Dynamics of human walking

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Abstract

The problem of biped locomotion at steady speeds is discussed through a Lagrangian formulation developed for velocity-dependent, body driving forces. Human walking on a level surface is analyzed in terms of the data on the resultant ground-reaction force and the external work. It is shown that the trajectory of the center of mass is due to a superposition of its rectilinear motion with a given speed and a backward rotation along a shortened hypocycloid. A stiff-to-compliant crossover between walking gaits is described and the maximum speed for human walking, given by an instability of the trajectory, is predicted.

Key words: locomotion, integrative biology, muscles, bipedalism, human walking.

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In biology, many fundamental discoveries come from studies of animal movement. The integrative approach to the locomotion of animals focuses on the interaction between the muscular, tendon and skeletal subsystems and the environment. [1]

During locomotion, the body as a whole performs several functions. Chemical energy released by muscles and mechanical elastic energy stored in passive muscles and tendons [2] are transformed into external and internal work, [3] and are partially lost as a heat. The muscular system provokes the ground-reaction forces applied to the animal body. The resultant force, including gravity and air resistance, accelerates and decelerates the body’s center of mass (COM). This body driving force is therefore involved in level walking and running, even when the average velocity remains constant. Over a complete step cycle, the driving force performs a certain external work to maintain a given speed. Studies of the mechanical efficiency [3] of animal locomotion at different steady speeds, determined through the external work and oxygen consumption, provide evidence that walking is more energetically economical than running. This finding corroborates the old idea that walking in humans, primates, and ground-dwelling birds can be understood as swings of an ideal pendulum. Indeed, the body vaults up and over each leg in an arc in each step, similarly to an inverted plane pendulum, and kinetic energy is transformed into gravitational energy when the body falls forward and downward. Such a stiff-legged mechanics of walking modeled by the compass-arc inverted pendulum is widely employed, [1,3,4] but by no means exhaustive. [1,5] Unlike the swing pendulum dynamics driven by constant gravitational force, the dynamics of animal biped walking (and quadruped trotting) is accompanied by body undulations, pulses, and peristaltic waves [1] generated by the ground-reaction force. In this Letter, we employ the fundamental principles of classical mechanics to approach to the problem of level locomotion. At very low speeds, animal movement is treated through the linear vibrations of the body near its quasistatic equilibrium given by quiet standing. [6] Nonlinear body motion effects, controlled by velocity-dependent forces, are taken into account at higher speeds.

All the three components (forward, vertical and lateral) of the resultant force applied to the ground are measured with good accuracy by means of the force-platform techniques.
The lateral body displacements are relatively small and, thus, the COM motion can be fairly described by the instantaneous polar vector $\mathbf{R}(t)$ defined in the ground coordinate system (see Fig. 1). For walking at a steady speed $V$, it is convenient to exclude the translational degree of freedom by introducing $\mathbf{r}(t)$, with $x(t) = Vt$ and $y(t) = H$, where $H$ is the height of the COM. In that way, the libration motion is given by $\Delta \mathbf{r}(t) = \mathbf{R}(t) - \mathbf{r}(t)$. The corresponding driving force $\Delta \mathbf{F}(t)$ follows from the force-platform records: the ground-reaction force $\mathbf{F}(t)$ is observed as the oscillating force near the body weight, thus, $\mathbf{F}(t) = -mg + \Delta \mathbf{F}(t)$. Taking into account that the muscle-tendon contractions are cyclic, the driving force must satisfy the steady-motion constraint: $<\Delta \mathbf{F}(t)>_c \equiv T_c^{-1} \int_0^{T_c} \Delta \mathbf{F}(t) dt = 0$, where $T_c(V)$ is the one-step ground-contact period (shown in Fig. 1).

