THEORETICAL ASPECTS OF THE PION-PION INTERACTION

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We give a brief review of the theoretical description of low energy pion-pion scattering by the combined use of Chiral Perturbation Theory and Roy equations, an update of the Regge parametrization of $\pi\pi$ cross sections at high energies, and a short discussion of the scalar radius of the pion.

Keywords: Chiral Perturbation Theory; Dispersion relations

1. Low-energy scattering: ChPT and Roy equations

The structure of the $\pi\pi$ interaction at low energies is strongly constrained by the fact that the pion is the Goldstone boson of a spontaneously broken approximate symmetry of QCD. The scattering amplitude can be calculated in a systematic way in the framework of Chiral Perturbation Theory (ChPT), by an expansion which converges very rapidly near the center of the Mandelstam triangle. However, the convergence becomes slow as one approaches the unitarity cuts, and already at threshold the direct application of chiral expansions is not satisfactory.

The dispersion relations determine the structure of the amplitude in terms of physical region absorptive parts and two subtraction constants, which can be identified with the S-wave scattering lengths, $a_0^0$ and $a_0^2$. By projecting the fixed-$t$ dispersion relations onto partial waves and using unitarity, one obtains a set of integral equations for the phase shifts, the Roy equations.

In the early applications of Roy equations the subtraction constants were taken from experiment, and had large uncertainties. An important step forward was to determine them theoretically, by combining ChPT inside the convergence region with dispersion relations, which led to a very precise theoretical prediction:

\[
\begin{align*}
  a_0^0 &= 0.22 \pm 0.005, \\
  a_0^2 &= -0.0444 \pm 0.0010.
\end{align*}
\]

This range is represented by the small red region in Fig. 1, where the black lines define the universal band imposed by Roy equations, the points are the ChPT

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Fig. 1. Theoretical and experimental status of the S-wave scattering lengths (details in the text).

calculations in the tree, one-loop, and two-loop approximation (see Ref. 4 for references), and the blue dashed lines are obtained by a Roy analysis \( a_0 \) that does not make use of chiral perturbation theory. We also show the recent, unquenched lattice result for \( a_0 \) obtained by NPLQCD \( a_0 \) as well as the range for \( a_0 \) and \( a_2 \) that corresponds to the values for the LECs given in Ref. 8. The experimental results are from BNL E865, CERN DIRAC, and CERN NA48/2 experiments.

Having precise values for the subtraction constants, the threshold parameters, the S and P-wave phase shifts below 0.8 GeV, and the coupling constants of the effective chiral \( SU(2) \times SU(2) \) Lagrangian relevant for \( \pi \pi \) scattering could be determined to high accuracy. Below 0.8 GeV the influence of the high energies in the Roy equations is very small. This was shown in Ref. 13, where we explored the sensitivity of the Roy solutions with respect to the high energy input. We solved Roy equations using at high energy, instead of the Regge model adopted in Ref. 5, an alternative Regge parametrization proposed recently. Our analysis shows that the predictions for the S-wave scattering lengths and the isoscalar S-wave and P-wave phase shifts are practically unchanged, while the exotic S-wave phase shift is modified by only 1.4° at 0.8 GeV. For a detailed discussion see Ref. 13.

2. High-energy scattering: Regge parametrization

The Regge parametrization of \( \pi \pi \) amplitudes was discussed in the seventies, but afterwards the interest in this subject diminished. Recently, a Regge analysis of \( \pi \pi \) amplitudes was presented in Ref. 12. We are currently performing a new analysis exploiting factorization of Regge residua and dispersive sum rules. The motivation of the study is to have an improved input for solving Roy equations above 0.8 GeV, in the validity region \( \sqrt{s} \leq 1.15 \text{GeV} \). Below we give only a few results on the Regge parametrization of the total cross sections.

Using the notations adopted in Ref. 15 for \( \pi^+N, NN \) and \( \bar{N}N \) scattering, we express the \( \pi \pi \) total cross sections as

\[
\sigma_{\pi^+\pi^-} = B \ln^2(s/s_0) + Z_{\pi\pi} + Y_{1\pi\pi}(s_1/s)^{\eta_1} \mp Y_{2\pi\pi}(s_1/s)^{\eta_2}.
\]
Fig. 2. Total cross section in the channels with $I_t = 0$ (left) and $I_t = 1$ (right).

