Decision-theoretic rough sets based on time-dependent loss function

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Abstract. A fundamental notion of decision-theoretic rough sets is the concept of loss functions, which provides a powerful tool of calculating a pair of thresholds for making a decision with a minimum cost. In this paper, time-dependent loss functions which are variations of the time are of interest because such functions are frequently encountered in practical situations, we present the relationship between the pair of thresholds and loss functions satisfying time-dependent uniform distributions and normal processes in light of bayesian decision procedure. Subsequently, with the aid of bayesian decision procedure, we provide the relationship between the pair of thresholds and loss functions which are time-dependent interval sets and fuzzy numbers. Finally, we employ several examples to illustrate that how to calculate the thresholds for making a decision by using time-dependent loss functions-based decision-theoretic rough sets.

Keywords: DTRS; Fuzzy number; Interval set; Normal process; Uniform distribution

1 Introduction

Rough set theory, proposed by Pawlak \cite{20} in 1982, is a powerful mathematical tool to deal with uncertainty, imprecise or incomplete knowledge for information systems. But the condition of the equivalence relation in Pawlak’s model is so strict that limits its applications. To generalize Pawlak’s rough sets, researchers have presented various kinds of probabilistic rough sets (PRS) such as decision-theoretic rough sets (DTRS) \cite{8,9,17,21,22,23,27}, bayesian rough sets (BRS) \cite{19,26} and game-theoretic rough sets (GTRS) \cite{1,2} for solving practical problems. To date, probabilistic rough set models have been successfully applied to many fields such as data mining, email spam filtering, investment management and web support.
Since PRS was proposed, the determination of a pair of thresholds has become a substantial challenge. Until now, researchers have presented some reasonable semantic interpretations for the pair of thresholds. For example, Cheng et al. [3] computed precision parameter values based on inclusion degree with variable precision rough set model. Deng and Yao [4, 5] presented an information-theoretic approach to the interpretation and determination of thresholds used in PRS and presented DTRS-based three-way approximations of fuzzy sets. Herbert and JT Yao [6, 7] proposed GTRS to determine the values of thresholds used in PRS by introducing game theory and investigated its capability of analyzing a major decision problem evident in existing PRS. Yao [24] proposed DTRS, which provides a new interpretation in the aspect of determining the threshold values by using loss functions, by combining bayesian decision theory with PRS. To trade off different types of classification error, three-way decision-theory was proposed by Yao for making a decision with the minimum cost on the basis of DTRS, whereas there are three choices of acceptance, deferment and rejection. Concretely, rules from the positive region are used for making a decision of acceptance, rules from the negative region are applied to make a decision of rejection, and rules from the boundary region are used for making a decision of deferment. More specially, the choice deferment reduces the loss of making a decision in DTRS. Therefore, DTRS provides a powerful tool for making a decision with a minimum cost ternary classifier.

In DTRS, three choices of acceptance, deferment and rejection are determined by loss functions. In recent years, many investigations have been done on loss functions for DTRS in literatures. For example, Jia et al. [9, 10] conducted the minimum cost attribute reduction in decision-theoretic rough set models and presented an optimization representation of decision-theoretic rough set model. Li and Zhou [11] gave two assumptions for the values of losses and proposed a multi-view DTRS decision model. Liu, Li and Liang [14] performed three-way government decision analysis with decision-theoretic rough sets. Liu, Yao and Li [18] proposed a profit-based three-way approach to the investment decision-making and utilized the objects to estimate the losses or carry out some questionnaires or behavioral experiments. Liu, Li and Ruan [16] investigated probabilistic model criteria with decision-theoretic rough sets. Yao [24] used relative values between losses to express the thresholds and reduced the variable amount of the thresholds. Furthermore, Liang et al. [12] presented triangular fuzzy decision-theoretic rough sets by considering bayesian decision procedure, in which loss functions are triangular fuzzy numbers. Liang and Liu [13] provided systematic studies on three-way decisions with interval-valued decision-theoretic rough sets, in which loss functions are interval-valued. Liu, Li and Liang [15] proposed dynamic decision-theoretic rough sets, in which loss functions are single-valued variations of time. Up to now DTRS has been successfully applied to expert system, medical diagnosis, environmental science, conflict analysis and economics. Accordingly, applications are increasingly being adopted with the development of DTRS.

In practical situations, time-dependent loss functions are of interest because such functions are frequently encountered. For example, if we intend to make omelets for breakfast with six eggs and have cracked five eggs into a bowl, then we will crack the sixth egg into the bowl. There are two situations
for the sixth egg: bad and good. If we crack a good egg into the bowl, then an omelet with six eggs is prepared for breakfast. If we crack a bad egg into the bowl, then five eggs will be lost. If the price of an egg is 1 unit now, then the loss is five units. If the price of an egg is 1.2 unit tomorrow, then the loss is 6 units. Clearly, the loss is the variation of the time since the price of an egg is varying with the time. If the bad egg is cracked into another bowl, then the loss is to wash one more bowl, and the expense of washing a bowel is also varying with the time. Furthermore, loss functions are not only variations of time but also satisfied some distributions such as uniform distributions, normal processes, interval sets and fuzzy numbers simultaneously. Therefore, it is urgent to further study time-dependent loss functions for making a decision by using three-way decision-theory.

The purpose of this paper is to further investigate time-dependent loss functions-based DTRS. Section 2 introduces the basic principles of DTRS. Section 3 calculates the values of thresholds when loss functions are satisfied time-dependent uniform distributions and normal processes. Section 4 is devoting to studying the relationship between the values of thresholds and loss functions which are time-dependent interval sets. Section 5 presents the relationship between the values of thresholds and loss functions which are time-dependent fuzzy numbers. The conclusion comes in Section 6.

2 Current research on DTRS

In this section, we review some concepts of decision-theoretic rough sets.

Suppose $S = (U, A, V, f)$ is an information system, $\forall X \subseteq U$ and $0 \leq \beta \leq \alpha \leq 1$, the $(\alpha, \beta)$- probabilistic lower and upper approximations of $X$ are defined as follows:

$$apr_{(\alpha, \beta)}(X) = \{x \in U : P(X|x) \geq \alpha\};$$

$$apr_{(\alpha, \beta)}(X) = \{x \in U : P(X|x) > \beta\},$$

where $P(X|x) = \frac{|X \cap [x]|}{|X|}$ is the conditional probability of an object $x$ belonging to $X$ when the object is described by its equivalence class $[x]$. On the basis of $(\alpha, \beta)$- probabilistic lower and upper approximation operators, we have the $(\alpha, \beta)$- probabilistic positive, boundary and negative regions as follows:

$$POS_{(\alpha, \beta)}(X) = apr_{(\alpha, \beta)}(X) = \{x \in U : P(X|x) \geq \alpha\};$$

$$BND_{(\alpha, \beta)}(X) = apr_{(\alpha, \beta)}(X) - apr_{(\alpha, \beta)}(X) = \{x \in U : \beta < P(X|x) < \alpha\};$$

$$NEG_{(\alpha, \beta)}(X) = U - apr_{(\alpha, \beta)}(X) = \{x \in U : P(X|x) \leq \beta\}.$$

In DTRS, rules from the positive region are used for making a decision of acceptance, rules from the negative region are used for making a decision of rejection, and rules from the boundary region are used for making a decision of deferment. In practical situations, it is hard to acquire the values of the parameters $\alpha$ and $\beta$ since they are subjective.

To determine the pair of thresholds objectively, Yao [24] proposed decision-theoretic rough sets by combining bayesian decision theory with PRS. Concretely, decision-theoretic rough sets model contains
2 states ($\Omega = \{X, \neg X\}$) and 3 actions ($\mathcal{A} = \{a_P, a_B, a_N\}$), where $X$ and $\neg X$ indicate that an object is in $X$ and not in $X$, respectively, and $a_P, a_B$ and $a_N$ denote three actions in classifying an object $x$ into $POS(X)$, $BND(X)$ and $NEG(X)$, respectively. In Table 1, $\lambda_{PP}$, $\lambda_{BP}$ and $\lambda_{NP}$ denote losses of taking actions of $a_P$, $a_B$ and $a_N$, respectively, when an object belongs to $X$; $\lambda_{PN}$, $\lambda_{BN}$ and $\lambda_{NN}$ denote losses of taking actions of $a_P$, $a_B$ and $a_N$, respectively, when an object belongs to $\neg X$.

| Action | $X(P)$ | $\neg X(N)$ |
|--------|--------|-------------|
| $a_P$  | $\lambda_{PP}$ | $\lambda_{PN}$ |
| $a_B$  | $\lambda_{BP}$ | $\lambda_{BN}$ |
| $a_N$  | $\lambda_{NP}$ | $\lambda_{NN}$ |

Table 1: Loss function.

Suppose $\lambda_{PP} \leq \lambda_{BP} \leq \lambda_{NP}$ and $\lambda_{NN} \leq \lambda_{BN} \leq \lambda_{PN}$, since $P(X|[x]) + P(\neg X|[x]) = 1$, the bayesian decision procedure suggests the following minimum-cost decision rules:

- $(P)$: If $P(X|[x]) \geq \alpha$, then $x \in POS(X)$;
- $(B)$: If $\beta < P(X|[x]) < \alpha$, then $x \in BND(X)$;
- $(N)$: If $P(X|[x]) \leq \beta$, then $x \in NEG(X)$, where

$$\alpha = \frac{\lambda_{PN} - \lambda_{BN}}{\lambda_{PN} - \lambda_{BN} + \lambda_{BP} - \lambda_{PP}}, \quad \beta = \frac{\lambda_{BN} - \lambda_{NN}}{\lambda_{BN} - \lambda_{NN} + \lambda_{NP} - \lambda_{BP}}.$$ 

In practice, loss functions are variations of the time, and it is of interest to study the relationship between the thresholds and time-dependent loss functions.

