Quantum Set Theory: 
Transfer Principle and De Morgan’s Laws *

Dedicated to the memory of Professor Gaisi Takeuti

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Abstract

In quantum logic introduced by Birkhoff and von Neumann, De Morgan’s Laws play an important role for the projection-valued truth value assignment of observational propositions in quantum mechanics. Takeuti’s quantum set theory extends this assignment to all the set-theoretical statements on the universe of quantum sets. However, Takeuti’s quantum set theory has a problem that De Morgan’s Laws do not hold between universal and existential bounded quantifiers. Here, we solve this problem by introducing a new truth value assignment for bounded quantifiers that satisfies De Morgan’s Laws. To justify the new assignment we prove the Transfer Principle showing that the assignment of the truth value for every bounded ZFC theorem has a lower bound determined by the commutator, a projection-valued degree of commutativity, of constants in the formula. We study the most general class of truth value assignments and obtain necessary and sufficient conditions for them to satisfy the Transfer Principle, to satisfy De Morgan’s Laws, and to satisfy both, respectively. For the class of assignments with polynomially definable logical operations, we determine exactly 36 assignments that satisfy the Transfer Principle and exactly 6 assignments that satisfy both the Transfer Principle and De Morgan’s Laws.

Key words and phrases: quantum set theory, orthomodular-valued models, Transfer Principle, De Morgan’s Laws, quantum logic, orthomodular lattices, commutator, implication, Boolean-valued models, ZFC

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1 Introduction

Quantum set theory originated from the methods of forcing introduced by Cohen [1,2] for independence proofs and quantum logic introduced by Birkhoff and von Neumann [3]. It crosses over two remote fields of mathematics, foundations of mathematics and foundations of quantum mechanics. After Cohen’s work, Scott and Solovay [4] reformulated the method of forcing by Boolean-valued models of set theory [5], which incorporated with various extensions of the notion of sets, such as sheaves [6], topos [7], and intuitionistic set theory [8]. As a successor of the above attempts, Takeuti [9] introduced quantum set theory, a set theory based on the Birkhoff-von Neumann quantum logic.

Takeuti constructed the universe $V^Q$ of quantum sets based on quantum logic $Q$ represented by projections on a Hilbert space $\mathcal{H}$, and to every formula $\phi(x_1, \ldots, x_n)$ in set theory assigned the $Q$-valued truth value $[\phi(u_1, \ldots, u_n)]$ for quantum sets $u_1, \ldots, u_n \in V^Q$ to satisfy $\phi(x_1, \ldots, x_n)$. For the well-known arbitrariness of implication in quantum logic, he adopted the Sasaki arrow for implication. In order to provide quantum counterparts of ZFC axioms, he introduced the notion of commutator of elements of the universe $V^Q$, a measure of the degree of commutativity, and he showed that the axioms of ZFC hold in the universe $V^Q$ if appropriately modified by the commutators. Based on his preceding work on Boolean-valued analysis [10], he pointed out that the real numbers in the universe $V^Q$ correspond to the self-adjoint operators on the underlying Hilbert space $\mathcal{H}$, suggesting rich applications to quantum physics and analysis.

Following Takeuti’s work, we explored the question how theorems of ZFC hold in the universe $V^Q$. We showed that the following Transfer Principle holds for Takeuti’s quantum set theory [11].

Transfer Principle. Every $\Delta_0$-formula $\phi(x_1, \ldots, x_n)$ in the language of set theory provable in ZFC holds for any elements $u_1, \ldots, u_n$ in the universe $V^Q$ with the $Q$-valued truth value $[\phi(u_1, \ldots, u_n)]$ at least the commutator $\bigvee(u_1, \ldots, u_n)$ of $u_1, \ldots, u_n \in V^Q$, i.e.,

$$[\phi(u_1, \ldots, u_n)] \geq \bigvee(u_1, \ldots, u_n).$$

This result was extended to general complete orthomodular lattices and to a general class of operations for implication [12]. Note that this generalization of formulation unifies quantum set theory with Boolean-valued models of set theory, which are included as the case where $Q$ is a Boolean algebra, and naturally incorporates the methods of Boolean-valued analysis [10,13–28] into various applications of quantum set theory. Quantum set theory was effectively applied to quantum mechanics to extend the probabilistic predictions from observational propositions to relations between observables such as commutativity, equality, and order relations [29,31] and applied to computer science [32]. Relations to paraconsistent set theory, intuitionistic set theory, and topos quantum mechanics are also studied recently [33–35].

In spite of the above successful development of the theory, one problem has eluded a solution. Takeuti’s assignment of the truth value does not satisfy De Morgan’s Laws for the universal–existential pair of bounded quantifiers. Since the inception of quantum logic due to Birkhoff and von Neumann [3], interpretations of connectives have

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been often polemical, but De Morgan’s Laws have played an important role. For instance, the meaning of disjunction is less obvious than those of conjunction and negation in quantum logic, and yet De Morgan’s Laws enable us to determine disjunction from conjunction and negation.

In this paper, we examine Takeuti’s truth value assignment of the truth value \([\phi]\) in the quantum logic \(Q\) to a set theoretic statement \(\phi\). In particular, Takeuti noted

In Boolean-valued universes, \([\forall x \in u) \phi(x)] = [\forall x (x \in u \rightarrow \phi(x)] \) and \([\exists x \in u) \phi(x)] = [\exists x (x \in u \land \phi(x)] \) [hold]. But this is not the case for \(V(Q)\). [9, p. 315]

and defined the truth values of bounded quantifications using the Sasaki arrow \(\rightarrow\) defined by \(P \rightarrow Q = P^\perp \lor (P \land Q)\) as follows.

1. \([\forall x \in u) \phi(x)] = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow [\phi(u')]).
2. \([\exists x \in u) \phi(x)] = \bigvee_{u' \in \text{dom}(u)} (u(u') \land [\phi(u')]).

However, it is problematic that the classical implication \(P \rightarrow Q = P^\perp \lor (P \land Q)\) was avoided in the bounded universal quantification, and yet the classical conjunction \(\land\) was used in the bounded existential quantification. Since the relation \(P \land Q = (P \rightarrow Q^\perp)^\perp\) does not hold for the classical conjunction \(\land\) and the Sasaki arrow \(\rightarrow\), De Morgan’s Laws,

3. \([\neg(\forall x \in u) \phi(x)] = [(\exists x \in u) \neg \phi(x)],
4. \([\neg(\exists x \in u) \phi(x)] = [(\forall x \in u) \neg \phi(x)],

do not hold. In fact, we shall show that there exists a predicate \(\phi(x)\) in Takeuti’s quantum set theory such that \([\exists x \in u) \neg \phi(x)] = 0\) but \([\neg(\forall x \in u) \phi(x)] > 0\).

In this paper, we introduce a new binary operation \(\ast\) by \(P \ast Q = (P \rightarrow Q^\perp)^\perp\) in quantum set theory and redefine the truth values of membership relation and bounded existential quantification as follows.

5. \([u \in v] = \bigvee_{u' \in \text{dom}(v)} (v(u') \ast [u' = u]).
6. \([\exists x \in u\) \phi(x)] = \bigvee_{u' \in \text{dom}(u)} (u(u') \ast [\phi(u')]).

Then, De Morgan’s Laws hold for bounded universal quantification and bounded existential quantification. Thus, for the language of quantum set theory we can assume only negation, conjunction, and bounded and unbounded universal quantification as primitive, while disjunction, bounded and unbounded existential quantification are considered to be introduced by definition.

The operation \(\ast\) was found by Sasaki [36], and has been studied as the Sasaki projection in connection with residuation theory [37], whereas up to our knowledge this operation has not been used for defining bounded quantifiers in quantum logic.

Because of the well-known arbitrariness of choosing the connective for implication in quantum logic [38], we previously introduced a general class of binary operations \(\rightarrow\) for implication on a general quantum logic represented by a complete orthomodular
lattice [12]. In this paper, we continue to explore those operations for the problem of the consistency between the Transfer Principle and De Morgan’s Laws. For this purpose, we introduce another general class of binary operations $\ast$ for conjunction. Then, we ask questions as to which pair $(\rightarrow, \ast)$ supports the Transfer Principle and which pair $(\rightarrow, \ast)$ supports both the Transfer Principle and De Morgan’s Laws and answers those questions. For polynomially definable operations, we determine all the 36 pairs $(\rightarrow, \ast)$ that admit the Transfer Principle, and we derive 6 out of 36 pairs that admit both the Transfer Principle and De Morgan’s Laws, including the pair of the Sasaki arrow $\rightarrow$ and the Sasaki projection $\ast$ and also the pair of the classical implication $\rightarrow$ and the classical conjunction $\ast$ as previously mentioned in Ref. [12].

This paper is organized as follows. Section 2 discusses general properties of quantum logic represented by a general complete orthomodular lattice (COML) $Q$. Section 3 discusses quantization of operations in classical logic including 96 polynomially definable operations found by Kotas [39] and also polynomially indefinable operations, which were introduced by Takeuti [9] and extensively studied in Ref. [12]. To set a sound general theory, we introduce a class of binary operations, called local binary operations, on a general COML $Q$, which share two local properties with polynomially definable ones. Section 4 studies quantum set theory based on the universe $V^{(Q)}$ constructed on an arbitrary COML $Q$ and $Q$-valued interpretations, $Q$-valued truth value assignments, $\mathcal{I}(\rightarrow, \ast)$, determined by arbitrary pairs $(\rightarrow, \ast)$ of local binary operations on $Q$. We characterize all the $Q$-valued interpretations $\mathcal{I}(\rightarrow, \ast)$ that admit the Transfer Principle and those that admit both the Transfer Principle and De Morgan’s Laws. For polynomially definable operations $\rightarrow$ and $\ast$ this result determines 6 $Q$-valued interpretations $\mathcal{I}(\rightarrow, \ast)$ that satisfy both the Transfer Principle and De Morgan’s Laws. We also discuss applications of the above results to the notion of spectral order in operator theory. Section 5 concludes the present paper. We also discuss new interpretations of quantum logical connectives using the commutator based direct product decomposition developed in Section 3.

2 Quantum Logic

2.1 Complete orthomodular lattices

A complete orthomodular lattice is a complete lattice $Q$ with an orthocomplementation, a unary operation $\perp$ on $Q$ satisfying (i) if $P \leq Q$ then $Q^\perp \leq P^\perp$, (ii) $P^{\perp \perp} = P$, (iii) $P \lor P^\perp = 1$ and $P \land P^\perp = 0$, where $0 = \bigwedge Q$ and $1 = \bigvee Q$, that satisfies the orthomodular law: if $P \leq Q$ then $P \lor (P^\perp \land Q) = Q$. In this paper, any complete orthomodular lattice is called a logic.

A non-empty subset of a logic $Q$ is called a sublattice iff it is closed under meet $\land$ and join $\lor$. A sublattice is called a subalgebra iff it is further closed under orthocomplementation $\perp$. A sublattice or a subalgebra $R$ of $Q$ is said to be complete iff it has the infimum $\bigwedge A$ and the supremum $\bigvee A$ in $Q$ of an arbitrary subset $A$ of $R$. For any subset $A$ of $Q$, the subalgebra generated by $A$ is denoted by $\Gamma_0 A$, and the complete subalgebra generated by $A$ is denoted by $\Gamma A$. We refer the reader to Kalmbach [40] for a standard reference on orthomodular lattices.

We say that $P$ and $Q$ in a logic $Q$ commute, in symbols $P \perp Q$, iff $P = (P \land
Q) \lor (P \land Q^\perp). All the relations P \perp Q, Q \perp P, P^\perp \perp Q, P \perp Q^\perp, and P^\perp \perp Q^\perp are equivalent. The distributive law does not hold in general, but the following useful proposition holds \cite[pp. 24–25]{40}.

**Proposition 2.1.** If P, Q \perp E, then the sublattice generated by P, Q, E is distributive.

When applying a distributive law under the assumption of Proposition 2.1, we shall say that we are focusing on E. From Proposition 2.1, a logic Q is a Boolean algebra if and only if P \perp Q for all P, Q \in Q. In this case, logic Q is called Boolean.

The following proposition is useful for later discussions \cite[Proposition 3.4]{40}; an elementary proof is given for the reader’s convenience.

**Proposition 2.2.** If P_\alpha, E \in Q and P_\alpha \perp E for all \alpha, then

\[(\bigvee_\alpha P_\alpha) \perp E, \quad \bigwedge_\alpha P_\alpha \perp E, \quad (\bigvee_\alpha P_\alpha) \land E = \bigvee_\alpha (P_\alpha \land E), \quad (\bigwedge_\alpha P_\alpha) \land E = \bigwedge_\alpha (P_\alpha \land E).\]

**Proof.** Suppose that P_\alpha, E \in Q and P_\alpha \perp E hold for every \alpha. From

\[\bigvee_\alpha (P_\alpha \land E) \leq E, \quad \bigvee_\alpha (P_\alpha \land E^\perp) \leq E^\perp,\]  

(1)

we have

\[\bigvee_\alpha (P_\alpha \land E) \perp E, \quad \bigvee_\alpha (P_\alpha \land E^\perp) \perp E.\]  

(2)

By assumption, we have P_\alpha = (P_\alpha \land E) \lor (P_\alpha \land E^\perp) for every \alpha. Since

\[\bigvee_\alpha P_\alpha = \bigvee_\alpha [(P_\alpha \land E) \lor (P_\alpha \land E^\perp)] = \bigvee_\alpha (P_\alpha \land E) \lor \bigvee_\alpha (P_\alpha \land E^\perp),\]

we conclude \bigvee_\alpha P_\alpha \perp E from Eq. (2). Focusing on E by Eq. (2), we have

\[\bigvee_\alpha P_\alpha \land E = [\bigvee_\alpha (P_\alpha \land E) \lor \bigvee_\alpha (P_\alpha \land E^\perp)] \land E = \bigvee_\alpha (P_\alpha \land E).\]

Thus, we conclude \bigvee_\alpha P_\alpha \land E = \bigvee_\alpha (P_\alpha \land E). The rest of the assertions follow similarly. \qed

For any subset \mathcal{A} \subseteq Q, we denote by \mathcal{A}^I the commutant of \mathcal{A} in Q \cite[p. 23]{40}, i.e.,

\[\mathcal{A}^I = \{P \in Q \mid P \perp Q \text{ for all } Q \in \mathcal{A}\}.\]

Then \mathcal{A}^I is a complete subalgebra of Q by Proposition 2.2 and satisfies \mathcal{A}^{III} = \mathcal{A}^I. A sublogic of Q is a subset \mathcal{A} of Q satisfying \mathcal{A} = \mathcal{A}^I. Thus, any sublogic of Q is a complete subalgebra of Q. A sublogic \mathcal{A} is called Boolean iff P \perp Q for all P, Q \in \mathcal{A}.

For any subset \mathcal{A} \subseteq Q, the smallest logic including \mathcal{A} is the logic \mathcal{A}'' called the logic generated by \mathcal{A}. We have \mathcal{A} \subseteq \Gamma \mathcal{A} \subseteq \mathcal{A}'''. Then it is easy to see that a subset \mathcal{A} is a Boolean sublogic, or equivalently a distributive sublogic, if and only if \mathcal{A} =
$A^!! \subseteq A^i$. If $A \subseteq A^i$, the subset $A^!!$ is the smallest Boolean sublogic including $A$. A maximal Boolean sublogic $B$ of $Q$ is characterized by $B^i = B$. By Zorn’s lemma, for every subset $A$ of $Q$ consisting of mutually commuting elements, there is a maximal Boolean sublogic of $Q$ including $A$.

For any logic $Q$, the set $Q^i$ is called the center of $Q$ and denoted by $Z(Q)$. Since $Z(Q) \subseteq Q = Z(Q)^i$, the center of $Q$ is a Boolean sublogic. For any subset $A$ of $Q$, the center of the logic $A^!!$ generated by $A$ is given by $Z(A^!!) = A^i \cap A^!!$.

### 2.2 Commutators

The commutator $\Downarrow(P, Q)$ of two elements $P$ and $Q$ of a logic $Q$ was introduced by Marsden [41] as

$$\Downarrow(P, Q) = (P \land Q) \lor (P \land Q^\perp) \lor (P^\perp \land Q) \lor (P^\perp \land Q^\perp).$$

(3)

This notion was generalized to finite subsets of $Q$ by Bruns & Kalmbach [42] as

$$\Downarrow(F) = \bigvee_{\theta:F\to\{id, \perp\}} \bigwedge_{P\in F} P^{\theta(P)}$$

(4)

for any finite subset $F$ of $Q$, where $\{id, \perp\}$ stands for the set consisting of the identity operation $id$ and the orthocomplementation $\perp$. Generalizing the notion of commutator to arbitrary subsets $A$ of $Q$, Takeuti [9] defined the commutator $\Downarrow(A)$ of $A$ by

$$\Downarrow(A) = \bigvee\{E \in A^i \mid P \land E \Downarrow Q \land E \text{ for all } P, Q \in A\}$$

(5)

for any subset $A$ of $Q$, which is consistent with Eq. (4) if $A$ is a finite subset [9, Proposition 4]. By Takeuti’s definition it is not clear whether the commutator $\Downarrow(A)$ is determined inside the logic $A^!!$ generated by $A$ or not, unlike the definition of $\Downarrow(F)$ for finite subsets $F$. To resolve this problem, we have shown the relation

$$\Downarrow(A) = \max\{E \in A^i \cap A^!! \mid P \land E \Downarrow Q \land E \text{ for all } P, Q \in A\}.$$

(6)

for any subset $A$ of $Q$ [30, Theorem 2.2]. From the above, we conclude $\Downarrow(A) \in A^i \cap A^!!$. Since every central element $E$ in a logic $R$ leads to the direct product decomposition $R = [0, E] \times [0, E^\perp]$ [40, Theorem 1.1], the above result leads to the following theorem [30, Theorem 2.4].

**Theorem 2.3 (Decomposition Theorem).** Let $A$ be a subset of a logic $Q$. Then, the sublogic $A^!!$ generated by $A$ is isomorphic to the direct product of the complete Boolean algebra $[0, \Downarrow(A)]_{A^!!}$ and the complete orthomodular lattice $[0, \Downarrow(A^\perp)]_{A^!!}$ without non-trivial Boolean factor.

We refer the reader to Pulmannová [43] and Chevalier [44] for further results about commutators in orthomodular lattices.
2.3 Logics on Hilbert spaces

Let $\mathcal{H}$ be a Hilbert space. For any subset $S \subseteq \mathcal{H}$, we denote by $S^\perp$ the orthogonal complement of $S$. Then, $S^\perp\perp$ is the closed linear span of $S$. Let $\mathcal{C}(\mathcal{H})$ be the set of all closed linear subspaces in $\mathcal{H}$. With the set inclusion ordering, the set $\mathcal{C}(\mathcal{H})$ is a complete lattice. The operation $M \mapsto M^\perp$ is an orthocomplementation on the lattice $\mathcal{C}(\mathcal{H})$, with which $\mathcal{C}(\mathcal{H})$ is a logic.

Denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$ and $\mathcal{Q}(\mathcal{H})$ the set of projections on $\mathcal{H}$. We define the operator ordering on $\mathcal{B}(\mathcal{H})$ by $A \leq B$ iff $(\psi, A\psi) \leq (\psi, B\psi)$ for all $\psi \in \mathcal{H}$. For any $A \in \mathcal{B}(\mathcal{H})$, denote by $\mathcal{R}(A) \in \mathcal{C}(\mathcal{H})$ the closure of the range of $A$, i.e., $\mathcal{R}(A) = (A\mathcal{H})^\perp\perp$. For any $M \in \mathcal{C}(\mathcal{H})$, denote by $\mathcal{P}(M) \in \mathcal{Q}(\mathcal{H})$ the projection operator of $\mathcal{H}$ onto $M$. Then, $\mathcal{R}\mathcal{P}(M) = M$ for all $M \in \mathcal{C}(\mathcal{H})$ and $\mathcal{P}\mathcal{R}(P) = P$ for all $P \in \mathcal{Q}(\mathcal{H})$, and we have $P \leq Q$ if and only if $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$ for all $P, Q \in \mathcal{Q}(\mathcal{H})$, so that $\mathcal{Q}(\mathcal{H})$ with the operator ordering is also a logic isomorphic to $\mathcal{C}(\mathcal{H})$. Any sublogic of $\mathcal{Q}(\mathcal{H})$ will be called a logic on $\mathcal{H}$. For any $P, Q \in \mathcal{Q}(\mathcal{H})$, we have $P \downarrow Q$ iff $PQ = QP$.

For any $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{A}'$ the commutant of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, i.e.,

$$\mathcal{A}' = \{ X \in \mathcal{B}(\mathcal{H}) | XA = AX \text{ for all } A \in \mathcal{A} \}.$$

A self-adjoint subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ is called a von Neumann algebra on $\mathcal{H}$ iff $\mathcal{M}'' = \mathcal{M}$. For any self-adjoint subset $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, $\mathcal{A}''$ is the von Neumann algebra generated by $\mathcal{A}$. We denote by $\mathcal{Q}(\mathcal{M})$ the set of projections in a von Neumann algebra $\mathcal{M}$. Then, a subset $\mathcal{Q} \subseteq \mathcal{Q}(\mathcal{H})$ is a logic on $\mathcal{H}$ if and only if $\mathcal{Q} = \mathcal{Q}(\mathcal{M})$ for some von Neumann algebra $\mathcal{M}$ on $\mathcal{H}$ [11, Proposition 2.1]. In this case, we have $\mathcal{Q} = \mathcal{Q}'' = \mathcal{Q}(\mathcal{Q}'')$.

3 Quantization of Logical Operations

3.1 Local operations

Let $\mathcal{Q}$ be a logic. A binary operation $f : \mathcal{Q}^2 \rightarrow \mathcal{Q}$ is said to be local iff the following conditions are satisfied.

(L1) $f(P, Q) \in \{ P, Q \}''$ for all $P, Q \in \mathcal{Q}$.

(L2) $f(P, Q) \wedge E = f(P \wedge E, Q \wedge E) \wedge E$ if $P, Q \downarrow E$ for all $P, Q, E \in \mathcal{Q}$.

Note that by property (L1) every sublogic of a logic $\mathcal{Q}$ is invariant under any local binary operation on $\mathcal{Q}$. The following theorem is useful for later discussions.

**Theorem 3.1.** Let $f$ be a local binary operation on a logic $\mathcal{Q}$. Let $P_\alpha, Q_\alpha, E \in \mathcal{Q}$ and suppose $P_\alpha, Q_\alpha \downarrow E$. Then the following relations hold.

(i) $\left( \bigwedge_\alpha f(P_\alpha, Q_\alpha) \right) \wedge E = \left( \bigwedge_\alpha f(P_\alpha \wedge E, Q_\alpha \wedge E) \right) \wedge E$.

(ii) $\left( \bigvee_\alpha f(P_\alpha, Q_\alpha) \right) \wedge E = \left( \bigvee_\alpha f(P_\alpha \wedge E, Q_\alpha \wedge E) \right) \wedge E$. 

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Proof. By assumption we have \( \{P_\alpha, Q_\alpha\}^\Pi \subseteq \{E\}^1 \). It follows from (L1) that \( f(P_\alpha, Q_\alpha) \in \{P_\alpha, Q_\alpha\}^\Pi \) so that \( f(P_\alpha, Q_\alpha) \downarrow E \). From Proposition \( \ref{prop:p-q} \) and (L2) we have

\[
\left( \bigvee_{\alpha} f(P_\alpha, Q_\alpha) \right) \wedge E = \bigvee_{\alpha} (f(P_\alpha, Q_\alpha) \wedge E) = \bigvee_{\alpha} (f(P_\alpha \wedge E, Q_\alpha \wedge E) \wedge E).
\]

Since \( P_\alpha, Q_\alpha \in \{E\}^1 \), we have \( P_\alpha \wedge E, Q_\alpha \wedge E \in \{E\}^1 \), and hence \( \{P_\alpha \wedge E, Q_\alpha \wedge E\}^\Pi \subseteq \{E\}^1 \). By (L1) we have \( f(P_\alpha \wedge E, Q_\alpha \wedge E) \in \{P_\alpha \wedge E, Q_\alpha \wedge E\}^\Pi \subseteq \{E\}^1 \), so that \( f(P_\alpha \wedge E, Q_\alpha \wedge E) \downarrow E \). From Proposition \( \ref{prop:p-q} \) and (L2) we have

\[
\bigvee_{\alpha} (f(P_\alpha \wedge E, Q_\alpha \wedge E) \wedge E) = \left( \bigvee_{\alpha} f(P_\alpha \wedge E, Q_\alpha \wedge E) \right) \wedge E.
\]

Thus, relation (ii) follows. Relation (i) follows similarly.

The following theorem provides an important properties of ortholattice-polynomials [12 Proposition 3.1].

**Theorem 3.2.** Every two-variable ortholattice-polynomial on a logic \( Q \) is a local binary operation.

### 3.2 Quantizations of classical connectives

In this section we introduce a new method for studying the properties of ortholattice polynomials as a simple application of the Decomposition Theorem.

Let \( P, Q \in Q \). By Theorem \( \ref{thm:p-q} \) the sublogic \( \{P, Q\}^\Pi \) generated by \( P, Q \) is factored into the complete Boolean algebra \( [0, \perp(P, Q)]_{\{P,Q\}^\Pi} \) and the complete orthomodular lattice \( [0, \perp(P, Q)^\perp]_{\{P,Q\}^\Pi} \) without non-trivial Boolean factor, where

\[
\perp(P, Q)^\perp = (P \lor Q) \land (P \lor Q^\perp) \land (P^\perp \lor Q) \land (P^\perp \lor Q^\perp),
\]

from Eq. \( \ref{eq:perp} \). For any \( X \in \{P, Q\}^\Pi \), define \( X_B \) and \( X_N \) by

\[
X_B = X \land \perp(P, Q),
\]

\[
X_N = X \land \perp(P, Q)^\perp.
\]

Then, any \( X \in \{P, Q\}^\Pi \) is uniquely decomposed as \( X = X_B \lor X_N \) with the condition that \( X_B \leq \perp(P, Q) \) and \( X_N \leq \perp(P, Q)^\perp \). By Eq. \( \ref{eq:1} \) and Eq. \( \ref{eq:7} \), we have \( P^\sigma \land Q^\tau \leq \perp(P, Q) \) and \( \perp(P, Q)^\perp \leq P^\sigma \lor Q^\tau \), where \( \sigma, \tau \in \{\text{id}, \perp\} \). Thus, we have

\[
(P^\sigma \land Q^\tau)_B = P^\sigma \land Q^\tau,
\]

\[
(P^\sigma \land Q^\tau)_N = 0,
\]

\[
(P^\sigma \lor Q^\tau)_B = \bigvee_{\sigma', \tau': \sigma' \neq \sigma \land \tau' \neq \tau} (P^{\sigma'} \land Q^{\tau'}),
\]

\[
(P^\sigma \lor Q^\tau)_N = \perp(P, Q)^\perp.
\]
A logic \( Q \) is said to be \textit{totally noncommutative} iff \( \parallel(P, Q) = 0 \), and \textit{extremely noncommutative} iff
\[
\bigvee \{ \parallel(P, Q) \mid Q \not\in \{ P, P^\perp \text{ and } P, Q \in Q \setminus \{0, 1\} \} = 0.
\]

\textbf{Proposition 3.3.} A logic \( Q \) is extremely noncommutative if and only if \( P \land Q = 0 \) for any \( P, Q \in Q \setminus \{1\} \) with \( P \neq Q \).

\textit{Proof.} Suppose \( Q \) is extremely noncommutative. Let \( P, Q \in Q \setminus \{1\} \) with \( P \neq Q \). If \( P = 0, Q = 0 \), or \( P = Q^\perp \), then \( P \land Q = 0 \), and otherwise \( \parallel(P, Q) = 0 \) by assumption, so that \( P \land Q \leq \parallel(P, Q) = 0 \). Conversely suppose that \( P \land Q = 0 \) for any \( P, Q \in \{1\} \) with \( P \neq Q \). Suppose \( 0 < P, Q < 1, Q \not\in \{1, P^\perp\} \). Then, \( P \neq Q, P \neq Q^\perp \neq Q \), so that \( P \land Q = P \land Q^\perp = P^\perp \land Q = P^\perp \land Q^\perp = 0 \), and hence \( \parallel(P, Q) = 0 \). Thus, \( Q \) is extremely noncommutative. \( \square \)

Two examples of extremely noncommutative logic are in order: (i) The modular lattice \( \text{MO2} = \{0, P, P^\perp, Q, Q^\perp, 1\} \) called the Chinese Lantern [40, p. 16]. (ii) The projection lattice \( Q(\mathbb{C}^2) \) of the 2-dimensional Hilbert space \( \mathbb{C}^2 \).

We obtain the following characterization of the two-variable ortholattice-polynomials on a logic, originally obtained by Kotas [39], as a straightforward consequence of the Decomposition Theorem (Theorem 2.3).

\textbf{Theorem 3.4.} Two-variable ortholattice-polynomials \( p(P, Q) \) in \( P, Q \) over a logic \( Q \) have the following form.

\[
p(P, Q) = (P \land Q \land \alpha) \lor (P \land Q^\perp \land \beta) \lor (P^\perp \land Q \land \gamma) \lor (P \land Q^\perp \land \delta) \lor (\epsilon \land \parallel(P, Q)^\perp),
\]

where \( \alpha, \beta, \gamma, \delta \in \{0, 1\} \) and \( \epsilon \in \{0, P, P^\perp, Q, Q^\perp, 1\} \). They define all the 16 Boolean operations

\[
p(P, Q) = (P \land Q \land \alpha) \lor (P \land Q^\perp \land \beta) \lor (P^\perp \land Q \land \gamma) \lor (P \land Q^\perp \land \delta)
\]

for \( \alpha, \beta, \gamma, \delta \in \{0, 1\} \) on \( Q \) if \( Q \) is Boolean, i.e., \( \parallel(Q) = 1 \). They define exactly 6 different monomials

\[
p(P, Q) = \epsilon,
\]

for \( \epsilon \in \{0, P, P^\perp, Q, Q^\perp, 1\} \) on \( Q \) if \( Q \) is extremely noncommutative. They define 96 different operations on \( Q \) if \( Q \) is not Boolean nor extremely noncommutative.

\textit{Proof.} Let \( p(P, Q) \) be an ortholattice-polynomial in \( P, Q \). Since \( p(P, Q) \in \{P, Q\}^n \), we have \( p(P, Q) = p(P, Q)_B \lor p(P, Q)_N \). Let \( q(P, Q) \) be the the disjunctive normal form of \( p(P, Q) \). Then, \( p(P, Q) \land \parallel(P, Q) = q(P, Q) \land \parallel(P, Q) \) by the distributive law and De Morgan’s Laws for the Boolean algebra \([0, \parallel(P, Q)]_{\{P, Q\}^n} \), and \( q(P, Q) \land \parallel(P, Q) = q(P, Q) \) by Eq. (3). Thus, we have \( p(P, Q)_B = q(P, Q) \). By De Morgan’s Laws, we can assume that \( p(P, Q) \) is a lattice polynomial in \( P, P^\perp, Q, Q^\perp \) without any loss of generality. Then, it follows from Eq. (11) and Eq. (13) that \( p(P, Q)_N = \epsilon \land \parallel(P, Q)^\perp \), where \( \epsilon \in \{0, P, P^\perp, Q, Q^\perp, 1\} \). Thus, Eq. (14) follows.
If $Q$ is Boolean, we have $p(P, Q) = q(P, Q)$ for all $P, Q \in Q$, so that $p(P, Q)$ defines at most 16 Boolean operations on $Q$. In every $Q$ the Boolean subalgebra $\{0, 1\}$ is invariant under any polynomials $p(P, Q)$, which define 16 different operations on $\{0, 1\}$. Thus, the polynomials $p(P, Q)$ defines all the 16 Boolean operations on $Q$ if $Q$ is Boolean.

Suppose that $Q$ is extremely noncommutative. Let $P, Q \in Q$. If $P, Q \in \{0, 1\}$ or $Q \in \{P, P^\perp\}$, the value of $p(P, Q)$ is constant or dependent only on $P$ or $Q$. Suppose $0 < P, Q < 1$ and $Q \not\in \{P, P^\perp\}$. Then, we have $\sqcap(P, Q) = 0$. Hence, $p(P, Q) = \epsilon \land \sqcap(P, Q)^\perp$ with $\epsilon \in \{0, P, P^\perp, Q, Q^\perp, 1\}$ defines at most 6 monomials on $Q$. In this case, the set $\{0, P, P^\perp, Q, Q^\perp, 1\}$ must have 6 different elements, otherwise we would have $\sqcap(P, Q) = 1$. Thus, $p(P, Q)$ defines exactly 6 operations on $Q$.

Suppose that $Q$ is not Boolean nor extremely noncommutative. In this case, there exists a pair $P, Q \in Q$ such that $P \not\in \{Q, Q^\perp\}$ and $0 < E = \sqcap(P, Q) < 1$; in fact, since $Q$ is not extremely noncommutative, if $P \not\in \{Q, Q^\perp\}$ then $0 < \sqcap(P, Q)$, and since $Q$ is not Boolean there exists a pair $P, Q \in Q$ such that $P \not\in \{Q, Q^\perp\}$ and $0 < \sqcap(P, Q) < 1$. By the Decomposition Theorem, in this case, $R = \{P, Q\}^{\perp \perp}$ is the direct product of the Boolean algebra $R_B$ isomorphic to $[0, \sqcap(Q)]$ and the complete orthomodular lattice $R_N$ isomorphic to $[0, \sqcap(Q)^\perp]$ such that $\sqcap(R_N)^\perp = 1_R$, where $1_R$ is the unit of $R$. According to the arguments already given above, $p(P, Q)$ defines 16 different operations on $B$ and 16 different operations on $R$. Therefore, $p(P, Q)$ defines 96 ($= 16 \times 6$) operations on $Q$.

A local binary operation $f(P, Q)$ on $Q$ is called a quantization of a Boolean polynomial $b(P, Q)$ iff $f(P, Q)_B = b_n(P, Q)$ for all $P, Q \in Q$, where $b_n(P, Q)$ is the disjunctive normal form of $b(P, Q)$, and moreover $f(P, Q)$ is called a polynomial quantization of $b(P, Q)$, iff $f(P, Q)$ is polynomially definable.

The following theorem holds.

**Proposition 3.5.** Let $f(P, Q)$ be a local binary operation on a logic $Q$ and $b(P, Q)$ a Boolean polynomial. The following statements are mutually equivalent.

(i) $f(P, Q)$ is a quantization of $b(P, Q)$.

(ii) If $P \perp Q$ then $f(P, Q) = b(P, Q)$ for all $P, Q \in Q$.

**Proof.** (i)$\Rightarrow$(ii): Let $f(P, Q)$ be a quantization of $b(P, Q)$, i.e., $f(P, Q)_B = b_n(P, Q)$. Suppose $P \perp Q$. Then $P, Q \in R^{\perp \perp}$, so that $f(P, Q) = f(P, Q)_B = b_n(P, Q) = b(P, Q)$. Thus, the assertion follows.

(ii)$\Rightarrow$(i): Let $E = \sqcap(P, Q)$. Since $P, Q \perp E$, from property (L2) we have

$$f(P, Q)_B = f(P, Q) \land E = f(P \land E, Q \land E) \land E.$$ 

Since $P \land E \perp Q \land E$, we have $f(P \land E, Q \land E) = b(P \land E, Q \land E)$ by assumption. Since the sublogic $\{P \land E, Q \land E\}^{\perp \perp}$ generated by $P \land E$ and $Q \land E$ is a Boolean sublogic, in which $b(P \land E, Q \land E)$ equals its disjunctive normal form $b_n(P \land E, Q \land E)$, i.e., $b(P \land E, Q \land E) = b_n(P \land E, Q \land E)$. Thus, we have

$$f(P \land E, Q \land E) \land E = b_n(P \land E, Q \land E) \land E.$$
From Theorem 3.2 we have $b_n(P \wedge E, Q \wedge E) \wedge E = b_n(P, Q) \wedge E$. Since $b_n(P, Q) \leq \perp(P, Q)$ by Eq. (3), we have $b_n(P, Q) \wedge E = b_n(P, Q)$. Thus, we have

$$f(P, Q)_B = b_n(P, Q)$$

and assertion (i) follows. \qed

It follows from Theorem 3.4 that for each two-variable Boolean-polynomials $b(P, Q)$ there are exactly 6 polynomial quantizations $p(P, Q)$ of $b(P, Q)$, which satisfies

$$p(P, Q) = b_n(P, Q) \lor (\epsilon \wedge \perp(P, Q) \perp),$$

where $b_n(P, Q)$ is the disjunctive normal form of $b(P, Q)$ and $\epsilon \in \{0, P, P \perp, Q, Q \perp, 1\}$.

### 3.3 Quantizations of implication

In classical logic, the implication connective $\to$ is defined by negation $\perp$ and disjunction $\lor$ as $P \to Q = P \perp \lor Q$. In quantum logic, several counterparts have been proposed. Hardegree [37] proposed the following requirements, as “minimal implicative conditions”, for the implication connective $\to$.

(LB) If $P \perp Q$, then $P \to Q = P \perp \lor Q$ for all $P, Q \in Q$.

(E) $P \to Q = 1$ if and only if $P \leq Q$ for all $P, Q \in Q$.

(MP) (modus ponens) $P \wedge (P \to Q) \leq Q$ for all $P, Q \in Q$.

(MT) (modus tollens) $Q \perp \wedge (P \to Q) \leq P \perp$ for all $P, Q \in Q$.

(NG) $P \wedge Q \perp \leq (P \to Q) \perp$ for all $P, Q \in Q$.

A local binary operation $\to$ on a logic $Q$ is called a quantized implication iff it is a quantization of classical implication $b(P, Q) = P \perp \lor Q$, or equivalently it satisfies (LB) by Proposition 3.5. A quantized implication $P \to Q$ on $Q$ is called a polynomially quantized implication or said to be polynomially definable iff there exists a two-variable ortholattice-polynomial $p(P, Q)$ in $P, Q$ such that $p(P, Q) = P \to Q$ for all $P, Q \in Q$. The Kotas theorem (Theorem 3.4) concludes.

**Theorem 3.6.** There exist exactly 6 two-variable ortholattice-polynomials $P \to_j Q$ for $j = 0, \ldots, 5$ satisfying (LB), given as follows.

(0) $P \to_0 Q = b_n(P, Q)$.

(1) $P \to_1 Q = b_n(P, Q) \lor (P \wedge \perp(P, Q) \perp)$.

(2) $P \to_2 Q = b_n(P, Q) \lor (Q \wedge \perp(P, Q) \perp)$.

(3) $P \to_3 Q = b_n(P, Q) \lor (P \perp \wedge \perp(P, Q) \perp)$.

(4) $P \to_4 Q = b_n(P, Q) \lor (Q \perp \wedge \perp(P, Q) \perp)$.
(5) \( P \rightarrow_5 Q = b_n(P, Q) \lor \bot(P, Q) \). 

