ON THE CAUCHY PROBLEM FOR THE DERIVATIVE NONLINEAR
SCHRÖDINGER EQUATION WITH PERIODIC BOUNDARY CONDITION

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ABSTRACT. It is shown that the Cauchy problem associated to the derivative nonlinear Schrödinger equation
\[ \partial_t u - i\partial_x^2 u = \lambda \partial_x (|u|^2 u) \]
is locally well-posed for initial data \( u(0) \in H^s(T) \), if \( s \geq \frac{1}{2} \) and \( \lambda \) is real. The proof is based on an adaption of the gauge transformation to periodic functions and sharp multi-linear estimates for the gauge equivalent equation in Fourier restriction norm spaces. By the use of a conservation law, the problem is shown to be globally well-posed for \( s \geq 1 \) and data which is small in \( L^2 \).

1. INTRODUCTION AND MAIN RESULT

We study the Cauchy problem associated to the derivative nonlinear Schrödinger (DNLS) equation with the periodic boundary condition
\[ \partial_t u - i\partial_x^2 u = \lambda \partial_x (|u|^2 u) \quad \text{in } (-T, T) \times T \]
\[ u(0) = u_0 \in H^s(T) \]  
where \( T = \mathbb{R} / 2\pi \mathbb{Z} \) and \( \lambda \in \mathbb{R} \). Our aim is to prove local and global well-posedness in low regularity Sobolev spaces.

In the case of the real line, local well-posedness in \( H^s(\mathbb{R}) \) for \( s \geq \frac{1}{2} \) was obtained by Takaoka \[18\] and this was shown to be sharp in the sense of the uniform continuity of the flow map by Biagioni and Linares \[1\] (the critical regularity for the scaling argument is \( L^2 \)). The local result was extended to global well-posedness for \( s > \frac{1}{2} \) by Colliander, Keel, Staffilani, Takaoka and Tao \[6\] for data which satisfies a \( L^2 \) smallness condition. Recently, Grünewick \[11\] obtained a local result in a \( \hat{L}^p \) setting. The DNLS equation found application as a model in plasma physics and it satisfies infinitely many conservation laws \[15\]. For a more detailed history and further references we refer the reader to these works.

The local result of Takaoka was proved by using the gauge transform developed by Hayashi and Ozawa \[13, 12, 14\] to derive a gauge equivalent equation. This equation still contains a tri-linear term with derivative of the form \( u^2 \partial_x \bar{u} \), but Takaoka \[18\] showed that this can be treated by the Fourier restriction norm method developed in \[2\], as long as \( s \geq \frac{1}{2} \). The proof of the main tri-linear estimate uses local smoothing and Strichartz estimates.

Here, we study the DNLS equation in the periodic setting. It is known that there exist global (weak) solutions in Sobolev spaces corresponding to \( H^1 \) subject to Dirichlet and generalized periodic boundary conditions due to the results from Chen \[16\] and Meškauskas \[16\], for initial data fulfilling a smallness condition.

We remark that the dispersive properties of solutions are weaker than in the non-periodic case. Of course, there are no local smoothing estimates available which could be used to control derivatives in nonlinear terms. Moreover, above \( L^4 \) the Strichartz estimates are only known to hold with a loss of \( \varepsilon > 0 \) derivatives. Therefore, the main question arising in the periodic case is whether a tri-linear estimate for \( u^2 \partial_x \bar{u} \) holds true.

2000 Mathematics Subject Classification. 35Q55 (Primary), 35B30 (Secondary).

Key words and phrases. derivative nonlinear Schrödinger equation, Cauchy problem, periodic boundary condition, gauge transformation, multi-linear estimates, well-posedness.
In the present work we will answer this question affirmatively. Our main ingredients are a point-wise estimate for the multiplier, suitable versions of Bourgain spaces [2][8][9][7][10] and the $L^4$ Strichartz estimate [2]. Combining this with a gauge transform [13][12][14] adapted to the periodic setting, it follows that the Cauchy problem (1) is locally well-posed in $H^s(\mathbb{T})$ for $s \geq \frac{1}{2}$ in the following sense:

**Theorem 1.1.** Let $s \geq \frac{1}{2}$ and $\lambda \in \mathbb{R}$. For all $r > 0$ there exists $T = T(r) > 0$ and a metric space $X^s$, such that for all $u_0 \in B_r = \{u_0 \in H^s(\mathbb{T}) \mid \|u_0\|_{H^s(\mathbb{T})} < r\}$ there exists a unique solution

$$u \in X^s \mapsto C([-T,T], H^s(\mathbb{T}))$$

of the Cauchy problem

$$\partial_t u - i\partial_x^2 u = \lambda \partial_x (|u|^2 u) \quad \text{in } (-T,T) \times \mathbb{T}$$

$$u(0) = u_0$$

which is a limit of smooth solutions in $X^s$.

Moreover, the flow map

$$F : H^s(\mathbb{T}) \supset B_r \rightarrow C([-T,T], H^s(\mathbb{T})) \quad , \quad u_0 \mapsto u$$

is continuous. For fixed $\mu > 0$ the restriction of $F$ to $\{u_0 \in B_r \mid \frac{1}{2\pi} \|u_0\|_{L^2} = \mu\}$ is Lipschitz continuous.

Due to a conservation law this extends to global well-posedness for $s \geq 1$ for data which is small in $L^2$.

**Corollary 1.1.** Let $s \geq 1$. There exists $\delta > 0$ such that under the additional hypothesis $\|u_0\|_{L^2} \leq \delta$, the time of existence $T > 0$ in Theorem 1.1 can be chosen arbitrary large.

