The trace formula for transversally elliptic operators on Riemannian foliations

Yuri A. KORDYUKOV
Department of Mathematics,
Ufa State Aviation Technical University,
K.Marx str. 12, Ufa 450000, Russia
e-mail: yurikor@math.ugatuc.ac.ru

September 1, 1999

1 Introduction

The main goal of the paper is to generalize the Duistermaat-Guillemin trace formula to the case of transversally elliptic operators on a compact foliated manifold. First, let us recall briefly the setting of the classical formula.

Let $P$ be a positive self-adjoint elliptic pseudodifferential operator of order one on a closed manifold $M$ (for example, $P = \sqrt{\Delta}$, where $\Delta$ is the Laplace-Beltrami operator of a Riemannian metric on $M$). For any function $f \in C^\infty_c(\mathbb{R})$, the operator $U_f = \int f(t)e^{itP}dt$ can be shown to be of trace class, and the mapping $\theta : f \mapsto \text{tr}U_f$ is a continuous linear functional on $C^\infty_c(\mathbb{R})$. Otherwise speaking, $\theta$ is a distribution on $\mathbb{R}$. The principal symbol $p$ of the operator $P$ is a smooth function on the symplectic manifold $T^*M \setminus 0$, and, by definition, the bicharacteristic flow $f_t$ of the operator $P$ is the Hamiltonian flow on $T^*M \setminus 0$ defined by the function $p$.

By the theorem due to Colin de Verdière and Chazarain [1, 2], the singularities of the distribution $\theta$ are contained in the period set of closed trajectories of the bicharacteristic flow $f_t$. Moreover, Duistermaat and Guillemin showed [3] that, under the assumption that the bicharacteristic flow is clean, one can write down an asymptotic expansion for the distribution $\theta$ near a given period of closed bicharacteristic. A formula for the leading term of this asymptotic expansion is the Duistermaat-Guillemin trace formula mentioned above. It involves the geometry of the bicharacteristic flow in the form of Poincaré map and Maslov indices and provides a far-reaching generalization of the classical Poisson formula and the Selberg trace formula on hyperbolic spaces.

In the paper, we prove a trace formula for an operator $P = \sqrt{A}$, where $A$ is a positive self-adjoint transversally elliptic pseudodifferential operator of order two with the positive, holonomy invariant transverse principal symbol on a compact foliated manifold $(M, \mathcal{F})$ (see Theorem [4]). One can consider such an operator as an elliptic operator on the singular
space $M/F$ of leaves of the foliation $F$ (this statement can be made more precise, using the language of noncommutative geometry, see [1]) the trace formula stated in this paper as an example of a trace formula for elliptic operators on singular spaces. We hope that this formula will be useful in further study of a general trace formula in noncommutative geometry (see, for instance, [2, 3] for discussion of a noncommutative trace formula).

It should be also noted that our trace formula can be viewed as a relative version of the Duistermaat-Guillemin trace formula.

2 Preliminaries and main results

Let $(M, F)$ be a compact, connected, oriented foliated manifold. We will use the following notation: $TF$ is the tangent bundle, $HF = TM/TF$ is the normal bundle and $N^*F$ is the conormal bundle to $F$. There is a short exact sequence

$$0 ightarrow TF ightarrow TM ightarrow HF ightarrow 0. \tag{2.1}$$

We will consider linear operators, acting on half-densities. Recall that an $\alpha$-density $(\alpha \in \mathbb{R})$ on a real vector space $V$ of dimension $n$ is a map $\phi : \Lambda^n V \rightarrow \mathbb{R}$ such that $\phi(\lambda v) = |\lambda|^\alpha \phi(v), v \in \Lambda^n V, \lambda \in \mathbb{R}$. For any real vector bundle $E$ over a smooth manifold $X$, we will denote by $|E|\alpha$ the $\alpha$-density bundle of $E$.

Given a pseudodifferential operator $A \in \Psi^m(M, |TM|^{1/2})$, the transversal principal symbol $\sigma_A$ of $A$ is defined to be the restriction of its principal symbol $a_m$ on $\tilde{N}^*F = N^*F \setminus 0$. An operator $A \in \Psi^m(M, |TM|^{1/2})$ is said to be transversally elliptic, if $\sigma_A(\nu) \neq 0$ for any $\nu \in \tilde{N}^*F$.

For any smooth leafwise path $\gamma$ from $x \in M$ to $y \in M$, sliding along leaves of the foliation defines the holonomy map $h_\gamma$, which associates to every germ of a local transversal to the foliation at the point $x$ a germ of a local transversal to the foliation at the point $y$ (this map is a natural generalization of the Poincaré first-return map for flows). The differential of this map (the linear holonomy map) is well-defined as a linear map $dh_\gamma : H_xF \rightarrow H_yF$ and the codifferential as a linear map $dh^*_\gamma : N^*_yF \rightarrow N^*_xF$.

The transversal principal symbol $\sigma_A$ of an operator $A \in \Psi^m(M, |TM|^{1/2})$ is said to be holonomy invariant, if $\sigma_A(dh^*_\gamma(\nu)) = \sigma_A(\nu)$ for any smooth leafwise path $\gamma$ from $x$ to $y$ and for any $\nu \in \tilde{N}^*_yF$.

Throughout in the paper, we will assume that $A$ is a linear operator in $C^\infty(M, |TM|^{1/2})$, satisfying the following conditions:

(A1) $A \in \Psi^2(M, |TM|^{1/2})$ is a transversally elliptic operator with the positive, holonomy invariant transversal principal symbol;

(A2) $A$ is an essentially self-adjoint positive operator in $L^2$ space of half-densities on $M$, $L^2(M)$ (with the initial domain $C^\infty(M, |TM|^{1/2})$).

Example 1. A geometrical example of an operator, satisfying the conditions (A1) and (A2), is given by the operator $A = I + \Delta_H$, where $\Delta_H$ is the transversal Laplacian of a bundle-like metric on a Riemannian foliation.
Recall that a foliation $\mathcal{F}$ on a smooth Riemannian manifold $(M, g_M)$ is Riemannian if it satisfies one of the following equivalent conditions (see, for instance, [7]):

1. $(M, \mathcal{F})$ locally has the structure of Riemannian submersion;
2. the transverse part of the Riemannian metric $g_M$ (that is, its restriction to $H = T\mathcal{F}^\perp$) is holonomy invariant;
3. the horizontal distribution $H$ is totally geodesic.

In this case, the metric $g_M$ is called bundle-like.

The Riemannian metric $g_M$ defines a decomposition of the cotangent bundle $T^*M$ into a direct sum $T^*M = F^* \oplus H^*$. With respect to this decomposition, the de Rham differential $d : C^\infty(M) \to C^\infty(M, T^*M)$ can be written as a sum $d = d_F + d_H$, where $d_F : C^\infty(M) \to C^\infty(M, F^*)$ and $d_H : C^\infty(M) \to C^\infty(M, H^*)$.

The transversal Laplacian is a second order transversally elliptic differential operator in the space $C^\infty(M)$ defined by the formula

$$\Delta_H = -d_H^*d_H.$$ 

Its principal symbol $a_2$ is given by the formula

$$a_2(x, \xi) = g_H(\xi, \xi), \quad (x, \xi) \in T^*M,$$

and the holonomy invariance of the transverse principal symbol $\sigma_{\Delta_H}$ is equivalent to the assumption on the Riemannian metric $g_M$ to be bundle-like.

From now on, we will assume that $A$ satisfies the assumptions (A1) and (A2). By the spectral theorem, the operator $P = \sqrt{A}$ generates a strongly continuous group $e^{itP}$ of bounded operators in $L^2(M)$. To define a distributional trace of the operator $e^{itP}$, one need an additional regularization. First, let us introduce some notation.

Recall that the holonomy groupoid $G = G_{\mathcal{F}}$ of the foliation $\mathcal{F}$ is the set of equivalence classes of leafwise paths $\gamma : [0, 1] \to M$ with respect to an equivalence relation $\sim_h$, setting $\gamma_1 \sim_h \gamma_2$ if $\gamma_1$ and $\gamma_2$ have the same initial and final points and the same holonomy maps. $G$ is equipped with maps $s, r : G \to M$ given by $s(\gamma) = \gamma(0)$ and $r(\gamma) = \gamma(1)$ and has a composition law given by the composition of paths. For any $\gamma_1, \gamma_2 \in G$, the composition $\gamma_1 \circ \gamma_2$ makes sense iff $r(\gamma_2) = s(\gamma_1)$. We will make use of standard notation: $G^x = r^{-1}(x)$, $G^x_x = s^{-1}(x) \cap r^{-1}(x), x \in M$. For any $x \in M$, $G^x_x$ is the holonomy group of the leaf $L_x$ through the point $x$ and the maps $s : G^x \to L_x$ and $r : G_x \to L_x$ are covering maps associated with $G^x_x$. We will identify a point $x \in M$ with the element in $G$ given by the constant path $\gamma(t) = x, t \in [0, 1]$.

