Is a typical bi-Perron algebraic unit a pseudo-Anosov dilatation?

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Abstract. In this note, we deduce a partial answer to the question in the title. In particular, we show that asymptotically almost all bi-Perron algebraic unit whose characteristic polynomial has degree at most $2n$ do not correspond to dilatations of pseudo-Anosov maps on a closed orientable surface of genus $n$ for $n \geq 10$. As an application of the argument, we also obtain a statement on the number of closed geodesics of the same length in the moduli space of area one abelian differentials for low genus cases.

Thurston’s celebrated theorem [19] classified self-homeomorphisms of compact orientable surfaces up to isotopy. As long as a homeomorphism does not admit a finite power which is isotopic to the identity and does not admit a finite union of disjoint simple closed curves which are preserved up to isotopy, it is isotopic to a so-called pseudo-Anosov map. Associated to a pseudo-Anosov map is a positive number called its dilatation. The logarithm of the dilatation of a pseudo-Anosov map is the topological entropy of the map. See [21] for details.

We call a number $\lambda$ to be a pseudo-Anosov dilatation if it is the dilatation constant for a pseudo-Anosov map on some compact orientable surface. We call an algebraic integer $\lambda$ bi-Perron if all the Galois conjugates of $\lambda$ are contained in an annulus of outer radius $\lambda$ itself and inner radius $1/\lambda$. Fried [7] showed that every pseudo-Anosov dilatation is a bi-Perron algebraic unit, and he conjectured that this is also a sufficient condition: every bi-Perron algebraic unit is realized as the dilatation constant of some pseudo-Anosov map.

This turned out to be a difficult question and there are many related works. Thurston [20] gave an answer to an Out($F_n$)-version of this question. In [20], Thurston also gave a few examples of constructions of pseudo-Anosov maps for given bi-Perron algebraic units (here the numbers are given as the slopes of some post-critically finite piecewise-linear self-homeomorphisms of the unit interval). These examples motivated the authors’ previous work to construct pseudo-Anosov maps in [2] for a bi-Perron algebraic unit given as the leading eigenvalue of a Perron-Frobenius matrix satisfying some additional properties. Both constructions in [20] and [2] are similar to the one given in [3]. See also, for instance, [1], [15], [17], [14], [11], [12], [13], [18] for various constructions of pseudo-Anosov maps.
In this short note, we formulate an easier version of Fried’s question, and give a statistical answer to it. To state our result precisely, we introduce some notation.

Let $n \geq 2$ be fixed, and $R$ be any positive real number. Define $\mathcal{B}_n(R)$ to be the set of bi-Perron algebraic units no larger than $R$ whose characteristic polynomial has degree at most $2n$. Here, by the characteristic polynomial of a bi-Perron algebraic unit $\lambda$, we mean the monic palindromic integral polynomial whose leading root is $\lambda$ and has the lowest degree among all such polynomials. And let $\mathcal{D}_n(R)$ be the set of dilatations no larger than $R$ of pseudo-Anosov maps with orientable invariant foliations on a closed orientable surface $S_n$ of genus $n$. Similarly, let $\mathcal{D}'_n(R)$ be the set of dilatations no larger than $R$ of pseudo-Anosov maps, not necessarily with orientable invariant foliation, on a closed orientable surface with genus $n$.

We remark that $\mathcal{D}_n(R)$ is contained in $\mathcal{B}_n(R)$, and $\mathcal{D}'_n(R)$ is contained in $\mathcal{B}_{3n-3}(R)$. A pseudo-Anosov dilatation $\lambda$ on a surface of genus $n$ is a root of an integral palindromic polynomial of degree at most $2n$ if its invariant foliations are orientable. This is because $\lambda$ is the leading eigenvalue of the induced symplectic action on the homology group of the surface, which is $\mathbb{Z}^{2n}$. If we do not require the invariant foliation to be orientable, the upper bound on degree is $6n - 6$: we can reduce this case to the case of orientable foliation by taking a double cover of the surface, and this bound follows from the fact that a quadratic differential on a surface of genus $n$ has at most $2n$ zeros which is due to Gauss-Bonnet together with the Riemann-Hurwitz formula.

