Disorder-Driven Superconductor-Insulator Transition in $d$-Wave Superconducting Ultrathin Films

Long He, Jian Sun, and Yun Song

Department of Physics, Beijing Normal University, Beijing 100875, China

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We study the superconductor-insulator transition (SIT) in $d$-wave superconducting ultrathin films. By means of the kernel polynomial method, the Bogoliubov-de Gennes equations are solved for square lattices with up to 360 × 360 unit cells self-consistently, making it possible to observe fully the nanoscale spatial fluctuations of the superconducting order parameters and discriminate accurately the localized quasiparticle states from the extended ones by the lattice-size scaling of the generalized inverse participation ratio. It is shown that Anderson localization can not entirely inhibit the occurrence of the local superconductivity in strongly-disordered $d$-wave superconductors. Separated by an insulating 'sea' completely, a few isolated superconducting 'islands' with significant enhancement of the local superconducting order parameters can survive across the SIT. The disorder-driven SIT, therefore, is a transition from a $d$-wave superconductor to a Bose insulator which consists of localized Cooper pairs. Unlike an $s$-wave superconductor which presents a robust single-particle gap across the SIT, the optical conductivity of a $d$-wave superconductor reveals a gapless insulating phase, where the SIT can be detected by observing the disappearance of the Drude weight with the increasing disorder.

I. INTRODUCTION

Recently, as huge amount of interest has been shown in the nanoscale superconductivity of thin films and interfaces between complex oxides, accompanied by the technological developments in the synthesis of high-quality nanostructures, people are more concerned about whether superconductivity can be enhanced in nanoscale structures with respect to the bulk limit. Because disorder is generated by reducing the film thickness, we should take much account of the effect of disorder in superconducting ultrathin films and heterointerfaces. The effect of disorder in two-dimensional (2D) superconductors has long been a subject of great interest. Because the 2D CuO$_2$ planes are characteristic of all the high-temperature cuprate superconductors, the analysis of disorder-induced inhomogeneities can aid in the understanding of the nature and origin of the $d$-wave superconducting state. At atomic limit, the cuprate superconductors can also be considered to have heterostructures with stacks of nanoscale superconducting planes intercalated by charge reservoir layers with nanoscale periodicity.

In 2D $d$-wave superconductors, the quasiparticle has a linear dispersion relation in the vicinity of nodal points. Accordingly, the elastic scattering of electrons caused by impurities is likely to destroy Cooper pairs and suppress strongly the order parameter in the vicinity of impurities. For this reason, the pseudogap state of cuprates is predicted to contain superconducting islands embedded in a normal metallic matrix, resulting from the scattering effects of the inhomogeneously-distributed pair breakers. Some theoretical studies have found that the pairing correlation of a disordered $d$-wave superconductor can be significantly enhanced near the impurities, leading to a strong spatial fluctuation of the coupling constant. Besides, the low-energy quasiparticle states in disordered $d$-wave superconductors are found to be localized with the presence of a mobility gap, bringing about a superconductor-insulator transition (SIT) at a critical disorder strength when the pair-breaking effect of disorder is taken into account.

A number of recent experiments demonstrate a direct disorder-driven transition between the superconducting and insulating phases in highly-disordered thin films. It has also been discovered that a superconducting state can be created in LaAlO$_3$/SrTiO$_3$ interface, accompanied with an increase of the relative disorder strength or the magnetic field across the SIT. Some theoretical researches return to the problem of the SIT in $d$-wave superconductors, but there is still debate about the nature of the insulating phase and the mechanism that drives the SIT. Recently, the insulating phase is predicted to be a Bose insulator, represented by localized Cooper pairs with a nonzero pairing amplitude.

