We show that a doubly degenerate thin-film equation obtained in modeling the flows of viscous coatings on spherical surfaces has a finite speed of propagation for nonnegative strong solutions and, hence, there exists an interface or a free boundary separating the regions, where the solution $u > 0$ and $u = 0$. By using local entropy estimates, we also establish the upper bound for the rate of propagation of the interface.

1. Introduction

In the present paper, we study a particular case of the following doubly degenerate fourth-order parabolic equation:

$$u_t + \left[u^n(1 - x^2)(a - bx + c(2u + ((1 - x^2)u_x)_x)_x)\right]_x = 0 \text{ in } Q_T, \tag{1.1}$$

where $u(x, t)$ represents the thickness of the thin film; the dimensionless parameters $a$, $b$, and $c$ describe the effects of gravity, rotation, and surface tension, respectively; $Q_T = \Omega \times (0, T)$, $n > 0$, $T > 0$, and $\Omega = (-1, 1)$. For $n = 3$ (no-slip mode), this equation describes the dynamics of a thin viscous liquid film on the outer surface of a solid sphere. For $n = 2$, the classical Navier slip condition is recovered. On the other hand, the range of parameter $n \in (0, 2)$ ($n \in (2, 3)$) in Eq. (1.1) corresponds to the strong (weak) wetting slip mode. A more general dynamics of the liquid film in the case where the draining of the film due to gravity was balanced by the centrifugal forces caused by the rotation of the sphere about the vertical axis and by capillary forces due to surface tension was considered in [11]. In addition, the Marangoni effects caused by temperature gradients were taken into account in [12]. A spherical model without surface tension and Marangoni effects was studied in [17, 18].

We are interested in the time evolution of the support of nonnegative strong solutions to

$$u_t + ((1 - x^2)|u|^n((1 - x^2)u_x)_{xx})_x = 0. \tag{1.2}$$

Equation (1.2) is a particular case of (1.1) with $a = b = 0$ but without the second-order diffusion term. The existence of weak solutions for (1.2) in a weighted Sobolev space was shown in [13], while the existence of more regular nonnegative strong solutions of (1.2) was recently proved in [16]. Unlike the classical thin-film equation

$$u_t + (|u|^n u_{xx})_x = 0, \tag{1.3}$$

the qualitative behavior of solutions for doubly degenerate thin-film equation (1.2) is still not well understood. Note that the model equation (1.3) describes the coating flow of a thin viscous film on a flat surface under the action of
surface tension. Depending on the value of the parameter $n$, the nonnegative solutions of this equation possess some interesting properties. Thus, in 1990, Bernis and Friedman [2] defined and constructed nonnegative weak solutions of Eq. (1.3) with $n \geq 1$. It was also shown that for $n \geq 4$ and a uniformly positive initial condition, there exists a unique positive classical solution. Later, in 1994, Bertozzi, et al. [6] generalized this positivity property to the case $n \geq \frac{7}{2}$. In 1995, Beretta, et al. [1] proved the existence of nonnegative weak solutions for Eq. (1.3) with $n > 0$ and the existence of strong solutions for $0 < n < 3$. They also showed that this positivity-preserving property holds at almost any time $t$ for $n \geq 2$. This positivity-preservation result was generalized to the case of a cylindrical surface in [7]. Furthermore, for $n \geq \frac{3}{2}$, the support of the solution to (1.3) is nondecreasing in time. Moreover, the support remains constant for $n \geq 4$. The existence (nonexistence) of a compactly supported spreading-source-type solution to (1.3) was demonstrated for $0 < n < 3$ ($n \geq 3$) in [5]. As one of interesting qualitative properties of nonlinear parabolic thin-film equations, we can mention a finite speed of propagation of the support, which is not the case if the parabolic equation is linear. This property was first established in [3] for $0 < n < 2$ and in [4, 10] for $2 \leq n < 3$ for nonnegative strong solutions of (1.3). A similar result for a cylindrical surface was obtained in [8].

Our main result for the thin-film equation on the spherical surface is the finite speed of the propagation of the interface in a special case of strong slip mode $n \in (1, 2)$. The proof of the property of finite speed of propagation is based on the local entropy estimate and Stampacchia’s lemma. Moreover, we establish the upper bound for the time evolution of the support as follows:

$$\Gamma(t) \leq C_0 t^{\frac{1}{n+4}}.$$ 

This bound coincides with the asymptotic behavior of the solutions of Eq. (1.3) of self-similar type (see [5]).

2. Main Result

We study a thin-film equation

$$u_t + \left| (1 - x^2)u_x \right|^n (1 - x^2) u_{xx} = 0 \quad \text{in} \quad Q_T$$

with the following no-flux boundary conditions

$$(1 - x^2)u_x = (1 - x^2) (1 - x^2) u_{xx} = 0 \quad \text{at} \quad x = \pm 1, \quad t > 0,$$

and the initial condition

$$u(x, 0) = u_0(x).$$

Here, $n > 0$, $Q_T = \Omega \times (0, T)$, $\Omega := (-1, 1)$, and $T > 0$.

