Sequential generation of Polynomial Invariants and N-body non-local correlations

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We report an inductive process that allows for a sequential construction of polynomial invariants of state coefficients for multipartite quantum states. The starting point can be a physically meaningful invariant of a smaller part of the system. The process is applied to construct a chain of invariants that quantify GHZ state like non-local N−way correlations in an N−qubit pure state and the sum of N−way and (N − 1)-way correlations. Analytic expressions for four and three-way correlation quantifiers for four qubits, as well as, five-way and four-way correlation quantifiers for a five qubit pure state are given.

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I. INTRODUCTION

Quantum entanglement, regarded as one of the most prominent features of quantum mechanics, is a nonlocal property of a quantum state determined by quantum correlations present in the system. Entanglement is a necessary ingredient of any quantum computation and a physical resource for quantum cryptography and quantum communication. It also plays a central role in attosecond molecular dynamics [1]. There is new evidence [2] that plants use quantum entanglement to get energy from photons and there exists a viewpoint that gravity might emerge from quantum entanglement [3]. The negativity [4, 5] of partial transpose [6] of the state operator was introduced to detect and measure entanglement of bipartite states. Multiparticle entanglement that comes into play in nature for quantum systems with more than two subsystems is a resource for multiuser quantum information tasks. Multipartite states have a mathematical structure which is much more complex than that of bipartite states, consequently, the structure of multipartite entanglement is far richer.

It is natural to expect that in a multipartite system intrinsic non-local correlations are a function of non-local correlations present in constituent sub-systems. One can construct unitary invariant polynomial functions of state coefficients of a multipartite state to quantify non-local correlations present in the state. Local unitary invariance implies that the value of invariant is independent of the choice of local basis for the subsystem. A clear distinction needs to be made between the local and non-local invariant functions. Local unitary invariant polynomial functions of state coefficients are known to discriminate between different entanglement types as in the case of three qubit states [7]. In this letter, we report a general inductive process which allows to sequentially generate polynomial unitary invariants for a multipartite quantum state in terms of known invariants of the subsystems. The process is described in the context of qubit systems but can be generalized to multilevel systems with \( d > 2 \). The physical meaning of set of polynomial invariants generated through the process is determined by the subsystem invariant which is selected as the starting point.

Usual approaches aim at obtaining all the polynomial invariants for a given number of qubits. For example, unitary invariants obtained by Luque and Thibon [8, 9] using classical invariant theory got geometrical meaning in the work of Levay [10, 11]. Relations between polynomial invariants and entanglement have been investigated in [12–13]. Interesting efforts to calculate the polynomial invariants of four and more qubits include refs. [16–28]. As the number of local unitary invariants grows exponentially with the number of constituent, it becomes a formidable task to identify physically meaningful invariants out of a maze of invariants. Using our process, one can focus on construction of invariants that detect and quantify a specific property of a multipartite system and the constituent subsystems in a quantum state. In particular, the polynomial invariants that have physical interpretation in terms of entanglement are of great interest as these may be used to quantify the entanglement resource content of a state. Entanglement measures based on polynomial invariants can quantify distinct entanglement types. It is known that local unitaries cannot link a state on which a given polynomial invariant is non-zero to another state on which the value of same invariant is null. Selective sequential generation of polynomial invariants, proposed in this letter, makes it possible to construct polynomial invariants that are relevant to detection of a specific type of entanglement.

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Global negativity \[4, 5\] or equivalently concurrence \[29\] of a two qubit pure state quantifies two-way correlations. Three tangle \[30\] of a three qubit pure state, is an entanglement monotone that quantifies GHZ-state like three-way correlations. Four-way correlations in a four qubit pure state are detected by four-tangle \[51\], which is a degree eight polynomial invariant. Three tangle and four tangle are moduli of specific local unitary invariants that are polynomial functions of state coefficients. A natural question is, can we write an invariant in terms of state coefficients to detect \(N\)-way non-local correlations in a general \(N\) qubit state? Furthermore, is it possible to find the exact mathematical relation between \(N\)-way polynomial invariants and \((N - 1)\)-way invariants of a quantum state? Effectiveness of our inductive process is demonstrated through construction of polynomial invariants that detect and quantify GHZ-like \(N\)-way non-local correlations. Our starting point is the negative eigenvalue of partially transposed two qubit pure state.

In section II, we present the formalism to obtain \(N\) qubit invariants in terms of \((N - 1)\) qubit invariants. The principal construction tools, local unitary (LU) invariance, selective partial transposition \[32, 33\] and notion of negativity fonts \[21, 22\], used in our earlier works are also needed for sequential construction of polynomial invariants that quantify \(N\)-way correlations. For the sake of completeness, we define the determinants of negativity fonts in section III. For a given polynomial invariant of an \((N - 1)\) qubit state, one may construct the set of \((N - 1)\) qubit invariants of an \(N\) qubit state by using the raising and lowering operators defined in section IV. In section V, we generate polynomial invariants with negativity of partial transpose of a two qubit pure state as the starting point. To illustrate the process, an invariant that can quantify five qubit GHZ state like correlations is written down. Although some five qubit invariants have been reported in refs \[3\], an analytical form in terms of state coefficients for a degree 16 invariant that detects genuine five-way GHZ-state like correlations is being reported for the first time. A summary of results is in section VI.

