CONFORMAL SCALAR PROPAGATION ON THE SCHWARZSCHILD BLACK-HOLE GEOMETRY

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Abstract. The vacuum activity generated by the curvature of the Schwarzschild black-hole geometry close to the event horizon is studied for the case of a massless, conformal scalar field. The associated approximation to the unknown, exact propagator in the Hartle-Hawking vacuum state for small values of the radial coordinate above \( r = 2M \) results in an analytic expression which manifestly features its dependence on the background space-time geometry. This approximation to the Hartle-Hawking scalar propagator on the Schwarzschild black-hole geometry is, for that matter, distinct from all other. It is shown that the stated approximation is valid for physical distances which range from the event horizon to values which are orders of magnitude above the scale within which quantum and backreaction effects are comparatively pronounced. An expression is obtained for the renormalised \(< \phi^2(x) >\) in the Hartle-Hawking vacuum state which reproduces the established results on the event horizon and in that segment of the exterior geometry within which the approximation is valid. In contrast to previous results the stated expression has the superior feature of being entirely analytic. The effect of the manifold’s causal structure to scalar propagation is also studied.

I. Introduction

Any reference to the absence of a preferred physical vacuum state in curved space-times is a statement of fact. An immediate consequence of that fact is that on curved manifolds the Feynman propagator - which in Minkowski space-time relates centrally to the dynamical behaviour of the associated quantum field - ceases to have central significance. Instead, the quantity which primarily reveals information about the dynamical behaviour of quantum fields in curved space-times as well as about the evolution of the space-time geometry is the vacuum expectation value of the stress-energy tensor \(< T_{\mu\nu}(x) >\). Also of importance, in this respect, is the - closely related - quantity \(< \phi^2(x) >\). These two quantities have the capacity to contain such information because they are local constructs. As a general consequence of that fact, although these two quantities depend on the specific choice of vacuum state, their divergence structure at the coincidence space-time limit \( x \to x' \) is independent of that choice at least in the wide class of space-times and boundary conditions in which these quantities admit a singularity structure of the Hadamard form. For the same reason the short-distance behaviour of the propagator at the same limit is just as independent of the choice of the vacuum

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This independence is the underlying reason for the relation between the un-renormalised $\langle T_{\mu\nu}(x) \rangle$ and the Feynman propagator at the coincidence space-time limit \cite{1, 2, 5}. The presence of event horizons in the space-time geometry lends additional importance to $\langle T_{\mu\nu}(x) \rangle$ and to $\langle \phi^2(x) \rangle$, not least, because of the information which they both contain on the evolution of such horizons. The presence of event horizons requires that boundary conditions based on the symmetries and causal structure of the space-time geometry be imposed for a vacuum state to be determined. In general, the choice of such boundary conditions is not unique. For that matter, $\langle T_{\mu\nu}(x) \rangle$ and $\langle \phi^2(x) \rangle$ are themselves defined by the vacuum state which is determined by the choice of such boundary conditions. At once, this choice of vacuum state also characterises the Feynman propagator in that space-time geometry.

It is primarily because of its relation to the stated local expressions that the Feynman propagator remains an essential mathematical construct for the study of vacuum activity on curved manifolds, especially in black-hole space-times. There are, nevertheless, intrinsic difficulties associated with the determination of the propagator on curved manifolds in any vacuum state. Even in the highly symmetric case of the Schwarzschild geometry the dependence of the Green functions on the radial variable is unknown. In turn, several approximation schemes have been devised the primary objective of which has been the development of general expressions for the scalar propagator as a prerequisite for the evaluation of the vacuum expectation value of the stress-energy tensor. Specifically, Candelas \cite{4} has used the complete set of normalised basis functions for a massless scalar field obtained by DeWitt in the static region of the Schwarzschild black-hole space-time in order to develop general expressions for the massless scalar propagator corresponding to the Hartle-Hawking, the Boulware and the Unruh vacuum states respectively. In these expressions the exact dependence on the Schwarzschild radial variable remains unknown. An exception is the expression which he has obtained for the scalar propagator associated with the Hartle-Hawking vacuum state when one end-point of propagation is specified on the bifurcation two-sphere of the event horizon on the maximally extended Schwarzschild manifold. In that case the exact radial dependence becomes explicit at the price of the stated severe restriction. In the context of the WKB approximation Anderson \cite{3, 5} has obtained a general expression of the scalar propagator in static, spherically symmetric space-times. With the exception of its asymptotic behaviour at radial infinity that approximation scheme also features the same lack of information as to the specific dependence of the scalar propagator on any region of the Schwarzschild black-hole geometry. By contrast, Page \cite{6} has obtained the Gaussian approximation to the path integral for the thermal, conformal scalar propagator in static metrics. The expression obtained in that context is explicit in terms of its dependence on the space-time variables but represents the propagator only at the semi-classical limit.

In what follows a distinctive approximation to the conformal scalar propagator associated with the Hartle-Hawking vacuum state - which corresponds to a black hole in thermal equilibrium with black-body radiation - will be developed. The ensuing calculation will eventuate in the analytic expression
\[ D(x_2 - x_1) = \]

\[
\frac{2}{\beta} \frac{1}{\sqrt{p_1 p_2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta_2, \phi_2) Y_{lm}^*(\theta_1, \phi_1) \sum_{p=0}^{\infty} e^{i\frac{2\beta}{\rho_1}(\tau_2 - \tau_1)} \int_{u_0[p]}^{\infty} \frac{du}{\pi^2 u^2 + 4(l^2 + l + 1)} \cos\left[\frac{\pi}{\beta}(4u + 2p + 3)(\rho_2 - \rho_1)\right] - \frac{2}{\beta^2} \frac{1}{\sqrt{\rho_1}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{p=0}^{\infty} \int_{u'_0[p]}^{\infty} \frac{du}{\pi^2 u^2 + 4(l^2 + l + 1)} J_p(\frac{2\beta}{\rho_1} \sqrt{l^2 + l + 1} \rho_2) J_p(2i\sqrt{l^2 + l + 1}) e^{i\frac{2\beta}{\rho_1}(\tau_2 - \tau_1)} Y_{lm}(\theta_2, \phi_2) Y_{lm}^*(\theta_1, \phi_1); \]

\[(I.1) \quad u_0 >> p; \quad u'_0 >> p; \quad \frac{\pi u}{\beta^2 \rho_2} >> p \]

for the conformal scalar propagator in the Euclidean sector of the Schwarzschild black-hole metric. Unlike all other expressions which have hitherto been obtained this result constitutes an explicit expression for the dependence of the propagator on the Schwarzschild black-hole geometry for values of the radial coordinate \( \rho \) much smaller than the value \( \rho = 4M \) which labels the non-trivial boundary of the Riemannian Schwarzschild black-hole manifold. It will be shown that the analytical extension of (I.1) in real time is a valid approximation to the Feynman conformal scalar propagator for values of the Schwarzschild radial coordinate ranging from the event horizon to an upper bound which is several orders of magnitude above the range within which particles are spontaneously created and backreaction effects are pronounced. It will be shown, for that matter, that there is an ample range of physical distances from the event horizon within which - in sharp contrast to all previous results - this Green function, whose space-time dependence is known and is explicitly featured, is a valid approximation to the exact conformal scalar propagator.

An essential advantage of the expression in (I.1) is that it renders the short-distance behaviour and the singularity structure of the propagator manifest. That expression is, for that matter, especially suited for an analytic evaluation of \( < T_{\mu\nu}(x) > \) and \( < \phi^2(x) > \). This aspect will be exploited in order to evaluate the renormalised value of \( < \phi^2(x) > \) in the Hartle-Hawking vacuum state. In the segment of the Schwarzschild black-hole space-time within which the approximation for the conformal scalar propagator is valid the result is

\[ < H|\phi^2(x)|H >_{ren} = \frac{1}{12(8\pi M)^2} \frac{1 - \frac{2M}{r}}{1 - \frac{2M}{r}} - \frac{1}{32\pi M^2} \frac{1}{(1 - \frac{2M}{r})^2} \times \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \int_{u'_0[p]}^{\infty} \frac{du}{\pi^2 u^2 + 4(l^2 + l + 1)} \cos\left[\frac{\pi}{4}(4u + 2p + 3)\left(1 - \frac{\sqrt{1 - \frac{2M}{r}}}{1 - \frac{2M}{r}}\right)\right] I_p(2\sqrt{l^2 + l + 1}) (2l + 1) \]

\[(I.2) \quad u'_0 >> p \]
and will be shown to be in very good agreement - if not in coincidence - with the corresponding result obtained in [12] as well as to reduce identically to the established result on the event horizon obtained in [4]. However, the result in (I.2) is entirely an analytic function of space-time. In this respect, within the stated range of validity, it signifies a better approximation to the renormalised expression of $<\phi^2(x)>$ than that obtained in [12], since the latter also involves a numerical component.

In Section II the necessary physical assumptions will be established and the asymptotic eigenfunctions and eigenvalues to the associated elliptic operator will be evaluated. In Section III the singular part which contains the singularity structure of the thermal scalar propagator will be derived on the basis of the results attained in Section II. In Section IV the finite, boundary part designed to cause the propagator to meet appropriate boundary conditions will be derived. In Section V the validity of the approximation will be analysed and its range will be established. The singularity structure of the scalar propagator will be analyzed and the effect which the causal structure of the Schwarzschild black-hole space-time has on propagation will be explored. Section VI exploits the results established in the previous sections in order to renormalise $<\phi^2(x)>$ on the black hole’s event horizon as well as in the entire region within which the approximation for the propagator in (I.1) is valid. The results will be shown to be in agreement with already established results. Section VII summarises the main results. As an integral aspect of the calculation expressions of divergent infinite series as pole structures are established in the Appendix.

II. Asymptotic Eigenfunctions and Asymptotic Eigenvalues

The Schwarzschild metric is

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The analytical extension $\tau = +it$ of the real-time coordinate $t$ in imaginary values results in a Euclidean, positive definite metric for $r > 2M$. The apparent singularity which persists at $r = 2M$ can be removed by introducing the new radial coordinate $[7]

$$\rho = 4M(1 - \frac{2M}{r})^{\frac{1}{2}}$$

Upon replacing $[3]

$$\beta = 4M$$

the metric in the new coordinates is

$$ds^2 = \rho^2(\frac{1}{\beta^2})d\tau^2 + (\frac{4r^2}{\beta^2})^2d\rho^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

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1The quantity $4M$ is denoted by $\beta$ eventhough it does not correspond to an angle in the Euclidean Schwarzschild geometry. As only the angle $8\pi M = 2\pi \beta$ which corresponds to the temperature of the thermal radiation will be used in this project this notation is not inconsistent.
For the purposes of the calculations herein the Schwarzschild radial coordinate $r$ in (4) will be replaced by the Euclidean radial coordinate $\rho$ through

$$r = \frac{1}{2} \frac{\beta^3}{\beta^2 - \rho^2}$$

obtained by (2). In effect, the metric becomes

$$ds^2 = \frac{1}{\beta^2} \rho^2 d\tau^2 + \frac{\beta^8}{(\beta^2 - \rho^2)^4} d\rho^2 + \frac{\beta^6}{4(\beta^2 - \rho^2)^2} (d\theta^2 + \sin^2 \theta d\phi^2)$$

The coordinate singularity at $r = 2M$ corresponds to the origin $\rho = 0$ of polar coordinates and is removed by identifying $\frac{\tau}{4M}$ with an angular coordinate of period $2\pi$. Equivalently, that coordinate singularity is removed if $\tau$ is identified as an angle of period $8\pi M$. In addition, although the curvature singularity at $r = 0$ can not be removed by coordinate transformations this procedure can be seen to avoid it altogether along with the entire $r < 2M$ region of the Schwarzschild geometry in real time. This procedure results, therefore, in a complete singularity-free positive definite, Euclidean metric which is periodic in the imaginary-time coordinate $\tau$. The period $8\pi M$ of the imaginary-time coordinate $\tau$ is the underlying cause of the thermal radiation at temperature $T = (8\pi M)^{-1}$ emitted by the Schwarzschild black hole.

