REGULAR AND BIREGULAR MODULE ALGEBRAS

CHRISTIAN LOMP

Abstract. Motivated by the study of von Neumann regular skew groups as carried out by Alfaro, Ara and del Rio in [1] we investigate regular and biregular Hopf module algebras. If $A$ is an algebra with an action by an affine Hopf algebra $H$, then any $H$-stable left ideal of $A$ is a direct summand if and only if $A^H$ is regular and the invariance functor $(-)^H$ induces an equivalence of $A^H$-Mod to the Wisbauer category of $A$ as an $A^H$-module. Analogously we show a similar statement for the biregularity of $A$ relative to $H$ where $A^H$ is replaced by $R = Z(A) \cap A^H$ using the module theory of $A$ as an $Ae \triangleleft H$-module, where $e$ is an enveloping Hopf algebroid of $A$ and $H$. We show that every two-sided $H$-stable ideal of $A$ is generated by a central $H$-invariant idempotent if and only if $R$ is regular and $A_m$ is $H$-simple for all maximal ideals $m$ of $R$. Further sufficient conditions are given for $A^H$ and $A^H$ to be regular.

1. Introduction

Motivated by the study of von Neumann regular of skew group rings by Alfaro et al. in [1] and by the studies of the regularity of fix rings by Goursad et all in [9] we look at the regularity of Hopf module algebras, their smash products and their subrings of invariants. To achieve our goal we will work in the following more general setting:

Let $k$ be a commutative ring. An extension $A \subseteq B$ of $k$-algebras is said to have an additional module structure if there exists a ring homomorphism $\Psi : B \rightarrow \text{End}_k(A)$ such that $\Psi(a) = L_a$ for all $a \in A$, where $L_a$ denotes the left multiplication of $a$ on $A$. Then $A$ is a cyclic left $B$-module with $B$-action $b \cdot a := \Psi(b)(a)$ for all $b \in B, a \in A$. Moreover $\alpha : B \rightarrow A$ with $(b)\alpha = b \cdot 1$ is an epimorphism of left $B$-modules. Note that we will write homomorphisms opposite of scalars. Furthermore $\phi : \text{End}_B(A) \rightarrow A$ with $\phi(f) = (1)f$ defines a ring homomorphism whose image is denote by $A^B$. In particular

$A^B = \{a \in A \mid \forall b \in B : b \cdot a = (b)a\alpha\} = \{a \in A \mid \forall b \in B \forall a' \in A : b \cdot (a'a) = (b \cdot a')a\}$.

Defining for any $B$-module $M$:

$M^B = \{m \in M \mid \forall b \in B \forall a \in A : b \cdot (am) = (b \cdot a)m\}$

one also has functorial isomorphisms

$\text{Hom}_B(A, M) \rightarrow M^B \ f \mapsto (1)f$

such that $\text{Hom}_B(A, -)$ and $(-)^B$ are isomorphic functors (see [14] for details). In the terminology of [3], $B$ is an $A$-ring with a right grouplike character.

Examples of the described situation are abundant in the theory of Hopf algebra actions where a Hopf algebra $H$ (or more general a weak Hopf algebra) acts on an algebra $A$ and $A \subseteq B = \cdots$

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$A\#H$ is an extension with additional module structure. This also includes group action and Lie actions. Further examples are given by the envelopping algebra $A \subseteq A^e$ or more generally by the envelopping Hopf algebroid $A^* \bowtie H$ as defined in [14] (see also [13] or [7]), $k$-algebras $A$ with an involution $*$ with $B = A^e * G$ where $G = \langle \sigma \rangle$ is group generated by the automorphism $\sigma$ of $A^e$ defined by $\sigma(a \otimes b) = b^* \otimes a^*$ or certain extensions $A \subseteq B$ arising in the study of Banach algebras (see Cabrera et al. [2]).

In this paper we will characterize regular and biregular $H$-module algebras, generalising some known results on the regularity skew group rings.

All rings will be associative and unital. Ring homomorphisms are supposed to respect the unit. Throughout the text $k$ will denote a commutative ring and $A$ a $k$-algebra. We denote by $A^e := A \otimes A^{op}$ the enveloping algebra of $A$ whose multiplication is defined as $(a \otimes b)(a' \otimes b') = aa' \otimes b'b$.

2. Regular Modules

John von Neumann defined a ring $R$ to be regular if for any element $a \in R$ there exists an element $b \in R$ such that $a = aba$. He showed in [17] that $R$ is regular if and only if every cyclic (finitely generated) left (right) ideal of $R$ is a direct summand. Later Auslander proved that the regularity of a ring can also be characterised by the property that any module is flat or equivalently that any submodule of a module is pure. Several author’s have transfered the regularity condition to modules. A.T uganbaev in [20] calls a left $R$-module $M$ regular if any cyclic (finitely generated) submodule is a direct summand using the lattice theoretical approach, while J.Zelmanowitz in [24] followed the original elementwise definition of von Neumann and called a left $R$-module $M$ regular if for any $m \in M$ there exists $f \in \text{Hom}_R(M, R)$ such that $(m)fm = m$.

The module theoretic version of Auslander’s characterisation had been carried out by Fieldhouse [8] where he called a left $R$-module regular if any of its submodule is pure in the sense of P.M.Cohen. R.Wisbauer [21] used his ideas to define regularity for nonassociative rings (see also [22, Chapter 34]): Let $R$ be an arbitrary ring and $M$ a left $R$-module. The Wisbauer category $\sigma[M]$ is the subcategory of $R$-Mod whose objects are the submodules of $M$-generated modules, i.e. submodules of factor modules of direct sums of copies of $M$. A module $P \in \sigma[M]$ is called finitely presented in $\sigma[M]$ if $P$ is finitely generated and every exact sequence in $\sigma[M]$:

$$0 \longrightarrow K \longrightarrow L \longrightarrow P \longrightarrow 0$$

with $L$ finitely generated implies $K$ to be finitely generated. Note that $P$ might be finitely presented in $\sigma[M]$ but not in $R$-Mod, for example take any simple module $P = M$. A short exact sequence in $\sigma[M]$ is called pure if any finitely presented module in $\sigma[M]$ is projective with respect to this sequence and a module $N \in \sigma[M]$ is called flat in $\sigma[M]$ if any short exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0$$

in $\sigma[M]$ is pure. Finally $M$ is called regular if any module in $\sigma[M]$ is flat or equivalently if any short exact sequence in $\sigma[M]$ is pure.

2.1. Relative regularity. Let $A \subseteq B$ be an extension with additional module structure. Our first aim will be to characterise $A$ as a regular $B$-module.

