Small area estimation using multiple imputation in three-parameter logistic models

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Abstract

We propose a novel methodology relating item response theory methods with small area estimation strategies in the presence of missing data. Specifically, we propose an unbiased estimator for the average ability parameter of three-parameter logistic models. Thus, we carry out an extensive simulation study in order to compare our estimator with the well-known Horvitz-Thompson estimator. According to our experiments with synthetic data, our proposal has substantial lower standard errors than its competitor. In addition, we perform an actual application by considering the Mathematics results of the 2015 Program for International Student Assessment (PISA), and also, compare our results with previous analyses. Our findings strongly suggest that our methodology is a high competitive alternative for generating compelling official statistics.

Keywords: Small Area Estimation, Item Response Theory, Missing data, PISA.

1 Introduction

Educational assessment can be understood as the process of using collected information about attitudes, beliefs, knowledge, and skills to improve learning in academic programs (Allen, 2004). These data are typically obtained from standardized tests applied to students for assessing planned learning goals (Kuh et al. 2014, ICFES 2015, OECD 2016, UNESCO 2019). However, standardized tests for educational assessment are currently facing a serious issue: A decreasing number of participants per application due to lack of access and financial restrictions.

A common problem is the presence of missing data in response strings. For example, in the Colombian institute for educational assessment, previous test applications as well as

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findings included in technical manuals (e.g., ICFES 2014), show that students would take more than fifteen hours to go through all the questions in the entire test. Exposing each student to a fifteen-hours test is not only a counterproductive strategy for the final scores, but also promotes cognitive conflicts from a pedagogical point of view (Álvarez et al., 2007). This is also the case with other tests, including Program for International Student Assessment (PISA, OECD 2014), Trends in the International Study of Mathematics and Sciences (TIMMS, Mullis et al. 2015), and Saber 3°, 5° y 9° tests (ICFES, 2015), among many others.

Estimation and analysis of standardized tests are mostly carried out employing Item Response Theory (IRT) methods (e.g., Lord 1980, Martínez Arias 1995, Muñiz 1997, Hambleton and Swaminathan 2013). IRT models intend to explain the relationship between latent traits, i.e., unobservable characteristics or attributes, and their manifestations, i.e., observed outcomes, responses or performance (Embretson and Reise, 2013). In particular, three-parameters logistic models (3PLMs, e.g., Paek and Cole 2019) are very popular for such and end. According to Sulis and Porcu (2017), these models characterize the probability of responses to any particular question (item) as a function of a location parameter (basal position along the latent trait), a discrimination parameter (item’s capability to discriminate individuals with different latent trait values), and finally, an ability parameter (intensity of the latent trait).

To the best of our knowledge, there is a substantial lack of research about non-response methods in the context of standardized tests (e.g., Adams and Darwin 1982, Baker and Kim 2004, Sulis and Porcu 2017). Therefore, our foundational aim relies on extending IRT methodologies accounting for missing data, while achieving similar quality indicators at lower costs. Such a task is displayed in Table 1. Rows represent students who are probabilistically chosen to take a given test. Students are grouped by domains, which may represent educational establishments. Columns are divided in two classes, namely, strings and plausible values. String columns register the response of each student as 1 if a given question was answered correctly, as 0 if not, and finally as ‘x’ for all those questions not applied to a particular student. Non-applied questions are drawn according to specific random schemes (e.g., the Instituto Colombiano para la Evaluación de la Educación, ICFES, uses balanced incomplete blocks for that end, ICFES 2017). Plausible values columns contain the realization of random variables that estimate each student’s ability to answer a given question accounting for the presence of missing data. Thus, there are available $k$ plausible values for each student and $n_d$ random variable realizations for each plausible value, which implies that $n_d \times k$ values are available in domain $d$, for $d = 1, \ldots, D$. Finally, within each domain $d$, we have $k$ estimates of the ability mean along with $k$ estimates of the corresponding variance.

In contrast with standard methods, our proposal incorporates estimates of the ability
Table 1: Information per domain and per student.

| Domain | Student | String | Plausible values |
|--------|---------|--------|------------------|
|        | 1 x 0 x · · · 1 | \( \hat{\theta}_{111} \) \( \hat{\theta}_{112} \) · · · \( \hat{\theta}_{11k} \) |
| 1      |         |        |                  |
|        | \vdots | \vdots | \vdots           |
|        | \( n_1 \) 1 x 0 · · · x | \( \hat{\theta}_{1n_11} \) \( \hat{\theta}_{1n_12} \) · · · \( \hat{\theta}_{1n_1k} \) |
| Direct estimation | \( \hat{\theta}_{11} \) | \( \hat{\theta}_{12} \) | · · · | \( \hat{\theta}_{1k} \) |
| Estimated variance | \( \hat{\text{Var}}(\hat{\theta}_{11}) \) | \( \hat{\text{Var}}(\hat{\theta}_{12}) \) | · · · | \( \hat{\text{Var}}(\hat{\theta}_{1k}) \) |
|        | 1 x 1 1 · · · 1 | \( \hat{\theta}_{d11} \) \( \hat{\theta}_{d12} \) · · · \( \hat{\theta}_{d1k} \) |
| d      |         |        |                  |
|        | \vdots | \vdots | \vdots           |
|        | \( n_d \) 0 x 0 · · · 1 | \( \hat{\theta}_{dn_d1} \) \( \hat{\theta}_{dn_d2} \) · · · \( \hat{\theta}_{dn_dk} \) |
| Direct estimation | \( \hat{\theta}_{d1} \) | \( \hat{\theta}_{d2} \) | · · · | \( \hat{\theta}_{dk} \) |
| Estimated variance | \( \hat{\text{Var}}(\hat{\theta}_{d1}) \) | \( \hat{\text{Var}}(\hat{\theta}_{d2}) \) | · · · | \( \hat{\text{Var}}(\hat{\theta}_{dk}) \) |
|        | 1 1 0 x · · · 1 | \( \hat{\theta}_{D11} \) \( \hat{\theta}_{D12} \) · · · \( \hat{\theta}_{D1k} \) |
| D      |         |        |                  |
|        | \vdots | \vdots | \vdots           |
|        | \( n_D \) 0 x 0 · · · 1 | \( \hat{\theta}_{Dn_D1} \) \( \hat{\theta}_{Dn_D2} \) · · · \( \hat{\theta}_{Dn_Dk} \) |
| Direct estimation | \( \hat{\theta}_{D1} \) | \( \hat{\theta}_{D2} \) | · · · | \( \hat{\theta}_{Dk} \) |
| Estimated variance | \( \hat{\text{Var}}(\hat{\theta}_{D1}) \) | \( \hat{\text{Var}}(\hat{\theta}_{D2}) \) | · · · | \( \hat{\text{Var}}(\hat{\theta}_{Dk}) \) |

parameter using the Fay-Herriot approach (Fay and Herriot, 1979) within the framework of IRT modeling where multiple imputation tasks are needed. Such a methodology has in itself two major contributions. On the one hand, it has profound practical implications to deal properly with problematic data structures involving missing data, and on the other, it resolves challenging theoretical issues associated with the production of reliable official statistics. We highlight that this is a complex task demanding auxiliary variables correlated with the ability of the students (such as parental socio-economic level, parental education, school infrastructure, among others, according to Treviño et al. 2010), as well as sophisticated statistical tools for computing the resulting estimator together with its Mean Square Error (MSE).

This article is structured as follows: Section 2 shows the theoretical development of our estimator along with its MSE under the restrictions presented above; Section 3 presents an exhaustive simulation study in which the proposed estimator is compared with existing estimators in the literature; Section 4 exhibits an application of our methodology...
regarding the 2015 PISA test. Finally, Section 5 discusses our findings as well as some relevant aspects for future research.

2 Theoretical development of the estimator

In this section, we present our approach for estimating the average in small areas using multiple imputation in three-parameter logistic models (3PLMs, e.g., Paek and Cole 2019). Firstly, we review the estimation of plausible values. Secondly, we use small area estimation theory as well as the Fay-Herriot model in order to obtain expressions of the ability mean estimator along with its Mean Square Error (MSE).

2.1 Ability estimation using plausible values and 3PLMs

We have a finite universe \( U = \bigcup_{d=1}^{D} U_d \) of size \( N = \sum_{d=1}^{D} N_d \) distributed in \( d \) domains, where \( U_d \) is the population in domain \( d \) of size \( N_d \). Also, let \( s = \bigcup_{d=1}^{D} s_d \) be the sample of size \( n = \sum_{d=1}^{D} n_d \) under consideration, where \( s_d \) is the sample in domain \( d \) of size \( n_d \) drawn under a particular sampling design \( p(\cdot) \). It may happen that for some domain \( d \), either \( s_d \) turns out to be empty or \( n_d \) is not large enough in order to enable reliable estimations. Moreover, indicator variables \( \xi_{ij} \) registering whether individual \( j \) answers item \( i \) correctly, \( \xi_{ij} = 1, \) or not, \( \xi_{ij} = 0, \) are observed, for \( j = 1, \ldots, n_d \) and \( i = 1, \ldots, I \).

IRT modeling assumes that individuals are endowed with an ability to answer items correctly. Let \( \theta \) be an array containing the ability parameters of all the individuals under consideration. In order to find the distribution of \( \theta \), there must be available known individual-level auxiliary information, typically stored in a vector of variables \( x_{ij} \) for each \( j = 1, \ldots, n_d \) (the sub-index \( i \) emphasizes the notion of auxiliary information at the level of individuals or subjects). Thus, the probability distribution of \( \theta \) in the population is not only conditional on the observed indicator variables \( \xi_{\text{obs}} \), but also on the auxiliary information \( X_I \), i.e.,

\[
p(\theta \mid \xi_{\text{obs}}, X_I) \propto p(\xi_{\text{obs}} \mid \theta, X_I) p(\theta \mid X_I),
\]

where \( X_I \) is a matrix storing all the individual-level auxiliary information. Assuming conditional independence between \( \xi_{\text{obs}} \) and \( X_I \) as in Rubin and Schenker (1991), which is quite reasonable in practice, it is easy to see that

\[
p(\theta \mid \xi_{\text{obs}}, X_I) \propto p(\xi_{\text{obs}} \mid \theta) p(\theta \mid X_I).
\]

Under this setting, the main goal relies on finding the conditional distribution of \( \theta \) given \( \xi_{\text{obs}} \) and \( X_I \), which in turn depends on two conditional distributions. Firstly,
\( \mathcal{P}(\xi_{\text{obs}} \mid \theta) \), the response chain distribution of students given their ability, considered here as a standard 3PLM (Bock and Aitkin, 1981). Secondly, \( \mathcal{P}(\theta \mid X_i) \), the ability distribution of students given the auxiliary information, regarded here as a multivariate Normal distribution with mean \( X_i \Gamma \) and covariance matrix \( \Sigma \), where both \( \Gamma \) and \( \Sigma \) need to be estimated. On this point, we are implicitly stating that the parameter space of each component of \( \theta \) is the real line, even though most of the ability mass lies between \(-3 \) and \( 3 \) (e.g., de Andrade et al. 2000).

