Abstract

We consider a brownian motion on a general graph, that starts at time \( t = 0 \) from some vertex \( O \) and stops at time \( t \) somewhere on the graph. Denoting by \( g \) the last time when \( O \) is reached, we establish a simple expression for the Laplace Transform, \( L \), of the probability density of \( g \). We discuss this result for some special graphs like star, ring, tree or square lattice. Finally, we show that \( L \) can also be expressed in terms of primitive orbits when, for any vertex, all the exit probabilities are equal.

Around the years 1930, P Levy \cite{1} got several arc-sine laws concerning the 1D Brownian Motion on an infinite line. Let us consider such a process starting at \( t = 0 \) from the origin \( O \) and stopping at time \( t \) and denote by \( g \) the last time when \( O \) is reached. The second arc-sine law discovered by Levy concerns the probability law of \( g \). It can be stated as follows:

\[
P(g < u) = \frac{2}{\pi} \arcsin \sqrt{\frac{u}{t}} \tag{1}
\]

with the probability density

\[
\mathcal{P}_t(u) \equiv \frac{dP(g < u)}{du} = \frac{1}{\pi} \frac{1}{\sqrt{u(t-u)}} \tag{2}
\]

In particular, the Laplace transform of \( \mathcal{P}_t(u) \) is written:

\[
\int_0^\infty dt \ e^{-\gamma t} \int_0^t du \ \mathcal{P}_t(u) \ e^{-\xi u} = \frac{1}{\sqrt{\gamma(\gamma + \xi)}} \tag{3}
\]

Levy also proved that the same law occurs when \( g \) represents the time spent in the region \( (x > 0) \) (first arc-sine law).

The infinite line with only one point, \( O \), specified, can be viewed as a kind of very simple graph consisting in one vertex, \( O \), and two semi-infinite lines originating from \( O \).

Our goal in this Letter is to get an analogue of the second arc-sine law \footnote{\label{footnote}4} but for a quite general graph.
Let us recall that graphs properties have interested for many years physicists \(^2\) as well as mathematicians \(^3\). For instance, the knowledge of spectral properties of the Laplacian operator is useful to understand superconducting networks \(^4\) as well as organic molecules \(^5\). We also know that spectral determinants play a central role in the computation of weak localization corrections \(^6\). Recently, graphs have also been used in the context of nonequilibrium statistical physics (see \(^7\) and references therein) and also in the field of quantum chaos \(^8\).

The study of Brownian motion on graphs is also an active area of research. For instance, occupation times \(^9\) \(^10\) or local time \(^11\) distributions have been recently computed for general graphs.

Let us consider a brownian motion starting at some vertex \(x\) with two arcs \((\alpha\beta)\). Sometimes we will also use the coordinate \(x\) \((\alpha\beta)\). An arc \((\alpha\beta)\) is defined as the oriented bond from \(x\) to \(x'\). On each bond \([\alpha\beta]\), of length \(l_{\alpha\beta}\), we define the coordinate \(x_{\alpha\beta}\) that runs from 0 (vertex \(\alpha\)) to \(l_{\alpha\beta}\) (vertex \(\beta\)). Sometimes we will also use the coordinate \(x_{\beta\alpha}\equiv l_{\alpha\beta}-x_{\alpha\beta}\). The set of coordinates \(\{x_{\alpha\beta}\}\) is simply noted \(x\).

An arc \((\alpha\beta)\) is defined as the oriented bond from \(\alpha\) to \(\beta\). Each bond \([\alpha\beta]\) is therefore associated with two arcs \((\alpha\beta)\) and \((\beta\alpha)\). In the sequel, we will consider the following ordering of the \(2B\) arcs: \((0\mu_1)(0\mu_2)\ldots(0\mu_{m_0})\ldots(\alpha\beta_1)\ldots(\alpha\beta_i)\ldots(\alpha\beta_{m_a})\ldots\)

