Weak Galerkin finite element method for linear poroelasticity problems

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Abstract
This paper is devoted to a weak Galerkin (WG) finite element method for linear poroelasticity problems where weakly defined divergence and gradient operators over discontinuous functions are introduced. We establish both the continuous and discrete time WG schemes, and obtain their optimal convergence order estimates in a discrete $H^1$ norm for the displacement and in an $H^1$ type and $L^2$ norms for the pressure. Finally, numerical experiments are presented to illustrate the theoretical error results in different kinds of meshes which shows the WG flexibility for mesh selections, and to verify the locking-free property of our proposed method.

Mathematics Subject Classification 2020: 65M60, 65M15, 76S05

Keywords: Weak Galerkin, finite element method, linear poroelasticity problem, optimal pressure error estimate, locking-free property

1. Introduction

In this paper, we consider the following two-field Navier-formed Biot’s consolidation model which depicts a quasi-static flow in a saturated deformable poroelastic medium. Let $\Omega$ be a convex polygonal or polyhedral domain in $\mathbb{R}^d$ ($d = 2, 3$) with smooth boundary $\Gamma = \Gamma_{p,D} \cup \Gamma_{p,N}$, where $\Gamma_{p,D}$ is nonempty and $T$ is a final time. The displacement of porous solid media $u(t) : \Omega \to \mathbb{R}^d$ and the pore pressure of fluid $p(t) : \Omega \to \mathbb{R}$ satisfy the system

\begin{align}
-(\lambda + \mu)\nabla(\nabla \cdot u) - \mu \Delta u + \alpha \nabla p &= f, \quad \text{in } \Omega, \ t \in (0, T], \\
\frac{\partial}{\partial t}(c_0 p + \alpha \nabla \cdot u) - \nabla \cdot (\kappa \nabla p) &= g, \quad \text{in } \Omega, \ t \in (0, T],
\end{align}

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Preprint submitted to Elsevier August 10, 2022
with the boundary conditions

\[
\begin{align*}
    u &= 0, & \text{on } \Gamma, \\
    p &= 0, & \text{on } \Gamma_{p,D}, \\
    \kappa \nabla p \cdot n &= \gamma, & \text{on } \Gamma_{p,N},
\end{align*}
\]

and the initial conditions

\[
\begin{align*}
    u(\cdot, 0) &= u^0, & \text{in } \Omega, \\
    p(\cdot, 0) &= p^0, & \text{in } \Omega.
\end{align*}
\]

Here \( f(t) : \Omega \to \mathbb{R}^d \) is the body force, \( g(t) : \Omega \to \mathbb{R} \) is the volumetric fluid source (or sink), \( \beta(t) : \Omega \to \mathbb{R}^d \) represents the prescribed surface traction, and \( \gamma(t) : \Omega \to \mathbb{R} \) states the prescribed discharge on the boundary. \( \varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T) \) stands for the strain tensor, \( \lambda \) and \( \mu \) are the Lamé constants, \( \alpha \) is the Biot-Willis parameter, \( c_0 \geq 0 \) represents the constrained specific storage coefficient, \( \kappa \) denotes the hydraulic conductivity, \( I \) states the identity tensor, and \( n \) is the unit outward normal vector. In this paper, for simplicity, we assume that \( \alpha = 1 \). If \( \alpha \neq 1 \), one may reduce the model to the one with \( \alpha = 1 \) by rescaling the equations.

As one of the classic poroelasticity problems, the Biot’s consolidation model has been treated by various numerical methods, such as finite element methods, finite difference methods, hybrid discontinuous Galerkin (HDG) methods, hybrid high-order (HHO) methods and WG methods. In [14], the authors present a finite element discretization preserving pointwise mass balance for the Biot’s consolidation model and provide numerical results to demonstrate the method. In [4], Chen and Yang design a three-variable weak form with mixed finite element and derive optimal convergence order estimates. Stability estimates and convergence analysis of finite difference methods for the Biot’s consolidation model are presented in [12]. In [11], the authors deal with the numerical solution of a secondary consolidation Biot model, and a family of finite difference methods on staggered grids in both time and spatial variables is considered. Fu employs high order HDG methods for the Biot’ consolidation equations and backward Euler methods are used for temporal discretization [10]. In [2], the authors discretize the displacements describing the elastic deformation by the HHO method [7], and the pressure representing the flow problem by the symmetric weighted interior penalty discontinuous Galerkin method [8]. In [13], the authors adopt WG linear finite elements for spatial discretization and backward Euler scheme for temporal discretization in order to obtain an implicit fully discretized scheme of the Biot’s consolidation model. The results of [13] are generalized to high order elements in [5], and degrees of freedom are reduced on element boundaries for the pressure approximation without compromising the accuracy. In [32], the authors apply a modified WG method to the Biot’s problem and derive the error estimates of semi-discrete and fully discrete schemes.

Generally, WG methods, first proposed and analyzed in [28], refers to finite element techniques for solving partial differential equations where differential operators (e.g., gradient, divergence, curl, etc.) are approximated by weak forms as distributions. Later WG is successfully extended to elliptic interface problems [16] [22], Helmholtz equations [18] [9],
linear parabolic equations \[15, 35, 34\], and further developed for other applications, such as
biharmonic problems \[17, 6\], Stokes problems \[23, 21\], Stokes-Darcy problems \[24, 24\] and
stochastic partial differential equations \[40, 41\], etc. The idea of parameter free stabilization
term is introduced in \[20\] to improve the flexibility of element construction and mesh gen-
eration. The resulting WG method is no longer limited to the RT \[26\] or BDM \[3\] elements
in the computation of discrete weak gradient. In analogy with discrete weak gradient, dis-
crete weak divergence is introduced in \[29\] where the proposed weak Galerkin mixed finite
element method (WGMFEM) is applicable for general finite element partitions consisting of
shape regular polygons in 2D or polyhedra in 3D. Additionally, WGMFEM is developed in
second-order elliptic equations with Robin boundary conditions \[37\], heat equations \[39, 38\],
Helmholtz equations with large wave numbers \[31\], and quasi-linear poroelasticity problems
\[36\], etc.

To our best knowledge, up to now, there have been two papers in total, i.e., \[13, 5\],
studying the linear two-field Biot model from the point of WG discretization. In \[13, 5\], the
authors adopt the piecewise polynomials with the same degree to discretize the displacements
and pressure in the interior of elements, which causes that they only obtain the suboptimal
error convergence rates for the pressure theoretically. In addition, the authors of both papers
consider only the case of spurious pressure oscillations in the numerical experiments, not the
locking problem.

Based on above, in this paper, we propose a WG method for the linear two-field (dis-
placement and pressure) Biot’s consolidation model in the Navier form and set up the con-
tinuous and discrete time WG schemes. We respectively design \(P_{j+1}^d - P_j^d\) and \(P_j - P_{j-1}^d\)
\((d = 2, 3 \text{ and } j \geq 1)\) WG combinations to gain the displacement and pressure approxima-
tions. With the use of these combinations satisfying the discrete inf-sup condition, we derive
the optimal order error estimates of the semi-discrete and fully discrete schemes in a discrete
\(H^1\) norm for the displacement and in an \(H^1\) type and \(L^2\) norms for the pressure. Finally, some
numerical examples are supplied to illustrate the advantages of our proposed method from
the two aspects, the good mesh flexibility and locking-free property for the system \((1.1)-(1.6)\).

The outline of this paper goes as follows. In Section 2, we establish the weak formulation
based on some necessary notations and definitions. The specific WG method is introduced
in Section 3 and we provide the semi-discrete and fully discrete numerical schemes. In Sec-
tion 4, the optimal order convergence estimates of two numerical schemes are derived, and
ultimately in Section 5 we supply numerical experiments to validate our theoretical findings
and expectation.

2. Notations and variational formulation

In this section, before bringing in the variational formulation of \((1.1)\) and \((1.2)\), we firstly
present some useful notations and definitions. In this paper, we utilize the standard definition
of Sobolev space \(H^s(\Omega)\) with \(s \geq 0\) (cf. \[II\]). The associated inner-product and norm in \(H^s(\Omega)\)
are denoted by \((\cdot, \cdot)_s\) and \(|| \cdot ||_s\), respectively. When \(s = 0\), \(H^0(\Omega)\) coincides with the space
of square-integrable functions $L^2(\Omega)$. In this case, the subscript $s$ is suppressed from the notation of inner product and norm. The above notations and definitions can easily be extended to vector-valued functions. The inner-product and norm for such functions shall follow the same naming convention. We also define two spaces

$$H^1_u(\Omega) := \{ \mathbf{v} \in [H^1(\Omega)]^d, \mathbf{v} = \mathbf{0} \text{ on } \Gamma \},$$

and

$$H^1_p(\Omega) := \{ q \in H^1(\Omega), q = 0 \text{ on } \Gamma_{p,D} \}.$$

In addition, the letter $C$ (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences throughout this paper.