To address the spatial inhomogeneity of high-$T_c$ superconducting thin films, it is increasingly important to have numerical methods that are capable of simulating both microscopic and mesoscopic length scales simultaneously. Beside, the inhomogeneity of the pair-breaking effect should be accounted for in a fully self-consistent manner. The exact diagonalization approach is one of the commonly adopted methods since it treats precisely the disorder-induced scattering and presents entirely the eigenvalues and eigenstates of a finite system. In addition, the correct low-energy density of states (DOS) of inhomogeneous $d$-wave superconductor produced by using the exact method cannot be obtained when the self-consistent $T$-matrix approximation is taken, although the correlation between impurity location and order parameter variation has been preserved. However, the exact calculation on large lattice is prevented primarily by the memory limitation of computer. Besides, the self-consistency of the Bogoliubov-de Gennes
(BdG) equations and the average over a large number of disorder configurations demand the fast processor speed strongly. Recently, the kernel polynomial method (KPM) is regarded as a distinct method for the disordered systems because it allows for the numerical calculations for dimensions of the order of \( D \approx 10^9 \). Therefore, the KPM approach has the potential to calculate lattices two or three orders of magnitude larger of the number of sites than the lattices studied by the exact diagonalization method. Consequently, the mesoscopic-scale inhomogeneity in 2D lattice can be fully presented by the KPM approach and the tight-binding BdG equations can significantly improved when the lattice-size scaling of a certain quantity is performed for the extrapolation of results of finite lattices to the infinite limitation.

In the present work, we apply the KPM approach to study the effect of disorder in high-\( T \) superconductors. For the large enough lattices, the disordered \( d \)-wave superconductors are investigated by observing rigorously the evolutions of the spatial fluctuations of the superconducting order parameters with increasing disorder in nanoscale. Since the KPM is designed to calculate the local density of states (LDOS) instead of the eigenfunctions, we introduce the generalized inverse participation ratio (GIPR) to study the Anderson localization of Bogoliubov quasiparticles. The GIPR, which relies only on the LDOS as defined in Eq. (1), has been proved to be a good measure of Anderson localization. We find that all Bogoliubov quasiparticles can be localized by weak disorder in the \( d \)-wave ultrathin films. Analyzing the spatial fluctuations of the order parameters, we show that several local superconducting islands can survive across the SIT, suggesting that the insulating phase is characterized by a Bose insulator. In addition, the carrier localization can be manifested by the disappearance of the Drude peak of the optical conductivity with increasing disorder. The calculated results agree with the observed experimental facts.

The paper is organized as follows: In Sec. II, we describe the KPM approach and the tight-binding BdG equations of the disordered \( d \)-wave superconductors in detail; results are presented in Sec. III. We first extend the GIPR to confirm the localization effect of disorder in \( d \)-wave superconductors; then we try to show a clear picture of the disorder-driven transition from a \( d \)-wave superconductor to a Bose insulator; finally, we discuss the experimental observation of the SIT in \( d \)-wave superconductor by the optical conductivity, where the SIT is accompanied by the disappearance of the Drude peak with the increasing disorder. A concluding summary is given in Sec. IV.

II. MODEL AND METHODOLOGY

We introduce a 2D mean-field Hamiltonian for the disordered \( d \)-wave superconducting films, which is expressed as

\[
H = -t \sum_{\langle ij \rangle} c_{i\sigma}^\dagger c_{j\sigma} + \sum_{i\sigma} \epsilon_i c_{i\sigma}^\dagger c_{i\sigma} - \sum_{\langle ij \rangle} \{ \Delta_{ij} c_{i\sigma}^\dagger c_{j\uparrow} + H.c. \},
\]

where \( c_{i\sigma} (c_{i\sigma}^\dagger) \) are the electronic annihilation (creation) operators at sites \( i \) with spin \( \sigma \) (\( \uparrow \) or \( \downarrow \)), \( t \) denote the hopping integrals between nearest neighbor (NN) sites \( i \) and \( j \), \( \epsilon_i \) represent the on-site disorder energies distributed evenly in the energy region \([-W/2, W/2]\), and \( \Delta_{ij} = -V (c_{ij} c_{rT}) \) are the superconducting order parameters under the NN attractive interactions \( V \) \((V \ll 0)\).

