Bicoloring covers for graphs and hypergraphs

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Monday 5th January, 2015

Abstract

Let the bicoloring cover number $\chi^c(G)$ for a hypergraph $G(V,E)$ be the minimum number of bicolorings of vertices of $G$ such that every hyperedge $e \in E$ of $G$ is properly bicolored in at least one of the $\chi^c(G)$ bicolorings. We establish a tight bound for $\chi^c(K^k_n)$, where $K^k_n$ is the complete $k$-uniform hypergraph on $n$ vertices. We investigate the relationship between $\chi^c(G)$, matchings, hitting sets, $\alpha(G)$ (independence number) and $\chi(G)$ (chromatic number).

We design a factor $O(\log n \log \log n - \log \log \log n)$ approximation algorithm for computing a bicoloring cover. We define a new parameter for hypergraphs - "cover-independence number $\gamma$" and prove that $\log |V| \gamma$ and $\frac{|V|^2}{\gamma}$ are lower bounds for $\chi^c(G)$ and $\chi(G)$, respectively. We show that $\chi^c(G)$ and $\chi(G)$ can be approximated by polynomial time algorithms achieving approximation ratios $\frac{1}{1-t}$ and $2n^t$ respectively, if $\gamma(G) = n^t$, where $t < 1$. We also construct a particular class of hypergraphs called cover-friendly hypergraphs where the gap between $\alpha(G)$ and $\gamma(G)$ is arbitrarily large. Let $m_x$ denote the minimum number of hyperedges such that some $k$-uniform hypergraph $G$ does not have a bicoloring cover of size $x$. We show that $2^{(k-1)x-1} < m_x \leq x \cdot k^2 \cdot 2^{k+1}x+2$. Let the dependency $d(G)$ of $G$ be the maximum number of neighbours of any hyperedge in $G$. We show using a Kolmogorov complexity argument that there is a bicoloring cover of size $x$ for $G$ if $d(G) \leq 2^{(k-1)-3}$. We improve this dependency bound further to $(\frac{2^{(k-3)}}{e} - 1)$ applying the local lemma for demonstrating a bicoloring cover of size $x$.

Keywords: Hypergraph bicoloring, local lemma, probabilistic method, Kolmogorov complexity, approximation

1 Introduction

We define the bicoloring cover number $\chi^c$ for a hypergraph $G(V,E)$ as the minimum number of bicolorings such that every hyperedge $e \in E$ of $G$ is properly bicolored in at least one of the $\chi^c$ bicolorings. Let $X$ be a set of bicolorings $\{X_1, X_2, ..., X_t\}$. Then $X$ is a bicoloring cover for $G$ if for each hyperedge $e$ of $G$, there is an integer $i(e) \in \{1, 2, ..., t\}$, such that $e$ is non-monochromatic with respect to bicoloring $X_{i(e)}$.

Consider the scenario where $n$ doctors can each be assigned one of two kinds of tasks; either he can see patients or perform operations. All doctors are equivalent and can perform
only one of the two tasks in each group. There are \( m \) groups made from this set of \( n \) doctors viz., \( S_1, S_2, \ldots, S_m \), where each group is of size \( k \). Any doctor can be a member of multiple groups. In order to provide proper treatment, all the \( k \) members of no group should be assigned the same task; each group must have at least one doctor seeing patients and at least one doctor performing operations. Given \( n \) doctors and \( m \) groups of doctors, is there a possible allocation of tasks to doctors such that none of the groups has all doctors allocated the same task? This problem can viewed as the hypergraph bicoloring problem for the \( k \)-uniform hypergraph \( G(V, E) \), where \( |V| = n, |E| = m \). Here, the doctors represent vertices, the groups represent \( k \)-uniform hyperedges, and the tasks assigned to doctors represent the two colors for bicoloring vertices. However, there exist hypergraphs that are not bicolorable. For such hypergraphs, it makes sense to use a bicoloring cover with \( \chi^c \) bicolorings. Instead of all \( m \) groups of doctors being deployed simultaneously, we could have a minimum number \( \chi^c \) of deployments, one for each of the bicolorings from a bicoloring cover for \( G(V, E) \). Note that in any of these bicolorings, the same doctor can serve in multiple groups. Observe that if we have to deploy each of the \( m \) group of doctors effectively, then we need at least \( \chi^c \) bicolorings, where each bicoloring yields one shift of duty assignments. The minimum number of shifts required for deploying all the \( m \) groups of doctors, is therefore the bicoloring cover number \( \chi^c(G) \). Throughout the paper, \( G \) denotes a \( k \)-uniform hypergraph with vertex set \( V \) and hyperedge set \( E \), unless otherwise stated. We use \( V(G) \) and \( V \), and \( E(G) \) and \( E \) interchangeably.

### 1.1 Related works

Graph decomposition is a widely studied problem in graph theory. The main idea of the problem is whether a given graph \( G(V, E) \) can it be decomposed into some family of smaller graphs i.e., is there a family of graphs \( \mathcal{H} = \{H_1, ..., H_j\} \) such that (1). \( V(H_i) \subseteq V(G) \) for all \( H_i \in \mathcal{H} \), (2). \( \bigcap_{H_i \in \mathcal{H}} E(H_i) = \emptyset \) and (3). \( \bigcup_{H_i \in \mathcal{H}} E(H_i) = E(G) \). In other words, the family of graphs \( \mathcal{H} \) covers \( G \), or partitions the edge set of \( G \). If such a \( \mathcal{H} \) exists, then splitting \( G \) into \( \{H_1, ..., H_j\} \) is called a \( \mathcal{H} \)-decomposition of \( G \). A kind of decomposition studied requires \( \mathcal{H} \) to a single graph (say \( \{H_1\} \)) and checks if \( G \) can be decomposed into multiple copies of \( H_1 \) with the disjoint intersection condition omitted. Such a decomposition is denoted by \( H_1|G \). The family \( \mathcal{H} \) may consist of paths, cycles, bipartite graphs or matchings. For instance, consider matching decomposition, where in a edge-coloring of \( G \), each color class is a matching. So, coloring edges of \( G \) by \( \chi_e(G) \) colors properly gives the minimum matching decomposition of the graph. Vizing’s theorem \([10]\) states that for all simple graphs \( G \), \( \chi_e(G) \leq \Delta(G) + 1 \). As a result, there is always a matching decomposition of \( G \) into \( \mathcal{H} \) of size \( |\mathcal{H}| = \Delta(G) + 1 \). A \( tK_2|G \) decomposition is splitting \( G \) into multiple copies of \( t \) \( K_2 \)'s i.e., matchings of size \( t \). Bialostocki and Roditty \([2]\) proved that \( 3K_2|G \) if and only if \( 3\|E(G)\| \) and \( \Delta(G) \leq \frac{|E(G)|}{3} \), with a finite number of exceptions. Alon \([1]\) shown that for every \( t > 1 \), if \( |E(G)| \geq \frac{8t^2 - 2t}{t} \), \( tK_2|G \) if and only if \( t\|E(G)\| \) and \( \Delta(G) \leq \frac{|E(G)|}{t} \). Along similar lines, a significant amount of study has been done and there is vast literature for various kinds of decomposition of graphs (see \([3]\)). In this paper, we aim to combine the concepts of decomposition and coloring graphs and hypergraphs.
1.2 Our contribution

We define \( \chi^c(G) \) for a hypergraph \( G(V, E) \) as the minimum number of bicolorings that guarantee every hyperedge \( e \in E \) of \( G \) is properly bicolored in at least one of the \( \chi^c(G) \) bicolorings. In section 2 (i) we derive a tight bound for \( \chi^c(G) \) for the complete \( k \)-uniform hypergraph \( G \), (ii) establish upper bounds for \( \chi^c(G) \) based on matchings and hitting sets of the hypergraph, and, (iii) design polynomial time algorithms for computing bicoloring covers. We also relate \( \chi^c(G) \) with independent sets and chromatic numbers and show that \( \lfloor \log \chi(G) \rfloor \leq \chi^c(G) \leq \lceil \log \chi(G) \rceil \).

In section 3 we present an inapproximability result about the impossibility of approximating the bicoloring cover of \( k \)-uniform hypergraphs on \( n \) vertices, to within an additive factor of \( (1 - \epsilon) \log n \), for any fixed \( \epsilon > 0 \) in time polynomial in \( n \). For a \( k \)-uniform hypergraph \( H(V, E) \), where \( |V| = n \), we show that the bicoloring cover number \( \chi^c(H) \) is \( O\left(\frac{\log n}{\log \log n - \log \log \log n}\right) \)-approximable.

Let \( C = \{C_i|C_i\} \) be a bicoloring cover of \( \chi^c(G) \) bicolorings that cover \( G(V, E) \). Let \( |C| = w, 1 \leq w \leq 2^{n\chi^c(G)} \). Let \( \gamma_i \) be the size of the largest set of vertices that receive the same color in each of the bicolorings in \( C_i \). Let \( \gamma(G) = \max_{1 \leq i \leq w} \gamma_i \) i.e., \( \gamma(G) \) is the size of the largest set of vertices that receive the same color in all the \( \chi^c(G) \) bicolorings in all the \( w \) possible bicoloring covers in \( C \). We call \( \gamma(G) \) the cover-independence number. In section 4 we show that for any \( k \)-uniform hypergraph \( G \), \( \gamma(G) \geq k - 1 \). We relate \( \gamma \) to \( \chi^c \) and \( \chi \) and derive the lower bounds of \( \log [\frac{|V'|}{w}] \) and \( \lceil \frac{|V'|}{2w} \rceil \) for \( \chi^c(G) \) and \( \chi(G) \), respectively. We show that \( \chi^c(G) \) and \( \chi(G) \) can be approximated in polynomial time by ratio factor \( \frac{1}{t-1} \) and \( 2^t \) algorithms respectively, if \( \gamma(G) = n^t \), where \( t < 1 \). We also construct a particular class of hypergraphs called cover-friendly hypergraphs where the gap between \( \alpha(G) \) and \( \gamma(G) \) is arbitrarily large.

Let \( H(V', E') \) be the largest \( k \)-uniform subhypergraph of a \( k \)-uniform hypergraph \( G(V, E) \), where \( V' \subseteq V \), \( E' \subseteq E \), and there is a hyperedge for every subset of \( k \) vertices in \( V' \), i.e., \( E' = \binom{V'}{k} \). We define \( \omega(G) = |V'| \). We prove that for any \( t \geq 1 \), there exist a \( k \)-uniform hypergraph \( G \) where \( \omega(G) = k \) and \( \chi^c(G) > t \). Observe that, for \( k = 2 \) (usual graphs), this result implies that triangle-free graphs can have arbitrarily large bicoloring cover numbers.

In sections 5 and 6 we correlate \( \chi^c(G) \) to the number \( |E| \) of hyperedges, and the dependency \( d(G) \), using probabilistic analysis, the Moser-Tardos algorithm [9], and the Kolmogorov complexity method. Let the dependency \( d(G) \) of \( G \) be the maximum number of neighboring hyperedges of a hyperedge in \( G \). We show that if \( |E| \leq 2^{k-1}x^{-1} \), then a bicoloring cover of size \( x \) can be computed in polynomial time. If \( d(G) \leq 2\left(\frac{2^{k-1}}{x} - 1\right) \), then we can compute a bicoloring cover of size two in randomized polynomial time, using an incremental method for cuts in hypergraphs for the first bicoloring and the Moser-Tardos algorithm [9] for the second bicoloring. We generalize the same algorithm for the computing the bicoloring cover of size \( x \) where \( d(G) \leq 2^{x-1}\left(\frac{2^{k-1}}{x} - 1\right) \). We use an incompressibility argument to obtain a cover of size \( x \) if the dependency \( d(G) \) is upper bounded by \( 2^{x(k-1)-3} \). We relax the dependency further to \( \left(\frac{2^{x(k-1)}}{x} - 1\right) \) for bicoloring covers of size \( x \). We also analyse the expected running time of the corresponding randomized algorithm that computes such bicoloring covers.
2 Preliminaries

We first present a fundamental result that precisely gives the bicoloring cover number for a complete \( k \)-uniform hypergraph of \( n \) vertices in Section 2.1. In Section 2.2 we study bicoloring cover for \( r \)-partite 2-uniform hypergraphs (usual graphs), and establish the relationship between bicoloring cover number, chromatic number and clique number.

