OSCILLATION CRITERIA FOR KERNEL FUNCTION
DEPENDENT FRACTIONAL DYNAMIC EQUATIONS

BAHAELDIN ABDALLA
Department of Mathematics and General Sciences, Prince Sultan University
P. O. Box 66833, Riyadh 11586, Saudi Arabia

THABET ABDELJAWAD*
Department of Mathematics and General Sciences
Prince Sultan University
P. O. Box 66833, Riyadh 11586, Saudi Arabia
Department of Medical Research
China Medical University, Taichung 40402, Taiwan
Department of Computer Science and Information Engineering
Asia University, Taichung, Taiwan

Abstract. In this work, we examine the oscillation of a class fractional differ-
etial equations in the frame of generalized nonlocal fractional derivatives with
function dependent kernel type. We present sufficient conditions to prove the
oscillation criteria in both of the Riemann-Liouville (RL) and Caputo types.
Taking particular cases of the nondecreasing function appearing in the kernel
of the treated fractional derivative recovers the oscillation of several proven
results investigated previously in literature. Two examples, where the kernel
function is quadratic and cubic polynomial, have been given to support the
validity of the proven results for the RL and Caputo cases, respectively.

1. Introduction. Fractional dynamical systems have attractive applications in
modeling of real world problems since they are good candidates most of the time
to describe complex systems with memory [16, 17]. On the other hand, the oscil-
lation theory is old as old its connection to spectral theory when was initiated by
Jacques Charles François Sturm when he investigated the Sturm—Liouville prob-
lems in 1836. After that huge number of research articles have been published. For
instance, we refer to [20, 9, 10] and the references therein.

Fractional calculus recently plays a critical important role in modelling. About
the advantages of the fractional differential equations over the models of integer-
order we refer to [18]. Recently, many researchers have started to study the oscilla-
tion of fractional differential and fractional difference equations as generalizations
to the corresponding aspects in the theory of ordinary differential and difference
equations [12, 11, 6, 1, 2, 3, 4]. For example, in [4] the authors obtained sufficient
conditions for the oscillation of fractional differential equations of the type (1) be-
low in the frame of Atangan-Baleanu Caputo fractional derivatives. Such fractional

2020 Mathematics Subject Classification. Primary: 34A08, 34C10; Secondary: 26A33.
Key words and phrases. General fractional integrals, derivative of a function with respect to a
function, fractional differential equations, oscillation theory, Young’s inequality, singular function
dependent kernel.
* Corresponding author: tabdeljawad@psu.edu.sa.
derivatives have non-singular Mittag-Leffler type kernels with fractional integral operators pieced into two parts [8]. It turned out to be clear that the structure of used kernel and the corresponding fractional integral operator obviously affected the oscillatory sufficient condition.

In this paper, we study the oscillation of general fractional differential equation of the form

\[
\begin{align*}
D_a^{\alpha,\psi} x(t) + f_1(t, x) &= r(t) + f_2(t, x), \quad t > a \geq 0 \\
\lim_{t \to a^+} D_a^{\alpha-j,\psi} x(t) &= b_j \quad (j = 1, 2, \ldots n),
\end{align*}
\]

where \( n = [\alpha] \), \( D_a^{\alpha,\psi} \) is the left-fractional general derivative of order \( \alpha \in \mathbb{C} \), \( \text{Re}(\alpha) \geq 0 \) in Riemann-Liouville setting. Such generalized fractional operators can be found in (section 2.5 in [16]). More on fractional derivatives of function with respect to another nondecreasing function and their Caputo type modification have been discussed in [14], where a generalized type of Laplace transform was developed to deal with such type of generalized fractional derivatives. We shall also study the oscillation of Eq. (1) in the setting of Caputo.

This paper is organized as follows. Section 2 introduces some notations and provides the definitions of general fractional integral and differential operators together with some basic properties and lemmas that are needed in the proofs of the main theorems. In section 3, the main theorems are presented. Section 4 is devoted to the results obtained for general fractional operators in Caputo setting. Examples are provided in section 5 to demonstrate the effectiveness of the main theorems.

2. Notations and preliminary assertions. We start this section by introducing the definition of the general fractional integrals and derivatives. Let \( \alpha > 0 \), \( I = [a, b] \) be a finite or infinite interval, \( f \) an integrable function defined on \( I \) and \( \psi \in C^1(I) \) an increasing function such that \( \psi'(x) \neq 0 \), for all \( x \in I \). Fractional integrals and fractional derivatives of a function \( f \) with respect to another function \( \psi \) are defined as [14, 17]

\[
(I_a^{\alpha,\psi} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} f(t) dt,
\]

and

\[
(D_a^{\alpha,\psi} f)(x) := \psi D_x^n I_a^{n-\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{n-\alpha-1} f(t) dt,
\]

respectively, where \( n = [\alpha] \) and

\[
\psi D_x^n = \left( \frac{d}{\psi'(x) dx} \right)^n.
\]

If we consider \( \psi(x) = x \) or \( \psi(x) = \ln x \), we obtain the RL and Hadamard fractional operators respectively.

