A DICHOTOMY FOR SIMPLE SELF-SIMILAR GRAPH $C^*$-ALGEBRAS

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Abstract. We investigate the pure infiniteness and stable finiteness of the Exel-Pardo $C^*$-algebras $O_{G,E}$ for countable self-similar graphs $(G,E,\varphi)$. In particular, we associate a specific ordinary graph $\tilde{E}$ to $(G,E,\varphi)$ such that some properties such as simpleness, stable finiteness or pure infiniteness of the graph $C^*$-algebra $C^*(\tilde{E})$ imply that of $O_{G,E}$. Among others, this follows a dichotomy for simple $O_{G,E}$: if $(G,E,\varphi)$ contains no $G$-circuits, then $O_{G,E}$ is stably finite; otherwise, $O_{G,E}$ is purely infinite.

Furthermore, Li and Yang recently introduced self-similar $k$-graph $C^*$-algebras $O_{G,\Lambda}$. We also show that when $|\Lambda^0| < \infty$ and $O_{G,\Lambda}$ is simple, then it is purely infinite.

1. Introduction

In [7], Exel and Pardo introduced self-similar graph $C^*$-algebras $O_{G,E}$ to give a unified framework like graph $C^*$-algebras for the Katsura’s [10] and Nekrashevych’s algebras [18, 19]. These $C^*$-algebras were initially considered in [7] only for countable discrete groups $G$ acting on finite graphs $E$ with no sources, and then generalized in [2, 8] for larger classes. Roughly speaking, Exel and Pardo attached an inverse semigroup $S_{G,E}$ and the tight groupoid $G_{\text{tight}}(S_{G,E})$ to $(G,E,\varphi)$ such that $O_{G,E} \cong C^*(G_{\text{tight}}(S_{G,E}))$, and then describe amenability [7, Corollary 10.18], minimality [7, Theorem 13.6], and effectivity (or topological principality) [7, Corollary 14.15] of $G_{\text{tight}}(S_{G,E})$, and thus simplicity and pure infiniteness of $O_{G,E}$ [7, Section 16], among others. Although only finite graphs are considered in [7], but many arguments and proofs work for countable row-finite graphs with no sources (see [8]).

The initial aim of this note comes from a dichotomy for simple groupoid $C^*$-algebras [21, 3]. According to [21, Theorem 4.7] and [3, Corollary 5.13], a simple reduced $C^*$-algebra $C^*_r(\mathcal{G})$ of ample groupoid $\mathcal{G}$ with an almost unperforated type semigroup is either purely infinite or stable finite. We explicitly describe this dichotomy for self-similar graph $C^*$-algebras $O_{G,E}$
by the underlying graphical properties. Here, we consider countable row-finite source-free graphs $E$ over an amenable (countable) group $G$ [2, 8]. However, our results may be generalized to any countable graph $E$ by the desingularization of [8].

We begin in Section 2 by reviewing necessary background on groupoid and self-similar graph $C^*$-algebras. Then, in Section 3, we generalize the Exel-Pardo’s characterization of purely infinite $O_{G,E}$ to countable self-similar graphs by the groupoid approach (for not necessarily simple cases). Moreover, for certain self-similar graphs $(G, E, \varphi)$, we show that the $C^*$-algebra $O_{G,E}$ is purely infinite and simple if and only if the additive monoid of nonzero Murray-von Neumann equivalent projections in $M_\infty(O_{G,E})$ is a group.

In Section 4, we focus on the stable finiteness of $O_{G,E}$. We attach a spacial graph $\tilde{E}$ to $(G, E, \varphi)$ such that some properties of $O_{G,E}$- such as simplicity, pure infiniteness, and stable infiniteness- can be derived from those of the graph $C^*$-algebra $C^*(\tilde{E})$. Then using known results about the graph $C^*$-algebras, we show that a simple $C^*$-algebra $O_{G,E}$ is stable infinite if and only if the underlying $(G, E, \varphi)$ contains no $G$-circuits. In particular, we deduce a dichotomy: A simple $O_{G,E}$ is purely infinite if $(G, E, \varphi)$ has a $G$-circuit; otherwise, it is stable finite.

As the $k$-graph version of Exel-Pardo $C^*$-algebras, Li and Yang introduced self-similar $k$-graphs $(G, \Lambda)$ and associated $C^*$-algebras $O_{G,\Lambda}$. Briefly, by a groupoid approach, they investigated their properties such as nuclearity [17, Theorem 6.6(i)], amenability [17, Theorem 5.9], and simplicity [17, Theorem 6.6(ii)]. In Section 5, We investigate the pure infiniteness of $O_{G,\Lambda}$ for the nonsimple cases. In particular, we modify and extend [17, Theorem 6.13].

Acknowledgement. The author appreciates Enrique Pardo for reviewing the initial version of the article and his helpful comments; in particular, for noting a gap in the proof of Theorem 3.8.

2. Preliminaries

2.1. Groupoid $C^*$-algebras. We give here a brief introduction to ample groupoids and associated $C^*$-algebras; for more details see [23, 1] for example. A groupoid is a small category $\mathcal{G}$ with inverses. The unit space of $\mathcal{G}$ is the set of identity morphisms, that is $\mathcal{G}^{(0)} := \{\alpha^{-1} \alpha : \alpha \in \mathcal{G}\}$. For each $\alpha \in \mathcal{G}$, we may define the range $r(\alpha) := \alpha \alpha^{-1}$ and the source $s(\alpha) := \alpha^{-1} \alpha$, which satisfy $r(\alpha) \alpha = \alpha = \alpha s(\alpha)$. Hence, for $\alpha, \beta \in \mathcal{G}$, the composition $\alpha \beta$ is well-defined in $\mathcal{G}$ if and only if $s(\alpha) = r(\beta)$. The isotropy subgroupoid of $\mathcal{G}$ is defined by

\[ \text{Iso}(\mathcal{G}) := \{ \alpha \in \mathcal{G} : s(\alpha) = r(\alpha) \}. \]

We work usually with groupoids $\mathcal{G}$ endowed with a topology such that the maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ are continuous (in this case, $\mathcal{G}$ is called a topological groupoid). A subset $B \subseteq \mathcal{G}$ is called a bisection if both restrictions $r|_B$ and
s|B are homeomorphisms. We say that \( \mathcal{G} \) is ample in case \( \mathcal{G} \) has a basis of compact and open bisections.

**Definition 2.1.** Let \( \mathcal{G} \) be a topological groupoid. We say that \( \mathcal{G} \) is effective if the interior of \( \text{Iso}(\mathcal{G}) \) is just \( \mathcal{G}^{(0)} \). Moreover, \( \mathcal{G} \) is called topologically principal if \( \{ u \in \mathcal{G}^{(0)} : s^{-1}(u) \cap r^{-1}(u) = \{ u \} \} \) is dense in \( \mathcal{G}^{(0)} \).

Note that, when \( \mathcal{G} \) is second-countable, [22, Proposition 3.3] implies that \( \mathcal{G} \) is effective if and only if it is topologically principal. In this paper, we will work frequently with second-countable effective ample groupoids.

