SUSY Gauge Theories on Squashed Three-Spheres

Naofumi Hama\textsuperscript{a}, Kazuo Hosomichi\textsuperscript{a} and Sungjay Lee\textsuperscript{b}

\textsuperscript{a}Yukawa Institute for Theoretical Physics, Kyoto University, Japan
\textsuperscript{b}Department of Applied Mathematics and Theoretical Physics, University of Cambridge, UK

Abstract: We study Euclidean 3D $\mathcal{N} = 2$ supersymmetric gauge theories on squashed three-spheres preserving isometries $SU(2) \times U(1)$ or $U(1) \times U(1)$. We show that, when a suitable background $U(1)$ gauge field is turned on, these squashed spheres support charged Killing spinors and therefore $\mathcal{N} = 2$ supersymmetric gauge theories. We present the Lagrangian and supersymmetry rules for general gauge theories. The partition functions are computed using localization principle, and are expressed as integrals over Coulomb branch. For the squashed sphere with $U(1) \times U(1)$ isometry, its measure and integrand are identified with the building blocks of structure constants in Liouville or Toda conformal field theories with $b \neq 1$.

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1. Introduction

In 3D supersymmetric gauge theories, there has been a remarkable technical breakthrough based on exact computation of partition functions of Euclidean theories on three-sphere $S^3$ using localization principle. It was first developed for superconformal theories by [1, 2], and recently extended to non-conformal theories by [3, 4]. The result is expected to provide us with a new quantitative tool to study strongly coupled IR physics of 3D supersymmetric theories. In particular, in [3] it has been argued that the 3D partition function can be used to determine the superconformal R-symmetry at the infrared fixed points, in a similar spirit to $a$-maximization in 4D gauge theories. It has also been attracting wide attention from various other fields, for example Wilson loops [5, 6], 3D dualities [2], large-$N$ duality of topological string [7] and the ABJM theory at strong coupling [8, 9, 10]. The localization technique can also be applied to the theory on $S^2 \times S^1$ and allows one to compute the superconformal index. Using this idea, the superconformal index of $\mathcal{N} = 6$ Chern-Simons matter theories was computed by [11]. It has recently been generalized to arbitrary $\mathcal{N} = 2$ supersymmetric gauge theories by [12]. See also [13, 14].

Another aspect, which we will focus on in this paper, is the application to the AGT relation. In [15] the exact partition function of 4D $\mathcal{N} = 2$ gauge theories on $S^4$ has been worked out, and it has led to a discovery of a remarkable correspondence between partition functions of 4D gauge theories and correlation functions of 2D Liouville or Toda conformal field theories [16, 17]. In this context, certain 3D supersymmetric gauge theories on $S^3$ play the role of domain walls [18, 19], and their partition functions were shown to correspond to transformation coefficients of conformal blocks under S-duality (or Moore-Seiberg groupoid operation).

In [19] it was observed that the building blocks of 3D partition functions on round $S^3$, namely the measure over Coulomb branch and the one-loop determinants, are precisely the
building blocks for the structure constants in Liouville or Toda CFTs with \( b = 1 \). Here \( b \) is the coupling which determines the central charge of Liouville or Toda theories, for example for Liouville theory one has \( c = 1 + 6(b + b^{-1})^2 \). In particular, the one-loop determinant of a charged matter multiplet is given by a double sine function \( s_b(x) \) at \( b = 1 \),

\[
s_b(x) \equiv \prod_{m,n \in \mathbb{Z}_{\geq 0}} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix}, \quad (Q \equiv b + \frac{1}{b})
\]

and \( x \) is related to a coordinate on the Coulomb branch. See \([20, 21]\) for more details on this function. This is plausible if we recall that 4D \( \mathcal{N} = 2 \) gauge theories on round \( S^4 \) are in correspondence with Liouville or Toda theories with \( b = 1 \). It is not yet known how to obtain theories with more general \( b \) though, as suggested in \([19]\), a reasonable guess would be that the background sphere should admit a continuous deformation to account for this.

In this paper we present the answer to this question for 3D theories.

We begin by presenting two kinds of squashed \( S^3 \) in Section 2. The first one appears frequently in the literature; the metric is written in terms of left-invariant one-forms and preserves \( SU(2) \times U(1) \) symmetry. The second one is less familiar and preserves only \( U(1) \times U(1) \) symmetry, but it has a simple definition as a hyper-ellipsoid embedded in flat \( \mathbb{R}^4 \). For both squashings we show that, if a suitable background \( U(1) \) gauge field is turned on, one can find a pair of Killing spinors with the \( U(1) \) charges \( \pm 1 \) which is necessary for defining \( \mathcal{N} = 2 \) supersymmetric gauge theories. We give the general construction of supersymmetric gauge theories in Section 3.

We then turn to the computation of partition functions. Section 4 discusses the case of \( SU(2) \times U(1) \) symmetric squashing, where we use the spherical harmonics to work out all the eigenmodes of the relevant Laplace and Dirac operators. We then use them to compute the one-loop determinant at each saddle point. Disappointingly, the partition function turns out to be essentially the same as that for round sphere, essentially due to \( SU(2) \) isometry. Next we discuss the less symmetric squashing in Section 5. Rather than working out the full spectrum of the Laplace and Dirac operators, we look closely into the structures in which the bosonic and fermionic modes are paired, and exhaust the modes unpaired which give nontrivial contributions to the one-loop determinant. The final expression for the integration measure and determinant are found to be precisely the building blocks for structure constants in Liouville or Toda theories with general \( b \).

We end the introduction by summarizing our conventions for bilinear products of spinors.

\[
\bar{\epsilon} \lambda = \bar{\epsilon}^a C_{\alpha\beta} \lambda^\beta, \quad \bar{\epsilon} \gamma^a \lambda = \bar{\epsilon}^a (C \gamma^a)_{\alpha\beta} \lambda^\beta, \quad \text{etc.}
\]

(1.2)

Here \( C \) is the charge conjugation matrix with nonzero elements \( C_{12} = -C_{21} = 1 \), and \( \gamma^a \) are Pauli’s matrices. Noticing that \( C \) is antisymmetric and \( C \gamma^a \) are symmetric, one finds

\[
\bar{\epsilon} \lambda = \lambda \bar{\epsilon}, \quad \bar{\epsilon} \gamma^a \lambda = -\lambda \gamma^a \bar{\epsilon}
\]

(1.3)

for all spinors \( \bar{\epsilon}, \lambda \) which we assume to be Grassmann odd.
2. Three-Spheres, Round and Squashed

Here we summarize our notations for various geometric quantities on round or squashed $S^3$. By squashing we mean certain deformations to its round metric. We will restrict to those preserving at least $U(1) \times U(1)$ isometry. The reason is that, when our $S^3$ is embedded as a domain wall in a 4D space on which an $\mathcal{N} = 2$ supersymmetric gauge theory is defined, the $U(1) \times U(1)$ isometry is necessary for the omega deformation. There are many metrics preserving this isometry, of which we take only two simple examples. After presenting the metrics for these two squashings we show that, with a suitable background $U(1)$ gauge field turned on, the two squashed spheres both admit a pair of charged Killing spinors.

The three-sphere is parametrized by an element $g$ of the Lie group $SU(2)$, and two copies of $SU(2)$ symmetry act on $g$ from the left and the right. We introduce the left-invariant (LI) and right-invariant (RI) one-forms

$$
\mu^a = \mu^a_{\nu} d\xi^\nu, \quad \tilde{\mu}^a = \tilde{\mu}^a_{\nu} d\xi^\nu,
$$

(2.1)

where $\gamma^a$ are Pauli matrices. These one-forms satisfy

$$
d\mu^a = \varepsilon^{abc} \mu^b \mu^c, \quad d\tilde{\mu}^a = -\varepsilon^{abc} \tilde{\mu}^b \tilde{\mu}^c.
$$

(2.2)

**Round sphere.** The left-right invariant metric on the round sphere with radius $\ell$ is

$$
ds^2 = \frac{1}{2} \ell^2 \text{tr}(dg dg^{-1}) = \ell^2 \mu^a \mu^a = \ell^2 \tilde{\mu}^a \tilde{\mu}^a.
$$

(2.3)

We define the vielbein in the “LI frame” as $e^a = e^a_{\mu} d\xi^\mu = \ell^a$. The spin connection in this frame is $\omega^{ab} = \varepsilon^{abc} \mu^c$ and satisfies the torsion-free condition $de^a + \omega^{ab} e^b = 0$. If we define the vielbein from $\tilde{\mu}^a$ (“RI frame”), the spin connection is $\tilde{\omega}^{ab} = -\varepsilon^{abc} \tilde{\mu}^c$.

Killing spinor $\epsilon$ satisfies the following equation

$$
D\epsilon \equiv d\epsilon + \frac{1}{2} \gamma^{ab} \omega^{ab} \epsilon = e^a \gamma^a \tilde{\epsilon},
$$

(2.4)

for a certain $\tilde{\epsilon}$. Here we used $\gamma^{ab} = i \varepsilon^{abc} \gamma^c$. There are two types of Killing spinors. The first one is constant in the LI frame,

$$
\epsilon = \epsilon_0 \text{ (constant)}, \quad \tilde{\epsilon} = + \frac{i}{2\ell} \epsilon.
$$

(2.5)

The second one reads

$$
\epsilon = g^{-1} \epsilon_0, \quad \tilde{\epsilon} = - \frac{i}{2\ell} \epsilon,
$$

(2.6)

and is constant in the RI frame.

