C∞ SPECTRAL RIGIDITY OF THE ELLIPSE

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Abstract. We prove that ellipses are infinitesimally spectrally rigid among C∞ domains with the symmetries of the ellipse.

An isospectral deformation of a plane domain Ω0 is a one-parameter family Ωε of plane domains for which the spectrum of the Euclidean Dirichlet (or Neumann) Laplacian Δε is constant (including multiplicities). We say that Ωε is a C1 curve of C∞ plane domains if there exists a C1 curve of diffeomorphisms φε of a neighborhood of Ω0 ⊂ ℝ2 with φ0 = id and with Ωε = φε(Ω0). The infinitesimal generator X = dX/dε is a vector field in a neighborhood of Ω0 which restricts to a vector field along ∂Ω0; we denote by Xν = ˙ρν its normal component.

With no essential loss of generality we may assume that φε|∂Ω0 is a map of the form

\[ x ∈ ∂Ω0 ↦ x + ρε(x)ν(x), \]

where ρε ∈ C1([0, ε0], C∞(∂Ω0)), and we put ˙ρ(x) = δρ(x) := dρ(x)/dε|ε=0. An isospectral deformation is said to be trivial if Ωε ≃ Ω0 (up to isometry) for sufficiently small ε. A domain Ω0 is said to be spectrally rigid if all isospectral deformations Ωε are trivial. The variation is called infinitesimally spectrally rigid if ˙ρ = 0 for all isospectral deformations.

In this article, we use the Hadamard variational formula of the wave trace (apparently for the first time) to study spectral rigidity problems (Theorem 2). Our main application is the infinitesimal spectral rigidity of ellipses among C1 curves of C∞ plane domains with the symmetries of an ellipse. We orient the domains so that the symmetry axes are the x-y axes. The symmetry assumption is then that each φε is invariant under (x, y) ↦ (±x, ±y).

**Theorem 1.** Suppose that Ω0 is an ellipse, and that Ωε is a C1 Dirichlet (or Neumann) isospectral deformation of Ω0 through C∞ domains with Z2 × Z2 symmetry. Then Xν = 0 or equivalently ˙ρ = 0.

As discussed in §0.2 and §3.2, Theorem 1 implies that ellipses admit no isospectral deformations for which the Taylor expansion of ρε at ε = 0 is non-trivial. A function such as \( e^{-1/ε} \) for which the Taylor series at ε = 0 vanishes is called ‘flat’ at ε = 0.

**Corollary 1.** Suppose that Ω0 is an ellipse, and that ε → Ωε is a C∞ Dirichlet (or Neumann) isospectral deformation through Z2 × Z2 symmetric C∞ domains. Then ρε must be flat at ε = 0. In particular, there exist no non-trivial real analytic curves ε → Ωε of Z2 × Z2 symmetric C∞ domains with the spectrum of an ellipse.

Spectral rigidity of the ellipse has been expected for a long time and is a kind of model problem in inverse spectral theory. Ellipses are special since their billiard flows and maps
are completely integrable. It was conjectured by G. D. Birkhoff that the ellipse is the only convex smooth plane domain with completely integrable billiards. We cannot assume that the deformed domains $\Omega_\epsilon$ have this property, although the results of [Sib2, Z2] come close to showing that they do. The results are somewhat analogous to the spectral rigidity of flat tori or the sphere in the Riemannian setting.

The main novel step in the proof is the Hadamard variational formula for the wave trace (Theorem 2), which holds for all smooth Euclidean domains $\Omega \subset \mathbb{R}^n$ satisfying standard ‘cleanliness’ assumptions. It is of independent interest and has applications to spectral rigidity beyond the setting of ellipses. We therefore present the proof in detail.

The main advance over prior results is that the domains $\Omega_\epsilon$ are allowed to be $C^\infty$ rather than real analytic. Much less than $C^\infty$ could be assumed for the domains $\Omega$, but we do not belabor the point. For real analytic domains a length spectral rigidity result for analytic domains with the symmetries of the ellipse was proved in [CdV]. The method does not apply directly to $\Delta$-isospectral deformations of ellipses since the length spectrum of the ellipse may have multiplicities and the full length spectrum might not be a $\Delta$-isospectral invariant. If it were, then Siburg’s results would imply that the marked length spectrum is preserved [Sib, Sib2, Sib3]. In [Z1, Z2] it is shown that analytic domains with one symmetry are spectrally determined if the length of the minimal bouncing ball orbit and one iterate is a $\Delta$-isospectral invariant.

The prior results on $\Delta$-isospectral deformations that we are aware of are contained in the articles [GM, PT, PT2] and concern deformations of boundary conditions. To our knowledge, the only prior results on $\Delta$-isospectral deformations of the domain are contained in [MM]. Marvizi-Melrose [MM] introduce new spectral invariants and prove certain rigidity results, but they do not apparently settle the case of the ellipse (see also [A, A2] for further attempts to apply them to the ellipse). It would be desirable to remove the symmetry assumption (to the extent possible), but symmetry seems quite necessary for our argument. Further discussion of prior results can be found in the earlier arXiv posting of this article.

0.1. Hadamard variation of the wave trace. We now state a general result on the variation of the wave trace on a domain with boundary under variations of the boundary.

To state the result, we need some notation. We denote by

\[ E_B(t) = \cos\left(t\sqrt{-\Delta_B}\right), \quad \text{resp.} \quad S_B(t) = \frac{\sin\left(t\sqrt{-\Delta_B}\right)}{\sqrt{-\Delta_B}} \]

the even (resp. odd) wave operators of a domain $\Omega$ with boundary conditions $B$. We recall that $E_B(t)$ has a distribution trace as a tempered distribution on $\mathbb{R}$. That is, $E_B(\hat{\rho}) = \int_\mathbb{R} \hat{\rho}(t)E_B(t)dt$ is of trace class for any $\hat{\rho} \in C_0^\infty(\mathbb{R})$; we refer to [GM2, PS] for background.

The Poisson relation of a manifold with boundary gives a precise description of the singularities of this distribution trace in terms of periodic transversal reflecting rays of the billiard flow, or equivalently periodic points of the billiard map. For the definitions of ‘billiard map’, ‘clean’, ‘transversal reflecting rays’ etc. we refer to [GM, GM2, PS]. A periodic point of the billiard map $\beta : B^*\partial\Omega \to B^*\partial\Omega$ on the unit ball bundle $B^*\partial\Omega$ of the boundary corresponds to a billiard trajectory, i.e an orbit of the billiard flow $\Phi^t$ on $S^*\Omega$. We define the ‘length’ of the periodic orbit of $\beta$ to be the length of the corresponding billiard trajectory in $S^*\Omega$. Note that the ‘period’ of a periodic point of $\beta$ is ambiguous since it could refer to this length or to
the power of $\beta$. We also denote by $\text{Lsp}(\Omega)$ the length spectrum of $\Omega$, i.e. the set of lengths of closed billiard trajectories. The perimeter of $\Omega$ is denoted by $|\partial \Omega|$.

In the following deformation theorem, the boundary conditions are fixed during the deformation and we therefore do not include them in the notation.

**Theorem 2.** Let $\Omega_0 \subset \mathbb{R}^n$ be a $C^\infty$ Euclidean domain with the property that the fixed point sets of the billiard map are clean. Then, for any $C^1$ variation of $\Omega_0$ through $C^\infty$ domains $\Omega_\epsilon$, the variation of the wave traces $\delta \text{Tr} \cos (t \sqrt{-\Delta})$, with Dirichlet (or Neumann) boundary conditions is a classical co-normal distribution for $t \neq m|\partial \Omega_0|$ ($m \in \mathbb{Z}$) with singularities contained in $\text{Lsp}(\Omega_0)$. For each $T \in \text{Lsp}(\Omega_0)$ for which the set $F_T$ of periodic points of the billiard map $\beta$ of length $T$ is a $d$-dimensional clean fixed point set consisting of transverse reflecting rays, there exist non-zero constants $C_\Gamma$ independent of $\dot{\rho}$ such that, near $T$, the leading order singularity is

$$\delta \text{Tr} \cos (t \sqrt{-\Delta}) \sim -\frac{t}{2} \Re \left\{ \left( \sum_{\Gamma \subset F_T} C_\Gamma \int_{\Gamma} \dot{\rho} \gamma_1 \, d\mu_\Gamma \right) (t - T + i0)^{-\frac{\delta}{2}} \right\},$$

modulo lower order singularities. The sum is over the connected components $\Gamma$ of $F_T$. (Here $\delta = \frac{d}{2}\epsilon |_{\epsilon = 0}$. See (28) for the definition of $\gamma_1$).

Here, the function $\gamma_1$ on $B^* \partial \Omega$ is defined in (28) and appeared earlier in [HZ]. The densities $d\mu_\Gamma$ on the fixed point sets of $\beta$ and its powers are the canonical densities defined in Lemma 4.2 of [DG], and further discussed in [GM, PT, PT2]. The constants $C_\Gamma$ are explicit and depend on the boundary conditions. We suppress the exact formulae since we do not need them, but their definition is reviewed in the course of the proof.

To clarify the dimensional issues, we note that there are four closely related definitions of the set of closed billiard trajectories (or closed orbits of the billiard map). The first is the fixed point set of the billiard flow $\Phi_T$ at time $T$ in $T^* \Omega$. The second is the set of unit vectors in the fixed point set. The third is the fixed point set of the billiard flow restricted to $T^*_{\partial \Omega} \Omega$, the set of covectors with foot points at the boundary. The fourth is the set of periodic points of the billiard map $\beta$ on $B^* \partial \Omega$ of length $T$, where as above the length is defined by the length of the corresponding billiard trajectory. The dimension $d$ refers to the dimension of the latter. In the case of the ellipse, for instance, $d = 1$; the periodic points of a given length form invariant curves for $\beta$.

To prove Theorem 2 we use the Hadamard variational formula for the Green’s kernel to give an exact formula for the wave trace variation (Lemma 1). We then prove that it is a classical conormal distribution and calculate its principal symbol.

It is verified in [GM] that the ellipse satisfies the cleanliness assumptions. We then have,

**Corollary 2.** For any $C^1$ variation of an ellipse through $C^\infty$ domains $\Omega_\epsilon$, the leading order singularity of the wave trace variation is,

$$\delta \text{Tr} \cos (t \sqrt{-\Delta}) \sim -\frac{t}{2} \Re \left\{ \left( \sum_{\Gamma \subset F_T} C_\Gamma \int_{\Gamma} \dot{\rho} \gamma_1 \, d\mu_\Gamma \right) (t - T + i0)^{-\frac{\delta}{2}} \right\},$$

modulo lower order singularities, where the sum is over the connected components $\Gamma$ of the set $F_T$ of periodic points of $\beta$ (and its powers) of length $T$. 
0.2. Flatness issues. We now discuss an apparently new flatness issue in isospectral deformations. The rather technical assumption that $\Omega_\epsilon$ is a $C^1$ family of $C^\infty$ domains rather than a $C^\infty$ family in the $\epsilon$ variable is made to deal with a somewhat neglected and obscure point about isospectral deformations. Isospectral deformations are curves in the ‘manifold’ of domains. The curve might be a non-trivial $C^\infty$ family in $\epsilon$ but the first derivative $\rho$ might vanish at $\epsilon = 0$. Thus, infinitesimal spectral rigidity is at least apparently weaker than spectral rigidity. We impose the $C^1$ regularity to allow us to reparameterize the family and show that the first derivative of any $C^1$ re-parametrization must be zero. This is not the primary focus of Theorem 1, but with no additional effort the proof extends to the $C^1$ case.

