Hölder Continuity and Differentiability Almost Everywhere of \((K_1, K_2)\)-Quasiregular Mappings

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Abstract. This paper deals with \((K_1, K_2)\)-quasiregular mappings. It is shown, by Morrey’s Lemma and isoperimetric inequality, that every \((K_1, K_2)\)-quasiregular mapping satisfies a Hölder condition with exponent \(\alpha\) on compact subsets of its domain, where

\[
\alpha = \begin{cases} 
1/K_1, & \text{for } K_1 > 1, \\
\text{any positive number less than } 1, & \text{for } K_1 = 1 \text{ and } K_2 > 0, \\
1, & \text{for } K_1 = 1 \text{ and } K_2 = 0, \\
1, & \text{for } K_1 < 1. 
\end{cases}
\]

Differentiability almost everywhere of \((K_1, K_2)\)-quasiregular mappings is also derived.

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§1 Introduction and Statement of Results

Let \(\Omega\) be an arbitrary open set in \(\mathbb{R}^n\), \(n \geq 2\). For any point \(x \in \Omega\) and \(r > 0\), we denote by \(B(x, r)\) the ball with radius \(r\) centered at \(x\) and \(S(x, r) = \partial B(x, r)\) the sphere of \(B(x, r)\). Let \(|B(x, r)| = \omega_n r^n\) be the \(n\)-dimensional Lebesgue measure of the ball \(B(x, r)\), where \(\omega_n\) be the volume of the unit ball in \(\mathbb{R}^n\). Denote by \(\rho_\Omega(x) = \text{dist}(x, \partial \Omega)\) the distance from \(x\) to \(\partial \Omega\), with the subscript \(\Omega\) omitted whenever no confusion can result.

For \(p \geq 1\), we denote by \(L^p(\Omega)\) the \(L^p\) space of functions on \(\Omega\), \(W^{1,p}(\Omega)\) will denote the corresponding Sobolev space of functions in \(L^p(\Omega)\) whose distributional first derivatives belong also to the space \(L^p(\Omega)\). Similarly, \(W^{1,p}(\Omega, \mathbb{R}^n)\) will be the space of functions \(f = (f^1, f^2, \cdots, f^n) : \Omega \to \mathbb{R}^n\) such that \(f^i \in W^{1,p}(\Omega)\) for \(i = 1, 2, \cdots, n\).

For \(A\) an \(n \times n\) matrix, we define the norm of \(A\) as \(|A| = \sup_{|\xi| = 1} |A\xi|\).

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We say that a function \( f : \Omega \to \mathbb{R}^n \) does not change sign in \( \Omega \) if either \( u(x) \geq 0 \) almost everywhere in \( \Omega \) or \( u(x) \leq 0 \) almost everywhere in \( \Omega \).

A mapping \( f : \Omega \to \mathbb{R}^n \) is said to be satisfy Hölder condition with exponent \( \alpha \) on compact subsets of \( \Omega \), where \( 0 < \alpha \leq 1 \), if for every compact set \( V \subset \subset \Omega \) there is a number \( M(V), 0 \leq M(V) < +\infty \), such that for any \( x_1, x_2 \in V, \)

\[
|f(x_1) - f(x_2)| \leq M(V)|x_1 - x_2|^\alpha.
\]

If \( f \) satisfies a Hölder condition with exponent \( \alpha = 1 \) on compact subsets of \( \Omega \), then \( f \) is said to satisfy a Lipschitz condition on compact subsets of \( \Omega \).

Let \( f = (f^1, f^2, \cdots, f^n) : \Omega \to \mathbb{R}^n \) be a mapping in \( W^{1,n}(\Omega, \mathbb{R}^n) \). The linear mapping

\[
Df(x) = \left( \frac{\partial f^i}{\partial x_j} \right)_{1 \leq i, j \leq n}
\]

is defined for almost all \( x \in \Omega \). Its determinate \( \det Df(x) \) is called the Jacobian of \( f \) at the point \( x \), and is denoted by \( J(x, f) \).

In [1], Zheng and Fang gave the definition for \((K_1, K_2)\)-quasiregular and quasiconformal mappings.

