AN EXAMPLE OF INTRINSIC RANDOMNESS IN DETERMINISTIC PDES

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Abstract. A new mechanism leading to a random version of Burgers’ equation is introduced: it is shown that the Totally Asymmetric Exclusion Process in discrete time (TASEP) can be understood as an intrinsically stochastic, non-entropic weak solution of Burgers’ equation on $\mathbb{R}$. In this interpretation, the appearance of randomness in the Burgers’ dynamics is caused by random additions of jumps to the solution, corresponding to the random effects in TASEP.

1. Introduction

Random solutions and stochastic versions of the Burgers’ equation
\begin{equation}
\partial_t u(t, x) + u(t, x) \partial_x u(t, x) = 0, \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}, t \geq 0,
\end{equation}
appear in various contexts, forms and applications. Relevant examples include [28, 11], where the (multi-dimensional) Burgers’ equation with Gaussian initial conditions was found in the study of the formation of large-scale structures in the Universe, [15] where the Burgers’ equation with a random forcing arises in the analysis of the dynamics of interfaces and in [19], where the Burgers’ equation with random flux appears in the analysis of mean field systems with common noise, related to mean field game systems.

In the present work, we uncover another mechanism along which randomness can enter the dynamics of the Burgers’ equation, by establishing a one-to-one correspondence between a certain class of solutions to Burgers’ equation and the discrete-time totally asymmetric simple exclusion process (TASEP): Sample functions are shown to be random (weak) solutions to the Burgers’ equation, stochasticity being introduced by a random creation of jumps in the solution, corresponding to jumps of TASEP particles.

Different concepts of solutions to Burgers’ equation (1) are known, which is due to the fact that weak solutions are non-unique. A unique characterization of weak solutions in form of entropy solutions can be given in the setting of vanishing diffusion approximations, that is, in the case that solutions to (1) are obtained as limits for ($\varepsilon \downarrow 0$) of
\begin{equation}
\partial_t u(t, x) + u(t, x) \partial_x u(t, x) = \varepsilon \partial_{xx} u(t, x).
\end{equation}

In contrast, in the setting of vanishing diffusion-dispersion approximations ($\varepsilon, \delta \downarrow 0$)
\begin{equation}
\partial_t u(t, x) + u(t, x) \partial_x u(t, x) = \varepsilon \partial_{xx} u(t, x) + \delta \partial_{xxx} u(t, x)
\end{equation}

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with scaling $\delta \approx \varepsilon^2$ non-classical shocks (i.e. violating entropy conditions for certain entropies) are known to appear \cite{13, 18}. The characterization of the class of weak solutions produced in this setting is an open problem. Similarly, limits of relaxation approximations \cite{14} to the Burgers' equation are known to converge to weak solutions of \eqref{1}, which are only known to be so-called quasi-solutions \cite{6}, i.e. which have finite, but not necessarily signed entropy production. Finally, in the last section of this work we comment on how the Burgers' equation heuristically appears to be related to the KPZ fixed point \cite{21}. Also in this setting, the correct concept of solutions does not seem to be entropy solutions, see \cite{21, Equation (1.3) ff.} and their identification is an open problem. The concept of a solution to \eqref{1} thus depends on the underlying application.

We will discuss how one can exploit non-uniqueness of weak solutions to perform random choices when extending discrete time dynamics to weak solutions to the Burgers' equation in continuous time, producing a stochastic process whose trajectories are non-entropic weak solutions to \eqref{1}, a so-called intrinsically random solution. Intrinsic stochasticity, that is, stochastic solutions of deterministic differential equations with deterministic initial conditions, is an interesting and challenging concept, for example arising in turbulence theory. For results and discussions about this notion we refer to \cite{8, 9, 16, 17, 20, 7, 27}.

Discrete-time TASEP consists in particles occupying sites of $\mathbb{Z}$, jumping at random to their right under the constraint that each site may not be occupied by more than one particle at once. We will define a closely related discrete-time particle model on $\mathbb{Z}$, which we call Active Bi-Directional Flow (ABDF), starting from TASEP and considering pairs of occupied and empty positions. ABDF model consists in particles constantly moving to their left or right, annihilating in pairs when colliding and being generated in pairs, at random, in certain positions. We will show that it is conjugated to TASEP as random dynamical systems in Theorem 7. The ABDF model shares features with other ones related to TASEP and the KPZ universality class to which the latter belongs, in particular the discrete-time polynuclear growth (PNG) process; however, to the best of our knowledge, the construction is original.

The behaviour of particles of ABDF model can be precisely mirrored by particular weak solutions of \eqref{1} composed of indicator functions of intervals, which we call quasi-particles, traveling to their left or right following characteristic lines until two of them meet. It is when quasi-particle collide that non-uniqueness is exploited to annihilate them, and also, by time-reversal, the same can be done to generate pairs of quasi-particles out of a null profile. The main result, Theorem 23, consists in showing that this close analogy between ABDF model and a random selection of Burgers’ weak solutions can be made precise with a bijection between samples of models.

Aside from the interest of “embedding” discrete-time random processes into weak solutions of Burgers’ equation, this study stems from an attempt of understanding possible links between non-entropic solutions of Burgers’ equation and the aforementioned KPZ universality class and KPZ fixed points. Hence the particular choice of TASEP as the “source” of intrinsic randomness, it being a most distinguished model in the study of KPZ universality. At this stage, what we can state to that end remains essentially conjectural, so we collect related observations and references to the last Section of the article.

The paper is organized as follows. In Section 2 we introduce the ABDF model; in Section 3 we link it to TASEP and finally, in Section 5 we link them to the Burgers’ equations. Preliminarily, in section 4, we introduce the class of weak solutions of
Burgers’ equations involved in the conjugacy. Some ideas about such solutions have been identified previously [1], but the link with TASEP described here is new.

2. ABDF model: Active Bi-Directional Flow

We begin with an informal description: a configuration of the ABDF model is made of particles and empty positions on \( \mathbb{Z} \), with no more than one particle at each position. Particles are divided into two classes: left and right particles, according to the direction in which they are allowed to move. Empty positions are also divided into two classes, active and inert positions, the former being allowed to generate couples of new particles as we will detail.

Relative positions of left and right particles is not arbitrary: two particles are consecutive if they occupy positions \( x_1 < x_2 \) such that no particle is in between, and we postulate that two consecutive particles of different type (one left and one right, independently of the order) are always separated by an odd number of empty positions. Moreover two consecutive particles of the same kind shall always be separated by an even number of empty positions.

Empty positions are active or inert depending on their distance from the first particle on their left or right, and the class of the latter. Precisely, assume the empty position, say \( x_0 \), lies between consecutive particles at \( x_1 < x_2 \). Let \( k = x_0 - x_1 \); if the particle at \( x_1 \) is a left-particle and \( k \) is odd, then the empty position at \( x_0 \) is active, otherwise it is inert. If the particle at \( x_1 \) is right and \( k \) is odd, then the empty position is inert, otherwise it is active. The same definition is given in terms of \( x_2 \): if \( h = x_2 - x_0 \) is odd and the particle at \( x_2 \) is of left-type, then the empty position at \( x_0 \) is inert; if the particle at \( x_2 \) is of right-type and \( h \) is odd, then it is active, otherwise the empty position is inert. It is easy to see that the two definitions coincide.

Finally, if an empty position is not between consecutive particles, either that we are dealing with an empty configuration, or such position is part of a half line of empty particles. In the second case, the rule concerning active or inert property is the same described above. The case of all empty positions is a very special one: active and inert positions should alternate, but they can do so in two ways, depending on the type of \( x = 0 \). It is important to distinguish between them, both possibly occurring during the evolution described in the next Definition 4. For \( x \in \mathbb{Z} \), we denote

\[
al_{0}(x) = x \mod 2, \quad a_{1}(x) = (x + 1) \mod 2,
\]

and the double sequences

\[
\overline{al}_{\alpha}(x) = (0, a_{\alpha}(x)), \quad \alpha = 0, 1,
\]

the first coordinate declaring that \( \overline{al}_{0} \) and \( \overline{al}_{1} \) are both zero sequences, namely they represent ABDF configurations with all empty sites.

For an empty position, being active or inert is a (nonlocal) consequence of the particle configuration. Therefore, in the formal definition of ABDF configurations we specify positions of particles –first component of the configuration– and deduce active or inert empty positions –second component of the configuration– by what we call activation map. The only exceptions are the ABDF configurations with all empty sites, where two different activation profiles are possible.

Let us move to the rigorous definition. We associate +1 to right particles, –1 to left particles, 0 to empty positions; then we introduce the “activation record” which associates 0 to any position where a new pair of particles cannot arise (empty inert positions and occupied positions), and 1 to active empty positions.
Definition 1. Let $\Lambda_0$ be the set of sequences

$$\theta : \mathbb{Z} \to \{-1, 0, 1\}$$

which are not identically zero (we write simply $\theta \neq 0$) such that

i) if $x_1 < x_2 \in \mathbb{Z}$ have the properties $\theta(x_1) \theta(x_2) = -1$ and $\theta(x) = 0$ for all $x \in (x_1, x_2) \cap \mathbb{Z}$, then the cardinality of $(x_1, x_2) \cap \mathbb{Z}$ is odd;

ii) if $x_1 < x_2 \in \mathbb{Z}$ have the properties $\theta(x_1) \theta(x_2) = 1$ and $\theta(x) = 0$ for all $x \in (x_1, x_2) \cap \mathbb{Z}$, then the cardinality of $(x_1, x_2) \cap \mathbb{Z}$ is even.

For every $\theta \in \Lambda_0$, introduce the activation record sequence

$$ar(\theta) : \mathbb{Z} \to \{0, 1\},$$
defined as:

iii) if $\theta(x_0) \in \{-1, 1\}$ then $ar(\theta)(x_0) = 0$;

iv) if $\theta(x_0) = 0$ and the set

$$L(x_0) := \{x < x_0 : x \in \mathbb{Z}, \theta(x) \in \{-1, 1\}\}$$

is not empty, taking $x_1 = \max L(x_0)$ and $k \in \mathbb{N}$ such that $x_0 = x_1 + k$,

$$ar(\theta)(x_0) = \left\lfloor \frac{\theta(x_1) + (-1)^k}{2} \right\rfloor$$

v) if $\theta(x_0) = 0$ and the set

$$R(x_0) := \{x > x_0 : x \in \mathbb{Z}, \theta(x) \in \{-1, 1\}\}$$

is not empty, taking $x_2 = \min R(x_0)$ and $h \in \mathbb{N}$ such that $x_0 = x_2 - h$,

$$ar(\theta)(x_0) = \left\lfloor \frac{\theta(x_2) + (-1)^{h+1}}{2} \right\rfloor.$$

The following results from a simple check.

Lemma 2. If both $L(x_0)$ and $R(x_0)$ are not empty, points (iv-v) above give the same definition of $ar(\theta)$.

Definition 3. A configuration of the ABDF model is a map

$$(\theta, act) : \mathbb{Z} \to \{-1, 0, 1\} \times \{0, 1\}$$

with the following properties:

a) if $\theta = 0$, then either $act = alt_0$ or $act = alt_1$ (in other words, either $(\theta, act) = alt_0$ or $(\theta, act) = alt_1$);

b) if $\theta \neq 0$, then $\theta \in \Lambda_0$ and $act = ar(\theta)$, where the set $\Lambda_0$ and the map $ar$ are introduced in Definition 1.

The set of all ABDF configurations will be denoted by $\Lambda$. 

Figure 1. A piece of an ABDF configuration. Numbers $\pm 1$ and 0 denote particle type or empty sites, subscripts are the values of activation record.
Figure 1 represents a piece of an ABDF configuration. The example makes it apparent how the number of empty positions between non-empty ones is regulated by the concordance of signs of the extremes.

