Volterra’s realization of the KM-system

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Abstract

We construct a symplectic realization of the KM-system and obtain the higher order Poisson tensors and commuting flows via the use of a recursion operator. This is achieved by doubling the number of variables through Volterra’s coordinate transformation. An application of Oevel’s theorem yields master symmetries, invariants and deformation relations.

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1 Introduction

The KM-system, also known as Volterra system, is defined by

\[ \dot{u}_i = u_i(u_{i+1} - u_{i-1}) \quad i = 1, 2, \ldots, n, \]

where \( u_0 = u_{n+1} = 0 \). It has been used as a model for predator-prey evolution systems [11], as well as a discretization of the Korteweg-de Vries equation. The integrability of the system was established in [7] and [9]. In [7] the inverse scattering technique is formulated in a discrete setting and applied on equations (1) to produce explicit solutions. Moser using a different method, namely continued fractions, has also integrated the model.

A diffeomorphism is established between the KM-system (1) and the celebrated Toda lattice equations

\[ \dot{a}_i = a_i(b_{i+1} - b_i) \quad i = 1, \ldots, n - 1 \]
\[ \dot{b}_i = 2(a_i^2 - a_{i-1}^2) \quad i = 1, \ldots, n, \]

via a transformation of Hénon

\[ a_i = \frac{1}{2} \sqrt{u_{2i}u_{2i-1}} \quad i = 1, \ldots, n - 1 \]
\[ b_i = \frac{1}{2}(u_{2i-1} + u_{2i-2}) \quad i = 1, \ldots, n. \]
Thus, the hierarchy of Poisson tensors, Hamiltonian functions, constants of motion and master symmetries known for the Toda lattice and expressed in Flaschka’s coordinates \((a, b)\), can be mapped to the corresponding ones for the KM-system in \(u-\)coordinates \([2]\). We note that the number of variables for the Toda lattice is odd and therefore we restrict our attention to the Volterra system with an odd number of variables.

A Hamiltonian description of the Volterra model can be found in the book of Fadeev and Takhtajan \([4]\). Later on, in \([2]\) two polynomial Poisson tensors of degree two and three are considered and placed in an infinite sequence of Poisson tensors that satisfy Lenard type relations. The quadratic Poisson bracket, \(\pi_2\), is defined by the formulas

\[
\{u_i, u_{i+1}\} = u_i u_{i+1},
\]

and all other brackets are zero. Using \(H = \sum_{i=1}^{2n-1} u_i\) as the Hamiltonian and the Poisson bracket \(\pi_2\), the Volterra equations are written in the form, \(\dot{u}_i = \{u_i, H\}\).

We will follow \([2]\) and use the Lax pair of that reference. It has the advantage of making the equations homogeneous polynomial. The Lax pair is given by

\[
\dot{L} = [B, L],
\]

where

\[
L = \begin{pmatrix}
    u_1 & 0 & \sqrt{u_1 u_2} & 0 & \cdots & 0 \\
    0 & u_1 + u_2 & 0 & \sqrt{u_2 u_3} & & \\
    \sqrt{u_1 u_2} & 0 & u_2 + u_3 & & & \\
    \vdots & \vdots & & \ddots & & \\
    \sqrt{u_1 u_2} & \ldots & & & u_{2n-2} + u_{2n-1} & 0 \\
    0 & \sqrt{u_2 u_3} & \ldots & & & u_{2n-2} + u_{2n-1} \\
    \sqrt{u_2 u_3} & \ldots & & & 0 & u_{2n-1}
\end{pmatrix},
\]

and

\[
B = \begin{pmatrix}
    0 & 0 & \frac{1}{2} \sqrt{u_1 u_2} & 0 & \cdots & 0 \\
    0 & 0 & 0 & \frac{1}{2} \sqrt{u_2 u_3} & & \\
    -\frac{1}{2} \sqrt{u_1 u_2} & 0 & 0 & 0 & \cdots & \\
    \vdots & \vdots & & \ddots & & \\
    -\frac{1}{2} \sqrt{u_2 u_3} & \ldots & & & \frac{1}{2} \sqrt{u_{2n-2} u_{2n-1}} & 0 \\
    0 & -\frac{1}{2} \sqrt{u_2 u_3} & \ldots & & & 0 \\
    0 & 0 & \ldots & & & 0
\end{pmatrix}.
\]
This is an example of an isospectral deformation; the entries of $L$ vary over time but the eigenvalues remain constant. It follows that the functions $H_i = \frac{1}{i} \text{Tr} L^i$ are constants of motion.