We now recall the definition of reduced \( C^\ast \)-algebra \( C^\ast_r(\mathcal{G}) \). Let \( \mathcal{G} \) be an ample groupoid. We write \( C_c(\mathcal{G}) \) for the complex vector space consisting of compactly supported continuous functions on \( \mathcal{G} \), which is an \( \ast \)-algebra with the convolution multiplication and the involution \( f^\ast(\alpha) := f(\alpha^{-1}) \). For each unit \( u \in \mathcal{G}^{(0)} \) and \( \mathcal{G}_u := s^{-1}(\{ u \}) \), let \( \pi_u : C_c(\mathcal{G}) \to B(\ell^2(\mathcal{G}_u)) \) be the left regular \( \ast \)-representation defined by

\[
\pi_u(f)\delta_\alpha := \sum_{s(\beta) = r(\alpha)} f(\beta)\delta_{\beta\alpha} \quad (f \in C_c(\mathcal{G}), \ \alpha \in \mathcal{G}_u).
\]

Then the reduced \( C^\ast \)-algebra \( C^\ast_r(\mathcal{G}) \) is the completion of \( C_c(\mathcal{G}) \) under the reduced \( C^\ast \)-norm

\[
\| f \|_r := \sup_{u \in \mathcal{G}^{(0)}} \| \pi_u(f) \|.
\]

Moreover, there is a full \( C^\ast \)-algebra \( C^\ast(\mathcal{G}) \) associated to \( \mathcal{G} \), which is the completion of \( C_c(\mathcal{G}) \) taken over all \( \| . \|_{C_c(\mathcal{G})} \)-decreasing representations of \( \mathcal{G} \). Hence, \( C^\ast_r(\mathcal{G}) \) is a quotient of \( C^\ast(\mathcal{G}) \), and [1, Proposition 6.1.8] shows that they are equal if the underlying groupoid \( \mathcal{G} \) is amenable.

**Definition 2.2 ([25]).** We say that a \( C^\ast \)-algebra \( A \) is purely infinite if every nonzero hereditary \( C^\ast \)-subalgebra of \( A \) contains an infinite projection.

The following is analogous to [4, Theorem 4.1] without the minimality assumption.

**Proposition 2.3.** Let \( \mathcal{G} \) be a second-countable Hausdorff ample groupoid and let \( \mathcal{B} \) be a basis of compact open sets for \( \mathcal{G}^{(0)} \). Suppose also that \( \mathcal{G} \) is effective. Then \( C^\ast_r(\mathcal{G}) \) is purely infinite if and only if \( 1_V \) is infinite in \( C^\ast_r(\mathcal{G}) \) for every \( V \in \mathcal{B} \) (\( 1_V \) is the characteristic function of \( V \)).

**Proof.** The “only if” implication is immediate. For the converse, suppose that every \( 1_V \) in \( C^\ast_r(\mathcal{G}) \) is infinite for \( V \in \mathcal{B} \). Let \( A \) be a nonzero hereditary \( C^\ast \)-subalgebra of \( C^\ast_r(\mathcal{G}) \) and take some positive element \( 0 \neq a \in A \). Using the hereditary property, we may follow the proof of [15, Proposition 5.2] to find a projection \( p \in A \) and some \( V \in \mathcal{B} \) such that \( p \sim 1_V \) in the Murray-von Neumann sense. Since the infiniteness is preserved under \( \sim \), then \( p \) is an infinite projection, concluding the result. \( \square \)
2.2. Graph \( C^\ast \)-algebras. Let \( E = (E^0, E^1, r, d) \) be a directed graph with the vertex set \( E^0 \), the edge set \( E^1 \), and the range and domain maps \( r, d : E^1 \to E^0 \). We say that \( E \) is row-finite if each vertex receives at most finitely many edges. A source in \( E \) is a vertex \( v \in E^0 \) which receives no edges, i.e. \( d^{-1}(v) = \emptyset \). We will write by \( E^* \) the set of finite paths in \( E \), that is

\[
E^* := \bigcup_{n \geq 0} E^n = \bigcup_{n \geq 0} \{ \alpha = e_1 \ldots e_n : e_i \in E^1, d(e_i) = r(e_{i+1}) \}.
\]

Then one may extend \( r, d : E^* \to E^0 \) by defining \( r(\alpha) = r(e_1) \) and \( d(\alpha) = d(e_n) \) for every path \( \alpha = e_1 \ldots e_n \in E^n \). Throughout the paper, we will consider only countable directed graphs.

Given a directed graph \( E \), a Cuntz-Krieger \( E \)-family is a collection \( \{ p_v, s_e : v \in E^0, e \in E^1 \} \) of pairwise orthogonal projections \( p_v \) and partial isometries \( s_e \) with the following relations

1. \( s^*_e s_e = p_{d(e)} \) for every \( e \in E^1 \),
2. \( s_e s^*_e \leq p_{r(e)} \) for every \( e \in E^1 \), and
3. \( p_v = \sum_{d(e)=v} s_e s^*_e \) for all vertices \( v \) with \( 0 < |d^{-1}(v)| < \infty \).

The graph \( C^\ast \)-algebra \( C^\ast(E) \) is the universal \( C^\ast \)-algebra generated by a Cuntz-Krieger \( E \)-family \( \{ p_v, s_e \} \) [20]. By the above relations, for \( e_1, \ldots, e_n \in E^1 \), \( s_{e_1} \ldots s_{e_n} \) is nonzero if and only if \( \alpha := e_1 \ldots e_n \) is a path in \( E \); in this case, we write \( s_\alpha := s_{e_1} \ldots s_{e_n} \).

2.3. Self-similar graphs and their \( C^\ast \)-algebras. Let \( G \) be a countable discrete group. An action \( G \curvearrowright E \) is a map \( G \times (E^0 \cup E^1) \to E^0 \cup E^1 \), denoted by \( (g, a) \mapsto ga \), such that the action of each \( g \in G \) on \( E \) gives a graph automorphism.

A self-similar graph is a triple \( (G, E, \varphi) \) such that

1. \( E \) is a directed graph,
2. \( G \) acts on \( E \) by automorphisms, and
3. \( \varphi : G \times E^1 \to G \) is a 1-cocycle for \( G \curvearrowright E \) satisfying \( \varphi(g,e)v = gv \) for every \( g \in G, e \in E^1 \), and \( v \in E^0 \).

Remark 2.4. According to [7, Proposition 2.4], we may extend inductively the action \( G \curvearrowright E \) and the cocycle \( \varphi \) on the finite path space \( E^* \) satisfying the desired relations [7, Equation 2.6]. Indeed, if \( \alpha = \alpha_1 \alpha_2 \in E^* \), then we define

\[
 ga = (g\alpha_1)(\varphi(g,\alpha_1)\alpha_2) \quad \text{and} \quad \varphi(g,\alpha) = \varphi(\varphi(g,\alpha_1),\alpha_2).
\]

Definition 2.5 ([7, 8]). Let \( (G, E, \varphi) \) be a (countable) self-similar graph. Then \( \mathcal{O}_{G,E} \) is the universal \( C^\ast \)-algebra generated by

\[
\{ p_v, s_e : v \in E^0, e \in E^1 \} \cup \{ u_g p_v : g \in G, v \in E^0 \}
\]

satisfying the following properties:

1. \( \{ p_v, s_e : v \in E^0, e \in E^1 \} \) is a Cuntz-Krieger \( E \)-family.
(2) \( u : G \to \mathcal{M}(\mathcal{O}_{G,E}), \ g \mapsto u_g \), is a unitary \( \ast \)-representation of \( G \) on the multiplier algebra \( \mathcal{M}(\mathcal{O}_{G,E}) \).

(3) \( u_g p_v = p_{v g} u_g \) for every \( g \in G \) and \( v \in E^0 \).

(4) \( u_g s_e = s_{g e} u_{\varphi(g,e)} \) for every \( g \in G \) and \( e \in E^1 \).

We usually use the notation \( \mathcal{O}_{G,E} \) instead of \( \mathcal{O}_{(G,E,\varphi)} \) for convenience. Also, we will write each \( u_g p_v \) by \( u_{g e} \). Then one may easily verify relations (b)-(e) of \([8, \text{Definition 2.2}]\).

**Standing assumption.** All self-similar graphs \((G, E, \varphi)\) considered in this paper will be countable, row-finite and source-free.