Let us come to the description of ABDF dynamics. All right particles move to the right by one position at every time step, all left particles to the left: unlike in exclusion processes, these jumps can not be prevented by an occupied arrival positions, since all particles move. All active empty positions $x_0$ may generate, at random with probability 1/2, a pair of particles: a left-particle in $x_0 - 1$ and a right-particle at $x_0 + 1$. It often happens that two particles meet at one position: a right particle which moved from $x - 1$ to $x$ and a left particle which moved from $x + 1$ to $x$ arrive at the same time $t$ at $x$. In such a case, the two particles disappear, annihilating each other, and position $x$ becomes empty.

We have to check that these rules are coherent and that they give rise to ABDF configurations described above.

**Definition 4.** Let $\Omega = \{0, 1\}^{\mathbb{N} \times \mathbb{Z}}$, with the $\sigma$-algebra $\mathcal{F}$ generated by cylinder sets, and the product probability measure $P$ of Bernoulli $p = \frac{1}{2}$ random variables. Given $\omega \in \Omega$, we write $\omega(t, x)$ for its $(t, x)$-coordinate, $(t, x) \in \mathbb{N} \times \mathbb{Z}$ and write $\kappa(t, x) := 1 - \omega(t, x)$ for the complementary value.

Then, based on the probability space $(\Omega, \mathcal{F}, P)$, we introduce a family of maps

$$T_{ABDF}(t, \omega, \cdot) : \Lambda \rightarrow (\{-1, 0, 1\} \times \{0, 1\})^\mathbb{Z}$$

indexed by $t \in \mathbb{N}$ and $\omega \in \Omega$, defined as follows. Denote

$$T_{ABDF}(t, \omega, (\theta, \text{act})) = (T_{ABDF}(t, \omega, (\theta, \text{act}))_1, T_{ABDF}(t, \omega, (\theta, \text{act}))_2),$$

where we recall that $\text{act} = \ar(\theta)$ unless $\theta = 0$. The map $T_{ABDF}(0, \omega, \cdot)$ is the identity. For $t > 0$, the first component is defined as

$$T_{ABDF}(t, \omega, (\theta, \text{act}))_1(x) := \max(\theta(x - 1), 0) + \text{act}(x - 1)\kappa(t - 1, x - 1) + \min(\theta(x + 1), 0) - \text{act}(x + 1)\kappa(t - 1, x + 1)$$

for every $x \in \mathbb{Z}$. For $t > 0$, if $T_{ABDF}(t, \omega, (\theta, \text{act}))_1$ is not the identically null sequence, the second component is defined by

$$T_{ABDF}(t, \omega, (\theta, \text{act}))_2 = \ar(T_{ABDF}(t, \omega, (\theta, \text{act}))_1)$$

for every $x \in \mathbb{Z}$.
with or (·) as in Definition 1. If \( T_{ABDF}(t, \omega, (\theta, act))_1 = 0 \) then \( T_{ABDF}(t, \omega, (\theta, act))_2 \) is either \( alt_0 \) or \( alt_1 \). It is equal to \( alt_1 \) in each one of the following three cases:

\[
\theta(0) = -1, \\
act(-1) = 1 \quad \text{and} \quad \omega(t - 1, -1) = 1, \\
act(0) = 1 \quad \text{and} \quad \omega(t - 1, 0) = 0,
\]

otherwise it is equal to \( alt_0 \).

It is not easy to see right away why we set \( T_{ABDF}(t, \omega, (\theta, act))_1 = 0 \) precisely in these three cases: this will become clear in the correspondence between a TASEP configuration \( \eta \) and \( \theta \).

The quantity \( T_{ABDF}(t, \omega, (\theta, act))_1(x) \) can only take values in \( \{-1, 0, 1\} \). We shall prove that it satisfies Definition 1, (i)-(ii) and therefore \( T_{ABDF}(t, \omega, \theta) \in \Lambda \).

To minimize double proofs, we postpone this fact to the verification of the link with TASEP (see Theorem 7 below).

Once this is proved, one can introduce the ABDF random dynamical system, of which \( T_{ABDF}(t, \omega, \cdot) \) is just the 1-step dynamics at time \( t \). We let \( \phi_{ABDF}(0, \omega) = id \), and for \( t > 0, t \in \mathbb{N} \),

\[
\phi_{ABDF}(t, \omega) := T_{ABDF}(t, \omega) \circ T_{ABDF}(t - 1, \omega) \circ \cdots \circ T_{ABDF}(1, \omega),
\]

so that it holds the random dynamical system property

\[
\phi_{ABDF}(t, \omega) \circ \phi_{ABDF}(s, \omega) = \phi_{ABDF}(t + s, \omega), \quad t, s \in \mathbb{N}, \omega \in \Omega.
\]

### 3. TASEP, ITS PAIRS AND ABDF

A TASEP configuration is a map

\[
\eta : \mathbb{Z} \to \{0, 1\}.
\]

When \( \eta(x) = 1 \), we say that \( x \) is occupied by a particle; when \( \eta(x) = 0 \), we say that \( x \) is empty.

TASEP dynamics in discrete time \( t \in \mathbb{N} \) consists in particles moving to the right by one position with probability \( \frac{1}{2} \), with simultaneous independent jumps, aborted when the arrival position is occupied. More precisely, given a configuration \( \eta \) at time \( t - 1 \in \mathbb{N} \), a particle at position \( x \in \mathbb{Z} \) (which means \( \eta_{t-1}(x) = 1 \)) has probability \( \frac{1}{2} \) to jump on the right at time \( t \) (namely \( \eta_t(x + 1) = 1 \)), but the jump is aborted if \( \eta_{t-1}(x + 1) = 1 \).

Using the probability space \((\Omega, \mathcal{F}, P)\) defined above, when a particle is at time \( t - 1 \in \mathbb{N} \) at positions \( x \in \mathbb{Z} \), it jumps if both \( \omega(t - 1, x) = 1 \) and the position \( x + 1 \) is free. Denote by \( T_{\text{TASEP}}(t, \omega, \cdot) \) the random map which associates to a given TASEP configuration \( \eta \) and a given random choice \( \omega \in \Omega \) the subsequent, one-time step, TASEP configuration. Heuristic prescriptions are summarized in

\[
T_{\text{TASEP}}(t, \omega, \eta)(x) = \begin{cases} 
\eta(x) & \text{if } \eta(x) = \eta(x + 1) = 1 \\
\omega(t - 1, x) & \text{if } \eta(x) = 1, \eta(x + 1) = 0 \\
\omega(t - 1, x - 1) & \text{if } \eta(x) = 0, \eta(x - 1) = 1 \\
\eta(x) & \text{if } \eta(x) = \eta(x - 1) = 0
\end{cases},
\]

or equivalently

\[
T_{\text{TASEP}}(t, \omega, \eta)(x) = \begin{cases} 
\omega(t - 1, x) \eta(x) + \omega(t - 1, x) \eta(x + 1) & \text{if } \eta(x) = 1 \\
\omega(t - 1, x - 1) \eta(x) + \omega(t - 1, x - 1) \eta(x - 1) & \text{if } \eta(x) = 0
\end{cases},
\]

which gives rise to the following rigorous Definition.
Definition 5. Let \((\Omega, F, P)\) be the probability space of Definition 4. We define the family of maps \(T_{\text{TASEP}}(t, \omega, \cdot)\) on \(\{0, 1\}^\mathbb{Z}\), indexed by \(t \in \mathbb{N}\) and \(\omega \in \Omega\), by

\[
T_{\text{TASEP}}(t, \omega, \eta)(x) = \begin{cases} 
\kappa(t-1, x) \eta(x) + \omega(t-1, x) \eta(x+1) \eta(x) \\
+ \kappa(t-1, x-1) \eta(x) + \omega(t-1, x-1) \eta(x-1) \eta(x-1) (1-\eta(x))
\end{cases}
\]

when \(t > 0\), \(T_{\text{TASEP}}(0, \omega, \cdot) = \text{id}\).

As for ABDF above, we may introduce TASEP random dynamical system by setting \(\phi_{\text{TASEP}}(0, \omega) = \text{id}\) and, for \(t > 0, t \in \mathbb{N}\),

\[
\phi_{\text{TASEP}}(t, \omega) := T_{\text{TASEP}}(t, \omega) \circ T_{\text{TASEP}}(t-1, \omega) \circ \cdots \circ T_{\text{TASEP}}(1, \omega),
\]

the latter satisfying the random dynamical system property

\[
\phi_{\text{TASEP}}(t, \omega) \circ \phi_{\text{TASEP}}(s, \omega) = \phi_{\text{TASEP}}(t+s, \omega), \quad t, s \in \mathbb{N}, \omega \in \Omega.
\]

We now turn our attention to pairs in TASEP configurations: pairs of particles and pairs of empty positions.

Definition 6. The pair operator

\[
P : \{0, 1\}^\mathbb{Z} \to (\{-1, 0, 1\} \times \{0, 1\})^\mathbb{Z}, \quad P(\eta)(x) = (P(\eta)_1(x), P(\eta)_2(x)), \quad x \in \mathbb{Z},
\]

is the function defined as follows:

- a) for every \(\eta \in \{0, 1\}^\mathbb{Z}\) and \(x \in \mathbb{Z}\),

\[
P(\eta)_1(x) = 1 - \eta(x) - \eta(x+1),
\]

- b) if \(P(\eta)_1 \neq 0\), then \(P(\eta)_2 = \text{ar} (P(\eta)_1)\), with \(\text{ar}\) as in Definition 1,

- c) if \(P(\eta)_1 = 0\), namely \(\eta = \text{alt}_\alpha\) for \(\alpha = 0\) or \(1\), then \(P(\eta)_2 = \eta\); in other words, \(P(\text{alt}_\alpha) = \overline{\alpha} \text{alt}_\alpha, \alpha = 0, 1\).

The link between TASEP pairs and ABDF configurations is the following conjugation result between the random dynamical systems \(\phi_{\text{ABDF}}(t, \omega)\) and \(\phi_{\text{TASEP}}(t, \omega)\).

Theorem 7. a) The pair map \(P\) is a bijection between \(\{0, 1\}^\mathbb{Z}\) and \(\Lambda\);

b) For every \(\eta \in \{0, 1\}^\mathbb{Z}, t \in \mathbb{N}, \omega \in \Omega\),

\[
T_{\text{ABDF}}(t, \omega, P(\eta)) = P(T_{\text{TASEP}}(t, \omega, \eta)),
\]

in particular, \(T_{\text{ABDF}}(t, \omega, \Lambda) \subset \Lambda\).
3.1. **Proof of Theorem 7, a).** The proof in itself can be made more concise than what follows, but we take the chance to introduce some more structure. We begin with a simple observation, that can be used to invert “by hand” the pair map $P$.

**Remark 8.** Given $\eta \in \{0, 1\}^\mathbb{Z}$, let $\theta(x) = 1 - \eta(x) - \eta(x + 1), x \in \mathbb{Z}$. Then, for every $n \in \mathbb{N}$

$$\theta(x + n) = 1 - \eta(x + n) - \eta(x + n + 1),$$

hence

$$\eta(x + n + 1) = 1 - \eta(x + n) - \theta(x + n)$$

$$= \eta(x + n - 1) + \theta(x + n - 1) - \theta(x + n)$$

$$= 1 - \eta(x + n - 2) - \theta(x + n - 2) + \theta(x + n - 1) - \theta(x + n),$$

and so on, which gives us

$$\eta(x + n + 1) = \frac{1 + (-1)^n}{2} - (-1)^n \eta(x) - \sum_{k=0}^{n} (-1)^k \theta(x + n - k).$$

A similar formula holds for negative integer $n$. Hence, we may reconstruct $\eta$ from $\theta$ at the price of fixing one value of $\eta$, say $\eta(0)$.

