MEDRoP: Memory-Efficient Dynamic Robust PCA

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Abstract

Robust PCA (RPCA) is the problem of separating a given data matrix into the sum of a sparse matrix and a low-rank matrix. The column span of the low-rank matrix gives the PCA solution. Dynamic RPCA is the time-varying extension of RPCA. It assumes that the true data vectors lie in a low-dimensional subspace that can change with time, albeit slowly. The goal is to track this changing subspace over time in the presence of sparse outliers. We propose an algorithm that we call Memory-Efficient Dynamic Robust PCA (MEDRoP). This relies on the recently studied recursive projected compressive sensing (ReProCS) framework for solving dynamic RPCA problems, however the actual algorithm is significantly different from, and simpler than, previous ReProCS-based methods. The main contribution of this work is a theoretical guarantee that MEDRoP provably solves dynamic RPCA under weakened versions of standard RPCA assumptions, a mild assumption on slow subspace change, and two simple assumptions (a lower bound on most outlier magnitudes and mutual independence of the true data vectors). Our result is important because (i) it removes the strong assumptions needed by the three previous complete guarantees for ReProCS-based algorithms; (ii) it shows that, it is possible to achieve significantly improved outlier tolerance compared to static RPCA solutions by exploiting slow subspace change and a lower bound on most outlier magnitudes; (iii) it is able to track a changed subspace within a delay that is more than the subspace dimension by only logarithmic factors and thus is near-optimal; and (iv) it studies an algorithm that is online (after initialization), fast, and, memory-efficient (its memory complexity is within logarithmic factors of the optimal).

1 Introduction

According to its modern definition [1], robust PCA (RPCA) is the problem of decomposing a given data matrix into the sum of a low-rank matrix (true data) and a sparse matrix (outliers). The column space of the low-rank matrix then gives the desired principal subspace (PCA solution). In recent years, the RPCA problem has been extensively studied, e.g., [1, 2, 3, 4, 5, 6]. A common application of RPCA is in video analytics in separating video into a slow-changing background image sequence (modeled as a low-rank matrix) and a foreground image sequence consisting of moving objects or people (sparse outliers) [1]. Dynamic RPCA refers to the time-varying extension of RPCA. It assumes that the true data lies in a low-dimensional subspace that can change with time, albeit slowly. The goal is to track this changing subspace over time in the presence of sparse outliers. Hence, this problem can also be referred to as robust subspace tracking.

Problem Statement. At each time $t$, we observe data vectors $y_t \in \mathbb{R}^n$ that satisfy

$$y_t := \ell_t + x_t, \text{ for } t = 1, 2, \ldots, d$$

(1)

where $x_t$ is the sparse outlier vector and $\ell_t$ is the true data vector that lies in a fixed or slowly changing low-dimensional subspace of $\mathbb{R}^n$. To be precise, $\ell_t = P(t) a_t$ where $P(t)$ is an $n \times r$ basis matrix\(^1\) with

\(^1\)tall matrix with mutually orthonormal columns
Also, \( \theta \) works \([7, 8, 9]\), we only used the projection distance and referred to it as the Subspace Error. Since the subspaces \( \hat{\mathbf{r}} \) mate of an of performing a vanilla much better than the static approaches. Thid fact is also backed up by comparisons on real videos shown the video application, this implies that it tolerates slow moving and occasionally static foreground objects.

Contributions. This work and its recent predecessor \([9]\) are the first works to provide performance guarantees for dynamic RPCA that hold under weakened versions of standard RPCA assumptions, slow subspace change, and two mild assumptions (a lower bound on outlier magnitudes and mutual independence of the \( \ell_t \)'s). We say “weakened” because our guarantee implies that the proposed algorithm, which we call MEDRoP (for Memory Efficient Dynamic Robust PCA), can tolerate an order-wise larger fraction of outliers per row than existing RPCA approaches \([2, 3, 4, 5]\), without requiring the outlier support to be uniformly randomly generated \([1]\) or without needing any other model on support change \([11, 8]\). For the video application, this implies that it tolerates slow moving and occasionally static foreground objects much better than the static approaches. Thid fact is also backed up by comparisons on real videos shown in Appendix F. Moreover, the proposed algorithm is provably fast, nearly memory-optimal, online (after initialization), and tracks a subspace change with a near-optimal delay. Its running time is equal to that of performing a vanilla \( r \)-SVD (computing top \( r \) singular vectors) on a data matrix with good eigen-gap.

We define the \( n \times d \) data matrix \( \mathbf{Y} := [\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_d}] = \mathbf{L} + \mathbf{X} \) where \( \mathbf{L}, \mathbf{X} \) are similarly defined. We use \( r_{\text{mat}} \) to denote the rank of \( \mathbf{L} \). If the subspace changes a total of \( J \) times, then in the worst case, \( r_{\text{mat}} = J r \). We use max-outlier-frac-col := \( s/n \) to denote the maximum fraction of outliers in any column of \( \mathbf{Y} \) (equivalently maximum fraction of nonzeros in any column of \( \mathbf{X} \)). We will use max-outlier-frac-row to denote the maximum fraction of outliers in any row of \( \alpha \)-consecutive-column sub-matrices of \( \mathbf{Y} \). We define this precisely later.

Given an initial subspace estimate, \( \hat{\mathbf{P}}_{\text{init}} \), the goal is to quickly and accurately estimate (track) \( \text{span}(\mathbf{P}(t)) \). A by-product of doing this is that the true data vectors \( \ell_t \), the sparse outliers \( x_t \), and their support sets \( T_t \) can also be tracked on-the-fly. Unlike past work \([7, 8, 9]\), the initial subspace estimate, \( \hat{\mathbf{P}}_{\text{init}} \), need not be very accurate. As we will show, it suffices to have an initial estimate that is within a constant projection distance (maximum principal angle) of the true initial subspace. This can be computed by applying only a few iterations of any static (batch) RPCA technique, e.g., PCP \([1]\) or AltProj \([4]\), on the first \( t_{\text{train}} \) data frames, \( \mathbf{Y}_{\text{init}} := \mathbf{Y}_{[1..t_{\text{train}}]} \). Here \([a, b]\) refers to all integers between \( a \) and \( b \), inclusive, \([a, b) := [a, b-1] \), and \( \mathbf{M}_T \) denotes a sub-matrix of \( \mathbf{M} \) formed by columns indexed by entries in the set \( T \).

We use \( n \times r \) basis matrices \( \hat{\mathbf{P}}, \mathbf{P} \), etc to represent the subspaces formed by their column spans. We use the following two metrics – projection distance (sine of the maximum principal angle) and chordal distance \([10]\) – to quantify the distance between subspaces:

\[
\sin \theta_{\max}(\hat{\mathbf{P}}, \mathbf{P}) := \|(\mathbf{I} - \hat{\mathbf{P}}\hat{\mathbf{P}}')\mathbf{P}\|, \quad \text{and} \\
\text{dist}(\hat{\mathbf{P}}, \mathbf{P}) := \left( \sum_{i=1}^{r} \sin^2 \theta_i(\hat{\mathbf{P}}, \mathbf{P}) \right)^{1/2}
\]

where \( \theta_i \) denotes the \( i \)-th largest principal angle, and is computed as \( \sin \theta_i(\hat{\mathbf{P}}, \mathbf{P}) = \sigma_i((\mathbf{I} - \hat{\mathbf{P}}\hat{\mathbf{P}}')\mathbf{P}) \). Also, \( \theta_{\max} := \theta_1 \) denotes the largest one while \( \theta_{\min} := \theta_r \) will be used to denote the smallest (\( r \)-th) one. Since the subspaces \( \hat{\mathbf{P}} \) and \( \mathbf{P} \) are of the same dimension, \( \sin \theta_{\max}(\hat{\mathbf{P}}, \mathbf{P}) = \sin \theta_{\max}(\hat{\mathbf{P}}, \mathbf{P}) \). In our earlier works \([7, 8, 9]\), we only used the projection distance and referred to it as the Subspace Error.

We also study the stable extension of dynamic RPCA where \( \mathbf{y}_t \) satisfies

\[
\mathbf{y}_t := \ell_t + x_t + v_t, \quad \text{for } t = 1, 2, \ldots, d
\]  \( \text{(2)} \)

and \( v_t \) is small bounded corruption/noise and everything else is as above.

Contributions. This work and its recent predecessor \([9]\) are the first works to provide performance guarantees for dynamic RPCA that hold under weakened versions of standard RPCA assumptions, slow subspace change, and two mild assumptions (a lower bound on outlier magnitudes and mutual independence of the \( \ell_t \)'s). We say “weakened” because our guarantee implies that the proposed algorithm, which we call MEDRoP (for Memory Efficient Dynamic Robust PCA), can tolerate an order-wise larger fraction of outliers per row than existing RPCA approaches \([2, 3, 4, 5]\), without requiring the outlier support to be uniformly randomly generated \([1]\) or without needing any other model on support change \([11, 8]\). For the video application, this implies that it tolerates slow moving and occasionally static foreground objects much better than the static approaches. Thid fact is also backed up by comparisons on real videos shown in Appendix F. Moreover, the proposed algorithm is provably fast, nearly memory-optimal, online (after initialization), and tracks a subspace change with a near-optimal delay. Its running time is equal to that of performing a vanilla \( r \)-SVD (computing top \( r \) singular vectors) on a data matrix with good eigen-gap. Also, its memory complexity differs from the minimum value of \( nr \) (memory needed to output an estimate of an \( r \)-dimensional subspace in \( \mathbb{R}^n \)) by only logarithmic factors. Here the term “online” means the
following: after each subspace change, the algorithm updates the subspace estimate every $\alpha = Cr \log n$ frames; and we can prove that the subspace recovery error bound decays exponentially with each such step and an $\epsilon$-accurate estimate is obtained with $K$ steps (see Theorem 2.1).

This work removes all limitations of the older two complete guarantees for ReProCS [11, 8] - all these required a very specific model on outlier support change; an unrealistic model on subspace change; and were much slower and memory-inefficient compared with the current algorithm. We explain these points in detail in Sec. 3. This work also removes a key limitation of [9]. Its guarantee required that exactly one direction change at each subspace change time (although different directions could change at different times). We remove that restriction in this work and instead allow all $r$ directions to change as long as the changes in all directions are of the same order. Because of this more general subspace model, we are also able to further simplify the subspace update step of MEDRoP. These two simple changes have many important implications. (a) The new model on subspace change is more general and realistic. (b) Our result implies that one can track the subspace change in near-optimal time. Even with noise-free data, $y_t = \ell_t$ a delay of $r$ frames will be needed to estimate the subspace. We show that, within a delay of $O(r \log n \log(1/\epsilon))$, which is more than $r$ by only logarithmic factors, it is possible to get an $\epsilon$-accurate estimate the changed subspace, with high probability (whp). Our required lower bound on delay between subspace change times is also the same and hence is also nearly-optimal. For this reason the current algorithm truly provides a robust subspace tracking solution. The older work [9] needed the same delays but for estimating only one (or $O(1)$) changed direction. (c) The approach that we use for tracking a subspace change can also be used to get an $\epsilon$-accurate estimate of the initial subspace starting from only a coarse initialization (computed using a few iterations of PCP or AltProj applied to a short initial dataset). The guarantees of all our earlier works needed the initial subspace estimate to be at least $C\epsilon$ accurate in order to be able to get the same accuracy for the changed subspaces.

Finally, because of (c), an easy corollary of our new approach is a different type of guarantee for an online, fast, and memory-efficient solution to the original RPCA problem. The same ideas can also be used to get guarantees for matrix completion and dynamic matrix completion without assuming uniformly randomly generated missing entries (as done in all other work). The tradeoff will be that the allowed number of missing entries will be lot fewer. For matrix completion it will also be possible to replace the coarse initialization requirement by random initialization. These are some of the future directions that we are investigating in ongoing work.

**Paper Organization.** We explain the main idea of the algorithm and give and discuss the guarantees for it in Sec. 2. We relate this to guarantees from prior work (for both static and dynamic RPCA) in Sec. 3. The actual algorithm is explained in Sec. 4. We give the proof outline for our main result in Sec. 5. The three lemmas stated in this section are proved in Sec. 6. Empirical evaluation is given in Sec. 7 and we conclude in Sec. 8.

## 2 Memory Efficient Dynamic Robust PCA (MEDRoP): Algorithm and Guarantee

In this section we give a basic version of the proposed algorithm, discuss the assumptions, and then state the main guarantee for it.

### 2.1 MEDRoP Algorithm

MEDRoP consists of two basic steps – (a) Projected Compressive Sensing (CS) in order to estimate the sparse outliers, $x_t$’s, and hence the $\ell_t$’s; and (b) Subspace Update to update the subspace estimate after each change. The projected CS step is borrowed from the old ReProCS algorithms [7], while the subspace update step is new and significantly simplified. MEDRoP starts with a “good” estimate of the initial subspace, which can be obtained by applying (a few iterations of) AltProj [4] or of the solver for PCP [1] on $Y_{\text{init}}$. Projected CS proceeds as follows. At time $t$, if the previous subspace estimate, $P_{(t-1)}$, is
accurate enough, because of slow subspace change, projecting \( y_t = x_t + \ell_t \) onto its orthogonal complement will nullify most of \( \ell_t \). We compute \( \hat{y}_t := \Psi y_t \) where \( \Psi := I - \hat{P}_{(t-1)}' \hat{P}_{(t-1)} \). Thus, \( \hat{y}_t = \Psi x_t + \Psi \ell_t \) and \( \| \Psi \ell_t \| \) is small. Recovering \( x_t \) from \( \hat{y}_t \) is thus a traditional CS / sparse recovery problem in small noise [12]. This is solvable because incoherence (denseness) of \( \hat{P}_{(t)} \)'s and slow subspace change implies that \( \Psi \) satisfies the restricted isometry property [7, Lemma 3.7]. We compute \( \hat{x}_{t,cs} \) using noisy \( l_1 \) minimization followed by thresholding based support estimation to get \( \hat{T}_t \). A Least Squares (LS) based debiasing step on \( \hat{T}_t \) returns the final \( \hat{x}_t \). We then estimate \( \ell_t \) as \( \hat{\ell}_t = y_t - \hat{x}_t \).

The \( \hat{\ell}_t \)'s are used for the Subspace Update step which involves (i) detecting subspace change; (ii) obtaining improved subspace estimates by \( K \) steps of SVD, each done with a new set of \( \alpha \) frames of \( \ell_t \). The subspace update step is designed assuming a piecewise constant subspace change model however, as can be seen from our experiments, the algorithm itself works even without this assumption (it works for real videos as well). The subspace change times are denoted by \( t \). To understand subspace update in a simple fashion, in Algorithm 1, we give a basic MEDRoP algorithm that assumes that the \( t \)'s are known. In this case, the \( k \)-th SVD step is done at time \( t_j + k\alpha - 1 \) and involves computing the top \( r \) left singular vectors of the matrix \([ \hat{\ell}_{t_j+(k-1)\alpha}, \hat{\ell}_{t_j+(k-1)\alpha+1}, \ldots, \hat{\ell}_{t_j+k\alpha-1} ]\).

The actual MEDRoP algorithm that automatically detects subspace changes is given and explained later; see Algorithm 2 and Sec. 4. This has five parameters – \( K, \alpha, \xi, \omega_{supp}, \omega_{evals} \). \( K \) and \( \alpha \) are described above; \( \xi \) is the noise bound used by the noisy \( l_1 \)-minimization step; \( \omega_{supp} \) is the support estimation threshold; and \( \omega_{evals} \) is an eigenvalue threshold for detecting subspace change. Algorithm 2 also contains an offline version of MEDRoP that obtains \( \epsilon \)-accurate estimates at all times \( t \). Its main idea is as follows. At \( t = \hat{t}_j + K\alpha \) offline MEDRoP (lines 26 - 30 of Algorithm 2) sets \( \hat{P}_t = [\hat{P}_{j-1}, (I - \hat{P}_{j-1} \hat{P}_{j-1}') \hat{P}_j] \) for all \( t \in [\hat{t}_{j-1} + K\alpha, \hat{t}_j + K\alpha) \). This is then used to get \( \epsilon \)-accurate estimates of \( \hat{x}_t \) and \( \hat{\ell}_t \) for all \( t \) in the above interval.
2.2 Assumptions and Main Result

Subspace change. We assume that the subspace changes every so often and use $t_j$ to denote the $j$-th change time, for $j = 1, 2, \ldots, J$. Define $t_0 := 1$ and $t_{j+1} := d$. Thus, $\ell_t = P_{(t)} a_t$, where $P_{(t)}$ is an $n \times r$ basis matrix with $P_{(t)} = P_j$ for $t \in [t_j, t_{j+1})$. Furthermore, the amount of change at each change time is small: we assume that all the $r$ principal angles of the change are upper bounded by $\Delta/\sqrt{r}$, i.e.,
\[
\sin \theta_{\text{max}}(P_{j-1}, P_j) \leq \Delta/\sqrt{r}.
\] (3)

It is immediate to see that this implies that the chordal distance between the two subspaces satisfies
\[
dist(P_{j-1}, P_j) \leq \Delta.
\]

In addition we also assume that all principal angles, $\sin \theta_i(P_{j-1}, P_j)$ are of the same order. We enforce this by assuming that
\[
\sin^2 \theta_{\min}(P_{j-1}, P_j) \geq 0.3 \sin^2 \theta_{\max}(P_{j-1}, P_j).
\] (4)

(2) The constant 0.3 can be made smaller. The assumption (4) is needed to ensure that a change in subspace can be automatically detected. If the change times, $t_j$, are known, it can be removed. Another point to note is that (4) along with $\dist(P_{j-1}, P_j) \leq \Delta$ implies that $\sin \theta_{\max}(P_{j-1}, P_j) \leq C\Delta/\sqrt{r}$ and thus this is another way our subspace change assumption could be stated.

