The General Solution of the Complex Monge-Ampère Equation in a space of arbitrary dimension

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Abstract

A general solution to the Complex Monge-Ampère equation in a space of arbitrary dimensions is constructed.
1 Introduction

The Complex Monge-Ampère equation in \(n\)-dimensional space takes the form:

\[
\det \begin{vmatrix}
\frac{\partial^2 \phi}{\partial y_1 \partial \bar{y}_1} & \cdots & \frac{\partial^2 \phi}{\partial y_1 \partial \bar{y}_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 \phi}{\partial y_n \partial \bar{y}_1} & \cdots & \frac{\partial^2 \phi}{\partial y_n \partial \bar{y}_n}
\end{vmatrix} = 0.
\]

(1)

Its real form, which arises from (1) under the assumption that the solution depends only upon \(n\) arguments \(x_i = y_i + \bar{y}_i\) was found before by different methods [1], [2]. But to the best of our knowledge the general solution of the complex M-A equation (1) is still wanting. The aim of the present paper is to fill this gap by using the method of our paper [2] (the solution of real M-A equation in a space of arbitrary dimension) to obtain and present the general exact solution of the complex version of this equation in implicit form. We assume that all functions which we introduce are twice differentiable.

2 Equivalent First Order Equations

The complex M-A equation, (1) is the eliminant of \(n + n\) linear equations which express the linear dependence between rows or columns of the determinantal matrix. They may be written as:

\[
\sum_{i=1}^{n} \alpha^i \phi_{y_i, y_k} = 0, \quad \sum_{i=1}^{n} \beta^i \phi_{\bar{y}_i, y_k} = 0
\]

(2)

where \(\phi_{y_i}\) denotes \(\frac{\partial \phi}{\partial y_i}\) etc.

The next few rows contain an obvious transformation (2) using only the rules of differentiation together with some new definitions:

\[
\sum_{i=1}^{n} \alpha^i \phi_{y_i \bar{y}_k} = \sum_{i=1}^{n} \alpha^i \phi_{y_i} \quad \text{(3)}
\]

\[
\sum_{i=1}^{n} \beta^i \phi_{\bar{y}_i \bar{y}_k} = \sum_{i=1}^{n} \beta^i \phi_{\bar{y}_i} \quad \text{(4)}
\]
Introducing two new functions

\[ R = \sum_{i=1}^{n} \alpha i \phi y_i, \quad \bar{R} = \sum_{i=1}^{n} \beta i \phi \bar{y}_i, \]

considering them as a functions of arguments \( R = R(\alpha, y), \bar{R} = \bar{R}(\beta, \bar{y}) \)
(under the assumption that \( \det J(\alpha, \bar{y}) \) and \( \det J(\beta, y) \) are different from zero), we rewrite the equations (3) and (4) in an equivalent form:

\[ R(\alpha, y), \quad \bar{R}(\beta, \bar{y}). \] (5)

Multiplying each equation (5) respectively by \( \alpha_i, \beta_i \), summing the results and recalling the definitions of the functions, \( R, \bar{R} \) we come to the conclusion that they are homogeneous functions of degree one with respect to the arguments \( \alpha, \beta \). Introducing the notation

\[ \frac{\alpha_i}{\alpha_n} = u^{\pi}, \quad \frac{\alpha_i}{\alpha_n} = v^{\pi} \]

with the convention that Greek indices take values from 1 to \( n - 1 \) we can represent the dependence of the functions \( R, \bar{R} \) in the following form:

\[ R = \alpha_n R(u, y), \quad \bar{R} = \beta_n \bar{R}(v, \bar{y}) \]

Substituting these expressions into equations (6) we arrive at the following relations which form the basis of our further investigations:

\[ \phi y_{\beta} = R u^{\beta}, \quad \phi y_{n} = R - \sum u^{\beta} R u^{\beta}, \] (6)

\[ \phi \bar{y}_{\beta} = \bar{R} v^{\beta}, \quad \phi \bar{y}_{n} = \bar{R} - \sum v^{\beta} \bar{R} v^{\beta}, \] (7)

