Degeneration of the spectral gap with negative Robin parameter

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Abstract
The spectral gap of the Neumann and Dirichlet Laplacians are each known to have a sharp positive lower bound among convex domains of a given diameter. Between these cases, for each positive value of the Robin parameter, an analogous sharp lower bound on the spectral gap is conjectured. In this paper, we show that the extension of this conjecture to negative Robin parameters fails by proving that the spectral gap of double cone domains are exponentially small, for each fixed parameter value.

KEYWORDS
Laplacian, Robin boundary conditions, spectral gap

1 INTRODUCTION

The Robin Laplacian eigenvalue problem with parameter \( \alpha \) on a bounded Lipschitz domain \( D \) is

\[
\begin{aligned}
-\Delta u &= \lambda u & \text{on } D, \\
\delta_{\partial} u + \alpha u &= 0 & \text{on } \partial D.
\end{aligned}
\]

It has a discrete spectrum of eigenvalues for each \( \alpha \in (-\infty, \infty) \):

\[\lambda_1(D) < \lambda_2(D) \leq \lambda_3(D) \leq \cdots \to \infty.\]

Recall that when \( \alpha = 0 \), this is the Neumann eigenvalue problem. The limiting case \( \alpha = \infty \) corresponds to Dirichlet boundary conditions. See [11, Chapter 4] for foundational material on the Robin problem.

In this paper, we prove that when \( \alpha < 0 \), the spectral gap

\[\lambda_2 - \lambda_1(D)\]

can be arbitrarily small among domains \( D \) of a given diameter. Investigation of this result for \( \alpha < 0 \) was motivated by the following conjecture of Andrews, Clutterbuck, and Hauer [3, section 10] for \( \alpha \geq 0 \):

**Conjecture 1.1** (Robin gap). If \( \alpha \geq 0 \) and \( D \subset \mathbb{R}^n \) is a bounded convex domain, then

\[\lambda_2 - \lambda_1(D) \geq (\lambda_2 - \lambda_1)(I),\]

where \( I \subset \mathbb{R} \) is an open interval of length equal to the diameter of \( D \).
The double cone domain $D_\theta$ in three dimensions with $\theta = \pi/5$.

The Robin gap conjecture is known to hold in the Neumann case ($\alpha = 0$) due to Payne and Weinberger [21]. A new proof was given by Andrews and Clutterbuck [2, p. 901]. In the Dirichlet case ($\alpha = \infty$), it is known, not only for the Laplacian, but also for Schrödinger operators with a convex potential, thanks to Andrews and Clutterbuck [2, Corollary 1.4].

To state our main result, define the family of convex double cone domains of angle $\theta$ by

$$D_\theta = \{(x, y) \in (-1,1) \times \mathbb{R}^{n-1} : |y| < \tan(\theta/2)(1 - |x|)\},$$

for each $\theta \in (0, \pi)$, as shown in Figure 1.

**Theorem 1.2** (Degeneration of the Robin gap). If $\alpha < 0$, then $(\lambda_2 - \lambda_1)(D_\theta) \to 0$ as $\theta \to 0$. In fact, if $\alpha < 0$, then for each $\epsilon > 0$, there exists a constant $C > 0$ such that

$$(\lambda_2 - \lambda_1)(D_\theta) \leq C \exp\{-4(1 - \epsilon)|\alpha|/\theta\},$$

for all $\theta$ sufficiently small. (1)

The scaling relation $\lambda_j(tD; \alpha) = t^{-2} \lambda_j(D; t\alpha)$ then shows that $(\lambda_2 - \lambda_1)(tD_\theta) \to 0$ as $\theta \to 0$, for each $t > 0$. Since $tD_\theta$ has diameter $2t$ when $\theta \in (0, \pi/2]$, Theorem 1.2 implies that among convex $D$ of any given diameter, the spectral gap $(\lambda_2 - \lambda_1)(D)$ can be made arbitrarily small when $\alpha < 0$. In particular, since $(\lambda_2 - \lambda_1)(I) > 0$, there exist convex domains $D$ such that $(\lambda_2 - \lambda_1)(D) < (\lambda_2 - \lambda_1)(I)$, and so the Robin gap conjecture fails to extend to $\alpha < 0$.

We believe the upper bound in Theorem 1.2 is sharp in the sense that estimate (1) fails when $\epsilon = 0$ as $\theta \to 0$. Although we do not prove this, the proof of our upper bound on $\lambda_2(D_\theta)$ and Lemma 4.2 suggest it is true.

The remainder of this paper is structured as follows: In Section 2, we discuss a heuristic explaining Theorem 1.2; in Section 3, we present a brief overview of relevant literature; in Section 4, we prove a lemma about convergence of Schrödinger eigenvalues; in Section 5, we prove Theorem 1.2; and in Section 6, we suggest a variant of Theorem 1.2 to a modification of the domains $D_\theta$ and discuss the possibility of extending Conjecture 1.1 for $\alpha < 0$ to a restricted class of domains. In the Appendix, we discuss results concerning the existence of the spectrum of the aforementioned Schrödinger eigenvalue problem and a weighted eigenvalue problem that is used extensively in the proof of Theorem 1.2.

## 2 A HEURISTIC FOR SMALL GAPS IN THEOREM 1.2: DISJOINT UNION OF CONES

The Robin eigenvalue problem with $\alpha < 0$ on the infinite cone of opening angle $\theta < \pi$,

$$C_\theta = \{(x, y) \in (0, \infty) \times \mathbb{R}^{n-1} : |y| < \tan(\theta/2)x\},$$

has at least one negative eigenvalue and a corresponding ground state (see Equation 16) that concentrates exponentially fast (in $L^2$-norm), as $\theta \to 0$. In the two-dimensional (2D) case, it is known that all the eigenfunctions concentrate in this fashion (see [14, Theorem 5.1]).

Because of this concentration, we expect that the eigenfunctions of $D_\theta$ concentrate at each vertex ($\pm 1, 0$) like the ground state of $C_\theta$ concentrates at the origin. Consequently, for small $\theta$, the Robin problem on $D_\theta$ should be approximated by the Robin problem on two copies of $C_\theta$. This approximation suggests $(\lambda_2 - \lambda_1)(D_\theta) \to 0$ as $\theta \to 0$, because the first eigenvalue on $C_\theta \cup C_\theta$ has multiplicity two and hence the spectral gap equals zero.

Proving the gap tends to zero is a difficult task because $\lambda_1(C_\theta) = -\alpha^2/\sin^2(\theta/2) \sim \theta^{-2}$ (see Equation 16). Thus, Theorem 1.2 says that even though $\lambda_1(D_\theta)$ and $\lambda_2(D_\theta)$ grow like $\theta^{-2}$, they are still exponentially close as $\theta \to 0$. Indeed, since
\( \lambda_1(D_\theta) \leq \lambda_2(D_\theta) \), estimate (15) in the proof of Theorem 1.2 actually shows that for each \( \varepsilon > 0 \),
\[
\lambda_1(D_\theta), \lambda_2(D_\theta) = -\frac{\alpha^2}{\sin^2(\theta/2)} + O(\exp(-4(1 - \varepsilon)|\alpha|/\theta)), \quad \text{as } \theta \to 0.
\]

### 3 | LITERATURE

For an overview of the Robin Laplacian and related shape optimization problems, see the article by Bucur, Freitas, and Kennedy [11, Chapter 4]. In the study of superconductors, Robin boundary conditions with negative parameters are also known for their connection to Ginzburg–Landau theory [12, section 1.4].

Recently, Ashbaugh and the current author [5], as well as Andrews, Clutterbuck, and Hauer [4], proved sharp lower bounds on the Robin spectral gap for 1D Schrödinger operators with various classes of single-well potentials. In contrast with Theorem 1.2, some of these lower bounds continue to hold for some \( \alpha < 0 \).

The Robin Laplacian on the infinite cone with small aperture will play a crucial role throughout this paper. Khalile and Pankrashkin [14] computed eigenvalue asymptotics for the small aperture limit of 2D infinite cones. Recently, this result was generalized to higher dimensional cones whose cross sections are a Lipschitz domain by Pankrashkin and Vogel [19]. This geometric limit is related to the \( \alpha \to -\infty \) limit, for which Khalile [13] proved that the first \( N = N(D) \) Robin eigenvalues of a fixed curvilinear polygon \( \theta \) are determined (to leading order in \( \alpha \)) by the angles of the vertices. For smooth domains, the second term in the \( \alpha \to -\infty \) asymptotic is determined by the mean curvature; see work of Pankrashkin and Popoff [20]. In related work, Bögli, Kennedy, and Lang [6] proved convergence of certain eigenvalues of the Robin Laplacian to those of the Dirichlet Laplacian for certain limits \( \alpha \to \infty \), for complex \( \alpha \).