In [9], we assumed a more restrictive model on subspace change: at each change time, only one direction could change (different directions could change at different change times though), and the sine of the change angle (the only nonzero principal angle) is bounded by $\Delta$. In this case, $\dist(P_{j-1}, P_j) = \sin \theta_{\max}(P_{j-1}, P_j) \leq \Delta$. In the current work, we relax the model significantly to allow all directions to change, but still assume that the “total” amount of change (as quantified by the chordal distance) is at most $\Delta$.

Assumption on the principal subspace coefficients $a_t$: mutual independence over time and element-wise boundedness. We assume that the $a_t$’s are zero mean and mutually independent random variables (r.v.) with diagonal covariance matrix $\Lambda$. We also assume that the $a_t$’s are element-wise bounded, i.e., there exists a numerical constant $\eta$, such that
\[
\max_{i=1,2,\ldots,r} \max_t \frac{(a_t)_i^2}{\lambda_i(\Lambda)} \leq \eta.
\] (5)

For most bounded distributions, $\eta$ is a little more than one, e.g., if the entries of $a_t$ are zero mean uniform, then $\eta = 3$. One can try to relate this assumption to the right incoherence assumption used by the other RPCA solutions [1, 3, 4], however, a rigorous one-to-one mapping is not possible because those works treat $L_j := L_{[t_j, t_{j+1})}$ as a deterministic matrix while we assume that $L_j = P_j A_j := P_j[a_{t_j}, a_{t_j+1}, \ldots a_{t_{j+1}-1}]$ where columns of $A_j$ are zero mean, mutually independent, and element-wise bounded.

Incoherence or denseness of left singular vectors of $L$. In order to separate the $\ell_t$’s from the sparse outliers $x_t$, we need to assume that the $\ell_t$’s are themselves not sparse (thus this property is also
referred to as “denseness”). Recall that $s := \max_t |T_t|$ denotes the maximum outlier support size. We need that

$$\max_{|T| \leq 2s} \|I_T'P_j\|^2 \leq 0.09.$$  \hspace{1cm} (6)

One way to ensure that this holds is by assuming incoherence/denseness of $P_j$’s as also done in all RPCA guarantees [1, 4], and then imposing a bound on max-outlier-frac-col. Incoherence is defined as follows: assume that

$$\max_{j=1,2,...,J} \|I_{T_j}'P_j\| \leq \mu \sqrt{\frac{r}{n}}.$$  \hspace{1cm} (7)

where $\mu$ is a numerical constant that is commonly referred to as the incoherence parameter (since columns of $P_j$ are unit norm, the smallest value $\mu$ can take is one). The LHS of the above inequality is the norm of any row of $P_j$. Clearly, incoherence along with a bound of $0.09/(2\mu^2r)$ on max-outlier-frac-col implies that (6) holds.

Outlier fractions. Similar to earlier RPCA works, we also need outlier fractions to be bounded. However, we need different bounds on this fraction per column and per row. The row bound can be much larger. Let max-outlier-frac-col := $\max_t |T_t|/n$ denote the maximum outlier fraction in any column of $Y$. As noted above, we will bound this by $0.09/(2\mu^2r)$. Because ReProCS is an online approach that updates the subspace estimate every $\alpha$ frames, we need the fraction of outliers per row of a sub-matrix of $Y$ with $\alpha$ consecutive columns to be bounded. To precisely define this, for a time interval, $J$, let

$$\gamma(J) := \max_{i=1,2,...,n} \frac{1}{|J|} \sum_{t \in J} 1_{\{i \in T_t\}}.$$  \hspace{1cm} (8)

where $1_S$ is the indicator function for event $S$. Thus $\gamma(J)$ is the maximum outlier fraction in any row of the sub-matrix $Y_J$ of $Y$. Let $J^\alpha$ denote a time interval of duration $\alpha$. We will bound

$$\max-outlier-frac-row := \max_{J^\alpha \subseteq [t_{\text{train}}, d]} \gamma(J^\alpha).$$  \hspace{1cm} (9)

It is not hard to see that [9, Lemma 6.7] $\gamma(J) = \frac{1}{\alpha} \sum_{t \in J} \|I_{T_t}I_{T_t}'\|$. This is how the bound on max-outlier-frac-row will be used in our proofs.

Main Result. With a few definitions given next, we are ready to state our main result.

1. Use $\lambda^-$ and $\lambda^+$ to denote the minimum and maximum eigenvalues of $\Lambda$ and let $f := \frac{\lambda^+}{\lambda^-}$ be its condition number.

2. Let $x_{\text{min}} := \min_t \min_{i \in T_t} |(x_t)_i|$ denote the minimum outlier magnitude.

3. Let

$$K(\varepsilon) := \left\lceil c_1 \log \left( \frac{c_2 \Delta}{\sqrt{\varepsilon}} \right) \right\rceil \text{ and } \alpha \geq \alpha_* := Cf^2(r \log n).$$  \hspace{1cm} (10)

Theorem 2.1 (MEDRoP). Consider Algorithm 2. Pick an $\varepsilon \leq 0.01 \min_j \sin^2 \theta_{\text{min}}(P_{j-1}, P_j)$. Let $\varepsilon_{\text{dist}} := \sqrt{\varepsilon}$. Assume the following.

1. (Subspace Change) (3) and (4) hold with $t_{j+1} - t_j > (K + 2)\alpha$, $\Delta \leq 0.3\sqrt{r}$ and

$$\left(\varepsilon \sqrt{r} + \Delta\right)\sqrt{n\lambda^+} < x_{\text{min}}/15.$$  \hspace{1cm} (11)

\footnotetext[3]{One practical application where this is useful is for slow moving or occasionally static video foreground moving objects. For a stylized example of this, see Model F.1 given in Appendix F.}
2. (outlier fractions and denseness) (7) holds, max-outlier-frac-col \leq \frac{0.09}{n\epsilon^r}, and max-outlier-frac-row \leq b_0 where b_0 := 0.02/f^2;

3. (statistical assumptions) assumptions on \(\mathbf{a}_t\) holds;

4. (algorithm parameters) set \(K = K(\epsilon)\) and \(\alpha\) as in (10), set \(\xi = x_{\text{min}}/15, \omega_{\text{supp}} = x_{\text{min}}/2, \omega_{\text{evals}} = 2\epsilon^2\lambda^+\);

5. (initialization) Assume that \(t_1 - t_{\text{train}} > K\alpha\) and that \(\hat{\mathbf{P}}_{\text{init}}\) satisfies \(\sin\theta_{\text{max}}(\hat{\mathbf{P}}_{\text{init}}, \mathbf{P}_0) \leq \Delta_{\text{init}}/\sqrt{r}\), with \(\Delta_{\text{init}} \leq 0.3\sqrt{r}\) and \(4\Delta_{\text{init}}n^{\alpha} \leq x_{\text{min}}/15\).

Then, with probability at least \(1 - 10dn^{-10}\), at all times, \(t, \hat{T}_t = T_t\) and the following hold

1. \(t_j \leq \hat{t}_j \leq t_j + 2\alpha\) for all \(j = 1, 2, \ldots, J\);

2. For \(t \in [t_j, \hat{t}_j]\),
   \[
   \sin\theta_{\text{max}}(\mathbf{P}_{(t)}, \hat{\mathbf{P}}_{(t)}) \leq (\epsilon + \Delta/\sqrt{r}) \text{ and } \text{dist}(\mathbf{P}_{(t)}, \hat{\mathbf{P}}_{(t)}) \leq (\epsilon_{\text{dist}} + \Delta)
   \]
   and \(\|\hat{x}_t - x_t\| = \|\hat{\ell}_t - \ell_t\| \leq 1.2(\epsilon + \Delta/\sqrt{r})\|\ell_t\|
   \]

3. For \(t \in [\hat{t}_j + (k - 1)\alpha, \hat{t}_j + k\alpha), k = 1, 2, \ldots, K\),
   \[
   \sin\theta_{\text{max}}(\mathbf{P}_{(t)}, \hat{\mathbf{P}}_{(t)}) \leq (0.3)^{k-1}(\epsilon + \Delta/\sqrt{r}) \text{ and } \text{dist}(\mathbf{P}_{(t)}, \hat{\mathbf{P}}_{(t)}) \leq (0.3)^{k-1}(\epsilon_{\text{dist}} + \Delta)
   \]
   and \(\|\hat{x}_t - x_t\| = \|\hat{\ell}_t - \ell_t\| \leq (0.3)^{k-1} \cdot 1.2(\epsilon + \Delta/\sqrt{r})\|\ell_t\|
   \]

4. For \(t \in [\hat{t}_j + K\alpha, t_{j+1}]\),
   \[
   \sin\theta_{\text{max}}(\mathbf{P}_{(t)}, \hat{\mathbf{P}}_{(t)}) \leq \epsilon \text{ and } \text{dist}(\mathbf{P}_{(t)}, \hat{\mathbf{P}}_{(t)}) \leq \epsilon_{\text{dist}}
   \]
   and \(\|\hat{x}_t - x_t\| = \|\hat{\ell}_t - \ell_t\| \leq \epsilon\|\ell_t\|
   \]

(using the upper bound on \(\hat{t}_j\), the last set of bounds definitely hold for \(t \in [t_j + 2\alpha + K\alpha, t_{j+1}]\)).

**Corollary 2.2** (Offline MEDRoP). Consider lines 26–30 of Algorithm 2. Under the assumptions of Theorem 2.1, at all times \(t\), \(\sin\theta_{\text{max}}(\hat{\mathbf{P}}_{(t)}, \mathbf{P}_{(t)}) \leq \epsilon\), and \(\|x_t - \hat{x}_t\| = \|\ell_t - \ell_t\| \leq 1.2\epsilon\|\ell_t\|\).

The dynamic RPCA problem studied so far assumes sparse outliers but no other small noise or corruption. This is clearly impractical since, in most real datasets, there is always some small noise or corruption. Also, under this idealized model, the lower bound on outlier magnitudes, \(x_{\text{min}}\), imposed by the subspace change assumption is counter-intuitive. At least very small magnitude corruptions should not be problematic. As we see next, this is indeed true. We have the following corollary.

**Corollary 2.3** (Stable Dynamic RPCA). Assume that \(\mathbf{y}_t\)’s satisfy (2) with the \(\mathbf{v}_t\)’s being zero-mean, mutually independent and identically distributed, and independent of \(\{\ell_t, x_t\}\). Let \(\Sigma_v := \mathbb{E}[\mathbf{v}_t\mathbf{v}_t']\). Further assume that \(\|\Sigma_v\| \leq c\epsilon^2\lambda^+\) and \(\max_i \|v_i\|^2 \leq Cr^2\lambda^+\). Then, under the assumptions of Theorem 2.1, all its conclusions hold with \(\epsilon\) replaced by \(2\epsilon\) everywhere.

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4This assumption can be satisfied by applying PCP [3] or AltProj [4] on the first \(t_{\text{train}} = Cr\log n\) data frames and assuming that \(\mathbf{Y}^{[t, t_{\text{train}}]}\) has outlier fractions in any row or column bounded by \(c/r\) (this, along with the other assumptions in the theorem, ensures that guarantees for PCP or AltProj hold). Since the PCP or AltProj guarantees place no explicit lower bound on the minimum number of columns (except the trivial lower bound of \(r\)), we have assumed that \(t_{\text{train}} = Cr\log n\) suffices for the guarantees.
While the above result is almost the same as Theorem 2.1, there is one important difference. It allows the corruptions to either have very small magnitude (this is the $v_t$ term does not significantly affect the subspace estimation) or be sparse and have a large enough magnitude (these are the outliers $x_t$ that are easy to detect).

**Remark 2.4.** Two points should be noted.

1. To keep our guarantee simple, we assume a single lower bound on outlier magnitudes at all times. Actually, as the changed subspace estimate improves, the required lower bound also decreases. If we define $x_{\min,k} := \min_{t \in [t_j + 2\alpha + (k-1)\alpha, t_j + 2\alpha + k\alpha]} \min_{i \in T} |(x_t)_i|$, we only need $x_{\min,k} \geq 0.3^{k-1}15(2\varepsilon / \sqrt{r} + \Delta)\sqrt{\eta \lambda^+}$.

2. In this work, we assume that all directions of the subspace can change. However, in practice, it may happen that only a few directions change. If this is the case, without knowing how many directions change or assuming any specific model on the change, we can in fact get a weaker lower bound on $x_{\min}$ (this bound will match that of s-ReProCS [9]) when only one direction changes. Suppose that only $r_{ch}$ directions change at each $t_j$. Let $P_{j-1}R_j = [P_{j-1,fix}, P_{j-1, chd}]$ where $R_j$ is a rotation matrix. This means that $P_j = [P_{j-1,fix}, P_{j-1, chd}]$ and so $\sin \theta_{\max}(P_{j-1}, P_j) = \sin \theta_{\max}(P_{j-1, chd}, P_{j, chd})$. We assume that $\sin \theta_{\max}(P_{j-1, chd}, P_{j, chd}) \leq \Delta / \sqrt{r_{ch}}$ and that (4) holds with $\theta_{\min}$ replaced by $\theta_{r_{ch}}$. With one simple change to our algorithm - set $P_t$ equal to an orthonormal basis matrix for span$([P_{j-1}, P_{j,k}])$ - we can get conclude that Theorem 2.1 holds with the following relaxed lower bound on $x_{\min}$: $x_{\min} \geq 15\sqrt{\eta}(\varepsilon / \sqrt{r} \lambda^+ + \Delta \sqrt{\lambda^+_{ch}})$. Here $\lambda^+_{ch}$ is the maximum eigenvalue along the changing directions (this will typically be much smaller than $\lambda^+$).

### 2.3 Discussion

**Subspace and outlier assumptions’ tradeoff.** When there are fewer outliers in the data or when outliers are easy to detect, one would expect to need weaker assumptions on the true data subspace or its rate of change. This is indeed true. For the static RPCA results, this is encoded in the condition $\text{max-outlier-frac} \leq c / (\mu^2 r)$ where $\mu$ quantifies not-denseness of both left and right singular vectors. Thus, when fewer outliers are present, the subspace dimension, $r$, can be larger and $\mu$ can be larger (the singular vectors can be less dense). From Theorem 2.1, this is also how max-outlier-frac-col, $\mu$ and $r$ are related for dynamic RPCA.

For our result, max-outlier-frac-row and the lower bound on $x_{\min}$ govern the allowed rate of subspace change which is measured by larger $\Delta$ and smaller lower bound on $(t_{j+1} - t_j)$. The latter relation is easily evident from the bound on $\Delta$. If $x_{\min}$ is larger (outliers are easier to detect), a larger $\Delta$ (faster changes) can be tolerated. The relation of max-outlier-frac-row to rate of change is not evident from the way the guarantee is stated in Theorem 2.1. The reason is we have assumed max-outlier-frac-row $\leq b_0 / f^2$ with $b_0 = 0.02$ (fixed numerical constant) and used this to get a simple expression for $K$. If we did not use this simplification, we would need (see Remark 6.11)

$$K = \left[ \frac{1}{\log(c_2 \sqrt{v_0})} \log \left( \frac{\Delta}{\varepsilon / \sqrt{r}} \right) \right].$$

Recall that we need $t_{j+1} - t_j \geq (K + 2)\alpha$. Thus, a smaller $b_0$ (tighter lower bound on max-outlier-frac-row) means one of two things: either a larger $\Delta$ (more change at each subspace change time) can be tolerated

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$^5$Changes to proof: In proof of the CS step, with $\hat{P}_t(t)$ set as above, it should be possible to show that $\|\Psi T_t\| \leq \varepsilon / \sqrt{r - r_{ch} \lambda^+} + 0.3^{k-1}(\varepsilon / \sqrt{r_{ch}})\sqrt{r_{ch} \lambda^+_{ch}}$. In the subspace update step, there will be no change: one can always ignore $\hat{P}_t$ term (since it is of no use for that part). In the subspace detection lemma, the decomposition of the term $T_t$ will be more involved: the zeros in (16) will get replaced by nonzero values and the approach of [9] for proving reliable subspace change detection will be used to show that those terms are small.
while keeping \( K \), and hence the lower bound on delay between change times, the same; or, for \( \Delta \) fixed, a smaller lower bound on delay between change times is needed.