The topology of the metric in (6) is $\mathbb{R}^2 \times S^2$. It is evident from (2) that the value $\rho = \beta$ is only the label which this coordinate system assigns to spatial infinity $r \to \infty$. As a consequence of (6) the proper distance between the origin and $\rho = \beta$ is infinite. That spatial infinity accommodates a compact non-trivial boundary of topology $S^2 \times S^1$.

The Euclidean Green functions for free fields are the unique solutions of the relevant Green equations which are regular on the Euclidean section and vanish as $r \to \infty$ and, consequently, as $\rho \to \beta$. The Green function of a massless, conformal scalar field is the solution to

$$[-\square_{x_2} + \xi R(x_2)]D(x_1, x_2) = \delta(x_1, x_2)$$

where for the conformally invariant theory it is $\xi = \frac{1}{6}$.

Since on the Schwarzschild geometry it is $R(x) = 0$ this Green function admits the expansion

$$D(x_2, x_1) = \sum_n \frac{\phi_n^*(x_1) \phi_n(x_2)}{\lambda_n}$$

with $\phi_n$ and $\lambda_n$ being respectively the eigenfunctions and eigenvalues of the elliptic operator $-\square$ on the Euclidean section. That is

$$-\square \phi_n = \lambda_n \phi_n$$

The eigenfunctions will be normalised herein by
\( \int d^4x \sqrt{g} \phi_n^*(x) \phi_m(x) = \pi \delta_{nm} \)

which differs from the usual orthonormality condition in that the right side has been scaled by \( \pi \). The reasons for this choice will be provided in what follows.

An immediate consequence of the fact that the Euclidean Schwarzschild black-hole geometry has a non-trivial boundary only at infinite proper distance from the origin is that the spectrum of the elliptic operator is continuous. For that matter, the summation in (8) corresponds essentially to an integral and the right side of (10), to a \( \delta \)-function. There are, in addition, no boundary-related contributions to the propagator for the same reason. However, in order to keep in line with certain assumptions made in [7] for numerical purposes the propagator \( D(x_1, x_2) \) will be accorded the discrete representation in (8) and will be properly replaced by an integral expression at the end of the calculation on the understanding that no boundary-related contributions need be calculated.

Separation of variables in (9) yields the eigenfunctions

\[ \phi_n = 2 \frac{\beta^2 - \rho^2}{\beta^3} \, e^{i \beta \tau} Y_{lm}(\theta, \phi) P_{pln}(\rho) \]

with the radial function satisfying the equation

\[ \frac{1}{\beta^4} \rho^2 (\beta^2 - \rho^2)^4 \frac{d^2 P_{pln}}{d \rho^2} + \frac{1}{\beta^4} \rho (\beta^2 - \rho^2)^3 (\beta^2 - 5 \rho^2) \frac{d P_{pln}}{d \rho} - V(\rho) P_{pln} = -\lambda_n \frac{\rho^2}{\beta^2} P_{pln} \]

where

\[ V(\rho) = (\frac{p}{\beta})^2 + \frac{\rho^2}{\beta^2} [4l(l + 1)(\frac{\beta^2 - \rho^2}{\beta^3})^2 + 4\beta(\frac{\beta^2 - \rho^2}{\beta^3})^3] \]

The parameter \( n \) labels radial eigenvalues. It is discrete if the Euclidean Schwarzschild black-hole manifold is - by assumption - bounded and continuous if that manifold is of infinite volume, as is indeed the case. As the imaginary-time coordinate \( \tau \) has an angular character the parameter \( p \) is

\[ p = 0, \pm 1, \pm 2, \ldots \]

The expressions given above differ from those in [7] in that \( r \) has been replaced by \( \rho \) through (5) and, accordingly, the radial eigenfunction \( R_{pln}(r) \) has been expressed as \( P_{pln}(\rho) \). The resulting

\[ \frac{d^2 P_{pln}}{d \rho^2} + \frac{(\beta^2 - 5 \rho^2)}{\rho (\beta^2 - \rho^2)} \frac{d P_{pln}}{d \rho} - [4l(l + 1)(\frac{\beta^2}{\beta^2 - \rho^2})^2 + \frac{4}{\beta^2 - \rho^2} + \frac{p^2}{\beta^2(\beta^2 - \rho^2)^4}] P_{pln} \]

\[ = -\lambda_n \frac{\beta^8}{(\beta^2 - \rho^2)^4} P_{pln} \]
can be seen to have irregular singular points at $\rho = \pm \beta$ and, as expected [7], is quite intractable.

The evaluation of the conformal scalar propagator in the Hartle-Hawking state commences at this point by developing an approximate solution to (15). To that effect, propagation on the Schwarzschild geometry will be examined in the range $2M < r < r_0$ which signifies a finite region of the geometry’s static segment. The mathematical consistency of this approximation as well as the range which $r_0$ signifies will be analysed and established in section V and, subsequently, reproduced by comparison to existing results in section VI. Equivalently, on the Euclidean section of the Schwarzschild metric the evaluation of the scalar propagator will be made in the context of the assumption

(16) \[ \rho^2 << \beta^2 \]

In effect, (15) reduces to the eigenvalue equation

(17) \[ \frac{d^2 P_{pln}}{d\rho^2} + \frac{1}{\rho} \frac{dP_{pln}}{d\rho} + [\lambda_n - \frac{4}{\beta^2}(l^2 + l + 1) - \frac{\rho^2}{\rho^2}]P_{pln}(\rho) = 0 \]

Imposing the condition of regularity at $\rho = 0$ and expanding in a power series about that regular singular point yields

\[ P_{pln}(\rho) = (c_{0})_{pln} \sum_{k=0}^{\infty} \left( -\right)^k \frac{1}{k!4^k2^p} \frac{1}{\Gamma(p + k + 1)\left[ \sqrt{\lambda_n - \frac{4(l^2 + l + 1)}{\beta^2}} \rho \right]^{2k+p}} \]

with $p = 0, 1, 2, \ldots$.

This series converges for all values of $\rho \in [0, 4M]$ and can be readily recognised as a Bessel function of the first kind. Consequently, the solution to (17) is

(18) \[ (c_{0})_{pln} \frac{2^p\Gamma(p + 1)}{\sqrt{\lambda_n - \frac{4(l^2 + l + 1)}{\beta^2}}} \sum_{k=0}^{\infty} \left( -\right)^k \frac{1}{k!4^k2^p} \frac{1}{\Gamma(p + k + 1)\left[ \sqrt{\lambda_n - \frac{4(l^2 + l + 1)}{\beta^2}} \rho \right]^{2k+p}} \]

(19) \[ P_{pln}(\rho) = (c_{0})_{pln} \frac{2^p\Gamma(p + 1)}{\sqrt{\lambda_n - \frac{4(l^2 + l + 1)}{\beta^2}}} J_p \left( \sqrt{\lambda_n - \frac{4(l^2 + l + 1)}{\beta^2} \rho} \right) \]

Replacing this expression in

(20) \[ \phi_n = \frac{2}{\beta} e^{i\frac{2\pi}{\beta}} Y_{lm}(\theta, \phi) P_{pln}(\rho) \]

- obtained from (11) in the context of (16) - yields

\[ \phi_n(\tau, \rho, \theta, \phi) = \]
These are the eigenfunctions to the elliptic operator $-\Box + \xi R$ on the Euclidean Schwarzschild black-hole geometry in the context of (16).

It should be remarked at this point that the normalisation condition \[ c^2 \int_0^{2\pi} |e^{i\beta \tau}|^2 d\tau = 1 \] can also be independently imposed on the temporal sector. This condition reflects the non-trivial topology of the temporal sector and does suggest that - at least in the context of (16) - (10) is more consistent than the usual orthonormality condition. Indeed, unlike the latter, (10) offsets the effect which the temporal period has and eventuates in a propagator which features the familiar power $\pi^{-2}$ in four dimensions.

It should also be remarked that, as is evident from (17), it is $[\lambda_n] \sim [\rho^{-2}]$. This is consistent with (9) as well as with the quantity which appears in the square root in (21) and shall also test the dimensional consistency of the result which will be obtained for the eigenvalues.

Since the eigenvalues $\lambda_n$ appear as a constant in (19) suitable boundary conditions on some hypersurface will, necessarily, impose restrictions on their acceptable values. Allowing for the mathematically trivial condition of regularity imposed throughout the Euclidean Schwarzschild black-hole geometry the only hypersurface on which physically meaningful boundary conditions can be associated with (17) is that at $\rho = \beta$. Specifically, since the Euclidean Green functions for free fields are required to vanish as $\rho \to \beta$ the exact eigenvalue equation (15) which determines the radial sector of the exact eigenvalue equation (9) must be associated with the boundary condition

\[ \lim_{\rho \to \beta} P_{pln}(\rho) = 0 \]

The eigenvalue equation (17) which emerges from (15) as a result of (16) is not, in principle, physically associated with (23). Mathematically, however, there is no reason that the independent radial variable $\rho$ be restricted. Although the solution $P_{pln}(\rho)$ to (17) is physically relevant only in the context of (16) the differential equation (17) itself is mathematically consistent for $\rho \in [0, \beta]$. If, for that matter, the boundary condition (23) is also imposed on the solution to (17) then the latter becomes a good approximation to the exact radial eigenfunctions which emerge as a solution to (15) also at $\rho = \beta$.

In effect, the boundary condition (23) imposed on the eigenfunctions in (21) is

\[ J_p \left( \sqrt{\lambda_n \beta^2 - 4(l^2 + l + 1)} \right) = 0 \]

This algebraic equation is quite intractable. However, in order to evaluate the propagator only an asymptotic solution for $\lambda_n$ is necessary. This is the case because the propagator
being a Green function - can always be expressed as the sum-total of a singular part and a smooth function which meets the boundary conditions.

The evaluation of the asymptotic expression for the eigenvalues necessitates the asymptotic expression

\[ J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos[z - (\nu + \frac{1}{2})\frac{\pi}{2}] ; |z| >> |\nu|, -\pi < \text{Arg}z < \pi \]

which - by virtue of the asymptotic condition \(|z| >> |\nu|\) - is, at once, a necessary and sufficient condition for the eigenfunctions in (21) to be themselves asymptotic.

