A monomorphism theorem for the inverse limit of nested retracts

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Abstract
Suppose a given space $X$ can be realized as the inverse limit of nested retracts $X_n$ where $X_n$ admits a universal cover or $X_n$ has a certain more general property.

In various ways this paper characterizes injectivity of the canonical homomorphism (into the inverse limit of the fundamental groups of the factors). For example injectivity is equivalent to the existence of a kind of generalized universal cover over $X$, a fibration whose fibres have trivial path components and whose total space is simply connected.

1 Introduction
A paper of Biss [1] shows quite generally that the based fundamental group of a topological space $X$ admits a canonical topology and moreover the topological group $\pi_1(X, p)$ is invariant under the homotopy type of $X$. If $X$ admits a universal cover in the usual sense then $\pi_1(X, p)$ has the discrete topology. However, both the algebraic and topological structure of $\pi_1(X, p)$ can be challenging to understand if $X$ fails to be locally contractible. This paper aims to explore $\pi_1(X, p)$ in the context of inverse limit spaces $X = \lim_{\to} X_n$, where $X$ can be approximated by factors $X_n$ where $\pi_1(X_n, p)$ is either discrete or totally disconnected.

For example if $X = HE$ is the Hawaiian earring, (the one point union of a null sequence of simple closed curves $X_n$ joined at a common point

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the canonical homomorphism into the inverse limit of free groups \( \phi : \pi_1(HE, p) \to \lim_\leftarrow \pi_1(X_n, p) \) is one to one (Theorem 4.1 Morgan/Morrison [13]). The injectivity of \( \phi \) ensures that \( \pi_1(HE, p) \) is a \( T_1 \) space, and since \( \pi_1(HE, p) \) is a topological group \( \pi_1(HE, p) \) is completely regular. Further analysis reveals that \( \phi \) is not surjective, and it is shown in [7] and [10] that \( \phi \) fails to be a homeomorphism onto its image, and that \( \pi_1(HE, p) \) fails to be a Baire space despite being the uncountable regular topological fundamental group of a Peano continuum.

In general the homomorphism \( \phi \) can fail to be one to one and \( \pi_1(X, p) \) can fail to be a \( T_1 \) space. For example the harmonic archipelago \( HA \) discussed in [1] [2] [6] has an uncountable topological fundamental group endowed with the course topology, despite the fact that \( HA \) is the inverse limit of nested absolute retracts.

The contrasting properties of the two examples \( HE \) and \( HA \) suggest the possibility of a useful theory underlying the investigation of topological fundamental groups.

For some positive results, it is shown in [11] that the extra topological structure of \( \pi_1(X, p) \) can sometimes distinguish the homotopy type of spaces with isomorphic homotopy groups, offsetting the general failure of the Whitehead theorem. It is shown in [2] that the condition \( \pi_1(X, p) \) is \( T_1 \) is equivalent to each retract of \( X \) determining a closed topological fundamental subgroup. This in turn yields ‘no retraction’ theorems such as: The Hawaiian earring cannot be embedded as a retract of any space whose topological fundamental group is completely metrizable.

The above discussion highlights the role that injectivity of \( \phi : \pi_1(X) \to \lim_\leftarrow \pi_1(X_n) \) can play in the investigation of \( \pi_1(X) \). This paper develops a list of characterizations of injectivity of \( \phi \) in terms of the topology of \( \pi_1(X, p) \) (Theorem 4). As an application we obtain a classification of the existence of a certain kind of generalized universal covering space: the injectivity of \( \phi \) characterizes the existence of a fibration \( q : X'_p \to X \) such that \( X'_p \) is simply connected and each fibre is totally path disconnected (Corollary 7).

The results in this paper are related to the work of a number of other authors including Cannon, Conner [3], de Smit [4], Eda [5], Fischer and Zastrow [12].

We close the paper with some related open questions in hopes of spurring further developments in the algebraic topology of nonlocally contractible spaces.
2 Definitions and Preliminaries

All of the following definitions are compatible with those found in Munkres [14].

Suppose $X$ is a metrizable space and $p \in X$. Let $S_p(X) = \{ f : [0, 1] \to X$ such that $f$ is continuous and $f(0) = p \}$. Endow $S_p(X)$ with the topology of uniform convergence.

Let $C_p(X) = \{ f \in S_p(X) | f(1) = p \}$.