Lemma 2.1. [17] Let \( \mathfrak{R}(\alpha) > 0 \) and \( \mathfrak{R}(\mu) > 0 \), then

\[
I_a^{\alpha,\psi}(\psi(x) - \psi(a))^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu + \alpha)} (\psi(x) - \psi(a))^{\mu+\alpha-1}.
\]

Lemma 2.2. [17] Let \( \mathfrak{R}(\alpha) > 0 \) and \( \mathfrak{R}(\mu) > 0 \), then

\[
D_a^{\alpha,\psi}(\psi(x) - \psi(a))^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu - \alpha)} (\psi(x) - \psi(a))^{\mu-\alpha-1}.
\]
Lemma 2.4. Let \( \alpha > 0, \, n = [\alpha], \, \beta > n, \, \beta \in \mathbb{R}, \) then

\[
(CD_a^{\alpha,\psi} f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi(t)(\psi(x) - \psi(t))^{n-\alpha-1} \psi D_a^{\beta} f(t) dt. \tag{5}
\]

If in (5), we set \( \psi(t) = t \) and \( \psi(t) = \ln t \), we obtain Caputo and Caputo-Hadamard fractional derivatives, respectively. Caputo-Hadamard fractional derivatives have been introduced and investigated thoroughly in [15, 5].

Lemma 2.5. [13] (Young’s inequality)

(i) Let \( L, M \geq 0, \, u > 1 \) and \( \frac{1}{u} + \frac{1}{v} = 1 \), then \( LM \leq \frac{1}{u} L^u + \frac{1}{v} M^v \), where the equality holds if and only if \( M = L^{u-1} \).

(ii) Let \( L \geq 0, \, M > 0, \, 0 < u < 1 \) and \( \frac{1}{u} + \frac{1}{v} = 1 \), then \( LM \geq \frac{1}{u} L^u + \frac{1}{v} M^v \), where the equality holds if and only if \( M = L^{u-1} \).

3. Oscillation of general fractional differential equations in the frame of Riemann. In this section, we study the oscillation theory for Eq. (1).

Lemma 3.1. [14] Let \( \text{Re}(\alpha) > 0, \, n = [-\text{Re}(\alpha)], \, f \in L(a, b) \) and \( (I_a^{\alpha,\psi} f)(x) \in AC_{v}^n[a, b] \). Then

\[
(f_a^{\alpha,\psi} D_a^{\alpha,\psi} f)(x) = f(x) - \sum_{j=1}^{n} \frac{(I_a^{\alpha,\psi} f)(x^+)}{\Gamma(\alpha-j+1)} (\psi(x) - \psi(a))^{\alpha-j}, \tag{6}
\]

Using Lemma 3.1, the solution of Eq. (1) can be represented by

\[
x(t) = \Theta(t) + I_a^{\alpha,\psi} F(t, x), \tag{7}
\]

where

\[
F(t, x) = r(t) + f_2(t, x) - f_1(t, x) \tag{8}
\]

and

\[
\Theta(t) = \sum_{j=1}^{n} \frac{(I_a^{\alpha,\psi} x)(x^+)}{\Gamma(\alpha-j+1)} (\psi(t) - \psi(a))^{\alpha-j}. \tag{9}
\]

A solution of Eq. (1) is said to be oscillatory if it has arbitrarily large zeros on \((0, \infty)\); otherwise, it is called nonoscillatory. An equation is said to be oscillatory if all of its solutions are oscillatory.

We prove our results under the following assumptions:

\[
x f_i(t, x) > 0 \quad (i = 1, 2), \quad x \neq 0, \quad t \geq 0, \tag{10}
\]

\[
|f_1(t, x)| \geq p_1(t) \quad |x| \beta \quad \text{and} \quad |f_2(t, x)| \leq p_2(t) \quad |x| \gamma, \quad x \neq 0, \quad t \geq 0, \tag{11}
\]

\[
|f_1(t, x)| \leq p_1(t) \quad |x| \beta \quad \text{and} \quad |f_2(t, x)| \geq p_2(t) \quad |x| \gamma, \quad x \neq 0, \quad t \geq 0, \tag{12}
\]

where \( p_1, p_2 \in C([0, \infty), (0, \infty)) \) and \( \beta, \gamma \) are positive constants.

Define

\[
\Xi(t, T_1) = \int_a^{T_1} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(s, x(s)) ds. \tag{13}
\]
Theorem 3.2. Let $f_2 = 0$ in Eq. (1) and assumption (10) holds. If
\[
\liminf_{t \to \infty}(\psi(t))^{1-\alpha} \int_T^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}r(s)ds = -\infty \quad (14)
\]
and
\[
\limsup_{t \to \infty}(\psi(t))^{1-\alpha} \int_T^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}r(s)ds = \infty, \quad (15)
\]
for every sufficiently large $T$, then every solution of Eq. (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1) with $f_2 = 0$. Suppose that $T_1 > a$ is large enough so that $x(t) > 0$ for $t \geq T_1$. Assumption (10) implies that $f_1(t, x) > 0$ for $t \geq T_1$. Using Eq. (2), we get from Eq. (7)
\[
\Gamma(\alpha)x(t) = \Gamma(\alpha)\Phi(t) + \int_a^{T_1} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}F(s, x(s))ds
\]
\[
+ \int_{T_1}^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) - f_1(s, x(s))]ds
\]
\[
\leq \Gamma(\alpha)\Phi(t) + \Xi(t, T_1) + \int_{T_1}^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}r(s)ds, \quad (16)
\]
where $\Phi$ and $\Xi$ are defined in Eq. (9) and Eq. (13) respectively.