Now, let us consider a brownian motion starting at some vertex \(O\) (label 0) of the graph at time \(t=0\) and stopped at time \(t\). Let \(g\) be the last time when \(O\) is visited. The probability, \(P_t(g < u)\) can be written:

\[
P_t(g < u) = \int_{\text{Graph}} dx' \, dx \, G_0(0,0;x,u) \, G_D(x,u;x',t)
\]

where \(G_0\) is the free propagator on the graph and \(G_D\) is the propagator with Dirichlet boundary conditions in \(O\). \(^4\) simply expresses the fact that the brownian particle will not hit \(O\) between the times \(u\) and \(t\). Let us define the probability density \(P_t(u) \equiv \frac{dP_t(g < u)}{du}\). In the sequel, we will be especially interested in the Laplace transforms:

\[
\mathcal{L} \equiv \int_0^\infty dt \, e^{-\gamma t} \int_0^t du \, P_t(g < u) \, e^{-\xi u}
\]

\[
L \equiv \int_0^\infty dt \, e^{-\gamma t} \int_0^t du \, P_t(u) \, e^{-\xi u} = \frac{1}{\gamma + \xi} + \xi \mathcal{L}
\]

With \(^4\), we see that:

\[
\mathcal{L} = \int_{\text{Graph}} dx' dx \left| \frac{1}{H + \gamma + \xi} \right| x \left| \frac{1}{H_D + \gamma} \right| x' \equiv \int_{\text{Graph}} dx' dx \, G_0(0,x;\gamma + \xi) \, G_D(x,x';\gamma)
\]

On the bond \([\alpha\beta]\), \(H\) and \(H_D\) are equal to \(-\frac{1}{2} \frac{d^2}{dx^2_{\alpha\beta}}\) and we define:

\[
G_0(\alpha\beta) = \lim_{x_{\alpha\beta} \to 0} G_0(0,x_{\alpha\beta};\gamma + \xi) \quad ; \quad G_0'(\alpha\beta) = \lim_{x_{\alpha\beta} \to 0} \frac{dG_0(0,x_{\alpha\beta};\gamma + \xi)}{dx_{\alpha\beta}}
\]

\[
G_D(\alpha\beta)(x) = \lim_{x_{\alpha\beta} \to 0} G_D(x,x'_{\alpha\beta};\gamma) \quad ; \quad G_D'(\alpha\beta)(x) = \lim_{x_{\alpha\beta} \to 0} \frac{dG_D(x,x_{\alpha\beta};\gamma)}{dx_{\alpha\beta}}
\]
The integration over \(x'\) in (7) is easily performed with the help of the equation

\[
\left( -\frac{1}{2} \frac{d^2}{dx_{\alpha\beta}^2} + \gamma \right) G_D(x, x_{\alpha\beta}; \gamma) = \delta(x - x_{\alpha\beta})
\]

We get

\[
\int_{\text{Graph}} dx' \ G_D(x, x'; \gamma) = \frac{1}{\gamma} - \frac{1}{2\gamma} \sum_{i=1}^{m_\alpha} G_D'(0\mu_i)(x)
\]

To go further, we have to specify the behaviours of the resolvants \(G_0\) and \(G_D\) in the neighbourhood of all the vertices.

Let us consider some vertex \(\alpha\) with its nearest neighbours \(\beta_i, i = 1, 2, \ldots m_\alpha\), on the graph. Suppose that the brownian particle reaches \(\alpha\). It will come out towards \(\beta_i\) with some exit probability  \(p_{\alpha\beta_i}\). This implies the following equations to be satisfied by \(G_0\):

\[
\frac{G_0^{(\alpha\beta_1)}}{p_{\alpha\beta_1}} = \frac{G_0^{(\alpha\beta_2)}}{p_{\alpha\beta_2}} = \ldots = \frac{G_0^{(\alpha\beta_{m_\alpha})}}{p_{\alpha\beta_{m_\alpha}}} \equiv f_{\alpha} \quad \forall\alpha
\]

Remark that the resolvant will be continuous in vertex \(\alpha\) only when the particle exits from \(\alpha\) with the same probability in all the directions.