Let $s^\ast(|T\mathcal{F}|^{1/2})$ and $r^\ast(|T\mathcal{F}|^{1/2})$ be the lifts of the vector bundle of leafwise half-densities $|T\mathcal{F}|^{1/2}$ to vector bundles on $G$ via the mappings $s$ and $r$ respectively, and $|TG|^{1/2} = r^\ast(|T\mathcal{F}|^{1/2}) \otimes s^\ast(|T\mathcal{F}|^{1/2})$. The line bundle $|TG|^{1/2}$ is the bundle of leafwise half-densities on $G$ with respect to the natural foliation $\mathcal{G}$.

The space $C_c^\infty(G, |TG|^{1/2})$ has the structure of involutive algebra (see, for instance, [8]). There is a natural $*$-representation $R$ of the involutive algebra $C_c^\infty(G, |TG|^{1/2})$ in $L^2(M)$. 

For any \( k \in C_c^\infty(G, |TG|^{1/2}) \), the operator \( R(k) \) in \( L^2(M) \) is defined as follows. According to the short exact sequence (2.1), the half-density vector bundle \( |TM|^{1/2} \) can be decomposed as

\[
|TM|^{1/2} \cong |T\mathcal{F}|^{1/2} \otimes |H\mathcal{F}|^{1/2}.
\]

For any \( \gamma \in G, s(\gamma) = x, r(\gamma) = y \), the corresponding linear holonomy map defines a map

\[
dh_\gamma^* : |H_y\mathcal{F}|^{1/2} \to |H_x\mathcal{F}|^{1/2}.
\]

Given \( u \in L^2(M) \) of the form \( u = u_1 \otimes u_2, u_1 \in L^2(M, |T\mathcal{F}|^{1/2}), u_2 \in L^2(M, |H\mathcal{F}|^{1/2}) \), \( R(k)u \in L^2(M) \) is defined by the formula

\[
R(k)u(x) = \int_{G^e} k(\gamma)s^*u_1(\gamma) \otimes dh_\gamma^*[u_2(s(\gamma))], \quad x \in M.
\]

**Proposition 2.** For any \( k \in C_c^\infty(G, |TG|^{1/2}) \) and \( f \in C_c^\infty(\mathbb{R}) \), the operator \( R(k) \int f(t)e^{it\xi} dt \) is of trace class. Moreover, for any \( k \in C_c^\infty(G, |TG|^{1/2}) \), the formula

\[
\langle \theta_k, f \rangle = \text{tr} R(k) \int f(t)e^{it\xi} dt, \quad f \in C_c^\infty(\mathbb{R}),
\]

defines a distribution \( \theta_k \) on the real line \( \mathbb{R} \), \( \theta_k \in \mathcal{D}'(\mathbb{R}) \).

Let \( \mathcal{F}_N \) be a foliation in \( \tilde{N}^*\mathcal{F} \), which is the horizontal foliation for the natural leafwise flat connection in \( \tilde{N}^*\mathcal{F} \) (the Bott connection). The leaf of the foliation \( \mathcal{F}_N \) through a point \( \nu \in \tilde{N}^*\mathcal{F} \) is the set of all \( dh_\gamma^*(\nu) \in \tilde{N}^*\mathcal{F} \) such that \( \gamma \in G, r(\gamma) = \pi(\nu) \), where \( \pi : N^*\mathcal{F} \to M \) is the natural projection.

Denote by \( H\mathcal{F}_N \) the normal bundle to \( \mathcal{F}_N \), \( H\mathcal{F}_N = T(N^*\mathcal{F})/T\mathcal{F}_N \). For any \( \nu \in N^*\mathcal{F} \), the space \( T_\nu N^*\mathcal{F} \) is a coisotropic subspace of the space \( T_\nu T^*M \) equipped with the canonical symplectic structure, and \( T_\nu \mathcal{F}_N \) its skew-orthogonal complement, therefore, the normal bundle \( H_\nu \mathcal{F}_N \) has a natural symplectic structure (see, for instance, [3]).

Given an operator \( A \) under the conditions (A1) and (A2) with the principal symbol \( a \), let \( \tilde{p} \) be a smooth function on \( \tilde{T}^*M \) homogeneous of degree one such that \( \tilde{p}(\xi) \neq 0 \) for \( \xi \in \tilde{T}^*M \), which is equal to \( p = a^{1/2} \) in some conical neighborhood of \( N^*\mathcal{F} \), and \( \tilde{f}_t \) the Hamiltonian flow of the function \( \tilde{p} \). Define \( \sigma_P \) to be the restriction of \( p \) on \( N^*\mathcal{F} \): \( \sigma_P = a^{1/2} \). The function \( \sigma_P \) coincides with the transverse principal symbol of any operator \( P_1 \in \Psi^1(M, |TM|^{1/2}) \) such that the principal symbols of \( P_1^2 \) and \( A \) are equal on \( N^*\mathcal{F} \).

The holonomy invariance assumption on \( \sigma_A \) implies

\[
d\tilde{p}(\nu)(X) = 0, \quad \nu \in \tilde{N}^*\mathcal{F}, \quad X \in T_\nu \mathcal{F}_N.
\]

Using (2.2) and the fact that \( \tilde{f}_t \) preserves the symplectic structure of \( T^*M \), one can easily check that the Hamiltonian flow \( \tilde{f}_t \) can be restricted on \( N^*\mathcal{F} \). The resulting flow will be denoted by \( f_t \). By definition, the flow \( f_t \) depends only on the 1-jet of the principal symbol \( a \) on \( N^*\mathcal{F} \), therefore, it doesn’t depend on a choice of \( \tilde{p} \) and can be naturally called the transverse bicharacteristic flow of the operator \( A \).
Since \( \tilde{f}_t \) preserves the symplectic structure of \( T^*M \) and \( TF_N \) is the skew-adjoint complement to \( TN^*F \), \( f_t \) maps leaves of the foliation \( F_N \) to leaves. In particular, the differential \( df_t \) defines a map \( T_{v}F_N \to T_{f_t(v)}F_N \) and a symplectic map \( H_{v}F_N \to H_{f_t(v)}F_N \).

We say that a point \( v \in N^*F \) is a relative fixed point of the diffeomorphism \( f_t : N^*F \to N^*F \) (with respect to the foliation \( F_N \)), if there exist \( \gamma \in G \) such that \( r(\gamma) = \pi(\nu) \) and \( f_{-t} \circ dh^*_\nu(\nu) = v \).

For any \( t \in \mathbb{R} \), denote by \( Z_t \) the set of relative fixed points of \( f_t \). We also introduce the corresponding set in the cospherical bundle \( SN^*F = \{ \nu \in N^*F : \sigma_p(\nu) = 1 \} : \) \( SZ_t = Z_t \cap SN^*F \). This set might be not closed, but, for any \( k \in C^\infty_c(G, |TG|^{1/2}) \), the corresponding part

\[
SZ_{t,k} = \{ \nu \in SN^*F : (\exists \gamma \in supp k, r(\gamma) = \pi(\nu)) f_{-t} \circ dh^*_\nu(\nu) = \nu \}
\]

is closed. By the transversality of \( \sigma_A \), the flow \( f_t \) is transverse to \( F_N \), therefore, the relative period set \( T_k = \{ t \in \mathbb{R} : SZ_{t,k} \neq \emptyset \} \) is a discrete subset of \( \mathbb{R} \).

The following theorem was proved in [10], but we will give an independent proof.

**Theorem 3.** Given an operator \( A \) under the conditions (A1) and (A2) and \( k \in C^\infty_c(G, |TG|^{1/2}) \), the distribution \( \theta_k \) is smooth outside of the relative period set \( T_k \) of the transverse bicharacteristic flow \( f_t \).

Let \( G_{F_N} \) denote the holonomy groupoid of the foliation \( F_N \). \( G_{F_N} \) consists of all pairs \( (\gamma, \nu) \in G \times N^*F \) such that \( r(\gamma) = \pi(\nu) \) with the source map \( s_{N} : G_{F_N} \to N^*F, s_{N}(\gamma, \nu) = dh^*_\nu(\nu) \), and the target map \( r_{N} : G_{F_N} \to N^*F, r_{N}(\gamma, \nu) = \nu \). There is a projection \( \pi_G : G_{F_N} \to G \) given by the formula \( \pi_G(\gamma, \nu) = \gamma \). Put also \( G_{SN^*F} = G_{F_N} \cap r_{N}^{-1}(SN^*F) \).

For any \( (\gamma, \nu) \in G_{F_N} \), denote by \( dH(\gamma, \nu) \) the associated linear holonomy map:

\[
dH(\gamma, \nu) : H_{dh^*_\nu(\nu)F_N} \to H_{\nu}F_N.
\]

It is easy to see that \( dH(\gamma, \nu) \) preserves the symplectic structure of \( HF_N \).

Denote by \( Q : TN^*F \to HF_N \) the projection map. The differential of the map \( (r_{N}, s_{N}) : G_{F_N} \to N^*F \times N^*F \) defines an isomorphism of the tangent space \( T_{(\gamma, \nu)}G_{F_N} \) with the set of all \( (v_1, v_2) \in T_{\nu}N^*F \oplus T_{dh^*_\nu(\nu)}N^*F \) such that the normal components of \( v_1 \) and \( v_2 \) are connected by the holonomy map: \( Q(v_1) = dH(\gamma, \nu)(Q(v_2)) \).

