$H^o$-type Riemannian metrics on the space of planar curves

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Abstract

Michor and Mumford have shown that the distances between planar curves in the simplest metric (not involving derivatives) are identically zero. We consider two conformally equivalent metrics for which the distances between curves are nontrivial. We show that in the case of the simpler of the two metrics, the only minimal geodesics are those corresponding to curve evolution in which the points of the curve move with the same normal speed. An equation for the geodesics and a formula for the sectional curvature are derived; a necessary and sufficient condition for the sectional curvature to be bounded is given.

1 Introduction

The purpose of this paper is to study the most basic properties of some of the simplest Riemannian metrics suggested by applications to Computer Vision. The problem is to understand and quantify similarities and differences between object shapes and their individual variations. At a fundamental level, the problem is to construct appropriate metrics on a space of closed surfaces in $\mathbb{R}^3$. A simpler version of the problem is the construction of Riemannian metrics on a space of closed planar curves. The choice of a metric depends on the type of similarity that is being considered. In their seminal paper [2], Michor and Mumford analyze two Riemannian metrics on a space of closed planar curves. Surprisingly, the Riemannian distance between any two curves in the simpler of the two metrics, an $H^o$-metric, turns out to be zero. To remedy this, they

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add a curvature term to the metric. Alternative construction of Riemannian metrics on the space of curves of constant length and parametrized by either the orientation of the tangent vector or the curvature is given by Klassen et al in [3]. A method for deforming one curve onto another by minimizing an approximate Hausdorff distance between them is described in [1]. Below, we analyse two conformal variants of the $H^o$-metric of Michor and Mumford. In §3, we derive upper and lower bounds for distances between curves and show that these metrics behave like $L^1$ metrics. A key issue is the existence of minimal geodesics. We show that for the simpler of the two conformal metrics, the minimal geodesics correspond exactly to those curve deformations in which the points of the curve move with the same normal speed. In the case of the second metric, no geodesic is minimal if the length of the curve is less than a certain threshold; the question of minimality when the length of the deforming curve is equal or greater than threshold is still open. We provide a partial answer in the form of a necessary and sufficient condition for the boundedness of the sectional curvature in §5. An equation for the geodesics is derived in §4.

This paper began with an analysis of the simpler of the two conformal metrics considered here. As it was being written, the author became aware of the work of A. Yezzi and A. Mennucci [4,5] in which they have proposed a more general formulation. Since our analysis applies to their formulation as well, we have included their formulation in the analysis below.

2 The Framework

The basic space considered by Michor and Mumford is the orbit space

$$B_c(S^1, \mathbb{R}^2) = \text{Emb}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$$

defined on the space of all $C^\infty$ embeddings of $S^1$ in the plane, under the action by composition from the right by diffeomorphisms of the unit circle. It is contained in the bigger space of immersions modulo diffeomorphisms:

$$B_i(S^1, \mathbb{R}^2) = \text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$$

Let $\pi : \text{Imm}(S^1, \mathbb{R}^2) \to B_i(S^1, \mathbb{R}^2)$ be the canonical projection. The simpler of the two metrics considered in [2] is an $H^o$-metric defined on $\text{Imm}(S^1, \mathbb{R}^2)$:

$$G^o_c(h, k) = \int_{S^1} (h \cdot k) |c_\theta| d\theta$$

where $c : S^1 \to \mathbb{R}^2$ is an immersion, defining a point in $\text{Imm}(S^1, \mathbb{R}^2)$, $h, k \in C^\infty(S^1, \mathbb{R}^2)$ are the vector fields along the image curve, defining two tangent vectors in $\text{Imm}(S^1, \mathbb{R}^2)$ at $c$, and $c_\theta = dc/d\theta$. $(h \cdot k)$ is the usual dot product in $\mathbb{R}^2$. Sometimes for the sake of clarity, we will use the notation $a \cdot b$ even when $a, b$ are scalars. Let $n_c$ denote the unit normal field along $c$. If we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, then, $n_c = ic_\theta/|c_\theta|$. For any $C_o, C_1 \in B_i$, consider
all liftings $c_o, c_1$ to $Imm(S^1, \mathbb{R}^2)$ and all smooth paths $t \rightarrow (\theta \mapsto c(t, \theta))$, $0 \leq t \leq 1$, in $Imm(S^1, \mathbb{R}^2)$ with $c(0, \cdot) = c_o$ and $c(1, \cdot) = c_1$. Let $c_t$ denote $\partial c/\partial t$ and $c_t^\perp = (c_t \cdot n_c) n_c$. The arc-length of such a path $c$ is given by

$$
\int_0^1 \sqrt{G_o^c(c_t, c_t)t} dt
$$

Michor and Mumford show that for any two curves in $B_i(S^1, \mathbb{R}^2)$,

$$
dist_{G_o}(C_1, C_2) = \inf_c \int_0^1 \sqrt{G_o^c(c_t^\perp, c_t^\perp)} dt = 0
$$

and strengthen $G_o$ by defining

$$
G_o^A(h, k) = \int_{S^1} (1 + A\kappa_c^2) (h \cdot k) |c_\theta| d\theta
$$

where $\kappa_c$ is the curvature, defined by the equation $\left(\frac{c_{\theta\theta}}{|c_\theta|}\right)_\theta = i\kappa_c c_\theta = \kappa_c |c_\theta| n_c$.

An alternative is to consider conformal transformations of $G_o$. They have the form

$$
G_o^\Phi(h, k) = \Phi(c) \int_{S^1} (h \cdot k) |c_\theta| d\theta
$$

where $\Phi$ is a $Diff(S^1)$-invariant function on $Imm(S^1, \mathbb{R}^2)$. For example,

$$
\Phi(c) = \int_{S^1} |c_\theta| d\theta \quad \text{or} \quad \Phi(c) = \int_{S^1} (1 + A\kappa_c^2) |c_\theta| d\theta
$$

Based on considerations of stability, Yezzi and Mennucci have proposed a conformal factor of the form $e^{A\ell}$ where $\ell = \int_{S^1} |c_\theta| d\theta$, or more generally, a function $\varphi(\ell)$ of $\ell$. In this paper, we consider only these conformal factors:

$$
G_o^\varphi(h, k) = \varphi(\ell) \int_{S^1} (h \cdot k) |c_\theta| d\theta \quad (2)
$$

If $c(t, \theta)$ is a smooth path in $Imm(S^1, \mathbb{R}^2)$ connecting $c_o, c_1$, let

$$
L_{G_o^\varphi}(c) = \int_0^1 \sqrt{G_o^\varphi(c_t^\perp, c_t^\perp)} dt \quad (3)
$$

If $C_o, C_1 \in B_i(S^1, \mathbb{R}^2)$, define $dist_{G_o^\varphi}(C_1, C_2)$ as follows. Consider their lifts $c_o, c_1$ to $Imm(S^1, \mathbb{R}^2)$ and all paths $c(t, \theta)$ such that $\pi(c_o) = C_o$ and $\pi(c_1) = C_1$. Then

$$
\tag{4}
dist_{G_o^\varphi}(C_1, C_2) = \inf_c L_{G_o^\varphi}(c)
$$
3 Bounds on $\text{dist}_{G^\varphi}$

If $c$ is a path connecting curves $C, E$, let $\alpha(c)$ denote the area swept out by $c$ in $\mathbb{R}^2$. For a path $c(t, \cdot)$, let $\ell_{\text{max}}(c) = \max_t \ell(c(t, \cdot))$. The following theorem characterizes the $L^1$-type behavior of the metrics $G^\varphi$.