Proposition. Let $A \subseteq B$ be an extension with additional module structure. The following statements are equivalent:

(a) $A$ is regular and finitely presented in $\sigma[B;A]$;
(b) Every left \( B \)-stable ideal that is finitely generated as left \( B \)-module is a direct summand and \( A \) is finitely presented in \( \sigma[BA] \).
(c) \( A^B \) is von Neumann regular and \( A \) is a generator in \( \sigma[BA] \).
(d) \( A^B \) is von Neumann regular and \((-)^B = \text{Hom}_B(A,-) : \sigma[BA] \to A^B\text{-Mod} \) is an equivalence of categories.

In this case \( A \) is a projective generator in \( \sigma[BA] \).

Proof. (a) \( \Leftrightarrow \) (b) follows from [22, 37.4]

(a) \( \Rightarrow \) (c) Since \( A \) is finitely presented and regular in \( \sigma[BA] \), it is projective in \( \sigma[BA] \). By As \( A \) is a cyclic \( B \)-module, by [22, 37.8], \( A \) is a (finitely generated) projective generator in \( \sigma[BA] \) and \( A^B \) is regular.

(c) \( \Rightarrow \) (d) is clear.

(c) \( \Rightarrow \) (b) Since \( BA \) is cyclic and a generator in \( \sigma[BA] \) and since \( A^B \cong \text{End}_B(A) \) is regular and thus \( A \) is a faithfully flat \( A^B \)-module, we have by [22, 18.5(2)] that \( A \) is self-projective (and hence projective in \( \sigma[BA] \)). This implies that \( A \) is also finitely presented in \( \sigma[BA] \). If \( U \) is a finitely generated \( B \)-stable left ideal of \( A \), then \( U = AI \) for \( I = U^B \). \( U \) being finitely generated as \( B \)-module, implies \( I \) being finitely generated as right ideal of \( A^B \). Thus \( I = A^B e \) for some idempotent \( e \) and \( U = AI = Ae \) is a direct summand of \( A \), i.e. \( A \) is regular by [22, 37.4].

\( \square \)

2.2. Relative biregularity. If \( A \subseteq B \) is an extension with additional modules structure \( \Psi : B \to \text{End}_k(A) \), we might identify \( B \) with its image in \( \text{End}_k(A) \) seeing it as an extension of the subalgebra generated by left multiplications of \( A \). In order to study the two-sided \( B \)-stable ideals we might enlarge \( B \) by considering \( B' = \langle B \cup M(A) \rangle \subseteq \text{End}_k(A) \). Note that all \( B \)-submodules of \( A \) are two-sided and \( A^B \subseteq Z(A) \) if \( M(A) \subseteq B \subseteq \text{End}_k(A) \). A \( B \)-stable ideal \( I \) is called prime if \( JK \subseteq I \) implies \( J \subseteq I \) or \( K \subseteq I \) for any \( B \)-stable ideals \( J \) and \( K \). \( I \) is semiprime if it is the intersection of prime \( B \)-stable ideals. \( A \) is \( B \)-semiprime if \( 0 \) is a prime as \( B \)-stable ideal or equivalently \( A \) does not contain any non-zero nilpotent \( B \)-stable ideal (see [13, 2.3]). If a cyclic \( B \)-stable ideal \( B \cdot a \) is a direct summand of \( A \), then there exists an idempotent \( e \in A^B \) with \( B \cdot a = B \cdot e = Ae \) and \( A = AE \oplus A(1 - e) \). A \( k \)-algebra \( A \) is called \( B \)-biregular if every cyclic \( B \)-stable ideal is a direct summand of \( A \). In particular Proposition [21] applies to get a characterisation of \( B \)-biregular algebras \( A \) in case \( A \) is finitely presented in \( \sigma[BA] \), namely that \( A \) is \( B \)-biregular if and only if \( A^B \) is a von Neumann regular ring with \((-)^B : \sigma[BA] \to A^B\text{-Mod} \) being an equivalence.

2.3. Properties of relative biregular algebras. In the next two subsections, we intend to characterise \( B \)-biregular algebras \( A \) without assuming that \( A \) is finitely presented in \( \sigma[BA] \).

Proposition. Let \( M(A) \subseteq B \subseteq \text{End}_k(A) \). Suppose that \( A \) is \( B \)-biregular. Then

1. \( A^B \) is von Neumann regular and \( A \) is \( B \)-semiprime.
2. \( A \) is a \( A^B \)-Ideal Algebra, i.e. the map \( I \mapsto IA \) is a bijection between the ideals \( I \) of \( A^B \) and the \( B \)-stable ideals of \( A \), whose inverse is given by \( N \mapsto \text{Ann}_{A^B}(A/N) \cong \text{Hom}_B(A/N,A) \).
3. Every finitely generated \( B \)-stable ideal of \( A \) is cyclic and is generated by some central idempotent in \( A^B \).
4. For any \( B \)-stable ideal \( I \) of \( A \), also \( A/I \) is \( B/I \)-biregular.
5. Every \( B \)-stable ideal of \( A \) is idempotent and equals the intersection of maximal \( B \)-stable ideals.
Every prime $B$-stable ideal is maximal.

Proof. (1) Let $f \in \text{End}_B(A)$, then $(A)f = B(1)f$ is a direct summand in $A$ by hypothesis, i.e. $(A)f = Ae$ with $e^2 = e \in A^B \subseteq Z(A)$. Since $A(1-e) \subseteq \ker (f) \subseteq \text{ann}(A)f = A(1-e)$ also the kernel of $f$ is a direct summand., Hence by [23, 7.6], $\text{End}_B(A)$ and thus $A^B$ is regular. Since no cyclic $B$-stable ideal is nilpotent, $A$ is $B$-semiprime.

(2) $A$ generates all cyclic $B$-stable ideals, i.e. $B A$ is a self-generator and since $A^B$ is regular by (1), $B A$ is intern-projective by [23 5.6]. Since $A$ is a cyclic $B$-module, the claim then follows by [23 5.9].

(3) Let $Ae$ and $Af$ be cyclic $B$-stable ideals with idempotents $e, f \in A^B$. Then $Ae + Af = A(e + f - ef) = A(e \oplus f)$, where $\oplus$ is the addition in the boolean ring of idempotents $B(A^B)$.

(4) By (2), every $B$-stable ideal $I$ can be written as $I = JA$ with $J$ ideal in $Z := A^B$. Hence the canonical projection $A = A \otimes_Z Z \to A/I \cong A \otimes_Z Z/J$ can be understood as the tensoring of the canonical projection of $Z \to Z/J$ by $A \otimes_Z -$, which respects direct sums.

(5) For every cyclic $B$-stable ideal $B \cdot x = Ae$ we have $(Ae)^2 = A^2 e^2 = Ae$. Hence $B \cdot x$ and thus any $B$-stable ideal is idempotent. Since there are no small $B$-submodules in $A$, we have $\text{Rad} (B A) = 0$ and 0 is the intersection of maximal $B$-stable ideals. By (4) we can use this argument to each $A/I$.

