The Orbital Chromatic Polynomial of a Cycle

Klaus Dohmen
Department of Mathematics
Mittweida University of Applied Sciences
09648 Mittweida, Germany

September 18, 2020

Abstract

Abstract. The orbital chromatic polynomial introduced by Cameron and Kayibi counts the number of proper λ-colorings of a graph modulo a group of symmetries. In this paper, we establish expansions for the orbital chromatic polynomial of the n-cycle for the group of rotations and its full automorphism group. Besides, we obtain a new proof of Fermat’s Little Theorem.

Keywords. chromatic polynomial, cycle, graph, automorphism group, dihedral group, totient function, cycle index, Fermat’s Little Theorem

Mathematics Subject Classification (2020). 05A05, 05A15, 05C15, 05C31, 05E18, 11A07, 20B05

1 Introduction

For more than one century the chromatic polynomial of a finite graph has gained considerable attention in combinatorial mathematics. It was originally introduced by Birkhoff [1] in 1912 to tackle the four-color problem. The idea was to count vertex colorings not just with four, but with an arbitrary number of available colors such that neighboring vertices receive different colors, and then to apply analytic tools.
In this paper, all graphs are considered as finite and undirected, and they may have loops and multiple edges. For any graph $\Gamma$, a *proper $\lambda$-coloring* of $\Gamma$ is a mapping $f$ from the vertex-set of $\Gamma$ to $\{1, \ldots, \lambda\}$ such that $f(v) \neq f(w)$ for any adjacent vertices $v$ and $w$ of $\Gamma$. Due to Birkhoff [1], the number of proper $\lambda$-colorings of $\Gamma$ is a polynomial in $\lambda$ of degree $n(\Gamma)$, where $n(\Gamma)$ denotes the number of vertices of $\Gamma$. This polynomial is called the *chromatic polynomial* of $\Gamma$, and denoted by $P_{\Gamma}(\lambda)$.

The chromatic polynomial has been computed for many classes of graphs. For some classes, it can even be expressed in closed-form, e.g., $\lambda(\lambda - 1)^{n-1}$ is the chromatic polynomial of a path on $n$ vertices, while for the cycle $\Gamma_n$ on $n$ vertices we have

$$P_{\Gamma_n}(\lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1).$$

The proofs of these statements are straightforward and can be found in [6]. We will take up these statements in Section 4 again where we consider Eq. (1) modulo a group of symmetries.

Chromatic polynomials modulo a group of symmetries were first investigated by Cameron and Kayibi in [2]. They introduced the so-called *orbital chromatic polynomial*, which generalizes the chromatic polynomial of a graph for any subgroup of its automorphism group. In [2, 3, 4], this new polynomial has been investigated for specific graphs, including the Petersen graph, complete graphs, null graphs, paths, and cycles of small length. So far, no general formula for the orbital chromatic polynomial of the $n$-cycle has been established. In this paper we present such formulae for the group of rotations and the full automorphism group of the $n$-cycle.

The paper is organised as follows: Section 2 recalls the definition of the orbital chromatic polynomial, following the exposition in [2] with a minor modification, along with the automorphism group of the $n$-cycle. In Section 3 we consider the structure of a graph which is derived from the $n$-cycle and each of its automorphisms by identifying the vertices in each cycle of the disjoint cycle decomposition of the respective automorphism, thus giving strict evidence to some unproved statements in [5], which are used in Section 4 to prove our main results on the orbital chromatic polynomial of the $n$-cycle. Along the way, and as a consequence of one of our main results, a new proof of Fermat’s Little Theorem is obtained.
2 The orbital chromatic polynomial

To prepare the definition of the orbital chromatic polynomial, for any graph \( \Gamma \) and any permutation \( g \) of the vertex-set of \( \Gamma \) we use \( \Gamma/g \) to denote the graph which arises from \( \Gamma \) by identifying the vertices within each cycle of the disjoint cycle decomposition of \( g \) (in order words, contracting them to a single vertex), and then replacing all parallel edges by single edges. The removal of parallel edges does not comply with the definition of \( \Gamma/g \) in [2, 5], but since parallel edges have no impact on the chromatic polynomial, we may safely remove them. We will benefit from this simplification in the next section.

