Fast Periodicity Estimation and Reconstruction of hidden components from noisy periodic signal

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Abstract—Periodicity estimation from an arbitrary length noisy signal is computationally very costly. A recently developed Ramanujan Fat Dictionary is one of the ways to find the hidden components from an arbitrary length (non-integral multiple of period) of the signal. This method suffers from high run time due to the lack of information about the period and effect of noise on the signal. We propose a new method that efficiently estimates the period of the signal and finding the hidden components thus becomes easy from it. Our method works well with significantly low SNR values and runs in $O(n)$ time complexity, $n$ being length of the signal. Comparison of run time analysis between our method for period estimation of a given signal and SVD method at various SNR values has been made and the corresponding hidden components are there by extracted by projecting onto the factor-Ramanujan Subspaces.

Index Terms—Period Estimation, Ramanujan Periodic Transform, Hidden Components.

I. INTRODUCTION

Most of the real time signals acquired by various sources are generally corrupted by noise. Extraction of weaker periodic components in presence of noise is a well-known challenge in the field of digital signal processing. A lot of available works have dealt with the component signal extraction from a composite signal while the components are sinusoidal. Although, in practical situations, in many a case, periodic signals that are non-sinusoidal are encountered. The period length, the structural patterns of periodic segments and the relative strengths of the respective segments are the three fundamental features of periodic signal.

In continuous domain, a signal $x(t)$ is said to be periodic with period $T$, if

$$x(t + T) = x(t), \forall t$$

(1)

where $T$ is the smallest positive interval that satisfies the Eq.1. Similarly, in discrete domain, periodicity is defined as follows:

$$x[n + P] = x[n], \forall n \in Z$$

(2)

where $P$ is the smallest positive integer that satisfies Eq.2. Over the years, many works have been done to estimate the period of a given signal along with the hidden periodic components of the signal.

Two or more periodic signals, when added, give rise to a periodic signal whose period is the least common multiple (lcm) or a factor of lcm of the two components [1]. Applying Fourier Transform (FT) is one of the simplest way to detect the periods where the inverse of the fundamental component gives the period of the signal. However, for composite signals (signals with more than one hidden period), some components may not be strong enough to be detected which indicates that spectral analysis is not a reliable method for such signals.

Periodic decomposition [2] provides an approach for determining the hidden periods from a composite signal. Setharas and Staley proposed an algorithm in which instead of choosing pre-determined basis, a data dependent non-orthogonal basis is introduced [3]. Their proposed method projected the signal onto subspaces of different periodicities. Comparison of the projection energies was used for period estimation. P. P. Vaidyanathan et.al. introduced the concept of the Ramanujan subspace ($S_q$) on the basis of Ramanujan sums [1]. The Ramanujan sums $C_q(n)$ and its circular shifts form an integer basis of the subspace $S_q$. Periodicity estimation of signals is achieved by representing the signals in terms of linear combination of its components belonging to a number of the Ramanujan sub-spaces. In the companion paper [4], Vaidyanathan et.al. showed that the Ramanujan—sum expansion method does not hold good for FIR signals. A linear combination of the Ramanujan sums, having number of terms equal to that of the signal length and a Ramanujan subspace based method were proposed to handle the FIR case. The Ramanujan Periodic Transform was also introduced, and a relatively easy calculation involving the scaled integer periodicity matrices yields the periodicities present in a signal from the non-zero energy projections.

However, the approach proposed in [5] has a limitation that the input signal must have a length which is an integral multiple of the hidden periods that are present in the signal. For example, if two signals having periods 7 and 13 respectively are super imposed to form a new signal then to apply the method proposed in [5], the length of the newly formed composite signal should be an integral multiple of $13 \times 7 = 91$ (i.e. 91, 182... etc.).