### 2.4. The groupoid associated to \((G, E, \varphi)\).

In \([7, \text{Section 4}]\), Exel and Pardo associated an inverse semigroup \( S_{G,E} \) to a self-similar graph \((G, E, \varphi)\) with finite graph \( E \). They then showed that \( \mathcal{O}_{G,E} \cong C^*_\text{tight}(S_{G,E}) \cong C^*(\mathcal{G}_{G,E}) \) where \( \mathcal{G}_{G,E} \) is the groupoid of germs for the action of \( S_{G,E} \) on \( E^\infty \) \([7, \text{Corollary 6.4 and Proposition 8.4}]\). Note that the constructions of \( S_{G,E} \) and \( \mathcal{G}_{G,E} \) in \([7]\) may be extended for countable row-finite, source-free self-similar graphs \((G, E, \varphi)\) with small modifications. We give a brief review of it here for convenience. So, fix a row-finite self-similar graph \((G, E, \varphi)\) without sources. Define the \( \ast \)-inverse semigroup \( S_{G,E} \) as

\[
S_{G,E} = \{ (\alpha, g, \beta) : \alpha, \beta \in E^*, g \in G, d(\alpha) = gd(\beta) \} \cup \{0\}
\]

with the operations

\[
(\alpha, g, \beta)(\gamma, h, \delta) := \begin{cases} 
(\alpha, g \varphi(h, \varepsilon), \delta h \varepsilon) & \text{if } \beta = \gamma \varepsilon \\
(\alpha \varepsilon, g \varphi(\varepsilon, h), \delta) & \text{if } \gamma = \beta \varepsilon \\
0 & \text{otherwise}
\end{cases}
\]

and \( (\alpha, g, \beta)^* := (\beta, g^{-1}, \alpha) \).

Let \( E^\infty \) be the space one-sided infinite paths of the form

\[
x = e_1 e_2 \ldots \text{ such that } d(e_i) = r(e_{i+1}) \text{ for } i \geq 1.
\]

By \([7, \text{Proposition 8.1}]\), there is a unique action \( G \curvearrowright E^\infty \) as follows: for each \( g \in G \) and \( x = e_1 e_2 \ldots \in E^\infty \), there is a unique infinite path \( gx = f_1 f_2 \ldots \) such that

\[
f_1 f_2 \ldots f_n = g(e_1 e_2 \ldots e_n) \text{ (for all } n \geq 1).\]

Moreover, we may consider the action of each \( (\alpha, g, \beta) \in S_{G,E} \) on \( x = \beta \hat{x} \in E^\infty \) by \((\alpha, g, \beta) \cdot x = \alpha(g \hat{x}) \). Then \( \mathcal{G}_{G,E} \) is the groupoid of germs of the action of \( S_{G,E} \) on \( E^\infty \), that is

\[
\mathcal{G}_{G,E} = \{ (\alpha, g, \beta; x) : x = \beta \hat{x} \}.
\]

Recall that two germs \([s; x], [t; y]\) in \( \mathcal{G}_{G,E} \) are equal if and only if \( x = y \) and there exists an idempotent \( 0 \neq e \in S_{G,E} \) such that \( e \cdot x = x \) and \( se = te \).

The unit space of \( \mathcal{G}_{G,E} \) is

\[
\mathcal{G}_{G,E}^{(0)} = \{ (\alpha, 1_G, \alpha; x) : x = \alpha \hat{x} \},
\]
which is identified with $E^\infty$ by $[\alpha, 1_G, \alpha; x] \mapsto x$. Then, the range and source maps are defined by

$$r([\alpha, g, \beta; \beta \hat{x}]) = \alpha(g \hat{x}) \quad \text{and} \quad s([\alpha, g, \beta; \beta \hat{x}]) = \beta \hat{x}.$$ 

Following [7, Section 10], we endow $G_{G,E}$ with the topology generated by compact open bisections of the form

$$\Theta(\alpha, g, \beta; Z(\gamma)) := \{[\alpha, g, \beta; y] \in G_{G,E} : y \in Z(\gamma)\}$$

where $\gamma \in E^*$ and $Z(\gamma) := \{\gamma x : x \in s(\gamma)E^\infty\}$. Hence, $G_{G,E}$ is an ample groupoid.

Definition 2.6. We say that $(G, E, \varphi)$ is pseudo free if for every $g \in G$ and $e \in E^1$,

$$ge = e \quad \text{and} \quad \varphi(g, e) = 1_G \implies g = 1_G.$$

In the end of this section, we recall briefly the following results from [7] for convenience. Although they are proved there for finite self-similar graphs with no sources, but we can obtain them for countable cases by a same way (see also [8]).

Proposition 2.7. Let $(G, E, \varphi)$ be a pseudo free self-similar graph without sources and let $G_{G,E}$ be the associated groupoid as above. Then

1. $G_{G,E} \cong G_{\text{tight}}(S_{G,E})$ [7, Theorem 8.19], $G_{G,E}$ is Hausdorff [7, Proposition 12.1], and $O_{G,E} \cong C^*(G_{G,E})$ [7, Theorem 9.6].

2. If moreover $G$ is an amenable group, then $G_{G,E}$ is an amenable groupoid in the sense of [1]. In particular, we have $O_{G,E} \cong C^*(G_{G,E}) \cong C_r^*(G_{G,E})$ by [1, Proposition 6.1.8].

Proposition 2.8 ([7, Corollary 14.13] and [8, Theorem 4.4]). Let $(G, E, \varphi)$ be a pseudo free self-similar graph with no sources. Then $G_{G,E}$ is effective\(^1\) if and only if the following properties hold:

1. Every $G$-circuit in $E$ has an entry, and
2. for every $v \in E^0$ and $1_G \neq g \in G$, the action of $g$ on $Z(v)$ is nontrivial (i.e., there is $x \in Z(v)$ such that $g.x \neq x$).

3. Purely infinite self-similar graph $C^*$-algebras

In [7, Corollary 16.3] and [8, Corollary 4.7], it is shown that when $O_{G,E}$ is simple and $(G, E, \varphi)$ contains a $G$-circuit, then $O_{G,E}$ is purely infinite. In this section, we study purely infinite $C^*$-algebras $O_{G,E}$ of countable self-similar graphs in the sense of [25] without the simplicity assumption. Our main result here is a generalization of [7, Theorem 16.2] to countable self-similar graphs. Note that there is another well-known notion of pure infiniteness from [11] which is equivalent to that of [25] for the simple cases. Moreover, our results in this section may be generalized for the Kirchberg-Rørdam’s notion using [11, Corollary 3.15] and the ideal structure [14, Corollary 6.15].

\(^1\)Note that the ‘effective’ property of groupoids is called essentially principal in [7, 8].
Theorem 3.1. Let \((G, E, \varphi)\) be a pseudo free self-similar graph over an amenable group \(G\). Suppose also that \((G, E, \varphi)\) satisfies conditions (1) and (2) of Proposition 2.8 (i.e., the groupoid \(G_{G,E}\) is effective). Then \(O_{G,E}\) is purely infinite if and only if every vertex projection \(s_v\) is infinite in \(O_{G,E}\).

Proof. We must prove the “if” implication only. So suppose that for every \(v \in E^0\), \(s_v\) is infinite in \(O_{G,E}\). Let \(\mathcal{G} = G_{G,E}\) be the groupoid associated to \((G, E, \varphi)\). By Proposition 2.7(2), \(\mathcal{G}\) is amenable, so \(C^*_r(\mathcal{G}) = C^*(\mathcal{G}) = O_{G,E}\).