The holonomy groupoid \( G_{F_N} \) has the natural foliation \( G_{F_N} \) such that the tangent bundle \( T_{(\gamma, \nu)}G_{F_N} \) corresponds to \( T_{\nu}F_N \oplus T_{dh^*_\nu(\nu)}F_N \) under the isomorphism described above. The normal space \( H_{(\gamma, \nu)}G_{F_N} \) is isomorphic to the set of all \( (v_1, v_2) \in H_{\nu}F_N \oplus H_{dh^*_\nu(\nu)}F_N \) such that \( v_1 = dH_{(\gamma, \nu)}(v_2) \), and therefore, the maps \( dr_{N} \circ ds_{N} \) define isomorphisms of \( H_{(\gamma, \nu)}G_{F_N} \) with \( H_{\nu}F_N \) \( (H_{dh^*_\nu(\nu)}F_N) \) respectively.

**Lemma 4.** The set \( Z_t \) is a saturated subset of \( N^*F \), that is, it is a union of leaves of the foliation \( F_N \).

**Proof.** By the holonomy invariance of \( p \), the Hamiltonian vector field \( \Xi_{p} \) with the Hamiltonian \( p \) satisfies the identity

\[
dH_{(\gamma, \nu)}(Q(\Xi_{p}(dh^*_\nu(\nu)))) = Q(\Xi_{p}(\nu)), \quad \nu \in N^*F, \quad \gamma \in G, \quad r(\gamma) = \pi(\nu),
\]
therefore, there exists a vector field $\hat{\Xi}_p$ on $G_{FN}$ such that
\[
ds_N(\hat{\Xi}_p(\gamma, \nu)) = \Xi_p(dh_\gamma^*(\nu)), \quad dr_N(\hat{\Xi}_p(\gamma, \nu)) = \Xi_p(\nu), \quad (\gamma, \nu) \in G_{FN}.
\] (2.3)

Let $\hat{F}_t$ be a flow on $G_{FN}$ generated by $\hat{\Xi}_p$. By (2.3), we have
\[
f_t \circ r_N = r_N \circ \hat{F}_t, \quad f_t \circ s_N = s_N \circ \hat{F}_t,
\]
or, if we write $\hat{F}_t : G_{FN} \to G_{FN}$ as $\hat{F}_t(\gamma, \nu) = (F_t(\gamma, \nu), f_t(\nu))$,
\[
f_t(dh_\gamma^*(\nu)) = dh_{F_t(\gamma, \nu)}^*(f_t(\nu)).
\] (2.4)

Take any $\nu \in Z_t$ with the corresponding $\gamma \in G$ such $r(\gamma) = \pi(\nu)$, $f_{-t} \ dh_\gamma^*(\nu) = \nu$. Let $(\gamma_1, \nu) \in G_{FN}$. Then we have
\[
f_t(dh_{\gamma_1}^*(\nu)) = dh_{F_t(\gamma_1, \nu)}^*(f_t(\nu)) \quad \text{by (2.4)}
\]
\[
= dh_{F_t(\gamma_1, \nu)}^*(dh_\gamma^*(\nu))
\]
\[
= (dh_{F_t(\gamma_1, \nu)}^* \circ dh_\gamma^* \circ dh_{\gamma_1}^*)(dh_\gamma^*(\nu))
\]
\[
= dh_{\gamma_1}^*(dh_\gamma^*(\nu)),
\]
where $\gamma' = \gamma_1^{-1} \circ \gamma \circ F_t(\gamma_1, \nu)$ that implies $dh_{\gamma_1}^*(\nu) \in Z_t$. \Box

The relative fixed point sets $Z_t$ can be naturally lifted to the holonomy groupoid $G_{FN}$:
\[
Z_t = \{(\gamma, \nu) \in G_{FN} : f_{-t} \ dh_\gamma^*(\nu) = \nu\}, \quad S\ Z_t = Z_t \cap G_{SN \times F}.
\]

By Lemma 4, $Z_t = r_N(Z_t) = s_N(Z_t)$.

Let us assume that $Z_t$ is a smooth submanifold of $G_{FN}$. By Lemma 4, the tangent space to $Z_t$ at a point $(\gamma, \nu) \in Z_t$ contains a subspace $F_{(\gamma, \nu)}Z_t$, which is the graph of the linear map $df_t(\nu) : T_{\gamma}F_N \to T_{dh_\gamma^*(\nu)}F_N = T_{f_t(\nu)}F_N$:
\[
F_{(\gamma, \nu)}Z_t = \{(v_1, v_2) \in T_{\gamma}F_N \times T_{dh_\gamma^*(\nu)}F_N : v_2 = df_t(\nu)(v_1)\}.
\]

Let
\[
H_{(\gamma, \nu)}Z_t = T_{(\gamma, \nu)}Z_t/F_{(\gamma, \nu)}Z_t, \quad H_{(\gamma, \nu)}SZ_t = T_{(\gamma, \nu)}SZ_t/F_{(\gamma, \nu)}Z_t.
\]

**Definition 5.** Let $t \in \mathbb{R}$ be a relative period of the flow $f_t$. We say that the flow $f_t$ is **clean** on $Z_t$, if:

1. $Z_t$ is a smooth submanifold of $G_{FN}$;
2. the normal space $H_{(\gamma, \nu)}Z_t$ at any point $(\gamma, \nu) \in Z_t$ coincides with the set of all $(v_1, v_2) \in H_{(\gamma, \nu)}G_{FN}$ such that $v_2 = df_t(\nu)(v_1)$.
Let $|T\mathcal{F}_N|^{1/2}$ be the vector bundle of leafwise half-densities on $N^*\mathcal{F}$, and $s_N^*(|T\mathcal{F}_N|^{1/2})$ and $r_N^*(|T\mathcal{F}_N|^{1/2})$ are the lifts of this vector bundle to vector bundles on $G_{\mathcal{F}_N}$ via the mappings $s_N$ and $r_N$ respectively. Let $|T\mathcal{G}_{\mathcal{F}_N}|^{1/2}$ be the vector bundle of leafwise half-densities on $G_{\mathcal{F}_N}$:

$$|T\mathcal{G}_{\mathcal{F}_N}|^{1/2} = r_N^*(|T\mathcal{F}_N|^{1/2}) \otimes s_N^*(|T\mathcal{F}_N|^{1/2}).$$

The projection $\pi_G : G_{\mathcal{F}_N} \to G$ defines a local diffeomorphism $\pi_G : G_{\mathcal{F}_N} \to \mathcal{G}$, that induces a map

$$\pi_G^* : C^\infty(G, |T\mathcal{G}_{\mathcal{F}_N}|^{1/2}) \to C^\infty(G_{\mathcal{F}_N}, |T\mathcal{G}_{\mathcal{F}_N}|^{1/2}).$$

Define a restriction map

$$R_{Z} : C^\infty(G_{\mathcal{F}_N}, |T\mathcal{G}_{\mathcal{F}_N}|^{1/2}) \to C^\infty(Z, |T\mathcal{F}_N|^{1})$$

as follows. If $\rho = f r_N^* \rho_1 \otimes s_N^* \rho_2$, $f \in C^\infty(G_{\mathcal{F}_N}), \rho_1, \rho_2 \in C^\infty(M, |T\mathcal{F}_N|^{1/2})$, then

$$R_{Z}\rho(\gamma, \nu) = f(\gamma, \nu) \rho_1(\gamma, \nu) df^*_t(\nu)[\rho_2(dh^*_t(\nu))], \quad (\gamma, \nu) \in Z_t,$$

where the map $df_t^*(\nu) : |Tf_t(\nu)\mathcal{F}_N|^{1/2} \to |Tf_t(\nu)\mathcal{F}_N|^{1/2}$ is induced by the linear map $df_t(\nu) : T_{f_t(\nu)}\mathcal{F}_N \to T_{f_t(\nu)}\mathcal{F}_N$.

If the flow $f_t$ is clean, there is defined a natural density $d\mu_Z$ on $H_{(\gamma,\nu)} Z_t$, being the fixed point set of the symplectic linear map $dH_{(\gamma,\nu)} \circ df_t(\nu)$ of the symplectic space $H_{\nu}\mathcal{F}_N$ (see, for instance, [3, Lemma 4.3]). Dividing $d\mu_Z$ by $d\sigma_P$, we get a density $d\mu_{SZ}$ on $H_{(\gamma,\nu)} S Z_t$.

Using the natural isomorphism

$$|TS Z_t| \cong |F Z| \otimes |HS Z_t|.$$

one can combine the densities $R_{Z}\pi_G^* k \in C^\infty(S Z_t, |FS Z_t|)$ and $d\mu_{SZ} \in C^\infty(S Z_t, |HS Z_t|)$ to get a smooth density $R_{Z}\pi_G^* k d\mu_{SZ}$ on $S Z_t$.

