Non-Abelian Flux Tubes in $\mathcal{N} = 1$ SQCD: Supersizing World-Sheet Supersymmetry

M. Shifman$^{a,b}$ and A. Yung$^{a,b,c}$

$^a$William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA
$^b$Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia
$^c$Institute of Theoretical and Experimental Physics, Moscow 117259, Russia

Abstract

We consider non-Abelian 1/2 BPS flux tubes (strings) in a deformed $\mathcal{N} = 2$ supersymmetric gauge theory, with mass terms $\mu_{1,2}$ of the adjoint fields breaking $\mathcal{N} = 2$ down to $\mathcal{N} = 1$. The main feature of the non-Abelian strings is the occurrence of orientational moduli associated with the possibility of rotations of their color fluxes inside a global SU($N$) group. The bulk four-dimensional theory has four supercharges; half-criticality of the non-Abelian strings would imply then $\mathcal{N} = 1$ supersymmetry on the world sheet, i.e. two supercharges. In fact, superalgebra of the reduced moduli space has four supercharges. Internal dynamics of the orientational moduli are described by two-dimensional $CP(N-1)$ model on the string world sheet. We focus mainly on the SU(2) case, i.e. $CP(1)$ world-sheet theory. We show that non-Abelian BPS strings exist for all values of $\mu_{1,2}$. The low-energy theory of moduli is indeed $CP(1)$, with four supercharges, in a wide region of breaking parameters $\mu_{1,2}$. Only in the limit of very large $\mu_{1,2}$, above some critical value, the $\mathcal{N} = 2$ world-sheet supersymmetry breaks down to $\mathcal{N} = 1$.

We observe “supersymmetry emergence” for the flux-tube junction (confined monopole): the kink–monopole is half-critical considered from the standpoint of the world-sheet $CP(1)$ model (i.e. two supercharges conserved), while in the bulk $\mathcal{N} = 1$ theory there is no monopole central charge at all.
## Contents

1 Introduction .................................................. 2

2 The bulk theory ............................................... 5
   2.1 Abelian bulk theory ...................................... 5
   2.2 Non-Abelian bulk theory ................................ 7

3 Non-Abelian strings .......................................... 12
   3.1 The non-Abelian string solution ....................... 13
   3.2 World-sheet effective theory ............................ 15

4 Fermion zero modes ........................................... 17
   4.1 $\mathcal{N} = 2$ limit .................................. 18
   4.2 Breaking $\mathcal{N} = 2$ supersymmetry ................ 21
   4.3 The large $\mu$ limit ................................... 24

5 Effective world-sheet theory in the large-$\mu$ limit .... 27

6 Limits of applicability ....................................... 28

7 Non-Abelian monopoles in $\mathcal{N} = 1$ ................. 29

8 Non-Abelian strings in $\mathcal{N} = 1$ SQCD ............... 32

9 Abelian strings ............................................... 33

10 Conclusions .................................................. 38
1 Introduction

Recently, significant progress has been achieved in obtaining non-Abelian strings in four-dimensional Yang–Mills theories (to be referred to as bulk theories), both $\mathcal{N} = 2$ supersymmetric and non-supersymmetric [1, 2, 3, 4, 5, 6, 7, 8]. A variety of models emerged which support non-Abelian magnetic flux tubes and non-Abelian confined magnetic monopoles at weak coupling. The non-Abelian strings are characterized by the presence of orientational moduli associated with the rotation of their color flux in the non-Abelian gauge group SU($N$). In supersymmetric bulk theories the non-Abelian strings are 1/2 BPS saturated. The low-energy world-sheet theory describing moduli dynamics turns out to be supersymmetric $CP(N - 1)$ model.

As well-known, critical solitons in field theory generally exhibit a moduli space $\mathcal{M}$ which (locally) admits the decomposition

$$\mathcal{M} \to \mathcal{M}_{\text{SUSY}} \times \tilde{\mathcal{M}} \quad (1.1)$$

where $\mathcal{M}_{\text{SUSY}}$ refers to the sector associated with bosonic generators in the superalgebra which are broken by the given soliton, by virtue of the introduction of central charges [9], plus their fermionic counterparts. In the case at hand, magnetic flux tubes, two translations are spontaneously broken. Realization of supersymmetry in this sector, associated with the unbroken generators (a half of translations and supertranslations are unbroken in the problem to be considered below) is fully fixed by flat geometry.

At the same time, $\tilde{\mathcal{M}}$ in Eq. (1.1), the reduced moduli space, associated with internal symmetries and the corresponding moduli, can have realizations of supersymmetry that are more contrived. A phenomenon of this type — supersymmetry enhancement — was discovered in Ref. [10] in the domain wall problem. The world-sheet dynamics on $\tilde{\mathcal{M}}$, at the level of two derivatives, were described [10] by a three-dimensional model which had twice more supercharges than one could have a priori expected.

In the present work we report similar results for the non-Abelian strings which emerge as topological defects in some $\mathcal{N} = 1$ four-dimensional super-Yang–Mills models with matter. The bulk model has four supercharges; the strings under consideration are 1/2 BPS. One could expect two supercharges in the world-sheet algebra. At the same time, the low-energy theory of moduli on the string world sheet — the $CP(N - 1)$ model — has four supercharges. Two extra (or “supernumerary”) supercharges which are realized on $\tilde{\mathcal{M}}$ cannot be lifted to supercharges of the bulk theory. Thus, the phenomenon of supersymmetry enhancement, or supersizing of the world-

\footnote{See Sect. 9 for more precise statements.}
sheet supersymmetry,\(^2\) is of a rather general nature and is not rare. It has a geometric origin and can be traced back to the Kähler structure of the reduced moduli space.\(^3\)

A particular bulk theory we will deal with is a deformed \(\mathcal{N} = 2\) supersymmetric \(SU(N) \times U(1)\) theory. This model has been already heavily exploited \([4]\) in the context of non-Abelian strings previously. Deformation discussed in \([4]\) was a linear in \(A\) superpotential term, where \(A\) is the adjoint superfield \(\in U(1)\). This deformation is known to be \(\mathcal{N} = 2\) preserving. Now, instead, we introduce mass terms \(\mu_{1,2}\) of the adjoint superfields \(A^a\) and \(A\) which certainly break \(\mathcal{N} = 2\) down to \(\mathcal{N} = 1\).

Thus, the bulk four-dimensional theory has four supercharges. Concentrating mainly on the simplest case of \(SU(2) \times U(1)\) we construct 1/2 BPS non-Abelian string solution exploiting techniques worked out previously. Because of half-criticality of our solution, \(a \text{ priori}\) we could expect two supercharges on the reduced moduli space, i.e. an \(\mathcal{N} = 1\) low-energy theory of moduli. This is not what actually happens. We show, by performing an explicit analysis of the zero modes, that the world-sheet theory on the reduced moduli space is the supersymmetric \(CP(1)\) model (at the level of two derivatives). This model has \(\mathcal{N} = 2\) i.e. four supercharges (for a review see e.g. \([12]\)).

The (real) dimension of the bosonic part of \(\tilde{M}\) is two. The necessary condition for the enhancement of supersymmetry is the occurrence of four fermion zero modes. Thus, the most crucial and most technically involved part of the analysis of the zero modes is that of the fermion zero modes. Their construction is carried out explicitly, including two extra modes. Once we obtain four fermion zero modes and introduce corresponding fermion moduli (four), combining this with the knowledge that \(\mathcal{N} = 1\) supersymmetry on the world sheet is automatic, the Kähler structure of \(\tilde{M}\) immediately implies the full-blown \(\mathcal{N} = 2\).

Then we address the issue of the evolution of the mass deformation. Indeed, as \(\mu_{1,2} \to \infty\) (in fact, we only need \(\mu \gg \sqrt{\xi}\), where \(\xi\) is a Fayet–Iliopoulos parameter), the adjoint fields \(A^a\) and \(A\) become very heavy and decouple from the bulk theory altogether, leading to \(\mathcal{N} = 1\) SQCD with the gauge group \(SU(2) \times U(1)\). It is known \([13]\) that \(\mathcal{N} = 1\) SQCD admits only Abelian BPS strings. The question is what happens with our non-Abelian 1/2 BPS strings as the parameters \(\mu_{1,2}\) grow.

This question turns out to be subtle. It turns out that the parameter \(\xi/\mu\) plays the role of an infrared regulator. Physically, at \(\mu \gg \sqrt{\xi}\) the adjoint fields do decouple. However, in the limit \(\mu \to \infty\), after the decoupling, the emerging \(\mathcal{N} = 1\) SQCD develops a Higgs branch, which is absent for any finite \(\mu\). At any finite \(\mu\) the vacuum

\(^2\)Supersizing supersymmetry is a part of the title of the talk delivered by Adam Ritz at Continuous Advances in QCD 2004, see \([11]\), devoted to supersymmetry enhancement in the problem of domain walls.

\(^3\)That supersymmetry enhancement could take place in flux-tube problems was conjectured \([11]\) shortly after publication \([10]\).
manifold is an isolated point, which makes the string solution, as well as zero modes, well-defined. If \( \mu \) is large but finite, the mass of the would-be moduli corresponding to the “motion” along the Higgs branch is \( \sim \frac{\xi}{\mu} \).

Thus, there is a seemingly irreconcilable contradiction. On the one hand, it is clear that at \( \mu \gg \sqrt{\xi} \) we must recover \( \mathcal{N} = 1 \) SQCD. On the other hand, in the BPS string analysis the limit \( \mu \to 0 \) seemingly cannot be taken.

A way out was in fact suggested in the literature in the context of a similar problem [14]. In Ref. [14] Abrikosov-Nielsen-Olesen (ANO) strings [15] were considered on the Higgs branch of an \( \mathcal{N} = 2 \) gauge theory (with massive fundamental matter). Common wisdom says [16] that there are no ANO strings in this case (it would be more accurate to say that they inflate and become infinitely thick), because of the same infrared problem. It was discovered, however, that strings of finite size \( L \) are perfectly well-defined, no matter how large \( L \) is. The role of \( L \) is to provide an infrared regularization. The string thickness was found [14] to be proportional to \( \ln L \), while the string mass \( \sim \frac{L}{\ln L} \) rather than pure \( L \) in the classical ANO case.

If we do the same thing in our problem — i.e. consider a finite-length string — the limit \( \mu \to \infty \) will become perfectly well-defined. The parameter \( \mu/\xi \) will be replaced by \( L \), which will provide infrared regularization. Unlike the problem considered in [14], in the present case the infrared divergence does not appear in the bosonic string solution per se. It is only the “extra” fermion zero mode normalization that is plagued by logarithmic divergence.

There is a price one has to pay for the finite-length regularization — the loss of “BPS-ness.” Since “BPS-ness” is a convenient feature, we find the finite-\( \mu \) regularization to be more appropriate, even though it requires inclusion of the adjoint fields in the bulk Lagrangian. This seems to be a smaller price. Once we stick to the finite-\( \mu \) regularization and normalizability of four fermion zero modes is achieved, the low-energy theory of moduli exhibits supersymmetry enhancement. The normalizing parameter, which depends on \( \mu \) logarithmically, can be absorbed in the definition of the moduli fields and does not show up explicitly.

Thus, if \( \mathcal{N} = 1 \) bulk theory has an isolated vacuum (no Higgs branch) we can state with certainty that the low-energy moduli theory on the world sheet of the non-Abelian BPS string is indeed \( CP(1) \), with four supercharges, as long as we limit ourselves to two-derivative terms in the world-sheet Lagrangian.

It is necessary to stress that although many features of the analysis reported here are parallel to those of the domain-wall problem [10], some important features are rather different. In particular, a Kähler structure for the moduli space, which appears automatic, is not sufficient now, generally speaking, for enhanced SUSY, since the Lorentz invariance in 1+1 dimensions imposes no useful constraints (as opposed to
the situation [10] in 1+2 dimensions). Indeed, in the pure $\mathcal{N} = 1$ limit (i.e with $\mu_{1,2} = \infty$), the Kähler structure for the bosonic moduli space persists. Then the minimal $\mathcal{N} = 1$ world-sheet SUSY will be realized in the chiral (0,2) form consistent with the complex structure.

In the flux-tube problem it is not the Lorentz invariance which ensures the one-to-two matching of bosonic versus fermionic zero modes, but, instead, the possibility of embedding the system within $\mathcal{N} = 2$ SQCD. This possibility was not available in the domain-wall case [10].

As a warm up exercise we will also consider a seemingly well-studied problem of the ANO strings in $\mathcal{N} = 1$ SQED. Of course, in this case, the internal moduli space $\tilde{M}$ is absent. However, following the same line of reasoning as in the case of non-Abelian strings above, we can start from $\mathcal{N} = 2$ SQED [17] (eight supercharges), construct the Abelian half-critical string which has four fermion moduli in $\mathcal{M}_{\text{SUSY}}$ and then make the adjoint mass deformation term very large effectively returning to $\mathcal{N} = 1$ SQED. For arbitrarily large but finite $\mu$ we will keep all four fermion zero modes: two natural and two “extra.” Correspondingly, we will keep the $\mathcal{N} = 2$ theory of moduli from $\mathcal{M}_{\text{SUSY}}$. Of course, in this methodical example it is a trivial free field theory (in 1+1 dimensions).

## 2 The bulk theory

In this section we will briefly describe the bulk theories we will deal with. $\mathcal{N} = 2$ SQED is discussed in detail in Ref. [17] while the version of SQCD we will focus on is thoroughly discussed in Refs. [3, 4].

