On the oscillation spectra of ultra compact stars

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ABSTRACT

Quasinormal modes of ultra compact stars with uniform energy density have been calculated. For less compact stars, there is only one very slowly damped polar mode (corresponding to the Kelvin f-mode) for each spherical harmonic index $l$. Further long-lived modes become possible for a sufficiently compact star (roughly when $M/R \geq 1/3$). We compare the characteristic frequencies of these resonant polar modes to the axial modes first found by Chandrasekhar and Ferrari \cite{Chandra}. We find that the two spectra approach each other as the star is made more compact. The oscillation frequencies of the corresponding polar and axial modes agree to within a percent for stars more compact than $M/R = 0.42$. At the same time, the damping times are slightly different. The results illustrate that there is no real difference between the origin of these axial and polar modes: They are essentially spacetime modes.
1. Introduction

In nonradial pulsations of a spherically symmetric star, the polar and the axial oscillation modes are quite different. Most importantly, the former couple to the density and pressure perturbations, while the latter do not. This means that an incident gravitational wave can only induce polar pulsation modes in the Newtonian limit. However, Chandrasekhar and Ferrari (1991b) have recently shown that this does not remain true in the relativistic case. Especially not if the surface of the star is located inside the Regge-Wheeler potential peak; when \( R < 3M \) or so. In such a case, gravitational waves can be temporarily trapped inside the barrier, and the system exhibits damped resonant oscillations also for axial perturbations. Chandrasekhar and Ferrari calculated eigenfrequencies for such resonant axial modes of highly compact stars, \( M/R > 0.41 \), where \( M \) and \( R \) are the mass and radius of the star (in geometrized units). They concluded that only a few modes were possible for each stellar model, but that the number of possible modes increased as the star became more compact. The extensive (and numerically reliable) calculations by Kokkotas (1994) do, however, make the existence of an infinite number of axial modes for each compact stellar model seem very likely. Most of these modes are rather rapidly damped and could not be distinguished using the resonance technique of Chandrasekhar and Ferrari.

In the stellar problem, axial modes depend only on the dynamical degree of freedom associated with gravitational waves. Polar perturbations, on the other hand, couple to the fluid oscillations and one would expect the polar and the axial spectra to be quite different. For example, the Newtonian p-modes adopt a small imaginary part (to account for radiation damping) when relativistic effects are included in the analysis. These polar oscillation modes can have no analogue among the axial modes. The situation is not at all that clear for the highly damped (w) modes (Kojima 1988, Kokkotas and Schutz 1992, Leins et al. 1993 and Andersson et al. 1994).
1995a). These are mainly spacetime modes, and there is no apparent reason why similar axial modes should not exist.

The situation prompts some interesting questions: Are there resonant polar modes analogous to the axial modes found by Chandrasekhar and Ferrari (1991b)? What is the relation between the branch of highly damped axial modes (cf. Kokkotas 1994) and the polar w-modes? Do the characteristic frequencies of the axial and the polar modes approach each other as the star is made more and more compact? One would certainly expect an affirmative answer to the first of these questions. There is no reason why such polar modes should not exist. In fact, one would expect the spectrum of polar modes to approach that of the axial modes for very compact stars. This expectation is based on the experience from studies of gravitational perturbations of a Schwarzschild black hole. In that case, the polar and axial perturbations are related by a certain transformation, and the corresponding quasinormal-mode frequencies are identical (Chandrasekhar 1983). In the case of stars, the equations that govern the perturbations in the exterior vacuum are identical to those for a black hole. It therefore seems likely that the two stellar spectra approach each other (in some sense) as the star becomes more compact, i.e., as more of the black-hole potential barriers come into play. Moreover, an assumption that the branch of highly damped axial modes (Kokkotas 1994) corresponds directly to the polar w-modes – that have so far only been studied for polytropic stellar models – is straightforwardly tested. If that is the case, highly damped axial modes should exist also for less compact stars ($M/R \geq 1/3$), even though such models cannot support “trapped” modes with a very slow damping. If these suggestions can be proved true it would illustrate beyond doubt that the axial modes – as well as the polar w-modes – are in all essential respects “spacetime” modes, the properties of which are determined only by the curvature of spacetime.

