An edge CLT for the log determinant of Laguerre beta ensembles

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Abstract

We obtain a CLT for \( \log | \det(M_n - s_n) | \) where \( M_n \) is a scaled Laguerre beta ensemble and \( s_n = d_+ + \sigma_n n^{-2/3} \) with \( d_+ \) denoting the upper edge of the limiting spectrum of \( M_n \) and \( \sigma_n \) a slowly growing function \( (\log \log^2 n \ll \sigma_n \ll \log^2 n) \). In the special cases of LUE and LOE, we prove that the CLT also holds for \( \sigma_n \) of constant order. A similar result was proved for Wigner matrices by Johnstone, Klochkov, Onatski, and Pavlyshyn. Obtaining this type of CLT of Laguerre matrices is of interest for statistical testing of critically spiked sample covariance matrices as well as free energy of bipartite spherical spin glasses at critical temperature.

1 Introduction

1.1 Background

As one of the most fundamental quantities in the study of matrices, determinants have been well studied in random matrix theory and there is a natural interest in how these determinants behave asymptotically as the size of the matrix grows. More specifically, a number of studies of the past decade have studied the log determinant, \( \log | \det(M_n) | \), for various random matrix ensembles, \( M_n \), and have established CLT results for this quantity as \( n \to \infty \). See papers by Nguyen and Vu for results non-Hermitian i.i.d. matrices [19] and Tao and Vu for results on Wigner matrices [23].

It is also of interest to study a log determinant away from the origin (i.e. \( \log | \det(M_n - s) | \) for \( s \neq 0 \)). We note that this quantity can also be written as \( \sum_{i=1}^{n} \log | \lambda_i - s | \) where \( \{\lambda_i\}_{i=1}^{n} \) are the eigenvalues of \( M_n \). For \( s \) outside the spectrum of \( M_n \), this is a special case of the well-studied linear spectral statistics, i.e. \( \sum_{i=1}^{n} f(\lambda_i) \) where \( f \) is a smooth function on the support of the spectrum of \( M_n \). Johansson proved a CLT for linear spectral statistics of Gaussian beta ensembles (with some generalization to other random matrices) [11] and Bai and Silverstein proved a similar result for Laguerre beta ensembles [1].

Recently, Johnstone, Klochkov, Onatski, and Pavlyshyn [12] considered a case in which \( M_n \) is a scaled Wigner ensemble (or Gaussian beta ensemble) and \( s \) is close to edge of the spectrum of \( M_n \) and approaches the edge as \( n \to \infty \). This is not covered by the studies of linear statistics, since \( \sum_{i=1}^{n} \log | \lambda_i - s | \) is singular for \( s \) at the edge of the spectrum. This work was motivated by high dimensional statistical testing and spin glasses. Johnstone et al derived a CLT for this case (see also a related result by Lambert and Paquette [15]). The goal of this paper is to derive an analogous result to [12] in the case where the matrix is from a Laguerre beta ensemble.

Laguerre beta ensembles: By Laguerre beta ensemble \((L_{\beta}E)\), we mean an \( n \times n \) random matrix \( M_{n,m} \) with joint eigenvalue density

\[
p(\lambda_1, \lambda_2, \ldots, \lambda_n) = C_{n,m,\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{n} \lambda_i^{\frac{2(m-n+1)}{\beta} - 1} e^{-\lambda_i/2}, \tag{1.1}
\]

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where \( m \geq n \) and \( \beta > 0 \) and \( C_{n,m,\beta} \) is the corresponding normalization constant. The cases of \( \beta = 1 \) and \( \beta = 2 \) correspond to the Laguerre Orthogonal Ensemble (LOE) and the Laguerre Unitary Ensemble (LUE) respectively, which can be constructed by setting \( M_n := AA^* \) where \( A \) is taken to be an \( n \times m \) matrix with i.i.d. entries that are real Gaussian (LOE) or complex Gaussian (LUE) with mean 0 and variance 1. We fix a parameter \( \lambda \) and take \( n, m \to \infty \) such that their ratio converges to \( \lambda \). More specifically, we require

\[
\frac{n}{m} = \lambda + O(n^{-1}), \quad 0 < \lambda \leq 1. \tag{1.2}
\]

Let \( \mu_1 \geq \mu_2 \geq \cdots \mu_n \geq 0 \) denote the eigenvalues of the scaled \( \beta \)E matrix \( \frac{1}{m} M_{n,m} \). It was shown by Marčenko and Pastur (for \( \beta = 1 \)) [18] and by Dumitriu and Edelman (for general \( \beta > 0 \)) [6] that, as \( n, m \to \infty \) with \( n/m \to \lambda \leq 1 \),

\[
\frac{1}{n} \sum_{i=1}^{n} \delta_{\mu_i} \to \frac{\sqrt{(d_+ - x)(x - d_-)}}{2\pi \lambda x} 1_{[d_-,d_+]}, \tag{1.3}
\]

where the convergence is weakly in distribution and \( d_\pm = (1 \pm \lambda^{1/2})^2 \).

Of particular importance for our purposes is the behavior of the largest eigenvalue. As \( n \to \infty \), this eigenvalue approaches the constant \( d_+ \) and displays Tracy-Widom type fluctuations of order \( n^{2/3} \) about \( d_+ \) (see [20] for the general \( \beta \) case):

\[
C_{\lambda,\beta}(\mu_1 - d_+) n^{2/3} \to TW_\beta, \tag{1.4}
\]

where the arrow denotes convergence in distribution, \( d_+ \) is as defined above, \( C_{\lambda,\beta} \) is a constant, and \( TW_\beta \) is the \( \beta \) version of the Tracy-Widom distribution.

**Motivation and recent related research:** In this paper we derive a CLT for the log determinant of \( \beta \)E matrices near the edge of the spectrum. More precisely, we study \( \log |\det(M_{n,m}/m - \gamma)| \) where \( \gamma := d_+ + \sigma_n n^{-2/3} \) for \( \sigma_n \) satisfying \(-\tau < \sigma_n \ll (\log n)^2 \) for some fixed \( \tau > 0 \). The motivation for this research question is two-fold, with applications in both statistics and spin glasses.

In high dimensional statistics, there is much interest in hypothesis testing for spiked models, i.e. matrices of the form \( M_n + hxx^* \) where \( M_n \) is a random matrix, \( h \) is a scalar, and \( x \) is a vector giving the direction of the spike (see, e.g. [14]). Laguerre beta ensembles are of particular interest in this context because of their connection to sample covariance matrices. The log determinant near the edge of the spectrum is useful in detecting the presence of a spike when \( h \) is small. Johnstone et al derive a CLT similar to ours for Gaussian beta ensembles (\( \beta \)E), which they also extend to Wigner ensembles with certain moment restrictions [12].

They used \( \beta \)E as a proxy for \( \beta \)E because they behave similarly but are less messy to analyze. Our paper confirms that, indeed, the CLT of the log determinant near the spectral edge of a \( \beta \)E matrix closely resembles that of a Wigner matrix, up to differences in the values of certain constants in the CLT formulas. Furthermore, in calculating these constants, we are able to make explicit the dependence of the CLT formula for \( \beta \)E on the parameter \( \lambda \).

Gaussian beta ensembles were also studied in this context by Lambert and Paquette [15], but via a different method. They prove that a rescaled version of the characteristic polynomial converges to a random function that can be characterized as a solution to the Stochastic Airy equation. From this convergence result, they obtain the CLT for the log determinant near the edge as a corollary.

In addition to the statistical motivation, this paper relates to questions of interest in spin glasses. Johnstone et al [13] and Landon [16] observe that the quantity \( \log |\det(M_n - s)| \) (with \( M_n \) being a scaled GOE matrix) appears in the calculations of the free energy of the spherical Sherrington-Kirkpatrick (SSK) spin glass model. Baik and Lee [2] showed in 2016 that the asymptotic fluctuations of the SSK free energy are Gaussian at high temperature but Tracy-Widom at low temperature. However, the nature of the free energy fluctuations near the critical temperature remained an open question, requiring a more detailed analysis of \( \log |\det(M_n - s)| \) in the case where \( s \) is near the spectral edge. The papers [13,16] analyze this critical case. Building on the findings of [12] and [15], they provide a free energy formula for SSK near the critical temperature that interpolates between the high temperature and low temperature cases.

Just as the edge CLT for the log determinant of GOE was needed to analyze the free energy of SSK at critical temperature, our result for Laguerre ensembles provides a necessary piece of information for the analysis of bipartite spherical spin glasses. As with the SSK model, the free energy of bipartite spherical spin
respectively), the CLT in Theorem 1.1 can be extended to hold for any $\sigma$ Theorem 1.2 (CLT at the edge)

As shown by Dumitriu and Edelman [6], the eigenvalue distribution of a $G$ in [12]. As shown by Dumitriu and Edelman [6], the eigenvalue distribution of a $G$ in [12]. As shown by Dumitriu and Edelman [6], the eigenvalue distribution of a $G$

Our contribution consists of two related Central Limit Theorems. Theorem 1.1 holds for general Laguerre beta ensembles and provides a CLT for the log determinant evaluated at a distance of $\sigma n^{-2/3}$ above the spectral edge where $\sigma$ is a slowly growing function (e.g. $\log n$). Theorem 1.2 extends this CLT all the way to the spectral edge in the cases of LUE and LOE.

**Theorem 1.1** (CLT slightly away from the edge). Let $M_{n,m}$ be a $L\beta E$ matrix where $n \leq m$ and $n/m = \lambda + O(n^{-1})$ as $n,m \to \infty$ for some $0 < \lambda < 1$. Define $\alpha = 2/\beta$. Let $D_n = \det(M_{n,m}/m - \gamma)$ where $\gamma = d_+ + \sigma n^{-2/3}$ with $(\log \log n)^2 \ll \sigma_n \ll (\log n)^2$ and $d_+$ denotes the upper edge of the limiting spectral distribution of $\frac{1}{m}M_{n,m}$. Then,

$$\log |D_n| - C_\lambda n - \frac{1}{3\lambda^{\beta/2}(1+\lambda^{\beta/2})^2\sigma_n^{1/3}} + \frac{2}{3\lambda^{\beta/2}(1+\lambda^{\beta/2})^2\sigma_n^{3/2}} + \frac{1}{6} (\alpha - 1) \log n \rightarrow N(0,1),$$

where

$$C_\lambda := (1 - \lambda^{-1}) \log(1 + \lambda^{1/2}) + \log(\lambda^{1/2}) + \lambda^{-1/2}.$$

**Theorem 1.2** (CLT at the edge). In the case where $M_{n,m}$ is from LUE or LOE ($\alpha = 1$ or $\alpha = 2$, respectively), the CLT in Theorem 1.1 can be extended to hold for any $\sigma_n$ satisfying $-\tau < \sigma_n \ll (\log n)^2$ for some fixed $\tau > 0$.

The majority of this paper is devoted to the proof of Theorem 1.1 after which the extension to Theorem 1.2 is accomplished in Section 7. Our proof of Theorem 1.1 is largely inspired by the proof of Theorem 2 in [12]. As shown by Dumitriu and Edelman [6], the eigenvalue distribution of a $G\beta E$ matrix is the same as that of a symmetric tridiagonal matrix. The key component of the proof of paper [12] is an analysis of a recurrence relations on the minors of the tridiagonal matrix. The recurrence relation is nonlinear with random coefficients. Johnstone et al were able to replace the nonlinear recurrence with a linear one with good error control and derived a CLT from the linear recurrence.

For $L\beta E$, the tridiagonal matrix representation is formed as a product of a bi-diagonal matrix and its transpose [6]. Similar to the proof of [12], we use this representation to arrive at a nonlinear recurrence relation, which we approximate by a linear one. However, unlike in the Gaussian case, our tridiagonal matrix has dependence between adjacent entries and the diagonal entries are not identically distributed. The more intricate structure of the matrix and the additional parameter $\lambda$ make the analysis of the recurrence significantly more technical. We outline the details of our proof of Theorem 1.1 in Subsection 2.2 after the set-up.

As in [12], the extension of Theorem 1.1 to Theorem 1.2 is first done in the case $\beta = 2$, relying on determinantal structures [10], then it is obtained for $\beta = 1$ using the inter-relationship between eigenvalues of unitary and orthogonal ensembles [8]. However, there is some subtlety in our case due to the singularity of the Marčenko–Pastur measure in the case $\lambda = 1$.

### 1.3 Organization of this paper and remarks on notations

The rest of the paper is organized as follows. Section 2 introduces key quantities, discusses sub-gamma random variables and concentration inequalities associated with them. In Section 3 we provide an asymptotic expression for the log determinant in terms of log of a rescaled determinant and a deterministic shift. In Section 4 we analyze a linear approximation of this log of the rescaled determinant. A CLT for the linear approximation is derived in Section 5. Error incurred from the linear approximation is shown to be negligible in Section 6. Taken together, Sections 2-6 complete the proof of Theorem 1.1. The extension of Theorem 1.1 to Theorem 1.2 is proved in Section 7. The Appendix contains proofs of some technical asymptotic estimates.
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2 Set-up and preliminary lemmas

2.1 Remarks on notation

We use several asymptotic notations throughout this paper and define our conventions here. Given a sequence \( \{a_n\} \) and a positive sequence \( \{b_n\} \), we write:

- \( a_n = O(b_n) \) if there exists some constant \( C \) such that \( |a_n| \leq C b_n \) for all \( n \),
- \( a_n = \Omega(b_n) \) if there exists some constant \( C \) such that \( |a_n| \geq C b_n \) for all \( n \),
- \( a_n = \Theta(b_n) \) if there exist constants \( C_1, C_2 \) such that \( C_1 b_n \leq |a_n| \leq C_2 b_n \) for all \( n \)
  (or, equivalently, \( a_n = O(b_n) \) and \( a_n = \Omega(b_n) \)),
- \( a_n \ll b_n \) if \( \lim_{n \to \infty} a_n / b_n = 0 \),
- \( a_n \gg b_n \) if \( \lim_{n \to \infty} b_n / a_n = 0 \).

Remark 2.1. Throughout the paper, we use \( C, C_1, C_2, c, c_1, c_2 \) in order to denote constants that are independent of \( N \). Even if the constant is different from one place to another, we may use the same notation \( C, C_1, C_2, c, c_1, c_2 \) as long as it does not depend on \( N \) for the convenience of the presentation.

Remark 2.2. Throughout the paper, we omit including \([\ ]\) and/or \([\ ]\) for floor and ceiling functions whenever a quantity that is seemingly not integer-valued is used as an integer. Instead, we implicitly apply floor function in all such cases. For example, \( \sum_{i=n/3}^{n/2} \) represents a sum over \( i \in \{ \lfloor n^{1/3} \rfloor, \lfloor n^{1/3} \rfloor + 1, \ldots, \lfloor n^{2/3} \rfloor \} \).

Remark 2.3. At various points throughout the paper, we replace \( n/m \) with \( \lambda \) without writing the \( O(n^{-1}) \) term to avoid cumbersome notation. This does not affect the computations as in all cases, the \( O(n^{-1}) \) term is small and gets absorbed into other error terms in the final approximation.

2.2 Set-up

As shown in [6], the eigenvalue distribution of a \( L \beta E \) matrix \( M_{n,m} \) is the same as that of the \( n \times n \) matrix \( T_n = BB^T \) where \( B \) is a bi-diagonal matrix of dimension \( n \times n \). More specifically,

\[
B = \begin{bmatrix}
  a_1 & b_1 & a_2 & b_2 & \cdots & a_n \\
  b_1 & b_2 & a_3 & b_3 & \cdots & b_{n-1} \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  b_1 & b_2 & b_3 & \cdots & b_{n-1} & a_n
\end{bmatrix}
\quad \text{so} \quad BB^T = \begin{bmatrix}
  a_1^2 & a_1 a_2 & a_2^2 & \cdots & a_n a_{n-1} \\
  a_1 a_2 & a_2^2 + b_1^2 & a_2 a_3 & \cdots & a_{n-1} a_{n-2} \\
  a_2 a_3 & a_3^2 + b_2^2 & a_3 a_4 & \cdots & a_{n-2} a_{n-3} \\
  \vdots & \ddots & \ddots & \cdots & \ddots \\
  a_{n-1} a_{n-2} & a_{n-1} a_{n-3} & a_{n-2} a_{n-4} & \cdots & a_2 a_1 \\
  a_n a_{n-1} & a_{n-1} a_{n-2} & a_{n-2} a_{n-3} & \cdots & a_1^2 + b_n^2
\end{bmatrix}
\] (2.1)

where the quantities \( \{a_i\}, \{b_i\} \) are all independent random variables with distributions satisfying

\[
a_i^2 \sim \frac{\alpha}{2} \chi^2 \left( \frac{2}{\alpha} (m-n+1) \right), \quad b_i^2 \sim \frac{\alpha}{2} \chi^2 \left( \frac{2}{\alpha} \right).
\] (2.2)

We observe that, while the entries of \( B \) are pairwise independent, \( T \) has dependence between adjacent entries. This is different from what occurs in the tridiagonalization of GOE/GUE matrices and it makes certain aspects of our computations more intricate than what is required in the Gaussian case.

We will find it useful to deal with a centered and rescaled version of the variables \( \{a_i\} \) and \( \{b_i\} \), so we introduce the notation

\[
d_i = \frac{a_i^2 - (m-n+1)}{\sqrt{m-n+i}}, \quad c_i = \frac{b_i^2 - i}{\sqrt{i}}.
\] (2.3)
To obtain our linear approximation of the recursion, we make the following observations:

\[ D_n := \det(T - \gamma m) \quad (2.4) \]

for \( \gamma \) as defined in the introduction. Let \( D_i \) be the determinant of the upper left \( i \times i \) minor of the matrix \( T - \gamma m \). Then the determinants satisfy the recursion

\[ D_i = (a_i^2 + b_{i-1}^2 - \gamma m)D_{i-1} - a_i^2 b_{i-1}^2 D_{i-2} \quad (2.5) \]

and, using our centered rescaled variables,

\[
D_i = (d_i \sqrt{m - n + i} + m - n + i + c_{i-1} \sqrt{i-1} + i - 1 - \gamma m)D_{i-1}
- (d_{i-1} \sqrt{m - n + i - 1} + m - n + i - 1)(c_{i-1} \sqrt{i-1} + i - 1)D_{i-2}.
\quad (2.6)
\]

We remark that the deterministic analog of this recursion is given by

\[ D_i' = (m - n + 2i - 1 - \gamma m)D_{i-1}' - (m - n + i - 1)(i - 1)D_{i-2}', \quad (2.7) \]

which has characteristic roots

\[ \rho_i^\pm = -\frac{1}{2} \left( \gamma m - (m - n + 2i - 1) \pm \sqrt{(\gamma m - (m - n + 2i - 1))^2 - 4(m - n + i - 1)(i - 1)} \right). \quad (2.8) \]

Observe that the roots \( \rho_i^+ \) and \( \rho_i^- \) are both negative for all \( i \). Their positive versions, \( |\rho_i^+| \) and \( |\rho_i^-| \), will be important throughout our analysis. To control the growth of \( D_i \), we introduce a normalized version of the recursion, following the approach used by Johnstone et al in the Gaussian case \cite{12}. In particular, we define

\[ E_i := \frac{D_i}{\prod_{j=1}^i |\rho_j^+|} \quad (2.9) \]

and obtain the recursion

\[
E_i = \frac{d_i \sqrt{m - n + i} + m - n + i + c_{i-1} \sqrt{i-1} + i - 1 - \gamma m}{|\rho_i^+|} E_{i-1}
- \frac{(d_{i-1} \sqrt{m - n + i - 1} + m - n + i - 1)(c_{i-1} \sqrt{i-1} + i - 1)}{|\rho_i^+| |\rho_{i-1}^-|} E_{i-2}.
\quad (2.10)
\]

We simplify this expression as

\[ E_i = \left( \alpha_i + \beta_i + \tau_i + \delta_i - \frac{\gamma m}{|\rho_i^+|} \right) E_{i-1} - (\alpha_{i-1} + \tau_{i-1})(\beta_i + \delta_i) E_{i-2}, \quad (2.11) \]

where

\[ \alpha_i = \frac{d_i \sqrt{m - n + i}}{|\rho_i^+|}, \quad \beta_i = \frac{c_{i-1} \sqrt{i-1}}{|\rho_i^+|}, \quad \tau_i = \frac{m - n + i}{|\rho_i^+|}, \quad \delta_i = \frac{i - 1}{|\rho_i^+|}. \quad (2.12) \]

We note that \( \tau_i \) and \( \delta_i \) are deterministic while \( \alpha_i \) and \( \beta_i \) are centered random variables with variance \( \alpha \tau_i/|\rho_i^+| \) and \( \alpha \delta_i/|\rho_i^+| \) respectively.

In the subsequent sections of this paper, we obtain a CLT for \( E_n \) and deduces a CLT for our original determinant. Our general approach, modeled after the methods in \cite{12}, is to approximate the recursion for \( E_i \) by a linear recursion. The authors \cite{12} observe that, in their setting, the ratio \( E_i/E_{i-1} \) is close to \(-1\) for all \( i \) when \( n \) is large. This observation holds in our setting as well (we note that \( E_i \) is non-zero since it is the rescaled characteristic polynomial of a minor of \( M_{n,m}/m \), evaluated at a point that is outside of the spectrum). Therefore, we define the quantity

\[ R_i := 1 + \frac{E_i}{E_{i-1}}, \quad (2.13) \]

and show is close to zero. Dividing the recursion \( (2.10) \) by \( E_{i-1} \) and rearranging terms, we obtain

\[ R_i = \left( \alpha_i + \beta_i + \tau_i + \delta_i + 1 - \frac{\gamma m}{|\rho_i^+|} \right) + (\alpha_{i-1} + \tau_{i-1})(\beta_i + \delta_i) \frac{1}{1 - R_{i-1}}. \quad (2.14) \]

To obtain our linear approximation of the recursion, we make the following observations:
• \( \frac{1}{1 - R_{i-1}} = 1 + \frac{R_{i-1}}{1 - R_{i-1}} = 1 + R_{i-1} + \frac{R_{i-1}^2}{1 - R_{i-1}} \).

• For any \( i \), we have \( \alpha_i, \beta_i, R_i \to 0 \) as \( m, n \to \infty \). This is easy to see for \( \alpha_i, \beta_i \) and not immediately obvious for \( R_i \), but we prove it later in the paper.

Using these observations, we rewrite the recursion for \( R_i \) as

\[
R_i = \xi_i + \omega_i R_{i-1} + \varepsilon_i,
\]

where

\[
\xi_i = \alpha_i + \beta_i (1 + \tau_{i-1}) + \alpha_{i-1} \delta_i,
\]

\[
\omega_i = \tau_{i-1} \delta_i,
\]

\[
\varepsilon_i = - \left( \gamma_i - \omega_i \right) + \alpha_{i-1} \beta_i + (\alpha_{i-1} \beta_i + \alpha_{i-1} \delta_i + \tau_{i-1} \beta_i) \frac{R_{i-1}}{1 - R_{i-1}} + \tau_{i-1} \delta_i \frac{R_{i-1}^2}{1 - R_{i-1}}.
\]

and \( \gamma_i = \frac{|\gamma_i|}{|\beta_i|} \) for \( 3 \leq i \leq n \).

We note that \( \{\xi_i\} \) are mean-zero random variables while \( \{\omega_i\} \) are deterministic and we will prove that \( \{\varepsilon_i\} \) are small. Thus, we can define a recursion on a new sequence of variables \( L_i \), which we will show are a good approximation of \( R_i \). We define \( L_i \) to satisfy

\[
L_i := \xi_i + \omega_i L_{i-1} \text{ for } i \geq 4, \quad L_3 := \xi_3.
\]

From this recursive definition,

\[
L_j = \sum_{i=3}^{j-1} \xi_i \omega_{i+1} \omega_{i+2} \cdots \omega_j + \xi_j, \quad \text{for } j \geq 4.
\]

It is important (in showing CLT) to express \( L_j \) as a sum of independent random variables, yet we have dependence between consecutive terms in the sequence \( \{\xi_i\} \). To address this issue, we expand \( \xi_i \) using (2.16) to have

\[
L_j = \sum_{i=3}^{j-1} \omega_{i+1} \cdots \omega_j X_i + X_j + \alpha_j - \omega_3 \cdots \omega_j \alpha_2,
\]

where

\[
X_i = (1 + \tau_{i-1}) (\delta_i \alpha_{i-1} + \beta_i), \quad 3 \leq i \leq n.
\]

Note that, unlike \( \xi_i \), the variables \( X_i \) are pairwise independent. In later calculations, it is more convenient to work with \( Y_i \) rather than with \( L_i \), where \( Y_i \) is given by

\[
Y_i = \sum_{j=3}^{i-1} \omega_{j+1} \cdots \omega_j X_j + X_i, \quad 3 \leq i \leq n.
\]

With this set-up, our proof of Theorem 1.1 consists of the following key steps:

1. First, we write the log determinant of \( T_n - \gamma_m \) in terms of log of the rescaled quantity \( |E_n| \), asymptotically as \( n \) goes to infinity.