Finally, sequences of domains formed by intersecting two infinite sectors with vertices that become far apart were studied by Helffer and Pankrashkin [10]. It was shown that the spectral gap of these domains is exponentially small as the distance between the vertices grows. Although these domains have a large diameter, the techniques used to study them may be applicable in studying eigenvalues of fixed diameter domains like \( D_\theta \) as well.

### 4 | CONVERGENCE OF SCHRODINGER EIGENVALUES ON EXPANDING BALLS

In this section, we prove the convergence and calculate the convergence rate as \( R \to \infty \) of the first eigenvalue \( E = E(R) \) of the Schrödinger eigenvalue problem
\[
\begin{cases}
(\Delta - \frac{k}{r})\varphi = E\varphi & \text{on } B(R),

(\partial_r + \gamma)\varphi = 0 & \text{on } \partial B(R),
\end{cases}
\]

where \( B(R) \) is the ball of radius \( R \) centered at the origin, \( k > 0 \) is constant, and \( \gamma \in (-\kappa/(n - 1), \infty] \). When \( \gamma = \infty \), we impose Dirichlet boundary conditions on \( \partial B(R) \). The following lemma characterizes the spectrum of problem (3) in the cases \( R < \infty \) and \( R = \infty \). When \( R = \infty \), we view problem (3) as the eigenvalue problem for the operator \(-\Delta - k/r \) on \( \mathbb{R}^n \).

The following lemma is proved in the Appendix and shows that the eigenvalues discussed above are well defined. In the following lemma, let \( H = H^1(B(R)) \) when \( \gamma \in (-\infty, \infty) \) and \( H = H^1_0(B(R)) \) when \( \gamma = \infty \).

**Lemma 4.1.** Assume the \( \gamma \in (-\infty, \infty] \) and \( k > 0 \) are fixed. If \( R < \infty \), then problem (3) has a discrete spectrum of eigenvalues
\[
E_1 < E_2 \leq E_3 \leq \cdots \to +\infty,
\]
with corresponding eigenfunctions in \( H \). Moreover, the eigenvalues have the following variational characterization,
\[
E_j = \min_F \max_{f \neq 0} \frac{\int_{B(R)} |\nabla f|^2 + \frac{k}{r} f^2 \, dV + \gamma \int_{\partial B(R)} f^2 \, dS}{\int_{B(R)} f^2 \, dV},
\]
where the infimum is taken over \( j \)-dimensional subspaces \( F \subset H \).
Note that the lemma does not readily follow from facts about problems without potentials because the $-\kappa/r$ potential is not bounded from below on $B(R)$.

Since the potential in (3) is radial, (3) reduces to an ordinary differential equation (ODE) eigenvalue problem that we analyze using special functions. In the proof of Theorem 1.2 in a later section, we will get a lower bound on $\lambda_1(D_0)$ by proving a lower bound on $\lambda_1(D_0) - \lambda_1(C_0)$ in terms of $E(R)$, where $R = \cot(\theta/2) \to \infty$ as $\theta \to 0$.

The aforementioned ODE eigenvalue problem gives rise to the confluent hypergeometric equation

$$\rho \frac{d^2 F}{d\rho^2} + (b - \rho) \frac{dF}{d\rho} - a F = 0,$$

for certain $a \in \mathbb{R}$ and $b \notin \{0, -1, -2, \ldots \}$, as we shall see in the proof of the next lemma. The solution $F = F(a, b; \rho)$ is a linear combination of the confluent hypergeometric functions of the first and second kind, $M(a, b; \rho)$ and $U(a, b; \rho)$, respectively. For more information on this ODE see Chapter 13 by Slater of the book edited by Abramowitz and Stegun [1] or Chapter 13 of the NIST Digital Library of Mathematical Functions [7].

In Lemma 4.2 below, we show that the first eigenvalue of (3) is determined by a positive solution $z$ of the transcendental equation

$$2 a_0 b_0 M(a_0 + 1, b_0 + 1; \rho_0) = (1 - 2\gamma z/\kappa) M(a_0, b_0; \rho_0),$$

where

$$m = (n - 3)/2, \quad a_0 = a_0(z) = m + 1 - z, \quad b_0 = 2m + 2, \quad \rho_0 = \rho_0(z) = \kappa R/z.$$  

When $\gamma = \infty$, the transcendental equation is $M(a_0, b_0; \rho_0) = 0$. To see this case formally, divide by $\gamma$ in (5) and let $\gamma \to \infty$.

**Lemma 4.2.** Assume that $\gamma \in (-\infty, \infty)$ and $\kappa > 0$ are fixed.

(i) If $R < \infty$, then the radial eigenfunctions of (3) with $E < 0$ are multiples of

$$M\left(a_0, b_0; 2\sqrt{|E|} R\right) e^{-\sqrt{|E|} R}$$

and have eigenvalues $E = -\kappa^2/4z^2$, where $z$ is a positive solution of (5).

(ii) If $R = \infty$, then the radial eigenfunctions of (3) are multiples of (6) with eigenvalues $E = -\kappa^2/4(j + (n - 3)/2)^2$ for $j \in \mathbb{N}$. Furthermore, $M\left(a_0, b_0; 2\sqrt{|E|} R\right)$ is a polynomial in $r$ of degree $j - 1$.

(iii) If $\gamma \in (-\kappa/(n - 1), \infty)$, then for each $\epsilon > 0$, there is a solution $z_\epsilon$ of (5) that corresponds to the first eigenvalue $E$ of (3) and satisfies $z_\epsilon(R) = (n - 1)/2 + O(\exp\{-2(1 - \epsilon)\kappa R/(n - 1)\})$, as $R \to \infty$. Consequently,

$$E(R) = -\kappa^2/(n - 1)^2 + O\left(\exp\left\{-\frac{2\kappa}{n - 1} R\right\}\right), \quad \text{as } R \to \infty.$$  

Part (iii) of the lemma shows that the first eigenvalue of (3) converges to $-\kappa^2/(n - 1)^2$ as $R \to \infty$, which is the first eigenvalue of $-\Delta - \kappa/r$ on $\mathbb{R}^n$.

Computing the spectrum of the Schrödinger operator $-\Delta - \kappa/r$ on $B(R)$ and $\mathbb{R}^n$ is closely related to computing the spectrum of the hydrogen atom from quantum mechanics. In fact, on $\mathbb{R}^3$, this calculation is precisely the hydrogen atom calculation (see the book by Landau and Lifschitz [16, Chapter V], for example). For general $n \geq 3$, the ODE in (7) below is the radial part of the Schrödinger operator for the 3D hydrogen atom with angular momentum $m$ (after changing variables). This physical interpretation breaks down when $n = 2$ since in this case $m < 0$. The operator in (7) also appears in the work of Khalile and Pankrashkin [14] as an effective operator for the Robin Laplacian with negative Robin parameter on $C_0$ as $\theta \to 0$, when $n = 2$. 

Proof of Lemma 4.2. Part (i):
Let \( \varphi = \varphi(r, \xi) \) be an arbitrary radial eigenfunction of (3) with eigenvalue \( E = E(R) \) written in \((r, \xi)\)-spherical coordinates. Thus, \( \varphi(r, \xi) \) is a constant times a function \( u(r) \). In this case, \( -\Delta - \kappa/r \) reduces to the radial part of the Laplacian plus the potential \(-\kappa/r\) so that
\[
\begin{cases}
  -u'' - \frac{n-1}{r} u' - \frac{\kappa}{r} u = Eu & \text{on } (0, R), \\
  u' + \gamma u = 0 & \text{at } r = R.
\end{cases}
\]
In the above problem and the ones below, we impose Dirichlet boundary conditions when \( \gamma = \infty \). Let \( m = (n-3)/2 \) and \( v(r) = r^{m+1} u(r) \) so that
\[
\begin{cases}
  -v'' + \frac{m(m+1)}{r^2} v - \frac{\kappa}{r} v = Eu & \text{on } (0, R), \\
  v' + \left( \gamma - \frac{m+1}{r} \right) v = 0 & \text{at } r = R.
\end{cases}
\]
(7)
Since \( E < 0 \), we can define \( z = \sqrt{2|E|} \) so that \( \rho = \frac{\kappa r}{z} \), and \( w(\rho) = v(\frac{2}{z} \rho) \). It follows that \( w \) satisfies
\[
\begin{cases}
  -w'' + \frac{m(m+1)}{\rho^2} w - \frac{\kappa}{\rho} w = 0 & \text{on } (0, \rho_0), \\
  w' + \frac{2\gamma z}{\rho} w = 0 & \text{at } \rho = \rho_0,
\end{cases}
\]
where \( \rho_0 = \rho_0(z) = \frac{\kappa R}{z} \).