**Definition 1** ([2]). For any finite graph \( \Gamma \) and any group \( G \) of automorphisms of \( \Gamma \), the orbital chromatic polynomial of \( \Gamma \) relative to \( G \) is defined as

\[
OP_{\Gamma,G}(\lambda) = \frac{1}{|G|} \sum_{g \in G} P_{\Gamma/g}(\lambda).
\]

**Remark 1.** For the trivial group \( G \) consisting only of the identity permutation, the orbital chromatic polynomial coincides with the chromatic polynomial.

The following proposition, due to Cameron and Kayibi [2], provides a combinatorial interpretation of the orbital chromatic polynomial. Its proof is based on Burnside’s lemma on orbit counting and the establishment of a one-to-one correspondence between the proper \( \lambda \)-colorings of \( \Gamma/g \) and the proper \( \lambda \)-colorings of \( \Gamma \) which are fixed by \( g \).

**Proposition 1** ([2]). Let \( \Gamma \) be a finite graph and \( G \) be a group of automorphisms of \( \Gamma \). Then, for any \( \lambda \in \mathbb{N} \), \( OP_{\Gamma,G}(\lambda) \) counts the number of equivalence classes of proper \( \lambda \)-colorings of \( \Gamma \) where \( f \) and \( f' \) are equivalent if \( f' = f \circ g \) for some \( g \in G \).

In the sequel of our paper, we consider the particular case where \( \Gamma \) is the cycle graph \( \Gamma_n \) on vertex-set \( \{0, \ldots, n-1\} \) (\( n \geq 1 \)). Note that for \( n = 1 \) resp. \( n = 2 \), \( \Gamma_n \) consists of one vertex with a loop attached to it resp. of two vertices joined by two parallel edges.

The content of the following proposition is well-known.
Proposition 2. For any $n \in \mathbb{N}$, the automorphism group of $\Gamma_n$ consists of the rotations $r_0, \ldots, r_{n-1}$, and of the reflections $s_0, \ldots, s_{n-1}$ if $n$ is odd, respectively $s_0, \ldots, s_{n/2-1}$, $s'_0, \ldots, s'_{n/2-1}$ if $n$ is even, where for $m = 0, \ldots, n-1$ and $v = 0, \ldots, n-1$,

$$
\begin{align*}
    r_m(v) &= (v + m) \mod n, \\
    s_m(v) &= (2m - v) \mod n, \\
    s'_m(v) &= (2m + 1 - v) \mod n.
\end{align*}
$$

For $n \geq 1$ we use $\text{Aut}(\Gamma_n)$ to denote the automorphism group of $\Gamma_n$, and $\text{Rot}(\Gamma_n) = \{r_0, \ldots, r_{n-1}\}$ its subgroup of rotations. For $n \geq 3$, $\text{Aut}(\Gamma_n)$ is known as the dihedral group $D_n$, where $|D_n| = 2n$. Note that $\text{Aut}(\Gamma_1) = \text{Rot}(\Gamma_1)$ and $\text{Aut}(\Gamma_2) = \text{Rot}(\Gamma_2)$.

3 The structure of $\Gamma_n/g$ for $g$ in $\text{Aut}(\Gamma_n)$

The statements of this section appear similarly in [5], but without proof and based on a different definition of $\Gamma/g$. For completeness, we provide strict evidence of them.

Proposition 3. For any $n \in \mathbb{N}$ and $m = 0, \ldots, n-1$ the graph $\Gamma_n/r_m$ is

(a) a cycle of length $\gcd(n, m)$ if $\gcd(n, m) \neq 2$;

(b) a path of length 1 if $\gcd(n, m) = 2$.