Integrating Eq. (2.1) by using the boundary conditions (2.2), we arrive at the mass conservation property

$$\int_{\Omega} u(x, t) \, dx = \int_{\Omega} u_0(x) \, dx =: M > 0.$$ 

The initial data $u_0(x) \geq 0$ are considered for all $x \in \bar{\Omega}$ satisfying

$$\int_{\bar{\Omega}} \left\{ u_0^2(x) + (1 - x^2) u_{0,x}^2(x) \right\} \, dx < \infty.$$
**Definition 2.1** (weak solution). Let \( n > 0 \). A function \( u \) is a weak solution of the problem (2.1)–(2.3) with initial data \( u_0 \) satisfying (2.5) if \( u(x, t) \) has the following properties:

\[
(1 - x^2)^\frac{\alpha}{2} u \in C^2_{x,t} (\bar{Q}_T), \quad 0 < \alpha < \beta \leq \frac{2}{n},
\]

\[
 u_t \in L^2(0,T; (H^1(\Omega))^*), \quad (1 - x^2)^\frac{\gamma}{2} u_x \in L^\infty(0,T; L^2(\Omega)),
\]

\[
 (1 - x^2)^\frac{1}{2} |u|^\frac{n}{2} (1 - x^2)_x x \in L^2(\mathcal{P}),
\]

and \( u \) satisfies Eq. (2.1) in a weak sense:

\[
\int_0^T \langle u_t, \phi \rangle_{(H^1)^*,H^1} \, dt - \iint_\mathcal{P} (1 - x^2)|u|^n((1 - x^2)_x)_x \phi \, dx \, dt = 0
\]

for all \( \phi \in L^2(0,T; H^1(\Omega)) \), where

\[
 P := \bar{Q}_T \setminus \{ \{u = 0\} \cup \{t = 0\} \},
\]

\[
 (1 - x^2)^\frac{1}{2} u_x(.,t) \to (1 - x^2)^\frac{1}{2} u_{0,x}(.) \quad \text{strongly in} \quad L^2(\Omega) \quad \text{as} \quad t \to 0,
\]

and the boundary conditions (2.2) hold at all points of the lateral boundary, where \( \{u \neq 0\} \).

Denote

\[
0 \leq G_0(z) := \begin{cases} 
\frac{z^{2-n} - A^{2-n}}{(n-1)(n-2)} - \frac{A^{1-n}}{1-n}(z - A) & \text{for } n \neq 1, 2, \\
z \ln z - z(\ln A + 1) + A & \text{for } n = 1, \\
\ln \left(\frac{A}{z}\right) + \frac{z}{A} - 1 & \text{for } n = 2,
\end{cases}
\]

where \( A = 0 \) for \( n \in (1, 2) \) and \( A > 0 \), otherwise.

**Theorem 2.1.** Assume that \( n \geq 1 \) and the initial data \( u_0 \) are such that

\[
\int_\Omega G_0(u_0) \, dx < +\infty.
\]

Then problem (2.1)–(2.3) has a nonnegative weak solution \( u \) in a sense of Definition 2.1 such that

\[
(1 - x^2)u_x \in L^2(0,T; H^1(\Omega)), \quad (1 - x^2)^\gamma u_x \in L^2(\bar{Q}_T), \quad \gamma \in (0,1],
\]

\[
u \in L^\infty(0,T; L^2(\Omega)), \quad (1 - x^2)^\mu u \in L^2(\bar{Q}_T), \quad \mu \in (-1,\beta].
\]
The solutions in a sense of Theorem 2.1 are called strong solutions. The existence of these solutions was proved in [16]. Our aim is to establish the property of finite speed of propagation for a strong solution \( u \) of (2.1).

**Theorem 2.2** (finite speed of propagation). Assume that \( 1 < n < 2 \), the initial data satisfies the hypotheses of Theorem 2.1 and the support of the initial data satisfies the inclusion \( \text{supp}(u_0) \subset \Omega \setminus (-r_0, r_0) \), where \( \Omega = (-1, 1) \) and \( r_0 \in (0, 1) \). Let \( u \) be a strong solution from Theorem 2.1. Then there exists a time \( T^* > 0 \) and a nondecreasing function \( \Gamma(t) \in C([0, T^*]) \), \( \Gamma(0) = 0 \) such that \( u \) has finite speed of propagation, i.e.,

\[
\text{supp}(u(\cdot, t)) \subseteq [-r_0 + \Gamma(t), r_0 - \Gamma(t)] \subset \Omega
\]

for all \( t \in [0, T^*] \). Moreover, \( \Gamma_{\text{opt}}(t) = C_0 t^{\frac{1}{n+4}} \) for all \( t \in [0, T^*] \).