As a consequence of (25) equation (24) admits the asymptotic form

\[ \sqrt{\frac{2}{\pi}} \frac{1}{[\lambda_n^2 - 4(l^2 + l + 1)]^{\frac{1}{4}}} \cos[\sqrt{\lambda_n^2 - 4(l^2 + l + 1)} - (p + \frac{1}{2})\frac{\pi}{2}] = 0 \]

for values of \(\lambda_n\) high enough to ensure

\[ \sqrt{\lambda_n^2 - 4(l^2 + l + 1)} >> p \]

so that for these values of \(\lambda_n\) it is

\[ \sqrt{\lambda_n^2 - 4(l^2 + l + 1)} - (p + \frac{1}{2})\frac{\pi}{2} = (n + \frac{1}{2})\pi ; n \in \mathbb{Z} \]

Consequently, the asymptotic solution is

\[ \lambda_n = \frac{\pi^2}{16\beta^2}(4n + 2p + 3)^2 + \frac{4}{\beta^2}(l^2 + l + 1) ; n \in \mathbb{Z} \]

and, as stated, features the expected dimensionality in length units.

It is evident from (29) that \(\lambda_n\) is asymptotically an increasing function of \(n\) and that the condition (27) is trivially satisfied for all values of \(p\) only if

\[ n >> p \]

Consequently, the asymptotic expression (29) essentially reduces to

\[ \lambda_n = \frac{\pi^2}{\beta^2}n^2 + \frac{4}{\beta^2}(l^2 + l + 1) ; n \in \mathbb{Z} \]

This is explicitly an asymptotic expression for the eigenvalues \(\lambda_n\) associated with (17). In fact, as the eigenvalues are independent of the radial variable \(\rho\) expression (31) also yields the asymptotic eigenvalues associated with the exact eigenvalue equation in (15). Consequently, (31) yields the asymptotic eigenvalues associated with the radial sector of the elliptic operator \(-\Box^2 + \xi R\) on the Euclidean Schwarzschild black-hole geometry. In this respect it is worth mentioning that if, in addition to (30), the assumption
is made then (31) results in

\[ \lambda_n = \frac{\pi^2}{\beta^2 n^2}; \quad n \in \mathbb{Z} \]

At the limit of infinite radius this expression coincides up to \(\pi^2\) with that obtained in \([9]\) by placing the black hole in a spherical box. The condition in (32) reveals that the set of asymptotic eigenvalues obtained in \([9]\) is more restricted than that which corresponds to (31).

In order to arrive at an expression for the asymptotic eigenfunctions corresponding to the asymptotic eigenvalues in (29) and (31) the latter must be, respectively, replaced in (21). Such an operation yields

\[ \phi_n(\tau, \rho, \theta, \phi) = (c_0)_{p,n} \frac{2^{p+\frac{3}{2}} \beta^{p-\frac{1}{2}} \Gamma(p+1)}{\sqrt{\pi} \sqrt{\rho}} \frac{1}{\sqrt{\rho}} e^{i\rho Y_{lm}(\theta, \phi)} \times \]

\[
\cos\left(\frac{\pi}{4\beta}(4n + 2p + 3)\rho - \frac{\pi}{2} - \frac{\pi}{4}\right); \quad n >> p; \quad \pi n / \beta >> p; \quad p = 0, 1, 2, ...
\]

on the understanding that the Bessel function in (21) must, itself, be expressed asymptotically through (25). The condition \(\pi n / \beta >> p\) is a direct consequence of the asymptotic condition \(|z| >> |\nu|\) in (25) and will be thoroughly analyzed in what follows.

In summary, the approximate eigenfunctions of the elliptic operator \(\Box + \xi R\) on the Euclidean black-hole geometry are expressed in terms of that operator’s eigenvalues in (21). This approximation is valid in the context of (16). The explicit expression for the asymptotic eigenvalues of the elliptic operator \(\Box + \xi R(x)\) is given in (31). The explicit expression for the asymptotic eigenfunctions of that operator in the context of (16) is given in (34).

III. The Scalar Thermal Propagator - Singular Part

In the range of propagation specified on the Euclidean Schwarzschild black-hole geometry by (16) the evaluation of the conformal scalar propagator will commence from (8). To that effect, it would appear that exact knowledge of the eigenvalues \(\lambda_n\) and eigenfunctions \(\phi_n(x)\) associated with (17) is necessary. This, however, is not the case. From the outset it is clear that the replacement of the asymptotic eigenfunctions and eigenvalues obtained in the preceding section will result in an asymptotic expression \(D_{as}\) for the propagator. That asymptotic expression is necessarily the singular part of the Euclidean Green function in (8). This is the case because the condition in (30) which characterises the asymptotic eigenfunctions and eigenvalues forces the lower bound for index \(n\) in the infinite series in (8) to correspond to some \(n_0 >> 1\), an operation which does not affect the divergence structure which that series has at \(x_2 \to x_1\). The complete Green function is
\[ D(x_2, x_1) = D_{as}(x_2, x_1) + D_b(x_2, x_1) \]

Apparently, away from the coincidence space-time limit \( x_2 \to x_1 \) the sum-total of the infinite number of terms corresponding to \( n = 1, n = 2, \ldots, n = n_0 - 1 \) for each different value of \( p \) respectively yields a finite result. This is also the case at the coincidence space-time limit since the singularity in the Green function is necessarily contained in its asymptotic part \( D_{as} \). Consequently, in the context of (8), the sum-total of the infinite number of terms corresponding to \( n = 1, n = 2, \ldots, n = n_0 - 1 \) for each different value of \( p \) respectively corresponds to the boundary part \( D_b \). In effect, \( D_{as} \) satisfies (7) without (necessarily) satisfying the boundary condition whereas \( D_b \) corresponds to an arbitrary smooth function which satisfies the homogeneous equation associated with (7) and which can be adjusted to enforce upon \( D(x_2, x_1) \) the boundary condition at infinity, \( \rho = \beta \).

The evaluation of the asymptotic expression \( D_{as} \) for the scalar propagator commences with the replacement of (21) into (8) which yields the expression

\[
\frac{4}{\beta^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta_2, \phi_2)Y_{lm}^*(\theta_1, \phi_1) \sum_{p=0}^{\infty} e^{i \frac{2\pi}{\beta_2}(\tau_2 - \tau_1)} \sum_{n=1}^{\infty} |(c_0)_pn_1|^2 \frac{2^{2p}[\Gamma(p + 1)]^2}{\left[ \lambda_n - \frac{4(l^2 + l + 1)}{\beta^2} \rho_2 \right]^{2p}} \times
\]

\[
\frac{1}{\lambda_n} J_p \left( \sqrt{\lambda_n - \frac{4(l^2 + l + 1)}{\beta^2} \rho_2} \right) J_p \left( \sqrt{\lambda_n - \frac{4(l^2 + l + 1)}{\beta^2} \rho_1} \right)
\]

(36)

This is the complete propagator associated with (17) expressed in terms of the eigenvalues \( \lambda_n \) to \(-\Box\).

In order to arrive at the explicit asymptotic expression for \( D(x_2, x_1) \) it is necessary to evaluate the coefficients \((c_0)_pn_1\). Such a task necessitates use of (10). Since the operator associated with (17) is self-adjointed the orthonormality relation (10) also applies to the eigenfunctions associated with (17). Replacing (34) in (10) yields in the asymptotic range given in (30)

\[
|\left(c_0\right)_pn_1|^2 \frac{2^{2p+1}[\Gamma(p + 1)]^2}{\left[ \pi \beta \right]^{2p}} \times
\]

(37) \[
\int_0^{2\pi} d\phi \int_0^{\pi} d\sin \theta |Y_{lm}(\theta, \phi)|^2 \int_0^{\beta} d\rho \left[ J_p \left( \frac{\pi}{4\beta}(4n + 2p + 3) \rho \right) \right]^2 = 1 ; \ n >> p
\]

where use has been made of the invariant-integration measure associated with the metric in (6).

The double angular integral in (37) is equal to unity on account of the orthonormality relation.
\[
\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta Y_{lm}'(\theta, \phi)Y_{lm}'(\theta, \phi) = \delta_{l'l}\delta_{mm'}
\]

The evaluation of the radial integral in (37) necessitates the relation 

\[
\int_0^a J_{\nu}(a\rho)\frac{\rho}{a}J_{\nu}(a\rho)\rho d\rho = \frac{1}{2}a^2[J_{\nu+1}(a\rho)]^2 \delta_{mm}
\]

which, as a result of (25), yields

\[
\int_0^\beta \rho d\rho \left[ J_p\left(\frac{\pi}{4\beta}(4n + 2p + 3)\rho\right)\right]^2 = \frac{\beta^2}{\pi^2 n}; \quad n >> p
\]

Replacing (40) in (37) yields

\[
|c_0|_{pnn}|^2 = \frac{2\pi[\pi n]^{2p+1}}{[2\beta]^{2p+2}[\Gamma(p+1)]^2}; \quad n >> p
\]

The asymptotic range for the scalar propagator is expected in the context of the same condition \(n >> p\). The lowest-order term \(n_0\) in the series over \(n\), for that matter, depends on \(p\). Using (29), (31) and (41) in (36) the asymptotic propagator is

\[
D_{as}(x_2, x_1) = \frac{2\pi^2}{\beta^2} \sum_{l=0}^\infty \sum_{m=-l}^l Y_{lm}(\theta_2, \phi_2)Y_{lm}^*(\theta_1, \phi_1) \times \sum_{p=0}^\infty e^{i\frac{p}{\pi}(\tau_2-\tau_1)} \frac{nJ_p\left(\frac{\pi}{4\beta}(4n + 2p + 3)\rho_2\right)J_p\left(\frac{\pi}{4\beta}(4n + 2p + 3)\rho_1\right)}{\pi^2 n^2 + 4(l^2 + l + 1)} ; \quad n_0 >> p
\]

on the additional condition that - as was the case in (34) - the Bessel functions in this expression be themselves expressed asymptotically through (25). Specifically, since - as stated - (25) is a necessary and sufficient condition for the eigenfunctions in (21) to be asymptotic it is, at once, a necessary and sufficient condition for the propagator in (36) to reduce to its asymptotic expression \(D_{as}(x_2, x_1)\). To that effect, the condition \(n_0 >> p\) which appears in (42) and which, as stated, ensures the asymptotic character of the eigenvalues in (31) must be supplemented with the condition \(\frac{2\pi n}{\pi} \rho >> p\). In the context of that condition expression (42) yields

\[
D_{as}(x_2, x_1) = \frac{4}{\beta} \frac{1}{\sqrt{\rho_1 \rho_2}} \sum_{l=0}^\infty \sum_{m=-l}^l Y_{lm}(\theta_2, \phi_2)Y_{lm}^*(\theta_1, \phi_1) \sum_{p=0}^\infty e^{i\frac{p}{\pi}(\tau_2-\tau_1)} \times
\]
(43) \[ \sum_{n=n_0}^{\infty} \frac{\cos(\frac{\pi}{\beta}(4n+2p+3)\rho_2-p\frac{\pi}{\beta}-\frac{\pi}{4})\cos(\frac{\pi}{\beta}(4n+2p+3)\rho_1-p\frac{\pi}{\beta}-\frac{\pi}{4})}{\pi^2 n^2 + 4(l^2 + l + 1)}; \quad n_0 >> p \]

Attention is invited to the fact that, as a result of the condition \( \frac{\pi}{\beta} \rho >> p \) the validity of (43) is contingent upon the radial variable \( \rho \). In fact, the lower bound \( n_0 \) in the series over \( n \) necessarily increases as either \( \rho_2 \) or \( \rho_1 \) decreases and, indeed, \( n_0 \) increases indefinitely as either radial variable approaches the event horizon at \( \rho = 0 \). The physical repercussions which this situation has to scalar propagation will be analyzed in due course.