With this “reconstruction algorithm” at hand, we can proceed with the proof. A crucial property is that a right pair and a left pair are always separated by an odd number (= 1,3,5,...) of empty pairs, two right pairs or two left pairs by an even number of empty pairs (= 0,2,4,...), see Figure 3.

For $x_1, x_2 \in \mathbb{Z}$ we set $[x_1, x_2]_\mathbb{Z} = (x_1, x_1 + 1, \ldots x_2 - 1, x_2)$

**Definition 9.** Given a segment $[x_1, x_2]_\mathbb{Z}$, we define

$$P_{[x_1, x_2]_\mathbb{Z}} : \{0, 1\}^{[x_1, x_2 + 1]_\mathbb{Z}} \to \{-1, 0, 1\}^{[x_1, x_2]_\mathbb{Z}},$$

$$P_{[x_1, x_2]_\mathbb{Z}}(\eta)(x) = 1 - \eta(x) - \eta(x + 1), \quad \eta \in \{0, 1\}^{[x_1, x_2 + 1]_\mathbb{Z}}, \ x \in [x_1, x_2]_\mathbb{Z}.$$
A segment \([x_1, x_2]_\mathbb{Z}\) of cardinality \(n + 2\) is called a maximal alternating segment of \(\eta \in \{0, 1\}_\mathbb{Z}\) if:

- \(\eta(x_1) = \eta(x_1 + 1)\) and \(\eta(x_2) = \eta(x_2 + 1)\), i.e. \(\mathcal{P}(\eta)(x_1), \mathcal{P}(\eta)(x_2) \in \{\pm 1\}\),
- \(\eta(x_1 + k) \neq \eta(x_1 + k + 1)\), i.e. \(\mathcal{P}(\eta)(x_1 + k) = 0\), for \(k = 1, 2, ..., n\) (not imposed if \(n = 0\)).

It is called a maximal alternating segment of \(\theta \in \{-1, 0, 1\}_\mathbb{Z}\) if

- \(\theta(x_1), \theta(x_2) \in \{-1, 1\}\),
- \(\theta(x_1 + k) = 0\) for \(k = 1, 2, ..., n\) (not imposed when \(n = 0\)).

Maximal alternating half lines \((-\infty, x_2]_\mathbb{Z}\) and \([x_1, \infty)_\mathbb{Z}\) are defined analogously.

Clearly, if \([x_1, x_2]_\mathbb{Z}\) is a maximal alternating segment of \(\eta \in \{0, 1\}_\mathbb{Z}\), then it is a maximal alternating segment of \(\theta := \mathcal{P}(\eta)_1\) (similarly for half lines). The two key facts on this concept are expressed by the following two lemmata.

**Lemma 10.** Let \([x_1, x_2]_\mathbb{Z}\) be a maximal alternating segment of \(\eta \in \{0, 1\}_\mathbb{Z}\) (thus of \(\theta := \mathcal{P}(\eta)_1\)) with cardinality \(n + 2\). Then:

- if \(\theta(x_1)\theta(x_2) = 1\), \(n\) is even;
- if \(\theta(x_1)\theta(x_2) = -1\), \(n\) is odd.

**Proof.** If \(\theta(x_1)\theta(x_2) = 1\), then \(\theta(x_1) = \theta(x_2) = \pm 1\). In the +1 case, \(\eta(x_1) = \eta(x_1 + 1) = 0\), \(\eta(x_2) = \eta(x_2 + 1) = 0\). Then, from identity (6),

\[
\eta(x_1 + n + 1) = \frac{1 + (-1)^n}{2} - (-1)^n \eta(x_1) - \sum_{k=0}^{n} (-1)^k \eta(x_1 + n - k) = \frac{1 + (-1)^n}{2} - (-1)^n \theta(x_1) = \frac{1 + (-1)^n}{2} - (-1)^n,
\]

where we have used \(\theta(x_1 + n - k) = 0\) for \(k = 0, 1, ..., n - 1\), in the second-last step. Since \(\eta(x_1 + n + 1) = 0\), this implies \(n\) even. Other cases are analogous. \(\square\)

Lemma 10 imposes a restriction to the sequences of \((-1, 0, 1)_\mathbb{Z}\) in the range of the first component of \(\mathcal{P}\).

**Lemma 11.** Let \(\theta \in \Lambda \setminus \{\alpha_0, \alpha_1\}\) and let \([x_1, x_2]_\mathbb{Z}\) be a maximal alternating segment of \(\theta\) of cardinality \(n + 2\). Then there exists a unique \(\eta \in \{0, 1\}^{[x_1, x_2+1]_\mathbb{Z}}\) such that

\[
\mathcal{P}_{[x_1, x_2]_\mathbb{Z}}(\eta) = \theta_{[x_1, x_2]_\mathbb{Z}}.
\]

The string \(\eta\) satisfies \(\eta(x_1) = \eta(x_1 + 1), \eta(x_2) = \eta(x_2 + 1),\) with the unique values determined by the values \(\theta(x_1), \theta(x_2)\) and in the middle it is given by (6), precisely

\[
\eta(x_1 + j + 1) = \frac{1 + (-1)^j}{2} - (-1)^j \eta(x_1) - \sum_{k=0}^{j} (-1)^k \theta(x_1 + j - k)
\]

for \(j = 1, ..., n - 1\).

**Proof.** If \(\eta \in \{0, 1\}^{[x_1, x_2+1]_\mathbb{Z}}\) satisfies \(\mathcal{P}_{[x_1, x_2]_\mathbb{Z}}(\eta) = \theta_{[x_1, x_2]_\mathbb{Z}}\), then the properties of the values of \(\eta\), including the formula for the intermediate values, are obvious or have been proved above. Thus uniqueness is clear. Proving the existence means proving that \(\eta(x_1 + n + 1)\) given by the formula coincides with the values of \(\eta(x_2) = \eta(x_2 + 1)\) prescribed by \(\theta(x_2)\). This must be checked case by case, and we only report \(\theta(x_1) = \theta(x_2) = 1: n\) is even since \(\theta \in \Lambda\), and \(\eta(x_1) = \eta(x_1 + 1), \eta(x_2) = \)
\( \eta(x_2 + 1) \), all equal to zero. We have from (6)

\[
\eta(x_1 + n + 1) = \frac{1 + (-1)^n}{2} - (-1)^n \eta(x_1) - \sum_{k=0}^{n} (-1)^k \theta(x_1 + n - k)
\]

\[
= \frac{1 + (-1)^n}{2} - (-1)^n = 0 = \eta(x_2). \quad \square
\]

**Proof of Theorem 7, a).** It is sufficient to prove that, given \( \theta \in \Lambda \setminus \{alt_0, alt_1\} \), there exists one and only one \( \eta \in \{0,1\}^\mathbb{Z} \setminus \{alt_0, alt_1\} \) such that \( \mathcal{P}(\eta) = \theta \).

Let \( \{x_n\} \) be the strictly increasing, possibly bi-infinite sequence of points of \( \mathbb{Z} \) such that \( [x_n, x_{n+1}]_\mathbb{Z} \) is an even or odd maximal segment of \( \theta \). There are four cases: \( \{x_n\} \) is bi-infinite, or infinite only to the left, or infinite only to the right, or finite.

The construction of \( \{x_n\} \) may proceed from the origin of \( \mathbb{Z} \): we denote by \( x_0 \) the first point \( \geq 0 \) with \( \theta(x_0) \neq 0 \); by \( x_1 \) the first point \( > x_0 \) such that \( \theta(x_1) \neq 0 \); and so on, obviously if they exist. And we denote by \( x_{-1} \) the first point \( < 0 \) such that \( \theta(x_{-1}) \neq 0 \) and so on.

For every \( n \) such that \( x_n, x_{n+1} \) exists, we construct the corresponding values of \( \eta(x_n), \ldots, \eta(x_{n+1}+1) \) as in (d) of the previous lemma. The construction is unique with the property that, locally, \( \mathcal{P}(\eta)_x = \theta \) on \( [x_n, x_{n+1}]_\mathbb{Z} \). However, in principle the definition for \( [x_n, x_{n+1}]_\mathbb{Z} \) may be in contradiction with the definition for \( [x_{n+1}, x_{n+2}]_\mathbb{Z} \) because the points \( x_{n+1}, x_{n+1}+1 \) are in common. But, based on \( [x_n, x_{n+1}]_\mathbb{Z} \), we have defined \( \eta(x_{n+1}) = \eta(x_{n+1}+1) \), equal to 1 if \( \theta(x_{n+1}) = -1 \), equal to 0 if \( \theta(x_{n+1}) = 1 \). And based on \( [x_{n+1}, x_{n+2}]_\mathbb{Z} \) we have given the same definition. Therefore there is no contradiction. The treatment of half lines is analogous.

3.2. Proof of Theorem 7. We already stressed the drawback of \( \ar(\theta) \) being nonlocal, but when \( \theta = \mathcal{P}(\eta) \), the expression of \( \ar(\theta)(x) \) becomes local when written in terms of \( \eta \) (which depends on \( \theta \)-locally). This is a key fact for the proof of Theorem 7.

**Lemma 12.** If \((\theta, act) = \mathcal{P}(\eta)\), then

\[
act(x) = \eta(x) (1 - \eta(x + 1)).
\]

In other words, \( act(x) \) is equal to one if and only if \( \eta(x) = 1, \eta(x + 1) = 0 \), namely there is a particle at \( x \) and the position \( x + 1 \) is free, so the particle can jump.

**Proof.** Let us treat separately the case when \( \eta = alt_\alpha, \alpha = 0, 1 \). In this case \( act = \eta \); and also \( \eta(x)(1 - \eta(x + 1)) = \eta(x) \), because if \( \eta(x + 1) = 0 \) it is true, while if \( \eta(x + 1) = 1 \) we necessarily have \( \eta(x) = 0 \) by alternation, which coincides with \( \eta(x)(1 - \eta(x + 1)) \). The formula of the lemma is proved in this particular case.

Assume now \( \eta \) different from \( alt_\alpha, \alpha = 0, 1 \), so that \( act = \ar(\theta) \). Recall the definition of \( \ar(\theta)(x) \) in Definition 3, points (iii)-(vi). Let \( x_0 \) be such that \( \theta(x_0) \neq 0 \). It means that \( \eta(x_0) = \eta(x_0 + 1) \), both equal to 0 or 1. In both cases \( \eta(x_0)(1 - \eta(x_0 + 1)) = 0 \), hence equal to \( \ar(\theta)(x_0) \) as defined in Definition 3 point (iii).

