Asymptotic Behavior and Stability in Linear Impulsive Delay Differential Equations with Periodic Coefficients

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Abstract: We study first order linear impulsive delay differential equations with periodic coefficients and constant delays. This study presents some new results on the asymptotic behavior and stability. Thus, a proper real root was used for a representative characteristic equation. Applications to special cases, such as linear impulsive delay differential equations with constant coefficients, were also presented. In this study, we gave three different cases (stable, asymptotic stable and unstable) in one example. The findings suggest that an equation that is in a way that characteristic equation plays a crucial role in establishing the results in this study.

Keywords: asymptotic behavior; characteristic equation; delay differential equation; stability

MSC: 34A37; 34K06; 34K20; 34K25; 34K40; 34K45

1. Introduction and Preliminaries

The impulsive delay differential equation is considered as:

\[ x'(t) = a(t)x(t) + \sum_{i \in I} b_i(t)x(t - \tau_i), \quad t \geq 0, \quad t \neq t_k, \quad (1) \]

\[ \triangle x(t_k) = \ell_k, \quad k \in \mathbb{Z}^+ = \{1, 2, \cdots \}, \quad (2) \]

where \( I \) is the initial segment of natural numbers, \( a \) and \( b_i \) for \( i \in I \) the continuous real-valued functions on the interval \([0, \infty)\), and \( \tau_i \) for \( i \in I \) positive real numbers with \( \tau_{i_1} \neq \tau_{i_2} \) for \( i_1, i_2 \in I \) such that \( i_1 \neq i_2 \). Suppose that the functions \( b_i \) for \( i \in I \) are not identically zero on \([0, \infty)\) and also the coefficients \( a \) and \( b_i \) for \( i \in I \) are the periodic functions with a common period \( T > 0 \) where \( \tau_i = m_i T \) for positive integers \( m_i \) for \( i \in I \). Furthermore, \( \ell_k \) for \( k \in \mathbb{Z}^+ \) are real constants and \( \triangle x(t_k) = x(t_k^+) - x(t_k^-) \). The impulsive positive points \( t_k \) satisfy

\[ 0 < t_1 < \cdots < t_k < t_{k+1} < \cdots \quad \text{and} \quad \lim_{k \to \infty} t_k = \infty. \]

Define

\[ \tau = \max_{i \in I} \tau_i. \]

Assume that the initial function \( \phi \) is a given continuous real-valued function on the interval \([-\tau, 0]\), then an initial condition is imposed, that is, along with (1):

\[ x(t) = \phi(t), \quad -\tau \leq t \leq 0. \quad (3) \]
The literature review we conducted comprehensively offers the behaviors based on the solutions of delay (non-impulsive) differential equations [1–4] and impulsive (non-delay) differential equations [5–8]. However, there are limited investigations concerning the corresponding theory of impulsive delay differential equations [9–19]. They also obtained very interesting results regarding the long time behavior of solutions of linear homogeneous or non-homogeneous impulsive delay differential equations [20–29]. The mathematical models are satisfying for different processes in science and technology [30–40]. There are additional resources containing further information on impulsive delay differential equations [41–43].

Gopalsamy and Zhang, in their extraordinary monograph [12], (Section 3), achieved significant results in linear impulsive delay differential equations in the form
\[
x'(t) + p(t)x(t - \tau) = 0, \quad t \neq t_i
\]
\[
x(t_i^+) - x(t_i^-) = b_i x(t_i^-),
\]
\[
0 < t_1 < \cdots < t_j \rightarrow \infty \quad \text{as} \quad j \rightarrow \infty
\]
where \( \tau \) is a positive real number, \( p \in C([0, \infty), [0, \infty)) \) and \( b_i \) are real numbers. Here, the authors conducted the first to study the oscillation behavior of solutions of linear impulsive delay differential equations. Also, the authors of Reference [44] have obtained a sufficient condition for the persistence of the non-oscillatory solutions of the same equation.

In the following years, Zhao and Yan [45,46] reached some important results by generalizing a little more as given in the form below
\[
x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i) = 0, \quad t \neq t_k
\]
\[
x(t_k^+) - x(t_k^-) = b_k x(t_k^-), \quad k = 1, 2, \cdots
\]
where \( p_i \in C([0, \infty), R), 0 \leq \tau_1 < \tau_2 < \cdots < \tau_n, 0 < t_1 < \cdots < t_k < t_{k+1} < \cdots, \lim_{k \rightarrow \infty} t_k = \infty, \) and \( b_k \) are constants. Here, the authors have achieved the asymptotic behavior of solutions for impulsive delay differential equations in Reference [45] and the existence of positive solutions for impulsive delay differential equations in Reference [46]. Next, the authors of Reference [47] examined the positive solutions and asymptotic behavior of the same equation with nonlinear Impulses. In the following years, some significant results were obtained behavior of solutions of the linear impulsive delay differential equations with variable delays [48,49].

In recent years, K.E.M. Church and X. Liu [50,51] have obtained very interesting results for linear and semilinear impulsive delay differential equations. The authors of Reference [50] (Section 7) proved that a periodic linear impulsive delay differential system is a periodic center manifold. Later, the authors of Reference [51] developed a computational framework for the center manifold reduction of periodic impulsive delay differential equations. Finally, the same authors of Reference [52] used the center manifold theory for the impulsive delay differential equations [50,51] to obtain information about the orbit structure in a particular pulsed SIR vaccination model involving delay.

Our aim in this paper is to present some new results on the asymptotic behavior and stability for linear impulsive delay differential equations with periodic coefficients. Note that the results in Reference [45] are that sufficient conditions are provided for the oscillatory and non-oscillatory solutions of linear impulsive delay differential equations to tend to zero. However, the article in Reference [45] has no information about unstable of solutions. Different from the article in Reference [45], in this article, we have obtained the stability, asymptotic stability and instability of solutions of (1)–(3) by making use of a suitable real root of the characteristic equation. A combination of several methods [28,53–57] is referred for the used techniques.