To overcome this problem, in [5] dictionary or fat matrix based approach has been implemented for periodicity estimation of a signal. An integer valued generalization of the complex valued Farey dictionary has been also proposed. RPT based dictionaries proved to yield better result in terms of noise immunity and complexity. The Farey Dictionary method in [5] was further generalized in [6] where a number of
other dictionaries have been constructed which provide faster calculations. Inspired from the RPT matrices a family of square matrices have been introduced that results in faster calculation. Some real valued alternatives of the DFT matrix is used to form the Nested Periodic Matrices fusing the concepts of many transforms that are used in periodicity estimation. Incorporation of the \( L_2 \) norm based methods further simplifies the calculation. In \cite{7} Ramanujan filter banks were introduced based on the Ramanujan-sums. In order to estimate periodicities with the help of filters, a filter must be designed which indicates presence of particular period in its input signal by a significant change in its output. Non adaptive comb filters based on the Ramanujan-sums serves the purpose. The Ramanujan filter banks also proved to be successful in representing the time varying periodicity of a signal. The advantages of period detection based on RFB over Short-Time Fourier Transform (STFT) is described in \cite{8}. Further researches suggested that the Nested Periodic Dictionaries proposed in \cite{9} are minimal in size \cite{9}. For proper integer period estimation, the highest lower boundary of the data length that is subjected to an algorithm was aimed to obtain in \cite{10}. The identifiability of the hidden periods has also been evaluated in terms of the input data length. After evolving the bound in data length that is applicable to any of the period estimation techniques, the results were further extended to the Farey dictionary based single and hidden period detection method. It was also shown that depending on the number of expected harmonics in a signal the required data length changes for signals with non integer periods.

A. Our Contribution

When the length of the signal (\( N \)) is not an integral multiple of the period of the signal then fat matrix technique presented in \cite{5} tries to figure out the period of the signal by projecting the signal on to all of the ramanujan subspaces \( S_q \)'s, \( q < N \). This is rather time consuming. In such cases, period of the signal has to be estimated. This estimation of period of the signal when the length of the signal is not an integral multiple of the period is dealt here. We proposed a method to estimate the period of the signal given the above restriction and compared it with the Singular Value Decomposition (SVD) method.
II. Period Estimation

Let us suppose we have a signal \( X \) of length \( N \) that is made up of \( k \) hidden components \( X_{p_1}, X_{p_2}, \ldots, X_{p_k} \) (where each component \( X_{p_i} \) \( (0 < i < k) \) is periodic with period \( p_i \)) is corrupted with some noise. Let the period of the composite signal be \( p \).

Now, when \( N \neq c \times p \) for some integer \( c \) then we need to apply the fat matrix [5] technique to figure out the hidden components. This involves in projection of the signal onto all the ramanujan spaces of size less than \( N \) which is rather time consuming. So, we have to estimate the period of the signal and then we need to project the signal only onto the ramanujan spaces of sizes which are factors to the estimated period.

We first discuss a little about the SVD method for period estimation and then discuss our technique for the estimation of the composite period.

In the SVD technique for period estimation we make a data matrix of size \( \left\lfloor\frac{N}{p}\right\rfloor \times P \) for an assumed period of \( P < N \). Period is estimated based on the \( \lambda_1/\lambda_2 \) values obtained through the SVD process for the data matrices obtained for each assumed period \( P \). When the assumed period \( P \) is the actual period \( p \) then we get maximum value for \( \lambda_1/\lambda_2 \) and \( p \) is identified.

**Data Matrix formation:**

Data Matrix \( D_P \) for an assumed period \( P \), is formed by dividing the signal into blocks of size \( P \) and omitting the last portion of the data if its size is less than \( P \) and then taking each block as a row of a \( \left\lfloor\frac{N}{P}\right\rfloor \times P \) size matrix.

**Example II.1:** For \( X = [1, 2, 3, 1, 2, 3, 1, 2] \) and for an assumed period of \( P = 3 \), the data matrix \( D_3 \) is as follows:

\[
D_3 = \begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3
\end{bmatrix}
\]

A. The Technique

Instead of running SVD on the obtained data matrices we employ a different technique as presented below for the estimation of the period. It is also shown later in II-B that information about the hidden periods as well can be obtained through this technique.

**Theorem II.1. Minimum Variance.**

The variance of each and every column vector of data matrix formed by chopping \( X \) at blocks of \( P \) is 0 iff \( P \) is the composite period \( p \) or a multiple of it.

