Solvability of a semilinear heat equation on Riemannian manifolds

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Abstract. We study the solvability of the initial value problem for the semilinear heat equation $u_t - \Delta u = u^p$ in a Riemannian manifold $M$ with a nonnegative Radon measure $\mu$ on $M$ as initial data. We give sharp conditions on the local-in-time solvability of the problem for complete and connected $M$ with positive injectivity radius and bounded sectional curvature.

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1. Introduction

The study of nonlinear parabolic equations on Riemannian manifolds is attractive in itself. Moreover, it has many applications in geometry and other areas. The existence, nonexistence and behavior of solutions usually reflect the geometric nature of the underlying Riemannian manifold. As the simplest possible model case, we study the following semilinear heat equation:

\[
\begin{aligned}
    u_t - \Delta u &= u^p \quad \text{in } M \times (0, T), \\
    u(\cdot, 0) &= \mu \quad \text{in } M.
\end{aligned}
\]  

(1.1)

Here, \( p > 1, \ T > 0, \ (M, g) \) is a possibly non-compact Riemannian \( N \)-manifold with bounded sectional curvature and \( \mu \) is a nonnegative Radon measure on \( M \). Our purpose in this paper is to give sharp conditions on the local-in-time solvability of (1.1).

We first recall known results for the necessary conditions and the sufficient conditions on \( \mu \) concerning the existence of solutions of (1.1) on \( M = \mathbb{R}^N \). Baras and Pierre [5, Proposition 3.2] gave necessary conditions on \( \mu \) for the existence of solutions of (1.1). By Andreucci and DiBenedetto [3, Theorem 4.3, Remark 4.7], if \( u \geq 0 \) satisfies \( u_t - \Delta u = u^p \) in \( \mathbb{R}^N \times (0, T) \), then there exists a unique initial trace \( \mu \). This particularly shows that the initial data of nonnegative solutions are nonnegative Radon measures. They [3, Proposition 4.3, Remark 4.9] also gave necessary conditions analogous to Baras and Pierre. The necessary conditions obtained in [3,5] were refined by Hisa and Ishige [17, Theorem 1.1] as follows. We write

\[
p_F := \frac{N + 2}{N}
\]

and \( B(x, \rho) := \{ y \in \mathbb{R}^N; |x - y| < \rho \} \) for \( x \in \mathbb{R}^N \) and \( \rho > 0 \). There exists a constant \( C > 0 \) depending only on \( N \) and \( p \) such that \( \mu \) must satisfy the following (i) and (ii):

(i) If \( p < p_F \), then \( \sup_{z \in \mathbb{R}^N} \mu(B(z, \sqrt{T})) \leq CT^{\frac{N}{2} - \frac{1}{p - 1}} \).

(ii) If \( p \geq p_F \), then for any \( 0 < \rho < \sqrt{T} \),

\[
\sup_{z \in \mathbb{R}^N} \mu(B(z, \rho)) \leq \begin{cases} 
    C \left( \log(e + \sqrt{T} \rho^{-1}) \right)^{-\frac{N}{2}} & \text{if } p = p_F, \\
    C \rho^{N - \frac{2}{p - 1}} & \text{if } p > p_F.
\end{cases}
\]
Moreover, by [17, Theorem 1.3], (i) with $C$ small is also a sufficient condition. By [17, Corollary 1.2], (ii) is optimal in the sense that there exist measures $\mu$ with the same growth rate as in the right-hand side such that (1.1) admits local-in-time solutions.

For sufficient conditions on $M = \mathbb{R}^N$, if $\mu$ is an $L^\infty$-function, it is easy to show the existence of a local-in-time solution of (1.1). In the case where $\mu$ is an $L^q$-function, the study of sufficient conditions dates back to Weissler [44, 45]. For the progress with functions as initial data, we refer recent papers [11, 13, 17, 18, 24, 29, 47], a book [37, Section 15] and the references therein. For sufficient conditions with measures as initial data, we refer [1–3, 5, 22, 23, 30, 39, 43]. The first author [42, Theorem 1] proved that the above (ii) is not a sufficient condition. More precisely, (1.1) does not admit any local-in-time nonnegative solutions for some $\mu$ satisfying

$$\sup_{z \in \mathbb{R}^N} \mu(B(z, \rho)) \leq C \rho^{N-\frac{2}{p-1}} \left(\log(e + \rho^{-1})\right)^{-\frac{1}{p-1}}$$

if $p \geq p_F$ for any $\rho > 0$ with a constant $C > 0$. The first author [42, Theorem 2] also proved that (1.1) admits a local-in-time solution if

$$\sup_{z \in \mathbb{R}^N} \mu(B(z, \rho)) \leq C \rho^{N-\frac{2}{p-1}} \left(\log(e + \rho^{-1})\right)^{-\frac{1}{p-1}-\varepsilon}$$

if $p \geq p_F$ for any $\rho > 0$ with constants $C > 0$ and $0 < \varepsilon < 1/(p - 1)$.

We next consider the solvability of (1.1) on a Riemannian manifold $M$. In [32, Theorems 4.2, 4.4], Punzo generalized the result of Weissler [45, Theorem 1] to bounded open subsets of $H^N$. Punzo [34, 35] also gave analogous results of [32] for open subsets of $S^N$ and manifolds with negative sectional curvature, respectively. For related results, see [4, 15, 16, 28, 31, 33, 36, 46]. It seems to the authors that, even for $S^N$ and $H^N$, the existence and estimates of initial traces and the conditions on solvability are not as well studied as the Euclidean case.

Intuitively, the local-in-time solvability of semilinear heat equations is determined by the local structure of the ambient space and of the initial data, and so one expects that the above results with measures on $\mathbb{R}^N$ as initial data also hold for the problem on $M$. In this paper, our main results show that the expectation is true under appropriate assumptions on $M$ involving $\mathbb{R}^N$, $S^N$, $H^N$ and more general manifolds with positive injectivity radius and bounded sectional curvature.

We summarize our main results. In what follows, we denote by $d = d(x, y)$ the distance function on $M$. For $x \in M$ and $\rho > 0$, we write $B(x, \rho) := \{y \in M; d(x, y) < \rho\}$. In Theorem 2.1, we show the existence and the uniqueness of an initial trace $\mu$. We also show that if (1.1) admits a solution, then $\mu$ must satisfy

$$\sup_{z \in M} \mu(B(z, \rho_T)) \leq C \rho_T^{N-\frac{2}{p-1}} \quad \text{if } p < p_F,$$

$$\sup_{z \in M} \mu(B(z, \rho)) \leq \begin{cases} C \left(\log(e + \rho_T \rho^{-1})\right)^{-\frac{N}{2}} & \text{if } p = p_F, \\
C \rho^{N-\frac{2}{p-1}} & \text{if } p > p_F, \end{cases}$$

for $\rho_T > 0$. The first author [42, Theorem 1] proved that the above (ii) is not a sufficient condition. More precisely, (1.1) does not admit any local-in-time nonnegative solutions for some $\mu$ satisfying

$$\sup_{z \in \mathbb{R}^N} \mu(B(z, \rho)) \leq C \rho^{N-\frac{2}{p-1}} \left(\log(e + \rho^{-1})\right)^{-\frac{1}{p-1}}$$

if $p \geq p_F$ for any $\rho > 0$ with a constant $C > 0$. The first author [42, Theorem 2] also proved that (1.1) admits a local-in-time solution if

$$\sup_{z \in \mathbb{R}^N} \mu(B(z, \rho)) \leq C \rho^{N-\frac{2}{p-1}} \left(\log(e + \rho^{-1})\right)^{-\frac{1}{p-1}-\varepsilon}$$

if $p \geq p_F$ for any $\rho > 0$ with constants $C > 0$ and $0 < \varepsilon < 1/(p - 1)$.
for any $0 < \rho < \rho_T$. Here $\rho_T$ is an explicit constant determined by $M$ and $T$, see (2.1) for the definition.

Theorem 2.2 shows that if $\mu$ satisfies (1.2) with $C$ small, then (1.1) admits a solution. We note that we can also construct solutions if $\mu$ satisfies (1.2) with $C$ large and $T > 0$ small. Thus, (1.2) is a necessary and sufficient condition for existence.

In Theorem 2.3, we prove that there exists $\mu$ satisfying

$$
\begin{align*}
C_1 \rho^N \frac{2}{p-1} \leq & \sup_{z \in M} \mu(B(z, \rho)) \leq C_2 \rho^N \frac{2}{p-1} \quad \text{if } p > p_F, \\
C_1 (\log(e + \rho_T \rho^{-1}))^{-\frac{N}{2}} \leq & \sup_{z \in M} \mu(B(z, \rho)) \leq C_2 (\log(e + \rho_T \rho^{-1}))^{-\frac{N}{2}} \quad \text{if } p = p_F,
\end{align*}
$$

for any $0 < \rho < \rho_T$ with small constants $C_1, C_2 > 0$ such that the problem (1.1) admits a solution. Hence, the condition (1.3) is sharp. We remark that $\mu$ in Theorem 2.3 can be written as $d\mu = u_0 dx$ with a suitable function $u_0$.

As proved in Theorem 2.2, the necessary condition (1.2) for $p < p_F$ is also a sufficient condition for the existence of local-in-time solutions. However, for $p \geq p_F$, Theorem 2.4 shows that the necessary condition (1.3), with $C$ replaced by a small constant, is not a sufficient condition. In the case $p \geq p_F$, for any constant $C > 0$, there exists a nonnegative Radon measure $\mu$ satisfying

$$
\sup_{z \in M} \mu(B(z, \rho)) \leq C \rho^N \frac{2}{p-1} \left(\log(e + \rho^{-1})\right)^{-\frac{1}{p-1}}
$$

for any $0 < \rho < \rho_\infty$ such that the problem (1.1) does not admit any solutions in $M \times [0, T)$ for all $T > 0$. Here $\rho_\infty$ is an explicit constant determined by $M$, see (2.2) for the definition.

For existence in the case $p \geq p_F$, Theorem 2.5 shows that if $\mu$ satisfies

$$
\sup_{z \in M} \mu(B(z, \rho)) \leq C \rho^N \frac{2}{p-1} \left(\log(e + \rho_T \rho^{-1})\right)^{-\frac{1}{p-1}}
$$

for any $0 < \rho < \rho_T$ with small constants $C$ and $\varepsilon$, then the problem (1.1) admits a solution.

As corollaries, we consider the solvability of (1.1) with $\mu$ replaced by a function $u_0$. Corollaries 2.1 and 2.3 give solvability results with $u_0$ in the uniformly local Lebesgue spaces. Fix $z_0 \in M$. Let $u_0$ satisfy

$$
u_0(x) = \begin{cases} 
Cd(z_0, x)^{-\frac{2}{p-1}} & \text{if } p > p_F, \\
Cd(z_0, x)^{-N} (\log(e + d(z_0, x)^{-1}))^{-\frac{N}{2} - 1} & \text{if } p = p_F,
\end{cases}
$$

for any $x \in B(z_0, \rho_\infty)$ with some constant $C > 0$ and $u_0 \in L^\infty(M \setminus B(z_0, \rho_\infty))$ if $M \setminus B(z_0, \rho_\infty) \neq \emptyset$. Then Corollaries 2.2 and 2.4 imply that there exists a constant $C_* > 0$ satisfying the following: If $C < C_*$, then a local-in-time solution starting from $u_0$ with (1.4) exists, and if $C > C_*$, then local-in-time solutions do not exist. We note that, up to the coefficients, the strongest singularity of $u_0$ for existence is
\[ d(z_0, \cdot)^{-2/(p-1)} \text{ if } p > p_F \] and \[ d(z_0, \cdot)^{-N}(\log(e + d(z_0, \cdot)^{-1}))^{-N/2-1} \text{ if } p = p_F. \]

We also note that the authors do not know the explicit representation of \( C_* \) and that specifying the representation is an open problem even for \( M = \mathbb{R}^N \), see Souplet and Weissler [41, Theorem 1] and Hisa and Ishige [17, Corollary 1.2].

This paper is organized as follows. In Sect. 2, we state assumptions on \( M \) and give the exact statements of main theorems and their corollaries. In Sect. 3, we study the necessary conditions on \( \mu \) for the solvability of (1.1) and prove Theorem 2.1. In Sect. 4, to see the sharpness of the necessary conditions, we construct solutions and prove Theorems 2.2 and 2.3. We also prove Theorem 2.5 in this section. In Sect. 5, we consider the nonexistence of solutions and prove Theorem 2.4. In Sect. 6, we show corollaries stated in Sect. 2. In Appendix A, we give some comparison theorems adjusted to the use in this paper. In Appendix B, we give covering theorems used in Sect. 3.

2. Main results

We list assumptions and notation in Sect. 2.1. In Sect. 2.2, we state main theorems rigorously. In Sect. 2.3, we summarize corollaries of the theorems.

2.1. Notation

Let \( N \geq 1 \). We always assume the following conditions:

\[
\begin{aligned}
&(M, g) \text{ is a connected and complete Riemannian } N\text{-manifold without boundary.} \\
&\text{The injectivity radius satisfies } 0 < \text{inj}(M) \leq \infty. \\
&\text{The sectional curvature satisfies } | \text{sec}(M) | \leq \kappa \text{ for some } 0 \leq \kappa < \infty \text{ when } N \geq 2.
\end{aligned}
\]

In the context of sufficient conditions for the existence of solutions, we may handle the problem (1.1) in \( \Omega \) with \( u = 0 \) on \( \partial \Omega \), where \( \Omega \) is a domain in \( M \). For more details, see Remark 2.3. For \( 0 < T \leq \infty \), we set

\[ \rho_T := \min \left\{ \sqrt{T}, \frac{1}{4} \text{inj}(M), \frac{\pi}{4\sqrt{\kappa}} \right\}, \tag{2.1} \]

where we interpret \( \pi/(4\sqrt{\kappa}) = \infty \) if \( \kappa = 0 \) or \( N = 1 \). We remark that \( M = \mathbb{R}^N \) if and only if \( \rho_\infty = \infty \), where we also interpret

\[ \rho_\infty = \min \left\{ \frac{1}{4} \text{inj}(M), \frac{\pi}{4\sqrt{\kappa}} \right\}. \tag{2.2} \]

Example 2.1. We list the possible values of \( \rho_\infty \) for some typical Riemannian manifolds.

- \( \rho_\infty = \infty \) for \( M = \mathbb{R}^N \) with the standard Euclidean metric.
- \( 0 < \rho_\infty \leq \pi/4 \) for \( M = S^N \), the sphere with the standard metric.
• $0 < \rho_{\infty} \leq \pi/4$ for $M = \mathbb{R}^{N-\ell} \times S^\ell$, the multi-cylinder with the standard metric ($\ell \geq 1$).

• $0 < \rho_{\infty} \leq \pi/4$ for $M = H^N$, the hyperbolic space with the standard metric.

• $0 < \rho_{\infty} < \infty$ for any compact $M$.

• $0 < \rho_{\infty} < \infty$ for any non-compact $M$ with bounded geometry, that is, the Riemannian curvature tensor and its derivatives are bounded and the injectivity radius is positive. For example, asymptotically locally Euclidean manifolds (the so-called ALE spaces) are in this class.

We denote by $d = d(x, y)$ the distance function on $(M, g)$. For $x \in M$ and $\rho > 0$, we write $B(x, \rho) := \{ y \in M; d(x, y) < \rho \}$. $C_0(M)$ denotes the space of compactly supported continuous functions on $M$. We denote by $C^{2,1}$ the space of functions which are twice continuously differentiable in the space variable and once in the time variable. $dV_g$ denotes the Riemannian volume form of $g$. Unless otherwise stated, we use the term “solution” in the following sense:

**Definition 2.1.** Let $T > 0$. A function $u$ is called a solution of (1.1) in $M \times [0, T)$ if $u \geq 0$, $u \in C^{2,1}(M \times (0, T))$ and $u$ satisfies

$$u_t - \Delta u = u^p \quad \text{in} \quad M \times (0, T),$$

$$\lim_{t \to 0} \int_M \psi u(\cdot, t) dV_g = \int_M \psi d\mu \quad \text{for any} \quad \psi \in C_0(M).$$

2.2. Statements of main results

Our first result gives necessary conditions on the solvability of (1.1).

**Theorem 2.1.** Let $N \geq 1$, $p > 1$ and $0 < T < \infty$. Let $u \in C^{2,1}(M \times (0, T))$ be a nonnegative function satisfying (2.3). Then there exists a unique nonnegative Radon measure $\mu$ such that $u$ is a solution of (1.1) in $M \times [0, T)$. Moreover, there exists a constant $C > 0$ depending only on $N$ and $p$ such that the following (i), (ii) and (iii) hold:

(i) If $p < p_F$, then $\sup_{z \in M} \mu(B(z, \rho_T)) \leq C \rho_T^{N-\frac{2}{p-1}}$.

(ii) If $p = p_F$, then $\sup_{z \in M} \mu(B(z, \rho)) \leq C (\log(e + \rho_T \rho^{-1}))^{\frac{N}{2}}$ for any $0 < \rho < \rho_T$.

(iii) If $p > p_F$, then $\sup_{z \in M} \mu(B(z, \rho)) \leq C \rho_T^{N-\frac{2}{p-1}}$ for any $0 < \rho < \rho_T$.

Theorem 2.1 is sharp in view of the following two theorems:

**Theorem 2.2.** Let $N \geq 1$, $1 < p < p_F$ and $0 < T < \infty$. Then there exists a constant $c > 0$ depending only on $N$ and $p$ such that the problem (1.1) admits a solution in $M \times [0, \tilde{\rho}^2)$ for any $\tilde{\rho} \in (0, \rho_T]$ and nonnegative Radon measure $\mu$ satisfying

$$\sup_{z \in M} \mu(B(z, \tilde{\rho})) \leq c \tilde{\rho}^{N-\frac{2}{p-1}}.$$
Remark 2.1. By taking $T$ small, we can construct solutions for $\mu$ satisfying (2.5) with $c$ large. Thus, (2.5) is a necessary and sufficient condition for the existence of local-in-time solutions.

Theorem 2.3. Let $N \geq 1$, $p \geq p_F$ and $0 < T < \infty$. Then there exist constants $C > 0$ and $\tilde{C} > 1$ such that the following holds: For any constant $0 < c < C$, there exists a nonnegative Radon measure $\mu$ satisfying

$$
\begin{cases}
\tilde{C}^{-1}c^{N-2/p-1} \leq \sup_{z \in M} \mu(B(z, \rho)) \leq \tilde{C}c^{N-2/p-1} & \text{if } p > p_F, \\
\tilde{C}^{-1}c^{(\log(e + \rho T \rho^{-1}))^{-N/2}} \leq \sup_{z \in M} \mu(B(z, \rho)) \leq \tilde{C}c^{(\log(e + \rho T \rho^{-1}))^{-N/2}} & \text{if } p = p_F,
\end{cases}
$$

for any $0 < \rho < \rho_T$ such that the problem (1.1) admits a solution in $M \times [0, \rho_T^2)$.

Here $C$ depends only on $N$ and $p$ if $p > p_F$ and only on $N$ and $T$ if $p = p_F$. On the other hand, $\tilde{C}$ depends only on $N$ and $p$ in each of the cases.

Remark 2.2. Each $\mu$ in Theorem 2.3 can be written by $d\mu = u_0 dx$ with a suitable function $u_0$. For the precise admissible singularity of $u_0$, see Corollary 2.4.

We show that the necessary conditions in Theorem 2.1 (ii) and (iii) are not always sufficient conditions even if $C$ is replaced by a small constant.

Theorem 2.4. Let $N \geq 1$ and $p \geq p_F$. Then, for any constant $c > 0$, there exists a nonnegative Radon measure $\mu$ satisfying

$$
\sup_{z \in M} \mu(B(z, \rho)) \leq c^{N-2/p-1}(\log(e + \rho^{-1}))^{-1/p-1} \text{ for any } 0 < \rho < \rho_\infty
$$

such that the problem (1.1) does not admit any solutions in $M \times [0, T)$ for any $T > 0$.

If we impose a slightly stronger condition than that of Theorem 2.4, we obtain a general sufficient condition for the solvability of (1.1) in the case $p \geq p_F$.

Theorem 2.5. Let $N \geq 1$, $p \geq p_F$ and $0 < T < \infty$. Then there exists a constant $c > 0$ depending only on $N$ and $p$ such that the following holds: For any $\varepsilon > 0$, $\tilde{\rho} \in (0, \rho_T]$ and nonnegative Radon measure $\mu$ satisfying

$$
\sup_{z \in M} \mu(B(z, \rho)) \leq c^{N-2/p-1}(\log(e + \tilde{\rho} \rho^{-1}))^{-1/p-1-\varepsilon} \text{ for any } 0 < \rho < \tilde{\rho},
$$

the problem (1.1) admits a solution in $M \times [0, \tilde{\rho}^2)$.

2.3. Corollaries

We summarize corollaries of our theorems for the following problem:

$$
\begin{cases}
u_t - \Delta u = u^p & \text{in } M \times (0, T), \\
u(\cdot, 0) = u_0 & \text{in } M,
\end{cases}
$$

(2.7)
where \( u_0 \) is a nonnegative function. A function \( u \) is called a solution of (2.7) in \( M \times [0, T] \) if \( u \) is a solution of (1.1) in the sense of Definition 2.1 with (2.4) replaced by

\[
\lim_{t \to 0} \int_M \psi(u(\cdot, t))dV_g = \int_M \psi u_0 dV_g \quad \text{for any } \psi \in C_0(M).
\]

Let us first consider the existence of solutions of (2.7) with functions in the uniformly local Lebesgue space \( L^q_{uloc, \rho} \) as initial data. Here, we define \( L^q_{uloc, \rho}(M) \) by

\[
L^q_{uloc, \rho}(M) := \{ f \in L^q_{loc}(M); \| f \|_{L^q_{uloc, \rho}(M)} < \infty \},
\]

\[
\| f \|_{L^q_{uloc, \rho}(M)} := \sup_{z \in M} \left( \int_{B(z, \rho)} |f|^q dV_g \right)^{\frac{1}{q}},
\]

for \( \rho > 0 \) and \( q > 1 \) (see [12,21,27] for \( M = \mathbb{R}^N \)). The following corollary generalizes the results by Weissler [44,45] which handle the case where \( u_0 \in L^q(\Omega) \) with \( \Omega \subset \mathbb{R}^N \).

**Corollary 2.1.** Let \( N \geq 1 \) and \( 0 < T < \infty \). Assume one of the following:

(i) \( p > p_F \) and \( q \geq N(p - 1)/2 \).
(ii) \( p = p_F \) and \( q > 1 \).
(iii) \( 1 < p < p_F \) and \( q \geq 1 \).

Then there exists a constant \( c > 0 \) such that the following holds: For any \( \tilde{\rho} \in (0, \rho_T] \) and nonnegative function \( u_0 \in L^q_{uloc, \tilde{\rho}}(M) \) with \( \| u_0 \|_{L^q_{uloc, \tilde{\rho}}(M)} \leq c \), the problem (2.7) admits a solution in \( M \times [0, \tilde{\rho}^2] \). Here, \( c \) depends only on \( N, p, q \) and \( \tilde{\rho}^{2-N(p-1)/q} \) if (i) holds and on \( N, p, q \) and \( \tilde{\rho}^{N(1-1/q)} \) if (ii) or (iii) holds.