**Definition 1.1.** A mapping \( f = (f^1, f^2, \cdots, f^n) : \Omega \to \mathbb{R}^n \) is called \((K_1, K_2)\)-quasiregular with \( 0 < K_1 < +\infty \), \( 0 \leq K_2 < +\infty \), if it satisfies the following conditions:

(i) \( f \) belongs to the class \( W^{1,n}(\Omega, \mathbb{R}^n) \),

(ii) the Jacobian \( J(x, f) \) does not change sign in \( \Omega \), and

\[
|Df(x)|^n \leq K_1|J(x, f)| + K_2
\]

for almost all \( x \in \Omega \).

A mapping \( f = (f^1, f^2, \cdots, f^n) : \Omega \to \mathbb{R}^n \) is said to be \((K_1, K_2)\)-quasiconformal if it satisfies (i), (ii), and

(iii) \( f \) is a homeomorphism.

The estimate of the modulus of continuity of \((K_1, 0)\)-quasiconformal mappings was first established by Kreines [2]. The Hölder property was first proved for a \((K_1, 0)\)-quasiregular mapping by Reshetnyak [3,4], and simultaneously by Callender [5]. Simon [6] established an estimate of Hölder continuity when he studied \((K_1, K_2)\)-quasiconformal mappings between two surfaces of the Euclidean space \( \mathbb{R}^3 \). This estimate has important applications to elliptic equations with two variables. In [7], Gilbarg and Trudinger obtained an \( a \text{ priori} \) \( C^{1,\alpha}_{loc} \) estimate for quasilinear elliptic equations with two variables by using the Hölder continuity method established in the study of plane
\((K_1, K_2)\)-quasiregular mappings, and then established the existence theorem of Dirichlet problem. Many results on quasiregular mappings and their applications to nonlinear PDEs and elasticity theory have been established recently, see [8-10] and the references therein.

Because of the importance of plane \((K_1, K_2)\)-quasiregular mappings to the \textit{a priori} estimates in nonlinear PDE theory, Zheng and Fang [1] developed the theory of \((K_1, K_2)\)-quasiregular mappings in 1998 by using the theory of outer differential forms and Grassman algebra, and obtained an \(L^p\)-integrability \((p > n)\) result for space \((K_1, K_2)\)-quasiregular mappings. For some other developments on \((K_1, K_2)\)-quasiregular mapping theory, see [11-15].

It is a typical situation in quasiconformal analysis that one wants to build up the Hölder continuity theory for \((K_1, K_2)\)-quasiregular mappings. In this paper, we generalize the results of [1,11,13], and the following Hölder continuity result is obtained.

**Theorem 1.1.** Let \(f: \Omega \rightarrow \mathbb{R}^n\) be a \((K_1, K_2)\)-quasiregular mapping. Assume that
\[
\int_{\Omega} |Df(x)|^n dx = M < +\infty.
\]
Then the function \(f\) satisfies a Hölder condition with exponent \(\alpha\) on compact subsets of \(\Omega\), where
\[
\alpha = \begin{cases} 
1/K_1, & \text{for } K_1 > 1, \\
\text{any positive number less than } 1, & \text{for } K_1 = 1 \text{ and } K_2 > 0, \\
1, & \text{for } K_1 = 1 \text{ and } K_2 = 0, \\
1, & \text{for } K_1 < 1.
\end{cases}
\] (1.2)

Further, if \(V\) is contained strictly inside \(\Omega\), then for any \(x, y \in V\)
\[
|f(x) - f(y)| \leq L|x - y|^\alpha,
\]
where the constant \(L\) depends only on \(V\), the constants \(K_1\) and \(K_2\), the dimension \(n\), the distance from \(V\) to the boundary of \(\Omega\), and the constant \(M\).

**A counterexample** The mapping \(f\) with \(f(0) = 0\) and \(f: x \mapsto x|x|^{\alpha-1}\) for \(x \neq 0\), where \(\alpha = 1/K_1\), shows that the exponent \(1/K_1\) in Theorem 1.1 is optimal. For this \(f\) we have \(|f(x) - f(0)| = |x|^\alpha\).

