Nonassociative Snyder $\phi^4$ Quantum Field Theory

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Abstract: In this article we define and quantize a truncated form of the nonassociative and noncommutative Snyder $\phi^4$ field theory by using the functional method in momentum space. More precisely, the action is approximated by expanding up to the linear order in the Snyder deformation parameter $\beta$, producing an effective model on commutative spacetime for the computation of the two-, four- and six-point functions. The two- and four-point functions at one loop have the same structure as at the tree level, with UV divergences faster than in the commutative theory. The same behavior appears in the six-point function, with a logarithmic UV divergence and renders the theory unrenormalizable at $\beta^1$-order except for the special choice of free parameters $s_1 = -s_2$. We would expect effects from nonassociativity on the correlation functions at $\beta^1$-order, but these are cancelled due to the average over permutations.

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1 Introduction

There is consensus in the theoretical and mathematical physics nowadays that at short distances spacetime has to be described by nonstandard geometrical structures, and that the very concept of point and localizability may no longer be adequate. Together with string theories \(^1\), this is one of the oldest motivations for the introduction of noncommutative (NC) geometry \(^2–8\). The simplest kind of noncommutative geometry is the so called canonical one \(^3,9–14\). Usually, the construction of a field theory on a noncommutative space is performed by deforming the product between functions (and hence between fields in general) with the introduction of a noncommutative star product. The noncommutative coordinates \(\hat{x}^\mu\) satisfy:

\[
[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu},
\]

with \(\hat{x}^\mu\) Hermitian operators.

The simplest case \(|\theta^{\mu\nu}| \sim \text{const.}\) is the well-known Moyal noncommutative spacetime \(^{11}\): \(|\theta^{\mu\nu}|\) does not depend on coordinates, and it scales like \(\text{length}^2 \sim \Lambda_{\text{NC}}^{-2}\), \(\Lambda_{\text{NC}}\) being the scale of noncommutativity with the dimension of energy. For Moyal geometry recently it was proven that there exists a \(\theta\)-exact formulation of noncommutative gauge field theory based on the Seiberg-Witten map \(^{1,14}\) that preserves unitarity \(^{15}\) and has improved UV/IR behavior at the quantum level by introducing supersymmetry \(^{16–19}\). All this could also have implications on cosmology, like for example through the determination
of the maximal decoupling temperature of the right-handed neutrino species in the early universe \[20\].

There are other important models, like the $\kappa$-Minkowski and the Snyder geometries, where we might expect the similar properties with similar cosmological consequences. Thus, Ref. \[20\] represents one of the strongest motivation for our investigation of Snyder spaces.

The $\kappa$-Minkowski models \[21–27\], are represented by:

\[
[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa} (\delta^{0}_\mu \hat{x}_\nu - \delta^{0}_\nu \hat{x}_\mu), \tag{1.2}
\]

where $\kappa$ is a mass parameter.

On the other hand, Snyder’s spacetime \[28\], the subject of this investigation, belongs to a rather different type of models \[29–32\], and is defined by the phase space commutation relations

\[
[\hat{x}_\mu, \hat{x}_\nu] = i\beta M_{\mu\nu}, \ [p^\mu, \hat{x}_\nu] = -i\delta^{\mu}_\nu - i\beta p^\mu p_\nu, \ [p_\mu, p_\nu] = 0, \tag{1.3}
\]

where $M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ are Lorentz generators, $x_\mu$ are the undeformed canonical coordinates and $p_\mu$ the momentum generators. Noncommutative coordinates $\hat{x}_\mu$ and momentum generators $p_\mu$ transform as vectors under Lorentz generators and $\beta$ is a real parameter $\beta \propto \ell^2_P$, where $\ell_P$ is the Planck length.

The Moyal and the $\kappa$-Minkowski geometries break the Lorentz invariance. Such effects are manifest in their star product. On the contrary, in his seminal paper Snyder \[28\] observed that assuming a noncommutative structure of spacetime and hence a deformation of the Heisenberg algebra it is possible to define a discrete spacetime without breaking the Lorentz invariance. It is therefore interesting to investigate the Snyder model from the point of view of noncommutative geometry.

Later the formalism of Hopf algebras has been applied to the study of noncommutative geometries \[4\]. The Snyder model has been studied in a series of papers \[30–36\] and the associated Hopf algebra was studied in \[30\] and \[36\], where the model has been generalized and the star product, coproducts and antipodes have been calculated using the method of realizations. A different approach was used in \[35\], where the Snyder model was considered in a geometrical perspective as a coset in momentum space, and the results are equivalent to those of refs. \[31, 32\]. A further generalization of Snyder spacetime deformations was recently introduced in \[36–38\]. Also several nonassociative star/cross product geometries and related quantum field theories have been discussed recently in \[39\].

In this paper we consider a Snyder-like quantum field theory, where the action is modified by truncating the model to first order in the deformation parameter $\beta$. The drawback of this truncation is the loss of the ultraviolet behavior of the original theory. In particular, we remark that the original theory could be ultraviolet finite. Moreover, any possible nonperturbative effect like the celebrated UV/IR mixing in \[14, 40, 41\] is also lost. Among other features, UV/IR mixing connects the noncommutative field theories with holography via UV and IR cutoffs in a model independent way \[42, 43\]. Holography and UV/IR mixing are known in the literature as possible windows to quantum gravity \[10, 43\]. In spite of this deficiency, we believe that our investigation is interesting as a starting point for further investigations on the properties of the full theory.
The paper is organized as follows: in the second section, we introduce the Hermitian realization of the model and the star product corresponding to this realization. The Snyder-deformed action for a $\phi^4$ theory based on the above formalism is introduced in Section 3. The quantization of the theory, including tree-level four-point function, as well as the one-loop two-, four-, and six-point functions, is discussed in Section 4. The effect of Snyder’s nonassociativity is presented in Section 5. Finally in Section 6 we discuss the UV/IR divergences and their effects.