### 2.1 Abelian bulk theory

Let us denote scalar and fermion fields in the “quark” hypermultiplets as $q$, $\tilde{q}$ and $\psi$, and $\tilde{\psi}$, respectively. Note that the scalars form a doublet under the action of global SU(2)$_R$ group, $q^I = (q, \tilde{q})$. In terms of these fields the action of $\mathcal{N} = 2$ SQED deformed by the ($\mathcal{N} = 2$)-breaking mass term $\mu$ of the adjoint field $a$ takes the form

$$
S_{\text{SQED}} = \int d^4x \left\{ \frac{1}{4e^2} F^2_{\mu\nu} + \frac{1}{e^2} |\partial_\mu a|^2 + \nabla_\mu q \nabla_\mu \tilde{q} + \nabla_\mu \bar{q} \nabla_\mu \bar{\tilde{q}} \right\}
$$

Here and below we use a formally Euclidean notation, e.g. $F^2_{\mu\nu} = 2F^2_{\mu0} + F^2_{ij}$, $(\partial_\mu a)^2 = (\partial_0 a)^2 + (\partial_i a)^2$, etc. This is appropriate since we are going to study static (time-independent) field configurations, and $A_0 = 0$. Then the Euclidean action is nothing but the energy functional. Furthermore, we define $\sigma^{\alpha\dot{\alpha}} = (1, -i\vec{\tau})$, $\tilde{\sigma}^{\dot{\alpha}\dot{\alpha}} = (1, i\vec{\tau})$. Lowing and raising of spinor indices is performed by virtue of the antisymmetric tensor defined as $\varepsilon_{12} = \varepsilon_{i\dot{i}} = 1$, $\varepsilon^{12} = \varepsilon^{i\dot{i}} = -1$. The same raising and lowering convention applies to the flavor SU(2) indices $f$, $g$, etc., see [4].
\[ + \frac{e^2}{8} \left( |q|^2 - |\bar{q}|^2 - 2\xi \right)^2 + \frac{e^2}{2} |\bar{q}q + \sqrt{2}\mu a|^2 \]
\[ + \frac{1}{2} \left( |q|^2 + |\bar{q}|^2 \right) |a|^2 \}
+ \text{fermion part}, \quad (2.1) \]

where
\[ \nabla_\mu = \partial_\mu - \frac{i}{2} A_\mu, \quad \bar{\nabla}_\mu = \partial_\mu + \frac{i}{2} A_\mu, \]

while \( \xi \) is the Fayet–Iliopoulos (FI) parameter. The vacuum in this theory is determined (up to gauge transformations) by the following vacuum expectation values (VEV’s):
\[ \langle q \rangle = \sqrt{\xi}, \quad \langle \bar{q} \rangle = 0, \quad \langle a \rangle = 0. \quad (2.2) \]

The nonvanishing VEV of the squark field breaks U(1) gauge symmetry giving mass to the photon.

The mass spectrum of the theory in the vacuum (2.2) was studied in Ref. [17], see also [18]. At non-zero \( \mu \), extended \( \mathcal{N} = 2 \) supersymmetry in (2.1) is broken down to \( \mathcal{N} = 1 \) and the states come in \( \mathcal{N} = 1 \) supermultiplets. The massive vector multiplet has the mass
\[ m_\gamma = \frac{g}{\sqrt{2}} \sqrt{\xi}, \quad (2.3) \]

while two chiral multiplets acquire masses
\[ m^\pm = \frac{g}{\sqrt{2}} \sqrt{\xi \lambda^\pm}, \quad (2.4) \]

where \( \lambda^\pm \) are two roots of the quadratic equation
\[ \lambda^2 - \lambda (2 + \omega^2) + 1 = 0 \quad (2.5) \]
and \( \omega \) is the \( \mathcal{N} = 2 \) breaking parameter
\[ \omega = \frac{\sqrt{2}g\mu}{\sqrt{\xi}}. \quad (2.6) \]

At \( \mu = 0 \) one gets
\[ \lambda^\pm = 1, \]
and all states listed above form the bosonic part of one long \( \mathcal{N} = 2 \) massive vector multiplet [17]. As we switch the parameter \( \mu \) on, this \( \mathcal{N} = 2 \) vector multiplet splits into one vector and two chiral multiplets of \( \mathcal{N} = 1 \) supersymmetric theory.

In the limit of \( \mu \to \infty \) the heavy neutral field \( a \) and its superpartners can be integrated out [19, 20, 17] leading to \( \mathcal{N} = 1 \) SQED
\[ S = \int d^4x \left\{ \frac{1}{4e^2} (F_{\mu\nu})^2 + |\nabla_\mu q|^2 + |\nabla_\mu \bar{q}|^2 + \frac{e^2}{8} (\bar{q}q - \bar{\bar{q}}\bar{q} - \xi)^2 \right\}. \quad (2.7) \]
This theory has a two-dimensional Higgs branch of a hyperbolic form. As we increase \( \mu \) in (2.1) we arrive, in the limit \( \mu \to \infty \), at a base point on this Higgs branch with \( \langle \tilde{q} \rangle = 0 \).

### 2.2 Non-Abelian bulk theory

The content of this section is a direct non-Abelian generalization of Sect. 2.1. The gauge symmetry of the model we will use is \( SU(2) \times U(1) \). Besides the gauge bosons, gauginos and their superpartners, it has a matter sector consisting of two “quark” hypermultiplets, with degenerate masses. In addition, we introduce a Fayet–Iliopoulos \( D \)-term for the \( U(1) \) gauge field which triggers the quark condensation.

Let us first discuss the undeformed theory with \( \mathcal{N} = 2 \). The superpotential has the form

\[
W_{\mathcal{N}=2} = \frac{1}{\sqrt{2}} \sum_{A=1}^{2} \left( \tilde{q}_A A^a q_A^A + \tilde{q}_A A^a \tau^a q_A^A \right)
\]

(2.8)

where \( A^a \) and \( A \) are chiral superfields, the \( \mathcal{N} = 2 \) superpartners of the gauge bosons of \( SU(2) \) and \( U(1) \), respectively. Furthermore, \( q_A \) and \( \tilde{q}_A \) \((A = 1, 2)\) represent two matter hypermultiplets. The flavor index is denoted by \( A \). Thus, in our model the number of colors equals the number of flavors.

Next we add a superpotential mass term which breaks supersymmetry down to \( \mathcal{N} = 1 \), namely,

\[
W_{\mathcal{N}=1} = \frac{\mu_1}{2} A^2 + \frac{\mu_2}{2} (A^a)^2,
\]

(2.9)

where \( \mu_1 \) and \( \mu_2 \) are mass parameters for the chiral superfields in \( \mathcal{N} = 2 \) gauge supermultiplets, \( U(1) \) and \( SU(2) \) respectively. Clearly, the mass term (2.9) splits these supermultiplets, breaking \( \mathcal{N} = 2 \) supersymmetry down to \( \mathcal{N} = 1 \).

The bosonic part of our \( SU(2) \times U(1) \) theory has the form

\[
S = \int d^4x \left[ \frac{1}{4g_1^2} (F_{\mu\nu}^a)^2 + \frac{1}{4g_2^2} (F_{\mu\nu})^2 + \frac{1}{g_1^2} |D_\mu a|^2 + \frac{1}{g_2^2} |\partial_\mu a|^2 + \left| \nabla_\mu q^A \right|^2 + \left| \nabla_\mu \tilde{q}^A \right|^2 + V(q^A, \tilde{q}_A, a^a, a) \right].
\]

(2.10)

Here \( D_\mu \) is the covariant derivative in the adjoint representation of \( SU(2) \), while

\[
\nabla_\mu = \partial_\mu - \frac{i}{2} A_\mu - i A^a_\mu \tau^a \frac{1}{2},
\]

(2.11)

and \( \tau^a \) are the \( SU(2) \) Pauli matrices. The coupling constants \( g_1 \) and \( g_2 \) correspond to the \( U(1) \) and \( SU(2) \) sectors, respectively. With our conventions the \( U(1) \) charges of the fundamental matter fields are \( \pm 1/2 \).
The potential \( V(q^A, \bar{q}_A, a^a, a) \) in the Lagrangian (2.10) is a sum of various \( D \) and \( F \) terms,

\[
V(q^A, \bar{q}_A, a^a, a) = \frac{g_2^2}{2} \left( \frac{1}{g_2^2} \varepsilon^{abc} \bar{a}^b q^c + \bar{q}_A \frac{\tau^a}{2} q^A - \bar{q}_A \frac{\tau^a}{2} \bar{q}_a \right)^2 \\
+ \frac{g_1}{8} \left( \bar{q}_A q^A - \bar{q}_A \bar{q}^A - 2\xi \right)^2 \\
+ \frac{g_2^2}{2} \left| \bar{q}_A \tau^a q^A + \sqrt{2} \mu_2 a^a \right|^2 + \frac{g_1^2}{2} \left| \bar{q}_A q^A + \sqrt{2} \mu_1 a \right|^2 \\
+ \frac{1}{2} \sum_{A=1}^2 \left\{ \left| (a + \tau^a a^a) q^A \right|^2 + \left| (a + \tau^a a^a) \bar{q}^A \right|^2 \right\}, \tag{2.12}
\]

where the sum over repeated flavor indices \( A \) is implied. The first and second lines here represent \( D \) terms, the third line the \( F_A \) terms, while the fourth line represents the squark \( F \) terms. We also introduced the Fayet–Iliopoulos \( D \)-term for the \( U(1) \) field, with the FI parameter \( \xi \) in (2.12), much in the same way as in Sect. 2.1. Note that the Fayet–Iliopoulos term does not break \( \mathcal{N} = 2 \) supersymmetry [21, 17]. The parameters which do break \( \mathcal{N} = 2 \) down to \( \mathcal{N} = 1 \) are \( \mu_1 \) and \( \mu_2 \).

The Fayet–Iliopoulos term triggers the spontaneous breaking of the gauge symmetry. The vacuum expectation values (VEV’s) of the squark fields can be chosen as

\[
\langle q^kA \rangle = \sqrt{\xi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \langle \bar{q}^kA \rangle = 0,
\]

\[
k = 1, 2, \quad A = 1, 2, \tag{2.13}
\]

up to gauge rotations, while the VEV’s of adjoint fields are given by

\[
\langle a^a \rangle = 0, \quad \langle a \rangle = 0. \tag{2.14}
\]

Here we write down \( q \) as a \( 2 \times 2 \) matrix, the first superscript \( (k = 1, 2) \) refers to \( SU(2) \) color, while the second \( (A = 1, 2) \) to flavor.

The color-flavor locked form of the quark VEV’s in Eq. (2.13) and the absence of VEV of the adjoint scalar \( a^a \) in Eq. (2.14) results in the fact that, while the theory is fully Higgsed, a diagonal \( SU(2)_{C+F} \) survives as a global symmetry. This is a particular case of the Bardakci-Halpern mechanism [22]. The presence of this symmetry leads to the emergence of orientational zero modes of \( Z_2 \) strings in the model (2.10) [3].

Note that VEV’s (2.13) and (2.14) do not depend on the supersymmetry breaking parameters \( \mu_1 \) and \( \mu_2 \). This is because our choice of parameters in (2.10) ensures
vanishing of the adjoint VEV’s, see (2.14). In particular, we have the same pattern of symmetry breaking all the way up to very large $\mu_1$ and $\mu_2$, where the adjoint fields decouple.

With two matter hypermultiplets, the SU(2) part of the gauge group is asymptotically free, implying generation of a dynamical scale $\Lambda$. If descent to $\Lambda$ were uninterrupted, the gauge coupling $g_2^2$ would explode at this scale. Moreover, strong coupling effects in the SU(2) subsector at the scale $\Lambda$ would break the SU(2) subgroup through the Seiberg-Witten mechanism [23]. Since we want to stay at weak coupling we assume that $\sqrt{\xi} \gg \Lambda$, so that the SU(2) coupling running is frozen by the squark condensation at a small value, namely,

$$\frac{8\pi^2}{g_2^2} = 2 \ln \frac{\sqrt{\xi}}{\Lambda} + \cdots \gg 1.$$  \hspace{1cm} (2.15)

Now let us discuss the mass spectrum in the theory (2.10). Since both U(1) and SU(2) gauge groups are broken by the squark condensation, all gauge bosons become massive. From (2.10) we get for the U(1) gauge boson

$$m_{U(1)} = g_1 \sqrt{\xi},$$  \hspace{1cm} (2.16)

while three gauge bosons of the SU(2) group acquire the same mass

$$m_{SU(2)} = g_2 \sqrt{\xi}.$$  \hspace{1cm} (2.17)

To get the masses of the scalar bosons we expand the potential (2.12) near the vacuum (2.13), (2.14) and diagonalize the corresponding mass matrix. The four components of the eight-component \footnote{We mean here eight real components.} scalar $q^{kA}$ are eaten by the Higgs mechanism for U(1) and SU(2) gauge groups. Another four components are split as follows: one component acquires the mass (2.16). It becomes a scalar component of a massive $\mathcal{N} = 1$ vector U(1) gauge multiplet. Other three components acquire masses (2.17) and become scalar superpartners of the SU(2) gauge boson in $\mathcal{N} = 1$ massive gauge supermultiplet.