### II. UNITARY INVARIANTS OF \(N\) QUBITS FROM \((N - 1)\) QUBIT INVARIANTS OF DEGREE \(k\)

A quantum system of \(N\) isolated qubits in a pure state is described by a wavefunction

\[
|\Psi^{A_1, A_2, ..., A_N}\rangle = \sum_{i_1, i_2, ..., i_N} a_{i_1 i_2 \ldots i_N} |i_1 i_2 \ldots i_N\rangle,
\]

where \(|e_\mu\rangle = |i_1 i_2 \ldots i_N\rangle\) is a basis vectors spanning a Hilbert space \(H = (\mathbb{C}^2)^\otimes N\) of dimension \(d = 2^N\), and \(A_p\) is the location of qubit number \(p\). The coefficients \(a_{i_1 i_2 \ldots i_N}\) are complex numbers. The local basis states of a single qubit are labelled by \(i_m = 0\) and \(1\), where \(m = 1, 2, \ldots, N\). How does a degree \(k\) function of state coefficients transform under a local unitary? The state space of \(k\) copies of the state, \(H^\otimes k\), is spanned by the basis vectors \(\{|e_{\mu_1} \otimes \ldots \otimes e_{\mu_k}\rangle : \mu_m = 1, d\}\). Under a unitary transformation on qubit \(q\), the set of elements \(\{|e_{\mu_1} \otimes \ldots \otimes e_{\mu_k}\rangle\}\) in which \(|i_q = 0\rangle\) appears the same number of times transform in the same way. Every such set of elements has a unique representative of the form \(|0^\otimes k-m \otimes 1^\otimes m\rangle\) \((m = 0 \text{ to } k)\). The number of elements in each set is \(k \choose m\). In the simplest case of a single qubit state \(|\Psi^A_q\rangle = a_0 |0\rangle + a_1 |1\rangle\), let \(f_{k-m, m} = (a_0)^{k-m} (a_1)^m\) be a function of degree \(k\) in coefficients of the state. If the state transforms under a local unitary as

\[
U^{A_q} |\Psi^A_q\rangle = \frac{1}{\sqrt{1 + |x|^2}} ((a_0 - xa^* a_1) |0\rangle + (a_1 + xa_0) |1\rangle),
\]

then the transformed function reads as

\[
f'_{k-m, m} = \frac{1}{\sqrt{1 + |x|^2}} (a_0 - xa^* a_1)^{k-m} (a_1 + xa_0)^m.
\]

For an \(N\) qubit state, an arbitrary function of degree \(k\) in state coefficients, \((F_{N-1,k})_{(A_q)}|_{k-m, m}\) \((m = 0 \text{ to } k)\), transforms under a local unitary \(U^{A_q}\) on a particular qubit \(A_q\) as

\[
(F'_{N-1,k})_{(A_q)}|_{k-m, m} = \frac{1}{\sqrt{1 + |x|^2}} \sum_{\mu_1 = 0}^{k-m} \sum_{\mu_2 = 0}^{m} \binom{k-m}{\mu_1} \binom{m}{\mu_2} \times (-x^*)^{\mu_1} x^{\mu_2} (F_k)_{(A_q)}|_{k-m-\mu_1 + \mu_2, m-\mu_2 + \mu_1}.
\]
Let $I_{N-1,k}$ be a known degree $k$ polynomial invariant of an $(N-1)$ ($N > 2$) qubit system and $(I_{N-1,k})_{(A_N)_{k-m,m}}$ represent an $(N-1)$ qubit polynomial invariant of degree $k$ in coefficients $a_{t_N}$ of $N$ qubit state, where $t_N \equiv \{i_{12}, \ldots, i_N \}$. The subscript $(A_N)_{k-m,m}$ indicates that $(I_{N-1,k})_{(A_N)_{k-m,m}}$ transforms in the same way as the coefficient of $|0^{\otimes k-m}1^{\otimes m}\rangle$ ($0 \leq m \leq k$) under a local unitary on qubit $A_N$. The space of degree $k$ invariants of $(N-1)$ qubits is a $k+1$ dimensional space. Consider a vector $|V\rangle$ in the state space of $k$ copies of $N^{th}$ qubit written as

$$|V\rangle = \sum_{m=0}^{k} \sqrt{\binom{k}{m}} (I_{N-1,k})_{(A_N)_{k-m,m}} |0^{\otimes k-m}1^{\otimes m}\rangle.$$  \hfill (4)

Local unitary transformation on $N^{th}$ qubit results in $k+1$ transformation equations for elements of the set. Polynomial invariants are obtained from the characteristic equation determined by the unitary that reduces one of the elements of the set to zero. Action of $U^{A_N} = \frac{1}{\sqrt{1+|x|^2}} \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$, on $|V\rangle$ generates a new vector

$$|V'\rangle = \sum_{m=0}^{k} \sqrt{\binom{k}{m}} (I_{N-1,k})'_{(A_N)_{k-m,m}} |0^{\otimes k-m}1^{\otimes m}\rangle.$$  \hfill (5)

Using $(F_{N-1,k})_{(A_N)_{k-m,m}} = (I_{N-1,k})_{(A_N)_{k-m,m}}$ in Eq. (3), we obtain

$$(I_{N-1,k})'_{(A_N)_{k-m,m}} = \frac{1}{(1+|x|^2)^{k/2}} \sum_{\mu_1=0}^{k-m} \sum_{\mu_2=0}^{m} \binom{k-m}{\mu_1} \binom{m}{\mu_2} \times (-x^*)^{\mu_1} (x)^{\mu_2} (I_{N-1,k})_{(A_N)_{k-m-\mu_1+m,\mu_1-\mu_2}}.$$  \hfill (6)