We next examine the nonexistence of solutions. Theorem 2.1 implies the following:

**Corollary 2.2.** Let \( N \geq 1, p \geq p_F \) and \( z_0 \in M \). Then there exists a constant \( c > 0 \) depending only on \( N \) and \( p \) such that the following holds: For any nonnegative function \( u_0 \) satisfying

\[
u_0(x) \geq \begin{cases} cd(z_0, x)^{-\frac{2}{p-1}} & \text{if } p > p_F, \\ cd(z_0, x)^{-N} (\log(e + d(z_0, x)^{-1}))^{-\frac{N}{2}} & \text{if } p = p_F, \end{cases}
\]

for any \( x \in B(z_0, \rho_\infty) \) and \( u_0 \in L^\infty(M \setminus B(z_0, \rho_\infty)) \) if \( M \setminus B(z_0, \rho_\infty) \neq \emptyset \), the problem (2.7) does not admit any local-in-time solutions.

By Corollary 2.2, we also obtain the nonexistence of solutions with functions in \( L^q_{uloc, \rho_\infty} \) as initial data. The following corollary generalizes the results by Weissler [45] and Brezis and Cazenave [6] for \( u_0 \in L^q(\Omega) \) with \( \Omega \subset \mathbb{R}^N \).

**Corollary 2.3.** Let \( N \geq 1 \). Assume one of the following:

(i) \( p > p_F \) and \( 1 \leq q < N(p - 1)/2 \).
(ii) \( p = p_F \) and \( q = 1 \).

Then there exists a nonnegative function \( u_0 \in L^q_{uloc,\rho_\infty}(M) \) such that the problem (2.7) does not admit any solutions in \( M \times [0, T) \) for any \( T > 0 \).

Finally, we give a pointwise condition for existence. The proof of Theorem 2.3 shows the following:

**Corollary 2.4.** Let \( N \geq 1 \), \( p \geq p_F \) and \( z_0 \in M \). Then there exists a constant \( c > 0 \) depending only on \( N \) and \( p \) such that the following holds: For any nonnegative function \( u_0 \) satisfying

\[
 u_0(x) \leq \begin{cases} 
 cd(z_0, x)^{-\frac{2}{p-1}} & \text{if } p > p_F, \\
 cd(z_0, x)^{-N} (\log(e + d(z_0, x)^{-1}))^{\frac{N}{2}-1} & \text{if } p = p_F,
\end{cases}
\]

for any \( x \in B(z_0, \rho_\infty) \) and \( u_0 \in L^\infty(M \setminus B(z_0, \rho_\infty)) \) if \( M \setminus B(z_0, \rho_\infty) \neq \emptyset \), the problem (2.7) admits a local-in-time solution.

**Remark 2.3.** Analogous results to Theorems 2.2, 2.3 and 2.5 and Corollaries 2.1 and 2.4 hold for the problem (1.1) in a smooth domain \( \Omega_1 \subset M \) with \( u = 0 \) on \( \partial \Omega_1 \), since the heat kernel on \( M \) dominates the Dirichlet heat kernel on \( \Omega_1 \) (see [7, Page 188, Theorem 4]). We note that the analog of Corollary 2.1 for \( \Omega_1 \) is stronger than the results of [32,34,35].

### 3. Necessary conditions

Fix \( 0 < T < \infty \). In Lemma 3.1, we show the existence and the uniqueness of the initial trace \( \mu \) of \( u \) based on the argument of [17, Lemma 2.3] due to the weak compactness of Radon measures and some limiting argument. In Lemma 3.2, to estimate \( \mu \), we substitute appropriate test functions into the following weak form of (1.1):

\[
 -\int_0^T \int_M u(\varphi_t + \Delta \varphi) dV_g dt = \int_0^T \int_M u^p \varphi dV_g dt + \int_M u(\cdot, \tau) \varphi(\cdot, \tau) dV_g,
\]

where \( \varphi \in C^{2,1}_0(M \times [0, T)) \) and \( 0 < \tau < T \). Then by the comparison theorems in Riemannian geometry (see Appendix A), we obtain estimates of \( \mu(B(z, c\rho)) \) with some \( 0 < c < 1 \). By showing a generalization of the Besicovitch type covering theorem (see Appendix B), we also obtain the desired estimates of \( \mu(B(z, \rho)) \). We note that the covering theorems used in this paper seem to be of independent interest.

**Lemma 3.1.** Let \( u \in C^{2,1}(M \times (0, T)) \) be a nonnegative function satisfying (2.3). Then there exists a unique nonnegative Radon measure \( \mu \) such that \( u \) is a solution of (1.1) in the sense of Definition 2.1.

**Proof.** For \( 0 < \tau < T \), we have \( u(\cdot, t) \rightarrow u(\cdot, \tau) \) and \( \int_M K(\cdot, y, t-\tau) u(y, \tau) dV_g(y) \rightarrow u(\cdot, \tau) \) in \( L^2_{uloc}(M) \) as \( t \rightarrow \tau \), where \( K = K(x, y, t) \) is the heat kernel on \( (M, g) \).
These together with \( u \ge 0, u_t \ge \Delta u \) and the minimality of \( K \) in [14, Theorem 8.1] show that

\[
u(x, t) \ge \int_{\mathcal{M}} K(x, y, t - \tau)u(y, \tau) d\nu_\tau(y) \quad (3.1)
\]

for \( x \in \mathcal{M} \) and \( 0 < \tau < t < T \). Fix \( z_0 \in \mathcal{M} \) and \((2/3)T < t_0 < T\). Let \( M_0 \) be a compact subset of \( \mathcal{M} \). Since \( \mathcal{M} \) is connected, by [14, Corollary 8.12], we have \( K(x, y, t) > 0 \) for \( x, y \in \mathcal{M} \) and \( t > 0 \). Thus,

\[
u(z_0, t_0) \ge \int_{M_0} K(z_0, y, t_0 - t)u(y, t) d\nu_\tau(y)
\]

\[
\ge \inf_{y \in M_0, \frac{1}{3}T < s < T} K(z_0, y, s) \int_{M_0} u(\cdot, t) d\nu_\tau
\]

for \( 0 < t < T/3 \), and so

\[
\int_{M_0} u(\cdot, t) d\nu_\tau \le C \quad \text{for } 0 < t < \frac{T}{3}, \quad (3.2)
\]

where \( C > 0 \) is a constant depending on \( z_0, t_0, M_0 \) and \( T \). Define a family of nonnegative Radon measures \( \{\mu_t\}_{t \in (0, T/3)} \) on \( \mathcal{M} \) by \( \mu_t(A) := \int_A u(\cdot, t) d\nu_\tau \). Let \( \{t_k\}_{k=1}^\infty \subset (0, T/3) \) satisfy \( t_k \to 0 \). Then, we obtain \( \sup_{k \ge 1} \mu_{t_k}(M_0) < \infty \) for any compact subset \( M_0 \subset \mathcal{M} \). By the weak compactness of Radon measures [40, Theorem 4.4], there exist a subsequence \( \{\mu_{t_k}\}_{k=1}^\infty \) and a nonnegative Radon measure \( \mu \) on \( \mathcal{M} \) such that

\[
\lim_{k \to \infty} \int_{\mathcal{M}} \psi d\mu_{t_k} = \int_{\mathcal{M}} \psi d\mu \quad \text{for any } \psi \in C_0(\mathcal{M}).
\]

We claim that \( \mu_{t_k} \to \mu \) without taking a subsequence. To prove this, take a subsequence \( \{\mu_{t''_k}\}_{k=1}^\infty \) of \( \{\mu_{t_k}\}_{k=1}^\infty \) arbitrary. Then, by the same reason as above, there exist a subsequence \( \{\mu_{t''_k}\}_{k=1}^\infty \) of \( \{\mu_{t''_k}\}_{k=1}^\infty \) and a nonnegative Radon measure \( \nu \) on \( \mathcal{M} \) such that \( \lim_{k \to \infty} \int_{\mathcal{M}} \psi d\mu_{t''_k} = \int_{\mathcal{M}} \psi d\nu \) for any \( \psi \in C_0(\mathcal{M}) \). If \( \mu = \nu \), then the limit \( \mu_{t_k} \to \mu \) immediately follows. Let \( \phi \in C_0(\mathcal{M}) \) with \( \phi \ge 0 \). We take a subsequence again if necessary and suppose \( t'_k > t''_k \) for \( k \ge 1 \). From (3.1) with \( \tau = t''_k \) and \( t = t'_k \), it follows that

\[
\int_{\mathcal{M}} \phi d\nu_{t'_k} = \int_{\mathcal{M}} \phi(x)u(x, t'_k) d\nu_\tau(x)
\]

\[
\ge \int_{\mathcal{M}} \phi(x) \int_{\mathcal{M}} K(x, y, t'_k - t''_k)u(y, t''_k) d\nu_\tau(y) d\nu_\tau(x)
\]

\[
= \int_{\mathcal{M}} \int_{\mathcal{M}} K(y, x, t'_k - t''_k) \phi(x) d\nu_\tau(y) d\mu_{t''_k}(y)
\]

\[
\ge \sup_{\phi} \int_{\mathcal{M}} K(y, x, t'_k - t''_k) \phi(x) d\nu_\tau(y) d\mu_{t''_k}(y)
\]
and that
\[
\int_M \phi d\mu_{t_k} \geq \int_{\text{supp} \phi} \phi d\mu_{t_k''} - \int_{\text{supp} \phi} \left| \int_M K(y, x, t_k' - t_k'') \phi(x) dV_g(x) - \phi(y) \right| d\mu_{t_k''}(y).
\]

By the uniform continuity of $\phi$ on $M$ and [14, Theorem 7.13], we have
\[
\lim_{k \to \infty} \int_M K(y, x, t_k' - t_k'') \phi(x) dV_g(x) = \phi(y)
\]
uniformly for $y \in M$. By the compactness of $\text{supp} \phi$ and (3.2), we have $\mu_{t_k''}(\text{supp} \phi) < \infty$. Therefore, letting $k \to \infty$ yields $\int_M \phi d\mu \geq \int_{\text{supp} \phi} \phi d\nu = \int_M \phi d\nu$. Interchanging $t_k'$ and $t_k''$ gives $\int_M \phi d\mu = \int_M \phi d\nu$ for any $\phi \in C_0(M)$ with $\phi \geq 0$. Thus, $\int_M \psi d\mu = \int_M \psi d\nu$ for any $\psi \in C_0(M)$. This implies $\mu = \nu$, and so $\mu_{t_k} \to \mu$ without taking a subsequence. Then (2.4) holds, and so $u$ is a solution of (1.1) in the sense of Definition 2.1. The uniqueness of $\mu$ also follows from the above argument. \hfill \square

We prepare an estimate for proving Theorem 2.1. In what follows, $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$ denote the positive part and the negative part of $f$, respectively.

**Lemma 3.2.** Let $p > 1$ and let $u$ be a solution of (1.1). Then there exists a constant $C > 0$ depending only on $p$ such that the initial data $\mu$ of $u$ satisfies
\[
\int_M \phi(\cdot, 0) \frac{2p}{p-1} d\mu \leq C \int_0^T \int_M \phi \frac{p}{p-1} \left( (\phi_t + \Delta \phi)^- \right) \frac{p}{p-1} dV_g dt
\]
for any $\phi \in C_0^2(M \times [0, T])$ with $\phi \geq 0$.

**Proof.** Let $\varphi \in C_0^2(M \times [0, T])$ and $0 < \tau < T$. Multiplying (2.3) by $\varphi$ and using integration by parts, we have
\[
- \int_\tau^T \int_M u(\varphi_t + \Delta \varphi) dV_g dt = \int_\tau^T \int_M u^p \varphi dV_g dt + \int_M u(\cdot, \tau) \varphi(\cdot, \tau) dV_g.
\]

Let $\phi \in C_0^\infty(M \times [0, T])$ with $\phi \geq 0$. We take $\varphi = \phi^{2p/(p-1)}$. Then,
\[
\varphi_t = \frac{2p}{p-1} \phi^{p+1/p-1} \phi_t,
\]
\[
\Delta \varphi = \frac{2p(p+1)}{(p-1)^2} \phi^{2/p-1} |\nabla \phi|^2 + \frac{2p}{p-1} \phi \frac{p+1}{p-1} \Delta \phi \geq \frac{2p}{p-1} \phi \frac{p+1}{p-1} \Delta \phi,
\]
and so
\[
\int_\tau^T \int_M u^p \phi \frac{2p}{p-1} dV_g dt + \int_M u(\cdot, \tau) \phi(\cdot, \tau) \frac{2p}{p-1} dV_g
\]
\[
\leq - \int_\tau^T \int_M u \phi \frac{p+1}{p-1} \frac{2p}{p-1} (\phi_t + \Delta \phi) dV_g dt
\]
\[
\leq \int_\tau^T \int_M u \phi \frac{p+1}{p-1} \frac{2p}{p-1} (\phi_t + \Delta \phi)^- dV_g dt.
\]
By Young’s inequality, we have
\[ u\phi^\frac{p+1}{p} \frac{2p}{p-1} (\phi_t + \Delta \phi)^- \leq u\phi^\frac{2p}{p-1} + C\phi^\frac{p}{p-1} ( (\phi_t + \Delta \phi)^-) \frac{p}{p-1}. \]
Thus,
\[ \int_M u(\cdot, \tau) \phi(\cdot, \tau)\frac{2p}{p-1} dV_g \leq C \int_T \int_{M} \phi^\frac{p}{p-1} ( (\phi_t + \Delta \phi)^-) \frac{p}{p-1} dV_g dt. \]
By using (2.4) and letting \( \tau \to 0 \), there exists a constant \( C > 0 \) depending only on \( p \) such that (3.3) holds. \( \square \)

In the rest of this section, we prove Theorem 2.1 by choosing appropriate \( \phi \) in (3.3).

**Proof of Theorem 2.1 (i) and (iii).** We take \( \eta \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \eta \leq 1, \eta' \geq 0, \eta(\tau) = 0 \) for \( \tau \leq 0 \) and \( \eta(\tau) = 1 \) for \( \tau \geq 1 \). Fix \( z \in M \). For \( 0 < \rho < \rho_T \), we set
\[ \phi(x, t) := \eta(\Phi(x, t)), \quad \Phi(x, t) := \frac{\rho^2}{d(x, z)^2 + t} - 1, \]
for \((x, t) \in M \times [0, \infty)\). Since \( \rho_T \leq \sqrt{T} \), we have \( \Phi(x, T) \leq \rho_T^2 T^{-1} - 1 \leq 0 \). Then \( \phi(\cdot, T) = 0 \), and so \( \phi \in C_0^\infty(M \times [0, T)) \). Let
\[ D := \{(x, t) \in M \times [0, \infty); \rho_T^2 < d(x, z)^2 + t < \rho_T^2\}. \]
Note that \( d(\cdot, z) \) is smooth on \( B(z, \text{inj}(M)) \setminus \{z\} \) and that
\[
(M \times [0, \infty)) \setminus D \subset \{(x, t) \in M \times [0, \infty); \eta'(\Phi(x, t)) = 0\},
\{(x, t) \in M \times [0, \infty); \eta'(\Phi(x, t)) \neq 0\} \subset D \subset B(z, \text{inj}(M)) \times [0, \infty). \tag{3.4}
\]
From \( \Phi_t = -\rho^2(d(x, z)^2 + t)^{-2} \), it follows that
\[ \phi_t = (\eta' \circ \Phi)\Phi_t \geq -C\rho^2(d(x, z)^2 + t)^{-2} \text{ for } (x, t) \in D. \tag{3.5} \]

Direct computations yield
\[ \nabla \Phi = -2\rho^2(d(x, z)^2 + t)^{-2}d(x, z)\nabla d(x, z), \]
\[ \Delta \Phi = 8\rho^2(d(x, z)^2 + t)^{-3}d(x, z)^2|\nabla d(x, z)|^2 - 2\rho^2(d(x, z)^2 + t)^{-2}|\nabla d(x, z)|^2 - 2\rho^2(d(x, z)^2 + t)^{-2}d(x, z)\Delta d(x, z). \]
Then by \(|\nabla d(\cdot, z)| = 1\), we have
\[
\Delta \phi = (\eta'' \circ \Phi)|\nabla \Phi|^2 + (\eta' \circ \Phi)\Delta \Phi
\geq -C|\nabla \Phi|^2 - C(\Delta \Phi)^-
\geq -C\rho^2(d(x, z)^2 + t)^{-2} - C\rho^2(d(x, z)^2 + t)^{-2}d(x, z)(\Delta d(x, z))^+
\text{ for } (x, t) \in D.
\]
We claim that
\[ \Delta \phi \geq -C \rho^2 (d(x, z)^2 + t)^{-2}. \] (3.6)

To show this, it suffices to prove
\[ \Delta d(x, z) \leq C d(x, z)^{-1} \quad \text{for } x \in B(z, \rho) \setminus \{z\}. \] (3.7)

Let \( r(x) := d(x, z) \). Remark that \( 0 < r < \rho < \rho_T \). By the assumption on \( M \), we can apply (A.1) and we have \( \Delta r(x) \leq 2(N - 1)r^{-1}(x) \) for \( x \in B(z, \rho) \setminus \{z\} \), where we used \( \rho_T \leq \pi/(4 \sqrt{k}) \). Hence, (3.6) follows.

By (3.5) and (3.6), we obtain \( (\phi + \Delta \phi)^- \leq C \rho^2 (d(x, z)^2 + t)^{-2} \). This together with Lemma 3.2 and \( \phi \leq 1 \) shows that
\[ \int_M \phi(\cdot, 0)^{\frac{2p}{p-1}} d\mu \leq C \rho^{\frac{2p}{p-1}} \int_D (d(x, z)^2 + t)^{-\frac{2p}{p-1}} dV_g(x)dt, \] (3.8)
where \( C > 0 \) is a constant depending only on \( N \) and \( p \). Since \( \phi(x, 0) = 1 \) for \( x \in B(z, \rho/2) \), the left-hand side of (3.8) can be estimated as
\[ \int_M \phi(\cdot, 0)^{\frac{2p}{p-1}} d\mu \geq \mu(B(z, \rho/2)). \]

We estimate the right-hand side of (3.8) by using Riemannian normal coordinates \((x_1, \ldots, x_N)\) centered at \( z \), where we identify \( x \in B(z, \text{inj}(M)) \) with \((x_1, \ldots, x_N)\). In these coordinates, we have
\[ \int_D (d(x, z)^2 + t)^{-\frac{2p}{p-1}} dV_g(x)dt = \int_{\tilde{D}} (|x|^2 + t)^{-\frac{2p}{p-1}} \sqrt{\det(g_{ij}(x))} dx dt \]
with \( \tilde{D} := \{(x, t) \in \mathbb{R}^N \times [0, \infty) ; \rho^2/2 \leq |x|^2 + t \leq \rho^2\} \). By the assumption on \( M \), we can apply the volume comparison theorem (A.2) and have \( \sqrt{\det(g_{ij}(x))} \leq 2^{N-1} \) for \( x \in B(z, \rho) \setminus \{z\} \), where we used \( \rho_T \leq \pi/(4 \sqrt{k}) \). Thus,
\[
\int_{\tilde{D}} (|x|^2 + t)^{-\frac{2p}{p-1}} \sqrt{\det(g_{ij})} dx dt \\
\leq C \int_{\tilde{D}} (|x|^2 + t)^{-\frac{2p}{p-1}} dx dt \\
\leq 2C \int_{\{|x|^2 + s^2 \leq \rho^2\}} (|x|^2 + s^2)^{-\frac{2p}{p-1}} s dx ds \\
\leq C \int_{\{|x|^2 + s^2 \leq \rho^2\}} (\sqrt{|x|^2 + s^2})^{-\frac{4p}{p-1} + 1} dx ds \leq C \rho^{N+2-\frac{4p}{p-1}}.
\]
Hence, for all \( p > 1 \), we obtain
\[ \mu(B(z, \rho/2)) \leq C \rho^{N-\frac{2}{p-1}} \quad \text{for any } 0 < \rho < \rho_T, \] (3.9)
where \( C > 0 \) is a constant depending only on \( N \) and \( p \). Fix \( z' \in M \). Since \( \rho_T \leq R' \) (see (B.3) for the definition of \( R' \)), we can use Theorem B.2 and say that there exists a
number $k_1$ depending only on $N$ such that $B(z', \rho)$ can be covered by at most $k_1$ balls with radii $\rho/2$. This together with (3.9) implies

$$\mu(B(z', \rho)) \leq k_1 C \rho^{N - \frac{2}{p-1}}$$

for any $0 < \rho < \rho_T$.

This shows (i) and (iii).

We improve (3.9) in the case $p = p_F$.

**Proof of Theorem 2.1 (ii).** We take $\eta \in C^\infty(R)$ as in the proof of Theorem 2.1 (i) and (iii). Fix $z \in M$. For $0 < \rho < \rho_T$, we set

$$\phi(x, t) := \eta(\Phi(x, t)), \quad \Phi(x, t) := 2 \left( \log \left( e + \frac{\rho_T^2}{d(x, z)^2 + t} \right) \right)^{-1}$$

for $(x, t) \in M \times [0, \infty)$. Remark that $\phi(\cdot, T) = 0$. Indeed, this follows from

$$\Phi(x, T) \leq 2(\log(e + 16))^{-1} \log(e + 1) - 1 \leq 0.$$

Let

$$D' := \left\{ (x, t) \in M \times [0, \infty); \rho^2/16 < d(x, z)^2 + t < \rho_T^2((e + 16 \rho_T^2 \rho^{-2})^{1/2} - e)^{-1} \right\}.$$

By $(e + 16 \rho_T^2 \rho^{-2})^{1/2} - e > 1$, we have $\rho_T((e + 16 \rho_T^2 \rho^{-2})^{1/2} - e)^{-1/2} < \rho_T \leq \inj(M)$. Therefore, (3.4) holds with $D$ replaced by $D'$. Note that $d(\cdot, z)$ is smooth on $B(z, \inj(M)) \setminus \{z\}$. By

$$\Phi_t = -2 \rho_T^2 (\log(e + 16 \rho_T^2 \rho^{-2}))^{-1} (e(d^2 + t) + \rho_T^2)^{-1} (d^2 + t)^{-1},$$

we have

$$\phi_t \geq -C|\Phi_t| \geq -C(\log(e + 16 \rho_T^2 \rho^{-2}))^{-1} (d(x, z)^2 + t)^{-1}$$

for some constant $C > 0$ independent of $T$. Straightforward computations yield

$$\nabla \Phi = -4 \rho_T^2 (\log(e + 16 \rho_T^2 \rho^{-2}))^{-1} (e(d^2 + t) + \rho_T^2)^{-1} (d^2 + t)^{-1} d \nabla d,$$

$$\Delta \Phi = 4 \rho_T^2 (\log(e + 16 \rho_T^2 \rho^{-2}))^{-1} (e(d^2 + t) + \rho_T^2)^{-1} (d^2 + t)^{-1} \times \left( 2(d^2 + t)^{-1} d^2 |\nabla d|^2 + 2e(e(d^2 + t) + \rho_T^2)^{-1} d^2 |\nabla d|^2 - |\nabla d|^2 - d \Delta d \right).$$

Then,

$$\Delta \phi = (\eta'' \circ \Phi)|\nabla \Phi|^2 + (\eta' \circ \Phi) \Delta \Phi$$

$$\geq -C|\nabla \Phi|^2 - C(\Delta \Phi)^-$$

$$\geq -C \rho_T^4 (\log(e + 16 \rho_T^2 \rho^{-2}))^{-2} (e(d^2 + t) + \rho_T^2)^{-2} (d^2 + t)^{-2} d^2 |\nabla d|^2$$

$$- C \rho_T^2 (\log(e + 16 \rho_T^2 \rho^{-2}))^{-1} (e(d^2 + t) + \rho_T^2)^{-1} (d^2 + t)^{-1} (|\nabla d|^2 + d(\Delta d)^+).$$
From $|\nabla d| = 1$ and (3.7), it follows that

$$\Delta \phi \geq -C \left( \log(e + 16\rho_T^2 \rho^{-2}) \right)^{-1} (d(x, z)^2 + t)^{-1},$$

where $C > 0$ is a constant depending only on $N$ and $p_F$.

