A numerical method for Hadamard finite-part integrals with an integral power singularity at an endpoint

Hidenori Ogata∗

September 20, 2019

Abstract

In this paper, we propose a numerical method for computing Hadamard finite-part integrals with an integral-power singularity at an endpoint, the part of the divergent integral which is finite as a limiting procedure. In the proposed method, we express the desired finite-part integral using a complex loop integral, and obtain the finite-part integral by evaluating the complex integral by the trapezoidal rule. Theoretical error estimate and some numerical examples show the effectiveness of the proposed method.

1 Introduction

The integral

\[ \int_0^1 x^{-1} f(x) dx, \]

where \( f(x) \) is an analytic function on the closed interval \([0, 1]\) such that \( f(0) \neq 0 \), is divergent. However, if \( f(x) \) is analytic on the closed interval \([0, 1]\), for \( \epsilon \) such that \( 0 < \epsilon \ll 1 \), we have the following using integral by part.

\[
\int_{\epsilon}^{1} x^{-1} f(x) dx = \int_{\epsilon}^{1} (\log x)' f(x) dx \\
= \left[ \log x f(x) \right]_{\epsilon}^{1} - \int_{\epsilon}^{1} \log x f'(x) dx \\
= - f(\epsilon) \log \epsilon - \int_{\epsilon}^{1} \log x f'(x) dx \\
= - f(0) \log \epsilon + (\text{terms finite as } \epsilon \downarrow 0).
\]

Therefore, we can define the so-called Hadamard finite-part (f.p.) integral by

\[
\text{f. p. } \int_0^1 x^{-1} f(x) dx = \lim_{\epsilon \downarrow 0} \left\{ \int_{\epsilon}^{1} x^{-1} f(x) + f(0) \log \epsilon \right\}
\]

∗Department of Computer and Network Engineering, Graduate School of Informatics and Engineering, The University of Electro-Communications, 1-5-1 Chofugaoka, Chofu, Tokyo 182-8585, Japan, (e-mail: ogata@im.uec.ac.jp)
Similarly, we can define the f.p. integrals
\[ \int_0^1 x^{-n} f(x) \, dx \quad (n = 1, 2, \ldots). \] (1)

We propose a numerical method for computing the f.p. integrals (1). In the proposed method, we express the f.p. integral by a complex loop integral, and we obtain the f.p. integral by evaluating the complex integral by the trapezoidal formula with equal mesh.

Previous works related to this paper are as follows. Ogata and Hirayama proposed a numerical integration method based on hyperfunction theory, a theory of generalized functions based on complex function theory, where they obtain ordinary integrals by expressing them as complex integrals and evaluating them by the conventional numerical integral formulas \[6\]. For Cauchy principal value integrals and Hadamard f.p. integrals with a singularity inside the integral interval
\[ \int_0^1 f(x) \frac{(x - \lambda)^n}{(x - \lambda)^n} \, dx \quad (n = 1, 2, \ldots), \] (2)
many approximation methods have been proposed. Elliot and Paget proposed a Gauss type numerical integration formula for Cauchy principal value integrals (2) with \( n = 1 \) \[4\], and Paget proposed a Gauss type formula for Hadamard finite-part integrals (2) with \( n = 2 \) \[8\]. Bialecki proposed approximation formulas for (2) based on the Sinc method \[1, 2\], that is, methods using the trapezoidal formula together with variable transforms as in the DE formula \[9\]. The author et al. improved them and proposed a DE-type numerical integration formula for Cauchy principal-value integrals and Hadamard finite-part integrals with an integral power singularity inside the integral interval \[7\].

The remainder of this paper is structured as follows. In Section 2 we define the f.p. integrals (1) and show the expression of them by complex loop integral. Then, we give an approximation formula for the desired f.p. integral. In Section 3 we show some numerical examples which show the effectiveness of the proposed method. In Section 4 we give a summary of this paper.