The following corollary is a direct consequence of Theorem 1.1.

**Corollary 1.1.** Let \(\Omega\) be an open domain in \(\mathbb{R}^n\), and \(F(\Omega, K_1, K_2, M)\) the collection of all \((K_1, K_2)\)-quasiregular mappings \(f\) on \(\Omega\) such that
\[
\int_{\Omega} |Df(x)|^n dx \leq M.
\]
Then the set of functions $f$ is equi-uniformly continuous on every compact subset of $\Omega$.

**Definition 1.2.** The mapping $f$ is said to have property $N$ if the image of every set $E \subset \Omega$ of measure zero is a set of measure zero.

**Corollary 1.2.** Let $f : \Omega \to \mathbb{R}^n$ be a $(K_1, K_2)$-quasiregular mapping with $0 < K_1 < 1$, or $K_1 = 1$ and $K_2 = 0$, then $f$ has property $N$.

**Proof.** [16, Theorem 2.2] states that every locally Lipschitz mapping has property $N$, which together with Theorem 1.1 yields the desired result.

**Definition 1.3.** A mapping $f : \Omega \to \mathbb{R}^n$ is said to be differentiable at a point $a \in \Omega$ if there exists a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(x) = f(a) + L(x - a) + \beta(x)|x - a|$$

for all $x \in \Omega$, where $\beta(x) \to 0$ as $x \to a$. The mapping $L$ is called the differential of $f$ at the point $a$.

The following theorem states that any $(K_1, K_2)$-quasiregular mapping $f$ is differential almost everywhere.

**Theorem 1.2.** Let $f$ be a $(K_1, K_2)$-quasiregular mapping. Then for almost all $x \in \Omega$ the linear mapping $Df(x)$ is the differential of $f$ at the point $x$.

**Proof.** The proof of Theorem 1.2 is almost line by line of the proof of [16, Theorem 1.2] by using Corollary 1.1. We omit the details.

§2 Preliminary Lemmas

The proof of Theorem 1.1 is based on two facts. The first is a lemma due to Morrey. The second is an isoperimetric inequality due to Reshetnyak.

**Lemma 2.1.** (Morrey’s Lemma [17]) Let $\Omega \subset \mathbb{R}^n$ be an open subset, and $f : \Omega \to \mathbb{R}^k$ a function of the class $W^{1,m}(\Omega, \mathbb{R}^k)$, where $1 \leq m \leq n$. Assume that there exist numbers $\alpha(0 < \alpha \leq 1)$, $M < +\infty$, and $\delta > 0$ such that

$$\int_{B(a,r)} |Df(x)|^m dx \leq M r^{n-m+\alpha}$$

for every ball $B(a,r) \subset \Omega$ with radius at most $\delta$. Then there exists a continuous function $\tilde{f}$ such that $f(x) = \tilde{f}(x)$ almost everywhere, and the oscillation of $\tilde{f}$ on any ball $B(x,r) \subset \Omega$ with $r \leq \delta/3$ and $r < \rho(x)/3$ does not exceed $CM^{1/m}r^{\alpha}$, where $C < +\infty$ is a constant.
Lemma 2.2. (Isoperimetric Inequality [16]) Suppose that $\Omega \subset \mathbb{R}^n$ and the mapping $f : \Omega \to \mathbb{R}^n$ is in the class $W^{1,n}(\Omega, \mathbb{R}^n)$. Then for any $a \in U$ and almost all $t \in (0, \rho(a))$

$$\int_{B(a,t)} \mathcal{J}(x, f) dx \leq \frac{t}{n} \int_{S(a,t)} |Df(z)|^n d\sigma(z),$$

(2.2)

where $d\sigma$ is the area element of the sphere $S(a,t)$.

With the Morrey’s Lemma and isoperimetric inequality in hands, we can now prove the following two lemmas, which will be used in the proof of Theorem 1.1.