**Proof.** If Part:

When the assumed period \( P \) is the composite period \( p \) or a multiple of \( p \) say \( m \times p \) then data matrix \( D_{mp} \) obtained is as follows:

\[
D_{mp} = \begin{bmatrix}
X[0] & \ldots & X[i] & \ldots & X[mp-1] \\
X[mp] & \ldots & X[mp+i] & \ldots & X[2mp-1] \\
\vdots & \ddots & \vdots & \ddots & \vdots
\end{bmatrix}
\]

Now, consider any column of \( D_{mp} \), say the \( i^{th} \) column, \( C_i \). We have,

\[
C_i = [X[i] \ X[mp+i] \ X[2mp+i] \ldots]^T \tag{4}
\]

As \( p \) is the composite period of the signal, we have \( X[mp+i] = X[i] \) for some integer \( n \). Thus, Eq.4 boils down to

\[
C_i = [X[i] \ X[i] \ X[i] \ldots]^T \tag{5}
\]

As all the elements of the columns are same, the variance is zero.

**Only If Part:**

Let us suppose there exists a \( P' \) for which all the columns of the data matrix \( D_{P'} \) formed by chopping \( X \) at blocks of \( P' \) is zero and \( P' \) is not a multiple of \( p \).

Since, each and every column of \( D_{P'} \) has variance zero we have that all the elements of any given column to be same. This is possible only if \( P' \) is a period or multiple of a period which is a contradiction.

Our idea is to form data matrix for each assumed period to the noisy signal \( P \) and find variance of all columns. The value of \( P \) for which the variance is minimized is our period.

B. Hidden Periods

The method described in II-A will tell us about the hidden periods as well.

**Theorem II.2. Hidden Components**

Let \( X_{p_i} \) be one of the component of \( X \) that has a period of \( p_i \). Now, when the data matrix \( D_{mp_i} \) formed by chopping at lengths of \( mp_i \) is considered the variance of each column has contributions due to the other component signals only.

**Proof.** We have data matrix \( D_{mp_i} \) as:

\[
D_{mp_i} = \begin{bmatrix}
X[0] & \ldots & X[i] & \ldots & X[mp_i-1] \\
X[mp_i] & \ldots & X[mp_i+i] & \ldots & X[2mp_i-1] \\
\vdots & \ddots & \vdots & \ddots & \vdots
\end{bmatrix}
\]
Now, consider any column of $D_{mp}$, say the $j^{th}$ column, $C_j$. Let us denote $X' = X - X_{mp}$. We have,

$$C_j = [X[j] X[m_1 + j] X[2m_1 + j] \ldots]^T$$
$$= [X_{mp}[j] X_{mp}[m_1 + j] X_{mp}[2m_1 + j] \ldots]^T$$
$$+ [X'[j] X'[m_1 + j] X'[2m_1 + j] \ldots]^T$$
$$= [X_{mp}[j] X_{mp}[m_1 + j] X_{mp}[2m_1 + j] \ldots]^T$$
$$+ [X'[j] X'[m_1 + j] X'[2m_1 + j] \ldots]^T$$
$$= [X_{mp}[j] X_{mp}[j] X_{mp}[j] \ldots]^T$$
$$+ [X'[j] X'[m_1 + j] X'[2m_1 + j] \ldots]^T$$

Variance of a data set is invariant to constant addition. This concludes the proof.

Let us call the plot of cumulative variance of all the columns against assumed periods as variance graph or variance plot.

**Definition II.1. Dip and Dip Magnitude**

We define dip as the point where the magnitude of variance is less than its immediate neighbours in the variance graph.

If there is a dip at an assumed period $P$, in the variance graph $\text{var}$, we defined the following dip magnitude measures on it:

1. $\max(\text{var}) - \text{var}[P]$
2. $\max(\text{var}) + \text{var}[P - 1] + \text{var}[P + 1] - 3\text{var}[P]$  \hspace{1cm} (8)

**Observation II.1.** Dips can be observed at assumed period lengths of integral multiples of hidden periods in a variance graph.

