Two-dimensional turbulent flow in a channel of constant width

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Abstract. The paper deals with the classical problem about a flow of viscous incompressible fluid in an infinite channel of constant width. The problem is solved in the two-dimensional case, both in the laminar and turbulent flow regimes, using as a mathematical model the Navier-Stokes equations in terms of vorticity-stream functions. The presented algorithm is a semi-analytical iterative method, based on the construction of special Green’s functions in the form of expansion in a Fourier series over the transverse coordinate for the channel.

1. Introduction

Last decades are characterized by rapid development of computer science. This affects the numerical methods applied for calculation of fluid flows. Generally, there are two different approaches to simulate flows of viscous incompressible fluids.

The first class of methods to simulate viscous fluid flow is a direct numerical simulation (DNS). Such methods are studied in detail in [1–4]. This is to solve Navier-Stokes in a discrete flow domain, by introducing a computational grid. Some grid functions are typically introduced on this grid, whose values are determined from discrete Navier-Stokes equations. Currently, such methods are being rapidly developed. This is not surprising since the results obtained by these methods agree very well with the results of physical experiments. Moreover, the DNS techniques can help to define new important physical principles of turbulence by non-physical experiments. Therefore, recently the DNS method has replaced physical experiments by numerical experiments. The basic defect is that there is a need to construct numerical grid over the domain under consideration, and then to define all grid functions. The problem is connected with the fact that correct simulation of physical parameters in a turbulent flow is connected with continuous scales of its motion. For example, to calculate near-wall flows it is necessary to take into account the quantities of the order Re^{21/8}, and to take into account the quantities of the order Re^{-7/2}, to calculate dynamically significant oscillations of the velocity at large Reynolds numbers [2]. This results in the fact that with the usage of the DNS for turbulent flows, one should restrict the study by the case of small Reynolds numbers and by simplest geometries like plates, channels, tubes of round, elliptic and rectangular cross-sections [3–7].

The second class of methods permits consideration of complex, real geometries in viscous fluid simulation. These methods are united by the common name — Immersed Boundary Methods (IBM). The basic idea of the IBM methods are some simplifications for numerical grid on the
boundary of the fluid and inside the medium at hand. For this aim, the medium is removed from the domain under consideration, and its influence to the flow is taken into account by introducing new force terms to the principal equations, in the modelling of the flow. The IBM methods thus permit solution of respective differential equations, which describe the flow of viscous fluid in the case of complex geometries, on simple grids. These methods are presented in more detail in works [8–11]. It should be noted that also these methods have their own defects. The key defect of these methods is, by our opinion, that it is impossible to study the flow near the immersed boundary, which is in fact the most interesting zone. This is because the immersed boundary is simulated by the Dirac’s delta-function, see works [12, 13], which leads to missing of the solution’s smoothness over the immersed boundary, and one obtains near the boundary some non-physical oscillations of the solution transmitted through the boundary.

In the present paper we propose a certain method which simulates the flow of incompressible viscous fluid in a channel of constant width, with a given fluid consumption. The proposed method is semi-analytical, i.e., in the contrast to the above discussed methods, this applies an analytical solution of the problem, at each temporal step. The application of the numerical methods in this case is not a part of numerical simulation, but only a technique allowing us to solve some boundary integral equations.

2. Problem formulation

Let us study the classical problem about a uniform turbulent flow of viscous incompressible fluid in an infinite channel. The problem is studied in the two-dimensional formulation. Let us assume that the width of the channel is $b$. The roughness of the channel walls as a factor, which affects the laminar-turbulent transition, is ignored. The walls are absolutely rigid, impenetrable for fluid, and the no-slip condition should be satisfied over the walls.