The cubic Poisson bracket, which corresponds to the second KdV bracket in the continuum limit, is defined by

$$\{u_i, u_{i+1}\} = u_i u_{i+1} (u_i + u_{i+1})$$
$$\{u_i, u_{i+2}\} = u_i u_{i+1} u_{i+2},$$

and all other brackets are zero. We denote this bracket by $\pi_3$. The Lenard relations take the form

$$\pi_3 \nabla H_i = \pi_2 \nabla H_{i+1}.$$ 

The higher order Poisson brackets are constructed using a sequence of master symmetries $Y_i$, $i = 0, 1, \ldots$. We define $Y_0$ to be the Euler vector field

$$Y_0 = \sum_{i=1}^{2n-1} u_i \frac{\partial}{\partial u_i},$$

and $Y_1$ the master symmetry

$$Y_1 = \sum_{i=1}^{2n-1} U_i \frac{\partial}{\partial u_i},$$

where

$$U_i = (i + 1) u_i u_{i+1} + u_i^2 + (2 - i) u_{i-1} u_i .$$

One can verify that the bracket $\pi_3$ is obtained from $\pi_2$ by taking the Lie derivative in the direction of $Y_1$.

The brackets $\pi_2$ and $\pi_3$ are just the beginning of an infinite family constructed in [2] using master symmetries. We quote the result:

**Theorem 1** There exists a sequence of Poisson tensors $\pi_j$ and a sequence of master symmetries $Y_j$ such that:

1. $\pi_j$ are all Poisson.
2. The functions $H_i$ are in involution with respect to all of the $\pi_j$.
3. $Y_i(H_j) = (i + j)H_{i+j}$.
4. $L_{Y_i} \pi_j = (j - i - 2) \pi_{i+j}$.
5. $[Y_i, Y_j] = (j - i) Y_{i+j}$.
6. $\pi_j \nabla H_i = \pi_{j-1} \nabla H_{i+1}$, where $\pi_j$ denotes the Poisson matrix of the tensor $\pi_j$. 

3
The KM-system is a special case of the more general Lotka-Volterra system which has the form,

\[ \dot{u}_i = \sum_{k=1}^{N} a_{ik} u_i u_k \quad i = 1, 2, \ldots, N, \quad (4) \]

where \((a_{ij})\) is a fixed matrix.

In the early work on (4), Volterra introduced a transformation from \(\mathbb{R}^{2N}\) to \(\mathbb{R}^N\), in his attempt to provide a Hamiltonian formulation; see for example [5]. Specifically he doubled the number of variables by defining

\[ q_i(t) = \int_0^t u_i(\tau) d\tau \quad (5) \]
\[ p_i(t) = \ln(\dot{q}_i) - \frac{1}{2} \sum_{k=1}^{N} a_{ik} q_k, \quad (6) \]

\(i=1, \ldots, N\), for a skew-symmetric \((a_{ij})\).

The explicit form of Volterra’s transformations from \(\mathbb{R}^{2N}\) to \(\mathbb{R}^N\), is

\[ u_i = e^{p_i + \frac{1}{2} \sum_{k=1}^{N} a_{ik} q_k} \quad i = 1, 2, \ldots, N. \quad (7) \]

The Hamiltonian function is given by

\[ H = \sum_{i=1}^{N} \dot{q}_i = \sum_{i=1}^{N} u_i, \quad (8) \]

which takes the form

\[ H = \sum_{i=1}^{N} e^{p_i + \frac{1}{2} \sum_{k=1}^{N} a_{ik} q_k}. \quad (9) \]