Now, we can define the variational equations of (1.1) and (1.2) as follows: For any $t \in (0, T]$, seek $u(t) \in H^1_u(\Omega)$ and $p(t) \in H^1_p(\Omega)$ such that

$$\begin{align*}
(\lambda + \mu)(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) + \mu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) &= (f, \mathbf{v}), \quad \forall \mathbf{v} \in H^1_u(\Omega), \\
\frac{\partial}{\partial t}(c_0 p + \nabla \cdot \mathbf{u}), q) + (\kappa \nabla p, \nabla q) &= (g, q) + (\gamma, q)_{\Gamma_{p,N}}, \quad \forall q \in H^1_p(\Omega),
\end{align*}$$

with the initial conditions (1.5) and (1.6).

3. WG method

In this section, the definitions of discrete weak divergence and weak gradient operators are firstly rendered. The key to the WG method is to use discrete weak differential operators in place of standard differential operators in the variational form of the original system (1.1) and (1.2). Then we supply the semi-discrete and fully discrete WG schemes used for our error analysis and numerical computation.

Let $\mathcal{T}_h$ be a finite element partition of the domain $\Omega \subset \mathbb{R}^d$ consisting of polygons ($d = 2$) or polyhedra ($d = 3$) satisfying the shape regularity requirements A1-A4 in [29]. Denote by $h_K$ the partition diameter of the element $K \in \mathcal{T}_h$ with boundary $\partial K$, and $h = \max_K h_K$. Let $\mathcal{E}_h$ be the set of all edges or faces in $\mathcal{T}_h$, and $\mathcal{E}^0_h = \mathcal{E}_h \setminus \Gamma$ be the set of all interior edges or faces. The sets of polynomials with degree no more than $j$ on each $K$ and $e \in \mathcal{E}_h$ are denoted by $P_j(K)$ and $P_j(e)$, respectively.

For the displacement $\mathbf{u}$, we define two weak vector-valued finite element spaces as, for any integer $j \geq 1$,

$$V_h := \{ \mathbf{v}_h = (\mathbf{v}_0, \mathbf{v}_b) : (\mathbf{v}_0, \mathbf{v}_b)|_K \in [P_{j+1}(K)]^d \times [P_j(e)]^d, \ K \in \mathcal{T}_h, \ e \subset \partial K \},$$

and

$$V^u_h := \{ \mathbf{v}_h = (\mathbf{v}_0, \mathbf{v}_b) \in V_h : \mathbf{v}_b = 0 \text{ on } \Gamma \}.$$
Based on the above definitions of spaces, we bring in discrete weak divergence and weak gradient operators. For \( v_h \in V_h \), define \( \nabla_{w,K} \cdot v_h \in P_j(K) \) and \( \nabla_{w,K} v_h \in [P_j(K)]^{d \times d} \) on each element \( K \) as follows,

\[
(\nabla_{w,K} \cdot v_h, \psi)_K = -(v_0, \nabla \psi)_K + \langle v_h \cdot n, \psi \rangle_{\partial K}, \quad \forall \psi \in P_j(K),
\]

\[
(\nabla_{w,K} v_h, \phi)_K = -(v_0, \nabla \cdot \phi) + \langle v_h \phi \cdot n \rangle_{\partial K}, \quad \forall \phi \in [P_j(K)]^{d \times d}.
\]

Then define the global weak divergence and weak gradient by patching the local ones, i.e.,

\[
(\nabla_w v_h)|_K = \nabla_{w,K} (v_h|_K), \quad \forall v_h \in V_h,
\]

\[
(\nabla_w v_h)|_K = \nabla_{w,K} (v_h|_K), \quad \forall v_h \in V_h.
\]

Similarly, for the pressure \( p \), we introduce its weak finite element spaces and discrete weak gradient operator. For any integer \( j \geq 1 \), define

\[
W_h := \{q_h = \{q_0, q_b\} : q_0, q_b|_K \in P_j(K) \times P_{j-1}(e), \ K \in T_h, \ e \subset \partial K\},
\]

and

\[
W_h^p := \{q_h = \{q_0, q_b\} \in W_h : q_b = 0 \text{ on } \Gamma_{p,D}\}.
\]

For \( q_h \in W_h \), define \( \nabla_{w,K} q_h \in [P_{j-1}(K)]^d \) on each element \( K \) as follows,

\[
(\nabla_{w,K} q_h, \zeta)_K = -(q_0, \nabla \cdot \zeta)_K + \langle q_b, \zeta \cdot n \rangle_{\partial K}, \quad \forall \zeta \in [P_{j-1}(K)]^d.
\]

Then define the global weak gradient by patching the local ones, i.e.,

\[
(\nabla_w q_h)|_K = \nabla_{w,K} (q_h|_K), \quad \forall q_h \in W_h.
\]

Before numerical schemes, we first present several definitions of \( L^2 \) projection operator. For each \( K \in T_h \), denote by \( Q_0 \) the \( L^2 \) projection operator from \([L^2(K)]^d\) onto \([P_{j+1}(K)]^d\), by \( Q_0 \) the \( L^2 \) projection operator from \( L^2(K) \) onto \( P_j(K) \). For each \( e \in E_h \), denote by \( Q_b \) the \( L^2 \) projection operator from \([L^2(e)]^d\) onto \([P_j(e)]^d\), by \( Q_b \) the \( L^2 \) projection operator from \( L^2(e) \) onto \( P_{j-1}(e) \). We shall combine \( Q_0 \) with \( Q_b \), and \( Q_0 \) with \( Q_b \), by writing \( Q_h = \{Q_0, Q_b\} \) and \( Q_h = \{Q_0, Q_b\} \), respectively.

Next we introduce several bilinear forms as follows: For \( v_h = \{v_0, v_b\} \in V_h, \ w_h = \{w_0, w_b\} \in V_h, \ q_h = \{q_0, q_b\} \in W_h, \ \eta_h = \{\eta_0, \eta_b\} \in W_h, \)

\[
s_u(v_h, w_h) = \sum_{K \in T_h} h^{-1}_K \langle Q_b v_0 - v_b, Q_b w_0 - w_b \rangle_{\partial K},
\]

\[
s_p(q_h, \eta_h) = \sum_{K \in T_h} h^{-1}_K \langle Q_b q_0 - q_b, Q_b \eta_0 - \eta_b \rangle_{\partial K},
\]

\[
a_u(v_h, w_h) = \sum_{K \in T_h} (\lambda + \mu) \langle \nabla_w v_h, \nabla_w \cdot w_h \rangle_K + \sum_{K \in T_h} \mu \langle \nabla_w v_h, \nabla_w w_h \rangle_K + s_u(v_h, w_h),
\]

\[
a_p(q_h, \eta_h) = \sum_{K \in T_h} \kappa \langle \nabla_w q_h, \nabla_w \eta_h \rangle_K + s_p(q_h, \eta_h),
\]

\[
b(v_h, q_h) = \sum_{K \in T_h} \langle \nabla_w \cdot v_h, q_0 \rangle_K.
\]
We also define two norms for $v_h \in V^u_h$ and $q_h \in W^p_h$ by
\[
\|v_h\|_V := \left\{ a_u(v_h, v_h) \right\}^{\frac{1}{2}},
\|q_h\|_W := \left\{ a_p(q_h, q_h) \right\}^{\frac{1}{2}}.
\]
From [32], we know that the bilinear forms $a_u(\cdot, \cdot)$ and $a_p(\cdot, \cdot)$ are bounded, symmetric and coercive in $V^u_h$ and $W^p_h$, respectively. And from [30], the bilinear form $b(\cdot, \cdot)$ is bounded in $V^u_h \times W^p_h$ and satisfies the inf-sup condition.

Now we can formulate the semi-discrete WG scheme. For any $t \in (0, T]$, find $u_h = \{u_0, u_b\} \in V^u_h$ and $p_h = \{p_0, p_b\} \in W^p_h$ such that
\[
a_u(u_h, v_h) - b(v_h, p_h) = (f, v_h), \quad \forall \, v_h = \{v_0, v_b\} \in V^u_h,
\]
\[
(c_0 p_0, q_0) + b(u_h, q_h) + a_p(p_h, q_h) = (g, q_0) + \langle \gamma, q_0 \rangle_{\Gamma_{p,u}, N}, \quad \forall \, q_h = \{q_0, q_b\} \in W^p_h,
\]
where $(c_0 p_0, q_0) = \sum_{K \in \mathcal{T}_h} (c_0 p_0, q_0)_K$. According to the properties of the bilinear forms $a_u(\cdot, \cdot)$, $a_p(\cdot, \cdot)$ and $b(\cdot, \cdot)$, the solution of problem (3.1) and (3.2) exists and is unique.

We turn our attention to the fully discrete numerical scheme. We introduce a time step size $\tau = \frac{T}{N}$ for some positive integer $N$ and $t_n = n\tau$ for $n = 0, 1, \ldots, N$. By $u^n_h$ and $p^n_h$, we denote the approximation of $u(t_n)$ and $p(t_n)$, respectively. We use the backward Euler method to approximate the time derivative in (3.2), and then the fully discrete scheme reads: For $n = 1, \ldots, N$, seek $u^n_h = \{u^n_0, u^n_b\} \in V^u_h$ and $p^n_h = \{p^n_0, p^n_b\} \in W^p_h$ such that
\[
a_u(u^n_h, v_h) - b(v_h, p^n_h) = (f(t_n), v_h), \quad \forall \, v_h = \{v_0, v_b\} \in V^u_h,
\]
\[
(c_0 \partial_r p_0^n, q_0) + b(\partial_r u^n_h, q_h) + a_p(p^n_h, q_h) = (g(t_n), q_0) + \langle \gamma(t_n), q_0 \rangle_{\Gamma_{p,u}, N}, \quad \forall \, q_h = \{q_0, q_b\} \in W^p_h,
\]
where $\partial_r p_0^n = \frac{p^n_0 - p_{n-1}^0}{\tau}$ and $\partial_r u^n_h = \frac{u^n_h - u_{n-1}^h}{\tau}$.