**Proof.** A ‘dip’ in variance is observed in the variance graph when data matrices are constructed by chopping the signal at block lengths of hidden periods or their multiples; as contribution to the variances due to corresponding signal component is zero by theorem II.2

As the composite period is the (least common) multiple of each and every hidden period a greater dip is expected at the composite period.

**C. The Algorithm**

In this section we discuss about two algorithms that we implemented based on the Theorems II.1 and II.2.

1) **Period Finder Algorithm:** First we create the data matrices $D_P$’s for various assumed periods $P$ and variance graph is obtained. This variance graph is processed to identify dips and from here a dip graph is obtained. A dip graph is the plot of dip-magnitude vs assumed period $P$ where ever a dip is found. (Here dip-magnitude is taken to be the second measure defined on the dip.)

As per the theorems II.1 and II.2 a dip corresponds to a multiple of hidden period or a hidden period itself. The value at which dip-graph has highest magnitude should correspond to the composite period $p$ provided the noise is not periodic.

Estimation of the composite period depends on the strength of the noise present in the signal. If the noise is too high then
it corrupts the value of the variance at the composite period so much that the estimated period may not be the composite period.

**Observation II.2.** The dip-magnitude at a given hidden period or a multiple of it depends on the relative strength of the hidden component w.r.t. the remaining other components apart from the noise present to the signal

Proof. From Theorem II.2 we have seen that magnitude of variance of the data matrix $D_j$ formed by chopping $X$ at lengths of $P$ which is a hidden period will have no contribution due to the component $X_i$. Stated otherwise, has contribution because of the other components only. Thus, if the relative strength of $X_i$ w.r.t to the remaining components as well as noise is less then it will have only a small dip-magnitude at that corresponding hidden period.

| Algorithm 1 Minimum Variance Period Finder |
|--------------------------------------------|
| 1: var = zeroes([N/2])                     |
| 2: dip = zeroes([N/2])                     |
| 3: procedure PERIOD FINDER($\hat{X}$)       |
| 4: for $j$ in 2 to $\lfloor N/2 \rfloor$ do|
| 5: Create data matrix $D_j$                |
| 6: for $i$ in 1 to $j$ do                  |
| 7: var[$j-1$] = var[$j-1$] + variance($D_j[:,i]$) |
| 8: end for                                  |
| 9: var[$j-1$] = var[$j-1$/j]                |
| 10: end for                                 |
| 11: prev = 1; cur = 2, nxt = 3;             |
| 12: for $j$ in 1 to $\lfloor N/2 \rfloor$ - 1 do |
| 13: if var[cur] < var[prev] and var[cur] < var[next] then |
| 14: dip[cur] = maximum(var) - var[cur]     |
| 15: dip[cur] += var[prev]+var[next] - 2*var[cur] |
| 16: dip[cur] = (dip[cur])$^4$               |
| 17: end if                                  |
| 18: end for                                 |
| 19: return maximum(dip)                     |
| 20: end procedure                           |

Step (5) of the algorithm creates a data matrix $D_j$ at an assumed period $j$. Steps (6-9) will calculate the cumulative variance of all columns of $D_j$ averaged over $j$. These steps (5-9) run in $O(n^2)$ time and step(4) loops over this $O(n^2)$ process. Thus steps (4-10) run in $O(n^3)$ time. Steps (11-19) extract the period with maximum dip magnitude. Here, the dip magnitude is taken as the fourth power of second dip-magnitude we measured. This extraction runs in $O(n)$. Thus entire procedure runs in $O(n^3)$ time.

2) Modified period finder - Monte Carlo Algorithm: We have another important observation to make here before going further

**Observation II.3.** The variance graph obtained by taking variance of a subset of every column vector of some constant length taken from the data matrix formed by chopping $X$ at various assumed period lengths would estimate the composite period correctly.