The exponentiation of the cosines in (43) reveals that, at the coincidence space-time limit at which \( \tau_2 \rightarrow \tau_1; \rho_2 \rightarrow \rho_1 \), the expression

\[
\sum_{p=0}^{\infty} \sum_{n=n_0}^{\infty} e^{i\frac{\pi}{\beta}(\tau_2-\tau_1)+i[\frac{\pi}{\beta}(4n+2p+3)(\rho_2+\rho_1)-p\pi-\frac{\pi}{2}]} \frac{\pi^2 n^2 + 4\frac{1}{\beta^2}(l^2 + l + 1)}{\pi^2 n^2 + 4\frac{1}{\beta^2}(l^2 + l + 1)} + \sum_{p=0}^{\infty} \sum_{n=n_0}^{\infty} e^{i\frac{\pi}{\beta}(\tau_2-\tau_1)+i[\frac{\pi}{\beta}(4n+2p+3)(\rho_2+\rho_1)+p\pi-\frac{\pi}{2}]} \frac{\pi^2 n^2 + 4\frac{1}{\beta^2}(l^2 + l + 1)}{\pi^2 n^2 + 4\frac{1}{\beta^2}(l^2 + l + 1)}
\]

diverges. Specifically, the first and the fourth double-series are convergent whereas the second and the third are divergent. Allowing for the inconsequential \( l \)-dependent constant the second and third series are, at the stated coincidence limit, of the form

\[
\sum_{p=0}^{\infty} \sum_{n=n_0}^{\infty} \frac{1}{n^2} = \frac{1}{n_0^2} + \frac{2}{(n_0+1)^2} + \frac{3}{(n_0+2)^2} + \ldots + \frac{n+1}{(n_0+n)^2} + \ldots =
\]

\[
\sum_{n=1}^{\infty} \frac{n+1}{(n_0+n)^2} ; \quad n_0 >> p
\]

In view of the fact that

\[
\int_{1}^{\infty} dx \frac{1+x}{(n_0+x)^2}
\]

diverges it follows that the second and third double-series also diverge.

As stated, the parameter \( n \) labels radial eigenvalues and is, for that matter, continuous because the Euclidean black-hole Schwarzschild geometry has a non-trivial boundary only at infinite proper distance from \( \rho = 0 \). Consequently, the logarithmically-divergent integral above replaces identically the preceding series over \( n \) at the limit at which the boundary’s proper distance from the origin goes to infinity. In effect, that integration is not merely a formal operation for testing the convergence properties of the series above but an essential consequence of the topology of the Euclidean Schwarzschild black-hole manifold.
The stated first and fourth double-series inherent in (43) can be absorbed in the finite boundary part $D_b$ of the scalar propagator. In effect, (43) results in

$$D_{as}(x_2, x_1) =$$

$$\frac{2}{\beta \sqrt{\rho_1 \rho_2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta_2, \phi_2) Y_{lm}^*(\theta_1, \phi_1) \sum_{p=0}^{\infty} e^{i \frac{n}{2}(\tau_2 - \tau_1)} \sum_{n=n_0[p]}^{\infty} \frac{\cos\left(\frac{\pi}{43}(4n + 2p + 3)(\rho_2 - \rho_1)\right)}{\pi^2 n^2 + 4(l^2 + l + 1)};$$

(44)

$$n_0 >> p$$

In the context of the stated assumption of a finite proper distance for the boundary (44) is the asymptotic expression of the massless conformal scalar propagator. It must be observed that - as a result of the spherical symmetry of the Schwarzschild manifold - at the coincidence space-time limit $x_2 \to x_1$ the ultraviolet domain $n \to \infty$ enforces the result

$$D_{as}(x_2 \to x_1) = \frac{2}{\pi^2 \beta \rho} \delta(\cos \theta_2 - \cos \theta_1) \delta(\phi_2 - \phi_1) \sum_{p=0}^{\infty} \sum_{n=n_0[p]}^{\infty} \frac{1}{n^2};$$

(45)

$$n_0 >> p$$

Use of

$$\delta(\cos \theta_2 - \cos \theta_1) \delta(\phi_2 - \phi_1) = \frac{1}{4\pi} \sum_{k=0}^{\infty} (2k + 1) P_k(\cos \gamma)$$

(46)

with

$$\cos \gamma = \cos \theta_2 \cos \theta_1 + \sin \theta_2 \sin \theta_1 \cos(\phi_2 - \phi_1)$$

in (45) reveals, by simple power counting, that the singularity contained in (44) in four dimensions at the coincidence space-time limit is a quadratic divergence. In real time this divergence will, of course, also occur when two distinct points $x_2$ and $x_1$ can be joined by a null geodesic. The quadratic divergence inherent in (44) is consistent with theoretical expectations of a scalar propagator on any manifold. This aspect of the scalar propagator on the Schwarzschild black-hole geometry will be further explored in Section V. It is also worth noting that - as stated in Sec II - the singular part of the propagator in (45), consistently, features the factor $\pi^{-2}$ which is also expected of a scalar propagator in four dimensions.

Since the actual spectrum of eigenvalues $\lambda_n$ is continuous as a result of the infinite volume of the Euclidean black-hole geometry the expression for the asymptotic propagator which corresponds to the actual geometry will be obtained by replacing the series over $n$ in (44) by an integral. Consequently,

$$D_{as}(x_2, x_1) =$$
\[ \frac{2}{\beta} \frac{1}{\sqrt{p_1 p_2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta_2, \phi_2) Y_{lm}^*(\theta_1, \phi_1) \sum_{p=0}^{\infty} e^{i \frac{\pi}{2} (\tau_2 - \tau_1)} \int_{0}^{\infty} du \frac{\cos \left[ \pi \left( 4u + 2p + 3 \right) \left( \rho_2 - \rho_1 \right) \right]}{\pi^2 u^2 + 4(l^2 + l + 1)} ; \]

(47)

where, for notational consistency, the discrete parameter \( n \) has been replaced by the continuous variable \( u \) the lower limit \( u_0 \) of which is also a function of \( \rho_2 \) or \( \rho_1 \) accordingly.

In the context of (16) this is the asymptotic expression for the massless conformal scalar propagator in the Hartle-Hawking state. Equivalently, this is the expression for the singular part of that propagator.

IV. The Scalar Thermal Propagator - Boundary Part

As stated, the boundary part \( D_b \) of the propagator satisfies the homogeneous equation associated with (7)

(48) \[ \Box_{x_2} D_b(x_2 - x_1) = 0 \]

and can be adjusted to enforce the boundary condition

(49) \[ D(x_2, x_1)|_{\rho_2 = \beta, \rho_1 = \beta} = 0 \]

of vanishing propagation on the boundary of the Euclidean Schwarzschild black-hole geometry. This boundary condition is physically relevant to the propagator \( D \) despite the range of propagation specified in (16) for the same reasons which lend validity to (24). Since, as analysed in the previous section, in the context of (8) the boundary part \( D_b \) is specified by the sum-total of all terms corresponding to \( n = 1, n = 2, ..., n = n_0 - 1 \) for all values of \( l \) and \( p \) it follows that

(50) \[ D_b(x_2 - x_1) = \sum_{lmp} \sum_{n=1}^{n_0 - 1} \frac{\phi_{nplm}(x_2) \phi_{nplm}^*(x_1)}{\lambda_{nplm}} ; \quad n_0 >> [p, l]_{\text{max}} \]

At once, (48) implies that \( D_b \) satisfies (9) with \( \lambda_n = 0 \). Consequently, the expression in (50) is independent of \( \lambda_n \). As a result of (48) and (21) the boundary part is

\[ D_b(x_2 - x_1) = \]

(51) \[ \frac{1}{\beta} \sum_{p=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=1}^{\infty} A_{plmn} \frac{2^{p+1} \Gamma(p+1)}{\left[ \frac{2i}{\beta \sqrt{l^2 + l + 1}} \right]_{\text{min}}^p} e^{i \frac{\pi}{2} \tau_2} Y_{lm}(\theta_2, \phi_2) J_p \left( \frac{2i}{\beta \sqrt{l^2 + l + 1}} \rho_2 \right) \]

It is worth mentioning that although (50) and (51) are both expressions for the boundary part \( D_b \) they are very different in terms of their mathematical essence. Specifically, (50) expresses \( D_b \) as the complement to the singular part \( D_{as} \) of the propagator whereas
(51) expresses $D_0$ as a linear combination of those eigenfunctions in (9) which correspond to $\lambda_n = 0$.

The coefficients $A_{plmn}$ are expected to manifest an explicit dependence on the space-time point $x_1$ and will naturally be determined by the boundary condition in (49) which, in view of (35), (44) and (51), becomes

\[
\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=0}^{\infty} A_{plmn} 2\beta^{p-1} \Gamma(p+1) \frac{J_p(2i\sqrt{l^2 + l + 1})}{i\sqrt{l^2 + l + 1}} \int_0^{2\pi} d\tau_2 e^{i\frac{\pi}{\beta} \tau_2} \int_0^{2\pi} d\phi_2 \int_0^{1} d\cos \theta_2 Y_{lm}(\theta_2, \phi_2) Y_{lm}^*(\theta_2, \phi_2) = 0 \quad n' > p
\]

It should be noted again that the bound $n_0$ which appears in the series over $n$ is, in general, dependent upon the radial variables, in addition to being dependent on $p$. Consequently, in (52) it is $n_0' \neq n_0$ since the bound $n_0$ which appears in the singular part of the propagator in (44) depends on $\rho_2$ whereas the bound $n_0'$ relates exclusively to the value $\rho_2 = \beta$. The physical significance of this difference will become obvious in the next section.

Multiplying both sides of (52) by $e^{-i\frac{\pi}{\beta} \tau_2} Y_{lm}^*(\theta_2, \phi_2)$ and integrating yields

\[
\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=0}^{\infty} A_{plmn} \frac{\beta^{p-1} \Gamma(p+1)}{i\sqrt{l^2 + l + 1}} J_p(2i\sqrt{l^2 + l + 1}) \int_0^{2\pi} d\tau_2 e^{i\frac{\pi}{\beta} \tau_2} \int_0^{2\pi} d\phi_2 \int_0^{1} d\cos \theta_2 Y_{lm}(\theta_2, \phi_2) Y_{lm}^*(\theta_2, \phi_2) =
\]

\[
-\frac{1}{\beta^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}^*(\theta_1, \phi_1) \int_0^{2\pi} d\phi_2 \int_0^{1} d\cos \theta_2 Y_{lm}(\theta_2, \phi_2) Y_{lm}^*(\theta_2, \phi_2) \times
\]

\[
\sum_{p=0}^{\infty} \sum_{n=n_0'}^{\infty} \frac{\cos \frac{\pi}{\beta}(4n + 2p + 3)(\beta - \rho_1)}{\pi^2 n^2 + 4(l^2 + l + 1)} \int_0^{2\pi} d\tau_2 e^{i\frac{\pi}{\beta} \tau_2}
\]

whereupon use of the orthonormality relation

\[
\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos \theta Y_{lm}^*(\theta, \phi) Y_{lm}(\theta, \phi) = \delta_{ll'} \delta_{mm'}
\]

as well as of

\[
\int_0^{L} dx e^{i\frac{\pi}{2}(n-m)x} = L \delta_{mn}
\]

results in
A consistent expression for the boundary part of the propagator for a massless conformal scalar field on the Euclidean Schwarzschild black-hole geometry is the sum total of (47) and (59). As announced in (I.1), that is

\[ D_b(x_2 - x_1) = \ldots \]

Replacing (57) in (51) results in

\[ D_b(x_2 - x_1) = \ldots \]

Again, in order to arrive at the expression which corresponds to the actual infinite volume the series over \( n \) in (58) must be replaced by an integral. Consequently, the consistent expression for the boundary part \( D_b \) on the Euclidean black-hole geometry is

\[ D_b(x_2 - x_1) = \ldots \]

In the context of (16) this is the expression for the boundary part of the scalar propagator on the Euclidean Schwarzschild black-hole geometry.

