Approximating with Gaussians

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Abstract

Linear combinations of translations of a single Gaussian, $e^{-x^2}$, are shown to be dense in $L^2(\mathbb{R})$. Two algorithms for determining the coefficients for the approximations are given, using orthogonal Hermite functions and least squares. Taking the Fourier transform of this result shows low-frequency trigonometric series are dense in $L^2$ with Gaussian weight function.

Key Words: Hermite series, Gaussian function, low-frequency trigonometric series

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1 Linear combinations of Gaussians with a single variance are dense in $L^2$

$L^2(\mathbb{R})$ denotes the space of square integrable functions $f : \mathbb{R} \to \mathbb{R}$ with norm $\|f\|_2 := \sqrt{\int_{\mathbb{R}} |f(x)|^2 \, dx}$. We use $f \approx g$ to mean $\|f - g\|_2 < \epsilon$. The following result was announced in [4].

Theorem 1 For any $f \in L^2(\mathbb{R})$ and any $\epsilon > 0$ there exists $t > 0$ and $N \in \mathbb{N}$ and $a_n \in \mathbb{R}$ such that

$$f \approx \frac{\epsilon}{\epsilon^2} \sum_{n=0}^{N} a_n e^{-(x-nt)^2}.$$

Proof. Since the span of the Hermite functions is dense in $L^2(\mathbb{R})$ we have for some $N$

$$f \approx \frac{\epsilon}{\epsilon^2} \sum_{n=0}^{N} b_n \frac{d^n}{dx^n} (e^{-x^2}). \tag{1}$$
Now use finite backward differences to approximate the derivatives. We have for some small $t > 0$

\[
\sum_{n=0}^{N} b_n \frac{d^n}{dx^n} \left( e^{-x^2} \right) \approx \frac{\epsilon}{2} b_0 e^{-x^2} + b_1 \frac{1}{4} \left[ e^{-x^2} - e^{-(x-t)^2} \right] + b_2 \frac{1}{\pi} \left[ e^{-x^2} - 2e^{-(x-t)^2} + e^{-(x-2t)^2} \right] \\
+ b_3 \frac{1}{\pi} \left[ e^{-x^2} - 3e^{-(x-t)^2} + 3e^{-(x-2t)^2} - e^{-(x-3t)^2} \right] + \cdots \\
= \sum_{n=0}^{N} b_n \frac{1}{4^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{-(x-kt)^2}.
\]

This result may be surprising; it promises we can approximate to any degree of accuracy a function such as the following characteristic function of an interval

\[
\chi([-11, -10]) (x) := \begin{cases} 
1 & \text{for } x \in [-10, -11] \\
0 & \text{otherwise}
\end{cases}
\]

with support far from the means of the Gaussians $e^{-(x-nt)^2}$ which are located in $[0, \infty)$ at the points $x = nt$. The graphs of these functions $e^{-(x-nt)^2}$ are extremely simple geometrically, being Gaussians with the same variance. We only use the right translates, and they all shrink precipitously (exponentially) away from their means.

\[
\sum a_n e^{-(x-nt)^2} \approx \text{characteristic function?}
\]

Surely there is a gap in this sketchy little proof?

No. We will, however, flesh out the details in section 2. The coefficients $a_n$ are explicitly calculated and the $L^2$ convergence carefully justified. But these details are elementary. We include them in the interest of appealing to a broader audience.

Then is this merely another pathological curiosity from analysis? We probably need impractically large values of $N$ to approximate any interesting functions.
No, $N$ need only be as large as the Hermite expansion demands. Certainly this particular approach depends on the convergence of the Hermite expansion, and for many applications Hermite series converge slower than other Fourier approximations—after all, Hermite series converge on all of $\mathbb{R}$ while, e.g., trigonometric series focus on a bounded interval. Hermite expansions do have powerful convergence properties, though. For example, Hermite series converge uniformly on finite compact subsets whenever $f$ is twice continuously differentiable (i.e., $C^2$) and $O\left(e^{-cx^2}\right)$ for some $c > 1$ as $x \to \infty$. Alternately if $f$ has finitely many discontinuities but is still $C^2$ elsewhere and $O\left(e^{-cx^2}\right)$ the expansion again converges uniformly on any closed interval which avoids the discontinuities [15], [16]. If $f$ is smooth and properly bounded, the Hermite series converges faster than algebraically [7].

