Lattice formulation of 2D $\mathcal{N} = (2,2)$ SQCD based on the B model twist

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Abstract

We present a simple lattice formulation of two-dimensional $\mathcal{N} = (2,2)$ $U(k)$ supersymmetric QCD (SQCD) with $N$ matter multiplets in the fundamental representation. The construction uses compact gauge link variables and exactly preserves one linear combination of supercharges on the two-dimensional regular lattice. Artificial saddle points in the weak coupling limit and the species doubling are evaded without imposing the admissibility. A perturbative power-counting argument indicates that the target supersymmetric theory is realized in the continuum limit without any fine tuning.

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1 Introduction

After seminal work by Kaplan et. al. [1–3], there have recently appeared various lattice formulations of extended supersymmetric gauge theories [4–20].\footnote{See Refs. [21–26] for relationships among these formulations and Refs. [27–30] for related study. See Ref. [31] for review.} A common feature of these lattice formulations is that at least one fermionic symmetry $Q$, that is a linear combination of supersymmetry charges, is manifestly preserved even with finite lattice spacings. This could be possible, if $Q$ is nilpotent and the continuum action $S$ is $Q$-exact, $S = QX$. Such $Q$ is thus naturally identified with the BRST supercharge in topological field theory [32,33]. In fact, formulations in Refs. [4–9,11,12,15,16,18–20] start with a topological field theoretical representation of a target continuum theory. In lower-dimensional systems in particular, because of this exact fermionic symmetry $Q$, one expects that the full supersymmetry is restored in the continuum limit without (or with a little) fine tuning. Quite recently, this expectation on supersymmetry restoration was clearly confirmed [34] (by means of a Monte Carlo simulation) in a lattice formulation of two-dimensional (2D) $\mathcal{N} = (2,2)$ supersymmetric Yang-Mills theory (SYM) of Ref. [5].

From an extended supersymmetric theory, one can construct a topological field theory by a procedure called topological twist, that is to define a new rotational group (the twisted rotation) by a particular combination of the original spacetime rotation and an internal $R$-symmetry. The above BRST charge $Q$ transforms as a scalar under the twisted rotation. However, if one does not regard the twisted rotation as a real spacetime rotation, as the standpoint we take here, the topological twist is nothing but simple relabeling of dynamical variables in the original supersymmetric action on a flat spacetime; it thus does not change the physical content of the theory. This procedure is nevertheless useful to find the above $Q$ transformation and a $Q$-exact form of the action in the continuum theory.

In this paper, we present a lattice formulation of 2D $\mathcal{N} = (2,2)$ $U(k)$ supersymmetric QCD (SQCD) with $N$ matter multiplets in the fundamental representation. For 2D $\mathcal{N} = (2,2)$ theories, there are two possible ways of topological twist. One is the so-called A model twist, with which the twisted rotation is defined as a diagonal $U(1)$ subgroup of the product of the original 2D rotation, $SO(2) \simeq U(1)$, and the internal $U(1)_V$ symmetry. Another is the B model twist, with which one takes the diagonal $U(1)$ part of the product of the 2D rotation $U(1)$ and the internal $U(1)_A$ symmetry. See, for example, Ref. [35]. For a different twist, a different combination of supercharges is regarded as the BRST charge $Q$. With our present standpoint, as noted above, these two twists are just different relabeling of dynamical variables and thus
they are completely equivalent in continuum theory. However, resulting lattice theories can differ and they have their own drawback and advantage. In this classification, lattice formulations of Refs. [1,2,6,11,13,17] can be regarded as those based on the B model twist (see also Ref. [36]), while formulations of Refs. [4,5,7,8,12,18,20] are regarded as those from the A model twist. This classification in terms of the topological twist is sometimes very useful. For example, this explains why it was rather nontrivial to incorporate the superpotential in lattice formulations of Refs. [13,17]; they correspond to the B model twist and, with the B model twist, the holomorphic part of the superpotential cannot be written as a $Q$-exact form, although it is $Q$-closed.

Here, our intention is to provide a simple lattice formulation of 2D $\mathcal{N} = (2, 2)$ $U(k)$ SQCD with $N$ fundamental matter multiplets (and no anti-fundamental multiplet) by using compact gauge link variables on the 2D regular lattice. For this, we adopt the B model twist picture; the point is the $Q$-transformation law in the matter sector.

Lattice formulations of 2D $\mathcal{N} = (2, 2)$ $U(k)$ SQCD in Refs. [18,20] (those also use compact gauge link variables) are based on the A model twist. With the A model twist, the superpotential is $Q$-exact and it is almost straightforward to incorporate the superpotential in the formulation; this is an advantage of the A model twist picture. However, as encountered in these references, it is rather tricky to avoid the species doubling in the matter sector with the A model twist. With the A model twist, the $Q$ transformation of chiral fields in the fundamental representation, for example, has a rather symmetrical structure as the $Q$ transformation of anti-chiral fields in the anti-fundamental representation. See Eq. (2.17) of Ref. [18]. For this reason, the Wilson term introduced to lift the species doublers inevitably mixes the fundamental fermions (their number is $n_+$) and the anti-fundamental fermions (their number is $n_-$) as long as the Wilson term is compatible with the $Q$-symmetry. This forced one to take $n_+ = n_-$ in Ref. [18] in which the conventional Wilson Dirac operator was used. In Ref. [20], to remove the restriction $n_+ = n_-$, the overlap Dirac operator [37,38] was utilized on the analogy of a complete chirality separation [39,40] based on the Ginsparg-Wilson relation [41,42]. The construction in Ref. [20] is so far a unique lattice formulation of 2D $\mathcal{N} = (2, 2)$ SQCD with $n_+ \neq n_-$ that can incorporate the superpotential.

The use of the overlap Dirac operator, however, requires the admissibility

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2 In the context of the topological field theory, one does not consider the B model twist of the 2D $\mathcal{N} = (2, 2)$ $U(k)$ SQCD with $N$ fundamental matter multiplets, because the $U(1)_A$ symmetry is anomalous in this system and consequently the twisted rotation becomes anomalous. The B model twist of this system is completely legitimate in our present context because we do not regard the twisted rotation as a real spacetime rotation.

3 See Refs. [16,21] for related issues.
condition [43–45] on gauge link variables and imposition of the admissibility complicates the lattice action. Of course, the implementation of the overlap Dirac operator itself (especially for odd \( n_+ \) or \( n_- \)) is practically cumbersome.

On the other hand, with the B model twist, left- and right-handed components of the fundamental fermions are transformed in a symmetrical way under \( Q \); see Eq. (2.7) below. It is thus expected that we can separately treat the fundamental fermions from the anti-fundamental fermions while keeping \( Q \)-invariance. This is the basic idea of this paper which allows a simple lattice action.

However, with the B model twist, the superpotential cannot be expressed as a \( Q \)-exact form and, quite unfortunately, we could not find a lattice discretization of the superpotential term that is invariant under our lattice \( Q \) transformation. This restricts the applicability of our lattice formulation. Nevertheless, even without the superpotential, it is discussed that 2D \( \mathcal{N} = (2, 2) \) SQCD with \( N \) fundamental matter multiplets possesses rich physical contents. The low energy effective theory would be given by the Grassmannian \( G(k, N) \) supersymmetric nonlinear sigma model [35, 46], in which one expects the spontaneous chiral symmetry breaking \( \mathbb{Z}_{2N} \to \mathbb{Z}_2 \) (the \( U(1)_A \) symmetry is anomalous to be broken to \( \mathbb{Z}_{2N} \)) and a dynamical generation of a mass gap [47–51]. It would be quite interesting to investigate these quantum phenomena by using Monte Carlo simulations on the basis of the present lattice formulation.

