Coverage Probability of Random Intervals *

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Abstract

In this paper, we develop a general theory on the coverage probability of random intervals defined in terms of discrete random variables with continuous parameter spaces. The theory shows that the minimum coverage probabilities of random intervals with respect to corresponding parameters are achieved at discrete finite sets and that the coverage probabilities are continuous and unimodal when parameters are varying in between interval endpoints. The theory applies to common important discrete random variables including binomial variable, Poisson variable, negative binomial variable and hypergeometrical random variable. The theory can be used to make relevant statistical inference more rigorous and less conservative.

1 Binomial Random Intervals

Let $X$ be a Bernoulli random variable defined in a probability space $(\Omega, \mathcal{F}, \Pr)$ such that $\Pr\{X = 1\} = p$ and $\Pr\{X = 0\} = 1 - p$ where $p \in (0, 1)$. Let $X_1, \cdots, X_n$ be $n$ identical and independent samples of $X$. In many applications, it is important to construct a confidence interval $(L, U)$ such that $\Pr\{L < p < U \mid p\} \approx 1 - \delta$ with $\delta \in (0, 1)$. Here $L = L(n, \delta, K)$ and $U = U(n, \delta, K)$ are multivariate functions of $n$, $\delta$ and random variable $K = \sum_{i=1}^n X_i$. To simply notations, we drop the arguments and write $L = L(K)$ and $U = U(K)$. Also, we use notation $\Pr\{L(K) < p < U(K) \mid p\}$ to represent the probability when the binomial parameter assumes value $p$. Such notation is used in a similar way throughout this paper. We would thus advise the reader to distinguish this notation from conventional notation of conditional probability.

Clearly, the construction of confidence interval is independent of the binomial parameter $p$. But, for fixed $n$ and $\delta$, the quantity $\Pr\{L(K) < p < U(K) \mid p\}$ is a function of $p$ and is conventionally referred to as the coverage probability. In many situations, it is desirable to know what is the worst-case coverage probability for $p$ belonging to interval $[a, b] \subseteq [0, 1]$. For this purpose, we have

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Theorem 1 Suppose that both \( L(k) \) and \( U(k) \) are monotone functions of \( k \in \{0, 1, \ldots, n\} \). Then, the minimum of \( \Pr\{L(K) < p < U(K) \mid p\} \) with respect to \( p \in [a, b] \) is attained at the discrete set \( \{a, b\} \cup \{L(k) \in (a, b) : 0 \leq k \leq n\} \cup \{U(k) \in (a, b) : 0 \leq k \leq n\} \).

We would like to emphasize that the only assumption in Theorem 1 is that both \( L(k) \) and \( U(k) \) are either non-decreasing or non-increasing with respect to \( k \). The interval \((L(K), U(K))\) can be general random interval without being restricted to the context of confidence intervals. This theorem can be generalized as Theorem 7 in Section 4. The application of the theorem is discussed in the full version of our paper [4]. Specially, Theorem 1 can be applied to the sample size problems studied in [1].

For closed confidence interval \([L, U]\), it is interesting to compute the infimum of \( \Pr\{L(K) \leq p \leq U(K) \mid p\} \) with respect to \( p \in [a, b] \subseteq [0, 1] \). For this purpose, we have

Theorem 2 Suppose that both \( L(k) \) and \( U(k) \) are monotone functions of \( k \in \{0, 1, \ldots, n\} \). Then, the infimum of \( \Pr\{L(K) \leq p \leq U(K) \mid p\} \) with respect to \( p \in [a, b] \) equals the minimum of the set \( \{C(a), C(b)\} \cup \{C_U(p) : p \in \mathcal{D}_U\} \cup \{C_L(p) : p \in \mathcal{D}_L\} \), where

\[
\mathcal{D}_U = \{U(k) \in (a, b) : 0 \leq k \leq n\}, \quad \mathcal{D}_L = \{L(k) \in (a, b) : 0 \leq k \leq n\}, \quad C(p) = \Pr\{L(K) \leq p \leq U(K) \mid p\},
\]

\[
C_U(p) = \Pr\{L(K) \leq p < U(K) \mid p\} \quad \text{and} \quad C_L(p) = \Pr\{L(K) < p \leq U(K) \mid p\}.
\]

It should be noted that the only assumption in the above theorem is that both \( L(k) \) and \( U(k) \) are either non-decreasing or non-increasing with respect to \( k \). The interval \([L(K), U(K)]\) can be general random interval without being restricted to the context of confidence intervals. This theorem can be considered as a specialized result of Theorem 7 in Section 4.

2 Poisson Random Intervals

Let \( X \) be a Poisson random variable defined in a probability space \((\Omega, \mathcal{F}, \Pr)\) such that

\[
\Pr\{X = k\} = \frac{\lambda^ke^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots
\]

where \( \lambda > 0 \) is called the Poisson parameter. Let \( X_1, \ldots, X_n \) be \( n \) identical and independent samples of \( X \). It is a frequent problem to construct a confidence interval \((L, U)\) such that \( \Pr\{L < \lambda < U \mid \lambda\} \approx 1 - \delta \) with \( \delta \in (0, 1) \). Here \( L = L(n, \delta, K) \) and \( U = U(n, \delta, K) \) are multivariate functions of \( n \), \( \delta \) and random variable \( K = \sum_{i=1}^{n} X_i \). For simplicity of notations, we drop the arguments and write \( L = L(K) \) and \( U = U(K) \). For fixed \( n \) and \( \delta \), the coverage probability \( \Pr\{L(K) < \lambda < U(K) \mid \lambda\} \) is a function of \( \lambda \). The worst-case coverage probability with respect to \( \lambda \) belonging to interval \([a, b] \subseteq [0, \infty)\) can be obtained by the following theorem.

Theorem 3 Suppose that both \( L(k) \) and \( U(k) \) are monotone functions of non-negative integer \( k \). Then, the minimum of \( \Pr\{L(K) < \lambda < U(K) \mid \lambda\} \) with respect to \( \lambda \in [a, b] \) is attained at the discrete set \( \{a, b\} \cup \{L(k) \in (a, b) : k \geq 0\} \cup \{U(k) \in (a, b) : k \geq 0\} \).
It should be emphasized that the interval \((L(K), U(K))\) can be general random interval without being restricted to the context of confidence intervals. The only assumption in the above theorem is that both \(L(k)\) and \(U(k)\) are either non-decreasing or non-increasing with respect to \(k\). This theorem can be generalized as Theorem 7 in Section 4. The application of the theorem is discussed in the full version of our paper [4] for the sample size problems studied in [2].