2 Hermitian realization of Snyder spaces

Following Refs. [36, 38], we consider the Hermitian realization of the Snyder spaces

$$\hat{x}_\mu = x_\mu + \beta \left[ s_1 M_{\mu \alpha} p^\alpha + (s_1 + s_2)(x \cdot p)p_\mu - i \left( s_1 + \frac{D + 1}{2} s_2 \right) p_\mu \right] + O(\beta^2),$$

(2.1)

with $D$ the dimension of the space-time we are considering,\(^1\) and $s_1, s_2$ real parameters. Generators $M_{\mu \nu}$, $p_\mu$, and $x_\mu$, $p_\mu$, generate the undeformed Poincaré and Heisenberg algebras, respectively. The commutation relations $[\hat{x}_\mu, \hat{x}_\nu]$, $[p^\mu, \hat{x}_\nu]$, are

$$[\hat{x}_\mu, \hat{x}_\nu] = i\beta(s_2 - 2s_1)M_{\mu \nu} + O(\beta^2), \quad [p^\mu, \hat{x}_\nu] = -i \left( \delta^\mu_\nu s_1 + \beta s_2 p^\mu p_\nu \right) + O(\beta^2),$$

(2.2)

which implies that the coordinates $\hat{x}_\mu$ become commutative for $s_2 = 2s_1$.

The corresponding star product takes the following form

$$e^{ikx} \star e^{iqx} = e^{iD(k,q)x} e^{iG(k,q)},$$

(2.3)

and it is in general nonassociative and noncommutative. However, for specific choice $s_2 = 2s_1$ in (2.2), the star product (2.3) becomes associative and commutative. The functions $D_\mu(k,q)$ and $G(k,q)$ are given up to first order in $\beta$ for arbitrary $s_1$ and $s_2$ by

$$D_\mu(k,q) = k_\mu + q_\mu + \beta \left[ k_\mu \left( s_1 q^2 + \left( s_1 + \frac{s_2}{2} \right) k \cdot q \right) + q_\mu s_2 \left( k \cdot q + \frac{k^2}{2} \right) \right] + O(\beta^2),$$

$$G(k,q) = -i\beta \left( s_1 + \frac{D + 1}{2} s_2 \right) k \cdot q + O(\beta^2),$$

(2.5)

and they satisfy relation

$$\det \left( \frac{\partial D_\mu(k,q)}{\partial k_\nu} \right) \bigg|_{k = -q} = \det \left( \frac{\partial D_\mu(k,q)}{\partial q_\nu} \right) \bigg|_{k = -q} = e^{iG(k,-k)} + O(\beta^2),$$

(2.6)

which induces the cyclicity of the star product under usual integration

$$\int f(x) \star g(x) = \int f(x)g(x) + O(\beta^2).$$

(2.7)

\(^1\)We write directly $D$ here since this factor later enters the loop computation and we use dimensional regularization when evaluating loop integrals. Dimensional regularization appears to be a natural choice because there is no tensor structure other than metric in our formulation of the Snyder theory. We only encounter scalar and vector objects and no pseudoscalars or pseudovectors.
3 The $\phi^4$ theory on Snyder spaces

The action for a Snyder-type $\phi^4$ theory on four-dimensional Euclidean space-time is given by

$$S = \int \frac{1}{2} \left( (\partial_\mu \phi) \star (\partial^\mu \phi) + m^2 \phi \star \phi \right) + S_{\text{int}},$$

where

$$S_{\text{int}} = -\frac{\lambda}{4!} \int \phi \star (\phi \star (\phi \star \phi)).$$

Up to the first order in $\beta$ we can remove the star product on the left using the cyclicity property of the star product, writing

$$S^1 = \int \frac{1}{2} \left( (\partial \phi)^2 + m^2 \phi^2 \right) - \frac{\lambda}{4!} \phi \star (\phi \star (\phi \star \phi)) + O(\beta^2).$$

The definition of the star product then allows us to write the interaction in momentum space as follows

$$S_{\text{int}}^1 = -\frac{\lambda}{4!} \int \phi \star (\phi \star (\phi \star \phi))$$

$$= -\frac{\lambda}{4!} \int d^4q_1 \ d^4q_2 \ d^4q_3 \ d^4q_4 \ (2\pi)^4 \delta^4(q_1 - q_2 - q_3 - q_4)$$

$$\cdot \left( \phi(q_1) \tilde{\phi}(q_2) \tilde{\phi}(q_3) \tilde{\phi}(q_4) + O(\beta^2) \right),$$

where

$$D_4(q_1, q_2, q_3, q_4) = q_1 + D(q_2, D(q_3, q_4)),$$

and

$$g_3(q_1, q_2, q_3, q_4) = 1 + iG(q_2, D(q_3, q_4)) + iG(q_3, q_4) + O(\beta^2).$$

This is our starting point for the following calculations.

