Triaccontagonal coordinates for the $E_8$ root system

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Abstract. This note gives an explicit formula for the elements of the $E_8$ root system. The formula is triacontagonally symmetric in that one may clearly see an action by the cyclic group with 30 elements. The existence of such a formula is due to the fact that the Coxeter number of $E_8$ is 30.

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1 Introduction

The Coxeter numbers of $H_4$ and $E_8$ are both equal to 30, [6]. Some artistically-inclined mathematicians have used this fact in order to depict these root systems. In so doing, each has necessarily produced a figure having triacontagonal symmetry, meaning the same symmetry as a regular 30-sided polygon. According to [3], where it is used as the frontispiece, van Oss first sketched such a projection of the regular polytope having Schläfli symbol \{3,3,5\}, also known as the “600-cell”. The vertices of \{3,3,5\} coincide with the elements of the (non-crystallographic) root system $H_4$. One may also find this sketch in the article [8] and a sketch of the dual polytope \{5,3,3\} in [1].

The roots of $E_8$ coincide with the vertices of a highly symmetric convex polytope 421 apparently discovered by Gosset, [3, 6], and also with the vertices of a regular complex polytope 3\{3\}3\{3\}3\{3\}3\{3\} discovered by Witting, [1] [2]. The book [4] displays a triacontagonal projection of 3\{3\}3\{3\}3\{3\}3\{3\} and [2] displays a triacontagonal projection of 421. According to [7], Peter McMullen sketched the triacontagonal projections of both of these polytopes by hand during a period lasting from August 18 until August 20 in 1964. Recently John Stembridge produced an 8-color computer sketch of 421, to supplement an announcement by a research group at the American Institute of Mathematics of having computed a particular class of representations of a real Lie group having the $E_8$ root system. A simplified sketch is offered here, showing only the relative locations of the 240 vectors, in 8 concentric cycles with 30 points per cycle.

Here is a streamlined version of the formula from [3] for the elements of the $H_4$ root system. Let $\omega = \exp \left( \frac{i\pi}{30} \right)$ and $a > b > c > d$ be the positive roots of the polynomial $45x^8 - 90x^6 + 60x^4 - 15x^2 + 1$ over $\mathbb{R}$. Then the 120 vectors

\[
A_n = (a\omega^{2n}, d\omega^{22n}), \quad B_n = (b\omega^{2n+1}, c\omega^{22n+11}),
\]
\[
C_n = (c\omega^{2n+1}, -b\omega^{22n+11}), \quad D_n = (d\omega^{2n}, -a\omega^{22n}),
\]

are elements of $H_4$. The remaining 120 elements are obtained by permuting the sign of elements and multiplying by $i$.
where \( n \in \{0, 1, 2, \ldots, 28, 29\} \), comprise the \( H_4 \) root system. This formula has the feature that projecting along the first coordinate, essentially forgetting the second coordinate, yields the triacontagonal projection of the \( H_4 \) roots. The purpose of this note is to give a similar formula for the \( E_8 \) root system.

The triacontagonal projections of the \( H_4 \) and \( E_8 \) root systems are closely related. The latter is the union of two scaled sizes of the former, where the ratio of the larger to the smaller is the golden ratio \( \tau = \frac{1}{2} (1 + \sqrt{5}) \). Given this, one expects that a similar formula should exist for the \( E_8 \) roots. Indeed, the formula given here was obtained by “guess-and-check”. Checking is tedious, but one may use Maple, for example, to compute as many inner products as are necessary to become convinced that the formula yields a root system isomorphic to \( E_8 \). (A more “ethical” way to obtain the formula would be to diagonalize a Coxeter element of the Coxeter group of \( E_8 \). However, this author believes that the ends could not possibly justify the enormity of the computations involved in such means.)

\[ \text{Figure 1. Triacontagonal projection of the } E_8 \text{ root system.} \]
2 The formula

Denote $\omega = \exp \left( \frac{2\pi i}{30} \right)$ and define $\{a, b, c, d\}$ as above. More explicitly, $a, b, c,$ and $d$ are the positive numbers satisfying

$$2a^2 = 1 + 3^{-1/2}25^{-1/4}3^{-3/2}, \quad 2b^2 = 1 + 3^{-1/2}5^{-1/4}2^{-3/2},$$

$$2c^2 = 1 - 3^{-1/2}25^{-1/4}3^{-3/2}, \quad 2d^2 = 1 - 3^{-1/2}5^{-1/4}2^{-3/2},$$

where $\tau$ is the golden ratio. For any integer $n$, denote $e_n = \omega^n + \omega^{-n} = 2\cos \left( \frac{n\pi}{30} \right)$. Next, denote

$$r_1 = a/c, \quad r_2 = b/c, \quad r_3 = c/d, \quad r_4 = d/c,$$

$$r_5 = a/c, \quad r_6 = b/c, \quad r_7 = c/c, \quad r_8 = d/c.$$

Then the 240 rows of the matrices

$$\begin{bmatrix}
A_n \\
B_n \\
C_n \\
D_n \\
E_n \\
F_n \\
G_n \\
H_n
\end{bmatrix} = \begin{bmatrix}
r_1 & r_4 & r_6\omega & r_7\omega \\
r_2\omega & r_1\omega^2 & -r_3\omega & -r_5\omega \\
r_3\omega^2 & -r_2\omega & r_4\omega^2 & -r_6\omega \\
r_4 & -r_1 & r_7\omega & -r_6\omega \\
r_5 & r_8 & -r_2\omega & -r_3\omega \\
r_6\omega^2 & r_7\omega^2 & r_4\omega^2 & -r_1\omega^2 \\
r_7\omega^3 & -r_6\omega^3 & -r_3\omega^3 & r_2\omega \\
r_8 & -r_5 & -r_3\omega & r_2\omega
\end{bmatrix} \cdot \begin{bmatrix}
\omega^{2n} \\
\omega^{22n} \\
\omega^{14n} \\
\omega^{26n}
\end{bmatrix},$$

where $n \in \{0, 1, 2, \ldots, 28, 29\}$, as regarded as elements of $\mathbb{C}^4$, comprise a root system isomorphic to $E_8$. Moreover, each of these vectors has norm equal to 1.

3 Remarks

Projecting along the first coordinate, by forgetting the last three coordinates, yields the image depicted above as a subset of $\mathbb{C} \cong \mathbb{R}^2$.

Each cycle of 30 roots, as denoted by $A_n$, $B_n$, and so on, is expressed using the trigonometric function $n \mapsto (\omega^{2n}, \omega^{22n}, \omega^{14n}, \omega^{26n})$, composed with some amplitude and phase adjustments. Each phase shift was chosen so that $\{A_n, B_n, C_n, D_n, E_n, F_n, G_n, H_n\}$ is a system of simple roots for any fixed value of $n$.

![Diagram of E8](image)

**Figure 2.** The Dynkin diagram of $E_8$. 

The amplitudes $r_k$ were chosen so that each vector has norm 1. Alternatively, one may use the amplitudes
\[ r_1 = 1, \quad r_2 = c_{11}, \quad r_3 = c_6 c_{13}, \quad r_4 = c_6 c_{14}, \]
\[ r_5 = c_{12}, \quad r_6 = c_{11} c_{12}, \quad r_7 = c_{13}, \quad r_8 = c_{14}. \]
In so doing, one avoids the complicated definition/formulae for the values $\{a, b, c, d\}$ given above. For example, using this choice of amplitudes facilitates the verification of the formula, for then all the coordinates lie in the cyclotomic field $\mathbb{Q}(\omega)$. The tradeoff is that the norms of the vectors are more complicated to express.

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