2 The continuum target theory in the B model twist picture

We extensively follow the notational convention of Ref. [18] for 2D \( \mathcal{N} = (2, 2) \) SQCD.\(^4\) From the \((U(1), U(1)_A)\) charges of spinorial fields, where the first \( U(1) \) is the spacetime rotation, it turns out that

\[
Q' \equiv -\frac{1}{\sqrt{2}}(\bar{Q}_L + \bar{Q}_R)
\]

(2.1)

can be taken as a BRST supercharge with the B model twist. We also use different linear combinations of variables in the \( \mathcal{N} = (2, 2) \) gauge multiplet.

\(^4\) In particular, we assume that the generators of \( U(k) \) are hermitian and normalized as \( \text{tr}(T^A T^B) = (1/2) \delta^{AB} \), where \( A \) and \( B \) run from 0 to \( k^2 - 1 \). The color components of adjoint fields are defined by \( (\text{field})(x) = \sum_A (\text{field})^A(x) T^A \). In the continuum theory, \( D_\mu \) denote the covariant derivatives with respect to the gauge potentials \( A_\mu; \)

\[
D_\mu = \partial_\mu + i[A_\mu, \cdot] \text{ for adjoint fields and } D_\mu \Phi^+_{+I} = \partial_\mu \Phi^+_{+I} + iA_\mu \Phi^+_{+I} \text{ and } D_\mu \Phi^+_{+I} = \partial_\mu \Phi^+_{+I} - i \Phi^+_{+I} A_\mu \text{ for a field } \Phi^+_{+I} \text{ in the fundamental representation.}
\[ \psi'_0 \equiv \frac{1}{\sqrt{2}}(\lambda_L - \lambda_R), \quad \psi'_1 \equiv \frac{i}{\sqrt{2}}(\lambda_L + \lambda_R), \]
\[ \chi' \equiv \frac{1}{\sqrt{2}}(\lambda_L + \bar{\lambda}_R), \quad \eta' \equiv \frac{i}{\sqrt{2}}(\bar{\lambda}_L - \lambda_R), \]
\[ X_0 \equiv -\frac{1}{2}(\phi + \bar{\phi}), \quad X_1 \equiv -\frac{i}{2}(\phi - \bar{\phi}), \]
\[ D' \equiv D - D_\mu X_\mu, \]
\[ A_\mu \equiv A_\mu - iX_\mu, \quad A_\mu^\dagger = A_\mu + iX_\mu, \quad (2.2) \]

where \( D_\mu X_\mu = \partial_\mu X_\mu + i[A_\mu, X_\mu] \). Compare these with Eq. (2.4) of Ref. [18]. With the B model twist, complexified gauge potentials in the last line naturally appear as we will see below. We thus introduce the covariant derivatives with respect to the complexified gauge potential \( A_\mu \). For the adjoint representation, it is defined by
\[ D_\mu \equiv \partial_\mu + i[A_\mu, \cdot]. \quad (2.3) \]

For a generic field in the fundamental representation of \( U(k), \Phi_{+I} \) \((I = 1, \ldots, N)\),
\[ D_\mu \Phi_{+I} \equiv \partial_\mu \Phi_{+I} + iA_\mu \Phi_{+I}, \]
\[ D_\mu \Phi_{+I}^\dagger \equiv (D_\mu \Phi_{+I})^\dagger = \partial_\mu \Phi_{+I}^\dagger - i\Phi_{+I}^\dagger A_\mu. \quad (2.4) \]
The corresponding 2D field strength is defined by
\[ F_{01} \equiv -i[D_0, D_1] = \partial_0 A_1 - \partial_1 A_0 + i[A_0, A_1]. \quad (2.5) \]

The \( Q' \)-transformation generated by the combination (2.1) can be obtained by setting \( \xi_L = \xi_R = 0 \) and \( \bar{\xi}_L = -\xi_L = -\bar{\xi}/\sqrt{2} \) in Eqs. (A.12) and (A.19) of Ref. [18] and removing \( \bar{\xi} \) from \( \delta \bar{\xi} \). For the gauge multiplet, we have
\[ Q'A_\mu = 0, \]
\[ Q'A_\mu^\dagger = 2\psi'_\mu, \quad Q'\psi'_\mu = 0, \]
\[ Q'\chi' = -iF_{01}, \]
\[ Q'\eta' = D', \quad Q'D' = 0. \quad (2.6) \]

Note that the complexified gauge potential, \( A_\mu = A_\mu - iX_\mu \), is \( Q' \)-invariant. The nilpotency of \( Q' \), \((Q')^2 = 0\), on the gauge multiplet is then obvious. For the fundamental matter multiplets (by using the same notation as Ref. [18],...
\[\tilde{m}_{+I} \text{ are the twisted masses [52]}, \text{ we have}\]

\[
\begin{align*}
Q'\phi_{+I} &= 0, \\
Q'\psi_{+IR} &= 2D_z\phi_{+I} + \tilde{m}^*_{+I}\phi_{+I}, \\
Q'\psi_{+IL} &= 2D_z\phi_{+I} + \tilde{m}_+\phi_{+I}, \\
Q'F_{+I} &= -2D_z\psi_{+IR} + 2D_z\psi_{+IL} - 2i\chi'\phi_{+I} - \tilde{m}_{+I}\psi_{+IR} + \tilde{m}_{+I}\psi_{+IL}, \\
Q'\phi^\dagger_{+I} &= -\bar{\psi}_{+IR} - \bar{\psi}_{+IL}, \\
Q'\bar{\psi}_{+IR} &= -F^\dagger_{+I}, \\
Q'\bar{\psi}_{+IL} &= F^\dagger_{+I}, \\
Q'F^\dagger_{+I} &= 0,
\end{align*}
\]

(2.7)

where \(D_{z,\bar{z}} \equiv \frac{1}{2}(D_0 \mp iD_1)\). Recalling \(Q'A_\mu = 0\) and Eq. (2.5), the nilpotency of \(Q'\) is again almost obvious. Note that the \(Q'\)-transformations of \(\psi_{+IR}\) and \(\psi_{+IL}\) have a symmetrical form as noted in Introduction. This is not the case with the A model twist; see Eq. (2.6) of Ref. [18].

Each term in the continuum action of 2D \(\mathcal{N} = (2, 2)\) SQCD except the superpotential (Eqs. (2.2), (2.23) and the Wick rotation of (A.17) of Ref. [18]) can be expressed in a \(Q'\)-exact form. The 2D SYM part is

\[
S_{2DSYM}^{(E)} = \frac{1}{g^2} \int d^2x \ tr \left( \frac{1}{2} F_{\mu\nu}F_{\mu\nu} + D_\mu\phi D_\mu\bar{\phi} + \frac{1}{4} [\phi, \bar{\phi}]^2 - D^2 \right. \\
+ \left. 4\bar{\lambda}_R D_\bar{z}\lambda_R + 4\bar{\lambda}_L D_\bar{z}\lambda_L + 2\bar{\lambda}_R[\bar{\phi}, \lambda_L] + 2\bar{\lambda}_L[\phi, \lambda_R] \right)
= Q' \frac{1}{g^2} \int d^2x \ tr \left[ -\eta' (D' + 2D_\mu X_\mu) + i\chi' F^\dagger_{01} \right].
\]

(2.8)

The FI term and the theta term are

\[
S_{FI,\theta}^{(E)} = \int d^2x \ tr \left( \kappa D - i \frac{\bar{\psi}}{2\pi} F_{01} \right)
= Q' \frac{1}{g^2} \int d^2x \ tr \left( \kappa\eta' + \frac{\bar{\psi}}{2\pi} \chi' \right).
\]