Assume now \( \theta(x_0) = 0 \), from which \( \eta(x_0) \neq \eta(x_0 + 1) \) and thus the pair \((\eta(x_0), \eta(x_0 + 1))\) is either \((1, 0)\) or \((0, 1)\). Assume that the set \( L(x_0) \) is non empty and let \( x_1 \) be its maximum and let \( k > 0 \) be such that \( x_1 + k = x_0 \). The proof can be divided into several cases depending on the value of \( \theta(x_1) \) and the parity of \( k \). For instance, assume \( \theta(x_1) = 1, k \) odd. Thus \( \eta(x_1) = \eta(x_1 + 1) = 0, \eta(x_1 + 2) = 1, \eta(x_1 + 3) = 0 \), and so on, hence \( \eta(x_0) = \eta(x_1 + k) = 0, \eta(x_0 + 1) = 1 \). In this case

\[
\eta(x_0)(1 - \eta(x_0 + 1)) = 0
\]
and (from Definition 3 point (iv))
\[ ar(\theta)(x_0) = \left\lfloor \frac{\theta(x_1) + (-1)^k}{2} \right\rfloor = 0 \]
so they coincide. If \( \theta(x_1) = 1, k \) even,
\[ \eta(x_0)(1 - \eta(x_0 + 1)) = 1, \quad ar(\theta)(x_0) = \left\lfloor \frac{\theta(x_1) + (-1)^k}{2} \right\rfloor = 1, \]
so they coincide. The reader can check the two cases with \( \theta(x_1) = 0 \). If \( L(x_0) \) is empty and \( R(x_0) \) is non empty, the arguments are similar. \( \square \)

**Proof of Theorem 7.** **Step 1.** We prove the identity between the first components:
\[ \mathcal{T}_{\text{ABDF}}(t, \omega, \mathcal{P}(\eta))_1 = \mathcal{P}(\mathcal{T}_{\text{TASEP}}(t, \omega, \eta))_1. \]
Let \( \eta \in \{0, 1\}^Z, t \in \mathbb{N}, \omega \in \Omega, \) be given and write \( \theta := \mathcal{P}(\eta)_1, \tilde{\eta} := \mathcal{T}_{\text{TASEP}}(t, \omega, \eta), \) 
\( \hat{\theta} := \mathcal{P}(\tilde{\eta})_1, \hat{\theta} := \mathcal{T}_{\text{ABDF}}(t, \omega, \theta)_1. \) We have to prove \( \hat{\theta} = \theta. \)

From Definition 6 and Definition 5,
\[ \hat{\theta}(x) = 1 - \tilde{\eta}(x) - \tilde{\eta}(x + 1) = 1 - [\kappa(t, x)\eta(x) + \omega(t, x)\eta(x + 1)]\eta(x) \]
- \[ -[\kappa(t, x - 1)\eta(x) + \omega(t, x - 1)\eta(x - 1)](1 - \eta(x)) \]
- \[ -[\kappa(t, x + 1)\eta(x + 1) + \omega(t, x + 1)\eta(t, x + 2)]\eta(x + 1) \]
- \[ -[\kappa(t, x)\eta(x + 1) + \omega(t, x)\eta(x)](1 - \eta(x + 1)) \].

It simplifies, for instance, to
\[ \hat{\theta}(x) = [1 - \omega(t, x - 1)\eta(x - 1)](1 - \eta(x)) \]
- \[ -[\kappa(t, x + 1) + \omega(t, x + 1)\eta(t, x + 2)]\eta(x + 1), \]
because \( \eta(x)\eta(x) = \eta(x), \eta(x)(1 - \eta(x)) = 0, \kappa(t, x) + \omega(t, x) = 1. \)

Concerning \( \hat{\theta} \), from Definitions 4 and 6,
\[ \hat{\theta}(x) = \max(\theta(x - 1), 0) + \text{act}(x - 1)\kappa(x, t - 1) \]
+ \[ \min(\theta(x + 1), 0) - \text{act}(x + 1)\kappa(x, t + 1), \]
where, by definition of \( \mathcal{P}(\eta)_1 \) and from Lemma 12,
\( \theta(x) = 1 - \eta(x) - \eta(x + 1), \quad \text{act}(x) = \eta(x)(1 - \eta(x + 1)). \)

We have,
\[ \max(1 - \eta(x - 1) - \eta(x), 0) = (1 - \eta(x - 1))(1 - \eta(x)), \]
\[ \min(1 - \eta(x + 1) - \eta(t, x + 2), 0) = -\eta(x + 1)\eta(t, x + 2). \]

Moreover,
\[ \text{act}(x - 1) = \eta(x - 1)(1 - \eta(x)), \quad \text{act}(x + 1) = \eta(x + 1)(1 - \eta(x + 2)). \]

Hence
\[ \hat{\theta}(x) = (1 - \eta(x - 1))(1 - \eta(x)) + \eta(x - 1)(1 - \eta(x))\kappa(x, t - 1) \]
- \[ \eta(x + 1)\eta(x + 2) - \eta(x + 1)(1 - \eta(x + 2))\kappa(x, t + 1) \]
= \[ 1 - \eta(x) - \eta(x - 1)(1 - \eta(x))\omega(t, x - 1) \]
- \[ \eta(x + 1)\kappa(t, x + 1) - \eta(x + 1)\eta(t, x + 2)\omega(t, x + 1) \]
which is equal to \( \hat{\theta}(x). \)

**Step 2.** We now prove the identity between second components:
\[ \mathcal{T}_{\text{ABDF}}(t, \omega, \mathcal{P}(\eta))_2 = \mathcal{P}(\mathcal{T}_{\text{TASEP}}(t, \omega, \eta))_2. \]
This identity is obviously true when the two elements of (7) are not zero, because both the terms of (8) are defined as the activation map of the corresponding terms of (7). Thus it remains to prove identity (8) when

\[ T_{\text{ABDF}} (t, \omega, P(\eta))_1 = 0, \quad P(T_{\text{TASEP}} (t, \omega, \eta))_1 = 0. \]

Condition \( P(T_{\text{TASEP}} (t, \omega, \eta))_1 = 0 \) implies \( T_{\text{TASEP}} (t, \omega, \eta) = \text{alt}_\alpha \) for some \( \alpha = 0, 1 \) and \( P(T_{\text{TASEP}} (t, \omega, \eta))_2 = \text{alt}_\alpha \). Therefore we have to prove

\[ T_{\text{ABDF}} (t, \omega, P(\eta))_2 = \text{alt}_\alpha, \]

and we split the proof in two more steps.

**Step 3.** We continue the proof of Step 2 assuming \( \alpha = 1 \). We have to prove that one of the following three conditions hold (see the three conditions at the end of Definition 4):

1. \( P(\eta)_1(0) = -1; \)
2. \( P(\eta)_2(-1) = 1 \) and \( \omega(t - 1, -1) = 1; \)
3. \( P(\eta)_2(0) = 1 \) and \( \omega(t - 1, 0) = 0. \)

If the first one is true, the proof is complete. Otherwise we have \( P(\eta)_1(0) = 1 \) or \( P(\eta)_2(0) = 0 \). Let us prove that \( P(\eta)_1(0) = 1 \) implies the second condition; and that \( P(\eta)_2(0) = 0 \) implies either the second or third conditions.

Thus assume \( P(\eta)_1(0) = 1 \). In this case \( \eta(0) = \eta(1) = 0 \), hence we need \( \eta(-1) = 1 \) and \( \omega(t - 1, -1) = 1 \) to get \( T_{\text{TASEP}} (t, \omega, \eta) = \text{alt}_1 \). But then, from \( \eta(-1) = 1 \), \( \eta(0) = 0 \) and \( \omega(t - 1, -1) = 1 \) we deduce \( P(\eta)_1(0) = 0 \) and \( P(\eta)_2(-1) = 1 \) (Lemma 12), so the second condition hold true.

If \( P(\eta)_1(0) = 0 \), then \( \eta(0) \neq \eta(1) \). It cannot be \( \eta(0) = 1, \eta(1) = 0, \omega(t - 1, 0) = 1 \), otherwise \( T_{\text{TASEP}} (t, \omega, \eta) (1) = 1 \), incompatible with \( T_{\text{TASEP}} (t, \omega, \eta) = \text{alt}_1 \). Thus: i) either \( \eta(0) = 0, \eta(1) = 1 \); ii) or \( \eta(0) = 1, \eta(1) = 0, \omega(t - 1, 0) = 0 \). In case (i), we must have \( \eta(-1) = 1 \) and \( \omega(t - 1, -1) = 1 \) to get \( T_{\text{TASEP}} (t, \omega, \eta) = \text{alt}_1 \); in this case the conclusion is, as above, that the second condition holds true. In case (ii) we have \( P(\eta)_1(0) = 0, P(\eta)_2(0) = 1 \) (Lemma 12) and of course \( \omega(t - 1, 0) = 0 \), hence the last of the three conditions hold. The case \( \alpha = 1 \) is solved.

**Step 4.** We continue the proof of Step 2 assuming \( \alpha = 0 \), namely \( T_{\text{TASEP}} (t, \omega, \eta) = \text{alt}_0 \). We have to prove that none of the conditions (9) hold. We argue by contradiction, observing that

\[
\begin{align*}
\{ P(\eta)_2(-1) = 1 & \Rightarrow \eta(-1) = 1, \eta(0) = 0 \Rightarrow T_{\text{TASEP}} (t, \omega, \eta) (0) = 1, \\
\omega(t - 1, -1) = 1 & \Rightarrow \eta(-1) = 1, \eta(1) = 0 \Rightarrow T_{\text{TASEP}} (t, \omega, \eta) (0) = 1,
\end{align*}
\]

\[
\begin{align*}
\{ P(\eta)_2(0) = 1 & \Rightarrow \eta(0) = 1, \eta(1) = 0 \Rightarrow \{ T_{\text{TASEP}} (t, \omega, \eta) (0) = 1, \\
\omega(t - 1, 0) = 0 & \Rightarrow T_{\text{TASEP}} (t, \omega, \eta) (1) = 0 \},
\end{align*}
\]

where conditions on the right are incompatible with \( T_{\text{TASEP}} (t, \omega, \eta) = \text{alt}_0 \). This completes the proof. \( \square \)

We complete this section with a simple corollary of Theorem 7 which is not easy to prove directly on ABDF dynamics. In plain words it says that the point \( x_0 \) where a pair coalesces, cannot be the origin of a pair at the same time of the coalescence.

**Corollary 13.** Let \( x_0 \in \mathbb{Z} \), \( (\theta, \text{act}) \in \Lambda \) be such that

\[ \theta(x_0 - 1) = 1, \theta(x_0) = 0, \theta(x_0 + 1) = -1. \]

Then, for every \( (t, \omega) \in \mathbb{N} \times \Omega \)

\[ T_{\text{ABDF}} (t, \omega, (\theta, \text{act}))_1 (x_0) = 0, \quad T_{\text{ABDF}} (t, \omega, (\theta, \text{act}))_2 (x_0) = 0. \]
The same result holds if one or both \( x_0 - 1, x_0 + 1 \) are arising pair point for \( \theta \).

Proof. Let \( \eta \) be such that \((\theta, \omega, \eta) = \mathcal{P}(\eta)\). By hypothesis

\[
\eta(x_0 - 1) = 0, \eta(x_0) = 0, \eta(x_0 + 1) = 1, \eta(x_0 + 2) = 1.
\]

TASEP dynamics cannot change the values at \( x_0 \) and \( x_0 + 1 \), hence

\[
\mathcal{T}_{\text{TASEP}}(t, \omega, \eta)(x_0) = 0, \mathcal{T}_{\text{TASEP}}(t, \omega, \eta)(x_0 + 1) = 1.
\]

This implies the result in the first case.

Now, assume \( x_0 - 1 \) is an arising pair point for \((\theta, \omega, \eta)\), and \( \theta(x_0) = 0, \theta(x_0 + 1) = -1 \). We now have

\[
\eta(x_0 - 1) = 1, \eta(x_0) = 0, \eta(x_0 + 1) = 1, \eta(x_0 + 2) = 1
\]

\[
\omega(t - 1, x_0 - 1) = 0.
\]

Again TASEP dynamics does not change the values at \( x_0 \) and \( x_0 + 1 \). The same argument applies to the case when \( x_0 + 1 \) is an arising pair point for \((\theta, \omega, \eta)\). \( \square \)

4. Pure-Jump Weak Solutions of Burgers’ Equation

We consider in this section Burgers’ equation (1),

\[
\partial_t u + u \partial_x u = 0.
\]

We are interested in bounded (non entropic!) weak solutions, so we restrict the definition to bounded functions, although it could be more general.