By the above computations, Lemma 3.2 and $p = p_F$, we obtain

$$\int_M \phi(\cdot, 0)^{N+2} d\mu \leq C \left( \log(e + \rho_T^2 \rho^{-2}) \right)^{-\frac{N}{2}} \int_M (d(x, z)^2 + t)^{-\frac{N}{2}} dV_g(x) dt. \quad (3.10)$$

Since $\phi(x, 0) = 1$ for $x \in B(z, \rho/4)$, the left-hand side of (3.10) can be estimated as

$$\int_M \phi(\cdot, 0)^{N+2} d\mu \geq \mu(B(z, \rho/4)).$$

We estimate the right-hand side of (3.10). In Riemannian normal coordinates $(x_1, \ldots, x_N)$ centered at $z$, we have

$$\int_{D'} (d(x, z)^2 + t)^{-\frac{N}{2}} dV_g(x) dt = \int_{D'} (|x|^2 + t)^{-\frac{N}{2}} \sqrt{\det(g_{ij})} dx dt,$$

$$\tilde{D'} := \left\{ (x, t) \in M \times [0, \infty); \theta_1^2 < |x|^2 + t < \theta_2^2 \right\},$$

$$\theta_1 := \rho/4, \quad \theta_2 := \rho_T((e + 16\rho_T^2 \rho^{-2})^{\frac{1}{2}} - e)^{-\frac{1}{2}}.$$

Then by the volume comparison theorem (A.2) and $\theta_2 < \rho_T$, we obtain

$$\int_{\tilde{D'}} (|x|^2 + t)^{-\frac{N}{2}} \sqrt{\det(g_{ij})} dx dt \leq C \int_{\tilde{D'}} (|x|^2 + t)^{-\frac{N}{2}} dx dt$$

$$= 2C \int_{\theta_1^2 < |x|^2 + t < \theta_2^2} (|x|^2 + s^2)^{-\frac{N}{2}} dx ds$$

$$\leq 2C \int_{\theta_1^2 < |x|^2 + s^2 < \theta_2^2} (\sqrt{|x|^2 + s^2})^{-N-1} dx ds$$

$$\leq C \log(\theta_2/\theta_1) \leq C \log(e + \rho_T \rho^{-1}),$$

where $C > 0$ is a constant independent of $\rho_T$. Hence, we deduce that

$$\mu(B(z, \rho/4)) \leq C \left( \log(e + \rho_T \rho^{-1}) \right)^{-\frac{N}{2}}$$

for any $0 < \rho < \rho_T$, where $C > 0$ is a constant depending only on $N$ and $p_F$. Fix $z' \in M$. Recall that $\rho_T \leq R'$, where $R'$ is given by (B.3). Then, applying Theorem B.2 twice, we obtain

$$\mu(B(z', \rho)) \leq k_T^2 C \left( \log(e + \rho_T \rho^{-1}) \right)^{-\frac{N}{2}}$$

for any $0 < \rho < \rho_T$. This is equivalent to (ii), and the proof is complete. \qed
4. Sharpness of necessary conditions

Let \( K = K(x, y, t) \) be the heat kernel on \((M, g)\). To construct solutions of (1.1), we construct integral solutions in the following sense:

**Definition 4.1.** Let \( 0 < T \leq \infty \).

(i) A function \( u \) is called an integral solution of (1.1) in \( M \times (0, T) \) if \( u \) is a measurable function on \( M \times (0, T) \) with \( 0 \leq u < \infty \) a.e. \( M \times (0, T) \) satisfying

\[
 u(x, t) = \Psi[u](x, t) \quad \text{for \ a.e.} \ (x, t) \in M \times (0, T),
\]

\[
 \Psi[u](x, t) := \int_M K(x, y, t)d\mu(y) + \int_0^t \int_M K(x, y, t-s)u(y, s)dV_g(y)ds.
\]

(ii) A function \( \overline{u} \) is called a supersolution of (4.1) in \( M \times [0, T) \) if \( \overline{u} \) is a measurable function on \( M \times (0, T) \) satisfying \( 0 \leq \overline{u} < \infty \) a.e. in \( M \times (0, T) \).

By applying a similar argument of [17] together with the Li-Yau Harnack inequalities [26] and the volume comparison, we give a sharp pointwise estimate of a solution \( \int_M K(x, y, t)d\mu(y) \) of the linear heat equation, where \( \mu \) is any nonnegative Radon measure on \( M \). Then we give candidates of supersolutions of the integral equation in the same spirit of [38, Section 4] and [17, Section 4]. By using the estimate of \( \int_M K(x, y, t)d\mu(y) \), we prove that the candidates are in fact appropriate supersolutions. Then the following lemma guarantees the existence of solutions:

**Lemma 4.1.** Assume that there exists a supersolution \( \overline{u} \) of (4.1) in \( M \times [0, T) \). Then there exists an integral solution of (1.1) in \( M \times [0, T) \).

**Proof.** Since \( M \) is connected, \( K \) is positive. Then this lemma follows from the same monotone iteration scheme as in [38, Theorem 1]. See also [17, Lemma 2.2]. \qed

In what follows, we first give estimates on the linear part in Sect. 4.1. Next, we prove Theorems 2.2, 2.3 and 2.5 in Sects. 4.2, 4.3 and 4.4, respectively.

4.1. Estimates of the linear part

We estimate solutions of the linear heat equation with measures as initial data. The following lemma is a generalization of [17, Lemma 2.1]:

**Lemma 4.2.** Let \( \mu \) be a nonnegative Radon measure on \( M \). Then there exists a constant \( C > 0 \) depending only on \( N \) such that

\[
 \int_M K(x, y, t)d\mu(y) \leq Ct^{-N/2} \sup_{z \in M} \mu(B(z, t^{1/2}))
\]

for \( (x, t) \in M \times (0, 16\rho_{\infty}^2) \).

**Remark 4.1.** If \( M = \mathbb{R}^N \), then (4.2) holds for \( (x, t) \in M \times (0, \infty) \).
Proof. Assume $\rho_\infty < \infty$. The case $\rho_\infty = \infty$ will be handled at the end of this proof. Let $0 < t < 16\rho_\infty^2$. Define $F := \bigcup_{x \in M} B(x, (t/4)^{1/2})$. Since $(t/4)^{1/2} < 2\rho_\infty < \mathrm{inj}(M)/2$, Corollary B.1 with $\xi := 2\rho_\infty$ implies the following: There exist points $x_{k,i} \in M$ for $k = 1, \ldots, k_0$ and $i \geq 1$ such that $B(x_{k,i}, (t/4)^{1/2}) \cap B(x_{k,j}, (t/4)^{1/2}) = \emptyset$ for $i \neq j$ and $M = \bigcup_{k=1}^{k_0} \bigcup_{i=1}^{\infty} B(x_{k,i}, (t/4)^{1/2})$. Here, the constant $k_0$ depends only on $N$.

Fix $x \in M$. Then,

$$\int_M K(x, y, t) d\mu(y) \leq \sum_{k=1}^{k_0} \sum_{i=1}^{\infty} \int_{B(x_{k,i}, (t/4)^{1/2})} K(x, y, t) d\mu(y)$$

$$\leq \sum_{k=1}^{k_0} \sum_{i=1}^{\infty} \mu(B(x_{k,i}, (t/4)^{1/2})) \sup_{y \in B(x_{k,i}, (t/4)^{1/2})} K(x, y, t)$$

$$\leq \sup_{z \in M} \mu(B(z, t^{1/2})) \sum_{k=1}^{k_0} \sum_{i=1}^{\infty} \sup_{y \in B(x_{k,i}, (t/4)^{1/2})} K(x, y, t).$$

To estimate $\sup_y K(x, y, t)$, we take $y, z \in \overline{B(x_{k,i}, (t/4)^{1/2})}$. Then, by $|\sec(M)| \leq \kappa$, $d(y, z) \leq \sqrt{t}$ and $t < \pi^2/\kappa$, the estimate of the heat kernel (A.8) implies the existence of a constant $C > 0$ depending only on $N$ such that

$$K(x, y, t) \leq CK(x, z, 2t).$$

Then there exists a constant $C > 0$ depending only on $N$ such that

$$\sup_{y \in B(x_{k,i}, (t/4)^{1/2})} K(x, y, t) \leq C \inf_{z \in B(x_{k,i}, (t/4)^{1/2})} K(x, z, 2t)$$

$$\leq \frac{C}{\mathrm{Vol}(B(x_{k,i}, (t/4)^{1/2}))} \int_{B(x_{k,i}, (t/4)^{1/2})} K(x, z, 2t) dV_g(z).$$

Since $(t/4)^{1/2} < 2\rho_\infty$, the volume comparison theorem (A.5) shows the existence of a constant $C > 0$ depending only on $N$ such that

$$\mathrm{Vol}(B(x_{k,i}, (t/4)^{1/2})) \geq Ct^N.$$

From the above computations, it follows that

$$\int_M K(x, y, t) d\mu(y) \leq Ct^{-N} \sup_{z \in M} \mu(B(z, t^{1/2})) \sum_{k=1}^{k_0} \sum_{i=1}^{\infty} \int_{B(x_{k,i}, (t/4)^{1/2})} K(x, z, 2t) dV_g(z)$$

$$\leq Ct^{-N} \sup_{z \in M} \mu(B(z, t^{1/2})) \sum_{k=1}^{k_0} \int_M K(x, z, 2t) dV_g(z)$$

$$\leq k_0 Ct^{-N} \sup_{z \in M} \mu(B(z, t^{1/2}))$$
for \((x, t) \in M \times (0, 16\rho_\infty^2)\), where \(C > 0\) depends only on \(N\). Hence, we obtain (4.2) in the case \(\rho_\infty < \infty\). Since \(C\) is independent of \(\rho_\infty\), (4.2) also holds in the case \(\rho_\infty = \infty\).

4.2. The case \(p < p_F\)

We prove Theorem 2.2.

**Proof of Theorem 2.2.** Let \(\mu\) and \(\tilde{\rho} \in (0, \rho_T]\) satisfy (2.5), that is,

\[
\sup_{z \in M} \mu(B(z, \tilde{\rho})) \leq c\tilde{\rho}^{N-2/p-1},
\]

where \(c > 0\) is a constant chosen later. Set

\[
\bar{u}(x, t) := 2U(x, t), \quad U(x, t) := \int_M K(x, y, t) d\mu(y).
\]

We check that \(\bar{u}\) is a supersolution of (4.1) in \(M \times [0, \tilde{\rho}^2]\). From Fubini’s theorem and the semigroup property of \(K\), it follows that

\[
\Psi[\bar{u}](x, t) = U(x, t) + 2p \int_0^t \int_M \bar{u}(x, y, t - s) U(y, s)^{p-1} U(y, s) dV_g(y) ds
\]

\[
\leq U(x, t) + 2p \int_0^t \|U(\cdot, s)\|_{L^\infty}^{p-1} \int_M \bar{u}(x, y, t - s) \int_M K(y, z) d\mu(z) dV_g(y) ds
\]

\[
= U(x, t) + 2p U(x, t) \int_0^t \|U(\cdot, s)\|_{L^\infty}^{p-1} ds
\]

(4.4)

for \((x, t) \in M \times (0, \tilde{\rho}^2)\). By Lemma 4.2 with \(\tilde{\rho} \leq 4\rho_\infty\), we have

\[
U(x, t) \leq Ct^{-\frac{N}{2}} \sup_{z \in M} \mu(B(z, \tilde{\rho}^2)) \leq Ct^{-\frac{N}{2}} \sup_{z \in M} \mu(B(z, \tilde{\rho}))
\]

for \((x, t) \in M \times (0, \tilde{\rho}^2)\). This together with (2.5) and \(1 - N(p - 1)/2 > 0\) gives

\[
\int_0^t \|U(\cdot, s)\|_{L^\infty}^{p-1} ds \leq Cc^{p-1} t^{1 - \frac{N(p-1)}{2}} \tilde{\rho}^{N(p-1) - 2} \leq Cc^{p-1}
\]

for \((x, t) \in M \times (0, \tilde{\rho}^2)\), where \(C > 0\) is a constant depending only on \(N\) and \(p\). Therefore, by choosing \(c\) satisfying \(2pCc^{p-1} \leq 1\), we see that \(\bar{u}\) is a supersolution of (4.1) in \(M \times [0, \tilde{\rho}^2]\).

By Lemma 4.1, we obtain an integral solution \(u\) of (1.1) in \(M \times [0, \tilde{\rho}^2]\) satisfying

\[
0 \leq u \leq \bar{u}. \quad \text{In particular, } u \in L_{\text{loc}}^\infty((0, \tilde{\rho}^2); L^\infty(M)).
\]

By \(u = \Psi[u]\), Fubini’s theorem and the semigroup property of \(K\), we can see that \(u\) satisfies

\[
u(x, t) = \int_M K(x, y, t - \tau) u(y, \tau) dV_g(y)
\]

\[
+ \int_0^t \int_M K(x, y, t - s) u(y, s)^p dV_g(y) ds
\]

(4.5)

for a.e. \(x \in M\) and \(0 < \tau < t < \tilde{\rho}^2\). Hence, we easily see that \(u \in C^{2,1}(M \times (0, \tilde{\rho}^2))\).
It remains to prove that \( u \) satisfies (2.4). To see this, it suffices to check (2.4) for any nonnegative function \( \psi \in C_0(M) \). We modify the argument of [17, Lemma 2.4]. Since \( u \) satisfies (4.1), we have
\[
\int_M u(\cdot, t)\psi dV_g = \int_M \int_M K(x, y, t)\psi(x)dV_g(x)d\mu(y) \\
+ \int_M \int_0^t \int_M K(x, y, t-s)u(y, s)^p dV_g(y)ds\psi(x)dV_g(x) \tag{4.6}
\]
for \( x \in M \) and \( 0 < t < \tilde{\rho}_2^2 \). We claim that the second term in the right-hand side of (4.6) converges to 0 as \( t \to 0 \). Write \( M_0 := \text{supp } \psi \). From (4.5) and Fubini’s theorem, it follows that
\[
\int_M u(\cdot, t)\psi dV_g \\
= \int_M \int_M K(y, x, t-\tau)\psi(x)dV_g(x)u(y, \tau)dV_g(y) \\
+ \int_M \int_0^t \int_M K(x, y, t-s)u(y, s)^p dV_g(y)ds\psi(x)dV_g(x) \\
\geq \int_{M_0} u(\cdot, \tau)\psi dV_g - \int_{M_0} u(\cdot, \tau)dV_g \left\| \int_M K(\cdot, x, t-\tau)\psi(x)dV_g(x) - \psi \right\|_{L^\infty(M_0)} \\
+ \int_M \int_0^t \int_M K(x, y, t-s)u(y, s)^p dV_g(y)ds\psi(x)dV_g(x)
\]
for \( x \in M \) and \( 0 < \tau < t < \tilde{\rho}_2^2/3 \). Since \( u \) satisfies (2.3) in \( M \times (0, \rho_2^2) \), by Lemma 3.1, there exists a nonnegative Radon measure \( \nu \) on \( M \) such that \( u \) satisfies (2.4) with \( \mu \) replaced by \( \nu \). Thus, using (3.2) and letting \( \tau \to 0 \) yield
\[
\int_M \psi d\nu \geq \int_{M_0} \psi d\nu - \sup_{0<\tau<\tilde{\rho}_2^2/3} \int_{M_0} u(\cdot, \tau) dV_g \left\| \int_M K(\cdot, x, t)\psi(x)dV_g(x) - \psi \right\|_{L^\infty(M_0)} \\
+ \int_M \int_0^t \int_M K(x, y, t-s)u(y, s)^p dV_g(y)ds\psi(x)dV_g(x)
\]
for \( x \in M \) and \( 0 < t < \tilde{\rho}_2^2/3 \). Since \( M_0 = \text{supp } \psi \), we see that
\[
\int_M \int_0^t \int_M K(x, y, t-s)u(y, s)^p dV_g(y)ds\psi(x)dV_g(x) \\
\leq \sup_{0<\tau<\tilde{\rho}_2^2/3} \int_{M_0} u(\cdot, \tau) dV_g \left\| \int_M K(\cdot, x, t)\psi(x)dV_g(x) - \psi \right\|_{L^\infty(M_0)} .
\]
By [14, Theorem 7.13], the right-hand side converges to 0 as \( t \to 0 \). Hence, by letting \( t \to 0 \), we can see that the second term in the right-hand side of (4.6) converges to 0 as \( t \to 0 \). Then the claim follows.

Finally, let us consider the first term in the right-hand side of (4.6). Fix \( z_0 \in M \) and \( R \geq \sqrt{3} \) so that \( M_0 = \text{supp } \psi \subset B(z_0, R) \). Since \( \mu \) is a Radon measure, we have \( \mu(B(z_0, R)) < \infty \). Then,
\[
\int_M \psi(y)d\mu(y) \leq \left\| \psi \right\|_{L^\infty(M_0)} \mu(B(z_0, R)) < \infty.
\]
Thus, $\psi \in L^1_{\mu}(M)$, that is, the nonnegative function $\psi$ is $\mu$-integrable. We check the $\mu$-integrability of $y \mapsto \int_{M} K(x, y, t)\psi(x)dV_{g}(x)$. In the case $y \in B(z_0, 2R)$, we see that

$$\int_{M} K(x, y, t)\psi(x)dV_{g}(x) \leq \|\psi\|_{L^{\infty}(M_0)} < \infty.$$ 

On the other hand, in the case $y \notin B(z_0, 2R)$, the upper bound of the heat kernel (A.9) yields

$$\int_{M} K(x, y, t)\psi(x)dV_{g}(x) = \int_{B(z_0, R)} K(x, y, t)\psi(x)dV_{g}(x)$$

$$\leq \|\psi\|_{L^{\infty}(M_0)} \int_{B(z_0, R)} Ct^{-N/2} \exp\left(-\frac{d(x, y)^2}{4(1/2)t}\right)dV_{g}(x),$$

where $C > 0$ depends only on $N, \kappa$ and $\text{inj}(M)$. Since $x \in B(z_0, R)$ and $y \notin B(z_0, 2R)$ with $R \geq \sqrt{3}$, we have $d(x, y)^2 \geq (3/4)[d(x, y)^2 + 1]$. Thus,

$$t^{-\frac{N}{2}} \exp\left(-\frac{d(x, y)^2}{4(1/2)t}\right)$$

$$\leq t^{-\frac{N}{2}} \exp\left(-\frac{1}{6t}\right) \exp\left(-\frac{d(x, y)^2}{6t}\right) \leq C \exp\left(-\frac{d(x, y)^2}{6t}\right)$$

$$\leq C \exp\left(-\frac{d(x, y)^2}{2\tilde{\rho}^2}\right)$$

$$= C(\tilde{\rho}^2)^{-\frac{N}{2}} (\tilde{\rho}^2)^{-\frac{N}{2}} \exp\left(-\frac{d(x, y)^2}{2\tilde{\rho}^2}\right)$$

for $0 < t < \tilde{\rho}^2/3$. Combining the above inequalities, we have

$$\int_{M} K(x, y, t)\psi(x)dV_{g}(x)$$

$$\leq C\|\psi\|_{L^{\infty}(M_0)} \int_{B(z_0, R)} (\tilde{\rho}^2)^{-\frac{N}{2}} \exp\left(-\frac{d(x, y)^2}{2\tilde{\rho}^2}\right)dV_{g}(x)$$

for some $C > 0$ depending only on $N, \kappa, \text{inj}(M)$ and $\tilde{\rho}$. Then, by the lower bound of the heat kernel (A.10), we have

$$\int_{M} K(x, y, t)\psi(x)dV_{g}(x) \leq C\|\psi\|_{L^{\infty}(M_0)} \int_{B(z_0, R)} K(x, y, \tilde{\rho}^2)dV_{g}(x). \quad (4.7)$$

Thus, dividing the integral with respect to $\mu$ into $B(z_0, 2R)$ and $B(z_0, 2R)^c$, we see that

$$\int_{M} \int_{M} K(x, y, t)\psi(x)dV_{g}(x)d\mu(y)$$

$$= \int_{y \in B(z_0, 2R)} \int_{x \in B(z_0, R)} K(x, y, t)\psi(x)dV_{g}(x)d\mu(y)$$

$$+ \int_{y \notin B(z_0, 2R)} \int_{x \in B(z_0, R)} K(x, y, t)\psi(x)dV_{g}(x)d\mu(y).$$
The first term is easily estimated from above as
\[
\int_{y \in B(z_0,2R)} \int_{x \in B(z_0,2R)} K(x, y, t) \psi(x) dV_g(x) d\mu(y) \leq \|\psi\|_{L^\infty(M_0)} \mu(B(z_0,2R)) < \infty.
\]

On the second term, we have
\[
\int_{y \notin B(z_0,2R)} \int_{x \in B(z_0,2R)} K(x, y, t) \psi(x) dV_g(x) d\mu(y)
\leq C \|\psi\|_{L^\infty(M_0)} \int_{x \in B(z_0,2R)} \int_{y \in M} K(x, y, \tilde{\rho}^2) d\mu(y) dV_g(x)
\leq C \|\psi\|_{L^\infty(M_0)} \int_{x \in B(z_0,2R)} (\tilde{\rho}^2)^{-\frac{N}{2}} \sup_{z \in M} \mu(B(z, \tilde{\rho})) dV_g(x),
\]
where the first inequality follows from (4.7) and the second one follows from Lemma 4.2. Thus, combining the assumption (2.5) for \(\mu\), we have
\[
\int_{x \in B(z_0,2R)} (\tilde{\rho}^2)^{-\frac{N}{2}} \sup_{z \in M} \mu(B(z, \tilde{\rho})) dV_g(x) \leq C (\tilde{\rho}^2)^{-\frac{N}{2}} \text{Vol}(B(z_0, R)) < \infty.
\]

Thus, we see that \(\int_M K(x, \cdot, t) \psi(x) dV_g(x) \in L^1_\mu(M)\). Since the pointwise convergence of \(\int_M K(x, y, t) \psi(x) dV_g(x)\) to \(\psi(y)\) as \(t \to 0\) is established (see [14, Theorem 7.13] for instance), by Lebesgue’s dominated convergence theorem, we can say that
\[
\int_M \left| \int_M K(y, x, t) \psi(x) dV_g(x) - \psi(y) \right| d\mu(y) \to 0 \quad \text{as} \quad t \to 0.
\]

This immediately implies that
\[
\int_M \int_M K(x, y, t) \psi(x) dV_g(x) d\mu(y)
= \int_M \psi d\mu + \int_M \left( \int_M K(y, x, t) \psi(x) dV_g(x) - \psi(y) \right) d\mu(y)
\to \int_M \psi d\mu \quad \text{as} \quad t \to 0.
\]

Hence, by letting \(t \to 0\) in (4.6), we obtain (2.4). The proof is complete.