(6) Suppose $A$ is $B$-prime and $B$-biregular. Let $0 \neq I = Ae$ be a cyclic $B$-stable ideal with idempotent $e$. As $A(1-e)$ is a $B$-stable ideal with $A(1-e)I = 0$, we have $A(1-e) = 0$, i.e. $I = A$ and $A$ is $B$-simple. \qedsymbol

2.4. Characterisation of relative biregularity. The next Proposition characterises biregular extensions $A \subseteq M(A) \subseteq B \subseteq \text{End}_k(A)$. Denote by $\text{Max}(A^B)$ the spectrum of maximal ideals of $A^B$ and by $A_m$ the localisation of $A$ by a maximal ideal $m$ of $A^B$. Note that if $A^B$ is regular, then $A_m = A/m A$ by [23 17.7] and in particular since $m A$ is $B$-stable, we might consider $B \subseteq \text{End}_k(A/m A) = \text{End}_k(A_m)$. We say that $A$ is $B$-simple if 0 and $A$ are the only $B$-stable ideals of $A$.

Theorem. The following statements are equivalent for an extension $M(A) \subseteq B \subseteq \text{End}_k(A)$.

(a) $A$ is $B$-biregular;
(b) $A^B$ is regular and every maximal $B$-stable ideal $M$ of $A$ is of the form $M = AM^B$.
(c) $A^B$ is regular and $A_m$ is $B$-simple for all $m \in \text{Max}(A^B)$.

Proof. (a) $\Rightarrow$ (b) the properties (i – iii) follow from Proposition [23] and (iv) follows from the fact if $A$ is $B$-biregular then for any $x \in A : \text{ann}_A(B x) = A(1-e)$ with $e^2 = e \in A^B$ is already a $B$-ideal.

(b) $\Rightarrow$ (c): Let $m$ be a maximal ideal of $A^B$ and let $M$ be a maximal $B$-stable ideal containing $m A \subseteq M$. Since $M = M^B A$ we have $m \subseteq (m A)^B \subseteq M^B$ which implies $M^B = m$ since $M \neq A$. Thus $m A = M$ and $A/M = A/m A = A_m$ is $B$-simple.

(c) $\Rightarrow$ (a) Let $I$ be any $B$-stable ideal $I$ of $A$. Then $I^B A \subseteq I$ and

$$(I^B A)_m = (I \cap A^B)_m A_m = (I_m \cap A^B_m) A_m.$$

If $I_m = A_m$ then $I^B_m = I_m \cap A^B_m = A^B_m$ and hence $(I^B A)_m = I_m$. If $I_m \neq A_m$, then $I_m = 0_m$ and therefore $I^B_m = 0_m$, i.e. $(I^B A)_m = I_m$. Since this holds for any maximal ideal $m$ of $A^B$, we get $I = I^B A$ which shows that $A$ is a self-generator as $B$-module.
2.5. **Regular subring of invariants.** Assume again that \( A \subseteq B \) is any extension with additional module structure. In order to determine when the subring of invariants \( A^B \) is regular, we need first to borrow another notion from module theory.

**Definition.** A left \( R \)-module \( M \) is called semi-projective, if every diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{g} & N \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

with \( N \subseteq M \) can be completed by an endomorphism \( h \in S := \text{End}_R(M) \) such that \( hf = g \). As it is easily seen: \( M \) is semi-projective if and only if \( \text{Hom}_R(M, Mf) = Sf \) for all \( f \in S \).

Hence \( A \) is semi-projective as left \( B \)-module if \( \forall x \in A^B : (Ax)^B = A^Bx \).

**Proposition.** Let \( A \subseteq B \) be an extension with additional module structure. Then \( A^B \) is von Neumann regular if and only if

1. \( A \) is semi-projective as left \( B \)-module and
2. every cyclic left ideal generated by an \( B \)-invariant element \( x \in A^B \) is a direct summand of \( A \) as \( B \)-module.

**Proof.** If \( A^B \simeq \text{End}_B(A) \) is regular, then \( B \cdot A \) is semi-projective by [23] 5.9. Furthermore since the images of \( B \)-linear maps are direct summands and are precisely the cyclic \( B \)-stable left ideals generated by a \( B \)-invariant element we are done.

On the other hand assume that \( B \cdot A \) is semi-projective. Let \( 0 \neq x \in A^B \) then \( B \cdot x = Ax \) is a direct summand of \( A \) as left \( B \)-module by hypothesis. Thus \( A = Ax \oplus I \) as left \( B \)-modules. But then

\[
A^B = (1)(\text{Hom}_B(A, Ax) \oplus \text{Hom}_B(A, I)) = (Ax)^B \oplus I^B = A^Bx \oplus I^B.
\]

Hence every cyclic left ideal of \( A^B \) is a direct summand, i.e. \( A^B \) is von Neumann regular. \( \square \)

2.6. **Large subring of invariants.** If \( A \) is finitely presented and regular in \( \sigma[B \cdot A] \), then by Proposition [2,1] it is a projective generator in \( \sigma[B \cdot A] \). Weakening the generator condition J.Zelmanowitz called a left \( R \)-module \( M \) retractable if \( \text{Hom}_R(M, N) \neq 0 \) for all non-zero submodules \( N \subseteq M \).

For a module algebra extension \( A \subseteq B \) we say that \( A^B \) is large in \( A \) if \( I \cap A^B \neq 0 \) for all \( B \)-stable left ideals of \( A \) or equivalently if \( A \) is a retractable \( B \)-module. A classical theorem of Bergmann and Isaacs says that if finite group \( G \) acts on an algebra \( A \) such that \( A \) is \( G \)-semiprime and has no \( |G| \)-torsion, then \( R^G \) is large in \( R \).

A purely module theoretical result by J.Zelmanowitz from [24] says now in our language:

**Lemma.** Let \( A \) be projective in \( \sigma[B \cdot A] \) and \( A^B \) large in \( A \), then

1. If \( A^B \) is left self-injective, then \( A \) is a self-injective left \( B \)-module.
2. If \( A^B \) is von Neumann regular, then \( A \) is a non-singular in \( \sigma[B \cdot A] \), i.e. if \( K \subseteq L \) is an essential extension in \( \sigma[B \cdot A] \), then \( \text{Hom}_B(L/K, A) = 0 \).

**Proof.** Zelmanowitz calls a left \( R \)-module \( M \) fully retractable if \( \text{Hom}_R(M, N)g \neq 0 \) for any \( 0 \neq g \in \text{Hom}_R(N, M) \) and submodule \( N \subseteq M \). It is easy to see that self-projective retractable modules are fully retractable. Zelmanowitz proves in [24] Proposition on page 567] that \( M \) is self-injective.
if \(M\) is fully retractable and left \(\text{End}_R (M)\) self-injective. Property (2) follows from [25] Corollary on page 568. \(\square\)

Note that a module \(M\) is non-singular in \(\sigma[M]\) if and only if it is “polyform” in the sense of J.Zelmanowitz (see [23]).

