Isospin corrections to super-allowed beta decays in nuclei

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Abstract: Isospin corrections to the super-allowed beta decay matrix elements are evaluated in perturbation theory using the notion of the giant isovector monopole state. The calculation avoids the separation into different contributions and therefore presents a consistent, systematic and more transparent approach. Explicit expressions for $\delta_c$ as a function of the mass number $A$ are derived. These corrections affect the values of the $V_{ud}$ matrix element in the Cabbibo-Maskawa-Kobayashi matrix. Also, it is pointed out that in some nuclei with a low number of excess neutrons (protons) Coulomb mixing with the anti-analog state can introduce significant isospin impurities in the isobaric analog state.

1. Introduction
Recently there is much activity to determine the corrections one has to introduce when one studies the beta-decay matrix elements for super-allowed decays in $T = 1$, $T_z = +1$ (or $T_z = -1$) nuclei [1,2]. This is important because using the measured $ft$ values one can relate these to the u-quark to d-quark transition matrix element (m.el.) $V_{ud}$ in the Cabibbo-Kobayashi-Maskawa (CKM) matrix. In the Standard Model (SM) this matrix satisfies the unitarity condition:

$$V_{ud}^2 + V_{us}^2 + V_{ub}^2 = 1$$

(1)

In order to use the experimental $ft$ values to determine $V_{ud}$ one has to introduce corrections [1, 2]. (There are radiative corrections which we will not treat here, discussions of these can be found abundantly in the literature [1,]). The second type of correction, which is usually termed as the isospin symmetry breaking term, denoted as $\delta_c$:

$$|M_F|^2 = |M_F^0|^2 (1 - \delta_c)$$

(2)

where $M_F$ is the physical Fermi matrix element:

$$M_F = \langle \Psi_1 | T_z | \Psi_2 \rangle$$

(3)

$|\Psi_1\rangle$ and $|\Psi_2\rangle$ are the parent and daughter physical states, $M_F^0$ stands for the Fermi matrix element obtained in the limit when in the Hamiltonian all the charge-dependent parts are put to zero, and the wave functions are eigenstates of the charge-independent Hamiltonian. In the
present approach we start from a charge-independent Hamiltonian so that the matrix element in eq. (1) is exactly $\sqrt{2}T$ and we then treat the Coulomb force in perturbation theory. In the way we approach the problem there is no need to break up the contribution of the Coulomb interaction into various separate components. All the effects of Coulomb mixing (such as isospin mixing, the change in the radial part of the wave functions, etc.) are taken into account in a single term.

2. Coulomb mixing

We start by introducing a nuclear charge independent Hamiltonian $H_0$. The eigenstates of this Hamiltonian with isospin $T$ and $T_z$ will be denoted as $|T, T_z\rangle$ and:

$$H_0 |T, T_z\rangle = E_T |T, T_z\rangle$$

(4)

The action of the isospin lowering and raising operators, $T_-, T_+$, gives:

$$T_- |T, T\rangle = \sqrt{2}T |T, T-1\rangle ; \quad T_+ |T, T-1\rangle = \sqrt{2}T |T, T\rangle$$

(5)

We now add to the charge independent Hamiltonian a charge dependent part $V_{CD}$. The dominant part in the charge dependent interaction is the charge asymmetric Coulomb force $V_C$. Because of the long range nature of the Coulomb force, the prevailing part will be in such cases the one-body part. Of interest to us here is the isovector part of the potential. For a uniform charge distribution of radius $R$ any off-diagonal matrix element between two states of the isovector part is:

$$\langle 0 | V_C | n \rangle = \frac{Ze^2}{2R^3} \sum_i r_i^2 t_i(i) \langle n \rangle = \frac{Ze^2}{2R^3} \langle 0 | M_i^{(0)} | n \rangle$$

(6)

where $M_i^{(0)}$ denotes the z-component of the isovector monopole operator. If state $|n\rangle$ is the giant isovector monopole [3], that is the state obtained by acting with the $M_i^{(0)}$ operator on the ground state $|0\rangle$ and normalizing, then the above matrix element will be proportional to $\langle 0 | M_i^{(0)} M_i^{(0)} | 0 \rangle$ and thus will exhaust the isovector part of $r_i^2 t_z$ of the Coulomb sum rule [3].

We now find in perturbation theory the effect of the charge-dependent part on the wave functions of the two members of the isomultiplet, $|T, T\rangle$ and $|T, T-1\rangle$:

$$\Psi_1 = \left( T, T \right) + \varepsilon_T | M_{1+1} \rangle + \varepsilon_{T+1} | M_{1+1} \rangle N_1$$

(7a)

$$\Psi_2 = \left( T, T-1 \right) + \eta_T | M_{-1+1} \rangle + \eta_T | M_{-1+1} \rangle + \eta_{T+1} | M_{1+1} \rangle N_2$$

(7b)

where $| M_{1+1} \rangle$, are the $T_-, T_z$ components of the isovector monopole, and where

$$N_1 = \sqrt{1 + \varepsilon_T^2 + \varepsilon_{T+1}^2} \quad \text{and} \quad N_2 = \sqrt{1 + \eta_T^2 + \eta_{T+1}^2 + \eta_{T-1}^2}$$

(8)

The admixtures are given in perturbation theory by the equations:
\[ \varepsilon_i = \frac{\langle T, T \| V^{(1)}_{C} \| M_{T+i,T} \rangle}{E_{M_{T+i,T}} - E_0}, \quad i = 0, 1 \]  

(9)

where \( E_0 \) is g.s. energy

\[ \eta_i = \frac{\langle T, T - 1 \| V^{(1)}_{C} \| M_{T+i,T-1} \rangle}{E_{M_{T+i,T-1}} - E_1}, \quad i = -1, 0, 1 \]  

(10)

Here \( E_1 \) is the energy of the analog state.