In the above, \( b_n(P, Q) \) is the disjunctive normal form of \( b(P, Q) = P \bot \lor Q \), i.e.
\[ b_n(P, Q) = (P \bot \land Q \bot) \lor (P \bot \land Q) \lor (P \land Q). \]

For \( j = 0, \ldots, 5 \), the above polynomials \( P \rightarrow_j Q \) are explicitly expressed as follows.

(0) \( P \rightarrow_0 Q = (P \bot \land Q \bot) \lor (P \bot \land Q) \lor (P \land Q). \)
(1) \( P \rightarrow_1 Q = (P \bot \land Q \bot) \lor (P \bot \land Q) \lor (P \land (P \bot \lor Q)). \)
(2) \( P \rightarrow_2 Q = (P \bot \land Q \bot) \lor Q. \)
(3) \( P \rightarrow_3 Q = P \bot \lor (P \land Q). \)
(4) \( P \rightarrow_4 Q = ((P \bot \lor Q) \land Q \bot) \lor (P \bot \land Q) \lor (P \land Q). \)
(5) \( P \rightarrow_5 Q = P \bot \lor Q. \)

The following characterizations of quantized implications hold \[12, \text{Proposition 3.2}. \]

**Proposition 3.7.** Let \( \rightarrow \) be a local binary operation on a logic \( Q \). Then the following conditions are equivalent.

(i) \( \rightarrow \) is a quantized implication, i.e., it satisfies (LB).

(ii) \( (P \rightarrow Q)_B = P \rightarrow_0 Q \) for all \( P, Q \in Q \).

(iii) \( (P \rightarrow Q) \lor \bot(P, Q) = P \rightarrow_5 Q \) for all \( P, Q \in Q \).

(iv) \( P \rightarrow_0 Q \leq P \rightarrow Q \leq P \rightarrow_5 Q \) for all \( P, Q \in Q \).

Note that every quantized implication \( \rightarrow \) satisfies that \( P \rightarrow Q = 1 \) if \( P \leq Q \), since if \( P \leq Q \) then \( P \bot Q \), so that \( P \rightarrow Q = P \bot \lor Q \geq P \bot \lor P = 1 \).

In classical logic, condition (E) uniquely determines \( \rightarrow = P \bot \lor Q \) up to Boolean equivalence. In quantum logic (E) implies (LB), whereas \( P \rightarrow_5 Q = P \bot \lor Q \) satisfies (LB) but does not satisfy (E), shown as follows.

**Theorem 3.8.** A two-variable ortholattice-polynomial \( P \rightarrow Q \) satisfies (E) if and only if it satisfies (LB) and \( (P \rightarrow Q)_N \in \{ 0_N, P_N, P_N \bot, Q_N, Q_N \bot \} \).

**Proof.** (only if part): Suppose that \( P \rightarrow Q \) satisfies (E). Suppose \( P \bot Q \). Then, \( \{ P, Q \} \) is a Boolean algebra. By the truth table argument, (E) implies \( P \rightarrow Q = P \bot \lor Q \). Thus, (LB) holds. From Theorem 3.4, for general \( P, Q \in Q \) we have
\[ P \rightarrow Q = (P \land Q) \lor (P \bot \land Q) \lor (Q \bot \land P) \lor (P \bot \land Q) \lor (P \land Q), \]

where \( \epsilon \in \{ 0, P, P \bot, Q, Q \bot, 1 \} \). Suppose \( \epsilon = 1 \), i.e., \( (P \rightarrow Q)_N = \bot(P, Q) \). In \( Q=MO2 \), for instance, there exist \( P, Q \in Q \) such that \( \bot(P, Q) = 0 \), for which \( P \rightarrow Q = 1 \).
holds but \( P \leq Q \) does not hold. This contradicts (E). Thus, (E) implies \((P \rightarrow Q)_N \in \{0, P_N, P_N^\perp, Q_N, Q_N^\perp\}\).

(if part): Conversely, suppose that \( \rightarrow \) satisfies (LB) and \((P \rightarrow Q)_N \in \{0, P_N, P_N^\perp, Q_N, Q_N^\perp\}\). If \( P \leq Q \), then \( P \perp Q \) and \( P \rightarrow Q = P^\perp \lor Q = 1 \), so that \( P \leq Q \) implies \( P \rightarrow Q = 1 \). Thus, it suffices to show that \( P \rightarrow Q = 1 \) entails \( P \leq Q \).

Suppose \( P \rightarrow Q = 1 \). Then \((P \rightarrow Q)_B = \uplus(P, Q)\) and \((P \rightarrow Q)_N = \uplus(P, Q)^\perp\).

Since \((P \rightarrow Q)_B = (P\perp \lor Q)_B\), it follows from \((P \rightarrow Q)_B = \uplus(P, Q)\) that \( P_B \leq Q_B \). Thus, it suffices to show that if either \((P \rightarrow Q)_N = 0\), \( P_N = P_N^\perp\), \( = Q_N\), or \( = Q_N^\perp\), the relation \((P \rightarrow Q)_N = \uplus(P, Q)^\perp\) entails \((P \rightarrow Q)_N = 0\). If \((P \rightarrow Q)_N = 0\), this is obvious. Suppose \((P \rightarrow Q)_N = P_N\). Since \((P \rightarrow Q)_N = \uplus(P, Q)^\perp\), we have \( P \land \uplus(P, Q)^\perp = \uplus(P, Q)^\perp\), and hence \( Q \land \uplus(P, Q)^\perp = Q \land P \land \uplus(P, Q)^\perp = 0 \) and \( Q^\perp \land \uplus(P, Q)^\perp = Q^\perp \land P \land \uplus(P, Q)^\perp = 0 \), so that \( \uplus(P, Q)^\perp = [Q \land \uplus(P, Q)^\perp] \lor [Q^\perp \land \uplus(P, Q)^\perp] = 0 \). Thus, if \((P \rightarrow Q)_N = P_N\) then \( \uplus(P, Q) = 1 \). Similarly, either \((P \rightarrow Q)_N = P_N^\perp\), \( (P \rightarrow Q)_N = Q_N\), or \((P \rightarrow Q)_N = Q_N^\perp\) implies \( \uplus(P, Q) = 1 \). It follows that \( P \rightarrow Q = 1 \) implies \( P \leq Q \).

Therefore, \((P \rightarrow Q)_B = (P \perp \lor Q)_B\) and \((P \rightarrow Q)_N \in \{0, P_N, P_N^\perp, Q_N, Q_N^\perp\}\) implies (E).

Up to our knowledge only an exhaustive proof has been known for the following fact [40] Theorem 15.3.

**Corollary 3.9.** There are exactly 5 two-variable ortholattice-polynomials \( P \rightarrow_j Q \) with \( j = 0, \ldots, 4 \) that satisfy (E), and yet \( P \rightarrow_5 Q = P^\perp \lor Q \) does not satisfy (E).

**Proof.** Among all the two-variable ortholattice-polynomials \( P \rightarrow_j Q \) with \( j = 0, \ldots, 5 \) that satisfy (LB), the condition \( (P \rightarrow_j Q)_N \in \{0, P_N, P_N^\perp, Q_N, Q_N^\perp\}\) is satisfied exactly by \( P \rightarrow_j Q \) with \( j = 0, \ldots, 4 \). Thus, Theorem 3.8 concludes that there are exactly 5 two-variable ortholattice-polynomials \( P \rightarrow_j Q \) with \( j = 0, \ldots, 4 \) that satisfy (E), but that \( P \rightarrow_5 Q = P^\perp \lor Q \) does not satisfy (E).

Quantized implications satisfying (MP), (MT), and (NG) are characterized, respectively, as follows.

**Proposition 3.10.** Let \( \rightarrow \) be a quantized implication on a logic \( Q \). Then the following statements hold.

(i) \( \rightarrow \) satisfies (MP) if and only if \( P \land (P \rightarrow Q)_N = 0 \) for all \( P, Q \in Q \).

(ii) \( \rightarrow \) satisfies (MT) if and only if \( Q^\perp \land (P \rightarrow Q)_N = 0 \) for all \( P, Q \in Q \).

(iii) \( \rightarrow \) always satisfies (NG).

**Proof.** (i) Suppose that (MP) holds. Then, we have \( P \land (P \rightarrow Q) \leq P \land Q \) and hence

\[
P \land (P \rightarrow Q)_N = P \land (P \rightarrow Q) \land \uplus(P, Q)^\perp \leq P \land Q \land \uplus(P, Q)^\perp = 0.
\]

Thus, \( P \land (P \rightarrow Q)_N = 0 \). Conversely, suppose \( P \land (P \rightarrow Q)_N = 0 \). Then we have

\[
P \land (P \rightarrow Q) = (P_B \land (P \rightarrow Q)_B) \lor (P_N \land (P \rightarrow Q)_N)
\]

\[
= P_B \land (P^\perp \lor Q)_B \leq Q_B \leq Q.
\]
Thus, (MP) holds, and assertion (i) follows.

(ii) Suppose that (MT) holds. Then, we have 
\[ Q \perp \land (P \rightarrow Q) \leq Q \perp \land P \perp, \]
and hence
\[ Q \perp \land (P \rightarrow Q)_N = Q \perp \land (P \rightarrow Q) \land \perp (P, Q) \perp \leq Q \perp \land P \perp \land \perp (P, Q) \perp = 0, \]
Thus, \[ Q \perp \land (P \rightarrow Q)_N = 0. \]
Conversely, suppose \[ Q \perp \land (P \rightarrow Q)_N = 0 \] holds. We have
\[ Q \perp \land (P \rightarrow Q) = (Q \perp \land (P \rightarrow Q))_B \lor (Q \perp \land (P \rightarrow Q))_N \]
\[ = [Q \perp \land (P \lor Q)]_B \leq P \perp \leq P \perp. \]
Thus, (MT) holds, and assertion (ii) follows.

(iii) From Theorem 3.6 we have \( P \rightarrow Q \leq b_n(P, Q) \lor \perp (P, Q) \perp \leq P \perp \lor Q. \)
Taking orthocomplement we conclude assertion (iii). \( \square \)

From the above polynomially quantized implications satisfying (MP), (MT), and (NG) are characterized, respectively, as follows.

**Theorem 3.11.** For any two-variable ortholattice polynomial \( P \rightarrow Q \) satisfying (LB), the following statements hold.

(i) \( P \rightarrow Q \) satisfies (MP) if and only if \( (P \rightarrow Q)_N \in \{0_N, P_N^\perp, Q_N, Q_N^\perp\} \).

(ii) \( P \rightarrow Q \) satisfies (MT) if and only if \( (P \rightarrow Q)_N \in \{0_N, P_N, P_N^\perp, Q_N\} \).

(iii) \( P \rightarrow Q \) always satisfies (NG).

**Proof.** The assertions easily follow from Proposition 3.10. \( \square \)

Hardegree [37, p. 189] called a two-variable ortholattice-polynomial that satisfies all the minimum implicative conditions, (E), (MP), (MT), and (NG), as a *material implication* and stated that there are exactly three material implications \( \rightarrow_j \) with \( j = 0, 2, 3 \), suggesting only an exhaustive proof. Here, we give an analytic proof for this statement.

**Corollary 3.12.** There are exactly three material implications \( \rightarrow_0, \rightarrow_2, \) and \( \rightarrow_3 \).

**Proof.** It follows from Theorems 3.8 and 3.11 that a polynomially definable operation \( P \rightarrow Q \) satisfies (E), (MP), and (MT) if and only if
\[ P \rightarrow Q = (P \perp \lor Q)_B \lor \epsilon_N \]
for \( \epsilon = \{0, P^\perp, Q\} \). They correspond to \( \rightarrow_0, \rightarrow_2, \) and \( \rightarrow_3 \). \( \square \)

We call \( \rightarrow_0 \) the minimum implication, or relevance implication [45], \( \rightarrow_2 \) the contrapositive Sasaki arrow, \( \rightarrow_3 \) the Sasaki arrow [36, 46], and \( \rightarrow_5 \) the classical implication. So far we have no general agreement on the choice from the above, although the majority view favors the Sasaki arrow [38].
3.4 Quantizations of conjunction

A local binary operation \( * \) on a logic \( Q \) is called a quantized conjunction iff it is a quantization of the classical conjunction \( b(P, Q) = b_n(P, Q) = P \wedge Q \), or equivalently, by Proposition 3.5, the following condition is satisfied.

(GC) If \( P \downarrow Q \) then \( P * Q = P \wedge Q \).

In Boolean logic, implication and conjunction are associated by the relation \( P \wedge Q = (P \to Q\downarrow)\downarrow \), and this relation plays an essential role in the duality between bounded universal quantification \( (\forall x \in u)\phi(x) \) and bounded existential quantification \( (\exists x \in u)\phi(x) \). In quantum logic, the truth value of the bounded universal quantification depends on the choice of implication \( \to \) as

\[
[(\forall x \in u)\phi(x)] = \bigwedge_{x \in \text{dom}(u)} (u(x) \to [\phi(x)]).
\]

In order to maintain the duality, the bounded existential quantification should be defined as

\[
[(\exists x \in u)\phi(x)] = [\neg (\forall x \in u)\neg \phi(x)]
\]

\[
= \left( \bigwedge_{x \in \text{dom}(u)} (u(x) \to [\phi(x)]\downarrow) \right) \downarrow
\]

\[
= \bigvee_{x \in \text{dom}(u)} (u(x) \to [\phi(x)]\downarrow)\downarrow
\]

\[
= \bigvee_{x \in \text{dom}(u)} (u(x) * [\phi(x)]),
\]

where \( * \) is defined by

\[
P * Q = (P \to Q\downarrow)\downarrow
\]

for all \( P, Q \in Q \). We call the operation \( * \) defined in Eq. (30) the dual conjunction of the quantized implication \( \to \).

For any \( j = 0, \ldots, 5 \) denote by \( *_j \) the dual conjunction of the polynomial implication \( \to_j \). Then, we have

0. \( P *_0 Q = (P \wedge Q) \lor \bot(P, Q)\downarrow. \)

1. \( P *_1 Q = (P \wedge Q) \lor (P\downarrow \land \bot(P, Q)\downarrow). \)

2. \( P *_2 Q = (P \wedge Q) \lor (Q \land \bot(P, Q)\downarrow). \)

3. \( P *_3 Q = (P \wedge Q) \lor (P \land \bot(P, Q)\downarrow). \)

4. \( P *_4 Q = (P \wedge Q) \lor (P\downarrow \land \bot(P, Q)\downarrow). \)

5. \( P *_5 Q = P \wedge Q. \)
We call $*_5$ the classical conjunction, and $*_3$ the Sasaki conjunction. If the implication $\rightarrow$ is the classical one, i.e., $P \rightarrow Q = P \rightarrow_5 Q = P^\perp \lor Q$, the dual conjunction $*_5$ is also the classical one, i.e., $P *_5 Q = P \land Q$. However, it is only in this case where the classical conjunction appears, e.g., the dual conjunction of the Sasaki arrow, $P \rightarrow_3 Q = P^\perp \lor (P \land Q)$, turns out to be the so called Sasaki projection, $P *_3 Q = P \land (P^\perp \lor Q)$ [36,46]. Some properties of $*_j$ for $j = 0, \ldots , 5$ were previously studied by D’Hooghe and Pykacz [47].

We have the following.

**Proposition 3.13.** A binary operation $*$ on a logic $Q$ is a quantized conjunction if and only if it is the dual conjunction of a quantized implication $\rightarrow$ on $Q$.

**Proof.** Let $*$ be the dual conjunction of a quantized implication $\rightarrow$ on $Q$. Since $\rightarrow$ is local, we have $P * Q = (P \rightarrow Q^\perp)^\perp \in \{P, Q\}^!!$ by property (L1). By the repeated use of property (L2) we have

$$
(P * Q) \land E = [(P \rightarrow Q^\perp)^\perp \land E] = [(P \rightarrow Q^\perp) \land E]^\perp \land E = \{(P \land E) \rightarrow (Q^\perp \land E)^\perp \land E \} = [(P \land E) \rightarrow (Q^\perp \land E)^\perp] \land E = \{(P \land E) \rightarrow (Q^\perp \land E)^\perp \land E \} = [(P \land E) \rightarrow (Q^\perp \land E)^\perp] \land E = [(P \land E) \rightarrow (Q^\perp \land E)^\perp] \land E = [(P \land E) \rightarrow (Q^\perp \land E)^\perp] \land E
$$

Thus, the operation $*$ is a local binary operation. Property (GC) of $*$ easily follows from property (LB) of $\rightarrow$. To show the converse part, let $*$ be a quantized conjunction. Let $\rightarrow$ be defined by $P \rightarrow Q = (P * Q^\perp)^\perp$ for all $P, Q \in Q$. Then, $P \rightarrow Q = (P * Q^\perp)^\perp \in \{P, Q\}^!!$, so that (L1) holds. We have

$$
(P \rightarrow Q) \land E = [(P * Q^\perp)^\perp \land E] = [(P * Q^\perp) \land E]^\perp \land E = [(P \land E) \rightarrow (Q \land E)^\perp]^\perp \land E = \{(P \land E) \rightarrow (Q \land E)^\perp \land E \} = [(P \land E) \rightarrow (Q \land E)^\perp] \land E = [(P \land E) \rightarrow (Q \land E)^\perp] \land E = [(P \land E) \rightarrow (Q \land E)^\perp] \land E,$$

and hence (L2) holds. Thus, $\rightarrow$ is a quantized implication. Since $(P \rightarrow Q^\perp)^\perp = (P * Q^\perp)^\perp \land = P * Q$, the operation $*$ is the dual conjunction of a quantized implication $\rightarrow$. This completes the proof.

We obtain the following characterizations of quantized conjunctions.
Proposition 3.14. Let * be a local binary operation on a logic Q. Then the following conditions are equivalent.

(i) * is a quantized conjunction, i.e., it satisfies (GC).

(ii) \((P \ast Q)_B = P \land Q\) for all \(P, Q \in Q\).

(iii) \((P \ast Q) \lor \perp(P, Q) = (P^\perp \lor Q) \land (P \lor Q) \land (P \lor Q)\) for all \(P, Q \in Q\).

(iv) \(P \land Q \leq P \ast Q \leq (P^\perp \lor Q) \land (P \lor Q) \land (P \lor Q)\) for all \(P, Q \in Q\).

In particular, a quantized conjunction * satisfies

\[ P \land Q \leq P \ast Q \leq P \lor Q. \tag{19} \]

Proof. Since every quantized conjunction is the dual conjunction of a quantized implication by Proposition 3.1, the assertion can be derived from Proposition 3.7 by duality; note that conditions (ii) and (iii) are the dual of conditions (iii) and (ii), respectively, in Proposition 3.7. Here, we alternatively give a direct proof.

(i) \(\Rightarrow\) (ii): Suppose (GC) is satisfied. Let \(P, Q \in Q\). Since \(P_B \lor Q_B\), we have \(P_B \ast Q_B = P_B \land Q_B\), and \((P_B \land Q_B) \land \perp(P, Q) = (P \land Q) \land \perp(P, Q) = P \land Q\).

Thus, from (L2) we have

\[ (P \ast Q) \land \perp(P, Q) = (P_B \ast Q_B) \land \perp(P, Q) = P \land Q, \]

and hence (i) \(\Rightarrow\) (ii) follows.

(ii) \(\Rightarrow\) (iii): Suppose (ii) holds. Note that \((P \ast Q) \lor \perp(P, Q) = (P \ast Q)_B \lor \perp(P, Q)\)^\perp. By taking the join with \(\perp(P, Q)\)^\perp in both sides of relation (ii), we have \((P \ast Q)_B \lor \perp(P, Q) = (P \land Q) \lor \perp(P, Q)\). Since \((P \land Q) \lor \perp(P, Q) = (P^\perp \lor Q) \land (P \lor Q) \land (P \lor Q)\) by calculation, we obtain (iii), and the implication (ii) \(\Rightarrow\) (iii) follows.