Throughout this work solution of a Cauchy problem

$$\partial_t u - i\partial_x^2 u = N(u) \quad \text{in } (-T,T) \times \mathbb{T}$$

$$u(0) = u_0$$

always means solution of the corresponding integral equation

$$u(t) = W(t)u_0 + \int_0^t W(t-t')N(u)(t')\, dt' , \quad t \in (-T,T)$$

at least in a limiting sense, see Sections 5 and 6 for the precise statements. For smooth functions, this notion of solutions coincides with the classical one. We also remark that the uniqueness statement in Theorem 1.1 could be sharpened, see Section 6.

In the exposition we focus on the DNLS equation but we remark that the same approach is also applicable to slightly more general nonlinearities, cp. [18], e.g.

$$\lambda_1 |u|^2 \partial_x u + \lambda_2 u^2 \partial_x \overline{u} + \text{polynomial}$$

We remark that the general strategy of proof of the tri-linear estimate is also applicable in the non-periodic case [18].

To illustrate the principle which allows us to gain the derivative on the complex conjugate wave, let us consider three solutions $u_1, u_2, u_3$ of the linear equation. Their Fourier transforms are supported on the parabola $\{(\tau, \xi) \mid \tau + \xi^2 = 0\}$. The Fourier transform of the interaction of the two linear waves $u_1, u_2$ at fixed frequencies $\xi_1, \xi_2$ with $\partial_x \overline{u_3}$ is supported on $\{(\tau, \xi) \mid \tau + \xi^2 = 2(\xi - \xi_1)(\xi - \xi_2)\}$. In the case where $\xi_1, \xi_2$ are small and the frequency $\xi_3$ is very large, the frequency of the resulting wave is also very large $\xi \sim \xi_3$. Hence its support is far away from the parabola, or more precisely $|\tau + \xi^2| \sim \xi_3^2$. This indicates that the Fourier restriction norm method allows us gain a factor of order $\xi_3$ and everything reduces to terms which can be treated by the $L^4$ Strichartz estimate [2]. Moreover, we are able to control all other possible nonlinear interactions, as long as $s \geq \frac{1}{2}$. 
The outline of the paper is as follows: We conclude this Section with some general notation. In Section 2 we introduce the gauge transform to link the DNLS with another derivative nonlinear Schrödinger equation. After the introduction of useful function spaces and linear estimates in Section 3 we are concerned with multi-linear estimates in Section 4, which are applied in Section 5 to derive the sharp well-posedness result for the gauge equivalent equation via the contraction mapping principle. The proof of well-posedness for the DNLS is carried out in Section 6. Finally, the Appendix provides proofs of some technical lemmata.

The author is indebted to Professor Herbert Koch, in particular for helpful discussions about the gauge transform. Moreover, the author is grateful to Axel Grünrock and Martin Hadac for useful remarks.

**Notation.** Let $S(\mathbb{R})$ be the space of Schwartz functions. We denote by $S_{\text{per}}$ the space of functions $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ such that for all $(t, x) \in \mathbb{R}^2$

$$f(t, x + 2\pi) = f(t, 2\pi), \ f(t, \cdot) \in C^\infty(\mathbb{R}), \ f(\cdot, x) \in S(\mathbb{R})$$

We write $f_x = \partial_x f$ or $f_t = \partial_t f$ for partial derivatives.

Throughout this work $\chi \in C^\infty_0((-2, 2))$ denotes a symmetric function with $\chi \equiv 1$ in $[-1, 1]$ and $\chi_T(t) = \chi(t/T)$.

The Fourier transform with respect to the spatial variable is defined by

$$\mathcal{F}_x f(\xi) = (2\pi)^{-\frac{1}{2}} \int_0^{2\pi} e^{-ix\xi} f(x) \, dx \quad , \xi \in \mathbb{Z}$$

and with respect to the time variable by

$$\mathcal{F}_t f(\tau) = (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} e^{-i\tau t} f(t) \, dt \quad , \tau \in \mathbb{R}$$

and $\mathcal{F} = \mathcal{F}_t \mathcal{F}_x$. For $1 \leq p, q \leq \infty$ we use the notation

$$\|f\|_{L^p_T L^q_x} := \left\| t \mapsto \|x \mapsto f(t, x)\|_{L^q(\mathbb{R})}\right\|_{L^p([-T, T])}$$

and if $T = \infty$ we write $\|f\|_{L^p_T L^q_x}$. Moreover, $\|f\|_{L^p(\mathbb{T})} = \|f\|_{L^p([0, 2\pi])}$.

We define the Sobolev spaces $H^s(\mathbb{T})$ as the completion of the space of $2\pi$-periodic $C^\infty$ functions $f$ with respect to the norm

$$\|f\|^2_{H^s(\mathbb{T})} := \|f\|^2_{H^s} := \sum_{\xi \in \mathbb{Z}} \langle \xi \rangle^{2s} |\mathcal{F}_x f(\xi)|^2$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.

The unitary group associated to $\partial_t - i\partial_x^2$ is defined via

$$\mathcal{F}_x W(t) u_0(\xi) = e^{-tit^2} \mathcal{F}_x u_0(\xi)$$

The Operator $J^s = J^s_T$ is defined by $\mathcal{F}_x J^s f(\xi) = \langle \xi \rangle^s \mathcal{F}_x f(\xi)$. We also use $J^s_T$ which is $J^s$ applied with respect to the $t$ variable.

For $u \in C([-T, T], L^2(\mathbb{T}))$ we define $\mu(u) = \frac{1}{2\pi} \|u(0)\|^2_{L^2_x}$. For $\mu \in \mathbb{R}$ we define translations $\tau_\mu u(t, x) := u(t, x + 2\mu t)$ for $u \in C([-T, T], L^2(\mathbb{T}))$.