**V. Scalar Propagation on the Black-Hole Geometry**

Within the range of radial values which are consistent with (16) the thermal scalar propagator for a massless conformal scalar field on the Euclidean Schwarzschild black-hole geometry is the sum total of (47) and (59). As announced in (I.1), that is

\[ D(x_2 - x_1) = \ldots \]
\[
2 \frac{1}{\beta \sqrt{\rho_1 \rho_2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta_2, \phi_2) Y_{lm}^*(\theta_1, \phi_1) \sum_{p=0}^{\infty} e^{i \frac{\pi}{4} (l^2 + 1)} \int_{u_0[p]}^{\infty} \cos \left( \frac{\pi}{4} (4u + 2p + 3)(\rho_2 - \rho_1) \right) \frac{du}{\pi^2 u^2 + 4(l^2 + l + 1)} \]

\[
- \frac{2}{\beta^2 \sqrt{\rho_1}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{p=0}^{\infty} \int_{u_0'[p]}^{\infty} \frac{du}{\pi^2 u^2 + 4(l^2 + l + 1)} \frac{J_p \left( \frac{2i}{\beta} \sqrt{l^2 + l + 1} \rho_2 \right)}{J_p \left( 2i \sqrt{l^2 + l + 1} \right)} e^{i \frac{\pi}{4} (l^2 + 1)} Y_{lm}(\theta_2, \phi_2) Y_{lm}^*(\theta_1, \phi_1) ;
\]

(60)

\[
u_0 >> p ; \quad u_0' >> p ; \quad \frac{\pi u}{\beta} \rho_{2,1} >> p
\]

The significance of the range of values in (16) is that for \( \rho^2 << \beta^2 \) the solution to (17) approximates the solution to (15) with an accuracy which increases with decreasing values of \( \rho \). For that matter, the closer to the Schwarzschild event horizon the propagation occurs the higher the accuracy with which the Green function in (60) approximates the exact scalar propagator is. Such considerations are qualitative. In order to make a quantitative estimation of the accuracy of (60) it must be observed that the operation of reducing (15) to (17) amounts essentially to treating \( \rho^2 \) as a perturbation applied to the unknown solution to (15). For that matter, the mathematical validity of such a reduction is contingent upon the stability of that solution. However, (17) is a linear differential equation whose solution (19) is bounded and stable. The linear character of (15) ensures, for that matter, that - at least in the range of values given by (16) and in the context of the same boundary conditions of regularity at the origin and of a vanishing amplitude at \( \rho = \beta \) - the solution to (15) is also stable. Moreover, the stability of the solution to (15) is independently expected on the physical grounds that it corresponds to the radial sector of the wave equation in the Schwarzschild space-time. In effect, the reduction of (15) to (17) is mathematically consistent.

In order to assess "how far" from the event horizon the radial values can be considered before the Green function in (60) begins to, substantially, detract from the expression of the exact propagator it must be noticed that since, in the context of (16), discarding \( \rho^2 \) in favour of \( \beta^2 \) does not affect the stability of the solution to (15) values of \( \rho^2 \) characterised by two orders of magnitude below the value of \( \beta^2 \) can be safely considered to be consistent with (16). Consequently, the range of validity for the Green function in (60) is

(61)

\[
0 \leq \rho_i^2 \leq \frac{\beta^2}{100} ; \quad i = 1, 2
\]

This estimate is physically reasonable. In fact, it will be independently rederived in the context of the renormalisation of \( \langle \phi^2(x) \rangle \) in section VI. That different approach will rigorously establish the range stated in (61).

The estimate for the upper bound given in (61) for the radial variable on the Euclidean sector of the Schwarzschild metric translates to
\begin{equation}
2M \leq r \leq 2.0050M \tag{62}
\end{equation}

for the Schwarzschild radial coordinate in real time. This range places an upper bound for \( r \) of the order of \( 10^{-3}r_S \) in excess of \( r_S \), the radius of the Schwarzschild event horizon. For a Schwarzschild black hole of one solar mass this amounts, in terms of coordinate radial distance, to \( r = 3,006km \) - that is, to \( 6m \) "above" the event horizon. In order to assess the physical significance of this range it must be recalled that the average wavelength of the quanta emitted in the vicinity of the horizon is \( \sim M \). For that matter, in the Hartle-Hawking vacuum state, the range in (62) explicitly refers to the segment of the static region which is of central importance to particle creation. It is desirable to know how large the range in (62) is with respect to the vacuum activity itself. Such a determination requires a comparison with a specific range of physical interest. The problem in this respect is that since it is impossible to localise a quantum to within one wavelength it also is meaningless to trace the origin of the emitted particles to any particular region near the event horizon \[1\]. Consequently, the range in (62) explored in the Hartle-Hawking vacuum state does not determine whether the scalar propagation in (60) is relevant to distances which extend far above the event horizon in the exterior geometry. A physically valid way to establish a comparison with a range of physical interest can be obtained by invoking instead the situation of an observer who accelerates to remain stationary at a finite \( r > 2M \). Unlike the observer in free fall - who, for that matter, registers the Hartle-Hawking vacuum state - the stated stationary observer will register the emission of particles in the vicinity of the event horizon. Since a black hole of one solar mass has a temperature of \( 6 \times 10^{-8}K \) the stated observer registers the emission of particles at proper distances which, for a black hole of one solar mass, correspond to \( r \sim 10^{-6}m \) above \( r = 2M = 3 \times 10^3m \). This range coincides with that within which gravitational back-reaction effects are substantial. The probability for particle detection above this range rapidly decreases \[10\]. Since - for a Schwarzschild black hole of one solar mass - the upper bound \( r \sim 2.0050M \) in (62) corresponds to a coordinate radial distance of \( 6m \) above the event horizon that bound signifies a range of validity for the propagator in (60) which corresponds to several orders of magnitude above the range within which particles are spontaneously created and back-reaction effects are pronounced. It can be seen, for that matter, that there is an ample range of values of \( r \) above \( r = 2M \) within which (61) and (62) signify respectively a good approximation to the exact propagator.

The result expressed in (60) is substantially different from all other approximate expressions which have been, hitherto, obtained for the scalar propagator in various contexts. It signifies the only approximation which specifically expresses the conformal scalar propagator in terms of its space-time dependence in a specific region of the Schwarzschild black-hole geometry.

In order to explore the implications of (60) to scalar propagation in the vicinity of the event horizon it is also necessary to, accurately, specify the range of possible values for the lower bounds \( u_0 \) and \( u'_0 \). In fact, as a consequence of the necessary condition

\begin{equation}
\frac{\pi u}{\beta} \rho_{1,2} \gg p \tag{63}
\end{equation}
the lower bound \( u_0 \) increases indefinitely as either \( \rho_2 \) or \( \rho_1 \) tends to zero. The precise manner in which \( u_0 \) depends on \( \rho_2 \) or \( \rho_1 \) is central to the singularity structure of the Green function in (60) and will be, accordingly, analyzed in this section. Such considerations are also valid for \( p = 0 \) since the replacement of \( p \) by \( p + 1 \) leaves (63) intact. Similar considerations also apply to \( u' \) as a function of \( \rho_1 \).

The consequences of such an effect to propagation in the vicinity of the event horizon can be revealed by first considering the integral in the singular part of the propagator in (60). It is

\[
(64) \quad \left| \int_{u_0[p]}^{\infty} du \frac{\cos[\frac{\pi}{4\beta}(4u + 2p + 3)(\rho_2 - \rho_1)]}{\pi^2 u^2 + 4(l^2 + l + 1)} \right| < \frac{1}{\pi^2} \int_{u_0[p]}^{\infty} du = \frac{1}{\pi^2 u_0[p]}
\]

The relations in (64), (63) and (46) imply that

\[
(65) \quad |D_{as}(x_2, x_1)| \ll \frac{1}{2\pi^2 \beta^2} \sum_{k=0}^{\infty} \sum_{p \neq 0} e^{i\tau_2 - \tau_1} \frac{2k + 1}{p} P_k(\cos\gamma)
\]

It can be seen, for that matter, that since \( \sqrt{\rho_2} \) multiplies each term in the series the singular part \( D_{as}(x_2 - x_1) \) of the propagator vanishes at \( \rho_2 \to 0 \) on condition that \( \rho_1 \) be fixed at any value above zero. At once, the boundary part \( D_b(x_2 - x_1)_{|\rho_2=0} \) can be seen to remain at a non-vanishing value.

The same vanishing effect for the singular part can also be, independently, obtained through use of the identity

\[
(66) \quad \int_{u_0[p]}^{\infty} du \frac{\cos[\frac{\pi u}{\beta}(\rho_2 - \rho_1)]}{u^2} = \frac{1}{2} \frac{\pi^2 \rho_2 - \rho_1}{\beta^2} + \frac{1}{u_0} \frac{1}{2} F_2[-\frac{1}{2} ; \frac{1}{2} ; \frac{1}{2} ; -\frac{\pi^2 (\rho_2 - \rho_1)^2}{4\beta^2 u_0^2}]
\]

with \( _1F_2 \) being the generalised hypergeometric function which admits the expansion

\[
_1F_2[-\frac{1}{2} ; \frac{1}{2} ; \frac{1}{2} ; -\frac{\pi^2 (\rho_2 - \rho_1)^2}{4\beta^2 u_0^2}] = 1 + \frac{(-\frac{1}{2})}{\frac{1}{2}} \frac{\pi^4 (\rho_2 - \rho_1)^4}{2! (2\beta)^4 u_0^4} + \ldots
\]

Again, at the limit \( \rho_2 \to 0 \) (66) yields

\[
(67) \quad \lim_{u_0 \to \infty} \int_{u_0[p]}^{\infty} du \frac{\cos[\frac{\pi u}{\beta} \rho_1]}{u^2} = 0
\]

which, through (64), signifies the same result as (65).

An immediate consequence of such an effect is that the Green function in (60) is not defined at the limit \( \rho_2 \to 0 \) when \( \rho_1 \neq 0 \). The propagator vanishes in its entirety in such a context eventhough its boundary part remains at a non-vanishing value. In the absence of the singular part such a non-vanishing expression is not, in any respect, associated
with propagation. In fact - as the preceding results reveal - the limit $\rho_1 \to 0$ causes that expression to, also, vanish.