(iii) \(\Rightarrow\) (iv): Suppose (iii) holds. Then, \(P \ast Q \leq \perp(P^\perp \lor Q) \land (P \lor Q) \land (P \lor Q)\).

By taking the meet with \(\perp(P, Q)\) in both sides of (iii), we have \((P \ast Q) \land \perp(P, Q) = (P \land Q) \land \perp(P, Q)\). Since \((P \land Q) \land \perp(P, Q) = P \land Q\), we have \(P \land Q \leq P \ast Q\).

Thus, the implication (iii) \(\Rightarrow\) (iv) follows.

(iv) \(\Rightarrow\) (i): Suppose (iv) holds. If \(P \perp Q\), we have \(P \land Q \leq P \ast Q \leq P \land Q\), so that \(P \ast Q = P \land Q\). Thus, the implication (iv) \(\Rightarrow\) (i) follows, and the proof is completed.

Eq. (19) follows from the relation \((P^\perp \lor Q) \land (P \lor Q \land (P \lor Q) \leq P \lor Q\).

The following proposition collects useful relations.

Proposition 3.15. Let Q be a logic with a quantized implication \(\to\) and a quantized conjunction *, and let \(P, Q, P_\alpha, Q_\alpha, E \in Q\). If \(P, Q, P_\alpha, Q_\alpha, \perp E\), then we have the following relations.

(i) \(P^\perp \land E = (P \land E)^\perp \land E\).

(ii) \((P \land Q) \land E = [(P \land E) \land (Q \land E)]\).

(iii) \((P \lor Q) \land E = [(P \land E) \lor (Q \land E)]\).
(iv) \((P \rightarrow Q) \land E = [(P \land E) \rightarrow (Q \land E)] \land E\).

(v) \((\bigwedge_{\alpha} (P_\alpha \rightarrow Q_\alpha)) \land E = \bigwedge_{\alpha} ((P_\alpha \land E) \rightarrow (Q_\alpha \land E)) \land E\).

(vi) \((\bigvee_{\alpha} (P_\alpha \ast Q_\alpha)) \land E = \bigvee_{\alpha} ((P_\alpha \land E) \ast (Q_\alpha \land E))\).

Proof. (i): The relation follows from focusing on \(E\) (cf. Proposition 2.1).

(ii): The relation follows from associativity.

(iii): The relation follows from focusing on \(E\) (cf. Proposition 2.1).

(iv): The relation follows from locality of \(\rightarrow\).

(v): The relation follows from locality of \(\rightarrow\) with Theorem 3.1 (i).

(vi): The relation follows from locality of \(\ast\) with Theorem 3.1 (ii) and the relation \([(P_\alpha \land E) \ast (Q_\alpha \land E)] \leq E\) obtained from Eq. (19).

\(\square\)

3.5 Polynomiably indefinable operations

Takeuti \([9]\) first introduced a polynomiably indefinable binary operation in quantum logic, for which he wrote:

We believe that we have to study this type of new operation in order to see the whole picture of quantum set theory including its strange aspects. \([9, \text{p. 303}]\)

In fact, Takeuti \([9]\) introduced a binary operation \(\circ_\theta\) on the logic \(Q(\mathcal{H})\) of projections on a Hilbert space \(\mathcal{H}\) by

\[P \circ_\theta Q = Q + (e^{i\theta} - 1)PQ + (e^{-i\theta} - 1)QP + 2(1 - \cos \theta)PQP\]

for all \(P, Q \in Q(\mathcal{H})\). It is easily seen that

\[P \circ_\theta Q = e^{i\theta P}Qe^{-i\theta P}\]

for all \(P, Q \in Q(\mathcal{H})\). If \(P \perp Q\), then \(P \circ_\theta Q = Q\). The binary operation \(f(P, Q) = P \circ_\theta Q\) is local, i.e., (L1) and (L2) hold. However, it is not in general definable as an ortholattice polynomial, since \(f(P, Q)\) is not generally in \(\Gamma\{P, Q\}\) \([12, \text{Proposition 4.2}]\).

Examples of polynomiably indefinable quantized implications \(\rightarrow\), which even satisfy (MP), were derived from Takeuti’s polynomiably indefinable operation \(\circ_\theta\) \([12]\). Those operations \(\rightarrow\) satisfy (L1), i.e., \(P \rightarrow Q \in \{P, Q\}!\), but do not satisfy the condition \(P \rightarrow Q \in \Gamma_0\{P, Q\}\), which all the polynomial implications satisfy; see §4 in Ref. \([12]\) for an extensive account on polynomiably indefinable quantized implications.

Examples of polynomiably indefinable quantized conjunctions \(\ast\) are given in the following. For \(j = 0, \ldots, 5\), for a real parameter \(\theta \in [0, 2\pi)\), and for \(i = 0, 1\), we define new binary operations \(*_{j, \theta, i}\) on \(Q = Q(\mathcal{H})\) by

\[
P *_{j, \theta, 0} Q = P *_{j} (P \circ_\theta Q),
\]

\[
P *_{j, \theta, 1} Q = (Q^\perp \circ_\theta P) *_{j} Q
\]

for all \(P, Q \in Q\). Obviously, \(*_{j, 0, i} = *_{j}\) for \(j = 0, \ldots, 5\) and \(i = 0, 1\). Then, we obtain the following relations (cf. Proposition 4.1 in Ref. \([12]\)).
(i) \( P_{*0,\theta,0} Q = P_{*0} Q \).

(ii) \( P_{*1,\theta,0} Q = P_{*1} Q \).

(iii) \( P_{*2,\theta,0} Q = (P_{*0} Q) \lor ((P \circ_\theta Q) \land \bot (P, Q) \lor) \).

(iv) \( P_{*3,\theta,0} Q = P_{*3} Q \).

(v) \( P_{*4,\theta,0} Q = (P_{*0} Q) \lor ((P \circ_\theta Q) \lor \bot (P, Q) \lor) \).

(vi) \( P_{*5,\theta,0} Q = P_{*5} Q \).

(vii) \( P_{*0,\theta,1} Q = P_{*0} Q \).

(viii) \( P_{*1,\theta,1} Q = (P_{*0} Q) \lor ((Q \lor_\theta P) \lor \bot (P, Q) \lor) \).

(ix) \( P_{*2,\theta,1} Q = P_{*2} Q \).

(x) \( P_{*3,\theta,1} Q = (P_{*0} Q) \lor ((Q \lor_\theta P) \lor \bot (P, Q) \lor) \).

(xi) \( P_{*4,\theta,1} Q = P_{*4} Q \).

(xii) \( P_{*5,\theta,1} Q = P_{*5} Q \).

The following theorem shows the existence of quantized conjunctions that are not polynomially definable.

**Theorem 3.16.** Quantized conjunctions \(*_{1,\theta,1}, *_{2,\theta,0}, *_{3,\theta,1}, and *_{4,\theta,0}\) are not polynomially definable for any \( \theta \in (0, 2\pi) \).

**Proof.** By duality the assertion follows immediate from Proposition 4.2 in Ref. [12]. □

4 Quantum Set Theory

4.1 Orthomodular-valued universe

We denote by \( V \) the universe of the Zermelo-Fraenkel set theory with the axiom of choice (ZFC). Let \( Q \) be a logic. For each ordinal \( \alpha \), let

\[
V_\alpha^Q = \{ u \mid u : \text{dom}(u) \rightarrow Q \text{ and } (\exists \beta < \alpha) \text{dom}(u) \subseteq V_\beta^Q \}.
\]

The \( Q \)-valued universe \( V^Q \) is defined by

\[
V^Q = \bigcup_{\alpha \in \text{On}} V_\alpha^Q,
\]

where On is the class of all ordinals.

In the case where \( Q \) is a Boolean algebra, \( V^Q \) is reduced to the Boolean-valued universe of set theory [5,48].

For every \( u \in V^Q \), the rank of \( u \), denoted by rank(\( u \)), is defined as the least \( \alpha \) such that \( u \in V^Q_{\alpha+1} \). It is easy to see that if \( u \in \text{dom}(v) \) then rank(\( u \)) < rank(\( v \)). An induction on rank argument leads to the following [5, p. 21].
Theorem 4.1 (Induction Principle for \( V(\mathcal{Q}) \)). For any predicate \( \phi(x) \),
\[
\forall u \in V(\mathcal{Q}) [\forall u' \in \text{dom}(u) \phi(u') \rightarrow \phi(u)] \rightarrow \forall u \in V(\mathcal{Q}) \phi(u)
\]

For \( u \in V(\mathcal{Q}) \), we define the support of \( u \), denoted by \( L(u) \), by transfinite recursion on the rank of \( u \) by the relation
\[
L(u) = \bigcup_{x \in \text{dom}(u)} L(x) \cup \{u(x) \mid x \in \text{dom}(u)\} \cup \{0\}.
\]

For \( \mathcal{A} \subseteq V(\mathcal{Q}) \) we write \( L(\mathcal{A}) = \bigcup_{u \in \mathcal{A}} L(u) \) and for \( u_1, \ldots, u_n \in V(\mathcal{Q}) \) we write \( L(u_1, \ldots, u_n) = L(\{u_1, \ldots, u_n\}) \). Then, we obtain the following characterization of subuniverses of \( V(\mathcal{Q}) \).

Proposition 4.2. Let \( \mathcal{R} \) be a sublogic of a logic \( \mathcal{Q} \) and \( \alpha \) an ordinal. For any \( u \in V(\mathcal{Q}) \), we have \( u \in V_\alpha^{(\mathcal{R})} \) if and only if \( u \in V_\alpha^{(\mathcal{Q})} \) and \( L(u) \subseteq \mathcal{R} \). In particular, \( u \in V^{(\mathcal{R})} \) if and only if \( u \in V(\mathcal{Q}) \) and \( L(u) \subseteq \mathcal{R} \). Moreover, for any \( u \in V^{(\mathcal{R})} \) the rank in \( V^{(\mathcal{R})} \) and that in \( V(\mathcal{Q}) \) are the same.

Proof. Immediate from transfinite induction on \( \alpha \). \( \square \)

4.2 Orthomodular-valued interpretations

Let \( \mathcal{L}(\in) \) be the language of first-order theory with equality consisting of the negation symbol \( \neg \), connectives \( \land, \lor, \rightarrow \), binary relation symbols \( =, \in \), bounded quantifier symbols \( \forall x \in y, \exists x \in y \), unbounded quantifier symbols \( \forall x, \exists x \), and no constant symbols. For any class \( U \), the language \( \mathcal{L}(\in, U) \) is the one obtained by adding a name for each element of \( U \).

To each statement \( \phi \) of \( \mathcal{L}(\in, U) \), the satisfaction relation \( \langle U, \in \rangle \models \phi \) is defined by the following recursive rules:

(i) \( \langle U, \in \rangle \models \neg \phi \) iff \( \langle U, \in \rangle \models \phi \) does not hold.

(ii) \( \langle U, \in \rangle \models \phi_1 \land \phi_2 \) iff \( \langle U, \in \rangle \models \phi_1 \) and \( \langle U, \in \rangle \models \phi_2 \).

(iii) \( \langle U, \in \rangle \models \phi_1 \lor \phi_2 \) iff \( \langle U, \in \rangle \models \phi_1 \) or \( \langle U, \in \rangle \models \phi_2 \).

(iv) \( \langle U, \in \rangle \models \phi_1 \rightarrow \phi_2 \) iff if \( \langle U, \in \rangle \models \phi_1 \) then \( \langle U, \in \rangle \models \phi_2 \).

(v) \( \langle U, \in \rangle \models (\forall x \in u) \phi(x) \) iff \( \langle U, \in \rangle \models \phi(u') \) for all \( u' \in u \).

(vi) \( \langle U, \in \rangle \models (\exists x \in u) \phi(x) \) iff there exists \( u' \in u \) such that \( \langle U, \in \rangle \models \phi(u') \).

(vii) \( \langle U, \in \rangle \models (\forall x) \phi(x) \) iff \( \langle U, \in \rangle \models \phi(u) \) for all \( u \in U \).

(viii) \( \langle U, \in \rangle \models (\exists x) \phi(x) \) iff there exists \( u \in U \) such that \( \langle U, \in \rangle \models \phi(u) \).

(ix) \( \langle U, \in \rangle \models u = v \) iff \( u = v \).

(x) \( \langle U, \in \rangle \models u \in v \) iff \( u \in v \).
Our assumption that $V$ satisfies ZFC means that if ZFC $\vdash \phi(x_1, \ldots, x_n)$, then $\langle V, \in, \rangle \models \phi(u_1, \ldots, u_n)$ for any formula $\phi(x_1, \ldots, x_n)$ of $\mathcal{L}(\in)$ provable in ZFC and for any $u_1, \ldots, u_n \in V$.

Denote by $\mathcal{S}(Q)$ the set of statements in $\mathcal{L}(\in, V^Q)$. A $Q$-valued interpretation of $\mathcal{L}(\in, V^Q)$ is a mapping $I(\rightarrow, *): \phi \in \mathcal{S}(Q) \mapsto [\phi]_Q \in Q$ determined with a pair $(\rightarrow, *)$ of local binary operations on $Q$ by the following rules, (R1)–(R10), recursive on the rank of elements of $V^Q$ and the complexity of formulas.

1. $[\neg \phi]_Q = [\phi]_Q$.
2. $[\phi_1 \land \phi_2]_Q = [\phi_1]_Q \land [\phi_2]_Q$.
3. $[\phi_1 \lor \phi_2]_Q = [\phi_1]_Q \lor [\phi_2]_Q$.
4. $[\phi_1 \rightarrow \phi_2]_Q = [\phi_1]_Q \rightarrow [\phi_2]_Q$.
5. $[\forall x \in u \phi(x)]_Q = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow [\phi(u')]_Q)$.
6. $[\exists x \in u \phi(x)]_Q = \bigvee_{u' \in \text{dom}(u)} (u(u') \land \neg [\phi(u')]_Q)$.
7. $[\forall x \in V^Q \phi(u)]_Q = \bigwedge_{u \in V^Q} [\phi(u)]_Q$.
8. $[\exists x \in V^Q \phi(u)]_Q = \bigvee_{u \in V^Q} [\phi(u)]_Q$.
9. $[u = v]_Q = [\forall x \in u (x \in v) \land (\forall x \in v) (x \in u)]_Q$.
10. $[u \in v]_Q = [\exists x \in u (x = u)]_Q$.

The following relations follow from the above rules.

1. $[u = v]_Q = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow [u']_Q) \land [u' \in v]_Q = [\forall v \in u]_Q$.
2. $[u \in v]_Q = \bigvee_{u' \in \text{dom}(u)} (v(u') \rightarrow [v']_Q)$.

For a sublogic $\mathcal{R}$ of a logic $Q$ with a $Q$-valued interpretation $I(\rightarrow, *)$, we denote by $[\phi]_\mathcal{R}$ the $\mathcal{R}$-valued truth value of a statement $\phi \in \mathcal{S}(Q)$ determined by the $\mathcal{R}$-valued interpretation $I(\rightarrow_\mathcal{R}, *_\mathcal{R})$, where $\rightarrow_\mathcal{R}$ and $*_\mathcal{R}$ are the restrictions of $\rightarrow$ and $*$ to $\mathcal{R}$, which are well-defined by the locality of $\rightarrow$ and $*$.

A formula in $\mathcal{L}(\in)$ is called a $\Delta_0$-formula iff it has no unbounded quantifiers $\forall x$ or $\exists x$. The following theorem holds.

**Theorem 4.3** ($\Delta_0$-Absoluteness Principle). Let $\mathcal{R}$ be a sublogic of a logic $Q$ with a $Q$-valued interpretation $I(\rightarrow, *)$ of $\mathcal{L}(\in, V^Q)$. For any $\Delta_0$-formula $\phi(x_1, \ldots, x_n) \in \mathcal{L}(\in)$ and $u_1, \ldots, u_n \in V^\mathcal{R}$, we have

$$[\phi(u_1, \ldots, u_n)]_\mathcal{R} = [\phi(u_1, \ldots, u_n)]_Q.$$
Proof. The assertion is proved by the induction on the complexity of formulas and the
rank of elements of $V^\langle Q \rangle$. Let $u,v \in V^\langle R \rangle$. By induction hypothesis, for any $u' \in \text{dom}(u)$ and $v' \in \text{dom}(v)$ we have $\llbracket u' \in w \rrbracket_R = \llbracket u' \in w \rrbracket_Q$, $\llbracket v' \in w \rrbracket_R = \llbracket v' \in w \rrbracket_Q$,
and $\llbracket v' = w \rrbracket_R = \llbracket v' = w \rrbracket_Q$ for all $w \in V^\langle Q \rangle$. Thus,
\[
\llbracket u = v \rrbracket_R = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow \llbracket u' \in v \rrbracket_R) \wedge \bigwedge_{v' \in \text{dom}(v)} (v(v') \rightarrow \llbracket v' \in u \rrbracket_R)
= \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow \llbracket u' \in v \rrbracket_Q) \wedge \bigwedge_{v' \in \text{dom}(v)} (v(v') \rightarrow \llbracket v' \in u \rrbracket_Q)
= \llbracket u = v \rrbracket_Q,
\]
and we also have
\[
\llbracket u \in v \rrbracket_R = \bigvee_{v' \in \text{dom}(v)} (v(v') \ast \llbracket v' = u \rrbracket_R)
= \bigvee_{v' \in \text{dom}(v)} (v(v') \ast \llbracket v' = u \rrbracket_Q)
= \llbracket u \in v \rrbracket_Q.
\]
Thus, the assertion holds for atomic formulas. Any induction step adding a logical
symbol works easily, even when bounded quantifiers are concerned, since the ranges
of the supremum and the infimum are common for evaluating $\llbracket \cdots \rrbracket_R$ and $\llbracket \cdots \rrbracket_Q$. □

Henceforth, for any $\Delta_0$-formula $\phi(x_1, \ldots, x_n) \in \mathcal{L}(\in)$ and $u_1, \ldots, u_n \in V^\langle Q \rangle$,
we abbreviate $\llbracket \phi(u_1, \ldots, u_n) \rrbracket = \llbracket \phi(u_1, \ldots, u_n) \rrbracket_Q$.

The universe $V$ can be embedded in $V^\langle Q \rangle$ by the following operation $\vee : v \mapsto \check{v}$
defined by the $\in$-recursion: for each $v \in V$, $\check{v} = \{ \check{u} \mid u \in v \} \times \{ 1 \}$. For any $P \in Q$,
define $\check{P} = \{ \langle \check{0}, P \rangle \} \in V^\langle Q \rangle$.

**Proposition 4.4.** In any $Q$-valued interpretation $\mathcal{I}(\rightarrow, \ast)$, the following relations hold.

(i) $\llbracket u \in 0 \rrbracket = 0$ for any $u \in V^\langle Q \rangle$.

(ii) $\llbracket 0 = 0 \rrbracket = 1$.

(iii) $\llbracket 0 = \check{P} \rrbracket = P \rightarrow 0$ for any $P \in Q$.

(iv) $\llbracket 0 \in \check{P} \rrbracket = P \ast 1$ for any $P \in Q$.