Multiplying inequality (16) by $(\psi(t))^{1-\alpha}$, we get
\[
0 < (\psi(t))^{1-\alpha}\Gamma(\alpha)x(t) \leq (\psi(t))^{1-\alpha}\Gamma(\alpha)\Phi(t) + (\psi(t))^{1-\alpha}\Xi(t, T_1)
\]
\[
+ (\psi(t))^{1-\alpha} \int_{T_1}^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}r(s)ds. \quad (17)
\]
Take $T_2 > T_1$. We consider two cases:

Case (1): Let $0 < \alpha \leq 1$, hence $n = 1$ and $h_1(t) = \left(1 - \frac{\psi(a)}{\psi(t)}\right)^{\alpha-1}$ is decreasing for $\psi(a) > 0$ and $t > T_2$. Then
\[
|\Gamma(\alpha)\Phi(t)| = |b_1(\psi(t))^{1-\alpha}(\psi(t) - \psi(a))^{\alpha-1}| = |b_1|\left(1 - \frac{\psi(a)}{\psi(t)}\right)^{\alpha-1}
\]
\[
\leq |b_1|\left(1 - \frac{\psi(a)}{\psi(T_2)}\right)^{\alpha-1} = c_1(T_2), \quad (18)
\]
and
\[
|\Xi(t, T_1)| = \left|\int_a^{T_1} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}F(s, x(s))ds\right|
\]
\[
\leq \int_a^{T_1} \psi'(s)\left(1 - \frac{\psi(s)}{\psi(t)}\right)^{\alpha-1}|F(s, x(s))|ds
\]
\[
\leq \int_a^{T_1} \psi'(s)\left(1 - \frac{\psi(s)}{\psi(T_2)}\right)^{\alpha-1}|F(s, x(s))|ds
\]
\[
:= c_2(T_1, T_2). \quad (19)
\]
Then inequality (17) implies
\[
(\psi(t))^{1-\alpha} \int_{T_1}^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}r(s)ds \geq -[c_1(T_2) + c_2(T_1, T_2)].
\]
Since the right hand side of the last inequality is a negative constant, we conclude that
\[ \liminf_{t \to \infty} (\psi(t))^{1-\alpha} \int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} r(s) ds \geq -[c_1(T_2) + c_2(T_1, T_2)] > -\infty, \]
which contradicts condition \((14)\).

**Case (2):** Let \( \alpha > 1 \), hence \( n \geq 2 \), \((1 - \frac{\psi(a)}{\psi(t)})^{\alpha-1} \leq 1 \) for \( \psi(a) > 1 \), and the function \( h_2(t) = (\psi(t) - \psi(a))^{1-j} \) is decreasing for \( j > 1 \). Then for \( t \geq T_2 \), we have
\[ |(\psi(t))^{1-\alpha} \Gamma(\alpha) \phi(t)| = \left| (\psi(t))^{1-\alpha} \Gamma(\alpha) \sum_{j=1}^{n} \frac{(I_{a}^{\alpha-j-1}(x))^{(\alpha-j)}}{\Gamma(\alpha-j+1)} (\psi(t) - \psi(a))^{\alpha-j} \right| \]
\[ \leq \Gamma(\alpha) \left(1 - \frac{\psi(a)}{\psi(t)}\right)^{\alpha-1} \sum_{j=1}^{n} |b_j| \frac{(\psi(t) - \psi(a))^{1-j}}{\Gamma(\alpha-j+1)} \]
\[ \leq \Gamma(\alpha) \sum_{j=1}^{n} |b_j| \frac{(\psi(T_2) - \psi(a))^{1-j}}{\Gamma(\alpha-j+1)} := c_3(T_2) \quad (20) \]
and
\[ |(\psi(t))^{1-\alpha} \Xi(t, T_1)| = \left| (\psi(t))^{1-\alpha} \int_{a}^{T_1} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(s, x(s)) ds \right| \]
\[ \leq \int_{a}^{T_1} \psi'(s) \left(1 - \frac{\psi(s)}{\psi(t)}\right)^{\alpha-1} |F(s, x(s))| ds \]
\[ \leq \int_{a}^{T_1} \psi'(s) |F(s, x(s))| ds := c_4(T_1). \quad (21) \]

Then inequality \((17)\) implies
\[ (\psi(t))^{1-\alpha} \int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} r(s) ds \geq -[c_3(T_2) + c_4(T_1)]. \]

Since the right hand side of the last inequality is a negative constant, we conclude that
\[ \liminf_{t \to \infty} (\psi(t))^{1-\alpha} \int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} r(s) ds \geq -[c_3(T_2) + c_4(T_1)] > -\infty, \]
which contradicts condition \((14)\).