**Theorem 1** If $\varphi(\ell) = \ell$, then,

$$\text{dist}_{G^\varphi}(C, E) = \inf_c \alpha(c)$$  \hspace{1cm} (5)

If $\varphi(\ell) = e^{A\ell}$, then,

$$\inf_c \sqrt{A}e^{\alpha(c)} \leq \text{dist}_{G^\varphi}(C, E) \leq \inf_c \sqrt{A}e^{A\ell_{\text{max}}(c)/2\alpha(c)}$$  \hspace{1cm} (6)

We first prove a series of lemmas.

**Lemma 2**

$$\text{dist}_{G^\varphi}(C, E) \geq \begin{cases} \inf_c \alpha(c) & \text{if } \varphi(\ell) = \ell, \\ \inf_c \sqrt{A}e^{\alpha(c)} & \text{if } \varphi(\ell) = e^{A\ell} \end{cases}$$

**Proof:** For any path $c$,

$$L_{G^\varphi}(c) = \int_0^1 \left[ \varphi(\ell) \left( \int_{S^1} (c^\perp_{\ell} \cdot c^\perp_{\ell}) |c_\theta|d\theta \right) \right] ^\frac{1}{2} dt$$

$$\geq \int_0^1 \left[ \left( \frac{\varphi(\ell)}{|\text{supp}(c^\perp_{\ell})|} \right) ^\frac{1}{2} \int_{S^1} |c^\perp_{\ell}||c_\theta|d\theta \right] dt$$

$$\geq \left[ \min_t \frac{\varphi(\ell)}{\ell} \right] ^\frac{1}{2} \int_{S^1 \times [0,1]} |\det dc(t, \theta)|d\theta dt$$

$$\geq \begin{cases} \alpha(c) & \text{if } \varphi(\ell) = \ell, \\ \sqrt{A}e^{\alpha(c)} & \text{if } \varphi(\ell) = e^{A\ell} \end{cases}$$

Q.E.D.

David Mumford observed from the formula for the sectional curvature that the geodesics along which $|\text{supp}(c^\perp_{\ell})| < \ell$ if $\varphi(\ell) = \ell$ and $< 1/A$ if $\varphi(\ell) = e^{A\ell}$ may not be minimal. Such a possibility can be heuristically seen from the inequality

$$L_{G^\varphi}(c) \geq \int_0^1 \left[ \left( \frac{\varphi(\ell)}{|\text{supp}(c^\perp_{\ell})|} \right) ^\frac{1}{2} \int_{S^1} |c^\perp_{\ell}||c_\theta|d\theta \right] dt$$

which suggests that while traversing a given area, one should try to minimize $\varphi(\ell)/|\text{supp}(c^\perp_{\ell})|$. The key point is that we can increase $|\text{supp}(c^\perp_{\ell})|$ indefinitely by replacing the part of the curve supporting $c^\perp_{\ell}$ by a saw-tooth shaped curve of high frequency and small amplitude. When $\varphi(\ell) = \ell$ and $|\text{supp}(c^\perp_{\ell})| < \ell$, 

we can increase $|\text{supp}(c_1^+)|$ so that $\ell/|\text{supp}(c_1^+)|$ tends to 1. When $\varphi(\ell) = e^{A\ell}$ and $|\text{supp}(c_1^+)| < 1/A$, we can force $e^{A|\text{supp}(c_1^+)|}/|\text{supp}(c_1^+)|$ to equal its unique minimum $A\ell$ by making $|\text{supp}(c_1^+)|$ equal $1/A$. (In the case of the metric $G^0$, $\varphi(\ell) = 1$ so that $\varphi(\ell)/|\text{supp}(c_1^+)|$ tends to 0.) In order to obtain an upper bound for a general path, we break it up into a series of tiny bumps. When $\varphi(\ell) = \ell$, this method gives an upper bound for $\text{dist}_{G^0}(C, E)$ which coincides with the lower bound. When $\varphi(\ell) = e^{A\ell}$, the larger the value of $|\text{supp}(c_1^+)|$, the greater the divergence between the upper bound obtained by this method and the lower bound since it is more efficient to create a large bump all at once instead a series of tiny bumps.

**Rectangular Bumps**
Let $c_o : S^1 \to \mathbb{R}^2$ be a smooth and free immersion. Let $C_o$ be the corresponding curve in $\mathbb{R}^2$. Let $c_o$ be parametrized by the arclength so that $\theta$ parametrizes the scaled circle $S^1_{\ell_o}$ where $\ell_o$ is the length of $C_o$. For any function $u(\theta)$, let $u'$ denote $du/d\theta$. Let $n_o$ denote the normal vector $ic'_o$. Let $\kappa_o$ denote the curvature of $c_o$.

Fix small positive numbers $\delta$ and $\epsilon$ such that $\delta < \ell_o$ and $\epsilon \|\kappa_o\|_{\infty,[0,\delta]} << 1$. Construct a "rectangular" bump, over $C_o$ as follows:

$$c_1(\theta) = \begin{cases} 
c_o(\theta) + \epsilon n_o & \text{if } 0 < \theta < \delta \\
c_o(\theta) + s n_o |0 \leq s \leq \epsilon & \text{if } \theta = 0, \delta \\
c_o(\theta) & \text{otherwise}
\end{cases}$$

Let $C_1$ be the corresponding curve in $\mathbb{R}^2$. The following lemma is inspired by a comment of David Mumford that the Michor-Mumford "teeth" construction [2] may be used to show that the obvious path for creating such a bump is not minimal.

**Lemma 3** (i) If $\varphi(\ell) = \ell$,

$$\text{dist}_{G^0}(C_o, C_1) \leq \left[1 - \epsilon \|\kappa_o\|_{\infty,[0,\delta]} \right]^{-2} (\text{area of the bump})$$

(ii) If $\varphi(\ell) = e^{A\ell}$ and $\delta < 1/A$,

$$\text{dist}_{G^0}(C_o, C_1) \leq \left[1 + \epsilon \|\kappa_o\|_{\infty,[0,\delta]} \right]^{-3/2} \sqrt{\frac{1 + \epsilon \|\kappa_o\|_{\infty,[0,\delta]} \sqrt{A e A(\ell_o + 2\epsilon - \delta)^2}}{\ell_o} (\text{bump area})}$$

**Proof:** We prove the lemma using a modification of the "teeth" construction of Michor and Mumford [2]. If $\varphi(\ell) = \ell$, choose $A < 1/\delta$. Approximate $C_1$ by a "trapezoidal" bump $C_1$ as follows. Replace $C_0$ in the interval $[0, \delta]$ by a saw-tooth curve of height $\eta$ and period $1/\delta$ such that its length equals $\frac{1}{\delta}$. This is done by growing teeth on $C_0$ in time $\eta$. Move the saw-tooth curve at
unit speed along the normals \( n_0 \) keeping its end-points fixed, until it touches the upper edge of the bump. Finally, retract the teeth in time \( \eta \). Formally, define a path \( c(t, \theta) = c_o(\theta) + f(t, \theta)n_0 \) where \( f(t, \theta) \) is defined as follows.

