Non-Archimedean meromorphic solutions of functional equations

Pei-Chu Hu & Yong-Zhi Luan∗

Abstract
In this paper, we discuss meromorphic solutions of functional equations over non-Archimedean fields, and prove analogues of the Clunie lemma, Malmquist-type theorem and Mokhon’ko theorem.

1 Introduction

Let $\kappa$ be an algebraically closed field of characteristic zero, complete for a non-trivial non-Archimedean absolute value $| \cdot |$. Let $\mathcal{A}(\kappa)$ (resp. $\mathcal{M}(\kappa)$) denote the set of entire (resp. meromorphic) functions over $\kappa$. As usual, if $R$ is a ring, we use $R[X_0, X_1, ..., X_n]$ to denote the ring of polynomials of variables $X_0, X_1, ..., X_n$ over $R$. We will use the following assumption:

(A) Fix a positive integer $n$. Take $a_i, b_i$ in $\kappa$ such that $|a_i| = 1$ for each $i = 0, 1, ..., n$, and such that
$$L_i(z) = a_i z + b_i \ (i = 0, 1, ..., n)$$
are distinct, where $a_0 = 1, b_0 = 0$. Let $f$ be a non-constant meromorphic function over $\kappa$ and write
$$f_i = f \circ L_i, \ i = 0, 1, ..., n$$
with $f_0 = f$. Take non-zero elements
$$B \in \mathcal{M}(\kappa)[X]; \ \Omega, \Phi \in \mathcal{M}(\kappa)[X_0, X_1, ..., X_n].$$

Under the assumption (A), there exist $\{b_0, ..., b_q\} \subset \mathcal{M}(\kappa)$ with $b_q \neq 0$ such that
$$B(X) = \sum_{k=0}^{q} b_k X^k. \quad (1)$$

Mathematics Subject Classification 2000 (MSC2000). Primary 11S80, 12H25; Secondary 30D35.

*The work of first author was partially supported by National Natural Science Foundation of China (Grant No. 11271227), and supported partially by PCSIRT (IRT1264).

Key words and phrases: meromorphic solutions, functional equations, Nevanlinna theory
Similarly, write
\[ \Omega(X_0, X_1, \ldots, X_n) = \sum_{i \in I} c_i X_0^{i_0} X_1^{i_1} \cdots X_n^{i_n}, \]  
(2)
where \( i = (i_0, i_1, \ldots, i_n) \) are non-negative integer indices, \( I \) is a finite set, \( c_i \in \mathcal{M}(\kappa) \), and
\[ \Phi(X_0, X_1, \ldots, X_n) = \sum_{j \in J} d_j X_0^{j_0} X_1^{j_1} \cdots X_n^{j_n}, \]  
(3)
where \( j = (j_0, j_1, \ldots, j_n) \) are non-negative integer indices, \( J \) is a finite set, \( d_j \in \mathcal{M}(\kappa) \).

In this paper, we will use the symbols from [8] on value distribution of meromorphic functions. For example, let \( \mu(r, f) \) denote the maximum term of power series for \( f \in \mathcal{A}(\kappa) \) and its fractional extension to \( \mathcal{M}(\kappa) \), \( m(r, f) \) the compensation (or proximity) function of \( f \), \( N(r, f) \) the valence function of \( f \) for poles, and the characteristic function of \( f \)
\[ T(r, f) = m(r, f) + N(r, f). \]

Now we can state our results as follows:

Theorem 1.1. Assume that the condition (A) holds. If \( f \) is a solution of the following functional equation
\[ B(f)\Omega(f, f_1, \ldots, f_n) = \Phi(f, f_1, \ldots, f_n) \]  
(4)
with \( \deg B \geq \deg \Phi \), then
\[ m(r, \Omega) \leq \sum_{i \in I} m(r, c_i) + \sum_{j \in J} m(r, d_j) + \log \left( r, \frac{1}{b_q} \right) + l \sum_{j=0}^{q} m(r, b_j), \]  
(5)
where \( l = \max\{1, \deg \Omega\} \), \( \Omega = \Omega(f, f_1, \ldots, f_n) \). Further, if \( \Phi \) is a polynomial of \( f \), we also have
\[ N(r, \Omega) \leq \sum_{i \in I} N(r, c_i) + \sum_{j \in J} N(r, d_j) + O \left( \sum_{j=0}^{q} N \left( r, \frac{1}{b_j} \right) \right). \]  
(6)