2. We then show that in the regime \( (\log \log n)^2 \ll \sigma_n \ll (\log n)^2 \), with probability \( 1 - O(n^{-1}) \), both \( \max_i |L_i| \) and \( \max_i |R_i| \) are \( o(n^{-1/3}) \). Thus Taylor’s approximation for logarithm is applied to obtain

\[
\log |E_n| = \sum_{i=3}^{n} \log |1 - R_i| + \log |E_2| = \sum_{i=3}^{n} (-R_i - R_i^2/2) + o(1),
\]

with probability \( 1 - O(n^{-1}) \).
3. With probability $1 - O(n^{-1})$, we have $\sum_{i=3}^{n} (-R_i - R_i^2/2)$ is $-\sum_{i=3}^{n} L_i$ plus a deterministic shift, up to an error of order $\sqrt{\log n}$.

4. Lastly, we show $-\sum_{i=3}^{n} L_i$ has variance of exact order $\log n$, and satisfies Lyapunov’s CLT.

While this general outline has close resemblance to that of the Gaussian case [12], each step involves more technical treatment due to the complicated structure of the recurrence relations. Before proceeding with these steps, we examine properties of the quantities introduced in this section.

## 2.3 Properties of sub-gamma random variables

It is central in our analysis that error due to linear approximation and similar reductions are negligible. In most instances, these error terms appear as sum of independent random variables that behave similarly to sub-gaussian random variables, known as sub-gamma families.

**Definition 2.4.** For $v, u > 0$, a real-valued centered random variable $X$ is said to belong to a sub-gamma family $SG(v, u)$ if for all $t \in \mathbb{R}$ such that $|t| < \frac{1}{u}$,

$$Ee^{tX} \leq \exp\left(\frac{t^2v}{2(1-tu)}\right).$$

(2.24)

The following properties of sub-gamma random variables are useful for our analysis.

- If $X \sim \chi^2(d)-d$, then $X \in SG(2d, 2)$
- Given a real number $c$ and $X \in SG(v_X, u_X)$, $cX \in SG(c^2v_X, |c|u_X)$
- If $X \in SG(v_X, u_X)$ and $Y \in SG(v_Y, u_Y)$ are independent, then $X + Y \in SG(v_X + v_Y, u_X \lor u_Y)$

We verify that for $i = 3, \ldots, n$, the random variables $\alpha_i$ and $\beta_i$ as defined in (2.12), and their linear combination $X_i$ belong to sub-gamma families.

**Lemma 2.5.** For $i = 3, \ldots, n$,

$$\alpha_i \in SG\left(\frac{\alpha\tau_i}{|\rho_i^+|^2}, \frac{\alpha}{|\rho_i^+|^2}\right), \quad \beta_i \in SG\left(\frac{\alpha\delta_i}{|\rho_i^-|^2}, \frac{\alpha}{|\rho_i^-|^2}\right), \quad X_i \in SG(v_i, u_i),$$

where

$$v_i = \frac{\alpha\delta_i}{|\rho_i^-|^2}(\omega_i + 1)(1 + \tau_{i-1})^2, \quad u_i = \frac{\alpha(1 + \tau_{i-1})}{|\rho_i^+|^2}.$$  

(2.25)

In the subsequent sections, both characterizations of sub-gamma random variables in terms of tail probabilities, and in terms of $p$-norms for $p \geq 1$ are used. In particular, we regularly apply the following result.

**Lemma 2.6.** (see Theorem 2.3 of [4])

If $X$ belongs to $SG(v, u)$, then for every $t > 0$,

$$\mathbb{P}(|X| > \sqrt{2vt + ut}) \leq 2e^{-t}.$$  

(2.26)

In addition, for every integer $p \geq 2$,

$$\|X\|_p = \mathbb{E}[|X|^p] \leq (p/2)!/(8v)^{p/2} + pl(4u)^p.$$  

(2.27)

## 2.4 Preliminary lemmas concerning the values of $\rho_i^+$, $\rho_i^-$, and $\omega_i$

We begin by observing that $|\rho_i^+|$ is a decreasing function of $i$ and $|\rho_i^-|$ is an increasing function of $i$. Other key properties are captured in the following lemma.

**Lemma 2.7.** The quantities $|\rho_i^+|$ and $|\rho_i^-|$ satisfy the following asymptotic bounds, uniformly in $i$:

(i) $|\rho_i^+| = \Theta(n)$,
(ii) \( |\rho_i^+| - |\rho_i^-| = \Omega(n^{2/3}\sigma_n^{1/2}) \),

(iii) \( |\rho_i^-| - |\rho_{i-1}^-| = O(n^{1/3}\sigma_n^{-1/2}) \) and \( |\rho_i^+| - |\rho_{i-1}^+| = O(n^{1/3}\sigma_n^{-1/2}) \),

(iv) \( \frac{|\rho_i^-|}{|\rho_{i-1}^-|} - \frac{|\rho_{i-1}^-|}{|\rho_i^-|} = O(n^{-2/3}\sigma_n^{-1/2}) \).

Proof. To show (iii), for the lower bound, we have

\[
|\rho_i^-| \geq |\rho_{n+1}^-| > \frac{1}{2} (\gamma m - (m + n - 1)) = \frac{1}{2} \left( 2\sqrt{mn} + \lambda^{-1}\sigma_n n^{1/3} + 1 \right) = \Omega(n).
\]

For the upper bound, we have

\[
|\rho_i^+| \leq |\rho_{n+1}^+| = \gamma m - (m + n - 1) = 2\sqrt{mn} + 2n + \lambda^{-1}\sigma_n n^{1/3} - 1 = O(n).
\]

For (ii), we have

\[
|\rho_i^+| - |\rho_i^-| > |\rho_{i-1}^+| - |\rho_{i-1}^-| = \sqrt{2\lambda^{-3/2}\sigma_n n^{4/3} + O(n)} = \Omega(n^{2/3}\sigma_n^{1/2}).
\]

For (i), it suffices to show that \( |\rho_i^-| - |\rho_{i-1}^-| + |\rho_i^+| - |\rho_{i-1}^+| = O(n^{1/3}\sigma_n^{-1/2}) \). This quantity can be rewritten as \( (|\rho_{i-1}^-| - |\rho_{i-1}^-|) - (|\rho_i^+| - |\rho_i^-|) \), which is the difference of two square root expressions. Thus,

\[
(\frac{(|\rho_{i-1}^+| - |\rho_{i-1}^-|)^2 - (|\rho_i^+| - |\rho_i^-|)^2}{|\rho_{i-1}^+| - |\rho_{i-1}^-| + |\rho_i^+| - |\rho_i^-|}) \leq \Omega \left( \frac{(|\rho_{i-1}^+| - |\rho_{i-1}^-|)^2 - (|\rho_i^+| - |\rho_i^-|)^2}{n^{2/3}\sigma_n^{1/2}} \right).
\]

Since the numerator inside the big-O term simplifies to \( 4\gamma m - 4 = O(n) \), part (iii) of the lemma follows. Lastly, since

\[
\frac{|\rho_i^-|}{|\rho_i^+|} = \frac{|\rho_{i-1}^-|}{|\rho_{i-1}^+|} = \frac{1}{|\rho_i^+|} \left( |\rho_i^-| - |\rho_{i-1}^-| \right) + \frac{|\rho_{i-1}^-|}{|\rho_i^+|} \left( |\rho_i^+| - |\rho_{i-1}^+| \right) < \frac{|\rho_i^-| - |\rho_{i-1}^-| + |\rho_i^+| - |\rho_{i-1}^+|}{|\rho_i^+|}.
\]

applying parts (ii) and (iii) of the lemma to this inequality, we obtain (iv).

Since \( \omega_i = |\rho_i^-|/|\rho_{i-1}^-| \) for \( i = 3, \ldots, n \), we know \( \omega_i \) takes values in \((0, 1)\) and is increasing in \( i \). Furthermore, the \( i \)-dependent asymptotic descriptions of \( \omega_i \) as \( n \to \infty \) can also be obtained from the equation, as in the following lemma.

**Lemma 2.8.** For \( i \leq n \) satisfying \( i \to \infty \) as \( n \to \infty \), the value of \( \omega_i \) satisfies the following asymptotic expressions.

(i) \( \text{If } n - i \ll n^{1/3}\sigma_n, \omega_i = 1 - 2\lambda^{-1/4}n^{-1/3}\sigma_n^{1/2} \left( 1 + O(n^{-1/3}\sigma_n^{1/2}) \right) \).

(ii) \( \text{If } n - i = \Theta(n^{1/3}\sigma_n), \omega_i = 1 - 2 \left( \lambda^{-1/2} + (\lambda^{1/2} + 1)^2 \cdot \frac{n-i}{n^{1/3}\sigma_n} \right)^{1/2} n^{-1/3}\sigma_n^{1/2} \left( 1 + O(n^{-1/3}\sigma_n^{1/2}) \right) \).

(iii) \( \text{If } n^{1/3}\sigma_n \ll n - i \ll n, \omega_i = 1 - 2(1 + \lambda^{1/2}) \left( \frac{n-i}{n} \right)^{1/2} \left( 1 + O((n-i)^{1/2}) \right) \).

(iv) \( \text{If } n - i = \Theta(n), \omega_i = \frac{\lambda^{-1/2} + n-i-(\lambda^{-1/2}+1)(n-i)^{1/2}}{\lambda^{-1/2} + n-i/(n^{1/3}\sigma_n) + (\lambda^{-1/2}+1)(n-i)^{1/2}} \left( 1 + o(1) \right) \).
Lemma 2.10. There exists constant $C > 0$ such that for sufficiently large $n$, we have

$$\gamma_i - \omega_i < \frac{C}{n(1 - \omega_i)}, \quad \text{for every } 3 \leq i \leq n.$$  \hspace{1cm} (2.38)

Proof. We have the relation

$$\gamma_i - \omega_i = \frac{\omega_i}{|\rho_i^+|} (|\rho_i^+| - |\rho_i^-|).$$  \hspace{1cm} (2.38)

Uniformly in $i \leq n$, $|\rho_i^+| = \Theta(n)$ and $\omega_i \in (0, 1)$, so it suffices to show $|\rho_i^+| - |\rho_i^-| = O(1/\omega_i)$. Define for $3 \leq i \leq n$,

$$U_i = (\gamma m - (m - n + 2i - 1))^2 - 4(i - 1)(m - n + i - 1).$$  \hspace{1cm} (2.39)

Then $U_{i-1} - U_i = 4(\gamma m - 1)$, and by (2.8),

$$|\rho_i^+| = \frac{1}{2} \left( \gamma m - (m - n + 2i - 1) + \sqrt{U_i} \right).$$  \hspace{1cm} (2.40)
We then note that \( \frac{\sqrt{\gamma_j}}{\gamma_j - (m_i n + 2i) - 1} = m_i^+ - 1 \) by (2.34) to arrive at
\[
|\rho_{i-1}^+| - |\rho_i^-| = 1 + \frac{2(\gamma_j m_i - 1)}{\sqrt{U_i} + \sqrt{U_i}} = 1 + \frac{2(\gamma_j m_i - 1)}{(m_i^+ - 1)(1 + \frac{4(\gamma_j m_i - 1)}{U_i})}.
\]

(2.41)

Using the asymptotics \( \gamma = (1 + \sqrt{\lambda})^2 + \sigma_n n^{-2/3} \) as \( n \to \infty \),
\[
U_i = 4(\lambda - \frac{1}{2}) + 1)^2 2^2 \left[ \frac{n - i}{n} + 4\lambda^{-1} \left( \frac{\gamma_i}{\lambda_i} + \frac{n - i}{n} \right) \sigma_n n^{-\frac{2}{3}} + o(\sigma_n n^{-\frac{2}{3}}) \right].
\]

(2.42)

Thus, the ratio on the right hand side of (2.41) satisfies that its numerator is \( O(1) \) while the expression under the square root in the denominator is \( 1 + O(n^{-1}) \). Both the big-O bounds are uniformly in \( i \). Hence, the right hand side of (2.41) is of order \( 1 + \frac{1}{m_i^+ - 1} \), where \( \frac{1}{m_i^+ - 1} \geq 1 \), by the definition of \( m_i^+ \). Therefore,
\[
|\rho_{i-1}^+| - |\rho_i^-| = O\left( \frac{1}{m_i^+ - 1} \right).
\]

Since \( m_i^+ + m_i^- = 2 \),
\[
\frac{1}{m_i^+ - 1} = \frac{2}{m_i^+ (1 - \gamma_i)} = \frac{2}{m_i^+ (1 - \gamma_i)} \left( \frac{1 - \gamma_i - \omega_j}{1 - \omega_i} \right) = \frac{2/\gamma_i^+}{1 - \omega_i} \left( 1 + O\left(n^{-\frac{2}{3}}\sigma_n^{-1}\right) \right),
\]

following from (2.37) and Corollary 2.9. We conclude \( |\rho_{i-1}^+| - |\rho_i^-| = O\left( \frac{1}{1 - \omega_i} \right) \). \( \square \)

One other quantity that comes up frequently throughout our calculations is the variance \( \text{Var}(X_i^2) \). In the following lemma, we give upper and lower bounds for this quantity.

**Lemma 2.11.** The variance of \( X_i^2 \) satisfies the following properties for \( 3 \leq i \leq n \):

(i) \( \text{Var}(X_i^2) = \Theta\left( \frac{\sigma_i}{n} \right) \) for all \( i \),

(ii) \( \text{Var}(X_i^2) = O(n^{-1}) \) uniformly in \( i \),

(iii) \( \text{Var}(X_i^2) = \Omega(n^{-2}) \) uniformly in \( i \).

**Proof.** From (2.22), we have
\[
\text{Var}(X_i^2) = (1 + \tau_{i-1})^2 \text{Var}(\alpha_i \gamma_{i-1} + \beta_i) = \alpha \delta_i (1 + \tau_{i-1})^2 \left( \frac{\omega_i}{|\rho_i^-|} + \frac{1}{|\rho_i^+|} \right).
\]

(2.43)

By Lemma 2.7 \( |\rho_i^-|^{-1} = \Theta(n^{-1}) \). Furthermore, it follows directly from definitions that \( \tau_i, \omega_i \) are positive and bounded above by a constant, uniformly in \( i \). This yields part (i) of the lemma. Parts (ii) and (iii) follow from the fact that \( \delta_i = \frac{\omega_i}{|\rho_i^-|} = \Theta\left( \frac{\omega_i}{|\rho_i^-|} \right) \). \( \square \)

### 3 Expressing \( \log |D_n| \) in terms of \( \log |E_n| \)

Our goal in this section is to obtain a closed form asymptotic expansion for the quantity \( \log |D_n| - \log |E_n| \), accurate down to order \( O(1) \). We will use this to obtain a CLT for \( \log |D_n| \) in terms of a CLT for \( \log |E_n| \).

**Lemma 3.1.** Assume \( \gamma = (1 + \sqrt{\lambda})^2 + \sigma_n n^{-2/3} \) for \( (\log \log n)^2 \ll \sigma_n \ll (\log n)^2 \). The quantity \( \log |D_n| - \log |E_n| \) has the asymptotic expansion
\[
\log |D_n| - \log |E_n| = C_\lambda n + \frac{1}{\lambda^{1/2}(1 + \lambda^{1/2})} \sigma_n n^{1/3} = \frac{2}{3\lambda^{1/4}(1 + \lambda^{1/2})^2} \sigma_n^{3/2} + O(1),
\]

(3.1)

where
\[
C_\lambda := (1 - \lambda^{-1}) \log(1 + \lambda^{1/2}) + \log(\lambda^{1/2}) + \lambda^{-1/2}.
\]

(3.2)
Proof. It follows from (2.9) that
\[ E_n = \frac{m^n D_n}{\prod_{i=1}^n |\rho_i^+|}. \] (3.3)

Expanding \(|\rho_i^+|\) using (2.8), we obtain
\[ D_n = E_n \prod_{i=1}^n \left( \frac{1}{2} (\gamma - (1 - \lambda)) - \frac{i-\frac{1}{2}}{m} + \sqrt{\left( \frac{1}{2} (\gamma - (1 - \lambda)) - \frac{i-\frac{1}{2}}{m} \right)^2 - \left( 1 - \lambda + \frac{i-1}{m} \right) \left( \frac{i-1}{m} \right)} \right). \] (3.4)

Thus,
\[ \log |D_n| - \log |E_n| = \sum_{i=1}^n \log \left( \frac{1}{2} (\gamma - (1 - \lambda)) - \frac{i-\frac{1}{2}}{m} + \sqrt{\left( \frac{1}{2} (\gamma - (1 - \lambda)) - \frac{i-\frac{1}{2}}{m} \right)^2 - \left( 1 - \lambda + \frac{i-1}{m} \right)} \right). \] (3.5)

Observe that the argument of the log is bounded away from zero, since it is equal to \(|\rho_i^+|/m\) where \(|\rho_i^+| = \Theta(n)\) by Lemma 2.7. For large \(n\), we approximate the above sum by the integral
\[ \frac{n}{\lambda} \int_0^\lambda \log \left( c - x + \sqrt{(c-x)^2 - (1-\lambda+x)x} \right) dx \] (3.6)
for \(c = \frac{1}{2} (\gamma - (1 - \lambda))\), incurring an error of order \(O(1)\) in the process. Note that
\[ c - x + \sqrt{(c-x)^2 - (1-\lambda+x)x} = \left( \sqrt{\frac{c^2}{\gamma} - x + r_+} \right) \left( \sqrt{\frac{c^2}{\gamma} - x + r_-} \right), \]
where \(r_\pm = \frac{1}{2} (\gamma \pm (1-\lambda)) \gamma^{-1/2}\). For every \(s \in \mathbb{R}\),
\[ \int \log(\sqrt{y} + s)dy = (y-s^2) \log(\sqrt{y} + s) - \frac{1}{2} y + s \sqrt{y} + C. \]

Thus, using \(s = r_\pm\) together with the change of variable \(y = \frac{c^2}{\gamma} - x\), we find that (3.6) is equal to
\[ \frac{n}{\lambda} A = \frac{n}{\lambda} (A_1 + A_2 + A_3 + A_4 + A_5), \] (3.7)
where
\[ A_1 = (a + \lambda - r_2) \log(\sqrt{a + \lambda} + r_+), \quad A_2 = -(a - r_+^2) \log(\sqrt{a} + r_+), \quad A_3 = (a + \lambda - r_2) \log(\sqrt{a + \lambda} + r_-), \quad A_4 = -(a - r_-^2) \log(\sqrt{a} + r_-), \quad A_5 = -\lambda + (r_+ + r_-)(\sqrt{a + \lambda} - \sqrt{a}), \] (3.8)
and \(a = \frac{c^2}{\gamma} - \lambda\). Therefore,
\[ \log |D_n| - \log |E_n| = \frac{n}{\lambda} A + O(1). \] (3.9)

We now evaluate each of \(A_i\) asymptotically, using \(\gamma = (1 + \sqrt{\lambda})^2 + \sigma_n n^{-2/3}\) as given. Setting \(\Delta_n := \frac{\sigma_n n^{-2/3}}{(1+\sqrt{\lambda})^{2/3}}\), we have
\[ r_+ = 1 + \frac{1}{2} \lambda^{1/2} \Delta_n + O(\Delta_n^2), \quad r_- = \lambda^{1/2} + \frac{1}{2} \Delta_n + O(\Delta_n^2), \quad a = \lambda^{1/2} \Delta_n + \frac{1}{4}(1 - \lambda^{1/2})^2 \Delta_n^2 + O(\Delta_n^3). \]
Therefore, \( A_3 = O(\Delta_n^2) \), and
\[
A_1 = (\lambda - 1) \left( \log(1 + \lambda^{1/2}) + \frac{1}{2} \Delta_n \right) + O(\Delta_n^2),
\]
\[
A_2 = \lambda^{1/4} \Delta_n^{1/2} + \left( \frac{1}{2} \lambda^{-1/4} - \frac{1}{2} \lambda^{1/4} - \frac{1}{24} \lambda^{3/4} \right) \Delta_n^{3/2} + O(\Delta_n^2),
\]
\[
A_4 = \lambda \log(\lambda^{1/2}) + \lambda^{3/4} \Delta_n^{1/2} + \left( \frac{11}{24} \lambda^{-1/4} - \frac{3}{2} \lambda^{3/4} + \frac{1}{8} \lambda^{5/4} \right) \Delta_n^{3/2} + O(\Delta_n^2),
\]
\[
A_5 = \lambda^{1/2} - \lambda^{1/4} (1 + \lambda^{1/2}) \Delta_n^{1/2} + \frac{1}{2} (1 + \lambda^{1/2})^2 \Delta_n = \frac{1}{8 \Delta_n^3} (1 + \lambda^{1/2})^3 \Delta_n^{3/2} + O(\Delta_n^2).
\]
Substituting the values of \( A_i \) into (3.7), then by (3.9), we obtain the statement (3.1) as in the lemma.

We now move to the step of approximating \( \log |E_n| \).

## 4 Linear approximation for \( \log |E_n| \)

Recall Definition 2.13 of \( R_i \). Assuming that \( R_i \) for \( 3 \leq i \leq n \) are \( o(n^{-1/3}) \) uniformly in \( i \), then Taylor expansion of the logarithm implies
\[
\log |E_n| = \sum_{i=3}^{n} \log |1 - R_i| + \log |E_2| = \sum_{i=3}^{n} (-R_i - R_i^2/2 + o(n^{-1})) + \log |E_2|. 
\]  (4.1)

The following lemma shows that the uniform bound of \( R_i \) indeed holds.

**Lemma 4.1.** Assume \( (\log \log n)^2 \ll \sigma_n \ll (\log n)^2 \). With probability \( 1 - O(\log^{-\delta} n) \),
\[
\max_{2 \leq i \leq n} |R_i| = o(n^{-1/3}).
\]

We include its proof in Section 6. Assuming the lemma, we rewrite (2.18) as
\[
\varepsilon_i = - (\gamma_i - \omega_i) + \alpha_i \beta_i + (\alpha_i \beta_i + \alpha_i \delta_i + \tau_i \beta_i) \frac{R_{i-1}}{1 - R_{i-1}} + \omega_i \frac{R_{i-1}^3}{1 - R_{i-1}} + \omega_i R_{i-1}^2, 
\]  (4.2)
and set for \( 3 \leq i \leq n \),
\[
R_i^{(1)} = \frac{R_{i-1}}{1 - R_{i-1}}, \quad R_i^{(2)} = \omega_i \frac{R_{i-1}^3}{1 - R_{i-1}}, \quad R_i^{(3)} = \omega_i R_{i-1}^2. 
\]  (4.3)

Then from the recursion (2.15), we obtain the decomposition
\[
R_i = L_i + \omega_i \ldots \omega_3 R_2 - A_{0i} + B_{0i} + B_{1i} + B_{2i} + B_{3i}, 
\]  (4.4)
where
\[
A_{0i} = \gamma_i - \omega_i + \omega_i (\gamma_i - \omega_i - 1) + \cdots + \omega_i \ldots \omega_i (\gamma_3 - \omega_3), 
\]  (4.5)
and
\[
B_{0i} = \left( \alpha_i - 1 + (\tau_i - 1 + \alpha_i) R_i^{(1)} \right) \beta_i + \omega_i \left( \alpha_i - 2 + (\tau_i - 2 + \alpha_i) R_i^{(1)} \right) \beta_i - 1 + \cdots + \omega_i \ldots \omega_4 (\alpha_2 + (\tau_2 + \alpha_2) R_3^{(1)}) \beta_3,
\]
\[
B_{1i} = \alpha_i \delta_i R_i^{(1)} + \omega_i \alpha_i \delta_i - 1 R_i^{(1)} + \cdots + \omega_i \ldots \omega_i \omega_i \alpha_i \delta_i R_3^{(1)},
\]
\[
B_{2i} = R_i^{(2)} + \omega_i R_i^{(1)} + \cdots + \omega_i \ldots \omega_i R_3^{(2)},
\]
\[
B_{3i} = R_i^{(3)} + \omega_i R_i^{(1)} + \cdots + \omega_i \ldots \omega_i R_3^{(3)}. 
\]
Substituting this into expression for \( \log |E_n| \), we have
\[
\log |E_n| = - \sum_{i=3}^{n} L_i + \sum_{i=3}^{n} A_{0i} - \sum_{i=3}^{n} B_{3i} - \sum_{i=3}^{n} (\omega_i \ldots \omega_3 R_2 + B_{0i} + B_{1i} + B_{2i}) - \frac{1}{2} \sum_{i=3}^{n} R_i^2 + \log |E_2| + o(1). 
\]  (4.6)

The following three lemmas state that the last three quantities in (4.6) are \( O(1) \) with probability \( 1 - o(1) \). Their proofs are included in the Appendix.
Lemma 4.2. \( \sum_{i=3}^n R_i^2 = O(1) \) with probability \( 1 - o(1) \).