\[
f(t, \theta) = 0, \quad 0 \leq t \leq \epsilon \text{ and } \delta \leq \theta \leq \ell_o
\]

For \( 0 \leq t \leq \eta \), \( 0 \leq k \leq m-1 \),

\[
f(t, \theta) = \begin{cases} 
  t \left( \frac{2n \theta}{2n} - 2k \right) & \frac{2k}{2n} \leq \frac{\theta}{\delta} \leq \frac{2k + 1}{2n} \\
  t \left( \frac{2n \theta}{2n} + 2 - 2n \theta \right) & \frac{2n}{2n} \leq \frac{\theta}{\delta} \leq \frac{3n}{2n} + 2
\end{cases}
\]

For \( \eta \leq t \leq \epsilon - \eta \),

\[
f(t, \theta) = \begin{cases} 
  \frac{\epsilon(t-n) + \eta(\epsilon-n-t)}{\epsilon - 2n} \cdot \frac{2m \theta}{2m} & 0 \leq \frac{\theta}{\delta} \leq \frac{1}{2n} \\
  \frac{\epsilon(t-n) + \eta(\epsilon-n-t)}{\epsilon - 2n} \cdot \frac{2m}{2m} \left( 1 - \frac{\theta}{\delta} \right) & 1 - \frac{1}{2m} \leq \frac{\theta}{\delta} \leq 1
\end{cases}
\]

For \( \epsilon - \eta \leq t \leq \epsilon \),

\[
f(t, \theta) = \begin{cases} 
  2m \epsilon \theta & 0 \leq \frac{\theta}{\delta} \leq \frac{1}{2m} \\
  2m \epsilon \left( 1 - \frac{\theta}{\delta} \right) & 1 - \frac{1}{2m} \leq \frac{\theta}{\delta} \leq 1
\end{cases}
\]

\[
c' = c'_o + f'n_o - f\kappa \kappa'_o = (1 - f\kappa)_o c'_o + f'n_o
\]

\[
|c'| = \sqrt{(1 - f\kappa)^2 + f'^2}
\]

\[
n = \frac{-f' c'_o + (1 - f\kappa)n_o}{|c'|}
\]

\[
c_t = f_t n_o
\]

\[
c_t^\perp \cdot c_t^\perp = (c_t \cdot n)^2 = \frac{(1 - f\kappa)^2 f^2}{|c'|^2}
\]

Let

\[
\beta = \sqrt{\left( 1 + \epsilon \|\kappa\|_{\infty, [0, \delta]} \right)^2 + f'^2}
\]

\[
1 \leq \beta \leq \frac{1 + \epsilon \|\kappa\|_{\infty, [0, \delta]}}{1 - \epsilon \|\kappa\|_{\infty, [0, \delta]}}
\]

Choose \( m \) and \( \eta \) such that \( \int_0^\delta |c'(\eta, \theta)|d\theta = \frac{1}{\lambda} \). Note that as \( m \to \infty \), \( \eta \to 0 \).
Estimates when \( 0 \leq t \leq \eta \): Since \( |f'| \) is independent of \( \theta \) and \( |f| \leq \epsilon \),

\[
\sqrt{(1 - \epsilon \|\kappa_0\|_{\infty,[0,\delta]})^2 + f^2} \leq |c'(\eta, \theta)| \leq \sqrt{(1 + \epsilon \|\kappa_0\|_{\infty,[0,\delta]})^2 + f^2}
\]

\[
\delta \sqrt{(1 - \epsilon \|\kappa_0\|_{\infty,[0,\delta]})^2 + f^2} \leq \int_0^\delta |c'(\eta, \theta)| d\theta = \frac{1}{A} \leq \delta \sqrt{(1 + \epsilon \|\kappa_0\|_{\infty,[0,\delta]})^2 + f^2}
\]

We also have

\[
\sqrt{\frac{(1 - \epsilon \|\kappa_0\|_{\infty,[0,\delta]})^2 + f^2}{\beta}} \leq |c'(t, \theta)| \leq \sqrt{(1 + \epsilon \|\kappa_0\|_{\infty,[0,\delta]})^2 + f^2}
\]

\[
\frac{1}{\beta} \sqrt{\frac{(1 + \epsilon \|\kappa_0\|_{\infty,[0,\delta]})^2 + f^2}{\delta}} \leq |c'(t, \theta)| \leq \frac{\beta}{A \delta}
\]

\[
\ell(c) = \int_0^\ell_o |c'| d\theta \leq (\ell_o - \delta) + \frac{\beta}{A}
\]

\[
e^{A \ell(c)} \leq e^{A(\ell_o - \delta)} e^\beta
\]

Since \( |f_i| \leq 1 \),

\[
\int_0^\ell_o |c^+_i|^2 |c'| d\theta \leq \int_0^\ell_o \frac{(1 - f \kappa_0)^2 f^2}{|c'|} d\theta \leq (1 + \epsilon \|\kappa_0\|_{\infty,[0,\delta]})^2 \beta A \delta^2
\]

\[
\lim_{m \to \infty} \int_0^\eta \left[ \varphi(\ell(c)) \int_o^{\ell_o} |c^+_i|^2 |c'| d\theta \right]^{1/2} dt = 0
\]

Estimates when \( \eta \leq t \leq \epsilon - \eta \): Estimate for \( |c'(t, \theta)| \) is the same as in the interval \([\delta(1 - \frac{1}{2\epsilon} \delta^2), \delta] \) since the curve has the same shape. In the intervals \([0, \frac{\delta}{2m}], \delta(1 - \frac{1}{2\epsilon} \delta^2) \), \( \frac{2m\eta}{\delta^2} \leq |f'(t, \theta)| \leq \frac{2m\epsilon}{\delta} \). Therefore, \( |c'(\eta, \theta)| \leq 1 + \epsilon \|\kappa_0\|_{\infty,[0,\delta]} + \frac{2m\epsilon}{\delta} \).

\[
\ell(c) = \int_0^\ell_o |c'| d\theta \leq (\ell_o - \delta) + \left( 1 + \epsilon \|\kappa_0\|_{\infty,[0,\delta]} + \frac{2m\epsilon}{\delta} \right) \frac{\delta}{m} + \frac{\beta}{A}
\]

\[
\lim_{m \to \infty} \ell(c) \leq (\ell_o - \delta) + 2\epsilon + \frac{1 + \epsilon \|\kappa_0\|_{\infty,[0,\delta]} - \frac{1}{A}}{1 + \epsilon \|\kappa_0\|_{\infty,[0,\delta]} - \frac{1}{A}} + \frac{\beta}{A}
\]

\[
\lim_{m \to \infty} e^{A \ell(c)} \leq e^{A(\ell_o - \delta)} e^{\frac{\epsilon - \eta}{\epsilon - 2\eta}}
\]

We also have \( |f_i| \leq \frac{\epsilon - \eta}{\epsilon - 2\eta} \). Therefore,

\[
\int_0^{\ell_o} |c^+_i|^2 |c'| d\theta \leq \left( 1 + \epsilon \|\kappa_0\|_{\infty,[0,\delta]} \right)^2 \left( \frac{\epsilon - \eta}{\epsilon - 2\eta} \right)^2 \beta A \delta^2
\]
Lemma 4

