The Cauchy Problem Of The Moment Theory Elasticity In $\mathbb{R}^n$
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Abstract
In this paper, we considered the problem of analytical continuation of the solution of the system equations of the moment theory of elasticity in spacious bounded domain from its values and values of its strains on part of the boundary of this domain, i.e., the Cauchy’s problem.

Key words: the Cauchy problem, system theory of elasticity, elliptic system, ill-posed problem, Carleman matrix, regularization.

1. Introduction
In this paper, we considered the problem of analytical continuation of the solution of the system equations of the moment theory of elasticity in spacious bounded domain from its values and values of its strains on part of the boundary of this domain, i.e., the Cauchy’s problem.

Since, in many actual problems, either a part of the boundary is inaccessible for measurement of displacement and tensions or only some integral characteristic are available. In experimental study of the stress-strain state of actual constructions, we can make measurements only on the accessible part of the surface.

In a practical investigation of experimental dates or diagnostic moving abject arise problems of estimation concerning deformed position of the object. Solution of the problems by using well known classical propositions is connected to difficulties of absence of experimental dates which is necessary for formulation of boundary value (classical) conditions.

Therefore it is necessary consider the problem of continuation for solution of elasticity system of equations to the domain by values of solutions and normal derivatives in the part of boundary of domain.

System equation of moment theory elasticity is elliptic. Therefore the problem Cauchy for this system is ill-posed. For ill-posed problems, one does not prove the existence theorem: the existence is assumed a priori. Moreover, the solution is assumed to belong to some given subset of the function space, usually a compact one [1]. The uniqueness of the solution follows from the general Holmgren theorem [2]. On establishing uniqueness in the article studio of ill-posed problems, one comes across important questions concerning the derivation of estimates of conditional stability and the construction of regularizing operators.

Our aim is to construct an approximate solution using the Carleman function method.

Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be points of the $n$-dimensional Euclidean space $\mathbb{R}^n$, $D$ a bounded simply connected domain in $\mathbb{R}^n$, with piecewise-smooth boundary consisting of a piece $\Sigma$ of the plane $y_n = 0$ and a smooth surface $S$ lying in the half-space $y_n > 0$.

Suppose that $2n$-component vector function $U(x) = (u_1(x), ..., u_n(x), w_1(x), ..., w_n(x)) = (u(x), w(x))$ satisfied in $D$ the system equations moments theory elasticity [3]:
\[
\begin{aligned}
\left\{ \begin{array}{l}
(\mu + \alpha)\Delta u + (\lambda + \mu - \alpha)\text{graddi}v + 2\alpha \text{rot}w + \rho \sigma^2 u = 0, \\
(\nu + \beta)\Delta w + (\varepsilon + \nu - \beta)\text{graddi}v + 2\alpha \text{rot}u - 4\alpha w + 4\theta \sigma^2 w = 0,
\end{array} \right. \\
\end{aligned}
\] (1),

where \(\lambda, \mu, \nu, \beta, \varepsilon, \alpha, \rho, \sigma\) is coefficients which characterizing medium, satisfying the conditions

\[
\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \alpha > 0, \quad 3\varepsilon + 2\nu > 0, \quad \beta > 0, \quad \rho > 0, \quad \sigma > 0.
\]

For brevity it is convenient to use matrix notation. Let us introduce the matrix differential operator

\[
M(\partial_x) = \begin{bmatrix}
M^{(1)} & M^{(2)} \\
M^{(3)} & M^{(4)}
\end{bmatrix},
\]

where

\[
M^{(i)} = \begin{bmatrix} M_{k,j}^{(i)} \end{bmatrix}_{n \times n}, \quad i = 1, 2, 3, 4,
\]

moreover

\[
M_{k,j}^{(1)} = \delta_{k,j}(\mu + \alpha)(\Delta + \sigma_1^2) + (\lambda + \mu - \alpha)\frac{\partial^2}{\partial x_k \partial x_j}, \quad k, j = 1, ..., n
\]

\[
M_{k,j}^{(2)} = M_{k,j}^{(3)} = -2\alpha \sum_{p=1}^{n} \varepsilon_{k,j,p} \frac{\partial}{\partial x_p}, \quad k, j = 1, ..., n,
\]

\[
M_{k,j}^{(4)} = \delta_{k,j} [(\nu + \beta)\Delta + \sigma_2^2] + (\varepsilon + \nu - \beta)\frac{\partial^2}{\partial x_k \partial x_j}, \quad k, j = 1, ..., n,
\]

\[
\sigma_1^2 = \frac{\rho \sigma^2}{\mu + \alpha}, \quad \sigma_2^2 = \frac{\theta \sigma^2 - 4\alpha}{\nu + \beta}, \quad \delta_{k,j} = \begin{cases} 1, & \text{if } k = j \\
0, & \text{if } k \neq j
\end{cases}
\]