2. THE GAUGE TRANSFORMATION

The Cauchy problem (11) is easily reduced to the case $\lambda = 1$ by the transformation

$$u(t, x) \mapsto \frac{1}{\sqrt{|\lambda|}} u(t, \text{sign}(\lambda)x)$$

From now on we only consider the case $\lambda = 1$ without further comments.
Let \( u \in C([-T,T], L^2(\mathbb{T})) \). We define the periodic primitive of \( |u|^2 - \frac{1}{2\pi} \| u(t) \|^2_{L^2} \) with zero mean by
\[
\mathcal{I}(u)(t,x) := \frac{1}{2\pi} \int_0^{2\pi} \int_\theta^x |u(t,y)|^2 - \frac{1}{2\pi} \| u(t) \|^2_{L^2(\mathbb{T})} dyd\theta
\]
and \( v = e^{-i\mathcal{I}(u)}u(t,x) \). Now suppose that \( u \) is a smooth solution to (1) and let us derive an equation for \( v 
\)
\[
v_t = \exp(-i\mathcal{I}(u))(-i\mathcal{I}(u)_{t}u + u_{t})
\]
\[
v_{xx} = \exp(-i\mathcal{I}(u))(-i\mathcal{I}(u)_{xx}u + i(\mathcal{I}(u)u_{x} + u_{x}))
\]
By the \( L^2 \) conservation law we have \( \| u(t) \|_{L^2(\mathbb{T})} = \| u(0) \|_{L^2(\mathbb{T})} \), see Appendix B

With
\[
\mu := \mu(u) = \frac{1}{2\pi} \| u(0) \|^2_{L^2(\mathbb{T})}
\]
we have
\[
\mu(u) = \mu(v) = \frac{1}{2\pi} \| v(0) \|^2_{L^2(\mathbb{T})} = \frac{1}{2\pi} \| v(t) \|^2_{L^2(\mathbb{T})}
\]
and \( \mathcal{I}(u)(t,x) = |u(t,x)|^2 - \mu \). Therefore,
\[
v_t - iv_{xx} = e^{-i\mathcal{I}(u)} \left( u_t - iu_{xx} - (\mathcal{I}(u)_{x}u_x) + i\mathcal{I}(u)_{xx}u - \mathcal{I}(u)_{x}u_x - i\mathcal{I}(u)_{t}u \right)
\]
\[
= e^{-i\mathcal{I}(u)} \left( \mu u_x + i(|u|^2 - \mu)^2 u - (|u|^2 - \mu)u_x - i\mathcal{I}(u)_{t}u \right)
\]
(2)
Moreover,
\[
\frac{d}{dt} \int_\theta^x |u(t,y)|^2 - \mu dy = \int_\theta^x u_t \pi(t,y) + u \pi_x(t,y) dy
\]
\[
= \int_\theta^x (iu_{xx} \pi - i\pi_{xx} u + \pi(|u|^2 u_x) + u(|u|^2 \pi_x)) (t,y) dy
\]
Integration by parts yields
\[
\int_\theta^x iu_{xx} \pi(t,y) - i\pi_{xx} u(t,y) dy = 2 \text{Im}(\pi_x u)(t,x) - 2 \text{Im}(\pi u_x)(t,\theta)
\]
and
\[
\int_\theta^x \pi(|u|^2 u_x)(t,y) + u(|u|^2 \pi_x)(t,y) dy = \frac{3}{2} |u|^4(t,x) - \frac{3}{2} |u|^4(t,\theta)
\]
which shows
\[
\mathcal{I}(u)_{t} = 2 \text{Im}(\pi_x u)(t,x) + \frac{3}{2} |u|^4(t,x)
\]
\[
- \frac{1}{2\pi} \int_0^{2\pi} 2 \text{Im}(\pi_x u)(t,\theta) + \frac{3}{2} |u|^4(t,\theta) d\theta
\]
Let us define
\[
\phi(u)(t) := \frac{1}{2\pi} \int_0^{2\pi} 2 \text{Im}(\pi_x u)(t,\theta) + \frac{3}{2} |u|^4(t,\theta) d\theta
\]
Plugging this into (2) we arrive at
\[
v_t - iv_{xx} = e^{-i\mathcal{I}(u)} \left( 2 \mu u_x + 2i\mu|u|^2 u + i|u|^2 u_x - \pi |u|^4 u + i\phi(u) \right)
\]
Using \(|u| = |v| \) as well as \( u_x = e^{i\mathcal{I}(u)}v_x + i(|u|^2 - \mu)u \) we get
\[
v_t - iv_{xx} = 2\mu v_x + 2i\mu(|v|^2 - \mu)v - 2i\mu|v|^2 v + i\mu^2 v - v^2 \pi_x
\]
\[
+ i|v|^2(|v|^2 - \mu) v - \frac{1}{2} |v|^4 v + i\phi(u) v
\]
\[
= 2\mu v_x - v^2 \pi_x + \frac{1}{2} |v|^4 v - i\mu|v|^2 v + i(\phi(u) - \mu^2)v
\]
With
\[ \psi(v)(t) = \frac{1}{2\pi} \int_0^{2\pi} 2 \Im(\pi_x v)(t, \theta) - \frac{1}{2} \left| v(t, \theta) \right| \, d\theta + \frac{1}{4 \pi^2} \left\| v(0) \right\|_{L^2(T)}^4 \]
we have
\[ \psi(v) = \phi(u) - \mu^2 \]
and therefore obtain
\[ v_t - iv_{xx} - 2\mu v_x = -v^2 \pi_x + \frac{i}{2} \left| v \right|^4 v - i\mu(v) \left| v \right|^2 v + i\psi(v)v \]
Now, we use the transformation \( w(t, x) := \tau_{-\mu} v(t, x) := v(t, x - 2\mu t) \) to cancel the linear term \( 2\mu v_x \) and arrive at
\[ w_t - iw_{xx} = -w^2 \pi_x + \frac{i}{2} \left| w \right|^4 w - i\mu(w) \left| w \right|^2 w + i\psi(w)w \] (3)
because \( (\tau_{-\mu} v)_t(t, x) = v_t(t, x - 2\mu t) - 2\mu v_x(t, x - 2\mu t) \) and \( \tau_{-\mu} \) commutes with partial differentiation in \( x \) as well as with \( \psi \) and is an isometry in \( L^2 \). The above calculation motivates the following definition.