This flatness issue does not seem to have arisen before in inverse spectral theory, even when the main conclusions are derived from infinitesimal rigidity. The main reason is that first order perturbation theory very often requires analytic perturbations (i.e. analyticity in the deformation parameter $\epsilon$), and so most (if not all) prior results on isospectral deformations assume that the deformation is real analytic. But our proof is based on Hadamard’s variational formula, which is valid for $C^1$ perturbations of domains and so we can study this more general situation. Further, the prior spectral rigidity results (e.g. [GK]) are proved for an open set of domains and metrics and therefore flatness at all points implies triviality of the deformations. We are only deforming the one-parameter family of ellipses and therefore cannot eliminate flat isospectral deformations by that kind of argument. We also note that there could exist continuous but non-differentiable isospectral deformations.

0.3. Pitfalls and complications. The route taken in the proof of Theorem 1 and the flatness issues just discussed, reflect certain technical issues that arise in the inverse problem. First is the issue of multiplicities in the eigenvalue spectrum or in the length spectrum. The multiplicities of the $\Delta$-eigenvalues of the ellipse (for either Dirichlet or Neumann boundary conditions) appear to be almost completely unknown. If a sufficiently large portion of the eigenvalue spectrum were simple (i.e. of multiplicity one), one could simplify the proof of Theorem 1 by working directly with the eigenfunctions and their semi-classical limits (as in the first arXiv posting of this article). The dual multiplicity of the length spectrum is also largely unknown for the ellipse. Without length spectral simplicity one cannot work with the wave trace invariants. Our proof relies on the observation in [GM] that the multiplicities have to be one (modulo the symmetry) for periodic orbits that creep close enough to the boundary.

Second is the issue of cleanliness. Theorem 2 and Corollary 2 would apply to any of the deformed domains $\Omega_\epsilon$ if the fixed points sets were known to be clean. One could then use the conclusion of Corollary 2 to rule out flat isospectral deformations. However, we do not know that the fixed point sets are clean for the deformed domains even though we do know that they have the same wave trace singularities as the ellipse. Equality of the wave traces for isospectral deformations of ellipses shows that the periodic points of $\beta$ of $\Omega_\epsilon$ can never be non-degenerate. Hence the deformations are very non-generic. It is plausible that equality of wave traces forces the sets of periodic points to be clean invariant curves of dimension one. But we do not know how to prove this kind of inverse result at this time.
1. Hadamard Variational Formula for Wave Traces

In this section we consider the Dirichlet (resp. Neumann) eigenvalue problems for a one parameter family of smooth Euclidean domains \( \Omega_\epsilon \subset \mathbb{R}^n \),

\[
\begin{align*}
-\Delta B_\epsilon \Psi_j(\epsilon) &= \lambda_j^2(\epsilon)\Psi_j(\epsilon) \quad \text{in} \ \Omega_\epsilon, \\
B_\epsilon \Psi_j(\epsilon) &= 0,
\end{align*}
\]

(3)

where the boundary condition \( B_\epsilon \) could be \( B_\epsilon \Psi_j(\epsilon) = \Psi_j(\epsilon)|_{\partial \Omega_\epsilon} \) (Dirichlet) or \( \partial_{\nu_\epsilon} \Psi_j(\epsilon)|_{\partial \Omega_\epsilon} \) (Neumann). Here, \( \lambda_j^2(\epsilon) \) are the eigenvalues of \(-\Delta B_\epsilon\), enumerated in order and with multiplicity, and \( \partial_{\nu_\epsilon} \) is the interior unit normal to \( \Omega_\epsilon \). We do not assume that \( \Psi_j(\epsilon) \) are smooth in \( \epsilon \). We now review the Hadamard variational formula for the variation of Green’s kernels, and adapt the formula to give the variation of the (regularized) trace of the wave kernel.

Our references are \( [G, \text{Pee}, \text{FTY}, \text{O}, \text{FO}] \).

We further denote by \( dq \) the surface measure on the boundary \( \partial \Omega \) of a domain \( \Omega \), and by \( ru = u|_{\partial \Omega} \) the trace operator. We further denote by \( rD = \partial_{\nu}|_{\partial \Omega} \) the analogous Cauchy data trace for the Dirichlet problem. We simplify the notation for the following boundary traces \( K^b(q', q) \in \mathcal{D}'(\partial \Omega \times \partial \Omega) \) of a Schwartz kernel \( K(x, y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n) \) (or more precisely a distribution defined in a neighborhood of \( \partial \Omega \times \partial \Omega \)):

\[
K^b(q', q) = \begin{cases}
(r_q r_q N_{\nu q} N_{\nu q} K)(q', q), & \text{Dirichlet,} \\
(\nabla_q^T \nabla_q r_q r_q K)(q', q) + (r_q r_q \Delta K)(q', q), & \text{Neumann.}
\end{cases}
\]

(4)

Here, the subscripts \( q', q \) refer to the variable involved in the differentiating or restricting. According to convenience, we also indicate this by subscripting with indices 1, 2 referring to the first, resp. second, variable in the kernel. For instance, \( \frac{\partial}{\partial q_i} K(q', q) = \frac{\partial}{\partial q_i} K(q', q) \). We also use the notations \( \partial_{\nu} \) and \( \frac{\partial}{\partial \nu} \) interchangeably to refer to the inward normal derivative. Also, \( N_\nu \) is any smooth vector field in \( \Omega \) extending \( \nu \).

We are principally interested in \( K(x, y) = S_B(t, x, y) \). In the Dirichlet, resp. Neumann, case then we have,

\[
S^b_B(t, q', q) = \begin{cases}
r_q r_q S_D(t, q', q), & \text{resp.} \\
\nabla_q^T \nabla_q r_q r_q S_N(t, q', q) + r_q r_q \Delta S_N(t, q', q).
\end{cases}
\]

(5)

**Lemma 1.** The variation of the wave trace with boundary conditions \( B \) is given by,

\[
\delta Tr \ E_B(t) = -\frac{t}{2} \int_{\partial \Omega_0} S^b_B(t, q, q) \hat{\rho}(q)dq.
\]

We summarize by writing,

\[
\delta Tr \ E_B(t) = -\frac{t}{2} Tr_{\partial \Omega_0} \hat{\rho} S^b_B.
\]

We prove the Lemma by relating the variation of the wave trace to the known variational formula for the Green’s function (resolvent kernel). We now review the latter.
1.1. Hadamard variational formulae. In the Dirichlet case, the classical Hadamard variational formulae states that, under a sufficiently smooth deformation $Ω_ε$,

$$\frac{δG_D(λ,x,y)}{δx} = -\int_{∂Ω_0} \frac{∂}{∂ν_2} G_D(λ,x,q) \frac{∂}{∂ν_1} G_D(λ,q,y) \dot{ρ}(q) dq.$$  

In the Neumann case,

$$\frac{δG_N(λ,x,y)}{δx} = \int_{∂Ω_0} \nabla_T^2 G_N(λ,x,q) \cdot \nabla_T^1 G_N(λ,q,y) \dot{ρ}(q) dq - λ^2 \int_{∂Ω_0} G_N(λ,x,q) G_N(λ,q,y) \dot{ρ}(q) dq.$$  

Above, the subscript refers to the variable with respect to which the derivative is taken and $∇^T$ denotes the derivative with respect to the unit tangent vector. We briefly review the proof of the Hadamard variational formula to clarify the definition of $δG(λ,x,y)$ and of the other kernels. Following [Pee], we write the inhomogeneous problem

$$\begin{cases}
  (-Δ + λ^2)u = f, & \text{in } Ω;
  u = 0 \text{ (resp. } ∂_ν u = 0) & \text{on } ∂Ω
\end{cases}$$

in terms of the energy integral

$$E(u,v) = \int_Ω \nabla u \cdot \nabla v dx + λ^2 \int_Ω uv dx = \int_Ω v(-Δ + λ^2)u dx + \int_{∂Ω} v∂_ν u dq,$$

where $∂_ν$ is the outer unit normal. The inhomogeneous problem is to solve

$$E(u,v) = \int_Ω fv dx,$$

where $v$ is a smooth test function which vanishes to order one (resp. 0) on $∂Ω$ for the Dirichlet (resp. Neumann) problem. We denote the energy density by $e(u,v) = \nabla u \cdot \nabla v + λ^2 uv$.

We now vary the problems over a one-parameter family of domains. As mentioned above, we use one-parameter families of diffeomorphisms $ϕ_ε$ of a neighborhood of $Ω_0 \subset \mathbb{R}^n$ to define the one-parameter families $Ω_ε = ϕ_ε(Ω_0)$ of domains. We assume $ϕ_ε$ to be a $C^1$ curve of diffeomorphisms with $ϕ_0 = id$.

The variational derivative of the solution is defined as follows: Let $u_ε \in H^s(Ω_ε)$. Then $ϕ_ε^* u_ε \in H^s(Ω_0)$. Put $X = \frac{d}{dε}|_{ε=0} ϕ_ε$, and put

$$θ_X u = \frac{d}{dε} ϕ_ε^* u_ε |_{ε=0}.$$

Assume that $θ_X u \in H^s(Ω_0)$ and that $u \in H^{s+1}(Ω_0)$. Then $\dot{u}$ exists and $θ_X u = \dot{u} + X u$. Further, let $v$ be a test function on $Ω_0$ and use $ϕ_ε^{-1*} v$ as a test function on $Ω_ε$. Now re-write the boundary problems as

$$\int_{Ω_ε} e(u, (ϕ_ε^{-1})^* v) dx = \int_{Ω_ε} f_ε((ϕ_ε^{-1})^* v) dx.$$  

Changing variables, one pulls back the equation to $Ω_0$ as

$$\int_{Ω_0} e_ε(ϕ_ε^* u_ε, v) ϕ_ε^* dx = \int_{Ω_0} (ϕ_ε^* f_ε) v ϕ_ε^* dx, \text{ where } e_ε(u,v) = ϕ_ε^*(e(ϕ_ε^{-1*} u, ϕ_ε^{-1*} v)).$$
Assuming that \( \dot{u}, \theta_X u \in H^s(\Omega_0) \) and that \( u \in H^{s+1}(\Omega_0) \), so that \( \theta_X u = \frac{d}{d\epsilon} \varphi^*_\epsilon u|_{\epsilon=0} \) exists as a limit in \( H^s(\Omega_0) \), we have (by the computations of [Pee] (8) and (10)) that

\[
\int_{\Omega_0} \dot{u}(-\Delta + \lambda^2) v \, dx = \int_{\Omega_0} \dot{f} v \, dx + \int_{\partial\Omega_0} f \dot{\nu} \, dq + \int_{\partial\Omega_0} (\nabla u \cdot \nabla v - \lambda^2 uv) \dot{\nu} \, dq
\]

(8)

\[
-\lambda^2 \int_{\partial\Omega_0} uv \dot{\nu} \, dq \begin{cases} \text{Dirichlet}, \\ 0 \quad \text{Neumann} \end{cases}
\]

To obtain (6)-(7), at least formally, one puts \( u_\epsilon(x) = G_{B,\epsilon}(\lambda, \varphi_\epsilon(x), y), \psi_\epsilon(x) = G_{B,0}(\lambda, y, x) \), and \( \varphi_\epsilon^* f_\epsilon = \delta_y(x) \). Thus, \( \delta G_B(\lambda, x, y) = \frac{d}{d\epsilon}|_{\epsilon=0} G_{B,\epsilon}(\lambda, \varphi_\epsilon(x), y) \). Assuming \( y \in \Omega^o \) (the interior), then \( y \in \Omega_\epsilon \) for sufficiently small \( \epsilon \) and one easily verifies that (8) implies (6)-(7). The Green’s kernel depends on \( \epsilon \) as smoothly as the coefficients of operator \( \tilde{\Delta}_\epsilon \) defined by the pulled back energy form. Indeed, the resolvent is an analytic function of the Laplacian.

