Operator perturbation weighing, from quasi-critical source system response

Pierre Albarède
rés. Valvert, 12 rue de la Fourane, 13090 Aix-en-Provence, France
palbarede@yahoo.com

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Abstract

In Hilbert space, a linear source-to-flux problem in the critical (zero eigenvalue) limit is ill-posed, but regularized by a constraint on a linear functional, fulfilled by tuning some control variable. For any exciting perturbation, I obtain, by spectral decomposition and perturbation theory, the regularized flux and the regularizing control variable non-linear responses.

May the exciting perturbation be obtained, inversely, from observable responses? Yes, in some cases, from the existence of a weight scale, a perturbation series, determined by recursion relations, involving well-posed source problems, and the possibility of obtaining this weight scale from observables of both the unconstrained and constrained systems.

key words: source, non-linear response, critical, regularization, perturbation, self-shielding, operator weighing, observable.

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1 Problem setup

In \( \mathcal{H} \), a real Hilbert space, with the scalar product \( \langle ., . \rangle \) and the norm \( \| \psi \| \equiv \langle \psi, \psi \rangle^{1/2} \), let \( L : \mathcal{F} \to \mathcal{Q} \equiv L(\mathcal{F}) \) be a linear (additive and homogeneous) endomorphism (\( \mathcal{F}, \mathcal{Q} \subset \mathcal{H} \) are linear manifolds).

I consider the source problem, finding, for some source \( Q \in \mathcal{Q} \), the flux \( \psi \in \mathcal{F} \) such that

\[
0 = L\psi + Q. \tag{1}
\]

I assume that the ‘flux-to-source’ operator \( L \) is invertible (on the left), hence

\[
\psi = \Phi(L, Q) \equiv -L^{-1}Q. \tag{2}
\]

It occurs that the flux operator \( \Phi \), defined in (2), is linear in \( Q \), but non-linear in \( L \). This non-linearity is formally eliminated by the change of variable \( L \mapsto L^{-1} \). However, expressing an inverse operator is often difficult, in particular, when the operator \( L \) has an eigenvalue close to zero (in this case, the non-linearity is somehow traded for a singularity). Therefore, I will keep the variable \( L \), and the non-linearity \( L \mapsto \Phi \).

The small eigenvalue critical limit of the source problem is conveniently defined by the following hypotheses:

- \( L, Q \) are functions of some scalar variable \( z \) (varying over an interval);
- \( \sigma(z) \) is a single eigenvalue of \( L(z) \) associated with an eigenvector \( \varphi(z) \);
- \( z_c \) is a critical value, such that

\[
\lim_{z \to z_c} \sigma(z) = 0; \tag{3}
\]

- for all \( z \neq z_c \), \( L(z) \) has an inverse on the left \( L^{-1}(z) \).

Even though the source problem has no more than one solution for \( z \neq z_c \), the source problem (1) is still ill-posed in the critical limit \( z \to z_c \), because \( \Phi \) is infinitely sensitive to perturbations of \( Q \) (essentially: \( z \mapsto \sigma(z)^{-1} \) has no derivative at \( z_c \)).

I take for regularizing condition the permanence of some linear continuous ‘gauge’ functional \( Q^\dagger \) of flux. This will be achieved by tuning the variable \( z \), thus appearing as a control variable. Equivalently, in the language of system analysis, a high-gain amplifier can be hardly used without feedback control, in order to maintain a reasonable output level (not too large, not too small).
As the Hilbert space is self-dual [1, §9.5, p. 264], the regularizing condition expresses as the constraint

\[ R \equiv \langle Q^\dagger, \Phi \rangle = R_0, Q^\dagger \in \mathcal{H}. \] (4)

The system parameters are \( T \equiv (L, Q, Q^\dagger) \), all functions of \( z \). \( T \) may be excited by some perturbation \( \delta T \), actually driven by some exciting variable \( \epsilon \), independent of \( z \).

Primary observables are the scalars outputs of the system, here

- the exciting variable \( \epsilon \),
- the control variable \( z \),
- the gauge output \( R \).

Secondary observables are parametric expressions thereof, independent of the system parameters \( T, \delta T \), for example, \( \ln R \), or the partial derivative \( (\partial R/\partial z)\epsilon \).

Observation is useless, without an interpretation relation, between \( T, \delta T \) and observables. The main issue of the paper is to prove the existence of an interpretation relation, of the form

\[ Z_1(T, \delta T) + Z_2(\text{prim. observables}) = 0, \] (5)

where \( Z_1, Z_2 \) are scalar weight functionals. The weight \( Z_1 \) is obtained by processing the primary observables through \( Z_2 \) and (3). I will explicitly construct \( Z_1, Z_2 \).

Ideally, the sources are unexcited (0 = \( \delta Q = \delta Q^\dagger \)) and the exciting perturbation is uncontrolled (\( \delta L \) does not depend on \( z \)). Solving (3) for \( \delta L \), if possible, is the principle of a direct operator perturbation measurement method.

2 The source system and the critical limit

2.1 Unconstrained system definition and properties

The source system is represented by (1). As the constraint is not used, the control variable \( z \) is left constant and implicit (‘asleep’) until the critical limit is considered (section 2.3). In the language of system analysis, the control loop is open.
The gauge output is
\[ R(T) \equiv \langle Q^\dagger, \Phi(L, Q) \rangle, Q^\dagger \in \mathcal{H}. \]  
(6)

I assume that \( L \) has an adjoint \( L^\dagger : \mathcal{F}^\dagger \to \mathcal{Q}^\dagger \equiv L(\mathcal{F}^\dagger) \) and \( Q^\dagger \in \mathcal{Q}^\dagger \). The adjoint source system is
\[ 0 = L^\dagger \psi^\dagger + Q^\dagger. \]  
(7)

I assume that \( L^\dagger \) has an inverse \( L^\dagger^{-1} : \mathcal{Q}^\dagger \to \mathcal{F}^\dagger \), solving the source problem (7):
\[ \forall Q^\dagger \in \mathcal{Q}^\dagger, \psi^\dagger = \Phi^\dagger(L, Q^\dagger) \equiv -L^\dagger^{-1}Q^\dagger = \Phi(L^\dagger, Q^\dagger). \]  
(8)

I introduce a restriction of the scalar product, noted \( \langle | \rangle \), to \((\mathcal{Q}^\dagger \times \mathcal{F}) \cup (\mathcal{F}^\dagger \times \mathcal{Q})\).

Considering, adjointly, the source as the gauge functional of the adjoint source system, the adjoint gauge output is
\[ R^\dagger(T^\dagger) \equiv \langle \Phi^\dagger(L, Q^\dagger)|Q \rangle, T^\dagger \equiv (L, Q^\dagger, Q) \]  
(9)

One shows that \( L^\dagger^{-1} \) is adjoint to \( L^{-1} \):
\[ L^\dagger^{-1} = L^{-1\dagger}. \]  
(10)

Using (10) in (3) yields the reciprocity relation [2],
\[ R(T) = \langle Q^\dagger|\Phi(L, Q) \rangle = \langle \Phi^\dagger(L, Q^\dagger)|Q \rangle = R^\dagger(T^\dagger). \]  
(11)

From (8, 11), the adjoint flux \( \Phi^\dagger(L, Q^\dagger) \), considered as a functional of \( Q \), produces the gauge output.

The linearity of the flux-to-source operator and the gauge functional imply scaling laws:
\[ \forall \alpha \in \mathbb{R}^*, \alpha R_0 = \langle \alpha Q^\dagger|\Phi(L, Q) \rangle = \langle \alpha \Phi^\dagger(L, Q^\dagger)|Q \rangle. \]  
(12)

Considering the source and the flux as the physical quantities, while the adjoint flux and the gain factor \( \alpha \) are mathematical and operational commodities, (12) shows that the physical quantities are independent of \( \alpha \), which corresponds to what physicists call gauge invariance [3, §VII.5].

The unconstrained system is represented by the parametric expression
\[ U(T) \equiv (R, \Phi, \Phi^\dagger)(T), \]  
(13)
mapping the independent variables, or parameters, onto the dependent variables, the gauge output and the fluxes. With (13), (12) becomes

$$\forall \alpha \in \mathbb{R}^*, U(L, Q, \alpha Q^\dagger) = (\alpha R, \Phi, \alpha \Phi^\dagger).$$

(14)

By definition (2), and adjointly, the direct and adjoint source systems are uncoupled:

$$0 = \frac{\partial \Phi}{\partial Q^\dagger} = \frac{\partial \Phi^\dagger}{\partial Q}.$$  

(15)

2.2 The spectral decomposition

I assume that $L^\dagger$ has an eigenvalue $\sigma^\dagger$, and that the associated eigenspace is of dimension one and eigenvector $\varphi^\dagger; \langle \varphi^\dagger, \varphi \rangle \neq 0$. Hence, the eigenvectors can be and are normalized, so that

$$\langle \varphi^\dagger | \varphi \rangle = 1.$$  

(16)

(As $\sigma \neq 0$, $\varphi \in Q$, and the use of the restricted product above is correct.) One shows (classically)

$$0 = (\sigma^\dagger - \sigma) \langle \varphi^\dagger | \varphi \rangle,$$

hence, from (16), $\sigma^\dagger = \sigma$. One shows that $\tilde{\mathcal{F}} \equiv \varphi^\dagger \perp \mathcal{F}$ is stable for $L$, and, adjointly, $\tilde{\mathcal{F}}^\dagger \equiv \varphi^\dagger \perp \mathcal{F}^\dagger$ is stable for $L^\dagger$.