3 Conditions of selfconsistency (part I)

Using the condition of equivalence of second mixed derivatives taken in different orders, we will be able to disentangle the main system (8), (7) and extract from it a very important system of equations connecting the functions \( u, v \) only. For this goal let us calculate the second mixed derivatives of the following pairs of variables \( (y_\alpha, \bar{y}_\beta), (y_\beta, \bar{y}_n), (y_\beta, y_n), (y_n, \bar{y}_n) \) and equate them.
We have in consequence for the pair \((\bar{y}_\beta, y_\alpha)\):

\[(R_u^\alpha)_{\bar{y}_\beta} = (\bar{R}_{v^\beta})_{y_\alpha}, \quad \sum R_{u^\alpha, u^\beta} u^\theta_{\bar{y}_\beta} = \sum \bar{R}_{v^\beta, u^\theta} v^\beta_{y_\alpha};\]

for the pair \((y_\beta, \bar{y}_n)\):

\[(R_u^\alpha)_{\bar{y}_n} = (\bar{R} - \sum v^\phi \bar{R}_{v^\phi})_{y_\beta}, \quad \sum R_{u^\beta, u^\theta} u^\theta_{\bar{y}_n} = -\sum v^\pi \sum \bar{R}_{v^\pi, u^\theta} v^\theta_{y_\beta};\]

for the pair \((y_\beta, y_n)\):

\[\sum \bar{R}_{v^\beta, u^\theta} v^\theta_{y_n} = -\sum u^\pi \sum \bar{R}_{u^\pi, u^\theta} u^\theta_{\bar{y}_n};\]

and finally the pair \((y_n, \bar{y}_n)\) leads to the equations

\[\sum u^\beta \sum R_{u^\beta, u^\theta} u^\theta_{\bar{y}_n} = \sum v^\beta \sum \bar{R}_{v^\beta, v^\theta} v^\theta_{y_n}\]

Multiplying the first equations respectively by \(u^\alpha, v^\beta\), summing the results and comparing with respectively the second and the third systems we arrive at the following systems of equations:

\[\sum R_{u^\beta, u^\theta} (u^\theta_{\bar{y}_n} + \sum v^\pi u^\theta_{y_n}) = 0, \quad \sum \bar{R}_{v^\beta, u^\theta} (v^\theta_{y_n} + \sum u^\pi v^\theta_{y_n}) = 0.\]

Assuming that \(R\) and \(\bar{R}\) are not solutions of the M-A equations in \((n - 1)\) \(u, v\) spaces respectively (this case of degeneracy demands additional consideration) we conclude that the functions \((u, v)\) satisfy the following separate system of equations:

\[u^\theta_{\bar{y}_n} + \sum v^\pi u^\theta_{y_n} = 0, \quad v^\theta_{y_n} + \sum u^\pi v^\theta_{y_n} = 0\] (8)

The system (8) was solved before [5] but for the convenience of the reader the next two sections will be devoted to its consideration.

The last comment is the following; the hydrodynamic system (8) is the result of only \(2(n - 1)\) equations of second mixed derivatives. Namely combinations of the first, second and third systems. It is not difficult to check that the equation for the pair \((y_n, \bar{y}_n)\) is automatically satisfied.

The \((n - 1)^2\) equations remaining unsolved connecting the pairs with barred and unbarred Greek indices will be considered in the section 6.
4 The system of hydrodynamic type

We understand by a system of hydrodynamic type the system (8) rewritten below:
\[ v^\nu y_n + \sum u^\mu v^\nu y^\mu = 0, \quad u^\mu y_n + \sum v^\nu u^\mu y^\nu = 0 \] (9)

Two propositions with respect to this system will be crucial in what follows.

Proposition 1. The pair of operators:
\[ D = \frac{\partial}{\partial y_n} + \sum u^\mu \frac{\partial}{\partial y^\mu}, \quad \bar{D} = \frac{\partial}{\partial \bar{y}_n} + \sum v^\nu \frac{\partial}{\partial \bar{y}^\nu} \] (10)

are mutually commutative if \((u^\mu, v^\nu)\) are solutions of the system (9).

Acting with the help of operators \((D, \bar{D})\) on the second and the first equations of (9) respectively we come to conclusion that \(2(n-1)\) functions:
\[ \bar{D}(v^\nu) = v^\nu y_n + \sum v^\mu v^\nu y^\mu, \quad D(u^\mu) = u^\mu y_n + \sum u^\nu u^\mu y^\nu \] (11)

are also solutions of the first and the second system of equations (9).