\(\varepsilon_{k,j,p}\) so-called \(\varepsilon\)-tensor or Levi-Civita’s symbol, which defend following equaliti’s

\[
\varepsilon_{k,j,p} = \begin{cases} 0, & \text{if at least two of three – subscripts } k, j, p \text{ are equal,} \\
1, & \text{if } (k, j, p) \text{ is an even permutation,} \\
-1, & \text{if } (k, j, p) \text{ is an odd permutation.}
\end{cases}
\]

Then system (??) maybe write in matrix from in the following way:

\[
M(\partial_x)U(x) = 0
\] (2)
A solution $U$ of system (2.8) in the domain $D$ is said to be regular if $U \in C^1(D) \cap C^2(D)$.

**Statement of the problem.** Find a regular solution $U$ of system (2.8) in the domain $D$ using its Cauchy data on the surface $S$:

$$U(y) = f(y), \quad T(\partial_y, n(y))U(y) = g(y), \quad y \in S, \quad (3)$$

where $T(\partial_y, n(y))$ is the stress operator, i.e.,

$$T(\partial_y, n(y)) = \begin{bmatrix} T^{(1)}(\partial_y, n) & T^{(2)}(\partial_y, n) \\ T^{(3)}(\partial_y, n) & T^{(4)}(\partial_y, n) \end{bmatrix},$$

$$T^{(i)}(\partial_y, n) = \left\| T^{(i)}(\partial_y, n) \right\|_{n \times n}, \quad i = 1, 2, 3, 4,$$

$$T^{(1)}(\partial_y, n) = \lambda n_k \frac{\partial}{\partial y_j} + (\mu - \alpha) n_j(y) \frac{\partial}{\partial y_k} + (\mu + \alpha) \delta_{kj} \frac{\partial}{\partial n(y)},$$

$$T^{(2)}(\partial_y, n) = 2\alpha \sum_{p=1}^{n} \varepsilon_{kj} n_p(y), \quad T^{(3)}(\partial_y, n) = 0,$$

$$T^{(4)}(\partial_y, n) = \varepsilon n_k(y) \frac{\partial}{\partial y_j} + (\nu - \beta) n_j(y) \frac{\partial}{\partial y_k} + (\nu + \beta) \frac{\partial}{\partial n(y)},$$

$n(y) = (n_1(y), \ldots, n_n(y))$ is the unit outward normal vector on $\partial D$ at a point $y$, $f = (f_1, \ldots, f_{2n})$, $g = (g_1, \ldots, g_{2n})$ are given continuous vector functions on $S$.

2. **Construction of the matrix Carleman and approximate solution for the domain type’s cap**

It is well known, that any regular solution $U(x)$ system (2.8) is specified by the formula

$$U(x) = \int_{\partial D} \Psi(y, x)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Psi(y, x)\}^*U(y)ds_y, \quad x \in D, \quad (4)$$

where symbol is denote of operation transposition, $\Psi(y, x)$ matrix of fundamental solutions system equation of steady-state oscillations of the couple-stress theory of elasticity:

$$\Psi(y, x) = \begin{bmatrix} \Psi^{(1)}(y, x) & \Psi^{(2)}(y, x) \\ \Psi^{(3)}(y, x) & \Psi^{(4)}(y, x) \end{bmatrix},$$

where

$$\Psi^{(i)}(y, x) = \left\| \Psi^{(i)}_{kj}(y, x) \right\|_{n \times n}, \quad i = 1, 2, 3, 4,$$

$$\Psi^{(1)}_{kj}(y, x) = \sum_{l=1}^{4} (\delta_{kj}\alpha_l + \beta l \frac{\partial^2}{\partial x_l \partial x_j}) \varphi_n(ikr), \quad k, j = 1, \ldots, n,$$
\[ \Psi_{kj}^{(2)}(y, x) = \Psi_{kj}^{(3)}(y, x) = \frac{2\alpha}{\mu + \alpha} \sum_{i=1}^{4} \sum_{p=1}^{n} \varepsilon k_{ijp} \frac{\partial}{\partial x_{ip}} \varphi_{m}(ik_{j}r), \quad k, j = 1, ..., n, \]