Other 16 real scalar components of fields $\tilde{q}_{Ak}$, $a^a$ and $a$ produce the following states: two states acquire mass

$$m_{U(1)}^+ = g_1 \sqrt{\xi \lambda_1^+},$$  \hspace{1cm} (2.18)

while the mass of other two states is given by

$$m_{U(1)}^- = g_1 \sqrt{\xi \lambda_1^-},$$  \hspace{1cm} (2.19)
where $\lambda_1^\pm$ are two roots of the quadratic equation
\begin{equation}
\lambda_i^2 - \lambda_i(2 + \omega_i^2) + 1 = 0 , \tag{2.20}
\end{equation}
for $i = 1$. Here we introduced two $N = 2$ supersymmetry breaking parameters associated with U(1) and SU(2) gauge groups, respectively,
\begin{equation}
\omega_1 = \frac{g_1 \mu_1}{\sqrt{\xi}} , \quad \omega_2 = \frac{g_2 \mu_2}{\sqrt{\xi}} . \tag{2.21}
\end{equation}
Furthermore, other $2 \times 3 = 6$ states acquire mass
\begin{equation}
m_{SU(2)}^+ = g_2 \sqrt{\xi \lambda_2^+} , \tag{2.22}
\end{equation}
while the rest $2 \times 3 = 6$ states also become massive, their mass is
\begin{equation}
m_{SU(2)}^- = g_2 \sqrt{\xi \lambda_2^-} . \tag{2.23}
\end{equation}
Here $\lambda_2^\pm$ are two roots of the quadratic equation (2.20) for $i = 2$. Note that all states come either as singlets or triplets of unbroken SU(2)$_{CF}$.

When the supersymmetry breaking parameters $\omega_i$ vanish, the masses (2.18) and (2.19) coincide with the U(1) gauge boson mass (2.16). The corresponding states form bosonic part of $N = 2$ long massive U(1) vector supermultiplet [17]. With non-zero $\omega_1$ this supermultiplet splits into massive $N = 1$ vector multiplet with mass (2.16), and two chiral multiplets with masses (2.18) and (2.19). The same happens to states with masses (2.22) and (2.23). If $\omega$’s vanish they combine into the bosonic parts of three $N = 2$ massive vector supermultiplets, with mass (2.17). At non-zero $\omega$’s these multiplets split to three $N = 1$ vector multiplets (for SU(2) group) with mass (2.17) and $2 \times 3$ chiral multiplets with masses (2.22) and (2.23). Note that essentially the same pattern of splitting was found in [17] for the Abelian case, see Sect. 2.1.

Now let us take a closer look at the spectrum obtained above in the limit of large $N = 2$ supersymmetry breaking parameters $\omega_i$,
\begin{equation}
\omega_i \gg 1 .
\end{equation}
In this limit the larger masses $m_{U(1)}^+$ and $m_{SU(2)}^+$ become
\begin{equation}
m_{U(1)}^+ = m_{U(1)} \omega_1 = g_1^2 \mu_1 , \quad m_{SU(2)}^+ = m_{SU(2)} \omega_2 = g_2^2 \mu_2 . \tag{2.24}
\end{equation}
Clearly, in the limit $\mu_i \to \infty$ these are the masses of the heavy adjoint scalars $a$ and $a^a$. At $\omega_i \gg 1$ these fields decouple and can be integrated out.
The low-energy bulk theory in this limit contains massive gauge $N = 1$ multiplets and chiral multiplets with lower masses $m^-$. Equation (2.20) gives for these masses

$$m^\text{U(1)}_{-} \frac{m^\text{U(1)}}{\omega_1} = \frac{\xi}{\mu_1}, \quad m^\text{SU(2)}_{-} \frac{m^\text{SU(2)}}{\omega_2} = \frac{\xi}{\mu_2}.$$  \hspace{1cm} (2.25)

In the limit of infinite $\mu$, these masses tend to zero. This fact reflects the emergence of a Higgs branch in $N = 1$ SQCD, see also Eq. (2.7). To observe the Higgs branch it is instructive to inspect the transition to $\mu = \infty$ in (2.10). Equation (2.10) flows to $N = 1$ SQCD with the gauge group $SU(2) \times U(1)$ and the Fayet--Iliopoulos $D$-term,

$$S = \int d^4x \left\{ \frac{1}{4g_2^2} (F^a_{\mu \nu})^2 + \frac{1}{4g_1^2} (F_{\mu \nu})^2 + \left| \nabla_{\mu} q^A \right|^2 + \left| \nabla_{\mu} \bar{q}^A \right|^2 + \frac{g_2^2}{2} \left( q^A_{\alpha} \tau^a_{\alpha} q^A - \bar{q}^A_{\alpha} \tau^a_{\alpha} \bar{q}^A \right)^2 + \frac{g_1^2}{8} \left( q^A_{\alpha} - \bar{q}^A_{\alpha} - 2\xi \right)^2 \right\}. \hspace{1cm} (2.26)$$

All $F$ terms disappear in this limit and we are left only with $D$ terms. For 16 real components of $q$ and $\bar{q}$ we have four $D$-term constraints in (2.26). Another four phases are eaten by the Higgs mechanism. Thus, the dimension of the Higgs branch in (2.26) is $16 - 4 - 4 = 8$. It can be described in terms of a gauge invariant meson matrix

$$M^B_A = \bar{q}^B q^A$$

plus baryon operators\(^6\)

$$B^{A B} = \frac{1}{2} \varepsilon_{k l} q^A k^A q^B l^B, \quad \tilde{B}_{A B} = \frac{1}{2} \varepsilon_{k l} \bar{q}^A k^A \bar{q}^B l^B,$$  \hspace{1cm} (2.28)

see [24] for a review. These operators are subject to a classical constraint

$$\det M - \tilde{B}_{A B} B^{A B} = 0$$  \hspace{1cm} (2.29)

which gets modified by instanton effects and becomes

$$\det M - \tilde{B}_{A B} B^{A B} = \Lambda^4_{N = 1}$$  \hspace{1cm} (2.30)

in the quantum theory [24]. Here $\Lambda_{N = 1}$ is the scale of $N = 1$ SQCD in terms of the scale $\Lambda$ of the deformed $N = 2$ theory (2.10); $\Lambda_{N = 1}$ has the form

$$\Lambda^4_{N = 1} = \mu_2^2 \Lambda^2.$$  \hspace{1cm} (2.31)

\(^6\)The baryon operators are not $U(1)$ gauge invariant in the $SU(2) \times U(1)$ theory. Their product is gauge invariant, however.
In order to keep the bulk theory in the weak coupling regime, in the limit of large $\mu_i$ we assume that

$$\sqrt{\xi} \gg \Lambda_{\mathcal{N}=1}.$$  \hspace{1cm} (2.32)

Note that the presence of the FI term cannot modify (2.30) because $\xi$ is not a holomorphic parameter.

The vacuum (2.13) corresponds to the base point of this Higgs branch with $\tilde{q} = 0$. In other words, flowing from $\mathcal{N} = 2$ theory (2.10) we do not recover the whole Higgs branch of $\mathcal{N} = 1$ SQCD (2.26). Instead, we arrive only at an isolated vacuum, a base point of the Higgs branch, no matter how large $\mu$ is.

What else is there to say? A question to be discussed is as follows: how our solution in which $\tilde{q} = 0$ can be compatible with the quantum constraint (2.30)? It seems apparent that the classical vacuum with $\tilde{q} = 0$ at the base of the Higgs branch no longer exists at the quantum level.

Our analysis is quasiclassical. We start with $\tilde{q} = 0$, so that the corresponding light moduli are not excited. Next we consider quantum corrections. What enters in the constraint (2.30) is the quantum average of the composite operator $\langle \tilde{q} q \rangle$. The above VEV does not factorize, and Eq. (2.30) can still hold in our solution. In fact, we expect it to hold. While the light modes fluctuate along the Higgs branch, the massive modes fluctuate in the “orthogonal” directions. Account of these latter fluctuations must modify the classical constraint (2.29) transforming it into (2.30).

Certainly, it would be instructive to check this explicitly. We leave this exercise for future studies. This issue is of a conceptual importance. Practically, though, it is rather unimportant since we work in the regime (2.32), so that the quantum deformation is parametrically small.

### 3 Non-Abelian strings

Recently, non-Abelian strings were shown to emerge at weak coupling [3, 4, 6, 7] in $\mathcal{N} = 2$ and deformed $\mathcal{N} = 4$ supersymmetric gauge theories (similar results in three dimensions were obtained in [2]). The main feature of the non-Abelian strings is the presence of orientational zero modes associated with rotation of their color flux in the non-Abelian gauge group, which makes such strings genuinely non-Abelian. As soon as the solution for the non-Abelian string suggested in [3, 4] for $\mathcal{N} = 2$ SQCD does not depend on the adjoint fields it can be easily generalized to our model (2.10) with the broken $\mathcal{N} = 2$ supersymmetry. We will carry out this program in Sect. 3.1.
3.1 The non-Abelian string solution

Here we generalize the string solutions found in [3, 4] to the model (2.10). Since this model includes a spontaneously broken gauge $U(1)$, it supports conventional Abrikosov-Nielsen-Olesen (ANO) strings [15] in which one can discard the $SU(2)_{\text{gauge}}$ part of the action. The topological stability of the ANO string is due to the fact that $\pi_1(U(1)) = \mathbb{Z}$. These are not the strings we are interested in. At first sight the triviality of the homotopy group, $\pi_1(SU(2)) = 0$, implies that there are no other topologically stable strings. This impression is false. One can combine the $Z_2$ center of $SU(2)$ with the elements $\exp(i\pi) \in U(1)$ to get a topologically stable string solution possessing both windings, in $SU(2)$ and $U(1)$. In other words,

$$\pi_1(SU(2) \times U(1)/Z_2) \neq 0. \quad (3.1)$$

It is easy to see that this non-trivial topology amounts to winding of just one element of matrix $q_{\text{vac}}$, say, $q^{11}$, or $q^{22}$, for instance,\footnote{As explained below, $\alpha$ is the angle of the coordinate $\vec{x}_{\perp}$ in the perpendicular plane.}

$$q_{\text{string}} = \sqrt{\xi} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}, \quad x \to \infty. \quad (3.2)$$

Such strings can be called elementary; their tension is $1/2$ of that of the ANO string. The ANO string can be viewed as a bound state of two elementary strings.

More concretely, the $Z_2$ string solution (a progenitor of the non-Abelian string) can be written as follows [3]:

$$q(x) = \begin{pmatrix} e^{i\alpha} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix},$$

$$A_3^i(x) = -\varepsilon_{ij} \frac{x_j}{r^2} \left( 1 - f_3(r) \right),$$

$$A_i(x) = -\varepsilon_{ij} \frac{x_j}{r^2} \left( 1 - f(r) \right) \quad (3.3)$$

where $i = 1, 2$ labels coordinates in the plane orthogonal to the string axis and $r$ and $\alpha$ are the polar coordinates in this plane. The profile functions $\phi_1(r)$ and $\phi_2(r)$ determine the profiles of the scalar fields, while $f_3(r)$ and $f(r)$ determine the $SU(2)$ and $U(1)$ gauge fields of the string solutions, respectively. These functions satisfy the following first order equations [3]:

$$r \frac{d}{dr} \phi_1(r) - \frac{1}{2} \left( f(r) + f_3(r) \right) \phi_1(r) = 0,$$
\[ r \frac{d}{dr} \phi_2(r) - \frac{1}{2} \left( f(r) - f_3(r) \right) \phi_2(r) = 0, \]
\[-\frac{1}{r} \frac{d}{dr} f(r) + \frac{g_1^2}{2} \left[ (\phi_1(r))^2 + (\phi_2(r))^2 - 2\xi \right] = 0, \]
\[-\frac{1}{r} \frac{d}{dr} f_3(r) + \frac{g_2^2}{2} \left[ (\phi_1(r))^2 - (\phi_2(r))^2 \right] = 0. \] (3.4)

Furthermore, one needs to specify the boundary conditions which would determine the profile functions in these equations. Namely,
\[ f_3(0) = 1, \quad f(0) = 1; \]
\[ f_3(\infty) = 0, \quad f(\infty) = 0 \] (3.5)

for the gauge fields, while the boundary conditions for the squark fields are
\[ \phi_1(\infty) = \sqrt{\xi}, \quad \phi_2(\infty) = \sqrt{\xi}, \quad \phi_1(0) = 0. \] (3.6)

Note that since the field \( \phi_2 \) does not wind, it need not vanish at the origin, and, in fact, it does not. Numerical solutions of the Bogomolny equations (3.4) for \( Z_2 \) strings were found in Ref. [3], see e.g. Figs. 1 and 2 in this paper.

The tension of this elementary string is
\[ T_1 = 2\pi \xi, \] (3.7)

to be compared with the tension of the ANO string,
\[ T_{\text{ANO}} = 4\pi \xi \] (3.8)
in our normalization.

The elementary strings are \textit{bona fide} non-Abelian. This means that, besides trivial translational moduli, they give rise to moduli corresponding to spontaneous breaking of a non-Abelian symmetry. Indeed, while the “flat” vacuum (2.13) is \( SU(2)_{C+F} \) symmetric, the solution (3.3) breaks this symmetry down to \( U(1) \). This means that the world-sheet (two-dimensional) theory of the elementary string moduli is the \( SU(2)/U(1) \) sigma model. This is also known as \( CP(1) \) model.

To obtain the non-Abelian string solution from the \( Z_2 \) string (3.3) we apply the diagonal color-flavor rotation preserving the vacuum (2.13). To this end it is convenient to pass to the singular gauge where the scalar fields have no winding at infinity, while the string flux comes from the vicinity of the origin. In this gauge we
have

\[ q = U \begin{pmatrix} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1}, \]

\[ A^a_i(x) = n^a \varepsilon_{ij} \frac{x_j}{r^2} f_3(r), \]

\[ A_i(x) = \varepsilon_{ij} \frac{x_j}{r^2} f(r), \]

where \( U \) is a matrix \( \in \text{SU}(2) \) and \( n^a \) is a moduli vector defined as

\[ n^a \tau^a = U \tau^3 U^{-1}, \quad a = 1, 2, 3. \]

It is subject to the constraint

\[ \vec{n}^2 = 1. \]

At \( n = \{0, 0, 1\} \) we get the field configuration quoted in Eq. (3.3).