For any pair

\[ C_1, C_2 \in B_i(S^1, \mathbb{R}^2), \quad \text{dist}_{G^*} (C_1, C_2) \leq d_{\infty}(C_1, C_2) \cdot \max \{ |\varphi(\ell_1)|, |\varphi(\ell_2)| \} \quad (7) \]

where \( \ell_i = \ell(C_i), i = 1, 2. \)
Proof: Let $c_1, c_2$ be lifts of $C_1, C_2$ to $Imm(S^1, \mathbb{R}^2)$. Let $c(t, \theta) = (1 - t)c_1(\theta) + tc_2(\theta)$ be a path connecting them. Then, $|c_\theta(t)| \leq (1 - t)|c_1, \theta| + t|c_2, \theta|$ and hence, $\ell(c(t)) \leq \max\{\ell(C_1), \ell(C_2)\}$. Moreover, $c_\ell = c_2 - c_1$. Therefore,

$$\text{dist}_{G^\theta}(C_1, C_2) \leq \inf_c L_{G^\theta}(c)$$

$$\leq \inf_{\{\text{pairs } c_1, c_2\}} \left\{ \sup_{\theta} |c_1(\theta) - c_2(\theta)| \right\} \max\{\varphi(\ell_1), \varphi(\ell_2)\}$$

Q.E.D.

The polygonal approximations used in the proof of the theorem lie on the boundary of $Imm(S^1, \mathbb{R}^2)$ and $B_i(S^1, \mathbb{R}^2)$, and Lemma 4 extends to them.

Proof of the theorem:

Consider a path $c(t, \theta)$ connecting $C$ and $E$. Since the absolute curvature of the curves $c(t, \cdot)$ is uniformly bounded by a constant $K$, each curve has a tubular neighborhood of width which is bounded from below. Choose $\epsilon$ and a sequence

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$$

such that $\epsilon K << 1$ and, for $0 \leq k \leq N$, $c(t_{k+1}, \cdot)$ is in a local chart of $c(t_k, \cdot)$:

$$c(t_{k+1}, \theta) = c(t_k, \theta) + f_k(\theta) n_k$$

where $c(t_k, \cdot)$ is parametrized by the arclength, $n_k$ is the normal vector field of $c(t_k, \cdot)$ and $|f_k| < \epsilon$. Let $F = \max\{\|f_k(\theta)\|_\infty |0 \leq k \leq N\}$. Let $C_k$ denote $\pi(c(t_k, \cdot))$. Let $\tilde{\ell}_k = \ell(C_k)$. Let $\bar{\ell}_k = (1 + \epsilon K + F)\ell_k$.

Choose $\delta$ such that

$$\max_k \sqrt{\bar{\ell}_k \varphi(\bar{\ell}_k)} \cdot F\delta < \frac{\epsilon}{N}$$

We now estimate the distances $\text{dist}_{G^\theta}(C_k, C_{k+1})$. Consider the path segment $[c_\alpha, c_\gamma]$ from $c_\alpha$ to $c_\gamma$ in the local chart at $c_\alpha$. Divide the range of $\theta$ into intervals of length $\delta$. Replace $f_\alpha$ by a piecewise constant function $\tilde{f}_\alpha$ whose value in each subinterval equals the average of $f_\alpha$ over that interval. Let $\tilde{C}_\alpha$ be the curve defined by $\tilde{f}_\alpha$. The Frechét distance between $C_1$ and $\tilde{C}_\alpha$ is $\leq F\delta$. The sum of the jumps in $\tilde{f}_\alpha$ is $\leq F\ell_\alpha$. Since $|\kappa'_\alpha| = |1 - \epsilon K| \leq 1 + \epsilon K$, $\ell(C_\alpha) \leq (1 + \epsilon K + F)\ell_\alpha$. Therefore,

$$\text{dist}_{G^\theta}(C_1, \tilde{C}_\alpha) \leq \max\{\tilde{\ell}_\alpha \varphi(\tilde{\ell}_\alpha), \tilde{\ell}_1 \varphi(\tilde{\ell}_1)\} \cdot F\delta \leq \frac{\epsilon}{N}$$

Let $\alpha([c_\alpha, c_\gamma])$ denote the area swept out by the path $c$ during $[0, t_2]$. The area between $C_\alpha$ and $\tilde{C}_\alpha$ equals the area between $C_\alpha$ and $C_1$ which in turn is less than or equal to $\alpha([c_\alpha, c_\gamma])$. The curve $\tilde{C}_\alpha$ consists of a series of bumps over $C_\alpha$. Traverse the bumps sequentially, taking care to retract the common edge of each bump with the previous bump before going to the next bump. Retracting a common edge can be done without incurring any cost.
By Lemma 3, if \( \varphi(\ell) = \ell \),

\[
\dist_{G^e} (C_o, \bar{C}_o) \leq \left( \frac{1 + \epsilon K}{1 - \epsilon K} \right)^2 \alpha([c_0, c_1])
\]

and if \( \varphi(\ell) = e^{A\ell} \),

\[
\dist_{G^e} (C_o, \bar{C}_o) \leq \left( \frac{1 + \epsilon K}{1 - \epsilon K} \right)^{3/2} e^{A(\ell_0 + 2\epsilon)/2} \sqrt{Ae^{1+\epsilon K}} \alpha([c_0, c_1])
\]

\[
\dist_{G^e} (C_o, C_1) \leq \dist_{G^e} (C_o, \bar{C}) + \frac{\epsilon}{N}
\]

Similar estimates hold for \( \dist_{G^e} (C_k, C_{k+1}) \) for \( 0 < k < N \).

Therefore, if \( \varphi(\ell) = \ell \),

\[
\dist_{G^e} (C, E) \leq \left( \frac{1 + \epsilon K}{1 - \epsilon K} \right)^2 \alpha(c) + \epsilon
\]

and if \( \varphi(\ell) = e^{A\ell} \),

\[
\dist_{G^e} (C, E) \leq \left( \frac{1 + \epsilon K}{1 - \epsilon K} \right)^{3/2} e^{A\ell_{\max}(c)/2} \sqrt{Ae^{1+\epsilon K}} \alpha(c) + \epsilon
\]

Since \( \epsilon \) is arbitrary, we have

\[
\dist_{G^e} (C, E) \leq \begin{cases} 
\alpha(c) & \text{if } \varphi(\ell) = \ell \\
\frac{\alpha(c)}{e^{A\ell_{\max}(c)/2} \sqrt{Ae^{1+\epsilon K}}} + \epsilon & \text{if } \varphi(\ell) = e^{A\ell}
\end{cases}
\]

Q.E.D.

For any oriented curve \( C^{or} \), define the integer-valued measurable function \( w_C \) on \( \mathbb{R}^2 \) by:

\[
w_C(x, y) = \text{the winding number of } C \text{ around } (x, y)
\]

and let

\[
d^p(C^{or}_1, C^{or}_2) = \int_{\mathbb{R}^2} |w_{C_1} - w_{C_2}| dxdy
\]

It is shown in [2] that for any two oriented curves \( C^{or}_1, C^{or}_2 \),

\[
d^p(C^{or}_1, C^{or}_2) \leq \min_{\text{all paths } c \text{ joining } C_1, C_2} \alpha(c)
\]

Therefore, we have

\[
\dist_{G^e} (C^{or}_1, C^{or}_2) \geq \begin{cases} 
d^p(C^{or}_1, C^{or}_2) & \text{if } \varphi(\ell) = \ell \\
\sqrt{Ae} d^p(C^{or}_1, C^{or}_2) & \text{if } \varphi(\ell) = e^{A\ell}
\end{cases}
\]

(8)
Corollary 5 (Existence of minimal geodesics) If \( \varphi(\ell) = \ell \), then the only minimal geodesics are the paths along which \( |c^t_\perp \ell| \) is constant.