It is straightforward to see that the conditional distribution \( \mathcal{P}(\theta \mid \xi_{\text{obs}}, X_i) \) is completely determined by handling the unknown parameters in \( \mathcal{P}(\xi_{\text{obs}} \mid \theta) \) as well as those in \( \mathcal{P}(\theta \mid X_i) \). Existing IRT methods typically deal with this setting by, first, using the Expectation–Maximization (EM) algorithm (e.g., Bock and Aitkin 1981) to estimate the unknown parameters in \( \mathcal{P}(\theta \mid X_i) \), and then, using the Metropolis-Hastings (MH) algorithm (e.g., Fox 2010) to draw \( L \) plausible values (i.e., abilities estimates) for each individual \( j \) in domain \( d \). In this spirit, let \( \theta_{dj\ell}^{pv} \) be the \( \ell \)-th MH plausible value (hence the upper-index \( pv \)) of individual \( j \) in domain \( d \), for \( \ell = 1, \ldots, L \), with \( j = 1, \ldots, J \) and \( d = 1, \ldots, D \). Also, let \( \gamma_d = g(\theta_{dj\ell}^{pv}) \) be an arbitrary function of \( \theta_{dj\ell}^{pv} \) characterizing a specific feature in the \( d \)-th domain (e.g., the mean). Thus, an estimate of \( \gamma_d \) and its corresponding variance can be found by noticing that

\[
\mathcal{P}(\gamma_d \mid \xi_{\text{obs}}) = \int \mathcal{P}(\gamma_d \mid \xi_{\text{nobs}}, \xi_{\text{obs}}) \mathcal{P}(\xi_{\text{nobs}} \mid \xi_{\text{obs}}) \, d\xi_{\text{nobs}},
\]

where \( \xi_{\text{nobs}} \) is composed of all those unobserved indicators variables not given in \( \xi_{\text{obs}} \), and the integral is carried out over the space parameter of \( \xi_{\text{nobs}} \). As a consequence, for the average of the individuals’ abilities \( \gamma_d \), taking \( g(\cdot) \) as the mean of the \( \theta_{dj\ell}^{pv} \), it follows that

\[
\begin{align*}
\mathbb{E}(\gamma_d \mid \xi_{\text{obs}}) &= \mathbb{E}[\mathbb{E}(\gamma_d \mid \xi_{\text{nobs}}, \xi_{\text{obs}}) \mid \xi_{\text{obs}}] \simeq \frac{1}{L} \sum_{\ell=1}^{L} \hat{\theta}_{dj\ell}^{pv} = \hat{\gamma}_d \\
\text{Var}(\gamma_d \mid \xi_{\text{obs}}) &= \mathbb{E}[\text{Var}(\gamma_d \mid \xi_{\text{nobs}}, \xi_{\text{obs}}) \mid \xi_{\text{nobs}}] + \text{Var}[\mathbb{E}(\gamma_d \mid \xi_{\text{nobs}}, \xi_{\text{obs}}) \mid \xi_{\text{obs}}] \\
&\simeq \frac{1}{L} \sum_{\ell=1}^{L} \text{Var}(\hat{\theta}_{dj\ell}^{pv}) + \left( 1 + \frac{1}{L} \right) \frac{1}{L-1} \sum_{\ell=1}^{L} (\hat{\theta}_{dj\ell}^{pv} - \hat{\gamma}_d)^2, \quad (1)
\end{align*}
\]

where \( \hat{\theta}_{dj\ell}^{pv} \) is the \( \ell \)-th plausible value in domain \( d \), and \( \text{Var}(\hat{\theta}_{dj\ell}^{pv}) \) depends on the specific formulation of the sampling design and corresponds to the average of the estimated variances for each plausible value.

### 2.2 Proposed estimator

Here, we take a step further and go beyond the exiting IRT literature as presented above in order to carry out our main task: Combining IRT methods with small area estimation
strategies accounting for missing data. Thus, inspired in Small Area Estimation (SAE) methods, we propose estimating \( \gamma = (\gamma_1, \ldots, \gamma_D) \), as a result of the model

\[
\hat{\gamma} = X_A \beta + Zu + e,
\]

where \( X_A \) is a fixed-effects matrix storing all the area-level auxiliary information (the sub-index A emphasizes the notion of auxiliary information at the level of areas or domains), \( \beta \) is a vector of unknown constants, \( Z \) is a random-effects design matrix, and finally, \( u \) and \( e \) are independent random vectors such that \( u \sim N(0, V_u) \) and \( e \sim N(0, V_e) \), with known \( V_u \) and \( V_e \). Thus, under the previous specification it follows directly that \( \text{Var}(\hat{\gamma}) = ZV_uZ^T + V_e = V \), and also, following standard results about linear mixed-effects models (e.g., McCulloch and Searle 2004), it can be shown that the optimal unbiased linear predictor of \( \tau = L \beta + Mu \) is \( \hat{\tau} = L \hat{\beta} + M \hat{u} \), with

\[
\hat{\beta} = \left( X_A^T V^{-1} X_A \right)^{-1} X_A^T V^{-1} \hat{\gamma} \quad \text{and} \quad \hat{u} = (ZV_u)^T V^{-1} \left( \hat{\gamma} - X_A \hat{\beta} \right).
\]

Now, building on ideas given in Harville (1977) and Morales and Molina (2015), we consider an unbiased linear predictor of the form \( \hat{\tau} = Q_0 + Q_1 \hat{\gamma} \), where \( Q_0 \) and \( Q_1 \) are two conformable matrices. Since \( \hat{\tau} \) is unbiased, we have that \( E(\hat{\tau} - \tau) = 0 \), but

\[
0 = E(\hat{\tau} - \tau) = Q_0 E(\hat{\gamma}) + Q_1 = Q_0 X_A \beta + Q_1 \quad \text{and} \quad E(\tau) = L \beta.
\]

Thus, the best predictor \( \hat{\tau} \) can be found by minimizing \( \text{Var}(\hat{\tau} - \tau) \) subject to \( Q_1 X_A = L \). Thus,

\[
\text{Var}(\hat{\tau} - \tau) = \text{Var}(Q_1 \hat{\gamma} - L \beta - Mu)
= \text{Var}(Q_1 \hat{\gamma}) + \text{Var}(Mu) - 2 \text{Cov}(Q_1 \hat{\gamma}, Mu)
= Q_1 VQ_1^T + MV_uM^T - 2Q_1 CM^T,
\]

with \( C = \text{Cov}(\hat{\gamma}, u) \). Since \( MV_uM^T \) does not depend on \( Q_1 \), the minimization problem given above can be restated as minimize \( Q_1 VQ_1^T - 2Q_1 CM^T \) subject to \( Q_1 X_A = L \), whose corresponding Lagrangian function is

\[
\ell(Q_1, \Lambda) = Q_1 VQ_1^T - 2Q_1 CM^T + 2(Q_1 X_A - L) \Lambda.
\]

Taking partial derivatives with respect to both \( Q_1 \) and \( \Lambda \), we get that

\[
\frac{\partial \ell(Q_1, \Lambda)}{\partial Q_1} = 2VQ_1^T - 2CM^T + 2X_A \Lambda = 0 \quad \Rightarrow \quad VQ_1^T + X_A \Lambda = CM^T,
\]

\[\Rightarrow\]
and
\[ \frac{\partial \ell(Q_1, \Lambda)}{\partial \Lambda} = 2(Q_1X_A - L) = 0 \quad \Rightarrow \quad X_A^TQ_1^T = L^T, \]
that in turn can be rewritten in matrix form as
\[
\begin{pmatrix}
V \\
X_A^T
\end{pmatrix}
\begin{pmatrix}
Q_1^T \\
\Lambda
\end{pmatrix}
= \begin{pmatrix}
CM^T \\
L^T
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
Q_1^T \\
\Lambda
\end{pmatrix}
= \begin{pmatrix}
V \\
X_A^T
\end{pmatrix}^{-1}
\begin{pmatrix}
VX_A^T
\end{pmatrix} \begin{pmatrix}
CM^T \\
L^T
\end{pmatrix}.
\]

In this way, using standard results from matrix algebra and letting
\[ G = \left( X_A^TVX_A \right)^{-1}, \]
we obtain that
\[
\begin{pmatrix}
Q_1^T \\
\Lambda
\end{pmatrix}
= \left( V^{-1} - V^{-1}X_AGX_A^TV^{-1} - V^{-1}X_AG \right)
\begin{pmatrix}
CM^T \\
L^T
\end{pmatrix},
\]
which means that
\[
Q_1 = \left[ V^{-1}X_AGL^T + V^{-1} \left( I - X_AGX_A^TV^{-1} \right) CM^T \right]^T
= LGX_A^TV^{-1} + MC^TV^{-1} \left( I - X_AGX_A^TV^{-1} \right),
\]
where \( I \) is the identity matrix.

Recall from our earlier discussion that the linear predictor takes the form \( \hat{\tau} = Q_1\hat{\gamma} \) because it is assumed to be unbiased from the beginning. Thus, substituting \( Q_1 \) by the expression found in Eq. (4), we get that the Best Linear Unbiased Prediction (BLUP) of \( \tau \) is given by
\[
\hat{\tau} = \left[ LGX_A^TV^{-1} + MC^TV^{-1} \left( I - X_AGX_A^TV^{-1} \right) \right] \hat{\gamma}
= LGX_A^TV^{-1}\hat{\gamma} + MC^TV^{-1}\hat{\gamma} - MC^TV^{-1}X_AGX_A^TV^{-1}\hat{\gamma}
= LGX_A^TV^{-1}\hat{\gamma} + MC^TV^{-1}\left( \hat{\gamma} - X_A\hat{\beta} \right)
= L\hat{\beta} + M\hat{u},
\]
where \( \hat{\beta} = GX_A^TV^{-1}\hat{\gamma} \) and \( \hat{u} = C^TV^{-1} \left( \hat{\gamma} - X_A\hat{\beta} \right) \).