4.3. The case \(p \geq p_F\)

In this subsection, let \(N\), \(p\) and \(T\) be as in Theorem 2.3. Let \(\eta \in C^\infty(\mathbb{R})\) satisfy
\[0 \leq \eta \leq 1, \eta' \leq 0, \eta(z) = 0 \text{ for } z \geq 1, \eta(z) = 1 \text{ for } z \leq 1/2 \text{ and } \sup_{z \in \mathbb{R}} |\eta'| = 1.\]

Fix \(z_0 \in M\) and define \(f \in L^1_{\text{loc}}(M)\) by \(f(x) := \tilde{f}(r(x))\), where
\[r(x) := d(z_0, x)\]
and
\[\tilde{f}(r) := \begin{cases} r^{-\frac{2}{p-1}} \eta(\rho_T^{-1} r) & \text{if } p > p_F, \\ (\rho_T^{-1} r)^N (\log(e^2 + \rho_T^{-1} r))^{-\frac{N}{2} - 1} \eta(\rho_T^{-1} r) & \text{if } p = p_F. \end{cases}\]

We remark that \(\tilde{f}(r)\) is strictly decreasing for \(r \in (0, \rho_T)\). We set a nonnegative Radon measure \(\mu_f\) by \(\mu_f(A) := \int_A f dV_g\). We prepare estimates of \(\mu_f\).
Lemma 4.3. There exists a constant $C > 1$ depending only on $N$ and $p$ such that the following (i) and (ii) hold:

(i) If $p > p_F$, then $C^{-1} \rho^{-\frac{2}{p-1}} \leq \sup_{z \in M} \mu_f(B(z, \rho)) \leq C \rho^{-\frac{2}{p-1}}$ for any $0 < \rho < \rho_T$.

(ii) If $p = p_F$, then $C^{-1} \rho_T^{-\frac{N}{p}} \leq \sup_{z \in M} \mu_f(B(z, \rho)) \leq C \rho_T^{-\frac{N}{p}}$ for any $0 < \rho < \rho_T$.

**Proof.** We prepare some estimates which hold for both (i) and (ii). Let $0 < \rho < \rho_T$. We estimate $\sup_{z \in M} \mu_f(B(z, \rho))$ from above. Fix $z \in M$ and $0 < \rho < \rho_T$. By the coarea formula, we have

$$
\mu_f(B(z, \rho)) = \int_{B(z, \rho)} f(x) dV_g(x) = \int_0^\infty \text{Vol} \left( f^{-1}((\lambda, \infty]) \cap B(z, \rho) \right) d\lambda.
$$

It is clear that

$$
\int_0^\infty \text{Vol} \left( f^{-1}((\lambda, \infty]) \cap B(z, \rho) \right) d\lambda = \int_0^{f(\rho)} \text{Vol} \left( f^{-1}((\lambda, \infty]) \cap B(z, \rho) \right) d\lambda + \int_{f(\rho)}^\infty \text{Vol} \left( f^{-1}((\lambda, \infty]) \cap B(z, \rho) \right) d\lambda.
$$

The first term is 0 if $\rho \geq \rho_T$. In the case $\rho < \rho_T$, the first term is estimated as

$$
\int_0^{f(\rho)} \text{Vol} \left( f^{-1}((\lambda, \infty]) \cap B(z, \rho) \right) d\lambda \leq \int_0^{f(\rho)} \text{Vol} (B(z, \rho)) d\lambda \leq C(N) \rho^N \tilde{f}(\rho),
$$

where the second inequality follows from the volume comparison theorem (A.3) with $C(N) := 2^{N-1} N^{-1} \text{Area}(S^{N-1})$. To estimate the second term, we note that $f^{-1}((\lambda, \infty]) = B(z_0, \tilde{f}^{-1}(\lambda))$ for $\lambda > 0$. Then, similarly to the estimate of the first term, we see that

$$
\int_{f(\rho)}^\infty \text{Vol} \left( f^{-1}((\lambda, \infty]) \cap B(z, \rho) \right) d\lambda \leq C(N) \int_{f(\rho)}^\infty (\tilde{f}^{-1}(\lambda))^N d\lambda = C(N) \int_0^\infty r^N \tilde{f}'(r) dr,
$$

where the last equality follows from the change of variable with $\lambda := \tilde{f}(r)$. We have

$$
C(N) \int_0^\infty r^N \tilde{f}'(r) dr = C(N) \left( r^N \tilde{f}(r) \right)_{r=\rho}^{r \to 0} - C(N) N \int_0^\rho r^{N-1} \tilde{f}(r) dr
$$

$$
= -C(N) \rho^N \tilde{f}(\rho) + C(N) N \int_0^\rho r^{N-1} \tilde{f}(r) dr.
$$

Thus, for $p \geq p_F$, we have

$$
\mu_f(B(z, \rho)) \leq C(N) \int_0^\rho r^{N-1} \tilde{f}(r) dr.
$$

(4.8)
(i) In the case \( p > p_F \), we have
\[
C(N) N \int_0^\rho r^{N-1} \bar{f}(r) dr \leq C(N) N \int_0^\rho r^{N-1} - \frac{2}{p-1} dr = \frac{C(N) N}{N - (2/(p-1))} \rho^{N-\frac{2}{p-1}}.
\]
This together with (4.8) shows the desired estimate from above. For the estimate from below, in Riemannian normal coordinates \((x_1, \ldots, x_N)\) centered at \(z_0\), we see that
\[
sup_{z \in M} \mu_f(B(z, \rho)) \geq \int_{B(z_0, \rho/2)} d(z_0, x)^{-\frac{2}{p-1}} dV_g(x) = \int_{B(0, \rho/2)} |x|^{-\frac{2}{p-1}} \sqrt{\det(g_{ij})} dx.
\]
By the volume comparison theorem (A.4), we have \( \sup_{z \in M} \mu_f(B(z, \rho)) \geq C^{-1} \rho^{N-2/(p-1)} \). Hence, (i) follows.

(ii) In the case \( p = p_F \), from the volume comparison theorem (A.4), it follows that
\[
\sup_{z \in M} \mu_f(B(z, \rho)) \geq \rho_T^N \int_{B(z_0, \rho/2)} d(z_0, x)^{-N} (\log(e^{2} + \rho_T d(z_0, x)^{-1}) - \frac{N}{2} - 1) dV_g(x)
\]
\[
\geq C^{-1} \rho_T^N \int_{B(0, \rho/2)} N \left( \frac{\rho_T}{e^2|x| + \rho_T} \right) |x|^{-N} (\log(e^{2} + \rho_T |x|^{-1}) - \frac{N}{2} - 1) dx
\]
\[
= C^{-1} \text{Area}(S^{N-1}) \rho_T^N (\log(e^{2} + 2\rho_T \rho^{-1}))^{-\frac{N}{2}} \geq C^{-1} \rho_T^N (\log(e + \rho_T \rho^{-1}))^{-\frac{N}{2}},
\]
where \( C > 1 \) depends only on \( N \). On the other hand, by (4.8), we have
\[
\sup_{z \in M} \mu_f(B(z, \rho)) \leq C \int_0^\rho r^{N-1} \bar{f}(r) dr \leq C \int_0^\rho r^{N-1}(\rho_T r^{-1})^N (\log(e^{2} + \rho_T r^{-1}) - \frac{N}{2} - 1) dr
\]
\[
= \frac{2C}{N} \rho_T^N \int_0^\rho \left( \frac{e^2 r + \rho_T}{\rho_T} \right) \left( \frac{\rho_T r^{-1}}{e^2 r + \rho_T} \right) (\log(e^{2} + \rho_T r^{-1}) - \frac{N}{2} - 1) dr
\]
\[
\leq C \rho_T^N (\log(e^{2} + \rho_T \rho^{-1}))^{-\frac{N}{2}} \leq C \rho_T^N (\log(e + \rho_T \rho^{-1}))^{-\frac{N}{2}}.
\]
Hence, (ii) follows. \( \square \)

We prepare auxiliary functions for proving Theorem 2.3. For \( 1 < \alpha < p \) and \( 0 < \beta < N/2 \), we define a convex and strictly increasing function \( h \) by
\[
h(z) := \begin{cases} 
  z^\alpha & \text{if } p > p_F, \\
  z(\log(A + z))^\beta & \text{if } p = p_F,
\end{cases} \tag{4.9}
\]
for \( z \geq 0 \). In the case \( p = p_F \), we fix \( A > e^N \) such that \( z \mapsto z^p / h(z) = z^{p-1} (\log(A + z))^{-\beta} \) is strictly increasing. Then, \( z \mapsto z^p / h(z) \) and \( z \mapsto h(z)/z \) are strictly increasing in each case. Note that the inverse function \( h^{-1} \) of \( h \) exists and is also strictly increasing.

For \( c > 0 \), define
\[
\bar{u}(x, t) := 2c U(x, t), \quad U(x, t) := h^{-1} \left( \int_M K(x, y, t) h(f(y)) dV_g(y) \right). \tag{4.10}
\]
Lemma 4.4. There exists a constant $C > 0$ depending only on $N$ and $p$ such that

$$U(x, t) \leq \begin{cases} 
C t^{-\frac{1}{p-1}} & \text{if } p > p_F, \\
C \rho_T^N t^{-\frac{N}{2}} (\log(e + \rho_T t^{-\frac{1}{2}}))^{-\frac{N}{2}} & \text{if } p = p_F,
\end{cases}$$

for $(x, t) \in M \times (0, \rho_T^2)$.

Proof. Let $0 < t < \rho_T^2$. We first estimate $h(U)$. Since $h(U(x, t)) = \int_M K(x, y, t) h(f(y))dV_g(y)$, Lemma 4.2 gives

$$h(U(x, t)) \leq Ct^{-\frac{N}{2}} \sup_{z \in M} \int B(\frac{1}{2}) h(f(y))dV_g(y).$$

Since $h(f(y)) = h(\tilde{f}(r(y)))$ and $h(\tilde{f}(r))$ is strictly decreasing for $r \in (0, \rho_T)$, we can obtain (4.8) with replacing $\tilde{f}$ with $h(\tilde{f})$ and have

$$\int_{B(\frac{1}{2})} h(f(y))dV_g(y) \leq C \int_0^{\frac{1}{2}} r^{N-1}h(\tilde{f}(r))dr$$

$$\leq \begin{cases} 
C \int_0^{\frac{1}{2}} r^{N-\frac{2N}{p-1}-1}dr & \text{if } p > p_F, \\
C \rho_T^N \left(\frac{e^{1/2}}{\rho_T} + 1\right) \int_0^{\frac{1}{2}} \frac{\rho_T}{er + \rho_T} r^{-1}(\log(e + \rho_T r^{-1}))^{-\frac{N}{2}}-1+\beta dr & \text{if } p = p_F.
\end{cases}$$

Then by $t < \rho_T^2$, we see that

$$\int_{B(\frac{1}{2})} h(f(y))dV_g(y) \leq \begin{cases} 
C t^{\frac{N}{2}-\frac{\alpha}{p-1}} & \text{if } p > p_F, \\
C \rho_T^N (\log(e + \rho_T t^{-\frac{1}{2}}))^{-\frac{N}{2}+\beta} & \text{if } p = p_F,
\end{cases}$$

where $C > 0$ is a constant depending only on $N$ and $p$. Thus,

$$h(U(x, t)) \leq \begin{cases} 
C t^{-\frac{\alpha}{p-1}} & \text{if } p > p_F, \\
C(\rho_T t^{-\frac{1}{2}})^N (\log(e + \rho_T t^{-\frac{1}{2}}))^{-\frac{N}{2}+\beta} & \text{if } p = p_F.
\end{cases} \tag{4.11}$$

Fundamental computations show that

$$h^{-1}(z) \begin{cases} 
z^{1/\alpha} & \text{if } p > p_F, \\
\leq C z(\log(A + z))^{-\beta} & \text{if } p = p_F.
\end{cases} \tag{4.12}$$

From this, it follows that the desired inequality holds. 

We are now in a position to prove Theorem 2.3.

Proof of Theorem 2.3. Fix $c > 0$. Define $\mu(A) := \int_A cf dV_g$ and $\overline{u}$ by (4.10). Since $h$ is convex, Jensen’s inequality gives

$$h \left( \frac{\int_M K(x, y, t)f(y)dV_g(y)}{\int_M K(x, y, t)dV(y)} \right) \leq \frac{\int_M K(x, y, t)h(f(y))dV_g(y)}{\int_M K(x, y, t)dV(y)}.$$
This together with $\int_M K(x, y, t)dV_g(y) = 1$ and $d\mu = cf dV_g$ yields

$$\int_M K(x, y, t)d\mu(y) \leq ch^{-1} \left(\int_M K(x, y, t)h(f(y))dV_g(y)\right) = cU(x, t),$$

and so

$$\Psi[\pi] \leq cU + 2^p c^p \int_0^t \int_M K(x, y, t-s)U(y, s)^p dV_g(y)ds =: cU + 2^p c^p I. \quad (4.13)$$

Let us estimate $I$. Write $\| \cdot \|_\infty := \| \cdot \|_{L^\infty(M)}$. From $h(U(y, s)) = \int_M K(y, z, s) h(f(z))dV_g(z)$, Fubini’s theorem and the semigroup property of $K$, it follows that

$$I = \int_0^t \int_M K(x, y, t-s) \frac{U(y, s)^p}{h(U(y, s))} h(U(y, s))dV_g(y)ds$$

$$\leq \int_0^t \left\| \frac{U(\cdot, s)^p}{h(U(\cdot, s))}\right\|_\infty \int_M K(x, y, t-s) K(y, z, s)h(f(z))dV_g(z)dV_g(y)ds$$

$$= \int_0^t \left\| \frac{U(\cdot, s)^p}{h(U(\cdot, s))}\right\|_\infty ds \int_M K(x, z, t)h(f(z))dV_g(z)$$

$$= h(U(x, t)) \int_0^t \left\| \frac{U(\cdot, s)^p}{h(U(\cdot, s))}\right\|_\infty ds.$$

Therefore,

$$I \leq U(x, t) \left\| \frac{h(U(\cdot, t))}{U(\cdot, t)} \right\|_\infty \int_0^t \left\| \frac{U(\cdot, s)^p}{h(U(\cdot, s))}\right\|_\infty ds =: U(x, t)J(t). \quad (4.14)$$

We estimate $J$ for $0 < t < \rho_T^2$. Since $z \mapsto z^p/h(z)$ and $z \mapsto h(z)/z$ are increasing, Lemma 4.4 and direct computations yield

$$J(t) \leq \left\{
\begin{array}{ll}
\|U(\cdot, t)\|_{L^\infty}^{p-1} \int_0^t \|U(\cdot, s)\|_{L^\infty}^{-\alpha} ds & (p > p_F) \\
(\log(A + \|U(\cdot, t)\|_\infty))^{\beta} \int_0^t \|U(\cdot, s)\|_{L^\infty}^{-1}(\log(A + \|U(\cdot, s)\|_\infty))^{-\beta} ds & (p = p_F) \\
C t^{-\frac{p-1}{p-\alpha}} \int_0^t s^{-\frac{p-\alpha}{p-1}} ds & (p > p_F) \\
C \rho_T^2 (\log(e + \rho_T t^{-1/2}))^{\beta} \int_0^t s^{-1}(\log(e + \rho_T s^{-1/2}))^{1-\beta} ds & (p = p_F) \\
C & (p > p_F) \\
C \rho_T^2 & (p = p_F)
\end{array}\right. \quad (4.15)$$

for $0 < t < \rho_T^2$, where $C > 0$ is a constant depending only on $N$ and $p$.

The above estimates together with $\rho_T \leq T^{1/2}$ show that

$$\Psi[\pi] \leq \left\{
\begin{array}{ll}
cU + 2^p C c^p U & if p > p_F, \\
cU + 2^p C T c^p U & if p = p_F,
\end{array}\right.$$
in $M \times (0, \rho_T^2)$. Hence, we deduce that $\bar{u}$ is a supersolution of (4.1) in $M \times [0, \rho_T^2)$ provided that $2^p C c^{p-1} \leq 1$ if $p > p_F$ and $2^p C T c^{2/N} \leq 1$ if $p = p_F$. Lemma 4.1 gives an integral solution $u$ of (1.1) in $M \times [0, \rho_T^2)$. Moreover, the estimates of $\mu$ in the statement (i) and (ii) follow from Lemma 4.3. The rest of the proof to see that $u$ is a solution in $M \times [0, \rho_T^2)$ is the same as that of Theorem 2.2. 

4.4. Sufficient conditions

In the previous subsections, we proved the sharpness of the necessary conditions. Even though Theorem 2.5 does not say the sharpness directly, we prove the theorem in this subsection, since all estimates for the proof have been prepared. The proof of Theorem 2.5 is similar to the proof of Theorems 2.2.

Proof of Theorem 2.5. Let $\mu$ and $\tilde{\rho} \in (0, \rho_T]$ satisfy (2.6), where $c$ is chosen later. Set $\bar{u}$ and $U$ be as in (4.3). By (4.4), we have

$$
\Psi[\bar{u}](x, t) = U(x, t) + 2^p \int_0^t \int_M K(x, y, t - s) U(y, s) p dV_g(y) ds
$$

$$
\leq U(x, t) + 2^p \left( \int_0^t \|U(\cdot, s)\|_{\infty}^{p-1} ds \right) U(x, t).
$$

Lemma 4.2 and (2.6) with $\tilde{\rho} (\leq \rho_T \leq 4 \rho_\infty)$ yield

$$
U(x, t) \leq C c t^{-\frac{1}{p-1}} (\log(e + \tilde{\rho} t^{-\frac{1}{2}}))^{-\frac{1}{p-1} - \varepsilon} \quad \text{for} \ (x, t) \in M \times (0, \tilde{\rho}^2),
$$

and so

$$
\int_0^t \|U(\cdot, s)\|_{\infty}^{p-1} ds \leq C c^{p-1} \int_0^t s^{-1} (\log(e + \tilde{\rho} s^{-\frac{1}{2}}))^{-1 - (p-1)\varepsilon} ds
$$

$$
\leq C c^{p-1} (\log(e + \tilde{\rho} t^{-\frac{1}{2}}))^{-(p-1)\varepsilon} \leq C c^{p-1}
$$

for $(x, t) \in M \times (0, \tilde{\rho}^2)$, where $C > 0$ is a constant depending only on $N$ and $p$. If $c$ satisfies $2^p C c^{p-1} < 1$, then $\bar{u}$ is a supersolution of (4.1) in $M \times [0, \tilde{\rho}^2)$. This together with Lemma 4.1 shows the existence of an integral solution $u$ of (1.1) in $M \times [0, \tilde{\rho}^2)$. The rest of the proof to see that $u$ is a solution in $M \times [0, \tilde{\rho}^2)$ is the same as that of Theorem 2.2. □

5. Nonexistence of solutions

To prove nonexistence, we use a Cantor type (fractal type) set and show the unboundedness of some fractional maximal operator. In the Euclidean case, such sets were introduced in Kan and the first author [20] for one dimension and were generalized by the first author [42] for higher dimension. In this paper, we construct such Cantor type sets on Riemannian manifolds.
Throughout this section, we assume $p \geq p_F$ and write
\[
\phi(\rho) := \log \left( e + \frac{1}{\rho} \right)^{-\frac{1}{p-1}} \quad \text{for } \rho > 0. \tag{5.1}
\]

We frequently use the following monotonicity properties for $\rho > 0$:
- $\phi(\rho)$ is strictly increasing.
- $\rho^{N-2/(p-1)} \phi(\rho)$ is strictly increasing.
- There exists $0 < \alpha < 2N/(p-1)$ such that $\rho^{-\alpha} \phi(\rho)$ is strictly decreasing.
- If $p > p_F$, there exists $0 < \beta < N - 2/(p-1)$ such that $\rho^\beta \phi(\rho)$ is strictly increasing.

5.1. A Cantor type set

In this subsection, we repeat the construction of a Cantor type set in [20,42]. For $0 < \sigma < 2^{-N}$, define a continuous function
\[
F(R) := R^{N-\frac{2}{p-1}} \phi(R) - \phi(2^{-1})\sigma \quad \text{for } R > 0.
\]

By $\lim_{R \to 0} F(R) = -\phi(2^{-1})\sigma < 0$ and
\[
F(2^{-1}) = ((2^{-1})^{N-\frac{2}{p-1}} - \sigma)\phi(2^{-1}) > (2^{-N} + 2^{N-2/(p-1)} - 2^{-N})\phi(1/2) > 0
\]
together with the monotonicity of $\rho^{N-2/(p-1)} \phi(\rho)$, the equation $F(R) = 0$ has a unique solution $0 < R(\sigma) < 1/2$ for each $\sigma$. Set
\[
R_0 := 1, \quad R_n := R(2^{-Nn}) \quad \text{for } n = 1, 2, \ldots.
\]

We observe that if $\sigma$ becomes small, then $F$ becomes large and $R(\sigma)$ becomes small. Therefore, we see that
\[
\frac{1}{2} > R_1 > R_2 > \cdots > R_n \to 0 \quad \text{as } n \to \infty.
\]

In addition, since $F(R(\sigma)) = 0$, we have
\[
2^{Nn} R_n^{N-\frac{2}{p-1}} \phi(R_n) = \phi(2^{-1}) \quad \text{for } n = 1, 2, \ldots.
\]

We denote the ratio of $R_n$ to $R_{n-1}$ by
\[
r_n := \frac{R_n}{R_{n-1}} \quad \text{for } n = 1, 2, \ldots.
\]

Let us estimate $r_n$. For $n \geq 2$, we have
\[
\phi(2^{-1}) = 2^{Nn} R_n^{N-\frac{2}{p-1}} \phi(R_n) = 2^{N(n-1)} R_{n-1}^{N-\frac{2}{p-1}} \phi(R_{n-1}).
\]
The monotonicity $R_n < R_{n-1}$ gives
\[
2^{Nn} R_n^{N-\frac{2}{p-1}} \phi(R_n) = 2^{N(n-1)} R_{n-1}^{N-\frac{2}{p-1}+\alpha} R_{n-1}^{-\alpha} \phi(R_{n-1}) \\
\leq 2^{N(n-1)} R_{n-1}^{N-\frac{2}{p-1}+\alpha} R_{n-1}^{-\alpha} \phi(R_n)
\]
for $n \geq 2$, and so
\[
\frac{R_n}{R_{n-1}} \leq 2^{-N/(2-(p-1))} < \frac{1}{2} \quad \text{for } n \geq 2.
\]
In addition, $r_1 = R_1/R_0 = R_1 < 1/2$. Thus, there exists $0 < \bar{r} < 1/2$ such that
\[
\frac{R_n}{R_{n-1}} \leq \bar{r} \quad \text{for } n \geq 1.
\]
Note that this upper bound is valid for all $p \geq p_F$. On the other hand, we give a lower bound only for $p > p_F$. By $R_{n-1} > R_n$ and similar computations to the above, we have
\[
2^{Nn} R_n^{N-\frac{2}{p-1}} \phi(R_n) = 2^{N(n-1)} R_{n-1}^{N-\frac{2}{p-1}+\alpha} R_{n-1}^{-\alpha} \phi(R_{n-1}) \\
\geq 2^{N(n-1)} R_{n-1}^{N-\frac{2}{p-1}+\alpha} R_{n-1}^{-\alpha} \phi(R_n)
\]
for $n \geq 2$, and so
\[
\frac{R_n}{R_{n-1}} \geq 2^{-N/(2-(p-1))} \quad \text{for } n \geq 2.
\]
Then there exists $0 < \underline{r} < 1/2$ such that
\[
\frac{R_n}{R_{n-1}} \geq \underline{r} \quad \text{for } n \geq 1 \text{ and } p > p_F.
\]
We define a Cantor type set $\{I_n\}_{n=0}^{\infty} \subset \mathbb{R}$ by
\[
I_n := \bigcup_{l=1}^{2^n} (a_{n,l}, b_{n,l}) \quad \text{for } n \geq 0,
\]
where
\[
\begin{align*}
a_{0,1} := 0, & \quad b_{0,1} := 1, \\
an_{n+1,2l-1} := a_{n,l}, & \quad b_{n+1,2l-1} := a_{n,l} + R_{n+1}, \\
an_{n+1,2l} := b_{n,l} - R_{n+1}, & \quad b_{n+1,2l} := b_{n,l}.
\end{align*}
\]
We note that $b_{n,l} - a_{n,l} = R_n$ and that
\[
I_0 = (0, 1), \\
I_1 = (0, R_1) \cup (1 - R_1, 1), \\
I_2 = (0, R_2) \cup (R_1 - R_2, R_1) \cup (1 - R_1 + R_2, 1 - R_1) \cup (1 - R_2, 1),
\]
and so on. By $r_n = R_n/R_{n-1} \leq \bar{r} < 1/2$, we also note that
\[
0 = a_{n,1} < b_{n,1} < a_{n,2} < b_{n,2} < \cdots < a_{n,2^n} < b_{n,2^n} = 1, \\
I_n \supset I_{n+1} \supset I_{n+2} \supset \cdots,
\]
for $n \geq 1$. Thus, each $I_n$ is a disjoint union of $2^n$ small subintervals of $(0, 1)$. 
5.2. Unboundedness of a fractional maximal operator

