Correlation functions of composite Ramond fields in deformed D1-D5 orbifold SCFT$_2$

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ABSTRACT

We study two families of composite twisted Ramond fields (made by products of two operators) in the $\mathcal{N} = (4,4)$ supersymmetric D1-D5 SCFT$_2$ deformed by a marginal moduli operator away from its $(T^4)^N/S_N$ free orbifold point. We construct the large $N$ contributions to the four-point functions of two such composite operators and two deformation fields. These functions allow us to derive various short-distance OPE limits and to calculate the anomalous dimensions of the composite operators. We demonstrate that one can distinguish two sets of composite Ramond states with twists $m_1$ and $m_2$: protected states, for which $m_1 + m_2 = N$, and “lifted” states for which $m_1 + m_2 < N$. The latter require an appropriate renormalisation. We also derive the leading order corrections to their two-point functions, and to their three-point functions with one marginal moduli operator.

Keywords: Microstate black hole geometries; Symmetric $\mathcal{N} = 4$ SUSY orbifold CFTs; Correlation functions.
1. Introduction

The scalar moduli deformation of the symmetric orbifold \((T^4)^N/S_N\) gives rise to a particular two-dimensional \(\mathcal{N} = (4, 4)\) superconformal theory with central charge \(c = 6N\), which for large values of \(N\) provides a fuzzball \([1]\) description of certain five-dimensional extremal supersymmetric black holes. Their IIB superstring counterparts are bound states of the D1-D5 brane system (see e.g. the review \([2]\)), which gave the first microscopical account for the Bekenstein-Hawking entropy \([3]\). There is strong evidence \([4–9]\) that appropriate coherent superpositions of twisted Ramond states (and certain products of them) reproduce the “microstate geometries” holographically dual to the semiclassical IIB supergravity 2-charge horizonless non-singular solutions of \(\text{AdS}_3 \times S^3 \times T^4\) type. Similar statements hold for the microstates of the more realistic near-extremal 3-charge 1/8-BPS black holes, the so-called D1-D5-P system, which can be realized as appropriate tensor products of the (left-right non-symmetric) descendants of twisted Ramond ground states of the same D1-D5 orbifold SCFT\(_2\) \([10–12]\). A more complete description of the quantum properties of such SUSY black holes requires further investigation of the spectra of conformal dimensions of composite fields, the construction of their correlation functions, and analysis of their renormalization as an effect of the interaction introduced by the marginal perturbation away from the free orbifold point.

Despite numerous results and achievements \([13–20]\), the super-conformal data concerning the effects of the interaction in the deformed D1-D5 SCFT\(_2\) remains incomplete. As we have demonstrated in a recent paper \([21]\), the simplest R-charged twisted Ramond fields \(R_{\pm m}^{\pm}(z, \bar{z})\) get renormalized, i.e. their conformal dimensions and certain structure constants acquire corrections in the perturbed theory. It is then natural to address the question of whether the simplest composite states \(R_{m_1}^{\pm} R_{m_2}^{\pm}(0)\), made by a product of two Ramond fields with twists \(m_1\) and \(m_2\), are BPS-protected or should be renormalized. If renormalization occurs to some fields,
what are, then, the conditions defining classes of “protected” and “lifted” Ramond states in the deformed theory?

The answer to the above questions requires the explicit construction of the large-$N$ contributions to the four-point correlation functions involving two composite Ramond fields and two deformation operators. This is what we compute in the present paper, using the ‘covering surface technique’ [22] together with the ‘stress-tensor method’ [23–26]. Our result allows us to examine certain short-distance limits, and to compute the structure constants as well as the conformal dimensions of the specific non-BPS descendants of twisted fields present in these OPEs.

We find that the four-point functions with two composite fields that we compute is given by a sum of “connected” and “disconnected” parts, which have different weights in the $1/N$ expansion. The former give the sub-leading contributions, of order $1/N^2$, while the latter give the leading terms of order $1/N$. This important observation seems to be quite generic and valid for more complicated products of twisted Ramond fields — say, composite fields made by three or more multipliers or powers of operators taken in coincident points.

Once we have the explicit form of the four-point functions, integrating over the positions of the interaction operators yields the correction to the conformal dimensions the composite operators $R_{m_1}^\pm R_{m_2}^\pm$, to second order in perturbation theory. The nature of the composite operators crucially depends on the properties of the twists $m_1$ and $m_2$ of their components. We demonstrate that the case when $m_1 + m_2 = N$ represents a family of protected states, whose conformal dimensions remain the same as in the free orbifold point because the correction vanishes. It turns out that all the remaining composite Ramond states (and fields) with $m_1 + m_2 < N$ suffer from certain UV divergences, hence they do require an appropriate renormalization and, as a result, their conformal dimensions get corrections in the considered large $N$ approximation. The separation of the protected from the “lifted” composite Ramond states is one of the main results of the present paper.

2. Symmetric orbifold D1-D5 SCFT$_2$

In this paper we are concerned with a symmetric orbifold model $(T_4)^N/S_N$ where $T_4$ is a four dimensional torus and $S_N$ is the corresponding symmetric group. This SCFT$_2$ orbifold model is considered as a “free orbifold point” of D1-D5 system (see for example [2,27]).

The theory contains $4N$ free scalar fields $X_I^i$, with $i = 1, \cdots, 4$ and $I = 1, \cdots, N$, and $4N$ free fermions $\psi_I^i$, with total central charge $c_{orb} = 6N$. The $N$ copies of the fields are identified by the action of the symmetric group: $X_I^i(e^{2\pi i}z,e^{-2\pi i}\bar{z}) = X_{g(I)}^i(z,\bar{z})$, where $g \in S_N$. These boundary conditions are realized by the so-called twist fields $\sigma_g(z)$, which are connected to the conjugacy classes of $S_N$. For example $\sigma_{(1\cdots n)}$ imposes the cyclic permutations of the fields corresponding to the cycle $(1\cdots n)$,

$$X_1^i \rightarrow X_2^i \rightarrow \cdots \rightarrow X_n^i \rightarrow X_1^i,$$  (1)
and similarly for the fermions belonging to the Ramond sector, i.e. with periodic boundary conditions. We denote by $\sigma_n$ the twist field corresponding to the conjugacy class obtained by summing over the orbits the whole symmetric group:

$$\sigma_n = \frac{1}{\sqrt{nN!(N-n)!}} \sum_{h \in S_N} \sigma_{h^{-1}(1\ldots n)h}; \quad (2)$$

We call attention for a notational convention that we use throughout the paper: a twist index without brackets, like in $\sigma_n$, indicates a sum over conjugacy classes of cycles of length $n$ as in the r.h.s. of Eq.(2). A twist index with brackets, like in $\sigma_{(n)}$, indicates one single twist corresponding to a specific permutation cycle $(n)$, of length $n$; e.g. $\sigma_{(2)}$ is a short notation for $\sigma_{(12)}$ or $\sigma_{(37)}$ or $\sigma_{(15)}$, etc. As expected, the dimension of $\sigma_n$ and of any non-$S_N$-invariant twist field $\sigma_{(n)}$ depends only on the length of the cycle and is given (for six bosons) by

$$\Delta_n^a = \frac{1}{4} \left( n - \frac{1}{n} \right). \quad (3)$$

The normalization factor

$$\mathcal{J}_n = \sqrt{nN!(N-n)!} \quad (4)$$

is responsible for the normalization of the two-point function of the $S_N$-invariant operators,

$$\langle \sigma_n(z, \bar{z})\sigma_m(0) \rangle = \frac{\delta_{mn}}{|z|^2 \Delta_n}. \quad (5)$$

We further pair the $4N$ real scalar fields into complex bosons $X^a_I$ and $X^{a\dagger}_I$, $a = 1, 2$. The Majorana fermions can also be combined into complex fermions and then bosonized by the use of $2N$ new free scalars: $\psi^{a}_I = e^{i\phi^a_I}$, $\psi^{a\dagger}_I = e^{-i\phi^a_I}$. The holomorphic sector possesses $\mathcal{N} = 4$ superconformal symmetry, generated by the stress-energy tensor $T(z)$, the SU(2) currents $J^i(z)$, $(i=1, 2, 3)$ and the supercurrents $G^a(z)$, $\hat{G}^a(z)$ $(a = 1, 2)$. These currents are expressed in terms of the free fields. For example, the stress tensor is given by

$$T(z) = -\frac{1}{2} \lim_{w \to z} \sum_{a=1}^{2N} \sum_{l=1}^{N} \left( \partial X^a_l(z) \partial X^{a\dagger}_l(w) + \partial \phi^a_l(z) \partial \phi^{a\dagger}_l(w) + \frac{6}{(z-w)^2} \right). \quad (6)$$

For the $J^3$ current of the SU(2) algebra defining the conserved R-charge we have

$$J^3(z) = \frac{i}{2} \sum_{l=1}^{N} (\partial \phi^1_l + \partial \phi^2_l)(z). \quad (7)$$

In the orbifold model, one has to consider distinct sectors: Ramond, Neveu-Schwarz (NS) and twisted, representing different boundary conditions for the constituent free fermions and bosons. The ground state twisted Ramond fields (those of dimension $c/24$) have a simple realization in terms of the free fields,

$$R^+_n = \frac{1}{\sqrt{nN!(N-n)!}} \sum_{h \in S_N} e^{\pm \frac{i}{2} \sum_{l=1}^{N} (\phi^1_l + \phi^2_l)} e_{h^{-1}(1\ldots n)h}; \quad (8)$$
which is an explicitly $S_N$-invariant construction, normalized to one by the combinatorial overall factor. Its dimension and R-charge are

$$\Delta^\pm = \frac{1}{4} n, \quad j^3 = \pm \frac{1}{2}. \quad (9)$$

By construction, these fields are doublets of the SU(2) R-symmetry algebra and singlets of the global SU(2)$_2$ one. In this paper we will be actually interested in composite fields made of products of two of these twisted Ramond fields. Composite Ramond fields play a role in the microstate description of the near-horizon and the interior of certain five-dimensional extremal supersymmetric black holes (or black rings) which can be realized semi-classically as $\text{AdS}_3 \times S^3 \times T^4$ solutions of type IIB supergravity. Within the AdS/CFT correspondence, they permit a particular dual holographic description in terms of a definite SCFT with (large) central charge $c = 6N$ [27, 28], cf. also [2].