Lemma 4.3. \( \sum_{i=3}^n \omega_i \ldots \omega_3 R_2 + B_{0i} + B_{1i} + B_{2i} = O(1) \) with probability \( 1 - o(1) \).

Lemma 4.4. \( \log |E_2| = O(1) \) with probability \( 1 - o(1) \).

The above lemmas imply that the main contribution to \( \log |E_n| \) comes from the first three sums \( \sum_{i=3}^n L_i \), \( \sum_{i=3}^n A_{0i} \), and \( \sum_{i=3}^n B_{0i} \). We turn to the tasks of computing the second and third sums in this section, and study the first sum \( \sum_{i=3}^n L_i \) in Section 5.

Definition 4.5. Given integer \( n \), define sequence \( \{g_i\}_{i=3}^{n+1} \) by the recurrence
\[
g_{n+1} = 1, \quad g_i = 1 + \omega_i g_{i+1}.
\]
That is, \( g_i = 1 + \omega_i + \omega_i \omega_{i+1} + \cdots + \omega_i \ldots \omega_n \) for \( 3 \leq i \leq n \).

Lemma 4.6.
\[
\sum_{i=3}^n A_{0i} = \frac{1}{6} \log n + O(\log \log n). \quad (4.7)
\]

Proof. We prove the lemma by computing an upper and a lower bound for the sum \( \sum_{i=3}^n A_{0i} \). Observe that
\[
\sum_{i=3}^n A_{0i} = \sum_{i=3}^n g_i (\gamma_i - \omega_i) > \sum_{i=n-n\nu_n^1}^{n} g_i (\gamma_i - \omega_i), \quad (4.8)
\]
for any slowing increasing sequence \( \nu_n \). For the purpose of this proof, it suffices to take \( \nu_n = \log \log n \).

The indices \( i \) in the sum on the right hand side of (4.8) satisfies \( i < n - n^{1/3} \), so by Lemma 5.1(i), \( g_{i+1} > \frac{1 - \log^2 n}{1 - \omega_{i+1}} \) for sufficiently large \( n \). We obtain
\[
\sum_{i=3}^n A_{0i} > (1 - \log^{-2} n) \sum_{i=n-n\nu_n^1}^{n} \frac{\gamma_i - \omega_i}{1 - \omega_{i+1}}. \quad (4.9)
\]

Recall that \( \gamma_i - \omega_i = \omega_i \frac{\rho_i^+ - \rho_i^-}{\rho_i^+} \). Since \( n^{1/3} \sigma_n \ll n - i \ll n \), Lemma 2.8(ii) implies \( \omega_i = 1 + O \left( \sqrt{\frac{n-i}{n}} \right) \) and
\[
(1 - \omega_{i+1})^{-1} = \frac{1}{2(\lambda^2 + 1)} \left( \frac{n-i}{n} \right)^{-\frac{1}{2}} \left( 1 + O \left( \sqrt{\frac{n-i}{n}} \right) \right). \quad (4.10)
\]

We now study the factor \( \frac{\rho_i^+ - \rho_i^-}{\rho_i^+} \). From the second display of (2.41),
\[
|\rho_{i-1}^+| - |\rho_i^+| = 1 + \frac{2(\gamma m - 1)}{U_{i-1} + \sqrt{U_i}} = 1 + \frac{2(\lambda + 1)^2 n (1 + O (\sigma_n n^{-2/3}))}{\sqrt{U_{i-1} + \sqrt{U_i}}},
\]
where \( U_i \) is defined in (2.39). By (2.42), we get
\[
\sqrt{U_i} = 2(\lambda + 1)^2 n \left( \frac{n-i}{n} \right)^{\frac{1}{2}} \left( 1 + O \left( \frac{n-i}{n} \right) \right), \quad (4.11)
\]
noting that \( (\frac{n-i}{n})^{-1} \geq \nu_n \). Together with the asymptotics
\[
\gamma m - (m-n+2i-1) = 2n \left( \lambda^{-1/2} + O \left( \frac{n-i}{n} \right) \right), \quad (4.12)
\]
it follows that
\[
|\rho_i^+| = \frac{1}{2} \left( \gamma m - (m-n+2i-1) + \sqrt{U_i} \right) = \lambda^{-1/2} n \left( 1 + O \left( \sqrt{\frac{n-i}{n}} \right) \right), \quad (4.13)
\]
Thus,
\[ \frac{|\rho^+_{i-1}| - |\rho^+_i|}{|\rho^+_i|} = \frac{1 + \lambda^{1/2}}{2} \frac{1}{n} \left( \frac{n-i}{n} \right)^{-\frac{1}{2}} \left( 1 + O \left( \sqrt{\frac{n-i}{n}} + n^{-\frac{2}{3}} \sigma_n \left( \frac{n-i}{n} \right)^{-1} \right) \right). \] (4.14)

Combine (4.10) and (4.14), we get
\[ \frac{\gamma_i - \omega_i}{1 - \omega_{i+1}} = \frac{1}{4n} \left( \frac{n-i}{n} \right)^{-1} \left( 1 + O \left( \sqrt{\frac{n-i}{n}} + n^{-\frac{2}{3}} \sigma_n \left( \frac{n-i}{n} \right)^{-1} \right) \right). \] (4.15)

Therefore, by (4.9),
\[ \sum_{i=3}^{n} A_{3i} > \sum_{i=n-n\nu}^{n-n^{1/3}\sigma_n\nu_n} \frac{1}{4n} \left( \frac{n-i}{n} \right)^{-1} \left( 1 + O \left( \sqrt{\frac{n-i}{n}} + n^{-\frac{2}{3}} \sigma_n \left( \frac{n-i}{n} \right)^{-1} \right) \right). \] (4.16)

Since
\[ \sum_{i=n-n\nu}^{n-n^{1/3}\sigma_n\nu_n} \frac{1}{n} \left( \frac{n-i}{n} \right)^{-\frac{1}{2}} = O(\nu^{-1/2}), \quad \text{and} \quad \sum_{i=n-n\nu}^{n-n^{1/3}\sigma_n\nu_n} \frac{1}{n} \left( \frac{n-i}{n} \right)^{-2} = O(\nu^{-1}), \]
we obtain the lower bound \( \sum_{i=3}^{n} A_{3i} > \frac{1}{6} \log n + O(\log \log n). \)

It remains to show \( \sum_{i=3}^{n} A_{3i} < \frac{1}{6} \log n + O(\log \log n). \) Since \( \omega_i \) and \( \frac{|\rho^+_{i-1}| - |\rho^+_i|}{|\rho^+_i|} \) are both increasing in \( i \), \( \gamma_i - \omega_i \) is also increasing in \( i \). Thus, it follows from (4.5) that \( A_{3i} < \frac{\gamma_i - \omega_i}{1 - \omega_i} \) for every \( i = 3, 4, \ldots, n \) and so, \( \sum_{i=3}^{n} A_{3i} < \sum_{i=3}^{n} \frac{\gamma_i - \omega_i}{1 - \omega_i}. \) By Lemma 2.10 and Corollary 2.9, the following three statements hold:
\[ \sum_{i=3}^{n-n\nu_n^{-1}} \frac{\gamma_i - \omega_i}{1 - \omega_i} = O(\log \nu_n), \] (4.17)
\[ \sum_{i=n-n^{1/3}\sigma_n\nu_n}^{n-n^{1/3}\sigma_n} \frac{\gamma_i - \omega_i}{1 - \omega_i} = O(\log \nu_n), \] (4.18)
\[ \sum_{i=n-n^{1/3}\sigma_n}^{n} \frac{\gamma_i - \omega_i}{1 - \omega_i} = O(1). \] (4.19)

Thus, we obtain
\[ \sum_{i=3}^{n} A_{3i} < \sum_{i=n-n\nu_n^{-1}}^{n-n^{1/3}\sigma_n\nu_n} \frac{\gamma_i - \omega_i}{1 - \omega_i} + O(\log \nu_n), \] (4.20)
where the sum over \( i \) on the right hand side is \( \frac{1}{6} \log n + O(\log \log n) \) by (4.15). This completes the proof of the lemma. \( \square \)

We now study contribution from the sum \( \sum_{i=3}^{n} B_{3i} \). The following lemma states that \( \sum_{i=3}^{n} B_{3i} \) is close to \( \sum_{i=3}^{n} B_{3i}^* \), where
\[ B_{3i}^* = (\omega_i L^2_{i-1}) + \omega_i(\omega_{i-1}L^2_{i-2}) + \cdots + \omega_i \omega_4 L^2_3. \]

**Lemma 4.7.** With probability \( 1 - o(1) \), \( \sum_{i=3}^{n} B_{3i} - B_{3i}^* = O(1). \)
The new sum is much simpler, and we turn now to the task of computing it. We begin by observing that $\sum_{i=3}^{n} B_{3i}$ can be rewritten as

$$\sum_{i=3}^{n} B_{3i} = \sum_{i=4}^{n} (g_i - 1)L_{i-1}^2$$

$$= \sum_{i=4}^{n} (g_i - 1)Y_{i-1}^2 + \sum_{i=4}^{n} (g_i - 1) \left[ 2Y_{i-1}(\alpha_{i-1} - \omega_3 \cdots \omega_{i-1} \alpha_2) + (\alpha_{i-1} - \omega_3 \cdots \omega_{i-1} \alpha_2)^2 \right].$$

(4.21)

The dominant contribution comes from the first sum while the second sum is bounded of constant order. We state this more precisely in the following two lemmas.

**Lemma 4.8.** For the second sum in (4.21), with probability $1 - o(1)$, we have the bound

$$\sum_{i=4}^{n} (g_i - 1) \left[ 2Y_{i-1}(\alpha_{i-1} - \omega_3 \cdots \omega_{i-1} \alpha_2) + (\alpha_{i-1} - \omega_3 \cdots \omega_{i-1} \alpha_2)^2 \right] = O(1).$$

(4.22)

**Lemma 4.9.** With probability $1 - o(1)$,

$$\sum_{i=4}^{n} (g_i - 1)Y_{i-1}^2 = \frac{\alpha}{6} \log n + O(\log \log n).$$

(4.23)

From (4.6) and (4.21), the Lemmas 4.2-4.9 together yield that

$$\log |E_n| = - \sum_{i=3}^{n} L_i + \frac{1 - \alpha}{6} \log n + O(\log \log n).$$

(4.24)

We postpone the proofs of Lemma 4.7 and Lemma 4.8 to the Appendix and turn now to proving Lemma 4.9.

In the proofs of Lemmas 4.8 and 4.9, we will need the following lemma, which is a Hanson Wright type inequality (see, for example Proposition 1.1 from Götze, Sambale, and Sinulis [9]). We employ this lemma in a similar manner to the way that Johnstone et al handle such quadratic forms in their paper [12].

**Lemma 4.10.** Let $x = (x_1, \ldots, x_n)^T$ be a vector with independent subgamma entries satisfying $x_i \in SG(v, u)$ where $v, u \leq Cn^{-1}$ for some $C > 0$. Then, for any symmetric matrix $A$,

$$|x^T Ax - \mathbb{E}x^T Ax| = O(\nu_n n^{-1} ||A||_{HS})$$

with probability at least $1 - \nu_n^{-1}$,

(4.25)

for any $\nu_n > 0$ (For the purposes of this paper we take $\nu_n$ to be a slowly growing function such as $\log \log n$).

**Proof.** It follows from Definition 2.4 that, there is a constant $c > 0$ such that $\|x_i\|_{\psi_1} \leq cn^{-\frac{1}{2}}$ for all $i = 1, \ldots, n$. Here,

$$\|X\|_{\psi_1} := \inf \{ t > 0 : \mathbb{E}\exp(\|X\|/t) \leq 2 \}$$

denotes the (exponential) Orlicz norm. Thus, by Proposition of 1.1 of [9], for some constants $C_1, C_2 > 0$,

$$\mathbb{E}|x^T Ax - \mathbb{E}x^T Ax| \leq \int_{0}^{\infty} 2 \exp \left( -\frac{1}{C_1} \min \left\{ \frac{nt^2}{\|A\|_{HS}^2}, \left( \frac{nt}{\|A\|_{op}} \right)^{1/2} \right\} \right) dt$$

$$= 2\frac{\|A\|_{HS}}{n} \int_{0}^{\infty} \exp \left( -\frac{1}{C_1} \min \left\{ u^2, \left( \frac{u \|A\|_{HS}}{\|A\|_{op}} \right)^{1/2} \right\} \right) du$$

(4.26)

$$\leq \frac{C_2\|A\|_{HS}}{n}.$$

Applying Markov’s inequality to $\mathbb{P}(|x^T Ax - \mathbb{E}x^T Ax| > C_3\nu_n n^{-1} ||A||_{HS})$ and using (4.26), we obtain the lemma with appropriate constant $C_3 > 0$ depending on $C_1$ and $C_2$. \qed
4.1 Proof of Lemma 4.9

We begin by showing that \( \sum (g_i - 1) Y_{i-1}^2 \) is close to its expectation with probability approaching one, then proceed to compute the leading order term of the expectation.

**Definition 4.11**. We define the following notations to be used in this proof and also in the Appendix.

\[
W = \begin{pmatrix}
1 & 1 & & & \\
\omega_4 & \omega_4 & \omega_5 & & \\
\omega_4 \omega_5 & \omega_5 & & & \\
& \vdots & \vdots & \ddots & \\
\omega_4 \ldots \omega_{n-2} & \omega_5 \ldots \omega_{n-2} & \ldots & \omega_{n-2} & 1 \\
\omega_4 \ldots \omega_{n-1} & \omega_5 \ldots \omega_{n-1} & \ldots & \omega_{n-2} \omega_{n-1} & \omega_{n-1}
\end{pmatrix},
\]

\( G = \text{diag}(g_4 - 1, \ldots, g_{n-1} - 1, g_n - 1), \)

\( D = \text{diag}(1 + \tau_2, 1 + \tau_3, \ldots, 1 + \tau_{n-2}), \)

\( Y = (Y_3, Y_4, \ldots, Y_{n-1})^T, \)

\( X = (X_3, X_4, \ldots, X_{n-1})^T. \)

Observe that \( Y = WX \) by Definition 2.23 and we can write

\[
\sum_{i=4}^{n} (g_i - 1) Y_{i-1}^2 = Y^TWY = X^TW^TGWX.
\]

(4.28)

Note that \( X \) is a vector of independent sub-gamma random variables satisfying the conditions of Lemma 4.10 and \( W^TGW \) is a symmetric, deterministic matrix. Thus, by the lemma, we conclude that, with probability \( 1 - O(\sigma_n^{-1/2}), \)

\[
|X^TW^GWX - E(X^TW^GWX)| = O\left(\sigma_n^{1/2}n^{-1}\|W\|_{\text{HS}}\right) = O\left(\sigma_n^{1/2}n^{-1}\|W\|\|GW\|_{\text{HS}}\right).
\]

(4.29)

To bound \( \|W\| \), we break \( W \) up as a sum of \( n \) matrices, each containing one of the subdiagonals of the matrix \( W \). The first such matrix contains the elements \( 1, 1, \ldots, 1 \), the second contains \( \omega_4, \omega_5, \ldots, \omega_{n-1}, \) and so forth. The norm of each of these matrices is equal to its largest element, so, using Lemma 2.8, we get

\[
\|W\| \leq 1 + \omega_n - 1 + \cdots + \omega_{n-1} - \omega_4 \leq \frac{1}{1 - \omega_{n-1}} = O(n^{1/3}\sigma_n^{-1/2}).
\]

(4.30)

To bound \( \|GW\|_{\text{HS}} \), we use Lemma 5.1 and Corollary 2.9 and conclude that

\[
\|GW\|_{\text{HS}} = \left(\sum_{i=4}^{n}(g_i - 1)^2(1 + \omega_2^2 - 1 + \cdots + \omega_{i-1}^2)\right)^{1/2} \]

\[
\leq \left(\sum_{i=4}^{n}(g_i - 1)^2\right)^{1/2} = O\left(\left(\sum_{i=4}^{n}(1 - \omega_{i-1}^{-3})^{-1/2}\right)^{1/2}\right) = O(n^{2/3}\sigma_n^{-1/4}).
\]

(4.31)

Thus with probability \( 1 - O(\sigma_n^{-1/2}), \)

\[
|X^TW^GWX - E(X^TW^GWX)| = O(\sigma_n^{-1/4}) = o(1).
\]

(4.32)

We now turn to the task of computing the leading term of

\[
E(X^TW^GWX) = \sum_{i=1}^{n-3}(W^TGW)_{i\cdot}EX_{i+2}^2 = \sum_{i=1}^{n-3} \sum_{j=i}^{n-3} (\omega_{i+3} \cdots \omega_{j+2})^2 \omega_{j+3} g_{j+4} E X_{i+2}^2.
\]

(4.33)
It will be convenient to switch the order of summation and rewrite this as
\[
\sum_{i=1}^{n-2} \sum_{j=i}^{n-3} (\omega_{i+3} \cdots \omega_{j+2})^2 \omega_{j+3} g_{j+4} \mathbb{E} X_{i+2}^2 = \sum_{j=1}^{n-3} \left( \sum_{i=1}^{j} (\omega_{i+3} \cdots \omega_{j+2})^2 \mathbb{E} X_{i+2}^2 \right) \left( \sum_{k=j}^{n-3} \omega_{j+3} \cdots \omega_{k+3} \right). \tag{4.34}
\]

It turns out that the dominant contribution comes from the portion of the sum where the indices are restricted to \( n - n \log n - 2 \leq i \leq n - n^{1/3} \sigma_n \log n \). We will begin by computing the sum on those indices and then show that the sum of the remaining terms is small. Thus, our first task is to compute
\[
\sum_{j=n-n \log n - 2}^{n-n^{1/3} \sigma_n \log n} \left( \sum_{i=n-n \log n - 2}^{j} (\omega_{i+3} \cdots \omega_{j+2})^2 \mathbb{E} X_{i+2}^2 \right) \left( \sum_{k=j}^{n-3} \omega_{j+3} \cdots \omega_{k+3} \right). \tag{4.35}
\]

For the purposes of calculating this, we begin by computing the asymptotics of a product of the form
\[
\prod_{i=i_1}^{i_2} \omega_i = \exp \left( \sum_{i=i_1}^{i_2} \log \omega_i \right),
\]
where \( n^{1/3} \sigma_n \ll n - i \ll n \) for indices \( i \) in the range \( i_1 \leq i \leq i_2 \). On this range of indices, it follows from the proof of Lemma 2.8 that \( \omega_i \) has a series expansion in powers of \( (\frac{n-i}{n})^{1/2} \) and \( n^{-1/3} \sigma_n^{1/2} \). Furthermore, using Taylor expansion of \( \log \), we can write
\[
\log \omega_i = -2(1 + \lambda^{1/2}) \left( \frac{n-i}{n} \right)^{1/2} \left( 1 + s \left( \frac{n-i}{n} \right) \right),
\]
where \( s(x) \) is a series in positive powers of \( x^{1/2} \) and \( n^{-1/3} \sigma_n^{1/2} \) (including mixed terms) and the series converges when \( x = \frac{n-i}{n} \) for \( i \) in the range of indices described above.

Using this representation, we have
\[
\prod_{i=i_1}^{i_2} \omega_i = \exp \left( \sum_{i=i_1}^{i_2} \left( -2(1 + \lambda^{1/2}) \left( \frac{n-i}{n} \right)^{1/2} \left( 1 + s \left( \frac{n-i}{n} \right) \right) \right) \right)
\]
\[
= \exp \left( -2(1 + \lambda^{1/2}) n \int_{1/n}^{i_2/n} (1-x)^{1/2} \left( 1 + s(1-x) \right) dx (1 + O(n^{-1})) \right)
\]
\[
= \exp \left( \frac{4}{3} (1 + \lambda^{1/2}) n \left( \left( \frac{n-i_1}{n} \right)^{3/2} \left( 1 + s \left( \frac{n-i_1}{n} \right) \right) - \left( \frac{n-i_2}{n} \right)^{3/2} \left( 1 + s \left( \frac{n-i_2}{n} \right) \right) \right) \right) \left( 1 + O \left( \left( \frac{n-i_2}{n} \right)^{3} \right) \right), \tag{4.36}
\]
where \((1-x)^{3/2}(1+s(1-x))\) is the series that emerges as the antiderivative of the integrand in the second line of the display above and \(s(1-x) = O((1-x)^{1/2})\). Next, we compute \( \mathbb{E} X_i^2 \) on the indices \( i < n - n^{1/3} \sigma_n \log n \).