2.1 Bicoloring cover for complete \( k \)-uniform hypergraphs

It is worthwhile analysing \( \chi^c \) for the complete \( k \)-uniform hypergraph \( K^k_n \). We establish the following theorem.

**Theorem 1** The bicoloring cover number \( \chi^c(K^k_n) = \lfloor \log \left( \frac{n}{k-1} \right) \rfloor \).

**Proof.** Any bicoloring of vertices of \( K^k_n \) splits the vertex set into two subsets, say \( B_1 \) and \( B_2 \), defining the cut \((B_1,B_2)\) where \( a = |B_1| \). Hyperedges consisting of at least one vertex each from each of the two subsets are properly colored by the bicoloring. So, the hyperedges that remain monochromatic consist of vertices exclusively from \( B_1 \) (\( B_2 \)). Observe that vertices in \( B_1 \) and \( B_2 \) can be bicolored independently in subsequent bicolorings. So, the problem of bicoloring a \( K^k_n \) reduces to the problem of bicoloring \( K^k_i \) and \( K^k_{n-i} \), where \( i = \max(a,n-a) \). Let \( T(n) \) denotes the number of bicolorings required for \( K^k_i \). Observe that \( T(2k-2) = 1 \) because we can arbitrarily split the vertex set into two sets of \( k-1 \) vertices and assign two different colors to the vertices of the two sets. Any hyperedge must have at least one vertex in either set. So, one bicoloring suffices. We have the following recurrence.

\[
T(n) = 1, \text{ for } n \leq 2k-2, \\
T(n) \leq \max\{T(a),T(n-a)\} + 1 = T(\max(a,n-a)) + 1, \text{ for } n > 2k-2,
\]

Replacing \( a = \frac{n}{2} \) and solving the recurrence, we get the desired upper bound.

The colors of the vertices in \( i \)th level of the recursion tree gives the \( i \)th bicoloring, starting with the entire vertex set at level 0 at the root, which is uncolored. After the first bicoloring, the only monochromatic hyperedges consist of vertices of either only \( A \) or only \( B \), but there is no monochromatic hyperedge containing vertices of both \( A \) and \( B \). So, in the subsequent bicoloring, the size of the problem becomes \( \left\lceil \frac{n}{2} \right\rceil \) as the hyperedges consisting of vertices of only \( A \) or only \( B \) becomes independent. Now we repeat the bicoloring algorithm again. The algorithm halts when the problem size is less than or equal to \( k-1 \), and hence no hyperedge can remain monochromatic.

In what follows, we show that \( \chi^c(K^k_n) \geq \lfloor \log \left( \frac{n}{k-1} \right) \rfloor \). Let \( i = \lfloor \log \left( \frac{n}{k-1} \right) \rfloor \). This implies that \( 2^{i-1} < \frac{n}{k-1} \leq 2^i \). Let \( h(i) \) denote the statement \( \chi^c \geq i \) for \( 2^{i-1} < \frac{n}{k-1} \). We establish the lower bound by showing \( h(i) \) holds for arbitrary \( i \). We proceed by strong induction on \( i \). For the base case \( h(1) \) holds because \( 1 < \frac{n}{k-1} \Rightarrow n > k-1 \) and we require at least one bicoloring to cover the edges. Now assume that \( h(j) \) holds for all \( 1 \leq j \leq i \) i.e. \( \chi^c \geq j \) for \( 2^{j-1} < \frac{n}{k-1} \). We show that \( h(i+1) \) holds i.e. \( \chi^c \geq i+1 \) if \( 2^i < \frac{n}{k-1} \). When we do a bicoloring of the complete hypergraph of \( n \) vertices, by pigeonhole principle, one of the color class contains at least \( \frac{n}{2} \) vertices, i.e. strictly greater than \( 2^{i-1} * (k-1) \) vertices. Let the set of vertices of the larger of the two color classes be \( A \). The \( k \)-uniform subgraph induced by vertices of \( A \) is a complete \( k \)-uniform hypergraph, hence by induction hypothesis, at least \( i \) bicolorings are required to the properly color the edges of \( A \). These \( i \) bicolorings combined with the first bicoloring makes \( \chi^c \geq i+1 \). This completes the inductive step. \( \Box \)
2.2 Bicoloring cover for graphs

Graphs are essentially 2-uniform hypergraphs. So, the trivial upper bound for bicoloring cover \( \chi^c(G) \) for any graph \( G(V,E) \), \(|V| = n\), is \([\log n]\) due to Theorem 1. We establish the following result for complete \( r \)-partite graphs.

**Theorem 2** For a complete \( r \)-partite graph \( G \), \( \chi^c(G) = [\log r] \).

**Proof.** Each partite set is a independent set. So all the edges go across the partite sets. Hence coloring every vertex of any partite set with same color does not make any edge monochromatic. Let the partite sets be \( p_1, p_2, \ldots, p_r \). In the first bicoloring, color each vertex of \( p_1 \) to \( p_1[z] \) with color 1 and \( p_1[z] \) to \( p_r \) with color 2. Now all edges running between \( p_1, p_2, \ldots, p[z] \) and \( p[z], p[z] + 1, \ldots p_r \) are properly colored. Only edges still not properly colored either lies inside \( p_1, p_2, \ldots, p[z] \) or \( p[z], p[z] + 1, \ldots p_r \). Hence the problem size is halved and we have the following recurrence. \( T(r) = T([\frac{r}{2}]) + 1 \), where \( T(x) \) denote the number of bicolorings to cover the \( x \) partite sets, \( T(2) = 1 \). Solving the recurrence gives the required upper bound.

Consider the mapping of the \( r \)-partite graph \( G \) to a complete graph \( H \) such that \( V(H) \) consists of exactly \( r \) vertices, one for each partite set, \( E(H) \) is the set of edges which denote existence of a edge between corresponding partite sets in \( G \). Now as \( H \) is complete, \( \chi^c(H) = [\log r] \). This implies \( \chi^c(G) \geq [\log r] \). This completes the proof of Theorem 2. \( \square \)

We also relate \( \chi^c(G) \) to the chromatic number \( \chi(G) \) for a simple undirected graph \( G(V,E) \). For any graph \( G \), let \( \omega(G) \) be the size of the maximum complete subgraph or clique.

**Theorem 3** For any simple undirected graph \( G(V,E) \), \( \log \omega(G) \leq \chi^c(G) \leq [\log \chi(G)] \).

**Proof.** As the complete subgraph of size \( \omega(G) \) has a cover of size \( \log \omega(G) \), the cover for the entire graph is at least of this much size, which proves the first inequality. The second inequality follows from the fact that each color class is a independent set and \( G \) is a subgraph of a complete \( \chi(G) \)-partite graph. \( \square \)

2.3 Bicoloring cover number and chromatic number for \( k \)-uniform hypergraphs

We establish the following result relating \( \chi^c(G) \) and \( \chi(G) \) for \( k \)-uniform hypergraphs.

**Theorem 4** Let \( G(V,E) \) be a \( k \)-uniform hypergraph. Let \( \chi^c(G) \) and \( \chi(G) \) be the bicoloring cover number and chromatic number of \( G \), respectively. Then, \([\log \chi(G)] \leq \chi^c(G) \leq [\log \chi(G)] \).

To show that \([\log \chi(G)] \leq \chi^c(G) \), choose a bicoloring cover \( C \) of size \( \chi^c(G) \) for \( G \). Each vertex \( v \) of \( G \) is assigned a set of \( \chi^c(G) \) colors (bits 0 or 1), by the \( \chi^c(G) \) bicolorings in the bicoloring cover \( C \). Assign the decimal equivalent of the \( \chi^c(G) \)-bit pattern for \( v \) as the color for \( v \) to get a vertex-coloring \( C' \) for \( G \). The total number of colors used is at most \( 2\chi^c(G) \). We claim \( C' \) is a proper vertex-coloring for \( G \), thereby enforcing the inequality \( \chi(G) \leq 2\chi^c(G) \) or \([\log \chi(G)] \leq \chi^c(G) \). For the sake of contradiction, assume that some hyperedge \( e \in E(G) \) is monochromatic under \( C' \). This means in each of the \( \chi^c(G) \) bicolorings, every vertex of \( e \) gets same color. As a result, \( e \) is not covered by the \( \chi^c(G) \) sized cover, which is a contradiction. Consequently, we have the following lemma.

**Lemma 1** For any \( k \)-uniform hypergraph \( G(V,E) \), \([\log \chi(G)] \leq \chi^c(G) \).
To prove the second inequality, consider a proper coloring \( C \) of the vertices of \( G \) with \( \chi(G) \) colors. Construct the bicoloring cover \( X \) of size \( \lceil \log \chi(G) \rceil \) by assigning the vertices with two colors determined by the 0/1 bits of the color they were assigned under proper coloring \( C \); a vertex \( v \) is assigned the \( i \)th bit of the color assigned to it under coloring \( C \) for the 0/1 bicoloring of \( v \) in the \( i \)th bicoloring of the bicoloring cover \( X \), \( 1 \leq i \leq \lceil \log \chi(G) \rceil \). Assume for the sake of contradiction that some \( e \in E(G) \) is not covered under bicoloring cover \( X \). This means every vertex of \( e \) has the same bit vector of length \( \lceil \log \chi(G) \rceil \), and therefore has the same color under coloring \( C \), a contradiction. Consequently, we have the following lemma.

**Lemma 2** For any \( k \)-uniform hypergraph \( G(V,E) \), \( \chi^c(G) \leq \lceil \log \chi(G) \rceil \).

Theorem 4 follows from Lemmas 1 and 2. The following lemma is immediate from Theorem 4.

**Lemma 3** Let \( G(V,E) \) be a \( k \)-uniform hypergraph, and \( \{I_1, I_2, \ldots, I_u\} \) be a partition of the the vertex set \( V \) into independent sets. Then there exist a bicoloring cover for \( G \) of size \( \lceil \log u \rceil \).

### 2.4 Matchings, hitting sets and bicoloring covers for hypergraphs

Let \( G(V,E) \) be a \( k \)-uniform hypergraph, with \( |V| = n \) and \( E = \{E_1, E_2, \ldots, E_m\} \), where \( E_i \subseteq V, 1 \leq i \leq n \). We have the following bounds for \( \chi^c(G) \) based on the size of maximal matchings and hitting sets.

**Theorem 5** For any \( k \)-uniform hypergraph \( G(V,E) \), \( \chi^c(G) \leq \log |M| + 2 \), where \( M \) is a maximal matching of \( G \).

**Algorithm 1** Hypergraph bicoloring cover using a matching

- **Input:** \( k \)-uniform hypergraph \( G(V,E) \) with \( |V| = n \), some maximal matching \( M \).
- **Output:** Set \( X \) of bicolorings of size \( |X| \leq \log_2|M| + 2 \).

1. Color every vertex in the hyperedges of \( M \) with color 1 and rest of the vertices with color 2.
2. **BICOLORCOVER1**\( (M) \).
3. Color the edges of the matching independently using one bicoloring.