Proof. At the composite period irrespective of the length of the subset of the column vector considered variance always depends on the noise. With in permissible levels of the noise, always the value of variance obtained at the composite period is minimum. □

To improve upon the run time bound of the II-C1, instead of accumulating and averaging over all the columns of the data matrix we consider only few (constant) number of columns. Also, by observation II.3 even in these columns, a subset of them is sufficient. So we choose only a fixed length subset of columns from this data. So effectively the algorithm runs in $O(n)$ To counter the effect of error that may enter due to selecting only few number of columns, we will process this using a Monte-Carlo type approach by asking the user to send the signal few(k) many times and only those assumed periods that are consistently present in all the k runs are considered for period estimation task.

| Algorithm 2 Monte Carlo Period Finder |
|---------------------------------------|
| 1: var = zeroes([N/2])                |
| 2: dip = zeroes([N/2])                |
| 3: count = zeroes([N/2])              |
| 4: resends = k                         |
| 5: procedure MONTE CARLO($\hat{X}$)   |
| 6: for $l$ in 1 to $\lfloor N/2 \rfloor$ do |
| 7: Create data matrix $D_j$            |
| 8: for $i$ in 1 to $\lfloor N/2 \rfloor$ do |
| 9: var[$j-1$] = var[$j-1$/j]           |
| 10: end for                             |
| 11: prev = 1; cur = 2, nxt = 3;        |
| 12: for $j$ in 1 to $\lfloor N/2 \rfloor$ - 1 do |
| 13: if var[cur] < var[prev] and var[cur] < var[next] then |
| 14: dip[cur] = maximum(var) - var[cur] |
| 15: dip[cur] += var[prev]+var[next] - 2*var[cur] |
| 16: dip[cur] = (dip[cur])$^4$           |
| 17: end if                              |
| 18: end for                             |
| 19: return maximum(dip)                 |
| 20: end procedure                       |

III. FINDING THE HIDDEN COMPONENTS

Let us suppose the hidden components in the given signal $X'$ be $X_{p_1}, X_{p_2}, X_{p_3}, \ldots, X_{p_k}$. Let us also denote the corrupted version of $X'$ by $X''$. Now using the process discussed as in [1], [4] we project $X''$ on to various ramanujan sub spaces that are factors of the estimated composite period $p$. Then $\{p_1,p_2,p_3,\ldots,p_k\}$ are factors of $p$ where $p_1 = 1$ and $p_k = p$ are trivial factors. As $p_1 = 1$ we have corresponding $X_{p_1} = X_1$ to be periodic with period 1. Hence $X_1$ corresponds to just a dc signal. Let the dc level be $d$. We know that ramanujan spaces are mutually orthogonal i.e.,

$$X_i^\dagger X_j = \delta_{ij}$$  (9)
Now, taking $i = 1$ in Eq.9 and $j \neq 1$, we have,

$$X_1^* X_j = \delta_{1j}$$  \hspace{1cm} (10)

$$\Rightarrow d \sum_{n=1}^{n=p} X_j[n] = 0$$  \hspace{1cm} (11)

$$\Rightarrow \sum_{n=1}^{n=p} X_j[n] = 0$$  \hspace{1cm} (12)

i.e., all $X_j$’s with $j \neq 1$ have dc value 0. So, the dc component present in each hidden component is transferred to the $X_1$ and all the rest of hidden components are zero mean components. Hence, hidden components obtained are shifted versions of the actual hidden components (the shift is so much that their dc value is zero) and the dc signal $X_1$ has dc value ($d$) such that

$$dc(X_{reconstructed}) = dc(X_1) = dc(X')$$  \hspace{1cm} (14)

When we have no information about the dc value of each hidden component then the reconstructed components are shifted versions of actual hidden components. Let us denote the actual hidden components by $X_{i_{act}}$. Then we have

$$X_i + \alpha_i X_1 = X_{i_{act}} \forall i \neq 1$$  \hspace{1cm} (15)

such that

$$\sum \alpha_i = 1$$  \hspace{1cm} (16)

When information about dc values of hidden components is not available then the $\alpha_i$’s cannot be determined. Let us suppose we have arbitrarily assigned some value ($\alpha'_i$) to each $\alpha_i$ in this case, i.e.,

$$X_i + \alpha'_i X_1 = X'_i \forall i \neq 1$$  \hspace{1cm} (17)

and let $X$ represent the sum of all the estimated hidden components. That is,

$$\sum_{p2}^{k} X'_i = X$$  \hspace{1cm} (18)

We show that all the $\alpha_i$’s should be chosen to be same for the correlation between $X'$ and $X$ to be maximum when no information on dc values of hidden components is provided.

