A programme to determine the exact interior of any connected digital picture

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Abstract

Region filling is one of the most important and fundamental operations in computer graphics and image processing. Many filling algorithms and their implementations are based on the Euclidean geometry, which are then translated into computational models moving carelessly from the continuous to the finite discrete space of the computer. The consequences of this approach is that most implementations fail when tested for challenging degenerate and nearly degenerate regions. We present a correct integer-only procedure that works for all connected digital pictures. It finds all possible interior points, which are then displayed and stored in a locating matrix. Namely, we present a filling and locating procedure that can be used in computer graphics and image processing applications.

Keywords: Raster Graphics, Region Filling, Point-based Modelling, Discrete Jordan Curve Theorem

1. Introduction

Region filling is a crucial operation in computer graphics and image processing. There are two basic approaches to region filling on raster systems. One of them is to start from a given interior position called seed, and paint outward from this point until we meet the specified boundary conditions. The other is to determine the overlap intervals of scan lines that cross the region. The seed-fill methods are mostly applied in interactive painting systems. Alternatively, the scan-conversion approach is typically used in hardware implementations and in general graphic packages to fill polygons, circumferences, ellipses and spline curves. These two approaches will be resumed in Section 2, but now we focus on their limitations: the non-robustness of scan conversion, and the high computational cost of seed fill.

As explained in Section 2, neither seed fill nor scan conversion can really provide perfect filling at the lowest computational cost. This is yet desirable in many circumstances. We list three of them.

The first is represented in Figure 1. It schematises the mould of an integrated circuit. In general, thousands of models are manufactured from the mould. A single flaw in it will produce useless lots, and therefore a great waste of material and time. Until nowadays, this kind of flaw detection still depends on handwork.

In fact, Figure 1 right is an output of our algorithms. They aim at filling regions perfectly and quickly. As an application, this helps detect imperceptible flaws. For the time being, however, they can detect only external flaws, but are adaptable for the whole job.

Our second example is of theoretical interest. Of course, there is already a valuable and extensive literature about region filling and point location. In fact, both problems are intimately related. Now, according to [1], none of them is free of minor failures, and the best ones are computationally expensive. Therefore, if there are algorithms that prove to be quick and faultless, they are really worth studying.

The third and most striking example is the technology of 3D-printing. This paper deals with a two-dimensional approach. But our algorithms were elaborated with ideas that are extendable to higher dimensions. In fact, we are already researching this problem for future works. The basic idea is to slice a 3-dimensional object into several 2-dimensional ones, which are then treated individually. In this case, least computational time becomes highly important. But also perfect filling and point location. The problem of printing plastic cups, for instance, is equivalent to what Figure 1 illustrates.

Of course, when performing region filling or point location we can content ourselves with flaws as we see in Figure 2, drawn in Xfig and saved as a PostScript file. If not, seed fill reduces them to a minimum but usually with a high computational cost.

Our research was motivated by the fact that scan-conversion algorithms are not always reliable. In the case of point location algorithms, they fail even for non-challenging regions, as demonstrated in [2]. Moreover, in [1] Schirra presents a complete comparison among all classical point-location algorithms.

In this paper, the difficulties in discretisation are acknowl-
2. Seed fill and scan conversion

Seed-fill algorithms can be subdivided into four steps [3]: a start procedure to choose an interior starting point, namely the seed; a propagation method to determine the next considered point; an inside test to check whether a point is in the region and if it should be marked; a set procedure to change the colour of a given picture element or pixel. The start procedure can be done automatically or user-guided. For complex objects, multiple seeds may be required and their automatic detection can be very difficult. This is one of the main reasons why this family of algorithms is more suitable for interactive region filling [4]. For the propagation method, the naive implementation of the algorithm searches for connected pixels recursively, and can be based either on pixel- or line-adjacency. In the former, the filling procedure is to move from a current pixel to all immediate neighbours in a certain order. In the latter, the interior of a region is regarded as a set of adjacent horizontal line segments. Usually, seed-fill algorithms need working memory in addition to the frame buffer, and are slower than scan-conversion algorithms. The amount of working memory cannot be constant, for it increases with both size and complexity of the region. Consequently, it cannot be stored in advance. Compared to seed-fill algorithms, scan-conversion algorithms are generally faster and require less or no additional working memory. In order to overcome the difficulties of seed-fill algorithms, different strategies like [5, 6] try to re-model the region and so get a fast image manipulation, processing and displaying.