Moreover, current conservation implies:

\[
\sum_{i=1}^{m_\alpha} G_0'(\alpha\beta_i) = 0 \quad \text{if} \ \alpha \neq 0
\]

\[
\sum_{i=1}^{m_0} G_0'(0\mu_i) = -2
\]

On the link \([\alpha\beta]\), \(G_0(0, x_{\alpha\beta}; \gamma + \xi)\) must satisfy:

\[
\left( -\frac{1}{2} \frac{d^2}{dx_{\alpha\beta}^2} + \gamma + \xi \right) G_0(0, x_{\alpha\beta}; \gamma + \xi) = 0
\]

with the solution:

\[
G_0(0, x_{\alpha\beta}; \gamma + \xi) = G_0^{(\alpha\beta)} \frac{\sinh \sqrt{2(\gamma + \xi)(l_{\alpha\beta} - x_{\alpha\beta})}}{\sinh \sqrt{2(\gamma + \xi)}l_{\alpha\beta}} + G_0^{(\beta\alpha)} \frac{\sinh \sqrt{2(\gamma + \xi)x_{\alpha\beta}}}{\sinh \sqrt{2(\gamma + \xi)}l_{\alpha\beta}}
\]

So, we deduce:

\[
G_0'(\alpha\beta) = \sqrt{2(\gamma + \xi)} \left( -c_{\beta\alpha}(\gamma + \xi) G_0^{(\alpha\beta)} + s_{\alpha\beta}(\gamma + \xi) G_0^{(\beta\alpha)} \right)
\]

with:

\[
c_{\alpha\beta}(\gamma + \xi) = \coth \sqrt{2(\gamma + \xi)}l_{\alpha\beta} = c_{\beta\alpha}(\gamma + \xi)
\]

\[
s_{\alpha\beta}(\gamma + \xi) = \frac{1}{\sinh \sqrt{2(\gamma + \xi)}l_{\alpha\beta}} = s_{\beta\alpha}(\gamma + \xi)
\]

Those considerations allow to write eqs. (12, 13, 14) in a matrix form:
\[ M(\gamma + \xi) f = \frac{1}{\sqrt{2(\gamma + \xi)}} l \]  

(20)

where \( f \) and \( l \) are two \((V \times 1)\) vectors. The components of \( f \) are the quantities \( f_\alpha \) and for the components of \( l \) we have \( l_j = 2\delta_{j0} \).

\( M(\gamma + \xi) \) is a \((V \times V)\) vertex matrix with the non vanishing elements (\( \alpha \) runs from 0 to \( V - 1 \)):

\[ M(\gamma + \xi)_{\alpha\alpha} = \sum_{i=1}^{m_\alpha} p_{\alpha\beta_i} c_{\alpha\beta_i}(\gamma + \xi) \]  

(21)

\[ M(\gamma + \xi)_{\alpha\beta} = -p_{\beta\alpha} s_{\alpha\beta}(\gamma + \xi) \quad \text{if } [\alpha\beta] \text{ is a bond} \]  

(22)

\[ = 0 \quad \text{otherwise} \]  

(23)

Similar considerations hold for \( G_D(x, x'; \gamma) \) and we get:

\[ \frac{G_D^{(\alpha\beta_1)}(x)}{p_{\alpha\beta_1}} = \ldots = \frac{G_D^{(\alpha\beta_{m_\alpha})}(x)}{p_{\alpha\beta_{m_\alpha}}} = F_\alpha(x) \quad \text{, if } \alpha \neq 0 \]  

(24)

\[ G_D^{(0\mu_1)}(x) = 0 \quad ; \quad i = 1, \ldots, m_0 \]  

(25)

\[ \sum_{i=1}^{m_\alpha} G'_D^{(\alpha\beta_i)}(x) = 0 \quad , \quad \text{if } \alpha \neq 0 \]  

(26)

and the matrix equation

\[ M_D(\gamma)F(x) = \frac{1}{\gamma} l_D(x) \]  

(27)

\( M_D \) is the matrix \( M \) with the first line and first column deleted. \( F(x) \) and \( l_D(x) \) are \((V - 1) \times 1\) vectors. The components of \( F(x) \) are the quantities \( F_\alpha(x), \alpha = 1, \ldots, V - 1 \). The nonvanishing components of \( l_D(x) \) are defined as follows.

