A general family of MSRD codes and PMDS codes with smaller field sizes from extended Moore matrices

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Abstract

We construct six new explicit families of linear maximum sum-rank distance (MSRD) codes, each of which has the smallest field sizes among all known MSRD codes for some parameter regime. Using them and a previous result of the author, we provide two new explicit families of linear partial MDS (PMDS) codes with smaller field sizes than previous PMDS codes for some parameter regimes. Our approach is to characterize evaluation points that turn extended Moore matrices into the parity-check matrix of a linear MSRD code. We then produce such sequences from codes with good Hamming-metric parameters. The six new families of linear MSRD codes with smaller field sizes are obtained using MDS codes, Hamming codes, BCH codes and three Algebraic-Geometry codes. The MSRD codes based on Hamming codes, of minimum sum-rank distance 3, meet a recent bound by Byrne et al.

Keywords: Linearized Reed-Solomon codes, locally repairable codes, Moore matrices, MDS codes, MRD codes, MSRD codes, PMDS codes, sum-rank metric.

MSC: 15B33; 11T71; 94B27; 94B65

1 Introduction

Maximum distance separable (MDS) codes are optimal in the sense that their minimum Hamming distance \[21\] attains the Singleton bound \[47\], which is independent of the alphabet size (i.e., field size). Thus, in erasure scenarios where alphabets need not be too small and for fixed block lengths, MDS codes offer the best erasure correction capability. One of such erasure scenarios is node repair in distributed storage. However, repairing a single node out of \(n\) nodes using an MDS code of rate \(k/n\) requires contacting \(k\) other nodes. Thus repairing a single node results in a high latency when using MDS codes. Locally repairable codes (LRCs) \[17\] \[20\] may repair one node (or more generally, \(\delta - 1\) nodes) by contacting a small number \(r\) (called locality) of other nodes. Simultaneously, they can correct many global erasures in catastrophic cases.

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Partial MDS (PMDS) codes are LRCs that can correct all the erasure patterns correctable by any other LRC over any alphabet but with the same information rate and locality constraints, if we assume that local repair sets are pair-wise non-intersecting (note that general LRCs do not require local repair sets to be non-intersecting). Singleton bounds for the global minimum distance of LRCs were given in [17, Eq. (2)] and [26, Th. 2.1]. Any PMDS code attains such Singleton bounds, but not all LRCs attaining such bounds are PMDS.

Several constructions of PMDS codes exist in the literature [3, 4, 5, 8, 9, 14, 16, 20, 24, 39, 41]. In Construction 1 in [39], it was shown that any maximum sum-rank distance (MSRD) code may be explicitly turned into a PMDS code [39, Th. 2]. Moreover, this Construction 1 enjoys further flexibility properties, such as enabling hierarchical PMDS codes (see [39]). As another application of such a flexibility, optimal LRCs with multiple disjoint repair sets were obtained based on MSRD codes in [7].

Apart from being used as PMDS codes for distributed storage [39], MSRD codes have applications in reliable and secure multishot network coding [42, 38], rate-diversity optimal space-time codes with multiple fading blocks [31, 46], multilayer crisscross error correction [37], and private information retrieval from locally repairable databases [34].

Codes over small fields are preferable, as they enjoy lower computational complexity for encoding and decoding. In contrast with MDS codes, PMDS codes and MSRD codes with linear field sizes in the code length do not exist for arbitrary dimensions [19, 6]. The problems of finding the smallest possible field sizes of PMDS codes and MSRD codes constitute two generalizations of the MDS conjecture. However, they are significantly harder since even the possible asymptotic field sizes are unknown for PMDS and MSRD codes, whereas it was known since [44, 47] that MDS codes exist if, and only if, the field sizes grow at least linearly in the code length.

In this work, we obtain a general family of MSRD codes that extends linearized Reed-Solomon codes, but also includes six new families of explicit MSRD codes (rows 2 to 7 in Table 1), each of which attains smaller field sizes than all other known MSRD codes for some parameter regime. For minimum sum-rank distance 3 (co-dimension 2), our MSRD codes meet a bound recently given in [6, Th. 6.12]. We also obtain two new families of explicit PMDS codes (rows 2 and 3 in Table 2), each of which attains smaller field sizes than all other known PMDS codes for some parameter regime. See Section 5 for a detailed summary and comparisons, and the Appendix for tables with concrete values of even field sizes and other parameters.

The manuscript is organized as follows. In Section 2 we collect preliminaries on MDS, MSRD and PMDS codes. In Section 3 we characterize sequences of evaluation points that turn an extended Moore matrix into the parity-check matrix of an MSRD code. In Section 4 we construct such sequences via tensor products and a range of Hamming-metric codes. In Section 5 we provide a summary of the obtained MSRD and PMDS codes and compare their parameters among themselves and with known codes.
2 Preliminaries

We will denote \( \mathbb{N} = \{0,1,2,\ldots\} \) and \( \mathbb{N}_+ = \{1,2,3,\ldots\} \). For positive integers \( m \leq n \), we denote \([n] = \{1,2,\ldots,n\}\) and \([m,n] = \{m,m+1,\ldots,n\}\). For a field \( \mathbb{F} \), we denote \( \mathbb{F}^* = \mathbb{F} \setminus \{0\} \) and we use \( \langle \cdot \rangle_{\mathbb{F}} \) and \( \dim_{\mathbb{F}}(\cdot) \) to denote \( \mathbb{F} \)-linear span and dimension over \( \mathbb{F} \), respectively. We denote by \( \mathbb{F}^{m \times n} \) the set of \( m \times n \) matrices with entries in \( \mathbb{F} \), and we denote \( \mathbb{F}^{n} = \mathbb{F}^{1 \times n} \). The group of invertible matrices in \( \mathbb{F}^{n \times n} \) is denoted by \( \text{GL}_n(\mathbb{F}) \). A code in \( \mathbb{F}^n \) is any subset \( C \subseteq \mathbb{F}^n \), and we say that \( C \) is a linear code if it is an \( \mathbb{F} \)-linear vector subspace of \( \mathbb{F}^n \). For matrices \( A_1, A_2, \ldots, A_g \in \mathbb{F}^{r \times s} \), for some positive integers \( g, r \) and \( s \), we define the block-diagonal matrix

\[
\text{diag}(A_1, A_2, \ldots, A_g) = \begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_g
\end{pmatrix} \in \mathbb{F}^{gr \times gs}.
\]

We will also denote by \( c \cdot d \in \mathbb{F} \) the conventional inner product of \( c, d \in \mathbb{F}^n \) (i.e., \( c \cdot d = cd^T \)), and we denote the dual of a linear code \( C \subseteq \mathbb{F}^n \) by \( C^\perp = \{d \in \mathbb{F}^n \mid c \cdot d = 0\} \), for all \( c \in C \) \( \subseteq \mathbb{F}^n \).

For a prime power \( q \), we denote by \( \mathbb{F}_q \) the finite field with \( q \) elements. Throughout this manuscript, we will fix a prime power \( q \) and a finite-field extension \( \mathbb{F}_q \subseteq \mathbb{F}_{q^m} \) for some positive integer \( m \). The field \( \mathbb{F}_q \) will be called the base field throughout the manuscript. Our target codes will be linear codes \( C \subseteq \mathbb{F}_{q^m}^n \), hence we will usually call \( \mathbb{F}_{q^m} \) the field of linearity of \( C \).

2.1 MDS codes

For a positive integer \( n \) and a field \( \mathbb{F} \), the Hamming weight \([21]\) of a vector \( c = (c_1, c_2, \ldots, c_n) \in \mathbb{F}^n \) is \( \text{wt}_H(c) = |\{i \in [n] \mid c_i \neq 0\}| \). We define the Hamming metric \( d_H : (\mathbb{F}^n)^2 \rightarrow \mathbb{N} \) by \( d_H(c,d) = \text{wt}_H(c-d) \), for all \( c,d \in \mathbb{F}^n \). For a (linear or non-linear) code \( C \subseteq \mathbb{F}^n \), its minimum Hamming distance is \( d_H(C) = \min \{d_H(c,d) \mid c,d \in C, c \neq d\} \).

We next revisit the Singleton bound and MDS codes \([47]\).

**Proposition 1** \([47]\). For any (linear or non-linear) code \( C \subseteq \mathbb{F}^n \), it holds that \( |C| \leq |\mathbb{F}|^{n-d_H(C)+1} \). If equality holds, then we say that \( C \) is a maximum distance separable (MDS) code.

2.2 MSRD codes

Fix positive integers \( m \) and \( r \), and an ordered basis \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{F}_{q^m}^m \) of \( \mathbb{F}_{q^m} \) over \( \mathbb{F}_q \). We define the matrix representation map \( M_\alpha : \mathbb{F}_{q^m}^m \rightarrow \mathbb{F}_{q^r}^{m \times r} \) by

\[
M_\alpha \left( \sum_{i=1}^m \alpha_i c_i \right) = \begin{pmatrix}
c_{1,1} & c_{1,2} & \cdots & c_{1,r} \\
c_{2,1} & c_{2,2} & \cdots & c_{2,r} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m,1} & c_{m,2} & \cdots & c_{m,r}
\end{pmatrix} \in \mathbb{F}_{q^r}^{m \times r}, \quad (1)
\]
where \( \mathbf{c}_i = (c_{i,1}, c_{i,2}, \ldots, c_{i,r}) \in \mathbb{F}_q^r \), for \( i = 1, 2, \ldots, m \). In order to define sum-rank weights on vectors with components in \( \mathbb{F}_{q^m} \), we will subdivide them into subvectors as \( \mathbf{c} = (\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(g)}) \in \mathbb{F}_{q^m}^{gr} \), where \( \mathbf{c}^{(i)} \in \mathbb{F}_{q^m}^r \), for \( i = 1, 2, \ldots, g \), for a positive integer \( g \). Using (11), we may consider \( \mathbf{c} \in \mathbb{F}_{q^m}^{gr} \) as a list of \( g \) matrices of size \( m \times r \) over \( \mathbb{F}_q \):

\[
\left( M_\alpha \left( c^{(1)} \right), M_\alpha \left( c^{(2)} \right), \ldots, M_\alpha \left( c^{(g)} \right) \right) \in \left( \mathbb{F}_q^{m \times r} \right)^g .
\]

(2)

We now define the sum-rank metric, which was explicitly defined in [32] Sec. III-D, but previously used implicitly in [31] Sec. III.

**Definition 2 (Sum-rank metric [31, 42]).** Let \( g \) be a positive integer, and let \( \mathbf{c} = (\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(g)}) \in \mathbb{F}_{q^m}^{gr} \), where \( \mathbf{c}^{(i)} \in \mathbb{F}_{q^m}^r \), for \( i = 1, 2, \ldots, g \). We define the sum-rank weight of \( \mathbf{c} \), for the partition \( (g, r) \) over the base field \( \mathbb{F}_q \), by

\[
\text{wt}_{SR}(\mathbf{c}) = \sum_{i=1}^{g} \text{Rk}(M_\alpha(\mathbf{c}^{(i)})) .
\]

We define the sum-rank metric \( d_{SR} : (\mathbb{F}_{q^m}^{gr})^2 \rightarrow \mathbb{N} \), for the partition \( (g, r) \) over the base field \( \mathbb{F}_q \), by \( d_{SR}(\mathbf{c}, \mathbf{d}) = \text{wt}_{SR}(\mathbf{c} - \mathbf{d}) \), for all \( \mathbf{c}, \mathbf{d} \in \mathbb{F}_{q^m}^{gr} \). For a code \( \mathcal{C} \subseteq \mathbb{F}_{q^m}^{gr} \) (linear or non-linear), we define its minimum sum-rank distance by \( d_{SR}(\mathcal{C}) = \min \{ d_{SR}(\mathbf{c}, \mathbf{d}) | \mathbf{c}, \mathbf{d} \in \mathcal{C}, \mathbf{c} \neq \mathbf{d} \} \). The number \( g \) will be called the number of matrix sets. If the context is clear, we will not specify the partition \( (g, r) \) nor the base field \( \mathbb{F}_q \).