4. Error analysis

In this section, we shall derive the optimal order error estimates for both continuous and discrete time WG methods.

4.1. Continuous time WG method

Firstly, we bring in two useful $L^2$ projection operators. In addition to the projection operators $Q_h = \{Q_0, Q_b\}$ and $Q_h = \{Q_0, Q_b\}$ mentioned above, for each element $K \in \mathcal{T}_h$, let $Q_0$ and $\hat{Q}_0$ be two local $L^2$ projection operators onto $[P_{j-1}(K)]^d$ and $[P_j(K)]^{d \times d}$, respectively. Then we have the following lemma.
Lemma 4.1. [1, 3] For any \( \mathbf{v} \in [H^1(\Omega)]^d \) and \( q \in H^1(\Omega) \), we have the following commutative properties of projection operators.

\[
\nabla_w \cdot (Q_h \mathbf{v}) = Q_0 (\nabla \cdot \mathbf{v}), \\
\nabla_w (Q_h \mathbf{v}) = \hat{Q}_0 (\nabla \mathbf{v}), \\
\nabla_w (Q_h q) = Q_0 (\nabla q).
\]

Based on these projection operators, the following results are presented as a preparation of error analysis.

Lemma 4.2. The solution \( \mathbf{u} \) and \( p \) to the model problem [1, 1] and [1, 2] satisfies

\[
a_u(Q_h \mathbf{u}, \mathbf{v}_h) - b(\mathbf{v}_h, Q_h p) = (f, \mathbf{v}_0) + s_u(Q_h \mathbf{u}, \mathbf{v}_h) + l_1(\mathbf{u}, \mathbf{v}_h) + l_2(\mathbf{u}, \mathbf{v}_h) - l_3(p, \mathbf{v}_h),
\]

for all \( \mathbf{v}_h \in V^u_h \) and \( q_h \in W^p_h \), where the linear functions \( l_1, l_2, l_3 \) and \( l_4 \) are defined as

\[
l_1(\mathbf{u}, \mathbf{v}_h) = \sum_{K \in T_h} (\lambda + \mu) (\nabla \cdot \mathbf{u} - Q_0 (\nabla \cdot \mathbf{u}), (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n})_{\partial K},
\]

\[
l_2(\mathbf{u}, \mathbf{v}_h) = \sum_{K \in T_h} \mu (\nabla \mathbf{u} - \hat{Q}_0 (\nabla \mathbf{u})) \cdot \mathbf{n}, (\mathbf{v}_0 - \mathbf{v}_b)_{\partial K},
\]

\[
l_3(p, \mathbf{v}_h) = \sum_{K \in T_h} (p - Q_0 p, (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n})_{\partial K},
\]

\[
l_4(p, q_h) = \sum_{K \in T_h} (\kappa (\nabla p - Q_0 (\nabla p)) \cdot \mathbf{n}, q_0 - q_b)_{\partial K}.
\]

Proof. For \( \mathbf{v}_h \in V^u_h \), together with Lemma 4.1, the definition of discrete weak divergence, integration by parts and the definition of \( Q_0 \), we acquire

\[
\sum_{K \in T_h} (\lambda + \mu) (\nabla_w \cdot (Q_h \mathbf{u}), \nabla_w \cdot \mathbf{v}_h)_{K} = \sum_{K \in T_h} (\lambda + \mu) (Q_0 (\nabla \cdot \mathbf{u}), \nabla_w \cdot \mathbf{v}_h)_{K} = \sum_{K \in T_h} \{ -(\lambda + \mu) (\nabla (Q_0 (\nabla \cdot \mathbf{u})), \mathbf{v}_0)_{K} + (\lambda + \mu) (Q_0 (\nabla \cdot \mathbf{u}), \mathbf{v}_b \cdot \mathbf{n})_{\partial K} \}
\]

\[
= \sum_{K \in T_h} \{ (\lambda + \mu) (Q_0 (\nabla \cdot \mathbf{u}), \nabla \cdot \mathbf{v}_0)_{K} - (\lambda + \mu) (Q_0 (\nabla \cdot \mathbf{u}), (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n})_{\partial K} \}
\]

\[
= \sum_{K \in T_h} \{ (\lambda + \mu) (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}_0)_{K} - (\lambda + \mu) (Q_0 (\nabla \cdot \mathbf{u}), (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n})_{\partial K} \},
\]

which implies that

\[
\sum_{K \in T_h} (\lambda + \mu) (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}_0)_{K} = \sum_{K \in T_h} (\lambda + \mu) (\nabla_w (Q_h \mathbf{u}), \nabla_w \cdot \mathbf{v}_h)_{K} + \sum_{K \in T_h} (\lambda + \mu) (Q_0 (\nabla \cdot \mathbf{u}), (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n})_{\partial K}.
\]

(4.1)
According to Lemma 4.1, the definition of discrete weak gradient, integration by parts and the definition of $\mathcal{Q}_0$, it follows that

\[
\sum_{K \in T_h} \mu(\nabla_w (Q_h u), \nabla_w v_h)_K = \sum_{K \in T_h} \mu(\mathcal{Q}_0(\nabla u), \nabla_w v_h)_K \\
= \sum_{K \in T_h} \left\{-\mu(\nabla \cdot (\mathcal{Q}_0(\nabla u)), v_0)_K + \mu(\mathcal{Q}_0(\nabla u) \cdot n, v_0)_{\partial K}\right\} \\
= \sum_{K \in T_h} \left\{\mu(\mathcal{Q}_0(\nabla u), \nabla v_0)_K - \mu(\mathcal{Q}_0(\nabla u) \cdot n, v_0 - v_b)_{\partial K}\right\} \\
= \sum_{K \in T_h} \left\{\mu(\nabla u, \nabla v_0)_K - \mu(\mathcal{Q}_0(\nabla u) \cdot n, v_0 - v_b)_{\partial K}\right\},
\]

which shows that

\[
\sum_{K \in T_h} \mu(\nabla u, \nabla v_0)_K = \sum_{K \in T_h} \mu(\nabla_w (Q_h u), \nabla_w v_h)_K + \sum_{K \in T_h} \mu(\mathcal{Q}_0(\nabla u) \cdot n, v_0 - v_b)_{\partial K}.
\]  

(4.2)

Because of the definition of discrete weak divergence, integration by parts and the definition of $Q_0$, we have

\[
\sum_{K \in T_h} (\nabla_w \cdot v_h, Q_0 p)_K = \sum_{K \in T_h} \left\{- (v_0, \nabla (Q_0 p))_K + (v_b \cdot n, Q_0 p)_{\partial K}\right\} \\
= \sum_{K \in T_h} \left\{\nabla \cdot v_0, Q_0 p)_K - ((v_0 - v_b) \cdot n, Q_0 p)_{\partial K}\right\} \\
= \sum_{K \in T_h} \left\{\nabla \cdot v_0, p)_K - ((v_0 - v_b) \cdot n, Q_0 p)_{\partial K}\right\},
\]

which leads to

\[
\sum_{K \in T_h} (\nabla \cdot v_0, p)_K = \sum_{K \in T_h} (\nabla_w \cdot v_h, Q_0 p)_K + \sum_{K \in T_h} ((v_0 - v_b) \cdot n, Q_0 p)_{\partial K}.
\]  

(4.3)

Now testing (1.1) with $v_0$ in $v_h = \{v_0, v_b\}$ and using integration by parts, we obtain

\[
\sum_{K \in T_h} (\lambda + \mu) (\nabla \cdot u, \nabla \cdot v_0)_K - \sum_{K \in T_h} (\lambda + \mu) \langle \nabla \cdot u, v_0 \cdot n \rangle_{\partial K} + \sum_{K \in T_h} \mu(\nabla u, \nabla v_0)_K \\
- \sum_{K \in T_h} \mu(\nabla u \cdot n, v_0)_{\partial K} - \sum_{K \in T_h} (p, \nabla \cdot v_0) + \sum_{K \in T_h} \langle p, v_0 \cdot n \rangle_{\partial K} = \langle f, v_0 \rangle.
\]  

(4.4)

Substituting (4.1), (4.2) and (4.3) into (4.4) and adding $s_u(Q_h u, v_h)$ to the both sides, together with the boundary conditions, we present the first equality of Lemma 4.2.