We know that the cylinders \(\{Z(\alpha) : \alpha \in E^*\}\) is a basis of compact open sets for the topology induced on \(E^\infty = G^{(0)}\). Moreover, Proposition 2.8 says that \(\mathcal{G}\) is effective. Hence, Proposition 2.3 implies that \(O_{G,E} = C^*_r(\mathcal{G})\) is purely infinite if and only if \(\{1_{Z(\alpha)} = s_\alpha s_\alpha^* : \alpha \in E^*\}\) are all infinite projections in \(O_{G,E}\). Now since \(s_\alpha s_\alpha^* \sim s_\alpha s_\alpha = s_{d(\alpha)}\) and the infiniteness passes through Murray-von Neumann equivalence, we conclude the result. \(\square\)

Definition 3.2. Let \(v, w \in E^0\). We say that \(v\) receives a \(G\)-path from \(w\) or \(w\) connects to \(v\) by a \(G\)-path, say \(v \triangleright w\), if there exist \(\alpha \in E^*\) and \(g \in G\) such that \(r(\alpha) = v\) and \(d(\alpha) = gw\). By [7, Proposition 13.2], this is equivalent to

\[\exists \alpha \in E^*, \exists g \in G \text{ such that } r(\alpha) = gw\text{ and } d(\alpha) = w.\]

Lemma 3.3. Let \((G, E, \varphi)\) be a self-similar graph. For \(v, w \in E^0\) and \(\alpha, \beta \in E^*\), we have

1. If \(v = gw\) for some \(g \in G\), then \(s_v \sim s_w\) in the Murray-von Neumann sense.
2. If \(v\) receives a \(G\)-path from \(w\), then \(s_v \triangleright s_w\).
3. If \(\beta = g\alpha\) for some \(g \in G\), then \(s_\beta s_\beta^* \sim s_\alpha s_\alpha^*\).

Proof. (1). If \(v = gw\), then we have \(s_v = (u_g s_v)^*(u_g s_v)\) and

\[(u_g s_v)(u_g s_v)^* = (s_{gw} u_g)(s_{gw} u_g)^* = s_{gw} u_g u_g^* s_{gw} = s_w,
\]

concluding \(s_v \sim s_w\).

For (2), suppose that there exist \(\alpha \in E^*\) and \(g \in G\) such that \(r(\alpha) = v\) and \(d(\alpha) = gw\). Then, by the Cuntz-Krieger relations,

\[s_v \geq s_\alpha s_\alpha^* \sim s_\alpha^* s_\alpha = s_{d(\alpha)} = s_{gw} \sim s_w,
\]

and consequently \(s_v \triangleright s_w\).

For (3), if \(\beta = g\alpha\), then by part (1) we have

\[s_\beta s_\beta^* \sim s_\alpha^* s_{d(\beta)} = s_{gw, d(\alpha)} \sim s_{d(\alpha)} = s_\alpha^* s_\alpha \sim s_\alpha s_\alpha^*,\]

giving \(s_\beta s_\beta^* \sim s_\alpha s_\alpha^*\). \(\square\)

Proposition 3.4. Let \((G, E, \varphi)\) be a pseudo free self-similar graph over an amenable group \(G\). Suppose that conditions (1) and (2) of Proposition 2.8 hold. Then

1. If every \(v \in E^0\) receives a \(G\)-path from a \(G\)-circuit, then \(O_{G,E}\) is purely infinite.
2. If the graph \(C^*-\text{algebra}\) \(C^*(E)\) is purely infinite, then so is \(O_{G,E}\).
Proof. (1). In view of Theorem 3.1, it suffices to prove that each $s_v$ is infinite in $\mathcal{O}_{G,E}$. So, fix some $v \in E^0$. By hypothesis, there is a $G$-circuit $\alpha$ connecting to $v$ by a $G$-path.

We first show that $s_{r(\alpha)}$ is infinite. For, let $\gamma$ be an entry for $\alpha$ by assumption. Since each of $\alpha$ nor $\gamma$ is not a subpath of the other, one may compute that $s_\alpha s_\alpha^*$ and $s_\gamma s_\gamma^*$ are orthogonal. Hence, the Cuntz-Krieger relations imply that

$$s_{r(\alpha)} \geq s_\alpha s_\alpha^* + s_\gamma s_\gamma^* > s_\alpha s_\alpha^* \sim s_\alpha s_\alpha^* = s_{d(\alpha)}.$$  

If $d(\alpha) = gr(\alpha)$, then $s_{d(\alpha)} \sim s_{r(\alpha)}$ by Lemma 3.3(1), and whence $s_{r(\alpha)}$ is infinite in $\mathcal{O}_{G,E}$ as claimed.

Now, because there is a $G$-path from $r(\alpha)$ to $v$, we have $s_v \succeq s_{r(\alpha)}$ by Lemma 3.3(2), and therefore $s_v$ is infinite as well. As $v \in E^0$ was arbitrary, Theorem 3.1 follows the result.

(2). If $C^*(E)$ is purely infinite, then each $s_v$ is infinite in $C^*(E)$, and so is in $\mathcal{O}_{G,E}$ as well. Now apply Theorem 3.1.

Remark 3.5. If $v \in E^0$ receives a $G$-path from a $G$-circuit with an entry but not a path from a circuit, then $s_v$ is infinite in $\mathcal{O}_{G,E}$ while not in $C^*(E)$. Therefore, the converse of Proposition 3.4(2) does not necessarily hold.

In the simple case we conclude the following.

Corollary 3.6. Let $(G, E, \varphi)$ be a pseudo free self-similar graph over an amenable group $G$. Suppose that $\mathcal{O}_{G,E}$ is simple. If $E$ contains a $G$-circuit, then $\mathcal{O}_{G,E}$ is purely infinite.

Proof. Note that the simplicity of $\mathcal{O}_{G,E}$ gives conditions (1) and (2) in Proposition 2.8 [8, Theorem 4.5]. So, by Theorem 3.1, it suffices to show that $s_v$ is infinite for each $v \in E^0$.

Let $(g, \alpha)$ be a $G$-circuit in $E$. By [7, Theorem 16.1], $(g, \alpha)$ has an entry, hence $s_{r(\alpha)}$ is infinite as seen in the proof of Proposition 3.4(1).

Fix an arbitrary $v \in E^0$. We may form the infinite path $\alpha^\infty = \alpha(g\alpha)(g^2\alpha) \cdots$, which is well-defined because

$$d(g^n\alpha) = g^n d(\alpha) = g^n gr(\alpha) = r(g^{n+1}\alpha).$$

Since $E$ is also weakly $G$-transitive by [8, Theorem 4.5], there is a $G$-path from $r(g^n\alpha)$ to $v$ for sufficiently large $n$. Note that as $r(g^n\alpha) = g^n r(\alpha)$, $s_{r(g^n\alpha)} = s_{g^n r(\alpha)}$ is infinite by Lemma 3.3(1). Also, Lemma 3.3(2) implies that $s_v \succeq s_{r(g^n\alpha)} \sim s_{r(\alpha)}$, and consequently $s_v$ is infinite too. As $v \in E^0$ was arbitrary, Theorem 3.1 concludes that $\mathcal{O}_{G,E}$ is purely infinite. $\square$

Remark 3.7. The converse of above corollary will be proved in Theorem 4.9 (1) $\iff$ (6).

The following result gives necessary and sufficient criteria for the purely infinite simple $C^*$-algebras by the monoid of equivalent projections. It is new even for the ordinary graph $C^*$-algebras. Before that we recall the definition of $K_0$-group of a unital $C^*$-algebra and establish some notations. Let
A be a unital $C^*$-algebra and write by $\mathcal{P}(A)$ the collection of all projections in $M_\infty(A) = \bigcup_{n \geq 1} M_n(A)$. We say that two projections $p \in M_n(A)$ and $q \in M_n(A)$ are equivalent, denoted by $p \sim q$, if

$$\exists v \in M_{m,n}(A) \text{ such that } p = v^*v \text{ and } q = v^*v.$$  

Note that, if $m \leq n$, then $p \sim q$ if and only if $p \oplus 0_{n-m}$ is Murray-von Neumann equivalent to $q$ in $M_n(A)$, where $x \oplus y := \text{diag}(x,y)$. Define $\mathcal{D}(A) := \mathcal{P}(A)/\sim = \{[p] : p \in \mathcal{P}(A)\}$, which is an abelian monoid with the operation $[p] + [q] := [p \oplus q]$. Then $K\mathcal{O}(A)$ is the Grothendieck group of $\mathcal{D}(A)$ endowed with a universal Grothendieck map $\phi : \mathcal{D}(A) \to K\mathcal{O}(A)$. The image of $\mathcal{D}(A)$ under $\phi$ is denoted by $K\mathcal{O}(A)^+$. It is known that when $\mathcal{D}(A) \setminus \{0\}$ is a group, then $K\mathcal{O}(A) = \mathcal{D}(A) \setminus \{0\}$.