One derives finally for \( T=1 \) isotriplets (the detailed derivation can be found in ref. [2]):

\[ \delta_\varepsilon = 8 \frac{V_1}{41 \xi^2 A^{2/3}} \]  

(11)

where \( V_s = \frac{V_1}{A} (\mathbf{i} \cdot \mathbf{T}) \) is the symmetry potential and \( \xi \) is a numerical factor which depends on the model used to describe the isovector monopole. The range of values for this parameter is between 3 and 4 [3].

### 3. Results

The isospin impurities in the ground state of a nucleus \( \varepsilon_i^2 \) where computed in the past [3] using collective models for the isovector monopole state. Using these impurities one finds the results shown in the table:

| Table 1. Values of \( \delta_\varepsilon \) in % for several mass numbers \( A \) for the four models discussed in ref. [2, 3]. |
|-----------------|-------|-------|-------|
| \( \delta_\varepsilon \) | 10    | 40    | 80    |
| Hydrodynamical  | 0.006%| 0.1%  | 0.4%  |
| NEWSR           | 0.001%| 0.04% | 0.20% |
| EWSR            | 0.005%| 0.09% | 0.4%  |
| Microscopic     | 0.009%| 0.06% | 0.25% |

These results are a factor 3-8 smaller than in ref. [1].

### 4. Isospin mixing and the role of the anti-analog state

In some nuclei with a low number of excess neutrons (protons) Coulomb mixing with the anti-analog state can introduce significant isospin impurities in the isobaric analog state. The corrections are particularly sizeable in those medium heavy nuclei that are off the stability line having a small number of excess neutrons. The enhancement of the isospin admixture in the analog can reach a few percent.
Consider a simple parent state in which \( n_1 \) excess neutrons occupy orbit \( j_1 \) and \( n_2 \) neutrons orbit \( j_2 \).

The parent state is:

\[
|\tau\rangle = \left| j_1^{n_1} (n) j_1^{n_1} (n) \right\rangle
\]  

(12)

The analog is:

\[
|\Lambda\rangle = \frac{1}{\sqrt{2T}} \left[ \sqrt{n_1} \left| j_1^{n_1-1} (n) j_1^{n_1} (n) \right\rangle + \sqrt{n_2} \left| j_2^{n_2-1} (n) j_2^{n_2} (n) \right\rangle \right]
\]  

(13)

The anti-analog \( \bar{\Lambda} \) is then:

\[
\bar{\Lambda} = \frac{1}{\sqrt{2T}} \left[ \sqrt{n_2} \left| j_1^{n_1-1} (n) j_1^{n_1} (n) \right\rangle - \sqrt{n_1} \left| j_2^{n_2-1} (n) j_2^{n_2} (n) \right\rangle \right]
\]  

(14)

The letters \( n \) and \( p \) denote neutrons and protons.

We will consider here parent nuclei with simple configurations: for even-even nuclei the \( n_1 \) and \( n_2 \) are even and in each orbit the excess nucleons are coupled to \( J = 0^+ \) and in odd-even nuclei \( n_1 \) is odd and \( n_2 \) is even.

The one-body Coulomb matrix element between the analog and anti-analog is then:

\[
\langle \bar{\Lambda} | V_c | A \rangle = \frac{\sqrt{n_1 n_2}}{2T} \left[ \langle j_1 | V_c | j_1 \rangle - \langle j_2 | V_c | j_2 \rangle \right]
\]  

(15)

If the excess neutrons occupy orbits belonging to different major shells, this matrix element is sizable. But even if this is not the case, because of binding energy effects in a finite potential well and angular momentum, this matrix element is of the order of several hundred keV [3,4].

The energy splitting between the analog and anti-analog is given by the symmetry potential. It is easy to show that in this case:

\[
\delta_c = c^2, \quad \text{the admixture of the anti-analog into the analog state.}
\]

For low \( T \) and large \( Z \) nuclei this contribution might be significant. For example: for \( T=3/2 \) nuclei with the excess of three nucleons occupying two orbits from different major shells we select \( ^{17}_7 N_{10} \) and find from \( \delta_c = 0.04\% \), for \( ^{45}_{22} Tl_{19} \), \( \delta_c = 0.7\% \) and for \( ^{38}_{36} Sr_{41} \), \( \delta_c = 3.3\% \).

For \( T=2 \) nuclei one can look at the example of \( ^{80}_{36} Sr_{42} \), here \( \delta_c = 2.1\% \). In the examples chosen we selected nuclei in which the excess nucleons occupy two orbits belonging to different harmonic oscillator shells.

Other examples for \( T=2 \) states, for mass \( A=40 \), \( \delta_c =0.4\% \), for \( A=60 \), \( \delta_c =1.1\% \) and for \( A=100 \), \( \delta_c =4.3\% \).

References

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[4] Bertsch G F and Mekjian A, 1972 Annu. Rev. Nucl. Sci. 22 25