2.7. As a consequence we have that if \(A\) is projective in \(\sigma[BA]\) and \(AB\) large in \(A\), then \(AB\) is regular and left self-injective if and only if \(A\) is injective and non-singular in \(\sigma[BA]\), because the endomorphism ring of any self-injective polyform module is self-injective and regular by [23, 11.1].

3. Relative semisimple extensions

Let \(A \subseteq B\) be an extension of \(k\)-algebras. An element \(c = \sum_i c_i \otimes c^i \in B \otimes_A B\) which is \(B\)-centralising, i.e. \(bc = cb\) for all \(b \in B\) is called a Casimir element for \(B\) over \(A\) (see [19] for the terminology). We say that a Casimir element acts unitarily on an element \(m\) of a left \(B\)-module \(M\) if \((\sum_i c_i c^i) \cdot m = m\).

**Proposition.** Let \(A \subseteq B\) be an extension with additional module structure and suppose that \(B\) has a Casimir element over \(A\) that acts unitarily on \(A\), then the following hold:

1. \(c\) acts unitarily on any module in \(\sigma[BA]\).
2. The \(k\)-linear map \(\text{Hom}_A (M, N) \rightarrow \text{Hom}_B (M, N)\) with \(f \mapsto \tilde{f} : [m \mapsto \sum_i c_i \cdot f(c^i \cdot m)]\)
   splits the embedding \(\text{Hom}_B (M, N) \subseteq \text{Hom}_A (M, N)\) for any \(N, M \in \sigma[BA]\).

**Proof.** Let \(\gamma := \sum_i c_i c^i\) and \(\alpha : B \rightarrow A\) with \((b)\alpha = b \cdot 1\). Then \(\alpha\) is left \(B\)-linear and \((\alpha)\alpha = a\) for any \(a \in A\). For all \(a \in A\) we have \(ac = \sum ac_i \otimes c^i = \sum c_i \otimes c^ia = ca\). Then also \((\sum ac_i c^i)\alpha = (\sum c_i c^ia)\alpha\) holds. Thus

\[
(\ast) \quad a = a(\gamma)\alpha = (\sum ac_i c^i)\alpha = (\sum c_i c^ia)\alpha = (\sum c_i c^i) \cdot (\alpha)\alpha = \gamma \cdot a.
\]

1. Let \(M \in \sigma[BA]\). Then there exists a set \(\Lambda\) and a \(B\)-submodule \(I \subseteq A^{(\Lambda)}\), such that \(M\) is isomorphic to a \(B\)-submodule of \(A^{(\Lambda)}/I\). We identify \(M\) with a submodule of \(A^{(\Lambda)}/I\). Let \(m \in M\), then there are elements \(\alpha_\lambda \in A\) for \(\lambda \in \Lambda\) such that \(m = (\alpha_\lambda)_\Lambda + I\). Now it follows with \((\ast)\):

\[
\gamma \cdot m = \gamma \cdot [(a_\lambda)_\Lambda + I] = (\gamma \cdot \alpha_\lambda)_\Lambda + I = (a_\lambda)_\Lambda + I = m.
\]

2. Obviously \(\tilde{f}\) is \(B\)-linear for all \(f : M \rightarrow N\) since \(c\) is a Casimir element. If \(f\) was already \(B\)-linear, then using (1) we get for all \(m \in M\):

\[
\tilde{f}(m) = \sum c_i \cdot f(c^i \cdot m) = (\sum c_i c^i) \cdot f(m) = f(m),
\]

i.e. \(\tilde{f} = f\) showing that the embedding splits. \(\square\)

3.1. \(M\) is a \((B, A)\)-semisimple \(B\)-module if any short exact sequence in \(\sigma[BM]\) that splits as left \(A\)-module, also splits as left \(B\)-module (see [23, page 170]). Recall that Hirata and Sugano called a ring extension \(A \subseteq B\) a semisimple extension if \(B\) is \((B, A)\)-semisimple (see [10]).

**Corollary.** If \(B\) has a Casimir element \(c\) which acts unitarily on \(A\), then \(A\) is a \((B, A)\)-semisimple \(B\)-module and for any \(M \in \sigma[BA]\)

- If \(M\) is \(N\)-projective as \(A\)-module for \(N \in \sigma[BA]\), then \(M\) is also \(N\)-projective as \(B\)-module.
- If \(M\) is \(N\)-injective as \(A\)-module for \(N \in B\text{-Mod}\), then \(M\) is also \(N\)-injective as \(B\)-module.

In particular \(A\) is projective in \(\sigma[BA]\).
Proof. Let \( \pi : M \to N \) be a projection in \( \sigma[BA] \) with \( \pi(n) = n \). Then for any \( n \in N \):
\[
\bar{\pi}(n) = \sum c_i \cdot \pi(c^i \cdot n) = \left( \sum c_i c^i \right) \cdot \pi(n) = \pi(n) = n.
\]
Thus \( \bar{\pi} \) splits the embedding of \( N \) into \( M \) as \( B \)-module. In the same way one proves the statements (1). For (2) note that if \( f : U \to M \) is \( B \)-linear, where \( U \) is a \( B \)-submodule of \( N \), then there exists an \( A \)-linear map \( g : N \to M \) such that \( g|_U = f \). Set as before \( \tilde{g} : N \to M \) which is \( B \)-linear. Then \( \tilde{g}(u) = \left( \sum c_i c^i \right) \cdot f(u) = f(u) \).

\[ \square \]

3.2. In [10] Hirata and Sugano called a ring extension \( A \subseteq B \) separable if there exists a Casimir element \( c = \sum_i c_i \otimes e^i \) such that \( \sum_i c_i e^i = 1 \).

**Corollary.** Let \( A \subseteq B \) be an extension with additional module structure such that there exists a Casimir element in \( B \) which acts unitarily on \( A \), then

1. If \( A \) is a semisimple artinian ring, then \( A \) is semisimple \( B \)-module.
2. If \( A \) is von Neumann regular and \( AB \) is finitely generated, then
   - \( A \) is a regular module in \( \sigma[BA] \);
   - \( AB \) is a regular ring and
   - \((-)^{B}\) defines a Morita equivalence between \( A^{B} \)-Mod and \( \sigma[BA] \).
3. If \( \sigma[BA] = B \)-Mod, then \( A \subseteq B \) is a semisimple extension.

Proof. (1) Is clear since \( A \) is \((B, A)\)-semisimple.