**Memory and time complexity.** Observe that the MEDRoP algorithm needs memory of order \( n\alpha \) in online mode and order \( K\alpha \) in offline mode. This is true even for initialization because \( t_{\text{train}} \) can be assumed to be the same as \( \alpha \) (since none of PCP, AltProj see [3, 4] require a lower bound on the matrix size). Setting \( \alpha = \alpha_s \) (its lower bound) and assuming \( f \) does not grow with \( n \), even in offline mode, the memory complexity is only \( O(nr \log n \log(1/\varepsilon)) \). This differs from the the lower bound of \( (nr) \) (space needed to store the output subspace estimate) by only log factors and hence is nearly optimal. The time complexity of our algorithm is \( O(ndr \log(1/\varepsilon)) \), which is equal to the cost of computing a vanilla rank \( r \)-SVD on a matrix of dimensions \( n \times d \). The detailed derivation is given in Appendix D. The above discussion assumes that the condition number \( f \) is constant (does not grow with \( n \)).

**Algorithm parameters.** Observe from Theorem 2.1 that we need knowledge of 4 model parameters \( r, \lambda^+, \lambda^- \) and \( x_{\min} \) - to set our algorithm parameters. The initial dataset used for estimating \( \hat{P}_{\text{init}} \) can be used to get an accurate estimate of \( r, \lambda^- \) and \( \lambda^+ \) using standard techniques. Thus one really only needs to set \( x_{\min} \). If continuity over time is assumed, a simple heuristic is to let it be time-varying and use \( \min_{i \in \hat{t}_{t-1}} |(\hat{x}_{t-1})_i| \) as its estimate at time \( t \).

### 3 Discussion of Prior Work

The first guarantee for dynamic RPCA given in [7] was a partial result: it imposed a condition on intermediate algorithm outputs. However, this work was important because it developed a nice framework for proving guarantees for dynamic RPCA solutions. All the later complete guarantees [11, 8, 9] as well as our current result build on this framework. In other work [13] by Feng et al., an online solver for PCP (called ORPCA) was developed and a partial guarantee was obtained for it: it required that the intermediate algorithm estimates be full rank. Moreover, the guarantee was only asymptotic.

Complete guarantees for dynamic RPCA first appeared in [11, 8]. However these needed the following very strong assumptions. (1) These needed a very specific model on outlier support change: the required model was inspired by an object that moves every so often. (2) These needed an impractical model of subspace change, e.g., in 3D, the model required that the subspace change from the x-y plane to the y-z plane; and these also needed upper bounds on the eigenvalues along the newly added direction for short periods after the change. All of these are restrictive and impractical requirements. (3) Furthermore, [8] required the eigenvalues of the covariance matrix of \( \ell_i \) to be clustered for certain periods of time while for the algorithm of [11], the subspace dimension could only keep increasing over time. (4) Finally, the algorithms studied in both these works had time and memory complexity that was much worse than MEDRoP: the reason was that the parameter \( \alpha \) depended on \( 1/\varepsilon^2 \) instead of on \( \log(1/\varepsilon) \) now.

Recently, in [9], we studied a ReProCS-based algorithm that we called simple-ReProCS (s-ReProCS). Its guarantee removed most of the above strong assumptions, however it needed that only one (or at most \( O(1) \)) direction of the subspace can change at each change time. For each subspace update, s-ReProCS consists of \( K \) iterations of projection-SVD to get an accurate estimate of the changed direction followed by an SVD based subspace deletion step. An easy corollary of the s-ReProCS result also analyzes a version of s-ReProCS that eliminates the SVD based deletion step (we can call this s-ReProCS-no-del). In Table 1, we compare guarantees for s-ReProCS and s-ReProCS-no-del with those for MEDRoP given here. MEDRoP and its guarantee allows all subspace directions to change at each change time. This simple fact has many useful advantages as mentioned earlier - the required delay to get an \( \varepsilon \)-accurate estimate of the changed subspace is nearly optimal (the minimum would be \( r \)) and the same is true for the required delay between change times. However, if we include the condition number \( f \) in our bounds, then, most of the time, s-ReProCS needs a weaker bound on outlier fractions per row. It also needs a weaker bound on \( \Delta \).

In terms of other solutions for provable dynamic RPCA, there is very little work. An approach called modified-PCP was proposed in earlier work to solve the problem of RPCA with partial subspace
knowledge. A corollary of its guarantee shows that it can also be used to solve dynamic RPCA [14]. Its advantage is that it does not need the outlier magnitude lower bound. However, for subspace change, it also only allows a few directions to change (not all). Moreover, it borrows most of the disadvantages of the PCP guarantee by [1] (henceforth called PCP(C)), e.g., it also needs uniformly randomly generated outlier support.

Other somewhat related work includes a streaming solution for the original (static) RPCA problem developed in very recent work [15] with a guarantee for only \( r_{\text{mat}} = r = 1 \) (one-dimensional RPCA).

**Brief Discussion of Static RPCA.** The first provable solution for static RPCA was the Principal Components Pursuit (PCP) convex program studied in [1, 2] and later in [3]. This had simple and nice performance guarantees as the best deterministic guarantee for PCP from [3], but are provably much faster than it. Of these, PG-RMC is the fastest, requiring \( O(\log(1/\epsilon)) \) time. However, since it is a solution for robust matrix completion (and uses deliberate undersampling of the complete data matrix to speed up robust PCA), it cannot recover the sparse outlier matrix \( X \). Also, it requires that \( d \) be of the same order as \( n \) which is a stringent requirement. PCP, AltProj and RPCA-GD have no such requirement, while ReProCS only needs \( d \geq Cr_{\text{mat}} \log n \log(1/\epsilon) \).

In terms of speed, the two best methods without the restriction \( d \approx n \) are RPCA-GD and AltProj. Assuming constant condition numbers, RPCA-GD has a run time of \( O(ndr_{\text{mat}} \log(1/\epsilon)) \) while requiring max-outlier-frac-row \( \in O(1/r_{\text{mat}}) \), while AltProj has a running time of \( O(ndr_{\text{mat}}^2 \log(1/\epsilon)) \) while requiring max-outlier-frac-row \( \in O(1/r_{\text{mat}}) \). In Table 1 we compare the result of MEDRoP with that of RPCA-GD and of s-reprocs. We use GD to represent the static RPCA literature because it is the fastest static RPCA solution that does not require that \( d \) be of the same order as \( n \) and that works for any value of \( r_{\text{mat}} \) (not just \( r_{\text{mat}} = 1 \) as for [15]).

To summarize, because MEDRoP uses extra assumptions (slow subspace change, lower bound on outlier magnitudes, and mutual independence of \( \ell_t \)'s), it can tolerate many more outliers per row compared with all static RPCA solutions discussed above. It is also online (after initialization), significantly more memory-efficient and, in fact, we show that its memory complexity is near optimal. In terms of speed, it is slower than only PG-RMC.

### 4 Automatic MEDRoP

We present the actual MEDRoP algorithm (automatic MEDRoP) in Algorithm 2. The main idea why automatic MEDRoP works is the same as that of the basic algorithm with the exception of the additional subspace detection step. The subspace detection idea is borrowed from [9], although its correctness proof has differences because we assume a much simpler subspace change model. In Algorithm 2, the subspace update stage toggles between the “detect” phase and the “update” phase. It starts in the “detect” phase. If the \( j \)-th subspace change is detected at time \( t \), we set \( \hat{t}_j = t \). At this time, the algorithm enters the “update” (subspace update) phase. We then perform \( K \) SVD steps with the following change: the \( k \)-th SVD step is now done at \( t = \hat{t}_j + k\alpha - 1 \) (instead of at \( t = t_j + k\alpha - 1 \)). Thus, at \( t = \hat{t}_j, fin = \hat{t}_j + K\alpha - 1 \), the subspace update is complete. At this time, the algorithm enters the “detect” phase again.

To understand the change detection strategy, consider the \( j \)-th subspace change. Assume that the previous subspace \( P_{j-1} \) has been accurately estimated by \( t = \hat{t}_{j-1}, fin = \hat{t}_{j-1} + K\alpha \) and that \( \hat{t}_{j-1}, fin < t_j \). Let \( \hat{P}_{j-1} \) denote this estimate. At this time, the algorithm enters the “detect” phase in order to detect the
Table 1: Comparison of assumptions and time taken: MEDRoP (this work), s-ReProCS [9] and RPCA-GD (best static RPCA solution) [5]. \( \kappa \) denotes condition number of \( L \) while \( f \) denotes condition number of \( E[L_j,L'_j] \) with \( L_j := L_{[t_j,t_{j+1}]} \). With our model the condition number of \( E[L,L'] \) will be much larger, in the worst case it will be \( f/(\min_j \sin^2(\theta_{\min}(P_{j-1},P_j))) \).

| Assumption                  | RPCA-GD (static RPCA sol) | s-ReProCS (no-del) | s-ReProCS (no-del) | MEDRoP (this work) |
|-----------------------------|---------------------------|--------------------|--------------------|-------------------|
| max-outlier-frac-row        | \( O(1/r_{mat}^{1.5}) \) | \( O(1) \) [most times*] | \( O(1) \)          | \( O(1/f^2) \)   |
| max-outlier-frac-col        | \( O(1/r_{mat}^{1.5}) \) | \( O(1/r) \)      | \( O(1/r_{mat}) \) | \( O(1/r) \)    |
| subspace change             | None                      | Strong             | Strong             | Weak              |
| amount of change, \( \Delta \) | None                      | \( \leq \frac{x_{min}}{15 \sqrt{q_{ch}}} \) | \( \leq \frac{x_{min}}{15 \sqrt{q_{ch}}} \) | \( \min \left( 0.3 \sqrt{r}, \frac{x_{min}}{15 \sqrt{q_{ch}}} \right) \) |
| time complexity             | \( O(\kappa ndr_{mat} \log(1/\varepsilon)) \) | \( O(f^2 ndr \log(1/\varepsilon)) \) | \( O(f^2 ndr \log(1/\varepsilon)) \) | \( O(f^2 ndr \log(1/\varepsilon)) \) |
| memory complexity           | \( O(nd) \)             | \( O(f^2 nr \log n \log(1/\varepsilon)) \) | \( O(f^2 nr \log n \log(1/\varepsilon)) \) | \( O(f^2 nr \log n \log(1/\varepsilon)) \) |

*This is \( O(1) \) at most times. For an \( \alpha \)-length time interval for every subspace change after the Projection-SVD steps are completed max-outlier-frac-row needs to be bounded by \( O(1/f^2) \). See [9, Section IV] for more details.

next \((j\text{-th})\) change. Let \( B_t := (I - \hat{P}_{j-1}\hat{P}'_{j-1})[\hat{e}_j - \alpha + 1, \ldots, \hat{e}_j] \). For every \( t = \hat{t}_{j-1..fin} + u\alpha, u = 1, 2, \ldots, \) we detect change by checking if the maximum singular value of \( B_t \) is above a pre-set threshold, \( \sqrt{\omega_{evals} \alpha} \), or not.

We claim that, whp, under assumptions of Theorem 2.1, this strategy has no “false subspace detections” and correctly detects change within a delay of at most 2\( \alpha \) frames. The former is true because, for any \( t \) for which \( [t - \alpha + 1, t] \subseteq [\hat{t}_{j-1..fin}, \hat{t}_j] \), all singular values of the matrix \( B_t \) will be close to zero (will be of order \( \varepsilon \sqrt{\lambda^+} \)) and hence its maximum singular value will be below \( \sqrt{\omega_{evals} \alpha} \). Thus, whp, \( \hat{t}_j \geq t_j \). To understand why the change is correctly detected within 2\( \alpha \) frames, first consider \( t = \hat{t}_{j-1..fin} + \lceil \frac{t_j - t_{j-1..fin}}{\alpha} \rceil \alpha = t_j^- \). Since we assumed that \( \hat{t}_{j-1..fin} < t_j \) (the previous subspace update is complete before the next change), \( t_j \) lie in the interval \([t_j^- - \alpha + 1, t_j^-] \). Thus, not all of the \( \hat{e}_t \)’s in this interval lie in the new subspace. Depending on where in the interval \( t_j \) lies, the algorithm may or may not detect the change at this time. However, in the next interval, i.e., for \( t \in [t_j^- + 1, t_j^- + \alpha] \), all of the \( \hat{e}_t \)’s lie in the new subspace. We can prove that \( B_t \) for this time \( t \) will have maximum singular value that is above the threshold. This proof is where the assumption of equal angles, (4) is needed. Thus, if the change is not detected at \( t_j^- \), whp, it will get detected at \( t_j^- + \alpha \). Hence one can show that, w.h.p., either \( \hat{t}_j = t_j^- \), or \( \hat{t}_j = t_j^- + \alpha \), i.e., \( t_j \leq \hat{t}_j \leq t_j + 2\alpha \). See the proof of our Theorem given in the Appendix.

5 Proof Outline of Theorem 2.1

The key results from other work that we use to prove these lemmas are summarized in Appendix A.

**Remark 5.5.** It is easy to see that (11) implies \( x_{min}/15 \geq (\varepsilon + \sin \theta_{\max}(P_{j-1},P_j)) \sqrt{\eta r \lambda^+} \). This follows using (3).

5.1 Algorithm 1: \( t_j \) known case

First we outline the proof for the case when \( t_j \)’s are known, i.e., we show correctness for Algorithm 1. This sets \( \hat{t}_j = t_j \).

**Definition 5.6.** We will use the following definitions in our proof.

1. Let \( q_0 := 1.2(\varepsilon + \sin \theta_{\max}(P_{j-1},P_j)) \), \( q_k = (0.3)^k q_0 \)
Algorithm 2 Automatic-MEDRoP. Let $\hat{t}_{t,\alpha} := [\hat{t}_{t-\alpha+1}, \ldots, \hat{t}_t]$. SVD$_r[M]$ computes the top of $r$ left singular vectors of the matrix $M$.

1: **Input:** $\hat{P}_0$, $y_t$; **Output:** $\hat{x}_t$, $\hat{t}_t$, $\hat{P}(t)$
2: **Params:** $\omega_{supp}$, $K$, $\alpha$, $\xi$, $r$, $\omega_{evals}$
3: $\hat{P}_{(t_{train})} \leftarrow \hat{P}_0$; $j \leftarrow 1$, $k \leftarrow 1$
4: phase $\leftarrow$ detect; $\hat{t}_{0,fin} \leftarrow t_{train}$;
5: **for** $t > t_{train}$ **do**
6: Lines 5 - 10 of Algorithm 1
7: **if** phase = detect and $t = \hat{t}_{j-1,fin} + u\alpha$ **then**
8: $\Phi \leftarrow (I - \hat{P}_{j-1}\hat{P}_{j-1}').$
9: $B \leftarrow \Phi \hat{L}_{t,\alpha}$
10: **if** $\lambda_{\max}(BB') \geq \alpha\omega_{evals}$ **then**
11: phase $\leftarrow$ update, $\hat{t}_j \leftarrow t$,
12: $\hat{t}_{j,fin} \leftarrow \hat{t}_j + K\alpha - 1$.
13: **end if**
14: **end if**
15: **if** phase = update **then**
16: **if** $t = \hat{t}_j + u\alpha$ for $u = 1, 2, \ldots$, **then**
17: $\hat{P}_{j,k} \leftarrow \text{SVD}_r[\hat{L}_{t,\alpha}], \hat{P}(t) \leftarrow \hat{P}_{j,k}$, $k \leftarrow k + 1$.  
18: **else**
19: $\hat{P}(t) \leftarrow \hat{P}(t-1)$
20: **end if**
21: **if** $t = \hat{t}_j + K\alpha$ **then**
22: $\hat{P}_j \leftarrow \hat{P}(t)$, $k \leftarrow 1$, $j \leftarrow j + 1$, phase $\leftarrow$ detect.
23: **end if**
24: **end if**
25: **end for**
26: **Offline MEDRoP:** At $t = \hat{t}_j + K\alpha$, for all $t \in [\hat{t}_{j-1} + K\alpha, \hat{t}_j + K\alpha - 1]$, 
27: $\hat{P}^{\text{offline}}(t) \leftarrow [\hat{P}_{j-1}, (I - \hat{P}_{j-1}\hat{P}_{j-1}')\hat{P}_j]$ 
28: $\Psi \leftarrow I - \hat{P}^{\text{offline}}(t)\hat{P}^{\text{offline}}(t)$
29: $\hat{x}_{t}^{\text{offline}} \leftarrow I_{T_t}(\Psi_{T_t}/\Psi_{T_t})^{-1}\Psi_{T_t}y_t$
30: $\hat{t}_t^{\text{offline}} \leftarrow y_t - \hat{x}_t^{\text{offline}}$.