(2.9)
The action of the fundamental matter multiplets with the twisted mass terms, \(^5\)

\[
S_{\text{mat, } \bar{m}}^{(E)} = \int d^2x \sum_{I=1}^{N} \left[ D_\mu \phi_{+I}^\dagger D_\mu \phi_{+I} + \frac{1}{2} \phi_{+I}^\dagger \left\{ \phi - \bar{m}_{+I}, \phi - \bar{m}_{+I}^* \right\} \phi_{+I} 
\right.
\]
\[
- F_{+I}^\dagger F_{+I} - \phi_{+I}^\dagger D_\mu \phi_{+I} 
+ 2 \bar{\psi}_{+IR} D_\mu \psi_{+IR} + 2 \bar{\psi}_{+IL} D_\mu \psi_{+IL} 
+ \bar{\psi}_{+IR} (\phi - \bar{m}_{+I}^*) \psi_{+IL} + \bar{\psi}_{+IL} (\phi - \bar{m}_{+I}) \psi_{+IR} 
\]
\[
- i \sqrt{2} \left( \phi_{+I}^\dagger (\lambda_L \psi_{+IR} - \lambda_R \psi_{+IL}) + (-\bar{\psi}_{+IR} \lambda_L + \bar{\psi}_{+IL} \lambda_R) \phi_{+I} \right) \bigg] 
= Q' \int d^2x \sum_{I=1}^{N} \left[ \left( D_\mu \phi_{+I}^\dagger \right) \psi_{+IR} + \left( D_\mu \phi_{+I}^\dagger \right) \psi_{+IL} 
+ \frac{1}{2} \left( \bar{\psi}_{+IR} - \bar{\psi}_{+IL} \right) F_{+I} - \phi_{+I}^\dagger \eta' \phi_{+I} 
+ \frac{1}{2} \phi_{+I}^\dagger \left( \bar{m}_{+I} \psi_{+IR} + \bar{m}_{+I}^* \psi_{+IL} \right) \right]. 
\]

(2.12)

On the basis of these representations of the continuum theory with the B model twist, we construct a lattice formulation in the next section.

3 Lattice model

3.1 Dynamical variables and lattice \(Q'\)-transformation

We start with to define a lattice analogue of the \(Q'\)-transformation. For the gauge multiplet, we define, on the analogy of Eq. (2.6),

\[
Q' U_\mu (x) = i \psi'_\mu (x) U_\mu (x), \quad Q' \psi'_\mu (x) = i \psi'_\mu (x) \psi'_\mu (x), \\
Q' V_\mu (x) = i \psi'_\mu (x) V_\mu (x), \\
Q' \chi' (x) = 1 - U_{01} (V_{-1} U) (x), \\
Q' \eta' (x) = D' (x), \quad Q' D' (x) = 0. 
\]

(3.1)

Here, \(U_\mu (x) \in U(k)\) are standard compact gauge link variables. The above definition of \(Q'\) on \(U_\mu (x)\), that is exactly nilpotent, is suggested from a naive corre-

\(^5\) To confirm this expression, it is useful to note

\[
\left( D_\mu \Phi_{+I}^\dagger \right) \Phi_{+I} + \Phi_{+I}^\dagger \left( D_\mu \Phi_{+I} \right) = \partial_\mu \left( \Phi_{+I}^\dagger \Phi_{+I} \right) + 2 \Phi_{+I}^\dagger X_\mu \Phi_{+I}, 
\]

(10.20)

and

\[
Q' D_\mu \phi_{+I}^\dagger = - D_\mu \left( \bar{\psi}_{+IR} + \bar{\psi}_{+IL} \right) - 2 i \phi_{+I}^\dagger \psi'_\mu. 
\]

(2.11)
spondence with the continuum field, $U_\mu(x) \sim e^{iaA_\mu}$, where $a$ denotes the lattice spacing. On the other hand, $V_\mu(x)$ are $k \times k$ hermitian positive (noncompact) matrices corresponding to the continuum scalar fields $X_\mu, V_\mu(x) \sim e^{-aX_\mu}$. More specifically, we introduce hermitian lattice variables $X_\mu(x)$ and define $V_\mu(x) \equiv e^{-aX_\mu(x)}$.

$$V_\mu(x) \equiv e^{-aX_\mu(x)}.$$ (3.2)

$U_{\mu\nu}(x)$ denote the standard plaquette variables

$$U_{\mu\nu}(x) \equiv U_{\mu\nu}(U)(x) \equiv U_\mu(x)U_\nu(x + a\hat{\mu})U_\nu(x+a\hat{\nu})^{-1}U_\mu(x)^{-1}$$ (3.3)

and $U_{01}(V^{-1}U)(x)$ in Eq. (3.1) is defined by the substitution $U_\mu(x) \to V_\mu(x)^{-1}U_\mu(x)$ in this expression. Note that in general $U_{01}(V^{-1}U)(x)$ are not unitary; they are elements of $GL(k, \mathbb{C})$.

We assume that $V_\mu(x)$ (for both $\mu = 0$ and 1) are site variables transforming as adjoint under lattice gauge transformations at the point $x$ (the same is assumed for $X_\mu(x), \psi'_\mu(x), \chi'(x), \eta'(x)$ and $D'(x)$). Then the both sides of each relation of Eq. (3.1) have the same gauge transformation property regarding $Q'$ as a gauge singlet.

With the above naive correspondence, $U_\mu(x) \sim e^{iaA_\mu}$ and $V_\mu(x) \sim e^{-aX_\mu}$, the combination $V_\mu(x)^{-1}U_\mu(x)$ would correspond to the exponential of the complexified gauge potential $A_\mu = A_\mu - iX_\mu, V_\mu(x)^{-1}U_\mu(x) \sim e^{iaA_\mu}$. In fact, the combination $V_\mu(x)^{-1}U_\mu(x)$ is invariant under lattice $Q'$-transformation (3.1).

Noting this, it is easy to see that lattice $Q'$-transformation (3.1) is nilpotent $(Q')^2 = 0$ on the gauge multiplet.

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6 Throughout this paper, as Ref. [18], all lattice field variables are taken to be dimensionless. These are related to dimensionful continuum fields as, $X_\mu(x) \to aX_\mu(x), \psi'_\mu(x) \to a^{3/2}\psi'_\mu(x), \chi'(x) \to a^{3/2}\chi'(x), \eta'(x) \to a^{3/2}\eta'(x), D'(x) \to a^2D'(x), \bar{\psi}+I_{L/R}(x) \to a^{1/2}\bar{\psi}+I_{L/R}(x), \tilde{\psi}+I_{L/R}(x) \to a^{1/2}\tilde{\psi}+I_{L/R}(x), F_{+I}(x) \to aF_{+I}(x)$, $F^I_{+I}(x) \to aF^I_{+I}(x)$, where all fields in the right-hand sides are continuum ones.
For the matter multiplets in the fundamental representation, we set

\[ Q' \phi_{+I}(x) = 0, \]
\[ Q' \psi_{+IR}(x) = 2aD_2 \phi_{+I}(x) + a\tilde{m}^*_{+I} \phi_{+I}(x), \]
\[ Q' \psi_{+IL}(x) = 2aD_2 \phi_{+I}(x) + a\tilde{m}_{+I} \phi_{+I}(x), \]
\[ Q' F_{+I}(x) = -2aD_2 \psi_{+IR}(x) + 2aD_2 \psi_{+IL}(x) \]
\[ - 2i\gamma'(x)V_1(x)^{-1}U_1(x)V_0(x + a\hat{1})^{-1}U_0(x + a\hat{0} + a\hat{1}) \]
\[ - a\tilde{m}_{+I} \psi_{+IR}(x) + a\tilde{m}^*_{+I} \psi_{+IL}(x), \]
\[ Q' \phi_{+I}(x)^\dagger = -\tilde{\psi}_{+IR}(x) - \tilde{\psi}_{+IL}(x), \]
\[ Q' \bar{\psi}_{+IR}(x) = -F_{+I}(x)^\dagger, \]
\[ Q' \bar{\psi}_{+IL}(x) = F_{+I}(x)^\dagger, \]
\[ Q' F_{+I}(x)^\dagger = 0. \quad (3.4) \]