Guessing a solution of the form \( w(\rho) = \rho^{m+1} e^{-\rho/2} F(\rho) \), we find that \( F(\rho) = F(a_0, b_0; \rho) \) is a solution of (4) on the interval \((0, \rho_0)\) with \( a_0(z) = m + 1 - z \), \( b_0 = 2m + 2 \), and boundary condition \( 2F'(a_0, b_0; \rho_0) = (1 - 2\gamma z/r) F(a_0, b_0; \rho_0) \).

When \( \gamma = \infty \), the boundary condition \( F(a_0, b_0; \rho_0) = 0 \) is immediate from the form of the ansatz. Thus, the original radial eigenfunctions \( \varphi \) are multiples of \( F(\rho) \).

The general solution of (4) is a linear combination of the confluent hypergeometric functions of the first and second kind, \( M(a_0, b_0; \rho) \) and \( U(a_0, b_0; \rho) \), respectively. It follows from formulas 13.4.21, 13.5.5, and 13.5.7 in [1] that
\[
U'(a_0, b_0; \rho) = -a_0 \frac{\Gamma(b_0)}{\Gamma(a_0 + 1)} \rho^{-b_0} + o(\rho^{-b_0}), \quad \text{as } \rho \to 0,
\]
so that \( U'(a_0, b_0; \rho) \not\in L^2(B(R)) \) since \( b_0 = n - 1 \). Thus, \( F(\rho) \) must have no \( U(a_0, b_0; \rho) \) component to guarantee that \( \varphi \in H^1(B(R)) \). This shows that \( F(\rho) \) is a polynomial of degree \(-a\) (see (8)). Thus, we must have \( a_0 \in \{0, -1, -2, \ldots\} \) to ensure \( \varphi \) tends to zero as \( r \to \infty \). It follows from the definition of \( a_0 = m + 1 - z \) that \( z = \kappa(2\sqrt{|E|})^{-1} \) and \( E = -\kappa^2/4(4j + m)^2 \) for \( j \in \mathbb{N} \), where \( m = (n-3)/2 \).

Part (ii): The same calculation shows that on \( \mathbb{R}^n \) the eigenfunctions \( \varphi \) are multiples of (6) with eigenvalue \(-\kappa^2/4z^2\). Note that if \( a \) and \( b \) are fixed and \( a \notin \{0, -1, -2, \ldots\} \), then \( M(a, b; \rho) \sim \rho^{a-b} e^\rho \) as \( \rho \to \infty \) (see [1, 13.1.4]). If \( a \in \{0, -1, -2, \ldots\} \), then \( M(a, b; \rho) \) is a polynomial of degree \(-a\) (see (8)). Thus, we must have \( a_0 \in \{0, -1, -2, \ldots\} \) to ensure \( \varphi \) tends to zero as \( r \to \infty \). Let \( 0 < \epsilon < 1 \).

Case 1: \( |\gamma| < \kappa/(2(m + 1)) \): Put
\[
z_-(R) = m + 1 - e^{-(1-\epsilon)\kappa R/(m+1)} \quad \text{and} \quad z_+ = m + 1.
\]
We show that there exists a \( z_* = z_*(R) \in (z_-(R), z_+) \) that solves (5) for all \( R \) sufficiently large, and \( z_* \not\in m + 1 \) exponentially fast, as \( R \to \infty \). We do this by applying the intermediate value theorem to the function
\[
M(z) = 2 \frac{a_0}{b_0} M(a_0 + 1, b_0 + 1; \rho_0) - (1 - 2\gamma z/\kappa) M(a_0, b_0; \rho_0),
\]
where recall that \( a_0 = a_0(z) = m + 1 - z, \ b_0 = 2m + 2, \) and \( \rho_0 = \rho_0(z) = xR/z. \) (Note that \( M(z) = 0 \) if and only if \( z \) satisfies the transcendental Equation 5.) The desired convergence for \( E(R) = -\kappa^2/4z_+(R)^2 \) as \( R \to \infty \) then follows immediately.

We show that \( M(z_+) < 0 \) and that \( M(z) > 0 \) for \( z \leq z_-(R) \) in order to apply the intermediate value theorem on the interval \([z_-, z_+]\) for each fixed \( R \) sufficiently large. To prove these estimates on \( M, \) we use the power series representation

\[
M(a, b; \rho) = \sum_{k=0}^{\infty} \frac{a^{(k)} b^{(k)}}{a_0^{(k)} b_0^{(k)}} \rho_0^k k!, \quad \text{for } a \in \mathbb{R}, \ b \notin \{0, -1, -2, \ldots\}, \tag{8}
\]

where \( a^{(k)} \) is the rising factorial. That is, \( a^{(0)} = 1 \) and \( a^{(k)} = a(a+1) \cdots (a+k-1) \) for \( k \geq 1. \) Observe that \( z \mapsto M(a_0(z), b; \rho_0(z)) \) is continuous for each fixed \( R, \) since the series converges uniformly in \( z \) on compact subsets of \((0, \infty)\) by the Weierstrass M-test.

Note that \( M(a_0, b_0; \cdot) \) is positive by (8) when \( 0 < z < m+1 \) because \( a_0(z) > 0. \) Thus, as soon as we prove that a solution \( 0 < z_* < m+1 \) exists, it corresponds to the first eigenvalue \( E \) of (3) since the ground state is the unique positive eigenfunction.

First, we prove \( M(z_+) < 0 \) for each \( R > 0. \) Indeed, since \( a_0(m+1) = 0, \) we know

\[
M(z_+) = M(m+1) = -(1 - 2\gamma(m+1)/\kappa) M(0, b_0; \rho_0) < 0
\]

because \( \gamma < \kappa/(2(m+1)) \) and \( M(0, b_0; \cdot) \) is identically 1.

Now we prove \( M(z) > 0 \) when \( z \leq z_-(R) \) and \( R \) is sufficiently large. First, choose \( \delta = \delta(\gamma) \in (0, 1) \) large enough that \( z \mapsto (\delta + 2\gamma z/\kappa) \) is uniformly positive on \( [0, m+1] \), which is possible since \( \gamma > -\kappa/(2(m+1)) \). Next, observe that there is a \( \delta = \delta(\gamma, m) \) independent of \( z \) such that

\[
2 \frac{k + m + 1 - z}{k + 2m + 2} \geq 1 + \delta, \quad \text{for each } k \geq K. \tag{9}
\]

The \( k \)th coefficient of the power series of the first term of \( M \) satisfies the following identity, by considering the first and last factors in the rising factorials:

\[
2 \frac{a_0 (a_0 + 1)^{(k)}}{b_0 (b_0 + 1)^{(k)}} = 2 \frac{m + 1 - z (m + 2 - z)^{(k)}}{(2m + 3)^{(k)}} = 2 \frac{k + m + 1 - z (m + 1 - z)^{(k)}}{k + 2m + 2 (2m + 2)^{(k)}},
\]

for each \( k \geq 0. \) Applying the lower bound (9), we have

\[
2 \frac{a_0 (a_0 + 1)^{(k)}}{b_0 (b_0 + 1)^{(k)}} \geq (1 + \delta) \frac{a_0^{(k)}}{b_0^{(k)}}, \quad \text{for each } k \geq K. \tag{10}
\]

Dropping the first \( K \) terms in the power series of \( M(a_0 + 1, b_0 + 1; \rho_0), \) applying (10), and adding and subtracting \( 1 + \delta \) times the first \( K \) terms in the series for \( M(a_0, b_0; \rho_0) \) shows

\[
2 \frac{a_0}{b_0} M(a_0 + 1, b_0 + 1; \rho_0) \geq (1 + \delta) \sum_{k=K}^{\infty} \frac{a_0^{(k)} \rho_0^k}{b_0^{(k)} k!} = (1 + \delta) M(a_0, b_0; \rho_0) - (1 + \delta) \sum_{k=0}^{K-1} \frac{a_0^{(k)} \rho_0^k}{b_0^{(k)} k!}. \]

