Sylvester’s question
and the Random Acceleration Process

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Abstract

Let $n$ points be chosen randomly and independently in the unit disk. “Sylvester’s question” concerns the probability $p_n$ that they are the vertices of a convex $n$-sided polygon. Here we establish the link with another problem. We show that for large $n$ this polygon, when suitably parametrized by a function $r(\phi)$ of the polar angle $\phi$, satisfies the equation of the random acceleration process (RAP), $d^2r/d\phi^2 = f(\phi)$, where $f$ is Gaussian noise. On the basis of this relation we derive the asymptotic expansion \( \log p_n = -2n \log n + n \log(2\pi e^2) - c_0 n^{1/5} + \ldots \), of which the first two terms agree with a rigorous result due to Bárány. The nonanalyticity in $n$ of the third term is a new result. The value $1/5$ of the exponent follows from recent work on the RAP due to Györgyi et al. \cite{PhysRevE.75.021123}. We show that the $n$-sided polygon is effectively contained in an annulus of width $\sim n^{-4/5}$ along the edge of the disk. The distance $\delta_n$ of closest approach to the edge is exponentially distributed with average $(2n)^{-1}$.

Keywords: random convex polygons, random points in convex position, random acceleration process, integrated Brownian motion

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1 Introduction

Let $n$ points be chosen independently according to a uniform distribution on a disk $D \subset \mathbb{R}^2$. We consider in this work the probability $p_n(D)$ that these points are the vertices of a convex $n$-sided polygon; an example is shown in figure 1. This question, with the disk $D$ replaced by an arbitrary convex domain $K \subset \mathbb{R}^2$, has a long history in mathematics. In 1864 Sylvester [1] asked about the value of $p_4(K)$.

For a parallelogram $S$ and a triangle $T$ the exact results

$$p_n(S) = \left[ \frac{1}{n!} \left( \frac{2n - 2}{n - 1} \right) \right]^2, \quad p_n(T) = \frac{2^n (3n - 3)!}{(n - 1)!^3 (2n)!}$$

were shown by Valtr [2, 3]. Since affine transformations may transform any parallelogram into a square and any triangle into an equilateral triangle while leaving the distribution uniform, expressions (1.1) do not depend on the particular choice of triangle or parallelogram. In the limit of large $n$, the probabilities $p_n(K)$ become very small and we will be interested in their asymptotic large-$n$ expansions. Those of $p_n(S)$ and $p_n(T)$ above are given by

$$\log p_n(S) = -2n \log n + n \log(16e^2) + O(\log n),$$
$$\log p_n(T) = -2n \log n + n \log(\frac{27}{2}e^2) + O(\log n),$$

in which only the coefficients of the terms linear in $n$ are different. These coefficients were subsequently explained by Bárány [4], who derived a general...
result that we will paraphrase as follows. Bárany showed that for an arbitrary convex domain $K$ of area $A_K$
\[
\log p_n(K) = -2n \log n + n \log \left(\frac{1}{2} e^{2P^3_K/K} \right) + o(n),
\]
(1.3)
in which the meaning of $P_K$ still has to be given. To do so we need the concept of the affine length $\mathcal{L}_C$ of a convex curve $C$, defined (see e.g. [4]) as
\[
\mathcal{L}_C = \int_C \kappa^{1/3} ds,
\]
(1.4)
with $ds$ a line element along the curve and $\kappa$ the local inverse radius of curvature. We note that the affine length has the physical dimensionality of a (length)$^{2/3}$. Let now a convex subdomain $K'$ of $K$ have a border $\partial K'$ of affine length $\mathcal{L}_{\partial K'}$ and let furthermore $\mathcal{L}_{\partial K'}$ attain its maximum for $K' = K'_{\max}$. Then
\[
P_K \equiv \max_{K' \subseteq K} \mathcal{L}_{\partial K'} = \mathcal{L}_{\partial K'_{\max}}
\]
(1.5)
is called the affine perimeter of $K$. This completes the definition of (1.3).

There is no simple way to calculate $P_K$ for general $K$. When $K$ is a disk $D$ of radius $R_D$, symmetry dictates that $K'_{\max} = K = D$. Then (1.5) and (1.4) with $\kappa = R_D^{-1}$ yield $P_K = \mathcal{L}_{\partial D} = 2\pi R_D^{2/3}$, so that (1.3) becomes
\[
\log p_n(D) = -2n \log n + n \log(2\pi e^2) + \mathcal{R}_n,
\]
(1.6)
in which the remainder $\mathcal{R}_n$ is $o(n)$. This statement about the remainder is weaker than that in equation (1.2), where the rest term is known to be $O(\log n)$. It will appear that there is a good reason for this difference.

In this work we represent the $n$ random points by polar coordinates $(R_m, \Phi_m)$ for $m = 1, \ldots, n$ in such a way that $0 < \Phi_1 < \Phi_2 < \ldots < \Phi_n \leq 2\pi$. We define the average radius, $R_{av}$, and the scaled deviation from average, $r_m$, by
\[
R_{av} \equiv n^{-1} \sum_{m=1}^{n} R_m, \quad R_m = R_{av}(1 + n^{-1/2} r_m).
\]
(1.7)
For $n \to \infty$ with $\phi = 2\pi mn^{-1}$ fixed, $r_m$ becomes a random function $r(\phi)$ on $[0, 2\pi]$. Our principal result is that in this limit the remainder $\mathcal{R}_n$ in (1.6) can be cast in the form
\[
\mathcal{R}_n = \log \left\langle \exp \left[-2n^{1/2} \max_{0 \leq \phi \leq 2\pi} r(\phi)\right]\right\rangle_0,
\]
(1.8)
in which $\langle \ldots \rangle_0$ denotes the average with respect to all $2\pi$-periodic zero-integral solutions $r(\phi)$ of
\[
\frac{d^2 r(\phi)}{d\phi^2} = f(\phi),
\]
(1.9)
where $f(\phi)$ is Gaussian noise of autocorrelation
\[
\langle f(\phi)f(\phi') \rangle_0 = \frac{3}{2} [2\pi \delta(\phi - \phi') - 1].
\] (1.10)

We arrive at these results by extending an analytic method that was developed originally in the context of planar Voronoi tessellations \[5, 6, 7\] and employed also to study Poisson line tessellations in general and the Crofton cell problem in particular \[5\]. The idea of applying this method here arose from the observation that the first two terms in expansion (1.6) are identical to those of the expansion of $\log p_{n}^{\text{Vor}}$, where $p_{n}^{\text{Vor}}$ is the probability for the typical Poisson-Voronoi cell to be $n$-sided. The work of reference \[9\] suggests that the two problems are related by an inversion of the radial coordinates with respect to the unit circle; this is indeed borne out by our analysis. Our method consists of a rather intricate but fully exact coordinate transformation, followed by an asymptotic expansion which, although nonrigorous, is of the kind routinely used in physics to obtain exact results.

Equation (1.9) is known as the random acceleration process; the function $r(\phi)$ is also referred to as Kolmogorov diffusion or integrated Brownian motion. We will briefly review some of the literature on this stochastic process in section \[4\]. Of particular relevance to us is recent work by Györgyi et al. (GMOR) \[10\]. Upon using their result for the right hand side of (1.8) we find that (1.6) becomes
\[
\log p_{n}(D) = -2n \log n + n \log(2\pi^2 \epsilon^2) - 2\epsilon_0 (3\pi^4 n)^{1/5} + \ldots,
\] (1.11)
where $\epsilon_0 > 0$ is the smallest eigenvalue of a linear eigenvalue problem, the only hypothesis being the existence of its solution. An immediate corollary is that in the large-$n$ limit the average (1.8) draws its main contribution from polygons that stay within a distance of order $\sim n^{-4/5} R_D$ from the edge of the disk $D$. For $n \to \infty$ the distance of closest approach to the disk is shown to be exponentially distributed with average $1/(2n)$.

The term proportional to $n^{1/5}$ in (1.11) is a new contribution to the answer to Sylvester’s question. Since such a nonanalytic term is absent from the expansions (1.2) for the triangle and the square, we are led to ask under which conditions such an $n^{1/5}$ term appears. Bárány \[4\] showed, essentially, that in the large-$n$ limit the $n$ points in convex position lie on the curve $\partial K_{\text{max}}'$, and that this curve is composed of
(i) arcs or isolated points that coincide with the domain boundary $\partial K$;
(ii) arcs of parabolas in the interior of $K$.

The present study provides strong indication that an $n^{1/5}$ term occurs whenever $\partial K_{\text{max}}'$ contains at least one arc coinciding with $\partial K$, that is, when $\partial K_{\text{max}}'$ sticks to the domain boundary over some nonzero angular interval. This is obviously the case for the circle, where $\partial K_{\text{max}}' = \partial K$, but not for the square and the triangle, where the limit curve $\partial K_{\text{max}}'$ touches the domain boundary $\partial K$ only in isolated points \[2, 3\].
In section 2 we carry out the exact coordinate transformation. In section 3 we perform the large-$n$ expansion and establish relation (1.8). In section 4.1 we use GMOR’s results to obtain (1.11). In section 4.2 we show how to obtain the power $\frac{1}{5}$ in (1.11) by heuristic arguments. In section 4.3 we return to the hypothesis of the existence of the eigenvalue $\epsilon_0$; we derive exact bounds for the right hand side of (1.8) and show that even without the existence hypothesis the conclusion remains valid that $R_n$ is nonanalytic in $n$. Section 5 contains the study of the distribution of the distance of closest approach of the edge. Section 6 is our conclusion.