Observe that the Hamming metric [21] and the rank metric [12, 13, 45] are recovered from the sum-rank metric by setting \( r = 1 \) and \( g = 1 \), respectively.

We have the following extension of the Singleton bound from the Hamming metric (Proposition [11]) to the sum-rank metric, given in [39] Cor. 2.

**Proposition 3 (Singleton bound [39]).** Let \( \mathcal{C} \subseteq \mathbb{F}_{q^m}^{gr} \) be a (linear or non-linear) code. For the partition \( (g, r) \) and the base field \( \mathbb{F}_q \), we have

\[
|\mathcal{C}| \leq q^{m(gr - d_{SR}(\mathcal{C}) + 1)} .
\]

(3)

Furthermore, equality holds in (3) if, and only if, \( \mathcal{C} \cdot \text{diag}(A_1, A_2, \ldots, A_g) \subseteq \mathbb{F}_{q^m}^{gr} \) is MDS, for all \( A_1, A_2, \ldots, A_g \in \text{GL}_r(\mathbb{F}_q) \).

The main objects of study in this manuscript are maximum sum-rank distance (MSRD) codes, introduced in [33] Th. 4], which extend MDS codes.

**Definition 4 (MSRD codes [33]).** For a positive integer \( g \), we say that a (linear or non-linear) code \( \mathcal{C} \subseteq \mathbb{F}_{q^m}^{gr} \) is maximum sum-rank distance (MSRD), for the partition \( (g, r) \) and the base field \( \mathbb{F}_q \), if equality holds in (3).

By [39] Cor. 3], \( m \geq r \) is required by any MSRD code of minimum sum-rank distance larger than 1. Thus we will assume \( m \geq r \) from now on.

The following result was proven in [34] Th. 5].
Lemma 5 ([35]). A linear code $C \subseteq \mathbb{F}_{q^m}^n$ is MSRD if, and only if, its dual $C^\perp \subseteq \mathbb{F}_{q^m}^{gr}$ is MSRD, in both cases for the partition $(g, r)$ and base field $\mathbb{F}_q$.

The previous lemma will be useful for our purposes, since we will construct MSRD codes by giving their parity-check matrices without worrying about computing their generator matrices, and proving that the parity-check matrices generate MSRD codes. Giving parity-check matrices will allow us to obtain higher information rates for smaller field sizes.

2.3 PMDS codes

We next revisit locally repairable codes [17, 26] and PMDS codes [3, 16].

Definition 6 (Locally repairable codes [17, 26]). Fix positive integers $g$, $r$, and $\delta$, and set $\nu = r + \delta - 1$. A code $C \subseteq \mathbb{F}_q^n$ is a locally repairable code (LRC) with $(r, \delta)$-localities if $n = g\nu$ and we may partition $[n] = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_g$ such that

$$\Gamma_i = [(i - 1)\nu + 1, iv]$$ and $d_H(C_{\Gamma_i}) \geq \delta$,

where $C_{\Gamma_i} \subseteq \mathbb{F}_q^n$ denotes the projection of $C$ onto the coordinates in $\Gamma_i$, for $i = 1, 2, \ldots, g$. The set $\Gamma_i$ is called the $i$th local set and $\nu$ is the local-set size. In many occasions, we only use the term locality for the number $r$, whereas $\delta$ is called the local distance.

LRCs as in [17, 26] do not require pair-wise disjoint local sets, but we consider only this case since it is required for PMDS codes. Moreover, local sets need not be of the same size for PMDS codes, but we consider only this case for simplicity. Partial MDS (PMDS) codes, introduced in [3, 16], are those LRCs that may correct any erasure pattern that is information-theoretically correctable given the locality constraints in Definition 6. Such patterns are exactly those with $\delta - 1$ erasures per local set and an extra $h = gr - k$ erasures anywhere else, where $k$ is the code dimension. This is equivalent to the following definition.

Definition 7 (PMDS codes [3, 16]). A linear code $C \subseteq \mathbb{F}_q^n$ is a partial MDS (PMDS) code with $(r, \delta)$-localities if it is an LRC with $(r, \delta)$-localities and, for any $\Delta_i \subseteq \Gamma_i$ with $|\Delta_i| = r$, for $i = 1, 2, \ldots, g$, the restricted code $C_\Delta \subseteq \mathbb{F}_{q^m}^{gr}$ is MDS, where $\Delta = \bigcup_{i=1}^g \Delta_i$.

The following is Construction 1 in [39].

Construction 1 ([39]). Fix positive integers $g$ and $r$, a base field size $q$ and an extension degree $m \geq r$. The field of linearity of our target codes is $\mathbb{F} = \mathbb{F}_{q^m}$. Choose:

1. Outer code: A linear MSRD code $C_{\text{out}} \subseteq \mathbb{F}_{q^m}^{gr}$ for the partition $(g, r)$ over $\mathbb{F}_q$.
2. Local code: A linear MDS code $C_{\text{loc}} \subseteq \mathbb{F}_q^\nu$ of dimension $r$ (over $\mathbb{F}_q$).
3. Global code: Let $C_{\text{glob}} \subseteq \mathbb{F}_q^n$, where $n = g\nu$, be given by

$$C_{\text{glob}} = C_{\text{out}} \cdot \text{diag}(A, A, \ldots, A),$$
g times, where $A \in \mathbb{F}_q^{r \times \nu}$ is an arbitrary generator matrix of $C_{\text{loc}}$. 

5
The following result is \cite[Th. 2]{39}.

**Proposition 8** (\cite[39]{39}). The linear code $C_{\text{glob}} \subseteq \mathbb{F}_{q^m}$ from Construction 7 has dimension $\dim(C_{\text{glob}}) = \dim(C_{\text{out}})$ and is a PMDS code with $(r, \delta)$-localities.

### 2.4 Field sizes in applications of MSRD codes

Before constructing MSRD codes, it is crucial to know what we want in an MSRD code. The parameters of the ambient space are $m$, $r$ (matrix sizes), $g$ (number of matrix sets) and $q$ (base field size). However, the computational complexity of encoding and decoding with a linear (over $\mathbb{F}_{q^m}$) code in $\mathbb{F}_{q^{gr/q^m}}$ is governed by the size of the field of linearity $q^m$. In the case of PMDS codes and multishot network coding \cite{42, 38}, the base field size $q$ is not as important. Thus, when comparing MSRD codes for such applications, $(m_1, q_1)$ is considered better than $(m_2, q_2)$ if $q_1^{m_1} < q_2^{m_2}$. However, in other applications, such as space-time coding \cite{31, 46} or crisscross error correction \cite{45, 37}, we may not have such a flexibility on the pair of parameters $(m, q)$, since $\mathbb{F}_q$ may be fixed (it corresponds to the constellation in space-time coding and the array alphabet in crisscross error correction). Thus in such applications, if we fix $q$, then it is desirable to obtain linear MSRD codes with smallest possible value of $m$. This is because of the next proposition, which implies that if we find a linear MSRD code for a pair $(q, m)$, then we may easily obtain a linear MSRD code for the pair $(q, mM)$, for any positive integer $M$. The proof is straightforward from the characterization in Proposition 8.

**Proposition 9.** For a linear code $C \subseteq \mathbb{F}_{q^m}$, define $C \otimes \mathbb{F}_{q^{mM}} \subseteq \mathbb{F}_{q^{gr/q^mM}}$ as the $\mathbb{F}_{q^{mM}}$-linear code with the same generator matrix as $C$ (which has entries in $\mathbb{F}_{q^m}$). Then $\dim_{\mathbb{F}_{q^m}}(C) = \dim_{\mathbb{F}_{q^{mM}}}(C \otimes \mathbb{F}_{q^{mM}})$ and $C$ is MSRD if, and only if, so is $C \otimes \mathbb{F}_{q^{mM}}$, in both cases for the length partition $(g, r)$ over the field $\mathbb{F}_q$.

A difficult research problem, open in most cases, is to determine constraints in $m$, $q$ and $q^m$ for the existence of MSRD codes and PMDS codes. This problem is a highly non-trivial extension of the MDS conjecture (not even the asymptotic order of the size $q^m$ of possible MSRD or PMDS codes is known in general, whereas we know that MDS codes exist if, and only if, the code length is at most linear in the field size \cite{14, 47}).

For PMDS codes, bounds on field sizes were given in \cite[Th. 3.5 and 3.8]{19}, and the case of one global parity is completely solved in \cite{23}. For MSRD codes, bounds on the parameters were given in \cite[Th. 6.12]{15}, which may be turned into bounds on field sizes. One of our MSRD codes meets the latter bounds, see Subsection 4.4.

### 3 Extended Moore matrices

This section contains the main method for constructing parity-check matrices of MSRD codes. The section concludes with a definition of a general family of MSRD codes (Definition 20). Such codes exist and are explicit as long as a certain sequence $(\beta_1, \beta_2, \ldots, \beta_{\mu r}) \in \mathbb{F}_{q^m}^{\mu r}$ is known. Explicit constructions of such sequences will be deferred to Section 4.
3.1 The definitions

We fix the field automorphism \( \sigma : \mathbb{F}_{q^m} \rightarrow \mathbb{F}_{q^m} \) given by \( \sigma(a) = a^q \), for \( a \in \mathbb{F}_{q^m} \). The following definition is a particular case of \([28]\) Eq. (2.5), but already appeared in \([24]\).

**Definition 10** ([27], [28]). We define the equivalence relation \( \sim_\sigma \) in \( \mathbb{F}_{q^m} \) as \( a \sim_\sigma b \) if there exists \( c \in \mathbb{F}_{q^m}^* \) such that \( b = \sigma(c) c^{-1} a = c^{q-1} a \), for \( a, b \in \mathbb{F}_{q^m} \).

It was shown in \([30]\) Cor. 1 that there are exactly \( q-1 \) non-zero equivalence classes in \( \mathbb{F}_{q^m} \) with respect to \( \sim_\sigma \), each of size \( (q^m - 1)/(q - 1) \). Furthermore, they are represented by powers of a primitive element, as observed in the paragraph after \([39\), Def. 2].

**Lemma 11** ([30], [39]). Let \( \gamma \in \mathbb{F}_{q^m}^* \) be a primitive element of \( \mathbb{F}_{q^m} \). Then \( \gamma^0, \gamma^1, \ldots, \gamma^{q-2} \) are pair-wise non-equivalent with respect to \( \sim_\sigma \).