The **topological fundamental group** $\pi_1(X, p)$ is the set of path components of $C_p(X)$ endowed with the quotient topology under the canonical surjection $q : C_p(X) \to \pi_1(X, p)$ satisfying $q(f) = q(g)$ if and only if $f$ and $g$ belong to the same path component of $C_p(X)$.

Thus a set $U \subset \pi_1(X, p)$ is open in $\pi_1(X, p)$ if and only if $q^{-1}(U)$ is open in $\pi_1(X, p)$.

If $X \subset Y$ then $X$ is a **retract** of $Y$ if there exists a map $f : Y \to X$ such that $f_X = id_X$. The space $X$ is $T_1$ if each one point subset of $X$ is closed.

A space $X$ is **semilocally simply connected** at $p \in X$ if there exists an open set $U$ such that $p \in U$ and $j_U : U \to X$ induces the trivial homomorphism $j_U^* : \pi_1(U, p) \to \pi_1(X, p)$.

A topological space $X$ is **discrete** if every subset is both open and closed. The space $X$ is **totally disconnected** if each the components of $X$ are the one point subsets.

If $A_1, A_2, ...$ are topological spaces and $f_n : A_{n+1} \to A_n$ is a continuous surjection then, (endowing $A_1 \times A_2..$ with the product topology) the **inverse limit space** $\lim_{\leftarrow} A_n = \{ (a_1, a_2, ...) \in (A_1 \times A_2...) | f_n(a_{n+1}) = a_n \}$.

A surjective map $q : E \to B$ is a **fibration** provided for each space $Y$, for each map $f : Y \to E$, and for each map $F : Y \times [0, 1] \to B$ such that $F(y, 0) = q(f)(0)$, there exists a map $F^- : Y \times [0, 1] \to E$ such that $q(F^-) = F$ and $F^-(y, 0) = q(y)$.

**Remark 1** The topological fundamental group $\pi_1(X, p)$ is a topological group under concatenation of paths. (Proposition 3.1[14]). A map $f : X \to Y$ determines a continuous homomorphism $f^* : \pi_1(X, p) \to \pi_1(Y, f(p))$ via $f^*([\alpha]) = [f(\alpha)]$ (Proposition 3.3 [14]). If the spaces $X$ and $Y$ have the same homotopy type then there is an isomorphism between $\pi_1(X, p)$ and $\pi_1(Y, p)$ which is a homeomorphism.
3 Admissible inverse limit spaces

The results in the paper apply to inverse limit spaces \( X \) whose factors have totally disconnected fundamental group. Suppose \( \{ p_1 \} \subset X_1 \subset X_2 \ldots \) is a nested sequence of path connected separable metric spaces and suppose \( r_n : X_{n+1} \to X_n \) is a retraction and suppose \( \pi_1(X_n, p) \) is totally disconnected. For the remainder of this paper the resulting inverse limit space \( X = \lim \leftarrow X_n \) is said to be admissible. Given such an admissible space \( X \) we also define \( p = (p_1, p_1, \ldots) \in X \).

Let \( R_n : X \to X_n \) denote the canonical map \( R_n(x_1, x_2, \ldots) = x_n \). Let \( R_n^* : \pi_1(X, p) \to \pi_1(X_n, p_1) \) denote the induced homomorphism. Let \( \phi : \pi_1(X, p) \to \lim \leftarrow \pi_1(X_n, p_n) \) denote the induced homomorphism \( \phi([f]) = (R_1^*([f]), R_2^*([f]), \ldots) \). Let \( P \) denote the constant map of \( C_p(X) \). Let \([P]\) denote the path component of \( P \) in \( C_p(X) \).

Remark 2 If \( X_n \) is both locally path connected and semilocally simply connected (for example if \( X_n \) is an ANR or if \( X_n \) is locally contractible) then \( \pi_1(X) \) is discrete and in particular \( \pi_1(X_n) \) is totally disconnected. [7].

Lemma 3 The map \( j_n : X_n \to \lim \leftarrow X_n \) defined via \( j_n(x_n) = (x_1, \ldots, x_n, x_n, \ldots) \) is an embedding of \( X_n \) onto a retract of \( X \). Henceforth we can treat \( X_n \) as a retract of \( X \).