In this subsection, we define some fractional maximal operator and show its unboundedness. To define the operator, we set up notation. We define a Morrey type space by

\[ Y := \{ f : f \in L^1_{\text{loc}}(M) \text{ with } \| f \|_Y < \infty \}, \]

\[ \| f \|_Y := \sup_{x \in M} \sup_{0 < \rho < \rho_\infty} \left( \phi(\rho)^{-1} \rho^{-(N-\frac{2}{p}-1)} \int_{B(x, \rho)} |f(y)| dV_g(y) \right). \] (5.3)

For \( \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \) and \( \rho > 0 \), we write

\[ Q(\xi, \rho) := \{ \eta = (\eta_1, \ldots, \eta_N) \in \mathbb{R}^N ; \xi_i - \rho < \eta_i < \xi_i + \rho \text{ for } 1 \leq i \leq N \}, \]

\[ Q_{\rho} := (0, \rho) \times \cdots \times (0, \rho). \]

Set

\[ P = (\rho_1, \ldots, \rho_N) \in Q_a, \quad a := \frac{1}{\sqrt{N}} \min\{\rho_\infty, 1\}, \]

\[ D_P(\xi) := \prod_{i=1}^N (\xi_i + (1 - 2\bar{r})\rho_i, \xi_i + \rho_i). \]

Let \( z_0 \in M \). Fix an orthonormal basis \((e_1, \ldots, e_N)\) of \( T_{z_0}M \). By using this basis, we identify \( T_{z_0}M \) and \( \mathbb{R}^N \). Recall that \( \rho_\infty \leq \text{inj}(M) \). Then the restriction of

\[ \mathbb{R}^N \ni (x_1, \ldots, x_N) \mapsto \exp_{z_0}(x_1 e_1 + \cdots + x_N e_N) \in M \]

to \( B(O, \rho_\infty) \subset \mathbb{R}^N \) is an injection, where \( O \) is the origin of \( \mathbb{R}^N \). We denote the inverse of this map by

\[ \varphi : B(z_0, \rho_\infty) \ni x \mapsto (x_1, \ldots, x_N) \in \mathbb{R}^N. \]

We also write \( \varphi(x) = \xi \). Let \( |\cdot|_{\mathbb{R}^N} \) be the Euclidean norm on \( \mathbb{R}^N \). We write \( |\cdot| = |\cdot|_{\mathbb{R}^N} \) when no confusion can arise.

For \( f \in Y \) and \( x \in \varphi^{-1}(Q_a) \), define a fractional maximal operator by

\[ \mathcal{H}[f](x) := \sup_{P=(\rho_1, \ldots, \rho_N) \in Q_a} |P|^{-N(1-\frac{2}{p})} \int_{D_P(\varphi(x))} f(\varphi^{-1}(\eta)) d\eta. \]

Our temporal goal is to show the unboundedness of \( \mathcal{H} \) from \( Y \) to \( L^p(\varphi^{-1}(Q_a)) \). More explicitly, we define functions \( f_n \) by

\[ f_n(x) := \begin{cases} \prod_{i=1}^N \chi_{I_n}(a^{-1} \varphi_i(x)) & \text{if } x \in \varphi^{-1}(Q_a), \\ 0 & \text{if } x \notin \varphi^{-1}(Q_a), \end{cases} \] (5.4)
and we prove that
\[ \frac{\|\mathcal{H}[f_n]\|_{L^p(\mathbb{R}^n)}}{\|f_n\|_Y} \to \infty \quad \text{as } n \to \infty. \]  
(5.5)

Here \( I_n \subset (0, 1) \) is given by the previous subsection, \( \chi_{I_n} \) is the characteristic function on \( I_n \) and \( \varphi_i(x) \) is the \( i \)-th component of \( \varphi(x) \).

First, we prove that there exists \( C > 0 \) independent of \( n \) such that
\[ \|f_n\|_Y \leq C(2^n R_n)^N. \]  
(5.6)

Since \( \text{supp } f_n \subset \{ x \in M; \varphi(x) \in Q_{\alpha} \} \), we have
\[ \sup_{x \in \text{supp } f_n} d(z_0, x) \leq d(z_0, \varphi^{-1}(a, \ldots, a)) = |(a, \ldots, a)| = \sqrt{Na} \leq \rho_\infty. \]  
(5.7)

Thus, for each \( x \in M \), we have
\[
\begin{align*}
\sup_{0 < \rho < \rho_\infty} \left( \int_{B(x, \rho)} |f_n(y)| dV_g(y) \right)^{\frac{1}{N}}
&= \sup_{0 < \rho < \rho_\infty} \left( \int_{B(x, \rho) \cap B(z_0, \rho_\infty)} |f_n(y)| dV_g(y) \right)^{\frac{1}{N}} \\
&= \sup_{0 < \rho < \rho_\infty} \left( \int_{\varphi(B(x, \rho) \cap B(z_0, \rho_\infty))} |f_n(\varphi^{-1}(\eta))| ((\varphi^{-1})^* dV_g)(\eta) \right)^{\frac{1}{N}}.
\end{align*}
\]

Remark that
\[ ((\varphi^{-1})^* dV_g)(\eta) = \sqrt{\det(g_{ij}(\eta))} d\eta. \]  
(5.8)

Thus, by the volume comparison theorem (A.2), we have
\[
\begin{align*}
&\int_{\varphi(B(x, \rho) \cap B(z_0, \rho_\infty))} |f_n(\varphi^{-1}(\eta))| ((\varphi^{-1})^* dV_g)(\eta) \\
&\leq 2^{N-1} \int_{\varphi(B(x, \rho) \cap B(z_0, \rho_\infty))} |f_n(\varphi^{-1}(\eta))| d\eta \\
&= 2^{N-1} \int_{\varphi(B(x, \rho) \cap B(z_0, \rho_\infty))} \prod_{i=1}^{N} \chi_{I_n}(a^{-1} \eta_i) d\eta.
\end{align*}
\]

We estimate the integral in the right-hand side from above. It suffices to consider the case where \( B(x, \rho) \cap B(z_0, \rho_\infty) \neq \emptyset \). Take \( \bar{x} \in B(x, \rho) \cap B(z_0, \rho_\infty) \) and set \( \bar{\xi} := \varphi(\bar{x}) \in B(O, \rho_\infty). \) Then, for any \( x' \in B(x, \rho) \cap B(z_0, \rho_\infty) \), by the distance comparison theorem (A.7) with \( \rho \leq \rho_\infty \), we have \( |\varphi_i(x') - \varphi_i(\bar{x})| \leq 2d(x', \bar{x}) \leq 4\rho \). This together with \( \xi = \varphi_i(x) \) implies \( \varphi(B(x, \rho) \cap B(z_0, \rho_\infty)) \subset Q(\bar{\xi}, 4\rho) \).

Thus, we have
\[
\begin{align*}
\int_{\varphi(B(x, \rho) \cap B(z_0, \rho_\infty))} \prod_{i=1}^{N} \chi_{I_n}(a^{-1} \eta_i) d\eta &\leq \int_{Q(\bar{\xi}, 4\rho)} \prod_{i=1}^{N} \chi_{I_n}(a^{-1} \eta_i) d\eta \\
&= a^N \int_{Q(a^{-1} \bar{\xi}, a^{-1} 4\rho)} \prod_{i=1}^{N} \chi_{I_n}(\zeta_i) d\zeta.
\end{align*}
\]
Thus,

\[
\|f_n\|_Y \leq C \sup_{\xi \in B(O, r_\infty)} \sup_{0 < \rho < r_\infty} \left( \phi(\rho)^{-1} \rho^{-(N-\frac{2}{p-1})} \int_{Q(a^{-1}\xi, a^{-1}\rho)} \prod_{i=1}^{N} \chi_{I_n}(\xi_i)d\xi \right)
\]

\[
\leq C \sup_{\xi \in \mathbb{R}^N} \sup_{\rho > 0} \left( \phi(\rho)^{-1} \rho^{-(N-\frac{2}{p-1})} \int_{Q(\xi, a^{-1}\rho)} \prod_{i=1}^{N} \chi_{I_n}(\xi_i)d\xi \right)
\]

\[
= C \sup_{\xi \in \mathbb{R}^N} \sup_{\rho > 0} \left( \phi(4^{-1}a\rho)^{-1} (4^{-1}a\rho)^{-(N-\frac{2}{p-1})} \right)
\]

Recall that there exists \(0 < \alpha < 2/(p-1)\) such that \(\rho^{-\alpha} \phi(\rho)\) is strictly decreasing. This together with \(4^{-1}a\rho < 2\rho\) yields

\[
\phi(4^{-1}a\rho)^{-1} (4^{-1}a\rho)^{-(N-\frac{2}{p-1})}
\]

\[
= \left( (4^{-1}a\rho)^{-\alpha} \phi(4^{-1}a\rho) \right)^{-1} (4^{-1}a\rho)^{-\alpha} (4^{-1}a\rho)^{-(N-\frac{2}{p-1})}
\]

\[
\leq (2\rho)^{-\alpha} (2\rho)^{-1} (4^{-1}a\rho)^{-\alpha} (4^{-1}a\rho)^{-(N-\frac{2}{p-1})}
\]

\[
= C \phi(2\rho)^{-1} \rho^{-(N-\frac{2}{p-1})}.
\]

From Fubini’s theorem, it follows that

\[
\|f_n\|_Y \leq C \sup_{\xi \in \mathbb{R}^N} \sup_{\rho > 0} \left( \phi(2\rho)^{-1} \rho^{-(N-\frac{2}{p-1})} \int_{Q(\xi, \rho)} \prod_{i=1}^{N} \chi_{I_n}(\xi_i)d\xi \right)
\]

\[
= C \sup_{\xi \in \mathbb{R}^N} \sup_{\rho > 0} \left( \phi(2\rho)^{-1} \rho^{-(N-\frac{2}{p-1})} \prod_{i=1}^{N} \int_{\xi_i - \rho}^{\xi_i + \rho} \chi_{I_n}(\eta)d\eta \right)
\]

\[
= C \sup_{\xi \in \mathbb{R}^N} \sup_{\rho > 0} \prod_{i=1}^{N} \left( \phi(2\rho)^{-1} \rho^{-(N-\frac{2}{p-1})} \int_{\xi_i - \rho}^{\xi_i + \rho} \chi_{I_n}(\eta)d\eta \right)
\]

Then by exactly the same computations as in [42, Lemma 2] (see page 263, line –6), we obtain (5.6).

Next, we show that there exists \(C > 0\) independent of \(n\) such that

\[
\|\mathcal{H}[f_n]\|_{L^p(\varphi^{-1}(Q_0))}^p \geq \begin{cases} \frac{1}{C} \left(2^n R_n\right)^{Np} \sum_{j=0}^{n-1} \phi(R_j)^{p-1} \quad & \text{if } p > p_F, \\ \frac{1}{C} \left(2^n R_n\right)^{Np} \left\{ \frac{1}{C} \sum_{j=0}^{n-1} \left( \phi(R_j)^{p-1} \log \frac{R_j}{R_{j+1}} \right) - C \right\} \quad & \text{if } p = p_F. \end{cases}
\]
Let $x \in \varphi^{-1}(Q_a)$. We write $\varphi(x) = \xi = (\xi_1, \ldots, \xi_N)$. Then,

$$\mathcal{H}[f_n](x) = \sup_{P=(\rho_1, \ldots, \rho_N) \in Q_a} |P|^{-N(1-\frac{2}{Np})} \int_{D_P(\xi)} f_n(\varphi^{-1}(\eta))d\eta.$$ 

Fubini’s theorem and the change of variables yield

$$\int_{D_P(\xi)} f_n(\varphi^{-1}(\eta))d\eta = \int_{D_P(\xi)} \prod_{i=1}^{N} \chi_{I_n}(a^{-1}\eta)d\eta$$

$$= \prod_{i=1}^{N} \int_{\xi_i + (1-2^j)\rho_i}^{\xi_i + \rho_i} \chi_{I_n}(a^{-1}\eta_i)d\eta_i = a^{-N} \prod_{i=1}^{N} \int_{a^{-1}(\xi_i + (1-2^j)\rho_i)}^{a^{-1}(\xi_i + \rho_i)} \chi_{I_n}(\xi_i)d\xi_i.$$

By writing $\tilde{P} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_N) := \langle a^{-1}\rho_1, \ldots, a^{-1}\rho_N \rangle$ and $\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_N) := \langle a^{-1}\xi_1, \ldots, a^{-1}\xi_N \rangle$, we see that

$$\mathcal{H}[f_n](x) = a^{-N} \sup_{P=(\rho_1, \ldots, \rho_N) \in Q_a} |P|^{-N(1-\frac{2}{Np})} \prod_{i=1}^{N} \int_{a^{-1}(\xi_i + (1-2^j)\rho_i)}^{a^{-1}(\xi_i + \rho_i)} \chi_{I_n}(\xi_i)d\xi_i$$

$$= a^{-N-N(1-\frac{2}{Np})} \sup_{\tilde{P}=(\tilde{\rho}_1, \ldots, \tilde{\rho}_N) \in Q_1} |\tilde{P}|^{-N(1-\frac{2}{Np})} \prod_{i=1}^{N} \int_{\tilde{\xi}_i + (1-2^j)\tilde{\rho}_i}^{\tilde{\xi}_i + \tilde{\rho}_i} \chi_{I_n}(\xi_i)d\xi_i.$$ 

Remark that $\tilde{\xi} \in Q_1$, since $x \in \varphi^{-1}(Q_a)$. The function defined by the supremum in the right-hand side has exactly the same form as the fractional maximal function defined in [42, (13)]. Thus, by exactly the same argument as in the first part of the proof of [42, Lemma 3] (see page 265, lines 3–15), we obtain

$$\mathcal{H}[f_n](\varphi^{-1}(a\tilde{\xi})) = \mathcal{H}[f_n](x) \geq a^{-N-N(1-\frac{2}{Np})}(2^{n-j-1}R_n)^N \left( \sum_{i=1}^{N} (b_{j,i} - \tilde{\xi}_i)^2 \right)^{-\frac{N}{2}(1-\frac{2}{Np})} \quad (5.10)$$

for $x \in \varphi^{-1}(Q_a)$ with $\tilde{\xi} \in J_{j,l_1,\ldots,l_N}$ and $a^{-1}\tilde{\xi} = a^{-1}\varphi(x)$, where $J_{j,l_1,\ldots,l_N}$ is defined by

$$J_{j,l_1,\ldots,l_N} := J_{j,l_1} \times \cdots \times J_{j,l_N}, \quad J_{j,l_i} := (a_{j,i} + R_{j+1}, b_{j,i} - (2^j)^{-1}R_{j+1}),$$

for $n \geq 1$, $0 \leq j \leq n-1$, $1 \leq i \leq N$ and $1 \leq l_i \leq 2^j$. Here $a_{j,i}$ and $b_{j,i}$ are given by (5.2). Note that $J_{j,l_1,\ldots,l_N}$ satisfies

$$\left( \bigcup_{1 \leq l_1, \ldots, l_N \leq 2^j} J_{j,l_1,\ldots,l_N} \right) \cap \left( \bigcup_{1 \leq l'_1, \ldots, l'_N \leq 2^j} J_{j',l'_1,\ldots,l'_N} \right) = \emptyset \quad \text{for} \ \ j \neq j',$$

$$\bigcup_{j=0}^{n-1} \bigcup_{1 \leq l_1, \ldots, l_N \leq 2^j} J_{j,l_1,\ldots,l_N} \subset Q_1. \quad (5.11)$$
The change of variables gives
\[ \| \mathcal{H}(f_n) \|^p_{L^p(\varphi^{-1}(Q_a))} = \int_{\varphi^{-1}(Q_a)} |\mathcal{H}(f_n)(x)|^p dV_g(x) \]
\[ = \int_{Q_a} \mathcal{H}(f_n)(\varphi^{-1}(\xi))^p ((\varphi^{-1})^*dV_g)(\xi). \]

By (5.8), we have \((\varphi^{-1})^*dV_g)(\xi) = \sqrt{\det(g_{ij}(\xi))}d\xi\). Since \(Q_a \subset B(O, \sqrt{Na}) \subset B(O, \rho_\infty)\), we also have \(\sqrt{\det(g_{ij}(\xi))} \geq 2^{1-N}\) by the volume comparison theorem (A.4). These together with (5.11) imply that
\[ \| \mathcal{H}(f_n) \|^p_{L^p(\varphi^{-1}(Q_a))} \geq 2^{1-N} \int_{Q_a} \mathcal{H}(f_n)(\varphi^{-1}(\xi))^p d\xi \]
\[ = 2^{1-N} a^N \int_{Q_a} \mathcal{H}(f_n)(\varphi^{-1}(a\tilde{\xi}))^p d\tilde{\xi} \]
\[ \geq 2^{1-N} a^N \sum_{j=0}^{n-1} \sum_{1 \leq l_1, \ldots, l_N \leq 2j} \int_{J_{j,l_1,\ldots,l_N}} \mathcal{H}(f_n)(\varphi^{-1}(a\tilde{\xi}))^p d\tilde{\xi}. \]

From (5.10), it follows that
\[ \| \mathcal{H}(f_n) \|^p_{L^p(\varphi^{-1}(Q_a))} \geq \frac{1}{C} \sum_{j=0}^{n-1} (2^{n-j-1}R_n)^{Np} \sum_{1 \leq l_1, \ldots, l_N \leq 2j} \int_{J_{j,l_1,\ldots,l_N}} \left( \sum_{i=1}^N (b_{j,l_i} - \tilde{\xi}_i)^2 \right)^{-\frac{Np}{2}(1-\frac{2}{pN})} d\tilde{\xi}, \]

where \(C > 0\) is a constant depending only on \(N, p\) and \(a\). The same expression in the right-hand side appears in the middle part of the proof of [42, Lemma 3] (see page 265, line –3). Then by exactly the same argument as in [42, Lemmas 3, 4 and 5], we obtain (5.9).

Finally, combining (5.6) and (5.9) gives
\[ \frac{\| \mathcal{H}(f_n) \|^p_{L^p(\varphi^{-1}(Q_a))}}{\| f_n \|^p_Y} \geq \begin{cases} \frac{1}{C} \sum_{j=0}^{n-1} \phi(R_j)^{p-1} & \text{if } p > p_F, \\ \frac{1}{C} \sum_{j=0}^{n-1} \left( \phi(R_j)^{p-1} \log \frac{R_j}{R_{j+1}} \right) - C & \text{if } p = p_F, \end{cases} \]

where \(\phi\) is given by (5.1). This together with \(\int_0^1 \eta^{-1} \phi(\eta)^{p-1}d\eta = \infty\) and [42, Lemma 6] shows (5.5), the unboundedness of \(\mathcal{H}\) from \(Y\) to \(L^p(\varphi^{-1}(Q_a))\).
5.3. Existence of a specific initial data

We prove the existence of a specific initial data in $Y$ based on the unboundedness of $H$ from $Y$ to $L^p(\varphi^{-1}(Q_\rho))$ and the closed graph theorem. The idea of using the theorem is due to Brezis and Cazenave [6, Theorem 11].

For a function $f$ on $M$, define

$$L[f](x, t) := \int_M K(x, y, t) f(y) dV_g(y).$$

We write $Q_\rho' := \varphi^{-1}(Q_\rho) \subset B(z_0, \text{inj}(M))$ for $0 < \rho < \text{inj}(M)/\sqrt{N}$.

**Lemma 5.1.** There exists a nonnegative function $\tilde{f} \in Y$ such that $\tilde{f} = 0$ a.e. in $M \setminus Q_a'$ and $\|L[\tilde{f}]\|_{L^p(Q_a' \times I_0)} = \infty$, where $I_0 := (0, 1)$.

**Proof.** Define a closed subspace $\tilde{Y}$ of $Y$ by

$$\tilde{Y} := \{ f \in Y; f = 0 \text{ a.e. in } M \setminus Q_a' \}.$$ 

To obtain a contradiction, suppose that $\|L[f]\|_{L^p(Q_a' \times I_0)} < \infty$ for any $f \in \tilde{Y}$ with $f \geq 0$.

Under this assumption, by dividing the positive part and the negative part for sign-changing functions, we also have $\|L[f]\|_{L^p(Q_a' \times I_0)} < \infty$ for any $f \in \tilde{Y}$. Then, the following linear operator

$$L : \tilde{Y} \ni f \mapsto L[f] \in L^p(Q_a' \times I_0)$$

is well-defined. Since $\tilde{Y}$ is a closed subspace of $Y$, $\tilde{Y}$ is a Banach space with $\| \cdot \|_Y$. Therefore, the boundedness of $L : \tilde{Y} \to L^p(Q_a' \times I_0)$ will follow from the closed graph theorem once we prove closedness.