2. **Events:** $\Gamma_{0,0} := \{\sin \theta_{\max}(\hat{P}_{\text{init}}, P_0) \leq \Delta_{\text{init}}/\sqrt{r}\}$, $\Gamma_{0,k} := \Gamma_{0,k-1} \cap \{\sin \theta_{\max}(\hat{P}_{0,k}, P_0) \leq 0.3^k\Delta_{\text{init}}/\sqrt{r}\}$, $\Gamma_{j,k} := \Gamma_{j,k-1} \cap \{\sin \theta_{\max}(\hat{P}_{j,k}, P_j) \leq q_k/4\}$ for $k = 1, 2, \ldots, K$ and $\Gamma_{j+1,0} := \Gamma_{j,K}$ for all $j \in \{0, 1, \ldots, J\}$.

3. **Note:** using the expression for $K$ given in (10), it follows that $\Gamma_{j,K}$ implies $\sin \theta_{\max}(\hat{P}_{j,K}, P_J) \leq \varepsilon$.

Observe that if we can show that $\Pr(\Gamma_{j,K}|\Gamma_{0,0}) \leq 1 - dn^{-10}$ we will be mostly done. The next two lemmas applied sequentially will help show that this is indeed true for Algorithm 1 ($t_j$ known). The proof of correctness of the actual algorithm (Algorithm 2) follows using the next two lemmas and Lemma 5.10.

**Lemma 5.7** (first Subspace Update). Under the conditions of Theorem 2.1, conditioned on $\Gamma_{j,0}$

1. for all $t \in [\hat{t}_j, \hat{t}_j + \alpha)$, the error $e_t = \hat{x}_t - x_t = \ell_t - \hat{\ell}_t$ satisfies

$$e_t = I_{T_t}(\Psi_{T_t}/\Psi_{T_t})^{-1}I_{T_t}/\Psi_{T_t}\ell_t,$$

and $\|e_t\| \leq 1.2(\varepsilon_{\text{dist}} + \Delta)\sqrt{\eta/\lambda_r}$. (12)
2. with probability at least $1 - 10n^{-10}$ the subspace estimate $\hat{P}_{j,1}$ satisfies $\sin \theta(\hat{P}_{j,1}, P_j) \leq (q_0/4)$, i.e., $\Gamma_{j,1}$ holds.

Lemma 5.8 ($k$-th Subspace Update). Under the conditions of Theorem 2.1, conditioned on $\Gamma_{j,k-1}$

1. for all $t \in [\hat{t}_j + (k-1)\alpha, \hat{t}_j + k\alpha - 1]$, the error $e_t = x_t - \hat{x}_t = \ell_t - \hat{\ell}_t$ satisfies (12) and for this interval, $\|e_t\| \leq (0.3)^{k-1} \cdot 1.2(\varepsilon_{dist} + \Delta)\sqrt{\eta\lambda^+}$.

2. with probability at least $1 - 10n^{-10}$ the subspace estimate $\hat{P}_{j,k}$ satisfies $\sin \theta(\hat{P}_{j,k}, P_j) \leq (q_{k-1}/4)$, i.e., $\Gamma_{j,k}$ holds.

Remark 5.9. For the case of $j = 0$, in both the lemmas above, $\Delta$ gets replaced with $\Delta_{init}$ and $\varepsilon$ by 0.

We prove these lemmas in Sec. 6. The proof relies on the following two results from earlier works: (i) The work of [7] relates $\delta_s(A)$: the order $s$-Restricted Isometry Constant (RIC) of $A$ as defined in [12] to the incoherence of the projection matrices as

$$\delta_s(I - PP') = \max_{|T| \leq s} \|I_T'P\|^2 \leq \mu \cdot r \cdot \max-outlier-frac-col;$$

(ii) The subspace update step proof uses the finite sample guarantee for PCA in sparse-data dependent noise from [16].

5.2 Algorithm 2: the actual $t_j$ unknown case

For the case when $t_j$’s are not known, the following lemma is used to show that when $\omega_{evals}$ is set according to Theorem 2.1, whp, we detect the subspace change within $2\alpha$ frames, i.e., for all $j$, $t_j \leq \hat{t}_j \leq t_j + 2\alpha$.

Lemma 5.10 (Subspace Change Detection). Consider an $\alpha$-length time interval $J^\alpha \subset [t_j, t_{j+1}]$. During this interval, $\ell_t = P_ja_t$.

1. If $\Phi := I - \hat{P}_{j-1}\hat{P}_{j-1}'$ and $\sin \theta(\hat{P}_{j-1}, P_j) \leq \varepsilon$, with probability at least $1 - 10n^{-10}$,

$$\lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \hat{\ell}_t \hat{\ell}_t' \Phi \right) \geq \lambda^+ [0.99 \sin^2 \theta_{\min} (P_{j-1}, P_j) - 0.3 \sin^2 \theta_{\max} (P_{j-1}, P_j) - 1.98\varepsilon] > \omega_{evals}$$

2. If $\Phi := I - \hat{P}_j\hat{P}_j'$ and $\sin \theta(\hat{P}_j, P_j) \leq \varepsilon$, with probability at least $1 - 10n^{-10}$,

$$\lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \hat{\ell}_t \hat{\ell}_t' \Phi \right) \leq 1.37\varepsilon^2\lambda^+ < \omega_{evals}$$

We prove this lemma in Sec. 6. The actual proof of Theorem 2.1 is an easy consequence of the above three lemmas. This is given in Appendix B.

6 Proof of Main Lemmas

In this section we prove Lemmas 5.7 5.8 and 5.10. Recall that $\varepsilon_{dist} = \varepsilon\sqrt{T}$.

Proof of Lemma 5.7.

Proof of item 1. The event $\Gamma_{j,0} := \Gamma_{j-1,\alpha}$ implies that $\sin \theta_{\max}(\hat{P}_{j-1}, P_{j-1}) \leq \varepsilon$. 

For the sparse recovery step, we wish to compute the 2s-RIP for the matrix $\Psi = I - \hat{P}_{j-1} \hat{P}_{j-1}'$. To do this, we first obtain bound on $\max_{|T| \leq 2s} \| I_T' \hat{P}_{j-1} \|$ as follows. Consider any set $T$ such that $|T| \leq 2s$. Then,

$$\| I_T' \hat{P}_{j-1} \| \leq \| I_T'(I - P_{j-1} P_{j-1}') \hat{P}_{j-1} \| + \| I_T' P_{j-1} P_{j-1}' \hat{P}_{j-1} \|$$

$$\leq \sin \theta_{\text{max}}(P_{j-1}, \hat{P}_{j-1}) + \| I_T' P_{j-1} \| = \sin \theta_{\text{max}}(P_{j-1}, \hat{P}_{j-1}) + \| I_T' P_{j-1} \|$$

Using the definition of $\mu$, item 1 of Lemma A.3 and the bound on max-outlier-frac-col (assumption 2 of Theorem 2.1),

$$\max_{|T| \leq 2s} \| I_T' P_{j} \|^2 \leq 2s \max_i \| I_i' P_{j} \|^2 \leq \frac{2s\mu r}{n} \leq 0.09 \quad (13)$$

Thus, using $\sin \theta_{\text{max}}(\hat{P}_{j-1}, P_{j-1}) \leq \epsilon$,

$$\max_{|T| \leq 2s} \| I_T' \hat{P}_{j-1} \| \leq \epsilon + \max_{|T| \leq 2s} \| I_T' P_{j-1} \| \leq \epsilon + 0.3$$

Finally, from Lemma A.3, it follows that $\delta_{2s}(\Psi_{j}) \leq 0.31^2 < 0.15$. Hence$^6$,

$$\left\| \left( \Psi_{\hat{T}}' \Psi_{\hat{T}} \right)^{-1} \right\| \leq \frac{1}{1 - \delta_{2s}(\Psi)} \leq \frac{1}{1 - 0.15} < 1.2 = \phi^+.$$ 

This gives

$$\| \Psi \ell_t \| = \left\| (I - \hat{P}_{j-1} \hat{P}_{j-1}') P_{j} a_t \right\| \leq \sin \theta_{\text{max}}(\hat{P}_{j-1}, P_{j}) \| a_t \|

\quad \leq (\epsilon + \sin \theta_{\text{max}}(P_{j-1}, P_{j})) \sqrt{\eta r \lambda^+} \leq (\epsilon_{\text{dist}} + \Delta) \sqrt{\eta \lambda^+} := b_b$$

where $(a)$ follows from Lemma A.1 with $Q_1 = \hat{P}_{j-1}$, $Q_2 = P_{j-1}$ and $Q_3 = P_{j}$, and $(b)$ follows from Remark 5.5. Under the assumptions of Theorem 2.1, $b_b < x_{\min}/15$. So, we can set $\xi = x_{\min}/15$. Using these facts, and $\delta_{2s}(\Psi) \leq 0.12 < 0.15$, [12, Theorem 1.2] implies that

$$\| \hat{x}_{t,cs} - x_t \| \leq 7\xi = 7x_{\min}/15$$

Thus,

$$\| (\hat{x}_{t,cs} - x_t)_i \| \leq \| \hat{x}_{t,cs} - x_t \| \leq 7x_{\min}/15 < x_{\min}/2$$

The Theorem sets $\omega_{\text{supp}} = x_{\min}/2$. Consider an index $i \in T_t$. Since $|(x_t)_i| \geq x_{\min}$,

$$x_{\min} - |(\hat{x}_{t,cs})_i| \leq |(x_t)_i| - |(\hat{x}_{t,cs})_i| \leq |(x_t - \hat{x}_{t,cs})_i| < \frac{x_{\min}}{2}$$

Thus, $|(\hat{x}_{t,cs})_i| > \frac{x_{\min}}{2} = \omega_{\text{supp}}$ which means $i \in \hat{T}_t$. Hence $T_t \subseteq \hat{T}_t$. Next, consider any $j \notin T_t$. Then, $(x_t)_j = 0$ and so

$$| (\hat{x}_{t,cs})_j | = | (x_{t,cs})_j | - | (x_t)_j | \leq | (x_{t,cs})_j | - (x_t)_j | \leq b_b < \frac{x_{\min}}{2}$$

which implies $j \notin \hat{T}_t$ and $\hat{T}_t \subseteq T_t$ implying that $\hat{T}_t = T_t$.

$^6$For the 1st subspace change, i.e., for $j = 1$, we similarly have $\sin \theta_{\text{max}}(P_{\text{init}}, P_0) \leq \Delta_{\text{init}}/\sqrt{n} \leq \Delta_{\text{init}} = 0.05$ which implies $\max_{|T| \leq 2s} \leq 0.35$ and thus using Lemma A.3 it follows that $\delta_{2s}(\Psi_0) \leq 0.35^2 < 0.15$. 

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With $\hat{T}_t = T_t$ and since $T_t$ is the support of $x_t$, $x_t = I_{\hat{T}_t}I_{\hat{T}_t}'x_t$, and so

$$\hat{x}_t = I_{\hat{T}_t} (\Psi_{\hat{T}_t}'\Psi_{\hat{T}_t})^{-1} \Psi_{\hat{T}_t}'(\Psi_{\ell_t} + \Psi x_t) = I_{\hat{T}_t} (\Psi_{\hat{T}_t}'\Psi_{\hat{T}_t})^{-1} I_{\hat{T}_t}'\Psi_{\ell_t} + x_t$$

since $\Psi_{\hat{T}_t}'\Psi = I_{\hat{T}_t}'\Psi'\Psi = I_{\hat{T}_t}'\Psi$. Thus $e_t = \hat{x}_t - x_t$ satisfies

$$e_t = I_{\hat{T}_t} (\Psi_{\hat{T}_t}'\Psi_{\hat{T}_t})^{-1} I_{\hat{T}_t}'\Psi_{\ell_t}$$

and

$$\|e_t\| \leq \left( \|\Psi_{\hat{T}_t}'\Psi_{\hat{T}_t}\|^{-1} \right) \|I_{\hat{T}_t}'\Psi_{\ell_t}\| \leq \phi^+ \|I_{\hat{T}_t}'\Psi_{\ell_t}\| \leq 1.2b_b$$

**Proof of Item 2:** Since $\hat{\ell}_t = \ell_t - e_t$ with $e_t$ satisfying the above equation, updating $\hat{P}(t)$ from the $\hat{\ell}_t$’s is a problem of PCA in sparse data-dependent noise (SDDN), $e_t$. To analyze this, we use the PCA-SDDN result of Theorem A.2 (this is taken from [16]). Recall from above that, for $t \in [\hat{t}_j, \hat{t}_j + \alpha]$, $\hat{T}_t = T_t$, and $\hat{\ell}_t = \ell_t - e_t$. Recall from the algorithm that we compute the first estimate of the $j$-th subspace, $\hat{P}_{j,1}$, as the top $r$ eigenvectors of $\frac{1}{\alpha} \sum_{t=t_j}^{t_j+\alpha} \hat{\ell}_t\hat{\ell}_t'$. In the notation of Theorem A.2, $y_t \equiv \hat{\ell}_t$, $w_t \equiv e_t$, $\ell_t \equiv \ell_t$, and $M_{s,t} = -(\Psi_{\hat{T}_t}'\Psi_{\hat{T}_t})^{-1} \Psi_{\hat{T}_t}$, and so $\|M_{s,t}P\| = \|\Psi_{\hat{T}_t}'\Psi_{\hat{T}_t}\|^{-1} \Psi_{\hat{T}_t}'P = \|P\| \leq \phi^+(\varepsilon + \sin\theta_{\max}(P_{j-1,1}, P_{j})) := q_0$. Also, using [9, Lemma 6.7], we have $b_0 \equiv \text{max-outlier-frac-row}$. Applying the PCA-SDDN result with $q \equiv q_0$, $b_0 \equiv \text{max-outlier-frac-row}$ and setting $\varepsilon_{\text{SE}} = q_0/4$, observe that we require

$$\sqrt{b_0}q_0 f \leq 0.9(q_0/4) \frac{1}{1 + (q_0/4)}.$$ 

This holds if $\sqrt{b_0}f \leq 0.12$ as provided by Theorem 2.1. Thus, from Corollary A.2, with probability at least $1 - 10n^{-10}$, $\sin\theta_{\max}(\hat{P}_{j,1,1}, P_{j}) \leq q_0/4$. \hfill \Box

**Proof of Lemma 5.8.** The proof of this lemma has many important differences with respect to Lemma 5.7. We first present the proof for $k = 2$ case and subsequently generalize it for an arbitrary $k$-th SVD step.

**(A) $k = 2$**

**Proof of Item 1:** The event $\Gamma_{j,1}$ implies that $\sin\theta_{\max}(\hat{P}_{j,1,1}, P_{j}) \leq q_0/4$.

For the sparse recovery step, we need to bound the 2s-RIC for the matrix $\Psi = I - \hat{P}_{j,1}\hat{P}_{j,1}'$. Consider any set $T$ such that $|T| \leq 2s$. Then,

$$\|I_T'\hat{P}_{j,1}\| \leq \|I_T'(I - P_jP_j')\hat{P}_{j,1}\| + \|I_T'P_jP_j'\hat{P}_{j,1}\| \leq \sin\theta_{\max}(P_j, \hat{P}_{j,1}) + \|I_T'P_j\| = \sin\theta_{\max}(\hat{P}_{j,1,1}, P_{j}) + \|I_T'P_j\|$$

The equality follows because $\sin\theta_{\max}$ is symmetric for two subspaces of the same dimension. Using $\sin\theta_{\max}(\hat{P}_{j,1,1}, P_{j}) \leq q_0/4$ and (13),

$$\max_{|T| \leq 2s} \|I_T'\hat{P}_{j,1}\| \leq q_0/4 + \max_{|T| \leq 2s} \|I_T'P_j\| \leq q_0/4 + 0.3$$

Finally, from using the assumptions of Theorem 2.1: $\varepsilon \leq 0.01$ and $\Delta \leq 0.3\sqrt{r}$ and Lemma A.3, it follows that $q_0 \leq 0.132$ and subsequently $\delta_{2s}(\Psi_{j}) \leq 0.333^2 < 0.15$. From this,

$$\left\| (\Psi_{\hat{T}_t}'\Psi_{\hat{T}_t})^{-1} \right\| \leq \frac{1}{1 - \delta_{s}(\Psi)} \leq \frac{1}{1 - \delta_{2s}(\Psi)} \leq \frac{1}{1 - 0.15} < 1.2 = \phi^+.$$ 

We also have that

$$\|\Psi_{\ell_t}\| = \left| (I - \hat{P}_{j,1}\hat{P}_{j,1}')P_ja_t \right| \leq \sin\theta_{\max}(\hat{P}_{j,1,1}, P_{j}) \|a_t\| \leq (q_0/4)\sqrt{n}r\lambda^+$$

$$\text{(a)} \leq 0.3(\varepsilon + \sin\theta_{\max}(P_{j-1,1}, P_{j}))\sqrt{n}r\lambda^+ \leq 0.3(\varepsilon_{\text{dist}} + \Delta)\sqrt{n}r\lambda^+ := 0.3b_b$$

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where (a) follows from using Lemma 5.7 and (b) follows from Remark 5.5. Now, under the condition of Theorem 2.1, 0.3b_{0} < b_{b} < x_{\min}/15 ensures exact support recovery exactly as in Lemma 5.7. Notice here that for the outliers in this interval we could a looser bound would suffice (as noted in Remark 2.4).