As a corollary we obtain the following

Proposition 2
\[ v^\nu y_n + \sum v^\mu v^\nu y^\mu = V^\nu(v; \bar{y}), \quad u^\mu y_n + \sum u^\nu u^\mu y^\nu = U^\mu(u; y) \] (12)

Indeed the \(n\) sets of variables \((1, u)\), and \((1, v)\) respectively satisfy a linear system of algebraic equations of \(n\) equations, the matrix of which coincides with the Jacobian matrix
\[ J = \det_n \begin{vmatrix} v^1 & \cdots & v^{n-1} & V^\nu \\ y_1 & \cdots & y_{n-1} & y_n \end{vmatrix} \]

which in the case of a non-zero solution of the linear system must vanish. So Proposition 2 is proved.

Compared with (9) (12) is an inhomogeneous system of hydrodynamic equations separated into functions \((u, v)\).

With respect to the generators \(D, \bar{D}\) all functions of \(2n\) dimensional space may be divided into the following subclasses: functions of general position \(F, DF \neq 0, \bar{D}F \neq 0\), the ”holomorphic” functions \(f, \bar{D}f = 0, Df \neq 0\), ”antiholomorphic” ones \(\bar{f}, D\bar{f} = 0, \bar{D}\bar{f} \neq 0\) and ”central” functions \(f^0\) which
are holomorphic and antiholomorphic simultaneously; \( \bar{D}f^0 = Df^0 = 0 \). Each central function may be represented in the form:

\[
f^0 = f^0(Q) = f^0(P) = g^0(\psi)
\]  

(13)

The reader will find the definitions of the functions \( Q, P, \psi \) in the next section.

5 General solution of the hydrodynamic system

Suppose we have the following system of equations defining implicitly \((n - 1)\) unknown functions \( (\psi) \) in \((2n)\) dimensional space co-ordinatized by \((y, \bar{y})\):

\[
Q^\nu(\psi; y) = P^\nu(\psi; \bar{y})
\]

(14)

The number of equations in (14) coincides with the number of unknown functions \( \psi^\alpha \).

With the help of the usual rules of differentiation of implicit functions we find from (14):

\[
\psi_y = (P_\psi - Q_\psi)^{-1}Q_y, \quad \psi_{\bar{y}} = -(P_\psi - Q_\psi)^{-1}P_{\bar{y}}
\]

(15)

Let us assume, that between \( n \) derivatives with respect to barred and unbarred variables there exists the linear dependence:

\[
\sum_{1}^{n} c_i \psi^\alpha_{y_i} = 0, \quad \sum_{1}^{n} d_i \psi^\alpha_{\bar{y}_i} = 0
\]

(16)

and analyse the consequences following from these facts.

Assuming that \( c_n \neq 0, d_n \neq 0 \), dividing each equation of the left and right systems respectively by them and introducing the notations \( u^\alpha = \frac{\psi^\alpha}{c_n}, v^\alpha = \frac{\psi^\alpha}{d_n} \) we rewrite the last systems in the form:

\[
\psi^\alpha_{y_n} + \sum_{1}^{n-1} u^\nu \psi^\alpha_{y_\nu} = 0, \quad \psi^\alpha_{\bar{y}_n} + \sum_{1}^{n-1} v^\nu \psi^\alpha_{\bar{y}_\nu} = 0
\]

(17)
Substituting the values of the derivatives from (15) and multiplying the result by the matrix \((P_\phi - Q_\phi)\) on the left we obtain:

\[
Q_\alpha y_n + \sum_1^{n-1} u^\nu Q_\alpha y_\nu = 0, \quad P_\alpha y_n + \sum_1^{n-1} v^\nu P_\alpha y_\nu = 0
\]  \tag{18}

From the last equations it immediately follows:

\[
u^\nu = -(Q_y)^{-1}Q_\alpha y_n, \quad v^\nu = -(P_\bar{g})^{-1}P_\alpha \bar{y}_n
\]  \tag{19}

We see that if we augment the initial system (14), by \((n-1)\) vector functions \((u, v)\) defined by (19) then operators of differentiation \(D, \bar{D}\) defined by (14) in connection with (17) annihilate each \(\psi\) either as a \(P\) or a \(Q\) function:

\[
D\psi = \bar{D}\psi = DQ = DP = DQ = DP = 0
\]  \tag{20}

These equations (20) explain the notations in the last formula (13) of the previous section.

This means that \(D\bar{f}(\phi, \bar{y}) = \bar{D}f(\phi, y) = 0\). As a direct corollary of this fact \(Dv = \bar{D}u = 0\), so the generators \(D, \bar{D}\) constructed above are mutually commutative. Thus we have the found general solution of the hydrodynamic system.