For the exact computation of the infimum of coverage probability \(\Pr\{L(K) \leq \lambda \leq U(K) | \lambda\}\) for the closed confidence interval \([L, U]\), we have

**Theorem 4** Suppose that both \(L(k)\) and \(U(k)\) are monotone functions of non-negative integer \(k\). Then, the infimum of \(\Pr\{L(K) \leq \lambda \leq U(K) | \lambda\}\) with respect to \(\lambda \in [a, b]\) equals the minimum of the set \(\{C(a), C(b)\} \cup \{C_U(\lambda) : \lambda \in \mathcal{D}_U\} \cup \{C_L(\lambda) : \lambda \in \mathcal{D}_L\}\) where

\[
\mathcal{D}_U = \{U(k) \in (a, b) : k \geq 0\}, \quad \mathcal{D}_L = \{L(k) \in (a, b) : k \geq 0\}, \quad C(\lambda) = \Pr\{L(K) \leq \lambda \leq U(K) | \lambda\},
\]

\[
C_U(\lambda) = \Pr\{L(K) \leq \lambda < U(K) | \lambda\}, \quad C_L(\lambda) = \Pr\{L(K) < \lambda \leq U(K) | \lambda\}.
\]

In Theorem 4, the interval \([L(K), U(K)]\) can be general random interval without being restricted to the context of confidence intervals. This theorem is a special case of Theorem 7 in Section 4.

### 3 Negative-Binomial Random Intervals

Let \(K\) be a negative binomial random variable such that

\[
\Pr\{K = k\} = \binom{k + r - 1}{k} p^r (1 - p)^k, \quad k = 0, 1, \ldots
\]

with parameter \(p \in (0, 1)\) and \(r > 0\). In the special case that \(r = 1\), a negative binomial random variable becomes a geometrical random variable. For the coverage probability of open random interval \((L(K), U(K))\) for a negative binomial random variable \(K\), we have

**Theorem 5** Suppose that both \(L(k)\) and \(U(k)\) are monotone functions of non-negative integer \(k\). Then, the minimum of \(\Pr\{L(K) < p \leq U(K) | \lambda\}\) with respect to \(p \in [a, b] \subset (0, 1)\) is attained at the discrete set \(\{a, b\} \cup \{L(k) \in (a, b) : k \geq 0\} \cup \{U(k) \in (a, b) : k \geq 0\}\).

This theorem can be readily obtained by applying Theorem 7 of Section 4. For the coverage probability of closed random interval \([L(K), U(K)]\) for a negative binomial random variable \(K\), we have

**Theorem 6** Suppose that both \(L(k)\) and \(U(k)\) are monotone functions of non-negative integer \(k\). Then, the infimum of \(\Pr\{L(K) \leq p \leq U(K) | \lambda\}\) with respect to \(p \in [a, b] \subset (0, 1)\) equals the minimum of the set \(\{C(a), C(b)\} \cup \{C_U(p) : p \in \mathcal{D}_U\} \cup \{C_L(p) : p \in \mathcal{D}_L\}\) where

\[
\mathcal{D}_U = \{U(k) \in (a, b) : k \geq 0\}, \quad \mathcal{D}_L = \{L(k) \in (a, b) : k \geq 0\}, \quad C(p) = \Pr\{L(K) \leq p \leq U(K) | \lambda\},
\]

\[
C_U(p) = \Pr\{L(K) \leq p < U(K) | \lambda\} \quad \text{and} \quad C_L(p) = \Pr\{L(K) < p \leq U(K) | \lambda\}.
\]

This theorem can be easily deduced from Theorem 7 of next section.
4 Fundamental Theorem of Random Intervals

In previous sections, we discuss coverage probability of random intervals for specific random variables. Actually, the results can be generalized to a large class of discrete random variables. In this direction, we have recently established in [4] the following fundamental theorem of random intervals.

**Theorem 7** Let $K$ be a discrete integer-valued random variable parameterized by $\theta \in \Theta$. Let $[a, b]$ be a subset of $\Theta$. Let $L(K)$ and $U(K)$ be functions of random variable $K$. Let $\mathcal{D}_U$ denote the intersection of $[a, b]$ and the support of $U(K)$. Let $\mathcal{D}_L$ denote the intersection of $[a, b]$ and the support of $L(K)$. Suppose the following assumptions are satisfied.

(i) For any integers $k$ and $l$, $\Pr\{k \leq K \leq l \mid \theta\}$ is a continuous and unimodal function of $\theta \in [a, b]$.

(ii) For any $\theta \in [a, b]$, there exist intervals $I_\theta$ and $I_\theta'$ of real numbers such that $\{L(K) \leq \theta \leq U(K)\} = \{K \in I_\theta\}$ and $\{L(K) < \theta < U(K)\} = \{K \in I_\theta'\}$.

Then, the following statements hold true.

(I) The minimum of $\Pr\{L(K) < \theta < U(K) \mid \theta\}$ with respect to $\theta \in [a, b]$ is attained at the discrete set $[a, b] \cup \mathcal{D}_U \cup \mathcal{D}_L$.

(II) The infimum of $\Pr\{L(K) \leq \theta \leq U(K) \mid \theta\}$ with respect to $\theta \in [a, b]$ equals the minimum of the set $\{C(a), C(b)\} \cup \{C_U(\theta) : \theta \in \mathcal{D}_U\} \cup \{C_L(\theta) : \theta \in \mathcal{D}_L\}$, where $C(\theta) = \Pr\{L(K) \leq \theta \leq U(K) \mid \theta\}$, $C_U(\theta) = \Pr\{L(K) \leq \theta < U(K) \mid \theta\}$ and $C_L(\theta) = \Pr\{L(K) < \theta \leq U(K) \mid \theta\}$.

(III) For both open and closed random intervals, the coverage probability is continuous and unimodal for $\theta \in (\theta', \theta'')$, where $\theta'$ and $\theta''$ are any two consecutive distinct elements of $[a, b] \cup \mathcal{D}_U \cup \mathcal{D}_L$.

This theorem is proved in Appendices A. The notion of unimodal functions used in Theorem 7 is described as follows:

A function of $\theta$ is said to be unimodal for $\theta \in [a, b]$ if there exists $\theta^* \in [a, b]$ such that the function is non-decreasing for $\theta \in [a, \theta^*]$ and non-increasing for $\theta \in [\theta^*, b]$.

It should be noted that the assumptions (i) and (ii) are satisfied for the following common discrete random variables:

- Binomial random variable;
- Poisson random variable;
- Geometrical random variable;
- Negative binomial random variable.

A sufficient but not necessary condition to guarantee assumption (ii) is that $L(.)$ and $U(.)$ are monotone functions.
5 Hypergeometrical Random Intervals

So far what we have addressed are random intervals of variables with continuous parameter spaces. In this section, we shall consider random intervals when the parameter space is discrete. We focus on the important hypergeometrical random variable.