Next, we derive the other equation of this lemma. Considering the definition of $Q_0$ and Lemma 4.1 we find

\[
(c_0 Q_0 p_t, q_0) = (c_0 p_t, q_0),
\]

and

\[
\sum_{K \in T_h} (\nabla_w \cdot (Q_h u_t), q_0)_K = \sum_{K \in T_h} (\nabla \cdot u_t, q_0)_K.
\]
Using Lemma 4.1, the definition of discrete weak gradient, integration by parts and the definition of $Q_0$, we get

$$
\sum_{K \in T_h} \kappa(\nabla_w Q_h p, \nabla_w q_h)_K = \sum_{K \in T_h} \kappa(Q_0(\nabla p), \nabla_w q_h)_K
$$

$$
= \sum_{K \in T_h} \{\kappa(Q_0(\nabla p) \cdot n, q_0)_{\partial K} - \kappa(\nabla Q_0(\nabla p), q_0)_K\}
$$

$$
= \sum_{K \in T_h} \{\kappa(Q_0(\nabla p), \nabla q_0)_K - \kappa(Q_0(\nabla p) \cdot n, q_0 - q_b)_{\partial K}\}
$$

which implies that

$$
\sum_{K \in T_h} \kappa(\nabla p, \nabla q_0)_K = \sum_{K \in T_h} \kappa(\nabla_w Q_h p, \nabla_w q_h)_K + \sum_{K \in T_h} \kappa(Q_0(\nabla p) \cdot n, q_0 - q_b)_{\partial K}.
$$

Testing (1.2) with $q_0$ in $q_h = \{q_0,q_b\}$, and utilizing integration by parts, the boundary conditions and the above estimates, we acquire the second equality of this lemma, which completes the proof.

Based on Lemma 4.2, we apply the Wheeler’s projection method in [33, 27] to study the optimal order of error estimates. For any $v_h \in V_h^u$ and $q_h \in W_h^p$, define two elliptic projections $\tilde{u}_h \in V_h^u$ and $\tilde{p}_h \in W_h^p$ such that

$$
a_u(\tilde{u}_h, v_h) - b(v_h, \tilde{p}_h) = a_u(Q_h u, v_h) - b(v_h, Q_h p) - s_u(Q_h u, v_h) - l_1(u, v_h) - l_2(u, v_h)
$$

$$
+ l_3(p, v_h),
$$

$$
a_p(\tilde{p}_h, q_h) = a_p(Q_h p, q_h) - s_p(Q_h p, q_h) - l_4(p, q_h).$$

For the numerical analysis of WG, we usually focus on the following error decomposition,

$$
Q_h u - u_h = (Q_h u - \tilde{u}_h) + (\tilde{u}_h - u_h) := e_h + e_h,
$$

and

$$
Q_h p - p_h = (Q_h p - \tilde{p}_h) + (\tilde{p}_h - p_h) := \theta_h + \rho_h.
$$

In order to bound the errors $e_h$ and $\theta_h$, we have the following results from [30, 13].

**Lemma 4.3.** [37, 13] Let $u \in H^1_u(\Omega) \cap H^{j+2}(\Omega)$ and $p \in H^1_p(\Omega) \cap H^{j+1}(\Omega)$ with any integer $j \geq 1$, for any $v_h \in V_h$ and $q_h \in W_h$, then

$$
|s_u(Q_h u, v_h)| \leq C h^{j+1} \|u\|_{j+2} \|v_h\|_V,
$$

$$
|s_p(Q_h p, q_h)| \leq C h^j \|p\|_{j+1} \|q_h\|_W,
$$

$$
|l_1(u, v_h)| \leq C h^{j+1} \|u\|_{j+2} \|v_h\|_V,
$$

$$
|l_2(u, v_h)| \leq C h^{j+1} \|u\|_{j+2} \|v_h\|_V,
$$

$$
|l_3(p, v_h)| \leq C h^{j+1} \|p\|_{j+1} \|v_h\|_V,
$$

$$
|l_4(p, q_h)| \leq C h^j \|p\|_{j+1} \|q_h\|_W.
$$
The estimates of $\epsilon_h$ and $\theta_h$ are provided as follows.

**Lemma 4.4.** Assume that $u \in H^1_u(\Omega) \cap H^{j+2}(\Omega)$ and $p \in H^1_p(\Omega) \cap H^{j+1}(\Omega)$ with any integer $j \geq 1$. For any $v_h \in V^u_h$ and $q_h \in W^p_h$, there hold

\[
\|\theta_h\|_W \leq Ch^j \|p\|_{j+1},
\]
\[
\|\epsilon_h\|_V \leq Ch^{j+1}(\|u\|_{j+2} + \|p\|_{j+1}).
\]

**Proof.** According to (4.6) and Lemma 4.3, we obtain

\[
a_p(\theta_h, q_h) = a_p(Q_h u, q_h) - a_p(\tilde{p}_h, q_h)
\]
\[
= s_p(Q_h p, q_h) + l_4(p, q_h)
\]
\[
\leq |s_p(Q_h p, q_h)| + |l_4(p, q_h)|
\]
\[
\leq C \|p\|_{j+1} \|q_h\|_W.
\]

Taking $q_h = \theta_h$ in the above inequality, then

\[
\|\theta_h\|_W \leq Ch^j \|p\|_{j+1}.
\]

Because of (4.5) and Lemma 4.3 we get

\[
a_u(\epsilon_h, v_h) = a_u(Q_h u, v_h) - a_u(\tilde{u}_h, v_h)
\]
\[
= b(v_h, \theta_h) + s_u(Q_h u, v_h) + l_1(u, v_h) + l_2(u, v_h) - l_3(p, v_h)
\]
\[
\leq C \|v_h\|_V \|\theta_0\| + Ch^{j+1} \|u\|_{j+2} \|v_h\|_V + Ch^{j+1} \|p\|_{j+1} \|v_h\|_V.
\]

Since (4.6) is a standard WG discretization of Poisson equation, by the duality argument [19], we acquire

\[
\|\theta_0\| \leq Ch^{j+1} \|p\|_{j+1}.
\]

Hence,

\[
a_u(\epsilon_h, v_h) \leq Ch^{j+1}(\|u\|_{j+2} + \|p\|_{j+1}) \|v_h\|_V.
\]

Taking $v_h = \epsilon_h$ in the above inequality, then

\[
\|\epsilon_h\|_V \leq Ch^{j+1}(\|u\|_{j+2} + \|p\|_{j+1}),
\]

which finishes the proof.

Next, we derive the error equations of semi-discrete WG method. For $v_h \in V^u_h$ and $q_h \in W^p_h$, noticing (4.5), Lemma 4.2 and (3.1), it follows that

\[
a_u(\epsilon_h, v_h) - b(v_h, \rho_h) = [a_u(\tilde{u}_h, v_h) - b(v_h, \tilde{p}_h)] - [a_u(u_h, v_h) - b(v_h, p_h)]
\]
\[
= (f, v_0) - (f, v_0)
\]
\[
= 0.
\]
By virtue of (3.2), (4.6) and Lemma 4.2, we get
\[ (c_0 \rho_{0,t}, q_0) + b(e_{h,t}, q_h) + a_p(\rho_h, q_h) \]
\[ = [(c_0 \tilde{p}_0, q_0) + b(\tilde{u}_{h,t}, q_h) + a_p(\tilde{p}_h, q_h)] - [(c_0 \tilde{p}_0, q_0) + b(u_{h,t}, q_h) + a_p(p_h, q_h)] \]
\[ = (c_0 \tilde{p}_0 - Q_0 p_t, q_0) + b(\tilde{u}_{h,t} - Q_h u_t, q_h) \]
\[ = -(c_0 \partial_{0,t}, q_0) - b(e_{h,t}, q_h). \]

Thus, we obtain the error equations
\[ a_u(e_h, v_h) - b(v_h, \rho_h) = 0, \tag{4.7} \]
\[ (c_0 \rho_{0,t}, q_0) + b(e_{h,t}, q_h) + a_p(\rho_h, q_h) = -(c_0 \partial_{0,t}, q_0) - b(e_{h,t}, q_h). \tag{4.8} \]

Before the error estimates of semi-discrete WG scheme, we need to supply the following useful lemma.

Lemma 4.5. Assume that the finite element partition \( T_h \) is shape regular. Then there exists a constant \( C \) such that
\[ ||q_0||^2 \leq C \| q_h \|_{V}^2, \quad \forall \; q_h = \{q_0, q_h\} \in W_h^p. \]

Now we are ready for the optimal order convergence estimates as follows.