Theorem 1.1 is a difference analogue of the Clunie lemma over non-Archimedean fields (cf. [8]). R. G. Halburd and R. J. Korhonen [4] obtained a difference analogue of the Clunie lemma over the field of complex numbers (cf. [2]). Theorem 1.1 has numerous applications in the study of non-Archimedean difference equations, and beyond. To state one of its applications, we need the following notation:

Definition 1.2. A solution \( f \) of (4) is said to be admissible if \( f \in \mathcal{M}(\kappa) \) satisfies (4) with
\[ \sum_{i \in I} T(r, c_i) + \sum_{j \in J} T(r, d_j) + \sum_{k=0}^{q} T(r, b_k) = o(T(r, f)), \]  
(7)
equivalently, the coefficients of \( B, \Phi, \Omega \) are slowly moving targets with respect to \( f \).

If all \( c_i, d_j, b_k \) are rational functions, each transcendental meromorphic function \( f \) over \( \kappa \) must satisfy (7), which means that each transcendental meromorphic solution \( f \) over \( \kappa \) is admissible.
**Theorem 1.3.** If $\Phi$ is of the form

$$\Phi(f, f_1, \ldots, f_n) = \Phi(f) = \sum_{j=0}^{p} d_j f^j,$$

and if (4) has an admissible non-constant meromorphic solution $f$, then

$$q = 0, \quad p \leq \deg(\Omega).$$

Theorem 1.3 is a difference analogue of a Malmquist-type theorem over non-Archimedean fields (cf. [8]). Malmquist-type theorems were obtained by Malmquist [10], Gackstatter-Laine [3], Laine [9], Toda [12], Yosida [13] (or see He-Xiao [5]) for meromorphic functions on $\mathbb{C}$, and Hu-Yang [7] or [6] for several complex variables.

**Corollary 1.4.** Assume that the condition (A) holds such that the coefficients of $B, \Omega, \Phi$ are rational functions over $\kappa$, and such that $\Phi$ has the form in Theorem 1.3. If (4) has a transcendental meromorphic solution $f$ over $\kappa$, then $\Phi/B$ is a polynomial in $f$ of degree $\leq \deg(\Omega)$.

Corollary 1.4 is a difference analogue of the non-Archimedean Malmquist-type theorem due to Boutabaa [1].

**Theorem 1.5.** Let $f \in \mathcal{M}(\kappa)$ be a non-constant admissible solution of

$$\Omega(f, f', \ldots, f^{(n)}) = 0,$$  \hspace{1cm} (8)

where the solution $f$ is called admissible if

$$\sum_{i \in I} T(r, c_i) = o(T(r, f)).$$

If a slowly moving target $a \in \mathcal{M}(\kappa)$ with respect to $f$, that is,

$$T(r, a) = o(T(r, f)),$$

does not satisfy the equation (8), then

$$m \left( r, \frac{1}{f - a} \right) = o(T(r, f)).$$

Theorem 1.5 is an analogue of a result due to A. Z. Mokhon’ko and V. D. Mokhon’ko [11] over non-Archimedean fields, which also has a difference analogue as follows:

**Theorem 1.6.** Assume that the condition (A) holds. Let $f \in \mathcal{M}(\kappa)$ be a non-constant admissible solution of

$$\Omega(f, f_1, \ldots, f_n) = 0,$$  \hspace{1cm} (9)

where the solution $f$ is called admissible if

$$\sum_{i \in I} T(r, c_i) = o(T(r, f)).$$

If a slowly moving target $a \in \mathcal{M}(\kappa)$ with respect to $f$ does not satisfy the equation (9), then

$$m \left( r, \frac{1}{f - a} \right) = o(T(r, f)).$$

A version of Theorem 1.6 over complex number field can be found in [4].
2 Difference analogue of the Lemma on the Logarithmic Derivative