2 Random convex polygon in a disk

We consider $n$ points $R_1, \ldots, R_n$ drawn randomly and independently from a uniform distribution on the disk of radius $R_D$, centered in the origin. We ask for the probability $p_n(D)$ that these $n$ points are the vertices of a convex polygon. A slightly different probability $p_n^\ast$ is defined the same way but with the additional condition that the polygon enclose the origin. When $n$ gets large, the ratio between $p_n(D)$ and $p_n^\ast$ will tend to unity exponentially rapidly with $n$, but $p_n^\ast$ will be easier to study. We begin by writing its definition. In terms of the polar coordinate representation $R_m = (R_m, \Phi_m)$ we have

$$p_n^\ast = \frac{1}{(\pi R_D^2)^n} \int_0^{R_D} R_1 dR_1 \ldots R_n dR_n \int_0^{2\pi} d\Phi_1 \ldots d\Phi_n \chi(R_1, \ldots, R_n), \quad (2.1)$$

in which $\chi$ is the indicator of the subdomain of phase space where the points form an origin-enclosing convex $n$-sided polygon; the explicit expression of $\chi$ will be discussed in section 2.2. Everywhere below we will scale the ‘radii’ $R_m$ such that $R_D = 1$.

We will now subject expression (2.1) to a series of coordinate transformations. The final result of these will be equation (2.22), together with (2.23) and (2.24).

2.1 Transformation of variables. I

In (2.1) we may set one of the angles, say $\Phi_n$, equal to $2\pi$ if we compensate by an extra factor $2\pi$; and the remaining $n - 1$ angles may be ordered such that

$$0 < \Phi_1 < \ldots < \Phi_{n-1} < \Phi_n = 2\pi \quad (2.2)$$

if a compensating factor $(n-1)!$ is introduced. Referring now to figure 2, we define the angle differences $\xi_m$ between two consecutive vertex vectors $R_{m-1}$ and $R_m$ by

$$\xi_m = \Phi_m - \Phi_{m-1}, \quad m = 1, \ldots, n, \quad (2.3)$$
with the convention $\Phi_0 = 0$. Then the $\xi_m$ are positive and satisfy the sum rule $\sum_{m=1}^{n} \xi_m = 2\pi$. In terms of these variables we can write (2.1) as

$$p_n^* = \frac{2\pi(n-1)!}{\pi^n} \int_0^1 R_1 dR_1 \ldots R_n dR_n \int_0^{2\pi} d\xi_1 \ldots d\xi_n \delta(\xi_1 + \ldots + \xi_n - 2\pi) \chi.$$  

(2.4)

Next we define the average radius $R_{\text{av}}$ and the reduced radii $\rho_m$ by

$$R_{\text{av}} = \frac{1}{n} \sum_{m=1}^{n} R_m, \quad R_m = \rho_m R_{\text{av}}.$$  

(2.5)

The $\rho_m$ therefore satisfy the sum rule

$$\frac{1}{n} \sum_{m=1}^{n} \rho_m = 1.$$  

(2.6)

We now rewrite the integrals on the $R_m$ in (2.4) as

$$\int_0^1 dR_1 \ldots dR_n = \int_0^1 dR_{\text{av}} R_{\text{av}}^{2n-1} \int_0^{R_{\text{av}}^{-1}} d\rho_1 \ldots d\rho_n \delta\left(1 - n^{-1} \sum_{m=1}^{n} \rho_m\right)$$

$$= \int_0^{\infty} d\rho_1 \ldots d\rho_n \delta\left(1 - n^{-1} \sum_{m=1}^{n} \rho_m\right)$$

$$\times \int_0^1 dR_{\text{av}} R_{\text{av}}^{2n-1} \prod_{m=1}^{n} \theta\left(R_{\text{av}}^{-1} - \rho_m\right)$$  

(2.7)

where $\theta(R_{\text{av}}^{-1} - \rho_m)$ is the Heaviside step function, equal to 1 if $R_{\text{av}}^{-1} - \rho_m > 0$ and to 0 if $R_{\text{av}}^{-1} - \rho_m < 0$. If (2.7) is applied to an integrand without $R_{\text{av}}$ dependence, then the integration on $R_{\text{av}}$ may be carried out according to

$$\int_0^1 dR_{\text{av}} R_{\text{av}}^{2n-1} \prod_{m=1}^{n} \theta(R_{\text{av}}^{-1} - \rho_m) = \int_0^{\min_{1 \leq m \leq n} R_{\text{av}}^{-1}} dR_{\text{av}} R_{\text{av}}^{2n-1}$$

$$= (2n)^{-1} \left[\max_{1 \leq m \leq n} \rho_m\right]^{-2n}.$$  

(2.8)

Hence, after we substitute (2.8) in (2.7) and (2.7) in (2.4) we get

$$p_n^* = \frac{(n-1)!}{n\pi^{n-1}} \int_0^{2\pi} d\xi_1 \ldots d\xi_n \delta\left(\sum_{m=1}^{n} \xi_m - 2\pi\right)$$

$$\times \int_0^{\infty} d\rho_1 \ldots d\rho_n \delta\left(1 - n^{-1} \sum_{m=1}^{n} \rho_m\right) \chi\left[\max_{1 \leq m \leq n} \rho_m\right]^{-2n}.$$  

(2.9)

This completes the conversion of the variables of integration from the $R_m$ and $\Phi_m$ to the $\rho_m$ and $\xi_m$. 

6
2.2 The convexity condition

We will now find an explicit expression for the convexity condition, enforced by the indicator $\chi$, in terms of the variables of integration $\rho_m$ and $\xi_m$. This is most easily done by means of the following geometric argument. In figure 2, the point $R$ is the intersection of the line segments $OR_m$ and $R_{m-1}R_{m+1}$. The point $R_m$ is in convex position with respect to $R_{m-1}$ and $R_{m+1}$ if $R < R_m$, or equivalently, if

$$\text{area}(OR_{m-1}R_m) + \text{area}(OR_{m+1}R_m) > \text{area}(OR_{m-1}R_{m+1}).$$  \hspace{1cm} (2.10)

The areas are readily expressed in terms of the two angles $\xi_m$ and $\xi_{m+1}$ and the three radii $R_{m-1}$, $R_m$, and $R_{m+1}$. After division by $R_{m-1}R_mR_{m+1}R_{av}^{-1}$ the length scale drops out and (2.10) becomes

$$\frac{\sin \xi_m}{\rho_{m+1}} + \frac{\sin \xi_{m+1}}{\rho_{m-1}} > \frac{\sin(\xi_m + \xi_{m+1})}{\rho_m}, \quad m = 1, \ldots, n, \hspace{1cm} (2.11)$$

where $\xi_m$ and $\rho_m$ are understood to be $n$-periodic in their index. A similar condition, differing only in that all radii are replaced by their inverses, was encountered in the case of the Voronoi cells [12, 6]; it there expresses the condition that the $m$th point (for $m = 1, \ldots, n$) contribute a nonzero segment to the perimeter of the Voronoi cell. This property is due to a duality argument: when $C$ is a convex set containing the origin in its interior, its
dual (or polar body) $C^*$ is the convex set \( \{ x \in \mathbb{R}^2 ; \langle x, y \rangle \leq 1 \ \forall \ y \in C \} \) where \( \langle \cdot, \cdot \rangle \) is the usual scalar product (see [13]). Formally, when applied to a convex \( n \)-sided polygon, the transformation provides a new convex \( n \)-sided polygon such that the projections of the origin onto its edges are given by the polar coordinates \( (R_m^{-1}, \Phi_m) \), \( 1 \leq m \leq n \).

Equations (2.9) and (2.11) represent an intermediate stage in the transformation of variables. Equation (2.11) couples the variables of integration nontrivially and makes the problem of evaluating (2.9) a hard one.

### 2.3 New angular variables

In this subsection we introduce the angular variables necessary for the remainder of our analysis. We refer to figure 3.

