SUBPRIME SOLUTIONS OF THE CLASSICAL YANG-BAXTER EQUATION

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Abstract. We introduce a new family of classical $r$-matrices for the Lie algebra $\mathfrak{sl}_n$ that lies in the Zariski boundary of the Belavin-Drinfeld space $\mathcal{M}$ of quasi-triangular solutions to the classical Yang-Baxter equation. In this setting $\mathcal{M}$ is a finite disjoint union of components; exactly $\phi(n)$ of these components are $SL_n$-orbits of single points. These points are the generalized Cremmer-Gervais $r$-matrices $r_{i,n}$ which are naturally indexed by pairs of positive coprime integers, $i$ and $n$, with $i < n$. A conjecture of Gerstenhaber and Giaquinto states that the boundaries of the Cremmer-Gervais components contain $r$-matrices having maximal parabolic subalgebras $p_{i,n} \subseteq \mathfrak{sl}_n$ as carriers. We prove this conjecture in the cases when $n \equiv \pm 1 \pmod{i}$. The subprime linear functionals $f \in p_{i,n}^*$ and the corresponding principal elements $H \in p_{i,n}$ play important roles in our proof. Since the subprime functionals are Frobenius precisely in the cases when $n \equiv \pm 1 \pmod{i}$, this partly explains our need to require these conditions on $i$ and $n$. We conclude with a proof of the GG boundary conjecture in an unrelated case, namely when $(i, n) = (5, 12)$, where the subprime functional is no longer a Frobenius functional.

1. Introduction and Main Results

Throughout this paper, we assume the ground field $\mathbb{F}$ has characteristic 0 although most results in this paper also hold for fields of nearly any other characteristic. In particular, we will fix a pair of positive integers $i$ and $n$ with $i < n$, and in some of the calculations that follow, the numbers 2 and $n$ appear in denominators. Thus we need $2n$ to be a nonzero element of the ground field $\mathbb{F}$. For vectors $u, v$ in an $\mathbb{F}$-vector space $V$, define $u \wedge v := \frac{1}{2}(u \otimes v - v \otimes u) \in V \wedge V \subseteq V \otimes V$. For a Lie algebra $\mathfrak{g}$ and an element in the tensor space $r = \sum a_i \wedge b_i \in \mathfrak{g} \wedge \mathfrak{g}$ we say that $r$ is a classical $r$-matrix if the Schouten bracket of $r$ with itself

\begin{equation}
\langle r, r \rangle := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]
\end{equation}

is $\mathfrak{g}$-invariant. Here, $r_{12} = r \otimes 1$, $r_{23} = 1 \otimes r$, and $r_{13} = \sigma(r_{23})$, where $\sigma$ is the linear endomorphism of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ that permutes the first two tensor components: $\sigma(x \otimes y \otimes z) = y \otimes x \otimes z$. Classical $r$-matrices arise naturally in the context of Poisson-Lie groups and Lie bialgebras (see e.g. [2, Chapter 1]). If $\langle r, r \rangle = 0$, then $r$ is said to be a solution to the classical Yang-Baxter equation (CYBE). On the other hand, if $\langle r, r \rangle$ is non-zero and $\mathfrak{g}$-invariant, $r$ is said to be a solution to the

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modified classical Yang-Baxter equation (MCYBE). Following [8], we let \( \mathcal{C} \) and \( \mathcal{M} \) denote the solution spaces of the CYBE and MCYBE respectively.

In the early 1980’s Belavin and Drinfeld [11] classified the solutions to the MCYBE for the finite-dimensional complex simple Lie algebras and showed that the solution space \( \mathcal{M} \) is a finite disjoint union of components of the projective space \( \mathbb{P}(g \wedge g) \). The components of \( \mathcal{M} \) are indexed by triples \( T = (T, S_1, S_2) \), where \( S_1 \) and \( S_2 \) are subsets of the set of simple roots of \( g \) and \( T : S_1 \to S_2 \) is a bijection that preserves the Killing form and satisfies a nilpotency condition. For any BD-triple \( (T, S_1, S_2) \), one can always produce another BD-triple \( T' \) by restricting \( T \) to a subset of \( S_1 \). This gives rise to the notion of a partial ordering on triples: \( T' < T \).

An interesting family of solutions of the MYCBE arises when considering maximal BD-triples with \( S_2 \) missing a single root. These occur only in the case when \( g = sl_n \), and in this setting there are exactly \( \phi(n) \) BD-triples of this type, where \( \phi \) is the Euler-totient function [8]. The component of \( \mathcal{M} \) corresponding to such a triple can each be described as the \( SL_n \)-orbit of a single point \( r \in sl_n \wedge sl_n \), called the Cremmer-Gervais \( r \)-matrix [3, 7]. Hence, throughout we let \( i \) and \( n \) be a pair of positive coprime integers with \( i < n \) and let \( r_{CG}(i, n) \) denote the corresponding Cremmer-Gervais \( r \)-matrix of type \((i, n)\). The BD-triple associated to \( r_{CG}(i, n) \) has the \( i \)-th simple root missing from \( S_2 \). The Cremmer-Gervais \( r \)-matrices have explicit formulas which we describe in Section 2.2. For example when \( i = 1 \) and \( n = 3 \), we have

\[
\begin{align*}
  r_{CG}(1, 3) &= 2e_{12} \wedge e_{32} + e_{12} \wedge e_{21} + e_{13} \wedge e_{31} + e_{23} \wedge e_{32} \\
  &\quad + \frac{1}{3}(e_{11} - e_{22}) \wedge (e_{22} - e_{33}) \in sl_3 \wedge sl_3.
\end{align*}
\]

A classification of solutions to the CYBE, on the other hand, is quite difficult and would entail classifying all quasi-Frobenius Lie algebras [14]. However, in certain cases there is a straightforward technique for constructing non-trivial solutions to the CYBE. For instance, if a Lie algebra \( \mathfrak{f} \) has a functional \( f \in \mathfrak{f}^* \) such that \( x \mapsto f([x, -]) \) is an isomorphism from \( \mathfrak{f} \) to \( \mathfrak{f}^* \), then, for an ordered basis \( x_1, x_2, \ldots, x_d \) of \( \mathfrak{f} \), the bilinear form \( (x_j, x_k) := f([x_j, x_k]) \) on \( \mathfrak{f} \) is invertible and \( \sum (f[x_j, x_k])^{-1} x_j \wedge x_k \) produces a non-degenerate solution to the CYBE. In such a setting, we call \( \mathfrak{f} \) a Frobenius Lie algebra and \( f \) is called a Frobenius functional. The inverse image of \( f \) under the map \( x \mapsto f([x, -]) \) is called the principal element [10]. The \( r \)-matrix produced from this construction is said to have carrier \( \mathfrak{f} \).

Frobenius algebras arise in several other contexts. For instance, in [12], Ooms shows that a universal enveloping algebra \( U(L) \) of a Lie algebra \( L \) over a field \( F \) of characteristic 0 is primitive if and only if \( L \) is Frobenius. Some interesting families of Frobenius subalgebras of \( sl_n \) include certain types of parabolic and biparabolic Lie algebras (see e.g. [3, 6, 13]). In this paper we focus on the maximal parabolic subalgebras \( \mathfrak{p}(i, n) \subseteq sl_n \), where the \( i \)-th negative root vector is deleted. In [6], Elashvili showed that \( \mathfrak{p}(i, n) \) is Frobenius if and only if \( i \) is relatively prime to \( n \). Hence, if \( f \in \mathfrak{p}(i, n)^* \) is a Frobenius functional, then a solution \( r_f \in sl_n \wedge sl_n \) to the CYBE having carrier subalgebra \( \mathfrak{p}(i, n) \) can be constructed from \( f \).