### 3 Time-dependent uniform distributions and normal processes-based DTRS

In this section, we investigate DTRS when loss functions are satisfied time-dependent uniform distributions and normal processes.

For a random variable $X$, there are two common probability density functions

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b; \\ 0, & \text{otherwise.} \end{cases} \quad \text{and } f(x, \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$ 

Then $X$ are said to be satisfied uniform distribution on $[a, b]$ and normal process which is satisfied mathematical expectations $\mu$ and variance $\sigma^2$, respectively, denoted as $X \sim U(a, b)$, and $X \sim U(\mu, \sigma^2)$, where $a$, $b$, $\mu$ and $\sigma$ are constants. In practice, the probability density functions are varying with time, and there is a need to study DTRS when loss functions are satisfied time-dependent probability density functions.

#### 3.1 Time-dependent uniform distributions-based DTRS

In this subsection, we introduce the concept of time-dependent uniform distribution for DTRS.
Definition 3.1 Let $X(t)$ be a variation of the time $t$, and the probability density function of $X(t)$ is

$$f(x, t) = \begin{cases} \frac{1}{b(t) - a(t)}, & \text{if } a(t) \leq x \leq b(t); \\ 0, & \text{otherwise.} \end{cases}$$

Then $X(t)$ is said to be satisfied time-dependent uniform distribution on $[a(t), b(t)]$, denoted as $X(t) \sim U(a(t), b(t))$.

In what follows, we employ Table 2 to illustrate a time-dependent loss function, where $\lambda_{PP}(t), \lambda_{BP}(t), \lambda_{NP}(t), \lambda_{PN}(t), \lambda_{BN}(t)$ and $\lambda_{NN}(t)$ are varying with the time. In Table 2, $\lambda_{PP}(t), \lambda_{BP}(t)$ and $\lambda_{NP}(t)$ denote losses of taking actions of $a_P, a_B$ and $a_N$, respectively, when an object belongs to $X$; $\lambda_{PN}(t), \lambda_{BN}(t)$ and $\lambda_{NN}(t)$ denote losses of taking actions of $a_P, a_B$ and $a_N$, respectively, when an object belongs to $\neg X$.

Subsequently, we discuss DTRS when loss function is satisfied time-dependent uniform distributions.

| Action | $X(P)$ | $\neg X(N)$ |
|--------|--------|------------|
| $a_P$  | $\lambda_{PP}(t)$ | $\lambda_{PN}(t)$ |
| $a_B$  | $\lambda_{BP}(t)$ | $\lambda_{BN}(t)$ |
| $a_N$  | $\lambda_{NP}(t)$ | $\lambda_{NN}(t)$ |

Theorem 3.2 Let $\lambda_{PP}(t) \sim U(a_{PP}(t), b_{PP}(t)), \lambda_{BP}(t) \sim U(a_{BP}(t), b_{BP}(t)), \lambda_{NP}(t) \sim U(a_{NP}(t), b_{NP}(t)), \lambda_{NN}(t) \sim U(a_{NN}(t), b_{NN}(t)), \lambda_{NP}(t) \sim U(a_{NP}(t), b_{NP}(t)), \lambda_{BN}(t) \sim U(a_{BN}(t), b_{BN}(t))$ and $\lambda_{PN}(t) \sim U(a_{PN}(t), b_{PN}(t))$, where $0 \leq a_{PP}(t) \leq b_{PP}(t), 0 \leq a_{NP}(t) \leq a_{BN}(t) \leq a_{PN}(t), 0 \leq a_{NN}(t) \leq a_{BP}(t) \leq b_{BP}(t) \leq b_{NP}(t), 0 \leq b_{NN}(t) \leq b_{BN}(t) \leq b_{BP}(t)$ and $t \in T$. Then we have the following rules:

1. If $P(X[x]) \geq \alpha(t)$, then $x \in POS(X)$;
2. If $\beta(t) < P(X[x]) < \alpha(t)$, then $x \in BND(X)$;
3. If $P(X[x]) \leq \beta(t)$, then $x \in NEG(X)$, where

$$\alpha(t) = \frac{|a_{NP}(t) + b_{NP}(t)| - |a_{BN}(t) + b_{BN}(t)|}{|a_{BN}(t) + b_{BN}(t)| - |a_{NN}(t) + b_{NN}(t)|};$$

$$\beta(t) = \frac{|a_{BN}(t) + b_{BN}(t)| - |a_{NN}(t) + b_{NN}(t)|}{|a_{BN}(t) + b_{BN}(t)| - |a_{NP}(t) + b_{NP}(t)| - |a_{BP}(t) + b_{BP}(t)|}.$$ 

Proof. Since $\lambda_{PP}(t) \sim U(a_{PP}(t), b_{PP}(t)), \lambda_{BP}(t) \sim U(a_{BP}(t), b_{BP}(t)), \lambda_{NP}(t) \sim U(a_{NP}(t), b_{NP}(t)), \lambda_{BN}(t) \sim U(a_{BN}(t), b_{BN}(t))$ and $\lambda_{PN}(t) \sim U(a_{PN}(t), b_{PN}(t))$, where $0 \leq a_{PP}(t) \leq b_{PP}(t), 0 \leq a_{NP}(t) \leq a_{BN}(t) \leq a_{PN}(t), 0 \leq a_{NN}(t) \leq a_{BP}(t) \leq b_{BP}(t) \leq b_{NP}(t), 0 \leq b_{NN}(t) \leq b_{BN}(t) \leq b_{BP}(t)$ and $t \in T$, we have

$$\frac{a_{PP}(t) + b_{PP}(t)}{2} \leq \frac{a_{BP}(t) + b_{BP}(t)}{2} \leq \frac{a_{NP}(t) + b_{NP}(t)}{2},$$

$$\frac{a_{NN}(t) + b_{NN}(t)}{2} \leq \frac{a_{BN}(t) + b_{BN}(t)}{2} \leq \frac{a_{PN}(t) + b_{PN}(t)}{2}.$$
By taking

\[
\lambda_{pp}(t) = \frac{a_{pp}(t) + b_{pp}(t)}{2}, \quad \lambda_{pn}(t) = \frac{a_{nn}(t) + b_{nn}(t)}{2}, \quad \lambda_{bp}(t) = \frac{a_{bp}(t) + b_{bp}(t)}{2}, \quad \lambda_{bn}(t) = \frac{a_{bn}(t) + b_{bn}(t)}{2},
\]

we have the expected losses \( R(a_p[x]) \), \( R(a_b[x]) \) and \( R(a_n[x]) \) associated with taking the individual actions for an object \( x \) and \( t \in T \) as follows:

\[
\begin{align*}
R(a_p[x]) &= \lambda_{pp}(t)P(X[x]) + \lambda_{pn}(t)P(\neg X[x]); \\
R(a_b[x]) &= \lambda_{bp}(t)P(X[x]) + \lambda_{bn}(t)P(\neg X[x]); \\
R(a_n[x]) &= \lambda_{np}(t)P(X[x]) + \lambda_{nn}(t)P(\neg X[x]).
\end{align*}
\]

The bayesian decision procedure suggests the local minimum-cost decision rules:

(P) : If \( R(a_p[x]) \leq R(a_b[x]) \) and \( R(a_p[x]) \leq R(a_n[x]) \), then \( x \in POS(X) \);

(B) : If \( R(a_b[x]) \leq R(a_p[x]) \) and \( R(a_b[x]) \leq R(a_n[x]) \), then \( x \in BND(X) \);

(N) : If \( R(a_n[x]) \leq R(a_p[x]) \) and \( R(a_n[x]) \leq R(a_b[x]) \), then \( x \in NEG(X) \).

Since \( P(X[x]) + P(\neg X[x]) = 1 \), we have \( \alpha(t) \) and \( \beta(t) \) as follows:

\[
\alpha(t) = \frac{\lambda_{pn}(t) - \lambda_{bn}(t)}{[\lambda_{pn}(t) - \lambda_{bn}(t)] + [\lambda_{bp}(t) - \lambda_{pp}(t)]} = \frac{a_{pn}(t) + b_{pn}(t) - a_{bn}(t) - b_{bn}(t)}{[a_{pn}(t) + b_{pn}(t)] - [a_{bn}(t) + b_{bn}(t)]};
\]

\[
\beta(t) = \frac{\lambda_{bn}(t) - \lambda_{nn}(t)}{[\lambda_{bn}(t) - \lambda_{nn}(t)] + [\lambda_{np}(t) - \lambda_{bp}(t)]} = \frac{a_{bn}(t) + b_{bn}(t) - a_{nn}(t) - b_{nn}(t)}{[a_{bn}(t) + b_{bn}(t)] - [a_{nn}(t) + b_{nn}(t)]}.
\]

On the basis of \( \alpha(t) \) and \( \beta(t) \), we simplify the rules as follows:

(P) : If \( P(X[x]) \geq \alpha(t) \), then \( x \in POS(X) \);

(B) : If \( \beta(t) < P(X[x]) < \alpha(t) \), then \( x \in BND(X) \);

(N) : If \( P(X[x]) \leq \beta(t) \), then \( x \in NEG(X) \). \( \square \)

In the following, we employ an example to illustrate that how to compute \( \alpha(t) \) and \( \beta(t) \) by using a loss function.