Let \( S_m = (S_m, \Psi_m) \) be the projection of the origin onto the line passing through \( R_{m-1} \) and \( R_m \); the \( S_m \) and \( \Psi_m \) may be expressed in terms of the \( R_m \) and \( \Phi_m \). The remaining angles needed in the discussion may then be defined in terms of the \( \Phi_m \) and \( \Psi_m \). The angle difference \( \eta_m \) between the projection vectors \( S_m \) and \( S_{m+1} \) is

\[
\eta_m = \Psi_{m+1} - \Psi_m, \quad m = 1, 2, \ldots, n,
\]

with the convention \( \Psi_{n+1} = \Psi_1 + 2\pi \). The \( \eta_m \) satisfy the sum rule \( \sum_{m=1}^n \eta_m = 2\pi \). Next we define \( 2n \) angles between projection and vertex vectors,

\[
\beta_m = \Psi_m - \Phi_{m-1}, \quad \gamma_m = \Phi_m - \Psi_m, \quad m = 1, \ldots, n.
\]

\( \beta_m \) and \( \gamma_m \) may be negative, as happens when the projection \( S_m \) falls outside the line segment connecting \( R_{m-1} \) and \( R_m \). In any case the geometry shows that we must have

\[
-\frac{\pi}{2} < \beta_m, \gamma_m < \frac{\pi}{2}.
\]

For fixed sets of angles \( \xi = \{ \xi_m \} \) and \( \eta = \{ \eta_m \} \) one may still jointly rotate the vertex vectors \( R_m \) with respect to the projection vectors \( S_m \), as this modifies only the relative angles \( \beta_m \) and \( \gamma_m \) (see figure 3) between the two sets. We may select any one of these relative angles and call it ‘the’ angle of rotation, since it will determine all others. We will select \( \beta_1 \) as this special degree of freedom and express the remaining \( \beta_m \) and \( \gamma_m \) as

\[
\beta_m = \beta_1 - \sum_{\ell=1}^{m-1} (\xi_\ell - \eta_\ell), \quad m = 2, \ldots, n,
\]

\[
\gamma_m = -\beta_1 + \sum_{\ell=1}^{m-1} (\xi_\ell - \eta_\ell) + \xi_m, \quad m = 1, \ldots, n.
\]
Now observe that the reduced radii $\rho_m$ satisfy the relations

$$\frac{\rho_{m-1}}{\rho_m} = \frac{\cos \gamma_m}{\cos \beta_m}, \quad m = 1, \ldots, n,$$

(2.16)

where $\rho_0 \equiv \rho_n$. This ratio (2.16) is the same as the one encountered in the Voronoi and Crofton cell problems [6, 8], except for an interchange of $\beta_m$ and $\gamma_m$. It follows that one may express the $\rho_m$ exclusively in terms of the angles by solving them from (2.16) together with (2.6).

We return now to the convexity condition (2.11). We may use (2.16) to eliminate $\rho_{m\pm 1}$ from (2.11) in favor of $\rho_m$, which subsequently divides out. After some trigonometry (2.11) appears to reduce to the condition $\tan \gamma_m + \tan \beta_{m+1} > 0$. Because of (2.14) this is equivalent to $\gamma_m + \beta_{m+1} > 0$. Still using (2.15) we see that

$$\eta_m > 0, \quad m = 1, 2, \ldots, n,$$

(2.17)

is, finally, the condition for the $n$ points to be in convex position.

### 2.4 Periodicity in the polar angle

Let us next define the function $G$ by

$$e^{2\pi G(\xi, \eta; \beta_1)} = \prod_{m=1}^n \frac{\cos \gamma_m}{\cos \beta_m},$$

(2.18)
with the $\beta_m$ and $\gamma_m$ given by (2.15). This function appears when we relate $\rho_n$ to $\rho_0$ by taking the product of (2.16) on $m = 1, \ldots, n$. Because of the periodicity condition $\rho_0 \equiv \rho_n$ the angle of rotation $\beta_1$ cannot be arbitrary: it must have the special value $\beta_1 = \beta_*(\xi, \eta)$ that is the solution of

$$G(\xi, \eta; \beta_*) = 0. \quad (2.19)$$

This equation is invariant under the interchange of $\beta_m$ and $\gamma_m$ and is therefore identical to the one that appeared in the Voronoi and Crofton cell problems. Indeed, as has already been said, the dual body of a convex $n$-sided polygon is a $n$-sided polygon such that the sets of angles $\{\xi_m; 1 \leq m \leq n\}$ and $\{\eta_m; 1 \leq m \leq n\}$ (as well as $\{\beta_m; 1 \leq m \leq n\}$ and $\{\gamma_m; 1 \leq m \leq n\}$) have been exchanged. It was shown in reference [14] that the solution $\beta_*(\xi, \eta)$ of (2.19), subject to (2.14), exists and is unique if and only if the $\xi_m$ and $\eta_m$ satisfy the condition

$$\max_{1 \leq m \leq n} \left[ \sum_{\ell=1}^{m-1} (\xi_\ell - \eta_\ell) + \xi_m \right] - \min_{1 \leq m \leq n} \left[ \sum_{\ell=1}^{m-1} (\xi_\ell - \eta_\ell) \right] < \pi. \quad (2.20)$$

We will let $\Theta(\xi, \eta)$ denote the indicator function, equal to zero or to unity, of the domain in phase space where (2.20) is fulfilled.

### 2.5 Transformation of variables. II

Finally we carry out the transformation that eliminates the $\rho_m$ occurring in (2.9) in favor of the $\eta_m$ defined in (2.12). The details of this transformation step have been described in [6] and the appendix of [14], and we will not reproduce them here. The result is an integral on the two sets of variables $\xi$ and $\eta$. It is described conveniently with the aid of a ‘zeroth order’ probability distribution $P^{(0)}(\xi, \eta)$ defined as

$$P^{(0)}(\xi, \eta) = \frac{1}{N} \left[ \prod_{m=1}^{n} \xi_m \right] \delta \left( \sum_{m=1}^{n} \xi_m - 2\pi \right) \delta \left( \sum_{m=1}^{n} \eta_m - 2\pi \right), \quad (2.21)$$

where $N = (2\pi)^{3n-2}/[(2n-1)!(n-1)!]$ is the appropriate normalization constant. The average of any quantity $X$ with respect to $P^{(0)}$ will be denoted as $\langle X \rangle_0$. The transformation then leads to

$$p_n^* = p_n^{(0)} \langle \Theta e^{-V} \rangle_0, \quad p_n^{(0)} = \frac{1}{4\pi^2} \frac{(8\pi^2)^n}{(2n)!}, \quad (2.22)$$

in which $\Theta$ is defined following (2.20) and $V$ is given by

$$e^{-V} = \frac{1}{G'(\xi, \eta; \beta_*)} \left[ \prod_{m=1}^{n} \frac{\rho_m^2 \sin \xi_m}{\xi_m \cos \beta_m \cos \gamma_m} \right] \left[ \max_{1 \leq m \leq n} \rho_m \right]^{-2n}, \quad (2.23)$$
where the prime on $G$ denotes differentiation with respect to its last argument. The factor $1/\xi_m$ in the product in (2.23) is compensated by the $\xi_m$ in the product in (2.21). This has been so arranged in order that $\exp(-V)$ do not diverge for any of the $\xi_m$ tending to zero. We note in passing that $p_m^{(0)}$ is the same zeroth order probability that occurs in the Voronoi problem \[6, 14\]. Equations (2.21)-(2.23) are fully exact and are at the basis of all further developments.

3 Large-$n$ limit and Random Acceleration Process

The initial problem (2.1) has been transformed into the evaluation of the average $\langle \Theta e^{-V} \rangle_0$ in (2.22). This, however, is still a formidable problem. From here on we will be able to proceed only by making a large-$n$ expansion. This is the subject of this section.

The average involves only angles and we make the hypothesis, to be confirmed by the calculation, that for large $n$ they all scale with negative powers of $n$. To begin with, we study the scaling that follows from $P^{(0)}(\xi, \eta)$. At a later stage we will then discuss how this scaling is modified by the presence of the integrand $\Theta e^{-V}$ in (2.22).

3.1 Scaling in the large-$n$ limit. I

We start with a preliminary. We note that according to $P^{(0)}$ the $\xi_m$ are independent apart from the sum rule constraint represented by the delta function; and a similar statement holds for the $\eta_m$. Furthermore, the $\xi_m$ and $\eta_l$ are mutually fully independent. More precisely, let $\{X_m; m \geq 1\}$ and $\{Y_l; l \geq 1\}$ be two independent sequences of i.i.d. random variables such that $Y_l$ is exponentially distributed with mean 1 and $X_m$ is distributed with law $u(x) = 4xe^{-2x}$, $x > 0$. For a fixed $n$, the variable $\xi_m$ (resp. $\eta_l$), $1 \leq m, l \leq n$, is equal in law to $(2\pi X_m)/(\sum_{k=1}^n X_k)$ (resp. to $(2\pi Y_l)/(\sum_{k=1}^n Y_k)$). Indeed, let us average any positive bounded test function $h(\eta_1, \ldots, \eta_n)$ on $[0, 2\pi]^n$ with respect to the set $\{Y_l; l \geq 1\}$. A direct change of variables provides

$$
\int_0^\infty dY_1 \ldots dY_n \ h\left(\frac{2\pi Y_1}{\sum_{k=1}^n Y_k}, \ldots, \frac{2\pi Y_n}{\sum_{k=1}^n Y_k}\right) \exp(-Y_1 - \ldots - Y_n) = \frac{(n-1)!}{(2\pi)^{n-1}} \int_0^{2\pi} d\eta_1 \ldots d\eta_n \ h(\eta_1, \ldots, \eta_n) \Theta\left(\sum_{k=1}^n \eta_k - 2\pi\right), \quad (3.1)
$$

which shows that the average with respect to the exponentially distributed i.i.d. $Y_k$ is the same as the average weighted with $P^{(0)}$. The same method
applies to the $X_m$ and $\xi_m$. We now have the following consequence. Let us consider the limit $n \to \infty$ with $k$ a fixed integer and $(m_1, m_2, \ldots, m_k)$ a not necessarily fixed subset of $\{1, 2, \ldots, n\}$. Then in this limit the marginal probability distribution of the vector $(n/2\pi)(\eta_{m_1}, \ldots, \eta_{m_k})$ converges to the probability distribution of $(Y_{m_1}, \ldots, Y_{m_k})$, i.e., to a product of $k$ exponentials. An analogous statement holds for the $\xi_m$ and $X_m$.