The vanishing effect expressed by

\begin{equation}
\lim_{\rho_2 \to 0} D(x_2 - x_1) = 0 ; \rho_1 \neq 0
\end{equation}

is a consequence of the causal structure of the Schwarzschild black-hole space-time. Specifically, at the semi-classical limit $\hbar \to 0$ all observers in the static region, regardless of their state of motion or choice of coordinates agree, that no particle reaches the hole’s event horizon within a finite advance of their proper time. Equivalently, at the semi-classical limit all observers in the static region agree that the frequency of a waveform tends to zero in the vicinity of the hole’s event horizon. Consequently, away from the semi-classical limit the transition amplitude for quantum propagation specified by one end-point of propagation being arbitrarily close to the event horizon is also expected to vanish for all observers in the static region. This situation is distinct from that expressed by the single limit $\rho \to 0 ; \rho_1 = \rho_2 \equiv \rho$ which is analysed below.

In order to explore the singularity structure of the propagator use will be made of the fact that - as simple power counting reveals - the singular part of the Green function in (60) contains a logarithmic divergence in $1 + 1$ dimensions and a quadratic divergence in four dimensions. Separating the space-time points in the radial direction by setting $x_1 = (\tau, \rho_1, \theta, \phi) ; x_2 = (\tau, \rho_1 + \epsilon\rho, \theta, \phi)$, and taking the limit $\rho_2 \to \rho_1 \Rightarrow \epsilon \rho \to 0$ it is

\begin{equation}
D_{as}(x_2 \to x_1) = \frac{1}{2\pi^3 \beta} \frac{1}{\rho} \sum_{k=0}^{\infty} (2k + 1) \left[ \sum_{p=0}^{p_0>1} \frac{1}{u_0} + \sum_{p=p_0}^{\infty} \frac{1}{v_0} \right]
\end{equation}

where, in transfer space, use has also been made of (46) in the ultra-violet domain $u \to \infty$. The significance of this domain is that, if the events $x_2$ and $x_1$ are arbitrarily close to each other, the dominant contribution to the integral over $u$ comes from $u >> l$ in each term in the series over $l$.

The demand for a quadratic divergence with a logarithmic divergence inherent in the radial-temporal sector of $D_{as}$ imposes the conditions

\begin{equation}
\begin{cases}
\frac{1}{u_0} = \frac{1}{p} f(\rho) + F_k^{(1)}(\rho) ; \text{ if } p > p_0 \\
\frac{1}{u_0} = \frac{1}{v_p(\rho)} f(\rho) + F_k^{(2)}(\rho) ; \text{ if } p < p_0 \\
v_p(\rho) \neq 0
\end{cases}
\end{equation}

for all values of $p$ and $\rho$ and on the understanding that power counting in the ultra-violet domain precludes any dependence of $f(\rho)$ and $v_p(\rho)$ on $k$.

Replacing (71) in (70) yields

\begin{equation}
D_{as}(x_2 \to x_1) = \frac{1}{2\pi^3 \beta} \frac{1}{\rho} \sum_{k=0}^{\infty} (2k + 1) \left[ \sum_{p=0}^{p_0>1} \frac{1}{v_p(\rho)} + \sum_{p=p_0}^{\infty} \frac{1}{p} f(\rho) + F(\rho) \right]
\end{equation}
and renders the quadratic divergence, which represents the propagator’s singularity at the coincidence space-time limit, manifest.

The arbitrary term $F(\rho)$ is finite. Its exclusive radial dependence is expected on the grounds that the manifold is static and spherically symmetric. As is also the case with the finite and arbitrary sum over $v_p^{-1}(\rho)$ the function $F(\rho)$ emerges as a necessary consequence of the operation of specifying the divergence in (70). Its exact expression in the range stated in (61) can only be independently determined by additional boundary conditions in the corresponding physical context. Examples of such a procedure will be given in what follows. It should be stressed, however, that the finite character of $F(\rho)$ is the exclusive consequence of the behaviour of the scalar propagator at the coincidence space-time limit and as such that character can not be guaranteed on the Schwarzschild black-hole’s event horizon where the regularity of a vacuum state - such as the Hartle-Hawking state $|H> - in a freely falling frame is ensured only as a consequence of the manner in which that state has been defined.

Since $F_{kp}^{(1)}(\rho)$ and $F_{kp}^{(2)}(\rho)$ are arbitrary the quantity $\frac{1}{u_0}$ can be redefined to be what it is in (71) with $F_{kp}^{(1)}(\rho) = F_{kp}^{(2)}(\rho) = 0$. In effect, (63) reduces the - as of yet - remaining, unspecified functions in (71) to being such that

$$
\begin{align*}
\frac{1}{u_0} &= \frac{\pi}{\rho} c p \rho + F_{kp}^{(1)}(\rho) \quad \text{if } p > p_0 \\
\frac{1}{u_0} &= \frac{\pi}{\rho} v_p(\rho) \rho + F_{kp}^{(2)}(\rho) \quad \text{if } p < p_0 \\
v_p(\rho) &\approx c p \quad \text{if } p >> 1
\end{align*}
$$

with $c > 1$; $v_p(\rho) > 1, \forall \rho$ and with $[v_p(\rho)] \sim [m^0]$; $[F_{kp}^{(1,2)}(\rho)] \sim [m^0]$. The conditions in (73) explicitly reveal the information which is inherent in (63) as to the singularity structure of the propagator in (60). Specifically, as stated, the lower bound $u_0$ increases indefinitely as either $\rho_2$ or $\rho_1$ tends to zero. It can be seen from (70) and (71), for that matter, that if - in the event that $F_{kp}^{(1)}(\rho) = 0$ - the bound $u_0$ behaves as $\rho^{-k}, k > 1; k \in Z$ or in any other respect which causes $\frac{1}{u_0}$ to be a non-linear function of $\rho$ then the inverse $\frac{1}{u_0}$ would cause each term in the divergent infinite series over $p$ in $D_{as}(x_2 \rightarrow x_1)$ to be multiplied by a function of $\rho$. This situation is physically inconsistent since a divergence - being the exclusive consequence of the coincidence space-time limit $x_2 \rightarrow x_1$ - can only depend on local geometric quantities. This is, in fact, an explicit feature of the representation of the Feynman propagator for short space-time separations which the Schwinger-DeWitt expansion yields. As a consequence, (63) imposes the physical demand that - in the absence of the arbitrary function $F_{kp}^{(1)}(\rho)$ - $k = 1$ and, hence, that $\frac{1}{u_0}$ be a linear function of $\rho$, on condition that $p$ receive values in the ultra-violet domain of transfer space. This demand is manifestly expressed by the first relation in (73).

Away from the coincidence space-time limit $x_2 \rightarrow x_1$ there is no reason that, in the absence of $F_{kp}^{(2)}(\rho)$, the expression of $\frac{1}{u_0}$ be a linear function of $\rho$. For that matter, as a consequence of (63), the general expression of that quantity for $p < p_0$ is that in the second relation in (73).
The third relation, \( v_p(\rho) \approx c p \) for very large values of \( p \), ensures that the second relation in (73) reduces to the first as the transfer space variable \( p \) makes the transition from values smaller than \( p_0 >> 1 \) to values bigger than \( p_0 >> 1 \).

The representation of the short distance behaviour of the exact propagator by the Schwinger-DeWitt expansion suggests that the underlying physical reason for the situation expressed in (73) is that in the Schwarzschild geometry it is \( R_{\mu\nu}(x) = 0 \) and \( R(x) = 0 \).

In effect, the redefinition in (73) reduces (72) to

\[
D_{as}(x_2 \to x_1) = \frac{1}{2\pi^2\beta^2} \sum_{k=0}^{\infty} (2k + 1) \left[ \sum_{p=0}^{p_0 > 1} \frac{1}{v_p(\rho)} + \frac{1}{c} \sum_{p=p_0}^{\infty} \frac{1}{p} \right] + F(\rho)
\]

It should be remarked, in passing, that the quadratic divergence which emerges by power counting at the coincidence space-time limit is - at least as a leading divergence - a characteristic feature of the scalar propagator on any manifold in four dimensions. This is readily understood by the fact that the coincidence space-time limit relates essentially to the Minkowski space tangent to the pseudo-Riemannian manifold at the associated point. Consequently, the background curvature does not have any effect on that divergence.

On the event horizon (74) is

\[
\text{lim}_{\rho \to 0} D(x_2 \to x_1) = \frac{1}{2\pi^2\beta^2} \sum_{k=0}^{\infty} (2k + 1) \left[ \sum_{p=0}^{p_0 > 1} \frac{1}{v_p(0)} + \frac{1}{c} \sum_{p=p_0}^{\infty} \frac{1}{p} \right] + F(0)
\]

since the boundary part vanishes at \( \rho_1 \to 0 \).

In contrast to (69) the physical significance of (75) is that of the dynamical behaviour of the quantum field at \( x_2 \to x_1 \) evaluated arbitrarily close to the event horizon in a freely falling frame.

In what follows these results will be exploited in order to demonstrate the merit of the Green function in (60) in calculations of local expressions.

**VI. Evaluation of \( \left< \phi^2(x) \right> \)**

In real time the expectation value \( \left< \phi^2(x) \right> \) for the square of the conformal scalar field in a specific vacuum state relates to the Feynman propagator \( G_F(x_2 - x_1) \) corresponding to the same vacuum state by

\[
\left< \phi^2 \right> = -i \text{lim}_{x_2 \to x_1} G_F(x_2 - x_1)
\]

In order to evaluate this expectation value in the Hartle-Hawking vacuum state the divergent series in (74) must be accorded a pole structure. This task has been accomplished in the Appendix. The results therein allow for the evaluation of \( \left< \phi^2 \right> \) on the black-hole’s event horizon. Replacing (A.10)

\[
\sum_{k=0}^{\infty} (2k + 1) = \frac{1}{2} \text{lim}_{\epsilon \to 0^+} \frac{1}{\epsilon^{2l}}
\]
and (A.9)

\[ \sum_{k=0}^{\infty} \sum_{p=p_0}^{\infty} \frac{2k+1}{p} = -\frac{1}{4} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon^{2l}} ; \ p_0 \gg 1 \]

in (75) yields

\[ \lim_{\rho \to 0} D(x_2 \to x_1) = \lim_{\epsilon \to 0^+} \frac{1}{32\pi^2 M^2} \frac{1}{c'(0)} \frac{1}{\epsilon^{2l}} + F(0) \]

where

\[ \frac{1}{c'(\rho)} = \frac{1}{2} \sum_{p=0}^{p_0 \gg 1} \frac{1}{v_p(\rho)} - \frac{1}{4c} ; \ [c'(\rho)] \sim [m^0] \]

The information necessary for determining \( c'(0) \) and the free parameter \( l \) in (77) is inherent in the nature of the DeWitt-Schwinger expansion as an approximation to the exact propagator for short separations. Specifically, in the context of point splitting in real time \( t \) the space-time points are separated along a geodesic in a non-null direction by an infinitesimal distance \[ \sigma = \frac{1}{2} \sigma_{\mu} \sigma^{\mu} ; \ \sigma^{\mu} = 2vt^{\mu} ; \ t^{\mu}t_{\mu} = \pm 1 \]

For such separations the DeWitt-Schwinger expansion is a valid approximation to the exact propagator - and, consequently, to the Feynman propagator

\[ G_F(x_2 - x_1) = iD(x_2 - x_1) \]

in the context of (62). For that matter, the divergent - at the coincidence space-time limit \( x_2 \to x_1 \) - terms in the DeWitt-Schwinger expansion coincide respectively, at the same limit, with the divergent terms in \( G_F(x_2 - x_1) \). These considerations constitute the basis for the renormalisation of the exact propagator. The DeWitt-Schwinger approximation for short separations is treated as a counterterm and subtracted from the unrenormalised propagator before the coincidence space-time limit is taken. In turn, the stated counterterm provides the basis for the one-loop effective action corresponding to free propagation on the curved manifold.