Proof. Let $\sigma_0 \sigma_1 \ldots \sigma_{k-1}$ be the disjoint cycle decomposition of $r_m$ where

$$
\sigma_i = (i, (i+m) \mod n, (i+2m) \mod n, \ldots, (i-m) \mod n) \quad (i = 0, \ldots, k-1).
$$

(3)

Following the construction of $\Gamma_n/r_m$ we identify distinct vertices $x$ and $y$ of $\Gamma_n$ if they belong to the same cycle in the cycle decomposition of $r_m$. In view of Eq. (3) this means that $x$ and $y$ are identified if for some $z \in \mathbb{Z}$, $z \equiv x \pmod{m}$ and $z \equiv y \pmod{n}$. By the generalized Chinese Remainder Theorem there is a simultaneous solution for $z$ if and only if $x \equiv y \pmod{\gcd(n, m)}$. Thus, each cycle $\sigma_i$ can be regarded as (an arrangement of) the intersection of $\{0, \ldots, n-1\}$ with the residue class of $i$ modulo $k = \gcd(n, m)$. 


Each cycle $\sigma_i$ of $r_m$ defines a vertex $\overline{\sigma_i}$ in $\Gamma_n/r_m$, and two (not necessarily distinct) vertices $\overline{\sigma_i}$ and $\overline{\sigma_j}$ are joined by an edge in $\Gamma_n/r_m$ if there are $v \in \sigma_i$ and $w \in \sigma_j$ such that $v$ and $w$ are adjacent in $\Gamma_n$, that is, $v \equiv i \pmod{k}$ and $w \equiv j \pmod{k}$ for some $v, w \in \{0, \ldots, n-1\}$ satisfying $v \equiv w \pm 1 \pmod{n}$, which implies $i \equiv j \pm 1 \pmod{k}$ since $k \mid n$. On the other hand, if $i \equiv j \pm 1 \pmod{k}$ we show that $\overline{\sigma_i}$ and $\overline{\sigma_j}$ are joined by an edge in $\Gamma_n/r_m$. Without loss of generality we may assume that $i \equiv j + 1 \pmod{k}$, otherwise we exchange $i$ and $j$. We distinguish two cases:

Case 1: If $i > 0$, then $i = j + 1$. Since $i$ and $j$ are adjacent in $\Gamma_n$, $\overline{\sigma_i}$ and $\overline{\sigma_j}$ are joined by an edge in $\Gamma_n/r_m$.

Case 2: If $i = 0$, then $j = k - 1 \equiv n - 1 \pmod{k}$ since $k \mid n$. Therefore, $0 \in \sigma_i$ and $n - 1 \in \sigma_j$. Since $0$ and $n - 1$ are adjacent in $\Gamma_n$, $\overline{\sigma_i}$ and $\overline{\sigma_j}$ are adjacent in $\Gamma_n/r_m$.

Hence, for $k = \gcd(n, m)$, $\Gamma_n/r_m$ consists of the cycle $(\overline{\sigma_0}, \overline{\sigma_1}, \ldots, \overline{\sigma_{k-1}}, \overline{\sigma_0})$ if $k > 2$, of the path $(\overline{\sigma_0}, \overline{\sigma_1})$ if $k = 2$, and of the loop on $\overline{\sigma_0}$ if $k = 1$. \hfill $\square$

**Proposition 4.** For any $n \in \mathbb{N}$ and $m = 0, \ldots, n - 1$, $\Gamma_n/s_m$ is a path of length $\lfloor n/2 \rfloor$, with a loop attached to one of its end vertices if $n$ is odd.

**Proof.** Let $\sigma_0 \sigma_1 \ldots \sigma_{\lfloor n/2 \rfloor}$ be the disjoint cycle decomposition of $s_m$ where $$\sigma_i = ((m - i) \bmod{n}, (m + i) \bmod{n}) \quad (i = 0, \ldots, \lfloor n/2 \rfloor).$$

Each cycle $\sigma_i$ defines a vertex $\overline{\sigma_i}$ in $\Gamma_n/s_m$, and two (not necessarily distinct) vertices $\overline{\sigma_i}$ and $\overline{\sigma_j}$ are adjacent in $\Gamma_n/s_m$ if there are adjacent vertices $v, w$ in $\Gamma_n$ such that $v \in \sigma_i$ and $w \in \sigma_j$, that is, $v \equiv w \pm 1 \pmod{n}$, $v = (m \pm i) \bmod{n}$, and $w = (m \pm j) \bmod{n}$. The conjunction of these three conditions is equivalent to $i = j \pm 1$ or $i = j = (n - 1)/2$, where for the second alternative $n$ is required to be odd. This shows that $\Gamma_n/s_m$ consists of the path $(\overline{\sigma_0}, \overline{\sigma_1}, \ldots, \overline{\sigma_{\lfloor n/2 \rfloor}})$ with an additional loop at $\overline{\sigma_{\lfloor n/2 \rfloor}}$ in case that $n$ is odd. \hfill $\square$

**Proposition 5.** For any even $n \in \mathbb{N}$ and $m = 0, \ldots, \frac{n}{2} - 1$, $\Gamma_n/s'_m$ is a path of length $\frac{n}{2} - 1$ with a loop attached to each of its end vertices.