6 Conditions of selfconsistency (part II)

The general solution of the hydrodynamic system depends upon \(2(n-1)\) arbitrary functions \((P, Q)\) each dependent upon \(2n-1\) independent arguments. This collection of arbitrary functions is more much (except for the case \(n = 2\)) than is necessary for the general solution of the M-A equation. So it is possible to expect that other conditions of selfconsistency (unused up to now) reduce it up to two functions each of \(2n-1\) indepent arguments.

We begin from the remaining unsolved \((n-1)^2\) equations of section 3, rewritten below:

\[
\sum R_{\alpha,\alpha} u^\theta_{\alpha,\alpha} = \sum \bar{R}_{\alpha,\alpha} v^\theta_{\alpha,\alpha}
\]  \tag{21}

For this and all calculations below knowledge of the explicit expressions for the derivatives of the functions \((u, v)\) functions will be necessary. We have in consequence:

\[
u_{\alpha} = -Q_y^{-1}(Q_{y_n,\alpha} + \sum Q_{y_n,\alpha}^\beta \bar{y}^\beta_{\alpha} - Q_{y,\alpha} Q_y^{-1} Q_{y_n} - \sum Q_{y,\alpha}^\beta \bar{y}^\beta_{\alpha} Q_y^{-1} Q_{y_n})
\]
\[-Q_y^{-1}(DQ_{yo}) + Q_y^{-1}(DQ_{\psi})(P_{\psi} - Q_{\psi})^{-1}Q_{yo}\]

\[v_{yo} = -P_y^{-1}(DP_{yo}) + P_y^{-1}(DP_{\psi})(P_{\psi} - Q_{\psi})^{-1}P_{yo}\]

By the same technique we calculate \(u_\bar{g}, v_y\) with the result:

\[u_\bar{g} = -Q_y^{-1}(DQ_{\psi})(P_{\psi} - Q_{\psi})^{-1}P_{\bar{g}}, \quad v_y = -P_y^{-1}(DP_{\psi})(P_{\psi} - Q_{\psi})^{-1}Q_y\]

Substituting the calculated values of derivatives into (21), we obtain in matrix notation:

\[R_{u,u}Q_y^{-1}(DQ_{\psi})(P_{\psi} - Q_{\psi})^{-1}P_{\bar{g}} = R_{v,v}P_{\bar{g}}^{-1}(DP_{\psi})(P_{\psi} - Q_{\psi})^{-1}Q_{yo}\]

or after moving the matrices \(P_{\bar{g}}^{-1}Q_{yo}\) to the left and right respectively we obtain the matrix equation:

\[(Q_y^T)^{-1}R_{u,u}Q_y^{-1}(DQ_{\psi})(P_{\psi} - Q_{\psi})^{-1} = [(P_y^T)^{-1}R_{v,v}P_{\bar{g}}^{-1}(DP_{\psi})(P_{\psi} - Q_{\psi})^{-1}]^T\]

where \(T\) denotes transpose.

Now let us consider results which follow from equating the second mixed derivatives with unbarred Greek indices. Calculations similar to the previous lead to the final result containing \(\frac{(n-1)(n-2)}{2}\) equations:

\[-(Q_y^T)^{-1}R_{u,u}Q_y^{-1}(DQ_{\psi})Q_y^{-1} + (Q_y^T)^{-1}R_{u,u}Q_y^{-1}(DQ_{\psi})(P_{\psi} - Q_{\psi})^{-1} +
\]

\[(Q_y^T)^{-1}R_{u,y}Q_y^{-1} = [...]^T\]

where by dots in the quadratic brackets of the right side we denote the matrix of the left hand side of the last equation.

The same calculations for mixed partial derivatives with barred Greek indices leads to result:

\[-(P_y^T)^{-1}R_{v,v}P_{\bar{g}}^{-1}(\bar{D}P_{\bar{g}})P_{\bar{g}}^{-1} + (P_y^T)^{-1}R_{v,v}P_{\bar{g}}^{-1}(\bar{D}P_{\psi})(P_{\psi} - Q_{\psi})^{-1} +
\]

\[(P_y^T)^{-1}R_{v,y}P_{\bar{g}}^{-1} = [...]^T\]