1.2. Proof of Lemma 1. Rather than the Green’s function, we are interested in the Hadamard variational formula for the wave kernels \( E_B(t), S_B(t) \) or more precisely, for their distribution traces. In fact, by definition of the distribution trace, we only need the variational formula for traces of variations \( \delta \int_\mathbb{R} e^{-i\lambda t}{\psi}(t)E_B(t)dt \) of integrals of these kernels against test functions \( \psi \in C^0(\mathbb{R}) \), which are simpler because the Schwartz kernels are smooth.

We derive the Hadamard variational formulae for wave traces from that of the Green’s function by using the identities,

\[
i\lambda R_B(\lambda) = \int_0^\infty e^{-i\lambda t} E_B(t) \, dt, \quad \frac{d}{dt} S_B(t) = E_B(t)
\]

integrating by parts and using the finite propagation speed of \( S_B(t) \) to eliminate the boundary contributions at \( t = 0, \infty \). It follows that

\[
R_B(\lambda) = \int_0^\infty e^{-i\lambda t} S_B(t) \, dt.
\]

As mentioned above, to obtain the variational formula for the singularity expansion of the wave trace, we only need variational formula for the smooth kernels

\[
\int_\mathbb{R} \hat{\psi}(t)e^{-i\lambda t} E_B(t) \, dt = \int_\mathbb{R} i\mu R_B(\mu) \psi(\mu - \lambda) \, d\mu,
\]

where \( R_B(\mu) = (-\Delta_B - \mu^2)^{-1} \) is the resolvent of \( -\Delta_B \). Here we assume that \( \hat{\psi} \) is supported in \( \mathbb{R}_+ \) since in the wave trace we localize its support to the length of a closed geodesic. In the Dirichlet case, it follows that

\[
\delta \int_\mathbb{R} \hat{\psi}(t)e^{-i\lambda t} E_B(t) \, dt = \delta \int_\mathbb{R} i\mu R_B(\mu) \psi(\mu - \lambda) \, d\mu.
\]
We then derive variational formulae for wave traces. In the Dirichlet case, it follows from (12) and (6) that
\[
\delta \int_R \hat{\psi}(t)e^{-i\lambda t} E_D(t)dt = -i \int_R \mu \psi(\mu - \lambda) \int_{\partial \Omega} \partial_{\nu_2} G_D(\mu, x, q) \partial_{\nu_1} G_D(\mu, q, y) \hat{\rho}(q)dq d\mu
\]
\[
= -\int_R \int_0^\infty e^{-i\mu t} \psi(\mu - \lambda) \int_{\partial \Omega} \partial_{\nu_2} E_D(t, x, q) \partial_{\nu_1} G_D(\mu, q, y) \hat{\rho}(q)dq d\mu dt
\]
\[
= -\int_R \int_0^\infty \int_0^\infty e^{-i\mu(t+t')} \psi(\mu - \lambda) \int_{\partial \Omega} \partial_{\nu_2} E_D(t, x, q) \partial_{\nu_1} S_D(t', q, y) \hat{\rho}(q)dq d\mu dt dt'
\]
\[
= -\int_0^\infty \int_{\partial \Omega} e^{-i\lambda(t+t')} \hat{\psi}(t + t') \int_{\partial \Omega} \partial_{\nu_2} E_D(t, x, q) \partial_{\nu_1} S_D(t', q, y) \hat{\rho}(q)dq dt dt'
\]
\[
= -\int_0^\infty \int_{\partial \Omega} e^{-i\lambda t} \hat{\psi}(t) \left( \int_{\partial \Omega} \partial_{\nu_2} E_D(\tau - t', x, q) \partial_{\nu_1} S_D(t', q, y) dt' \right) \hat{\rho}(q)d\tau dq.
\]
The inner integral is the same if we change the argument of $E_D$ to $t'$ and that of $S_D$ to $\tau - t'$. We then average the two, set $x = y$, integrate over $\Omega$ and use the angle addition formula for $\sin$ to obtain
\[
\delta \int_R \hat{\psi}(t)e^{-i\lambda t} E_D(t)dt = -\frac{1}{2} \int_R t \hat{\psi}(t)e^{-i\lambda t} \int_{\partial \Omega} \partial_{\nu_1} \partial_{\nu_2} S_D(t, q, q) \hat{\rho}(q)dq dt.
\]
This is the real part of wave trace variational formula stated in the Lemma in the Dirichlet case, i.e. the variational formula for $\delta \ Tr E_D(t)$. The proof in the Neumann case is similar and left to the reader.

This concludes the proof of Lemma 1. To navigate the formulae, we also give a derivation based on the Hadamard variational formulae for eigenvalues. When $\lambda_j^2(0)$ is a simple eigenvalue (i.e. of multiplicity one), then Hadamard’s variational formula for Dirichlet eigenvalues of Euclidean domains states that
\[
\delta(\lambda_j^2) = \int_{\partial \Omega_0} (\partial_{\nu} \Psi_j |_{\partial \Omega_0})^2 \hat{\rho}(q) dq,
\]
where $\Psi_j$ is an $L^2$ normalized eigenfunction for the eigenvalue $\lambda_j^2(0)$ and $dq$ is the induced surface measure. See [1]. The same comparison shows that if the eigenvalue $\{\lambda_{j,k}^2(\epsilon)\}_{k=1}^{m(\lambda_j(0))}$ is multiple and if $\{\lambda_{j,k}^2(\epsilon)\}_{k=1}^{m(\lambda_j(0))}$ is the perturbed set of eigenvalues, then
\[
\delta \sum_{k=1}^{m(\lambda_j(0))} \lambda_{j,k}^2 = \sum_{k=1}^{m(\lambda_j(0))} \int_{\partial \Omega_0} (\partial_{\nu} \Psi_{j,k} |_{\partial \Omega_0})^2 \hat{\rho}(q) dq = \int_{\partial \Omega_0} \partial_{\nu_1} \partial_{\nu_2} \Pi_{\lambda_j(0)}(q, q) \hat{\rho}(q)dq,
\]
where $\{\Psi_{j,k} \}_{k=1}^{m(\lambda_j(0))}$ is an orthonormal basis for the eigenspace of the multiple eigenvalue $\lambda_j^2(0)$ and $\Pi_{\lambda_j(0)}(x, y)$ is its spectral projections kernel. Since $\delta \lambda_j = \frac{\delta(\lambda_j^2)}{2\lambda_j^2}$, by (14) we have
\[
\delta \ Tr E_B(t) = \delta \sum_{j,k} \cos(t \lambda_{j,k}) = -t \sum_{j} \left( \sum_{k=1}^{m(\lambda_j(0))} \delta(\lambda_{j,k}^2) \frac{\sin(t \lambda_{j,k}^2)}{2 \lambda_{j,k}^2} \right) \frac{\sin(t \lambda_{j,k}^2)}{2 \lambda_{j,k}^2} = \frac{1}{2} \int_{\partial \Omega_0} \partial_{\nu_1} \partial_{\nu_2} S_B(t, q, q) \hat{\rho}(q)dq.
\]
Hence Lemma 1 follows in the Dirichlet case.

There exist similar Hadamard variational formulae in the Neumann case. When the eigenvalue is simple, we have
\[
\delta(\lambda_j^2) = \int_{\partial \Omega_0} (|\nabla_\Omega^T (\Psi_j |_{\partial \Omega_0}(q))|^2 - \lambda_j^2(0)(\Psi_j |_{\partial \Omega_0}(q))) \hat{\rho}(q) dq,
\]
For a multiple eigenvalue we sum over the expressions over an orthonormal basis of the eigenspace. The result does not depend on a choice of orthonormal basis. Similar computation using (15) follows to show Lemma 1 for the Neumann case.

2. Proof of Theorem 2

We now study the singularity expansion of $\delta Tr \cos(t\sqrt{-\Delta_B})$ and prove Theorem 2. At first sight, one could do this in two ways: by taking the variation of the spectral side of the formula, or by taking the variation of the singularity expansion. It seems simpler and clearer to do the former since we do not know how the invariant tori of the integrable elliptical billiard deform under an isospectral deformation. In this section we will drop the subscript 0 in $\Omega_0$.

The variational formula for $\delta Tr \cos(t\sqrt{-\Delta_B})$ is given in Lemma 1. In the Dirichlet case, by (4)-(5),

$$\begin{equation}
Tr_{\partial \Omega} \hat{\rho} S^b_N = \pi_* \Delta^* \hat{\rho} \left( r_1 r_2 N_{\nu_1} N_{\nu_2} S_D(t, x, y) \right),
\end{equation}$$

where $\nu$ is any smooth vector field in $\Omega$ extending $\nu$, and where the subscripts indicate the variables on which the operator acts. In the Neumann case by (4)-(5),

$$\begin{equation}
Tr_{\partial \Omega} \hat{\rho} S^b_N = \pi_* \Delta^* \hat{\rho} \left( (\nabla_1^2 \nabla_2^2 r_1 r_2 - r_1 r_2 \Delta) S_N(t, x, y) \right).
\end{equation}$$

Here, $\Delta : \partial \Omega \to \partial \Omega \times \partial \Omega$ is the diagonal embedding $g \to (q, q)$ and $\pi_*$ (the pushforward of the natural projection $\pi : \partial \Omega \times \mathbb{R} \to \mathbb{R}$) is the integration over the fibers with respect to the surface measure $dq$. The duplication in notation between the Laplacian and the diagonal is regrettable, but both are standard and should not cause confusion. Since $S_B(t, x, y)$ is microlocally a Fourier integral operator near the transversal periodic reflecting rays of $F_T$, it will follow from (16) that the trace is locally a Fourier integral distribution near $t = T$.