Levoy and Whitted suggest points rather than triangle meshes as display primitives [5, 6, 7]. Similarly, we propose points as low-level modelling primitives, and so get rid of the well-known discretisation problems of the polygon and of all general curved boundaries too. Points are building bricks of geometry discretisation, and they can also provide a common format at the lowest level for both constructed and scanned geometric data. Namely, for both computer graphics and image processing.

3. The mathematical structure

A dominant reason for which raster graphics algorithms fail in robustness is the absence of formal specifications. Raster systems are discrete devices, which generate image by displaying the intensity value of each pixel in a finite two-dimensional matrix of pixels. But discrete definitions are implicitly assumed to be like their respective counterparts of the continuous Euclidean space. Commonly, a formal definition of interior/exterior of digital objects is omitted. This can cause difficulties in differing between the rules of odd-parity and nonzero winding number, and mainly in comparing different algorithms. Figure 2 shows a typical ambiguity that arises when such terms are treated only implicitly.

Now we define some basic terms, and the first is our space of points:
Definition 3.1. Given any two positive natural numbers \( N \) and \( M \), a canvas in \( \mathbb{Z}^2 \) is the set \( \{(p_x, p_y) : 1 \leq y \leq N, 1 \leq x \leq M\} \), where \( N \times M \) indicates its dimension.

In a computing context, a digital image is a two-dimensional array of values reflecting the brightness and colours of the pixels.

Definition 3.2. Let \( S = S_{N \times M} \) be a canvas. For any \( f : S \rightarrow \{0, 1\} \), we say that the set \( f^{-1}(1) \) is a digital picture in the canvas.

In Definition 3.2, our convention is to think of 0 as white and 1 as black. Therefore, the digital picture is a set of black dots over a white background. As remarked in the introduction, our input is a digital picture \( f^{-1}(1) \), or just a matrix with entries 0 and 1. Our filling algorithm does not require any other mathematical expression for the function \( f \).

Definition 3.3. A discrete curve in a canvas \( S \) is an 8-connected list \( L : \{0 \ldots n\} \rightarrow S \). We say that \( L \) is monotone and closed when \( L(i) = L(j) \iff i = j \mod n \).

Definition 3.4. Consider a digital picture \( P \) with its corresponding \( f : S \rightarrow \{0, 1\} \). If any two points \( p, q \in f^{-1}(1) \) always admit a discrete curve \( L : \{0 \ldots n\} \rightarrow S \) with \( L(0) = p, L(n) = q \) and \( f(L(i)) = 1 \forall i \), then we say that \( P \) is connected.

These definitions are comparable to those given in works by Rosenfeld and Kong [10, 11]. Notice that the same digital picture can correspond to distinct discrete curves. By looking at Figure 3 the black pixels on either case form exactly the same digital picture \( P \). However, we can track these pixels in different ways. Each image comes from a different discrete curve that represents \( P \).

Figure 3 helps understand the following definition:

Definition 3.5. Given \( p \in L([0 \ldots n]) \), consider the biggest square \( Q \) such that \( p \in Q \subset L([0 \ldots n]) \). Its dimension is \( k \times k \) for a certain \( k \geq 1 \). Then \( k \) is the thickness of the discrete curve \( L \) at \( p \).

Definition 3.6. We say that a discrete curve \( L \) is thin when it has thickness 1 everywhere.

In practice, Definition 3.6 can be replaced by a weaker condition:

Definition 3.7. An injective discrete curve \( L : \{0 \ldots n\} \rightarrow S \) is locally thin when the following holds. There are intervals \( I_1 \cup \ldots \cup I_k = [0 \ldots n] \) such that \( |I_j| \geq 4 \) and \( L|_{I_j} \) is thin, \( \forall j \in \{1, \ldots, k\} \).

For instance, the black pixels in Figure 4 cannot represent the image of an \( L \) that is locally thin. We shall give more details in Section 5.

Our purpose is to consider any digital picture \( P \), solely restricted to Definition 3.4. For instance, take a more general example like in Figure 4. Now the question is: How to describe our input, a digital picture \( P \) as in Figure 4 in terms of a function \( L \) in accordance with Definition 3.3?