The linear autonomous impulsive delay differential equation is a special version of the impulsive delay differential Equation (1)
\[ x'(t) = ax(t) + bx(t - \tau), \quad t \neq t_k, \quad t \geq 0, \]

where \( \tau \) is positive constant, \( a, b \) the real constants for \( x(t) \in \mathbb{R} \). The impulse times \( t_k \) satisfy \( 0 < t_1 < \cdots < t_k < t_{k+1} < \cdots \) and \( \lim_{k \to \infty} t_k = \infty \). (see Reference [28] (Chapter 4) In Reference [28], the asymptotic and stability criteria were determined for linear impulsive delay differential equations, including constant coefficients and constant delays.

Providing that

(i) \( x(t) \) is a continuous function at any point except \( t_k, \quad k \in \mathbb{Z}^+; \)
(ii) \( x(t) \) is continuously differentiable for \( t \geq 0 \) and \( t \neq t_k, \quad k \in \mathbb{Z}^+; \)
(iii) for any \( t_k, \quad k \in \mathbb{Z}^+, \) \( x(t_k^-) \) and \( x(t_k^+) \) are available and \( x(t_k^-) = x(t_k^+); \)
(iv) \( x(t) \) satisfies (1) for almost any point in \( [0, \infty) \setminus \{t_k\}, \) (2) for all \( t = t_k, \quad k \in \mathbb{Z}^+ \) and (3).

Then, the function \( x(t) \) is described as a solution of the initial value problem (1)–(3) on \([\tau, \infty).\)

Clearly, any \( x(t) \) of (1)–(2) is right-hand continuous at \( t = t_k, \quad t \geq 0. \) This paper uses the notation

\[
A = \frac{1}{T} \int_0^T a(t)dt, \quad B_i = \frac{1}{T} \int_0^T b_i(t)dt \quad \text{and} \quad \tilde{B}_i = \frac{1}{T} \int_0^T |b_i(t)|dt \quad \text{for} \quad i \in I.
\]

(Clearly, \( A, B_i \) and \( \tilde{B}_i \) for \( i \in I \) are real constants.) We obviously have

\[
|B_i| \leq \tilde{B}_i \quad \text{for} \quad i \in I.
\]

Furthermore, we associate the following equation with the differential Equation (1)

\[
\lambda = A + \sum_{i \in I} B_i e^{-\lambda t_k}, \quad (4)
\]

specified as the characteristic equation of (1). There were given sufficient conditions to obtain a unique real root of the characteristic Equation (4) in Philos [53].

In what follows, the T-periodic extensions are denoted by \( \overline{a} \) and \( \overline{B}_i \) for \( i \in I \) for the coefficients \( a \) and \( b_i \) for \( i \in I, \) respectively, on the interval \([\tau, \infty).\) To construct a suitable mapping for the asymptotic criterion of the solutions, we should reach a finding as follows. Suppose that \( \lambda_0 \) is a real root of (4). We can now write

\[
h_{\lambda_0}(t) = \overline{a}(t) + \sum_{i \in I} \overline{B}_i(t) e^{-\lambda_0 t_k} \quad \text{for} \quad t \geq -\tau, \quad (5)
\]

and set

\[
y(t) = x(t) \exp \left[ -\int_0^t h_{\lambda_0}(u)du \right] \quad \text{for} \quad t \geq -\tau. \quad (6)
\]

Therefore, by (1) we obtain for all \( t \geq 0 \) and \( t \neq t_k \) for \( k \in \mathbb{Z}^+: \)

\[
y'(t) = (a(t) - h_{\lambda_0}(t))y(t) + \sum_{i \in I} b_i(t) e^{-\lambda_0 t_k} y(t - t_k)
\]

or, using (5), from the last equation

\[
y'(t) = -\sum_{i \in I} b_i(t) e^{-\lambda_0 t_k} [y(t) - y(t - t_k)]. \quad (7)
\]

Furthermore, the initial condition (3) is equivalent to

\[
y(t) = \phi(t) \exp \left[ -\int_0^t h_{\lambda_0}(u)du \right], \quad t \in [-\tau, 0], \quad (8)
\]
Lemma 1. Suppose that \( \lambda_0 \) is a real root of the characteristic Equation (4) and set (5). Thus \( y \) is the single solution of the initial value problem (7)–(9) if and only if \( y \) is a solution of the following system

\[
y(t) = \phi(t) \exp \left[ -\int_0^t h_{\lambda_0}(u) du \right] \quad \text{if} \quad -\tau \leq t \leq 0
\]

and

\[
y(t) = \Phi(0) + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{-\tau_i}^0 b_i(s) \phi(s) \exp \left[ -\int_0^s h_{\lambda_0}(u) du \right] ds
- \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{\tau_i}^t b_i(s) y(s) ds + \sum_{j=1}^{n(t)} \ell_j \exp \left[ -\int_{t_j}^t h_{\lambda_0}(u) du \right], \quad \text{if} \quad t \geq 0
\]

where \( n(t) = \max \{ k \in \mathbb{Z}^+ : t_k \leq t \} \) and \( n(0) = 0 \) if \( t < t_1 \).

Proof. Assume that \( y \) is an appropriate solution of (7)–(9). Considering (7) for \( t \in [t_k, t_{k+1}) \), we get

\[
y(t) = y(t_k) - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t_k}^t b_i(s) \left[ y(s) - y(s - \tau_i) \right] ds.
\]

If we take into account (9) from the last equation, for \( t \in [t_k, t_{k+1}) \), we obtain

\[
y(t) = y(t_k^-) + \ell_k \exp \left[ -\int_0^t h_{\lambda_0}(u) du \right] - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t_k}^t b_i(s) \left[ y(s) - y(s - \tau_i) \right] ds.
\]

We remark that the equation is satisfied only when \( t \in [t_k, t_{k+1}) \); however, we use the analogous formula to (11) through backstepping to express \( y(t_k^-) \) for \( t \in [t_{k-1}, t_k) \) as \( t \to t_k^- \). When we step back similarly, we obtain:

\[
y(t_k^-) = y(t_{k-1}^-) + \ell_{k-1} \exp \left[ -\int_0^{t_{k-1}} h_{\lambda_0}(u) du \right] - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t_{k-1}}^{t_k} b_i(s) \left[ y(s) - y(s - \tau_i) \right] ds,
\]

\[
\vdots
\]

\[
y(t_1^-) = y(0) - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_0^{t_1} b_i(s) \left[ y(s) - y(s - \tau_i) \right] ds.
\]