3. Proof of Theorem 2.2

3.1. Local Entropy Estimate.

**Lemma 3.1.** Assume that \( 1 < n < 2 \) and \( \nu > 1 \) and that \( \zeta \in C^{1,2}_{t,x}(\bar{Q}_T) \) is such that its support satisfies \( \text{supp}(\zeta) \subset \Omega \) and \( (\zeta^4)_x = 0 \) on \( \partial \Omega \). Then there exist positive constants \( C_1 \) and \( C_2 \) independent of \( \Omega \) and such that, for all \( T > 0 \), the strong solution \( u \) from Theorem 2.1 satisfies the inequality

\[
\int_{\Omega} (1 - x^2)\nu \zeta^4(x, T) G_0(u) \, dx - \int_{Q_T} (1 - x^2)^\nu (\zeta^4_x) G_0(u) \, dx \, dt + \frac{1}{4} \int_{Q_T} (1 - x^2)^\nu + 2 u^2_{xx} \zeta^4 \, dx \, dt
\]

\[
\leq \int_{\Omega} (1 - x^2)\nu \zeta^4(x, 0) G_0(u_0) \, dx
\]

\[
+ C_1 \int_{Q_T} (1 - x^2)\nu u^2_x [\zeta^4 + \zeta^2 \zeta^2_x + \zeta^3 \zeta_{xx}] \, dx \, dt
\]

\[
+ C_2 \int_{Q_T} (1 - x^2)^{-2} u^2 [\zeta^4 + \zeta^2 + \zeta^2 \zeta^2_{xx}] \, dx \, dt \tag{3.1}
\]

**Proof.** Equation (2.1) is doubly degenerate for \( u = 0 \) and \( x = \pm 1 \). Therefore, for any \( \varepsilon > 0 \) and \( \delta > 0 \) we consider the following two-parameter regularized equations:

\[
u_{\varepsilon,\delta,t} + \left[(1 - x^2 + \delta) \left(|\nu_{\varepsilon,\delta}|^n + \varepsilon\right) (1 - x^2 + \delta) u_{\varepsilon,\delta,x}ight]_{xx} = 0 \quad \text{in} \quad Q_T \tag{3.2}
\]

with boundary conditions

\[
u_{\varepsilon,\delta,x} = ((1 - x^2 + \delta) u_{\varepsilon,\delta,x})_{xx} = 0 \quad \text{at} \quad x = \pm 1,
\]

and initial data

\[
u_{\varepsilon,\delta}(x, 0) = u_{0,\varepsilon,\delta}(x) \in C^{1+\gamma}(\bar{\Omega}), \quad \gamma > 0,
\]
where
\[
u_{0,\varepsilon}(x) \geq u_0(x) + \varepsilon^\theta, \quad \theta \in \left(0, \frac{1}{2(n-1)}\right),
\]

\[u_{0,\varepsilon} \to u_0 \text{ strongly in } H^1(\Omega) \text{ as } \varepsilon \to 0,
\]

\[(1 - x^2 + \delta)^{1/2} u_{0,x,\varepsilon} \to (1 - x^2)^{1/2} u_{0,x} \text{ strongly in } L^2(\Omega) \text{ as } \delta \to 0.
\]

The parameters \(\varepsilon > 0\) and \(\delta > 0\) in (3.2) make the problem regular up to the boundary (i.e., uniformly parabolic). The existence of a local (in time) solution of (3.2) is guaranteed by the classical Schauder estimates (see [9]). We now suppose that \(u_{\varepsilon,\delta}\) is a solution of equation (3.2) and that it is continuously differentiable with respect to the time variable and four times continuously differentiable with respect to the space variable. For the full detailed proof of existence of strong solutions, we refer the reader to [16].

Multiplying Eq. (3.2) by \(\phi(x,t)G'_\varepsilon(u_{\varepsilon,\delta})\), integrating over \(\Omega\), and then integrating by parts, we find

\[
\frac{d}{dt} \int_\Omega \phi G'_\varepsilon(u_{\varepsilon,\delta}) \, dx - \int_\Omega \phi_t G'_\varepsilon(u_{\varepsilon,\delta}) \, dx
\]

\[
= \int_\Omega (1 - x^2 + \delta) u_{\varepsilon,\delta,x} [(1 - x^2 + \delta) u_{\varepsilon,\delta,x}]_x \phi \, dx
\]

\[
+ \int_\Omega (1 - x^2 + \delta)(|u_{\varepsilon,\delta}|^n + \varepsilon) G'_\varepsilon(u_{\varepsilon,\delta}) [(1 - x^2 + \delta) u_{\varepsilon,\delta,x}]_x \phi_x \, dx
\]

\[
= - \int_\Omega [(1 - x^2 + \delta) u_{\varepsilon,\delta,x}]_x^2 \phi \, dx - \int_\Omega (1 - x^2 + \delta) u_{\varepsilon,\delta,x} [(1 - x^2 + \delta) u_{\varepsilon,\delta,x}]_x \phi_x \, dx
\]

\[
- \int_\Omega [(1 - x^2 + \delta)(|u_{\varepsilon,\delta}|^n + \varepsilon) G'_\varepsilon(u_{\varepsilon,\delta}) \phi_x]_x [(1 - x^2 + \delta) u_{\varepsilon,\delta,x}]_x \phi_x \, dx
\]