**function** **BICOLORCOVER1**\( (M) \)

```text
if (|M| > 1) then
    Split \( M \) two sets \( A, B \) of size \( \lfloor |M|/2 \rfloor \) and \( \lceil |M|/2 \rceil \) respectively.
    Color every vertex in \( A \) with color 1 and every vertex in \( B \) with color 2.
    **BICOLORCOVER1**\( (A) \).
    **BICOLORCOVER1**\( (B) \).
end if
end function
```

**Proof.** In the first bicoloring, we color every vertex in \( M \) with white and all the remaining vertices with the black. As every hyperedge not in \( M \) shares at least one vertex with with some edge in \( M \), every edge not in \( M \) is certainly properly bicolored. So, the only remaining edges that are not properly bicolored in the first bicoloring are those belonging to \( M \). Now in the second bicoloring, we color every vertex of \( \lfloor |M|/2 \rfloor \) hyperedges with black and the rest of the \( \lceil |M|/2 \rceil \) hyperedges with white. So after the first bicoloring, the problem size is \( |M| \) and
the problem size gets halved in each subsequent bicoloring step. So, after \( \log |M| \) bicolorings, the problem reduces to only one hyperedge, which can be bicolored in the next bicoloring. Collecting the colors of the vertices in \( i \)th level of the recursion tree gives the \( (i+1) \)th bicoloring, starting with \( M \) at level 0 at the root, which is uncolored. These \( \log M + 1 \) bicolorings combined with the first bicoloring gives the desired cover. \( \square \)

**Algorithm 2** Hypergraph bicoloring cover using a hitting set

Input: \( k \)-uniform hypergraph \( G(V, E) \) with \( |V| = n \), some hitting set \( H \).
Output: Set \( X \) of bicolorings of size \( |X| = \log_2 \lceil \frac{|H|}{k-1} \rceil + 1 \).

Color every vertex in \( H \) with color 1 and rest of the vertices with color 2.

\( \text{BICOLORCOVER2}(H) \).

function \( \text{BICOLORCOVER2}(H) \)
  if \( |H| > k - 1 \) then
    Split \( H \) two sets \( A, B \) of size \( \left\lfloor \frac{|H|}{2} \right\rfloor \) and \( \left\lceil \frac{|H|}{2} \right\rceil \) respectively.
    Color every vertex in \( A \) with color 1 and every vertex in \( B \) with color 2.
    \( \text{BICOLORCOVER2}(A) \).
    \( \text{BICOLORCOVER2}(B) \).
  end if
end function

**Theorem 6** For any \( k \)-uniform hypergraph \( G(V, E) \), \( \chi^c(G) \leq \log_2 \left\lceil \frac{|H|}{k-1} \right\rceil + 1 \), where \( H \) is a hitting set of \( G \).

**Proof.** In the first bicoloring, we color every vertex in \( H \) with white and all the remaining vertices with the black. So, the only remaining edges that are not properly bicolored in the first coloring are those belonging to \( H \). Now using Theorem [1] we need \( \log \left\lceil \frac{|H|}{k-1} \right\rceil \) more bicolorings to cover the remaining hyperedges. Collecting the colors of the vertices in the \( i \)th level of the recursion tree gives the \( (i+1) \)th bicoloring. These \( \log \left\lceil \frac{|H|}{k-1} \right\rceil \) bicolorings combined with the first bicoloring gives the desired cover. \( \square \)

As the union of vertices of some maximal matching \( M \) gives a hitting set, replacing \( |H| \) by \( |M|/k \), yields the same bound as in Theorem 5. As the effectiveness of the algorithm followed in proof of Theorem 5 depends on the size of the maximal matching, finding the smallest maximal matching is useful.

**3 Approximate bicoloring covers**

Lovász [7] showed that the decision problem of bicolourability of hypergraphs is NP-complete. Feige and Killian [4] showed that if NP does not have efficient randomized algorithms, i.e. \( NP \not \subseteq ZPP \), then there is no polynomial time algorithm for approximating the chromatic number of a \( n \) vertex graph within a factor of \( n^{1-\varepsilon} \), for any fixed \( \varepsilon > 0 \). Using the above result, Krivelevich [6] demonstrated that for any fixed \( k \geq 3 \), it is impossible to approximate the chromatic number of \( k \)-uniform graphs on \( n \) vertices within a factor of \( n^{1-\varepsilon} \) for any fixed \( \varepsilon > 0 \) in time polynomial in \( n \). In what follows, we show that it is impossible to approximate the bi-coloring cover of \( k \)-uniform hypergraphs on \( n \) vertices within a additive factor of \( (1 - \varepsilon) \log n \) for any fixed \( \varepsilon > 0 \) in time polynomial in \( n \). We design approximation algorithms for computing bicoloring covers using the methods developed in [6].
3.1 Inapproximability result for computing $\chi^c(G)$

Let $G(V,E)$ be a $k$-uniform hypergraph, and $\chi^c(G)$ and $\chi(G)$ be the bicoloring cover number and the chromatic number of $G$, respectively. Assume that $G$ has a bicoloring cover of size $x$, i.e. $\chi^c(G) \leq x$. By Theorem\[4\] $\chi(G) \leq 2^x$. Let $R$ be an algorithm that computes a bicoloring cover of size $x$ for graph $G$. If $R$ is a factor $\alpha$ additive approximation algorithm (i.e. for any input instance $G$, $x < \chi^c(G) + \alpha$), then, using $L$, we can design an approximation algorithm for proper coloring that uses $\chi(G) \leq 2^x \leq 2^{\chi^c(G) + \alpha} < \chi(G)2^{x+1}$ colors. But, from (\ref{thm:bicolinear}), we know that no polynomial time algorithm can approximate $\chi(G)$ within a factor of $n^{1-\epsilon}$, for some fixed $\epsilon > 0$. So, setting $2^{x+1} = n^{1-\epsilon}$, we get $\alpha = (1-\epsilon)\log n - 1$. So we have the following theorem.

**Theorem 7** Under the assumption that $NP \not\subset ZPP$, there exist an $\epsilon_0$ such that for any $\epsilon_0 > \epsilon > 0$, no polynomial time algorithm can approximate bicoloring cover number $\chi^c$ within an additive approximation factor of $(1-\epsilon)\log n - 1$.

3.2 An approximation scheme for bicoloring cover

Krivelevich and Sudakov [6] have developed an algorithm $D(G,p)$ for approximating the chromatic number of the hypergraph $G(V,E)$ under the assumption that the chromatic number of $G$ is known in advance i.e $\chi(G) = p$. The algorithm uses two algorithms $C_1$ and $C_2$ which colors the hypergraph $G$ using at most $8n^{1-\frac{1}{k+1}-\frac{1}{k-1}}$ and $\frac{2n}{\log n}$ colors respectively and succeeds if $p \geq \chi(G)$. Since actual value of the chromatic number value is unknown, $D(G,p)$ is run with all possible values of $p$ in range 1 to $|V|$. We can use a similar idea for approximating bicoloring cover. Let $\chi^c(G) = s$. Since $\chi(G) \leq 2^{\chi^c(G)}$, if we pass $p = 2^s$ as input to $D(G,p)$, the algorithms $C_1$ and $C_2$ properly colors the hypergraph $G$ using at most $8n^{1-\frac{1}{k+1}-\frac{1}{k-1}}$ and $\frac{2n}{\log n}$ colors respectively. Since $s$ is actually not known in advance, we can try all possible values of $s$ in range 1 through $\lceil \log \frac{n}{\epsilon} \rceil$ (as this is the upper bound for $s$ according to Theorem\[1\] and choose the minimum value of $s$ for which $D(G,2^s)$ outputs a proper coloring. Now from the above proper coloring, we can get a bicoloring cover using the reduction in proof of Lemma\[2\]. Let $C_{12}$ be the algorithm that (i) takes a $k$-uniform hypergraph $G(V,E)$ and an integer $s$ as input; (ii) runs $D(G,2^s)$ for different values of $s$, and (iii) obtains a bicoloring cover from the proper coloring output of $D(G,2^s)$ and outputs the bicoloring cover. From Lemma\[2\] it is clear that $C_{12}$ produces a bicoloring cover of size $\min \left( \frac{\log(n)}{\log(n+\frac{\log n}{\log \log n})}, \frac{2n}{\log n} \right)$. So the approximation ratio for Algorithm $C_{12}$ is at most $\min \left( \frac{\log(n)}{\log(n+\frac{\log n}{\log \log n})}, \frac{2n}{\log n} \right)$. Note that the first term of the minimization increases with increase in $s$, whereas the second term decreases with increase in $s$. So we choose the value of $s$ that makes both the terms of the same order. Setting $s = \log \left( \frac{\log n}{\log \log n} \right)$, the reduction to bicoloring cover via the proper coloring of $C_1$ produces a cover of size $\log(n) = \log(8n^{1-\frac{1}{k+1}-\frac{1}{k-1}}) = \log(8n^{\frac{\log n}{\log n+\frac{\log n}{\log \log n}}}) \leq \log(8n^{\frac{\log n}{\log n}}) = \log(8n^{\frac{\log n}{\log \log n} + \log n}) = O(\log(n^{\frac{\log n}{\log \log n} + \log n}))$. Size of the cover produced by the reduction to bicoloring cover via the proper coloring of $C_2$ is $\log(\frac{2n}{\log n}) = O(\log(n^{\frac{\log n}{\log \log n}}))$. As a result, $C_{12}$ has a approximation ratio of $\frac{\log n+\log \log n+\log \log n}{\log n+\log \log n+\log \log n} = 1$. Consequently, we get the following theorem.

**Theorem 8** For a $k$-uniform hypergraph $H(V,E)$, $|V| = n$, the bicoloring cover $\chi^c(H)$ is
\[ O(\log n) \approx \log n \] approximable.

### 4 Independent sets and the bicoloring cover number

In this section we study the the relationship between bicoloring cover number, independent sets and related concepts. A set \( I \) of vertices of any hypergraph \( G \) is called an independent set if there is no edge of \( G \) in \( I \), i.e. for any edge \( e \in E(G) \), \( e \not\subseteq I \). The maximum size of any independent set is called the independence number \( \alpha(G) \). Note that \( \chi(G) \geq \frac{|V|}{\alpha} \geq \frac{n}{\alpha} \).

Combined with Lemma 1, we have the following observations.

**Observation 1** For a \( k \)-uniform hypergraph \( G \), \( \chi^c(G) \geq \log \left[ \frac{|V(G)|}{\alpha} \right] \).

**Proposition 1** For a \( k \)-uniform hypergraph \( G \), \( \chi^c(G) \) can be approximated in polynomial time by a ratio factor \( \frac{1}{t-1} \) algorithm if \( \alpha(G) = n^t \), where \( t < 1 \).

**Proof.** The algorithm used in Theorem 6 computes a bicoloring cover of size \( \log \left[ \frac{|H|}{\alpha} \right] + 1 \) in polynomial time. Following Observation 1, we observe that the approximation ratio is at most \( \frac{\log |H|}{\log |V(G)|} + 1 \), which is at most \( \frac{1}{t-1} \) if \( \alpha(G) = n^t \) and \( t < 1 \).

Though the chromatic number \( \chi \) and bicoloring cover number \( \chi^c \) are lower bounded by \( \frac{|V|}{\alpha} \) and \( \log \frac{|V|}{\alpha} \) respectively, this bound may be not useful for cases where there is a single large independent set and all other independent sets are of much smaller size. This is the motivation of the next section, where we wish to obtain partition of vertices into independent sets in a way that reduces the approximation factor drastically for certain classes of hypergraphs.