In particular $(I_{N-1,k})_{(A_N)_{k,0}}' = 0$, is given by

$$(I_{N-1,k})_{(A_N)_{k,0}}' = \frac{1}{(1+|x|^2)^{k/2}} \sum_{\mu=0}^{k} \binom{k}{\mu} (-x^*)^{\mu} (I_{N-1,k})_{(A_N)_{k-\mu,\mu}}.$$  \hfill (6)

A unitary that makes $(I_{N-1,k})_{(A_N)_{k,0}}' = 0$, is determined by the condition

$$\sum_{\mu=0}^{k} \binom{k}{\mu} (-x^*)^{\mu} (I_{N-1,k})_{(A_N)_{k-\mu,\mu}} = 0,$$  \hfill (7)

which is a polynomial in variable $x^*$ with $N-1$ qubit invariants as coefficients. Invariants of the polynomial (7) are the $N$ qubit invariants we are looking for. Two of the degree $2k$ invariants of Eq. (7) are found to have the form

$$I_{N,2k} = \sum_{m=0,(m\neq k/2)}^{k/2-1} (-1)^m \binom{k}{m} \times (I_{N-1,k})_{(A_N)_{k-m,m}} (I_{N-1,k})_{(A_N)_{m,k-m}}$$
$$+ (-1)^{k/2} \frac{1}{2^k} \binom{k}{k/2} \left((I_{N-1,k})_{(A_N)_{k/2,k/2}}\right)^2,$$  \hfill (8)

and

$$(N_{N,2k})^{A_1,A_2 \ldots A_{N-1}(A_N)} = \left[ \left(\binom{k}{k/2}\right)^2 \left|(I_{N-1,k})_{(A_N)_{k/2,k/2}}\right|^2 \right.$$
$$+ \sum_{m=0,(m\neq k/2)}^{k/2-1} \binom{k}{m} \left(|(I_{N-1,k})_{(A_N)_{k-m,m}}|^2 + |(I_{N-1,k})_{(A_N)_{m,k-m}}|^2 \right) \left(\binom{k}{k/2}\right)^2 \right].$$  \hfill (9)
Here $\langle N_{N,2k}\rangle^{A_1 A_2 \ldots A_{N-1}(A_N)}$ is the norm of vector $|V\rangle$. Since $N - 1$ qubits including qubit $A_1$ may be selected in $N - 1$ distinct ways, a normalized sum of invariants (such that $N_{N,2k}^{A_1} \leq 1$) may be constructed as

$$N_{N,2k}^{A_1} = C_N \left[ \langle N_{N,2k}\rangle^{A_1 A_2 \ldots A_{N-1}(A_N)} + \sum_{q=2}^{N-1} \langle N_{N,2k}\rangle^{A_1 A_2 \ldots A_q - 1 A_{q+1} \ldots A_N (A_N)} \right].$$

(10)

Using the analytical expression for $I_{N-1,k}$ one can write down $\langle I_{N-1,k}\rangle_{(A_N)^{k,o}}$ and $\langle I_{N-1,k}\rangle_{(A_N)^{k,m}}$. Other elements of the set of $(N - 1)$ qubit polynomial invariants $\langle I_{N-1,k}\rangle_{(A_N)^{k,m}}$ with $m > 0$ for a given $N$-qubit state can be easily generated from $\langle I_{N-1,k}\rangle_{(A_N)^{k,m}}$ or $\langle I_{N-1,k}\rangle_{(A_N)^{k,b}}$ by using index switching operators defined in section IV. In the following sections, we take negativity of partial transpose of a two-qubit pure state as the starting point and apply the process outlined above to construct polynomial invariants to detect 3-way, 4-way, ..., $N$-way non-local correlations in a pure state.

### III. DETERMINANT OF A $K$-WAY NEGATIVITY FONT

In refs. [21, 22], the concept of negativity fonts was used to construct invariants that quantify three and four-way correlation for three and four qubit pure states, respectively. Negativity fonts are the elementary units of entanglement in a quantum superposition. Determinants of negativity fonts are linked to way correlation for three and four qubit pure states, respectively. Negativity fonts are the elementary units of correlations. Since the most general multiqubit state may have $N - 1$ correlation types, it has as many types of negativity fonts. Local unitary transformations on qubits result in a state with a new set of negativity fonts. Our object is to identify combinations of determinants of negativity fonts that are invariant with respect to local unitary transformations.