**Proof.** Let $\sigma_0 \sigma_1 \ldots \sigma_{n/2 - 1}$ be the disjoint cycle decomposition of $s'_m$ where $$\sigma_i = ((m - i) \bmod{n}, (m + i + 1) \bmod{n}) \quad (i = 0, \ldots, n/2 - 1).$$

Similar to the preceding proof, $\Gamma_n/s'_m$ has vertices $\overline{\sigma_0}, \ldots, \overline{\sigma_{n/2-1}}$, and two of them, $\overline{\sigma_i}$ and $\overline{\sigma_j}$ (not necessarily distinct) are adjacent in $\Gamma_n/s'_m$ if there are
adjacent vertices $v, w \in \Gamma_n$ such that $v \in \sigma_i$ and $w \in \sigma_j$, that is, $v \equiv w \pm 1 \pmod{n}$, $v = (m-i) \pmod{n}$ or $v = (m+i+1) \pmod{n}$, and $w = (m-j) \pmod{n}$ or $w = (m+j+1) \pmod{n}$. The conjunction of these three conditions is equivalent to $i = j \pm 1$ or $i = j = 0$ or $i = j = n/2 - 1$. Therefore, $\Gamma_n/s_m'$ consists of the path $(\sigma_0, \sigma_1, \ldots, \sigma_{n/2-1})$ with loops at $\sigma_0$ and $\sigma_{n/2-1}$. □

4 Results on the $n$-cycle

In the sequel, $\phi(n)$ denotes the Euler totient of $n$, that is, the number of positive integers less than or equal to $n$ and coprime with $n$. The symbol $|$ denotes the divisibility relation.

**Theorem 1.** For any odd $n \in \mathbb{N}$,

$$OP_{\Gamma_n, \text{Rot}(\Gamma_n)}(\lambda) = \frac{1}{n} \sum_{1 \leq d \leq n \atop d | n} \phi(n/d)(\lambda - 1)^d - \lambda + 1.$$  \hfill (4)

For any even $n \in \mathbb{N}$,

$$OP_{\Gamma_n, \text{Rot}(\Gamma_n)}(\lambda) = \frac{1}{n} \sum_{1 \leq d \leq n \atop d | n} \phi(n/d)(\lambda - 1)^d.$$  \hfill (5)

**Proof.** By Definition 4 and Proposition 3, and since the chromatic polynomial of a path of length 1 coincides with the chromatic polynomial of cycle of length 2,

$$OP_{\Gamma_n, \text{Rot}(\Gamma_n)}(\lambda) = \frac{1}{n} \sum_{m=0}^{n-1} P_{\Gamma_n/\sigma_m}(\lambda) = \frac{1}{n} \sum_{m=0}^{n-1} P_{\Gamma_{\gcd(n,m)}}(\lambda).$$  \hfill (6)

Using the identity $\gcd(n, m) = \gcd(n, n - m)$ and rearranging terms we obtain

$$OP_{\Gamma_n, \text{Rot}(\Gamma_n)}(\lambda) = \frac{1}{n} \sum_{m=0}^{n-1} P_{\Gamma_{\gcd(n,n-m)}}(\lambda) = \frac{1}{n} \sum_{m=1}^{n} P_{\Gamma_{\gcd(n,m)}}(\lambda).$$