Let us next introduce the Killing vector fields $\mathcal{L}^a = \mathcal{L}^a_{\mu} \partial_{\mu}$ and $\mathcal{R}^a = \mathcal{R}^{au} \partial_{\nu}$ which generate the left and the right actions of $SU(2)$. They can be determined from

$$
\mathcal{L}^a g = ig \gamma^a, \quad \mathcal{R}^a g = ig \gamma^a.
$$

(2.7)

The vector fields $\frac{i}{2} \mathcal{L}^a$ and $-\frac{i}{2} \mathcal{R}^a$ satisfy the standard commutation relations of $SU(2)$ Lie algebra. It is also easy to find $\mathcal{R}^{au} \mu^b_{\nu} = \mathcal{L}^{au} \tilde{\mu}^b_{\nu} = \delta^{ab}$, in other words $\mathcal{R}^{au}$ and $\mathcal{L}^{au}$ are
proportional to the inverse-violbeins in LI or RI frames. The action of these Killing vector fields on the LI and RI one-forms is given by

\[ \mathcal{L}^a \tilde{\mu}^b = 2\varepsilon^{abc} \tilde{\mu}^c, \quad \mathcal{R}^a \mu^b = -2\varepsilon^{abc} \mu^c, \quad \mathcal{L}^a \mu^b = \mathcal{R}^a \tilde{\mu}^b = 0. \quad (2.8) \]

It therefore follows that \( \mu^1 \mu^2 \mu^3 = d^3 \xi \det(\mu^\alpha_\nu) \) can be used to define the invariant volume form.

**Squashed sphere (familiar one).** Squashed spheres are defined by the metric or vielbein one-forms

\[ ds^2 = \ell^2(\mu^1 \mu^1 + \mu^2 \mu^2) + \tilde{\ell}^2 \mu^3 \mu^3, \quad (e^1, e^2, e^3) = (\ell \mu^1, \ell \mu^2, \tilde{\ell} \mu^3). \quad (2.9) \]

The squashed metric preserves the \( SU(2)_L \times U(1)_R \) symmetry. The torsion-free condition determines the spin connection \( \omega^{ab} \) as follows,

\[ \omega^{12} = (2\tilde{\ell}^{-1} - f^{-1}) e^3, \quad \omega^{23} = f^{-1} e^1, \quad \omega^{31} = f^{-1} e^2, \quad (2.10) \]

where \( f \equiv \ell^2 \tilde{\ell}^{-1} \) is a constant. Then one can show that any constant spinor \( \psi \) satisfies

\[ d\psi + \frac{1}{4} \gamma^{ab} \omega^{ab} \psi = \frac{i}{2f} \gamma^a e^a \psi + i \gamma^3 V \psi, \quad V \equiv e^3 \left( \frac{1}{\ell} - \frac{1}{\tilde{\ell}} \right). \quad (2.11) \]

This is the Killing spinor equation were it not for the last term in the right hand side.

Now, let us think of the one-form \( V \) as a background gauge field for a certain \( U(1) \) symmetry. Let \( \epsilon \) be a complex spinor field with charge +1 under this \( U(1) \), so that its covariant derivative becomes

\[ D\epsilon \equiv d\epsilon + \frac{1}{4} \gamma^{ab} \omega^{ab} \epsilon - i V \epsilon. \quad (2.12) \]

Then the constant spinor \( \epsilon = (1, 0)^t \) satisfies the \( (U(1))-\text{covariant version of} \) Killing spinor equation,

\[ D\epsilon = \frac{i}{2f} \gamma^a e^a \epsilon. \quad (2.13) \]

Likewise, a spinor \( \bar{\epsilon} \) carrying the \( U(1) \) charge \(-1\) is a Killing spinor if \( \bar{\epsilon} = (0, 1)^t \). One also finds

\[ \bar{\epsilon}\epsilon = -1, \quad \bar{\epsilon} \gamma^a \epsilon = -\delta^{a3}. \quad (2.14) \]

The latter represents nothing but the Killing vector \( \mathcal{R}^3 \). The squashed \( S^3 \) can be regarded as an \( S^1 \) fibration over \( S^2 \), where each fiber is an orbit of the action of \( \mathcal{R}^3 \).
Squashed sphere (less familiar one). Next we consider the less familiar version of squashed sphere which preserve only $U(1)_L \times U(1)_R$ symmetry. We take one of the simplest metrics with this property,

\[
d s^2 = \ell^2 (dx_0^2 + dx_1^2) + \tilde{\ell}^2 (dx_2^2 + dx_3^2).
\]

\[(x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1) \quad (2.15)\]

Inserting $(x_0, x_1, x_2, x_3) = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, \sin \theta \cos \chi, \sin \theta \sin \chi)$ one obtains

\[
d s^2 = f(\theta)^2 d\theta^2 + \ell^2 \cos^2 \theta d\varphi^2 + \tilde{\ell}^2 \sin^2 \theta d\chi^2,
\]

\[f(\theta) \equiv \sqrt{\ell^2 \sin^2 \theta + \tilde{\ell}^2 \cos^2 \theta}. \quad (2.16)\]

We set the vielbein as follows,

\[e^1 = \ell \cos \theta d\varphi, \quad e^2 = \tilde{\ell} \sin \theta d\chi, \quad e^3 = f(\theta) d\theta. \quad (2.17)\]

The spin connection then becomes

\[\omega^{12} = 0, \quad \omega^{13} = -\frac{\ell}{f} \sin \theta d\varphi, \quad \omega^{23} = \frac{\tilde{\ell}}{f} \cos \theta d\chi. \quad (2.18)\]

The Killing spinor equation for a spinor field $\psi$ can be reduced to

\[if \partial_\varphi \psi = \ell \gamma^2 (\frac{1}{2} \sin \theta + \cos \theta \partial_\theta) \psi, \quad if \partial_\chi \psi = \tilde{\ell} \gamma^1 (\frac{1}{2} \cos \theta - \sin \theta \partial_\theta) \psi. \quad (2.19)\]

For round sphere one has $\ell = \tilde{\ell} = f$, and the above equation is easily shown to have four independent solutions,

\[\psi_{st} = \begin{pmatrix} e^{\frac{t}{2}(s\chi + t\varphi - st\theta)} \\ -se^{\frac{t}{2}(s\chi + t\varphi + st\theta)} \end{pmatrix}, \quad (s, t = \pm) \quad (2.20)\]

satisfying

\[D_\mu \psi_{st} = -\frac{ist}{2f} \gamma_\mu \psi_{st}. \quad (2.21)\]

For squashed spheres with $\ell \neq \tilde{\ell}$, the same spinor fields $\psi_{st}$ fail to satisfy the Killing spinor equations.

\[-\frac{ist}{2f} \gamma_\mu \psi_{st} = D_\mu \psi_{st} - iV^{(st)}_{\mu} \psi_{st}, \quad (2.22)\]

where

\[V^{(st)} = \frac{t}{2} \left(1 - \frac{\ell}{f}\right) d\varphi + \frac{s}{2} \left(1 - \frac{\tilde{\ell}}{f}\right) d\chi. \quad (2.23)\]

Again, one can reinterpret the unwanted term in the right hand side of (2.22) as the coupling to a background $U(1)$ gauge field. For the discussions in later sections, we choose to turn on the gauge field $V = V^{(+-)}$, so that the spinors $\epsilon = \psi_{+-}$ and $\tilde{\epsilon} = \psi_{-+}$ satisfy the Killing spinor equation with $U(1)$ charges $\pm 1$. 

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3. SUSY Theories on Squashed $S^3$

Here we present the Euclidean 3D $\mathcal{N} = 2$ gauge theories on manifolds with generalized Killing spinors. The formulae for the action and supersymmetry rules are almost the same as those for round $S^3$ (see e.g. [8]), but now contain the background $U(1)$ gauge field we have turned on. Since the $U(1)$ symmetry rotates the Killing spinors, it is the R-symmetry which has been gauged. In the following we show that the closure of supersymmetry algebra and invariance of Lagrangian are achieved if the background gauge field is coupled to the fields according to their R-charge.

**Vector multiplets.** Vector multiplet fields obey the following transformation laws,

$$
\delta A_\mu = -\frac{1}{2}(\bar{\epsilon}\gamma^\mu \lambda - \bar{\lambda}\gamma^\mu \epsilon),
$$

$$
\delta \sigma = \frac{1}{2} (\bar{\epsilon} \lambda - \bar{\lambda} \epsilon),
$$

$$
\delta \lambda = \frac{1}{2} \gamma^{\mu\nu} \epsilon F_{\mu\nu} - D\epsilon + i \gamma^\mu \epsilon D_\mu \sigma + \frac{2i}{3} \sigma \gamma^\mu D_\mu \epsilon,
$$

$$
\delta \bar{\lambda} = \frac{1}{2} \gamma^{\mu\nu} \bar{\epsilon} F_{\mu\nu} + D\bar{\epsilon} - i \gamma^\mu \bar{\epsilon} D_\mu \sigma - \frac{2i}{3} \sigma \gamma^\mu D_\mu \bar{\epsilon},
$$

$$
\delta D = -\frac{1}{2} \bar{\epsilon} \gamma^\mu D_\mu \lambda - \frac{1}{2} D_\mu \bar{\lambda} \gamma^\mu \epsilon + \frac{1}{2} (\bar{\epsilon} \lambda, \sigma) + \frac{1}{2} [\bar{\lambda} \epsilon, \sigma] - \frac{i}{6} (D_\mu \bar{\epsilon} \gamma^\mu \lambda + \bar{\lambda} \gamma^\mu D_\mu \epsilon). \tag{3.1}
$$