Assuming the displacements $\Delta \mathbf{r}$ to be small, we introduce the librational part of the potential energy $\Delta U[\Delta \mathbf{r}(t)]$ in the harmonic approximation: $\Delta U_0 = k_0(\Delta x^2 + \Delta y^2)/2$ through the body stiffness coefficient $k_0(V)$. Combining this with the kinetic energy $\Delta K_0 = m(\Delta \dot{x}^2 + \Delta \dot{y}^2)/2$ and employing Lagrangian formalism, one deduces the Newton equations $m\Delta \ddot{\mathbf{r}} + k_0 \Delta \mathbf{r} = 0$. The free COM motion is therefore a superposition of the two linear oscillations:

$$\Delta x_0(t) = \Delta l_0 \cos(\omega_0 t - \frac{\pi}{2}); \quad \Delta y_0(t) = \Delta h_0 \sin(\omega_0 t - \frac{\pi}{2}),$$

These are solutions $\Delta \mathbf{r}_0(t)$ given by the harmonic amplitudes $\Delta l_0(V)$ and $\Delta h_0(V)$ and the one-step angular frequency $\omega_0(V)$, with $\omega_0 = 2\pi/T_c = \sqrt{k_0/m}$. The backward elliptical COM rotation in Eq.(1) is due to the harmonic part of the driving force $\Delta \mathbf{F}_0(t) = -k_0 \Delta \mathbf{r}_0$, derived from the experiment and treated as an inertial force. Its components:

$$\Delta F_{0x}(t) = -m\omega_0^2 \Delta l_0 \sin(\omega_0 t); \quad \Delta F_{0y}(t) = m\omega_0^2 \Delta h_0 \cos(\omega_0 t),$$

are shown by solid lines in Fig. 1. With increasing speed, anharmonic displacements become important and therefore $\Delta U = \Delta U_0 + \Delta U_1$. Without loss of generality, the anharmonic part of the mechanical potential energy $\Delta U_1$ is parametrized in terms of the anharmonic force amplitudes $\Delta l_1$ and $\Delta h_1$, namely
\[ \Delta U_1(\Delta r) = \frac{k_0}{\Delta h_0} \left[ -\frac{\Delta l_1}{\Delta l_0} \Delta x^2 \Delta y + \frac{\Delta h_1}{3\Delta h_0} \Delta y^3 + O(\Delta x \Delta y^2) + O(\Delta x^3) \right]. \] (3)

Within the perturbation scheme, the nonlinear forces are defined by the derivatives \( \Delta F_1 = -d\Delta U_1/d\Delta r \) taken at \( \Delta r = \Delta r_0 \) given in Eq.(1). This results in

\[
F_x(t) = -m\omega_0^2 [\Delta l_0 \sin(\omega_0 t) + \Delta l_1 \sin(2\omega_0 t)]; \\
F_y(t) = mg + m\omega_0^2 [\Delta h_0 \cos(\omega_0 t) - \Delta h_1 \cos(2\omega_0 t)].
\] (4)

The third and the fourth terms in Eq.(3) correspond to the terms of \( O[\cos(2\omega_0 t)] \) and of \( O[\sin(2\omega_0 t)] \), which formally should appear in \( F_x \) and \( F_y \), respectively. Both the terms are omitted in Eqs.(4) because they are not observed in the available data on human walking. [7,8] Also, the steady-motion constraint provides the force-amplitude relation \( \Delta l_0 \Delta l_1 = \Delta h_0 \Delta h_1 \) that is nevertheless violated, even in the case of the small \( V \). As a matter of fact, the theory behind this relation presumes that \( \Delta F_1 \) is a conservative force, which disagrees with the experimental data.

Let us introduce a generalized velocity-dependent Lagrangian [9] \( \Delta L(\Delta r, \Delta \cdot r) = \Delta K_0 - \Delta U_{eff} \), where \( \Delta U_{eff} = \Delta U_0 + \Delta U_1 - \Delta K_1 \) and \( \Delta K_1 \) is the anharmonic kinetic energy. [10] Within the scope of this analysis, Eqs. (4) are not altered and the steady-motion constraint is satisfied by new kinetic terms. With the help of the frictional coefficient \( \gamma(V) \), we also introduce the resistance force \( \Delta F_{res}(t) = -\gamma \Delta \cdot r_1 \), associated with the anharmonic displacements \( \Delta r_1 = \Delta r - \Delta r_0 \). The latter obey the equations