To take into account the \( d \)-wave gap symmetry in high-temperature superconductors, we constrain \( \Delta_{ij} = \frac{1}{2} |\Delta_{ij}| \{ (-1)^{x_{ij} - y_{ij}} \} \) with \((x_{ij}, y_{ij})\) connecting sites \( i \) and \( j \). Using the Bogoliubov transformation, the eigenvalues \( E_\alpha \) of Hamiltonian Eq. (1) are obtained by solving the BdG equations self-consistently, which are given by

\[
\sum_j \left( \frac{\xi_{ij}}{\Delta_{ij}} - \xi_{ij} \right) \left( u_{\alpha j} (r_j) v_{\alpha r} (r_i) \right) = E_\alpha \left( u_{\alpha r} (r_i) v_{\alpha j} (r_j) \right),
\]

with \( \xi_{ij} = -t_{ij} + \epsilon_i \delta_{ij} \). Here \( u_{\alpha j} (r_j) \) and \( v_{\alpha r} (r_i) \) denote the Bogoliubov coefficients of sites \( i \). For a lattice with \( N \) sites, the Hamiltonian of BdG equations can be represented by a \( 2N \times 2N \) Hermite matrix with \( 2N \) eigenvalues, requiring \( E_{N+k} = -E_k \) \((k = 1, 2, ..., N)\).

It is convenience to introduce two real-space Green’s functions \( G_1^{11}(\omega) \) and \( G_1^{12}(\omega) \) for Hamiltonian Eq. (1), which are denoted by \( N \times N \) matrices,

\[
G_1^{11}(\omega) = \sum_{l=1}^{N} \left( \frac{u_{il} u_{jl}^{*}}{\omega - E_l} + \frac{v_{il} v_{jl}^{*}}{\omega + E_l} \right), \quad G_1^{12}(\omega) = \sum_{l=1}^{N} \left( \frac{u_{il} v_{jl}^{*}}{\omega - E_l} + \frac{v_{il} u_{jl}^{*}}{\omega + E_l} \right).
\]

As a result, LDOS with respect to sites \( i \) can be expressed as

\[
\rho (r_i, \omega) = \frac{1}{\pi} \text{Im} G_{ii}^{11}(\omega) = \sum_{l=1}^{N} \{|u_{il}|^2 \delta(\omega - E_l) + |v_{il}|^2 \delta(\omega + E_l)|.
\]

And similarly, the NN superconducting order parameters \( \Delta_{ij} \) can be obtained by

\[
\Delta_{ij} = -\frac{V}{\pi} \int_{-\infty}^{+\infty} \text{Im} G_{ij}^{12}(\omega) f(\omega) d\omega = -V \sum_{l=1}^{N} \{ u_{il} v_{jl}^{*} f(E_l) + v_{il} u_{jl}^{*} (1 - f(E_l)) \},
\]

where \( f(E) \) represents the Fermi-Dirac distribution function.
Instead of directly diagonalizing the Hamiltonian matrix of BdG equations, the physical quantities, such as LDOS and superconducting order parameters are calculated by expanding the single particle Green’s functions in terms of Chebyshev polynomials within the KP approach.\cite{19,20} Considering that both the first and second kinds of Chebyshev polynomials should be defined in the interval [-1, 1], a simple linear transformation should be introduced to rescale the Hamiltonian and all energies

\[
\tilde{H} = \frac{H - b}{a}, \quad \tilde{\omega} = \frac{\omega - b}{a},
\]

where \(a = (E_{\text{max}} - E_{\text{min}})/2\), and \(b = (E_{\text{max}} + E_{\text{min}})/E_{\text{max}}\) and \(E_{\text{min}}\) are the extremal eigenvalues of Hamiltonian, which can be approximately obtained on a small lattice by using the Lanczos algorithm.\cite{21} Thus we can expand the single particle Green’s functions shown Eq. (3) into series of Chebyshev polynomials as

\[
\text{Im} \tilde{G}_{ij}^{\alpha\beta}(\tilde{\omega}) = -\frac{1}{\sqrt{1 - \tilde{\omega}^2}} \left( g_0 \rho_{ij}^{\alpha\beta} + 2 \sum_{n=1}^{M-1} g_n \rho_n^{\alpha\beta} T_n(\tilde{\omega}) \right),
\]

