Gradient-Free Optimization for Non-Smooth Saddle Point Problems under Adversarial Noise

Abstract

We consider non-smooth saddle point optimization problems. To solve these problems, we propose a zeroth-order method under bounded or Lipschitz continuous noise, possible adversarial. In contrast to the state-of-the-art algorithms, our algorithm is optimal in terms of both criteria: oracle calls complexity and the maximum value of admissible noise. The proposed method is simple and easy to implement as it is built on zeroth-order version of the stochastic mirror descent. The convergence analysis is given in terms of the average and probability. We also pay special attention to the duality gap $r$-growth condition ($r \geq 1$), for which we provide a modification of our algorithm using the restart technique. We also comment on infinite noise variance and upper bounds in the case of Lipschitz noise. The results obtained in this paper are significant not only for saddle point problems but also for convex optimization.

1 Introduction

In this paper, we consider stochastic non-smooth saddle point problems of the following form

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y),$$

(1)

where $f(x, y) \triangleq \mathbb{E}_\xi [f(x, y, \xi)]$ is the expectation, w.r.t. $\xi \in \Xi$. $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is convex-concave and Lipschitz continuous, and $\mathcal{X} \subseteq \mathbb{R}^d_x$, $\mathcal{Y} \subseteq \mathbb{R}^d_y$ are convex compact sets. The standard interpretation of such min-max problems is the antagonistic game between a learner and an adversary, where the equilibria are the saddle points Neumann (1928). Now the interest in saddle point problems is renewed due to the popularity of generative adversarial networks (GANs), whose training involves solving min-max problems Goodfellow et al. (2014); Chen et al. (2017).

Motivated by many applications in the field of reinforcement learning Choromanski et al. (2018); Mania et al. (2018) and statistics, where only a black-box access to objective values is available, we consider zeroth-order oracle (also known as gradient-free oracle). Particularly, we mention the classical problem of adversarial multi-armed bandit Flaxman et al. (2004); Bartlett et al. (2008); Bubeck and Cesa-Bianchi (2012), where a learner receives a feedback given by the function evaluations from an adversary. Thus, zeroth-order methods Conn et al. (2009) are the workhorse technique when the
Related Work. Zeroth-order methods in the non-smooth setup were developed in a wide range of works, such as Polyak (1987), Spall (2003), Conn et al. (2009), Duchi et al. (2015), Shamir (2017), Nesterov and Spokoiny (2017), Gasnikov et al. (2017), Bezkosikov et al. (2020), Gasnikov et al. (2022). Particularly, in Shamir (2017), an optimal algorithm was provided as an improvement to Duchi et al. (2015) for a non-smooth case but Lipschitz continuous in stochastic convex optimization problems. However, this algorithm uses the exact function evaluations that can be infeasible in some applications. Indeed, objective $f(z, \xi)$ can be not directly observed but instead, its noisy approximation $\varphi(z, \xi) \equiv f(z, \xi) + \delta(z)$ can be queried, where $\delta(z)$ is some noise, possibly adversarial. This noisy-corrupted setup was considered in many works, however, such an algorithm that is optimal in terms of the number of oracle calls complexity and the maximum value of the noise has not been proposed. For instance, in Bayandina et al. (2018), Bezkosikov et al. (2020), optimal algorithms in terms of oracle calls complexity were proposed, however, they are not optimal in terms of the maximum value of the noise. In papers Risteski and Li (2016), Vasin et al. (2021), algorithms are optimal in terms of the maximum value of the noise, however, they are not optimal in terms of the oracle calls complexity. This paper presents a new algorithm which is optimal both in terms of the inexact oracle calls complexity and the maximum value of admissible noise. The method is built on a gradient-free version of the mirror descent with an inexact oracle. We consider two possible scenarios for the nature of the noise arising in different applications: the noise is bounded or is Lipschitz continuous. Table 1 demonstrates our contribution by comparing our results with the existing optimal bounds, where $\epsilon$ is the desired accuracy to solve problem (1) and $d$ is the problem dimension. We notice that the results obtained for saddle point problems are also valid for convex optimization.

Table 1: Summary of the Contribution

| PAPER                | PROBLEM       | EXPECTATION OR LARGE DEVIATION | IS THE NOISE LIPSCHITZ? | NUMBER OF ORACLE CALLS | MAXIMUM VALUE OF THE NOISE |
|----------------------|---------------|-------------------------------|-------------------------|-------------------------|-----------------------------|
| Bayandina et al. 2018| convex        | $E$                           | $\times$                | $d^{1/2}$               | $\epsilon^2/d^{3/2}$        |
| Bezkosikov et al. 2020| saddle point | $E$                           | $\times$                | $d^{1/2}$               | $\epsilon^2/d$             |
| Vasin et al. 2021    | convex        | $E$                           | $\times$                | $\text{Poly } (d^{1/4})$| $\epsilon^2/\sqrt{d}$      |
| Risteski and Li 2016 | convex        | $E$                           | $\times$                | $\text{Poly } (d^{1/4})$| $\max \{\epsilon^2/\sqrt{d}, d^{3/4}\}$ |
| THIS WORK            | saddle point  | $E$ and $P$                   | $\times$                | $d^{1/2}$               | $\epsilon^2/\sqrt{d}$      |
| THIS WORK            | saddle point  | $E$ and $P$                   | $\checkmark$            | $d^{1/2}$               | $\epsilon^2/\sqrt{d}$      |

(1) This bound is also the upper bound up to a logarithmic factor. In the large-scale setup ($\epsilon^{-2} \ll d$), the maximum is reached on the second term, namely $\epsilon^2/\sqrt{d}$.

(2) All of the estimates, except this one, in this column are for the maximum value of the noise. This estimate is the estimate of the Lipschitz constant as now the noise is Lipschitz continuous.

Contribution. Now we list our contribution as follows

- We provide an algorithm which is optimal in terms of number of oracle calls and maximum value of admissible noise. We state the results about its convergence in expectation and probability
- For the $r$-growth condition, we restate the results for the proposed algorithm run with restarts
- We comment on how the results can be modified under infinite noise variance
- We comment on ‘upper’ bound in the case of Lipschitz noise

Paper Organization. This paper is organized as follows. In Section 2, we present the main algorithm of the paper and analysis of its convergence. In Section 3, under additional assumption of $r$-growth condition we restate the results for the proposed algorithm run with restarts. In Section 4, we comment on the case of infinite noise variance. Finally, Section 5 gives some ideas about upper bounds in the case of Lipschitz noise.
2 Zeroth-order algorithm

In this section, we present an algorithm (see Algorithm 1) that is optimal in terms of the number of inexact zeroth-order oracle calls and the maximum value of adversarial noise. The algorithm is based on a gradient-free version of the stochastic mirror descent (SMD) [Ben-Tal and Nemirovski, 2013]. We start with some key notation, background material and assumptions.

2.1 Notation and assumptions

We use $\langle x, y \rangle \triangleq \sum_{i=1}^d x_i y_i$ to define the inner product of $x, y \in \mathbb{R}^d$, where $x_i$ is the $i$-th component of $x$. By norm $\| \cdot \|_p$, we mean the $\ell_p$-norm. Then the dual norm of the norm $\| \cdot \|_p$ is $\| \lambda \|_q \triangleq \max \{ \langle x, \lambda \rangle | \| x \|_p \leq 1 \}$. Operator $\mathbb{E}[\cdot]$ is the full expectation and operator $\mathbb{E}_\xi[\cdot]$ is the conditional expectation, w.r.t. $\xi$. Let us introduce the embedding space $\mathcal{Z} \triangleq \mathcal{X} \times \mathcal{Y}$, and then some $z \in \mathcal{Z}$ means $z \triangleq (x, y)$, where $x \in \mathcal{X}, y \in \mathcal{Y}$. On this embedding space, we introduce the $\ell_p$-norm and a prox-function $\omega(z)$ compatible with this norm. Then we define the Bregman divergence associated with $\omega(z)$ as

$$V_z(v) \triangleq \omega(z) - \omega(v) - \langle \nabla \omega(v), z - v \rangle \geq \| z - v \|^2_p / 2, \quad \text{for all } z, v \in \mathcal{Z}. $$

We also introduce a prox-operator as follows

$$ \text{Prox}_z(\xi) \triangleq \arg \min_{v \in \mathcal{Z}} (V_z(v) + \langle \xi, v \rangle), \quad \text{for all } z \in \mathcal{Z}. $$

Finally, we denote the $\omega$-diameter of $\mathcal{Z}$ by $\mathcal{D} \triangleq \max_{z, v \in \mathcal{Z}} \sqrt{2V_z(v)} = \tilde{O} \left( \max_{z, v \in \mathcal{Z}} \| z - v \|_p \right)$. Here $\tilde{O} (\cdot)$ is $O (\cdot)$ up to a $\sqrt{\log d}$-factor.

**Assumption 1** (Lipshitz continuity of the objective). Function $f(z, \xi)$ is $M_2$-Lipschitz continuous in $z \in \mathcal{Z}$ w.r.t. the $\ell_2$-norm, i.e., for all $z_1, z_2 \in \mathcal{Z}$ and $\xi \in \mathbb{X}$,

$$|f(z_1, \xi) - f(z_2, \xi)| \leq M_2(\xi) \| z_1 - z_2 \|_2. $$

Moreover, there exists a positive constant $M_2$ such that $\mathbb{E}[M_2(\xi)] \leq M_2^2$.

**Assumption 2** (Boundedness of the noise). For all $z \in \mathcal{Z}$, it holds $|\delta(z)| \leq \Delta$.

**Assumption 3** (Lipschitz continuity of the noise). Function $\delta(z)$ is $M_{2, \delta}$-Lipschitz continuous in $z \in \mathcal{Z}$ w.r.t. the $\ell_2$-norm, i.e., for all $z_1, z_2 \in \mathcal{Z}$,

$$|\delta(z_1) - \delta(z_2)| \leq M_{2, \delta} \| z_1 - z_2 \|_2. $$

2.2 Black-box oracle and gradient approximation.

We assume that we can query zeroth-order oracle corrupted by an adversarial noise $\delta(z)$:

$$\varphi(z, \xi) \triangleq f(z, \xi) + \delta(z). $$ (2)

The gradient of $g(z, \xi, e)$ from (2), w.r.t. $z$, can be approximated by the function evaluations in two random points closed to $z$. To do so, we define vector $e$ picked uniformly at random from the Euclidean unit sphere $\{e: \|e\|_2 = 1\}$. Let $e \triangleq (e_x^T, -e_y^T)^T$, where $\dim(e_x) \triangleq d_x, \dim(e_y) \triangleq d_y$ and $\dim(e) \triangleq d = d_x + d_y$. Then the gradient of $g(z, \xi, e)$ can be estimated by the following approximation with a small variance [Shamir, 2017]:

$$g(z, \xi, e) = \frac{d}{2\tau} \left( \varphi(z + \tau e, \xi) - \varphi(z - \tau e, \xi) \right) \left( e_x \atop -e_y \right), $$ (3)

where $\tau > 0$ is some constant.