In this short paper we examine compact stellar models with uniform energy
density. Although astrophysically unrealistic, such models have the advantage that we need not worry about fluid oscillations. Moreover, these models can easily be made very compact. The present discussion concerns stellar models that approach the limit of compactness imposed by General Relativity: $M/R \leq 4/9$. It should be remembered that stars as compact as that, almost certainly, do not exist in our universe. A useful comparison is provided by the values $R = 10$ km and $M = 1.4M_\odot$ often used in rough calculations involving neutron stars. These values correspond to $M/R \approx 0.21$. Nevertheless, the ultracompact uniform density star serves as a reasonably simple model problem that can help us understand better the origin of the various oscillation modes of relativistic stars.

2. Comments on stars with uniform energy density

We assume that the energy density is constant throughout the star in its equilibrium state. We also assume that the Eulerian change of the energy density vanishes. In Newtonian pulsation theory, this approximation leads to a single oscillation mode; the Kelvin f-mode. In fact, we use the uniform density approximation to avoid “uninteresting” fluid oscillations that give rise to the p- and g-modes. These are well understood and there is no reason why we should include them here.

The equations that govern axial and polar perturbations of uniform density stars are easily derived from the equations used for more realistic stellar models. Consequently, we will not give many details here. Rather, we will outline the approach that we have used in each case, and refer the reader to the original papers for more details.

In the case of polar perturbations, the relevant equations can be obtained by imposing $\delta \rho = 0$, or equivalently $C^{-2} = 0$ in eqs. (38)–(39) of Kojima (1992). Explicit expressions for the remaining variables (such as the pressure) for a uniform density
The basic equations inside the star become two second-order differential equations for certain components of the metric perturbations \((H_0,l\) and \(K_l\)). Physical solutions to these two coupled equations must be regular at the centre of the star. One must also ensure that the Lagrangian change in pressure vanishes at the stellar surface.

Outside the star – in vacuum – the perturbation equations reduce to a single second-order differential equation. In the case of polar perturbations, this is the Zerilli equation familiar from studies of Schwarzschild black holes (Chandrasekhar 1983). A physically acceptable solution to the problem for the stellar interior generally corresponds to a linear combination of outgoing and incoming waves at infinity;

\[ K_l \rightarrow A_{\text{out}} \exp(-i\sigma r^*) + A_{\text{in}} \exp(i\sigma r^*) , \quad \text{as } r^* \rightarrow +\infty , \]

where \(r^*\) denotes the standard tortoise coordinate. \(A_{\text{in}}\) and \(A_{\text{out}}\) are the constant amplitudes of the incoming and outgoing gravitational waves at infinity, respectively. We assume that all perturbations have time-dependence \(\exp(i\sigma t)\). Quasinormal modes of the stellar system are distinguished by purely outgoing waves at spatial infinity, i.e., \(A_{\text{in}} = 0\). This condition is satisfied only for a discrete set of complex frequencies.

We use the resonance method developed by Chandrasekhar and Ferrari (1991a) to determine the slowest damped polar modes for the uniform density model. Assuming that the imaginary part \((\sigma_I)\) of the eigenvalue is small, we calculate \(|A_{\text{in}}|\) for real values of \(\sigma\). The curve of \(|A_{\text{in}}|\) then exhibits a deep minimum as \(|A_{\text{in}}|^2 = \text{const} \times \{(\sigma - \sigma_R)^2 + \sigma_I^2\}\). From the position and shape of such minima it is straightforward to deduce \(\sigma_R\) and \(\sigma_I\), which correspond to the real and imaginary parts of our complex eigenfrequencies.

The resonance method can only be trusted for modes which are slowly damped. For highly damped modes it must be replaced by an iterative method. Such a method must be able to deal with the problem that the quasinormal-mode eigenfunctions
diverge at spatial infinity. (It is clear from (11) that \( K_l \) diverges when the imaginary part of \( \sigma \) is positive.) Methods that have been developed to handle this difficulty include the WKB method used by Kokkotas (1994) and the numerical integration scheme of Andersson et al. (1994a). However, in the present analysis we do not wish to study highly damped modes. We are primarily interested in the “trapped” modes that occur as the star becomes extremely compact. These are going to be slowly damped so we are not much restricted by the limitations of the resonance method.

In the case of axial perturbations the problem has been described in detail by Chandrasekhar and Ferrari (1991b). Since there is no coupling to the fluid, the interior problem can be formulated as a single second-order differential equation analogous to the equation for the exterior vacuum; the Regge-Wheeler equation (Chandrasekhar 1983). As in the polar case, the physically acceptable solution – that is regular at the centre of the star – generally corresponds to a combination of out- and ingoing waves at infinity, and axial quasinormal modes can be computed in exactly the same way as the polar modes. In our calculations for axial modes we use the approach of Andersson et al. (1995b). This scheme was specially developed for highly damped modes (it finds the modes by iteration), but it works equally well for slowly damped modes.