By (2.43) and Lemma 2.7,
\[
\mathbb{E} X_i^2 = \alpha \frac{\delta_i}{|\rho_i^+|} (1 + \tau_{i-1})^2 \left( \omega_i + 1 + O(n^{-2/3} \sigma_n^{-1/2}) \right). \tag{4.37}
\]

We then consider each factors on the right hand side individually. Using Lemma 2.8 to obtain asymptotics for \( \omega_i \) with \( i < n - n^{1/3} \sigma_n \log n \), we obtain \( \omega_i + 1 + O(n^{-2/3} \sigma_n^{-1/2}) = 2 + O \left( \sqrt{\frac{n-i}{n}} \right) \) and
\[
\frac{\delta_i}{|\rho_i^+|} = \frac{\omega_i}{\tau_{i-1} |\rho_i^+|} = \frac{1 - O \left( \sqrt{\frac{n-i}{n}} \right)}{n(\lambda^{1-1/2} - \frac{n-i-1}{n})} = \frac{\lambda}{n} \left( 1 + O \left( \sqrt{\frac{n-i}{n}} \right) \right). \tag{4.38}
\]

Finally, we have
\[
(1 + \tau_{i-1})^2 = \left( 1 + \frac{m-n+i-1}{|\rho_i^-|} \right)^2 = \left( 1 + \frac{|\rho_i^-|}{n \left( \frac{n-i-1}{n} \right)} \right) \left( 1 + O(n^{-1}) \right)^2. \tag{4.39}
\]
Since $\sigma_n n^{-2/3} \ll \frac{n}{n}$, computations in the proof of Lemma 4.6 (in particular, (4.11) and (4.12)) imply that
\[
|\rho_{i-1}^{-1}| = \frac{1}{2} \left( \gamma m - (m - n + 2i - 3) - \sqrt{U_{i-1}} \right) = \lambda^{-\frac{1}{2}} n \left( 1 + O \left( \sqrt{\frac{n-i}{n}} \right) \right). \tag{4.40}
\]
Thus, by (4.39),
\[
(1 + \tau_{i-1})^2 = \left( \lambda^{-\frac{1}{2}} + 1 \right)^2 + O \left( \sqrt{\frac{n-i}{n}} \right). \tag{4.41}
\]
Putting all factors of (4.37) together, we conclude that, for $i < n - n^{1/3} \sigma_n \log n$,
\[
\text{EX}_i^2 = 2\alpha n^{-1} (1 + \lambda^{1/2})^2 \left( 1 + O \left( \sqrt{\frac{n-i}{n}} \right) \right). \tag{4.42}
\]
We use the notation $C_0 = \frac{4}{3} (1 + \lambda^{1/2})$ and $C_1 = 2(\lambda^{1/2} + 1)^2$ and combine (4.36) and (4.42) to obtain
\[
\sum_{i=n-n(\log n)^{-2}}^j (\omega_{i+3} \cdots \omega_{j+2})^2 \text{EX}_{i+2}^2 = \\
\sum_{i=n-n(\log n)^{-2}}^j \exp \left( -2C_0 n \left[ \left( \frac{n-i-3}{n} \right)^{\frac{3}{2}} (1 + \tilde{s}(\frac{n-i-3}{n})) - \left( \frac{n-j-2}{n} \right)^{\frac{3}{2}} (1 + \tilde{s}(\frac{n-j-2}{n})) \right] \right) \frac{\alpha C_1 + O((\frac{n-i}{n})^{1/2})}{n}. \tag{4.43}
\]
This gives us
\[
\sum_{i=n-n(\log n)^{-2}}^j (\omega_{i+3} \cdots \omega_{j+2})^2 \text{EX}_{i+2}^2 = \\
= \alpha C_1 \left( 1 + O((\log n)^{-1}) \right) \int_{1 - \frac{i}{n}}^{(\log n)^{-2}} \exp \left( -2C_0 n \left[ x^{\frac{3}{2}} (1 + \tilde{s}(x)) - (1 - \frac{j}{n})^{\frac{3}{2}} (1 + \tilde{s}(1 - \frac{j}{n})) \right] \right) dx. \tag{4.44}
\]
Next, we make the change of variables $u = x(1 + \tilde{s}(x))^{2/3} (2C_0 n)^{2/3}$. Noting that $du = (2C_0 n)^{2/3} (1 + O(\tilde{s}(x))) dx$, the right hand side of the display above becomes
\[
\frac{\alpha C_1 (1 + O((\log n)^{-1}))}{(2C_0 n)^{2/3}} \exp \left( 2C_0 n \left( 1 - \frac{j}{n} \right)^{\frac{3}{2}} (1 + \tilde{s}(1 - \frac{j}{n})) \right) \int_{(2C_0 n)^{2/3} (1 + \tilde{s}(1 - \frac{j}{n}))^{2/3}}^{(2C_0 n)^{2/3} (1 + \tilde{s}(\log n)^{1/2})^{2/3}} \exp(-u^{3/2}) du. \tag{4.45}
\]
The integrand $\exp(-u^{3/2})$ has antiderivative $-\frac{2}{3} \Gamma \left( \frac{2}{3}, u^{3/2} \right)$. Furthermore, the asymptotics of the incomplete Gamma function (see Digital Library of Mathematical Functions 8.11.2) are
\[
\Gamma(a, z) = z^{a-1} e^{-z} (1 + O(z^{-1})) \text{ for fixed } a \text{ and } z \to \infty. \tag{4.46}
\]
Applying this to the preceding equation, we get
\[
\sum_{i=n-n(\log n)^{-2}}^j (\omega_{i+3} \cdots \omega_{j+2})^2 \text{EX}_{i+2}^2 = \frac{\alpha C_1}{3C_0 n} \left( 1 - \frac{j}{n} \right)^{-1/2} (1 + O((\log n)^{-1})). \tag{4.47}
\]
It remains to calculate $\omega_{j+3} g_{j+4} = \sum_{k=j}^{n-3} \omega_{j+3} \cdots \omega_{k+3}$ and then compute the outer sum in the expression (4.35). Using Lemmas 5.1 and 2.8 for indices $j$ in our desired range, we have the lower bound
\[
\omega_{j+3} g_{j+4} \geq \frac{\omega_{j+3}(1 + (\log n)^{-1})}{1 - \omega_{j+4}} = \frac{1 + O \left( \left( \log n \right)^{-1} + \left( \frac{n-(j+4)}{n} \right)^{1/2} \right) \frac{2(1 + \lambda^{1/2}) \left( \frac{n-(j+4)}{n} \right)^{1/2}}{1 + O \left( \left( \log n \right)^{-1} + \left( \frac{n-j}{n} \right)^{1/2} \right)}}{1 + O \left( \left( \log n \right)^{-1} + \left( \frac{n-j}{n} \right)^{1/2} \right)} = \frac{3}{2} C_0 \left( \frac{n-j}{n} \right)^{1/2}. \tag{4.48}
\]
The analogous upper bound obtained from Lemma 5.1 is not tight enough. Instead, we upper bound $\omega_{j+3}g_{j+4}$ by rewriting it as two sums

$$\omega_{j+3}g_{j+4} = \sum_{k=j}^{n-n^{1/3}\sigma_n} \omega_{j+3} \cdot \omega_{k+3} + \sum_{k=n-n^{1/3}\sigma_n+1}^{n-3} \omega_{j+3} \cdot \omega_{k+3} =: S_1 + S_2.$$  (4.49)

We will show that $S_1 \leq 1 + O\left(\frac{1}{\log^2 \frac{n}{\log n}}\right) \frac{O}{4C_0 \frac{\log^2 n}{\log (\log n)}}$ while $S_2 = o(1)$. For $S_2$, we have

$$S_2 = (\omega_{j+3} \cdot \omega_{n-n^{1/3}\sigma_n+4}) \left( 1 + \sum_{k=n-n^{1/3}\sigma_n+2}^{n-3} \omega_{n-n^{1/3}\sigma_n+5} \cdots \omega_{k+3} \right).$$  (4.50)

Equation (4.36) implies that, for some $C > 0$, the product above has the bound

$$\omega_{n-n^{1/3}\sigma_n} \cdots \omega_{n^{-1/3}\sigma_n} \log n \leq \exp(-C(\sigma_n \log n)^{3/2}).$$  (4.51)

Meanwhile, Lemmas 5.1 and 2.8 imply that $g_{n-n^{1/3}\sigma_n+5} = O(n^{1/3}\sigma_n^{-1/2})$. Thus, we conclude that $S_2 = o(1)$.

For $S_1$, we will make use of the asymptotic in (4.36) to bound the product $\omega_{j+3} \cdots \omega_{k+3}$. Although (4.36) is only a valid asymptotic expression for indices satisfying $n - i \gg n^{1/3}\sigma_n$, it is nevertheless a valid upper bound for all indices in the range covered by $S_1$. This is because (4.36) was obtained using the approximation of $\omega_i$ in Lemma 2.8(iii) which is an upper bound for the approximation in Lemma 2.8(ii) when $n - i = \Theta(n^{1/3}\sigma_n)$. Thus we obtain

$$S_1 \leq \sum_{k=j}^{n-n^{1/3}\sigma_n} \exp \left( -C_0 n \left[ \left( \frac{n-j-3}{n} \right)^{1/2} \left( 1 + \tilde{s}(\frac{n-j-3}{n}) \right) - \left( \frac{n-k-3}{n} \right)^{1/2} \left( 1 + \tilde{s}(\frac{n-k-3}{n}) \right) \right] \right) \left( 1 + O\left( \frac{n-j-3}{n} \right) \right).$$  (4.52)

The asymptotics of this sum can be computed using a similar approach to the computation of the sum in (4.43). Using this method, we arrive at

$$S_1 \leq \frac{2}{3C_0} \left( \frac{n-j}{n} \right)^{-1/2} \left( 1 + O\left( \frac{n-j-3}{n} \right)^{1/2} \right).$$  (4.53)

Combining this with (4.48) and (4.50) along with the fact that $j \geq n - n(\log n)^{-2}$, we conclude that

$$\omega_{j+3}g_{j+4} = \frac{2}{3C_0} \left( \frac{n-j}{n} \right)^{-1/2} \left( 1 + O\left( \log(n) \right) \right).$$  (4.54)

Finally, plugging the results from (4.47) and (4.54) into the summation (4.35), we get

$$\sum_{j=n-n^{1/3}\sigma_n \log n}^{j=\infty} \sum_{i=n-n(\log n)^{-2}}^{j} (\omega_{i+3} \cdots \omega_{j+2})^2 \bar{X}_{i+2} \cdot \omega_{j+3}g_{j+4}$$

$$= \frac{\alpha C_1}{3C_0 n} \left( 1 - \frac{2}{n} \right)^{-1/2} \left( \frac{2}{C_0} \left( 1 - \frac{2}{n} \right)^{-1/2} \left( 1 + O\left( \log(n) \right) \right) \right)$$

$$= \frac{2\alpha C_1}{9C_0^2} \int_{n^{-2/3}\sigma_n \log n}^{(\log n)^{-2}} z^{-1}dz \left( 1 + O\left( \log(n)^{-1} \right) \right) = \frac{2\alpha C_1}{9C_0^2} \left( \frac{2}{3} \log n + O\left( \log(n) \right) \right) \leq \frac{\alpha}{6} \log n + O\left( \log\log n \right).$$  (4.55)

Next, we must consider the terms in (4.34) whose indices do not satisfy $n - n(\log n)^{-2} < i < j < n - n^{1/3}\sigma_n \log n$ and we must show that the sum over those indices is $O(\log\log n)$. More specifically, we will show that
(a) \[ \sum_{j=n-n^{1/3}\sigma_n}^{n-3} \sum_{i=1}^{j} (\omega_i+3 \cdots \omega_{j+2}) \omega_j+3 g_{j+4} \mathbb{E} X_i^2 = O(\log \log n), \]

(b) \[ \sum_{j=1}^{n-n^{1/3}\sigma_n} \sum_{i=1}^{j} (\omega_i+3 \cdots \omega_{j+2}) \omega_j+3 g_{j+4} \mathbb{E} X_i^2 = O(\log n), \]

(c) \[ \sum_{j=n-n^{1/3}\sigma_n}^{n-n^{1/3}\sigma_n} \sum_{i=1}^{j} (\omega_i+3 \cdots \omega_{j+2}) \omega_j+3 g_{j+4} \mathbb{E} X_i^2 = O(1). \]

To prove (a), we begin by using Lemma 5.1 and the fact that \( \mathbb{E} X_i^2 = O(n^{-1}) \) to observe that, for some constant \( C \),

\[ \sum_{i=1}^{j} (\omega_i+3 \cdots \omega_{j+2}) \omega_j+3 g_{j+4} \mathbb{E} X_i^2 \leq \sum_{i=1}^{j} \omega_{j+i}^2 \cdot \frac{C}{n(1-\omega_{j+i})} \leq \frac{1}{1-\omega_{j+2}} \cdot \frac{C}{n(1-\omega_{j+3})}. \]  

(4.56)

Using Lemma 2.8 and the fact that \( \omega_j \) is increasing in \( j \), we conclude

\[ \sum_{j=n-n^{1/3}\sigma_n}^{n-3} \sum_{i=1}^{j} (\omega_i+3 \cdots \omega_{j+2}) \omega_j+3 g_{j+4} \mathbb{E} X_i^2 \leq \sum_{j=n-n^{1/3}\sigma_n}^{n-n^{1/3}\sigma_n} \frac{1}{n-\omega_j^2} \cdot \frac{C}{n(1-\omega_j)} \leq \sum_{j=n-n^{1/3}\sigma_n}^{n-n^{1/3}\sigma_n} \frac{C}{n(1-\omega_j)^2} + \sum_{j=n-n^{1/3}\sigma_n+1}^{n-n^{1/3}\sigma_n} \frac{C}{n(1-\omega_j)^2} \]

\[ = O \left( \int_{n^{1/3}\sigma_n}^{n^{1/3}\sigma_n} x^{-1} dx \right) + O(1) = O(\log n). \]

To prove (b) we observe that inequality (4.56) still holds. Using this, we obtain the following result, where \( C_1 \) is the constant that comes from applying Corollary 2.9

\[ \sum_{j=1}^{n-n^{1/3}\sigma_n} \sum_{i=1}^{j} (\omega_i+3 \cdots \omega_{j+2}) \omega_j+3 g_{j+4} \mathbb{E} X_i^2 \leq \sum_{j=1}^{n-n^{1/3}\sigma_n} \frac{1}{1-\omega_{j+2}^2} \cdot \frac{C}{n(1-\omega_{j+4})} \]

\[ \leq \sum_{j=1}^{n-n^{1/3}\sigma_n} \frac{C}{n \cdot C_1 \left( \frac{n-(j+4)}{n} \right)} = O \left( \int_{(\log n)^{-1}}^{1} \frac{1}{x} dx \right) = O(\log \log n). \]

To prove (c) we observe that, on the indices we consider,

\[ (\omega_{i+3} \cdots \omega_{j+2})^2 = (\omega_{i+3} \cdots \omega_{n-n^{1/3}\sigma_n} \cdots \omega_{j+2})^2 \]

\[ < (\omega_{i+3} \cdots \omega_{n-n^{1/3}\sigma_n})^2 < (\omega_{n-n^{1/3}\sigma_n}^2)^{(n-n^{1/3}\sigma_n)^{-1}} \cdot \omega_{n-n^{1/3}\sigma_n}^{-1}. \]  

(4.59)

Using this, along with Lemma 2.8 and Corollary 2.9 we conclude

\[ \sum_{j=n-n^{1/3}\sigma_n}^{n-n^{1/3}\sigma_n} \sum_{i=1}^{j} (\omega_i+3 \cdots \omega_{j+2}) \omega_j+3 g_{j+4} \mathbb{E} X_i^2 \leq \sum_{j=n-n^{1/3}\sigma_n}^{n-n^{1/3}\sigma_n} \sum_{i=1}^{j} \left( \omega_{n-n^{1/3}\sigma_n}^2 \right)^{(n-n^{1/3}\sigma_n)^{-1}-(i+2)} \frac{1}{1-\omega_{j+4}^2} \cdot \frac{1}{n} \cdot \frac{1}{n \cdot C_1 \left( \frac{n-(j+4)}{n} \right)} \]

\[ \leq \sum_{j=n-n^{1/3}\sigma_n}^{n-n^{1/3}\sigma_n} \frac{1}{1-\omega_{n-n^{1/3}\sigma_n}^2} \cdot \frac{1}{(n-1)^{1/2}} \cdot \frac{1}{n} \cdot O \left( \frac{1}{n} \right). \]

(4.60)
To simplify this we use the fact that \( \frac{1}{n-n[(\log n)^{-2}]} = O(\log n) \) and we rewrite the summation as an integral, so the entire expression above becomes

\[
O(\log n) \cdot \int_{n^{-2/3} \sigma n \log n}^{\log n - 1} \frac{1}{x^{1/2}} dx = O(1).
\]

(4.61)

5 CLT for \( \sum_{i=3}^{n} L_i \)

From [2.21] and Definition 4.5

\[
\sum_{i=3}^{n} L_i = \sum_{i=3}^{n} g_{i+1}X_i + \sum_{i=3}^{n} \alpha_i - g_3\alpha_2.
\]

In this section, we show that \( \sum_{i=3}^{n} L_i \) satisfies the CLT

\[
\frac{\sum_{i=3}^{n} L_i}{\left( \sum_{i=3}^{n} g_{i+1}^2 \mathbb{E}X_i^2 \right)^{1/2}} \xrightarrow{d} \mathcal{N}(0, 1)
\]

(5.1)

by showing the following claims.

1. The mean-zero random variable \( \sum_{i=3}^{n} g_{i+1}X_i \) satisfies Lyapunov condition

\[
\frac{\sum_{i=3}^{n} g_{i+1}^4 \mathbb{E}X_i^4}{\left( \sum_{i=3}^{n} g_{i+1}^2 \mathbb{E}X_i^2 \right)^2} \to 0 \quad \text{as } n \to \infty.
\]

(5.2)

2. \( \frac{\sum_{i=3}^{n} \alpha_i - g_3\alpha_2}{\left( \sum_{i=3}^{n} g_{i+1}^2 \mathbb{E}X_i^2 \right)^{1/2}} \) converges to 0 in probability.

Here, knowing the order of the variance \( \sum_{i=3}^{n} g_{i+1}^2 \mathbb{E}X_i^2 \) is sufficient for both claims, so we delay the computation of its leading term to the end of the section.

We now verify (5.2). It follows from (2.43) and (5.3) uniformly in \( i \) that

\[
\mathbb{E}X_i^2 \geq \frac{C\alpha_i}{n}.
\]

(5.3)

Meanwhile, the uniform lower bound \( |\rho^+_i| = \Omega(n) \) and Lemma 2.6 with \( p = 4 \) imply

\[
\mathbb{E}X_i^4 \leq C\alpha^4 (1 + \tau_i - 1)^4 \left( \frac{\delta^2_1}{|\rho^+_i|^2} + \frac{1}{|\rho^+_i|^4} \right) = O \left( \frac{n^4\delta^2_1}{n^2} \right).
\]

(5.4)

We also need to estimate powers of \( g_i \) for \( 3 \leq i \leq n \). The following lemma shows that for most indices \( i \), \( g_i \) is of order \( (1 - \omega_i)^{-1} \).

**Lemma 5.1.** Let \( \{g_i\}_{i=3}^{n+1} \) be as above (see Definition 4.5). Then,

(i) for any \( k > 0 \) and sufficiently large \( n \), \( g_i > \frac{1 - \log^{-k} n}{1 + \omega^2_i} \) for all \( 3 \leq i \leq n - n^{1/3} \),

(ii) for sufficiently large \( n \), \( g_i < \frac{1 + n^{-3/2}}{1 + \omega_i} \) for all \( 3 \leq i \leq n \).

**Proof.** Fix \( k > 0 \). Since \( \{\omega_i\}_{i=3}^{n} \) is increasing in \( i \),

\[
g_i > 1 + \omega_i + \omega_i^2 + \cdots + \omega_i^{n+i+1} = \frac{1 - \omega_i^{n+i+2}}{1 - \omega_i}.
\]

(5.5)

By Corollary 2.9 \( \omega_i \leq 1 - cn^{-1/3} \sigma_n^{1/2} \) for \( 3 \leq i \leq n \). If in addition, \( i \leq n - n^{1/3} \), then

\[
\omega_i^{n+i+2} \leq \left( 1 - cn^{-1/3} \sigma_n^{1/2} \right)^{n+1/3} < e^{-c\sigma_n^{1/2}}.
\]

(5.6)
The right hand side is less than $e^{-k \log \log n} = \log^{-k} n$ for sufficiently large $n$, so we obtain $[1]$. 

For the upper bound, define $N_i = (1 - \omega_i)g_i - 1 - \sigma_n^{-3/2}$. Then $g_i = \frac{N_i + 1 + \sigma_n^{-3/2}}{1 - \omega_i}$ and it suffices to show $N_i < 0$ for every $3 \leq i \leq n$. The case $i = n$ is clear, as

$$N_n = (1 - \omega_n)(1 + \omega_n) - 1 - \sigma_n^{-3/2} = -\omega_n^2 - \sigma_n^{-3/2} < 0. \tag{5.7}$$

Suppose $N_i < 0$. From the definition of $g_i$,

$$N_{i-1} = (1 - \omega_{i-1})(1 + \omega_{i-1}g_i) - 1 - \sigma_n^{-3/2}$$

$$= (1 - \omega_{i-1}) \left( 1 + \frac{\omega_{i-1}}{1 - \omega_i} (N_i + 1 + \sigma_n^{-3/2}) \right) - 1 - \sigma_n^{-3/2}. \tag{5.8}$$

By the induction hypothesis,

$$N_{i-1} < -\omega_{i-1} - \sigma_n^{-3/2} + \frac{\omega_{i-1}}{1 - \omega_i} \left( 1 + \sigma_n^{-3/2} \right) \left( 1 - \omega_{i-1} \right) + \left( 1 - \omega_{i-1} \right)^2 \sigma_n^{-3/2}$$

$$= \left( \omega_{i-1} + \sigma_n^{-3/2} \right) (\omega_i - \omega_{i-1}) - \left( 1 - \omega_{i-1} \right)^2 \sigma_n^{-3/2}$$

$$\leq \frac{\omega_{i-1} + \sigma_n^{-3/2} \omega_i - \omega_{i-1}) - \left( 1 - \omega_{i-1} \right)^2 \sigma_n^{-3/2}}{1 - \omega_i}. \tag{5.9}$$

Note that $\omega_{i-1} + \sigma_n^{-3/2} \leq 2$. We then provide bounds for $\omega_i - \omega_{i-1}$ and $1 - \omega_n$, in order to show that the numerator is negative. We approach the first quantity by bounding the growth of $\omega_i$, where $\omega_i$ is considered as function of $i/m \in [3/m, \lambda]$. For brevity of the presentation, we define for $x \in [3/m, \lambda]$,

$$f(x) = \frac{x (m-n + x)}{(C_n - x)^2} \quad \text{and} \quad g(x) = \frac{1 - \sqrt{1 - f(x)}}{1 + \sqrt{1 - f(x - \frac{1}{m})}}, \tag{5.10}$$

where $C_n = \frac{2m - (m-n+1)}{2n}$. Then $\frac{1}{m} \omega_i = \left( C_n - \frac{i-1}{m} \right) \left( 1 \pm \sqrt{1 - f \left( \frac{i-1}{m} \right) \sqrt{1 - f \left( \frac{i-1}{m} \right)} \right)$, which implies

$$\omega_i = \frac{|\rho_{i+1} \rho_{i+1}^\prime|}{|\rho_{i+1}|} = \left( 1 - \frac{1/m}{C_n - \frac{i-1}{m}} \right) g \left( \frac{i-1}{m} \right). \tag{5.11}$$

Since both $f(x)$ and $f'(x) = \frac{m-n+2x}{(C_n-x)^3} + \frac{x (m-n+x)}{(C_n-x)^2}$ are increasing in $x$, so is

$$g'(x) = \frac{\frac{1}{2}f'(x)/\sqrt{1 - f(x)}}{1 + \sqrt{1 - f(x - \frac{1}{m})}} + \frac{(1 - \sqrt{1 - f(x)})\frac{1}{2}f'(x - \frac{1}{m})/\sqrt{1 - f(x - \frac{1}{m})}}{(1 + \sqrt{1 - f(x - \frac{1}{m})})^2}. \tag{5.12}$$

Therefore,

$$\omega_i - \omega_{i-1} \leq \left( 1 - \frac{1/m}{C_n - \frac{i-1}{m}} \right) \int_{\frac{i-1}{m}}^{\frac{i}{m}} g'(x) dx < \frac{1}{m} g' \left( \frac{n-1}{m} \right). \tag{5.13}$$

Set $y_i = 1 - f \left( \frac{i-1}{m} \right)$. From (5.12), for sufficiently large $n$,

$$g' \left( \frac{n-1}{m} \right) = \frac{f' \left( \frac{n-1}{m} \right)}{\sqrt{y_n(1 + \sqrt{y_{n-1}})}^2} \cdot \frac{1 + \sqrt{y_{n-1}} + \sqrt{y_n(1 - \sqrt{y_n})}}{2} < \frac{f' \left( \frac{n-1}{m} \right)}{\sqrt{y_n(1 + \sqrt{y_{n-1}})}^2}. \tag{5.14}$$

We verify the above inequality by showing that for sufficiently large $n$,

$$1 + \sqrt{1 - \frac{y_{n-1} - y_n}{y_{n-1}}} + \frac{y_{n-1} - y_n}{\sqrt{y_{n-1}}} < 2. \tag{5.15}$$
We note the identity $y_i = \left(\gamma m - (m - \frac{n}{i} - 2)\right)^2$ for $i = 3, \ldots, n$, where $U_i$ is defined in (2.39). By (2.42) and the fact $\gamma = (1 + \lambda^2)^2 + \sigma_n n^{-3}$, there are constants $1 < c_1 < c_2$ such that $c_1 \sigma_n n^{-3} < y_n < y_{n-1} < c_2 \sigma_n n^{-3}$. In addition, $U_{n-1}$ and $U_n$ are $\Theta(n^{1/3} \sigma_n)$ by (2.42), and $U_{n-1} - U_n = 4(\gamma - 1) = \Theta(n)$. Thus,

$$y_{n-1} - y_n = \frac{(\gamma m - (m + n - 1))^2 (U_{n-1} - U_n) - 4(\gamma m - (m + n - 2)U_n)}{(\gamma m - (m + n - 1))^2 (\gamma m - (m + n - 3))^2} = \Theta(n^{-1}).$$

(5.15)

Therefore, the left hand side of (5.14) has asymptotics $2 - c_3 n^{-1/2} + O(n^{-2} \sigma_n^{-1/2})$ as $n \to \infty$ for some $c_3 > 0$, and we obtain the claim.

We now consider $f'(\frac{n-1}{m})$. Note that $C_n - \frac{n-1}{m} = \sqrt{X} + \frac{1}{2} n^{-1/3} \sigma_n + O(n^{-1})$, so using expression of $f'(x)$ as above, we have

$$f'(\frac{n-1}{m}) = \frac{\lambda^{3/2} (1 + \lambda^{-1/2})^2 + 1 + 4n^{-2/3} \sigma_n + O(n^{-1})}{\sqrt{\lambda + \frac{1}{2} n^{-2/3} \sigma_n + O(n^{-1})}} < \frac{1}{\sqrt{2}} (1 + \lambda^{-1/2}).$$

(5.16)

We obtain

$$\omega_i - \omega_i - 1 < \frac{1}{m} \sqrt{\frac{1}{\lambda}} \frac{(1 + \lambda^{-1/2})^2 / \sqrt{2}}{\sqrt{y_n(1 + \sqrt{y_n})^2}}.$$

(5.17)

On the other hand,

$$1 - \omega_i > 1 - g \left(\frac{n-1}{m}\right) = 1 - \frac{1 - \sqrt{y_n}}{1 + \sqrt{y_n}} > \frac{2 \sqrt{y_n}}{1 + \sqrt{y_n}}.$$

(5.18)

Displays (5.17) and (5.18) together imply

$$N_{i-1} < \frac{1}{m} \cdot \frac{(1 + \sigma_n^{-3/2})^2 (1 + \lambda^{-1/2})^2}{\sqrt{y_n(1 + \sqrt{y_n})^2} \sqrt{2} m(1 - \omega_i) \sqrt{y_n(1 + \sqrt{y_n})^2} - \frac{4 \frac{y_n}{n} (1 + \lambda^{-1/2})^2 \sigma_n^{-3/2}}{(1 + \sqrt{y_n}) \sigma_n^{-3/2}}}.$$

(5.19)

Since $y_n > n^{-2/3} \sigma_n$ and $0 < \lambda \leq 1$ for all $n$, the numerator is less than $\sqrt{2} (1 + \lambda^{-1/2})^2 - 4 \lambda^{-1} + O(\sigma_n^{-3/2})$, which is negative for sufficiently large $n$. Therefore $N_{i-1} < 0$ for sufficiently large $n$.