**Example 3.3** Let \( \lambda_{pp}(t) = 0, \lambda_{nn}(t) = 0, \lambda_{bp}(t) \sim U(2t + 2, 4t + 4), \lambda_{np}(t) \sim U(3t + 6, 5t + 12), \lambda_{pn}(t) \sim U(4t + 8, 6t + 12) \).
confidence intervals functions in decision-theoretic rough set theory are satisfied normal processes as follows: a need to study DTRS when loss functions are satisfied normal processes. Suppose loss function and suppose

\[ U(2t + 14, 4t + 20) \text{ and } \lambda_{BN}(t) \sim U(t + 2, 3t + 10). \text{ Then we have} \]

\[
\begin{align*}
\lambda_{PP}(t) &= \lambda_{NN}(t) = 0, \\
\lambda_{PN}(t) &= \frac{a_{NN}(t) + b_{NN}(t)}{2} = \frac{2t + 14 + 4t + 20}{2} = 3t + 17, \\
\lambda_{BP}(t) &= \frac{a_{BP}(t) + b_{BP}(t)}{2} = \frac{2t + 2 + 4t + 4}{2} = 3t + 3, \\
\lambda_{BN}(t) &= \frac{a_{BN}(t) + b_{BN}(t)}{2} = \frac{t + 2 + 3t + 10}{2} = 2t + 6, \\
\lambda_{NP}(t) &= \frac{a_{NP}(t) + b_{NP}(t)}{2} = \frac{3t + 6 + 5t + 12}{2} = 4t + 9.
\end{align*}
\]

By Theorem 3.2, we have

\[
\begin{align*}
\alpha(t) &= \frac{\lambda_{PN}(t) - \lambda_{BN}(t)}{\lambda_{PN}(t) - \lambda_{BN}(t) + \lambda_{BP}(t) - \lambda_{PP}(t)} = \frac{t + 11}{4t + 14}, \\
\beta(t) &= \frac{\lambda_{BN}(t) - \lambda_{NN}(t)}{\lambda_{BN}(t) - \lambda_{NN}(t) + \lambda_{NP}(t) - \lambda_{BP}(t)} = \frac{2t + 6}{3t + 12}.
\end{align*}
\]

### 3.2 Normal processes-based DTRS

In this subsection, we introduce the concept of normal processes for DTRS.

**Definition 3.4** Let \( X(t) \) be a variation of the time \( t \), the probability density function of \( X(t) \) is

\[ f(x, \mu(t), \sigma^2(t)) = \frac{1}{\sigma(t) \sqrt{2\pi}} e^{-\frac{(x-\mu(t))^2}{2\sigma^2(t)}}. \]

Then \( X(t) \) is called a normal process which is satisfied mathematical expectations \( \mu(t) \) and variance \( \sigma^2(t) \), denoted as \( X(t) \sim U(\mu(t), \sigma^2(t)) \).

In [15], Liu et al. discussed DTRS when loss functions are satisfied normal distributions, and there is a need to study DTRS when loss functions are satisfied normal processes.

In what follows, we discuss DTRS when loss functions are satisfied normal processes. Suppose loss functions in decision-theoretic rough set theory are satisfied normal processes as follows:

\[
\begin{align*}
\lambda_{PP}(t) &\sim U(\mu_{PP}(t), \sigma^2_{PP}(t)), \lambda_{BP}(t) \sim U(\mu_{BP}(t), \sigma^2_{BP}(t)), \lambda_{NP}(t) \sim U(\mu_{NP}(t), \sigma^2_{NP}(t)), \\
\lambda_{NN}(t) &\sim U(\mu_{NN}(t), \sigma^2_{NN}(t)), \lambda_{BN}(t) \sim U(\mu_{BN}(t), \sigma^2_{BN}(t)), \lambda_{PN}(t) \sim U(\mu_{PN}(t), \sigma^2_{PN}(t)).
\end{align*}
\]

Suppose \( \lambda(t) \sim U(\mu(t), \sigma^2(t)) \), we have \( P(\mu(t) - \sigma(t) \leq \lambda(t) \leq \mu(t) + \sigma(t)) \approx 0.6827, P(\mu(t) - 2\sigma(t) \leq \lambda(t) \leq \mu(t) + 2\sigma(t)) \approx 0.9545 \) and \( P(\mu(t) - 3\sigma(t) \leq \lambda(t) \leq \mu(t) + 3\sigma(t)) \approx 0.9973 \). If we take three confidence intervals \([\mu(t) - n\sigma(t), \mu(t) + n\sigma(t)](n = 1, 2, 3)\) instead of \( \lambda(t) \) for loss function and suppose \( \mu(t) - n\sigma(t) \geq 0 \) for \( \mu(t) \), then the expected losses \( R(\alpha_P[x]), R(\alpha_B[x]) \) and \( R(\alpha_N[x]) \) associated with taking
the individual actions for an object \(x\) and \(t \in T\) are shown as follows:

\[
R(a_p[x]) = \lambda_{pp}(t)P(X[x]) + \lambda_{pn}(t)P(\neg X[x]);
\]

\[
R(a_b[x]) = \lambda_{bp}(t)P(X[x]) + \lambda_{bn}(t)P(\neg X[x]);
\]

\[
R(a_n[x]) = \lambda_{np}(t)P(X[x]) + \lambda_{nn}(t)P(\neg X[x]).
\]

The bayesian decision procedure suggests the local minimum-cost decision rules:

\((P)\) : If \(R(a_p[x]) \leq R(a_b[x])\) and \(R(a_p[x]) \leq R(a_n[x])\), then \(x \in POS(X)\);

\((B)\) : If \(R(a_p[x]) \leq R(a_p[x])\) and \(R(a_p[x]) \leq R(a_n[x])\), then \(x \in BND(X)\);

\((N)\) : If \(R(a_n[x]) \leq R(a_p[x])\) and \(R(a_n[x]) \leq R(a_b[x])\), then \(x \in NEG(X)\).

Since \(P(X[x]) + P(\neg X[x]) = 1\), we take the values of \(\alpha(t)\) and \(\beta(t)\) as follows:

\[
\alpha(t) = \frac{\lambda_{pn}(t) - \lambda_{bn}(t)}{\lambda_{pn}(t) - \lambda_{bn}(t) + \lambda_{bp}(t) - \lambda_{pp}(t)}, \beta(t) = \frac{\lambda_{bn}(t) - \lambda_{nn}(t)}{\lambda_{bn}(t) - \lambda_{nn}(t) + \lambda_{np}(t) - \lambda_{bp}(t)}.
\]

Concretely, on the basis of \(\alpha^{\min}(t), \alpha^{\max}(t), \beta^{\min}(t)\) and \(\beta^{\max}(t)\), we have the following results:

\[
\alpha(t) \in [\max(\alpha^{\min}(t), 0), \min(\alpha^{\max}(t), 1)], \beta(t) \in [\max(\beta^{\min}(t), 0), \min(\beta^{\max}(t), 1)],
\]

where

\[
\alpha^{\min}(t) = \frac{[\mu_{pn}(t) - n\sigma_{pn}(t)] - [\mu_{bn}(t) + n\sigma_{bn}(t)]}{[\mu_{pn}(t) + n\sigma_{pn}(t)] - [\mu_{bn}(t) - n\sigma_{bn}(t)]};
\]

\[
\alpha^{\max}(t) = \frac{[\mu_{pn}(t) - n\sigma_{pn}(t)] - [\mu_{bn}(t) + n\sigma_{bn}(t)]}{[\mu_{pn}(t) + n\sigma_{pn}(t)] - [\mu_{bn}(t) - n\sigma_{bn}(t)]};
\]

\[
\beta^{\min}(t) = \frac{[\mu_{bn}(t) + n\sigma_{bn}(t)] - [\mu_{nn}(t) - n\sigma_{nn}(t)]}{[\mu_{bn}(t) - n\sigma_{bn}(t)] - [\mu_{nn}(t) + n\sigma_{nn}(t)]};
\]

\[
\beta^{\max}(t) = \frac{[\mu_{bn}(t) + n\sigma_{bn}(t)] - [\mu_{nn}(t) - n\sigma_{nn}(t)]}{[\mu_{bn}(t) - n\sigma_{bn}(t)] - [\mu_{nn}(t) + n\sigma_{nn}(t)]}.
\]

By using the thresholds \(\alpha(t)\) and \(\beta(t)\), we simplify the rules as follows:

\((P)\) : If \(P(X[x]) \geq \alpha(t)\), then \(x \in POS(X)\);

\((B)\) : If \(\beta(t) < P(X[x]) < \alpha(t)\), then \(x \in BND(X)\);

\((N)\) : If \(P(X[x]) \leq \beta(t)\), then \(x \in NEG(X)\).

**Example 3.5** Let \(\lambda_{pp}(t) \sim U\left(\frac{3t+2}{2}, \frac{t^2+4n^2+4}{4n^2}\right)\), \(\lambda_{bp}(t) \sim U\left(\frac{5t+8}{2}, \frac{t^2+4n^2+4}{4n^2}\right)\), \(\lambda_{np}(t) \sim U\left(\frac{7t+14}{2}, \frac{t^2+4n^2+4}{4n^2}\right)\), \(\lambda_{bn}(t) \sim U\left(\frac{3t+2}{2}, \frac{t^2+4n^2+4}{4n^2}\right)\), \(\lambda_{bn}(t) \sim U\left(\frac{5t+8}{2}, \frac{t^2+4n^2+4}{4n^2}\right)\) and \(\lambda_{pn}(t) \sim U\left(\frac{7t+18}{2}, \frac{t^2+4n^2+4}{4n^2}\right)\). Then we have

\[
\alpha^{\min}(t) = \frac{1}{4t+12}, \beta^{\min}(t) = \frac{1}{4t+10}, \alpha^{\max}(t) = \frac{2t+7}{4}, \beta^{\max}(t) = \frac{2t+5}{1}.
\]