Therefore, we conclude that \( x(t) \) is oscillatory. In case \( x(t) \) is eventually negative, similar arguments lead to a contradiction with condition \((15)\). \( \square \)

**Theorem 3.3.** Suppose that assumptions \((10)\) and \((11)\) hold with \( \beta > \gamma \). If
\[ \liminf_{t \to \infty} (\psi(t))^{1-\alpha} \int_{T}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} [r(s) + H(s)] ds = -\infty \quad (22) \]
and
\[ \limsup_{t \to \infty} (\psi(t))^{1-\alpha} \int_{T}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} [r(s) - H(s)] ds = \infty, \quad (23) \]
for every sufficiently large $T$, where
\begin{equation}
H(s) = \frac{\beta - \gamma}{\gamma} [p_1(s)]^{\frac{\beta}{\beta - \gamma}} \left[ \frac{\gamma p_2(s)}{\beta} \right]^{\frac{\beta}{\beta - \gamma}}, \tag{24}
\end{equation}
then every solution of Eq. (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1), say, $x(t) > 0$ for $t \geq T > a$. Let $s \geq T_1$. Using assumptions (10) and (11), we get
\[ f_2(s, x) - f_1(s, x) \leq p_2(s)x^\gamma(s) - p_1(s)x^\beta(s). \]

Let $X = x^\gamma(s)$, $Y = \frac{\gamma p_2(s)}{\beta p_1(s)}$, $u = \frac{\beta}{\gamma}$ and $v = \frac{\beta}{\beta - \gamma}$, then from part (i) of Lemma 2.5 we get
\[ p_2(s)x^\gamma(s) - p_1(s)x^\beta(s) = \frac{\beta p_1(s)}{\gamma} \left[ x^\gamma(s) \left( \frac{\gamma p_2(s)}{\beta p_1(s)} - \frac{\gamma}{\beta} (x^\gamma(s))^{\frac{\beta}{\gamma}} \right) \right] \]
\[ = \frac{\beta p_1(s)}{\gamma} \left[ XY - \frac{1}{u} Y^u \right] \leq \frac{\beta p_1(s)}{\gamma} \frac{1}{v} Y^v = H(s), \tag{25} \]
where $H$ is defined by Eq. (24). Then from Eq. (7), we obtain
\begin{align*}
\Gamma(\alpha)x(t) &= \Gamma(\alpha)\Phi(t) + \Xi(t, T_1) \\
&\quad + \int_{T_1}^t \psi'(s)(\psi(t) - \psi(s))^{\alpha - 1} \left[ r(s) + f_2(s, x(s)) - f_1(s, x(s)) \right] ds \\
&\leq \Gamma(\alpha)\Phi(t) + \Xi(t, T_1) \\
&\quad + \int_{T_1}^t \psi'(s)(\psi(t) - \psi(s))^{\alpha - 1} \left[ r(s) + p_2(s)x^\gamma(s) - p_1(s)x^\beta(s) \right] ds \\
&\leq \Gamma(\alpha)\Phi(t) + \Xi(t, T_1) \\
&\quad + \int_{T_1}^t \psi'(s)(\psi(t) - \psi(s))^{\alpha - 1} \left[ r(s) + H(s) \right] ds
\end{align*}
The rest of the proof is the same as in that of Theorem 3.2. \hfill \Box

**Theorem 3.4.** Let $\alpha \geq 1$ and suppose that assumptions (10) and (12) hold with $\beta < \gamma$. If
\begin{equation}
\limsup_{t \to \infty} (\psi(t))^{1 - \alpha} \int_T^t \psi'(s)(\psi(t) - \psi(s))^{\alpha - 1} [r(s) + H(s)] ds = \infty \tag{27}
\end{equation}
and
\begin{equation}
\liminf_{t \to \infty} (\psi(t))^{1 - \alpha} \int_T^t \psi'(s)(\psi(t) - \psi(s))^{\alpha - 1} [r(s) - H(s)] ds = -\infty, \tag{28}
\end{equation}
for every sufficiently large $T$, where $H$ is defined by Eq. (24), then every bounded solution of Eq. (1) is oscillatory.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of Eq. (1). Then there exist constants $M_1$ and $M_2$ such that
\[ M_1 \leq x(t) \leq M_2 \text{ for } t \geq a. \tag{29} \]
Assume that $x$ is a bounded eventually positive solution of Eq. (1). Then there exists $T_1 > a$ such that $x(t) > 0$ for $t \geq T_1 > a$. Using assumptions (10) and (12),
we get \( f_2(s, x) - f_1(s, x) \geq p_2(s)x^\alpha(s) - p_1(s)x^\beta(s) \). Using (ii) of Lemma 2.5 and similar to the proof of Eq. (25), we get
\[ p_2(s)x^\alpha(s) - p_1(s)x^\beta(s) \geq H(s) \text{ for } s \geq T_1. \]