Then is the method unstable?

Yes, there are two serious drawbacks to using Theorem 1.

1. Numerical differentiation is inherently unstable. Fortunately we are estimating the derivatives of Gaussians, which are as smooth and bounded as we could hope, and so we have good control with an explicit error formula. It is true, though, that dividing by $t^n$ for small $t$ and large $n$ will eventually lead to huge coefficients $a_n$ and round-off error. There are quite a few general techniques available in the literature for combatting round-off error in numerical differentiation. We review the well-known $n$-point difference formulas for derivatives in section 6.

2. The surprising approximation is only possible because it is weaker than the typical convergence of a series in the mean. Unfortunately

$$f(x) \neq \sum_{n=0}^{\infty} a_n e^{-\left(x-nt\right)^2}$$

Theorem 1 requires recalculating all the $a_n$ each time $N$ is increased. Further, the $a_n$ are not unique. The least squares best choice of $a_n$ are calculated in section 3 but this approach gives an ill-conditioned matrix. A different formula for the $a_n$ is given in Theorem 3 which is more computationally efficient.

Despite these drawbacks the result is worthy of note because of the new and unexpected opportunities which arise using an approximation method with such simple functions. In this vein, section 4 details an interesting corollary of Theorem 1: apply the Fourier transform to see that low-frequency trigonometric series are dense in $L^2(\mathbb{R})$ with Gaussian weight function.

2 Calculating the coefficients with orthogonal functions

In this section Theorem 3 gives an explicit formula for the coefficients $a_n$ of Theorem 1. Let’s review the details of the Hermite-inspired expansion

$$f(x) = \sum_{n=0}^{\infty} b_n \frac{d^n}{dx^n} \left(e^{-x^2}\right)$$
claimed in the proof. The formula for these coefficients is

\[ b_n := \frac{1}{n!2^n \sqrt{\pi}} \int_{\mathbb{R}} f(x) e^{x^2} \frac{d^n}{dx^n} \left( e^{-x^2} \right) dx. \]

Be warned this is not precisely the standard Hermite expansion, but a simple adaptation to our particular requirements. Let’s check this formula for the \( H_n(x) \) using the techniques of orthogonal functions.

Remember the following properties of the Hermite polynomials \( H_n(x) \), e.g.). Define \( H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \). The set of Hermite functions

\[
\left\{ h_n(x) := \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} H_n(x) e^{-x^2/2} : n \in \mathbb{N} \right\}
\]

is a well-known basis of \( L^2(\mathbb{R}) \) and is orthonormal since

\[
\int_{\mathbb{R}} H_m(x) H_n(x) e^{-x^2} dx = n!2^n \sqrt{\pi} \delta_{m,n}. \tag{3}
\]

This means given any \( g \in L^2(\mathbb{R}) \) it is possible to write

\[
g(x) = \sum_{n=0}^{\infty} c_n \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} H_n(x) e^{-x^2/2} \tag{4}
\]

(equality in the \( L^2 \) sense) where

\[
c_n := \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} \int_{\mathbb{R}} g(x) H_n(x) e^{-x^2/2} dx \in \mathbb{R}.
\]

The necessity of this formula for \( c_n \) can easily be checked by multiplying both sides of \( (4) \) by \( H_n(x) e^{-x^2/2} \), integrating and applying \( (3) \). However, we want

\[
f(x) = \sum_{n=0}^{\infty} b_n \frac{d^n}{dx^n} e^{-x^2}
\]

so apply this process to \( g(x) = f(x) e^{x^2/2} \). But \( f(x) e^{x^2/2} \) may not be \( L^2 \) integrable. If it is not, we must truncate it: \( f(x) e^{x^2/2} \chi_{[-M,M]}(x) \) is \( L^2 \) for any \( M < \infty \) and \( f \cdot \chi_{[-M,M]} \approx f \) for a sufficiently large choice of \( M \). Now we get new \( c_n \) as follows

\[
f(x) e^{x^2/2} \chi_{[-M,M]}(x) = \sum_{n=0}^{\infty} c_n \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} H_n(x) e^{-x^2/2} \quad \text{so}
\]