Proof. Note the function \( j_n \) is well defined since \( r_{n+k} \) fixes \( X_n \) pointwise. Moreover \( j_n \) is continuous since \( \Pi_k(j_n) \) is continuous with \( \Pi_k : \lim \leftarrow X_n \to X_k \) the projection map. The map \( j_n \) is one to one since if \( x_n \neq y_n \) then \( \Pi_n(x_n) \neq \Pi_n(y_n) \). To check that \( j_n \) is a homeomorphism onto its image note if \( \lim_{k \to \infty} (x_1^k, \ldots x_n^k, x_n^k, \ldots) = (x_1, \ldots, x_n, x_n, \ldots) \) then convergence is coordinatewise and in particular \( x_n^k \to x_n \). Finally the map \( R_n : \lim \leftarrow X_n \to \text{im}(j_n) \) defined via \( R_n = (\Pi_1, \Pi_2, \ldots, \Pi_n, \Pi_n, \ldots) \) is the desired retraction. ■

4 A monomorphism theorem

For admissible spaces \( X \) Theorem offers various characterizations of injectivity of the canonical homomorphism \( \phi : \pi_1(X, p) \to \lim \leftarrow \pi_1(X_n, p) \). It is interesting to note that whether or not \( \phi \) is injective is completely determined by topological properties of \( \pi_1(X, p) \).
Theorem 4 Suppose $X$ is admissible as demonstrated by the retracts $r_n : X_{n+1} \to X_n$ and suppose $\phi : \pi_1(X,p) \to \lim_{\to} \pi_1(X_n,p)$ is the canonical homomorphism. The following are equivalent.

1. $\phi$ is one to one.
2. $\pi_1(X,p)$ is $T_1$.
3. $\pi_1(X,p)$ is normal.
4. $\pi_1(X,p)$ is totally disconnected.
5. $\pi_1(X,p)$ is totally path disconnected.

Proof. First we establish the equivalence of 1, 2, and 3.

1 $\Rightarrow$ 2. Suppose $\phi$ is one to one. Since $\pi_1(X_n,p)$ is $T_1$, the product $\Pi_{n=1}^\infty \pi_1(X_n,p)$ is $T_1$. Since $\phi$ is continuous and one to one $\phi^{-1}(\phi(x)) = \{x\}$ is closed. Thus $\pi_1(X,p)$ is $T_1$.

2 $\Rightarrow$ 3. Suppose $\pi_1(X,p)$ is $T_1$. Since $\pi_1(X,p)$ is a topological group and since $\pi_1(X,p)$ is $T_1$, $\pi_1(X,p)$ is regular (ex 6 p.145 [14]). Since $X$ is a separable metric space $X$, the Urysohn metrization theorem shows $X$ can be embedded in the Hilbert cube (Thm 4.1 p. 217 [14]). Hence $C_p(X)$ is a separable metric space and in particular $C_p(X)$ is Lindelof. Thus $\pi_1(X,p)$ is Lindelof since the quotient map $q : C_p(X) \to \pi_1(X,p)$ is surjective. Hence $\pi_1(X,p)$ is paracompact since $\pi_1(X,p)$ is both regular and Lindelof (Ex 2 p. 259 [14]). Thus, since $\pi_1(X,p)$ is paracompact, $\pi_1(X,p)$ is normal (Thm. 4.1 p. 255 [14]).

3 $\Rightarrow$ 2 by definition.

To show 2 $\Rightarrow$ 1 suppose $\pi_1(X,p)$ is $T_1$ and $[f] \in \ker(\phi)$. Let $f = (f_1, f_2, \ldots)$. The retracts $r_{n+k} : X_{n+k+1} \to X_{n+k}$ enable us to define $f^n \in C_p(X)$ via $f^n = (f_1, \ldots, f_n, f_n, \ldots)$. Lemma 3 shows $j_n(R_n) : X \to im(j_n)$ is a retraction such that $j_n(R_n)(f) = f^n$. Suppose $[f] \in \ker(\phi)$. Since $[f] \in \ker(\phi)$, $f^n$ is inessential in $j_n(X_n)$ for each $n$ and in particular $f^n$ is inessential in $X$. Note $f^n \to f$ in $C_p(X)$. Since $\pi_1(X)$ is a $T_1$ space the trivial element is closed in $\pi_1(X,p)$. Since $g : C_p(X) \to \pi_1(X,p)$ is a quotient map the path component $[P]$ is closed in $C_p(X)$. Since $f^n \to f$ in $C_p(X)$ and since $f^n$ is path homotopic to $P$ it follows that $f$ is path homotopic to $P$. Hence $\phi$ is one to one.