System (4) can then be endowed with the following symplectic realization

\[ \dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\} \quad (10) \]
\[ \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\}, \quad (11) \]

where the Poisson bracket in \((q,p)\) coordinates in \(\mathbb{R}^{2N}\) is the canonical one. We note that for the KM-system in \(u\)-space both Poisson tensors \(\pi_2\) and \(\pi_3\) are degenerate. Therefore, an application of the theory of recursion operators is hindered.
In this paper we consider the Volterra model in $\mathbb{R}^{2n-1}$ and obtain a symplectic realization of the system by increasing the dimension of the space. Namely, the number of variables is doubled through Volterra’s coordinate transformation. We rediscover the higher order Poisson tensors and flows for the system via the use of a recursion operator. We define a conformal symmetry in symplectic space and apply Oevel’s theorem to produce deformation relations, which can then be projected to give the deformation relations for the Volterra system in $\mathbb{R}^{2n-1}$.

2 Master Symmetries and Recursion Operators

Let us consider the differential equation $\dot{x} = \mathcal{X}(x)$ on a manifold $M$ defined by the Hamiltonian vector field $\mathcal{X}$. Below we give the definition of master symmetries, due to Fokas and Fuchssteiner, and briefly mention their basic properties. A vector field $Z$ is a symmetry of the equation if $[Z, \mathcal{X}] = 0$. In the case that $Z = Z(t, x)$, $Z$ is a time-dependent symmetry if

$$\frac{\partial Z}{\partial t} + [Z, \mathcal{X}] = 0.$$  

A more general definition is that of a generator of symmetries. $Z$ is called a generator of degree zero if $[Z, \mathcal{X}] = 0$, and a generator of degree one if $[[Z, \mathcal{X}], \mathcal{X}] = 0$. A generator of degree $k$ is the one that satisfies $[[\ldots [Z, \mathcal{X}], \ldots] = 0$, where there are $k + 1$ nested Lie brackets. We remark that if $Z$ is a generator of degree $k$ then $[Z, \mathcal{X}]$ is a generator of degree $k - 1$. Also, if $Z$ is a generator of degree $k$ then $Z$ is a generator of degree $i \geq k$. A symmetry is a generator of degree zero. A generator of degree one that is not a generator of degree zero is called a master symmetry. Oevel’s theorem provides a useful method for constructing master symmetries.

Suppose that we have a bi-Hamiltonian system with a symplectic Poisson tensor. Namely, a pair of Poisson tensors $J_0$ and $J_1$, with $J_0$ symplectic and a pair of Hamiltonian functions $H_1, H_2$ that give rise to the same system, i.e,

$$J_0 \nabla H_2 = J_1 \nabla H_1.$$  

Then a recursion operator $\mathcal{R}$ is defined by $\mathcal{R} = J_1 J_0^{-1}$, and gives rise to a family of Hamiltonian vector fields that are defined recursively as,

$$\mathcal{X}_i = \mathcal{R}^{i-1} \mathcal{X}_1,$$

and higher order Poisson tensors

$$J_i = \mathcal{R}^i J_0.$$  

The Hamiltonians $H_i$ corresponding to the vector fields $\mathcal{X}_i$ are given by $\nabla H_i = (\mathcal{R}^*)^i \nabla H_0$. 

5
These higher order flows have a multi-Hamiltonian formulation
\[ X_{i+j} = J_i \nabla H_j. \]  \(14\)

Magri’s theorem \[8\] states that the flows \(X_i\) pairwise commute. Also the functions \(H_i\) are constants of motion for each flow and commute with respect to all higher order Poisson tensors. We thus have an infinite sequence of involutive Hamiltonian flows. Furthermore, Oevel’s theorem provides a method for constructing master symmetries \[10\]. We quote the theorem.