Consider the state operator of $N$ qubit pure state, $\rho = \langle \Psi^{A_1 A_2 \ldots A_N} | (\Psi^{A_1 A_2 \ldots A_N}) \rangle$. A specific $K$-way negativity font with $K = \sum_{m=1}^N (1 - \delta_{i_m,j_m})$ that belongs to partially transposed state operator $\rho^{T_{A_1}}$ is represented by

$$\nu_{S_{2,T}}^{i_1 i_2 \ldots i_N} = \begin{bmatrix} a_{i_1 i_2 \ldots i_N} & a_{j i_1 i_2 \ldots i_N} & a_{j i_2 i_1 i_3 \ldots i_N} & \ldots \end{bmatrix}. $$

(11)

Here, the symbol $\oplus$ represents addition modulo 2, $\delta_{i_m,j_m} = 0$ for $i_m \neq j_m$ and $\delta_{i_m,j_m} = 1$ for $i_m = j_m$. To understand further the meaning of $K$-way in the context of a negativity font, the set of $N$ qubits with their locations and local basis indices given by, $T = \{(A_1)_{i_1}, (A_2)_{i_2} \ldots (A_N)_{i_N}\}$, is split into two subsets. Subset $S_{1,T}$ contains $K$ qubits with $i_m \neq j_m$, and $S_{2,T}$ the remaining $N - K$ qubits for which $i_m = j_m$. The determinant of a negativity font is written as $D_{S_{2,T}}^{S_{1,T}} = \det (\nu_{S_{2,T}}^{i_1 i_2 \ldots i_N})$, where $S_{1,T}$ represents the sequence of local basis indices for qubits in set $S_{1,T}$. We notice that $D_{S_{2,T}}^{S_{1,T}}$ is a two qubit invariant. Under a local unitary on qubit $(A_p)_{i_p} (p \neq 1)$ in set $S_{2,T}$ determinant $D_{S_{2,T}}^{S_{1,T}}$ transforms as the coefficient of $(i_p) \oplus 1$. However, if qubit $(A_p)_{i_p} (p \neq 1)$ belongs in set $S_{1,T}$ then $D_{S_{2,T}}^{S_{1,T}}$ transforms as coefficient of $|0 \oplus 1\rangle$. Elementary negativity fonts that quantify the negativity of $\rho^{T_{A_p}}$ for $p \neq 1$ are defined in an analogous fashion. Determinant of a given negativity font measures the potential entanglement present in a specific four by four subspace of Hilbert space $\mathcal{H}$. Determinant of a $K$-way negativity font detects potential $K$-way quantum correlations in an $N$ qubit state. For even $K$, proper combinations of determinants of $K$-way negativity fonts are found to be degree two invariants under the action of local unitary operations on $K$ qubits [22].

### IV. INDEX RAISING OPERATOR

We define an index raising operator $T_{A_N}^+$ for qubit $A_N$ through the action

$$T_{A_N}^+ a_{t_{N-1}0} = a_{t_{N-1}1}, \quad T_{A_N}^+ a_{t_{N-1}1} = 0,$$

(12)

where $t_N = i_1 i_2 \ldots i_N$. The operator acts on a product of state coefficients as

$$T_{A_N}^+ (a_{t_{N-1}1} a_{t_{N-1}1} u_N) = (T_{A_N} a_{t_{N-1}1} u_N) a_{t_{N-1}1} u_N + a_{t_{N-1}1} T_{A_N} a_{t_{N-1}1} u_N.$$

(13)
We can verify that
\[
(T_{AN}^+)^r (a_{1N-0})^k = \frac{k!}{(k-r)!} (a_{1N-0})^{k-r} (a_{1N-1})^r.
\]

Index lowering operator may be defined, analogously.

How does the index raising operator relate the determinants of $N$ and $(N-1)$ qubit negativity fonts? Let $D_{S2,T}^{(s_1,T)} = \det \left( v_{S2,T}^{i_1 i_2 \ldots i_{N-1}} \right)$ be the determinant of an arbitrary negativity font in an $(N-1)$ qubit state. Two $N$ qubit determinants obtained by adding a subscript $(A_N)_{i_N}$ to set $S_{2,T}$ are
\[
D_{(S_2,T)(A_N)_{i_N}}^{(s_1,T)} = \det \left( v_{S_{2,T}(A_N)_{i_N}}^{i_1 i_2 \ldots i_N} \right); \quad i_N = 0, 1.
\]

Determinants $D_{(S_2,T)(A_N)_{i_N}}^{(s_1,T)}$ transform under local unitaries as coefficient of $|i_N\rangle\langle i_N|$. Two determinants obtained by adding $i_N$ to set $s_{1,T}$ transform as $|0_N \otimes 1_N\rangle$. The combination
\[
\frac{1}{2} \left( D_{S_2,T}^{(s_1,T)0} - D_{S_2,T}^{(s_1,T)1} \right) = \frac{1}{2} \left( \det \left( v_{S_{2,T}^{0}}^{i_1 i_2 \ldots i_{N-1}} \right) - \det \left( v_{S_{2,T}^{1}}^{i_1 i_2 \ldots i_{N-1}} \right) \right)
\]
is invariant with respect to local unitaries on qubits $A_1$ and $A_N$, whereas the sum $\frac{1}{2} \left( D_{S_2,T}^{(s_1,T)0} + D_{S_2,T}^{(s_1,T)1} \right)$ is not so. It is easily verified that
\[
T_{AN}^+ D_{(S_2,T)(A_N)_{0}}^{(s_1,T)} = \left( D_{S_2,T}^{(s_1,T)0} + D_{S_2,T}^{(s_1,T)1} \right),\]
\[
T_{AN}^+ \left( D_{S_2,T}^{(s_1,T)0} + D_{S_2,T}^{(s_1,T)1} \right) = 2D_{(S_2,T)(A_N)_{1}},\]
\[
T_{AN}^+ D_{(S_2,T)(A_N)_{1}}^{(s_1,T)} = 0,
\]

On a product of two qubit invariants $D_1 D_2$, we have
\[
T \left( D_1 D_2 \right) = (TD_1) D_2 + D_1 (TD_2).
\]