(2) Since \( AB \) is finitely generated, \( A \) is finitely presented in \( \sigma[BA] \). If \( B \cdot a \) is a cyclic \( B \)-submodule of \( A \), then by hypothesis \( B \cdot a \) is also finitely generated as left \( A \)-module and hence a direct summand of \( A \) as left \( A \)-module. Thus \( B \cdot a \) is also a direct summand of \( A \) as left \( B \)-module since \( A \) is \((B, A)\)-semisimple. By [2.1] \( A \) is a regular module in \( \sigma[BA] \). Also by [2.1] we have that \( \text{End}_B(A) \simeq A^B \) is regular and \( A \) is a progenerator in \( \sigma[BA] \) with equivalence \( \text{Hom}_B(A, -) \simeq (-)^{B} : \sigma[BA] \to \text{End}_B(A) \simeq A^B \).

(3) If \( A \) is a subgenerator in \( B \)-Mod, then \( B \in \sigma[BA] \) is itself \((B, A)\)-semisimple. \( \square \)

### 4. Applications to Hopf algebra actions

Let \( H \) be a Hopf algebra over \( k \) acting on an algebra \( A \), i.e. \( A \) is a left \( H \)-module algebra. The smash product of \( A \) and \( H \) is denoted by \( A\#H \) whose underlying \( k \)-module is \( A \otimes_k H \) and whose multiplication is defined by
\[
(a\#h)(b\#g) = \sum_{(h)} a(h_1 \cdot b)\#h_2g,
\]
where \( \Delta(h) = \sum_{(h)} h_1 \otimes h_2 \) is the comultiplication of \( h \). Then \( A \subseteq A\#H =: B \) is an extension with additional module structure whose module action is given by \( a\#h \cdot b = a(h \cdot b) \). The subring of invariants is \( A^B = A^H = \{ a \in A \mid h \cdot a = e(h)a \ \forall h \in H \} \). For more details on Hopf algebra action we refer to [110].

#### 4.1. Regularity of the subring of invariants

From [2.5] we get a characterisation of the regularity of the subring of invariants of \( A \).

**Proposition.** Let \( A \) be a \( k \)-algebra with Hopf action \( H \). Then \( A^H \) is regular if and only if \( A \) is a semi-projective left \( A\#H \)-module such that any cyclic left ideal generated by a \( H \)-invariant element is generated by an \( H \)-invariant idempotent.
4.2. In order to ensure that \( A \) is a finitely presented object in \( \sigma[A_{\#}H A] \) we will assume some finiteness conditions on \( H \) or on its action. We say that a Hopf algebra \( H \) acts finitely on a \( k \)-algebra \( A \) if the image of the defining action \( H \to \text{End}_k(A) \) is a finitely generated \( k \)-module or equivalently if \( A_{\#}H/\text{Ann}_{A_{\#}H}(A) \) is finitely generated as left \( A \)-module. Recall that a \( k \)-algebra \( A \) is called affine if it is finitely generated as \( k \)-algebra.

Denote by \( \epsilon : H \to k \) the counit of \( H \). We need the following Lemma:

**Lemma.** Let \( H \) be a Hopf algebra over \( k \) that is affine as \( k \)-algebra, then \( \text{Ker}(\epsilon) \) is a finitely generated left ideal.

**Proof.** Suppose that \( H \) is affine and let \( B \subseteq H \) be a finite set of elements which generate \( H \) as a \( k \)-algebra. We will show that \( \text{Ker}(\epsilon) = \sum_{b \in B} H(b - \epsilon(b)) \). Obviously the right hand side is included in the left hand side. Note that for any word (=product) \( \omega = b_1 \cdots b_m \) with \( b_i \in B \) we might set \( a_0 = \epsilon(\omega) \), \( a_m = \omega \) and \( a_i = b_1 \cdots b_i \epsilon(b_{i+1} \cdots b_m) \) for \( 0 < i < m \) and conclude that \( \omega - \epsilon(\omega) \in \sum_{i=1}^m H(b_i - \epsilon(b_i)) \), since as a telescopic sum we have

\[
\omega - \epsilon(\omega) = \sum_{i=1}^m a_i - a_{i-1} = \sum_{i=1}^m b_1 \cdots b_{i-1} \epsilon(b_{i+1} \cdots b_m)(b_i - \epsilon(b_i)).
\]

Take any element \( h \in \text{Ker}(\epsilon) \). Then there exist \( \lambda_i \in k \) and words \( \omega_i \) in \( B \) such that

\[
h = h - \epsilon(h) = \sum_i \lambda_i [\omega_i - \epsilon(\omega_i)] \in \sum_{b \in B} H(b - \epsilon(b)).
\]

Thus \( \text{Ker}(\epsilon) \) is finitely generated. \( \square \)

In the telescopic sum argument in the proof of the last Lemma we made use of the fact that the counit \( \epsilon \) of a Hopf algebra is an algebra homomorphism. We do not know whether this Lemma holds true for affine weak Hopf algebras.

4.3. From Proposition 2.1 we deduce the next result:

**Theorem.** If \( H \) is an affine \( k \)-algebra or acts finitely on \( A \), then \( A \) is a finitely presented in \( \sigma[A_{\#}H A] \) and the following statements are equivalent:

1. \( A \) is \( H \)-regular, i.e. any finitely generated \( H \)-stable left ideal is generated by an \( H \)-invariant element;

2. \( A \) is a projective generator in \( \sigma[A_{\#}H A] \) and any cyclic left ideal generated by an \( H \)-invariant element is generated by an \( H \)-invariant idempotent.

3. \( A^H \) is von Neumann regular and \( (-)^H : \sigma[A_{\#}H A] \to A^H \text{-Mod} \) is an equivalence.

4. \( A^H \) is von Neumann regular and \( A \) is a projective generator in \( \sigma[A_{\#}H A] \).

5. \( A \) is a regular module in \( \sigma[A_{\#}H A] \).

**Proof.** Once we showed that \( A \) is finitely presented in \( \sigma[A_{\#}H A] \), the result follows from 2.1. If \( H \) acts finitely on \( A \), then we might substitute \( A^H \) by \( B = A_{\#}H/\text{Ann}_{A_{\#}H}(A) \) which is finitely generated as left \( B \)-module. Also \( \alpha \) lifts to a map \( \overline{\alpha} : B \to A \) and splits as left \( A \)-module map. Thus \( \text{Ker}(\overline{\alpha}) \) is finitely generated as left \( A \)-module and thus as left ideal of \( B \), i.e. \( A \) is finitely presented in \( B \text{-Mod} \) and hence in \( \sigma[B A] = \sigma[A_{\#}H A] \).

On the other hand suppose that \( H \) is affine, then \( \text{Ker}(\epsilon) \) is a finitely generated left ideal by Lemma 4.2. For the module algebra \( A \) and \( \text{Ker}(\alpha : A_{\#}H \to A) \), we have that if \( x = \sum_{i=1}^n a_i_{\#}h_i \in \)}
Corollary. \( Z \) of a Hopf module algebra \( A \).