Lemma 2.3. Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and $f$ a $(K_1, K_2)$-quasiregular mapping. For $x \in \Omega$ and $r < \rho(x)$ let

$$v(x, K_1, K_2, n, r) = \left\{ \begin{array}{ll}
\frac{w(r)}{r^{n/K_1}} + \frac{K_2 \omega_n r^{n-K_1}}{K_1-1}, & \text{for } K_1 \neq 1, \\
\frac{w(r)}{r^n} + K_2 \omega_n \ln r, & \text{for } K_1 = 1,
\end{array} \right.$$  

(2.3)

where

$$w(r) = \int_{B(x,r)} |Df(x)|^n dx.$$

Then the function $r \mapsto v(x, K_1, K_2, n, r)$ is nondecreasing.

Proof. For $r < \rho(x)$, (1.1) leads to

$$\int_{B(x,r)} |Df(x)|^n dx \leq K_1 \int_{B(x,r)} \mathcal{J}(x, f) dx + K_2 \int_{B(x,r)} dx$$

$$= K_1 \int_{B(x,r)} \mathcal{J}(x, f) dx + K_2 |B(x, r)|$$

(2.4)

because $\mathcal{J}(x, f)$ does not change sign in $\Omega$. On the basis of Lemma 2.2

$$\int_{B(x,r)} \mathcal{J}(x, f) dx \leq \frac{r}{n} \int_{S(x,r)} |Df(x)|^n d\sigma(x)$$

(2.5)

for almost all $r \in (0, \rho(x))$. From (2.4) and (2.5) we get

$$\int_{B(x,r)} |Df(x)|^n dx \leq \frac{K_1 r}{n} \int_{S(x,r)} |Df(x)|^n d\sigma(x) + K_2 \omega_n r^n.$$

(2.6)

Let

$$\int_{S(x,r)} |Df(x)|^n d\sigma(x) = s(r).$$

Applying Fubini’s theorem, we get that $w(r) = \int_0^r s(t) dt$ for all $r \in (0, \rho(x))$. This leads us to conclude that the function $w$ is absolutely continuous and $w'(r) = s(r)$ for almost all $r \in (0, \rho(x))$. From (2.6) we have that

$$w(r) \leq \frac{K_1 r w'(r)}{n} + K_2 \omega_n r^n$$
for almost all \( r \). This is equivalent to
\[
\frac{K_1 r w'(r)}{n} - w(r) + K_2 \omega_n r^n \geq 0.
\]
Multiplying both sides of this inequality by \( r^{-(n/K_1) - 1} \) yields
\[
\frac{K_1 w'(r)}{n r^{n/K_1}} - \frac{w(r)}{r^{n/K_1 + 1}} + K_2 \omega_n r^{n-n/K_1 - 1} \geq 0.
\]
We get after obvious transformations that
\[
\frac{\partial v(x, K_1, K_2, n, r)}{\partial r} \geq 0,
\]
where \( v(x, K_1, K_2, n, r) \) is defined by (2.3). Consequently, the function \( r \mapsto v(x, K_1, K_2, n, r) \) is nondecreasing, as desired.

Lemma 2.4. Suppose that \( \Omega \subset \mathbb{R}^n \) is an open set and \( f \) a \((K_1, K_2)\)-quasiregular mapping. Let \( \int_{\Omega} |Df(x)|^n dx = M^n \). Then the vector-valued function \( f \) is equivalent, in the sense of the theory of integral, to some continuous function \( \tilde{f} \). Further, for every set \( V \) lying strictly inside \( \Omega \) the oscillation of \( \tilde{f} \) on any ball \( B(a, r) \) of radius \( r < 2d/3 \) about an \( a \in V \) does not exceed \( C r^\alpha \), where \( d = \text{dist}(V, \partial \Omega) \).

Proof. Let \( a \in V \) and
\[
w(a, r) = \int_{B(a, r)} |Df(x)|^n dx \leq M^n.
\]
According to Lemma 2.3, the function \( r \mapsto v(a, K_1, K_2, n, r) \) is nondecreasing. We now divide the proof into four cases.