\[ m\Delta \cdot \cdot r_1 + \gamma\Delta \cdot r_1 + k_0 \Delta r_1 = \Delta F_1(t), \] (5)

where \( \Delta F_1(t) \) is given by the last terms in Eqs. (4). Solutions of inhomogeneous differential equations (5) provide the desired description for the body’s COM motion in the ground coordinate system:

\[
X(t) = V t + \Delta x_0(t) + \frac{\Delta l_1}{3} \frac{\sin(2\omega_0 t + \varphi)}{\sqrt{1 + \tan(\varphi)^2}}, \\
Y(t) = H + \Delta y_0(t) + \frac{\Delta h_1}{3} \frac{\cos(2\omega_0 t + \varphi)}{\sqrt{1 + \tan(\varphi)^2}}. \] (6)
They are found [9,10] for the steady motion regimes in the weak friction approximation, with

\[ \varphi(V) = \arctan \left( \frac{2\omega_1}{3\omega_0} \right) < \frac{\pi}{4}; \quad \omega_1(V) = \frac{\gamma}{m}. \]  

(7)

As seen from Eqs.(6), these regimes are established not only by the frequencies, but also by the amplitudes. The latter statement follows from the force-time fitting analysis given in Fig. 1 that can be explicit in the force-amplitude ratio: \( \Delta f_0^{(\text{exp})}/\Delta f_1^{(\text{exp})} = \Delta h_0^{(\text{exp})}/\Delta h_1^{(\text{exp})} \), which equals 2. As can be recognized from Eqs. (1), (6) and the force-amplitude ratio, a trajectory of the human body’s COM, in the moving inertial coordinate system, is a closed orbit given by a shortened hypocycloid \( (\Delta r_1 < \Delta r_0 \ll H) \), passing through three turning points in the backward direction. [11] For qualitative analysis, we describe this closed orbit by the characteristic ellipse, which crosses the same turning points and is introduced by the axes: \( \Delta l = \Delta l_0 + \Delta l_1/3\sqrt{1 + \tan(\varphi)^2} \) and \( \Delta h = \Delta h_0 + \Delta h_1/3\sqrt{1 + \tan(\varphi)^2} \), in the forward and vertical directions, respectively. One infers that the COM moves on the height \( H \) with the speed \( V \), and simultaneously rotates along the hypocycloid circumscribed by a shrunken (or a flattened) ellipse of eccentricity \( e_+ \) (or \( e_- \)), with \( e_\pm(V) = \sqrt{1 - (\Delta l_1/\Delta h_1)^\pm_2} \).

The step-cycle external work \( W_{\text{tot}} \) is performed by the COM to maintain the forward speed \( V \) and the height \( H \), thus, \( W_{\text{tot}} = W_f + W_v \). This work can be estimated through the instant power averaged over the cycle period: \( W_{\text{tot}}(V) = 2\pi \omega_0^{-1} < \dot{W}_{\text{tot}}(t) >_c \), where \( \dot{W}_{\text{tot}}(t) = \mathbf{F} \cdot \ddot{\mathbf{R}} \). Nevertheless, not all the components of the ground-reaction force, produced by active and passive muscles, contribute to the cyclic work. Bearing in mind the conditions i) of the periodicity of the driving force, ii) of the orthogonality between the linear and nonlinear displacements, and iii) of the conservative nature of the harmonic force, one deduces that the only nonzero contribution to \( W_{\text{tot}} \) is due to the anharmonic part of the power \( \dot{W}_1(t) = \Delta F_1 \Delta \dot{r}_1 \). This power follows from Eqs. (6) and provides

\[ W_{\text{tot}}(V) = \frac{4\pi \gamma \omega_0 (\Delta l_1^2 + \Delta h_1^2)}{9 \left( 1 + (2\gamma/3m\omega_0)^2 \right)}. \]

(8)

On the one hand, the total external work corresponds to that part of the mechanical energy that is lost as heat. It must be therefore restored in the next step through chemical energy
by oxygen consumption. On the other hand, the external work is realized through the positive and negative contributions: 

\[ W_{\text{tot}} = W^+ - |W^-| \]