2 Hadamard finite-part integrals and its approximation

We define the Hadamard finite-part integrals (1) by
\[ \int_0^1 f(x) \, dx = \lim_{\epsilon \downarrow 0} \left\{ \int_0^1 x^{-n} f(x) \, dx - \sum_{k=0}^{n-2} \frac{x^{k+1-n}}{k!(n-1-k)} f^{(k)}(0) + \frac{\log \epsilon}{(n-1)!} f^{(n-1)}(0) \right\} \quad (n = 1, 2, \ldots), \] (3)
where the integrand \( f(x) \) is analytic on the closed interval \([0, 1]\), and the second term on the right-hand side is zero if \( n = 1 \). We can show that it is well-defined.
using integral by part as follows.

\[
\int_\epsilon^1 x^{-n} f(x) \, dx \\
= -\frac{1}{n-1} \int_\epsilon^1 (x^{-(n-1)})' f(x) \, dx \\
= -\frac{1}{n-1} \left\{ \left[ x^{-(n-1)} f(x) \right]_\epsilon^1 - \int_\epsilon^1 x^{-(n-1)} f'(x) \, dx \right\} \\
= -\frac{f(1)}{n-1} + \frac{\epsilon^{1-n}}{n-1} f(\epsilon) + \frac{1}{n-1} \int_\epsilon^1 x^{-(n-1)} f'(x) \, dx \\
= \frac{\epsilon^{1-n} n^{-2}}{n-1} \sum_{k=0}^n \frac{\epsilon^k}{k!} f^{(k)}(\epsilon) - \frac{1}{(n-1)(n-2)} \int_\epsilon^1 (x^{-(n-2)})' f'(x) \, dx \\
+ \text{(terms finite as } \epsilon \downarrow 0, \text{ which is denoted by “...” below)} \\
= \frac{\epsilon^{1-n} n^{-2}}{n-1} \sum_{k=0}^n \frac{\epsilon^k}{k!} f^{(k)}(\epsilon) + \frac{\epsilon^{2-n}}{(n-1)(n-2)} \sum_{k=0}^{n-1} \frac{\epsilon^k}{k!} f^{(k+1)}(\epsilon) \\
- \frac{1}{(n-1)(n-2)(n-3)} \int_\epsilon^1 (x^{-(n-3)})' f''(x) \, dx + \ldots \\
= \ldots \\
= \frac{\epsilon^{1-n} n^{-2}}{n-1} \sum_{k=0}^n \frac{\epsilon^k}{k!} f^{(k)}(\epsilon) + \frac{\epsilon^{2-n}}{(n-1)(n-2)} \sum_{k=0}^{n-1} \frac{\epsilon^k}{k!} f^{(k+1)}(\epsilon) \\
+ \frac{\epsilon^{3-n}}{(n-1)(n-2)(n-3)} \sum_{k=0}^{n-2} \frac{\epsilon^k}{k!} f^{(k+2)}(\epsilon) + \ldots + \frac{\epsilon^{-1}}{(n-1)!} f^{(n-1)}(0) - \frac{\log \epsilon}{(n-1)!} f^{(n-1)}(0) \\
- \frac{1}{(n-1)!} \int_\epsilon^1 \log x f^{(n)}(x) \, dx + \ldots \\
= \frac{\epsilon^{1-n}}{n-1} f(0) + \frac{\epsilon^{2-n}}{n-1} f'(0) \left\{ \frac{1}{n-1} + \frac{1}{(n-1)(n-2)} \right\} \\
+ \frac{\epsilon^{1-n} n^{-2}}{n-1} \sum_{k=2}^n \frac{\epsilon^k}{k!} f^{(k)}(\epsilon) + \frac{\epsilon^{2-n}}{(n-1)(n-2)} \sum_{k=2}^{n-2} \frac{\epsilon^k}{(k-1)!} f^{(k)}(\epsilon) \\
+ \frac{\epsilon^{3-n}}{(n-1)(n-2)(n-3)} \sum_{k=2}^{n-2} \frac{\epsilon^k}{(k-2)!} f^{(k)}(\epsilon) + \ldots + \frac{\epsilon^{-1}}{(n-1)!} f^{(n-2)}(0) - \frac{\log \epsilon}{(n-1)!} f^{(n-1)}(0) \\
- \frac{1}{(n-1)!} \int_\epsilon^1 \log x f^{(n)}(x) \, dx + \ldots \\
\]
\[
\begin{align*}
\frac{\epsilon^{1-n}}{n-1} f(0) & + \frac{\epsilon^{2-n}}{n-2} f'(0) + \frac{\epsilon^{3-n}}{n-3} f''(0) \left\{ \frac{1}{2!} + \frac{1}{n-2} + \frac{1}{(n-2)(n-3)} \right\} \\
& + \frac{\epsilon^{1-n}}{n-1} \sum_{k=3}^{n-2} \frac{\epsilon^k}{k!} f^{(k)}(0) + \frac{\epsilon^{2-n}}{n-1} \sum_{k=3}^{n-2} \frac{\epsilon^{k-1}}{(k-1)!} f^{(k)}(0) \\
& + \frac{\epsilon^{3-n}}{(n-1)(n-2)(n-3)} \sum_{k=3}^{n-2} \frac{\epsilon^{k-2}}{(k-2)!} f^{(k)}(0) + \ldots + \frac{\epsilon^{1-n}}{(n-1)!} f^{(n-1)}(0) + \ldots
\end{align*}
\]