4 Quantizing the Snyder field theory

Since the quadratic part of the classical action is undeformed, it is convenient to adopt the functional method in momentum space, previously used in similar problems like for example \[41\]. Our starting point is the generating functional

$$Z[J] = e^{W[J]} = \exp \left[ -S + \int J \phi \right],$$

which we shall evaluate perturbatively. The generating functional for the free theory is

$$Z_0[J] = \exp \left[ \int d^4xd^4y J(x)G(x - y)J(y) \right].$$

Since the free Euclidean Green’s function is simply

$$G(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2},$$

\[\text{In order to avoid complications we choose to work directly on the Euclidean space-time.}\]
the free generating functional can be reduced to the momentum space expression

\[ Z_0[J] = \exp \left[ \int d^4x d^4y J(x) G(x - y) J(y) \right] = \exp \left[ \int \frac{d^4k}{(2\pi)^4} \tilde{J}(k) \frac{1}{k^2 + m^2} \tilde{J}(-k) \right]. \]  

(4.4)

The generating functional of the interacting theory is obtained by introducing the interaction through functional derivatives of the free generating functional, i.e.

\[ Z[J] = \mathcal{N} \exp \left[ \frac{\lambda}{4!} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} g_3(q_1, q_2, q_3, q_4) \cdot (2\pi)^4 \delta(D_4(q_1, q_2, q_3, q_4)) \right] \delta J(q_1) \delta J(q_2) \delta J(q_3) \delta J(q_4)] Z_0[J]. \]

(4.5)

The functional derivative \( \frac{\delta}{\delta J(q_2)} \) satisfies

\[ \frac{\delta}{\delta J(q)} \tilde{J}(p) = (2\pi)^4 \delta(p - q), \]

(4.6)

where the factor \((2\pi)^4\) follows from the normalization adopted for the Fourier transformation,

\[ \phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} \tilde{\phi}(p). \]

(4.7)

The Green’s function obtained from the generating functional contains in principle a number of \(\delta\)-functions, in particular the composite ones on the vertices, so we need a strategy to handle them properly. We choose the following prescription: first we work on the position space connected correlation functions

\[ G(x_1, x_2, ..., x_n) = \int \prod_{i=1}^{n} \frac{d^Dp_i}{(2\pi)^D} e^{ip_i x_i} \frac{\delta}{\delta J(p_i)} W[J] \bigg|_{J=0}, \]

(4.8)

because all external and internal momenta are integrated over and consequently all \(\delta\)-functions can be evaluated as well. We then integrate over one specific fixed external momentum \(p_i\) in order to remove the final (composite) \(\delta\)-function that describes the modified overall momentum conservation. This is not the only possible choice one could make, but we will stick with it and construct both tree and 1-loop level integrals accordingly.

### 4.1 Tree-level four-point function

As an example of the method described in the last section as well as an explanation to the further computations, we evaluate first the tree-level four-point correlation function, \(G_{\text{tree}}(x_1, x_2, x_3, x_4)\) (corresponding to Fig.1), which is defined as follows

\[ G_{\text{tree}}(x_1, x_2, x_3, x_4) = \frac{\lambda}{4!} \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} \frac{d^4p_4}{(2\pi)^4} e^{ip_1 x_1} e^{ip_2 x_2} e^{ip_3 x_3} e^{ip_4 x_4} \cdot (2\pi)^4 \sum_{\sigma \in S_4} \delta(D_4(\sigma(p_1, p_2, p_3, p_4))) \cdot g_3(\sigma(p_1, p_2, p_3, p_4)), \]

(4.9)
where $\sigma \in S_4$ denotes sum over all four momenta permutations, i.e.

$$
\delta\left(D_4(\sigma(p_1, p_2, p_3, p_4))\right) = \delta\left(D_4(q_1 = p_{\sigma(1)}, q_2 = p_{\sigma(2)}, q_3 = p_{\sigma(3)}, q_4 = p_{\sigma(4)})\right),
$$

(4.10)

$$
g_3(\sigma(p_1, p_2, p_3, p_4)) = g_3(q_1 = p_{\sigma(1)}, q_2 = p_{\sigma(2)}, q_3 = p_{\sigma(3)}, q_4 = p_{\sigma(4)}).
$$

(4.11)

The composite $\delta$-function $\delta\left(D_4(\sigma(p_1, p_2, p_3, p_4))\right)$ is then evaluated with respect to $p_4$:³

$$
\delta\left(D_4(\sigma(p_1, p_2, p_3, p_4))\right) = \frac{\delta\left(p_4 - p_4(p_1, p_2, p_3)\right)}{\det\left(\frac{\partial D_4(\sigma(p_1, p_2, p_3, p_4))}{\partial p_4}\right)\bigg|_{p_4 = p_4(p_1, p_2, p_3)}},
$$

(4.12)

where $p_4(p_1, p_2, p_3)$ is the solution to the equation

$$
D_4(\sigma(p_1, p_2, p_3, p_4)) = 0.
$$

(4.13)

At $\beta^1$ order, this equation can be solved iteratively noting that

$$
D_4(\sigma(p_1, p_2, p_3, p_4)) = p_1 + p_2 + p_3 + p_4 + \beta D_4^1(\sigma(p_1, p_2, p_3, p_4)) + \mathcal{O}(\beta^2).
$$

(4.14)

Thus the iterative solution of $p_4$ takes the following form:

$$
p_4(p_1, p_2, p_3) = p_4^0(p_1, p_2, p_3) + \beta p_4^1(p_1, p_2, p_3) + \mathcal{O}(\beta^2)
\quad = -p_1 - p_2 - p_3 - \beta D_4^1(\sigma(p_1, p_2, p_3, p_4^0 = -p_1 - p_2 - p_3)) + \mathcal{O}(\beta^2).
$$

(4.15)

Similarly, in order to obtain $G_{\text{tree}}(x_1, x_2, x_3, x_4)$ up to $\beta^1$-order, we have to expand the $g_3$ factor and the Jacobian determinant in (4.12) up to first order in $\beta$ around the solution $p_4(p_1, p_2, p_3)$. This is straightforward since both of them have a constant value 1 at $\beta^0$.