In [8], it was shown that the Zariski boundary of \( \mathcal{M} \) is contained in \( \mathcal{C} \). Thus, one would hope that there might be a simple description of boundary solutions analogous to the BD-classification, but this seems to be a very difficult problem.
In an effort to better understand the boundary of \( \mathcal{M} \), Gerstenhaber and Giaquinto conjectured that the Zariski boundary of the component of \( \mathcal{M} \) containing \( r_{CG}(i,n) \) contains an \( r \)-matrix having parabolic carrier \( p(i,n) \). They prove their conjecture in the case when \( i = 1 \) [8]. In Section 5 we extend these results by proving their conjecture also holds in the cases when \( n \equiv \pm 1 \pmod{i} \). As it turns out, the subprime functional

\[
 f_{\text{subprime}} := \sum_{i<j \leq n} e_{j-i,j}^* + \sum_{1 \leq j < i} e_{j+1,j}^* \in p(i,n)^*
\]

is Frobenius if and only if \( n \equiv \pm 1 \pmod{i} \) (see e.g. [9] Section 9.1)]. Due to the key role that the subprime functional plays in our proofs, we refer to the cases when \( n \equiv \pm 1 \pmod{i} \) as subprime cases.

On the way towards proving that the boundary conjecture holds in the subprime cases, we define in Section 4.4 a rational map \( \Phi : \mathbb{F}^x \to SL_n \) (that depends on \( i \) and \( n \)), where \( \mathbb{F}^x = \mathbb{F} \setminus \{0\} \). By rational, we mean the matrix entries in \( \Phi(t) \) are polynomials in \( t^{\pm 1} \). The principal element associated to the subprime functional is needed in the construction of \( \Phi \), and, in fact, all entries of \( \Phi(t) \) have the form \( ct^p \) with \( c \) and \( p \) integers. For example, when \( i = 3 \) and \( n = 7 \) our formula for \( \Phi \) is given by

\[
 \Phi : t \mapsto \begin{bmatrix}
 t^{-2} & 0 & 0 & -2t^{-2} & 0 & 0 & t^{-2} \\
 -2t^{12} & t^{12} & 0 & t^{12} & -t^{12} & 0 & t^{12} \\
 t^{26} & -t^{26} & t^{26} & -2t^{26} & t^{26} & -t^{26} & t^{26} \\
 0 & 0 & 0 & t^{-16} & 0 & 0 & -t^{-16} \\
 0 & 0 & 0 & -2t^{-2} & t^{-2} & 0 & -t^{-2} \\
 0 & 0 & 0 & t^{12} & -t^{12} & t^{12} & -t^{12} \\
 0 & 0 & 0 & 0 & 0 & 0 & t^{-30}
\end{bmatrix} \in SL_n.
\]

We use the map \( \Phi \) to produce a one-parameter family of solutions to the MYCBE lying in \( SL_n \)-orbit of the Cremmer-Gervais \( r \)-matrix. More precisely, we show that \( \Phi(t).r_{CG}(i,n) \) is of the form \( r' + t^2nh \), where \( r' \in \mathcal{M} \) and \( b \in \mathcal{C} \). Here \( \Phi(t) \in SL_n \) acts on \( r_{CG}(i,n) \) via the adjoint action. In a sense the map \( \Phi \) is used to deform the Cremmer-Gervais \( r \)-matrix, and \( t \) can be viewed as a deformation parameter. Under mild conditions (which are met in our case) it follows that the coefficient of the highest degree term in \( t \) is a boundary solution to the CYBE [8] Proposition 5.1]. Hence, \( b \) is a boundary solution, which we refer to as a subprime boundary solution of the classical Yang-Baxter equation. Our main result is the following theorem that proves the Gerstenhaber-Giaquinto boundary conjecture holds in the subprime cases.

**Main Theorem.** Let \( i \) and \( n \) be a pair of positive integers with \( i < n \) so that \( n \equiv \pm 1 \pmod{i} \) and let \( r = r_{CG}(i,n) \) denote the Cremmer-Gervais \( r \)-matrix of type \((i,n)\). Then

1. \( \Phi(t).r \) is of the form \( r' + t^2nb \), where \( r' \in \mathcal{M} \) and \( b \in \mathcal{C} \),
2. the \( r \)-matrices \( r' \) and \( b \) appearing in part (1) lie in the Zariski closure of the component of \( \mathcal{M} \) containing \( r \).
3. the \( r \)-matrix \( b \) is a boundary solution to the classical Yang-Baxter equation,
4. the carrier of \( b \) is the maximal parabolic subalgebra \( p(i,n) \subseteq \mathfrak{sl}_n \).
In Section 5 we give explicit formulas for the \( r \)-matrices \( r' \) and \( b \) appearing in the statement of our main result. Interestingly, the principal element \( H \) associated to the subprime linear functional plays an important role in our construction of \( \Phi \). In Proposition 4.4 we prove a curious relationship between \( r' \) and \( H \), specifically that \( r' \) is in the kernel of the adjoint action of \( H \). Since the subprime functional is a Frobenius functional in \( p(i, n)^* \) if and only if \( n \equiv \pm 1 \pmod{i} \), this partly explains the need for us to assume these conditions on \( i \) and \( n \) in our proof.

In Section 6 we conclude with a proof of the Gerstenhaber-Giaquinto boundary conjecture in a case unrelated to the subprime cases, namely when \( i = 5 \) and \( n = 12 \), where the subprime functional is no longer Frobenius.

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2. Solutions of the MCYBE

2.1. Belavin-Drinfeld Classification. Solutions to the modified classical Yang-Baxter equation (MCYBE) for a finite dimensional complex simple Lie algebra \( g \) were constructively classified by Belavin and Drinfeld [1]. Their classification result says that the solution space \( M \) is a finite disjoint union of quasi-projective subvarieties of \( \mathbb{P}(g \wedge g) \). The components of \( M \) are indexed by combinatorial objects, called BD-triples, on the Dynkin diagram of \( g \). A BD-triple \( T \) puts a particular partial ordering \( \preceq \) on the positive root vectors of \( g \). This ordering is used to define

\[
\alpha = 2 \sum_{p < j} e_p \wedge e_j \in g \wedge g
\]

and a subvariety \( \beta(T) \) of \( h \wedge h \), where \( h \) is a Cartan subalgebra of \( g \). The BD classification result states that each \( r \in M \) is equivalent, up to scaling and applying inner automorphisms of \( g \), to a unique \( r \)-matrix of the form \( \alpha + \beta + \gamma \), where \( \beta \in \beta(T) \) and \( \gamma \) is a certain fixed element of \( g \wedge g \) independent of \( T \). For further details on the BD-classification, see e.g. [2, Chapter 1].

For our purposes, rather than give a full description of the Belavin-Drinfeld classification, we turn our attention to the case when \( g = sl_n \), the Lie algebra of traceless \( n \times n \) matrices. Let \( g = n^- \oplus h \oplus n^+ \) denote the triangular decomposition. In this setting, a BD-triple \( T = (T, S_1, S_2) \) is a bijection \( T : S_1 \rightarrow S_2 \), where \( S_1 \) and \( S_2 \) are subsets of \([1, n - 1]\) (here and below we use the notation \([a, b]\) for \(\{a, a+1, \ldots, b\}\), satisfying the conditions

\[
\begin{align*}
(1) & \quad \text{(adjacency preserving)} \ |T(j) - T(k)| = 1 \text{ if and only if } |j - k| = 1, \\
(2) & \quad \text{(local nilpotency)} \text{ for every } j \in S_1 \text{ there exists } N \in \mathbb{N} \text{ so that } T^N(j) \notin S_1.
\end{align*}
\]