From Eq. (7) and similar to the proof of inequality (46), we obtain
\[ \Gamma(\alpha)x(t) \geq \Gamma(\alpha)\Phi(t) + \Xi(t, T_1) + \int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\{r(s) + H(s)\}ds \]

Multiplying by \((\psi(t))^{1-\alpha}\), we get
\[
(\psi(t))^{1-\alpha}\Gamma(\alpha)x(t) \geq (\psi(t))^{1-\alpha}\Gamma(\alpha)\Phi(t) + (\psi(t))^{1-\alpha}\Xi(t, T_1) + (\psi(t))^{1-\alpha}\int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + H(s)]ds. \tag{30}
\]

Take \( T_2 > T_1 \). We consider two cases:

**Case (1):** Let \( \alpha = 1 \). Then
\[
|(\psi(t))^{1-\alpha}\Gamma(\alpha)\Phi(t)| = |\Gamma(\alpha)\Phi(t)| = |b_1|
\]
and
\[
|(\psi(t))^{1-\alpha}\Xi(t, T_1)| = |\Xi(t, T_1)| = \left| \int_{a}^{T_1} \psi'(s)F(s, x(s))ds \right|
\leq \int_{a}^{T_1} \psi'(s)|F(s, x(s))|ds = c_5(T_1). \tag{31}
\]

Hence from inequality (30) and using inequality (29), we find that
\[
M_2 \geq x(t) \geq |b_1| - c_5(T_1) + \int_{T_1}^{t} \psi'(s)[r(s) + H(s)]ds,
\]
for \( t \geq T_2 \). Thus, we get
\[
\limsup_{t \to \infty} (\psi(t))^{1-\alpha}\int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + H(s)]ds
= \limsup_{t \to \infty} \int_{T_1}^{t} \psi'(s)[r(s) + H(s)]ds \leq |b_1| + c_5(T_1) + M_2 < \infty,
\]
which contradicts condition (27).

**Case (2):** Let \( \alpha > 1 \). Then inequality (20) and inequality (21) are still true. Hence from inequality (30) and using inequality (29) we find that
\[
M_2\Gamma(\alpha)(\psi(t))^{1-\alpha} \geq -c_3(T_2) - c_4(T_1) + (\psi(t))^{1-\alpha}\int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + H(s)]ds,
\]
for \( t \geq T_2 \). Since \( \psi(t) \) is an increasing function, we have \( \lim_{t \to \infty} (\psi(t))^{1-\alpha} = 0 \) for \( \alpha > 1 \). Then
\[
\limsup_{t \to \infty} (\psi(t))^{1-\alpha}\int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + H(s)]ds \leq c_3(T_2) + c_4(T_1) < \infty,
\]
which contradicts condition (27). Therefore, we conclude that \( x(t) \) is oscillatory. In case \( x(t) \) is eventually bounded negative, similar arguments lead to a contradiction with condition (28). \( \square \)
Remark 1. If we let $\psi(t) = t$, $\psi(t) = \ln t$ and $\psi(t) = \frac{(t-a)^n}{n!}$ then we recover the RL and Hadamard fractional oscillation results in [12], [3] and [2], respectively.

4. Oscillation of general fractional differential equations in the frame of Caputo. In this section, we study the oscillation of the General fractional differential equations in the Caputo setting of the form

$$
\left\{
\begin{array}{l}
CD_a^{\alpha,\psi}x(t) + f_1(t,x) = r(t) + f_2(t,x), t > a \\
\psi D^k_a x(a) = b_k \ (k = 0,1,...,n-1),
\end{array}
\right.
$$

(32)

where $n = [\alpha]$, $\psi D^k$ is defined by Eq. (4) and $CD_a^{\alpha,\psi}$ is defined by Eq. (5).

Lemma 4.1. [14] Let $f \in C^n[a,b]$ and $\alpha > 0$. Then

$$
I_a^\alpha(\psi D^k_a)\psi f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(\psi(x) - \psi(a))^k}{k!} \psi D^k_a f(a),
$$

(33)

Using Lemma 4.1, the solution representation of Eq. (32) can be written as

$$
x(t) = \chi(t) + I_a^\alpha F(t,x),
$$

(34)

where $F(t,x) = r(t) + f_2(t,x) - f_1(t,x)$ and

$$
\chi(t) = \sum_{k=0}^{n-1} \frac{(\psi(t) - \psi(a))^k}{k!} \psi D^k_a x(a).
$$

(35)

Theorem 4.2. Let $f_2 = 0$ in Eq. (32) and condition (10) holds. If

$$
\lim_{t \to \infty} \frac{1}{(t-a)^{1-n}} \int_a^t \psi(s)(\psi(t) - \psi(s))^{\alpha-1} r(s) ds = -\infty
$$

(36)

and

$$
\lim_{t \to \infty} \frac{1}{(t-a)^{1-n}} \int_a^t \psi(s)(\psi(t) - \psi(s))^{\alpha-1} r(s) ds = \infty,
$$