**Proof:** Since the inequality
\[
L_{G^*}(c) \geq \int_0^1 \left[ \int_{S^1} |c^t_\perp| |c_\theta| d\theta \right] dt = \alpha(c)
\]
is an equality if and only if \( |c^t_\perp| \) does not depend on \( \theta \), that is, \( \frac{\partial}{\partial \theta} |c^t_\perp| = 0 \) (the case of "grassfire"), if \( c(t) \) is a minimal geodesic, \( |c^t_\perp| \) must be independent of \( \theta \). Let \( c(t, \theta) \) be a path connecting \( C_1, C_2 \) such that \( |c^t_\perp| \) is independent of \( \theta \). After reparametrization if necessary, we may assume that \( c_t \cdot c_\theta = 0 \). Following [2], we let \( w(c) \) be the 2-current defined by the path \( c(t, \theta) \). Since \( c(t, \theta) \) is an immersion,
\[
d^p(C_{or}^1, C_{or}^2) = \int_{\mathbb{R}^2} |w(c)| dx dy = \int_{S^1 \times [0,1]} |\det dc(t, \theta)| d\theta dt = \alpha(c)
\]
Therefore, \( L_{G^*}(c) \) is the minimal distance between \( C_1, C_2 \). For \( c(t, \theta) \) to be a geodesic path, reparametrize \( t \) such that the infinitesimal arc-length \( |c^t_\perp \ell| \) is constant along the path. Q.E.D.

Corollary 6 Suppose \( \varphi(\ell) = e^{At} \) and \( c(t, \theta), 0 \leq t \leq 1 \), is a path connecting \( C_1, C_2 \). Assume that \( |c^t_\perp| \) does not depend on \( \theta \) and \( \ell(t) < 1/A \) for all \( t \). Then,
\[
dist_{G^*}(C_{or}^1, C_{or}^2) = \sqrt{Aed^p(C_{or}^1, C_{or}^2)}
\]
It follows that the path \( c(t, \theta) \) is not minimal.

**Proof:** Break up the interval \([0,1]\) into small segments of length \( \varepsilon \) and apply Lemma 3 with \( \delta = \varepsilon \). (The proof of the lemma extends to this case after minor modifications.) Calculate the length of the new path \( \tilde{c}_\varepsilon \) applying the construction of Lemma 3 to each of the segments. We get \( \lim_{\varepsilon \to 0} L_{G^*}(\tilde{c}_\varepsilon) = \sqrt{Aed^p(c_{or}^1, c_{or}^2)} \) as in the proof of Theorem 2. On the other hand,
\[
L_{G^*}(c) = \int_0^1 \left[ \int_{S^1} |c^t_\perp| |c_\theta| d\theta \right] dt = \int_0^1 \sqrt{\frac{\varphi(\ell)}{\ell}} |c^t_\perp| \ell dt > \sqrt{\min \frac{\varphi(\ell)}{\ell}} \alpha(c) = \sqrt{Aed^p(c)}
\]
Q.E.D.

### 4 Geodesic Equations

We reproduce calculations in [2] mutatis mutandis. Let \( t \mapsto c(t, \cdot) \) be a path in \( Imm(S^1, \mathbb{R}^2) \).
Equation of Geodesic in $\text{Imm}(S^1, \mathbb{R}^2)$:

$$(\varphi(\ell)|c_0|c_t) = -\frac{1}{2} \left[ \left( \int_{S^1} |c_t|^2 |c_0| d\theta \right) i\varphi'(\ell)\kappa_c c_0 + \varphi(\ell) \left( \frac{|c_t|^2 c_0}{|c_0|} \right) \right]_\theta$$

(9)

where $\varphi'$ denotes $d\varphi/d\ell$.

**Proof:** We calculate the first variation of the energy of the path to obtain the geodesics:

$$E_{G^*} (c) = \frac{1}{2} \int_a^b \left[ \varphi(\ell) \left( \int_{S^1} (c_t \cdot c_t) |c_0| d\theta \right) \right] dt$$

$$\partial_s|_0 \varphi(\ell) = \varphi'(\ell)\partial_s|_0 \int_{S^1} (c_0 \cdot c_0) \partial_s|_\theta d\theta$$

$$= \varphi'(\ell) \int_{S^1} \frac{(c_{\theta} \cdot c_0)}{|c_0|} d\theta$$

$$= -\varphi'(\ell) \int_{S^1} (c_s \cdot \left( \frac{c_0}{|c_0|} \right)_\theta) d\theta$$

$$= -\varphi'(\ell) \int_{S^1} (c_s \cdot i\kappa_c c_0) d\theta$$

$$\partial_s|_0 \frac{1}{2} \int_{S^1} (c_t \cdot c_t) |c_0| d\theta = \int_{S^1} (c_{st} \cdot c_t) |c_0| d\theta + \frac{1}{2} \int_{S^1} |c_t|^2 \frac{(c_{\theta} \cdot c_0)}{|c_0|} d\theta$$

$$= \int_{S^1} (c_{st} \cdot c_t) |c_0| d\theta - \frac{1}{2} \int_{S^1} (c_s \cdot \left( \frac{|c_t|^2 c_0}{|c_0|} \right)_\theta) d\theta$$

$$\partial_s|_0 E_{G^*} (c) = -\frac{1}{2} \varphi'(\ell) \int_a^b \left( \int_{S^1} (c_s \cdot i\kappa_c c_0) d\theta \right) \left( \int_{S^1} |c_t|^2 |c_0| d\theta \right) dt$$

$$- \frac{1}{2} \int_a^b \varphi(\ell) \left( \int_{S^1} (c_s \cdot \left( \frac{|c_t|^2 c_0}{|c_0|} \right)_\theta) d\theta \right) dt$$

$$+ \int_a^b \varphi(\ell) \left( \int_{S^1} (c_{st} \cdot c_t) |c_0| d\theta \right) dt$$

The last term $= - \int_a^b (c_s \cdot \varphi(\ell)c_t |c_0|_t) d\theta dt$
The normal component of the right hand side is the point of the function \( r \), where we have used the formulae \( \frac{\partial}{\partial s} A \) vanishes identically. The normal component of the left-hand-side is \( c \) we may assume that \( (c_t \cdot c_\theta) = 0 \). Write \( c_t \) as \( i a c_\theta / |c_\theta| \). After substituting this in Eq. (9), split the equation into its component along \( c_\theta \) and \( n_\nu \). The former vanishes identically. The normal component of the left-hand-side is

\[
(\varphi(\ell)|c_\theta|)_t = \varphi(\ell)a_t |c_\theta| - \varphi(\ell)\kappa_c a^2 |c_\theta| - \varphi'(\ell) a |c_\theta| \int_{S^1} a\kappa_c |c_\theta| d\theta
\]

where we have used the formulae \( |c_\theta|_t = -a\kappa_c |c_\theta| \) and \( \ell_t(c) = -\int_{S^1} a\kappa_c |c_\theta| d\theta \).