Now, our goal is to see through the previous expression in order to adapt the Fay-Herriot model,
\[
\hat{\gamma}_d = x_{A,d}^T \beta + u_d + e_d, \quad d = 1, \ldots, D,
\]
where \( x_{A,d}^T \beta = \sum_{k=1}^p x_{A,dk} \beta_k \) is a linear predictor of fixed effects and \( u_d \) is a domain-specific random effect. As usual, all the random effects are assumed to be independent and identically distributed with zero mean and variance \( \sigma_u^2 \). Furthermore, \( e_d \) is the
Therefore, the BLUP proposed in this article is

\[ \hat{\beta} = (X_A^T V^{-1} X_A)^{-1} X_A^T V \hat{\gamma} \]

and

\[ \hat{u} = C^T V^{-1} (\hat{\gamma} - X_A \hat{\beta}) \]

Using the results provided in this section and rewriting model (5) in matrix form as in Eq. (2), it follows that

\[
\begin{pmatrix}
\hat{\gamma}_1 \\
\vdots \\
\hat{\gamma}_D
\end{pmatrix} =
\begin{pmatrix}
x_{A11} & \cdots & x_{Ap}
\vdots & \ddots & \vdots \\
x_{AD1} & \cdots & x_{ADp}
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\vdots \\
\beta_p
\end{pmatrix} +
\begin{pmatrix}
u_1 \\
\vdots \\
u_D
\end{pmatrix} +
\begin{pmatrix}
e_1 \\
\vdots \\
e_d
\end{pmatrix},
\]

which means that \( Z = I_D \) and \( V = \text{diag}(\sigma_u^2 + \sigma_1^2, \ldots, \sigma_u^2 + \sigma_D^2) \). As a consequence, we have that the Best Linear Unbiased Estimator (BLUE) of \( \beta = (\beta_1, \ldots, \beta_p) \) and \( u = (u_1, \ldots, u_D) \) are respectively

\[
\tilde{\beta} = \left( X_A^T V^{-1} X_A \right)^{-1} X_A^T V \hat{\gamma}
\]

and

\[
\tilde{u} = C^T V^{-1} \left( \hat{\gamma} - X_A \tilde{\beta} \right)
\]

Therefore, the BLUP proposed in this article is

\[
\hat{\gamma}_d^B = \frac{\sigma_u^2}{\sigma_d^2 + \sigma_d^2} \hat{\gamma}_d + \frac{\sigma_d^2}{\sigma_d^2 + \sigma_d^2} x_{Ad}^T \tilde{\beta}.
\]

Also, the Empirical Best Linear Unbiased Predictor (EBLUP) of the mean of the small area \( \gamma_d \) under model (5) can be obtained just by replacing \( \sigma_u^2 \) by its corresponding estimate \( \hat{\sigma}_u^2 \) in Eq. (7), i.e.,

\[
\hat{\gamma}_d^P = \frac{\hat{\sigma}_u^2}{\hat{\sigma}_d^2 + \hat{\sigma}_d^2} \hat{\gamma}_d + \frac{\hat{\sigma}_d^2}{\hat{\sigma}_d^2 + \hat{\sigma}_d^2} x_{Ad}^T \tilde{\beta} = (1 - B_d) \hat{\gamma}_d + B_d x_{Ad}^T \tilde{\beta},
\]

with \( B_d = \sigma_d^2 / (\hat{\sigma}_u^2 + \hat{\sigma}_d^2) \).

The proposed estimator \( \hat{\gamma}_d^P \) is unbiased for \( \gamma_d \), for \( d = 1, \ldots, D \). Indeed, consider the difference

\[
\hat{\gamma}_d^P - \gamma_d = (1 - B_d) \hat{\gamma}_d + B_d x_{Ad}^T \tilde{\beta} - [B_d \gamma_d + (1 - B_d) \gamma_d].
\]

Now, by letting \( \alpha_d = 1 - B_d \), \( \gamma_d = x_{Ad}^T \beta + u_d \), and \( \hat{\gamma}_d = \gamma_d + e_d \), we have that

\[
\begin{align*}
\hat{\gamma}_d^P - \gamma_d &= \alpha_d (\gamma_d + e_d) + (1 - \alpha_d) x_{Ad}^T \tilde{\beta} - \alpha_d \gamma_d - (1 - \alpha_d) \gamma_d \\
&= \alpha_d e_d + (1 - \alpha_d) x_{Ad}^T \tilde{\beta} - (1 - \alpha_d) [x_{Ad}^T \beta + u_d] \\
&= \alpha_d e_d + (1 - \alpha_d) [x_{Ad}^T (\tilde{\beta} - \beta)] - (1 - \alpha_d) u_d.
\end{align*}
\]

Then, taking expected values,

\[
\mathbb{E} \left( \hat{\gamma}_d^P - \gamma_d \right) = \mathbb{E} \left( \alpha_d e_d + (1 - \alpha_d) [x_{Ad}^T (\tilde{\beta} - \beta)] - (1 - \alpha_d) u_d \right) = 0,
\]

because \( \mathbb{E}(e_d) = \mathbb{E}(u_d) = 0 \), and also, \( \mathbb{E}(\tilde{\beta} - \beta) = 0 \) since \( \tilde{\beta} \) is an unbiased estimator for \( \beta \).
2.3 Mean square error of the proposed estimator

Here, we follow very closely Kackar and Harville (1984), Prasad and Rao (1990) as well as Ghosh and Rao (1994), in order to obtain an expression for the Mean Square Error (MSE) of the proposed estimator $\hat{\gamma}_d$. Specifically, we consider the variance component estimation method, which require us to find three quantities explicitly, namely, $g_{1d}(\sigma_u^2)$, $g_{2d}(\sigma_u^2)$, and $g_{3d}(\sigma_u^2)$, for $d = 1, \ldots, D$. We refer the reader to the previous references for details about such a method. However, we outline below some fundamental details.

First, in order to calculate $g_{1d}(\sigma_u^2)$, we need to take into account that $V_u = \sigma_u^2 I_D$ and $V_e = \sigma_e^2 W_N^{-1} V_s^{-1} = \text{diag} (V_{s1}^{-1}, \ldots, V_{sD}^{-1})$, where $W_N$ is a $N \times N$ diagonal matrix of weights induced by the sampling design $p(\cdot)$, with $N$ the population size, $s_d$ the sample in domain $d$, and $V_{s_d}^{-1} = \frac{1}{\sigma_d^2} \left( W_{s_d} - \frac{B_d}{w_d} w_n w_{n_d}^T \right)$, where $w_d$ is the weight of domain $d$, for $d = 1, \ldots, D$. Thus,

$$V_s = \frac{\sigma_u^4}{\sigma_d^2} \text{diag} \left( 1_{n_1}, \ldots, 1_{n_D} \right) \cdot \text{diag} \left( W_{n_1} - \frac{B_1}{n_1} w_{n_1} 1_{n_1}^T, \ldots, W_{n_D} - \frac{B_D}{n_D} w_{n_D} 1_{n_D}^T \right) \cdot \text{diag} \left( 1_{n_1}, \ldots, 1_{n_D} \right) \cdot \text{diag} \left( B_1, \ldots, B_D \right),$$

and therefore,

$$T_s = V_u (I_D - Z_s^T V_e^{-1} Z_s V_u) = \sigma_u^2 \text{diag} (1 - B_1, \ldots, 1 - B_D),$$

with $Z_s = \text{diag} (1_{n_1}, \ldots, 1_{n_D})$ and

$$V_{e_s} = \text{diag} \left( W_{n_1} - \frac{B_1}{n_1} w_{n_1} 1_{n_1}^T, \ldots, W_{n_D} - \frac{B_D}{n_D} w_{n_D} 1_{n_D}^T \right).$$

The previous result is quite useful because we want to estimate the average of the plausible values for all the individuals within domain $d$, i.e., $\eta = a^T \gamma_U$, with

$$a^T = \frac{1}{N_d} \left( 0_{N_1}^T, \ldots, 0_{N_{d-1}}^T, 1_{N_d}^T, 0_{N_{d+1}}^T, \ldots, 0_{N_D}^T \right),$$

with $\gamma_U$ is the population parameter, leading us directly to consider all those individuals included the sample as well as those that did not, i.e, $s$ and $r$, respectively. As a
consequence,
\[
g_{1d} \left( \sigma^2_u \right) = a_r^\top Z_s T_s Z_s^\top a_r = \left( 0^\top, \ldots, 0^\top, 1_{N_d-n_d}^\top, 0^\top, \ldots, 0^\top \right) \cdot \text{diag} \left( 1_{N_1-n_1}, \ldots, 1_{N_d-n_d} \right) \cdot \frac{\sigma^2_u}{N_d^2} \text{diag} \left( 1 - B_1, \ldots, 1 - B_D \right) \cdot \text{diag} \left( 1_{N_1-n_1}^\top, \ldots, 1_{N_D-n_D}^\top \right) \left( 0^\top, \ldots, 0^\top, 1_{N_d-n_d}^\top, 0^\top, \ldots, 0^\top \right)
\]
\[
= \frac{\sigma^2_u}{N_d^2} (1 - B_d) (N_d - n_d)^2
\]
\[
\simeq \sigma^2_u (1 - B_d),
\]
for \( n_d << N_d \).

Now, in order to compute \( g_{2d} \left( \sigma^2_u \right) \), we need to get
\[
Z_s T_s Z_s^\top = \text{diag} \left( 1_{N_1-n_1}, \ldots, 1_{N_D-n_D} \right) \cdot \sigma^2_u \text{diag} \left( 1 - B_1, \ldots, 1 - B_D \right) \cdot \text{diag} \left( 1_{n_1}^\top, \ldots, 1_{n_D}^\top \right) = \sigma^2_u \text{diag} \left( (1 - B_1) 1_{N_1-n_1} 1_{n_1}^\top, \ldots, (1 - B_D) 1_{N_D-n_D} 1_{n_D}^\top \right),
\]
which means that,
\[
a_r^\top Z_s T_s Z_s^\top V_s^{-1} X_{A_s} = \frac{1}{N_d} \frac{\sigma^2_u}{\sigma_d^2} \left( 0^\top, \ldots, 0^\top, 1_{N_d-n_d}^\top, 0^\top, \ldots, 0^\top \right) \cdot \text{diag} \left( (1 - B_1) 1_{N_1-n_1} 1_{n_1}^\top, \ldots, (1 - B_D) 1_{N_D-n_D} 1_{n_D}^\top \right) \cdot W_s X_{A_s}
\]
\[
= \frac{1}{N_d} \frac{\sigma^2_u}{\sigma_d^2} (1 - B_d) (N_d - n_d) \left( 0^\top, \ldots, 0^\top, w_{n_d}^\top, 0^\top, \ldots, 0^\top \right) X_{A_s}
\]
\[
= (1 - f_d) B_d \hat{x}_d,
\]
with \( f_d = n_d/N_d \) is the sampling fraction and \( \hat{x}_d = \frac{1}{\sum_{wd}} \sum_{k \in s_d} x_{dk} w_{dk} \). Moreover,
\[
a_r^\top X_{A_r} = \frac{1}{N_d} \left( 0^\top, \ldots, 0^\top, 1_{N_d-n_d}^\top, 0^\top, \ldots, 0^\top \right) X_{A_r} = (1 - f_d) \bar{x}_d,
\]
with \( \bar{x}_d = \frac{1}{N_d} \sum_{k \in d} x_{dk} \). Using together Eqs. (10) and (11), along with \( n_d << N_d \), we finally get that
\[
g_{2d} \left( \sigma^2_u \right) = \left( \bar{x}_d - B_d \hat{x}_d \right) \left( X_{A_s}^\top V_s^{-1} X_{A_s} \right)^{-1} \left( \bar{x}_d - B_d \hat{x}_d \right)^\top.
\]
Lastly, in order to \( g_{3d}(\sigma_u^2) \), we need to get
\[
\mathbf{b}^T = \frac{1}{N_d} \frac{\sigma_u^2}{\sigma_d^2} \left( 0^T, \ldots, 0^T, 1_{N_d-n_d}^T, 0^T, \ldots, 0^T \right) \cdot \\
\text{diag} \left( 1_{N_1-n_1}, \ldots, 1_{N_D-n_D} \right) \cdot \\
\text{diag} \left( (\mathbf{W} - \frac{B_1}{w_1} \mathbf{w}_n \mathbf{w}_n^T, \ldots, \mathbf{W} - \frac{B_D}{w_D} \mathbf{w}_n \mathbf{w}_n^T) \right) \cdot \\
\frac{1}{N_d} \left( 0^T, \ldots, 0^T, 1_{N_d-n_d}^T, 0^T, \ldots, 0^T \right).
\]