Let $\{f_n\} \subset \tilde{Y}$. We assume that there exists $f_\infty \in \tilde{Y}$ satisfying $f_n \to f_\infty$ in $Y$. We also assume the existence of $\tilde{f}_\infty \in L^p(Q_a' \times I_0)$ such that $L[f_n] \to \tilde{f}_\infty$ in $L^p(Q_a' \times I_0)$. To see the closedness of $L : \tilde{Y} \to L^p(Q_a' \times I_0)$, we check $L[f_\infty] = \tilde{f}_\infty$. From Fubini’s theorem and $f_n, f_\infty \in \tilde{Y}$, it follows that

$$\|L[f_n] - L[f_\infty]\|_{L^1(Q_a' \times I_0)}$$

$$\leq \int_{I_0} \int_{Q_a'} \int_M K(x, y, t) |f_n(y) - f_\infty(y)| dV_g(y) dV_g(x) dt$$

$$\leq \int_{I_0} \int_M \int_{Q_a'} K(x, y, t) |f_n(y) - f_\infty(y)| dV_g(y) dV_g(x) dt$$

$$= \int_M |f_n(y) - f_\infty(y)| dV_g(y) = \int_{Q_a'} |f_n(y) - f_\infty(y)| dV_g(y).$$
Since \( \overline{Q}_a \) is compact, there exist \( k \) and \( x_1', \ldots, x_k' \in M \) such that \( \overline{Q}_a \subset \bigcup_{i=1}^k B(x_i', 1/4) \), where \( k \) is independent of \( n \). Thus,

\[
\int_{\overline{Q}_a} |f_n(y) - f_\infty(y)|dV_g(y) \leq \sum_{i=1}^k \int_{B(x_i', 1/4)} |f_n(y) - f_\infty(y)|dV_g(y)
\]

\[
\leq \sum_{i=1}^k \phi(1/4)(1/4)^N \|f_n - f_\infty\|_Y = C\|f_n - f_\infty\|_Y,
\]

and so \( \|\mathcal{L}[f_n] - \mathcal{L}[f_\infty]\|_{L^1(Q_a' \times I_0)} \leq C\|f_n - f_\infty\|_Y \to 0 \) as \( n \to \infty \). On the other hand, by \( \mathcal{L}[f_n] \to \tilde{f}_\infty \) in \( L^p(Q_a' \times I_0) \), we have \( \|\mathcal{L}[f_n] - \tilde{f}_\infty\|_{L^1(Q_a' \times I_0)} \to 0 \) as \( n \to \infty \). Hence, \( \mathcal{L}[f_\infty] = \tilde{f}_\infty \), and so \( \mathcal{L} : \tilde{Y} \to L^p(Q_a' \times I_0) \) is closed. By the closed graph theorem, \( \mathcal{L} \) is also bounded. In particular, there exists a constant \( C \) such that

\[
\|\mathcal{L}[f]\|_{L^p(Q_a' \times I_0)} \leq C\|f\|_Y \quad \text{for any } f \in \tilde{Y}.
\]

(5.12)

Fix a nonnegative function \( f \in \tilde{Y} \). We claim that

\[
\|\mathcal{L}[f](x, \cdot)\|_{L^p(I_0)} \geq C\mathcal{H}[f](x) \quad \text{for any } x \in Q_a'.
\]

(5.13)

Fix a point \( P = (\rho_1, \ldots, \rho_N) \in Q_a \). By \( a \leq 1/\sqrt{N} \), we have \( |P| < \sqrt{Na} \leq 1 \), and so \( (|P|^2/2, |P|^2) \subset I_0 \). Let \( x \in Q_a' \) and \( t \in (|P|^2/2, |P|^2) \). Then,

\[
\mathcal{L}[f](x, t) = \int_M K(x, y, t)f(y)dV_g(y) = \int_{Q_a'} K(x, y, t)f(y)dV_g(y).
\]

The lower bound of the heat kernel (A.10) gives

\[
K(x, y, t) \geq Ct^{-\frac{N}{2}}e^{-\frac{d(x,y)^2}{4t}} \quad \text{for } x, y \in Q_a' \text{ and } t \in (0, 1),
\]

where \( C > 0 \) is a constant depending only on \( N \) and \( \kappa \). This together with \( Q_a' = \varphi^{-1}(Q_a) \), \( \xi = \varphi(x) \), (5.8) and the volume comparison theorem (A.4) yields

\[
\mathcal{L}[f](x, t) \geq C \int_{Q_a'} t^{-\frac{N}{2}}e^{-\frac{d(x,y)^2}{4t}}f(y)dV_g(y)
\]

\[
= C \int_{Q_a'} t^{-\frac{N}{2}}e^{-\frac{d(\varphi^{-1}(\xi), \varphi^{-1}(\eta))^2}{4t}}f(\varphi^{-1}(\eta))(\varphi^{-1})^*dV_g(\eta)
\]

\[
\geq C \int_{Q_a'} t^{-\frac{N}{2}}e^{-\frac{d(\varphi^{-1}(\xi), \varphi^{-1}(\eta))^2}{4t}}f(\varphi^{-1}(\eta))d\eta.
\]

By (A.6), we see that \( d(\varphi^{-1}(\xi), \varphi^{-1}(\eta)) \leq 2|\xi - \eta| \) for any \( \xi, \eta \in Q_a \). Recall that \( P = (\rho_1, \ldots, \rho_N) \in Q_a \) and \( \xi = \varphi(x) \in Q_a \) for \( x \in Q_a' \). Then \( \xi_i + \rho_i < 2a \) for \( 1 \leq i \leq N \), and so

\[
Q_{2a} \supset D_P(\xi) \quad \text{for any } P \in Q_a \text{ and } \xi \in Q_a.
\]
Therefore, since $f = 0$ a.e. in $M \setminus \overline{Q_a'}$, we see that

\[
\mathcal{L}[f](x, t) \
\geq C |P|^{-N} \int_{Q_a'} e^{-\frac{2|\xi - \eta|^2}{t}} f(\varphi^{-1}(\eta)) d\eta \
= C |P|^{-N} \int_{Q_{2a}} e^{-\frac{2|\xi - \eta|^2}{t}} f(\varphi^{-1}(\eta)) d\eta \geq C |P|^{-N} \int_{D_p(\xi)} e^{-\frac{2|\xi - \eta|^2}{t}} f(\varphi^{-1}(\eta)) d\eta
\]

for $x \in Q_a'$ and $t \in (|P|^2/2, |P|^2) \subset I_0$. By

\[
\frac{2|\xi - \eta|^2}{t} = \frac{2}{t} \left( (\xi_1 - \eta_1)^2 + \cdots + (\xi_N - \eta_N)^2 \right) \leq \frac{4}{|P|^2} \left( \rho_1^2 + \cdots + \rho_N^2 \right) = 4
\]

for $\eta \in D_p(\xi)$ and $t \in (|P|^2/2, |P|^2)$. Integrating over $t \in I_0$ and using $(|P|^2/2, |P|^2) \subset I_0$ yield

\[
\| \mathcal{L}[f](x, \cdot) \|_{L^p(I_0)} \geq \left( \int_{|P|^2/2} |P|^{-N} \int_{D_p(\xi)} f(\varphi^{-1}(\eta)) d\eta \right)^{\frac{1}{p}} \left( \int_{|P|^2/2} dt \right)^{\frac{1}{p}} \geq C |P|^{-N} \int_{D_p(\xi)} f(\varphi^{-1}(\eta)) d\eta \left( \int_{|P|^2/2} dt \right)^{\frac{1}{p}} = 2^{-\frac{1}{p}} C |P|^{-N(1 - \frac{2}{np})} \int_{D_p(x)} f(\varphi^{-1}(\eta)) d\eta
\]

for $x \in Q_a'$. Then by $P = (\rho_1, \ldots, \rho_N) \in Q_a, x \in Q_a'$ and the definition of $\mathcal{H}$, we obtain (5.13).

Taking the $L^p(Q_a')$-norm in (5.13) gives

\[
\| \mathcal{H}[f] \|_{L^p(Q_a')} \leq C \left( \int_{Q_a'} \| \mathcal{L}[f](x, \cdot) \|_{L^p(I_0)}^p dV_g(x) \right)^{\frac{1}{p}} = C \| \mathcal{L}[f] \|_{L^p(Q_a' \times I_0)}.
\]

This together with (5.12) shows that $\| \mathcal{H}[f] \|_{L^p(Q_a')} \leq C \| f \|_{Y}$. By $Q_a' = \varphi^{-1}(Q_a)$, we obtain

\[
\| \mathcal{H}[f] \|_{L^p(\varphi^{-1}(Q_a))} \leq C \| f \|_{Y} \quad \text{for any } f \in \tilde{Y} \text{ with } f \geq 0.
\]

Define $f_n$ by (5.4). By definition, we have $f_n \geq 0$ and supp $f_n \subset Q_a'$. The upper estimate (5.6) gives $f_n \in Y$. Thus, $f_n \in \tilde{Y}$ with $f_n \geq 0$. However, $f_n$ satisfies (5.5), a contradiction. \qed
By Lemma 5.1 and a scaling argument, we prove the following:

**Lemma 5.2.** Let $0 < T < a^2$. Then there exists a nonnegative function $f_T \in Y$ such that $f_T = 0$ a.e. in $M \setminus Q'_{\sqrt{T}}$ and $\|\mathcal{L}[f_T]\|_{L^p(Q'_{\sqrt{T}} \times (0,bT))} = \infty$, where $b := 9/a^2$.

**Proof.** We write $\lambda := \sqrt{T}/a$ in this proof. By using $\tilde{f}$ in Lemma 5.1, we set

$$f_T(x) := \begin{cases} \tilde{f} \left( \varphi^{-1} \left( \lambda^{-1} \varphi(x) \right) \right) & \text{for } x \in Q'_{\sqrt{T}}, \\ 0 & \text{for } x \in M \setminus Q'_{\sqrt{T}}. \end{cases}$$

Remark that $\varphi^{-1}(\lambda^{-1} \varphi(x)) \in Q'_a$ for $x \in Q'_{\sqrt{T}}$. Then the properties of $\tilde{f}$ imply that $f_T \geq 0$, $f_T \in Y$ and $f_T = 0$ a.e. in $M \setminus Q'_{\sqrt{T}}$. For simplicity of notation, we write $A_T := \|\mathcal{L}[f_T]\|_{L^p(Q'_{\sqrt{T}} \times (0,bT))}^p$ in this proof. In what follows, we compute

$$A_T = \int_0^{bT} \int_{Q'_a} \left| \int_M K(x, y, t) f_T(y) dV_g(y) \right|^p dV_g(x) dt.$$

By the change of variables and (5.8), we have

$$\int_M K(x, y, t) f_T(y) dV_g(y) = \int_{Q'_{\sqrt{T}}} K(x, y, t) \tilde{f} \left( \varphi^{-1} \left( \lambda^{-1} \varphi(y) \right) \right) dV_g(y)$$

$$= \int_{Q'_{\sqrt{T}}} K(x, \varphi^{-1}(\eta), t) \tilde{f} \left( \varphi^{-1} \left( \lambda^{-1} \eta \right) \right) \sqrt{\det(g_{ij}(\eta))} d\eta$$

$$= \int_{Q_a} K \left( x, \varphi^{-1}(\lambda \tilde{\eta}), t \right) \tilde{f} (\varphi^{-1}(\tilde{\eta})) \lambda^N \sqrt{\det \left( g_{ij}(\lambda \tilde{\eta}) \right)} d\tilde{\eta}$$

for $(x, t) \in Q'_{\sqrt{T}} \times (0, bT)$. Since $|\lambda \tilde{\eta}| \leq |\tilde{\eta}| < \sqrt{N} a \leq \rho_\infty$ for $\tilde{\eta} \in Q_a$, the volume comparison theorem (A.4) implies the existence of a constant $C > 0$ independent of $T$ such that

$$\int_M K(x, y, t) f_T(y) dV_g(y) \geq C \lambda^N \int_{Q_a} K \left( x, \varphi^{-1}(\lambda \tilde{\eta}), t \right) \tilde{f} (\varphi^{-1}(\tilde{\eta})) d\tilde{\eta}$$

for $(x, t) \in Q'_{\sqrt{T}} \times (0, bT)$. Thus, (5.8) gives

$$A_T \geq C \lambda^{Np} \int_0^{bT} \int_{Q'_a} \left| \int_{Q_a} K \left( x, \varphi^{-1}(\lambda \tilde{\eta}), t \right) \tilde{f} (\varphi^{-1}(\tilde{\eta})) d\tilde{\eta} \right|^p dV_g(x) dt$$

$$= C \lambda^{Np} \int_0^{bT} \int_{Q'_a} \left| \int_{Q_a} K \left( \varphi^{-1}(\xi), \varphi^{-1}(\lambda \tilde{\eta}), t \right) \tilde{f} (\varphi^{-1}(\tilde{\eta})) d\tilde{\eta} \right|^p \sqrt{\det(g_{ij}(\xi))} d\xi dt.$$
By the change of variables and the volume comparison theorem (A.4) again, we see that

\[ A_T \geq C\lambda^N p \int_0^{b_T} \int_{Q_a} \int_{Q_a} K \left( \varphi^{-1} \left( \lambda \xi \right), \varphi^{-1} \left( \lambda \eta \right), t \right) \tilde{f} \left( \varphi^{-1} \left( \eta \right) \right) d\eta \bigg| \lambda^N \sqrt{\det (g_{ij} (\lambda \xi))} d\xi dt \]

\[ \geq C\lambda^{N(p+1)} T \int_0^{b} \int_{Q_a} \int_{Q_a} K \left( \varphi^{-1} \left( \lambda \xi \right), \varphi^{-1} \left( \lambda \eta \right), Tt \right) \tilde{f} \left( \varphi^{-1} \left( \eta \right) \right) d\eta \bigg| \lambda^N \sqrt{\det (g_{ij} (\lambda \xi))} d\xi dt. \]

We estimate the heat kernel in the integrand. By \( \lambda = \sqrt{T} / a < 1 \), we can apply (A.11) with \( \alpha = \lambda \) and \( \beta = T \), where we remark that \( \beta / (9\alpha^2) = 1 / b \). Then there exists \( C > 0 \) depending only on \( N, \kappa, \text{inj}(M), a \) and \( T \) such that

\[ K \left( \varphi^{-1} \left( \lambda \xi \right), \varphi^{-1} \left( \lambda \eta \right), Tt \right) \geq CK \left( \varphi^{-1} \left( \xi \right), \varphi^{-1} \left( \eta \right), b^{-1} T \right) \]

for \( 0 < \sqrt{t} < \min\{\text{inj}(M), \pi/(4\sqrt{\kappa}), \sqrt{bT} \} \). This together with the volume comparison theorem (A.2) yields

\[ A_T \geq C \int_0^{b} \int_{Q_a} \int_{Q_a} K \left( \varphi^{-1} \left( \lambda \xi \right), \varphi^{-1} \left( \lambda \eta \right), b^{-1} t \right) \tilde{f} \left( \varphi^{-1} \left( \eta \right) \right) d\eta \bigg| \lambda^N \sqrt{\det (g_{ij} (\lambda \xi))} d\xi dt \]

\[ = C \int_0^{1} \int_{Q_a} \int_{Q_a} K \left( \varphi^{-1} \left( \lambda \xi \right), \varphi^{-1} \left( \lambda \eta \right), t' \right) \tilde{f} \left( \varphi^{-1} \left( \eta \right) \right) d\eta \bigg| \lambda^N \sqrt{\det (g_{ij} (\lambda \xi))} d\xi dt' \]

\[ \geq C \int_0^{1} \int_{Q_a} \int_{Q_a} K \left( \varphi^{-1} \left( \lambda \xi \right), \varphi^{-1} \left( \lambda \eta \right), t' \right) \tilde{f} \left( \varphi^{-1} \left( \eta \right) \right) \sqrt{\det (g_{ij} (\eta))} d\eta \bigg| \lambda^N \sqrt{\det (g_{ij} (\lambda \xi))} d\xi dt' \]

\[ = C \int_0^{1} \int_{Q_a} \int_{Q_a} K (x, y, t) \tilde{f} (y) dV_g (y) \bigg| \lambda^N \sqrt{\det (g_{ij} (\eta))} d\xi dt'. \]

Recall that \( \tilde{f} = 0 \) a.e. in \( M \setminus \overline{Q'_a} \) and \( \| \mathcal{L}[\tilde{f}] \|_{L^p(Q'_a \times I_0)} = \infty \) by Lemma 5.1. Then,

\[ A_T \geq C \int_0^{1} \int_{Q_a} \int_{M} K (x, y, t) \tilde{f} (y) dV_g (y) \bigg| \lambda^N \sqrt{\det (g_{ij} (\eta))} d\xi dt \]

\[ = C \| \mathcal{L}[\tilde{f}] \|_{L^p(Q'_a \times I_0)} = \infty, \]

and so \( \| \mathcal{L}[f_T] \|_{L^p(Q'_T \times (0,bT))} = A_T^{1/p} = \infty. \)

Lemma 5.2 shows the existence of a specific initial data \( f_T \) depending on the choice of \( T \). By using this lemma, we construct the desired initial data \( f_0 \) independent of \( T \).

Lemma 5.3. There exists a nonnegative function \( f_0 \in Y \) such that \( \| f_0 \|_Y \leq 1, f_0 = 0 \) a.e. in \( M \setminus \overline{Q'_a} \) and \( \| \mathcal{L}[f_0] \|_{L^p(Q'_T \times (0,bT))} = \infty \) for all \( 0 < T < a^2 \).
Proof. The proof is the same as in [20, Theorem 2.4] and [42, Proposition 1]. We give an outline. Let \( \{q_j\}_{j=1}^{\infty} \) be the all rational numbers in \((0, a^2)\). By Lemma 5.2, for each \( j \), there exists \( 0 \leq f_{\sqrt{q_j}} \in Y \) such that \( f_{\sqrt{q_j}} = 0 \) a.e. in \( M \setminus Q_{\sqrt{q_j}} \) and \( \| \mathcal{L}[f_{\sqrt{q_j}}]\|_{L^p(Q_{\sqrt{q_j}} \times (0,bq_j))} = \infty \). Set

\[
 f_0(x) := \sum_{j=1}^{\infty} 2^{-j} f_{\sqrt{q_j}}(x) / \| f_{\sqrt{q_j}} \|_Y.
\]

We can see that \( f_0 \geq 0 \), \( f_0 \in Y \), \( \| f_0 \|_Y \leq 1 \) and \( f_0 = 0 \) a.e. in \( M \setminus Q_a \). For \( 0 < T < a^2 \), there exists \( j' \) such that \( Q_{\sqrt{q_j}} \subset (0, bq_j) \subset Q_{\sqrt{T}} \times (0, bT) \). Then

\[
 f_0(x) \geq 2^{-j'} \| f_{\sqrt{q_j}} \|_Y^{-1} f_{\sqrt{q_j}}(x) \text{ and}
\]

\[
 \| \mathcal{L}[f_0]\|_{L^p(Q_{\sqrt{T}} \times (0,bT))} \geq 2^{-j'} \| f_{\sqrt{q_j}} \|_Y^{-1} \| \mathcal{L}[f_{\sqrt{q_j}}]\|_{L^p(Q_{\sqrt{T}} \times (0,bq_j))} = \infty.
\]

Hence, \( f_0 \) is the desired function. \( \square \)

5.4. Completion of proof

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. Fix \( c > 0 \). Set \( u_0 := cf_0 \), where \( f_0 \) is given by Lemma 5.3. We define a nonnegative Radon measure \( \mu_0 \) by \( \mu_0(A) := \int_A u_0(x) dV_g(x) \). Then by \( u_0 \geq 0 \), \( \| u_0 \|_Y \leq c \) and the definition of \( Y \) in (5.3), we have

\[
 \sup_{z \in M} \mu_0(B(z, \rho)) = \sup_{z \in M} \int_{B(z, \rho)} u_0(y) dV_g(y)
\]

\[
 \leq c \rho^{N-\frac{2}{p-1}} (\log(\rho^{-1}))^{-\frac{1}{p-1}} \quad \text{for any } 0 < \rho < \rho_\infty.
\]

Hence, \( \mu_0 \) satisfies the growth condition in Theorem 2.4. It suffices to prove that the problem (1.1) with \( u(\cdot,0) = \mu_0 \) does not admit any solutions in \( M \times [0, T) \) for all \( T > 0 \).

To obtain a contradiction, suppose that there exists \( T > 0 \) such that (1.1) with \( u(\cdot,0) = \mu_0 \) admits a solution \( u \) in \( M \times [0, T) \). Without loss of generality, we assume \( 0 < T < a^2 \). By Lemma 5.3, we have \( \| \mathcal{L}[u_0]\|_{L^p(Q_{\sqrt{T}} \times (0,bT))} = \infty \). Thus, we will reach a contradiction once we obtain

\[
 \| \mathcal{L}[u_0]\|_{L^p(Q_{\sqrt{T}} \times (0,bT))} < \infty. \tag{5.14}
\]

Let us prove (5.14). The same computations as (5.7) give \( d(z_0,x) \leq \sqrt{Na} \) for \( x \in Q_a' \). Then by \( Q_{\sqrt{T}} \subset Q_a' \), we observe that \( \text{supp} \, u_0 \subset Q_a' \subset B(z_0, \sqrt{Na}) \). Based on this observation, we choose \( \psi \in C^\infty_0(M) \) such that \( 0 \leq \psi \leq 1, \psi = 1 \) in \( B(z_0, \sqrt{Na}) \), \( \psi = 0 \) in \( M \setminus B(z_0, 2\sqrt{Na}) \). Let \( 0 < \varepsilon < bT \). Integrating the equation
\begin{align*}
(2.3) \text{over } M \times (\varepsilon, bT) \text{ and integrating by parts, we see that}
\int_{\varepsilon}^{bT} \int_M u^p \psi dV_g dt &= \int_{\varepsilon}^{bT} \int_M (u_t - \Delta u) \psi dV_g dt \\
&= \int_M (u(x, bT) - u(x, \varepsilon)) \psi dV_g - \int_{\varepsilon}^{bT} \int_M u \Delta \psi dV_g dt \\
&\leq \int_M u(x, bT) \psi dV_g + \int_{\varepsilon}^{bT} \int_M u |\Delta \psi| dV_g dt.
\end{align*}

By the same argument as in the proof of Theorem 2.2 (proof of (2.4)), we see that
\begin{align*}
\int_M u(x, t) \tilde{\psi}(x) dV_g(x) \to \int_M u_0(x) \tilde{\psi}(x) dV_g(x)
\end{align*}
for any $\tilde{\psi} \in C_0(M)$ as $t \to 0$. Then there exists $0 < t_0 < bT$ such that
\begin{align*}
\int_M |u(x, t) - u_0(x)| |\Delta \psi(x)| dV_g(x) \leq 1 \quad \text{for } 0 < t < t_0.
\end{align*}
From this, it follows that
\begin{align*}
\int_M u(x, t) |\Delta \psi(x)| dV_g(x) \leq 1 + \int_M u_0(x) |\Delta \psi(x)| dV_g(x) \leq C
\end{align*}
for $0 < t < t_0$, where $C > 0$ is a constant independent of $\varepsilon$. Thus,
\begin{align*}
\int_{\varepsilon}^{bT} \int_M u |\Delta \psi| dV_g dt \leq \int_0^{t_0} \int_M u |\Delta \psi| dV_g dt + \int_{t_0}^{bT} \int_M u |\Delta \psi| dV_g dt \leq C,
\end{align*}
and so
\begin{align*}
\int_{\varepsilon}^{bT} \int_M u^p \psi dV_g dt \leq \int_M u(x, bT) \psi dV_g + C.
\end{align*}
Since the right-hand side is independent of $\varepsilon$ and $u^p \psi \geq 0$, we obtain
\begin{align*}
\int_{0}^{bT} \int_M u^p \psi dV_g dt \leq \int_M u(x, bT) \psi dV_g + C < \infty.
\end{align*}
In particular, $u \in L^p \left( B(z_0, \sqrt{Na}) \times (0, bT) \right)$.