This question will be answered in Section 5. In fact, that section introduces what we call the Lego curve. Its set of pixels will not always coincide with \( P \). However, starting from the Lego curve it is then possible to obtain a function \( L : \{0 \ldots n\} \rightarrow S \) as in Definition 3.3 such that \( L([0 \ldots n]) = P \).

Notice that Definition 3.3 does not require a discrete curve to be simple. Namely, it can have self-intersections defined as follows:

Definition 3.8. Based on Definition 3.3, we say that \( L \) has
- a self-crossing when there are two disjoint intervals \( I, J \subset [0 \ldots n] \), \( |I| = |J| = 2 \), such that \( L(I) \neq L(J) \) and all pixels \( L(I \cup J) \) are 8-connected.
- an overlapping when we have \( I, J \subset [0 \ldots n] \), \( \min(|I|, |J|) \geq 2 \), such that \( I \cap J \) is unitary, and either \( L(I) \subset L(J) \) or \( L(J) \subset L(I) \).

If any of these two cases occur, then \( L \) has a self-intersection. Otherwise, it is a simple curve.

Examples in \( \mathbb{R}^2 \) can clarify Definition 3.8. For instance, the so-called Bernoulli’s Lemniscate is a closed curve, of which...
the parametric equations are \( x(t) = 2 \cos t/(1 + \sin^2 t), \ y(t) = \sin 2t/(1 + \sin^2 t) \), with \( t \in [-\pi, \pi] \). Its trace is \( \infty \) (the symbol of infinity). In this case we just have a self-crossing at the origin. For the curve \((0, \sin t), \ t \in [0, \pi], \) overlapping occurs at \( t = 0 \) and at \( t = \pi/2 \). Notice that \( t \) could vary in a bigger domain, but we always take the smallest for which the curve is closed.

The reader must have noticed that we treat some technical terms as synonyms: point and pixel, discrete and digital, etc. In fact, this started to happen right at Definition 3.1, because our constructions are devoted to both theoretical and practical contexts. The reader is free to choose any of these contexts, and then think in accordance with the corresponding terms.

4. The main programme

In this section we discuss the algorithm of our main programme, named loci.m after a mnemonic to “location of curve interior”. In Section 2 the reader will find details about downloading and running it. The programme loci.m is an implementation of the “Filling Up Algorithm” (FUA). Before discussing FUA, it is important to explain some hypotheses under which we claim that loci.m will always give a correct answer.

We recall the difference between two similar terms: connectedness and connectivity. Connectedness of a digital picture \( P \) is a global property implied by Definition 3.4. Connectivity is a local property that applies to each pixel with respect to the ones that surround it in a digital picture \( P \).

Namely, given any \( p \in P \), if either \( p \pm (1,0) \) or \( p \pm (0,1) \) is again in \( P \), then it has connectivity 4 with \( p \). If any of \( p + (\pm 1, \pm 1) \) is in \( P \), then it has connectivity \( D \) with \( p \). Finally, when both 4- and \( D \)-cases apply to \( p \), we say in short that surrounding pixels have connectivity 8 with \( p \).

Now we summarise the three conditions on \( L \) that will make FUA correct. The third condition will be explained in the sequel:

- **C1.** \( L \) is locally thin;
- **C2.** \( L \) can only be tracked horizontally and vertically. In other words, \( L : [0 \ldots n] \rightarrow S \) is such that \( L(i) \) and \( L(i + 1) \) are 4-connected \( \forall i < n \);
- **C3.** \( L \) is “spike-free”.

Before discussing C3, we need to introduce a concept called \( i - o \) pixels (see Definition 3.1) which is analogous regarding scan-conversion algorithms to the usual marking phase required to compute the intervals of points contained in the region.

From upside down, the scanning of a digital picture gives an \( N \times M \) matrix of black and white pixels. Typical rows are represented in Figure 5. We track each row from the left to the right. In this process, \( i \)-pixels indicate the entry to and \( o \)-pixels the exit from \( P \). When a pixel is simultaneously \( i \) and \( o \), we mark it with an \( x \).

In Section 5 we are going to introduce CoTRA, Connectivity and Thickness Reduction Algorithm. CoTRA produces our “Lego curve” \( L \).

Figure 6 gives an example of the input (top) and the output (bottom) by CoTRA applied to a digital picture \( P \). In fact, \( P \) is a detail of Figure 9 obtained from the drawing of a non-self-intersecting polygon in Xfig. The spurious self-intersection at the left middle occurs due to discretization errors when moving from the ideal mathematical model to the digital picture.