$\varphi^\dagger \perp$ is the hyper-plane, orthogonal to $\varphi^\dagger$, a subspace of codimension one. From (16), $\varphi \notin \varphi^\dagger \perp$, so that $\mathcal{H} = \varphi \oplus \varphi^\dagger \perp$ and one shows

$$\mathcal{F} = \varphi \oplus \tilde{\mathcal{F}}, \mathcal{F}^\dagger = \varphi^\dagger \oplus \tilde{\mathcal{F}}^\dagger,$$

which are equivalent to ‘closure relations’,

$$1_F = \varphi \varphi^\dagger + \tilde{\pi}, 1_F^\dagger = \varphi^\dagger \varphi + \tilde{\pi}^\dagger,$$  

(17)

where

- $\varphi \varphi^\dagger$ stands for the tensor product $|\varphi \rangle \langle \varphi^\dagger|$,
- $\varphi^\dagger \varphi$ stands for the tensor product $|\varphi^\dagger \rangle \langle \varphi|$, adjoint to the latter,
- $\tilde{\pi}$ is the projection, in $\mathcal{F}$, on $\tilde{\mathcal{F}}$, along $\varphi$,
- $\tilde{\pi}^\dagger$ is the projection, in $\mathcal{F}^\dagger$, on $\tilde{\mathcal{F}}^\dagger$, along $\varphi^\dagger$, adjoint to the latter.
Condensed notations will be necessary:
\[
\forall \psi \in \mathcal{F}, \bar{\psi} \equiv \bar{\pi} \psi, \forall \psi^\dagger \in \mathcal{F}^\dagger, \bar{\psi}^\dagger \equiv \bar{\pi}^\dagger \psi^\dagger, \bar{L} \equiv \bar{\pi} L, \bar{L}^\dagger \equiv \bar{\pi}^\dagger L^\dagger.
\]

Applying \(\bar{\pi}\) to (1),
\[
0 = \bar{L} \psi + \bar{Q} = \bar{L} \bar{\psi} + \bar{Q},
\]
which is solved for the harmonic flux:
\[
\bar{\Phi} = -\bar{L}^{-1} \bar{Q},
\]
where \(\bar{L}\) has been restricted to the stable subspace \(\bar{\mathcal{F}}\). Applying \(\langle \varphi^\dagger \rvert \) to (1),
\[
\langle \varphi^\dagger \rvert \bar{\Phi} \rangle = -\langle \varphi^\dagger \rvert \bar{Q} \rangle \sigma.
\]
Combining (17, 20, 19),
\[
\Phi = -\langle \varphi^\dagger \rvert \bar{Q} \rangle \sigma \varphi - \bar{\pi}^\dagger \bar{\pi} \bar{Q},
\]
\[
R = -\frac{\langle Q^\dagger \varphi \varphi^\dagger \rvert Q \rangle}{\sigma} + \langle Q^\dagger \rvert - \bar{\pi} L^{-1} \bar{Q} \rangle.
\]
On (21) appears the (unconstrained) source problem critical ill-posedness. Eliminating \(\sigma\) from (22) and inserting the result into (21),
\[
\omega \equiv \frac{\langle Q^\dagger - \bar{L}^{-1} \bar{Q} \rangle}{R} = \frac{\langle Q^\dagger \bar{\Phi} \rangle}{\langle Q^\dagger \bar{\Phi} \rangle},
\]
\[
\sigma = -\frac{\langle Q^\dagger \varphi \varphi^\dagger \rvert Q \rangle}{R(1 - \omega)},
\]
\[
\bar{\Phi} = \frac{R(1 - \omega)}{\langle Q^\dagger \varphi \rangle} \varphi - \bar{L}^{-1} \bar{Q}.
\]
\(\omega\) is the ‘harmonicity’. The negative source-to-flux operator is
\[
-\bar{L}^{-1} = \frac{R(1 - \omega)}{\langle Q^\dagger \varphi \varphi^\dagger \rvert Q \rangle} \varphi \varphi^\dagger - \bar{\pi} \bar{\pi}^\dagger.
\]
Some useful relations are, adjointly,
\[
\Phi^\dagger = \frac{R(1 - \omega^\dagger)}{\langle \varphi^\dagger \rvert Q \rangle} \varphi^\dagger - \bar{L}^\dagger \bar{\pi}^\dagger \bar{Q}^\dagger,
\]
\[
\omega^\dagger \equiv \frac{-\langle \bar{L}^\dagger \bar{\pi}^\dagger \bar{Q} \rangle}{R} = \frac{\langle \bar{\Phi}^\dagger \rvert Q \rangle}{\langle \Phi^\dagger \rvert Q \rangle} = \omega,
\]
and

\[ \tilde{\pi}^2 = \tilde{\pi}, \]
\[ \tilde{\pi}L = L\tilde{\pi}, \]
\[ \tilde{L}^{-1} = \tilde{L}^{-1}, \]

\[ \langle Q^\dagger| - \tilde{L}^{-1}\tilde{\pi} = \langle \Phi^\dagger|\tilde{\pi}, \tag{28} \]

and adjointly.

### 2.3 The constant gauge output critical limit

I show up the (smooth) scalar control variable \( z \). Acknowledging the functional character of the parameters \( T \), (13) must be corrected to show chained functions:

\[ U \circ T = (R, \Phi, \Phi^\dagger) \circ T. \]

I assume that observables are bounded in the critical limit (3). Considering moreover (24, 25, 27), the direct flux or the adjoint flux must diverge.

I assume

\[ \langle \varphi^\dagger|Q|\varphi(z) = O(\sigma(z)) \right. \]
\[ \left. \rightarrow z_c, \quad \exists m_1, \forall z, 0 < m_1 \leq \| \langle Q^\dagger|\varphi \rangle \varphi^\dagger(z) \|, \right. \tag{29} \]

(while the source is turned off, the gauge is not) so that the direct flux does not diverge and the adjoint flux does, though only by a scalar factor.

Following the comments of section 2.1 on gauge invariance, the direct flux boundedness is physically motivated.

I assume spectral separation, uniform in \( z \):

\[ \exists m, \forall z, \forall (\tilde{\psi} \in \tilde{F}, \| \tilde{\psi} \|= 1), \forall (\tilde{\psi}^\dagger \in \tilde{F}^\dagger, \| \tilde{\psi}^\dagger \|= 1), \]
\[ 0 < m \leq \min(\| \tilde{L}(z)\tilde{\psi} \|, \| \tilde{L}^\dagger(z)\tilde{\psi}^\dagger \|). \tag{30} \]

Consequently, any eigenvalue of \( \tilde{L} \) has a modulus greater than \( m \). Only the eigenvalue \( \sigma(z) \) gets close to zero. \( \sigma(z) \) and \( \varphi(z) \) are the ‘fundamental’ eigenvalue and eigenvector (and adjointly). The maximum of all \( m \) in (30) is the ‘spectral gap’. The harmonicity and the ratios of the second terms over the first terms in the r. h. s. of (25, 27) are \( O(\sigma(z)/m) \). The constant gauge
output critical limits are
\[
\omega \circ T(z) = \mathcal{O}(\sigma(z)/m),
\]
\[
\Phi \circ T(z) \sim \frac{R_0}{\langle Q|\varphi_c(z_c)\rangle} \varphi_c, \quad \varphi_c \equiv \varphi(z_c),
\]
\[
\Phi_\dagger \circ T(z) \sim \frac{R_0}{\langle \varphi_\dagger|Q\rangle(z_c)} \varphi_\dagger_c, \quad \varphi_\dagger_c \equiv \varphi_\dagger(z_c).
\]

3 Perturbation theory with constraint

3.1 The constraint operator

The constraint consists in tuning the control variable \( z \) to the ‘balancing’ value \( \tilde{z} \), such that the gauge output remains equal to a reference value \( R_0 \), which is a new independent variable, while \( \tilde{z} \) is a new dependent variable.

I assume that the problem has a unique solution:
\[
\exists! \tilde{z}(T, R_0), R(T(\tilde{z})) = R_0.
\]

Remark 1. As \( T \) is a function (of \( z \)), \( \tilde{z}(., R_0) \) is a (non-linear) functional (of the function \( T \)).

For all function \( t \) of \( z \), like \( T \), is defined the constrained value
\[
t(T, R_0) \equiv t(\tilde{z}(T, R_0)).
\]

The (linear) ‘constraint operator’,
\[
\mathcal{J} : t \mapsto \tilde{t} = t \circ \tilde{z}.
\]
transforms a function of \( z \) into an operator on \( (T, R_0) \). (33) is formally the commutation relation
\[
\mathcal{J} t = t \mathcal{J}.
\]

For all operator \( u \) on \( T(z) \) (like \( \text{Id}, U \)), \( \tilde{u}(T) \equiv u \circ T = u(T) \triangleright u \). The symbol \( \triangleright \) introduces abbreviations (to be used with care).