Consequently, we compute the thresholds \(\alpha(t)\) and \(\beta(t)\) when loss functions are two special cases as follows.
(1) Considering \( \lambda_{PP}^1(t) = \mu_{PP}(t) - n\sigma_{PP}(t), \lambda_{BP}^1(t) = \mu_{BP}(t) - n\sigma_{BP}(t), \lambda_{NP}^1(t) = \mu_{NP}(t) - n\sigma_{NP}(t), \lambda_{NN}^1(t) = \mu_{NN}(t) - n\sigma_{NN}(t), \lambda_{BN}^1(t) = \mu_{BN}(t) - n\sigma_{BN}(t) \) and \( \lambda_{PN}^1(t) = \mu_{PN}(t) - n\sigma_{PN}(t) \), then we have

\[
\alpha^1(t) = \frac{[\mu_{PN}(t) - n\sigma_{PN}(t)] - [\mu_{BN}(t) - n\sigma_{BN}(t)]}{[\mu_{BN}(t) - n\sigma_{BN}(t)] - [\mu_{NN}(t) - n\sigma_{NN}(t)]};
\]

\[
\beta^1(t) = \frac{[\mu_{BN}(t) - n\sigma_{BN}(t)] - [\mu_{NN}(t) - n\sigma_{NN}(t)] + [\mu_{NP}(t) - n\sigma_{NP}(t)] - [\mu_{BP}(t) - n\sigma_{BP}(t)]}{[\mu_{BN}(t) - n\sigma_{BN}(t)] - [\mu_{NN}(t) - n\sigma_{NN}(t)]}.
\]

(2) Considering \( \lambda_{PP}^2(t) = \mu_{PP}(t) + n\sigma_{PP}(t), \lambda_{BP}^2(t) = \mu_{BP}(t) + n\sigma_{BP}(t), \lambda_{NP}^2(t) = \mu_{NP}(t) + n\sigma_{NP}(t), \lambda_{NN}^2(t) = \mu_{NN}(t) + n\sigma_{NN}(t), \lambda_{BN}^2(t) = \mu_{BN}(t) + n\sigma_{BN}(t) \) and \( \lambda_{PN}^2(t) = \mu_{PN}(t) + n\sigma_{PN}(t) \), then we have

\[
\alpha^2(t) = \frac{[\mu_{PN}(t) + n\sigma_{PN}(t)] - [\mu_{BN}(t) + n\sigma_{BN}(t)]}{[\mu_{BN}(t) + n\sigma_{BN}(t)] - [\mu_{NN}(t) + n\sigma_{NN}(t)]};
\]

\[
\beta^2(t) = \frac{[\mu_{BN}(t) + n\sigma_{BN}(t)] - [\mu_{NN}(t) + n\sigma_{NN}(t)] + [\mu_{NP}(t) + n\sigma_{NP}(t)] - [\mu_{BP}(t) + n\sigma_{BP}(t)]}{[\mu_{BN}(t) + n\sigma_{BN}(t)] - [\mu_{NN}(t) + n\sigma_{NN}(t)]}.
\]

On the basis of the above results, we have that \( \alpha^1(t), \alpha^2(t) \in [max[a_{\min}(t), 0], min[a_{\max}(t), 1]] \) and \( \beta^1(t), \beta^2(t) \in [max[\beta_{\min}(t), 0], min[\beta_{\max}(t), 1]] \).

4 Time-dependent interval sets-based DTRS

In this section, we discuss DTRS when loss functions are time-dependent interval sets. Firstly, we present the concept of time-dependent interval sets for DTRS.

**Definition 4.1** Let \( \lambda(t) = [a(t), b(t)] \), where \( a(t) \) and \( b(t) \) are variations of the time \( t \), then \( \lambda(t) \) is called a time-dependent interval set.

In what follows, we employ Table 3 to illustrate loss functions which are time-dependent interval sets.

| Action | \( \lambda_{PP}(t) = [\lambda_{\min}^{PP}(t), \lambda_{\max}^{PP}(t)] \) | \( \lambda_{BP}(t) = [\lambda_{\min}^{BP}(t), \lambda_{\max}^{BP}(t)] \) | \( \lambda_{NP}(t) = [\lambda_{\min}^{NP}(t), \lambda_{\max}^{NP}(t)] \) |
|---|---|---|---|
| \( \alpha \) | \( \lambda_{PN}(t) = [\lambda_{\min}^{PN}(t), \lambda_{\max}^{PN}(t)] \) | \( \lambda_{BN}(t) = [\lambda_{\min}^{BN}(t), \lambda_{\max}^{BN}(t)] \) | \( \lambda_{NN}(t) = [\lambda_{\min}^{NN}(t), \lambda_{\max}^{NN}(t)] \) |

In Table 3, \( \lambda_{PP}(t), \lambda_{BP}(t), \lambda_{NP}(t), \lambda_{PN}(t), \lambda_{BN}(t) \) and \( \lambda_{NN}(t) \) are time-dependent interval sets. Below, we discuss DTRS when loss functions are the lower and upper bounds of time-dependent interval sets.

On one hand, \( \lambda_{\min}^{PP}(t), \lambda_{\min}^{BP}(t), \lambda_{\min}^{NP}(t), \lambda_{\min}^{PN}(t), \lambda_{\min}^{BN}(t) \) and \( \lambda_{\min}^{NN}(t) \) denote the lower bounds of time-dependent interval sets in Table 3. The expected losses \( R^{opt}(\alpha_{P}[x]), R^{opt}(\alpha_{B}[x]) \) and \( R^{opt}(\alpha_{N}[x]) \) as-
sociated with taking the individual actions for an object $x$ and $t \in T$ are shown as follows:

$$R_{op}^P(a_P[x]) = \lambda_{PP}^{min}(t)P(X[x]) + \lambda_{PN}^{min}(t)P(\neg X[x]);$$

$$R_{op}^B(a_B[x]) = \lambda_{BP}^{min}(t)P(X[x]) + \lambda_{BN}^{min}(t)P(\neg X[x]);$$

$$R_{op}^N(a_N[x]) = \lambda_{NP}^{min}(t)P(X[x]) + \lambda_{NN}^{min}(t)P(\neg X[x]).$$

The bayesian decision procedure suggests the local minimum-cost decision rules:

(P) If $R_{op}^P(a_P[x]) \leq R_{op}^P(a_B[x])$ and $R_{op}^P(a_P[x]) \leq R_{op}^P(a_N[x])$, then $x \in POS(X)$;

(B) If $R_{op}^P(a_B[x]) \leq R_{op}^P(a_P[x])$ and $R_{op}^P(a_B[x]) \leq R_{op}^P(a_N[x])$, then $x \in BND(X)$;

(N) If $R_{op}^P(a_N[x]) \leq R_{op}^P(a_P[x])$ and $R_{op}^P(a_N[x]) \leq R_{op}^P(a_B[x])$, then $x \in NEG(X)$.

Suppose $0 \leq \lambda_{PP}^{min}(t) \leq \lambda_{BP}^{min}(t) \leq \lambda_{NP}^{min}(t)$ and $0 \leq \lambda_{BN}^{min}(t) \leq \lambda_{NN}^{min}(t) \leq \lambda_{PN}^{min}(t)$ for $t \in T$. Since $P(X[x]) + P(\neg X[x]) = 1$, we simplify the rules as follows:

(P) If $P(X[x]) \geq \alpha_{op}^{min}(t)$, then $x \in POS(X)$;

(B) If $\beta_{op}^{min}(t) < P(X[x]) < \alpha_{op}^{min}(t)$, then $x \in BND(X)$;

(N) If $P(X[x]) \leq \beta_{op}^{min}(t)$, then $x \in NEG(X)$, where

$$\alpha_{op}^{min}(t) = \frac{\lambda_{PN}^{min}(t) - \lambda_{BP}^{min}(t)}{\lambda_{BN}^{min}(t) - \lambda_{BP}^{min}(t)} \cdot \lambda_{PP}^{min}(t) - \lambda_{BN}^{min}(t),$$

$$\beta_{op}^{min}(t) = \frac{\lambda_{PN}^{min}(t) - \lambda_{BP}^{min}(t)}{\lambda_{BN}^{min}(t) - \lambda_{BP}^{min}(t)} \cdot \lambda_{PN}^{min}(t) - \lambda_{BN}^{min}(t).$$

Example 4.2 Let $\lambda_{PP}(t) = [t, 2t + 2], \lambda_{PN}(t) = [4t + 8, 4t + 10], \lambda_{BP}(t) = [2t + 3, 2t + 5], \lambda_{BN}(t) = [3t + 2, 3t + 6], \lambda_{NP}(t) = [3t + 6, 3t + 8]$ and $\lambda_{NN}(t) = [2t, 2t + 2]$. Then we have

$$\alpha_{op}^{max}(t) = \frac{\lambda_{PN}^{max}(t) - \lambda_{BN}^{max}(t)}{\lambda_{BN}^{max}(t) - \lambda_{BP}^{max}(t)} = \frac{t + 6}{2t + 9},$$

$$\beta_{op}^{max}(t) = \frac{\lambda_{PN}^{max}(t) - \lambda_{BN}^{max}(t)}{\lambda_{BN}^{max}(t) - \lambda_{BP}^{max}(t)} = \frac{t + 2}{2t + 5}.$$

On the other hand, $\lambda_{PP}^{max}(t), \lambda_{BP}^{max}(t), \lambda_{NP}^{max}(t), \lambda_{BN}^{max}(t)$ and $\lambda_{NN}^{max}(t)$ are upper bounds of time-dependent interval sets in Table 3. We have the expected losses $R_{pes}^{op}(a_P[x]), R_{pes}^{op}(a_B[x])$ and $R_{pes}^{op}(a_N[x])$ associated with taking the individual actions for an object $x$ and $t \in T$ as follows:

$$R_{pes}^{op}(a_P[x]) = \lambda_{PP}^{max}(t)P(X[x]) + \lambda_{PN}^{max}(t)P(\neg X[x]);$$

$$R_{pes}^{op}(a_B[x]) = \lambda_{BP}^{max}(t)P(X[x]) + \lambda_{BN}^{max}(t)P(\neg X[x]);$$

$$R_{pes}^{op}(a_N[x]) = \lambda_{NP}^{max}(t)P(X[x]) + \lambda_{NN}^{max}(t)P(\neg X[x]).$$

The bayesian decision procedure suggests the local minimum-cost decision rules:

(P') If $R_{pes}^{op}(a_P[x]) \leq R_{pes}^{op}(a_B[x])$ and $R_{pes}^{op}(a_P[x]) \leq R_{pes}^{op}(a_N[x])$, then $x \in POS(X)$;

(B') If $R_{pes}^{op}(a_B[x]) \leq R_{pes}^{op}(a_P[x])$ and $R_{pes}^{op}(a_B[x]) \leq R_{pes}^{op}(a_N[x])$, then $x \in BND(X)$. 
(N′) : If \( R_{pes}(a_N||x) \leq R_{pes}(a_P||x) \) and \( R_{pes}(a_N||x) \leq R_{pes}(a_B||x) \), then \( x \in NEG(X) \).