We are assuming that the set of periodic points of the billiard map corresponding to space-time billiard trajectories of length $T \in Lsp(\Omega)$ is a submanifold $F_T$ of $B^s \partial \Omega$. We thus fix $T \in Lsp(\Omega)$ consisting only of periodic reflecting rays, i.e. we assume $T \neq m|\partial \Omega|$ ($|\partial \Omega|$ being the perimeter) for $m \in \mathbb{Z}$. In order to study the singularity of the boundary trace near a component $F_T$ of the fixed point set, we construct a pseudo-differential cutoff $\chi_T = \chi_T(t, D_t, q, D_q) \in \Psi^0(\mathbb{R} \times \partial \Omega)$ whose complete symbol $\chi_T(t, \tau, q, \zeta)$ has the form $\chi_T(q, \zeta)$ with $\chi_T(y, \zeta)$ supported in a small neighborhood of the fixed point set $F_T \subset B^s \partial \Omega$, equals one in a smaller neighborhood, and in particular vanishes in a neighborhood of the glancing directions in $S^* \partial \Omega = \partial (B^s \partial \Omega)$. Since the symbol of $\chi_T$ is independent of $t$ we will instead use $\chi_T(D_t, q, D_q)$. We may assume that the support of the cutoff is invariant under the billiard map $\beta$. Therefore we need to study the operator

$$\begin{equation}
\pi_* \Delta^* \hat{\rho} \chi_T(D_t, q', D_{q'}) \chi_T(D_t, q, D_q) S^b_B(t, q', q),
\end{equation}$$

and compute its symbol. To do this we first study the operators $r$ and $S_B(t)$ and review their basic properties. Next we study the composition

$$\chi_T(D_t, q', D_{q'}) \chi_T(D_t, q, D_q) S^b_B(t, q', q)$$

and compute its symbol. Finally in Lemma 7 we take composition with $\pi_* \Delta^* \hat{\rho}$ and calculate the symbol of (18).
2.1. FIOs and their symbol. We recall that the principal symbol $\sigma_I$ of a Fourier integral distribution

$$I = \int_{\mathbb{R}^N} e^{i\varphi(x,\theta)} a(x, \theta) d\theta, \quad I \in I^m(M, \Lambda_\varphi),$$

of order $m$ is defined in terms of the parametrization

$$\iota_\varphi : C_\varphi = \{(x, \theta) : d_\theta \varphi = 0 \} \to (x, d_x \varphi) \in \Lambda_\varphi \subset T^* M$$

of the associated Lagrangian $\Lambda_\varphi$. It is a half density on $\Lambda_\varphi$ given by $\sigma_I = (\iota_\varphi)_*(a_0|dC_\varphi|^\frac{1}{2})$ where $a_0$ is the leading term of the classical symbol $a \in S^{m+\frac{1}{4}}(M \times \mathbb{R}^N)$, $n = \dim M$ and

$$d_{C_\varphi} := \frac{dc}{|D(c, \varphi'_c)/D(x, \theta)|}$$

is the Gelfand-Leray form on $C_\varphi$ where $c$ is a system of coordinates on $C_\varphi$. For notation and background we refer to [Ho]. When $I(x, y) \in I^m(X \times Y, \Lambda)$ is the kernel of an FIO it is very standard to use the symplectic form $\omega_X - \omega_Y$ on $X \times Y$ and define

$$\iota_\varphi : C_\varphi = \{(x, y, \theta) : d_\theta \varphi = 0 \} \to (x, d_x \varphi, y, -d_y \varphi) \in \Lambda_\varphi \subset T^* X \times T^* Y.$$

We will call $\Lambda_\varphi$ the canonical relation of $I(x, y)$.

2.2. The restriction operator $r$ as an FIO. The restriction $r$ to the boundary satisfies,

$$r \in I^\frac{1}{4}(\partial \Omega \times \mathbb{R}^n, \Gamma_{\partial \Omega}),$$

with the canonical relation

$$(19) \quad \Gamma_{\partial \Omega} = \{(q, \zeta, q, \xi) \in T^* \partial \Omega \times T^*_\partial \Omega \mathbb{R}^n ; \xi|_{T_q \partial \Omega} = \zeta \}.$$

The adjoint then satisfies $r^* \in I^\frac{1}{4}(\mathbb{R}^n \times \partial \Omega, \Gamma^*_{\partial \Omega})$, where

$$(20) \quad \iota_{\Gamma_{\partial \Omega}}(q, \zeta) = (q, \zeta|_{T_q (\partial \Omega)}, q, \zeta).$$

To prove these statements, we introduce Fermi normal coordinates $(q, x_n)$ along $\partial \Omega$, i.e. $x = \exp_q(x_n\nu_q)$ where $\nu_q$ is the interior unit normal at $q$. Let $\zeta = (\zeta, \xi_n) \in T_{(q, x_n)} \mathbb{R}^n$ denote the corresponding symplectically dual fiber coordinates. In these coordinates, the kernel of $r$ is given by

$$(21) \quad r(q, (q', x'_n)) = C_n \int_{\mathbb{R}^n} e^{i(q-q', \zeta) - ix'_n\xi_n} d\xi_n d\zeta.$$

The phase $\varphi(q, (q', x'_n), (\zeta, \xi_n)) = \langle q - q', \zeta \rangle - x'_n\xi_n$ is non-degenerate and its critical set is $C_\varphi = \{(q, q', x'_n, \xi_n, \zeta) : q' = q, x'_n = 0 \}$. The Lagrange map $\iota_\varphi : (q, q, 0, \xi_n, \zeta) \to (q, \zeta, q, \zeta, \xi_n)$ embeds $C_\varphi \to T^* \partial \Omega \times T^* \mathbb{R}^n$ and maps onto $\Gamma_{\partial \Omega}$. The adjoint kernel has the form $K^*(x, q) = K(q, x)$ and therefore has a similar oscillatory integral representation. It is clear from (21) that the order of $r$ as an FIO is $\frac{1}{4}$. Also, in the parametrization (20), the principal symbol of $r$ is $\sigma_r = |dq \wedge d\zeta \wedge d\xi_n|^\frac{1}{2}$. 

2.3. Background on parametrices for \( S_B(t) \). We first review the Fourier integral description of \( E_B(t) \), \( S_B(t) \) microlocally near transversal reflecting rays. This is partly for the sake of completeness, but mainly because we need to compute their principal symbols (and related ones) along the boundary. Although the principal symbols are calculated in the interior in [GM2, PS] and elsewhere (see Proposition 5.1 of [GM2], section 6 of [MM] and section 6 of [PS]), the results do not seem to be stated along the boundary (i.e. the symbols are not calculated at the boundary). The statements we need are contained in Theorem 3.1 of [Ch2] (and its proof), and we largely follow its presentation.

We need to calculate the canonical relation and principal symbol of the wave group, its derivatives and their restrictions to the boundary. We begin by recalling that the propagation of singularities theorem for the mixed Cauchy-Dirichlet (or Neumann) problem for the wave equation states that the wave front set of the wave kernel satisfies,

\[
WF(S_B(t, x, y)) \subset \bigcup_{\pm} \Lambda_{\pm},
\]

where \( \Lambda_{\pm} = \{(t, \tau, x, \xi, y, \eta) : (x, \xi) = \Phi^t(y, \eta), \ \tau = \pm |\eta| y \} \subset T^* (\mathbb{R} \times \Omega \times \Omega) \) is the graph of the generalized (broken) geodesic flow, i.e. the billiard flow \( \Phi^t \). For background we refer to [GM2, PS, Ch2] and to [Ho] (Vol. III, Theorem 23.1.4 and Vol. IV, Proposition 29.3.2). For the application to spectral rigidity, we only need a microlocal description of wave kernels away from the glancing set, i.e. in the hyperbolic set microlocally near periodic transversal reflecting rays. In these regions, there exists a microlocal parametrix due to Chazarain [Ch2], which is more fully analyzed in [GM2, PS] and applied to the ellipse in [GM].

The microlocal parametrices for \( E_B \) and \( S_B \) are constructed in the ambient space \( \mathbb{R} \times \mathbb{R}^n \times \Omega \). Since \( E_B = \frac{d}{dt} S_B \) it suffices to consider the latter. Then there exists a Fourier integral (Lagrangian) distribution,

\[
\tilde{S}_B(t, x, y) = \sum_{j = -\infty}^{\infty} S_j(t, x, y), \text{ with } S_j \in I^{-\frac{1}{4}-1}(\mathbb{R} \times \mathbb{R}^n \times \Omega, \Gamma^j_{\pm})
\]

which microlocally approximates \( S_B(t, x, y) \) modulo a smooth kernel near a transversal reflecting ray. The sum is locally finite hence well-defined. The canonical relation of \( \tilde{S}_B \) is contained in a union

\[
\Gamma = \bigcup_{\pm, j \in \mathbb{Z}} \Gamma^j_{\pm} \subset T^* (\mathbb{R} \times \mathbb{R}^n \times \Omega)
\]

of canonical relations \( \Gamma^j_{\pm} \) corresponding to the graph of the broken geodesic flow with \( j \) reflections. Notice we let \( j \in \mathbb{Z} \) which is different from [Ch2] where \( j \) goes from 0 to \( \infty \) and where the two graphs \( \Gamma^j_{\pm} \) and \( \Gamma^{-j}_{\pm} \) are combined.

To describe \( \Gamma^j_{\pm} \), we introduce some useful notation from [Ch2] with a slight adjustment. We have two Hamiltonian flows \( g^{\pm t} \) corresponding to the Hamiltonians \( \pm |\eta| \). For \((y, \eta)\) in \( T^* \Omega \) we define the first impact times with the boundary,

\[
\begin{align*}
t^1_{\pm}(y, \eta) &= \inf \{ t > 0 : \pi g^{\pm t}(y, \eta) \in \partial \Omega \}, \\
t^{-1}_{\pm}(y, \eta) &= \sup \{ t < 0 : \pi g^{\pm t}(y, \eta) \in \partial \Omega \}.
\end{align*}
\]

The impact times are related by \( t^{-1}_{\pm} = -t^1_{\pm} \). We define \( t^j_{\pm} \) inductively for \( j > 0 \) res. \( j < 0 \) to be the time of \( j \)-th reflection (i.e. impact with the boundary) for the flow \( g^{\pm t} \) as \( t \) increases.
res. decreases from $t = 0$. Then we put
\[
\begin{align*}
\lambda_\pm^1(y, \eta) &= g^{\pm t_\pm^1(y, \eta)}(y, \eta) \in T_{\partial \Omega}^* \Omega, \\
\lambda_\pm^{-1}(y, \eta) &= g^{\mp t_\pm^{-1}(y, \eta)}(y, \eta) \in T_{\partial \Omega}^* \Omega.
\end{align*}
\]

Next we define $\lambda_\pm^1(y, \eta)$ to be the reflection of $\lambda_\pm^1(y, \eta)$ at the boundary. For any $(q, \xi) \in T_q^* \mathbb{R}^n, q \in \partial \Omega$, the reflection $\xi \to \hat{\xi}$ has the same tangential projection as $\xi$ but opposite normal component. Similarly we define $\lambda_\pm^{-1}(y, \eta)$. Flowing $\lambda_\pm^1(y, \eta)$ (resp. $\lambda_\pm^{-1}(y, \eta)$) by $g^{\pm t}$ as $t$ increases (resp. decreases) and continuing the same procedure we get $t_\pm^j(y, \eta)$ and $\lambda_\pm^j(y, \eta)$ for all $j \in \mathbb{Z}$. We also set $T_\pm^j = \sum_{k=1}^j t_\pm^k$ for $j > 0$ and $T_\pm^j = \sum_{k=-j}^0 t_\pm^k$ for $j < 0$.

The canonical graph $\Gamma_\pm^j$ can now be written as
\[
(22) \quad \Gamma_\pm^j = \begin{cases}
\left\{(t, \tau, g^{\pm t}(y, \eta), y, \eta) : \tau = \pm|\eta|_y\right\} & j = 0, \\
\left\{(t, \tau, g^{\pm (t-T_\pm^j(y, \eta))}(y, \eta), y, \eta) : \tau = \pm|\eta|_y\right\} & j \in \mathbb{Z}, j \neq 0.
\end{cases}
\]

For each $j \in \mathbb{Z}, \bigcup_{\pm} \Gamma_\pm^j$ is the union of two canonical graphs, which we refer to as its ‘branches’ or ‘components’ (see figure 3.2 of [GM2] for an illustration). These two branches arise because $S_B(t) = \frac{1}{2\sqrt{-\Delta_B}}(e^{it\sqrt{-\Delta_B}} - e^{-it\sqrt{-\Delta_B}})$ is the sum of two Fourier integral operators whose canonical relations are respectively the graphs of the forward/backward broken geodesic flow and which correspond to the two halves $\tau > 0, \tau < 0$ of the characteristic variety $\tau^2 - |\eta|^2 = 0$ of the wave operator.