Combining Lemma 5.1 and Corollary 2.9 we obtain

$$\sum_{i=3}^{n} \gamma_i^2 \mathbb{E}X_i^4\left(\mathbb{E}X_i^2\right)^2 = \frac{O}{\left(\sum_{i=3}^{n} \gamma_i^2 \mathbb{E}X_i^4\left(\mathbb{E}X_i^2\right)^2\right)^2} = \frac{O}{\left(\sum_{i=3}^{n} \gamma_i^2 \mathbb{E}X_i^4\left(\mathbb{E}X_i^2\right)^2\right)^2} = O\left(\sum_{i=3}^{n} \frac{\sigma_n}{n-i} \left(\frac{n}{n-i} \right)^2 + \sum_{i>n^{-1/3} \sigma_n} \frac{(n^{-1/3} \sigma_n)^2 (n^{-1/3} \sigma_n)^2}{2} \left(\sum_{i=3}^{n} \frac{n^{-1/3} \sigma_n}{n-i} \left(\frac{n}{n-i} \right)^2 \right)^2\right) \leq o(1).$$

(5.20)

Therefore, Lyapunov condition (5.2) holds.

The above computations suggest the variance $\sum_{i=3}^{n} \gamma_i^2 \mathbb{E}X_i^4$ is increasing in $n$, with lower bound $C \log n$ for some constant $C > 0$. As $\sum_{i=3}^{n} \alpha_i - g_3 \alpha_2$ belongs to some sub-gamma family $SG(v, u)$, Lemma 2.6 with $t = \sqrt{\log n}$ implies

$$\mathbb{P}\left(\sum_{i=3}^{n} \alpha_i - g_3 \alpha_2 > \sqrt{2ut + ut}\right) \leq 2n^{-1/2}.$$

(5.21)
Hence, the claim on convergence to zero in probability holds as long as the parameters \( v, u \) satisfy \( v = o(\sqrt{\log n}) \) and \( u = o(1) \). Indeed, by Lemma 2.5

\[
v = \alpha \left( \frac{g_i^2 \tau_i^2}{\rho_i^2} + \sum_{i=n}^{n} \frac{\tau_i}{|\rho_i|^2} \right) = O(1), \quad u = \alpha \max \left\{ \frac{g_i}{\rho_i^2}, \frac{1}{|\rho_i|^2} : 3 \leq i \leq n \right\} = O(n^{-1}). \tag{5.22}
\]

We now provide the asymptotics for \( \sum_{i=3}^{n} g_{i+1}^2 \mathbb{E}X_i^2 \). We will show that dominant contribution to the sum comes from indices \( i \leq n - n^{1/3} \sigma_n \sqrt{\log n} \), while the sum over the remaining indices is at most order \( \sqrt{\log n} \).

**Lemma 5.2.**

\[
\sum_{i=3}^{n} g_{i+1}^2 \mathbb{E}X_i^2 = \frac{\alpha}{3} \log n + o(\log n). \tag{5.23}
\]

**Proof.** We begin by showing that the terms with indices \( n - n^{1/3} \sigma_n \sqrt{\log n} \leq i \leq n - n^{1/3} \sigma_n \) and \( n - n^{1/3} \sigma_n \leq i \leq n \), together, contribute only \( O(\sqrt{\log n}) \) to the sum. In these calculations, we use the fact that \( \mathbb{E}X_i^2 = O(n^{-1}) \) uniformly in \( i \) and we bound \( g_i \) using Lemma 5.1 and Corollary 2.9. In particular, we obtain

\[
\sum_{i=n-n^{1/3} \sigma_n \sqrt{\log n}}^{n-n^{1/3} \sigma_n} g_{i+1}^2 \mathbb{E}X_i^2 = O \left( \sum_{i=n-n^{1/3} \sigma_n \sqrt{\log n}}^{n-n^{1/3} \sigma_n} \frac{n-n^{1/3} \sigma_n}{n-i} \frac{1}{n} \right) = O(\sqrt{\log n}), \tag{5.24}
\]

\[
\sum_{i=n-n^{1/3} \sigma_n}^{n} g_{i+1}^2 \mathbb{E}X_i^2 = O \left( \sum_{i=n-n^{1/3} \sigma_n}^{n} \frac{n^{2/3}}{\sigma_n} \frac{1}{n} \right) = O(1).
\]

We now compute the sum over the indices \( i < n - n^{1/3} \sigma_n \sqrt{\log n} \). Using Lemma 5.1, we obtain

\[
g_{i+1}^2 = \frac{1}{(1 - \omega_{i+1})^2(1 + o(1))} = \frac{1}{4(1 + \lambda^{1/2})^2} \left( \frac{n-i}{n} \right)^{-1} (1 + o(1)). \tag{5.25}
\]

Combining this with the computation of \( \mathbb{E}X_i^2 \) in (4.42), we get

\[
g_{i+1}^2 \mathbb{E}X_i^2 = \frac{1}{n} \left( \frac{n-i}{n} \right)^{-1} \left[ \frac{\alpha}{2} + o(1) \right]. \tag{5.26}
\]

Therefore,

\[
\sum_{i=3}^{n-n^{1/3} \sigma_n \sqrt{\log n}} g_{i+1}^2 \mathbb{E}X_i^2 = \int_{n-n^{1/3} \sigma_n \sqrt{\log n}}^{1} \frac{1}{x} \left[ \frac{\alpha}{2} + o(1) \right] dx + O \left( \frac{1}{n} \cdot \frac{1}{n-n^{1/3} \sigma_n \sqrt{\log n}} \right) \tag{5.27}
\]

\[= \frac{\alpha}{2} \cdot \frac{2}{3} \log n(1 + o(1)).
\]

Since the indices \( i > n - n^{1/3} \sigma_n \sqrt{\log n} \) only contribute \( O(\sqrt{\log n}) \) to the sum, the lemma is proved. \( \square \)

6 Proof of Lemma 4.1: Uniform bounds for \( R_i \)

Rather than working directly with \( \{R_{i} \}_{i=3}^{n} \), we consider the alternative process \( \{\tilde{R}_{i} \}_{3 \leq i \leq n} \) given below.

Let \( \tilde{R}_{i} = \phi_{n-1/2}((R_{i})), \) where \( \phi_{u} \) for \( u > 0 \) is given by

\[
\phi_{u}(x) = \begin{cases} x, & |x| \leq u, \\ \frac{x}{|x|}u, & |x| > u. \end{cases}
\]
We also set
\[ \hat{R}_i^{(1)} = \frac{\bar{R}_{i-1}}{1 - \bar{R}_{i-1}}, \quad \hat{R}_i^{(2)} = \omega_i \frac{\bar{R}_{i-1}^3}{1 - \bar{R}_{i-1}}, \quad \hat{R}_i^{(3)} = \omega_i \bar{R}_{i-1}^2. \]

Consider the process
\[
\begin{align*}
\hat{R}_2 &= R_2, \\
\hat{R}_i &= L_i + \omega_i \ldots \omega_3 R_2 - A_{0i} + \hat{B}_{0i} + \hat{B}_{1i} + \hat{B}_{2i} + \hat{B}_{3i}, \quad 3 \leq i \leq n,
\end{align*}
\]
where
\[ A_{0i} = \gamma_i - \omega_i + \omega_i (\gamma_{i-1} - \omega_{i-1}) + \cdots + \omega_i \ldots \omega_4 (\gamma_3 - \omega_3), \]
and
\[
\hat{B}_{0i} = \left( \alpha_{i-1} + (\tau_{i-1} + \alpha_{i-1}) \hat{R}_i^{(1)} \right) \beta_i + \omega_i \left( \alpha_{i-2} + (\tau_{i-2} + \alpha_{i-2}) \hat{R}_i^{(1)} \right) \beta_{i-1}
+ \cdots + \omega_i \ldots \omega_4 \left( \alpha_2 + (\tau_2 + \alpha_2) \hat{R}_3^{(1)} \right) \beta_3,
\]
\[
\hat{B}_{1i} = \alpha_{i-1} \delta_i \hat{R}_i^{(1)} + \omega_i \alpha_{i-2} \delta_i \hat{R}_i^{(1)} + \cdots + \omega_i \ldots \omega_4 \alpha_2 \delta_3 \hat{R}_3^{(1)},
\]
\[
\hat{B}_{2i} = \hat{R}_i^{(2)} + \omega_i \hat{R}_i^{(2)} + \cdots + \omega_i \ldots \omega_4 \hat{R}_4^{(2)},
\]
\[
\hat{B}_{3i} = \hat{R}_i^{(3)} + \omega_i \hat{R}_i^{(3)} + \cdots + \omega_i \ldots \omega_4 \hat{R}_4^{(3)}.
\]

On the event \( \max_{2 \leq i \leq n} |\hat{R}_i| = o(n^{-1/3}) \), observe that \( \hat{R}_i = \bar{R}_i \) for all \( i \geq 2 \). In particular, \( |\hat{R}_2| \leq n^{-1/3}/2 \), and \( \hat{R}_2 = \bar{R}_2 = R_2 \). This implies \( \hat{R}_3^{(\ell)} = \hat{R}_3^{(\ell)} \) for \( \ell = 1, 2, 3 \). As a result, \( \hat{R}_3 = R_3 \), which induces \( \hat{R}_4 = R_4 \) and so on. Therefore, by showing that
\[
\max_{2 \leq i \leq n} |\hat{R}_i| = o(n^{-1/3}), \quad \text{with probability } 1 - O(\log^{-5} n),
\]
we will obtain Lemma 4.1.

We check (6.3) by showing, uniformly in \( i \), each term in the decomposition (6.2) is sufficiently small with probability \( 1 - O(\log^{-5} n) \). First, we have \( \gamma_i - \omega_i = O \left( \frac{1}{n^{1-\omega_i}} \right) \) by Lemma 2.10, and \( (1 - \omega_i)^{-1} = O(n^{1/3} \sigma_n^{-1/2}) \) uniformly in \( i \) by Corollary 2.9. Thus,
\[
A_{0i} < \frac{\max_{3 \leq j \leq i} (\gamma_j - \omega_j)}{1 - \omega_j} = O \left( \frac{1}{n(1 - \omega_i)^2} \right) = o(n^{-1/3}).
\]

At the same time, a direct computation shows that
\[
R_2 = 1 + \alpha_2 + \tau_2 + \beta_2 + \delta_2 = \frac{\gamma_2}{\delta_2}, \quad \bar{R}_2 = \frac{(\alpha_1 + \tau_1) (\beta_2 + \delta_2)}{\alpha_1 - \gamma_2 - (m-n+1)},
\]
where in the last equality, we apply the identity
\[
\tau_i + \delta_i (1 + \tau_{i-1}) + 1 - \frac{\gamma_m}{|\beta_i|} = - (\gamma_i - \omega_i).
\]

By Lemma 2.10 and Corollary 2.9, \( |\omega_2 - \gamma_2| = O \left( \frac{1}{n^{1-\omega_2}} \right) = O(n^{-1}) \). Moreover, by Lemma 2.6, \( |\alpha_1| = O(n^{-1/2} \log^{1/2} n) \) with probability \( 1 - O(n^{-1}) \) and \( |\alpha_2| \vee |\beta_2| = O(n^{-1/2} \log^{1/2} n) \) with probability \( 1 - O(n^{-1}) \). Therefore,
\[
|\bar{R}_2| = O(n^{-1/2} \log^{1/2} n), \quad \text{with probability } 1 - O(n^{-1}).
\]

We now show uniform bounds for all four sequences \( \{\bar{B}_j\} \), \( 0 \leq j \leq 3 \) in Subsection 6.1. The uniform bound for \( L_i \) is provided in Subsection 6.2. Throughout these subsections, all the Big-O bounds are uniformly in \( i \), for \( 3 \leq i \leq n \).
6.1 Uniform bound for $\tilde{B}_{ji}$, $0 \leq j \leq 3$

For fixed $i$, $\tilde{B}_{0i}$ is a sum of random variables

$$Z_j := \omega_{j+1} \ldots \omega_i \left( \alpha_{j-1} + (\tau_{j-1} + \alpha_{j-1}) \tilde{R}_i^{(1)} \right) \beta_j, \quad 3 \leq j \leq i. \tag{6.7}$$

Let $\mathcal{F}_i$ be the $\sigma$-algebra generated by $\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i$. Observe that $\tilde{R}_i^{(1)}$ is $\mathcal{F}_{i-1}$-measurable and $\mathbb{E}[Z_j | \mathcal{F}_{j-1}] = 0$ a.s. for all $j$. By Theorem 2.1 of [21] (Marcinkiewicz–Zygmund type inequality), for any integer $p > 2$,

$$\|\tilde{B}_{0i}\|_p^2 \leq (p - 1)(\|Z_i\|_p^2 + \|Z_{i-1}\|_p^2 + \cdots + \|Z_3\|_p^2). \tag{6.8}$$

By Lemma 2.6, there exists absolute constant $C > 0$ such that for all integers $p > 2$ and all $3 \leq j \leq n$,

$$\|\alpha_{j-1}\|_p < C \alpha pn^{-1/2} \quad \text{and} \quad \|\beta_j\|_p < C \alpha pn^{-1/2}. \quad \text{Also, } |\tilde{R}_i^{(1)}| \leq n^{-1/3}. \quad \text{Hence, for } p = [2 \log n],

$$\|\tilde{R}_i^{(1)}\|_p^2 = \|\alpha_{j-1} + (\tau_{j-1} + \alpha_{j-1}) \tilde{R}_i^{(1)}\|_p^2 \leq \left( \|\alpha_{j-1}\|_p + n^{-1/3}(\tau_{j-1} + \|\alpha_{j-1}\|_p) \right)^2 \leq C \alpha^2 p^2 n^{-5/3}.$$ 

From (6.8),

$$\|\tilde{B}_{0i}\|_p^2 < C(p - 1)p^2 \alpha^2 n^{-5/3} \leq 8C \alpha^2 n^{-4/3} \sigma_n^{-1/2} \log^3 n. \tag{6.9}$$

Apply Markov’s inequality and take union bound, we obtain that with probability at least $1 - \frac{1}{n}$,

$$|\tilde{B}_{0i}| \leq e \|\tilde{B}_{0i}\|_2 \log n = o(n^{-1/2}) \quad \text{for every } 3 \leq i \leq n. \tag{6.10}$$

Since $\mathbb{E}[\alpha_{i-1} \delta_i \tilde{R}_i^{(1)} | \mathcal{F}_{j-1}] = \alpha_{i-1} \delta_i \tilde{R}_i^{(1)}$ for all $i \leq n$, which is nonzero with positive probability, we cannot apply Theorem 2.1 of [21] to bound $|\tilde{B}_{1i}|$. Instead, we use Minkowski’s inequality. Let $p = 2 \log n$ as before.

$$\|\tilde{B}_{1i}\|_p \leq n^{-1/3}(\delta_i \|\alpha_{i-1}\|_p + \sum_{j=3}^{i-1} \omega_{j+1} \ldots \omega_i \delta_j \|\alpha_{j-1}\|_p) < \frac{\delta_i n^{-1/3}}{1 - \omega_i} \max_{2 \leq j \leq i-1} \|\alpha_j\|_p < C \alpha pn^{-1/2} \sigma_n^{-1/2} = O(n^{-1/2} \sigma_n^{-1/2} \log n). \tag{6.11}$$

Thus, with probability at least $1 - \frac{1}{n}$,

$$|\tilde{B}_{1i}| \leq e \|\tilde{B}_{1i}\|_2 \log n = O(n^{-1/2} \sigma_n^{-1/2} \log n) \quad \text{for every } 3 \leq i \leq n. \tag{6.12}$$

Lastly, $|\tilde{R}_i^{(2)}| = \omega_i |\tilde{R}_i^{(3)}| \leq n^{-1}$, $|\tilde{R}_i^{(3)}| = \omega_i |\tilde{R}_{i-1}| \leq n^{-2/3}$, so uniformly in $i$,

$$|\tilde{B}_{2i}| < \frac{n^{-1}}{1 - \omega_i} = O(n^{-2/3} \sigma_n^{-1/2}), \quad \text{and} \quad |\tilde{B}_{3i}| < \frac{n^{-2/3}}{1 - \omega_i} = O(n^{-1/3} \sigma_n^{-1/2}). \tag{6.13}$$

We have now bounded all the terms of $\tilde{R}_i$, except for $L_i$. We provide a uniform bound in $i$ for this quantity in the following subsection. This will conclude the proof of Lemma 2.1.

6.2 Uniform bound for $L_i$

Recall that each $L_i$ is $Y_i$ plus a small term $\alpha_i - \omega_3 \ldots \omega_i \alpha_2$, where $Y_i$ is a weighted sum of independent random variables,

$$Y_i = \sum_{j=3}^{i} \omega_{j+1} \ldots \omega_i X_j + X_i, \quad 3 \leq i \leq n. \tag{6.14}$$

We first show $Y_i$ is small, uniformly in $i$, in the lemma below.
Lemma 6.1. Assume \((\log \log n)^2 \ll \sigma_n \ll (\log n)^2\). Then with probability \(1 - O(\log^{-5} n)\),

\[
\max_{3 \leq i \leq n} |Y_i| = O \left( \frac{\log \log n}{(n^{1/3} \sigma_n^{1/2})} \right).
\]

In the course of the proof of Lemma 6.1, we need the following lower bound of the product \(\omega_{j+1}\ldots\omega_i\).

Lemma 6.2. If \(i \geq n - n^{1/3} \log^3 n\) and \(i < j \leq i + n^{1/3} \log^{-2} n\), then \(\omega_{i+1}\ldots\omega_j \geq \frac{1}{2}\).

Proof of Lemma 6.2. Since \(\omega_i\) is increasing in \(i\), \(\log(\omega_{i+1}\ldots\omega_j) \geq (j - i) \log \omega_{i+1}\). We have \(\omega_i \geq 1 - Cn^{-1/3} \log^{3/2} n\) for some constant \(C > 0\). There exists \(C_1 > 0\) such that \(\log(1 - x) \geq -C x\) for all \(x \in (0, 1)\), so

\[
\log \omega_{i+1} \geq -C_1 n^{-1/3} \log^{3/2} n
\]

for some \(C_1 > 0\). If \(i < j \leq i + n^{1/3} \log^{-2} n\), then

\[
\log(\omega_{i+1}\ldots\omega_j) \geq (j - i) \log \omega_{i+1} \geq -C \log^{1/2} n \geq \log(1/2).
\]

Proof of Lemma 6.1. By Lemma 2.5, \(Y_i \in \text{SG}(v_{Y_i}, u_{Y_i})\) where

\[
v_{Y_i} = \sum_{j=3}^{i} (\omega_{j+1} \ldots \omega_j)^{2} v_j + v_i \leq \frac{v_i}{1 - \omega_i^2} = \frac{\alpha(i - 1)(1 + \tau_i - 1)^2}{|\rho_i^+|^2(1 - \omega_i)},
\]

\[
u_{Y_i} = \max \left\{ \frac{\alpha}{|\rho_i^+|}, \omega_{j+1} \ldots \omega_i(1 + \tau_i - 1) \frac{\alpha}{|\rho_j^+|} : 3 \leq j \leq i - 1 \right\} \leq \frac{\alpha(1 + \tau_i - 1)}{|\rho_i^+|}.
\]

There exists a constant \(C > 0\) is such that \(1 + \tau_j \leq C\) for all \(n\) and all \(3 \leq j \leq n\). Thus, by Lemma 2.6, for each \(i\),

\[
P \left( |Y_i| > \sqrt{\frac{C^2 \alpha(i - 1)t}{|\rho_i^+|^2(1 - \omega_i)} + \frac{C \alpha t}{|\rho_i^+|}} \right) \leq 2e^{-t}.
\]

(6.16)

Change variable \(t \mapsto t + \log 2n\) and take union bound, we have

\[
P \left( \forall 3 \leq i \leq n : |Y_i| > \sqrt{\frac{C^2 \alpha(i - 1)(t + \log 2n)}{|\rho_i^+|^2(1 - \omega_i)} + \frac{C \alpha(t + \log 2n)}{|\rho_i^+|}} \right) \leq e^{-t}.
\]

(6.17)

Fix \(\eta > 0\) and consider \(i \leq n - n^{1/3} \log^{2+\eta} n\). By Corollary 2.9(i), \(1 - \omega_i > C_1 \left( \frac{n+i}{n} \right)^{1/2}\). Therefore, for \(t = \log n\),

\[
\sqrt{\frac{C^2 \alpha(i - 1)(t + \log 2n)}{|\rho_i^+|^2(1 - \omega_i)}} + \frac{C \alpha(t + \log 2n)}{|\rho_i^+|} = O \left( \sqrt{\frac{(i - 1) \log n}{n^2(n^{1/3} \log^{2+\eta} n)^{1/2}}} \right) = O(n^{-1/3} \log^{-n/4} n).
\]

Take \(\eta = 1/2\). We have shown that with probability \(1 - O(n^{-1})\),

\[
\max_{3 \leq i \leq n - n^{1/3} \log^{2+\eta} n} |Y_i| = O(n^{-1/3} \log^{-1/2} n).
\]

(6.18)

Now consider \(i > n - n^{1/3} \log^{2+\eta} n\). By Corollary 2.9 there exists \(c > 0\) such that for all \(3 \leq i \leq n\),

\[
1 - \omega_i > cn^{-1/3} \sigma_n^{1/2}.
\]

(6.19)

By (6.16), this implies that for some \(c_1 > 0\),

\[
P \left( |Y_i| > c_1 \frac{\sqrt{\log \log n}}{n^{1/3} \sigma_n^{1/4}} \right) \leq \frac{2}{\log^{10} n}.
\]

(6.20)
That is, we have \( |Y_i| = o(n^{-1/3}) \) for each \( i > n - n^{1/3} \log^{2+n} n \), but the probability bound is too large to apply union bound over this range of indices. Instead, we apply (6.20) to a small number of indices \( i > n - n^{1/3} \log^{2+n} n \), say \( K \) of them. We then bound the maximum \( Y_i \) over the \( K + 1 \) subsets partitioned by these indices.

Define for \( 2 \leq i < j \leq n \),
\[
\tilde{Y}_j^i = \frac{Y_j}{\omega_{i+1} \ldots \omega_j} - Y_i. \tag{6.21}
\]

Note that \( \tilde{Y}_i^i = 0 \). As \( \{Y_j\} \) satisfies \( Y_j = \omega_j Y_{j-1} + X_j \), we have the recursion
\[
\tilde{Y}_j^i = \tilde{Y}_{j-1}^{i+1} + \frac{X_j}{\omega_{i+1} \ldots \omega_j}. \tag{6.22}
\]

Thus, \( \tilde{Y}_j^i \) for fixed \( i \) is a sum of independent random variables \( \frac{X_k}{\omega_{i+1} \ldots \omega_k} \) for \( k = i+1, \ldots, j \). We now show that \( \tilde{Y}_j^i \) is also subgamma. Let \( i \) and \( j \) be as given in Lemma 6.2. By Lemma 2.5, \( X_k \in \text{SG}(v_k, u_k) \) where \( v_k \) and \( u_k \) are increasing in \( k \). Moreover, \( u_j = \frac{\alpha (1+\tau_j - 1)}{\rho_j} \leq C \frac{n}{\log n} \) and
\[
\sum_{k=i+1}^j v_k \leq (j-i)v_j \leq 2C^2 \alpha n^{1/3} \frac{\log \log n}{\rho_j} \leq O \left( n^{-2/3} \log^{-2} n \right). \tag{6.23}
\]

Hence, for some \( C > 0 \),
\[
\tilde{Y}_j^i = \frac{X_{i+1}}{\omega_{i+1}} + \cdots + \frac{X_j}{\omega_{i+1} \ldots \omega_j} \in \text{SG} \left( \frac{C \alpha}{n^{2/3} \log^2 n}, \frac{C \alpha}{n} \right). \tag{6.24}
\]

Applying Lemma 2.6 with \( t = 10 \log \log n \), we have for some \( C > 0 \) and sufficiently large \( n \),
\[
\max_{1 \leq j \leq n} \mathbb{P} \left( |\tilde{Y}_j| > C \frac{\sqrt{\log \log n}}{n^{1/3} \log n} \right) \leq \frac{2}{\log^{10} n}. \tag{6.25}
\]

By Etemadi’s theorem [7],
\[
\mathbb{P} \left( \max_{1 \leq j \leq n} |\tilde{Y}_j| > 3C \frac{\sqrt{\log \log n}}{n^{1/3} \log n} \right) \leq 3 \max_{1 \leq j \leq n} \mathbb{P} \left( |\tilde{Y}_j| > C \frac{\sqrt{\log \log n}}{n^{1/3} \log n} \right) \leq \frac{6}{\log^{10} n}. \tag{6.26}
\]

Here, the power 10 can be made larger by choosing sufficiently large \( C \).