The constrained system is thus represented by \( (\mathcal{U}, \tilde{z}) \), where \( \Phi \) should be evaluated from (24), where \( R \rightarrow R_0 \). On the constrained system, (14) becomes
\[
\forall \alpha \in \mathbb{R}^*, (\mathcal{U}, \tilde{z})((L, Q, \alpha Q^\dagger), \alpha R_0) = ((\alpha R, \Phi, \alpha \Phi_\dagger), \tilde{z})((L, Q, Q^\dagger), R_0)).
\]
The adjoint source $Q^\dagger$, although it does not appear in (1), nevertheless affects the flux, because the flux-to-source operator depends on $\hat{z}$, that depends on $Q^\dagger$; the constraint couples the direct and adjoint source systems; as opposed to (15),

$$\frac{\partial \Phi}{\partial Q^\dagger} \neq 0, \frac{\partial \Phi^\dagger}{\partial Q} \neq 0.$$  \hfill (36)

### 3.2 Functional variational definitions

The perturbation of any vector $x$ is noted, as usual,

$$(x, \delta x)^* \equiv x + \delta x \triangleright x^*.$$  

$x, x^*$ are respectively the reference and perturbed values. The definition is extended to any parametric expression $f(x)$:

$$(f(x), \delta f, \delta x)^* \equiv (f + \delta f)(x + \delta x) \triangleright f^*(x^*) \triangleright f(x)^*,$$

$$\delta(f(x), \delta f, \delta x) \equiv (f + \delta f)(x + \delta x) - f(x) \triangleright \delta(f(x)) \triangleright \Delta f.$$  \hfill (37)

In particular:

$$\Delta f = f(x^*) - f(x) + \delta f(x^*),$$  \hfill (38)

e. g., for $f : x \mapsto y.x$, where ‘.’ stands for a bilinear product,

$$\delta(y.x) = y.\delta x + \delta y.x^*.$$  \hfill (39)

I assume that the source system remains of the form (1, 6), hence the 'law invariance' statement (on the operator $U$, not its value!):

$$\delta U = 0.$$  \hfill (40)

and, consequently, $\delta(U, \hat{z}) = 0$. On the contrary, (the function) $T$ may vary by (the smooth function) $\delta T$. Control is linear if and only if the second derivatives vanish:

$$0 = T'' = \delta T''.$$  \hfill (41)

Definition: control is remote (from excitation) if and only if

$$\delta T' = 0.$$  \hfill (42)
A perturbation of the unconstrained system $U(T(z))$ is, from (37, 40),

$$U(T(z)) = U(T(z) + \Delta T) \triangleright U^*,$$

$$T(z^*) - T(z) + \delta T(z^*) \triangleright \Delta T. \quad (43)$$

Taking advantage of (35), the perturbed constrained system $U(T^*, R_0^*)$ is rescaled, according to the gauge transform

$$(Q^{i*}, R_0^*, \Phi^{i*}) \rightarrow \alpha(Q^{i*}, R_0^*, \Phi^{i*}), \alpha = \frac{R_0}{R_0^*}, \quad (44)$$

so that

$$R_0 = R_0^*. \quad (45)$$

Whenever the gauge output is constrained, it is taken as constant, without loss of generality, which allows to drop the independent variable $R_0$.

From (38), for all operator $u$ on $T(z)$,

$$\forall(u, \delta u = 0), \Delta u = u(T^*) - u(T) \triangleright u^* - u \triangleright \delta u,$$

$$\Delta = \delta.: \quad (46)$$

the result of constraint and perturbation on $u(T(z))$ does not depend on the order of these operations. In particular, with $u \in \{\text{Id}, U\}$, considering (40),

$$\Delta T = \delta T, \Delta U = \delta U. \quad \text{I will also often use } \delta T, \text{ and one must be careful that}$$

**Remark 2.** $\delta T \neq \delta T = \Delta T$.

### 3.3 Perturbative expressions of the gauge output and the fluxes

Using (39) on (1, 3), and noticing that $U$ depends on $T$, that depends on $z$,

$$\Delta R = \langle \Delta Q^i|\Phi^*\rangle + \langle Q^i|\Delta \Phi\rangle, \quad (47)$$

$$-L\Delta \Phi = \Delta L\Phi^* + \Delta Q. \quad (48)$$

Multiplying (38) by $\langle \Phi^i|\rangle$ and using (2),

$$\langle Q^i|\Delta \Phi\rangle = \langle \Phi^i|\Delta L\Phi^* + \Delta Q\rangle. \quad (49)$$

Combining (17, 49),

$$\Delta R = dR_T(\Delta T) + \langle \Phi^i|\Delta L\Delta \Phi\rangle + \langle \Delta Q^i|\Delta \Phi\rangle, \quad (50)$$
where $dR$ is the differential of $R(T)$:

$$dR_T(dT) = \langle \Phi^\dagger | dL\Phi + dQ \rangle + \langle dQ^\dagger | \Phi \rangle. \tag{51}$$

The harmonic part of $\Delta \Phi$ is obtained by projecting (18) on the harmonic subspace and inverting:

$$\Delta \tilde{\Phi} = -\tilde{L}^{-1} (\Delta \tilde{L} \Phi + \Delta \tilde{Q}) - \tilde{L}^{-1} \Delta \tilde{L} \Delta \Phi. \tag{52}$$

I abbreviate $\varphi_c \triangledown \varphi$ (and adjointly). The fundamental amplitude $\langle \varphi^\dagger | \Delta \Phi \rangle$ is determined, independently of the fundamental eigenvalue, from (17, 17):

$$\Delta R = \langle Q^\dagger | \Delta \Phi \rangle + \langle \Delta Q^\dagger | \Phi \rangle + \langle \Delta Q^\dagger | \Delta \Phi \rangle,$n
$$\langle Q^\dagger | \Delta \Phi \rangle = \langle Q^\dagger | \varphi \varphi^\dagger | \Delta \Phi \rangle + \langle Q^\dagger | \Delta \tilde{\Phi} \rangle. \tag{52}$$

From (52, 28, 51) and after rearranging,

$$\Delta R = dR_T(\Delta \tilde{T}) + \langle Q^\dagger | \varphi \varphi^\dagger | \Delta \Phi \rangle + \langle \Phi^\dagger | \Delta \tilde{L} \Delta \Phi \rangle + \langle \Delta Q^\dagger | \Delta \Phi \rangle, \tag{55}$$
$$\tilde{T} \equiv (\tilde{L}, \tilde{Q}, Q^\dagger). \tag{53}$$

**Remark 3.** $\tilde{T} \neq (\tilde{L}, \tilde{Q}, \tilde{Q}^\dagger)$. The constraint is applied to (50, 52, 53). In particular, from (10, 46, 45),

$$0 = \Delta R = \delta R, \tag{54}$$

$$0 = dR_T(\delta \tilde{T}) + \langle \Phi^\dagger | \delta L \delta \Phi \rangle + \langle \delta Q^\dagger | \delta \Phi \rangle, \tag{55}$$
$$\delta \tilde{L} \delta \Phi = \delta \tilde{L} \Phi + \delta \tilde{Q} + \delta \tilde{L} \delta \Phi, \tag{56}$$
$$-\langle Q^\dagger | \varphi \varphi^\dagger | \delta \Phi \rangle = dR_T(\delta \tilde{T}) + \langle \Phi^\dagger | \delta \tilde{L} \delta \Phi \rangle + \langle \delta Q^\dagger | \delta \Phi \rangle. \tag{57}$$

In agreement with (36), not only the direct source, as expected from (1), but also the adjoint source, affects the flux, by ways of (45). The similarity between the r. h. s. of (55, 56, 57) helps calculations.

(53, 50, 57) do not yield perturbed quantities in a closed form; but, with $\delta \tilde{T} = \mathcal{O}(\epsilon), \epsilon \to 0$, they yield $\delta \Phi$ up to $\mathcal{O}(\epsilon^2)$: they are ‘perturbative’ [4], which allows to solve the perturbation problem by perturbation theory [5, 6], as follows.
3.4 Perturbation series of the fluxes and the control variable

Perturbation theory transforms $f(x)^*$ into a function of a perturbation variable $\epsilon$, according to

$$\delta x \rightarrow \epsilon \delta x.$$  \hfill (58)

This is applied to the constrained perturbed system. The response is sought as a power series of $\epsilon$:

$$(U, \dot{z})(T + \epsilon \delta T) \triangleright (U, \dot{z})^* = \sum_{n=0}^{\infty} (U, \dot{z})_n(T, \delta T)\epsilon^n.$$  \hfill (59)

By definition, $(U, \dot{z})_n(T, .)$ is homogeneous:

$$\forall \alpha \in \mathbb{R}^*, (U, \dot{z})_n(T, \alpha \delta T) = \alpha^n (U, \dot{z})_n(T, \delta T).$$

I do not study the convergence of the perturbation series: I only find ‘formal’ power series solutions. Applying (38, 58, 43) on $T(z)$,

$$T^* = T(z^*) + \epsilon \delta T(z^*).$$  \hfill (60)

I assume that $(T, \delta T)(z)$ have power series at $\dot{z}(T)$. (106, 107) are used to develop in powers of $\epsilon$ the functions of $z^*$ in the r. h. s. of (60)

$$T^* = \sum_{n=0}^{\infty} (\dot{z}_n T' + \dot{z}_{n-1} \delta T') + T_n(\dot{z}_1, \ldots, \dot{z}_{n-1}) + \delta T_{n-1}(\dot{z}_1, \ldots, \dot{z}_{n-2}))\epsilon^n,$$ \hfill (61)

where the term $n = 0$ is correct from (109) and a variable shift on $T$, so that $0 = \dot{z}_0 = \dot{z}_{-1} = \delta T_{-1}$.