\[
= - \int_\Omega [(1 - x^2 + \delta) u_{\varepsilon,\delta,x}]_x^2 \chi \, dx + \frac{1}{2} \int_\Omega [(1 - x^2 + \delta) u_{\varepsilon,\delta,x}]_x^2 \phi_{xx} \, dx
\]

\[
- \int_\Omega [(1 - x^2 + \delta) u_{\varepsilon,\delta,x}]_x (|u_{\varepsilon,\delta}|^n + \varepsilon) G'_\varepsilon(u_{\varepsilon,\delta}) [(1 - x^2 + \delta) \phi_x]_x \, dx
\]

\[
- \int_\Omega [(1 - x^2 + \delta) u_{\varepsilon,\delta,x}]_x (1 - x^2 + \delta) [(|u_{\varepsilon,\delta}|^n + \varepsilon) G'_\varepsilon(u_{\varepsilon,\delta})]_x u_{\varepsilon,\delta,x} \phi_x \, dx. \quad (3.3)
\]

We now integrate (3.3) with respect to time and let the regularizing parameter \(\varepsilon \to 0\). As a result, by applying the Young inequality and the relation

\[z^n G'_\varepsilon(z) = \frac{1}{1 - n} z,
\]
we finally get

\[
\int_{\Omega} \phi G_0(u_\delta) \, dx - \iint_{Q_T} \phi_t G_0(u_\delta) \, dx dt + \iint_{Q_T} [(1 - x^2 + \delta) u_{\delta,x}]^2 \phi \, dx dt
\]

\[
\leq \int_{\Omega} \phi G_0(u_{0,\delta}) \, dx + \frac{1}{2} \iint_{Q_T} [(1 - x^2 + \delta) u_{\delta,x}]^2 \phi_{xx} \, dx dt
\]

\[- \frac{1}{1 - n} \iint_{Q_T} [(1 - x^2 + \delta) u_{\delta,x}]^2 u_\delta (1 - x^2 + \delta) \phi_x \, dx dt
\]

\[- \frac{1}{1 - n} \iint_{Q_T} [(1 - x^2 + \delta) u_{\delta,x}]^2 (1 - x^2 + \delta) u_{\delta,x} \phi_x \, dx dt
\]

\[
\leq \int_{\Omega} \phi G_0(u_{0,\delta}) \, dx + \mu \iint_{Q_T} [(1 - x^2 + \delta) u_{\delta,x}]^2 \phi \, dx dt
\]

\[+ \frac{2 - n}{2(1 - n)} \iint_{Q_T} [(1 - x^2 + \delta) u_{\delta,x}]^2 \phi_{xx} \, dx dt
\]

\[+ \frac{1}{4\mu(1 - n)^2} \iint_{Q_T} u_\delta^2 \frac{(1 - x^2 + \delta) \phi_x}{\phi} \, dx dt,
\]

(3.4)

where \( \mu > 0 \). Choosing \( \mu = \frac{1}{2} \) in (3.4), we obtain

\[
\int_{\Omega} \phi G_0(u_\delta) \, dx - \iint_{Q_T} \phi_t G_0(u_\delta) \, dx dt + \frac{1}{2} \iint_{Q_T} [(1 - x^2 + \delta) u_{\delta,x}]^2 \phi \, dx dt
\]

\[
\leq \int_{\Omega} \phi G_0(u_{0,\delta}) \, dx + \frac{2 - n}{2(1 - n)} \iint_{Q_T} [(1 - x^2 + \delta) u_{\delta,x}]^2 |\phi_{xx}| \, dx dt
\]

\[+ \frac{1}{2(1 - n)^2} \iint_{Q_T} u_\delta^2 \frac{(1 - x^2 + \delta) \phi_x}{\phi} \, dx dt.
\]

(3.5)

Letting \( \delta \to 0 \) in (3.5), we conclude that

\[
\int_{\Omega} \phi(T) G_0(u) \, dx - \iint_{Q_T} \phi_t G_0(u) \, dx dt + \frac{1}{2} \iint_{Q_T} [(1 - x^2) u_x]_x^2 \phi \, dx dt
\]
\[ \leq \int_{\Omega} \phi(0) G_0(u_0) \, dx + \frac{2 - n}{2(1 - n)} \int_{Q_T} \left[ (1 - x^2)u_x \right]^2 |\phi_{xx}| \, dx \, dt \]
\[ + \frac{1}{2(1 - n)^2} \int_{Q_T} u^2 \frac{(1 - x^2)\phi_x}{\phi} x \, dx \, dt. \]  

(3.6)

