The Triton in a Finite Volume

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Overview

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   • Partial Waves

3 Results
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Introduction

- **Aim:** Understanding of the volume dependence of the triton binding energy

- Two-body systems: Volume-dependence well known
  [Lüscher 86, 91; Beane et al. 03]

- Extraction of infinite volume scattering parameters possible

- NN-scattering lengths found to be of natural size for unphysical pion masses [Beane et al., 10]
First Lattice results in the triton channel have recently become available, but no properties were extracted [Beane et al. 09]

Three-body systems with unnaturally large scattering length show universal properties

Described by EFT using only contact interactions

Three identical bosons previously studied as a model case [SK & Hammer 09, 10]

Calculate changes to the bound state spectrum in finite volume
Formalism
Lagrangian

- Effective Lagrangian for a system of three nucleons in zero-range limit (cf., e.g., [Bedaque, Hammer, van Kolck 99])

\[
\mathcal{L} = N^\dagger \left( i\partial_t + \frac{1}{2} \nabla^2 \right) N + \frac{g_t}{2} t_j^\dagger t_j + \frac{g_s}{2} s_A^\dagger s_A \\
- \frac{g_t}{2} \left( t_j^\dagger (N^T \tau_2 \sigma_j \sigma_2 N) + \text{h.c.} \right) - \frac{g_s}{2} \left( s_A^\dagger (N^T \sigma_2 \tau_A \tau_2 N) + \text{h.c.} \right) \\
+ \mathcal{L}_3
\]

- \( t_j, s_A \): auxiliary dinucleon fields in the \( ^3S_1 (^1S_0) \) channel

- \( \mathcal{L}_3 \sim (N^\dagger N)^3 \): Wigner SU(4)-symmetric 3-body contact interaction with a cutoff dependent coupling constant \( H(\Lambda) \) needed to renormalize the \( S = \frac{1}{2} \) sector
Renormalization

- Infrared finite-volume physics vs. renormalization in UV → Perform renormalization in infinite volume

- Regulate loop integrals via a cutoff $\Lambda$

- Match 2-body couplings $g_{s,t}$ with 2-body scattering lengths $a_{s,t}$

- Log-periodic dependence of $H(\Lambda)$ on the cutoff due to limit cycle

- One additional 3-body input is needed

- Renormalization in finite volume will be explicitly verified
Bound State Amplitude

- Compare residues of bound state poles in scattering amplitude
- Two coupled integral equations for two bound state amplitudes

\[
\frac{A_1}{A_2} = \frac{A_3}{A_4} + \frac{A_5}{A_6} = \frac{A_7}{A_8} + \frac{A_9}{A_10}
\]

- Energies for which integral equations are solvable $\rightarrow$ binding energies
- Calculation of the finite volume dinucleon propagator:
  \[
  D_{s,t}(E) = \frac{32\pi}{g^2_{s,t}} \left[ \frac{1}{a_{s,t}} - \sqrt{-E} + \frac{1}{L} \sum_{\vec{j} \neq \vec{0}} \frac{1}{|\vec{j}|^2} e^{-|\vec{j}|\sqrt{-EL}} \right]^{-1}
  \]
  - Reduces indeed to the infinite volume dinucleon propagator for $L \rightarrow \infty$
- Loop momenta are quantized $\rightarrow$ Rewrite using Poisson’s sum equation
Partial Waves

Integral equation for bound state amplitude $\mathcal{F}$

\[
\begin{pmatrix}
\mathcal{F}_t(\vec{p}) \\
\mathcal{F}_s(\vec{p})
\end{pmatrix}
= \frac{1}{\pi^2} \sum_{\vec{n} \in \mathbb{Z}^3} \int_0^\Lambda d^3\vec{y} \ e^{i\vec{n} \cdot \vec{y}} \left[ M(\vec{y}) \mathcal{Z}(\vec{p}, \vec{y}) + N(\vec{y}) \frac{2H(\Lambda)}{\Lambda^2} \right] \begin{pmatrix}
\mathcal{F}_t(\vec{y}) \\
\mathcal{F}_s(\vec{y})
\end{pmatrix}
\]

$M(\vec{y}) = \begin{pmatrix}
-d_t(\vec{y}) & 3d_s(\vec{y}) \\
3d_t(\vec{y}) & -d_s(\vec{y})
\end{pmatrix}$, \hspace{1cm} $N(\vec{y}) = \begin{pmatrix}
-d_t(\vec{y}) & d_s(\vec{y}) \\
d_t(\vec{y}) & -d_s(\vec{y})
\end{pmatrix}$

$\mathcal{Z}(\vec{p}, \vec{y}) = (p^2 + \vec{p} \cdot \vec{y} + y^2 - E)^{-1}$, \hspace{1cm} $d_{s,t}(y) = \frac{g_{s,t}^2}{8\pi} D_{s,t}(E - \frac{y^2}{2}, \vec{y})$