Let $I_{N-1,k}$ be a degree $k$ invariant of state
\[
|\Phi_{N-1}\rangle = \sum_{i_1 i_2 \ldots i_{N-1}} a_{i_1 i_2 \ldots i_{N-1}} |i_1 i_2 \ldots i_{N-1}\rangle,
\]
which is known in terms of state coefficients or determinants of negativity fonts. Then we can write down two of the $N-1$ qubit invariants $(I_{N-1,k})_{(A_N)_{0,k}}$ and $(I_{N-1,k})_{(A_N)_{0,0}}$ for $N$ qubit state
\[
|\Psi_N\rangle = |\Phi_{N-1}^{0}\rangle + |\Phi_{N-1}^{1}\rangle.
\]
The invariants $(I_{N-1,k})_{m,k-m}$ for $1 < m < (k-1)$ are new $(N-1)$ qubit invariants with no counterpart in $N-1$ qubit state. Starting from the invariant $(I_{N-1,k})_{(A_N)_{0,k}}$ we can construct the invariants $(I_{N-1,k})_{(A_N)_{k-m,m}}$ by the action of raising operator $T_{AN}^+$ on determinants of negativity fonts that appear in the analytical form of $(I_{N-1,k})_{(A_N)_{0,k}}$, more specifically
\[
(T_{AN}^+)^m (I_{N-1,k})_{(A_N)_{0,k}} = \frac{k!}{(k-m)!} (I_{N-1,k})_{(A_N)_{k-m,m}}.
\]

V. POLYNOMIAL INVARIANTS OF AN N QUBIT STATE BASED ON NEGATIVITY OF A TWO QUBIT PURE STATE

A two qubit pure state shared by Alice (qubit $A_1$) and Bob (qubit $A_2$),
\[
|\Psi_{A_1 A_2}\rangle = \sum_{i_1 i_2} a_{i_1 i_2} |i_1 i_2\rangle, i_m = 0, 1,
\]

(23)
has a single two-way negativity font \( \nu^{00} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \). Determinant of negativity font is a two qubit unitary invariant of degree two that is \( I_{2,2} = D^{00} = (a_{00}a_{11} - a_{10}a_{01}) \). Global negativity of partial transpose of state operator, \( N_{G}^{A_1} = 2 |I_{2,2}| \), is an entanglement monotone and quantifies the entanglement of qubit pair \( A_1A_2 \).

For a three qubit state

\[
|\Psi^{A_1A_2A_3}\rangle = \sum_{i_1i_2} (a_{i_1i_20} |i_1i_20\rangle + a_{i_1i_21} |i_1i_21\rangle),
\]

(24)
determinants of negativity fonts are defined as \( D_{(A_3)}^{00} = a_{000}a_{111} - a_{101}a_{011} \) (two-way fonts), and \( D_{(A_3)}^{00i} = a_{001}a_{11i} - a_{10i}a_{011} \) (three-way fonts). These correspond to partial transposition with respect to qubit \( A_i \). Two qubit invariants, corresponding to \( I_{2,2} \) are

\[
(I_{2,2})_{(A_3)2,0} = D_{(A_3)0}^{00}, \quad \text{and} \quad (I_{2,2})_{(A_3)0,2} = D_{(A_3)1}^{00},
\]

(25)
such that \( T_{A_3}^{+}D_{(A_3)0}^{00} = (D^{000} + D^{001}) \). Therefore, the third two qubit invariant reads as

\[
(I_{2,2})_{(A_3)1,1} = \frac{1}{2} T_{A_3}^{+} (I_{2,2})_{(A_3)2,0} = \frac{1}{2} \left( D_{(A_3)0}^{00} + D_{(A_3)1}^{00} \right).
\]

(26)

On applying a unitary \( U^{A_3} \) to \( |\Psi^{A_1A_2A_3}\rangle \), two qubit invariant \( (I_{2,2})_{(A_3)2,0} \) transforms as

\[
(I_{2,2})_{(A_3)2,0}' = \frac{1}{1 + |x|^2} \left[ (I_{2,2})_{(A_3)2,0} - 2x^2 (I_{2,2})_{(A_3)1,1} + (x^*)^2 (I_{2,2})_{(A_3)0,2} \right].
\]

(27)

Using Eq. (25), the three qubit polynomial invariant is identified as

\[
I_{3,4} = (I_{2,2})_{(A_3)2n} (I_{2,2})_{(A_3)0,2} - (I_{2,2})_{(A_3)1,1}^2
= D_{(A_3)0}^{00} D_{(A_3)1}^{00} - \left( \frac{D_{(A_3)0}^{00} + D_{(A_3)1}^{00}}{2} \right)^2.
\]

(28)

This is the invariant which defines three tangle \( \tau_{3,4} \) through \( \tau_{3,4} = 16 |I_{3,4}| \), an entanglement monotone which quantifies 3-way correlations in a three qubit state. The second invariant corresponding to Eq. (29) is

\[
\mathcal{N}_{3,4}^{A_1A_2(A_3)} = \left( \frac{2}{1} \right) \left| (I_{2,2})_{(A_3)1,1}\right|^2 + \left| (I_{2,2})_{(A_3)2,0}\right|^2 + \left| (I_{2,2})_{(A_3)0,2}\right|^2
= 2 \left( \frac{D_{(A_3)0}^{00} + D_{(A_3)1}^{00}}{2} \right)^2 + \left| D_{(A_3)0}^{00} \right|^2 + \left| D_{(A_3)1}^{00} \right|^2.
\]

(29)

The difference

\[
4 \left( \mathcal{N}_{3,4}^{A_1A_2(A_3)} - 2 |I_{3,4}| \right) = \left( \tau_{2}^{A_1A_2} \right)^2,
\]

(30)
defines \( \tau_{2}^{A_1A_2} \), which determines the amount of two-way correlations in state \( \rho^{A_1A_2} = \text{tr}_{A_3} \left( |\Psi^{A_1A_2A_3}\rangle \langle \Psi^{A_1A_2A_3}| \right) \).