Setting \( \phi(x, t) = (1 - x^2)^{\nu} \zeta^4(x, t) \) in (3.6) for \( \nu > 1 \), we find

\[ \int_{\Omega} (1 - x^2)^{\nu} \zeta^4(T) G_0(u) \, dx - \int_{Q_T} (1 - x^2)^{\nu} (\zeta^4)_t G_0(u) \, dx \, dt \]
\[ + \frac{1}{2} \int_{Q_T} (1 - x^2)^{\nu} [ (1 - x^2)u_x ]^2 \zeta^4 \, dx \, dt \]
\[ \leq \int_{\Omega} (1 - x^2)^{\nu} \zeta^4(0) G_0(u_0) \, dx \]
\[ + \tilde{C}_1 \int_{Q_T} \left[ (1 - x^2)u_x \right]^2 \left[ (1 - x^2)^{\nu} - 2 \zeta^4 + (1 - x^2)^{\nu} \left( \zeta^2 \zeta_x + \zeta^2 |\zeta_{xx}| \right) \right] \, dx \, dt \]
\[ + C_2 \int_{Q_T} u^2 \left[ (1 - x^2)^{\nu} - 2 \zeta^4 + (1 - x^2)^{\nu} + 2 \zeta^2 \zeta_x + (1 - x^2)^{\nu} \right] \, dx \, dt \]
\[ \leq \int_{\Omega} (1 - x^2)^{\nu} \zeta^4(0) G_0(u_0) \, dx + \tilde{C}_1 \int_{Q_T} (1 - x^2)^{\nu} u_x^2 \left[ \zeta^4 + \zeta^2 \zeta_x + \zeta^2 |\zeta_{xx}| \right] \, dx \, dt \]
\[ + C_2 \int_{Q_T} (1 - x^2)^{\nu - 2} u^2 \left[ \zeta^4 + \zeta_x^4 + \zeta^2 \zeta_{xx} \right] \, dx \, dt, \]

where

\[ \tilde{C}_1 = \frac{2 - n}{1 - n} \max \{ 5\nu, 2\nu(\nu + 1), 2(3 + 2\nu) \} \quad \text{and} \quad C_2 = \frac{16(\nu + 1)^4}{(1 - n)^2}. \]

In view of the fact that

\[ \frac{1}{2} \int_{Q_T} (1 - x^2)^{\nu} [ (1 - x^2)u_x ]^2 \zeta \, dx \, dt \]
\[ = \frac{1}{2} \int_{Q_T} (1 - x^2)^{\nu + 1} u_x^2 \zeta^4 \, dx \, dt - 2 \int_{Q_T} x (1 - x^2)^{\nu + 1} u_x \zeta^4 \, dx \, dt \]
\[ + 2 \int_{Q_T} x^2 (1 - x^2)^\nu u^2_\xi x^4 \, dx \, dt \]
\[ \geq \frac{1}{4} \int_{Q_T} (1 - x^2)^{\nu + 2} u^2_x \xi x^4 \, dx \, dt - 2 \int_{Q_T} (1 - x^2)^\nu u^2_x \xi x^4 \, dx \, dt, \]

this yields (3.1) with \( C_1 = \tilde{C}_1 + 2 \).

Lemma 3.1 is proved.

### 3.2. Finite Speed of Propagation

For any \( s > 0 \) and \( 0 < \delta \leq s \), we consider the families of sets

\[ \Omega(s) := \{ x \in \Omega : |x| \leq s \}, \quad Q_T(s) = (0, T) \times \Omega(s), \quad \text{and} \quad K_T(s, \delta) = Q_T(s) \setminus Q_T(s - \delta). \]

We now introduce a nonnegative cutoff function \( \eta(\tau) \) from the space \( C^2(\mathbb{R}^1) \) with the following properties:

\[ \eta(\tau) = \begin{cases} 
1 & \text{for } \tau \leq 0, \\
-\tau^3(6\tau^2 - 15\tau + 10) + 1 & \text{for } 0 < \tau < 1, \\
0 & \text{for } \tau \geq 1.
\end{cases} \]

Further, we introduce our main cutoff functions \( \eta_{s,\delta}(x) \in C^2(\Omega) \) such that \( 0 \leq \eta_{s,\delta}(x) \leq 1 \) for all \( x \in \Omega \) with the following properties:

\[ \eta_{s,\delta}(x) = \eta\left(\frac{|x| - (s - \delta)}{\delta}\right) = \begin{cases} 
1, & x \in \Omega(s - \delta), \\
0, & x \in \Omega \setminus \Omega(s),
\end{cases} \]

\[ |(\eta_{s,\delta})_x| \leq \frac{15}{8\delta}, \quad |(\eta_{s,\delta})_{xx}| \leq \frac{5(\sqrt{3} - 1)}{\delta^2} \]

for all \( s > 0 \) and \( 0 < \delta \leq s \). Choosing \( \zeta^4(x,t) = \eta_{s,\delta}(x) e^{-\tau} \) in (3.1), we arrive at the formula