Applying recursive substitution into (11), we obtain in general, the solution \( y(t) \) necessarily fulfills:

\[
y(t) = \phi(0) - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_0^t b_i(s) \left[ y(s) - y(s - \tau_i) \right] ds + \sum_{j=1}^{n(t)} \ell_j \exp \left[ -\int_{t_j}^t h_{\lambda_0}(u) du \right].
\]
or
\[
    y(t) = \phi(0) - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_0^t b_i(s) y(s) \, ds + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t-\tau}^t b_i(s) y(s) \, ds + \sum_{j=1}^{n(t)} \ell_j \exp \left[ - \int_0^{t_j} h_{\lambda_0}(u) \, du \right], \quad \text{for } t \geq 0.
\]

Because of (8), the above equation is equivalently written as
\[
    y(t) = \phi(0) + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_0^t b_i(s) \phi(s) \exp \left[ - \int_0^s h_{\lambda_0}(u) \, du \right] ds - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t-\tau}^t b_i(s) y(s) \, ds + \sum_{j=1}^{n(t)} \ell_j \exp \left[ - \int_0^{t_j} h_{\lambda_0}(u) \, du \right], \quad \text{for } t \geq 0.
\]

Hence, \( y \) is a solution for (10). Assume that \( y \) solves the integral Equation (10), so \( y \) will become the solution of (7)–(9). In fact, using (10), we get
\[
    y'(t) = - \sum_{i \in I} b_i(t) e^{-\lambda_0 \tau_i} [y(t) - y(t - \tau_i)] \quad \text{for } t \geq 0 \quad \text{and} \quad t \neq t_k.
\]

It can be easily showed as follows
\[
    y(t_k) - y(t_k^-) = \ell_k \exp \left[ - \int_0^{t_k} h_{\lambda_0}(u) \, du \right], \quad k \in \mathbb{Z}^+.
\]

Thus, we have already proved Lemma 1. \( \square \)

By Lemma 1 the following corollary can be derived.

**Corollary 1.** Suppose that \( \lambda_0 \) is a real root of the characteristic Equation (4) and set (5). Thus, \( x \) is the single solution of the initial value problem (1)–(3) if and only if the function \( y \) defined by
\[
    y(t) = x(t) \exp \left[ - \int_0^t h_{\lambda_0}(u) \, du \right] \quad \text{for } t \geq -\tau
\]
is the solution of the integral Equation (10) which gives the initial condition
\[
    y(t) = \phi(t) \exp \left[ - \int_0^t h_{\lambda_0}(u) \, du \right] \quad \text{for } t \in [-\tau, 0].
\]

2. The Asymptotic Behaviour of Solutions

We give a fundamental asymptotic criterion as a theorem to solve the problem (1)–(3).

**Theorem 1.** Assume that Lemma 1 is valid and that the root \( \lambda_0 \) satisfies
\[
    \mu(\lambda_0) = \sum_{j=1}^{\infty} |\ell_j| \exp \left[ - \int_0^{t_j} h_{\lambda_0}(u) \, du \right] + \sum_{i \in I} |B_i| \tau_i e^{-\lambda_0 \tau_i} < 1. \quad (12)
\]

Thus, the solution \( x \) of (1)–(3) fulfills
\[
    \lim_{t \to \infty} \left\{ x(t) \exp \left[ - \int_0^t h_{\lambda_0}(u) \, du \right] \right\} = \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \quad (13)
\]
where
\[
L(\lambda_0; \phi) = \phi(0) + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{-\tau_i}^{0} \overline{b}_i(s) \phi(s) \exp \left[ - \int_{0}^{s} h_{\lambda_0}(u) du \right] ds
\]
(14)

and
\[
\beta(\lambda_0) = B_1 \tau_i e^{-\lambda_0 \tau_i}.
\]
(15)

Note: It is guaranteed by the property (12) that \(0 < 1 + \beta(\lambda_0) < 2\).

Proof. By (12), we have \(|\beta(\lambda_0)| \leq \mu(\lambda_0) < 1\). Thus, this yields that \(0 < 1 + \beta(\lambda_0) < 2\).

Assume that \(x\) a solution of (1)–(3). Identify the function \(y\) using (6). Afterwards, \(x\) will be the solution of (1)–(3), and \(y\) the solution of the integral Equation (10) yielding the initial condition (8). Therefore, by (14), using (10), we obtain
\[
y(t) = L(\lambda_0; \phi) + \sum_{j=1}^{n(t)} \ell_j \exp \left[ - \int_{0}^{t} h_{\lambda_0}(u) du \right] - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t-\tau_i}^{t} \overline{b}_i(s) y(s) ds.
\]
(16)

Now, for \(t \geq -\tau\) we construct
\[
z(t) = y(t) - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)}.
\]

Hence, from the Equation (16), it is reduced to the equation as below:
\[
z(t) = \sum_{j=1}^{n(t)} \ell_j \exp \left[ - \int_{0}^{t} h_{\lambda_0}(u) du \right] - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t-\tau_i}^{t} \overline{b}_i(s) z(s) ds
\]
(17)

for \(t \geq 0\).