If \( x \in [ab] \) with \( a \neq 0 \) and \( b \neq 0 \) then

\[ (l_D(x_{ab}))_a = s_{ab}(\gamma) \sinh(\sqrt{2\gamma} x_{ba}) \]  

(28)

\[ (l_D(x_{ab}))_b = s_{ab}(\gamma) \sinh(\sqrt{2\gamma} x_{ab}) \]  

(29)

If \( x \in [0\mu_j] \) then

\[ (l_D(x_{0\mu_j}))_{\mu_j} = s_{0\mu_j}(\gamma) \sinh(\sqrt{2\gamma} x_{0\mu_j}) \]  

(30)

Performing the integration over \( x \) in (7), we finally get the following simple expression for \( L \):

\[ L = \frac{1}{\sqrt{\gamma(\gamma + \xi)}} \frac{(M^{-1}(\gamma + \xi))_{00}}{(M^{-1}(\gamma))_{00}} \]  

(31)

This is the central result of this paper.
Remark that, for the special case when \( G_0 \) and \( G_D \) are continuous Green’s functions, we can recast (31) in the form

\[
L = \frac{1}{\gamma} \frac{G_0(0, 0; \gamma + \xi)}{G_0(0, 0; \gamma)}
\]

(32)

To go further and deduce from (31) some general properties of the density \( P_t(u) \), let us consider the following identity

\[
\int_0^\infty dt \ e^{-\gamma t} \int_0^t du \ e^{-\xi u} r(u) s(t - u) = R(\gamma + \xi)S(\gamma)
\]

(33)

where \( R \) and \( S \) are the Laplace Transforms of \( r \) and \( s \).

Setting \( R(p) = \frac{(M^{-1}(p))_{00}}{\sqrt{p}} \) and \( S(p) = \frac{1}{\sqrt{p} (M^{-1}(p))_{00}} \) in (33), we realize from (31) and (6) that the probability density factorizes in the following way:

\[
P_t(u) = r(u) s(t - u)
\]

(34)

This result holds for any graph.

It is now easy to analyze the limiting behaviours of \( P_t(u) \) when \( u \to 0^+ \) and when \( u \to t^- \).

In the first case, we can write \( P_t(u) \approx r(u)s(t) \). Moreover, for large \( p \) values, \( M(p) \to 1 \) and \( R(p) \sim \frac{1}{\sqrt{p}} \). So

\[
P_t(u) \sim \frac{1}{\sqrt{\pi u}} s(t) \quad \text{, when } u \to 0^+
\]

(35)

On the other hand, when \( u \to t^- \), \( P_t(u) \approx r(t)s(t - u) \). For large \( p \) values, \( S(p) \sim \frac{1}{\sqrt{p}} \) and finally

\[
P_t(u) \sim \frac{1}{\sqrt{\pi (t-u)}} r(t) \quad \text{, when } u \to t^-
\]

(36)

Let us now develop some examples.

For the graph of Figure 1, we get:

\[
L = \frac{1}{\gamma} \sum_{i=1}^n p_{0i} \ \text{th}(\sqrt{2}\gamma l_{0i}) \sum_{i=1}^n p_{0i} \ \text{th}(\sqrt{2}(\gamma + \xi) l_{0i})
\]

\[
\to \frac{1}{\sqrt{\gamma(\gamma + \xi)}} \quad \text{, when } l_{0i} \to \infty \quad i = 1, \ldots, n
\]

(37)

(38)

This is readily done with the help of the Dirichlet Green’s function:

\[
G_D(x, x'; \gamma) = G_0(x, x'; \gamma) - \frac{G_0(x, 0; \gamma)G_0(0, x'; \gamma)}{G_0(0, 0; \gamma)}
\]

Using completeness and also the relationship

\[
\int_{\text{Graph}} dx' \ G_0(x, x'; \lambda) = \frac{1}{\lambda}
\]

all the integrations in (37) can be performed, leading to (38).
Figure 1: a star-graph with \( n \) legs of lengths \( l_{0i} \), \( i = 1, \ldots, n \), originating from \( O \).

When all the legs become infinite, \( L \) does not depend on the exit probabilities from \( O \), \( p_{0i} \), because in that case all the legs are equivalent. The same behaviour occurs for a finite symmetric star \( (l_{01} = \ldots = l_{0n} = l) \) where we get

\[
L = \frac{1}{\sqrt{\gamma(\gamma + \xi)}} \frac{\text{th}(\sqrt{2\gamma})}{\text{th}(\sqrt{2(\gamma + \xi)l})}
\]

that leads to the density

\[
\mathcal{P}_t(u) = \frac{1}{\pi \sqrt{u(t-u)}} \left( \sum_{p=-\infty}^{+\infty} (-1)^p e^{-\frac{2p^2 l^2}{t-u}} \right) \left( \sum_{q=-\infty}^{+\infty} e^{-\frac{2q^2 l^2}{t-u}} \right)
\]

\[
= \frac{1}{2l^2} \left( \sum_{m=-\infty}^{+\infty} e^{-\frac{(2m-1)^2 \pi^2}{8l^2(t-u)}} \right) \left( \sum_{n=-\infty}^{+\infty} e^{-\frac{n^2 \pi^2}{2t^2}} \right)
\]