4.6. As before we need to ensure that \( \text{idempotent} \).

\[ x = \sum_{i=1}^{n} a_i \# h_i \left( \sum_{i=1}^{n} a_i \epsilon(h_i) \right) \# 1 = \sum_{i=1}^{n} a_i \# (h_i - \epsilon(h_i)) \in A \# \text{Ker} \left( \alpha \right). \]

Thus \( \text{Ker} \left( \alpha \right) = A \# \text{Ker} \left( \epsilon \right) = \sum_{b \in B} A \# H (1 \# (b - \epsilon(b))) \) is a finitely generated left ideal of \( A \# H \) and therefore \( A \) is finitely presented.

4.4. Note that the notion of regularity used here is different from the concept of an \( H \)-regular module algebra as defined by [22]. There the author define an element \( a \) of an \( H \)-module algebra \( A \) to be \( H \)-regular if \( a \in (H \cdot a)A(H \cdot a) \) and calls \( A \) \( H \)-regular if every element is \( H \)-regular.

4.5. The envelopping Hopf algebroid. In general a Hopf action does not extend to the envelopping algebra \( A^e \) unless \( H \) is cocommutative. In order to study the two-sided \( H \)-stable ideals of a Hopf module algebra \( A \) with Hopf action \( H \), one defines a new product on the tensor product \( A^e \otimes H \) as follows:

\[ [(a \otimes b) \triangleright h][(a' \otimes b') \triangleright h'] = \sum_{(h)} a(h_1 \cdot a') \otimes (h_3 \cdot b') b \triangleright h_2 h' \]

for all \( a \otimes b, a' \otimes b' \in A^e \) and \( h, h' \in H \). This construction had been used by the author in [14] (see also [13]) in order to define the central closure of a module algebra \( A \) as the self-injective hull of \( A \) as \( A^e \triangleright H \)-module and had also been used by Connes and Moscovici in [7]. A similar construction had been used by L.Kadison in [11] which in [18] was shown to be isomorphic to the construction by Connes-Moscovici. Following Kadison, we denote this algebra on \( A^e \otimes H \) by \( A^e \triangleright H \) and call it the envelopping Hopf algebroid of \( A \) and \( H \). For any left \( A^e \triangleright H \)-module \( M \) denote by

\[ Z(M)^H := \{ m \in M \mid am = ma \land hm = \epsilon(h)m \forall a \in A, h \in H \} \]

Then since \( A \subseteq A^e \triangleright H \) is again an extension with additional module structure we have that \( Z(-)^H \) is a functor from \( A^e \triangleright H \rightarrow Z(A)^H \text{-Mod} \) and that

\[ \text{Hom}_{A^e \triangleright H} (A, M) \rightarrow Z(M)^H \ f \mapsto (1)f \]

is a functorial isomorphism. Note that \( Z(A)^H := Z(A) \cap A^H \cong \text{End}_{A^e \triangleright H} (A) \).

From [13] we get a characterisation of the regularity of the subring of central invariants of \( A \).

Corollary. \( Z(A)^H \) is regular if and only if \( A \) is a semi-projective left \( A^e \triangleright H \)-module such that any cyclic ideal generated by a central \( H \)-invariant element is generated by a central \( H \)-invariant idempotent.

4.6. As before we need to ensure that \( A \) is a finitely presented object in \( \sigma_{[A^e \triangleright H \cdot A]} \) in order to apply [21].

Lemma. If \( A \) and \( H \) are affine \( k \)-algebras, then \( A \) is a finitely presented module in \( \sigma_{[A^e \triangleright H \cdot A]} \).

Proof. Consider \( \alpha : A^e \triangleright H \rightarrow A \) by \( a \otimes b \triangleright h \mapsto ac(h)b \). For any \( x = \sum_{i=1}^{n} a_i \otimes b_i \triangleright h_i \in \text{Ker} \left( \alpha \right) \) we have \( \sum_{i=1}^{n} a_i \epsilon(h_i)b_i = 0 \). Hence

\[ x = \sum_{i=1}^{n} a_i \otimes b_i \triangleright h_i - \left( \sum_{i=1}^{n} a_i b_i \epsilon(h_i) \right) \otimes 1 \otimes 1 + \left[ \sum_{i=1}^{n} a_i \otimes b_i \triangleright \epsilon(h_i) - \sum_{i=1}^{n} a_i \otimes b_i \triangleright \epsilon(h_i) \right] \]

\[ = \sum_{i=1}^{n} a_i \otimes b_i \triangleright [h_i - \epsilon(h_i)] + \sum_{i=1}^{n} a_i \epsilon(h_i)[1 \otimes b_i - b_i \otimes 1] \otimes 1 \]

\[ \in A^e \triangleright \text{Ker} \left( \epsilon \right) + A \text{Ker} \left( \mu \right) \triangleright 1 \]
where \( \text{Ker} \ (\mu) \) is the augmentation ideal of the enveloping algebra, i.e. the kernel of the multiplication map \( \mu : A^e \rightarrow A \). Hence we see that \( \text{Ker} \ (\alpha) \) is generated as left ideal of \( A^e \rtimes H \) by elements of \( 1 \otimes \text{Ker} \ (\epsilon) \) and \( \text{Ker} \ (\mu) \rtimes 1 \). It is well-known that \( \text{Ker} \ (\mu) \) is finitely generated left ideal of \( A^e \) if \( A \) is affine and by \( \ref{1.2} \) it follows that \( \text{Ker} \ (\epsilon) \) is finitely generated if \( H \) is affine. \( \square \)

4.7. \( H \)-biregular module algebras. The last statement, \( \ref{2.1} \) and \( \ref{2.4} \) yield the main result in this section which generalises \( \ref{1.2} \) from group actions to Hopf actions.

**Corollary.** Let \( A \) and \( H \) be affine \( k \)-algebras, then the following statements are equivalent:

(a) \( A \) is \( H \)-biregular, i.e. every finitely generated \( H \)-stable two-sided ideal of \( A \) is generated by a central \( H \)-invariant idempotent.

(b) \( A \) is a projective generator in \( \sigma[A \rtimes_{\sigma} H] \) and any ideal generated by a central \( H \)-invariant element is generated by an idempotent central \( H \)-invariant element.

(c) \( A \) is a regular module in \( \sigma[A \rtimes_{\sigma} H] \).

(d) \( Z(A)^H \) is von Neumann regular and one of the following statements hold:

(i) the functor \( Z(-)^H : \sigma[A \rtimes_{\sigma} H] \rightarrow Z(A)^H \text{-Mod} \) is an equivalence.

(ii) \( A \) is a projective generator in \( \sigma[A \rtimes_{\sigma} H] \).