Consider a finite population of $N$ units, among which $M$ units have a certain attribute. Let $K$ be the number of units found to have the attribute in a sample of $n$ units obtained by sampling without replacement. The number $K$ is known to be a random variable of hypergeometrical distribution.

It is a basic problem to construct a confidence interval $(L, U)$ with $L = L(N, n, \delta, K)$ and $U = U(N, n, \delta, K)$ such that $\Pr\{L < M < U \mid M\} \approx 1 - \delta$. Here, $U$ and $L$ only assume integer values. For notational simplicity, we write $L = L(K)$ and $U = U(K)$. In practice, it is useful to know the minimum of coverage probability $\Pr\{L < M < U \mid M\}$ with respect to $M \in [a, b]$, where $a$ and $b$ are integers taken values in between 0 and $N$. For this purpose, we have

**Theorem 8** Suppose that $L(0) \leq L(1) \leq \cdots \leq L(n)$ and $U(0) \leq U(1) \leq \cdots \leq U(n)$. Then, the minimum of $\Pr\{L(K) < M < U(K) \mid M\}$ with respect to $M \in [a, b]$ is attained at the discrete set $I_{UL}$, where $I_{UL} = \{a, b\} \cup \{L(k) \in (a, b) : 0 \leq k \leq n\} \cup \{U(k) \in (a, b) : 0 \leq k \leq n\}$. Moreover, $\Pr\{L(K) < M < U(K) \mid M\}$ is unimodal with respect to $M$ in between consecutive distinct elements of $I_{UL}$.

For a proof, see Appendix B. In Theorem 8, the interval $(L(K), U(K))$ can be general random interval without being restricted to the context of confidence intervals. This theorem can be applied to the sample size problems discussed in [3].

A Proof of Theorem 7

We need some preliminary results.

**Lemma 1** Suppose that $\{\theta' \leq L(K) < \theta''\} = \{\theta' < U(K) < \theta''\} = \emptyset$. Then,

$$\{L(K) < \theta < U(K)\} = \{L(K) \leq \theta \leq U(K)\} = \{L(K) \leq \theta' < U(K)\} = \{L(K) < \theta'' \leq U(K)\}$$

for any $\theta \in (\theta', \theta'')$.

**Proof.** By the assumption of the lemma, we have $\{L(K) < \theta''\} = \{L(K) \leq \theta'\} \cup \{\theta' < L(K) < \theta''\} = \{L(K) \leq \theta'\}$ and $\{\theta' < L(K) < \theta\} \subseteq \{\theta' < L(K) \leq \theta\} \subseteq \{\theta' < L(K) < \theta''\} = \emptyset$ for any $\theta \in (\theta', \theta'')$. Consequently,

$$\{L(K) \leq \theta\} = \{L(K) \leq \theta'\} \cup \{\theta' < L(K) \leq \theta\} = \{L(K) \leq \theta'\} = \{L(K) < \theta''\},$$

$$\{L(K) < \theta\} = \{L(K) \leq \theta'\} \cup \{\theta' < L(K) < \theta\} = \{L(K) \leq \theta'\}$$

(1) $\{L(K) \leq \theta\} = \{L(K) \leq \theta'\}$

(2) $\{L(K) \leq \theta\}$
By taking intersection of events and making use of (3) and (6), we have that
\[ \Pr\{\theta < U(K) \leq \theta'\} = \{L(K) \leq \theta'\} = \{L(K) < \theta''\}, \quad \forall \theta \in (\theta', \theta''). \] (3)

By the assumption of the lemma, we have \(\{U(K) > \theta'\} = \{U(K) \geq \theta''\} \cup \{\theta' < U(K) < \theta''\} = \{U(K) \geq \theta''\}\) and \(\{\theta < U(K) < \theta''\} \subseteq \{\theta < U(K) < \theta''\} \subseteq \{\theta < U(K) < \theta''\} = \emptyset\) for any \(\theta \in (\theta', \theta'')\). Consequently,
\[ \{U(K) \geq \theta\} = \{U(K) \geq \theta''\} \cup \{\theta < U(K) < \theta''\} = \{U(K) \geq \theta''\} = \{U(K) > \theta'\}, \] (4)
\[ \{U(K) > \theta\} = \{U(K) \geq \theta''\} \cup \{\theta < U(K) < \theta''\} = \{U(K) \geq \theta''\} \] (5)
for any \(\theta \in (\theta', \theta'')\). Combining (3) and (5) yields
\[ \Pr\{U(K) \geq \theta\} = \Pr\{U(K) > \theta\} = \{U(K) > \theta'\} = \{U(K) \geq \theta''\}, \quad \forall \theta \in (\theta', \theta''). \] (6)

By taking intersection of events and making use of (3) and (6), we have
\[ \{L(K) < U(K) \leq \theta\} = \{L(K) \leq \theta < U(K)\} = \{L(K) \leq U(K)\} = \{L(K) < \theta'' \leq U(K)\} \]
for any \(\theta \in (\theta', \theta'')\). This completes the proof of the lemma. \(\square\)

Now we are in a position to prove Theorem 7. First, we shall show statement (I). Let \(\theta' < \theta''\) be two consecutive distinct elements of \([a, b] \cup 2U \cup 2L\). Then, \(\{\theta' < U(K) < \theta''\} = \{\theta' < U(K) < \theta''\} = \emptyset\) and by Lemma 1, we have
\[ \Pr\{L(K) < \theta < U(K) \mid \theta\} = \Pr\{L(K) < \theta' < U(K) \mid \theta\} = \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\} \]
for any \(\theta \in (\theta', \theta'')\). By assumption (ii), for any \(\theta \in [a, b]\), event \(\{L(K) < \theta < U(K)\}\) can be expressed as an event that \(K\) is included in an interval. This implies that \(\{L(K) < \theta'' \leq U(K)\}\) can be expressed as an event that \(K\) is included in an interval. It follows from assumptions (i) that \(\Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}\) is continuous and unimodal function of \(\theta \in [a, b]\). Hence, for \(\theta \in (\theta', \theta'')\), letting \(0 < \eta < \min(\theta - \theta', \theta'' - \theta, \frac{\theta'' - \theta'}{2})\), we have \(\theta' + \eta < \theta < \theta'' - \eta\) and
\[ \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\} \geq \min\{\Pr\{L(K) < \theta'' \leq U(K) \mid \theta' + \eta\}, \Pr\{L(K) < \theta'' \leq U(K) \mid \theta'' - \eta\}\}. \]