(37)

for every sufficiently large $T$, then every solution of Eq. (32) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (32) with $f_2 = 0$. Suppose that $T_1 > a$ is large enough so that $x(t) > 0$ for $t \geq T_1$. Hence, assumption (10) implies that $f_1(t,x) > 0$ for $t \geq T_1$. Similar to inequality (16) we get from Eq. (34)

$$
\Gamma(\alpha)x(t) = \Gamma(\alpha)\chi(t) + \int_a^{T_1} \psi(s)(\psi(t) - \psi(s))^{\alpha-1} F(s,x(s)) ds
$$

+ \int_{T_1}^t \psi(s)(\psi(t) - \psi(s))^{\alpha-1} [r(s) - f_1(s,x(s))] ds
$$

\leq \Gamma(\alpha)\chi(t) + \Xi(t,T_1) + \int_{T_1}^t \psi(s)(\psi(t) - \psi(s))^{\alpha-1} r(s) ds,
$$

(38)

where $\chi$ and $\Xi$ are defined in Eq. (35) and Eq. (13) respectively.

Multiplying inequality (38) by $(\psi(t))^{1-n}$, we get

$$
0 < (\psi(t))^{1-n} \Gamma(\alpha)x(t) \leq (\psi(t))^{1-n} \Gamma(\alpha)\chi(t) + (\psi(t))^{1-n} \Xi(t,T_1)
$$

+ (\psi(t))^{1-n} \int_{T_1}^t \psi(s)(\psi(t) - \psi(s))^{\alpha-1} r(s) ds.
$$

(39)
Take \( T_2 > T_1 \). We consider two cases:

**Case (1):** Let \( 0 < \alpha \leq 1 \). Then \( n = 1 \) and

\[
|\psi(t)|^{1-n}\Gamma(\alpha)\chi(t) = |\Gamma(\alpha)x(a)| \leq c_6.
\] (40)

Using inequality (31), we get

\[
|\psi(t)|^{1-n}\Xi(t, T_1) = |\Xi(t, T_1)| \leq c_5(T_1).
\] (41)

Then using inequality (39), we get

\[
(\psi(t))^{1-n}\int_{T_1}^{T} \psi'(s)(\psi(t) - \psi(s))^{n-1}r(s)ds \geq -[c_6 + c_5(T_1)].
\]

Since the right hand side of the last inequality is a negative constant, we conclude that

\[
\liminf_{t \to \infty}(\psi(t))^{1-n}\int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{n-1}r(s)ds \geq -[c_6 + c_5(T_1)] > -\infty,
\]

which contradicts condition (36).

**Case (2):** Let \( \alpha > 1 \). Then \( n \geq 2 \). Also, \( (1 - \frac{\psi(a)}{\psi(t)})^{n-1} \leq 1 \) and the function \( h_2(t) = (\psi(t) - \psi(a))^{1-j} \) is decreasing for \( j > 1 \). Then for \( t \geq T_2 \), we have

\[
|\psi(t)|^{1-n}\Gamma(\alpha)\chi(t) = |\Gamma(\alpha)\sum_{k=0}^{n-1} \frac{\psi(t) - \psi(a)}{k!} D_t^k x(a) |
\leq \Gamma(\alpha)\sum_{k=0}^{n-1} \frac{(\psi(t) - \psi(a))^{k-n+1}}{k!} \psi D_t^k x(a)
\leq \Gamma(\alpha)\sum_{k=0}^{n-1} \frac{\psi(T_2) - \psi(a))^{k-n+1}}{k!} \psi D_t^k x(a) : = c_7(T_2)
\]

Also, since \( (1 - \frac{\psi(a)}{\psi(t)})^{n-1} \leq 1 \) for \( n \geq 2 \) and similar to inequality (21) we get

\[
|\psi(t)|^{1-n}\Xi(t, T_1) \leq c_4(T_1).
\] (43)

Then, using inequality (39) we get a contradiction with condition (36).

Therefore, we conclude that \( x(t) \) is oscillatory. In case \( x(t) \) is eventually negative, similar arguments lead to a contradiction with condition (37).

**Theorem 4.3.** Suppose that assumptions (10) and (11) hold with \( \beta > \gamma \). If

\[
\liminf_{t \to \infty}(\psi(t))^{1-n}\int_{T}^{t} \psi'(s)(\psi(t) - \psi(s))^{n-1}[r(s) + H(s)]ds = -\infty
\] (44)

and

\[
\limsup_{t \to \infty}(\psi(t))^{1-n}\int_{T}^{t} \psi'(s)(\psi(t) - \psi(s))^{n-1}[r(s) - H(s)]ds = \infty,
\] (45)

for every sufficiently large \( T \), where \( H \) is defined by Eq. (24), then every solution of Eq. (32) is oscillatory.
Proof. Let \( x(t) \) be a nonoscillatory solution of equation (32), say, \( x(t) > 0 \) for \( t \geq T_1 > a \). Let \( s \geq T_1 \). Using assumptions (10) and (11) and applying Eq. (25), we obtain

\[
f_2(s, x) - f_1(s, x) \leq p_2(s)x^\gamma(s) - p_1(s)x^\beta(s) \leq H(s).
\]

Then Eq. (7) implies

\[
\Gamma(\alpha)x(t) = \Gamma(\alpha)\chi(t) + \Xi(t, T_1) + \int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + f_2(s, x(s)) - f_1(s, x(s))]ds
\]

\[
\leq \Gamma(\alpha)\chi(t) + \Xi(t, T_1) + \int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + H(s)]ds
\]

(46)

The rest of the proof is the same as in that of Theorem 4.2.