Then,
\[
\nabla \mathbf{b}^T = \left( 0^T, \ldots, 0^T, (1 - f_{d}) \frac{\partial B_d}{\partial \sigma_d^2} \frac{1}{w_d} \mathbf{w}_n \mathbf{w}_n^T, 0^T, \ldots, 0^T \right)
\]

and therefore,
\[
g_{3d}(\sigma_u^2) = (1 - f_d)^2 \left( \sigma_u^2 + \frac{\sigma_d^2}{w_d} \right). \\
\text{tr} \left\{ \left( \frac{\partial B_d}{\partial \sigma_d^2} \frac{2}{\sigma_u^2 + \sigma_d^2} \right) \left( \begin{array}{cc} \text{Var}(\hat{\sigma}_d^2) & \text{Cov}((\hat{\sigma}_d^2, \hat{\sigma}_u^2)) \\ \text{Cov}((\hat{\sigma}_d^2, \hat{\sigma}_u^2)) & \text{Var}(\hat{\sigma}_u^2) \end{array} \right) \right\}
\]

which means that,
\[
g_{3d}(\sigma_u^2) = \left( \sigma_u^2 + \frac{\sigma_d^2}{w_d} \right)^{-3} \frac{1}{w_d^3} \text{Var}(\hat{\sigma}_d^2 - \sigma_d^2 \hat{\sigma}_u^2) \quad \text{(13)}
\]

for \( n_d << N_d \).

The estimation of the MSE for the proposed estimator \( \hat{\gamma}_d^{\text{P}} \) can be now obtained taking into account our findings given in Eqs. (9), (12), and (13). Specifically, considering \( \sigma_u^2 \) instead of \( \hat{\sigma}_u^2 \), it follows that
\[
g_{1d}(\hat{\sigma}_u^2) = \frac{\sigma_u^2 \sigma_d^2}{\sigma_u^2 + \sigma_d^2} = \sigma_d^2 (1 - B_d) \quad \text{and} \quad g_{2d}(\hat{\sigma}_u^2) = \left( \frac{\sigma_d^2}{\sigma_u^2 + \sigma_d^2} \right)^2 a = B_d^2 a,
\]
where \( B_d = \sigma_d^2 / (\sigma_u^2 + \sigma_d^2) \) and \( a = x_d A^{-1} x_d^T A \), with \( F = (\sigma_u^2 + \sigma_d^2)^{-1} \sum_{d=1}^D x_d A x_d^T A \). The component \( g_{3d}(\hat{\sigma}_u^2) \) depends on the estimation method of the variance components.
Either way, \( g_{3d}(\hat{\sigma}_u^2) \) takes the form

\[
g_{3d}(\hat{\sigma}_u^2) = \frac{\sigma_d^4}{(\hat{\sigma}_u^2 + \sigma_d^2)^3} \ \text{var}(\hat{\sigma}_u^2),
\]

where

\[
\text{Var}(\hat{\sigma}_u^2) = \frac{2}{D} \left[ \sigma_u^4 + \frac{2\hat{\sigma}_u^2}{D} \sum_{d=1}^{D} \sigma_d^2 + \frac{1}{D} \sum_{d=1}^{D} \sigma_d^4 \right] \quad \text{or} \quad \text{Var}(\hat{\sigma}_u^2) = \frac{2}{D} \left[ \sum_{d=1}^{D} (\hat{\sigma}_u^2 + \sigma_d^2)^{-2} \right]^{-1},
\]

using the Prasad-Rao moment estimator (Prasad and Rao, 1990), or either the Maximum Likelihood (ML) estimator or the Restricted Maximum Likelihood (REML) estimator, respectively. If the latter, \( g_{3d}(\hat{\sigma}_u^2) \) simplifies to

\[
g_{3d}(\hat{\sigma}_u^2) = \left( \frac{1}{\sigma_u^2 + \sigma_d^2} \right) \frac{2D^2}{\sum_{d=1}^{D} (\hat{\sigma}_u^2 + \sigma_d^2)^2}. \]

Finally, on the one hand, if either Prasad-Rao or REML are used, then the MSE estimator is given by

\[
\widehat{\text{MSE}}(\hat{\gamma}_d^P) = g_{1d}(\hat{\sigma}_u^2) + g_{2d}(\hat{\sigma}_u^2) + 2g_{3d}(\hat{\sigma}_u^2),
\]

but on the other, if ML is used, then

\[
\widehat{\text{MSE}}(\hat{\gamma}_d^P) = g_{1d}(\hat{\sigma}_u^2) + g_{2d}(\hat{\sigma}_u^2) + 2g_{3d}(\hat{\sigma}_u^2) - b \nabla g_1,
\]

with

\[
b = - \left[ \sum_{d=1}^{D} (\hat{\sigma}_u^2 + \sigma_d^2)^{-2} \right]^{-1}.
\]

\[
\text{tr} \left\{ \left( \sum_{d=1}^{D} (\hat{\sigma}_u^2 + \sigma_d^2)^{-1} x_d A x_d^T A \right)^{-1} \left( \sum_{d=1}^{D} (\hat{\sigma}_u^2 + \sigma_d^2)^{-2} x_d A x_d^T A \right) \right\}
\]

and \( \nabla g_1 = \sigma_d^4 (\hat{\sigma}_u^2 + \sigma_d^2)^{-2} \).

### 3 Simulation study

In this section, we carry out a simulation study in order to assess the statistical properties of the proposed estimators. The aim of this simulation is to analyze the Relative Bias, \( SB_d = \frac{\gamma_d - \hat{\gamma}_d}{\gamma_d} \), the Relative Standard Error \( EER_d = \frac{\text{RMSE}(\hat{\gamma}_d)}{\gamma_d} \times 100\% \), as well as the Relative Mean Square Error \( \text{RMSE}(\hat{\gamma}_d) = \sqrt{\text{MSE}(\hat{\gamma}_d)} \) associated with the proposed estimator versus the Horvitz-Thompson estimator (Narain 1951, Horvitz and Thompson 1952), the calibration estimator, and the composite estimator.

Our simulation study follows the following steps:
1. Simulate a population of $N = 100,000$ students with 150 items per student and $D = 500$ domains, along with two auxiliary variables associated with the ability $\theta$ for each student.

2. Set 10%, 20% and 30% of missing responses per student completely at random in the population.

3. Define two (2) auxiliary variables at the domain level correlated with $\theta_d$ on three levels (high ($> 80\%$), medium ($60\%$, $80\%$) and low ($< 60\%$)).

4. Estimate five (5) plausible values for each student according to Equation ??.

5. Consider a different number of domains in the sample using different sampling fractions as $f_d = 30\%$, 50%, and 70% on the total number of domains in the population.

6. For the domains selected in step 5., select a random sample from the population using simple random sampling with sampling fraction $f_n = 5\%$, 10%, and 20%, considering the Horvitz-Thompson estimate, $\hat{\gamma}_d^{Dir}$, the calibration estimate, $\hat{\gamma}_d^{Cal}$, and the composite estimator, $\hat{\gamma}_d^{Comp}$, the proposed estimator, $\hat{\gamma}_d^{P}$ using the REML method, for each selected domain, along with its standard deviation.

7. Calculate the relative biases, $SB_d$, and the relative standard errors, $EER_d$, by domain, for each of the estimators $\hat{\gamma}_d^{Dir}$, $\hat{\gamma}_d^{Cal}$, $\hat{\gamma}_d^{Comp}$, and $\hat{\gamma}_d^{P}$, and compute:
   (a) $SB = \frac{\sum_{D}^{D} SB_d}{D}$ (average relative bias).
   (b) $EER = \frac{\sum_{D}^{D} EER_d}{D}$ (average relative standard error).

8. Repeat steps 5. to 7. for 100,000 random samples and compute:
   (a) $SBR = \frac{\sum_{r=1}^{100000} SB_{P_r}}{100000}$ (mean of the average relative biases).
   (b) $EERP = \frac{\sum_{r=1}^{100000} EERP_r}{100000}$ (mean of the average relative standard error).
### 3.1 Simulation Study Results

| $f_d$ (%) | $f_n$ (%) | $\text{EERP}^{\gamma}_{4n}$ (%) | $\text{EERP}^{\gamma}_{Cal}$ (%) | $\text{EERP}^{\gamma}_{\text{Comp}}$ (%) | $\text{EERP}^{\gamma}_{P}$ (%) | $\text{SBR}^{\gamma}_{P}$ (%) |
|-----------|-----------|-------------------------------|-------------------------------|---------------------------------|-----------------|-----------------|
| 30%       | 5%        | 1.53                          | 1.20                          | 1.03                            | 0.85            | 0.08            |
| 30%       | 10%       | 1.23                          | 1.07                          | 1.00                            | 0.85            | 0.02            |
| 30%       | 20%       | 1.03                          | 1.00                          | 0.98                            | 0.85            | -0.01           |
| 50%       | 5%        | 1.87                          | 1.38                          | 1.09                            | 0.88            | 0.14            |
| 50%       | 10%       | 1.45                          | 1.16                          | 1.02                            | 0.86            | 0.05            |
| 50%       | 20%       | 1.17                          | 1.04                          | 0.99                            | 0.87            | 0.01            |
| 70%       | 5%        | 2.16                          | 1.56                          | 1.15                            | 0.96            | 0.22            |
| 70%       | 10%       | 1.63                          | 1.24                          | 1.05                            | 0.87            | 0.09            |
| 70%       | 20%       | 1.29                          | 1.09                          | 1.01                            | 0.89            | 0.03            |

*Table 2: Missing 10% and high correlation*

Table 2 considers the simulation scenario when the percentage of missing values is 10% and there is a high correlation between the auxiliary variables and the mean ability. Our findings make evident that the proposed estimator is unbiased as it was theoretically shown. In all the scenarios, the proposed estimator has a lower $\text{EERP}$ as well as a lower mean squared error compared to the alternative estimators. One of the scenarios where the $\text{EERP}$ is higher for all the considered estimators is when the percentage of domains is 70% (350 domains) and sample fraction of 5% (5,000 individuals). Such a configuration is not very convenient in practical terms, since, on average, each selected domain will have fifteen observations. Also, under this particular scenario, the Horvitz-Thompson estimator becomes less efficient than both the calibration and composite ones. However, the proposed estimator has the property of high efficiency with a low $\text{EERP}$ for all the simulation scenarios.