\[ \int_{\Omega(s-\delta)} (1 - x^2)^\nu u^{2-n}(T) \, dx \]
\[ + \frac{1}{T} \int_{Q_T(s-\delta)} (1 - x^2)^\nu u^{2-n} \, dx \, dt + C \int_{Q_T(s-\delta)} (1 - x^2)^{\nu + 2} u_x^2 \, dx \, dt \leq e \int_{\Omega(s)} (1 - x^2)^\nu u_0^{2-n}(x) \, dx \]
\[
\leq \frac{C}{\delta^2} \int \int (1 - x^2)^\nu u_x^2 \, dx \, dt + \frac{C}{\delta^2} \int \int (1 - x^2)^{\nu - 2} u^2 \, dx \, dt \quad (3.7)
\]

for all \(0 < \delta \leq s\). Here and throughout the proof, \(C\) denotes a positive constant independent of \(\Omega\). By (3.7), we obtain

\[
(1 - (s - \delta)^2)^\nu \int_{\Omega(s-\delta)} u^{2-n}(T) \, dx + \frac{(1 - (s - \delta)^2)^\nu}{T} \int_{Q_T(s-\delta)} u^{2-n} \, dx \, dt
\]

\[
+ C(1 - (s - \delta)^2)^\nu \int_{Q_T(s-\delta)} (1 - x^2)^2 u_{xx}^2 \, dx \, dt
\]

\[
\leq \frac{C(1 - (s - \delta)^2)^\nu}{\delta^2} \int \int u_x^2 \, dx \, dt
\]

\[
+ \frac{C(1 - (s - \delta)^2)^\nu}{\delta^4} \int \int (1 - x^2)^{-2} u^2 \, dx \, dt,
\]

whence

\[
\int_{\Omega(s-\delta)} u^{2-n}(T) \, dx + \frac{1}{T} \int \int_{Q_T(s-\delta)} u^{2-n} \, dx \, dt + C(1 - r_0^2)^2 \int_{Q_T(s-\delta)} u_{xx}^2 \, dx \, dt
\]

\[
\leq \frac{C}{\delta^2} \int \int u_x^2 \, dx \, dt + \frac{C(1 - r_0^2)^{-2}}{\delta^4} \int \int u^2 \, dx \, dt =: R(s) \quad (3.8)
\]

for all \(0 < \delta \leq s \leq r_0\). We apply Lemma A.1 in the region \(\Omega(s - \delta)\) to a function \(v := u\) with

\[
a = d = j = 2, \quad b = 2 - n, \quad k = 0 \quad (\text{or } k = 1), \quad N = 1,
\]

and

\[
\theta_1 = \frac{n}{8 - 3n} \quad (\text{or } \theta_2 = \frac{4 - n}{8 - 3n}).
\]

Integrating the resulting inequalities with respect to time and taking into account (3.8), we arrive at the following relations:

\[
A(s - \delta) \leq C(1 - r_0^2)^{-\alpha_1} T^{\beta_1} (R(s))^{1+\kappa_1} + C T (R(s))^{1+\kappa_3}, \quad (3.9)
\]

\[
B(s - \delta) \leq C(1 - r_0^2)^{-\alpha_2} T^{\beta_2} (R(s))^{1+\kappa_2} + C T (R(s))^{1+\kappa_3}, \quad (3.10)
\]
where

\[ A(s) := \int_{Q_T(s)} u^2 dx dt, \quad B(s) := \int_{Q_T(s-\delta)} u^2 dx dt, \]

\[ \alpha_1 = \frac{4(n + 4)}{8 - 3n}, \quad \alpha_2 = \frac{4(n - 6)}{8 - 3n}, \]

\[ \beta_1 = \frac{4(2 - n)}{8 - 3n}, \quad \beta_2 = \frac{2(2 - n)}{8 - 3n}, \]

\[ \kappa_1 = \frac{4n}{8 - 3n}, \quad \kappa_2 = \frac{2n}{8 - 3n}, \quad \kappa_3 = \frac{n}{2 - n}. \]

Since all integrals on the right-hand sides of (3.9), (3.10) vanish as \( T \to 0 \) and \( u \in L^2(0, T; H^1(-r_0, r_0)) \), then, for sufficiently small \( T \), we get

\[ A(s - \delta) \leq C_3(1 - r_0^2)^{-\alpha_1 T^{\beta_1}}(\delta^{-4} A(s) + \delta^{-2} B(s))^{1 + \kappa_1}, \]  

(3.11)

\[ B(s - \delta) \leq C_4(1 - r_0^2)^{-\alpha_2 T^{\beta_2}}(\delta^{-4} A(s) + \delta^{-2} B(s))^{1 + \kappa_2}, \]  

(3.12)

where \( C_3 \) and \( C_4 \) are positive constants depending on all known parameters and independent of \( \Omega \). We denote

\[ D(s) := A^{1 + \kappa_2}(s) + B^{1 + \kappa_1}(s), \quad \kappa = (1 + \kappa_1)(1 + \kappa_2), \]

\[ C_5(T) := 2^{\kappa - 1} \max \left\{ \left[ C_3(1 - r_0^2)^{-\alpha_1 T^{\beta_1}} \right]^{1 + \kappa_2}, \left[ C_4(1 - r_0^2)^{-\alpha_2 T^{\beta_2}} \right]^{1 + \kappa_1} \right\}. \]