\[
\text{Re} \tilde{G}_{ij}^{\alpha\beta}(\tilde{\omega}) = \frac{1}{\sqrt{1 - \tilde{\omega}^2}} \left( g_0 \rho_{ij}^{\alpha\beta} + 2 \sum_{n=1}^{M-1} g_n \rho_n^{\alpha\beta} U_n(\tilde{\omega}) \right),
\]

where \(\alpha\beta = 11\) or \(12\), \(g_n = \sin[\lambda(1 - n/M)]/\sin(\lambda)\) are the Lorenz Kernel with a free parameter \(\lambda\), \(M\) is the order of the series, \(\rho_{ij}\) represent the coefficients of the expansions, and \(T_n(x) = \cos[n \arccos(x)]\) and \(U_n(x) = \sin[(n + 1) \arcsin(x)]/\sin[\arccos(x)]\) are the Chebyshev polynomials of the first and second kind respectively.

Now the pivotal issue has turned to how to calculate the coefficients \(\rho_{ij}\) of the expansions. According to the proposal given by Covaci et al.,\cite{22} the moments \(\mu_n^{11}\) and \(\mu_n^{12}\) can be obtained efficiently through a recursive procedure. Since the Chebyshev polynomials have recursion relations

\[
T_0(\tilde{H}) = 1, \quad T_{-1}(\tilde{H}) = T_1(\tilde{H}) = \tilde{H},
\]

\[
T_{m+1}(\tilde{H}) = 2\tilde{H}T_m(\tilde{H}) - T_{m-1}(\tilde{H}),
\]

we can obtain the coefficients by calculating the matrix elements of the Chebyshev polynomial as

\[
\mu_n^{11}(i, j) = \langle \text{vac}| c_i T_n(\tilde{H}) c_j^\dagger |\text{vac}\rangle, \]

\[
\mu_n^{12}(i, j) = \langle \text{vac}| c_i c_j^\dagger T_n(\tilde{H}) |\text{vac}\rangle, \]

where \(\text{vac}\) is the vacuum state of Hamiltonian Eq. (1). As a result, the whole Green’s function is extracted in absence of obtaining all the eigenvalues and eigenstates.

### III. DISORDER-DRIVEN SIT IN d-WAVE SUPERCONDUCTORS

The one-parameter scaling theory\cite{23} predicts that, in a simple 2D Anderson model, arbitrarily weak disorder would localize all electronic states. But a 2D superconductor is believed to transition into an insulator beyond a certain critical value of the disorder strength. There remains a dispute about whether the insulating phase is characterized by a Bose insulator or a Fermi insulator.\cite{24} Here, we investigate the disorder-driven SIT in d-wave superconductors from two aspects, i.e. the localization of Bogoliubov quasiparticles and the Cooper pairs breaking across the SIT.

#### A. The Localization of Bogoliubov quasiparticles

The localization length of a particle can be obtained by the GIPR, which is defined as\cite{40,41}

\[
G_2(\omega) = \frac{\sum_i \rho(r_i, \omega)^2}{\left[ \sum_i \rho(r_i, \omega) \right]^2},
\]

where \(\rho(r_i, \omega)\) represent the local density of states (LDOS) at sites \(i\). The GIPR is a proven way in testing the Anderson localization of electronic states.\cite{40,41} Since an energy broadening \(\gamma\) has to be introduced to get a continuum LDOS of a finite lattice, a relation of \(\gamma = W/L^2\) should be satisfied to guarantee that we average over the same number of states to calculate the GIPR for lattices of different sizes.\cite{42} For the Green’s functions obtained by KPM, the parameter \(\lambda\) in Lorenz kernel plays a similar role as parameter \(\gamma\), except that the order \(M\) of the Chebyshev polynomial series also need to be taken in to account. Therefore, we introduce the relation of \(\lambda = W/L^2\).