Definition 14. We say that a bounded measurable function \( u : [t_0, t_1] \times \mathbb{R} \rightarrow \mathbb{R} \) is a weak solution on \([t_0, t_1]\) if:

i) for every smooth test function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) with compact support in \( \mathbb{R} \) the function \( t \mapsto \int_{\mathbb{R}} u(t, x) \varphi(x)dx \) is continuous on \([t_0, t_1]\);

ii) for every smooth test function \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R} \) with compact support in \((t_0, t_1) \times \mathbb{R}\), we have

\[
\int_{t_0}^{t_1} \int_{\mathbb{R}} \left( u(t, x) \partial_t \phi(t, x) + \frac{1}{2} u^2(t, x) \partial_x \phi(t, x) \right) dx dt = 0.
\]

Given a test function \( \varphi \), the function \( t \mapsto \int_{\mathbb{R}} u(t, x) \varphi(x)dx \) is always defined almost everywhere, by Fubini-Tonelli theorem. We require that it is continuous, for a minor reason appearing in the next Proposition. It is not restrictive in our examples.

In the sequel we shall piece together weak solutions defined on different space-time domains: let us see two rules allowing us to do so. When we say that \( u(T, \cdot) = v(T, \cdot) \) for a certain \( T \in [t_0, t_1] \) we mean that \( \int_{\mathbb{R}} u(T, x) \varphi(x)dx = \int_{\mathbb{R}} v(T, x) \varphi(x)dx \) for all test functions \( \varphi \) of the class above.

Proposition 15. Assume \( u(t, x) \) is a weak solution on \([t_0, t_1]\) and \( v(t, x) \) a weak solution on \([t_1, t_2]\), with \( u(t_1, \cdot) = v(t_1, \cdot) \). Then the function \( w \), defined on \([t_0, t_2]\), equal to \( u \) on \([t_0, t_1]\) and \( v \) on \([t_1, t_2]\), is a weak solution on \([t_0, t_2]\).

Proof. Let \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) with compact support in \( \mathbb{R} \). Consider the function \( t \mapsto \int_{\mathbb{R}} u(t, x) \varphi(x)dx \), defined a.s. by Fubini-Tonelli theorem. By the continuity of the function \( t \mapsto \int_{\mathbb{R}} u(t, x) \varphi(x)dx \) on \([t_0, t_1]\) and of \( t \mapsto \int_{\mathbb{R}} v(t, x) \varphi(x)dx \) on \([t_1, t_2]\) and by the property \( u(t_1, \cdot) = v(t_1, \cdot) \), we deduce that \( t \mapsto \int_{\mathbb{R}} w(t, x) \varphi(x)dx \) is continuous.

Given \( \phi \) as in the definition, part (ii), introduce

\[
\phi_n(t, x) = \phi(t, x) (1 - \chi_n(t - t_1))
\]
Let us recall the definition of the Heaviside function and its formal weak derivative
\[ \chi_n(t) = \chi(n t), \]
where \( \chi \) is smooth, \( \chi(x) = \chi(-x) \), with values in \([0, 1]\), equal to 1 in \([-1, 1]\), to zero outside \([-2, 2]\), and take \( n \) large enough. The function \( \phi_n(t, x) \), restricted to \((t_0, t_1) \times \mathbb{R}\), is a good test function for \( u \), hence
\[
\int_{t_0}^{t_1} \int_{\mathbb{R}} \left( u(t, x) \partial_t \phi_n(t, x) + \frac{1}{2} u^2(t, x) \partial_x \phi_n(t, x) \right) \, dx \, dt = 0.
\]
Similarly on \((t_1, t_2) \times \mathbb{R}\) for \( v \), hence
\[
\int_{t_0}^{t_2} \int_{\mathbb{R}} \left( w(t, x) \partial_t \phi_n(t, x) + \frac{1}{2} w^2(t, x) \partial_x \phi_n(t, x) \right) \, dx \, dt = 0.
\]
The same identity holds for \( \phi \), completing the proof, if we show that
\[
\lim_{n \to \infty} \int_{t_0}^{t_2} \int_{\mathbb{R}} w(t, x) \partial_t \chi_n(t - t_1) \, dx \, dt = 0,
\]
\[
\lim_{n \to \infty} \int_{t_0}^{t_2} \int_{\mathbb{R}} w(t, x) \partial_x \chi_n(t - t_1) \, dx \, dt = 0,
\]
\[
\lim_{n \to \infty} \int_{t_0}^{t_2} \int_{\mathbb{R}} w^2(t, x) \chi_n(t - t_1) \partial_x \phi(t, x) \, dx \, dt = 0.
\]
The second and third limits are clear. The first claim is equivalent to
\[
\lim_{n \to \infty} \int_{t_1 - \frac{1}{n}}^{t_1 + \frac{1}{n}} n \chi'(n(t - t_1)) \left( \int_{\mathbb{R}} w(t, x) \phi(t, x) \, dx \right) \, dt = 0.
\]
It is easy, using also the boundedness of \( w \), to show that the function
\[
t \mapsto \int_{\mathbb{R}} w(t, x) \phi(t, x) \, dx
\]
is continuous (approximate \( \phi \) by functions piecewise constant in \( t \)). With a similar argument we can replace this function by a constant in the previous limit and thus reduce us to check the property
\[
\lim_{n \to \infty} \int_{t_1 - \frac{1}{n}}^{t_1 + \frac{1}{n}} \chi'(nt) \, dt = 0
\]
(we have also changed variables). But this means \( \lim_{n \to \infty} \int_{-2}^{2} \chi'(s) \, ds = 0 \), which is true by symmetry of \( \chi \).

Proposition 16. Assume that \( u, v \) are two weak solutions, on \([t_0, t_1]\), with disjoint supports, namely there are sets \( S_u, S_v \subset [t_0, t_1] \times \mathbb{R} \), disjoint, Borel measurable, such that \( u = 0 \) a.s. outside \( S_u \) and \( v = 0 \) a.s. outside \( S_v \). Define
\[
w = u + v.
\]
Then \( w \) is a weak solution. The result remains true when the intersection of the supports has Lebesgue measure zero.

Proof. All properties are easily checked. Notice that \( w^2 = u^2 + v^2 \) almost everywhere.

Let us recall the definition of the Heaviside function and its formal weak derivative
\[
H(x) = 1_{[0, \infty)}(x) = 1_{\{x \geq 0\}}, \quad H'(x) = \delta_0(x) = \delta(x = 0).
\]
Given \( t_0 \in \mathbb{R} \), the simplest example of a pure-jump weak solution — which is not an entropy solution — of Burgers’ equation on \([t_0, t_1]\), for any \( t_1 > t_0 \), is
\[
u(t, x) = 1_{\{x \geq x_0 + v(t - t_0)\}} w = 1_{\{x - x_0 - v(t - t_0) \geq 0\}} w = H(x - x_0 - v(t - t_0)) w
\]
for \( t \in [t_0, t_1] \), and with \( v > 0 \) and \( w = 2v \). Part (i) of Definition 14 comes from
\[
\int_{\mathbb{R}} u(t, x) \varphi(x) dx = \int_{x_0 + v(t-t_0)}^{\infty} w \varphi(x) dx,
\]
and this will be the case in all examples below, hence we shall not repeat it. Checking condition (ii) of Definition 14 is elementary but quite lengthy. However, we can perform a formal computation:
\[
\partial_t u = -\delta_0(x - x_0 - v(t - t_0)) w v, \\
u^2(t, x) = u(t, x) w, \\
\partial_x u^2 = \delta_0(x - x_0 - v(t - t_0)) w^2,
\]
which, by \( w^2 = 2vw \), implies \( \partial_x u^2 = -2\partial_t u \). In what follows we will deal with analogous computations in more complicated situations: the above formal computation with Dirac’s deltas is both concise and transparent, so we will keep on making use of it, but any such computation can be easily made rigorous in terms of couplings with test functions.

4.1. Isolated quasi-particles. Given \( h > 0 \) (it will be typically small in our main results), \( v > 0 \), \( t_1 > t_0 \), we call right-quasi-particle on \([t_0, t_1]\) a function of the following form: for \((t, x) \in \mathbb{R} \times [t_0, t_1]\), and \( w = 2v \),
\[
u(t, x) = 1_{\{x_0 + v(t-t_0) - h \leq x < x_0 + v(t-t_0)\}} w \\
= 1_{\{x \geq x_0 + v(t-t_0) - h\}} w - 1_{\{x > x_0 + v(t-t_0)\}} w \\
= H(x - x_0 - v(t - t_0) + h) w - H(x - x_0 - v(t - t_0)) w.
\]
The latter is a weak solution of Burgers’ equation on \([t_0, t_1]\):
\[
\partial_t u = -\delta_0(x - x_0 - v(t - t_0) + h) w v + \delta_0(x - x_0 - v(t - t_0)) w v, \\
u^2(t, x) = u(t, x) w, \\
\partial_x u^2 = \delta_0(x - x_0 - v(t - t_0) + h) w^2 - \delta_0(x - x_0 - v(t - t_0)) w^2,
\]
hence \( \partial_x u^2 = -2\partial_t u \). Regarded as a soliton, a right-quasi-particle moves to the right with velocity \( v \).

A left-quasi-particle on \([t_0, t_1]\) has the form (with \( v = w/2 > 0 \) as above)
\[
u(t, x) = 1_{\{x_0 - v(t-t_0) \leq x < x_0 - v(t-t_0) + h\}} w, \\
= 1_{\{x \geq x_0 - v(t-t_0)\}} w - 1_{\{x < x_0 - v(t-t_0) + h\}} w, \\
= -H(x - x_0 + v(t - t_0)) - H(x - x_0 + v(t - t_0) - h)) w.
\]
Again, as a soliton, a left-quasi-particle moves to the left, again with velocity \( v \).

4.2. Arising pairs of quasi-particles. In our construction, the traveling solitons just defined usually emerge somewhere and disappear somewhere else. In this subsection we describe the creation mechanism: quasi-particles appear in pairs, a positive and a negative one, moving in opposite directions. After a short time of order \( h \) they become isolated solitons of the form described in the previous subsection. But at the beginning, when they emerge and develop, they are made of two pieces with increasing size smaller than \( h \), see Figure 5.