\[
f(x) \chi_{[-M,M]}(x) = \sum_{n=0}^{\infty} c_n \frac{(-1)^n}{\sqrt{n!2^n \sqrt{\pi}}} (-1)^n H_n(x) e^{-x^2} = \sum_{n=0}^{\infty} b_n \frac{d^n}{dx^n} e^{-x^2}
\]

where

\[
c_n = \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} \int_{\mathbb{R}} f(x) e^{x^2/2} \chi_{[-M,M]}(x) H_n(x) e^{-x^2/2} dx \]

\[
= \frac{1}{\sqrt{n!2^n \sqrt{\pi}}} \int_{\mathbb{R}} f(x) \chi_{[-M,M]}(x) H_n(x) dx
\]
so we must have
\[ b_n = c_n \frac{(-1)^n}{\sqrt{n}2^n \sqrt{2\pi}} = \frac{1}{n^{1/2} \sqrt{2\pi}} \int_{\mathbb{R}} f(x) \chi_{[-M,M]}(x) e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \, dx. \]  

(5)

Now the second step of the proof of Theorem 1 claims that the Gaussian’s derivatives may be approximated by divided backward differences
\[ \frac{d^n}{dx^n} e^{-x^2} \approx \frac{1}{t^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{d^n}{dx^n} e^{-(x-kt)^2} \]
in the \( L^2(\mathbb{R}) \) norm. We’ll use the “big oh” notation: for a real function \( \Psi(t) \) the statement “\( \Psi(t) = O(t) \) as \( t \to 0 \)” means there exist \( K > 0 \) and \( \delta > 0 \) such that \( |\Psi(t)| < K |t| \) for \( 0 < |t| < \delta \).

**Proposition 2** For each \( n \in \mathbb{N} \) and \( p \in (0, \infty) \)

\[ \left( \int_{\mathbb{R}} \left| \frac{d^n}{dx^n} e^{-x^2} - \frac{1}{t^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{-(x-kt)^2} \right|^p \, dx \right)^{1/p} = O(t). \]

**Proof.** In Appendix 3 the pointwise formula is derived:
\[ \frac{d^n}{dx^n} g(x) = \frac{1}{t^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(x-kt) - \frac{t}{(n+1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+1} g^{(n+1)}(\xi_k) \]
where all of the \( \xi_k \) are between \( x \) and \( x + nt \). Therefore the proposition holds with \( g(x) = e^{-x^2} \) since \( g^{(n+1)}(\xi_k) \) is integrable for each \( k \). This is not perfectly obvious because we don’t have explicit formulæ for the \( \xi_k \). But the tails of \( g^{(n+1)} \) vanish exponentially, the continuity of \( g^{(n+1)} \) guarantees a finite maximum on the bounded interval between the tails, and \( |\xi_k - x| < k |t| \).

Continuing the derivation of the coefficients \( a_n \) we now have for sufficiently small \( t \neq 0 \)
\[ f \approx \sum_{n=0}^{N} b_n \frac{1}{t^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{-(x-kt)^2} = \sum_{k=0}^{N} b_n \frac{1}{t^n} \sum_{n=k}^{N} \binom{n}{k} (-1)^k e^{-(x-kt)^2} \]
(6)

In the last equality we just switched the order of summation (see [9], section 2.4 for an overview of such tricks). Combining (5) and (6) we have

**Theorem 3** For any \( f \in L^2(\mathbb{R}) \) and any \( \epsilon > 0 \) there exist \( N \in \mathbb{N} \) and \( t_0 > 0 \) such that for any \( t \neq 0 \) with \( |t| < t_0 \)
\[ f \approx \sum_{n=0}^{N} a_n e^{-(x-nt)^2} \]
for some choice of \( a_n \in \mathbb{R} \) dependent on \( N \) and \( t \).
If \( f(x) e^{x^2/2} \) is integrable, then one choice of coefficients is,

\[
a_n = \frac{(-1)^n}{n! \sqrt{\pi}} \sum_{k=0}^{N} \frac{1}{(k-n)! (2\pi)^{k/2}} \int_{\mathbb{R}} f(x) e^{x^2} \frac{d^k}{dx^k} \left( e^{-x^2} \right) dx.
\]

If \( f(x) e^{x^2/2} \) is not integrable, replace \( f \) in the above formula with \( f \cdot \chi_{[-M,M]} \) where \( M \) is chosen large enough that \( \| f - f \cdot \chi_{[-M,M]} \|_2 < \epsilon \).