Subtracting and adding \( (2\gamma z/\kappa) M(a_0, b_0; \rho_0) \) to the right side shows

\[
2 \frac{a_0}{b_0} M(a_0 + 1, b_0 + 1; \rho_0) \geq (1 - 2\gamma z/\kappa) M(a_0, b_0; \rho_0) + \mathcal{E}(\rho_0, z), \tag{11}
\]

where

\[
\mathcal{E}(\rho_0, z) = (\delta + 2\gamma z/\kappa) M(a_0, b_0; \rho_0) - (1 + \delta) \sum_{k=0}^{K-1} \frac{(m + 1 - z)^{(k)} \rho_0^k}{(2m + 2)^{(k)} k!}.
\]
To conclude that \( M(z) > 0 \) from (11), we will show that \( \mathcal{E}(\rho_0, z) > 0 \) for \( z \leq z_-(R) \), when \( R \) is sufficiently large. In what follows, \( C \) is a positive constant that is independent of both \( k \) and \( z \) and may change from line to line. Since \( z \mapsto (\delta + 2\gamma z / \kappa) \) is uniformly positive on \([0, m + 1]\) by our choice of \( \delta \) and the second term of \( \mathcal{E}(\rho_0, z) \) satisfies

\[
(1 + \delta) \sum_{k=0}^{K-1} \frac{(m + 1 - z)^{(k)}}{(2m + 2)^{(k)}} \frac{\rho_0^k}{k!} \leq C(1 + \rho_0^K),
\]  

(12)

to obtain \( \mathcal{E}(\rho_0, z) > 0 \), it suffices to show that

\[
M(a_0, b_0; \rho_0) \geq C e^{(\epsilon/2)\rho_0}.
\]

(Note \( \rho_0 = x R / z \geq x R / m \to \infty \), as \( R \to \infty \) and so \( e^{(\epsilon/2)\rho_0} \) dominates \( \rho_0^K \).)

We estimate the factor \( a_0^{(k)} / b_0^{(k)} \) from below using that \( m + 1 - z > 0 \) and converting the rising factorials to ordinary factorials so that for \( k \geq 1 \),

\[
\frac{a_0^{(k)}}{b_0^{(k)}} = \frac{(m + 1 - z)^{(k)}}{(2m + 2)^{(k)}} \geq \frac{(m + 1 - z)}{(2m + 2)(k)} = \frac{(m + 1 - z)(k - 1)!}{(2m + 2 + (k - 1))!}.
\]

Noting that \( (2m + 1)! \) only depends on the dimension \( n \) and making cancellations leads to the lower bound

\[
\frac{a_0^{(k)}}{b_0^{(k)}} \geq (m + 1 - z) \frac{C}{(k + 2m + 1)^{2m+2}}, \quad \text{for } k \geq 0.
\]

Using this lower bound, observing \((k + 2m + 1)^{-2m+2} \geq C(1 - \epsilon / 2)^k\), and using the Taylor series for \( e^\rho \), we have

\[
M(a_0, b_0; \rho_0) \geq C(m + 1 - z) \sum_{k=0}^{\infty} \frac{1}{(k + 2m + 1)^{2m+2}} \rho_0^k
\]

\[
\geq C(m + 1 - z) \sum_{k=0}^{\infty} \frac{(1 - \epsilon / 2)^k}{k!} 
\]

\[
= C(m + 1 - z)e^{(1-\epsilon/2)\rho_0}.
\]

Finally, using that \( z < z_-(R) \leq m + 1 - e^{-(1-\epsilon)\rho_0} \) shows that

\[
(m + 1 - z)e^{(1-\epsilon/2)\rho_0} \geq e^{(\epsilon/2)\rho_0},
\]

and hence (12) holds.

Since \( z \mapsto M(a_0(z), b_0; \rho_0(z)) \) is continuous, the intermediate value theorem shows that there exists a solution \( z_* \) to (5) that satisfies \( z_-(R) < z_* < z_+ = m + 1 \). Using \( m + 1 = (n - 1)/2 \), we have

\[
\frac{n - 1}{2} - \exp \left\{ -(1 - \epsilon) \frac{2\kappa}{n - 1} R \right\} \leq z_* \leq \frac{n - 1}{2},
\]

for all \( R \) sufficiently large. This shows the first estimate in part (iii) of the lemma for \( |\gamma| < \kappa / (2(m + 1)) \). Hence,

\[
E(R) = - \frac{\kappa^2}{4z_*(R)^2} = - \frac{\kappa^2}{(n - 1)^2} + O \left( \exp \left\{ -(1 - \epsilon) \frac{2\kappa}{n - 1} R \right\} \right), \quad \text{as } R \to \infty.
\]

(13)

This completes the proof of part (iii) of the lemma for \( |\gamma| < \kappa / (2(m + 1)) \).
**Case 2:** $\gamma \in [\pi/(2(m + 1)), \infty]$: In the remainder of the proof, let $E(R, \gamma)$ denote the first eigenvalue of the Schrödinger problem (3). To prove the asymptotic for $E(R, \gamma)$ when $\gamma \in [\pi/(2(m + 1)), \infty]$, notice that $\gamma \mapsto E(R, \gamma)$ is increasing for $\gamma \in (-\infty, \infty)$ by the variational characterization. In particular, $E(R, 0) \leq E(R, \gamma) \leq E(R, \infty)$ for each $\gamma > 0$. Thus, applying formula (13) with $\gamma = 0$ gives the desired lower bound for $E(R, \gamma)$ and it suffices to prove that the right-hand side of (13) is also an upper bound for the first Dirichlet eigenvalue $E(R, \infty)$.

By part (ii) of the lemma, the ground state of $-\Delta - \alpha/r$ on $\mathbb{R}^n$ is $\exp\{-\sqrt{|E(\mathbb{R}^n)|} r\}$ with eigenvalue $E(\mathbb{R}^n) = -\alpha^2/(n - 1)^2$. Let $\chi$ be a smooth radial cutoff function that is 1 for $r < 1 - \epsilon/2$, decreases to zero over $1 - \epsilon/2 \leq r \leq 1$, and is zero for $r > 1$. The function $f(r)$ defined by $\chi(r/R) \exp\{-\sqrt{|E(\mathbb{R}^n)|} r\}$ restricted to the ball $B(R)$ is a valid trial function for the first eigenvalue $E(R, \infty)$ of (3) so that Lemma 4.1 shows

$$E(R, \infty) \leq \int_{B(R)} f(-\Delta - \alpha/r) f \, dV \int_{B(R)} f^2 \, dV = E(\mathbb{R}^n) + O\left(\exp\left\{-2(1-\epsilon)\frac{\alpha}{\sin(\epsilon)}\right\}\right), \quad \text{as } R \to \infty.$$

The last equality follows from using the decay of $\exp\{-\sqrt{|E(\mathbb{R}^n)|} r\}$ and that $\chi(r/R)$ has bounded $r$-derivatives. These facts show that the second terms in the numerator and denominator are the same size as the error term in (13) and give the desired upper bound since $E(\mathbb{R}^n) = -\alpha^2/(n - 1)^2$.

This completes the proof of the asymptotic for $E(R, \gamma)$ for all $\gamma \in (-\pi/(2(m + 1)), \infty]$. Note that the asymptotic for $z_*$ in part (iii) of the lemma follows from the formula $z_* = \alpha \left(2\sqrt{|E(\mathbb{R}^n)|} \right)^{-1}$. \qed

### 5 \quad PROOF OF THEOREM 1.2

We prove Theorem 1.2 by getting an upper bound on $\lambda_2(D_\theta)$ and a lower bound on $\lambda_1(D_\theta)$ in terms of the first eigenvalue of the infinite cone $C_\theta$ (defined in (2)). Specifically, for each $\epsilon > 0$, we show that there exists a $C > 0$ such that

$$\lambda_2(D_\theta) \leq \lambda_1(C_\theta) + C \exp\{2(1-\epsilon)\alpha/\sin(\theta/2)\}, \quad \text{as } \theta \to 0,$$

and

$$\lambda_1(D_\theta) = \lambda_1(C_\theta) + O(\exp\{2(1-\epsilon)\alpha/\sin(\theta/2)\}), \quad \text{as } \theta \to 0.$$

Combining these bounds and using $\sin(\theta/2) \leq \theta/2$, we conclude that there is a $C > 0$ such that $(\lambda_2 - \lambda_1)(D_\theta) \leq C \exp\{-4(1-\epsilon)\alpha/\theta\}$ for all $\theta$ sufficiently small. This concludes the proof of Theorem 1.2.