Another result in Table 2 is that if the sample percentage increases ($f_n \uparrow$), being the number of domains fixed, then the $\text{EERP}$ in all the estimators decreases. Since the sampling fraction inside each domain increases, therefore the variance of the estimates tends to decrease. On the other hand, if the number of domains increases ($f_d \uparrow$) being the sample percentage fixed, the $\text{EERP}$ for all estimators increases. This is because, as the number of domains increases, the sample size per domain decreases and makes the estimates less efficient. The results obtained for the remaining scenarios are consistent with the ones in Table 2 and they can be found in the Appendix.
4 Application

The information considered in this section corresponds to the one published in the OECD website https://www.oecd.org/pisa/data/. Plausible values for the PISA 2015 Mathematics Test are available in this dataset for each country and they will be used to estimate their average results. In addition, the auxiliary information that we consider to carry out this application is composed of variables related to the learning context that, according to Treviño et al. (2016), have a direct relationship with academic achievement. In particular, the auxiliary variables considered for each country in this case are: i) Gross Domestic Product (GDP), ii) Expenditure per student at secondary level (% of GDP per capita), iii) Unemployment total (% of total participation in the labour force as a national estimate), iv) Number of articles in scientific and technical publications, v) Expenditure on research and development (% of GDP), vi) Public expenditure on education total (% of GDP), vii) Gini index, viii) Percentage of schools with access to drinking water service, ix) Percentage of schools with access to electric service. The model that PISA uses to obtain the estimated ability of the students is shown below,

\[ P_i(\xi_{ik} = 1 \mid \theta_k, a_i, b_i, ) = \frac{e^{1.7a(\theta_k-b_i)}}{1+e^{1.7a(\theta_k-b_i)}}. \]  

(14)

Also, the significant auxiliary variables and their associated coefficients, \( \beta \), at a 10% significance level are shown in Table 3.

| Covariate                                    | \( \hat{\beta} \)      | p-values       |
|----------------------------------------------|------------------------|----------------|
| Intercept                                    | -5669.07               | \( 2.79 \times 10^{-5} \) |
| Public spending on education, total (% of GDP) | 14.52                  | \( 9.7310^{-3} \)   |
| % of schools with access to electricity service | 54.60                  | \( 8.53 \times 10^{-5} \) |
| % of schools with access to drinking water service | 6.26                   | \( 5.16 \times 10^{-2} \) |
| Research and development expenditure (% of GDP) | 10.41                  | \( 2.91 \times 10^{-2} \) |
| Unemployment, total (% of total labor force participation) | -1.50                  | \( 9.6 \times 10^{-2} \) |

Table 3: Coefficients estimates, \( \beta \), and their p-values

In order to compute the estimated ability mean using the proposed estimator, we use directly Eq. (8), and doing so, first, we estimate the variance of the random effect by means of Eq. (6), obtaining that \( \hat{\sigma}_u = 986.58 \). The variances \( \sigma_d^2 \) and the direct estimates \( \hat{\gamma}_d \) are those reported by PISA 2015. Thus, in order to calculate the \( MSE \) of \( \hat{\gamma}_d^P \), we must compute \( g_1d(\hat{\sigma}_u^2) \), \( g_2d(\hat{\sigma}_u^2) \), and \( g_3d(\hat{\sigma}_u^2) \) as shown in Section 2.3. The corresponding \( MSE \) values of the proposed estimator \( \hat{\gamma}_d^P \) are shown in Table 5.
| Countries             | $\sigma_d$ | $B_d$ | $1 - B_d$ | $x_d^T A \hat{\beta}$ | $\gamma_d$ | $\gamma_d^P$ |
|-----------------------|-----------|-------|-----------|--------------------------|------------|-------------|
| Albania               | 11.90     | 0.01  | 0.99      | 430.00                   | 413.00     | 413.00      |
| Germany               | 8.35      | 0.01  | 0.99      | 510.00                   | 506.00     | 506.00      |
| Australia             | 2.59      | 0.00  | 1.00      | 476.00                   | 494.00     | 494.00      |
| Austria               | 8.18      | 0.01  | 0.99      | 519.00                   | 497.00     | 497.00      |
| Belgium               | 5.52      | 0.01  | 0.99      | 496.00                   | 507.00     | 507.00      |
| Brazil                | 8.18      | 0.01  | 0.99      | 417.00                   | 377.00     | 377.00      |
| Bulgaria              | 15.60     | 0.02  | 0.98      | 458.00                   | 441.00     | 441.00      |
| Canada                | 5.34      | 0.01  | 0.99      | 502.00                   | 516.00     | 516.00      |
| Qatar                 | 1.61      | 0.00  | 1.00      | 472.00                   | 402.00     | 402.00      |
| Chile                 | 6.45      | 0.01  | 0.99      | 464.00                   | 423.00     | 423.00      |
| Colombia              | 5.24      | 0.01  | 0.99      | 359.00                   | 390.00     | 390.00      |
| Korea                 | 13.76     | 0.01  | 0.99      | 529.00                   | 524.00     | 524.00      |
| Costa Rica            | 6.10      | 0.01  | 0.99      | 416.00                   | 400.00     | 400.00      |
| Croatia               | 7.67      | 0.01  | 0.99      | 465.00                   | 464.00     | 464.00      |
| Denmark               | 4.71      | 0.00  | 1.00      | 513.00                   | 511.00     | 511.00      |
| Arab Emirates         | 5.81      | 0.01  | 0.99      | 461.00                   | 427.00     | 427.00      |
| Slovakia              | 7.08      | 0.01  | 0.99      | 478.00                   | 475.00     | 475.00      |
| Slovenia              | 1.59      | 0.00  | 1.00      | 495.00                   | 510.00     | 510.00      |
| Spain                 | 4.62      | 0.00  | 1.00      | 458.00                   | 486.00     | 486.00      |
| United States         | 10.05     | 0.01  | 0.99      | 505.00                   | 470.00     | 470.00      |
| Estonia               | 4.16      | 0.00  | 1.00      | 497.00                   | 520.00     | 520.00      |
| Russian Federation    | 9.67      | 0.01  | 0.99      | 456.00                   | 494.00     | 494.00      |
| Finland               | 5.34      | 0.01  | 0.99      | 507.00                   | 511.00     | 511.00      |
| France                | 4.41      | 0.00  | 1.00      | 470.00                   | 493.00     | 493.00      |
| Greece                | 14.06     | 0.01  | 0.99      | 447.00                   | 454.00     | 454.00      |
| Hong Kong-China       | 8.88      | 0.01  | 0.99      | 505.00                   | 548.00     | 548.00      |
| Hungary               | 6.40      | 0.01  | 0.99      | 487.00                   | 477.00     | 477.00      |
| Indonesia             | 9.49      | 0.01  | 0.99      | 420.00                   | 386.00     | 386.00      |
| Ireland               | 4.20      | 0.00  | 1.00      | 453.00                   | 504.00     | 504.00      |
| Iceland               | 3.96      | 0.00  | 1.00      | 542.00                   | 488.00     | 488.00      |
| Israel                | 13.18     | 0.01  | 0.99      | 512.00                   | 470.00     | 471.00      |
| Italy                 | 8.12      | 0.01  | 0.99      | 469.00                   | 490.00     | 490.00      |
| Japan                 | 9.00      | 0.01  | 0.99      | 519.00                   | 532.00     | 532.00      |
| Jordan                | 7.02      | 0.01  | 0.99      | 420.00                   | 380.00     | 380.00      |
| Latvia                | 3.50      | 0.00  | 1.00      | 461.00                   | 482.00     | 482.00      |
| Lithuania             | 5.43      | 0.01  | 0.99      | 454.00                   | 478.00     | 478.00      |
| Luxembourg            | 1.61      | 0.00  | 1.00      | 477.00                   | 486.00     | 486.00      |
| Macao-China           | 1.23      | 0.00  | 1.00      | 491.00                   | 544.00     | 544.00      |
| Mexico                | 5.02      | 0.01  | 0.99      | 398.00                   | 408.00     | 408.00      |
| Country         | $\hat{\gamma}_d$ | $\Delta$ | $\bar{P}$ | $\hat{P}_d$ | $\bar{P}_d$ | $\hat{P}_d$ |
|-----------------|-------------------|----------|-----------|-------------|-------------|-------------|
| Montenegro      | 2.13              | 0.00     | 1.00      | 437.00      | 418.00      | 418.00      |
| Norway          | 4.97              | 0.01     | 0.99      | 482.00      | 502.00      | 502.00      |
| New Zealand     | 5.15              | 0.01     | 0.99      | 470.00      | 495.00      | 495.00      |
| Netherlands     | 4.88              | 0.00     | 1.00      | 504.00      | 512.00      | 512.00      |
| Peru            | 7.34              | 0.01     | 0.99      | 424.00      | 387.00      | 387.00      |
| Poland          | 5.71              | 0.01     | 0.99      | 480.00      | 504.00      | 504.00      |
| Portugal        | 6.20              | 0.01     | 0.99      | 482.00      | 492.00      | 492.00      |
| Czech Republic  | 5.76              | 0.01     | 0.99      | 513.00      | 492.00      | 492.00      |
| Romania         | 14.36             | 0.01     | 0.99      | 457.00      | 444.00      | 444.00      |
| Singapore       | 2.16              | 0.00     | 1.00      | 511.00      | 564.00      | 564.00      |
| Sweden          | 10.05             | 0.01     | 0.99      | 491.00      | 494.00      | 494.00      |
| Switzerland     | 8.53              | 0.01     | 0.99      | 519.00      | 521.00      | 521.00      |
| Thailand        | 9.18              | 0.01     | 0.99      | 456.00      | 415.00      | 415.00      |
| Tunisia         | 8.70              | 0.01     | 0.99      | 355.00      | 367.00      | 367.00      |
| Turkey          | 17.06             | 0.02     | 0.98      | 473.00      | 420.00      | 421.00      |
| Vietnam         | 19.89             | 0.02     | 0.98      | 439.00      | 495.00      | 494.00      |