Let $x \in M$, $0 < t < bT$ and $0 < \tau < bT - t$. Again by the same argument as in the proof of Theorem 2.2 (proof of (4.5)), $u$ satisfies
\begin{align*}
u(x, t + \tau) &= \int_M K(x, y, t) u(y, \tau) dV_g(y) \\
&\quad + \int_0^{t+\tau} \int_M K(x, y, t + \tau - s) u(y, s) dV_g(y) ds.
\end{align*}
Then by $K$, $u \geq 0$ and $\psi \leq 1$, we have
\begin{align*}
u(x, t + \tau) \geq \int_M K(x, y, t) \psi(y) u(y, \tau) dV_g(y).
\end{align*}
6. Proofs of corollaries

We give proofs of Corollaries 2.1, 2.2, 2.3 and 2.4. The existence part is a corollary of the proof of Theorem 2.3 and the nonexistence part is a corollary of Theorem 2.1.

Proof of Corollary 2.1. We first assume (i). Let $1 < \alpha < q$. For a nonnegative function $u_0 \in L^q_{uloc,\bar{\rho}}(M)$ with $\bar{\rho} \in (0, \bar{\rho}_T]$, set

$$\bar{u}(x,t) := 2U(x,t), \quad U(x,t) := \left( \int_M K(x,y,t)u_0(y)^\alpha dV_g(y) \right)^{\frac{1}{\alpha}},$$

as in (4.10). Then by $\int_M K(x,y,t)u_0(y)dV_g(y) \leq U(x,t)$, (4.13) and (4.14), we have

$$\Psi[\bar{u}] \leq U + 2^p U(x,t) \left\| U(\cdot,t)^{\alpha-1} \right\|_{\infty} \int_0^t \left\| U(\cdot,s)^{p-\alpha} \right\|_{\infty} ds.$$

Lemma 4.2 and the Hölder inequality show that, for $(x,t) \in M \times (0, \bar{\rho}_T^2)$,

$$U(x,t) \leq \left( Ct^{-\frac{N}{\alpha}} \sup_{z \in M} \int_{B(z,t^{\frac{1}{2}})} u_0^\alpha dV_g \right)^{\frac{1}{\alpha}} \leq Ct^{-\frac{N}{\alpha}} \sup_{z \in M} \left( \int_{B(z,t^{\frac{1}{2}})} u_0^\alpha dV_g \right)^{\frac{1}{\alpha}} \text{Vol}(B(z,t^{\frac{1}{2}}))^{\frac{1}{\alpha} - \frac{1}{q}},$$

where $C > 0$ depends only on $N$ and $\alpha$. Note that the volume comparison theorem (A.3) yields $\text{Vol}(B(z,t^{1/2})) \leq 2^{N-1} \text{Area}(S^{N-1}) N^{-1} t^{N/2}$, where we used $t^{1/2} < \frac{1}{\rho}$.
\[ \tilde{\rho} \leq \rho_\infty. \] Then by \( \alpha < q, \) we have \[ U \leq C t^{-N/(2q)} \| u_0 \|_{L_{uloc, \tilde{\rho}}^q(M)} \leq C c t^{-N/(2q)}, \] and so

\[ \Psi[\tilde{u}] \leq U + C c t^{1 - N/(2q)} - U \leq (1 + C c \tilde{\rho}^{-2N/(p-1)}) U \]

for \( 0 < t < \tilde{\rho}^2. \) Hence, \( \tilde{\rho} \) is a supersolution of (4.1) in \( M \times [0, \tilde{\rho}^2] \) if \( C c \tilde{\rho}^{-2N/(p-1)/q} \leq 1. \) The rest is the same as that of Theorem 2.2.

We next assume (ii). For a nonnegative function \( u_0 \in L_{uloc, \tilde{\rho}}^q(M) \) with \( \tilde{\rho} \in (0, \rho_T], \) again by the volume comparison theorem (A.3), we have

\[
\sup_{x \in M} \int_{B(z, \rho)} u_0 dV_g \leq \| u_0 \|_{L_{uloc, \tilde{\rho}}^q(M)} \mathrm{Vol}(B(z, \rho))^{1 - \frac{1}{q}} \leq C \| u_0 \|_{L_{uloc, \tilde{\rho}}^q(M)} \rho^{N(1 - \frac{1}{q})}.
\]

(6.1)

for any \( 0 < \rho < \tilde{\rho}, \) where \( C > 0 \) is a constant depending only on \( N \) and \( q. \) From \( X^{-N/(1-q)} \leq C (\log(e + X^{-1}))^{-N/2} \) for \( 0 < X < 1, \) it follows that

\[
\sup_{x \in M} \int_{B(z, \rho)} u_0 dV_g \leq C c \tilde{\rho}^{-N(1 - \frac{1}{q})} (\log(e + \rho \rho^{-1}))^{-\frac{N}{2}}.
\]

Define \( \mu \) by \( \mu(A) := \int_A u_0 dV_g. \) Then \( \mu \) satisfies (2.6) with \( \varepsilon = 1 \) provided that \( C c \tilde{\rho}^{-N(1 - \frac{1}{q})} \) is small enough. Therefore, we can apply Theorem 2.5.

Finally, we assume (iii). Define \( \mu \) as above. By (6.1), \( \sup_{x \in M} \mu(B(z, \tilde{\rho})) \leq C c \tilde{\rho}^{-N(1 - \frac{1}{q})}, \) and so \( \mu \) satisfies (2.5) if \( C c \tilde{\rho}^{-N(1 - \frac{1}{q})} \) is small. Hence, we can apply Theorem 2.2.

**Proof of Corollary 2.2.** Let \( z_0 \in M \) and let \( u_0 \in L_{loc}^\infty(M \setminus \{z_0\}) \) satisfy (2.8). Set \( \mu(A) := \int_A u_0 dV_g. \) By using Riemannian normal coordinates \( (x_1, \ldots, x_N) \) centered at \( z_0, \) the volume comparison theorem (A.4) and \( 1 \geq (e|x| + 1)^{-1}, \) we have

\[
\mu(B(z_0, \rho)) \geq \begin{cases} 
\frac{c}{p} \int_{B(0, \rho)} |x|^{-\frac{2}{p-1}} \sqrt{\det(g_{ij}(x))} dx & (p > p_F) \\
\frac{c}{p} \int_{B(0, \rho)} |x|^{-N(e|x| + 1)^{-1}} \sqrt{\det(g_{ij}(x))} dx & (p = p_F)
\end{cases}
\]

\[
\geq \begin{cases} 
\tilde{c} \rho^{N-\frac{2}{p-1}} & (p > p_F) \\
\tilde{c} (\log(e + \rho^{-1}))^{-\frac{N}{2}} & (p = p_F)
\end{cases}
\]

for any \( 0 < \rho < \rho_\infty, \) where \( \tilde{c} > 0 \) is a constant depending only on \( N \) and \( p. \) We denote by \( \tilde{C} \) the constant guaranteed by Theorem 2.1, where \( \tilde{C} \) depends only on \( N \) and \( p. \)

Let \( c > \tilde{C}/\tilde{c}. \) To obtain a contradiction, we suppose that there exists \( T > 0 \) such that (2.7) has a solution on \( M \times [0, T). \) Then by Theorem 2.1, we have

\[
\mu(B(z_0, \rho)) \leq \begin{cases} 
\tilde{C} \rho^{N-\frac{2}{p-1}} & (p > p_F) \\
\tilde{C} (\log(e + \rho_T \rho^{-1}))^{-\frac{N}{2}} & (p = p_F)
\end{cases}
\]
for any $0 < \rho < \rho_T$. If $p > p_F$, then $c \leq \tilde{C}/\tilde{c}$. If $p = p_F$, since $\rho_T \leq \rho_\infty$, we have

$$c \leq \left(\frac{\log(e + \rho^{-1})}{\log(e + \rho_T \rho^{-1})}\right)^{\frac{N}{2}}$$

for any $0 < \rho < \rho_T$.

Letting $\rho \to 0$ gives $c \leq \tilde{C}/\tilde{c}$. Hence, we obtain $c \leq \tilde{C}/\tilde{c}$ for $p \geq p_F$, contrary to $c > \tilde{C}/\tilde{c}$.

**Proof of Corollary 2.3.** Let $\tilde{c}$ and $\tilde{C}$ be as in the proof of Corollary 2.2. Fix $c > \tilde{C}/\tilde{c}$ and $z_0 \in M$. We set $u_0(x) := \tilde{u}_0(r(x))$ with $r(x) := d(z_0, x)$ and

$$\tilde{u}_0(r) := \begin{cases} cr^{-\frac{2}{p-1}} & \text{if } p > p_F, \\ cr^{-N}(\log(e + r^{-1})^{-\frac{N}{2}-1} & \text{if } p = p_F, \end{cases}$$

for $x \in B(z_0, \rho_\infty)$ and $u_0(x) := 0$ for $x \in M \setminus B(z_0, \rho_\infty)$. Let $1 \leq q < N(p-1)/2$ if $p > p_F$ and $q = 1$ if $p = p_F$. Since $\tilde{u}_0$ is decreasing, we see that (4.8) with $f$ replaced by $\tilde{u}_0^q$ holds. Thus,

$$\|u_0\|_{L^q_{uloc, \rho_\infty}(M)} = \sup_{z \in M} \left(\int_{B(z, \rho_\infty)} u_0^q dV_g\right)^{\frac{1}{q}} \leq C(N) \left(\int_0^{\rho_\infty} r_{N-1}^{\frac{q}{p} - 1} \tilde{u}_0^q(r) dr\right)^{\frac{1}{q}} < \infty,$$

so that $u_0 \in L^q_{uloc, \rho_\infty}(M)$. On the other hand, by $c > \tilde{C}/\tilde{c}$, the proof of Corollary 2.2 shows that (2.7) does not admit any local-in-time solutions. Hence, $u_0$ is the desired function. □

**Proof of Corollary 2.4.** Let $\eta \in C^\infty(\mathbb{R})$ be as in Sect. 4.3. Since $u_0 \in L^\infty(M \setminus B(z_0, \rho_\infty))$ if $M \setminus B(z_0, \rho_\infty) \neq \emptyset$, we can assume that $u_0(x) \leq cf(x) + \tilde{C}$ for $x \in M$, where $\tilde{C} \geq 0$ is determined by $u_0$, and $f$ is given by

$$f(x) := \begin{cases} d(z_0, x)^{-\frac{2}{p-1}} \eta(\rho_\infty^{-1} d(z_0, x)) & \text{if } p > p_F, \\ d(z_0, x)^{-N}(\log(e + d(z_0, x)^{-1})^{-\frac{N}{2}-1} \eta(\rho_\infty^{-1} d(z_0, x)) & \text{if } p = p_F. \end{cases}$$

Define $h$ by (4.9). Set

$$\bar{u}(x, t) := 2cU(x, t) + 2\tilde{C}, \quad U(x, t) := h^{-1}\left(\int_M K(x, y, t)h(f(y))dV_g(y)\right).$$

From $\int_M K(x, y, t)dV_g(y) = 1$, it follows that

$$\Psi[\bar{u}] \leq cU + \tilde{C} + Cc^p \int_0^t \int_M K(x, y, t - s)U(y, s)p dV_g(y)ds + C\tilde{C}^p t$$

$$\leq (c + Cc^p J(t))U + \tilde{C} + C\tilde{C}^p t,$$

where $J$ is given by (4.14). We estimate $J$ as follows. By computations similar to the derivation of (4.11) (with $\rho_T$ replaced by 1), we have

$$h(U(x, t)) \leq \begin{cases} Ct^{-\frac{\alpha}{p-1}} & \text{if } p > p_F, \\ C(1 + r^2)^{-\frac{N}{2}} (\log(e + r^{-1}))^{-\frac{N}{2} + \beta} & \text{if } p = p_F. \end{cases}$$
Recall (4.12). Then we obtain

\[
U(x,t) \leq \begin{cases} 
C t^{-\frac{1}{p-1}} & \text{if } p > p_F, \\
C (1 + t^{\frac{1}{2}}) t^{-\frac{N}{2}} (\log(e + t^{-\frac{1}{2}}))^{-\frac{N}{2}} & \text{if } p = p_F,
\end{cases}
\]

for \((x,t) \in M \times (0, \rho_2^\infty)\), where \(C > 0\) depends only on \(N\) and \(p\). Similar computations to (4.15) show that

\[
J(t) \leq \begin{cases} 
C t^{-\frac{p-1}{p-1}} \int_0^t s^{-\frac{p}{p-1}} ds & (p > p_F) \\
C (\log(e + t^{-\frac{1}{2}}))^{\beta} (1 + t^{\frac{1}{2}})^{p-1} \int_0^t s^{-1} (\log(e + s^{-\frac{1}{2}}))^{-1-\beta} ds & (p = p_F)
\end{cases}
\]

for \(0 < t < \rho_2^\infty\), where \(C > 0\) depends only on \(N\) and \(p\).

Let \(0 < T < \rho_2^\infty\). Then,

\[
\Psi[\eta] \leq \begin{cases} 
(c + Cc^p)U + \tilde{C} + C\tilde{C}^p T & \text{if } p > p_F, \\
(c + Cc^p (1 + T^{\frac{1}{2}})^p)U + \tilde{C} + C\tilde{C}^p T & \text{if } p = p_F,
\end{cases}
\]

for \(0 < t < T\). By choosing \(T \leq \min\{1, C^{-1} \tilde{C}^{-(p-1)}\}\) and \(c \leq 2^{-p/(p-1)} C^{-1/(p-1)}\), we see that \(\eta\) is a supersolution of (4.1) in \(M \times [0, T)\). We remark that the choice of \(c\) depends only on \(N\) and \(p\). The rest of the proof is the same as that of Theorem 2.2.

\[\Box\]

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Appendix A. Comparison theorems

In this appendix, we prepare estimates which are needed in this paper and strongly related to the so-called comparison theorems in Riemannian geometry. Those include bounds of the Laplacian of the distance function, Jacobi fields, the volume of balls and the heat kernel. In this section, we assume that \((M, g)\) is a connected \(N\)-dimensional complete Riemannian manifold with \(0 < \text{inj}(M) \leq \infty\) and \(|\text{sec}(M)| \leq \kappa\) for some \(0 \leq \kappa < \infty\).

Remark A.1. We remark that the bound of \(\text{sec}(M)\) is not imposed for \(N = 1\), since the sectional curvature is not defined. It is clear that a 1-dimensional connected manifold is diffeomorphic to \(\mathbb{R}^1\) or \(S^1\). Moreover, by a suitable coordinate change, we can say that any 1-dimensional connected complete Riemannian manifold is isometric to \(\mathbb{R}^1\) with the standard metric \(g_{\mathbb{R}^1}\) or \(S^1\) with the standard metric \(g_{S^1}\). This means that it suffices to consider \((\mathbb{R}^1, g_{\mathbb{R}^1})\) or \((S^1, g_{S^1})\) when \(N = 1\) in the category of connected complete Riemannian manifolds. Thus, in the following, we implicitly assume that \(g\) is \(g_{\mathbb{R}^1}\) or \(g_{S^1}\) in the case \(N = 1\).

A.1. Laplacian comparison

We start with the Laplacian comparison. Fix a point \(z_0 \in M\) and put \(r(x) := d(z_0, x)\) for \(x \in M\). Then, \(r\) is a smooth function on \(B(z_0, \text{inj}(M))\) with \(|\nabla r| = 1\). Moreover, since \(\text{Ric} \geq (N-1)(-\kappa)\) in the case \(N \geq 2\), from the Laplacian comparison theorem (see [25, Theorem 11.15] for instance), it follows that

\[
\Delta r(x) \leq \begin{cases} 
(N - 1) r^{-1}(x) & \text{if } \kappa = 0, \\
(N - 1) \sqrt{\kappa} \coth(\sqrt{\kappa} r(x)) & \text{if } \kappa > 0,
\end{cases}
\]

for \(x \in B(z_0, \text{inj}(M))\) with \(0 < r(x) < \pi / \sqrt{\kappa}\). Since \(\sqrt{\kappa} \coth(\sqrt{\kappa} r) \leq 2 r^{-1}\) for \(r \in (0, 1 / \sqrt{\kappa})\) if \(\kappa > 0\), we have

\[
\Delta r(x) \leq 2(N - 1) r^{-1}(x) \tag{A.1}
\]

for \(x \in B(z_0, \text{inj}(M))\) with \(0 < r(x) < \pi / (4 \sqrt{\kappa})\). This also holds for \(x \in B(z_0, \text{inj}(M))\) in the case \(N = 1\), since \(\Delta r(x) = 0\).

A.2. Volume comparison

We explain the upper and lower bound of the volume form. Fix a point \(z_0 \in M\) and put \(r(x) := d(z_0, x)\) for \(x \in M\). Let \((e_1, \ldots, e_N)\) be an orthonormal basis of \(T_{z_0}M\). Then, the map which sends \((x_1, \ldots, x_N) \in \mathbb{R}^N\) to \(\exp_{z_0}(x_1 e_1 + \cdots + x_N e_N) \in M\) gives Riemannian normal coordinates centered at \(z_0\) on \(B(z_0, \text{inj}(M))\), where \(\exp_{z_0} :\)
$T_{z_0}M \to M$ is the exponential map at $z_0$. Denote by $g_{ij}$ the coefficients of $g$ with respect to this normal coordinates. For a while, assume $N \geq 2$. Then, by the proof of [25, Theorem 11.16], we have

$$\sqrt{\det(g_{ij}(x))} \leq \begin{cases} 1 & \text{if } \kappa = 0, \\ (\sqrt{\kappa}r(x))^{1-N} \sinh^{N-1}(\sqrt{\kappa}r(x)) & \text{if } \kappa > 0, \end{cases}$$

for $x \in B(z_0, \text{inj}(M))$, where we used $\text{Ric} \geq (N - 1)(-\kappa)$. Since $(\sqrt{\kappa}r)^{1-N} \sinh^{N-1}(\sqrt{\kappa}r) \leq 2^{N-1}$ for $r \in [0, 1/\sqrt{\kappa})$ if $\kappa > 0$, we have

$$\sqrt{\det(g_{ij}(x))} \leq 2^{N-1}$$

for $x \in B(z_0, \text{inj}(M))$ with $0 \leq r(x) < \pi/(4\sqrt{\kappa})$. This also holds for $x \in B(z_0, \text{inj}(M))$ in the case $N = 1$, since $g_{11} = 1$. Especially, we have

$$\text{Vol}(B(x, r)) \leq 2^{N-1}N^{-1} \text{Area}(S^{N-1})r^N$$

(A.3)

for all $x \in M$ and $0 \leq r < \min\{\text{inj}(M), \pi/(4\sqrt{\kappa})\}$ when $N \geq 2$, and for all $x \in M$ and $0 \leq r < \text{inj}(M)$ when $N = 1$. Here, $\text{Area}(S^{N-1})$ is the area of the unit sphere in $\mathbb{R}^N$.

We can also derive the lower bound of the volume form, see for instance the proof of [25, Theorem 11.14]. From $\text{sec}(M) \leq \kappa$ in the case $N \geq 2$, it follows that

$$\sqrt{\det(g_{ij}(x))} \geq \begin{cases} 1 & \text{if } \kappa = 0, \\ (\sqrt{\kappa}r(x))^{1-N} \sinh^{N-1}(\sqrt{\kappa}r(x)) & \text{if } \kappa > 0, \end{cases}$$

for $x \in B(z_0, \text{inj}(M))$ with $0 \leq r(x) < \pi/\sqrt{\kappa}$. Since $(\sqrt{\kappa}r)^{1-N} \sinh^{N-1}(\sqrt{\kappa}r) \geq (1/2)^{N-1}$ for $r \in [0, \pi/(2\sqrt{\kappa}))$ if $\kappa > 0$, we have

$$\sqrt{\det(g_{ij}(x))} \geq 2^{1-N}$$

(A.4)

for $x \in B(z_0, \text{inj}(M))$ with $0 \leq r(x) < \pi/(4\sqrt{\kappa})$, and this also holds for $x \in B(z_0, \text{inj}(M))$ in the case $N = 1$, since $g_{11} = 1$. Especially, we have

$$\text{Vol}(B(x, r)) \geq 2^{1-N}N^{-1} \text{Area}(S^{N-1})r^N$$

(A.5)

for all $x \in M$ and $0 \leq r < \min\{\text{inj}(M), \pi/(4\sqrt{\kappa})\}$ when $N \geq 2$, and for all $x \in M$ and $0 \leq r < \text{inj}(M)$ when $N = 1$

A.3. Jacobi field comparison

Based on the estimates of Jacobi fields, we derive the upper and lower bound of the derivative of the exponential map and deduce some applications. Fix a point $z_0 \in M$ and put $r(x) := d(z_0, x)$ for $x \in M$. Let $(e_1, \ldots, e_N)$ be an orthonormal basis of $T_{z_0}M$. Denote by $\varphi : B(z_0, \text{inj}(M)) \to B(O, \text{inj}(M)) \subset \mathbb{R}^N$ the inverse of the map
which sends \((x_1, \ldots, x_N) \in \mathbb{R}^N\) to \(\exp_{z_0}(x_1e_1 + \cdots + x_Ne_N) \in M\). Then, in the case \(N \geq 2\), the Rauch comparison theorem, see [19, Corollary 5.6.1] for instance, implies that

\[
|D_\eta \exp_{z_0}(w)|^2 \leq \begin{cases} 
|w_1|^2 + |w_2|^2 & \text{if } \kappa = 0, \\
|w_1|^2 + |w_2|^2(\sqrt{\kappa} |\eta|)^{-2} \sinh^2(\sqrt{\kappa} |\eta|) & \text{if } \kappa > 0,
\end{cases}
\]

for \(\eta \in T_{z_0}M\), where \(w_1\) and \(w_2\) are vectors in \(T_{z_0}M\) which are uniquely determined by \(w = w_1 + w_2\) with \(w_1 \in \mathbb{R}\eta\) and \(w_2 \in (\mathbb{R}\eta)^\perp\). Then, similarly as (A.4), we can say that

\[
|D_\eta \exp_{z_0}(w)| \leq 2|w|
\]

for \(\eta \in T_{z_0}M\) with \(|\eta| \leq \pi/(4\sqrt{\kappa})\). As a corollary of this estimate, we can compare the distance on \(M\) and \(\mathbb{R}^N\). Let \(a, b \in B(z_0, \rho_\infty)\), where \(\rho_\infty\) is given by (2.2). Put \(a' := \exp_{z_0}^{-1}(a)\) and \(b' := \exp_{z_0}^{-1}(b)\). Then, \(c(t) := \exp_{z_0}(1 - t)a' + tb'\) is a curve in \(B(z_0, \rho_\infty)\) joining \(a\) and \(b\). Thus, we have

\[
d(a, b) \leq \int_0^1 |\dot{c}(t)|dt = \int_0^1 |D_{c(t)} \exp_{z_0}(b' - a')|dt \leq 2|b' - a'|. \tag{A.6}
\]

We remark that this also holds for \(N = 1\).