There can be multiple sets of bicolorings of size \( \chi^c(G) \) that covers \( G \). Let \( w \) be the number of distinct bicoloring covers of size \( \chi^c(G) \), for some \( 1 \leq w \leq 2^{n/\chi^c} \); the set \( C = \{C_1, ..., C_w\} \) be the set of all the bicoloring covers of size \( \chi^c(G) \). Let \( C_i = \{X_1, ..., X_{\chi^c}\} \), \( 1 \leq i \leq w \), be a bicoloring cover of size \( \chi^c(G) \), where \( X_j \) denote the \( j^{th} \) bicoloring of vertices of \( G \). The \( \chi^c \) bicolorings in \( C_i \) assigns a bit vector \( B_v \) of \( \chi^c \) bits to each vertex \( v \in V \), where the \( j^{th} \) bit of \( B_v \) corresponds to the color of \( v \) in the \( j^{th} \) bicoloring \( X_j \). Partition the vertex set into parts \( \{V_1, V_2, ...\} \) such that \( u, v \in V_r \iff B_u = B_v \). Let \( \gamma_i \) be the largest set of vertices that receive the same color in each of the bicolorings (i.e. have the same bit vector) in \( C_i \).

\[
\gamma_i = \max_{1 \leq i \leq w} |V_r|
\]

i.e. \( \gamma_i \) is the size of largest part among \( \{V_1, V_2, ...\} \).

Let \( \gamma(G) = \max_{1 \leq i \leq w} \gamma_i \), i.e. \( \gamma(G) \) is the largest set of vertices that receive the same color in all bicolorings among all possible \( \chi^c \) bicolorings of some \( C_i, C_i \in C \). We call \( \gamma(G) \) as the cover-independence number.

In order to get a better understanding of \( \gamma \), we study the parameter for specific hypergraphs. For a bipartite graph \( G \), there can be multiple ways to split the vertex set such that no edges of the graph lies inside a partite set. Let \( \gamma_i \) be the size of the larger of the two partite sets of \( i^{th} \) bipartition of \( G \) such that no edges of the graph lies inside a partite set. \( \gamma \geq \gamma_i \geq \frac{n}{2} \).

If \( G \) is connected, then there is a unique bipartition and \( \gamma \) becomes equal to the order of the larger partition.
For a complete graph $G$, $\gamma(G)$ is equal to 1. Observe that between any two vertices there is an edge and the bicoloring cover that covers each of the edges must color every pair of vertices with different color in some bicoloring in order to cover the edge. For example, consider the bicoloring covers of a $K_4$. From Fig. 1 it is clear that the largest set of vertices colored with same color in both the bicolorings is 1 in all the three covers $C_1$, $C_2$ and $C_3$, i.e. $\gamma_1 = \gamma_2 = \gamma_3 = 1$, hence $\gamma \geq 1$. To see that $\gamma(K_4) < 2$, observe that if any pair of vertices(say 1 and 2) are colored with the same color in both the bicolorings, then the edge (1,2) remains uncovered by the set of bicolorings. For an odd cycle $G'(V,E), V \{v_1, \ldots, v_n\}, \gamma(G')$ is equal to $\frac{n+1}{2}$ and $\chi^c(G') = 2$. As any odd cycle is non-bicolorable, $\chi^c(G') \geq 2$. Consider a bicoloring $X_1$, where every odd vertex is colored 0, every even vertex is colored 1. Only edge that is not properly colored is $(v_1,v_n)$. A second bicoloring $X_2$, which is exactly same as $X_1$ except the color of $v_n$ which is 1 in $X_2$. $X_2$ properly colors $(v_1,v_n)$: $C = \{X_1, X_2\}$ covers $G'$. So, $\chi^c(G') = 2$. The vertices $\{v_1, v_3, \ldots, v[n-2]\}$ are colored with 0 in both $X_1$ and $X_2$ in $C$. Consequently, $\gamma(G') \geq \frac{n-1}{2}$ (see Fig. 1). As any set of vertices of size greater than $\frac{n-1}{2}$ must contain two consecutive vertices, and as a result at least one edge, $\gamma(G') \leq \frac{n-1}{2}$.

For any bicolorable hypergraph, $\chi^c = 1$, set $C$ consists of all the proper bicolorings of vertices, $\gamma_i$ is the size of the larger of the two color classes of $i^{th}$ proper bicoloring. $\gamma \geq \gamma_i \geq \frac{\delta}{2}$. For example, consider the bicoloring of $H(V,E)$, where $V = \{1, 2, 3, 4, 5\}$, and $E$ consists of all the 3-uniform edges except $\{1, 2, 3\}$ and $\{1, 2, 4\}$ (see Fig. 2). Certainly $H$ is bicolorable with bicolorings $X_1$ and $X_2$: $X_1 = red\{1, 2, 3\}, \ blue\{4, 5\}, X_2 = red\{1, 2, 4\}, \ blue\{3, 5\}$. $\gamma_1 = \gamma_2 = 3$. Coloring any four vertices with same color in a bicoloring does not cover all the edges. Hence $\gamma = 3 \geq \frac{\delta}{2}$.

For arbitrary $k$-uniform hypergraph $G$, $\gamma(G) \geq k - 1$. We can prove the statement by contradiction. Assume that $\gamma(G) = l$ for some $1 \leq l \leq k - 2$. This implies there exists a bicoloring cover $C_l$ such that $\gamma_l = l$ and no bicoloring cover with $\gamma_l = l + 1$. We show that there exists a bicoloring cover $C_j$ with $\gamma_j = l + 1$. Consider $C_l$. Let $V'$ be the set of $l$ vertices and $V''$ be some other set, which are colored with same color in every bicoloring of $C_l$. $|V'| \leq l$. Let us move a vertex $s$ from $V''$ to $V'$ to get another set of bicolorings $C_j$. If we can show that $C_j$ also covers the hypergraph, then it proves that $\gamma_j = l + 1$ and we are done. Any hyperedge that does not contain any vertex from $V'$ and $V''$ is covered by $C_j$ (using the same bicolorings as in $C_l$). If certain hyperedge includes at least one vertex from $V''$ along with $s$, it is also still covered by $C_j$ as $|V''| \leq k - 2$ in $C_l$. If certain hyperedge includes only $s$
Two bicolorings of $V = \{1, 2, 3, 4, 5\}$: $X_1 = \text{red}\{1, 2, 3\}, \text{blue}\{4, 5\}$, $X_2 = \text{red}\{1, 2, 4\}, \text{blue}\{3, 5\}$. $C = \{X_1, X_2\}$ is a bicoloring cover of $K_5^2$. $\gamma(K_5^2) \geq |\{1, 2\}| = 2$. $C_1 = \{X_1\}$ and $C_2 = \{X_2\}$ are two distinct bicoloring covers for $H = K_3^2 \setminus \{\{1, 2, 3\}, \{1, 2, 4\}\}$. $\gamma_1 \geq |\{1, 2, 3\}| = 3$. $\gamma_2 \geq |\{1, 2, 4\}| = 3$. $\gamma(H) \geq \max(\gamma_1, \gamma_2) = 3$.

from $V''$ but includes some vertex $u$ outside $V'$, it is also still covered by $C_j$ as $u$ and $s$ receive different colors in at least one bicoloring of $C_j$. As $|V' \cup s| \leq k - 1$, there is always a $u$ outside $V'$ for every edge in $G$. As these three exhaustive cases covers every edge in $G$, $C_j$ is a cover for $G$, hence $\gamma(G) \geq l + 1$. Consequently, the relation $\gamma(G) \geq k - 1$ follows.

**Theorem 9** For any $k$-uniform hypergraph $G$, $\gamma(G) \geq k - 1$.

But how $\gamma$ is significant to the size of bicoloring cover? Observe that $\gamma$ is an upper bound on any set of number of vertices that receive the same color in all bicoloring. As a result, $\gamma$ can be used to bound number of such sets of vertices that receive the same color in all bicolorings in each of the $C_i$’s as shown in the analysis below.

Consider some minimum bicoloring cover $C$ of a $k$-uniform hypergraph $G$. $C$ splits $V(G)$ into at least $\lceil \frac{|V(G)|}{\gamma} \rceil$ sets $V_1, \ldots, V_{\lceil \frac{|V(G)|}{\gamma} \rceil}$, each of size at most $\gamma$ such that each set receives same color in each of the bicolorings. For any $i, j, i \neq j$, $V_i$ and $V_j$ receives different colors in at least one bicoloring. If for any $i, j, i < j$, there is no hyperedge that shares at least a vertex each with $V_i$ and $V_j$ in $G$, merge the two into $V_i$. Repeat the process till for every $i, j, i < j$, there is at least one hyperedge that shares at least a vertex each with $V_i$ and $V_j$. Let this new bicoloring cover be $C_1$ and let $V_1, \ldots, V_p$ be the partition of the vertices of $G$ due to $C_1$, where $p$ denotes the number of sets in the partition due to $C_1$. Let $C_1 = \{X_1, \ldots, X_{\gamma(G)}\}$, where $X_i$ denote the $i$th bicoloring. By definition of $\gamma$, $|V_i| \leq \gamma$, $1 \leq i \leq p$, so $p \geq \lceil \frac{|V(G)|}{\gamma} \rceil$.

The partitioning in $C_1$ can be mapped to a complete graph $H$ such that $V(H)$ consists of $p$ vertices $\{1, 2, \ldots, p\}$, $p \geq \lceil \frac{|V(G)|}{\gamma} \rceil$, one for each set $V_i$ in $C_1$. $E(H)$ is the set of edges which denote existence of an edge that shares at least a vertex each with corresponding sets $V_i$ and $V_j$ in $G$.

**Proposition 2** $H$ is a complete graph.

**Proof.** According to the definition of $C_1$, for every $i, j, 1 \leq i < j \leq p$, there is at least one hyperedge that shares at least a vertex each with $V_i$ and $V_j$. So, for every $i, j, 1 \leq i < j \leq p$, there is an edge between vertices $i$ and $j$ in $H$. So, the proposition follows. $\square$

Now as $H$ is complete, using Theorem 1

$$\chi^c(H) \geq \log \left\lceil \frac{|V(G)|}{\gamma} \right\rceil.$$  (1)
Lemma 4 $\chi^c(H) \leq \chi^c(G)$.

Proof. We show that the bicoloring cover $C1$ that covers $G$ can always be modified into a bicoloring cover $C1'$ for $H$. We construct $C1'$ in the following way:

- For each $X_i \in C1$, add a bicoloring $X_i'$ to $C1'$, $-X_i$ is a mapping of $V(H)$ to $\{0, 1\}$.
- Assign the color of vertices of $V_i$ in $X_i$ to vertex $i$ in $X_i'$.

From the above construction, $C1' = \{X_1', ..., X_{\chi^c(G)}', \}$, $|C1'| = \chi^c(G)$. We need to show that $C1'$ is a valid bicoloring cover for $H$. Let $e' = (i, j) \in E(H)$. This implies that there exists an hyperedge $e$ that shares at least a vertex each with $V_i$ and $V_j$. Let $e$ is covered in bicoloring $X_i$ of $C1$. This implies that $V_i$ and $V_j$ are colored with different colors in $X_i$. Consequently, $i$ and $j$ are colored with different colors in $X_i'$, which covers $e'$. So $C1'$ is a valid bicoloring cover for $H$ and $\chi^c(H) \leq |C1'| = \chi^c(G)$.

Using Lemma 4 and Equation 1, we have the following theorem.

Theorem 10 For a $k$-uniform hypergraph $G$, $\chi^c(G) \geq \log \left( \frac{|V(G)|}{\gamma} \right)$.

Again, since $\chi > 2^\chi - 1$ (using Lemma 2), we have the following lower bound for $\chi$.

Theorem 11 For a $k$-uniform hypergraph $G$, $\chi(G) > \left[ \frac{|V(G)|}{2^\gamma} \right]$.