Now we prove the equivalence of 2, 4, and 5.

4 $\Rightarrow$ 5 trivially.
5 ⇒ 1. Suppose 5 holds. Let \( \overline{\{e\}} \) denote the closure of the trivial element \( \{e\} \) in \( \pi_1(X,p) \). Notice every function \( \alpha : [0,1] \to \overline{\{e\}} \) is continuous. Thus if \( \pi_1(X,p) \) fails to be \( T_1 \) then \( \overline{\{P\}} \neq \{P\} \) and hence 5 would fail to hold since selecting \( f \in \{P\}\setminus\{P\} \) be may define a (continuous) function \( \alpha : [0,1] \to \{e,f\} \) such that \( \alpha(t) = e \) if \( 0 \leq t < 1 \) and \( \alpha(1) = f \). Thus 5 ⇒ 1.

1 ⇒ 4. If \( \pi_1(X,p) \) is \( T_1 \) then \( \phi \) is continuous and one to one and maps \( \pi_1(X,p) \) into the totally disconnected space \( \Pi \pi_1(X_n,p) \). Hence \( \pi_1(X,p) \) is totally disconnected. 4 ⇒ 5 trivially.

This completes the proof.

5 Application to generalized universal covers

Corollary 7 offers another characterization of injectivity of \( \phi \) in terms of the existence of a kind of generalized universal cover over \( X \).

The paper [1] develops the following generalization of the familiar covering maps.

A fibration \( q : E \to X \) is a **rigid covering fibration** provided the following 3 properties hold.

1. For each \( x \in X \) the fibre \( q^{-1}(x) \) is totally path disconnected.

2. If \( n \neq 1 \) then the induced homomorphism \( q_n^* : \pi_n(E) \to \pi_1(X) \) is an isomorphism.

3. If \( n \geq 2 \) then the induced homomorphism \( q_1^* : \pi_1(E) \to \pi_1(X) \) is one to one.

**Remark 5** It shown in Spanier [1] that a fibration has unique path lifting if and only if each fibre is totally path disconnected (Lemma 4 p.68). For such fibrations (Theorem 4 p.72, Corollary 11 p.377) \( q_n^* \) is one to one and \( q_n^* \) is an isomorphism for \( n \geq 2 \). Consequently conditions 2 and 3 are consequences of condition 1.

The familiar universal cover can be seen as a special case of a **generalized universal cover**, a rigid fibration whose total space is simply connected. Locally path connected spaces which are semilocally simply connected spaces admit a universal cover. Which path connected spaces admit a generalized universal cover? Theorem 5 provides a complete answer. We should point
out that Theorem 6 also follows by combining Theorems 4.3 and 4.6 and Corollary 4.7 of [1]. For the sake of clarity we include proof of Theorem 6.

**Theorem 6** Suppose $X$ is any path connected space. Then $\pi_1(X, p)$ is totally path disconnected if and only if there exists a rigid covering fibration $q : E \to X$ such that $E$ is simply connected.