Note that Fried’s conjecture is equivalent to $\mathcal{B}_n(R)$ being contained in $\mathcal{D}_m(R)$ for some large enough $m$. But a priori, $m$ could be arbitrarily large and we do not know how to prove or disprove the claim. Instead we show the following.

**Theorem 1.** Let $\mathcal{B}_n(R)$, $\mathcal{D}'_n(R)$ and $\mathcal{D}_n(R)$ be as above. Then

1. \[
\lim_{{R \to \infty}} \frac{|\mathcal{D}_m(R) \cap \mathcal{B}_n(R)|}{|\mathcal{B}_n(R)|} = 0,
\]
   where $4m - 3 \leq n(n+1)/2$. In particular, $\lim_{{R \to \infty}} \frac{|\mathcal{D}_n(R)|}{|\mathcal{B}_n(R)|} = 0$, $\forall n \geq 6$.

2. \[
\lim_{{R \to \infty}} \frac{|\mathcal{D}'_m(R) \cap \mathcal{B}_n(R)|}{|\mathcal{B}_n(R)|} = 0,
\]
   where $6m - 6 \leq n(n+1)/2$. In particular, $\lim_{{R \to \infty}} \frac{|\mathcal{D}'_n(R) \cap \mathcal{B}_n(R)|}{|\mathcal{B}_n(R)|} = 0$, $\forall n \geq 10$.

Here $|A|$ means the cardinality of $A$ for a finite set $A$.

Theorem 1 says that asymptotically almost all bi-Perron algebraic units whose characteristic polynomial has degree at most $2n$ do not correspond to dilatations of pseudo-Anosov maps on a surface of genus $n$ for all $n \geq 10$. 


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and for \( n \geq 6 \) if the invariant foliation is further assumed to be orientable. It would be interesting to see if the statement still holds for lower genera.

Let \( \Gamma_n \) be the set of all periodic orbits for the Teichmüller flow on the moduli space of area one abelian differentials on \( S_n \). Then there exists a surjective map \( \Gamma_n \to \cup_{R>0} D_n(R) \) defined by \( \gamma \mapsto e^\ell(\gamma) \) where \( \ell(\gamma) \) is the length of the orbit \( \gamma \in \Gamma_n \). Let \( \Gamma_n(R) \) be the preimage of \( D_n(R) \) of this map.

By \( f \sim g \) we mean \( \exists C \) such that \( \frac{1}{C}f(x) \leq g(x) \leq Cf(x) \) when \( x \gg 0 \). By \( f \preceq g \) we mean \( f = O(g) \) when \( x \to \infty \). At our best knowledge, the following theorem is independently due to Eskin-Mirzakhani-Rafi [5] and Hamenstädt [9] (which is rephrased for our purpose).

**Theorem 2** ([5], [9]). \( |\Gamma_n(R)| \sim \frac{R^{4n-3}}{(4n-3) \log R} \).

We give a brief explanation of the above statement. In [5] and [9], it was stated that when restricted to each connected component of the strata of the moduli space of area one abelian differentials on \( S_n \),

\[
|\Gamma_n(R)| = |\{ \gamma \in \Gamma_n : e^\ell(\gamma) \leq R \}|
\]

\[
= |\{ \gamma \in \Gamma_n : \ell(\gamma) \leq \log R \}|
\]

\[
\sim \frac{e^{(2n+\ell-1) \log R}}{(2n+\ell-1) \log R} = \frac{R^{2n+\ell-1}}{(2n+\ell-1) \log R}.
\]

Here \( \ell \) is the maximum number of zeros of an area one abelian differential on \( S_n \), so it is \( 2n - 2 \). See [16], [22], [4] and [5] for the relevant background of this theorem.

As a result, we have \( |\Gamma_n(R)| \sim \frac{R^{4n-3}}{(4n-3) \log R} \) on each connected component of the strata. But for fixed \( n \), there exists only finite number of such components. Therefore we get \( |\Gamma_n(R)| \sim \frac{R^{4n-3}}{(4n-3) \log R} \) without restricting to the components. Note that since \( n \) is fixed, we can just say \( |\Gamma_n(R)| \sim \frac{R^{4n-3}}{\log R} \).

As a direct corollary, we have:

**Corollary 3.** \( |D_n(R)| \preceq \frac{R^{4n-3}}{\log R} \).

In the exactly same way, we can obtain an analogue of Corollary 3 for \( |D'_n(R)| \) from the following theorem which is due to Eskin and Mirzakhani [4].