More precisely, we will consider two types of composite Ramond fields,

$$R^\pm_{m_1}, R^\mp_{m_2}(z, \bar{z}), \quad R^\mp_{m_1}, R^\pm_{m_2}(z, \bar{z}), \quad (10)$$

which are, respectively, charged and neutral under R-symmetry. Under the action of the “isospin” SU(2) algebra, these products of $j = 1/2$ representations form a triplet with $j_3 = \{0, \pm 1\}$, given by

$$R^+_m R^-_m, \quad \frac{1}{\sqrt{2}}(R^+_m R^-_m + R^-_m R^+_m), \quad R^+_m R^+_m \quad (11)$$

and a singlet $\frac{1}{\sqrt{2}}(R^+_m R^-_m - R^-_m R^+_m)$ as well. A composite operator must be constructed with cycles $(m_1)$ and $(m_2)$ without coincident elements, such that they constitute a conjugacy class of $S_N$ (see discussion in [29]). Thus we define the composite twist

$$\sigma_{m_1, m_2} : = \frac{1}{\mathcal{C}_{m_1 m_2}} \sum_{h \in S_N} \mathcal{C}(1, \ldots, m_1) h \mathcal{C}(m_1 + 1, \ldots, m_1 + m_2) h, \quad (12)$$

and the composite Ramond operators (10) are defined in a similar way.

We can give a counting argument (generalizing the one given in [30] to find (4)) to obtain the normalization factor $\mathcal{C}_{m_1 m_2}$ which will be important later. Consider the two-point function

$$\langle \sigma_{m_1, m_2}; (0) : \mathcal{C}_{m_1, m_2} \mathcal{C}_{m'_1, m'_2}; (z) \rangle$$

$$= \frac{1}{\mathcal{C}_{m_1, m_2} \mathcal{C}_{m'_1, m'_2}} \sum_{h \in S_N} \sum_{k \in S_N} \langle \sigma_{h(1, \ldots, m_1) k h} \sigma_{h(1, \ldots, m_1 + m_2) k} (0) \rangle$$

$$\times \langle \mathcal{C}_{m_1, m_2} \mathcal{C}_{m'_1, m'_2}; (z) \rangle \quad (13)$$

We must have $m_1 = m'_1$ and $m_2 = m'_2$ because the cycles in each term have to compose to the identity, $(m_1)(m_2) = 1$, otherwise the function vanishes. The (non-vanishing) terms in the r.h.s all give a contribution of $1/|z|^{\Delta_{m_1 + m_2}}$. It remains to see how many of such terms there are. The permutaitons in each of the terms have $s = m_1 + m_2$ distinct elements undergoing non-trivial permutations. One can choose $s$ elements among $N$ in $N!/s!$(!) different ways. Also, for each cycle $(m_r)$ appearing in the twists there are $N - m_r$ implicit elements undergoing trivial permutations, and which can be arranged in any way; there are $(N - m_r)!$
ways of arranging $N - m_r$ objects. We can also make cyclic rotations of the $m_r$
elements of the cycle. Hence each twist $\sigma^{(m_r)}$ appearing in (13) carries a factor of $m_r(N - m_r)!$. We have thus found

$$\frac{N!}{(N - m_1 - m_2)!} \times [m_1(N - m_1)!]^2 \times [m_2(N - m_2)!]^2$$

contributions, which should equal $g_{m_1,m_2}^2$ if the two-point function of the composite twists is normalized. So, finally,

$$\frac{1}{g_{m_1,m_2}} = \frac{1}{m_1(N - m_1)!m_2(N - m_2)!} \sqrt{\frac{(N - m_1 - m_2)!}{N!}}$$

(14)
determines the proper normalization of the composite Ramond fields as well.

3. Correlation functions of composite Ramond fields

We are interested in the two- and three-point functions of composite Ramond fields in the marginally perturbed theory,

$$S_{\text{def}}(\lambda) = S_{\text{orb}} + \lambda \int d^2u O_2^{(\text{int})}(u, \bar{u})$$

(15)

where $\lambda$ is a dimensionless coupling constant, and the deformation operator $O_2^{(\text{int})}$ is an $S_N$-invariant SU(2) scalar, preserving $\mathcal{N} = (4, 4)$ supersymmetry. Its explicit form

$$O_2^{(\text{int})}(u, \bar{u}) = \left( \hat{G}_{1/2}^1 G_{-1/2}^1 - G_{1/2}^1 \hat{G}_{-1/2}^1 \right) O_2(u, \bar{u}) + c.c.$$  

(16)
is a sum of descendants of the twist-two chiral field $O_2$ with conformal dimensions $\Delta_2 + \Delta_2 = 1$ and SU(2) charges $j_3 = 1/2 = j_{3}$. See e.g. [13].

The conformal dimension of the composite operator $R_{m_1}^{\pm} R_{m_2}^{\pm}(z, \bar{z})$, at the free orbifold point, is given by the sum of the dimensions of its constituents, i.e.

$$\left( \frac{m_1 + m_2}{4}, \frac{m_1 + m_2}{4} \right).$$

The first nontrivial correction to the two-point function

$$\langle R_{m_1}^- R_{m_2}^- (\infty) R_{m_2}^+ R_{m_1}^+ (0) \rangle_{\lambda}$$

appears at second order in perturbation theory,

$$\frac{\lambda^2}{2} \int d^2z_2 d^2z_3 \langle R_{m_1}^- R_{m_2}^- (z_1, \bar{z}_1) O_2^{(\text{int})}(z_2, \bar{z}_2) O_2^{(\text{int})}(z_3, \bar{z}_3) R_{m_2}^+ R_{m_1}^+ (z_4, \bar{z}_4) \rangle.$$

(18)

Conformal invariance fixes the form of the four-point functions up to an arbitrary function $G(u, \bar{u}) = G(u)\tilde{G}(\bar{u})$ of the anharmonic ratio $u = z_{12}z_{34}/z_{13}z_{24}$ and its complex conjugate $\bar{u}$,

$$\langle R_{m_1}^- R_{m_2}^- (z_1, \bar{z}_1) O_2^{(\text{int})}(z_2, \bar{z}_2) O_2^{(\text{int})}(z_3, \bar{z}_3) R_{m_2}^+ R_{m_1}^+ (z_4, \bar{z}_4) \rangle = \frac{|z_{14}|^{m_2-m_1}}{|z_{13}z_{24}|^4} G(u, \bar{u}).$$

(19)

One can further make a suitable change of variables and factorize the integral. As a result we get for the first nontrivial correction to the two-point function,

$$\frac{\lambda^2\pi}{|z_{14}|^{m_1+m_2}} \log \frac{\Lambda}{|z_{14}|} \int d^2u G(u, \bar{u}),$$

(20)
where $\Lambda$ is an ultraviolet cutoff, and we have used $\text{SL}(2, \mathbb{C})$ invariance to fix three points in the correlation function, so that

$$G(u, \bar{u}) = \langle R_{m_1}^{-} R_{m_2}^{-} (\infty) O^{(\text{int})}_2 (1) O^{(\text{int})}_2 (u, \bar{u}) R_{m_2}^{+} R_{m_1}^{+} (0) \rangle. \quad (21)$$

### 3.1. Connected and disconnected functions

As defined in (8), the $S_N$-invariant function (21) is a sum over the group orbits of the various twists,

$$G(u, \bar{u}) = \sum_{S_N} \langle R_{h_\infty}^{-} R_{h_1}^{-} R_{h_u}^{-} (\infty) O^{(\text{int})}_{h_1} (2) h_2 (1) O^{(\text{int})}_{h_u} (u, \bar{u}) R_{h_0}^{+} R_{h_0}^{+} (0) \rangle \quad (22)$$

where the sum is over every $h_\infty, h_1, h_u, h_0 \in S_N$. Each individual term in this sum corresponds to one of the possible individual permutations resulting from the composition of the six permutation cycles $(n_i)$, ordered by (the radial order of) the points $z_i$ where the twists $\sigma(n_i)(z_i)$ are located. Following [30], we will denote the permutation of the twist field $\sigma(n_1)(z_1)$ by the cycle $(n_1)z_1$, labeled by a position index. The cycles in Eq. (22) are accordingly denoted as $(m_1)_{\infty}(m_2)_{\infty}(2) (1) (2) (m_2)_0 (m_1)_0$. The ordering of the labels, $\infty > 1 > u > 0$, is crucial, since $S_N$ is non-abelian. Every permutation contributing to the sum (22) must satisfy the condition

$$(m_1)_{\infty}(m_2)_{\infty}(2) (1) (2) (m_2)_0 (m_1)_0 = 1, \quad (23)$$

otherwise the correlation function vanishes. Some of the correlators in the r.h.s. of Eq. (22) factorize in different ways, and some will be completely connected.

A term in the sum (22) will be completely connected when one of the elements of $(2)_1 = (k, \ell)$, say $k$, overlaps with $(m_1)_{\infty}$, and the other element, $\ell$, overlaps with $(m_2)_{\infty}$. Because of (23), a similar overlap will happen for $(2)_u, (m_1)_0$ and $(m_2)_0$.

In this case, there is always a number

$$s_c = m_1 + m_2 \quad (24)$$

of different elements entering the permutation $(m_1)_{\infty}(m_2)_{\infty}(2) (1) (2) (m_2)_0 (m_1)_0$.

A four-point function in the sum (22) can factorize in three qualitatively different ways which do not vanish. Factorization depends on the existence of cycles commuting with all the others, which is regulated by the different possibilities of overlapping the elements of the cycles $(2)_1$ and $(2)_u$ with the other cycles, since $(m_1)$ and $(m_2)$ are always disconnected. The first possibility is that $(2)_1$ and $(2)_u$ commute with every Ramond-operator cycles. Then the four-point function splits into

$$\langle O^{(\text{int})}_2 (1) O^{(\text{int})}_2 (u, \bar{u}) \rangle \langle R_{(m_1)}^{(\infty)} R_{(m_2)}^{-} (\infty) R_{(m_2)}^{+} R_{(m_1)}^{+} (0) \rangle \quad (25)$$

with $(m_1)_{\infty}(m_2)_{\infty}(m_1)_0 (m_2)_0 = 1$. In this case, the integral (18) is over the “vacuum bubbles” $\langle O^{(\text{int})}_2 (1) O^{(\text{int})}_2 (u, \bar{u}) \rangle$, which diverge. These divergences are natural in perturbation theory, and can be eliminated by proper normalization of the correlation functions,

$$\langle R_{m_1}^{-} R_{m_2}^{-} (\infty) O^{(\text{int})}_2 (1) O^{(\text{int})}_2 (u, \bar{u}) R_{m_2}^{+} R_{m_1}^{+} (0) \rangle_{\lambda} \quad (26)$$
We will assume this normalization from now on but omit the \( \langle 1 \rangle_\lambda \), so terms like (26) are henceforth excluded from (21).