2.3 Randomized smoothing.

Unfortunately, this standard zeroth-order approximation (3) is a poor estimator for subgradients of a non-smooth objective. To support this, let us consider the following example.
Example 1 (one-dimensional case). $f(x) = |x|$. Zeroth-order approximation of subgradients of $f(x)$ can be simplified as

$$g(x) = \frac{1}{2\tau}(f(x + \tau) - f(x - \tau)).$$

(4)

For point $x \in [-\tau, \tau]$, $g(x) = x/\tau$. However, for all $x > 0$, $\nabla f(x) = 1$ and for all $x < 0$, $\nabla f(x) = -1$.

Since the problem is non-smooth, we introduce the following smooth approximation of a non-smooth function (see Figure 1 as an example)

$$f^\tau(z) \equiv \mathbb{E}_e f(z + \tau \hat{e}),$$

where $\tau > 0$ and $\hat{e}$ is a vector picked uniformly at random from the Euclidean unit ball: $\{ \hat{e} : \|\hat{e}\|_2 \leq 1 \}$. Function $f^\tau(z)$ can be referred as a smooth approximation of $f(z)$ and it will be used only for deriving the convergence rate of the proposed algorithm. Here $f(z) \equiv \mathbb{E} f(z, \xi)$.

The next lemma presents the quality of such an approximation.

Lemma 1. Let $f(z)$ be $M_2$-Lipschitz continuous function. Then for $f^\tau(z)$ from (5), it holds

$$\sup_{z \in \mathbb{Z}} |f^\tau(z) - f(z)| \leq \tau M_2.$$ 

Proof. By the definition of $f^\tau(z)$ from (5) and Assumption we have

$$|f^\tau(z) - f(z)| = |\mathbb{E}_e [f(z + \tau e)] - f(z)| = \mathbb{E}_e [|f(z + \tau e) - f(z)|] \leq \mathbb{E}[M_2\|e\|_2] = M_2 \tau.$$

Lemma 2. Function $f^\tau(z)$ is differentiable with the following gradient

$$\nabla f^\tau(z) = \mathbb{E}_e \left[ \frac{d}{\tau}f(z + \tau e)e \right].$$

2.4 Algorithm and its convergence rate

Now we present zeroth-order algorithm to solve problem (1) (see Algorithm 1). The stepsize $\gamma_k = \frac{\pi}{M} \sqrt{\frac{2}{N}}$, where positive constant $M$ is chosen as:

1. under Assumption

$$M^2 \equiv \mathcal{O} \left( da_0^2 M_2^2 + d^2 a_0^2 \Delta^2 \tau^{-2} \right),$$

(6)

2. under Assumption

$$M^2 \equiv \mathcal{O} \left( da_0^2 \left( M_2^2 + M_2^2 \beta \right) \right),$$

(7)

where $N$ is the number of algorithm iterations and $a_0^2 \equiv \mathcal{O} \left( \sqrt{\mathbb{E}[\|e\|_2^4]} \right) = \mathcal{O} \left( \min\{q, \log d\} d^{2/q - 1} \right)$.

Gorbunov et al. (2019).

The next theorem presents the convergence rate of the Algorithm 1 in terms of the expectation.

Theorem 4. Let $\epsilon$ be the desired accuracy to solve problem (1) and $\tau$ from randomized smoothing (5) be chosen as $\tau = \mathcal{O} \left( \epsilon / M_2 \right)$. Let function $f(x, y, \xi)$ satisfy the Assumption and one of the two following statements is true:

1. Assumption holds with $\Delta = \mathcal{O} \left( \frac{\epsilon^2}{2dM_2}\sqrt{\alpha} \right)$.

Algorithm 1 Zeroth-order SMD

**Input:** iteration number $N$;

**Output:** $\hat{z}^{\ast} \leftarrow \arg \min_{z \in \mathbb{Z}} d(z)$

1: $z^1 \leftarrow \arg \min_{z \in \mathbb{Z}} d(z)$
2: Sample $e^k, \xi^k$ independently
3: Initialize $\gamma_k \to \frac{\pi}{M} \sqrt{\frac{2}{N}}$ with $M$ defined by (6) or (7)
4: Calculate $g(z^k, \xi^k, e^k)$ via (3)
5: $z^{k+1} \leftarrow \text{Prox}_{\gamma_k} \left( \gamma_k g(z^k, \xi^k, e^k) \right)$
6: end for

Theorem 1. Let $\epsilon$ be the desired accuracy to solve problem (1) and $\tau$ from randomized smoothing (5) be chosen as $\tau = \mathcal{O} \left( \epsilon / M_2 \right)$. Let function $f(x, y, \xi)$ satisfy the Assumption and one of the two following statements is true:

1. Assumption holds with $\Delta = \mathcal{O} \left( \frac{\epsilon^2}{2dM_2}\sqrt{\alpha} \right).$
2. Assumption 3 holds with $M_{2,\delta} = O\left(\frac{\tau}{\Delta^2}a_q^2\right)$.

Then for $\epsilon_{sad} \triangleq \max_{y \in Y} f(\hat{x}^N, y) - \min_{x \in \mathcal{X}} f(x, \hat{y}^N)$ with the output of Algorithm 2 $z^N \triangleq (\hat{x}^N, \hat{y}^N)$, it holds $\mathbb{E}\left[\epsilon_{sad}(z^N)\right] \leq \epsilon$ after the following number of iterations

$$N = O\left(\frac{dM_2^2D^2a_q^2}{\epsilon^2}\right).$$

Next we specify the Theorem 4 in the two following special setups: the $\ell_2$-norm and the $\ell_1$-norm in the two following examples.

Example 2 ($\ell_2$-norm). Let $p = 2$, then $q = 2$ and $\sqrt{\mathbb{E}\left[\|e\|_2^2\right]} = 1$. Thus, $a_2^2 = 1$ and $D^2 = \max_{z, v \in Z} \|z - v\|_2^2$. Consequently, the number of iterations in the Corollary 4 can be rewritten as follows

$$N = O\left(\frac{dM_2^2}{\epsilon^2} \max_{z, v \in Z} \|z - v\|_2^2\right).$$

Example 3 ($\ell_1$-norm). [Shamir 2017, Lemma 4] Let $p = 1$ then, $q = \infty$ and $\sqrt{\mathbb{E}\left[\|e\|_\infty^2\right]} = O\left(\frac{\log d}{d}\right)$. Thus, $a_\infty^2 = O\left(\frac{\log d}{d}\right)$ and $D^2 = O\left(\log d \max_{z, v \in Z} \|z - v\|_1^2\right)$. Consequently, the number of iterations in the Corollary 4 can be rewritten as follows

$$N = O\left(\frac{(\log d)^2M_2^2}{\epsilon^2} \max_{z, v \in Z} \|z - v\|_1^2\right).$$

Remark 1 (Variable separation). Hereafter we assumed that the proximal setups for spaces $\mathcal{X}$ and $\mathcal{Y}$ are the same. In some applications, this is not the case. For instance, when spaces $\mathcal{X}$ and $\mathcal{Y}$ require different Bregman divergences. In this case, we can replace the proximal step $z^{k+1} \leftarrow \text{Prox}_{z_k} (\gamma_k g(z^k, \xi^k, e^k))$ in Algorithm 2 by two proximal steps on spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively

$$x^{k+1} \leftarrow \text{Prox}_{z_k} (\gamma_k g_x(z^k, \xi^k, e^k)), \quad y^{k+1} \leftarrow \text{Prox}_{y_k} (\gamma_k g_y(z^k, \xi^k, e^k)).$$

Proof of Theorem 4. For brevity, we provide the proof only under Assumption 2. The convergence rate under Assumption 3 can be obtained similarly. By the definition $z^{k+1} = \text{Prox}_{z_k} (\gamma_k g(z^k, e^k, \xi^k))$ we get [Ben-Tal and Nemirovski 2013], for all $u \in Z$

$$\gamma_k (g(z^k, e^k, \xi^k), z^k - u) \leq V_{z^k}(u) - V_{z^{k+1}}(u) + \frac{\gamma_d^2}{2} \|g^d(z^k, e^k, \xi^k)\|_q^2/2.$$

Taking the conditional expectation w.r.t. $\xi, e$ and summing for $k = 1, \ldots, N$ we obtain, for all $u \in Z$

$$\sum_{k=1}^N \gamma_k \mathbb{E}_{e^k, \xi^k}\left[\|g(z^k, e^k, \xi^k, z^k - u)\| \right] \leq V_{z^k}(u) + \sum_{k=1}^N \frac{\gamma_d^2}{2} \mathbb{E}_{e^k, \xi^k}\left[\|g(z^k, e^k, \xi^k)\|_q^2\right].$$

Lemma 3. For $g(z, \xi, e)$ from (3), the following holds under Assumption 7 for $c > 0$

1. and Assumption 7 $\mathbb{E}_{e^k, \xi^k}\left[\|g(z^k, e^k, \xi^k)\|_q^2\right] \leq cdM_2^2a_q^2 + d^2\Delta^2\tau^{-2}a_q^2$,
2. and Assumption 3 $\mathbb{E}_{e^k, \xi^k}\left[\|g(z^k, e^k, \xi^k)\|_q^2\right] \leq cd(M_2^2 + M_{2,\delta})a_q^2$.

Step 1. For the second term in the r.h.s of (8) we use Lemma 3 and get under Assumption 2

$$\mathbb{E}_{e^k, \xi^k}\left[\|g(z^k, e^k, \xi^k)\|_q^2\right] \leq cdM_2^2a_q^2 + d^2\Delta^2\tau^{-2}a_q^2,$$

where $c$ is some numerical constant and $\sqrt{\mathbb{E}\left[\|e^k\|_q^2\right]} \leq a_q^2$.

Lemma 4. For $g(z, \xi, e)$ from (3) and $f^\ast(z)$ from (5), the following holds

1. under Assumption 2 $\mathbb{E}_{e, r}\left[\langle g(z, \xi, e), r \rangle \right] \geq \langle \nabla f^\ast(z), r \rangle - d\Delta\tau^{-1}\mathbb{E}_{e}\left[\|e\|_r\right]$,
2. under Assumption 3 $\mathbb{E}_{e, r}\left[\langle g(z, \xi, e), r \rangle \right] \geq \langle \nabla f^\ast(z), r \rangle - dM_{2,\delta}\mathbb{E}_{e}\left[\|e\|_r\right]$,
Step 2. For the l.h.s. of (8) and \( u \overset{\Delta}{=} (x^T, y^T)^T \), we use Lemma 4 under Assumption 2
\[
\sum_{k=1}^{N} \gamma_k \mathbb{E}_{e_k} \left[ \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \geq \sum_{k=1}^{N} \gamma_k \langle \nabla f^r(z^k), z^k - u \rangle
- \sum_{k=1}^{N} \gamma_k \mathbb{E}_{e_k} \left[ \langle d\Delta^{-1} e^k, z^k - u \rangle \right].
\tag{10}
\]

For the first term of the r.h.s. of (10) we have
\[
\sum_{k=1}^{N} \gamma_k \langle \nabla f^r(z^k), z^k - u \rangle = \sum_{k=1}^{N} \gamma_k \left( \langle \nabla_x f^r(x^k, y^k), x^k - x \rangle - \langle \nabla_y f^r(x^k, y^k), y^k - y \rangle \right)
\geq \sum_{k=1}^{N} \gamma_k \left( f^r(x^k, y^k) - f^r(x, y^k) - f^r(x^k, y^k) + f^r(x^k, y^k) \right) = \sum_{k=1}^{N} \gamma_k (f^r(x^k, y^k) - f^r(x, y^k))
\tag{11}
\]

Then we use the fact function \( f^r(x, y) \) is convex in \( x \) and concave in \( y \) and obtain
\[
\left( \sum_{i=1}^{N} \gamma_k \right)^{-1} \sum_{k=1}^{N} \sum_{k=1}^{N} \gamma_k (f^r(x^k, y^k) - f^r(x, y^k)) \geq f^r(\hat{x}^N, y) - f^r(x, \hat{y}^N),
\tag{12}
\]
where \((\hat{x}^N, \hat{y}^N)\) is the output of the Algorithm 1. Using (12) for (11) we get
\[
\sum_{k=1}^{N} \gamma_k \langle \nabla f^r(z^k), z^k - u \rangle \geq \sum_{k=1}^{N} \gamma_k (f^r(\hat{x}^N, y) - f^r(x, \hat{y}^N)).
\tag{13}
\]

The next lemma is the key moment of the proof giving optimal convergence result.