As the bicoloring cover covers every edge of the graph in some bicoloring, there is no edge between the $\gamma(G)$ vertices in $G$. It is an independent set and hence is less than or equal to $\alpha(G)$, i.e., $\alpha(G) \geq \gamma(G)$. For the hypergraphs where $\alpha(G)$ and $\gamma(G)$ are far apart, inequality given in Theorem 10 gives a much tighter lower bound for $\chi^c(G)$ and, consequently, for $\chi(G)$.

Corollary 1 For a $k$-uniform hypergraph $G$, $\chi^c(G)$ can be approximated in polynomial time by a ratio factor $\frac{1}{t-\delta}$ algorithm if $\gamma(G) = n^t$, where $t < 1$.

Proof. The algorithm used in Theorem 6 computes a bicoloring cover of size $\log \frac{|H|}{\alpha} + 1$ in polynomial time. Following Theorem 10 we observe that the approximation ratio is at most $\log \frac{|H|}{\alpha} - \log (k-1) + 1 = \log 2$, which is at most $\frac{1}{t-\delta}$ if $\gamma(G) = n^t$ and $t < 1$.

Corollary 2 For a $k$-uniform hypergraph $G$, $\chi(G)$ can be approximated in polynomial time by a ratio factor $2n^t$ algorithm if $\gamma(G) = n^t$, where $t < 1$.

Proof. Using Observation 1, we can compute a bicoloring cover of size $\frac{1}{t-\delta} \chi^c(G)$ for any hypergraph $G$ with bicoloring cover number $\chi^c(G)$. Using Lemma 1, we can compute a proper coloring of $G$ using $2^{\frac{1}{t-\delta} \chi^c(G)}$ colors. If $2^{\frac{1}{t-\delta} \chi^c(G)} \leq n$, then we follow the algorithm used in Theorem 6 to compute a bicoloring cover and get a proper coloring with less than or equal to $n$ colors; otherwise, we can use algorithm used in proof of Theorem 1 to compute a cover of $\left[ \frac{n}{n^t} \right]$ size and then compute a proper coloring of $G$ using $2^{2t n^t} \leq n$ colors. So, $\chi(G) \leq n$. From Theorem 11 $\chi(G) > \frac{n}{2^{n^t}} = n^{\frac{1}{2^{n^t}}}$. Consequently, approximation factor for $\chi(G)$ becomes at most $\frac{2n}{2^{n^t}} = 2n^t$.

We now need to show that there exists hypergraphs where there is an arbitrary gap between $\alpha$ and $\gamma$. Consider the 3-uniform hypergraph $G1(V, E), V = \{1, 2, ..., 12\}$. There are two types of edges:

- $E_1 = \{(u, v, w) | 1 \leq u < v < w \leq 8\}$. 

\[ E_2 = \{ (u, v, w) | 1 \leq u < v \leq 8, 9 \leq w \leq 12 \} \setminus \{ \{1, 5, 9\}, \{2, 6, 10\}, \{3, 7, 11\}, \{4, 8, 12\} \}. \]

Let \( E = E_1 \cup E_2 \). \( G_1 \) is certainly non-bicolorable, since it contains a \( K_5^3 \) as a subgraph and from Theorem 1.

\[ \chi^e(G_1) \geq \lceil \log_\frac{5}{3} \rceil = 2. \] \hspace{1cm} (2)

Firstly, we claim that the maximum sized independent sets of \( G_1 \) are \( \{ Z_i = \{ i, 9, 10, 11, 12 \} \} \) \( 1 \leq i \leq 8 \) and hence \( \alpha = 5 \). Observe that any maximum independent set can contain at most two vertices from \( \{1, \ldots, 8\} \), since otherwise it introduces at least one edge \( e \in E_1 \). Assume that \( u < v \) if \( u = v-4 \), then we cannot add any more vertex to that independent set; if \( u = v-4 \), then we can add only one vertex \( v+4 \) to that independent set. This generates an independent set of size 3. If, however, we restrict only one vertex \( u \) from \( \{1, \ldots, 8\} \) in the independent set, we can add \( \{9, 10, 11, 12\} \) to the independent set, which generates the independent set \( \{u, 9, 10, 11, 12\} \).

Consider the bicolorings of vertices:

- \( X_1 = \{0, 0, 1, 0, 0, 1, 1, 0, 1, 1\} \).
- \( X_2 = \{0, 1, 0, 1, 0, 1, 0, 1, 0, 1\} \), where \( j^{th} \) entry in \( X_i \) denote the color of the vertex \( j \) in \( i^{th} \) bicoloring.

Each vertex \( j, 1 \leq j \leq 12 \), receives a pair of bits \( (b_{j,2}, b_{j,1}) \), where \( b_{j,2} \) and \( b_{j,1} \) denote the color of \( j \) in \( X_2 \) and \( X_1 \) respectively. We claim \( C_1 = \{X_1, X_2\} \) is a bicoloring cover for \( G_1 \). In order to prove \( C_1 \) is indeed a bicoloring cover, split \( V \) into partition \( \{V_1, V_2, V_3, V_4\} \) such that: vertex \( j \) is added to part \( V_1, V_2, V_3 \) or \( V_4 \) if \( j \) receives bits \( (0, 0), (0, 1), (1, 0), (1, 1) \), respectively. So \( V_1 = \{1, 5, 9\}, V_2 = \{2, 6, 10\}, V_3 = \{3, 7, 11\}, V_4 = \{4, 8, 12\} \). Note that each \( V_r, 1 \leq r \leq 4, \) is an independent set and has a distinct bit vector associated with it due to \( C_1 \).

Every hyperedge of \( G_1 \) consists of vertices from at least two partitions. Consider a hyperedge \( e = \{u, v, w\}, 1 \leq u \leq v \leq w \leq 12 \). Without loss of generality, let \( u \in V_1 \) and \( v \in V_2 \). Since \( u \) and \( v \) receive different colors in \( X_1 \), \( e \) is covered by \( C_1 \). So \( C_1 \) is a valid bicoloring cover for \( G_1 \) and \( \chi^e(G_1) \leq |C_1| = 2 \). Combined with Equation 2 we get, \( \chi^e(G_1) = 2 \). So \( C_1 \) is a bicoloring cover of minimum size and from definition, \( \gamma_1 = \max(|V_r|, 1 \leq r \leq 4) = 3 \).

We claim that \( \gamma \) is also equal to 3. Note that each of the parts are independent; independent sets of size greater than equal 4 for \( G_1 \) are obtained only by adding at most one vertex from \( \{1, \ldots, 8\} \) to the set \( \{9, 10, 11, 12\} \). Let this set be \( V_0 \). The hypergraph induced by the vertices \( V \setminus V_0 \) (say \( G_1' \)) is either a \( K_5^3 \) or \( K_3^3 \). In order to partition the remaining vertices into independent sets, note that the size of any independent set in a \( K_5^3 \) or \( K_3^3 \) is at most 2. So we need at least \( \lceil \frac{4}{2} \rceil = 4 \) independent sets. These parts combined with \( V_0 \) gives rise to a partitioning with 5 distinct parts and each part needs a distinct bit pattern, which needs \( \lceil \log 5 \rceil = 3 \) bits and, consequently, 3 distinct bicolorings. So if \( \gamma > 3 \), then the bicoloring cover producing the independent set of size 3 is not minimum, which contradicts the definition of \( \gamma \). So, \( \gamma(G_1) = 3 \). Hence, \( \alpha(G_1) = 5 > 3 = \gamma(G_1) \).

In order to give an general construction,

- choose a \( t, 0 < t < 0.5 \).
- choose a \( n \) such that \( \lceil \log n^{1-t} \rceil < \lceil \log n^{1-t} + 1 \rceil \) and \( n^t \) is an integer.
We generate a $n$ vertex $k$-uniform hypergraph $G(V, E)$ with $\alpha(G) > n^{1-t}$, $k = n^t$ and $\gamma = n^t$ in following way:

- partition the vertices of $V = \{1, ..., n\}$ into $n^{1-t}$ parts $\{V_1, ..., V_{n^{1-t}}\}$, such that the $i^{th}$ vertex is placed in $V_{(i-1) \mod n^{1-t} + 1}$. Since $n^{1-t}$ divides $n$, $1 \leq r \leq n^{1-t}$, $|V_r| = n^t$.

- $E_1 = \{(u_1, ..., u_k)|n^{1-t} + 1 \leq u_1 < ... < u_k \leq n\}$. $E_2 = \{(u_1, ..., u_k)|1 \leq u_1 \leq n^{1-t}, n^{1-t} + 1 \leq u_2 < ... < u_k \leq n\}$. $E_3 = \{(u_1, ..., u_k)|(u_1 < ... < u_k)\}$ and for a fixed $r, 1 \leq r \leq n^{1-t}, (u_1, ..., u_k) \in V_r \}. \ E = E_1 \cup E_2 \setminus E_3$. Since $k = n^t$, each $V_r$ is an independent set.

Firstly, we claim that the maximum sized independent sets of $G$ are set $\{\{1, ..., n^{1-t}\} \cup \{i_1, ..., i_{k-2}\}|n^{1-t} + 1 \leq i_1 < ... < i_{k-2} \leq n\}$ and hence $\alpha(G) = n^{1-t} + n^t - 2$. Observe that any maximum independent set cannot contain at most $k - 1$ from $\{n^{1-t} + 1, ..., n\}$, since otherwise it introduces at least one edge $e \in E_1$. Assume that $u_1, ..., u_{k-1}$ any $k - 1$ vertices from $\{n^{1-t} + 1, ..., n\}$ that are in some maximum independent set. If every $u_j, 1 \leq j \leq k - 1$, belongs to the same part $V_r$, $1 \leq r \leq n^{1-t} - 1$, then we can add at most one more vertex $v, 1 \leq v \leq n^{1-t}$, that belongs to the same $V_r$ to the independent set; otherwise, we cannot add any more vertex to that independent set. This generates an independent set of size $n^t$. If, however, we restrict only $k - 2$ vertices $u_1, ..., u_{k-2}$ from $\{n^{1-t} + 1, ..., n\}$ in the independent set, we can add $\{1, ..., n^{1-t}\}$ to the independent set, which generates the independent set $\{i_1, ..., i_{k-2}, 1, ..., n^{1-t}\}$.

$G$ has a complete $k$-uniform subgraph induced by vertices $\{n^{1-t} + 1, ..., n\}$ due to hyper-edges of $E_1$. So, using \[1\]

$\chi^c(G) \geq \lceil \log\frac{n - n^{1-t}}{k - 1} \rceil = \lceil \log\frac{n - n^{1-t}}{n^t - 1} \rceil = \lceil \log n^{1-t} \rceil$. \hspace{1cm} (3)

By construction, $\{V_1, ..., V_{n^{1-t}}\}$ is a partition of $V$ into independent set. So, using Lemma \[3\] $G$ has a bicoloring cover of size $\lceil \log (n^{1-t}) \rceil$, i.e.

$\chi^c(G) \leq \lceil \log n^{1-t} \rceil$. \hspace{1cm} (4)

From Equations \[3\] and \[4\] it is clear that the set of bicolorings that partitions $V$ into $\{V_1, ..., V_{n^{1-t}}\}$ is a bicoloring cover of optimal size. Let this bicoloring cover be $C_i$. From definition, $\gamma_i = \max\{|V_r| \leq r \leq n^{1-t}\} = n^t$.

Claim 1 $\gamma(G) = n^t$.