**Remark 4** The approximation in Theorem 3 also holds on \( C[a,b] \) with the uniform norm since the Hermite expansion is uniformly convergent on \( C^2[a,b] \) (see [15], [16]) and the finite difference formula’s error term from Appendix 6 converges to 0 uniformly as \( t \to 0^+ \). The Stone-Weierstrass Theorem does not apply in this situation because linear combinations of Gaussians with a single variance do not form an algebra.

**Remark 5** As a consequence of Theorem 3, for any \( \epsilon > 0 \) the closed linear span of \( \left\{ e^{-(x-s)^2} : s \in [0,\epsilon] \right\} \) is \( L^2(\mathbb{R}) \). It is even sufficient to replace \([0,\epsilon] \) with \( \left\{ \frac{i}{2^j} : i, j \in \mathbb{N} \right\} \cap [0,\epsilon] \).

Let’s explore some concrete examples in applying Theorem 3. Choose an interesting function with discontinuities and some support negative:

\[
f(x) := (x-1)^2 \chi_{[-1,2]}(x) := \begin{cases} (x-1)^2 & \text{for } x \in [-1,2] \\ 0 & \text{otherwise} \end{cases}
\]

and observe graphically:

- **Hermite series** \( N = 20 \)
- **Hermite** \( N = 40 \)

- **Theorem 3** \( N = 20, t = .05 \)
- **Theorem 3** \( N = 20, t = .01 \)
- **Theorem 3** \( N = 40, t = .01 \)
The Hermite approximation is slowed by discontinuities, but does converge. The next choice of $f$ is continuous but not smooth.

$$f(x) := (\sin x) \chi_{[-\pi,\pi]}(x)$$

Hermite expansion $N = 10$

Hermite expansion $N = 20$

In section 6 we review a standard technique accelerating this convergence in $t$. In our experiments, though, we’ve found the Hermite expansion is generally the bottleneck, not the round-off error of the derivative approximations for $e^{-x^2}$.

We need about 120 terms before visual accuracy is achieved for this simple function. There is a host of methods in the literature for improving convergence of the Hermite expansion, but generally we have better success with functions that are smooth and bounded. Our last examples in this section illustrate how convergence is faster for functions which are smooth and “clamped off”,

7
meaning multiplied by $(x - a)^n (x + a)^n \chi_{[-a,a]}$ whether or not they are positive or symmetric.

3 Calculating the coefficients with least squares

Theorem 1 promises any $L^2$ function can be approximated $f(x) \approx \sum_{n=0}^{N} a_n e^{-(x-nt)^2}$.

Theorem 3 gives a formula for the coefficients $a_n$ but this formula is not unique, and in fact is not “best” according to the classical continuous least squares technique.
In least squares we minimize the error function

\[ E_2(a_0, ..., a_N) := \int_\mathbb{R} \left| f(x) - \sum_{n=0}^{N} a_n e^{-(x-nt)^2} \right|^2 dx \]

by setting \( \frac{\partial E_2}{\partial a_j} = 0 \) for \( j = 0, ..., N \) and solving for the \( a_n \). These \( N + 1 \) linear equations are called the normal equations. The matrix form of this system is \( M \overrightarrow{v} = \overrightarrow{b} \) where \( M \) is the matrix

\[
M = \left[ \sqrt{\frac{\pi}{2}} e^{-\left(\frac{k^2+j^2-(k+j)^2}{2}\right)} \right]_{j,k=0}^{N}
\]

and

\[
\overrightarrow{v} = [a_j]_{j=0}^{N} \quad \text{and} \quad \overrightarrow{b} = \left[ \int_\mathbb{R} f(x) e^{-(x-jt)^2} dx \right]_{j=0}^{N}
\]

\( M \) is symmetric and invertible, so we can always solve for the \( a_n \). But these least squares matrices are notorious for being ill-conditioned when using non-orthogonal approximating functions. The Hilbert matrix is the archetypical example. The current application is no exception since the matrix entries are very similar for most choices of \( N \) and \( t \), so round-off error is extreme. Choosing \( N = 7 \) instead of 5 in the graphed example above requires almost 300 significant digits.