Let us introduce the Cartesian coordinate system so that axis $Ox$ is directed along the channel, axis $Oy$ is directed transversally to the channel (see figure 1). Let us denote the velocity vector as $\vec{v} = \{v_x, v_y\} = \{u, v\}$, where $u(x, y, t)$ is the longitudinal component of the velocity, and $v(x, y, t)$ is its transverse component. Let us perform the simulation of the fluid flow in the channel, by using the Navier-Stokes equations in terms of stream and vorticity functions [13]:

$$\frac{\partial \zeta}{\partial t} = -u \frac{\partial \zeta}{\partial x} - v \frac{\partial \zeta}{\partial y} + \nu \Delta \zeta, \quad \zeta = \Delta \psi. \quad (1)$$

The components of the velocity vector of any fluid particle $u(x, y, t)$, $v(x, y, t)$ are related with the stream function $\psi(x, y, t)$ and the vorticity $\zeta(x, y, t)$, in the following way

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad \zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}. \quad (2)$$
Let us assume that the fluid consumption is known to be constant along variable $x$ and along time $t$:

$$Q = \int_0^b u(x, y, t) \, dy = \psi(x, b, t) - \psi(x, 0, t) \equiv \text{const},$$  \hspace{1cm} (3)

Correct solution of the formulated problem requires to take into account the small-scale vortices that results in increasing number of nodes of the computational grid.

Relations (1)–(3) imply the following boundary conditions over the lower and the upper boundary walls of the channel:

$$\psi(x, 0, t) = 0, \quad \psi(x, b, t) = Q, \quad \frac{\partial \psi}{\partial y}(x, 0, t) = 0, \quad \frac{\partial \psi}{\partial y}(x, b, t) = 0.$$  \hspace{1cm} (4)

Various methods to solve the formulated problem are discussed in [14–20]. In the present work we develop a new approach which is based upon the following assumptions. First, it is experimentally confirmed that the choice of the initial velocity field does not change the qualitative character of the turbulent flow. Therefore, it is assumed that at the initial moment of time there is accepted a certain initial condition for the kinematic quantities. Second, it is assumed that the uniform turbulent flow in the infinite channel has the same properties along the flow, if the characteristics of these flow are registered with a sufficiently long period $L$ along axis $Ox$. This assumption makes possible to write out periodic boundary conditions for the stream function and the vorticity, as well as for their derivatives.

By taking into account the above assumptions and using the simplest Euler’s finite-difference implicit scheme along time for the viscous term in (1) and explicit scheme for the convective term, the problem may be reduced at each temporal step to the pair of the following elliptic problems at $n$th temporal step ($\theta$ is the step over time):

$$\zeta_n - \varepsilon \Delta \zeta_n = g_{n-1}, \quad \Delta \psi_n = \zeta_n, \quad \varepsilon = \nu \theta, \quad g_{n-1} = \zeta_{n-1} + \theta \left( \frac{\partial \psi}{\partial x} \frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \zeta}{\partial x} \right)_{n-1},$$  \hspace{1cm} (5)

considered in the rectangle $(x, y) \in [(0, L) \times (0, b)]$, with the boundary condition (4) over the walls of the channel, and the following condition of periodicity over the side faces of the rectangle:

$$\psi\big|_{x=0} = \psi\big|_{x=L}, \quad \zeta\big|_{x=0} = \zeta\big|_{x=L}, \quad \frac{\partial \psi}{\partial x}\big|_{x=0} = \frac{\partial \psi}{\partial x}\big|_{x=L}, \quad \frac{\partial \zeta}{\partial x}\big|_{x=0} = \frac{\partial \zeta}{\partial x}\big|_{x=L}.$$  \hspace{1cm} (6)

It should also be noted that if one applies a certain more advanced finite-difference scheme instead of Euler’s one, say the Adams-Bashforth scheme in a combination with the Crank-Nicolson scheme [4], then the system of two elliptic equations keeps its form (5), with a slight modification of expression for function $g$, and with another form of expression for parameter $\varepsilon$.