A BD-triple defines a partial ordering \( \prec \) on \([1, n - 1]\) by \( j \prec k \) if and only if there exists \( N \in \mathbb{N} \) so that \( T^N(j) = k \). This ordering extends to the set \(\{e_{jk} \in sl_n \mid 1 \leq j < k \leq n\}\) of positive root vectors of \( sl_n \): put \( e_{jk} \prec e_{\ell m} \) if and only if \( m = k - j + \ell \) and \( j + s \prec \ell + s \) for all \( s \in [0, k - j - 1] \). We define

\[
(2.1) \quad \alpha := 2 \sum e_{jk} \wedge e_{m\ell} \in n^+ \wedge n^-,
\]

where the sum is over all tuples \((j, k, \ell, m) \in [1, n]^4\) such that \( e_{jk} \prec e_{\ell m} \). Next let \( h_j := e_{jj} - e_{j+1,j+1} \) for \( j \in [1, n - 1] \) and let \( e_{jk}^* \in g^* \) be the linear functional that returns the \((j, k)\)-entry. Next define the subvariety \( \beta(T) \subseteq h \wedge h \) by
\begin{eqnarray}
\beta(T) := \left\{ \beta \in \mathfrak{h} \cap \mathfrak{h} \mid (1 \otimes (e_T^* \cdot T + e_j^* \cdot j)) \beta = \frac{1}{2} (h_T + h_j) \text{ for all } j \in S_1 \right\}.
\end{eqnarray}

As a variety $\beta(T)$ has dimension $d(d-1)/2$, where $d = n-1 - |S_1|$. Finally define
\begin{eqnarray}
\gamma := \sum_{1 \leq k < \ell \leq n} e_{k\ell} \wedge e_{\ell k} \in \mathfrak{n}^+ \wedge \mathfrak{n}^-.
\end{eqnarray}

Observe that the solution space $\mathcal{M}$ is stable under the adjoint action of the special linear group $SL_n$ and also by rescaling by any nonzero scalar $\lambda$. We call $r, r' \in \mathcal{M}$ equivalent if there exists a nonzero scalar $\lambda$ and $g \in SL_n$ so that $\lambda g \cdot r = r'$. Belavin and Drinfeld’s classification result (specialized to the setting $\mathfrak{g} = \mathfrak{sl}_n$) asserts that $r = \alpha + \beta + \gamma \in \mathfrak{sl}_n \wedge \mathfrak{sl}_n$ is a solution to the MCYBE for every BD-triple $T = (T, S_1, S_2)$ and $\beta \in \beta(T)$, and conversely any solution to the MCYBE is equivalent to a unique $r$-matrix of this form.

2.2. Cremmer-Gervais $r$-matrices. We turn our attention to a particularly interesting family of $r$-matrices called Cremmer-Gervais $r$-matrices (see [4,7]). These are precisely the solutions to the MCYBE for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ associated to maximal BD-triples, i.e. those with $S_1$ having the maximum possible cardinality, $|S_1| = n-2$.

Our aim in this section is to explicitly describe the Cremmer-Gervais $r$-matrices. First of all there are exactly $\phi(n)$ maximal BD-triples, where $\phi$ is the Euler-totient function. Hence, throughout the rest of this section we fix a pair of positive coprime integers $i$ and $n$ with $i < n$ and let $r_{CG}(i, n) \in \mathcal{M}$ denote the Cremmer-Gervais $r$-matrix of type $(i, n)$. We define a bijection $T_{i,n}$ from the set $S_1 := [1, n-i-1] \cup [n-i+1, n-1]$ to the set $S_2 := [1, i-1] \cup [i+1, n-1]$ by $T_{i,n}(j) = (j+i) \mod n$ and construct a directed graph $\Gamma_{CG}(i, n)$ having vertices labelled $1, 2, \ldots, n-1$ and edges $j \rightarrow T_{i,n}(j)$ for every $j \in S_1$. The graph $\Gamma_{CG}(i, n)$ is, in fact, isomorphic to $i \rightarrow 2i \rightarrow \cdots \rightarrow n-i$, where the numbers are assumed to be reduced modulo $n$.

We will refer to the edges $j \rightarrow k$ with $j < k$ as the forward arrows, whereas the edges $j \rightarrow k$ with $j > k$ are called the backward arrows. In Figure 1 we draw the forward arrows in the top half of the graph and the backward arrows in the bottom half. Recall the map $T_{i,n}$ defines an ordering $\prec$ on $[1, n-1]$. Equivalently $j \prec k$ if and
only if the path in $\Gamma_{CG}(i, n)$ that starts at vertex $j$ eventually reaches vertex $k$ by following the directed edges. In fact, the ordering is simply $i < 2i < 3i < \cdots < n-i$, where the numbers are assumed to be reduced modulo $n$.

Let $\alpha_{\rightarrow}$ denote the $\alpha$-part of $r_{CG}(i, n)$ obtained by considering only the forward arrows, and let $\alpha_{\leftarrow}$ denote the terms introduced to $\alpha$ by including the backward arrows. Thus, $\alpha = \alpha_{\rightarrow} + \alpha_{\leftarrow}$. To describe the $\alpha$-part, we find it convenient to first define the following elements of the general linear Lie algebra $\mathfrak{gl}_n$:

\[
\xi_{k\ell} := \sum_{p \in \mathbb{Z}} e^{k+(ip,\ell+ip)} \in \mathfrak{gl}_n, \quad \eta_{k\ell} := \sum_{p \geq 0} e^{k+(ip,\ell+ip)} \in \mathfrak{gl}_n, \tag{2.4}
\]

for all $k, \ell \in \mathbb{Z}$. Throughout, we treat $\mathfrak{sl}_n$ as a subset of $\mathfrak{gl}_n$ and follow the convention $e_{jk} := 0$ if either subscript is out of range, i.e. if either $j$ or $k$ is greater than $n$ or less than 1. For integers $p, q \in \mathbb{Z}$ with $q > 0$, we write $p \mod q$ to denote the reduction of $p$ modulo $q$ in $[0, q-1]$.

The $\alpha$-parts of the Cremmer-Gervais $r$-matrix $r_{CG}(i, n)$ are

\[
\alpha_{\rightarrow} = 2 \sum_{1 \leq k < \ell \leq n-i} e_{k\ell} \wedge \eta_{\ell+i,k+i} \in \mathfrak{n}^+ \wedge \mathfrak{n}^-, \tag{2.5}
\]

and

\[
\alpha_{\leftarrow} = 2 \sum_{(j, k, \ell, m) \in [1, i]^4} e_{j+(n \mod i),k+(n \mod i)} \wedge e_{m\ell} \in \mathfrak{n}^+ \wedge \mathfrak{n}^-, \tag{2.6}
\]

where the sum above runs over all tuples $(j, k, \ell, m) \in [1, i]^4$ such that $e_{jk} \preceq e_{\ell m}$ in the partial ordering on the positive root vectors of $\mathfrak{sl}_i$ (in particular, $1 \leq j < k \leq i$ and $1 \leq \ell < m \leq i$) induced by the Cremmer-Gervais graph $\Gamma_{CG}(i - (n \mod i), i)$.

Equation (2.6) provides a way to iteratively construct $\alpha_{\leftarrow}$. The number of steps needed to construct $\alpha_{\leftarrow}$ equals the number of steps until the Euclidean algorithm, starting with the integers $n$ and $i$, terminates.

The varieties $\beta(\mathcal{T}_{i,n})$ associated to maximal BD-triples each reduce to a single point. In [8] a formula for this point is given by

\[
\beta := \sum_{1 \leq j < \ell \leq n} \left(-1 + \frac{2}{n} \left((j - \ell)i^{-1} \mod n\right)\right) e_{jj} \wedge e_{\ell\ell} \in \mathfrak{h} \wedge \mathfrak{h}. \tag{2.7}
\]

The Cremmer-Gervais $r$-matrix $r_{CG}(i, n)$ is equal to $r_{CG}(i, n) = \alpha_{\rightarrow} + \alpha_{\leftarrow} + \beta + \gamma \in \mathcal{M}$.