The fact that the interior problem can be formulated as a single Schrödinger-like differential equation for axial perturbations inspired Chandrasekhar and Ferrari (1991b) to the discovery of slowly damped axial modes. The general idea is that, when the surface of the star lies inside the peak of the Regge-Wheeler potential barrier, the system can support “quasi-bound” states that slowly leak out through the barrier to spatial infinity. That similar modes should exist for polar perturbations is not as easily made apparent.
3. Discussion of numerical results

We have calculated the slowest damped quasinormal-mode frequencies for both axial and polar perturbations of uniform density models with a varying degree of compactness. All calculations discussed here are for quadrupole modes ($l = 2$). In Figure 1 we show the amplitude of incoming waves at infinity ($|A_{in}|$) as a function of real frequency $\sigma$ for four stellar models. The figure adheres to polar modes and contains information relevant for mode-calculation by the resonance method. One resonance minimum is evident in the stellar model with $M/R = 0.2$. The oscillation frequency is $\sigma_R(R^3/M)^{1/2} = 0.887$ and the damping rate is governed by $\sigma_I(R^3/M)^{1/2} = 3.6 \times 10^{-4}$. We have verified that this mode tends to the frequency of the Kelvin f-mode; $\sigma_R(R^3/M)^{1/2} = \sqrt{2l(l-1)/(2l+1)} = 0.894$ (cf. Tassoul 1978) in the Newtonian limit (as $M/R \to 0$). It is clear that, there is only one really slowly damped oscillation mode in less compact stars. This is expected from Newtonian pulsation theory. We know, of course, that there will also be an infinite number of highly damped (w) modes, but as the star gets less relativistic the damping of these modes increases (Kokkotas & Schutz 1992). Hence, such modes cannot be identified by the resonance method.

For the model with $M/R = 0.40$ one can see a tiny dent at $\sigma(R^3/M)^{1/2} \sim 1.7$ in Figure 1. The resonance minimum becomes evident in the model with $M/R = 0.42$, although the frequency has then shifted to $\sigma(R^3/M)^{1/2} \sim 1.3$. It is possible to determine this mode numerically for $M/R > 0.4$. This criterion is almost the same as that obtained in the axial-mode study of Chandrasekhar and Ferrari (1991b). The resonance point near $\sigma(R^3/M)^{1/2} \sim 1.7$ in the model with $M/R = 0.40$ is so ambiguous that reliable calculation by the resonance method is impossible. The
need for an alternative approach, for this and less compact models, is clear. As the star becomes more compact, further resonant modes become evident – as shown in the model with $M/R = 0.44$ in Figure 1. The number of distinguishable resonances clearly increases with the compactness.

Kokkotas (1994) calculated many higher overtone axial modes. His results have recently been confirmed as reliable by Andersson et al. (1995b). The imaginary part of most of those modes is so large that the resonance method can not be used to calculate them, however. But it is important to remember that each stellar model supports an (almost certainly) infinite number of axial modes.

In Figure 2 we compare the quasinormal-mode frequencies for polar perturbations to those for axial perturbations. The real and imaginary part of each mode-frequency are studied separately as a function of the compactness of the star, $M/R$. Our numerical calculation of polar modes is limited to $\sigma_I/\sigma_R \leq 0.01$ and it should be remembered that the minima of the amplitude $|A_{in}|$ become less clear as the star becomes less compact: The imaginary part of each mode increases. We have also limited the study to $M/R \leq 0.44$. This value is slightly smaller than the extreme case allowed by General Relativity; $M/R = 4/9 = 0.444\ldots$. As the star becomes extremely compact, there seems to be a mode-crossing between the f-mode and the first spacetime mode. This occurs near $M/R = 0.44$ in Figure 2. When this happens two minima of $|A_{in}|$ merge, and it is very difficult to calculate eigenfrequencies near the crossing point with the resonance method.

As can be seen from Figure 2, the agreement between the polar and the axial modes is good for all the studied models. In fact, the physical oscillation frequencies of the axial and polar modes agree to within a percent for models more compact than
$M/R = 0.42$. This close agreement does, indeed, support the idea that the axial and polar modes approach each other as the star gets very compact (when it, in a loose sense, “approaches” a black hole). In fact, the discrepancy is at the same level as the numerical uncertainties expected in the resonance method. It is also apparent from Figure 2 that the discrepancy between the imaginary parts is, in general, larger. We believe that this is to some extent due to numerical inaccuracies. When the imaginary part is many orders of magnitude smaller than the corresponding real part it is difficult to determine $\sigma_I$ with great accuracy.