At the boundary, we have four modes of propagation: in addition to the two $\pm$ branches corresponding to $\tau > 0$ and $\tau < 0$, there are two modes of propagation corresponding to the two ‘sides’ of $\partial \Omega$. To illustrate this we first discuss the simple model of the upper half space.

2.3.1. Upper half space; a local model for one reflection. Let $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n \geq 0\}$ be the upper half space. Denote by $S_0(t, x, y)$ the kernel of $\sin(t\sqrt{-\Delta})/\sqrt{-\Delta}$ of Euclidean $\mathbb{R}^n$. By the classical method of images,
\[
\begin{align*}
S_D(t, x, y) &= S_0(t, x, y) - S_0(t, x, y^*), \\
S_N(t, x, y) &= S_0(t, x, y) + S_0(t, x, y^*)
\end{align*}
\]

where $y^* \in \mathbb{R}^n_+$ is the reflection of $y$ through the boundary $\mathbb{R}^{n-1} \times \{0\}$.

The canonical relation associated to $S_N$ and $S_D$ is the union of the canonical relations of $S_0$ and of $S_0^* = S_0(t, x, y^*)$. More precisely by our notation in (22)
\[
WF(S_B(t, x, y)) \subset \Gamma_\pm^0 \cup \Gamma_\pm^1 \cup \Gamma_\pm^{-1}.
\]

Note that this example is asymmetric in past and future: the forward trajectory may intersect boundary, but then backward one does not. Also, in this example for $j > 1$ and $j < -1$ the graphs $\Gamma_\pm^j$ are empty.
2.3.2. Symbol of $S_B(t, x, y)$ in the interior. In the boundaryless case of \[DG\], the half density symbol of $e^{it\sqrt{-\Delta g}}$ is a constant multiple (Maslov factor) of the canonical graph volume half density $\sigma_{can} = |dt \wedge dy \wedge d\eta|^\frac{1}{2}$ on $\Gamma_+$ in the graph parametrization $(t, y, \eta) \to \Gamma_+ = (t, |\eta|_g, g^t(y, \eta), y, \eta)$. In the boundary case for $E_B(t)$ the symbol in the interior is computed in Corollary 4.3 of \[GM2\] as a scalar multiple of the graph half-density. It is a constant multiple of the graph half-density

$$\sigma_{can, \pm} = |dt \wedge dy \wedge d\eta|^\frac{1}{2}$$

in the obvious graph parametrization of $\Gamma^j_{\pm}$ in \[22\]; the constant equals $\frac{1}{2}$ in the Neumann case and $\frac{(-1)^j}{2}$ in the Dirichlet case. However in \[GM2\] the symbols are not calculated at the boundary.

2.3.3. Symbol of $S_B(t, x, y)$ at the boundary. Since we want to restrict kernels and symbols to the boundary, we introduce further notation for the subset of the canonical relations lying over boundary points. Following \[Ch2\], we denote by $A^0_{\pm} = \{ (0, \tau, y, \eta, y, \eta) : \tau = \pm |\eta|_g \}$ the subset of $\Gamma^0_{\pm}$ with $t = 0$. Under the flow $\psi^t_{\pm}$ of the Hamiltonian $\tau \pm |\xi|_x$ on $\mathbb{R} \times \mathbb{R}^n$, it flows out to the graph $\Gamma^0_{\pm}$ (denoted $C_{\pm}$ in \[Ch2\], \[2.11\]). One then defines $A^1_{\pm} \subset \Gamma^0_{\pm}$ resp. $A^{j}_{\pm} \subset \Gamma^j_{\pm}$ as the subset lying over $\mathbb{R} \times \partial \Omega \times \Omega$ resp. $\mathbb{R} \times \partial \Omega \times \Omega$. We then have

$$\Gamma^1_{\pm} = \bigcup_{t \in \mathbb{R}} \psi^t_{\pm} \hat{A}^1_{\pm},$$

and

$$\Gamma^{-1}_{\pm} = \bigcup_{t \in \mathbb{R}} \psi^t_{\pm} \hat{A}^{-1}_{\pm},$$

as the flow out under the Euclidean space-time geodesic flow of $\hat{A}^1_{\pm}$ and $\hat{A}^{-1}_{\pm}$. Thus, along the boundary, for $t > 0$ (resp. $t < 0$) $A^1_{\pm}$ and $\hat{A}^1_{\pm}$ (resp. $A^{-1}_{\pm}$ and $\hat{A}^{-1}_{\pm}$) both lie in the canonical relation of $E_B(t), S_B(t)$. In a similar way one defines $A^j_{\pm}$ to be the subset of $\Gamma^j_{\pm}$ lying over $\mathbb{R} \times \partial \Omega \times \Omega$ and $\hat{A}^j_{\pm}$ to be its reflection. Then also $A^2_{\pm} \cup \hat{A}^2_{\pm}$ lies in the canonical relation. Similarly one defines $A^j_{\pm}$ and $\hat{A}^j_{\pm}$ for all $j \in \mathbb{Z}$.

Remark: Since we are interested in the singularity of the trace at $t = T > 0$ we will only consider the graphs $\Gamma^j_{\pm}$ for $j \geq 0$. Regardless of this, because $\delta Tr E_B(t)$ is even in $t$ it has the same singularity at in $t = L$ and $t = -T$.

The symbols of $E_B(t)$ and $S_B(t)$ are half-densities on the associated canonical relations, and therefore are sums of four terms at boundary points, i.e. there is a contribution from each of $A^j_{\pm}$ and $\hat{A}^j_{\pm}$. In the interior, there is only a contribution from the $\pm$ components.

The following Lemma gives formulas for the principal symbol of $S_B$ (and therefore $E_B$) on $\Gamma^j_{\pm}$ and its restriction to $\Gamma_{\partial \Omega} \circ (A^j_{\pm} \cup \hat{A}^j_{\pm})$.

**Lemma 2.** Let $e^\pm$ be the principal symbol of $\hat{S}_B$ when restricted to $\Gamma^j_{\pm} = \bigcup_j \Gamma^j_{\pm}$. Let $\sigma_r$ be the principal symbol of the boundary restriction operator $v$. Then

1. In the interior, on $\Gamma^j_{\pm}$, up to Maslov factors we have $e^\pm = \frac{(-1)^j}{2^r} \sigma_{can, \pm} = \pm \frac{(-1)^j}{2^{2r}} \sigma_{can, \pm}$ in the Dirichlet case, and $e^\pm = \frac{1}{2^r} \sigma_{can, \pm} = \pm \frac{1}{2^{2r}} \sigma_{can, \pm}$ in the Neumann case.
2. At the boundary, on \( \Gamma_\partial \cap A_\perp^j = \Gamma_\partial \cap \tilde{A}_\perp^j \) we have

In the Dirichlet case:

\[
\sigma_r \circ e_\pm(t_\perp^j, \pm \tau, \lambda_\perp^j(y, \eta), y, \eta) = -\sigma_r \circ e_\pm(t_\perp^j, \pm \tau, \lambda_\perp^j(y, \eta), y, \eta),
\]

In the Neumann case:

\[
\sigma_r \circ e_\pm(t_\perp^j, \pm \tau, \lambda_\perp^j(y, \eta), y, \eta) = \sigma_r \circ e_\pm(t_\perp^j, \pm \tau, \lambda_\perp^j(y, \eta), y, \eta).
\]

Proof. These formulas are obtained from the transport equations in [Ch2], \((b_0') - (e_0')\) (page 175). We now sketch the proof.

The transport equations for the symbols of \( E_B, S_B \) determine how they propagate along broken geodesics. As in the boundaryless case, the principal symbol has a zero Lie derivative, \( L_{H^r_t + t} \sigma_E = 0 \), in the interior along geodesics. The important point for us is the rule by which they are reflected at the boundary. Let \( \sigma_B \) be the principal symbol of the boundary restriction operator \( B \) defined in [3] \((B = r \text{ resp. } B = r N \text{ when we have Dirichlet resp. Neumann boundary condition})\) and let \( \sigma_0 \) be the principal symbol of the restriction operator to \( t = 0 \). Then,

\[
(b_0) : (\frac{d}{dt} - \Delta_B) \tilde{S}_B \sim 0 \implies (b_0') : L_{\psi^I} e_\pm = 0
\]

\[
(c_0) : \tilde{S}_B|_{t=0} \sim 0 \implies (c_0') : \sigma_0 \circ e_\pm(0, \tau, y, \eta, y, \eta) = \sigma_0 \circ e_\pm(0, -\tau, y, \eta, y, \eta) = 0
\]

\[
(d_0) : \frac{d}{dt}|_{t=0} \tilde{S}_B \sim \delta(x - y) \implies (d_0') : \tau(\sigma_0 \circ e_\pm(0, \tau, y, \eta, y, \eta) - \sigma_0 \circ e_\pm(0, -\tau, y, \eta, y, \eta)) = \sigma_I
\]

\[
(e_0) : B \tilde{S}_B \sim 0 \implies (e_0') : \sigma_B \circ e_\pm = \sigma_B \circ (e_\pm|_{A_\perp^j}) + \sigma_B \circ (e_\pm|_{\tilde{A}_\perp^j}) = 0.
\]

Here \( \sigma_I \) is the principal symbol of the identity operator. The implication \((b_0) \implies (b_0')\) follows for example from Theorem 5.3.1 of [DH]. The other implications are obvious. From \((e_0')\) and \((d_0')\) we get

\[
(\sigma_0 \circ e_\pm)(y, \eta, y, \eta) = \frac{1}{2\tau} \sigma_I, \quad \text{on } T^*\Omega.
\]

But by \((b_0')\), the symbol \( e_\pm \) is invariant under the flow \( \psi^I \) and therefore the first part of the Lemma follows but only on \( \Gamma_\perp^0 \). The second part of the Lemma follows from \((e_0')\). The first term of \((e_0')\) is known from the previous transport equations. Hence \((e_0')\) determines the ‘reflected symbol’ at the \( j \)th impact time and impact point. In the Dirichlet case, \( B \) is just \( r \) the restriction to the boundary and so the reflected principal symbol is simply the opposite of the direct principal symbol. In the Neumann case, \( B \) is the product of the symbol \( \langle \lambda_\perp^j(y, \eta) \rangle \) of the inward normal derivative times restriction \( r \). The reflected symbol thus equals the direct symbol since the sign is canceled by the sign of the \( \langle \lambda_\perp^j(y, \eta), \nu_y \rangle = -\langle \lambda_\perp^j(y, \eta), \nu_y \rangle \) factor. Thus, the volume half-density is propagated unchanged in the Neumann case and has a sign change at each impact point in the Dirichlet case. It follows that, on \( \Gamma_\perp^j \) and after \( j \) reflections, the Dirichlet wave group symbol is \((-1)^j\) times \( \frac{1}{2\tau} \) times the graph half-density [23] and the Neumann symbol is \( \frac{1}{2\tau} \) times the graph half-density.