As shown in Fig. (a), the scaling of the GIPR satisfies...
a linear relationship as

$$G_2(\omega, L) = \alpha + \beta \frac{1}{L^d},$$  \hspace{1cm} (11)

where $\alpha$ represents the intercept in the limitation $L \to \infty$, and $\beta$ is the slope of the straight scaling line. A localized state is predicted to have a nonzero intercept, from which we can achieve the localization length by

$$\xi_{loc}(\omega) = \frac{1}{G_2(\omega, \infty)^{1/d}} = \frac{1}{\sqrt{\alpha}}.$$  \hspace{1cm} (12)

where $d$ is the lattice dimension. It is also shown in Fig. (a) that the intercept $\alpha$ increases significantly with the strengthening of the disorder strength $W$, demonstrating that the localization of the quasiparticles is enhancement with the decreasing localization lengths.

In Fig. (b) and (c), we compare the localization lengths of Bogoliubov quasiparticles of 2D $d$-wave superconductors with that of the electrons in 2D Anderson model, and find that the localization lengths increase by an order of magnitude or more when the attractive interactions $V = 4t$ are taken into account. Despite this, the scaling of the GIPR predicts that disorder can still localize all quasiparticles in 2D superconductors, which is enhanced with the decreasing localization lengths.

However, to discriminate a Fermi insulator from a Bose insulator, the presence of both the completely suppression to the superconductivity and the entirely vanishing of the pair amplitude by disorder is essential. Next, we study the effects of disorder on the superconducting order parameters across SIT.

![FIG. 2: (Color online) The disorder induced spatial fluctuations of the superconducting order parameters $|\Delta_{i,i\pm \hat{x}}|$ at different disorder strength $W = 4$ (a) and $W = 16$ (c). The fluctuations of $|\Delta_{i,i\pm \hat{y}}|$ are also shown for $W = 4$ (b) and $W = 16$ (d), respectively. The other parameters are: $L = 360$, $N = L^2$, and $V = -4$. Energies are in unit of $t$.](https://example.com/fig2)

B. Superconducting Blobs

It is well known that the suppression of superconductivity by non-magnetic disorder is much stronger in the $d$-wave superconductor than in the $s$-wave superconductor. Here we study the competition and correlations between disorder and the $d$-wave superconductivity in large-sized lattices. The NN superconducting order parameters are acquired self-consistently by

$$\Delta_{ij} = -V \int_{E_c}^{E_c} \text{Im}G_{ij}^{12}(E)(1-2f(E))dE,$$  \hspace{1cm} (13)

where $G_{ij}^{12}$ is the off-diagonal Green’s function, and $E_c (-E_c)$ represents the band-edge energy. As expected, the average of order parameters along one of the bond directions, $\Delta_\eta = 1/N \sum_i \Delta_{i,i+\eta}$ with $\eta = \hat{x}$ or $\hat{y}$, drops with the increasing disorder monotonously. In spite of the maintenance of the $d$-wave symmetry of the average order parameter of the whole lattice, the superconductor is locally no longer $d$-wave because of the violation of the translational invariance by disorder. In Fig. we plot the spatial distribution of the NN superconducting order parameters $\Delta_{i,i+\hat{x}}$ and $\Delta_{i,i+\hat{y}}$ in $d$-wave superconductors in weak ($W = 4t$) and strong ($W = 16t$) disorder cases respectively. Fig. (a) and (b) show clearly that the system is still a superconductor in the weak disorder cases. However, the system transitions into an insulator as shown in Fig. (c) and (d) when $W = 16t$, where the great majority of the NN bonds are not superconducting at all but a few local superconducting bonds can survive across the disorder-driven SIT.