This last formula allows for studying the large \( u \) and \( t \) behaviours of \( \mathcal{P}_t(u) \). For instance:

\[
\mathcal{P}_t(u) \sim \frac{1}{l^2} e^{-\frac{\pi^2 (t-u)}{8l^2}} \quad \text{when} \quad u \gg l^2 \quad \text{and} \quad (t-u) \gg l^2
\]

In Figure 2, we consider a ring where the vertices \( O \) and 1 are linked by two bonds of lengths \( l_{01}^{(1)} \) and \( l_{01}^{(2)} \) \( (l = l_{01}^{(1)} + l_{01}^{(2)}) \). The exit probabilities from \( O \) (resp. 1) are \( p_{01}^{(1)} \) and \( p_{01}^{(2)} \) (resp. \( p_{10}^{(1)} \) and \( p_{10}^{(2)} \)) and we define \( c_{01}^{(i)}(\gamma) = \coth \sqrt{2\gamma l_{01}^{(i)}} \), \( \delta_{01}^{(i)}(\gamma) = \frac{1}{\sinh \sqrt{2\gamma l_{01}^{(i)}}}, \ i = 1, 2. \)

Figure 2: a ring with two vertices, \( O \) and 1, linked by two bonds (1) and (2).

Adding intermediate vertices on each bond, we easily show that the matrix \( M \) writes:
\[
M = \left( \begin{array}{cc}
\sum_{i=1}^{2} p_{01}^{(i)} c_{01}^{(i)} - \sum_{i=1}^{2} p_{10}^{(i)} s_{01}^{(i)} \\
- \sum_{i=1}^{2} p_{01}^{(i)} s_{01}^{(i)} + \sum_{i=1}^{2} p_{10}^{(i)} c_{01}^{(i)}
\end{array} \right)
\]

When \( p_{10}^{(1)} = p_{10}^{(2)} = 1/2 \), \( L \) does not depend on \( p_{01}^{(i)} \) and we get:

\[
L = \frac{1}{\sqrt{\gamma(\gamma + \xi)}} \coth(\sqrt{2\gamma l}) + F(\gamma + \xi)
\]

In that case, the ring can be viewed as a star with two legs of equal lengths, \( l/2 \), glued at their endpoints. Thus, (43) is simply (39) where \( l \) has been replaced by \( l/2 \).

Remark, however, that we should obtain a quite different result when taking \( p_{01}^{(1)} = p_{01}^{(2)} = 1/2 \) :

\[
L = \frac{1}{\sqrt{\gamma(\gamma + \xi)}} \coth(\sqrt{2\gamma l}) + F(\gamma + \xi) + F(\gamma)
\]

with

\[
F(\gamma) = \left( p_{10}^{(1)} - p_{10}^{(2)} \right) \frac{\sinh \sqrt{2\gamma \left( \ell_{01}^{(2)} - \ell_{01}^{(1)} \right)}}{\cosh \sqrt{2\gamma l} - 1}
\]

Let us now turn to a brownian motion starting at the root \( O \) of a regular tree of coordination number \( z \) and depth \( n \) (see Figure 3 part a) for \( z = 3 \) and \( n = 2 \). We suppose that all the links have the same length \( l \) and also that

\[
p_{\alpha\beta_i} = \frac{1}{m_\alpha} = \frac{1}{z} \quad \text{for any vertex inside the graph} \quad (46)
\]

\[
= 1 \quad \text{otherwise} \quad (47)
\]

Figure 3: part a) a tree with coordination number \( z = 3 \) and depth \( n = 2 \); part b) the equivalent linear graph with exit probabilities given in (48) and (49).
For the last passage problem in $O$ this tree is equivalent to a $(n+1)$ linear graph with the exit probabilities

$$p_{01} = 1 = p_{n,n-1} \quad (48)$$

$$p_{i,i-1} = \frac{1}{z}, \quad p_{i,i+1} = 1 - \frac{1}{z}, \quad i = 1, \ldots, n-1 \quad (49)$$