The power series (59, 61) are introduced into (53, 56, 57). By convention, $\sum_0 = 0$, and I define, for all endomorphism $D$ on $T$, and also for $D = \delta$,

$$\forall p \geq 0, \langle DT \rangle_{0p} \equiv \langle DL^i \Phi^j + DQ^i | \Phi_p \rangle,$$

$$\langle DT \rangle_{p0} \equiv \langle \Phi_p | DL \Phi + DQ \rangle,$$

$$\langle DT \rangle_{0p+p0} \equiv \langle DT \rangle_{0p} + \langle DT \rangle_{p0},$$

$$\langle DT \rangle \equiv \langle DT \rangle_{00} = dR_T(DT),$$

$$\forall p, q \geq 1, \langle DT \rangle_{pq} \equiv \langle \Phi^i_p | DL \Phi^q \rangle.$$ \hfill (62)

$$\forall p, q \geq 1, \langle DT \rangle_{pq} \equiv \langle \Phi^i_p | DL \Phi^q \rangle.$$ \hfill (63)
By identification at order $\epsilon^n$:

$$\forall n \geq 1, -\mathcal{Z}_n(T') = \mathcal{Z}_{n-1}(\delta T') + (T_n + \delta T_{n-1}) + \sum_{1 \leq p \leq n-1} (\mathcal{Z}_{n-p}(T')_0p + \mathcal{Z}_{n-p-1}(\delta T')_0p + (T_{n-p} + \delta T_{n-p-1})_0p), \quad (64)$$

$$\forall n \geq 1, -\mathcal{L}_{\Phi_n} = \mathcal{Z}_n(L' + \bar{\Phi}) + \mathcal{Z}_{n-1}(\delta L' \Phi + \bar{Q}) + (L_n + \delta L_{n-1}) \Phi + \bar{Q} + \delta \bar{Q}_{n-1} + \sum_{1 \leq p \leq n-1} (\mathcal{Z}_{n-p}(L'_0p + \mathcal{Z}_{n-p-1}(\delta L')_0p + (L_{n-p} + \delta L_{n-p-1})_0p), \quad (65)$$

Furthermore, using (64), $\mathcal{Z}_n$ is eliminated from (65, 66) to obtain recursion relations. After some rearrangements:

$$\forall n \geq 1, -\langle Q^\dagger \varphi_{\mathcal{P}}^\dagger | \Phi_n \rangle = \mathcal{Z}_n(L' \Phi + \bar{Q}) + \mathcal{Z}_{n-1}(\delta L' \Phi + \bar{Q}) + (L_n + \delta L_{n-1}) \Phi + \bar{Q} + \delta \bar{Q}_{n-1} + \sum_{1 \leq p \leq n-1} (\mathcal{Z}_{n-p}(L'_0p + \mathcal{Z}_{n-p-1}(\delta L')_0p + (L_{n-p} + \delta L_{n-p-1})_0p), \quad (67)$$

$$\forall n \geq 1, -\langle Q^\dagger | \varphi_{\mathcal{P}}^\dagger | \Phi_n \rangle =$$

$$\mathcal{Z}_n-1((\delta T') - (\delta T') \frac{\langle T' \rangle}{\langle T' \rangle}) + (\langle T_n + \delta T_{n-1} \rangle + (\langle T_n + \delta T_{n-1} \rangle \frac{\langle T' \rangle}{\langle T' \rangle}) +$$

$$\sum_{1 \leq p \leq n-1} (\mathcal{Z}_{n-p}(\langle T' \rangle)_0p - (\langle T' \rangle)_0p \frac{\langle T' \rangle}{\langle T' \rangle}) + \mathcal{Z}_{n-p-1}((\delta T')_0p - (\delta T')_0p \frac{\langle T' \rangle}{\langle T' \rangle}) +$$

$$(\langle T_{n-p} + \delta T_{n-p-1} \rangle_0p - (T_{n-p} + \delta T_{n-p-1} \rangle_0p \frac{\langle T' \rangle}{\langle T' \rangle})) \quad (68)$$
Finally, the coefficients \((U, z)_n\) can be constructed, up to any order, from the reference flux-to-source operator, source and constraint, and their exciting perturbations, with the recursion relations \(64, 67, 68\), and adjoint relations. First and second order results are

\[
\dot{z}_1 = -\frac{\langle \delta T \rangle}{\langle T' \rangle},
\]

\[
\Phi_1 = -\frac{1}{\langle Q^{\dagger}|\varphi \rangle}(\langle \delta \tilde{T} \rangle - \langle \delta T \rangle \langle \tilde{T}' \rangle) \varphi - \tilde{L}^{-1}(-\langle \delta T \rangle \langle \dot{\tilde{T}} \rangle + \tilde{L} \dot{\Phi} + \delta \tilde{Q});
\]

\[
-\dot{z}_2 \langle T' \rangle = \dot{z}_1(\langle \delta T' \rangle + \langle T'_0 \rangle) + \langle T_2 \rangle + \langle \delta T \rangle_0
\]

In \((71)\) appear the various contributions to control variable response non-linearity: \(\langle T_2 \rangle, \langle \delta T' \rangle\) represent, respectively, the intrinsic control non-linearity and the cross effect of control and excitation, vanishing if control is, respectively, linear \((41)\) or remote \((42)\). \(\langle T'_0 \rangle, \langle \delta T \rangle_0\) represent control and exciting perturbation shielding, as they involve the flux shielding coefficient \(\Phi_1\), a global consequence of both exciting and control perturbations.

**Remark 4.** Because of shielding, the control variable response \(\epsilon \mapsto \dot{z}\) may be non-linear, notwithstanding linear and remote control.

\[
-\tilde{L} \dot{\Phi}_2 = \dot{z}_1(-\frac{\langle \delta T' \rangle + \langle T'_0 \rangle}{\langle T' \rangle} \langle \tilde{T}' \rangle \dot{\Phi} + \tilde{Q}' + \delta \tilde{L} \dot{\Phi} + \tilde{L} \dot{\Phi}_1 + \delta \tilde{Q}) - \frac{\langle T_2 \rangle + \langle \delta T \rangle_0}{\langle T' \rangle} \langle \tilde{T}' \rangle \dot{\Phi} + \tilde{Q}' + \tilde{L} \dot{\Phi}_1 + \dot{\Phi}_2,
\]

\[
-\langle Q^{\dagger}|\varphi \phi^{\dagger} \rangle |\Phi_2 \rangle = \dot{z}_1(\langle \delta \tilde{T} \rangle - \langle \delta T' \rangle \langle \tilde{T}' \rangle - \langle \tilde{T}' \rangle \langle T'_0 \rangle) + (\langle T_2 \rangle - \langle \delta T \rangle_0 \langle T'_0 \rangle) + (\langle \tilde{T}_2 \rangle - \langle \tilde{T}' \rangle \langle T'_0 \rangle) + (\langle \delta \tilde{T} \rangle_0 - \langle \delta T \rangle_0 \langle T'_0 \rangle).
\]

For linear control, using \((111), (64, 67, 68)\) become

\[
\forall n \geq 1, -\dot{z}_n \langle T' \rangle = \dot{z}_{n-1} \langle \delta T' \rangle + \delta_{n,1} \langle \delta T \rangle + \langle \delta T \rangle_0, n-1 + \sum_{1 \leq p \leq n-1} (\dot{z}_{n-p} \langle T'_p \rangle_0 + \dot{z}_{n-p-1} \langle \delta T' \rangle_0),
\]