³We find necessary to evaluate the composite $\delta$-functions during the formulation of correlation functions because in loop calculation the loop momenta on the vertex should stay fixed (for example in a tadpole diagram). All we can generate through the composite $\delta$-function(s) is then the way how certain external momentum becomes dependent to the other/others.
order, and hence the expansion involves only expansions of these two objects up to $\beta^1$ order at the place $p_4 = p_1^0 = -p_1 - p_2 - p_3$. Moreover, at $\beta^1$-order the determinant reduces to

$$\det \left( \frac{\partial D_{4\nu}(\sigma(p_1, p_2, p_3, p_4))}{\partial p_{4\nu}} \right) \bigg|_{p_4 = p_1^0} = 1 + \text{tr} \left( \frac{\partial D_{4\nu}(\sigma(p_1, p_2, p_3, p_4))}{\partial p_{4\nu}} \right) \bigg|_{p_4 = p_1^0} + O(\beta^2).$$

Finally, we also notice that the momentum in the last external propagator is shifted from the commutative solution $p_1^0$. We therefore expand it to $\beta^1$-order too, obtaining

$$\frac{\epsilon(p_1, p_2, p_3) x_4}{p_4(p_1, p_2, p_3)^2 + m^2} = \frac{e^{-i(p_1 + p_2 + p_3)x_4}}{(p_1 + p_2 + p_3)^2 + m^2} \cdot \left( 1 + \beta p_4^1(p_1, p_2, p_3) \cdot (ix_4 + \frac{2(p_1 + p_2 + p_3)}{(p_1 + p_2 + p_3)^2 + m^2}) \right) + O(\beta^2).$$

Now we collect all $\beta^1$-order contributions and sum over the $S_4$ permutations to obtain

$$G_{\text{tree}}(x_1, x_2, x_3, x_4) = G_{\text{tree}}^0(x_1, x_2, x_3, x_4) + \beta G_{\text{tree}}^1(x_1, x_2, x_3, x_4) + O(\beta^2)$$

$$= \lambda \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} \frac{e^{i p_1 x_1}}{p_1^2 + m^2} \frac{e^{i p_2 x_2}}{p_2^2 + m^2} \frac{e^{i p_3 x_3}}{p_3^2 + m^2} e^{-i(p_1 + p_2 + p_3)x_4}$$

$$\cdot \left( 1 + \frac{\beta}{3} \left( \Sigma_1 + \Sigma_2 \cdot (ix_4 + \frac{2(p_1 + p_2 + p_3)}{(p_1 + p_2 + p_3)^2 + m^2}) \right) \right) + O(\beta^2),$$

where

$$\Sigma_1(p_1, p_2, p_3) = (D + 2)(s_1 + s_2)(p_1 \cdot p_2 + p_2 \cdot p_3 + p_3 \cdot p_1),$$

$$\Sigma_2(p_1, p_2, p_3) = -(s_1 + s_2)(p_1((p_1 + p_2 + p_3)^2 - p_1^2) + p_2((p_1 + p_2 + p_3)^2 - p_2^2) + p_3((p_1 + p_2 + p_3)^2 - p_3^2)).$$

### 4.2 One-loop two-point function

Following the same procedure as for the tree-level four-point function, we can now evaluate the one-loop two-point function of Fig. 2

$$G_{\text{1-loop}}(x_1, x_2) = \frac{1}{2^4} \lambda \int \frac{d^Dp_1}{(2\pi)^D} \frac{d^Dp_2}{(2\pi)^D} \frac{d^D\ell}{(2\pi)^D} \frac{\epsilon^{ip_1 x_1}}{p_1^2 + m^2} \frac{\epsilon^{ip_2 x_2}}{p_2^2 + m^2} \frac{\epsilon^{i\ell \cdot \ell}}{\ell^2 + m^2}$$

$$\cdot \left( 2\pi^4 \sum_{\sigma \in S_4} \delta(D_4(\sigma(p_1, p_2, \ell, -\ell))) \cdot g_3(\sigma(p_1, p_2, \ell, -\ell)) \right).$$

A peculiar property of Snyder field theory is that, due to the law addition of momenta, $p_1$ and $p_2$ are in general different, so the momenta are not strictly conserved due to loop effects.
Figure 2. Tadpole contribution to the 2-point function.

All δ-functions in (4.21) can be evaluated using the iterative procedure of § 4.1. After summing over all these permutation channels, we observe that the structures Σ_1 and Σ_2 emerge as expected. Using Σ_1 and Σ_2 we can rewrite $G_{1-loop}(x_1, x_2)$ as follows

$$G_{1-loop}(x_1, x_2) = \frac{\lambda}{2} \int \frac{d^D p_1}{(2\pi)^D} \frac{e^{ip_1 x_1} - e^{-ip_1 x_2}}{p_1^2 + m^2} \left( \frac{1}{4\pi^2} \right) \left( \frac{1 - \beta}{\beta^2} \frac{\delta m^2}{D} \right) + O(\beta^2).$$