Let us define the scaled zero-average variables $\delta x_m$ and $\delta y_m$ by

$$
n^{-1} \delta x_m = \xi_m - 2\pi n^{-1},
$$

$$
n^{-1} \delta y_m = \eta_m - 2\pi n^{-1}, \quad m = 1, \ldots, n. \tag{3.2}
$$

They satisfy the sum rules $\sum_{m=1}^n \delta x_m = \sum_{m=1}^n \delta y_m = 0$ and one readily calculates their variances and covariances,

$$
\langle \delta x_\ell \delta x_m \rangle_0 = \frac{(2\pi)^2 n}{2n+1} \left( \delta_{\ell m} - \frac{1}{n} \right),
$$

$$
\langle \delta y_\ell \delta y_m \rangle_0 = \frac{(2\pi)^2 n}{n+1} \left( \delta_{\ell m} - \frac{1}{n} \right),
$$

$$
\langle \delta x_\ell \delta y_m \rangle_0 = 0, \quad \ell, m = 1, \ldots, n, \tag{3.3}
$$

which will be needed later. The necessary calculations may be carried out directly via (2.21) or with the use of the explicit realization of $\xi_m$ and $\eta_m$ as functions of $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$. Equations (3.3) show that $\delta x_m$ and $\delta y_m$ are of order $n^0$ in the large $n$ limit.

We define $u_m$ by

$$
n^{-\frac{1}{2}} u_m = \frac{1}{2} (\beta_m - \gamma_m)
$$

$$
= \beta_1 - \sum_{\ell=1}^{m-1} (\xi_\ell - \eta_\ell) - \frac{1}{2} \xi_m, \quad m = 1, \ldots, n, \tag{3.4}
$$

where to obtain the second line we used (2.15). It follows from (3.4) together with the scaling of the $\delta x_m$ and $\delta y_m$ found in (3.2) that in the large-$n$ limit the $\beta_m$ and $\gamma_m$, being sums of independent random variables, have Gaussian distributions of average $\pi/n$ and of width $\sim n^{-1/2}$. The $u_m$ therefore remain of order $n^0$ when $n$ gets large [15]. We define $r_m$ by

$$
\rho_m = 1 + n^{-\frac{1}{2}} r_m, \quad m = 1, \ldots, n. \tag{3.5}
$$

From the scaling of $\beta_m$ and $\gamma_m$ together with (2.16) and (3.5) it follows that the $r_m$ are of order $n^0$. Due to the sum rule on the $\rho_m$ in (2.6) they satisfy $\sum_{m=1}^n r_m = 0$. 

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3.2 Second order recursion

We are now ready to perform the large-$n$ expansion of equation (2.16), $\rho_m/\rho_{m-1} = \cos \gamma_m/\cos \beta_m$, using the scaling of the preceding subsection. This yields a recursion relation which in terms of the scaled variables $r_m$, $u_m$, and $\delta x_m$ takes the form

\[ r_m - r_{m-1} + \ldots = \frac{1}{2}(\beta_m^2 - \gamma_m^2) + \ldots \]

(3.6)

where the dots represent terms of higher order. Using that $\beta_m + \gamma_m = \xi_m$ and $\beta_m - \gamma_m = 2n^{-1/2}u_m$ we get from this in the large-$n$ limit

\[ r_m - r_{m-1} = \frac{2\pi}{n} u_m + \frac{1}{n} \delta x_m u_m, \]

(3.7)

where now on both sides we have kept only the lowest order. Here and throughout the remainder it is understood that all quantities are $n$-periodic in $m$. In (3.7) the first and the second term on the right hand side come from the average and the random part of the angle $\xi_m$, respectively. Both are of order $n^{-1}$, but there is a difference. To see this we imagine to sum (3.7) on an $m$ interval of order $n^{\delta}$ for some $\delta > 0$. Since $u_m$ has long-range correlations we may consider it as effectively constant over this interval. Then, because of the $\delta x_m$, the second term will become a sum of independent zero-average random variables and hence its contribution will be of relative order $n^{-\delta/2}$ with respect to that coming from the first term. We may therefore neglect the $\delta x_m u_m$ term in (3.7) when our purpose is to study the variation of $r_m$ on a scale that increases as a power of $n$. Taking now second order differences we find the recursion

\[ r_{m-1} - 2r_m + r_{m+1} = 2\pi n^{-\frac{3}{2}} f_m, \]

(3.8)

with the right hand member defined by

\[ f_m = -\frac{1}{2}(\delta x_m - 2\delta y_m + \delta x_{m+1}). \]

(3.9)

This quantity satisfies the sum rule $\sum_{m=1}^{n} f_m = 0$.

3.3 Second order differential equation

We may pass to the following continuum description. We define the polar angle $\phi = 2\pi m/n$ and the functions $r(\phi) = r_m$ and $dr/d\phi = u(\phi) = u_m$. In the large-$n$ limit $r$ becomes a continuous function of a continuous variable. For the second order difference on the left hand side of (3.8) we get

\[ r_{m-1} - 2r_m + r_{m+1} \rightarrow (2\pi/n)^2 d^2r/d\phi^2. \]

On the right hand side we put $f_m \rightarrow 2\pi n^{-\frac{3}{2}} f(\phi)$. This converts (3.8) into the second order differential equation

\[ \frac{d^2r(\phi)}{d\phi^2} = f(\phi), \quad 0 \leq \phi \leq 2\pi, \]

(3.10)
valid for angle differences $|\phi - \phi'|$ on the scale $n^{-1+\delta}$, that is, large on the scale of the discrete index $m$. In the same way as in the previous subsection we may imagine to sum (3.10) on an $m$ interval of length $\sim n^\delta$. In the large-$n$ limit $\sum_{m=m_0}^{m_0+n^\delta} f_m$ is a Gaussian variable centered at zero, whose correlation with similar variables on neighboring intervals is small. Hence on scales $\lesssim n^\delta$ the right hand side of (3.10) becomes Gaussian noise. Its autocorrelation function follows from definition (3.9) and the correlations (3.3), namely
\[
\langle f(\phi)f(\phi') \rangle_0 = \frac{2}{\pi} \left[ 2\pi \delta(\phi - \phi') - 1 \right].
\] (3.11)

The noise is white except for the extra $-1$ term inside the brackets. This term leaves all Fourier components of $f$ with nonzero wavenumber unaffected: they behave as under white Gaussian noise. Only the zero-wavenumber component of $f$ is exceptional. By integrating (3.11) on $\phi$ and $\phi'$ one finds that $\int_0^{2\pi} d\phi f(\phi) = 0$, in agreement with the sum rule in the last line of section 3.2.

Now the differential equation (3.10) shows that instead of having this sum rule on $f$ one may equivalently impose on $r$ that its derivative be periodic, $r'(0) = r'(2\pi)$. The functions $r(\phi)$ that we must deal with are therefore, in summary, those that satisfy equation (3.10) with boundary conditions
\[
r(0) = r(2\pi), \quad r'(0) = r'(2\pi)
\] (3.12)

and that, due to the relation in the last line of section 3.1 obey moreover the zero-integral constraint $\int_0^{2\pi} d\phi r(\phi) = 0$. For this class of functions the noise $f$ is effectively white.

In the statistical physics literature equation (3.10) represents what is called the Random Acceleration Process (RAP). Some time ago this process appeared [6] (but without being named explicitly) in a similar way in the study of many-sided Voronoi cells. In section 4.1 of this work we will see that, through the connection that we have discovered here, known facts about the RAP provide further answers to Sylvester’s question.