Moreover, the elements in \( \mathbb{F}_q \) represent the \( q - 1 \) equivalence classes in \( \mathbb{F}_{q^m} \) with respect to \( \sim_\sigma \) if, and only if, \( q - 1 \) and \( m \) are coprime \([36]\) Remark 27.

We next define truncated norms. Again, the following definition is a particular case of \([28]\) Eq. (2.3), but already appeared in \([27]\).

**Definition 12** ([27], [28]). For \( a \in \mathbb{F}_{q^m} \) and \( i \in \mathbb{N} \), we define \( N_i(a) = \sigma^{-1}(a) \cdots \sigma(a) a \).

We may now define extended Moore matrices.

**Definition 13** (Extended Moore matrices). Let \( a = (a_1, a_2, \ldots, a_\ell) \in (\mathbb{F}_{q^m}^*)^\ell \) be a vector of \( \ell \) pair-wise non-equivalent elements in \( \mathbb{F}_{q^m} \) with respect to \( \sim_\sigma \). Let \( \beta = (\beta_1, \beta_2, \ldots, \beta_\eta) \in \mathbb{F}_{q^m}^\eta \) be an arbitrary vector, for some positive integer \( \eta \). For \( h = 1, 2, \ldots, \ell \eta \), we define the extended Moore matrix \( M_h(a, \beta) \in \mathbb{F}_{q^m}^{h \times (\ell \eta)} \) by

\[
M_h(a, \beta) = \begin{pmatrix}
\beta_1 & \ldots & \beta_\eta & \ldots & \beta_1 & \ldots & \beta_\eta \\
\beta_1 a_1 & \ldots & \beta_\eta a_1 & \ldots & \beta_1 a_\ell & \ldots & \beta_\eta a_\ell \\
\beta_1^2 N_2(a_1) & \ldots & \beta_\eta^2 N_2(a_1) & \ldots & \beta_1^2 N_2(a_\ell) & \ldots & \beta_\eta^2 N_2(a_\ell) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\beta_1^{h-1} N_{h-1}(a_1) & \ldots & \beta_\eta^{h-1} N_{h-1}(a_1) & \ldots & \beta_1^{h-1} N_{h-1}(a_\ell) & \ldots & \beta_\eta^{h-1} N_{h-1}(a_\ell)
\end{pmatrix}
\]

Such matrices extend the well known Moore matrices \([28]\) Lemma 3.51] from one to several equivalence classes of \( \sim_\sigma \). They extend the matrices in \([33]\), p. 604] in the sense that the \( \eta \) components of \( \beta \in \mathbb{F}_{q^m}^\eta \), over \( \mathbb{F}_{q^m} \), need not be linearly independent over \( \mathbb{F}_q \). The vector \( \beta \in \mathbb{F}_{q^m}^\eta \) may be different (and possibly of different lengths \( \eta \) at each of the \( \ell \) blocks. All the results in this manuscript also hold in such a generality. However, we assume equal vectors \( \beta \in \mathbb{F}_{q^m}^\eta \) at different blocks for simplicity.

We would like to turn the matrix \( M_h(a, \beta) \) into the parity-check matrix of an MSRD code. To that end, we will use Proposition \([3]\) and a characterization of when an extended Moore matrix is the parity-check matrix of an MDS code.
3.2 MDS extended Moore matrices

In this subsection, we characterize when an extended Moore matrix is the parity-check matrix of an MDS code. We need the concept of \( h \)-wise independence from [16] Def. 9.

**Definition 14** ([16]). We say that a subset \( T \subseteq \mathbb{F}_q^m \) is \( h \)-wise independent over \( \mathbb{F}_q \) if any subset of at most \( h \) elements of \( T \) is linearly independent over \( \mathbb{F}_q \). Analogously, for a positive integer \( \eta \), we say that a vector \( \beta = (\beta_1, \beta_2, \ldots, \beta_\eta) \in \mathbb{F}_q^\eta \) is \( h \)-wise independent if \( T = \{\beta_1, \beta_2, \ldots, \beta_\eta\} \) has size \( \eta \) and is \( h \)-wise independent.

We will also need the following four auxiliary lemmas. The first one is trivial.

**Lemma 15.** Fix integers \( 1 \leq \eta \leq h \) and \( a \in \mathbb{F}_q^\eta \). Assume that there exist \( \lambda_1, \lambda_2, \ldots, \lambda_\eta \in \mathbb{F}_q \) such that \( \lambda_1 \beta_1 + \lambda_2 \beta_2 + \cdots + \lambda_\eta \beta_\eta = 0 \), for \( \beta_1, \beta_2, \ldots, \beta_\eta \in \mathbb{F}_q^\eta \). Then

\[
\begin{pmatrix}
\beta_1 & \beta_2 & \cdots & \beta_\eta \\
\beta_1^q a & \beta_2^q a & \cdots & \beta_\eta^q a \\
\beta_1^{q^2} N_2(a) & \beta_2^{q^2} N_2(a) & \cdots & \beta_\eta^{q^2} N_2(a) \\
\vdots & \vdots & \ddots & \vdots \\
\beta_1^{q^{h-1}} N_{h-1}(a) & \beta_2^{q^{h-1}} N_{h-1}(a) & \cdots & \beta_\eta^{q^{h-1}} N_{h-1}(a)
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_\eta
\end{pmatrix} = 0.
\]

The next lemma follows immediately from the invertibility of Moore matrices [29] Lemma 3.51 and Lemma [15]

**Lemma 16.** Fix integers \( 1 \leq \eta \leq h \) and \( a \in \mathbb{F}_q^\eta \). The dimension of the \( \mathbb{F}_q \)-linear subspace generated by \( \beta_1, \beta_2, \ldots, \beta_\eta \in \mathbb{F}_q^\eta \) equals the rank of

\[
\begin{pmatrix}
\beta_1 & \beta_2 & \cdots & \beta_\eta \\
\beta_1^q a & \beta_2^q a & \cdots & \beta_\eta^q a \\
\beta_1^{q^2} N_2(a) & \beta_2^{q^2} N_2(a) & \cdots & \beta_\eta^{q^2} N_2(a) \\
\vdots & \vdots & \ddots & \vdots \\
\beta_1^{q^{h-1}} N_{h-1}(a) & \beta_2^{q^{h-1}} N_{h-1}(a) & \cdots & \beta_\eta^{q^{h-1}} N_{h-1}(a)
\end{pmatrix} \in \mathbb{F}_q^{h \times \eta}.
\]

The next lemma may be easily derived by simplifying telescopic products.

**Lemma 17.** With notation as in Definition [13] it holds that

\[
M_h(a, \beta) \cdot \text{diag} \left( \beta_1^{-1}, \ldots, \beta_\eta^{-1} \right) =
\begin{pmatrix}
1 & \cdots & 1 \\
\beta_1^{q-1} a_1 & \cdots & \beta_\eta^{q-1} a_1 \\
N_2(\beta_1^{q-1} a_1) & \cdots & N_2(\beta_\eta^{q-1} a_1) \\
\vdots & \ddots & \vdots \\
N_{h-1}(\beta_1^{q-1} a_1) & \cdots & N_{h-1}(\beta_\eta^{q-1} a_1)
\end{pmatrix}
\begin{pmatrix}
1 & \cdots & 1 \\
\beta_1^{q-1} a_\ell & \cdots & \beta_\eta^{q-1} a_\ell \\
N_2(\beta_1^{q-1} a_\ell) & \cdots & N_2(\beta_\eta^{q-1} a_\ell) \\
\vdots & \ddots & \vdots \\
N_{h-1}(\beta_1^{q-1} a_\ell) & \cdots & N_{h-1}(\beta_\eta^{q-1} a_\ell)
\end{pmatrix}
\]

The next lemma is a particular case of [27] Th. 23 (1)].
Theorem 1. \( \beta \) \( \eta \) independent over \( F \) positive integers \( \mu \) and set \( g = \eta_1 + \eta_2 + \cdots + \eta_\ell \). For \( h = 1, 2, \ldots, g \), the extended Moore matrix \( M_h(a, \beta) \in \mathbb{F}_q^{h \times (\ell \eta)} \) from Definition \( \mathbb{F}_q^{h \times \eta} \) is a full-rank parity-check matrix of an MDS code if, and only if, \( \beta = (\beta_1, \beta_2, \ldots, \beta_\ell) \in \mathbb{F}_q^{\ell \times \eta} \) is \( h \)-wise independent over \( \mathbb{F}_q \).

Proof. First, assume that \( (\beta_1, \beta_2, \ldots, \beta_\eta) \) is not \( h \)-wise independent over \( \mathbb{F}_q \). Then \( M_h(a, \beta) \) contains an \( h \times h \) submatrix that is not invertible by Lemma \( \mathbb{F}_q^{h \times \eta} \).

Conversely, assume that \( (\beta_1, \beta_2, \ldots, \beta_\eta) \) is \( h \)-wise independent over \( \mathbb{F}_q \). Take an arbitrary \( h \times h \) submatrix \( M' \in \mathbb{F}_q^{h \times h} \) of \( M_h(a, \beta) \), and let \( 0 \leq \eta_i \leq \min\{h, \eta \} \) be the number of columns from the \( i \)th block of \( \eta_i \) columns in \( M_h(a, \beta) \) appearing in \( M' \), for \( i = 1, 2, \ldots, \ell \). Note that \( h = \eta_1 + \eta_2 + \cdots + \eta_\ell \). Since \( (\beta_1, \beta_2, \ldots, \beta_\eta) \) is \( h \)-wise independent over \( \mathbb{F}_q \) and \( \eta_i \leq h \), then the \( i \)th block of \( \eta_i \) columns in \( M' \) forms an \( \eta_i \times h \) matrix of full rank \( \eta_i \) by Lemma \( \mathbb{F}_q^{h \times \eta} \) for \( i = 1, 2, \ldots, \ell \). Finally, by combining Lemmas \( \mathbb{F}_q^{h \times \eta} \) and \( \mathbb{F}_q^{h \times \eta} \), we conclude that \( \text{Rk}(M') = \eta_1 + \eta_2 + \cdots + \eta_\ell = h \), and therefore \( M' \in \mathbb{F}_q^{h \times h} \) is invertible. Hence \( M_h(a, \beta) \) is MDS and we are done. \( \square \)

3.3 MSRD extended Moore matrices

In this subsection, we characterize when an extended Moore matrix is the parity-check matrix of an MSRD code. We start by combining Proposition \( \mathbb{F}_q^{h \times \eta} \) Lemma \( \mathbb{F}_q^{h \times \eta} \) and the \( \mathbb{F}_q^{h \times \eta} \) -linearity of the map \( \sigma \).

Proposition 19. \( \eta \) \( (a_1, a_2, \ldots, a_\ell) \in \mathbb{F}_q^{\ell \times \eta} \) be a vector of \( \ell \) pair-wise non-equivalent elements in \( \mathbb{F}_q^{\ell \times \eta} \) with respect to \( \sim \). Let \( \beta = (\beta_1, \beta_2, \ldots, \beta_\mu) \in \mathbb{F}_q^{\mu \times r} \), for positive integers \( \mu \) and \( r \), and set \( g = \ell \mu \). For \( h = 1, 2, \ldots, g \), the extended Moore matrix \( M_h(a, \beta) \in \mathbb{F}_q^{\ell \times (\ell \mu \times r)} \) from Definition \( \mathbb{F}_q^{h \times (\ell \mu \times r)} \) is a full-rank parity-check matrix of an MSRD
code for the partition \((g,r)\) over \(\mathbb{F}_q\) if, and only if, for all \(A_1, A_2, \ldots, A_\mu \in \text{GL}_r(\mathbb{F}_q)\), the vector

\[
(\beta_1, \beta_2, \ldots, \beta_\mu) \cdot \text{diag}(A_1, A_2, \ldots, A_\mu) \in \mathbb{F}_q^{m_r}
\]

is \(h\)-wise independent over \(\mathbb{F}_q\).