Once we evaluate Σ_1 and Σ_2 explicitly, an intriguing cancellation happens to send Σ_2 to be zero and erases the effect of momentum non-conservation completely. The 1-loop 2-point function then boils down to

$$G_{1-loop}(x_1, x_2) = \frac{\lambda}{2} \int \frac{d^D p_1}{(2\pi)^D} \frac{e^{ip_1 x_1} - e^{-ip_1 x_2}}{p_1^2 + m^2} \left( \frac{1}{4\pi^2} \right) \left( \frac{1 - \beta}{\beta^2} \frac{\delta m^2}{D} \right) + O(\beta^2) + \text{counter-term}. \quad (4.23)$$

While the integral is quartic divergent, the Green function has the same structure as at tree level, thus one could in principle renormalize it using a mass counter-term $\delta m^2$. 

\[ -8 - \]
4.3 One loop four point function

As their commutative counterpart, one-loop four-point functions can still be split into three Mandelstam-variable channels, as depicted in Figs. 3-5

\[ G_{1\text{-loop}}(x_1, x_2, x_3, x_4) = I_s + I_t + I_u, \]  

(4.24)
but each of them now splits into two, depending on which of the two vertices is evaluated
the usual $\beta^0$ order, as we choose once again to integrate over the external momentum $p_4$ only.
Note that this procedure creates an additional momentum shift within the loop-integral
when $p_4$ is attached to the $\beta^0$ vertex which is not explicitly shown in the diagrams. By
realizing that the $\beta^0$ vertex is totally symmetric with respect to all momenta attached, we
are able, from Fig.3, to obtain the following expression

$$I_s = I_s^0 + \beta (I_1 + I_2) + \mathcal{O}(\beta^2),$$

(4.25)

with

$$I_s^0 = \frac{\lambda^2}{2} \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{d^D p_3}{(2\pi)^D} \frac{e^{i p_1 x_1} e^{i p_2 x_2} e^{i p_3 x_3}}{p_1^2 + m^2 p_2^2 + m^2 p_3^2 + (p_1 + p_2 + p_3)^2 + m^2} \ \frac{e^{-i(p_1 + p_2 + p_3) x_4}}{2}$$

(4.26)

the usual $\beta^0$-order loop contribution, while

$$I_1^0 = \frac{\lambda^2}{6} \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{d^D p_3}{(2\pi)^D} \frac{e^{i p_1 x_1} e^{i p_2 x_2} e^{i p_3 x_3}}{p_1^2 + m^2 p_2^2 + m^2 p_3^2 + (p_1 + p_2 + p_3)^2 + m^2} \ \frac{e^{-i(p_1 + p_2 + p_3) x_4}}{2}$$

(4.27)

and

$$I_2^0 = \frac{\lambda^2}{6} \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{d^D p_3}{(2\pi)^D} \frac{e^{i p_1 x_1} e^{i p_2 x_2} e^{i p_3 x_3}}{p_1^2 + m^2 p_2^2 + m^2 p_3^2 + (p_1 + p_2 + p_3)^2 + m^2} \ \frac{e^{-i(p_1 + p_2 + p_3) x_4}}{2}$$

(4.28)

are the $\beta^1$-order corrections from Snyder-type deformations. Once we work out all the
objects explicitly, the s-channel integral boils down to

$$I_s = G_{tree}^0(x_1, x_2, x_3, x_4) \cdot (I_s + \beta (I_1 + I_2)) + \beta G_{tree}^1(x_1, x_2, x_3, x_4) \cdot I_s,$$

(4.29)

where

$$I_s = \int \frac{d^D \ell}{(2\pi)^D} \frac{\lambda}{(\ell^2 + m^2)((\ell + p_1 + p_2)^2 + m^2)},$$

(4.30)

is the usual s-channel scalar loop integral while

$$I_1 = \frac{\lambda}{6} \int \frac{d^D \ell}{(2\pi)^D} \frac{(D + 2)(s_1 + s_2) \ell^2}{(\ell^2 + m^2)((\ell + p_1 + p_2)^2 + m^2)}.$$

(4.31)
and

\[ \mathcal{I}_2 = \frac{\lambda}{6} \int \frac{d^D \ell}{(2\pi)^D} \frac{2(\ell + p_1 + p_2) \Sigma_2(p_1, p_2, \ell)}{((\ell + p_1 + p_2)^2 + m^2)^2}. \] (4.32)

presents Snyder-type deformation effects within the loop integral at \( \beta^1 \)-order. We are particularly interested in the UV divergence within these two integrals. It is easy to see that \( \mathcal{I}_2 \) is quadratic UV divergent. Explicit computation shows that at \( D \to 4 - \epsilon \) limit this integral reduces to

\[ \mathcal{I}_1 = \lambda(s_1 + s_2) \frac{m^2}{(4\pi)^2} \left( \frac{4}{\epsilon} + \frac{1}{3} - 2\gamma_E - \int_0^1 dz \log \frac{m^2(z(1-z)(p_1 + p_2)^2 + m^2)}{(4\pi)^2} \right) + \mathcal{O}(\epsilon). \] (4.33)