Here and throughout this paper, $D_\mu$ denotes the covariant derivative with respect to gauge, local Lorentz and background gauged R-symmetries as well as general covariance. $\gamma^\mu = e^a_\mu \gamma^a$ is the Dirac matrix with curved index. The spinors $\epsilon, \bar{\epsilon}$ are assumed to satisfy Killing spinor equation. Namely there are spinors $\bar{\epsilon}, \tilde{\epsilon}$ which satisfy

$$
D_\mu \epsilon \equiv (\partial_\mu + \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} - i V_\mu) \epsilon = \gamma_\mu \bar{\epsilon},
$$

$$
D_\mu \bar{\epsilon} \equiv (\partial_\mu + \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} + i V_\mu) \bar{\epsilon} = \gamma_\mu \tilde{\epsilon}. \tag{3.2}
$$

Denoting $\delta$ as the sum of unbarred and barred parts, $\delta = \delta_\epsilon + \delta_{\bar{\epsilon}}$, one can show that two unbarred or two unbarred supersymmetries commute. Also, on most of the fields the commutator $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ is a sum of translation by $i v^\mu$, gauge transformation by $\Lambda$, Lorentz rotation by $\Theta^{\mu\nu}$, dilation by $\rho$ and R-rotation by $\alpha$, where

$$
v^\mu = \tilde{\epsilon} \gamma^\mu \epsilon,
$$

$$
\Theta^{\mu\nu} = i D^{[\mu} v^{\nu]} + i v^\lambda \omega^{\mu\nu}_\lambda,
$$

$$
\Lambda = -i A_\mu v^\mu + \sigma \epsilon \bar{\epsilon},
$$

$$
\rho = \frac{i}{3} (\bar{\epsilon} \gamma^\mu D_\mu \epsilon + D_\mu \bar{\epsilon} \gamma^\mu \epsilon),
$$

$$
\alpha = \frac{i}{3} (D_\mu \bar{\epsilon} \gamma^\mu \epsilon - \bar{\epsilon} \gamma^\mu D_\mu \epsilon) + v^\mu V_\mu. \tag{3.3}
$$

The only exception is that

$$
[\delta_\epsilon, \delta_{\bar{\epsilon}}] D = i v^\mu \partial_\mu D + i [\Lambda, D] + 2 \rho D
$$

$$
+ \frac{i}{3} \sigma (\bar{\epsilon} \gamma^\mu \gamma^\nu D_\mu D_\nu \epsilon - \epsilon \gamma^\mu \gamma^\nu D_\mu D_\nu \bar{\epsilon}). \tag{3.4}
$$

The last term in the right hand side vanishes provided that $\epsilon$ and $\bar{\epsilon}$ satisfy, in addition to (3.2), the following equations

$$
\gamma^\mu \gamma^\nu D_\mu D_\nu \epsilon = -\frac{3}{8} (R + 2 i V_{\mu\nu} \gamma^{\mu\nu}) \epsilon,
$$

$$
\gamma^\mu \gamma^\nu D_\mu D_\nu \bar{\epsilon} = -\frac{3}{8} (R - 2 i V_{\mu\nu} \gamma^{\mu\nu}) \bar{\epsilon}. \tag{3.5}
$$
for a certain set of functions \((R, V_{\mu\nu})\). By arguing in a similar way to [4] one finds that \(R\) is the scalar curvature of the 3D manifold and \(V_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu\) is the field strength of the background gauge field. Note that the Killing spinors on the squashed \(S^3\) of our interest actually all satisfy a stronger condition \((3.14)\), so the supersymmetry is not reduced by the above condition \((3.5)\). For later convenience, we give here some additional formulae which are related to \((3.5)\).

\[
\begin{align*}
\gamma^\mu \gamma^\nu D_\mu D_\nu \epsilon &= 3D_\mu D^\mu \epsilon, \\
\gamma^\mu \gamma^\nu D_\mu D_\nu \bar{\epsilon} &= 3D_\mu D^\mu \bar{\epsilon},
\end{align*}
\]

**Matter multiplets.** The fields in a chiral multiplet coupled to a gauge symmetry transform as follows,

\[
\begin{align*}
\delta \phi &= \bar{\epsilon} \psi, \\
\delta \bar{\phi} &= \epsilon \bar{\psi}, \\
\delta \psi &= i \gamma^\mu \epsilon D_\mu \phi + i \epsilon \sigma \phi + \frac{2q}{3} \gamma^\mu D_\mu \epsilon \phi + \bar{\epsilon} F, \\
\delta \bar{\psi} &= i \gamma^\mu \bar{\epsilon} D_\mu \bar{\phi} + i \bar{\epsilon} \sigma \bar{\phi} + \frac{2q}{3} \bar{\epsilon} \gamma^\mu D_\mu \bar{\epsilon} + \bar{F} \epsilon, \\
\delta F &= \epsilon (i \gamma^\mu D_\mu \psi - i \sigma \psi - i \lambda \phi) + \frac{1}{3} (2q - 1) D_\mu \epsilon \gamma^\mu \psi, \\
\delta \bar{F} &= \bar{\epsilon} (i \gamma^\mu D_\mu \bar{\psi} - i \bar{\psi} \sigma + i \phi \bar{\lambda}) + \frac{1}{3} (2q - 1) D_\mu \bar{\epsilon} \gamma^\mu \bar{\psi}.
\end{align*}
\]

Here we assumed the fields \(\phi, \psi, F (\bar{\phi}, \bar{\psi}, \bar{F})\) to be column vectors (resp. row vectors) on which the vector multiplet fields act as matrices from the left (right). The lowest components \((\phi, \bar{\phi})\) are assigned the dimension \(q\) and R-charge \((-q, +q)\), as one can obtain from the supersymmetry algebra realized on these fields. The supersymmetry algebra can be easily shown to close off-shell, except that two unbarred supersymmetries do not simply commute on \(F\),

\[
[\delta_\epsilon, \delta_{\epsilon'}] F = \epsilon \gamma^\mu \epsilon' (2D_\mu D_\nu \phi + i F_{\mu\nu} \phi) + \frac{2q}{3} \phi \gamma^\mu D_\mu \epsilon \epsilon' - \epsilon' \gamma^\mu \gamma^\nu D_\mu D_\nu \epsilon \epsilon'.
\]

The right hand side does not vanish for general Killing spinors \(\epsilon, \epsilon'\). With an additional condition \((3.7)\) one finds

\[
[\delta_\epsilon, \delta_{\epsilon'}] F = \epsilon \gamma^\mu \epsilon' (2D_\mu D_\nu \phi + i F_{\mu\nu} \phi - i q \phi V_{\mu\nu}).
\]

The right hand side vanishes if \(\phi\) couples to \(V_\mu\) according to its R-charge \(-q\), namely,

\[
D_\mu \phi \equiv (\partial_\mu - i A_\mu + i q V_\mu) \phi.
\]

Likewise, two barred supersymmetry commute on \(\bar{F}\) provided that the Killing spinors satisfy \((3.7)\) and \(\bar{\phi}\) couples with \(V_\mu\) according to its charge \(+q\).

**Supersymmetric Lagrangians.** The Chern-Simons Lagrangian for \(\mathcal{N} = 2\) vector multiplet is invariant under supersymmetry.

\[
\mathcal{L}_{CS} = \text{Tr} \left[ \frac{1}{\sqrt{g}} \varepsilon^{\mu\nu\lambda} (A_\mu \partial_\nu A_\lambda - \frac{2q}{3} A_\mu A_\nu A_\lambda) - \bar{\lambda} \lambda + 2D \sigma \right].
\]

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The $F$-term of gauge-invariant chiral multiplets of R-charge $q = 2$ is invariant under supersymmetry up to total derivatives.

$$\delta F = iD_\mu (\gamma^\mu \psi), \quad \delta \bar{F} = iD_\mu (\overline{\epsilon} \gamma^\mu \bar{\psi}).$$ (3.12)

These terms are invariant under $\delta$ for any Killing spinors $\epsilon, \bar{\epsilon}$. In addition, chiral matter multiplets with canonical dimensions have the kinetic Lagrangian,

$$L = D_\mu \bar{\phi} D^\mu \phi - \frac{R}{8} \bar{\phi} \phi + i \bar{\psi} \gamma^\mu D_\mu \psi + i \bar{\psi} \sigma \psi + i \bar{\psi} \lambda \phi - i \bar{\phi} \lambda \psi + i \bar{\psi} D \phi + \bar{\phi} \sigma^2 \phi + \bar{F} F,$$ (3.13)

which is invariant under supersymmetry if the Killing spinors $\epsilon, \bar{\epsilon}$ satisfy (3.5).