The other two possibilities are of a very different nature. If the pairs of cycles with lengths \( m_1 \) or \( m_2 \) commute with the other cycles, than we have the factorizations

\[
\begin{align}
\langle R_{(m_2)}^-(\infty) R_{(m_2)-1}^+(0) \rangle & \langle R_{(m_1)}^-(\infty) O_{(2)}^{(\text{int})} (1) O_{(2)}^{(\text{int})} (u, \bar{u}) R_{(m_1)}^+(0) \rangle, \quad (27a) \\
\langle R_{(m_1)}^-(\infty) R_{(m_1)-1}^+(0) \rangle & \langle R_{(m_2)}^-(\infty) O_{(2)}^{(\text{int})} (1) O_{(2)}^{(\text{int})} (u, \bar{u}) R_{(m_2)}^+(0) \rangle, \quad (27b)
\end{align}
\]

where \( (m_1)_\infty (2)_1 (2)_u (m_1)_0 = 1 \) in (27a), and \( (m_2)_\infty (2)_1 (2)_u (m_2)_\infty = 1 \) in (27b), so as to satisfy (23). Note that, if a term in (27) factorizes further, it has the form (25) and is canceled by (26). Denote by \( k, \ell \) the elements of \( (2)_1 = (k, \ell)_1 \), then look at the permutation \( (m_1)_\infty (m_2)_\infty (2)_1 \). There are two qualitatively different ways in which the factorizations (27) happen, as follows.

1) Only one of the elements of \( (2)_1 \), say \( k \), overlaps with \( (m_2)_\infty \), while the other element, \( \ell \), does not overlap with any of the \( (m_1) \) nor the \( (m_2) \) cycles. This gives a factorization (27b).

A factorization (27a) happens when one of the elements of \( (2)_1 \), say \( k \), overlaps with \( (m_1)_\infty \), and the other element, \( \ell \), does not overlap with any of the \( (m_1) \) nor the \( (m_2) \) cycles. In any case, there is always a number

\[ s = m_1 + m_2 + 1 \quad (28) \]

of distinct elements entering the permutation (23).

2) Both \( k \) and \( \ell \) overlap with \( (m_2)_\infty \) or, instead, both overlap with \( (m_1)_\infty \). These possibilities are mutually exclusive, since \( (m_1)_\infty \) and \( (m_2)_\infty \) do not share elements.

Concerning the number of different elements appearing in the permutation, in Case 2 there are two different situations. For simplicity, let us drop indices and call the “non-factorized” permutation simply \( (m)_\infty (2)_1 (2)_u (m)_0 \). We can use \( S_N \) symmetry to fix \( (m)_\infty = (1, 2, 3, \cdots, m) \) and \( (2)_1 = (1, \ell) \).

2a) In the generic case, we have \( \ell \neq 2 \) and \( \ell \neq m \), i.e. \( \ell \). Then the permutation splits into \( (1, 2, \cdots, \ell, \cdots, m)_\infty (1, \ell)_1 = (1, \cdots, \ell - 1) (\ell, \cdots, m) \). Hence there is a number \( m \) of distinct elements which should also appear in \( (2)_u (m)_0 \) so that \( (m)_\infty (2)_1 (2)_u (m)_0 = 1 \). Counting these elements together with the other “factorized” ones, we find

\[ s = m_1 + m_2 \quad (29) \]

distinct elements entering the r.h.s. of (23).

2b) However, if \( \ell = 2 \) or \( \ell = m \), then the permutation \( (1, 2, 3, \cdots, m)_\infty (1, \ell)_1 \) collapses to a cycle with length \( m - 1 \). For example, if \( \ell = 1 \), then

\[ (1, 2, 3, \cdots, m)_\infty (1, m)_1 = (1, 2, \cdots, m - 1). \]

Now the permutation \( (2)_u (m)_0 \), which must equal the inverse cycle, can accommodate one more distinct element, which is not in \( \{1, 2, \cdots, m\} \), because

\[ (r, 1)_u (r, m - 1, \cdots, 2, 1)_0 = (m - 1, \cdots, 2, 1) \]

because \[ (r, 1)_u (r, m - 1, \cdots, 2, 1)_0 = (m - 1, \cdots, 2, 1) \]
for any \( r \in [1, N] \), not only for \( r = m \). There are, therefore, \( m + 1 \) elements entering the “non-factorized” permutation, hence \( s = m_1 + m_2 + 1 \) distinct elements entering the permutation (23), the same number (28).

The sum over orbits preserves the cycle structure of factorized functions, hence the function (22), normalized as (26), splits into three terms:

\[
G(u, \bar{u}) = G_c(u, \bar{u}) + G_{m_1}(u, \bar{u}) + G_{m_2}(u, \bar{u}),
\]

where

\[
G_c(u, \bar{u}) = \langle R_{m_1}^- R_{m_2}^- (\infty) O_2^{(\text{int})}(1) O_2^{(\text{int})} (u, \bar{u}) R_{m_1}^+ R_{m_2}^+(0) \rangle_{\text{conn}}
\]

(31)

\[
G_{m_1}(u, \bar{u}) = \langle R_{m_1}^- (\infty) O_2^{(\text{int})}(1) O_2^{(\text{int})} (u, \bar{u}) R_{m_1}^+(0) \rangle
\]

(32)

\[
G_{m_2}(u, \bar{u}) = \langle R_{m_2}^- (\infty) O_2^{(\text{int})}(1) O_2^{(\text{int})} (u, \bar{u}) R_{m_2}^+(0) \rangle
\]

(33)

(Note the twist indices without parenthesis, indicating that each of the correlators are (multiple) sums over orbits.) We emphasize that all correlators are normalized as (26), and the ‘conn’ in (31) indicates that there is no factorization of the composite operators. The Ramond two-point functions in (27) have disappeared because of the normalization (8) — after summing over orbits, the factored two-point functions are \( \langle (R_{m_1}^\pm)^1 R_{m_2}^\pm \rangle = 1 \). The functions \( G_{m_1} \) and \( G_{m_2} \) are four-point functions of non-composite operators, and have been considered in [21]. The integral (20) over these terms does not vanish, hence renormalization of the Ramond fields is required to cancel the logarithmic divergence in Eq. (20). We will return to this point later. For the remaining of this section, we focus on function \( G_c \).

3.2. Large-\(N\) limit. We are interested in the approximation where \( N \gg 1 \). To find the \( N \)-dependence of the correlation functions, we can organize the sum (31) according to the conjugacy classes of the symmetric group. \( S_N \)-invariance implies that every term belonging to the same equivalence class \( \alpha \) must give the same result, hence the connected function \( G_c \) is given by

\[
G_c(u, \bar{u}) = \sum_{\alpha} C_\alpha(N) \langle R_{g_\infty}^- R_{g_\infty}^- (\infty) O_{g_1}^{(\text{int})}(1) O_{g_2}^{(\text{int})} (u, \bar{u}) R_{g_0}^+ R_{g_0}^+(0) \rangle_{\text{conn}},
\]

(34)

where the permutation given by the ordered cycle composition \( g_\infty^\alpha g_\infty^\alpha g_\infty g_\infty g_0^\alpha g_0^\alpha g_0^\alpha \) is any representative of the class \( \alpha \), and the class-dependent numerical factor \( C_\alpha(N) \) is given by the number of elements in \( \alpha \) (and some normalization factors). As shown in [30], it is very convenient to further organize the sum (34) by separating, inside the equivalence classes \( \alpha \), permutations \( g_\infty^\alpha g_\infty^\alpha g_\infty^\alpha g_\infty^\alpha g_0^\alpha g_0^\alpha g_0^\alpha \) with a definite number \( s \) of distinct ‘active’ elements, i.e. elements which undergo non-trivial permutations.\(^4\)

Then (we omit the ‘conn’ hereafter)

\[
G_c(u, \bar{u}) = \sum_s \sum_{\alpha_s} C_{s, \alpha_s}(N) \langle R_{g_\infty}^- R_{g_\infty}^- (\infty) O_{g_1}^{(\text{int})}(1) O_{g_2}^{(\text{int})} (u, \bar{u}) R_{g_0}^+ R_{g_0}^+(0) \rangle,
\]

(35)

where now \( \alpha_s \) is the set of permutations belonging to class \( \alpha \) and involving \( s \) different elements. Let us follow [30] to determine the numerical coefficients \( C_{s, \alpha_s}(N) \). First we need to find the number of terms equivalent to

\[
\langle R_{g_\infty}^- R_{g_\infty}^- (\infty) O_{g_1}^{(\text{int})}(1) O_{g_2}^{(\text{int})} (u, \bar{u}) R_{g_0}^+ R_{g_0}^+(0) \rangle.
\]

(36)

\(^4\)For example, the permutation (259)(3)(14) has five ‘active’ elements: 1, 2, 4, 5 and 9.
Then we need to account for any $N$-dependence in the term itself. We have the following contributions:

1. There are $N!/(N - s)!$ ways to pick $s$ different elements from $N$ to be the specific active numbers appearing inside the permutation

$$g^{α_s} g^{α_s} g^{α_s} g^{α_s} g^{α_s} g^{α_s} = \prod_{r=1}^{6} (n_r)$$

where $n_1 = m_1 = n_6$

$$n_2 = m_2 = n_5$$

$$n_3 = 2 = n_4$$

(37)

2. The permutation (37) is only fixed up to cyclic relabeling of the individual cycles. There are $n_r$ ways to write the cycle $(n_r)$. Hence for each of the choices in Item 1 there are $\prod_r n_r$ terms.