Let $\sigma_{sub}(A)$ denote the subprincipal symbol of $A$. Define $\sigma_{sub}(P) = \frac{1}{2} a^{-\frac{1}{2}} \sigma_{sub}(A)$ in some conic neighborhood of $N^*\mathcal{F}$. The restriction of $\sigma_{sub}(P)$ on $N^*\mathcal{F}$ is equal to the restriction on $N^*\mathcal{F}$ of the subprincipal symbol of any operator $P_1 \in \Psi^1(M, |TM|^{1/2})$ such that the complete symbols of $P_1^2$ and $A$ are equal mod $S^{-\infty}$ in some neighborhood of $N^*\mathcal{F}$.

**Theorem 6.** Let $t \in \mathbb{R}$ be a relative period of the flow $f_t$. Assume that the relative fixed point set $Z_t$ is clean. Then, for any $k \in C^\infty(G, |TG|^{1/2})$ and for any $\tau$ in some neighborhood of $t$, we have

$$\theta_k(\tau) = \sum_{Z_j} \int_{-\infty}^{+\infty} \alpha_j(s, k)e^{is(\tau-t)}ds, \quad (2.5)$$

where:

1. $Z_j$ are all connected components of the set $S Z_t$ in $G_{SN^*\mathcal{F}}$ of dimensions $d_j = \dim Z_j$.
2. \( \alpha_j \) has an asymptotic expansion
\[
\alpha_j(s, k) \sim \left( \frac{s}{2\pi i} \right)^{(d_j - p - 1)/2} i^{-\sigma_j} \sum_{r=0}^{+\infty} \alpha_{j,r}(k) s^{-r}, \quad s \to +\infty
\] (2.6)
with \( \alpha_{j,0} \) given by the formula
\[
\alpha_{j,0}(k) = \int_{\mathbb{Z}_j} e^{i\int_0^t \sigma_{\text{sub}}(P)(f_{-r}dh\zeta(\nu))d\tau} R_{\mathbb{Z}} \pi^*_G k(\gamma, \nu) d\mu_{\mathbb{Z}_j}(\gamma, \nu),
\] (2.7)
where \( \sigma_j \) denotes the Maslov index associated with the connected component \( \mathbb{Z}_j \) (see below for the definition).

3 Reduction to the case when \( A \) is elliptic

In this section, we will assume that \( A \) is an operator under the assumptions (A1) and (A2). We will use the classes \( \Psi^{m,-\infty}(M, \mathcal{F}, |TM|^{1/2}) \) of transversal pseudodifferential operators (see [4] for the definition) and the Sobolev spaces \( H^s(M) \) of half-densities on \( M \). Put also \( \Psi^{*,-\infty}(M, \mathcal{F}, |TM|^{1/2}) = \bigcup_m \Psi^{m,-\infty}(M, \mathcal{F}, |TM|^{1/2}) \).

By [4], the operator \( P = A^{1/2} \) satisfies the following conditions:

(H1) \( P \) has the form
\[
P = P_1 + R_1,
\]
where:
(a) \( P_1 \in \Psi^1(M, |TM|^{1/2}) \) is a transversally elliptic operator with the positive, holonomy invariant transversal principal symbol such that the complete symbols of \( P_1^2 \) and \( A \) are equal mod \( S^{-\infty} \) in some neighborhood of \( N^*\mathcal{F} \);
(b) \( R_1 \) is a bounded operator from \( L^2(M) \) to \( H^{-1}(M) \) and for any \( K \in \Psi^{*,-\infty}(M, \mathcal{F}, |TM|^{1/2}) \) the operator \( KR_1 \) is a smoothing operator in \( L^2(M) \), that is, it defines a bounded operator from \( L^2(M) \) to \( C^\infty(M, |TM|^{1/2}) \).

(H2) \( P \) is essentially self-adjoint in \( L^2(M) \) (with the initial domain \( C^\infty(M, |TM|^{1/2}) \)).

Lemma 7 ([10]). Any operator \( P \), satisfying the conditions (H1) and (H2), can be represented in the form
\[
P = P_2 + R_2,
\] (3.1)
where:
(a) \( P_2 \in \Psi^1(M, |TM|^{1/2}) \) is an essentially self-adjoint, elliptic operator with the positive principal symbol and the holonomy invariant transversal principal symbol such that the complete symbols of \( P_1 \) and \( P_2 \) are equal mod \( S^{-\infty} \) in some neighborhood of \( N^*\mathcal{F} \);
(b) \( R_2 \) is a bounded operator from \( L^2(M) \) to \( H^{-1}(M) \) and, for any \( K \in \Psi^{*,-\infty}(M, \mathcal{F}, |TM|^{1/2}) \), the operator \( KR_2 \) is a smoothing operator in \( L^2(M) \).
Proof. Take a foliated coordinate chart $\Omega$ on $M$ with coordinates $(x, y) \in I^p \times I^q$ ($I$ is the open interval $(0, 1)$) such that the restriction of $\mathcal{F}$ on $U$ is given by the sets $y = \text{const}$. Let $p_1 \in S^1(I^p \times \mathbb{R}^n)$ be the complete symbol of the operator $P_1$ in this chart. Assume that $p_1(x, y, \xi, \eta)$ is invertible for any $(x, y, \xi, \eta) \in U, |\xi|^2 + |\eta|^2 > R^2$, where $R > 0, U$ is a conic neighborhood of the set $\eta = 0$. Take any function $\phi \in C^\infty(I^n \times \mathbb{R}^n), \phi = \phi(x, y, \xi, \eta), x \in I^p, y \in I^q, \xi \in \mathbb{R}^p, \eta \in \mathbb{R}^q$, homogeneous of degree 0 in $(\xi, \eta)$ for $|\xi|^2 + |\eta|^2 > 1$, which is supported in some conic neighborhood of $\eta = 0$ and is equal to 1 in $U$, and put

$$p_2(x, y, \xi, \eta) = \phi p_1(x, y, \xi, \eta) + (1 - \phi)(1 + |\xi|^2 + |\eta|^2)^{1/2}.$$ 

Take $P_2$ to be the operator $p_2(x, y, D_x, D_y)$ with the complete symbol $p_2$ (or, more precisely, $p_2(x, y, D_x, D_y) + p_2(x, y, D_x, D_y)^*$ to provide self-adjointness) and put $R_2 = P - P_2$. The operator $P_1 - P_2$ has order $-\infty$ in some conic neighbourhood of $N^*\mathcal{F}$, therefore, for any $K \in \Psi^{*-\infty}(M, \mathcal{F}, |TM|^{1/2})$ the operator $K(P_1 - P_2)$ is a smoothing operator [4], that completes immediately the proof. 

Denote by $W(t) = e^{itP_2}$ the wave group generated by the elliptic operator $P_2$. It is well-known that $W(t)$ is a Fourier integral operator (see below for more details). Put also $R(t) = e^{itP} - W(t)$.

Proposition 8. For any $K \in \Psi^{*-\infty}(M, \mathcal{F}, |TM|^{1/2})$, the family $KR(t), t \in \mathbb{R}$, is a smooth family of bounded operators from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$.

Proof. Since $P^2 = A \in \Psi^2(M, |TM|^{1/2})$, by interpolation and duality, $P$ defines a bounded operator from $H^1(M)$ to $L^2(M)$ and from $L^2(M)$ to $H^{-1}(M)$ and, for any natural $N$, $P^N$ defines a bounded operator from $H^N(M)$ to $L^2(M)$ and from $L^2(M)$ to $H^{-N}(M)$. Since $R_2 = P - P_2$ and $P_2 \in \Psi^1(M, |TM|^{1/2}), R_2$ also defines a bounded operator from $H^1(M)$ to $L^2(M)$ and from $L^2(M)$ to $H^{-1}(M)$.

By assumption, for any $K \in \Psi^{*-\infty}(M, \mathcal{F}, |TM|^{1/2})$, the operator $KR_2$ is a smoothing operator, therefore, the operator $KR_2P^N$ is defined as an operator from $H^N(M)$ to $C^\infty(M, |TM|^{1/2})$.

Lemma 9. For any $K \in \Psi^{*-\infty}(M, \mathcal{F}, |TM|^{1/2})$ and for any $N \in \mathbb{N}$, the operator $KR_2P^N$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$.

Proof. We will prove the lemma by induction on $N$. For $N = 0$, the statement is true by assumption. Let us assume that it is true for some $N$, that is, for any $K \in \Psi^{*-\infty}(M, \mathcal{F}, |TM|^{1/2})$, the operator $KR_2P^N$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$.