The vector \( n^a \) parametrizes orientational zero modes of the string associated with flux rotation in \( \text{SU}(2) \). The presence of these modes makes the string genuinely non-Abelian. We stress that the orientational moduli encoded in the vector \( n^a \), first observed in \([2, 3]\), are not gauge artifacts.

### 3.2 World-sheet effective theory

In this subsection we briefly review derivation of the effective world-sheet theory for the orientational collective coordinates \( n^a \) of the non-Abelian string. We follow Ref. [3, 4]. (Generalization to the case of \( \text{SU}(N) \times \text{U}(1) \) gauge group is done in [8].) As was already mentioned, this macroscopic theory is \( CP(1) \) model (\( CP(N-1) \) model for the general case of \( \text{SU}(N) \times \text{U}(1) \) gauge group) \([2, 3, 4, 6, 8]\).

Assume that the orientational collective coordinates \( n^a \) are slowly varying functions of the string world-sheet coordinates \( x_k, k = 0, 3 \). Then the moduli \( n^a \) become fields of a \((1+1)\)-dimensional sigma model on the world sheet. Since the vector \( n^a \) parametrizes the string zero modes, there is no potential term in this sigma model. We begin with the kinetic term \([3]\).

To obtain the kinetic term we substitute our solution, which depends on the moduli \( n^a \), in the action (2.10) assuming that the fields acquire a dependence on the coordinates \( x_k \) via \( n^a(x_k) \). Then we arrive at the \( O(3) \) sigma model

\[ S^{(1+1)} = \frac{\beta}{2} \int dt \, dz \, (\partial_k n^a)^2, \quad \vec{n}^2 = 1, \]

\[ \text{We skip here some details of derivation. The interested reader is referred to Refs. [3, 4].} \]
where the coupling constant $\beta$ is given by a normalizing integral

$$\beta = \frac{2\pi}{g_2^2} \int_0^\infty dr \left\{ -\frac{d}{dr} f_3 + \left( \frac{2}{r} f_3^2 + \frac{d}{dr} f_3 \right) \frac{\phi_1^2}{\phi_2^2} \right\}.$$  \hspace{1cm} (3.13)

Using the first-order equations for the string profile functions (3.4) one can see that the integral here reduces to a total derivative and given by the flux of the string determined by $f_3(0) = 1$. This allows us to conclude that the sigma-model coupling $\beta$ does not depend on the ratio of the U(1) and SU(2) coupling constants and is given by

$$\beta = \frac{2\pi}{g_2^2}.$$  \hspace{1cm} (3.14)

The two-dimensional coupling constant is determined by the four-dimensional non-Abelian coupling.

In summary, the effective world-sheet theory describing dynamics of the string orientational moduli is the celebrated $O(3)$ sigma model (which is the same as $CP^1$). The symmetry of this model reflects the presence of the global $SU(2)_{C+F}$ symmetry in the bulk theory.

The relation between the four-dimensional and two-dimensional coupling constants (3.14) is obtained at the classical level. In quantum theory both couplings run. So we have to specify a scale at which the relation (3.14) takes place. The two-dimensional $CP(1)$ model (3.12) is an effective low-energy theory good for the description of internal string dynamics at low energies, much lower than the inverse thickness of the string which, in turn, is given by $\sqrt{\xi}$. Thus, $\sqrt{\xi}$ plays the role of a physical ultraviolet (UV) cutoff in (3.12). This is the scale at which Eq. (3.14) holds. Below this scale, the coupling $\beta$ runs according to its two-dimensional renormalization-group flow.

The sigma model (3.12) is asymptotically free [25]; at large distances (low energies) it gets into the strong coupling regime. The running coupling constant as a function of the energy scale $E$ at one loop is given by

$$4\pi \beta = 2 \ln \left( \frac{E}{\Lambda_{CP(1)}} \right) + \cdots,$$  \hspace{1cm} (3.15)

where $\Lambda_{CP(1)}$ is the dynamical scale of the $CP(1)$ model. As was mentioned above, the ultraviolet cut-off of the sigma model at hand is determined by $\sqrt{\xi}$. Hence,

$$\Lambda_{CP(1)}^2 = \xi e^{\frac{\pi^2}{g_2^2}}.$$  \hspace{1cm} (3.16)

Note that in the bulk theory, due to the VEV’s of the squark fields, the coupling constant is frozen at $\sqrt{\xi}$. There are no logarithms in the bulk theory below this scale. Below $\sqrt{\xi}$ the logarithms of the world-sheet theory take over.
At small values of the deformation parameter \( \mu_2 \),
\[
\mu_2 \ll \sqrt{\xi},
\]
the coupling constant \( g_2 \) of the four-dimensional bulk theory is determined by the scale \( \Lambda \) of the \( \mathcal{N} = 2 \) theory. Then Eq. (3.16) gives [4]
\[
\Lambda_{CP(1)} = \Lambda,
\]
where we take into account that the first coefficient of the \( \beta \) function equals to 2 both in \( \mathcal{N} = 2 \) limit of the four-dimensional bulk theory and in the two-dimensional \( CP(1) \) model.

Instead, in the limit of large \( \mu_2 \),
\[
\mu_2 \gg \sqrt{\xi},
\]
the coupling constant \( g_2 \) of the bulk theory is determined by the scale \( \Lambda_{\mathcal{N}=1} \) of the \( \mathcal{N} = 1 \) SQCD (2.26), as shown in Eq. (2.31). In this limit Eq. (3.16) gives
\[
\Lambda_{CP(1)} = \frac{\Lambda_{\mathcal{N}=1}^2}{\sqrt{\xi}},
\]
where we take into account that the first coefficient of the \( \beta \) function in \( \mathcal{N} = 1 \) SQCD equals to four. The renormalization group flow in our theory at \( \mu_2 \gg \sqrt{\xi} \) is schematically presented in Fig. 1.

4 Fermion zero modes

Technically, this is a key section of the present work. Let us start from the \( \mathcal{N} = 2 \) theory (2.10) with the breaking parameters set to zero, \( \mu_i = 0 \). Our string solution is 1/2 BPS-saturated. This means that four supercharges, out of eight of the four-dimensional theory (2.10), act trivially on the string solution (3.9). The remaining four supercharges generate four fermion zero modes which we call supertranslational modes because they are superpartners to two translational zero modes. The corresponding four fermionic moduli are superpartners to the coordinates \( x_0 \) and \( y_0 \) of the string center. The supertranslational fermion zero modes were found in Ref. [17]. As a matter of fact, they were found for the U(1) ANO string in \( \mathcal{N} = 2 \) theory but the transition to the model at hand is absolutely straightforward. We will not dwell on this procedure here.

\[9\]We will review them and study their deformation in \( \mu \)-deformed QED (2.1) in Sect. 9.
Figure 1: Renormalization-group (RG) flow in the theory (2.10) at large $\mu_2$. Flow in four dimensions is represented by $N = 1$ and $N = 2$ curves, while $CP(1)$ marks the RG flow in the two-dimensional $CP(1)$ model.

Instead, we will focus below on four additional fermion zero modes which arise only for the non-Abelian strings. They are superpartners of the bosonic orientational moduli $n^a$; therefore, we will refer to these modes as superorientational. In the $N = 2$ limit these modes were obtained in [4]. If we switch on supersymmetry (SUSY) breaking parameters $\mu_i$ the number of supercharges in the four-dimensional bulk theory drops to four. The 1/2 BPS string would have two superorientational fermion zero modes in this theory. However, our string is a descendant of $N = 2$ theory where it has four superorientational zero modes. Clearly the number of zero modes cannot jump as we switch on parameters $\mu_i$, at least at small $\mu$. This number is determined by index theorems. Thus, it is clear that (at least at small $\mu$) our string has a set of superorientational fermion zero modes twice bigger than algebra tells. In this section we elaborate the issue of four zero modes explicitly at small and large $\mu$ while in Sect. 5 we will study the impact of their presence on the $CP(1)$ model on the string world sheet. To begin with, in Sect. 4.1 we review these modes in the $N = 2$ limit and then examine what happens to them in the deformed bulk theory.

4.1 $N = 2$ limit

The fermionic part of the action of the model (2.10) is

$$S_{\text{ferm}} = \int d^4 x \left\{ \frac{i}{g_2^2} \bar{\lambda}_i^a \tilde{D}^a \lambda^i j + \frac{i}{g_1^2} \bar{\lambda}_j^a \tilde{D}^a \lambda^i j + \text{Tr} \left[ \bar{\psi}_i \tilde{\nabla}_\psi \right] + \text{Tr} \left[ \bar{\psi}_i \tilde{\nabla}_\bar{\psi} \right] \right\}$$
\[ + \frac{i}{\sqrt{2}} \text{Tr} \left[ \bar{q}_f (\lambda^f \psi) + (\bar{\psi} \lambda_f) q^f \right] + \frac{i}{\sqrt{2}} \text{Tr} \left[ \bar{q}_f \tau^a (\lambda^a \psi) \right] + \frac{i}{\sqrt{2}} \text{Tr} \left[ \bar{q}_f \tau^a (\lambda^a \bar{\psi}) \right] + \frac{i}{\sqrt{2}} \text{Tr} \left[ \bar{\psi} (a + a^a \tau^a) \psi \right] \]

where the matrix color-flavor notation is used for matter fermions \((\psi^\alpha)^k_A\) and \((\bar{\psi}^\alpha)^A_k\) and the traces are performed over the color–flavor indices. Contraction of the spinor indices is assumed inside all parentheses, for instance,

\[(\lambda \psi) \equiv \lambda^\alpha \psi^\alpha.\]

We write the squark fields in (4.1) as doublets of SU(2)_R group which is present in \(\mathcal{N} = 2\) theory, \(q^f = (q, \bar{q})\). Here \(f = 1, 2\) is the SU(2)_R index which labels two supersymmetries of the bulk theory in the \(\mathcal{N} = 2\) limit. Moreover, \(\lambda^a_f\) and \((\lambda^a)_f\) stand for the gauginos of the U(1) and SU(2) groups, respectively. Note that the last two terms are \(\mathcal{N} = 1\) deformations in the fermion sector of the theory induced by the breaking parameters \(\mu_i\). They involve only \(f = 2\) components of \(\lambda\)'s explicitly breaking the SU(2)_R invariance.

Next, we put \(\mu_i = 0\) and apply the general method which was designed in [4] to generate superorientational fermion zero modes of the non-Abelian string in the \(\mathcal{N} = 2\) case. In Ref. [17] it was shown that the four supercharges selected by the conditions

\[\epsilon^{11} = 0, \quad \epsilon^{22} = 0\]

act trivially on the BPS string in the theory with the Fayet–Iliopoulos D term. Here \(\epsilon^{a_f}\) are parameters of the SUSY transformation.

Now, to generate the superorientational fermion zero modes the following method was used in [4]. Assume that the orientational moduli \(n^a\) in the string solution (3.9) have a slow dependence on the world-sheet coordinates \(x_0\) and \(x_3\) (or \(t\) and \(z\)). Then the four supercharges selected by the conditions (4.2) (namely, \(\epsilon^{12}\), \(\epsilon^{21}\) and their complex conjugates) no longer act trivially. Instead, their action now gives fermion fields proportional to \(n^a\) derivatives of \(n^a\). This is exactly what one expects from the residual \(\mathcal{N} = 2\) supersymmetry in the world-sheet theory. The above four supercharges generate the world-sheet supersymmetry in the \(\mathcal{N} = 2\) two-dimensional \(CP(1)\) model,

\[\delta \chi_a^1 = i\sqrt{2} \left[ (\partial_0 + i\partial_3) n^a \varepsilon_2 + \varepsilon^{abc} n^b (\partial_0 + i\partial_3) n^c \eta_2 \right],\]
\[
\delta \chi^a_2 = i\sqrt{2} \left[ (\partial_0 - i\partial_3) n^a \varepsilon_1 + \varepsilon^{abc} n^b (\partial_0 - i\partial_3) n^c \eta_1 \right],
\]

where \( \chi^a_\alpha (\alpha = 1, 2 \) is the spinor index) are real two-dimensional fermions of the \( CP(1) \) model. They are superpartners of \( n^a \) and subject to orthogonality condition \( n^a \chi^a_\alpha = 0 \). Real parameters of \( \mathcal{N} = 2 \) two-dimensional SUSY transformation \( \varepsilon_\alpha \) and \( \eta_\alpha \) are identified with the parameters of the four-dimensional SUSY transformations (with the constraint (4.2)) as

\[
\varepsilon_1 - i\eta_1 = \epsilon^{21},
\]
\[
\varepsilon_2 + i\eta_2 = -\epsilon^{12}.
\]

The world-sheet supersymmetry was used to reexpress the fermion fields obtained upon the action of these four supercharges in terms of the \((1+1)\)-dimensional fermions. This procedure gives us the superorientational fermion zero modes [4],

\[
\bar{\psi}_{Ak2} = \left( \frac{\tau^a}{2} \right)_{Ak} \frac{1}{2\phi_2} (\phi^2_1 - \phi^2_2) \left[ \chi^a_2 + i\varepsilon^{abc} n^b \chi^c_2 \right],
\]
\[
\bar{\psi}_1 = \left( \frac{\tau^a}{2} \right)_{1A} \frac{1}{2\phi_2} (\phi^2_1 - \phi^2_2) \left[ \chi^a_1 - i\varepsilon^{abc} n^b \chi^c_1 \right],
\]
\[
\bar{\psi}_{Ak1} = 0, \quad \bar{\psi} = 0,
\]
\[
\lambda^{a22} = \frac{i}{\sqrt{2}} \frac{x_1 + ix_2}{r^2} f_3 \phi_1 \left[ \chi^a_1 - i\varepsilon^{abc} n^b \chi^c_1 \right],
\]
\[
\lambda^{a11} = \frac{i}{\sqrt{2}} \frac{x_1 - ix_2}{r^2} f_3 \phi_1 \left[ \chi^a_2 + i\varepsilon^{abc} n^b \chi^c_2 \right],
\]
\[
\lambda^{a12} = 0, \quad \lambda^{a21} = 0,
\]

where the dependence on \( x_i \) is encoded in the string profile functions, see (3.9).