For the covariant differences for a generic lattice field \( \Phi_{+I}(x) \) in the fundamental representation, we adopt the forward differences

\[ aD_\mu \Phi_{+I}(x) \equiv V_\mu(x)^{-1}U_\mu(x)\Phi_{+I}(x + a\hat{\mu}) - \Phi_{+I}(x), \]
\[ aD_\mu \Phi_{+I}(x)^\dagger \equiv (aD_\mu \Phi_{+I}(x))^\dagger = \Phi_{+I}(x + a\hat{\mu})^1U_\mu(x)^{-1}V_\mu(x)^{-1} - \Phi_{+I}(x)^\dagger \quad (3.5) \]

and, as in the continuum theory, \( D_{\bar{z},z} \equiv \frac{1}{2}(D_0 \mp iD_1) \). The nilpotency of \( Q' \) in Eq. (3.4) is almost obvious except that for the auxiliary field \( F_{+I}(x) \). This nilpotency \( (Q')^2F_{+I}(x) = 0 \) follows from the identity

\[ (1 - U_{01}(V^{-1}U)(x))V_1(x)^{-1}U_1(x)V_0(x + a\hat{1})^{-1}U_0(x + a\hat{0} + a\hat{1}) \]
\[ = 2ia^2[D_{\bar{z}}, D_z] \phi_{+I}(x), \quad (3.6) \]

that is a lattice analogue of the relation (2.5).

We thus defined \( Q' \)-transformation on the lattice that is completely nilpotent on all lattice variables. Note that the nilpotency \( (Q')^2 = 0 \) holds without referring to any equivalence under the gauge or flavor rotations; this is quite different from the cases in the A model twist [18,20].

### 3.2 Lattice action

Next, we define the lattice action. The SYM part is defined by, on the analogy of Eq. (2.8),

\[ S_{2DSYM}^{\text{LAT}} = Q' \frac{1}{a^2 g^2} \sum_x \text{tr} \left[ -\eta'(x) (D'(x) + W(x)) + \chi'(x) \left( 1 - U_{01}(V^{-1}U)(x) \right)^\dagger \right] . \quad (3.7) \]
As a possible choice of $W(x)$, that is a lattice counterpart of $2D_\mu X_\mu$, we take

\[ W(x) \equiv 2 \sum_\mu \left[ V_\mu(x)^{-1} + U_\mu(x - a\hat{\mu})^{-1} V_\mu(x - a\hat{\mu}) U_\mu(x - a\hat{\mu}) - 2 \right] \quad (3.8) \]

and this in fact reduces to $2a^2D_\mu X_\mu$ in the naive continuum limit, $U_\mu(x) \sim e^{iaA_\mu}$, $V_\mu(x) \sim e^{-aX_\mu}$ and $a \to 0$. Note that $W(x)$ are hermitian matrices.

The FI term and the theta term are defined by (see Eq. (2.9))\(^7\)

\[ S_{\text{FI},\vartheta}^{\text{LAT}} = Q' \sum_x \text{tr} \left[ \kappa \eta'(x) + \frac{\vartheta}{2\pi} \chi'(x) \right]. \quad (3.9) \]

The lattice action for matter multiplets is almost the same as the continuum one:

\[ S_{\text{mat.},+m}^{\text{LAT}} = Q' \sum_x \sum_{I=1}^N \left[ (aD_\phi \phi_{+I}(x)\dagger) \psi_{+IR}(x) + (aD_\bar{\phi} \phi_{+I}(x)\dagger) \bar{\psi}_{+IL}(x) \right. \]
\[ + \frac{1}{2} \left( \bar{\psi}_{+IR}(x) - \bar{\psi}_{+IL}(x) \right) F_{+I}(x) - \phi_{+I}(x)\dagger \eta'(x) \phi_{+I}(x) \]
\[ \left. + \frac{1}{2} \phi_{+I}(x)\dagger \left( am_{+I} \psi_{+IR}(x) + am^*_{+I} \bar{\psi}_{+IL}(x) \right) \right]. \quad (3.10) \]

Note, however, that the lattice covariant differences appearing in these expressions are forward ones (3.5).

From the above construction and from the nilpotency $(Q')^2 = 0$, it is clear that our lattice action is invariant under gauge and fermionic $Q'$ transformations.

The total lattice action possesses also some global symmetries; one is the $U(1)_V$ symmetry, under which

\[ \psi_{+I}(x) \to e^{i\alpha} \psi_{+I}(x), \quad \chi'(x) \to e^{-i\alpha} \chi'(x), \quad \eta'(x) \to e^{-i\alpha} \eta'(x), \]
\[ \psi_{+I,L/R}(x) \to e^{-i\alpha} \psi_{+I,L/R}(x), \quad \bar{\psi}_{+I,L/R}(x) \to e^{i\alpha} \bar{\psi}_{+I,L/R}(x), \]
\[ F_{+I}(x) \to e^{-2i\alpha} F_{+I}(x), \quad F_{+I}(x)\dagger \to e^{2i\alpha} F_{+I}(x)\dagger. \quad (3.11) \]

\(^7\) This theta term is not topologically invariant with finite lattice spacings. One could instead use $S_{\vartheta}^{\text{LAT}} = -\vartheta/(2\pi) \sum_x \text{tr} \ln U_{01}(U)(x)$, that is topologically invariant if configurations with an eigenvalue of $U_{01}(U)(x)$ being $-1$ for a certain $x$ are excised (see Ref. [53] and references cited therein), as precisely the case when the admissibility is imposed [18,20]. However, even without imposing the admissibility, such configurations should not contribute to functional integrals in the continuum limit (see Sec. 3.3) and this term would practically work as a topological (and thus $Q'$) invariant in the continuum limit.
and other variables are kept intact. Another is $U(1)^N$ symmetry that rotates each fundamental multiplet independently.\(^8\)

$$\Phi_{+I}(x) \rightarrow e^{i\alpha_I} \Phi_{+I}(x), \quad \Phi_{+I}(x)^\dagger \rightarrow e^{-i\alpha_I} \Phi_{+I}(x)^\dagger. \quad \text{(3.12)}$$

Besides symmetry under discrete translations by the lattice unit, the present lattice action does not possess further (fermionic as well as bosonic) symmetries that were present in the continuum theory.

### 3.3 Weak coupling saddle point of the lattice action

In the naive continuum limit $a \to 0$, in which one assumes $U_\mu(x) \sim e^{iA_\mu}$ and $V_\mu(x) \sim e^{-aX_\mu}$, our lattice action reproduces the continuum action of 2D $\mathcal{N} = (2,2)$ $U(k)$ SQCD with $N$ fundamental matter multiplets. A perturbative argument in Sec. 3.6 then indicates that the continuum limit of the present lattice model is given by the weak coupling limit $\beta \to \infty$, where $1/(a^2 g^2) \equiv \beta/(2k)$, $\kappa = N/(4\pi) \ln \beta + \text{const.}$, $\tilde{m}_{+I} = (\tilde{m}_{+I}/g) \sqrt{2k/\beta}$ and $\tilde{m}_{+I}^\ast = (\tilde{m}_{+I}^\ast/g) \sqrt{2k/\beta}$.