In the present context, where the mass of the black hole sets a characteristic scale for the Schwarzschild metric, dimensional analysis allows for

\[ v = M\epsilon \]

and reveals that in (77) it is, necessarily, \( l = 1 \) since the singular term is already proportional to \( v^{-2} \). In fact, for infinitesimal radial separations \( \epsilon_r = r_2 - 2M \equiv M\epsilon^2 \) from \( r_1 = 2M \) the demand that the divergence contained in \( G_F(x_2 - x_1) \) coincide - at \( r_2 \to 2M \) - with the divergent part \[ \[ \]$
stemming from the DeWitt-Schwinger expansion makes it obvious that the limit $\epsilon \to 0^+$ reduces to the limit $\rho \to 0 \Rightarrow r_2 \to 2M$ only when $l = 1$ and $c'(0) = 1$.

As a result, (77) and (78) imply that

$$(80) \quad \lim_{x_2 \to x_1} G_F(x_2 - x_1)_{|r_1=2M} = \lim_{r_2 \to 2M} \frac{i}{32\pi^2 M(r_2 - 2M)} + \lim_{r_2 \to 2M} \tilde{F}(r_2)$$

with $\tilde{F}(r) = F[\rho(r)]$.

Replacing (80) in (76) yields

$$(81) \quad <H|\phi^2(x_2; r_2 = 2M)|H> = \lim_{r_2 \to 2M} \frac{1}{32\pi^2 M(r_2 - 2M)} + \lim_{r_2 \to 2M} \tilde{F}(r_2)$$

It has already been stressed in section V that additional, independent physical conditions are required on $G_F(x_2 - x_1)$ in order to determine the arbitrary term $\tilde{F}(r_2)$. This task can best be accomplished through the renormalisation of $<H|\phi^2(x)|H>$ within the entire range given in (62).

The choice of temporal separations $x_1 = (\tau, \rho, \theta, \phi)$ and $x_2 = (\tau + \epsilon, \rho, \theta, \phi)$ for $\rho \neq 0$ in the Euclidean sector of the metric followed by the limit $\epsilon \tau \to 0$ taken on the expression in (60) yields

$$<H|\phi^2(x)|H> = \frac{1}{2\pi^2 \beta^2} \sum_{k=0}^{\infty} (2k + 1) \left[ \sum_{p=0}^{p_0 \gg 1} \frac{1}{v_p(\rho)} + \frac{1}{c} \sum_{p=p_0}^{\infty} \frac{1}{p} \right] + F(\rho)$$

$$- \frac{2}{\beta^2} \frac{1}{\sqrt{\rho}} \times$$

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{p=0}^{\infty} \int_{u_0[p]}^{\infty} du \frac{\cos[\frac{\pi}{12}(4u + 2p + 3)(\beta - \rho)]}{\pi^2 u^2 + 4(l^2 + l + 1)} \frac{J_p(2u \sqrt{l^2 + l + 1})}{J_p(2i \sqrt{l^2 + l + 1})} |Y_{lm}(\theta, \phi)|^2 ;$$

$$(82) \quad u_0 \gg p$$

where (74), (76) and (78) have been used.

On the lines of the calculation which eventuated in (77) the singular part reduces to

$$(83) \quad D_{as}(x_2 \to x_1) = \lim_{\epsilon \to 0^+} \frac{1}{32\pi^2 M^2} \frac{1}{c'(\rho) \epsilon^2} + F(\rho)$$

Repeating the point-splitting procedure for infinitesimal temporal separations in real time $\nu = \epsilon \tau = M \epsilon -$ in which procedure, pursuant to (1), it is $t^\mu t_\mu = -1$ - the demand that the divergence contained in $G_F(x_2 - x_1)$ at $\epsilon \tau \to 0$ coincide with the divergence contained in the DeWitt-Schwinger expansion to the relevant order [4]
(84) \[
\frac{1}{4\pi^2} \frac{1}{2\sigma} = \frac{1}{4\pi^2} \frac{1}{(1 - \frac{2M}{r})^2} + \frac{1}{4\pi^2} \frac{M^2}{12r^4(1 - \frac{2M}{r})}
\]

at the same limit reveals that, in this context, it is \( k = 1 \) and \( \bar{c}'(r) = c'[\rho(r)] = \frac{1}{8}(1 - \frac{2M}{r}) \), respectively. With these values (82) reduces to

\[
< H | \phi^2(x) | H > = \frac{1}{4\pi^2} \lim_{\epsilon t \to 0} \frac{1}{(1 - \frac{2M}{r})\epsilon_t^2} + \tilde{F}(r)
\]

\[
- \frac{1}{8M^2} \frac{1}{(1 - \frac{2M}{r})^2} \times \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{p=0}^{\infty} \int_{u'_{0}[p]}^{\infty} \frac{\cos[\frac{\pi}{4}(4u + 2p + 3)(1 - \sqrt{1 - \frac{2M}{r}})]}{\pi^2u^2 + 4(l^2 + l + 1)} \frac{I_p[2\sqrt{(l^2 + l + 1)(1 - \frac{2M}{r})}]}{I_p(2\sqrt{l^2 + l + 1})} |Y_{lm}(\theta, \phi)|^2
\]

(85)

\[
u'_{0} \gg p
\]

where use has been made of the relation

\[
I_p(x) = i^{-p} J_p(ix)
\]

between the modified Bessel function of the first kind \( I_p(x) \) and the Bessel function \( J_p(ix) \).

The renormalisation of the divergent expression in (85) is accomplished through the operation

\[
< H | \phi^2(x) | H >_{\text{ren}} = \lim_{\epsilon t \to 0} \left[ < H | \phi^2(x) | H > - \frac{1}{8\pi^2\sigma} \right]
\]

to the finite expression

\[
< H | \phi^2(x) | H >_{\text{ren}} = \frac{1}{12(8\pi M)^2} \frac{1 - \frac{2M}{r}}{1 - \frac{2M}{r}} - \frac{1}{12(8\pi M)^2} \frac{1}{1 - \frac{2M}{r}} + \tilde{F}(r) - \frac{1}{32\pi M^2} \frac{1}{(1 - \frac{2M}{r})^2} \times \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \int_{u'_{0}[p]}^{\infty} \frac{\cos[\frac{\pi}{4}(4u + 2p + 3)(1 - \sqrt{1 - \frac{2M}{r}})]}{\pi^2u^2 + 4(l^2 + l + 1)} \frac{I_p[2\sqrt{(l^2 + l + 1)(1 - \frac{2M}{r})}]}{I_p(2\sqrt{l^2 + l + 1})} (2l + 1)
\]

(87)

\[
u'_{0} \gg p
\]

where following [12] the finite part of the counterterm in (84) has been expressed as the difference of the first two terms, the first of which remains finite throughout the range of radial values stated in (62) whereas the other diverges at \( r = 2M \). In (87) use has also been made of
\[
\sum_{m=-l}^{l} Y_{lm}^{*}(\Omega')Y_{lm}(\Omega) = \frac{1}{4\pi}(2l + 1)P_l(cos\gamma)
\]

which is an immediate consequence of (46).

The underlying physical condition necessary to determine \(< H|\phi^2(x)|H >_{ren}\) is inherent in the Hartle-Hawking vacuum state \(|H >\) itself as a consequence of whose definition the renormalised stress-energy tensor \(< H|T^\mu_\nu(x)|H >_{ren}\) is regular on both the past and the future horizon of the maximally extended Kruskal manifold \([1], [4]\). For that matter, it immediately follows from (87) that

\[
(88) \tilde{F}(r) = \frac{1}{12(8\pi M)^2} \frac{1}{1 - \frac{2M}{r}} + \tilde{F}'(r) ; \tilde{F}'(r) < \infty \forall r \in [2M, \infty)
\]

and, consequently, that

\[
\sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \int_{u_0'}^{\infty} \frac{du}{\pi^2 u^2 + 4(l^2 + l + 1)^2} cos \left[ \frac{\pi}{2}(4u + 2p + 3)(1 - \sqrt{1 - \frac{2M}{r}}) \right] I_p \left[ 2\sqrt{(l^2 + l + 1)(1 - \frac{2M}{r})} \right] \frac{I_p(2\sqrt{l^2 + l + 1})}{(2l + 1)}\]

\[
\sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \int_{u_0'}^{\infty} \frac{du}{\pi^2 u^2 + 4(l^2 + l + 1)^2} cos \left[ \frac{\pi}{2}(4u + 2p + 3)(1 - \sqrt{1 - \frac{2M}{r}}) \right] I_p \left[ 2\sqrt{(l^2 + l + 1)(1 - \frac{2M}{r})} \right] \frac{I_p(2\sqrt{l^2 + l + 1})}{(2l + 1)}\]

(89)

\[
u_0' >> p ; 2M \leq r \leq 2.0050M
\]

The remaining \(\tilde{F}'(r)\) term is a regular function of \(r \in [2M, \infty)\). For that matter, it is, itself, determined by the fact that, at \(x_2 \to x_1\), (7) and (48) imply that the boundary part of \(G_F(x_2, x_1)\) - which is featured in the third term of (89) - necessarily corresponds to the addition of two functions of \(r\) one of which is \(\tilde{F}'(r)\) and the other is the finite sum in (50). Since the boundary part, as a solution to (48), always satisfies (51) it follows that in (89) it is

\[
(90) \tilde{F}'(r) = 0 ; 2M \leq r \leq 2.0050M
\]

Together (89) and (90) constitute the result announced in (1.2).

Similar considerations apply in the context of radial separations. In the context of point splitting such separations, of course, constitute the exclusive regulating approach when the coincidence space-time limit is specified on the event horizon. Again, in view of the counterterm in (79) the renormalisation of the divergent expression in (81) is accomplished through the operation

\[
(91) \quad < H|\phi^2(x; r = 2M)|H >_{ren} = lim_{r_2 \to 2M} \left[ < H|\phi^2(x_2; r_2)|H > - \frac{1}{8\pi^2\sigma} \right]
\]

to the finite expression
The renormalised expression in (92) is regular on the event horizon on condition that $\tilde{F}(2M) < \infty$. In view of the fact that

$$\lim_{r \to 2M} \frac{1}{12(8\pi M)^2} \frac{1 - \left(\frac{2M}{r}\right)^4}{1 - \frac{2M}{r}} = \frac{1}{192\pi^2 M^2}$$

of the fact that the boundary part of the propagator vanishes at $r = 2M$ as well as of the fact that the expressions respectively obtained for $< H|\phi^2(x)|H >_{ren}$ through radial and temporal separations are necessarily identical it follows from (89), (90) and (92) that

(93) $\tilde{F}(2M) = 0$

and that consequently

(94) $< H|\phi^2(x; r = 2M)|H >_{ren} = \frac{1}{192\pi^2 M^2}$

The result in (94) is the renormalised value of $< \phi^2(x) >$ on the event horizon of the Schwarzschild black hole in the Hartle-Hawking vacuum state. This result coincides with the corresponding one cited in [4].