Suppose \( 0 \leq \lambda_{PP}^{max}(t) \leq \lambda_{BN}^{max}(t) \leq \lambda_{NN}^{max}(t) \) and \( 0 \leq \lambda_{NP}^{max}(t) \leq \lambda_{BN}^{max}(t) \leq \lambda_{PN}^{max}(t) \) for \( t \in T \). Since \( P(X||x) + P(\neg X||x) = 1 \), we simplify the rules as follows:

\[(P′) : \text{If } P(X||x) \geq \alpha_{pes}(t), \text{ then } x \in POS(X);\]
\[(B′) : \text{If } \beta_{pes}(t) < P(X||x) < \alpha_{pes}(t), \text{ then } x \in BND(X);\]
\[(N′) : \text{If } P(X||x) \leq \beta_{pes}(t), \text{ then } x \in NEG(X), \text{ where}\]

\[
\alpha_{pes}(t) = \frac{\lambda_{BN}^{max}(t) - \lambda_{PN}^{max}(t)}{\lambda_{BN}^{max}(t) - \lambda_{BN}^{max}(t) + \lambda_{BN}^{max}(t) - \lambda_{PP}^{max}(t)} \beta_{pes}(t) = \frac{\lambda_{BN}^{max}(t) - \lambda_{PP}^{max}(t)}{\lambda_{BN}^{max}(t) - \lambda_{NP}^{max}(t) + \lambda_{BN}^{max}(t) - \lambda_{BP}^{max}(t)}.
\]

**Example 4.3** Let \( \lambda_{PP}(t) = [t, t+2], \lambda_{PP}(t) = [4t + 8, 4t + 10], \lambda_{BP}(t) = [2t + 3, 2t + 5], \lambda_{BN}(t) = [3t + 2, 3t + 6], \lambda_{NP}(t) = [3t + 6, 3t + 8] \) and \( \lambda_{NN}(t) = [2t, 2t + 2] \). Then we have

\[
\alpha_{pes}(t) = \frac{\lambda_{NP}^{max}(t) - \lambda_{BN}^{max}(t)}{\lambda_{NP}^{max}(t) - \lambda_{BN}^{max}(t) + \lambda_{BN}^{max}(t) - \lambda_{PP}^{max}(t)} = \frac{t + 4}{t + 7};
\]
\[
\beta_{pes}(t) = \frac{\lambda_{BN}^{max}(t) - \lambda_{NP}^{max}(t)}{\lambda_{BN}^{max}(t) - \lambda_{NP}^{max}(t) + \lambda_{NP}^{max}(t) - \lambda_{BP}^{max}(t)} = \frac{t + 4}{2t + 7}.
\]

In general, by taking \( \lambda_{PP}^{*}(t) \in \lambda_{PP}(t), \lambda_{BP}^{*}(t) \in \lambda_{BP}(t), \lambda_{NP}^{*}(t) \in \lambda_{NP}(t), \lambda_{PN}^{*}(t) \in \lambda_{PN}(t), \lambda_{BN}^{*}(t) \in \lambda_{BN}(t) \) and \( \lambda_{NN}^{*}(t) \in \lambda_{NN}(t) \), we have the expected losses \( R(a_P||x), R(a_B||x) \) and \( R(a_N||x) \) associated with taking the individual actions for an object \( x \) and \( t \in T \) as follows:

\[
R(a_P||x) = \lambda_{PP}^{*}(t)P(X||x) + \lambda_{PN}^{*}(t)P(\neg X||x);
\]
\[
R(a_B||x) = \lambda_{BP}^{*}(t)P(X||x) + \lambda_{BN}^{*}(t)P(\neg X||x);
\]
\[
R(a_N||x) = \lambda_{NP}^{*}(t)P(X||x) + \lambda_{NN}^{*}(t)P(\neg X||x).
\]

The bayesian decision procedure suggests the local minimum-cost decision rules:

\[(P′′) : \text{If } R(a_P||x) \leq R(a_B||x) \text{ and } R(a_P||x) \leq R(a_N||x), \text{ then } x \in POS(X);\]
\[(B′′) : \text{If } R(a_B||x) \leq R(a_P||x) \text{ and } R(a_B||x) \leq R(a_N||x), \text{ then } x \in BND(X);\]
\[(N′′) : \text{If } R(a_N||x) \leq R(a_P||x) \text{ and } R(a_N||x) \leq R(a_B||x), \text{ then } x \in NEG(X).\]

Suppose \( 0 \leq \lambda_{PP}^{*}(t) \leq \lambda_{BP}^{*}(t) \leq \lambda_{NP}^{*}(t) \) and \( 0 \leq \lambda_{NP}^{*}(t) \leq \lambda_{BN}^{*}(t) \leq \lambda_{PN}^{*}(t) \) for \( t \in T \). Since \( P(X||x) + P(\neg X||x) = 1 \), we simplify the rules as follows:

\[(P′′) : \text{If } P(X||x) \geq \alpha(t), \text{ then } x \in POS(X);\]
\[(B′′) : \text{If } \beta(t) < P(X||x) < \alpha(t), \text{ then } x \in BND(X);\]
\[(N′′) : \text{If } P(X||x) \leq \beta(t), \text{ then } x \in NEG(X), \text{ where}\]

\[
\alpha(t) = \frac{\lambda_{NP}^{*}(t) - \lambda_{BN}^{*}(t)}{\lambda_{NP}^{*}(t) - \lambda_{BN}^{*}(t) + \lambda_{BN}^{*}(t) - \lambda_{PP}^{*}(t)} \beta(t) = \frac{\lambda_{BN}^{*}(t) - \lambda_{NN}^{*}(t)}{\lambda_{BN}^{*}(t) - \lambda_{NP}^{*}(t) + \lambda_{NP}^{*}(t) - \lambda_{BP}^{*}(t)}.
\]
Theorem 4.4 Let \(0 \leq \lambda_{BP}^{min}(t) \leq \lambda_{BP}^{max}(t) \leq \lambda_{BN}^{min}(t) \leq \lambda_{BN}^{max}(t) \leq \lambda_{NN}^{min}(t) \leq \lambda_{NN}^{max}(t)\) and \(0 \leq \lambda_{NP}^{min}(t) \leq \lambda_{NP}^{max}(t)\), where \(t \in T\). Then

\[
\alpha(t) \in \left[ \frac{\lambda_{BN}^{min}(t) - \lambda_{BN}^{max}(t)}{\lambda_{BN}^{max}(t) - \lambda_{BN}^{min}(t)}, \min\left\{ \frac{\lambda_{BN}^{max}(t) - \lambda_{BN}^{min}(t)}{\lambda_{BN}^{max}(t) - \lambda_{BN}^{min}(t)}, 1 \right\} \right]; \tag{1}
\]

\[
\beta(t) \in \left[ \frac{\lambda_{BP}^{min}(t) - \lambda_{BP}^{max}(t)}{\lambda_{BP}^{max}(t) - \lambda_{BP}^{min}(t)}, \min\left\{ \frac{\lambda_{BP}^{max}(t) - \lambda_{BP}^{min}(t)}{\lambda_{BP}^{max}(t) - \lambda_{BP}^{min}(t)}, 1 \right\} \right]. \tag{2}
\]

Proof. (1) Since \(0 \leq \lambda_{PN}^{min}(t) \leq \lambda_{PN}^{max}(t) \leq \lambda_{BN}^{min}(t) \leq \lambda_{BN}^{max}(t) \leq \lambda_{NP}^{min}(t) \leq \lambda_{NP}^{max}(t)\) and \(0 \leq \lambda_{NP}^{min}(t) \leq \lambda_{NP}^{max}(t)\), we have

\[
\lambda_{PN}^{min}(t) - \lambda_{BP}^{min}(t) \leq \lambda_{PN}^{max}(t) - \lambda_{BP}^{max}(t),
\]

\[
\lambda_{BP}^{min}(t) - \lambda_{PP}^{min}(t) \leq \lambda_{BP}^{max}(t) - \lambda_{PP}^{max}(t).
\]

It implies that

\[
\lambda_{PN}^{min}(t) - \lambda_{BN}^{max}(t) + \lambda_{BP}^{min}(t) - \lambda_{PP}^{max}(t) \leq \lambda_{PN}^{max}(t) - \lambda_{BN}^{max}(t) + \lambda_{BP}^{max}(t) - \lambda_{PP}^{max}(t)
\]

\[
\leq \lambda_{PN}^{min}(t) - \lambda_{BN}^{max}(t) + \lambda_{BP}^{max}(t) - \lambda_{PP}^{max}(t).
\]