The scheme (5) possesses linear convergence with the time step $\theta$, and it is conditionally convergent. The Adams-Bashforth plus Crank-Nicolson combination is quadratically convergent, but still conditionally. This means that convergence of the process in time is attained only for sufficiently small $\theta$.

3. Expressions for stream and vorticity functions

It is stated above that, to find these two functions, one should solve two elliptic equations: the Helmholtz equation for vorticity function $\zeta$ and the Poisson equation for stream function $\psi$. It is obvious that despite these are two different equations, their solutions are closely connected with each other.

The classical approach to these two elliptic equations is to discretize them, and then – to reduce them to systems of linear algebraic equations (SLAE). For example, for the Helmholtz
equation it is quite efficient to use the $LU$ decomposition of the matrix $A$ of the initial SLAE, in particular the $LDL^T$ method, where $L$ is a low-triangle matrix, $D$ is a diagonal matrix, $L^T$ is an upper-triangle matrix. The main advantage of this method is that if such a decomposition is obtained only once, this further permits solution of the SLAE for many right-hand sides. However, this advantage of the method is at the same time its disadvantage, since a storage of this $LDL^T$ requires too much memory.

Regarding Poisson’s equation, there have been proposed various methods to solve it. The most popular one is now its reduction to a three-diagonal SLAE and further reduction to a certain code parallelization, by using a graphical accelerator of the Nvidia CUDA graphics cards. A more detailed survey can be found in [21]. The evident difficulty of such an approach is the complicated program code and the need to apply special kinds of computer platforms.

The method proposed in the present paper is based not on numerical but on analytical solution, for both Helmholtz and Poisson equation. A Boundary Integral Equation (BIE) method is applied, to solve both the equations. The general detailed description of the BIE method is presented in [22]. It should be noted that application of the BIE method requires two Green’s functions – for stream and vorticity functions, which satisfy the homogeneous boundary conditions on the walls, i.e. at $h = 0, b$ and the periodic conditions in the horizontal direction. The detailed derivation of these functions can be found in our recent work [23]. Here we demonstrate their expressions:

$$G_\psi(\xi, \eta, x, y) = -\sum_{m=1}^{\infty} \frac{e^{-b_m(L-|\xi-x|)} + e^{-b_m|\xi-x|}}{\pi m (1 - e^{-b_mL})} \sin(b_m \eta) \sin(b_m y), \quad b_m = \frac{\pi m}{b}, \quad (7)$$

$$G_\zeta(\xi, \eta, x, y) = \sum_{m=1}^{\infty} \frac{e^{-\lambda_m(L-|\xi-x|)} + e^{-\lambda_m|\xi-x|}}{b \varepsilon \lambda_m (1 - e^{-\lambda_mL})} \sin(b_m \eta) \sin(b_m y), \quad \lambda_m = \sqrt{b_m^2 + \frac{1}{\varepsilon}}. \quad (8)$$

By applying the BIE method to the elliptic boundary value problems (5) one obtains the following expressions:

$$\psi^{(n)}(x, y) = \int_0^L \int_0^b \zeta^{(n)}(\xi, \eta) G_\psi(\xi, \eta, x, y) \, d\xi \, d\eta + Q \int_0^L \frac{\partial G_\psi}{\partial \eta}(\xi, b, x, y) \, d\xi, \quad (9)$$

$$\zeta^{(n)}(x, y) = \int_0^L \int_0^b g^{(n-1)}(\xi, \eta) G_\zeta(\xi, \eta, x, y) \, d\xi \, d\eta - \int_0^L \left[ \zeta^{(n)}(\xi, 0) \frac{\partial G_\zeta}{\partial \eta}(\xi, 0, x, y) - \zeta^{(n)}(\xi, b) \frac{\partial G_\zeta}{\partial \eta}(\xi, b, x, y) \right] \, d\xi. \quad (10)$$