**Definition 2.1.** For \( f \in L^2(\mathbb{T}) \) we define
\[ G(f)(x) = e^{-i\mathcal{I}(f)} f(x) \]
where
\[ \mathcal{I}(f)(x) := \frac{1}{2\pi} \int_0^{2\pi} \int_0^x \left| f(y) \right|^2 - \frac{1}{2\pi} \left\| f \right\|_{L^2(\mathbb{T})}^2 \, dy \, d\theta \]
For \( u \in C([-T, T], L^2(\mathbb{T})) \) we define
\[ G(u)(t, x) := G(u(t))(x - 2\mu(u)t) \] (4)

**Remark 1.** The function \( G(f) \) is \( 2\pi \)-periodic, since
\[ \left| f(y) \right|^2 - \frac{1}{2\pi} \left\| f \right\|_{L^2(\mathbb{T})}^2 \]
has zero mean value and therefore
\[ \int_0^{2\pi} \int_0^x \left| f(y) \right|^2 - \frac{1}{2\pi} \left\| f \right\|_{L^2(\mathbb{T})}^2 \, dy \, d\theta \]
is \( 2\pi \)-periodic.

In the next Lemma, we summarize important properties of the nonlinear operator \( G \).

**Lemma 2.1.** For \( s \geq 0 \) the map
\[ G : C([-T, T], H^s(\mathbb{T})) \to C([-T, T], H^s(\mathbb{T})) \]
is a homeomorphism. Moreover, for \( r > 0 \) there exists \( c > 0 \), such that for
\[ u, v \in B_{r, \mu} = \left\{ u \in C([-T, T], H^s(\mathbb{T})) \mid \sup_{|t| \leq T} \left\| u(t) \right\|_{H^s(\mathbb{T})} \leq r, \mu(u) = \mu \right\} \]
the map \( G \) satisfies
\[ \left\| G(u)(t) - G(v)(t) \right\|_{H^s(\mathbb{T})} \leq c \left\| u(t) - v(t) \right\|_{H^s(\mathbb{T})}, \, t \in [-T, T] \] (5)
for all \( \mu \geq 0 \). The inverse map is given by
\[ G^{-1}(v)(t, x) = e^{i\mathcal{I}(\tau_{\mu(v)} v)} \tau_{\mu(v)} v(t, x) \]
and \( G^{-1} \) satisfies the same estimate \( \mathbb{S} \) on subsets \( B_{r, \mu} \). Hence, \( G \) is locally bi-Lipschitz on subsets with prescribed \( \left\| u(0) \right\|_{L^2} \).
Lemma 2.2. Let
\[ \|e^{\pm i\mathcal{L}}(f) - e^{\pm i\mathcal{L}}(g)\|_{H^s} \leq c\|f\|_{H^s} + \|g\|_{H^s} \] for fixed \( f, g \in H^s(\mathbb{T}) \)

This is proved in Appendix A. To show the Lipschitz estimate, let \( u, v \in B_{r,\mu} \). We observe that for fixed \( t \) in \( x \) is an isometric isomorphism on \( H^s(\mathbb{T}) \). Using (5) we get
\[
\|G(u)(t) - G(v)(t)\|_{H^s(\mathbb{T})} \leq \left( e^{t^2c^2r^2} + c^2c^2r^2 + 1 \right) \|u(t)\|_{H^s}
\]
which shows the Lipschitz continuity on \( B_{r,\mu} \).

If \( v = G(u) \), then \( \|v(0)\|_{L^2} = \|u(0)\|_{L^2} \) and therefore \( \mu(u) = \mu(v) \). Moreover, \( |v(t, x + 2\mu t)| = |u(t, x)| \) for a.e. \( x \). Now, the inversion formula is obvious and for \( G^{-1} \) the Lipschitz estimate on subsets \( B_{r,\mu} \) follows as above by replacing \( - \) by \( + \) in the exponential.

Now, the continuity of
\[ \mathcal{G}, G^{-1} : C([-T, T], H^s(\mathbb{T})) \to C([-T, T], H^s(\mathbb{T})) \]
on the whole space follows from the Lipschitz continuity of \( \mathcal{G}, G^{-1} \) on subsets \( B_{r,\mu} \) and the continuity of the translations
\[ \tau_\mu : \mathbb{R} \to C([-T, T], H^s(\mathbb{T})), \quad \tau_\mu u(t, x) = u(t, x + 2\mu t) \]

The inversion formula is obvious and for \( G^{-1} \) the Lipschitz estimate on subsets \( B_{r,\mu} \) follows as above by replacing \( - \) by \( + \) in the exponential.