The f.p. integral is expressed using a complex loop integral as in the following theorem.
**Theorem 1** We suppose that \( f(z) \) is analytic in a complex domain \( D \) containing the closed interval \([0, 1]\) in its interior. Then, the f.p. integral (3) is expressed as

\[
\text{f.p. } \int_0^1 x^{-n} f(x) dx = \frac{1}{2\pi i} \oint_C z^{-n} f(z) \log \left( \frac{z}{z-1} \right) dz - \sum_{k=0}^{n-2} \frac{f^{(k)}(0)}{k!(n-1-k)} (n = 1, 2, \ldots),
\]

where \( C \) is a closed complex integral path in \( D \) encircling the interval \([0, 1]\) in the positive sense, and the second term on the right-hand side of (4) is zero if \( n = 1 \).

**Proof of Theorem** Using Cauchy’s integral theorem, we have

\[
\frac{1}{2\pi i} \oint_C z^{-n} f(z) \log \left( \frac{z}{z-1} \right) dz
= \frac{1}{2\pi i} \left( \int_{C_0^0} + \int_{C_0^1} + \int_{\Gamma^{(+)}_1} + \int_{\Gamma^{(-)}_1} \right) z^{-n} f(z) \log \left( \frac{z}{z-1} \right) dz,
\]

where the integral paths \( C_0^0, C_0^1, \Gamma^{(+)}_1 \) and \( \Gamma^{(-)}_1 \) are respectively

\[
\begin{align*}
C_0^0 &= \{ e^{i\theta} | 0 \leq \theta \leq \pi \}, \\
C_0^1 &= \{ 1 + e^{i\theta} | 0 \leq \theta \leq \pi \}, \\
\Gamma^{(+)}_1 &= \{ x \in \mathbb{R} | 1 - \epsilon \geq x \geq \epsilon \}, \\
\Gamma^{(-)}_1 &= \{ x \in \mathbb{R} | \epsilon \leq x \leq 1 - \epsilon \}
\end{align*}
\]

with small \( \epsilon > 0 \) (see Figure 1). From Table 1, we have

![Figure 1: The integral paths.](image)