To understand whether the spatial fluctuations of $\Delta_{i,i+\hat{x}}$ and $\Delta_{i,i+\hat{y}}$ are correlated or independent, we illustrate in partial views of the effects of box distributed disorder on the local superconducting order parameters when the disorder strength is comparative with the attractive interactions ($V = W = 4t$). Concurrently, we also display the local effects of disorder on the on-site $d$-wave superconducting order parameters $\Delta_i$, which is expressed as

$$\Delta_i = \frac{1}{4}(\Delta_{i,i+\hat{x}} - \Delta_{i,i+\hat{y}} + \Delta_{i,i-\hat{x}} - \Delta_{i,i-\hat{y}}).$$  \hspace{1cm} (14)

As shown in Fig. it is obvious that the regions of weakened and strengthen superconductivity are separated completely, suggesting that the spatial fluctuations of the superconducting order parameters along the two directions ($\Delta_{i,i+\hat{x}}$ and $\Delta_{i,i+\hat{y}}$) are not independent. Consequently, the disorder energy of a certain site can not determine completely the local on-site order parameters accordingly.

Because of the existence of correlation effects presented above, it worth clarifying the relationship between the disorder energy and the local superconducting order parameter. For the case without disorder ($W = 0$), the homogenous superconducting order parameters are $\Delta_0 = 0.62t$ when $V = 4t$. As the disorder is introduced,
there appears very strong spatial fluctuation of the superconducting order parameters with the minimum value $|\Delta_{ij}| = 0.01t$ and the maximum value $|\Delta_{ij}|^\text{max} = 0.71t$ as shown in Fig. 3. To define the system to be a Bose insulator, we need to figure out why the superconducting order parameters can be enhanced by disorder.

C. A Bose insulator with localized Cooper pairs

In Fig. 4 we plot the distributions of the NN order parameters $\Delta_{i,i+\eta}$ (Fig. 4(a)-(c)) and the on-site order parameters $\Delta_i$ (Fig. 4(d)-(f)) over the disorder energies. It is shown that the lower the site energy $\epsilon_i$ or the energy difference $|\epsilon_i - \epsilon_{i+\eta}|$ are, the stronger the suppression of the superconducting order parameters can be observed. On the contrary, the constrain effect of disorder is much weaker in the lower-energy region, but the fluctuation of the order parameters is getting stronger as shown in Fig. 4. To those sites with zero disorder energy, the fluctuation is the strongest, suggesting that the non-local effect of disorder can not be ignored. In the strong disorder regime, we even find that the disorder potential can change the sign of $\Delta_i$ locally. As expected, the fluctuations are getting stronger with the increasing of disorder strength, and $\Delta_{ij}^\text{max}/\Delta_0$ increases to 2.43 when $W = 16t$ and $V = 4t$. Therefore, separated completely by the very large non-superconducting regions, there still exist several bonds which have comparatively large superconducting order parameters. Anderson’s theorem proposes that the transition temperature and gap are insensitive to impurity scattering when the coherent length $\xi_{\text{coh}}$ is much larger than the lattice spacing $a$. Whereas we find strong fluctuations of the superconducting order parameters in $d$-wave superconductor, suggesting that Anderson’s theorem is invalid. As a result, the coherence length should be of the order of the lattice spacing $a$, implying that the cooper pairs are strongly localized.

It has been proposed that the presence of localized Cooper pairs will lead to the disappearance of the coherence peaks in the one-particle DOS whereas the superconducting gap remains intact. It is obvious that this judgement is currently only applicable to the SIT driven by Cooper pair localization in some disordered $s$-wave superconductors. For the $d$-wave superconductors, it has been demonstrated that even weak disorder can significantly alter the DOS at low energy. As a result, the DOS near the Fermi level increase with the increasing disorder (Fig. 4(c)), accompanied with the vanish of the coherence peaks of $d$-wave Cooper pairs. In spite of this, a Bose insulator may be determined by the detection of the localized Cooper pairs using energy distribution analysis of the LDOS. As shown in Fig. 4(a) and 5(b), there is significant difference of the LDOS around the Fermi
Next we study the disorder effect on the optical conductivity, which can be calculated directly by

$$\sigma(\omega) = \sum_{k,q} \frac{|\langle k|J|q\rangle|^2 [f(E_k) - f(E_q)]}{2ZL^2} \delta(\omega - (E_k - E_q))$$

$$= \frac{1}{\omega} \int_{-\infty}^{\infty} j(x, x + \omega)[f(x) - f(x + \omega)]dx,$$  \hspace{1cm} (15)