It follows that

\[
\frac{\lambda_{PN}^{min}(t) - \lambda_{BN}^{max}(t)}{\lambda_{BN}^{max}(t) - \lambda_{BN}^{min}(t) + \lambda_{BP}^{max}(t) - \lambda_{PP}^{min}(t)} \leq \frac{\lambda_{PN}^{max}(t) - \lambda_{BN}^{max}(t)}{\lambda_{BN}^{max}(t) - \lambda_{BN}^{min}(t) + \lambda_{BP}^{max}(t) - \lambda_{PP}^{min}(t)}.
\]

Obviously, we have

\[
\alpha(t),\quad \beta(t) \in [0, 1].
\]

Therefore,

\[
\alpha(t) \in \left[ \frac{\lambda_{BN}^{min}(t) - \lambda_{BN}^{max}(t)}{\lambda_{BN}^{max}(t) - \lambda_{BN}^{min}(t) + \lambda_{BP}^{max}(t) - \lambda_{PP}^{min}(t)}, \min\left\{ \frac{\lambda_{BN}^{max}(t) - \lambda_{BN}^{min}(t)}{\lambda_{BN}^{max}(t) - \lambda_{BN}^{min}(t) + \lambda_{BP}^{max}(t) - \lambda_{PP}^{min}(t)}, 1 \right\} \right]. \tag{1}
\]

(2) Since \(0 \leq \lambda_{NP}^{min}(t) \leq \lambda_{NP}^{max}(t) \leq \lambda_{BP}^{min}(t) \leq \lambda_{BP}^{max}(t) \leq \lambda_{NP}^{min}(t) \leq \lambda_{NP}^{max}(t)\) and \(0 \leq \lambda_{NP}^{min}(t) \leq \lambda_{NP}^{max}(t)\), we have

\[
\lambda_{NP}^{min}(t) - \lambda_{BP}^{min}(t) \geq 0, \quad \lambda_{BN}^{min}(t) - \lambda_{BN}^{max}(t) \geq 0,
\]

\[
\lambda_{NP}^{min}(t) - \lambda_{BP}^{max}(t) \leq -\lambda_{NP}^{max}(t) - \lambda_{NP}^{min}(t) \leq \lambda_{NP}^{min}(t) - \lambda_{NP}^{max}(t),
\]

\[
\lambda_{BN}^{min}(t) - \lambda_{BN}^{max}(t) \leq \lambda_{BN}^{min}(t) - \lambda_{BN}^{max}(t) \leq \lambda_{BN}^{min}(t) - \lambda_{BN}^{max}(t).
\]
It implies that
\[
A_{\min}\text{ }_{1N}\text{ }_{4N}(t) - \lambda_{1B}^{\text{max}}(t) + \lambda_{1N}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t) \leq \lambda_{1B}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t) + \lambda_{1B}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t).
\]
It follows that
\[
\frac{\lambda_{1N}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)}{\lambda_{1B}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)} \leq \frac{\lambda_{1B}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)}{\lambda_{1B}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)} \leq \frac{\lambda_{1N}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)}{\lambda_{1B}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)}.
\]
Obviously, we have
\[
\beta(t), \frac{\lambda_{1N}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)}{\lambda_{1B}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)} + \frac{\lambda_{1N}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)}{\lambda_{1B}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)} \in [0, 1].
\]
Therefore,
\[
\beta(t) \in \left[\min\left(\frac{\lambda_{1N}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)}{\lambda_{1B}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)} + \frac{\lambda_{1N}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)}{\lambda_{1B}^{\text{max}}(t) - \lambda_{1N}^{\text{max}}(t)}\right), 1\right].
\]

5 Time-dependent fuzzy numbers-based DTRS

In this section, we investigate DTRS when loss functions are time-dependent fuzzy numbers. We introduce the concepts of time-dependent fuzzy numbers and cut sets for DTRS.

**Definition 5.1** Let \( \mu_{\tilde{A}(t)} \) be a mapping from \( U \) to \([0, 1]\) such as \( \mu_{\tilde{A}(t)} : U \rightarrow [0, 1] : x \rightarrow \mu_{\tilde{A}(t)} \), where \( t \in T, \mu_{\tilde{A}(t)} \) is the membership function of \( \tilde{A}(t) \), \( \mu_{\tilde{A}(t)}(x) \) is the membership degree of \( x \) to \( \tilde{A}(t) \), denoted as \( \tilde{A}(t) = \{(x, \mu_{\tilde{A}(t)}(x))|x \in U\} \), then \( \tilde{A}(t) \) is called a time-dependent fuzzy number.

**Example 5.2** Let \( \tilde{A}(t) \) be a time-dependent fuzzy number, where \( \tilde{A}(t) = \frac{4t+1}{1-t^2} + \frac{4t^2+1}{1-t^3} + \frac{2t^2-1}{1-t^4} + \frac{4t^2-1}{1-t^5} + \frac{2t^2+1}{1-t^6} + \frac{4t+1}{1-t^7} + \frac{2t+1}{1-t^8} + \frac{4t^2+1}{1-t^9} + \frac{2t^2+1}{1-t^{10}} + \frac{2t^2+1}{1-t^{11}} + \frac{2t^2+1}{1-t^{12}} + \frac{2t^2+1}{1-t^{13}} + \frac{2t^2+1}{1-t^{14}} + \frac{2t^2+1}{1-t^{15}}. \) By Definition 5.1, we have that \( \mu_{\tilde{A}(0)}(t+1) = 1 - \frac{1}{t^2} \) and \( \mu_{\tilde{A}(0)}(2t^2+1) = 1 - \frac{2}{2^t+1}. \)

**Definition 5.3** Let \( \tilde{A}(t) \in F(X), \forall \eta(t) \in [0, 1], where t \in T, then
\[
(1) \tilde{A}_\eta(t) = \{x| x \in U, \mu_{\tilde{A}(t)} \geq \eta(t)\} is referred to as a \eta(t)-cut set of \tilde{A}(t);
\]
\[
(2) \tilde{A}_\eta^>(t) = \{x| x \in U, \mu_{\tilde{A}(t)} > \eta(t)\} is referred to as a strong \eta(t)-cut set of \tilde{A}(t).
\]

**Example 5.4** Let \( \tilde{A}(t) \) be a time-dependent fuzzy number, where \( \tilde{A}(t) = \frac{4t+1}{1-t^2} + \frac{4t^2+1}{1-t^3} + \frac{2t^2-1}{1-t^4} + \frac{4t^2-1}{1-t^5} + \frac{2t^2+1}{1-t^6} + \frac{4t^2+1}{1-t^7} + \frac{2t+1}{1-t^8} + \frac{4t^2+1}{1-t^9} + \frac{2t^2+1}{1-t^{10}} + \frac{2t^2+1}{1-t^{11}} + \frac{2t^2+1}{1-t^{12}} + \frac{2t^2+1}{1-t^{13}} + \frac{2t^2+1}{1-t^{14}} + \frac{2t^2+1}{1-t^{15}}. \) By Definition 5.3, we have that \( \tilde{A}_\frac{1}{3}(t) = \{2t+1, 4t+1, 4t-1, 2t-1, 2t^2+1\} \) and \( \tilde{A}_\{\frac{1}{3}\}^>(t) = \{2t-1, 2t^2+1\}. \)
In what follows, we employ Table 4 to illustrate loss functions which are time-dependent fuzzy numbers.

Table 4: Time-dependent fuzzy loss function.

| Action | \( X(P) \) | \( -X(N) \) |
|--------|----------|---------|
| \( a_P \) | \( \overline{\lambda}_{PP}(t) = [PP_{\lambda}^L(t), PP_{\lambda}^U(t)] \) | \( \overline{\lambda}_{PN}(t) = [PN_{\lambda}^L(t), PN_{\lambda}^U(t)] \) |
| \( a_B \) | \( \overline{\lambda}_{BP}(t) = [BP_{\lambda}^L(t), BP_{\lambda}^U(t)] \) | \( \overline{\lambda}_{BN}(t) = [BN_{\lambda}^L(t), BN_{\lambda}^U(t)] \) |
| \( a_N \) | \( \overline{\lambda}_{NP}(t) = [NP_{\lambda}^L(t), NP_{\lambda}^U(t)] \) | \( \overline{\lambda}_{NN}(t) = [NN_{\lambda}^L(t), NN_{\lambda}^U(t)] \) |

In Table 4, \( \overline{\lambda}_{PP}(t), \overline{\lambda}_{BP}(t), \overline{\lambda}_{NP}(t), \overline{\lambda}_{PN}(t), \overline{\lambda}_{BN}(t) \) and \( \overline{\lambda}_{NN}(t) \) are time-dependent fuzzy numbers.

Below, we discuss DTRS when loss functions are the lower and upper bounds of time-dependent fuzzy numbers.

On one hand, \( PP_{\lambda}^L(t), BP_{\lambda}^L(t), NP_{\lambda}^L(t), PN_{\lambda}^L(t), BN_{\lambda}^L(t) \) and \( NN_{\lambda}^L(t) \) denote the lower bounds of time-dependent fuzzy numbers in Table 4. We have the expected losses \( R_{opt}(a_P[x]), R_{opt}(a_B[x]) \) and \( R_{opt}(a_N[x]) \) associated with taking the individual actions for an object \( x \) and \( t \in T \) as follows:

\[
R_{opt}(a_P[x]) = PP_{\lambda}^L(t)P(X[x]) + PN_{\lambda}^L(t)P(\neg X[x]);
\]

\[
R_{opt}(a_B[x]) = BP_{\lambda}^L(t)P(X[x]) + BN_{\lambda}^L(t)P(\neg X[x]);
\]

\[
R_{opt}(a_N[x]) = NP_{\lambda}^L(t)P(X[x]) + NN_{\lambda}^L(t)P(\neg X[x]).
\]

The bayesian decision procedure suggests the local minimum-cost decision rules:

(\( P \)) : If \( R_{opt}(a_P[x]) \leq R_{opt}(a_B[x]) \) and \( R_{opt}(a_B[x]) \leq R_{opt}(a_N[x]) \), then \( x \in POS(X) \);

(\( B \)) : If \( R_{opt}(a_B[x]) \leq R_{opt}(a_P[x]) \) and \( R_{opt}(a_P[x]) \leq R_{opt}(a_N[x]) \), then \( x \in BND(X) \);

(\( N \)) : If \( R_{opt}(a_N[x]) \leq R_{opt}(a_P[x]) \) and \( R_{opt}(a_P[x]) \leq R_{opt}(a_B[x]) \), then \( x \in NEG(X) \).