There are Lagrangians which are not superconformal but are still invariant under some supersymmetry. In the following we look for the quantities which are invariant if the parameters $\epsilon, \bar{\epsilon}$ satisfy

$$D_\mu \epsilon = \frac{1}{2f} \gamma_\mu \epsilon, \quad D_\mu \bar{\epsilon} = \frac{1}{2f} \gamma_{\mu} \bar{\epsilon}$$ (3.14)

for some function $f$. Note that by combining these with (3.5) one finds

$$(R + 2i \gamma^{\mu \nu} V_{\mu \nu}) \epsilon = (6f^{-2} - 4i \gamma^\mu \partial_\mu f^{-1}) \epsilon,$$

$$(R - 2i \gamma^{\mu \nu} V_{\mu \nu}) \bar{\epsilon} = (6f^{-2} - 4i \gamma^\mu \partial_\mu f^{-1}) \bar{\epsilon}.$$ (3.15)

One example for such Lagrangian is the kinetic Lagrangian for matter fields with non-canonical R-charges.

$$\mathcal{L}_{\text{mat}} = D_\mu \bar{\phi} D^\mu \phi + \bar{\phi} \sigma^2 \phi + \frac{i(2q-1)}{f} \bar{\phi} \sigma \phi - \frac{q(2q-1)}{2f} \bar{\phi} \phi + \frac{q}{4} R \bar{\phi} \phi + i \bar{\psi} \gamma^\mu D_\mu \psi + i \bar{\psi} \sigma \psi - \frac{(2q-1)}{2f} \bar{\psi} \psi + i \bar{\psi} \lambda \phi - i \bar{\phi} \lambda \psi.$$ (3.16)

Another example is the Yang-Mills Lagrangian for vector multiplet.

$$\mathcal{L}_{\text{YM}} = \text{Tr} \left(\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} D_\mu \sigma D^\mu \sigma + \frac{1}{8} (D + \frac{q}{2})^2 + \frac{5}{4} \bar{\lambda} \gamma^\mu D_\mu \lambda + \frac{5}{4} \bar{\lambda} \lambda \bar{\lambda} \right).$$ (3.17)

Note that $\mathcal{L}_{\text{mat}}$ and $\mathcal{L}_{\text{YM}}$ can be expressed as total-supерderivatives,

$$\bar{\epsilon} \epsilon \cdot \mathcal{L}_{\text{mat}} = \delta_\epsilon \delta_\bar{\epsilon} \left( \psi \psi - 2i \delta \phi \sigma \phi + \frac{2q-1}{f} \bar{\phi} \phi \right),$$

$$\bar{\epsilon} \epsilon \cdot \mathcal{L}_{\text{YM}} = \delta_\epsilon \delta_\bar{\epsilon} \text{Tr} \left( \frac{1}{2} \bar{\lambda} \lambda - 2D \sigma \right).$$ (3.18)

Finally, there is an analogue of FI D-term for abelian vector multiplet. The auxiliary field $D$ in abelian vector multiplet is also invariant up to total derivative,

$$\mathcal{L}_{\text{FI}} \equiv D - \frac{q}{2}, \quad \delta \mathcal{L}_{\text{FI}} = -\frac{1}{2} D_\mu (\bar{\epsilon} \gamma^\mu \lambda + \bar{\lambda} \gamma^\mu \epsilon).$$ (3.19)
4. Partition function on Squashed $S^3$ (Familiar One)

Here we compute partition functions of supersymmetric gauge theories on squashed $S^3$ based on localization principle. As has been explained in [1, 2], the path integral localizes onto the saddle points where the supersymmetry variation of all the fermions vanish. They are characterized by

$$A_\mu = \phi = 0, \quad \sigma = -f D = \text{constant}. \quad (4.1)$$

The integration over all the modes transverse to the locus of saddle points can be made finite by introducing arbitrary weight $e^{-S}$, where the regulator action $S$ should be supersymmetry exact. Our Lagrangians $\mathcal{L}_{\text{mat}}$ and $\mathcal{L}_{\text{YM}}$ are both supersymmetry exact so they can be included in $S$ with arbitrary coefficient. When this coefficient is taken larger and larger, then the saddle-point (Gaussian) approximation for the path integral becomes more and more accurate. The partition function can therefore be computed by truncating the regulator action up to quadratic order at each saddle point, evaluating the Gaussian path integral (one-loop determinant) and then integrating over the space of saddle points parametrized by $\sigma$.

In the following we calculate these determinants for chiral and vector multiplets on two different versions of squashed $S^3$. In this section we focus on the familiar squashing preserving $SU(2)_\xi \times U(1)_R$ symmetry, for which the metric is given by (2.8) and the background $U(1)$ gauge field $V$ is given in (2.11).

Matter multiplets. Let us evaluate the determinant from matter chiral multiplets first. For simplicity, we focus on the simplest example of a single chiral multiplet of R-charge $q$ coupled to an abelian vector multiplet. The matter kinetic term on the saddle points is $\mathcal{L}_\phi + \mathcal{L}_\psi$, with

$$\mathcal{L}_\phi = g^{\mu\nu}D_\mu \bar{\phi} D_\nu \phi + \bar{\phi} \sigma^2 \phi + \frac{2i(q-1)}{f} \bar{\phi} \sigma \phi + \left\{ \frac{q(1-2q)}{2f^2} + \frac{q R}{4} \right\} \bar{\phi} \phi,$$

$$\mathcal{L}_\psi = -i \bar{\psi} \gamma^\mu D_\mu \psi + i \bar{\psi} \sigma \phi - 2 \bar{q} \frac{1-q}{2f} \bar{\psi} \phi. \quad (4.2)$$

We rewrite them using the differential operators $\mathcal{R}^a$ as well as $R = \frac{8}{f \ell} - \frac{2}{f}$ and get

$$\mathcal{L}_\phi = \frac{1}{f^2} \left\{ \mathcal{R}_1^1 \phi \mathcal{R}_1^1 \phi + \mathcal{R}_2^3 \phi \mathcal{R}_2^3 \phi \right\} + \frac{1}{f^2} \left\{ \mathcal{R}_1^3 \phi - iq(1 - \frac{\ell}{f}) \phi \right\} \left\{ \mathcal{R}_1^3 \phi + iq(1 - \frac{\ell}{f}) \phi \right\} + \bar{\phi} \left( \sigma^2 + \frac{2i(q-1)\sigma}{f^2} - \frac{q^2}{f^2} + \frac{2q}{f} \right) \phi,$$

$$\mathcal{L}_\psi = \bar{\psi} \left\{ -\frac{i}{f} (\gamma^1 \mathcal{R}_1 + \gamma^2 \mathcal{R}_2) - i \gamma^3 \mathcal{R}_3 + \frac{1}{2f} \frac{1}{\ell} + (q-1)(\frac{1}{\ell} - \frac{1}{f}) \gamma^3 + \bar{\psi} \right\}. \quad (4.3)$$

We further rewrite them in terms of $J^a = \frac{1}{f^2} \mathcal{R}^a$ and $S^a = \frac{1}{f^2} \gamma^a$ satisfying standard $SU(2)$ commutation relations, and obtain the Laplace operator $\Delta_\phi$ for scalar field and the Dirac operator $\Delta_\psi$ for spinor field,

$$\Delta_\phi = \frac{4}{f^2} (J^1 J^1 + J^2 J^2) + \frac{4}{f^2} \left\{ J^3 + \frac{q}{2} (1 - \frac{\ell}{f}) \right\}^2 + \sigma^2 + \frac{2i(q-1)\sigma}{f} - \frac{q^2}{f^2} + \frac{2q}{f} \frac{1}{\ell},$$

$$\Delta_\psi = \frac{4}{f^2} (S^1 J^1 + S^2 J^2) + \frac{4}{f^2} S^3 J^3 + \frac{1}{2f^2} + 2(q-1)(\frac{1}{\ell} - \frac{1}{f}) \mathcal{S}^3. \quad (4.4)$$
The one-loop determinant can thus be computed from the spectrum of these operators.

The Laplace operator is diagonalized by scalar spherical harmonics which belong to the representations \((j, j)\) of \(SU(2)_R \times SU(2)_L\). There are therefore \(2j + 1\) scalar wave functions with angular momentum \(j\) and \(J^3 = m\) corresponding to the eigenvalue

\[
\Delta_\phi = \frac{4j(j + 1) - 4m^2}{\ell^2} + \frac{2m + q(1 - \frac{j}{j})}{\ell^2} + \sigma^2 + \frac{2i(q - 1)\sigma}{f} - \frac{q^2 + 2q}{f\ell},
\]

(4.5)

Note that for \(m = j\) this can be factorized as follows,

\[
\Delta_\phi = \left( \frac{2j + q}{\ell} + \frac{2 - 2q}{f} + i\sigma \right) \left( \frac{2j + q}{\ell} - i\sigma \right).
\]

(4.6)

Next we turn to the spectrum of Dirac operator \(\Delta_\psi\). Its generic eigenstates are suitable linear combinations of two states \(|j; m, \frac{1}{2}\rangle\) and \(|j; m + 1, -\frac{1}{2}\rangle\), where \(j\) labels the orbital angular momentum and

\[
J^3|j; m, s\rangle = m|j; m, s\rangle, \quad (J^1 \pm iJ^2)|j; m, s\rangle = (j \mp m)|j; m \pm 1, s\rangle,
\]

\[
S^3|j; m, s\rangle = s|j; m, s\rangle, \quad (S^1 \pm iS^2)|j; m, \mp \frac{1}{2}\rangle = |j; m, \pm \frac{1}{2}\rangle.
\]

The action of \(\Delta_\psi\) on the states of the form

\[
x_+|j; m, \frac{1}{2}\rangle + x_-|j, m + 1, -\frac{1}{2}\rangle,
\]

(4.7)

can be translated into the following \(2 \times 2\) matrix acting on \((x_+, x_-)^t\).

\[
\begin{pmatrix}
\frac{2m+1}{\ell} + \frac{q-1}{2}(\frac{1}{\ell} - \frac{1}{q}) + \frac{1-q}{f} + i\sigma & \frac{2}{\ell}(j+m+1) \\
\frac{1}{2}(j-m) & \frac{2m+1}{\ell} - \frac{q-1}{2}(\frac{1}{\ell} - \frac{1}{q}) + \frac{1-q}{f} + i\sigma
\end{pmatrix}.
\]

Its determinant is precisely \((-1)^2\) times the expression (1.5) for the eigenvalue of \(\Delta_\phi\). Note that \(m\) can take values between \(-j\) to \(j\). In addition to these generic eigenstates, the states \(|j, \pm \frac{1}{2}\rangle\) are themselves the eigenstates of \(\Delta_\psi\) for the eigenvalues

\[
\Delta_\psi = \frac{2j + q}{\ell} + \frac{2 - 2q}{f} + i\sigma, \quad \frac{2j + 2 - q}{\ell} + i\sigma.
\]

(4.8)

Since \(SU(2)_L\) is unbroken, all these spectra acquire the multiplicity \((2j + 1)\).