We have the equality $P^2 - P_2^2 = R_2P + P_2R_2$ as operators from $H^1(M)$ to $H^{-1}(M)$, therefore, for any $K \in \Psi^{*-\infty}(M, \mathcal{F}, |TM|^{1/2})$,

$$KR_2P^{N+1} = K(R_2P)P^N = K(P^2 - P_2^2)P^N - KP_2R_2P^N.$$ 

The operator $P^2 - P_2^2 \in \Psi^2(M, |TM|^{1/2})$ has order $-\infty$ in some conic neighbourhood of $N^*\mathcal{F}$, therefore, for any $K \in \Psi^{*-\infty}(M, \mathcal{F}, |TM|^{1/2})$ the operator $K(P^2 - P_2^2)$ extends to a bounded operator from $H^s(M)$ to $C^\infty(M, |TM|^{1/2})$ for any $s$ and $K(P^2 - P_2^2)P^N$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$. 

9
Since $P_2 \in \Psi^1(M, |TM|^{1/2})$ and $K \in \Psi^{*, -\infty}(M, \mathcal{F}, |TM|^{1/2})$, by the composition theorem $\Psi_{\mathbb{R}}$, $KP_2 \in \Psi^{*, -\infty}(M, \mathcal{F}, |TM|^{1/2})$ and, by induction hypothesis, $KP_2 R_2 P^N$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$. 

By the Duhamel formula, we have
\[
R(t)u = i \int_0^t e^{i\tau P_2} R_2 e^{i(t-\tau)P} u \, d\tau, \quad u \in H^1(M) \subset D(P),
\]
therefore, for any $K \in \Psi^{*, -\infty}(M, \mathcal{F}, |TM|^{1/2})$,
\[
KR(t) = i \int_0^t Ke^{i\tau P_2} R_2 e^{i(t-\tau)P} \, d\tau = i \int_0^t e^{i\tau P_2} e^{-i\tau P_2} Ke^{i\tau P_2} R_2 e^{i(t-\tau)P} \, d\tau.
\]

Any operator $K \in \Psi^{*, -\infty}(M, \mathcal{F}, |TM|^{1/2})$ is a Fourier integral operator (see below for more details) and, using the composition theorem for Fourier integral operators, one can check that $e^{-i\tau P_2} Ke^{i\tau P_2} \in \Psi^{*, -\infty}(M, \mathcal{F}, |TM|^{1/2})$. Therefore, the operator $e^{-i\tau P_2} Ke^{i\tau P_2} R_2$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$. Since $e^{i\tau P_2}$ maps $C^\infty(M, |TM|^{1/2})$ to $C^\infty(M, |TM|^{1/2})$ and, by the spectral theorem, $e^{i(t-\tau)P}$ is a bounded operator in $L^2(M)$, for any $K \in \Psi^{*, -\infty}(M, \mathcal{F}, |TM|^{1/2})$, the operator $KR(t)$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$. Moreover, one can be easily seen from above arguments that the function $KR(t)$ is continuous as a function on $\mathbb{R}$ with values in the space $\mathcal{L}(L^2(M), C^\infty(M, |TM|^{1/2}))$ of bounded operators from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$.

For any $u \in H^1(M)$, the function $\mathbb{R} \to H : t \mapsto KR(t)u$ is differentiable, and
\[
\frac{d}{dt} KR(t)u = iK(P e^{itP} u - P_2 e^{itP_2} u) = i(KP_2 R(t) + KR_2 e^{itP})u.
\]
The operator $KP_2 R(t) + KR_2 e^{itP}$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$, and, moreover, the function $t \mapsto KP_2 R(t) + KR_2 e^{itP}$ is a continuous function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M), C^\infty(M, |TM|^{1/2}))$. Using this, one can be easily seen that the function $t \mapsto KR(t)$ is differentiable as a function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M), C^\infty(M, |TM|^{1/2}))$ and
\[
\frac{d}{dt} KR(t) = i(KP_2 R(t) + KR_2 e^{itP}).
\]

Let us proceed by induction. Assume that, for any $K \in \Psi^{*, -\infty}(M, \mathcal{F}, |TM|^{1/2})$ and for some natural $n$, the function $KR(t)$ is $n$-times differentiable as a function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M), C^\infty(M, |TM|^{1/2}))$ and the derivative $KR^{(n)}(t), t \in \mathbb{R}$ satisfies the equation
\[
KR^{(n)}(t) = iKP_2 R^{(n-1)}(t) + i^n KR_2 P^{n-1} e^{itP}.
\] (3.2)

To prove that the function $t \mapsto KR^{(n)}(t)$ is differentiable as a function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M), C^\infty(M, |TM|^{1/2}))$, as above, it suffices to prove that the derivative $(d/dt)KR^{(n)}(t)u$ exists for any $u$ from a dense subspace of $L^2(M)$, it extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$, and its extension is continuous as a function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M), C^\infty(M, |TM|^{1/2}))$. 

10
From (3.2), one can easily see that the derivative $(d/dt)KR^{(n)}(t)u$ exists for any $u \in H^1(M)$ and satisfies the equation

$$
\frac{d}{dt}KR^{(n)}(t)u = iKP_2R^{(n)}(t)u + i^{n+1}KR_2P^n e^{itP}u. \tag{3.3}
$$

The first term in the right-hand side of (3.3), $iKP_2R^{(n)}(t)$, is a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$ by the induction hypothesis. By Lemma 9, the operator $KR_2P^n$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$, and, by the spectral theorem, $e^{itP}$ is a bounded operator in $L^2(M)$, therefore, the second term in the right-hand side of (3.3), the operator $i^{n+1}KR_2P^n e^{itP}$, extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$. It is also clear the right-hand side of (3.3) is continuous as a function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M), C^\infty(M, |TM|^{1/2}))$. This completes the proof of the existence of the derivative $KR^{(n+1)}(t) = (d/dt)KR^{(n)}(t)$ and the induction arguments. 

**Proof of Proposition 3.** Let $W(t)$ and $R(t)$ be as in Proposition 8 and $k \in C^\infty_c(G, |TG|^{1/2})$. Define $\theta_k(t)$ by the formula

$$
\theta_k(t) = \text{tr} R(k)W(t) + \text{tr} R(k)R(t).
$$

Since $P_2$ is an elliptic operator, the operator $\int f(t)e^{itP}dt$ is a smoothing operator in $\mathcal{D}'(M)$ and the trace of the operator $R(k)W(t)$ is well-defined as a distribution on $\mathbb{R}$. Since any bounded operator $T$ in $L^2(M)$, which extends to a bounded operator from $L^2(M)$ to $H^s(M)$ with $s > n = \dim M$, is a trace class operator, the trace $\text{tr} R(k)R(t)$ is a well-defined smooth function on $\mathbb{R}$ by Proposition 8.

**Corollary 10.** For any $k \in C^\infty_c(G, |TG|^{1/2})$, the function $\text{tr} R(k)R(t) = \theta_k(t) - \text{tr} R(k)W(t)$ is a smooth function on $\mathbb{R}$.

It should be noted that, without any additional assumption about the operator $P$ in question, the corresponding distribution on $G \times \mathbb{R}$, $k \mapsto \text{tr} R(k)R(t)$, might be very singular, but the singularities of the distribution $k \mapsto \text{tr} R(k)W(t)$ can be described rather explicitly under the clear intersection assumption.

## 4 The case of an elliptic operator

Let $P_2 \in \Psi^1(M, |TM|^{1/2})$ be an essentially self-adjoint, elliptic operator with the positive principal symbol and the holonomy invariant transversal principal symbol and $W(t) = e^{itP_2}$. The singularities of the distribution $t \mapsto \text{tr} R(k)W(t)$ can be studied in a standard manner, using microlocal analysis.

Fix $k \in C^\infty_c(G, |TG|^{1/2})$. We will consider the operator family $R(k)W(t)$ as a single operator $R(k)W$ from $L^2(M)$ to $L^2(\mathbb{R} \times M)$. We will prove that this operator is a Fourier integral operator. At first, let us recall well-known facts about the structure of the operators $R(k)$ and $W$. 

11
As above, let $\tilde{p}$ be a smooth function on $\tilde{T}^*M$ homogeneous of degree one such that $\tilde{p}(\xi) \neq 0$ for $\xi \in \tilde{T}^*M$, which is equal to $p = a^{1/2}$ in some conic neighborhood of $N^*F$ and $\tilde{f}_t$ the Hamiltonian flow of $\tilde{p}$. Without loss of generality, we may assume that $\tilde{p}$ is the principal symbol of the operator $\tilde{P}_2$. Let $\Lambda_{\tilde{p}}$ be the Lagrangian submanifold in $\tilde{T}^*\mathbb{R} \times \tilde{T}^*M \times \tilde{T}^*M$:

$$\Lambda_{\tilde{p}} = \{((t, \tau), (x, \xi), (y, \eta)) \in \tilde{T}^*\mathbb{R} \times \tilde{T}^*M \times \tilde{T}^*M : \tau = \tilde{p}(x, \xi), (x, \xi) = \tilde{f}_{-t}(y, \eta)\}.$$

Then $W$ is a Fourier integral operator associated with the canonical relation $\Lambda'_{\tilde{p}}$, $W \in I^{-1/4}(\mathbb{R} \times M \times M, \Lambda'_{\tilde{p}})$.