3. Solutions of the CYBE

3.1. Quasi-Frobenius and Frobenius subalgebras. The solution space $\mathcal{C}$ to the classical Yang-Baxter equation (CYBE), on the other hand, is difficult to describe as there is not a constructive classification analogous to the classification result of Belavin and Drinfeld for the solution space $\mathcal{M}$. However we recall a homological description of solutions to the CYBE [14].

For $r \in \mathcal{C}$, let $\mathfrak{f} \subseteq \mathfrak{g}$ be the carrier of $r$. The carrier is the Lie subalgebra of $\mathfrak{g}$ spanned by $\{(\xi \otimes 1)r \mid \xi \in \mathfrak{g}^*\}$. The map $\tilde{r} : \mathfrak{f}^* \to \mathfrak{f}$ defined by $\xi \mapsto (\xi \otimes 1)r$ is
a linear isomorphism and induces a Lie algebra 2-cocycle \( B : \mathfrak{f} \times \mathfrak{f} \to \mathbb{F} \) given by \((x, y) \mapsto \langle r^{-1}(x), y \rangle\). Thus, \( \mathfrak{f} \) is a quasi-Frobenius Lie algebra. This process be can inverted to give a one-to-one correspondence between quasi-Frobenius Lie algebras \((\mathfrak{f}, B)\) and \(r\)-matrices in \( C \) having carrier \( \mathfrak{f} \). Thus classifying all solutions to the CYBE would entail classifying all quasi-Frobenius subalgebras \( \mathfrak{f} \subseteq \mathfrak{g} \).

If the cocycle \( B \) corresponding to \( r \) is a coboundary, this means that \( B \) is of the form \( B(x, y) = f([x, y]) \) for some linear functional \( f \in \mathfrak{f}^* \). In such cases we say that \( r \) admits the functional \( f \) and we call \( \mathfrak{f} \) a Frobenius Lie algebra and call \( f \) a Frobenius functional. Here, the linear map \( x \mapsto f([x, -]) \) is an isomorphism from \( \mathfrak{f} \) to its dual space \( \mathfrak{f}^* \). The inverse image of \( f \) under this map is called the principal element, which is the unique \( H \in \mathfrak{f} \) satisfying \( f([H, x]) = f(x) \) for all \( x \in \mathfrak{f} \) (see e.g. [11]).

**Example 3.1.** The maximal parabolic subalgebra \( \mathfrak{p}(i, n) \subseteq \mathfrak{sl}_n \) obtained by deleting the \( i \)-th negative simple root is Frobenius if and only if \( i \) and \( n \) are relatively prime [6] (see Figure 2).

\[
\begin{align*}
\mathfrak{p}(2, 5) &= \begin{bmatrix}
* & * & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & * & * & *
\end{bmatrix} & \mathfrak{p}(4, 9) &= \begin{bmatrix}
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & *
\end{bmatrix}
\end{align*}
\]

**Figure 2.** Maximal parabolic Frobenius Lie algebras \( \mathfrak{p}(2, 5) \) and \( \mathfrak{p}(4, 9) \)

**Example 3.2.** The subprime functional \( f = \sum_{i<j\leq n} e_{j-i,j}^* + \sum_{1\leq j<i} e_{j+1,j}^* \in \mathfrak{p}(i, n)^* \) is Frobenius if and only if \( n \equiv \pm 1 \pmod{i} \) (see e.g. [9, Section 9.1]). The associated principal element is

\[
(3.1) \quad H = \text{diag}(0, 1, 2, \ldots, i-1, -1, 0, 1, \ldots, i-2, -2, -1, 0, \ldots) + \Theta I_n \in \mathfrak{p}(i, n).
\]

where \( \Theta = \frac{n-1}{2n} \left( \frac{n+1}{2} - i \right) \) is a scalar making \( H \) traceless. When \( i = 1 \) the subprime functional reduces to the prime functional and, up to a scalar multiple, \( H \) reduces to the semisimple element of Konstant’s principal three-dimensional subalgebra of \( \mathfrak{sl}_n \) [14].

### 3.2. Boundary Solutions of the CYBE

In [8] Gerstenhaber and Giaquinto introduce boundary solutions to the CYBE and show, among other things, that all points lying in the Zariski boundary of \( \mathcal{M} \) are contained in \( C \). Thus one would hope that there might be a fairly simple description of boundary \( r \)-matrices analogous to the Belavin-Drinfeld classification. We will make use of the following theorem and corollary found in [8], which together provide a general technique for constructing a large number of examples of boundary \( r \)-matrices. In the following theorem and
corollary the variable $t$ is treated as a formal parameter; the base field $\mathbb{F}$ is enlarged to $\mathbb{F}[t]$.

**Theorem 3.3.** If $r \in \mathcal{M}$ and $r_t = r + r_1 t + r_2 t^2 + \cdots + r_m t^m \in \mathcal{M}$ with $\langle r, r \rangle = \langle r_1, r_1 \rangle$, then $r_m$ is a boundary solution to the CYBE.

**Corollary 3.4.** Suppose $x \in \mathfrak{g}$ is nilpotent and $r \in \mathcal{M}$. Then $\exp(tx)r$ has the form $\exp(tx)r = r + t r_1 + \cdots + t^m r_m$ with $r_m \in \mathcal{C}$ a boundary solution to the CYBE.

**Example 3.5.** Let $e = e_{12} \in \mathfrak{sl}_2$, $f = e_{21} \in \mathfrak{sl}_2$, and $h = e_{11} - e_{22} \in \mathfrak{sl}_2$ be the standard basis of $\mathfrak{sl}_2$. We have $r := e \wedge f \in \mathcal{M}$ and $\exp(t \cdot e)r = r + t(e \wedge h)$. Therefore, $e \wedge h \in \mathcal{C}$ is a boundary $r$-matrix.

4. **Towards a Proof of the Main Result**

We next introduce four objects that play important roles in our proof of the Gerstenhaber-Giaquinto boundary conjecture in the subprime cases, namely the cases when $n \equiv \pm 1 \pmod{m}$. These objects are (1) a Lie subalgebra $\mathfrak{n}$ of $\mathfrak{sl}_n$, (2) an $\mathfrak{n}$-module $M$, (3) the principal element $H \in \mathfrak{p}(i, n)$ corresponding to the subprime functional, and (4) a map $\Phi : \mathbb{F}^\times \to SL_n$. After a discussion of some key properties of $\mathfrak{n}$, $M$, $H$, and $\Phi$ we will then be in a position to prove our main result.

- From now on, unless stated otherwise, we assume $i$ and $n$ are a pair of positive integers with $i < n$ such that $n \equiv \pm 1 \pmod{m}$.

4.1. **The Lie Algebra $\mathfrak{n}$**. We define $\epsilon := 1$ if $n \mod i = 1$ and $\epsilon := -1$ if $n \mod i \neq 1$. For $k \in \mathbb{Z}$ we let $c_k := \lfloor \frac{n-k}{i} \rfloor$ and $a_k := \epsilon((-nk) \mod i)$. Next define the matrices

\begin{equation}
X := \sum_{1 \leq j \leq n-i} c_j e_{j,j+i} \in \mathfrak{sl}_n, \quad Z := \sum_{1 \leq j < i} a_j \xi_{j+1,j} \in \mathfrak{sl}_n.
\end{equation}

Let $\mathfrak{n}$ be the Lie subalgebra of $\mathfrak{sl}_n$ generated by the matrices $X$ and $Z$. The Lie algebra $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{sl}_n$. When $i = 1$, $\mathfrak{n} = \mathbb{F} X$. For $i > 1$ the matrices $Z, X, (ad Z)X, (ad Z)^2 X, \ldots, (ad Z)^{i-1} X$ form a linear $\mathbb{F}$-basis for $\mathfrak{n}$, the $\mathbb{F}$-span of $X, (ad Z)X, (ad Z)^2 X, \ldots, (ad Z)^{i-1} X$ is an abelian Lie subalgebra of $\mathfrak{n}$ of codimension 1, and $(ad Z)^i X = 0$.