**Theorem 4.4.** Suppose that assumptions (10) and (12) hold with \( \beta < \gamma \). If

\[
\limsup_{t \to \infty}(\psi(t))^{1-n} \int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + H(s)]ds = \infty
\]

(47)

and

\[
\liminf_{t \to \infty}(\psi(t))^{1-n} \int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) - H(s)]ds = -\infty,
\]

(48)

for every sufficiently large \( T \), where \( H \) is defined by Eq. (24), then every bounded solution of Eq. (32) is oscillatory.

**Proof.** Let \( x(t) \) be a bounded nonoscillatory solution of Eq. (32). Then inequality (29) is valid for some constants \( M_1 \) and \( M_2 \). Assume that \( x \) is a bounded eventually positive solution of Eq. (1). Then there exists \( T_1 > a \) such that \( x(t) > 0 \) for \( t \geq T_1 > a \). Following the first part of the proof of Theorem 3.4, the following inequality is still valid.

\[
f_2(s, x) - f_1(s, x) \geq p_2(s)x^\gamma(s) - p_1(s)x^\beta(s) \geq H(s).
\]

Similar to the inequality (30), we have the following

\[
(\psi(t))^{1-n}\Gamma(\alpha)x(t) \geq (\psi(t))^{1-n}\Gamma(\alpha)\chi(t) + ((\psi(t))^{1-n}\Xi(t, T_1) +
\]

\[
+ (\psi(t))^{1-n} \int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + H(s)]ds
\]

(49)

Take \( T_2 > T_1 \). We consider two cases:

**Case (1):** Let \( 0 < \alpha \leq 1 \). Similar to case(1) in Theorem 4.2, Eq. (40) and Eq. (41) are still valid. Now, using inequality (29), we get

\[
M_2 \geq x(t) \geq -c_6 - c_5(T_1) + \int_{T_1}^{t} \psi'(s)[r(s) + H(s)]ds,
\]

for \( t \geq T_2 \). Thus, we have

\[
\limsup_{t \to \infty}(\psi(t))^{1-n} \int_{T_1}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + H(s)]ds =
\]

\[
\limsup_{t \to \infty} \int_{T_1}^{t} \psi'(s)[r(s) + H(s)]ds \leq c_6 + c_5(T_1) + M_2 < \infty,
\]

which contradicts condition (47).

**Case (2):** Let \( \alpha > 1 \). Then (42) and (43) are still true. Hence using inequality
(29), we obtain from inequality (49)
\[
\Gamma(\alpha)(\psi(t))^{1-n}M_2 \geq \Gamma(\alpha)(\psi(t))^{1-n}x(t) \geq -c_7(T_2) - c_4(T_1) + (\psi(t))^{1-n} \int_{T_1}^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + H(s)]ds,
\]
for \( t \geq T_2 \). Since \( \psi(t) \) is an increasing function, we have \( \lim_{t \to \infty} (\psi(t))^{1-n} = 0 \) for \( n > 1 \). Then
\[
\limsup_{t \to \infty} (\psi(t))^{1-n} \int_{T_1}^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + H(s)]ds \leq c_7(T_2) + c_4(T_1) < \infty,
\]
which contradicts condition (47).
Therefore, we conclude that \( x(t) \) is oscillatory. In case \( x(t) \) is eventually bounded negative, similar arguments lead to a contradiction with condition (48). \( \square \)

**Remark 2.** If we let \( \psi(t) = t \), \( \psi(t) = \ln t \) and \( \psi(t) = \frac{(t-a)^\gamma}{\rho} \), then we recover the RL and Hadamard fractional oscillation results in the frame of Caputo in [12], [3] and [2], respectively.

5. **Examples.** In this section, we construct numerical examples to illustrate the effectiveness of our theoretical results.