Lemma 5. Let vector \( e \) be a random unit vector from the Euclidean unit sphere \( \{e: \|e\|_2 = 1\} \). Then it holds for all \( r \in \mathbb{R}^d \)
\[
\mathbb{E}_{e} \left[ \|e, r\| \right] \leq \|r\|_2 / \sqrt{d}.
\]

Using this Lemma 5 we estimate the term \( \mathbb{E}_{e_k} \left[ \langle e^k, z^k - u \rangle \right] \) in (10)
\[
\mathbb{E}_{e_k} \left[ \langle e^k, z^k - u \rangle \right] \leq \|z^k - u\|_2 / \sqrt{d}.
\tag{14}
\]

Now we substitute (13) and (14) to (10), and get under Assumption 2
\[
\sum_{k=1}^{N} \gamma_k \mathbb{E}_{e_k, \xi_k} \left[ \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \geq \sum_{k=1}^{N} \gamma_k \left( f^r(\hat{x}^N, y) - f^r(x, \hat{y}^N) - \sqrt{d\Delta^{-1}}\|z^k - u\|_2 \right).
\tag{15}
\]

Step 3 (under Assumption 2). Now we combine (15) with (9) for (8) and obtain under Assumption 2 the following
\[
\sum_{k=1}^{N} \gamma_k \left( f^r(\hat{x}^N, y) - f^r(x, \hat{y}^N) - \sqrt{d\Delta^{-1}}\|z^k - u\|_2 \right) \leq V_{\hat{x}}(u) + \sum_{k=1}^{N} \gamma_k \left( cdM_2^2 a_q^2 + d^2\Delta^2\tau^{-1} a_q^2 \right).
\tag{16}
\]

Using Lemma 1 we obtain
\[
f^r(\hat{x}^N, y) - f^r(x, \hat{y}^N) \geq f(\hat{x}^N, y) - f(x, \hat{y}^N) - 2\tau M_2.
\]

Using this we can rewrite (16) as follows
\[
f(\hat{x}^N, y) - f(x, \hat{y}^N) \leq \frac{V_{\hat{x}}(u)}{\sum_{k=1}^{N} \gamma_k} + \frac{cdM_2^2 a_q^2 + d^2\Delta^2\tau^{-1} a_q^2}{\sum_{k=1}^{N} \gamma_k} \sum_{k=1}^{N} \gamma_k \frac{2}{2} 
+ \sqrt{d\Delta^{-1}} \max_{k} \|z^k - u\|_2 + 2\tau M_2.
\tag{17}
\]
For the r.h.s. of (17) we use the definition of the $\omega$-diameter of $Z$:
$$D \triangleq \max_{z,v \in Z} \sqrt{2V_z(v)}$$ and estimate $\|z^k - u\|_2 \leq D$ for all $z^1, \ldots, z^k$ and all $u \in Z$. Using this for (17) and taking the maximum in $(x, y) \in (\mathcal{X}, \mathcal{Y})$, we obtain
$$\max_{y \in \mathcal{Y}} f(\hat{x}^N, y) - \min_{x \in \mathcal{X}} f(x, \hat{y}^N) \leq \frac{D^2 + (cdM_2^2a_0^2 + d^2\Delta^2\tau^{-2}a_0^2)\sum_{k=1}^{N} \gamma_k}{\sum_{k=1}^{N} \gamma_k} + \sqrt{d}\Delta D\tau^{-1} + 2\tau M_2.$$

Taking the expectation of this and choosing stepsize $\gamma_k = \frac{D}{M\sqrt{2N}}$ with $M^2 \triangleq cdM_2^2a_0^2 + d^2\Delta^2\tau^{-2}a_0^2$ we get
$$\mathbb{E} \left[ \max_{y \in \mathcal{Y}} f(\hat{x}^N, y) - \min_{x \in \mathcal{X}} f(x, \hat{y}^N) \right] \leq MD\sqrt{2/N} + \sqrt{d}\Delta D\tau^{-1} + 2\tau M_2. \quad (18)$$

3 Zeroth-order algorithm with restarts

In this section, we assume that we additionally have the $r-$growth condition for duality gap (see, Shapiro et al. [2021] for convex optimization problems). For such a case, we apply the restart technique Juditsky and Nesterov (2014) to Algorithm 1.

Assumption 5 ($r-$growth condition). There is $r \geq 1$ and $\mu_r > 0$ such that for all $z = (x, y) \in Z$
$$\frac{\mu_r}{2}\|z - z^*\|_p \leq f(x, y) - f(x^*, y),$$
where $(x^*, y^*)$ is a solution of problem (1).

The next theorem states that if additionally Assumption 5 holds, then the convergence results of Theorem 4 can be improved.

Theorem 6. Let $\epsilon$ be the desired accuracy to solve problem (1) and $\tau$ from randomized smoothing (5) be chosen as $\tau = \mathcal{O}(\epsilon/M_2)$. Let function $f(x, y, \xi)$ satisfy the Assumption 7 and Assumption 3 with $r \geq 2$. Let one of the two following statements be true

1. Assumption 2 holds with $\Delta \lesssim \frac{\mu^{1/r}e^{2-1/r}}{M_2\sqrt{d}}$;
2. Assumption 3 holds with $M_{2,\Delta} \lesssim \frac{\mu^{1/r}e^{2-1/r}}{\sqrt{d}}$.

Then for $\hat{z}_{sad} \triangleq f(\hat{z}^N, y^*) - f(x^*, \hat{y}^N)$, where $\hat{z}^N \triangleq (\hat{x}^N, \hat{y}^N)$ is the output of Algorithm 1 with restarts, it holds $\mathbb{E} \left[ \hat{z}_{sad}(\hat{z}^N) \right] \leq \epsilon$ after the following number of iterations
$$N = \tilde{O} \left( \frac{dM_2^2a_0^2d}{\mu^{2/r}e^{2(r-1)/r}} \right) \quad (19)$$

3.1 Convergence rate in high-probability bound

Heretofore, all the results were stated in the average, now we provide the convergence results in terms of probability. To do so, we need the following assumption.

Assumption 7 (Uniformly Lischitz continuity of the objective). Function $f(z, \xi)$ is uniformly $M_2$-Lipschitz continuous in $z \in Z$ w.r.t. the $\ell_2$-norm, i.e., for all $z_1, z_2 \in Z$ and $\xi \in \Xi$,
$$|f(z_1, \xi) - f(z_2, \xi)| \leq M_2\|z_1 - z_2\|_2.$$

The next theorem is stated in the Euclidean proximal setup ($p = q = 2, a_2 = 1$).

Theorem 8. Let $\epsilon$ be the desired accuracy to solve problem (1) and $\tau$ be chosen as $\tau = \mathcal{O}(\epsilon/M_2)$. Let the Assumption 2 holds with $\Delta \lesssim \frac{\mu^{1/r}e^{2-1/r}}{M_2\sqrt{d}}$ and let function $f(x, y, \xi)$ satisfy Assumption 7.

Then for the output $\hat{z}^N \triangleq (\hat{x}^N, \hat{y}^N)$ of Algorithm 1, it holds $\mathbb{P} \{ \hat{z}_{sad}(\hat{z}^N) \leq \epsilon \} \geq 1 - \sigma$ after
$$N = \mathcal{O} \left( dM_2^2D^2/\epsilon^2 \right)$$
iterations. Moreover if Assumption 5 is satisfied with \( r \geq 1 \), then for the output \( \hat{z}^N \triangleq (\hat{x}^N, \hat{y}^N) \) of Algorithm 1 with restarts, it holds \( \mathbb{P} \{ \epsilon_{\text{sad}}(\hat{z}^N) \leq \epsilon \} \geq 1 - \sigma \) after the following number of iterations
\[
N = \tilde{O}\left( \frac{dM_2^2}{\mu r^{2/r} \sigma^{2(r-1)/r}} \right).
\] (20)

Remark 2. In the case \( r = 2 \) it is probably possible to improve the bound (20) in \( \log \epsilon^{-1} \) factor by using alternative algorithm [Harvey et al. 2019].

4 Infinite noise variance

Now we comment on the case when the second moment of the stochastic subgradient \( \nabla f(z, \xi) \) is unbounded. In this case the rate of convergence may changes dramatically. For such a case, we modify the Assumptions 1.

Assumption 9 (Lipschitz continuity of the objective under infinite noise variance). Function \( f(z, \xi) \) is \( M_2 \)-Lipschitz continuous in \( z \in Z \) w.r.t. the \( \ell_2 \)-norm, i.e., for all \( z_1, z_2 \in Z \) and \( \xi \in \Xi \),
\[
|f(z_1, \xi) - f(z_2, \xi)| \leq M_2(\xi) \|z_1 - z_2\|_2.
\]
Moreover, there exists a positive constant \( \hat{M}_2 \) such that \( \mathbb{E} [M_2(\xi)^{1+\kappa}] \leq \hat{M}_2^{1+\kappa} \), where \( \kappa \in (0, 1] \).