By repeating computation from the beginning of this section in reverse order, we prove

**Lemma 2.2.** Let \( u, v \in C([-T, T], H^3(\mathbb{T})) \cap C^1([-T, T], H^1(\mathbb{T})) \) for \( u = G(u) \). Then, \( u \) is a solution of
\[
\partial_t u - i\partial_x^2 u = \partial_x(|u|^2u) \quad \text{in} \quad (-T, T) \times \mathbb{T}
\]
with \( u(0) = u_0 \)

if and only if \( v \) is a solution of
\[
\partial_t v - i\partial_x^2 v = -v^2\partial_x v + \tfrac{1}{2}|v|^4v - i\mu(v)|v|^2v + i\psi(v)v \quad \text{in} \quad (-T, T) \times \mathbb{T}
\]
with \( v(0) = G(u_0) \)

where
\[
\psi(v)(t) = \frac{1}{2\pi} \int_0^{2\pi} 2\text{Im}(\mathcal{P}_x v)(t, \theta) - \frac{1}{2}|v|^4(t, \theta) \, d\theta + \mu(v)^2
\]

Finally, we show an estimate for \( \psi \).

**Lemma 2.3.** Let \( \psi \) be defined by (7). Then,
\[
|\psi(u)(t) - \psi(v)(t)| \leq \left( 1 + \|u(t)\|_{H^\frac{3}{2}} + \|v(t)\|_{H^\frac{3}{2}} \right) \|u - v\|_{H^\frac{3}{2}} + 2\|u(0)\|_{L^2}^{\frac{3}{2}} \|v(0)\|_{L^2}^{\frac{3}{2}} \|u(0) - v(0)\|_{L^2}
\]

**Proof.** We suppress the \( t \) dependence and just write \( u = u(t), v = v(t) \).

\[
\int_0^{2\pi} |\text{Im}(\mathcal{P}_x u) - \text{Im}(\mathcal{P}_x v)| \, dx \leq \|u - v, u_x - v_x\|_{L^2}
\]
Since $J^+_x$ is formally self-adjoint with respect to $(\cdot, \cdot)_{L^2}$ we get
\[
\|(u - v, u_x)_{L^2} + ((v, u_x - v_x)_{L^2}
\leq c(\|u\|_{H^\frac{1}{6}} + \|v\|_{H^\frac{1}{6}})\|u - v\|_{H^\frac{1}{6}}
\]
Moreover,
\[
\left| \int_0^{2\pi} (|u|^4 - |v|^4) \, dx \right| \leq \int_0^{2\pi} |u| - |v| (|u|^3 + |u|^2|v| + |u||v|^2 + |v|^3) \, dx
\leq 2(\|u\|_{L^6}^4 + \|v\|_{L^6}^4)\|u - v\|_{L^2}
\]
Finally,
\[
\|u(0)\|_{L^2}^4 - \|v(0)\|_{L^2}^4 \leq 2(\|u(0)\|_{L^6}^3 + \|v(0)\|_{L^6}^3)\|u(0) - v(0)\|_{L^2}
\]
These three estimates together with the Sobolev embedding $H^\frac{1}{6} \hookrightarrow L^6$ prove (8). \hfill \Box

3. Definition of the spaces and linear estimates

The following spaces are well-known from [2, 8, 9, 7, 10].

**Definition 3.1.** Let $s, b \in \mathbb{R}$. The Bourgain space $X_{s,b}$ associated to the Schrödinger operator $\partial_t - i\partial_x^2$ is defined as the completion of the space $S_{\text{per}}$ with respect to the norm
\[
\|f\|^2_{X_{s,b}} := \sum_{\xi \in \mathbb{Z}} \int_{\mathbb{R}} (\tau + \xi^2)^{2b}|\mathcal{F} f(\tau, \xi)|^2 \, d\tau
\]  

Similarly we define $X_{s,b}^{-}$ by replacing $(\tau + \xi^2)$ with $(\tau - \xi^2)$.

Moreover, we define $Y_{s,b}$ as the completion of the space $S_{\text{per}}$ with respect to
\[
\|f\|^2_{Y_{s,b}} := \sum_{\xi \in \mathbb{Z}} \left( \int_{\mathbb{R}} (\tau + \xi^2)^b (\xi^s) |\mathcal{F} f(\tau, \xi)| \, d\tau \right)^2
\]  

and the space $Z_s := X_{s,\frac{1}{6}} \cap Y_{s,0}$ with norm
\[
\|u\|_{Z_s} := \|u\|_{X_{s,\frac{1}{6}}} + \|u\|_{Y_{s,0}}
\]  

For $T > 0$ we define the restriction norm space
\[
Z^T_s := \{u|_{[-T,T]} \mid u \in Z_s \}
\]
with norm
\[
\|u\|_{Z^T_s} = \inf \{\|\tilde{u}\|_{Z_s} \mid u = \tilde{u}|_{[-T,T]}, \tilde{u} \in Z_s \},
\]
Finally, we define the metric space
\[
\mathcal{X}^T_s := \{u \in C([\tau, \sigma]) \mid \mathcal{F}(u(\tau, \xi)) \in Z^T_s \}
\]
with metric $d_{\mathcal{X}^T_s}(u, v) = \|\mathcal{G}(u) - \mathcal{G}(v)\|_{Z^T_s}$.

**Remark 2.**
1. For complex conjugation we observe $\|\bar{u}\|_{X_{s,b}} = \|u\|_{X_{s,b}^{-}}$.
2. The metric space $\mathcal{X}^T_s$ is complete.

Now, we start with frequently used embedding theorems.