- Bound state amplitudes in $G_1$ representation of the double cover of the cubic group
- Expand angular dependence of amplitudes in terms of basis functions $|jm\rangle$ (see, e.g., [Bernard et al. 08])
- Perform angular integration and project on $J$th partial wave
Partial Waves

Coupled integral equations for partial waves

\[
\begin{pmatrix}
F_t^{(J)}(y) \\
F_s^{(J)}(y)
\end{pmatrix} = \frac{4}{\pi} \int_0^\Lambda dy \, y^2 \sum_j \left[ M(y) \, Z^{(\ell(J))}(p, y) + N(y) \, \frac{2H(\Lambda)}{\Lambda^2} \delta_{\ell(J),0} \right]
\]

\[
\times \left[ \delta_{Jj} + \sum_{\vec{n} \in \mathbb{Z}^3 \backslash \{0\}} \sum_{\ell'} i^{\ell'} j_{\ell'}(L|\vec{n}|y) \sqrt{\frac{(2\ell(j) + 1)(2\ell' + 1)}{2\ell(J) + 1}} \right]
\]

\[
\times \sum_{m(\ell(j)), s(\frac{1}{2})} \frac{\tilde{C}_{j,m+s}}{\tilde{C}_{JM}} Y^*_{\ell'}(M-s-m)(\hat{n})(C-G \text{ coeffs}) \left( \begin{array}{c}
F_t^{(j)}(y) \\
F_s^{(j)}(y)
\end{array} \right)
\]

\[
Z^{(\ell)}(p, y) = \frac{(2\ell+1)}{py} Q_{\ell} \left( \frac{p^2 + y^2 - E}{py} \right), \quad Q_{\ell}: \text{Legendre function of the 2nd kind}
\]

\[
j_{\ell}: \text{spherical Bessel function}
\]
Specialization to $J = \frac{1}{2}$

Result for $J = \frac{1}{2}$

\[
\begin{pmatrix}
F_{t}^{(\frac{1}{2})}(y) \\
F_{s}^{(\frac{1}{2})}(y)
\end{pmatrix}
= \frac{4}{\pi} \int_{0}^{\Lambda} dy \, y^2 \sum_{j} \left( G_{1}^{+} \right)
\left[
M(y) \, Z^{(0)}(p, y) + N(y) \, \frac{2H(\Lambda)}{\Lambda^2}
\right]
\begin{pmatrix}
F_{t}^{(j)}(y) \\
F_{s}^{(j)}(y)
\end{pmatrix}
\times \left[ \delta_{j, \frac{1}{2}} + \sum_{\vec{n} \in \mathbb{Z}^3 \, \vec{n} \neq \vec{0}} \sqrt{4\pi i} \, j_{\ell(j)}(L | \vec{n}| y)
\right]
\times \sum_{m(\ell(j))} (-1)^m \tilde{C}_{j,m} C_{\ell(j)m}^{j M + \frac{1}{2} M} Y_{\ell(j)m}(\hat{n})
\]

\[
Z^{(0)}(p, y) = \frac{1}{2py} \ln \left( \frac{p^2 + py + y^2 - E}{p^2 - py + y^2 - E} \right)
\]

- First approach: Neglect higher partial waves
Results
The Triton in Finite Volume

- $E_3^{\infty} = -8.4818$ MeV
- Size of the triton $\sim 2$ fm
- Results are renormalized
- Shift at volumes typical in Lattice QCD already more than 100%

- Fit of the form $E_3(L) = E_3(L = \infty) \left[ 1 + \frac{c}{L} e^{-L/L_0} \right]$  

- Comparison to data from Chiral EFT on the lattice [Epelbaum et al, 10]: study higher partial waves, higher orders
Corrections – Higher Partial Waves

- Corrections from higher partial waves: More important for smaller volumes
- Inclusion straightforward but numerically expensive
- Estimate from case of three identical bosons: 20% for volumes three times larger than the state itself

| $L/a$ | $E_3(L)\, ma^2$, s-wave only | $E_3(L)\, ma^2$, $\ell = 0, 4$ | $\delta_{rel}$ |
|-------|-------------------------------|---------------------------------|---------------|
| $\infty$ | -5.05 | N/A | N/A |
| 1 | -11.1 | -11.8 | 6% |
| 0.7 | -19.0 | -20.7 | 9% |

[SK & Hammer, EPJ A43 (2010) 229]
Corrections – EFT

- NLO of the EFT: Corrections of types $r_e/a$ and $kr_e$
- First type dominated by spin-triplet channel, about 30%
- Estimate for typical momentum in three-body bound states:
  \[ \frac{2}{3} \sqrt{mB_3} \rightarrow 90 \text{ MeV for infinite volume triton} \]
- Second type about 40% for large volumes, growing with binding energy
- Inclusion of higher orders needed for precise extrapolations
- Infinite volume: Corrections up to $N^2$LO under control
  \[ \rightarrow \text{Extension of finite volume framework straightforward} \]
Pion-Mass Dependence – Introduction

- Lattice QCD calculations are performed at unphysical pion masses
- Conjectured closeness of QCD to an infrared limit cycle in 3N-sector
- $a_s(M_\pi) = a_t(M_\pi) = \infty$ compatible w/ $\chi$EFT near $M_\pi = 197$ MeV
- Efimov effect: Excited states of the triton appear

![Graph](image)

[Hammer, Phillips, Platter, 2007]

- Pion-mass dependence of observables under control in $\chi$EFT
  → Obtain input data by chiral extrapolation
Pion-Mass Dependence – Ground State

Shift to more negative energies, small effect on the slopes
Pion–Mass Dependence – Spectrum

Excited state crosses threshold!
“Crossing volume” predictable from universality?