Take $a(\neq 0), b \in \kappa$ and consider the linear transformation

$$L(z) = az + b$$

over $\kappa$. For a positive integer $m$, set

$$\Delta_L f = f \circ L - f, \quad \Delta^m_L f = \Delta_L(\Delta^{m-1}_L f).$$

**Lemma 2.1.** Take $f \in \mathcal{A}(\kappa)$ and assume $|a| \leq 1$. When $r > |b|/|a|$, we have

$$\mu(r, f \circ L) \leq \mu(r, f).$$

Moreover, we obtain

$$\mu\left(r, \frac{f \circ L}{f}\right) \leq 1, \quad \mu\left(r, \frac{\Delta^m_L f}{f}\right) \leq 1.$$ 

**Proof.** We can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

since $f \in \mathcal{A}(\kappa)$. Therefore

$$f(L(z)) = \sum_{n=0}^{\infty} a_n (az + b)^n.$$

First of all, we take $r \in |\kappa|$, that is, $r = |z|$ for some $z \in \kappa$. When $r > |b|/|a|$, we find (cf. [5])

$$\mu(r, f \circ L) = |f(L(z))| \leq \max_{n \geq 0} |a_n||az + b|^n = \max_{n \geq 0} |a_n||az|^n \leq \max_{n \geq 0} |a_n||z|^n = \mu(r, f).$$

In particular,

$$\mu\left(r, \frac{f \circ L}{f}\right) = \frac{\mu(r, f \circ L)}{\mu(r, f)} \leq 1,$$

and hence

$$\mu\left(r, \frac{\Delta_L f}{f}\right) = \frac{\mu(r, f \circ L - f)}{\mu(r, f)} \leq \frac{1}{\mu(r, f)} \max\{\mu(r, f \circ L), \mu(r, f)\} \leq 1.$$ 

By induction, we can prove

$$\mu\left(r, \frac{\Delta^m_L f}{f}\right) \leq 1.$$ 

Since $|\kappa|$ is dense in $\mathbb{R}_+ = [0, \infty)$, by using continuity we easily see that these inequalities hold for all $r > |b|/|a|$.

\[ \square \]
Note that (cf. [8])
\[ m(r, f) = \log^+ \mu(r, f) = \max\{0, \log \mu(r, f)\}. \] (10)

Lemma 2.1 implies immediately the following difference analogue of the Lemma on the Logarithmic Derivative:

**Corollary 2.2.** Take \( f \in \mathcal{A}(\kappa) \) and assume \(|a| \leq 1\). When \( r > |b|/|a| \), we have
\[
m \left( r, \frac{f \circ L}{f} \right) = 0, \quad m \left( r, \frac{\Delta^m f}{f} \right) = 0.
\]

**Lemma 2.3.** Take \( f \in \mathcal{M}(\kappa) - \{0\} \) and assume \(|a| = 1\). When \( r > |b|/|a| \), we have
\[
\mu(r, f \circ L) = \mu(r, f).
\] (11)

Moreover, we obtain
\[
\mu \left( r, \frac{f \circ L}{f} \right) = 1, \quad \mu \left( r, \frac{\Delta^m f}{f} \right) \leq 1.
\]

**Proof.** Since \( f \in \mathcal{M}(\kappa) - \{0\} \), there are \( g, h \neq 0 \in \mathcal{A}(\kappa) \) with \( f = \frac{g}{h} \). Thus (cf. [8])
\[
\mu(r, f \circ L) = \frac{\mu(r, g \circ L)}{\mu(r, h \circ L)}.
\] (12)

Take \( r \in |\kappa| \). Since \(|a| = 1\), we have
\[
|L(z)| = |az + b| = |z| = r
\]
when \( r > |b|/|a| \), and so
\[
\mu(r, g \circ L) = \mu(r, g).
\]

Similarly, we have \( \mu(r, h \circ L) = \mu(r, h) \). Thus the formula (11) holds. By using continuity we easily see that the inequality holds for all \( r > |b| \).