**Proof of Item 2:** Again, updating \( \hat{P}_{(t)} \) using \( \hat{\ell}_{t} \)'s is a problem of PCA in sparse data-dependent noise (SDDN), \( e_{t} \). We use the result of Theorem A.2. Recall from the proof of item 1 that for \( t \in [\hat{t}_{j} + \alpha, \hat{t}_{j} + 2\alpha] \), \( \hat{T}_{t} = T_{t} \), and \( \hat{\ell}_{t} = \ell_{t} - e_{t} \). We compute \( \hat{P}_{j,2} \) as the top \( r \) eigenvectors of \( \frac{1}{\alpha} \sum_{t=\hat{t}_{j}+\alpha}^{\hat{t}_{j}+2\alpha-1} \hat{\ell}_{t} \hat{\ell}_{t}' \). In notation of Theorem A.2, \( y_{t} \equiv \hat{\ell}_{t}, w_{t} \equiv e_{t}, \ell_{t} \equiv \hat{\ell}_{t}, \) and, \( M_{s,t} = - (\Psi_{T_{t}}' \Psi_{T_{t}})^{-1} \Psi_{T_{t}} \) and so \( \|M_{s,t}P_{j}\| = \| (\Psi_{T_{t}}' \Psi_{T_{t}})^{-1} \Psi_{T_{t}}' P_{j} \| \leq (\phi^+/4)q_{0} := q_{1} \). Also, using [9, Lemma 6.7], we have \( b_{0} \equiv \max\text{-outlier-frac-row} \).

Now, applying the PCA-SDDN result with \( q \equiv q_{1}, b_{0} \equiv \max\text{-outlier-frac-row} \), and setting \( \varepsilon_{SE} = q_{1}/4 \), observe that we require

\[
\sqrt{b_{0}}q_{1}f \leq \frac{0.9(q_{1}/4)}{1 + (q_{1}/4)}
\]

which holds if \( \sqrt{b_{0}}f \leq 0.12 \). Thus, from Corollary A.2, with probability at least \( 1 - 10n^{-10} \), \( \sin \theta_{\max}(\hat{P}_{j,2}, P_{j}) \leq (q_{1}/4) = 0.25 \cdot 0.3q_{0} \). In other words, with probability at least \( 1 - 10n^{-10} \), conditioned on \( \Gamma_{j,1}, \Gamma_{j,2} \) holds.

(B) **General k**

**Proof of Item 1:** Now consider the interval \( [\hat{t}_{j} + (k - 1)\alpha, \hat{t}_{j} + k\alpha] \). Using the same idea as for the \( k = 2 \) case, we have that for the \( k \)-th interval, \( q_{k-1} = (\phi^+/4)^{(k-1)}q_{0} \) and \( \varepsilon_{SE} = (q_{k-1}/4) \). From this it is easy to see that

\[
\delta_{2s}(\Psi) \leq \left( \max_{|T| \leq 2s} \|I_{T}' P_{j,k-1}\| \right)^{2} \leq (\sin \theta_{\max}(\hat{P}_{j,k-1}, P_{j}) + \max_{|T| \leq 2s} \|I_{T}' P_{j}\|)^{2} \leq (\sin \theta_{\max}(\hat{P}_{j,k-1}, P_{j}) + 0.3)^{2} \leq ((\phi^+/4)^{(k-1)}(\varepsilon + \sin \theta_{\max}(P_{j-1}, P_{j}) + 0.3)^{2} < 0.15
\]

where (a) follows from (13). Using the approach Lemma 5.7,

\[
\|\Psi \hat{\ell}_{t}\| \leq \sin \theta_{\max}(\hat{P}_{j,k-1}, P_{j}) \|a_{t}\| \leq (\phi^+/4)^{(k-1)}(\varepsilon + \sin \theta_{\max}(P_{j-1}, P_{j}) \sqrt{\eta r \lambda^{+}}) \leq (\phi^+/4)^{(k-1)}(\varepsilon + \Delta) \sqrt{\eta r \lambda^{+}} := (\phi^+/4)^{(k-1)}b_{0}
\]

where (a) follows from Remark 5.5.

**Proof of Item 2:** Again, updating \( \hat{P}_{(t)} \) from \( \hat{\ell}_{t} \)'s is a problem of PCA in sparse data-dependent noise given in Theorem A.2. From proof of Item 1 for \( t \in [\hat{t}_{j} + (k - 1)\alpha, \hat{t}_{j} + k\alpha] \), \( \hat{T}_{t} = T_{t} \), and \( \hat{\ell}_{t} = \ell_{t} - e_{t} \). We update the subspace, \( \hat{P}_{j,k} \) as the top \( r \) eigenvectors of \( \frac{1}{\alpha} \sum_{t=\hat{t}_{j}+\alpha}^{\hat{t}_{j}+k\alpha-1} \hat{\ell}_{t} \hat{\ell}_{t}' \). In the setting above \( y_{t} \equiv \hat{\ell}_{t}, w_{t} \equiv e_{t}, \ell_{t} \equiv \hat{\ell}_{t}, \) and, \( M_{s,t} = - (\Psi_{T_{t}}' \Psi_{T_{t}})^{-1} \Psi_{T_{t}} \) and so \( \|M_{s,t}P_{j}\| = \| (\Psi_{T_{t}}' \Psi_{T_{t}})^{-1} \Psi_{T_{t}}' P_{j} \| \leq (\phi^+/4)^{(k-1)}q_{0} := q_{k-1} \). Now applying the PCA-SDDN result with \( q \equiv q_{k-1}, b_{0} \equiv \max\text{-outlier-frac-row} \), and setting \( \varepsilon_{SE} = q_{k-1}/4 \), observe that we require

\[
\sqrt{b_{0}}q_{k-1}f \leq \frac{0.9(q_{k-1}/4)}{1 + (q_{k-1}/4)}
\]

which holds if \( \sqrt{b_{0}}f \leq 0.12 \) as provided by Theorem 2.1. Thus, from Corollary A.2, with probability at least \( 1 - 10n^{-10} \), \( \sin \theta_{\max}(\hat{P}_{j,k}, P_{j}) \leq (\phi^+/4)^{(k-1)}g_{1} \). In other words, with probability at least \( 1 - 10n^{-10} \), conditioned on \( \Gamma_{j,k-1}, \Gamma_{j,k} \) holds.

**Remark 6.11** (Deriving the long expression for \( K \) given in the Discussion). We have used \( \max\text{-outlier-frac-row} \leq b_{0} \) with \( b_{0} = 0.02/f^{2} \) throughout the analysis in order to simplify the proof. If we were not to do this, and if we used [16, Corollary 2.12], it is possible to show that the “decay rate” \( q_{k} \) is of the form \( q_{k} = (c_{2} \sqrt{b_{0}})^{k}q_{0} \) from which it follows that to obtain an \( \varepsilon \)-accurate approximation of the subspace it suffices to have \( K \geq \left[ \frac{1}{-\log(c_{2} \sqrt{b_{0}})} \log \left( \frac{\Delta}{\varepsilon \sqrt{r}} \right) \right] \).
Proof of Lemma 5.10. Proof of Item (a): Recall that
\[
\sigma_{\min}((I - P_{j-1}P_{j-1}')P_j) = \sin \theta_{\min}(P_{j-1}, P_j) \geq 0.14 \sin \theta_{\max}(P_{j-1}, P_j) \tag{14}
\]
Now, consider
\[
\lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi [P_j a_t a_t' P_j' + e_t e_t' + \ell_t e_t' + e_t e_t'] \Phi \right)
\]
\[
\overset{(a)}{\geq} \lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi P_j a_t a_t' P_j' \Phi \right) + \lambda_{\min} \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi [e_t e_t' + \ell_t e_t' + e_t e_t'] \Phi \right)
\]
\[
\overset{(b)}{\geq} \lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi P_j a_t a_t' P_j' \Phi \right) - \left\| \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi e_t e_t' \Phi \right\| - 2 \left\| \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \ell_t e_t' \Phi \right\|
\]
\[
:= \lambda_{\max}(T) - \sqrt{b_0 q_0^2} \lambda^+(1 + \epsilon_2) - 2 \sqrt{b_0 q_0}(\sin \theta_{\max}(P_{j-1}, P_j) + \epsilon) \lambda^+(1 + \epsilon_1)
\tag{15}
\]
where (a) follows from Weyl’s Inequality; (b) follows from Lemma A.7, and holds with probability at least $1 - 10n^{-10}$, and using $\sin \theta_{\max}(P_{j-1}, P_j) \leq \epsilon + \sin \theta_{\max}(P_{j-1}, P_j)$. Now we bound the first term as follows.
Define $\Phi P_j^Q = E_j R_j$ as the reduced QR decomposition\footnote{$E_j$ is an $n \times r$ matrix with orthonormal columns and $R_j$ is an $r \times r$ upper diagonal matrix} and let
\[
A := R_j \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} a_t a_t' \right) R_j'.
\]
and observe that $T$ can also be written as
\[
T = [E_j E_j, 0] \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_j' \\ E_{j, \perp}' \end{bmatrix}
\tag{16}
\]
and thus $\lambda_{\max}(A) = \lambda_{\max}(T)$. Working with $\lambda_{\max}(A)$ in the sequel,
\[
\lambda_{\max}(A) = \lambda_{\max} \left( R_j \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} a_t a_t' \right) R_j' \right) \overset{(a)}{\geq} \lambda_{\min}(R_j R_j') \lambda_{\max} \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} a_t a_t' \right)
\]
where (a) uses [17, Theorem 4.5.9]. We now bound $\lambda_{\min}(R_j R_j')$. Note that for all $i = 1, 2, \cdots, r$, $\sigma_i(\Phi P_j) = \sigma_i(R_j)$ and thus
\[
\lambda_{\min}(R_j R_j') = \lambda_{\min}(P_j'(I - \hat{P}_{j-1} \hat{P}_{j-1}) P_j) = 1 - \| \hat{P}_{j-1}' P_j \|^2 \tag{17}
\]
further,
\[
\| \hat{P}_{j-1}' P_j \| \leq \| \hat{P}_{j-1}' (I - P_{j-1} P_{j-1}') P_j \| + \| \hat{P}_{j-1}' P_{j-1} P_{j-1} P_j \| \\
\leq \sin \theta_{\max}(\hat{P}_{j-1}, P_{j-1}) + \| P_{j-1}' P_j \| \leq \epsilon + \| P_{j-1}' P_j \|, \tag{18}
\]
furthermore, using (14),
\[
1 - \| P_{j-1}' P_j \|^2 = \lambda_{\min}(P_j'(I - P_{j-1} P_{j-1}') P_j) = \sigma_r^2((I - P_{j-1} P_{j-1}') P_j) \geq \sin^2 \theta_{\min}(P_{j-1}, P_j)
\]
\[
\implies \| P_{j-1}' P_j \| \leq \sqrt{1 - \sin^2 \theta_{\min}(P_{j-1}, P_j)} \leq 1 - \frac{\sin^2 \theta_{\min}(P_{j-1}, P_j)}{2} \tag{19}
\]
thus from (17), (18) and (19)
\[
\lambda_{\min}(R_j R'_j) \geq 1 - \left( \epsilon + 1 - \frac{\sin^2 \theta_{\min}(P_{j-1}, P_j)}{2} \right)^2 \geq \sin^2 \theta_{\min}(P_{j-1}, P_j) - 2\epsilon
\]

Now, using \( \epsilon \leq 0.01 \sin^2 \theta_{\min}(P_{j-1}, P_j) \), (15), \( \epsilon_0 = \epsilon_1 = \epsilon_2 = 0.01 \), with probability at least \( 1 - 10n^{-10} \)
\[
\lambda_{\max}\left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \hat{\ell}_t \hat{\ell}_t' \Phi \right) \geq \lambda^+ [0.99(\sin^2 \theta_{\min}(P_{j-1}, P_j) - 2\epsilon) - \\
\sqrt{b_0 q_0 (1.01 q_0 + 2.02(\sin \theta_{\max}(P_{j-1}, P_j) + \epsilon))} := B_l
\]

**Proof of Item (b):** Consider \( \lambda_{\max}\left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \hat{\ell}_t \hat{\ell}_t' \Phi \right) \),
\[
\leq \lambda_{\max}\left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \ell_t \ell_t' \Phi \right) + \lambda_{\max}\left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi [\ell_t e_t' + e_t \ell_t' + e_t e_t'] \Phi \right) \\
\leq \lambda_{\max}\left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \ell_t \ell_t' \Phi \right) + \left\| \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi e_t e_t' \Phi \right\| + 2 \left\| \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \ell_t e_t' \Phi \right\| \\
\leq \lambda_{\max}\left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \ell_t \ell_t' \Phi \right) + \sqrt{b_0 q_K \lambda^+ (q_K (1 + \epsilon_2) + 2(1 + \epsilon_1)\epsilon)} \\
:= \lambda_{\max}(T) + \sqrt{b_0 q_K \lambda^+ (q_K (1 + \epsilon_2) + 2(1 + \epsilon_1)\epsilon)}
\]

where (a) follows from Lemma A.7 and using \( \sin \theta_{\max}(\hat{P}_j, P_j) \leq \epsilon \). Further, proceeding as before, define \( \Phi P_j^{QR} E_j R_j \) as the reduced Q.R. decomposition\(^8\) and defining \( A \) as before, we know \( \lambda_{\max}(T) = \lambda_{\max}(E_j'TE_j) = \lambda_{\max}(A) \). Further,
\[
\lambda_{\max}(A) = \lambda_{\max}\left( R_j \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} a_t a_t' \right) R_j' \right) \leq \lambda_{\max}\left( \frac{1}{\alpha} \sum_{t \in J^\alpha} a_t a_t' \right) \lambda_{\max}(R_j R'_j)
\]

where (a) uses [17, Theorem 4.5.9]. We bound the last term above as follows
\[
\lambda_{\max}(R_j R'_j) = \sigma_{\max}^2(R_j) = \sigma_{\max}^2((I - \hat{P}_j \hat{P}_j')P_j) \leq \epsilon^2
\]

Now, using \( \epsilon_0 = \epsilon_1 = \epsilon_2 = 0.01 \), when the subspace has not changed, with probability at least \( 1 - 10n^{-10} \),
\[
\lambda_{\max}\left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \hat{\ell}_t \hat{\ell}_t' \Phi \right) \leq \lambda^+[1.01 \epsilon^2 + \sqrt{b_0 q_K (1.01 q_K + 2.01 \epsilon)}] := B_h
\]

**Remark 6.12 (Simplifying bounds).** From above,
\[
B_l = \lambda^+[0.99(\sin^2 \theta_{\min}(P_{j-1}, P_j) - 2\epsilon) - 1.01 \sqrt{b_0 q_0 (q_0 + 2(\sin \theta_{\max}(P_{j-1}, P_j) + \epsilon))}] \\
B_h = \lambda^+[1.01 \epsilon^2 + \sqrt{b_0 q_K (1.01 q_K + 2.02 \epsilon)}]
\]

recalling that \( q_0 = 1.2(\epsilon + \sin \theta_{\max}(P_{j-1}, P_j)) \), \( q_K = \epsilon, \epsilon \leq 0.01 \sin^2 \theta_{\min}(P_{j-1}, P_j)^2 \) and (4) holds. Plugging in these values, and \( b_0 = 0.02/f^2 \leq 0.02 \) in (20) we obtain
\[
B_h \leq 1.37 \epsilon^2 \lambda^+ < \omega_{\text{evals}} < (0.03 \sin^2 \theta_{\max}(P_{j-1}, P_j) - 1.98 \epsilon) \lambda^+ \leq B_l
\]
For the low-rank matrix $L$ command in MATLAB. With these parameters, $\sin \theta$ random from the interval $[0, \pi]$ for $t > t_{\text{train}}$ we used $s/n = 0.01$, $b_0 = 0.01$ and $b_0 = 0.3$. For the Bernoulli model we set $\rho = 0.01$ for the first $t_{\text{train}}$ frames and $\rho = 0.3$ for the subsequent frames. The sparse outlier magnitudes are generated uniformly at random from the interval $[x_{\text{min}}, x_{\text{max}}]$ with $x_{\text{min}} = 10$ and $x_{\text{max}} = 20$ in both experiments. The results are averaged over 50 independent trials. The results are shown in Fig. 1.