If Assumption 9 holds, the stepsize in Algorithm 1 is replaced by
\[
\gamma_k = \left( (1 + \kappa) V_{s^2}(z^*)/\kappa \right) M^{-1} N^{-\tau/\kappa},
\]
where \( V_{s^2}(z^*) \) is the Bregman divergence determined by the following prox-function with \( q \in [1+\kappa, \infty) \) and \( 1/p + 1/q = 1 \)
\[
\omega(x) = K_q^{1+\kappa}/(1+\kappa) \|x\|_p^{1+\kappa} \text{ where } K_q = 10 \max \left\{ 1, (q - 1)^{(1+\kappa)/2} \right\},
\]
and constant \( \hat{M} \) is determined as (can be obtained from Shamir (2017) Lemmas 9 – 11)

1. under Assumption 2 \( \mathbb{E} [\|g(z, \xi, e)\|_q^{1+\kappa}] \leq \tilde{c} a_2^2 d \|\tau\|^2/2 \hat{M}_2^{1+\kappa} + 2^{1+\kappa} d^{1+\kappa} a_2 \delta \Delta^2 r^{-2} = \hat{M}_2^{1+\kappa}, \)
2. under Assumption 5 \( \mathbb{E} [\|g(z, \xi, e)\|_q^{1+\kappa}] \leq \tilde{c} a_2^2 d \|\tau\|^2/2 (\hat{M}_2^{1+\kappa} + \hat{M}_2^{1+\kappa}) = \hat{M}_2^{1+\kappa}, \)

where \( \tilde{c} \) is some numerical constant and \( \sqrt{\mathbb{E}_\nu \|e\|_q^{2+2\kappa}} \leq \tilde{a}_2^2. \) As a particular case: \( \tilde{a}_2^2 = 1, \)
\( \tilde{a}_\infty^2 = O\left( \frac{\log d}{d^{(1+\kappa)/2}} \right). \)

Based on Vural et al. (2022), one can prove that convergence rate of Algorithm 1 changes dramatically in comparison with Theorem 6 (see [18]), namely under Assumption 2
\[
\mathbb{E} \left[ \max_{y \in Y} f(\hat{x}^N, y) - \min_{x \in X} f(x, \hat{y}^N) \right] \leq \hat{M} \left( \frac{1 + \kappa}{\kappa} V_{s^2}(z^*) \right)^{1+\kappa} N^{-\tau/\kappa} + \Delta D/\tau + 2\tau \hat{M}_2.
\]
These results can be further generalized to \( r \)-growth condition for duality gap (\( r \geq 2 \)).

5 ‘Upper’ bound in the case of Lipschitz noise

Now let us consider a stochastic convex optimization problem of the form
\[
\min_{x \in X} F(x) \triangleq \mathbb{E}_\xi f(x, \xi), \quad (21)
\]
where \( X \subseteq \mathbb{R}^d \) is a convex set, and for all \( \xi \), \( f(x, \xi) \) is convex in \( x \in X \) and satisfies Assumption 1.

The empirical counterpart of this problem (21) is
\[
\min_{x \in X} \hat{F}(x) \triangleq \frac{1}{N} \sum_{k=1}^N f(x, \xi^k). \quad (22)
\]
The exact solution ($\epsilon/2$-solution) of (22) is an $\epsilon$-solution of (21) if the sample size $N$ is taken as follows [Shapiro and Nemirovski 2005; Feldman 2016]

\[ N = \Omega \left( D_2^2 / \epsilon^2 \right), \]  

(23)

where $D_2$ is the diameter of $X$ in the $\ell_2$-norm. This lower bound is tight [Shapiro and Nemirovski 2005; Shapiro et al. 2021]. On the other hand, $\tilde{F}(x)$ from (22) can be considered as an inexact zeroth-order oracle for $F(x)$ from (21). If $\delta(x) = F(x) - \tilde{F}(x)$ then in Poly $(d, 1/\epsilon)$ points $y, x$ with probability $1 - \beta$ the following holds

\[ |\delta(y) - \delta(x)| = O \left( M_2 \| y - x \|_2 \sqrt{\frac{N}{\ln \left( \frac{\text{Poly} \left( d, 1/\epsilon \right)}{\beta} \right)} \right) = \tilde{O} \left( M_2 \| y - x \|_2 \sqrt{\frac{N}{\ln \left( \frac{\text{Poly} \left( d, 1/\epsilon \right)}{\beta} \right)} \right), \]

i.e., $\delta(x)$ is a Lipschitz function with Lipschitz constant

\[ M_{2,\delta} = \tilde{O} \left( M_2 / \sqrt{N} \right). \]

Let us assume there exists a zeroth-order algorithm that can solve (21) with accuracy $\epsilon$ in Poly $(d, 1/\epsilon)$ oracle calls, where an oracle returns an inexact value of $F(x)$ with a noise that has the following Lipschitz constant

\[ M_{2,\delta} \gg \frac{\epsilon}{D_2 \sqrt{d}}, \]

We can use this algorithm to solve problem (21) with $N$ determines from (see (24))

\[ \frac{M_2}{\sqrt{N}} \gg \frac{\epsilon}{D_2 \sqrt{d}}, \quad \text{i.e.} \quad N \ll d M_2^2 D_2^2 / \epsilon^2, \]

that contradicts the lower bound (23). Thereby it is impossible in general to solve with accuracy $\epsilon$ (in function value) Lipschitz convex optimization problem via Poly $(d, 1/\epsilon)$ inexact zero-order oracle calls if Lipschitz constant of noise is greater than

\[ \epsilon / (D_2 \sqrt{d}). \]

(25)

Unfortunately, we obtain this upper bound assuming that Poly $(d, 1/\epsilon)$ points were chosen regardless of $\{\xi^k\}_{k=1}^N$. That is not the case for the most of practical algorithms, in particular, considered above. But the dependence of these points from $\{\xi^k\}_{k=1}^N$ is significantly weakened by randomization we use in zero-order methods. So we may expect that nevertheless this upper bound still takes place.

For arbitrary Poly $(d, 1/\epsilon)$ points (possibly that could depend on $\{\xi^k\}_{k=1}^N$) we can guarantee only (see [Shapiro et al. 2021])

\[ M_{2,\delta} = \tilde{O} \left( \sqrt{d} M_2 / \sqrt{N} \right). \]

(26)

That is large than (24). Consequently, the upper bound (25) should be rewritten as

\[ \epsilon / D_2. \]

(27)

We believe that (27) is not tight upper bound, rather than (25). That is, there exists algorithm (see Algorithm 1) that can solve (21) with accuracy $\epsilon$ (in function value) via $\approx d M_2^2 D_2^2 / \epsilon^2$ inexact zero-order oracle calls if Lipschitz constant of noise is bounded from above by $M_{2,\delta} \lesssim \epsilon / (D_2 \sqrt{d})$. But there are no algorithms reaching the same accuracy $\epsilon$ by using Poly $(d, 1/\epsilon)$ inexact zero-order oracle calls if Lipschitz constant of noise is bounded from above by $M_{2,\delta} \gg \epsilon / (D_2 \sqrt{d})$, in particular, for $M_{2,\delta}$ given by (27). Note, that (23) holds also if $f(x, \xi)$ has Lipschitz $x$-gradient [Feldman 2016]. Hence we can expect that obtained lower bounds take place also for smooth problems.

**Conclusion**

In this paper, we demonstrate how to solve non-smooth stochastic convex-concave saddle point problems with two-point gradient-free oracle. In the Euclidean proximal setup, we obtain optimal oracle complexity bound $\tilde{O}(d / \epsilon^2)$. We also generalize this result for an arbitrary proximal setup and obtain a tight upper bound $\tilde{O}(\epsilon^2 / \sqrt{d})$ on maximal level of additive adversary noise in oracle calls. We generalize this result for the class of saddle point problems satisfying $r$-growth condition for duality gap.
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A Proofs of auxiliary lemmas

Proof of Lemma 3. Let us consider

\[
E_{\xi,e} \left[ \|g(z, \xi, e)\|_q^2 \right] = E_{\xi,e} \left[ \left\| \frac{d}{2\tau} (\phi(z + \tau e, \xi) - \phi(z - \tau e, \xi)) e \right\|_q^2 \right]
\]

\[
= \frac{d^2}{4\tau^2} E_{\xi,e} \left[ \|e\|_q^2 \left( f(z + \tau e, \xi) + \phi(z + \tau e) - f(z - \tau e, \xi) - \phi(z - \tau e) \right)^2 \right]
\]

\[
\leq \frac{d^2}{2\tau^2} \left( E_{\xi,e} \left[ \|e\|_q^2 (f(z + \tau e, \xi) - f(z - \tau e, \xi))^2 \right] + E_e \left[ \|e\|_q^2 (\phi(z + \tau e) - \phi(z - \tau e))^2 \right] \right),
\]

where we used that for all \( a, b, (a - b)^2 \leq 2a^2 + 2b^2 \). For the first term in the r.h.s. of (28), the following holds with an arbitrary parameter \( \alpha \)

\[
\frac{d^2}{2\tau^2} E_{\xi,e} \left[ \|e\|_q^2 (f(z + \tau e, \xi) - f(z - \tau e, \xi))^2 \right]
\]

\[
= \frac{d^2}{2\tau^2} E_{\xi,e} \left[ \|e\|_q^2 ((f(z + \tau e, \xi) - \alpha) - (f(z - \tau e, \xi) - \alpha))^2 \right]
\]

\[
\leq \frac{d^2}{\tau^2} E_{\xi,e} \left[ \|e\|_q^2 (f(z + \tau e, \xi) - \alpha)^2 + (f(z - \tau e, \xi) - \alpha)^2 \right] \quad / \ast \forall a, b, (a - b)^2 \leq 2a^2 + 2b^2 \ast /
\]

\[
= \frac{d^2}{\tau^2} \left( E_{\xi,e} \left[ \|e\|_q^2 (f(z + \tau e, \xi) - \alpha)^2 \right] + E_{\xi,e} \left[ \|e\|_q^2 (f(z - \tau e, \xi) - \alpha)^2 \right] \right)
\]

\[
= \frac{2d^2}{\tau^2} E_{\xi,e} \left[ \|e\|_q^2 (f(z + \tau e, \xi) - \alpha)^2 \right]. \quad / \ast \text{ the distribution of } e \text{ is symmetric } \ast /
\]

Applying the Cauchy–Schwarz inequality for (29) and using \( \sqrt{E \left[ \|e\|_q^2 \right]} \leq a_q^2 \) we obtain

\[
\frac{d^2}{\tau^2} E_{\xi,e} \left[ \|e\|_q^2 (f(z + \tau e, \xi) - \alpha)^2 \right] \leq \frac{d^2}{\tau^2} E_{\xi,e} \left[ \sqrt{E \left[ \|e\|_q^2 \right]} \sqrt{E_e \left[ (f(z + \tau e, \xi) - \alpha)^4 \right]} \right]
\]

\[
\leq \frac{d^2 a_q^2}{\tau^2} E_{\xi,e} \left[ \sqrt{E_e \left[ (f(z + \tau e, \xi) - \alpha)^4 \right]} \right]. \quad (30)
\]

Next we use the following lemma.