Moreover, (8) is defined as for \(t \in [-\tau, 0]\)
\[
z(t) = \phi(t) \exp \left[ - \int_{0}^{t} h_{\lambda_0}(u) du \right] - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)}.
\]
(18)

Using \(y\) and \(z\), we should prove the equality (13), that is,
\[
\lim_{t \to -\tau} z(t) = 0.
\]
(19)

Put
\[
W(\lambda_0; \phi) = \max \left\{ 1, \max_{t \in [-\tau, 0]} \left| \phi(t) \exp \left[ - \int_{0}^{t} h_{\lambda_0}(u) du \right] - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \right| \right\}.
\]

Thus, by (18) we obtain
\[
|z(t)| \leq W(\lambda_0; \phi) \quad \text{for} \quad -\tau \leq t \leq 0.
\]
(20)

Now, the following inequality will be proved
\[
|z(t)| \leq W(\lambda_0; \phi) \quad \text{for} \quad t \geq -\tau.
\]
(21)
On the contrary, assume that a point is found where \( \tilde{t} > 0 \) such that \(|z(\tilde{t})| > W(\lambda_0; \phi)\). Let

\[
t^* = \inf \{ \tilde{t} : |z(\tilde{t})| > W(\lambda_0; \phi) \}.
\]

According to the continuity from right, either \(|z(t^*)| = W(\lambda_0; \phi)\) without impulsive point at \(t^*\), or \(|z(t^*)| \geq W(\lambda_0; \phi)\) with a jump at \(t^*\). In both cases, by the right continuity, we obtain \(|z(t)| \leq W(\lambda_0; \phi)\) for \(-\tau \leq t < t^*\), where \(|z(t^*)| = W(\lambda_0; \phi)\) provided that this satisfies at a non-impulsive point. Therefore, considering (12), by the integral representation of \(z(t)\), which all solutions to (17), we obtain

\[
|z(t^*)| = \left| \sum_{j=1}^{n(t^*)} \ell_j \exp \left[ - \int_0^t h_{\lambda_0}(u)du - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t_{i-1}}^{t_i} \overline{B}_i(s)z(s)ds \right] \right|
\leq \left| \sum_{j=1}^{n(t^*)} \ell_j \exp \left[ - \int_0^{t_j} h_{\lambda_0}(u)du + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t_{i-1}}^{t_i} |\overline{B}_i(s)||z(s)|ds \right] \right|
\leq \left\{ \sum_{j=1}^{n(t^*)} |\ell_j| \exp \left[ - \int_0^{t_j} h_{\lambda_0}(u)du + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t_{i-1}}^{t_i} |\overline{B}_i(s)|ds \right] \right\} W(\lambda_0; \phi)
\leq \mu(\lambda_0)W(\lambda_0; \phi) < W(\lambda_0; \phi)
\]

which contradicts the definition of \(t^*\) because we showed \(|z(t^*)| < W(\lambda_0; \phi)\), and we suppose \(|z(t^*)| = W(\lambda_0; \phi)\) where \(t^*\) is continuous, or \(|z(t^*)| \geq W(\lambda_0; \phi)\) where \(t^*\) is discontinuous. Hence, the inequality (21) holds.

Next, by (21), considering (17) we obtain for \(t \geq 0\),

\[
|z(t)| = \left| \sum_{j=1}^{n(t)} \ell_j \exp \left[ - \int_0^{t_j} h_{\lambda_0}(u)du - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t_{i-1}}^{t_i} \overline{B}_i(s)z(s)ds \right] \right|
\leq \left| \sum_{j=1}^{n(t)} |\ell_j| \exp \left[ - \int_0^{t_j} h_{\lambda_0}(u)du + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t_{i-1}}^{t_i} |\overline{B}_i(s)||z(s)|ds \right] \right|
\leq \left\{ \sum_{j=1}^{n(t)} |\ell_j| \exp \left[ - \int_0^{t_j} h_{\lambda_0}(u)du + \sum_{i \in I} e^{-\lambda_0 \tau_i} |B_i|_{\tau_i} \right] \right\} W(\lambda_0; \phi)
\leq \mu(\lambda_0)W(\lambda_0; \phi).
\]

In other words, we have

\[
|z(t)| \leq \mu(\lambda_0)W(\lambda_0; \phi) \quad \text{for} \quad t \geq 0.
\]

By (12), (21) and (22), using an easy induction, from (17) it can be proved that

\[
|z(t)| \leq [\mu(\lambda_0)]^n W(\lambda_0; \phi) \quad \text{for} \quad t \geq n\tau - \tau \quad (n = 0, 1, \cdots).
\]

Due to (12), we obtain \(\lim_{n \to \infty} [\mu(\lambda_0)]^n = 0\). Thus, from (23) we obtain

\[
\lim_{t \to \infty} z(t) = \lim_{t \to \infty} \left\{ x(t) \exp \left[ - \int_0^t h_{\lambda_0}(u)du - \frac{L(\lambda_0; \phi)}{1 + B(\lambda_0)} \right] \right\} = 0
\]

that is, (13) satisfies. Theorem 1 has been already proven. \(\square\)
Corollary 2. Assume that
\[ a(t) + \sum_{i \in I} b_i(t) = 0 \quad \text{for} \quad t \in [0, \infty) \]  
(24)
and
\[ \sum_{i \in I} |B_i| \tau_i + \sum_{j=1}^\infty |\ell_j| < 1. \]
(25)

Thus, the solution \( x \) of (1)–(3) satisfies for any \( \phi \in \left( [-\tau, 0], \mathbb{R} \right) \),
\[ \lim_{t \to \infty} x(t) = \frac{\phi(0) + \sum_{i \in I} \int_{-\tau_i}^0 b_i(s) \phi(s) \, ds}{1 + \sum_{i \in I} B_i \tau_i}. \]

Note: It is guaranteed by (25) that \( 2 > 1 + \sum_{i \in I} B_i \tau_i > 0 \).