where $Z$ is coordination number of the system. The matrix element density $j(x, y)$ can be expanded as

$$j(x, y) = \sum_{n,m=0}^{M-1} \frac{\mu_{nm} h_{nm} g_n g_m T_n(x) T_m(y)}{\pi^2 (1 - x^2) (1 - y^2)}$$  \hspace{1cm} (16)

where $g_n = 1/(1 + \delta_{0,n})$ is the kernel damping factors, $h_{nm}$ accounts for the correct normalization, and the moments $\mu_{nm}$ are obtained from

$$\mu_{nm} = \int_{-1}^{1} \int_{-1}^{1} \tilde{j}(x, y) T_n(x) T_m(y) dx dy$$

$$= \frac{1}{D} Tr \left[ T_n(\hat{H}) J T_n(\hat{H}) J \right],$$  \hspace{1cm} (17)

where $\tilde{j}(x, y)$, which is transformed linearly from the matrix element density $j(x, y)$, is defined within the energy interval $[-1, 1]$.

The disorder effects on the optical conductivity and matrix element density $j(x, y)$ are plotted in Fig. 6 respectively. We find firstly that the optical conductance diverges near $\omega = 2\Delta_0 = 1.24t$ in the weakly disordered $d$-wave superconductor. Meanwhile, the matrix element density is concentrated in four small regions located symmetrically on the two diagonal lines $x = \pm y$ with the so called "shark fan" structure as shown in Fig. 6(b). Eq. (15) presents that the Drude weight is obtained by calculating the integral of the density of $j(x, y)$ along the diagonal line $x = y$. Since there exist the low-energy quasiparticle excitations in the pure $d$-wave superconductor, we find two peaks with "shark fin" structure along the diagonal line $x = y$, corresponding to a significant Drude weight. When the effects of disorder is considered, the the matrix element density appears to be dispersing towards the regions surrounded as shown in Fig. 6(c). As a result, the Drude weight tends to zero since the matrix element density along the diagonal line $x = y$ drops very quickly with the increasing of disorder strength. When the disorder strength is large than the critical value $W_c \approx 5t$ for the attractive interactions $V = -4t$, we find that the spreading of the density of $j(x, y)$ in the energy plan have six centers. As a result, no density of $j(x, y)$ can stay in the diagonal line $x = y$, leading to the completely suppression of the Drude weight by disorder in the high disorder region. Therefore, it is convenient to probe the SIT in $d$-wave superconductors by

\[\text{FIG. 6: (Color online) (a) The effect of disorder on the optical conductivity. Insert: the corresponding DOS; (b) and (c) the matrix element density } j(x, y) \text{ for the pure (}W = 0\text{) and disordered (}W = 4\text{) systems, respectively. The other parameters are: } L = 360, V = -4, M = 2000, \text{ Energies are in unit of } t.\]

\[\text{FIG. 5: (Color online) The local density of states at sites with different local superconducting order parameters } \Delta_i = 0.75 \text{ (a) and } \Delta_i = 0.02 \text{ (b) when } W = 8. \text{ (c) The density of states at different disorder strength } W = 2, \text{ } W = 4, \text{ and } W = 8. \text{ The other parameters are: } L = 320, V = -4, M = 2000, \text{ and } \lambda = 2. \text{ Energies are in unit of } t.\]
observing the Drude weight in the optical conductivity.

IV. CONCLUSIONS

In summary, we extend the kernel polynomial method to investigate the superconductor-insulator transition in inhomogeneous two dimensional \( d \)-wave superconductors. We have improved on previous numerical results by calculating very big cluster, as well as come up with explanations to the effects of disorder on nanoscale superconductivity of thin films and interfaces between complex oxides. It is manifest by the optical conductivity that disorder can drive a transition from a superconductor to a gapless insulator in the \( d \)-wave superconducting ultrathin films. The insulating phase is characterized as a Bose insulator, where all Bogoliubov quasiparticles are localized but a few of superconducting blobs can survive strong disorder.

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* Electronic address: yumsong@bnu.edu.cn

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