To prove both inequality (14) and (15), we will need to use the ground state of the infinite cone $C_\theta$ to construct trial functions. A direct calculation shows the $L^2$-normalized ground state of the Robin Laplacian on $C_\theta$ is

$$\phi_\theta(x, y) = A_\theta e^{\alpha x/\sin(\theta/2)}, \quad \text{when } \alpha < 0 \text{ and } \theta < \pi,$$

with eigenvalue

$$\lambda_1(C_\theta) = -\alpha^2/\sin^2(\theta/2),$$

where the constant in (16) satisfies

$$A_\theta^{-1} = 2^{-n/2} |\alpha|^{-n} \omega_{n-1} \Gamma(n) \tan^{n-1}(\theta/2) \sin^n(\theta/2).$$

Here, $\omega_{n-1}$ is the volume of the $(n - 1)$-dimensional unit sphere and $\Gamma$ is the Gamma function.
5.1 Upper bound on $\lambda_2(D_\theta)$

In order to get an upper bound on $\lambda_2(D_\theta)$, we transplant two copies of $\phi_\theta$ to construct a trial function. Fix $0 < \epsilon < 1/2$ and let $\chi = \chi(x, y)$ denote a smooth nonnegative cutoff function on $\mathbb{R}^n$ that is independent of $y$, equals 1 for $x \leq 1 - \epsilon$, 0 for $x \geq 1$, and is less than or equal to 1 everywhere. Let $\psi_\theta = \chi \phi_\theta$ be a cutoff ground state and

$$ F(x, y) = (x + 1, y) \quad \text{and} \quad G(x, y) = (1 - x, y) $$

be rigid motions. Define the trial function $\tilde{\phi}_\theta : D_\theta \to \mathbb{R}$ by putting a cutoff ground state at each vertex of $D_\theta$:

$$ \tilde{\phi}_\theta = (\psi_\theta \circ F) - (\psi_\theta \circ G) \sqrt{2}. $$

Notice that $\tilde{\phi}_\theta$ is odd with respect to $x$, by construction.

Since $D_\theta$ is reflection symmetric in $x$, each eigenfunction with simple eigenvalue is either even or odd in $x$. The ground state must be even since it is nonnegative. Thus, the odd function $\tilde{\phi}_\theta$ is orthogonal to the ground state, and so it is a valid trial function for $\lambda_2(D_\theta)$. By the variational characterization,

$$ \lambda_2(D_\theta) \leq \frac{\int_{D_\theta} |\nabla \tilde{\phi}_\theta|^2 dV + \alpha \int_{\partial D_\theta} \tilde{\phi}_\theta^2 dS}{\int_{D_\theta} \tilde{\phi}_\theta^2 dV} = \frac{\int_{D_\theta} |\nabla \psi_\theta|^2 dV + \alpha \int_{\Sigma_\theta} \psi_\theta^2 dS}{\int_{D_\theta} \psi_\theta^2 dV}, $$

where we define

$$ T_\theta = C_\theta \cap \{x < 1\} \quad \text{and} \quad \Sigma_\theta = \partial T_\theta \cap \{0 \leq x < 1\} $$

and the truncated cone and the curved part of its boundary.

A straightforward calculation (see below) will show there are remainder terms

$$ \text{Rem}_i(\theta) = O(\exp(2(1 - 2\epsilon)\alpha / \sin(\theta/2))), \quad \text{as} \ \theta \to 0, \quad \text{for} \ i = 1, 2, 3, $$

such that

$$ \int_{T_\theta} |\nabla \psi_\theta|^2 dV = \int_{C_\theta} |\nabla \phi_\theta|^2 dV + \text{Rem}_1(\theta), \quad \int_{\Sigma_\theta} \psi_\theta^2 dS = \int_{\partial C_\theta} \phi_\theta^2 dS + \text{Rem}_2(\theta), $$

and

$$ \int_{T_\theta} \psi_\theta^2 dV = \int_{C_\theta} \phi_\theta^2 dV + \text{Rem}_3(\theta) = 1 + \text{Rem}_3(\theta), $$

as $\theta \to 0$, where the final equality uses the $L^2$-normalization of $\phi_\theta$. Combining these estimates with the upper bound (18) shows we have a remainder term $\text{Rem}(\theta) = (\text{Rem}_1(\theta) + \alpha \text{Rem}_2(\theta))/(1 + \text{Rem}_3(\theta)) = O(\exp(2(1 - 2\epsilon)\alpha / \sin(\theta/2)))$ such that

$$ \lambda_2(D_\theta) \leq \frac{\int_{C_\theta} |\nabla \phi_\theta|^2 dV + \alpha \int_{\partial C_\theta} \phi_\theta^2 dS}{\int_{T_\theta} \phi_\theta^2 dV} + \text{Rem}(\theta) = \frac{\lambda_1(C_\theta)}{\int_{T_\theta} \phi_\theta^2 dV} + \text{Rem}(\theta) \leq \frac{\lambda_1(C_\theta)}{\int_{T_\theta} \phi_\theta^2 dV} + \text{Rem}(\theta), $$

as $\theta \to 0$, since $\int_{T_\theta} \psi_\theta^2 dV \leq 1$ and $\lambda_1(C_\theta) < 0$. This proves (14) since $\epsilon$ is arbitrary.

We estimate $\text{Rem}_3(\theta)$; the other remainder terms are similar. The inequality $\int_{T_\theta} \psi_\theta^2 dV \leq \int_{C_\theta} \phi_\theta^2 dV$ follows from $\chi$ being less than 1. For an inequality in the opposite direction, since

$$ \int_{T_\theta} \psi_\theta^2 dV \geq \int_{C_\theta \cap \{x < 1 - \epsilon\}} \phi_\theta^2 dV = \int_{C_\theta} \phi_\theta^2 dV - \int_{C_\theta \cap \{x \geq 1 - \epsilon\}} \phi_\theta^2 dV, $$

we estimate $\text{Rem}_3(\theta)$ by $O(\exp{2(1 - 2\epsilon)\alpha / \sin(\theta/2)})$. Combining these with (18), we obtain the upper bound on $\lambda_2(D_\theta)$.

We finish this section with the following results.
it suffices to show the second term on the right is exponentially small. Since \( \theta_0 \) is constant in \( y \), we have that

\[
\int_{C_0 \cap \{ x \geq 1 - \epsilon \}} \phi_0^2 \, dV = A_0^2 \int_{1-\epsilon}^{\infty} \exp\{2\alpha x / \sin(\theta/2)\} \text{Vol}_{n-1}(B(\tan(\theta/2)x)) \, dx
\]

\[
= \Gamma(n)^{-1} \int_{2(1-\epsilon)\alpha / \sin(\theta/2)}^{\infty} \exp\{-z\}z^{n-1} \, dz,
\]

where we have made the change of variables \( z = 2\alpha x / \sin(\theta/2) \) for all sufficiently large \( z \), by integrating, we have

\[
\int_{C_0 \cap \{ x \geq 1 - \epsilon \}} \phi_0^2 \, dV \lesssim \exp\{2(1 - 2\epsilon)\alpha / \sin(\theta/2)\}, \quad \text{as } \theta \to 0,
\]

which shows that \( \text{Rem}_3(\theta) = O(\exp\{2(1 - 2\epsilon)\alpha / \sin(\theta/2)\}) \).

### 5.2 Lower bound on \( \lambda_1(D_\theta) \)

We prove the lower bound by viewing \( \lambda_1(D_\theta) - \lambda_1(C_\theta) \) as the first eigenvalue of a weighted problem on the truncated cone \( T_\theta \). Then, we “push out” the problem on \( T_\theta \) to a radial problem on a spherical sector \( S_\theta \). Finally, we transform the problem on \( S_\theta \) into the Schrödinger eigenvalue problem (3) whose solutions were analyzed in Lemma 4.2. We break up the proof into four steps.

Throughout Steps 1, 2, and 3 below, we use the weighted operator \( \Delta_\tau(\cdot) = \tau^{-1} \text{div}(\tau \nabla(\cdot)) \), called the \( \tau \)-Laplacian for a function \( \tau \). In the Appendix, Lemma A.1 states the existence of the spectrum and the variational characterization of the eigenvalues for problems involving the \( \tau \)-Laplacian.

**Step 1:** Let \( u \) denote the \( L^2 \)-normalized ground state of \( D_\theta \) restricted and translated to \( T_\theta = C_\theta \cap \{ x < 1 \} \). Since the ground state of \( D_\theta \) is even in \( x \), we know \( u \) has Neumann boundary conditions on the flat part of the boundary defined by

\[
\Gamma_\theta = T_\theta \cap \{ x = 1 \}
\]

and retains its Robin boundary conditions on the complement \( \Sigma_\theta = \partial T_\theta \cap \{ 0 \leq x < 1 \} \).

For notational convenience, write \( \phi \) instead of \( \phi_\theta \) for the Robin ground state of \( C_\theta \). As we will show shortly, the ratio \( v = u / \phi \) is the first eigenfunction of \( \Delta_\tau \) with \( \tau = \phi^2 \) and appropriate boundary conditions. Specifically, \( v \) satisfies

\[
\begin{cases}
-\Delta_\tau v = \mu_1 v & \text{on } T_\theta, \\
\partial_\nu v + (\alpha / \sin(\theta/2))v = 0 & \text{on } \Gamma_\theta, \\
\partial_\nu v = 0 & \text{on } \Sigma_\theta,
\end{cases}
\]

(19)

where \( \mu_1 = \mu_1(T_\theta) = \lambda_1(D_\theta) - \lambda_1(C_\theta) \) is the first eigenvalue of (19) since \( v \) is positive.