Suppose \( 0 \leq PP_{\lambda}^L(t) \leq BP_{\lambda}^L(t) \leq NP_{\lambda}^L(t) \leq BN_{\lambda}^L(t) \leq PN_{\lambda}^L(t) \) for \( t \in T \). Since \( P(X[x]) + P(\neg X[x]) = 1 \), we simplify the rules as follows:

(\( P \)): If \( P(X[x]) \geq \alpha_{opt}(t) \), then \( x \in POS(X) \);

(\( B \)) : If \( \beta_{opt}(t) < P(X[x]) < \alpha_{opt}(t) \), then \( x \in BND(X) \);

(\( N \)) : If \( P(X[x]) \leq \beta_{opt}(t) \), then \( x \in NEG(X) \), where

\[
\alpha_{opt}(t) = \frac{PN_{\lambda}^L(t) - BN_{\lambda}^L(t)}{PN_{\lambda}^L(t) - BN_{\lambda}^L(t) + BP_{\lambda}^L(t) - PP_{\lambda}^L(t)},
\]

\[
\beta_{opt}(t) = \frac{BN_{\lambda}^L(t) - NN_{\lambda}^L(t)}{BN_{\lambda}^L(t) - NN_{\lambda}^L(t) + NP_{\lambda}^L(t) - BP_{\lambda}^L(t)}.
\]

Example 5.5 Let \( \overline{\lambda}_{PP}(t), \overline{\lambda}_{BP}(t), \overline{\lambda}_{NP}(t), \overline{\lambda}_{NN}(t), \overline{\lambda}_{BN}(t) \) and \( \overline{\lambda}_{PN}(t) \) be shown as follows:

\[
\overline{\lambda}_{PP}(t) = \frac{t}{1 - \frac{1}{3}} + \frac{t + 1}{1 - \frac{1}{3}} + \frac{2t + 2}{1 - \frac{1}{3}} + \frac{3t + 3}{1 - \frac{1}{7}} + \frac{5t + 3}{1 - \frac{1}{7}} + \frac{4t + 6}{1 - \frac{2}{7}} + \frac{4t + 3}{1 - \frac{1}{7}},
\]

\[
\overline{\lambda}_{BP}(t) = \frac{2t + 3}{1 - \frac{1}{3}} + \frac{2t + 4}{1 - \frac{1}{3}} + \frac{2t + 5}{1 - \frac{1}{3}} + \frac{3t + 6}{1 - \frac{1}{7}} + \frac{3t + 7}{1 - \frac{1}{7}} + \frac{4t + 6}{1 - \frac{2}{7}} + \frac{4t + 9}{1 - \frac{1}{7}} + \frac{4t + 10}{1 - \frac{1}{7}},
\]
\[ \bar{\lambda}_{NP}(t) = \frac{3t + 6}{1 - \frac{1}{7}} + \frac{3t + 7}{1 - \frac{1}{7}} + \frac{3t + 8}{1 - \frac{1}{7}} + \frac{4t + 9}{1 - \frac{1}{7}} + \frac{5t + 9}{1 - \frac{1}{7}} + \frac{4t + 10}{1 - \frac{1}{7}} + \frac{4t + 11}{1 - \frac{1}{7}} + \frac{4t + 12}{1 - \frac{1}{7}} + \frac{5t + 10}{1 - \frac{1}{7}}, \]
\[ \bar{\lambda}_{NN}(t) = \frac{2t}{1 - \frac{1}{7}} + \frac{2t + 1}{1 - \frac{1}{7}} + \frac{2t + 2}{1 - \frac{1}{7}} + \frac{3t + 3}{1 - \frac{1}{7}} + \frac{5t + 3}{1 - \frac{1}{7}} + \frac{4t + 6}{1 - \frac{1}{7}} + \frac{4t + 8}{1 - \frac{1}{7}} + \frac{4t + 9}{1 - \frac{1}{7}} + \frac{4t + 10}{1 - \frac{1}{7}}. \]
\[ \bar{\lambda}_{BN}(t) = \frac{3t + 2}{1 - \frac{1}{7}} + \frac{3t + 4}{1 - \frac{1}{7}} + \frac{3t + 6}{1 - \frac{1}{7}} + \frac{4t + 6}{1 - \frac{1}{7}} + \frac{5t + 6}{1 - \frac{1}{7}} + \frac{4t + 7}{1 - \frac{1}{7}} + \frac{4t + 8}{1 - \frac{1}{7}} + \frac{4t + 9}{1 - \frac{1}{7}} + \frac{4t + 10}{1 - \frac{1}{7}}. \]
\[ \bar{\lambda}_{PN}(t) = \frac{4t + 8}{1 - \frac{1}{7}} + \frac{4t + 9}{1 - \frac{1}{7}} + \frac{4t + 10}{1 - \frac{1}{7}} + \frac{4t + 11}{1 - \frac{1}{7}} + \frac{4t + 12}{1 - \frac{1}{7}} + \frac{5t + 11}{1 - \frac{1}{7}} + \frac{4t + 15}{1 - \frac{1}{7}} + \frac{5t + 10}{1 - \frac{1}{7}}. \]

By taking \( \eta = 1 - \frac{1}{7} \), we have
\[ \bar{\lambda}_{PP}(t) = [t, 2t + 2], \bar{\lambda}_{PN}(t) = [4t + 8, 4t + 10], \bar{\lambda}_{BP}(t) = [2t + 3, 2t + 5], \]
\[ \bar{\lambda}_{BN}(t) = [3t + 2, 3t + 6], \bar{\lambda}_{NP}(t) = [3t + 6, 3t + 8], \bar{\lambda}_{NN}(t) = [2t, 2t + 2]. \]

Consequently, we have
\[ \alpha^{opt}(t) = \frac{\lambda_{PP}^{\min}(t) - \lambda_{BN}^{\min}(t)}{\lambda_{BN}^{\min}(t) - \lambda_{PP}^{\min}(t) + \lambda_{BP}^{\min}(t) - \lambda_{PP}^{\min}(t)} = \frac{t + 6}{2t + 9}; \]
\[ \beta^{opt}(t) = \frac{\lambda_{BN}^{\min}(t) - \lambda_{NN}^{\min}(t)}{\lambda_{BN}^{\min}(t) - \lambda_{NN}^{\min}(t) + \lambda_{NP}^{\min}(t) - \lambda_{BP}^{\min}(t)} = \frac{t + 2}{2t + 5}. \]

On the other hand, \( PP_{\eta}^{U}(t), BP_{\eta}^{U}(t), NP_{\eta}^{U}(t), PN_{\eta}^{U}(t), BN_{\eta}^{U}(t) \) and \( NN_{\eta}^{U}(t) \) denote the upper bounds of time-dependent fuzzy numbers in Table 4. We show the expected losses \( R^{pes}(a_{P}[x]), R^{pes}(a_{B}[x]) \) and \( R^{pes}(a_{N}[x]) \) associated with taking the individual actions for an object \( x \) and \( t \in T \) as follows:
\[ R^{pes}(a_{P}[x]) = PP_{\eta}^{U}(t)P(X[x]) + PN_{\eta}^{L}(t)P(-X[x]); \]
\[ R^{pes}(a_{B}[x]) = BP_{\eta}^{U}(t)P(X[x]) + BN_{\eta}^{L}(t)P(-X[x]); \]
\[ R^{pes}(a_{N}[x]) = NP_{\eta}^{L}(t)P(X[x]) + NN_{\eta}^{L}(t)P(-X[x]). \]

The bayesian decision procedure suggests the local minimum-cost decision rules:
(P'): If \( R^{pes}(a_{P}[x]) \leq R^{pes}(a_{B}[x]) \leq R^{pes}(a_{N}[x]) \), then \( x \in POS(X) \);
(B'): If \( R^{pes}(a_{B}[x]) \leq R^{pes}(a_{P}[x]) \leq R^{pes}(a_{N}[x]) \), then \( x \in BND(X) \);
(N'): If \( R^{pes}(a_{N}[x]) \leq R^{pes}(a_{P}[x]) \leq R^{pes}(a_{B}[x]) \), then \( x \in NEG(X) \).