4 Low-frequency trig series are dense in \( L^2 \) with Gaussian weight

For \( f \in L^2(\mathbb{R}, \mathbb{C}) \) define the norm

\[ \|f\|_{2,G} := \left( \int_\mathbb{R} |f(x)|^2 e^{-x^2} dx \right)^{1/2} \]

Write \( f \approx_{\epsilon,G} g \) to mean \( \|f - g\|_{2,G} < \epsilon \).
Theorem 6  For every \( f \in L^2(\mathbb{R}, \mathbb{C}) \) and \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) and \( t_0 > 0 \) such that for any \( t \neq 0 \) with \( |t| < t_0 \)

\[
f(x) \approx_{\epsilon,G} \sum_{n=0}^{N} a_n e^{-intx}
\]

for some choice of \( a_n \in \mathbb{C} \) dependent on \( N \) and \( t \).

Proof. We use the Fourier transform with convention

\[
\mathcal{F}[f](s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-isx} \, dx.
\]

\( \mathcal{F} \) is a linear isometry of \( L^2(\mathbb{R}, \mathbb{C}) \) with

\[
\mathcal{F}[e^{-\alpha x^2}] = \frac{1}{\sqrt{2\alpha}} e^{-s^2/4},
\]

\( \mathcal{F}[f(x+r)] = e^{-irt} \mathcal{F}[f(x)] \quad \text{and} \quad \mathcal{F}[g * h] = \sqrt{2\pi} \mathcal{F}[g] \mathcal{F}[h]. \)

where * is convolution.

Let \( f \in L^2 \) and we now show \( f_2(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2} * \mathcal{F}^{-1}[f](x) \in L^2 \). Notice \( g := \mathcal{F}^{-1}[f] \in L^2 \) and

\[
\|f_2\|^2_2 = \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x-y) e^{-y^2} \, dy \right|^2 ds \leq \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} |g(x-y)|^2 e^{-2y^2} \, dyds
\]

\[
= c \left\| \mathcal{W}_t \right\|_1 \left\| g \right\|_2^2 \leq c \|g\|_1 \|g\|_2^2 = c \|f\|_2^2 < \infty
\]

for some \( c > 0 \). Here \( \mathcal{W}_t[h] \) is the solution to the diffusion equation for time \( t \) and initial condition \( h \). (The notation \( \mathcal{W} \) refers to the Weierstrass transform.) The reason for the third equality in the previous calculation is that \( \mathcal{W}_t \) maintains the \( L^1 \) integral of any positive initial condition \( h \) for all time \( t > 0 \).

Now approximate the real and imaginary parts of \( f_2 \) with Theorem 3. Then we get

\[
\frac{1}{\sqrt{2\pi}} e^{-x^2} * \mathcal{F}^{-1}[f](x) \approx_{\epsilon} \sum_{n=0}^{N} a_n e^{-(x-nt)^2} \quad a_n \in \mathbb{C}
\]

and applying \( \mathcal{F} \) gives

\[
\frac{1}{\sqrt{2\pi}} e^{-s^2/4} f(s) \approx_{\epsilon} \sum_{n=0}^{N} a_n e^{-ints} \frac{1}{\sqrt{2\pi}} e^{-s^2/4}
\]

Hence

\[
f(s) \approx_{\sqrt{2\pi,G}} \sum_{n=0}^{N} a_n e^{-ints}
\]

using the fact that \( e^{-s^2/4} > e^{-s^2} \). □
This result is surprising, even in the context of this paper, because for instance, series of the form \( \sum_{n=-N}^{N} a_n e^{-i(x+nt)} \) for all \( t \) and \( a_n \) are not dense in \( L^2 \) and in fact only inhabit a 4-dimensional subspace of the infinite dimensional Hilbert space \( \mathbb{R}^4 \).

**Corollary 7** On any finite interval \([a,b]\) for any \( \omega > 0 \) the finite linear combinations of sine and cosine functions with frequency lower than \( \omega \) are dense in \( L^2 ([a,b], \mathbb{R}) \).