By virtue of the continuity of \(\Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}\) with respect to \(\theta \in [a, b]\), we have
\[ \lim_{\eta \to 0} \Pr\{L(K) < \theta'' \leq U(K) \mid \theta' + \eta\} = \Pr\{L(K) < \theta'' \leq U(K) \mid \theta'\}, \]
\[ \lim_{\eta \to 0} \Pr\{L(K) < \theta'' \leq U(K) \mid \theta'' - \eta\} = \Pr\{L(K) < \theta'' \leq U(K) \mid \theta''\}. \]
Therefore,
\[ \Pr\{L(K) < \theta < U(K) \mid \theta\} = \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\} \]
\[ \geq \min\{\Pr\{L(K) < \theta'' \leq U(K) \mid \theta'\}, \Pr\{L(K) < \theta'' \leq U(K) \mid \theta''\}\} \]
\[ = \min\{\Pr\{L(K) \leq \theta' < U(K) \mid \theta'\}, \Pr\{L(K) < \theta'' \leq U(K) \mid \theta''\}\} \]
\[ \geq \min\{\Pr\{L(K) < \theta' < U(K) \mid \theta'\}, \Pr\{L(K) < \theta'' \leq U(K) \mid \theta''\}\} \]
for any $\theta \in (\theta', \theta'')$. This implies that the minimum of $\Pr\{L(K) < \theta < U(K) \mid \theta\}$ with respect to $\theta \in [\theta', \theta'']$ is achieved at either $\theta'$ or $\theta''$. It follows that statement (I) is true.

Next, we shall show statement (II). Let $\theta' < \theta''$ be two consecutive distinct elements of $\{a, b\} \cup \mathcal{D}_U \cup \mathcal{D}_L$. Then, $\{\theta' < L(K) < \theta''\} = \{\theta < U(K) < \theta''\} = \emptyset$ and, by Lemma 1, we have $\Pr\{L(K) \leq \theta \leq U(K) \mid \theta\} = \Pr\{L(K) \leq \theta' < U(K) \mid \theta\} = \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}$ for any $\theta \in (\theta', \theta'')$. By assumption (ii), for any $\theta \in [a, b]$, event $\{L(K) \leq \theta \leq U(K)\}$ can be expressed as an event that $K$ is included in an interval. This implies that $\{L(K) < \theta'' \leq U(K)\}$ can be expressed as an event that $K$ is included in an interval. It follows from assumptions (i) that $\Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}$ is continuous and unimodal function of $\theta \in [a, b]$. In the course of proving statement (I), we have shown that

$$
\Pr\{L(K) < \theta'' \leq U(K) \mid \theta\} \geq \min\{\Pr\{L(K) < \theta' < U(K) \mid \theta\}, \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}\}
$$

for any $\theta \in (\theta', \theta'')$, where the lower bound in the right side is no greater than $\min(\Pr\{L(K) \leq \theta' \leq U(K) \mid \theta\}, \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\})$. Hence,

$$
\inf_{\theta \in [\theta', \theta'']} \Pr\{L(K) \leq \theta \leq U(K) \mid \theta\} \geq \min(\Pr\{L(K) < \theta' < U(K) \mid \theta\}, \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}).
$$

Now we show that the equality must hold. For simplicity of notations, let the left and right sides in the above inequality be denoted by $\gamma$ and $\rho$ respectively. Suppose $\gamma > \rho$. Then,

$$
\Pr\{L(K) \leq \theta \leq U(K) \mid \theta\} > \frac{\gamma + \rho}{2}, \quad \forall \theta \in [\theta', \theta''],
$$

which implies that

$$
\Pr\{L(K) \leq \theta' < U(K) \mid \theta\} = \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\} > \frac{\gamma + \rho}{2}, \quad \forall \theta \in (\theta', \theta'').
$$

Recalling that both $\Pr\{L(K) \leq \theta' < U(K) \mid \theta\}$ and $\Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}$ are continuous for $\theta \in [a, b]$, we have

$$
\rho = \min\{\Pr\{L(K) \leq \theta' < U(K) \mid \theta\}, \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}\} \geq \frac{\gamma + \rho}{2},
$$

leading $\rho \geq \gamma$, which contradicts $\gamma > \rho$. Therefore, it must be true that $\gamma = \rho$. That is,

$$
\inf_{\theta \in [\theta', \theta'']} \Pr\{L(K) \leq \theta \leq U(K) \mid \theta\} = \min(\Pr\{L(K) \leq \theta' < U(K) \mid \theta\}, \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}).
$$

It follows that the infimum of $\Pr\{L(K) \leq \theta \leq U(K) \mid \theta\}$ with respect to $\theta \in [a, b]$ equals the minimum of the set $\{C(a), C(b)\} \cup \{C_U(\theta) : \theta \in \mathcal{D}_U \cup \mathcal{D}_L\} \cup \{C_L(\theta) : \theta \in \mathcal{D}_L \cup \mathcal{D}_U\}$. Let

$$
\mathcal{D}' = \mathcal{D}_U \cap \mathcal{D}_L, \quad \mathcal{D}'_U = \mathcal{D}_U \setminus \mathcal{D}', \quad \mathcal{D}'_L = \mathcal{D}_L \setminus \mathcal{D}'.
$$

Then,

$$
\{C_U(\theta) : \theta \in \mathcal{D}_U \cup \mathcal{D}_L\} \cup \{C_L(\theta) : \theta \in \mathcal{D}_U \cup \mathcal{D}_L\} = \{C_U(\theta) : \theta \in \mathcal{D}'_U\} \cup \{C_U(\theta) : \theta \in \mathcal{D}'\} \cup \{C_L(\theta) : \theta \in \mathcal{D}'_L\} \cup \{C_L(\theta) : \theta \in \mathcal{D}'\}.
$$

(7)
For \( \theta \in \mathcal{U}' \), we have \( 0 \leq \Pr\{L(K) = \theta < U(K) \mid \theta\} \leq \Pr\{L(K) = \theta \mid \theta\} = 0 \) and

\[
C_U(\theta) - C_L(\theta) = \Pr\{L(K) = \theta < U(K) \mid \theta\} - \Pr\{L(K) < \theta \leq U(K) \mid \theta\} = \Pr\{L(K) = \theta < U(K) \mid \theta\} - \Pr\{L(K) < \theta = U(K) \mid \theta\} = \Pr\{L(K) < \theta = U(K) \mid \theta\} \leq 0,
\]

which implies that

\[
\min \{C_U(\theta) : \theta \in \mathcal{U}'\} \leq \min \{C_L(\theta) : \theta \in \mathcal{U}'\}.
\]