Combining everything together, we obtain the one-loop determinant,

\[
\frac{\det \Delta_\psi}{\det \Delta_\phi} = \prod_{\substack{2j \in \mathbb{Z}_{>0}}} \left[ (-1)^{2j} \left( \frac{2j+q}{\ell} + \frac{2 - 2q}{f} + i\sigma \right) \left( \frac{2j + 2 - q}{\ell} - i\sigma \right) \right]^{2j+1}
\]

\[
= \prod_{n>0} \left( \frac{n + 1 - q + i\bar{l}\sigma}{n - 1 + q - i\bar{l}\sigma} \right)^n = s_{b=1}(i - iq - \bar{l}\sigma).
\]

(4.9)

This is essentially the same as the result for round \(S^3\) (see, e.g. \[\text{[4]}\]) except that the radius of round \(S^3\) is replaced by \(\bar{l}\).
**Vector multiplets.** Next we study vector multiplets. We denote by $\varphi$ the fluctuation mode of the scalar field away from its classical value $\sigma$, and consider the path integral with linearized Lagrangian $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$,

$$
\mathcal{L}_B = \text{Tr} \left( \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{i}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} [A_\mu, \sigma] [A^\mu, \sigma] - i[A_\mu, \sigma] \partial^\mu \varphi \right),
$$

$$
\mathcal{L}_F = \text{Tr} \left( \frac{1}{2} \lambda \gamma^\mu D_\mu \lambda + \frac{1}{2} \bar{\lambda} \sigma, \lambda \right) - \frac{1}{4 \alpha^2} \lambda \lambda \right),
$$

(4.10)

where $\tilde{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. We decompose all the adjoint-valued fields with respect to the Cartan-Weyl basis $(H_i, E_\alpha, E_{-\alpha})$ satisfying the commutation relations,

$$
[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha_i E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \frac{2}{|\alpha|^2} \alpha_i H_i.
$$

We may assume that $\sigma$ takes values in the Cartan subalgebra, i.e. $\sigma = \sigma_i H_i$. Then the Lagrangian for fermions can be rewritten into the form

$$
\mathcal{L}_F = \sum_i \lambda_i (i \gamma^\mu D_\mu - \frac{1}{2j} \lambda_i) + \sum_{\alpha \in \Delta} \bar{\lambda}_{-\alpha} (i \gamma^\mu D_\mu + i \sigma \alpha - \frac{1}{2j} \lambda_\alpha),
$$

(4.11)

where $\sigma \alpha \equiv \sigma_i \alpha_i$. Note that $D_\mu \lambda$ contains the couplings to spin connection as well as the background $U(1)$ vector field according to the R-charge of $\sigma$. The Dirac operator for these fermions is thus the same as that of matter fermions in a chiral multiplet with $q = 0$. The determinant of Dirac operator for the fields $\lambda_{\pm \alpha}, \bar{\lambda}_{\pm \alpha}$ is thus given by

$$
\prod_{j \geq 0} \left[ \left( \frac{2i}{\ell} + \frac{j}{j} - i \sigma \alpha \right) \left( \frac{2i}{\ell} + \frac{j}{j} + i \sigma \alpha \right) \left( \frac{2i+2}{\ell} - i \sigma \alpha \right) \left( \frac{2i+2}{\ell} + i \sigma \alpha \right) \right]^{2j+1}
$$

$$
\times \prod_{j \geq 1/2 m = -j} \left[ \Delta_0(j, m) + (\sigma \alpha)^2 + \frac{2j}{\ell} \sigma \alpha \right]^{2j+1} \left[ \Delta_0(j, m) + (\sigma \alpha)^2 - \frac{2j}{\ell} \sigma \alpha \right]^{2j+1},
$$

(4.12)

where

$$
\Delta_0(j, m) \equiv \frac{4j(j+1) - 4m^2}{\ell^2} + \frac{4m^2}{\ell^2}.
$$

(4.13)

The bosonic Lagrangian is easier to handle once written in terms of differential forms,

$$
d^3 \xi \sqrt{g} \mathcal{L}_B = \frac{1}{2} \text{Tr} \left( dA \wedge dA + d\varphi \wedge d\varphi - [\sigma, A] \wedge [\sigma, A] + i[\sigma, A] \wedge d\varphi \right).
$$

(4.14)

This leads to the Laplace operator acting on a pair $(A, \varphi)$ as follows,

$$
\Delta_B : \begin{pmatrix} A \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} *dA + [\sigma, [\sigma, A]] - i[\sigma, d\varphi] \\ -i[\sigma, *dA] - *d \varphi \end{pmatrix}.
$$

(4.15)

Here the Hodge star $*$ is a linear map from $d$-forms to $(3-d)$-forms satisfying $** = 1$. It is defined by the equations

$$
*1 = e^1 e^2 e^3, \quad *e^1 = e^2 e^3, \quad *e^2 = e^3 e^1, \quad *e^3 = e^1 e^2.
$$

(4.16)

Let us hereafter focus on the components of $(A, \varphi)$ proportional to the Lie algebra element $E_\alpha$, so that the commutator with $\sigma$ becomes just the multiplication by $\sigma \alpha$. 
Generic eigenmodes of $\Delta_B$ take the form
\[
A_\alpha = x^+ Y_{j,m+1} e^+ + x^- Y_{j,m-1} e^- + x^3 Y_{j,m} e^3, \quad \varphi_\alpha = x Y_{j,m}, \quad (4.17)
\]
where $e^\pm \equiv \frac{1}{2}(e^i \pm i e^j)$, and the spherical harmonics $Y_{j,m}$ is normalized to satisfy
\[
J^3 Y_{j,m} = m Y_{j,m}, \quad (J^1 \pm i J^2) Y_{j,m} = (j \mp m) Y_{j,m \pm 1}.
\]
The Laplace operator $\Delta_B$ then translates into a matrix $X$ acting on $(x^+, x^3, x^-, x^t)$. It takes the $(3 + 1) \times (3 + 1)$ block decomposed form
\[
X = \begin{pmatrix}
U + (\sigma \alpha)^2 & (\sigma \alpha) \vec{v} \\
(\sigma \alpha) \vec{w} & \Delta_0
\end{pmatrix}, \quad (4.18)
\]
with
\[
U = \begin{pmatrix}
-\frac{2m}{\ell} & \frac{2(j-m)}{\ell} & 0 \\
\frac{j+m+1}{\ell} & \frac{2}{\ell} & -\frac{j-m+1}{\ell} \\
0 & -\frac{2(j+m)}{\ell} & \frac{2m}{\ell}
\end{pmatrix},
\]
\[
\vec{v} = \begin{pmatrix}
\frac{2(j+m)}{m} & \frac{2m}{\ell} & \frac{2(j+m)}{\ell}
\end{pmatrix}^t,
\]
\[
\vec{w} = \begin{pmatrix}
\frac{j+m+1}{\ell} & \frac{2m}{\ell} & \frac{j-m+1}{\ell}
\end{pmatrix}, \quad (4.19)
\]
and $\Delta_0$ given in (4.13). Noticing that $U \vec{v} = \vec{w} U = 0$ and $\Delta_0 = \vec{w} \cdot \vec{v}$, one can easily find two eigenvectors of $X$,
\[
(\vec{v}, -(\sigma \alpha))^t : X = 0, \quad (4.20)
\]
\[
((\sigma \alpha) \vec{v}, \Delta_0)^t : X = \Delta_0 + (\sigma \alpha)^2.
\]
These two modes are longitudinal in the sense that $A$ is a total derivative. The above two eigenvalues do not enter the formula for one-loop determinant, as we explain later. The other two eigenvalues of $X$ correspond to transverse modes, for which $\varphi \equiv 0$ and $A$ is divergenceless. They have eigenvalues $X = \zeta_1^2 + (\sigma \alpha)^2$ and $\zeta_2^2 + (\sigma \alpha)^2$, where $\zeta_1, \zeta_2$ are the two nonzero eigenvalues of $U$. They can therefore be found as two nonzero roots of the characteristic equation
\[
0 = \text{Det}(\zeta - U) = \zeta^3 - \frac{2}{j} \zeta^2 - \Delta_0 \zeta. \quad (4.21)
\]
The product of the eigenvalues of $X$ from the two transverse modes is
\[
\{ \zeta_1^2 + (\sigma \alpha)^2 \} \{ \zeta_2^2 + (\sigma \alpha)^2 \} = \{ \Delta_0 - \frac{2j}{f}(\sigma \alpha) + (\sigma \alpha)^2 \} \{ \Delta_0 + \frac{2j}{f}(\sigma \alpha) + (\sigma \alpha)^2 \}. \quad (4.22)
\]
Up to now it was assumed that $|m| \leq j - 1$ and therefore $j \geq 1$. When $m = j$ and $j \geq 1/2$, one is interested in the eigenmodes for which $A$ has no component proportional to $e^-$, so that now $X$ becomes a $3 \times 3$ matrix. Its three eigenvalues are
\[
0, \quad \Delta_0 + (\sigma \alpha)^2, \quad \left( \frac{2}{f} + \frac{2j}{f} \right)^2 + (\sigma \alpha)^2. \quad (4.23)
\]
Similarly, when $m = -j$ and $j \geq 1/2$ the matrix $X$ becomes $3 \times 3$, and obtains the same three eigenvalues as above. When $m = \pm(j+1)$ and $j \geq 1/2$ one finds that $X$ becomes one-dimensional,

$$X = \left(\frac{2j+2}{\ell}\right)^2 + (\sigma \alpha)^2.$$  

(4.24)

