We study the geometry of interacting knotted solitons. The interaction is local and advances either as a three-body or as a four-body process, depending on the relative orientation and a degeneracy of the solitons involved. The splitting and adjoining is governed by a four-point vertex in combination with duality transformations. The total linking number is preserved during the interaction. It receives contributions both from the twist and the writhe, which are variable. Therefore solitons can twine and coil and links can be formed.
Knotlike configurations appear in a variety of physical, chemical and biological scenarios. Examples include Early Universe cosmology, the structure of elementary particles, magnetic materials, turbulent fluid dynamics, polymer folding, and DNA replication, transcription and recombination. Whenever knots occur, it becomes important to understand how they twist, coil, split and adjoin.

Here we outline a geometrical first principles approach for understanding knots in interaction. For definiteness we shall consider Hamiltonian field theories where knots appear as solitons i.e. as stable finite energy solutions to the pertinent nonlinear equations of motion [1]-[4]. Even though our conclusions are independent of any detailed structure of the equations of motion, for concreteness we specify to a three component vector field \( \mathbf{n}(\mathbf{x}) \) with unit length \( \mathbf{n} \cdot \mathbf{n} = 1 \) [5]. We note that \( \mathbf{n} \) appears as an order parameter in a variety of applications. It can for example describe magnetization in a ferromagnet, velocity distribution \( \mathbf{n} = \mathbf{v}/|\mathbf{v}| \) in fluids, the director field in a nematic liquid crystal, and quantum fluctuations of a Higgs field around a spontaneously broken vacuum in a grand unified field theory.

A soliton configuration is localized, and \( \mathbf{n} \) goes to a constant vector \( \mathbf{n}(\mathbf{x}) \rightarrow \mathbf{n}_0 \) at spatial infinity. Therefore it is a map from the compactified three-space \( \mathbb{R}^3 \sim S^3 \) to the internal unit two-sphere \( S^2_{\mathbf{R}} \). Such maps fall into nontrivial homotopy classes \( \pi_3(S^2) \approx \mathbb{Z} \) labelled by the Hopf invariant [6] which can be evaluated by noting that the pre-image in \( R^3 \) of a point in \( S^2_{\mathbf{R}} \) is an embedding of a circle \( S^1 \in \mathbb{R}^3 \). This embedding is a knot in the mathematical sense [3], and the Hopf invariant is the linking number of any two such knots. We select the asymptotic vector \( \mathbf{n}_0 \) to point down, along the negative \( z \)-axis in three-space. At the core (center) of the knotted soliton \( \mathbf{n} \) then points up, and the pre-image of the core is a mathematical knot \( \mathbf{x}(\sigma) \) parametrized by \( \sigma \in [0, 2\pi] \) so that \( \mathbf{x}(\sigma + 2\pi) = \mathbf{x}(\sigma) \).

We consider a short strand of a knotted soliton, a tubular distribution of energy density that surrounds the core and does not overlap with any other strand. We inspect it with a planar cross sectional disk \( D_\tau \in \mathbb{R}^3 \) that cuts the strand at right angle to its core at \( \mathbf{x}(\sigma) \). The center of \( D_\tau \) coincides with the core, and its boundary is a circle around the core where \( \mathbf{n} \rightarrow \mathbf{n}_0 = -\hat{e}_z \), see figure 1. Note that topologically each \( D_\tau \) is a (Riemann) two-sphere \( S^2_\sigma \), and \( \mathbf{n} \) has the structure of a map \( S^1 \times S^2_\sigma \rightarrow S^2_{\mathbf{R}} \) that can be locally parametrized by

\[
\mathbf{n}(\mathbf{x}) = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} \cos(k\tau + l\varphi) \sin \theta \\ \sin(k\tau + l\varphi) \sin \theta \\ \cos \theta \end{pmatrix}
\]

(1)

Here \( \tau \) is a modular coordinate for the self-linking along the core, it is proportional to \( \sigma \). The integer \( k \) measures the net number of twists in our strand when we proceed along the core. The coordinates \( \varphi \) and \( \theta \) are defined at each \( \sigma \), and cover the disk \( D_\tau \). The angle \( \varphi \in [0, 2\pi] \) is the azimuth around the center of \( D_\tau \) and \( \theta \in [0, \pi] \) is the polar angle with \( \theta = 0 \) at the core and \( \theta = \pi \) at the boundary of \( D_\tau \). The integer \( l \) labels
the $\pi_2(S^2) \simeq \mathbb{Z}$ homotopy class of $\vec{n}$ as a map from the disk $D_\tau \sim S^2_\tau$ to the internal two-sphere $S^2_\bar{n}$.