This can be formulated as follows:

**Maximize:** $X'^t X' \\
**Subjected to:**

$$X_i + \alpha'_i X_1 = X'_i \forall i \neq 1$$  \hspace{1cm} (19)

$$\sum \alpha'_i = 1$$  \hspace{1cm} (20)

This optimization problem is solved using Lagrange method of undetermined multipliers and it yields the result that all $\alpha_i$’s should be same.

Thus when we have no information about the dc value of the hidden components then to maximize the correlation we need to distribute $X_1$ equally to the rest of the components constructed.

### IV. Case Studies

Here we show various results obtained using the period estimation technique and comparison between our proposed method and SVD method for period estimation as well as signal estimation based on the estimated period are shown.

We consider the hidden periods of a composite signal are $p_1 = 8$, $p_2 = 11$, $p_3 = 16$. A random signal is added to the composite signal and finally it is considered to be a noisy signal. Let the data length is 4119 whereas the composite period of the signal is 176.

![Fig. 8. Mean Zero triangular periodic signal with period 8](image1)

![Fig. 9. Mean Zero cosine signal with period 11](image2)
From the above noisy signal, to find the composite periods, below is two processes:

In this figure, only hidden or multiple of hidden and composite or multiple of composite period is present rest of all the points are zero.

Another advantage of using our proposed method is to compute the composite period length or an integral multiple of composite period length from a very long data length, is very fast, i.e. an order of $O(n)$ algorithm is proposed in this study.

Here the green line is the plot of our proposed method time taken curve and the blue one is the SVD method time taken curve.
This is the curve of Hit-Miss of finding actual composite period. It shows out of 20 iterations, only 2\textsuperscript{nd} time it failed when resends is 2. So, here the hit-miss ratio is 19 : 1.

This the Fat-Matrix solution of the whole signal. So, from the fat matrix solution, it is impossible to comment about the hidden periods. Whereas, if we project the signal truncating it at the composite period length or the multiple of composite period length, it is easy to identify the actual hidden components present in a signal. After following our process, we truncated the signal at the multiple of composite periods and then projected this signal at the factor ramanujan subspace of that composite period to find the hidden components. Below is the figure of the highest error valued component strengths w.r.t the SNR values.
This is the plot of 4th highest component normalized strength vs. SNR (in dB) is plotted. Which shows the highest noisy component strength is very low that it won’t effect the actual signal.

After getting the information about the hidden components, reconstructed hidden periodic signal forms are found. Those are below:

This is the reconstructed 1st periodic component \( p_1 = 8 \). From the noisy signal of noise margin, SNR of 7.39 dB.

This is the 2nd reconstructed periodic component \( p_2 = 11 \). From the noisy signal of noise margin, SNR of 7.39 dB.

This is the 3rd reconstructed periodic component \( p_3 = 16 \). From the noisy signal of noise margin, SNR of 7.39 dB. The ‘resends’ value in algorithm depends on the noise distribution.

V. CONCLUSION

We proposed a method to find the composite period length of an one dimensional signal in \( O(n) \) order. The suggested method is working very well till the SNR of 5dB. The biggest advantage of using this randomized process, is less computational time. If we have provided with the data length in the order of millions, SVD process will get stuck while running in the present available computational resources as the time complexity is in the order of \( O(m^2n) \) where \( m < n \) for a data matrix of dimension \( m \times n \). That plot is shown in the result section. Once we get the composite period length, we are truncating the signal at the multiple of the composite period length to find the hidden components. Once we get the truncated signal, we are projecting it in the factor ramanujan subspaces of the composite period. Hence, exact hidden components are being able to be found. If the data length is very high, these steps are found to work extremely good. From these information, the shape or form of the individual hidden components can be approximated even in the presence of noise.

Another contribution of this study is to identify the DC component strength of the individual hidden periods such that the auto-correlation between the reconstructed and the actual hidden component is maximum.

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