With $c(\gamma) = \coth \sqrt{2\gamma}l$ and $s(\gamma) = \frac{1}{\sinh \sqrt{2\gamma}l}$, we can write the relations

$$\det M(\gamma) = c(\gamma) \det M_n(\gamma) - \frac{1}{z} (s(\gamma))^2 \det M_{n-1}(\gamma) \quad (50)$$

$$\det M_{n-k}(\gamma) = c(\gamma) \det M_{n-k-1}(\gamma) + \frac{1}{z} \left( \frac{1}{z} - 1 \right) (s(\gamma))^2 \det M_{n-k-2}(\gamma), \quad 0 \leq k \leq n-3 \quad (51)$$

$$\det M_2(\gamma) = 1 + \frac{1}{z} (s(\gamma))^2 \quad (52)$$

$$\det M_1(\gamma) = c(\gamma) \quad (53)$$

$M_{n-k}$ is the matrix $M$ where the $(k+1)$ first lines and first columns have been deleted. Recursion relations can also be written for $(M^{-1})_{00} \equiv \frac{\det M_n}{\det M}$. In the limit $n \to \infty$, we get:

$$L = \frac{1}{\sqrt{\gamma(\gamma + \xi)}} c(\gamma) \left( 1 - \frac{1}{z} \right) + \frac{1}{\sqrt{2(\gamma + \xi)l}} \frac{K \left( \sqrt{2(\gamma + \xi)l} \right)}{K \left( \sqrt{2\gamma l} \right)} \quad (54)$$

As expected, $z = 2$ leads to the second arc-sine Levy’s law.

As a final example, let us consider an infinite 2D square lattice of stepsize $l$ with exit probabilities equal to $1/4$ in any vertex. A standard computation based on the tight binding model [12] leads to:

$$L = \frac{1}{\sqrt{\gamma(\gamma + \xi)}} \frac{\text{th} \sqrt{2(\gamma + \xi)l}}{\text{th} \sqrt{2\gamma l}} \frac{K \left( \sqrt{\text{cosh} \sqrt{2(\gamma + \xi)l}} \right)}{K \left( \sqrt{\text{cosh} \sqrt{2\gamma l}} \right)} \quad (55)$$

$K(\lambda)$ is a complete elliptic integral of first kind [13].

To complete this work, let us come back to a general graph and choose equal exit probabilities in each vertex:

$$\forall \alpha, \quad p_{\alpha\beta_i} = \frac{1}{m_\alpha}; \quad i = 1, \ldots, m_\alpha \quad (56)$$

In that case, $L$ can be expressed in terms of primitive orbits on the graph. Recall that an orbit $\tilde{C}$ is said to be primitive if it cannot be decomposed as a repetition of any smaller orbit. From [14], we get:

$$\prod_{\tilde{C}} \left( 1 - \mu(\tilde{C}) e^{-\sqrt{2\gamma l}}(\tilde{C}) \right) = \left[ \prod_{[\alpha\beta]} \left( 1 - e^{-2\sqrt{2\gamma l}[\alpha\beta]} \right) \right] \det M(\gamma) \quad (57)$$

The product on the left hand-side is taken over all the primitive orbits $\tilde{C}$ of the graph. $l(\tilde{C})$ is the length of $\tilde{C}$ and the weight $\mu(\tilde{C})$ is the product of all the reflection – or transmission – factors along $\tilde{C}$. (An orbit is a closed succession of arcs $\ldots (\tau\alpha)(\alpha\beta) \ldots$ and a reflection (transmission)
occurs in $\alpha$ if $\tau = \beta$ ($\tau \neq \beta$). The corresponding factors are \( \left( \frac{2}{m_\alpha} - 1 \right) \) for a reflection and \( \left( \frac{2}{m_\alpha} \right) \) for a transmission.

Now, imposing Dirichlet boundary conditions in $O$, we get, still with (14):

$$
\prod_{\tilde{C}} \left( 1 - \mu^{(0)}(\tilde{C})e^{-\sqrt{2\gamma}(\tilde{C})} \right) = \prod_{[\alpha\beta]} \left( 1 - e^{-2\sqrt{2\gamma}(\alpha\beta)} \right) \det M_D(\gamma)
$$

(58)

$\mu^{(0)}(\tilde{C})$ is the same as $\mu(\tilde{C})$ except for orbits visiting $O$ where the reflection and transmission factors are respectively $-1$ and $0$.