\( \square \)
2.4. \( \chi_T(D_t, q', D_q')\chi_T(D_t, q, D_q)S_B^b(t, q', q) \) is a Fourier integral operator.

**Lemma 3.** We have,

\[
\chi_T(D_t, q', D_q')\chi_T(D_t, q, D_q)S_B^b(t, q', q) \in \mathcal{I}^{1\frac{1}{2}+\frac{1}{4}}(\mathbb{R} \times \partial\Omega \times \partial\Omega, \Gamma_{\partial, \pm}).
\]

Here, \( \Gamma_{\partial, \pm} = \bigcup_{j \in \mathbb{Z}} \Gamma_{j, \partial, \pm} \), with

\[
\Gamma_{j, \partial, \pm} := \{(t, \tau, q', \zeta', q, \zeta) \in T^*(\mathbb{R} \times \partial\Omega \times \partial\Omega) : \exists \zeta' \in T_q^*\mathbb{R}^n, \xi \in T_q^*\mathbb{R}^n : (t, \tau, q', \zeta', q, \zeta, \xi, \eta) \in \Gamma_{j, \partial, \pm} \}
\]

\[
(t, \tau, q', \zeta', q, \zeta, \xi) \in \Gamma_{j, \partial, \pm}, \xi|_{\tau \cdot \partial\Omega} = \zeta', \xi|_{\tau \cdot \partial\Omega} = \zeta.
\]

**Proof.** We only show the proof in the Dirichlet case. The Neumann case is very similar. The kernel \( \chi_T(D_t, q', D_q')\chi_T(D_t, q, D_q)S_B^b(t, q', q) \) for fixed \( t \) is the Schwartz kernel of the composition

\[
\chi_T \circ (r \cdot N) \circ S_D(t) \circ (N^*r^*) \circ \chi_T : L^2(\partial\Omega) \to L^2(\partial\Omega),
\]

where \( r^* \) is the adjoint of \( r : H^\frac{1}{2}(\Omega) \to L^2(\partial\Omega) \).

To prove the Lemma, we use that \( r \) is a Fourier integral operator with a folding canonical relation, and that the composition \( \chi_T \) is transversal away from the tangential directions to \( \partial\Omega \), where \( S_B(t) \) fails to be a Fourier integral operator. The cutoff \( \chi_T \) removes the part of the canonical relation near the fold locus, hence the composition is a standard Fourier integral operator.

By the results cited above in [Ch2, GM2, PS, MM, Ch2], microlocally away from the gliding directions, the wave operator \( S_B(t) \) is a Fourier integral operator associated to the canonical relations \( \Gamma_{\pm}^j \). Since \( \Gamma_{\pm}^j \) is a union of graphs of canonical transformations, its composition with the canonical relation of \( r^D = r \cdot N \) is automatically transversal. The further composition with the canonical relation of \( r^D \cdot r^* \) is also transversal. Hence, the composition is a Fourier integral operator with the composed wave front relation and the orders add. Taking into account that we have two boundary derivatives, we need to add \( \frac{1}{2} \) to the order.

To determine the composite relation, we note that

\[
\Phi_{\pm} : \mathbb{R} \times T^*_{\partial\Omega}\mathbb{R}^n \to T^*\mathbb{R} \times T^*\mathbb{R} \times T^*_{\partial\Omega}\mathbb{R}^n,
\]

\[
\Phi_{\pm}(t, q, \zeta, \xi_n) := (t, \pm|\zeta + \xi_n|, \Phi'(q, \zeta, \xi_n), q, \zeta, \xi_n)
\]

parameterizes the graph of the (space-time) billiard flow with initial condition on \( T^*_{\partial\Omega}\mathbb{R}^n \). Here, \( \zeta \in T^*\mathbb{R} \) and \( \xi_n \in N^*_{\partial\Omega}\mathbb{R}^n \), the inward pointing (co-)normal bundle. \( \Phi_{\pm} \) is a homogeneous folding map with folds along \( \mathbb{R} \times T^*\mathbb{R} \) (see e.g. [Ho] (volume III) for background). It follows that \( S_D(t) \circ (N^*r^*)\chi_T^* \) is a Fourier integral operator of order one associated to the canonical relation

\[
\{(t, \pm|\xi|, \Phi'(q, \xi), q, \xi|_{T^*\partial\Omega}) \subset T^*(\mathbb{R} \times \Omega \times \partial\Omega),
\]

and is a local canonical graph away from the fold singularity along \( T^*\partial\Omega \). Composing on the left by the restriction relation produces a Fourier integral operator with the stated canonical relation. The two normal derivatives \( N \) of course do not change the relation.

\[\square\]
2.5. Symbol of $\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S_B^t(t, q', q)$. The next step is to compute the principal symbols of the operators in Lemma [3].

To state the result, we need some further notation. We denote points of $T^*_\partial \Omega \mathbb{R}^n$ by $(q, 0, \zeta, \xi_n)$ as above, and put $\tau = \sqrt{\lvert \xi \rvert^2 + \xi^2}$. We note that $\xi_n$ is determined by $(q, \zeta, \tau)$ by $\xi_n = \sqrt{\tau^2 - \lvert \zeta \rvert^2}$, since it is inward pointing. The coordinates $q, \zeta$ are symplectic, so the symplectic form on $T^*\partial \Omega$ is $d\sigma = dq \wedge d\zeta$. We then relate the graph of the billiard flow $[20]$ with initial and terminal point on the boundary to the billiard map (after $j$ reflections) by the formula

$$\tag{27} \Phi^j(q, 0, \zeta, \xi_n) = (\tau \beta_j^j(q, \frac{\zeta}{\tau}), \xi^j_n(q, \zeta, \xi_n)),$$

where $\xi^j_n = \tau \sqrt{1 - |\beta_j(q, \frac{\zeta}{\tau})|^2}$. We also put

$$\tag{28} \gamma(q, \zeta, \tau) = \sqrt{1 - \frac{|\zeta|^2}{\tau^2}}, \text{ and } \gamma_1(q, \zeta) = \sqrt{1 - |\zeta|^2}.$$

It is the homogeneous (of degree zero) analogue of the function denoted by $\gamma$ in [HZ].

Further, we parameterize the canonical relation $\Gamma^j_{\partial^+}$ of Lemma [3] using the billiard map $\beta$ and its powers. We define the $j$th return time $T^j(q, \zeta)$ of the billiard trajectory in a codirection $(q, \zeta) \in T^*_q \Omega$ to be the length the $j$-link billiard trajectory starting at $(q, \zeta)$ and ending at a point $\Phi^j(q, \zeta) \in T^*_\partial \Omega$. It is the same as $T^j_+(q, \zeta)$. Then we define

$$\tag{29} \iota_{\partial^+, j} : \mathbb{R}_+ \times T^*\partial \Omega \to T^*(\mathbb{R} \times \partial \Omega \times \partial \Omega), \quad \iota_{\partial, j}(\tau, q, \zeta) = (T^j(q, \zeta(q, \zeta, \tau)), \tau, (\tau \beta_j^j(q, \frac{\zeta}{\tau})), q, \zeta),$$

where

$$\zeta(q, \zeta, \tau) = \zeta + \xi_n \nu_q, \quad |\zeta|^2 + |\xi_n|^2 = \tau^2.$$

The map (29) parameterizes $\Gamma^j_{\partial^+}$ of Lemma [3].

**Proposition 4.** In the coordinates $(\tau, q, \zeta) \in \mathbb{R}_+ \times T^*\partial \Omega$ of (29), the principal symbol of $\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S_B^t(t, q', q)$ on $\Gamma^j_{\partial^+}$ is as follows:

- in the Dirichlet case:
  $$\sigma_{j, +}(q, \zeta, \tau) = C_{j, +}^D \chi_T(q, \frac{\zeta}{\tau}) \chi_T(\beta_j^j(q, \frac{\zeta}{\tau})) \gamma^j(\zeta(q, \zeta, \tau)) \gamma^j(\tau \beta_j^j(q, \frac{\zeta}{\tau})) \tau|dq \wedge d\zeta \wedge d\tau|^\frac{1}{2},$$

- in the Neumann case:
  $$\sigma_{j, +}(q, \zeta, \tau) = C_{j, +}^N \chi_T(q, \frac{\zeta}{\tau}) \chi_T(\beta_j^j(q, \frac{\zeta}{\tau})) \gamma^j(\zeta(q, \zeta, \tau)) \gamma^j(\tau \beta_j^j(q, \frac{\zeta}{\tau})) \frac{1}{2} \sigma(\zeta, \beta_j^j(q, \frac{\zeta}{\tau})) \tau|dq \wedge d\zeta \wedge d\tau|^\frac{1}{2},$$

where $C_{j, +}^B$ are certain constants (Maslov factors).

**Proof.** We only show the computations in the Dirichlet case. The Neumann case is very similar and uses [5] which will produce an additional factor of $\tau(\zeta, \beta_j^j(q, \frac{\zeta}{\tau})) - \tau^2$.

By Lemma [2] the principal symbol of $S_B^t(t)$ consists of four pieces at the boundary, one for each mode $A_j^k, A_j^\pm$. The symbol for the $-$ mode of propagation is equal to that for the $+$ mode of propagation under the time reversal map $\xi \to -\xi$. Further by part 2 of Lemma [2]
Since the composition is transversal, \( \alpha \) the map \( f : \gamma \) of \( \Gamma \) on \( \Gamma_j \)
(normal coordinates), \( f \) We recall that a map \( \xi \) multiply the symbol by \( \gamma \) of the canonical relation of the \( j \)(30) \( \Gamma \)
composition is equivalent to the pullback of the symbol under the pullback
\( r^D = r \) \( N \) and \( r^D \) of the adjoint \( r^{\ast}N \). Therefore we compute the restriction of the \( \Gamma \) component onto \( \Gamma_{\partial^+} \) and we remember to multiply the symbol by \( \xi_n \gamma = \tau_2 \gamma(q, \xi, \tau) \gamma(\tau_2^{\beta}(q, \xi, \tau)) \) and also by \( \frac{1}{2\pi} \) at the end.

It is simplest to use symbol algebra and pullback formulae to calculate it (see [DG]). The composition is equivalent to the pullback of the symbol under the pullback
\[
\Gamma^j_{\partial} = (i_{\partial \Omega} \times i_{\partial \Omega})^{\ast} \Gamma^j,
\]
of the canonical relation of the \( S_B \) by the canonical inclusion map
\[
i_{\partial \Omega} \times i_{\partial \Omega} : \mathbb{R} \times \partial \Omega \times \partial \Omega \to \mathbb{R} \times \mathbb{R}^n \times \Omega.
\]
We recall that a map \( f : X \to Y \) is transversal to \( W \subset T^*Y \) if \( df^{\ast} \eta \neq 0 \) for any \( \eta \in W \). If \( f : X \to Y \) is smooth and \( \Gamma \subset T^*Y \) is Lagrangian, and if \( f \) and \( \pi : T^*Y \to Y \) are transverse then \( f^{\ast} \Gamma \) is Lagrangian. Since
\[
(i_{\partial \Omega} \times i_{\partial \Omega})^{\ast}(t, \tau, \Phi^i(q, \xi, q, \xi) = (t, \tau, \Phi^i(q, \xi)|_{\partial \Omega}, q, \xi|_{\partial \Omega})
\]
at a point over \((i_{\partial \Omega} \times i_{\partial \Omega})(t, q', q)\), and since \( \tau = |\xi| \neq 0 \), it is clear that \( i_{\partial \Omega} \times i_{\partial \Omega} \) is transversal to \( \pi \).