*Table 4: Estimate of the estimator $\hat{\gamma}_d$ by country.*
| Countries          | $g_{1d}(\hat{\sigma}_u^2)$ | $g_{2d}(\hat{\sigma}_u^2)$ | $g_{3d}(\hat{\sigma}_u^2)$ | MSE $(\hat{\gamma}_d)$ |
|-------------------|-----------------------------|-----------------------------|-----------------------------|--------------------------|
| Albania           | 11.7606                     | 0.0149                      | 0.0051                      | 11.7810                  |
| Germany           | 8.2820                      | 0.0038                      | 0.0025                      | 8.2880                   |
| Australia         | 2.5853                      | 0.0006                      | 0.0002                      | 2.5860                   |
| Austria           | 8.1123                      | 0.0048                      | 0.0024                      | 8.1200                   |
| Belgium           | 5.4018                      | 0.0011                      | 0.0011                      | 5.4940                   |
| Brazil            | 8.1123                      | 0.0096                      | 0.0024                      | 8.1240                   |
| Bulgaria          | 15.3596                     | 0.0167                      | 0.0087                      | 15.3850                  |
| Canada            | 5.3074                      | 0.0013                      | 0.0010                      | 5.3100                   |
| Qatar             | 1.6103                      | 0.0004                      | 0.0001                      | 1.6110                   |
| Chile             | 6.4097                      | 0.0031                      | 0.0015                      | 6.4140                   |
| Colombia          | 5.2164                      | 0.0145                      | 0.0010                      | 5.2320                   |
| Korea             | 13.5747                     | 0.0281                      | 0.0068                      | 13.6100                  |
| Costa Rica        | 6.0634                      | 0.0045                      | 0.0014                      | 6.0690                   |
| Croatia           | 7.6137                      | 0.0050                      | 0.0022                      | 7.6210                   |
| Denmark           | 4.6865                      | 0.0013                      | 0.0008                      | 4.6890                   |
| Arab Emirates     | 5.7741                      | 0.0016                      | 0.0012                      | 5.7770                   |
| Slovakia          | 7.0252                      | 0.0020                      | 0.0018                      | 7.0290                   |
| Slovenia          | 1.5850                      | 0.0001                      | 0.0001                      | 1.5850                   |
| Spain             | 4.6009                      | 0.0040                      | 0.0008                      | 4.6060                   |
| United States     | 9.9476                      | 0.0050                      | 0.0037                      | 9.9560                   |
| Estonia           | 4.1441                      | 0.0007                      | 0.0006                      | 4.1450                   |
| Russian Federation| 9.5782                      | 0.0077                      | 0.0034                      | 9.5890                   |
| Finland           | 5.3074                      | 0.0018                      | 0.0010                      | 5.3100                   |
| France            | 4.3904                      | 0.0021                      | 0.0007                      | 4.3930                   |
| Greece            | 13.8649                     | 0.0496                      | 0.0071                      | 13.9220                  |
| Hong Kong-China   | 8.8012                      | 0.0101                      | 0.0029                      | 8.8140                   |
| Hungary           | 6.3596                      | 0.0013                      | 0.0015                      | 6.3620                   |
| Indonesia         | 9.3961                      | 0.0079                      | 0.0033                      | 9.4070                   |
| Ireland           | 4.1847                      | 0.0010                      | 0.0007                      | 4.1860                   |
| Iceland           | 3.9443                      | 0.0045                      | 0.0006                      | 3.9490                   |
| Israel            | 13.0032                     | 0.0311                      | 0.0062                      | 13.0410                  |
| Italy             | 8.0562                      | 0.0024                      | 0.0024                      | 8.0610                   |
| Japan             | 8.9186                      | 0.0063                      | 0.0029                      | 8.9280                   |
| Jordan            | 6.9729                      | 0.0048                      | 0.0018                      | 6.9790                   |
| Latvia            | 3.4845                      | 0.0005                      | 0.0005                      | 3.4850                   |
| Lithuania         | 5.3992                      | 0.0023                      | 0.0011                      | 5.4030                   |
| Luxembourg        | 1.6103                      | 0.0001                      | 0.0001                      | 1.6100                   |
| Macao-China       | 1.2306                      | 0.0002                      | 0.0001                      | 1.2310                   |
| Mexico            | 4.9922                      | 0.0048                      | 0.0009                      | 4.9980                   |
| Country            | $\hat{\gamma}_d$ | $\sigma^2$ | $\sigma^3$ | $\hat{\gamma}_d$ |
|--------------------|-------------------|------------|------------|-------------------|
| Montenegro         | 2.1270            | 0.0005     | 0.002      | 2.1280            |
| Norway             | 4.9480            | 0.0020     | 0.0009     | 4.9510            |
| New Zealand        | 5.1261            | 0.0025     | 0.0010     | 5.1300            |
| Netherlands        | 4.8600            | 0.0010     | 0.0009     | 4.8620            |
| Peru               | 7.2898            | 0.0083     | 0.0020     | 7.3000            |
| Poland             | 5.6792            | 0.0011     | 0.0012     | 5.6820            |
| Portugal           | 6.1614            | 0.0021     | 0.0014     | 6.1650            |
| Czech Republic     | 5.7266            | 0.0023     | 0.0012     | 5.7300            |
| Romania            | 14.1580           | 0.0259     | 0.0074     | 14.1910           |
| Singapore          | 2.1562            | 0.0002     | 0.0002     | 2.1570            |
| Sweden             | 9.9476            | 0.0123     | 0.0037     | 9.9640            |
| Switzerland        | 8.4533            | 0.0058     | 0.0026     | 8.4620            |
| Thailand           | 9.0963            | 0.0093     | 0.0031     | 9.1090            |
| Tunisia            | 8.6264            | 0.0234     | 0.0028     | 8.6530            |
| Turkey             | 16.7670           | 0.0109     | 0.0103     | 16.7880           |
| Vietnam            | 19.4985           | 0.1504     | 0.0139     | 19.6630           |

*Table 5: MSE of $\hat{\gamma}_d$ by country.*
| Countries          | $\tilde{t}_d$ | $CVE\ (100\%)$ | $\tilde{t}_d^P$ | $EER_d\ (100\%)$ | $Di_{rel}\ (100\%)$ |
|-------------------|--------------|----------------|---------------|------------------|------------------|
| Albania           | 413.0000     | 0.8354         | 413.0000      | 0.8308           | 0.9810           |
| Germany           | 506.0000     | 0.5711         | 506.0000      | 0.5690           | 0.7332           |
| Australia         | 494.0000     | 0.3259         | 494.0000      | 0.3256           | 0.2202           |
| Austria           | 497.0000     | 0.5755         | 497.0000      | 0.5732           | 0.7043           |
| Belgium           | 507.0000     | 0.4635         | 507.0000      | 0.4624           | 0.4953           |
| Brazil            | 377.0000     | 0.7586         | 377.0000      | 0.7555           | 0.6457           |
| Bulgaria          | 441.0000     | 0.8957         | 441.0000      | 0.8891           | 1.3383           |
| Canada            | 516.0000     | 0.4477         | 516.0000      | 0.4467           | 0.4747           |
| Qatar             | 402.0000     | 0.3159         | 402.0000      | 0.3156           | 0.1256           |
| Chile             | 423.0000     | 0.6005         | 423.0000      | 0.5984           | 0.5547           |
| Colombia          | 390.0000     | 0.5872         | 390.0000      | 0.5868           | 0.2132           |
| Korea             | 524.0000     | 0.7080         | 524.0000      | 0.7041           | 1.0728           |
| Costa Rica        | 400.0000     | 0.6175         | 400.0000      | 0.6158           | 0.4957           |
| Croatia           | 464.0000     | 0.5970         | 464.0000      | 0.5950           | 0.6502           |
| Denmark           | 511.0000     | 0.4247         | 511.0000      | 0.4238           | 0.4118           |
| Arab Emirates     | 427.0000     | 0.5644         | 427.0000      | 0.5627           | 0.5146           |
| Slovakia          | 475.0000     | 0.5600         | 475.0000      | 0.5582           | 0.6316           |
| Slovenia          | 510.0000     | 0.2471         | 510.0000      | 0.2469           | 0.1436           |
| Spain             | 486.0000     | 0.4424         | 486.0000      | 0.4417           | 0.3465           |
| United States     | 470.0000     | 0.6745         | 470.0000      | 0.6710           | 0.8854           |
| Estonia           | 520.0000     | 0.3923         | 520.0000      | 0.3916           | 0.3724           |
| Russian Federation| 494.0000     | 0.6296         | 494.0000      | 0.6274           | 0.8207           |
| Finland           | 511.0000     | 0.4521         | 511.0000      | 0.4510           | 0.4652           |
| France            | 493.0000     | 0.4260         | 493.0000      | 0.4253           | 0.3645           |
| Greece            | 454.0000     | 0.8260         | 454.0000      | 0.8222           | 0.9518           |
| Hong Kong-China   | 548.0000     | 0.5438         | 548.0000      | 0.5422           | 0.7138           |
| Hungary           | 477.0000     | 0.5304         | 477.0000      | 0.5288           | 0.5770           |
| Indonesia         | 386.0000     | 0.7979         | 386.0000      | 0.7941           | 0.7996           |
| Ireland           | 504.0000     | 0.4067         | 504.0000      | 0.4062           | 0.3698           |
| Iceland           | 488.0000     | 0.4078         | 488.0000      | 0.4071           | 0.2573           |
| Israel            | 470.0000     | 0.7723         | 471.0000      | 0.7676           | 0.9874           |
| Italy             | 490.0000     | 0.5816         | 490.0000      | 0.5797           | 0.7283           |
| Japan             | 532.0000     | 0.5639         | 532.0000      | 0.5619           | 0.7682           |
| Jordan            | 380.0000     | 0.6974         | 380.0000      | 0.6948           | 0.5869           |
| Latvia            | 482.0000     | 0.3880         | 482.0000      | 0.3874           | 0.3140           |
| Lithuania         | 478.0000     | 0.4874         | 478.0000      | 0.4864           | 0.4650           |
| Luxembourg        | 486.0000     | 0.2613         | 486.0000      | 0.2611           | 0.1439           |
| Macao-China       | 544.0000     | 0.2040         | 544.0000      | 0.2040           | 0.0966           |
| Mexico            | 408.0000     | 0.5490         | 408.0000      | 0.5481           | 0.3736           |
Table 6: $\hat{\gamma}_d$ and $\hat{\gamma}_d^P$ along with their quality measures by country.