**Proof.** Since $\text{dom}(\check{0}) = \emptyset$, relations (i) and (ii) follow from
\[
\llbracket u \in 0 \rrbracket = \bigvee_{v \in \text{dom}(\check{0})} (\check{0}(v) \ast \llbracket v = u \rrbracket) = 0,
\]
\[
\llbracket 0 = 0 \rrbracket = \bigwedge_{u \in \text{dom}(\check{0})} (\check{0}(u) \rightarrow \llbracket u \in 0 \rrbracket) \wedge \bigwedge_{u \in \text{dom}(\check{0})} (\check{0}(u) \rightarrow \llbracket u \in 0 \rrbracket) = 1.
\]
Since \( \text{dom}(\tilde{P}) = \{\tilde{0}\} \), relations (iii) and (iv) follow from

\[
\begin{align*}
[\tilde{0} = \tilde{P}] & = \bigwedge_{u \in \text{dom}(\tilde{0})} (\tilde{0}(u) \rightarrow [u \in \tilde{P}]) \land \bigwedge_{v \in \text{dom}(\tilde{P})} (\tilde{P}(v) \rightarrow [v \in \tilde{0}]) \\
& = 1 \land (\tilde{P}(\tilde{0}) \rightarrow 0) = P \rightarrow 0.
\end{align*}
\]

\[
[\tilde{0} \in \tilde{P}] = \bigvee_{u \in \text{dom}(\tilde{P})} (\tilde{P}(u) \ast [u = \tilde{0}])
\]

\[
= \tilde{P}(\tilde{0}) \ast [\tilde{0} = \tilde{0}]
\]

\[
= P \ast 1.
\]

4.3 Transfer Principle: Necessity

In this section, we investigate the Transfer Principle that gives any \( \Delta_0 \)-formula provable in ZFC a lower bound for its truth value, which is determined by the degree of the commutativity of the elements of \( V(Q) \) appearing in the formula as constants.

Let \( Q \) be a logic. Let \( A \subseteq V(Q) \). The commutator of \( A \), denoted by \( \bigvee(A) \), is defined by

\[
\bigvee(A) = \bigwedge(L(A)).
\]  \hspace{1cm} (23)

For any \( u_1, \ldots, u_n \in V(Q) \), we write \( \bigvee(u_1, \ldots, u_n) = \bigvee(\{u_1, \ldots, u_n\}) \).

Let \( I(\rightarrow, \ast) \) be a \( Q \)-valued interpretation. We denote by \( [\phi] \) the \( Q \)-valued truth value of a statement \( \phi \in S(Q) \) determined by the \( Q \)-valued interpretation \( I(\rightarrow, \ast) \). Then, the Transfer Principle for the \( Q \)-valued interpretation \( I(\rightarrow, \ast) \) is formulated as follows.

**Transfer Principle.** Any \( \Delta_0 \)-formula \( \phi(x_1, \ldots, x_n) \) in \( L(\in) \) provable in ZFC satisfies

\[
[\phi(u_1, \ldots, u_n)] \geq \bigvee(u_1, \ldots, u_n)
\]  \hspace{1cm} (24)

for any \( u_1, \ldots, u_n \in V(Q) \).

A \( Q \)-valued interpretation \( I(\rightarrow, \ast) \) is called the Takeuti interpretation iff \( \rightarrow \) is the Sasaki arrow and \( \ast \) is the classical conjunction, i.e., \( P \rightarrow Q = P \rightarrow_{3} Q = P \perp \vee (P \land Q) \) and \( P \ast Q = P \ast_{5} Q = P \land Q \) for all \( P, Q \in Q \). It was shown that if \( Q \) is the projection lattice of a von Neumann algebra, then the \( Q \)-valued Takeuti interpretation \( I(\rightarrow_{3}, \ast_{5}) \) satisfies the Transfer Principle \[11\]. This result was extended to an arbitrary logic \( Q \) and arbitrary quantized implication \( \rightarrow \) on \( Q \) to show that any \( Q \)-valued interpretation \( I(\rightarrow, \ast_{5}) \) satisfies the Transfer Principle \[12\]. In the present paper we consider the problem to find all the interpretations that satisfy the Transfer Principle.

In order to eliminate uninteresting interpretations from our consideration, we call a \( Q \)-valued interpretation \( I(\rightarrow, \ast) \) non-trivial iff for any \( P \in Q \) there exist a \( \Delta_0 \)-formula \( \phi(x_1, \ldots, x_n) \in L(\in) \) and \( u_1, \ldots, u_n \in V(Q) \) such that \( \bigvee(u_1, \ldots, u_n) = 1 \) and \( [\phi(u_1, \ldots, u_n)] = P \). Simple sufficient conditions for non-triviality are given as follows.
**Proposition 4.5.** If a $Q$-valued interpretation $I(\rightarrow, \ast)$ satisfies

(i) $P \rightarrow 0 = P^\bot$ for all $P \in Q$, or

(ii) $P \ast 1 = P$ for all $P \in Q$.

then $I(\rightarrow, \ast)$ is non-trivial.

**Proof.** Suppose condition (i) holds. Let $\phi(x_1, x_2) := \neg(x_1 = x_2)$, $u_1 = \bar{0}$, and $u_2 = \bar{P}$. Then $\forall(u_1, u_2) = 1$ and $\llbracket \phi(u_1, u_2) \rrbracket = \llbracket 0 = \bar{P} \rrbracket = (P \rightarrow 0)^\bot = P$ from Proposition 4.4 (iii). Thus, the interpretation $I(\rightarrow, \ast)$ is non-trivial. Suppose condition (ii) holds. Let $\phi(x_1, x_2) := (x_1 \in x_2), u_1 = \bar{0}$, and $u_2 = \bar{P}$. Then $\forall(u_1, u_2) = 1$ and $\llbracket \phi(u_1, u_2) \rrbracket = \llbracket 0 \in \bar{P} \rrbracket = P \ast 1 = P$ from Proposition 4.4 (iv). Thus, the interpretation $I(\rightarrow, \ast)$ is non-trivial.

In what follows, we introduce the connective $\leftrightarrow$ in the language $\mathcal{L}(\in V(Q))$ as an abbreviation for $\phi \leftrightarrow \psi := (\phi \land \psi) \lor (\neg \phi \land \neg \psi)$ for any $\phi, \psi \in \mathcal{L}(\in, V(Q))$ and the corresponding operation $\leftrightarrow$ on $Q$ by $P \leftrightarrow Q := (P \land Q) \lor (P^\bot \land Q^\bot)$ for all $P, Q \in Q$.

Then, we have the following theorem showing that in order for a non-trivial $Q$-valued interpretation $I(\rightarrow, \ast)$ to satisfy the Transfer Principle it is necessary that $\rightarrow$ satisfies (LB) and that $\ast$ satisfies (GC).

**Theorem 4.6.** If a non-trivial $Q$-valued interpretation $I(\rightarrow, \ast)$ of $\mathcal{L}(\in, V(Q))$ satisfies the Transfer Principle, then the operation $\rightarrow$ is a quantized implication and the operation $\ast$ is a quantized conjunction.

**Proof.** Let $P \in Q$. By assumption, there exist a $\Delta_0$-formula $\phi(x_1, \ldots, x_n) \in \mathcal{L}(\in)$ and $u_1, \ldots, u_n \in V(Q)$ such that $\forall(u_1, \ldots, u_n) = 1$ and $\llbracket \phi(u_1, \ldots, u_n) \rrbracket = P$. Since

$$\llbracket \phi(x_1, \ldots, x_n) \rightarrow \neg(x_{n+1} = x_{n+1}) \rrbracket \leftrightarrow \neg\phi(x_1, \ldots, x_n)$$

is provable in ZFC, by the Transfer Principle we have

$$\llbracket [\phi(u_1, \ldots, u_n) \rightarrow \neg(0 = \bar{0})] \leftrightarrow \neg\phi(u_1, \ldots, u_n) \rrbracket \geq \forall(u_1, \ldots, u_n, \bar{0}).$$

Since $\text{dom}(\bar{0}) = \emptyset$, we have $\forall(u_1, \ldots, u_n, \bar{0}) = \forall(u_1, \ldots, u_n) = 1$, so that we obtain

$$\llbracket \phi(u_1, \ldots, u_n) \rrbracket \rightarrow \llbracket \bar{0} = \bar{0} \rrbracket \downarrow = \llbracket \phi(u_1, \ldots, u_n) \rrbracket \downarrow.$$

Since $\llbracket \bar{0} = \bar{0} \rrbracket = 1$ by Proposition 4.4 (ii), we have $P \rightarrow 0 = P^\bot$ for all $P \in Q$. Recall $\bar{P} = \{\langle \bar{0}, P \rangle\} \in V(Q)$. Since $\forall(\bar{P}, \bar{0}) = 1$, from the Transfer Principle we obtain

$$\llbracket (\exists x \in \bar{P})(x = \bar{0}) \leftrightarrow \neg(\forall x \in \bar{P})\neg(x = \bar{0}) \rrbracket = 1.$$
Since $\text{dom}(\tilde{P}) = \{\tilde{0}\}$, we obtain
\[
[\tilde{0} \in \tilde{P}] = [(\exists x \in \tilde{P})(x = \tilde{0})]
= [\neg(\forall x \in \tilde{P})(x = \tilde{0})]
= \left( \bigwedge_{u' \in \text{dom}(\tilde{P})} (\tilde{P}(u') \rightarrow [u' = \tilde{0}]^\perp) \right)^\perp
= (\tilde{P}(\tilde{0}) \rightarrow [\tilde{0} = \tilde{0}]^\perp)^\perp
= (P \rightarrow 0)^\perp.
\]

Since $P \rightarrow 0 = P^\perp$, we have $[\tilde{0} \in \tilde{P}] = P$. Let $\varphi(x_1, x_2, x_3)$ be the $\Delta_0$-formula in $\mathcal{L}(\in)$ such that
\[
\varphi(x_1, x_2, x_3) := (x_1 \in x_2 \rightarrow x_1 \in x_3) \iff (\neg(x_1 \in x_2) \lor (x_1 \in x_3)).
\]
Then $\text{ZFC} \vdash \varphi(x_1, x_2, x_3)$. Let $P, Q \in \mathcal{Q}$ with $P \perp P$. We have $\bigvee(\tilde{0}, \tilde{P}, \tilde{Q}) = \perp(P, Q) = 1$. By the Transfer Principle, we have $[[\varphi(\tilde{0}, \tilde{P}, \tilde{Q})]] \geq \bigvee(\tilde{0}, \tilde{P}, \tilde{Q}) = 1$. Thus, we have
\[
[\tilde{0} \in \tilde{P}] \rightarrow [\tilde{0} \in \tilde{Q}] = [\tilde{0} \in \tilde{P}]^\perp \lor [[\tilde{0} \in \tilde{Q}]],
\]
and hence we conclude
\[
P \rightarrow Q = P^\perp \lor Q.
\]
Since $P, Q \in \mathcal{Q}$ are arbitrary elements with $P \perp Q$, the operation $\rightarrow$ satisfies (LB), and hence it is a quantized implication.

By the definition of the interpretation $I(\rightarrow, \ast)$, we have relation (A2), so that
\[
[\tilde{P} \in \tilde{Q}] = \bigvee_{v \in \text{dom}(\tilde{Q})} (\tilde{Q}(v) \ast \tilde{P})]
= \tilde{Q}(\tilde{0}) \ast [\tilde{0} = \tilde{P}]
= Q \ast P^\perp.
\]

On the other hand, since
\[
\phi(x_1, x_2) := x_1 \in x_2 \iff \neg(\forall y_1 \in x_2) \neg(y_1 = x_1)
\]
is a $\Delta_0$-formula provable in ZFC and $\bigvee(\tilde{P}, \tilde{Q}) = 1$, by the Transfer Principle we have $[[\phi(\tilde{P}, \tilde{Q})]] = 1$, so that
\[
[\tilde{P} \in \tilde{Q}] = [\neg(\forall v \in \tilde{Q}) \neg(v = \tilde{P})]
= \left( \bigwedge_{v \in \text{dom}(\tilde{Q})} (\tilde{Q}(v) \rightarrow [v = \tilde{P}]^\perp) \right)^\perp
= (\tilde{Q}(\tilde{0}) \rightarrow [\tilde{0} = \tilde{P}]^\perp)^\perp
= (Q \rightarrow P)^\perp.
\]
Since \( P \downarrow Q \) and \( \to \) satisfies (LB), we have \((Q \to P)^\perp = Q \land P^\perp\). Thus, \([\hat{P} \in \hat{Q}] = Q \land P^\perp\). It follows that \( Q \ast P^\perp = Q \land P^\perp \). Since \( P, Q \in \mathcal{Q} \) were arbitrary pair of commuting elements, condition (GC) holds for the operation \( \ast \). Thus, \( \ast \) is a quantized conjunction.

A \( \mathcal{Q} \)-valued interpretation \( \mathcal{I}(\to, \ast) \) is called normal if \( \to \) is a quantized implication and \( \ast \) is a quantized conjunction. It is easy to see that all normal interpretations are non-trivial. It follows from Theorem 4.6 that all the non-trivial \( \mathcal{Q} \)-valued interpretations satisfying the Transfer Principle are normal.

4.4 Transfer Principle: Sufficiency

In what follows, suppose that for any \( \phi \in \mathcal{L}(\in, V^{(\mathcal{Q})}) \) the truth value \([\phi] \in \mathcal{Q}\) is assigned by a fixed but arbitrary normal \( \mathcal{Q} \)-valued interpretation \( \mathcal{I}(\to, \ast) \). In this section, we shall prove that all the normal interpretations admit the Transfer Principle.

The following theorem is known as the fundamental theorem of Boolean-valued models of set theory.

**Theorem 4.7.** If \( \mathcal{Q} \) is a Boolean logic, all the normal interpretations define the unique \( \mathcal{Q} \)-valued interpretation and satisfies the following statements.

(i) \([\exists x \in u] \phi(x) = \bigvee_{u' \in \text{dom}(u)} (u'(u') \land [\phi(u')])\) for every formula \( \phi(x) \) in \( \mathcal{L}(\in, V^{(\mathcal{Q})}) \) with one free variable \( x \) and \( u \in V^{(\mathcal{Q})} \).

(ii) \([\forall x \in u] \phi(x) = [\bigwedge_{x \in u} \neg \phi(x)]\) for every formula \( \phi(x) \) in \( \mathcal{L}(\in, V^{(\mathcal{Q})}) \) with one free variable \( x \) and \( u \in V^{(\mathcal{Q})} \).

(iii) \([\phi] = 1\) for any statement in \( \mathcal{L}(\in, V^{(\mathcal{Q})}) \) provable in ZFC.

**Proof.** Let \( \mathcal{Q} \) be a Boolean logic. Let \( \mathcal{I}(\to, \ast) \) be a normal \( \mathcal{Q} \)-valued interpretation of \( \mathcal{L}(\in, V^{(\mathcal{Q})}) \). By normality we have

\[
[\exists x \in u] \phi(x) = \bigvee_{u' \in \text{dom}(u)} (u'(u') \land [\phi(u')]).
\]

Then, statement (i) follows from the relation

\[
[\exists x \in u] \phi(x) = \bigvee_{u' \in \text{dom}(u)} (u'(u') \land [\phi(u')])
\]

well-known for Boolean-valued models [48, Theorem 13.13], [5, Corollary 1.18]. Statement (ii) follows from (i) by duality between \( \to \) and \( \ast \) on Boolean algebras. Thus, the interpretation is uniquely determined by the part of the language without bounded quantifiers. Then, statement (iii) follows from the fundamental theorem of Boolean-valued models [48 Theorems 13.12 and 14.25], [5, Theorem 1.33].

Denote by \( \mathbf{2} \) the sublogic \( \mathbf{2} = \{0, 1\} \) in any logic \( \mathcal{Q} \). We have the following.
Theorem 4.8 ($\Delta_0$-Elementary Equivalence Principle). Let $\phi(x_1, \ldots, x_n)$ be a $\Delta_0$-formula of $\mathcal{L}(\in)$. For any $u_1, \ldots, u_n \in V$, we have
\[ \langle V, \in \rangle \models \phi(u_1, \ldots, u_n) \text{ if and only if } [\phi(\bar{u}_1, \ldots, \bar{u}_n)] = 1. \]

Proof. By induction it is easy to see that $\langle V, \in \rangle \models \phi(u_1, \ldots, u_n)$ if and only if $[\phi(\bar{u}_1, \ldots, \bar{u}_n)] = 1$ for any $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}(\in)$, and this is equivalent to $[\phi(\bar{u}_1, \ldots, \bar{u}_n)] = 1$ for any $\Delta_0$-formula $\phi(x_1, \ldots, x_n)$ by the $\Delta_0$-absoluteness principle.

The following proposition is useful in applications.

Proposition 4.9. If $\text{dom}(u) \subseteq \text{dom}(\bar{X})$ for some $X \in V$, then $[x \in u] = u(x)$ for any $x \in \text{dom}(u)$ in any normal $\mathcal{Q}$-valued interpretation $\mathcal{I}(\rightarrow, *)$.

Proof. Let $x \in \text{dom}(u)$. Since $\text{dom}(u) \subseteq \text{dom}(\bar{X})$, there is some $x' \in X$ such that $x = x'$. We have
\[ [x \in u] = \bigvee_{u' \in \text{dom}(u)} u(u') * [u' = x] \]
\[ = \bigvee_{u' \in u} u(\bar{u}') * [\bar{u}' = \bar{x}'] \]
\[ = \bigvee_{u' \in u} u(\bar{u}') \land [\bar{u}' = \bar{x}'] \]
\[ = \bigvee_{u' = x'} u(\bar{u}') \]
\[ = u(\bar{x}') \]
\[ = u(x). \]

Thus, the assertion follows.

Let $\mathcal{A} \subseteq V(\mathcal{Q})$. The logic generated by $\mathcal{A}$, denoted by $\mathcal{Q}(\mathcal{A})$, is defined by
\[ \mathcal{Q}(\mathcal{A}) = L(\mathcal{A})''. \] (25)

For $u_1, \ldots, u_n \in V(\mathcal{Q})$, we write $\mathcal{Q}(u_1, \ldots, u_n) = \mathcal{Q}(\{u_1, \ldots, u_n\})$.

The following theorem shows that the Transfer Principle partially holds if $\bigvee(u_1, \ldots, u_n) = 1$.

Theorem 4.10. For any $u_1, \ldots, u_n \in V(\mathcal{Q})$ with $\bigvee(u_1, \ldots, u_n) = 1$, every $\Delta_0$-formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}(\in)$ provable in ZFC holds with the truth value 1, i.e.,
\[ [\phi(u_1, \ldots, u_n)] = 1. \]

Proof. Since $\bigvee(u_1, \ldots, u_n) = 1$, $\mathcal{Q}(u_1, \ldots, u_n)$ is a Boolean algebra. Let $\mathcal{B} = \mathcal{Q}(u_1, \ldots, u_n)$. Apply Theorem 4.7 (iii) to the $\mathcal{Q}$-valued interpretation $\mathcal{I}(\rightarrow, *)$ restricted to $V(\mathcal{B})$, we have $[\phi(u_1, \ldots, u_n)] = 1$. By the $\Delta_0$-Absoluteness Principle we have
\[ [\phi(u_1, \ldots, u_n)] = [\phi(u_1, \ldots, u_n)] = 1, \]
and the proof is completed.
Let \( u \in V^\mathcal{Q} \) and \( p \in \mathcal{Q} \). The restriction \( u|_p \) of \( u \) to \( p \) is defined by the following transfinite recursion:

\[
u|_p = \{ \langle x|_p, u(x) \land p \rangle \mid x \in \text{dom}(u) \} \cup \{ \langle u, 0 \rangle \}.
\]

The last term \( \{ \langle u, 0 \rangle \} \) has no essential role but ensures that the function \( u|_p : \text{dom}(u|_p) \to \mathcal{Q} \) is well-defined, i.e., if \( u|_p = v|_p \) then \( u = v \) and \( u(x) \land p = v(x) \land p \) for all \( x \in \text{dom}(u) = \text{dom}(v) \). Note that our definition of restriction is simpler than the corresponding notion given by Takeuti [9]. We shall develop the theory of restriction along with a different line.