However, for the above perturbative picture on the basis of expansion around $U_\mu(x) = 1$ and $V_\mu(x) = 1$ to be consistent, the configuration $U_\mu(x) = 1$ and $V_\mu(x) = 1$ (up to gauge transformations) should give the unique saddle point in the weak coupling limit. It suffices if this is so for an infinite lattice. Whether this is really the case or not, however, could generally be a nontrivial issue. In fact, in Refs. [5,12,18,20], the admissibility [44] was incorporated in the lattice action to remove weak coupling saddle points that have no continuum counterpart.

In the present lattice model, at least for $\vartheta = 0$, we can see that the unique saddle point in the weak coupling limit for an infinite lattice is $U_\mu(x) = 1$ and $V_\mu(x) = 1$ up to gauge transformations. The argument proceeds as follows.\(^9\)

\(^8\) If some twisted masses are degenerated, this symmetry enhances accordingly.

\(^9\) In usual lattice gauge theory with compact gauge link variables, such as lattice QCD, the weak coupling saddle point is not affected by the presence of fermions, because the fermion determinant would be a bounded function of link variables and consequently it cannot modify saddle points for $\beta \to \infty$. Strictly speaking, this reasoning cannot be applied to our present system because the fermion determinant could be an unbounded function of noncompact scalar fields. The fermion determinant in principle could balance with bosonic action (3.13) and modify the saddle points for $\beta \to \infty$. We do not consider this possibility below because this could occur only at the “boundary” of the field space, such as $V_\mu = 0$ or $V_\mu = +\infty$, and if this occurs, our lattice formulation would be meaningless in any case.
We first seek saddle points for $\beta \to \infty$ on a lattice with a finite number of lattice points, $N\mu$ in the $\mu$-direction, to avoid a subtlety associated with an infinite lattice. We assume periodic boundary conditions for bosonic fields. After obtaining all saddle points on this finite lattice, we send $N\mu$ to infinity yielding saddle points for an infinite lattice.

The bosonic part of the lattice action, after integrating over the auxiliary fields, takes the form

$$S_{2DSYM}^{LAT} + S_{FI,\vartheta}^{LAT} + S_{\text{mat},+\tilde{m}}^{LAT} = \frac{\beta}{2k} \sum_x \text{tr} \left[ \frac{1}{4} \left\{ W(x) + \frac{2k}{\beta} \left( \sum_{I=1}^N \phi_{+I}(x) \phi_{+I}(x)^\dagger - \kappa \right) \right\}^2 
+ \left( 1 - U_{01}(V^{-1}U)(x) \right) \left( 1 - U_{01}(V^{-1}U)(x) + \frac{2k}{\beta} \frac{\vartheta}{2\pi} \right)^\dagger \right] 
+ \sum_x \sum_{I=1}^N \left( aD_\mu \phi_{+I}(x) + a\tilde{m}_{+I,\mu}\phi_{+I}(x) \right)^\dagger \left( aD_\mu \phi_{+I}(x) + a\tilde{m}_{+I,\mu}\phi_{+I}(x) \right),$$

(3.13)

where

$$\tilde{m}_{+I,\mu} \equiv \begin{cases} \text{Re } \tilde{m}_{+I}, & \text{for } \mu = 0, \\ -\text{Im } \tilde{m}_{+I}, & \text{for } \mu = 1, \end{cases}$$

(3.14)

and we have assumed that $\tilde{m}_{+I}$ and $\tilde{m}_{+I}^*$ are complex conjugate to each other. Therefore, after integrating over the auxiliary fields, the bosonic part of the lattice action is real and positive semi-definite for $\vartheta = 0$ (recall that $W(x)$ are hermitian); this is certainly a desired property. From the above expression, for $\vartheta = 0$, saddle points for $\beta \to \infty$ are specified by

$$W(x) + \frac{2k}{\beta} \sum_{I=1}^N \phi_{+I}(x) \phi_{+I}(x)^\dagger = \frac{2k}{\beta} \kappa,$$

(3.15)

$$U_{01}(V^{-1}U)(x) = 1.$$  (3.16)

We then apply $\sum_x \text{tr}$ to both sides of Eq. (3.15) to yield

$$\sum_x \text{tr} W(x) + \frac{2k}{\beta} \sum_x \sum_{I=1}^N \phi_{+I}(x)^\dagger \phi_{+I}(x) = \frac{2k}{\beta} \kappa \sum_x \text{tr} 1.$$  (3.17)

We further note that $\sum_x \text{tr} W(x)$ is positive semi-definite

$$\sum_x \text{tr} W(x) = \sum_x \sum_{\mu} \text{tr} \left[ V_\mu(x)^{-1} + V_\mu(x - a\tilde{\mu}) - 2 \right] 
= \sum_x \sum_{\mu} \text{tr} \left[ V_\mu(x)^{-1} + V_\mu(x) - 2 \right] 
= \sum_x \sum_{\mu} \text{tr} \left[ (V_\mu(x)^{-1/2} - V_\mu(x)^{1/2})^2 \right] \geq 0$$

(3.18)
(recall that $V_\mu(x)$ are positive and thus the square root can always be defined).

Now, in the $\beta \to \infty$ limit, the right-hand side of Eq. (3.17) vanishes because $\kappa$ grows at most $\sim \ln \beta$. In the left-hand side of Eq. (3.17), there cannot occur cancellation between the first and the second terms because both are positive semi-definite. These imply that $\sum_x \text{tr} W(x) = 0$ for $\beta \to \infty$. Then, from Eq. (3.18), we have $V_\mu(x) = 1$ at saddle points for $\beta \to \infty$. Plugging this $V_\mu(x) = 1$ into Eq. (3.16), we see that at saddle points, the gauge plaquette is unity $U_{01}(U)(x) = 1$. This is an identical condition for the weak coupling saddle point with the standard plaquette action. As shown in Appendix A, the most general form of such flat connections satisfying $U_{01}(U)(x) = 1$ on a periodic lattice is given by $U_\mu(x) = g(x)T_\mu g(x+a\hat{\mu})^{-1}$, where the gauge transformation $g(x) \in U(k)$ is periodic on the lattice and the constant factor $T_\mu$ is given by Eq. (A.4). Therefore, up to gauge transformations, the weak coupling saddle points are given by $U_\mu(x) = T_\mu$ and $V_\mu(x) = 1$.

Finally, in the infinite lattice limit $N_\mu \to \infty$, $T_\mu \to 1$ as Eq. (A.4) shows. Therefore, the saddle point for $\beta \to \infty$ on an infinite lattice is given by $U_\mu(x) = 1$ and $V_\mu(x) = 1$ up to gauge transformations. This completes our argument for the weak coupling saddle point.