**Theorem 4** (Theorem 1.1, [4]). The number of geodesics in the moduli space of genus \( n \) surface of length at most \( \log(R) \) \( \sim \frac{R^{6n-6}}{(6n-6) \log(R)} \).

And as a direct corollary, just like Corollary 3 we have:

**Corollary 5.** \( |D'_n(R)| \preceq \frac{R^{6n-6}}{\log R} \).

We remark that this does not imply \( |D_n(R)| \sim \frac{R^{4n-3}}{\log R} \) or \( |D'_n(R)| \sim \frac{R^{6n-6}}{\log R} \), since each element in \( D_n(R) \) or \( D'_n(R) \) may correspond to a lot of different closed geodesics in the moduli space.
Now we study how $|\mathcal{B}_n(R)|$ grows. Let $P_n(R)$ be the set of Perron polynomials of degree $n$ with roots no larger than $R$, ($x > 1$ is Perron if it is the root of a monic irreducible polynomial with integer coefficients, so that the other roots of the polynomial are (strictly) less than $x$ in absolute value).

**Lemma 6.**  $|P_n(R)| \sim R^{n(n+1)/2}$.

**Proof.** $|P_n(R)| \lesssim R^{n(n+1)/2}$: Due to Vieta’s formula the absolute value of the coefficient of the $x^k$ of a monic, degree $n$ polynomial with all roots no larger than $R$ is $\lesssim R^{n-k}$. Hence, the total number of such polynomials must be $\lesssim \prod_k R^{n-k} = R^{n(n+1)/2}$.

$R^{n(n+1)/2} \lesssim |P_n(R)|$: Let $a_k$ be the coefficient of $x^k$ in a degree $n$ monic polynomial (so $a_n = 1$). By Rouché’s theorem, when

$$1 > \left| \frac{a_0}{R^n} + \frac{a_1}{R^{n-1}} + \cdots + \frac{a_{n-1}}{R} \right|$$

(1)

$$\left( \frac{1}{2} \right)^{n-1} \frac{|a_{n-1}|}{R} > \left| \frac{a_0}{R^n} + \cdots + \frac{a_{n-2}}{R^2} \right| \left( \frac{1}{2} \right)^{n-2} + \left( \frac{1}{2} \right)^n$$

(2)

and

$$\left( \frac{1}{3} \right)^{n-1} \frac{|a_{n-1}|}{R} > \left| \frac{a_0}{R^n} + \cdots + \frac{a_{n-2}}{R^2} \right| \left( \frac{1}{3} \right)^{n-2} + \left( \frac{1}{3} \right)^n$$

(3)

one root $\lambda$ of this polynomial has magnitude between $R/2$ and $R$ while all other roots have magnitude smaller than $R/3$. Hence $\lambda$ must be real. If $\lambda < 0$ one can multiply $(-1)^{n-k}$ to $a_k$ to get a polynomial with a root $-\lambda$. Hence, half of those polynomials have a leading positive real root, so they are in $P_n(R)$.

The inequalities (1), (2) and (3) are satisfied if and only if the point $(a_0/R^n, \ldots, a_{n-1}/R)$ lies in a non-empty open subset $U \subset [-1,1]^n$. The number of such points as $R \to \infty$ converges to the volume of this open subset divided by the co-volume of the lattice $\mathbb{Z}/R^n \times \mathbb{Z}/R^{n-1} \times \cdots \times \mathbb{Z}/R$, and the co-volume of this lattice is $R^{-n(n+1)/2}$.

**Lemma 7.** $\lim_{R \to \infty} \frac{|\text{reducible elements in } P_n(R)|}{|P_n(R)|} = 0$

**Proof.** Because any reducible monic integer polynomial can be written as the product of two monic integer polynomials of lower degree, we have:

$$|\{\text{reducible elements in } P_n(R)\}| \leq \sum_k |b_k(R)||b_{n-k}(R)|$$

where $b_k(R)$ is the set of monic polynomials with roots bounded by $R$. The first part of the proof of the previous lemma implies that $|b_k(R)| \lesssim R^{k(k+1)/2}$, hence $|\{\text{reducible elements in } P_n(R)\}| \sim o(R^{n(n+1)/2})$. $\Box$
Let $\mathcal{P}_n(R)$ be the set of Perron numbers of degree $n$ no larger than $R$. They have one-one correspondence with irreducible elements in $P_n(R)$ which by Lemma 4 constitute almost all of $P_n(R)$ asymptotically. Hence by Lemma 6, $|\mathcal{P}_n(R)| \sim R^{n(n+1)/2}$.