4.2. **The $\mathfrak{n}$-module $M$**. Let $r := r_{CG}(i, n)$ be the Cremmer-Gervais $r$-matrix of type $(i, n)$ and let $M$ be the $\mathfrak{n}$-module generated by $r$, i.e. $M = \mathcal{U}(\mathfrak{n}).r \subseteq \mathfrak{sl}_n$, where $\mathcal{U}(\mathfrak{n})$ is the universal enveloping algebra of $\mathfrak{n}$. Our aim is to explicitly describe the $\mathfrak{n}$-module $M$ by giving a list of basis vectors of $M$ and providing formulas for the actions of $X$ and $Z$ on each basis vector. In computing these formulas we sometimes find it convenient to write $\alpha_+$ and $Z$ as

\begin{equation}
\alpha_+ = 2 \sum_{1 \leq j < k \leq i} E_{jk} \wedge E_{k-j+i-1,t-1}, \quad Z = \sum_{1 \leq j < i} (i-j)E_{j,j-1}.
\end{equation}

where, for $k, \ell \in \mathbb{Z}$, $E_{k\ell} := \xi_{k+1,\ell+1}$ if $n \mod i = 1$ and $E_{k\ell} := -\xi_{i-\ell,i-k}$ if $n \mod i \neq 1$. Next define the following elements of $\mathfrak{p}(i, n) \wedge \mathfrak{p}(i, n)$:
Proof. If proving Proposition 4.1 we provide two lemmas, both computational in nature. 

\[ V_0 := \frac{2}{n} X \cap I - 2 \sum_{1 \leq k < \ell \leq n} e_{k\ell} \wedge \eta_{k,k+i} - 2 \sum_{1 \leq k < \ell \leq i} E_{k\ell} \wedge E_{\ell-k,i}, \]

\[ W := 2 \left( \sum_{1 \leq \ell \leq i} d_{\ell} \wedge E_{\ell,\ell-1} + \sum_{1 \leq \ell < j \leq i} E_{j-1,\ell-1} \wedge E_{\ell-j+\ell-1} \right), \]

\[ V_\ell := -2 \left( \sum_{1 \leq \ell \leq i} d_{\ell} \wedge E_{\ell+i} + \sum_{\ell \leq j < k \leq i} E_{j\ell} \wedge E_{k-j+i} - \sum_{0 \leq k < j \leq \ell} E_{jk} \wedge E_{k-j+i} \right) \]

for \( \ell \in [1,i] \), where \( d_{\ell} \) is the diagonal matrix in \( \mathfrak{sl}_n \) defined by \( d_{\ell} := \sum_{\ell \leq j < i} E_{jj} + \Theta I_n \). The scalar \( \Theta \in \mathbb{F} \) is determined by the condition that \( d_{\ell} \) is traceless. The following proposition characterizes the \( n \)-module \( M \).

**Proposition 4.1.**

1. \( Z.r = W \),
2. \( X.r = V_0 \),
3. \( Z.W = 0 \),
4. \( Z.V_\ell = (\ell + 1)V_{\ell+1} \) for all \( \ell \in [0,i] \),
5. \( X.W = \frac{1}{2} V_1 \), and
6. \( X.V_\ell = 0 \)

with the convention that \( V_{i+1} = 0 \).

When \( i > 1 \), the set of vectors \( \{ r, W, V_0, V_1, \ldots V_i \} \) is an \( \mathbb{F} \)-basis of \( M \). However if \( i = 1 \), the vectors \( W \) and \( V_1 \) vanish; in this case \( \{ r, V_0 \} \) is an \( \mathbb{F} \)-basis of \( M \). Before proving Proposition 4.1 we provide two lemmas, both computational in nature.

**Lemma 4.2.**

1. For \( k, \ell \in [1,n] \),
   a. \( [X, \eta_{k\ell}] = (\left\lceil \frac{n-k}{\ell} \right\rceil - \left\lceil \frac{n-\ell}{k} \right\rceil) \eta_{k,\ell+i} + \left( \left\lceil \frac{n-k}{\ell} \right\rceil + 1 \right) e_{k-i,\ell} \)
   b. \( [X, e_{k\ell}] = (\left\lceil \frac{n-k}{\ell} \right\rceil + 1) e_{k-i,\ell} - \left\lceil \frac{n-k}{\ell} \right\rceil e_{k,\ell+i} \)
   c. \( [Z, \eta_{k\ell}] = a_k \eta_{k+1,\ell} - a_\ell \eta_{k,\ell-1} + \delta_{(n-\ell) \mod i, i-1} a_\ell a_{\ell-1} e_{n+k-1,\ell+i} \)
   d. \( [Z, e_{k\ell}] = a_k e_{k+1,\ell} - a_{\ell-1} e_{k,\ell-1} \)
2. For \( k \in [1,i-1] \) and \( \ell \in [0,i] \),
   a. \( [Z, E_{k\ell}] = (i - 1 - k)E_{k+1,\ell} - ((\ell \mod i) E_{k,\ell-1}) \)
   b. \( [X, E_{k\ell}] = -\delta_{i,0} E_{k\ell}, \)
3. For \( \ell \in [1,i] \),
   a. \( [X, d_{\ell}] = 0, \)
   b. \( [Z, d_{\ell}] = -(i - \ell) E_{\ell,\ell-1}. \)

Proof. For convenience, we first recall the relevant definitions: \( \xi_{k\ell} := \sum_{p \in \mathbb{Z}} e_{k+i,p,\ell+i} \),

\( \eta_{k\ell} := \sum_{p \geq 0} e_{k+i,p,\ell+i} \),

\( X := \sum_{1 \leq j \leq n-i} c_j e_{j,j+i} \),

\( Z := \sum_{1 \leq j < i} a_j \xi_{j+1,j} \),

where \( c_j = \left\lceil \frac{n-j}{i} \right\rceil, a_j := \epsilon((-n^2) \mod i) \) (where \( \epsilon = 1 \) if \( n \mod i = 1 \) and \( \epsilon = -1 \) if \( n \mod i = 1 \)). Recall also \( E_{k\ell} := \xi_{k+1,\ell+1} \) if \( n \mod i = 1 \) and \( E_{k\ell} := -\xi_{i-1,i-k} \) if \( n \mod i = 1 \). Finally, \( d_{\ell} := \sum_{\ell \leq j < i} E_{jj} + \Theta I_n \), where the scalar \( \Theta \in \mathbb{F} \) is uniquely determined by the condition that \( d_{\ell} \) is traceless.
The identities \(1a\) and \(1c\) can be proved directly from these definitions. For instance, we compute

\[
[X, \eta_{ki}] = \left[ \sum_{1 \leq j \leq n-i} c_j e_{j,j+i}, \sum_{p \geq 0} e_{k+ip,\ell+ip} \right]
\]

\[
= \sum_{1 \leq j \leq n-i} c_j (\delta_{j+i,k+ip} e_{j,j+i} - \delta_{j,\ell+ip} e_{k+ip,j+i})
\]

\[
= \sum_{p \geq -1} c_{k+ip} e_{k+ip,\ell+(p+1)i} - \sum_{p \geq 0} c_{\ell+ip} e_{k+ip,\ell+(p+1)i}
\]

\[
= c_k e_{k-i,\ell} + \sum_{p \geq 0} (c_k+ip - c_{\ell+ip}) e_{k+ip,\ell+(p+1)i}
\]

\[
= (c_k + 1) e_{k-i,\ell} + \sum_{p \geq 0} (c_k - c_{\ell}) e_{k+ip,\ell+(p+1)i}
\]

and

\[
[Z, \eta_{ki}] = \sum_{1 \leq j < i} a_j \left[ e_{j+1,j}, \eta_{ki} \right]
\]

\[
= \sum_{1 \leq j < i} a_j \left[ e_{j+1+pi,j+pi}, e_{k+qi,\ell+qi} \right]
\]

\[
= \sum_{1 \leq j < i} a_j (\delta_{j+pi,k+qi} e_{k+1+qi,\ell+qi} - \delta_{j+1+pi,\ell+qi} e_{k+qi,\ell+1+qi})
\]

\[
= a_k \eta_{k+1,\ell} - a_{\ell-1} \eta_{k,\ell-1} + \delta_{(n-\ell) \mod i,i-1} a_{\ell-1} e_{n+k-\ell+1,n-i}
\]