3. The term (36) includes two non-$S_N$-invariant twist operators $σ_{(2)}$, which carry a normalization factor of $1/\mathcal{F}_2$ because of the normalization used in the $S_N$-invariant combination (2), and two non-$S_N$-invariant composite twist operators $:σ_{(m_1)}σ_{(m_2)}: \phantom{\Sigma}$, which carry a normalization factor of $1/\mathcal{C}_{m_1,m_2}$ because of the normalization used in (12).

A twist operator $σ_{(n_r)}$ (including the ones inside the composite operators) also has an implicit part made of trivial one-cycles with the other $N - n_r$ elements that are not “active” in the cycle, and these trivial permutations/elements can be arranged in whatever order; there $(N - n_r)!$ different arrangements.

In conclusion, each non-composite twist field $σ_{(n_r)}$ appearing in the term carries a factor $(N - n_r)!/\mathcal{F}_n$, and every composite twist $:σ_{m_1}σ_{m_2}:$ carries a factor $(N - m_1)!(N - m_2)!/\mathcal{C}_{m_1,m_2}$. We get an overall factor of

$$\left( \frac{(N - 2)!}{\mathcal{F}_2} \right)^2 \left( \frac{(N - m_1)!(N - m_2)!}{\mathcal{C}_{m_1,m_2}} \right)^2$$

(38)

The conclusion from Items 1 and 2 is that there are $(N! \prod_{r=1}^{6} n_r)/(N - s)!$ terms, and from (3) each term has a factor (38). This gives

$$C_{s,α_s}(N) = \frac{N!}{(N - s)!} \prod_{r=1}^{4} n_r \prod_{s=3}^{4} (N - n_s) \left( \frac{(N - m_1)!(N - m_2)!}{\mathcal{C}_{m_1,m_2}} \right)^2$$

(39)

We are refraining from inserting $n_3 = n_4 = 2$ to better keep track of the contributions from each term. From (39) we can see that $C_{s,α_s}(N)$ does not depend on the class $α_s$, only on the number of active elements $s$ and on the length of the cycles entering the $S_N$-invariant correlation function. Thus the $C_{s,α_s}(N)$ can be taken outside the sum over $α_s$ in (35). To find the leading terms of the sum over $s$ in the large-$N$ limit, we use Strirling’s formula for $n! \approx \sqrt{2πn}(n/e)^n$,

$$C_{s,α_s}(N) = N^{s - \frac{1}{2} \sum_{r=1}^{s} n_r} \left( \frac{(2m_1 m_2)^2}{\sqrt{4m_1 m_2}} + O(1/N) \right)$$

(40)

Thus the leading terms are those with smaller $s$. Compare (39) and (40) with the formula for the $q$-point function involving only non-composite operators of twists
\[ C_{s,\alpha s}(N) = \frac{N!}{(N-s)!} \prod_{s=1}^{q} \sqrt{n_s \frac{(N-n_s)!}{(N-n_s)!}} = N^{s-\frac{1}{2} \sum s n_s} \left( \prod_{s=1}^{q} \sqrt{n_s} + O(1/N) \right). \]

The exponent in (40) can be recast into another interesting form using the Riemann-Hurwitz formula

\[ g = \frac{1}{2} \sum_{p=1}^{q} (n_p - 1) - s + 1 = \frac{1}{2} \sum_{p=1}^{q} n_p - \frac{1}{2} q - s + 1 \] (41)

which gives the genus \( g \) of a surface \( \Sigma \) which is a ramified covering of the sphere possessing \( s \) sheets and with \( q \) ramification points with ramification orders \( n_p - 1 \) \([22,30]\). Using the surface \( \Sigma \) as a ‘covering surface’ of the base sphere is the standard way of calculating correlation functions in the orbifold theory \([31]\), as we will do later. The sum appearing in the exponent of \( N^{s-\frac{1}{2} \sum r n_r} \),

\[ \frac{1}{2} \sum_{r=1}^{6} n_r = \frac{1}{2} [2(m_1 + m_2 - 1) + 2(2 - 1)] - \frac{1}{2} q \] (42)

is the same sum appearing in the r.h.s. of (41) for \( q = 4 \). This corresponds to the four ramification points \( z = 1, z = 0, z = \infty \), with ramification orders \( 2 - 1 \) and \( m_1 + m_2 - 1 \), respectively, appearing in the four-point function (34). In conclusion, to leading order, \( C_{s,\alpha s}(N) \sim N^{-g-1} \), which is the same behavior appearing in \( q \)-point functions of non-composite operators \([30]\), and we can write (40) as

\[ C_{s,\alpha s}(N) \sim N^{-g-1}. \] (43)

The number \( s \) of sheets of the covering surface \( \Sigma \) is equal to the number of distinct elements entering non-trivially in the permutations twists of the correlation function. For our four-point functions, Eq.(41) relates it to the genus as

\[ g = m_1 + m_2 + 1 - s. \] (44)

Now we see that terms of type (1) and of type (2b), which have \( s \) given in (28), are described by surfaces of \( g = 0 \), for which \( C_{s,\alpha s}(N) \sim N^{-1} \). The terms of type (2a), which have (29), and the fully connected functions, which have (24), both can be described by surfaces of \( g = 1 \), for which \( C_{s,\alpha s}(N) \sim N^{-2} \).

### 3.3. Covering maps

The most effective method for calculating multi-point functions in the orbifold theory is the ‘covering surface technique’ of Lunin and Mathur \([31]\). A covering surface \( \Sigma \) of the base sphere \( S^2_{\text{base}} \) where \( G(u, \bar{u}) \) is defined, is given by a map \( z(t) \), with \( t \in \Sigma \) and \( z \in S^2_{\text{base}}, \) and with multiple inverses \( t_a(z) \) corresponding to the branches introduced by the twist operators in \( G(u, \bar{u}) \). The ramification points replace the twist operators, so on \( \Sigma \), and there is only one single untwisted copy of the fields \( X^i(t), X^{\dagger i}(t), \phi^i(t) \).
Here we want to construct the covering map with the correct monodromies for connected function $G_c$ in (31). It must be a function $z(t)$ such that

\begin{align}
  z(t) &\approx b_1 t^{m_1} (t - t_0)^{m_2} & \text{as } z \to 0 \quad (45a) \\
  z(t) &\approx 1 + b_2 (t - t_1)^2 & \text{as } z \to 1 \quad (45b) \\
  z(t) &\approx u + b_3 (t - x)^2 & \text{as } z \to u \quad (45c) \\
  z(t) &\approx b_4 t^{m_1} & \text{as } z \to \infty \quad (45d)
\end{align}

The powers impose the correct monodromies of the inverse maps $t_a(z)$ around the position of the twists in $z = \{0, 1, u, \infty\} \in S^2_{\text{base}}$. Note how each part of the composite operator $R^{+}_{m_1} R^{+}_{m_2}(0)$ has been lifted to a different point on $\Sigma$, viz. $R^{+}_{m_1}(0)$ goes to $t = 0$ and $R^{+}_{m_2}(0)$ goes to $t = t_0$. Because of the branching points, $\Sigma$ will have multiple sheets when covering $S^2_{\text{base}}$. The number $s$ of sheets, equal to the number of distinct elements entering the permutations in twists, is given by (24),

\[ s_c = m_1 + m_2. \quad (46) \]

Following [22], we now make the ansatz $z(t) = f_1(t)/f_2(t)$, where $f_1$ and $f_2$ are polynomials of degrees $d_1, d_2 \in \mathbb{N}$. From condition (45d), we know that $d_1 - d_2 = m_1$, hence $d_1 > d_2$. On the other hand, the larger degree $d_1$ is equal to the number of inverse maps $t_a(z)$, hence to the number of sheets $s_c$ of the covering surface given in (46). We have thus found that $d_1 = m_1 + m_2$. To be consistent with (45a), we thus must have $f_1 = A t^{m_1} (t - t_0)^{m_2}$. Also $d_2 = m_1 - s_c = m_2$, so $f_2 = B (t - t_\infty)^{m_2}$. Adjusting the constants $A$ and $B$ so that, as required by (45b), $z(t_1) = 1$, we thus have

\[ z(t) = \left( \frac{t}{t_1} \right)^{m_1} \left( \frac{t - t_0}{t_1 - t_0} \right)^{m_2} \left( \frac{t_1 - t_\infty}{t - t_\infty} \right)^{m_2}. \quad (47) \]

Note how the second composite operator $R^{-}_{m_1} R^{-}_{m_2}(\infty)$ has also been lifted to different points on $\Sigma$, viz. $R^{-}_{m_1}(\infty)$ goes to $t = \infty$ and $R^{-}_{m_2}(\infty)$ goes to $t = t_\infty$.

Imposing that the map (47) locally satisfies the conditions (45b) and (45c) near the points $t_1$ and $x$ implies that

\[ \frac{1}{z} \left. \frac{dz}{dt} \right|_{(t_1, x)} = \frac{m_1 t_1^2 + [(m_2 - m_1)t_0 - (m_1 + m_2)t_\infty] t + m_1 t_0 t_\infty}{t(t - t_0)(t - t_\infty)} = 0. \quad (48) \]

In other words, $x$ and $t_1$ are the roots of the quadratic equation in the numerator. Using the relation between the coefficients and of this equation and its two roots, we find two relations between the parameters $t_1, t_0, t_\infty$ and $x$. We have the choice of fixing one of the $t_1, t_0, t_\infty$, and the two relations fix the other two as a function of $x$, which is the image of the “free” point $u$. We choose

\[ t_0 = x - 1, \quad t_1 = \frac{(x - 1)(m_1 + m_2 x - m_2)}{m_1 + m_2 x}, \quad t_\infty = x - \frac{m_2 x}{m_2 x + m_1} \quad (49) \]

leading to the map $u(x) = z(x)$

\[ u(x) = \left( \frac{x + \frac{m_1}{m_2}}{x - 1} \right)^{m_1 + m_2} \left( \frac{x - 1 + \frac{m_1}{m_2}}{x} \right)^{m_2 - m_1}. \quad (50) \]

\[ ^5 \text{See, e.g. the discussion in [32].} \]
The form of a ratio of polynomials is similar to the ‘Arutyunov-Frolov map’ [24]. Note that when \( m_1 = m_2 \), the map simplifies considerably:

\[
u(x) = \left(\frac{x + 1}{x - 1}\right)^{2m} \quad (m_1 = m_2 = m).
\] (51)