**Proof.** On \([a,b]\) the Gaussian is bounded and so the norms with or without weight function are equivalent. Apply Theorem 6 to \( f \in L^2 ([a,b], \mathbb{R}) \) and choose \( t \) such that \( Nt < \omega \) to get

\[
f \approx \epsilon \sum_{n=0}^{N} \text{Re} (a_n) \cos (ntx) + \text{Im} (a_n) \sin (ntx)
\]

where

\[
a_n = \frac{(-1)^n}{nt2\pi} \sum_{n=0}^{N} a_n e^{-i \left( k-n \right)! \left( 2 \right)^{k} \int_{\mathbb{R}} e^{-x^2} \ast \mathcal{F}^{-1} [f] (x) e^{x^2} \frac{d^k}{dx^k} \left( e^{-x^2} \right) dx}.
\]

Applying Remark 5 to this result shows even discrete sets of positive frequencies that approach 0 make the span of the corresponding sine and cosine functions equal to \( L^2 ([a,b], \mathbb{R}) \).

Finally, low-frequency cosines span the even functions:

**Proposition 8** On any finite interval \([0,b]\) for any \( \omega > 0 \) the finite linear combinations of cosine functions with frequency lower than \( \omega \) are dense in \( L^2 ([0,b], \mathbb{R}) \).

**Proof.** Let \( f \in L^2 ([0,b], \mathbb{R}) \) and extend it as an even function on \([-b,b]\). Now use the previous corollary to write

\[
f \approx \epsilon \sum_{n=0}^{N} a_n \cos (ntx) + b_n \sin (ntx).
\]

We’d like to conclude right now that the \( b_n = 0 \) or \( b_n \approx 0 \), but that is not true. However, every function \( g \) on \([-b,b]\) may be written uniquely as a sum of even and odd functions

\[
g = g_e + g_o
\]

\[
g_e (x) = \frac{g (x) + g (-x)}{2}
\]

\[
g_o (x) = \frac{g (x) - g (-x)}{2}
\]
and so
\[ g \approx_h \epsilon \Rightarrow g_\epsilon \approx h_\epsilon. \]

Therefore
\[
f = f_\epsilon \approx \epsilon \left[ \sum_{n=0}^{N} a_n \cos (ntx) + b_n \sin (ntx) \right] = \sum_{n=0}^{N} a_n \cos (ntx).
\]

\[ \Box \]

Beware this last result; it’s not as strong as Fourier approximation. The coefficients for the sine functions calculated above may be large; the proposition merely promises the linear combination of the sine terms is small. Using least squares, however, will have vanishing sine coefficients.

5 Origins and generalizations

The mathematical inspiration for Theorem 1 comes from geometrical investigations in infinite dimensional control theory. We noticed that function translation and vector translation in \( L^2(\mathbb{R}) \) do not commute. Specifically, “function translation” is a flow on the infinite dimensional vector space \( L^2(\mathbb{R}) \) given by the map \( F_t : L^2(\mathbb{R}) \times \mathbb{R} \to L^2(\mathbb{R}) \) where \( F_t(f)(x) := f(x+t) \). “Vector translation” in the direction of \( g \in L^2(\mathbb{R}) \) is the flow \( G_t : L^2(\mathbb{R}) \times \mathbb{R} \to L^2(\mathbb{R}) \) where \( G_t(f) := f + tg \). Taking for example \( g(x) := e^{-x^2} \) and composing \( F \) and \( G \) we see \( F_t \circ G_t \neq G_t \circ F_t \) since for \( f \equiv 0 \)
\[
F_t \circ G_t(f)(x) = te^{-(x+t)^2} \quad \text{while} \quad G_t \circ F_t(f)(x) = te^{-x^2}.
\]

Notice however the key fact
\[
\frac{F_t \circ G_t - G_t \circ F_t}{t^2} (f) \to \frac{d}{dx} \left(e^{-x^2}\right) \quad \text{as } t \to 0
\]

In finite dimensions the commutator quotient above gives the Lie bracket \([X,Y]\) of the vector fields \(X\) and \(Y\) which generate the flows \(F\) and \(G\), respectively. A fundamental result in finite-dimensional control theory states that the reachable set via \(X\) and \(Y\) is given by the integral surface to the distribution made up of iterated Lie brackets starting from \(X\) and \(Y\) (Chow’s Theorem, which is an interpretation of Frobenius’ Foliation Theorem, see \([13]\), e.g.). The idea we are exploiting is that iterated Lie brackets for our flows \(F\) and \(G\) will give successive derivatives of the Gaussian, whose span is dense in \(L^2(\mathbb{R})\). Consequently, the reachable set via \(F \) and \(G \) from \(f \equiv 0\) should be all of \(L^2(\mathbb{R})\). That is to say, sums of translates and multiples of one Gaussian (with fixed variance) can approximate any integrable function.