For \( \theta \in \mathcal{L}' \), we have \( 0 \leq \Pr\{L(K) < \theta = U(K) \mid \theta\} \leq \Pr\{U(K) = \theta \mid \theta\} = 0 \) and

\[
C_U(\theta) - C_L(\theta) = \Pr\{L(K) = \theta < U(K) \mid \theta\} - \Pr\{L(K) < \theta = U(K) \mid \theta\} = \Pr\{L(K) = \theta < U(K) \mid \theta\} \geq 0,
\]

which implies that

\[
\min \{C_U(\theta) : \theta \in \mathcal{L}'\} \geq \min \{C_L(\theta) : \theta \in \mathcal{L}'\}.
\]

Combing (7), (8) and (9), we have

\[
\min \{C_U(\theta) : \theta \in \mathcal{U} \cup \mathcal{L}\} \cup \{C_U(\theta) : \theta \in \mathcal{U} \cup \mathcal{L}\} = \min \{C_U(\theta) : \theta \in \mathcal{U}'\} \cup \{C_U(\theta) : \theta \in \mathcal{U}'\} \cup \{C_U(\theta) : \theta \in \mathcal{L}'\} \cup \{C_U(\theta) : \theta \in \mathcal{L}'\},
\]

which implies that the minimum of the set \( \{C(a), C(b) \cup \{C_U(\theta) : \theta \in \mathcal{U} \cup \mathcal{L}\} \cup \{C_L(\theta) : \theta \in \mathcal{U} \cup \mathcal{L}\} \) equals the minimum of \( \{C(a), C(b) \cup \{C_U(\theta) : \theta \in \mathcal{U}\} \cup \{C_L(\theta) : \theta \in \mathcal{L}\} \). This proves statement (II) of Theorem 7.

Clearly, statement (III) of Theorem 7 is already justified in the course of proving statements (I) and (II). This concludes the proof of Theorem 7.

### B Proof of Theorem 8

For the simplicity of notations, define

\[
\binom{m}{z} = \begin{cases} 
\frac{m!}{z!(m-z)!} & \text{if } 0 \leq z \leq m, \\
0 & \text{if } z < 0 \text{ or } z > m
\end{cases}
\]

for non-negative integer \( m \) and arbitrary integer \( z \). We now establish some preliminary results.

**Lemma 2** Let \( 0 \leq M < N \). Define \( T(k,M,n) = \binom{M}{k}\binom{N-M-1}{n-k-1} \). Then, \( \Pr\{K \leq k \mid M\} - \Pr\{K \leq k \mid M + 1\} = T(k,M,n) \) for any integer \( k \).
\textbf{Proof.} We first show the equation for $0 \leq k \leq M$. We perform induction on $k$. For $k = 0$, we have

\[
\Pr\{K \leq k \mid M\} - \Pr\{K \leq k \mid M + 1\} = \Pr\{K = 0 \mid M\} - \Pr\{K = 0 \mid M + 1\} = \binom{M}{0} \binom{N - M}{n} - \binom{M}{0} \binom{N - M - 1}{n - 1} = \binom{N}{n} - \binom{N}{n} = 0, \tag{10}
\]

where (10) follows from the fact that, for non-negative integer $m$,

\[
\binom{m + 1}{z + 1} = \binom{m}{z} + \binom{m}{z + 1}, \tag{11}
\]

for any integer $z$.

Now suppose the lemma is true for $k - 1$ with $1 \leq k \leq M$, i.e.,

\[
\Pr\{K \leq k - 1 \mid M\} - \Pr\{K \leq k - 1 \mid M + 1\} = \binom{M}{k-1} \binom{N-M-1}{n-k}.
\]

Then,

\[
\Pr\{K \leq k \mid M\} - \Pr\{K \leq k \mid M + 1\} = \Pr\{K \leq k - 1 \mid M\} - \Pr\{K \leq k - 1 \mid M + 1\} + \binom{M}{k} \binom{N-M}{n-k} - \binom{M}{k} \binom{N-M-1}{n-k} = \binom{M}{k} \binom{N-M}{n-k} - \binom{M}{k} \binom{N-M-1}{n-k} - \binom{M}{k} \binom{N-M}{n-k} + \binom{M}{k} \binom{N-M-1}{n-k} = \binom{M}{k} \binom{N-M}{n-k} - \binom{M}{k} \binom{N-M-1}{n-k} + \binom{M}{k} \binom{N-M}{n-k} - \binom{M}{k} \binom{N-M-1}{n-k} = \binom{M}{k} \binom{N-M}{n-k} - \binom{M}{k} \binom{N-M-1}{n-k} - \binom{M}{k} \binom{N-M}{n-k} + \binom{M}{k} \binom{N-M-1}{n-k} = \binom{M}{k} \binom{N-M}{n-k} - \binom{M}{k} \binom{N-M-1}{n-k} - \binom{M}{k} \binom{N-M}{n-k} + \binom{M}{k} \binom{N-M-1}{n-k} = \binom{M}{k} \binom{N-M}{n-k} - \binom{M}{k} \binom{N-M-1}{n-k} = \binom{M}{k} \binom{N-M}{n-k} - \binom{M}{k} \binom{N-M-1}{n-k} \tag{12}
\]

where (12) and (13) follows from (11). Therefore, we have shown the lemma for $0 \leq k \leq M$.

For $k > M$, we have $\Pr\{K \leq k \mid M\} = \Pr\{K \leq k \mid M + 1\} = 1$ and $T(k, M, N, n) = 0$. For $k < 0$, we have $\Pr\{K \leq k \mid M\} = \Pr\{K \leq k \mid M + 1\} = 0$ and $T(k, M, N, n) = 0$. Thus, the lemma is true for any integer $k$.

\[
\square
\]

\textbf{Lemma 3} \textit{Let $1 \leq M \leq N$ and $k \leq l$. Then,}

\[
\Pr\{k \leq K \leq l \mid M\} - \Pr\{k \leq K \leq l \mid M - 1\} = T(k - 1, M - 1, N, n) - T(l, M - 1, N, n).
\]
Proof. To show the lemma, it suffices to consider 6 cases as follows.

Case (i): $0 < n < k \leq l$. In this case, $\Pr\{k \leq K \leq l \mid M\} = \Pr\{k \leq K \leq l \mid M-1\} = 0$ and $T(k-1, M-1, N, n) = T(l, M-1, N, n) = 0$.

Case (ii): $k \leq l < 0$. In this case, $\Pr\{k \leq K \leq l \mid M\} = \Pr\{k \leq K \leq l \mid M-1\} = 0$ and $T(k-1, M-1, N, n) = T(l, M-1, N, n) = 0$.

Case (iii): $k \leq 0 < n \leq l$. In this case, $\Pr\{k \leq K \leq l \mid M\} = \Pr\{k \leq K \leq l \mid M-1\} = 1$ and $T(k-1, M-1, N, n) = T(l, M-1, N, n) = 0$.