These can be exemplified by the accelerated and decelerated forward body’s displacements, respectively. A special case is ideal oscillational motion, when the two contributions are equal and, thus, 

\[ W_{\text{tot}} = 0 \]

Also, the recovery coefficient, defined [3] for arbitrary cyclic motion as 

\[ r = \frac{W^+ - W_{\text{tot}}}{W^+} \]

equals one. To estimate \( r(V) \), we specify the positive work by 

\[ W^+(V) = T_c < P[\dot{W}_1(t)] >_c \]

where the auxiliary function \( P(x) = xH(x) \), with \( H(x) \) is the Heavyside step function. This leads to the recovery coefficient

\[
r(\omega_0, \omega_1) = 1 - \frac{\left[P[\cos(2\omega_0 t) \sin(2\omega_0 t + \varphi)]\right]}{2\sqrt{((3\omega_0/2\omega_1)^2 + 1)}}.
\]

If one employs the human-walk data on \( \omega_0^{(\text{exp})}(V) \) [12] and \( r^{(\text{exp})}(V) \), Eq. (9) can be read as

\[ r(\omega_0^{(\text{exp})}, \omega_1) = r^{(\text{exp})} \]

and solved for the mass-specific frictional coefficient (see Fig. 2). Additionally, a straightforward estimation of the positive work performed in the forward \( (W_f^+) \) and vertical \( (W_v^+) \) directions, yields the anharmonic amplitudes

\[
\Delta l_1(V) = \sqrt{\frac{(1 - r) W_f^+ + 4 \omega_0^2}{8\pi m \omega_0 \omega_1^2}}; \quad \Delta h_1(V) = \Delta l_1 \sqrt{\frac{W_v^+}{W_f^+}}.
\]

As follows from a numerical analysis of Eqs. (10) given in Fig. 3, the instability of the vertical COM librations occurs at the maximum speed for human walking with \( V_{\text{max}} = 3.4 \) m/s, in accord with the recent experimental data [5] \( V_{\text{max}}^{(\text{exp})} = 3.2 \) m/s. The eccentricity of the orbit-characteristic ellipses (see Fig. 4) specifies this instability by the critical condition 

\[ e_-(V_{\text{max}}) = 1 \]

associated with a dynamical transition from walking to running. There is also a dynamical crossover at the speed \( V_{\text{cr}} = 1.7 \) m/s, which separates slow walking from fast walking. At this speed, \( e_\pm(V_{\text{cr}}) = 0 \) and the shrunken ellipses \( (\Delta l < \Delta h) \) transform into the flatter ellipses \( (\Delta h < \Delta l) \). In reality, the crossover in locomotion is attributed to changes in performance of the human legs in the stiff-legged \( (\Delta l < \Delta h) \) and the compliant \( (\Delta h < \Delta l) \) walking. These two walking gaits are distinguished through distinct postures (and reaction-force records) and identified with a modern human walk and a walk of nonhuman primates, respectively. [5]
In conclusion, we have discussed the problem of animal locomotion on level ground in view of human data on reaction force and external work for walking at steady speeds. Application of the standard Lagrangian formalism developed for nonconservative forces, permits one to introduce a generalized equation for animal locomotion in the point-body-mass approximation. Such a description involves only integrative properties of body, given through the inertial, elastic, and resistance characteristics, and is therefore expected to be helpful for comparative studies of quadruped trotting of animals where lateral effects are also negligible. For the case of human walking, the locomotion for the body’s COM is a superposition of the rectilinear motion with its backward rotation along a shortened hypocycloid.

Records of the ground-reaction force elucidate a variety of body’s functions. In animal locomotion, the observed reaction force acts as a motor-brake force, which additionally supports the body weight and controls the stability of forward advancement. The efficiency of the employed above body-support function is restricted by anatomical adaptation of the long-bone limbs for peak force-ground contacts. This adaptation was proven [13] to be universal for all terrestrial mammals, with evolution of their body mass. As to motor-brake effects, we have deduced that the velocity-independent forces, which produce harmonic body vibrations, are responsible for an effective-exchange mechanism of elastic mechanical energy, attributed to passive muscles and parametrized by the body stiffness coefficient. The nonlinear anharmonic librations, which are due to the nonconservative part of the driving force, produce the main part of the external work. The feedback between linear and nonlinear COM librations is revealed through the phase, the amplitude and the frequency constraints imposed on cyclic human walking. Finally, our analysis of the external work given without recourse to a swing-pendulum or spring-mass modeling provides insights into the two principal gaits of biped walking. We have seen that the stiff-to-compliant crossover in human walking arises from the changes of body-resistance performance and not from the body-elastic adjustment, prescribed earlier by the pendulum dynamics.