Unfortunately this program doesn’t automatically work on the infinite dimensional vector space \(L^2(\mathbb{R})\) since the function translation flow is not generated by a simple vector field on \(L^2(\mathbb{R})\). So instead of studying vector fields, we consider flows as primary. The fundamental results can be rewritten and
still hold in the general context of a metric space \[3\]. Then other functions besides \(g(x) = e^{-x^2}\) can be checked to be derivative generating and other flows may be used in place of translation. E.g., Fourier approximation is achieved using dilation \(F : L^2(\mathbb{R}, \mathbb{C}) \times \mathbb{R} \to L^2(\mathbb{R}, \mathbb{C})\) where \(F_t(f)(x) := f(\epsilon x)\) and \(G_t(f)(x) := f(x) + te^{ix}\). This gives us a general tool for determining the density of various families of functions.

Another opportunity for generalizing the results of this paper presents itself with the observation that Hermite expansions are valid for functions defined on \(\mathbb{C}\) or \(\mathbb{R}^n\) and in spaces of tempered distributions; and divided differences works in all of these spaces as well.

Note also that while the results of section 2 work for uniform approximations of continuous functions on finite intervals (Remark 4), this is an open question for low-frequency trigonometric approximations.

The results of this paper can be ported to the language of control theory where we can then conclude the system

\[u_t = c_1(t) u_x + c_2(t)e^{-x^2}\]  

is bang-bang controllable with controls of the form \(c_1, c_2 : \mathbb{R}^+ \to \{-1, 0, 1\}\). Theorem 3 drives the initial condition \(f \equiv 0\) to any state in \(L^2\) under the system 7, but may be nowhere near optimal for approximating a function such as \(e^{-(x+10)^2}\), since it uses only Gaussians \(e^{-(x+s)^2}\) with choices of \(s << 10\).

Finally, interpreting Theorem 1 in terms of signal analysis, we see a Gaussian filter is a universal synthesizer with arbitrarily short load time. Let \(G(x) := \frac{1}{\sqrt{\pi}} e^{-x^2}\). A Gaussian filter is a linear time-invariant system represented by the operator

\[W(f)(x) := (f * G)(x) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(y) e^{-(s-x)^2} dy.\]

Notice if you feed \(W\) a Dirac delta distribution \(\delta_t\) (an ideal impulse at time \(x = t\)) you get \(W(\delta) = G(x-t)\). Then Theorem 1 gives

**Corollary 9** For any \(f \in L^2(\mathbb{R})\) and any \(\epsilon > 0\) and any \(\tau > 0\) there exists \(t > 0\) and \(N \in \mathbb{N}\) with \(tN < \tau\) such that

\[f \approx W\left(\sum_{n=0}^{N} a_n \delta_{nt}\right)\]

for some choice of \(a_n \in \mathbb{R}\).

Feed a Gaussian filter a linear combination of impulses and we can synthesize any signal and arbitrarily small load time \(\tau\). The design of physical approximations to an analog Gaussian filter are detailed in \[6, 11]\.

### 6 Appendix: Approximating higher derivatives

The results in this paper may be much improved with voluminous techniques available from numerical analysis. E.g., \[8\] gives an algorithm which speeds the
calculation of sums of Gaussians, and [10] explores Hermite expansion acceleration useful in step 1 of the proof of Theorem 1. This section is devoted to reviewing methods which improve the error in step 2, approximating derivatives of the Gaussian with finite differences. We also derive the error formula used in Proposition 2.