The normal component of the right hand side is

\[
-\frac{1}{2} \varphi(\ell)\kappa_c a^2 |c_\theta| - \frac{1}{2} \varphi'(\ell) \kappa_c \left( \int_{S^1} a^2 |c_\theta| d\theta \right) |c_\theta|,
\]

Therefore, the equation of the geodesic in \( B_i(S^1, \mathbb{R}^2) \) is

\[
a_t = \frac{\kappa_c}{2} \left( a^2 - \varphi'(\ell) \int_{S^1} a^2 |c_\theta| d\theta \right) + \frac{\varphi'(\ell)}{\varphi(\ell)} a \int_{S^1} a\kappa_c |c_\theta| d\theta
\]

\[
= \begin{cases} 
\frac{\kappa_c}{2} \left( a^2 - \frac{\varphi'(\ell)}{\varphi(\ell)} \right) + a \cdot \frac{a\kappa_c}{\varphi(\ell)} & \text{if } \varphi(\ell) = \ell \\
\frac{\kappa_c}{2} \left( a^2 - (A\ell)^2 \right) + (A\ell) a \cdot \frac{a\kappa_c}{\varphi(\ell)} & \text{if } \varphi(\ell) = e^{A\ell} \quad (10)
\end{cases}
\]

where for any function \( f(t,\theta) \), \( \overline{f} \) denotes the average \( \frac{1}{2} \int_{S^1} f |c_\theta| d\theta \).

As an example, consider the case of concentric circles [2], \( c(t,\theta) = r(t)e^{i\theta}, r_o = r(0), r_1 = r(1) \) We have \( \kappa_c = 1/r \) and \( a = -r_t \). Substituting these in Eq. (10) when \( \varphi(\ell) = \ell \), we get \( -r_{tt} = r_t^2/r \) or \( (r^2)_{tt} = 0 \). Therefore, \( r^2(t) = tr_1^2 + (1-t)r_o^2 \). This example is a special case of the curve evolution by "grassfire" in which \( a \) is independent of \( \theta \). We have \( a^2 = \overline{a^2} \) and the equation of the geodesic reduces to \( a_t = a \cdot \overline{a\kappa_c} = -a\ell_t/\ell \) and hence \( (a\ell)_t = 0 \). Therefore, \( a\ell \) is a constant. By substituting in the equation for the length of the geodesic, we find that \( a\ell = \) the length of the geodesic. When \( \varphi(\ell) = e^{A\ell} \), the equation of the geodesic in the case of concentric circles is \( (r^2)_{tt} = r_t^2(1 - 2\pi rA) \) which is zero when the perimeter of the circle equals \( 1/A \), marking the unique inflection point of the function \( r^2(t) \).
5 Sectional Curvature

A formula for the sectional curvature may be derived exactly as in [2] by means of local charts. Let \( c : S^1 \to \mathbb{R}^2 \) be a smooth and free immersion, an element in the space of free immersions, \( \text{Imm}_f(S^1, \mathbb{R}^2) \). Let \( \pi \circ \psi : C^\infty(S^1, (-\epsilon, \epsilon)) \to \text{Imm}_f(S^1, \mathbb{R}^2) \) be a local chart centered at \( C \). As before, let \( c \) be parametrized by the arclength so that \( \theta \) parametrizes the scaled circle \( S^1 \ell \) where \( \ell \) is the length of \( c \). Let

\[
\psi : C^\infty(S^1, (-\epsilon, \epsilon)) \to \text{Imm}_f(S^1, \mathbb{R}^2) \\
\psi(f)(\theta) = c(\theta) + f(\theta)n_c(\theta) \\
\pi \circ \psi : C^\infty(S^1, (-\epsilon, \epsilon)) \to \text{Imm}_f(S^1, \mathbb{R}^2)
\]

be a local chart centered at \( C \). For any function \( u \), let \( u' \) denote its derivative so that \( \varphi' = d\varphi/d\ell \) and if \( u \) is a function on \( S^1 \), \( u' = du/d\theta \). We have the following formulae from [2]:

\[
\psi(f)' = (1 - f\kappa_c)c' + f'n_c \\
n_{\psi(f)} = \frac{(1 - f\kappa_c)n_c - f'c'}{\sqrt{(1 - f\kappa_c)^2 + f'^2}}
\]

For \( h \in C^\infty(S^1, \mathbb{R}) \), \( h \cdot n_c \in T_{\psi(f)}\text{Imm}_f(S^1, \mathbb{R}^2) \) and

\[
(h \cdot n_c)\perp = \frac{(1 - f\kappa_c)h}{\sqrt{(1 - f\kappa_c)^2 + f'^2}}n_{\psi(f)}
\]

Let

\[
< h, k >_f = \int_{S^1} \frac{(1 - f\kappa_c)^2hk}{\sqrt{(1 - f\kappa_c)^2 + f'^2}} d\theta = \int_{S^1} hk(1 - f\kappa_c - \frac{1}{2}f'^2 + O(f^3)) d\theta \\
\ell_f = \int_{S^1} \sqrt{(1 - f\kappa_c)^2 + f'^2} d\theta = \int_{S^1} (1 - f\kappa_c + \frac{1}{2}f'^2 + O(f^3)) d\theta
\]

\(< h, k >_o \) will be denoted simply as \( < h, k > \) and \(||h||^2 = < h, h >\). The metric at the point in \( B_{i, f}(S^1, \mathbb{R}^2) \) corresponding to \( f \) is given by

\[
G^\varphi_f(h, k) = \varphi(\ell_f) < h, k >_f
\]

We now outline the calculation of the Christoffel Symbol and the sectional curvature. (For more details, see [2].) Differentiating the metric in direction \( j \), we get

\[
d_j G^\varphi_f(h, k) = (\varphi'(\ell_f)d_j\ell_f) < h, k >_f + \varphi(\ell_f)d_j < h, k >_f
\]
\[
\begin{align*}
d_j f &= - \langle j, \kappa_c \rangle + \langle f', j' \rangle + O(f^2) = - \langle \kappa_c + f'', j \rangle + O(f^2) \\
d_j < h, k > &= - \langle h k, j \kappa_c + f' j' \rangle + O(f^2)
\end{align*}
\]

Therefore,
\[
\begin{align*}
d_j G^\rho_f (h, k) \\
&= \varphi'(\ell_f) \left( - \langle j, \kappa_c \rangle + \langle f', j' \rangle \right) < h, k > f - \varphi(\ell_f) < h k, j \kappa_c + f' j' > + O(f^2) \\
&= - \varphi'(\ell_f) < \kappa_c + f'', j > < h, k > f - \varphi(\ell_f) < h k, j \kappa_c + f' j' > + O(f^2)
\end{align*}
\]

The Christoffel symbol satisfies the identity
\[
2G^\rho_f (\Gamma_f (h, k), j) = d_j G^\rho_f (h, k) - d_k G^\rho_f (k, j) - d_k G^\rho_f (h, j)
\]