(72)
\[ \forall n \geq 1, -\overline{L} \Phi_n = z_{n-1}(-\frac{\langle \delta T \rangle^2}{\langle T \rangle} (\overline{L}' \Phi + \overline{Q}') + \delta \overline{L}' \Phi + \delta \overline{Q}') + \]
\[ \delta_{n,1}(-\frac{\langle \delta T \rangle}{\langle T \rangle} (\overline{L}' \Phi + \overline{Q}') + \delta \overline{L} \Phi + \delta \overline{Q}) - \frac{\langle \delta T \rangle_{0,n-1}}{\langle T \rangle} (\overline{L}' \Phi + \overline{Q}') + \delta \overline{L} \Phi_{n-1} + \]
\[ \sum_{1 \leq p \leq n-1} (z_{n-p}(-\frac{\langle T \rangle_0^p}{\langle T \rangle} (\overline{L}' \Phi + \overline{Q}') + \overline{L}' \Phi_p) + z_{n-p-1}(-\frac{\langle \delta T \rangle_0^p}{\langle T \rangle} (\overline{L}' \Phi + \overline{Q}') + \delta \overline{L}' \Phi_p)), \] (73)

\[ \forall n \geq 1, -\langle Q \mid \varphi \varphi^\dagger \mid \Phi_n \rangle = z_{n-1}(\langle \delta T' \rangle - \langle \delta T \rangle^2) + \]
\[ \delta_{n,1}(\langle \delta T' \rangle - \langle \delta T \rangle^2) + (\langle \delta T \rangle_{0,n-1} - \langle \delta T \rangle_{0,n-1}^2) + \]
\[ \sum_{1 \leq p \leq n-1} (z_{n-p}(\langle T' \rangle_0^p - \langle T \rangle_0^p \frac{T'}{\langle T \rangle} + z_{n-p-1}(\langle \delta T' \rangle_0^p - \langle \delta T \rangle_0^p \frac{T'}{\langle T \rangle})). \] (74)

### 3.5 Perturbation series of a bilinear form of the fluxes

I seek the power series of \( \langle DT \rangle^* \), where \( D \) is any endomorphism on \( T \), like \( D = d/dz \), such that \( \delta D = 0 \).

**Remark 5.** By (51, 62), the operator \( \langle D. \rangle \) depends implicitly on \( T \), through the fluxes, so that, in general \( \delta(\langle DT \rangle, \delta(\langle D. \rangle, \delta T) \triangleright \Delta \langle DT \rangle \neq \langle \delta DT \rangle). \)

I evaluate \( \Delta \langle DT \rangle \) by applying twice the formula for the perturbation of a bilinear product (37); after rearranging, I obtain the perturbative expression

\[ \Delta \langle DT \rangle - \langle \delta DT \rangle = \langle D L^\dagger \Phi^\dagger + D Q^\dagger \mid \delta \Phi \rangle + \langle \delta \Phi^\dagger \mid D L^\dagger \Phi + D Q^\dagger \rangle + \langle \delta \Phi^\dagger \mid D L^\dagger \delta \Phi \rangle. \] (75)

I assume that \( D \) is linear. Applying \( D \) and (58) on (53), noticing that \( D \) commutes, by linearity, with \( \delta \) and \( \delta' \), I obtain that (54), where \( T \rightarrow DT \), still holds, giving the power series of \( DT^* \). This power series, and that of \( (U, z)^* \) (58), are inserted into (75), yielding the power series

\[ \delta(\langle DT \rangle, \delta(\langle D. \rangle, \epsilon \delta T) \triangleright \Delta \langle DT \rangle \equiv \sum_{n=1}^{\infty} \langle DT \rangle_n(T, \delta T) \epsilon^n, \] (76)

the coefficients of which are determined by identification. Eventually,

\[ \langle (\langle DT \rangle, \delta(\langle D. \rangle, \epsilon \delta T)^* \triangleright \langle DT \rangle^* \rangle \equiv \sum_{n=0}^{\infty} \langle DT \rangle_n(T, \delta T) \epsilon^n, \] (77)
\[\forall n \geq 0, \langle DT \rangle_n(T, \delta T) = z_n \langle DT' \rangle + z_{n-1} \langle D\delta T' \rangle + \langle DT_n + D\delta T_{n-1} \rangle + \]
\[
\sum_{1 \leq p \leq n} (z_{n-p} \langle DT' \rangle_{0p+p0} + z_{n-p-1} \langle D\delta T' \rangle_{0p+p0} + \langle DT_{n-p} + D\delta T_{n-p-1} \rangle_{0p+p0}) +
\]
\[
\sum_{1 \leq p_1, p_2 \leq n} (z_{n-p_1-p_2} \langle DT' \rangle_{p_1 p_2} + z_{n-p_1-p_2-1} \langle D\delta T' \rangle_{p_1 p_2} + \langle DT_{n-p_1-p_2} + D\delta T_{n-p_1-p_2-1} \rangle_{p_1 p_2}).
\]

(78)

In (78), \(z_n\) is known and need not be eliminated, which allows to group terms nicely:

\[\forall n \geq 0, \langle DT \rangle_n(T, \delta T) = \]
\[
\sum_{0 \leq p_1, p_2 \leq n} (z_{n-p_1-p_2} \langle DT' \rangle_{p_1 p_2} + z_{n-p_1-p_2-1} \langle D\delta T' \rangle_{p_1 p_2} + \langle DT_{n-p_1-p_2} + D\delta T_{n-p_1-p_2-1} \rangle_{p_1 p_2}).
\]

(79)

First and second order results are

\[\langle DT \rangle_1 = z_1 \langle DT' \rangle + \langle D\delta T \rangle + \langle DT \rangle_{01+10},\]

\[\langle DT \rangle_2 = z_2 \langle DT' \rangle + \langle DT_2 \rangle + z_1 (\langle D\delta T' \rangle + \langle DT' \rangle_{01+10}) + \langle D\delta T \rangle_{01+10} + \langle DT \rangle_{02+20} + \langle DT \rangle_{11}.
\]

### 4 Weight definition and observability

#### 4.1 The balance equation and weight definition

For an infinitesimal perturbation of the constrained system, (55) becomes:

\[0 = dR_T(dT).
\]

(80)

Applying (38) to \(d(T(z))\), then the constraint, with (46):

\[d(T(z)) = T'(z)dz + dT(z); d_T = T'dz + dT.
\]

(81)

I introduce (81) into (80), use (62), where \(D \rightarrow d\), to obtain the balance equation

\[0 = \langle T' \rangle dz + \langle dT \rangle.
\]

(82)
\( \langle T' \rangle \) is the ‘differential weight’ of the exciting variable \( z \). (82) is indeed equivalent to (69).

From the chain rule, the differential weight is just the derivative of the unconstrained gauge output (at constant \( T \)), with respect to \( z \), hence an observable:

\[
\langle T' \rangle = (R \circ T)'(z) \tag{83}
\]

I show up the exciting variable \( z_1 \), driving \( \delta T \), while the original control variable is noted \( z_2 \):

\[
\exists T_2, (T + \delta T)(z_2) = T_2(z_1, z_2). \tag{84}
\]

For a given exciting perturbation \( \delta T \), (the function) \( T_2 \) remains constant.

**Remark 6.** *An experiment that is not reproducible is still represented by a constant \( T_2 \), taking for \( z_1 \) the quasi-stationary time.*

I assume for \( T_2(\cdot, z_2) \) the same regularity properties as for \( T_2(z_1, \cdot) \). Thus, the variables \((z_1, z_2)\) are exchangeable, and either may serve as the control variable. I define an exchange operator:

\[
ET_2(z_2, z_1) \equiv T_2(z_1, z_2). \tag{85}
\]

The constraint operator applies to each partial function \( T_2(\cdot, z_2), T_2(z_1, \cdot) \). For example, with \( z_2 \) as the control variable,

\[
u(T_2(z_1, \cdot)) = u(T_2(z_1, \tilde{z}_2)), \tilde{z}_2(T_2(z_1, \cdot)) \triangleright \tilde{z}_2.
\]

The functions \( z_2 \mapsto \tilde{z}_2(T_2(\cdot, z_2)) \) and \( z_1 \mapsto \tilde{z}_1(T_2(z_1, \cdot)) \) are inverse to each other and the constrained value of \( u(T_2(z_1, z_2)) \) does not depend on the choice of the control variable. For all \((z_1, z_2)\), such that \( R(T_2(z_1, z_2)) = R_0 \),

\[
(z_1, \tilde{z}_2) = (z_1, z_2) = (\tilde{z}_1, z_2),
\]

\[
u(T_2(z_1, \cdot)) = u(T_2(z_1, z_2)) = u(T_2(\cdot, z_2)).
\]

With (84), the balance equation (82) (with \( z_2 \) as the control variable) takes the nearly symmetric form

\[
0 = \langle \partial_1 T_2(\cdot, \tilde{z}_2) \rangle dz_1 + \langle \partial_2 T_2(z_1, \cdot) \rangle d\tilde{z}_2. \tag{86}
\]

The differential weight of \( z_1 \) (controlled by \( z_2 \)) is

\[
w_1(T_2, z_1) = \langle \partial_1 T_2(z_1, \cdot) \rangle, \tag{87}
\]
and symmetrically, so that (80) becomes

\[0 = w_1(T_2, z_1)dz_1 + w_2(T_2, \bar{z}_2)d\bar{z}_2.\] (88)

I define the (integral) weight (of \(z_1\)),

\[Z_1(T_2, z_1) \equiv \int_0^{z_1} w_1(T_2, z_1')dz_1'.\] (89)

The function \(Z_1(T_2, .)\) is the (non-linear) weight scale. By integration of (88), the sum of weight perturbations, over both variables, is zero:

\[Z_1(T_2, z_1) + Z_2(T_2, \bar{z}_2(T_2(z_1, .))) - Z_2(T_2, \bar{z}_2(T_2(0, .))) = 0.\] (90)

(88, 90) are the basis of the weighing method, announced by (3).