Above we approximated derivatives with the formula

\[
\frac{d^n}{dx^n} f(x) = \frac{1}{t^n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kt) + O(t) \quad \text{gives round-off error as } t \to 0^+ \tag{8}
\]

The Nörlund-Rice integral may be of interest for extremely large \( n \) as it avoids the calculation of the binomial coefficient by evaluating a complex integral. In this section, though, we devote our attention to deriving \( n \)-point formulas; these formulas decrease round-off error by increasing the number of evaluations \( f(x + kt) \)–this shrinks the truncation error without sending \( t \to 0 \).

In approximating the \( k \)th derivative with an \( n + 1 \) point formula

\[
f^{(k)}(x) \approx \frac{1}{t^k} \sum_{i=0}^{n} c_i f(x + k_i t)
\]

we wish to calculate the coefficients \( c_i \). In the forward difference method, the \( k_i = i \), but keeping these values general allows us to find the coefficients for the central or backward difference formulas just as easily. The following method for finding the \( c_i \) was shown to us by our student Jeffrey Thornton who rediscovered the formula.

Taylor’s Theorem has

\[
f(x + k_i t) = \sum_{j=0}^{n} \frac{(k_i t)^j}{j!} f^{(j)}(x) + \frac{(k_i t)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_i)
\]

for some \( \xi_i \) between \( x \) and \( x + k_i t \). From this it follows

\[
\sum_{i=0}^{n} c_i f(x + k_i t)
\]

\[
= \begin{bmatrix} f(x) \\ tf'(x) \\ \vdots \\ t^n f^{(n)}(x) \\ t^{n+1} \end{bmatrix}^T \begin{bmatrix} 1 & 1 & \cdots & 1 \\ k_0 & k_1 & \cdots & k_n \\ k_0^2 & k_1^2 & \cdots & k_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ k_0^n & k_1^n & \cdots & k_n^n \\ k_0^{n+1} f^{(n+1)}(\xi_0) & k_1^{n+1} f^{(n+1)}(\xi_1) & \cdots & k_n^{n+1} f^{(n+1)}(\xi_n) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}
\]

14
Now pick \( c = [c_i] \) as a solution to

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
k_0 & k_1 & \cdots & k_n \\
k_0^2 & k_1^2 & \cdots & k_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
k_0^n & k_1^n & \cdots & k_n^n \\
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_n \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
1 \\
\vdots \\
0 \\
\end{bmatrix}
\]

(9)

which is possible since the \( k_i \) are different, so the matrix is invertible, as is seen using the Vandermonde determinant

\[
\det = \prod_{0 \leq i < j \leq n} (k_j - k_i) \prod_{2 \leq i \leq n} i!.
\]

Then we must have

\[
\sum_{i=0}^{n} c_i f(x + k_i t) =
\begin{bmatrix}
f(x) \\
tf'(x) \\
\vdots \\
t^n f^{(n)} (x) \\
\end{bmatrix}^T
\begin{bmatrix}
0 \\
1 \text{ (k-th position)} \\
\vdots \\
0 \\
\end{bmatrix}
\]

\[
= t^k f^{(k)} (x) + \frac{t^{n+1}}{(n+1)!} \sum_{i=0}^{n} c_i k_i^{n+1} f^{(n+1)}(\xi_i).
\]

Therefore

\[
f^{(k)} (x) = \frac{1}{tk} \sum_{i=0}^{n} c_i f(x + k_i t) + \text{Error}
\]

for \( c_i \) which satisfy (9) where

\[
\text{Error} = -\frac{t^{n+1-k}}{(n+1)!} \sum_{i=0}^{n} c_i k_i^{n+1} f^{(n+1)}(\xi_i).
\]

This \text{Error} formula shows how truncation error may be decreased by increasing \( n \) without shrinking \( t \), thus combating round-off error at the expense of increased computation of sums.

The coefficients in (8) are obtained by solving \( M \) for the \( c_i \) with \( k_i \) chosen as \( k_i = i \).

Thornton also points out that the \( k_i \) may be chosen as complex values when \( f \) is analytic (as is the case with our Gaussians). This gives us another opportunity to mitigate round-off error, since a greater quantity of regularly-spaced nodes \( k_i \) can be packed into an epsilon ball around zero in the complex plane than on the real line.
As final note we mention there have been numerous advances to the present
day in inverting the Vandermonde matrix. We mention only the earliest appli-
cation to numerical differentiation [14] which gives a formula in terms of the
Stirling numbers.

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