Remark 17. An arising pair comes from the identically zero solution. Hence, before the birth time \( t_0 \), we have \( u = 0 \), which is a weak solution. In the time interval \([t_0, t_0 + \frac{h}{2}]\) the pair develops. After time \( t_0 + \frac{h}{2} \) we have two disjoint isolated quasi-particles, which is again a weak solution, by Proposition 16 and the examples of the previous subsection. Thus, thanks to Proposition 15, it is sufficient to define the arising pair in \([t_0, t_0 + \frac{h}{2}]\) and to prove that it is a weak solution there.
Given \( h > 0, v > 0 \) and \( t_0 \), we call arising pair of quasi-particles at \((t_0, x_0)\), the function defined for \((t, x) \in [t_0, t_0 + \frac{h}{v}] \times \mathbb{R}\) by

\[
\begin{align*}
  u(t, x) &= 1\{x_0 \leq x < x_0 + v(t - t_0)\} w - 1\{x_0 - v(t - t_0) \leq x < x_0\} w \\
  w &= 2v.
\end{align*}
\]

An expression for continuing the motion after \( t_0 + \frac{h}{v} \) is

\[
\begin{align*}
  u(t, x) &= 1\{x_0 + v(t - t_0) - \min(h, v(t - t_0)) \leq x < x_0 + v(t - t_0)\} w \\
  -1\{x_0 - v(t - t_0) \leq x < x_0 - v(t - t_0) + \min(h, v(t - t_0))\} w.
\end{align*}
\]

Let us treat only the case \( t \in [t_0, t_0 + \frac{h}{v}] \). It can also be written as

\[
\begin{align*}
  u(t, x) &= 1\{x \geq x_0\} w - 1\{x \geq x_0 + v(t - t_0)\} w - (1\{x \geq x_0 - v(t - t_0)\} - 1\{x \geq x_0\}) w \\
  &= H(x - x_0) w - H(x - x_0 - v(t - t_0)) w \\
  - (H(x - x_0 + v(t - t_0)) - H(x - x_0)) w.
\end{align*}
\]

It is a weak solution of Burgers’ equation: indeed it holds

\[
\partial_t u = \delta_0(x - x_0 - v(t - t_0)) w v - \delta_0(x - x_0 + v(t - t_0)) w v
\]

and, using the fact that the two pieces have disjoint support,

\[
\begin{align*}
  u^2(t, x) &= 1\{x_0 \leq x \leq x_0 + v(t - t_0)\} w^2 + 1\{x_0 - v(t - t_0) \leq x \leq x_0\} w^2 \\
  &= H(x - x_0) w^2 - H(x - x_0 - v(t - t_0)) w^2 \\
  + H(x - x_0 + v(t - t_0)) w^2 - H(x - x_0) w^2, \\
  \partial_x u^2 &= \delta_0(x - x_0) w^2 - \delta_0(x - x_0 - v(t - t_0)) w^2 \\
  &\quad + \delta_0(x - x_0 + v(t - t_0)) w^2 - \delta_0(x - x_0) w^2 = -2\partial_t u.
\end{align*}
\]
4.3. Coalescing pairs of quasi-particles. As anticipated above, usually a quasi-particle meets after a short time another quasi-particle traveling in the opposite direction: in this case they annihilate each other. This process is described by the following solution: given \( h > 0, v = w/2 > 0 \) and \( t_1 \), we call *coalescing pair of quasi-particles* at \((t_1, x_0)\), the function defined for \((t, x) \in [t_1 - \frac{h}{2}, t_1] \times \mathbb{R}\) by
\[
u(t, x) = 1_{\{x_0 - v(t - t_1) \leq x < x_0\}} w - 1_{\{x_0 \leq x < x_0 + v(t - t_1)\}} w.
\]
The proof that it is a weak solution is the same as for arising quasi-particles: in fact one can also argue by time-reversal of Burgers’ equation.

**Remark 18.** The content of the present section is easily adapted to produce weak solutions of
\[
\partial_t u = \lambda \partial_x u^2
\]
with \( \lambda \neq 0 \), the latter being the formal derivative of (12). Indeed, it suffices to replace the relation \( w = 2v \) between parameters \( v, w \) with \( w = -\lambda v \).

5. TASEP pairs, ABDF and Burgers quasi-particles

In this section we describe a bijection between TASEP sample (and therefore ABDF samples) and realizations of a random weak solution of Burgers’ equation. We associate special solutions of Burgers’ equation with ABDF configurations, using integer times \( t_0 \in \mathbb{N} \) for this correspondence; then we interpolate for \( t \in [t_0, t_0 + 1) \) using the special quasi-particle weak solutions defined in the previous section.

This idea however is complicated by a tricky detail. In the ABDF model, creation of new pairs happens at integer times \( t_0 - 1 \) (without being visible) and is observed only at time \( t_0 \): discrete time allows to do so. We refer to Figure 2 for an example: new pairs arise from empty sites (diamond-shaped in the picture). On the contrary, due to continuous time, for Burgers’ equation we need to create new pairs before integer times, so that the pair is fully formed at integer time. The creation instant of a new pair will be at times \( t_0 - 1 - \frac{h}{2}, t_0 \in \mathbb{N}_0 \).

Strictly speaking, the correspondence between TASEP and Burgers’ samples is not a conjugation of random dynamical systems, as opposed to the one between TASEP and ABDF, because in order to define the configuration at time \( t_0 \in \mathbb{N} \) of the random weak solution of Burgers’ equation we need two pieces of information: the configuration of TASEP or ABDF at time \( t_0 \) and the noise values \( \{\omega(t_0, x) : x \in \mathbb{Z}\} \). Nevertheless, it is a bijection of samples of stochastic processes, and thus it may allow to study the behavior of each one of the two processes starting from the other one.

**Definition 19.** Given \( t_0 \in \mathbb{N}, (\theta, \text{act}) \in \Lambda, \omega \in \Omega \), let us introduce the following sets:
\[
M^+ (\theta) = \{ z \in \mathbb{Z} : \theta (z) = 1 \}
M^- (\theta) = \{ z' \in \mathbb{Z} : \theta (z') = -1 \},
A (\theta, \text{act}, \omega, t_0) = \{ z \in \mathbb{Z} : \text{act} (z) = 1, \omega (t_0, z) = 0 \},
MA^+ (\theta, \text{act}, \omega, t_0) = M^+ (\theta) \cup A (\theta, \text{act}, \omega, t_0),
MA^- (\theta, \text{act}, \omega, t_0) = M^- (\theta) \cup A (\theta, \text{act}, \omega, t_0),
C (\theta, \text{act}, \omega, t_0) = \{ z \in \mathbb{Z} : z - 1 \in MA^+ (\theta, \text{act}, \omega, t_0), z + 1 \in MA^- (\theta, \text{act}, \omega, t_0) \}.
\]
(In plain words they are the sets of, respectively, particles Moving to the right, particles Moving to the left, sites of Arising pairs, sites of Moving or Arising, sites of Coalescence).
Given \((\theta, \text{act}) \in \Lambda, t_0 \in \mathbb{N}\) and \(\omega \in \Omega\), we define, for \(x \in \mathbb{R}\),
\[
u(t_0, x, \omega) = 2 \sum_{z \in M^+(\theta)} 1\{z \leq x < z + \frac{1}{2}\} - 2 \sum_{z \in M^-(\theta)} 1\{z - \frac{1}{2} \leq x < z\} + 2 \sum_{z \in A(\theta, \text{act}, \omega, t_0)} \left(1\{z \leq x < z + \frac{1}{2}\} - 1\{z - \frac{1}{2} \leq x < z\}\right).
\]

To explain the definition, assume \((\theta, \text{act}) = \mathcal{P}(\eta)\). The formula for \(\nu(t_0, x)\) includes three summands:

- a right-quasi-particle
  \[
  2 \cdot 1\{z \leq x < z + \frac{1}{2}\}
  \]
  at each point \(z\) where \(\theta(z) = 1\), namely where there is a right-particle of ABDF, or equivalently where TASEP has two consecutive empty spaces;

- a left-quasi-particle
  \[
  -2 \cdot 1\{z - \frac{1}{2} \leq x < z\}
  \]
  at each point \(z\) where \(\theta(z) = -1\), namely where there is a left-particle of ABDF, or equivalently where TASEP has two consecutive particles;

- an arising pair of quasi-particles
  \[
  2 \left(1\{z \leq x < z + \frac{1}{2}\} - 1\{z - \frac{1}{2} \leq x < z\}\right)
  \]
at each point $z$ where $\theta (z) = 0$, $\text{act} (z) = 1$ (or equivalently $\eta (z) = 1$, $\eta (z + 1) = 0$) and $\omega (t_0, x) = 0$, namely at each active empty position of ABDF, where the noise prescribes the creation of two particles.

We have chosen $h = \frac{1}{2}$, $v = 1$, and thus $w = 2$, in the definitions of quasi-particles of section 4; the value of $h$ can be changed to any value in $(0, 1)$ without consequences, while the value of $v$ is coordinated with the scheme.

5.1. Bijection with TASEP at Integer Times. In the remainder of this section we need to show several facts. The first one is the bijection property between TASEP (or ABDF) realizations and these particular functions $u (t_0, x, \omega)$. This can be formalized in different ways; we limit ourselves to state that, given the function $u (t_0, x, \omega)$, above, we can reconstruct the values of $(\theta, \text{act})$. The proof is straightforward, just noticing that each $z \in \mathbb{Z}$ appears at most in one of the sums defining $u (t_0, x, \omega)$.

**Proposition 20.** Let $u (t_0, x, \omega)$ be given by Definition 19, with respect to $(\theta, \text{act}) \in \Lambda$ and $\omega \in \Omega$. Then

$$\theta (z) = \int_{z-\frac{1}{2}}^{z+\frac{1}{2}} u (t_0, x, \omega) \, dx.$$ 

If $\theta \neq 0$, $\text{act}$ is ar $(\theta)$ and thus can be reconstructed from $u$. If $\theta = 0$, for a.e. $\omega$ there are infinitely many active points $z$ where $\omega (t_0, z) = 0$; this means that $u (t_0, x, \omega)$ contains infinitely many points $z$ where the jump

$$[u (t_0, \cdot, \omega)]_z := \lim_{x \to z^-} u (t_0, x, \omega) - \lim_{x \to z^+} u (t_0, x, \omega)$$

is equal to 4. If these points are even, $\text{act} = \text{act}_1$, otherwise $\text{act} = \text{act}_0$.

5.2. Continuation Shortly After Integer Times. Next, we have to interpolate the functions $u (t_0, x, \omega)$ between integer times. The particular weak solutions introduced in section 4 prescribe a unique continuation of $u (t_0, x, \omega)$ from the “initial” value at time $t_0 \in \mathbb{N}$ to all values of $t$ in $[t_0, t_0 + \frac{1}{2}]$ (the value $\frac{1}{2}$ is related to the choice $h = \frac{1}{2}$):

$$u (t, x, \omega) = 2 \sum_{z \in \mathcal{M}A^+ (\theta, \text{act}, \omega, t_0)} 1 \{ z + (t-t_0) \leq x < z + (t-t_0) + \frac{1}{4} \} - 2 \sum_{z' \in \mathcal{M}A^- (\theta, \text{act}, \omega, t_0)} 1 \{ z' - (t-t_0) - \frac{1}{4} \leq x < z' - (t-t_0) \}.$$

**Proposition 21.** The function thus defined for $t \in [t_0, t_0 + \frac{1}{2}]$, $x \in \mathbb{R}$ is a weak solution of Burgers’ equation.

**Proof.** Coincidence of the last formula at time $t_0$ with the initial condition $u (t_0, x, \omega)$ above is obvious. The statement is a consequence of a simple fact: every pair of terms taken from the two sums defining $u (t, x, \omega)$ is made of quasi-particles with disjoint supports on $[t_0, t_0 + \frac{1}{2}]$, and thus the sum solves Burgers’ equation in weak sense –by Proposition 16 –on the interval $[t_0, t_0 + \frac{1}{2}]$.

Let us check that supports are disjoint. Quasi-particles moving to the right (those corresponding to the first sum) are clearly isolated between themselves, having a “support” of size $\frac{1}{4}$ of the form $[x (t), x (t) + \frac{1}{4}]$ with $x (t)$ of the form $z + (t-t_0)$ with $z$ of distance at least one from each other. The same holds for left-quasi-particles, among themselves. Thus the problem is only about the interaction between a right-quasi-particle

$$2 \cdot 1 \{ z + (t-t_0) \leq x < z + (t-t_0) + \frac{1}{4} \}$$
Figure 7. Evolution of $u$ considered in Proposition 21 and Proposition 22 built upon ABDF evolution of Figure 2. Two shades of gray denote, as above, right and left quasi-particles. The dotted grid has side length 1, so $u = \pm 2$ respectively on dark and light gray areas.

and a left-quasi-particle

$$-2 \cdot 1 \left\{ x' - (t - t_0) - \frac{1}{2} \leq x < x' - (t - t_0) \right\}$$

with $z$ in the first sum and $z'$ in the second one. The supports of these two solitons have size $\frac{1}{2}$ and are of the form $[x(t), x(t) + \frac{1}{2}]$ with $x(t) = z + (t - t_0)$ and $[x'(t) - \frac{1}{2}, x'(t)]$ with $x'(t) = z' - (t - t_0)$, respectively. We claim that these supports are disjoint, for $t \in [t_0, t_0 + \frac{1}{2}]$. If $z' \leq z$ this is clear, since $x'(t)$ is decreasing and $x(t)$ is increasing. When $z' > z$ we claim that sets $[x(t), x(t) + \frac{1}{2}]$ and $[x'(t) - \frac{1}{2}, x'(t)]$ are disjoint because $(t - t_0) + \frac{1}{2} \leq 1$ and $z' \geq z + 2$ (to be shown below) and thus

$$z' - (t - t_0) - \frac{1}{2} \geq z + (t - t_0) + \frac{1}{2}$$

The key fact $z' \geq z + 2$ requires inspection into the conditions that $z$ and $z'$ belong to two different sums. Recall we are treating the case $z' > z$, hence the two solitons are not the result of an arising pair.