4. Concluding remarks

Outside a star, both polar and axial perturbations are described by a second-order differential equation with an effective potential barrier, the peak of which lies at roughly $r \approx 3M$. When the stellar surface lies inside the peak of this barrier, gravitational waves can be temporarily trapped in a way that is reminiscent of “quasibound” resonance states in quantum scattering. These trapped gravitational-wave modes then decay gradually. The modes studied here correspond to this situation. As the star becomes more compact the decay rate, i.e., the imaginary part of the mode-frequency, decreases because the potential “well” inside the barrier gets deeper. Moreover, as the star is made increasingly compact more of the black-hole potentials – that govern the perturbation in the exterior spacetime – is unveiled. Since the axial and polar-mode spectra are manifestly identical for Schwarzschild black holes, one might expect the two spectra to approach each other for stars of increasing compactness. The results obtained in the present study support this

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1 It should, of course, be emphasized that the spectra studied here are considerably different from that of a black hole. This is, however, to be expected; There is simply no continuous transformation from a stellar model to a black hole. The inner boundary condition, i.e., the existence of a horizon is but one crucial factor that causes the differences.
view. It is clear that, there are considerable similarities between the quasinormal modes for polar and axial perturbations of compact stellar models. Since there are no fluid motion in a uniform density star the oscillation modes studied here are related to the dynamical degree of freedom of gravitational waves. With exception of the f-mode, there are no real differences between the mechanisms behind the polar and the axial modes here: They are all essentially “spacetime” modes.

In many ways, the modes studied here resemble the w-modes that has been found for polar perturbations of polytropic stellar models (Kojima 1988, Kokkotas & Schutz 1992, Leins et al. 1993, Andersson et al. 1995a). For these modes, the gravitational-wave degrees of freedom play an essential role. All w-mode calculations to date have been for much less relativistic stellar models than those considered here. However, as can be inferred from Figure 2, the imaginary part of each “trapped” mode increases (roughly) exponentially as $M/R$ decreases. If we recall that the slowest damped w-mode for a polytropic model with $M/R = 0.297$ is $(2.910 + 0.346i)(R^3/M)^{1/2}$ (Andersson et al. 1995a), this makes it, indeed, plausible that the modes studied here are intimately connected to the w-modes in a less compact star. If that is the case, one should be able to confirm the existence of very long-lived w-modes for compact polytropes.

Furthermore, it seems logical to predict that highly damped axial modes should exist for much less relativistic stars. No one has actually searched for such modes, but there is no apparent reason why they should not exist. The argument has been that quasinormal modes rely upon coupling to the fluid for their existence. We believe that this argument is outdated, and that the results discussed here (and also the recent ones by Kokkotas (1994) and Andersson et al. (1995b)) demonstrate that these are modes which, in all essential respects, are due to the curvature of spacetime in the neighbourhood of the star. To verify the existence of highly damped axial modes for less relativistic models, and test the correspondence between the axial
and polar spectra further, one must calculate eigenfrequencies with large imaginary parts. That task is beyond the scope of the present work, but we aim to address it in the future.

Acknowledgements

This work was inspired by a discussion at the Seventh Marcel Grossman meeting at Stanford in July 1994. During that discussion Prof. S. Chandrasekhar advanced the view that the axial and the polar spectra would approach each other as a star is made more compact. At that time we had preliminary results that supported this view, but the discussion prompted us to study the problem in more detail.

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**Figure caption**

Fig.1. The amplitude of the incoming gravitational wave at spatial infinity, $|A_{in}|$, as a function of frequency, $\sigma(R^3/M)^{1/2}$, for different uniform density stellar models. This figure is for polar perturbations. From the minima quasinormal frequencies can be calculated.

Fig.2. The quasinormal-mode frequencies for polar and axial perturbations as functions of the compactness $M/R$. The upper panel shows the imaginary part and the lower one the real part. The full-drawn curve is for polar modes and the dashed one for axial modes. The circles represent the values of the lowest mode of the axial perturbation obtained by Chandrasekhar and Ferrari (1991b) and the triangles those of the higher modes by Kokkotas (1994). The line indicated by f represents the frequency of the Kelvin f-mode.
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