Finally, for $j = 0$ the four eigenmodes and eigenvalues of $\Delta_B$ are given by

$$(A = e^\pm, \varphi = 0) \quad \Delta_B = \frac{4}{\ell^2} + (\sigma \alpha)^2,$$

$$(A = e^3, \varphi = 0) \quad \Delta_B = \frac{4}{\ell^2} + (\sigma \alpha)^2,$$

$$(A = 0, \varphi = 1) \quad \Delta_B = 0.$$  

(4.25)

Almost all the zero eigenvalues of $\Delta_B$ correspond to gauge symmetry, so they should be excluded from the physical determinant. The only exception is the constant mode of $\varphi$, which correspond to shifting the saddle point and therefore should also be excluded. To evaluate the partition function correctly, one also has to take proper account of Faddeev-Popov determinant. Instead of introducing ghost fields and modifying the supersymmetry by BRST transformation, we choose to take a quicker route which takes advantage of saddle point approximation.

Now that we worked out the spectrum and eigenmode decomposition of the operator $\Delta_B$, one can regard the path integral (before gauge fixing) as an integral over the mode variables $x_i$ and $\tilde{x}_j$ corresponding to zero and nonzero eigenmodes of $\Delta_B$,

$$DAD\varphi = \prod_{(\Delta_B=0)} dx_i \times \prod_{(\Delta_B\neq0)} d\tilde{x}_j.$$  

(4.26)

We assume the mode variables are normalized to satisfy

$$\frac{1}{2} \int \text{Tr}(A \wedge *A + \varphi \wedge *\varphi) = \pi \sum_i x_i^2 + \pi \sum_j \tilde{x}_j^2,$$  

(4.27)

so that the following equality can be reproduced using variables $x_i, \tilde{x}_j$,

$$\int DAD\varphi \exp \left( -\frac{1}{2} \int \text{Tr}(A \wedge *A + \varphi \wedge *\varphi) \right) = 1.$$  

(4.28)

Excluding the zero eigenvalues from the determinant of $\Delta_B$ corresponds to the insertion of $\prod \delta(x_i)$. But the correct gauge fixing is given by $\prod \delta(\omega_i) = J \prod \delta(x_i)$, where $\omega_i$ are the mode variables of gauge transformation which acts on the fields $A$ and $\varphi$ as

$$\delta_\omega A = d\omega, \quad \delta_\omega \varphi = i[\omega, \sigma]$$  

(4.29)

on the saddle point labelled by $\sigma$. The variables $\omega_i$ also satisfy the normalization condition similar to (4.27). The Jacobian $J$ for the change of variables is called Faddeev-Popov determinant. Noticing $\prod dx_i = JD\omega$, one can determine $J$ by inserting (4.29) into (4.28),

$$1 = J \cdot \int D\omega \exp \left( -\frac{1}{2} \int d^3 \xi \sqrt{g} \text{Tr}(\partial_\mu \omega \partial^\mu \omega - [\sigma, \omega]^2) \right).$$  

(4.30)
Note that the constant mode should be excluded from the measure $D'\omega$ as explained in the previous paragraph. This path integral can be easily worked out using spherical harmonics. The mode $\omega \sim Y_{j,m}E_\alpha$ is an eigenmode of the Laplace operator with the eigenvalue

$$\Delta_0(j, m) + (\sigma_\alpha)^2. \quad (4.31)$$

This precisely cancels with the contribution of longitudinal vector eigenmode to the determinant. The path integral over bosons $A_{\pm \alpha}, \varphi_{\pm \alpha}$ modulo gauge equivalence finally becomes

$$\left(\frac{4}{\ell^2} + (\sigma_\alpha)^2\right)^{-1} \prod_{j \geq 1/2} \left[\left(\frac{j}{2} + \frac{2j}{\ell}\right)^2 + (\sigma_\alpha)^2\right]^{-2(2j+1)} \prod_{j \geq 0} \left[\left(\frac{(2j+2)}{2} + (\sigma_\alpha)^2\right)^{-2(2j+1)} \right. $$

$$\times \prod_{j \geq 1} \prod_{m = 1-j}^{j-1} \left(\Delta_0(j, m) - \frac{2j}{\ell}\sigma_\alpha + (\sigma_\alpha)^2\right)^{-2j-1} \left(\Delta_0(j, m) + \frac{2j}{\ell}\sigma_\alpha + (\sigma_\alpha)^2\right)^{-2j-1}. \quad (4.32)$$

where $j$ runs over half-integers.

Combining the bosonic and fermionic contributions together, we obtain the one-loop determinant of vector multiplet.

$$\prod_{\alpha \in \Delta_+} \prod_{n > 0} \left[\frac{n^2}{\ell^2} + (\sigma_\alpha)^2\right]^2 \sim \prod_{\alpha \in \Delta_+} \left(\frac{\sinh(\pi \ell \sigma_\alpha)}{\pi \ell \sigma_\alpha}\right)^2. \quad (4.33)$$

Here the product runs over all positive roots $\alpha$. The factors in the denominator cancel against the Vandermonde determinant which arises when the integration over the Lie algebra is reduced to that on Cartan subalgebra. Again, the result is essentially the same as for round $S^3$ except now $\ell$ is playing the role of the radius of round $S^3$.

Thus, after all these tedious computations, we found a rather disappointing result that the familiar squashing of $S^3$ with $SU(2)_L \times U(1)_R$ gives nothing new. We notice here that there is a simple reason for this. Since our squashed $S^3$ has an unbroken $SU(2)$ symmetry, the eigenmodes of Laplace or Dirac operators naturally form multiplets with the same eigenvalues. The exponent $n$ in the formula $\left(\right.4.9\left.)\right.$ simply reflects the fact that the multiplicity becomes larger as the angular momentum $j$ gets larger. This is tied to the degeneration of zeroes and poles of $s_b(x)$ for $b = 1$. Therefore, in order to find generalization to $b \neq 1$, we need to look for a less symmetric squashing of $S^3$.

5. Partition function on Squashed $S^3$ (Less Familiar One)

In this section we study the partition function on less familiar version of squashed $S^3$ which preserves only $U(1)_{L^3} \times U(1)_{R^3}$ symmetry. The vielbein, spin connection, Killing spinors and the background $U(1)$ gauge field $V$ are summarized in section 2. In the following discussion we regard the Killing spinors $\epsilon, \bar{\epsilon}$ to be Grassmann-even. For convenience, we also renormalize them by constants so that they satisfy

$$\bar{\epsilon}\epsilon = 1, \quad \psi^\mu \psi_\mu = \bar{\epsilon}\gamma^\mu \epsilon \cdot \bar{\epsilon}\gamma_\mu \epsilon = 1. \quad (5.1)$$
More explicitly, we choose
\[ \epsilon = \frac{1}{\sqrt{2}} \left( -e^{i(\chi - \phi + \theta)} \right), \quad \bar{\epsilon} = \frac{1}{\sqrt{2}} \left( e^{i(\chi - \phi - \theta)} \right), \tag{5.2} \]
so that
\[ \bar{\epsilon} \gamma^\mu \epsilon = (-\cos \theta, \sin \theta, 0), \]
\[ \epsilon \gamma^\mu \epsilon = (i \sin \theta, i \cos \theta, +1) e^{i(\chi - \phi)}, \]
\[ \bar{\epsilon} \gamma^\mu \bar{\epsilon} = (i \sin \theta, i \cos \theta, -1) e^{-i(\chi - \phi)}. \tag{5.3} \]
Note also the equalities
\[ v_\mu \bar{\epsilon} \gamma^\mu = \bar{\epsilon}, \quad \gamma^\mu \epsilon v_\mu = \epsilon, \tag{5.4} \]
which will be frequently used in what follows.

Since the squashed $S^3$ of our interest here has a reduced symmetry and the metric cannot be written in terms of LI or RI forms, it will be a tedious task to work out all the eigenmodes of the bosonic and fermionic kinetic operators. In fact, the precise form of most of the eigenmodes and eigenvalues is irrelevant since, as we have seen in the previous section, they give cancelling contributions to the determinant due to the pairing of bosonic and fermionic modes by supersymmetry. Therefore, to compute the one-loop determinant, it will be useful to understand how this cancellation happens.