The integral \( \mathcal{I}_2 \) requires a more detailed investigation. Writing down explicitly the numerator

\[ \mathcal{I}_2 = \frac{\lambda}{3} (s_1 + s_2) \int \frac{d^D \ell}{(2\pi)^D} \frac{(\ell + p_1 + p_2)^4 - (\ell + p_1 + p_2) \cdot (p_1 p_2^2 + p_2 p_1^2 + \ell^2)}{((\ell + p_1 + p_2)^2 + m^2)^2} \]

\[ = \frac{\lambda}{3} (s_1 + s_2) \frac{1}{(4\pi)^2} \int_0^1 dz \int \frac{d^D \ell}{(2\pi)^D} \frac{2z(3z - 2)(1 + \frac{z^2}{4})}{(\ell^2 + z(1-z)(p_1 + p_2)^2 + m^2)^2} + \text{finite terms}, \] (4.34)

where \( z \) is the usual Feynman variable. At the \( D \to 4 - \epsilon \) limit integral reduces to

\[ \mathcal{I}_2 = \frac{\lambda}{3} (s_1 + s_2) \frac{1}{(4\pi)^2} \left( 1 + \frac{1}{2} + \frac{\epsilon}{8} \right) (p_1 + p_2)^2 \int_0^1 \frac{dz}{2z(3z - 2)} \]

\[ \cdot \left( \frac{2}{\epsilon} - \gamma_E + \log 4\pi - \log \left( z(1-z)(p_1 + p_2)^2 + m^2 \right) + \mathcal{O}(\epsilon) \right) + \text{finite terms}. \] (4.35)

We can then find that the \( 1/\epsilon \) divergence vanishes because

\[ \int_0^1 \frac{dz}{2z(3z - 2)} = 2(z^3 - z^2) \bigg|_0^1 = 0, \] (4.36)

therefore the whole integral remains finite in dimensional regularization.

The t and u channels, corresponding to Figs. 4 and 5, can be obtained from the s-channel formulae above by the permutations \( p_2 \leftrightarrow p_3 \) and \( p_1 \leftrightarrow p_3 \), respectively.

The 1-loop structure (4.29) suggests us to renormalize the 4-point function by introducing a \( \beta \)-expansion of the coupling constant counter-term

\[ \delta \lambda = \delta \lambda^0 + \beta \delta \lambda^1 + \mathcal{O}(\beta^2). \] (4.37)

We see that the UV divergence in \( \mathcal{I}_s \) can be absorbed by \( \delta \lambda^0 \), and the new divergence from \( I_1 \) by \( \delta \lambda^1 \). The latter is valid since the \( 1/\epsilon \) term is proportional to the mass only.
Figure 6. Typical diagram contribution to the 6-point function. The $\beta^1$-order contribution has to be considered as running over all three vertices in order to complete each channel.

4.4 UV divergence in the 1-loop 6-point function

Our experience with two- and four-point function shows that the degree of divergence of each of them is higher than its commutative counterpart, which suggests that the 1-loop 6-point function can pick up UV divergent contributions also from the triangle diagram of Fig. 6, where the black dot represents the $\beta^1$ vertex which contains $\Sigma_1(p_1, p_2, \ell)$ term. Explicit evaluation, starting from (3.4), gives the following form of the divergent integral in one channel:

$$
I_{6UV} = \int \frac{d^D \ell}{(2\pi)^D} \frac{\Sigma_1(p_1, p_2, \ell) + \Sigma_1(\ell + p_1 + p_2, p_3, p_4) + \Sigma_1(\ell + p_1 + p_2 + p_3 + p_4, p_5, -\ell)}{(\ell^2 + m^2)((\ell + p_1 + p_2)^2 + m^2)((\ell + p_1 + p_2 + p_3 + p_4)^2 + m^2)} \left( D + 2 \right) (s_1 + s_2) \left( -\ell^2 + \sum_{1 \leq i < j \leq 5} p_i \cdot p_j \right)
$$

\[= \int \frac{d^D \ell}{(2\pi)^D} \frac{(\ell^2 + m^2)((\ell + p_1 + p_2)^2 + m^2)((\ell + p_1 + p_2 + p_3 + p_4)^2 + m^2)}{(D + 2)(s_1 + s_2)(-\ell^2 + \sum_{1 \leq i < j \leq 5} p_i \cdot p_j)}.
\]

(4.38)

with $\Sigma_1$ being defined in (4.19). The sum of three $\Sigma_1$’s contains also contributions from additional two diagrams obtained from diagram in Fig.6 by shifting the black dot to the other two available positions in the diagram. Other channels can be reached by an appropriate permutation of the external momenta. As we can see the first term in the numerator creates a logarithmic UV divergence. However we can of course still remove this divergence by demanding $s_1 + s_2 = 0$. In this case all nontrivial $\beta^1$-order quantum corrections are removed and we are dealing with exactly the same renormalization procedure as in the commutative theory.
5 The effect of Snyder nonassociativity

The Snyder-type star products discussed in Section 2 are, in general, nonassociative, except in the case $s_2 = 2s_1$, which means that the ordering of the products matters. Taking into account integration by parts, from (3.4) we obtain two additional types of $\phi^4$ interactions, giving altogether the following:

$$S_{\text{int}}^1 = (S_{\text{int}}^1)_1 = -\frac{\lambda}{4!} \int \phi(\phi \ast (\phi \ast \phi)), \quad (5.1)$$

$$S_{\text{int}}^2 = (S_{\text{int}}^2)_2 = -\frac{\lambda}{4!} \int \phi((\phi \ast \phi) \ast \phi), \quad (5.2)$$

$$S_{\text{int}}^3 = (S_{\text{int}}^3)_3 = -\frac{\lambda}{4!} \int (\phi \ast \phi)(\phi \ast \phi). \quad (5.3)$$

Repeating the computation above (4.38), using $(S_{\text{int}}^1)_2$ and $(S_{\text{int}}^1)_3$ in place of $(S_{\text{int}}^1)_1$, we find that all three variants of the Snyder-type $\phi^4$ interaction give the same results at the first order in $\beta$. This result is rather surprising. Each of the permutation channels contains different inputs, yet the average over all permutations totally cancels all these effects. It is however possible that going to higher orders in $\beta$, this degeneracy is lost.