In the present lattice model on the finite lattice, moreover, one can see that the space of zeros of the bosonic action is, if it is not empty, always compact even for finite $\beta$ and $\kappa$; this follows from Eq. (3.17), the condition that such zeros must satisfy. In Eq. (3.17), the first term defines the square of a distance

It is possible to tailor a $W(x)$ such that there are infinite (gauge inequivalent) weak coupling saddle points for the scalar fields $V_\mu(x)$. For example, with the choice

$$W(x) \equiv -2 \sum_\mu \left[V_\mu(x) - U_\mu(x-a\hat{\mu})^{-1}V_\mu(x-a\hat{mu})U_\mu(x-a\hat{\mu})\right] + \frac{1}{\kappa} L(x) \quad (3.19)$$

and

$$L(x) = \text{tr} \left[ 2 - U_{01}(U)(x) - U_{01}(U)(x)^{-1} + \sum_\mu a^2 D_\mu \phi(x) D_\mu \bar{\phi}(x) + \frac{1}{4} [\phi(x), \bar{\phi}(x)]^2 \right] \quad (3.20)$$

($\phi$ and $\bar{\phi}$ are given by Eq. (2.2) and $X_\mu(x) \equiv -\ln V_\mu(x)$; $D_\mu$ are forward covariant differences for adjoint fields with gauge link variables $U_\mu(x)$ used), by an argument similar to that in the main text, one sees that the saddle points (for an infinite lattice) are given by $U_\mu(x) = 1$ and $X_\mu(x) =$ const. and $[X_0, X_1] = 0$ up to gauge transformations. These configurations (with $\phi_{+1}(x) = 0$) provide also a noncompact set of zeros of the lattice bosonic action when $\kappa = 0$, corresponding to the Coulomb branch in the continuum theory. The free kinetic terms of fermions resulting from the above $W(x)$ are identical to those in the main text and thus this choice does not lead to the species doubling.
between $V_\mu(x)$ and $V_\mu(x) \equiv 1$ and similarly the second term defines a norm of $\phi_{+I}$, both are positive semi-definite. It is then obvious that any solution of the relation (3.17) cannot grow indefinitely because the right-hand side remains finite for any nonzero $\beta$; this shows that the space of zeros of the bosonic action is compact.\footnote{This property is shared also by lattice formulations of 2D $\mathcal{N} = (2, 2)$ SYM in Refs. \[12\] and \[54\] which use compact lattice scalar fields. The former formulation possesses a manifest fermionic symmetry.}

### 3.4 Absence of the species doubling

Once the expansion around $U_\mu(x) = 1$ and $V_\mu(x) = 1$ is justified, it is straightforward to see that the present lattice formulation is free from the species doubling. Setting $U_\mu(x) = V_\mu(x) = 1$, we have $S_{2 \text{DSYM}}^{\text{LAT}} = -2/(a^2 g^2) \times \sum_x \text{tr}[\bar{\psi}(x) aD \psi(x)]$, where $\psi \equiv (\bar{\lambda}_L, \bar{\lambda}_R)$ and $\psi^T \equiv (\lambda_R, \lambda_L)$, and

$$aD \equiv \sum_\mu \gamma_\mu \frac{1}{2} (a\partial_\mu + a\partial^*_\mu) - \frac{1}{2} \left(a^2 \partial_0 \partial_0 - i \gamma_5 a^2 \partial_1 \partial_1\right).$$

(3.21)

In this expression, $\gamma_0 \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $\gamma_1 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\gamma_5 \equiv \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and $\partial_\mu$ and $\partial^*_\mu$ are the forward and the backward difference operators, respectively. The second term in Eq. (3.21) acts as a Wilson term and, since $a^2 D^\dagger D = -\sum_\mu a^2 \partial_\mu \partial^*_\mu$, the free Dirac operator $D$ vanishes only at the origin of the Brillouin zone. That is, there is no species doubling.

Similarly, for the matter sector, for $U_\mu(x) = V_\mu(x) = 1$ and $\phi_{+I}(x) = 0$, we have

$$S_{\text{mat,}+\bar{m}}^{\text{LAT}} = \sum_x \sum_{I=1}^N \bar{\psi}_{+I} \gamma_5 \left(aD - a\bar{m}_{+I} \frac{1 + \gamma_5}{2} - a\bar{m}^*_{+I} \frac{1 - \gamma_5}{2}\right) \gamma_5 \psi_{+I},$$

(3.22)

where $\bar{\psi}_{+I} \equiv (\bar{\psi}_{+IL}, \bar{\psi}_{+IR})$ and $\psi^T_{+I} \equiv (\psi_{+IR}, \psi_{+IL})$. This kinetic operator reproduces the correct dispersion relation for massive fermions near the origin of the Brillouin zone.

### 3.5 Invariant integration measure

For our lattice formulation to be invariant under gauge, $Q', U(1)_V$ and $U(1)^N$ transformations, not only the lattice action but also the integration measure must be invariant under these transformations. Except for the scalar
fields $V_\mu(x)$, the integration measure is standard:

$$\prod_x \left[ \prod_{\mu=0}^1 dU_\mu(x) \prod_A d\psi^A_\mu(x) d\bar{\psi}^A_\mu(x) d\chi^A(x) d\eta^A(x) dD^A(x) \prod_{I=1}^N (d\mu_{\text{mat.}+I}) \right],$$

(3.23)

where

$$(d\mu_{\text{mat.}+I})$$

$$\equiv \prod_x \prod_{i=1}^k d\phi_{+I_i}(x) d\phi_{+I_i}^*(x) d\bar{\psi}_{+I_iL_i}(x) d\psi_{+I_iR_i}(x) d\bar{\psi}_{+I_iL_i}(x) d\psi_{+I_iR_i}(x)$$

$$\times dF_{+I_i}(x) dF_{+I_i}^*(x).$$

(3.24)

In Eq. (3.23), $dU_\mu(x)$ is the conventional Haar measure on $U(k)$. It can be seen that the above measure is invariant under gauge, $Q', U(1)_V$ and $U(1)^N$ transformations. The argument is essentially the same as that of Ref. [12]. In particular, $Q'$-invariance of the Haar measure follows from the fact that the $Q'$ transformation on link variables $U_\mu(x)$ can be regarded as a left-multiplication of a group element $U_\mu(x) \to g(x)U_\mu(x)$, where $g(x) \in U(k)$, as shown in Eq. (2.3) of Ref. [12].

On the other hand, the definition of an invariant integration measure for the scalar fields $V_\mu(x)$ is somewhat intricate. We start with the following norm of a variation of $V_\mu(x)$

$$\|\delta V_\mu(x)\|^2 \equiv \text{tr} \left[ V_\mu(x)^{-1}\delta V_\mu(x)V_\mu(x)^{-1}\delta V_\mu(x) \right].$$

(3.25)

This norm is positive semi-definite, because $\|\delta V_\mu(x)\|^2 = \text{tr}[(V_\mu(x)^{-1/2}\delta V_\mu(x)V_\mu(x)^{-1/2})^2]$. An integration measure associated with this norm, according to a standard recipe, is given by

$$\prod_x \prod_\mu \left[ \prod_A dV^A_\mu(x) \right] \sqrt{\frac{\text{det} \text{tr} [V_\mu(x)^{-1}T^A V_\mu(x)^{-1}T^B]]}{\|\delta V_\mu(x)\|^2}}.$$

(3.26)

The point is that the norm (3.25) is invariant under the substitutions (I) $V_\mu(x) \to u_\mu(x)V_\mu(x)u_\mu(x)^{-1}$ and (II) $V_\mu(x) \to h_\mu(x)V_\mu(x)h_\mu(x)$, where $u_\mu(x)$ are unitary matrices and $h_\mu(x)$ are invertible hermitian matrices. From this invariance of the norm, it follows that also the measure (3.26) is invariant under these substitutions, as can be verified explicitly by using $\sum_A (T^A)^{ij}(T^A)^{kl} = (1/2)\delta_{il}\delta_{jk}$. From the invariance under (I), gauge invariance of the measure is obvious because gauge transformations take the form of (I). Furthermore, the measure is invariant also under the $Q'$-transformation as follows.