We now pick \( K \) indices as proposed previously. Choose \( n_0 < n_1 < \cdots < n_K = n \) where \( K \leq 2 \log^5 n \) so that \( n_0 \leq n - n^{1/3} \log^3 n \) and
\[
\frac{n^{1/3}}{2 \log^2 n} \leq n_k - n_{k-1} \leq \frac{n^{1/3}}{2 \log^2 n}. \tag{6.27}
\]

Take union bound of (6.20) over the set \( \{n_k\}_{k=0}^K \), and take union bound of (6.25) over \( K \) pairs \( \{(n_{k-1}, n_k)\}_{k=0}^K \) to have
\[
|Y_{n_k-1}| \leq C \frac{\sqrt{\log \log n}}{n^{1/3} \sigma_n^{1/4}} \quad \text{and} \quad \max_{n_{k-1} \leq j \leq n_k} \left| \tilde{Y}_j^{n_k-1} \right| \leq 4C \frac{\sqrt{\log \log n}}{n^{1/3} \log n}. \tag{6.28}
\]

for all \( K > 0 \) with probability \( 1 - O(\log^5 n) \). On this event, for every \( k = 0, \ldots, K \), if \( j \in [n_{k-1}, n_k] \) then
\[
|Y_j| < |Y_{n_k-1}| + \left| \tilde{Y}_j^{n_k-1} \right| \leq 5C \frac{\sqrt{\log \log n}}{n^{1/3} \sigma_n^{1/2}}. \tag{6.29}
\]

Together with (6.18), we conclude
\[
\max_{3 \leq i \leq n} |Y_i| = O \left( \frac{\sqrt{\log \log n}}{n^{1/3} \sigma_n^{1/2}} \right) \quad \text{with probability} \quad 1 - O(\log^{-5} n). \tag{6.30}
\]
Lastly, note that $L_i = Y_i + s_i$, where
\[
s_i := \alpha_i - \omega_3 \ldots \omega_i \alpha_2 \in \text{SG} \left( \frac{2\alpha \sigma_i}{|\rho_i^2|}, \frac{\alpha}{|\rho_i^2|} \right) \subset \text{SG} \left( \frac{C\alpha}{n}, \frac{\alpha}{n} \right).
\] (6.29)

The rightmost sub-gamma family is independent of $i$. Apply Lemma \[2.6\] with $t = n^{1/3-\epsilon}$ for small $\epsilon > 0$ and take the union bound,
\[
\mathbb{P} \left( \max_{3 \leq i \leq n} |s_i| > \frac{n^{1/6-\epsilon}/2}{n^{1/2}} \right) \leq \sum_{i=3}^{n} \mathbb{P} \left( |s_i| > \frac{n^{1/6-\epsilon}/2}{n^{1/2}} \right) \leq Cn \exp \left( -n^{1/3-\epsilon} \right),
\] (6.30)
for some $C > 0$. This completes our proof of Lemma \[6.1\].

We now combine the bounds from all previous subsections. For $t = n^{-1/3}(\log \log n)^{-1/2}$,
\[
\mathbb{P}( \max_{2 \leq i \leq n} |\tilde{R}_i| > 12t) \leq \mathbb{P}( |R_2| \geq 6t) + \mathbb{P}( \max_{3 \leq i \leq n} |\tilde{R}_i| > 6t) \\
\leq \frac{1}{n} + \mathbb{P}( \max_{3 \leq i \leq n} |L_i| \geq t) + \mathbb{P}( |R_2| \geq t) + \mathbb{P}( \max_{3 \leq i \leq n} |A_{0i}| \geq t) + \mathbb{P}( \max_{3 \leq i \leq n} |\tilde{B}_{0i}| \geq t) \\
+ \mathbb{P}( \max_{3 \leq i \leq n} |\tilde{B}_{1i}| \geq t) + \mathbb{P}( \max_{3 \leq i \leq n} |\tilde{B}_{2i}| \geq t) + \mathbb{P}( \max_{3 \leq i \leq n} |\tilde{B}_{3i}| \geq t) \\
= O(\log^{-5} n),
\]
and we obtain Lemma \[4.1\].

7 Extension all the way to the edge (Theorem 1.2)

We now consider the case where the sequence $\{\sigma_n\}_n$ satisfies
\[
\text{for some constant } \tau > 0, \quad -\tau < \sigma_n \ll (\log n)^2 \quad \text{for all } n \in \mathbb{N},
\] (7.1)
and restrict the matrix ensemble to LUE or LOE. We begin by extending Theorem \[1.1\] to this broader range of $\sigma_n$ in the case of LUE, utilizing spectral properties of LUE derived from its determinantal representation (see in particular \[10\]) in our proof. We then extend the result to LOE matrices using the relationship between eigenvalues of unitary and orthogonal ensembles (see \[8\]).

Remark 7.1. Using a similar technique (drawing on results in \[8\]), this result could be extended to the symplectic ensemble ($\beta = 4$). In fact, we expect it to hold for all $\beta > 0$, although proving this would require a substantially different set of techniques that does not rely on determinantal structures (perhaps similar to techniques used in \[15\]). Here, we restrict our proof to LUE and LOE, which are the relevant cases for statistical and spin glass applications.

7.1 Set-up

Define for $x \in \mathbb{R}$,
\[
S_n(x) = \sum_{i=1}^{n} \log |d_+ + x n^{-2/3} - \mu_i| - C_n - \frac{1}{\lambda^{1/2}(1 + \lambda^{1/2})} \sigma_n n^{1/3} + \frac{2}{3 \lambda^{3/4}(1 + \lambda^{1/2})^2} \sigma_n^{3/2} + \frac{\alpha - 1}{6} \log n.
\] (7.2)

Theorem \[1.1\] implies that for $\tilde{\sigma}_n = (\log n)^3$,
\[
\frac{S_n(\tilde{\sigma}_n)}{\sqrt{\tilde{\sigma}_n \log n}} \xrightarrow{d} \mathcal{N}(0, 1).
\]

We will show the exact CLT holds for $S_n(\sigma_n)$ by showing that with probability $1 - o(1)$,
\[
S_n(\tilde{\sigma}_n) - S_n(\sigma_n) = o(\sqrt{\log n}).
\] (7.3)
Let
\[ \varepsilon_n = n^{-2/3}(\bar{\sigma}_n - \sigma_n), \]
\[ \ell_i = \log((\gamma - \mu_i) + \varepsilon_n) - \log|\gamma - \mu_i| - (\gamma - \mu_i)^{-1}\varepsilon_n. \]

Note that \( \varepsilon_n \) as above is not the same as \( \varepsilon_n \) in (2.18) that arises from the three-term recurrence. We then write
\[ S_n(\bar{\sigma}_n) - S_n(\sigma_n) = \sum \{ \log(\gamma - \mu_i + \varepsilon_n) - \log|\gamma - \mu_i| \} - \frac{n^{1/3}(\bar{\sigma}_n - \sigma_n)}{\lambda^{1/2}(1 + \lambda^{1/2})} + \frac{2(\bar{\sigma}_n^3 - \sigma_n^3)}{3\lambda^{3/4}(1 + \lambda^{1/2})^2} \]
\[ = \sum \ell_i + \varepsilon_n \left( \sum_{i=1}^{n} \frac{1}{\gamma - \mu_i} - \frac{1}{\lambda^{1/2}(1 + \lambda^{1/2})^2} \right) + O(\bar{\sigma}_n^2). \quad (7.4) \]

The first sum \( \sum \ell_i \) can be approximated by a linear eigenvalue statistics, using the following two lemmas. The proof of Lemma 7.2 is included in Subsection 7.4. For Lemma 7.2 we let \( \eta_1, \ldots, \eta_n \) be unordered eigenvalues of \( \frac{1}{n}M_{n,m} \). Let \( p_{n,LUE}(x) \) and \( p_{n,LOE}(x) \) be the normalized one-point correlation functions of \( \eta_1, \ldots, \eta_n \) in the case where \( M_{n,m} \) is from LUE and LOE respectively.

**Lemma 7.2.** Let \( z_\lambda = d_+^1 \lambda^{-1/6} \). Given \( s_0 \in \mathbb{R} \), there exists \( C = C(s_0) > 0 \) such that for sufficiently large \( n \), for all \( s \geq s_0 \), both of the following statements hold.
\[ p_{n,LUE}(d_+ + sn^{-2/3}) \leq Cn^{-1/3} \exp(-2s_\lambda s). \]
\[ p_{n,LOE}(d_+ + sn^{-2/3}) \leq Cn^{-1/3} \exp(-s_\lambda s). \]

**Lemma 7.3.** Let \( M_{n,m} \) be a scaled LOE/LUE. Assume \( \sigma_n > -\tau \) for all \( n \). Let \( \gamma = d_+ + \sigma_n^{-2/3} \) and \( \bar{\gamma} = d_+ + \bar{\sigma}_n^{-2/3} \). For \( \epsilon > 0 \), there exists \( k = k(\epsilon, \tau) > 0 \) such that for sufficiently large \( n \),
\[ \mathbb{P}(\mu_1 > \bar{\gamma} - n^{-2/3}) < \epsilon, \quad \mathbb{P}(\mu_k > \gamma) < \epsilon. \]
Furthermore, there exist \( c_i = c_i(\epsilon, \tau), i = 1, 2 \) such that for sufficiently large \( n \),
\[ \mathbb{P}(\min_{i \leq n} |\gamma - \mu_i| < c_1n^{-2/3}) < \epsilon, \quad \mathbb{P}(\max_{i \leq k} |\gamma - \mu_i| > (c_2 + |\sigma_n|)n^{-2/3}) < \epsilon. \]

**Proof.** Lemma 7.3 of this paper is the LUE/LOE version of Lemma 4 of [12]. There, letting \( E = 2 + \sigma_n^{-2/3} \) and \( \bar{E} = 2 + \bar{\sigma}_n^{-2/3} \), the probability bounds on the distance between location of singularities \( E, \bar{E} \) to the eigenvalues of scaled GUE/GOE take the exact form as in (7.5) and (7.6). The key ingredient to the proof is the convergence to the Tracy-Widom law \( F_2 \) (of type 2 and 1 for the unitary and orthogonal case, respectively) of the \( j \)th largest eigenvalues (after properly shifted and scaled) for all \( j \leq k \) for some fixed \( k \). Since the \( k \)th largest eigenvalues of LUE matrices also satisfy Tracy-Widom convergence, the same proof argument applies. In particular, by replacing their notations with the analogous ones provided in the Table 1, we obtain a proof for our lemma.

By Lemma 7.3 there exists a \( k > 0 \) such that with probability at least \( 1 - \epsilon \), for \( i \leq k \),
\[ |\ell_i| = \left| \log(n^{2/3}(\gamma - \mu_i) + \bar{\sigma}_n - \sigma_n) - \log(n^{2/3}(\gamma - \mu_i)) - (\gamma - \mu_i)^{-1}\varepsilon_n \right| \]
\[ \leq \log(3\bar{\sigma}_n) + \log(c_2 + |\sigma_n|) + \frac{n^{-2/3}}{c_1} \bar{\sigma}_n \leq c_3 \bar{\sigma}_n, \]
for some constant \( c_3 = c_3(\epsilon, \tau) > 0 \). In the case \( i > k \), by the fact \( |\log(1 + x) - x| \leq x^2/2 \) for \( x \geq 0 \), we obtain
\[ |\ell_i| = |\log(1 + (\gamma - \mu_i)^{-1}\varepsilon_n) - (\gamma - \mu_i)^{-1}\varepsilon_n| \leq \frac{1}{2} \frac{\varepsilon_n^2}{(\gamma - \mu_i)^2}. \]

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depending on $\epsilon > 0$ and prove the following lemma about the Marˇ cenko–Pastur measure (see Proposition 7.4).

As this section focuses solely on LUE matrices, we denote $p_{\text{LUE}}$ simply by $p_n$ throughout the section. For our proofs below, we will need the following result from Götze and Tikhomirov:

**Proposition 7.4.** Consider $\gamma = d_+ + \sigma_n n^{-2/3}$ where $\sigma_n$ satisfies (7.1), and $\alpha = 1$ or $\alpha = 2$. Then for any $\epsilon > 0$, with probability at least $1 - \epsilon$, the following two equations hold.

$$
\sum_{i=1}^{n} (\gamma - \mu_i)^{-1} - \frac{n}{\lambda^{1/2} (1 + \lambda^{1/2})} = O \left( \left( 1 + |\sigma_n|^{1/2} \right) n^{2/3} \right),
$$

$$
\sum_{i=1}^{n} (\gamma - \mu_i)^{-2} = O(n^{4/3}).
$$

The proof of Proposition 7.4 is first provided for the LUE case in Section 7.2 and the proof of the LOE case is included in Section 7.3. Applying Proposition 7.4 and the bounds (7.7) to (7.4), we obtain

$$
S_n(\sigma_n) - S_n(\sigma_n) = O(\sigma_n^2) = o(\sqrt{\log n})
$$
as claimed, and this completes the proof of Theorem 1.2.

### 7.2 Proof of Proposition 7.4 for LUE

As this section focuses solely on LUE matrices, we denote $p_{\text{LUE}}$ simply by $p_n$ throughout the section. For our proofs below, we will need the following result from Götze and Tikhomirov:

**Lemma 7.5 (Theorems 1.5 and 1.6, [10]).** Let $M_{n,m}$ denote an LUE matrix where $\frac{n}{m} \rightarrow \lambda \leq 1$ as $n, m \rightarrow \infty$. Let $p_n$ denote the expected spectral density of the empirical spectral measure on $M_{n,m}$, and let $p_{MP}$ be that of the Marˇ cenko–Pastur measure (see (1.3) for definition of these measures). There exist constants $C, a > 0$ depending on $\lambda$ such that, for $x \in [d_- + a n^{-2/3}, d_+ - a n^{-2/3}],$

$$
|p_n(x) - p_{MP}(x)| \leq \frac{C}{n(d_+ - x)(x - d_-)}. \quad (7.8)
$$

Furthermore, for $\lambda = 1$, this holds on the larger interval $x \in [d_- + an^{-2}, d_+ - an^{-2/3}].$

As an initial step toward proving Proposition 7.4, we define

$$
f_c(x) = \frac{1}{\gamma - x} 1\{|\gamma - x| > cn^{-2/3}\} \quad (7.9)
$$

and prove the following lemma about $f_c(x)$.
Lemma 7.6. Let $\sigma_n$ be in the range $-\tau \leq \sigma_n \leq (\log \log n)^3$. Then, for each $c > 0$, we have

$$
\mathbb{E} \frac{1}{n} \sum_{j=1}^{n} f^l_n(\mu_j) = \begin{cases} 
\frac{1}{\lambda^{l/2}(1 + \lambda^{1/2})} + O \left( (1 + |\sigma_n|^{1/2})n^{-1/3} \right), & l = 1 \\
O(\lambda^{-l/2}), & l = 2.
\end{cases} 
$$

(7.10)

Proof. This lemma is analogous to Lemma 18 in [12] and we follow a similar proof method. We have

$$
\frac{1}{n} \sum_{j=1}^{n} f^l_n(\mu_j) = \int f^l_n(x)p_n(x)dx.
$$

(7.11)

This integral with respect to $p_n$, is well approximated by the integral with respect to $p_{MP}$ from the Marčenko–Pastur measure, so our first task is to bound the error in making this change of measure. More specifically, we will bound the difference by considering the integral over disjoint intervals:

$$
\left| \int f^l_n p_n - \int f^l_n p_{MP} \right| \leq \int_{I_n} \left| f^l_n \cdot (p_n - p_{MP}) \right|(x)dx + \int_{J_n^{-} \cup J_n^{+}} \left| f^l_n \cdot (p_n - p_{MP}) \right|(x)dx,
$$

(7.12)

where the intervals $J_n^{-}, I_n, J_n^{+}$ are defined differently for the case of $\lambda < 1$ and $\lambda = 1$ such that the middle interval, $I_n$, corresponds to range on which we can apply the bounds in Lemma 7.5. In particular, for any $a > 0$ and for $\lambda < 1$, we define

$$
J_n^{-} = (0, d_- + an^{-2/3}), \quad I_n = [d_- + an^{-2/3}, d_+ - an^{-2/3}], \quad J_n^{+} = (d_+ - an^{-2/3}, \infty).
$$

(7.13)

If $\lambda = 1$, we set

$$
J_n^{-} = (0, an^{-2}), \quad I_n = [an^{-2}, d_+ - an^{-2/3}], \quad J_n^{+} = (d_+ - an^{-2/3}, \infty).
$$

(7.14)

For the integral over $J_n^{-} \cup J_n^{+}$, we use the upper bound

$$
\sup_{J_n^{\pm}} |f^l_n| \int_{J_n^{\pm}} (p_n + p_{MP}) + \sup_{J_n^{\pm}} |f^l_n| \int_{J_n^{\pm}} (p_n + p_{MP}).
$$

(7.15)

On $J_n^{+}$, we have $|f^l_n| = O(n^{2l/3})$. Direct computation shows that $\int_{J_n^{+}} p_{MP} = O(n^{-1})$ and, using the edge bounds from Lemma 7.2, we see that,

$$
\int_{J_n^{+}} p_n(x)dx = n^{-2/3} \int_{-\frac{an}{2}}^{\infty} p_n(2 + sn^{-2/3})ds = O(n^{-1}).
$$

(7.16)

Thus, we conclude that

$$
\sup_{J_n^{+}} |f^l_n| \int_{J_n^{+}} (p_n + p_{MP}) = O(n^{2l-1}).
$$

(7.17)

On the interval $J_n^{-}$, the function $|f^l_n|$ is bounded above by a constant. Two separate computations for $\lambda = 1$ and $\lambda < 1$ show that $\int_{J_n^{-}} p_{MP} = O(n^{-1})$. For $p_n$, we observe that

$$
\int_{J_n^{-}} p_n \leq 1 - \int_{I_n} p_n \leq 1 - \int_{I_n} p_{MP} + \int_{I_n} |p_n - p_{MP}|.
$$

(7.18)

We have $1 - \int_{I_n} p_{MP} = O(n^{-1})$ based on our computations of $\int_{I_n} p_{MP}$ and $\int_{J_n^{-}} p_{MP}$. For the difference of measures, we apply Lemma 7.5 and obtain

$$
\int_{J_n^{-}} |p_n - p_{MP}| \leq C \int_{J_n^{-}} \frac{1}{(d_+ - x)(x - d_-)}dx \leq C \int_{J_n^{-}} \frac{1}{(d_+ - x)(x - d_-)}dx \leq \frac{2C}{n} \int_{\frac{d_+ + d_-}{2}}^{\frac{d_+ + d_-}{2}} \frac{1}{(d_+ - x)(x - d_-)}dx = O(n^{-1} \log n).
$$

(7.19)
We conclude that
\[ \sup_{J^c_n} |f^l_c| \int_{J^c_n} (p_n + p_{MP}) = O(n^{-1}\log n). \]  
(7.20)

For the integral over \( I_n \), we consider separately the intervals \( I_n^- := I_n \cap [0,1] \) and \( I_n^+ := I_n \cap (1,\infty) \). On \( I_n^- \), the function \(|f^l_c|\) is bounded above by a constant. Combining this with line (7.19), we get
\[ \int_{I_n^-} |f^l_c| \cdot |p_n - p_{MP}| = O(n^{-1}\log n). \]  
(7.21)

Next, we bound the integral on \( I_n^+ \). Making the substitution \( x = d_+ - u n^{-2/3} \), we obtain
\[
\int_{I_n^+} |f^l_c| \cdot |p_n - p_{MP}| \leq \int_1^{d_+ - an^{-2/3}} \frac{1}{|\gamma - x|} \cdot \frac{C}{n(d_+ - x)(x-d_-)} dx
\]
\[= O(n^{\frac{2}{3}l-1}) \cdot \int_a^{(d_+ - u n^{-2/3})^{1/3}} \frac{1}{|u + \sigma_n|^l} \cdot \frac{du}{u} \]
\[\leq O(n^{\frac{2}{3}l-1}) \int_{\min(a,c)}^{\infty} \frac{1}{u^{l+1}} du = O(n^{\frac{2}{3}l-1}) \]
(7.22)

Putting together the results from (7.17), (7.20), (7.21) and (7.22), we have shown that
\[ \left| \int f^l_c p_n - \int f^l_c p_{MP} \right| = O(n^{\frac{2}{3}l-1}). \]  
(7.23)

It remains to compute the integral of \( f^l_c \) with respect to the Marčenko–Pastur measure.

For \( \sigma_n \geq c \), the integral \( \int f^l_c p_{MP} \) is \(-s_{MP}(\gamma)\), where \( s_{MP}(z) \) denotes the Stieltjes transform of \( p_{MP} \), given by
\[ s_{MP}(z) := \int \frac{1}{x-z} p_{MP}(x) dx = \frac{-z - \lambda + 1 + \sqrt{(z - \lambda - 1)^2 - 4\lambda}}{2\lambda z}. \]  
(7.24)

Likewise, \( \int f^l_c p_{MP} \) is the derivative of the Stieltjes transform, evaluated at \( \gamma \). Using this and \( \gamma = (1 + \sqrt{\lambda})^2 + \sigma_n n^{-2/3} \), we conclude for \( \sigma_n \geq c \),
\[ \int f^l_c(x)p_{MP}(x)dx = \left\{ \begin{array}{ll}
-s_{MP}(\gamma) = \frac{1}{\lambda^{1/2}(1+\lambda^{1/2})} + O((\sigma_n n^{-2/3})^{1/2}) & l = 1, \\
s'_{MP}(\gamma) = O((\sigma_n n^{-2/3})^{-1/2}) & l = 2.
\end{array} \right. \]  
(7.25)

In the case of \( \sigma_n < c \), we consider the integral over two sub-intervals \((d_-, d_c)\) and \((d_c, d_+)\), where we set \( d_c = d_+ + (\sigma_n - c)n^{-2/3} \). Observe that, on the first interval, \( f^l_c(x) = \frac{1}{(\gamma-x)} \) and, on the second interval, \( |f^l_c(x)| \leq c^{-l} n^{2/3} \). Thus, the integral on \((d_c, d_+)\) has the bound
\[ \left| \int_{d_c}^{d_+} f^l_c(x)p_{MP}(x)dx \right| = O \left( n^{2/3} \int_{d_c}^{d_+} \sqrt{d_+ - x} dx \right) = O(n^{\frac{2}{3}l-1}). \]  
(7.26)

For the integral on \((d_-, d_c)\), we consider the cases of \( l = 1 \) and \( l = 2 \) separately. For \( l = 1 \), we have
\[ \int_{d_+}^{d_-} f_c \cdot p_{MP} = \int_{d_+}^{d_-} \left( f_c(x) - \frac{1}{d_+ - x} \right) p_{MP}(x)dx + \int_{d_-}^{d_+} \frac{1}{d_+ - x} p_{MP}(x)dx - \int_{d_-}^{d_+} \frac{1}{d_+ - x} p_{MP}(x)dx \]
\[= \int_{d_+}^{d_-} \left( f_c(x) - \frac{1}{d_+ - x} \right) p_{MP}(x)dx + \frac{1}{\lambda^{1/2}(1+\lambda^{1/2})} + O(n^{-1/3}), \]
(7.27)

where the second equality holds by the fact that the middle term is equal to \(-s_{MP}(d_+)\). To bound the remaining integral on the right side, we have
\[ \int_{d_-}^{d_+} \left| f_c(x) - \frac{1}{d_+ - x} \right| p_{MP}(x)dx = |\sigma_n| n^{-2/3} \int_{d_-}^{d_+} \frac{1}{(\gamma-x)(d_+ - x)} p_{MP}(x)dx \]
\[= O \left( n^{-2/3} \int_{d_-}^{d_+} \frac{dx}{(\gamma-x)(\sqrt{x(d_+ - x)})} \right). \]  
(7.28)
Note that when \( \lambda = 1, d_- = 0 \) and the integrand contains a singularity at \( x = 0 \). However, the integral still remains bounded near that singularity, so we can replace the last line with \( O\left( n^{-2/3} \int_{d_-}^{d_+} \frac{dx}{(y-x)^2} \right) \). By the change of variable \( x = d_- - y^{-2/3} \), this becomes
\[
\int_{d_-}^{d_+} \frac{dx}{(y-x)^2} = n^{-1/3} \int_0^{d_+ - d_-} \frac{dy}{(y + c)^2} = O(n^{-1/3}).
\]
Thus, we have shown that, for \( \sigma_n < c \) and \( l = 1 \),
\[
\int f_{l, PM}^3 = \frac{1}{\lambda^{1/2}(1 + \lambda^{1/2})} + O(n^{-1/3}).
\]
It remains to bound the integral \( \int f_{l, PM}^3 \) in the case \( \sigma_n < c \) and \( l = 2 \). A bound on the portion over \( (d_-, d_+) \) is already obtained in (7.26). The the portion over \( (d_-, d_c) \) is
\[
\int_{d_-}^{d_c} f_{l, PM}^2(x) dx = O\left( \int_{d_-}^{d_c} \frac{\sqrt{d_+ - x}}{(y-x)^2} dx \right) = O\left( \int_{d_-}^{d_c} \frac{\sqrt{d_+ - x}}{(y-x)^2} dx \right),
\]
where the second equality follows by similar reasoning as above. Again, making the substitution \( x = d_- - y^{-2/3} \),
\[
\int_{d_-}^{d_c} \frac{\sqrt{d_+ - x}}{(y-x)^2} dx = n^{1/3} \int_0^{(d_c - d_-) n^{2/3}} \frac{\sqrt{y + (c - \sigma_n)}}{(y + c)^2} dy = O(n^{1/3}).
\]
This completes the proof of Lemma 7.6

Besides estimation of the expectation, we also need the following bound on the variance.