Using the same logic one finds that

\[
\mathcal{N}_{3,4}^{A_1A_3(A_2)} = \left( \frac{1}{2} \right) \left| D_{(A_3)1}^{00} - D_{(A_2)0}^{00} \right|^2 + \left| D_{(A_3)0}^{00} \right|^2 + \left| D_{(A_2)1}^{00} \right|^2,
\]

(31)

and

\[
4 \left( \mathcal{N}_{3,4}^{A_1A_3(A_2)} - 2 |I_{3,4}| \right) = \left( \tau_{2}^{A_1A_3} \right)^2.
\]

(32)
The entanglement of qubit $A_1$ with qubits $A_2$ and $A_3$ due to two-way and three-way correlations is quantified by

$$N_{3,4}^{A_1} = 4 \left( N_{3,4}^{A_1,A_2(A_3)} + N_{3,4}^{A_1,A_3(A_2)} \right)$$

$$= 4 \left[ |D^{000}_{(A_3)_{0}}|^2 + |D^{001}_{(A_3)_{0}}|^2 + |D^{000}_{(A_3)_{1}}|^2 + |D^{000}_{(A_2)_{0}}|^2 \right].$$  \( (33) \)

The invariant $N_{3,4}^{A_1}$ is equal to the square of global negativity of three qubit pure state with respect to qubit $A_1$. The difference

$$N_{3,4}^{A_1} - \tau_{3,4} = \left( \tau_{2,1}^{A_1} \right)^2 + \left( \tau_{2,1}^{A_3} \right)^2$$  \( (34) \)

determines the sum of pairwise entanglement of qubit pairs $A_1A_2$ and $A_1A_3$. For three qubits the well known CKW inequality \(30\) states that

$$N_{3,4}^{A_1} - \tau_{3,4} \geq (C_{A_1A_2})^2 + (C_{A_1A_3})^2,$$

where squared concurrence, \( (C_{A_iA_j})^2 \), \( (29) \) is a calculable measure of bipartite entanglement of qubits $A_i$ and $A_j$ in the reduced two qubit state.

In a general four qubit pure state written as

$$|\Psi^{A_1A_2A_3A_4}\rangle = \sum_{i_1i_2i_3i_4} (a_{i_1i_2i_3i_4} |i_1i_200\rangle + a_{i_1i_2i_3i_4} |i_1i_211\rangle), \quad (35)$$

we identify $D^{000}_{(A_3)_{0}} = a_{000i_3}a_{111i_4} - a_{101i_3}a_{011i_4}$ (two-way), $D^{001}_{(A_3)_{0}} = a_{000i_3}a_{111i_4} - a_{101i_3}a_{011i_4}$ (three-way), $D^{000}_{(A_3)_{1}} = a_{000i_3}a_{111i_4} - a_{101i_3}a_{011i_4}$ (three-way), and $D^{001}_{(A_3)_{1}} = a_{000i_3}a_{111i_4} - a_{101i_3}a_{011i_4}$ (four-way) as the determinants of negativity fonts. Transformation of three qubit invariant

$$(I_{3,4})_{(A_4)_{00}} = D^{000}_{(A_3)_{0}}(A_4)_{0}D^{000}_{(A_3)_{1}}(A_4)_{0} - \frac{D^{000}_{(A_3)_{0}} + D^{001}_{(A_3)_{0}}}{2}^2,$$  \( (36) \)

under $U^{A_4}$ yields (using Eq. \(16\))

$$(I_{3,4})'_{(A_4)_{00}} = \frac{1}{\left( 1 + |x|^2 \right)^2} \times \sum_{\mu=0}^4 \binom{4}{\mu} (-x^*)^\mu (I_{3,4})_{(A_4)_{00}}, \quad (37)$$

Exact expressions for additional degree four three qubit invariants of four qubit state coefficients, obtained by successive applications of index raising operator (Eqs. \(14, 15\) and \(22\)) to $(I_{3,4})_{(A_4)_{00}}$ are given in Appendix A. Degree eight invariant that detects genuine four-way entanglement of a four qubit state is

$$I_{4,8} = 3 \left( (I_{3,4})_{(A_4)_{12}} \right)^2 + (I_{3,4})_{(A_4)_{10}} (I_{3,4})_{(A_4)_{04}} - 4 (I_{3,4})_{(A_4)_{11}} (I_{3,4})_{(A_4)_{11}}, \quad (38)$$