On top, \( P \) is given as a set of black pixels in the white canvas. At the bottom, we mark blue pixels that are originally from \( P \), and \( i - o \) pixels in magenta. The blue and magenta pixels form the sought after Lego curve \( L \). However, notice that not all pixels were marked by CoTRA. For example, close to \((y, x) = (130, 70)\) it found a stretch interpreted as a self-intersection of \( P \). Moreover, close to \((120, 90)\) and \((150, 90)\) we have turning points. CoTRA cuts them by keeping local thickness 1. We assert that FUA will be correct for Lego curves.
L (see Section 6 for details).

In Section 5, we formalise “interior” of $\mathcal{P}$ by means of Definition 5.3.

Now the reader may be curious about the turning points of Figure 6. Of course, the one close to (120, 90) could also have thickness 1 and look like a thinner “spike”. In technical terms, we have

**Definition 4.1.** Let $L : [0 \ldots n] \to S$ be a monotone discrete curve with $L(i) = (L_x(i), L_y(i)) \forall i$. For all $3 < k < 3$, if there is $i$ such that $|L_x(i + 1) - L_x(i)| < 1$ and either $L_y(i) \geq L_y(i + 1)$ or $L_y(i) \leq L_y(i + 1)$, then $L$ has a vertical spike at $i$. If there is $i$ such that $|L_x(i) - L_x(i - 1)| < 1$ and either $L_y(i) \geq L_y(i + 1)$ or $L_y(i) \leq L_y(i + 1)$, then $L$ has a horizontal spike at $i$.

Handling a picture with spikes proved to needlessly increase the complexity of FUA. Indeed, we are going to obtain the interior of $\mathcal{P}$ as the interior of $L$ minus the set $\mathcal{P}$ itself. CoTRA was programmed in order to eliminate spikes and then correct FUA.

As in any improvement, CoTRA adds a computational cost. But the users can choose: if the first output is not satisfactory, they call CoTRA for a second answer.

Now we present the basic algorithm to fill up $\mathcal{P}$ by marking its entry-pixels (see Algorithm 1).

**Algorithm 1** The “Filling Up Algorithm” (FUA)

**Input:** Digital Picture $\mathcal{P} = f^{-1}(1) \subset \text{img}(\cdot, \cdot, 1)$

**Output:** Interior of $\mathcal{P}$

1. for $i = 2 : \text{length}(\text{img}(\cdot, 1, 1)) - 1$ do
2. $\text{bps} \leftarrow \text{Find}_{\text{joPixels}}$
3. for $k = 1 : 2 : \text{length}(\text{bps}) - 1$ do
   
   5. $\text{shorten bps}$
6. for $k = 2 : 4 : \text{length}(\text{bps}) - 1$ do
7. $\text{img}(i, \text{bps}(k)) : \text{bps}(k + 1), 3) = 0$

Notice that the “spike-free” condition is important in lines 4-5, where we “jump” the $i - o$ pixels of a local maximum (or minimum) stretch of the picture. This is because the status of being inside/outside the picture does not change at passing through such stretches.

For the rest, lines 6-7 simply mark the actual interior points of the $i$th row in yellow ($R = 255$, $G = 255$, $B = 0$). In the next section, we shall formalise several terms already used herein.

5. **Definition of the Lego curve**

At any row of pixels, the digital picture has white and black ones. Fix any row and consider the pixels in it. Up to scaling, each of them has an $x$-coordinate that ranges from a natural number to its successor.

**Definition 5.1.** Given a row $R$ of black and white pixels, we say that $p \in R$ is an $i$-pixel when $p$ is white and $p + 1$ is black. We say that $p \in R$ is an $o$-pixel when $p$ is white and $p - 1$ is black. If $p$ is simultaneously $i$ and $o$, we call it an $x$-pixel.

The letters $i$ and $o$ stand for the process of getting inside/outside the black horizontal segments from Figure 4, as depicted in Figure 5. Definition 5.1 implies that the $i$-$o$ sequence is always alternated, if not empty. By adding a frame of white pixels around the picture, we guarantee the sequence to begin with $i$ and end with $o$. We have added this frame in Figure 7.