Our main characterization is the following theorem.

**Theorem 2.** Let \(\mathbf{a} = (a_1, a_2, \ldots, a_\ell) \in (\mathbb{F}_q^*)^\ell\) be a vector of \(\ell\) pair-wise non-equivalent elements in \(\mathbb{F}_q^m\) with respect to \(\sim_\sigma\). Let \(\mathbf{\beta} = (\beta_1, \beta_2, \ldots, \beta_\mu) \in \mathbb{F}_q^{m_r}\), for positive integers \(\mu\) and \(r\), and set \(g = \ell \mu\). Define the \(\mathbb{F}_q\)-linear subspace

\[
\mathcal{H}_i = \langle \beta_{(i-1)r+1}, \beta_{(i-1)r+2}, \ldots, \beta_{ir} \rangle_{\mathbb{F}_q} \subseteq \mathbb{F}_q^m,
\]

for \(i = 1, 2, \ldots, \mu\). Given \(1 \leq h \leq gr\), the extended Moore matrix \(M_h(\mathbf{a}, \mathbf{\beta}) \in \mathbb{F}_q^{h \times (gr)}\) from Definition 12 is a full-rank parity-check matrix of an MSRD code for the partition \((g,r)\) over \(\mathbb{F}_q\) if, and only if, the following two conditions hold for all \(i = 1, 2, \ldots, \mu\):

1. \(\dim_{\mathbb{F}_q}(\mathcal{H}_i) = r\), i.e., \(\beta_{(i-1)r+1}, \beta_{(i-1)r+2}, \ldots, \beta_{ir}\) are \(\mathbb{F}_q\)-linearly independent, and
2. \(\mathcal{H}_i \cap \left( \sum_{j \in \Gamma} \mathcal{H}_j \right) = \{0\}\), for any set \(\Gamma \subseteq [\mu]\), such that \(i \notin \Gamma\) and \(|\Gamma| \leq \min\{h, \mu\} - 1\).

**Proof.** We prove both implications separately.

\(\iff\): Take matrices \(A_1, A_2, \ldots, A_\mu \in \text{GL}_r(\mathbb{F}_q)\). Condition 1 implies that \(\beta'_{(i-1)r+1}, \beta'_{(i-1)r+2}, \ldots, \beta'_{ir} \in \mathbb{F}_q^m\) are linearly independent over \(\mathbb{F}_q\), where

\[
(\beta'_{(i-1)r+1}, \beta'_{(i-1)r+2}, \ldots, \beta'_{ir}) = (\beta_{(i-1)r+1}, \beta_{(i-1)r+2}, \ldots, \beta_{ir}) \cdot A_i \in \mathbb{F}_q^m,
\]

for all \(i = 1, 2, \ldots, \mu\). Next, fix an index \(i = 1, 2, \ldots, \mu\), and take a subset \(\Gamma \subseteq [\mu]\), such that \(i \notin \Gamma\) and \(|\Gamma| \leq \min\{h, \mu\} - 1\). Condition 2 and the \(\mathbb{F}_q\)-linear independence of each set \(\{\beta'_{(j-1)r+1}, \beta'_{(j-1)r+2}, \ldots, \beta'_{jr}\}\) imply that the set

\[
\bigcup_{j \in \Gamma \setminus \{i\}} \left\{ \beta'_{(j-1)r+1}, \beta'_{(j-1)r+2}, \ldots, \beta'_{jr} \right\} \subseteq \mathbb{F}_q^m
\]

is linearly independent over \(\mathbb{F}_q\). Since every subset of size at most \(h\) of \(\{\beta'_1, \beta'_2, \ldots, \beta'_{m_r}\}\) is contained in a set of the form \(5\), we deduce that the vector \((\beta'_1, \beta'_2, \ldots, \beta'_{m_r})\) is \(h\)-wise linearly independent over \(\mathbb{F}_q\). Hence the extended Moore matrix \(M_h(\mathbf{a}, \mathbf{\beta}) \in \mathbb{F}_q^{h \times N}\) is MSRD by Proposition 12.

\(\implies\): Assume first that Condition 1 does not hold for some \(i = 1, 2, \ldots, \mu\). Without loss of generality, we may assume that there exist \(\lambda_1, \lambda_2, \ldots, \lambda_{r-1} \in \mathbb{F}_q\) such that

\[
\sum_{j=1}^{r-1} \lambda_j \beta_{(i-1)r+j} + \beta_{ir} = 0.
\]
Thus if we define the invertible matrix
\[ A_i = \begin{pmatrix} I_{r-1} & \lambda_1 \\ 0 \ldots 0 & \lambda_{r-1} \end{pmatrix} \in \text{GL}_r(\mathbb{F}_q), \]
where \( I_{r-1} \in \text{GL}_{r-1}(\mathbb{F}_q) \) denotes the \((r-1) \times (r-1)\) identity matrix, then it holds that
\[(\beta_{i-1}r+1, \ldots, \beta_{ir-1}, \beta_{ir}) \cdot A_i = (\beta_{i-1}r+1, \ldots, \beta_{ir-1}, 0).\]
Clearly, \((\beta_{i-1}r+1, \ldots, \beta_{ir-1}, 0) \in \mathbb{F}_q^{m}\) is not \(h\)-wise independent, thus \(M_h(a, \beta)\) is not MSRD by Proposition 19.

Next, assume that Condition 2 does not hold for some \(i = 1, 2, \ldots, \mu\). Then we may assume, without loss of generality, that there exists a subset \(\Gamma \subseteq [\mu]\) such that \(i \in \Gamma\), \(|\Gamma| \leq h\), and there exist \(\lambda_{j,u} \in \mathbb{F}_q\), for \(u = 1, 2, \ldots, r\), for \(j \in \Gamma\), such that \(\lambda_{j,r} = 1\), for \(j \in \Gamma\), and
\[ \sum_{j \in \Gamma} \sum_{u=1}^r \lambda_{j,u} \beta_{(j-1)r+u} = 0. \]
Define, for each \(j \in \Gamma\), the invertible matrix
\[ A_j = \begin{pmatrix} I_{r-1} & \lambda_{j,1} \\ 0 \ldots 0 & \lambda_{j,r-1} \end{pmatrix} \in \text{GL}_r(\mathbb{F}_q), \]
and define, for convenience, \(A_j = I_r \in \text{GL}_r(\mathbb{F}_q)\) if \(j \notin \Gamma\). If we set
\[(\beta_{1}', \beta_{2}', \ldots, \beta_{\mu}') = (\beta_1, \beta_2, \ldots, \beta_{\mu}) \cdot \text{diag}(A_1, A_2, \ldots, A_\mu),\]
then it holds that
\[ \sum_{j \in \Gamma} \beta_{jr}' = \sum_{j \in \Gamma} \sum_{u=1}^r \lambda_{j,u} \beta_{(j-1)r+u} = 0. \]
Since \(|\Gamma| \leq h\), then the vector \((\beta_{1}', \beta_{2}', \ldots, \beta_{\mu}')\) is not \(h\)-wise independent over \(\mathbb{F}_q\), hence \(M_h(a, \beta)\) is not MSRD by Proposition 19.

Therefore, we have the following general family of MSRD codes. Since we may puncture a linear MSRD code to obtain a shorter linear MSRD code [35, Cor. 7], we will assume from now on that \(\ell = q - 1\), which is the maximum value of \(\ell\) that we may choose as explained in Subsection 3.1.

**Definition 20.** Let \(a = (a_1, a_2, \ldots, a_{q-1}) \in (\mathbb{F}_q^*)^{q-1}\) be a vector of \(q - 1\) pair-wise non-equivalent elements in \(\mathbb{F}_q^*\) with respect to \(\sim_\sigma\). Let \(\beta = (\beta_1, \beta_2, \ldots, \beta_{\mu r}) \in \mathbb{F}_{qr}\) satisfy Conditions 1 and 2 in Theorem 7 and set \(g = (q - 1)\mu\). For \(h = 1, 2, \ldots, gr\), we define the following \((gr - h)\)-dimensional linear MSRD code for the partition \((g, r)\) over the base field \(\mathbb{F}_q\):
\[ C_k(a, \beta) = \{ y \in \mathbb{F}_q^r \mid M_h(a, \beta)y^T = 0 \}. \]
4 Explicit constructions of MSRD codes

What is missing in Definition 20 is finding the sequence \((\beta_1, \beta_2, \ldots, \beta_{\mu r}) \in \mathbb{F}_{q^m}^{\mu r}\) satisfying Conditions 1 and 2 in Theorem 2. In this section, we provide a technique for explicitly constructing such sequences. This method provides several explicit subfamilies of the codes in Definition 20 where the vector \(a \in (\mathbb{F}_{q^m})^{r-1}\) can be explicitly chosen as in Lemma 11 or using all of the elements of \(\mathbb{F}_q^t\) when \(q - 1\) and \(m\) are coprime [36, Remark 27].

4.1 Tensor products or field reduction

In this subsection, we explore tensor products of sequences over \(\mathbb{F}_{q^r}\) and \(\mathbb{F}_{q^m}\). This technique is inspired by [14, Sec. IV-B]. However, the codes obtained in [14, Sec. IV-B] and in this work are not equivalent (by inspecting their parameters). Our technique is also connected to finite geometry. Families of subspaces satisfying Conditions 1 and 2 as in Theorem 2 are called arcs or pseudo-arcs in finite geometry [11, Sec. 4]. The technique that we use here to construct them is equivalent to field reduction [11, Sec. 5].

For the remainder of this section, we will fix an ordered basis \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{F}_{q^r}\) of \(\mathbb{F}_{q^r}\) over \(\mathbb{F}_q\). We will also assume from now on that \(m = r\rho\) (hence \(\mathbb{F}_{q^r} \subseteq \mathbb{F}_{q^m}\)), for some positive integer \(\rho\). Choose a vector

\[
\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\mu) \in \mathbb{F}_{q^m}^\mu.
\]

Define the tensor product of \(\alpha\) with \(\gamma\)

\[
(\beta_1, \beta_2, \ldots, \beta_{\mu r}) = \alpha \otimes \gamma = (\alpha_1 \gamma_1, \alpha_2 \gamma_1, \ldots, \alpha_r \gamma_1) \cdots (\alpha_1 \gamma_\mu, \ldots, \alpha_r \gamma_\mu) \in \mathbb{F}_{q^m}^{\mu r}.
\]

In other words, \((\beta_{(i-1)r+1}, \beta_{(i-1)r+2}, \ldots, \beta_{ir}) = \gamma_i \alpha \in \mathbb{F}_{q^m}^\mu\), for \(i = 1, 2, \ldots, \mu\).

The proof of the following theorem is straightforward and is left to the reader.