Similarly, one can use the definitions provided above to prove identities \(2a, 2b, 3a\) and \(3b\). To prove identities \(2a\) and \(3a\) one can use an alternative formula for \(Z\), namely \(Z = \sum_{1 \leq j < i} (i - j) E_{j,j-1}\) together with the observation that \([E_{kt}, E_{rs}] = \delta_{k,r} E_{ks} - \delta_{s,k} E_{rt}\) when \(k, r \in [1, i - 1]\) and \(\ell, s \in [0, \ell]\). \(\Box\)

Since \(\alpha\), \(\alpha\), \(\beta\), and \(\gamma\) are composed of terms only involving \(e_{k\ell}, \eta_{k\ell}, E_{k\ell}\), and \(d_{k\ell}\) with subscripts \(k\) and \(\ell\) in the ranges given in Lemma 4.2, we can readily compute the adjoint actions of \(X\) and \(Z\) on these constituent parts of \(r\) to obtain the following lemma.

**Lemma 4.3.**

\[X.\alpha = -2 \sum_{1 \leq k \leq \ell < n} e_{k\ell} \land \eta_{k,k+i} + \sum_{1 \leq j \leq n-i} \left( \sum_{1 \leq j \leq \ell+i} \frac{n-j}{i} e_{js} \land e_{s,j+i} \right)\]

\[X.\alpha = -2 \sum_{1 \leq k \leq \ell < i} E_{k\ell} \land E_{\ell-k,i}\]
(3) $X, \beta = \sum_{1 \leq j \leq n} \left[ \frac{n-j}{i} \right] e_{j,j+i} \wedge \left( \frac{2}{n} I - e_{j,j} - e_{j,j+i} \right)$

(4) $X, \gamma = \sum_{1 \leq j \leq n} \left[ \frac{n-j}{i} \right] \left( e_{j,j+i} \wedge (e_{j,j} - e_{j,j+i}) - 2 \sum_{j < s < j+i} e_{js} \wedge e_{s,j+i} \right)$

(5) $Z, \alpha \rightarrow = 2 \sum_{1 \leq j < n} a_j \eta_{j+i+1,j+i} \wedge h_j$

(6) $Z, \alpha \rightarrow = 2 \left( \sum_{1 \leq j < i} (i-j) E_{j,j-1} \wedge E_{jj} + \sum_{1 \leq \ell < j \leq k \leq i} E_{j-1,k-1} \wedge E_{k-j+\ell,t-1} \right)$

(7) $Z, \beta = -2 \sum_{1 \leq j < i} (i-j) E_{j,j-1} \wedge (d_j - d_{j+1}) - \sum_{1 \leq \ell < n} a_{\ell} \left( \eta_{\ell+1,\ell} + \eta_{\ell+1,1+i} \right) \wedge h_\ell$

(8) $Z, \gamma = \sum_{1 \leq j < n} a_j e_{j+1,j} \wedge h_j$

Proof. Part 1 follows from identities (13) and (15) of Lemma 4.2. Part 2 follows from (2a) of Lemma 4.2. Parts 3 and 4 follow from (11). Part 5 follows from (1c) and (1d). Part 6 follows from (2a). Parts 7 and 8 follow from (1d).

Proof of Proposition 4.1. Part 1 follows directly from the identities (5) (6) (7) and (8) of Lemma 4.3. Part 2 follows directly from the identities (11) (13) and (15) of Lemma 4.3. For parts 3 and 4 (in the case when $\ell \neq 0$) use the identities (2a) and (3b) of Lemma 4.2. To prove part 4 in the case when $\ell = 0$, we use the identities (1c) and (1d) of Lemma 4.2 to directly obtain

$$Z \left( \sum_{1 \leq k < \ell \leq n} e_{k \ell} \wedge \eta_{\ell,k+i} \right) = \sum_{1 \leq k < \ell \leq n} a_k e_{k+1,\ell} \wedge \eta_{\ell,k+i} - a_{k-1} e_{k \ell} \wedge \eta_{\ell,(k-1)+i} + a_{\ell-1} e_{k,\ell-1} \wedge \eta_{\ell,k+i} + c_{k} = \delta_{(n-k) \mod i,i-1} a_{k-1} e_{k \ell} \wedge \eta_{\ell+1,\ell-k-i,n}.\quad \text{We observe that the } C_{k\ell}'s \text{ all equal 0 because the subscript } n+\ell-k-i \text{ appearing in the expression for } C_{k\ell} \text{ is greater than } n \text{ whenever } (n-k) \mod i = i-1 \text{ and } c_k > c_\ell. \text{ We shift the indices of summation so that the terms involved in the } A_{k\ell}'s \text{ and } B_{k\ell}'s \text{ combine together into a single sum. This gives us}

$$Z \left( \sum_{1 \leq k < \ell \leq n} e_{k \ell} \wedge \eta_{\ell,k+i} \right) = a_n \left( \sum_{1 \leq k < \ell \leq n} e_{k+1,\ell} \wedge \eta_{\ell,k+i} - \sum_{1 \leq k < \ell \leq n} e_{k \ell} \wedge \eta_{\ell+1,k+i} \right)

= ca_n \sum_{1 \leq k \leq i} E_{1k} \wedge E_{ki} = (i-1) \sum_{1 \leq k \leq i} E_{1k} \wedge E_{ki},$$
Finally apply identity \(2a\) of Lemma \(4.2\) and it becomes a straightforward computation to verify that \(Z.V_0 = V_1\). To prove parts \(5\) and \(6\) use identities \(1a\), \(1b\), \(2b\), and \(3a\) of Lemma \(4.2\).

\[\tag{4.6} r' := r - \alpha_0 = r + V_i \in M.\]

Notice that \(r'\) is a solution to the MCYBE.

**Proposition 4.4.** The \(n\)-module \(M\) decomposes into \(H\)-eigenspaces \(M = \oplus M_\lambda\), where \(M_\lambda = \{v \in M \mid H.v = \lambda v\}\). The nonzero eigenspaces \(M_\lambda\) occur only if \(\lambda \in [0, i + 1]\). Let \(r'\) be as given above. We have

1. \(M_0\) is spanned by \(r'\),
2. \(M_1\) is spanned by \(V_0\) and \(W\),
3. for \(\lambda \in [2, i + 1]\), \(M_\lambda\) is spanned by \(V_{\lambda - 1}\).

**Proof.** First, compute to obtain the identity \([H, E_{kl}] = (k - \ell + \delta_{ik}(i + 1))E_{kl}\) for all \(k \in [1, i - 1]\) and \(\ell \in [0, i]\). We also have the identities \([H, X] = X\) and \([H, Z] = Z\).