![Figure 7: A digital picture and its corresponding track of the Lego curve.](image)

We recall Definition 5.7. In Figure 7 the black pixels form a picture $\mathcal{P}$. Notice that $\mathcal{P}$ cannot be the image of a discrete curve that is locally thin. This is only possible if we remove at least 9 pixels from $\mathcal{P}$, namely (7, 7) and (8, 7 – 14).

In order to define the Lego curve, we first need to introduce two concepts as follows:

**Definition 5.2.** Consider a pixel canvas $S$ with dimension $N \times M$, and also a digital picture $\mathcal{P}$ contained therein. Suppose $\mathcal{P}$ is surrounded by the white pixel frame $F := [1, N] \times [1, M] \cup [1, N] \times [1, M]$. In this case, the exterior of $\mathcal{P}$ is the set $E$, $F \subset E \subset (S \setminus \mathcal{P})$, such that for any two points $p, q \in E$ there exists a discrete curve $L : [0 \ldots n] \to (S \setminus \mathcal{P})$ with $L(i)$ and $L(i + 1)$ 4-connected $\forall i < n$.

**Definition 5.3.** Under the hypotheses of Definition 5.2 if $E$ is the exterior of $\mathcal{P}$, then its interior is $S \setminus (\mathcal{P} \cup E)$.

**Definition 5.4.** Let $\mathcal{P}$ be a digital picture and $p$ a corresponding $i$ or $o$-pixel, such that $p$ belongs to the exterior of $\mathcal{P}$. If $\mathcal{P}$ has two pixels that are D-connected to each other, and also 4-connected to $p$, then we say that $p$ is an $L$-pixel of $\mathcal{P}$.

In Definition 5.4 if we write the pixel coordinates as (line, column) in Figure 7 then $\mathcal{P}$ has exactly the following L-pixels: $(2, 5)$, $(4, 5)$, $(7, 2)$, $(10, 4)$, $(10, 8)$, $(9, 20)$, $(6, 22)$, $(4, 23)$, (3, 20), (3, 15), (5, 10), (3, 9) and (2, 8).

**Definition 5.5.** Given a digital picture $\mathcal{P}$ with exterior $E$, let $L$ be the set of L-pixels of $\mathcal{P}$ that are 4-connected with $E$. The Lego curve of $\mathcal{P}$ is a digital curve of minimal length that satisfies $C_2$, and has an exterior $E' \subset E$ such that $E \setminus E' \equiv L$.

We conclude this section by remarking that Definition 5.5 does not require the Lego curve to be locally thin, not even free of self-intersections. However, they can only occur in the special form of overlapping, not self-crossing. The FUA does work in the former case, but can fail in the latter.
6. Existence and uniqueness of the Lego curve

In this section we present our main theorem, which translates to the open source code lego.m. We have written our proof like a pseudocode.

**Theorem 6.1.** Any digital picture $\mathcal{P}$ admits a unique corresponding Lego curve $L$.

**Proof:** We are going to construct a list $L : [0 \ldots n] \to \mathbb{Z}^2$. Take $\mathcal{P}$ as in Definition 5.2. Its pixels will be called black. However, we shall say that a pixel is white exactly when it belongs to the exterior of $\mathcal{P}$. In what follows, both the words increase/decrease mean “by one”.

Consider the $i$-th pixels of $\mathcal{P}$ with least ordinate $y$. Among them, name $p$ the one with the least abscissa $x$. It has coordinates $(p_x, p_y), 1 < p_x < N$ and $1 < p_y < M$. We start with $n = 0$ and either $L(0) = p$ or $L(0) = p + (0, 1)$, depending on whether $p$ is an L-pixel or not, respectively. At any step, the construction interrupts when $L(n)$ coincides with $L(0)$.

**Step I.** Increase $n$ and define $L(n)$ as follows:

(a) if at the southwest of $L(n-1)$ we have a black or an L-pixel, then it will be $L(n+1)$, and $L(n)$ will be the southern pixel;
(b) otherwise, if the south is black or an L-pixel then it will be $L(n)$;
(c) if neither (a) nor (b) then take $L(n) = L(n-1) + (0, 1)$.

**Step II.** Now $L(n)$ has just been defined.

If $L(n+1)$ is already defined, interchange the roles $x \leftrightarrow y$ by rotating the canvas clockwise. Increase $n$, redefine $p$ as $L(n)$ and repeat Steps I-II.