We initialized all the ReProCS algorithms using AltProj applied to $Y_{[1, t_{\text{train}}]}$. The smaller outlier fraction helped achieve $\sin \theta_{\text{max}}(\hat{P}_{\text{init}}, P_0) \approx 10^{-3}$. For the batch methods used in the comparisons – PCP, AltProj and RPCA-GD, we implement the algorithms on $Y_{[1, r]}$ every $t = t_{\text{train}} + k\alpha - 1$ frames. Further, we set the regularization parameter for PCP $1/\sqrt{n}$ in accordance with [1]. The other known parameters, $r$ for Alt-Project, outlier-fraction for RPCA-GD, are set using the true values. For online methods we implement ORPCA by [13] and GRASTA by [18]. The regularization parameter for ORPCA was set as with $\lambda_1 = 1/\sqrt{n}$ and $\lambda_2 = 1/\sqrt{d}$ according to [13]. A more detailed discussion of numerical evaluation along with results on videos is presented in Appendix F.

8 Conclusions

We obtained the first complete guarantee for any online, streaming or dynamic RPCA algorithm that holds under weakened versions of standard RPCA assumptions, a simple and general model on slow subspace change (entire subspace can change every so often, as long as changes in all directions are of the same order), and two other mild assumptions (outlier magnitudes lower bounded and the $\ell_i$'s mutually independent). We analyzed an algorithm that we call MEDRoP (memory efficient dynamic robust PCA) that is based on the ReProCS framework [7, 8, 9]. The algorithm itself is significantly simpler than all previously studied ReProCS-based methods, is provably faster, obtains an $\epsilon$-accurate estimate of the changed subspace with a delay that is near-optimal (is more than the subspace dimension by only log

\[ E_j \] is an $n \times r$ matrix with orthonormal columns and $R_j$ is an $r \times r$ upper diagonal matrix.

\[ s \] is the number of outliers.

\[ t \] is the time index.

\[ x \] is the sample size.

\[ t_{\text{train}} \] is the training period.

\[ \alpha \] is the fracture rate.

\[ \theta \] is the angle between the target and the estimated subspace.

\[ \rho \] is the outlier fraction.

\[ \lambda \] is the regularization parameter.

\[ P \] is the projection matrix.
A question for future work is whether the lower bound on outlier magnitudes can be removed if we use the stronger assumption on maximum outlier fractions per row (assume they are of order 1/r). We will look at borrowing ideas from the AltProj [4] or NO-RMC [6] proof to do this. If this can be done, it may also help us remove all statistical assumptions on the $\ell_1$‘s.

A Preliminaries

In this section we state some preliminary results upon which the proof of our main lemmas are based. First we state and prove a simple lemma.

**Lemma A.1.** Let $Q_1, Q_2$ and $Q_3$ be $r$-dimensional subspaces in $\mathbb{R}^n$ such that $\sin \theta_{\max}(Q_1, Q_2) \leq \Delta_1$ and $\sin \theta_{\max}(Q_2, Q_3) \leq \Delta_2$. Then, $\sin \theta_{\max}(Q_1, Q_3) \leq \Delta_1 + \Delta_2$.

**Proof of Lemma A.1.** The proof follows from triangle inequality as

\[
\sin \theta_{\max}(Q_1, Q_3) = \|(I - Q_1Q_1')(Q_3)\| = \| (I - Q_1Q_1')(I - Q_2Q_2' + Q_2Q_2')Q_3\| \\
\leq \| (I - Q_1Q_1')(I - Q_2Q_2')Q_3\| + \| (I - Q_1Q_1')Q_2Q_2'Q_3\| \\
\leq \| (I - Q_1Q_1')\| \sin \theta_{\max}(Q_2, Q_3) + \sin \theta_{\max}(Q_1, Q_2) \|Q_2'Q_3\| \leq \Delta_1 + \Delta_2
\]

[16, Remark 4.18] states the following

**Theorem A.2.** (PCA-SDDN) Given data vectors $y_t := \ell_t + w_t = \ell_t + I_{\bar{T}_t}M_{s,t}\ell_t$, $t = 1, 2, \ldots, \alpha$, where $\bar{T}_t$ is the support set of $w_t$, and $\ell_t$ satisfying the model detailed above. Further, $\max_t \|M_{s,t}P\|_2 \leq q < 1$, for any $\alpha \geq \alpha_0$ where

\[
\alpha_0 := C\eta_1^{-2}f^2r \log n
\]

the fraction of nonzeros in any row of the noise matrix $[w_1, w_2, \ldots, w_\alpha]$ is bounded by $b_0$, and $3\sqrt{b_0} q f \leq 0.9 \varepsilon_{SE} / (1 + \varepsilon_{SE})$. For an $\alpha \geq \alpha_0$, let $\hat{P}$ be the matrix of top $r$ eigenvectors of $D := \frac{1}{\alpha} \sum_i y_iy_i'$. With probability at least $1 - 10n^{-10}$, $\sin \theta_{\max}(\hat{P}, P) \leq \varepsilon_{SE}$.

**Lemma A.3 ([7]).** For an $n \times r$ basis matrix $P$,

1. $\max_{|\bar{T}| \leq s} \|I_{\bar{T}}'P\|^2 \leq s \max_i = 1, 2, \ldots, r \|I_i'P\|^2$

2. $\delta_s(I - PP') = \max_{|\bar{T}| \leq s} \|I_{\bar{T}}'P\|^2$
3. If $P = [P_1, P_2]$ then $\|I\_T^\tau P\|^2 \leq \|I\_T^\tau P_1\|^2 + \|I\_T^\tau P_2\|^2$.

Cauchy-Schwartz for sums of matrices says the following [7].

\textbf{Theorem A.4. (Cauchy-Schwartz)} For matrices $X$ and $Y$ we have

$$\left\| \frac{1}{\alpha} \sum_t X_t Y_t' \right\|^2 \leq \left\| \frac{1}{\alpha} \sum_t X_t X_t' \right\| \left\| \frac{1}{\alpha} \sum_t Y_t Y_t' \right\|$$

(21)

The following theorem is adapted from [20, Theorem 1.6].

\textbf{Theorem A.5. (Matrix Bernstein)} Given an $\alpha$-length sequence of $n_1 \times n_2$ dimensional random matrices and a r.v. $X$. Assume the following holds. For all $X \in C$, (i) conditioned on $X$, the matrices $Z_t$ are mutually independent, (ii) $\Pr(\|Z_t\| \leq R_t X) = 1$, and (iii) $\max \left\{ \left\| \frac{1}{\alpha} \sum_t \mathbb{E}[Z_t' Z_t] \right\|, \left\| \frac{1}{\alpha} \sum_t \mathbb{E}[Z_t Z_t'] \right\| \right\} \leq \sigma^2$. Then, for an $\epsilon > 0$ and for all $X \in C$,

$$\Pr\left( \left\| \frac{1}{\alpha} \sum_t Z_t \right\| \leq \left\| \frac{1}{\alpha} \sum_t \mathbb{E}[Z_t X] \right\| + \epsilon X \right) \geq 1 - (n_1 + n_2) \exp \left( \frac{-\alpha \epsilon^2}{2(\sigma^2 + R\epsilon)} \right).$$

(22)

The following theorem is adapted from [21, Theorem 5.39].

\textbf{Theorem A.6. (Sub-Gaussian Rows)} Given an $N$-length sequence of sub-Gaussian random vectors $w_i$ in $\mathbb{R}^{n_w}$, an r.v $X$, and a set $C$. Assume the following holds. For all $X \in C$, (i) $w_i$ are conditionally independent given $X$; (ii) the sub-Gaussian norm of $w_i$ is bounded by $K$ for all $i$. Let $W := [w_1, w_2, \ldots , w_N]'$. Then for an $\epsilon \in (0, 1)$ and for all $X \in C$

$$\Pr\left( \left\| \frac{1}{N} W' W - \frac{1}{N} \mathbb{E}[W' W | X] \right\| \leq \epsilon |X| \right) \geq 1 - 2 \exp \left( n_w \log 9 - \frac{c^2 N}{4K^4} \right).$$

(23)

We now state the concentration bounds needed for proving the main subspace change detection lemma.

\textbf{Lemma A.7.} Assume that the assumptions of Theorem 2.1 hold. For the sake of this lemma assume that $\Phi := I - \hat{P} \hat{P}'$ and $\ell_t = Pa_t$. Then,

$$\Pr\left( (1 - \epsilon_0) \lambda^+ \leq \lambda_{\max} \left( \frac{1}{\alpha} \sum_t a_t a_t' \right) \leq (1 + \epsilon_0) \lambda^+ | \mathcal{E}_0 \right) \geq 1 - 10n^{-10}$$

$$\Pr\left( \left\| \frac{1}{\alpha} \sum_t \Phi \ell_t w_t' \Phi \right\| \leq (1 + \epsilon_1) \sin \theta_{\max}(\hat{P}, P) \sqrt{b_0 q} \lambda^+ | \mathcal{E}_0 \right) \geq 1 - 10n^{-10}$$

$$\Pr\left( \left\| \frac{1}{\alpha} \sum_t \Phi w_t w_t' \Phi \right\| \leq (1 + \epsilon_2) \sqrt{b_0 q^2} \lambda^+ | \mathcal{E}_0 \right) \geq 1 - 10n^{-10}$$

$$\Pr\left( \left\| \frac{1}{\alpha} \sum_t \Phi \ell_t v_t \Phi \right\| \leq \epsilon_{\ell,v} \lambda^+ | \mathcal{E}_0 \right) \geq 1 - 2n^{-10}$$

$$\Pr\left( \left\| \frac{1}{\alpha} \sum_t \Phi v_t v_t' \Phi \right\| \leq (\epsilon^2 + \epsilon_{v,v}) \lambda^+ | \mathcal{E}_0 \right) \geq 1 - 2n^{-10}$$

The proof of this lemma is straightforward and given in Appendix E. Similar bounds are also proved in [9].
B Proof of Theorem 2.1

Here we first state two simple observations which will be used in various places in our proofs.

Remark B.1 (Derivation of $K$). Here we show how setting $K$ as specified in (10) implies the guarantees of Theorem 2.1. From Lemma 5.8 the subspace error bound in the $k$-th SVD step is given as $0.3^k (\varepsilon + \sin \theta_{\text{max}}(P_{j-1}, P_j))$. To ensure that after $K$ steps, an $\varepsilon$-approximate estimate is found, we need

\[
0.3^K (\varepsilon + \sin \theta_{\text{max}}(P_{j-1}, P_j)) < \varepsilon
\]

\[
\implies K \geq \lceil c_1 \log((\sin \theta_{\text{max}}(P_{j-1}, P_j-1) + \varepsilon)/\varepsilon) \rceil
\]

Using $\sin \theta_{\text{max}}(P_{j-1}, P_j) \leq \Delta/\sqrt{r}$, we obtain the expression for $K$.

We first prove Theorem 2.1 for the case when $t_j$’s are known, i.e., correctness of Algorithm 1.

Proof of Theorem 2.1 with assuming $t_j$ known. In this case $\hat{t}_j = t_j$. The proof is an easy consequence of Lemmas 5.7 and 5.8. Recall that $\Gamma_{j,K} \subseteq \Gamma_{j,K-1} \subseteq \cdots \Gamma_{0,0}$ and $\Gamma_{j,K} \subseteq \Gamma_{j-1,K} \subseteq \cdots \subseteq \Gamma_{1,K}$ follow from definitions. Furthermore, to show that the conclusions of the Theorem hold, it suffices to show that $Pr(\Gamma_{j,K} | \Gamma_{0,0}) \geq 1 - 10dn^{-10}$. Using the chain rule of probability the following follows

\[
Pr(\Gamma_{j,K} | \Gamma_{1,0}) = Pr(\Gamma_{j,K}, \Gamma_{j-1,K}, \cdots, \Gamma_{1,K} | \Gamma_{1,0})
\]

\[
= \prod_{j=1}^J Pr(\Gamma_{j,K} | \Gamma_{j,0}) = \prod_{j=1}^J \prod_{k=1}^K Pr(\Gamma_{j,k} | \Gamma_{j,k-1})
\]

\[
\geq (1 - 10n^{-10})^{JK} \geq 1 - 10JKn^{-10}.
\]

where (a) used $Pr(\Gamma_{j,1} | \Gamma_{j,0}) \geq 1 - 10n^{-10}$ from Lemma 5.7 and $Pr(\Gamma_{j,k} | \Gamma_{j,k-1}) \geq 1 - 10n^{-10}$ from Lemma 5.8.

Proof of Theorem 2.1. Here we provide the proof of the general case when $t_j$’s are detected by Algorithm 2. Define

\[
\hat{t}_{j-1,\text{fin}} := \hat{t}_{j-1} + K\alpha, \quad t_{j,*} = \hat{t}_{j-1,\text{fin}} + \left\lfloor \frac{t_j - \hat{t}_{j-1,\text{fin}}}{\alpha} \right\rfloor \alpha
\]

Thus, $\hat{t}_{j-1,\text{fin}}$ is the time at which the $(j-1)$-th subspace update is complete; w.h.p., this occurs before $t_j$. With this assumption, $t_{j,*}$ is such that $t_j$ lies in the interval $[t_{j,*} - \alpha + 1, t_{j,*}].$

Recall from the algorithm that we increment $j$ to $j + 1$ at $t = \hat{t}_j + K\alpha := t_{j,\text{fin}}$. Define the events

1. $\text{Det0} := \{\hat{t}_j = t_{j,*}\} = \{\lambda_{\text{max}}(\frac{1}{\alpha} \sum_{t=j_*-\alpha + 1}^{t_{j,*}} \Phi \hat{\ell}_t \hat{\ell}_t^T \Phi) > \omega_{\text{evals}}\}$ and

$\text{Det1} := \{\hat{t}_j = t_{j,*} + \alpha\} = \{\lambda_{\text{max}}(\frac{1}{\alpha} \sum_{t=j_*+\alpha}^{t_{j,*}+\alpha} \Phi \hat{\ell}_t \hat{\ell}_t^T \Phi) > \omega_{\text{evals}}\},$

2. $\text{SubUpd} := \bigcap_{k=1}^K \text{SubUpd}_{k}$ where $\text{SubUpd}_{k} := \{\sin \theta_{\text{max}}(\hat{P}_{j,k}, P_j) \leq \zeta_{j,k}^{+}\}$,

3. $\text{NoFalseDets} := \{\text{for all } J^\alpha \subseteq [\hat{t}_{j,\text{fin}}, t_{j+1}], \lambda_{\text{max}}(\frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \hat{\ell}_t \hat{\ell}_t^T \Phi) \leq \omega_{\text{evals}}\}$

4. $\Gamma_{0,\text{end}} := \{\text{dist}(\hat{P}_{\text{init}}, P_0) \leq \Delta_{\text{init}}\},$

5. $\Gamma_{j,\text{end}} := \Gamma_{j-1,\text{end}} \cap (\text{Det0} \cap \text{SubUpd} \cap \text{NoFalseDets}) \cup (\text{Det1} \cap \text{SubUpd} \cap \text{NoFalseDets})$.
Let $p_0$ denote the probability that, conditioned on $\Gamma_{j-1,\text{end}}$, the change got detected at $t = t_j, \ast$, i.e., let

$$p_0 := \Pr(\det_{0, j-1, \text{end}}).$$

Thus, $\Pr(\det_{0} | \Gamma_{j-1, \text{end}}) = 1 - p_0$. It is not easy to bound $p_0$. However, as we will see, this will not be needed. Assume that $\Gamma_{j-1, \text{end}} \cap \det_{0}$ holds. Consider the interval $J^\alpha := [\hat{t}_j, t_j, \ast + \alpha]$. This interval starts at or after $t_j$, so, for all $t$ in this interval, the subspace has changed. For this interval, $\Phi = I - \hat{P}_{j-1} \hat{P}_{j-1}'$.