Theorem 4.6. For any \( t \in (0, T) \) and \( j \geq 1 \), let
\[ (u(t), p(t)) \in L^\infty(0, T; H^1 \cap H^{j+2}(\Omega)) \times L^\infty(0, T; H^j(\Omega) \cap H^{j+1}(\Omega)) \]
be the solution of (1.1) and (1.2), and \( (u_h(t), p_h(t)) \in V^u_h \times W^p_h \) be the solution of (3.1) and (3.2). Assume that
\[ u(t) \in L^2(0, T; H^{j+2}(\Omega)), \quad p(t) \in L^2(0, T; H^{j+1}(\Omega)), \]
then
\[ \| Q_h u(t) - u_h(t) \|_V^2 + \| Q_0 p(t) - p_0(t) \|_V^2 \leq C h^{2j+2} \left( \| u_0 \|_{j+2}^2 + \| p_0 \|_{j+1}^2 + \| u(t) \|_{j+2}^2 + \| p(t) \|_{j+1}^2 \right) \]
\[ + \int_0^t \| u_\tau(t) \|_{j+2}^2 d\tau + \int_0^t \| p_\tau(t) \|_{j+1}^2 d\tau. \]

Proof. Choosing \( v_h = e_{h,t} \) and \( q_h = \rho_h \) in (4.7) and (4.8), respectively, and adding these two equalities yield
\[ a_u(e_h, e_{h,t}) + (c_0 \rho_{0,t}, \rho_0) + \| \rho_h \|_V^2 = -(c_0 \partial_{0,t}, \rho_0) - b(e_{h,t}, \rho_h). \]

It follows from Lemma 4.5 and the Cauchy-Schwarz inequality that
\[ \frac{1}{2} \frac{d}{dt} \| e_h \|_V^2 + \frac{c_0}{2} \frac{d}{dt} \| \rho_0 \|_V^2 + \| \rho_h \|_V^2 = -(c_0 \partial_{0,t}, \rho_0) - b(e_{h,t}, \rho_h) \]
\[ \leq c_0 \| \partial_{0,t} \| \| \rho_0 \| + C \| e_{h,t} \|_V \| \rho_0 \| \]
\[ \leq c_0 \| \partial_{0,t} \| \| \rho_h \|_W + C \| e_{h,t} \|_V \| \rho_h \|_W \]
\[ \leq C \| \theta_{0,t} \|^2 + C \| e_{h,t} \|_V^2 + C_i \rho_h \|_W^2. \]
Supposing $0 < C_1 \leq 1$ and integrating the both sides of inequality with respect to $t$, together with Lemma 4.4, we write

$$
\|e_h(t)\|_V^2 + \|\rho_0(t)\|^2 \leq \|e_h(0)\|_V^2 + \|\rho_0(0)\|^2 + C \int_0^t \|\theta_{0,t}(\tau)\|_V^2 d\tau + C \int_0^t \|e_{h,t}(\tau)\|_V^2 d\tau \\
\leq \|e_h(0)\|_V^2 + \|\rho_0(0)\|^2 + Ch^{2j+2} (\int_0^t \|u_t(\tau)\|_{j+2}^2 d\tau + \int_0^t \|p_t(\tau)\|_{j+1}^2 d\tau).
$$

Because of the error estimate of $L^2$ projection operator and Lemma 4.4, we arrive at

$$
\|e_h(0)\|_V^2 = \|(Q_h u(0) - u_h(0)) - e_h(0)\|_V^2 \\
\leq C \|u(0) - u_h(0)\|_V^2 + C \|u(0) - Q_h u(0)\|_V^2 + C \|e_h(0)\|_V^2 \\
\leq Ch^{2j+2}(\|u(0)\|_{j+2}^2 + \|p(0)\|_{j+1}^2).
$$

Similarly, there holds

$$
\|\rho_0(0)\|^2 = \|(Q_0 p(0) - p_0(0)) - \theta_0(0)\|^2 \\
\leq C \|p(0) - p_0(0)\|^2 + C \|p(0) - Q_0 p(0)\|^2 + C \|\theta_0(0)\|^2 \\
\leq Ch^{2j+2}\|p(0)\|_{j+1}^2.
$$

Therefore,

$$
\|Q_h u(t) - u_h(t)\|_V^2 + \|Q_0 p(t) - p_0(t)\|^2 \\
\leq C(\|e_h(t)\|_V^2 + \|e_{h,t}(t)\|_V^2 + \|\theta_{0,t}(t)\|^2 + \|\rho_0(t)\|^2) \\
\leq Ch^{2j+2}(\|u(0)\|_{j+2}^2 + \|p(0)\|_{j+1}^2 + \|u(t)\|_{j+2}^2 + \|p(t)\|_{j+1}^2 + \int_0^t \|u_t(\tau)\|_{j+2}^2 d\tau + \int_0^t \|p_t(\tau)\|_{j+1}^2 d\tau),
$$

where Lemma 4.4 is applied, and the proof is completed.

**Theorem 4.7.** Under the assumption of Theorem 4.6 with $c_0 > 0$, we have

$$
\|Q_h p(t) - p_h(t)\|_W^2 \leq Ch^{2j} (\|p(0)\|_{j+1}^2 + \|p(t)\|_{j+1}^2 + Ch^{2j+2} (\int_0^t \|p_t(\tau)\|_{j+1}^2 d\tau + \int_0^t \|u_t(\tau)\|_{j+2}^2 d\tau).
$$

**Proof.** First, we differentiate (4.7) with respect to $t$,

$$
a_u(e_{h,t}, v_h) - b(v_h, p_{h,t}) = 0. \tag{4.11}
$$

Taking $v_h = e_{h,t}$ and $q_h = p_{h,t}$ in (4.11) and (4.8), respectively, and adding,

$$
\|e_{h,t}\|_V^2 + c_0 \|\rho_{0,t}\|^2 + \frac{1}{2} \frac{d}{dt} \|\rho_{0,t}\|_W^2 = -(c_0 \theta_{0,t}, \rho_{0,t}) - b(e_{h,t}, p_{h,t}) \\
\leq c_0 \|\theta_{0,t}\| \|\rho_{0,t}\| + C \|e_{h,t}\|_V \|\rho_{0,t}\| \\
\leq C \|\theta_{0,t}\|^2 + C \|e_{h,t}\|_V^2 + C_2 \|\rho_{0,t}\|^2.
$$
Assuming $0 < C_2 \leq c_0$ and integrating with respect to $t$, it follows from Lemma 4.4 that
\[
\|\rho_h(t)\|^2_W \leq \|\rho_h(0)\|^2_W + C \int_0^t \|\theta_{0,t}(\tau)\|^2 d\tau + C \int_0^t \|\epsilon_{0,t}(\tau)\|^2 d\tau \\
\leq \|\rho_h(0)\|^2_W + Ch^{2j+2}(\int_0^t \|p_t(\tau)\|^2_{j+1} d\tau + \int_0^t \|u_t(\tau)\|^2_{j+2} d\tau).
\]
Using the error estimate of $L^2$ projection operator and Lemma 4.4 we provide
\[
\|\rho_h(0)\|^2_W \leq C(\|p(0) - p_h(0)\|^2_W + \|p(0) - Q_h p(0)\|^2_W + \|\theta(0)\|^2_W) \\
\leq Ch^{2j}\|p(0)\|^2_{j+1}.
\]
Hence,
\[
\|Q_h p(t) - p_h(t)\|^2_W \leq C(\|\theta_h(t)\|^2_W + \|\rho_h(t)\|^2_W) \\
\leq Ch^{2j}\|p(t)\|^2_{j+1} + C\|\rho_h(0)\|^2_W + Ch^{2j+2}(\int_0^t \|p_t(\tau)\|^2_{j+1} d\tau + \int_0^t \|u_t(\tau)\|^2_{j+2} d\tau) \\
\leq Ch^{2j}(\|p(0)\|^2_{j+1} + \|p(t)\|^2_{j+1} + Ch^{2j+2}(\int_0^t \|p_t(\tau)\|^2_{j+1} d\tau + \int_0^t \|u_t(\tau)\|^2_{j+2} d\tau),
\]
where Lemma 4.4 is utilized, and we finish the proof. \hfill \Box

4.2. Discrete time WG method

In this section, we estimate the errors of fully discrete WG method. Similarly to the semi-discrete problem, we separate $Q_h u(t_n) - u^n_h$ and $Q_h p(t_n) - p^n_h$ into two parts, respectively,
\[
Q_h u(t_n) - u^n_h = (Q_h u(t_n) - \tilde{u}_h(t_n)) + (\tilde{u}_h(t_n) - u^n_h) \\
:= \epsilon_h(t_n) + e^n_h,
\]
and
\[
Q_h p(t_n) - p^n_h = (Q_h p(t_n) - \tilde{p}_h(t_n)) + (\tilde{p}_h(t_n) - p^n_h) \\
:= \theta_h(t_n) + \rho^n_h.
\]
Then we obtain the discrete time error equations,
\[
a_u(e^n_h, v_h) - b(v_h, \rho^n_h) = [a_u(\tilde{u}_h(t_n), v_h) - b(v_h, \tilde{p}_h(t_n))] - [a_u(u^n_h, v_h) - b(v_h, p^n_h)] \\
= (f(t_n), v_0) - (f(t_n), v_0) \\
= 0,
\]
and
\[
\begin{align*}
(c_0 \partial_{\tau} p^n_0, q_0) + b(\partial_{\tau} e^n_h, q_h) + a_p(\rho^n, q_h) \\
= [(c_0 \partial_{\tau} \tilde{p}_0(t_n), q_0) + b(\partial_{\tau} \tilde{u}_h(t_n), q_h) + a_p(\tilde{p}_h(t_n), q_h)] - [(c_0 \partial_{\tau} p^n_0, q_0) + b(\partial_{\tau} u^n_h, q_h) + a_p(p^n_h, q_h)] \\
= [(c_0 \partial_{\tau} \tilde{p}_0(t_n), q_0) + b(\partial_{\tau} \tilde{u}_h(t_n), q_h) + a_p(\tilde{p}_h(t_n), q_h)] - [(g(t_n), q_0) + (\gamma(t_n), q_0)_{\Gamma_{h,n}}] \\
= c_0 (\partial_{\tau} \tilde{p}_0(t_n) - Q_0 p_0(t_n), q_0) + b(\partial_{\tau} \tilde{u}_h(t_n) - Q_h u_h(t_n), q_h) \\
= [c_0 (\partial_{\tau} Q_0 p(t_n) - Q_0 p(t_n), q_0) + b(\partial_{\tau} Q_h u(t_n) - Q_h u(t_n), q_h)].
\end{align*}
\]
For convenience, we set $J_{p^0}^n := \partial_t Q_0 p(t_n) - Q_0 p(t_n)$ and $J_u^n := \partial_t Q_h u(t_n) - Q_h u(t_n)$, then the error equations for the fully discrete problem are given as follows,

$$a_u(e_h^n, v_h) - b(v_h, \rho_h^n) = 0, \quad (4.13)$$

$$(c_0 \partial_t \rho_0^n, q_0) + b(\partial_t e_h^n, q_h) + a_p(p^n, q_h) = -c_0(\partial_t \theta_0(t_n), q_0) - b(\partial_t e_h(t_n), q_h) + [c_0(J_{p^0}^n, q_0) + b(J_u^n, q_h)]. \quad (4.14)$$

The optimal order error estimates for the fully discrete scheme are given in the next two theorems.