Problem (19) has a discrete spectrum by applying Lemma A.1 with domain \( \Omega = T_\theta \), weight \( \tau = \phi^2 \), and boundary parameter that is equal to \( \alpha / \sin(\theta/2) \) on \( \Gamma_\theta \) and 0 on \( \Sigma_\theta \), since the weight \( \tau = \phi^2 \) is bounded and uniformly positive on \( T_\theta \). Lemma A.1 also shows that problem (19) has Rayleigh quotient

\[
\frac{\int_{T_\theta} |V f|^2 \, dx + (\alpha / \sin(\theta/2)) \int_{\Gamma_\theta} f^2 \phi^2 \, dS}{\int_{T_\theta} f^2 \phi^2 \, dx}, \quad \text{for } f \in H^1(T_\theta) \setminus \{0\}.
\]

In particular, the Rayleigh quotient shows that \( \mu_1(T_\theta) < 0 \) by applying the variational characterization with trial function \( f \) equal to a constant.
To see that $v$ is an eigenfunction of $\Delta_\tau$, we compute derivatives and use that $u$ and $\phi$ are eigenfunctions of $-\Delta$:

\[-\Delta_\tau v = -\phi^{-2} \text{div}\left(\phi^2 \frac{\nabla u \phi - u\nabla \phi}{\phi^2}\right)\]

\[= -\phi^{-2}(\Delta u \phi + \nabla u \cdot \nabla \phi - \nabla u \cdot \nabla \phi - u \Delta \phi)\]

\[= (\lambda_1(D_\sigma) - \lambda_1(C_\sigma)) u.\]

The boundary conditions in (19) follow from computing

\[\partial_\nu v = \phi^{-2}[\partial_\nu u \phi - u \partial_\nu \phi],\]

and applying the boundary conditions of $u$ discussed at the beginning of Step 1, as well as the definition of $\phi$.

**Step 2:** Consider the spherical coordinates $(r, \xi)$, where $\xi = (\xi_1, ..., \xi_{n-1})$ is the vector of angle coordinates and $\xi_1$ is the angle between the point $(x,y)$ and the x-axis. In this step, we show $\mu_1(S_\delta) \leq (1 + o(1))\mu_1(T_\delta)$, as $\delta \to 0$, where $\mu_1(S_\delta)$ is the first eigenvalue of a certain radial problem on the spherical sector

\[S_\delta = \{(r, \xi) : 0 < r < \cos(\theta/2)^{-1}, |\xi_1| < \theta/2\}.\]

Note that this sector is obtained by “pushing out” the flat part of the boundary of $T_\delta$. To define this problem on $S_\delta$, let the “push in” diffeomorphism $P : S_\delta \to T_\delta$ be defined in spherical coordinates by

\[P(r, \xi) = (p_r, \xi),\]

where $p = p(\xi_1) = \cos(\theta/2) \sec(\xi_1)$.

More specifically, this problem is a $\sigma$-Laplacian eigenvalue problem with domain $\Omega = S_\delta$, $\sigma = \phi^2 \circ P$, and boundary parameter equal to $\beta/\sin(\theta/2)$ on the spherical part of the boundary $P^{-1}(\Gamma_\delta) = \delta S_\delta \cap \delta B(\cos(\theta/2)^{-1})$ and 0 on $\Sigma_\delta$. The parameter $\beta = \beta(\theta)$ is equal to $(1 + o(1))\alpha$ and will be defined precisely at the end of Step 2. This problem has a discrete spectrum by applying Lemma A.1 in a similar fashion to (19) and Rayleigh quotient

\[\frac{\int_{S_\delta} |
abla f|^2(\phi^2 \circ P)\,dV + (\beta/\sin(\theta/2)) \int_{P^{-1}(\Gamma_\delta)} f^2(\phi^2 \circ P)\,dS}{\int_{S_\delta} f^2(\phi^2 \circ P)\,dV}, \quad \text{for } f \in H^1(S_\delta) \setminus \{0\}.\]

Now we take $f = v \circ P$ as a trial function for $\mu_1(S_\delta)$ so that

\[\mu_1(S_\delta) \leq \frac{\int_{S_\delta} |
abla [v \circ P]|^2(\phi^2 \circ P)\,dV + (\beta/\sin(\theta/2)) \int_{P^{-1}(\Gamma_\delta)} [v \circ P]^2(\phi^2 \circ P)\,dS}{\int_{S_\delta} [v \circ P]^2(\phi^2 \circ P)\,dV}.\]

(20)

Let $J_P$ and $J_{P^{-1}}$ be the Jacobian matrices of $P$ and $P^{-1}$, $I_{S_\delta}$ and $I_{T_\delta}$ be the identities on $S_\delta$ and $T_\delta$, and $I$ be the identity matrix. After writing $P$ and $P^{-1}$ in rectangular coordinates, a calculation shows that

\[\|P - I_{S_\delta}\|_{\infty}, \|J_P - I\|_{opo,\infty}, \|P^{-1} - I_{T_\delta}\|_{\infty}, \|J_{P^{-1}} - I\|_{opo,\infty} \to 0, \quad \text{as } \theta \to 0.\]

(21)

Here, $\| \cdot \|_{\infty}$ is the supremum norm and $\| \cdot \|_{opo,\infty}$ is the supremum norm of the (pointwise) operator norm of a matrix valued function.

We calculate the denominator of (20) by changing variables and using (21) so that

\[\int_{S_\delta} [v \circ P]^2(\phi^2 \circ P)\,dV = \int_{T_\delta} v^2 \phi^2 |\det(J_{P^{-1}})|\,dV = (1 + o(1)) \int_{T_\delta} v^2 \phi^2 \,dV, \quad \text{as } \theta \to 0.\]
Now we prove an upper bound on the gradient term in (20): The chain rule gives
\[ \int_{S_0} |\nabla [v \circ P]_\star |^2 (\phi^2 \circ P) \, dV \leq |J_P|_\infty^2 \int_{S_0} |(\nabla v) \circ P|_\star^2 (\phi^2 \circ P) \, dV \leq M(\Theta) \int_{T_0} |\nabla v| \phi^2 \, dV, \]
where \( M(\Theta) = |J_P|_\infty^2 |\det(J_{P^{-1}})|_\infty = 1 + o(1) \), as \( \Theta \to 0 \) by applying the estimates in (21).

Finally, to handle the boundary term, we prove the lower bound
\[ \int_{P^{-1}(\Gamma_\Theta)} [v \circ P]_\star (\phi^2 \circ P) \, dS \geq m(\Theta) \int_{\Gamma_\Theta} v^2 \phi^2 \, dS, \]
where \( m(\Theta) = 1 + o(1) \) as \( \Theta \to 0 \) by applying the estimates in (21).

Applying the estimates on each part of the Rayleigh quotient to (20) shows that \( \mu_1(\Theta) \leq (1 + o(1)) \left[ M(\Theta) \int_{T_0} |\nabla v| \phi^2 \, dV + (\beta(\Theta)m(\Theta)/\sin(\Theta/2)) \int_{\Gamma_\Theta} v^2 \phi^2 \, dS \right] / \int_{T_0} v^2 \phi^2 \, dV, \) as \( \Theta \to 0 \).

Choosing \( \beta(\Theta) = M(\Theta)m(\Theta)^{-1} \alpha = (1 + o(1))\alpha \) and factoring out the \( M(\Theta) \) from the numerator shows that \( \mu_1(S_\Theta) \leq (1 + o(1))\mu_1(T_\Theta) \), as \( \Theta \to 0 \), completing the proof of Step 2.

**Step 3:** Lemma A.1 can be used one final time to see that the weighted Laplacian problem (23), as well as the rescaled version (24), have discrete spectrum. (Note that their weak formulation is given by (A4).)

In addition to being defined as the first eigenvalue of the problem on \( S_\Theta \), we claim that \( \mu_1(S_\Theta) \) is also the first eigenvalue of the problem
\[ \begin{cases} -\Delta w = \mu w & \text{on } B(\cos(\Theta/2)^{-1}), \\ \partial_r w + (\beta / \sin(\Theta/2)) w = 0 & \text{on } \partial B(\cos(\Theta/2)^{-1}), \end{cases} \]
where \( \sigma = \phi^2 \circ P \) and \( \partial_r \) is the radial derivative (i.e., the normal derivative). Indeed, \( \sigma = \exp\{2\alpha \cot(\Theta/2)r\} \) is radial, and so the first eigenfunction \( w \) is also radial (by uniqueness), and therefore the restriction \( w|_{S_\Theta} \) has Neumann boundary conditions on \( \Sigma_\Theta \). The restriction is still an eigenfunction, and since it has the correct boundary conditions and is nonnegative, it must be a ground state of the problem on \( S_\Theta \) and so \( \mu_1(S_\Theta) \) is the first eigenvalue of (23).