There is an evident asymmetry in the maps (47) and (50) when we exchange \( m_1 \) and \( m_2 \). This is because we have constructed \( G \) the stress-tensor method. Here and \( m \) obviously related by the change

\[
G_{m_1} \text{ mapped to } 0 \text{ while } R_{m_2}^+(0) \text{ is mapped to } t = 0, \text{ as already stated. We can choose instead to map } R_{m_1}^+(0) \text{ to } t = 0 \text{ and } R_{m_2}^-(0) \text{ to } t = 0, \text{ and following the same reasoning above (of imposing the corresponding monodromies, etc.) we find new maps equivalent to making } m_1 \leftrightarrow m_2 \text{ in the previous ones. Hence both maps}
\]

\[
z_{21}(t) = \left(\frac{t}{t(1)}\right)^{m_1} \left(\frac{t - t_0^{(1)}}{t_1 - t_0^{(1)}}\right)^{m_2} \left(\frac{t_1 - t_0^{(1)}}{t_1 - t_0^{(1)}}\right)^{m_2} (52a)
\]

\[
z_{12}(t) = \left(\frac{t}{t(2)}\right)^{m_2} \left(\frac{t - t_0^{(2)}}{t_1 - t_0^{(2)}}\right)^{m_1} \left(\frac{t_1 - t_0^{(2)}}{t_1 - t_0^{(2)}}\right)^{m_1} (52b)
\]

correspond to covering surfaces of genus zero and with the correct monodromies. Here \( t_1^{(1)}, t_0^{(1)}, t_\infty^{(1)} \) are the functions of \( x \) given by (49), and \( t_1^{(2)}, t_0^{(2)}, t_\infty^{(2)} \) are these same function after the change \( m_1 \leftrightarrow m_2 \). By the same token, we have two maps

\[
u_{21}(x) = \left(\frac{x + m_1}{x - 1}\right)^{m_1 + m_2} \left(\frac{x - 1 + m_1}{x}\right)^{m_2 - m_1} (53a)
\]

\[
u_{12}(x) = \left(\frac{x + m_1}{x - 1}\right)^{m_2 + m_1} \left(\frac{x - 1 + m_1}{x}\right)^{m_1 - m_2} (53b)
\]

The covering maps will be used below to compute the correlation function \( G_c(u, \bar{v}) \). Using each of the maps (47) or (52b) gives us two different functions which are obviously related by the change \( m_1 \leftrightarrow m_2 \).

3.4. The stress-tensor method. We now use the covering maps to compute \( G_c(u, \bar{v}) \), following the ‘stress tensor method’ [23–25]. The Ward identities for the stress-energy tensor give

\[
F_{m_1 m_2}(z, u) = \frac{\langle T(z) R_{m_1}^- R_{m_2}^- (\infty) O_{2}^{(\text{int})}(1) O_{2}^{(\text{int})}(u) R_{m_2}^+ R_{m_1}^+ (0) \rangle}{\langle R_{m_1}^- R_{m_2}^- (\infty) O_{2}^{(\text{int})}(1) O_{2}^{(\text{int})}(u) R_{m_2}^+ R_{m_1}^+ (0) \rangle}
= \frac{1}{(z - u)^2} + \frac{H_{m_1 m_2}(u)}{z - u} + \cdots
\] (54)

If one is able to obtain independently the function \( H_{m_1 m_2}(u) \), then (54) leads to a simple differential equation for our function \( G_{m_1 m_2}(u) \):

\[
\partial_u \log G_{m_1 m_2}(u) = H_{m_1 m_2}(u).
\] (55)

Of course, this determines just the holomorphic part of

\[
G_{m_1 m_2}(u, \bar{v}) = G_{m_1 m_2}(u) \tilde{G}_{m_1 m_2}(\bar{v}),
\] (56)
with the anti-holomorphic part entering as a multiplicative “integration constant” in (55), but repeating an analogous procedure with the anti-holomorphic stress-tensor $T(z)$ we can find $\tilde{G}_{m_1,m_2}(u)$, instead of $G_{m_1,m_2}(u)$. It is evident from the non-chiral structure of the operators that $G_{m_1,m_2}(u)$ and $\tilde{G}_{m_1,m_2}(\bar{u})$ are complex conjugates of each other. The very same procedure can be used to find $G_{m_2m_1}(u,\bar{u})$, and it is actually clear that it suffices to make $m_1 \leftrightarrow m_2$ at every step of the calculation.

The interaction operator (16), which involves NS modes of the supercharges, is given by contour integrals in the base sphere. However, on the covering surface this integrals can be evaluated trivially, and one is able to express the operator simply in terms of the orbifold fields, without any integrals. The expression obtained for $O^{(\text{int})}(x,\bar{x})$ on the covering is just a sum terms containing $\partial X^i(t)$ or $\partial X^\dagger(t)$ and exponentials $\exp[i \frac{1}{2}(\phi^1(t) + \phi^2(t))]$.\footnote{See for example §2.3 of [13] or [32].} It is therefore easier to first compute the function

$$
F_{\text{cover}}(t, x) = \frac{\langle T(t) R^-(\infty) R^- (t_{\infty}) O^{(\text{int})}(t_1) O^{(\text{int})}(x) R^+(t_0) R^+(0) \rangle}{\langle R^-(\infty) R^- (t_{\infty}) (O^{(\text{int})}(t_1) O^{(\text{int})}(x) R^+(t_0) R^+(0) \rangle}\tag{57}
$$
on the covering surface. The absence of indices $m_1, m_2, 2$ is because the twists are trivialized on $\Sigma$, $\sigma(n) \mapsto 1$, and also $\sum I \phi^I \mapsto n \phi^a$. Thus, for example, from (8), we have $R^\pm(t) = \exp(\pm \frac{1}{2} [\phi^1(t) - \phi^2(t)])$. Taking into account the explicit form of the stress-tensor (6), we find

$$
F_{\text{cover}}(t, x) = \frac{(t_1 - x)^2}{(t - t_1)^2(t - x)^2} - \frac{1}{8} \left[ \left( \frac{1}{t - t_\infty} - \frac{1}{t - t_0} - \frac{1}{t} \right)^2 + \left( \frac{1}{t - t_1} - \frac{1}{t - x} \right)^2 \right].\tag{58}
$$

The first term in the r.h.s. is the contribution from the bosons $X^i, X^\dagger$; the terms in the second line are the fermionic contributions, with the first squared expression corresponding to the Ramond operators, and the second one to the (bosonized) fermions inside $O^{(\text{int})}$.

We now must map this result from $t$ to $z$. It is this mapping that restores the twisted structure of the operators which had been erased on the covering. Inverting (52a) we find the function $G_{m_2m_1}$, while using the map (52b) will give us $G_{m_1m_2}$. Let us write in detail the latter. There are multiple maps $t_a(z_{12}), a = 1, \ldots, 2m_1$ obtained from inverting (52a); corresponding to the 2$m_1$ maps satisfying the correct monodromies, which is the number of solutions of the equation $u_{12}(x) - u_* = 0$ for fixed $u_*$. (There are other 2$m_2$ maps obtained from inverting $u_{21}(x) - u_* = 0$ and contributing to $G_{m_2m_1}$.)

The function $F_{m_1m_2}(z, u)$ is the sum of the functions obtained from $F_{\text{cover}}(t, x)$ after the transformations $z \mapsto t_a(z_{12})$; each map $t_a$ gives the terms corresponding to terms in (22) in an entire conjugacy class; see [25, 26, 30]. For brevity, we will omit the indices of $z_{12}$. Among the inverse maps $t_a(z)$ of (52b), only two contribute as $z \rightarrow u$, the relevant limit for (54), because 2 is the number of sheets locally near the points where the $O_2^{(\text{int})}$ operators are inserted. We must sum over these two maps, which gives an overall factor of 2 because the functions turn out to give the same
result. Taking into account the anomalous transformation of $T$,

$$F_{mj}^m(z, u) = \{t, z\} + \left(\frac{dt}{dz}\right)^2 \left(\frac{(t_1 - x)^2}{(t(z) - t_1)^2 (t(z) - x)^2}\right) - \frac{1}{4} \left(\frac{dt}{dz}\right)^2 \left[ \frac{1}{t(z) - t_\infty} - \frac{1}{t(z) - t_0} - \frac{1}{t(z)} \right]^2 + \left(\frac{1}{t(z) - t_1} - \frac{1}{t(z) - x}\right)^2$$  \tag{59}

where $\{t, z\}$ is the Schwarzian derivative, $\{t, z\} = (\frac{dt}{dz})' - \frac{1}{2}(\frac{dt}{dz})^2$. Here $t(z)$ is any of the two inverse maps of (52b) near $z = u$. They can be found by expanding $z(t)$ in a power series, and inverting the series $t - x = \sum_{k=1}^{\infty} c_k (z - u)^{k/2}$. For the computation up to the singular terms which determine $H_{mj}^m(u)$, it is sufficient to obtain only the first three coefficients $c_k(x)$. See [24, 26, 32]. In order to find $H_{mj}^m(u)$ we have to further extract the coefficient in front of $(z - u)^{-1}$. The function $H_{mj}^m(u)$ is actually very complicated, since one should replace every $x$ by the multiple inverses of the function (53b). This can be avoided by making a change of variables from $u$ to $x$ in (55),

$$\partial_a \log G_{mj}^m(x) = u_{12}(x) \partial_a \log G_{mj}^m(u) = u_{12}(x) H_{mj}^m(u_{12}(x)).$$  \tag{60}

The result is ratio of polynomials, and the integral gives

$$G_{mj}^{++}(x) = C_{mj}^{++} \frac{x^{m_2 - m_1 + 1} - 1}{m_1 + m_2 + 2(x + \frac{m_1}{m_2}) - m_1 - m_2 + 2(x + \frac{m_1 - m_2}{m_2})} \frac{m_1 - m_2 + 4}{(x + m_1 - m_2)^4}$$  \tag{61}