Since the \(\beta\) and \(\gamma\)-parts of \(r\) are annihilated by the adjoint action of \(H\) (in fact, \(\beta\) and \(\gamma\) are annihilated by the adjoint action of any diagonal matrix) and additionally \(H.\alpha \to = 0\), this implies \(H.r = H.\alpha \to\). As \(\alpha \to\) can be written in terms of the \(E_{kl}\)'s, we obtain

\[
H.r = H.\alpha \to = 2 \sum_{1 \leq \ell < j < k \leq i} \delta_{kj}(i + 1)E_{jk} \wedge E_{k-j+\ell-1, j-\ell-1} = 2 \sum_{1 \leq \ell < j < i} (i + 1)E_{ij} \wedge E_{i-j+\ell-1, j-\ell-1} = -(i + 1)V_i
\]

Now that we have established how \(H\ acts on \(X, Z,\) and \(r\), Proposition \(4.4\) provides a way to compute the adjoint action of \(H\) on the \(V_i\)'s and \(W\). We obtain \(H.V_j = (j + 1)V_j\) (for \(j \in [0, i]\)) and \(H.W = W\). Therefore \(H.r' = H(r + V_i) = H.r + H.V_i = 0.\)

\[\tag{4.7} g := e^{-Z}e^{-X}exp\left(\sum_{1 \leq k < i} \left(\begin{array}{c} i \\ k \end{array}\right) E_{ki}\right) \in SL_n.\]

The matrix \(g \in SL_n\) is obtained by seeking a matrix of the form \(e^X\) with \(X \in \mathfrak{n}\) so that \(e^X.r \in M_0 \oplus M_1\). These conditions significantly restrict plausible candidates for \(g\). They, in fact, uniquely determine \(g\) up to a single non-zero free parameter in
After obtaining such a matrix $g (= e^X)$, we observe that it can be factored into the product form as we have defined it above. The factored form of $g$ highlights the roles of the matrices $X$ and $Z$ and allows us to easily prove the following proposition.

**Proposition 4.5.** All entries in the matrix $g$ are integers.

**Proof.** From the definitions of the matrices $X$ and $Z$, it is easy to see that for every $k > 0$, all entries in $X^k$ and $Z^k$ are products of $k$ consecutive integers. Hence the matrices $X^k/k!$ and $Z^k/k!$ have integer entries. Therefore $e^X$ and $e^Z$ have integer entries. Next, we note that $e^{E_{ki}} = I - \sum_{1 \leq k < i} \binom{i}{k} E_{ki}$. Thus, the matrix $e^{\sum_{1 \leq k < i} \binom{i}{k} E_{ki}}$ has integer entries. Since $g$ is written as the product of three matrices, each of them having integer entries, $g$ also has integer entries.

□

We observe that multiplying the principal element $H$ by $2n$ gives us a diagonal matrix having integer entries. This implies that $t^{2nH}$ is a diagonal matrix having polynomial entries in $t^{1/2}$. In fact, $t^{2nH} = \text{diag}(t^{2nH_1}, t^{2nH_2}, \ldots, t^{2nH_n})$ whenever $H = \text{diag}(H_1, H_2, \ldots, H_n)$. By rescaling we avoid having to consider whether or not roots of $t$ exist. For instance $t^{1/j}$ may not exist for every $t \in \mathbb{F}$ if $\mathbb{F}$ is not algebraically closed. In short, rescaling is the reason why we do not need to require that $\mathbb{F}$ be algebraically closed.

Finally, define the map $\Phi : \mathbb{F}^* \to SL_n$ by $\Phi(t) := t^{2nH}g$. Since $t^{2nH}$ is a diagonal matrix with nonzero entries of the form $t^m$ with $m$ an integer, Proposition 4.5 implies that all entries in $\Phi(t)$ have the form $ct^p$ with $c$ and $p$ integers.

5. **Main Result**

We are now in a position to prove the main result. Define

\[(5.1) \quad b := -(X + Z)r = -V_0 - W \in p(i, n) \land p(i, n).\]

**Theorem 5.1.** Suppose $n \equiv \pm 1 \pmod{i}$ and let $r = r_{CG}(i, n)$ be the corresponding Cremmer-Gervais $r$-matrix of type $(i, n)$. Let $r'$, $b$, and $\Phi(t)$ be as defined above, then

1. $\Phi(t)r = r' + t^{2n}b$.
2. $b$ is a boundary solution to the CYBE having carrier $p(i, n)$.
3. $b$ lies in the closure of the component of $\mathcal{M}$ containing the Cremmer-Gervais $r$-matrix $r$.

**Proof.** To prove part 1, observe first that for $k \in [1, i-1]$ we have the identity $(\text{ad}Z)^kX = \frac{(i-1)!}{(i-k-1)!} E_{ki}$. Thus we can use the identities in Proposition 4.1 to compute the adjoint action of $e^{\sum_{1 \leq k < i} \binom{i}{k} E_{ki}}$ on $r$. We compute

\[
g.r = e^{-Z}e^{-X}\left(r - \sum_{1 \leq k < i} V_k \right) = e^{-Z}\left(r - \sum_{0 \leq k < i} V_k \right) = r - W - V_0 + V_i = r' + b
\]
Since \( t^{2nH}v = t^{2n\lambda}v \) for every \( H \)-eigenvector \( v \in M_\lambda \) and we have, in particular, that \( r' \in M_0 \) and \( b \in M_1 \), it follows that \( t^{2nH}g.r = t^{2nH}(r' + b) = r' + t^{2n}b \). Part 3 follows as a direct consequence of part 1. For part 2 we note that \( \langle r, r \rangle = \langle r', r' \rangle \), therefore Theorem 3.3 implies that \( b \) is a boundary solution to the CYBE. The definitions of \( V_0 \) and \( W \) show that the carrier of \( b \) is \( p(i,n) \).

\[ \Box \]

Part 2 of Theorem 5.1 can be slightly generalized. In fact, for all scalars \( \rho, \mu \in \mathbb{F} \), \( b_{\rho, \mu} := (\rho X + \mu Z).r \) is a boundary solution to the CYBE. Furthermore if \( \rho \) and \( \mu \) are both nonzero, the carrier of \( b_{\rho, \mu} \) is the maximal parabolic subalgebra \( p(i,n) \subseteq \mathfrak{sl}_n \) and \( b_{\rho, \mu} \) admits the Frobenius functional

\[
(5.2) \quad f_{\rho, \mu} := -\rho^{-1} \left( \sum_{i < j \leq n} e_{j-i,j} \right) + \epsilon \mu^{-1} \left( \sum_{1 \leq j < i} e_{j+1,j} \right) \in p(i,n)^*.
\]

The principal element corresponding to \( f_{\rho, \mu} \) is equal to \( H \) (from Example 3.2), independent of the scalars \( \rho \) and \( \mu \). Notice that when the scalars are specialized to \( \rho = -1 \) and \( \mu = \epsilon \), the functional \( f_{\rho, \mu} \) reduces to the subprime functional.

6. Closing Remarks

The results in the previous sections prove all cases of the Gerstenhaber-Giaquinto boundary conjecture when \( i = 1, 2, 3, \) and 4. The next smallest case to consider is when \( i = 5 \) and \( n = 7 \). However, this particular case is handled by applying the non-trivial Dynkin diagram automorphism of \( \mathfrak{sl}_7 \) to the \((2,7)\)-case. Similarly, the case when \( i = 5 \) and \( n = 8 \) is handled by applying Dynkin diagram automorphism to the \((3,8)\)-case. After these cases, the next smallest case to consider is when \( i = 5 \) and \( n = 12 \).