With $\phi_d(n) = \#\{m \in \{1, \ldots, n\} \mid \gcd(n, m) = d\}$ (1 \leq d \leq n) we conclude that

$$OP_{\Gamma_n, \text{Rot}(\Gamma_n)}(\lambda) = \frac{1}{n} \sum_{1 \leq d \leq n \atop d | n} \phi_d(n)P_{\Gamma_d}(\lambda) = \frac{1}{n} \sum_{1 \leq d \leq n \atop d | n} \phi(n/d)P_{\Gamma_d}(\lambda).$$
and hence by Eq. (1),

\[ OP_{\Gamma_n, \text{Rot}(\Gamma_n)}(\lambda) = \frac{1}{n} \sum_{\substack{1 \leq d \leq n \\ \text{divides} \ n}} \phi(n/d) ((\lambda - 1)^d + (-1)^d(\lambda - 1)) . \]

From this, the result follows since

\[ \sum_{\substack{1 \leq d \leq n \\ \text{divides} \ n}} (-1)^d \phi(n/d) = \begin{cases} -n, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even}. \end{cases} \]

**Remark 2.** For a prime \( p \), Eq. (4) states that \( OP_{\Gamma_p, \text{Rot}(\Gamma_p)}(\lambda) = \frac{1}{p}((\lambda - 1)^p + (p-1)(\lambda-1))-\lambda+1 \), which implies Fermat’s Little Theorem: \((\lambda - 1)^p \equiv \lambda - 1 \pmod{p} \).

**Theorem 2.** For any odd \( n \in \mathbb{N} \),

\[ OP_{\Gamma_n, \text{Aut}(\Gamma_n)}(\lambda) = \frac{1}{2n} \sum_{\substack{1 \leq d \leq n \\ \text{divides} \ n}} \phi(n/d)(\lambda - 1)^d - \frac{\lambda}{2} + \frac{1}{2}. \]  

(7)

For any even \( n \in \mathbb{N} \),

\[ OP_{\Gamma_n, \text{Aut}(\Gamma_n)}(\lambda) = \frac{1}{2n} \sum_{\substack{1 \leq d \leq n \\ \text{divides} \ n}} \phi(n/d)(\lambda - 1)^d + \frac{\lambda}{4}(\lambda - 1)^{n/2}. \]  

(8)

**Proof.** Since \( \text{Aut}(\Gamma_n) = \text{Rot}(\Gamma_n) \) for \( n = 1, 2 \) and the right-hand sides of Eqs. (4) and (7) resp. (5) and (8) agree for \( n = 1 \) resp. \( n = 2 \), we may assume that \( n \geq 3 \). Thus, if \( n \) is odd, \( \text{Aut}(\Gamma_n) = \{r_0, \ldots, r_{n-1}, s_0, \ldots, s_{n-1}\} \) is of order \( 2n \), and hence by Definition 1

\[ OP_{\Gamma_n, \text{Aut}(\Gamma_n)}(\lambda) = \frac{1}{2n} \left( \sum_{m=0}^{n-1} P_{\Gamma_n/r_m}(\lambda) + \sum_{m=0}^{n-1} P_{\Gamma_n/s_m}(\lambda) \right). \]

By Proposition 4 \( \Gamma_n/s_m \) contains a loop, so \( P_{\Gamma_n/s_m}(\lambda) = 0 \). Hence, by Eqs. (4) and (6),

\[ OP_{\Gamma_n, \text{Aut}(\Gamma_n)}(\lambda) = \frac{1}{2n} n \cdot \frac{1}{2} \left( \frac{1}{n} \sum_{\substack{1 \leq d \leq n \\ \text{divides} \ n}} \phi(n/d)(\lambda - 1)^d - \lambda + 1 \right), \]
which proves the odd case. For even $n \geq 4$, $\text{Aut}(\Gamma_n) = \{r_0, \ldots, r_{n-1}, s_0, \ldots, s_{n/2-1}, s'_0, \ldots, s'_{n/2-1}\}$, which is of order $2n$, and hence by Definition 1,

$$OP_{\Gamma_n, \text{Aut}(\Gamma_n)}(\lambda) = \frac{1}{2n} \left( \sum_{m=0}^{n-1} P_{\Gamma_n/r_m}(\lambda) + \sum_{m=0}^{n/2-1} P_{\Gamma_n/s_m}(\lambda) + \sum_{m=0}^{n/2-1} P_{\Gamma_n/s'_m}(\lambda) \right).$$