Note that each of the parts in the partitioning of $V$ are independent sets, independent sets of size greater than equal $n^t$ for $G$ are obtained only by adding at most $k - 2$ vertices from $\{n^{1-t} + 1, ..., n\}$ to the set $\{1, ..., n^{1-t}\}$. Let this set be $V_0$. $|V_0| \leq n^{1-t} + n^t - 2$. The hypergraph induced by the vertices $V' = V \setminus V_0$ (say $G'$) is complete $k$-uniform graph with $V' \geq n - n^{1-t} = n^t + 2$. In order to partition the remaining vertices into independent sets, note that the size of any independent set in a $G'$ is at most $n^t - 1$. So with $(n^{1-t} - 1)$ independent sets, we can cover at most $(n^t - 1) * (n^{1-t} - 1) = n - n^{1-t} - n^t + 1 < V'$ vertices. So, we need at least $n^{1-t}$ independent sets to partition $V'$ into independent sets. These parts combined with $V_0$ gives rise to a partitioning with $n^{1-t} + 1$ distinct parts and each part needs a distinct bit pattern, which needs $\lceil \log n^{1-t} \rceil + 1 > \lceil \log n^{1-t} \rceil = \chi^c(G)$ bits and, consequently, $\lceil \log n^{1-t} \rceil$
distinct bicolorings. So if $\gamma > n^t$, then the bicoloring cover producing the independent set of size $\gamma$ is not minimum, which contradicts the definition of $\gamma$. So, $\gamma(G) = n^t$. Hence, $\alpha(G) > n^{1-t} > n^t = \gamma(G)$. From the above discussion, we get the following theorem.

**Theorem 12** There exist $k$-uniform hypergraphs such that the gap between the independence number $\alpha$ and the cover independence number $\gamma$ is arbitrarily large.

As discussed above, there exists $k$-uniform hypergraphs $G(V, E)$ where $\alpha(G) \geq n^{1-t}$ where as $\gamma(G) = n^t$, for some small fraction $0.5 > t > 0$. We call this special class of hypergraphs cover-friendly hypergraphs. For cover-friendly hypergraphs, using Corollary 1 we get an approximation ratio of $\frac{1}{1-t}$ for approximating $\chi^c(G)$ using $\gamma$; but, using Proposition 1 we get an approximation ratio of at least $\frac{1}{t}$ using $\alpha$. Hence we get an improvement for approximation ratio of $\chi^c$ by a factor of at least $\frac{1-t}{1-t}$ using $\gamma$ than $\alpha$. For chromatic number, the approximation factor using $\alpha$ is at least $n^{1-t}$; but, using $\gamma$, the factor becomes $2n^t$. The reduction in approximation factor is at least $\frac{n^{1-t}}{2n^t} = n^{1-2t}$ using $\gamma$ than using $\alpha$. Note that if $t = 0.1$, we get an improvement of approximation factor for $\chi^c$ by 9 times and improvement of approximation factor for $\chi$ by $n^{1-2t}$ times using $\gamma$ over $\alpha$. We state these results as a theorem below.

**Theorem 13** For cover-friendly graph $G(V, E)$, i.e. $\gamma(G) = n^t$ and $\alpha(G) \geq n^{1-t}$, there is an improvement by a factor of at least $\frac{1-t}{1-t}$ and $n^{1-2t}$ using $\gamma(G)$ over $\alpha(G)$ for approximation of $\chi^c(G)$ and $\chi(G)$ respectively.

### 4.1 Clique number

Let $H(V', E')$ be the largest sub-hypergraph of $G$, $V' \subseteq V, E' \subseteq E$, such that between every $k$ vertices, there exists a hyperedge, i.e. $|E'| = \binom{|V'|}{k}$. Let $\omega(G) = |V'|$. Theorem 1 implies that $\chi^c(G) \geq \lceil \log \left( \frac{\omega(G)}{k} \right) \rceil$. But, in fact, $\chi^c(G)$ can be arbitrarily apart from $\omega(G)$. The analysis is similar to existential proof of triangle free graphs of arbitrary chromatic numbers (see [8]). From Theorem 1 we know that $\chi^c(G) \geq \log(\frac{|V|}{t})$. So if we can show that $\log(\frac{|V|}{t}) > t$, then $\chi^c(G) > t$ automatically holds. The condition $\log(\frac{|V|}{t}) > t$ is always satisfied if we can ensure that there is no independent set of size $\frac{|V|}{2^t}$. We use this fact in the proof of the following theorem.

**Theorem 14** For any $t \geq 1$, there exist a hypergraph with $\omega < k + 1$ and $\chi^c > t$.

**Proof.** We choose a random $k$-uniform graph $G$ from the standard $G_{n,p}$ model with $p = n^{-\frac{t}{k+1}}$. We need to show $\chi^c > t$; using Theorem 4 this condition can be enforced by proving that $G$ contains no independent set of size $\left\lceil \frac{n}{2^t} \right\rceil$. We will show a stronger claim that $G$ does not contain any independent set of size $\left\lceil \frac{kn}{(k+1)^2} \right\rceil$ with high probability.

Let $C_l$ denote the number of independent sets of size $\left\lceil \frac{kn}{(k+1)^2} \right\rceil$ in $G$. Let $F$ be some set of $\left\lceil \frac{kn}{(k+1)^2} \right\rceil$ vertices. The probability that $F$ is an independent set is $(1-p)^{\binom{n}{\frac{kn}{(k+1)^2}}}$. The expectation $\mathcal{E}(C_l)$ is just the sum of the above probability over all possible $\left\lceil \frac{kn}{(k+1)^2} \right\rceil$ sets of $G$.

\[
\mathcal{E}(C_l) = \left(\binom{n}{\frac{kn}{(k+1)^2}}\right)(1-p)^{\binom{n}{\frac{kn}{(k+1)^2}}} \leq \left(\frac{n}{\frac{kn}{(k+1)^2}}\right)(1-p)^{\binom{n}{\frac{kn}{(k+1)^2}}} < 2^n e^{-p\frac{kn}{(k+1)^2}} \cdot \frac{2^n e^{-p\frac{kn}{(k+1)^2}}}{(k+1)^2} = 2^n e^{-p\frac{kn}{(k+1)^2}} < \frac{1}{2}, \text{ for } n \geq \frac{(k+1)^2 \log 2 + 2}{k}.
\]

Now using Markov’s inequality, $P(C_l > 0) \leq \mathcal{E}(C_l) < \frac{1}{2}$ for large values of $n$. Next we need to show that the probability of any complete subgraph of order $k + 1$ is small. Let $C_\omega$ denote the number of complete subgraph of
order \( k+1 \) in \( G \). Let \( W \) be some set of \( k+1 \) vertices. \( W \) is a complete subgraph with probability \( p^{k+1} \). The expectation \( E(C_\omega) \) is given by \( E(C_\omega) = \binom{n}{k+1} p^{k+1} \). Again using Markov’s inequality, \( P(C_\omega \geq \frac{1}{\omega+1}) < \frac{1}{\omega+1} \). Since \( P(C_\omega > 1) + P(C_\omega \geq \frac{1}{\omega+1}) < 1 \), there exists some graph \( G \) for which \( C_\omega = 0 \) and \( C_\omega < \frac{1}{\omega+1} \). Now from each of the \( k+1 \) sized complete subgraphs, remove one vertex each to remvoe that complete subgraph. This transformation results in a subhypergraph \( G'(V', E') \) of \( G \) such that \( G' \) does not contain any \( k+1 \) sized complete subgraphs and \( |V'| \geq n - \frac{n}{k+1} = \frac{kn}{k+1} \). Again \( G' \) does not contain any independent set of size \( \lceil \frac{kn}{(k+1)^2} \rceil = \lceil \frac{|V'|}{2} \rceil \) and so \( \chi'(G') > t \).

5 Bicoloring cover numbers for sparse hypergraphs

In a random bicoloring a \( k \)-uniform hyperedge is rendered monochromatic with probability \( \frac{2}{2^k} = 2^{-(k-1)} \). So if the number of hyperedges \( |E| \) is less than or equal to \( 2^{k-2} \), the probability that some hyperedge is monochromatic is upper bounded \( \frac{2^{k-2}}{2^k} < \frac{1}{2} \). As the probability that none of the hyperedges are monochromatic exceeds \( \frac{1}{2} \), we have a algorithm as follows: randomly color the vertices and check if all the hyperedges are properly colored. If so, then, we have a bicoloring. Otherwise repeat the bicoloring step. We can easily verify the expected steps of failure is less than 2. The same argument gives the following relation between number of edges and cover size.

Theorem 15 A \( k \)-uniform hypergraph \( G(V, E) \) with \( |E| \leq 2^{(k-1)x-1} \) has a bicoloring cover of size \( x \).

Proof. As as all the \( x \) bicolorings are independent, the probability that any edge becomes monochromatic in each of the bicoloring is \( \left( \frac{2^{k-2}}{2^k} \right)^x \). Choosing the number of hyperedges \( |E| \leq 2^{(k-1)x-1} \), the probability that any edge becomes monochromatic in each of the \( x \) bicolorings is less than or equal to \( \frac{1}{2} \). Consequently, the hypergraph has a cover of size \( x \) and that can be obtained by random coloring of vertices in expected two iterations.

Let \( m_x \) denote the minimum number of hyperedges in a \( k \)-uniform hypergraph \( G \) such that \( G \) does not have a cover of size \( x \). The significance of this quantity is that \( m_x \) is the function of only \( k \) i.e. \( m_x \) is independent of the number of vertices in the hypergraph. If the number of edges in the \( k \)-uniform hypergraph is less than \( m_x \), then it is certainly has a cover of size \( x \). In other words, for any hypergraph of size less than \( m_x \), there exist at least one set of \( x \) bicolorings of vertices that properly colors every hyperedge of the hypergraph. However, if the number of hyperedges is greater than or equal \( m_x \), then we cannot guarantee whether the hypergraph has a cover of size \( x \) or not. Alternatively, there exist at least one \( k \)-uniform hypergraph of size \( m_x \) such that no set of \( x \) bicolorings can properly bicolor every hyperedge in the hypergraph. From Theorem 15 it is obvious that \( m_x > 2^{(k-1)x-1} \). In order to show an upper bound on \( m_x \), we fix the number of vertices as per our requirement and give combinatorial arguments to find the minimum number of hyperedges in any \( k \)-uniform hypergraph which is sufficient to make it uncoverable by any set of \( x \) bicolorings.

We set \(|V(G)| = n = k^2 + k\), fix \( x \) independent bicolorings on \( G \) and pick up \( m \) hyperedges independently. Any bicoloring colors some vertices with color 1 and rest with color 2. Let the set of color 1 vertices has size \( a \). Then total number of monochromatic hyperedges is \( \binom{a}{k} + \binom{n-a}{k} \). This sum is minimized at \( a = \lceil \frac{n}{2} \rceil \). The probability that some hyperedge \( e \) is
monochromatic in one bicoloring is at least \( 2 \left( \frac{k}{n} \right) \geq 2 \sum_{i=0}^{k} \frac{2^{-(n-k+1)} \ldots (n-k+1)}{n(n-1) \ldots (n-k)} > \frac{1}{2^{k-1}} \left( \frac{n-2k}{n-k} \right)^k = \frac{1}{2^{k-1}} \left( 1 - \frac{1}{2} \right)^k \) (since \( \frac{n}{n-1} > \frac{n-1}{n-2} > \ldots > \frac{2-k}{n-k} \)). The probability that \( e \) becomes monochromatic in each of the \( x \) bicolorings is at least \( \left( \frac{1}{2^{k-1}} \left( 1 - \frac{1}{2} \right)^k \right)^x \). Probability that \( m \) independently chosen hyperedges become non-monochromatic in at least one of the \( x \) bicolorings is at most \( \left( 1 - \left( \frac{1}{2^{k-1}} \left( 1 - \frac{1}{2} \right)^k \right)^x \right)^m \). As there are \( 2^{nx} \) ways to do \( x \) independent bicolorings, the probability that none of the \( m \) hyperedges are monochromatic in each of the \( x \) bicolorings is at most \( 2^{nx} \left( 1 - \left( \frac{1}{2^{k-1}} \left( 1 - \frac{1}{2} \right)^k \right)^x \right)^m \). For \( k \geq 2 \), \( \left( 1 - \frac{1}{2} \right)^k \geq \frac{1}{2} \). After replacement, the expression is upper bounded by \( 2^{nx} \left( 1 - \left( \frac{1}{2^{k-1}} \right)^x \right)^m < 2^{nx}e^{-(x+1)/2} \). Replacing \( m \) by \( 2^{(k+1)x} \cdot n \cdot x \ln 2 \), rhs becomes 1 which denotes the existence of some monochromatic hyperedge in each of the \( x \) bicolorings. As \( n = k^2 + k, 2^{(k+1)x} \cdot n \cdot x \ln 2 < 2^{(k+1)x} \cdot 2k^2 \cdot x \cdot 2 = xk^2 2^{(k+1)x+2} \) and we get the following theorem.