\[ \Psi_{kj}^{(4)}(y, x) = \sum_{i=1}^{4} (\delta_{kj} \gamma_{i} + \delta_{i} \frac{\partial^{2}}{\partial x_{k} \partial x_{j}}) \varphi_{n}(ik_{j}r), \quad k, j = 1, ..., n, \]

here \( \varphi_{n} \) - fundamental solution Helmholtz equation, \( r = |x - y| \),

\[ \alpha_{l} = \frac{(-1)^{l} \alpha^{2} - k_{l}^{2}}{2\pi(\mu + \alpha)(k_{l}^{2} - k_{l}^{2})}, \quad \beta_{l} = -\frac{\delta_{l} \alpha}{2\pi \beta^{2} - 4\alpha} + \frac{\alpha_{l}}{k_{l}^{2}}, \quad \sum_{l=1}^{4} \beta_{l} = 0, \]

\[ \gamma_{l} = \frac{(-1)^{l} \alpha^{2} - k_{l}^{2}}{2\pi(\beta + \nu)(k_{l}^{2} - k_{l}^{2})}, \quad \delta_{l} = -\frac{\delta_{l} \alpha}{2\pi \beta^{2} - 4\alpha} + \frac{\gamma_{l}}{k_{l}^{2}}, \quad \sum_{l=1}^{4} \beta_{l} = 0, \]

\[ \varepsilon_{l} = \frac{(-1)^{l} (\alpha^{2} + k_{l}^{2}) (\beta_{l} + \alpha_{l})}{2\pi (\beta + \nu) (k_{l}^{2} - k_{l}^{2})}, \quad \sum_{l=1}^{4} \varepsilon_{l} = 0, \quad k_{l} = \frac{\rho \sigma^{2}}{\lambda + 2\mu}, \quad k_{l}^{2} = \frac{\theta \sigma^{2} - 4\alpha}{\varepsilon + 2\nu}. \]

\[ k_{1}^{2} = k_{2}^{2} = \sigma_{1}^{2} + \sigma_{2}^{2} + \frac{4\alpha^{2}}{(\mu + \alpha)(\beta + \nu)}, \quad k_{1}^{2}k_{2}^{2} = \sigma_{1}^{2}\sigma_{2}^{2}. \]

Easily we can verify, that \( u = \Psi_{j}^{(??)}(y, x), \ w = \Psi_{j}^{(??)}(y, x) \) or \( u = \Psi_{j}^{(??)}(y, x), \ w = \Psi_{j}^{(??)}(y, x) \) are solution system (??), where \( \Psi_{j}^{(??)}(y, x) \) - \( j \) - vector tuple \( i \) - matrix.

**Definition.** By the Carleman matrix of problem (??),(??) we mean an \( 2n \times 2n \) matrix \( \Pi(y, x, \tau) \) depending on the two points \( y, x \) and positive numerical number parameter \( \tau \) satisfying the following two conditions:

1) \( \Pi(y, x, \tau) = \Psi(y, x) + G(y, x, \tau), \)

where matrix \( G(y, x, \tau) \) satisfies system (??) with respect to the variable \( y \) in the domain \( D \), and \( \Psi(y, x) \) is a matrix of the fundamental solutions of system (??);

2) \( \int_{\partial D \setminus S} (|\Pi(y, x, \tau)| + |T(\partial_{y}, n)\Pi(y, x, \tau)|) ds_{y} \leq \varepsilon(\tau), \)

where \( \varepsilon(\tau) \to 0, \) as \( \tau \to \infty; \) here \( ||\Pi|| \) is the Euclidean norm of the matrix \( \Pi = \sum_{i,j=1}^{2n} \Pi_{ij}^{2n}, \) i.e., \( ||\Pi|| = (\sum_{i,j=1}^{2n} \Pi_{ij}^{2n})^{\frac{1}{2}}. \) In particular, \( ||U|| = (\sum_{m=1}^{n} (u_{m}^{2} + w_{m}^{2}))^{\frac{1}{2}}. \)
It is well known, that for the regular vector functions \(v(y)\) and \(u(y)\) holds formula [4]:

\[
\int_D [v(y)\{M(\partial_y)u(y)\} - u(y)\{M(\partial_y)v(y)\}]dy =
\]

\[
= \int_{\partial D} [v(y)\{T(\partial_y, n)u(y)\} - u(y)\{T(\partial_y, n)v(y)\}]ds_y.
\]

Substituting in this equality \(v(y) = G(y, x, \tau)\) and \(u(y) = U(y)\) is solution system (??), we have

\[
0 = \int_{\partial D} [G(y, x, \tau)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)G(y, x, \tau)\}^*U(y)]ds_y.\tag{5}
\]

Now adding (??) and (??), we have

**Theorem 1.** Any regular solution \(U(x)\) of system (??) in the domain \(D\) is specified by the formula

\[
U(x) = \int_{\partial D} (\Pi(y, x, \tau)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \tau)\}^*U(y))ds_y, \quad x \in D,\tag{6}
\]

where \(\Pi(y, x, \tau)\) is matrix Carleman.