By Proposition 4, $\Gamma_n/s_m$ is a path of length $n/2$, so $P_{\Gamma_n/s_m}(\lambda) = \lambda(\lambda - 1)^{n/2}$. By Proposition 5, $\Gamma_n/s'_m$ contains a loop, so $P_{\Gamma_n/s'_m}(\lambda) = 0$. Therefore, by Eqs. (5) and (6),

$$OP_{\Gamma_n, \text{Aut}(\Gamma_n)}(\lambda) = \frac{1}{2n} \left( n OP_{\Gamma_n, \text{Rot}(\Gamma_n)}(\lambda) + \frac{n}{2} \lambda(\lambda - 1)^{n/2} \right)$$

$$= \frac{1}{2} \left( \frac{1}{n} \sum_{1 \leq d \leq n \atop d|n} \phi(n/d)(\lambda - 1)^d \right) + \frac{1}{4} \lambda(\lambda - 1)^{n/2},$$

which proves the even case. \( \square \)

**Remark 3.** The cycle index polynomial of the group of rotations is known to satisfy

$$Z(\text{Rot}(\Gamma_n); x, \ldots, x) = \frac{1}{n} \sum_{1 \leq d \leq n \atop d|n} \phi(n/d)x^d. \quad (9)$$

Thus, our results can be rephrased in terms of the cycle index polynomial:

$$OP_{\Gamma_n, \text{Rot}(\Gamma_n)}(\lambda) = Z(\text{Rot}(\Gamma_n), \lambda - 1, \ldots, \lambda - 1) - \lambda + 1, \quad (4')$$

$$OP_{\Gamma_n, \text{Aut}(\Gamma_n)}(\lambda) = \frac{1}{2} Z(\text{Rot}(\Gamma_n), \lambda - 1, \ldots, \lambda - 1) - \frac{\lambda}{2} + \frac{1}{2}, \quad (7')$$

if $n$ is odd, respectively

$$OP_{\Gamma_n, \text{Rot}(\Gamma_n)}(\lambda) = Z(\text{Rot}(\Gamma_n), \lambda - 1, \ldots, \lambda - 1), \quad (5')$$

$$OP_{\Gamma_n, \text{Aut}(\Gamma_n)}(\lambda) = \frac{1}{2} Z(\text{Rot}(\Gamma_n), \lambda - 1, \ldots, \lambda - 1) + \frac{1}{4} \lambda(\lambda - 1)^{n/2}, \quad (8')$$

if $n$ is even.
5 Examples

The polynomials in Theorems 1 and 2 can easily be computed using a computer algebra system. We use Sage [7] to perform the necessary calculations:

```python
def s(n):
    return sum(euler_phi(n//d)*(x -1)^d \
                        for d in range(1,n+1) if n%d==0)

def Theorem1(n):
    if n %2 == 1:
        return factor(s(n)/n -x +1)
    else:
        return factor(s(n)/n)

def Theorem2(n):
    if n %2 == 1:
        return factor(s(n) /(2*n)-x /2+1/2)
    else:
        return factor(s(n) /(2*n)+x*(x -1)^(n /2) /4)

Listing 1: Sage code for Theorems 1 and 2
```

For \( n \leq 10 \) the resulting polynomials are displayed in Tables 1 and 2.