Lemma 6. [Shamir 2017, Lemma 9] For any function \( f(e) \) which is \( M \)-Lipschitz w.r.t. the \( \ell_2 \)-norm, it holds that if \( e \) is uniformly distributed on the Euclidean unit sphere, then for some constant \( c \)

\[
\sqrt{E \left[ (f(e) - \mu f(e))^4 \right]} \leq cM_2^2 / d.
\]

Then we use this Lemma 6 along with the fact that \( f(z + \tau e, \xi) \) is \( \tau M_2(\xi) \)-Lipschitz, w.r.t. \( e \) in terms of the \( \ell_2 \)-norm. Thus for (30) and \( \alpha \triangleq E_e \left[ f(z + \tau e, \xi) \right] \), it holds

\[
\frac{d^2 a_q^2}{\tau^2} E_{\xi,e} \left[ \sqrt{E_e \left[ (f(z + \tau e, \xi) - \alpha)^4 \right]} \right] \leq \frac{d^2 a_q^2}{\tau^2} c \frac{\tau^2 E \left[ M_2^2(\xi) \right]}{d} = cdM_2^2 a_q^2. \quad (31)
\]

1. Under the Assumption 2 for the second term in the r.h.s. of (28), the following holds

\[
\frac{d^2}{4\tau^2} E_e \left[ \|e\|_q^2 (\delta(z + \tau e) - \delta(z - \tau e))^2 \right] \leq \frac{d^2}{\tau^2} E_e \left[ \|e\|_q^2 \right] \leq \frac{d^2}{\tau^2} \Delta^2 a_q^2. \quad (32)
\]
2. Under the Assumption \[ n \] for the second term in the r.h.s. of (28), the following holds with an arbitrary parameter \( \beta \)

\[
\frac{d^2}{4\tau^2} \mathbb{E}_e \left[ \|e\|_q^2 \left( \delta(z + \tau e) - \delta(z - \tau e) \right)^2 \right] 
\leq \frac{d^2}{\tau^2} \mathbb{E}_e \left[ \|e\|_q^2 \left( \left( \delta(z + \tau e) - \beta \right)^2 + \left( \delta(z - \tau e) - \beta \right)^2 \right) \right] 
\leq \frac{d^2}{\tau^2} \mathbb{E}_e \left[ \|e\|_q^2 \left( \delta(z + \tau e) - \beta \right)^2 \right].
\]

Then we use Lemma \[ \ref{lemma} \] together with the fact that \( \delta(z + \tau e) \) is \( \tau M_{2,\delta} \)-Lipschitz continuous, w.r.t. \( e \) in terms of the \( \ell_2 \)-norm. Thus, for (33) and \( \beta \triangleq \mathbb{E}_e [\delta(z + \tau e)] \), the following holds

\[
\frac{d^2 a_q^2}{\tau^2} \mathbb{E}_e \left[ \left( \delta(z + \tau e) - \beta \right)^4 \right] \leq \frac{d^2 a_q^2}{\tau^2} \cdot \frac{c\tau^2 M_{2,\delta}^2}{d} \leq ca_q^2 dM_{2,\delta}^2.
\]

Using (31) and (32) (or (34)) for (28), we get the statement of the lemma.

\[ \square \]

**Proof of Lemma \[ \ref{lemma} \]** Let us consider

\[
g(z, e, \xi) = \frac{d}{2\tau} (\varphi(z + \tau e, \xi) - \varphi(z - \tau e, \xi)) e
= \frac{d}{2\tau} ((f(z + \tau e, \xi) - f(z - \tau e, \xi)) e + (\delta(z + \tau e) - \delta(z - \tau e)) e).
\]

Using this we have the following

\[
\mathbb{E}_{\xi, e} \left[ (g(z, \xi, e), r) \right] = \frac{d}{2\tau} \mathbb{E}_{\xi, e} \left[ ((f(z + \tau e, \xi) - f(z - \tau e, \xi)) e, r) \right]
+ \frac{d}{2\tau} \mathbb{E}_e \left[ (\delta(z + \tau e) - \delta(z - \tau e)) e, r) \right].
\]

Taking the expectation, w.r.t. \( e \), from the first term of the r.h.s. of (35) and using the symmetry of the distribution of \( e \), we have

\[
\frac{d}{2\tau} \mathbb{E}_{\xi, e} \left[ ((f(z + \tau e, \xi) - f(z - \tau e, \xi)) e, r) \right]
= \frac{d}{2\tau} \mathbb{E}_{\xi, e} \left[ (f(z + \tau e, \xi) e, r) \right] + \frac{d}{2\tau} \mathbb{E}_{\xi, e} \left[ (f(z - \tau e, \xi) e, r) \right]
= \frac{d}{\tau} \mathbb{E}_e \left[ (\mathbb{E}_\xi [f(z + \tau e, \xi)] e, r) \right] = \frac{d}{\tau} \mathbb{E}_e \left[ (\nabla f^\tau(z), r) \right].
\]

\[ \square \]

1. For the second term of the r.h.s. of (35) under the Assumption \[ \ref{assumption} \] we obtain

\[
\frac{d}{2\tau} \mathbb{E}_e \left[ (\delta(z + \tau e) - \delta(z - \tau e)) e, r) \right] \geq - \frac{d}{2\tau} 2\Delta \mathbb{E}_e \left[ |\langle e, r \rangle| \right] = - \frac{d\Delta}{\tau} \mathbb{E}_e \left[ |\langle e, r \rangle| \right].
\]

2. For the second term of the r.h.s. of (35) under the Assumption \[ \ref{assumption} \] we obtain

\[
\frac{d}{2\tau} \mathbb{E}_e \left[ (\delta(z + \tau e) - \delta(z - \tau e)) e, r) \right] \geq - \frac{d}{2\tau} M_{2,\delta}^2 2\tau \mathbb{E}_e \left[ |\langle e, r \rangle| \right] = -dM_{2,\delta} \mathbb{E}_e \left[ |\langle e, r \rangle| \right].
\]

Using (36) and (37) (or (38)) for (35) we get the statement of the lemma.

\[ \square \]
Complete proof of Theorem 4

Proof of Theorem 4. By the definition $z^{k+1} = \text{Prox}_{z^k}(g(z^k, e^k, \xi^k))$ we get [Ben-Tal and Nemirovski (2013)] for all $u \in \mathcal{Z}$

$$\gamma_k \langle g(z^k, e^k, \xi^k), z^k - u \rangle \leq V_{z^k}(u) - V_{z^{k+1}}(u) + \gamma_k^2 \|g(z^k, e^k, \xi^k)\|^2_q / 2.$$ 

Taking the conditional expectation w.r.t. $\xi, e$ and summing for $k = 1, \ldots, N$ we obtain, for all $u \in \mathcal{Z}$

$$\sum_{k=1}^N \gamma_k \mathbb{E}_{e^k, \xi^k} \left[ \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \leq V_{z^k}(u) + \sum_{k=1}^N \frac{\gamma_k^2}{2} \mathbb{E}_{e^k, \xi^k} \left[ \|g(z^k, e^k, \xi^k)\|^2_q \right]. \quad (39)$$

Step 1.

For the second term in the r.h.s of inequality (39) we use Lemma 3 and obtain

1. under Assumption 2

$$\mathbb{E}_{e^k, \xi^k} \left[ \|g(z^k, \xi^k, e^k)\|^2_q \right] \leq c d M_2^2 a_q^2 + d^2 \Delta^2 \tau^{-2} a_q^2, \quad (40)$$

2. under Assumption 3

$$\mathbb{E}_{e^k, \xi^k} \left[ \|g(z^k, \xi^k, e^k)\|^2_q \right] \leq c (M_2^2 + M_{2, \delta}^2) a_q^2, \quad (41)$$

where $c$ is some numerical constant and $\sqrt{\mathbb{E} \left[ \|e^k\|^2_q \right]} \leq a_q.$

Step 2.

For the l.h.s of (39), we use Lemma 4 with $u \equiv (x^T, y^T)^T$

1. under Assumption 2

$$\sum_{k=1}^N \gamma_k \mathbb{E}_{e^k, \xi^k} \left[ \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \geq \sum_{k=1}^N \gamma_k \langle \nabla f(x^k), z^k - u \rangle$$

$$+ \sum_{k=1}^N \gamma_k \mathbb{E}_{e^k} \left[ \langle d \Delta^{-1} e^k, z^k - u \rangle \right]. \quad (42)$$

2. under Assumption 3

$$\sum_{k=1}^N \gamma_k \mathbb{E}_{e^k, \xi^k} \left[ \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \geq \sum_{k=1}^N \gamma_k \langle \nabla f(x^k), z^k - u \rangle$$

$$+ \sum_{k=1}^N \gamma_k \mathbb{E}_{e^k} \left[ \langle d M_{2, \delta} e^k, z^k - u \rangle \right]. \quad (43)$$

For the first term of the r.h.s of (42) and (43) we have

$$\sum_{k=1}^N \gamma_k \langle \nabla f(x^k), z^k - u \rangle = \sum_{k=1}^N \gamma_k \left( \langle \nabla_x f^T(x^k, y^k), x^k - x \rangle - \langle \nabla_y f^T(x^k, y^k), y^k - y \rangle \right)$$

$$= \sum_{k=1}^N \gamma_k \left( \langle \nabla_x f^T(x^k, y^k), x^k - x \rangle - \langle \nabla_y f^T(x^k, y^k), y^k - y \rangle \right)$$

$$\geq \sum_{k=1}^N \gamma_k (f^T(x^k, y^k) - f^T(x, y^k)) - (f^T(x^k, y^k) - f^T(x^k, y))$$

$$= \sum_{k=1}^N \gamma_k (f^T(x^k, y) - f^T(x, y^k)). \quad (44)$$
Then we use the fact function $f^\tau(x, y)$ is convex in $x$ and concave in $y$ and obtain
\[
\frac{1}{\sum_{i=1}^{N} \gamma_k} \sum_{i=1}^{N} \gamma_k \left( f^\tau(x^k, y) - f^\tau(x, y^k) \right) \geq f^\tau \left( \frac{\sum_{k=1}^{N} \gamma_k x^k}{\sum_{k=1}^{N} \gamma_k}, y \right) - f^\tau \left( x, \frac{\sum_{k=1}^{N} \gamma_k y^k}{\sum_{k=1}^{N} \gamma_k} \right) = f^\tau \left( \hat{x}^N, y \right) - f^\tau \left( x, \hat{y}^N \right),
\] (45)
where $(\hat{x}^N, \hat{y}^N)$ is the output of the Algorithm. Using (44) for (44) we get
\[
\sum_{k=1}^{N} \gamma_k \langle \nabla f^\tau(z^k), z^k - u \rangle \geq \sum_{k=1}^{N} \gamma_k \left( f^\tau \left( \hat{x}^N, y \right) - f^\tau \left( x, \hat{y}^N \right) \right).
\] (46)
Next we estimate the term $E_{e^k} \left[ \| (e^k, z^k - u) \| \right]$ in (42) and (43), by the Lemma
\[
E_{e^k} \left[ \| (e^k, z^k - u) \| \right] \leq \| z^k - u \|_2 / \sqrt{d}.
\] (47)
Now we substitute (46) and (47) to (42) and (43), and get

1. under Assumption 2
\[
\sum_{k=1}^{N} \gamma_k E_{e^k, \xi^k} \left[ \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \geq \sum_{k=1}^{N} \gamma_k \left( f^\tau \left( \hat{x}^N, y \right) - f^\tau \left( x, \hat{y}^N \right) - \sqrt{d} \Delta_k \| z^k - u \|_2 \tau^{-1} \right). (48)
\]