Doing the same procedure, but using the maps (52a) and (53a), we find $H_{m_2 m_1}$ and solve the analogous of (60) to obtain a function $G_{mj}^{++}(x)$, which is of course the same as (61), but with $m_1 \leftrightarrow m_2$. As discussed above, we have therefore found that

$$G_c^{++}(u(x), \bar{u}(x)) = G_{mj}^{++}(u(x)) \bar{G}_{mj}^{++}(\bar{u}(x)) + G_{m_2 m_1}^{++}(u(x)) \bar{G}_{m_2 m_1}^{++}(\bar{u}(x)).$$  \tag{62}

We have introduced indices $++$ on the functions (62) because we now want to distinguish it from the other possible composite Ramond field, $R_{mj}^{++} R_{m_2}^{-}$. The second-order correction of the two-point function of this neutral field is given by the same expression (19) where now the function $G(u)$ has the form

$$G^{-+}(u) = \langle R_{mj}^{+} R_{m_2}^{+} (\infty) G_2^{int}(1) G_2^{int}(u) R_{mj}^{+} R_{m_2}^{+} (0) \rangle.$$  \tag{63}

The discussion concerning the covering maps, which only depends on the structure of the twists, still holds as above. In the stress-tensor method calculation, there is only a change of signs inside the first term in parenthesis in the second line of Eq.(59). The final result for the maps (52b) and (53b) is

$$G_{mj}^{++}(x) = C_{mj}^{++} \frac{x^{m_2 - m_1 + 2} - 1}{m_1 + m_2 + 1(x + \frac{m_1}{m_2}) - m_1 - m_2 + 1(x + \frac{m_1 - m_2}{m_2})} \frac{m_1 - m_2 + 4}{(x + m_1 - m_2)^4}$$  \tag{64}
and there is an equivalent solution for the maps (52a) and (53a), with \( m_1 \leftrightarrow m_2 \), such that
\[
G_c^{+}\left( u(x), \bar{u}(x) \right) = G_{m_1m_2}^{+}\left( u(x) \right) \bar{G}_{m_1m_2}^{+}\left( \bar{u}(x) \right) + G_{m_2m_1}^{-}\left( u(x) \right) \bar{G}_{m_2m_1}^{-}\left( \bar{u}(x) \right). \tag{65}
\]
Again, there are other contributions \( G_{m_1}^{+} \) and \( G_{m_2}^{-} \), coming from factorizations like in (32)-(33). These non-composite four-point functions again reduce to what has been computed in [21].

3.5. The non-composite contributions. One can use the stress-tensor method allied with the covering surface technique to compute the non-composite functions \( G_{mp}(u, \bar{u}), p = 1, 2 \), as well. See [21,32]. For that, one needs new covering maps, since the monodromy conditions are not (45) anymore. Taking the large-\( N \) limit, the covering map and its corresponding Arutyunov-Frolov map are
\[
z_p(t) = \left( \frac{t}{t_1} \right)^{m_p} \left( \frac{t - t_0}{t_1 - t_0} \right) \left( \frac{t_1 - t}{t - t_\infty} \right), \tag{66}
\]
\[
u_p(x) = \frac{x^{m_p} - (x + m_p)^{m_p + 1}}{(x - 1)^{m_p + 1}(x + m_p - 1)^{m_p - 1}}. \tag{67}
\]
where \( t_0 = x - 1, t_\infty = x - x(x + m_p)^{-1} \) and \( t_1 = t_0 t_\infty/x \). Proceeding with the stress-tensor method, we get the correlators (32)-(33) to be
\[
G_{mp}(x) = C_{mp} \frac{x^{5(2-mp)}(x-1)^{5(2+mp)}(x+m_p)^{2-3mp}(x+m_p-1)^{2+3mp}}{(x+m_p^{-1})^4} \tag{68}
\]
where
\[
C_{mp} = \frac{1}{16m_p^2}. \tag{69}
\]
This can be found by looking at OPE channels [21,32].

The non-composite functions (68) correspond to the factorized correlators (27). As discussed in §3.1, the factorization occurs in two cases.

The map (66) corresponds to a covering surface of the base sphere with genus zero and a number
\[
s_p = m_p + 1 \tag{70}
\]
of sheets. This corresponds to Cases 1 and 2b) discussed in §3.1. From Eq.(28), there are \( s = m_1 + m_2 + 1 \) distinct elements in the cycles inside the correlation function in the r.h.s. of (22). But, after the factorization, only \( s_p \) elements remain inside the four-point functions in Eqs.(27). In Case 2a), Eq.(29) shows there are initially \( s = m_1 + m_2 \) distinct permutation elements inside the correlator, but after factorization only \( \bar{s} = m_p \) elements remain inside the four-point function. By the Riemann-Hurwitz formula (41), these terms must be calculated with a genus-one map (which is not (66)), but will give a sub-leading contribution at order \( N^{-2} \), while \( G_{mp} \) captures the leading behavior of the correlators, at order \( N^{-1} \).

4. OPEs and structure constants

In this section, we would like to use the formulae for \( G_c(x) \) we found above, Eqs.(61) and (64), to examine various possible OPEs, by taking the coincidence limit of the operators in the four-point functions. Expressing the functions \( G_c(x) \)
as explicit functions of \(u\) is impossible in general, because the inverse map \(x(u)\) is multi-valued: one would need to know all the solutions \(x_j(u)\) of (50) and then sum them up. However, to find the OPEs we only need to invert the functions locally, which can be done by expanding the functions near the singular points.

### 4.1. Contributions from connected functions.

We first consider the contributions from the fully connected functions \(G_c^{++}(u(x))\) and \(G_c^{-+}(u(x))\) given in Eqs. (61) and (64). For each function, we analyze two short-distance behaviors: the limit \(u \to 1\) corresponding to the OPE \((\text{int})_2(u)(\text{int})_2(1)\), and the limit \(u \to 0\), corresponding to the OPE between \((\text{int})_2(u)\) and the composite Ramond operator.

#### 4.1.1. OPEs from the four-point function with charged composite operators.

Let us start with the OPE of two interaction terms \((\text{int})_2(u)O_2^{(\text{int})}(1)\). This corresponds to taking the limit \(u \to 1\) in the correlation function (31). Each of two functions in (61) correspond to one of the two maps (53). For each term, we thus have to invert the corresponding map by solving the equations \(u_{12}(x) = 1\) and \(u_{21}(x) = 1\) for \(x\). The solutions will be related by swapping \(m_1\) and \(m_2\); we give the calculation explicitly for \(u_{21}(x)\). Direct check of Eq. (53a) shows that \(u_{21}(x) \to 1\) corresponds to two possibilities: \(x \to \infty\) or \(x \to \frac{m_2 - m_1}{2m_2}\) (here we impose also the condition that simultaneously \(x \to t_1^{(1)}\)). Let us consider the first possibility. From (53a) it follows that in this case

\[
x = -\frac{4m_1}{1-u} + \frac{1}{2} \left(1 + 4m_1 - \frac{m_1}{m_2}\right) + \cdots
\]

An equivalent solution with \(m_1 \leftrightarrow m_2\) is found by inverting \(u_{12}(x)\) for \(x \to \infty\).

At the same time \(G_c^{++}(u)\) scales as

\[
G_c^{++}(u) = C_{m_1m_2}^{++} x^2 \left[1 - \left(1 + 4m_1 - \frac{m_1}{m_2}\right) + \cdots\right] + \left(m_1 \leftrightarrow m_2\right)
\]

\[
= \frac{16(C_{m_1m_2}^{++} m_1^2 + C_{m_2m_1}^{++} m_2^2)}{(1-u)^2} + 0 \times \frac{1}{1-u} + \cdots
\]

From counting the dimensions, it is clear that this channel corresponds to the identity operator, i.e. \((\text{int})_2(u)O_2^{(\text{int})}(1) \sim 1 + \cdots\). The absence of the subleading term ensures that there is no operator of dimension 1 in this OPE, as it should be for a truly marginal deformation.

Now let us consider the terms that appear when \(u_{21} \to 1\) while \(x \to \frac{m_2 - m_1}{2m_2}\),

\[
x - \frac{m_2 - m_1}{2m_2} = \left(\frac{3}{64} \left(m_1^2 - m_2^2\right)^2\right)^{\frac{1}{2}} (1-u)^{\frac{1}{2}} + \cdots
\]

Expanding (61) around \(x = \frac{m_2 - m_1}{2m_2}\), and doing the same for \(m_1 \leftrightarrow m_2\), we get to the following behavior of the function in this channel,

\[
G_c^{++}(u) = C \left(\frac{1}{(1-u)^{\frac{1}{2}}} + 0 \times \frac{1}{1-u} + \cdots\right)
\]

for a constant \(C\) which can be calculated in terms of \(C_{m_1m_2}^{++}\) and \(C_{m_2m_1}^{++}\). Dimensional analysis determines that the first term corresponds to the OPE \((\text{int})_2O_2^{(\text{int})}\) ∼
The appearance of the twist field \( \sigma_3 \) is not surprising because the interaction \( O^{(\text{int})}_2 \) is constructed using \( \sigma_2 \), and the above OPEs follow the \( S_N \) group multiplication rule \( \sigma_2 \sigma_2 \sim 1 + \sigma_3 \). The sub-leading term would correspond to an operator of dimension one, and its absence is again a confirmation of the correct behavior of the function \( G_c^{++}(u) \). Let us comment that taking the OPE of the two interactions in this channel in the correlation function leads to

\[
\langle R_{m_1}^- R_{m_2}^- (\infty) O^{(\text{int})}_2(1) O^{(\text{int})}_2(u) R_{m_1}^+ R_{m_2}^+ (0) \rangle \\
\sim \frac{1}{(1-u)^{4/3}} C_{223} \langle R_{m_1}^- R_{m_2}^- (\infty) \sigma_3(1) R_{m_1}^+ R_{m_2}^+ (0) \rangle + \cdots
\]

Therefore the coefficient \( C \) contributes to the product of structure constants

\[
C_{223} \langle R_{m_1}^- R_{m_2}^- (\infty) \sigma_3(1) R_{m_1}^+ R_{m_2}^+ (0) \rangle.
\]  

(75)

Let us turn to the channel \( u \to 0 \). It corresponds to the OPE of the interaction field with the composite Ramond field: \( O^{(\text{int})}_2(u) R_{m_1}^+ R_{m_2}^+ (0) \). Solving \( u_{21}(x) = 0 \), we find the channels

\[
x \to 0 \quad \text{for} \quad m_1 > m_2 \quad (76a)
\]

\[
x \to (m_2 - m_1)/m_2 \quad \text{for} \quad m_1 < m_2 \quad (76b)
\]

\[
x \to -m_1/m_2, \quad \text{for} \quad m_1 \leq m_2 \quad (76c)
\]