6 Discussion and Conclusion

In this article we have studied Snyder field theory with the action truncated at first order in the deformation parameter $\beta$, producing an effective model on commutative spacetime. The study is performed by using the functional method in momentum space at one loop.

We recall the main points of our analysis: we have proposed a simple perturbative quantization for the $\phi^4$ theory on Snyder-type spaces with Hermitian realizations and have evaluated the 1-loop 2- and 4-point functions at $\beta^1$-order, showing that they give raise to UV divergences. They are faster than in the commutative theory, thus they can be absorbed by the tree level counter-terms.

However, the $\beta^1$-order 1-loop 6-point function receives a logarithmic UV divergent quantum correction in general, which renders the theory unrenormalizable. Remarkably, at $\beta^1$-order all information about nonassociativity in the definition of $\phi^4$ interaction is canceled, namely one obtains identical results for both tree and one loop correlation functions, independently of the ordering of the products.

Inspecting the $\beta^1$-order equations (4.19,4.20,4.33,4.34,4.38) we find that the correlation functions depend on the free parameters $s_1$ and $s_2$ only through their sum $s_1 + s_2$. In other words, one can turn off all nontrivial $\beta^1$-order effects by setting $s_1 = -s_2$, which corresponds to the removal of the dependence on the dilatation operator $(x \cdot p)$ from the definition of the noncommutative coordinates $\tilde{x}_\mu$ in (2.1).

Our investigation has been limited to the first order in the $\beta$-deformation parameter. The full theory has of course different properties, especially in the ultraviolet limit, which could be finite for some choices of the defining commutation relations. However our results are certainly valid for energies much less than the Planck scale, and constitute a good starting point for the investigation of the full theory.
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References

[1] N. Seiberg and E. Witten, String theory and noncommutative geometry, JHEP 9909 (1999) 032, doi:10.1088/1126-6708/1999/09/032 [hep-th/9908142].
[2] A. Connes, Noncommutative Geometry, Academic Press, (1994).
[3] S. Doplicher, K. Fredenhagen and J. E. Roberts, The Quantum structure of space-time at the Planck scale and quantum fields, Commun. Math. Phys. 172 (1995) 187 doi:10.1007/BF02104515 [hep-th/0303037].
[4] S. Majid, Foundations of quantum group theory, Cambridge University Press 1995.
[5] G. Landi, An introduction to noncommutative spaces and their geometry, Springer, (1997), Lect. Notes Phys. Monogr. 51 (1997) 1 doi:10.1007/3-540-14949-X [hep-th/9701078].
[6] J. Madore, Noncommutative geometry for pedestrians, gr-qc/9906059.
[7] J. Madore, An introduction to noncommutative differential geometry and its physical applications, Lond. Math. Soc. Lect. Note Ser. 257 (2000) 1.
[8] J. M. Gracia-Bondia, J. C. Varilly and H. Figueroa, Elements Of Noncommutative Geometry, Boston, USA: Birkhaeuser (2001) 685 p.
[9] R. J. Szabo, Quantum field theory on noncommutative spaces, Phys. Rept. 378 (2003) 207 doi:10.1016/S0370-1573(03)00059-0 [hep-th/0109162].
[10] R. J. Szabo, Quantum Gravity, Field Theory and Signatures of Noncommutative Spacetime, Gen. Rel. Grav. 42 (2010) 1-29 [arXiv:0906.2913 [hep-th]].
[11] J. E. Moyal, Quantum mechanics as a statistical theory, Proc. Cambridge Phil. Soc. 45 (1949) 99, doi:10.1017/S0305004100000487.
[12] M. R. Douglas and N. A. Nekrasov, Noncommutative field theory, Rev. Mod. Phys. 73 (2001) 977 doi:10.1103/RevModPhys.73.977 [hep-th/0106048].
[13] P. Schupp, J. Trampetic, J. Wess and G. Raffelt, The Photon neutrino interaction in noncommutative gauge field theory and astrophysical bounds, Eur. Phys. J. C 36 (2004) 405, doi:10.1140/epjc/s2004-01874-5, [hep-ph/0212292].
[14] P. Schupp and J. You, UV/IR mixing in noncommutative QED defined by Seiberg-Witten map, JHEP 0808 (2008) 107, doi:10.1088/1126-6708/2008/08/107, [arXiv:0807.4886].
[15] O. Aharony, J. Gomis and T. Mehen, On theories with lightlike noncommutativity, JHEP 0009, 023 (2000), [hep-th/0006236].
[16] R. Horvat, A. Ilakovac, J. Trampetic and J. You, On UV/IR mixing in noncommutative gauge field theories, JHEP 1112 (2011) 081, doi:10.1007/JHEP12(2011)081, [arXiv:1109.2485].

[17] C. P. Martin, J. Trampetic and J. You, Super Yang-Mills and θ-exact Seiberg-Witten map: absence of quadratic noncommutative IR divergences, JHEP 1605 (2016) 169, doi:10.1007/JHEP05(2016)169 [arXiv:1602.01333 [hep-th]].

[18] C. P. Martin, J. Trampetic and J. You, Equivalence of quantum field theories related by the θ-exact Seiberg-Witten map, Phys. Rev. D 94 (2016) no.4, 041703, doi:10.1103/PhysRevD.94.041703 [arXiv:1606.03312 [hep-th]].

[19] C. P. Martin, J. Trampetic and J. You, Quantum duality under the θ-exact Seiberg-Witten map, JHEP 1609 (2016) 052, arXiv:1607.01541 [hep-th].