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Fig. 1. Analysis of force-platform data on human level walking. Records for the horizontal ($F_x$) and vertical ($F_y$) forces are taken from Fig. 1 in Ref. [8] (the case of mass 59 kg and of speed 3.9 km/h) and given for the two-step period $2T_c$. Solid lines describe the force-time linear harmonics at distinct phases (denoted by letters) explicit in Eq.(2) along with the fitting amplitudes $\Delta l_0 = 0.012$ m and $\Delta h_0 = 0.016$ m. The lines indicated by hatched areas correspond to the double-frequency harmonics given in Eqs.(4) and adjusted with $\Delta l_1 = 0.006$ m and $\Delta h_1 = 0.008$ m; $H (\approx 1m)$ stands for the body’s COM height. Asymmetric deviations are due to the differences between i) the exertions by left and right feet and ii) the time intervals for the single-foot ($T_{sc}$) and double-foot ($T_{dc}$) ground contacts, with $T_c = T_{sc} + T_{dc}$. Rotations of the driving force and the corresponding velocity are shown by dotted lines in the coordinates $\Delta F_y(t)$ vs $\Delta F_x(t)$ and $\Delta V_y(t)$ vs $\Delta V_x(t)$, respectively.

Fig. 2. Human-walk characteristic frequencies against steady speed. Points for the cyclic frequency $\omega_0(V)$ reproduce cinematographic data (open circles) taken from Fig. 3 in Ref. [12]. They are extended by the case analyzed in Fig. 1 (open square) and fitted by $\omega_0(V) = 4.94 + 4.02V$ (solid line). Points for the mass-specific frictional coefficient $\gamma(V)/m$ (shaded circles) are found through Eq.(9) with the help of $\omega_0(V)$ and $r^{(exp)}(V)$ (reproduced in the insert from Fig. 2 in Ref. [3]) and fitted by $\omega_1(V) = 6.37 - 6.15V + 2.38V^2$ (given by the solid curve).

Fig. 3. Anharmonic amplitudes for the driving force against speed in human walking. The horizontal and vertical amplitudes $\Delta l_1(V)$ and $\Delta h_1(V)$, reduced by $\Delta l_0(1.1) = 0.012$ m, are given by open and shaded circles, respectively. They are estimated through Eqs.(10) with the help of the parameters obtained in Fig. 2 and of data on the positive work $W_f^{+(exp)}$ and $W_v^{+(exp)}$ (taken from Fig. 2 in Ref. [3]). The positive forward and vertical works are
adjusted at $V = 1.1 \text{ m/s}$ (shown by open and shaded squares, respectively). The curves are third-order polynomial fits extrapolated to the maximum speed $V_{\text{max}} = 3.4 \text{ m/s}$ (shown by the arrow).

Fig. 4. The human center-of-mass body orbit characteristics for slow and fast level walking. Inserts show ellipses, which circumscribe the closed COM trajectories in the inertial coordinate system moving with different speeds $V$ (indicated by arrows). The elliptic axes are reduced by the amplitude $\Delta l_0 = 0.012 \text{ m}$. Experimental data on the elliptic eccentricities (shown by open circles) for the shrunken and flatter ellipses obtained, respectively, through

$$\sqrt{1 - \frac{W_f^{+\text{(exp)}}}{W_u^{+\text{(exp)}}}}$$

and

$$\sqrt{1 - \frac{W_u^{+\text{(exp)}}}{W_f^{+\text{(exp)}}}},$$

with the help of the experimental data reported in Ref. [3]. The fitting curve corresponds to the analysis given in Fig. 3.