Since the subprime functional \( f \in p(i,n)^* \) is Frobenius if and only if \( n \equiv \pm 1 \) (mod \( i \)) it seems unlikely that \( f \) will play as important a role, if any, in proving the Gerstenhaber-Giaquinto boundary conjecture in cases when \( n \not\equiv \pm 1 \) (mod \( i \)). The subprime functional is, first of all, an example of a small functional. This means that if we write the subprime functional as \( f = \sum_{(j,k) \in E} e_{jk} \), with \( E = \{(1, i + 1), (2, i + 2), \ldots, (n-i, n), (2, 1), (3, 2), \ldots, (i + 1, i)\} \), the corresponding directed graph having vertices 1, \ldots, \( n \) and edges \( j \to k \) for each \((j,k) \in E\) has an underlying undirected graph a tree. In fact any functional \( f \) of the form \( \sum_{(j,k) \in S} e_{jk} \), where \( S \) is the edge set, with corresponding graph a tree is called a small functional (see e.g. \[9\] Section 3). The edge set \( S \) is the support of \( f \). The smallest possible size for the support of a Frobenius functional \( f \in p(i,n)^* \) is \( n - 1 \).

In proving the Gerstenhaber-Giaquinto boundary conjecture for the \((5, 12)\) case it seems that we must look beyond the small functionals. To be more precise, first recall that for any Frobenius Lie algebra \( \mathfrak{f} \) and Frobenius functional \( f \in \mathbb{F} \) we can produce a solution to the classical Yang-Baxter equation \( r_f \in \mathcal{C} \) having carrier \( \mathfrak{f} \) by inverting the matrix \((B_{jk})\), where \( B_{jk} = f([x_j, x_k]) \) with respect to a basis \( x_1, \ldots, x_d \) of \( \mathfrak{f} \). We simply put \( r_f = \sum (B_{jk})^{-1} x_j \wedge x_k \). In this paper, part of the main result illustrates that when \( f \in p(i,n)^* \) is the subprime functional and \( n \equiv \pm 1 \) (mod \( i \)), there exists \( x \in \mathfrak{sl}_n \) so that \( x.r_{CG}(i,n) = r_f \), namely \( x = -X + \epsilon Z \). In view
of this, one approach to proving the boundary conjecture in the (5, 12)-case would be to find \( x \in \mathfrak{sl}_2 \) and a Frobenius functional \( f \in \mathfrak{p}(5, 12)^* \) so that \( x.r_{\text{CG}}(5, 12) = r_f \).

With this in mind, consider the 12 \( \times \) 12 matrix \( x = (a_{jk}) \) where the entry in the \((j, k)\)-position is a variable named \( a_{jk} \). Thus, with a fixed functional \( f \), the equation \( x.r_{\text{CG}}(5, 12) = r_f \) represents a linear system of 12\(^4\) equations in 144 variables. However computer calculations indicate that this system is inconsistent for each small Frobenius functional \( f \in \mathfrak{p}(5, 12)^* \).

Rather than searching first for a plausible functional \( f \) that solves the system \( x.r_{\text{CG}}(5, 12) = r_f \), we could instead first find \( x \in \mathfrak{sl}_2 \) so that \( x.r_{\text{CG}}(5, 12) \in \mathcal{C} \). However this amounts to solving a system of 144\(^4\) quadratic equations in 144 variables, which is a difficult task. Instead we only look for candidates for \( x \) and let \( \mathcal{C} \) be the Cremmer-Gervais \( n \)-matrix of type \( (5, 12) \). Computer calculations indicate that this system is inconsistent with the scalars \( a_{jk} \in \mathbb{F} \) nonzero only when \( j - k \) belongs to a sufficiently “small” subset of integers. This restriction significantly reduces the number of equations and variables of the quadratic system. This approach likely leads to several plausible candidates for \( x \), but we only want to consider those with \( x.r_{\text{CG}}(5, 12) \) having carrier equal to \( \mathfrak{p}(5, 12) \). For example, define

\[
(6.1) \quad x := X_7 + X_4 + X_1 + X_{-2} \in \mathfrak{p}(5, 12),
\]

where \( X_7, X_4, X_1, X_{-2} \in \mathfrak{p}(5, 12) \) are defined as

\[
(6.2) \quad X_7 := -\sum_{1 \leq j \leq 5} e_{j,j+7}, \quad X_1 := -(e_{1,2} + e_{3,4} + 2e_{5,6} + e_{6,7} + e_{8,9} + e_{10,11}) \),
\]

\[
(6.3) \quad X_4 := -e_{2,6} - e_{3,7}, \quad X_{-2} := \sum_{1 \leq j \leq 5} \left\lfloor \frac{7-j}{2} \right\rfloor (e_{j,j-2} + e_{j+7,j+5}),
\]

and let \( r = r_{\text{CG}}(5, 12) \) be the Cremmer-Gervais \( n \)-matrix of type \( (5, 12) \). Computer calculations verify that \( x.r \) is a solution to the CYBE having carrier \( \mathfrak{p}(5, 12) \) and admits the Frobenius functional

\[
(6.4) \quad f = \left( \sum_{1 \leq j \leq 5} e^*_{j,j+7} \right) + \left( \sum_{1 \leq j \leq 5} e^*_{j+7,j+5} \right) + e^*_{6,7} + e^*_{7,8} + e^*_{6,10} + e^*_{7,11} \in \mathfrak{p}(5, 12)^*.
\]

Observe that \( f \) is not a small Frobenius functional because its support has size greater than 11. The principal element associated to \( f \) is

\[
(6.5) \quad H = \frac{1}{2} \cdot \text{diag}(1, -1, 3, 1, 5, -3, -5, -1, -3, 1, -1, 3) + 3(e_{3,6} + e_{4,7} + e_{1,7}) \in \mathfrak{p}(5, 12).
\]

Similar to our previous constructions, we now let \( \mathfrak{n} \) denote the Lie subalgebra of \( \mathfrak{sl}_2 \) generated by the matrices \( X_7, X_4, X_1, \) and \( X_{-2} \). The \( \mathfrak{n} \)-module \( M := \text{U}(\mathfrak{n}).r \) decomposes into \( H \)-eigenspaces \( M = M_0 \oplus M_1 \oplus \cdots \oplus M_{12} \), where \( M_\lambda := \{ v \in M \mid H.v = \lambda v \} \), having respective dimensions 1, 4, 5, 9, 10, 12, 10, 8, 4, 2. Interestingly, the subspace \( M_0 \) is spanned by

\[
(6.6) \quad r' := \exp(-e_{3,6} - e_{4,7} - e_{1,7}) \cdot (r - \alpha_0),
\]
where $\alpha_0$ is the $\alpha$-part of $r$ obtained by adding the edges $5 \to 10$ and $6 \to 11$ to the Cremmer-Gervais graph $\Gamma_{CG}(5,12)$. We note that $r'$ is a solution to the MCYBE equivalent to $r - \alpha_0$.

We also have $x.r \in M_1$. Thus $t^{2H}(x,r) = t^2(x,r)$. We remark that $t^{2H}$ has polynomial entries in $t^{\pm 1}$. The matrix $t^{2H}$ can be obtained by diagonalization: $2H = PDP^{-1}$ with $D$ a diagonal matrix having integer entries $D = \text{diag}(H_1,H_2,\ldots,H_{12})$. Next put $t^{2H} = P\text{diag}(t^{2H_1},\ldots,t^{2H_{12}})P^{-1}$. Define $Y := -2e_{1,6} - e_{2,6} + 3e_{3,6} + 2e_{4,6} + e_{5,6} - e_{4,7} - e_{4,7} - e_{5,7} \in n$ and put

$$
(6.7) \quad g := e^{X_7}e^{-2}e^{[X_2,X_1]}e^{X_1}e^Y \in SL_{12}.
$$

As before, the matrix $g$ is obtained by seeking a matrix of the form $g = e^X$ with $X \in n$ so that $e^X.r \in M_0 \oplus M_1$ and observing that $g$ factors into the form written above. We have $g.r = r' + x.r$. Therefore $t^{2H}g.r = r' + t^2x.r$. Since $\langle t^{2H}g.r, t^{2H}g.r \rangle = \langle r', r' \rangle$, Theorem 3.3 implies $x.r$ is a boundary solution to the CYBE lying in the closure of the Belavin-Drinfeld component of $M$ containing $r$.

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