Using the matrix Carleman, easily conclude the estimate stability of solution of the problem (??), (??) and also indicate effective method decision this problem.

With a view to construct approximate solution of the problem (??), (??) we construct the following matrix:

\[
\Pi(y, x) = \begin{bmatrix}
\Pi^{(1)}(y, x) & \Pi^{(2)}(y, x) \\
\Pi^{(3)}(y, x) & \Pi^{(4)}(y, x)
\end{bmatrix},
\]

\[
\Pi^{(i)}(y, x) = \left\| \Pi^{(i)}_{kj}(y, x) \right\|_{n \times n}, \quad i = 1, 2, 3, 4,
\]

\[
\Pi^{(1)}_{kj}(y, x) = \sum_{l=1}^{4} (\delta_{kj} \alpha_l + \beta_l \frac{\partial^2}{\partial x_k \partial x_j})\Phi(y, x, k_l), \quad k, j = 1, ..., n
\]

\[
\Pi^{(2)}_{kj}(y, x) = \Pi^{(3)}_{kj}(y, x) =
\]

\[
= \frac{2\alpha}{\mu + \alpha} \sum_{l=1}^{4} \sum_{p=1}^{n} \epsilon_{lkj} \frac{\partial}{\partial x_p} \Phi(y, x, k_l), \quad k, j = 1, ..., n,
\]

\[
\Pi^{(4)}_{kj}(y, x) = \sum_{l=1}^{4} (\delta_{kj} \gamma_l + \delta_l \frac{\partial^2}{\partial x_k \partial x_j})\Phi(y, x, k_l), \quad k, j = 1, ..., n \tag{8}
\]
where

\[ C_n K(x_n) \Phi(y, x, k) = \int_0^\infty I_m \left[ \frac{K(\sqrt{u^2 + s} + y_n)}{\sqrt{u^2 + s} + y_n - x_n} \right] \psi(ku) \, du \]  

(9)

\[ \psi(ku) = \begin{cases} u J_0(ku), & n = 2m, \ m \geq 1, \\ \cos ku, & n = 2m + 1, \ m \geq 1, \end{cases} \]

\[ J_0(u) - \text{Bessel function of order zero,} \]

\[ s = (y_1 - x_1)^2 + ... + (y_{n-1} - x_{n-1})^2, \]

\[ C^2 = 2\pi \]

\[ C_n = \begin{cases} (-1)^m \cdot 2^{-n}(n-2)! \pi \omega_n(m-2)!, & n = 2m \\ (-1)^m \cdot 2^{-n}(n-2)! \pi \omega_n(m-1)!, & n = 2m + 1. \end{cases} \]

\[ K(\omega), \ \omega = u + iv \quad (u, v \ \text{are real}), \]

is an entire function taking real values on the real axis and satisfying the conditions

\[ K(u) \neq \infty, \quad |u| < \infty, \]

\[ K(u) \neq 0, \quad \sup_{v \geq 1} |\exp \nu |K^{(p)}(\omega)| = M(p, u) < \infty, \]

\[ p = 0, ..., m, \ u \in R^1. \] In work [4] proved.

**Lemma 1.** For function \( \Phi(y, x, k) \) the formula is valid

\[ C_n \Phi(y, x, k) = \varphi_n(ikr) + g_n(y, x, k), \quad r = |y - x|, \]  

(10)

where \( \varphi_n \) - fundamental solution Helmholtz equation, \( g_n(y, x, k) \) is a regular function that is defined for all \( y \) and \( x \) satisfies Helmholtz equation: \( \Delta(\partial_y)g_n - k^2 g_n = 0. \)

In (10) we assume the function \( K(\omega) = \exp(\tau \omega) \). Then

\[ \Phi(y, x, k) = \Phi_\tau(y - x, k), \]

\[ C_n \Phi_\tau(y - x, k) = \frac{\partial^{m-1}}{\partial s^{m-1}} \int_0^\infty I_m \left[ \frac{\exp \tau(\sqrt{u^2 + s} + y_n - x_n)}{\sqrt{u^2 + s} + y_n - x_n} \right] \psi(ku) \, du = \]

\[ = \exp \tau(y_n - x_n) \frac{\partial^{m-1}}{\partial s^{m-1}} \int_0^\infty \left[ -\cos \tau \sqrt{u^2 + \alpha^2} + \right. \]

\[ \left. (y_n - x_n) \sin \tau \sqrt{u^2 + s} \right] \psi(ku) \, du, \]  