**Theorem 3.8.**  
(1) Let $E$ be an arbitrary directed graph (non necessarily row-finite, source-free, or even countable) with $|E^0| < \infty$. Then $C^*(E)$ is purely infinite and simple if and only if $\mathcal{D}(C^*(E)) \setminus \{0\}$ is a group (or equivalently, $\mathcal{D}(C^*(E)) \setminus \{0\} = K\mathcal{O}(C^*(E))$).

(2) Let $(G,E,\varphi)$ be a pseudo free self-similar graph over an amenable group $G$. Suppose also that $|E^0| < \infty$ and conditions (1) and (2) of Proposition 2.8 hold. Then $\mathcal{O}_{G,E}$ is purely infinite simple if and only if $\mathcal{D}(\mathcal{O}_{G,E})$ is a group.

**Proof.** Note that the “only if” implications hold for every unital purely infinite simple $C^*$-algebra. Indeed, if $A$ is a purely infinite simple $C^*$-algebra, then nonzero projections of $A$ are all infinite. Thus, combining Proposition 1.5 and Theorem 1.4 of [5] implies that $\mathcal{D}(A) \setminus \{0\}$ is a group ($= K\mathcal{O}(A)$).

So it is enough to prove the “if” part. We first show that every projection $p$ in $A$ is infinite for any unital $C^*$-algebra $A$ with $\mathcal{D}(A) \setminus \{0\}$ a group. Indeed, if $[f]$ is the identity of $\mathcal{D}(A) \setminus \{0\}$, then

$$[p] = [p] + [f] = [p \oplus f],$$

thus we have

$$p \sim p \oplus 0 < p \oplus f \sim p,$$

where $0$ is a zero matrix in $M_\infty(A)$. Therefore, $p$ is an infinite projection in $A$, as claimed.

In the case of statement (1), this follows that $E$ satisfies Condition (L). In fact if there exists a circuit in $E$ with no entries, then $C^*(E)$ contains an ideal Morita equivalent to $C(T)$, hence it has a finite projection. Recall that by Condition (L) every ideal of $C^*(E)$ has a (vertex) projection. Now take a nonzero ideal $I$ of $C^*(E)$ and some projection $0 \neq p \in I$. As $|E^0| < \infty$, write $1 := \sum_{v \in E^0} s_v$ the unit of $C^*(E)$. Then $[p] + [1 - p] = [1]$ and we have

$$[p] = [1] + [q] = [1 \oplus q],$$

where $[q]$ is the inverse of $[1-p]$ in $\mathcal{D}(C^*(E)) \setminus \{0\}$. Therefore, $p \sim 1 \oplus q$ which says that there is $x = [x_1 \ldots x_n] \in M_1(\mathcal{O}_{G,E})$ such that $x^*px = 1 \oplus q$. In particular, $1 = x_1^*px_1 \in I$, concluding $I = C^*(E)$. Therefore $C^*(E)$ is simple.
For the pure infiniteness, let $B$ be a nonzero hereditary $C^*$-subalgebra of $C^*(E)$. Again, Condition (L) gives a nonzero projection $p$ in $B$. If $[f]$ is the identity of $D(C^*(E)) \setminus \{0\}$, then
\[
[p] = [p] + [f] = [p \oplus f],
\]
and we have
\[
p \sim p \oplus 0 < p \oplus f \sim p
\]
where $0$ is a zero matrix in $M_\infty(O_G,E)$, and consequently $p$ is infinite. Therefore, $C^*(E)$ is purely infinite.

For statement (2), note that $G_{G,E}$ is effective by Proposition 2.8, and $O_{G,E} \cong C^*_r(G_{G,E})$ by Proposition 2.7. This implies that every ideal of $O_{G,E}$ contains a projection (see [6, Theorem 4.4] for example). Now we may follow the proof of statement (1) to obtain the result. □

4. Stable finiteness and a dichotomy

In this section, we associate a special graph $\tilde{E}$ to any self-similar graph $(G, E, \varphi)$. We show that if the graph $C^*$-algebra $C^*(\tilde{E})$ is either simple, purely infinite, or stable finite then so is $O_{G,E}$ respectively. Then we will conclude a dichotomy for simple self-similar graph $C^*$-algebras.

Definition 4.1. Let $\mathbb{K}$ denote the $C^*$-algebra of compact operators on a separable, infinite dimensional Hilbert space. A (simple) $C^*$-algebra $A$ is called stably finite if $A \otimes \mathbb{K}$ contains no infinite projections.

Fix a self-similar graph $(G, E, \varphi)$. In the following we define a graph $\tilde{E}$ associated to $(G, E, \varphi)$. Define $\approx$ on $E^* = \bigsqcup_{n=0}^{\infty} E^n$ by
\[
\alpha \approx \beta \iff \exists g \in G \text{ such that } \beta = g\alpha,
\]
which is an equivalent relation on each $E^n$ (and so on $E^*$). The vertex set of $\tilde{E}$ is $\tilde{E}^0 := E^0/\approx \simeq$ the collection of vertex classes. In each class $[v] \in \tilde{E}^0$ pick exactly one vertex up and collect them in the set $\Omega$. Hence, $\tilde{E}^0 = \{[v] : v \in \Omega\}$, and we have $[v] \neq [w]$ for $v \neq w \in \Omega$. For every $v \in \Omega$ and $e \in r^{-1}(v)$ draw an edge $\tilde{e}$ from $[d(e)]$ to $[v]$. Hence we obtain the graph $\tilde{E}$ so that
\[
\tilde{E}^0 := \{[v] : v \in \Omega\}, \quad \text{and}
\tilde{E}^1 := \bigcup_{v \in \Omega} r^{-1}(v) = \bigcup_{v \in \Omega} \{\tilde{e} : r(\tilde{e}) = v\},
\]
with the range $\tilde{r}(\tilde{e}) = [r(e)]$ and domain $\tilde{d}(\tilde{e}) = [d(e)]$ for every $\tilde{e} \in \tilde{E}^1$.

Example 4.2. For $n \geq 1$, let $\mathbb{Z}_{modn}$ be the additive group $\{1, 2, \ldots, n\}$. Let $(\mathbb{Z}_{modn}, E, \varphi)$ be a triple with the cyclic graph $E$.
Proof. Statement (1) is clear by the definition of $\alpha$ and the action $\mathbb{Z}/n \mathbb{Z} \curvearrowright E$ defined by

$$kv := v \quad \text{and} \quad k\alpha_i := \alpha_{k+i} \quad (1 \leq k, i \leq n),$$

for every $\alpha_i \in \{w_i, e_i, f_i, g_i\}$. Since $w_i \approx w_j$, for any $1 \leq i, j \leq n$, we may select $w_1$ of the class $[w_1] = \{w_1, \ldots, w_n\}$. As $r^{-1}(v) = \{e_1, \ldots, e_n\}$ and $r^{-1}(w_1) = \{f_1, g_n\}$, then the graph $\tilde{E}$ would be

and the action $\mathbb{Z}/n \mathbb{Z} \curvearrowright E$ defined by

$$kv := v \quad \text{and} \quad k\alpha_i := \alpha_{k+i} \quad (1 \leq k, i \leq n),$$

for every $\alpha_i \in \{w_i, e_i, f_i, g_i\}$. Since $w_i \approx w_j$, for any $1 \leq i, j \leq n$, we may select $w_1$ of the class $[w_1] = \{w_1, \ldots, w_n\}$. As $r^{-1}(v) = \{e_1, \ldots, e_n\}$ and $r^{-1}(w_1) = \{f_1, g_n\}$, then the graph $\tilde{E}$ would be

$$\tilde{g}_n \quad \ldots \quad \tilde{f}_1 \quad \ldots \quad \tilde{e}_n \quad \tilde{e}_1 \quad \tilde{e}_2 \quad [v]$$