**Proposition 4.11.** For any \( \mathcal{A} \subseteq V^\mathcal{Q} \) and \( p \in \mathcal{Q} \), we have

\[L(\{u|_p \mid u \in \mathcal{A}\}) = L(\mathcal{A}) \land p.\]  

**Proof.** By induction, it is easy to see the relation \( L(u|_p) = L(u) \land p \), so that the assertion follows easily. \( \square \)

**Proposition 4.12.** For any \( \Delta_0 \)-formula \( \phi(x_1, \ldots, x_n) \) in \( L(\in) \) and \( u_1, \cdots, u_n \in V^\mathcal{Q} \), the following statements hold.

(i) \( \langle \phi(u_1, \ldots, u_n) \rangle \in \mathcal{Q}(u_1, \ldots, u_n) \).

(ii) If \( p \in L(u_1, \ldots, u_n)^1 \), then \( p \downarrow [\phi(u_1, \ldots, u_n)] \) and \( p \downarrow [\phi(u_1|_p, \ldots, u_n|_p)] \).

**Proof.** (i): Let \( \mathcal{A} = \{ u_1, \ldots, u_n \} \). Since \( L(\mathcal{A}) \subseteq \mathcal{Q}(\mathcal{A}) \), it follows from Proposition 4.2 that \( u_1, \ldots, u_n \in V^\mathcal{Q}(\mathcal{A}) \). By the \( \Delta_0 \)-absoluteness principle, we have \([\phi(u_1, \ldots, u_n)] = [\phi(u_1, \ldots, u_n)]_{\mathcal{Q}(\mathcal{A})} = \mathcal{Q}(\mathcal{A}) \).

(ii) Let \( u_1, \ldots, u_n \in V^\mathcal{Q} \). If \( p \in L(u_1, \ldots, u_n)^1 \), then \( p \in \mathcal{Q}(u_1, \ldots, u_n)^1 \). From (i), \([\phi(u_1, \ldots, u_n)] \in \mathcal{Q}(u_1, \ldots, u_n) \), so that \( p \downarrow [\phi(u_1, \ldots, u_n)] \). From Proposition 4.11 \( L(u_1|_p, \ldots, u_n|_p) = L(u_1, \ldots, u_n) \land p \), and hence \( p \in L(u_1|_p, \ldots, u_n|_p)^1 \), so that \( p \downarrow [\phi(u_1|_p, \ldots, u_n|_p)] \). \( \square \)

We define the binary relation \( x_1 \subseteq x_2 \) by \( \forall x \in x_1( x \in x_2 ) \). Then, by definition for any \( u, v \in V^\mathcal{Q} \) we have

\[[u \subseteq v] = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow [u' \in v]),\]  

and we have \([u = v] = [u \subseteq v] \land [v \subseteq u] \).

**Proposition 4.13.** For any \( u, v \in V^\mathcal{Q} \) and \( p \in L(u, v)^1 \), the following relations hold.

(i) \([u|_p \subseteq v|_p] \land p = [u \subseteq v] \land p \).

(ii) \([u|_p \subseteq v|_p] \land p = [u \subseteq v] \land p \).

(iii) \([u|_p = v|_p] \land p = [u = v] \land p \).
Proof. We prove the relations by induction on the rank of \( u, v \). If \( \text{rank}(u) = \text{rank}(v) = 0 \), then \( \text{dom}(u) = \text{dom}(v) = \emptyset \), so that the relations trivially hold. Let \( u, v \in V'(Q) \) and \( p \in L(u, v) \). To prove (i), let \( v' \in \text{dom}(v) \). Then, we have \( p \upharpoonright v(v') \) by the assumption on \( p \). By induction hypothesis, we have also \( [u_p = v'_p] \wedge p = [v' = u] \wedge p \). By Proposition 4.12(ii), we have \( p \upharpoonright [v' = u] \), so that \( v(v'), [v' = u] \in \{p\} \), and hence \( v(v') \ast [v' = u] \in \{p\} \) by locality. From Proposition 3.15(vii) we have

\[
[u_p \in v_p] \wedge p = \bigg( \bigvee_{v' \in \text{dom}(v)} (v_p(v') \ast [u_p = v']) \bigg) \wedge p
\]

Thus, by induction hypothesis and Proposition 3.15(vii) we have

\[
[u_p \in v_p] \wedge p = \bigg( \bigvee_{v' \in \text{dom}(v)} (v(v') \ast [v' = u]) \bigg) \wedge p
\]

\[
= \bigg( \bigvee_{v' \in \text{dom}(v)} (v(v') \ast [v' = u]) \bigg) \wedge p
\]

Thus, relation (i) has been proved. To prove (ii), let \( u' \in \text{dom}(u) \). Then, we have \( [u'_p \in v_p] \wedge p = [u' \in v] \wedge p \) by induction hypothesis. Thus, by Proposition 3.15(v) we have

\[
[u_p \subseteq v_p] \wedge p = \bigg( \bigwedge_{u' \in \text{dom}(u)} (u_p(u') \rightarrow [u' \in v_p]) \bigg) \wedge p
\]

\[
= \bigg( \bigwedge_{u' \in \text{dom}(u)} (u_p(u'_p) \rightarrow [u'_p \in v_p]) \bigg) \wedge p
\]

\[
= \bigg( \bigwedge_{u' \in \text{dom}(u)} ([u_p(u') \wedge p] \rightarrow ([u'_p \in v_p] \wedge p)) \bigg) \wedge p
\]

\[
= \bigg( \bigwedge_{u' \in \text{dom}(u)} ([u_p(u') \wedge p] \rightarrow ([u' \in v] \wedge p)) \bigg) \wedge p.
\]

We have \( p \upharpoonright u(u') \) by assumption on \( p \), and \( p \upharpoonright [u' \in v] \) by Proposition 4.12(ii), so that \( p \upharpoonright u(u') \rightarrow [u' \in v] \) and \( p \upharpoonright (u(u') \wedge p) \rightarrow ([u' \in v] \wedge p) \). From Proposition
We have the following theorem.

**Theorem 4.14 (Δ₀-Restriction Principle).** For any Δ₀-formula φ(x₁, ..., xₙ) in L(∈) and u₁, ..., uₙ ∈ V(Q), if p ∈ L(u₁, ..., uₙ)¹, then

\[ [φ(u₁, ..., uₙ)] ∧ p = [φ(u₁|p, ..., uₙ|p)] ∧ p. \]

**Proof.** We shall write \( \vec{u} = (u₁, ..., uₙ) \) and \( \vec{u}|p = (u₁|p, ..., uₙ|p) \). We prove the assertion by induction on the complexity of \( φ(x₁, ..., xₙ) \). From Proposition 4.13, the assertion holds for atomic formulas. Thus, it suffices to consider the following induction steps:

1. \( φ \Rightarrow ¬φ \),
2. \( φ₁, φ₂ \Rightarrow φ₁ ∧ φ₂ \),
3. \( φ₁, φ₂ \Rightarrow φ₁ ∨ φ₂ \),
4. \( φ₁, φ₂ \Rightarrow φ₁ → φ₂ \),
5. \( \{φ(u') | u' ∈ \text{dom}(u)\} \Rightarrow (∀x ∈ u)φ(x) \),
6. \( \{φ(u') | u' ∈ \text{dom}(u)\} \Rightarrow (∃x ∈ u)φ(x) \).

(i): Let \( p ∈ L(\vec{u})¹ \). Suppose \[ [φ(\vec{u})] ∧ p = [φ(\vec{u}|p)] ∧ p. \] From Proposition 3.15 (i) we have

\[ [φ(\vec{u})] ⊥ ∧ p = ([φ(\vec{u})] ∧ p) ⊥ ∧ p = ([φ(\vec{u}|p)] ∧ p) ⊥ ∧ p = [φ(\vec{u}|p)] ⊥ ∧ p, \]
so that we have
\[ [-\phi(\vec{u})] \land p = [-\phi(\vec{u}_j)] \land p. \]

(ii)–(iii): Let \( p \in L(\vec{u})^1 \). Suppose \([\phi_j(\vec{u})] \land p = [\phi_j(\vec{u}_j)] \land p \) for \( j = 1, 2 \). Then, from Proposition 3.15 (ii)–(iii), we have
\[ [\phi_1(\vec{u}) \land \phi_2(\vec{u})] \land p = [\phi_1(\vec{u}_j) \land \phi_2(\vec{u}_j)] \land p, \]
\[ [\phi_1(\vec{u}) \lor \phi_2(\vec{u})] \land p = [\phi_1(\vec{u}_j) \lor \phi_2(\vec{u}_j)] \land p. \]

(iv) Let \( p \in L(\vec{u})^1 \). Suppose \([\phi_j(\vec{u})] \land p = [\phi_j(\vec{u}_j)] \land p \) for \( j = 1, 2 \). It follows from Proposition 3.15 (iv) and the induction hypothesis that
\[ [\phi_1(\vec{u}) \rightarrow \phi_2(\vec{u})] \land p = \left([\phi_1(\vec{u}_j) \land p] \rightarrow ([\phi_2(\vec{u}_j)] \land p) \land p \right) \]
\[ = \left([\phi_1(\vec{u}_j) \land p] \rightarrow ([\phi_2(\vec{u}_j)] \land p) \land p \right) \]
\[ = ([\phi_1(\vec{u}_j)] \rightarrow [\phi_2(\vec{u}_j)] \land p) \land p, \]
so that we have
\[ [\phi_1(\vec{u}) \rightarrow \phi_2(\vec{u})] \land p = [\phi_1(\vec{u}_j) \rightarrow \phi_2(\vec{u}_j)] \land p. \]

(v)–(vi): Suppose \([\phi_j(u)] \land p = [\phi_j(u|u_j)] \land p \) for \( j = 1, 2 \) for any \( u \in V(Q) \) and \( p \in L(u)^1 \). Suppose \( u \in V(Q) \) and \( p \in L(u)^1 \). Let \( u' \in \text{dom}(u) \). Since \( L(u') \subseteq L(u) \), we have \( p \in L(u')^1 \). It follows that
\[ [\phi_j(u')] \land p = [\phi_j(u'|u')] \land p \quad \text{and} \quad p \downarrow [\phi(u')], [\phi(u'|u')], \]
for all \( u' \in \text{dom}(u) \). Thus, from Proposition 3.15 (v) we have
\[ [\forall x \in u] \phi(x) \land p = \left( \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow [\phi(u')]) \right) \land p \]
\[ = \bigwedge_{u' \in \text{dom}(u)} [(u(u') \rightarrow [\phi(u')]) \land p] \]
\[ = \bigwedge_{u' \in \text{dom}(u)} \{[(u(u') \land p) \rightarrow ([\phi(u')] \land p)] \land p \} \]
\[ = \bigwedge_{u' \in \text{dom}(u|u)} \{[u|u(u') \land p \rightarrow ([\phi(u')] \land p)] \land p \} \]
\[ = \bigwedge_{u' \in \text{dom}(u|u)} \{[u|u(u') \rightarrow ([\phi(u')] \land p)] \land p \} \]
\[ = \left( \bigwedge_{u' \in \text{dom}(u|u)} (u|u(u') \rightarrow [\phi(u')]) \right) \land p. \]

It follows that
\[ [\forall x \in u] \phi(x) \land p = [\forall x \in u|u] \phi(x) \land p. \]

The relation
\[ [\exists x \in u] \phi(x) \land p = [\exists x \in u|u] \phi(x) \land p \]
follows similarly from Proposition 3.15 (vi).
Now, we obtain the following theorem showing that for any $Q$-valued interpretation $I(\rightarrow, \ast)$ to satisfy the Transfer Principle it suffices for $\rightarrow$ and $\ast$ to satisfy $(LB)$ and $(GC)$, respectively.

**Theorem 4.15** (Transfer Principle). *Any normal interpretation of $L(\in, V^Q)$ satisfies the Transfer Principle.*

**Proof.** Let $p = \bigvee(u_1, \ldots, u_n)$. Then, we have $a \land p \not\land b \land p$ for any $a, b \in L(u_1, \ldots, u_n)$, and hence there is a Boolean sublogic $B$ such that $L(u_1, \ldots, u_n) \land p \subseteq B$. From Proposition 4.11 we have $L(u_1|p, \ldots, u_n|p) \subseteq B$. It follows that $\bigvee(u_1|p, \ldots, u_n|p) = 1$. By Theorem 4.10 we have $[\phi(u_1|p, \ldots, u_n|p)] = 1$. From Proposition 4.14 we have $[\phi(u_1, \ldots, u_n)] \land p = [\phi(u_1|p, \ldots, u_n|p)] \land p = p$, and the assertion follows. \qed

We call a normal $Q$-valued interpretation $I(\rightarrow, \ast)$ ***polynomially definable*** iff the local operations $\rightarrow$ and $\ast$ are both polynomially definable. The following theorem characterizes non-trivial $Q$-valued interpretations that satisfy the Transfer Principle.

**Theorem 4.16.** *A non-trivial $Q$-valued interpretation $I(\rightarrow, \ast)$ satisfies the Transfer Principle if and only if it is normal. Non-trivial polynomially definable $Q$-valued interpretations $I(\rightarrow, \ast)$ of $L(\in, V^Q)$ satisfying the Transfer Principle are unique if $Q$ is a Boolean algebra, and exactly 36 $Q$-valued interpretations $I(\rightarrow_j, \ast_k)$ for $j, k = 0, \ldots, 5$ if $Q$ is not a Boolean algebra.*

**Proof.** The first statement is an immediate consequence of Theorems 4.6 and 4.15. If $Q$ is a Boolean algebra, normal interpretations are unique by Theorem 4.7. If $Q$ is not a Boolean algebra, there are at most 36 polynomially definable normal $Q$-valued interpretations $I(\rightarrow_j, \ast_k)$ for $j, k = 0, \ldots, 5$. If $Q$ is not extremely noncommutative, there exist a non-commuting pair $P, Q \in Q$ such that the algebra $\Gamma\{P, Q\}$ generated by $P, Q$ is a direct product of a non-trivial Boolean algebra and the six-element Chinese Lantern \[MO2\{0, P_N, Q_N, P_N^\perp, Q_N^\perp, 1_N\}\] [42], where $\perp(P, Q) = \perp = E > 0$, and $P_N = P \land E$, $P_N^\perp = P \land \perp$, $Q_N = Q \land E$, and $Q_N^\perp = Q \land \perp$, on which the polynomially definable quantized implications $\rightarrow_j$ and the polynomially definable quantized conjunctions $\ast_k$...
for $j, k = 0, \ldots, 5$ actually define 36 different interpretations as shown below.

$$[	ilde{P} \subseteq \tilde{Q}]_{j=0} = P \to_0 Q = (P \land Q) \lor (P \land Q) \lor (P \land Q).$$
$$[	ilde{P} \subseteq \tilde{Q}]_{j=1} = P \to_1 Q = (P \to_0 Q) \lor P_N.$$
$$[	ilde{P} \subseteq \tilde{Q}]_{j=2} = P \to_2 Q = (P \to_0 Q) \lor Q_N.$$
$$[	ilde{P} \subseteq \tilde{Q}]_{j=3} = P \to_3 Q = (P \to_0 Q) \lor P_N.$$
$$[	ilde{P} \subseteq \tilde{Q}]_{j=4} = P \to_4 Q = (P \to_0 Q) \lor Q_N.$$
$$[	ilde{P} \subseteq \tilde{Q}]_{j=5} = P \to_5 Q = (P \to_0 Q) \lor 1_N.$$

$$[\tilde{Q}^\perp \in \tilde{P}]_{k=0} = P \ast_0 Q = (P \land Q) \lor 1_N.$$  
$$[\tilde{Q}^\perp \in \tilde{P}]_{k=1} = P \ast_1 Q = (P \land Q) \lor P_N^\perp.$$  
$$[\tilde{Q}^\perp \in \tilde{P}]_{k=2} = P \ast_2 Q = (P \land Q) \lor Q_N.$$  
$$[\tilde{Q}^\perp \in \tilde{P}]_{k=3} = P \ast_3 Q = (P \land Q) \lor P_N.$$  
$$[\tilde{Q}^\perp \in \tilde{P}]_{k=4} = P \ast_4 Q = (P \land Q) \lor Q_N^\perp.$$  
$$[\tilde{Q}^\perp \in \tilde{P}]_{k=5} = P \ast_5 Q = P \land Q.$$

For instance, the interpretation $\mathcal{I}(\rightarrow_3, \ast_5)$ is characterized by the unique relations

$$[	ilde{P} \subseteq \tilde{Q}]_{j=3} = P \to_3 Q \lor P_N^\perp \quad \text{and} \quad [\tilde{Q}^\perp \in \tilde{P}]_{k=5} = P \land Q.$$ 

In the case where $Q$ is extremely noncommutative, any $P, Q \in \mathbb{Q}$ with $0 < P, Q < 1$ generate Chinese Lantern $\text{MO}_2 = \{0, P, P^\perp, Q, Q^\perp, 1\}$ since $\land(P, Q) = 0$. Thus, $\rightarrow_j$ and $\ast_k$ for $j, k = 0, \ldots, 5$ define 36 different interpretations as follows.

$$[	ilde{P} \subseteq \tilde{Q}]_{j=0} = P \to_0 Q = 0.$$  
$$[	ilde{P} \subseteq \tilde{Q}]_{j=1} = P \to_1 Q = P.$$  
$$[	ilde{P} \subseteq \tilde{Q}]_{j=2} = P \to_2 Q = Q.$$  
$$[	ilde{P} \subseteq \tilde{Q}]_{j=3} = P \to_3 Q = P^\perp.$$  
$$[	ilde{P} \subseteq \tilde{Q}]_{j=4} = P \to_4 Q = Q^\perp.$$  
$$[	ilde{P} \subseteq \tilde{Q}]_{j=5} = P \to_5 Q = 1.$$  

$$[\tilde{Q}^\perp \in \tilde{P}]_{k=0} = P \ast_0 Q = 1.$$  
$$[\tilde{Q}^\perp \in \tilde{P}]_{k=1} = P \ast_1 Q = P^\perp.$$  
$$[\tilde{Q}^\perp \in \tilde{P}]_{k=2} = P \ast_2 Q = Q.$$  
$$[\tilde{Q}^\perp \in \tilde{P}]_{k=3} = P \ast_3 Q = P.$$  
$$[\tilde{Q}^\perp \in \tilde{P}]_{k=4} = P \ast_4 Q = Q^\perp.$$  
$$[\tilde{Q}^\perp \in \tilde{P}]_{k=5} = P \ast_5 Q = 0.$$  

Thus, if $Q$ is not Boolean, there exist exactly 36 $Q$-valued interpretations $\mathcal{I}(\rightarrow_j, \ast_k)$ for $j, k = 0, \ldots, 5$ that satisfy the Transfer Principle.  

\[\square\]
As shown in Theorem 4.7, if the logic \( Q \) is a Boolean logic, any formula \( \varphi(x_1, \ldots, x_n) \) in \( L(\in) \) provable in ZFC holds true for any \( u_1, \ldots, u_n \in V(Q) \), i.e.,

\[
[\varphi(u_1, \ldots, u_n)] = 1.
\]

We show that the lower bound 1 is possible only in this case.

**Theorem 4.17.** In any normal \( Q \)-valued interpretation \( \mathcal{I}(\to, \ast) \), if the relation

\[
[\varphi(u_1, \ldots, u_n)] = 1
\]

holds for any \( \Delta_0 \)-formula \( \varphi(x_1, \ldots, x_n) \in L(\in) \) provable in ZFC and \( u_1, \ldots, u_n \in V(Q) \), then \( Q \) is a Boolean logic.