**Table 1:** The arguments of the functions appearing in the complex integral (5).

| \( z \)       | \( \Gamma^{(+)}_\epsilon \) | \( C_\epsilon^{(0)} \) | \( C_\epsilon^{(1)} \) | \( \Gamma^{(-)}_\epsilon \) | \( C_\epsilon^{(2)} \) | \( \Gamma^{(+)}_\epsilon \) |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \arg z \) | 0 | 2\pi | 2\pi | \( \arg(z - 1) \) | \( \pi \) | \( \pi \) | 3\pi | \( \arg(z/(z - 1)) \) | \( -\pi \) | \( \pi \) | \( -\pi \) |
Then, we have
\[
\frac{1}{2\pi i} \left( \int_{C_1^{(1)}} + \int_{C_1^{(-)}} \right) z^{-n} f(z) \log \left( \frac{z}{z-1} \right) \, dz
\]
\[
= - \frac{1}{2\pi i} \int_{\epsilon}^{1-\epsilon} x^{-n} f(x) \left\{ \log \left( \frac{x}{1-x} - i\pi \right) \right\} \, dx + \frac{1}{2\pi i} \int_{\epsilon}^{1-\epsilon} x^{-n} f(x) \left\{ \log \left( \frac{x}{1-x} + i\pi \right) \right\} \, dx
\]
\[
= \int_{\epsilon}^{1} x^{-n} f(x) \, dx = \int_{\epsilon}^{1} x^{-n} f(x) \, dx + O(\epsilon).
\]
The integral on \( C_1^{(0)} \) is written as
\[
\frac{1}{2\pi i} \int_{C_1^{(0)}} z^{-n} f(z) \log \left( \frac{z}{z-1} \right) \, dz
\]
\[
= \frac{1}{2\pi i} \int_{0}^{2\pi} e^{-n} e^{-i\theta} f(e^{i\theta}) \log \left( \frac{-e^{i\theta}}{1-e^{i\theta}} \right) i\epsilon e^{i\theta} \, d\theta
\]
\[
= \frac{\epsilon^{1-n}}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) \log(e^{i(\theta-\pi)}) e^{-i(n-1)\theta} \, d\theta - \frac{\epsilon^{1-n}}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) \log(1-e^{i\theta}) e^{-i(n-1)\theta} \, d\theta.
\]
The first integral on the right-hand side is written as
\[
\frac{\epsilon^{1-n}}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) \log(e^{i(\theta-\pi)}) e^{-i(n-1)\theta} \, d\theta
\]
\[
= \frac{\epsilon^{1-n}}{2\pi} \int_{0}^{2\pi} \left\{ \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} f^{(k)}(0)e^{ik\theta} \right\} \log \epsilon + i(\theta - \pi) \right) e^{-i(n-1)\theta} \, d\theta
\]
\[
= \frac{\epsilon^{1-n} \log \epsilon}{2\pi} \sum_{k=0}^{\infty} \int_{0}^{2\pi} e^{i(n-1-k)\theta} \, d\theta + \frac{i(n-1)\epsilon^{1-n}}{2\pi} \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} f^{(k)}(0) \int_{0}^{2\pi} (\theta - \pi) e^{i(k-n+1)\theta} \, d\theta
\]
\[
= \frac{\epsilon^{1-n} \log \epsilon}{(n-1)!} f^{(n-1)}(0) - \sum_{k=0}^{n-2} \frac{\epsilon^{k-n+1}}{k!(n-1-k)} f^{(k)}(0) + O(\epsilon),
\]
where we exchanged the order of the integral and the infinite summation on the second equality since the infinite sum is uniformly convergent on \( 0 \leq \theta \leq 2\pi \).

Similarly, the second integral is written as
\[
\frac{\epsilon^{1-n}}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) \log(1-e^{i\theta}) e^{-i(n-1)\theta} \, d\theta = -\sum_{k=0}^{n-2} \frac{f^{(k)}(0)}{k!(n-1-k)}.
\]

Then, we have
\[
\frac{1}{2\pi i} \int_{C_1^{(1)}} z^{-n} f(z) \log \left( \frac{z}{z-1} \right) \, dz
\]
\[
= -\sum_{k=0}^{n-2} \frac{\epsilon^{k-n+1}}{k!(n-1-k)} f^{(k)}(0) + \frac{\log \epsilon}{(n-1)!} f^{(n-1)}(0) + \sum_{k=0}^{n-2} \frac{f^{(k)}(0)}{k!(n-1-k)} + O(\epsilon).
\]

As to the integral on \( C_1^{(2)} \), we have
\[
\frac{1}{2\pi i} \int_{C_1^{(2)}} z^{-n} f(z) \log \left( \frac{z}{z-1} \right) \, dz = O(\epsilon \log \epsilon)
\]
since $z^{-n}f(z) \log z$ is analytic near $z = 1$. Summarizing the above calculations, we have

$$\frac{1}{2\pi i} \oint_C z^{-n}f(z) \log \left( \frac{z}{z-1} \right) dz = \int_{\epsilon}^1 x^{-n}f(x)dx - \sum_{k=0}^{n-2} \frac{\epsilon^{k-n+1}}{k!(n-1+k)} f^{(k)}(0)$$

$$+ \log \epsilon \left( \frac{f^{(n-1)}(0)}{(n-1)!} + \sum_{k=0}^{n-2} \frac{f^{(k)}(0)}{k!(n-1+k)} + O(\epsilon \log \epsilon) \right).$$

Taking the limit $k \downarrow 0$, we have (4).