The operator $R(k)$ belongs to $\Psi^{0,-\infty}(M, F, |TM|^{1/2})$ and, therefore, is a Fourier integral operator associated with an immersed canonical relation, which is the image of $G_{F_N}$ under the mapping

$$(r_N, s_N) : G_{F_N} \to T^*M \times T^*M, \quad (\gamma, \nu) \mapsto (\nu, dh^*_\gamma(\nu)),$$

given by the source and the target mappings of the groupoid $G_{F_N}$ (see [4]). More precisely, $R(k) \in I^{-n/2}(M \times M, \Gamma_{F_N})$.

By the transversal ellipticity of $\tilde{p}$, the intersection of $\Lambda'_{\tilde{p}}$ with $G'_{F_N}$ is transverse, and, by the composition theorem of Fourier integral operators [1], the operator $R(k)W$ is a Fourier integral operator associated with an immersed canonical relation from $T^*M$ to $T^*(\mathbb{R} \times M)$ given by the map

$$\Pi : \mathbb{R} \times G_{F_N} \to T^*\mathbb{R} \times T^*M \times T^*M, \quad (t, \gamma, \nu) \mapsto (t, p(\nu), \nu, f_{-t}dh^*_\gamma(\nu)).$$

More precisely, $R(k)W \in I^{-n/2-1/4}(\mathbb{R} \times M \times M; \mathbb{R} \times G_{F_N}, \Pi)$.

Recall that the trace functional can be treated from the point of microlocal analysis as follows [3]. Let $\Delta : \mathbb{R} \times M \to \mathbb{R} \times M \times M$ be the diagonal map, $\Delta(t,x) = (t, x, x), (t, x) \in \mathbb{R} \times M$, and $\pi : \mathbb{R} \times M \to M$ the projection map. Then

$$\text{tr} R(k)W = \pi_* \Delta^* W_k, \quad (4.1)$$

where $W_k \in C^\infty(\mathbb{R} \times M \times M, |T(\mathbb{R} \times M \times M)|^{1/2})$ is the Schwartz kernel of the operator $R(k)W$, $\Delta^* : C^\infty(\mathbb{R} \times M \times M, |T(\mathbb{R} \times M \times M)|^{1/2}) \to C^\infty(\mathbb{R}, |T\mathbb{R}|^{1/2}) \otimes C^\infty(M, |TM|)$ is defined by the formula

$$\Delta^*(s_1 \otimes s_2 \otimes s_3)(t,x) = s_1(t) \otimes (s_2(x) \otimes s_3(x)), \quad t \in \mathbb{R}, \quad x \in M,$$

where $s_1 \in C^\infty(\mathbb{R}, |T\mathbb{R}|^{1/2}), s_2 \in C^\infty(M, |TM|^{1/2}), s_3 \in C^\infty(M, |TM|^{1/2})$, and

$$\pi_* : C^\infty(\mathbb{R}, |T\mathbb{R}|^{1/2}) \otimes C^\infty(M, |TM|) \to C^\infty(\mathbb{R}, |T\mathbb{R}|^{1/2})$$

is given by integration along fibers of the projection $\pi$.

It is known that $\pi_* \Delta^* \in I^0(\mathbb{R} \times M \times M \times \mathbb{R}, \Gamma)$, where $\Gamma$ is the conormal bundle to the diagonal in $\mathbb{R} \times M \times M \times \mathbb{R}$:

$$\Gamma = \{(t, \tau_1, \nu_1, \nu_2, t, \tau_2) \in T^*\mathbb{R} \times T^*M \times T^*M \times T^*\mathbb{R} : \nu_1 = -\nu_2, \tau_1 = -\tau_2\}.$$
There is a commutative diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{p_1} & Z \\
\varphi \downarrow & & \downarrow p_2 \\
T^* (\mathbb{R} \times M \times M) & \xleftarrow{\Pi} & \mathbb{R} \times G_{\mathcal{F}_N}
\end{array}
\] (4.2)

where

\[p_1(t, \gamma, \nu) = (\Pi(t, \gamma, \nu), t, -p(\nu)) = (t, p(\nu), \nu, -t, -p(\nu)), \quad (t, \gamma, \nu) \in Z,\]

\[p_2\] is a natural inclusion and

\[\varphi(t, \tau, \nu, -\nu, t, -\tau) = (t, \tau, \nu, -\nu), \quad (t, \tau, \nu, -\nu, t, -\tau) \in \Gamma.\]

It is easy to see that (4.2) is a fiber product diagram, that is,

\[Z \cong \{ (x, y) \in (\mathbb{R} \times G_{\mathcal{F}_N}) \times \Gamma : \Pi(x) = \varphi(y) \}.\]

Using this fact and the functoriality properties of the wave-front sets (see, for instance, [12]), one get immediately the description of the singularities of the distribution \(\theta_k\), given by Theorem 3.

To finish the proof of Theorem 3, we will state under the assumption on the flow \(f_i\) to be clean in the sense of Definition 5 that \(\pi_* \Delta^k W_k\) is a Lagrangian distribution and compute its symbol. We begin with computation of the symbol of the operator \(R(k)W\). Recall first the description of the principal symbol of the operator \(R(k)\).

According to the short exact sequence

\[0 \to TG_{\mathcal{F}_N} \to TG_{\mathcal{F}_N} \to HG_{\mathcal{F}_N} \to 0,\]

the half-density vector bundle on \(G_{\mathcal{F}_N}\) can be decomposed as

\[|TG_{\mathcal{F}_N}|^{1/2} \cong |TG_{\mathcal{F}_N}|^{1/2} \otimes |HG_{\mathcal{F}_N}|^{1/2},\]

where \(|HG_{\mathcal{F}_N}|^{1/2}\) is the transverse half-density bundle on \(G_{\mathcal{F}_N}: |H(\gamma, \nu)G_{\mathcal{F}_N}|^{1/2} \cong |H_\nu G_{\mathcal{F}_N}|^{1/2} \cong |H_{d\omega_\gamma(\nu)} F_N|^{1/2}\).

Let \(|dy \wedge d\eta|^{1/2} \in C^\infty(N^* \mathcal{F}, |H \mathcal{F}_N|^{1/2})\) be given by the Liouville form of the canonical transverse symplectic structure on the foliated manifold \((N^* \mathcal{F}, \mathcal{F}_N)\), and \(r_N^* (|dy \wedge d\eta|^{1/2}) \in C^\infty(G_{\mathcal{F}_N}, |HG_{\mathcal{F}_N}|^{1/2})\) its pull back via the map \(r_N: G_{\mathcal{F}_N} \to N^* \mathcal{F}\).

Recall that the space \(S^m(G_{\mathcal{F}_N}, |TG_{\mathcal{F}_N}|^{1/2})\) is defined to be the space of all smooth sections \(s\) of the vector bundle \(|TG_{\mathcal{F}_N}|^{1/2}\) on \(G_{\mathcal{F}_N}\) homogeneous of degree \(m\) such that \(\pi_G(\text{supp} \ s)\) is compact in \(G_{\mathcal{F}}\).

The half-density principal symbol of \(R(k)\) is an element of \(S^0(G_{\mathcal{F}_N}, |TG_{\mathcal{F}_N}|^{1/2})\) given by the formula

\[\sigma(R(k))(\gamma, \nu) = \pi^*_G k(\gamma, \nu) \otimes r_N^* (|dy \wedge d\eta|^{1/2}), \quad (\gamma, \nu) \in G_{\mathcal{F}_N}.\]
The Maslov bundle $M(\mathbb{R} \times G_{\mathcal{F}_N}, \Pi)$ of the immersed canonical relation $(\mathbb{R} \times G_{\mathcal{F}_N}, \Pi)$ restricted to $t = 0$ is isomorphic to the Maslov bundle $M(G_{\mathcal{F}_N})$ of $G_{\mathcal{F}_N}$, therefore, it has a canonical constant section, which extends to a global section $s$ of $M(\mathbb{R} \times G_{\mathcal{F}_N}, \Pi)$ by requiring it to be constant along each bicharacteristic $(t, \tau, \nu_1, \nu_2)$, $\nu_1 = f_\tau(\nu_2), t \in \mathbb{R}$.

Using the description of the principal symbol of the operator $W(t)$ given, for instance, in [3] and the composition theorem of Fourier integral operators, we get immediately that the principal symbol of the operator $R(k)W$ is an element of $S^0(\mathbb{R} \times G_{\mathcal{F}_N}, M(\mathbb{R} \times G_{\mathcal{F}_N}, \Pi) \otimes |T(\mathbb{R} \times G_{\mathcal{F}_N})|^{1/2})$, whose value at a point $(t, \gamma, \nu) \in \mathbb{R} \times G_{\mathcal{F}_N}$ is given by
\[
\sigma(R(k)W)(t, \gamma, \nu) = e^{i \int_0^t \sigma_{\text{sub}(P)}(f_s^{dh_\tau}(\nu))} ds \otimes |dt|^{1/2} \otimes \pi_G^* k(\gamma, \nu) \otimes r_N^*(|dy \land d\eta|^{1/2}).
\]

Now let us turn to the composition (4.1). First, we check the corresponding cleanness assumption.