Applying the first item of Lemma 5.10, w.p. at least $1 - 10n^{-10}$,

$$\lambda_{\text{max}} \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \ell_t \ell_t' \Phi \right) \geq \omega_{\text{evals}}$$

and thus $\hat{t}_j = t_j, \ast + \alpha$. The last inequality follows using 5.5. In other words,

$$\Pr(\det_{1} | \Gamma_{j-1, \text{end}} \cap \det_{0}) \geq 1 - 10n^{-10}.$$

Conditioned on $\Gamma_{j-1, \text{end}} \cap \det_{0} \cap \det_{1}$, the first SVD step is done at $t = \hat{t}_j + \alpha = t_j, \ast + 2\alpha$ and the subsequent steps are done every $\alpha$ frames. We can prove Lemma 5.7 with $\Gamma_{j, \ast}$ replaced by $\Gamma_{j, \text{end}} \cap \det_{0} \cap \det_{1}$ and Lemma 5.8 with $\Gamma_{j, k-1}$ replaced by $\Gamma_{j, \text{end}} \cap \det_{0} \cap \det_{1} \cap \text{SubUpd}_1 \cap \cdots \cap \text{SubUpd}_{k-1}$ and with the $k$-th SVD interval being $J^\alpha_k := [\hat{t}_j + (k-1)\alpha, \hat{t}_j + k\alpha]$. Applying Lemmas 5.7, and 5.8 for each $k$, we get

$$\Pr(\text{SubUpd} | \Gamma_{j-1, \text{end}} \cap \det_{0} \cap \det_{1}) \geq (1 - 10n^{-10})^{K+1}.$$

We can also do a similar thing for the case when the change is detected at $t_j, \ast$, i.e. when Det0 holds. In this case, we replace $\Gamma_{j, \ast}$ by $\Gamma_{j, \text{end}} \cap \det_{0}$ and $\Gamma_{j, k}$ by $\Gamma_{j, \text{end}} \cap \det_{0} \cap \text{SubUpd}_1 \cap \cdots \cap \text{SubUpd}_{k-1}$ and conclude that

$$\Pr(\text{SubUpd} | \Gamma_{j-1, \text{end}} \cap \det_{0}) \geq (1 - 10n^{-10})^K.$$

Finally consider the NoFalseDets event. First, assume that $\Gamma_{j-1, \text{end}} \cap \det_{0} \cap \text{SubUpd}$ holds. Consider any interval $J^\alpha \subseteq [\hat{t}_{j, \text{fin}}, t_{j, 1}]$. In this interval, $\hat{P}_j, \Phi = I - \hat{P}_j \hat{P}_j'$ and sin $\theta_{\text{max}}(\hat{P}_j, P_j) \leq \varepsilon$. Using the second part of Lemma 5.10, and Remark 6.12, we conclude that w.p. at least $1 - 10n^{-10}$,

$$\lambda_{\text{max}} \left( \frac{1}{\alpha} \sum_{t \in J^\alpha} \Phi \ell_t \ell_t' \Phi \right) < \omega_{\text{evals}}$$

Since Det0 holds, $\hat{t}_j = t_j, \ast$. Thus, we have a total of $\left\lceil \frac{t_{j, 1} - t_j, \ast - K\alpha - \alpha}{\alpha} \right\rceil$ intervals $J^\alpha$ that are subsets of $[\hat{t}_{j, \text{fin}}, t_{j, 1}]$. Moreover, $\left[ \frac{t_{j, 1} - t_j, \ast - K\alpha - \alpha}{\alpha} \right] \leq \left[ \frac{t_{j, 1} - t_j, \ast - K\alpha - \alpha}{\alpha} \right] \leq \left[ \frac{t_{j, 1} - t_j, \ast}{\alpha} \right] - (K + 1)$ since $\alpha \leq \alpha$. Thus,

$$\Pr(\text{NoFalseDets} | \Gamma_{j-1, \text{end}} \cap \det_{0} \cap \text{SubUpd}) \geq (1 - 10n^{-10}) \left[ \frac{t_{j, 1} - t_j, \ast}{\alpha} \right] - (K + 1)$$

On the other hand, if we condition on $\Gamma_{j-1, \text{end}} \cap \det_{0} \cap \det_{1} \cap \text{SubUpd}$, then $\hat{t}_j = t_j, \ast + \alpha$. Thus,

$$\Pr(\text{NoFalseDets} | \Gamma_{j-1, \text{end}} \cap \det_{0} \cap \det_{1} \cap \text{SubUpd}) \geq (1 - 10n^{-10}) \left[ \frac{t_{j, 1} - t_j, \ast}{\alpha} \right] - (K + 1)$$

We can now combine the above facts to bound $\Pr(\Gamma_{j, \ast} | \Gamma_{j-1, \text{end}})$. Recall that $p_0 := \Pr(\det_{0} | \Gamma_{j-1, \text{end}})$. Clearly, the events $(\det_{0} \cap \text{SubUpd} \cap \text{NoFalseDets})$ and $(\det_{0} \cap \det_{1} \cap \text{SubUpd} \cap \text{NoFalseDets})$ are disjoint. Thus,

$$\Pr(\Gamma_{j, \ast} | \Gamma_{j-1, \text{end}})$$

$$= p_0 \Pr(\text{SubUpd} \cap \text{NoFalseDets} | \Gamma_{j-1, \text{end}} \cap \det_{0})$$

$$+ (1 - p_0) \Pr(\det_{1} | \Gamma_{j-1, \text{end}} \cap \det_{0}) \Pr(\text{SubUpd} \cap \text{NoFalseDets} | \Gamma_{j-1, \text{end}} \cap \det_{0} \cap \det_{1})$$

$$\geq p_0 (1 - 10n^{-10})^K (1 - 10n^{-10}) \left[ \frac{t_{j, 1} - t_j, \ast}{\alpha} \right] - (K + 1)$$

$$+ (1 - p_0)(1 - 10n^{-10})(1 - 10n^{-10})^K (1 - 10n^{-10}) \left[ \frac{t_{j, 1} - t_j, \ast}{\alpha} \right] - (K + 1)$$

$$= (1 - 10n^{-10}) \left[ \frac{t_{j, 1} - t_j, \ast}{\alpha} \right] \geq (1 - 10n^{-10})^{t_{j, 1} - t_j}.$$
Since the events $\Gamma_{j,\text{end}}$ are nested, the above implies that
\[
\Pr(\Gamma_{j,\text{end}}|\Gamma_{0,\text{end}}) = \prod_j \Pr(\Gamma_{j,\text{end}}|\Gamma_{j-1,\text{end}}) \geq \prod_j (1 - 10n^{-10})^{t_{j+1} - t_j} = (1 - 10n^{-10})^d \\
\geq 1 - 10dn^{-10}.
\]
\[\square\]

We now provide the proof of the Offline Algorithm (lines 26 – 30 of Algorithm 2).

**Proof of Theorem 2.2 (Offline MEDRoP).** The proof of this follows from the conclusions of the online counterpart. Note that the subspace estimate in this case is not necessarily $r$ dimensional. This is essentially done to ensure that in the time intervals when the subspace has changed, but has not yet been updated, the output of the algorithm is still an $\varepsilon$-approximate solution to the true subspace. In other words, for $t \in [\hat{t}_j + K \alpha, t_j]$, the true subspace is $P_{\hat{t} - 1}$ and so in this interval
\[
\sin \theta_{\max}(P_{(t)}^{\text{offline}}, P_{\hat{t} - 1}) = \sin \theta_{\max}(\hat{P}_{\hat{t} - 1}, (I - \hat{P}_{\hat{t} - 1} \hat{P}_{\hat{t} - 1}')(I - P_{\hat{t} - 1})) \\
= \left(\begin{array}{c}
(I - (I - \hat{P}_{\hat{t} - 1} \hat{P}_{\hat{t} - 1}')(I - \hat{P}_{\hat{t} - 1} \hat{P}_{\hat{t} - 1}')(I - \hat{P}_{\hat{t} - 1} \hat{P}_{\hat{t} - 1}')(I - \hat{P}_{\hat{t} - 1} \hat{P}_{\hat{t} - 1}')) \\
\sin \theta_{\max}(P_{\hat{t} - 1}, P_{\hat{t} - 1}) \leq \varepsilon
\end{array}\right)
\]

where (a) follows because for orthogonal matrices $P_1$ and $P_2,$
\[
I - P_1 P_1' - P_2 P_2' = (I - P_1 P_1')(I - P_2 P_2') = (I - P_2 P_2')(I - P_1 P_1').
\]

Now consider the interval $t \in [t_j, \hat{t}_j + K \alpha]$. In this interval, the true subspace is $P_j$ and we have back propagated the $\varepsilon$-approximate subspace $\hat{P}_j$ in this interval. We first note that $\text{span}([\hat{P}_{\hat{t} - 1}, (I - \hat{P}_{\hat{t} - 1} \hat{P}_{\hat{t} - 1}')(I - \hat{P}_{\hat{t} - 1} \hat{P}_{\hat{t} - 1}'))] = \text{span}([\hat{P}_j, (I - \hat{P}_j \hat{P}_j')(I - \hat{P}_j \hat{P}_j')])$. And so we use the latter to quantify the error in this interval as
\[
\sin \theta_{\max}(\hat{P}_j, P_j) = \sin \theta_{\max}(\hat{P}_j, (I - \hat{P}_j \hat{P}_j')(I - \hat{P}_j \hat{P}_j') P_j) \\
= \left(\begin{array}{c}
\sin \theta_{\max}(\hat{P}_j, P_j) \leq \varepsilon
\end{array}\right)
\]
\[\square\]

### C Proof of Stable Dynamic RPCA: Main Ideas

**Proof of Corollary 2.3.** The proof is very similar to that of the Theorem 2.1 but there are two differences due to the additional noise term. The first is the effect of the noise on the sparse recovery step. The approach to address this is straightforward. We note that the error now seen in the sparse recovery step is bounded by $||\Psi(\ell_t + \nu_t)||$ and using the bound on $||\nu_t||$, we observe that the error only changes by a constant factor. In particular, we can show that $||e_t|| \leq 2.4(\varepsilon_{\text{dist}} + 2\Delta)\sqrt{\eta \lambda^+}$. The other crucial difference is that in updating subspace estimate. To deal with the additional uncorrelated noise, we use the following result.

**Corollary C.1 (Noisy PCA-SDDN).** Given data vectors $y_t := \ell_t + w_t + z_t = \ell_t + I_{T_t} M_{s,t} \ell_t + z_t$, $t = 1, 2, \ldots, \alpha$, where $T_t$ is the support set of $w_t$, and $\ell_t$ satisfying the model detailed above. Furthermore, $\max_t ||M_{s,t} P||_2 \leq q < 1$. $z_t$ is small uncorrelated noise such that $E[z_t z_t^\top] = \Sigma_z$, $\max_t ||z_t||^2 := b_z^2 < \infty$. 24
Define $\lambda_z^+ := \lambda_{\max}(\Sigma_z)$ and $r_z$ as the “effective rank” of $z_t$ such that $b_z^2 = r_z \lambda_z^+$. Then for any $\alpha \geq \alpha_0$, where

$$
\alpha_0 := \frac{C}{\varepsilon_{SE}} \max\left\{ \eta q^2 f^2 r \log n, \frac{b_z^2}{\lambda_z^+} f \log n \right\}
$$

the fraction of nonzeros in any row of the noise matrix $[w_1, w_2, \ldots, w_\alpha]$ is bounded by $b_0$, and

$$
3 \sqrt{b_0 q f + \lambda_z^+ / \lambda^-} \leq \frac{0.9 \varepsilon_{SE}}{1 + \varepsilon_{SE}}
$$

For an $\alpha \geq \alpha_0$, let $\hat{P}$ be the matrix of top $r$ eigenvectors of $\hat{D} := \frac{1}{\alpha} \sum_t y_t y_t'$. With probability at least $1 - 10n^{-10}$, $\sin \theta_{\max}(\hat{P}, P) \leq \varepsilon_{SE}$.

We illustrate how applying Corollary C.1 changes the subspace update step. Consider the first subspace estimate, i.e., we are trying to get an estimate $\hat{P}_{j,1}$ in the $j$-th subspace change time interval. Define $(e_t)_t = I_{\gamma}^{-1}(\Psi \gamma \Psi)^{-1} \Psi e_t$ and $(e_v)_t = I_{\gamma}^{-1}(\Psi \gamma \Psi)^{-1} \Psi v_t$. We estimate the new subspace, $\hat{P}_{j,1}$ as the top $r$ eigenvectors of $\frac{1}{\alpha} \sum_{t=i}^{i+\alpha-1} \ell_t \ell_t'$. In the setting above, $y_t \equiv \ell_t$, $w_t \equiv (e_t)_t$, $z_t \equiv (e_v)_t$, $\ell_t \equiv e_t$ and $M_{s,t} = -(\Psi \gamma \Psi)^{-1} \Psi e_t$ and so $\|M_{s,t} P\| = \| (\Psi \gamma \Psi)^{-1} \Psi e_t P \| \leq \phi^+ (e + \sin \theta_{\max}(P_{j-1}, P_t)) := q_0$. Applying Corollary C.1 with $q \equiv q_0$, and recalling that the support, $\ell_t$ satisfies the assumptions similar to that of the noiseless case and hence $b_0 \equiv \max$-outlier-frac-row. Now, setting $\varepsilon_{SE,1} = q_0/4$, observe that we require

$$(i) \sqrt{b_0 q f} \leq \frac{0.5 \cdot 0.9 \varepsilon_{SE,1}}{1 + \varepsilon_{SE,1}}, \quad \text{and} \quad (ii) \frac{\lambda_z^+}{\lambda^-} \leq \frac{0.5 \cdot 0.9 \varepsilon_{SE,1}}{1 + \varepsilon_{SE,1}}.$$ 

which holds if (i) $\sqrt{b_0} f \leq 0.12$, and (ii) is satisfied as follows from using the assumptions on $v_t$ as follows. It is immediate to see that $\lambda_z^+ / \lambda^- \leq \varepsilon^2 f \leq 2 \varepsilon_{SE,1}$. Furthermore, the sample complexity term remains unchanged due to the choice of $v_t$. To see this, notice that the only extra term in the $\alpha_0$ expression is $b_z^2 f \log n / (\varepsilon_{SE} \lambda^-)$ which simplifies to $\varepsilon^2 f^2 r \log n / \varepsilon_{SE}^2$, which is what was required even in the noiseless case. Thus, from Corollary C.1, with probability at least $1 - 10n^{-10}$, $\sin \theta_{\max}(\hat{P}_{j,1}, P_j) \leq \varepsilon_{SE,1} = q_0/4$. Consequently $\text{dist}(\hat{P}_{j,1}, P_j) \leq \sqrt{q_0} / 4 \leq (\phi^+/4)(\varepsilon_{dist} + 42)$. The argument in other subspace update stages will require the same changes and follows without any further differences.

The final difference is in the subspace detection step. Notice that here too there will be some extra assumption required to provably detect the subspace change. However, due to the careful selection of the bounds on $\|v_t\|$ and the bounds on using $\varepsilon_{tv} = 0.01$, we see that (i) the extra sample complexity term is the same as that required in the noiseless case, and (ii) the condition of equal angles guarantees subspace detection within $2\alpha$ frames with the same probability.

\[\square\]

### D  Running Time of Algorithm 2

The time complexity is calculated as (a) Init: Performing a batch Robust PCA (eg. Alt Proj) on an $n \times t_{train}$ matrix with rank $r$ to achieve an error $\epsilon$ takes $O(nt_{train}^2 \log(1/\epsilon))$ time. Typically $t_{train} \ll d$ and so it is perfectly valid to assume that $t_{train} r \leq d$ and so run-time of initialization is $O(n dr \log(1/\epsilon))$; (b) Proj-CS: Performing $\ell_1$ minimization of an $n$-dimensional vector is equal to the cost of matrix-multiply times the dependence on the desired accuracy. In Algorithm 2 the matrix under consideration, $I - \widehat{P}_j \widehat{P}_j'$ and so $O(nr \log(1/\epsilon))$ suffices. Since there are $(d - t_{train})$ vectors, the total run-time is $O(n (d - t_{train}) r \log(1/\epsilon)) = O(n dr \log(1/\epsilon))$; (c) Subspace Update: The subspace update involves at most $((d - t_{train}) / \alpha)$ rank $r$-SVD's on $n \times \alpha$ matrices. Thus the total run-time of this operation is $O((d - t_{train}) nr \log(1/\epsilon) + Jn \alpha \log(1/\epsilon))$.