Proof. It immediately follows from (24) that \( A + \sum_{i \in I} B_i = 0 \). Hence, \( \lambda_0 = 0 \) is a real root of (4). By using again (24), we see that for \( \lambda_0 = 0 \), we have \( h_{\lambda_0} = 0 \) on the interval \( [-\tau, \infty) \). Moreover, (25) facilitates the verification of which the root \( \lambda_0 = 0 \) of (4) has the property (12). Therefore this can be applied Theorem 1. \( \square \)

3. Stability Criterion

Theorem 2. Assume that Theorem 1 is satisfied and \( \mu(\lambda_0) < 1 \), where \( \mu(\lambda_0) \) is identified by (12), and \( h_{\lambda_0}(t) \) and \( \beta(\lambda_0) \) are specified by (5) and (15), respectively. Set
\[ R(\lambda_0; \phi) = \max \left\{ 1, \max_{-\tau \leq t \leq 0} |\phi(t)|, \max_{-\tau \leq t \leq 0} \left[ |\phi(t)| \exp \left[ -\int_0^t h_{\lambda_0}(u) \, du \right] \right] \right\}. \]
(26)

Accordingly, the solution \( x \) of (1)–(3) meets the following condition
\[ |x(t)| \leq N(\lambda_0) R(\lambda_0; \phi) \exp \left[ \int_0^t h_{\lambda_0}(u) \, du \right], \quad \text{for all} \quad t \geq 0, \]
(27)
where
\[ N(\lambda_0) = \mu(\lambda_0) + k(\lambda_0) \left( \frac{1 + \mu(\lambda_0)}{1 + \beta(\lambda_0)} \right) \]
(28)
and
\[ k(\lambda_0) = 1 + \sum_{i \in I} |B_i| \tau_i e^{-\lambda_0 \tau_i}. \]
(29)

Note: It is guaranteed by the property (12) that \( 0 < 1 + \beta(\lambda_0) < 2 \).

Furthermore, the trivial solution of (1)–(2) is:
(i) stable if
\[ \limsup_{t \to \infty} \int_0^t h_{\lambda_0}(u) \, du < \infty \]
(30)
or, equivalently, providing that the conditions (24) and (25) are met,
(ii) asymptotically stable if

\[
\lim_{t \to \infty} \int_0^t h_{\lambda_0}(u)\,du = -\infty, \quad (31)
\]

(iii) unstable if

\[
\lim_{t \to \infty} \int_0^t h_{\lambda_0}(u)\,du = \infty. \quad (32)
\]

**Proof.** Suppose that \(x\) is the solution of (1)–(3) and \(y, z\) are defined as above, that is, for \(t \geq -\tau\)

\[
y(t) = x(t) \exp \left[ -\int_0^t h_{\lambda_0}(u)\,du \right] \quad \text{and} \quad z(t) = y(t) - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)},
\]

where \(L(\lambda_0; \phi)\) is defined as in (14). Therefore, we specify \(W(\lambda_0; \phi)\) as in the proof of Theorem 1, that is,

\[
W(\lambda_0; \phi) = \max \left\{ 1, \max_{t \in [-\tau, 0]} |\phi(t)| \exp \left[ -\int_0^t h_{\lambda_0}(u)\,du \right] - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \right\}.
\]

Hence, as in Theorem 1, it can be also proved that \(z\) satisfies inequality (22). Thus, for \(t \geq 0\) we get

\[
|y(t)| \leq \mu(\lambda_0) W(\lambda_0; \phi) + \frac{|L(\lambda_0; \phi)|}{1 + \beta(\lambda_0)}. \quad (33)
\]

Using (26) and (29), from (14), we obtain

\[
|L(\lambda_0; \phi)| \leq |\phi(0)| + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{-\tau_i}^0 |\overline{B}_i(s)||\dot{\phi}(s)| \exp \left[ -\int_0^s h_{\lambda_0}(u)\,du \right] ds
\]

\[
\leq \left( 1 + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{-\tau_i}^0 |\overline{B}_i(s)| ds \right) R(\lambda_0; \phi)
\]

\[
= \left( 1 + \sum_{i \in I} |B(i)| e^{-\lambda_0 \tau_i} \right) R(\lambda_0; \phi)
\]

\[
= k(\lambda_0) R(\lambda_0; \phi).
\]

Furthermore, using the definition of \(W(\lambda_0; \phi)\), we have

\[
W(\lambda_0; \phi) \leq \max \left\{ 1, R(\lambda_0; \phi) + \frac{|L(\lambda_0; \phi)|}{1 + \beta(\lambda_0)} \right\} = R(\lambda_0; \phi) + \frac{|L(\lambda_0; \phi)|}{1 + \beta(\lambda_0)}
\]

\[
\leq R(\lambda_0; \phi) + \frac{k(\lambda_0) R(\lambda_0; \phi)}{1 + \beta(\lambda_0)} = \left( 1 + \frac{k(\lambda_0)}{1 + \beta(\lambda_0)} \right) R(\lambda_0; \phi).
\]

Thus, using (26) and (28), by (33), we reach for \(t \geq 0\)

\[
|y(t)| \leq \mu(\lambda_0) \left( 1 + \frac{k(\lambda_0)}{1 + \beta(\lambda_0)} \right) R(\lambda_0; \phi) + \frac{k(\lambda_0) R(\lambda_0; \phi)}{1 + \beta(\lambda_0)}
\]

\[
= \left\{ \mu(\lambda_0) \left( 1 + \frac{k(\lambda_0)}{1 + \beta(\lambda_0)} \right) + \frac{k(\lambda_0)}{1 + \beta(\lambda_0)} \right\} R(\lambda_0; \phi)
\]

\[
= N(\lambda_0) R(\lambda_0; \phi).
\]

Last of all, using the definition of \(y\), we get

\[
|x(t)| \leq N(\lambda_0) R(\lambda_0; \phi) \exp \left[ \int_0^t h_{\lambda_0}(u)\,du \right], \quad \text{for all} \quad t \geq 0.
\]
Therefore, the first part of this theorem has been proven. Now, we can start to establish a proof for the second part (stability criterion).