We now claim that on the pullback of \( \Gamma^j \), using the parametrization [29],
\[
(i_{\partial \Omega} \times i_{\partial \Omega})^{\ast}[dt \wedge dx \wedge d\xi]^{\frac{1}{2}} = \gamma^{-\frac{1}{2}}(q, \xi, \tau) \gamma^{-\frac{1}{2}}(\tau_2^{\beta}(q, \xi, \tau))|dq \wedge d\xi \wedge \tau|^{\frac{1}{2}},
\]
where \( \gamma \) is defined in (29). To see this, we use the pullback diagram
\[
\begin{array}{ccc}
\Gamma^j & \xleftarrow{\pi} & F \\
& \downarrow i & \alpha \\
& \pi \downarrow & \end{array}
\quad (i_{\partial \Omega} \times i_{\partial \Omega})^{\ast} \Gamma^j \subset T^*(\mathbb{R} \times \partial \Omega \times \partial \Omega)
\]
Here, \( F \) is the fiber product, \( N^{\ast}\text{graph}(i_{\partial \Omega} \times i_{\partial \Omega}) \) is the co-normal bundle to the graph, and the map \( \alpha : F \to (i_{\partial \Omega} \times i_{\partial \Omega})^{\ast} \Gamma^j \) is the natural projection to the composition (see [DG]). Since the composition is transversal, \( D \alpha \) is an isomorphism (loc. cit.). The graph of \( i_{\partial \Omega} \times i_{\partial \Omega} \) is the set \( \{(t, q, q', t, q, q') : (t, q, q') \in \mathbb{R} \times \partial \Omega \times \partial \Omega \} \) and its conormal bundle is (in the Fermi normal coordinates),
\[
N^{\ast}\text{graph}(i_{\partial \Omega} \times i_{\partial \Omega}) = \{(t, t, q, \xi, q', \xi', t, -\tau, q, -\xi, q', -\xi' + \xi'_n),
\]
\[
(q, \xi, \xi_n), (q', \xi', \xi'_n) \in T^{\ast}_{\partial \Omega} \mathbb{R}^n \}
\subset \ T^{\ast}(\mathbb{R} \times \partial \Omega \times \partial \Omega \times \mathbb{R}^n \times \mathbb{R}^n),
\]
The half density produced by the pullback diagram takes the exterior tensor product of the canonical half density
\[ |dt \wedge dq \wedge d\zeta \wedge d\xi_n \wedge d\xi'_n \wedge dq' \wedge d\zeta'|^{\frac{1}{2}} \]
on \mathcal{N}^*(\text{graph}(i_{\partial\Omega} \times i_{\partial\Omega})) and
\[ |dt' \wedge dx' \wedge d\xi'|^{\frac{1}{2}}, \quad \text{on } \Gamma^j \subset T^*\left(\mathbb{R} \times \mathbb{R}^n \times \Omega\right) \]
at a point of the fiber product (where the \( T^*(\mathbb{R} \times \mathbb{R}^n \times \Omega) \) components are equal) and divides by the canonical half density
\[ |dt' \wedge dx' \wedge d\zeta' \wedge d\xi'_n \wedge dx'_n \wedge d\zeta' \wedge d\xi'|^{\frac{1}{2}} \]
on the common \( T^*\mathbb{R} \times T^*\Omega \) component.
Since \( \tau' = \tau \), the factors of \( |dt' \wedge dx' \wedge dq' \wedge d\zeta' \wedge d\xi'_n \wedge dx'_n \wedge d\zeta' \wedge d\xi'|^{\frac{1}{2}} \) cancel in the quotient half-density, leaving the half density
\[ \frac{|dt \wedge dq \wedge d\zeta \wedge d\xi_n|^{\frac{1}{2}}}{|dx'_n|^{\frac{1}{2}}} \]
on the composite. The numerator is a half-density on \( \mathbb{R} \times T^*_{\partial\Omega}\mathbb{R}^n \). We write it more intrinsically in the following Lemma. Note that it explains the first of our two \( \gamma \) factors.

**Lemma 5.** Let \( \Phi = \Phi_+ \) be the parametrization \((26)\) and \( \omega_{T^*\mathbb{R}^n} \) be the canonical symplectic form of \( T^*\mathbb{R}^n \). Then \( |dt \wedge dq \wedge d\zeta \wedge d\xi_n|^{\frac{1}{2}} = \left| \frac{\xi_n}{\sqrt{(\xi^2 + \xi^2_n)}} \right|^{\frac{1}{2}} |\Phi^* \omega_{T^*\mathbb{R}^n}|^{\frac{1}{2}} \) as half-densities on \( \mathbb{R} \times T^*_{\partial\Omega}\mathbb{R}^n \).

**Proof.** We have,
\[
\frac{\Phi^* \omega_{T^*\mathbb{R}^n}}{dt \wedge dq \wedge d\zeta \wedge d\xi_n} = \omega_{T^*\mathbb{R}^n}(\frac{d}{dt} \Phi^t(q, \zeta, \xi_n), d\Phi^t \frac{\partial}{\partial q}, d\Phi^t \frac{\partial}{\partial \zeta}, d\Phi^t \frac{\partial}{\partial \xi_n})
\]
\[ = \omega_{T^*\mathbb{R}^n}(H_g, \frac{\partial}{\partial q}, \frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \xi_n})
\]
\[ = \frac{\xi_n}{\sqrt{(\xi^2 + \xi^2_n)}} \omega_{T^*\mathbb{R}^n}(\frac{\partial}{\partial q}, \frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \xi_n})
\]
\[ = \frac{\xi_n}{\sqrt{(\xi^2 + \xi^2_n)}}
\]
since \( \frac{d}{dt} \Phi^t(q, \eta, \xi_n) = H_g = \frac{\xi_n}{\sqrt{(\xi^2 + \xi^2_n)}} \frac{\partial}{\partial q} + \cdots \) is the Hamilton vector field of \( g = \sqrt{g^2}, g^2 = \xi^2 + (g')^2 \) where \( \cdots \) represent vector fields in the span of \( \frac{\partial}{\partial q}, \frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \xi_n} \). Finally, we use that \( d\Phi^t \) is symplectic linear and that \( q, x_n, \zeta, \xi_n \) are symplectic coordinates. Note that we are evaluated the symplectic volume form at the domain point, not the image point.
\[ \square \]

Next we take consider the points in the image of \( \Phi \) on \( \mathbb{R} \times T^*_{\partial\Omega}\mathbb{R}^n \) where \( x'_n = 0 \) and take the quotient by \( |dx'_n|^{\frac{1}{2}} \), resulting in a half density on \( \Gamma^j \). The next Lemma explains the origin of the second \( \gamma \) factor.
Lemma 6. In the subset $\Gamma^j_\partial \subset \Phi(\mathbb{R} \times T^*_\partial \mathbb{R}^n)$ where $x'_n = 0$ and where $t = T^j$, we have (in the parameterizing coordinates (29)),

$$\frac{|dt \wedge dq \wedge d\zeta \wedge d\xi_n|}{|dx'_n|} = \left|((\beta^j)^*-\gamma^{-1})dq \wedge d\eta \wedge d\tau\right|^{\frac{1}{2}}.$$ 

Proof. By the previous Lemma, it suffices to rewrite

$$|dx'_n|^{-\frac{1}{2}} |\Phi^* \omega_{T^* \mathbb{R}^n}|^{\frac{1}{2}}$$

in the coordinates $(\tau, q, \eta)$ of $\iota_\partial,j$ in (29). We observe that $x'_n = \Phi^* x_n$. Hence

$$|dx'_n|^{-\frac{1}{2}} |\Phi^* \omega_{T^* \mathbb{R}^n}|^{\frac{1}{2}} = \left|((\beta^j)^*-\gamma^{-1})dq \wedge d\zeta \wedge d\tau\right|^{\frac{1}{2}}.$$ 

In the last line, we use (27), that

$$\omega_{T^* \mathbb{R}^n} \left|\frac{dx_n^*}{dx'_n}\right| = dq \wedge d\zeta \wedge d\xi_n,$$

and that $\beta$ is symplectic. Indeed, by (27),

$$\Phi^* (dq \wedge d\zeta \wedge d\xi_n) = \tau (\beta^j)^*(dq \wedge d\zeta) \wedge \Phi^* d\xi_n$$

$$= \tau (\beta^j)^*(dq \wedge d\zeta) \wedge \Phi^* d\sqrt{\tau^2 - |\zeta|^2}$$

$$= dq \wedge d\zeta \wedge \Phi^* \frac{\tau d\tau}{\sqrt{\tau^2 - |\zeta|^2}} = ((\beta^j)^*-\gamma^{-1})dq \wedge d\zeta \wedge d\tau.$$ 

Note that $\tau (\beta^j)^*(dq \wedge d\zeta) = dq \wedge d\zeta|_{\beta(q,\zeta)}$.

Combining Lemma 6 with Lemma 5 completes the proof of (31) and Proposition 4.

2.6. Trace along the boundary: composition with $\pi_* \Delta^*$. We now take the trace along the boundary of this operator. Analogously to [DG, GM, MM], we define $\Delta : \mathbb{R} \times \partial \Omega \rightarrow \mathbb{R} \times \partial \Omega \times \partial \Omega$ to be the diagonal embedding and $\pi_*$ to be integration over $\partial \Omega$.

Lemma 7. If the fixed point sets of period $T$ of $\beta^k$ are clean for all $k$ and form a submanifold $F_T$ of $B^* \partial \Omega$ of dimension $d$ (with connected components $\Gamma$), then

$$\pi_* \Delta^* \rho \chi_T(D_t, q', D_{q'}) \chi_T(D_t, q, D_q) S_B^b(t, q', q) \in I^{\frac{d+3}{2} + 1 - \frac{1}{4}}(\mathbb{R}, T^*_T \mathbb{R}),$$

where

$$T^*_T \mathbb{R} = \bigcup_{\pm} \Lambda^T, \pm = \bigcup_{\pm} \{(T, \pm \tau) : \tau \in \mathbb{R}_+\},$$

and its principal symbol on $\Lambda^T, \pm$ is given by

$$c^\pm_{\tau} \frac{d+2}{2} \sqrt{d\tau},$$

where

$$c^\pm = \sum_{\Gamma \subset F_T} C^\pm_\Gamma \int_{\Gamma} \rho \gamma_1 d\mu_\Gamma$$
and \( c^- = \overline{c^+} \) the complex conjugate of \( c^+ \).

**Proof.** The calculation of the principal symbol of the trace of a Fourier integral operator in \([DG]\) is valid for the boundary restriction of the wave kernel, since it only uses that it is \( \pi_*\Delta^* \) composed with a Fourier integral kernel with a known symbol and canonical relation. Hence we follow the proof closely and refer there for further details.