The measure (3.26) is of course invariant under an infinitesimal version of the substitutions, (i) $V_\mu(x) \to V_\mu(x) + i\theta^A_\mu(x)[T^A,V_\mu(x)]$ and (ii) $V_\mu(x) \to
\[ V_\mu(x) + \zeta^A_\mu(x)\{T^A, V_\mu(x)\}, \] where \( \theta^A_\mu(x) \) and \( \zeta^A_\mu(x) \) are infinitesimal real parameters. However, this invariance holds even if we regard \( \theta^A_\mu(x) \) and \( \zeta^A_\mu(x) \) as infinitesimal complex parameters, because no complex conjugation is involved for the invariance. In particular, we may set \( \theta^A_\mu(x) = (1/2)\xi \psi^A_\mu(x) \) and \( \zeta^A_\mu(x) = (i/2)\bar{\psi}^A_\mu(x) \), where \( \xi \) is a Grassmann parameter. Then a combination of the above (i) and (ii) becomes \( V_\mu(x) \to V_\mu(x) + i\xi \psi^A_\mu(x)V_\mu(x) \), the \( Q' \)-transformation on \( V_\mu(x) \). This shows that the measure (3.26) is invariant also under the \( Q' \)-transformation. Thus, the above defined measure (3.26) has desired invariance properties.

However, if integration variables are \( V_\mu(x) \), one has to take into account the fact that \( V_\mu(x) \) are positive matrices. Practically, an integration measure for hermitian variables \( X_\mu(x) \) in Eq. (3.2) should be more useful. By rewriting invariant norm (3.25) in terms of a variation of \( X_\mu(x) \), we have

\[
\prod_x \prod_\mu \left[ \prod_A dX^A_\mu(x) \right] \sqrt{\det M^{AB}_\mu(x)},
\] (3.27)

where

\[
M^{AB}_\mu(x) \equiv \int_0^1 d\alpha \int_0^1 d\beta \tr \left[ e^{(\alpha-\beta)X_\mu(x)}T^A e^{-((\alpha-\beta)X_\mu(x)}T^B \right],
\] (3.28)

and the integration region of each variable \( X^A_\mu(x) \) is \((-\infty, +\infty)\).

From integration variables \( X^A_\mu(x) \), one can construct the matrices \( V_\mu(x) \) by

\[
V_\mu(x) = u_\mu(x)e^{-\lambda_\mu(x)}u_\mu(x)^{-1}, \quad \text{where} \quad \lambda_\mu(x) \equiv \text{diag}(\lambda_{\mu 1}(x), \ldots, \lambda_{\mu k}(x)) \text{ and } \lambda_{\mu i}(x) (i = 1, 2, \ldots, k) \text{ are eigenvalues of } X_\mu(x) = \sum_A X^A_\mu(x)T^A; \text{ } u_\mu(x) \text{ are unitary matrices that diagonalize } X_\mu(x), \text{ } X_\mu(x) = u_\mu(x)\lambda_\mu(x)u_\mu(x)^{-1}. \text{ In terms of these eigenvalues, the volume element is expressed as }
\]

\[
\sqrt{\det M^{AB}_\mu(x)} = \frac{1}{\sqrt{2^k}} \prod_{i<j} \cosh((\lambda_{\mu i}(x) - \lambda_{\mu j}(x)) - \frac{1}{(\lambda_{\mu i}(x) - \lambda_{\mu j}(x))^2} \geq \frac{1}{\sqrt{2^k}}.
\] (3.29)

One can directly confirm that the measure (3.27) with Eq. (3.29) is invariant under the above substitutions (i) and (ii), and thus under the \( Q' \)-transformation.

Also, in the hybrid Monte Carlo algorithm, one needs to compute variations of \( V_\mu(x) \) and of the volume element with respect to the integration variables \( X^A_\mu(x) \). They are given by

\[
(\delta V_\mu(x))_{ij} = \sum_{k,l} \left( u_\mu(x) \right)_{ik} \frac{e^{-\lambda_{\mu k}(x)} - e^{-\lambda_{\mu l}(x)}}{\lambda_{\mu k}(x) - \lambda_{\mu l}(x)} \left( u_\mu(x)^{-1}T^A u_\mu(x) \right)_{kl} \left( u_\mu(x)^{-1} \right)_{lj} \delta X^A_\mu(x)
\] (3.30)
and

\[ \delta \ln \sqrt{\det M^A_{\mu B}}(x) = \sum_{i \neq j} f(\lambda_{\mu i}(x) - \lambda_{\mu j}(x)) \left( u_\mu(x)^{-1} T^A u_\mu(x) \right)_{ii} \delta X^A_\mu(x), \]  

(3.31)

where

\[ f(x) \equiv \frac{\sinh x}{\cosh x - 1} - \frac{2}{x}. \]  

(3.32)

These expressions should be useful in actual Monte Carlo simulations.\(^\dagger\)

### 3.6 Continuum limit

In the present super-renormalizable gauge theory, the continuum limit is given by \( \beta \to \infty \), where \( \beta \equiv 2k/(a^2g^2) \). In this section, we argue that (within perturbation theory) all symmetries broken by lattice regularization are restored in the continuum limit without any fine tuning. For this argument, it is convenient to rescale continuum matter multiplets as \( \Phi^I \to (1/g)\Phi^I \) so that the mass dimensions of fields in matter multiplets become the same as the gauge multiplet (that is, bosonic fields have mass dimension 1, fermionic have 3/2, the auxiliary fields 2).

Generally speaking, symmetries broken by UV regularization could be recovered by supplementing appropriately chosen local counterterms. The most general form of local terms in the effective action, from the dimensional consideration, is

\[ \left( c_0 a^{p-4} g^2 + c_1 a^{p-2} g^2 + \cdots \right) \int d^2x \varphi^a \partial^b \psi^{2c} A^d, \quad p \equiv a+b+3c+2d \geq 0, \]  

(3.34)

up to some powers of possible logarithmic (\( \ln a \)) factors. In this expression, \( \varphi \) symbolically denotes bosonic fields in the continuum theory except the auxiliary fields, \( \psi \) denotes fermionic fields, and \( A \) denotes the auxiliary fields; \( \partial \) a derivative. Abbreviated terms in the parentheses are of strictly positive powers in \( a \), so they are irrelevant in the continuum limit. The coefficients \( c_0, \ldots, c_\infty \).

\(^\dagger\)For gauge groups \( U(1) \) and \( U(2) \), \( V_\mu(x) \) and the volume element can directly be expressed by \( X_\mu(x) \): For \( U(1) \), \( V_\mu(x) = e^{-X_0^0(x)/\sqrt{2}} \) and \( \sqrt{\det_{A,B} M^A_{\mu B}}(x) = 1/\sqrt{2} \). For \( U(2) \),

\[ V_\mu(x) = e^{-X_0^0(x)/2} \left( \cosh(|\vec{X}_\mu(x)|/2) - \frac{\sinh(|\vec{X}_\mu(x)|/2)}{|\vec{X}_\mu(x)|/2} \vec{T} \cdot \vec{X}_\mu(x) \right), \]  

(3.33)

where \( \vec{T} \equiv (\sigma^1/2, \sigma^2/2, \sigma^3/2) \) and \( \vec{X}_\mu(x) \equiv (X^1_\mu(x), X^2_\mu(x), X^3_\mu(x)) \), and \( \sqrt{\det_{A,B} M^A_{\mu B}}(x) = [\cosh(|\vec{X}_\mu(x)|) - 1]/2|\vec{X}_\mu(x)|^2 \).
$c_1$ and $c_2$ are some dimensionless combinations of the parameters $\kappa$, $\vartheta$, $g/\tilde{m}_{+I}$ and $g/\tilde{m}_{+I}$.

Now, local operators that are proportional to the first term in the parentheses of Eq. (3.34) arise only at the tree-level approximation; that is, from the naive continuum limit of the lattice action. Our lattice action reproduces, in this limit, the classical action of 2D $\mathcal{N} = (2,2)$ SQCD. Those local terms are simply the terms in the classical action $S^{(E)}_{2DSYM} + S^{(E)}_{FI,\vartheta} + S^{(E)}_{mat,\vartheta,\tilde{m}}$.