**Lemma 3.1.**

If $2 \leq p < \infty, b \geq \frac{1}{2} - \frac{1}{p}$, $\|u\|_{L^p_t H^s} \leq c\|u\|_{X_{s,b}}$ \hspace{1cm} (12)

If $2 \leq p, q < \infty, b \geq \frac{1}{2} - \frac{1}{p}, s \geq \frac{1}{2} - \frac{1}{q}$, $\|u\|_{L^p_t L^q_x} \leq c\|u\|_{X_{s,b}}$ \hspace{1cm} (13)

If $1 < p \leq 2, b \leq \frac{1}{2} - \frac{1}{p}$, $\|u\|_{X_{s,b}} \leq c\|u\|_{L^p_t H^s}$ \hspace{1cm} (14)
We may replace $X_{s,b}$ by $X_{s,-b}$. Moreover,
\begin{equation}
\|u\|_{C([0,T];H^s(T))} \leq c \|u\|_{Z_s}, \ s \in \mathbb{R} 
\tag{15}
\end{equation}
\begin{equation}
\|u\|_{Y_{s,b_1}} \leq c \|u\|_{X_{s,b_2}}, \ b_2 > b_1 + \frac{1}{2} 
\tag{16}
\end{equation}

**Proof.** We consider $v = W(-t)J_x^s u$ for $u \in S_{\text{per}}$. Then, by Minkowski’s and Sobolev’s inequality
\begin{equation}
\|u\|_{L_t^p H_x^s} = \|v\|_{L_t^p L_x^2} \leq \|v\|_{L_t^2 L_x^p} \leq c \|J_x^s v\|_{L_t^2 L_x^p} = c \|u\|_{X_{s,b}}
\end{equation}
and the claim (12) follows. Combining this with another application of Sobolev’s inequality in the space variable $\|v(t)\|_{L_x^p} \leq c \|J_x^s v(t)\|_{L_x^2}$ gives (13). Estimate (14) follows by duality from (12). The estimates for $X_{s,b}$ follow from the invariance of $L_t^p H_x^s$ and $L_t^p L_x^q$ under complex conjugation. To prove (15) it suffices to prove an estimate for the sup norm for $u \in S_{\text{per}}$ by density. We write for $t \in \mathbb{R}$
\begin{equation}
F_x u(t, \xi) = c \int_{\mathbb{R}} e^{it\tau} F u(\tau, \xi) \ d\tau
\end{equation}
by the Fourier inversion formula. This yields
\begin{equation}
\|u(t)\|_{H^s} = c \left\| \int_{\mathbb{R}} e^{it\tau} (\xi)^s F u(\tau, \xi) \ d\tau \right\|_{L_x^2} \leq c \|\langle \xi \rangle^s F u(\tau, \xi)\|_{L_x^2}
\end{equation}
Now we take the supremum with respect to $t$. The last estimate follows from the Cauchy-Schwarz inequality in $\tau$:
\begin{align*}
\|u\|_{Y_{s,b_1}} &= \sum_{\xi \in \mathbb{Z}} \left( \int_{\mathbb{R}} (\tau + \xi^2)^{b_1} \langle \xi \rangle^s |F f(\tau, \xi)| \ d\tau \right)^2
\leq \sum_{\xi \in \mathbb{Z}} \int_{\mathbb{R}} (\tau + \xi^2)^{2b_1 - 2b_2} d\tau \int_{\mathbb{R}} (\tau + \xi^2)^{2b_2} \langle \xi \rangle^{2s} |F f(\tau, \xi)|^2 \ d\tau
\end{align*}
Since by assumption $2b_1 - 2b_2 < -1$, there exists $c > 0$, such that for all $\xi$
\begin{equation}
\int_{\mathbb{R}} (\tau + \xi^2)^{2b_1 - 2b_2} d\tau \leq c
\end{equation}
which finishes the proof.}

**Lemma 3.2.** For $-b', b > \frac{3}{2}$ there exists $c > 0$, such that
\begin{equation}
\|u\|_{L_t^p} \leq c \|u\|_{X_{0,b}} 
\tag{17}
\end{equation}
\begin{equation}
\|u\|_{X_{0,b'}} \leq c \|u\|_{L_t^{3/2}} 
\tag{18}
\end{equation}
In these estimates we may also replace $X_{0,b}$ by $X_{0,-b}$.

**Proof.** The estimate (17) is essentially Bourgain’s $L^4$ Strichartz estimate [2], but in a version which is global in time. This can be found in [10], Lemma 2.1. Using duality this also shows (18). That the estimates hold both for $X_{s,b}$ and $X_{s,-b}$ results from the invariance of $L_t^p L_x^q$ spaces under complex conjugation. \qed

We summarize the behavior of the $X_{s,b}, Y_{s,0}$ norms under multiplication with cutoffs in time. This will be frequently used in the sequel without further remarks.

**Lemma 3.3.** Let $s \in \mathbb{R}$ and $0 < T \leq 1$. There exists $c > 0$, such that
\begin{equation}
\|\chi_T u\|_{Y_{s,0}} \leq c \|u\|_{Y_{s,0}}
\end{equation}
Moreover, for $0 \leq b_1 < b_2 < \frac{1}{2}$ or $-\frac{1}{2} < b_1 < b_2 \leq 0$ there exists $c > 0$, such that
\begin{equation}
\|\chi_T u\|_{X_{s,b_1}} \leq c T^{b_2 - b_1} \|u\|_{X_{s,b_2}}
\end{equation}
and for any \( \delta > 0 \) there exists \( c > 0 \), such that
\[
\| \chi_{T} u \|_{X_{s, \frac{1}{2}}} \leq c T^{-\delta} \| u \|_{X_{s, \frac{1}{2}}}
\]