Firstly, suppose that (30) holds and set

\[ p(\lambda_0) = \sup_{t \geq 0} \left\{ \exp \left[ \int_0^t h_{\lambda_0}(u) du \right] \right\}. \]

Obviously \( p(\lambda_0) \) is a real constant such that \( p(\lambda_0) \geq 1 \). Furthermore, we set \( P(\lambda_0) = p(\lambda_0)N(\lambda_0) \). Since \( N(\lambda_0) > 1 \), we also obtain \( P(\lambda_0) > 1 \). Let \( \phi \) be an arbitrary function in \( C\left([-\tau, 0], \mathbb{R}\right) \) and \( x \) be the solution of (1)-(3). Thus, (27) gives

\[ |x(t)| \leq P(\lambda_0)R(\lambda_0; \phi) \quad \text{for all} \quad t \geq 0. \]

When \( \| \phi \| = \max_{-\tau \leq t \leq 0} |\phi(t)| \leq R(\lambda_0; \phi) \) and \( P(\lambda_0) > 1 \), it gives that

\[ |x(t)| \leq P(\lambda_0)R(\lambda_0; \phi) \quad \text{for all} \quad t \geq -\tau. \]

For any \( \epsilon > 0 \), choosing \( \delta = \frac{\epsilon}{P(\lambda_0)} \) with \( R(\lambda_0; \phi) < \delta \), we get that \( \| \phi \| < \delta \). Hence,

\[ |x(t)| \leq P(\lambda_0)R(\lambda_0; \phi) < P(\lambda_0)\delta = \epsilon. \]

As a result, we obtain the stability of the trivial solution of (1)-(2). In particular, let us consider the case where conditions (24) and (25) hold. Then, we identified that \( \lambda_0 = 0 \) and \( h_{\lambda_0} = 0 \) on the interval \([-\tau, \infty)\) as mentioned in Section 2. In the case, (30) is always satisfied.

Next, let us suppose that (31) is fulfilled. Then, (30) is also satisfied. Thus, the trivial solution of (1)-(2) is stable. Moreover, since \( \lim_{t \to \infty} x(t) = 0 \), it is guaranteed by the inequality (27) that the trivial solution of (1)-(2) is asymptotically stable.

Finally, let (32) be satisfied and we will prove that the trivial solution of (1)-(2) is unstable. On the contrary, assume that it is stable. Hence, we can choose \( \delta > 0 \) such that for each \( \phi \in C\left([-\tau, 0], \mathbb{R}\right) \) with

\[ |x(t)| < 1 \quad \text{for all} \quad t \geq -\tau. \quad (34) \]

Define

\[ \phi_0(t) = \exp \left[ \int_0^t h_{\lambda_0}(u) du \right] \quad \text{for} \quad t \in [-\tau, 0]. \]

We see \( \phi_0 \in C\left([-\tau, 0], \mathbb{R}\right) \) and \( \phi_0 \neq 0 \). From (14), we have

\begin{align*}
L(\lambda_0; \phi_0) &= \phi_0(0) + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{-\tau_i}^0 \varphi_i(s) \phi_0(s) \exp \left[ -\int_0^s h_{\lambda_0}(u) du \right] ds \\
&= 1 + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{-\tau_i}^0 \varphi_i(s) ds = 1 + \sum_{i \in I} \beta_i \tau_i e^{-\lambda_0 \tau_i} \\
&= 1 + \beta(\lambda_0) > 0. \quad (35)
\end{align*}

Now, we take a number \( \delta_0 > 0 \) with \( 0 < \delta_0 < \delta \) and we define

\[ \phi = \frac{\delta_0}{\| \phi_0 \|} \phi_0. \]
Clearly, $\phi \in C \left( [-\tau, 0], \mathbb{R} \right)$ and $\| \phi \| = \delta_0 < \delta$. Therefore, the solution $x$ of (1)–(3) fulfills (34), that is, $x$ is always bounded on $[-\tau, \infty)$. Thus, from (32), we may obtain
\[
\lim_{t \to \infty} \left\{ x(t) \exp \left[ - \int_0^t h_{\lambda_0}(u) du \right] \right\} = 0.
\]
Furthermore, since the operator $L(\lambda_0; \cdot)$ is linear and by (35), using (13) we obtain
\[
\lim_{t \to \infty} \left\{ x(t) \exp \left[ - \int_0^t h_{\lambda_0}(u) du \right] \right\} = \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} = \frac{\delta_0}{\| \phi \|} > 0.
\]
We consequently reached a contradiction here that the trivial solution of (1)–(2) is unstable. Thus, the second part of the theorem has been proved, which yields the required result. \qed

4. The Special Case of Linear Impulsive Delay Differential Equations with Constant Coefficients

It can be found here a unique case of first order autonomous linear impulsive delay differential equations. Suppose that it is
\[
x'(t) = ax(t) + \sum_{i \in I} b_i x(t - \tau_i), \quad t \geq 0, \quad t \neq t_k, \quad k \in \mathbb{Z}^+
\]
\[
\Delta x(t_k) = \ell_k, \quad k \in \mathbb{Z}^+ = \{1, 2, \ldots \},
\]
where $I$ is the initial segment of natural numbers, $a, b_i$ for $i \in I$ are the real constants, and $\tau_i$ for $i \in I$ the positive real numbers with $\tau_{i_1} \neq \tau_{i_2}$ for $i_1, i_2$ with $i_1 \neq i_2$. Furthermore, $\ell_k$ for $k \in \mathbb{Z}^+$ are real constants and $\Delta x(t_k) = x(\ell_k) - x(t_k^-)$. The impulsive positive points $t_k$ satisfy
\[
0 < t_1 < \cdots < t_k < t_{k+1} < \cdots \quad \text{and} \quad \lim_{k \to \infty} t_k = \infty.
\]
Let $\tau$ be defined by $\tau = \max_{i \in I} \tau_i$, and the initial function be given as in (3). The characteristic equation of (36) is
\[
\lambda = a + \sum_{i \in I} b_i e^{-\lambda \tau_i}.
\]

There were given sufficient conditions to obtain a unique real root of characteristic equation (37) in Philos [53] (Chapter 5).

The following observation will be made to get a mapping appropriate to the asymptotic criterion for the solutions. Let $\lambda_0$, be a real root of (37). Define
\[
y(t) = x(t)e^{-\lambda_0 t}, \quad \text{for all} \quad t \in [-\tau, \infty).
\]
Thus, from (36) we get
\[
y'(t) = -\sum_{i \in I} b_i e^{-\lambda_0 \tau_i} [y(t) - y(t - \tau_i)], \quad \text{for all} \quad t \geq 0.
\]
It can also be written that the initial condition (3) is equivalent to
\[
y(t) = e^{-\lambda_0 t} \phi(t) \quad \text{for} \quad t \in [-\tau, 0].
\]
In addition, if $x(t)$ meets the impulsive conditions (2), it is evident that
\[ y(t_k) - y(t_k^-) = \ell_k e^{-\lambda_0 t_k}, \quad k \in \mathbb{Z}^+. \tag{40} \]

The constant coefficients \(a\) and \(b_i\) of (36) can be considered as \(T\)-periodic functions, for each real number \(T > 0\). Moreover, as it concerns the autonomous delay differential Equation (36), the hypothesis that there exists positive integers \(m_i\) for \(i \in I\) such that \(\tau_i = m_i T\) holds by itself. After these observations, it is not difficult to apply the main results of this paper, that is, Lemma 1, Theorems 1 and 2, to the special case of the autonomous linear impulsive delay differential Equation (36). For the impulsive delay differential system (36) and (3), we have the proofs presented below.