| \( n \) | \( \text{OP}_{\Gamma_n, \text{Rot}(\Gamma_n)}(\lambda) \) |
|-------|--------------------------------------------------|
| 1     | 0                                                |
| 2     | \( \frac{1}{2} (\lambda - 1) \lambda \)        |
| 3     | \( \frac{1}{3} (\lambda - 1)(\lambda - 2) \lambda \) |
| 4     | \( \frac{1}{4} (\lambda^2 - 3 \lambda + 4)(\lambda - 1) \lambda \) |
| 5     | \( \frac{1}{5} (\lambda^2 - 2 \lambda + 2)(\lambda - 1)(\lambda - 2) \lambda \) |
| 6     | \( \frac{1}{6} (\lambda^2 - \lambda + 1)(\lambda^2 - 4 \lambda + 5)(\lambda - 1) \lambda \) |
| 7     | \( \frac{1}{7} (\lambda^2 - \lambda + 1)(\lambda^2 - 3 \lambda + 3)(\lambda - 1)(\lambda - 2) \lambda \) |
| 8     | \( \frac{1}{8} (\lambda^6 - 7 \lambda^5 + 21 \lambda^4 - 35 \lambda^3 + 36 \lambda^2 - 24 \lambda + 12)(\lambda - 1) \lambda \) |
| 9     | \( \frac{1}{9} (\lambda^6 - 6 \lambda^5 + 16 \lambda^4 - 24 \lambda^3 + 22 \lambda^2 - 12 \lambda + 6)(\lambda - 1)(\lambda - 2) \lambda \) |
| 10    | \( \frac{1}{10} (\lambda^8 - 9 \lambda^7 + 36 \lambda^6 - 84 \lambda^5 + 126 \lambda^4 - 125 \lambda^3 + 80 \lambda^2 - 30 \lambda + 9)(\lambda - 1) \lambda \) |

Table 1: \( \text{OP}_{\Gamma_n, \text{Rot}(\Gamma_n)}(\lambda) \) for \( n = 1, \ldots, 10 \) (Theorem 1)
Table 2: $OP_{\Gamma_n, \text{Aut}(\Gamma_n)}(\lambda)$ for $n = 1, \ldots, 10$ (Theorem 2)

| $n$ | $OP_{\Gamma_n, \text{Aut}(\Gamma_n)}(\lambda)$ |
|-----|-----------------------------------------------|
| 1   | 0                                             |
| 2   | $\frac{1}{2} (\lambda - 1)\lambda$           |
| 3   | $\frac{1}{6} (\lambda - 1)(\lambda - 2)\lambda$ |
| 4   | $\frac{1}{8} (\lambda^2 - \lambda + 2)(\lambda - 1)\lambda$ |
| 5   | $\frac{1}{12} (\lambda^2 - 2\lambda + 2)(\lambda - 1)(\lambda - 2)\lambda$ |
| 6   | $\frac{1}{12} (\lambda^4 - 5\lambda^3 + 13\lambda^2 - 15\lambda + 8)(\lambda - 1)\lambda$ |
| 7   | $\frac{1}{12} (\lambda^2 - \lambda + 1)(\lambda^2 - 3\lambda + 3)(\lambda - 1)(\lambda - 2)\lambda$ |
| 8   | $\frac{1}{12} (\lambda^6 - 7\lambda^5 + 21\lambda^4 - 31\lambda^3 + 24\lambda^2 - 12\lambda + 8)(\lambda - 1)\lambda$ |
| 9   | $\frac{1}{18} (\lambda^6 - 6\lambda^5 + 16\lambda^4 - 24\lambda^3 + 22\lambda^2 - 12\lambda + 6)(\lambda - 1)(\lambda - 2)\lambda$ |
| 10  | $\frac{1}{20} (\lambda^8 - 9\lambda^7 + 36\lambda^6 - 84\lambda^5 + 131\lambda^4 - 145\lambda^3 + 110\lambda^2 - 50\lambda + 14)(\lambda - 1)\lambda$ |

References

[1] G. D. Birkhoff. A determinant formula for the number of ways of coloring a map. *Ann. Math.*, 14:42–46, 1912.

[2] P. J. Cameron and K. K. Kayibi. Orbital chromatic and flow roots. *Comb. Probab. Comput.*, 16:401–407, 2007.

[3] P. J. Cameron and B. J. J. Rudd. Orbit-counting polynomials for graphs and codes. *Discrete Math.*, 308:920–930, 2008.

[4] P. J. Cameron and J. Semeraro. The cycle polynomial of a permutation group. *Electron. J. Combin.*, 25, #P1.14, 2018.

[5] D. H. Kim, A. H. Mun, and M. Omar. Chromatic bounds on orbital chromatic roots. *Electron. J. Combin.*, 21, #P4.17, 2014.

[6] R. C. Read. An introduction to chromatic polynomials. *J. Combin. Theory*, 4:52–71, 1968.

[7] SageMath, the Sage Mathematics Software System (Version 9.0). The Sage Developers, 2020. [https://www.sagemath.org](https://www.sagemath.org)