Now we will directly verify that the zero modes (4.5) satisfy the Dirac equations of motion. From the fermion action of the model (4.1) we get the relevant Dirac equations for \( \lambda^a \),

\[
\frac{i}{g_2^2} \slashed{D} \lambda^a + \frac{i}{\sqrt{2}} \text{Tr} \left( \bar{\psi}_\tau q^f + \bar{q}^f \tau^a \psi \right) - \mu_2 \delta^f_2 \bar{\lambda}^a_2 = 0,
\]

while for the matter fermions

\[
\frac{i}{\sqrt{2}} \nabla \bar{\psi} + \frac{i}{\sqrt{2}} \left[ \bar{q}_f \lambda^f - (\tau^a \bar{q}_f) \lambda^a_f + (a - a^a \tau^a) \bar{\psi} \right] = 0,
\]
Next, we substitute the orientational fermion zero modes (4.5) into these equations and take the limit $\mu_2 = 0$. After some algebra one can check that (4.5) do satisfy the Dirac equations (4.6) and (4.7) provided the first-order equations for string profile functions (3.4) are fulfilled.

It is instructive to check that the zero modes (4.5) do produce the fermion part of the $\mathcal{N} = 2$ two-dimensional $CP(1)$ model. To this end we return to the usual assumption that the fermion collective coordinates $\chi^a_\alpha$ in Eq. (4.5) have an adiabatic dependence on the world-sheet coordinates $x_k$ ($k = 0, 3$). This is quite similar to the procedure of Sect. 3.2. Substituting Eq. (4.5) in the fermion kinetic terms in the bulk theory (4.1), and taking into account the derivatives of $\chi^a_\alpha$ with respect to the world-sheet coordinates we arrive at

$$\beta \int dt dz \left\{ \frac{1}{2} \chi^a_1 (\partial_0 - i \partial_3) \chi^a_1 + \frac{1}{2} \chi^a_2 (\partial_0 + i \partial_3) \chi^a_2 \right\},$$  \hfill (4.8)

where $\beta$ is given by the same integral (3.13) as for the bosonic kinetic term, see Eq. (3.12).

We can use the world-sheet $\mathcal{N} = 2$ supersymmetry to reconstruct the four-fermion interactions inherent to $CP(1)$. The SUSY transformations in the $CP(1)$ model have the form (see [12] for a review)

$$\delta \chi^a_1 = i \sqrt{2} (\partial_0 + i \partial_3) n^a \varepsilon_2 + \sqrt{2} \varepsilon_1 n^a (\chi^a_1 \chi^a_2),$$

$$\delta \chi^a_2 = i \sqrt{2} (\partial_0 - i \partial_3) n^a \varepsilon_1 - \sqrt{2} \varepsilon_2 n^a (\chi^a_1 \chi^a_2),$$

$$\delta n^a = \sqrt{2} (\varepsilon_1 \chi^a_2 + \varepsilon_2 \chi^a_1),$$  \hfill (4.9)

where for simplicity we put $\eta_\alpha = 0$. Imposing this supersymmetry leads to the following effective theory on the string world sheet:

$$S_{CP(1)} = \beta \int dt dz \left\{ \frac{1}{2} (\partial_k n^a)^2 + \frac{1}{2} \chi^a_1 i(\partial_0 - i \partial_3) \chi^a_1 \\
+ \frac{1}{2} \chi^a_2 i(\partial_0 + i \partial_3) \chi^a_2 - \frac{1}{2} (\chi^a_1 \chi^a_2)^2 \right\},$$  \hfill (4.10)

This is indeed the action of the $\mathcal{N} = 2 CP(1)$ sigma model.

### 4.2 Breaking $\mathcal{N} = 2$ supersymmetry

Now let us switch on our breaking parameters $\mu_i$. As was discussed in Sect. 3, the bosonic solution for the non-Abelian string does not change at all. It is still given by
Eq. (3.9). However, the fermion zero modes do change. Now only four supercharges survive in the four-dimensional bulk theory. They are associated with the parameters $\epsilon^{a1}$ for $f = 1$. Nevertheless, we still can use the method of Ref. [4] reviewed in Sect. 4.1 to generate superorientational fermion zero modes. Condition (4.2) tells us that we now have only one complex parameter $\epsilon^{21}$ of SUSY transformations unbroken by the string. This leads to the presence of two supercharges associated with two real parameters $\varepsilon_1$ and $\eta_1$, according to identification (4.4), in the world-sheet theory. Following the same steps which led us to (4.5) and taking into account that the bosonic string solution (3.9) does not depend on $\mu_i$ we then obtain

$$\bar{\psi}_{Ak2} = \left( \frac{r^a}{2} \right)_{Ak} \frac{1}{2\phi_2} (\phi_1^2 - \phi_2^2) \left[ \lambda_2^a + i\epsilon^{abc} n^b \lambda_2^c \right],$$

$$\bar{\psi}_{Aki} = 0,$$

$$\lambda^{a11} = \frac{i}{\sqrt{2}} \frac{x_1 - ix_2}{r^2} f_3 \frac{\phi_1}{\phi_2} \left[ \lambda_2^a + i\epsilon^{abc} n^b \lambda_2^c \right];$$

$$\lambda^{a21} = 0. \quad (4.11)$$

We see that reduced supersymmetry generates for us only two fermion superorientational modes parametrized by the two-dimensional fermion field $\chi_2^a$. This was expected, of course. The modes proportional to $\chi_1^a$ do not appear. This is because $\chi_1^a$ is related to the SUSY transformations generated by $\epsilon^{12}$ (see (4.3) and (4.4)) which is no longer present in the deformed bulk theory. One can easily check that zero modes (4.11) still satisfy the Dirac equations of motion (4.6), (4.7) just because the parameter $\mu_2$ does not enter the equations for $\lambda^{a1}$ and $\bar{\psi}$.

It is clear, however, that the other two fermion zero modes proportional to $\chi_1$ do not disappear. They are just modified and can no longer be obtained by supersymmetry. To find them we have to actually solve the Dirac equations (4.6), (4.7). In this section we consider small $\mu_2$ and develop perturbation theory for (4.6), (4.7). In Sect. 4.3 we treat the large $\mu_2$ limit.

We can solve (4.6), (4.7) order by order in $\mu_2$. Say, if we take (4.5) for the zeroth-order approximation and substitute $\lambda^{22}$ from (4.5) into the last term in Eq. (4.6) we generate fermion zero modes to the first order in $\mu_2$. Let us actually do this.

First we note that

$$z^{kA}_{\psi_2} = 0, \quad \lambda^{a12} = 0. \quad (4.12)$$

They vanish in the zeroth order (see (4.5)) and, as follows from Eqs. (4.6) and (4.7), are not generated in any order in $\mu_2$. It is also easy to check that the remaining
fermion fields have the following form:

\[
\lambda^a_{22} = \frac{x_1 + ix_2}{r} \lambda_+(r) \left[ \chi^a_1 - i\varepsilon^{abc} n^b \chi^c_1 \right] + \lambda_-(r) \left[ \chi^a_1 + i\varepsilon^{abc} n^b \chi^c_1 \right],
\]

\[
\bar{\psi}_1 \dot{=}_k = \psi_1 + \left( \frac{r^2}{2} \right) \left[ \chi^a_1 i\varepsilon^{abc} n^b \chi^c_1 \right] + \frac{x_1 - ix_2}{r} \psi_-(r) \left( \frac{r^a}{2} \right) \left[ \chi^a_1 + i\varepsilon^{abc} n^b \chi^c_1 \right].
\]

Here we introduced four profile functions \(\lambda_{\pm}\) and \(\psi_{\pm}\) parametrizing the fermion fields \(\lambda^{22}\) and \(\bar{\psi}_1\). The functions \(\lambda_+\) and \(\psi_+\) are expandable in even powers of \(\mu_2\) while the functions \(\lambda_-\) and \(\psi_-\) in odd powers of \(\mu_2\).

Substituting (4.13) into the Dirac equations (4.6), (4.7) we get following equations for fermion profile functions:

\[
\frac{d}{dr} \psi_+ - \frac{1}{2r} (f - f_3) \psi_+ + i\sqrt{2} \phi_1 \lambda_+ = 0,
\]

\[
- \frac{d}{dr} \lambda_+ - \frac{1}{r} \lambda_+ + \frac{f_3}{r} \lambda_+ + i\frac{g_2^2}{\sqrt{2}} \phi_1 \psi_+ + g_2^2 \mu_2 \lambda_- = 0,
\]

\[
\frac{d}{dr} \psi_- + \frac{1}{2r} (f + f_3) \psi_- + i\sqrt{2} \phi_2 \lambda_- = 0,
\]

\[
- \frac{d}{dr} \lambda_- + \frac{f_3}{r} \lambda_- + i\frac{g_2^2}{\sqrt{2}} \phi_2 \psi_- + g_2^2 \mu_2 \lambda_+ = 0.
\]

The leading contributions to the \(\mu\) even solutions to these equations is

\[
\lambda_+ = \frac{i}{\sqrt{2}} \frac{f_3}{r} \phi_1 + O(\mu_2^3), \quad \psi_+ = \frac{1}{2\phi_2} \left( \phi_1^2 - \phi_2^2 \right) + O(\mu_2^3),
\]

where we express the zeroth-order fermion modes \(\lambda^{22}\) and \(\bar{\psi}_1\) (4.5) in terms of the fermion profile functions. Substituting (4.15) into the last equation in (4.14) we can solve for the leading contributions to the \(\mu\) odd profile functions. They can be expressed in terms of the string profile functions as follows:

\[
\lambda_- = \mu_2 \frac{i}{2\sqrt{2}} \left[ (f_3 - 1) \frac{\phi_2}{\phi_1} + \frac{\phi_1}{\phi_2} \right] + O(\mu_2^3),
\]

\[
\psi_- = \mu_2 \frac{r}{4\phi_1} \left( \phi_1^2 - \phi_2^2 \right) + O(\mu_2^3).
\]

Using the boundary conditions (3.5) and (3.6) for the string profile functions it is easy to check that these solutions vanish at \(r \to \infty\) and are non-singular at \(r = 0\).
We conclude that the number of the superorientational zero modes of the non-Abelian string does not jump as we switch on the deformation parameters $\mu_i$. We keep all four zero modes parametrized by $\chi^a_1$ and $\chi^a_2$. The modes proportional to $\chi^a_1$ are now modified. Still, we can find them order by order in $\mu_2$ by solving the Dirac equations (4.14). As was mentioned in Sect. 1 (and will be explained in detail in Sect. 5) the four fermion zero modes imply $N = 2$ supersymmetry in the two-dimensional world-sheet sigma model (four supercharges). On the other hand, $N = 2$ supersymmetry in the bulk theory is broken down to $N = 1$ (four supercharges). Thus, we do observe enhancement of supersymmetry on the string world sheet.

On general grounds one might expect a breaking of the enhanced world-sheet supersymmetry at some critical value $\mu^*_i$. What could happen is the fermion zero modes associated with $\chi^a_1$ could become non-normalizable at some value of $\mu_2$. Clearly, one would not be able to see the loss of normalizability in perturbation theory in $\mu_2$. In Sect. 4.3 we will examine the limit of large $\mu_i$ and show that the fermion modes (4.13) become non-normalizable only at $\mu_2 \to \infty$.

### 4.3 The large $\mu$ limit

Let us dwell on the limit of large $\mu_2$, or, more explicitly\(^\text{10}\)

\[ \omega_2 \gg 1, \tag{4.17} \]

see (2.21). As was explained in Sect. 2 the fields $a^a$ (as well as their fermion counterparts $\lambda^{a2\alpha}$) become heavy and can be integrated out. The low-energy theory for the SU(2) sector contains one massive SU(2) gauge multiplet, with mass

\[ m_0 \equiv m_{\text{SU}(2)} = g_2 \sqrt{\xi}, \tag{4.18} \]

see (2.17), and three $(a = 1, 2, 3)$ chiral light multiplets, with mass

\[ m_L \equiv m_{\text{SU}(2)} = \frac{\xi}{\mu_2}. \tag{4.19} \]

Integrating out heavy fields can be carried out in superpotentials (2.8), (2.9), as in [19, 20, 17], or directly in the component Lagrangian. One just drops the kinetic terms for the heavy fields and solves algebraic equations for these fields. We do it in the fermion sector of the theory in the Dirac equations (4.6) for $\lambda^{a2\alpha}$. More exactly, we get expressions for the $\lambda$-profile functions in terms of the $\psi$-profile functions from

\(^{10}\)The parameter $\omega_1$ does not enter Eqs. (4.14), therefore we can ignore it.
the first and the third equations in (4.14). Namely,

\[
\lambda_+ = \frac{i}{\sqrt{2}\phi_1} \left[ \frac{d}{dr} \psi_+ - \frac{1}{2r} (f - f_3) \psi_+ \right],
\]

\[
\lambda_- = \frac{i}{\sqrt{2}\phi_2} \left[ \frac{d}{dr} \psi_- + \frac{1}{r} \psi_- - \frac{1}{2r} (f + f_3) \psi_- \right].
\] (4.20)

Dropping the kinetic term for \(\lambda\)'s in the second and the fourth equations in (4.14) and substituting (4.20) in these equations we arrive at

\[
\frac{d}{dr} \psi_+ - \frac{1}{2r} (f - f_3) \psi_+ + m_L \frac{\phi_1 \phi_2}{\xi} \psi_- = 0,
\]

\[
\frac{d}{dr} \psi_- + \frac{1}{r} \psi_- - \frac{1}{2r} (f + f_3) \psi_- + m_L \frac{\phi_1 \phi_2}{\xi} \psi_+ = 0,
\] (4.21)

where \(m_L\) is the light mass given in Eq. (4.19).