**Matter multiplets.** We study the matter fields first, focusing on the case with a single chiral multiplet coupled to an abelian vector multiplet. What we need is the spectrum of the kinetic operators $\Delta_\phi$ and $\Delta_\psi$ for bosons and fermions. In this section we define them from the regulator Lagrangian
\[ L_{\text{reg}} = \delta_\epsilon \delta_\epsilon \left( \bar{\psi} \psi - 2i \bar{\phi} \sigma \phi \right) \]
\[ = D_\mu \bar{\psi} D^\mu \phi + \frac{2(q-1)}{f} v^\mu D_\mu \bar{\phi} \phi + 2i \bar{\phi} \left( \frac{\sigma}{f} + D \right) \phi + \frac{2q^2 - 3q}{2f^2} \bar{\phi} \phi + \frac{q^4}{4} R \bar{\phi} \phi \]
\[ - i \bar{\psi} \gamma^\mu D_\mu \psi + i \bar{\psi} \sigma \psi - \frac{1}{2f} \bar{\psi} \psi + \frac{q-1}{2f} \bar{\psi} \gamma^\mu v_\mu \psi + i \bar{\psi} \lambda \phi - i \bar{\phi} \lambda \psi + F. \tag{5.5} \]
This leads to the kinetic operators
\[ \Delta_\phi = -D_\mu D^\mu - \frac{2(q-1)}{f} v^\mu D_\mu + \sigma^2 + \frac{2q^2 - 3q}{2f^2} + \frac{q^4}{4}, \]
\[ \Delta_\psi = -i \gamma^\mu D_\mu + i \sigma - \frac{1}{2f} + \frac{q-1}{2f} \gamma^\mu v_\mu, \tag{5.6} \]
acting on scalars and spinors of R-charges $-q, 1-q$ respectively. Note that in deriving $\Delta_\phi$ we used
\[ D_\mu v^\mu = 0, \quad v^\mu \partial_\mu f = 0, \tag{5.7} \]
which can be shown using Killing spinor equation.

Now, it is a tedious but straightforward computation to show the following. First, if $\Psi$ be a spinor eigenmode for $\Delta_\psi = M$, then $\bar{\epsilon} \Psi$ is a scalar eigenmode for $\Delta_\phi = M(M - 2i \sigma)$. 

\[ -15 - \]
Second, let $\Phi$ be a scalar eigenmode for $\Delta \phi = M(M - 2i\sigma)$. Then if we define a pair of spinor wave functions as

$$\Psi_1 = \epsilon \Phi, \quad \Psi_2 = i\gamma^\mu \epsilon D_\mu \Phi + i\epsilon \sigma \Phi - \frac{q}{f} \epsilon \Phi,$$  \hspace{1cm} (5.8)

then $\Delta \psi$ acts on them as follows.

$$\begin{pmatrix} \Delta \psi \Psi_1 \\ \Delta \psi \Psi_2 \end{pmatrix} = \begin{pmatrix} 2i\sigma & -1 \\ -M(M - 2i\sigma) & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

\hspace{1cm} (5.9)

The $2 \times 2$ matrix on the right hand side has eigenvalues $M$ and $2i\sigma - M$. We thus found a pairing between a scalar eigenmode with $\Delta \phi = M(M - 2i\sigma)$ and two spinor eigenmodes with $\Delta \psi = M, 2i\sigma - M$. Any modes which take part in this pairing can be neglected when computing the one-loop determinant. Note that part of this pairing, namely $\Phi = \bar{\epsilon} \Psi$ and the construction (5.8) of $\Psi_2$ from $\Phi$, could be guessed from the supersymmetry transformation rules (3.7).

Had we chosen the Lagrangian (3.16) for the regulator, we would find that the relation between scalar and spinor eigenmodes contains $f$ and therefore becomes coordinate-dependent. This can be traced to the third term in the right hand side of the first equation in (3.18). Since one may use arbitrary regulator Lagrangian as long as it regulates the integral over all the modes transverse to the saddle point locus, we may choose it so that the relation between bosonic and fermionic eigenmodes becomes as simple as possible. This is why we chose the Lagrangian (5.5).

Nontrivial contributions to one-loop determinant come from the modes which do not fall into the multiplet structure explained in the last paragraph. There are two types of such modes.

The first is unpaired spinor eigenmodes $\Psi$, which vanish when contracted with $\bar{\epsilon}$ and therefore do not have scalar partners. Such modes can be expressed as $\Psi = \bar{\epsilon} F$, where $F$ is a scalar with R-charge $2 - q$. The eigenmode equation $\Delta \psi \Psi = M \Psi$ can be rewritten into

$$(M + \frac{2 - 2 - i\sigma}{2}) F = -i \cos \theta D_1 F + i \sin \theta D_2 F,$$

$$D_3 F = +i \sin \theta D_1 F + i \cos \theta D_2 F,$$

where we have used the explicit form of $\bar{\epsilon}$, and

$$D_1 F = \frac{1}{\ell \cos \theta} \left\{ \partial_\phi - \frac{i}{2} (q - 2)(1 - \frac{\ell}{f}) \right\} F,$$

$$D_2 F = \frac{1}{\ell \cos \theta} \left\{ \partial_\chi + \frac{i}{2} (q - 2)(1 - \frac{\ell}{f}) \right\} F,$$

$$D_3 F = \frac{1}{f} \partial_\theta F.$$

\hspace{1cm} (5.10)

If we assume $F \sim e^{im\phi - in\chi}$, then the first equation in (5.10) determines the eigenvalue $M$, 

$$M = i\sigma + \frac{m}{\ell} + \frac{n}{\ell} - \frac{q - 2}{2} \left( \frac{1}{\ell} + \frac{1}{f} \right),$$

\hspace{1cm} (5.12)
while the second equation determines the $\theta$-dependence of $F$.

\[
\frac{1}{f} \partial_\theta F = -\frac{\sin \theta}{\ell \cos \theta} \left\{ m - \frac{q^2}{2} (1 - \frac{\tilde{\ell}}{\ell}) \right\} F + \frac{\cos \theta}{\ell \sin \theta} \left\{ n - \frac{q^2}{2} (1 - \frac{\tilde{\ell}}{\ell}) \right\} F.
\]

(5.13)

The precise form of the solutions to this differential equation is not important, but we need to know what values of $(m, n)$ leads to normalizable eigenmodes. We recall here that the squared norm of $F$ is defined by an integral over $\theta \in [0, \pi/2]$ with a measure which is proportional to $d\theta f(\theta) \sin \theta \cos \theta$. The solution to (5.13) may develop singularities at the two ends of the integration domain. The behavior of $F$ there is easily found to be

\[
F \sim \cos^m \theta \sin^n \theta. \quad (\theta \sim 0 \text{ or } \pi/2)
\]

(5.14)

Therefore the normalizability requires $m, n$ to be nonnegative.

The second is the missing spinor eigenmodes. This is the case where the map (5.8) does not give two independent spinor eigenmodes from one scalar eigenmode. Namely $\Psi_1, \Psi_2$ are proportional to each other. Let us put $\Psi_2 = M \Psi_1$, i.e.

\[
i\gamma^\mu \epsilon D_\mu \Phi + i\epsilon \sigma \Phi - \frac{q}{f} \epsilon \Phi = M \epsilon \Phi.
\]

(5.15)

Then one can show that $\Psi_2 = M \Psi_1$ is a spinor eigenmode for $\Delta_\psi = 2i\sigma - M$, and moreover $\Phi$ has the eigenvalue $\Delta_\phi = M(M - 2i\sigma)$. That is to say, a spinor eigenmode for $\Delta_\psi = M$ is missing. To work out the spectrum of missing eigenvalues, we rewrite (5.15) into the following form,

\[
(M + \frac{q}{f} - i\sigma) \Phi = -i \cos \theta D_1 \Phi + i \sin \theta D_2 \Phi,
\]

\[
D_3 \Phi = -i \sin \theta D_1 \Phi - i \cos \theta D_2 \Phi.
\]

(5.16)

The rest of the computation is the same as for unpaired spinor eigenmodes. The missing spinor eigenvalues are thus given by

\[
M = i\sigma - \frac{m}{\ell} - \frac{n}{\ell} - \frac{q^2}{2} \left( \frac{1}{\ell} + \frac{1}{\tilde{\ell}} \right), \quad m, n \geq 0.
\]

(5.17)

The one-loop determinant is the product of all the unpaired spinor eigenvalues divided by the product of all the missing spinor eigenvalues. Ignoring the sign factors we find

\[
\frac{\det \Delta_\psi}{\det \Delta_\phi} = \prod_{m,n \geq 0} \frac{m^2 + n^2 + i\sigma - \frac{q^2}{2} \left( \frac{1}{\ell} + \frac{1}{\tilde{\ell}} \right)}{m^2 + n^2 - i\sigma + \frac{q^2}{2} \left( \frac{1}{\ell} + \frac{1}{\tilde{\ell}} \right)}.
\]

(5.18)

Introducing

\[
b \equiv \left( \frac{\tilde{\ell}}{\ell} \right) \frac{2}{\ell}, \quad Q \equiv b + b^{-1}, \quad \hat{\sigma} \equiv (\ell \tilde{\ell}) \frac{2}{\ell} \sigma,
\]

(5.19)

one can rewrite this using the double sine function $s_b(x)$ given in the Introduction,

\[
\frac{\det \Delta_\psi}{\det \Delta_\phi} = \prod_{m,n \geq 0} \frac{mb + nb^{-1} + \frac{Q}{2} + i\hat{\sigma} + \frac{Q}{2} (1 - q)}{mb + nb^{-1} + \frac{Q}{2} - i\hat{\sigma} - \frac{Q}{2} (1 - q)} = s_b \left( \frac{Q}{2} (1 - q) - \hat{\sigma} \right).
\]

(5.20)
Now $b$ is determined from the shape of the background squashed sphere and it can take any values, which is precisely what we have been seeking for!

