Laurent Expansions for Vertex Operators

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Abstract
A method is presented for using coherent vectors to calculate the explicit form of Schur polynomials which are the coefficients of Laurent expansion of a vertex operator.

1 Preliminaries

Let $\Gamma_0 D$ be a Bose algebra (cf. [4]) i.e. a commutative graded algebra generated by a pre-Hilbert space $D, \langle \cdot, \cdot \rangle$ (the so-called one-particle space) and the unity $\phi$ (the vacuum) provided with the extension $\langle \cdot, \cdot \rangle$ of the scalar product of $D$ making $\phi$ a unit vector and fulfilling the property that for every $x \in D$, the adjoint $x^*$ to the operator of multiplication by $x$ is defined on the whole $\Gamma_0 D$ and constitutes a derivation (i.e. fulfils the Leibniz rule). We make the space $\tilde{\Gamma} D$ of all antilinear functionals on $\Gamma_0 D$ the extension of $\Gamma_0 D$ by identifying $f \in \Gamma_0 D$ with the antilinear functional $\langle \cdot, f \rangle$. The space $\tilde{\Gamma} D$ can be naturally made into an algebra containing $\Gamma_0 D$ as a subalgebra. We consider $\tilde{\Gamma} D$ as a locally convex space with the weak topology $\sigma (\tilde{\Gamma} D, \Gamma_0 D)$. The weak closure $\tilde{D}$ of $D$ is a subspace of $\tilde{\Gamma} D$. It is easy to show that $\Gamma_0 D, \langle \cdot, \cdot \rangle$ admits the completion $\Gamma \tilde{D}$ within $\tilde{\Gamma} D$.

We shall use the exponentials of elements $w \in D$,

$$e^w = \sum_{n=0}^{\infty} \frac{1}{n!} w^n \in \Gamma \tilde{D},$$

which are called coherent vectors. In [4] the following relations are verified:

$$\langle a, b \rangle^j = \frac{1}{j!} \langle a^j, b \rangle$$

(1)
\[ (x^n)^* e^w = \langle x, w \rangle^n e^w \]  
\[ \langle e^u, fg \rangle = \langle e^u, f \rangle \langle e^u, g \rangle \]  
\[ e^{a(w)} e^v = e^{(w,v)} e^v. \]

Also a proof that the set \( \{ e^x : x \in \mathcal{D} \} \) of coherent vectors is total in \( \Gamma \overline{\mathcal{D}} \) can be found in [4].

## 2 The Laurent Expansion for a Vertex operator

Let \( \mathcal{D} \) be spanned by an orthonormal system \( \{ f_n \} \) and by an orthonormal system \( \{ g_n \} \) as well. The operator valued functions of \( z \)
\[ V(z) = e^{\sum_{n=1}^{\infty} z^n f_n} e^{\sum_{n=1}^{\infty} z^{-n} g_n^*} : \Gamma_0 \mathcal{D} \to \overline{\Gamma} \mathcal{D}, \]
shall be called a vertex operator (cf. [2], [3], [1]).

Write \( (p, q) \) for tuples of non-negative integers
\[ (p, q) = (p_1, q_1, p_2, q_2, \ldots, p_k, q_k, \ldots) \]
and define
\[ \mathcal{N}_m = \left\{ (p, q) : \sum_{k=1}^{\infty} (p_k + q_k) = m \right\} \]
and
\[ \mathcal{N}^w = \left\{ (p, q) : \sum_{k=1}^{\infty} (p_k + q_k) < \infty, \sum_{j=1}^{\infty} j (p_j - q_j) = w \right\} \]
For \( s = (s_1, s_2, \ldots) \), write
\[ s! = \prod_{k=1}^{\infty} s_k! \]

We prove the following
THEOREM  Vertex operators admit the weak evaluation on $\Gamma_0 D$ and the weak convergent Laurent expansion

$$V(z) = e^{\sum_{n=1}^{\infty} z^n f_n} e^{\sum_{n=1}^{\infty} z^{-n} g^*_n} = \sum_{w \in \mathbb{Z}} S_w \{ f_n, g^*_n \} z^w$$

with coefficients

$$S_w \{ f_n, g^*_n \} = \sum_{m=0}^{\infty} \sum_{(p,q) \in \mathbb{N}_m \cap \mathbb{N}_w} \frac{1}{p!q!} \left( \prod_{k=1}^{\infty} f_k^{p_k} \right) \left( \prod_{k=1}^{\infty} g_k^{q_k} \right)^*$$

called the Schur polynomials (cf. [3]).