### 3.4 Large-$n$ expansion of $p_n^*$

We return to the evaluation of the probability $p_n^*$ given by (2.22),
\[
p_n^* = p_n^{(0)} \langle \Theta e^{-V} \rangle_0, \quad p_n^{(0)} = \frac{1}{4\pi^2} \frac{(8\pi^2)^n}{(2n)!},
\] (3.13)
in which $V$ is given by (2.23). All operations that led to these equations were exact for any finite $n$. We will now see what happens if we perform a large-$n$ expansion assuming the scaling of section 3.1. In the large-$n$ limit the condition imposed by $\Theta(\xi,\eta)$ in (2.22) will be satisfied with a probability that goes exponentially fast to unity; hence we have $\Theta \simeq 1$ and
\[
p_n^* \simeq p_n^{(0)} \langle e^{-V} \rangle_0,
\] (3.14)
where the $\simeq$ sign denotes an equality up to corrections that are exponentially small in $n$. We consider now the $n \to \infty$ limit of $V$. In (2.23) we have $G'(\xi, \eta; \beta_s) = 1 + O(n^{-1})$. Expanding the other factors for small angles we get

$$e^{-V} = \exp \left[ -\frac{1}{n} \sum_{m=1}^{n} (r_m^2 - u_m^2) - 2n^{1/2} \max_{1 \leq m \leq n} r_m + \left( \max_{1 \leq m \leq n} r_m \right)^2 + \ldots \right]$$

$$\simeq \exp \left[ -\frac{1}{2\pi} \int_0^{2\pi} d\phi \left( r^2(\phi) - \left( \frac{dr}{d\phi} \right)^2 \right) - 2n^{1/2} \max_{0 \leq \phi \leq 2\pi} r(\phi) \right. \left. + \left( \max_{0 \leq \phi \leq 2\pi} r(\phi) \right)^2 + \ldots \right],$$

(3.15)

where the dots indicate terms of higher order in the angles. As $n \to \infty$ with the scaling discussed in section 3.1, the dot terms in the exponential in (3.15) tend to zero; the first term (which is a sum of $n$ terms compensated by a factor $n^{-1}$) and the third term stay of order $n^0$; and the second term is of order $n^{1/2}$.

### 3.5 Scaling in the large-$n$ limit. II

As shown by (3.14), expression (3.15) has to be submitted to an average, denoted $\langle \ldots \rangle_0$, with respect to all solutions $r(\phi)$ of the random acceleration process satisfying the constraints stated in section 3.3. The law for $r(\phi)$ follows, via equation (3.10), from the law for $f(\phi)$, which in turn follows, via (3.9), from the law (2.21) for the $\xi_m$ and $\eta_\ell$. When $n$ is large, the second term in the exponential in (3.15) will suppress large excursions of $r(\phi)$ from zero in a severe but as yet quantitatively unknown way. Let us therefore suppose that the average $\langle e^{-V} \rangle_0$ draws its principal contribution from those $r(\phi)$ that stay within a tube of a width $w_n \sim n^{-\alpha}$ with an as yet unknown $\alpha$. Hence the effectively contributing $r(\phi)$ will scale as

$$r(\phi) \sim n^{-\alpha}, \quad \max_{0 \leq \phi \leq 2\pi} r(\phi) \sim n^{-\alpha}$$

(3.16)

for some as yet unknown $\alpha > 0$. We must ask ourselves, first of all, if we are allowed to have the scaling already introduced above followed by this new scaling. In fact we are, because it only restricts us more narrowly to the center of the realm where the first scaling holds. The same results are arrived at if the full combined scaling is used from the start. The reason for the procedure we chose is that this is the best way of showing the connection with the Random Acceleration Process, of which finally only a subprocess plays a role, namely the one consisting of trajectories that stay within a tube.

Secondly, if indeed the main contributing $r(\phi)$ scale with a negative power of $n$, then the first and the third term in (3.15) are negligible with respect
to the second one in a large-$n$ expansion. So finally the problem becomes to calculate the average $\langle e^{-V} \rangle_0$ with
\[
\langle e^{-V} \rangle_0 \simeq \left\langle \exp \left[ -2n^{1/2} \max_{0 \leq \phi \leq 2\pi} r(\phi) \right] \right\rangle_0,
\] (3.17)
where in the last line the average is with respect to the process $r(\phi)$ defined by (3.10) and (3.11). In equation (3.17) the only reference to $n$ is the one explicitly visible in the exponential on the right hand side. Hence here $n$ has become an “external” parameter coupling to the maximum value of an $n$ independent random acceleration process. Finally, the remainder $R_n$ in the expansion of the convexity probability $p_n(D)$ in a disk is related to (3.17) by
\[
R_n = \log \langle e^{-V} \rangle_0.
\] (3.18)

The connection, in the large-$n$ limit, between Sylvester’s question and the Random Acceleration Process is the principal achievement of this paper. We will now see that, when combined with what is known about this process, it leads to the results announced in the introduction.

## 4 Properties of the Random Acceleration Process

### 4.1 The work by Györgyi et al.

The random acceleration process is the $k = 2$ member of the wider class of equations
\[
d^k \tilde{r}/d\phi^k = \tilde{f}(\phi)
\] (4.1)
with $k = 1, 2, \ldots$, and where $\tilde{f}(\phi)$ is white Gaussian noise. Because of their relevance in physics and in applied statistics, many examples of such processes have been studied.

In early work on the $k = 1$ case, where $\tilde{r}(\phi)$ is Brownian motion, Foltin et al. \[16\] analyzed the distribution of the square width of $\tilde{r}(\phi)$. Burkhardt \[17\] considered the $k = 2$ problem in a half-space. More recently, Majumdar and Comtet \[18, 19\], were interested in the distribution of the “maximum height” $r_{\text{max}}$, defined as the maximum of $\tilde{r}(\phi)$ relative to its interval average,
\[
r_{\text{max}} \equiv \max_{0 \leq \phi \leq 2\pi} \tilde{r}(\phi) - \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \tilde{r}(\phi).
\] (4.2)

Using powerful path integral methods, Majumdar and Comtet showed that for $k = 1$ and periodic boundary conditions the probability law of the maximum height is the Airy distribution. They pointed out the importance of this problem as an instance of extreme value statistics of a set of strongly
correlated random variables. The equation with general $k$ was studied very recently by Györgyi et al. [10] (GMOR), who also focused on the distribution $P(r_{\text{max}})$ of the maximum height (4.2). The authors of reference [10] calculate this distribution for the class of functions $\tilde{r}(\phi)$ that satisfy (for $k = 2$) the periodicity conditions

$$\tilde{r}(0) = \tilde{r}(2\pi), \quad \tilde{r}'(0) = \tilde{r}'(2\pi).$$

(4.3)

Hence, after subtraction of the interval average, this is the same class that appears in the averages $\langle \ldots \rangle_0$ in sections 3.4 and 3.5. This establishes the applicability of GMOR’s results to the problem of section 3.

By extending Majumdar and Comtet’s [18] methods GMOR found an expression for the distribution $P(r_{\text{max}})$ of $r_{\text{max}}$. Still more recently, Burkhardt et al. considered the distribution of the maximum relative to the initial value [20].

Expression (3.17) is the Laplace transform with Laplace variable $2n^{1/2}$ of the maximum height distribution of the random function $r(\phi)$. No mathematically rigorous results for this quantity seem to exist; however, closely related properties, involving in particular the maximum of the absolute value $|r(\phi)|$, were studied, e.g., by Khoshnevisan and Shi [21] and more recently by Chen and Li [22]. The Laplace transformed maximum height distribution did figure among the many properties of the RAP studied by GMOR [10]. These authors express this quantity by means of a trace formula involving the second order differential operator

$$L = \frac{1}{2} \frac{\partial^2}{\partial u^2} - u \frac{\partial}{\partial r} - r,$$

(4.4)

which is the generator of the Markov process $(r(\phi), u(\phi))$. They consider the linear eigenvalue problem

$$L \psi(r, u) = -\epsilon \psi(r, u), \quad 0 < r < \infty, \quad -\infty < u < \infty,$$

$$\psi(0, u) = 0, \quad u > 0,$$

(4.5)

and assume that it has a well-defined solution with lowest eigenvalue $\epsilon_0$. When converted to our notation [23], the result of GMOR, obtained essentially by a scaling of all variables in the trace formula, is that in the limit of large $n$

$$- \log \langle e^{-V} \rangle_0 \simeq 2 \epsilon_0 (3\pi^4 n)^{1/5}, \quad n \to \infty.$$  

(4.6)

With (4.6) we have obtained our main result, equation (1.11) of section 1.

### 4.2 Heuristic argument for the power 1/5

We will derive the power $1/5$ found by GMOR by a heuristic scaling argument. This is not meant as additional evidence for this result, but as a physical way
of understanding it. We focus on the $n$ dependence of $\langle e^{-V}\rangle_0$. If we agree to consider the expression for $V$ in the exponential in (3.17) as an ‘energy’, then what we will present is basically an energy-versus-entropy argument.

We begin by supposing that due to the factor $n^{1/2}$ included in the ‘energy’ in (3.17) the contributions to $\langle e^{-V}\rangle_0$ will come essentially from trajectories $r(\phi)$ with a maximum less than some small but as yet unknown width $w_n$. These contributing trajectories will typically be confined to a tube of width $2w_n$.

We refer now to figure 4. We imagine the tube divided into sections such that inside each section the trajectory evolves freely, but at the sections’ ends it is ‘reflected’ by the tube walls. Let $\lambda_n$ be the angular interval of a typical section (and hence $\lambda_nR_{av}$ is its length). We may estimate the $n$ dependence of $\lambda_n$ and $w_n$ as follows. If $u(\phi) = dr/d\phi$ is the ‘radial speed’, then (3.10) may be rewritten $du/d\phi = f(\phi)$. In an angular interval $\lambda_n$ this speed will change by $\delta u = \int_{\phi}^{\phi+\lambda_n} du d\phi \sim \lambda_n^{1/2}$, so that the typical radial speed inside the tube will be of order $|\delta u| \sim \lambda_n^{1/2}$. It follows that in this interval the value $r(\phi)$ of the radius itself will change by $\delta r \sim \lambda_n \delta u \sim \lambda_n^{3/2}$. This means that we have typically $|r| \sim \max_{0 \leq \phi \leq 2\pi} r(\phi) \sim w_n \sim \lambda_n^{3/2}$. Hence a typical contributing trajectory $r(\phi)$ will contribute to the average on the right hand side of (3.17) an amount $\exp(-\text{cst} \times n^{1/2} \times \lambda_n^{3/2})$. This is our result for the ‘energy’ factor.