**Proof.** The first estimate follows from Young’s inequality in \( \tau \): For fixed \( \xi \) we have
\[
\| \mathcal{F}(\chi_{T} u)(\tau, \xi) \|_{L^{1}_{\tau}} = c \| \int_{\mathbb{R}} \mathcal{F}_{\chi}(\tau - \tau_{1}) \mathcal{F}u(\tau_{1}, \xi) d\tau_{1} \|_{L^{1}_{\tau}} \\
\leq c \| \mathcal{F}_{\chi} \|_{L^{1}_{\tau}} \| \mathcal{F}u(\tau, \xi) \|_{L^{1}_{\tau}}
\]
Because \( \| \mathcal{F}_{\chi} \|_{L^{1}_{\tau}} = \| \mathcal{F}_{\chi} \|_{L^{1}_{\tau}} \) the estimate follows by taking the \( L^{2}_{\tau} \) norms on both sides. The second estimate is proved in [10], Lemma 1.2 and for the third estimate we refer to the proof of [9], Lemma 2.5 and the subsequent remark, which remain true in the periodic setting. \( \square \)

The next Lemma contains the classical (cp. [2, 3, 9, 7]) estimates for the linear homogeneous and inhomogeneous problem.

**Lemma 3.4.** Let \( s \in \mathbb{R} \). There exists \( c > 0 \), such that for all \( u_{0} \in H^{s}(\mathbb{T}) \)
\[
\| \chi W(t) u_{0} \|_{Z_{s}} \leq c \| u_{0} \|_{H^{s}}
\]
and for all \( f \in \mathcal{S}_{\text{per}} \) with \( \text{supp}(f) \subset \{ (t, x) \mid |t| \leq 2 \} \)
\[
\left\| \chi \int_{0}^{t} W(t-t') f(t') dt' \right\|_{Z_{s}} \leq c \| f \|_{Y_{s, -1}} + c \| f \|_{X_{\frac{s}{2}, -\frac{1}{2}}}
\]

**Proof.** It suffices to consider smooth \( u_{0} \). Let us write
\[
\mathcal{F}(\chi W(\cdot) u_{0})(\tau, \xi) = \mathcal{F}_{\chi}(\tau + \xi^{2}) \mathcal{F}u_{0}(\xi)
\]
Then, because \( \mathcal{F}_{\chi} \) is a Schwartz function the estimate [19] follows. Now we turn to the estimate [20] for the linear inhomogeneous equation and we follow the argumentation from [7], Lemma 3.1. We have
\[
\chi(t) \int_{0}^{t} W(t-t') f(t') dt' = I(t) + J(t)
\]
with
\[
I(t) = \frac{1}{2} \chi(t) W(t) \int_{\mathbb{R}} \varphi(t') W(-t') f(t') dt' \\
J(t) = \frac{1}{2} \chi(t) \int_{\mathbb{R}} \varphi(t-t') W(t-t') f(t') dt'
\]
and \( \varphi(t') = \chi(t'/10) \text{sign}(t') \). Moreover,
\[
|\mathcal{F}_{\chi} \varphi(\tau)| \leq c |\tau|^{-1}
\]
Now, by estimate [19]
\[
\| I \|_{Z_{s}} \leq c \left\| \int_{\mathbb{R}} \varphi(t') W(-t') f(t') dt' \right\|_{H^{s}(\mathbb{T})}
\]
and by Parseval’s equality
\[
\mathcal{F}_{\chi} \left( \int_{\mathbb{R}} \varphi(t') W(-t') f(t') dt' \right)(\xi) = \int_{\mathbb{R}} \mathcal{F}_{\chi} \varphi(\tau + \xi^{2}) \mathcal{F} f(\tau, \xi) d\tau
\]
which implies
\[
\left\| \int_{\mathbb{R}} \varphi(t') W(-t') f(t') dt' \right\|_{H^{s}(\mathbb{T})} \leq c \| f \|_{Y_{s, -1}}
\]
by [21]. To show the estimate for \( J \) we first apply Lemma 3.3 with \( T = 1 \)
\[
\| J \|_{Z_{s}} \leq c \left\| \int_{\mathbb{R}} \varphi(t-t') W(t-t') f(t') dt' \right\|_{Z_{s}}
\]
and observe
\[ \mathcal{F} \left( \int_{\mathbb{R}} \varphi(t - t') W(t - t') f(t') dt' \right) (\tau, \xi) = \mathcal{F}_1 \varphi(\tau + \xi^2) \mathcal{F} f(\tau, \xi) \]
The claim then follows from (21). \qed

4. Multi-linear estimates

We start with an elementary bound for the multi-linear multiplier in the spirit of [7].

Lemma 4.1. Let \( \tau_j \in \mathbb{R} \) and \( \xi_j \in \mathbb{Z} \), \( j = 1, 2, 3 \) and define
\[ M(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) = \frac{\langle \xi \rangle \langle \xi_1 \rangle \langle \xi_2 \rangle \langle \xi_3 \rangle}{\langle \tau + \xi^2 \rangle \langle \tau_1 + \xi_1^2 \rangle \langle \tau_2 + \xi_2^2 \rangle \langle \tau_3 + \xi_3^2 \rangle} \]
where \((\tau, \xi) = (\tau_1, \xi_1) + (\tau_2, \xi_2) + (\tau_3, \xi_3)\). Moreover let
\[ M_0(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) = \frac{\chi_{A_0}}{\langle \xi_1 \rangle \langle \xi_2 \rangle \langle \tau_1 + \xi_1^2 \rangle \langle \tau_2 + \xi_2^2 \rangle \langle \tau_3 + \xi_3^2 \rangle} \]
\[ M_j(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) = \frac{\chi_{A_j}}{\langle \xi_1 \rangle \langle \xi_2 \rangle \langle \tau + \xi^2 \rangle \langle \tau_1 + \xi_1^2 \rangle \langle \tau_2 + \xi_2^2 \rangle \langle \tau_3 + \xi_3^2 \rangle} \]
for \( j, k \in \{1, 2\} \) and \( k \neq j \), as well as
\[ M_3(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) = \frac{1}{\langle \tau + \xi^2 \rangle \langle \tau_1 + \xi_1^2 \rangle \langle \tau_2 + \xi_2^2 \rangle \langle \tau_3 + \xi_3^2 \rangle} \]
and the subregions \( A_j \) for \( j = 1, 2, 3 \) are defined analogously. Then, the estimate
\[ |M| \leq 16 \left( \sum_{j=0}^{3} M_j + N \right) \quad (22) \]
holds true.