**Corollary 2.4.** Take \( f \in \mathcal{M}(\kappa) - \{0\} \) and assume \(|a| = 1\). When \( r > |b| \), we have
\[
m \left( r, \frac{f \circ L}{f} \right) = 0, \quad m \left( r, \frac{\Delta^m f}{f} \right) = 0.
\]

3 Proof of Theorem 1.1

To prove (5), take \( z \in \kappa \) with
\[
f(z) \neq 0, \infty; \quad b_k(z) \neq 0, \infty \quad (0 \leq k \leq q);
\]
\[
c_i(z) \neq 0, \infty \quad (i \in I); \quad d_j(z) \neq 0, \infty \quad (j \in J).
\]

Write
\[
b(z) = \max_{0 \leq k < q} \left\{ 1, \left( \frac{|b_k(z)|}{|b_{q}(z)|} \right)^{\frac{1-k}{q}} \right\}.
\]
If $|f(z)| > b(z)$, we have

$$|b_k(z)||f(z)|^k \leq |b_q(z)|b(z)^{q-k}|f(z)|^k < |b_q(z)||f(z)|^q,$$

and hence

$$|B(f)(z)| = |b_q(z)||f(z)|^q.$$

Then

$$|\Omega(f, f_1, \ldots, f_n)(z)| = \frac{|\Phi(f, f_1, \ldots, f_n)(z)|}{|B(f)(z)|} \leq \max_{j \in J} \left| d_j(z) \right| \frac{|f_1(z)|^{j_1}}{|f(z)|} \cdots \frac{|f_n(z)|^{j_n}}{|f(z)|}.$$

If $|f(z)| \leq b(z)$,

$$|\Omega(f, f_1, \ldots, f_n)(z)| \leq b(z)^{\text{deg}(\Omega)} \max_{i \in I} \left| c_i(z) \right| \frac{|f_1(z)|^{i_1}}{|f(z)|} \cdots \frac{|f_n(z)|^{i_n}}{|f(z)|}.$$

Therefore, in any case, the inequality

$$\mu(r, \Omega) \leq \max_{j \in J, i \in I} \left\{ \frac{\mu(r, d_j)}{\mu(r, b_q)} \prod_{k=1}^n \mu \left( r, \frac{f_k}{f} \right)^{j_k}, \mu(r, c_i) \prod_{k=1}^n \mu \left( r, \frac{f_k}{f} \right)^{i_k} \right\} \cdot \max_{0 \leq k < q} \left\{ 1, \mu \left( r, \frac{b_k}{b_q} \right)^{\frac{\text{deg}(\Omega)}{q-k}} \right\},$$

holds where $r = |z|$, which also holds for all $r > 0$ by continuity of the functions $\mu$. By using Lemma 2.3, we find

$$\mu(r, \Omega) \leq \max_{j \in J, i \in I} \left\{ \frac{\mu(r, d_j)}{\mu(r, b_q)} \cdot \mu(r, c_i) \cdot \max_{0 \leq k < q} \left\{ 1, \mu \left( r, \frac{b_k}{b_q} \right)^{\frac{\text{deg}(\Omega)}{q-k}} \right\} \right\},$$

and hence (5) follows from this inequality.

According to the proof of (4.9) in [8], we easily obtain the inequality (6).

4 Proof of Theorem 1.3

By using the algorithm of division, we have

$$\Phi(f) = \Phi_1(f)B(f) + \Phi_2(f)$$

with $\text{deg}(\Phi_2) < q$. Thus, the equation (11) can be rewritten as follows:

$$\Omega(f, f_1, \ldots, f_n) - \Phi_1(f) = \frac{\Phi_2(f)}{B(f)}.$$

Applying Theorem 1.1 to this equation, we obtain

$$m(r, \Omega - \Phi_1) = o(T(r, f)),$$
\[ N(r, \Omega - \Phi_1) = o(T(r, f)), \]

and hence
\[ T(r, \Omega - \Phi_1) = o(T(r, f)). \]

Theorem 2.12 due to Hu-Yang \cite{8} implies
\[ T(r, \Omega - \Phi_1) = T \left( r, \frac{\Phi_2}{B} \right) = qT(r, f) + o(T(r, f)). \]

It follows that \( q = 0 \), and (4) assumes the following form
\[ \Omega(f, f_1, ..., f_n) = \Phi(f). \]