Now we transfer the \( \Theta \) dependence of \( \sigma \) to the domain by rescaling the radial coordinate by \( r \mapsto \cot(\Theta/2)r \) and letting \( (24) \) becomes the Schrödinger eigenvalue problem (3) with a shifted potential. A direct calculation shows that the radial function \( \varphi(r) = \bar{w}(r)e^{\alpha r} \) satisfies the Schrödinger eigenvalue problem
\[ \begin{cases} (-\Delta - \frac{\kappa}{r}) \varphi = (\nu_1 - \alpha^2) \varphi & \text{on } B(R), \\ \partial_r \varphi + (\beta - \alpha) \varphi = 0 & \text{on } \partial B(R), \end{cases} \]
where \( \kappa = (n - 1)|\alpha| \). Since \( \varphi \geq 0 \), we know that \( \nu_1 - \alpha^2 \) is the first eigenvalue of this problem.
The definition of $\tilde{\beta}$ (at the end of Step 3) shows that $\tilde{\beta} - \alpha = o(1)$ as $R \to \infty$. Thus, $\gamma < \tilde{\beta} - \alpha$ for all $R$ sufficiently large, where $\gamma \in (-|\alpha|, 0)$ is the fixed Robin parameter in the Schrödinger problem (3) with eigenvalue $E(R)$. Because the eigenvalues of Schrödinger operators are increasing functions of the Robin parameter, we have that $E(R) \leq \nu_1(R) - \alpha^2$ for $R$ sufficiently large. Using $R = \sin(\theta/2)^{-1}$ and combining this inequality with those from Steps 1, 2, and 3, there is a constant $C > 0$ such that for all small $\theta$,

$$C(R^2 - 1)(E(R) + \alpha^2) \leq C(R^2 - 1)\nu_1(R) \leq C\mu_1(S_\theta) \leq \mu_1(T_\theta) = \lambda_1(D_\theta) - \lambda_1(C_\theta) < 0.$$

Fix an $\epsilon > 0$. Using part (iii) of Lemma 4.2, we have that $E(R) + \alpha^2 = O(\exp[-2(1-\epsilon)|\alpha|R])$ as $R \to \infty$. Hence,

$$\lambda_1(D_\theta) - \lambda_1(C_\theta) = O(\exp[-2(1-\epsilon)|\alpha|/\sin(\theta/2)]) \quad \text{as} \quad \theta \to 0,$$

where this last estimate follows by using $R = \sin(\theta/2)^{-1}$ and that the factor of $R^2 - 1$ multiplying $E(R) + \alpha^2$ can be absorbed into the error term since $\epsilon$ is arbitrary. Thus, (15) is proven.

Remark 5.1. To prove the lower bound for $\lambda_1(D_\theta)$, we used the map $P^{-1}$ to “push out” the truncated cone $T_\theta$ to the spherical sector $S_\theta$. This idea was used to compute bounds on eigenvalues of thin isosceles triangles by Freitas in [8, section 3] for Dirichlet boundary conditions and by Laugesen and Siudeja in [18, proof of Lemma 5.2] for Neumann boundary conditions. Here, we adapt the technique to Robin boundary conditions.

6 | QUESTIONS

Theorem 1.2 shows that Conjecture 1.1 fails to extend to $\alpha < 0$ for general convex $\mathcal{D}$. In a positive direction, Laugesen [17, Theorem III.8] recently showed that Conjecture 1.1 holds and is sharp for all $\alpha \in (-\infty, \infty)$ for rectangular boxes (i.e., products of intervals). This inspires the following question: Is there a general class of convex domains such that the gap inequality in Conjecture 1.1 holds when $\alpha < 0$? Since the proof of Theorem 1.2 appears to rely on the fact that the domain $D_\theta$ has vertices with small opening angle, we might consider minimizing $(\lambda_2 - \lambda_1)(\cdot)$ over convex domains with a uniformly bounded Lipschitz constant.

Surprisingly, it seems that the gap still fails to have a positive lower bound over this restricted class of domains. To see this, we sketch an approximation argument. Define the truncated double cone domains

$$D_{\theta,\epsilon} = t_\epsilon((x, y) \in D_\theta : |x| < 1 - \epsilon), \quad \text{for} \quad \epsilon \in [0, 1),$$

where the rescaling $t_\epsilon$ is such that $D_{\theta,\epsilon}$ has diameter 2 for each $\epsilon$ sufficiently small. (Note that $t_\epsilon = 1 + o(1)$, as $\epsilon \to 0$.) For $\theta$ fixed, $\{D_{\theta,\epsilon}\}_{\epsilon \in [0, 1)}$ has a Lipschitz constant that is uniform in $\epsilon \in [0, 1)$ since the angle between the vertical faces at $x = \pm t_\epsilon(1 - \epsilon)$ of $\partial D_{\theta,\epsilon}$ and the remainder of the boundary does not depend on $\epsilon$. In addition, $D_{\theta,\epsilon} \to D_\theta$ in the Hausdorff distance and $|\partial D_{\theta,\epsilon}| \to |\partial D_\theta|$ as $\epsilon \to 0$ so we expect that

$$\lambda_j(D_{\theta,\epsilon}) \to \lambda_j(D_\theta), \quad \text{as} \quad \epsilon \to 0,$$

for each $j \in \mathbb{N}$; see [11, Remark 4.33] for a discussion of Robin continuity results.

Assuming the above limit, we can use Theorem 1.2 to find a “diagonal sequence” $c(\theta)$ converging to zero such that

$$(\lambda_2 - \lambda_1)(D_{\theta,\epsilon(\theta)}) \to 0, \quad \text{as} \quad \theta \to 0,$$

and $D_{\theta,\epsilon(\theta)}$ has diameter 2 for each $\theta$. Observe that the truncated domains $\{D_{\theta,\epsilon(\theta)}\}_{\theta \in (0, \pi/2)}$ also have uniformly bounded Lipschitz constants. This shows that $\lambda_2 - \lambda_1$ can be made arbitrarily small among convex domains of a given diameter and with a uniformly bounded Lipschitz constant. This example demonstrates the difficulty of determining a general class of domains where Conjecture 1.1 holds when $\alpha < 0$, even with a weaker lower bound.
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APPENDIX A: SCHröDINGER AND τ-LAPLACIAN—EXISTENCE OF SPECTRUM
In this appendix, we prove Lemma 4.1, which states that the Schrödinger eigenvalue problem has discrete spectrum. We also discuss Lemma A.1 below, which gives a rigorous formulation for the weighted Laplacian eigenvalue problems that appear in this paper. We do not prove A.1 because it is quite similar to an analogous proof for the unweighted problem due to the assumptions on the weight.

The weak formulation of a linear problem with discrete spectrum can be written in terms of a triple $(H, a, b)$ consisting of a suitable Hilbert space $H$ and bilinear forms $a, b : H \times H \to \mathbb{R}$ as follows: Find eigenvalue–eigenvector pairs $(\lambda, f) \in \mathbb{R} \times H \setminus \{0\}$ such that

\[ a(f, g) = \lambda b(f, g), \quad \text{for every } g \in H. \] (A1)
To prove that problems of this form have discrete spectrum, it suffices to show that \((H, a, b)\) satisfies the following three conditions:

(C1): \(a(\cdot, \cdot)\) is a coercive on \(H\), meaning \(k\| \cdot \|^2 \leq a(\cdot, \cdot)\) for some \(k > 0\).

(C2): \(a(\cdot, \cdot)\) is continuous on \(H \times H\).

(C3): \(b(\cdot, \cdot)\) is weakly (sequentially) continuous on \(H \times H\).

This result can be found in [15, Theorem 2.2], for example.

**Proof of Lemma 4.1.** We begin by defining the appropriate triple \((H, a, b)\) for the weak formulation of problem (3). Write \(B = B(R)\) for ease of notation. Let \(H = H^1(B)\) when \(\gamma \in \mathbb{R}\) and \(H = H^1_0(B)\) when \(\gamma = \infty\), both with norm defined by \(\| f \|^2 = \int_B |\nabla f|^2 + f^2 \, dV\). We define \(a\) and \(b\) by

\[
a(f, g) = \int_B \nabla f \cdot \nabla g - \frac{\kappa}{r} fg \, dV + \gamma \int_{\partial B} fg \, dS,
\]

and \(b(f, g) = \int_B fg \, dV\). The bilinear form \(b(\cdot, \cdot)\) is well defined on \(H \times H\) since it is the \(L^2\)-inner product. We will see shortly that \(a(\cdot, \cdot)\) is also well defined on \(H \times H\).