It is a function of terms that involve either the products of invariants that detect 3-way correlations or involve determinants of four-way negativity fonts. The invariant $I_{4,8}$ is non-zero on states having 4-way correlations. Four tangle based on degree eight invariant is defined \(31\) as

$$\tau_{4,4} = \left( |6(I_{4,8})| \right)^{\frac{1}{2}}, \quad (39)$$

Four qubit invariant that quantifies three and four way correlations reads as

$$N_{4,8}^{A_1} = 4 \left( (N_{4,8})^{A_1A_2A_3(A_4)} + (N_{4,8})^{A_1A_2A_4(A_3)} + (N_{4,8})^{A_1A_3A_4(A_2)} \right), \quad (40)$$
The difference \( (N_{4,8})^{A_1, A_2, A_3(A_4)} - |2(I_{4,8})| \), determines the three-way correlations in the reduced state, \( \rho^{A_1, A_2, A_3} = \text{tr}_{A_4} \left( \langle \Psi^{A_1, A_2, A_3, A_4} | \Psi^{A_1, A_2, A_3, A_4} \rangle \right) \). Combining these results and using proper normalization, the difference \( N_{4,8}^{A_1} - (\tau_{4,4})^2 \) determines the amount of total three-way correlations in the state. We define
\[
\left( \tau_{3,4}^{A_1, A_2} \right)^2 = 16 \left( (N_{4,8})^{A_1, A_2, A_3(A_4)} - |2(I_{4,8})| \right),
\]
to be a measure of three-way correlations in the state \( \rho^{A_1, A_2, A_3} = \text{tr}_{A_4}(\rho^{A_1, A_2, A_3, A_4}) \), where \( \rho^{A_1, A_2, A_3, A_4} \) is the state operator for a four qubit pure state. For four qubits, the sum of 3-way and 4-way correlations satisfies the equality
\[
N_{4,8}^{A_1} - (\tau_{4,4})^2 = \left( \tau_{3,4}^{A_1, A_2, A_3} \right)^2 + \left( \tau_{3,4}^{A_1, A_2, A_4} \right)^2 + \left( \tau_{3,4}^{A_1, A_3, A_4} \right)^2,
\]

Four qubit invariant \( (I_{4,8})^{(A_5)} \), which is a function of five qubit state coefficients, transforms under \( U^{A_5} \) as
\[
(I_{4,8})^{(A_5)}_{s,0} = \frac{1}{(1 + |x|^2)^4} \sum_{\mu=0}^{8} \left( \frac{8}{\mu} \right) (-x^\mu) (I_{4,8})^{(A_5)}_{s, \mu}.
\]
and for a five qubit state yields the invariants
\[
I_{5,16} = (I_{4,8})_{(A_5)_{s,0}} + (I_{4,8})_{(A_5)_{s,0,8}} + 28 (I_{4,8})_{(A_5)_{s,2}} + (I_{4,8})_{(A_5)_{s,26}} + 35 \left( (I_{4,8})_{(A_5)_{4,4}} \right)^2 - 8 (I_{4,8})_{(A_5)_{4,1}} (I_{4,8})_{(A_5)_{1,7}} - 56 (I_{4,8})_{(A_5)_{5,3}} (I_{4,8})_{(A_5)_{5,5}},
\]
and
\[
(N_{5,16})^{A_1} = (4)^4 \left[ (N_{5,16})^{A_1, A_2, A_3, A_4(A_5)} + (N_{5,16})^{A_1, A_2, A_3, A_5(A_4)} + (N_{5,16})^{A_1, A_2, A_4, A_5(A_3)} + (N_{5,16})^{A_1, A_3, A_4, A_5(A_2)} \right],
\]
where
\[
(N_{5,16})^{A_1, A_2, A_3, A_4(A_5)} = \sum_{m=0}^{8} \left( \frac{8}{m} \right) \left| (I_{4,8})^{(A_5)}_{s, -m} \right|^2.
\]
Being a function of all possible four qubit invariants of five qubit state, the invariant \( (N_{5,16})^{A_1} \) quantifies four-way and five-way correlations. It satisfies the equality
\[
(N_{5,16})^{A_1} - (4)^4 (8I_{5,16}) = \left( \frac{\tau_{4,4}^{A_1, A_2, A_3, A_4}}{4} \right)^4 + \left( \frac{\tau_{4,4}^{A_1, A_2, A_3, A_5}}{4} \right)^4 + \left( \frac{\tau_{4,4}^{A_1, A_2, A_4, A_5}}{4} \right)^4 + \left( \frac{\tau_{4,4}^{A_1, A_3, A_4, A_5}}{4} \right)^4,
\]
\[
\left( \frac{\tau_{4,4}^{A_1, A_2, A_3, A_4}}{4} \right)^4 = (4)^4 \left( (N_{5,16})^{A_1, A_2, A_3, A_4(A_5)} - 2I_{5,16} \right),
\]
determines the four-way correlations in the reduced state \( \rho^{A_1, A_2, A_3, A_4} = \text{tr}_{A_4}(\rho^{A_1, A_2, A_3, A_4, A_5}) \). The four qubit invariant \( \tau_{4,4}^{A_1, A_2, A_3, A_4} \) is a known function of four qubit invariants \( (I_{4,8})^{(A_5)}_{s, -m} \) \( (m = 0 \text{ to } 8) \) of five qubit state.