**Proof.** Suppose $\pi_1(X, p)$ is totally path disconnected. Let $E$ denote the following quotient space of $S_\theta(X)$. Declare $f^*g$ if $f$ and $g$ are path homotopic in $X$. Define $Q : E \to X$ such that $Q(f) = f(1)$. To prove $Q : E \to X$ is a fibration suppose $f : Y \to E$ is any map and $F : Y \times [0,1] \to X$ is any homotopy such that $F(y, 0) = Q(f(y))$. For each $y \in Y$ select $\beta_y \in f(y)$. For each $y \in Y$ and $t \in [0,1]$ define $\alpha_{y,t} : [0,1] \to X$ such that $\alpha_{y,t}(s) = F(y, st)$. Define $F^* : Y \times [0,1] \to E$ such that $F^*(y, t) = [\beta_y * \alpha_{y,t}]$ where $*$ denotes the familiar concatenation of paths. To check that $F^*$ is well defined note $\beta_y(1) = F(y, 0) = \alpha_{y,0}(s)$. Since $\alpha_{y,0}$ is constant $\beta_y * \alpha_{y,0}$ is path homotopic to $\beta_y$. Thus $F^*(y, 0) = f(y)$. To check that $F^*$ is continuous let $A = \{(f, g) \in S_\theta(X) \times S_\theta(X) | f(1) = g(0)\}$. Let $B = \{[f], [g] \in E \times E | (f, g) \in A\}$. Note $(Q \times Q) : A \to B$ is a quotient map. Since we are using the uniform topology on $S_\theta(X)$ the standard path concatenation function $* : A \to E$ is continuous. To check that $*$ induces a map from $B \to E$ it suffices to note that $f * g$ is path homotopic to $f' * g'$ if $[f] = [f']$ and $[g] = [g']$. Thus $Q : E \to X$ is a fibration. By definition $Q^{-1}(p)$ is canonically homeomorphic to the topological fundamental group $\pi_1(X, p)$. If $x \in X$ and $\alpha$ is a path from $p$ to $x$ then the fibre over $x$ is homeomorphic to $\pi_1(X, p)$ via $\pi_1(X, p)*[\alpha]$. Thus $Q$ is a rigid covering fibration with unique path lifting. To prove that $E$ is simply connected note if $\alpha : [0,1] \to X$ is an inessential loop then the (unique) lift of $\alpha$ based at $[P]$ is the function $\alpha^* : [0,1] \to E$ satisfying $\alpha^*(t) = \beta^*(t)$ and $\beta^* : [0,1] \to X$ satisfies $\beta^*(s) = \alpha(st)$). In particular $\alpha^*(1) = [\alpha] \neq [P]$. Thus if $\gamma : [0,1] \to E$ is a loop based at $P$ then $Q(\gamma)$ is inessential. Hence $Q(\gamma)$ bounds a disk and the (unique) lift of the disk bounding $Q(\gamma)$ bounds $\gamma$. Thus $E$ is simply connected.

Conversely, suppose $q : E \to X$ is a rigid covering fibration such that $E$ is simply connected. Consider the following homotopy $F : C_\theta(X) \times [0,1] \to X$. Let $F(\alpha, t) = \alpha(t)$. Select $e \in E$ such that $q(e) = p$. Let $F^* : C_\theta(X) \times [0,1] \to E$ denote the unique lift satisfying $F^*(\alpha, 0) = e$.

Since $E$ is simply connected $F^*(\alpha, 1) = F^*(\beta, 1)$ if and only if $\alpha$ and $\beta$ are path homotopic in $X$. Since $\pi_1(X, p)$ inherits the quotient topology, $F^*$
induces a continuous injection $g : \pi_1(X,p) \to q^{-1}(p)$ determined the rule $g([\alpha]) = F^-(\alpha,1)$. Since $\text{im}(g)$ is totally path disconnected, and since $g$ is one to one, $\pi_1(X,p)$ must totally path disconnected. \[\square\]

Combining Theorems 4 and 6 in the context of admissible spaces, the existence of a simply connected rigid covering fibration is characterized by injectivity of the canonical homomorphism.

**Corollary 7** Suppose the inverse limit space $X$ is admissible as demonstrated by the retracts $r_n : X_{n+1} \to X_n$ and suppose $\phi : \pi_1(X,p) \to \lim_{\leftarrow} \pi_1(X_n,p)$ is the canonical homomorphism. Then the following are equivalent:

1. $\phi$ is one to one.
2. There exists a rigid covering fibration $q : E \to X$ such that $E$ is simply connected.

**Example 8** The Hawaiian earring $HE$ is locally path connected but not semilocally simply connected. Consequently $\pi_1(HE)$ fails to be discrete (Thm 2 [8]) and $HE$ does not admit a universal cover in the usual sense. However the canonical homomorphism $\phi : \pi_1(HE) \to \lim_{\leftarrow} \pi_1(F_n)$ is one to one (Thm. 4.1 [13]). Consequently $HE$ admits a generalized universal cover.

**6 Questions**

Here are some natural questions relevant to the results of this paper.

**Problem 9** Which path connected separable metric spaces $X$ are admissible?

**Problem 10** Suppose $X$ is any path connected space such that $\pi_1(X,p)$ is $T_1$. Must $\pi_1(X,p)$ be totally disconnected?

**Problem 11** Theorem 4.8 [1] asserts that if $q : E \to X$ is a rigid covering fibration such that $E$ is locally path connected and simply connected then $\pi_1(X,p)$ is isomorphic to the group of homeomorphisms $f : E \to E$ such that $q(f) = q$. To what extent can the assumption that $E$ is locally path connected be dropped? Under what conditions is the isomorphism also a homeomorphism with respect to the compact open topology?
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