We have $z \in MA^+ (\theta, \text{act}, \omega, t_0)$. This is the union of two cases. Consider the case $\theta(z) = 1$ and assume by contradiction that $z' = z + 1$. By the rules of $\Lambda$, $\theta(z')$ cannot be $-1$ (because the number of integer points strictly between $z$ and $z'$ is even); by the rules of the map $ar$, if $\theta(z') = 0$, $ar(\theta(z'))$ is zero. Hence we have found a contradiction.

Consider the case $\text{act}(z) = 1$ and assume by contradiction that $z' = z + 1$. Again by the rules of $\Lambda$ we cannot have $\theta(z') = -1$ and we cannot have $\text{act}(z') = 1$. Hence, we get a contradiction also in this case, and this proves $z' \geq z + 2$. □

5.3. Continuation Shortly Before Integer Times. To continue the solution in time intervals $t \in [t_0 + \frac{1}{2}, t_0 + 1]$ is somewhat trickier because of two phenomena: coalescence of quasi-particles, and growth of new pairs. We are now at time $t_0 + 1/2$, namely

$$u \left( t_0 + \frac{1}{2}, x, \omega \right) = 2 \sum_{z \in MA^+(\theta, \text{act}, \omega, t_0)} 1 \left\{ z + \frac{1}{2} \leq x < z + 1 \right\} - 2 \sum_{z' \in MA^-(\theta, \text{act}, \omega, t_0)} 1 \left\{ z' - 1 \leq x < z' - \frac{1}{2} \right\}.$$
The continuation depends on this configuration and on the section of the noise 
\[ \{ \omega(t_0 + 1, x) : x \in \mathbb{Z} \}. \]
Indeed, at time \( t_0 + 1 \) we could observe the result of arising pairs, as it was above at time \( t_0 \). These pairs started existing at time \( t_0 + \frac{1}{2} \).

Let us also write explicitly where we want to arrive at at time \( t_0 + 1 \): called 
\[ (\theta', \text{act}') = T_{\text{ABDF}}(t_0, \omega, (\theta, \text{act})) \]
we want to have 
\[ u(t_0 + 1, x, \omega) := 2 \sum_{z \in M^+(\theta')} \frac{1}{2} \left\{ \frac{1}{2} \leq x < z + \frac{1}{2} \right\} - 2 \sum_{z \in M^-(\theta')} \frac{1}{2} \left\{ \frac{1}{2} < x \leq z \right\} + 2 \sum_{z \in A(\theta', \text{act}', \omega, t_0 + 1)} \left( 1 \left\{ \frac{1}{2} \leq x < \frac{1}{2} + z \right\} - 1 \left\{ \frac{1}{2} < x \leq z \right\} \right). \]

**Proposition 22.** Given the sets defined above, let us introduce also 
\[ MA_{\text{iso}}^{+}(\theta, \text{act}, \omega, t_0) = \left\{ z \in MA^+(\theta, \text{act}, \omega, t_0) : z + 1 \notin C(\theta, \text{act}, \omega, t_0) \right\}, \]
\[ MA_{\text{iso}}^{-}(\theta, \text{act}, \omega, t_0) = \left\{ z \in MA^-(\theta, \text{act}, \omega, t_0) : z - 1 \notin C(\theta, \text{act}, \omega, t_0) \right\}. \]

Consider the functions \( u(t_0 + 1, x, \omega) \) and \( u(t_0 + 1, x, \omega) \) we just defined. The following function \( u(t, x, \omega), t \in [t_0 + \frac{1}{2}, t_0 + 1] \), interpolates between them and is a weak solution of Burgers’ equation: we set, for \( t \in [t_0 + \frac{1}{2}, t_0 + 1] \),
\[ u(t, x, \omega) = u_{\text{iso}}(t, x, \omega) + u_{\text{coa}}(t, x, \omega) + u_{\text{ari}}(t, x, \omega), \]
where \( u_{\text{iso}} \) collects isolated quasi-particles,
\[ u_{\text{iso}}(t, x, \omega) = \sum_{z \in MA_{\text{iso}}^+(\theta, \text{act}, \omega, t_0)} u_{\text{iso}}^{(z,+)}(t, x, \omega), \]
\[ + \sum_{z' \in MA_{\text{iso}}^-(\theta, \text{act}, \omega, t_0)} u_{\text{iso}}^{(z,-)}(t, x, \omega), \]
\[ u_{\text{iso}}^{(z,+)}(t, x, \omega) = 2 \cdot 1 \left\{ \frac{1}{2} \leq t < \frac{1}{2} z < (t - t_0) + \frac{1}{2} \right\}, \]
\[ u_{\text{iso}}^{(z,-)}(t, x, \omega) = -2 \cdot 1 \left\{ \frac{1}{2} < z < \frac{1}{2} - (t - t_0) \right\}, \]
\( u_{\text{coa}} \) the coalescing quasi-particles,
\[ u_{\text{coa}}(t, x, \omega) = \sum_{x_0 \in C(\theta, \text{act}, \omega, t_0)} u_{\text{coa}}^{(x_0)}(t, x, \omega), \]
\[ u_{\text{coa}}^{(x_0)}(t, x, \omega) = 2 \cdot 1 \left\{ x_0 - (t_0 + 1 - t) \leq x \leq x_0 \right\} - 2 \cdot 1 \left\{ x_0 \leq x < x_0 + (t_0 + 1 - t) \right\}, \]
and finally, with \( (\theta', \text{act}') = T_{\text{ABDF}}(t_0, \omega, (\theta, \text{act})) \), \( u_{\text{ari}} \) corresponds to arising pairs,
\[ u_{\text{ari}}(t, x, \omega) = \sum_{x_0 \in A(\theta', \text{act}', \omega, t_0 + 1)} u_{\text{ari}}^{(x_0)}(t, x, \omega), \]
\[ u_{\text{ari}}^{(x_0)}(t, x, \omega) = 2 \cdot 1 \left\{ x_0 \leq x < x_0 + (t_0 - \frac{1}{2}) \right\} - 2 \cdot 1 \left\{ x_0 - (t_0 - \frac{1}{2}) \leq x < x_0 \right\}. \]

**Proof.** The fact that, for almost all \( x \), \( (u_{\text{iso}} + u_{\text{coa}} + u_{\text{ari}})(t, x, \omega) \) coincides with the functions \( u(t_0 + \frac{1}{2}, x, \omega) \) and \( u(t_0 + 1, x, \omega) \) defined above can be easily checked - notice that \( u_{\text{coa}}(t_0 + 1, x, \omega) = 0 \) and \( u_{\text{ari}}(t_0 + \frac{1}{2}, x, \omega) = 0 \). Considered by itself, each term of \( u_{\text{iso}} \) is a weak solution on \([t_0 + \frac{1}{2}, t_0 + 1]\): similarly, each \( u_{\text{coa}}^{(x_0)}(t, x, \omega) \) is a coalescing pair on \([t_0 + \frac{1}{2}, t_0 + 1] \) and each \( u_{\text{ari}}^{(x_0)}(t, x, \omega) \) is an arising pair on \([t_0 + \frac{1}{2}, t_0 + 1] \). Thus the sum of all these functions is a weak solution if they have disjoint supports.
Elements $u_{iso}^{(z, +)}(t, x, \omega)$ have supports of the form $[x(t), x(t) + \frac{1}{2}]$ with $x(t) = z + (t - t_0)$; and $[x'(t) - \frac{1}{2}, x'(t)]$ with $x'(t) = z' - (t - t_0)$ for $u_{iso}^{(z', -)}(t, x, \omega)$. Supports of functions $u_{iso}^{(z, +)}(t, x, \omega)$ cannot intersect each other, since quasi-particles move in parallel; the same holds for $u_{iso}^{(z', -)}(t, x, \omega)$, among themselves.

The support of a function $u_{iso}^{(z, +)}(t, x, \omega)$ cannot intersect the support of a function $u_{iso}^{(z', -)}(t, x, \omega)$ for the following reason. It is not possible if $z \geq z'$, because they move in opposite directions. Keeping in mind that we consider $t \in [t_0 + \frac{1}{2}, t_0 + 1]$, the same argument applies when $z = z' - 1$. If $z = z' - 2$, then $x_0 := z + 1$ is of class $C(\theta, \text{act}, \omega, t_0)$, hence $z$ and $z'$ cannot be in $MA_\text{iso}^+(\theta, \text{act}, \omega, t_0)$ and $MA_\text{iso}^-(\theta, \text{act}, \omega, t_0)$ respectively. It remains to discuss the case $z \leq z' - 3$. But now the supports $[x(t), x(t) + \frac{1}{2}]$ and $[x'(t) - \frac{1}{2}, x'(t)]$ do not have sufficient time to meet, for $t \in [t_0 + \frac{1}{2}, t_0 + 1]$. Summarizing, we have proved that all terms of $u_{iso}(t, x, \omega)$ have disjoint supports.

Let $x_0 \in C(\theta, \text{act}, \omega, t_0)$. Its corresponding coalescing pair $u_{iso}^{(x_0)}(t, x, \omega)$; its support has the form $[x_0 - (t_0 + 1 - t), x_0 + (t_0 + 1 - t)]$, contained in $[x_0 - \frac{1}{2}, x_0 + \frac{1}{2}]$ for $t \in [t_0 + \frac{1}{2}, t_0 + 1]$. These supports are clearly disjoint when $x_0$ varies in $C(\theta, \text{act}, \omega, t_0)$. They are also disjoint from any element of $u_{iso}(t, x, \omega)$: let us see why, in the case of a function $u_{iso}^{(z, +)}(t, x, \omega)$. Since $x_0 \in C(\theta, \text{act}, \omega, t_0)$, $x_0 - 1$ cannot be of class $MA_\text{iso}^+(\theta, \text{act}, \omega, t_0)$. Thus we need to have $z < x_0 - 1$ and $u_{iso}^{(z, +)}(t, x, \omega)$ cannot reach the coalescing pair in the time interval $[t_0 + \frac{1}{2}, t_0 + 1]$, for the same reason why different points of $MA_\text{iso}^+(\theta, \text{act}, \omega, t_0)$ cannot lead to intersections.

Finally, let us consider a point $x_0 \in A'(\theta, \text{act'}, \omega, t_0 + 1)$ where a new arising pair starts to exists at time $t_0 + \frac{1}{2}$, and the associated function $u_{ari}^{(x_0)}(t, x, \omega)$. Let us first discuss the case of $z \in MA_\text{iso}^+(\theta, \text{act}, \omega, t_0)$. If $z \geq x_0$ or $z \leq x_0 - 2$ there is no intersection: the difficult case is $z = x_0 - 1$. But in such a case, having excluded by $MA_\text{iso}^+(\theta, \text{act}, \omega, t_0)$ the possibility of coalescing points, we should have $\theta'(x_0) = 1 \neq 0$, in contradiction with $x_0 \in A'(\theta, \text{act'}, \omega, t_0 + 1)$. Hence this case does not exist. Points $z \in MA_\text{iso}^+(\theta, \text{act}, \omega, t_0)$ are similar.