**Theorem 3.** The vector \((\beta_1, \beta_2, \ldots, \beta_{\mu r}) \in \mathbb{F}_{q^m}^{\mu r}\) in (4) satisfies Conditions 1 and 2 in Theorem 2, for \(i = 1, 2, \ldots, \mu\), if and only if, the vector \(\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\mu) \in \mathbb{F}_{q^m}^\mu\) is t-wise independent over \(\mathbb{F}_{q^r}\), for \(t = \min\{h, \mu\}\).

We may construct t-wise independent vectors by using the following equivalence, which has been used previously in the PMDS literature in [16, Th. 17], [14, Lemma 7] and [20]. We will assume from now on that \(\rho \leq \mu\). The following result follows by combining Definition 14, the \(\mathbb{F}_{q^r}\)-linearity of the map \(M_\delta\) and [32, Th. 10, p. 33].

**Lemma 21.** Let \(\delta \in \mathbb{F}_{q^m}^\mu\) be an ordered basis of \(\mathbb{F}_{q^m}\) over \(\mathbb{F}_{q^r}\). Consider the matrix representation map \(M_\delta : \mathbb{F}_{q^m}^\mu \rightarrow \mathbb{F}_{q^r}^{\rho \times \mu}\), as in (1), and define \(H_\gamma = M_\delta(\gamma) \in \mathbb{F}_{q^r}^{\rho \times \mu}\). The vector \(\gamma \in \mathbb{F}_{q^m}^\mu\) is t-wise independent over \(\mathbb{F}_{q^r}\) if, and only if, \(d_H(C_\gamma) \geq t + 1\), for

\[
C_\gamma = \{y \in \mathbb{F}_{q^r}^\mu \mid H_\gamma y^T = 0\} \subseteq \mathbb{F}_{q^r}^\mu.
\]

In conclusion, to construct \(\gamma \in \mathbb{F}_{q^m}^\mu\), we may choose any \(\mathbb{F}_{q^r}\)-linear code \(C_\gamma \subseteq \mathbb{F}_{q^r}^\mu\) of dimension \(\mu - \rho\) and minimum Hamming distance at least \(t + 1\). As in related works [14, 16, 20], the field size \(q^m = (q^r)^\rho\) has as exponent \(\rho\) the co-dimension of the code \(C_\gamma\). Therefore such a co-dimension needs to be as small as possible.
4.2 Using trivial codes: Recovering linearized RS codes

As a first choice of \( C_\gamma \), we choose a trivial code \( C_\gamma = \{0\} \) and recover duals of linearized Reed-Solomon codes \([33]\). This is the only case where the MSRD codes that we obtain are not new. This section is included as a remark.

**Theorem 4.** Choose \( \mu = \rho = 1 \). Thus \( m = r \), \( C_\gamma = \{0\} \subseteq \mathbb{F}_q^1 \) and

\[
(\beta_1, \beta_2, \ldots, \beta_r) = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{F}_q^r.
\]

In particular, \( g = q - 1 \). Then the MSRD code \( C_k(a, \beta) \subseteq \mathbb{F}_q^r \) is the dual of a linearized Reed-Solomon code \([33, \text{Def. 31}]\). The co-dimension \( h \) is arbitrary with \( 1 \leq h \leq qr - 1 \), and the field of linearity has size

\[
|\mathbb{F}_{q^m}| = q^r.
\]  

By \([33, \text{Th. 4}]\), such duals are precisely linearized Reed-Solomon codes for \( a \in (\mathbb{F}_q^*)^{g-1} \) as in Lemma \([11]\) (see also \([36, \text{Prop. 38}]\) for other cases and \([11]\) in general).

Notice that the PMDS codes from \([39]\) are precisely those obtained from Construction \([1]\) when taking the MSRD codes from Theorem \([4]\) as outer codes.

4.3 Using MDS codes

In this subsection, we explore the case where \( C_\gamma \) is an MDS code.

**Theorem 5.** Choose \( \mu = q^r + 1 \), \( g = (q - 1)(q^r + 1) \) and \( \rho = t = \min\{h, \mu\} \), being \( h \) arbitrary with \( 1 \leq h \leq qr - 1 \). Choose \( C_\gamma \subseteq \mathbb{F}_q^r \) in \([33]\) as an MDS code of dimension \( \mu - t \), thus \( d_H(C_\gamma) = t + 1 \). For instance, \( C_\gamma \) can be chosen as the projective extension \([25, \text{Th. 5.3.4}]\) of a classical Reed-Solomon code \([44]\). Then the field of linearity of the MSRD code \( C_k(a, \beta) \subseteq \mathbb{F}_{q^m}^g \) has size

\[
|\mathbb{F}_{q^m}| = \left(\frac{g}{q - 1} - 1\right)^{\min\{h, \frac{\mu}{q - 1}\}}.
\]  

We now plug the MSRD codes from Theorem \([5]\) into Construction \([1]\). The following corollary holds by Proposition \([8]\).

**Corollary 22.** In Construction \([1]\) choose \( C_{out} = C_k(a, \beta) \subseteq \mathbb{F}_{q^m}^g \) to be the MSRD code in Theorem \([5]\) and let \( q \) be a power of 2 satisfying \( q > \nu \). Then \( C_{glob} \subseteq \mathbb{F}_{q^m}^{gr} \) in Construction \([1]\) is a PMDS code with \((r, \delta)\)-localities, and its field of linearity has size

\[
|\mathbb{F}_{q^m}| = q^r^{\min\{h, \mu\}} \leq \max\left\{(2\nu)^r, \left\lceil \frac{g}{\nu} \right\rceil - 1\right\}^{\min\{h, \frac{\mu}{\nu}\}}.
\]  

4.4 Using Hamming codes

We now investigate the case \( h = 2 \). As we show next, we obtain MSRD codes with arbitrary parameters except for \( h = 2 \) and with field sizes \( q^m \) that are linear in \( g \). In addition, such MSRD codes meet the bounds \([6, \text{Th. 6.12}]\).
Theorem 6. Consider \( h = 2 \) and \( 1 \leq \rho < \mu \), and choose \( C_\gamma \subseteq \mathbb{F}_{q^r}^\mu \) in (8) as a \((\mu - \rho)\)-dimensional Hamming code. In other words, choose the vector \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\mu) \in (\mathbb{F}_{q^r})^\mu \) in (4) such that its components form the projective space \( \mathbb{P}_{\mathbb{F}_{q^r}}(\mathbb{F}_{q^r}) = \{[\gamma_1], [\gamma_2], \ldots, [\gamma_\mu]\} \), where \( [\gamma] = \{\lambda \gamma \in \mathbb{F}_{q^r}^\mu \mid \lambda \in \mathbb{F}_{q^r}^*\} \), for \( \gamma \in \mathbb{F}_{q^r}^\mu \). The MSR code \( C_k(a, \beta) \subseteq \mathbb{F}_{q^m}^{\rho \eta} \) satisfies that \( d_{SR}(C_k(a, \beta)) = 3 \) and its field of linearity has size
\[
|\mathbb{F}_{q^m}| = \frac{q^\gamma - 1}{q - 1} \cdot g + 1. \tag{12}
\]

For \( r \geq 2 \), the linear MSR codes from Theorem 6 meet the following bound on field sizes, which follows from [32, Th. 6.12].

Proposition 23 ([6]). For positive integers \( m, r \) and \( g \), let \( C \subseteq \mathbb{F}_{q^r}^\eta \) be a (linear or non-linear) MSR code. If \( d_{SR}(C) = 3 \), \( r \) divides \( m \) and \( r \geq 2 \), then
\[
g \leq (q - 1) \cdot \frac{q^m - 1}{q^r - 1}, \quad \text{or} \quad q^m \geq \frac{q^r - 1}{q - 1} \cdot g + 1. \tag{13}
\]

We will not provide the corresponding construction of PMDS codes via Construction 1 as there exist linear PMDS codes for \( h = 2 \) with smaller field sizes [4, 5].

4.5 Using BCH codes

In this subsection, we explore the case where \( C_\gamma \subseteq \mathbb{F}_{q^r}^\mu \) is a BCH code. We will set in this subsection \( \mu = q^s - 1 \), for an arbitrary positive integer \( s \). Consider the code \( C_\gamma \subseteq \mathbb{F}_{q^r}^\mu \) in (8) to be a BCH code, see [32, Sec. 7.6] [24, Sec. 4.5 & Ch. 5]. By the BCH bound [32, Sec. 7.6, Th. 8] [24, Th. 4.5.3], we have that \( d_H(C_\gamma) \geq \partial \) if the minimal generator polynomial of \( C_\gamma \) vanishes in \( 1, a, a^2, \ldots, a^{\partial - 2} \in \mathbb{F}_{q^r}^\mu \), for an integer \( 2 \leq \partial \leq \mu + 1 \), where \( a \in \mathbb{F}_{q^r}^\mu \) is a primitive element. If we choose \( C_\gamma \) to be the largest BCH code whose minimal generator polynomial has such roots, then by [24, Th. 4.2.1], we have that
\[
\rho = \mu - \dim(C_\gamma) = |C_0 \cup C_1 \cup C_2 \cup \ldots \cup C_{\partial - 2}|,
\]
where \( C_i = \{i, iq^r, iq^{2r}, iq^{3r}, \ldots\} \) modulo \( \mu \), see [32, Sec. 7.5, p. 197] [24, Sec. 4.1], for \( i = 0, 1, \ldots, \mu - 1 \). The integer \( \partial \) is called the prescribed distance of the BCH code \( C_\gamma \), and the set \( C_0 \cup C_1 \cup C_2 \cup \ldots \cup C_{\partial - 2} \) is called the defining set of \( C_\gamma \).

Set \( \partial = t + 1 = \min\{h, \mu\} + 1 \). To upper bound \( \rho \), we only need to upper bound the size of the defining set \( C_0 \cup C_1 \cup C_2 \cup \ldots \cup C_{\partial - 1} \). It is trivial and well known that \( C_0 = \{0\}, \ |C_i| \leq s \) and \( C_{iq^r} = C_i \), for \( i = 0, 1, 2, \ldots, \mu - 1 \). Therefore, \( |C_0| = 1 \) and we may remove from \( C_0 \cup C_1 \cup C_2 \cup \ldots \cup C_{\partial - 1} \) each cyclotomic coset \( C_i \) where \( i \) is a multiple of \( q^r \). Hence
\[
|C_0 \cup C_1 \cup C_2 \cup \ldots \cup C_{\partial - 1}| \leq 1 + s \cdot \left[ \frac{q^r - 1}{q^r} \cdot (t - 1) \right].
\]

Therefore, we have proven the following theorem.
Theorem 7. Let $s \in \mathbb{N}_+$, $\mu = q^{rs} - 1$, $g = (q - 1)(q^{rs} - 1)$ and let $h$ be arbitrary with $1 \leq h \leq gr - 1$. Choose $C_\gamma \subseteq \mathbb{F}_{q^\mu}$ in (8) to be a BCH code, as above, with defining set $C_0 \cup C_1 \cup C_2 \cup \ldots \cup C_{t-1}$. Then the field of linearity of the MSRD code $C_k(a, \beta)$ has size
\[
|\mathbb{F}_{q^m}| = q^{r\rho} \leq q^r \cdot \left(\frac{g}{q-1} + 1\right)^{\left\lceil \frac{g-1}{\sqrt{q-1}}(h-1)\right\rceil}.
\]

(14)

We now plug the MSRD codes from Theorem 7 into Construction 1. The following corollary holds by Proposition 8.