(1)

\[ \Phi'_\tau(y - x, k) = \frac{\partial \Phi_\tau}{\partial \tau}. \]

\[ C_n \Phi'_\tau(y - x, k) = \exp \tau(y_n - x_n) \frac{\partial^{m-1}}{\partial s^{m-1}} \int_0^\infty \frac{\sin \tau \sqrt{u^2 + s}}{\sqrt{u^2 + s}} \psi(ku) \, du, \]  

(1)
\[ C_n \Phi'_\tau(y - x, k) = \exp \left( \tau(y_n - x_n) \frac{\partial^{m-1}}{\partial s^{m-1}} \psi'_\tau(k, s) \right), \]  

(2)

\[ \psi'_\tau = \begin{cases} 
0, & \tau < k \\
\cos \sqrt{s(\tau^2 - k^2)}, & n = 2m \\
\frac{1}{2} \pi J_0(\sqrt{s(\tau^2 - k^2)}), & \tau > k 
\end{cases} \]

Now in formul (??), (??) and (??) to take \( \Phi(y, x, k) = \Phi_\tau(y - x, k) \), we construct matrix \( \Pi(y, x) = \Pi(y, x, \tau) \).

From Lemma 1 we obtain.

**Lemma 2.** The matrix \( \Pi(y, x, \tau) \) given by (??) and (??) is Carleman’s matrix for problem (??),(??).

**Proof.** By (??), (??), (??) and Lemma 1 we have

\[ \Pi(y, x, \tau) = \Psi(y, x) + G(y, x, \tau), \]

where

\[ G(y, x, \tau) = \begin{bmatrix} G^{(1)}(y, x, \tau) & G^{(2)}(y, x, \tau) \\
G^{(3)}(y, x, \tau) & G^{(4)}(y, x, \tau) \end{bmatrix}, \]

\[ G^{(i)}(y, x, \tau) = \begin{bmatrix} G^{(i)}_{k,j}(y, x, \tau) \end{bmatrix}_{n \times n}, \quad i = 1, 2, 3, 4, \]

\[ G^{(1)}_{k,j}(y, x, \tau) = \sum_{l=1}^{4} (\delta_{k,j} \alpha_l + \beta_l \frac{\partial^2}{\partial x_k \partial x_j}) g_{n}(y, x, k_l, \tau), \quad k, j = 1, ..., n \]

\[ G^{(2)}_{k,j}(y, x, \tau) = G^{(3)}_{k,j}(y, x, \tau) = \]

\[ = \frac{2\alpha}{\mu + \alpha} \sum_{l=1}^{4} \sum_{p=1}^{n} \varepsilon_{k,j} \frac{\partial^2}{\partial x_k \partial x_j} g_{n}(y, x, k_l, \tau), \quad k, j = 1, ..., n, \]

\[ G^{(4)}_{k,j}(y, x, \tau) = \sum_{l=1}^{4} (\delta_{k,j} \gamma_l + \delta_l \frac{\partial^2}{\partial x_k \partial x_j}) g_{n}(y, x, k_l, \tau), \quad k, j = 1, ..., n \]

By a straightforward calculation, we can verify that the matrix \( G(y, x, \tau) \) satisfies system (??) with respect to the variable \( y \) everywhere in \( D \). By using (??), (??), and (??) we obtain

\[ \int_{\partial D \setminus S} (|\Pi(y, x, \tau)| + |T(\partial y, n)\Pi(y, x, \tau)|) \, ds_y \leq C_1(x) \tau^m \exp(-\tau x_n), \]  

(13)

where \( C_1(x) \) some bounded function inside \( D \). The lemma is thereby proved.

Let us set
\[ U_\tau(x) = \int_S [\Pi(y, x, \tau)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \tau)\}^*U(y)]ds_y. \quad (14) \]

The following theorem holds.

**Theorem 1.** Let \( U(x) \) be a regular solution of system (14) in \( D \) such that

\[ |U(y)| + |T(\partial_y, n)U(y)| \leq M, \quad y \in \partial D \setminus S. \quad (15) \]

Then for \( \tau \geq 1 \) the following estimate is valid:

\[ |U(y) - U_\tau(y)| \leq MC_2(x)\tau^m \exp(-\tau x_n). \]

**Proof.** By formula (14) and (15), we have

\[ |U(x) - U_\tau(x)| = \int_{\partial D \setminus S} [\Pi(y, x, \tau)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \tau)\}^*U(y)]ds_y. \]

Now on the basis of (15) and (15) we obtain required inequality. The theorem is thereby proved.