Our interpretations become more apparent when we substitute (1) in the integral representation of the Hopf invariant [1],

$$Q_H = \frac{k \cdot l}{8\pi^2} \int \sin \theta d\theta \wedge d\phi \wedge d\tau$$

(2)

Notice that a knotted soliton has either a right-handed or a left-handed orientation which is determined by $d\theta \wedge d\phi \wedge d\tau$. Furthermore, for any regular Hamiltonian the Hopf invariant is conserved even when we have interactions where strands split and adjoin.

We combine $n_1$ and $n_2$ in (1) into a two component vector field $\vec{w} = (n_1, n_2)$. The core $\theta = 0$ is a fixed point of $\vec{w}$. Therefore we can define an index $Ind_\varphi(\vec{w})$ that counts how many times $\vec{w}$ rotates around the core when we go once around it along some curve on the disk $D_\tau$ in the right-handed direction. This index is a topological invariant which is independent both of the curve on $D_\tau$ and of the point $\sigma$ along the core, and from (1) we find $Ind_\varphi(\vec{w}) = l$. This index is also additive, if we evaluate it along a curve that surrounds $N$ fixed points of $\vec{w}$ the result coincides with the sum of the $N$ indices for the individual fixed points.

The third component $n_3$ in (1) is a continuous function on $D_\tau$ with values between $[-1, +1]$. It has a maximum at the core $\theta = 0$ and its minimum occurs on the boundary of the disk $D_\tau$ where $\theta \to \pi$. In general the maximum at $\theta = 0$ is a degenerate critical point, and for our knotted solitons we expect that its degeneracy has a definite multiplicity given by the absolute value of the index $Ind_\varphi(\vec{w})$,

$$deg(n_3)_{\theta=0} = |Ind_\varphi(\vec{w})| \sim |l|$$

(3)

In particular, when $Ind_\varphi(\vec{w}) = \pm 1$ the core is non-degenerate. In the model studied in [1]-[4] we can verify (3) explicitly: If $\rho$ measures distance from the core in local polar coordinates $(\vartheta, \rho)$ on $D_\tau$, we find from the equations of motion that the core is truly a degenerate critical point of $n_3$ on $D_\tau$ with $l$-fold multiplicity,

$$n_3(\vartheta, \rho) \xrightarrow{\rho \to 0} 1 - \mathcal{O}(\rho^{2|l|})$$

(4)

This result is related to analyticity near $\theta = 0$: The disks $D_\tau$ have a natural complex structure $\sin \theta d\theta \wedge d\phi \sim dz \wedge d\bar{z}$ suggesting that (3) should be valid whenever the equations of motion are elliptic. Indeed, in our physical applications (3) makes sense. It means that a knotted soliton with $l$-fold degeneracy is a bound state of $l$ non-degenerate ones.

Since $\theta = 0$ is a critical point of $n_3$, we have

$$\frac{\partial n_3}{\partial x_i}(\theta = 0) = 0$$

(5)
and since the core is a curve in $\mathbb{R}^3$, the $3 \times 3$ symmetric matrix
\[
\frac{\partial^2 n_3}{\partial x_i \partial x_j}(\theta = 0)
\]
has one vanishing eigenvalue $\lambda_1 = 0$, the corresponding eigenvector is tangent to the core. Since $\theta = 0$ is a maximum, the two additional eigenvalues of (6) are non-positive $\lambda_{2,3}(\tau) \leq 0$ at the core. If both are non-vanishing the core is a non-degenerate critical point of $n_3$ on $D_\tau$. But if the core is a degenerate critical point then either $\lambda_2$ or $\lambda_3$ (or both) should vanish.