**Proof.** Let \( P, Q \in Q \). Since the formula

\[
z \in x \Leftrightarrow [(z \in x \land z \in y) \lor (z \in x \land \neg(z \in y))]
\]

is provable in ZFC, by assumption we have

\[
[[0 \in \bar{P}] \Leftrightarrow [(0 \in \bar{P} \land 0 \in \bar{Q}) \lor (0 \in \bar{P} \land \neg(0 \in \bar{Q})]] = 1.
\]

Thus, we obtain

\[
\left( [[0 \in \bar{P}] \Leftrightarrow (0 \in \bar{P} \land [[0 \in \bar{Q}]] \lor (0 \in \bar{P} \land [[0 \in \bar{Q}]]^\perp)) \right] = 1.
\]

Therefore, the relation \( P = (P \land Q) \lor (P \land Q^\perp) \) follows, and we conclude \( P \not\models Q \). Since \( P, Q \in Q \) were arbitrary, we conclude that \( Q \) is a Boolean logic.

4.5 De Morgan’s Laws

Every \( Q \)-valued interpretation \( \mathcal{I}(\to, \ast) \) with arbitrary pair \( (\to, \ast) \) of local binary operations satisfies De Morgan’s Laws for conjunction-disjunction connectives and for universal-existential quantifiers simply according to the duality between supremum and infimum as follows.

\begin{align*}
\text{(M1)} \quad & [[\neg(\phi_1 \land \phi_2)]] = [[\neg\phi_1 \lor \neg\phi_2]], \\
\text{(M2)} \quad & [[\neg(\phi_1 \lor \phi_2)]] = [[\neg\phi_1 \land \neg\phi_2]], \\
\text{(M3)} \quad & [[\neg(\forall x \, \phi(x))]] = [[\exists x \, (\neg\phi(x))]], \\
\text{(M4)} \quad & [[\neg(\exists x \, \phi(x))]] = [[\forall x \, (\neg\phi(x))]].
\end{align*}

However, De Morgan’s Laws for bounded quantifiers

\begin{align*}
\text{(M5)} \quad & [[\neg(\forall x \, \phi(x))]] = [[\exists x \, (\neg\phi(x))]], \\
\text{(M6)} \quad & [[\neg(\exists x \, \phi(x))]] = [[\forall x \, (\neg\phi(x))]].
\end{align*}
are not generally satisfied, even for normal interpretations as shown below.

A $Q$-valued interpretation $\mathcal{I}(\to, \ast)$ of $\mathcal{L}(\in, V(Q))$ is called the Takeuti interpretation iff $\to = \to_3$ and $\ast = \ast_5 = \land$. The Takeuti interpretation was introduced by Takeuti [9] for the projection lattice $Q = Q(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, extended to the projection lattice $Q = Q(\mathcal{M})$ of a von Neumann algebra $\mathcal{M}$ [11], and extended to a general complete orthomodular lattice $Q$ [12]. It is only one interpretation for quantum set theory having been studied seriously so far [30–34]. However, the Takeuti interpretation does not satisfy De Morgan’s Laws for bounded quantifications as follows.

**Theorem 4.18.** Let $Q$ be a logic. For the Takeuti interpretation $(Q, \to_3, \ast_5)$, we have the following statements:

(i) The relation

$$\left[ (\exists x \in u) \neg \phi(x) \right] \leq \left[ \neg (\forall x \in u) \phi(x) \right].$$

holds for any formula $\phi(x)$ in $\mathcal{L}(\in, V(Q))$.

(ii) The equality holds in (i) if $u(u')$ and $\left[ \phi(u') \right]$ commute for all $u' \in \text{dom}(u)$.

(iii) If $Q$ is not Boolean, there exists a formula $\phi(x)$ in $\mathcal{L}(\in, V(Q))$ such that

$$\left[ (\exists x \in u) \neg \phi(x) \right] = 0 \text{ but } \left[ \neg (\forall x \in u) \phi(x) \right] > 0.$$

**Proof.** Assertions (i) and (ii) follow from the relations below, where $\ast_3$ denotes the dual conjunction of the Sasaki arrow $\to_3$.

$$\left[ (\exists x \in u) \neg \phi(x) \right] = \bigvee_{u' \in \text{dom}(u)} (u(u') \land [\phi(u')]^\perp).$$

$$\left[ \neg (\forall x \in u) \phi(u') \right] = \left( \bigwedge_{u' \in \text{dom}(u)} (u(u') \to_3 [\phi(u')]^\perp) \right)^\perp$$

$$= \bigvee_{u' \in \text{dom}(u)} (u(u') \ast_3 [\phi(u')]^\perp)$$

$$= \bigvee_{u' \in \text{dom}(u)} [(u(u') \land [\phi(u')]^\perp) \lor (u(u') \land [\phi(u')]^\perp)].$$

To show assertion (iii), suppose that $Q$ is not Boolean. Then, there exists a pair $P_0, Q_0 \in Q$ such that $P_0$ does not commute with $Q_0$, so that $\perp(P_0, Q_0)^+ > 0$. Let $E = \perp(P_0, Q_0)^{-}, P = P_0 \land E$, and $Q = Q_0 \land E$. If $P = 0$ then $P_0 = P_0 \land \perp(P_0, Q_0)$ so that $P_0 \perp Q_0$, a contradiction. Thus, $P \neq 0$. We also have that $P \land Q = P_0 \land Q_0 \land \perp(P_0, Q_0)^+ = 0$, so that $P \land Q = 0$. Recall $\bar{P} = \{\langle 0, P \rangle\}$ and $\bar{Q} = \{\langle 0, Q \rangle\}$. Consider the formula $\phi(x) := \neg(x \in \bar{Q})$. Then, we have

$$\left[ (\exists x \in \bar{P}) \neg \phi(x) \right] = \bigvee_{u' \in \text{dom}(\bar{P})} (\bar{P}(u') \land [\neg \phi(u')]^\perp)$$

$$= \bar{P}(0) \land [0 \in \bar{Q}]$$

$$= P \land Q$$

$$= 0.$$
On the other hand, we have

\[
\lnot \forall x \in \tilde{P} \phi(x) = \left( \forall x \in \tilde{P} \phi(x) \right)^\perp
\]

\[
= \left( \bigwedge_{u \in \text{dom}(\tilde{P})} (\tilde{P}(u') \rightarrow \exists \tilde{Q} \phi(u')) \right)^\perp
\]

\[
= (\tilde{P}(\tilde{0}) \rightarrow \exists \tilde{Q} [\tilde{0} \in \tilde{Q}])^\perp
\]

\[
= \tilde{P}(\tilde{0}) \ast_3 [\tilde{0} \in \tilde{Q}]^\perp
\]

\[
= P \ast_3 Q
\]

\[
= (P \land Q) \lor (P \land \top(P, Q)^\perp)
\]

\[
= P
\]

Therefore, assertion (iii) follows. \(\Box\)

A \(Q\)-valued interpretation \(\mathcal{I}(\rightarrow, \ast)\) of \(L(\in, V(Q))\) is said to be self-dual iff

\[
P \ast Q = (P \rightarrow Q)^\perp
\]

for all \(P, Q \in Q\).

**Theorem 4.19.** A non-trivial \(Q\)-valued interpretation \(\mathcal{I}(\rightarrow, \ast)\) of \(L(\in, V(Q))\) satisfies De Morgan’s Laws if and only if it is self-dual.

**Proof.** Suppose that for any \(\phi \in L(\in, V(Q))\) the truth value \(\llbracket \phi \rrbracket\) is assigned by a \(Q\)-valued interpretation \(\mathcal{I}(\rightarrow, \ast)\). Let \(\phi(x) \in L(\in, V(Q))\). By definition, we have

\[
\llbracket (\exists x \in u) \phi(x) \rrbracket = \bigvee_{x \in \text{dom}(u)} (u(x) \ast \llbracket \phi(x) \rrbracket),
\]

(28)

and

\[
\llbracket \lnot (\forall x \in u) \lnot \phi(x) \rrbracket = \left( \bigwedge_{x \in \text{dom}(u)} (u(x) \rightarrow \llbracket \phi(x) \rrbracket)^\perp \right)^\perp
\]

\[
= \bigvee_{x \in \text{dom}(u)} (u(x) \rightarrow \llbracket \phi(x) \rrbracket)^\perp)^\perp.
\]

(29)

Thus, if the interpretation is self-dual, De Morgan’s Laws holds.

Conversely, suppose that the \(Q\)-value \(\llbracket \phi \rrbracket\) is assigned for all \(\phi \in L(\in, V(Q))\) by a non-trivial \(Q\)-valued interpretation \(\mathcal{I}(\rightarrow, \ast)\) satisfying De Morgan’s Laws. Let \(\phi(x) := (x \in \tilde{P}) \in L(\in, V(Q))\). Then, we have

\[
\llbracket (\exists x \in \tilde{Q}) \lnot \phi(x) \rrbracket = \bigvee_{u \in \text{dom}(\tilde{Q})} (\tilde{Q}(u) \ast \llbracket \phi(u) \rrbracket)^\perp
\]

\[
= \tilde{Q}(\tilde{0}) \ast [\tilde{0} \in \tilde{P}]^\perp
\]

\[
= Q \ast P.
\]
Thus, if De Morgan’s Laws hold we have
\[ P \ast Q = (P \rightarrow Q^\perp)^\perp \]
for all \( P, Q \in Q \), so that the \( Q \)-valued interpretation \( I(\rightarrow, \ast) \) is self-dual. \( \square \)

Now, we conclude:

**Corollary 4.20.** A \( Q \)-valued interpretation \( I(\rightarrow, \ast) \) satisfies both the Transfer Principle and De Morgan’s Laws if and only if \( \rightarrow \) is a quantized implication and \( \ast \) is its dual conjunction, namely, the \( Q \)-valued interpretation \( I(\rightarrow, \ast) \) is normal and self-dual.

For a normal self-dual \( Q \)-valued interpretation \( I(\rightarrow, \ast) \) of \( L(\in, V(\mathcal{R})) \), we can take the symbols \( \neg, \land, \rightarrow, \forall x \in y \), and \( \forall x \) as primitive, and the symbols \( \lor, \exists x \in y \), and \( \exists x \) as derived symbols by defining:

(D1) \( \phi \lor \psi = \neg(\neg \phi \land \neg \psi) \),

(D2) \( \exists x \in y \phi(x) = \neg(\forall x \in y \neg \phi(x)) \),

(D3) \( \exists x \phi(x) = \neg(\forall x \neg \phi(x)) \).

To each statement \( \phi \) of \( L(\in, V(\mathcal{R})) \) we assign the \( Q \)-valued truth value \( \llbracket \phi \rrbracket \) by the following rules.

(R1) \( \llbracket \neg \phi \rrbracket = \llbracket \phi \rrbracket^\perp \).

(R2) \( \llbracket \phi_1 \land \phi_2 \rrbracket = \llbracket \phi_1 \rrbracket \land \llbracket \phi_2 \rrbracket \).

(R4) \( \llbracket \phi_1 \rightarrow \phi_2 \rrbracket = \llbracket \phi_1 \rrbracket \rightarrow \llbracket \phi_2 \rrbracket \).

(R5) \( \llbracket (\forall x \in u) \phi(x) \rrbracket = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow \llbracket \phi(u') \rrbracket) \).

(R7) \( \llbracket (\forall x) \phi(x) \rrbracket = \bigwedge_{u \in V(\mathcal{R})} \llbracket \phi(u) \rrbracket \).

The truth values of atomic formulas are determined by the following rules with recursion on the rank of \( u \) and \( v \).

(R9) \( \llbracket u = v \rrbracket = \llbracket \forall x \in u(x \in v) \land \forall x \in v(x \in u) \rrbracket \),

(R10) \( \llbracket u \in v \rrbracket = \llbracket \neg(\forall x \in v)(\neg x = u) \rrbracket \).
By definitions of derived logical symbols, (D1)–(D3), we have the following relations.

(R3) \([\phi_1 \lor \phi_2] = [\phi_1] \lor [\phi_2]\).

(R5) \([\exists x \in u] \phi(x) = \bigvee_{u' \in \text{dom}(u)} (u(u') \ast_j [\phi(u')]).\)

(R8) \([\exists x \phi(x) = \bigvee_{u \in V(\mathcal{I})} [\phi(u)].\)

(A1) \([u = v] = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow [u' \in v]) \land \bigwedge_{v' \in \text{dom}(v)} (v(v') \rightarrow [v' \in u]).\)

(A2) \([u \in v] = \bigvee_{v' \in \text{dom}(v)} (v(v') \ast_j [u = v']).\)

In addition to (M1)–(M4), De Morgan’s Laws for bounded quantifications, (M5)–(M6),

(M5) \([- (\forall x \in u \phi(x))] = \exists x \in u (\neg \phi(x)),\)

(M6) \([- (\exists x \in u \phi(x))] = \forall x \in u (\neg \phi(x)),\)

hold.

Now we conclude the following characterization of polynomially definable interpretations that satisfy both the Transfer Principle and De Morgan’s Laws.

**Theorem 4.21.** Let \(Q\) be a logic and \((\to, \ast)\) be a pair of two-variable ortholattice polynomials. Then, \(Q\)-valued interpretation \(\mathcal{I}(\to, \ast)\) of \(\mathcal{L}(\in, V(\mathcal{Q}))\) satisfying both the Transfer Principle and De Morgan’s Laws are unique if \(Q\) is a Boolean algebra and exactly six, i.e., \(\mathcal{I}(\to_j, \ast_j)\) for \(j = 0, \ldots, 5\), if \(Q\) is not Boolean.

**Proof.** A \(Q\)-valued interpretation \(\mathcal{I}(\to, \ast)\)s of \(\mathcal{L}(\in, V(\mathcal{Q}))\) satisfies both the Transfer Principle and De Morgan’s Laws if and only if it is normal and self-dual. If \(Q\) is a Boolean algebra, normal interpretations are automatically self-dual and unique. If \(Q\) is not Boolean, there exist exactly 36 polynomially definable normal interpretations \(\mathcal{I}(\to_j, \ast_k)\) for \(j, k = 0, \ldots, 5\), and out of them there exist exactly six polynomially definable normal and self-dual interpretations \(\mathcal{I}(\to_j, \ast_j)\) for \(j = 0, \ldots, 5\). Thus, if \(Q\) is not Boolean, there exist exactly six interpretations \(\mathcal{I}(\to_j, \ast_j)\) for \(j = 0, \ldots, 5\) that satisfy both the Transfer Principle and De Morgan’s Laws.

4.6 Calculus of quantum subsets

In what follows we consider the interplay between the Transfer Principle and De Morgan’s Laws in the calculus of quantum subsets of a classical set.

Let \(Q\) be a non-Boolean logic. Let \(X\) be a non-empty set, i.e., \(X \in V\) and \(X \neq \emptyset\). Recall that a copy \(\tilde{X}\) of \(X\) in \(V(\mathcal{Q})\) is defined by \(\tilde{X} = \{\langle \tilde{x}, 1 \rangle \mid x \in X\}\). To define the power set of \(\tilde{X}\) in \(V(\mathcal{Q})\) let \(\mathcal{P}(\tilde{X})(\mathcal{Q})\) be such that

\[
\mathcal{P}(\tilde{X})(\mathcal{Q}) = \{u \in V(\mathcal{Q}) \mid \text{dom}(u) = \text{dom}(\tilde{X})\}. \tag{31}
\]

Any \(A \in \mathcal{P}(\tilde{X})(\mathcal{Q})\) is called a **quantum subset** of a classical set \(X\). The power set \(\mathcal{P}(\tilde{X})\) of \(\tilde{X}\) in \(V(\mathcal{Q})\) is defined by

\[
\mathcal{P}(\tilde{X}) = \mathcal{P}(\tilde{X})(\mathcal{Q}) \times \{1\}. \tag{32}
\]
For any \( A \in \mathcal{P}(\bar{\mathcal{X}}) \) define its complement \( A^\perp \in \mathcal{P}(\bar{\mathcal{X}}) \) by \( A^\perp(\bar{x}) = A(\bar{x})^\perp \) for all \( x \in X \). For any \( A, B \in \mathcal{P}(\bar{\mathcal{X}}) \) define their meet \( A \cap B \in \mathcal{P}(\bar{\mathcal{X}}) \) and join \( A \cup B \in \mathcal{P}(\bar{\mathcal{X}}) \) by \( (A \cap B)(\bar{x}) = A(\bar{x}) \land B(\bar{x}) \) and \( (A \cup B)(\bar{x}) = A(\bar{x}) \lor B(\bar{x}) \) for all \( x \in X \). Recall that the set inclusion relation is defined as \( A \subseteq B := (\forall x \in A)(x \in B) \).

Since 
\[
\phi(u, v) := (\forall x \in u)(x \in v) \iff \neg(\exists x \in u)(x \in v)
\]
is provable in ZFC, by the Transfer Principle the relation
\[
\[ A \subseteq B \implies A \cap B^\perp = \emptyset \] \geq \lor (A, B)
\]
holds in any normal \( Q \)-valued interpretation \( \mathcal{I}(\rightarrow, \ast) \), where \( \phi_1 \iff \phi_2 \) is abbreviation for \( (\phi_1 \land \phi_2) \lor (\neg \phi_1 \land \neg \phi_2) \). Then whether a stronger relation
\[
\[ A \subseteq B \] = \[ A \cap B^\perp = \emptyset \],
\]
holds or not is an interesting problem.

Consider the case where \( X = \{0\} \), \( A = \tilde{P} \), and \( B = \tilde{Q} \). In any normal interpretation, we have the following.

\[
\[ A \subseteq B \] = \[ (\forall x \in \tilde{P})(x \in \tilde{Q}) \]
= \[
\bigwedge_{x \in \text{dom}(\tilde{P})} \tilde{P}(x) \to [x \in \tilde{Q}]
\]
= \( \tilde{P}(\emptyset) \to \tilde{Q}(\emptyset) \).
\[
\[ A \cap B^\perp = \emptyset \] = \[ \tilde{P} \cap \tilde{Q}^\perp = \emptyset \]
= \[
\bigwedge_{x \in \text{dom}(\tilde{P} \cap \tilde{Q}^\perp)} [(\tilde{P} \cap \tilde{Q}^\perp)(x) \to [x \in \emptyset]] \land \bigwedge_{x \in \text{dom}(\tilde{Q})} (\emptyset(x) \to [x \in (\tilde{P} \cap \tilde{Q}^\perp)])
\]
= \( [\tilde{P} \cap \tilde{Q}^\perp](\emptyset) \to [\emptyset \in \emptyset] \) \land 1
= \( [\tilde{P} \cap \tilde{Q}^\perp](\emptyset) \to 0 \) \land 1
= \( \tilde{P}(\emptyset) \land \tilde{Q}(\emptyset)^\perp \)
= \( \tilde{P}(\emptyset)^\perp \lor \tilde{Q}(\emptyset). \)

Consequently, we have

\[
\[ A \subseteq B \] = P \to Q,
\]
\[
\[ A \cap B^\perp = \emptyset \] = P^\perp \lor Q.
\]

Thus, Eq. (35) holds only if \( P \to Q = P^\perp \lor Q \), namely \( \rightarrow = \rightarrow_5 \).

It follows that Eq. (35) does not hold in the Takeuti interpretation \( \mathcal{I}(\rightarrow_3, \ast_5) \). To see more precisely, suppose \( \bot_5(P, Q) = 0 \). In this case \( \lor(A, B) = 0 \) and Eq. (34)
gives no constraint. From Theorem 3.6 for \( j = 0, \ldots, 5 \) we have
\[
[A \subseteq B]_0 = 0, \quad (38) \\
[A \subseteq B]_1 = P, \quad (39) \\
[A \subseteq B]_2 = Q, \quad (40) \\
[A \subseteq B]_3 = P^\perp, \quad (41) \\
[A \subseteq B]_4 = Q^\perp, \quad (42) \\
[A \subseteq B]_5 = 1, \quad (43)
\]
but we have
\[
[A \cap B]_j = P^\perp \lor Q \geq \perp(P, Q)^\perp = 1
\]
for all \( j = 0, \ldots, 5 \), where \([ \cdots ]_j = \cdots \) denotes the \( Q \)-valued truth value in a normal \( Q \)-valued interpretation \( \mathcal{I}(\rightarrow, \star) \) with \( \rightarrow = \rightarrow_5 \). Thus, \([ A \subseteq B]_j = [A \cap B] = \hat{0} \) does not hold for \( j = 0, \ldots, 4 \), while \([ A \subseteq B]_5 = [A \cap B]_5 = [A \cap B]_5^\perp = 1 \) holds. However, the above relations do not mean \( P \leq Q \) since \( \rightarrow_5 \) violates (E): \( P \rightarrow Q = 1 \) if and only if \( P \leq Q \). On this ground the implication \( \rightarrow_5 \) has been abandoned in the conventional approach.