To prove the Theorem we shall need the following

LEMMA  Take any pair of elements $u, v \in D$. Then the element $V(z) e^u$ is well defined in $\tilde{\Gamma} D$ and we have

$$\langle e^u, V(z) e^v \rangle = e^{\sum_{w \in \mathbb{Z}} S_w \{ f_n, g^*_n \} z^w} e^v,$$

where

$$S_w \{ f_n, g^*_n \} = \sum_{m=0}^{\infty} \sum_{(p,q) \in \mathbb{N}_m \cap \mathbb{N}_w} \frac{1}{p!q!} \left( \prod_{k=1}^{\infty} f_k^{p_k} \right) \left( \prod_{k=1}^{\infty} g_k^{q_k} \right)^*.$$

Proof.  Take $u,v \in D$. By virtue of (4) we get

$$\langle e^u, e^x e^y \rangle = e^{\sum_{w \in \mathbb{Z}} S_w \{ f_n, g^*_n \} z^w} e^v,$$

and consequently

$$\langle e^u, V(z) e^v \rangle = e^{\sum_{w \in \mathbb{Z}} S_w \{ f_n, g^*_n \} z^w} e^{\sum_{n=1}^{\infty} \langle f_n, u \rangle z^n + \langle v, g_n \rangle z^{-n}}.$$

Since $u$ and $v$ are linear combinations of $f_k$ and $g_k$ respectively, $\langle f_n, u \rangle z^n = \langle v, g_n \rangle z^{-n} = 0$ for large $n$. Due to (2) we get

$$\langle e^u, \left( \prod_{k=1}^{\infty} f_k^{p_k} \right) \left( \prod_{k=1}^{\infty} g_k^{q_k} \right)^* e^v \rangle = \langle \left( \prod_{k=1}^{\infty} f_k^{p_k} \right) \left( \prod_{k=1}^{\infty} g_k^{q_k} \right)^* \rangle e^{\sum_{n=1}^{\infty} \langle f_k, u \rangle^{p_k} \langle v, g_k \rangle^{q_k}} e^{\langle u, v \rangle}.$$
where all the products are finite and they are non-zero only when $p_k$ and $q_k$ are zeros for $f_k$ and $g_k$ orthogonal to $v$ and $u$ respectively. Consequently

$$\frac{1}{m!} \left( \sum_{n=1}^{\infty} \langle f_n, u \rangle z^n + \sum_{n=1}^{\infty} \langle v, g_n \rangle z^{-n} \right)^m$$

$$= \sum_{(p,q) \in \mathcal{N}_m} \frac{1}{p!q!} \prod_{k=1}^{\infty} \left( \langle f_k, u \rangle^{p_k} \langle v, g_k \rangle^{q_k} z^{k(p_k-q_k)} \right)$$

$$= \sum_{w \in \mathbb{Z}} \sum_{(p,q) \in \mathcal{N}_m \cap \mathcal{N}_w} \frac{1}{p!q!} \left( \prod_{k=1}^{\infty} \langle f_k, u \rangle^{p_k} \langle v, g_k \rangle^{q_k} \right) z^w$$

$$= \left\langle e^u, \sum_{w \in \mathbb{Z}} \left( \sum_{(p,q) \in \mathcal{N}_m \cap \mathcal{N}_w} \frac{1}{p!q!} \left( \prod_{k=1}^{\infty} f_k^{p_k} \right) \left( \prod_{k=1}^{\infty} g_k^{q_k} \right)^* \right) z^w e^v \right\rangle e^{-(u,v)}.$$ 

Hence

$$\frac{1}{m!} \left( \sum_{n=1}^{\infty} \langle f_n, u \rangle z^n + \sum_{n=1}^{\infty} \langle v, g_n \rangle z^{-n} \right)^m$$

$$= \left\langle e^u, \sum_{w \in \mathbb{Z}} \sum_{m=0}^{\infty} \sum_{w \in \mathbb{Z}} \frac{1}{p!q!} \left( \prod_{k=1}^{\infty} f_k^{p_k} \right) \left( \prod_{k=1}^{\infty} g_k^{q_k} \right)^* \right) z^w e^v \right\rangle e^{-(u,v)},$$

and finally

$$\langle e^u, V(z) e^v \rangle = e^{(u,v)} \sum_{n=1}^{\infty} \left( \langle f_n, u \rangle z^n + \langle v, g_n \rangle z^{-n} \right)$$

$$= \left\langle e^u, \sum_{w \in \mathbb{Z}} \sum_{m=0}^{\infty} \sum_{w \in \mathbb{Z}} \frac{1}{p!q!} \left( \prod_{k=1}^{\infty} f_k^{p_k} \right) \left( \prod_{k=1}^{\infty} g_k^{q_k} \right)^* \right) z^w e^v \right\rangle$$

which concludes the proof of the Lemma.

**Proof of the Theorem**

Since $\Gamma_0 D$ is the linear span of the set $\{ x^k : x \in D, k = 1, 2, \ldots \}$ (cf. [1]), it is sufficient to show that for any $u, v \in D$ and any natural numbers $k, j$ we have

$$\langle u^k, V(z) v^j \rangle = \left\langle u^k, \left( \sum_{w \in \mathbb{Z}} S_w \{ f_n, g_n^* \} z^w \right) v^j \right\rangle$$

which follows by differentiating respectively $k$ and $j$ times at 0 the variables $t$ and $s$ of the identity [3] with $tu$ and $sv$ substituted for $u$ and $v.$ ■
References

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