Lemma 2. Suppose that \(\lambda_0\) is a real root of the characteristic Equation (37). Thus \(y\) is the unique solution of (38)–(40) if and only if \(y\) is a solution of the following system

\[ y(t) = e^{-\lambda_0 t} \phi(t), \quad \text{if} \quad t \in [-\tau, 0] \]

and

\[ y(t) = \phi(0) + \sum_{i \in I} b_i e^{-\lambda_0 \tau_i} \int_{-\tau_i}^t e^{-\lambda_0 s} \phi(s) ds - \sum_{i \in I} b_i e^{-\lambda_0 \tau_i} \int_{t-\tau_i}^t y(s) ds + \sum_{j=1}^{n(t)} \ell_j e^{-\lambda_0 \tau_j}, \quad \text{if} \quad t \in [0, \infty) \tag{41} \]

where \(n(t) = \max\{k \in \mathbb{Z}^+: t_k \leq t\}\) and \(n(t) = 0\) if \(t < t_1\).

By Lemma 2 the following corollary can be derived.

Corollary 3. Let \(\lambda_0\) be a real root of the characteristic Equation (37). Then \(x\) is the unique solution of the system (36) and (3) if and only if the function \(y\) defined by

\[ y(t) = e^{-\lambda_0 t} x(t) \quad \text{for all} \quad t \in [-\tau, \infty) \]

a solution of the integral Equation (41) giving the initial condition

\[ y(t) = e^{-\lambda_0 t} \phi(t) \quad \text{for all} \quad t \in [-\tau, 0]. \]

Theorem 3. Suppose that \(\lambda_0\) be a real root of (37) with

\[ \mu(\lambda_0) = \sum_{j=1}^{\infty} |\ell_j| e^{-\lambda_0 \tau_j} + \sum_{i \in I} |b_i| \tau_i e^{-\lambda_0 \tau_i} < 1. \tag{42} \]

Thus the solution \(x\) of the system (36) and (3) satisfies

\[ \lim_{t \to \infty} e^{-\lambda_0 t} x(t) = \frac{L(\lambda_0; \phi)}{1 + \sum_{i \in I} b_i \tau_i e^{-\lambda_0 \tau_i}}, \]

where

\[ L(\lambda_0; \phi) = \phi(0) + \sum_{i \in I} b_i e^{-\lambda_0 \tau_i} \int_{-\tau_i}^0 \phi(s) e^{-\lambda_0 s} ds. \]

Note: It is guaranteed by the property (42) that \(0 < 1 + \sum_{i \in I} b_i \tau_i e^{-\lambda_0 \tau_i} < 2\).

Application of the Theorem 3 with \(\lambda_0 = 0\) leads to the following corollary.
Corollary 4. Assume that
\[ a + \sum_{i \in I} b_i = 0 \quad \text{and} \quad \sum_{i \in I} |b_i| \tau_i + \sum_{j=1}^{\infty} |\ell_j| < 1. \] (43)

The solution \( x(t) \) of the system (36) and (3) satisfies
\[ \lim_{t \to \infty} x(t) = \frac{\phi(0) + \sum_{i \in I} b_i \int_{-\tau_i}^{0} \phi(s) ds}{1 + \sum_{i \in I} b_i \tau_i}. \]

Theorem 4. Assume that Theorem 3 is satisfied and let \( \lambda_0 \) be a real root of (37) satisfying (42) and set
\[ R(\lambda_0; \phi) = \max \left\{ 1, \max_{-\tau \leq t \leq 0} |\phi(t)|, \max_{-\tau \leq t \leq 0} \left| e^{-\lambda_0 t} |\phi(t)| \right| \right\}. \]

Thus the solution \( x(t) \) of the system (36) and (3) satisfies
\[ |x(t)| \leq N(\lambda_0) R(\lambda_0; \phi) e^{\lambda_0 t} \quad \text{for} \quad t \geq 0, \]

if
\[ N(\lambda_0) = \mu(\lambda_0) + (1 + \mu(\lambda_0)) \left( \frac{1 + \sum_{i \in I} |b_i| \tau_i e^{-\lambda_0 \tau_i}}{1 + \sum_{i \in I} b_i \tau_i e^{-\lambda_0 \tau_i}} \right). \]

Moreover, the trivial solution of (36) and (2) is:

(i) unstable if \( \lambda_0 > 0 \),
(ii) stable if \( \lambda_0 = 0 \) or, equivalently, providing that the conditions (43) are met, and
(iii) asymptotically stable if \( \lambda_0 < 0 \).