**Theorem 16** \( 2^{(k-1)x-1} < m_x \leq x \cdot k^2 \cdot 2^{(k+1)x+2} \).

We note that the notion of 2-coloring can be easily extended to a \( r \)-coloring and on similar lines, it is easy to see that \( r^{(k-1)x-1} < m_x \leq x \cdot k^2 \cdot r^{(k+1)x+2} \).

### 6 Bicoloring cover numbers for hypergraphs with bounded hyperedge dependency

We define **dependency graph** as follows. The **dependency graph** for a set of events \( E_1, E_2, \ldots, E_n \) is a directed graph \( H(V, S) \), where \( V = \{ E_1, E_2, \ldots, E_n \} \), and event \( E_i \) is mutually dependent only on events in the set \( \{ E_j | \{ E_i, E_j \} \in S \} \). The **dependency** of an event \( E_j \) is the number of events, including itself, with which it shares an edge in the dependency graph. Suppose \( E_1, E_2, \ldots, E_n \) is the set of bad events in some probability space \( \Omega \). We wish to know whether it is possible that in a random assignment to **decision variables**, none of the bad events \( E_i \), \( 1 \leq i \leq n \) occur. Here, **decision variables** are assigned randomly and independently, and events are defined over these variables. It may however be the case that some bad events may occur in any random sample of the decision variables. So, we may adopt the strategy of guaranteeing or forcing the non-occurrence of some events, and correct/modify decision variables, over several iterations, so that finally no bad event occurs, as we keep correcting decision variables. Lovász Local lemma ensures existence of proper bicoloring of any hypergraph provided the maximum dependency in the dependency graph is bounded by \( \frac{d^{k-1}}{e} - 1 \). The Kolmogorov complexity proof of Lovász Local lemma achieves this dependency bound upto a constant factor (see [5]). The Moser-Tardos algorithm [9] achieves the exact dependency bound of \( \frac{d^{k-1}}{e} - 1 \). In what follows we use similar techniques for establishing bounds on the maximum dependency of the hypergraph as a function of the size of a bicoloring cover. Firstly, we prove a lemma called the **Dependency Halving Lemma** [5] that directly implies a relation between the size of bicoloring cover and dependency. We subsequently improve the dependency bound further using Kolmogorov and Lovász Local lemma arguments.

#### 6.1 Dependency Halving Lemma

**Lemma 5** Let \( G(V, S) \) be a \( k \)-uniform hypergraph where the dependency of any hyperedge is upper bounded by \( d \). For any bicoloring of vertices of \( G \) (say \( X_1 \)), let \( G_{1[X_1]}(V, S') \) be the
hypergraph such that \( e \in S' \) implies that \( e \in S \) and \( e \) is monochromatic with \( X_1 \). Then, there exists a bicoloring of vertices of \( G \) (let \( X' \)) such that the dependency of any hyperedge in \( G[X'] \) is upper bounded by \( \frac{d}{2} \). Such a bicoloring can be computed in time polynomial in \(|S|\).

For any hyperedge \( S_1 \in S \), let \( \Gamma(S_1) \) be the neighbourhood of hyperedge \( S_1 \), i.e., the set of other hyperedges that share at least one vertex with \( S_1 \). Perform a random bicoloring of vertices of \( G \). Certain hyperedges may become monochromatic in the first random bicoloring, let \( S_1 \) be one such monochromatic hyperedge. Let \( \Gamma_m(S_1) \) be the set of hyperedges in the exclusive neighbourhood of hyperedge \( S_1 \); these edges are monochromatic in the same color as that of vertices in \( S_1 \). \( \Gamma_m^+(S_1) = \Gamma_m(S_1) \cup \{S_1\} \). Let \( \Gamma_p(S_1) \) be all the properly bicolored hyperedges in the neighbourhood of \( S \). Let \( MV = \{S|\Gamma_m(S) > \frac{d}{2} \} \) and \( S_i \in S \).

We establish the following lemma.

**Lemma 6** Let \( G(V,S) \) be a \( k \)-uniform hypergraph with dependency upper bounded by \( d \). Suppose after a single random bicoloring of vertices, there is a monochromatic hyperedge \( S_1 \), such that \( \Gamma_m(S_1) > \frac{d}{2} \). Then, reassigning every vertex \( v \in S_1 \) the opposite color, strictly increases the size of the cut across the two monochromatic sets of vertices.

**Proof.** After the initial random bicoloring of vertices, let us group the vertices colored White as the set \( B_1 \) and the remaining Black vertices as \( B_2 \). So the hyperedges that belong to \( MV \) consist of either only vertices from \( B_1 \) or only vertices from \( B_2 \). Any hyperedge that contains at least one vertex from \( B_1 \) and at least one vertex from \( B_2 \) is properly bicolored. Let \( x \) be the number of hyperedges across the cut \([B_1,B_2] \) i.e., the number of properly bicolored hyperedges. Let \( S_1 \) be some monochromatic hyperedge with \( \Gamma_m(S_1) > \frac{d}{2} \). Let \( |\Gamma_m(S_1)| = \frac{d}{2} + 1 + r \), where \( 0 \leq r \leq d - \frac{d}{2} - 1 \). As \( |\Gamma_m(S_1)| + |\Gamma_p(S_1)| \leq d \), \( |\Gamma_m(S_1)| > |\Gamma_p(S_1)| \).

Observe that moving all the vertices of \( S_1 \) to the other side, i.e., coloring every vertex in \( S_1 \) with the opposite color, increases the number of hyperedges across the cut \([B_1,B_2] \) by at least \( |\Gamma_m(S_1)| - |\Gamma_p(S_1)| \). So, there is a non-zero increase in the cut size during such a correction of any hyperedge \( S_1 \).

It’s important to note that we use \( \Gamma_m(S) > \frac{d}{2} \) instead of \( \Gamma_m^+(S) > \frac{d}{2} \) in the premise of Lemma 6 as the later may not increase the cut after switching. For example, let \( d = 2a + 1 \), \( S \) be some monochromatic hyperedge and \( \Gamma_m(S) = a \). This implies \( \Gamma_p(S) \) also equals \( a \). Hence premise of Lemma 6 is not satisfied. But if we consider \( \Gamma_m^+(S) > \frac{d}{2} \) as our premise, then the premise is satisfied. But if we switch the colors of \( S \), \( \Gamma_m(S) \) and \( \Gamma_p(S) \) still remains \( a \), there is no increase in cut size.

**Algorithm 3** Dependency Halving Algorithm

Input: \( k \)-uniform hypergraph \( G(V,S) \) with dependency \( d \).
Output: a bicoloring \( X_1 \) producing \( G[X_1](V,S') \) with dependency bounded by \( \frac{d}{2} \).

1. Randomly assign color \{White, Black\} to vertices of \( G \).
   
   if \( \forall S, S \in M, \Gamma_m(S) \leq \frac{d}{2} \), i.e. \( MV = \phi \) then
   
   Goto 3.
   
   end if

2. while \( MV \neq \phi \) For some \( S \in MV \)
   
   Assign every vertex \( v_i \in S \) the opposite color.
   
   Update \( MV \).

3. Remove all the properly colored hyperedges from \( G \) to form \( G[X_1] \). Output the colors of vertices as the bicoloring \( X_1 \).
To prove that the Dependency Halving Algorithm 3 actually works, we need to show that Step 2 terminates in the algorithm. Let \( S_1 \in MV \) be chosen for the correction Step 2 of the algorithm. When we move all the vertices of \( S_1 \) to the other side in the correctional step, by Lemma 6, there is a strict increase in the cut across the two monochromatic sets of vertices. As the number of hyperedges across the cut cannot exceed \(|S|\), this process can continue for only a finite number of steps. This completes the proof of Lemma 5.

Lemma 5 directly implies the following upper bound on \( \chi_c(G) \) based on the dependency of the hypergraph.

**Theorem 17** If dependency of any hyperedge in the \( k \)-uniform hypergraph \( G(V, S) \) is less than or equal to \( 2^{x-1}(\frac{2^{k-1}}{e} - 1) \), then \( \chi_c(G) \leq x \), where \( X \) is the set of \( x \) bicolorings.

**Proof.** Using the Dependency Halving Algorithm 3 we can obtain a bicoloring \( X_1 \) that generates a hypergraph \( G_{|X_1}(V, S') \) with maximum dependency upper bounded by \( 2^{x-1}(\frac{2^{k-1}}{e} - 1) \). Repeating the algorithm \( x - 1 \) times, we get a set of bicolorings \( \{X_1, \ldots, X_{x-1}\} \) and a hypergraph \( G_{|X_1, \ldots, X_{x-1}}(V, S) \) with maximum dependency upper bounded by \( \frac{2^{k-1}}{e} - 1 \). Now \( G_{|X_1, \ldots, X_{x-1}}(V, S) \) can be properly colored with a single bicoloring \( X_x \) using the Moser-Tardos algorithm (see [9]).

Within each bicoloring, there is always an increase in cut size during a correctional step. Whenever a hyperedge appears across the cut in any bicoloring, it becomes properly colored. Hence the total number of correctional steps (i.e. switching color of all the vertices of any hyperedge) combined across all the bicolorings cannot exceed the total number of hyperedges. Hence this algorithm runs in polynomial time. We note that the bound can be further improved by direct application of Lovász local lemma (Section 6.3). But the novelty of this approach is that all the steps excluding the last step (that uses randomized correction of [9]) can be made deterministic simply by starting with a particular split of vertices into two partitions and continuing the corrections.

**Algorithm 4** Hypergraph cover algorithm 3

Input: \( k \)-uniform hypergraph \( G(V, S) \) with \( d \leq 2^{x-1} \ast (\frac{2^{k-1}}{e} - 1) \).

Output: Set \( X \) of bicolorings of size less than or equal to \( x \).

While \( d > \frac{2^{k-1}}{e} - 1 \) do

1.1. Randomly assign color \{White, Black\} to vertices of \( G \). \( M_{violate} = \phi \).

if \( \forall S', S'' \in M, \Gamma(m(S')) > \frac{d}{2} \), then

Update \( M_{violate} = M_{violate} \cup S' \).

end if

1.2. If \( M_{violate} \neq \phi \)

for all \( S' \in M_{violate} \)

Assign every vertex \( v_i \in S' \) the opposite color.