The complex integral in (4) is the integral of an analytic function over an interval of the length of one period, and it is accurately approximated by the trapezoidal formula with equal mesh. Using a parameterization of the closed integral path

$$C : z = \varphi(u), \quad 0 \leq u \leq u_p,$$

where $\varphi(u)$ is a periodic function of period $u_p$, we obtain the following approximation formula for the f.p. integral.

$$\text{f.p.} \int_0^1 x^{-n}f(x)dx \simeq I^{(n)}_{N}[f]$$

$$\equiv \frac{\hbar}{2\pi i} \sum_{k=0}^{N-1} \varphi(kh)^{-n}f(\varphi(kh)) \log \left( \frac{\varphi(kh)}{\varphi(kh)-1} \right) \varphi'(kh) - \sum_{k=0}^{n-2} \frac{f^{(k)}(0)}{k!(n-1-k)}$$

$$\left( h = \frac{u_p}{N} \right), \quad (6)$$

where the second term on the right-hand side is zero if $n = 1$.

We remark here that, if the integrand $f(x)$ is real valued on $[0, 1]$ and the integral path $C$ is symmetric with respect to the real axis, we can reduce the number of sampling points $N$ by half. In fact, in this case, we have $I(x) = I(z)$ due to the reflection principle, and $\varphi(-u) = \varphi(u), \varphi'(-u) = -\varphi'(u)$. Then, we have

$$\text{f.p.} \int_0^1 x^{-n}f(x)dx \simeq I^{(n)}_{\frac{N}{2}}[f]$$

$$\equiv \frac{\hbar}{\pi} \text{Im} \left\{ \varphi(0)^{-n}f(\varphi(0)) \log \left( \frac{\varphi(0)}{1-\varphi(0)} \right) \varphi'(0) \right. $$

$$+ \varphi\left( \frac{u_p}{2} \right)^{-n}f\left( \varphi\left( \frac{u_p}{2} \right) \right) \log \left( \frac{\varphi(u_p/2)}{1-\varphi(u_p/2)} \right) \varphi'\left( \frac{u_p}{2} \right) \right\}$$

$$+ \frac{\hbar}{\pi} \text{Im} \left\{ \sum_{k=1}^{N-1} \varphi(kh)^{-n}f(\varphi(kh)) \log \left( \frac{\varphi(kh)}{\varphi(kh)-1} \right) \varphi'(kh) \right\}$$

$$- \sum_{k=0}^{n-2} \frac{f^{(k)}(0)}{k!(n-1-k)} \left( h = \frac{u_p}{2N} \right), \quad (7)$$

7
Applying the theorem in §4.6.5 in [3] to the approximation of the complex integral by the trapezoidal formula in (6), we have the following theorem on the error estimate of the approximation formula (6).

**Theorem 2** We suppose that

- the strip domain
  \[ D_d = \{ w \in \mathbb{C} \mid |\text{Im} w| < d \} \quad (d > 0) \]
  is contained in \( \mathbb{C} \setminus [0,1] \),
- the parameterization function \( \varphi(w) \) of \( C \) is analytic in \( D_d \), and
- the integrand \( f(z) \) is analytic in \( \varphi(D_d) = \{ \varphi(w) \mid w \in D_d \} \).

Then, we have the following inequality for arbitrary \( 0 < d' < d \).

\[
\left| \text{f.p.} \int_0^1 x^{-n} f(x) dx - I_N^{(n)}[f] \right| \leq \frac{d}{\pi} \mathcal{N}(f,n,d') \frac{\exp(-2\pi d'N/u_p)}{1 - \exp(-2\pi d'N/u_p)},
\]

where

\[
\mathcal{N}(f,n,d') = \max_{|\text{Im} w| = d'} \left| \varphi(w)^{-n} f(\varphi(w)) \log \left( \frac{\varphi(w)}{1 - \varphi(w)} \right) \right|.
\]

This theorem says that the approximation (6) converges exponentially as \( N \) increases if the integrand function \( f(x) \) is analytic on \([0,1]\) and the integral path \( C \) is an analytic curve.