Without loss of generality, we can define a function

\[ \tilde{D}(s) = D(s) \text{ for } s \in (0, r_0) \quad \text{and} \quad \tilde{D}(s) = 0 \text{ for } s > r_0. \]

Thus, by (3.11) and (3.12) we arrive at the inequality

\[ \tilde{D}(s - \delta) \leq C_5(T)(\delta^{-4\kappa} \tilde{D}^{1 + \kappa_1}(s) + \delta^{-2\kappa} \tilde{D}^{1 + \kappa_2}(s)) \]  

(3.13)

for all \( s \in \mathbb{R}^+ \) and \( \delta \in (0, r_0] \). Choosing

\[ \delta(s) = \max \left\{ \left[ 4C_5(T) \tilde{D}^{\kappa_1}(s) \right]^{\frac{1}{4\kappa}}, \left[ 4C_5(T) \tilde{D}^{\kappa_2}(s) \right]^{\frac{1}{2\kappa}} \right\} \]

in (3.13), we find

\[ \tilde{D}(s - \delta(s)) \leq \frac{1}{2} D(s), \]

whence it follows that

\[ \delta(s - \delta(s)) \leq \gamma \delta(s) \quad \forall s \in \mathbb{R}^+, \]  

(3.14)
where\[ \gamma = \max \{2^{-\frac{\nu_1}{4\nu}}, 2^{-\frac{\nu_2}{2\nu}}\} < 1. \]

Applying Stampacchia’s lemma (Lemma A.2) to (3.14), we obtain\[ \delta(s) = 0 \quad \text{for all } s \leq r_0 - \frac{\delta(r_0)}{1 - \gamma}. \]

Further, we find the upper bound for \( \delta(r_0) \). In view of Theorem 2.1,\[ (1 - x^2)^{\frac{\nu-2}{2}} \in L^2(Q_T) \quad \text{and} \quad (1 - x^2)^{\frac{\nu}{2}} u_x \in L^2(Q_T) \]
for any \( \nu > 1 \). Thus, the right-hand side of (3.7) is bounded for all \( T > 0 \). Hence, taking \( s = 2r_0 \) and \( \delta = r_0 \) in (3.9) and (3.10), we find \( \bar{D}(r_0) \leq C_6 C_5(T) \), whence it follows that\[ \delta(r_0) \leq C_7 (1 - r_0^2)^{-2(6-n) \frac{2-n}{n-4}} T^{\frac{2-n}{n-4}}. \]

This implies the upper bound for the speed of propagation of the solution support, i.e.,\[ \Gamma(T) \leq r_0 - C_8 T^{\frac{2-n}{n-4}} \quad \text{for all } T \leq T^* := \left( \frac{r_0}{C_8} \right)^{\frac{8-n}{2-n}} \] (3.15)
for any \( r_0 \in (0, 1) \), where\[ C_8 = \frac{C_7}{1 - \gamma} (1 - r_0^2)^{-2(6-n) \frac{2-n}{n-4}}. \]

### 3.3. Exact Upper Bound for the Speed of Propagation.

In this section, we improve estimate (3.15). Throughout the section, \( C \) denotes a positive constant independent of \( \Omega \). Applying Lemma A.1 in the region \( \Omega(s) \setminus \Omega(s - \delta) \) to a function \( v := u \) with\[ a = d = j = 2, \quad b = 1, \quad k = 0 \text{ (or } k = 1), \quad N = 1, \]
and\[ \theta_1 = \frac{1}{5} \quad \text{(or } \theta_2 = \frac{3}{5}), \]
integrating the resulting inequalities with respect to time, and taking into account the mass conservation law (2.4), we arrive at the following estimates:

\[ \iint_{K_T(s, \delta)} u^2 \, dx \, dt \leq C T^{1 - \theta_1} M^{2(1 - \theta_1)} \left( \iint_{K_T(s, \delta)} u_{xx}^2 \, dx \, dt \right)^{\theta_1} + C \delta^{-1} T M^2, \] (3.16)