Terms being proportional to the second term in the parentheses of Eq. (3.34) arise at the one-loop level or lower. For them to be relevant or marginal, we have to have $p \leq 2$. Most of possible local operators from the dimensional grounds are excluded by the gauge invariance. Further requiring $Q'$-invariance, that is manifest in the present lattice formulation, possibilities such as $\text{tr} X_\mu$ are excluded. Then only possible combinations are, 1 (identity), $\text{tr} \mathcal{F}_{01}$ and $\text{tr} D'$. The identity operator 1 has no dynamical effect, while $\int d^2 x \text{tr} \mathcal{F}_{01} = \int d^2 x \text{tr} F_{01}$ and $\int d^2 x \text{tr} D'$ are simply the theta term and the FI term, respectively.

Finally, terms being proportional to the third term in the parentheses arise at the two-loop level or lower. A unique local operator with $p \leq 0$ is the identity 1 and thus this has no dynamical effect.

In this way, we observe that only nontrivial local terms that can radiatively be generated in the effective action are the FI term and the theta term. These are terms already present in the continuum classical action and of course invariant under all symmetries of the target continuum theory, especially under supersymmetry. Therefore, there is no need to supplement local counterterms to restore symmetries of the continuum theory; this shows that in perturbation theory symmetries are restored in the continuum limit without any fine tuning.

With the present lattice regularization, the radiative correction to the FI parameter $\kappa$ up to the one-loop order is given by

$$\kappa_R = \kappa + \frac{1}{4\pi} \left[ N \left( \ln a^2 - \ln 32 \right) + \sum_{I=1}^{N} \ln (\tilde{m}_{+I}^* \tilde{m}_{+I}) \right], \quad (3.35)$$

for $a \to 0$. Higher order corrections are UV finite and of $O(g^2/m^2)$, where $m^2$ is a linear combination of $\tilde{m}_{+I}^* \tilde{m}_{+I}$. Thus, in order to take the continuum limit while keeping $\kappa_R$ fixed, one has to set $\kappa = (N/4\pi) \ln \beta + \text{const.}$.\footnote{A possible renormalization scheme for $\kappa$ is given by setting $\tilde{m}_{+I}^* \tilde{m}_{+I} = 2\mu^2$ in one-loop expression (3.35), where $\mu$ is a renormalization scale. If one defines the lambda parameter in this scheme (this scheme corresponds to the choice in Ref. [46]) by $\Lambda^{\text{MASS}} \equiv \mu e^{-2\pi/N} e^{(N/4\pi) \kappa_R (\mu)}$, then the ratio to the lambda parameter on the lattice $\Lambda^{\text{LAT}} \equiv (1/a) e^{-2\pi/N} e^{\kappa}$ is given by $\Lambda^{\text{MASS}}/\Lambda^{\text{LAT}} = 4$ for any $N \neq 0$, while}
On the other hand, according to a standard argument on the basis of the $U(1)_A$ anomaly, the theta parameter would radiatively be corrected as

$$\vartheta_R = \vartheta + \frac{1}{2i} \sum_{l=1}^{N} \ln \left( \frac{\tilde{m} + I}{\tilde{m}^* + I} \right),$$

(3.36)

for $a \to 0$. Since this $\vartheta_R$ is UV finite, $\vartheta$ can be taken to be independent of the lattice spacing in the continuum limit.

4 Conclusion

In this paper, we presented a lattice formulation of 2D $\mathcal{N} = (2, 2)$ $U(k)$ SQCD with $N$ fundamental matter multiplets. The formulation uses compact gauge link variables and respects one exact fermionic symmetry $Q'$ on the 2D regular lattice. Our lattice action is considerably simpler compared with the lattice action of Ref. [20]. In particular, the lattice action is polynomial in bosonic variables $U_\mu(x), V_\mu(x)$ and their inverse matrices. We think that this point can be a practical advantage in actual implementations on the computer. In the near future, we hope to use the present lattice formulation to investigate physical questions (as in Refs. [55–59]) in 2D $\mathcal{N} = (2, 2)$ $U(k)$ SQCD with $N$ fundamental multiplets.

Still, at the present moment, the range of applicability of the present formulation is rather limited compared with the formulation of Ref. [20]; we could not incorporate the superpotential and we do not know how to truncate the gauge group $U(k)$ to $SU(k)$. Also, although it is almost straightforward to define a nilpotent lattice $Q'$-transformation and a $Q'$-exact lattice action for matter multiplets in other gauge representations (such as the anti-fundamental and the adjoint), we could not find a rigorous argument that shows that $U_\mu(x) = 1$ (up to gauge transformations) is a unique weak coupling saddle point. Further study is needed on these possible generalizations of the present lattice formulation.

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$$\frac{\Lambda_{\text{MS}}}{\Lambda_{\text{MASS}}} = \sqrt{2}.$$
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A Flat connections on a periodic lattice

In this appendix, we give the most general solution of $U_{01}(U)(x) = 1$ on a periodic lattice.

Let $W_\mu$ be the products of gauge link variables along nontrivial cycles on a periodic lattice (Wilson lines):

$$W_0 \equiv U_0(0,0)U_0(a,0)\ldots U_0((N_0-1)a,0),$$
$$W_1 \equiv U_1(0,0)U_1(0,a)\ldots U_1(0,(N_1-1)a).$$  \hspace{1cm} (A.1)

Since $[W_0, W_1] = 0$ from $U_{01}(x) = 1$, there exists $\Omega \in U(k)$ such that

$$\Omega^{-1}W_0\Omega = \begin{pmatrix} e^{i\alpha_1} \\ & \ddots \\ & & e^{i\alpha_k} \end{pmatrix}, \hspace{1cm} \Omega^{-1}W_1\Omega = \begin{pmatrix} e^{i\beta_1} \\ & \ddots \\ & & e^{i\beta_k} \end{pmatrix},$$  \hspace{1cm} (A.2)

where $0 \leq \alpha_i, \beta_i < 2\pi$ ($i = 1, \ldots, k$). Set the gauge transformation function at the origin

$$g(0,0) = \Omega,$$  \hspace{1cm} (A.3)

and let $T_\mu$ be the $N_\mu$-th root of the right-hand side of Eq. (A.2),

$$T_0 \equiv \begin{pmatrix} e^{i\alpha_1/N_0} \\ & \ddots \\ & & e^{i\alpha_k/N_0} \end{pmatrix}, \hspace{1cm} T_1 \equiv \begin{pmatrix} e^{i\beta_1/N_1} \\ & \ddots \\ & & e^{i\beta_k/N_1} \end{pmatrix}. $$  \hspace{1cm} (A.4)

Clearly, $[T_0, T_1] = 0$. Finally, we define the gauge transformation function $g(x)$ by

$$g(x)^{-1} \equiv T_0^{-C_0} T_1^{-C_1} g(0,0)^{-1} U(C),$$  \hspace{1cm} (A.5)

where $C$ denotes a certain path on the periodic lattice that connects the origin $(0,0)$ and the point $x$, and $U(C)$ is the path ordered product of link variables along $C$. $C_\mu$ are integers defined by

$$C_\mu \equiv (\sharp \text{ of } U_\mu \text{ in } U(C)) - (\sharp \text{ of } U_\mu^{-1} \text{ in } U(C)).$$  \hspace{1cm} (A.6)

Because of $U_{01}(x) = 1$, $g(x)$ defined above does not depend on the chosen path from $(0,0)$ to $x$. It is then straightforward to confirm that $g(x)$ is periodic on
the lattice and
\[ g(x)^{-1}U_{\mu}(x)g(x + a\hat{\mu}) = T_\mu, \] (A.7)
which shows that the most general solution of \[ U_{01}(U)(x) = 1 \] is given by
\[ U_\mu(x) = g(x)T_\mu g(x + a\hat{\mu})^{-1}. \]

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