**Lemma 8.** $|B_n(R)| \sim R^{n(n+1)/2}$.

**Proof.** When $x > 1$, $1/x < 1$. Hence, $x \in B_n(R)$ implies that $x + 1/x \in P_n(R + 1)$, hence $|B_n(R)| \leq (R + 1)^{n(n+1)/2} \sim R^{n(n+1)/2}$.

On the other hand, from the proof of Lemma 6 and the discussion above, the number of Perron numbers of degree $n$ which lie between $R$ and $R/2$ and have all conjugates smaller than $R/3$ is $\sim R^{n(n+1)/2}/2$. When $R$ is sufficiently large, each such Perron number $y$ corresponds to a bi-Perron number $x$ by the relation $x + 1/x = y$. And in fact $x$ is an algebraic unit (so it is in $B_n(R)$). Note that both roots of the polynomial $x^2 - yx + 1$ are algebraic integers, since it is a monic polynomial with algebraic integer coefficients. Furthermore, the product of these roots is 1, so they must be algebraic units. Hence $B_n(R) \gtrsim R^{n(n+1)/2}$. □

Now we are ready to prove our main theorem.

**Proof of Theorem 7** By Corollary 5 and Lemma 8, we get

$$\frac{|D_n(R) \cap B_n(R)|}{|B_n(R)|} \sim \frac{R^{4m-3}}{\log R \cdot R^{n(n+1)/2}}.$$ 

The right-hand side goes to 0 as $R$ goes to $+\infty$ as long as we have $4m - 3 \leq n(n+1)/2$. When $m = n$, this inequality is satisfied if and only if $n \geq 6$ (recall that $n$ is always assumed to be at least 2), which proved part (1).

Part (2) follows from the same argument above but using Corollary 5 instead of Corollary 3. □

Recall that $\Gamma_n$ is the set of all closed geodesics in the moduli space of area one abelian differentials on the surface $S_n$, and $\Gamma_n(R)$ is the subset of $\Gamma_n$ which consists of the closed geodesics of length no larger than $\log R$. For each $\gamma \in \Gamma_n$, let $m_\gamma$ be the number $|\{\gamma' \in \Gamma_n : \ell(\gamma') = \ell(\gamma)\}|$.

We remark that if $m_\gamma$ were uniformly bounded, one could have obtained Theorem 1 (1) for $n \geq 2$ instead of $n \geq 6$ using Theorem 1 (1) of [10]. But at least in the low genus cases, this is not true. As an application of our argument, we obtain the following theorem.

**Theorem 9.** Suppose $n \leq 5$. For any positive integer $k$, the set $\{\gamma \in \Gamma_n : m_\gamma \geq k\}$ is typical, i.e.,

$$\lim_{R \to \infty} \frac{|\{\gamma \in \Gamma_n(R) : m_\gamma \geq k\}|}{|\Gamma_n(R)|} \to 1.$$

**Proof.** Suppose not. Then for some $k$, we have

$$\limsup \frac{|\{\gamma \in \Gamma_n(R) : m_\gamma < k\}|}{|\Gamma_n(R)|} > 0.$$
But this implies that
\[ \limsup \frac{|D_n(R)|}{|\Gamma_n(R)|} \geq \limsup \frac{\frac{1}{k}|\{\gamma \in \Gamma_n(R) : m_\gamma < k\}|}{|\Gamma_n(R)|} > 0. \]

On the other hand, we know that \( D_n(R) \subset B_n(R) \). As a consequence,
\[ \lim \frac{|B_n(R)|}{|\Gamma_n(R)|} \geq \lim \sup \frac{|D_n(R)|}{|\Gamma_n(R)|}. \]
By Corollary 3 and Lemma 8 we get
\[ \frac{|B_n(R)|}{|\Gamma_n(R)|} \sim \frac{R^{n(n+1)/2} \log R}{R^{4n-3}} \to 0, \text{ for } n \leq 5, \]
a contradiction. □

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