If channel \((76a)\) is calculated with \( u_{21}(x) \), then channel \((76b)\) has to be calculated with \( u_{12}(x) \), and vice-versa. Let us consider the common channel \((76c)\) first,

\[
x + m_1/m_2 = c_1 u^{1+m_1/m_2} + c_2 u^{2+m_1/m_2} + \cdots
\]

where the coefficients \( c_i \) are readily computable. From here one gets for the correlation function in this channel,

\[
G_c^{++}(u) = C u^{-1+\frac{2}{m_1+m_2}} + \cdots
\]  

(77)

with \( C \) (another) constant. Dimensional analysis of \((77)\) shows that the OPE in question has the following possible forms:

\[
O^{(\text{int})}_2(u) R_{m_1}^+ R_{m_2}^+ (0) \sim X \sigma_{m_1+m_2}(0)
\]

where \( X \) is some operator of dimension \( \Delta_X = \frac{9}{4(m_1+m_2)} \) and R-charge 1, acting on the twist field, or

\[
O^{(\text{int})}_2(u) R_{m_1}^+ R_{m_2}^+ (0) \sim \hat{X} R_{m_1+m_2}^+(0)
\]

where \( \hat{X} \) has dimension \( \Delta_{\hat{X}} = \frac{2}{m_1+m_2} \) and R-charge 1/2. This second form should be connected to previous results \([16,33]\) where similar three-point functions, but with the chiral field \( O_2 \), instead of its descendent \( O^{(\text{int})}_2 \), were considered. In both cases, the coefficient \( C \) in \((77)\) plays the role of (the square of) the structure constant. In the channel \((76a)\), we have

\[
x = c_1 u^{\frac{1}{m_1-m_2}} + c_2 u^{\frac{2}{m_1-m_2}} + \cdots
\]

leading to

\[
G_c^{++}(u) = C u^{-1+\frac{1}{m_1-m_2}} + \cdots
\]  

(78)
It follows that the OPE in this channel has the possible form

$$O^{(\text{int})}_2(u)R^+_{m_1}R^+_{m_2}(0) \sim X \sigma_{m_1-m_2}(0)$$

where $X$ is now some operator of dimension $\Delta_X = \frac{5}{4(m_1-m_2)} + \frac{m_2}{2}$ and R-charge 1, or alternatively

$$O^{(\text{int})}_2(u)R^+_{m_1}R^+_{m_2}(0) \sim \tilde{X} R^+_{m_1-m_2}(0)$$

and $\Delta_{\tilde{X}} = \frac{1}{m_1-m_2} + \frac{m_2}{2}$, R-charge 1/2. Since we used $u_{21}(x)$ to compute the channel (76a), we should use $u_{12}(x)$ to compute the final channel (76b), as already stated. But this actually implies that, for consistency, using the same map $u_{21}(x)$ to compute both channels should give equivalent results, related by $m_1 \leftrightarrow m_2$. It is easy to check explicitly that this is indeed the case.

Now let us consider the behavior of our correlation function when $m_1 = m_2 = m$. The highly simplified function $u(x)$ is now given by Eq.(51). We can compute the correlation function with the same procedure as before, and find simply

$$G^{++}_{c}(x) = C^{++} x^{-2}(x-1)^{2m+2}(x+1)^{-2m+2}.$$ 

In the limit $u \to 1$ with $x \to \infty$ we find again a behavior showing that the identity appears in the product of interaction fields, and in the other limit, $u \to 1$ with $x \to 0$, the coefficient in front of the contribution of the field $\sigma_3$ vanishes, so in this case there is no such channel in the OPE of two interaction terms. When $u \to 0$, one single solution survives: $x \to -1$, and the function scales as

$$G^{++}_{c}(u) = c \ u^{-1 + \frac{2}{2m}} + \cdots$$

This means that, if one accepts our suggestions above, only descendants of $\sigma_{2m}$ or $R_{2m}$ appear on the r.h.s. of the OPE, and the term like $\sigma_0$ is absent, as it should be, of course.

4.1.2. OPEs from the four-point function with neutral composite operators. We turn next to consider the short-distance behavior of the two-point function (64) of the neutral composite fields $R^+_{m_1}R^-_{m_2}$. Let us consider its behavior as $u \to 1$, corresponding to the OPE of the two interaction terms. In the identity channel, where $x \to \infty$, there is, again, no contact term of dimension one, as it should be. The second channel, where $x \to \frac{m_2-m_1}{2m_2}$, yields the twist field $\sigma_3$.

The limit $u \to 0$ accounts for the OPE $O^{(\text{int})}_2(u)R^+_{m_1}R^-_{m_2}(0)$. In the channel $x \to -m_1/m_2$,

$$G^{+-}_{c}(u) \sim C' u^{-1 + \frac{1}{m_1+m_2}} + \cdots$$

This result leads to the following possible suggestions for the OPE:

$$O^{(\text{int})}_2(u)R^+_{m_1}R^-_{m_2}(0) \sim X \sigma_{m_1+m_2}(0)$$

where $X$ is some operator acting on the twist field of dimension $\Delta_X = \frac{5}{4(m_1+m_2)}$ and R-charge zero, or

$$O^{(\text{int})}_2(u)R^+_{m_1}R^-_{m_2}(0) \sim \tilde{X} R^+_{m_1+m_2}(0)$$

with $\tilde{X}$ having $\Delta_{\tilde{X}} = \frac{1}{m_1+m_2}$, and R-charge $\mp 1$. The channel $x \to 0$ leads to

$$G^{+-}_{c}(u) \sim C' u^{-1 + \frac{2}{m_1+m_2}} + \cdots$$
and one possible interpretation of this scaling for the form of the OPE is
\[ O_2^{\text{int}}(u) R_m^+ R_m^-(0) \sim X \sigma_{m_1-m_2}(0) \]
where \( X \) has \( \Delta_X = \frac{9}{4(m_1-m_2)} + \frac{m}{2} \) and R-charge zero. Finally, we would like to notice that, here, the case \( m_1 = m_2 = m \) is a bit subtle, due to the fact that in the OPE \( R_m^-(z_1) R_m^-(z_2) \) there should appear also the identity operator, so one should be more careful in the precise definition of the composite field.

4.2. Contributions from the non-composite parts. The complete OPEs discussed above include terms not only from \( G_c \), but also from the functions \( G_{m_p} \), \( p = 1, 2 \), given in Eq.(68), which are correlators with non-composite Ramond operators. These are described in detail in [32]. It turns out that both \( G_{m_p}(u) \) have the same structure as \( G_c(u) \) in the coincident limit contributing to \( O_2^{\text{int}}(u) O_2^{\text{int}}(1) \),
\[
G_{m_p}(u) = \frac{1}{(1-u)^2} + 0 \times \frac{1}{1-u} + \frac{C_{4/3}}{(1-u)^3} + \frac{C_{2/3}}{(1-u)^2} + \frac{C_{1/3}}{(1-u)} + \text{Finite terms}
\]
(79)
Note the absence of terms \((1-u)^{-1}\), hence of operators of dimension 1. The factor of 1 in the identity term is obtained by imposition of Eq.(69).

5. Renormalization and anomalous dimensions

The two-point function of the composite Ramond fields \( R_{m_1}^\pm R_{m_2}^\pm \), evaluated at second order in the deformed orbifold SCFT \( (20) \), contains, in the large-\( N \) limit, a \( \log|z_{14}| \) correction term together with the logarithmic divergence
\[
\lambda^2 \pi \log \Lambda \int d^2 x \left[ |u'_{21}(x) G_{m_1 m_2}(u(x))|^2 + |u'_{12}(x) G_{m_2 m_1}(u(x))|^2 \right.
\]
\[
+ \left. |u'_{m_1}(x) G_{m_1}(u_{m_1}(x))|^2 + |u'_{m_2}(x) G_{m_2}(u_{m_2}(x))|^2 \right].
\]
(80)
The first two terms in brackets corresponds to \( G_c = G_{m_1 m_2} + G_{m_2 m_1} \), and the integrand — the sum of \( G_c \), \( G_{m_1} \) and \( G_{m_2} \) given in Eqs.(31)-(33) — forms the complete \( S_N \)-invariant four-point function \( G \) in Eq.(21). For each function, we made a change of integration variables \( d^2 u = d^2 x |u'(x)|^2 \) with the four maps \( u_4(x) \) given in Eqs.(53) and (67). We are forced to do this change of variables, since we have calculated in (61), (64) and (68) the explicit form of the correlation functions \( G(u(x)) \) parameterized by \( x \), and it is not (always) possible to invert these maps \( u(x) \) and to find \( G(u) \) explicitly.

We start with the integral of \( G_c(u, \bar{u}) \):
\[
I_c^{++} = \int d^2 u \ G_c^{++}(u, \bar{u})
\]
\[
= \int d^2 x \left[ |u'_{21}(x) G_{m_1 m_2}^{++}(x)|^2 + |u'_{12}(x) G_{m_1 m_2}^{++}(x)|^2 \right]
\]
\[
\sim \int d^2 x \left[ \frac{(x-1)(x+\frac{m_1}{m_2})}{(x+\frac{m_1-m_2}{2m_2})^2} \right]^2 + \int d^2 x \left[ \frac{(x-1)(x+\frac{m_2}{m_1})}{(x+\frac{m_2-m_1}{2m_1})^2} \right]^2.
\]
(81)
Then, in the first integral, we make one more change of variables,

\[ y = -4m_2(m_1 + m_2)^{-2}(x - 1)(m_2x + m_1), \]

while in the second - an equivalent change with \( m_1 \leftrightarrow m_2 \) in \( y(x) \). Both integrals then become equals,

\[ I_c^{++} \sim \int d^2y \frac{|y|^2}{|1 - y|^3} = \frac{1}{\Gamma(-1)} = 0. \] (82)

The same happens with the connected part of the contributions in the case of R-neutral composite Ramond field \( R_{m_1}^+ R_{m_2}^- \). Now \( G_{c}^{+-}(x) \) is given by Eq.(64) and its integral has a rather similar form:

\[ I_c^{+-} = \int d^2u G_{c}^{+-}(u, \bar{u}) \]

\[ = \int d^2x \left[ |u_{12}(x) G_{m_1m_2}^{+-}(x)|^2 + |u_{21}'(x) G_{m_1m_2}^{+-}(x)|^2 \right] \] (83)

\[ \sim \int d^2x \frac{x(x + m_1 - m_2)}{(x + m_1 - m_2)^2} \]

Again by a further change of the variables

\[ y(x) = -4m_2(m_1 - m_2)^{-2} \left( x + \frac{m_1 - m_2}{m_2} \right), \]

in the first one (and \( y(x) = (m_1 \leftrightarrow m_2) \) in the second) we get exactly the same result as before,

\[ I_c^{+-} \sim \int d^2y \frac{|y|^2}{|1 - y|^3} = 0. \] (84)

Hence the fully connected part \( G_c \) of the four-point function (19) do not contribute to the anomalous dimensions of none of the considered composite operators.