**Lemma 4.3.** Let $(G, E, \varphi)$ be a self-similar graph, and consider an associated graph $\tilde{E}$ as above. Then

1. If $E$ is row-finite, then so is $\tilde{E}$.
2. For each finite path $\tilde{\alpha} = \tilde{\alpha}_1 \ldots \tilde{\alpha}_n \in \tilde{E}^n$, there is a path $\gamma = \gamma_1 \ldots \gamma_n$ in $E^n$ such that $\gamma \approx \alpha_i$ for $1 \leq i \leq n$. Conversely, if $\gamma = \gamma_1 \ldots \gamma_n \in E^n$, then there exists $\tilde{\alpha} = \tilde{\alpha}_1 \ldots \tilde{\alpha}_n \in \tilde{E}^n$ such that $\gamma \approx \alpha_i$ for $1 \leq i \leq n$.
3. If $\tilde{\alpha} \in \tilde{E}^n$ and $\gamma \in E^n$ are two paths as in statement (2), then $\tilde{\alpha}$ is a circuit in $\tilde{E}$ if and only if $\gamma$ is a $G$-circuit in $E$. Moreover, $\tilde{\alpha}$ has an entry if and only if $\gamma$ does.

**Proof.** Statement (1) is clear by the definition of $\tilde{E}$. For (2), let first $\tilde{\alpha} = \tilde{\alpha}_1 \ldots \tilde{\alpha}_n \in \tilde{E}^n$ be a path in $\tilde{E}$. Then, for each $1 \leq i < n$, we have

$$[d(\alpha_i)] = \tilde{d}(\tilde{\alpha}_i) = \tilde{r}(\tilde{\alpha}_{i+1}) = [r(\alpha_{i+1})],$$

and so there exists $g_i \in G$ such that $d(\alpha_i) = g_i r(\alpha_{i+1})$. Now set $\gamma_1 := \alpha_1$ and $\gamma_i := g_1 \ldots g_{i-1} \alpha_i$ for every $2 \leq i \leq n$. Then

$$d(\gamma_i) = d(g_1 \ldots g_{i-1} \alpha_i) = g_1 \ldots g_{i-1} d(\alpha_i) = g_1 \ldots g_{i-1} g_i r(\alpha_{i+1}) = r(\gamma_{i+1}),$$

and hence $\gamma = \gamma_1 \ldots \gamma_n$ is a desired path in $E$. 


Conversely, let \( \gamma = \gamma_1 \ldots \gamma_n \) be a finite path in \( E^n \). For each \( 1 \leq i \leq n \), there is \( v_i \in \Omega \) such that \( v_i = g_i r(\gamma_i) \) for some \( g_i \in G \). Hence, we have \( \bar{\alpha} = (\bar{g_1} \gamma_1) \ldots (\bar{g_n} \gamma_n) \in \bar{E} \) with \( \alpha \approx \gamma \).

For statement (3), given \( \bar{\alpha} \) and \( \gamma \) as in part (2), we have
\[
\bar{\alpha} \text{ is a circuit in } E \iff [d(\alpha_n)] = [r(\alpha_1)] \\
\iff d(\alpha_n) \approx r(\alpha_1) \\
\iff d(\gamma_n) \approx d(\alpha_n) \approx r(\alpha_1) \approx r(\gamma_1) \\
\iff \gamma \text{ is a } G-\text{circuit.}
\]

Moreover, since \( |r^{-1}(r(\gamma_i))| = |\bar{r}^{-1}(\bar{r}(\alpha_i))| \) for each \( 1 \leq i \leq n \), we have
\[
\gamma \text{ has an entry } \iff |r^{-1}(r(\gamma_i))| > 1 \text{ for some } 1 \leq i \leq n \\
\iff |\bar{r}^{-1}(\bar{r}(\alpha_i))| > 1 \text{ for some } 1 \leq i \leq n \\
\iff \bar{\alpha} \text{ has an entry in } \bar{E}.
\]

\( \square \)

**Definition 4.4.** Let \((G, E, \varphi)\) be a self-similar graph. Following [7, Definition 3.4], we say that \( E \) is weakly \( G \)-transitive if for every \( v \in E^0 \) and \( x \in E^\infty \), there exists a path \( \alpha \) such that \( d(\alpha) = x(n, n) \) for some \( n \geq 0 \) and \( r(\alpha) = gv \) for some \( g \in G \). If we have an ordinary graph \( E \) (with the trivial group action), we say simply that \( E \) is weakly transitive. Note that the weakly transitive is called cofinal in [20].

**Lemma 4.5.** Let \((G, E, \varphi)\) be a self-similar graph, and associate a graph \( \bar{E} \) as above. Then

1. Every \( G \)-circuit in \( E \) has an entry if and only if every circuit in \( \bar{E} \) does.
2. \( E \) is weakly \( G \)-transitive if and only if \( \bar{E} \) is weakly transitive.

**Proof.** Statement (1) follows from items (2) and (3) of Lemma 4.3. For (2), let \( \bar{E} \) be transitive. Take an arbitrary infinite path \( x \in E^\infty \) and some \( v \in E^0 \). By item (2) in Lemma 4.3, there is \( \bar{y} \in \bar{E}^\infty \) such that \( y(0, n) \approx x(0, n) \) for every \( n \geq 0 \). By transitivity, there exists \( \bar{\gamma} \in \bar{E}^* \) such that \( \bar{r}(\bar{\gamma}) = [v] \) and \( \bar{d}(\bar{\gamma}) = [y(n, n)] \) for some \( n \). Hence, \( v \approx r(\gamma) \) and \( d(\gamma) \approx y(n, n) \approx x(n, n) \). This follows that \( E \) is \( G \)-transitive. The converse is analogous. \( \square \)

**Proposition 4.6.** Let \((G, E, \varphi)\) be a self-similar graph over an amenable group \( G \), and let \( \bar{E} \) be an associated graph.

1. In case the groupoid \( G_{G,E} \) is Hausdorff (see [8, Theorem 4.2]), then \( \mathcal{O}_{G,E} \) is simple if and only if
   a. the graph \( C^* \)-algebra \( C^*(\bar{E}) \) is simple, and
   b. for \( v \in E^0 \) and \( g \in G \), if the action of \( g \) on the cylinder \( Z(v) \) is trivial (i.e., \( gx = x \) for every \( x \in Z(v) \)), then \( g \) is slack at \( v \).
(2) Suppose that \((G, E, \varphi)\) is pseudo free and for any \(v \in E^0\) and \(1_G \neq g \in G\), the action of \(g\) on \(Z(v)\) is nontrivial. If \(C^*(\tilde{E})\) is purely infinite, then so is \(\mathcal{O}_{G,E}\).

**Proof.** Statement (1) follows from Lemma 4.5 and [8, Theorem 4.5]. For (2), if the graph \(C^*\)-algebra \(C^*(\tilde{E})\) is purely infinite, then every circuit in \(\tilde{E}\) has an entry and every vertex \([v] \in \tilde{E}^0\) can be reached from a circuit. By Lemma 4.5, every \(G\)-circuit has an entry and every \(v \in E^0\) receives a \(G\)-path from a \(G\)-circuit. Now, Proposition 3.4(1) concludes that \(\mathcal{O}_{G,E}\) is purely infinite. \(\square\)

**Example 4.7.** The graph \(\tilde{E}\) in Example 4.2 is weakly transitive and every circuit in \(\tilde{E}\) has an entry. Then \(C^*(\tilde{E})\) is simple and purely infinite, and so is the \(C^*\)-algebra \(\mathcal{O}_{G,E}\) by Proposition 4.6.