If $L(n+1)$ is not defined yet, then we look at $L(n)$. In the case it is $L(n-1)_y + 1$, redefine $p$ as $L(n)$ and repeat Steps I-II. In the case it is $L(n-1)_x$, we can have

(a) $L(n)$ and $L(n) + (1, 0)$ both white; this situation is predicted in Step III.
(b) $L(n) + (1, 0)$ black or an L-pixel; take $L(n+1) = L(n) + (1, 0)$, increase $n$, redefine $p = L(n)$ and repeat Steps I-II.
(c) Otherwise redefine $p = L(n)$ and repeat Steps I-II.

**Step III.** In the present orientation you cannot increase $L(n)_y$ any more. Then interchange the roles $x \leftrightarrow y$ by rotating the canvas clockwise. Clear $L(n)$, decrease $n$ and repeat Steps I-II. Note: if this step happens three times consecutively, then $\mathcal{P}$ consists of a single pixel. This case can be excluded beforehand.

**Remark:** If $\mathcal{P}$ does not consist of a single row of black pixels, then there will be a least value $n_1 \leq n$ such that $L(n_1)$ has a greater ordinate than $L(0)$ with respect to the initial orientation.

The above construction is in fact what we called CoTRA. But the reader will notice a gap in it, as illustrated in Figure 8.

It is easily identified by “trapped” 4-connected L-pixels. Compare it with Figure 6 But we easily go round this gap by identifying non-marked L-pixels and locally repeating the above construction.

![Figure 8: False self-intersections interpreted by CoTRA.]

However, our list $L$ may still have spikes. In this case, we recall Section II and see that $C_2$ trivially holds for $L$. Regarding $C_1$, it easily follows from $C_2$ and $C_3$. In order to get $C_3$, we have to adapt the above construction as follows: if the roles $x \leftrightarrow y$ are interchanged consecutively twice, then lop off some newly defined $L(n)$ by checking if either $x$- or $y$-coordinates remained constant.

Of course, there remain two important questions that were not answered yet. The first is: why does the construction ever finish? In other words, why does $L(n)$ coincide with $L(0)$ for a certain positive $n$?

Notice that any two 8-connected pixels of $\mathcal{P}$, that are both connected to $E$, can either project onto $Ox$ or $Oy$ injectively. We start with $Oy$ and keep on using it until injectivity fails. Then we change to $Ox$ and do the same to this new axis. This is exactly when the roles $x \leftrightarrow y$ are interchanged in the above construction. It begins with a local graph $x = x(y)$ so that pixels right under the graph belong to the exterior $E$ of $\mathcal{P}$. As a matter of fact, this reasoning holds for the whole construction. Thus no pixel of $E$, that is right under a local graph, is left out. Hence $L(n)$ will eventually coincide with $L(0)$ for a certain $n$.

By the way, the same reasoning explains why $L$ contains the set $L$ from Definition 5.3. This partially answers the second question: why $L$ fits Definition 5.3? It remains to check that it has minimal length among any other curve $L'$ with exterior $E'' \subseteq E$ and $E \setminus E'' = L$.

We know $E = E' \cup L = E'' \cup L$, thus $E' = E''$. Suppose there exists $p \in L \setminus L'$. Consequently, $p \in E$ and is 4-connected with $E$. Indeed, it cannot be only D-connected because then an L-pixel would have replaced it in the above construction. Either $p$ is 4-connected with $E' = E''$ or $p$ is simultaneously 4-connected with two different pixels $q, r \in L$. This latter is the only possibility for $p \notin E'$, since $p \notin E$ and both $L, L'$ have to satisfy $C_2$. But then $q, r \in L'$ have to be connected by a longer path than through $p$. Hence $L'$ cannot have minimal length. q.e.d.

We finish this section by stating and proving a corollary, which is a weaker version of the discrete Jordan curve theorem. This is because we use an additional hypothesis $H$, described below. In Appendix A we discuss some other theoretical results that can be derived from ours.
**Corollary 6.1.** Let \( C : \{0, \ldots, n\} \rightarrow S \) be a simple closed curve and denote its image by \( P \). Suppose \( S \) and \( P \) are such that Definition 5.2 applies. Additionally, the following holds:

**H:** Any \( p \in P \) is connected to exactly two other elements of \( P \)

Hence \( S \setminus P = E \cup J \), where \( E \) is the exterior of both \( P \) and \( C \), and \( J \) the interior of both \( P \) and \( C \). Moreover, \( E \) is connected, and the same holds for \( J \).