Note that our condition on the principal angles implies that the $r$-th and $(r + 1)$-th singular values are well separated and thus we can estimate the singular vectors in $O(nar)$ time.
Since $Jo \leq d$, it follows that run time here is $O(ndr \log(1/\epsilon))$. This is equal to the cost of computing a vanilla rank $r$-SVD on a matrix of dimensions $n \times d$. Up to constant factors, the running time of offline MEDRoP is equal to that of the online version.

### E Proof of Lemma A.7

The proof follows using Theorems A.5 and A.6.

**Proof of Lemma A.7. Item 1**: Recall that the $(a_i)_t$ are bounded r.v.’s satisfying $|(a_i)_t| \leq \sqrt{n}$. Thus, the vectors, $a_i$ are sub-Gaussian with $\|a_i\|_{\psi_2} = \max_i \|a_i\|_{\psi_2} = \sqrt{n}$. We now apply Theorem A.6 with $K = \sqrt{n}$, $\epsilon = \epsilon_0$, $N = \alpha$ and $n_w = r$. Now, for an $\alpha \geq \alpha_0 := \frac{C(r \log 9 + 10 \log n)}{\epsilon_0^2}$,

$$
\Pr \left( \| \frac{1}{\alpha} \sum a_i a_i' - \Lambda \| \leq \epsilon_0 | \mathcal{E}_0 \right) \geq 1 - 10n^{-10}
$$

now noting that for a Hermitian matrix $A$, $\|A\| = \max(\lambda_{\max}(A), -\lambda_{\min}(A))$ the result follows.

**Item 2**: For the second term, we proceed as follows. Since $\|\Phi\| = 1$,

$$
\left\| \frac{1}{\alpha} \sum \Phi_{\ell_i} w_i \right\| \leq \left\| \frac{1}{\alpha} \sum \Phi_{\ell_i} w_i \right\|
$$

To bound the RHS above, we will apply Theorem A.5 with $Z_t = \Phi_{\ell_i} w_i'$. Conditioned on $\{\hat{P}, Z\}$, the $Z_t$’s are mutually independent. We first bound a bound on the expected value of the time average of the $Z_t$’s and then compute $R$ and $\sigma^2$. By Cauchy-Schwartz,

$$
\left\| \mathbb{E} \left[ \frac{1}{\alpha} \sum \Phi_{\ell_i} w_i' \right] \right\|^2 = \left\| \frac{1}{\alpha} \sum \Phi_{\ell_i} P_{\Lambda} P' M_{1,i} M_{2,t} \right\|^2 \\
\leq \left\| \frac{1}{\alpha} \sum \left( \Phi_{\ell_i} P_{\Lambda} P' M_{1,i} \right) \left( M_{1,i} P_{\Lambda} P' \Phi_{\ell_i} \right) \right\| \left\| \frac{1}{\alpha} \sum M_{2,i} M_{2,t} \right\| \\
\leq b_0 \left[ \max_i \| \Phi_{\ell_i} P_{\Lambda} P' M_{1,i} \|^2 \right] \\
\leq b_0 \sin \theta_{\max}(\hat{P}, P) q^2 (\lambda^+)^2
$$

(24)

where (a) follows by Cauchy-Schwartz (Theorem A.4) with $X_t = \Phi_{\ell_i} P_{\Lambda} P' M_{1,i}'$ and $Y_t = M_{2,t}$, (b) follows from the assumption on $M_{2,t}$. To compute $R$

$$
\|Z_t\| \leq \|\Phi_{\ell_i} \| \| w_i \| \leq \sin \theta_{\max}(\hat{P}, P) q \eta r \lambda^+ := R
$$

Next we compute $\sigma^2$. Since $w_i$’s are bounded r.v.’s, we have

$$
\left\| \frac{1}{\alpha} \sum \mathbb{E}[Z_t Z_t'] \right\| = \left\| \frac{1}{\alpha} \sum \mathbb{E}[\Phi_{\ell_i} w_i' w_i' \Phi_{\ell_i}] \right\| = \left\| \frac{1}{\alpha} \mathbb{E}[\| w_i \|^2 \Phi_{\ell_i} \Phi_{\ell_i}] \right\| \\
\leq \left( \max_{w_t} \| w_t \|^2 \right) \left\| \frac{1}{\alpha} \sum \mathbb{E}[\Phi_{\ell_i} \Phi_{\ell_i}] \right\| \\
\leq q^2 \sin \theta_{\max}(\hat{P}, P) q \eta r (\lambda^+)^2 := \sigma^2
$$
it can also be seen that $\| \frac{1}{\alpha} \sum_t \mathbb{E}[Z_t' Z_t] \|$ evaluates to the same expression. Thus, applying Theorem A.5

$$
\Pr \left( \left\| \frac{1}{\alpha} \sum_t \Phi \ell_t w_t' \right\| \leq \sin \theta_{\max}(\hat{P}, P) \sqrt{b_0 q \lambda^+ + \epsilon} \big| \mathcal{E}_0 \right) \geq 1 - 2n \exp \left( \frac{-\alpha}{4 \max \left\{ \frac{\alpha^2}{\epsilon^2}, \frac{R}{\epsilon} \right\}} \right).
$$

Let $\epsilon = \epsilon_1 \lambda^-$, then $\sigma^2/\epsilon^2 = cnf^2 r$ and $R/\epsilon = cnf r$. Hence, for the probability to be of the form $1 - 2n^{-10}$ we require that $\alpha \geq \alpha_{(1)} := C \cdot \eta f^2 (r \log n)$.

Item 3: We use Theorem A.5 with $Z_t := \Phi w_t w_t' \Phi$. The proof is analogous to the previous item. First we bound the norm of the expectation of the time average of $Z_t$:

$$
\left\| \mathbb{E} \left[ \frac{1}{\alpha} \sum_t \Phi w_t w_t' \Phi \right] \right\| = \left\| \frac{1}{\alpha} \sum_t \Phi M_{2,t} M_{1,t} P \Delta P' M_{1,t}' M_{2,t}' \Phi \right\|
\leq \left\| \frac{1}{\alpha} \sum_t M_{2,t} M_{1,t} P \Delta P' M_{1,t}' M_{2,t}' \right\|
\leq (a) \left( \left\| \frac{1}{\alpha} \sum_t M_{2,t} M_{1,t} \right\| \left[ \max_t \left\| M_{2,t} M_{1,t} P \Delta P' M_{1,t}' \right\|^2 \right] \right)^{1/2}
\leq (b) \sqrt{b_0} \left[ \max_t \left\| M_{1,t} P \Delta P' M_{1,t}' M_{2,t}' \right\| \right] \leq \sqrt{b_0} q^2 \lambda^+.
$$

where (a) follows from Theorem A.4 with $X_t = M_{2,t}$ and $Y_t = M_{1,t} P \Delta P' M_{1,t}' M_{2,t}'$ and (b) follows from the assumption on $M_{2,t}$. To obtain $R$,

$$
\|Z_t\| = \| \Phi w_t w_t' \Phi \| \leq \max_t \| \Phi M_t P a_t \|^2 \leq q^2 \eta \lambda^+ := R
$$

To obtain $\sigma^2$,

$$
\left\| \frac{1}{\alpha} \sum_t \mathbb{E} \left[ \Phi w_t (\Phi w_t)' (\Phi w_t) w_t' \Phi \right] \right\|
\leq \left( \max_{w_t} \| \Phi w_t \|^2 \right) \| \Phi M_t P \Delta P' M_t' \Phi \|
\leq q^2 \eta \lambda^+ \cdot q^2 \lambda^+ := \sigma^2
$$

Applying Theorem A.5, we have

$$
\Pr \left( \left\| \frac{1}{\alpha} \sum_t \Phi w_t w_t' \Phi \right\| \leq \sqrt{b_0} q^2 \lambda^+ + \epsilon \big| \mathcal{E}_0 \right) \geq 1 - n \exp \left( \frac{-\alpha \epsilon^2}{2(\sigma^2 + R\epsilon)} \right)
$$

Letting $\epsilon = \epsilon_2 \lambda^-$ we get $R/\epsilon = c\eta f$ and $\sigma^2/\epsilon^2 = c\eta f^2$. For the success probability to be of the form $1 - 2n^{-10}$ we require $\alpha \geq \alpha_{(2)} := C\eta \cdot 11f^2 (r \log n)$.

The proof of the last two items follow from using [16, Lemma A.20].

**F Detailed Experimental Results**

In this section we present a detailed discussion of the models used to generate the synthetic data, and also present the results on videos.
F.1 Synthetic Data

**Moving Object Model.** One practical instance where outlier fractions per row can be larger than those per column is in the case of video moving objects that are either occasionally static or slow moving. The outlier support model for our first and second experiments is inspired by this example. It models a 1D video consisting of a person/object of length \( s \) pacing up and down in a room with frequent stops. The object is static for \( \beta \) frames at a time and then moves down. It keeps moving down for a period of \( \tau \) frames, after which it turns back up and does the same thing in the other direction. We let \( \beta = \lceil c_0 \tau \rceil \) for a \( c_0 < 1 \). With this model, for any interval of the form \((k_1 - 1)\tau + 1, k_2\tau\) for \( k_1, k_2 \) integers, the outlier fraction per row is bounded by \( c_0 \). For any general interval of length \( \alpha \geq \tau \), this fraction is still bounded by \( 2c_0 \). The fraction per column is \( s/n \).

**Assumption F.1.** Let \( \beta = \lceil c_0 \tau \rceil \). Assume that the \( T_t \) satisfies the following. For the first \( \tau \) frames (downward motion),

\[
T_t = \begin{cases} 
[1, s], & t \in [1, \beta] \\
[s + 1, 2s], & t \in [\beta + 1, 2\beta] \\
\vdots \\
[(1/c_0 - 1)s + 1, s/c_0], & t \in [\tau - \beta + 1, \tau] 
\end{cases}
\]

for the next \( \tau \) frames (upward motion),

\[
T_t = \begin{cases} 
[(1/c_0 - 1)s + 1, s/c_0], & t \in [\tau + 1, \tau + \beta] \\
[(1/c_0 - 2)s + 1, (1/c_0 - 1)s], & t \in [\tau + \beta + 1, \tau + 2\beta] \\
\vdots \\
[1, s], & t \in [2\tau - \beta + 1, 2\tau]. 
\end{cases}
\]

Starting at \( t = 2\tau + 1 \), the above pattern is repeated every \( 2\tau \) frames until the end, \( t = d \).

The above model is one practically motivated way to simulate data that is not not generated uniformly at random (or as i.i.d. Bernoulli, which is approximately the same as the uniform model for large \( n \)). It also provides a way to generate data with a different bounds on outlier fractions per row and per column. The maximum outlier fraction per column is \( s/n \). For any time interval of length \( \alpha \geq \tau \), the outlier fraction per row is bounded by \( 2c_0 \). A snapshot of the above model is shown in Figure 2.

![Figure 2](image.png)

**Figure 2:** Illustrating the sparse matrix under the moving-object model (Model F.1). For this figure, \( t_{\text{train}} = 200 \). The first \( t_{\text{train}} \) frames have fewer outliers than the rest.
Bernoulli Model. For this model we assume that every entry of a matrix $G$ is chosen independently with a probability $\rho$, i.e., if we let the sampling operator be denoted by $P_{\Omega}$, so that $S = P_{\Omega}(G)$

$$(S)_{ij} = \begin{cases} G_{ij}, & \text{with probability } \rho \\ 0, & \text{with probability } 1 - \rho \end{cases}$$

Under this model, the expected fraction-outliers-per-row and fraction-outliers-per-column is the same and can be verified to be $\rho$. Full Rotation Model. We again assume that the subspace changes every so often and use $t_j$ to denote the $j$-th change time, for $j = 1, 2, \ldots, J$. We let $t_0 := 1$ and $t_{J+1} := d$. Thus, $\ell_t = P(t) a_t$ where $P(t)$ is an $n \times r$ basis matrix with $P(t) = P_j$ for $t \in [t_j, t_{j+1})$, $j = 0, 1, 2, \ldots, J$. The basis matrix $P_j$ changes by the action of a rotation matrix on the left of $P_j$. To be precise, we have

$$P_j = e^{\delta_j B_j} P_{j-1}$$

where $\delta_j$ controls the subspace error and $B_j$ is a skew-Hermitian matrix which ensures that $P_j' P_j = I_r$.

All time comparisons are performed on a Desktop Computer with Intel® Xeon E3-1240 8-core CPU @ 3.50GHz and 32GB RAM.

In this section, we used the following parameters. $n = 1000, d = 10,000, J = 2, t_1 = 3000, t_2 = 6000, r = 30, \delta_1 = 0.01, \delta_2 = 1.5 \delta_1$ and the matrices $B_1$ and $B_2$ are generated using the `skewdec` command in MATLAB. We set $\alpha = 300$. This gives us the basis matrices $P(t)$ for all $t$. To obtain the low-rank matrix $L$ from this we generate the coefficients $a_t \in \mathbb{R}^n$ as independent zero-mean, bounded random variables. They are $(a_t)_i \overset{i.i.d.}{\sim} \text{unif}[-q, q]$ where $q_i = \sqrt{f_j - \sqrt{f_j} r}$ for $i = 1, 2, \ldots, r - 1$ and $q_r = 1$. Thus the condition number is $f$ and we selected $f = 50$. We used the first $t_{train} = 200$ frames as the training data, where we generated a smaller fraction of outliers with parameters $s/n = 0.01, b_0 = 0.01$ and for $t > t_{train}$ we used $s/n = 0.05$ and $b_0 = 0.3$. The sparse outlier magnitudes are generated uniformly at random from the interval $[x_{min}, x_{max}]$ with $x_{min} = 10$ and $x_{max} = 20$.

We initialized all the ReProCS algorithms using AltProj applied to $Y_{[1,t_{train}]}$. For the batch methods used in the comparisons - PCP, AltProj and RPCA-GD, we implement the algorithms on $Y_{[1,t]}$ every $t = t_{train} + k \alpha - 1$ frames. Further, we set the regularization parameter for PCP $1/\sqrt{n}$ in accordance with [1]. The other known parameters, $r$ for Alt-Proj, outlier-fraction for RPCA-GD, are set using the true values. For online methods we implemented the algorithms without modifications. The regularization parameter for ORPCA was set as with $\lambda_1 = 1/\sqrt{n}$ and $\lambda_2 = 1/\sqrt{d}$ according to [13].

For the Bernoulli model we used the same parameters as above for the subspace change while for the sparse outliers we set $\rho = 0.01$ for the first $t_{train}$ frames and $\rho = 0.3$ for the subsequent frames. The sparse outlier magnitudes are generated exactly as above.

F.2 Real Video Experiments

In this section we provide simulation results for on real video, specifically the Meeting Room (MR) sequence. The meeting room sequence is set of 1964 images of resolution $64 \times 80$. The first 1755 frames consists of outlier-free data. Henceforth, we consider only the last 1209 frames. For the MEDRoP algorithm, we used the first 400 frames as the training data. In the first 400 frames, a person wearing a black shirt walks in, writes something on the board and goes back. In the subsequent frames, the person walks in with a white shirt. This is a challenging video sequence because the color of the person and the color of the curtain are hard to distinguish. MEDRoP is able to perform the separation at around 43 frames per second.

We obtained an estimate using the Alt Proj algorithm. For the Alt Proj algorithm we set $r = 40$. The remaining parameters were used with default setting. For the MEDRoP algorithm, we set $\alpha = 60, K = 3$, $\xi_t = \|\Psi \ell_{t-1}\|_2, \theta^- = 20^\circ$. We found that these parameters work for most videos that we verified our algorithm on. For RPCA-GD we set the “corruption fraction” $\alpha = 0.2$ as described in the paper. Lobby dataset: This dataset contains 1555 images of resolution $128 \times 160$. The first 341 frames are outlier free.
Here we use the first 400 “noisy” frames as training data. The Alt Proj algorithm is used to obtain an initial estimate with rank, $r = 40$. The parameters used in all algorithms are exactly the same as above. MEDRoP achieves a “test” processing rate of 16 frames-per-second.

Switch Light dataset: This dataset contains 2100 images of resolution $120 \times 160$. The first 770 frames are outlier free. Here we use the first “noisy” 400 frames as training data. The Alt Proj algorithm is used to obtain an initial estimate with rank, $r = 40$. The parameters used in all algorithms are exactly the same as in the MR and LB dataset. MEDRoP achieves a “test” processing rate of 12 frames-per-second.

Figure 4: Comparison of visual performance in Foreground Background separation for the Lobby (LB) dataset. The recovered background images are shown at $t = t_{\text{train}} + 260, 545, 610$. 

...
Figure 5: Comparison of visual performance in Foreground Background separation for the SwitchLight (SL) dataset. The recovered background images are shown at $t = t_{\text{train}} + 60, 200, 688, 999$. Note that because of the drastic change in environment, none of the techniques are able to detect the object properly.

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