We now estimate which fraction of all trajectories is ‘contributing’ in the above sense, that is, stays within the tube of width $2w_n$. This confinement costs a constant amount of entropy per section, due to the ‘reflection’ at its ends, the trajectory being essentially free everywhere else. Since the number of sections is $2\pi/\lambda_n$, our result for the ‘entropy’ factor is $\exp(-\text{cst} \times 2\pi \lambda_n^{-1})$. 

Figure 4: Heavy solid arc: edge of the disk of radius $R_D = 1$. Solid curve: the convex polygon $R(\phi)$ in the limit $n \to \infty$. Dashed arcs: annular tube of width $W_n \sim n^{-4/5}$. The two dotted transverse line segments mark the ends of a typical tube section of length $\lambda_n$ between two consecutive ‘reflections’ of the polygon at the inner and the outer tube walls; the section length scales as $\lambda_n \sim n^{-1/5}$. The distance of closest approach of the edge scales as $\delta_n \sim n^{-1}$. 

$$W_n \sim n^{-4/5}, \quad \lambda_n \sim n^{-1/5}, \quad \delta_n \sim n^{-1}$$
Upon multiplying the energy and entropy estimates we arrive at the heuristic estimate for $\langle e^{-V} \rangle_0$. Taking logarithms yields
\[- \log \langle e^{-V} \rangle_0 \simeq C_1 \lambda_n^{-1} + C_2 n^{1/2} \lambda_n^{3/2}, \quad n \to \infty, \quad (4.7)\]
where $C_1, C_2 > 0$. In (4.7) the $n$ dependence of the section length $\lambda_n$ is still unspecified. By setting $\lambda_n \sim n^{-\alpha}$ and minimizing the right hand side with respect to $\alpha$ we find $\alpha = \frac{1}{5}$; whence
\[\lambda_n \sim n^{-1/5}, \quad w_n \sim n^{-3/10}. \quad (4.8)\]
We may return now to the original unscaled variables. By transposing (1.7) to continuum notation we see that the radius of the polygon, $R(\phi) = R_{av}[1 + n^{-1/2}r(\phi)]$, runs through a tube of width $W_n = R_{av}n^{-1/2}w_n \sim n^{-4/5}$, as depicted in figure 4.

### 4.3 Exact bounds

The mathematical methods used in section 2 are fully exact. Those of section 3 are basically a systematic expansion in negative powers of $n$; although we did not rigorously prove its validity, it has been obtained by methods that commonly lead to exact results. The argument of section 4.1 due to GMOR, involves an assumption of a different kind, namely the existence of a lowest-energy solution of (4.4) and (4.5) with an eigenvalue $\epsilon_0$. We have no reasons to doubt that this solution exists. However, we wish to point out here that without this existence assumption one may derive by elementary methods a weaker but still very meaningful statement. We have relegated the proof to Appendix A. We there show by straightforward methods starting from (3.17) that we have the bounds
\[c_6 n^{1/6} \leq - \log \langle e^{-V} \rangle_0 \leq c_4 n^{1/4}, \quad n \to \infty, \quad (4.9)\]
with positive constants $c_4$ and $c_6$.

Hence, with or without the existence assumption of $\epsilon_0$, it is established in either case that the large-$n$ expansion of $\log \langle e^{-V} \rangle_0$ depends nonanalytically on $n$. Consequently, because of (3.18) and (1.6), the expansion of $\log p_n(D)$ for a disk contains this same nonanalytic term and is clearly distinct from the expansions (1.2) for squares and triangles.

### 5 Closest approach of the edge

Let us call $\delta_n$ the distance of closest approach between the perimeter of the polygon and the edge of the unit disk, that is,
\[\delta_n = 1 - \max_{1 \leq m \leq n} R_m. \quad (5.1)\]
Given the results of the preceding sections, it is now fairly easy to study this quantity.

We now first turn to the study of the distribution of $R_{av}$, which we will call $P(R_{av})$. We have

$$P(R_{av}) = \frac{p(R_{av})}{p^*_n}. \tag{5.2}$$

where the denominator $p^*_n$ is given by integral (2.1) and the numerator $p(R_{av})$ by this same integral except that we put primes on the variables of integration and insert a factor $\delta(R_{av}' - R_{av})$. Evaluation of the numerator proceeds as it did for the denominator in section 2.1. Instead of (2.8) we have to calculate

$$\int_{\min}^{\max} \rho_{m}^{-1} dR_{av}' R_{av}^{2n-1} \delta(R_{av}' - R_{av}) = R_{av}^{2n-1} \theta([\max_{1\leq m\leq n} \rho_{m}]^{-1} - R_{av}). \tag{5.3}$$

It then follows that

$$P(R_{av}) = \langle \Theta e^{-v_{R_{av}}} \rangle_{0} \theta(\langle \Theta e^{-v} \rangle_{0}^{-1} - \rho_{m}^{-1} \delta_{m} - \max_{1\leq m\leq n} \rho_{m}^{-1} - R_{av}). \tag{5.4}$$

in which $\exp(-v_{R_{av}})$ is the same expression as (2.23) except that the factor $[\max_{1\leq m\leq n} \rho_{m}]^{-2n}$ has been replaced with $2nR_{av}^{2n-1}([\max_{1\leq m\leq n} \rho_{m}]^{-1} - R_{av})$. Expression (5.4) for $P(R_{av})$ is still fully exact.

We cannot go beyond this point unless we make again a large-$n$ expansion. We define a variable $d_{av}$ by

$$R_{av} = 1 - n^{-1/2} d_{av}, \tag{5.5}$$

where we tacitly suppose that $d_{av}$ is at most of order $n^{0}$ but possibly smaller. Using in (5.1) that $R_{m} = R_{av}\rho_{m}$, employing (5.5), and expanding, we obtain between $\delta_{n}$ and $d_{av}$ the relationship

$$\delta_{n} = n^{-1/2} [d_{av} - \max_{1\leq m\leq n} r_{m}] + \ldots, \tag{5.6}$$

where the dots indicate terms of higher order.

Using expression (3.15) for $\rho_{m}$ and expanding as in section 3.4 we get an expression for $P(R_{av})$ in terms of the variable $d_{av}$,

$$P(R_{av}) = 2n \langle e^{-v} \rangle_{0}^{-1} \exp \left[ -\frac{1}{n} \sum_{m=1}^{n} (r_{m}^{2} - u_{m}^{2}) - 2n^{1/2} d_{av} - d_{av}^{2} + \ldots \right] \tag{5.7}$$

For the same reasons as in section 3.4 we neglect now the sum and the $d_{av}^{2}$ term in the exponential in (5.7). Finally, using (5.6), we eliminate the variable $d_{av}$ from (5.4) in favor of $\delta_{n}$ whose distribution we will call $p(\delta_{n})$. We get in the limit $n \to \infty$ the final result of this section, namely the probability distribution $p$ of the distance of closest approach of the edge,

$$p(\delta_{n}) = 2n \ e^{-2n \delta_{n}}, \quad \delta_{n} > 0. \tag{5.8}$$

Not only does this show that $\delta_{n}$ scales as $1/n$, but also that it is exponentially distributed.
6 Conclusion

In this work we considered Sylvester’s question: given \( n \) points chosen randomly from a uniform distribution on a disk, what is the probability that they are the vertices of a convex \( n \)-sided polygon? For \( n \) large this probability becomes very small and one may consider the asymptotic expansion of \( \log p_n \). The first two terms of the expansion were known; they are proportional to \( n \log n \) and to \( n \). In this work we establish, first of all, a relation between Sylvester’s question and the Random Acceleration Process. We then show that the third term in the expansion (in absolute value) is asymptotically bounded from above and below by \( \sim n^{1/6} \) and \( \sim n^{1/4} \), respectively, so that it must be nonanalytic in \( n \). If one accepts the hypothesis underlying the work of Györgyi et al., which we easily do, it follows that this term must be proportional to \( \sim n^{1/5} \) with a well-defined coefficient given in (1.11). Along with this expansion of \( p_n \) we harvest a variety of results concerning the most probable way that the \( n \) points are distributed along the edge of the disk.

The subject has not been fully exhausted. Remaining questions in the present work concern, for example, correlations between the angles. Besides, a natural generalization would be to consider more general convex domains \( K \) and in particular convex polygons. In this context, the work [24] provides precise results on convex chains for deriving exact distributional results on the number of vertices of the convex envelope. Extensions of this work could deal with the crossover phenomenon that must exist when the convex domain \( K \) varies between finite-sided polygons and the disk: for example, what happens when one considers \( n \) points randomly chosen in the interior of a regular \( M_n \)-gon when \( n \) and \( M_n \) tend to infinity in a specified way? Another extension would be the study of the convex envelope of \( N_n \) randomly chosen points knowing that this envelope is \( n \)-sided, again in the limit of \( n \) and \( N_n \) tending to infinity. However, we leave these and other matters for future investigation.