Update \( M_{violate} \).

end for

1.3. \( d = \frac{d}{2} \). Remove all the properly colored hyperedges from \( G \).

end while

Perform the random assignment of values \((0, 1)\) to vertices of the remaining hyperedges using Sequential Solver Moser-Tardos algorithm [9].
6.2 Kolmogorov incompressibility for bounding bicoloring cover number

Kolmogorov complexity and the incompressibility theory provide an important tool for relaxing the dependency beyond the existing limits. Lance Fortnow (see [5]) demonstrates a proof of local lemma using Kolmogorov complexity. We use a similar idea for finding small 2-covers for high dependency hypergraphs. A 2-cover is a bicoloring cover with two independent bicolorings. The algorithm for a 2-cover cover goes as follows: "Start with two random bicolorings of vertices of \( V \). Let \( S = \{S_1, \ldots, S_l\} \) be the set of hyperedges monochromatic with both bicolorings. We select a random hyperedge \( S_i \in S \) and do two random bicoloring assignments for each of its vertices by the procedure \( \text{FIX}(S_i) \), one random assignment for each of the bicolorings. If there exists a neighbour \( S_j \) of \( S_i \) such that \( S_j \) becomes monochromatic due to the random assignment of \( S_i \), then we do random assignments for vertices of \( S_j \) by invoking \( \text{FIX}(S_j) \). We repeat until all such \( S_j \)’s are exhausted. We modify \( S \) accordingly. We repeat the same process until \( S = \emptyset \). In order to prove that this algorithm halts, we need to show that the number of \( \text{FIX} \) calls on the set \( S \) is finite and each recursive \( \text{FIX} \) call also terminates.

Let \( n, m \) and \( d \) denote the number of vertices, the number of hyperedges and the maximum allowable dependency, respectively.

Algorithm 5 Hypergraph cover algorithm 2

Input: \( k \)-uniform hypergraph \( G(V, S) \) with \( d' \leq 2^{2k-5} \).

Output: Set \( X \) of bicolorings of size \( |X| = 2 \).

function \( \text{BICOLORINGCOVER}(V, S) \)
{
    Pick two independent random bicolorings of all the vertices, namely \( X_1 \) and \( X_2 \)
    while there exist a hyperedge \( S_i \) that is monochromatic with both \( X_1 \) and \( X_2 \) do
        \( \text{FIX}(S_i) \).
    end while
}
end function

function \( \text{FIX}(S_i) \)
{
    For every vertex \( v \in S_i \), choose two independent random evaluations(color), first one modifies \( X_1 \) and second one modifies \( X_2 \).
    while \( \exists S_j : S_j \cap S_i \neq \emptyset \) and \( S_j \) is monochromatic with both \( X_1 \) and \( X_2 \) do
        \( \text{FIX}(S_j) \).
    end while
}
end function

Assume that \( \text{FIX}(S_i) \) calls made inside the \( \text{BICOLORINGCOVER} \) function always terminates. Let at some intermediate step of algorithm, \( \text{FIX} \) is called on hyperedge \( S_i \) inside \( \text{BICOLORINGCOVER} \) function. When \( \text{FIX}(S_i) \) terminates, all the hyperedges that were covered with \( X_1 \) and \( X_2 \) before this \( \text{FIX}(S_i) \) are still covered and now \( S_i \) is also covered. Hence as there are \( m \) hyperedges, at most \( m \) calls are made to \( \text{FIX} \) inside \( \text{BICOLORINGCOVER} \) function.

Now we need to show that \( \text{FIX}(S_i) \) always terminates. Let us assume that there are exactly \( C \) \( \text{FIX} \) calls during the algorithm(including the recursive ones). We use Kolmogorov complexity(the incompressibility argument) to show that \( C \) is bounded. Let us fix a Kolmogorov
random string $x$ of length $2n + 2Ck$. Let the algorithm uses the bits of $x$ for assignments to vertex(0 and 1 mapped to different colors): first $2n$ bits for the initial assignment to the vertices in $X_1$ and $X_2$ and then $k$ bits each for $X_1$ and $X_2$ in the correctional random evaluations during each of the FIX calls.

Now a alternative description goes like this. There is a log $L$ that records the resampled hyperedges in the order of algorithm execution. Each entry of $L$ looks like $\langle b^1, b^2, S \rangle$, where $b^1$ and $b^2$ denote whether $S$ was monochromatic with 0 or 1 in $X_1$ and $X_2$ respectively.

**Claim 2** There is a one to one correspondence between the log $L$ and the final assignments to the variables with the random bits used during the algorithm execution.

**Proof.** Let log $L$ contains $\{\langle b^1_1, b^2_1, S_1 \rangle, \langle b^1_2, b^2_2, S_2 \rangle, ..., \langle b^1_i, b^2_i, S_i \rangle\}$ at some some instant of the algorithm and we know the values of the variables at this instant. As $S_i$ is in the log, it must be monochromatic with color $b^1_i$ at step $i − 1$ with assignments in $X_1$ and monochromatic with color $b^2_i$ with assignments in $X_2$. So using these values and the values of the variables at step $i$, we can get the values of the variables in step $i − 1$ in both $X_1$ and $X_2$. And continuing in this fashion, we can recover all the $2n + 2k$ random bits used.

In order to get a alternative description, we need to specify the log $L$ effectively, which can be described by the recursion tree, whose nodes are the hyperedges in the log and an edge between two nodes $S_i$ and $S_j$ specify the correctional step in the algorithm in which correctional step in $S_i$ makes $S_j$ monochromatic. In order to specify a call from $S_i$ to $S_j$, we need $\log d$ bits to distinguish $S_j$ among all neighbours of $S_i$ and 1 bit to denote that it is a recursive call downwards, 1 bit each to denote the color of $S_i$ before resampling in $X_1$ and in $X_2$ and while returning from $S_j$ to $S_i$ we need only 1 bit to specify that this is a return edge(termination). So in total we consume $\log d + 4$ bits per single recursive call. In order to specify the hyperedges which are monochromatic after the first random assignment in BICOLOR function, we need $m$ bits, 1 per hyperedge denoting whether it is monochromatic in both $X_1$ and $X_2$ or not. These combined with the final $2n$ values of the variables provide an alternative description of the algorithm, which uses $m + C(\log d + 4) + 2n$ bits.

As $x$ is incompressible,

$$m + C(\log d + 4) + 2n \geq 2n + 2Ck$$

$$\implies C(2k - \log d - 4) \leq m$$

Hence if we restrict $(2k - \log d - 4) > 0$, $C$ value has to be small, as otherwise we get a compression for large values of $C$. Hence $d \leq 2^{2k-5}$. As $C = O(m)$, this algorithm runs in polynomial time in $m$ and $n$. Consequently, we have the following theorem.

**Theorem 18** Let $G(V, S)$ be a $k$-uniform hypergraph with vertex set $V$, $|V| = n$, and hyperedge set $S$, $|S| = m$ with dependency of any hyperedge $d'$ upper bounded by $2^{2k-5}$. Then the hypergraph has a 2-cover and such pair of bicolorings can be obtained in polynomial time in $n$ and $m$.

This can be generalized for a cover $X$ of size $|X|$. Note that instead of two extra bits, additional $|X|$ bits are required for the individual entries of the log $L$. The proof is similar to one in the previous case.
Theorem 19 Let $G(V, S)$ be a $k$-uniform hypergraph with $|V| = n$, and $|S| = m$, with dependency of any hyperedge $d$ upper bounded by $2^{X(k-|X|)-3}$, for some integer $|X|$. Then the hypergraph has a bicoloring cover $X$ of size $|X|$ and such set of bicolorings can be obtained in polynomial time in $n$ and $m$.

6.3 Bicoloring cover number: Local lemma application

The dependencies can be further improved by using an extension of the following original local lemma.

Lemma 7 (Lovász local lemma) Let $G(V, E)$ be the dependence graph of events $E_1, E_2, ..., E_n$ in a probability space. Suppose there exists $x_i \in [0, 1]$ for $1 \leq i \leq n$ such that $P(E_i) \leq x_i \prod_{(E_i, E_j) \in E} (1 - x_j)$, then $P(\bigwedge_{i=1}^n E_i) \geq \prod_{i=1}^n (1 - x_i)$.

Let the bad event $E_i$ correspond to the hyperedge $S_i$ colored becoming monochromatic in each of the $|X|$ independent bicolorings. The probability $p(E_i)$ is at most $1/2k-1/k$. So, by the direct application of local lemma corollary, the maximum allowable dependency $d$ of a hyperedge becomes $2^{X(k-1)} - 1$. Consequently, we have the following theorem.

Theorem 20 If $\forall E_i \in E$, $d \leq 2^{X(k-1)} - 1$, then there exists a $|X|$-cover, which can be obtained in $\frac{1}{2}$ expected resamplings per hyperedge and $\frac{m}{2}$ resamplings in total.

The algorithm differs from the Moser-Tardos algorithm [9] in only the number of random evaluation of variables per bad event: instead of choosing 1 random evaluation of variables for a monochromatic hyperedge, we choose $|X|$ random evaluations for each variable, one each for each of the bicoloring in $X$, for any hyperedge monochromatic with every bicoloring in $X$.

Algorithm 6 Hypergraph cover algorithm 2

Input: $k$-uniform hypergraph $G(V, S)$ with $d \leq 2^{X(k-1)} - 1$.
Output: Set $X$ of bicolorings of size $|X|$.

function SEQUENTIAL-LLL-COVER($V$, $S$)
{
    for $v \in V$ do
        for $i \in \{1, \ldots, |X|\}$ do
            $r_{vi} \leftarrow$ a random evaluation of $v$ in $i^{th}$ bicoloring of $X$.
        end for
    end for
    while $\exists E_i \in E$: $E_i$ happens i.e. every bicoloring in $X$ renders $S_i$ monochromatic do
        Pick a arbitrary violated event $E_i \in E$.
        for $v \in S_i$ do
            for $i \in \{1, \ldots, |X|\}$ do
                $r_{vi} \leftarrow$ a random evaluation of $v$ in $i^{th}$ bicoloring of $X$.
            end for
        end for
    end while
}
end function
As the dependency grows beyond a certain limit, the bicoloring cover size guaranteed by local lemma also increases, but we know that for any $k$-uniform graph $G(V,E)$, $|V| = n$, $\chi_c(G) \leq \lceil \log(\frac{n}{e^{k-1}}) \rceil$. So when the cover guaranteed by local lemma $|X| > \lceil \log(\frac{n}{e^{k-1}}) \rceil$, we can switch to the simpler algorithm given in the proof of Theorem 1 to get a desired cover. Hence the application of this algorithm is practical for the case when it guarantees a cover of size less than or equal to $\lceil \log(\frac{n}{e^{k-1}}) \rceil$. Now we can find the maximum dependency for which this algorithm is applicable by simply replacing $|X|$ in the dependency bound. $d \leq \frac{2^{X((k-1)}}{e} - 1 \leq \frac{1}{2} \lceil \log(\frac{n}{e^{k-1}}) \rceil (k-1) - 1 \implies d \leq \frac{1}{2} (\frac{n}{e^{k-1}})^{(k-1)} - 1$.

7 Concluding remarks

Bounds for bicoloring cover numbers established in this paper are supported by algorithms that generate the bicoloring covers of the corresponding sizes. The algorithms and bounds can be generalized for multicolorings, where more than two colors are used. In such natural extensions to multicolorings, the constraint imposed on every hyperedge is that at least $c > 2$ vertices of the hyperedge must be distinctly colored in at least one of the multicoloring. Throughout the paper, we have used independent bicolorings in our analysis. Whether the use of mutually dependent bicolorings would lead to discovery of better bounds for bicoloring cover numbers, remains an open question. We have shown that for certain classes of hypergraphs, approximating $\chi_c(G)$ and $\chi(G)$ using the cover-independence number $\gamma$ improves the approximation bounds. Finding the exact value of $\gamma$ without exhaustively evaluating every set of $\chi_c(G)$ bicolorings still remains open.

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