It is now easy to realize that the ratio $\frac{\det M_D(\gamma)}{\det M(\gamma)}$ only involves the primitive orbits $\tilde{C}_0$ that visit $O$ at least once. Thus:

$$
(M^{-1}(\gamma))_{00} = \prod_{\tilde{C}_0} \left( 1 - \mu^{(0)}(\tilde{C}_0)e^{-\sqrt{2\gamma}(\tilde{C}_0)} \right) / \prod_{\tilde{C}_0} \left( 1 - \mu(\tilde{C}_0)e^{-\sqrt{2\gamma}(\tilde{C}_0)} \right)
$$

(59)

Finally:

$$
L = \frac{1}{\sqrt{\gamma}(\gamma + \xi)} \prod_{\tilde{C}_0} \left( \frac{1 - \mu^{(0)}(\tilde{C}_0)e^{-\sqrt{2(\gamma + \xi)l}(\tilde{C}_0)}}{1 - \mu^{(0)}(\tilde{C}_0)e^{-\sqrt{2\gamma}(\tilde{C}_0)}} \right) \prod_{\tilde{C}_0} \left( \frac{1 - \mu(\tilde{C}_0)e^{-\sqrt{2(\gamma + \xi)l}(\tilde{C}_0)}}{1 - \mu(\tilde{C}_0)e^{-\sqrt{2\gamma}(\tilde{C}_0)}} \right)
$$

(60)

$\prod_{\tilde{C}_0}$ means that we only consider primitive orbits with reflections in $O$.

As an example, let us see how (60) works with the star-graph of Figure 1 with $n = 2$ legs. Only one primitive orbit, with length $2(l_{01} + l_{02})$ and two reflection factors equal to 1, contributes to the product $\prod_{\tilde{C}_0}$.

Concerning $\prod_{\tilde{C}_0}$, two orbits, with lengths $2l_{01}$ and $2l_{02}$, will contribute. For each one, the two reflection factors are $(-1)$ and 1. For $L$, we get the result:

$$
L = \frac{1}{\sqrt{\gamma}(\gamma + \xi)} \frac{\left( 1 + e^{-2\sqrt{2(\gamma + \xi)}l_{01}} \right) \left( 1 + e^{-2\sqrt{2(\gamma + \xi)}l_{02}} \right)}{\left( 1 + e^{-2\sqrt{2\gamma}l_{01}} \right) \left( 1 + e^{-2\sqrt{2\gamma}l_{02}} \right)} \frac{\left( 1 - e^{-2\sqrt{2(\gamma + \xi)}l_{01} + l_{02}} \right)}{\left( 1 - e^{-2\sqrt{2(\gamma + \xi)}l_{01} + l_{02}} \right)}
$$

(61)

$$
= \frac{1}{\sqrt{\gamma}(\gamma + \xi)} \frac{\cosh \sqrt{2(\gamma + \xi)}l_{01} \cosh \sqrt{2(\gamma + \xi)}l_{02}}{\cosh \sqrt{2\gamma}l_{01} \cosh \sqrt{2\gamma}l_{02}} \frac{\sinh \sqrt{2(\gamma + \xi)}(l_{01} + l_{02})}{\sinh \sqrt{2(\gamma + \xi)}(l_{01} + l_{02})}
$$

(62)

in agreement with (37).

For the ring of Figure 2, two (time-reversed) orbits of length $l$ should contribute to $\prod_{\tilde{C}_0}$ and only one of length $2l$ should contribute to $\prod_{\tilde{C}_0}$. So:
\[ L = \frac{1}{\sqrt{\gamma(\gamma + \xi)}} \frac{(1 - e^{-2\sqrt{(\gamma + \xi)l}})}{(1 - e^{-2\sqrt{\gamma l}})} \frac{(1 - e^{-\sqrt{2\gamma l}})^2}{(1 - e^{-\sqrt{2(\gamma + \xi)l}})^2} \]  

(63)

\[ L = \frac{1}{\sqrt{\gamma(\gamma + \xi)}}\frac{\mathrm{th}\sqrt{2\gamma l}}{\mathrm{th}\sqrt{2(\gamma + \xi)l}}\]  

(64)

as given by (43).

References

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