Similarly, we can estimate the distance from below. The upper bound of the sectional curvature with the Rauch comparison theorem (see [19, Corollary 5.6.1]) implies that

\[
|D_\eta \exp_{z_0}(w)|^2 \geq \begin{cases} 
|w_1|^2 + |w_2|^2 & \text{if } \kappa = 0, \\
|w_1|^2 + |w_2|^2(\sqrt{\kappa} |\eta|)^{-2} \sin^2(\sqrt{\kappa} |\eta|) & \text{if } \kappa > 0,
\end{cases}
\]

for \(\eta \in T_{z_0}M\) with \(|\eta| \leq \pi/\sqrt{\kappa}\) in the case \(N \geq 2\), where \(w_1\) and \(w_2\) are as above. Then, similarly as (A.4), we can say that

\[
|D_\eta \exp_{z_0}(w)| \geq 2^{-1}|w|
\]

for \(\eta \in T_{z_0}M\) with \(|\eta| \leq 7\pi/(12\sqrt{\kappa})\). By this estimate, we can also bound \(d(a, b)\) from below. Fix \(a, b \in B(z_0, \rho_\infty)\) and let \(c(t) (t \in [0, 1])\) be the geodesic from \(a\) to \(b\). Then, we see that \(c(t) \in B(z_0, 7\rho_\infty/3)\) for all \(t \in [0, 1]\). Actually, if not, there exists \(t' \in (0, 1)\) such that \(c(t') \notin B(z_0, 7\rho_\infty/3)\). Then, \(d(a, c(t')) > 4\rho_\infty/3\) and \(d(b, c(t')) > 4\rho_\infty/3\), and this implies that the length of \(c(t)\) is bigger than \(8\rho_\infty/3\). This contradicts the fact that the length of \(c(t)\) is now \(d(a, b)\) and it is smaller than \(2\rho_\infty\). We remark that \(7\rho_\infty/3 < \text{inj}(M)\). Put \(\xi(t) := \exp_{z_0}^{-1}(c(t)) \in B(O, 7\rho_\infty/3)\). Then, we have \(c(t) = \exp_{z_0}(\xi(t))\) and

\[
d(a, b) = \int_0^1 |\dot{c}(t)|dt = \int_0^1 |D_{\xi(t)} \exp_{z_0}(\xi(t))|dt \geq 2^{-1} \int_0^1 |\xi(t)|dt \geq 2^{-1}|b' - a'|. \tag{A.7}
\]

where \(a' := \exp_{z_0}^{-1}(a)\) and \(b' := \exp_{z_0}^{-1}(b)\). We remark that this also holds for \(N = 1\).
A.4. Estimates of the heat kernel

Let $K(x, y, t)$ be the heat kernel on $(M, g)$. Thanks to many established works on the heat kernel on a Riemannian manifold (for instance Cheeger and Yau [8] and Li and Yau [26]), we can assume that $K(x, y, t)$ is similar to the heat kernel on the Euclidean space especially in the case where $d(x, y)$ and $t$ are very small. We explain those estimates necessary in this paper.

First, we introduce a direct corollary form [26, Theorem 2.2]. For a while, assume $N \geq 2$. Since $\sec(M) \geq -\kappa$ implies $\Ric \geq (N - 1)(-\kappa)$, we can use it. Set $q = y = y_0 = \theta = 0$, $t_1 = t$, $t_2 = 2t$ and $\alpha = 3/2$ in that theorem. Then, we can say that there exists a constant $C > 0$ depending only on $N$ such that

$$K(x, y, t) \leq \begin{cases} 2^{3N/4} \exp \left( \frac{3d(y, z)^2}{8t} \right) K(x, z, 2t) & \text{if } \kappa = 0, \\ 2^{3N/4} \exp \left( C(N - 1)\kappa t + \frac{3d(y, z)^2}{8t} \right) K(x, z, 2t) & \text{if } \kappa > 0, \end{cases}$$

for all $x, y, z \in M$ and $t \in (0, \infty)$. Thus, if $d(y, z) \leq \sqrt{t}$ and $t < \pi^2/\kappa$, we can say that there exists a constant $C > 0$ depending only on $N$ such that

$$K(x, y, t) \leq CK(x, z, 2t). \quad (A.8)$$

This also holds for $d(y, z) \leq \sqrt{t}$ when $N = 1$ by the explicit formula of the heat kernel on $(\mathbb{R}^1, g_{\mathbb{R}^1})$ or $(S^1, g_{S^1})$.

Next, we introduce an upper bound of $K(x, y, t)$ by $C_1 t^{-N/2} e^{-d(x, y)^2/(C_2 t)}$ as an application of [26, Corollary 3.1]. They proved that, when $N \geq 2$, if the Ricci curvature is bounded from below by $-A$, for some $A \geq 0$, then for $1 < \alpha < 2$ and $0 < \varepsilon < 1$, the heat kernel satisfies

$$K(x, y, t) \leq \left( \frac{C(\varepsilon)^{2\alpha}}{\text{Vol}(B(x, \sqrt{t})) \text{Vol}(B(y, \sqrt{t}))} \right)^{1/2} \exp \left( \frac{C(N)\varepsilon At}{\alpha - 1} - \frac{d(x, y)^2}{(4 + \varepsilon) t} \right).$$

Thus, we can apply the above upper bound with $A = (N - 1)\kappa$. Then, letting $\alpha = 3/2$ and $\varepsilon = 1/2$ and applying the volume comparison theorem (A.5), we can say that the heat kernel satisfies

$$K(x, y, t) \leq Ct^{-N/2} \exp \left( -\frac{d(x, y)^2}{(4 + (1/2)) t} \right) \quad (A.9)$$

for $0 < \sqrt{t} < \min\{\text{inj}(M), \pi/(4\sqrt{\kappa})\}$, where $C > 0$ depends only on $N$, $\kappa$ and $\text{inj}(M)$. Of course, we can check that this holds for $0 < \sqrt{t} < \text{inj}(M)$ when $N = 1$ by the explicit formula of the heat kernel.

A lower bound of $K(x, y, t)$ can be induced from a result of Cheeger and Yau [8]. Put

$$\lambda := \begin{cases} -1 & \text{if } \kappa = 0, \\ -\kappa & \text{if } \kappa > 0, \end{cases}$$
for $N \geq 2$. Then, $\lambda < 0$ and $\text{Ric} \geq (N - 1)\lambda$. By [8, Theorem 3.1] (see also Theorem 7 of Section 4 of Chapter VIII in [7]), we can say that the heat kernel satisfies

$$K(x, y, t) \geq K_\lambda(d(x, y), t)$$

for all $x, y \in M$ and $t \in (0, \infty)$, where $K_\lambda : [0, \infty) \times (0, \infty) \to \mathbb{R}$ be a smooth function for $(r, t)$ so that $K_\lambda(d_\lambda(x, y), t)$ becomes the heat kernel on the $N$-dimensional simply connected space form of constant sectional curvature $\lambda$ with its induced distance function $d_\lambda$. A lower bound of $K_\lambda$ is obtained by Davies and Mandouvalos [9, Theorem 3.1]. Even though their estimate is for the case $\lambda = -1$, by scaling appropriately for $\lambda < 0$, we can see that

$$K_\lambda(r, t) \geq C t^{-\frac{N}{2}} \exp \left( -\frac{r^2}{4t} + \frac{(N - 1)^2\lambda t}{4} - \frac{(N - 1)\sqrt{-\lambda}r}{2} \right) \times \left( 1 + \sqrt{-\lambda}r - \lambda t \right)^{\frac{N-1}{2}} \left( 1 + \sqrt{-\lambda}r \right),$$

where $C > 0$ depends only on $N$ and $\kappa$. By Young’s inequality, we have

$$\frac{(N - 1)\sqrt{-\lambda}r}{2} = 2 \frac{r}{2\sqrt{t}} \frac{(N - 1)\sqrt{-\lambda}t}{2} \leq \frac{r^2}{4t} + \frac{(N - 1)^2(-\lambda)t}{4}.$$ 

Thus, if $t$ is restricted to $[0, T)$ with $0 < T < \infty$, we have

$$\exp \left( -\frac{r^2}{4t} + \frac{(N - 1)^2\lambda t}{4} - \frac{(N - 1)\sqrt{-\lambda}r}{2} \right) \geq C \exp \left( -\frac{r^2}{2t} \right),$$

where $C > 0$ depends only on $N, T$ and $\kappa$. Also, one can see that, for $t \in [0, T)$ with $0 < T < \infty$,

$$\left( 1 + \sqrt{-\lambda}r - \lambda t \right)^{\frac{N-1}{2}} \left( 1 + \sqrt{-\lambda}r \right) \geq C$$

for some $C > 0$ depends only on $N, T$ and $\kappa$. Thus, we see that there exists $C > 0$ depending only on $N, T$ and $\kappa$ such that for $t \in (0, T)$ and $x, y \in M$ the heat kernel satisfies

$$K(x, y, t) \geq C t^{-N/2} \exp \left( -\frac{d(x, y)^2}{2t} \right). \tag{A.10}$$

Of course, this holds for $N = 1$.

Combining (A.9) and (A.10), we can deduce the following scaling property of the heat kernel. Assume $N \geq 2$ for a while. Fix a base point $z_0 \in M$ and an orthonormal basis $(e_1, \ldots, e_N)$ of $T_{z_0} M$. Denote by $\varphi : B(z_0, \text{inj}(M)) \to B(O, \text{inj}(M)) \subset \mathbb{R}^N$ the inverse of the map which sends $(x_1, \ldots, x_N) \in \mathbb{R}^N$ to $\exp_{z_0} (x_1 e_1 + \cdots + x_N e_N) \in M$. Take $\xi, \eta \in B(O, \rho_\infty) \subset \mathbb{R}^N$, $0 < \alpha < 1$ and $0 < \beta < \infty$. Put, for simplicity,

$$\alpha \xi' := \varphi^{-1}(\alpha \xi), \quad \alpha \eta' := \varphi^{-1}(\alpha \eta), \quad \xi' := \varphi^{-1}(\xi), \quad \eta' := \varphi^{-1}(\eta).$$
Fix $0 < T < \infty$. Then, by (A.10) and (A.6), there exists $C > 0$ depending only on $N, \beta T$ and $\kappa$ such that
\[ K(\alpha \xi', \alpha \eta', \beta t) \geq C(\beta t)^{-N/2} \exp \left( -\frac{\alpha^2 |\xi - \eta|^2}{\beta t} \right) \]
for all $t \in (0, \beta T)$. On the other hand, by (A.9) and (A.7), there exists $C' > 0$ depending only on $N, \kappa$ and $\text{inj}(M)$ such that
\[ K(\xi', \eta', t) \leq C't^{-N/2} \exp \left( -\frac{|\xi - \eta|^2}{2(4 + (1/2))t} \right) \]
for $0 < \sqrt{t} < \min\{\text{inj}(M), \pi/(4\sqrt{\kappa})\}$. Comparing the above two inequalities, we can see that there exists $C > 0$ depending only on $N, \kappa, \text{inj}(M), \alpha, \beta$ and $T$ such that
\[ K(\alpha \xi', \alpha \eta', \beta t) \geq CK(\xi', \eta', \frac{\beta}{9\alpha^2}t) \quad (A.11) \]
for $0 < \sqrt{t} < \min\{\text{inj}(M), \pi/(4\sqrt{\kappa}), \sqrt{\beta T}\}$. Without the above argument, we can directly prove (A.11) in the case $N = 1$ by the explicit formula of the heat kernel.

Appendix B. Covering theorems

In this appendix, we prove two facts on Riemannian manifolds. One is a kind of the Besicovitch covering theorem and the other is an upper bound of the number which we need to cover a ball with half balls. Recall that $(M, g)$ is a connected $N$-dimensional complete Riemannian manifold with $\text{inj}(M) > 0$. We also assume $|\sec(M)| \leq \kappa$ for some $0 \leq \kappa < \infty$ when $N \geq 2$.

Roughly speaking, by [10, Theorem 2.8.14], the Besicovitch covering theorem holds if a metric space $(X, d)$ is $(\xi, \eta, \zeta)$-limited in the sense of [10, Definition 2.8.9]. First, we introduce the definition of a $(\xi, \eta, \zeta)$-limited space. Let $(X, d)$ be a metric space. Fix $A \subset X, \xi > 0$ and $0 < \eta \leq 1/3$. For $a \in A$ and $B \subset B(a, \xi) \setminus \{a\}$ with $\# B \geq 2$, we define
\[ \text{Width}_{a, B}(b, c) := \inf \left\{ \frac{d(x, c)}{d(x, a)} : x \in X, d(a, x) = d(a, c), \frac{d(a, x)}{d(a, c)} \cdot d(a, x) + d(x, b) = d(a, b) \right\} \quad (B.1) \]
for all $b, c \in B$ with $b \neq c$ and $d(a, b) \geq d(a, c)$.

**Definition B.1.** For $\zeta \in \mathbb{N}$, $(X, d)$ is directionally $(\xi, \eta, \zeta)$-limited at $A$ if
\[ \sup\{\# B; a \text{ and } B \text{ satisfy Width}_{a, B} \geq \eta\} \leq \zeta. \]

**Theorem B.1.** ([10, Theorem 2.8.14]). Assume that $(X, d)$ is directionally $(\xi, \eta, \zeta)$-limited at $A$. If $\mathcal{F}$ is a family of closed balls with radii less than $\mu$ with $\mu < \xi/2$ and each point of $A$ is the center of some member of $\mathcal{F}$, then $A$ is contained in the union of $2\zeta + 1$ disjointed subfamilies of $\mathcal{F}$. 
For our use, it suffices to prove that \( (M, g) \) is directionally \((\xi, \eta, \zeta)\)-limited at \( M \) for some \((\xi, \eta, \zeta)\). Take \( \xi \in (0, \text{inj}(M)) \). Fix \( a \in M \) and \( B \subset B(a, \xi) \setminus \{a\} \) with \( \#B \geq 2 \). We denote by \( UT_a M := \{ v \in T_a M; |v|_g = 1 \} \) the unit sphere in \( T_a M \) and define a map \( \theta : B(a, \text{inj}(M)) \setminus \{a\} \to UT_a M \) by \( \theta(b) := \exp_{a}^{-1}(b) / | \exp_{a}^{-1}(b) | \).

**Lemma B.1.** If \( \text{Width}_{a,B} > 0 \), then \( \theta \) restricted to \( B \) is injective.

**Proof.** If not, there exist \( b, c \in B \) with \( b \neq c \) such that \( \theta(b) = \theta(c) \). Assume \( d(a, b) \geq d(b, c) \) if necessary. Then \( x := c \in M \) satisfies \( d(a, x) = d(a, c) \) and \( d(a, x) + d(x, b) = d(a, b) \), since \( b \) and \( c \) are on the same unit speed shortest geodesic \( \gamma \) emerging from \( a \) with \( \gamma'(0) = \theta(b) = \theta(c) \). Then, the right-hand side of (B.1) should be 0 and this contradicts Width \( \Box \)

Considering \( \theta(B) \) as a subset of the metric space \( T_a M \) with metric \( g_a \), we can also define Width \( 0, \theta(B) \) for \( \theta(B) \). Put a positive constant

\[
A := \begin{cases} 
2 & \text{if } \kappa = 0 \text{ or } N = 1, \\
2(\sqrt{\kappa} \xi)^{-1} \sinh(\sqrt{\kappa} \xi) & \text{if } \kappa > 0.
\end{cases}
\]

We remark that \( A \to \infty \) as \( \sqrt{\kappa} \xi \to \infty \) if \( \kappa > 0 \).

**Lemma B.2.** If \( \text{Width}_{a,B} \geq \eta \) for some \( 0 < \eta < \infty \), then \( \text{Width}_{0,\theta(B)} \geq A^{-1} \eta \).

**Proof.** Take \( \tilde{b}, \tilde{c} \in \theta(B) \) with \( \tilde{b} \neq \tilde{c} \). Note that \( |\tilde{b}| = |\tilde{c}| = 1 \). There exist \( b, c \in B \) with \( b \neq c \) such that \( \tilde{b} = \theta(b) \) and \( \tilde{c} = \theta(c) \). We remark that \( \tilde{c} = \theta(c) \) means \( c = \exp_{a}(d(a, c) \tilde{c}) \), since \( | \exp_{a}^{-1} c | = d(a, c) \). We suppose \( d(a, b) \geq d(a, c) \) if necessary. Since

\[
\{ x \in T_a M; |x| = 1, |x| + |x - b| = |\tilde{b}| \} = \{ \tilde{b} \},
\]

we have

\[
\text{Width}_{0,\theta(B)}(\tilde{b}, \tilde{c}) = \frac{|\tilde{b} - \tilde{c}|}{|\tilde{c}|} = |\tilde{b} - \tilde{c}|.
\]

On the other hand, \( x := \exp_{a}(d(a, c) \tilde{b}) \in M \) satisfies \( d(a, x) = d(a, c) \) and \( d(a, x) + d(x, b) = d(a, b) \). Thus, \( d(x, c)/d(a, c) \geq \text{Width}_{a,B} \geq \eta \). It suffices to estimate \( d(x, c)/d(a, c) \) from above by \( A|\tilde{b} - \tilde{c}| \). Then, by the distance comparison theorem (A.6) with replacing 2 by \( A \), we have

\[
d(x, c) \leq A|d(a, c) \tilde{b} - d(a, c) \tilde{c}| \leq A|\tilde{b} - \tilde{c}|d(a, c).
\]

Thus, \( \text{Width}_{0,\theta(B)}(\tilde{b}, \tilde{c}) \geq A^{-1} \eta \) and the proof is complete. \( \Box \)

**Definition B.2.** Let \( S^{N-1} \) be a unit sphere in \( R^N \). For \( w \in (0, \infty) \), define \( \text{Dis}(N, w) \) as the positive integer such that the following property holds: There exists \( P \subset S^{N-1} \) with \( \#P \leq \text{Dis}(N, w) \) such that \( |p_1 - p_2| \geq w \) for all \( p_1, p_2 \in P \) \((p_1 \neq p_2)\), and there does not exist \( P \subset S^{N-1} \) with \( \#P \geq \text{Dis}(N, w) + 1 \) such that \( |p_1 - p_2| \geq w \) for all \( p_1, p_2 \in P \) \((p_1 \neq p_2)\). We remark that \( \text{Dis}(1, w) = 2 \) for \( w \in (0, 2) \).
Roughly speaking, \( \text{Dis}(N, w) \) is the maximum number of points distributed on the unit sphere keeping distance over \( w \). One can see that \( \text{Dis}(N, w) \to \infty \) as \( w \to 0 \).

**Proposition B.1.** A connected \( N \)-dimensional complete Riemannian manifold \((M, g)\) with \( \text{inj}(M) > 0 \) satisfying \( |\text{sec}(M)| \leq \kappa \) for some \( 0 \leq \kappa < \infty \) when \( N \geq 2 \) is directionally \((\xi, \eta, \zeta)\)-limited at \( M \) for \( \xi \in (0, \text{inj}(M)), \eta \in (0, 1/3) \) and \( \zeta \geq \text{Dis}(N, A^{-1}\eta) \), where \( A \) is defined by (B.2).

**Proof.** Fix \( a \in M \) and \( B \subset B(a, \xi) \setminus \{a\} \) with \( \#B \geq 2 \). Assume that \( \text{Width}_{a, B} \geq \eta \). Then, by Lemma B.2, we have \( \text{Width}_{0, \theta(B)} \geq A^{-1}\eta \). Thus, \( \tilde{b}, \tilde{c} \in \theta(B) \) with \( \tilde{b} \neq \tilde{c} \) satisfy \( |\tilde{b} - \tilde{c}| \geq A^{-1}\eta \), and so \( \#\theta(B) \leq \text{Dis}(N, A^{-1}\eta) \). Since \( \#\theta(B) = \#B \) by Lemma B.1, the proof is complete.

This proposition together with Theorem B.1 shows the following:

**Corollary B.1.** Let \((M, g)\) be a connected \( N \)-dimensional complete Riemannian manifold with \( \text{inj}(M) > 0 \) satisfying \( |\text{sec}(M)| \leq \kappa \) for some \( 0 \leq \kappa < \infty \) when \( N \geq 2 \). For \( \xi \in (0, \text{inj}(M)/2) \), there exists a positive integer \( k_0 \) such that the following holds: If \( F \) is a family of closed balls with radii less than \( \rho \) with \( \rho < \xi \) and each point of \( M \) is the center of some member of \( F \), then \( M \) is covered by the union of \( k_0 \) disjointed subfamilies of \( F \). Here, the constant \( k_0 \) depends only on \( N \) if \( \kappa = 0 \) and on \( N \) and \( \sqrt{\kappa}\xi \) if \( \kappa > 0 \). Moreover, \( k_0 \to \infty \) as \( \sqrt{\kappa}\xi \to \infty \).

We next give a bound on how many balls with radii \( \rho/2 \) are necessary to cover a ball with radius \( \rho \) for sufficiently small \( \rho \). Put

\[
R' := \begin{cases}
\frac{4}{5} \text{inj}(M) & \text{if } N = 1, \\
\frac{4}{5} \min\left\{ \text{inj}(M), \frac{\pi}{2\sqrt{\kappa}} \right\} & \text{if } N \geq 2.
\end{cases}
\]  

(B.3)

For \( a \in M \) and \( \rho \in (0, R') \), we set the following property (P) for a subset \( B \subset B(a, \rho) \) with \( \#B \geq 2 \).

\((P): B(b_1, \rho/4) \cap B(b_2, \rho/4) = \emptyset \) for all \( b_1, b_2 \in B \) with \( b_1 \neq b_2 \).

Then, we define

\[
\text{Pac}(a, \rho) := \sup\{\#B; B \subset B(a, \rho) \text{ satisfies (P)}\}.
\]

**Lemma B.3.** Assume \( \text{Pac}(a, \rho) < \infty \). Take \( B \) such that \( \#B = \text{Pac}(a, \rho) \). Then,

\[
B(a, \rho) \subset \bigcup_{b \in B} B(b, \rho/2).
\]

**Proof.** If not, there exists \( c \in B(a, \rho) \setminus (\bigcup_{b \in B} B(b, \rho/2)) \). Then, one can easily see that \( B' := B \cup \{c\} \) also satisfies (P). This contradicts the maximality of \( \#B \).  

□
Thus, it suffices to bound $P_a(a, \rho)$ from above. Fix $B \subset B(a, \rho)$ with $\#B \geq 2$ satisfying (P). We remark that

$$\bigcup_{b \in B} B(b, \rho/4) \subset B(a, 5\rho/4). \quad \text{(B.4)}$$

We estimate the volume of $B(b, \rho/4)$ and $B(a, 5\rho/4)$ from below and above, respectively. By the volume comparison theorem from above (A.3), we have

$$\text{Vol}(B(a, 5\rho/4)) \leq 2^{N-1} N^{-1} \text{Area}(S^{N-1})(5\rho/4)^{N},$$

where we used $5\rho/4 \leq \pi/(2\sqrt{\kappa})$ when $N \geq 2$ and $\kappa > 0$. On the other hand, by the volume comparison theorem from below (A.5), we have

$$\text{Vol}(B(b, \rho/4)) \geq 2^{1-N} N^{-1} \text{Area}(S^{N-1})(\rho/4)^{N},$$

where we used $\rho/4 \leq \pi/(2\sqrt{\kappa})$ when $N \geq 2$ and $\kappa > 0$. Since the left-hand side of (B.4) is the disjoint union, by taking the volume of the both-hand sides, we see that $\#B$ is bounded from above by the ratio of the right-hand sides of the above two inequalities. Then, by Lemma B.3, we obtain the following:

**Theorem B.2.** Let $(M, g)$ be a connected $N$-dimensional complete Riemannian manifold with $\text{inj}(M) > 0$ satisfying $|\text{sec}(M)| \leq \kappa$ for some $0 \leq \kappa < \infty$ when $N \geq 2$. Then, there exists a number $k_1$ depending only on $N$ such that any ball with radius $\rho < R'$ can be covered by at most $k_1$ balls with radii $\rho/2$, where $R'$ is given by (B.3).

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