4.2 Weight perturbation theory

From law invariance (10) (on functionals):

\[0 = \delta w_1 = \delta Z_1.\] (91)

A general differential weight perturbation is

\[\delta(w_1(T_2, z_1), 0, (\delta T_2, \delta z_1)) \triangleright \Delta w_1.\] (92)

By definition (87),

\[\Delta w_1 = \delta(\partial T_2(z_1, .), \delta(\partial_1, .), \delta(T_2(z_1, .)))\]

which is just (76), where

\[T \rightarrow T_2(z_1, .); D \rightarrow \partial_1'; \partial_2' \rightarrow \epsilon \rightarrow 1.\] (93)

Using the properties of definite integral in (89), an arbitrary weight perturbation is

\[\delta(Z_1(T_2, z_1), 0, \delta(T_2, z_1)) = \int_{z_1}^{z_1'} w_1(T_2, z_1)dz_1' + \int_0^{z_1'} \delta(w_1(T_2, z_1'), 0, (\delta T_2, 0))dz_1'.\] (94)

The last integrand is just (92), where

\[\delta z_1 \rightarrow 0; z_1 \rightarrow z_1'.\]
has another important application. I write \( \epsilon \) as
\[
Z_1(T_2, z_1) = \int_0^{z_1} (w_1(T_2, 0), 0, (0, z'_1)) dz'_1 = w_1(T_2, 0) z_1 + \int_0^{z_1} \delta(w_1(T_2, 0), 0, (0, z_1)).
\] (95)

The last integrand is just (92), where
\[
z_1 \to 0; \delta T_2 \to 0; \delta z_1 \to z_1.
\]

Therefore, the weight scale itself is obtained as a perturbation series, which is, more precisely, (76), where
\[
\delta T \to T_1(z_1, .) - T_2(0, .); T \to T_2(0, .); D \to \partial_1'; \to \partial_2; \epsilon \to 1.
\]

I complete the determination of the weight scale only in the case of a linear exciting variable: \( \partial^2_1 T = 0 \), equivalent to
\[
T_2(z_1, z_2) = T(z_2) + z_1 \delta T(z_2); \delta T(z_2) \equiv \partial_2 T.
\] (96)

With (97), the remote control condition (12) becomes \( \partial^2_1 T_2 = 0 \). In (95), \( z'_1 \) is used as the perturbation variable:
\[
(w_1(T_2, 0), 0, (0, z'_1)) = (\langle \partial_1 T_2 \rangle, \delta \langle \partial_1 \rangle, z'_1 \delta T) \to \langle \delta T \rangle^n z_1^n,
\]
and the coefficients \( \langle \delta T \rangle_n \) are obtained from (78, 79), where
\[
\epsilon \to z'_1; DT \to \delta T; D\delta T \to 0;
\]

\[
\forall n \geq 0, \langle \delta T \rangle_n = z_n \langle \delta T' \rangle + \langle \delta T_n \rangle + \sum_{1 \leq p \leq n} (\langle z_{n-p} \langle \delta T' \rangle_{0p} + \langle \delta T_{n-p} \rangle_{0p} + \langle \delta T_{n-p} \rangle_{0p} + \langle \delta T_{n-p} \rangle_{0p}) + \sum_{1 \leq p_1 \leq n-1} \langle z_{n-p_1} \langle \delta T' \rangle_{p_1p_2} + \langle \delta T_{n-p_1} \rangle_{p_1p_2};
\]

\[
\forall n \geq 0, \langle \delta T \rangle_n = \sum_{0 \leq p_1, p_2 \leq n} z_{n-p_1-p_2} \langle \delta T' \rangle_{p_1p_2} + \langle \delta T_{n-p_1-p_2} \rangle_{p_1p_2}.
\] (97)

First and second order results are
\[
\langle \delta T \rangle_1 = z_1 \langle \delta T' \rangle + \langle \delta T \rangle_{01+10},
\]
\[
\langle \delta T \rangle_2 = z_2 \langle \delta T' \rangle + \langle \delta T_{02} \rangle + z_1 \langle \delta T' \rangle_{01+10} + \langle \delta T \rangle_{02+20} + \langle \delta T \rangle_{11}.
\]

Evaluating the integral in (95), and with \( z_1 \to \epsilon \), to remember that the exciting variable is linear,
\[
Z_1(T_2, \epsilon) = \sum_{n=0}^{\infty} \langle \delta T \rangle_n \frac{\epsilon^{n+1}}{n+1}.
\] (98)
4 WEIGHT DEFINITION AND OBSERVABILITY

4.3 Operator perturbation weighing

I define an ideal instrument by its parameters $T$, consisting of linear functions of the control variable (there is no interest in non-linear ideal control); the measured object is a perturbation $\delta T$, with remote control. The combination of the instrument and the measured object is represented by the functional parameters $T_2$:

$$T_2(\epsilon, z) \equiv T(z) + \epsilon \delta T, \Phi(T_2(0, .)) \equiv 0.$$  \hfill (99)

Observables are not available for $T_2$, because of its ideal part $T$, but on some approximate realization $T_2^*$. $\delta T_2 = T_2^* - T_2$ is the realization error, meaning it is impossible to realize exactly the ideal experiment. From (99),

$$-Z_1(T_2, \epsilon) = Z_2(T_2, \Phi(T_2, \epsilon)) - Z_2(T_2, 0) \approx Z_2(T_2^*, \Phi(T_2^*, \epsilon)) - Z_2(T_2^*, 0). \hfill (100)$$

The error analysis is left for the next section. From (89, 83),

$$Z_2(T_2^*, \Phi(T_2, \epsilon)) - Z_2(T_2^*, 0) = \int_0^{\Phi(T_2^*, \epsilon)} \partial_2 R \circ T_2^*(\xi(T_2^*, z), z)dz,$$  \hfill (101)

which I write, more lightly, by impliciting $T_2^*$:

$$Z_2(\Phi(\epsilon)) - Z_2(0) = \int_0^{\Phi(\epsilon)} \partial_2 R^*(\xi(z), z)dz.$$

(102)

As explained in section 4.1, the functions $\xi$ and $\Phi$ (at constant $T_2$) are inverse to each other.

(102) explains how to map observables onto weight, by universal operations, like summation, independent of the system parameters. (98) expresses what is weight and how to compute weight, by universal operations, from the system parameters, representing the instrument and the measured object. The function $Z_1(T_2, .)$ is a power series, bijectively related to its coefficients $\langle \delta T \rangle_n$. As control in $T_2$ is linear and remote, (101) becomes

$$\forall n \geq 0, \langle \delta T \rangle_n = \sum_{0 \leq p_1, p_2 \leq n \atop p_1 + p_2 = n} \langle \delta T \rangle_{p_1 p_2}, \hfill (103)$$

In particular, from (82, 71),

$$\langle \delta T \rangle_{00} = \langle \delta T \rangle_0 = \langle \delta T \rangle = \langle \Phi^\dagger | \delta L \Phi \rangle + \langle \Phi^\dagger | \delta Q \rangle + \langle \delta Q^\dagger | \Phi \rangle.$$
There are simplifications also in the recursion relations \(I\). If the sequence of perturbed fluxes \(\Phi_n\) is a basis (and adjointly) and 

\[ 0 = \delta Q = \delta Q^\dagger, \]

then \(\delta L\) is completely determined (except for a scalar factor) by its matrix elements \(\langle \delta T \rangle_{p,p}^n\), defined by \(I\). From \(J\), \(\langle \delta T \rangle_n\) is the finite sum, invariant by transposition, of the diagonal \(p \mapsto \langle \delta T \rangle_{n-p,p}\). The \(\langle \delta T \rangle_n\) do not fully determine \(\delta L\), except, for example, if \(\delta L\) has only one coefficient on each diagonal.

If \(H\) is not separable (has no basis [1, §3.7, p. 95]), then \(\delta L\) cannot be obtained from observables. We must content ourselves with weight, which does not completely determine \(\delta L\), but can be used to check postulated \(\delta L\), given the perturbed fluxes. Or, knowing \(\delta L\), perturbed flux calculations can be checked, from measured weights.

### 4.4 Error analysis

From (38), the error on the measured weight scale is

\[
Z_2^*(T_2^*,.) - Z_2(T_2,.) = (Z_2(T_2^*,.) - Z_2(T_2,.) + \delta Z_2(T_2^*,.)
\]

The first term in the r. h. s. is the realization error, that may be evaluated, just like any perturbation, from weight perturbation formulae \(I\), after the variable exchange \(Z_2(T_2,.) = Z_1(ET_2,.)\) \(J\).

The realization error could be removed by taking \(T_2 \equiv T_2^*\), thus attaching the weight definition to a real (non-ideal) standard ('old-style' metrology).