A Bounds for \( \langle e^{-V} \rangle_0 \) as \( n \to \infty \)

In order to derive upper and lower bounds on the average \( \langle e^{-V} \rangle_0 \) we begin by expressing this quantity in terms of the Fourier transforms of the variables involved.

A.1 Fourier transforms

We define the Fourier transform

\[
\hat{f}_q = \frac{1}{2\pi n^2} \sum_{m=1}^{n} e^{2\pi i q m / n} f_m, \quad \hat{r}_q = n^{-1} \sum_{m=1}^{n} e^{2\pi i q m / n} r_m, \tag{A.1}
\]
where, if for convenience we take $n$ odd, $q = 0, \pm 1, \pm 2, \ldots, \pm (\frac{1}{2}n - \frac{1}{2})$. The sum rules imply that $\hat{f}_0 = \hat{r}_0 = 0$. In Fourier language recursion (3.8) becomes

$$\hat{r}_q = -\frac{n^2}{n^2 \sin^2 \frac{\pi q}{n}} \hat{\phi}_q \simeq -\frac{1}{q^2} \hat{\phi}_q, \quad q \neq 0. \quad (A.2)$$

where the $\simeq$ sign indicates the limit $n \to \infty$ at fixed $q$. This amounts to neglecting higher orders in $q/n$, which are small if in agreement with our preceding discussion we restrict ourselves to $q$ on a scale $\sim n^{1-\delta}$, that is, to spatial distances which in units of the index $m$ scale as $\sim n^\delta$. From definition (A.1) together with (3.9) we have furthermore

$$\langle \hat{f}_q \hat{f}_{q'} \rangle_0 = \frac{3}{2} \delta_{q+q',0}, \quad q, q' \neq 0. \quad (A.3)$$

The $\hat{f}_q$ are distributed according to a marginal distribution $P^{(0)}(f)$ of $P^{(0)}(\xi, \eta)$. In the limit $n \to \infty$ all $\hat{f}_q$ become Gaussian distributed and therefore $P^{(0)}$ is given by

$$P^{(0)}(f) = \prod_{q \neq 0} \frac{1}{\sqrt{3\pi}} \exp \left(-\frac{1}{3} \hat{f}_q \hat{f}_{-q} \right). \quad (A.4)$$

This distribution may be used for averaging quantities that depend essentially on the long wavelength properties of the process.

It will be easier to deal with real, as opposed to complex, quantities. Choosing a convenient normalization we define the radial and angular components $F_q$ and $\omega_q$ of $\hat{f}_\pm q$ by

$$\hat{f}_\pm q = \frac{1}{2} \sqrt{6} F_q (\cos \omega_q \pm i \sin \omega_q), \quad q > 0. \quad (A.5)$$

Letting as before $\phi = 2\pi m/n$, we have that in terms of these Fourier transforms the process $r(\phi)$ becomes

$$r(\phi) = \sum_q e^{-iq\phi} \hat{r}_q = -\sqrt{6} \sum_{q=1}^{\infty} \frac{F_q}{q^2} \cos (q\phi - \omega_q). \quad (A.6)$$

Substituting (A.6) in (3.17) and transforming to the $F_q$ and $\omega_q$ as new variables of integration we get

$$\langle e^{-V} \rangle_0 \simeq \int_0^\infty \prod_{q=1}^{\infty} 2F_q dF_q \int_{0}^{2\pi} \prod_{q=1}^{\infty} \frac{d\omega_q}{2\pi} \exp \left[- \sum_{q=1}^{\infty} F_q^2 \right]$$

$$\times \exp \left[-2\sqrt{6n} \max_{0 \leq \phi \leq 2\pi} \sum_{q=1}^{\infty} \frac{F_q}{q^2} \cos (q\phi - \omega_q) \right]. \quad (A.7)$$

In the right hand side of (A.7) the only $n$ dependence that comes in is the one explicitly exhibited in the argument of the exponential. We note that the
Fourier expression (A.6) shows that \( r(\phi) \), due to the factor \( 1/q^2 \) in the sum on \( q \), is precisely a quantity of the type that receives its main contribution from small \( q \) values and whose properties may therefore be calculated by averaging with respect to (A.4). The feature that makes this problem hard to solve exactly is the maximum that is required in (A.7). In what follows we will obtain upper and lower bounds for \( \langle e^{-V} \rangle_0 \).

### A.2 Two inequalities

We recall that the argument of the second exponential in (A.7) is equal to \(-2\sqrt{n} \max_{0 \leq \phi \leq 2\pi} r(\phi)\). Upper and lower bounds for \( \langle e^{-V} \rangle_0 \) are based on bounds for this maximum expressed in terms of the Fourier amplitudes of \( r(\phi) \). These bounds, valid asymptotically for \( n \) large, read

\[
\frac{\sigma^2}{M} \leq \max_{0 \leq \phi \leq 2\pi} r(\phi) \leq M, \tag{A.8}
\]

where

\[
M = \sqrt{6} \sum_{q=1}^{\infty} \frac{F_q}{q^2}, \quad \sigma^2 = 3 \sum_{q=1}^{\infty} \frac{F_q^2}{q^4}, \tag{A.9}
\]

the \( F_q \) are defined by (A.6), and in which it is assumed that both sums on \( q \) converge. The upper bound given by (A.8)-(A.9) is obvious and leads to a lower bound for \( \langle e^{-V} \rangle_0 \) derived in Appendix A.3. The lower bound given by (A.8)-(A.9) is proved in Appendix A.4.1; it leads to an upper bound for \( \langle e^{-V} \rangle_0 \) which is derived in Appendix A.4.2.

### A.3 Lower bound for \( \langle e^{-V} \rangle_0 \)

In this section of the appendix we abbreviate

\[
N = 2\sqrt{6n}. \tag{A.10}
\]

We will find a lower bound for \( \langle e^{-V} \rangle_0 \) valid in the limit of large \( N \). We replace the maximum in (A.7) by its upper bound given in (A.8). Since this bound is independent of the \( \omega_q \), the integrals on these variables reduce to a factor unity. The integrals on the \( F_q \) factorize and we get

\[
\langle e^{-V} \rangle_0 \geq \prod_{q=1}^{\infty} \int_0^{\infty} 2x dx \exp \left( -x^2 - Nq^{-2}x \right) = \exp \left[ -\sum_{q=1}^{\infty} F(qN^{-\frac{1}{2}}) \right] \tag{A.11}
\]

where

\[
F(\kappa) = -\log \int_0^{\infty} 2x dx \exp(-x^2 + \kappa^{-2}x). \tag{A.12}
\]
This function is positive for all $\kappa > 0$ and it is integrable since it behaves as $F(\kappa) \approx 4 \log \kappa^{-1}$ for $\kappa \to 0$ and as $F(\kappa) \approx \frac{1}{2} \sqrt{\pi} \kappa^{-2}$ for $\kappa \to \infty$. The sum on $q$ in (A.11) makes the argument of $F$ vary by steps of spacing $N^{-\frac{1}{2}}$ and we may therefore in the limit $N \to \infty$ replace it by an integral, which gives

$$\langle e^{-V} \rangle_0 \geq \exp \left[ - N^{\frac{1}{2}} \int_0^{\infty} d\kappa \, F(\kappa) \right] = \exp \left[ - c_4 n^{\frac{1}{4}} \right], \quad (A.13)$$

where in the last step we used the relation between $N$ and $n$ given in (A.10), and where $c_4$ is a positive constant.

### A.4 Upper bound for $\langle e^{-V} \rangle_0$

In order to prove the upper bound for $\langle e^{-V} \rangle_0$, we first need to prove the lower bound on $r(\phi)$ stated in (A.8).

#### A.4.1 Lower bound for $\max_{0 \leq \phi \leq 2\pi} r(\phi)$

We prove the following property.