2. under Assumption 3
\[
\sum_{k=1}^{N} \gamma_k E_{e^k, \xi^k} \left[ \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \geq \sum_{k=1}^{N} \gamma_k \left( f^\tau \left( \hat{x}^N, y \right) - f^\tau \left( x, \hat{y}^N \right) - \sqrt{d} M_2 \| z^k - u \|_2 \right). (49)
\]

Step 3. (under Assumption 2)
Now we combine (48) with (49) for (39) and obtain under Assumption 2, the following
\[
\sum_{k=1}^{N} \gamma_k \left( f^\tau \left( \hat{x}^N, y \right) - f^\tau \left( x, \hat{y}^N \right) - \sqrt{d} \Delta \| z^k - u \|_2 \tau^{-1} \right) \leq V_{z^1} (u) + \sum_{k=1}^{N} \gamma_k \left( \frac{1}{2} \left( cdM_2 a_2^2 + d^2 \Delta^{2} \tau^{-2} a_4^2 \right) \right). (50)
\]
Using Lemma 1 we obtain
\[
f^\tau \left( \hat{x}^N, y \right) - f^\tau \left( x, \hat{y}^N \right) \geq f \left( \hat{x}^N, y \right) - f \left( x, \hat{y}^N \right) - 2 \tau M_2.
\]
Using this we can rewrite (50) as follows
\[
f \left( \hat{x}^N, y \right) - f \left( x, \hat{y}^N \right) \leq \frac{V_{z^1} (u)}{\sum_{k=1}^{N} \gamma_k} + \frac{1}{2} \sum_{k=1}^{N} \gamma_k \frac{\gamma_k}{2} + \sqrt{d} \Delta \max_k \| z^k - u \|_2 \tau^{-1} + 2 \tau M_2. (51)
\]
For the r.h.s. of (51) we use the definition of the $\omega$-diameter of $Z$:
\[
D \triangleq \max_{x, y \in Z} \sqrt{2 V_x} \left( v \right) \text{ and estimate } \| z^k - u \|_2 \leq D \text{ for all } z^1, \ldots, z^k \text{ and all } u \in Z. \text{ Using this for (51) and taking the maximum in } (x, y) \in (X, Y)$, we obtain
\[
\max_y f \left( \hat{x}^N, y \right) - \min_x f \left( x, \hat{y}^N \right) \leq \frac{D^2 + \left( cdM_2 a_2^2 + d^2 \Delta^{2} \tau^{-2} a_4^2 \right) \sum_{k=1}^{N} \gamma_k / 2}{\sum_{k=1}^{N} \gamma_k} + \sqrt{d} D \tau^{-1} + 2 \tau M_2. (52)
\]
Taking the expectation of (52) and choosing learning rate $\gamma_k = \frac{D}{M_{case1}} \sqrt{\frac{1}{N}}$ with $M_{case1} \triangleq cdM_2 a_2^2 + d^2 \Delta^{2} \tau^{-2} a_4^2$ in (52) we get
\[
E \left[ \max_y f \left( \hat{x}^N, y \right) - \min_x f \left( x, \hat{y}^N \right) \right] \leq M_{case1} D \sqrt{2/N} + \sqrt{d} D \tau^{-1} + 2 \tau M_2.
\]
Step 4. (under Assumption 3)
Now we combine (49) with (41) for (39) and obtain under Assumption 3
\[
\sum_{k=1}^{N} \gamma_k f^r(\hat{x}^N, y) - f^r(x, \hat{y}^N) - \sum_{k=1}^{N} \gamma_k \sqrt{d} M_{2, \delta} \|z^k - u\|_2 \leq V_{z,1}(u) + \sum_{k=1}^{N} \frac{\gamma_k^2}{2} c_d^2 d (M_2^2 + M_{2, \delta}^2). \tag{53}
\]
Using Lemma 1 we obtain
\[
f^r(\hat{x}^N, y) - f^r(x, \hat{y}^N) \geq f(\hat{x}^N, y) - f(x, \hat{y}^N) - 2\tau M_2.
\]
Using this we can rewrite (53) as follows
\[
f(\hat{x}^N, y) - f(x, \hat{y}^N) \leq \frac{V_{z,1}(u)}{\sum_{k=1}^{N} \gamma_k} + \frac{cd(M_2^2 + M_{2, \delta}^2)a_q^2}{\sum_{k=1}^{N} \gamma_k^2} \sum_{k=1}^{N} \gamma_k^2 / 2
\]
\[
+ \sqrt{d} M_{2, \delta} \max_k \|z^k - u\|_2 + 2\tau M_2. \tag{54}
\]
For the r.h.s. of (54) we use the definition of the \( \omega \)-diameter of \( Z \):
\[D \triangleq \max_{z \in Z} \sqrt{2} V_z(v)\] and estimate \( \|z^k - u\|_2 \leq D \) for all \( z^1, \ldots, z^k \) and all \( u \in Z \). Using this for (54) and taking the maximum in \( (x, y) \in (\mathcal{X}, \mathcal{Y}) \), we obtain
\[
\max_{y \in \mathcal{Y}} f(\hat{x}^N, y) - \min_{x \in \mathcal{X}} f(x, \hat{y}^N) \leq \frac{D^2 + cd(M_2^2 + M_{2, \delta}^2)a_q^2 \sum_{k=1}^{N} \gamma_k^2 / 2}{\sum_{k=1}^{N} \gamma_k}
\]
\[
+ \sqrt{d} M_{2, \delta} D + 2\tau M_2. \tag{55}
\]
Taking the expectation of (55) and choosing learning rate \( \gamma_k = \frac{D}{M_{\text{case}2}} \sqrt{\frac{2}{N}} \) with \( M_{\text{case}2} \triangleq cd(M_2^2 + M_{2, \delta}^2)a_q^2 \) in (55) we get
\[
\mathbb{E} \left[ \max_{y \in \mathcal{Y}} f(\hat{x}^N, y) - \min_{x \in \mathcal{X}} f(x, \hat{y}^N) \right] \leq M_{\text{case}2} D \sqrt{2/N} + \sqrt{d} M_{2, \delta} D + 2\tau M_2.
\]
\[
\square
\]
C Sketch of the proof of Theorem 6

Sketch of the proof of Theorem 6. We repeat the proof of Theorem 4 except that now \( z^1 \) can be chosen in a stochastic way. Moreover, now we use a rougher inequality instead of (47)
\[
\mathbb{E}_{\omega \in \mathcal{W}} [\langle \omega^k, z^k - u \rangle] \leq D/\sqrt{d}. \tag{56}
\]
Step 1. (under Assumption 2)
Taking the expectation in (51), choosing \( (x, y) = (x^*, y^*) \), and learning rate \( \gamma_k = \sqrt{\mathbb{E}[V_{z,1}(z^*)]} \sqrt{\frac{2}{N}} \) with \( M_{\text{case}1} \triangleq cdM_2^2a_q^2 + d^2 D^2 r^{-2} a_q^2 \) we get
\[
\mathbb{E} \left[ f(\hat{x}^N, y^*) - f(x^*, \hat{y}^N) \right] \leq \sqrt{\frac{2}{N}} M_{\text{case}1} \sqrt{\mathbb{E}[V_{z,1}(z^*)]} + \sqrt{d} D \tau^{-1} + 2\tau M_2. \tag{57}
\]
Step 2. (under Assumption 3)
Taking the expectation in (54), choosing \( (x, y) = (x^*, y^*) \), and learning rate \( \gamma_k = \sqrt{\mathbb{E}[V_{z,1}(z^*)]} \sqrt{\frac{2}{N}} \) with \( M_{\text{case}2} \triangleq cd(M_2^2 + M_{2, \delta}^2)a_q^2 \) we obtain
\[
\mathbb{E} \left[ f(\hat{x}^N, y^*) - f(x^*, \hat{y}^N) \right] \leq \sqrt{\frac{2}{N}} M_{\text{case}2} \sqrt{\mathbb{E}[V_{z,1}(z^*)]} + \sqrt{d} M_{2, \delta} D + 2\tau M_2. \tag{58}
\]
Step 3. (Restarts)
Now let \( \tau \) be chosen as \( \tau = \mathcal{O}(\epsilon/M_2) \), where \( \epsilon \) is the desired accuracy to solve problem (1). If one of the two following statement holds
1. Assumption 2 holds and \( \Delta = O \left( \frac{\epsilon^2}{DM \sqrt{d}} \right) \)
2. Assumption 3 holds and \( M_{2, \delta} = O \left( \frac{\epsilon^2}{D \sqrt{d}} \right) \)

then we obtain the convergence rate of the following form

\[
\mathbb{E} \left[ f \left( \hat{x}^{N_1}, y^* \right) - f \left( x^*, \hat{y}^{N_1} \right) \right] = O \left( \frac{\sqrt{dM_2 a_q}}{\sqrt{N_1}} \sqrt{\mathbb{E}[V_1(z^*)]} \right).
\] (59)

In this step we will employ the restart technique that is a generalization of the technique proposed in [Juditsky and Nesterov 2014]. For the l.h.s. of (59) we use the Assumption 5. For the r.h.s. of (59) we use the fact \( V_1(z^*) = O(\|z^*\|_p^2) \) from [Gasnikov and Nesterov 2018 Remark 3]

\[
\frac{\mu_r}{2} \mathbb{E} \left[ \|z^{N_1} - z^*\|_p^2 \right] \leq \mathbb{E} \left[ f \left( \hat{x}^{N_1}, y^* \right) - f \left( x^*, \hat{y}^{N_1} \right) \right] = O \left( \frac{\sqrt{dM_2 a_q}}{\sqrt{N_1}} \sqrt{\mathbb{E} \left[ \|z^1 - z^*\|_p^2 \right]} \right).
\] (60)

Then for the l.h.s. of (60) we use the Jensen inequality and get the following

\[
\frac{\mu_r}{2} \left( \mathbb{E} \left[ \|z^{N_1} - z^*\|_p^2 \right] \right)^{r/2} \leq \frac{\mu_r}{2} \mathbb{E} \left[ \|z^{N_1} - z^*\|_p^2 \right] \leq \mathbb{E} \left[ f \left( \hat{x}^{N_1}, y^* \right) - f \left( x^*, \hat{y}^{N_1} \right) \right]
\]

\[
= O \left( \frac{\sqrt{dM_2 a_q}}{\sqrt{N_1}} \sqrt{\mathbb{E} \left[ \|z^1 - z^*\|_p^2 \right]} \right).
\] (61)

Finally, let us introduce \( R_k \triangleq \sqrt{\mathbb{E} \left[ \|z^{N_k} - z^*\|_p^2 \right]} \) and \( R_0 \triangleq \sqrt{\mathbb{E} \left[ \|z^1 - z^*\|_p^2 \right]} \). Then we take \( N_1 \) so as to halve the distance to the solution and get

\[ N_1 = O \left( \frac{dM_2 a_q^2}{\mu_r^2 R_1^{(r-1)}} \right). \]

Next, after \( N_1 \) iterations, we restart the original method and set \( z^1 = z^{N_1} \). We determine \( N_2 \) similarly: we halve the distance \( R_1 \) to the solution, and so on. Thus, after \( k \) restarts, the total number of iterations will be

\[ N = N_1 + \cdots + N_k = O \left( \frac{2^{(r-1)} dM_2 a_q^2}{\mu_r^2 R_0^{(r-1)}} \left( 1 + 2^{(r-1)} + \cdots + 2^{(k-1)(r-1)} \right) \right). \] (62)