| Country       | $\hat{\gamma}_d$ | $\hat{\gamma}_d^P$ | $\hat{\gamma}_d$ | $\hat{\gamma}_d^P$ |
|---------------|-------------------|---------------------|-------------------|---------------------|
| Montenegro    | 418.0000          | 0.3493              | 418.0000          | 0.3489              |
| Norway        | 502.0000          | 0.4442              | 502.0000          | 0.4434              |
| New Zealand   | 495.0000          | 0.4586              | 495.0000          | 0.4577              |
| Netherlands   | 512.0000          | 0.4316              | 512.0000          | 0.4307              |
| Peru          | 387.0000          | 0.7003              | 387.0000          | 0.6978              |
| Poland        | 504.0000          | 0.4742              | 504.0000          | 0.4731              |
| Portugal      | 492.0000          | 0.5061              | 492.0000          | 0.5048              |
| Czech Republic| 492.0000          | 0.4878              | 492.0000          | 0.4865              |
| Romania       | 444.0000          | 0.8536              | 444.0000          | 0.8483              |
| Singapore     | 564.0000          | 0.2606              | 564.0000          | 0.2604              |
| Sweden        | 494.0000          | 0.6417              | 494.0000          | 0.6391              |
| Switzerland   | 521.0000          | 0.5605              | 521.0000          | 0.5584              |
| Thailand      | 415.0000          | 0.7301              | 415.0000          | 0.7267              |
| Tunisia       | 367.0000          | 0.8038              | 367.0000          | 0.8019              |
| Turkey        | 420.0000          | 0.9833              | 421.0000          | 0.9738              |
| Vietnam       | 495.0000          | 0.9010              | 494.0000          | 0.8981              |

From Table 6, we see that the $EER_d$, in all those countries that participated in the test, are lower than the $CVE$ published by PISA. However, it is not surprising that using the proposed estimator, the estimation error decreased, as shown in the $Dif_{rel}$ (100%) column, even though they were already very small.

5 Discussion

This paper shows practical and methodological advantages of integrating small area estimation, item response theory, and multiple imputation. It is possible to show from a theoretical and empirical perspective that the proposed estimator for the ability mean is unbiased. In addition, in all of the scenarios where the proposed estimator is tested,
it has a lower average relative standard error than its competitors, considering simple random sampling. On the one hand, by varying the sample fractions and the domain fractions, the proposed estimator has lower mean relative standard errors compared to the other estimators used in the simulation. This behaviour is achieved under all the correlation scenarios.

Finally, when the percentage of missing data varies, the relative standard errors increases as the rate of missing data does. The proposed estimator always obtains the lowest relative standard errors compared to the other estimators studied in the simulation. Moreover, in the scenario with 30% of missing data and a small sampling fraction, the relative standard errors did not exceed 5% on average, implying that these estimates are quite low.

References

Adams, L. M. and Darwin, G. (1982). Solving the quandary between questionnaire length and response rate in educational research. Research in Higher Education, 17:231–240.

Allen, T. M. (2004). Assessing Academic Programs in Higher Education. Jossey-Bass.

Álvarez, L., González-Castro, P., Núñez, J. C., González-Pienda, J. A., Álvarez, D., and Bernardo, A. (2007). Desarrollo de los procesos atencionales mediante «actividades adaptadas». Papeles del psicólogo, 28(3):211–217.

Baker, F. B. and Kim, S. H. (2004). Item Response Theory: Parameter Estimation Techniques. New York: Dekker.

Bock, R. D. and Aitkin, M. (1981). Marginal maximum likelihood estimation of item parameters: Application of an EM algorithm. Psychometrika, 46(4):443–459.

de Andrade, D. F., Tavares, H. R., and da Cunha-Valle, R. (2000). Teoria da Resposta ao Iten: Conceitos e Aplicações. ABE, Sao Paulo, Brasil.

Embretson, S. E. and Reise, S. P. (2013). Item response theory. Psychology Press.

Fay, R. E. and Herriot, R. A. (1979). Estimates of income for small places: An application of James-Stein procedures to census data. Journal of the American Statistical Association, 74(366):269–277.

Fox, J.-P. (2010). Bayesian Item Response Modeling: Theory and Applications. Springer Science & Business Media.

Ghosh, M. and Rao, J. N. K. (1994). Small area estimation: An appraisal. Statistical Science, 9(1):55–83.

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Hambleton, R. K. and Swaminathan, H. (2013). *Item response theory: Principles and applications*. Springer Science & Business Media.

Harville, D. A. (1977). Maximum likelihood approaches to variance component estimation and to related problems. *Journal of the American Statistical Association*, 72(358):320–338.

Horvitz, D. G. and Thompson, D. J. (1952). A generalization of sampling without replacement from a finite universe. *Journal of the American Statistical Association*, 47(260):663–685.

ICFES (2014). Pruebas saber 3º, 5º y 9º. Lineamientos para la aplicación muestral y censal 2014. Technical report, ICFES.

ICFES (2015). Manual de calificación. prueba cognitiva saber 3, 5, 7 y 9. Technical report, ICFES.

ICFES (2017). El armado de las pruebas saber y la comparabilidad en el tiempo. *en Breve, Saber*.

Kackar, R. N. and Harville, D. A. (1984). Approximations for standard errors of estimators of fixed and random effects in mixed linear models. *Journal of the American Statistical Association*, 79(388):853–862.

Kuh, G. D., Jankowski, N., and Ikenberry, S. O. (2014). Knowing what students know and can do: The current state of learning outcomes assessment in u.s. colleges and universities. Technical report, University of Illinois and Indiana University, National Institute for Learning Outcomes Assessment.

Lord, F. M. (1980). *Applications of Item Response Theory to Practical Testing Problems*. Routledge.

Martínez Arias, R. (1995). Psicometría: Teoría de los tests psicológicos y educativos.

McCulloch, C. E. and Searle, S. R. (2004). *Generalized, linear, and mixed models*. John Wiley & Sons.

Morales, D. and Molina, I. (2015). Estimación en áreas pequeñas: Métodos Basados en Modelos. Unpublished.

Mullis, I., Martin, M. O., Kennedy, A. M., Trong, K. L., and Sainsbury, M. P. (2015). Assessment frameworks. *TIMMS and Pirls International Study Center, Boston College*.

Muñiz, F. J. (1997). *Introducción a la Teoría de Respuesta a los Ítems*. Ediciones Pirámide.
Narain, R. D. (1951). On sampling without replacement with varying probabilities. *Journal of the Indian Society of Agricultural Statistics*, 3:169–174.

OECD (2014). Pisa 2012 Technical Report. url:https://www.oecd.org/pisa/pisaproduc ts/PISA-2012-technical-report-final.pdf.

OECD (2016). Pisa 2015 Resultados Claves. url:https://www.oecd.org/pisa/pisa-2015-results-in-focus-ESP.pdf.

Paek, I. and Cole, K. (2019). *Using R for item response theory model applications*. Routledge.

Prasad, N. G. N. and Rao, J. N. K. (1990). The estimation of the mean squared error of small-area estimators. *Journal of the American Statistical Association*, 85(409):163–171.

Rubin, D. B. and Schenker, N. (1991). Multiple imputation in health-care databases: An overview and some applications. *Statistics in Medicine*, 10(4):585–598.

Sulis, I. and Porcu, M. (2017). Handling missing data in item response theory. assessing the accuracy of a multiple imputation procedure based on latent class analysis. *Journal of Classification*, 34:327–359.

Treviño, E., Fraser, P., Meyer, A., Morawietz, L., Hinostrosa, P., and Naranjo, E. (2016). Informe de Resultados del Tercer Estudio Regional Comparativo y Explicativo. Factores Asociados 2015.

Treviño, E., Valdés, H., Castro, M., Costilla, R., Pardo, C., and Donoso-Rivas, F. (2010). *Factores asociados al logro cognitivo de los estudiantes de América Latina y el Caribe*. OREALC/UNESCO.

UNESCO (2019). The promise of large-scale learning assessments: Acknowledging limits to unlock opportunities. Technical report, United Nations Educational, Scientific and Cultural Organization.
A  More on simulation study results

| \(f_d\) (%) | \(f_u\) (%) | \(\text{EERP}_{\hat{\gamma}}^{\text{Dir}}\) (%) | \(\text{EERP}_{\hat{\gamma}}^{\text{Cal}}\) (%) | \(\text{EERP}_{\hat{\gamma}}^{\text{Comp}}\) (%) | \(\text{EERP}_{\hat{\gamma}}^{P}\) (%) | \(\text{SBR}_{\hat{\gamma}}^{P}\) (%) |
|---------|---------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 30%     | 5%      | 1.48             | 1.47             | 1.01             | 0.92             | 0.06             |
| 30%     | 10%     | 1.17             | 1.15             | 0.94             | 0.88             | 0.05             |
| 30%     | 20%     | 0.96             | 0.96             | 0.90             | 0.87             | 0.01             |
| 50%     | 5%      | 1.84             | 1.84             | 1.11             | 0.96             | 0.09             |
| 50%     | 10%     | 1.38             | 1.37             | 0.99             | 0.88             | 0.07             |
| 50%     | 20%     | 1.09             | 1.09             | 0.92             | 0.86             | 0.04             |
| 70%     | 5%      | 2.11             | 2.17             | 1.22             | 1.11             | 0.11             |
| 70%     | 10%     | 1.58             | 1.57             | 1.02             | 0.92             | 0.08             |
| 70%     | 20%     | 1.22             | 1.21             | 0.95             | 0.87             | 0.05             |

*Table 7: Missing 10% and medium correlation*

| \(f_d\) (%) | \(f_u\) (%) | \(\text{EERP}_{\hat{\gamma}}^{\text{Dir}}\) (%) | \(\text{EERP}_{\hat{\gamma}}^{\text{Cal}}\) (%) | \(\text{EERP}_{\hat{\gamma}}^{\text{Comp}}\) (%) | \(\text{EERP}_{\hat{\gamma}}^{P}\) (%) | \(\text{SBR}_{\hat{\gamma}}^{P}\) (%) |
|---------|---------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 30%     | 5%      | 1.49             | 2.84             | 1.33             | 1.22             | -0.06            |
| 30%     | 10%     | 1.19             | 1.93             | 1.10             | 1.08             | -0.03            |
| 30%     | 20%     | 1.00             | 1.24             | 0.97             | 1.01             | 0.01             |
| 50%     | 5%      | 1.84             | 3.85             | 1.61             | 1.45             | -0.14            |
| 50%     | 10%     | 1.40             | 2.60             | 1.26             | 1.15             | -0.04            |
| 50%     | 20%     | 1.13             | 1.72             | 1.06             | 1.01             | 0.01             |
| 70%     | 5%      | 2.12             | 4.71             | 1.95             | 1.86             | -0.19            |
| 70%     | 10%     | 1.58             | 3.12             | 1.40             | 1.23             | -0.11            |
| 70%     | 20%     | 1.24             | 2.11             | 1.14             | 1.03             | -0.02            |