Corollary 24. In Construction 1, choose $C_{\text{out}} = C_k(a, \beta) \subseteq \mathbb{F}_{gr}$ to be the MSRD code in Theorem 7. We further assume that $q$ is the smallest power of 2 such that $q > \nu = r + \delta - 1$. Then $C_{\text{glob}} \subseteq \mathbb{F}_{\nu}^{gr}$ in Construction 1 is a PMDS code with $(r, \delta)$-localities, and its field of linearity has size
\[
|\mathbb{F}_{\nu}| \leq (2\nu)^r \cdot \left(\frac{\nu}{2^r} + 1\right)^{\left\lceil \frac{\nu-1}{\sqrt{\nu}}(h-1)\right\rceil}.
\]

(15)

4.6 Using Algebraic-Geometry (AG) codes

In this subsection, we explore the case where $C_\gamma \subseteq \mathbb{F}_{q^\mu}$ is an Algebraic-Geometry code, or AG code for short. AG codes have been proposed to construct PMDS codes in [5, 20], but they have not yet been used to construct MSRD codes.

We will only need to describe the parameters of the considered AG codes. For further details, the reader is referred to [48]. Consider an irreducible projective curve $X$ over $\mathbb{F}_{q^r}$ with algebraic function field $\mathbb{F}$, and let $g = g(X) = g(\mathbb{F})$ be its genus. Points in the curve $X$ with coordinates over $\mathbb{F}_{q^r}$ will be called rational. We will mainly use the following lemma on the parameters of AG codes, which follows from [48, Cor. 2.2.3].

Lemma 25 ([48]). Let $\mu \geq 2g$ such that the number of rational points of $X$ is at least $\mu + 1$. For any $g \leq k \leq \mu - g$, we may construct a $k$-dimensional linear AG code $C \subseteq \mathbb{F}_{q^\mu}$ from the curve $X$ such that $d_H(C) \geq \mu - k - g + 1$.

From this lemma, we may deduce the following theorem.

Theorem 8. Assume that $\mu - h \geq 2g$, in particular $t = \min\{h, \mu\} = h$, and assume that the number of rational points of $X$ is at least $\mu + 1$. Define $\rho = h + g$, which satisfies that $g \leq \mu - \rho \leq \mu - g$. Choose the code $C_\gamma \subseteq \mathbb{F}_{q^\mu}$ in (8) to be the $(\mu - \rho)$-dimensional linear AG code from Lemma 25. Then the field of linearity of the MSRD code $C_k(a, \beta) \subseteq \mathbb{F}_{q^m}$ has size
\[
|\mathbb{F}_{q^m}| = q^{r\rho} = (q^r)^{h+\theta}.
\]

(16)

We next particularize Theorem 8 to three concrete choices of the curve $X$ to construct linear MSRD codes with smaller field sizes.
4.7 Using Hermitian AG codes

We start by exploring Hermitian curves $X$ (see [48, Sec. 8.3]). Fix a positive integer $s$ such that $q^s = p^{2s}$, where $p$ is prime. The Hermitian curve is the projective plane curve with homogeneous equation

$$x^{q^2} + y^{q^2}z - yz^{q^2} = 0.$$  

This curve has $q^{3r} + 1$ rational points and genus

$$g = \frac{q^2(q^2 - 1)}{2}.$$  

Therefore, we may choose $\mu = q^{3r/2}$ in Theorem 8, and we deduce the following.

**Corollary 26.** Let the notation and assumptions be as in Theorem 8, but where $X$ is the Hermitian curve above, and where $\mu = q^{3r/2}$. Then the field of linearity of the MSRD code $C_k(\alpha, \beta) \subseteq \mathbb{F}_{q^m}$ has size

$$|\mathbb{F}_{q^m}| = (q^r)^{h+g} = \mu^{\frac{1}{3}(h+g)} = \mu^{\left(2h + \mu^{2/3} - \mu^{1/3}\right)},$$

that is,

$$m = r \left( h + \frac{1}{2} \left( \mu^{2/3} - \mu^{1/3} \right) \right),$$

where $\mu = \frac{g}{q-1}$.  

4.8 Using Suzuki AG codes

In this subsection, we explore Suzuki curves $X$ (see [22]). Let $r$ and $s$ be positive integers such that $r$ divides $2s + 1$, and consider $q^r = 2^{2s+1}$. The Suzuki curve is the projective plane curve with homogeneous equation

$$x^{2s} (y^{q^r} + yx^{q^r-1}) = z^{2s} (z^{q^r} + zx^{q^r-1}).$$

This curve has $q^{2r} + 1$ rational points over $\mathbb{F}_{q^r}$ by [22, Prop. 2.1], and genus $g = 2^s(q^r - 1)$ by [22, Lemma 1.9]. Therefore, we may choose $\mu = q^{2r}$ in Theorem 8 hence

$$g = 2^s \left( \mu^{1/2} - 1 \right) \leq \mu^{1/2} \left( \mu^{1/2} - 1 \right) = \mu^{3/4} - \mu^{1/4},$$

and we deduce the following consequence.

**Corollary 27.** Let the notation and assumptions be as in Theorem 8, but where $X$ is the Suzuki curve above, and where $\mu = q^{2r}$. Then the field of linearity of the MSRD code $C_k(\alpha, \beta) \subseteq \mathbb{F}_{q^m}$ has size

$$|\mathbb{F}_{q^m}| = (q^r)^{h+g} = \mu^{\frac{1}{2}(h+g)} \leq \mu^{\left(2h + \mu^{3/4} - \mu^{1/4}\right)},$$

that is,

$$m = r \left( h + 2^s \left( \mu^{1/2} - 1 \right) \right) \leq r \left( h + \mu^{3/4} - \mu^{1/4} \right),$$

where $\mu = \frac{g}{q-1}$.  

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4.9 Using García-Stichtenoth’s AG codes

In this subsection, we explore the second sequence of curves \((X_i)_{i=1}^\infty\) given by García and Stichtenoth (see [15] or [48, Sec. 7.4]). Fix a positive integer \(s\) such that \(q^r = p^{2s}\), where \(p\) is prime. For these curves, it is more convenient to define recursively the associated sequence of algebraic function fields \((F_i)_{i=1}^\infty\). First we define \(F_1 = F_{q^r}(x_1)\), where \(x_1\) is transcendental over \(F_{q^r}\), and then we define recursively \(F_{i+1} = F_i(x_{i+1})\), where \(x_{i+1}\) is algebraic over \(F_i\) satisfying the equation

\[
x_{i+1}^q + x_{i+1} = \frac{x_i^q}{x_i^{q^2-1} + 1},
\]

for all \(i \in \mathbb{N}_+\). The \(i\)th curve \(X_i\) has \(q^\frac{\mu_i}{2r} \left(q^\frac{\mu_i}{r} - 1\right) + 1\) rational points, and its genus is

\[
g(X_i) = \begin{cases} \left(q^\frac{\mu_i}{2r} - 1\right)^2 & \text{if } i \text{ is even}, \\ \left(q^\frac{\mu_i}{2r} \cdot \left(q^\frac{(i+1)r}{4} - 1\right) - 1\right) & \text{if } i \text{ is odd}, \end{cases}
\]

by [15] Remark 3.8. We will choose \(\mu_i = q^\frac{\mu_i}{2r} \left(q^\frac{\mu_i}{r} - 1\right)\) in Theorem 8, hence

\[
g_i = g(X_i) \leq q^\frac{\mu_i}{2r} = \frac{\mu_i}{q^\frac{\mu_i}{r} - 1},
\]

for all \(i \in \mathbb{N}_+\), and we deduce the following consequence.

**Corollary 28.** Let the notation and assumptions be as in Theorem 8 but where \(X_i\) and \(\mu_i\) are as above, for \(i \in \mathbb{N}_+\). Then, for \(i \in \mathbb{N}_+\), the field of linearity of the MSRD code \(C_{\mathbf{a}_i, \mathbf{b}_i} \subseteq \mathbb{F}_{q^{m_i}}\) has size

\[
|\mathbb{F}_{q^{m_i}}| = (q^r)^{h_i + \psi_i} \leq (q^r)^{h_i + q^\frac{\mu_i}{2r}} = \left(\frac{\mu_i}{q^\frac{\mu_i}{r} - 1}\right)^{h_i + \frac{\mu_i}{q^\frac{\mu_i}{r} - 1}}.
\]

that is, \(m_i \leq r \left(h_i + q^\frac{\mu_i}{2r}\right) = r \left(h_i + \frac{\mu_i}{q^\frac{\mu_i}{r} - 1}\right)\), where \(\mu_i = q^\frac{\mu_i}{2r} - 1\).

5 Summary of results and comparisons

In this final section, we will summarize the parameters of the MSRD codes and PMDS codes obtained throughout this work, and compare them to those from the literature.

The parameters of the MSRD codes obtained in Subsections 4.2, 4.3, 4.4, 4.5, 4.7, 4.8, and 4.9 are summarized in Table 1. The parameters of the PMDS codes obtained in Subsections 4.2, 4.3, and 4.5 are summarized in Table 2.
| Code $C_\gamma$ | $q, r, h$ | No. matrix sets $g$ | Field of linearity $q^m$ |
|-----------------|-----------|-----------------|-----------------|
| Trivial $C_\gamma = \{0\}$ | Any | $q - 1$ | $q^r = (g + 1)^r, m = r$ |
| MDS | Any | $(q - 1)(q^r + 1)$ | $\left(\frac{q}{q - 1} - 1\right)\min\{h, \frac{q}{q - 1}\}$ |
| Hamming, $\rho \in \mathbb{N}_+$ | $h = 2$ | $(q - 1) \cdot \frac{q^{\rho - 1}}{q - 1}$ | $q^{\rho} = \frac{q^r - 1}{q - 1} \cdot g + 1$ |
| Pr. BCH, $s \in \mathbb{N}_+$ | Any | $(q - 1)(q^{rs} - 1)$ | $\leq q^r \cdot \left(\frac{q}{q - 1} + 1\right)\left[\frac{q^r - 1}{q} (h - 1)\right]$ |
| Hermitian AG | $q^r = p^{2s}$ | $(q - 1)q^{2r}$ | $\mu^{\frac{1}{2}}(2h + \mu^{2/3} - \mu^{1/3}), \mu = \frac{q}{q - 1}$ |
| Suzuki AG | $q^r = 2^{2s+1}$ | $(q - 1)q^{2r}$ | $\leq \mu^{\frac{1}{2}}(h + \mu^{3/4} - \mu^{1/4}), \mu = \frac{q}{q - 1}$ |
| AG [15], $i \in \mathbb{N}_+$ | $q^r = p^{2s}$ | $(q - 1)\left(q^{\frac{r}{2}} - 1\right)q^{\frac{r^2}{2}}$ | $\leq \left(\frac{\mu_i}{q^{r - 1}}\right)^{\frac{1}{2}}(h_i + \mu_i^{1/2})$, $\mu_i = \frac{q^r}{q - 1}$ |

Table 1: Table summarizing the code parameters of the linear MSRD codes obtained in this work throughout Subsections 4.2, 4.3, 4.4, 4.5, 4.7, 4.8, and 4.9. They are $\mathbb{F}_q^m$-linear codes in $\mathbb{F}_{q^m}$ with code length $N = gr$, dimension $k = gr - h$, and minimum sum-rank distance $d = h + 1$. Their codewords can be seen as lists of $g$ matrices over $\mathbb{F}_q$ of size $m \times r$, where $m = r\rho, \rho \in \mathbb{N}_+$. The linear MSRD code in the first row was obtained in [33], and later independently in [10] and [43]. The codes in the other six rows are new. The symbol $\leq$ in the last column implies that the given expression is an upper bound on the smallest field size $q^m$ possible for the corresponding codes.