Simon Kreuzer (Uni Bonn)
Universality of Finite Volume Corrections?

- Form dimensionless number \( r = -mE_3 \infty L^2_{100\%} \)

- Results for ground state:

| \( m_\pi \) | \( E_3 \infty / \text{MeV} \) | \( E_3 \infty / B_2 \) | \( L_{100\%} / \text{fm} \) | \( r \) |
|---|---|---|---|---|
| phys. | 8.48 | 3.8 | 5.62 | 6.47 |
| 190 | 4.31 | 93 | 7.62 | 6.03 |
| 195 | 3.85 | 613 | 8.05 | 6.01 |
| 196 | 3.74 | 1,407 | 8.13 | 5.97 |
| 197 | 3.65 | 6,374 | 8.25 | 5.98 |
| 200 | 3.41 | 26,368 | 8.52 | 5.97 |
| 205 | 3.14 | 2,092 | 8.82 | 5.90 |

- Hints for universal behavior away from threshold

- Formula for binding energies in infinite volume [Efimov 79]

\[
mE_3 + 1/a^2 = (515)^{n-n^*} \exp[\Delta(\xi)/s_0] \kappa^2, \quad \tan \xi = -a \sqrt{mE_3}
\]

- Is there a similar formula for finite volume states with a \( \Delta(\xi, L) \)?
Summary

- Derivation of integral equations for partial waves of 3-nucleon bound state amplitude

- Numerical solution of the equations → modifications of the spectrum

- Renormalization in finite volume verified explicitly

- Infinite volume extrapolation for the triton is possible

- Calculated pion-mass dependence of triton ground and excited state in finite volume
  → Ground state: dominant shift to more negative energies, small effect on slopes
  → Excited states: Cross threshold

- Access to scattering phase shifts a la Lüscher implicit in the results
Outlook

- Include higher partial waves
- Include next-to-leading order of the EFT
- Universality in the finite volume?
  - Efimov equation for binding energies in finite volume?
  - “Crossing volume” for shallow states predictable?
- Extend formalism to include nucleon-deuteron scattering
Outlook

- Include higher partial waves
- Include next-to-leading order of the EFT
- Universality in the finite volume?
  - Efimov equation for binding energies in finite volume?
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- Extend formalism to include nucleon-deuteron scattering

Thank you for your attention!
Bonus material
Introduction (2)

- Three-body systems with large scattering length show universal properties
- Described by EFT using only contact interactions
- Efimov effect: Sequence of three-body bound states [Efimov 70]
  Signature of ultraviolet limit cycle in renormalization of EFT
  
  ![Diagram showing three-body states](image)

- Calculate changes to the bound state spectrum in finite volume
Pion-Mass Dependence – Excited state

- $M_\pi = 190$ MeV:

| $L$/fm | $B^{(1)}_3$/MeV | $B_2$/MeV |
|--------|----------------|-----------|
| $\infty$ | 0.052 | 0.047 |
| 19.7 | 0.742 | 0.486 |
| 14.8 | 0.890 | 0.823 |
| 14.4 | 0.884 | 0.865 |
| 14.2 | N/A | 0.888 |

- $M_\pi = 195$ MeV:

| $L$/fm | $B^{(1)}_3$/MeV | $B_2$/MeV |
|--------|----------------|-----------|
| $\infty$ | 0.016 | 0.006 |
| 19.7 | 0.686 | 0.431 |
| 14.8 | 0.761 | 0.753 |
| 14.4 | N/A | 0.794 |

- $M_\pi = 197$ MeV:

| $L$/fm | $B^{(1)}_3$/MeV | $B_2$/MeV |
|--------|----------------|-----------|
| $\infty$ | 0.009 | $5.7 \times 10^{-4}$ |
| 39.5 | 0.184 | 0.105 |
| 27.6 | 0.240 | 0.211 |
| 26.4 | 0.233 | 0.231 |
| 26.2 | N/A | 0.234 |

- $M_\pi = 200$ MeV:

| $L$/fm | $B^{(1)}_3$/MeV | $B_2$/MeV |
|--------|----------------|-----------|
| $\infty$ | 0.038 | $1.3 \times 10^{-4}$ |
| 29.6 | 0.355 | 0.182 |
| 19.7 | 0.625 | 0.407 |
| 15.6 | 0.662 | 0.651 |
| 15.4 | N/A | 0.668 |