5.1. **Example 1.** This is an example about Theorem 3.3. Consider the following General fractional differential equation in Riemann setting
\[
\begin{align*}
\left\{ \begin{array}{l}
D_0^\alpha \psi(x(t)) + x^\alpha(t) \ln(t + e) &= \frac{\Gamma(1.5)}{\Gamma(1.5 - \alpha)}(\psi(t) - \psi(a))^{0.5-\alpha} \\
+ (\psi(t) - \psi(a))^{5/2} - (\psi(t) - \psi(a))^{1/6}] \ln(t + e) + x^{1/3}(t) \ln(t + e),
\end{array} \right.
\end{align*}
\]
where \( 0 < \alpha < 1 \), \( \psi(t) = t^2 \), \( t > 0 \) and \( a = 0 \). Then comparing with Eq. (1), we have \( n = 1 \), \( f_1(t, x) = x^2(t) \ln(t + e) \), \( f_2(t, x) = x^{1/3}(t) \ln(t + e) \) and \( r(t) = \frac{\Gamma(1.5)}{\Gamma(1.5 - \alpha)}(\psi(t) - \psi(a))^{0.5-\alpha} + (\psi(t) - \psi(a))^{5/2} - (\psi(t) - \psi(a))^{1/6}] \ln(t + e) \).
It is easy to verify that conditions (10) and (11) are satisfied for \( \beta = 5 \), \( \gamma = \frac{1}{6} \) and \( p_1(t) = p_2(t) = \ln(t + e) \). However, we show in the following that condition (22) does not hold. For every sufficiently large \( T \geq 1 \) and all \( t \geq T \), we have \( r(t) > 0 \).
Calculating \( H(s) \) as defined by Eq. (24), we find that \( H(s) = 14(15)^{-1/4} \ln(s + e) \geq 0.77 \). Then using Eq. (2) and Lemma 2.1 for \( \mu = 1 \) we get
\[
\begin{align*}
\liminf_{t \to \infty} (\psi(t))^{1-\alpha} \int_T^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[r(s) + H(s)]ds & \geq \\
\liminf_{t \to \infty} (\psi(t))^{1-\alpha} \int_T^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}H(s)ds & \geq \\
\liminf_{t \to \infty} 0.77(\psi(t))^{1-\alpha} \int_T^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}(\psi(s) - \psi(a))^{\alpha}ds & = \\
\liminf_{t \to \infty} 0.77(\psi(t))^{1-\alpha} \Gamma(\alpha) H_{a, t}^\alpha(\psi(t) - \psi(a))^{\alpha} & = \\
\liminf_{t \to \infty} 0.77 t^{2-2\alpha} \frac{\Gamma(\alpha)}{\Gamma(1 + \alpha)} t^{2\alpha} & = \liminf_{t \to \infty} 0.77 t^{2-2\alpha} \frac{\Gamma(\alpha)}{\Gamma(1 + \alpha)} = \infty.
\end{align*}
\]
However, for $\psi(t) = t^2$ and $a = 0$, $x(t) = (\psi(t) - \psi(a))^{1/2}$ is a nonoscillatory solution of Eq. (50). The initial condition is also satisfied because

$$D^\alpha_a \psi(t) = D^\alpha_a (t^2 - 0)^{1/2} = \frac{\Gamma(1.5)}{\Gamma(2.5 - \alpha)} t^{3 - 2\alpha}.$$

### 5.2. Example 2

This example supports Theorem 3.4. Consider the following General fractional differential equation of Caputo type

$$\left\{ \begin{array}{l}
C^\alpha D^\alpha_a \psi x(t) + e^t x^3(t) = \frac{2}{\Gamma(3 - \alpha)} t^{6 - 3\alpha} + t^{18} e^t, \quad t > 0 \\
x(a) = 0, \quad 0 < \alpha < 1,
\end{array} \right.$$  \hspace{1cm} (51)

where $\psi(t) = t^3$ and $a = 0$. Comparing with Eq. (32), we have $n = 1$, $f_1(t, x) = e^t x^3(t)$, $r(t) = \frac{2}{\Gamma(3 - \alpha)} t^{6 - 3\alpha} + t^{18} e^t$ and $f_2(t, x) = 0$. Then, condition (10) is satisfied. Since $r(s) > 0$ and using Eq. (2), we get

$$\liminf_{t \to \infty} (\psi(t))^{1-n} \int_T^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} r(s) ds \geq$$

$$\liminf_{t \to \infty} (\psi(t))^{1-n} \int_T^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} (\psi(s) - \psi(a))^{\alpha} ds =$$

$$\liminf_{t \to \infty} (\psi(t))^{1-n} \Gamma(\alpha) L^\alpha_a (\psi(t) - \psi(a))^{\alpha} =$$

$$\liminf_{t \to \infty} t^{3(1-n)} \Gamma(\alpha) \left( \frac{\Gamma(1)}{\Gamma(1 + \alpha)} (\psi(t) - \psi(a))^{\alpha} \right) =$$

$$\liminf_{t \to \infty} \frac{t^{3\alpha}}{\alpha} = \infty.$$

Which means that condition (36) does not hold. However, $x(t) = (\psi(t) - \psi(a))^{2}$ is a nonoscillatory solution of Eq. (51).

### 6. Conclusion

In this article, the oscillation theory for general fractional differential equations was studied. Sufficient conditions for the oscillation of solutions of general fractional differential equations in Riemann setting Eq. (1) were given in three theorems in Section 3. The main approach is based on applying Young’s inequality which will help us in obtaining sharper conditions. The oscillation for the general fractional differential equations in Caputo setting have been investigated as well. Examples have presented to demonstrate the effectiveness of the obtained results. In the examples we have chosen the function $\psi$ of quadratic and cubic polynomial type in order to treat fractional derivatives other than RL ($\psi(t) = t$) and Hadamard ($\psi(t) = \ln t$) or their Caputo modifications.

### Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Acknowledgments

The authors would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.
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Received November 2019; revised March 2020.

E-mail address: babdallah@psu.edu.sa
E-mail address: tabdeljawad@psu.edu.sa