Now observe that long-range tails of the solutions to these equations are determined by the small mass \(m_L\), while the string profile functions \(f\) and \(f_3\) are important at much smaller distances \(R \sim 1/m_0\). This key observation allows us to solve Eqs. (4.21) analytically. We will treat separately two domains: (i) large \(r\),

\[ r \gg 1/m_0 \]

and (ii) intermediate \(r\),

\[ r \leq 1/m_0. \]

Large-\(r\) domain, \(r \gg 1/m_0\)

In this domain we can drop the terms in (4.21) containing \(f\) and \(f_3\) and use the first equation to express \(\psi_-\) in terms of \(\psi_+\). We then get

\[
\psi_- = -\frac{1}{m_L} \frac{d}{dr} \psi_+.
\] (4.22)

Substituting this into the second equation in (4.21) we obtain

\[
\frac{d^2}{dr^2} \psi_+ + \frac{1}{r} \frac{d}{dr} \psi_+ - m_L^2 \psi_+ = 0.
\] (4.23)

This is a well-known equation for a free field with mass \(m_L\) in the radial coordinates. Its solution is well-known too\(^{11}\)

\[
\psi_+ = m_L K_0(m_L r),
\] (4.24)

\(^{11}\)Equation (4.23) determines the profile function \(\psi_+\) up to an overall normalization constant. This constant is included in the normalization of the two-dimensional fermion field \(\chi_i^a\). We will discuss this normalization in Sect. 5.
where $K_0(x)$ is the imaginary argument Bessel function. At infinity it falls-off exponentially,
\[ K_0(x) \sim \frac{e^{-x}}{\sqrt{x}}, \quad (4.25) \]
while at $x \to 0$ it has the logarithmic behavior,
\[ K_0(x) \sim \ln \frac{1}{x}. \quad (4.26) \]

Taking into account (4.22) we get the solutions for the fermion profile functions at $r \gg 1/m_0$,
\[ \psi_+ = m_L K_0(m_L r), \quad \psi_- = -\frac{d}{dr} K_0(m_L r). \quad (4.27) \]
In particular, at $r \ll 1/m_L$ we have
\[ \psi_+ \sim m_L \ln \frac{1}{m_L r}, \quad \psi_- \sim \frac{1}{r}. \quad (4.28) \]

**Intermediate-$r$ domain, $r \leq 1/m_0$**

In this domain we neglect small mass terms in (4.21). We then arrive at
\[ \frac{d}{dr} \psi_+ - \frac{1}{2r} (f - f_3) \psi_+ = 0, \]
\[ \frac{d}{dr} \psi_- + \frac{1}{r} \psi_- - \frac{1}{2r} (f + f_3) \psi_- = 0. \quad (4.29) \]
These equations are identical to those for the string profile functions, see (3.4). Therefore, their solutions are known,
\[ \psi_+ = c_1 \phi_2, \quad \psi_- = \frac{c_2}{r} \phi_1, \quad (4.30) \]
up to normalization constants $c_{1,2}$. To fix these constants we match the long-distance behavior in (4.30) with the short-distance behavior of the solutions in the domain $r \gg 1/m_0$ given in (4.28). This gives the fermion profile functions at intermediate $r$,
\[ \psi_+ = \frac{m_L \ln (m_0/m_L)}{\sqrt{\xi}} \phi_2, \quad \psi_- = \frac{1}{r \sqrt{\xi}} \phi_1. \quad (4.31) \]

Equations (4.27) and (4.31) present our final result for the fermion profile functions in the limit of large $\mu_2$. They determine two fermion superorientational zero modes proportional to $\chi^a_1$ via Eq. (4.13). The main feature of these modes is the presence of the long-range tails determined by the small mass $m_L$. Neither bosonic string solution (3.9) nor two other superorientational fermion zero modes (4.11) determined by $\mathcal{N} = 1$ supersymmetry have these logarithmic long-range tails\(^{12}\).

\(^{12}\)Here the word *logarithmic* is used in a somewhat Pickwick sense. More precisely one should say
5 Effective world-sheet theory in the large-$\mu$ limit

To fully specify the fermion sector of the world-sheet sigma model we substitute the fermion zero modes (4.27), (4.31) and (4.11) into the fermion action (4.1), much in the same way we did in Sect. 4.1 in the $\mathcal{N} = 2$ limit. Then instead of Eq. (4.10) we get

$$S_{1+1} = \beta \int dtdz \left\{ \frac{1}{2} \left( \partial_k n^a \right)^2 + I_f \frac{1}{2} \chi_1^a i(\partial_0 - i\partial_3) \chi_1^a \right. $$

$$+ \frac{1}{2} \chi_2^a i(\partial_0 + i\partial_3) \chi_2^a - I_f \frac{1}{2} (\chi_1^a \chi_2^a)^2 \right\}.$$  \hspace{1cm} (5.1)

Here $I_f$ is the normalization integral for the deformed fermion zero modes (4.27) and (4.31). Its leading behavior at large $\mu_2$ is given by

$$I_f = 2g_2^2 \int r dr \left( |\psi_+|^2 + |\psi_-|^2 \right) \sim g_2^2 \ln \left( \frac{m_0}{m_L} \right)$$  \hspace{1cm} (5.2)

coming from $\psi_-$. Substituting the mass values from (4.18) and (4.19) we then obtain

$$I_f \sim g_2^2 \ln \left( \frac{g_2 \mu_2}{\sqrt{\xi}} \right).$$  \hspace{1cm} (5.3)

Note that the calculation actually gives us only the bilinear fermion terms in (5.1). We fix the coefficient in front of the quartic term using $\mathcal{N} = 1$ supersymmetry on the world sheet generated by parameters $\varepsilon_1$ and $\eta_1$, see (4.9). This supersymmetry is necessarily present in our world-sheet theory. In particular, it relates the coefficient in front of the kinetic term for $\chi_1^a$ and the one in front of the quartic term.

Next, we absorb the normalization integral $I_f$ in the definition of the fermion fields $\chi_1^a$. As a result, we arrive at the $CP(1)$ model (4.10). This model has $\mathcal{N} = 2$ supersymmetry in two dimensions (four supercharges). We thus confirm enhancement of supersymmetry in our effective theory on the string world sheet. As was explained in Sect. 1, this result could be expected on general grounds. The target space of the $CP(1)$ model ($S_2$ sphere) is the Kähler manifold. Supersymmetry on the Kähler manifolds requires four supercharges. If our string is BPS and the world-sheet theory is local, the world-sheet supersymmetry must be enhanced. The same reasoning was recently used [10] to prove enhanced supersymmetry on the world volume of domain walls.

If we started directly from $\mathcal{N} = 1$ SQCD (2.26) we would have never obtained enhanced supersymmetry on the world sheet of the non-Abelian string. We would that the large-distance behavior of the long-range tails is such that the corresponding normalization factors diverge logarithmically. This divergence is cut off at $m_L^{-1}$. 

27
find only two fermion zero modes \((4.11)\), while the other two are non-normalizable. The reason for this is the presence of the Higgs branch in (2.26). Embedding (2.26) in the deformed \(\mathcal{N} = 2\) theory (2.10) lifts the Higgs branch and makes the second pair of the fermion zero modes normalizable at any finite \(\mu_2\). This infrared (IR) regularization allows us to obtain \(\mathcal{N} = 2\) supersymmetric \(CP(1)\) model \((4.10)\) as an effective theory on the world sheet of the non-Abelian string.

### 6 Limits of applicability

Although the two-derivative term we derived above is \(\mathcal{N} = 2\) supersymmetric for any finite \(\mu\) one should expect the enhanced \(\mathcal{N} = 2\) supersymmetry to be broken at some (large) value of \(\mu_2\) due to induced terms with four or more derivatives. Let us determine this critical value. To this end let us note that higher derivative corrections run in powers of

\[
\Delta \partial_k, \quad (6.1)
\]

where \(\Delta\) is a string transverse size. At small \(\mu_2\),

\[
\Delta \sim \frac{1}{\sqrt{\xi}}.
\]

The typical energy scale on the string world sheet is given by the scale \(\Lambda_{CP(1)}\) of the \(CP(1)\) model which is given by \((3.17)\) at small \(\mu_2\). Thus,

\[
\partial \rightarrow \Lambda
\]

and higher derivative corrections in fact run in powers of \(\Lambda/\sqrt{\xi}\). At small \(\mu_2\) higher derivative corrections are suppressed as \(\Lambda/\sqrt{\xi} \ll 1\), and we can ignore them. However, as we increase \(\mu_2\) the fermion zero modes \((4.27), (4.31)\) acquire long-range tails. This means that an effective “fermion” thickness of the string grows and becomes

\[
\Delta \sim \frac{1}{m_L} = \frac{\mu_2}{\xi}, \quad (6.2)
\]

Higher derivative terms are small if \(\Delta \Lambda_{CP(1)} \ll 1\). Substituting here the scale of the \(CP(1)\) model given by \((3.18)\) at large \(\mu_2\) and the scale of \(\mathcal{N} = 1\) SQCD \((2.31)\) we arrive at

\[
\mu_2 \ll \mu_2^*, \quad (6.3)
\]

where the critical value of \(\mu_2\) is given by

\[
\mu_2^* = \frac{\xi}{\Lambda_{CP(1)}} = \frac{\xi^{3/2}}{\Lambda_{\mathcal{N}=1}^{3/2}}. \quad (6.4)
\]
If the condition (6.3) is met, the $\mathcal{N} = 2$ CP(1) model gives a good description of the
world-sheet physics. A spectrum of relevant scales in our theory is shown in Fig. 2.

If we increase $\mu_2$ above the critical value (6.4) the non-Abelian strings become
effectively thick and their world-sheet dynamics is no longer described by $\mathcal{N} = 2$ $CP(1)$
sigma model. The higher derivative corrections on the world sheet explode. Since
the higher derivative sector does not respect the enhanced $\mathcal{N} = 2$ supersymmetry the
latter gets broken down to $\mathcal{N} = 1$ (two supercharges).

Note that the physical reason for the growth of the string thickness $\Delta$ is the
presence of the Higgs branch in $\mathcal{N} = 1$ SQCD (2.26). Although the classical string so-
lution (3.9) stays compact, the presence of the Higgs branch shows up at the quantum
level. In particular, the fermion zero modes feel its presence and acquire long-range
logarithmic tails.

Summarizing, the $\mathcal{N} = 2$ CP(1) model with enhanced supersymmetry is a valid
description of the world-sheet physics of the non-Abelian string if the condition (6.3) is
met. Otherwise the $\mathcal{N} = 2$ world-sheet supersymmetry is broken down to $\mathcal{N} = 1$ by
higher derivative terms. Simultaneously, the string at hand becomes “thick.” By
thick we mean that its transverse dimension is determined by the large parameter
$\mu_2/\xi \to \infty$ rather than by $\xi^{-1/2}$.

7 Non-Abelian monopoles in $\mathcal{N} = 1$

Since the $\mathcal{N} = 2$ CP(1) model is the effective low-energy theory describing the
world-sheet physics of the non-Abelian string all consequences of this model ensue,
in particular, two degenerate vacua and a kink which interpolates between them —
the same kink that we had in $\mathcal{N} = 2$ [4] and interpreted as a (confined) non-Abelian
monopole, the descendent of the ’t Hooft–Polyakov monopole [26].

Let us briefly review the reason for this interpretation [5, 4]. We first set to zero
the $\mathcal{N} = 2$ breaking parameters $\mu_i$ in (2.10) and introduce a mass difference $\Delta m$
for two quark supermultiplets, see [4] for details. Let us start from the vanishing FI
parameter $\xi$ (i.e. start from the Coulomb branch). At $\Delta m \neq 0$ the gauge group SU(2)
is broken down to U(1) by a VEV of the SU(2) adjoint scalar $\langle a^3 \rangle \sim \Delta m$. Thus, there
are ’t Hooft-Polyakov monopoles of broken gauge SU(2). Classically, on the Coulomb
branch their mass is proportional to $|\Delta m|/g_2^2$. In the limit $\Delta m \to 0$ they become massless, formally, in the classical approximation. Simultaneously their size become infinite [27]. The mass and size are stabilized by confinement effects which are highly quantum. The confinement of monopoles occurs in the Higgs phase, at $\xi \neq 0$.

A qualitative evolution of the monopoles under consideration as a function of the relevant parameters is presented in Fig. 3.

The 't Hooft–Polyakov monopole

The 't Hooft–Polyakov monopole

Almost free monopole

Confined monopole, quasiclassical regime

Confined monopole, highly quantum regime

Figure 3: Various regimes for the monopoles and flux tubes. The latter case corresponds to the vanishing $\Delta m$.

We begin with the limit $\xi \to 0$ while $\Delta m$ is kept fixed. Then the corresponding microscopic theory supports the conventional (unconfined) 't Hooft-Polyakov monopoles [26] due to the spontaneous breaking of the gauge SU(2) down to U(1), (the upper left corner of Fig. 3).