**Theorem 4.8.** For $j \geq 1$, let $(u(t); p(t)) \in L^\infty(0, T; H^1_u(\Omega) \cap H^{j+2}(\Omega)) \times L^\infty(0, T; H^1_p(\Omega) \cap H^{j+1}(\Omega))$ be the solution of (1.1) and (1.2), and $(u_h^n; p_h^n) \in V_h^n \times W_h^n$ be the solution of (3.3) and (3.4). Suppose that

$$u(t) \in L^2(0, T; H^{j+2}(\Omega)), \quad u_t(t) \in L^2(0, T; H^1(\Omega)), \quad p(t) \in L^2(0, T; H^1_p(\Omega)).$$

Then there holds

$$\|Q_h u(t_n) - u_h^n\|_V^2 + \|Q_0 p(t_n) - p_h^n\|_W^2 \leq C \tau^2 \int_0^{t_n} (\|u_t\|_1^2 + \|p_t\|_1^2) \, d\tau + Ch^{2j+2}(\|u(0)\|_{j+2}^2 + \|p(0)\|_{j+1}^2$$

$$+ \|u(t_n)\|_{j+2}^2 + \|p(t_n)\|_{j+1}^2) + \int_0^{t_n} (\|u_t\|_{j+2}^2 + \|p_t\|_{j+1}^2) \, d\tau).$$

Proof. Choosing $v_h = \partial_t e_h^n$ and $q_h = \rho_h^n$ in (4.13) and (4.14) and counting up,

$$a_u(e_h^n, \partial_t e_h^n) + c_0(\partial_t \rho_0^n, \rho_0^n) + \|\rho_h^n\|^2_W = [-c_0(\partial_t \theta_0(t_n), \rho_0^n) - b(\partial_t e_h(t_n), \rho_h^n)] + [c_0(J_{p^0}^n, \rho_0^n) + b(J_u^n, \rho_h^n)].$$

Since

$$a_u(e_h^n, \partial_t e_h^n) = \frac{1}{2} \partial_t a_u(e_h^n, e_h^n) + \frac{\tau}{2} a_u(\partial_t e_h^n, \partial_t e_h^n),$$

and

$$\partial_t \rho_0^n + \rho_0^n = \frac{1}{2} \partial_t (\rho_0^n, \rho_0^n) + \frac{\tau}{2} (\partial_t \rho_0^n, \partial_t \rho_0^n),$$

together with the Cauchy-Schwarz inequality and Lemma 4.5 then

$$\frac{1}{2} \partial_t \|e_h^n\|^2_V + \frac{\tau}{2} \|\partial_t e_h^n\|^2_V + \frac{c_0}{2} \|\partial_t \rho_0^n\|^2 + \frac{c_0 \tau}{2} \|\partial_t \rho_0^n\|^2 + \|\rho_h^n\|^2_W \leq C(\|\partial_t \theta_0(t_n)\|_W^2 + \|\partial_t \theta_h(t_n)\|_W^2) + \|J_{p^0}^n\|_W^2 + \|J_u^n\|_V^2 + C_3 \|\rho_h^n\|^2_W.$$

Let $0 < C_3 \leq 1$, and we find

$$\frac{1}{2} \partial_t \|e_h^n\|^2_V + \frac{c_0}{2} \|\partial_t \rho_0^n\|^2 \leq C(\|\partial_t \theta_0(t_n)\|^2 + \|\partial_t \theta_h(t_n)\|^2 + \|J_{p^0}^n\|^2 + \|J_u^n\|^2_V),$$

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Similarly, we have
\[
\| e_h^n \|^2_{V} + \| \rho_0^n \|^2 \leq \| e_h^{n-1} \|^2_{V} + \| \rho_0^{n-1} \|^2 + C \tau (\| \partial_t \theta_0(t_n) \|^2 + \| \partial_t e_h(t_n) \|^2_{V} + \| J_p^n \|^2 + \| J_u^n \|^2_{V}).
\]

It follows by induction that
\[
\| e_h^n \|^2_{V} + \| \rho_0^n \|^2 \leq \| e_h^0 \|^2_{V} + \| \rho_0^0 \|^2 + \sum_{i=1}^{n} C \tau (\| \partial_t \theta_0(t_i) \|^2 + \| \partial_t e_h(t_i) \|^2_{V} + \| J_p^n \|^2 + \| J_u^n \|^2_{V}).
\] (4.15)

Next, we estimate the four terms \(\sum_{i=1}^{n} C \tau \| \partial_t \theta_0(t_i) \|^2\), \(\sum_{i=1}^{n} C \tau \| \partial_t e_h(t_i) \|^2_{V}\), \(\sum_{i=1}^{n} C \tau \| J_p^n \|^2\) and \(\sum_{i=1}^{n} C \tau \| J_u^n \|^2_{V}\), respectively. Firstly, let us focus on \(\sum_{i=1}^{n} C \tau \| \partial_t \theta_0(t_i) \|^2\).

\[
\partial_t \theta_0(t_i) = \frac{1}{\tau} \int_{t_{i-1}}^{t_i} \theta_0(t_i) d\tau,
\]
combined with Lemma 4.4 we present
\[
\sum_{i=1}^{n} C \tau \| \partial_t \theta_0(t_i) \|^2 \leq \sum_{i=1}^{n} C \int_{t_{i-1}}^{t_i} \| \theta_0(t_i) \|^2_2 d\tau \leq C h^{2j+2} \int_0^{t_n} \| p_i \|^2_{J_{j+1}} d\tau.
\] (4.16)

Similarly, we have
\[
\sum_{i=1}^{n} C \tau \| \partial_t e_h(t_i) \|^2_{V} \leq \sum_{i=1}^{n} C \int_{t_{i-1}}^{t_i} \| e_h(t_i) \|^2_{V} d\tau \leq C h^{2j+2} \int_0^{t_n} (\| u_t \|^2_{J_{j+2}} + \| p_t \|^2_{J_{j+1}}) d\tau.
\] (4.17)

In order to bound \(\sum_{i=1}^{n} C \tau \| J_p^n \|^2\), we use
\[
J_p^n = \partial_t Q_0 p(t_i) - Q_0 p_i(t_i) = \frac{1}{\tau} Q_0 (p(t_i) - p(t_{i-1}) - \tau p_t(t_i)) = \frac{1}{\tau} Q_0 \int_{t_{i-1}}^{t_i} (-\tau + t_{i-1}) p_t(t) d\tau,
\]
therefore,
\[
\sum_{i=1}^{n} C \tau \| J_p^n \|^2 \leq \sum_{i=1}^{n} C \int_{t_{i-1}}^{t_i} \| (-\tau + t_{i-1}) p_t(t) \|^2 d\tau \leq C \tau^2 \int_0^{t_n} \| p_t \|^2 d\tau.
\] (4.18)

Likewise, we obtain
\[
\sum_{i=1}^{n} C \tau \| J_u^n \|^2_{V} \leq \sum_{i=1}^{n} C \int_{t_{i-1}}^{t_i} \| (-\tau + t_{i-1}) u_t(t) \|^2_{V} d\tau \leq C \tau^2 \int_0^{t_n} \| u_t \|^2_{V} d\tau.
\] (4.19)

Substituting (4.16), (4.17), (4.18) and (4.19) into (4.15), we get
\[
\| e_h^n \|^2_{V} + \| \rho_0^n \|^2 \leq \| e_h^0 \|^2_{V} + \| \rho_0^0 \|^2 + C \tau^2 \int_0^{t_n} (\| u_t \|^2_{V} + \| p_t \|^2_{V}) d\tau
\]
\[
+ C h^{2j+2} \int_0^{t_n} (\| u_t \|^2_{J_{j+2}} + \| p_t \|^2_{J_{j+1}}) d\tau.
\]
Making use of Lemma 4.4, (4.9) and (4.10), we arrive at

\[
\|Q_h u(t_n) - u^n_s\|^2_V + \|Q_0 p(t_n) - p_0^n\|^2 \\
\leq C(\|\varepsilon_h(t_n)\|^2_V + \|\theta_0(t_n)\|^2 + \|\rho_0^n\|^2) \\
\leq C h^{2j+2}(\|u(t_n)\|^2_{j+2} + \|p(t_n)\|^2_{j+1}) + C(\|\varepsilon_h^n\|^2_V + \|\theta_0^n\|^2) + C \tau^2 \int_0^{t_n} (\|u_tt\|^2_1 + \|p_{tt}\|^2) d\tau \\
+ C h^{2j+2} \int_0^{t_n} (\|u_t\|^2_{j+2} + \|p_t\|^2_{j+1}) d\tau \\
\leq C \tau^2 \int_0^{t_n} (\|u_{tt}\|^2_1 + \|p_{tt}\|^2) d\tau + C h^{2j+2}(\|u(0)\|^2_{j+2} + \|p(0)\|^2_{j+1} + \|u(t_n)\|^2_{j+2} + \|p(t_n)\|^2_{j+1} \\
+ \int_0^{t_n} (\|u_t\|^2_{j+2} + \|p_t\|^2_{j+1})d\tau),
\]

which completes the proof. \(\square\)