We next compute the contributions coming from the last two terms, \( G_{m_1}(x) \) and \( G_{m_2}(x) \), in Eq.(80). As we have explained in Sect.3.5, these functions, defined by Eqs.(32)-(33), are four-point functions of specific non-composite operators, that naturally arise in the \( S_N \) invariant sum (22) over the conjugacy classes. Using (68), the last two integrals in Eq.(80) take the form [21]

\[ J_R(n) = \int d^2v \left< R_n^{-}O_2^{(\text{int})}(1)O_2^{(\text{int})}(v, \bar{v})R_n^{+}(0) \right> \]

\[ = \left( \frac{n + 1}{16n} \right)^2 \int d^2y \frac{|y|^{2a}|1 - y|^{2b}|y - w_n|^{2c}, \quad w_n \equiv \frac{4n}{(n + 1)^2}, \right) \] (85)

where \( n = m_1 \) or \( n = m_2 \), and \( a = \frac{1}{2} + \frac{1}{4}n \), \( b = \frac{3}{2} \), \( c = \frac{1}{2} - \frac{1}{4}n \). Evaluation of the above integrals \( J_R(m) \) can be performed by applying the Dotsenko-Fateev method [34,35]. The final result can be written in terms of combinations of hypergeometric functions which asymptote to finite, small numbers when \( n \) is large [21].

The first consequence of the existence of finite non-vanishing terms in Eq. (80) is the renormalisation of the conformal dimensions of the composite twisted Ramond fields. In order to cancel the \( \log \Lambda \) divergent terms, we follow the standard QFT...
rules, i.e. dressing each one of the “bare” Ramond fields to get their renormalized counterparts

\[ R_{mp}^{\pm}(\text{ren}) = \Lambda \frac{1}{2} \pi \lambda^2 |J_R(m_p)| R_{mp}^{\pm}. \]  

(86)

Therefore the $\lambda^2$-corrected conformal dimensions of the composite Ramond fields in deformed orbifold SCFT\(_2\) takes the form

\[ \Delta(\lambda, m_1, m_2) = \frac{m_1 + m_2}{2} + \frac{1}{2} \pi \lambda^2 (|J_R(m_1)| + |J_R(m_2)|), \]  

(87)

and the two-point functions of the composite Ramond fields can be rewritten as

\[ \frac{\langle R_{m_1}^- R_{m_2}^- (z_1, \bar{z}_1) R_{m_1}^+ R_{m_2}^+ (z_4, \bar{z}_4) \rangle_{\lambda}^{\text{ren}}}{\langle 1 \rangle_{\lambda}^{\text{ren}}} = \frac{1}{|z_{14}|^{m_1 + m_2 + \pi \lambda^2 |J_R(m_1)| + |J_R(m_2)|}} \cdot \left[ 1 - \pi \lambda^2 (|J_R(m_1)| + |J_R(m_2)|) \log |z_{14}| + O(\lambda^4) \right]. \]  

(88)

A similar statement is valid for the case of the R-neutral composite Ramond fields $R_{m_1}^+, R_{m_2}^-$; in fact both type of composite Ramond fields (charged and neutral) turn out to have equal conformal dimensions, but different R-charges.

We have to note another important implication of the above result, concerning the non-vanishing finite parts in the integral in Eq. (80). It allows one to also derive the non-zero correction to the three-point function

\[ \langle R_{m_1}^- R_{m_2}^- (\infty) G_{2}^{(\text{int})(1)} R_{m_1}^+ R_{m_2}^+ (0) \rangle_{\lambda}^{\text{ren}} = \lambda \left( |J_R(m_1)| + |J_R(m_2)| \right) + \cdots, \]  

(89)

which in fact is providing the value of the structure constant at the first order in perturbation theory in $\lambda$.

The fact that at the second order in perturbation theory the purely connected part $G_c(x)$ of the $S_N$ invariant 4-point function (19) gives no contributions to the two-point function of the composite Ramond fields $R_{m_1}^+, R_{m_2}^-$, while those of the so-called “disconnected” parts $G_{m_p}(x)$ yield non-vanishing contributions raises the question:

Could one impose appropriate restrictions on the values of the twists $m_p$ that select the BPS-protected from the lifted (non-protected) composite Ramond states?

The answer is hidden in the conditions on the properties of the cycles $(m_1)_0, (m_2)_0$ and (2), that select the terms in the sum (22) contributing to the disconnected four-point functions $G_{m_1}$ and $G_{m_2}$ involving non-composite Ramond fields. As explained in Sect.3.1 these are the cases when the cycles of deformation operator (2)_1 = (k\ell) made by a set of two numbers $k, \ell \in [1, N]$ such that one of them, say $\ell$, coincides with one of the elements of the cycle $(m_1)$ and the second one, $k$, does not belong neither to $(m_1)$ nor to $(m_2)$. This condition selects the terms contributing to $G_{m_1}(x)$ in the large $N$ approximation. Similarly, in the case of the function $G_{m_2}(x)$ we have to take into account the terms where now $\ell \in (m_2)$. It is then clear that when the twists $m_1$ and $m_2$ are such that $m_1 + m_2 = N$ number $k \in [1, N]$ that does not belong to either cycle $m_p$ does not exist. Hence, in this family of pair of twists $m_1, m_2$, we have no disconnected contributions and, as a consequence, such composite Ramond fields $R_{(m_1)}^\pm, R_{(m_2)}^\pm$ are protected. They do not receive any corrections to their “free orbifold point” conformal dimensions $\Delta_R = \frac{m_1 + m_2}{2}$. Notice that in all other cases
one is able to choose $k \in [1,N]$ that is not in $(m_1)$ nor in $(m_2)$. Then we have both “connected” as well as “disconnected” contributions to the corresponding four-point functions and, as a result these composite Ramond states (and fields) are lifted, i.e. they get $\lambda^2$ dependent corrections (87) to their conformal dimensions.

6. Concluding Remarks

In the marginal deformation (15) of the orbifold SCFT$_2$, the twisted Ramond fields $R^\pm_n$ and products such as $R^\pm_{m_1}R^\pm_{m_2}$ appear to be part of specific coherent superpositions of twisted Ramond states [8]. Such superpositions are an important ingredient in the verification of the holographic duality between limits of the two-charge extremal horizonless black hole solutions in type IIB supergravity and the VEVs of certain “light” (low-twist) chiral NS fields $O_n$, which enter the superpositions together with products of twisted Ramond states. Comparison between the bulk SUGRA solutions and the D1-D5 orbifold SCFT$_2$ data is based on the conjecture that every chiral NS field and certain BPS twisted Ramond ground states are not affected by the marginal interaction (15), i.e. the values of such VEVs are $\lambda$-independent. Our results concerning the selection rule that separates the protected from the “lifted” (renormalized) states, in the case of the simplest composite Ramond states $R^\pm_{m_1}R^\pm_{m_2}$ and $(R^\pm_m)^2$, provide indications that only the specific protected part we have selected could contribute to the coherent superposition of Ramond states. Indeed, what are the renormalization properties of the products (and higher powers) of twisted Ramond states made by more than two operators, say $R^+_m(R^-_n)^2$, is still an open question. One should also be able to discover whether and how one is to separate them into protected and renormalized states. Another open question is about the eventual changes due to the interaction, say, up to second-order in $\lambda$, of three-point functions which are the appropriate generalizations of the simplest ones $(R^-_{m_1}R^-_{m_2}(\infty)O_2(1)R^{+}_{m_1+m_2}(0))_\lambda$, as for example those considered in the recent papers [16,36].

It is worthwhile mentioning that the renormalization properties of the R-neutral (but SU(2)$_2$ doublet) twisted Ramond field $R^0_n$, along with composite operators such as powers, e.g. $(R^0_n)^2$, and products with R-charged fields, e.g. $R^+_mR^0_n$, is another interesting unaddressed problem. These composite fields and their (left-right asymmetric) descendants are the main ingredient in construction of the microstates of the three-charge extremal black hole (with a horizon) in the D1-D5-P system [12].

To conclude, the problems solved in the present paper are based on the construction of the appropriate covering maps and the derivation of the renormalization of the two- and three-point functions involving composite twisted Ramond fields in the deformed D1-D5 orbifold SCFT$_2$. An important byproduct of our investigations appears to be a simple selection rule that allows us to separate between protected and “lifted” states. These results can be easily generalized for composite twist fields $\sigma_{m_1}\sigma_{m_2}$ and for chiral NS fields $O_mO_n$, since the covering map to be used is the same as the one we have constructed (50). Our preliminary results indicate that the case of twist fields seems to be identical to the Ramond case, while composite chiral NS
fields, similarly to the simple $O_n$ field, seem to be free of any renormalization [32]. In fact, the most important problem behind all these questions is the lack of a complete description of the (super)symmetry algebra of the considered deformed orbifold SCFT$_2$, of knowledge of the structure of its null vectors and the eventual classification of its unitary representations. Many partial recent results [37–40] provide important hints about different aspects of this problem. We consider that the information extracted from the specific 3-, 4- and 5-point functions of (composed) twisted Ramond fields in the free orbifold point, together with the developments of the methods of the calculations of certain integrals of them, also might provide relevant indications about the spectra of the representations of the deformed D1-D5 orbifold model.

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