If we allow $\xi$ be non-vanishing but

$$|\Delta m| \gg \sqrt{\xi}$$

then the effect which comes into play first is the above spontaneous breaking of the gauge SU(2). Further gauge symmetry breaking, due to $\xi \neq 0$, which leads to complete Higgsing of the model and the string formation (confinement of monopoles) is much weaker. Thus, we deal here with the formation of “almost” 't Hooft-Polyakov monopoles, with a typical size $\sim |\Delta m|^{-1}$. Only at much larger distances, $\sim \xi^{-1/2}$,
the charge condensation enters the game, and forces the magnetic flux, rather than spreading evenly a la Coulomb, to form flux tubes (the upper right corner of Fig. 3). There will be two such flux tubes, with the distinct orientation of the color-magnetic flux ($Z_2$ strings discussed in Sect. 3.1). The monopoles, albeit confined, are weakly confined.

Now, if we further reduce $|\Delta m|$, 
\[ \Lambda_{CP(1)} \ll |\Delta m| \ll \sqrt{\xi}, \]  
(7.2)
the size of the monopole ($\sim |\Delta m|^{-1}$) becomes larger than the transverse size of the attached strings. The monopole gets squeezed in earnest by the strings — it becomes a bona fide confined monopole (the lower left corner of Fig. 3). A macroscopic description of such monopoles is provided by the twisted-mass $CP(1)$ model on the string world sheet [5, 4]. Namely two $Z_2$ strings are interpreted as two vacua of the $CP(1)$ model while the monopole (string junction of two $Z_2$ strings) is interpreted as a kink interpolating between these two vacua.

The value of the twisted mass equals $\Delta m$ while the size of the twisted-mass sigma-model kink/confined monopole is of order of $|\Delta m|^{-1}$. As we further diminish $|\Delta m|$ approaching $\Lambda_{CP(1)}$ and then getting below $\Lambda_{CP(1)}$, the size of the monopole grows, and, classically, it would explode. This is where quantum effects in the world-sheet theory take over. It is natural to refer to this domain of parameters as the “regime of highly quantum dynamics.” While the thickness of the string (in the transverse direction) is $\sim \xi^{-1/2}$, the $z$-direction size of the kink representing the confined monopole in the highly quantum regime is much larger, $\sim \Lambda_{CP(1)}^{-1}$, see the lower right corner of Fig. 3.

In [4] the first order equations for 1/4 BPS string junction of two $Z_2$ strings were explicitly solved and the solution shown to correspond to a kink solution of the two-dimensional $CP(1)$ model. Moreover, it was shown that the mass of the monopole matches the mass of the $CP(1)$-model kink both in the quasiclassical ($\Delta m \gg \Lambda_{CP(1)}$) and quantum ($\Delta m \ll \Lambda_{CP(1)}$) limits.

Thus, at zero $\Delta m$ we still have a confined “monopole” stabilized by quantum effects in the world-sheet $CP(1)$ model (interpreted as a kink). Now we can switch on the $N = 2$ breaking parameters $\mu_i$. If we keep $\mu_2$ less than the critical value (6.4) the effective world-sheet description of the non-Abelian string is still given by the $N = 2$ $CP(1)$ model. This model obviously still has two vacua which should be interpreted as two elementary non-Abelian strings in the quantum regime, and a BPS kink can interpolate between these vacua. This kink should still be interpreted as a non-Abelian confined monopole/string junction. Its mass and inverse size is determined by $\Lambda_{CP(1)}$ which in the limit of large $\mu_2$ is given by Eq. (3.18).
This kink–monopole is half-critical considered from the standpoint of the $CP(1)$ model (i.e. two supercharges conserved). Thus, we observe supersymmetry enhancement at the next level too. In fact, this is “supersymmetry emergence” rather than enhancement, since in the bulk $\mathcal{N} = 1$ theory there is no such thing as the monopole central charge! Indeed, in the $\mathcal{N} = 2$ model [23] there exists a “monopole” central charge [28] which implies, in turn, the critical nature of the ’t Hooft–Polyakov monopole. By appropriately varying parameters of the model one can trace continuous evolution of the conventional (unconfined) ’t Hooft–Polyakov monopole into a weakly confined monopole and then into 1/2-BPS non-Abelian confined kink–monopole in a highly quantum regime.

In the $\mathcal{N} = 1$ model at hand the monopole central charge cannot exist for symmetry reasons, and one cannot expect BPS-saturated ’t Hooft–Polyakov monopoles. On the other hand, the kink central charge certainly exists in the two-dimensional superalgebra [29] pertinent to the $CP(1)$ model. Here we encounter the notion of a central charge that exist in the low-energy moduli theory but cannot be lifted to the bulk theory as a matter of principle. A similar phenomenon does actually occur in the domain-wall system [10].

In the model discussed in [10] two central charges — of the domain wall and domain line types — are allowed [30]. But what we focus on now, is a different central charge. The relevant world-volume central charge in the domain-wall case corresponds to $CP(1)$ “lumps.” Although the existence of such states was not explicitly verified in Ref. [10] and the corresponding solution not found due to strong coupling issues, but the very fact that composites carrying this charge do exist in the domain-wall problem is beyond doubt. Indeed, since the 1/4-BPS (bulk quarter-criticality) wall junctions (domain lines) correspond to $CP(1)$ kinks, of which there are two inequivalent kinds, one could in principle construct a system with the two domain lines (on the wall) joined at a single point in 1+3 dimensions. This single point is a “junction of junctions.” There is no central charge for this localized junction of junctions in the 1+3 dimensional bulk theory, but it should nonetheless be a 1/4-BPS state on the wall world volume saturating both the kink and lump central charges.

8 Non-Abelian strings in $\mathcal{N} = 1$ SQCD

The IR problems we encounter in $\mathcal{N} = 1$ SQCD emerging at $\mu \rightarrow \infty$ are quite similar to those discussed in [14, 18]. In these papers strings on the Higgs branches were studied. In particular, in [18] the Abelian strings in $\mathcal{N} = 1$ SQED were considered.\footnote{We elaborate on supertranslational fermion zero modes in this theory in Sect. 9.}
This theory has a Higgs branch which can be lifted by embedding the theory in the deformed $\mathcal{N} = 2$ SQED (2.1).

In Ref. [18] strings at an arbitrary point on the (lifted) Higgs branch were considered, with both $\langle q \rangle$ and $\langle \tilde{q} \rangle$ nonvanishing (cf. Eq. (2.13), where $\langle \tilde{q} \rangle = 0$). In this case the string appears to be non-BPS. The string solution consists of a “BPS core” and a long-range logarithmic tail of size $\sim 1/m_L$. To take the limit $\mu \to \infty$ one can proceed as follows [14, 18]. Let us consider a string of a large but finite length $L$. Then at a very large $r$,

$$r \sim L,$$

the problem is no longer two-dimensional and logarithmic tails are cut off. In other words, the scale $1/L$ plays the role of the IR cut-off instead of $m^{-1}$. Now one can safely take the limit $\mu \to \infty$.

Let us follow a similar approach to the problem at hand. Consider the string of a finite length $L$. Then the scale $1/L$ will play the role of an IR regularization for the fermion zero modes (4.28) and the normalization integral $I_f$ becomes finite. (Unfortunately, taking the length of a string to be finite destroys the BPS nature of a string).

Now we can safely take the limit $\mu_2 \to \infty$. The normalization integral for the fermion zero modes (5.3) stays finite,

$$I_f \sim g_2^2 \ln \left(g_2 \sqrt{\xi L}\right). \quad (8.1)$$

It still can be absorbed into the definition of the field $\chi_1$.

The world-sheet theory become non-local containing powers of higher derivative corrections, all of the same order. The non-locality arises because the string becomes thick. Note, that this effect does not affect the string tension.

9 Abelian strings

In this section we briefly review Abelian BPS strings solutions and their fermion zero modes in $\mathcal{N} = 2$ SQED obtained in [17], and then elaborate on the issue of fermion zero modes in the U(1) theory (2.1), with broken $\mathcal{N} = 2$ supersymmetry. In particular, we will focus on the large $\mu$-limit when the theory (2.1) reduces to $\mathcal{N} = 1$ SQED.

The Abelian string solution with the minimal winding number in the model (2.1) has the form

$$q(x) = e^{i\alpha} \phi(r),$$
\[ A_i(x) = -\varepsilon_{ij} \frac{x_j}{r^2} (1 - f(r)) , \]  

(9.1)

where \( f(r) \) and \( \phi(r) \) are profile functions for gauge and scalar fields, respectively. These functions satisfy the following first-order equations:

\[ r \frac{d}{dr} \phi(r) - f(r) \phi(r) = 0 , \]

\[ -\frac{1}{r} \frac{d}{dr} f(r) + \frac{e^2}{4} \left[ (\phi(r))^2 - \xi \right] = 0 . \]  

(9.2)

The boundary conditions for these functions are

\[ f(0) = 1 , \quad f(\infty) = 0 \]  

(9.3)

for the gauge field, while the boundary conditions for the squark field are

\[ \phi(\infty) = \sqrt{\xi} , \quad \phi(0) = 0 . \]  

(9.4)

Equations (9.2) can be solved numerically. The tension of the string with the minimal winding is

\[ T_1 = 2\pi \xi . \]  

(9.5)

Note that the string solution does not depend on the deformation parameter \( \mu \), much in the same way as in the non-Abelian case. This is because the neutral scalar field \( a \) vanishes on the solution.

Consider first the \( \mathcal{N} = 2 \) limit \( \mu = 0 \). The string is half-critical, so 1/2 of supercharges (related to SUSY transformation parameters \( \epsilon^{12} \) and \( \epsilon^{21} \), see Sect. 4.1) act trivially on the string solution. The remaining four (real) supercharges parametrized by \( \epsilon^{11} \) and \( \epsilon^{22} \) generate four supertranslational fermion zero modes. They have the form [17]

\[ \bar{\psi}_2 = -2\sqrt{2} \frac{x_1 + ix_2}{r^2} f \phi \zeta_2 , \]

\[ \bar{\psi}_1 = 2\sqrt{2} \frac{x_1 - ix_2}{r^2} f \phi \zeta_1 , \]

\[ \bar{\psi}_1 = 0 , \quad \bar{\psi}_2 = 0 , \]

\[ \lambda^{22} = ig^2 (\phi^2 - \xi) \zeta_1 , \]

\[ \lambda^{11} = -ig^2 (\phi^2 - \xi) \zeta_2 , \]

\[ \lambda^{12} = 0 , \quad \lambda^{21} = 0 , \]  

(9.6)
where the modes proportional to complex Grassmann parameters $\zeta_1$ and $\zeta_2$ are generated by $\epsilon^{22}$ and $\epsilon^{11}$ transformations, respectively.

It is quite straightforward to check that these modes satisfy the Dirac equations,

$$\frac{i}{e^2} \bar{\psi} \lambda^f + \frac{i}{\sqrt{2}} (\bar{\psi} q^f + \bar{q}^f \psi) - \mu_2 \bar{\delta}_2 \lambda_2 = 0,$$

$$i \nabla \bar{\psi} + \frac{i}{\sqrt{2}} [\bar{q}^f \lambda^f + a \bar{\psi}] = 0,$$

$$i \nabla \bar{\psi} + \frac{i}{\sqrt{2}} [\lambda^f q^f + a \psi] = 0$$

(9.7)

for the U(1) model (2.1) at $\mu = 0$.

Now, we switch on the breaking parameter $\mu$,

$$\mu \neq 0.$$

The number of supercharges in the bulk theory drops to four which means that we have only two supercharges, associated with the complex parameter $\epsilon^{11}$ acting non-trivially on the string solution. If we apply these supercharges to the string solution (9.1) we generate only half of the modes in (9.6) proportional to $\zeta_2$. We get

$$\bar{\psi}_2 = -2\sqrt{2} \frac{x_1 + ix_2}{r^2} f \phi \zeta_2,$$

$$\bar{\psi}_1 = 0,$$

$$\lambda^{11} = -ig^2 (\phi^2 - \xi) \zeta_2,$$

$$\lambda^{21} = 0.$$  

(9.8)

As in the non-Abelian case the other two zero modes proportional to $\zeta_1$ do not disappear. They just get modified and can no longer be obtained by SUSY transformation. We derive them below by explicitly solving the Dirac equations (9.7) following the same steps as in Sects. 4.2 and 4.3.