(iii) every maximal \( H \)-stable ideal \( M \) of \( A \) can be written as \( M = [Z(A)^H \cap M]A \).

(iv) \( A_m \) is \( H \)-simple for any maximal ideal \( m \) of \( Z(A)^H \).

4.8. Relative semisimple extension. Let \( G \) be a finite group acting on an algebra \( A \). The condition that \( |G| \) is invertible is frequently used in the study of group actions because it implies that \( A \subseteq A \rtimes G \) is a separable extension. The weaker condition on \( A \) of having an element of trace 1, i.e. an element \( z \in A \) such that \( t \cdot a = \sum_{g \in G} (g \cdot a) = 1 \), where \( t = \sum_{g \in G} g \), implies at least the projectivity of \( A \) as \( A \rtimes G \)-module. Here we will analyse those concepts and carry them over to Hopf algebra actions.

The antipode of a Hopf algebra \( H \) is denoted by \( S \). An element \( t \in H \) is called a right(resp. left) integral in \( H \) if \( th = te(h) \) (resp. \( ht = e(h)t \) for all \( h \in H \). It is well-known that \( \sum_{(t)} S(t_1) \otimes t_2 h = \sum_{(t)} hS(t_1) \otimes t_2 \) for all \( h \in H \).

**Proposition.** Let \( H \) be a Hopf algebra and \( A \) a left \( H \)-module algebra. Suppose \( H \) has a non-zero right integral \( t \) and \( A \) admits a central element \( z \) such that \( S(t) \cdot z = 1 \), then

\[
    c := \sum_{(t)} (1 \# S(t_1)) \otimes (z \# t_2)
\]

is a Casimir element of \( A \# H \) that acts unitarily on \( A \). Hence \( A \) is a semisimple \( (A \# H, A) \)-module.

(1) \( A \subseteq A \# H \) is a semisimple extension if \( A^H \subseteq A \) is a \( H^* \)-Galois extension, i.e. \( A \) is a generator in \( A \# H \text{-Mod} \).

(2) \( A \subseteq A \# H \) is separable if \( z \in A^H \) or \( H \) is cocommutative

**Proof.** The element \( \sum_{(t)} 1 \# S(t_1) \otimes z \# t_2 \) is a Casimir element in \( (A \# H) \otimes_A (A \# H) \) because for all \( a \# h \in A \# H \):
4.10. The observation that condition of 4.8 is fulfilled.

4.9. If the antipode is bijective and of non-finitely generated ones (see [12]).

4.12. First note the following Corollary that we get from 4.8:

4.11. Regularity of smash products. In [11] the authors studied the regularity of skew group rings. They showed in particular that a skew group ring exists a right integral for a left integral $t$ of $A$. They showed in particular that a skew group ring $A^H$ is separable if $A$ is a Frobenius $k$-algebra and thus finitely generated and projective as $k$-module. Note that the existence of a left (or right) integral forces a Hopf algebra in many cases to be finitely generated although there are examples of non-finitely generated ones (see [12]).

4.10. The observation that $A \subseteq A^H$ is separable if $z \in Z(A)^H$ or $H$ cocommutative, can also be found in [3] Theorem 1.11] or [13], but under the hypothesis of $H$ being a Frobenius $k$-algebra and thus finitely generated and projective as $k$-module. Note that the existence of a left (or right) integral forces a Hopf algebra in many cases to be finitely generated although there are examples of non-finitely generated ones (see [12]).

4.9. If the antipode is bijective and $A$ has a central element of trace 1, i.e. $z \in Z(A)$ with $t \cdot z = 1$ for a left integral $t$ of $H$, then $t' = S^{-1}(t)$ is a right integral, and $S(t') \cdot z = 1$ holds, i.e. the condition of 4.8 is fulfilled.

Moreover

$$\sum_{(t)} (1#S(t_1))(z#t_2) \cdot 1 = \sum_{(t)} S(t_1) \cdot (z(t_2 \cdot 1)) = S(t) \cdot z = 1.$$ 

Thus $c$ acts unitarily on $A$. By [3] $A$ is a semisimple $(A^#H, A)$-module.

If $A/A^H$ is a $H^*$-Galois extension, then $\sigma_{A^#HA} = A^#H$ and the claim follows from [12].

Note that $\mu(c) = \sum_{(t)} S(t_2)z#S(t_1)t_3$. If $z \in Z(A)^H$, then $\mu(c) = \sum_{(t)} z#S(t_1)c(t_2)t_3 = c(S(t))z#1 = S(t)z#1 = 1#1$. If $H$ is cocommutative, then $\mu(c) = \sum_{(t)} S(t_1)z#S(t_2)t_3 = S(t)x#1 = 1#1$. Hence in both cases $A^#H$ is separable over $A$.}

\[c(a#h) = \sum_{(t)} (1#S(t_1)) \otimes (z#t_2)(a#h)\]
\[= \sum_{(t)} (1#S(t_1)) \otimes (z(t_2 \cdot a)#t_3h)\]
\[= \sum_{(t)} (S(t_2) \cdot (t_3 \cdot a))#S(t_1) \otimes z#t_4h\]
\[= (a#1) \left( \sum_{(t)} 1#S(t_1) \otimes z#t_3h \right)\]
\[= (a#1) \left( \sum_{(t)} 1#hS(t_1) \otimes z#t_2 \right)\]
\[= (a#h) \left( \sum_{(t)} 1#hS(t_1) \otimes z#t_2 \right) = (a#h)c\]
Proof. (1) If we substitute $A\# H$ by $B = A\# H/\text{Ann}(A)$, then $AB$ is finitely generated and $c = \sum_{i} a_{i}\# h_{i} \in A\# H$ can be lifted to $c' \in B \otimes_{A} B$ which still acts unitarily on $A$. By Corollary 3.2, $A$ has the properties stated above.

(2) if $z \in A^{H}$ or $H$ is cocommutative, then by 4.8 $A \subseteq A\# H$ is separable and hence $A\# H$ is regular as $A$ was regular. In case $A/A^{H}$ is $H^{\ast}$-Galois we have that $\sigma_{[A\# H, A]} = A\# H\text{-Mod}$ and by 4.8 $A \subseteq A\# H$ is a semisimple extension. Since $kH$ is finitely generated, $A\# A^{H}$ is finitely generated. Hence any cyclic left ideal $I$ of $A\# H$ is finitely generated as left $A$-submodule of $A\# H$.