Thus, Eq. (35) is not satisfied by any normal interpretations \( \mathcal{I}(\rightarrow, \star) \) that satisfy (E). In this paper, we have explored a way to have both condition (E) and the essence of Eq. (35). Here, we should note that De Morgan’s Laws ensure the relation
\[
[(\forall x \in A) (x \in B)] = [\neg(\exists x \in A) \neg(x \in B)] = 1 \quad (44)
\]
stronger than the relation
\[
[(\forall x \in A) (x \in B)] \leftrightarrow \neg(\exists x \in A) \neg(x \in B)] \geq \bigvee(A, B), \quad (45)
\]
which follows from the Transfer Principle. Thus, in any normal self-dual \( Q \)-valued interpretation \( \mathcal{I}(\rightarrow, \star) \) we have
\[
[(\forall x \in A) (x \in B)] = [\neg(\exists x \in A) \neg(x \in B)]. \quad (46)
\]
Since the relation
\[
[(\forall x \in A) (x \in B)] = [A \subseteq B], \quad (47)
\]
holds in any normal interpretation, Eq. (35) is equivalent to the relation
\[
[\neg(\exists x \in A) \neg(x \in B)] = [A \cap B] = \hat{0}], \quad (48)
\]
which does not hold except for the case where \( \rightarrow_5 = \rightarrow_5 \). Thus, in order to extend Eq. (35) to the interpretations \( \mathcal{I}(\rightarrow_5, \star_5) \) for \( \neq 5 \), which satisfy (E), we have to introduce a new set calculus. For any quantized conjunction \( \star \) on \( Q \) we define the quantized meet \( A \cap_\star B \in \mathcal{P}(X)(Q) \) of \( A, B \in \mathcal{P}(X)(Q) \) by \( (A \cap_\star B)(x) = A(x) \star B(x) \) for all \( x \in X \). Then, in any normal self-dual interpretation \( \mathcal{I}(\rightarrow, \star) \) we can derive the relation
\[
[A \subseteq B] = [A \cap_\star B] = \hat{0}]. \quad (49)
\]
In fact, we have

\[ [A \subseteq B] = [[(\forall x \in A)(x \in B)]] \]
\[ [A \cap_\ast B^\bot = \tilde{0}] = [[-(\exists x \in A)\neg(x \in B)]] \]

in any normal interpretation \(\mathcal{I}(\to, \ast)\). Here, relation (51) follows from

\[
[A \cap_j B^\bot = \tilde{0}]_j = \bigwedge_{x \in X} [(A \cap_j B^\bot)(\bar{x}) \to [\bar{x} \in \tilde{0}]_j] \land \bigwedge_{x \in \text{dom}(\tilde{0})} (\tilde{0}(x) \to [x \in (A \cap_j B^\bot)]_j)
\]

\[
= \left( \bigwedge_{x \in X} [(A \cap_j B^\bot)(\bar{x}) \to 0] \right) \land 1
\]

\[
= \bigwedge_{x \in X} (A \cap_j B^\bot)(\bar{x})^\bot
\]

\[
= \left( \bigvee_{x \in X} (A \cap_j B^\bot)(\bar{x}) \right)^\bot
\]

\[
= \left( \bigvee_{x \in X} [A(\bar{x}) \ast B(\bar{x})^\bot] \right)^\bot
\]

\[
= \left( \bigvee_{x \in \text{dom}(A)} A(x) \ast [x \in B]_j \right)^\bot
\]

\[
= [[-(\exists x \in A)\neg(x \in B)]_j.
\]

Thus, Eq. (49) is equivalent to \([\phi(A, B)] = 1\). Since \([\phi(A, B)] = 1\) follows from De Morgan’s Laws, we conclude that Eq. (49) holds in all the normal self-dual \(Q\)-valued interpretations including polynomially definable interpretations \(\mathcal{I}(\to_j, \ast_j)\) with \(j = 0, \ldots, 5\). We also conclude that Eq. (35) holds for and only for the interpretation \(\mathcal{I}(\to_5, \ast_5)\), since \(A \cap_\ast B = A \cap B\) holds for any \(A, B \in \mathcal{P}(X)^{(Q)}\) if and only if \(\ast = \ast_5\).

To see more precisely, suppose, for instance, \(A = \tilde{P}, B = \tilde{Q}\), and \(\sqsubseteq(P, Q) = 0\). Then we have

\[
[A \subseteq B]_0 = [A \cap_1 B^\bot = \tilde{0}]_0 = 0,
\]
\[
[A \subseteq B]_1 = [A \cap_2 B^\bot = \tilde{0}]_1 = P,
\]
\[
[A \subseteq B]_2 = [A \cap_3 B^\bot = \tilde{0}]_2 = Q,
\]
\[
[A \subseteq B]_3 = [A \cap_4 B^\bot = \tilde{0}]_3 = P^\bot,
\]
\[
[A \subseteq B]_4 = [A \cap_5 B^\bot = \tilde{0}]_4 = Q^\bot,
\]
\[
[A \subseteq B]_5 = [A \cap_6 B^\bot = \tilde{0}]_5 = 1,
\]

where \([\cdots]_j\) denotes the \(Q\)-value in the interpretation \(\mathcal{I}(\to_j, \ast_j)\) and \(\cap_j\) abbreviates \(\cap_\ast\) for \(j = 0, \ldots, 5\). If we drop the condition \(\sqsubseteq(P, Q) = 0\), the relation \([A \subseteq B]_j = 1\) or \([A \cap B^\bot = \tilde{0}]_j = 1\) implies \(P \leq Q\) by condition (E) except for \(j = 5\).
4.7 Applications to operator theory

We continue the consideration on calculus of quantum subsets. In Ref. [31] the case where $X = \mathbb{Q}$, the set of rational numbers, was investigated in the interpretations $\mathcal{I}(\rightarrow_j, \ast_5)$ with $j = 0, 2, 3$, and it was shown that quantum subset calculus on $\mathcal{P}(\mathbb{Q})$ can be effectively applied to quantum theory and the theory of self-adjoint operators on a Hilbert space $\mathcal{H}$.

Suppose $\mathbb{Q} = \mathbb{Q}(\mathcal{H})$. The real numbers in the $\mathbb{Q}$-valued universe $V(\mathbb{Q})$ is defined as Dedekind cuts of the set $\mathbb{Q}$ of rational numbers, represented by upper segments with endpoints if exist. Thus, the set $\mathbb{R}(\mathbb{Q})$ of quantized real numbers in $V(\mathbb{Q})$ is defined as

$$\mathbb{R}(\mathbb{Q}) = \{ u \in \mathcal{P}(\mathbb{Q}) | [\mathbb{R}(u)] = 1 \},$$

$$\mathbb{R}(x) := \forall y \in x \forall y \in \mathbb{Q} \Rightarrow \exists y \in \mathbb{Q} \wedge \exists y \in \mathbb{Q} (y \notin x)$$

$$\wedge y \in \mathbb{Q} (y \in x \iff \forall z \in \mathbb{Q} (y < z \rightarrow z \in x)).$$

Then, the set $\mathbb{R}(\mathbb{Q})$ is in one-to-one correspondence $u \leftrightarrow A$ with the set $SA(\mathcal{H})$ of self-adjoint operators on $\mathcal{H}$ in such a way that $u \leftrightarrow A$ if and only if

$$u(\hat{r}) = E^A(r)$$

for all $r \in \mathbb{Q}$, where $\{ E^A(\lambda) | \lambda \in \mathbb{R} \}$ is the right-continuous spectral family of the self-adjoint operator $A$. The above one-to-one correspondence is called the Takeuti correspondence. In what follows we shall write $u = A$ and $A = u$ iff $u \leftrightarrow A$.

For any self-adjoint operators $A, B \in SA(\mathcal{H})$ we write $A \ll B$ iff $E^B(\lambda) \leq E^A(\lambda)$ for all $\lambda \in \mathbb{R}$. The relation, originally introduced by Olson [49], is called the spectral order. With the spectral order the set $SA(\mathcal{H})$ is a conditionally complete lattice. The spectral order coincides with the usual linear order on projections and mutually commuting operators, and for any $0 \leq A, B \in SA(\mathcal{H})$, we have $A \ll B$ if and only if $A^n \leq B^n$ for all $n \in \mathbb{N}$ [49][50].

The $\mathbb{Q}$-valued order relation over $\mathbb{R}(\mathbb{Q})$ is defined by the set inclusion in reverse, i.e., $u \leq v := v \subseteq u$, so that

$$[u \subseteq v] = [v \subseteq u]$$

holds for any $u, v \in \mathcal{P}(\mathbb{Q})$. Then interestingly it was shown that $[\hat{A} \leq \hat{B}] = 1$ if and only if $A \ll B$ holds for any $A, B \in SA(\mathcal{H})$. Thus, the investigation on the order relation of quantized reals in $V(\mathbb{Q})$ provides a new method for studying the spectral order of self-adjoint operators. In particular, $\mathbb{Q}$-values $[\hat{A} \leq \hat{B}]$ for self-adjoint operators $A, B \in SA(\mathcal{H})$ provide more precise information on the spectral order. In fact, in Ref. [31] it was shown that the $\mathbb{Q}$-values $[\hat{A} \leq \hat{B}]_j$ for $\mathcal{I}(\rightarrow_j, \ast_5)$ have different operational meanings for different interpretations for $j = 0, 2, 3$ on the joint probability of outcomes of successive measurements.

Now we apply our discussions above on De Morgan’s Laws. For any self-adjoint operators $A, B \in SA(\mathcal{H})$, we have the corresponding elements $\hat{A}, \hat{B} \in \mathcal{P}(\mathbb{Q})$ and $\mathbb{Q}$-values $[\hat{A} \leq \hat{B}] = [\hat{B} \subseteq \hat{A}]$. In our previous investigations we considered only interpretations $\mathcal{I}(\rightarrow_j, \ast_5)$ so that the relation

$$[A \subseteq B] = [A \cap_\ast B^\perp = \emptyset].$$

(60)
does not hold. However, the results in this paper suggests that interpretations \( I(\rightarrow_j, \ast_j) \) for \( j = 0, \ldots, 4 \) would be more useful. In those interpretations we have

\[
[\hat{A} \leq \hat{B}]^{\perp} = [(\exists r \in \hat{B}) (r \in \hat{A})]
\]

or

\[
= \bigvee_{r \in Q} [r \in \hat{B}] \ast [r \in \hat{A}]^{\perp}
\]

or

\[
= \bigvee_{r \in Q} E^B(r) \ast E^A(r)^{\perp}.
\]

In particular, we have that \( A \preceq B \) if and only if \( E^B(r) \ast_j E^A(r)^{\perp} = 0 \) for all \( r \in Q \), where \( j = 0, \ldots, 4 \). Interestingly, it is not sufficient for \( A \preceq B \) that \( E^B(r) \land E^A(r)^{\perp} = 0 \) for all \( r \in Q \), since the interpretation \( I(\rightarrow_5, \ast_5) \) is excluded because of the violation of condition (E).

More systematic applications of the order relation of the real numbers in \( V(\mathbb{Q}) \) to spectral order of self-adjoint operators will be discussed elsewhere.

5 Discussion

In quantum logic the meaning of logical connectives have been often polemical, and yet conjunction and negation have considered to have firm bases. As pointed out by Husimi [51] the conjunction \( P \land Q \) of two quantum propositions \( P, Q \in Q \) holds exactly in the states where both \( P \) and \( Q \) hold simultaneously. Also, the proposition \( P \) and its negation \( P^{\perp} \) are commuting to have classical interpretation as negation. However, the disjunction \( P \lor Q \) has a difficulty, since \( P \lor Q \) holds even in the case where there exist no simultaneous eigenstates. De Morgan’s Laws provide the simplest solution to determine the disjunction for quantum logic to have operational but mathematically tractable structure. The operational meaning of \( P \lor Q \) is as follows (cf. Section 5; note that \( P \lor Q = P^{\perp} \rightarrow_5 Q \)). For any state vector \( \Psi \), the disjunction \( P \lor Q \) holds with probability \( \| (P \lor Q) \Psi \|^2 = \| (P \lor Q)_{B} \Psi \|^2 + \| (P \lor Q)_{N} \Psi \|^2 \). Here, \( P \) and \( Q \) are simultaneously determinate with probability \( \| \perp (P, Q) \Psi \|^2 \), in which \( P \) holds or \( Q \) holds with probability \( \| (P \lor Q)_{B} \Psi \|^2 \), and \( P \) and \( Q \) are simultaneously indeterminate with probability \( \| \perp (P, Q)^{\perp} \Psi \|^2 \), which equals \( \| (P \lor Q)_{N} \Psi \|^2 \). De Morgan’s Laws determine how to distribute the probability of indeterminacy of the pair \( P, Q \) to the two dual connectives.

In the case of the \( (\land, \lor) \)-pair, \( \phi \land \psi \) means that \( \phi \) and \( \psi \) are simultaneously determinate, and \( \phi \) holds and \( \psi \) holds, whereas \( \phi \lor \psi \) means that \( (\phi \text{ and } \psi \text{ are simultaneously determinate, and } \phi \text{ holds or } \psi \text{ holds) or (} \phi \text{ and } \psi \text{ are simultaneously indeterminate}) \). Similar duality holds for the pair of quantized implications and quantized conjunction. For instance, in the \( I(\rightarrow_3, \ast_3) \)-interpretation \( \phi \rightarrow \psi \) means that \( (\phi \text{ and } \psi \text{ are simultaneously determinate, and } \phi \text{ does not hold or } \psi \text{ holds}) \) or \( (\phi \text{ and } \psi \text{ are simultaneously indeterminate, and } \phi \text{ does not hold}) \), whereas \( \phi \ast \psi \) means that \( (\phi \text{ and } \psi \text{ are simultaneously determinate, and } \phi \text{ holds and } \psi \text{ holds}) \) or \( (\phi \text{ and } \psi \text{ are simultaneously indeterminate, and } \phi \text{ holds}) \). Thus, \( \neg(\phi \rightarrow \psi) \) means \( (\phi \text{ and } \psi \text{ are simultaneously determinate, and } \phi \text{ holds and } \psi \text{ does not hold}) \) or \( (\phi \text{ and } \psi \text{ are simultaneously indeterminate, and } \phi \text{ hold}) \), which equals what \( \phi \ast \neg \psi \) means. Consequently, De Morgan’s
Law \( \neg(\phi \rightarrow \psi) \Leftrightarrow (\phi \ast \neg\psi) \) holds, and yet \( \neg(\phi \rightarrow \psi) \Leftrightarrow (\phi \land \neg\psi) \) does not hold in this interpretation. This hidden duality exists between bounded universal quantifiers and bounded existential quantifiers.

Takeuti’s quantum set theory has been successfully applied to quantum theory to extend the Born formula for atomic observational propositions to relations between two observables \([29-31]\). Historically, the Born formula was originally formulated for the atomic formula \( A = a \) for an observable \( A \) and a real number \( a \) as \( \Pr\{A = a\|\Psi\} = \|E^A(a)\Psi\|^2 \), i.e., the probability of the observable \( A \) taking the value \( a \) on the measurement in the state \( \Psi \) equals the squared length of its projection to the eigenspace of the operator \( A \) belonging to the eigenvalue \( a \). Then, Birkhoff–von Neumann \([3]\) extended this to observational propositions \( \phi \) as \( \Pr\{\phi\|\Psi\} = \|\phi\Psi\|^2 \), where quantum logical (projection-valued) truth value \( \llbracket\phi\rrbracket \) is determined by the Birkhoff-von Neumann rule. However, even by the Birkhoff-von Neumann rule, we could not determine the probability of the equality relation \( A = B \) for arbitrary two observables \( A \) and \( B \). Takeuti’s quantum set theory enabled us to determine this probability \( \Pr\{A = B\|\Psi\} = \|\llbracket A = B \rrbracket\Psi\|^2 \) for the first time by determining the projection-valued truth value \( \llbracket A = B \rrbracket \) of the equality for two real numbers in the universe \( V(Q) \), which correspond bijectively to quantum observables. The operational meaning of this probability has been studied extensively to show that this is the probability that \( A \) and \( B \) are simultaneously determinate and they have the same value \([30]\).

This paper studies and proposes a solution to the violation of De Morgan’s Laws in Takeuti’s quantum set theory. To be more precise, in Takeuti’s quantum set theory and the later generalizations of his theory, De Morgan’s Law for bounded quantifiers, or the duality between \( (\exists x \in u) \) and \( (\forall x \in u) \), does not hold. This causes a difficulty, for instance, in defining the complement \( A^c \) of a set \( A \), since \( x \in A^c \) and \( \neg(x \in A) \) are not equivalent under the violation to De Morgan’s Laws. The problem is whether this difficulty is inherent to quantum logic, just like intuitionistic logic, or not, just like classical logic. We have shown that this problem can be solved to eliminate the above difficulty by reformulating quantum set theory on a more natural basis to satisfy De Morgan’s Laws for bounded quantifiers.

In quantum logic, there is still well-known arbitrariness of the choice of implication connective. The choice of implication immediately affects the interpretation of bounded universal quantifiers. What is the right choice of implication or bounded universal quantifiers may depend on the problem to apply \([31]\). However, what is the right choice of bounded existential quantifiers should be determined through De Morgan’s Laws by our choice of implication in the bounded universal quantifiers to avoid the ambiguity of the truth value assignment.

Our conclusion is as follows. As long as polynomially definable operations are concerned, we have only 6 interpretations \( \mathcal{I}(\rightarrow_j, *_j) \) for \( j = 0, \ldots, 5 \) that satisfy the Transfer Principle and De Morgan’s Laws. According to Hardegree \([37]\) three interpretations \( \mathcal{I}(\rightarrow_j, *_j) \) for \( j = 0, 2, 3 \) are more desirable, since the implication \( \rightarrow_j \) satisfies his minimum implicative condition only for \( j = 0, 2, 3 \). The majority view favors \( \rightarrow_3 \), and, in fact, Takeuti and his followers adopted the interpretation \( \mathcal{I}(\rightarrow_3, *_5) \), although this choice causes the violation of De Morgan’s Laws between universal and existential bounded quantifications. Our research recommend the interpretation \( \mathcal{I}(\rightarrow_3, *_3) \) instead of \( \mathcal{I}(\rightarrow_3, *_5) \), whenever \( \rightarrow_3 \) is chosen at all for implication, and then both
the Transfer Principle and De Morgan’s Laws hold. Despite of the majority view, the other two choices would be worth investigating. We have studied the real numbers in the interpretations $\mathcal{I}(\rightarrow_j, \ast_5)$ for $j = 0, 2, 3$ [31]. We have shown that the reals in the universe and the truth values of their equality are the same for the above three interpretations. Interestingly, however, the order relation between quantum reals significantly depends on the underlying implications. We have characterize the operational meanings of those order relations in terms of joint probability distributions obtained by successive measurement.

As discussed in Section [4.7] De Morgan’s Laws would play an important role in this subject. It is naturally expected that the new interpretations will give a firm basis for and enhance the power of quantum set theory in theory and application.

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