In passing, it should be remarked that the physical condition of regularity which eventuated in (88) is absent on the past event horizon in the Unruh vacuum state and on both horizons in the Boulware vacuum state. It is seen yet again, for that matter, that the function $F(\rho)$ which emerged in (72) at the coincidence space-time limit necessarily depends on the choice of vacuum state. In fact, the procedure followed above in the context of temporal and radial separations is indicative of the manner in which the function $F(\rho)$ depends on both, the physical conditions set on a hypersurface and the choice of regulating scheme. The extension of the present results obtained in the context of the Hartle-Hawking vacuum state to the corresponding contexts of the two other vacuum states will be the object of future work.

Finally, following the coincidence of the result in (94) with that in [4] consistency with established results requires that the, analytically evaluated, renormalised expression in (89) be compared with the results in [12] which have been obtained through a combination of analytical and numerical techniques and constitute an extension of the stated result in [4] to the static region of the Schwarzschild black-hole space-time. The result in [12] is of the form

(95) $< H|\phi^2(x)|H >_{ren} = \frac{1}{12(8\pi M)^2} \frac{1 - \left(\frac{2M}{r}\right)^4}{1 - \frac{2M}{r}} + \frac{\Delta(r)}{(8\pi M)^2}$

where the second term has been evaluated numerically.

In view of the fact that the first term in (95) identically coincides with the first term in (89) and of the fact that, pursuant to (90), the second term in (89) vanishes the second
term in (95) must be compared with the third term in (89). In order to accomplish such a comparison it is necessary to change, in that term, the independent variable $r$ to the variable

\begin{equation}
\xi = \frac{r}{M} - 1
\end{equation}

used in [12]. The result is

\begin{equation}
\frac{\tilde{\Delta}(r)}{(8\pi M)^2} = - \frac{2\pi}{(8\pi M)^2} \left[ \frac{\xi + 1}{\xi - 1} \right]^4 \times \\
\sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \int_{u_0[p]}^{\infty} \frac{\cos \left[ \frac{\pi}{4} (4u + 2p + 3)(1 - \sqrt{\frac{\xi - 1}{\xi + 1}}) \right]}{\pi^2 u^2 + 4(l^2 + l + 1)} \frac{I_p[2\sqrt{(l^2 + l + 1)\frac{\xi - 1}{\xi + 1}}]}{I_p[2\sqrt{(l^2 + l + 1)}] (2l + 1)} \ du
\end{equation}

In view of the fact that $I_p(x)$ is an increasing function of $x$, for all values of $p$ and of the fact that $0 \leq \frac{\xi - 1}{\xi + 1} < 1$; \( \lim_{\xi \to \infty} \frac{\xi - 1}{\xi + 1} = 1 \) convergence of the expression in (97) can be rigorously established by examining it in the context of the general asymptotic expression [11].

\begin{equation}
I_{\nu}(z) \sim \frac{e^z}{\sqrt{2\pi z}} ; \ |z| >> \nu ; \ -\frac{\pi}{2} < \text{Arg}z < \frac{\pi}{2}
\end{equation}

![Figure 1. \( \tilde{\Delta} \) as a function of \( \xi \). The approximation with \( \Delta \) emerges for values close to 1](image)

The two graphs which will be analyzed in what follows have been obtained through a program formulated in MATLAB. Although the graph in Fig. 1 yields insufficient information as to the exact behaviour of \( \tilde{\Delta} \) close to 1 it manifestly features the stated
limit \( \lim_{\xi \to 1} \tilde{\Delta}(\xi) = 0 \). It can be seen that there is an ample range of values of \( \xi \) for which the values of \( \tilde{\Delta} \) are in conformity with the claim made in [12] to the effect that for no values of \( \xi \) does the second term in (95) exceed 1\% of the first. In fact, inspection of the results cited in [12] for \( \Delta(\xi) \) reveals that the magnitude of the latter is of order of, at most, \( 10^{-5} \) within the range \( 1 \leq \xi \sim 1.005 \) which corresponds to that in (62).

The graph in Fig. 2 manifestly expresses the behaviour of \( \tilde{\Delta} \) close to 1. It can be seen that the range of values for \( \tilde{\Delta}(\xi) \) corresponding to values of \( \xi \) between 1.003 and \( \sim 1.004 \) is in very good agreement with the results indicated for \( \Delta(\xi) \) in [12] in the same range of \( \xi \). In addition, the absence of any oscillatory behaviour in that range suggests that \( \tilde{\Delta}(\xi) \) remains an increasing function also for values of \( \xi \) between 1 and 1.003 and that, consequently, there is very good agreement with the results in [12] for values of \( \xi \) between 1 and - at least - 1.004. In effect, Fig. 2 suggests that \( \tilde{\Delta}(\xi) \) increases from zero to a value of order of, at most, \( 10^{-5} \) for values of \( \xi \) between 1 and \( \sim 1.005 \), a range which corresponds to the physically reasonable estimate in (62) for the validity of the approximation to the scalar propagator. It is a limitation of the computer program used herein that, for values of \( \xi \) between 1 and 1.003, Fig. 2 reveals the stated information as to the behaviour of \( \tilde{\Delta}(\xi) \) only qualitatively. The exact behaviour of that function in that range requires a more elaborate computer program. Work in that direction is already in progress. However, the graphs in Fig. 1 and Fig. 2 reveal a very good agreement - if not a coincidence - between \( \tilde{\Delta} \) and \( \Delta \) at least within the range of values specified in (62). Within the same range, for that matter, the renormalised vacuum expectation value in (89) is in very good agreement with the corresponding expression in (95). This comparison renders the advantage which the Green function in (60) has in the
calculation of local expressions manifest. All the information on the singularity structure of that Green function is inherent in that function’s singular part in (60). By properly exploiting the universality of the quadratic divergence which a scalar propagator has, at least as a leading divergence, in any space-time that structure was rendered manifest in (74). At once, the Green function in (60) is entirely analytic. In turn, through a consistent renormalisation procedure $< H|\phi^2(x)|H >_{\text{ren}}$ was evaluated to the entirely analytic expression of the space-time geometry in (89) with considerably less effort than that required for the evaluation of the same physical quantity through the combination of analytical and numerical approaches in [12].

In view of the fact that the range of the exterior geometry within which (89) essentially coincides with (95) extends to the black hole’s event horizon the vacuum expectation value in (89) has the superior feature of being entirely analytic in the most physically important segment of the static region of the black-hole space-time geometry. At once, the stated agreement between (89) and (95) rigorously establishes the physically reasonable estimate made in (61) for the same range in the Euclidean sector of the Schwarzschild metric. That, is the range of physical relevance of the expression obtained in (60) for the massless, conformal, scalar propagator on the Schwarzschild black-hole geometry.

VII. Conclusions

The approximation to the conformal scalar propagator associated with the Hartle-Hawking vacuum state which has been developed herein eventuated in an analytic expression which is, for that matter, explicit in its dependence on the Schwarzschild black-hole space-time. In effect, this approximation is sharply distinct from all others. Although valid near the event horizon its range of validity extends to several orders of magnitude above the range within which quantum and backreaction effects are comparatively pronounced. As a result of such an approximation certain aspects of propagation related to the causal structure of the black-hole space-time have been studied.

An essential advantage of the Feynman propagator which has been developed herein is that its short-distance behaviour and singularity structure are manifest. This propagator is, for that matter, especially suited for an analytic evaluation of $< T_{\mu\nu}(x) >$ and $< \phi^2(x) >$. This aspect has been exploited in order to reproduce established results for the renormalised value of $< \phi^2(x) >$ on the event horizon as well as in the segment of the static region of the black-hole space-time which corresponds to the range of validity of this approximation.

Although the evaluation of $< T_{\mu\nu}(x) >$ in the static region of the Schwarzschild black-hole and the manner in which this compares to established results [9], [13] is an interesting enterprise an immediate priority is the extension of the present results in the interior region of the Schwarzschild black hole.

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APPENDIX
In this section the divergent infinite series in (74) will be expressed as poles.
The logarithmically divergent series can be expressed as

\[ \sum_{p=1}^{\infty} \frac{1}{p} = \lim_{x \to \infty} \left[ (1 - \frac{1}{x}) + \frac{1}{2}(1 - \frac{1}{x})^2 + \frac{1}{3}(1 - \frac{1}{x})^3 + \ldots \right] \]

In view of

\[ -\ln(1-u) = u + \frac{u^2}{2} + \frac{u^3}{3} + \ldots ; \quad 0 \leq u < 1 \]

and of the fact that, prior to taking the limit \( x \to \infty \), the condition for convergence in (A.2) is trivially satisfied (A.1) yields

\[ \sum_{p=1}^{\infty} \frac{1}{p} = -\lim_{x \to \infty} \ln[1 - (1 - \frac{1}{x})] = \lim_{x \to \infty} \ln x \]

Since the omission of a finite number of terms does not affect the convergence properties of a series it, also, is

\[ \sum_{p=p_0}^{\infty} \frac{1}{p} = \lim_{x \to \infty} \ln x \]

The quadratically divergent series in (74) requires a somewhat different approach. Expressing it as

\[ \sum_{k=0}^{\infty} (2k+1) = \left[ 1+3x^2+5x^4+\ldots \right]_{x'=1} \]

and setting

\[ g(x') = 1 + 3x'^2 + 5x'^4 + \ldots \]

it is

\[ \int_0^{<1} dx' g(x') = \frac{x'}{1-x'^2} \]

so that

\[ g(x') = \frac{1 + x'^2}{(1-x'^2)^2} \]

as a result of which (A.5) becomes

\[ \sum_{k=0}^{\infty} (2k+1) = \lim_{x' \to 1} \frac{1 + x'^2}{(1-x'^2)^2} = \frac{1}{2} \lim_{x' \to 1} \frac{1}{(1-x')^2} \]
Further, setting

\[ x' = 1 - \frac{1}{x} \]

results in

\[(A.7) \quad \sum_{k=0}^{\infty} (2k+1) = \frac{1}{2} \lim_{x \to \infty} x^2 \]

Since (A.4) and (A.7) respectively imply that for an arbitrarily large, but finite, value of \( x \) and for accordingly large values of \( p_0 \) and \( k_0 \) it is

\[ p'_0 >> p_0 \quad \frac{1}{p} \sim \ln x \]

and

\[ k_0 >> 1 \quad \sum_{k=0}^{(2k+1)} \sim \frac{1}{2} x^2 \]

it follows that

\[(A.8) \quad \sum_{k=0}^{\infty} \sum_{p=p_0}^{\infty} \frac{2k+1}{p} = \frac{1}{2} \lim_{x \to \infty} x^2 \ln x ; \quad p_0 >> 1 \]

which through

\[ x = \frac{1}{e^l} \]

finally becomes

\[(A.9) \quad \sum_{k=0}^{\infty} \sum_{p=p_0}^{\infty} \frac{2k+1}{p} = -\frac{1}{4} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon^2} ; \quad p_0 >> 1 \]

where \( l \) is a parameter to be determined.

At once, (A.7) also implies

\[(A.10) \quad \sum_{k=0}^{\infty} (2k+1) = \frac{1}{2} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon^2} \]
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