\[ \iint_{K_T(s, \delta)} u_x^2 \, dx \, dt \leq C T^{1 - \theta_2} M^{2(1 - \theta_2)} \left( \iint_{K_T(s, \delta)} u_{xx}^2 \, dx \, dt \right)^{\theta_2} + C \delta^{-3} T M^2. \] (3.17)
Using (3.16), (3.17), and Young’s inequality, in view of (3.8), we find
\[
\frac{1}{T} \int_{Q_T(s-\delta)} u^{2-n}(T) \, dx + \frac{1}{T} \int_{Q_T(s-\delta)} u^{2-n} \, dx \, dt + C(1-r_0^2)^2 \int_{Q_T(s-\delta)} u_x^2 \, dx \, dt \\
\leq \varepsilon (1-r_0^2)^2 \int_{K_T(s,\delta)} u_x^2 \, dx \, dt + C\varepsilon \delta^{-5} (1-r_0^2)^{-3} TM^2,
\]
where \( \varepsilon > 0 \). Selecting sufficiently small \( \varepsilon \in (0,2^{-5}) \), as a result of the standard iterative process, we get
\[
\frac{1}{T} \int_{Q_T(s-\delta)} u^{2-n}(T) \, dx + \frac{1}{T} \int_{Q_T(s-\delta)} u^{2-n} \, dx \, dt + C(1-r_0^2)^2 \int_{Q_T(s-\delta)} u_x^2 \, dx \, dt \\
\leq C \delta^{-5} (1-r_0^2)^{-3} TM^2. \tag{3.18}
\]
Setting \( s = 2\Gamma(T) \) and \( \delta = \Gamma(T) \) in (3.18), we obtain
\[
\iint_{Q_T(\Gamma(T))} u_{xx}^2 \, dx \, dt \leq C \Gamma^{-5}(T)(1-r_0^2)^{-5} TM^2,
\]
whence, by analogy with (3.16) and (3.17), we find
\[
A(\Gamma(T)) \leq C \Gamma^{-1}(T)(1-r_0^2)^{-1} TM^2,
\]
\[
B(\Gamma(T)) \leq C \Gamma^{-3}(T)(1-r_0^2)^{-3} TM^2.
\]
Therefore,
\[
\delta(\Gamma(T)) \leq C \max \left\{ \left[ \Gamma^{-\kappa_1}(T) (1-r_0^2)^{-\kappa_1+2\alpha_1} M^{2\kappa_1} \right]^\frac{1}{4(1+\kappa_1)}, \right. \\
\left. \left[ \Gamma^{-3\kappa_2}(T) (1-r_0^2)^{-3\kappa_2+2\alpha_2} M^{2\kappa_2} \right]^\frac{1}{2(1+\kappa_2)} \right\}
\]
\[
= C_9 \max \left\{ \Gamma^{-\frac{n}{n+s}}(T) T^{\frac{2}{n+s}}, \Gamma^{-\frac{3n}{8-n}}(T) T^{\frac{2}{8-n}} \right\}.
\]
Thus, we get
\[
\Gamma(T) + C_{10} \max \left\{ \Gamma^{-\frac{n}{n+s}}(T) T^{\frac{2}{n+s}}, \Gamma^{-\frac{3n}{8-n}}(T) T^{\frac{2}{8-n}} \right\} \leq r_0, \tag{3.19}
\]
where \( C_{10} = \frac{C_9}{1-\gamma} \).
We now use the following calculus result: Let \( a > 0 \) and \( b > 0 \). Then the function \( f(x) = x + ax^{-b} \) for \( x \geq 0 \) takes its minimum value
\[
f(x_{\text{min}}) = \frac{1 + b}{b} x_{\text{min}}
\]
at \( x_{\text{min}} = (ab)^{\frac{1}{1+b}} \). Hence, minimizing the right-hand side, we obtain
\[
\Gamma_{\text{opt}}(T) = C_0 T^{\frac{1}{n+4}} \quad \text{for all} \quad T \leq T^*.
\]

Theorem 2.2 is proved.

Appendix A

**Lemma A.1** [14]. If \( \Omega \subset \mathbb{R}^N \) is a bounded domain with piecewise-smooth boundary, \( a > 1, \quad b \in (0, a), \quad d > 1, \quad \text{and} \quad 0 \leq k < j, \quad k, j \in \mathbb{N} \), then there exist positive constants \( d_1 \) and \( d_2 \) (\( d_2 = 0 \) if \( \Omega \) is unbounded) depending only on \( \Omega, \quad d, \quad j, \quad b, \quad \text{and} \quad N \) such that the following inequality is true for every \( v(x) \in W^{j,d}(\Omega) \cap L^b(\Omega) \):
\[
\left\| D^k v \right\|_{L^p(\Omega)} \leq d_1 \left\| D^j v \right\|_{L^\theta(\Omega)} \left\| v \right\|_{L^b(\Omega)}^{1-\theta} + d_2 \left\| v \right\|_{L^b(\Omega)},
\]
where
\[
\theta = \frac{1}{b} + \frac{k}{N} - \frac{1}{a} \in \left[ \frac{k}{j}, 1 \right).
\]

Note that if \( \Omega = B(0, R) \setminus B(0, r) \), where \( B(0, x) \) is a ball with radius \( x \) and origin at 0, then
\[
d_2 = c(R - r)^{-\frac{(a-b)N}{ab} - k}.
\]

**Lemma A.2** [15]. Assume that \( f(s) \) is nonnegative nondecreasing function satisfying the following inequality:
\[
f(s - f(s)) \leq \varepsilon f(s) \quad \forall s \leq s_0,
\]
where \( \varepsilon \in (0, 1) \). Then \( f(s) = 0 \) for all \( s \leq s_0 - \frac{f(s_0)}{1 - \varepsilon} \).

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