Now we need to determine the number of restarts. To do so, we fix the desired accuracy and using the inequality (60) we obtain

\[ \mathbb{E} [\epsilon_{\text{sad}}] = O \left( \frac{\mu_r R_k^2}{2} \right) = \hat{O} \left( \frac{a_q M_2 \sqrt{d}}{\sqrt{N_k}} \right) = \hat{O} \left( \frac{\mu_r R_0^2}{2^{k \epsilon}} \right) \leq \epsilon. \] (63)

Then to fulfill this condition, one can choose \( k = \log_2 \left( \frac{\hat{O} (\mu_r R_0^2 / \epsilon)}{\epsilon} \right) / r \) and using (62) we get the total number of iterations

\[ N = \hat{O} \left( \frac{2^{k(r-1)} dM_2 a_q^2}{\mu_r^2 R_0^{(r-1)}} \right) = \hat{O} \left( \frac{dM_2 a_q^2}{\mu_r^{2/r} \epsilon^{2(r-1)/r}} \right). \]

If in Theorem 6 we use a tighter inequality (47) instead of (56) (as in Theorem 4), then the estimations on the \( \Delta \) and \( M_{2, \delta} \) can be improved. Choosing \( u = (x^*, y^*) \) we can provide exponentially decreasing sequence of \( D^k = E \|z^k - u\|_2 \) in (59) and get

1. under Assumption 2 \( \Delta \lesssim \frac{\|z^k - u\|_2}{M_2 \sqrt{d}} \)
2. under Assumption 3 \( M_{2, \delta} \lesssim \frac{\|z^k - u\|_2}{d} \)

\[ \square \]
D Sketch of the proof of Theorem 8

Sketch of the proof of Theorem 8

By the definition \( z^{k+1} = \text{Prox}_{z^k} (\gamma_k g(z^k, e^k, \xi^k)) \) we get [Ben-Tal and Nemirovski (2013)], for all \( u \in \mathcal{Z} \)

\[
\gamma_k g(z^k, e^k, \xi^k), z^k - u) \leq V_{z^k}(u) - V_{z^{k+1}}(u) + \gamma_k^2 \|g(z^k, e^k, \xi^k)\|^2 / 2
\]

Summing for \( k = 1, \ldots, N \) we obtain, for all \( u \in \mathcal{Z} \)

\[
\sum_{k=1}^{N} \gamma_k g(z^k, e^k, \xi^k), z^k - u) \leq V_{z^1}(u) + \sum_{k=1}^{N} \gamma_k^2 \|g(z^k, e^k, \xi^k)\|^2 / 2. \tag{64}
\]

Next we provide the definition of zeroth-order gradient approximation similarly to (3) but under zero noise

\[
g_f(z, \xi, e) = \frac{d}{2\tau} (f(z + \tau e, \xi) - f(z - \tau e, \xi)) \begin{pmatrix} e_x \\ -e_y \end{pmatrix}, \tag{65}
\]

where \( \tau > 0 \) is some constant.

Lemma 7 (Concentration of Lipschitz functions on the Euclidean unit sphere). (Ledoux, 2001) [proof of Proposition 2.10 and Corollary 2.6] For any function \( g(e) \) which is \( L \)-Lipschitz w.r.t. the \( \ell_2 \)-norm, it holds that if \( e \) is uniformly distributed on the Euclidean unit sphere, then

\[
\mathbb{P} (|g(e) - \mathbb{E}[g(e)]| > t) \leq 2 \exp \left( -\frac{c t^2}{L^2} \right),
\]

where \( c' \) is some numerical constant.

Step 1.

Now we estimate the second term in the r.h.s of inequality (64) under Assumption 2

\[
\|g(z^k, e^k, \xi^k)\|^2_2 = \frac{d^2}{4\tau^2} (f(z + \tau e, \xi) + \delta(z + \tau e) - f(z - \tau e, \xi) - \delta(z - \tau e))^2 \\
\leq \|g_f(z^k, e^k, \xi^k)\|^2_2 + \frac{d^2 \Delta^2}{2\tau^2}. \tag{66}
\]

To estimate the first term in the r.h.s of (66), we notice that function \( g_f(z^k, e^k, \xi^k) \) is uniformly \( dM_2 \)-Lipschitz continuous and \( \mathbb{E}_{e^k} g_f(z^k, e^k, \xi^k) = 0 \). Thus, we use Lemma 7 and obtain for some constant \( c \):

\[
\mathbb{P} (\|g_f(z^k, e^k, \xi^k)\|_2 > t) \leq 2 \exp \left( -\frac{ct^2}{M_2^2 d} \right) \tag{67}
\]

Let us denote \( \phi_k = \gamma_k^2 \|g_f(z^k, e^k, \xi^k)\|^2_2/2 \) and \( \sigma_k = 2\gamma_k^2 dM_2^2 \). Then we consider conditional expectation

\[
\mathbb{E}_{\phi_k} \left[ \exp \left( \frac{\phi_k}{\sigma_k} \right) \right] = \mathbb{E}_{\phi_k} \left[ \exp \left( \frac{\|g_f(z^k, e^k, \xi^k)\|^2_2}{4dM_2^2} \right) \right] = \int_0^\infty \mathbb{P} \left( \exp \left( \frac{\|g_f(z^k, e^k, \xi^k)\|^2_2}{4dM_2^2} \right) \geq \tilde{t} \right) d\tilde{t} \\
\leq \int_0^1 1 d\tilde{t} + \int_1^\infty \mathbb{P} \left( \exp \left( \frac{\|g_f(z^k, e^k, \xi^k)\|^2_2}{4dM_2^2} \right) \geq \tilde{t} \right) d\tilde{t} \leq 1 + 1 \\
\leq \exp(1).
\]

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Thus, we can use (Lan et al., 2012, Lemma 2, case B) and get

\[ \mathbb{P} \left( \sum_{k=1}^{N} \gamma_k^2 \|g_f(z^k, e^k, \xi^k)\|^2_2 > 2dM^2_2 \sum_{k=1}^{N} \gamma_k^2 + 2\Omega dM^2_2 \right) \leq \exp \left( -\frac{\Omega^2}{12} \right) + \exp \left( -\frac{3\sqrt{N}\Omega}{4} \right) \]

Thus using this and (66) we can estimate the second term in the r.h.s of inequality (64) as follows

\[ \mathbb{P} \left( \sum_{k=1}^{N} \gamma_k^2 \|g(z^k, e^k, \xi^k)\|^2_2 / 2 > 2dM^2_2 \sum_{k=1}^{N} \gamma_k^2 + 2\Omega dM^2_2 \left( \sum_{k=1}^{N} \gamma_k^4 + \frac{d^2\Delta^2}{2\tau^2} \sum_{k=1}^{N} \gamma_k^2 \right) \right) \leq \exp \left( -\frac{\Omega^2}{12} \right) + \exp \left( -\frac{3\sqrt{N}\Omega}{4} \right) \] (68)

**Step 2.**
Using the notation (65) we rewrite the l.h.s. of (64) as following:

\[ \sum_{k=1}^{N} \gamma_k \langle g(z^k, e^k, \xi^k), z^k - u \rangle = \sum_{k=1}^{N} \gamma_k \langle g_f(z^k, e^k, \xi^k) - \nabla f^* (z^k), z^k - u \rangle + \]

\[ + \sum_{k=1}^{N} \gamma_k \langle \frac{d}{2\tau} (\delta(z^k + \tau e) - \delta(z^k - \tau e)) e, z^k - u \rangle \]

\[ + \sum_{k=1}^{N} \gamma_k \langle \nabla f^* (z^k), z^k - u \rangle. \] (69)

1. For the first term in the r.h.s. of (69), we provide the following notions:

\[ \sigma_k = 2\gamma_k \sqrt{dM^2_2 \sqrt{V_{z^1}(u)\Omega}}, \]

\[ \phi_k(\xi_k) = \gamma_k \langle g_f(z^k, e^k, \xi^k) - \nabla f^* (z), z^k - u \rangle. \]

For applying the case A of Lemma 2 from (Lan et al., 2012) we need to estimate the module of function \( \gamma_k \langle g_f(z^k, e^k, \xi^k), z^k - u \rangle \). Using Assumption 1 we obtain:

\[ \gamma_k \langle g_f(z^k, e^k, \xi^k), z^k - u \rangle \leq \gamma_k \frac{d}{2\tau} \left| f(z + \tau e, \xi) - f(z - \tau e, \xi) \right| \left| \frac{e_x - e_y}{\sqrt{d}} \right|, \]

\[ \leq \gamma_k dM_2 \left| \langle e, z^k - u \rangle \right|. \] (70)

Now we need to estimate the term \( \left| \langle e, z^k - u \rangle \right| \). To do this, using the Poincaré’s lemma from (Lifshits, 2012) paragraph 6.3 we rewrite \( e \) in different form:

\[ e \overset{D}{=} \frac{\eta}{\sqrt{\eta_1^2 + \cdots + \eta_d^2}}, \] (71)

where \( \eta = (\eta_1, \eta_2, \ldots, \eta_d)^T = \mathcal{N}(0, I_d) \). Using Lemma 1 (Laurent and Massart, 2000) it follows:

\[ \mathbb{P} \left( \sum_{k=1}^{d} |\eta_k|^2 \leq d - 2\sqrt{\Omega d} \right) \leq \exp(-\Omega). \] (72)

Using the definition of \( \eta \) we obtain:

\[ \langle \eta, z^k - u \rangle \overset{D}{=} \mathcal{N}(0, ||z^k - u||_2^2). \] (73)
Using (72), (75) and (71) we can estimate the r.h.s of (70):

\[
P \left( \frac{|\langle e, z^k - u \rangle|}{\sqrt{d}} > \frac{\Omega V_{z^k}(u)}{\sqrt{d}} \right) \leq 3 \exp(-\Omega/4).
\]  

(74)

Moreover, using (70), (74) and the fact that \( V_{z^k}(u) = O(V_{z^k}(u)) \) for all \( k \geq 1 \) from Gorbunov et al. [2021] we have:

\[
P \left( \exp \left( (\gamma_k g_f(z^k, e^k, \xi^k) - \nabla f^*(z), z^k - u) \frac{2}{\sigma_k^2} \right) \geq \exp(1) \right) \leq 3 \exp(-\Omega/4).
\]

From Lan et al. [2012] case A of Lemma 2 it holds

\[
P \left( \sum_{k=1}^{N} \gamma_k \langle g_f(z^k, e^k, \xi^k) - \nabla f^*(z), z^k - u \rangle > 4\sqrt{d} \Omega^2 \sqrt{V_{z^k}(u) M_2} \sum_{k=1}^{N} \gamma_k \right)
\]

\[
\leq 2 \exp(-\Omega^2/3) + 3 \exp(-\Omega/4).
\]

(75)