The hardest case is when $x_0 \in A'(\theta, \text{act'}, \omega, t_0 + 1)$ also belongs to $C(\theta, \text{act}, \omega, t_0)$. In plain words, the question is whether a pair may arise in a point of coalescence. This case is solved by Corollary 13, implying that if $x_0 \in C(\theta, \text{act}, \omega, t_0)$, then $x_0 \notin A'(\theta, \text{act'}, \omega, t_0 + 1)$. Indeed, assuming the former, by definition $x_0 - 1, x_0 + 1 \in A(\theta, \text{act}, \omega, t_0)$, and by Corollary 13 this implies $\text{act'}(x_0) = 0$, so $x_0 \notin A(\theta', \text{act'}, \omega, t_0 + 1)$. This rules out the last possible intersection of supports, and the proof is complete.

\section*{5.4. Main Result.} Merging the statements of Proposition 21 and Proposition 22, along with the simple claim of Proposition 20, we finally get the main result of this work:

\textbf{Theorem 23.} Given, at time $t_0 = 0$, an element $(\theta, \text{act}) \in \Lambda$ and the section \{$(0, x) : x \in \mathbb{Z}$\}, define $u_0(x, \omega) := u(0, x, \omega)$, following Definition 19.

Construct the stochastic process $(\theta(t_0), \omega, \text{act}(t_0), \omega)$, $t_0 \in \mathbb{N}$, by setting 

$$(\theta(t_0), \omega, \text{act}(t_0), \omega) := \phi_{ABDF}(t_0, \omega)(\theta, \text{act})$$

namely by performing the ABDF random dynamics.

Define the stochastic process $u(t, x, \omega)$, $t \in [0, \infty)$, $x \in \mathbb{R}$ as follows. For every $t_0 \in \mathbb{N}$, define $u(t_0, x, \omega)$ from Definition 19 with respect to $(\theta, \text{act})$ given by $(\theta(t_0), \omega, \text{act}(t_0), \omega)$; define $u(t, x, \omega)$ for $t \in [t_0, t_0 + \frac{1}{2}]$ by Proposition 21; finally define $u(t, x, \omega)$ for $t \in [t_0 + \frac{1}{2}, t_0 + 1]$ by Proposition 22.
Then $u(t, x, \omega)$ is a weak solution of Burgers’ equation.

We have thus shown that, given a realization of the ABDF process, we construct a weak solution of Burgers’ equation; for almost every $\omega$, from this weak solution it is possible to reconstruct the underlying ABDF realization, by Proposition 20.

Remark 24. The stochastic process $u$ so defined is adapted to the noise filtration shifted by $\frac{1}{2}$. Namely, if $\mathcal{F}_t$ is the natural filtration of the noise, $u(t, \cdot)$ is $\mathcal{F}_{t+\frac{1}{2}}$ adapted, due to the creation mechanism that starts at half-integer times. This anticipation is just instrumental, and not a deep phenomenon. One can develop an alternative construction such that $u(t, \cdot)$ is $\mathcal{F}_t$-adapted, just shifting time by $\frac{1}{2}$, or more precisely starting to create new particles at integer times and completing annihilation at half-integer times. However, we deem the construction just described more elegant.

6. Final Remarks on TASEP, Burger’s Equation and KPZ Universality Class

Among the most striking recent results on stochastic systems is the first quite complete understanding of the KPZ fixed point as a scaling limit of fluctuations of the height function associated to TASEP. Height functions of models in the KPZ universality class are conjectured to converge in the 1:2:3 scaling limit,

$$ h(t, x) \to h^*(t, x) = \varepsilon^{1/2} h(\varepsilon^{-3/2} t, \varepsilon^{-1} x) - C\varepsilon t, \quad \varepsilon \downarrow 0, C\varepsilon \uparrow \infty, $$

to a universal, scale invariant limit process, characterized as a Markov process by its transition probabilities in [21]. Although we did not discuss scaling limits, it is essential to refer to the recent works [22, 23] on transition probabilities of KPZ fixed point obtained as limits of the ones of TASEP, see also [2] for the discrete-time setting. The 1 : 2 : 3 scaling we referred to above identifies fluctuations, and it is worth recalling that macroscopic limits of models such as TASEP are classically known to be solutions of nonlinear conservation laws, [26, 25].

The Kardar-Parisi-Zhang (KPZ) equation, introduced in [15],

$$ \partial_t h = \nu \partial_x^2 h + \lambda (\partial_x h)^2 + \sigma \xi, \quad \nu, \lambda, \sigma > 0, $$

where $\xi$ denotes space-time white noise, is not invariant under the 1:2:3 scaling. Its solution theory, initiated in [4], has been the starting point of recent breakthrough developments in stochastic PDE theory, [12, 10]. Solutions to the KPZ equation are special models in the KPZ universality class, as they are expected to describe the unique heteroclinic orbit between the KPZ fixed point and the Gaussian Edwards-Wilkinson fixed point, [24]. Under the 1:2:3 scaling, the diffusion and noise terms of (11) vanish, formally leading to the Hamilton-Jacobi equation

$$ \partial_t h = \lambda (\partial_x h)^2. $$

This, informally, suggests that the KPZ fixed point can be understood as a (stochastic) solution to (12), corresponding to Burgers’ equation (1) with $\lambda = -1$ and $u = \partial_x h$. However, as pointed out in [21], entropy solutions to (12) given by the Hopf-Lax formula,

$$ h(t, x) = \sup_y \left( h(0, y) - \frac{(x - y)^2}{4M} \right), $$

are not suited to describe the KPZ fixed point. Indeed, entropy solutions would not preserve the regularity of Brownian motion, unlike the KPZ fixed point. In addition, since the KPZ fixed point has the space regularity of Brownian motion, the nonlinear term of (12) is ill-posed. The possibility of a different kind of weak solutions to (12) describing the KPZ fixed point was left open in [21].
Our result might thus be regarded as hinting to a relation between the weak, non-entropic, intrinsically stochastic solutions of Burgers’ equation we built and linked with discrete-time TASEP, and the KPZ fixed point. However, our arguments yield a bijection of models before any scaling limit is considered, and even conjecturing how non-entropic, intrinsically stochastic Burgers’ solution might describe—or simply relate to—the KPZ fixed point, seems very difficult. We do mention it since the problem of finding an equation satisfied by the KPZ fixed point remains completely open, and non-entropic weak solutions might be the right objects to consider (cf. [3] and remarks in the introduction of [5]).

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References

[1] M. Mariani A. Boritchev, C.-E. Bréhier. Personal communication.
[2] Yuta Arai. The KPZ fixed point for discrete time TASEPs. Journal of Physics. A. Mathematical and Theoretical, 53(41):415202, 33, 2020.
[3] Yuri Bakhtin and Konstantin Khanin. On global solutions of the random Hamilton-Jacobi equations and the KPZ problem. Nonlinearity, 31(4):R93–R121, 2018.
[4] Lorenzo Bertini and Giambattista Giacomin. Stochastic Burgers and KPZ equations from particle systems. Communications in Mathematical Physics, 183(3):571–607, 1997.
[5] Ivan Corwin, Jeremy Quastel, and Daniel Remenik. Renormalization fixed point of the KPZ universality class. Journal of Statistical Physics, 160(4):815–834, 2015.
[6] Camillo De Lellis, Felix Otto, and Michael Westdickenberg. Structure of entropy solutions for multi-dimensional scalar conservation laws. Archive for Rational Mechanics and Analysis, 170(2):137–184, 2003.
[7] Theodore D. Drivas, Alexei A. Mailybaev, and Artem Raibekas. Statistical determinism in non-lipschitz dynamical systems. April 2020.
[8] Weinan E, K. Khanin, A. Mazel, and Ya. Sinai. Invariant measures for Burgers equation with stochastic forcing. Annals of Mathematics. Second Series, 151(3):877–960, 2000.
[9] Weinan E and Eric Vanden Eijnden. Generalized flows, intrinsic stochasticity, and turbulent transport. Proceedings of the National Academy of Sciences of the United States of America, 97(15):8200–8205, 2000.
[10] Massimiliano Gubinelli and Nicolas Perkowski. KPZ reloaded. Communications in Mathematical Physics, 349(1):165–269, 2017.
[11] Sergei N Gurbatov, Aleksandr I Saichev, and Sergey F Shandarin. Large-scale structure of the universe. the zeldovich approximation and the adhesion model. Physics-Uspekhi, 55(3):223, 2012.
[12] Martin Hairer. Solving the KPZ equation. Annals of Mathematics. Second Series, 178(2):559–664, 2013.
[13] Brian T. Hayes and Philippe G. LeFloch. Non-classical shocks and kinetic relations: scalar conservation laws. Archive for Rational Mechanics and Analysis, 139(1):1–56, 1997.
[14] Seok Hwang and Athanasios E. Tzavaras. Kinetic decomposition of approximate solutions to conservation laws: application to relaxation and diffusion-dispersion approximations. Communications in Partial Differential Equations, 27(5-6):1229–1254, 2002.
[15] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang. Dynamic scaling of growing interfaces. Physical Review Letters, 56(9):889, 1986.
[16] A. Kupiainen. Nondeterministic dynamics and turbulent transport. Annales Henri Poincaré. A Journal of Theoretical and Mathematical Physics, 4(suppl. 2):S713–S726, 2003.
[17] Yves Le Jan and Olivier Raimond. Integration of Brownian vector fields. The Annals of Probability, 30(2):826–873, 2002.
[18] Philippe G. LeFloch. An introduction to nonclassical shocks of systems of conservation laws. In An introduction to recent developments in theory and numerics for conservation laws (Freiburg/Littenweiler, 1997), volume 5 of Lect. Notes Comput. Sci. Eng., pages 28–72. Springer, Berlin, 1999.
[19] Pierre-Louis Lions, Benoît Perthame, and Panagiotis E. Souganidis. Scalar conservation laws with rough (stochastic) fluxes. *Stochastic Partial Differential Equations. Analysis and Compu-
tations*, 1(4):664–686, 2013.

[20] Alexei A. Mailybaev. Spontaneous stochasticity of velocity in turbulence models. *Multiscale Modeling & Simulation. A SIAM Interdisciplinary Journal*, 14(1):96–112, 2016.

[21] Konstantin Matetski, Jeremy Quastel, and Daniel Remenik. The kpz fixed point. December 2016.

[22] Mihai Nica, Jeremy Quastel, and Daniel Remenik. Solution of the Kolmogorov equation for TASEP. *The Annals of Probability*, 48(5):2344–2358, 2020.

[23] Jeremy Quastel and Konstantin Matetski. From the totally asymmetric simple exclusion process to the KPZ fixed point. In *Random matrices*, volume 26 of *IAS/Park City Math. Ser.*, pages 251–301. Amer. Math. Soc., Providence, RI, 2019.

[24] Jeremy Quastel and Sourav Sarkar. Convergence of exclusion processes and kpz equation to the kpz fixed point. August 2020.

[25] Fraydoun Rezakhanlou. Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$. *Communications in Mathematical Physics*, 140(3):417–448, 1991.

[26] Hermann Rost. Non-equilibrium behaviour of a many particle process: Density profile and local equilibria. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 58(1):41–53, 1981.

[27] Simon Thalabard, Jérémie Bec, and Alexei A Mailybaev. From the butterfly effect to spontaneous stochasticity in singular shear flows. *Communications Physics*, 3(1):1–8, 2020.

[28] Ya B Zel’Dovich. Gravitational instability: An approximate theory for large density perturbations. *Astronomy and astrophysics*, 5:84–89, 1970.