Let us calculate the second integral in (9). If one substitutes there expression for Green’s function $G_\psi$ taken from (7), then one obtains

$$Q \int_0^L \frac{\partial G_\psi}{\partial \eta}(\xi, b, x, y) \, d\xi = -\frac{2Q}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m \sin(b_m y)}{m} = \frac{Qy}{b}. \quad (11)$$

Let us substitute expressions (7), (8), (11) to formulas (9) and (10), extracting slowly convergent part of the functional series and applying summation. Then we obtain exact solutions of the
elliptic equations (5) in the following form:

\[ \psi^{(n)}(x, y) = \varepsilon \left\{ \sinh((b - y)/\sqrt{\varepsilon}) \sinh(b/\sqrt{\varepsilon}) + \frac{y}{b} - 1 \right\} \zeta^{(n)}(x, 0) + \left[ \frac{\sinh(y/\sqrt{\varepsilon})}{\sinh(b/\sqrt{\varepsilon})} - \frac{y}{b} \right] \zeta^{(n)}(x, b) \]

\[ + \frac{\varepsilon}{b} \sum_{m=1}^{\infty} \frac{1}{\lambda_m} \int_{0}^{L} \left[ \zeta''(\tau, 0) - (-1)^m \zeta_n(\tau, b) \right] \text{sign}(\tau - x) \frac{\lambda_m e^{-b_m(L-|\tau-x|)} - e^{-b_m|\tau-x|}}{1 - e^{-b_mL}} d\tau(b_my) + \frac{Qy}{b} \]

\[ = \frac{1}{\lambda_m} \int_{0}^{L} \left[ \zeta''(\tau, 0) - (-1)^m \zeta_n(\tau, b) \right] \text{sign}(\tau - x) \frac{e^{-\lambda_m(L-|\tau-x|)} - e^{-\lambda_m|\tau-x|}}{1 - e^{-\lambda_mL}} d\tau(b_my) \]

where

\[ J_m(\tau, x) = 2 \varepsilon \left\{ \lambda_m(1 - e^{-\lambda_mL})[e^{-b_m(L-|\tau-x|)} + e^{-b_m|\tau-x|}] \right\} + b_m(1 - e^{-b_mL})[e^{-\lambda_m(L-|\tau-x|)} + e^{-\lambda_m|\tau-x|}] \}

\[ \lambda_m = \frac{1}{b \lambda^2_m(1 - e^{-\lambda_mL})} \int_{0}^{L} g_m^{(n-1)}(\tau, x) \sin(b_m \eta) d\eta \]

It should be noted that to find (12) and (13) on the nth temporal layer, it is necessary to know their values on the (n - 1)st temporal layer, as well as the values of functions \( \zeta^{(n)}(x, 0) \), \( \zeta^{(n)}(x, b) \), \( \zeta_n(x, 0) \), and \( \zeta'_n(x, b) \). Hence, the presented expressions (12) and (13) are only some implicit representations for stream and vorticity functions.

It should also be noted that all terms in the series in expressions (12), (13) have the order \( O(1/m^3) \) as \( m \to \infty \), which is quite sufficient for their differentiation applied to variables \( x \) and \( y \), and so for correct treatment of function \( g_m^{(n-1)}(x, y) \).

In order to calculate stream and vorticity functions, one should treat efficiently the integrals of the following type:

\[ f(x) = \int_{0}^{L} \varphi(\tau)e^{-c|\tau-x|} d\tau \]

It can easily be seen that the direct treatment for all \( x \) belonging to a certain grid of dimension \( N \) requires \( O(N^2) \) arithmetic operations. Since integral (15) is of convolution type, then by using the FFT this can be treated in \( O(N \log N) \) operations. Now we will show that a refined approach permits its treatment in \( O(N) \) operations. In fact, integral in (15) is a sum of two ones:

\[ f(x) = f_t(x) + f_r(x), \quad f_t(x) = \int_{x}^{L} \varphi(\tau)e^{-c|\tau-x|} d\tau, \quad f_r(x) = \int_{x}^{L} \varphi(\tau)e^{-c|\tau-x|} d\tau \]