The second term in the r. h. s. of (104), \(\delta Z_2(T_2^*,.)\), is a processing error, meaning that observables are not processed exactly as demanded by (101), for example, because of discretization errors. Processing errors such that

\[
\exists f, Z_2^*(T_2,.) = Z_2(f(T_2),.)
\]

are reducible, by definition \(I\), to realization errors. Not all processing errors are reducible.

Errors in the l. h. s. of (100) are mathematically treated just as errors in the r. h. s. (except for variable exchange). Although \(T\) is ideally defined, one may have to compare weight scales resulting from different definitions of \(T\). An error on \(Z_1\) may also occur in computation, e. g. a power series truncation. From (11), an error on \(Z_1\), which is not reducible in the sense of (107), breaks the law of flux-to-source operator and constraint linearity.

The weight scale is actually obtained approximately, and only on a finite set of \(\epsilon\). Something must be said on error propagation, from the weight scale points to its power series coefficients.
Firstly, I consider only the effect of discreteness of $\epsilon$. I assume that $Z_1(T_2, \epsilon)$ is exactly known, but only for $N + 2, N \geq 0$ discrete values of $\epsilon$. An interpolating polynomial of degree $N + 1$ can be constructed, giving (approximately) the coefficients $\langle \delta T \rangle_{0 \leq n \leq N}$; higher order coefficients are completely undetermined: there is a cut-off between known and unknown coefficients, at $n = N$. If $N = 0$ (only two values of $\epsilon$ are realized), then only $\langle \delta T \rangle_0$ is obtained.

Secondly, I take into account, not only discreteness, but also errors on weight values. Errors propagate non-uniformly to the coefficients $\langle \delta T \rangle_n$, the error on $\langle \delta T \rangle_n$ increasing with $n$, at given $N$, and decreasing with $N$ at given $n$. Hence the interest of taking large $N$, even if only $\langle \delta T \rangle_0$ is sought. $N \geq 1$ reveals and allows to correct the shielding error on $\langle \delta T \rangle_0$ obtained with $N = 0$.

5 Conclusion

I applied perturbation theory to a linear source problem, allowing for a critical limit, the consequences of which were non-linear response, ill-posedness, a regularizing constraint, and the functional character of parameters.

The main hypotheses were the constraint linearity and the existence of a stable, spectrally separated, harmonic hyperplane. (In spatially extended systems, spectral separation usually follows from boundary conditions.)

The basic tool was linear (or multi-linear) vector algebra. The perturbation of functional parameters and the constraint application were delicate and required specific notations. Progress was achieved by the thorough use of symmetries (the last three being familiar in theoretical physics):

- duality (‘adjointness’),
- commutation relations (between the constraint operator and variational symbols),
- exchange (of the exciting and control variables),
- gauge invariance (results are independent of the gauge output unit).

Perturbation theory was then used to propose a weight definition and measurement method. More precisely, the weight defining functional $Z_1$ and the weighing functional $Z_2$, needed in (4), were constructed, showing weight as a secondary observable, except for a realization error, which was analyzed, from the same perturbation analysis.
The weighing functional is an integro-differential processor (filter) on both the unconstrained (open loop) and constrained (closed loop) system responses.

The weight scale $Z_1(T_2, \ldots)$ is a power series, whose coefficients are (in the simplest case) diagonal sums of the flux-to-source perturbation matrix $\delta L$. As this relation is generally not invertible, $\delta L$ cannot be fully determined.

The present work is not mathematically complete, and does neither treat in detail any particular application. Emphasis was put into discovering and solving a new problem, operator weighing.

The mathematical model and the weighing method may actually apply, more or less easily, to high-gain feedback linear amplifiers, with any number of degrees of freedom: electronic amplifiers, photomultipliers, nuclear fission chain reactors [4, 2] and computational models thereof.

On the mathematical side, progress may be sought in the formulation of the abstract problem, the analytic aspects of its solution (the conditions of convergence of power series), and carrying out large symbolic calculations.

Systems are often studied with the approximation of linear response, which fails for non-linear systems, but also, as I pointed out, for a linear system (especially quasi-critical). Measurement methods based on the linear response approximation lack of an essentially non-linear error analysis, and hardly have any quantitative interest.

### A Composition of power series

The power series of the compound function $T \circ z$ is sought. $z$ has a power series at zero and $T$ has a Taylor series at $z_0 = z(0)$:

$$T(z) = \sum_{n=0}^{\infty} \frac{1}{n!} T^{(n)}(z_0)(z - z_0)^n.$$

The multinomial formula,

$$\forall n \geq 0, (z(\epsilon) - z_0)^n = \left( \sum_{p=1}^{\infty} z_p \epsilon^p \right)^n = \sum_{q_0, \sum q_p = n} n! \prod_{p \geq 1} \frac{z_p^{q_p}}{q_p!} \epsilon^{\sum q_p},$$

is multiplied by $T^{(n)}/n!$, summed over $n$, and the terms of same power in $\epsilon$
are gathered:

\[ T \circ z(\epsilon) = \sum_{n=1}^{\infty} (T \circ z)_n(z_0)\epsilon^n, \]

\[ (T \circ z)_n = \sum_{q_p \geq 0, \sum pq_p = n} T^{(\Sigma q_p)} \prod_{p \geq 1} \frac{z_p^{q_p}}{q_p!}. \]

The sum in the r. h. s. is split into a term corresponding to the sequence \( q_p = \delta_{p,n} \), and a remainder \( T_n \) (\( T_1 = 0 \)):

\[ \forall n \geq 1, (T \circ z)_n = z_n T' + T_n(z_1, \ldots, z_{n-1}), \]

\[ \forall n \geq 2, T_n(z_1, \ldots, z_{n-1}) = \sum_{p<n, q_p \geq 0, \sum pq_p = n} T^{(\Sigma q_p)} \prod_{1 \leq p < n} \frac{z_p^{q_p}}{q_p!}. \]

\[ T_0 = T, \]
\[ T_1 = 0, \]
\[ T_2(z_1) = T'''z_1^2/2, \]
\[ T_3(z_1, z_2) = T'''z_1z_2 + T'''z_1^3/3!. \]

If \( T \) is linear, then

\[ \forall n \geq 2, T_n = 0. \]

Proof. If \( T \) is linear and \( T^{(\Sigma q_p)} \) in (108) is not zero, then

\[ \sum_{1 \leq p < n} q_p = 1; \exists! m < n, q_p = \delta_{p,m}; \sum_{1 \leq p < n} pq_p = m = n, \]

which is inconsistent. \( \square \)

## B Finite-dimensional examples

The reference gauge output \( R_0 \) is set to unity.

I assume, for simplicity, linear and remote control:

\[ L(z') = A + z'B, \delta B = 0, \delta L(z') = C = \delta A, \]
\[ L_2(\epsilon, z') = L(z') + \epsilon \delta L(z') = A + z'B + \epsilon C, \]

where \( A, B, C \) are constant matrices; \( A \) has the eigenvalue zero; sources are unexcited (\( 0 = \delta Q = \delta Q^\dagger \)) and uncontrolled (\( 0 = Q' = Q'^\dagger \)); there is no realization error (\( \delta T_2 = 0 \)).

I will obtain directly the fluxes, the gauge output, the control variable and flux response, and check (90, 98, 102). The functional operations, like perturbation and constraint, will appear concretely.
B.1 One dimension

A = 0, the harmonic subspace is \{0\}.

\[
\Phi^*(\epsilon, z') = -\frac{Q}{z'B + \epsilon C};
\]
\[
R^*(\epsilon, z') = -\frac{Q^\dagger Q}{z'B + \epsilon C}, 1 = -\frac{Q^\dagger Q}{z'B + \epsilon C}.
\]

With the new variable \(z \equiv z' + Q^\dagger Q/B\),

\[
\Phi^*(\epsilon) = -\frac{C\epsilon}{B}, z(0) = 0,
\]
\[
\Phi^*(\epsilon, z) = \frac{Q}{Q^\dagger Q - zB - \epsilon C},
\]
\[
\Phi^*(\epsilon) = \frac{1}{Q^\dagger}, \Phi^*(\epsilon) = \frac{1}{Q},
\]
\[
R^*(\epsilon, z) = \frac{Q^\dagger Q}{Q^\dagger Q - zB - \epsilon C}.
\]

From (87, 89, 102):

\[
Z_1(\epsilon) = \Phi^\dagger C\Phi^* \epsilon = \frac{C\epsilon}{Q^\dagger Q};
\]
\[
\partial_2 R^*(\epsilon, z) = \frac{BQ^\dagger Q}{(Q^\dagger Q - zB - \epsilon C)^2}, \partial_2 R^*(\epsilon, z) = \frac{B}{Q^\dagger Q};
\]
\[
Z_2(\epsilon) - Z_2(0) = \int_0^{\epsilon(\epsilon)} \partial_2 R^*(\epsilon(z), z) dz = \frac{Bz}{Q^\dagger Q} = -Z_1(\epsilon).
\]

\(\delta L/(Q^\dagger Q)\) is the secondary observable, resulting from the weighing method.