**Theorem 2** Suppose that \(X_0\) is a conformal symmetry for both \(\pi_1, \pi_2\) and \(H_1\), i.e. for some scalars \(\lambda, \mu,\) and \(\nu\) we have
\[ L_{X_0} \pi_1 = \lambda \pi_1, \quad L_{X_0} \pi_2 = \mu \pi_2, \quad L_{X_0} H_1 = \nu H_1. \]  \(15\)

Then the vector fields \(X_i = \mathcal{R}^i X_0\) are master symmetries and we have,
\[ (a) \quad L_{X_i} H_j = (\nu + (j - 1 + i)(\mu - \lambda))H_{i+j} \]
\[ (b) \quad L_{X_i} \pi_j = (\mu + (j - i - 2)(\mu - \lambda))\pi_{i+j} \]
\[ (c) \quad [X_i, X_j] = (\mu - \lambda)(j - i)X_{i+j}. \]

As a corollary to Oevel’s theorem we have the existence of the following time-dependent symmetries for each flow in the hierarchy,
\[ Y_{X_i} = X_i + t(\mu + \nu + (j - 1)(\mu - \lambda))X_{i+j}, \quad i, j = 1, 2, \ldots \]  \(16\)

In the next section we will formulate the bi-Hamiltonian Volterra system in a symplectic setting so that we can apply the theory described in this section and obtain the results stemming out of the theorems of Magri and Oevel.

### 3 Symplectic setting

We consider the Volterra map
\[ \Psi : \mathbb{R}^{2(2n-1)} \mapsto \mathbb{R}^{2n-1} \]
\[ u_i = e^{p_i + \frac{1}{2}(q_{i+1} - q_{i-1})}, \quad i = 1, \ldots, 2n - 1, \]  \(17\)

where \(q_0 = q_{2n} = 0\). We note that \(u_0 = u_{2n} = 0\). The Hamiltonian in \((q, p)\) coordinates is given by
\[ h_1 = \sum_{i=1}^{2n-1} e^{p_i + \frac{1}{2}(q_{i+1} - q_{i-1})}, \]  \(18\)
and together with the canonical symplectic bracket in $\mathbb{R}^{2(2n-1)}$, call it $J_2$, corresponds to the Volterra system (11) under the mapping (17). In particular, the degenerate quadratic Poisson tensor $\pi_2$ defined in Section 1 is lifted to the symplectic bracket $J_2$ via transformation (17).

To find the pre-image of the cubic bracket $\pi_3$, we will lift the master symmetry $Y_1$ of Section 1 from the $u$-space in $\mathbb{R}^{2n-1}$ to a master symmetry $X_1$ in the symplectic space $(q, p) \in \mathbb{R}^{2(2n-1)}$. In fact, $\mathcal{L}_{X_1} J_2 = J_3$, where $X_1$ projects to $Y_1$ using the Volterra map. One possible definition for $X_1$ is the following:

$$X_1 = \sum_{i=1}^{2n-1} A_i \frac{\partial}{\partial q_i} + \sum_{i=1}^{2n-1} B_i \frac{\partial}{\partial p_i}$$  \hspace{1cm} (19)

where,

$$A_i = \sum_{j=1}^{2n-1} c_{j,i} e^{p_{j+\frac{1}{2}(q_{j+1}-q_{j-1})}};$$

$$B_i = (i + 1) e^{p_{i+\frac{1}{2}(q_{i+2}-q_{i})}} + e^{p_{i+\frac{1}{2}(q_{i+1}-q_{i-1})}} + (2 - i) e^{p_{i-\frac{1}{2}(q_{i}-q_{i-2})}} + \frac{1}{2} \sum_{j=1}^{2n-1} (c_{j,i-1} - c_{j,i+1}) e^{p_{j+\frac{1}{2}(q_{j+1}-q_{j-1})}},$$

for $i = 1, 2, \ldots, 2n - 1$. The constants $c_{i,j}$ are given by

$$c_{i,j} = 0, \quad i = 1, \ldots, 2n - 2, \quad j > i$$

$$c_{i,j} = -1, \quad i = 2, \ldots, 2n - 1, \quad j < i$$

$$c_{i,i} = i - 1, \quad i = 1, \ldots, 2n - 1.$$

We note that $c_{j,0} = c_{j,2n} = 0$. The constant matrix $C := (c_{i,j})$ takes the form,

$$C := \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
-1 & 1 & 0 & 0 & \cdots & 0 \\
-1 & -1 & 2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-1 & \cdots & \cdots & -1 & 2n - 3 & 0 \\
-1 & \cdots & \cdots & -1 & 2n - 2 & 0
\end{pmatrix}.$$  