As in \([GM]\), the composition of \( \pi_*\Delta^* \) with

\[
\dot{\rho}\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S^b_B(t, q, q')
\]

is clean if and only if the fixed point set of \( \beta^k \) corresponding to periodic orbits of period \( T \) is clean. When the fixed point set has dimension \( d \) in the ball bundle \( B^*\partial\Omega \), composition with \( \pi_*\Delta^* \) adds \( \frac{d}{2} \) to the order (see \([DG]\), (6.6)). Combining with Lemma 3, we obtain the order \( \frac{d}{2} + 1 - \frac{1}{4} \).

Hence under the cleanliness assumption, it follows that \( \delta \text{Tr} \cos t\sqrt{-\Delta_B} \) is a Lagrangian distribution on \( \mathbb{R} \) with singularities at \( t \in Lsp(\Omega) \). As discussed in \([DG]\) (loc. cit.) for the upper/lower half lines \( \Lambda^T, \pm \) in \( T^*_T\mathbb{R}, I^*_T\mathbb{R}, \Lambda^T, \pm \) consists of multiples of the distribution

\[
\int_0^\infty t^{d/2} e^{\pm it(t-T)} dt = (t - T \pm i0)^{-\frac{d+4}{2}}.
\]

The principal symbol of this Fourier integral distribution is \( t^{d/2} \sqrt{dt} \). Therefore to conclude the Lemma we only need to compute the coefficients of this symbol in the trace.

This coefficient is computed in a universal way from the principal symbol of (32) computed from Proposition 4. Following the proof of \([DG]\), the coefficient of \( t^{d/2} \sqrt{dt} \) is

\[
c^\pm = \sum_{\Gamma \subset F_T} C^{\pm}_\Gamma \int_{\Gamma} \dot{\rho} \gamma_1 d\mu_\Gamma,
\]

where \( F_T \) is the fixed point set of \( \beta \) (and its powers) in \( B^*\partial\Omega \). The sum is over the connected components \( \Gamma \) of \( F_T \) and \( d\mu_\Gamma \) is the canonical density on the fixed point component \( \Gamma \) defined in Lemma 4.2 of \([DG]\). We note that the distribution \( c^+(t-T+i0)^{-\frac{d+4}{2}} + c^-(t-T-i0)^{-\frac{d+4}{2}} \) is real only if \( c^- = \overline{c^+} \). This completes the proof of the Lemma.

The Lemma also completes the proof of the Theorem. We close the section with a remark:

**Remark:** As a check on the order, we note that for the wave trace in the interior and for non-degenerate closed trajectories, the singularities are of order \( (t - T + i0)^{-1} \). When the periodic orbits are degenerate and the unit vectors in the fixed point sets have dimension \( d \), the singularity increases to order \( (t - T + i0)^{-1 - \frac{d}{2}} \). If we formally take the variation of the wave trace, the singularity should increase to order \( (t - T + i0)^{-1 - \frac{d}{2} - 1} \).

In comparison, the boundary trace in the Dirichlet case involves two extra derivatives of the wave kernel and composition with \( (-\Delta)^{-\frac{1}{2}} \). Compared to the interior trace, this adds one net derivative and order to the trace singularity. We claim that the restriction to the boundary does not further change the order compared to the interior trace. This can be seen by considering the method of stationary phase for oscillatory integrals with Bott-Morse phase functions, whose non-degenerate critical manifolds are transverse to the boundary. If we restrict the integral to the boundary, we do not change the number of phase variables.
in the integral, but we simultaneously decrease the number of variables by one and the dimension of the fixed point set by one. The number of non-degenerate directions stays the same. It follows that the singularity order of the variational trace goes up by one overall unit compared to the interior trace, consistently with the formal variational calculation.

3. Case of the ellipse and the proof of Theorem 1

In this section we let \( \Omega_0 \) be an ellipse. In this case, the fixed point sets are clean fixed point sets for \( \Phi_t \) in \( T^*\Omega_0 \), resp. for \( \beta \) in \( B^*\partial\Omega_0 \) (See [GM] Proposition 4.3). In fact the fixed point sets \( F_T \) of \( \beta \) in \( B^*\partial\Omega_0 \) form a one dimensional manifold. Thus \( d = 1 \) and corollary 2 follows.

As is well-known, both the billiard flow and billiard map of the ellipse are completely integrable. In particular, except for certain exceptional trajectories, the periodic points of period \( T \) form a Lagrangian tori in \( S^*\Omega_0 \), and the homogeneous extensions of the Lagrangian tori are cones in \( T^*\Omega_0 \). The exceptions are the two bouncing ball orbits through the major/minor axes and the trajectories which intersect the foci or glide along the boundary. The fixed point sets of \( \Phi_T \) intersect the co-ball bundle \( B^*\partial\Omega_0 \) of the boundary in the fixed point sets of the billiard map \( \beta : B^*\partial\Omega_0 \to B^*\partial\Omega_0 \) (for background we refer to [PS, GM, GM2, HZ, TZ2] for instance). Except for the exceptional orbits, the fixed point sets are real analytic curves. For the bouncing ball rays, the associated fixed point sets are non-degenerate fixed points of \( \beta \).

Since the final step of the proof uses results of [GM], we briefly review the description of the billiard map of the ellipse \( \Omega_0 := \frac{x^2}{a} + \frac{y^2}{b} = 1 \) (with \( a > b > 0 \)) in that article. In the interior, there exist for each \( 0 < Z \leq b \) a caustic set given by a confocal ellipse

\[
\frac{x^2}{\epsilon + Z} + \frac{y^2}{Z} = 1
\]

where \( \epsilon = a - b \), or for \( -\epsilon < Z < 0 \) by a confocal hyperbola. Let \((q, \zeta)\) be in \( B^*\partial\Omega_0 \) and let \((q, \xi)\) in \( S^*\Omega_0 \) be the unique inward unit normal to boundary that projects to \((q, \zeta)\). The line segment \((q, r\xi)\) will be tangent to a unique confocal ellipse or hyperbola (unless it intersects the foci). We then define the function \( Z(q, \zeta) \) on \( B^*\partial\Omega_0 \) to be the corresponding \( Z \). Then \( Z \) is a \( \beta \)-invariant function and its level sets \( \{Z = c\} \) are the invariant curves of \( \beta \). The invariant Leray form on the level set is denoted \( du_Z \) (see [GM], (2.17), i.e. the symplectic form of \( B^*\partial\Omega_0 \) is \( dq \wedge d\zeta = dZ \wedge du_Z \). A level set has a rotation number and the periodic points live in the level sets with rational rotation number. As it is explained in [GM] (page 143) the Leray form \( du_Z \) restricted to a connected component \( \Gamma \) of \( F_T \) is a constant multiple of the canonical density \( d\mu_T \).

As mentioned in the introduction, the well-known obstruction to using trace formula calculations such as in Proposition 2 is multiplicity in the length spectrum, i.e. existence of several connected components of \( F_T \). A higher dimensional component is not itself a problem, but there could exist cancellations among terms coming from components with different Morse indices, since the coefficients \( C_T \) are complex. This problem arose earlier in the spectral theory of the ellipse in [GM]. Their key Proposition 4.3 shows that there are is a sufficiently large set of lengths \( T \) for which \( F_T \) has one component up to \((q, \zeta) \to (q, -\zeta)\) symmetry. Since it is crucial here as well, we state the relevant part:
Proposition 8. (see [GM], Proposition 4.3): Let $T_0 = |\partial \Omega_0|$. Then for every interval $(mT_0 - \epsilon, mT_0)$ for $m = 1, 2, 3, \ldots$ there exist infinitely many periods $T \in Lsp(\Omega_0)$ for which $F_T$ is the union of two invariant curves which are mapped to each other by $(q, \zeta) \to (q, -\zeta)$.

Since for an isospectral deformation $\delta \text{ Tr} \cos(t\sqrt{-\Delta}) = 0$, we obtain from Proposition 2 the following

Corollary 9. Suppose we have an isospectral deformation of an ellipse $\Omega_0$ with velocity $\dot{\rho}$. Then for each $T$ in Proposition 8 for which $F_T$ is the union of two invariant curves $\Gamma_1$ and $\Gamma_2$ which are mapped to each other by $(q, \zeta) \to (q, -\zeta)$ we have

$$\int_{\Gamma_j} \dot{\rho} \gamma_1 \, du_Z = 0, \quad j = 1, 2.$$  

Proof. From Proposition 2 we get

$$\mathbb{R}\left\{ \sum_{j=1}^{2} C_{j} \int_{\Gamma_j} \dot{\rho} \gamma_1 \, d\mu_{j}, \right\} (t - T + i0)^{-2 - \frac{4}{\pi}} = 0.$$  

Since $\dot{\rho}$ and $\gamma_1$ are invariant under the time reversal map $(q, \zeta) \to (q, -\zeta)$, the two integrals are identical. Also by directly looking at the stationary phase calculations it can be shown that the Maslov coefficients $C_{\Gamma_1}$ and $C_{\Gamma_2}$ are also the same. Thus the corollary follows. □

3.1. Abel transform. The remainder of the proof of Theorem 1 is identical to that of Theorem 4.5 of [GM] (see also [PT]). For the sake of completeness, we sketch the proof.

Proposition 10. The only $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant function $\dot{\rho}$ satisfying the equations of Corollary 9 is $\dot{\rho} = 0$.

Proof. First, we may assume $\dot{\rho} = 0$ at the endpoints of the major/minor axes, since the deformation preserves the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry and we may assume that the deformed bouncing ball orbits will not move and are aligned with the original ones. Thus $\dot{\rho}(\pm \sqrt{a}) = \dot{\rho}(\pm \sqrt{b}) = 0$.

The Leray measure may be explicitly evaluated (see 2.18 in [GM]). By a change of variables with Jacobian $J$, and using the symmetric properties of $\dot{\rho}$, the integrals become

$$A(Z) = \int_{a}^{b} \frac{\dot{\rho}(t) \gamma_1 J(t) \, dt}{\sqrt{t - (b - Z)}}.$$  

for an infinite sequence of $Z$ accumulating at $b$. Since $0 < a < b$, the function $A(Z)$ is smooth in $Z$ for $Z$ near $b$. It vanishes infinitely often in each interval $(b - \epsilon, b)$, hence is flat at $b$. The $k$th Taylor coefficient at $b$ is

$$A^{(k)}(b) = \int_{a}^{b} \dot{\rho}(t) \gamma_1 J(t) t^{k-\frac{4}{\pi}} \, dt = 0.$$  

Since the functions $t^{-k}$ span a dense subset of $C[a, b]$, it follows that $\dot{\rho} \equiv 0$. □
3.2. **Infinitesimal rigidity and flatness.** We now show that infinitesimal rigidity implies flatness and prove Corollary 1. As mentioned, the Hadamard variational formula is valid for any $C^1$ parametrization $\Omega_{\epsilon(\alpha)}$ of the domains $\Omega_{\epsilon}$. For each one we have $\delta \rho_{\epsilon(\alpha)}(x) \equiv 0$.

Assume $\rho_{\epsilon}(x)$ is not flat at $\epsilon = 0$ and let $\epsilon^k$ be the first non-vanishing term in the Taylor expansion of $\rho_{\epsilon}(x)$ at $\epsilon = 0$. Then

$$
\rho_{\epsilon}(x) = \epsilon^k \rho^{(k)}(x) + \epsilon^{k+1} \rho^{(k+1)}(x) \frac{1}{(k+1)!} + \cdots.
$$

By Hadamard’s variational formulae we get $\delta \rho_{\epsilon(\alpha)}(x) = \rho^{(k)}(x) \equiv 0$, a contradiction.

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