**Proof:** Since any \( p \in P \) is connected to exactly two other elements of \( P \), then \( C \) is thin and spike-free. The same will happen to \( L \), constructed from \( P \) by means of Theorem 6.1. Moreover, \( L \) only can be tracked horizontally and vertically. Hence, it is not difficult to conclude that its image \( P' \) has the following property: \( S \setminus P' = E' \cup J' \), where \( E' \) is connected, and the same holds for \( J' \). But \( E = E' \cup L \), where \( L \) is given by Definition 5.3. Moreover, \( E \) is connected. Now, \( J = S \setminus (P \cup E) = S \setminus (P' \cup E') = J' \). Therefore, \( J \) is also connected. q.e.d.

Notice that neither lists of Figure 3 fulfill the hypothesis of Corollary 6.1.

7. Results

This present section is like a manual to the programme. Download legoloci.zip from

https://sites.google.com/site/aefabris/codes

and extract it in a folder. You will get loci.p, lego.p and some test-files. In order to draw our figures, we have chosen Xfig for Linux Ubuntu 12.04. They all have extension FIG, and the reader can either change them or even create new tests. However, our programmes do not read FIG-files. Whenever you create a figure, please export it in a format like TIF or JPG.

We shall make the programmes loci.m and lego.m available in future. For the time being, users can run the p-codes in Matlab. To test the robustness of our algorithms, one can either edit the presented test files or even create new ones.

The source codes run both in Matlab and Octave. Table 1 shows the average performance of these programmes in each software, providing you use 4GB of RAM, microprocessor Intel Core i5 3.2GHz, and operational system Linux Ubuntu 12.04. Table 1 refers to Matlab 7.8.0 (R2009a) and Octave 3.2.4 (R2009). All test files have extension JPG, M stands for Matlab and O for Octave.

Type loci at the Matlab prompt. You will be asked to give a filename, for instance test3.tif. Please write the full name with extension. After pressing the Enter-key, loci will display the figure and also its inside according to FUA. The elapsed computational time is printed in the Matlab terminal window. It counts all commands that come right after having stored the image in the variable img (see Algorithm 1), and right after having displayed its interior on the screen.

Now the user gets the following message:

Try CoTRA? Yes = 1; No = any key

| Table 1: Comparison to Computational Times in Seconds | loci-M | lego-M | loci-O | lego-O |
|------------------------------------------------------|--------|--------|--------|--------|
| test1                                                | 0.30   | 0.80   | 0.34   | 2.18   |
| test2                                                | 0.36   | 1.17   | 0.51   | 4.03   |
| test3                                                | 0.29   | 0.63   | 0.34   | 2.91   |
| test4                                                | 0.29   | 0.76   | 0.39   | 3.46   |
| test5                                                | 0.56   | 2.65   | 1.25   | 8.40   |
| test6                                                | 0.30   | 0.89   | 0.41   | 2.32   |
| test7                                                | 0.36   | 0.59   | 0.43   | 7.51   |
| test8                                                | 0.30   | 0.93   | 0.39   | 3.86   |

By choosing 1, CoTRA is called to create the Lego curve as in the proof of Theorem 6.1. A new figure will be displayed, but now its inside matches Definition 5.3. Once again, the elapsed computational time is printed in the Matlab terminal window. It counts all commands the same way as done in the previous case.

Now we illustrate some outputs.

**Figure 9:** Filling up a challenging picture with CoTRA.

At the Introduction, we mentioned that the marking phase of scan conversion requires careful attention to both geometric and numerical details. Figure 9 left was obtained by drawing a polygonal test4.fig in Xfig. Like many other drawing softwares in computer graphics, the continuous mathematical structure is then discretized as a digital picture.

A detail of Figure 9 is zoomed in Figure 8, which shows a self-intersection that occurs when we move from the ideal simple (i.e. non-self-intersecting) continuous polygon to the digital picture: the digital picture test4.jpg is obtained by exporting test4.fig to the JPEG-format. As commented in Figure 8, CoTRA finds a self-intersection at \((y, x) = (130, 70)\). So test4.jpg is filled in accordance with Definition 5.3.

Figure 10 shows that parts with empty or non-connected interiors are correctly detected.