**Definition 4.8 ([9]).** Let \((G, E, \varphi)\) be a self-similar graph. A **graph trace** on \(E\) is map \(T : E^0 \rightarrow \mathbb{R}^+\) such that

1. \(T(\tau(e)) \geq T(d(e))\) for every \(e \in E^1\), and
2. \(T(v) = \sum_{e \in v} T(d(e))\) for every \(v \in E^0\).

A **graph \(G\)-trace** in \(E\) is a graph trace \(T : E^0 \rightarrow \mathbb{R}^+\) such that \(T(v) = T(w)\) for every \(v \approx w\) in \(E^0\).

**Theorem 4.9.** Let \((G, E, \varphi)\) be a pseudo free self-similar graph over an amenable group \(G\). Suppose that \(\mathcal{O}_{G,E}\) is simple. Then the following are equivalent.

1. \(\mathcal{O}_{G,E}\) is stably finite.
2. \(\mathcal{O}_{G,E}\) is quasi diagonal.
3. \((G, E, \varphi)\) has a nonzero graph \(G\)-trace.
4. \(\tilde{E}\) has a nonzero graph trace.
5. \(\tilde{E}\) contains no circuits.
6. \(E\) contains no \(G\)-circuits.

**Proof.** Statements (1) and (2) are equivalent by [21, Corollary 6.6].

(1) \(\Rightarrow\) (6). If \(E\) has a \(G\)-circuit, then \(\mathcal{O}_{G,E}\) is purely infinite by Corollary 3.6. In particular, \(\mathcal{O}_{G,E}\) is not stably finite, a contradiction.

(6) \(\Rightarrow\) (5). Suppose that \(\tilde{E}\) has no circuits. Arrange \(\tilde{E}^0 = \{[v_1], [v_2], \ldots\}\).

For each \(n \geq 1\), let \(F_n\) be the full subgraph of \(\tilde{E}\) containing all \(\bigcup_{i=1}^{n-1} \tilde{F}^{-1}([v_i])\). Since \(F_n\)’s have no circuits, [13, Corollary 2.3] implies that \(C^*(F_1) \subseteq C^*(F_2) \subseteq \ldots\) is a sequence of finite dimensional \(C^*\)-subalgebras of \(C^*(\tilde{E})\) such that \(C^*(\tilde{E}) = \lim C^*(F_n)\) (i.e., \(C^*(\tilde{E})\) is AF). Thus there exist bounded traces \(\tau_n : C^*(F_n) \rightarrow \mathbb{C}\) such that \(\tau_n|_{C^*(F_i)} = \tau_i\) for \(i \leq n\). This induces a semifinite trace \(\tau = \lim \tau_n\) on \(C^*(\tilde{E})\). Therefore, if \(C^*(\tilde{E}) = C^*(t_e, q_{[v]}\rangle\), we obtain the nonzero graph trace \(T : E^0 \rightarrow \mathbb{R}^+\), by \(T([v]) = \tau(q_{[v]}\rangle\), on \(\tilde{E}\).

(4) \(\Rightarrow\) (3). Suppose that \(T\) is a nonzero graph trace on \(\tilde{E}\). Note that, since the action of \(G\) on \(E^1\) gives automorphisms respecting to the range
and domain, for any \( v \neq w \in E^0 \) with \( w = gv \), the map \( e \mapsto ge \) is a bijection from \( r^{-1}(v) \) onto \( r^{-1}(w) \). In particular, \( |r^{-1}(w)| = |r^{-1}(v)| \). Being this fact in mind, one may easily see that the map \( T : E^0 \to \mathbb{R}^+ \), defined by \( T'(v) := T([v]) \), is a nonzero graph \( G \)-trace on \( E \), as desired.

(3) \( \Rightarrow \) (1). By [23, Proposition II.4.8], there exists a faithful conditional expectation \( \pi : C^*(G,G,E) \to C_0(G,G,E) \) such that \( \pi(f) = f|_{G,G,E} \) for all \( f \in C_c(G,G,E) \). Note that the isomorphism \( \psi : O_{G,E} \to C^*(G,G,E) \) in Proposition 2.7(1) maps the core \( O_{G,E}^0 \) onto \( C_0(G,G,E) \). Hence \( \varphi := \psi^{-1} \circ \pi \circ \psi \) is a faithful conditional expectation from \( O_{G,E} \) onto \( O_{G,E}^0 \) such that

\[
\phi(s_\alpha u_g s_\beta^*) = \begin{cases} 
    s_\alpha s_\beta^* & \beta = \alpha, \ g = 1_G \\
    0 & \text{otherwise}
\end{cases}
\]

for every \( \alpha, \beta \in E^* \) and \( g \in G \).

Now suppose that \( T \) is a nonzero graph \( G \)-trace on \( E \). Define \( t : O_{G,E}^0 \to \mathbb{C} \) by \( t(s_\alpha s_\beta^*) = T(d(\alpha)) \), which is a linear functional on \( O_{G,E}^0 \). So, we may easily verify that \( \tau := t \circ \phi \) is a semifinite trace on \( O_{G,E} \) such that \( 0 < \tau(s_v) < \infty \) for all \( v \in E^0 \). Moreover, \( \tau \) is faithful because \( O_{G,E} \) is simple. Thus [21, Corollary 6.6] yields that \( O_{G,E} \) is stably finite. \( \square \)

Recall from [7, Corollary 10.16] that if \( G \) is amenable, then \( O_{G,E} \) is a nuclear \( C^* \)-algebra. So, combining Corollary 3.6 and Theorem 4.9 implies the following dichotomy for simple \( O_{G,E} \).

**Corollary 4.10.** Let \((G,E,\varphi)\) be a pseudo free self-similar graph over an amenable group \( G \). Suppose that \( O_{G,E} \) is simple. Then

1. If \( E \) has a \( G \)-circuit, then \( O_{G,E} \) is purely infinite. In this case, \( O_{G,E} \) is a Kirchberg algebra, and we have \( K_0(O_{G,E}) = D(O_{G,E}) \setminus \{0\} \) whenever \( |E^0| < \infty \).
2. Otherwise, \( O_{G,E} \) is stably finite. In this case, \((K_0(O_{G,E}), K_0(O_{G,E})^+)\) is an ordered abelian group (see [24, Proposition 5.1.5(iv)])

**Remark 4.11.** Note that in case \( O_{G,E} \) is stably finite, the embedding \( \iota : C^*(E) \hookrightarrow O_{G,E} \) of [7, Section 11] induces an embedding \( K_0(\iota) : K_0(C^*(E)) \hookrightarrow K_0(O_{G,E}) \) defined by \( K_0(\iota)([p]) = [\iota(p)]_0 \), where the map \( \iota \) is naturally extended on \( M_\infty(C^*(E)) \) into \( M_\infty(O_{G,E}) \). Indeed, if \( p \in M_\infty(C^*(E)) \) is a projection with \( [\iota(p)]_0 = 0 \), then we must have \( \iota(p) = 0 \) because \( M_\infty(O_{G,E}) \) has no infinite projection, and hence \( p = 0 \).

5. **Pure infiniteness of self-similar \( k \)-graph \( C^* \)-algebras**

In this section, we consider the pure infiniteness of self-similar \( k \)-graph \( C^* \)-algebras. Let us first recall the definitions of self-similar \( k \)-graphs and their \( C^* \)-algebras from [17]. Fix \( k \in \mathbb{N} \cup \{\infty\} \) and let \( \Lambda = (\Lambda^0, \Lambda, r, s) \) be a row-finite \( k \)-graph with no sources (we refer the reader to [20] for basic definitions and concepts about \( k \)-graphs and associated \( C^* \)-algebras).
Consider $\mathbb{N}^k$ as a category with a single object 0 and the coordinatewise partial order $\leq$. Let $\Omega_k := \{(p,q) : p, q \in \mathbb{N}^k, p \leq q\}$. An infinite path in $\Lambda$ is a morphism $x : \Omega_k \to \Lambda$ with the range $r(x) := x(0,0)$. We write by $\Lambda^\infty$ the set of infinite paths in $\Lambda$.