Case 1 \( K_1 > 1 \). In this case,
\[
v(a, K_1, K_2, n, r) = w(a, r) + \frac{K_2 \omega_n}{K_1 - 1} r^{n-n/K_1} \leq w(a, K_1, K_2, n, 2d/3) \leq M^n (2d/3)^{n-n/K_1} + \frac{K_2 \omega_n}{K_1 - 1} (2d/3)^{n-n/K_1} \leq C_1(K_1, K_2, M, d, n),
\]
for all \( r \in (0, 2d/3) \); hence
\[
w(a, r) \leq C_1 r^{n/K_1} - \frac{K_2 \omega_n}{K_1 - 1} r^n \leq C_1 r^{n/K_1}.
\]

Case 2 \( K_1 = 1 \) and \( K_2 > 0 \). It is no loss of generality to assume that \( d < 3/2 \),
\[
v(a, K_1, K_2, n, r) = w(a, r) + K_2 \omega_n \ln r \leq w(a, K_1, K_2, n, 2d/3) \leq M^n (2d/3)^{-n} + K_2 \omega_n \ln(2d/3).
\]

(2.8)
This implies, for any $0 < \alpha < 1$,

$$w(a,r) \leq [M^n(2d/3)^{-n} + K_2\omega_n\ln(2d/3)]r^n - K_2\omega_n r^n \ln r$$
$$= M^n(2d/3)^{-n}r^n + K_2\omega_n r^n \ln(2d/3r)$$
$$= \left[M^n(2d/3)^{-n}r^n(1-\alpha) + K_2\omega_n r^n(1-\alpha) \ln(2d/3r)\right] r^n \alpha. \quad (2.9)$$

since $\lim_{r \to 0^+} r^n(1-\alpha) \ln(2d/3r) = 0$ for any $0 < \alpha < 1$, then we take $\delta$ such that $\delta^n(1-\alpha) \ln(2d/3) \leq 1$. When $0 < r \leq \delta$, we have from (2.9) that

$$w(a,r) \leq \left[\left(\frac{3M}{2}\right)^n d^{-n} + 1\right] r^n \alpha := C_2 r^n \alpha.$$

Case 3 $K_1 = 1$ and $K_2 = 0$. (2.8) implies

$$w(a,r) \leq M^n(2d/3)^{-n}r^n := C_3 r^n.$$

Case 4 $K_1 < 1$. In this case, (2.7) also holds for all $r \in (0, 2d/3)$, thus

$$w(a,r) \leq C_1 r^{n/K_1} + \frac{K_2\omega_n}{1-K_1} r^n = \left[C_1 r^{n/K_1} + \frac{K_2\omega_n}{1-K_1}\right] r^n \leq \left[C_1 (2d/3)^{n(1-K_1)/K_1} + \frac{K_2\omega_n}{1-K_1}\right] r^n := C_4 r^n.$$

In all the cases we have derived that for $0 < r \leq \delta$,

$$\int_{B(a,r)} |Df(x)|^n dx \leq C r^\alpha,$$

where $\alpha$ is defined as (1.2) and $C$ depends only on $K_1, K_2, M, d, n$. The required result follows directly from Lemma 2.1.

\section*{§3 Proof of Theorem 1.1}

\textbf{Proof.} Let $V$ be a compact subset of $\Omega$, and let $\gamma$ be the smaller of the numbers $\delta/3$ and $d/3$, where $\delta$ is the constant in Lemma 2.1 and $d = \text{dist}(V, \partial \Omega)$. We consider the function $h$ defined as follows on the product $V \times V : h(x,y) = |f(x) - f(y)|/|x - y|^{\alpha}$ for $x \neq y$, and $h(x,x) = 0$. Let $H$ be the set of pairs $(x,y) \in V \times V$ such that $|x - y| \geq \gamma$, and let $G = (V \times V) \setminus H$. The set $H$ is compact, and thus $h$ is bounded on $H$ by continuity. The conclusion of Lemma 2.4 enables us to deduce that $h$ is bounded also on $G$. Consequently, $h$ is bounded on $V \times V$, and thus $|f(x) - f(y)| \leq L|x - y|^{\alpha}$ for any $x, y \in V$.

\hfill $\square$
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