The first two terms are the contribution of the Pomeron, the last two the contribution of the $f$ and $\rho$ Regge poles, respectively. $Z$ and $Y$ denote the Regge residues supposed to satisfy factorization, and $\eta_j$ are related to the trajectories intercepts.

In Figs. 2 we present our results for the total cross sections with definite isospin in the t-channel: $I_t = 0$, which receives contribution from the Pomeron and $f$, and $I_t = 1$, dominated by the $\rho$ Regge pole. The sum of the low partial waves at lower energies is indicated for comparison. The red bands are obtained by using the fits of $\pi N$ and $NN$ data above 5 GeV and the factorization of Regge residua. The blue and green bands represent the parametrizations considered in Ref. 5 and Ref. 12, respectively. The bands denoted as “our estimates” are obtained, for $I_t = 0$ channel, by ascribing an extrapolation error to the parameters derived from factorization above 5 GeV and, for $I_t = 1$ channel, by applying Olsson sum rule (in this case the results given by factorization have large uncertainties). Our results show that in Ref.5 the $I_t = 0$ contribution was slightly underestimated, while in Ref. 12 the authors take too small a value for the $\rho$ residue. Details will be reported elsewhere.

3. Scalar radius of the pion

The scalar radius of the pion is an important quantity in ChPT, because it is related to an effective coupling constant, $\ell_4$, that determines the first nonleading contribution in the chiral expansion of the pion decay constant. A first crude estimate for the scalar radius in ChPT reads $\langle r^2 \rangle^\pi_s = 0.55 \pm 0.15 \text{fm}^2$. An improved result, $\langle r^2 \rangle^\pi_s = 0.61 \pm 0.04 \text{fm}^2$, was obtained from dispersion theory and two-channel unitarity for the scalar form factor $\Gamma_\pi(s)$ (see Ref. 17 for a recent discussion and references to earlier works). This result was questioned in Ref. 18, where the author used a single channel Omnès representation of the scalar radius in terms of the phase $\delta_T(s)$ of the form factor, to advocate a larger value: $\langle r^2 \rangle^\pi_s = 0.75 \pm 0.07 \text{fm}^2$. However, as we emphasized in Ref. 17, the estimate of the phase $\delta_T(s)$ ignores an ambiguity of $\pm \pi$ in the Watson theorem, which can be the resolved only by the explicit inclusion of inelastic channels in the Mushkeshvili-Omnès formalism.
In Ref.19 the author invokes perturbative QCD in favor of a large phase $\delta_\Gamma(s)$. To leading order in $\alpha_s$, neglecting quark and pion masses in the propagators, one obtains\(^a\) for large spacelike momenta $Q^2 > 0$:

$$\Gamma_\pi(Q^2) \sim \frac{4\pi f_\pi^2 Q^2}{Q^2} \int_0^1 d\xi \int_0^1 d\eta \left[ \bar{m}_u^2(Q^2) \frac{\phi(\xi) \phi(\eta)}{\xi(1-\eta)^2} + M_\pi^2 \frac{\phi(\xi) \phi_p(\eta)}{\xi(1-\eta)} \right], \quad (3)$$

where $\phi(\xi) = 6\xi(1-\xi)$ and $\phi_p(\xi) = 1$ are the twist-2 and twist-3 light-cone distribution amplitudes, respectively.\(^b\) Both terms in Eq. (3) contain an end-point logarithmic divergence, which is usually replaced by $\ln(Q^2/\Lambda_{QCD}^2)$.\(^c\) In Ref. 19 the author keeps only the first term in (3) and claims that around 1 GeV the phase $\delta_\Gamma(s)$ is much larger than its asymptotic limit $\pi$. But the first term in (3) vanishes faster than the second one for $m_u \to 0$ and represents, in comparison, a small correction. Therefore, the arguments put forward in Ref. 19 are based on an incomplete calculation. Moreover, the logarithmic singularity makes the specific predictions from QCD doubtful in this case. A complete discussion will be given elsewhere.

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