Let $G$ be a (discrete and countable) group. An action $G \actson \Lambda$ is a map $G \times \Lambda \to \Lambda$, $(g, \lambda) \mapsto g\lambda$, which gives a graph automorphism preserving the degree map for every $g \in G$.

**Definition 5.1 ([17])**. A self-similar $k$-graph is a triple $(G, \Lambda, \varphi)$, where $\Lambda$ is a $k$-graph, $G$ is a group acting on $\Lambda$, and $\varphi : G \times \Lambda \to \Lambda$ is a cocycle for $G \actson \Lambda$ with the property

$$\varphi(g, \lambda) \cdot v = gv \quad (g \in G, v \in \Lambda^0, \lambda \in \Lambda).$$

Following [17], we consider only self-similar $k$-graphs $(G, \Lambda, \varphi)$ for row-finite and source-free $k$-graphs with $|\Lambda^0| < \infty$. We will write $(G, \Lambda, \varphi)$ by $(G, \Lambda)$ for simplicity. Note that $\varphi$ was called the restriction map in [17] and each $\varphi(g, \lambda)$ was denoted by $g|_\lambda$ there.

**Definition 5.2.** Let $(G, \Lambda)$ be a self-similar $k$-graph. We say that

1. $(G, \Lambda)$ is pseudo free, if $g\lambda = \lambda$ and $\varphi(g, \lambda) = 1_G$ imply $g = 1_G$.
2. $(G, \Lambda)$ is $G$-aperiodic if for any $v \in \Lambda^0$, there exists $x \in v\Lambda^\infty$ such that $x(p, \infty) = gx(q, \infty)$ implies $g = 1_G$ and $p = q$ for $p, q \in \mathbb{N}^k$ and $g \in G$.
3. $(G, \Lambda)$ is $G$-cofinal if for every $x \in \Lambda^\infty$ and $v \in \Lambda^0$, there exist $p \in \mathbb{N}^k$, $\mu \in \Lambda$, and $g \in G$ such that $s(\mu) = x(p, p')$ and $r(\mu) = gv$.

**Definition 5.3.** Let $(G, \Lambda)$ be a self-similar $k$-graph as in Definition 5.1 with $|\Lambda^0| < \infty$. The $C^*$-algebra $O_{G,\Lambda}$ associated to $(G, \Lambda)$ is the universal $C^*$-algebra generated by $\{s_\lambda : \lambda \in \Lambda\}$ and $\{u_g : g \in G\}$ such that

1. $\{s_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger $\Lambda$-family in the sense of [12].
2. $u : G \to O_{G,\Lambda}$, given by $g \mapsto u_g$, is a unitary $*$-representation of $G$.
3. $u_gs_\lambda = s_\lambda u_{\varphi(g, \lambda)}$ for every $g \in G$ and $\lambda \in \Lambda$.

Similar to the construction of $G_{G,E}$ in Section 2.4, Li and Yang associated an ample groupoid $G_{G,\Lambda}$ in [17, Section 5.1] such that $O_{G,\Lambda} \cong C^*(G_{G,\Lambda}) \cong C_r^*(G_{G,\Lambda})$ when $G$ is amenable and $(G, \Lambda)$ is pseudo free [17, Theorem 5.9]. In particular, the unit space $G_{G,\Lambda}^{(0)}$ is homeomorphic to $\Lambda^\infty$ endowed with the topology generated by cylinders $Z(\lambda) := \{\lambda x : x \in \Lambda^\infty\}$.

Recall that a circuit in $\Lambda$ is a path $\alpha \in \Lambda$ with $r(\alpha) = s(\alpha)$. $\tau \in \Lambda$ is called an entry for $\alpha$ if $r(\tau) = r(\alpha)$ and there are no common extensions for $\alpha$ and $\tau$ (i.e., $\alpha \mu \neq \tau \nu$ for all $\mu, \nu \in \Lambda$).

**Theorem 5.4.** Let $(G, \Lambda)$ be a pseudo free self-similar $k$-graph with $|\Lambda^0| < \infty$ over an amenable group $G$. If $\Lambda$ is $G$-aperiodic, then $O_{G,\Lambda}$ is purely infinite. In particular, if $\Lambda$ is also $G$-cofinal, then $O_{G,\Lambda}$ is a Kirchberg algebra.
Proof. Let $\mathcal{G}_{G,A}$ be the groupoid associated to $(G, \Lambda)$. Then $\mathcal{G}_{G,A}$ is amenable and effective [17, Proposition 6.5], and we thus have $C^*(\mathcal{G}_{G,A}) = C^*_r(\mathcal{G}_{G,A}) = \mathcal{O}_{G,A}$ by [17, Theorem 5.9]. We know that the cylinders $\{Z(\lambda) : \lambda \in \Lambda\}$ form a basis of compact open sets for the topology on $\Lambda^\infty = G^{(0)}_{G,A}$. So, in light of Proposition 2.3, it suffices to prove that each $1_{Z(\lambda)}$ is an infinite projection for $\lambda \in \Lambda$. For this, since

$$1_{Z(\lambda)} = s_\Lambda s_\lambda^* s_\lambda = s_\lambda(\lambda),$$

we show all $s_v$'s are infinite in $\mathcal{O}_{G,A}$ for $v \in \Lambda^0$.

So fix an arbitrary $v \in \Lambda^0$. We claim that $v$ reaches from a circuit with an entry. To see this, take some $x \in v\Lambda^\infty$. For any $t \in \mathbb{N}$, write $t := (t_0, t_1, \ldots) \in \mathbb{N}^k$. Since $\{x(t, t) : t \geq 1\} \subseteq \Lambda^0$ is finite, there are $t_1 < t_2$ such that $x(t_1, t_1) = x(t_2, t_2)$. Hence $x(t_1, t_2)$ is a circuit in $\Lambda$, which connects to $v$ by $x(0, t_1) \in \Lambda$. Note that the $G$-aperiodicity yields clearly the periodicity of $\Lambda$. Hence, one may follow [16, Lemma 6.1] to find an (initial) circuit $\alpha$ with an entry $\tau$ connecting to $v$, as claimed.

Since $\alpha$ and $\tau$ have no common extensions, one may compute that $s_\alpha s_\alpha^*$ and $s_\tau s_\tau^*$ are orthogonal (by applying [20, Lemma 9.4]). Thus, by the Cuntz-Krieger relations we have

$$s_{\tau(\alpha)} \geq s_\alpha s_\alpha^* + s_\tau s_\tau^* > s_\alpha s_\alpha^* \sim s_\alpha^* s_\alpha = s_\alpha = s_{\tau(\alpha)};$$

so $s_{\tau(\alpha)}$ is infinite. Moreover, if $\lambda$ connects $r(\alpha)$ to $v$, then

$$s_v \geq s_\lambda s_\lambda^* \sim s_\lambda^* s_\lambda = s_\lambda = s_{\tau(\alpha)},$$

which says that $s_v$ is an infinite projection in $\mathcal{O}_{G,A}$ as well. Since $v \in \Lambda^0$ was arbitrary, this deduces that $\mathcal{O}_{G,A}$ is purely infinite by Proposition 2.3.

For the last statement, if moreover $\Lambda$ is $G$-cofinal, then [17, Theorem 6.6] implies that $\mathcal{O}_{G,A}$ is nuclear and simple, which satisfies UCT. Hence, $\mathcal{O}_{G,A}$ is a Kirchberg algebra. \hfill \Box

**Corollary 5.5** (See [17, Theorem 6.13]). Let $(G, \Lambda)$ be a pseudo free self-similar $k$-graph with $|\Lambda^0| < \infty$ over an amenable group $G$. Whenever $\mathcal{O}_{G,A}$ is simple, then it is purely infinite too.

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