Summing the systems (23) and (24) and taking into account (22) we eliminate terms with the factors \((P_{\psi} - Q_{\psi})^{-1}\) and come to the equation:

\[-(Q_y^T)^{-1}R_{u,u}Q_y^{-1}(DQ_{\psi})Q_y^{-1} + (Q_y^T)^{-1}R_{u,y}Q_y^{-1} - [...]^T =
\]

\[-(P_y^T)^{-1}R_{v,v}P_{\bar{g}}^{-1}(\bar{D}P_{\psi})P_{\bar{g}}^{-1} + (P_y^T)^{-1}R_{v,y}P_{\bar{g}}^{-1} - [...]^T \equiv A^0\]
Indeed the left hand side of the last equality depends upon the arguments $(u, y)$, while the right hand side depends upon the arguments $(v, \bar{y})$ and in view of the comments in the end of the section 4 the antisymmetrical $(n - 1) \times (n - 1)$ matrix $A^0$ is a central function.

The combination of the equations (22) and (23) leads to:

$$(Q_y^T)^{-1} R_{u,u} Q_y^{-1} (DQ_\psi)(P_\psi - Q_\psi)^{-1} - (P_y^T)^{-1} \bar{R}_{v,v} P_{\bar{y}}^{-1} (\bar{D}P_\psi(P_\psi - Q_\psi)^{-1} = A^0$$

(26)

After multiplication of last equation by the matrix $(P_\psi - Q_\psi)$ on the right and the observation that the left hand side of the equation arising is the difference of holomorphic and antiholomorphic functions we solve it with the final result:

$$(Q_y^T)^{-1} R_{u,u} Q_y^{-1} (DQ_\psi) = A^0 Q_\psi, \quad (P_y^T)^{-1} \bar{R}_{v,v} P_{\bar{y}}^{-1} (\bar{D}P_\psi) = A^0 P_\psi$$

(27)

This last system together with equation defining $A^0$ is now the subject of further investigation.

7 Solution of selfconsistency equations

At first we consider in detail the simplest examples $n = 2$ solved before [3], which will serve as a guess for a form of solution in the general case of arbitrary $n$.

7.1 The case $n=2$

In this case Greek index takes only one value 1, the antisymmetrical matrix $A^0$ is equal to zero. In this sense it escapes from from the general case. In spite of this the calculations in this case make a good exercise useful for further consideration. In this case the system of equations (3), (4) takes the following form:

$$\phi_{y_1} = R_u, \quad \phi_{y_2} = R - uR_u, \quad \phi_{\bar{y}_1} = \bar{R}_v, \quad \phi_{\bar{y}_2} = \bar{R} - v\bar{R}_v$$

The second mixed partial derivatives of the pairs $(y_1, \bar{y}_2), (y_1, y_2)$, and $(\bar{y}_2, y_2)$ have as their corollary the hydrodynamical system of equations:

$$v_{y_2} + uv_{\bar{y}_1} = 0, \quad v_{\bar{y}_2} + vu_{y_1} = 0$$
as in the general case; the general solution of which is given in connection
with the results of section 5 by the formulae [19]:

\[ u = \frac{Q_{y_2}}{Q_{y_1}}, \quad v = \frac{P_{y_2}}{P_{y_1}} \]

There is only one equation of selfconsistency, connecting the barred and
unbarred index 1:

\[ (R_u)_{\bar{y}_1} = (\bar{R}_v)_{y_1}, \quad R_u u_{\bar{y}_1} = \bar{R}_v v_{y_1}, \quad R_u u_{\psi} \psi_{\bar{y}_1} = \bar{R}_v v_{\psi} \psi_{y_1} \quad (28) \]

Substituting into (28) the known values of the derivatives of the func-
tion \( \psi \) (15) we pass to the final equation of interest:

\[ \frac{R_{uu} u_{\psi}}{Q_{y_1}} = -\frac{\bar{R}_{vv} v_{\psi}}{P_{y_1}} = A_0^\psi \]

Indeed the left hand side of the last equality is a holomorphic function, the
right hand side antiholomorphic one. Thus \( A_0^\psi \) is a central function.