The process can be continued on to obtain invariants for desired value of \( N \). While for few qubit systems the process yields exact analytical expressions for polynomial invariants in terms of functions of state coefficients, for larger systems it can be implemented, numerically. We notice that when negativity of partial transpose of a two qubit pure state is the starting point, then invariant \( I_{N, k} \) of degree \( k = 2^N - 1 \) \( (N > 2) \) quantifies \( N \)-way GHZ-like non-local correlations. The invariant \( (N_{N, k})^{A_1} \) is a measure of \( N \)-way and \( (N - 1) \)-way correlations of qubit \( A_1 \) with the rest of the system.
Local unitary invariance is the basic principle used to formulate an inductive process that generates a chain of polynomial invariants of state coefficients starting from a known subsystem invariant of a multipartite system. Our approach of selective sequential construction allows for construction of physically meaningful invariant functions for a large system in terms of known properties of subsystems. The process is applied to obtain the chain of invariants based on global negativity which is an entanglement monotone for a two-qubit pure state. The resulting degree $k$ invariant $I_{N,k}$ ($k = 2^{N-1}$) detects entanglement due to GHZ like $N$-way non-local correlations in an $N$ qubit state and is used to define a quantifiers of such correlations. Another interesting polynomial invariant that quantifies the sum of $N$-way and $(N - 1)$-way correlations is also obtained. Currently, the work is on to establish exact monogamy of quantum correlations by using analytical expressions for relevant invariants. Invariants for mixed states can be constructed through convex roof extension.

Since the form of $N$-qubit invariants is directly linked to the underlying structure of the composite system state, it can throw light on the suitability of a given state for a specific information processing task. Polynomial invariants that identify the nature of correlations in a state are useful to apply qubits to classify $N$-way non-local correlations in an $N$ qubit state, we obtain

APPENDIX A: SET OF DEGREE FOUR THREE QUBIT INVARIANTS OF A FOUR QUBIT STATE

Elements of the set of degree four three qubit invariants of four qubit state coefficients are related to each other through index raising operator (Eqs. 17, 18 and 23). Starting with three qubit invariant

$$I_{3,4}(A_4)_{4,0} = D_{(A_3)_1(A_4)_0}^{00} D_{(A_3)_0(A_4)_4}^{00} - \left( \frac{D_{(A_3)_1(A_4)_0}^{00} + D_{(A_3)_0(A_4)_4}^{00}}{2} \right)^2,$$

we obtain

$$I_{3,4}(A_4)_{3,1} = \frac{1}{4} T_{A_4}^+ (I_{3,4})_{(A_4)_{4,0}}$$

$$= \frac{1}{4} \left[ D_{(A_3)_1(A_4)_0}^{00} \left( D_{(A_3)_0(A_4)_0}^{00} + D_{(A_3)_0(A_4)_4}^{01} \right) + D_{(A_3)_0(A_4)_4}^{00} \left( D_{(A_3)_1(A_4)_0}^{00} + D_{(A_3)_1(A_4)_4}^{01} \right) \right]$$

$$- \frac{1}{8} \left( D_{(A_4)_0}^{00} + D_{(A_4)_4}^{01} \right) \left( D_{(A_3)_0}^{00} + D_{(A_3)_0}^{01} + D_{(A_3)_4}^{00} + D_{(A_3)_4}^{01} \right),$$

$$I_{3,4}(A_4)_{2,2} = \frac{1}{12} \left( T_{A_4}^+ \right)^2 (I_{3,4})_{(A_4)_{4,0}}$$

$$= \frac{1}{6} \left( D_{(A_3)_1(A_4)_0}^{00} + D_{(A_3)_1(A_4)_0}^{01} \right) \left( D_{(A_3)_0(A_4)_4}^{00} + D_{(A_3)_0(A_4)_4}^{01} \right)$$

$$+ \frac{1}{6} \left( D_{(A_3)_1(A_4)_0}^{00} D_{(A_3)_0(A_4)_4}^{00} + D_{(A_3)_0(A_4)_4}^{00} D_{(A_3)_1(A_4)_4}^{00} \right)$$

$$- \frac{1}{24} \left( D_{(A_3)_0}^{0000} + D_{(A_3)_0}^{0011} + D_{(A_3)_4}^{0000} + D_{(A_3)_4}^{0011} \right)^2$$

$$- \frac{1}{12} \left( D_{(A_4)_0}^{0000} + D_{(A_4)_4}^{0011} \right) \left( D_{(A_3)_0}^{0000} + D_{(A_3)_0}^{0011} \right).$$

VI. CONCLUSIONS

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\[
(I_{3,4})_{(A_4)_{1,3}} = \frac{1}{24} (T_{A_4}^+) 3 (I_{3,4})_{(A_4)_{4,0}}
\]

\[
= \frac{1}{4} \left[ D^{00}_{(A_3)_1(A_4)_1} \left( D^{000}_{(A_3)_0} + D^{001}_{(A_3)_1} \right) + \left( D^{000}_{(A_3)_1} + D^{001}_{(A_3)_1} \right) D^{00}_{(A_3)_0(A_4)_1} \right] - \frac{1}{8} \left( D^{0000} + D^{0011} + D^{0010} + D^{0011} \right) \left( D^{000}_{(A_4)_1} + D^{001}_{(A_4)_1} \right),
\]

(A4)

and

\[
(I_{3,4})_{(A_4)_{04}} = \frac{1}{24} (T_{A_4}^+) 4 (I_{3,4})_{(A_4)_{4,0}}
\]

\[
= D^{00}_{(A_3)_1(A_4)_1} D^{00}_{(A_3)_0(A_4)_1} - \frac{1}{2} \left( D^{000}_{(A_4)_1} + D^{001}_{(A_4)_1} \right)^2.
\]

(A5)

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