Now we write out a result that allows us to calculate \( U(x) \) approximately if, instead of \( U(y) \) and \( T(\partial_y, n)U(y) \), their continuous approximations \( f_\delta(y) \) and \( g_\delta(y) \) are given on the surface \( S \):

\[ \max_S |f(y) - f_\delta(y)| + \max_S |T(\partial_y, n)U(y) - g_\delta(y)| \leq \delta, \quad 0 < \delta < 1. \quad (16) \]

We define a function \( U_{\tau, \delta}(x) \) by setting

\[ U_{\tau, \delta}(x) = \int_S [\Pi(y, x, \tau)g_\delta(y) - \{T(\partial_y, n)\Pi(y, x, \tau)\}^*f_\delta(y)]ds_y, \quad (17) \]

where

\[ \tau = \frac{1}{x_n^0} \ln \frac{M}{\delta}, \quad x_n^0 = \max_D x_n, \quad x_n > 0. \]

**Theorem 2.** Let \( U(x) \) be a regular solution of system (14) in \( D \) satisfying condition (15). Then the following estimate is valid:

\[ |U(x) - U_{\tau, \delta}(x)| \leq C_3(x)\delta^m \left( \ln \frac{M}{\delta} \right)^m, \quad x \in D. \]

**Proof.** From formula (15) and (15) we have

\[ U(x) - U_{\tau, \delta}(x) = \int_S [\Pi(y, x, \tau)\{T(\partial_y, n)U(y) - g_\delta(y)\} - \]

8
−\{T(\partial_y, n)\Pi(y, x, \tau)(U(y) - f_0(y))\}ds_y + 
\int_{\partial D \setminus S} [\Pi(y, x, \tau)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \tau)\}^*U(y)]ds_y.

By the assumption of the theorem and inequalities (7), (8) and (9) for any \(x \in D\), we obtain

\[|U(x) - U_\tau(x)| = C_2(x)\delta \tau^m \exp \tau(x^0_n - x_n) + C_1(x)\tau^m \exp(-\tau x_n) \leq\]

\[\leq C_3(x)\tau^m (M + \delta \exp \tau x^0_n) \exp(-\tau x_n).\]

Now, it to take \(\tau = \frac{1}{x_n} \ln \frac{M}{\delta}\), then we obtain to proof theorem. The theorem is thereby proved.

**Theorem 3.** Let \(U(x)\) be a regular solution of system (7) in \(D\) satisfying conditions

\[|U(y)| + |T(\partial_y, n)U(y)| \leq M, \ y \in \partial D \setminus S,\]

\[|U(y)| + |T(\partial_y, n)U(y)| \leq \delta, \ 0 < \delta < 1, \ y \in S.\]

Then

\[|U(x)| \leq C_4(x)\delta \tau^m \left(\ln \frac{M}{\delta}\right)^m,
\]

where \(C_4(x) = \tilde{C} \int_{\partial D} \frac{1}{\tau} ds_y\), \(\tilde{C}\) – constant depending on \(\lambda, \mu, \varepsilon, \beta, \nu\).

**Proof.** On the basis of Theorem 2 we obtain

\[|U(x)| \leq |U_\tau(x)| + MC_2(x)\tau^m \exp(-\tau x_n).\]

Next from the condition theorem and (7), (2) we obtain

\[|U_\tau(x)| = \int_S [\Pi(y, x, \tau)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \tau)\}^*U(y)]ds_y \leq\]

\[\leq \int_S (|\Pi(y, x, \tau)| + |T(\partial_y, n)\Pi(y, x, \tau)|) (|U(y)| + |T(\partial_y, n)U(y)|) ds_y \leq\]

\[\leq \delta \int_S (|\Pi(y, x, \tau)| + |T(\partial_y, n)\Pi(y, x, \tau)|) ds_y \leq C_3(x)\delta \tau^m \exp(\tau x^0_n - \tau x_n).\]
Then

$$|U(x)| \leq C_4(x)\tau^m \exp(-\tau x_n)(M + \delta \exp \tau x_n^0).$$

Next if we take $\tau = \frac{1}{x_n^0} \ln M$, then we obtain stability estimate:

$$|U(x)| \leq C_4(x)\delta^{\frac{x_n}{m}} \left(\ln \left(\frac{M}{\delta}\right)\right)^m.$$

The theorem is thereby proved.