**Theorem 4.9.** Under the assumption of Theorem 4.8 together with \(c_0 > 0\), we have the following estimate

\[
\|Q_h p(t_n) - p_h^n\|^2_W \leq C h^{2j}(\|p(0)\|^2_{j+1} + \|p(t_n)\|^2_{j+1}) + C h^{2j+2} \int_0^{t_n} (\|u_t\|^2_{j+2} + \|p_t\|^2_{j+1}) d\tau \\
+ C \tau^2 \int_0^{t_n} (\|u_{tt}\|^2_1 + \|p_{tt}\|^2) d\tau.
\]

**Proof.** Applying the backward Euler method to approximate the time derivative in (4.11),

\[
a_u(\partial_t e_h^n, v_h) - b(v_h, \partial_t \rho_h^n) = 0.
\]

(4.20)

Taking \(v_h = \partial_t e_h^n\) and \(q_h = \partial_t \rho_h^n\) in (4.20) and (4.14), and adding,

\[
\|\partial_t e_h^n\|^2_V + c_0 \|\partial_t \rho_h^n\|^2 + a_p(\rho_h^n, \partial_t \rho_h^n) = [-c_0(\partial_t \theta_0(t_n), \partial_t \rho_h^n) - b(\partial_t e_h(t_n), \partial_t \rho_h^n)] \\
+ [c_0(J^n_{\rho_0}, \partial_t \rho_h^n) + b(J^n_u, \partial_t \rho_h^n)].
\]

Since

\[
a_p(\rho_h^n, \partial_t \rho_h^n) = \frac{1}{2} \partial_t a_p(\rho_h^n, \rho_h^n) + \frac{\tau}{2} a_p(\partial_t \rho_h^n, \partial_t \rho_h^n),
\]

it follows from the Cauchy-Schwarz inequality that

\[
\|\partial_t e_h^n\|^2_V + c_0 \|\partial_t \rho_h^n\|^2 + \frac{1}{2} \partial_t \|\rho_h^n\|^2_W + \frac{\tau}{2} \|\partial_t \rho_h^n\|^2_W \\
= [-c_0(\partial_t \theta_0(t_n), \partial_t \rho_h^n) - b(\partial_t e_h(t_n), \partial_t \rho_h^n)] + [c_0(J^n_{\rho_0}, \partial_t \rho_h^n) + b(J^n_u, \partial_t \rho_h^n)] \\
\leq C(\|\partial_t \theta_0(t_n)\|^2 + \|\partial_t e_h(t_n)\|^2_V + \|J^n_{\rho_0}\|^2 + \|J^n_u\|^2_V) + C_4 \|\partial_t \rho_0^n\|^2.
\]

Let \(0 < C_4 \leq c_0\), then

\[
\|\rho_h^n\|^2_W \leq \|\rho_h^{n-1}\|^2_W + C \tau (\|\partial_t \theta_0(t_n)\|^2 + \|\partial_t e_h(t_n)\|^2_V + \|J^n_{\rho_0}\|^2 + \|J^n_u\|^2_V).
\]

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Utilizing the iteration method, (4.12), (4.16), (4.17), (4.18) and (4.19), we obtain
\[
\|\rho_h^n\|_W^2 \leq \|\rho_h^0\|_W^2 + \sum_{i=1}^n C\tau(\|\partial_t\theta_0(t_i)\|^2 + \|\partial_t\varepsilon_h(t_i)\|^2_\mathcal{V} + \|J_{\rho_b}\|^2 + \|J_u\|^2_\mathcal{V})
\]
\[
\leq Ch^{2j}\|p(0)\|^2_{j+1} + Ch^{2j+2}\int_0^{t_n}(\|u_t\|^2_{j+2} + \|p_t\|^2_{j+1})d\tau + C\tau^2\int_0^{t_n}(\|u_{tt}\|^2_1 + \|p_{tt}\|^2)d\tau,
\]
which gives, combined with Lemma 4.4
\[
\|Q_h p(t_n) - p_h^n\|_W^2 \leq C(\|\theta_h(t_n)\|_W^2 + \|\rho_h^n\|_W^2)
\]
\[
\leq Ch^{2j}(\|p(0)\|^2_{j+1} + \|p(t_n)\|^2_{j+1}) + Ch^{2j+2}\int_0^{t_n}(\|u_t\|^2_{j+2} + \|p_t\|^2_{j+1})d\tau
\]
\[
+ C\tau^2\int_0^{t_n}(\|u_{tt}\|^2_1 + \|p_{tt}\|^2)d\tau.
\]
The proof is finished. \(\square\)

5. Numerical experiments
In this section, we carry out some numerical examples from two aspects: (1) Our proposed methods are flexible in the selections of mesh; (2) The locking problem is overcome by the presented WG methods. Throughout this section, we consider the system (1.1) and (1.2) on a two-dimensional domain \(\Omega = (0, 1)^2\), with the Dirichlet boundary conditions (1.3) and (1.4) for \(u\) and \(p\) on the entire boundary, respectively. The parameters are taken as \(\varepsilon_0 = 1\), \(\kappa = 1\), \(\mu = 1\) and the final time \(T = 1\). For \(\lambda\), we separately test two cases that \(\lambda = 1\) and \(\lambda = 10^4, 10^8\) in the following two subsections. For the weak finite element spaces, we choose \(j = 1\). Specifically, we adopt the following discrete spaces,
\[
V_h := \{v_h : \{v_0, v_b\} : \{v_0, v_b\}_K \in [P_2(K)]^2 \times [P_1(e)]^2, K \in \mathcal{T}_h, e \subset \partial K\},
\]
\[
W_h := \{q_h : \{q_0, q_b\} : \{q_0, q_b\}_K \in P_1(K) \times P_0(e), K \in \mathcal{T}_h, e \subset \partial K\},
\]
and the weak differential operators are computed by
\[
(\nabla_{w,K} \cdot v_h, \psi)_K = -(v_0, \nabla \psi)_K + (v_b \cdot n, \psi)_{\partial K}, \quad \forall \psi \in P_1(K),
\]
\[
(\nabla_{w,K} v_h, \phi)_K = -(v_0, \nabla \phi)_K + (v_b \cdot n, \psi)_{\partial K}, \quad \forall \phi \in [P_1(K)]^{2 \times 2},
\]
\[
(\nabla_{w,K} q_h, \zeta)_K = -(q_0, \nabla \zeta)_K + (q_b \cdot n, \psi)_{\partial K}, \quad \forall \zeta \in [P_0(K)]^2.
\]
As in the previous section, we use \(u_h^n = \{u^n_0, u^n_b\} \in V_h^u\) and \(p_h^n = \{p^n_0, p^n_b\} \in W_h^p\), given by (3.3) and (3.4), to denote the approximate solution of \(u(t_n)\) and \(p(t_n)\), respectively. The \(L^2\)-norm for \(Q_0 u(t_n) - u^n_h\), the \(\| \cdot \|_V\)-norm for \(Q_h u(t_n) - u^n_h\), the \(L^2\)-norm for \(Q_0 p(t_n) - p^n_h\) and the \(\| \cdot \|_W\)-norm for \(Q_h p(t_n) - p^n_h\) are utilized to illustrate the numerical results.
5.1. Tests for convergence orders on different meshes

In this subsection, we accomplish the numerical computations and estimate the convergence orders on triangular meshes, rectangular meshes and hybrid polygonal meshes, respectively. The Lamé constant $\lambda = 1$ is chosen. The right-hand side terms $f$ and $g$ of (1.1) and (1.2) are selected according to the analytical solution which is given as follows,

\[ u = \left( \begin{array}{c} 10x^2(1 - x)^2y(1 - y)(1 - 2y)\exp(-t) \\ -10x(1 - x)(1 - 2x)y^2(1 - y)^2\exp(-2t) \end{array} \right), \]

\[ p = 10x^2(1 - x)^2y(1 - y)(1 - 2y)\exp(-3t). \]

5.1.1. Triangular meshes

A uniform triangular mesh is considered on the two-dimensional domain $\Omega = (0, 1)^2$, and we test the convergence orders with $h = \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{2}{8}}, \sqrt{\frac{2}{16}}, \sqrt{\frac{2}{32}}, \sqrt{\frac{2}{64}}$ and the time step $\tau = h^2$. Table 1 and 2 describe that the error convergence orders of the $L^2$-norm and the $\|\cdot\|_V$-norm for $u$ are severally $O(h^3)$ and $O(h^2)$, and the ones of the $L^2$-norm and the $\|\cdot\|_W$-norm for $p$ are $O(h^2)$ and $O(h)$, respectively. These optimal convergence orders verify our theoretical results in Theorem 4.8 and 4.9.