A comment on $N = 4$ extended supersymmetry is in order. Recall a $N = 4$ hypermultiplet is a pair of $N = 2$ chiral multiplets $(q, \tilde{q})$ with R-charge 1/2 and opposite gauge charges. Its one-loop determinant becomes very simple on round sphere ($b = 1$),

$$
\left. s_b \left( \frac{iQ}{2} - \hat{\sigma} \right) s_b \left( \frac{iQ}{2} + \hat{\sigma} \right) \right|_{b=1} = s_1 \left( \frac{i}{2} - \hat{\sigma} \right) s_1 \left( \frac{i}{2} + \hat{\sigma} \right) = \frac{1}{2 \cosh \pi \hat{\sigma}}. \tag{5.21}
$$

This follows from an identity of double sine function,

$$
\left. s_b \left( i\frac{Q}{2} - x \right) s_b \left( i\frac{Q}{2} + x \right) \right|_{b=1} = \frac{1}{2 \cosh \pi bx}. \tag{5.22}
$$

However, this simplification does not happen on squashed sphere, unless one turns on a mass deformation by gauging the global symmetry under which $q, \tilde{q}$ have the same charge. Therefore, matrix models on the squashed $S^3$ will be much more complicated than on round $S^3$.

**Vector multiplets.** Next we study the vector multiplets. As in the previous section we denote by $\varphi$ the fluctuation mode of the scalar field away from its classical value $\sigma$, and use the original Lagrangian for the regulator which becomes (4.10) after Gaussian approximation. We also decompose all the adjoint fields into Cartan-Weyl basis. The modes proportional to the root $\alpha$ get a mass $\sigma\alpha$, and we denote them with suffix $\alpha$. We thus need to find out what kind of multiplet structure is formed by the bosonic eigenmode of $(A_\alpha, \varphi_\alpha)$ and the fermionic eigenmodes of $\lambda_\alpha$.

We saw in the previous section that, for generic quantum number of $SU(2)_L \times SU(2)_R$, the four bosonic eigenmodes for $(A_\alpha, \varphi_\alpha)$ split into two transverse modes with $dA_\alpha = 0$ and two longitudinal modes with $A_\alpha \sim d\varphi_\alpha$. After fixing a gauge and combining with the volume of the gauge group, the longitudinal modes were shown to yield no net contribution to the one-loop determinant. This property can be shown to be independent of the 3D metric. Our problem is therefore how the remaining two transverse modes are paired with the spinor eigenmodes.

The eigenmodes are generically paired in the following manner. Let $\Lambda$ and $A$ be spinor and transverse vector eigenmodes for the eigenvalue $M$ satisfying

$$
MA = (i\gamma^\mu D_\mu + i\sigma \alpha - \frac{1}{2f}) \Lambda, \tag{5.23}
$$

$$
MA = i\sigma \alpha A - *dA. \tag{5.24}
$$

Then, up to constant multiplication, they are mapped to each other by

$$
A \equiv d(\bar{\epsilon}\Lambda) + (iM + \sigma \alpha)\bar{\epsilon}\gamma_\mu Ad\bar{\epsilon}^\mu, \quad \Lambda \equiv \gamma^\mu \epsilon A_\mu. \tag{5.25}
$$

This relation could again be guessed from supersymmetry transformation rules (3.3). First, inserting a transverse vector eigenmode $A$ satisfying (5.24) and a constant $\sigma$ into the right hand side of $\delta \lambda$ in (3.1) gives

$$
\delta \lambda = \frac{1}{2} \gamma^{\mu \nu} \epsilon (\partial_\mu A_\nu - \partial_\nu A_\mu) - (\sigma \alpha) \gamma^\mu \epsilon A_\mu, \tag{5.26}
$$

\[\text{– 18 –}\]
which is proportional to $\Lambda = \gamma^\mu \epsilon A_\mu$. On the other hand, the naive map from spinor to vector eigenmodes is $A \simeq \bar{\epsilon} \gamma_\mu \Lambda d \xi^\mu$, but it can be shown to satisfy

$$
-i \ast d(\bar{\epsilon} \gamma_\mu \Lambda d \xi^\mu) = d(\epsilon \Lambda) + (i M + \sigma \alpha) \bar{\epsilon} \gamma_\mu \Lambda d \xi^\mu,
$$

namely it fails to satisfy (5.24) due to a nonzero divergence term. The right hand side of this equality is by construction divergenceless, and can be used to define the map from $\Lambda$ to $A$.

The modes which have a superpartner under the relation (5.25) are irrelevant in computing the one-loop determinant. Unpaired spinor eigenmodes satisfy (5.23) and are annihilated by (5.25), and contribute the eigenvalue $M$ to the numerator of the determinant. Missing spinor eigenmodes correspond to $A$ satisfying (5.24) and are annihilated by (5.25), and contribute $M$ to the denominator.

We solve the equation for unpaired spinor eigenmodes by putting

$$
\Lambda = \epsilon \Phi_0 + \bar{\epsilon} \Phi_2,
$$

where $\Phi_0$ and $\Phi_2$ are scalars with R-charges 0 and 2. The equations for $\Phi_0$ and $\Phi_2$ at first look overdetermined but turn out to have solutions. If we put the ansatz

$$
\Phi_0 = \varphi_0(\theta) e^{im\varphi - in\chi}, \quad \Phi_2 = \varphi_2(\theta) e^{i(m-1)\varphi - i(n-1)\chi},
$$

then the solution is determined by

$$
M = \frac{m}{\ell} + \frac{n}{\ell} + i\sigma\alpha, \quad \partial_\theta \varphi_0 = i f \left( \frac{m}{\ell} + \frac{n}{\ell} \right) \varphi_2,
$$

$$
\left( \frac{1}{f} \partial_\theta - \frac{m \sin \theta}{\ell \cos \theta} - \frac{n \cos \theta}{\ell \sin \theta} \right) \varphi_0 = 0.
$$

Normalizability of the solution requires $m, n$ to be nonnegative.

Also, note that for $m = n = 0$ or $M = i\sigma\alpha$ the second term in the first equation of (5.25) vanishes, so that it does not give a map from spinor eigenmode to transverse vector eigenmode. By a direct check one finds for $m = n = 0$ there is no normalizable unpaired spinor eigenmode.

To find the transverse vector eigenmodes with missing spinor partners, we begin by solving $A_a \gamma^a \epsilon = 0$ by

$$
A_1 = iY \sin \theta, \quad A_2 = iY \cos \theta, \quad A_3 = Y.
$$

It is straightforward to solve the eigenmode equation (5.24) in components. Putting the ansatz $Y = y(\theta) e^{im\varphi - in\chi}$ one finds

$$
M = \frac{m}{\ell} + \frac{n}{\ell} + i\sigma\alpha,
$$

$$
0 = \left( \frac{1}{f} \partial_\theta - \frac{\sin \theta}{\cos \theta} \left( \frac{m}{\ell} + \frac{1}{f} \right) + \frac{\cos \theta}{\sin \theta} \left( \frac{n}{\ell} + \frac{1}{f} \right) \right) y.
$$

Normalizability of the solution requires $m, n \leq -1$. 

\[ \text{– 19 –} \]
Now we combine all the contributions to compute the one-loop determinant.

\[
\prod_{\alpha \in \Delta} \left[ \frac{1}{i \sigma \alpha} \prod_{m,n \geq 0} \left( \frac{m}{\ell} + \frac{n}{\tilde{\ell}} + i \sigma \alpha \right) \right] = \prod_{\alpha \in \Delta_+} \prod_{n > 0} \left( \frac{\pi^2 + (\sigma \alpha)^2}{\pi^2 + (\sigma \alpha)^2} \right) = \prod_{\alpha \in \Delta_+} \frac{\sinh(\pi b \sigma \alpha) \sinh(\pi b^{-1} \sigma \alpha)}{(\pi \sigma \alpha)^2}. 
\]

(5.33)

Here the product is over all the positive roots \( \alpha \), and we used the parameters of (5.19). The factors in the denominator in the right hand side cancel against the Vandermonde determinant when reducing the integral from Lie algebra to its Cartan subalgebra.

We thus found that, by putting the 3D theory on the \( U(1) \times U(1) \) symmetric squashed \( S^3 \) (or the hyper-ellipsoid in \( \mathbb{R}^4 \) with the four axis-length parameters \( \ell, \ell, \tilde{\ell}, \tilde{\ell} \)), the partition function is expressed in terms of the double sine function with \( b = (\tilde{\ell}/\ell)^{1/2} \), namely the formula (5.21). Also, the integration measure over the Coulomb branch (5.33) is given by products of pairs of sinh functions. These are both identified with the building blocks of structure constants in Liouville or Toda CFTs.

6. Concluding Remarks

As was studied in [18, 19], certain 3D supersymmetric gauge theories arise on domain walls in 4D \( \mathcal{N} = 2 \) gauge theories. When the fields in the 4D theories on the two sides are connected on the wall via S-duality, the AGT relation relates the 3D partition functions on the wall with the kernels of the corresponding S-duality transformation acting on conformal blocks. By replacing the round metric of the wall \( S^3 \) by a squashed one, one obtains the kernels for general \( b \). Also, the AGT relation was interpreted in [22] as the equivalence between the space of supersymmetric ground states of 4D \( \mathcal{N} = 2 \) gauge theories on \( \mathbb{R} \times S^3 \) (with omega deformation) and the space of conformal blocks. Our results should give a generalization of the AGT relation to \( b \neq 1 \) along this line, too.

We conclude with one immediate conjecture: there should be a similar squashing of \( S^4 \), with some background gauge field turned on, which gives the generalization of AGT relation to \( b \neq 1 \). The metric and gauge field should be such that, when we view \( S^4 \) as a bundle of \( S^3 \) fibred over a line segment, they take precisely the forms given in this paper when restricted to each fiber.

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