T(u))v - (1)^{|u||v|} \psi_1, (T(v))u + \lambda [u, v]),
\]

\[(6.80)\]

\[
T(u) \cdot T(v) = T(\psi_\bullet (T(u))v + (1)^{|u||v|} \psi_\bullet (T(v))u + \lambda u \cdot_1 v).
\]

\[(6.81)\]

When \((V,\{,\}_1,\bullet_1,\psi_1,\psi_\bullet) = (\mathcal{A},\{,\},\bullet, ad, L\), Eqs \((6.80)\) and \((6.81)\) become

\[
[T(u), T(v)] = T([T(u),v] + [u, T(v)] + \lambda [u, v]),
\]

\[(6.82)\]

\[
T(u) \cdot T(v) = T(T(u) \cdot v + u \cdot T(v) + \lambda u \cdot v),
\]

\[(6.83)\]

respectively. Equations \((6.82)\) and \((6.83)\) imply that \(T : \mathcal{A} \rightarrow \mathcal{A}\) is a Rota-Baxter operator of weight \(\lambda \in K\) on the Lie superalgebra \((\mathcal{A},\{,\})\) and on the commutative associative superalgebra \((\mathcal{A},\bullet)\), respectively.

**Theorem 6.2.** Let \((\mathcal{A},\{,\},\bullet)\) be a Poisson superalgebra and \((V,\{,\}_1,\bullet_1,\psi_1,\psi_\bullet)\) be an \(\mathcal{A}\)-module Poisson superalgebra. Let the even linear map \(T : V \rightarrow \mathcal{A}\) be an \(O\)-operator of weight \(\lambda \in K\) associated to \((V,\{,\}_1,\bullet_1,\psi_1,\psi_\bullet)\). Define four new even bilinear operations \(\{,\}, \circ, \cdot, \succ : V \otimes V \rightarrow V\) as follows:

\[
[u,v] = \lambda [u,v], \quad u \circ v = \psi_1 (T(u))v, \quad u \cdot v = \lambda u \cdot_1 v \quad u \succ v = \psi_\bullet (T(u))v, \quad \forall u,v \in V.
\]

\[(6.84)\]

Then \((V,\{,\},\circ, \cdot, \succ)\) is a post-Poisson superalgebra and \(T\) is a homomorphism of Poisson superalgebras from the associated Poisson superalgebra \(P(V)\) of \((V,\{,\},\circ, \cdot, \succ)\) to \((\mathcal{A},\{,\},\bullet)\).

**Proof.** First we check that \((V,\{,\},\circ)\) is a post-Lie superalgebra. For any \(u,v,w \in V\), it is easy to obtain

\[
[u,v] = -(1)^{|u||v|} [v,u],
\]

\[
(1)^{|u||w|} [u,v,w] + (1)^{|u||v|} [w,u,v] + (1)^{|v||w|} [v,w,u] = 0.
\]

So it is sufficient to verify the following conditions:

\[
(1)^{|u||v|} w \circ (v \circ u) - v \circ (w \circ u) + (v \circ w) \circ u - (1)^{|u||v|} (w \circ v) \circ u + [v,w] \circ u = 0,
\]

\[
w \circ [u,v] - [w \circ u,v] - (1)^{|u||v|} [u,w \circ v] = 0.
\]

In fact, we have

\[
(1)^{|u||v|} w \circ (v \circ u) - v \circ (w \circ u) + (v \circ w) \circ u - (1)^{|u||v|} (w \circ v) \circ u + [v,w] \circ u = 0,
\]

\[
= (1)^{|u||w|} \psi_1 (T(w))(\psi_1 (T(v))u - \psi_1 (T(v))\psi_1 (T(w))u + \psi_1 (T\psi_1 (T(v))w)u
\]

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Definition 6.8. A coboundary Poisson superbialgebra \((A, \{\cdot,\cdot\}, \delta, \Delta, r)\) is called quasitriangular if \(r\) is a solution of PYBE, that is, it satisfies both CYBE and AYBE.

\(2\) If \((A, \{\cdot,\cdot\}, \delta, \Delta, r)\) are both quasitriangular, then 
\[
\psi_{11}(T(u))\psi_{11}(T(v))u - \psi_{11}(T(v))\psi_{11}(T(u))u + \psi_{11}(T(v), T(u))u = 0,
\]
and
\[
w \circ [u, v] - [w \circ u, v] - (-1)^{|u||v|}[u, w \circ v] = \lambda w \circ [u, v]_1 - [w \circ u, v]_1 - (-1)^{|u||v|}[u, w \circ v]_1 = \lambda [\psi_{11}(T(u))u, v]_1 - [\psi_{11}(T(v))u, v]_1 - (-1)^{|u||v|}[u, \psi_{11}(T(u))v]_1 = 0,
\]
as required.

Next we will check that \((V, \cdot, >)\) is a commutative dendriform superbialgebra. For any \(u, v, w \in V\), it is easy to obtain that \(u \cdot v = (-1)^{|u||v|}v \cdot u\), and \((u \cdot v) \cdot w = u \cdot (v \cdot w)\). So we need to check the following conditions:

\[
(u > v > w) = (u > v + (-1)^{|u||v|}v > u + u \cdot v) > w,
\]
\[
(u \circ v) \cdot w = u > (v \cdot w).
\]

For the first equality, we calculate
\[
(u \circ v + (-1)^{|u||v|}v \circ u + u \cdot v) > w
\]
\[
= \left(\psi_{11}(T(u))v + (-1)^{|u||v|}\psi_{11}(T(v))u + \lambda u \circ_1 v\right) > w
\]
\[
= \psi_{11}(T(\psi_{11}(T(u))v + (-1)^{|u||v|}\psi_{11}(T(v))u + \lambda u \circ_1 v))w
\]
\[
= \psi_{11}(T(u))\psi_{11}(T(v))w
\]
\[
= u > (v \circ w).
\]