Case (iv): $k \leq 0 \leq l < n$. In this case, $T(k-1, M-1, N, n) = 0$ and, by Lemma 2

$$\Pr\{k \leq K \leq l \mid M\} - \Pr\{k \leq K \leq l \mid M-1\} = \Pr\{K \leq K \leq l \mid M\} - \Pr\{K \leq K \leq l \mid M-1\} = T(k-1, M-1, N, n) - T(l, M-1, N, n).$$

Case (v): $0 \leq k \leq n < l$. In this case, $T(l, M-1, N, n) = 0$ and, by Lemma 2

$$\Pr\{k \leq K \leq l \mid M\} - \Pr\{k \leq K \leq l \mid M-1\} = \Pr\{K < k \mid M\} - \Pr\{K < k \mid M-1\} = T(k-1, M-1, N, n) - T(l, M-1, N, n).$$

Case (vi): $0 < k \leq l < n$. In this case, by Lemma 2

$$\Pr\{k \leq K \leq l \mid M\} - \Pr\{k \leq K \leq l \mid M-1\} = \Pr\{K \leq K \leq l \mid M\} - \Pr\{K \leq K \leq l \mid M-1\} = T(k-1, M-1, N, n) - T(l, M-1, N, n).$$

\[\square\]

Lemma 4 Let $l \geq 0$ and $k < n$. Then, $\left\lfloor \frac{nM}{N+1} \right\rfloor \geq l$ for $M \geq 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$, and $\left\lfloor \frac{nM}{N+1} \right\rfloor \leq k-1$ for $M \leq 1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor$.

Proof. To show the first part of the lemma, observe that $(N+1-n)l \geq 0$, by which we can show $\frac{nN}{n-1} \geq (N+1)l$. Hence, $n \left(1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor \right) > \frac{nN}{n-1} \geq (N+1)l$. That is, $\frac{n}{N+1} \left(1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor \right) > l$. It follows that $\left\lfloor \frac{n}{N+1} \left(1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor \right) \right\rfloor \geq l$. Since the floor function is non-decreasing, we have $\left\lfloor \frac{nM}{N+1} \right\rfloor \geq l$ for $M \geq 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$.

To prove the second part of the lemma, note that $(N+1-n)(n-k) > 0$, from which we can deduce $1 + \frac{N(k-1)}{n-1} < \frac{(N+1)k}{n}$. Hence, $1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor < \frac{(N+1)k}{n}$, i.e., $\frac{n}{N+1} \left(1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor \right) < k$, leading to $\left\lfloor \frac{n}{N+1} \left(1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor \right) \right\rfloor \leq k-1$. Since the floor function is non-decreasing, we have $\left\lfloor \frac{nM}{N+1} \right\rfloor \leq k-1$ for $M \leq 1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor$.

\[\square\]
Lemma 5 Let $0 \leq r \leq n$. Then, the following statements hold true.

(I) 
\[ T(r-1, M-1, N, n) \leq T(r, M-1, N, n) \quad \text{for} \quad 1 \leq r \leq \left\lfloor \frac{nM}{N+1} \right\rfloor; \]
\[ T(r+1, M-1, N, n) \leq T(r, M-1, N, n) \quad \text{for} \quad \left\lfloor \frac{nM}{N+1} \right\rfloor \leq r \leq n-1. \]

(II) 
\[ T(r, M-2, N, n) \leq T(r, M-1, N, n) \quad \text{for} \quad 1 < M \leq 1 + \left\lfloor \frac{Nr}{n-1} \right\rfloor; \]
\[ T(r, M, N, n) \leq T(r, M-1, N, n) \quad \text{for} \quad 1 + \left\lfloor \frac{Nr}{n-1} \right\rfloor \leq M < N. \]

Proof. To show statement (I), note that $T(r, M-1, N, n) = 0$ for $\min(M-1, n-1) < r \leq n$. Our calculation shows that
\[ \frac{T(r-1, M-1, N, n)}{T(r, M-1, N, n)} = \frac{r}{M} \frac{N - M + 1 - (n-r)}{n-r} \leq 1 \quad \text{for} \quad 1 \leq r \leq \frac{nM}{N+1} \]
and
\[ \frac{T(r-1, M-1, N, n)}{T(r, M-1, N, n)} > 1 \quad \text{for} \quad \frac{nM}{N+1} < r \leq \min(M-1, n-1). \]

To show statement (II), note that $T(r, M-1, N, n) = 0$ for $1 \leq M < r+1$, and $T(r, M-1, N, n) \geq T(r, M-2, N, n) = 0$ for $M = r+1$. Direct computation shows that
\[ \frac{T(r, M-1, N, n)}{T(r, M-2, N, n)} = \frac{M-1}{M-1-r} \frac{N - M + 2 - (n-r)}{N - M + 1} \geq 1 \quad \text{for} \quad r+1 < M \leq 1 + \frac{Nr}{n-1}, \]
and
\[ \frac{T(r, M-1, N, n)}{T(r, M-2, N, n)} < 1 \quad \text{for} \quad 1 + \frac{Nr}{n-1} < M \leq N. \]

Lemma 6 Let $0 \leq \mathcal{L} \leq \mathcal{U} \leq N$. Then, for any integers $k$ and $l$, $\Pr\{k \leq \mathcal{L} \leq l \mid M\}$ is unimodal with respect to $M$ for $\mathcal{L} \leq M \leq \mathcal{U}$.

Proof. Clearly, the lemma is trivially true if $k > l$. Hence, to show the lemma, it suffices to consider 6 cases as follows.

Case (i): $0 < n < k \leq l$. In this case, $\Pr\{k \leq \mathcal{L} \leq l \mid M\} = 0$ for any $M \in [\mathcal{L}, \mathcal{U}]$.

Case (ii): $k \leq l < 0 < n$. In this case, $\Pr\{k \leq \mathcal{L} \leq l \mid M\} = 0$ for any $M \in [\mathcal{L}, \mathcal{U}]$.

Case (iii): $k \leq 0 < n \leq l$. In this case, $\Pr\{k \leq \mathcal{L} \leq l \mid M\} = 1$ for any $M \in [\mathcal{L}, \mathcal{U}]$.

Case (iv): $k \leq 0 \leq l < n$. In this case, $\Pr\{k \leq \mathcal{L} \leq l \mid M\} = \Pr\{K \leq l \mid M\}$ is non-increasing with respect to $M \in [\mathcal{L}, \mathcal{U}]$ as can be seen from Lemma 2.