Therefore,
\[
\begin{align*}
2G^\rho_f (\Gamma_f (h, k), j) &= \varphi'(\ell_f) < \kappa_c + f'', h > < k, j > f + \varphi(\ell_f) < h k, j \kappa_c + f' h' > \\
&\quad + \varphi'(\ell_f) < \kappa_c + f'', k > < h, j > f + \varphi(\ell_f) < h j, k \kappa_c + f' k' > \\
&\quad - \varphi'(\ell_f) < \kappa_c + f'', j > < h, k > f - \varphi(\ell_f) < h k, j \kappa_c + f' j' > + O(f^2)
\end{align*}
\]

At the center \( f = 0 \), this simplifies to
\[
\begin{align*}
2G^\rho_o (\Gamma_o (h, k), j) &= \varphi(\ell) \kappa c h + \varphi'(\ell) \left[ < \kappa_c, h > k + < \kappa_c, k > h - < h, k > \kappa c \right], j > \\
Since \( G^\rho_o (\Gamma_o (h, k), j) = \varphi(\ell) < \Gamma_o (h, k), j > \), we have
\end{align*}
\]

\[
\begin{align*}
\Gamma_o (h, k) &= \frac{1}{2} \kappa c h k + \frac{\varphi'(\ell)}{2 \varphi(\ell)} \left[ < \kappa_c, h > k + < \kappa_c, k > h - < h, k > \kappa c \right]
\end{align*}
\]

Next, we calculate the second derivative:
\[
\begin{align*}
d_m d_j G^\rho_f (h, k) \\
&= d_m \left( \varphi'(\ell_f) \left( - \langle j, \kappa_c \rangle + \langle f', j' \rangle \right) < h, k > f - \varphi(\ell_f) < h k, j \kappa_c + f' j' > + O(f^2) \right) \\
&= \varphi''(\ell_f) \left( < m, \kappa_c > - < f', m' > \right) \left( < j, \kappa_c > - < f', j' > \right) < h, k > f \\
&\quad + \varphi'(\ell_f) < m', j' > < h, k > f + \varphi'(\ell_f) < j, \kappa_c > - < f', j' > < h k, m \kappa_c + f' m' > \\
&\quad + \varphi'(\ell_f) < m, \kappa_c > - < f', m' > < h k, j \kappa_c + f' j' > - \varphi(\ell_f) < h k, m' j' > + O(f)
\end{align*}
\]

At the center,
\[
\begin{align*}
d_m d_j G^\rho_o (h, k) \\
&= \varphi''(\ell) < m, \kappa_c > < j, \kappa_c > < h, k > \\
&\quad + \varphi'(\ell) \left( < m', j' > < h, k > + < j, \kappa_c > < h k, m \kappa_c > + < m, \kappa_c > < h k, j \kappa_c > \right) \\
&\quad - \varphi(\ell) < h k, m' j' >
\end{align*}
\]
Let $P(m, h)$ be a tangent plane at $c$, spanned by normal vector fields $m(\theta), h(\theta)$ along $C$. Assume that $m, h$ have been normalized so that

$$\varphi(t) \|m\|^2 = \varphi(t) \|h\|^2 = 1$$

and $<m, h> = 0$

The sectional curvature at $C$ is given by the formula

$$k_c(P(m, h)) = d_m d_h G_o^c(h, m) - \frac{1}{2} [d_m d_m G_o^c(h, h) + d_h d_h G_o^c(m, m)]$$

$$+ G_o^c(\Gamma_o(h, m), \Gamma_o(h, m)) - G_o^c(\Gamma_o(m, m), \Gamma_o(h, h))$$

$$d_m d_h G_o^c(h, m) - \frac{1}{2} [d_m d_m G_o^c(h, h) + d_h d_h G_o^c(m, m)]$$

$$= \varphi'( <m, \kappa_c > < mh, h\kappa_c > + < h, \kappa_c > < mh, m\kappa_c >) - \varphi(t) < mh, m'h'>$$

$$- \frac{\varphi''}{2} \left[ < m, \kappa_c >^2 \|h\|^2 < h, \kappa_c >^2 \|m\|^2 \right]$$

$$- \frac{\varphi'}{2} \left[ \|m'\|^2 \|h\|^2 + 2 < m, \kappa_c > < h^2, m\kappa_c > \right] + \frac{\varphi}{2} < m^2, h^2 >$$

$$- \frac{\varphi'}{2} \left[ \|m\|^2 \|h'\|^2 + 2 < h, \kappa_c > < m^2, \kappa_c > \right] + \frac{\varphi}{2} < m^2, h^2 >$$

$$= \frac{\varphi}{2} \|m' h - mh'\|^2 - \frac{\varphi'}{2\varphi} (\|m'\|^2 + \|h'\|^2) - \frac{\varphi''}{2\varphi} ( < m, \kappa_c >^2 + < h, \kappa_c >^2)$$

4 $\left[ \Gamma_o^2 (h, m) - \Gamma_o (m, m) \Gamma_o (h, h) \right]$

$$= \left[ \frac{\varphi'}{\varphi} ( < m, \kappa_c > h + < h, \kappa_c > m ) + mh\kappa_c \right]^2$$

$$- \left[ \frac{\varphi'}{\varphi} (2 < m, \kappa_c > m - \kappa_c \|m\|^2) + m^2\kappa_c \right] \left[ \frac{\varphi'}{\varphi} (2 < h, \kappa_c > h - \kappa_c \|h\|^2) + h^2\kappa_c \right]$$

$$= \left( \frac{\varphi'}{\varphi} \right)^2 ( < m, \kappa_c >^2 h^2 + < h, \kappa_c >^2 m^2)$$

$$+ \left( \frac{\varphi'}{\varphi} \right)^2 (2 < m, \kappa_c > \kappa_c \|h\|^2 + 2 < h, \kappa_c > \kappa_c \|m\|^2 - \kappa_c^2 \|h\|^2 \|m\|^2)$$

$$- 2 \left( \frac{\varphi'}{\varphi} \right)^2 < m, \kappa_c > < h, \kappa_c > mh + \frac{\varphi'}{\varphi} (m^2\kappa_c^2 \|h\|^2 + h^2\kappa_c^2 \|m\|^2)$$

Taking into account that $h, m$ form an orthonormal basis,

$$G_o^c(\Gamma_o(h, m), \Gamma_o(h, m)) - G_o^c(\Gamma_o(m, m), \Gamma_o(h, h))$$

$$= \frac{1}{4} \left( \frac{\varphi'}{\varphi} \right)^2 \left( 3 < m, \kappa_c >^2 + 3 < h, \kappa_c >^2 - \frac{\|\kappa_c\|^2}{\varphi} \right) + \frac{1}{4} \frac{\varphi'}{\varphi} (\|m\kappa_c\|^2 + \|h\kappa_c\|^2)$$
Putting all of this together, we get the sectional curvature

\[ k_c(P(m, h)) = \frac{\varphi}{2} \|m'\|^2 - \frac{\varphi'}{2\varphi} \left( \|m'\|^2 + \|h'\|^2 \right) + \frac{\varphi'}{4\varphi} \left( \|\kappa_c\|^2 + \|h\kappa_c\|^2 \right) \]

\[ + \frac{3\varphi'^2 - 2\varphi\varphi''}{4\varphi^2} \left( <m, \kappa_c> + <h, \kappa_c>^2 \right) - \frac{\varphi'^2}{4\varphi^3} \|\kappa_c\|^2 \]

Each of the last three terms on the right-hand side is bounded:

\[ \|\kappa_c\|^2 + \|h\kappa_c\|^2 \leq \frac{2\|\kappa_c\|_\infty^2}{\varphi}, \quad <m, \kappa_c> + <h, \kappa_c>^2 \leq \frac{2\ell\|\kappa_c\|_\infty^2}{\varphi}, \quad \|\kappa_c\|^2 \leq \ell\|\kappa_c\|_\infty^2 \]

where \( \|\kappa_c\|_\infty = \max_\theta |\kappa_c(\theta)| \). Therefore, the boundedness of the sectional curvature from above and hence the minimality of a geodesic crucially depend on the first two terms. For a fixed \( m \), the magnitude of each of the two terms depends on \( \|h\| \) which can be made arbitrarily large while keeping \( \|h\| \) fixed by making \( h \) highly wiggly.