Taking the Lie derivative of the symplectic bracket $J_2$ in the direction of $X_1$ we obtain
the Poisson bracket \( J_3 \),

\[
\begin{align*}
\{q_i, q_j\} &= e^{p_j + \frac{3}{2}(q_j+1-q_j-1)} \\
\{q_i, p_1\} &= e^{p_1 + \frac{1}{2}q_2} + \frac{1}{2} e^{p_2 + \frac{1}{2}(q_3-q_1)} \\
\{q_i, p_i\} &= \frac{1}{2} e^{p_i + \frac{1}{2}(q_i+1-q_i-1)} + \frac{1}{2} e^{p_i+1 + \frac{1}{2}(q_i+2-q_i)} \\
\{q_i, p_{i+1}\} &= \frac{1}{2} e^{p_{i+2} + \frac{1}{2}(q_{i+3}-q_{i+1})} \\
\{q_1, p_1\} &= e^{p_2 + \frac{3}{4}(q_3-q_1)} \\
\{q_i, p_{i-1}\} &= \frac{1}{2} e^{p_i + \frac{1}{2}(q_i+1-q_i-1)} \\
\{q_i, p_j\} &= -\frac{1}{2} e^{p_j - 1 + \frac{1}{2}(q_j-q_j-2)} + \frac{1}{2} e^{p_j+1 + \frac{1}{2}(q_j+2-q_j)} \\
\{q_i, p_1\} &= \frac{1}{2} e^{p_1 + \frac{1}{2}(q_1+1-q_1-1)} \\
\{p_1, p_2\} &= \frac{1}{2} e^{p_1 + \frac{1}{2}q_2} + \frac{1}{4} e^{p_2 + \frac{3}{2}(q_3-q_1)} - \frac{1}{4} e^{p_3 + \frac{3}{2}(q_3-q_2)} \\
\{p_i, p_{i+1}\} &= \frac{1}{4} e^{p_i + \frac{1}{2}(q_i+1-q_i-1)} + \frac{1}{4} e^{p_{i+1} + \frac{1}{2}(q_{i+2}-q_i)} \\
\{p_1, p_3\} &= \frac{1}{2} e^{p_2 + \frac{1}{2}(q_3-q_1)} - \frac{1}{4} e^{p_4 + \frac{1}{2}(q_4-q_3)} \\
\{p_1, p_{i+2}\} &= \frac{1}{4} e^{p_{i+2} + \frac{1}{2}(q_{i+3}-q_{i+1})} \\
\{p_1, p_j\} &= -\frac{1}{4} e^{p_{j} + \frac{1}{2}(q_{j+2}-q_j)} + \frac{1}{4} e^{p_{j-1} + \frac{1}{2}(q_{j}-q_{j-2})} \\
\end{align*}
\]

and all other brackets are zero. We recall that \( e^{p_2n + \frac{3}{2}(q_{2n+1}-q_{2n-1})} = u_{2n} = 0 \). The Jacobi identity for the bracket \( J_3 \) can be rigorously checked by considering the following four cases: (a) three \( q \), (b) three \( p \), (c) two \( p \) and one \( q \), and (d) two \( q \) and one \( p \). For example, the Jacobi identity for \( q_i, q_j, q_k \) for \( 1 \leq i < j < k \leq 2n - 1 \) can be broken up to two subcases: 
(a1) \( k = j + 1 \), and (a2) \( k \geq j + 2 \). In a similar manner one can consider the other three cases.

Under the Volterra transformation, \( J_2 \) maps to \( \pi_2 \) and \( J_3 \) to \( \pi_3 \). The function

\[
h_2 = \frac{1}{2} \sum_{i=1}^{2n-1} e^{2p_i + q_{i+1} - q_{i-1}} + \sum_{i=1}^{2n-2} e^{p_{i} + q_{i+1} + \frac{3}{2}(q_{i+2} + q_{i+1} - q_i - q_{i-1})}
\]

corresponds under mapping (17) to a constant multiple of \( H_2 = \frac{1}{2} \text{Tr}(L)^2 \). We recall that \( H_1, H_2 \) and \( \pi_2, \pi_3 \) constitute a bi-Hamiltonian pair,

\[
\pi_2 \nabla H_2 = \pi_3 \nabla H_1.
\]
However, both Poisson tensors are degenerate. The Volterra map places this bi-Hamiltonian pair in a symplectic setting. That is,

$$J_2 \nabla h_2 = J_3 \nabla h_1. \quad (22)$$

Therefore, the definition of a recursion operator, $R = J_3 J_2^{-1}$ is possible. $J_3$ is by construction compatible with $J_2$ since it is generated from a master symmetry; see [3]. We note the absence of a negative recursion operator as in [1] using this method, since the matrix representing $J_3$ is not invertible.