Case (v): $0 < k \leq n \leq l$. In this case, $\Pr\{k \leq \mathcal{L} \leq l \mid M\} = 1 - \Pr\{K < k \mid M\}$ is non-decreasing with respect to $M \in [\mathcal{L}, \mathcal{U}]$ as can be seen from Lemma 2.
Clearly, the lemma is true for the above five cases.

Case (vi): $0 < k \leq l < n$. Define $\Delta(k, l, M, n) = \Pr\{k \leq K \leq l \mid M\} - \Pr\{k \leq K \leq l \mid M - 1\}$. By Lemma 3, $\Delta(k, l, M, n) = \Delta(k, l, M - 1, N, n) - T(l, M - 1, N, n)$. By statement (I) of Lemma 5, $T(r, M - 1, N, n)$ is non-decreasing with respect to $r \leq l$. Consequently, $T(k - 1, M - 1, N, n) \leq T(l, M - 1, N, n)$, leading to $\Delta(k, l, M, n) \leq 0$ for $M \geq 1 + \left\lceil \frac{N}{n} \right\rceil$. Similarly, applying Lemma 4, for $M \leq 1 + \left\lceil \frac{N(k-1)}{n-1} \right\rceil$, we have that $\frac{nM}{N+1} \leq k - 1$ and thus, by statement (I) of Lemma 5, $T(r, M - 1, N, n)$ is non-increasing with respect to $r \geq k - 1$. Consequently, $T(k - 1, M - 1, N, n) \geq T(l, M - 1, N, n)$, leading to $\Delta(k, l, M, n) \geq 0$ for $M \leq 1 + \left\lceil \frac{N(k-1)}{n-1} \right\rceil$.

By statement (II) of Lemma 5 for $1 + \left\lceil \frac{N(k-1)}{n-1} \right\rceil \leq M \leq 1 + \left\lceil \frac{N}{n} \right\rceil$, we have that $T(l, M - 1, N, n)$ is non-decreasing with respect to $M$ and that $T(k - 1, M - 1, N, n)$ is non-increasing with respect to $M$. It follows that $\Delta(k, l, M, N, n)$ is non-increasing with respect to $M$ in this range. Therefore, there exists an integer $M^*$ such that $1 + \left\lceil \frac{N(k-1)}{n-1} \right\rceil \leq M^* \leq 1 + \left\lceil \frac{N}{n} \right\rceil$ and that $\Delta(k, l, M, N, n) \geq 0$ for $0 \leq M \leq M^*$, and $\Delta(k, l, M, N, n) \leq 0$ for $M^* \leq M \leq N$. This implies that $\Pr\{k \leq K \leq l \mid M\}$ is non-decreasing for $0 \leq M \leq M^*$ and non-increasing for $M^* \leq M \leq N$. This concludes the proof of the lemma.

\begin{lemma}
Let $0 \leq M < N$. Then, $\Pr\{g \leq K \leq h + 1 \mid M + 1\} \geq \Pr\{g \leq K \leq h \mid M\}$ for any integers $g$ and $h$.
\end{lemma}

\begin{proof}
Clearly, the lemma is trivially true if $g > h$. Hence, to show the lemma, it suffices to consider the case $g \leq h$. Note that, by Lemma 3

$$
\Pr\{g \leq K \leq h + 1 \mid M + 1\} - \Pr\{g \leq K \leq h \mid M\} = \left(\frac{M+1}{h+1}\right)\left(\frac{N-M-1}{n-h-1}\right) / \left(\frac{N}{n}\right) + \Pr\{g \leq K \leq h + 1 \mid M + 1\} - \Pr\{g \leq K \leq h \mid M\}
$$

$$
= \left(\frac{M+1}{h+1}\right)\left(\frac{N-M-1}{n-h-1}\right) / \left(\frac{N}{n}\right) + T(g-1, M, N, n) - T(h, M, N, n)
$$

$$
= \left[\left(\frac{M+1}{h+1}\right)\left(\frac{N-M-1}{n-h-1}\right) - \left(\frac{M}{h}\right)\left(\frac{N-M-1}{n-h-1}\right)\right] / \left(\frac{N}{n}\right) + T(g-1, M, N, n)
$$

$$
= \left(\frac{M}{h+1}\right)\left(\frac{N-M-1}{n-h-1}\right) / \left(\frac{N}{n}\right) + T(g-1, M, N, n) \geq 0,
$$

where the last equality follows from (11).
\end{proof}

\begin{lemma}
Let $0 < M \leq N$. Then, $\Pr\{g - 1 \leq K \leq h \mid M - 1\} \geq \Pr\{g \leq K \leq h \mid M\}$ for any integers $g$ and $h$.
\end{lemma}

\begin{proof}

\end{proof}
Proof. Clearly, the lemma is trivially true if \( g > h \). Hence, to show the lemma, it suffices to consider the case \( g \leq h \). Note that, by Lemma 9

\[
\Pr\{g - 1 \leq K \leq h \mid M - 1\} = \Pr\{g \leq K \leq h \mid M\}
\]

\[
= \left(\frac{M - 1}{g - 1}\right) \left(\frac{N - M + 1}{n - g + 1}\right) / \left(\frac{N}{n}\right) + \Pr\{g \leq K \leq h \mid M - 1\} - \Pr\{g \leq K \leq h \mid M\}
\]

\[
= \left(\frac{M - 1}{g - 1}\right) \left(\frac{N - M + 1}{n - g + 1}\right) / \left(\frac{N}{n}\right) + T(h, M - 1, N, n) - T(g - 1, M - 1, N, n)
\]

\[
= \left[\left(\frac{M - 1}{g - 1}\right) \left(\frac{N - M + 1}{n - g + 1}\right) - \left(\frac{M - 1}{g - 1}\right) \left(\frac{N - M}{n - g}\right)\right] / \left(\frac{N}{n}\right) + T(h, M - 1, N, n)
\]

where the last equality follows from (11).

\[\square\]

Lemma 9 Suppose that \( \{M' < L(K) < M''\} = \{M' < U(K) < M''\} = \emptyset \). Then, \( \Pr\{L(K) < M < U(K) \mid M\} \) is unimodal with respect to \( M \) for \( M' \leq M \leq M'' \).

Proof. First, we shall show the following facts:

(i) If \( \{L(K) = M'\} = \emptyset \), then \( \{L(K) < M\} = \{L(K) < M'\} = \{L(K) < M''\} \) for \( M' \leq M \leq M'' \).

(ii) If \( \{L(K) = M'\} \neq \emptyset \), then \( \{L(K) < M\} = \{L(K) \leq M'\} = \{L(K) < M''\} \) for \( M' < M \leq M'' \).

(iii) If \( \{U(K) = M''\} = \emptyset \), then \( \{U(K) > M\} = \{U(K) > M'\} = \{U(K) > M''\} \) for \( M' \leq M \leq M'' \).