Table 1: WG error convergence orders for $u$ with $\lambda = 1$ and $\tau = h^2$ on uniform triangular meshes

| $h$       | $\|Q_0u(t_n) - u_0^n\|$ | Order | $\|Q_hu(t_n) - u_h^n\|_V$ | Order |
|-----------|------------------------|-------|-------------------------|-------|
| $\sqrt{\frac{2}{3}}$ | 9.1014E-03        | -     | 7.0190E-02               | -     |
| $\sqrt{\frac{2}{4}}$ | 1.3362E-03        | 2.7679| 2.1489E-02               | 1.7077|
| $\sqrt{\frac{2}{8}}$ | 1.7620E-04        | 2.9229| 5.7772E-03               | 1.8952|
| $\sqrt{\frac{2}{16}}$ | 2.2603E-04        | 2.9626| 1.4845E-03               | 1.9604|
| $\sqrt{\frac{2}{32}}$ | 2.9438E-05        | 2.9408| 3.7534E-04               | 1.9837|
| $\sqrt{\frac{2}{64}}$ | 4.1461E-07        | 2.8278| 9.4302E-05               | 1.9928|

Table 2: WG error convergence orders for $p$ with $\lambda = 1$ and $\tau = h^2$ on uniform triangular meshes

| $h$       | $\|Q_0p(t_n) - p_0^n\|$ | Order | $\|Q_hp(t_n) - p_h^n\|_W$ | Order |
|-----------|------------------------|-------|-------------------------|-------|
| $\sqrt{\frac{2}{3}}$ | 4.5721E-03        | -     | 1.9844E-02               | -     |
| $\sqrt{\frac{2}{4}}$ | 8.9546E-04        | 2.3521| 7.2165E-03               | 1.4593|
| $\sqrt{\frac{2}{8}}$ | 2.2088E-04        | 2.0194| 3.4149E-03               | 1.0794|
| $\sqrt{\frac{2}{16}}$ | 5.5267E-05        | 1.9987| 1.6864E-03               | 1.0180|
| $\sqrt{\frac{2}{32}}$ | 1.3825E-05        | 1.9992| 8.4064E-04               | 1.0044|
| $\sqrt{\frac{2}{64}}$ | 3.4568E-06        | 1.9998| 4.2001E-04               | 1.0011|
5.1.2. Rectangular meshes

In this test, we make use of a uniform rectangular mesh $T_h$ with $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$ and the time step $\tau = h^2$, and the convergence rates are depicted in Tables 3 and 4. From the two tables, it can be seen that the four norms for $u$ and $p$ all achieve the optimal error convergence orders which are accordance with our theoretical analysis.

Table 3: WG error convergence orders for $u$ with $\lambda = 1$ and $\tau = h^2$ on uniform rectangular meshes

| $h$   | $\|Q_h u(t_n) - u_0^n\|$ Order | $\|Q_h u(t_n) - u_h^n\|_V$ Order |
|-------|-------------------------------|-----------------------------------|
| $\frac{1}{2}$ | 2.1907E-02 - | 1.2383E-01 - |
| $\frac{1}{4}$ | 3.3989E-03 2.6883 | 3.9438E-02 1.6507 |
| $\frac{1}{8}$ | 4.5465E-04 2.9022 | 1.1434E-02 1.7863 |
| $\frac{1}{16}$ | 5.6820E-05 2.9022 | 1.1434E-02 1.7863 |
| $\frac{1}{32}$ | 7.1510E-06 2.9022 | 1.1434E-02 1.7863 |
| $\frac{1}{64}$ | 9.8402E-07 2.9022 | 1.1434E-02 1.7863 |

Table 4: WG error convergence orders for $p$ with $\lambda = 1$ and $\tau = h^2$ on uniform rectangular meshes

| $h$   | $\|Q_0 p(t_n) - p_0^n\|$ Order | $\|Q_h p(t_n) - p_h^n\|_W$ Order |
|-------|-------------------------------|-----------------------------------|
| $\frac{1}{2}$ | 8.3611E-03 - | 2.9900E-02 - |
| $\frac{1}{4}$ | 1.8531E-03 1.1653E-02 | 1.3594 |
| $\frac{1}{8}$ | 4.7047E-04 1.9778 | 5.5779E-03 1.0630 |
| $\frac{1}{16}$ | 1.1930E-04 1.9795 | 2.7740E-03 1.0077 |
| $\frac{1}{32}$ | 2.9973E-05 1.9929 | 1.3862E-03 1.0008 |
| $\frac{1}{64}$ | 7.5036E-06 1.9980 | 6.9304E-04 1.0001 |

5.1.3. Hybrid polygonal meshes

In this subsection, we partition the two-dimensional domain $\Omega = (0, 1)^2$ into hybrid polygonal meshes which are shown as Figure 1, where $N_h$ is the number of complete subdivisions on each boundary of $\Gamma$. Numerical tests are conducted with the time step $\tau = N_h^{-2}$. Table 5 and 6 render all errors and convergence results of optimal orders, which agree with our expectation.

5.2. Tests for locking-free when $\lambda \to \infty$

The aim of this subsection is to validate the locking-free property of our WG method. The right-hand side terms are chosen so that the exact solution is

$$u = \begin{pmatrix} \exp(-t)(\sin(2\pi y)(-1 + \cos(2\pi x)) + \frac{1}{\mu + \lambda} \sin(\pi x) \sin(\pi y)) \\ \exp(-t)(\sin(2\pi x)(1 - \cos(2\pi y)) + \frac{1}{\mu + \lambda} \sin(\pi x) \sin(\pi y)) \end{pmatrix},$$

$$p = \exp(-t) \sin(\pi x) \sin(\pi y).$$
Figure 1: Polygonal grids with $N_h = 1, 2, 4$.

Table 5: WG error convergence orders for $u$ with $\lambda = 1$ and $\tau = N_h^{-2}$ on hybrid polygonal meshes

| $N_h$ | $\|Q_0 u(t_n) - u_n^0\|$ | Order | $\|Q_h u(t_n) - u_n^h\|_V$ | Order |
|-------|----------------------------|-------|-----------------|-------|
| 2     | 1.3017E-02                 | -     | 8.1600E-02      | -     |
| 4     | 2.6812E-03                 | 3.1095| 3.2019E-02      | 1.8411|
| 8     | 4.3683E-04                 | 3.1545| 1.0204E-02      | 1.9882|
| 16    | 6.1513E-05                 | 3.1400| 2.8763E-03      | 2.0282|
| 32    | 8.2731E-04                 | 3.0607| 7.6434E-04      | 2.0218|
| 64    | 1.1501E-06                 | 2.9303| 1.9708E-04      | 2.0129|

Table 6: WG error convergence orders for $p$ with $\lambda = 1$ and $\tau = N_h^{-2}$ on hybrid polygonal meshes

| $N_h$ | $\|Q_0 p(t_n) - p_n^0\|$ | Order | $\|Q_h p(t_n) - p_n^h\|_W$ | Order |
|-------|----------------------------|-------|-----------------|-------|
| 2     | 3.5652E-03                 | -     | 1.7364E-02      | -     |
| 4     | 1.3776E-03                 | 1.8715| 9.5444E-03      | 1.1778|
| 8     | 4.1133E-04                 | 2.1013| 5.0863E-03      | 1.0942|
| 16    | 1.1304E-04                 | 2.0690| 2.6650E-03      | 1.0353|
| 32    | 2.9658E-05                 | 2.0412| 1.3699E-03      | 1.0152|
| 64    | 7.5972E-06                 | 2.0226| 6.9530E-04      | 1.0071|

For the verification of locking-free of our method, we compare the WG method with the lowest order Taylor-Hood element, i.e., $[P_2]^2 \times P_1$ element, for $u$ and $p$ with the three choices of $\lambda = 1, 10^4, 10^8$. We denote the finite element solution of $u$ and $p$ by $u_{h,fem}^n$ and $p_{h,fem}^n$, respectively. The error convergence rates on uniform triangulation are computed with the time step $\tau = h^2$.

Figure 2-4 depict the errors of our method and the conforming finite element method for the different $\lambda$. As can be observed from these figures, the locking problem does not affect the finite element approximation to $p$, but only the one to $u$. As $\lambda$ goes to infinity, the orders of error convergence of finite element solution $u_{h,fem}^n$ degenerate from the optimal ones to the lower ones. While WG method has no significant fluctuations and keeps the optimal convergence orders during the entire change in $\lambda$, which clearly shows the advantage of our
method.

Figure 2: Errors for $u$ (Left) and $p$ (Right) with $\lambda = 1$.

Figure 3: Errors for $u$ (Left) and $p$ (Right) with $\lambda = 10^4$.

Acknowledgments

This work was sponsored by the Research Foundation for Beijing University of Technology New Faculty Grant No. 006000514122516. The computations here were partly done
Figure 4: Errors for $u$ (Left) and $p$ (Right) with $\lambda = 10^8$.

on the high performance computers of State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences. The authors sincerely thank Dr. Hui Peng for the discussion about mathematical models and their solving algorithms.

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