A multi-Hamiltonian structure of the form, $\mathcal{X}_i + J = J_i \nabla h_j$, is provided by the higher order Poisson tensors and Hamiltonian vector fields

$$J_i = R^{i-2} J_2, \quad i = 3, 4, \ldots, \quad (23)$$

$$\mathcal{X}_i = R^{i-1} \mathcal{X}_1, \quad i = 2, 3, \ldots, \quad (24)$$

where $\mathcal{X}_i$ stands for $\mathcal{X}_{h_i}$.

Theorem 2 requires the existence of a conformal symmetry $X_0$ such that

$$\mathcal{L}_{X_0} J_2 = \lambda J_2, \quad \mathcal{L}_{X_0} J_3 = \mu J_3, \quad \mathcal{L}_{X_0} (h_1) = \nu h_1. \quad (25)$$

We define the conformal symmetry

$$X_0 = \sum_{i=1}^{2n-1} \frac{\partial}{\partial p_i}, \quad (26)$$

and one can check that relations (25) are satisfied with $\lambda = 0, \mu = 1, \nu = 1$. Therefore, in addition to the infinite family of commuting Hamiltonian flows, we have the following deformation relations:

$$[X_i, h_j] = (i + j) h_{i+j} \quad (27)$$

$$L_{X_i} J_j = (j - i - 2) J_{i+j} \quad (28)$$

$$[X_i, X_j] = (j - i) X_{i+j}. \quad (29)$$

Using the Volterra map we can project these to the $u$--space and provide an alternative proof of the statements of Theorem [1].
4 Discussion

A different symplectic realization for the KM-system has been achieved recently in \[1\] using the map

\[
\Phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n-1}
\]

\[
u_{2i-1} = -e^{p_i}, \quad i = 1, \ldots, n
\]

\[
u_{2i} = e^{q_{i+1} - q_i}, \quad i = 1, \ldots, n-1.
\]

The Hamiltonian is defined as

\[
H = -\sum_{i=1}^{n} e^{p_i} + \sum_{i=1}^{n-1} e^{q_{i+1} - q_i}
\]

and the standard symplectic bracket in \((q, p)\)-space maps to the degenerate quadratic Poisson tensor \(\pi_2\) via transformation (30). A second symplectic bracket is obtained by lifting the cubic bracket \(\pi_3\).

In this paper we consider the Volterra map

\[
\Psi : \mathbb{R}^{2(2n-1)} \rightarrow \mathbb{R}^{2n-1}
\]

\[
u_i = e^{p_i + \frac{1}{2}(q_{i+1} - q_{i-1})}, \quad i = 1, \ldots, 2n-1,
\]

in order to lift the bi-Hamiltonian structure of the KM-system to a symplectic space in \(\mathbb{R}^{2(2n-1)}\). The big difference between the dimensions of the source and the target space in (32) impedes the application of the methodology used in [1]. However, the existence of a pair of Poisson tensors, at least one of which is non-degenerate, is possible. A second bracket \(J_3\) in \(\mathbb{R}^{2(2n-1)}\) is obtained so that its image under mapping (32) is the cubic bracket \(\pi_3\). Since \(J_3\) is symplectic, a recursion operator is defined as \(\mathcal{R} = J_3 J_2^{-1}\), and used to give rise to an infinite hierarchy of commuting Hamiltonian flows and Poisson tensors. The conformal symmetry of the KM-system in \(u\)-space is lifted to the symplectic \((q, p)\)-space, and an application of Oevel’s theorem leads to an infinite number of master symmetries, Poisson tensors and invariants. We note the absence of a negative recursion operator using this realization since \(J_3\) is non-invertible.

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