[20] R. Horvat, J. Trampetic and J. You, Spacetime Deformation Effect on the Early Universe and the PTOLEMY Experiment, arXiv:1703.04800 [hep-ph].

[21] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoi, Q deformation of Poincare algebra, Phys. Lett. B 264 (1991) 331.

[22] J. Lukierski, A. Nowicki and H. Ruegg, New quantum Poincare algebra and k deformed field theory, Phys. Lett. B 293 (1992) 344.

[23] S. Meljanac, A. Samsarov, M. Stojic and K. S. Gupta, Kappa-Minkowski space-time and the star product realizations, Eur. Phys. J. C 53 (2008) 295 doi:10.1140/epjc/s10052-007-0450-0 [arXiv:0705.2471 [hep-th]].

[24] T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac and D. Meljanac, Twisted statistics in kappa-Minkowski spacetime, Phys. Rev. D 77 (2008) 105010, doi:10.1103/PhysRevD.77.105010, [arXiv:0802.1576 [hep-th]].

[25] S. Meljanac and S. Kresic-Juric, Differential structure on kappa-Minkowski space, and kappa-Poincare algebra, Int. J. Mod. Phys. A 26 (2011) 3385, doi:10.1142/S0217751X11053948, [arXiv:1004.4647 [math-ph]].

[26] S. Meljanac, A. Samsarov, J. Trampetic and M. Wohlgenannt, Scalar field propagation in the φ⁴κ-Minkowski model, JHEP 1112 (2011) 010 doi:10.1007/JHEP12(2011)010 [arXiv:1111.5553 [hep-th]].

[27] T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac and D. Meljanac, Deformed Oscillator Algebras and QFT in κ-Minkowski Spacetime, Phys. Rev. D 80 (2009) 025014, [arXiv:0903.2355 [hep-th]].

[28] H. S. Snyder, Quantized space-time, Phys. Rev. 71, 38 (1947), doi:10.1103/PhysRev.71.38.

[29] M. Maggiore, The Algebraic structure of the generalized uncertainty principle, Phys. Lett. B 319 (1993) 83 doi:10.1016/0370-2693(93)90785-G, [hep-th/9309034].

[30] M. V. Battisti and S. Meljanac, Scalar Field Theory on Non-commutative Snyder Space-Time, Phys. Rev. D 82 (2010) 024028 doi:10.1103/PhysRevD.82.024028 [arXiv:1003.2108 [hep-th]].

[31] S. Mignemi, Classical dynamics on Snyder spacetime, Int. J. Mod. Phys. D 24 (2015) no.6, 1550043 doi:10.1142/S0218271815500431 [arXiv:1308.0673 [hep-th]].

[32] S. Mignemi and R. Straijn, Path integral in Snyder space, Phys. Lett. A 380 (2016) 1714 doi:10.1016/j.physleta.2016.03.005 [arXiv:1509.05311 [hep-th]].
[33] L. Lu and A. Stern, *Snyder space revisited*, Nucl. Phys. B **854** (2012) 894 doi:10.1016/j.nuclphysb.2011.09.022 [arXiv:1108.1832 [hep-th]].

[34] L. Lu and A. Stern, *Particle Dynamics on Snyder space*, Nucl. Phys. B **860** (2012) 186 doi:10.1016/j.nuclphysb.2012.02.012 [arXiv:1110.4112 [hep-th]].

[35] F. Girelli and E. R. Livine, *Scalar field theory in Snyder space-time: Alternatives*, JHEP **1103** (2011) 132 doi:10.1007/JHEP03(2011)132 [arXiv:1004.0621 [hep-th]].

[36] S. Meljanac, D. Meljanac, S. Mignemi and R. Štrajn, *Snyder-type spaces, twisted Poincaré algebra and addition of momenta*, arXiv:1608.06207 [hep-th].

[37] S. Meljanac, D. Meljanac, F. Mercati and D. Pikutic, *Noncommutative spaces and Poincaré symmetry*, Phys. Lett. B **766** (2017) 181, doi:10.1016/j.physletb.2017.01.006 [arXiv:1610.06716 [hep-th]].

[38] S. Meljanac, D. Meljanac, S. Mignemi and R. Štrajn, *Quantum field theory in generalised Snyder spaces*, Phys. Lett. B **768** (2017) 321, doi:10.1016/j.physletb.2017.02.059 arXiv:1701.05862 [hep-th].

[39] V. G. Kupriyanov and R. J. Szabo, *$G_2$-structures and quantization of non-geometric M-theory backgrounds*, JHEP **1702** (2017) 099 doi:10.1007/JHEP02(2017)099 [arXiv:1701.02574 [hep-th]].

[40] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, JHEP **0002** (2000) 020 doi:10.1088/1126-6708/2000/02/020, [hep-th/9912072].

[41] H. Grosse and M. Wohlgenannt, *On $\kappa$-deformation and UV/IR mixing*, Nucl. Phys. B **748** (2006) 473, [arXiv:hep-th/0507030].

[42] R. Horvat, J. Trampetic, *Constraining noncommutative field theories with holography*, JHEP **1101**, 112 (2011) [arXiv:1009.2933 [hep-ph]].

[43] A. G. Cohen, D. B. Kaplan and A. E. Nelson, *Effective field theory, black holes, and the cosmological constant*, Phys. Rev. Lett. **82**, 4971 (1999) [arXiv:hep-th/9803132].

[44] Wolfram Research, Inc., *Mathematica*, Version 8.0, Champaign, IL (2010).

[45] J. Martin-Garcia, *xAct*, http://www.xact.es/