**Lemma 7.7.** If \( \eta_1, \ldots, \eta_n \) are the unordered eigenvalues of \( \frac{1}{m} M_{n,m} \), where \( M_{n,m} \) is sampled from LUE, then
\[
\operatorname{Var}\left[ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i) \right] \leq \frac{1}{n} \int f^2(x) p_{n,LUE}(x) dx.
\]

**Proof.** In Chapter V of [22], Laguerre polynomials \( L_n^{(a)} \) where \( a = m - n > -1 \) (for general \( \beta, a = \frac{\beta}{2}(m-1) - 1 \)) are given by two conditions:

1. \( \int_0^\infty L_j^{(a)}(x)L_k^{(a)}(x) dx = \Gamma(a+1)\binom{k+a}{k} \delta_{jk} \),
2. coefficient of \( x^k \) in \( L_n^{(a)}(x) \) has sign \((-1)^k\).

Let \( \phi_k(x; a) = h_k^{-1/2} x^{a/2} e^{-x/2} L_k^{(a)}(x) \), where \( h_k = \int_0^\infty L_k^{(a)}(x)^2 x^a e^{-x} dx \). Then \( \phi_k \) are orthonormal functions with respect to \((0, \infty)\).

Let \( f_n(x_1, \ldots, x_n) \) be the joint density of unordered eigenvalues \( \eta_1, \ldots, \eta_n \) of scaled LUE matrix \( \frac{1}{m} M_{n,m} \), \( m \geq n \). Let \( R_k(x_1, \ldots, x_n) \) for \( k \geq 1 \) be the corresponding \( k \)-point correlation function, and \( S_{n,LUE}(x) \) be the correlation kernel. Then,
\[
R_k(x_1, \ldots, x_k) = \frac{n!}{(n-k)!} \int \cdots \int f_n(x_1, \ldots, x_n) dx_{k+1} \ldots dx_n.
\]

Moreover, for any integrable function \( g \) that is symmetric in \( k \) variables,
\[
\mathbb{E} g(\eta_1, \ldots, \eta_k) = \frac{(n-k)!}{n!} \int \cdots \int g(x_1, \ldots, x_k) R_k(x_1, \ldots, x_k) dx_1 \ldots dx_k.
\]

Note that the \( k \)-point correlation function for unordered eigenvalues of the unscaled LUE, denoted by \( \tilde{R}_k \), is related to \( R_k \) by
\[
R_k(x_1, \ldots, x_k) = m^k \tilde{R}_k(mx_1, \ldots, mx_k).
\]
The normalized one-point correlation of (scaled) eigenvalues then satisfies
\[ p_n(x) = \frac{1}{n} R_1(x) - \frac{1}{\lambda} \hat{R}_1(mx). \] (7.35)

By the determinantal structure of the eigenvalues (see for example, Section 5.4 of [5]), \( \hat{R}_k \) satisfies
\[ \hat{R}_k(y_1, \ldots, y_k) = \text{det}(S_{n,LUE}(y_i, y_j))_{i,j=1,\ldots,k}, \]
where \( S_{n,LUE}(x, y) = \sum_{j=0}^{n-1} \phi_j(x; a) \phi_j(y; a) \) and \( a = m - n \). Thus \( R_1(x) = m S_{n,LUE}(x, x) \) and
\[ R_2(x, y) = m^2 [ R_1(mx) R_1(my) - S_{n,LUE}^2(mx, my) ] \]
\[ = n^2 (p_n(x)p_n(y) - \lambda^{-2} S_{n,LUE}^2(mx, my)). \]

Set \( I = E \left[ n^{-1} \sum_{i=1}^n f(\eta_i) \right]^2 \). We have
\[ I = n^{-2} E \left[ \sum_{i=1}^n f^2(\eta_i) \right] + n^{-2} E \left[ \sum_{i \neq j} f(\eta_i)f(\eta_j) \right] \]
\[ = n^{-1} E f^2(\eta_i) + n^{-2} n(n-1) E [ f(\eta_1) f(\eta_2) ] \]
\[ = n^{-2} \int f^2(x) R_1(x) dx + n^{-2} \int \int f(x)f(y) R_2(x, y) dxdy \quad \text{by (7.34)} \]
\[ = n^{-1} \int f^2(x) p_n(x) dx + \left( \int f(x) p_n(x) dx \right)^2 - \frac{1}{\lambda} \int \int f(x)f(y) S_{n,LUE}^2(mx, my) dxdy. \]

Write \( S_{n,LUE}(x, y) \) as a sum of products \( \phi_j(x) \phi_j(y) \), the last integral on the right hand side of (7.36) is a sum of squares (of integrals) so it is positive. In addition, recall the definition of \( I \) and that \( \left( \int f(x)p_n(x)dx \right)^2 = (n^{-1} E \sum_{i=1}^n f(\eta_i))^2 \). The last equality of (7.36) then implies
\[ \text{Var} \left[ \frac{1}{n} \sum_{i=1}^n f(\eta_i) \right] \leq n^{-1} \int f^2(x)p_n(x)dx. \]

\[ \square \]

We now combine Lemmas 7.6 and 7.7 to obtain Proposition 7.4. For the \( l = 1 \) case,
\[ \text{Var} \left( \frac{1}{n} \sum f_c(\mu_i) \right) \leq \frac{1}{n} E \left( \frac{1}{n} \sum f_c^2(\mu_i) \right) = O(n^{-2/3}), \] (7.37)
where the inequality follows from Lemma 7.7 and the big-O term follows from Lemma 7.6. For the \( l = 2 \) case, we observe that \( f_c^2 \) is strictly positive, so \( \frac{1}{n} \sum f_c^2(\mu_i) = O \left( E(\frac{1}{n} \sum f_c^2(\mu_i)) \right) \) with high probability. These observations along with the expectations in Lemma 7.6 imply
\[ \sum_{i=1}^n f_c(\mu_i) - \frac{n}{\lambda^{1/2}(1 + \lambda^{1/2})} = O \left( \left( 1 + |\sigma_n|^{1/2} \right) n^{2/3} \right), \]
(7.38)
\[ \sum_{i=1}^n f_c^2(\mu_i) = O(n^{4/3}). \]

Finally, from Lemma 7.3, we know that, for any \( \varepsilon > 0 \), there exists \( c \) such \( \sum f_c^2(\mu_i) = \sum (\gamma - \mu_i)^{-l} \) with probability \( 1 - \varepsilon \). Since (7.38) holds for any \( c \), we obtain Proposition 7.4.
7.3 Extension of Proposition 7.4 to LOE

We extend Proposition 7.4 from the LUE case to the LOE case using the same method that the authors of [12] use to extend their result from the GUE case to the GOE case. Since the proof is nearly identical, we do not repeat it here, but rather summarize the key steps in the proof and provide the translation between their setting and ours.

In both our setting and that of [12], a key tool to extend results from \( \alpha = 1 \) to the \( \alpha = 2 \) is a result from Forrester and Rains about the relationships between eigenvalues of orthogonal, unitary, and symplectic ensembles [8]. Among other findings, their Theorem 5.2 states that

\[
\text{even}(\text{GOE}_n \cup \text{GOE}_{n+1}) = \text{GUE}_n, \tag{7.39}
\]
\[
\text{even}(\text{LOE}_{n,m} \cup \text{LOE}_{n+1,m+1}) = \text{LUE}_{n,m}. \tag{7.40}
\]

Here \( \text{LOE}_{n,m} \) denotes the set of eigenvalues of the LOE matrix that we previously called \( M_{n,m} \) (with the notations LUE, GOE, GUE defined similarly). The notation \( \text{even}(\cdot) \) denotes the set containing only the even numbered elements among the ordered list of elements in the original set.

The other key tool in the extension from \( \alpha = 1 \) to \( \alpha = 2 \) is Cauchy’s eigenvalue interlacing theorem. This theorem states that, if a symmetric \((n+1) \times (n+1)\) matrix and its principal minor have eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1} \) and \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \) respectively, then the eigenvalues satisfy the relation

\[
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \mu_n \geq \lambda_{n+1}. \tag{7.41}
\]

The authors of [12] use this to relate the eigenvalues of a GOE matrix \( M_{n+1} \) to the eigenvalues of its principal minor, which is distributed as an \( n \times n \) GOE matrix. We can also do this for an LOE matrix, provided that we use the tridiagonal representation of LOE (this guarantees that the principal minor is also distributed as an LOE matrix).

Using these two tools, the authors of [12] prove a theorem about \( n \times n \) GUE and GOE matrices \( M^C_n \) and \( M^R_n \) (see Theorem 19 of [12]). We state below the analogous theorem in our setting, which follows from the same proof.

**Theorem 7.8.** Let \( M^C_{n,m} \) and \( M^R_{n,m} \) denote LUE and LOE matrices respectively. If \( f_n \) is a sequence of functions such that

\[
f_n(M^C_{n,m}) = a_n + O(b_n) \tag{7.42}
\]

for some sequences \( a_n \) and \( b_n \), then

\[
f_n(M^R_{n,m}) = a_n + O(b_n + TV(f_n)) \tag{7.43}
\]

where \( TV(f_n) \) denotes the total variation of \( f_n \) and the big-O bounds hold with probability converging to 1.

In this theorem, the functions \( f_n \) are taken to be single-variable functions where the notation \( f_n(M_{n,m}) \) is shorthand for \( \sum_{i=1}^{n} f_n(\mu_i) \). The proof is for the unscaled version of these matrices, but it holds for the scaled version as well since scaling the argument does not change the total variation of the function. Using this theorem, and noting that \( TV(f^l_1) = O(n^{l-1}) \) for \( f_x \) as defined in (7.9), we can extend Lemma 7.6 from the LUE case to the LOE case. We can further use this theorem to obtain a weaker version of Lemma 7.7 for the LOE case, namely

\[
\text{Var} \left[ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i) \right] \leq O \left( \frac{1}{n} \int f^2(x)p_{n,\text{LOE}}(x)dx + TV^2(f) \right). \tag{7.44}
\]

These LOE versions of Lemmas 7.6 and 7.7 are enough to extend Proposition 7.4 from the LUE case to the LOE case.

7.4 Proof of Lemma 7.2

We use the same notations as in the proof of Lemma 7.7. The following equations follow from displays (11) to (15) of [17]. To begin, note that the one-point correlation function \( \tilde{R}_1(x) \) of unordered eigenvalues of
unscaled LUE matrix has integral representation

\[ \tilde{R}_1(x) = \sum_{i=0}^{n-1} \phi_k(x; a)^2 = 2 \int_0^\infty \phi(x + z; a) \psi(x + z; a) dz, \]

where

\[ \phi(x; a) := (-1)^n \sqrt{\frac{n(n + a)}{2}} \phi_n(x; a - 1) x^{-1/2} 1_{\{x \geq 0\}}, \]

\[ \psi(x; a) := (-1)^n \sqrt{\frac{n(n + a)}{2}} \phi_{n-1}(x; a + 1) x^{-1/2} 1_{\{x \geq 0\}}. \]

Throughout the remaining of the proof, we write \( \phi(x) \) and \( \psi(x) \) when the the parameter \( a \) is clear from the context. Given integer \( k \), let \( k_- = k - \frac{1}{2} \). Set

\[ u_n = (\sqrt{n_-} + \sqrt{m_-})^2, \quad r_n = (\sqrt{n_-} + \sqrt{m_-}) \left( \frac{1}{\sqrt{n_-}} + \frac{1}{\sqrt{m_-}} \right)^{1/3}, \]

and define \( z_n = \tilde{z}_n(s) \) by \( d_+ m + \lambda^{-2/3} m^{1/3} = u_n + z_n r_n \). Then \( z_n = z_n s + O(n^{-1/3}) \), where the big-O term is uniformly in \( s \). We also define

\[ \eta(z) = u_n + z r_n, \]

\[ \phi^{(n)}(z) = r_n \phi(\eta(z)), \quad \psi^{(n)}(z) = r_n \psi(\eta(z)). \]

From (7.35), \( p_{n,LUE}(d_+ + sn^{-2/3}) \) is in fact

\[ \frac{1}{\lambda} \tilde{R}_1(u_n + z r_n) = 2r_n^{-2} \int_0^\infty \phi^{(n)}(z_n + z r_n^{-1}) \psi^{(n)}(z_n + z r_n^{-1}) dz = \frac{2r_n^{-1}}{\lambda} \int_{z_n}^\infty \phi^{(n)}(z) \psi^{(n)}(z) dz. \]

By Proposition 2 of [17],

\[ \forall z_0 \in \mathbb{R}, \exists N_0 = N_0(z_0, \lambda), \quad n \geq N_0 \implies |\phi^{(n)}(z)|, |\psi^{(n)}(z)| \leq C(z_0) e^{-z} \quad \forall z \geq z_0. \quad (7.45) \]

Apply (7.45) with \( z_0 = z_n \), then for sufficiently large \( n \), for all \( s > s_0 \),

\[ p_{n,LUE}(d_+ + sn^{-2/3}) \leq \frac{2r_n^{-1}}{\lambda} C(z_n)^2 \exp(-2 z_n) = O \left( n^{-1/3} \exp(-2 z_n \lambda s) \right), \quad n \to \infty. \quad (7.46) \]

We now verify the edge bound for \( p_{n,LOE}(d_+ + sn^{-2/3}) \). Equation (15) of [17], in our notations, states that for \( x, y > 0 \),

\[ S_{n,LOE}(x, y) = S_{n,LUE}(x, y) + \psi(x) \frac{1}{2} \int_0^\infty \phi(u) sgn(y - u) du \]

\[ = S_{n,LUE}(x, y) + \psi(x) \left[ \frac{1}{2} I_\phi - \int_y^\infty \phi(u) du \right], \]

where \( I_\phi = \int_0^\infty \phi(u) du \). Recall \( S_{n,LUE}(x, y) = \tilde{R}_1(x) \) and the relation (7.35). The above display implies

\[ p_{n,LOE}(x) = p_{n,LUE}(x) + \psi(mx) \left[ \frac{1}{2} I_\phi - \int_{mx}^\infty \phi(u) du \right]. \]

Substitute \( x = d_+ + \sigma_n n^{-2/3} \) and use notation \( mx = u_n + z_n r_n \), we obtain

\[ p_{n,LOE}(d_+ + sn^{-2/3}) = p_{n,LUE}(d_+ + sn^{-2/3}) + \frac{r_n^{-1} \psi^{(n)}(s)}{2} \left[ \frac{1}{2} I_\phi - \int_{z_n}^\infty \phi^{(n)}(z) dz \right]. \quad (7.47) \]

By (7.45), \( \int_{z_n}^\infty \phi^{(n)}(z) dz \leq C e^{-\beta_1 s} \) for some \( C = C(s_0, \lambda) > 0 \), for all \( s \geq s_0 \). In addition, the quantity \( I_\phi \) is denoted by \( \beta_N \) in [17], where it is shown to satisfies \( I_\phi = \frac{1}{\sqrt{2}} + O(n^{-1}) \). Thus, the second term on the right hand side of (7.47) is \( O(n^{-1/3} e^{-\beta_1 s}) \) uniformly for \( s \geq s_0 \). We conclude

\[ p_{n,LOE}(d_+ + sn^{-2/3}) \leq C n^{-1/3} e^{-\beta_1 s}, \quad \forall s \geq s_0. \]
A Technical lemmas

Consider the following process

$$\hat{R}_2 = R_2$$
$$\hat{R}_i = L_i + \omega_1 \ldots \omega_3 \hat{R}_2 - A_{0i} + \hat{B}_{0i} + \hat{B}_{1i} + \phi_{\frac{2}{n(1-\omega_i)}} (B_{2i}) + \hat{B}_{3i}, \quad 3 \leq i \leq n$$

where

$$\hat{B}_{0i} = \left( \alpha_{i-1} + (\tau_{i-1} + \alpha_{i-1}) \hat{R}_i^{(1)} \right) \beta_i + \omega_1 \left( \alpha_{i-2} + (\tau_{i-2} + \alpha_{i-2}) \hat{R}_i^{(1)} \right) \beta_{i-1}$$
$$+ \cdots + \omega_i \ldots \omega_4 \left( \alpha_2 + (\tau_2 + \alpha_2) \hat{R}_i^{(1)} \right) \beta_3, \quad \hat{R}_i^{(1)} = \frac{\hat{R}_{i-1}}{1 - \phi_{1/2}(\hat{R}_{i-1})},$$
$$\hat{B}_{1i} = \alpha_{i-1} \delta_i \hat{R}_i^{(1)} + \omega_i \alpha_i \hat{R}_i^{(1)} + \cdots + \omega_i \ldots \omega_4 \alpha_2 \beta_3 \hat{R}_3^{(1)},$$
$$\hat{B}_{3i} = \hat{R}_i^{(3)} + \omega_1 \hat{R}_i^{(3)} + \cdots + \omega_i \ldots \omega_4 \hat{R}_3^{(3)}, \quad \hat{R}_i^{(3)} = \omega_i \phi_{n-1/3}(\hat{R}_{i-1}) \hat{R}_{i-1}.$$

The event that $2|1 - \alpha_j|^{-1} > 1$ and $|R_i| \leq n^{-1/3}$ and $|B_{2i}| \leq \frac{2}{n(1-\omega_i)}$ for all $3 \leq i \leq n$ occurs with probability $1 - O(\log^{-5} n)$. The bound for $|R_i|$ holds by Lemma 4.1, and bound for $|B_{2i}|$ follows from inequality (6.13) for $\hat{B}_{2i}$ in the proof of Lemma 4.1. Thus on this event, $\hat{R}_2 = R_2$ and $\hat{R}_3^{(\ell)} = R_3^{(\ell)}$ for $\ell = 1, 3$, and $\phi_{\frac{2}{n(1-\omega_i)}} (B_{2i}) = B_{2i}$. Thus $\hat{R}_3 = R_3$. Repeat the argument with increasing $i$, we obtain that

$$\hat{R}_i = R_i \quad \text{for every } 2 \leq i \leq n \quad \text{with probability } 1 - O(\log^{-5} n).$$

(A.1)

A.1 Proof of Lemma 4.2

Consider $\sum_{i=3}^n \hat{R}_i^2$. From the inequality

$$\sum_{i=3}^n \|\hat{R}_i^2\|_1 \leq \sum_{i=3}^n \|\hat{R}_i\|_2^2 \leq \sum_{i=3}^n \|\hat{R}_i\|_1^2,$$

(A.2)

and Markov’s inequality, it suffices to show the last sum is of order $1$. Lemma 2.6 implies that if $X \in SG(v, u)$, then $\|X\|_p \leq C_p (v^\frac{p}{2} + u^p)^\frac{1}{p}$. By (6.15) and (6.29),

$$\|L_i\|_4 \leq \|Y_i\|_4 + \|s_i\|_4 \leq \frac{C \alpha^{\frac{1}{2}}}{\sqrt{n(1-\omega_i)}}.$$  

(A.3)

Also, $\|\alpha_i\|_4, \|\beta_i\|_4 = O(n^{-\frac{1}{2}})$ uniformly in $i$. Hence, by (6.5),

$$\|\hat{R}_2\|_4 \leq \|R_2\|_4 \leq |\omega_2 - \gamma_2| + \|\alpha_2\|_4 + (1 + C \|\alpha_1\|_4) \|\beta_2\|_4 + \frac{1 + C \|\alpha_1\|_4}{\|\hat{R}_2\|_1} = O\left(n^{-\frac{1}{2}}\right).$$  

(A.4)

Thus $|\omega_3 \ldots \omega_4 \hat{R}_2|_4 \leq \omega_3 \|\hat{R}_2\|_4 = O(n^{-\frac{1}{2}})$. Observe that $\|\phi_{\frac{2}{n(1-\omega_i)}} (B_{2i})\|_4 \leq \frac{2}{n(1-\omega_i)}$, and $|A_{0i}| < \frac{1}{n(1-\omega_i)^2}$ from (6.4). Now, for each $i$,

$$\|\left( \alpha_{i-1} + (\tau_{i-1} + \alpha_{i-1}) \hat{R}_i^{(1)} \right) \beta_i\|_4 \leq \|\alpha_{i-1}\|_4 \|\beta_i\|_4 + C \|\beta_i\|_4 \|\hat{R}_i^{(1)}\|_4 \leq C n^{-1} + C n^{-1/2} \|\hat{R}_i^{(1)}\|_4.$$  

Hence,

$$\|\hat{B}_{0i}\|_4 \leq \frac{n^{-\frac{1}{2}}}{n(1-\omega_i)} + \max_{3 \leq j \leq i-1} \|\hat{R}_j\|_4.$$  

(A.5)

Similarly, $\|\alpha_{i-1} \delta_i \hat{R}_i^{(1)}\|_4 \leq C n^{-\frac{1}{2}} \|\hat{R}_{i-1}\|_4$ and $\|\hat{R}_i^{(3)}\|_4 \leq n^{-\frac{1}{2}} \|\hat{R}_{i-1}\|_4$ so

$$\|\hat{B}_{1i}\|_4 \leq \frac{C n^{-1/2}}{1 - \omega_i} \max_{3 \leq j \leq i-1} \|\hat{R}_j\|_4 \quad \text{and} \quad \|\hat{B}_{3i}\|_4 \leq \frac{n^{-1/3}}{1 - \omega_i} \max_{3 \leq j \leq i-1} \|\hat{R}_j\|_4.$$  

(A.6)
Combining all the estimates, we have
\[
\|\hat{R}_i\|_4 \leq \frac{C_\alpha^{1/2} + o(1)}{\sqrt{n(1 - \omega_i)}} + o(1) \max_{3 \leq j \leq i - 1} \|\hat{R}_j\|_4, \quad 3 \leq i \leq n. \tag{A.7}
\]

Since \(\|\hat{R}_2\|_4 = O(n^{-1/2})\), by induction we obtain for sufficiently large \(n\),
\[
\|\hat{R}_i\|_4 \leq \frac{C_\alpha^{1/2}}{\sqrt{n(1 - \omega_i)}}, \quad \text{for } i = 3, \ldots, n. \tag{A.8}
\]

Therefore,
\[
\sum_{i=3}^{n} \|\hat{R}_i^2\|_1 = O\left(\sum_{i=3}^{n} \frac{C_\alpha^{1/2}}{\sqrt{n(1 - \omega_i)}}\right) = O\left(\frac{1}{n} \sum_{i=3}^{n} \left(\frac{n - 1}{n}\right)^{-1/2} + \frac{1}{n} \sum_{i>n-n^{-1/2}\sigma_n} n^{-1/2}\sigma_n\right) = O(1), \tag{A.9}
\]

and we obtain \(\sum_{i=3}^{n} \hat{R}_i^2 = O(1)\) with probability \(1 - o(1)\). By (A1), the same statement applies to \(\sum_{i=3}^{n} R_i^2\).