Thus, Theorem 2.12 in \cite{8} implies
\[ T(r, \Omega) = T(r, \Phi) = pT(r, f) + o(T(r, f)). \] (14)

On other hand, it is easy to find the following estimate
\[ N(r, \Omega) \leq \deg(\Omega)N(r, f) + \sum_{i \in I} N(r, c_i). \] (15)

Obviously, we also have
\[ m(r, \Omega) \leq \deg(\Omega)m(r, f) + \max_{i \in I} \left\{ m(r, c_i) + \sum_{\alpha=1}^n i_\alpha m \left( r, \frac{f_\alpha}{f} \right) \right\}. \] (16)

By Lemma 2.3, we obtain
\[ T(r, \Omega) \leq \deg(\Omega)T(r, f) + \sum_{i \in I} T(r, c_i) + O(1). \] (17)

Our result follows from (14) and (17).

5 Proof of Theorem 1.5, 1.6

By substituting \( f = g + a \) into (8), we obtain
\[ \Psi + P = 0, \]

where
\[ \Psi \left( g, g', ..., g^{(n)} \right) = \sum_i C_i g^{i_0} (g')^{i_1} \cdots (g^{(n)})^{i_n} \]

is a differential polynomial of \( g \) such that all of its terms are at least of degree one, and
\[ T(r, P) = o(T(r, f)). \]

Also \( P \neq 0 \), since \( a \) does not satisfy (8).
Take \( z \in \kappa \) with
\[
g(z) \neq 0, \infty; \quad C_i(z) \neq \infty; \quad P(z) \neq 0, \infty.
\]
Set \( r = |z| \). If \( |g(z)| \geq 1 \), then
\[
m\left( r, \frac{1}{g} \right) = \max \left\{ 0, \log \frac{1}{|g(z)|} \right\} = 0.
\]
It is therefore sufficient to consider only the case \( |g(z)| < 1 \). But then,
\[
\left| \frac{\Psi\left(g(z), g'(z), \ldots, g^{(n)}(z)\right)}{g(z)} \right| = \frac{1}{|g(z)|} \left| \sum_i C_i(z) g(z)^{i_0} g'(z)^{i_1} \cdots g^{(n)}(z)^{i_n} \right|
\]
\[
\leq \max_i \left| C_i(z) \right| \left| \frac{g'(z)}{g(z)} \right|^{i_1} \cdots \left| \frac{g^{(n)}(z)}{g(z)} \right|^{i_n}
\]
since \( i_0 + \cdots + i_n \geq 1 \) for all \( i \). Therefore,
\[
m\left( r, \frac{1}{g} \right) = \log \frac{1}{|g(z)|} = \log \frac{|P(z)|}{|g(z)|} + \log \frac{1}{|P(z)|}
\]
\[
= \log \frac{|\Psi\left(g(z), g'(z), \ldots, g^{(n)}(z)\right)|}{|g(z)|} + \log \frac{1}{|P(z)|}
\]
\[
\leq \sum_i \left\{ m(r, C_i) + i_1 m\left( r, \frac{g'}{g} \right) + \cdots + i_n m\left( r, \frac{g^{(n)}}{g} \right) \right\} + m\left( r, \frac{1}{P} \right)
\]
\[
= o(T(r, f)).
\]
Since \( g = f - a \), the assertion follows.

Obviously, according to the method above, we can prove Theorem 1.6 similarly.

6 Final notes

We will use the following assumption:

\( \text{(B)} \) Fix a positive integer \( n \). Take \( a_i, b_i \) in \( \kappa \) such that \( |a_i| = 1 \) for each \( i = 1, \ldots, n \), and such that
\[
L_i(z) = a_i z + b_i \quad (i = 1, \ldots, n)
\]
satisfy \( L_i(z) \neq z \) for each \( i = 1, \ldots, n \). Let \( f \) be a non-constant meromorphic function over \( \kappa \) and let \( \{ f_1, \ldots, f_m \} \) be a finite set consisting of the forms \( \Delta_{L_i} f \). Take
\[
B \in \mathcal{M}(\kappa)[f]; \quad \Omega, \Phi \in \mathcal{M}(\kappa)[f, f_1, \ldots, f_m].
\]

According to the methods in this paper, we can prove easily the following results:

**Theorem 6.1.** Assume that the condition \( \text{(B)} \) holds. If \( f \) is a solution of the following equation
\[
B(f)\Omega(f_1, \ldots, f_m) = \Phi(f, f_1, \ldots, f_m)
\] (18)
with $\deg B \geq \deg \Phi$, then

$$m(r, \Omega) \leq \sum_{i \in I} m(r, c_i) + \sum_{j \in J} m(r, d_j) + \log \left( r, \frac{1}{b_q} \right) + l \sum_{j=0}^q m(r, b_j), \quad (19)$$

where $l = \max\{1, \deg \Omega\}$, $\Omega = \Omega(f, f_1, \ldots, f_m)$. Further, if $\Phi$ is a polynomial of $f$, we also have

$$N(r, \Omega) \leq \sum_{i \in I} N(r, c_i) + \sum_{j \in J} N(r, d_j) + O \left( \sum_{j=0}^q N \left( r, \frac{1}{b_j} \right) \right). \quad (20)$$

**Theorem 6.2.** If $\Phi$ is of the form

$$\Phi(f, f_1, \ldots, f_m) = \Phi(f) = \sum_{j=0}^p d_j f^j,$$

and if (18) has an admissible non-constant meromorphic solution $f$, then

$$q = 0, \quad p \leq \deg(\Omega).$$

**Theorem 6.3.** Assume that the condition (B) holds. Let $f \in M(\kappa)$ be a non-constant admissible solution of

$$\Omega(f, f_1, \ldots, f_m) = 0, \quad (21)$$

where the solution $f$ is called admissible if

$$\sum_{i \in I} T(r, c_i) = o(T(r, f)).$$

If a slowly moving target $a \in M(\kappa)$ with respect to $f$ does not satisfy the equation (21), then

$$m \left( r, \frac{1}{f-a} \right) = o(T(r, f)).$$

**References**

[1] Boutabaa, A., Applications de la théorie de Nevanlinna $p$-adic, Collect. Math. 42(1991), 75-93.

[2] Clunie, J., On integral and meromorphic functions, J. London Math. Soc. 37(1962), 17-27.

[3] Gackstatter, F. and Laine, I., Zur Theorie der gewöhnlichen Differentialgleichungen im Komplexen, Ann. Polon. Math. 38(1980), 259-287.

[4] Halburd, R. G. and Korhonen, R. J., Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, J. Math. Anal. Appl. 314(2006), 477-487.
[5] He, Y. Z. and Xiao, X. Z., Algebroid functions and ordinary differential equations (Chinese), Science Press, Beijing, 1988.

[6] Hu, P. C. and Yang, C. C., The Second Main Theorem for algebroid functions of several complex variables, Math. Z. 220(1995), 99-126.

[7] Hu, P. C. and Yang, C. C., Further results on factorization of meromorphic solutions of partial differential equations, Results in Mathematics 30(1996), 310-320.

[8] Hu, P. C. and Yang, C. C., Meromorphic functions over non-Archimedean fields, Mathematics and Its Applications 522, Kluwer Academic Publishers, 2000.

[9] Laine, I., Admissible solutions of some generalized algebraic differential equations, Publ. Univ. Joensun. Ser. B 10(1974).

[10] Malmquist, J., Sur les fonctions á un nombre fini de branches satisfaisant á une érentielle du premier order, Acta Math. 42(1920), 433-450.

[11] Mokhon’ko, A. Z. and Mokhon’ko, V. D., Estimates for the Nevanlinna characteristics of some classes of meromorphic functions and their applications to differential equations, Sibirsk. Mat. Zh. 15(6) (1974), 1305-1322 (in Russian); English translation: Sib. Math. J. 15(6) (1974), 921-934.

[12] Toda, N., On the growth of meromorphic solutions of an algebraic differential equations, Proc. Japan Acad. 60 Ser. A (1984), 117-120.

[13] Yosida, K., On algebroid-solutions of ordinary differential equations, Japan J. Math. 10(1934), 119-208.

School of Mathematics
Shandong University
Jinan 250100, Shandong, China
E-mail: pchu@sdu.edu.cn
E-mail: huanyongzhi@gmail.com