Then it is obvious that the simple iteration process:

\[ f_t(x + \Delta x) = e^{-c\Delta x}f_t(x) + (1 - e^{-c\Delta x})\varphi(x + \Delta x), \quad f_t(0) = 0, \]

\[ f_r(x - \Delta x) = e^{-c\Delta x}f_r(x) + (1 - e^{-c\Delta x})\varphi(x - \Delta x), \quad f_r(L) = 0, \]

(17)
permits calculation of all values of function $f$ in linear time at once.

4. Defining vorticity function on the walls of the channel

It is well known that the basic obstacle to satisfy the no-slip condition is that the latter is formulated for the stream function only [4]. Therefore, formally there is no boundary condition in the elliptic boundary value problem for the vorticity in (4). To find expressions (12) and (13), one should know the value of the vorticity and its derivative on the channel’s walls as functions of coordinate $x$ along the stream. This is a classical obstacle. To overcome the obstacle, various methods are described in literature [13]. The most popular and frequently used approach is to apply the Tom condition for the vorticity over the walls. The results on the basis of Tom’s condition well agree with experimental data. However, the Tom boundary condition has no physical meaning, being only a result of a mathematical discretization of the vortex near the rigid boundary.

In the present work we directly satisfy the remaining boundary conditions for quantities $(\partial \psi / \partial y)(x, 0) = 0$ and $(\partial \psi / \partial y)(x, b) = 0$. With the approach proposed, the problem to find the quantities $\zeta^n(x, 0)$, $\zeta^n(x, b)$, $\zeta^n_0(x, 0)$, and $\zeta^n(x, \tau, b)$ is reduced to the following system of integro-differential equations:

\[
\left( \frac{\varepsilon}{b} - \sqrt{\varepsilon} \coth \frac{b}{\sqrt{\varepsilon}} \right) \zeta^n(x, 0) + \left[ \frac{\sqrt{\varepsilon}}{\sinh(b/\sqrt{\varepsilon})} - \frac{\varepsilon}{b} \right] \zeta^n(x, b) + \frac{Q}{b} + \frac{\varepsilon}{b} \sum_{m=1}^{\infty} \frac{b_m}{\lambda_m} \int_0^L \left[ \zeta^n_0(\tau, 0) \right. \\
- (-1)^m \zeta^n_0(\tau, b) \left| \text{sign}(\tau - x) \right] \left[ \frac{\lambda_m e^{-b_m(L-|\tau-x|)} - e^{-b_m|\tau-x|}}{1 - e^{-b_mL}} - \frac{b_m e^{-\lambda_m(L-|\tau-x|)} - e^{-\lambda_m|\tau-x|}}{1 - e^{-\lambda_mL}} \right] d\tau \\
- \sum_{m=1}^{\infty} \frac{1}{4\varepsilon \lambda_m (1 - e^{-b_m|L|})(1 - e^{-\lambda_m|L|})} \int_0^L g^{(n-1)}(\tau) J_m(x, \tau) d\tau = 0, \quad (18a)
\]

\[
\left[ \frac{\varepsilon}{b} - \frac{\sqrt{\varepsilon}}{\sinh(b/\sqrt{\varepsilon})} \right] \zeta^n(x, 0) + \left( \frac{\sqrt{\varepsilon}}{\sinh(b/\sqrt{\varepsilon})} - \frac{\varepsilon}{b} \right) \zeta^n(x, b) + \frac{Q}{b} + \frac{\varepsilon}{b} \sum_{m=1}^{\infty} \frac{b_m}{\lambda_m} \int_0^L \left[ (-1)^m \zeta^n_0(\tau, 0) \right. \\
- \zeta^n_0(\tau, b) \left| \text{sign}(\tau - x) \right] \left[ \frac{\lambda_m e^{-b_m(L-|\tau-x|)} - e^{-b_m|\tau-x|}}{1 - e^{-b_mL}} - \frac{b_m e^{-\lambda_m(L-|\tau-x|)} - e^{-\lambda_m|\tau-x|}}{1 - e^{-\lambda_mL}} \right] d\tau \\
- \sum_{m=1}^{\infty} \frac{(-1)^m}{4\varepsilon \lambda_m (1 - e^{-b_m|L|})(1 - e^{-\lambda_m|L|})} \int_0^L g^{(n-1)}(\tau) J_m(x, \tau) d\tau = 0. \quad (18b)
\]