**Proposition 7** For a given \( m \), the sectional curvature is bounded from above if and only if \( \|m\|_\infty^2 \leq \frac{\varphi'}{\varphi^2} \) or, equivalently, \( m^2(\theta) \leq \frac{\varphi'\ell}{\varphi} m^2(\theta) \) since \( \frac{\varphi'}{\varphi^2} = \frac{\varphi'}{\varphi^2} \).

**Proof:** We need to estimate only the first two terms on the right-hand side of Eq. (11).

Suppose \( \|m\|_\infty^2 \leq \frac{\varphi'}{\varphi^2} \). Then,

\[ \frac{\varphi}{2} \|m'h - mh'\|^2 - \frac{\varphi'}{2\varphi} \left( \|m'\|^2 + \|h'\|^2 \right) \]

\[ = \varphi \int_{S^1} \left[ \frac{\varphi}{2} (m'h)^2 - \frac{\varphi'}{2\varphi} m'^2 - \varphi mm'h + \frac{\varphi}{2} \left( m^2 - \frac{\varphi'}{\varphi^2} \right) h'^2 \right] d\theta \]

\[ \leq \frac{\varphi}{2} \|m'\|_\infty^2 - \frac{\varphi^2}{4} \int_{S^1} (m'^2)'(h'^2)' d\theta \]

\[ \leq \frac{\varphi}{2} \|m'\|_\infty^2 + \frac{\varphi^2}{4} \int_{S^1} (m'^2)''(h'^2) d\theta \]

\[ \leq \frac{\varphi}{2} \|m'\|_\infty^2 + \frac{\varphi^2}{4} (m'^2)_\infty < \infty \]

Conversely, suppose \( \|m\|_\infty^2 > \frac{\varphi'}{\varphi^2} \). Choose \( \epsilon \) such that \( U = \{ \theta : m^2(\theta) > \varphi'/\varphi^2 + \epsilon \} \) is not empty. Let \( h \) be a high frequency wave function with
If \( \text{supp}(h) \subset U \). Then,
\[
\frac{\varphi}{2} \| m'h - mh' \|^2 - \frac{\varphi'}{2\varphi} \left( \| m' \|^2 + \| h' \|^2 \right)
\]
\[
= \varphi \int_{S^1} \left[ \frac{\varphi}{2} (m'h)^2 - \frac{\varphi'}{2\varphi} m'^2 - \varphi mm'h' + \frac{\varphi}{2} \left( m^2 - \frac{\varphi'}{\varphi^2} \right) h'^2 \right] d\theta
\]
\[
\geq \frac{\varphi'}{2\varphi} \| m' \|^2 + \frac{\varphi^2}{4} \int_{S^1} (m'^2)(h'^2)d\theta + \frac{\varphi\epsilon}{2} \| h' \|^2
\]
\[
\geq \frac{\varphi'}{2\varphi} \| m' \|^2 - \frac{\varphi^2}{4} \|(m'^2)''\|_{\infty} + \frac{\varphi\epsilon}{2} \| h' \|^2
\]
which tends to \( \infty \) as the frequency of the wave function \( h \) tends to \( \infty \). Q.E.D.

If \( U = \{ \theta : m^2(\theta) > \varphi'/\varphi^2 + \epsilon \} \) is not empty,
\[
1 = \varphi \left[ \int_U m^2 d\theta + \int_{[0,\ell]/U} m^2 d\theta \right]
\]
\[
\geq \varphi \int_U m^2 d\theta \geq \left( \frac{\varphi'}{\varphi} + \epsilon \varphi \right) |U|
\]
and hence, \( |U| < \varphi'/\varphi' \). If \( \varphi = \ell, |U| < \ell \) and if \( \varphi = e^{A\ell}, |U| < 1/A \). Thus, the case when \( |m|^2_{\infty} > \varphi'/\varphi^2 \) may be seen as a generalization of the rectangular bump considered in §3.

If \( \varphi = \ell \), the sectional curvature is bounded if and only if \( m^2(\theta) \leq m^2 \) which is true if and only if \( m = 1/\ell \). Setting \( m = 1/\ell \), we get
\[
k_c(P(1/\ell,h)) = \frac{\| h\kappa_c \|^2}{4\ell} + \frac{3}{4\ell^2} \left( < 1/\ell, \kappa_c \rangle >^2 + < h, \kappa_c >^2 \right)
\]
which is always positive. If \( h \) and \( \kappa_c \) additionally have disjoint supports, the sectional curvature equals \( 3\pi^2 N^2/\ell^4 \) where \( N \) is the rotation index of \( C \).

If \( \varphi = e^{A\ell} \), the sectional curvature is bounded if and only if \( m^2(\theta) \leq (A\ell)m^2 \). In particular, the sectional curvature is unbounded for every \( m \) if \( \ell < 1/A \). The analysis of rectangular bumps in §3 suggests the conjecture that when \( \varphi = e^{A\ell} \), a geodesic is locally minimal if and only if \( a^2(\theta) \leq (A\ell)a^2 \) where \( a = |c|_t \) (notation of Eq. 10).

For an example of a negative sectional curvature, consider the unit square with slightly rounded corners. Choose \( m, h \) such that \( \text{supp}(m) \) and \( \text{supp}(h) \) are disjoint and concentrated along the straight portions of the square. Then,
\[
k_c(P(m,h)) = -\frac{\varphi'}{2\varphi} \left( \| m' \|^2 + \| h' \|^2 \right) - \frac{\varphi'^2}{4\varphi^3} \| \kappa_c \|^2
\]

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6 References

1. G. Charpiat, O. Faugeras and R. Keriven, "Approximations of shape metrics and application to shape warping and empirical shape statistics", To appear in J. of Foundations of Computational Mathematics.

2. P. Michor and D. Mumford, "Riemannian geometries on spaces of plane curves", arXiv:math.DG/0312384, v2, Sep. 22, 2004.

3. E. Klassen, A. Srivastava, W. Mio and S.H. Joshi, "Analysis of planar shapes using geodesic paths on shape spaces", IEEE Trans. PAMI, 26(3), pp. 372-383, 2004.

4. A. Yezzi and A. Mennucci, "Conformal Riemannian metrics in space of curves", EUSIPCO04, MIA, 2004.

5. A. Yezzi and A. Mennucci, "Metrics in the space of curves", arXiv:math.DG/0412454 v2, May 25, 2005.