### 3 Numerical examples

We computed the integrals

(1) \( \text{f.p.} \int_0^1 x^{-n} e^{x} dx = \sum_{k=0}^{\infty} \frac{1}{k!(k-n+1)} \)

(2) \( \text{f.p.} \int_0^1 x^{-n} 1 + x dx = (-1)^n \left\{ \log 2 + \sum_{l=1}^{n-1} \frac{(-1)^l}{l} \right\} \)

where the second term on the right-hand side of the integral (2) is zero if \( n = 1 \), by the proposed method. All the computations were performed using programs coded in C++ with double precision working. The complex integral path \( C \) was taken as the ellipse

\[ C : z = \frac{1}{2} + \frac{1}{4} \left( \rho + \frac{1}{\rho} \right) \cos u + \frac{1}{4} \left( \rho - \frac{1}{\rho} \right) \sin u, \quad 0 \leq u \leq 2\pi \quad (\rho > 1), \]

where the parameter \( \rho \) is taken as \( \rho = 10 \) for the integral (1) and \( \rho = 2 \) for the integral (2). Figure [2] shows the relative errors of the approximation formula (6) applied to the integrals (1) and (2) as functions of the number of sampling points \( N \). From these figures, the errors decay exponentially as \( N \) increases. Table [2] shows the decay rates of the errors of the proposed method.
Figure 2: The relative errors of the proposed method for f.p. integrals applied to the integrals (1) and (2).

Table 2: The decay rates of the errors of the proposed method for f.p. integrals applied to the integrals (1) and (2).

| n  | relative error integral (1) | integral (2) |
|----|--------------------------|-------------|
| 1  | O(0.024^N)  | O(0.25^N)  |
| 2  | O(0.025^N)  | O(0.29^N)  |
| 3  | O(0.021^N)  | O(0.32^N)  |
| 4  | O(0.029^N)  | O(0.35^N)  |
| 5  | O(0.039^N)  | O(0.38^N)  |
4 Summary

We proposed a numerical method for Hadamard finite-part integrals with an integral order power singularity at an endpoint over a finite interval. In the proposed method, we express the desired f.p. integral using a complex loop integral, and we obtain the f.p. integral by evaluating the complex integral by the trapezoidal formula. Theoretical error estimate and numerical examples show the exponential convergence of the proposed method.

We can also give approximation methods for f.p. integrals with a non-integral power singularity and f.p. integrals over a half-infinite interval in a way similar to this paper. They will be reported in other papers.

References

[1] B. Bialecki. A sinc-hunter quadrature rule for cauchy principal value integrals. Math. Comput., 55:665–681, 1990.

[2] B. Bialecki. A sinc quadrature rule for hadamard finite-part integrals. Numer. Math., 57:263–269, 1990.

[3] P. J. Davis and P. Rabinowitz. Methods of Numerical Integration, Second Ed. Academic Press, San Diego, 1984.

[4] D. Elliot and D. F. Paget. Gauss type quadrature rules for cauchy principal value integrals. Math. Comput., 33:301–309, 1979.

[5] R. Estrada and R. P. Kanwal. Regularization, pseudofunction, and hadamard finite part. J. Math. Anal. Appl., 141:195–207, 1989.

[6] H. Ogata and H. Hirayama. Numerical integration based on hyperfunction theory. J. Comput. Appl. Math., 327:243–259, 2018.

[7] H. Ogata, M. Sugihara, and M. Mori. De-type quadrature formulae for cauchy principal-value integrals and for hadamard finite-part itnegrals. In Proceedings of the Second ISAAC Congress, volume 1, pages 357–366, 2000.

[8] D. F. Paget. The numerical evaluation of hadamard finite-part integrals. Numer. Math., 36:447–453, 1981.

[9] H. Takahasi and M. Mori. Double exponential formulas for numerical integration. Publ. Res. Inst. Math. Sci., Kyoto Univ., 339:721–741, 1978.