We now proceed to apply these considerations to geometrically describe the interactions between strands of knotted solitons. We first inspect the situation with two initial strands that both have a non-degenerate core. We then explain how the results generalize when strands have degenerate cores.

When both initial strands have a non-degenerate core, the corresponding eigenvalues $\lambda_{2,3}(\tau)$ are non-vanishing. Therefore a nontrivial interaction such as splitting and adjoining can not occur unless at least one of the $\lambda_{2,3}(\tau)$ vanishes. At that point the two cores must coincide.

We first assume that the two initial strands have a parallel relative orientation. From (3) we then conclude that when the cores coincide $\theta = 0$ becomes a doubly degenerate critical point of $n_3$ with $\text{Ind}_\varphi(\vec{w}) = \pm 2$, depending on whether the two strands are right-handed or left-handed. The interaction is pointwise with two outgoing non-degenerate strands, and it can be described by a four-point vertex.

To visualize the vertex we draw plane projections of the strands. For this we trace the three gradient vectors $d\theta$, $d\varphi$ and $d\tau$ along the cores. The gradient $d\tau$ is (co)tangent to a core. In a plane projection we describe it by drawing an arrow along the direction of the core. For each $\tau$ the two additional gradients $d\theta$ and $d\varphi$ span the cross sectional planes $D_\tau$. We describe them by drawing two oriented lines in the vicinity of the core.

We recall that the self-linking number $Lk$ of a knotted soliton can be computed from its plane projection using [3]
\[
Lk = Tw + Wr
\]
where $Tw$ is the twist and $Wr$ is the writhe in the plane projection. During an interaction neither $Tw$ nor $Wr$ is in general separately conserved. Only their sum is since it coincides with the Hopf invariant which is conserved.

To picture the interaction vertex, we first employ continuity to translate all twists and writhes in the initial strands away from the vicinity of the interaction point. The plane projections of the initial configurations then become well-groomed lines with either antiparallel (A) or parallel (P) alignment. For example, in the case of two (right-handed) planar strands with antiparallel (A) alignment, we have the four-point interaction vertex $V_A$ in figure 2. Here both the twist and the writhe are separately conserved. In the case of two (right-handed) planar strands with parallel (P) alignment we have the four-point
interaction vertex $V_P$ in figure 3. Here neither the twist nor the writhe are conserved but their sum is, since the interaction preserves the Hopf invariant. Note that a difference in the final twist and writhe between the two vertices has important physical consequences, it suggests that the strands will twine, coil and link. Indeed, by repeating $V_P$ twice as in figure 4, we either continue coiling or form a link, depending on the global geometry. This is not equally obvious in the case $V_A$. However, we note that in $R^3$ a writhe can always be continuously deformed into a twist. Furthermore, two non-coplanar strands can exhibit an antiparallel alignment in one plane projection, but parallel alignment in another plane projection. Consequently we expect that the two vertices $V_P$ and $V_A$ should be related. For this we first vertically flip the plane projection of one of the two initial strands in figure 3. This yields two strands with antiparallel alignment in the plane projection. We then implement an interaction described by the vertex $V_A$ in figure 2, followed by another vertical flip. The result coincides with that obtained by implementing the vertex $V_P$ alone. This means that the two vertices $V_A$ and $V_P$ are dual to each other, with vertical flip defining the duality transformation.

Depending on the dynamical details of the Hamiltonian, besides splitting and adjoining the two initial strands may also combine into a single strand but with a doubly degenerate core. This can be described using the topologically invariant index $Ind_\phi(\vec{w})$. As an example we consider two initial right-handed strands both with $Ind_\phi(\vec{w}) = +1$ (figure 5). Since the index is additive, we conclude that for a curve which encircles both cores we have $Ind_\phi(\vec{w}) = +2$. This is also the index of the final, doubly degenerate strand. The interaction is a process where two homotopically nontrivial maps $S^1_\phi \to S^1$ each with a $\pi_1(S^1)$ winding number $+1$ combine into a single map with $\pi_1(S^1)$ winding number $+2$. Dynamically, this leads to the formation of a doubly degenerate knotted soliton, a bound state, in a manner which is consistent with the conservation of the Hopf invariant. We visualize the interaction as a three-body process in figure 5.