5.1 Comparison among MSRD codes

We start by discussing MSRD codes. First of all, smaller values of $g$ and $r$ in Table I may be obtained in each row, while keeping the same values of $q$, $m$ and $h$, by puncturing or shortening the corresponding MSRD codes, see [33, Cor. 7].

As discussed in the Introduction, any MRD code [12, 13, 45] is an MSRD code, however, their fields of linearity have size $q^m \geq q^gr \geq 2^gr$, thus exponential in the code length $N = gr$. All of the field sizes $q^m$ in Table I are much smaller than $2^gr$.

The first known construction of MSRD codes with sub-exponential field sizes is that of linearized Reed-Solomon codes, introduced in [33], and later independently in [10, 43]. They correspond to the first row in Table I. The other six rows in Table I correspond to new MSRD codes, all of which attain larger values of $g$ (relative to the other parameters) than linearized Reed-Solomon codes, which require $g < q$. We next show that each of these six MSRD codes attains the smallest field sizes $q^m$ for some parameter regime.

The field sizes $q^m$ of the MSRD codes in row 3 are the smallest among all MSRD codes relative to $q$, $r$ and $g$, in the sense that they meet the bound (13). However, these codes are restricted to $h = 2$ (dimension or co-dimension 2).

The MSRD codes in rows 2 and 4 in Table I attain smaller field sizes $q^m$ than
linearized Reed-Solomon codes for small $h$ relative to $r$, by looking at the exponents (the base is roughly $g$ in rows 1, 2 and 4). Consider the parameter regime $q - 1 < g \leq (q - 1)(q^r - 1)$ (i.e., setting $s = 1$ in row 4), which are unattainable values of $g$ for linearized Reed-Solomon codes. In this regime, the MSRD codes in row 2 attain smaller field sizes $q^m$ than those in row 4 if

$$h \geq \frac{(q^r + 1)^2}{q^r - 1} = \Omega(q^r)$$

(for code length $N = gr = (q - 1)(q^r - 1)r = \Theta(q^{r+1}r)$, the relative co-dimension is thus required to be $h/N = \Omega(q^{r-1}r^{-1})$). When $h < q^r$ (thus relative co-dimension $h/N < (q - 1)^{-1}r^{-1}$), the MSRD codes in row 4 attain smaller field sizes than those in row 2. Notice that in this parameter regime, i.e., $q - 1 < g \leq (q - 1)(q^r - 1)$, the MSRD codes in rows 5, 6 and 7 require larger field sizes than those in rows 1, 2, 3 or 4.

Now consider the parameter regime $q > (q - 1)(q^r + 1)$ and $h > 2$, thus we may only consider the codes in the last four rows of Table 1. Consider first the MSRD codes in row 5, assume $h < \mu$ and let $\varepsilon > 0$ be such that $\varepsilon h > \mu^{2/3} - \mu^{1/3}$. Then the field sizes in row 5 are $q^m = \mu^{h(2+\varepsilon)/3}$. Since only $\varepsilon h > \mu^{2/3} - \mu^{1/3}$ is required, we may consider $\varepsilon > 0$ to be as small as desired for large enough $\mu$ and $h$. In that case, the field sizes in row 5 are $\mu^{O(2h/3)}$, which are asymptotically smaller than the field sizes $\mu^{\Theta(h)}$ from row 4. Consider now the MSRD codes in row 6, assume $h < \mu$ and let $\varepsilon > 0$ be such that $\varepsilon h > \mu^{3/4} - \mu^{1/4}$. Again, $\varepsilon > 0$ may be as small as desired for large enough $h$ and $\mu$. Then, the field sizes in row 6 are $\mu^{O(h/2)}$, which are asymptotically smaller than the field sizes $\mu^{O(2h/3)}$ from row 5, however with stronger restrictions, as we require $\varepsilon h > \mu^{3/4} - \mu^{1/4}$ instead of $\varepsilon h > \mu^{2/3} - \mu^{1/3}$. Finally, consider the MSRD codes in row 7 and assume that $\mu/(q^r/2 - 1) < h < \mu$. Then the field sizes in row 7 are $\mu^{O(h/i)}$, where $i \in \mathbb{N}_+$ may be arbitrary. Such field sizes are asymptotically smaller than the field sizes from rows 5 and 6, however with stronger restrictions, as we require $h > \mu/(q^r/2 - 1)$ instead of $\varepsilon h > \mu^{2/3} - \mu^{1/3}$ or $\varepsilon h > \mu^{3/4} - \mu^{1/4}$.

Finally, we mention two recent works where new MSRD codes are constructed. Twisted linearized Reed-Solomon codes have been introduced in [40]. However, their field sizes are not smaller than those of linearized Reed-Solomon codes (row 1 in Table 1). Some constructions of MSRD codes were recently given in [6]. However, such codes are only $\mathbb{F}_q$-linear, and have minimum sum-rank distance equal to 2 or $\sum_{i=1}^g r_i - 1$ (total number of columns, across all matrices, minus 1), or require the number of rows or columns to be 1 at certain positions in the matrices in [2].

5.2 Comparison among PMDS codes

We now turn to discussing PMDS codes. For $h \in \{0, 1, 2, 3\}$, the PMDS codes from [3] [19] have smaller field sizes than the PMDS codes in Table 2. For dimension $r + 1$, the PMDS codes from [2] also have smaller field sizes than those in Table 2.

To the best of our knowledge, the PMDS codes for general parameters with the smallest known field sizes are those in [8] [15] [39], being those from [39] exactly the
Restrictions on $r$, $\delta$, $g$, $h$, $q$ and Field size $q^m$

| Code $C_{\gamma}$ | Restrictions on $r$, $\delta$, $g$, $h$, $q$ | Field size $q^m$ |
|-------------------|---------------------------------------------|------------------|
| Trivial $C_{\gamma} = \{0\}$ | $\max\{\nu, g\} < q \leq 2 \max\{\nu, g\}$ | $\leq (2 \max\{\nu, g\})^r$ |
| MDS | $g = (q - 1) (q^r + 1)$ or $(2\nu)^r > \frac{g}{\nu}$ | $\leq \max\{(2\nu)^r, \lfloor \frac{g}{\nu} \rfloor - 1\}^{\min\{h, \lfloor \frac{g}{\nu} \rfloor\}}$ |
| Primitive BCH | $g = (q - 1) (q^r - 1)$ and $q > \nu$ | $\leq (2\nu)^r \cdot (\lfloor \frac{g}{\nu} \rfloor + 1)^{h-1}$ |

Table 2: Table summarizing the code parameters of the linear PMDS codes obtained in this work throughout Subsections 4.2, 4.3, and 4.5. They are $\mathbb{F}_{q^m}$-linear codes in $\mathbb{F}_{q^\nu}$, where $r$ is the locality, $\delta$ is the local distance, $g$ is the number of local sets, $h$ is the number of global parities, $\nu = r + \delta - 1$ is the local-set size and $q$ is a power of 2. The field size of the local codes is a subfield of $\mathbb{F}_q$. The linear PMDS code in the first row was obtained in [39]. The codes in the other two rows are new. The symbol $\leq$ in the last column implies that the given expression is an upper bound on the smallest field size $q^m$ possible for the corresponding codes.

PMDS codes in row 1 in Table 2. The constructions from [8, 18, 39], put together, yield PMDS codes with field sizes $q^m$ such that

$$\max\{\nu, g\}^{\min\{r,h\}} < q^m \leq (2 \max\{\nu, g\})^{\min\{r,h\}}.$$  \hfill (20)

If $g > (2\nu)^r\nu$, then the field sizes of the PMDS codes in row 2 in Table 2 would be

$$q^m \leq \left(\left\lfloor \frac{g}{\nu} \right\rfloor - 1\right)^h,$$

which would be strictly smaller than those in row 3 in Table 2 and in (20) when $h < r$.

Asymptotically, if $\nu = \mathcal{O}(1)$ and $g > \nu$ grows unboundedly, the field sizes of the PMDS codes in row 3 in Table 2 are

$$q^m = \mathcal{O}\left(\left(\frac{g}{\nu}\right)^{h-1}\right),$$

which would be asymptotically smaller than those of the PMDS codes in row 2 in Table 2 and those in (20) when $h < r$.

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Appendix: Tables with even field sizes for MSRD codes

In this appendix, we provide several tables of attainable field sizes $q^m$, divisible by 2, among the linear MSRD codes obtained in this work.

First, we give a summary in Table 3 which is similar to Table 1, but where the field size $q^m$ is not compared to $g$, but written as a function of $q$, $r$ and $h$, excluding $g$. The reason behind this is that typically the maximum attainable value of $g$ is quite large for most of these codes, and in most cases we would puncture them in order to have a much smaller number of matrix sets $g$.

In Tables 4, 5 and 6 we fix $g$ and let other parameters vary. In contrast, in Tables 8 and 9, we fix the code length $N = gr$ and let other parameters vary. In these tables, bold numbers indicate field sizes that are the smallest among MSRD codes of the same parameters. As linearized Reed-Solomon codes have the same field sizes for all $h$, a bold number in that row means that the field size is the smallest for the corresponding parameters for some $h$.

The field sizes attained by linear MSRD codes based on AG codes (Subsections 4.7, 4.8 and 4.9) are quite larger than those obtained by the other linear MSRD codes for small parameters. In general, MSRD codes based on AG codes (as in Subsection 4.6) are mostly of asymptotic interest. For this reason, they are not included in Tables 4, 5, 6, 7, 8 and 9.

Finally, at the end of each table we consider the smallest field size attainable by an MRD code for the corresponding parameters. As it can be seen, MRD codes always require significant larger field sizes than the MSRD codes from this work, for the same parameters.

| Code $C_\gamma$ | $q$, $r$, $h$ | No. matrix sets $g$ | Field of linearity $q^m$ |
|-----------------|--------------|-----------------|------------------|
| Trivial $C_\gamma = \{0\}$ (Lin. RS) | Any | $q - 1$ | $q^r$ |
| MDS | Any | $(q - 1) (q^r + 1)$ | $q^r \min\{h, q^r+1\}$ |
| Hamming, $\rho \in \mathbb{N}_+$ | $h = 2$ | $(q - 1) \cdot \frac{q^r - 1}{q - 1}$ | $q^r \rho$ |
| Pr. BCH, $s \in \mathbb{N}_+$ | Any | $(q - 1) (q^rs - 1)$ | $\leq q^r \left(1 + s \left[\frac{q^r - 1}{q^r - q^r - q^r - 1}\right]\right)$ |
| Hermitian AG | $q^r = p^{2s}$ | $(q - 1)q^{\frac{3s}{2}}$ | $q^r \left(h + s \left(q^r - q^r\right)\right)$ |
| Suzuki AG | $q^r = 2^{2s+1}$ | $(q - 1)q^{2r}$ | $q^r \left(h + 2r (q^r - 1)\right)$ |
| AG [15], $i \in \mathbb{N}_+$ | $q^r = p^{2s}$ | $(q - 1) \left(q^r - 1\right)q^{\frac{i}{2}}$ | $\leq q^r \left(h + q^r\right)$ |

Table 3: Table summarizing the code parameters of the linear MSRD codes obtained in this work. In contrast with Table 1, field sizes are described in terms of $q$, $r$ and $h$, excluding $g$. 