The equations of selfconsistency for the pairs \((y_1, y_2), (\bar{y}_1, \bar{y}_2)\) may be ma-
nipulated to the following attractive form:

\[ DR_u = R_{y_1}, \quad D\bar{R}_v = R_{\bar{y}_1} \quad (29) \]

Considering now \( R_u = R_u(\psi; y_1, y_2) \) and \( \bar{R}_v = \bar{R}_v(\psi; \bar{y}_1, \bar{y}_2) \) we resolve equations containing \( A_0^\psi \) function in the form:

\[ R_u = \Theta_{y_1}(A; y_1, y_2), \quad Q = \Theta_A(A; y_1, y_2) \]
\[ R_v = \bar{\Theta}_{y_1}(A; \bar{y}_1, \bar{y}_2), \quad P = -\bar{\Theta}_{A}(A; \bar{y}_1, \bar{y}_2) \quad (30) \]

It remains only to check the equalities (29). Let us distinguish by the
upper indices \( u, A \) the derivatives \( \frac{\partial^u}{\partial y_1}, \frac{\partial^A}{\partial y_1} \) corresponding to the partial deriva-
tives of the space coordinates \((y, \bar{y})\) keeping \( u, (v) \) constant in first case and
\( A \) at a constant value in the second. The equality which has to be checked
in this notations is

\[ \frac{\partial^u}{\partial y_1} R_u = \frac{\partial}{\partial u} \frac{\partial^u}{\partial y_1} R \]
Keeping in mind that $D^uA = 0$ ($A$ is a central function) and the definition of all values involved in terms of the function $\Theta$, we obtain in consequence for the right hand side of the last equality:

$$(D^u R_u)_u = (D^u \frac{\partial A}{\partial y_1} \Theta)_u = (D^u \frac{\partial A}{\partial y_1} \Theta)_u A_u =$$

$$= (\frac{\partial^2 A}{\partial y_1 \partial y_2} + u \frac{\partial^2 A}{\partial y_1 \partial y_1})_u A_u = \Theta_{y_1,y_1} + (\Theta_{A,y_1,y_2} - \frac{\Theta_{A,y_1,y_1}}{\Theta_{A,y_1}} A_u) A_u =$$

$$= \Theta_{y_1,y_1} + \Theta_{A,y_1} A_u \frac{\partial A_u}{\partial y_1}$$

In all transformations above we have not written the upper index $A$ with respect to derivatives of the space coordinates $y$ applied to the function $\Theta$.

Similar calculations for the left hand side leads to:

$$\frac{\partial^u}{\partial y_1} \Theta_{y_1} = \Theta_{y_1,y_1} + \Theta_{A,y_1} A_u \frac{\partial A_u}{\partial y_1}$$

which shows that equalities (29) are satisfied.

But (29) in its turn is an equation of second order with respect to the unknown function $R$. We rewrite it in explicit form substituting instead of $R_u$ its value from (30):

$$(\frac{\partial^2}{\partial y_1 \partial y_2} + u \frac{\partial^2}{\partial y_1 \partial y_1}) \Theta = R^u_{y_1} \equiv R_{y_1} + R_A A_{y_1} =$$

$$R_{y_1} - \frac{u}{u_A} R_A = R_{y_1} - u_{y_1} R_u = R_{y_1} - u_{y_1} \Theta_{y_1}$$

(31)

In the process of the evaluation of the last expression the crucial element was the calculation of $A_{y_1}$ maintaining the value of $u$ fixed. It was achieved by direct differentiation of the definition of $u$ rewritten in the form:

$$\Theta_{A,y_2} + u \Theta_{A,y_1} = 0$$

with respect to the argument $y_1$ (under fixed $u$) and regrouping of the terms arising.

Preserving in the last equality the first and the last terms we arrive at the equation for the function $R$ in integrable form. The result of its integration
determines the function $R$ in terms of the function $\Theta$ in a very attractive form:

$$ R = D\Theta, \quad \bar{R} = \bar{D}\bar{\Theta} $$

Substituting these expressions in the equations connected derivatives of the solution of the M-A equation (6), (7) with $R, \bar{R}$ functions we obtain finally

$$ \phi_{y_1} = \Theta_{y_1}, \quad \phi_{y_2} = \Theta_{y_2}, \quad \phi_{\bar{y}_1} = \bar{\Theta}_{\bar{y}_1}, \quad \phi_{\bar{y}_2} = \bar{\Theta}_{\bar{y}_2} $$

Theorem:
Let the function $A$ be determined implicitly by the equation:

$$ \Theta_A(A; y_1, y_2) = -\bar{\Theta}_A(A; \bar{y}_1, \bar{y}_2) $$

where $\Theta, \bar{\Theta}$ are arbitrary functions of their 3 arguments, then selfconsistent derivatives of the function $\phi$ satisfying the complex M-A equation in two dimension are determined with the help of the formulae

$$ \phi_{y_1} = \Theta_{y_1}, \quad \phi_{y_2} = \Theta_{y_2}, \quad \phi_{\bar{y}_1} = \bar{\Theta}_{\bar{y}_1}, \quad \phi_{\bar{y}_2} = \bar{\Theta}_{\bar{y}_2} $$

8 General case of arbitrary $n$

We will not present the calculations which are not very simple allowing us to obtain the main result for arbitrary $n$ but give its formulation and prove it directly. This turns out to be much easier.