B.2 Two dimensions, linear control variable response

The complementary matrix (or transposed comatrix) \(\overline{A}\) of a matrix \(A\) is

\[
\forall A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \overline{A} \equiv \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.
\]

\[
\overline{AA} = A\overline{A} = \det(A)1, \overline{A} = A, \overline{A}^\dagger = \overline{A}^\dagger.
\]

(112)

The operator \(A \mapsto \overline{A}\) is linear and the non-linearity of \(A \mapsto A^{-1} = \det(A)^{-1}\) \(\overline{A}\) lays fully in the inverse determinant.
Let the ‘codeterminant’ $A * B$ of two matrices $A, B$ be

$$A * B \equiv \begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$ 

Properties: * is a bilinear product, symmetric and, for all $A, C$,

$$A * A = 2 \det A, A * C = A^\dagger * C^\dagger,$$

$$\overline{AC}A = (C * A)\overline{A} - \det(A)\overline{C}.$$  \hfill (113)

The reference flux-to-source operator has a stable subspace and the eigenvalue scale is arbitrary. Therefore, it is no restriction to take

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hfill (114)

Property: for $A$ defined by (114) and for all $B, C$,

$$\overline{ACB} + \overline{BCA} = (C * B)\overline{A} + b_{11}\overline{C} - c_{11}\overline{B}.$$  \hfill (115)

Let $Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ and adjointly $Q^\dagger = \begin{pmatrix} q_1^\dagger \\ q_2^\dagger \end{pmatrix}$.

$$-\det L_2(\epsilon, z') = -\det(B)z'^2 - \det(C)\epsilon^2 - B * C'\epsilon + b_{11}z' + c_{11}\epsilon,$$  \hfill (116)

$$-\det L_2\Phi^*(\epsilon, z') = (\overline{A} + z'\overline{B} + \epsilon\overline{C})Q,$$  \hfill (117)

$$-\det L_2R^*(\epsilon, z') = -q_1^\dagger q_1 + \langle Q^\dagger |\overline{B}Q \rangle z' + \langle Q^\dagger |\overline{C}Q \rangle \epsilon,$$  \hfill (118)

$$-\det L_2(\epsilon) = -q_1^\dagger q_1 + \langle Q^\dagger |\overline{B}Q \rangle z' + \langle Q^\dagger |\overline{C}Q \rangle \epsilon.$$  \hfill (119)

I assume, for simplicity, $b_{11} \neq 0$ and

$$0 = \det B = \det C = B * C,$$

which makes the control variable response linear. (119) still leaves five degrees of freedom in $B, C$, out of eight generally.

With the new variable

$$z \equiv z' - \alpha_1, \alpha_1 \equiv \frac{q_1^\dagger q_1}{\langle Q^\dagger |\overline{B}Q \rangle - b_{11}}$$

and (119), (116, 117, 118) yield

$$-\det L_2(\epsilon, z) = b_{11}z + c_{11}\epsilon + b_{11}\alpha_1,$$

$$R^*(\epsilon, z) = \frac{\langle Q^\dagger |\overline{B}Q \rangle z + \langle Q^\dagger |\overline{C}Q \rangle \epsilon + \alpha_1\langle Q^\dagger |\overline{B}Q \rangle - q_1^\dagger q_1}{b_{11}z + c_{11}\epsilon + b_{11}\alpha_1},$$  \hfill (120)

$$z(\epsilon) = \alpha_2\epsilon, \alpha_2 \equiv -\frac{\langle Q^\dagger |\overline{C}Q \rangle - c_{11}}{\langle Q^\dagger |\overline{B}Q \rangle - b_{11}}.$$
\[ \Phi^*(\epsilon) = \frac{\alpha_3}{1 - \epsilon \alpha_4} (\overline{A} + \alpha_1 \overline{B} + \epsilon (\mathcal{C} + \alpha_2 \mathcal{B})) Q, \]  
(121)

\[ \alpha_3 \equiv \frac{1}{b_{11} \alpha_1}, \alpha_4 \equiv \frac{b_{11} \langle Q^\dagger \mathcal{C} Q \rangle - c_{11} \langle Q^\dagger \mathcal{B} \rangle}{b_{11} q_1^\dagger q_1}, \]

\[ \alpha_1 \alpha_4 + \alpha_2 = -\frac{c_{11}}{b_{11}}. \]  
(122)

To avoid the determination of primitives, I will rather check the differential form of (90):

\[ -\langle \Phi^\dagger \mathcal{C} \Phi^* \rangle (\epsilon) = \partial_2 R^*(\epsilon, z) \frac{d \dot{z}}{d \epsilon}(\epsilon). \]  
(123)

The r. h. s. of (123) is evaluated by applying on (120)

\[ \forall (A, \det A \neq 0), \forall (\dot{z}, \frac{a_{11} \dot{z} + a_{12}}{a_{21} \dot{z} + a_{22}} = 1), \frac{d}{d \epsilon} \frac{a_{11} \dot{z} + a_{12}}{a_{21} \dot{z} + a_{22}}(\dot{z}) = \frac{(a_{11} - a_{21})^2}{\det A} : \]

\[ \partial_2 R^*(\epsilon, z) \frac{d \dot{z}}{d \epsilon}(\epsilon) = -\frac{\langle Q^\dagger \mathcal{B} Q \rangle - b_{11}(\langle Q^\dagger \mathcal{C} Q \rangle - c_{11})}{b_{11} q_1^\dagger q_1 (1 - \epsilon \alpha_4)} + \frac{1}{1 - \epsilon \alpha_4} = \sum_{n=0}^{\infty} \alpha_n^4 \epsilon^n. \]  
(124)

The l. h. s. of (123) involves the differential weight of the exciting variable:

\[ \langle \Phi^\dagger \mathcal{C} \Phi^* \rangle (\epsilon) = w_1(\epsilon) = Z_1'(\epsilon) = \sum_{n=0}^{\infty} \delta T_n \epsilon^n. \]

The \( \delta T_n \) are given by (103), which appears directly by substituting in \( \langle \Phi^\dagger \mathcal{C} \Phi^* \rangle \) the power series of the perturbed flux (121) (and adjointly). The flux and weight power series coefficients are

\[ \Phi_0 = \alpha_3 (\overline{A} + \alpha_1 \overline{B}) Q, \quad \Phi_0^\dagger = \alpha_3 (\overline{A}^\dagger + \alpha_1 \overline{B}^\dagger) Q^\dagger, \]

\[ \Phi_1 = \alpha_3 (\alpha_4 \overline{A} - \frac{c_{11}}{b_{11}} \overline{B} + \mathcal{C}) Q, \quad \Phi_1^\dagger = \alpha_3 (\alpha_4 \overline{A}^\dagger - \frac{c_{11}}{b_{11}} \overline{B}^\dagger + \mathcal{C}^\dagger) Q^\dagger, \]

\[ \forall p \geq 1, \Phi_p = \alpha_4^{p-1} \Phi_1, \quad \Phi_p^\dagger = \alpha_4^{p-1} \Phi_1^\dagger. \]

\[ \delta T_0 = \langle \Phi_0^\dagger | C \Phi_0 \rangle, \]

\[ \forall n \geq 1, \delta T_n = \alpha_4^{n-1} \langle \Phi_0^\dagger | C \Phi_1 \rangle + \langle \Phi_1^\dagger | C \Phi_0 \rangle + \alpha_4^{n-2} (n - 1) \langle \Phi_1^\dagger | C \Phi_1 \rangle. \]  
(125)
\[ \delta T_0 = \alpha_2^3 Q^\dagger (AC\bar{A} + \alpha_1(AC\bar{B} + BC\bar{A}) + \alpha_1^2 BC\bar{B})Q. \]

From (113, 115, 119), and after a few lines of algebra,
\[ \delta T_0 = \frac{\langle Q^\dagger |BQ \rangle - b_{11}}{q_1^i q_1 b_{11}} \langle Q^\dagger |CQ \rangle - c_{11} \rangle. \]  \hspace{1cm} (126)

\[ \langle \Phi_1^\dagger |C_1 \Phi_1 \rangle = \alpha_2^3 \langle Q^\dagger |(\alpha_4\bar{A} - \frac{c_{11}}{b_{11}}\bar{B} + \bar{C})(\alpha_4\bar{A} - \frac{c_{11}}{b_{11}}\bar{B} + \bar{C})Q \rangle \]
\[ = \langle Q^\dagger |(\alpha_4^2 AC\bar{A} + (\frac{c_{11}}{b_{11}})^2 BC\bar{B} - \alpha_4\frac{c_{11}}{b_{11}}(AC\bar{B} + BC\bar{A}))Q \rangle = 0. \]  \hspace{1cm} (127)

Taking (127) in (125) shows that \( w_1 \) is a geometric series. The ratio \( \alpha_4 \) and the first term given by (126) are consistent with (124, 123). The radius of convergence is \( \alpha_4^{-1} \). Other results can be checked: the spectral properties of \( L(z) \), the recursion relations on perturbation series coefficients . . .

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