When the two initial strands have an opposite orientation, splitting and adjoining cannot occur since two strands with different orientation should not connect. Instead the two strands can annihilate each other. As in figure 5, this can also be described using the topologically invariant index $Ind_\phi(\vec{w})$. For example, if one of the initial strands corresponds to $Ind_\phi(\vec{w}) = +1$ and the other strand to $Ind_\phi(\vec{w}) = -1$, since the index is additive we conclude that for a curve that encircles both cores we have $Ind_\phi(\vec{w}) = 0$. As in figure 5, the interaction is a process where two homotopically nontrivial maps $S^1_\phi \to S^1$ combine. But now one of the initial maps has $\pi_1(S^1)$ winding number $+1$ and the other $-1$, and the combined map is homotopically trivial. This corresponds to an annihilation between the two strands, that proceeds dynamically in a manner which is consistent with the conservation of the Hopf invariant. The annihilation can be visualized as a three-body process, much like in figure 5.

In the general case where the initial strands are degenerate the interaction can proceed in a multitude of different fashions, details depending on the degrees of degeneracy
and the relative orientations of the interacting strands. But according to (3) degenerate strands can always be viewed as bound states of non-degenerate strands. Thus there is no need to discuss their interactions separately. We can always describe their interactions using various combinations of the interactions that occur between non-degenerate strands, in a rather obvious manner.

Finally, the interaction processes that we have outlined here all conserve the Hopf invariant. This follows from the condition that the order parameter $\vec{n}$ is a unit vector. But in a number of applications one may wish to relax the constraint $\vec{n} \cdot \vec{n} = 1$ so that $|\vec{n}|$ can vanish at some points in $R^3$. One way to achieve this is to replace the constraint by a term $\lambda(\vec{n} \cdot \vec{n} - 1)^2$ in the Hamiltonian, with $\lambda$ a coupling constant. When $\lambda \to \infty$ we recover $\vec{n} \cdot \vec{n} = 1$. But for finite (large) coupling we expect $|\vec{n}|$ to fluctuate in the vicinity of $|\vec{n}| \approx 1$ even though we can also have $|\vec{n}(\vec{x})| \to 0$ at some points. This generalizes the models discussed here in a manner that should be of interest in a variety of applications. For example in quantum field theory it allows us to recover renormalizability, while in molecular biology it enables us to account for effects of enzymes in DNA. Obviously study of interaction processes with a finite $\lambda$ should be of wide interest.

We have found that depending on the relative orientation and degeneracy of the solitons involved, an actual interaction can involve various combinations of splitting and adjoining, twisting, coiling and linking. It can lead to the formation of bound states and annihilation between strands. Our description of the interaction processes is quite model independent and based entirely on elementary concepts of continuity and differentiability. Consequently we hope that our results can form a basis for detailed investigations of interacting knots. Indeed, we expect that numerical studies in the model discussed in [1]-[5] (and its generalization to the case where $|\vec{n}|$ can vanish) provide prolific tests for various aspects of knot interaction.

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Figure Caption

**Figure 1**: A cross-sectional disk $D_\tau$ cuts a strand at a right angle to its core. The center of the disk coincides with the core where $\vec{n}$ points up, and its boundary is a circle where $\vec{n}$ points down.

**Figure 2**: The plane projected interaction vertex $V_A$ between two right-handed, mutually antiparallel strands.

**Figure 3**: The plane projected interaction vertex $V_P$ between two right-handed, mutually parallel strands.

**Figure 4**: If repeated twice, the vertex $V_P$ can lead to further coiling or to the formation of a link.

**Figure 5**: Two right-handed non-degenerate strands can combine into a single doubly-degenerate strand. The process leads to a doubly degenerate knotted soliton.
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