Theorem:
Let the the set of the functions $\psi^\alpha(y; \bar{y})$ be determined implicitly by the following set of equations, the number of which coincides with the number of $\psi$ functions:

$$ \Theta^\alpha_\psi(\psi; y) = -\bar{\Theta}^\alpha_\psi(\psi; \bar{y}) $$

where $\Theta, \bar{\Theta}$ are arbitrary functions of their $(2n - 1)$ arguments. Then selfconsistent derivatives of the function $\phi$ satisfying the complex M-A equation in $n$ dimension are determined with the help of the formulae:

$$ \phi_y = \Theta_y, \quad \phi_{\bar{y}} = \bar{\Theta}_{\bar{y}} $$

Let us first check the conditions of selfconsistency of the second mixed partial derivatives with the same (barred, unbarred) indices. We have consequently ($(y_1, y_2)$ - arbitrary two coordinates):

$$ (\phi_{y_1})_{y_2} = \Theta_{y_1,y_2} + \sum \Theta_{y_1,\psi^\nu} \psi^{\nu}_{y_2} = \Theta_{y_1,y_2} + \sum \Theta_{y_1,\psi} (\Theta_{\psi,\psi} + \bar{\Theta}_{\psi,\psi})^{-1} \Theta_{y_2,\psi} $$

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The matrix \((\Theta_{\psi,\psi} + \bar{\Theta}_{\psi,\psi})^{-1}\) is obviously symmetric so the last expression is symmetric with respect to a permutation of the indices \((1,2)\). Thus second mixed partial derivatives with the same kind of indices are self consistent.

Now let us calculate the mixed derivatives with indices of different kinds:

\[
(\phi_{y_i})_{\bar{y}_k} = \sum \Theta_{y_i,\psi} \psi_{\bar{y}_k} = \sum \Theta_{y_i,\psi} (\Theta_{\psi,\psi} + \bar{\Theta}_{\psi,\psi})^{-1} \bar{\Theta}_{\bar{y}_k,\psi}
\]

The result of the calculation in the opposite order gives exactly the same result also as corollary of symmetry of the same matrix.

Finally let us multiply the second mixed partial derivatives calculated above by \(\beta^k\) and sum the result. We see that from right hand side the term \(\bar{D}(\bar{\Theta}_{\psi})\), which is equal to zero always arises.

Thus we have proved that between the the rows (and columns) of the determinantal matrix linear dependence occurs and the so equation of M-A in \(n\) dimensional space is satisfied.

9 Outlook

The main result of the present paper is in the Thorem of the previous section, giving the possibility of finding a general solution of the homogeneous Complex M-A equation (1) in implicit form. We specially emphasize that we can’t say that we have found all solutions of this equation but only those in which the number of arbitrary functions and their functional dependence are sufficient for the statement of the problem of the solution of the M-A equation in terms of initial data in the sense of Cauchy-Kovalevski. This solution is the most nondegenerate and excludes solutions of the shock wave type. Moreover only after detailed analysis of all the many assumptions made and the precise consideration of the corollaries which follow from the results of section 6 (which we have ommitted in this paper) it will be possible to clarify the situation and increase our understanding of what collection of solutions is contained in the construction of the present paper.

No less important and interesting is the general solution of the hydrodynamic (8) system of equations. It turns out that, using it as a basis, it is possible to generalise two-dimensional theory of integrable systems (based upon representation theory of semisimple algebras) [4] in the multidimensional case [4].
Using the hydrodynamic system of equations it is possible to solve the Complex Bateman (Complex Universal) equation \([6]\). (This equation serves as a Lagrangian for the complex M-A) As the present solution of the \(M - A\) equation and that of the Complex Bateman correspond to different reductions of the general solution of hydrodynamic system, we can’t exclude the possibility that other interesting reductions exist which are connected with multidimensional (in the sense of the number of unknown functions) systems of equations of M-A and Universal equations hitherto unknown.

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