*Table 8: Missing 10% and low correlation*
| $f_d$ (%) | $f_n$ (%) | $\text{EERP}^{\gamma}_{d,\text{Dir}}$ (%) | $\text{EERP}^{\gamma}_{d,\text{Cal}}$ (%) | $\text{EERP}^{\gamma}_{d,\text{Comp}}$ (%) | $\text{EERP}^{\gamma}_{d,\text{P}}$ (%) | $\text{SBR}^{\gamma}_{d,\text{P}}$ (%) |
|----------|----------|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| 30%      | 5%       | 1.57                                 | 1.25                                 | 1.09                                 | 0.92                                 | 0.07                                 |
| 30%      | 10%      | 1.27                                 | 1.12                                 | 1.05                                 | 0.91                                 | 0.02                                 |
| 30%      | 20%      | 1.10                                 | 1.06                                 | 1.04                                 | 0.91                                 | -0.01                                |
| 50%      | 5%       | 1.90                                 | 1.42                                 | 1.14                                 | 0.94                                 | 0.03                                 |
| 50%      | 10%      | 1.48                                 | 1.20                                 | 1.08                                 | 0.93                                 | 0.06                                 |
| 50%      | 20%      | 1.22                                 | 1.10                                 | 1.05                                 | 0.92                                 | 0.01                                 |
| 70%      | 5%       | 2.18                                 | 1.59                                 | 1.20                                 | 1.02                                 | 0.24                                 |
| 70%      | 10%      | 1.67                                 | 1.29                                 | 1.10                                 | 0.94                                 | 0.1                                  |
| 70%      | 20%      | 1.33                                 | 1.14                                 | 1.06                                 | 0.95                                 | 0.03                                 |

Table 9: Missing 20% and high correlation.

| $f_d$ (%) | $f_n$ (%) | $\text{EERP}^{\gamma}_{d,\text{Dir}}$ (%) | $\text{EERP}^{\gamma}_{d,\text{Cal}}$ (%) | $\text{EERP}^{\gamma}_{d,\text{Comp}}$ (%) | $\text{EERP}^{\gamma}_{d,\text{P}}$ (%) | $\text{SBR}^{\gamma}_{d,\text{P}}$ (%) |
|----------|----------|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| 30%      | 5%       | 1.53                                 | 1.52                                 | 1.09                                 | 0.98                                 | 0.07                                 |
| 30%      | 10%      | 1.22                                 | 1.22                                 | 1.02                                 | 0.95                                 | 0.04                                 |
| 30%      | 20%      | 1.04                                 | 1.04                                 | 0.98                                 | 0.94                                 | 0.00                                 |
| 50%      | 5%       | 1.85                                 | 1.89                                 | 1.18                                 | 1.04                                 | 0.09                                 |
| 50%      | 10%      | 1.44                                 | 1.43                                 | 1.08                                 | 0.97                                 | 0.07                                 |
| 50%      | 20%      | 1.17                                 | 1.16                                 | 1.01                                 | 0.94                                 | 0.03                                 |
| 70%      | 5%       | 2.13                                 | 2.24                                 | 1.29                                 | 1.19                                 | 0.09                                 |
| 70%      | 10%      | 1.62                                 | 1.61                                 | 1.11                                 | 0.98                                 | 0.09                                 |
| 70%      | 20%      | 1.28                                 | 1.27                                 | 1.03                                 | 0.95                                 | 0.05                                 |

Table 10: Missing 20% and correlation medium.

| $f_d$ (%) | $f_n$ (%) | $\text{EERP}^{\gamma}_{d,\text{Dir}}$ (%) | $\text{EERP}^{\gamma}_{d,\text{Cal}}$ (%) | $\text{EERP}^{\gamma}_{d,\text{Comp}}$ (%) | $\text{EERP}^{\gamma}_{d,\text{P}}$ (%) | $\text{SBR}^{\gamma}_{d,\text{P}}$ (%) |
|----------|----------|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| 30%      | 5%       | 1.52                                 | 2.83                                 | 1.34                                 | 1.16                                 | -0.07                                |
| 30%      | 10%      | 1.20                                 | 1.93                                 | 1.12                                 | 1.02                                 | 0.01                                 |
| 30%      | 20%      | 1.02                                 | 1.26                                 | 1.00                                 | 0.96                                 | 0.02                                 |
| 50%      | 5%       | 1.83                                 | 3.83                                 | 1.61                                 | 1.41                                 | -0.07                                |
| 50%      | 10%      | 1.41                                 | 2.57                                 | 1.27                                 | 1.08                                 | -0.04                                |
| 50%      | 20%      | 1.15                                 | 1.73                                 | 1.08                                 | 0.96                                 | 0.01                                 |
| 70%      | 5%       | 2.12                                 | 4.70                                 | 1.92                                 | 1.86                                 | -0.16                                |
| 70%      | 10%      | 1.60                                 | 3.11                                 | 1.41                                 | 1.19                                 | -0.08                                |
| 70%      | 20%      | 1.26                                 | 2.11                                 | 1.16                                 | 0.99                                 | -0.01                                |

Table 11: Missing 20% and low correlation.
| $f_d$ (%) | $f_n$ (%) | $\text{EERP}_{\gamma_d}^{\text{Dir}}$ (%) | $\text{EERP}_{\gamma_d}^{\text{Cal}}$ (%) | $\text{EERP}_{\gamma_d}^{\text{Comp}}$ (%) | $\text{EERP}_{\gamma_d}^{P}$ (%) | SBR$_{\gamma_d}^{P}$ (%) |
|----------|----------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|-----------------|
| 30%      | 5%       | 1.57                          | 1.26                          | 1.11                          | 0.90                          | 0.1             |
| 30%      | 10%      | 1.29                          | 1.13                          | 1.07                          | 0.89                          | 0.04            |
| 30%      | 20%      | 1.11                          | 1.07                          | 1.05                          | 0.90                          | 0.01            |
| 50%      | 5%       | 1.89                          | 1.41                          | 1.16                          | 0.92                          | 0.17            |
| 50%      | 10%      | 1.48                          | 1.21                          | 1.10                          | 0.90                          | 0.07            |
| 50%      | 20%      | 1.23                          | 1.11                          | 1.07                          | 0.92                          | 0.03            |
| 70%      | 5%       | 2.16                          | 1.58                          | 1.21                          | 1.01                          | 0.25            |
| 70%      | 10%      | 1.67                          | 1.29                          | 1.12                          | 0.92                          | 0.11            |
| 70%      | 20%      | 1.34                          | 1.15                          | 1.08                          | 0.93                          | 0.05            |

Table 12: Missing 30% and High correlation

| $f_d$ (%) | $f_n$ (%) | $\text{EERP}_{\gamma_d}^{\text{Dir}}$ (%) | $\text{EERP}_{\gamma_d}^{\text{Cal}}$ (%) | $\text{EERP}_{\gamma_d}^{\text{Comp}}$ (%) | $\text{EERP}_{\gamma_d}^{P}$ (%) | SBR$_{\gamma_d}^{P}$ (%) |
|----------|----------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|-----------------|
| 30%      | 5%       | 1.64                          | 1.65                          | 1.24                          | 1.04                          | 0.09            |
| 30%      | 10%      | 1.36                          | 1.36                          | 1.18                          | 1.04                          | 0.04            |
| 30%      | 20%      | 1.20                          | 1.20                          | 1.15                          | 1.03                          | 0.02            |
| 50%      | 5%       | 1.94                          | 2.00                          | 1.33                          | 1.11                          | 0.12            |
| 50%      | 10%      | 1.54                          | 1.57                          | 1.22                          | 1.04                          | 0.07            |
| 50%      | 20%      | 1.31                          | 1.31                          | 1.17                          | 1.03                          | 0.04            |
| 70%      | 5%       | 2.22                          | 2.34                          | 1.42                          | 1.25                          | 0.13            |
| 70%      | 10%      | 1.72                          | 1.74                          | 1.26                          | 1.07                          | 0.09            |
| 70%      | 20%      | 1.41                          | 1.42                          | 1.19                          | 1.05                          | 0.06            |

Table 13: Missing 30% and correlation medium

| $f_d$ (%) | $f_n$ (%) | $\text{EERP}_{\gamma_d}^{\text{Dir}}$ (%) | $\text{EERP}_{\gamma_d}^{\text{Cal}}$ (%) | $\text{EERP}_{\gamma_d}^{\text{Comp}}$ (%) | $\text{EERP}_{\gamma_d}^{P}$ (%) | SBR$_{\gamma_d}^{P}$ (%) |
|----------|----------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|-----------------|
| 30%      | 5%       | 1.60                          | 2.95                          | 1.45                          | 1.24                          | -0.07           |
| 30%      | 10%      | 1.32                          | 2.03                          | 1.24                          | 1.08                          | -0.01           |
| 30%      | 20%      | 1.16                          | 1.39                          | 1.14                          | 1.03                          | 0.02            |
| 50%      | 5%       | 1.91                          | 3.93                          | 1.70                          | 1.46                          | -0.14           |
| 50%      | 10%      | 1.51                          | 2.67                          | 1.38                          | 1.14                          | -0.05           |
| 50%      | 20%      | 1.26                          | 1.84                          | 1.21                          | 1.04                          | 0.00            |
| 70%      | 5%       | 2.18                          | 4.83                          | 2.00                          | 1.94                          | -0.14           |
| 70%      | 10%      | 1.68                          | 3.20                          | 1.51                          | 1.26                          | -0.08           |
| 70%      | 20%      | 1.37                          | 2.20                          | 1.28                          | 1.07                          | -0.01           |

Table 14: Missing 30% and low correlation
B Notation

Matrices and vectors with entries consisting of subscripted variables are denoted by a boldfaced version of the letter for that variable. For example, $\mathbf{x} = (x_1, \ldots, x_n)$ denotes an $n \times 1$ column vector with entries $x_1, \ldots, x_n$. We use $\mathbf{0}$ and $\mathbf{1}$ to denote the column vector with all entries equal to 0 and 1, respectively, and $\mathbf{I}$ to denote the identity matrix. A subindex in this context refers to the corresponding dimension; for instance, $\mathbf{I}_n$ denotes the $n \times n$ identity matrix. The transpose of a vector $\mathbf{x}$ is denoted by $\mathbf{x}^T$; analogously for matrices. Moreover, if $\mathbf{X}$ is a square matrix, we use $\text{tr}(\mathbf{X})$ to denote its trace and $\mathbf{X}^{-1}$ to denote its inverse. The norm of $\mathbf{x}$, given by $\sqrt{\mathbf{x}^T \mathbf{x}}$, is denoted by $\| \mathbf{x} \|$. 