Notice that the quadratic form \(f \mapsto a(f, f)\) cannot be coercive since it is negative for certain \(f\). Thus, we show \((H, a, b)\) has a discrete spectrum of eigenvalues by studying the auxiliary problem

\[
\tilde{a}(f, g) = \tilde{E} b(f, g), \quad \text{for every } g \in H,
\]

(A2)

where the bilinear form \(\tilde{a}(\cdot, \cdot)\) is defined by

\[
\tilde{a}(f, g) = a(f, g) + cb(f, g).
\]

The constant \(c > 0\) will be chosen large enough that \(f \mapsto \tilde{a}(f, f)\) coercive on \(H\). Note that the \(j\)th eigenvalues \(E_j\) and \(\tilde{E}_j\) of the two problems are related by the equation \(E_j = \tilde{E}_j - c\). Therefore, to prove \((H, a, b)\) has a discrete spectrum, it suffices to show (A2) has discrete spectrum by checking that \((H, \tilde{a}, b)\) satisfy conditions (C1), (C2), and (C3).

We begin by showing condition (C2) holds by proving each term of \(\tilde{a}(\cdot, \cdot)\) is a bounded bilinear form. It will follow that \(\tilde{a}(\cdot, \cdot)\) is well defined. First, observe that the gradient term in \(\tilde{a}(f, g)\) is bounded by \(\| f \| \| g \|\) by Cauchy–Schwarz. To bound the potential term, we use Hölder’s inequality with exponents \(p\) and \(q\) (depending on the dimension \(n\)) followed by the Sobolev inequalities

\[
\int_B (-\frac{\kappa}{r} + c) fg \, dV \leq \| -\frac{\kappa}{r} + c \|_{L^p} \| f \|_{L^{2n}} \| g \|_{L^{2n}} \leq \| f \| \| g \|.
\]

When \(n \geq 3\), the last inequality follows from taking \(p = n/2\) and \(q = n/n - 2\). When \(n = 2\), we can take \(1 < p < 2\) and \(q = p/(p-1)\). Lastly, recall that the boundary integral is defined using the trace operator \(T : H^1(B) \rightarrow L^2(\partial B)\). Since \(T\) is a bounded operator, Cauchy–Schwarz shows that

\[
|\gamma \int_{\partial B} fg \, dS| = |\gamma \int_{\partial B} T(f) T(g) \, dS| \leq |\gamma| \|T(f)\|_{L^2(\partial B)} \|T(g)\|_{L^2(\partial B)} \leq C_T^2 \| f \| \| g \|,
\]

where \(C_T\) is the operator norm of \(T\). This shows \(\tilde{a}(\cdot, \cdot)\) is bounded and (C2) holds.

To prove coercivity (C1), we bound the absolute value of the potential term by using the decomposition \(\kappa/r = V_1 + V_2\), where

\[
V_1 = \frac{\kappa}{r} \chi_{B(r)}, \quad \text{and} \quad V_2 = \frac{\kappa}{r} \chi_{B(r)}^c.
\]
and \( \chi_{B(r)} \) denotes the indicator function of the ball of radius \( r \in (0, R) \). Notice that \( \| V_1 \|_{L^\infty} < \infty \) for each \( \epsilon > 0 \). Using this decomposition and Hölder’s inequality with exponents \( p \) and \( q \), we find that
\[
\left| \int_B \left( -\frac{\kappa}{r} \right) f^2 \, dV \right| \leq \| V_1 \|_{L^\infty} \| f \|_{L^2}^2 + \| V_2 \|_{L^p} \| f \|_{L^q}^2 \\
\leq \| V_1 \|_{L^\infty} \| f \|_{L^2}^2 + \| V_2 \|_{L^p} S_{n,2q}(\| \nabla f \|_{L^2}^2 + |f|_{L^2}^2),
\]
where \( S_{n,2q} \) is the Sobolev constant. We make the same choices of \( p \) and \( q \) as above depending on the dimension \( n \). Since \( \| V_2 \|_{L^p} < \infty \), we know \( \| V_2 \|_{L^p} \to 0 \) as \( r \to 0 \) by the monotone convergence theorem. Thus, we can choose \( r \) so that \( \| V_2 \|_{L^p} S_{n,2q} < 1/2 \). Applying the above inequality to the potential term in \( a(f, f) \) and collecting the gradient terms shows that
\[
\tilde{a}(f, f) \geq \frac{1}{2} \| \nabla f \|_{L^2}^2 + \left( c - \frac{1}{2} - |\gamma| C^2_T \right) \| f \|_{L^2}^2.
\]
Thus, choosing \( c = 1 + \| V_1 \|_{L^\infty} + |\gamma| C^2_T \) proves coercivity of \( \tilde{a}(\cdot, \cdot) \) with constant \( k = \frac{1}{2} \).

Finally, we must show that \( b(\cdot, \cdot) \) is weakly continuous to prove condition (C3) holds. Let \( f_n \to f \) and \( g_n \to g \) in \( H^1(B) \). Consider an arbitrary subsequence of each sequence, and note that since weakly convergent sequences are bounded, we can extract a further subsequence that converges weakly in \( H^1(B) \) and strongly in \( L^2(B) \). Since weak limits are unique, the strong limits must be \( f \) and \( g \), respectively. This shows that \( b(f_n, g_n) \to b(f, g) \) since \( b(\cdot, \cdot) \) is the \( L^2 \)-inner product and the initial subsequences extracted were arbitrary.

This shows that problem (A2) has a discrete spectrum of eigenvalues and so problem (A1) has the same spectrum shifted by \( -c \).

It can be deduced from [15, Theorem 5.8] that the variational characterization for the eigenvalues holds.

The second result of this appendix will focus on eigenvalue problems for the weighted operator called the \( \tau \)-Laplacian \( \Delta_\tau(\cdot) = \tau^{-1} \text{div}(\tau \nabla(\cdot)) \). In this paper, we are only concerned with weights \( \tau : \overline{\Omega} \to \mathbb{R} \) that are bounded and (uniformly) positive functions. The problem
\[
\begin{cases}
-\Delta f = \mu f & \text{on } \Omega, \\
\partial_\nu f + \eta(x)f = 0 & \text{on } \partial \Omega, 
\end{cases}
\quad (A3)
\]
where the boundary parameter \( \eta : \partial \Omega \to \mathbb{R} \) is a bounded function, has weak formulation \( a(f, g) = \mu b(f, g) \) for all \( g \in H \).

\[
a(f, g) = \int_\Omega f g \tau \, dV + \int_{\partial \Omega} f g \eta \tau \, dS,
\]
and \( b(f, g) = \int_\Omega f g \tau \, dV \).

The following lemma summarizes the existence of the spectrum and eigenfunctions for the above problem.

**Lemma A.1.** Let \( \Omega \) be a bounded Lipschitz domain. If \( \tau \) is a bounded and (uniformly) positive function on \( \overline{\Omega} \), then the eigenvalue problem (A3) has a discrete spectrum of eigenvalues
\[
\mu_1 < \mu_2 \leq \mu_3 \leq \cdots \to \infty,
\]
with an orthonormal basis of eigenfunctions in $H^1(\Omega)$. Moreover, the eigenvalues are characterized by the variational characterization

$$\mu_j = \min_{\mathcal{P}} \max_{f \neq 0} \frac{\int_\Omega |\nabla f|^2 \tau \, dV + \int_{\partial \Omega} f^2 \eta \tau \, dV}{\int_\Omega f^2 \tau \, dV},$$

where the minimum is taken over all $j$-dimensional subspaces $\mathcal{P} \subset H^1(\Omega)$.

While it seems natural to consider the $\tau$-Laplacian problem on a weighted Sobolev space constructed from $L^2(\Omega, \tau \, dV)$, we choose to work with $H^1(\Omega)$ for simplicity. This is possible because the weighted Sobolev space is actually equivalent to $H^1(\Omega)$ in the sense that they are equal as vector spaces with comparable norms. The norms are comparable with constants equal to the $\sqrt{\inf_\Omega \tau}$ and $\sqrt{\sup_\Omega \tau}$.

Since this problem can be formulated on an unweighted Sobolev space, the proof is quite similar to that of the Robin Laplacian with constant $\tau$ and can be proved via the same framework used in Lemma 4.1. Alternatively, one can find a complete proof of a slightly more general result in the work of Laugesen and Girouard in [9, Proposition 15].