**Lemma 11.** The assumption on the flow $f_t$ to be clean on $Z_t$ in the sense of Definition guarantees that the composition of $\mathbb{R} \times G_{\mathcal{F}_N}$ with $\Gamma$ is clean.

**Proof.** By definition, the composition of $\mathbb{R} \times G_{\mathcal{F}_N}$ with $\Gamma$ is clean iff $Z_t$ is a submanifold of $\mathbb{R} \times G_{\mathcal{F}_N}$ and in addition the fiber product diagram (4.2) is clean at any point $(t, \gamma, \nu) \in Z$, that is, the linearized diagram
\[
\begin{array}{cccc}
T_{p_1(t,\gamma,\nu)} \Gamma & \overset{dp_1}{\leftarrow} & T_{(t,\gamma,\nu)} Z \\
\downarrow dp_2 & & \downarrow dp_1 \\
T_{(t,p(\nu),\nu,\nu)}(T^*(\mathbb{R} \times M \times M)) & \overset{d\Pi}{\leftarrow} & T_{(t,\gamma,\nu)}(\mathbb{R} \times G_{\mathcal{F}_N})
\end{array}
\]

is a fiber product diagram. Since $T_{(t,\gamma,\nu)} Z$ is always contained in $T_{\nu} N^* \mathcal{F} \oplus T_{dh_\tau(\nu)} N^* \mathcal{F}$, this is true iff the diagram
\[
\begin{array}{cccc}
T_{p_1(t,\gamma,\nu)}(\Gamma \cap T^* \mathbb{R} \times N^* \mathcal{F} \times N^* \mathcal{F} \times T^* \mathbb{R}) & \overset{dp_1}{\leftarrow} & T_{(t,\gamma,\nu)} Z \\
\downarrow dp_2 & & \downarrow dp_1 \\
T_{(t,p(\nu),\nu,\nu)}(T^* \mathbb{R} \times N^* \mathcal{F} \times N^* \mathcal{F}) & \overset{d\Pi}{\leftarrow} & T_{(t,\gamma,\nu)}(\mathbb{R} \times G_{\mathcal{F}_N})
\end{array}
\]

is a fiber product diagram.

The diagram (4.3) has a subdiagram
\[
\begin{array}{cccc}
L_{p_1(t,\gamma,\nu)} \Gamma & \overset{dp_1}{\leftarrow} & L_{(t,\gamma,\nu)} Z \\
\downarrow dp_2 & & \downarrow dp_1 \\
0 \oplus T_\nu \mathcal{F}_N \oplus T_{-\nu} \mathcal{F}_N & \overset{d\Pi}{\leftarrow} & L_{(t,\gamma,\nu)}(\mathbb{R} \times G_{\mathcal{F}_N})
\end{array}
\]

where $L_{(t,\gamma,\nu)}(\mathbb{R} \times G_{\mathcal{F}_N}) \subset T_{(t,\gamma,\nu)}(\mathbb{R} \times G_{\mathcal{F}_N})$ is given by
\[
L_{(t,\gamma,\nu)}(\mathbb{R} \times G_{\mathcal{F}_N}) = \{(U, V_1, V_2, W) \in T_t(\mathbb{R}) \oplus T_\nu \mathcal{F}_N \oplus T_{dh_\tau(\nu)} \mathcal{F}_N \oplus H_{(\gamma,\nu)} \mathcal{G}_{\mathcal{F}_N} : U = 0, W = 0 \}
\]

\[\cong T_\nu \mathcal{F}_N \oplus T_{dh_\tau(\nu)} \mathcal{F}_N\]
\[ L_{(t,\gamma,\nu)}Z \subset T_{(t,\gamma,\nu)}Z \] by

\[ L_{(t,\gamma,\nu)}Z = \{(U, V_1, V_2, W) \in T_t(\mathbb{R}) \oplus T_\nu F_N \oplus T_{dh^*_\nu}F_N \oplus H_{(\gamma,\nu)}G_{F_N} : U = 0, V_1 = -df_{-\nu}(dh^*_\nu)(V_2), W = 0 \} \cong T_\nu F_N \]

and \( L_{p_1(t,\gamma,\nu)}\Gamma \subset T_{p_1(t,\gamma,\nu)}\Gamma \) by

\[ L_{p_1(t,\gamma,\nu)}\Gamma \cong \{(U_1, V_1, V_2, U_2) \in T_{(t,p(\nu))}(T^*\mathbb{R}) \oplus T_\nu F_N \oplus T_{-\nu}F_N \oplus T_{(t,-p(\nu))}(T^*\mathbb{R}) : U_1 = U_2 = 0, V_1 = -V_2 \} \]

which can be easily seen to be a fiber diagram.

Therefore, the diagram (4.4) is a fiber product diagram iff the quotient diagram is a fiber product diagram:

\[ \begin{array}{ccc}
H_{p_1(t,\gamma,\nu)}\Gamma & \xleftarrow{dp_1} & H_{(t,\gamma,\nu)}Z \\
\downarrow{d\varphi} & & \downarrow{dp_2} \\
T_{(t,p(\nu))}(T^*\mathbb{R}) \oplus H_\nu F_N \oplus H_{-\nu}F_N & \xleftarrow{dp_1} & H_{(t,\gamma,\nu)}(\mathbb{R} \times G_{F_N})
\end{array} \] (4.6)

where

\[ H_{(t,\gamma,\nu)}(\mathbb{R} \times G_{F_N}) = \{(U, V_1, V_2, W) \in T_t(\mathbb{R}) \oplus T_\nu F_N \oplus T_{dh^*_\nu}F_N \oplus H_{(\gamma,\nu)}G_{F_N} : V_1 = 0, V_2 = 0 \} \cong T_t(\mathbb{R}) \oplus H_{(\gamma,\nu)}G_{F_N}, \]

\[ H_{(t,\gamma,\nu)}Z = T_{(t,\gamma,\nu)}Z / L_{(t,\gamma,\nu)}Z \cong T_\nu Z_t / T_\nu F_N, \]

\[ H_{p_1(t,\gamma,\nu)}\Gamma \cong \{(U_1, V_1, V_2, U_2) \in T_{(t,p(\nu))}(T^*\mathbb{R}) \oplus H_\nu F_N \oplus H_{-\nu}F_N \oplus T_{(t,-p(\nu))}(T^*\mathbb{R}) : U_1 = -U_2, V_1 = -V_2 \}. \]

In its turn, the diagram (4.6) is a fiber product diagram iff the flow \( f_t \) is clean on \( Z_t \) in the sense of Definition 5. \( \square \)

For any connected component \( Z_j \) of \( Z_t \), the excess of the clean diagram (4.2) equals \( d_j \), the dimension of the relative fixed point set \( S Z_j \) in \( G_{SN \cdot F} \). By the composition theorem of Fourier integral operators, \( \theta_k \) belongs to \( \bigoplus_j I^{\frac{d_j - p}{2} - \frac{1}{4}}(\Lambda_t) \), where \( \Lambda_t = \{(t, \tau) \in T^*\mathbb{R} : \tau \in \mathbb{R}_- \} \), that proves the desired representation of \( \theta_k \) in the form (2.5) and the existence of the asymptotic expansion (2.6) for \( \alpha_j(s, k) \). ^4

To obtain the explicit formula for the leading coefficients \( \alpha_{j,0} \), we compute the principal symbol of \( \theta_k \), \( \sigma(\theta_k) \), following the arguments in (4.3).

^4Note that the appearance of the term \( -p/2 \) in the exponent is due to the fact \( R(k)W \in I^{-p/2-1/4}(M \times M; \mathbb{R} \times G_{F_N}, \Pi) \).
Fix a connected component $\mathcal{Z}_j$ of $\mathcal{Z}_t$ and $(t, \gamma, \nu) \in \mathcal{Z}_j$. The fiber product diagram (4.3) defines a composition map \([3, 11]\)

$$
*: |T_{p_1(t,\gamma,\nu)}\Gamma|^{1/2} \otimes |T_{(t,\gamma,\nu)}(\mathbb{R} \times G_{\mathcal{F}_N})|^{1/2} \rightarrow |T_{(t,\nu)}(\Lambda_t)|^{1/2} \otimes |T_{(t,\gamma,\nu)}\mathcal{Z}_j| \tag{4.7}
$$

and, due to \([13, \text{proof of Lemma 4.6}]\) and the composition theorem for Fourier integral operators, the value of $\sigma$ in $\mathcal{Z}_j$ defined by the coisotropic subspace $\Gamma = \pi_\ast \Delta^\top \ast \pi_\ast \Delta^\top \ast (\nu, t, \gamma, \nu)$ is given by the computation of

$$
\sigma(\pi_\ast \Delta^\top \ast \pi_\ast (W_k)) \in C^\infty(\Lambda_t \times \mathcal{Z}_j, |T\Lambda_t|^{1/2} \otimes |T\mathcal{Z}_j|).$
$$