5. Example

In the following example, we will apply the stability criteria of the Theorem 2. For simplicity of example we consider the problem as follows:
\[ x'(t) = a(t)x(t) + b(t)x(t - \tau), \quad t \geq 0, \quad t \neq t_k, \] (44)
\[ \Delta x(t_k) = \ell_k, \quad k \in \mathbb{Z}^+ = \{1, 2, \ldots \}, \]
\[ x(t) = \phi(t), \quad -\tau \leq t \leq 0, \]
where \( a(t) = c_1 + \sin 2\pi t \), \( b(t) = c_2 - \sin 2\pi t \) (\( c_1 \) and \( c_2 \) are fixed constants) with period \( T = 1 \), \( t_k = k \) and \( \phi(t) \) is an arbitrary continuous initial function on \([-\tau, 0].\) The characteristic equation of (44) is from (4)
\[ \lambda = A + Be^{-\lambda \tau}, \] (45)
where \( A = \frac{1}{T} \int_{0}^{T} a(t) dt \) and \( B = \frac{1}{T} \int_{0}^{T} b(t) dt. \) We will find the real roots of the characteristic Equation (45) in all three cases below. Note that for each real root \( \lambda_0 \) the characteristic Equation (45), there is from (5) \( h_{\lambda_0}(t) = a(t) + b(t)e^{-\lambda_0 \tau}, \) for \( t \geq 0, \) and from (12), it becomes
\[ \mu(\lambda_0) = \sum_{j=1}^{\infty} |\ell_j| \exp \left[ -\int_{0}^{T} h_{\lambda_0}(u) du \right] + |B| \tau e^{-\lambda_0 \tau} < 1. \] (46)

Case 1: Choose \( c_1 = \frac{1}{3}, c_2 = -\frac{1}{3}, \) \( \ell_k = \left(-\frac{1}{3}\right)^k \) and \( \tau = 1. \) Since \( A = \int_{0}^{1} \left(\frac{1}{3} + \sin 2\pi t\right) dt = \frac{1}{3} \) and \( B = \int_{0}^{1} \left(-\frac{1}{3} - \sin 2\pi t\right) dt = -\frac{1}{3}, \) from (45) we obtain
\[ \lambda = \frac{1}{3} \left( 1 - e^{-\lambda} \right). \]

We have \( \lambda_1 \approx -1.9 \) and \( \lambda_2 = 0 \) are real roots of characteristic equation. Let \( \lambda_0 \approx -1.9 \). Then, the last term in (46) \( \frac{t}{e^t} \approx 2.23 \). Therefore, Theorem 2 cannot be applied to Equation (44). But, let \( \lambda_0 = 0 \). We check the condition for Theorem 2 as follows: Since \( \mu(0) = 0 \), we check the condition for Theorem 2 as follows: Since \( \mu(0) = 0 \), we obtained easily

\[ \mu(0) = \mu(0) = \sum_{j=1}^{\infty} \left| \left( -\frac{1}{3} \right)^j \right| + \frac{1}{3} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1. \]

Therefore, the condition of Theorem 2 is satisfied. Since

\[ \limsup_{t \to \infty} \int_0^t h_{\lambda_0}(u)du = 0 < \infty, \]

the trivial solution of (44) is stable.

**Case 2:** Choose \( c_1 = -1, \ c_2 = 0, \ \ell_k = \frac{1}{e^t} \) and \( \tau = 2 \). Since \( A = \int_0^1 (-1 + \sin 2\pi t)dt = -1 \) and \( B = \int_0^1 (-\sin 2\pi t)dt = 0 \), from (45) it is easy to see \( \lambda_0 = -1 \). Hence, \( h_{\lambda_0}(t) = -1 + (1 - e^2) \sin 2\pi t \). We check the condition for Theorem 2 as follows: From (46), we obtained

\[ \mu(0) = \mu(0) = \sum_{j=1}^{\infty} \frac{1}{e^{2j}} \left[ \exp \left[ -\int_0^j (-1 + (1 - e^2) \sin 2\pi t)dt \right] \right] = \sum_{j=1}^{\infty} \frac{1}{e^j} = \frac{1}{e - 1} < 1. \]

Thus, the condition of Theorem 2 is satisfied. Since

\[ \lim_{t \to \infty} \int_0^t h_{\lambda_0}(u)du = \lim_{t \to \infty} \int_0^t \left[ -1 + (1 - e^2) \sin 2\pi u \right] du = -\infty, \]

the trivial solution of (44) is asymptotically stable.

**Case 3:** Choose \( c_1 = \frac{\varepsilon - 1}{\varepsilon}, \ c_2 = 1, \ \ell_k = (-1)^k \) and \( \tau = 1 \). Since \( A = \int_0^1 \left( \frac{\varepsilon - 1}{\varepsilon} + \sin 2\pi t \right)dt = \frac{\varepsilon - 1}{\varepsilon} \) and \( B = \int_0^1 (1 - \sin 2\pi t)dt = 1 \), from (45) we get

\[ \lambda = \frac{e - 1}{e} + e^{-\lambda}. \]

We see easily that \( \lambda_0 = 1 \) is unique real root of the above characteristic equation. Hence, \( h_{\lambda_0}(t) = 1 + (1 - e^{-1}) \sin 2\pi t \). We check the condition for Theorem 2 as follows: From (46), we obtained

\[ \mu(1) = \mu(1) = \sum_{j=1}^{\infty} \left| (-1)^j \right| \exp \left[ -\int_0^j \left( 1 + (1 - e^{-1}) \sin 2\pi t \right) dt \right] + \frac{1}{\varepsilon} \]

\[ = \sum_{j=1}^{\infty} \frac{1}{e^j} + \frac{1}{\varepsilon} = \frac{1}{e - 1} + \frac{1}{\varepsilon} \approx 0.95 < 1. \]

Thus, the condition of Theorem 2 is satisfied. Since

\[ \lim_{t \to \infty} \int_0^t h_{\lambda_0}(u)du = \lim_{t \to \infty} \int_0^t \left[ 1 + (1 - e^{-1}) \sin 2\pi u \right] du = \infty, \]

the trivial solution of (44) is unstable.
6. Conclusions

In this study, firstly, we created a very useful lemma to prove our fundamental asymptotic criterion. Later, we proved that there was a basic asymptotic criterion for the solutions of the initial value problem (1)–(3). Finally, using this asymptotic criterion, we obtained a useful exponential boundary for solutions of (1)–(3) and showed the stability of trivial solutions. In other words, we determined the stability of the trivial solution by converting the constructed equation into two integral equations. These results were obtained using a suitable real root for the characteristic equation. Namely that, this real root played an important role in establishing the results of the article. We have also presented the application in the special case of constant coefficients of the results obtained. In addition, we gave three different cases in one example.

It would be interesting to use the same method for the stability of first order linear impulsive neutral delay differential equations, which can be the subject of a future study to extend the current results to first order linear impulsive neutral delay differential equations with periodic coefficients to provide valuable insights into the literature.

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