4.13. Locally finite Hopf algebras. Call an extension $A \subseteq B$ locally separable if every element of $B$ is contained in an intermediate algebra $A \subseteq C \subseteq B$ such that $C$ is a separable extension of $A$ (see also A.Magid’s definition 15). Of course, if $A \subseteq B$ is locally separable and $A$ is regular, then $B$ is regular, because any element $x \in B$ is contained in a separable extension $C$ of $A$. And if $A$ is regular, then also $C$. Thus $x = xyx$ for some $y \in C \subseteq B$. Hence $B$ is regular. The characterisation of regular group rings $k[G]$ can actually be stated as $k \subseteq k[G]$ being locally separable and $k$ being regular. Alfar, Ara and del Rio proved in [1] Theorem 1.3] that if $G$ is a locally finite group acting on a regular ring $A$ such that for every finite subgroup $H$ there exists a central element of trace 1 with respect to $H$, then the skew group ring $A \ast G$ is regular. We will slightly generalize there result to Hopf algebra actions by showing that the hypotheses of their result imply that $A \subseteq A \ast G$ is locally separable.

A group is called locally finite if any finitely generated subgroup is finite.

Definition. Let $H$ be a Hopf algebra over $k$. A Hopf algebra is called locally finite if any finite set $X \subseteq H$ is contained in a Hopf subalgebra of $H$ which contains a non-zero right integral.

Note that any Hopf algebra $H$ which is free as a module over $k$ is finitely generated as $k$-module if and only if it contains a non-zero right integral (see [12]). A group ring $k[G]$ is of course locally finite if $G$ is locally finite.

4.14. We are now in position to generalize [1 Theorem 1.3]:

Corollary. Let $H$ be a locally finite Hopf algebra over $k$ acting on a $k$-algebra $A$ such that for any Hopf subalgebra $K$ of $H$ that contains a non-zero right integral $t$, there exists a central element $z \in A$ with $S(t) \cdot z = 1$. If $H$ is cocommutative or $z \in Z(A)^{H}$ for all right integrals $t$, then $A \subseteq A\# H$ is locally separable. Hence if $A$ is regular, so is $A\# H$.

Proof. Let $x = \sum a_{i}\# h_{i} \in A\# H$. By hypothesis $K := \langle \{h_{1}, \ldots, h_{k}\} \rangle$ contains a non-zero right integral $t$. By 4.8(2), $A \subseteq A\# K$ is separable, and hence regular if $A$ was regular.

4.15. As a consequence we have that if $H$ is a cocommutative Hopf algebra acting on a commutative regular $k$-algebra having a central element of trace 1, then $A\# H$ is regular (which partly generalizes [1 Corollary 2.5]). It had been shown in [1, 2.4], that if a skew-group ring $A \ast G$ is regular, then is also $A$. This is not anymore true for smash products as it is easily seen by the fact, that for any finite dimensional Hopf algebra $H$ over a field $k$, the smash product $H\# H^{\ast} \cong M_{n}(k)$ is isomorphic to a semisimple artinian ring, whether $H$ is semisimple or not. However we have that if $H$ is an $n$-dimensional cosemisimple Hopf algebra over a field $k$ acting on an algebra $A$ such that $A\# H$ is regular, then $A$ is regular. Simply because by the Blattner-Montgomery duality one
has \((A\#H)\#H^* \simeq M_n(A)\) and since \(H^*\) is separable over \(k\), we have \(A\#H\#H^*\) being separable over \(A\#H\) inducing regularity on \(M_n(A)\) and hence on \(A\).

5. Regularity and injectivity of the subring of invariants

Note that from Zelmanowitz result 2.6, we get

**Corollary.** Let \(A\) be an \(H\)-module algebra that is projective in \(\sigma[A]\). If \(A^H\) is large in \(A\) then \(A^H\)

is left self-injective and von Neumann regular if and only if \(A\) is a self-injective left \(A\#H\)-module which is non-singular in \(\sigma[A\#H]\). In this case \(A\) is also \(H\)-semiprime.

**Proof.** The equivalence of the statements follows verbatim from 2.6. If \(I\) is an \(H\)-stable ideal of \(A\) with \(I^2 = 0\), then \((I^H)^2 = 0\). But since \(A^H\) is regular, \(I^H = 0\) and since \(A^H\) is large \(I = 0\).

5.1. To compare the injectivity of \(A\) and its subring of invariants, we need the following Lemma which is probably known:

**Lemma.** Let \(S \subseteq T\) be rings such that \(T_S\) is flat. If \(T\) is left self-injective, then so is \(S\).

**Proof.** Let \(I\) be a left ideal of \(S\), denote the inclusion map by \(f : I \rightarrow S\) and let \(g : I \rightarrow S\) be an \(S\)-linear map. Let \(\gamma : T \rightarrow T \otimes_S S\) be the canonical isomorphism. Then \(\gamma\) is left \(T\)-linear and \(\gamma|_{TI} : TI \rightarrow T \otimes_S I\) is also an isomorphism of left \(T\)-modules. Let \(\tilde{f} := \gamma(1 \otimes f)\gamma^{-1} : TI \rightarrow T\). As \(T_S\) is flat, \(\tilde{f}\) is injective. Also set \(\tilde{g} := \gamma(1 \otimes g)\gamma^{-1} : TI \rightarrow T\). Then we can consider the following diagram with exact rows, where \(\iota : S \rightarrow T\) denotes the inclusion map (which is of course just \(S\)-linear):

\[
\begin{array}{ccc}
0 & \longrightarrow & I \\
\downarrow & & \downarrow \\
0 & \longrightarrow & TI \\
\downarrow & & \downarrow \\
\iota & \longrightarrow & T \\
\downarrow & & \\
& & T
\end{array}
\]

As \(T\) is left self-injective, there there exists a \(T\)-linear map \(\tilde{h} : T \rightarrow T\) such that \(\tilde{f}\tilde{h} = \tilde{g}\). Hence the outer trapezoid also commutes, i.e \(f\tilde{h} = \tilde{g}\). Since for all \(x \in I\) : \((x)\tilde{g} = (x)\tilde{g}\) we may identify \(\iota\tilde{g}\) with \(g\) and take \(h := \tilde{h}\) to be the desired \(S\)-linear map.

5.2. We will finish with the following result on the transfer of regularity and injectivity to the subring of invariants of a module algebra which should be compared to [9, Theorem A].

**Corollary.** Let \(H\) be Hopf algebra acting on \(A\). Suppose \(H\) has a right integral \(t\) and \(A\) has a central element \(z\) such that \(S(t) \cdot z = 1\). If \(A\) is regular and left self-injective ring, then \(A^H\) is regular and left self-injective.

**Proof.** Since \(A\) is \((A\#H,A)\)-semisimple by 4.8, \(A\) is semi-projective as \(A\#H\)-module. Take any \(x \in A^H\), then \(Ax\) is a direct summand in \(A\) and by relative semisimplicity also a direct summand as \(A\#H\)-submodule. Thus by 5.1 \(A^H\) regular. Now it follows from (1) that \(A^H\) is also left self-injective.

□
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University of Porto, Department of pure Mathematics, Rua Campo Alegre 687, 4169-007 Porto (PORTUGAL), clomp@fc.up.pt