(iv) If \( \{U(K) = M''\} \neq \emptyset \), then \( \{U(K) > M\} = \{U(K) > M'\} = \{U(K) \geq M''\} \) for \( M' < M < M'' \).

To show statement (i), making use of \( \{L(K) = M'\} = \{M' < L(K) < M''\} = \emptyset \), we have \( \{M' \leq L(K) < M\} = \{M' < L(K) < M\} \subseteq \{M' < L(K) < M''\} = \emptyset \) and \( \{L(K) < M\} = \{L(K) < M'\} \cup \{M' \leq L(K) < M\} = \{L(K) < M'\} \) for \( M' \leq M \leq M'' \). On the other hand, \( \{L(K) < M\} = \{L(K) < M''\} \setminus \{M \leq L(K) < M''\} = \{L(K) < M''\} \) for \( M' \leq M \leq M'' \).

To show statement (ii), making use of \( \{M' < L(K) < M''\} = \emptyset \), we have \( \{M' < L(K) < M\} \subseteq \{M' < L(K) < M''\} = \emptyset \) and \( \{L(K) < M\} = \{L(K) \leq M'\} \cup \{M' < L(K) < M\} = \{L(K) \leq M'\} \) for \( M' \leq M \leq M'' \). On the other hand, \( \{L(K) < M\} = \{L(K) < M''\} \setminus \{M \leq L(K) < M''\} = \{L(K) < M''\} \) for \( M' < M \leq M'' \).

To show statement (iii), using \( \{U(K) = M''\} = \{M' < U(K) < M''\} = \emptyset \), we have \( \{M' < U(K) \leq M\} \subseteq \{M' < U(K) < M''\} = \emptyset \) and \( \{U(K) > M\} = \{U(K) > M'\} \setminus \{M' < U(K) \leq M\} \) for \( M' < U(K) \leq M \).
\( M \} = \{ U(K) > M' \} \) for \( M' \leq M \leq M'' \). On the other hand, \( \{ U(K) > M \} = \{ U(K) > M'' \} \cup \{ M < U(K) \leq M'' \} = \{ U(K) > M'' \} \) for \( M' \leq M \leq M'' \).

To show statement (iv), note that \( \{ U(K) > M \} = \{ U(K) > M' \} \) for \( M' \leq M < M'' \). On the other hand, \( \{ U(K) > M \} = \{ U(K) \geq M'' \} \cup \{ M < U(K) < M'' \} = \{ U(K) \geq M'' \} \) for \( M' \leq M < M'' \).

Now, to show the lemma, it suffices to consider four cases as follows.

Case (i): \( \{ L(K) = M' \} = \emptyset, \{ U(K) = M'' \} = \emptyset \).

Case (ii): \( \{ L(K) = M' \} = \emptyset, \{ U(K) = M'' \} \neq \emptyset \).

Case (iii): \( \{ L(K) = M' \} \neq \emptyset, \{ U(K) = M'' \} = \emptyset \).

Case (iv): \( \{ L(K) = M' \} \neq \emptyset, \{ U(K) = M'' \} \neq \emptyset \).

In Case (i), making use of facts (i) and (iii), we have \( \{ L(K) < M < U(K) \} = \{ L(K) < M' < U(K) \} \) for \( M' \leq M \leq M'' \). Invoking Lemma 6, we have that \( \Pr(L(K) < M < U(K) | M) \) is unimodal with respect to \( M \) for \( M' \leq M \leq M'' \).

In Case (ii), making use of facts (i) and (iv), we have \( \{ L(K) < M < U(K) \} = \{ L(K) < M' < U(K) \} \) for \( M' \leq M < M'' \). Invoking Lemma 6, we have that \( \Pr(L(K) < M < U(K) | M) \) is unimodal with respect to \( M \) for \( M' \leq M < M'' \). Since \( \{ M'' = U(K) \} \neq \emptyset \) and \( U(K) \) is monotonically increasing, we have \( \{ M'' \leq U(K) \} = \{ K \geq \bar{k} \} \) and \( \{ M'' < U(K) \} = \{ K > \bar{k} + 1 \} \), where \( \bar{k} = \min\{ k : U(k) \geq M'' \} \leq \bar{k} = \max\{ k : U(k) \leq M'' \} \).

Therefore, as a result of Lemma 6,

\[
\Pr(L(K) < M'' \leq U(K) | M'' - 1) \geq \Pr(L(K) < M'' < U(K) | M'').
\] (14)

It follows that \( \Pr(L(K) < M < U(K) | M) \) is unimodal with respect to \( M \) for \( M' \leq M \leq M'' \).

In Case (iii), making use of facts (ii) and (iii), we have \( \{ L(K) < M < U(K) \} = \{ L(K) \leq M' < U(K) \} \) for \( M' \leq M \leq M'' \). Invoking Lemma 6, we have that \( \Pr(L(K) < M < U(K) | M) \) is unimodal with respect to \( M \) for \( M' < M \leq M'' \). Since \( \{ M' = L(K) \} \neq \emptyset \) and \( L(K) \) is monotonically increasing, we have \( \{ M' \geq L(K) \} = \{ K \leq \bar{k} \} \) and \( \{ M' > L(K) \} = \{ K \leq \bar{k} - 1 \} \), where \( \bar{k} = \min\{ k : L(k) \geq M' \} \leq \bar{k} = \max\{ k : L(k) \leq M' \} \).

Therefore, as a result of Lemma 6,

\[
\Pr(L(K) < M' < U(K) | M') \leq \Pr(L(K) \leq M' < U(K) | M' + 1).
\] (15)

It follows that \( \Pr(L(K) < M < U(K) | M) \) is unimodal with respect to \( M \) for \( M' \leq M \leq M'' \).

In Case (iv), making use of facts (ii) and (iv), we have \( \{ L(K) < M < U(K) \} = \{ L(K) \leq M' < U(K) \} \) for \( M' < M < M'' \). Invoking Lemma 6, we have that
Pr\{L(K) < M < U(K) \mid M\} is unimodal with respect to M for M' < M < M''. Recalling (14) and (15), we have that Pr\{L(K) < M < U(K) \mid M\} is unimodal with respect to M for M' ≤ M ≤ M''.

Finally, we are in a position to prove the theorem. Let M' < M'' be two consecutive distinct elements of I_{UL}. Then, \{M' < L(K) < M''\} = \{M' < U(K) < M''\} = ∅. By Lemma 9, we have that Pr\{L(K) < M < U(K) \mid M\} is unimodal with respect to M for M' ≤ M ≤ M''. Since this argument holds for any consecutive distinct elements of the set I_{UL}, Theorem 8 is established.

References

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