Suppose \( 0 \leq PP_{\eta}^{U}(t) \leq BP_{\eta}^{U}(t) \leq NP_{\eta}^{U}(t) \) and \( 0 \leq NN_{\eta}^{U}(t) \leq BN_{\eta}^{U}(t) \leq PN_{\eta}^{U}(t) \) for \( t \in T \). Since \( P(X[x]) + P(-X[x]) = 1 \), we simplify the rules as follows:
(P'): If \( P(X[x]) \geq \alpha^{pes}(t) \), then \( x \in POS(X) \);
(B'): If \( \beta^{pes}(t) < P(X[x]) < \alpha^{pes}(t) \), then \( x \in BND(X) \);
(N'): If \( P(X[x]) \leq \beta^{pes}(t) \), then \( x \in NEG(X) \), where
\[ \alpha^{pes}(t) = \frac{PN_{\eta}^{U}(t) - BN_{\eta}^{U}(t)}{PN_{\eta}^{U}(t) - BN_{\eta}^{U}(t) + BP_{\eta}^{U}(t) - PP_{\eta}^{U}(t)}, \beta^{pes}(t) = \frac{BN_{\eta}^{U}(t) - NN_{\eta}^{U}(t)}{BN_{\eta}^{U}(t) - NN_{\eta}^{U}(t) + NP_{\eta}^{U}(t) - BP_{\eta}^{U}(t)}. \]
Example 5.6  (Continuation of Example 5.5) On the basis of Theorem 5.7, with taking the individual actions for an object $P$ and $PNN$, we have

$$
\alpha^{pes}(t) = \frac{\lambda_{PN}^{max}(t) - \lambda_{BN}^{max}(t)}{\lambda_{PN}^{max}(t) - \lambda_{BN}^{max}(t) + \lambda_{BP}^{max}(t) - \lambda_{PP}^{max}(t)} = \frac{t + 4}{t + 7};
$$

$$
\beta^{pes}(t) = \frac{\lambda_{BN}^{max}(t) - \lambda_{NN}^{max}(t) + \lambda_{NP}^{max}(t) - \lambda_{BP}^{max}(t)}{\lambda_{BN}^{max}(t) - \lambda_{NN}^{max}(t) + \lambda_{NP}^{max}(t) - \lambda_{BP}^{max}(t)} = \frac{t + 4}{2t + 7}.
$$

In general, by taking $PP_\eta(t) \in \bar{\lambda}_{PP}(t), BP_\eta(t) \in \bar{\lambda}_{BP}(t), NP_\eta(t) \in \bar{\lambda}_{NP}(t), PN_\eta(t) \in \bar{\lambda}_{PN}(t), BN_\eta(t) \in \bar{\lambda}_{BN}(t)$ and $NN_\eta(t) \in \bar{\lambda}_{NN}(t)$, we show the expected losses $R(aP||x), R(aB||x)$ and $R(aN||x)$ associated with taking the individual actions for an object $x$ and $t \in T$ as follows:

$$
R(aP||x) = PP_\eta(t)P(X||x) + PN_\eta(t)P(\neg X||x);
$$

$$
R(aB||x) = BP_\eta(t)P(X||x) + BN_\eta(t)P(\neg X||x);
$$

$$
R(aN||x) = NP_\eta(t)P(X||x) + NN_\eta(t)P(\neg X||x).
$$

The bayesian decision procedure suggests the local minimum-cost decision rules:

$(P''')$: If $R(aP||x) \leq R(aB||x), R(aP||x) \leq R(aN||x)$, then $x \in POS(X)$;

$(B''')$: If $R(aB||x) \leq R(aP||x), R(aB||x) \leq R(aN||x)$, then $x \in BND(X)$;

$(N''')$: If $R(aN||x) \leq R(aP||x), R(aN||x) \leq R(aB||x)$, then $x \in NEG(X)$.

Suppose $0 \leq PP_\eta(t) \leq BP_\eta(t) \leq NP_\eta(t)$ and $0 \leq NN_\eta(t) \leq BN_\eta(t) \leq PN_\eta(t)$ for $t \in T$. Since $P(X||x) + P(\neg X||x) = 1$, we simplify the rules as follows:

$(P'')$: If $P(X||x) \geq \alpha(t)$, then $x \in POS(X)$;

$(B'')$: If $\beta(t) < P(X||x) < \alpha(t)$, then $x \in BND(X)$;

$(N'')$: If $P(X||x) \leq \beta(t)$, then $x \in NEG(X)$, where

$$
\alpha(t) = \frac{PN_\eta(t) - BN_\eta(t)}{PN_\eta(t) - BN_\eta(t) + BP_\eta(t) - PP_\eta(t)}, \beta(t) = \frac{BN_\eta(t) - NN_\eta(t)}{BN_\eta(t) - NN_\eta(t) + NP_\eta(t) - BP_\eta(t)}.
$$

On the basis of the above results, we have the following theorem for DTRS when loss functions are time-dependent fuzzy numbers.

Theorem 5.7 Let $0 \leq PP_\eta^L(t) \leq PP_\eta^U(t) \leq BP_\eta^L(t) \leq BP_\eta^U(t) \leq NP_\eta^L(t) \leq NP_\eta^U(t)$ and $0 \leq NN_\eta^L(t) \leq NN_\eta^U(t) \leq BN_\eta^L(t) \leq BN_\eta^U(t) \leq PN_\eta^L(t) \leq PN_\eta^U(t)$, where $t \in T$. Then

1. $\alpha(t) \in \left[ \frac{PN_\eta(t)^L - BN_\eta(t)^L}{PN_\eta(t)^U - BN_\eta(t)^U + BP_\eta(t)^U - PP_\eta(t)^U}, \min \left\{ \frac{PN_\eta(t)^U - BN_\eta(t)^U}{PN_\eta(t)^L - BN_\eta(t)^L + BP_\eta(t)^L - PP_\eta(t)^L}, 1 \right\} \right];$

2. $\beta(t) \in \left[ \frac{BN_\eta(t)^L - NN_\eta(t)^L}{BN_\eta(t)^U - NN_\eta(t)^U + NP_\eta(t)^U - BP_\eta(t)^U}, \min \left\{ \frac{BN_\eta(t)^U - NN_\eta(t)^U}{BN_\eta(t)^L - NN_\eta(t)^L + NP_\eta(t)^L - BP_\eta(t)^L}, 1 \right\} \right].$
Proof. (1) Since \(0 \leq PP^L(t) \leq PP^U(t) \leq BP^L(t) \leq BP^U(t) \leq NP^L(t) \leq NP^U(t)\) and \(0 \leq NN^L(t) \leq NN^U(t)\) \(\leq BN^L(t) \leq BN^U(t) \leq PN^L(t) \leq PN^U(t)\), we have

\[
PN^U(t) - BN^U(t) > 0, BP^U(t) - PP^U(t) \geq 0.
\]

It implies that

\[
PN^L(t) - BN^L(t) \leq PN^U(t) - BN^U(t) \leq PN^L(t) - BN^L(t),
BP^L(t) - PP^L(t) \leq BP^U(t) - PP^L(t).
\]

It follows that

\[
PN^L(t) - BN^L(t) + BP^L(t) - PP^L(t) \leq PN^U(t) - BN^L(t) + BP^U(t) - PP^L(t).
\]

Obviously, we have

\[
\frac{PN^L(t) - BN^L(t)}{PN^U(t) - BN^L(t) + BP^L(t) - PP^L(t)} \leq \frac{PN^U(t) - BN^L(t)}{PN^U(t) - BN^L(t) + BP^L(t) - PP^L(t)}.
\]

Therefore,

\[
\alpha(t) \in \left[\frac{PN^L(t) - BN^L(t)}{PN^U(t) - BN^L(t) + BP^L(t) - PP^L(t)}, \min\left\{\frac{PN^U(t) - BN^L(t)}{PN^U(t) - BN^L(t) + BP^L(t) - PP^L(t)}, 1\right\}\right].
\]

(2) Since \(0 \leq PP^L(t) \leq PP^U(t) \leq BP^L(t) \leq BP^U(t) \leq NP^L(t) \leq NP^U(t)\) and \(0 \leq NN^L(t) \leq NN^U(t)\) \(\leq BN^L(t) \leq BN^U(t) \leq PN^L(t) \leq PN^U(t)\), we have

\[
BN^U(t) - NN^U(t) > 0, NP^U(t) - BP^U(t) \geq 0.
\]

It implies that

\[
BN^L(t) - NN^L(t) \leq BN^U(t) - NN^U(t) \leq BN^L(t) - NN^L(t),
NP^U(t) - BP^U(t) \leq NP^U(t) - BP^U(t) \leq NP^U(t) - BP^U(t).
\]

It follows that

\[
BN^L(t) - NN^U(t) + NP^L(t) - BP^U(t) \leq BN^U(t) - NN^U(t) + NP^U(t) - BP^U(t)
\]

\[
\leq BN^L(t) - NN^U(t) + NP^U(t) - BP^U(t).
\]
Obviously, we have

\[
\begin{align*}
\frac{BN^U_\eta(t) - NN^U_\eta(t)}{BN^U_\eta(t) - NN^L_\eta(t) + NP^U_\eta(t) - BP^L_\eta(t)} & \leq \frac{BN_\eta(t) - NN_\eta(t)}{BN_\eta(t) - NN^L_\eta(t) + NP_\eta(t) - BP_\eta(t)} \\
& \leq \frac{BN^U_\eta(t) - NN^L_\eta(t)}{BN^U_\eta(t) - NN^L_\eta(t) + NP^U_\eta(t) - BP^U_\eta(t)}.
\end{align*}
\]

Therefore,

\[
\beta(t) \in \left[ \frac{BN^L_\eta(t) - NN^U_\eta(t)}{BN^U_\eta(t) - NN^L_\eta(t) + NP^U_\eta(t) - BP^L_\eta(t)}, \min\left\{ \frac{BN^U_\eta(t) - NN^L_\eta(t)}{BN^U_\eta(t) - NN^L_\eta(t) + NP^U_\eta(t) - BP^U_\eta(t)}, 1 \right\} \right]. \square
\]

6 Conclusions

Many researchers have focused on investigations of loss functions in DTRS. In this paper, we have investigated DTRS when loss functions are satisfied time-dependent uniform distributions and normal processes. Furthermore, we have studied DTRS when loss functions are time-dependent interval sets. Consequently, we have investigated DTRS when loss functions are time-dependent fuzzy numbers. Finally, we have employed several examples to illustrate that how to make decisions by using time-dependent loss functions-based DTRS.

There are still many interesting topics deserving further investigations on DTRS. For example, there are many types of loss functions which are satisfied stochastic processes, and it is of interest to investigate time-dependent loss functions-based DTRS. In the future, we will further investigate time-dependent loss functions and discuss the application of DTRS in knowledge discovery.

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