2. For the second term in r.h.s of (69) under the Assumption 2 we obtain:

\[
\sum_{k=1}^{N} \gamma_k \frac{d}{2\tau} (\delta(z^k + \tau e) - \delta(z^k - \tau e)) e, z^k - u \geq - \sum_{k=1}^{N} \gamma_k \frac{d\Delta}{\tau} |\langle e, z^k - u \rangle|
\]

Using (74) we can estimate the r.h.s of (76):

\[
P \left( \sum_{k=1}^{N} \gamma_k \frac{d\Delta}{\tau} |\langle e, z^k - u \rangle| > \frac{\Omega \sqrt{d} \Delta}{\tau} \sum_{k=1}^{N} \gamma_k \right) \leq 3 \exp(-\Omega/4).
\]

(77)

3. We rewrite the third term in r.h.s. of (69) in the following form:

\[
\sum_{k=1}^{N} \gamma_k \langle \nabla f^*(z^k), z^k - u \rangle = \sum_{k=1}^{N} \gamma_k \left( \langle \nabla_x f^*(x^k, y^k), x^k - x \rangle - \langle \nabla_y f^*(x^k, y^k, y^k - y) \rangle \right)
\]

\[
\geq \sum_{k=1}^{N} \gamma_k \left( f^*(x^k, y^k) - f^*(x, y^k) - f^*(x^k, y^k) - f^*(x^k, y^k) \right)
\]

\[
= \sum_{k=1}^{N} \gamma_k \left( f^*(x^k, y) - f^*(x, y^k) \right).
\]

(78)

Then we use the fact function \( f^*(x, y) \) is convex in \( x \) and concave in \( y \) and obtain

\[
\frac{1}{\gamma_k} \sum_{k=1}^{N} \gamma_k \left( f^*(x^k, y) - f^*(x, y^k) \right) \leq f^*(\hat{x}^N, y) - f^*(x, \hat{y}^N),
\]

(79)

where \( (\hat{x}^N, \hat{y}^N) \) is the output of the Algorithm 1. Using (79) for (78) we get

\[
\sum_{k=1}^{N} \gamma_k \langle \nabla f^*(z^k), z^k - u \rangle \geq \sum_{k=1}^{N} \gamma_k \left( f^*(\hat{x}^N, y) - f^*(x, \hat{y}^N) \right).
\]

(80)

Step 4.
Then we use this with (69), (76) and (80) for (64) and obtain

\[
f^*(\hat{x}^N, y) - f^*(x, \hat{y}^N) \leq \frac{V_{x^k}(u)}{\sum_{k=1}^{N} \gamma_k} + \frac{1}{\sum_{k=1}^{N} \gamma_k} \frac{\gamma_k}{2} \|g(z^k, e^k, \xi^k)\|^2
\]

\[
- \frac{1}{\sum_{k=1}^{N} \gamma_k} \sum_{k=1}^{N} \gamma_k \langle g_f(z^k, e^k, \xi^k) - \nabla f^*(z^k), z^k - u \rangle + \frac{1}{\sum_{k=1}^{N} \gamma_k} \sum_{k=1}^{N} \gamma_k \frac{d\Delta}{\tau} |\langle e, z^k - u \rangle|.
\]

(81)
For the l.h.s. of (81) we use Lemma 1 and obtain
\[ f^* (\hat{x}^N, y) - f^* (x, \hat{y}^N) \geq f (\hat{x}^N, y) - f (x, \hat{y}^N) - 2\tau M_2. \]  
(82)

Using (68), (75), (77), (82) and taking the maximum in \((x, y) \in (\mathcal{X}, \mathcal{Y})\), we obtain we can rewrite (81) as follow:
\[
\mathbb{P} \left( \max_{y \in \mathcal{Y}} f (\hat{x}^N, y) - \min_{x \in \mathcal{X}} f (x, \hat{y}^N) \geq \frac{4\Omega^2 \sqrt{d} \sqrt{V_{z^i} (u) M_2 \sqrt{\sum_{k=1}^N \gamma_k^2}}}{\sum_{k=1}^N \gamma_k} + \frac{2M_2^2 \sqrt{\sum_{k=1}^N \gamma_k^2}}{c \sum_{k=1}^N \gamma_k} \right)
\]
\[
- \frac{\Omega \sqrt{d} \Delta}{\tau} + \frac{2 \sqrt{2} \Delta^2 \sum_{k=1}^N \gamma_k}{c \gamma_k} + \frac{\sqrt{V_{z^i} (u) M_2^2 \sqrt{d}}}{c \sqrt{N \sqrt{d}}} + \frac{\Delta^2 \sum_{k=1}^N \gamma_k}{2 \tau^2} \leq \exp(-\Omega^2/12) + \exp(-3\sqrt{N \Omega}/4) + 2 \exp(-\Omega^2/3) + 6 \exp(-\Omega/4).
\]
(83)

Choosing the stepsize \(\gamma_k = \frac{\sqrt{V_{z^i} (u)}}{M} \sqrt{\frac{\tau}{N}}\) with \(M^2 = \frac{dM_2^2 + d^2 \Delta^2 \tau^{-2}}{\sqrt{N}}\) in (83) we obtain
\[
\mathbb{P} \left( \max_{y \in \mathcal{Y}} f (\hat{x}^N, y) - \min_{x \in \mathcal{X}} f (x, \hat{y}^N) \geq \frac{4\Omega^2 \sqrt{d} \sqrt{V_{z^i} (u) M_2 \sqrt{\sum_{k=1}^N \gamma_k^2}}}{\sum_{k=1}^N \gamma_k} + \frac{2M_2^2 \sqrt{\sum_{k=1}^N \gamma_k^2}}{c \sum_{k=1}^N \gamma_k} \right)
\]
\[
- \frac{\sqrt{d} \Delta}{\tau} + \frac{2 \sqrt{2} \Delta^2 \sum_{k=1}^N \gamma_k}{c \gamma_k} + \frac{\sqrt{V_{z^i} (u) M_2^2 \sqrt{d}}}{c \sqrt{N \sqrt{d}}} + \frac{\Delta^2 \sum_{k=1}^N \gamma_k}{2 \tau^2} \leq \exp(-\Omega^2/12) + \exp(-3\sqrt{N \Omega}/4) + 2 \exp(-\Omega^2/3) + 6 \exp(-\Omega/4).
\]
(84)

Using the notation of \(M\) we obtain:
\[
\mathbb{P} \left( \max_{y \in \mathcal{Y}} f (\hat{x}^N, y) - \min_{x \in \mathcal{X}} f (x, \hat{y}^N) \geq \left( \frac{4\Omega^2 + \frac{2\sqrt{2}}{c}}{c \gamma_k} \right) \frac{\sqrt{V_{z^i} (u) M_2 \sqrt{d}}}{\sqrt{N \sqrt{d}}} + \frac{\Omega \sqrt{d} \Delta}{\tau} + 2\tau M_2 + \right.
\]
\[
+ \frac{\sqrt{2} \Delta^2 \sum_{k=1}^N \gamma_k}{c \gamma_k} \leq \exp(-\Omega^2/12) + \exp(-3\sqrt{N \Omega}/4) + 2 \exp(-\Omega^2/3) + 6 \exp(-\Omega/4).
\]
(85)

Now we need to get a more compact convergence result in the form \(\mathbb{P} \left\{ \epsilon_{\text{sad}} (\hat{z}^N) \leq \epsilon \right\} \geq 1 - \sigma\). For this we fix \(\sigma > 0\) and let \(\tau\) be chosen as \(\tau = \mathcal{O} (\epsilon / M_2)\). If moreover Assumption 2 holds true with \(\Delta = \mathcal{O} (\frac{\epsilon^2}{\Delta M_2 \sqrt{d}})\), then taking \((x, y) = (x^*, y^*)\) in (85) we can choose \(\Omega\) small enough to obtain:
\[
\mathbb{P} \left( f (\hat{x}^N, y^*) - f (x^*, \hat{y}^N) = \mathcal{O} \left( M_2 \sqrt{d} \frac{\|z^1 - z^\ast\|_p}{\sqrt{N}} \right) \right) \geq 1 - \sigma.
\]
(86)

We note that now the notation \(\mathcal{O}\) contains the factor \(\log \sigma^{-1}\).

**Step 3. (Restarts)**

In this step we will employ the restart technique that is generalization of technique proposed in Juditsky and Nesterov (2014). For the l.h.s. of (86) we use Assumption 3 then with probability at least \(1 - \sigma_1\)
\[
\frac{\mu_r}{2} \left\| z_{N_1} - z^\ast \right\|_p^p \leq f (\hat{x}^N_1, y^*) - f (x^*, \hat{y}^N_1) = \mathcal{O} \left( M_2 \sqrt{d} \frac{\|z^1 - z^\ast\|_p}{\sqrt{N_1}} \right).
\]
(87)

Then taking \(N_1\) so as to reduce the distance to the solution by half, we obtain
\[
N_1 = \mathcal{O} \left( \frac{M_2^2 d}{\mu_r^2 R_1^2 (\tau - 1)} \right).
\]
Next, after $N_1$ iterations, we restart the original method and set $z^1 = z^{N_1}$. We determine $N_2$ from a similar condition for reducing the distance $R_1$ to the solution by a factor of 2, and so on.

We remark that at each restart step $i \in \{1, k\}$, the resulting formula (87) is valid only with probability $1 - \sigma_i$. Thus, we choose $\sigma_i = \sigma/k$ and then by the union bound inequality all inequalities are satisfied simultaneously with probability at least $1 - \sigma$. We will determine the number of restarts further, but at this stage we use the fact that $k$ depends on the accuracy only logarithmically, which entails that notations $\tilde{O}$, and $\tilde{O}$ are equivalent $\forall i \in \{1, k\}$. Thus, after $k$ of such restarts, the total number of iterations will be

$$N = N_1 + \ldots + N_k = \tilde{O}\left(\frac{2^{2(r-1)}dM_2^2}{\mu_r^2R_0^2(r-1)} \left(1 + 2^{2(r-1)} + \ldots + 2^{2(k-1)(r-1)}\right)\right). \quad (88)$$

It remains for us to determine the number of restarts, for this we fix the desired accuracy in terms of $P\{\epsilon_{sad}(z^N) \leq \epsilon\} \geq 1 - \sigma$ and using the inequality (87) we obtain

$$\epsilon_{sad} = \tilde{O}\left(\frac{\mu_r R_0^r}{2} \right) = \frac{\sqrt{dM_2 \tilde{O}(R_{k-1})}}{\sqrt{N_k}} = \tilde{O}\left(\frac{\mu_r R_0^r}{2^{k(r)}}\right) \leq \epsilon. \quad (89)$$

Then to fulfill this condition one can choose $k = \log_2(\tilde{O}(\mu_r R_0^r/\epsilon))/r$ and using (62) we get the total number of iterations

$$N = \tilde{O}\left(\frac{2^{2k(r-1)}dM_2^2}{\mu_r^2R_0^2(r-1)}\right) = \tilde{O}\left(\frac{dM_2^2}{\mu_r^{2/r}\epsilon^{2(r-1)/r}}\right). \quad \square$$