First we note that certain components of the fermion fields are not generated, namely,

$$\bar{\psi}_2 = 0, \quad \lambda^{12} = 0.$$  

(9.9)

Other components can be parametrized by fermion profile functions $\lambda_\pm(r)$ and $\psi_\pm(r)$ via

$$\lambda^{22} = \lambda_+(r) \zeta_1 + \frac{x_1 + ix_2}{r} \lambda_-(r) \bar{\zeta}_1,$$

$$\bar{\psi}_1 = \frac{x_1 - ix_2}{r} \psi_+(r) \zeta_1 + \psi_-(r) \bar{\zeta}_1.$$  

(9.10)
The above profile functions satisfy the following Dirac equations:

\[
\frac{d}{dr} \psi_+ + \frac{1}{r} \psi_+ - \frac{1}{r} f \psi_+ + \frac{i}{\sqrt{2}} \phi \lambda_+ = 0 , \\
- \frac{d}{dr} \lambda_+ + i \frac{e^2}{\sqrt{2}} \phi \psi_+ + e^2 \mu \lambda_+ = 0 , \\
\frac{d}{dr} \psi_- - \frac{1}{r} f \psi_- + \frac{i}{\sqrt{2}} \phi \lambda_- = 0 , \\
- \frac{d}{dr} \lambda_- - \frac{1}{r} \lambda_- + i \frac{e^2}{\sqrt{2}} \phi \psi_- + e^2 \mu \lambda_- = 0 .
\] (9.11)

Parallelizing the derivation in Sect. 4.3 let us consider the large \( \mu \)-limit,

\[ \mu \gg \sqrt{\xi} . \]

In this limit we can integrate out the heavy \( \lambda^2 \) field. In particular, the first and third equations in (9.11) give

\[
\lambda_+ = \frac{i \sqrt{2}}{\phi} \left[ \frac{d}{dr} \psi_+ + \frac{1}{r} \psi_+ - \frac{f}{r} \psi_+ \right] , \\
\lambda_- = \frac{i \sqrt{2}}{\phi} \left[ \frac{d}{dr} \psi_- - \frac{f}{r} \psi_- \right] .
\] (9.12)

Neglecting the kinetic terms for \( \lambda \) fields in the second and last equations in (9.11) we get

\[
\frac{d}{dr} \psi_+ + \frac{1}{r} \psi_+ - \frac{f}{r} \psi_+ + \frac{\phi^2}{2 \mu} \psi_- = 0 , \\
\frac{d}{dr} \psi_- - \frac{f}{r} \psi_- + \frac{\phi^2}{2 \mu} \psi_+ = 0 .
\] (9.13)

The large-\( r \) behavior in these equations is determined by the mass of the \( \text{U}(1) \) gauge multiplet

\[ \tilde{m}_0 = \frac{g}{\sqrt{2}} \sqrt{\xi} \] (9.14)

and the mass of the light chiral multiplet

\[ \tilde{m}_L = \frac{\xi}{2 \mu} , \] (9.15)
see Sect. 2.2. The light mass (9.15) is determined by the smaller root of the quadratic equation (2.5). In particular, in the limit $\mu \to \infty$ it tends to zero. The corresponding massless states become moduli on the Higgs branch of $N = 1$ SQED (2.7).

Given this hierarchy of masses we use the same method as in Sect. 4.3 to solve equations (9.13). Consider first the large-$r$ region,

$$r \gg 1/m_0.$$  

Repeating the same steps which lead us to Eq. (4.27) we get

$$
\psi_+ = \tilde{m}_L \sqrt{\xi} K_0(\tilde{m}_L r), \quad \psi_- = -\sqrt{\xi} \frac{d}{dr} K_0(\tilde{m}_L r). \tag{9.16}
$$

In particular, at $r \ll 1/\tilde{m}_L$ we have

$$
\psi_+ \sim \tilde{m}_L \sqrt{\xi} \ln \frac{1}{\tilde{m}_L r}, \quad \psi_- \sim \sqrt{\xi} \frac{1}{r}. \tag{9.17}
$$

Passing to the intermediate region of $r$,

$$r \leq 1/\tilde{m}_0,$$

we now obtain

$$
\psi_+ = \tilde{m}_L \ln (\tilde{m}_0/\tilde{m}_L) \phi, \quad \psi_- = \frac{1}{r} \phi. \tag{9.18}
$$

Equation (9.16) shows that the supertranslational fermion zero modes of the Abelian string in the model (2.1) acquire long-range tails too. In particular, in the limit $\mu \to \infty$ they become logarithmically non-normalizable. Still at any finite $\mu$ we can absorb the normalization integral into the definition of the two-dimensional fermion fields $\zeta_1$, exactly in the same way this was done for the superorientational modes in Sect. 5. This leads us to the following effective theory on the world sheet of the Abelian string:

$$
S_{1+1} = 2\pi \xi \int dt dz \left\{ \frac{1}{2} (\partial_k x_{0i})^2 + \frac{1}{2} \tilde{\zeta}_1 i (\partial_0 - i \partial_3) \zeta_1 + \frac{1}{2} \tilde{\zeta}_2 i (\partial_0 + i \partial_3) \zeta_2 \right\}, \tag{9.19}
$$

where $x_{0i}$ ($i = 1, 2$) denote the coordinates of the string position in $(1, 2)$-plane.

This is a free theory with two real bosonic and four fermionic fields of $t, z$. Counting the number of degrees of freedom we observe the enhanced $N = 2$ supersymmetry in two dimensions (four supercharges): the fields at hand form a supermultiplet of $\mathcal{N} = 2$. 

37
We see that the phenomenon of the enhanced world-sheet supersymmetry is quite general and occurs both for Abelian and non-Abelian strings. It can be traced back to strings in \( \mathcal{N} = 2 \) supersymmetric bulk theory from which our strings are descendants.

The two-dimensional theory (9.19) is a trivial free-field theory and it does not generate its own scale. Therefore, we cannot estimate the critical value of \( \mu \) when the enhanced \( \mathcal{N} = 2 \) supersymmetry breaks down to \( \mathcal{N} = 1 \) in this case. The theory (9.19) is a low-energy effective theory which describes the string at small energies \( E, E \ll m_L \). At larger energies higher derivative corrections to (9.19) become important. The higher derivative sector does not respect \( \mathcal{N} = 2 \) supersymmetry and at large energies supersymmetry breaking effects take over. As we increase \( \mu \), the region of validity of (9.19) becomes exceedingly narrower. In the \( \mathcal{N} = 1 \) SQED limit of \( \mu \to \infty \) the string becomes thick and the effective theory on the string world sheet becomes non-local. It is worth stressing again that this happens due to the presence of the Higgs branch in \( \mathcal{N} = 1 \) SQED.

There is one more thing we must emphasize. The translational sector of the U(1) gauge theory (2.1) is discussed in this section just for the sake of simplicity. The generalization to the translational sector of the non-Abelian string in theory (2.10) is absolutely straightforward. We get the same results for the translational sector of the non-Abelian string.

### 10 Conclusions

This concluding section could have been entitled “How extended supersymmetry dynamically emerges from Kählerian geometry.” After the phenomenon is identified, it seems to be rather trivial and transparent. Indeed, if we start from a bulk theory with \(^{14} \nu \) supercharges and obtain half-critical solitons with a nontrivial moduli space, a linear realization of \( \nu/2 \) supercharges in the low-energy world-sheet theory of moduli is guaranteed. If, in addition, the geometry of the moduli space is Kählerian, and the numbers of the boson and fermion zero modes appropriately match, \( \nu/2 \) extra “supernumerary” supercharges emerge with necessity. Apparently this is not a rare occurrence, since we encounter one and the same situation, enhancement of supersymmetry, in two most widely discussed problems — domain walls in \( \mathcal{N} = 1 \) SQCD with \( N_f = N_c \) [10], and in the current problem of non-Abelian strings. It is worth stressing, however, that the reasons lying behind enhancement of supersymmetry in these two problems are not quite the same, as was explained in Sect. 1.

\(^{14}\)Although \( \nu = 4 \) in the 4D cases under consideration we would like to stick to a more general formulation.
We also observe “supersymmetry emergence” for the flux-tube junctions (confined monopoles): our kink–monopole is half-critical considered from the standpoint of the world-sheet $CP(1)$ model (i.e. two supercharges conserved), while in the bulk $\mathcal{N} = 1$ theory there is no monopole central charge at all. A similar phenomenon was also noted in Ref. [10].

A number of interesting questions remains unanswered or not answered in full. Let us list some of them.

(i) In Sect. III.B3 of Ref. [10] it was shown that a mass deformation removing the continuous moduli space of the world-volume theory leaves the enhanced $\mathcal{N} = 2$ supersymmetry intact, at least for small mass deformations. The lifting of the moduli space occurred through a generation of a Killing vector potential. More precisely, it was verified that, at leading order in the unequal mass deformation, the effect of the mass deformation reduced to a potential which is the norm-squared of a U(1) Killing vector on $CP(1)$ (the so-called real mass deformation), see [31]. Such a potential preserves $\mathcal{N} = 2$ as it maintains the complex structure. It was unclear what symmetry ensures this form for the potential. It was also unclear whether this particular form holds beyond the leading order in the deformation.

It would be extremely interesting to explore whether or not a similar structure persists in the flux-tube case.

(ii) It seems imperative to understand the necessary (rather than just sufficient) conditions for supersymmetry enhancement more precisely. In the context of this question it would be nice to find a symmetry argument which would explain why turning on a (finite) adjoint mass has no impact (up to field rescalings) on the flux-tube world-sheet theory.

(iii) Another interesting question is: what happens with our “kink–monopole” state in $\mathcal{N} = 1$ theory when we vary parameters moving towards weaker confinement? In other words a challenging and illuminating problem is: what happens when two scales in Eq. (2.32) are of the same order? We are not aware of any discussion of this regime in the literature.

(iv) The issue of supersymmetry emergence seems intriguing. Is it promising from the standpoint of applications?

We hope to return to the above issues elsewhere.
Acknowledgments

We are very grateful to Adam Ritz for thorough discussions and communications, and careful reading of a preliminary version of the present paper which resulted in a number of valuable remarks. We would like to thank Nathan Seiberg for posing stimulating questions.

The work of M.S. was supported in part by DOE grant DE-FG02-94ER408. The work of A.Y. was supported in part by the RFBR grant No. 05-02-17360 and by Theoretical Physics Institute at the University of Minnesota.

References

[1] M. Shifman and A. Yung, Phys. Rev. D 67, 125007 (2003) [hep-th/0212293]; Phys. Rev. D 70, 025013 (2004) [hep-th/0312257].
[2] A. Hanany and D. Tong, JHEP 0307, 037 (2003) [hep-th/0306150].
[3] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B 673, 187 (2003) [hep-th/0307287].
[4] M. Shifman and A. Yung, Phys. Rev. D 70, 045004 (2004) [hep-th/0403149].
[5] D. Tong, Phys. Rev. D 69, 065003 (2004) [hep-th/0307302].
[6] A. Hanany and D. Tong, JHEP 0404, 066 (2004) [hep-th/0403158].
[7] V. Markov, A. Marshakov and A. Yung, Nucl. Phys. B 709, 267 (2005) [hep-th/0408235].
[8] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D 71, 045010 (2005) [hep-th/0412082].
[9] E. Witten and D. I. Olive, Phys. Lett. B 78, 97 (1978).
[10] A. Ritz, M. Shifman and A. Vainshtein, Phys. Rev. D 70, 095003 (2004) [hep-th/0405175].
[11] A. Ritz, Supersizing Worldvolume Supersymmetry: BPS Domain Walls and Junctions in SQCD, in Continuous Advances in QCD 2004, Ed. T. Gherghetta, (World Scientific, Singapore, 2004), p. 428.
[12] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Phys. Rept. 116, 103 (1984).
[13] A. Gorsky and M. A. Shifman, Phys. Rev. D 61, 085001 (2000) [hep-th/9909015].
[14] A. Yung, Nucl. Phys. B 562, 191 (1999) [hep-th/9906243].
[15] A. Abrikosov, Sov. Phys. JETP 32 1442 (1957) [Reprinted in Solitons and Particles, Eds. C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 356]; H. Nielsen and P. Olesen, Nucl. Phys. B61 45 (1973) [Reprinted in Solitons and Particles, Eds. C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 365].

40
[16] A. A. Penin, V. A. Rubakov, P. G. Tinyakov and S. V. Troitsky, Phys. Lett. B 389, 13 (1996) [hep-ph/9609257].
[17] A. I. Vainshtein and A. Yung, Nucl. Phys. B 614, 3 (2001) [hep-th/0012250].
[18] K. Evlampiev and A. Yung, Nucl. Phys. B 662, 120 (2003) [hep-th/0303047].
[19] D. Kutasov, A. Schwimmer and N. Seiberg, Nucl. Phys. B 459, 455 (1996) [hep-th/0510222].
[20] A. Gorsky, A. I. Vainshtein and A. Yung, Nucl. Phys. B 584 (2000) 197 [hep-th/0004087].
[21] A. Hanany, M. J. Strassler and A. Zaffaroni, Nucl. Phys. B 513, 87 (1998) [hep-th/9707244].
[22] K. Bardakci and M. B. Halpern, Phys. Rev. D 6, 696 (1972).
[23] N. Seiberg and E. Witten, Nucl. Phys. B 426, 19 (1994), (E) B 430, 485 (1994) [hep-th/9407087]; Nucl. Phys. B 431, 484 (1994) [hep-th/9408099].
[24] K. Intrilligator and N. Seiberg, Nucl. Phys. (Proc. Suppl.) 45BC, 1 (1996) [hep-th/9509066].
[25] A. M. Polyakov, Phys. Lett. B 59, 79 (1975).
[26] G. ’t Hooft, Nucl. Phys. B 79, 276 (1974); A. M. Polyakov, Pisma Zh. Eksp. Teor. Fiz. 20, 430 (1974) [JETP Lett. 20, 194 (1974)].
[27] E. J. Weinberg, Nucl. Phys. B 167, 500 (1980); Nucl. Phys. B 203, 445 (1982).
[28] R. Haag, J. T. Lopuszanski and M. Sohnius, Nucl. Phys. B 88, 257 (1975) [Reprinted in Supersymmetry, Ed. S. Ferrara, (North-Holland/World Scientific, 1987) Vol. 1, p. 51].
[29] A. Losev and M. Shifman, Phys. Rev. D 68, 045006 (2003) [hep-th/0304003].
[30] B. Chibisov and M. A. Shifman, Phys. Rev. D 56, 7990 (1997), (E) D 58, 109901 (1998) [hep-th/9706141].
[31] L. Alvarez-Gaumé and D. Z. Freedman, Commun. Math. Phys. 91, 87 (1983); S. J. Gates, Nucl. Phys. B 238, 349 (1984); S. J. Gates, C. M. Hull and M. Roček, Nucl. Phys. B 248, 157 (1984).