**Property.** Let $f(x)$ be a function on $[0, 2\pi]$ such that

$$\int_0^{2\pi} dx f(x) = 0, \quad (A.14)$$

$$\int_0^{2\pi} dx f^2(x) = 2\pi \sigma^2, \quad (A.15)$$

$$\min_{0 \leq x \leq 2\pi} f(x) = -M, \quad (A.16)$$

where $\sigma$ and $M$ are given positive constants. Then

$$\max_{0 \leq x \leq 2\pi} f(x) \geq \frac{1}{2} M \left[ \left( 1 + \frac{4\sigma^2}{M^2} \right)^{1/2} - 1 \right]. \quad (A.17)$$

In the limit $M \gg \sigma$ this yields

$$\max_{0 \leq x \leq 2\pi} f(x) \geq \frac{\sigma^2}{M} \left[ 1 + O\left( \frac{\sigma^2}{M^2} \right) \right]. \quad (A.18)$$

**Proof.** We set

$$f(x) = f_+(x) - f_-(x) \quad (A.19)$$
in which \( f_\pm(x) = \max\{0, \pm f(x)\} \). In terms of these Eqs. (A.14)-(A.16) become
\[
\int_0^{2\pi} dx f_+(x) - \int_0^{2\pi} dx f_-(x) = 0, \quad (A.20)
\]
\[
\int_0^{2\pi} dx f_+^2(x) + \int_0^{2\pi} dx f_-^2(x) = 2\pi \sigma^2, \quad (A.21)
\]
\[
\max_{0 \leq x \leq 2\pi} f_-(x) = M. \quad (A.22)
\]

We abbreviate
\[
m = \max_{0 \leq x \leq 2\pi} f_+(x). \quad (A.23)
\]

We now have the estimate
\[
m \geq \frac{1}{2\pi} \int_0^{2\pi} dx f_+(x)
= \frac{1}{2\pi} \int_0^{2\pi} dx f_-(x), \quad (A.24)
\]
where in the second step we used (A.20). Next,
\[
\int_0^{2\pi} dx f_-(x) \geq \frac{1}{M} \int_0^{2\pi} dx f_+^2(x)
= \frac{1}{M} [2\pi \sigma^2 - \int_0^{2\pi} dx f_+^2(x)]
\geq \frac{2\pi}{M} [\sigma^2 - m^2], \quad (A.25)
\]
where in the first step we used (A.22), in the second step (A.21), and in the third step (A.23).

We now combine (A.24) and (A.25) to get
\[
m \geq \frac{\sigma^2 - m^2}{M}. \quad (A.26)
\]

By solving the associated second-order equation for \( m \) and using the definition (A.23) we are led directly to (A.17). This completes the proof of property (A.17).

Application. We apply inequality (A.17) to the function \( r(\phi) \) given in (A.6). Its \( M \) satisfies \( M < M_0 \) with \( M_0 \) given in equation (A.9) and its \( \sigma^2 \) is given in that same equation. Since (as is implied by the arguments of section A.4.2) for \( n \to \infty \) the ratio \( \sigma/M_0 \) tends to zero, we may replace (A.17) by (A.18) and obtain the desired lower bound stated in (A.8).
A.4.2 Proof of upper bound for \( \langle e^{-V} \rangle_0 \)

We will next find an upper bound for \( \langle e^{-V} \rangle_0 \) valid in the limit of large \( n \).
We replace the maximum in (A.7) by its lower bound given in (A.8). The integrals on the variables \( \omega_q \) reduce again to a factor unity. The integrals on the \( x_q \) do not factorize in this case. In order to make them do so we introduce integral representations and write

\[
\langle e^{-V} \rangle_0 < \int_0^\infty \prod_{q=1}^{\infty} 2F_q dF_q \exp \left[ -\sum_{q=1}^{\infty} F_q^2 - \frac{1}{2} N \sum_{q=1}^{\infty} \frac{F_q^2}{q^4} / \sum_{q=1}^{\infty} F_q \right]
\]

\[
= \int_0^\infty d\mu \int_{-\infty}^\infty \frac{ds}{2\pi} \int_0^\infty \prod_{q=1}^{\infty} 2F_q dF_q \exp \left[ is \left( \sum_{q=1}^{\infty} F_q - \mu \right) - \sum_{q=1}^{\infty} F_q^2 - \frac{N}{2\mu} \sum_{q=1}^{\infty} F_q^2 \right]
\]

\[
= \int_0^\infty d\mu \int_{-\infty}^\infty \frac{ds}{2\pi} e^{is\mu} \exp \left[ \sum_{q=1}^{\infty} \log \int_0^\infty 2xdx \left( -x^2 - \frac{N}{2\mu q^4} x^2 + \frac{is}{q^2} x \right) \right]
\]

\[
= \int_0^\infty d\mu \exp \left[ -G_1 \left( \frac{N}{2\mu} \right) \right] \int_{-\infty}^\infty \frac{ds}{2\pi} e^{is\mu} \exp \left[ -G_2 \left( s, \frac{N}{2\mu} \right) \right]
\] (A.27)

in which

\[
G_1(z) = 2 \sum_{q=1}^{\infty} \log \alpha_q(z), \quad \alpha_q^2(z) = 1 + \frac{z}{q^4},
\]

\[
G_2(s, z) = -\sum_{q=1}^{\infty} \log \int_0^\infty 2ydy \exp \left( -y^2 + \frac{is}{q^2 \alpha_q(z)} y \right).
\] (A.28)

We will need the large \( z \) behavior of \( G_1(z) \). Upon scaling \( q = \kappa z^{1/4} \) in (A.28) we find

\[
G_1(z) \simeq a_0 z^{1/4}, \quad a_0 = \int_0^\infty d\kappa \log(1 + \kappa^{-4}), \quad z \to \infty.
\] (A.29)

In order to study \( G_2(s, z) \) we expand the integrand in its definition (A.28) in a power series in \( is \), do the \( y \) integration term by term, and expand the logarithm in powers of \( is \). This gives

\[
G_2(s, z) = -\sum_{q=1}^{\infty} \log \left[ \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + 1)}{k!} \left( \frac{is}{q^2 \alpha_q} \right)^k \right]
\]

\[
= -\pi^{1/2} A_1(z) is - \frac{1}{2} (1 + \pi) A_2(z) (is)^2 + \ldots
\] (A.30)

where

\[
A_k(z) = \sum_{q=1}^{\infty} \frac{1}{(q^2 \alpha_q(z))^k}, \quad k = 1, 2, \ldots.
\] (A.31)
After again scaling $q = \kappa z^{1/4}$ we get by the same method as above

$$A_k(z) = a_k z^{1/4 - 1/2k} [1 + \ldots], \quad a_k = \int_0^\infty \mathrm{d}\kappa (1 + \kappa^4)^{-1/2k}, \quad z \to \infty, \quad (A.32)$$

where the dots stand for a series in powers of $z^{-1/2}$. Let us denote by $J_n(\mu, N^{2\mu})$ the integral on $s$ in the last line of (A.27). Substitution of the above expansion of $G_2$ yields

$$J_n(\mu, N^{2\mu}) = \int_0^\infty \frac{ds}{2\pi} \exp \left[ -is \left\{ \mu - \pi^{1/2} a_1 \left( \frac{N}{2\mu} \right)^{1/8} + \ldots \right\} \right]$$

$$-\frac{1}{2} s^2 (1 + \pi) \left\{ a_2 \left( \frac{N}{2\mu} \right)^{-1/2} + \ldots \right\} + O\left( \left( \frac{N}{2\mu} \right)^{-4/3} s^3 \right),$$

(A.33)

in which the dots come from expansion (A.32). Let us scale $s = (N^{2\mu})^{3/8} t$. We then get

$$J_n(\mu, N^{2\mu}) = \left( \frac{N}{2\mu} \right)^{3/8} \int_0^\infty \frac{dt}{2\pi}$$

$$\times \exp \left[ -i \left\{ \mu \left( \frac{N}{2\mu} \right)^{3/8} - \pi^{1/2} a_1 \left( \frac{N}{2\mu} \right)^{1/8} \right\} t - \frac{1}{2} (1 + \pi) a_2 t^2 \right] + \ldots$$

$$= \left[ \frac{2\pi}{(1 + \pi) a_2} \right]^{3/8} \left( \frac{N}{2\mu} \right)^{3/8} \exp \left[ -\left\{ \mu \left( \frac{N}{2\mu} \right)^{3/8} - \pi^{1/2} a_1 \left( \frac{N}{2\mu} \right)^{1/8} \right\}^2 \right] + \ldots$$

(A.34)

where the dots stand for terms proportional to negative powers of $N/(2\mu)$. It is now possible to see how we should scale $\mu$ with $N$. We will introduce the scaled variable $\nu$ defined by $2\mu = \nu^4 N^{-1/4}$, where the fourth power on $\nu$ is just a matter of convenience. Substituting (A.34) in (A.27), using the large $z$ expansion (A.29), and neglecting terms that vanish as $N \to \infty$ we then get

$$\langle e^{-V} \rangle_0 < \text{cst} \times N^{3/8} \int_0^\infty \mathrm{d}\nu \exp^{-N^{3/8} g(\nu)}$$

(A.35)

with

$$g(\nu) = c_1 \nu^{-1} + \frac{\left( \frac{3}{2} \nu^{3/2} - \pi^{1/2} \nu^{-1/2} \right)^2}{2(1 + \pi) a_2}.$$ 

(A.36)

The function $g(\nu)$ increases as $\sim \nu^{-1}$ for $\nu \to 0$ and as $\sim \nu$ for $\nu \to \infty$, and hence has a minimum value for some $\nu = \nu_{\text{min}}$. Upon doing the integral
by steepest descent we get

\[
\langle e^{-V} \rangle_0 < \text{cst} \times N^{\frac{3}{2}} \exp \left[ -N^{\frac{1}{3}} g(\nu_{\text{min}}) \right]
\]

\[
= \text{cst} \times n^{\frac{1}{2}} \exp \left[ -c_6 n^{\frac{1}{2}} \right],
\]

(A.37)

where \( c_6 \) is a positive constant.

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