8. Conclusions

In this paper, the difficulties in discretisation are acknowledged and tackled by casting the problem to be solved as a discrete integer problem from the very outset. Integer spaces
are more amenable to analysis and proof. There the commonly employed geometric operations are translated into discrete versions, which allow a better control of robustness.

This approach enables the development of an integer-only algorithm, of which the main characteristics are: simple structure, no special cases to forecast, no extra memory allocation, and proved robustness. Namely, it can handle arbitrarily complex inputs. The source codes loci.m and lego.m were implemented without either recursive calls or repeat/while-loops. Only for-loops were allowed in the programmes. These characteristics can be forwarded to hardware implementation.

In fact, the main purpose of our work was to make faultless and fast algorithms that could also be implemented in hardware.

Regarding the efficiency of our present implementations, FUA alone has linear complexity with respect to $N \cdot M$. Our observations regarding CoTRA on numerous test scenes suggest quasi-linearity, but before any formalisation we need to check for improvements. Indeed, although CoTRA is relatively fast, we could improve its speed by exploring coherence tests in several ways.

Moreover, FUA and CoTRA require input data that can either be scanned or constructed by a mathematical model. Therefore, these algorithms can be interactively used for applications in both Image Processing and Computer Graphics.

See Table 2 for a summary of seed-fill, scan-conversion and our approach.

### Table 2: Comparison to Seed-fill, Scan-conversion and Our approach

|                     | Seed-fill               | Scan Conversion                                         | Our approach                             |
|---------------------|-------------------------|---------------------------------------------------------|------------------------------------------|
| Input data          | scanned/math defined    | math defined                                            | scanned/math defined                     |
| Implementation      | naive recursive: simple | complex                                                 | simple                                   |
| Modelling space     | screen space            | object space                                            | screen space                             |
| Boundary modelling  | points/polygons         | polygons                                                | points                                   |
| Algorithm           | integer/floating point  | floating point                                          | integer                                  |
| Applications areas  | image processing        | computer graphics                                       | computer graphics and image processing   |
| Additional memory   | very large stack        | small/large stack                                       | no/small stack                           |
| Hard/software design| unsuitable hardware     | suitable hard/software                                   | suitable hard/software                    |
| Device dependency   | requires GetPixelValue  | device independent                                       | device independent                       |
| Theoretical background| simple                  | simple                                                  | relatively complex                       |
| Robustness          | to be more robust       | most algorithms fail                                     | robust                                   |
| Efficiency          | relatively slow         | most are fast                                           | relatively fast                           |

### Appendix A. Some Theory After This Work

Any seed-fill algorithm easily finds the exterior $E$ of a picture $P \subset S$ directly from Definition 5.2. Hence we get the interior as $S \setminus (P \cup E)$. But the computational cost is huge because seed-fill approaches are recursive.

Theorem 6.1 constructs the Lego curve, which after some pruning is then applied to FUA (see Section 6).

However, the Lego curve is much more than just a way to get correct digital answers. It can help prove theorems in the continuous Euclidean plane. For instance, consider this strong version of Jordan’s theorem, called Schöenflies Theorem:

**Theorem Appendix A.1.** Let $S^1$ be the unitary circumference in $\mathbb{R}^2$ centred at the origin. Let $f : S^1 \to C \subset \mathbb{R}^2$ be a homeomorphism between $S^1$ and $C$. Hence, there exists a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F|_{S^1} = f$.

An immediate consequence of Theorem Appendix A.1 is that the interior of $C$, namely the bounded component of $\mathbb{R}^2 \setminus C$, is simply connected. It is a powerful result that, however, does not seem to admit a proof accessible to graduate students. But here we outline some ideas of which details could be carried out in future works.

As depicted in Figure 7 we could get a continuous Lego curve out of any grid in which the original $C$ is covered by black pixels. Ideally, resolution becomes infinite when the pixel dimension $(1/n) \times (1/n)$ goes to zero, where $n$ is a positive integer. For each $n$, we get the continuous Lego curve $L_n$ that corresponds to $C$. It is not difficult to describe $f_n : S^1 \to L_n$ such that $f_n \to f$, because $L_n$ is tracked only horizontally and vertically. This same reason allows us to construct a homeomorphism $F_n : \mathbb{R}^2 \to \mathbb{R}^2$, with $F_n|_{S^1} = f_n$. Finally, a limit process will then result in the general proof.

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