From proved above theorems we obtain

Corollary 1. The limit relation

$$\lim_{\tau \to \infty} U_\tau(x) = U(x), \quad \lim_{\delta \to 0} U_{\tau \delta}(x) = U(x)$$

hold uniformly on each compact subset of $D$.

3. Regularization of solution of the problem (??), (??) for a domain of cone type

Let $x = (x_1, x_n)$ and $y = (y_1, y_n)$ be points in $E^n$, $D_\rho$ be a bounded simply connected domain in $E^n$ whose boundary consists of a cone surface

$$\Sigma : \quad \alpha_1 = \tau y_n, \quad \alpha_2 = y^2_1 + \cdots + y^2_{n-1}, \quad \tau = \frac{\pi}{2\rho}, \quad y_n > 0, \quad \rho > 1$$

and a smooth surface $S$, lying in the cone. Assume $x_0 = (0, \ldots, 0, x_n) \in D_\rho$.

We construct Karleman matrix. In formula (??), (??) and (??) to take

$$K(\omega) = E_\rho[\tau (\omega - x_n)], \quad \tau > 0, \quad \rho > 1.$$  

Then

$$\Phi(y, x, k) = \Phi_\tau(y - x, k), k > 0$$

$$C_n\Phi_\tau(y - x, k) = \frac{\partial^{m-1}}{\partial s^{m-1}} \int_0^\infty \text{Im} \left\{ E_\rho \left[ \tau (i\sqrt{u^2 + s} + y_n - x_n) \right] \right\} \frac{\psi(ku) du}{\sqrt{u^2 + s}}$$  \hspace{1cm} (18)

$$\Phi'_\tau(y - x, k) = \frac{\partial \Phi_\tau}{\partial \tau}$$

$$C_n\Phi'_\tau(y - x, k) = \frac{\partial^{m-1}}{\partial s^{m-1}} \int_0^\infty \text{Im} \left\{ E_\rho \left[ \tau (i\sqrt{u^2 + s} + y_n - x_n) \right] \right\} \frac{\psi(ku) du}{\sqrt{u^2 + s}}$$  \hspace{1cm} (19)

where $E_\rho(w)$— Mittag-Liffer’s a entire function [5]. For the functions $\Phi_\tau(y - x, k)$ holds Lemma 1 and Lemma 2.

Now again to denote by $U_\tau(x)$, $U_{\tau \delta}(x)$ as (??) and (??). Then holds analogical theorem as Theorem 1, 2, 3.
For \( n = 3 \) we reduce entirely.

Suppose that \( D_\rho \) is bounded simple connected domain from \( E^3 \) with boundary consisting of part \( \Sigma \) of the surface of the cone

\[
y_1^2 + y_2^2 = \tau y_3^2, \quad \tau = \frac{\pi}{2\rho}, \quad \rho > 1, \quad y_3 > 0,
\]

and of a smooth portion of the surface \( S \) lying inside the cone. Assume \( x_0 = (0, 0, x_3) \in D_\rho \).

We construct Carleman’s matrix. In formula (??), (??) we take

\[
\Phi_{\tau}(y, x, k) = \frac{1}{4\pi^2 E_\rho(\tau\frac{\pi}{2}x_3)} \int_0^\infty \frac{E_\rho(\frac{\pi}{2}u)}{i\sqrt{u^2 + s + y_3 - x_3}} \cos ku \, du,
\]

where \( w = i\sqrt{u^2 + s + y_3}, \quad E_\rho(w) - \text{Mittag-Leffler’s a entire function.} \)

For the functions \( \Phi_{\tau}(y, x, k) \) holds Lemma 1.

If follows from the properties of \( E_\rho(w) \) that for \( y \in \Sigma, \quad 0 < u < \infty \) the function \( \Phi_{\tau}(y, x, k) \) defined by (??) its gradient and second partial derivatives tend to zero as \( \tau \to \infty \) for a fixed \( x \in D_\rho \).

Then from (??) we find that the matrix \( \Pi(y, x, \tau) \) and its stresses \( T(\partial_y, n)\Pi(y, x, \tau) \) also tend to zero as \( \tau \to \infty \) on \( y \in \Sigma \), i.e., \( \Pi(y, x, \tau) \) is the Carleman matrix for the domain \( D_\rho \) and the part \( \Sigma \) of the boundary.