### A.2 Proof of Lemma 4.3

By (A1), it suffices to show that, with probability \(1 - o(1)\),
\[
\sum_{i=3}^{n} \omega_3 \ldots \omega_i R_2 + \hat{B}_0i + \hat{B}_1i + \phi_{\frac{n^{-1/2}}{\sigma_n}}(B_2i) = O(1). \tag{A.10}
\]

This holds as long as the \(L_1\) norm of this sum is of order 1. By Definition 4.5 and \(\|R_2\|_1 \leq \|R_2\|_4 = O(n^{-1/2})\) (which is a consequence of (A4)),
\[
\sum_{i=3}^{n} \omega_3 \ldots \omega_i \|R_2\|_1 = \omega_3 g_4 \|R_2\|_1 = O(\omega_3 \|R_2\|_4) = O(n^{-1/2}).
\]

Here, \(g_4 = O(1)\) follows from Lemmas 5.1 and 2.9 and a direct computation gives \(\omega_3 = O(n^{-1})\). By (A5) and (A8) and Corollary 2.9,
\[
\sum_{i=3}^{n} \|\hat{B}_0i\|_1 \leq \sum_{i=3}^{n} \frac{C_\alpha}{n(1 - \omega_i)} + \sum_{i=3}^{n} \frac{C_\alpha}{n(1 - \omega_i)^{1/2}} = O(1).
\]

Similarly, by (A6),
\[
\sum_{i=3}^{n} \|\hat{B}_1i\|_1 \leq \sum_{i=3}^{n} \frac{C_\alpha}{n(1 - \omega_i)^{1/2}} = O(1).
\]

Lastly,
\[
\sum_{i=3}^{n} \left\|\phi_{\frac{n^{-1/2}}{\sigma_n}}(B_2i)\right\|_1 \leq \sum_{i=3}^{n} \frac{2}{n(1 - \omega_i)} = O(1).
\]

### A.3 Proof of Lemma 4.4

Observe that \(E_1 = \frac{\sigma^2 - \gamma_m}{|\rho|} = \alpha_1 - 1\). Hence,
\[
E_2 = E_1 (R_2 - 1) = (1 - \alpha_1)(1 - R_2).
\]

By Lemma 2.6 we have with probability \(1 - O(n^{-1})\), \(|\alpha_1| = O(n^{-1/2} \log^{1/2} n)\) and \(|R_2| = O(n^{-1/2})\). Thus there exists \(C_1 < 0 < C_2\) such for sufficiently large \(n\),
\[
\log |E_2| = \log |1 - \alpha_1| + \log |1 - R_2| \in (C_1, C_2)
\]

with probability \(1 - O(n^{-1})\).
A.4 Proof of Lemma 4.7

We apply (A.1) to replace $B_{3i}$ by $\hat{B}_{3i}$ for every $i = 3, \ldots, n$, then show that $\sum_{i=3}^{n} \hat{B}_{3i} - B_{3i}^* = O(1)$ with probability $1 - o(1)$. Recall

$$\hat{B}_{3i} = \hat{R}_{i}^{(3)} + \omega_i \hat{R}_{i-1}^{(3)} + \omega_i \ldots \omega_4 \hat{R}_{3}^{(3)},$$

where $\hat{R}_{i}^{(3)} = \omega_i \phi_{n-1/3}(\hat{R}_{i-1}) \hat{R}_{i-1}$. Consider

$$\hat{C}_{3i} = (\omega_i \hat{R}_{i-1}^{2}) + \omega_i (\omega_{1-1} \hat{R}_{i-2}^{2}) + \cdots + \omega_i \ldots \omega_4 (\omega_3 \hat{R}_{2}^{2}).$$

By Lemma 4.1 and the fact $R_i = \hat{R}_i$ for all $2 \leq i \leq n$ with probability $1 - o(1)$, it holds with probability $1 - o(1)$ that $\hat{B}_{3i} = C_{3i}$ for all $3 \leq i \leq n$. Hence, it is sufficient to show $\sum_{i=3}^{n} \|\hat{C}_{3i} - B_{3i}^*\|_1 = O(1)$.

$$\|\hat{C}_{3i} - B_{3i}^*\|_1 \leq \sum_{j=3}^{i-1} \omega_j \ldots \omega_3 \|\hat{R}_{j-1}^{2} - L_{j-1}^{2}\|_1 \leq \frac{\omega_i}{1 - \omega_i} \max_{2 \leq j \leq i-1} \|\hat{R}_{j-1}^{2} - L_{j-1}^{2}\|_1. \quad (A.11)$$

By Hölder’s inequality,

$$\|\hat{R}_{i}^{2} - L_{i}^{2}\|_1 \leq \|\hat{R}_{i} - L_{i}\|_2 \|\hat{R}_{i} + L_{i}\|_2.$$

Apply triangle inequality, we have

$$\|\hat{R}_{i} - L_{i}\|_2 \leq \|\omega_i \ldots \omega_3 \hat{R}_{2}\|_2 + \|A_{0i}\|_2 + \|\hat{B}_{0i}\|_2 + \|\hat{B}_{1i}\|_2 + \|\phi_{n(1-1/3)}(B_{2i})\|_2 + \|\hat{B}_{3i}\|_2. \quad (A.12)$$

Since $\|X\|_2 \leq \|X\|_4$ for all random variables $X \in L_4(\mathbb{F})$, we can apply the bounds on $L_4$-norms obtained in the proof of Lemma 3.3. Thus, for some $C > 0$ and sufficiently large $n$, the following four inequalities hold.

$$\|\omega_i \ldots \omega_3 \hat{R}_{2}\|_2 \leq C n^{-\frac{3}{2}}, \quad \|A_{0i}\|_2 \leq \frac{C}{n(1 - \omega_i)}, \quad \|\hat{B}_{0i}\|_2 \leq \frac{C}{n(1 - \omega_i)^{\frac{3}{2}}} + \frac{C}{n(1 - \omega_i)^{\frac{3}{2}}}, \quad \|\hat{B}_{1i}\|_2 \leq \frac{C}{n(1 - \omega_i)^{\frac{3}{2}}}.$$

At the same time, since $\|\hat{R}_{i}^{(3)}\|_2 \leq \|\hat{R}_{i}^{2}\|_2 = \|\hat{R}_{i}\|_4 = O \left( \frac{1}{n(1 - \omega_i)} \right),$

$$\|\hat{B}_{3i}\|_2 \leq \frac{1}{n - \omega_i}, \quad \max_{3 \leq j \leq i} \|\hat{R}_{i}^{(3)}\|_2 \leq \frac{C}{n(1 - \omega_i)^{\frac{3}{2}}}.$$

Hence, $\|\hat{R}_{i} - L_{i}\|_2 = O \left( \frac{1}{n(1 - \omega_i)^{\frac{3}{2}}} \right)$. Similarly, by (A.3) and (A.8),

$$\|\hat{R}_{i} + L_{i}\|_2 \leq \|\hat{R}_{i}\|_4 + \|L_{i}\|_4 = O \left( \frac{1}{\sqrt{n}} \right). \quad (A.13)$$

Therefore,

$$\|\hat{R}_{i}^{2} - L_{i}^{2}\|_1 \leq \|\hat{R}_{i} - L_{i}\|_2 \|\hat{R}_{i} + L_{i}\|_2 = O \left( \frac{1}{n^{\frac{3}{2}}(1 - \omega_i)^{\frac{3}{2}}} \right),$$

and $\|\hat{C}_{3i} - B_{3i}^*\|_1 = O \left( \frac{1}{n^{\frac{3}{2}}(1 - \omega_i)^{\frac{3}{2}}} \right)$. By Lemma 2.8, we conclude

$$\sum_{i=3}^{n} \|\hat{C}_{3i} - B_{3i}^*\|_1 = O \left( \sum_{i=3}^{n} \frac{1}{n^{\frac{3}{2}}} \left( \frac{n - i}{n} \right)^{-\frac{3}{2}} + \sum_{i=n-n^{\frac{3}{2}}\sigma_n}^{n} \frac{n^{-\frac{3}{2}}(n^{\frac{3}{2}}\sigma_n)^{-\frac{3}{2}}} \right) = O \left( \sigma_n^{-\frac{3}{2}} \right) = o(1). \quad (A.14)$$
A.5 Proof of Lemma 4.8

We expand the terms inside the sum to have
\[
\sum_{i=4}^{n} (g_i - 1) \left[ 2Y_{i-1}(\alpha_{i-1} - \omega_3 \cdots \omega_{i-1} \alpha_2) + (\alpha_{i-1} - \omega_3 \cdots \omega_{i-1} \alpha_2)^2 \right] = P_1 - P_2 + P_3,
\]
where
\[
P_1 := \sum_{i=4}^{n} 2(g_i - 1)Y_{i-1} \alpha_{i-1},
\]
\[
P_2 := \sum_{i=4}^{n} 2(g_i - 1)Y_{i-1} \omega_3 \cdots \omega_{i-1} \alpha_2,
\]
\[
P_3 := \sum_{i=4}^{n} (g_i - 1)(\alpha_{i-1} - \omega_3 \cdots \omega_{i-1} \alpha_2)^2.
\]

Recalling that
\[
Y_{i-1} = \sum_{j=3}^{i-2} X_j \omega_{j+1} \cdots \omega_{i-1} + X_{i-1}, \quad X_j = (1 + \tau_{j-1}) (\delta_j \alpha_{j-1} + \beta_j),
\]
we further decompose \( P_1 \) into
\[
P_1 = \sum_{i=4}^{n} 2(g_i - 1) \alpha_{i-1} \left( \sum_{j=3}^{i-2} (1 + \tau_{j-1}) \alpha_{j-1} \delta_j \omega_{j+1} \cdots \omega_{i-1} + (1 + \tau_{i-2}) \alpha_{i-2} \delta_{i-1} \right)
+ \sum_{i=4}^{n} 2(g_i - 1) \alpha_{i-1} \left( \sum_{j=3}^{i-2} (1 + \tau_{j-1}) \beta_j \omega_{j+1} \cdots \omega_{i-1} + (1 + \tau_{i-2}) \beta_{i-1} \right)
= : P_{11} + P_{12}.
\]

We now rewrite each part of the summation as quadratic forms and apply Lemma 4.10 as follows. Define
\[
a = (\alpha_2, \alpha_3, \ldots, \alpha_{n-1})^T, \quad b = (\beta_3, \beta_4, \ldots, \beta_{n-1})^T,
\]
\[
a^{(l)} = (\alpha_3, \alpha_4, \ldots, \alpha_{n-1})^T, \quad a^{(l)} = (\alpha_2, \alpha_3, \ldots, \alpha_{n-2})^T.
\]

**Bound for \( P_{11} \) part:** Observe that
\[
P_{11} = (a^{(l)})^T Z a^{(l)}
\]
where \( Z \) is the lower triangular matrix
\[
Z = 2 \begin{pmatrix}
(4_1-1)(1+\tau_2)\delta_3 & (g_5-1)(1+\tau_3)\delta_4 & \cdots & (g_{n-1}-1)(1+\tau_{n-2})\delta_{n-1} \\
(g_5-1)(1+\tau_2)\delta_4 & (g_6-1)(1+\tau_3)\delta_5 & \cdots & \\
(g_6-1)(1+\tau_2)\delta_5 & (g_7-1)(1+\tau_3)\delta_6 & \cdots & \\
\vdots & \vdots & \ddots & \\
(g_8-1)(1+\tau_2)\delta_6 & \cdots & & (g_{n-1}-1)(1+\tau_{n-2})\delta_{n-1}
\end{pmatrix}.
\]

Alternatively, we can express this as a quadratic form
\[
a^T \tilde{Z} a
\]
where \( \tilde{Z} \) is the matrix \( Z \) with a row of zeros appended at the top and a column of zeros appended at the right. That is,
\[
\tilde{Z}_{ij} = 2(g_{i+2} - 1)(1 + \tau_{j+1}) \delta_{j+2} \omega_{j+3} \cdots \omega_{i+1} \quad \text{for } i \geq j + 1.
\]
Next, we observe that
\[
a^T \tilde{Z} a = \frac{1}{2} a^T (\tilde{Z} + \tilde{Z}^T) a.
\]
Since \( \tilde{Z} + \tilde{Z}^T \) is a symmetric matrix and \( \mathbf{a} \) is a vector of independent random variables satisfying \( \alpha_i \in SG(c_1 n^{-1}, c_2 n^{-1}) \) we can apply Lemma 4.10. Furthermore, since \( \tilde{Z} \) has zeros along the diagonal, \( \mathbb{E} \mathbf{a}^T \tilde{Z} \mathbf{a} = 0 \) so we conclude that with probability \( 1 - o(1) \),

\[
\mathbf{a}^T \tilde{Z} \mathbf{a} = O \left( \frac{\nu_n}{n} \| \tilde{Z} + \tilde{Z}^T \|_{HS} \right) = O \left( \frac{\nu_n}{n} \| \tilde{Z} \|_{HS} \right),
\]

where \( \nu_n \) is some slowly growing function of \( n \) (for example, \( \sqrt{\log n} \)). We observe that

\[
\| \tilde{Z} \|_{HS} = \sqrt{\sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \tilde{Z}_{ij}^2}.
\]

We bound the quantity \( \tilde{Z}_{ij}^2 \) as follows:

\[
\tilde{Z}_{ij}^2 = 4(g_{i+2} - 1)^2 (1 + \tau_{j+1})^2 \sigma_j^2 \omega_{j+1} \cdots \omega_{i+1} \leq C \left( \frac{1}{1 - \omega_{i+2}} \right)^2 \left( \frac{j + 1}{n} \right)^2 \omega_{j+1}^2 \cdots \omega_{i+1}^2.
\]

For fixed \( i \),

\[
\sum_{j=1}^{i-1} \tilde{Z}_{ij}^2 \leq \sum_{j=1}^{i-1} \frac{C(i + 2)}{n(1 - \omega_{i+2})^3} \omega_{j+1}^2 \cdots \omega_{i+1}^2 < \frac{C(i + 2)}{n(1 - \omega_{i+2})^2} \cdot \frac{1}{1 - \omega_{i+1}} < \frac{C(i + 2)}{n(1 - \omega_{i+2})^3}.
\]

We now sum this quantity over the indices \( i \), treating separately the indices \( i \leq n - n^{1/3} \sigma_n \) and \( i \geq n - n^{1/3} \sigma_n \). Since we care only about the order of this quantity we omit the initial constant \( C \), although \( c \) will show up later denoting some other constant. For the sum over indices less than \( n - n^{1/3} \sigma_n \), we get

\[
\sum_{i=2}^{n - n^{1/3} \sigma_n} \frac{i + 2}{n(1 - \omega_{i+2})^3} \leq \sum_{i=2}^{n - n^{1/3} \sigma_n} \frac{1}{n - (i + 2)} \left( \frac{cn}{n(i + 2)} \right)^{3/2} = O \left( n \int_{n^{-2/3} \sigma_n}^1 (1 - x)x^{-3/2} dx \right) = O \left( n \int_{n^{-2/3} \sigma_n}^1 x^{-3/2} dx \right) = O(n^{4/3} \sigma_n^{-1/2}).
\]

Meanwhile,

\[
\sum_{i=n-n^{1/3} \sigma_n}^{n-2} \frac{i + 2}{n(1 - \omega_{i+2})^3} < \sum_{i=n-n^{1/3} \sigma_n}^{n-2} \frac{1}{n - (i + 2)} \leq \sum_{i=n-n^{1/3} \sigma_n}^{n-2} \left( \frac{cn^{1/3} \sigma_n^{-1/2}}{n} \right)^3 = O \left( n^{1/3} \sigma_n^{-3/2} \right) = O \left( n^{4/3} \sigma_n^{-1/2} \right).
\]

Putting the two sums together, we get

\[
\| \tilde{Z} \|_{HS} = \sqrt{\sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \tilde{Z}_{ij}^2} = O(\sqrt{n^{4/3} \sigma_n^{-1/2}}) = O(n^{2/3} \sigma_n^{-1/4}),
\]

and thus, with probability \( 1 - o(1) \),

\[
\mathbf{a}^T \tilde{Z} \mathbf{a} = O \left( \frac{1}{n} \cdot n^{2/3} \sigma_n^{-1/4} \nu_n \right) = O(n^{-1/3} \sigma_n^{-1/4} \nu_n),
\]

where \( \nu_n \) is, again, some slowly growing function.
Bound for $P_{12}$ part: Using the vectors defined above and the matrices $W, G, D$ from Definition 4.11, we write
\[ P_{12} = 2(a^{(i)})^T GWDb = ((a^{(i)})^T b^T) \left( \begin{array}{c} O \\ (GW)^T \\ O \end{array} \right) \left( \begin{array}{c} a^{(i)} \\ b \end{array} \right). \] (A.33)

Since the matrix has zeros on the diagonal, $\mathbb{E}P_{12} = 0$. By Lemma 4.10 with probability $1 - o(1)$,
\[ 2(a^{(i)})^T GWDb = O \left( \left\| \frac{\nu_n}{n} \right\| \left\| \frac{O}{(GW)^T} \right\|_{HS} \right) = O \left( \frac{\nu_n}{n} \left\| GW \right\|_{HS} \right). \] (A.34)

We have
\[ \left\| GW \right\|_{HS}^2 = \sum_{i=1}^{n-3} \sum_{j=1}^{i} (GW)^2_{ij} = \sum_{i=1}^{n-3} \sum_{j=1}^{i} (g_{i+3} - 1)^2 (1 + \tau_{j+1})^2 \omega_{j+3}^2 \cdots \omega_{i+2}^2 \] (A.35)

For indices $1 \leq i \leq n - 1/3 \sigma_n - 3$,
\[ (g_{i+3} - 1)^2 \leq \frac{n}{n - (i + 3)}, \quad \text{and} \quad \frac{1}{1 - \omega_i} \leq \sqrt{\frac{cn}{n - (i + 3)}}. \]

Thus,
\[ \sum_{i=1}^{n-1/3 \sigma_n - 3} \sum_{j=1}^{i} (g_{i+3} - 1)^2 \omega_{j+3}^2 \cdots \omega_{i+2}^2 \leq \sum_{i=1}^{n-1/3 \sigma_n - 3} \frac{n}{n - (i + 3)} \frac{1}{1 - \omega_{i+2}} \] (A.36)

The contribution from the remaining terms is
\[ \sum_{i=n-n^{1/3} \sigma_n - 2}^{n-3} \sum_{j=1}^{i} (g_{i+3} - 1)^2 \omega_{j+3}^2 \cdots \omega_{i+2}^2 \leq \sum_{i=n-n^{1/3} \sigma_n - 2}^{n-3} \sum_{j=1}^{i} (C n^{1/3} \sigma_n^{-1/2})^2 \omega_{j+3}^2 \cdots \omega_{i+2}^2 \] (A.37)

Thus, we get $\left\| GW \right\|_{HS}^2 = O(n^{4/3} \sigma_n^{-1/2})$. By [A.34], we conclude that, with probability $1 - o(1)$,
\[ 2(a^{(i)})^T GWDb = O \left( \frac{\nu_n}{n} \left\| GW \right\|_{HS} \right) = O \left( \frac{\nu_n}{n} \left\| GW \right\|_{HS} \right) = O(1). \]

Bound for $P_2$ part: We recall two facts. First, from Lemma 6.1 $\max_{3 \leq i \leq n} |Y_i| = o(n^{-1/3})$ with probability $1 - o(1)$. Second, $\alpha_2 \in SG(v, u)$ with $v, u = O(n^{-1})$ so, by Lemma 2.6 $\alpha_2 = O(n^{-1/2 + \epsilon})$ with probability $1 - o(1)$ for any $\epsilon > 0$. Combining these two facts, we deduce that, with probability $1 - o(1)$,
\[ P_2 = o \left( n^{-1/3} n^{-1/2 + \epsilon} \sum_{i=4}^{n} (g_i - 1) \omega_3 \cdots \omega_{i-1} \right). \] (A.38)

The above sum can be crudely bounded as
\[ \sum_{i=1}^{n} (g_i - 1) \omega_3 \cdots \omega_{i-1} < \sum_{i=1}^{n} (g_i - 1) \omega_3 \omega_4 = O \left( n \cdot n^{1/3} \sigma_n^{-1/2} \cdot n^{-2} \right) = O(n^{-2/3} \sigma_n^{-1/2}), \] (A.39)
where we use Lemmas 5.1 and 2.8 to bound $g_i$ and the definition of $\omega_i$ to bound $\omega_3, \omega_4$. We obtain $P_2 = o(1)$.

**Bound for $P_3$ part:** We now bound $\sum_{i=4}^{n}(g_i - 1)(\alpha_{i-1} - \omega_3 \cdots \omega_{i-1} \alpha_2)^2$. This can be expressed as a quadratic form

$$a^T Q a$$

where $a$ is the same vector from before and $Q$ is a symmetric matrix with non-zero entries only in the first row, first column, and along the diagonal. More specifically, these entries are

$$Q_{ij} = \begin{cases} 
  g_{i+2} - 1 & i = j \geq 2, \\
  -(g_{j+2} - 1)\omega_j \cdots \omega_{j+1} & i = 1, j \geq 2, \\
  -(g_{i+2} - 1)\omega_1 \cdots \omega_{i+1} & i \geq 2, j = 1, \\
  -\sum_{k=2}^{n-2}(g_{k+2} - 1)(\omega_3 \cdots \omega_{k+1})^2 & i = j = 1.
\end{cases} \quad (A.41)$$

Again, by Lemma 4.10 with probability $1 - o(1)$,

$$a^T Q a - \mathbb{E} a^T Q a = O \left( \frac{\nu_n}{n} \| Q \|_{\text{HS}} \right),$$

where

$$\| Q \|_{\text{HS}}^2 = Q_{11}^2 + \sum_{i=2}^{n-2} Q_{ii}^2 + 2 \sum_{i=2}^{n-2} Q_{i1}^2 < \left( \omega_3^2 \sum_{i=2}^{n-2} (g_{i+2} - 1) \right)^2 + 3 \sum_{i=2}^{n-2} (g_{i+2} - 1)^2$$

$$< C \left( \frac{1}{n^2} \sum_{i=2}^{n-2} \frac{1}{1 - \omega_{i+2}} \right)^2 + \sum_{i=2}^{n-2} \frac{1}{(1 - \omega_{i+2})^2}. \quad (A.43)$$

The first sum satisfies

$$\left( \frac{1}{n^2} \sum_{i=2}^{n-2} \frac{1}{1 - \omega_{i+2}} \right)^2 = O \left( \frac{1}{n^2} \cdot n \cdot n^{1/3} \sigma_n^{-1/2} \right)^2 = o(1), \quad (A.44)$$

while the second one is

$$\sum_{i=2}^{n-2} \frac{1}{(1 - \omega_{i+2})^2} = \sum_{i=2}^{n-n^{1/3} \sigma_n} \frac{1}{(1 - \omega_{i+2})^2} + \sum_{n-n^{1/3} \sigma_n}^{n-2} \frac{1}{(1 - \omega_{i+2})^2}$$

$$= O \left( n \int_{n-n^{1/3} \sigma_n}^1 x^{-1} dx + n^{1/3} \sigma_n \cdot n^{2/3} \sigma_n^{-1} \right) = O(n \log n). \quad (A.45)$$

We conclude

$$a^T Q a - \mathbb{E} a^T Q a = O \left( \frac{\nu_n}{n} \sqrt{n \log n} \right) = O \left( n^{-1/2} \nu_n \sqrt{\log n} \right). \quad (A.46)$$

It remains to evaluate the expectation.

$$\mathbb{E} a^T Q a = \mathbb{E} \sum_{i=4}^{n}(g_i - 1)(\alpha_{i-1} - \omega_3 \cdots \omega_{i-1} \alpha_2)^2 = \sum_{i=4}^{n}(g_i - 1)\mathbb{E}(\alpha_{i-1}^2 + \omega_3^2 \cdots \omega_{i-1}^2 \alpha_2^2). \quad (A.47)$$

We note that $\mathbb{E}(\alpha_{i-1}^2 + \omega_3^2 \cdots \omega_{i-1}^2 \alpha_2^2) = O(n^{-1})$ and, in the course of the proof above (see (A.43) and (A.44)), we showed that $\frac{1}{n} \sum_{i=4}^{n}(g_i - 1) = O(1)$. Therefore, $P_3 = O(1)$ with probability $1 - o(1)$.
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