For the second equality, we have
\[
(u \circ v) \cdot w = \lambda \psi_{11}(T(u))v \circ_1 w = \psi_{11}(T(u))(v \circ w) = u > (v \circ w).
\]

The proof of other cases is similar. Then \((V, [\cdot,\cdot], \cdot, >)\) becomes a post-Poisson superalgebra and \(T\) is a homomorphism of Poisson superalgebras from the associated Poisson superalgebra \(P(V)\) of \((V, [\cdot,\cdot], \cdot, >)\) to \((A, [\cdot,\cdot], \cdot)\).

\[\square\]

**Definition 6.8.** A coboundary Poisson superbialgebra \((A, \{\cdot,\cdot\}, \bullet, \delta, \Delta, r)\) is called quasitriangular if \(r\) is a solution of PYBE, that is, it satisfies both CYBE and AYBE.

**Remark 6.1.** 1) Obviously, a coboundary Poisson superbialgebra \((A, \{\cdot,\cdot\}, \bullet, \delta, \Delta, r)\) is quasitriangular if and only if \((A, \{\cdot,\cdot\}, \delta, \Delta, r)\) as a coboundary Lie superbialgebra and \((A, \bullet, \Delta, r)\) as a coboundary infinitesimal superbialgebra are both quasitriangular.

2) If \((A, \{\cdot,\cdot\}, \bullet, \delta, \Delta, r)\) is a quasitriangular Poisson superbialgebra. Then by Theorem 5.7, \(r\) also satisfies the following equations:

\[
(ad_{L_{\bullet}}(x) \otimes id + (-1)^{|x||r|}id \otimes ad_{L_{\bullet}}(x))(r + \tau(r)) = (L_{\bullet}(x) \otimes id + (-1)^{|x||r|}id \otimes L_{\bullet}(x))(r + \tau(r)) = 0,
\]
for all \(x \in A\).
Let $V = V_0 \oplus V_1$ be a finite-dimensional superspace and $r \in V \otimes V$. Then $r$ can be identified as an even linear map from $V^*$ to $V$ which we still denote by $r$ through
\[
\langle b, r(a) \rangle = \langle a \otimes b, r \rangle, \quad \forall \ a, b \in V^*.
\] (6.86)

Define a even linear map $r' : V^* \to V$ by
\[
\langle a, r'(b) \rangle = \langle r, a \otimes b \rangle, \quad \forall \ a, b \in V^*.
\] (6.87)

We call
\[
\alpha = \alpha_r = \frac{r - r'}{2}, \quad \beta = \beta_r = \frac{r + r'}{2},
\] (6.88)

the skew-supersymmetry part and the supersymmetric part of $r$ respectively. Therefore, we have the following theorem.

**Theorem 6.3.** Let $(\mathcal{P}, [,], \bullet, \delta, \Delta, r)$ be a quasitriangular Poisson superbialgebra. Let the even linear map $\beta$ be defined by Eq. 6.88. Define two new even bilinear operations $[\cdot, \cdot], \cdot : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P}$ as follows:
\[
[a, b] = -2ad_{\cdot|\cdot}^r(\beta(a))b, \quad a \cdot b = 2L^\bullet_r(\beta(a))b, \quad \forall \ a, b \in \mathcal{P}^*.
\] (6.89)

Then $(\mathcal{P}^*, [,], \cdot, ad_{\cdot|\cdot}^r, -L^\bullet_r)$ becomes a $\mathcal{P}$-module Poisson superalgebra of $(\mathcal{A}, [,], \bullet)$. Moreover, regarded as an even linear map from $\mathcal{P}^* \to \mathcal{P}$ through Eq. $\text{fin} 6.86$ is an $O$-operator of weight 1 associated to $(\mathcal{P}^*, [,], \cdot, ad_{\cdot|\cdot}^r, -L^\bullet_r)$ that is,
\[
[r(a), r(b)] = r(ad_{\cdot|\cdot}^r(r(a))b - (-1)^{|a||b|}ad_{\cdot|\cdot}^r(r(b))a + [a, b]), \quad \forall \ a, b \in \mathcal{P}^*,
\] (6.90)
\[
r(a) \cdot r(b) = r(-L^\bullet_r(r(a))b - (-1)^{|a||b|}L^\bullet_r(r(b))a + a \cdot b). \quad \forall \ a, b \in \mathcal{P}^*,
\] (6.91)

The following result establishes a close relation between a post-Poisson superalgebra and a quasitriangular Poisson superbialgebra.

**Theorem 6.4.** With the conditions and notations in Theorem 6.3, define four new even bilinear operations $[\cdot, \cdot], \circ, \cdot, > : \mathcal{P}^* \otimes \mathcal{P}^* \to \mathcal{P}^*$ on $\mathcal{P}^*$ as follows:
\[
[a, b] = -2ad_{\cdot|\cdot}^r(\beta(a))b, \quad a \circ b = ad_{\cdot|\cdot}^r(r(a))b,
\] (6.92)
\[
a \cdot b = 2L^\bullet_r(\beta(a))b, \quad a > b = -L^\bullet_r(r(a))b, \quad \forall \ a, b \in \mathcal{P}^*.
\] (6.93)

Then $(\mathcal{P}^*, [\cdot, \cdot], \circ, \cdot, >)$ becomes a post-Poisson superalgebra and $r$ is a homomorphism of Poisson superalgebras from the associated Poisson superalgebra $\mathcal{P}(\mathcal{P}^*)$ of $(\mathcal{P}^*, [,], \circ, \cdot, >)$ to $(\mathcal{P}, [,], \bullet)$.

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