For the \( U(x) \)– regular solution system (??) following integral formula holds

\[
U(x) = \int_{\partial D_\rho} [\Pi(y, x, \tau)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \tau)\}^* U(y)] \, ds_y.
\]

By \( x \in D_\rho \) we denote \( U_\tau(x) \) follows:

\[
U_\tau(x) = \int_S [\Pi(y, x, \tau)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \tau)\}^* U(y)] \, ds_y.
\]

The following theorem holds.

**Theorem 4.** Let \( U(x) \) be a regular solution of system (??) in \( D_\rho \) such that

\[
|U(y)| + |T(\partial_y, n)U(y)| \leq M, \quad y \in \Sigma.
\]

Then for \( \tau \geq 1 \) the following estimate is valid:

\[
|U(x_0) - U_\tau(x_0)| \leq MC_\rho(x_0)\tau^3 \exp(-\tau x_3''),
\]

where \( x_0 = (0, 0, x_3) \in D_\rho, \quad x_3 > 0, \)
\[ C_\rho(x_0) = C_\rho \int_{\Sigma} \frac{1}{r_0} ds_y, \quad r_0 = |y - x_0|, \quad C_\rho - constant. \]

**Proof.** By analogy with proved Theorem 2 and Theorem 3 from (??) and (??) we obtain

\[ |U(x_0) - U_\tau(x_0)| \leq M \int_{\Sigma} [||\Pi(y, x_0, \tau)| + |T(\partial y, n)\Pi(y, x_0, \tau)||] ds_y. \]

By formula (??) we have following inequality:

\[ |\Phi_\tau(y, x, k)| \leq C^{(1)}_\rho \tau E^{-1} (\tau^{1/4}) r^{-1}, \]

\[ \left| \frac{\partial \Phi_\tau(y, x, k)}{\partial y_i} \right| \leq C^{(2)}_\rho \tau \tau E^{-1} (\tau^{1/4}) r^{-2}, \]

\[ \left| \frac{\partial^2 \Phi_\tau(y, x, k)}{\partial y_k \partial y_j} \right| \leq C^{(3)}_\rho \tau^2 E^{-1} (\tau^{1/4}) r^{-3}. \]

Then from (??)

\[ |\Pi(y, x, \tau)| \leq C^{(4)}_\rho \tau^2 E^{-1} (\tau^{1/4}) r^{-3}, \]

\[ |T(\partial y, n)\Pi(y, x, \tau)| \leq C^{(5)}_\rho \tau^3 E^{-1} (\tau^{1/4}) r^{-4}. \]

Therefore we obtain

\[ |U(x_0) - U_\tau(x_0)| \leq MC_\rho(x_0) \tau^3 \exp(-\tau x^3_0), \]

where

\[ C_\rho(x_0) = C_\rho \int_{\Sigma} \frac{1}{r_0} ds_y, \quad r_0 = |y - x_0|, \quad C_\rho - constant. \]

The theorem is thereby proved.

Suppose that instead of \( U(y) \) and \( T(\partial y, n)U(y) \) gives their continuous approximations \( f_\delta(y) \) and \( g_\delta(y) \) such that

\[ \max_S |U(y) - f_\delta(y)| + \max_S |T(\partial y, n)U(y) - g_\delta(y)| \leq \delta, \quad 0 < \delta < 1. \]

Define the function \( U_{\tau \delta}(x) \) by

\[ U_{\tau \delta}(x) = \int_S [\Pi(y, x, \tau)g_\delta(y) - \{T(\partial y, n)\Pi(y, x, \tau)\}^* f_\delta(y)] ds_y. \]

The following theorem holds
Theorem 5. Let $U(x)$ is a regular solution of system (??) in the domain $D_\rho$ satisfying the condition (??), then

$$|U(x_0) - U_{\tau \delta}(x_0)| \leq C_\rho(x_0)\delta^q(\ln \frac{M}{\delta})^3,$$

where $\tau = (\tau R)^{-p} \ln \frac{M}{\delta}$, $R^p = \max \frac{\Re(\sqrt{s} + y\delta)\rho}{s}$,

$$q = (\frac{\gamma}{R})^p, \quad C_\rho(x_0) = C_\rho \int_\Sigma \left[ \frac{1}{r_0^3} + \frac{1}{r_0^4} \right] ds_y.$$

The proof theorem is similar to those of Theorem 3 and 4.

Corollary 2. The limit relation

$$\lim_{\tau \to \infty} U_\tau(x) = U(x), \quad \lim_{\delta \to 0} U_{\tau \delta}(x) = U(x)$$

hold uniformly on each compact subset of $D_\rho$.

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