The solution to the system (18a), (18b) is constructed by the collocation technique, with further cut of the infinite series in the kernel. To solve the arising SLAEs, various numerical techniques may be applied.

5. The results of the calculations and general conclusions

The method proposed here is related to conditionally convergent methods [4]. This converges for sufficiently small time step $\theta$. If the Reynolds number is defined as $Re = Q/\nu$, then for a given flow consumption $Q$ the Reynolds number is defined by the kinematic viscosity. In our calculations we set $Q = 1 \text{m}^2/\text{s}$, then $Re = (1 \text{m}^2/\text{s})/\nu$. In practice, the method converges if the time step decreases proportionally with the decrease of viscosity, or with the increase of the Reynolds number, i.e. $\theta \sim 1/Re$. In literature there is known the theoretical critical Reynolds number for the two-dimensional channel. With our notations this is equal to $Re^* = 7696$ [24]. For two figures 2 and 3 the initial distribution of physical fields is taken as a strong perturbation of the Poiseuille flow — over both longitudinal and transverse coordinate. The initial profile of the longitudinal velocity is strongly asymmetric. The width of the channel $b = 1 \text{m}$. The elongation of the channel is taken $L/b = 7$. Figures 2 and 3 show the results of the calculations.
with the following parameters: the number of terms in the infinite series \( M = 4096 \), the time step is \( \theta = 5 \times 10^{-4} \) s, the number of nodes along the channel \( N = 210 \), the number of iterations \( 6 \times 10^4 \), full time in the physical space is \( t = 30 \) s.

**Figure 2.** Longitudinal velocity diagram in the laminar flow: \( \nu = 2 \times 10^{-4} \text{ m}^2/\text{s} \), \( \text{Re} = 5000 \)

**Figure 3.** Longitudinal velocity diagram in the turbulent flow: \( \nu = 1 \times 10^{-4} \text{ m}^2/\text{s} \), \( \text{Re} = 10000 \)

It should be noted that solid lines are related to the Poiseuille parabola, and dotted lines are obtained by the proposed method. It is clear from figure 2 that in the laminar flow a strongly asymmetric velocity diagram at the initial step then rapidly becomes symmetric with iterations, approaching with the time steps to the Poiseuille parabola. At the same time, as can be seen from figure 3, in the turbulent flow the velocity diagram remains asymmetric with time. The calculations performed show that the turbulent flow is oscillating, and one can clearly extract from the flow a principal harmonic oscillation mode with a certain period. With so doing, the maximum value on the diagram of the longitudinal velocity does periodically float from the left to the right and vice versa, with the periodic law described above.

Generally, when tracking the computations in both the laminar and turbulent regimes for the two cases discussed above, the initial steps over time are similar in both the regimes. Namely, the process starting from strongly asymmetric diagram for the longitudinal velocity component tends then to a symmetrization. This phase lasts for a short time, around few seconds in the physical time. Further, during a relatively long time, around ten seconds in the physical space, the laminar diagram is being explicitly “attracted” to a shape coinciding with the Poiseuille parabola. The turbulent diagram continues to remain periodically asymmetric, in the sense that its maximum point performs harmonic oscillations from left to right with a certain period. However, the shape of the turbulent diagram, apart that this is no more symmetric, remains visibly very similar to a certain shape close to parabola.

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