Using the functoriality of the $*$ operation on half densities with respect to reduction (cf. \([13, \text{proof of Lemma 4.6}]\)), the computation of the $*$-product $\sigma(\pi_\ast \Delta^\top \ast \pi_\ast (W_k))$ can be reduced to the computation of a $*$-product defined by the transverse fiber product diagram (4.6):

$$
*: |H_{p_1(t,\gamma,\nu)}\Gamma|^{1/2} \otimes |H_{(t,\gamma,\nu)}(\mathbb{R} \times G_{\mathcal{F}_N})|^{1/2} \rightarrow |T_{(t,\nu)}(\Lambda_t)|^{1/2} \otimes |H_{(t,\gamma,\nu)}\mathcal{Z}_j|.
$$

More precisely, we apply the result stated in \([3, \text{proof of Lemma 4.6}]\) with a symplectic vector space $\mathcal{V} = T_{(t,p,\nu, -\nu, t, -\nu)}(T^* \mathbb{R} \times T^* M \times T^* M \times T^* \mathbb{R})$, two Lagrangian subspaces in $\mathcal{V}$: $\Lambda_1 = T_{p_1(t,\gamma,\nu)}\Gamma$ and $\Lambda_2$, which is the image of $T_{(t,\gamma,\nu)}(\mathbb{R} \times G_{\mathcal{F}_N})$ in $\mathcal{V}$ and the reduction given by the coisotropic subspace $\Gamma = T_{(t,p,\nu, -\nu, t, -\nu)}(T^* \mathbb{R} \times N^* \mathcal{F} \times N^* \mathcal{F} \times T^* \mathbb{R})$.

By this result, the leafwise component of $\sigma(\theta_k)$, $\sigma_t(\theta_k) \in |L_{(t,\gamma,\nu)}\mathcal{Z}_j|$, is obtained from the leafwise component of $\sigma(R(k)W)$:

$$
\sigma_t(R(k)W) = e^{i \int_0^t \sigma_{sub}(P)(f_{-\nu}dh^*_\nu)(ds)} \pi^*_G k
$$

by application of the restriction map $R_\mathcal{Z}$:

$$
\sigma_t(\theta_k) = e^{i \int_0^t \sigma_{sub}(P)(f_{-\nu}dh^*_\nu)(ds)} R_\mathcal{Z} \pi^*_G k,
$$

and the transversal component of $\sigma(\theta_k)$, $\sigma_t(\theta_k) \in |T_{(t,\nu)}(\Lambda_t)|^{1/2} \otimes |H_{(t,\gamma,\nu)}\mathcal{Z}_j|$, is equal to the $*_{\gamma}$-product of the transverse component of $\sigma(R(k)W)$,

$$
\sigma_t(R(k)W) = |dt|^{1/2} \otimes r_N^*(|dy \wedge d\eta|^{1/2}),
$$

and the transverse component of $\sigma(\pi_\ast \Delta^\top)$. By \([3]\), we get

$$
\sigma_t(\theta_k) = |dt|^{1/2} \otimes d\mu_{\mathcal{Z}_j}
$$

that completes the calculation of the half-density principal symbol of $\theta_k$ and implies the formula (2.7) for the leading coefficients $\alpha_{j,0}$ as in \([3]\).

5 Maslov indices

In this section, we define the Maslov factors $\sigma$, corresponding to $(\gamma, \nu) \in \mathcal{Z}_t$. For this goal, we will use local coordinates on the holonomy groupoid $G$ described, for instance, in \([3, 11]\). Choose a pair of compatible foliated charts near the points $\pi(\nu)$ and $\pi(f_t(\nu))$ with the coordinates $(x, y)$ and $(x', y)$, corresponding to $\gamma \in G$. Then we have the corresponding coordinates in $H_\nu \mathcal{F}_N = T_v N^* \mathcal{F} / T_v \mathcal{F}_N$ defined as $(\delta y, \delta \eta)$, and the vertical and horizontal.
subspaces $V_\nu$ and $H_\nu$, given by the equations $\delta y = 0$ and $\delta \eta = 0$ accordingly. The linear holonomy map $dH_{(\gamma, \nu)}$ of the foliation $(N^*F, F_N)$ defines an isomorphism of the symplectic spaces $H_{f_t(\nu)}F_N$ and $H_\nu F_N$, which preserves the vertical and horizontal subspaces. Due to this isomorphism, we can obtain a closed curve $\omega_{(\gamma, \nu)}$ in the Lagrangian Grassmannian $\mathcal{G}$ of the symplectic space $H_\nu F_N$, pulling back, via $df_t$, the vertical subspace at $f_t(\nu)$ for $t$ between 0 and $T$. Denote by $\kappa_{(\gamma, \nu)}$ the intersection number of $\omega$ with the horizontal subspace $H_\nu$:

$$\kappa_{(\gamma, \nu)} = [\omega_{(\gamma, \nu)} : H_\nu].$$

Let $\chi(t, x, y, \xi, \eta)$ be the generating function of the canonical transformation $f_t$ in the chosen coordinates. Recall that $\chi$ is the solution of the Cauchy problem

$$d_t \chi = p(x, y, dx\chi, dy\chi), \quad \chi(0, x, y, \xi, \eta) = x\xi + y\eta. \quad (5.1)$$

By the holonomy invariance of $p$, it can be easily seen that $\chi(t, x, y, 0, \eta)$ is independent of $x$ and $\xi$: $\chi(t, x, y, 0, \eta) = \chi(t, y, \eta)$. Let

$$R_{(\gamma, \nu)} = \begin{bmatrix} d_{yy}\chi & d_{y\eta}\chi & -1 \\ d_{\eta y}\chi & d_{\eta\eta}\chi & 0 \\ -1 & 0 & 0 \end{bmatrix}. $$

We define the Maslov factor $\sigma(\gamma, \nu)$ as

$$\sigma(\gamma, \nu) = \text{sgn} \, R_{(\gamma, \nu)} + 2\kappa_{(\gamma, \nu)}, \quad (\gamma, \nu) \in \mathbb{Z}.$$

It is clear that $\sigma(\gamma, \nu)$ is a locally constant function on $\mathbb{Z}$.

To handle with Maslov factors in the proof of Theorem 6, let us write the Schwartz kernel of the operator $R(k)$ in a foliated coordinate chart on $G$ given by a pair of compatible coordinate systems as $k(x, x_1, y)\delta(y - y_1)$, and the Schwartz kernel $W(t)$ microlocally as an oscillatory integral of the form

$$\int e^{i\alpha(t, x_1, y_1, x_2, y_2, \xi, \eta)} a(t, x_1, y_1, x_2, y_2, \xi, \eta) d\xi d\eta,$$

where

$$\alpha(t, x_1, y_1, x_2, y_2, \xi, \eta) = \chi(t, x_1, y_1, \xi, \eta) - x_2\xi - y_2\eta$$

and $\chi(t, x_1, y_1, \xi, \eta)$ is the generating function of the canonical transformation $f_t$ given by (5.1). Then the Schwartz kernel of the operator $R(k)W$ is given by the formula

$$\int e^{i\alpha(t, x_1, y_1, x_2, y_2, \xi, \eta)} k(x, x_1, y_1)a(t, x_1, y_1, x_2, y_2, \xi, \eta) dx_1 d\xi d\eta,$$

from where one can easily derive the desired assertion, following the arguments of [3].

17
References

[1] Y. Colin de Verdière. Spectre du Laplacien et longeurs des géodésiques periodiques. Comp. Math., 27:159–184, 1973.

[2] J. Chazarain. Formule de Poisson pour le variétés riemanniennes. Invent. Math., 24:65–82, 1974.

[3] J.J. Duistermaat and V. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. Invent.Math., 29:39–79, 1975.

[4] Yu. A. Kordyukov. Noncommutative spectral geometry of Riemannian foliations. Manuscripta Math., 94:45–73, 1997.

[5] A. Connes. Trace formula in noncommutative geometry and the zeros of the Riemann zeta function. Selecta Math. (N.S.), 5:29–106, 1999.

[6] F. Golse and E. Leichtnam. Applications of Connes’ geodesic flow to trace formulas in noncommutative geometry. J. Funct. Anal., 160:408–436, 1998.

[7] B. L. Reinhart. Differential Geometry of Foliations. Springer, Berlin Heidelberg New York, 1983.

[8] A. Connes. Sur la théorie non-commutative de l’intégration. In Algèbres d’opérateurs, volume 725 of Lecture Notes Math., pages 19–143. Springer, Berlin Heidelberg New York, 1979.

[9] V. Guillemin and S. Sternberg. Some problems in integral geometry and some related problems in microlocal analysis. Amer. J. Math., 101:915–959, 1979.

[10] Yu. A. Kordyukov. The transversal wave equation and the noncommutative geodesic flow in Riemannian foliations. Preprint ETH Zürich, 1997; [dg-ga/9703013]

[11] L. Hörmander. The analysis of linear partial differential operators IV. Springer, Berlin Heidelberg New York, 1986.

[12] V. Guillemin and S. Sternberg. Geometric Asymptotics. American Mathematical Society, Providence, R. I., 1977.

[13] V. Guillemin and A. Uribe. Circular symmetry and the trace formula. Invent. Math., 96:385–423, 1989.