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| Code \(C_\gamma\) | \(r = 2\) | \(r = 3\) | \(r = 4\) | \(r = 5\) | \(r = 6\) |
|---|---|---|---|---|---|
| Trivial \(C_\gamma = \{0\}\) (Lin. RS) | \(2^2\) | \(2^3\) | \(2^4\) | \(2^5\) | \(2^6\) |
| MDS, \(h = 2\) | \(2^4\) | \(2^6\) | \(2^8\) | \(2^{10}\) | \(2^{12}\) |
| \(h = 3\) | \(2^6\) | \(2^9\) | \(2^{12}\) | \(2^{15}\) | \(2^{18}\) |
| \(h = 4\) | \(2^8\) | \(2^{12}\) | \(2^{16}\) | \(2^{20}\) | \(2^{24}\) |
| Hamming, \(\rho = 3, h = 2\) | \(2^6\) | \(2^{21}\) | \(2^{12}\) | \(2^{15}\) | \(2^{18}\) |
| Pr. BCH, \(s = 2, h = 2\) | \(2^{10}\) | \(2^{15}\) | \(2^{20}\) | \(2^{25}\) | \(2^{30}\) |
| \(h = 3\) | \(2^{14}\) | \(2^{21}\) | \(2^{25}\) | \(2^{30}\) | \(2^{35}\) |
| \(h = 4\) | \(2^{20}\) | \(2^{30}\) | \(2^{50}\) | \(2^{50}\) | \(2^{50}\) |

Table 4: Table for fixed \(q = 2\), while other parameters vary.

| Code \(C_\gamma\) | \(r = 2\) | \(r = 3\) | \(r = 4\) | \(r = 5\) | \(r = 6\) |
|---|---|---|---|---|---|
| Trivial \(C_\gamma = \{0\}\) (Lin. RS), \(\forall h \geq 1\) | \(2^6\) | \(2^3\) | \(2^{12}\) | \(2^3\) | \(2^{18}\) |
| MDS, \(h = 2\) | \(2^3\) | \(2^6\) | \(2^8\) | \(2^{10}\) | \(2^{12}\) |
| \(h = 3\) | \(2^{12}\) | \(2^9\) | \(2^{12}\) | \(2^{15}\) | \(2^{18}\) |
| \(h = 4\) | \(2^{16}\) | \(2^{12}\) | \(2^{16}\) | \(2^{20}\) | \(2^{24}\) |
| Hamming, \(\rho = 3, h = 2\) | \(2^6\) | \(2^{29}\) | \(2^{12}\) | \(2^{15}\) | \(2^{18}\) |
| Pr. BCH, \(s = 1, 2, h = 2\) | \(2^{10}\) | \(2^{29}\) | \(2^{12}\) | \(2^{15}\) | \(2^{18}\) |
| \(h = 3\) | \(2^{14}\) | \(2^{12}\) | \(2^{16}\) | \(2^{20}\) | \(2^{24}\) |
| \(h = 4\) | \(2^{14}\) | \(2^{28}\) | \(2^{42}\) | \(2^{56}\) | \(2^{70}\) |

Table 5: Table for fixed \(g = 7\), while other parameters vary.
| Code $C_\gamma$ | $r = 2$ | $r = 3$ | $r = 4$ | $r = 5$ | $r = 6$ |
|-----------------|---------|---------|---------|---------|---------|
| Trivial $C_\gamma = \{0\}$ (Lin. RS), $\forall h \geq 1$ | $2^8$ | $2^4$ | $2^{12}$ | $2^4$ | $2^{16}$ | $2^4$ | $2^{20}$ | $2^4$ | $2^{24}$ | $2^4$ |
| MDS, $h = 2$ | $2^8$ | $2^{12}$ | $2^8$ | $2^{10}$ | $2^{12}$ | $2^8$ | $2^{10}$ | $2^{12}$ | $2^8$ | $2^{10}$ | $2^{12}$ |
| $h = 3$ | $2^{12}$ | $2^2$ | $2^{18}$ | $2^2$ | $2^{12}$ | $2^2$ | $2^{18}$ | $2^2$ | $2^{12}$ | $2^2$ |
| $h = 4$ | $2^{16}$ | $2^{24}$ | $2^{16}$ | $2^{20}$ | $2^{24}$ | $2^{16}$ | $2^{20}$ | $2^{24}$ | $2^{16}$ | $2^{20}$ | $2^{24}$ |
| Hamming, $\rho = 3$, $h = 2$ | $2^6$ | $2^2$ | $2^9$ | $2^2$ | $2^{12}$ | $2^2$ | $2^{15}$ | $2^2$ | $2^{18}$ | $2^2$ |
| $h = 3$ | $2^{10}$ | $2^2$ | $2^{15}$ | $2^2$ | $2^{12}$ | $2^2$ | $2^{15}$ | $2^2$ | $2^{18}$ | $2^2$ |
| $h = 4$ | $2^{14}$ | $2^{21}$ | $2^{16}$ | $2^2$ | $2^{20}$ | $2^{24}$ | $2^{16}$ | $2^{20}$ | $2^{24}$ | $2^{16}$ | $2^{20}$ | $2^{24}$ |
| Best MRD code possible, $\forall h \geq 1$ | $2^{30}$ | $2^2$ | $2^{45}$ | $2^2$ | $2^{60}$ | $2^2$ | $2^{75}$ | $2^2$ | $2^{90}$ | $2^2$ |

Table 6: Table for fixed $g = 15$, while other parameters vary.

| Code $C_\gamma$ | $r = 2$ | $r = 3$ | $r = 4$ | $r = 5$ | $r = 6$ |
|-----------------|---------|---------|---------|---------|---------|
| Trivial $C_\gamma = \{0\}$ (Lin. RS), $\forall h \geq 1$ | $2^{10}$ | $2^5$ | $2^{15}$ | $2^5$ | $2^{20}$ | $2^5$ | $2^{25}$ | $2^5$ | $2^{30}$ | $2^5$ |
| MDS, $h = 2$ | $2^8$ | $2^{12}$ | $2^8$ | $2^{10}$ | $2^{12}$ | $2^8$ | $2^{10}$ | $2^{12}$ | $2^8$ | $2^{10}$ | $2^{12}$ |
| $h = 3$ | $2^{12}$ | $2^2$ | $2^{18}$ | $2^2$ | $2^{12}$ | $2^2$ | $2^{18}$ | $2^2$ | $2^{12}$ | $2^2$ |
| $h = 4$ | $2^{16}$ | $2^{24}$ | $2^{16}$ | $2^{20}$ | $2^{24}$ | $2^{16}$ | $2^{20}$ | $2^{24}$ | $2^{16}$ | $2^{20}$ | $2^{24}$ |
| Hamming, $\rho = 3$, $h = 2$ | $2^8$ | $2^2$ | $2^9$ | $2^2$ | $2^{12}$ | $2^2$ | $2^{15}$ | $2^2$ | $2^{18}$ | $2^2$ |
| $h = 3$ | $2^{14}$ | $2^2$ | $2^{15}$ | $2^2$ | $2^{12}$ | $2^2$ | $2^{15}$ | $2^2$ | $2^{18}$ | $2^2$ |
| $h = 4$ | $2^{20}$ | $2^{21}$ | $2^{28}$ | $2^2$ | $2^{20}$ | $2^{24}$ | $2^{28}$ | $2^2$ | $2^{24}$ | $2^2$ |
| Best MRD code possible, $\forall h \geq 1$ | $2^{62}$ | $2^2$ | $2^{93}$ | $2^2$ | $2^{124}$ | $2^2$ | $2^{155}$ | $2^2$ | $2^{186}$ | $2^2$ |

Table 7: Table for fixed $g = 31$, while other parameters vary.
Table 8: Table for fixed $N = gr = 30$, while other parameters vary.

| Code $C_{\gamma}$ | $r = 2$ | $r = 3$ | $r = 4$ | $r = 5$ | $r = 6$ |
|-------------------|---------|---------|---------|---------|---------|
| Trivial $C_{\gamma} = \{0\}$ (Lin. RS), $\forall h \geq 1$ | $q^m$ | $q^m$ | $q^m$ | $q^m$ | $q^m$ |
| MDS, $h = 2$ | $2^8$ | $2^{12}$ | $2^{16}$ | $2^{15}$ | $2^{18}$ |
| $h = 3$ | $2^{12}$ | $2^{18}$ | $2^{12}$ | $2^{15}$ | $2^{18}$ |
| $h = 4$ | $2^{16}$ | $2^{24}$ | $2^{16}$ | $2^{20}$ | $2^{24}$ |
| Hamming, $\rho = 3$, $h = 2$ | $2^6$ | $2^{9}$ | $2^{12}$ | $2^{15}$ | $2^{18}$ |
| $h = 3$ | $2^{10}$ | $2^{15}$ | $2^{12}$ | $2^{15}$ | $2^{18}$ |
| $h = 4$ | $2^{14}$ | $2^{21}$ | $2^{16}$ | $2^{20}$ | $2^{24}$ |
| Best MRD code possible, $\forall h \geq 1$ | $2^{30}$ | $2^{30}$ | $2^{30}$ | $2^{30}$ | $2^{30}$ |

Table 9: Table for fixed $N = gr = 62$, while other parameters vary.

| Code $C_{\gamma}$ | $r = 2$ | $r = 3$ | $r = 4$ | $r = 5$ | $r = 6$ |
|-------------------|---------|---------|---------|---------|---------|
| Trivial $C_{\gamma} = \{0\}$ (Lin. RS), $\forall h \geq 1$ | $q^m$ | $q^m$ | $q^m$ | $q^m$ | $q^m$ |
| MDS, $h = 2$ | $2^{10}$ | $2^{15}$ | $2^{20}$ | $2^{20}$ | $2^{24}$ |
| $h = 3$ | $2^{12}$ | $2^{18}$ | $2^{24}$ | $2^{24}$ | $2^{24}$ |
| $h = 4$ | $2^{16}$ | $2^{24}$ | $2^{32}$ | $2^{20}$ | $2^{24}$ |
| Hamming, $\rho = 3$, $h = 2$ | $2^8$ | $2^{9}$ | $2^{12}$ | $2^{15}$ | $2^{18}$ |
| $h = 3$ | $2^{14}$ | $2^{15}$ | $2^{20}$ | $2^{15}$ | $2^{18}$ |
| $h = 4$ | $2^{20}$ | $2^{21}$ | $2^{28}$ | $2^{20}$ | $2^{24}$ |
| Best MRD code possible, $\forall h \geq 1$ | $2^{62}$ | $2^{62}$ | $2^{62}$ | $2^{62}$ | $2^{62}$ |