Proof. The key for the proof will be the observation that
\[ \tau + \xi^2 - (\tau_1 + \xi_1^2 + \tau_2 + \xi_2^2 + \tau_3 - \xi_3^2) = 2(\xi - \xi_1)(\xi - \xi_2) \quad (23) \]
which implies
\[ 2(\xi - \xi_1)(\xi - \xi_2) \leq \langle \tau + \xi^2 \rangle \langle \tau_1 + \xi_1^2 \rangle \langle \tau_2 + \xi_2^2 \rangle \langle \tau_3 + \xi_3^2 \rangle \]
\[ \leq 4 \left( \chi_{A_0} \langle \tau + \xi^2 \rangle \chi_{A_1} \langle \tau_1 + \xi_1^2 \rangle \chi_{A_2} \langle \tau_2 + \xi_2^2 \rangle \chi_{A_3} \langle \tau_3 + \xi_3^2 \rangle \right) \quad (24) \]
We distinguish 4 different cases.

1. \( |\xi| > 2|\xi_1| \) and \( |\xi| > 2|\xi_2| \): In this case \( |\xi_3| \leq 2|\xi| \) and \( 4(|\xi - \xi_1|)(|\xi - \xi_2|) \geq |\xi|^2 \)
and we conclude \( |M| \leq 16 \sum_{j=0}^{3} M_j \).

2. \( |\xi| \leq 2|\xi_1| \) and \( |\xi| \leq 2|\xi_2| \): In this case we have \( |\xi_3| \leq 4 \) and \( |\xi - \xi_1| \leq 2|\xi| \)
and \( |\xi - \xi_2| \leq 2|\xi| \), which shows \( |M| \leq 4N \).

3. \( |\xi| > 2|\xi_1| \) and \( |\xi| \leq 2|\xi_2| \): We have
\[ |\xi| \leq 2(|\xi - \xi_1| + |\xi - \xi_2|) \leq 2((\xi - \xi_1)(\xi - \xi_2)) \]
Due to the fact that
\[ |\xi_3|^2 \leq |\xi - \xi_2|^2 + |\xi_1|^2 \]
we have
\[ |\xi|^2 |\xi_3|^2 \leq 2 |\xi|^2 |\xi_3|^2 \leq 2 |\xi|^2 |\xi - \xi_2|^2 + 2 |\xi|^2 |\xi_1|^2 \]
The first term is bounded by $4\langle(\xi - \xi_1)(\xi - \xi_2)\rangle^{\frac{1}{2}}$ and we apply (24). The second term is bounded by $4\langle \xi_2 \rangle^{\frac{1}{2}}\langle \xi_1 \rangle$, which proves $|M| \leq 16\sum_{j=0}^{N} M_j + 4N$.

4. $|\xi| \leq 2|\xi_1|$ and $|\xi| > 2|\xi_2|$: By the symmetry of $M$ in $\xi_1, \xi_2$ we find the same estimate as in case 3.

\[ \square \]

In the sequel we will use the abbreviations

$$\int \sum_{k} \prod_{j=1}^{k} f_j(\tau_j, \xi_j) := \int \sum_{\tau = \tau_1 + \cdots + \tau_k} \prod_{j=1}^{k} f_j(\tau_j, \xi_j)$$

$$:= \int \sum_{\xi = \xi_1 + \cdots + \xi_k \in 2^{k-1}} \prod_{j=1}^{k-1} f_j(\tau_j, \xi_j) f_k(\tau - \sum_{j=1}^{k-1} \tau_j, \xi - \sum_{j=1}^{k-1} \xi_j) d\tau_1 \ldots d\tau_k$$

which is nothing else but the convolution $f_1 \ast \ldots \ast f_k(\tau, \xi)$.

### Theorem 4.1.

There exists $c, \varepsilon > 0$, such that for $T \in (0, 1]$ and $u_j \in \mathcal{S}_{\text{per}}$ with $\text{supp}(u_j) \subset \{(t, x) | |t| \leq T, j = 1, 2, 3\}$, we have

$$\|u_1 u_2 \partial_x u_3\|_{X_{\frac{1}{2}, \frac{1}{2}}} \leq c T^\varepsilon \|u_1\|_{X_{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X_{\frac{1}{2}, \frac{1}{2}}} \quad (25)$$

#### Proof.

We define

$$f_j(\tau_j, \xi_j) = \langle \tau_j + \xi_j^2 \rangle^{\frac{1}{2}} \langle \xi_j \rangle^{\frac{1}{2}} \mathcal{F} u_j(\tau_j, \xi_j)$$

for $j = 1, 2$ and

$$f_3(\tau_3, \xi_3) = \langle \tau_3 - \xi_3^2 \rangle^{\frac{1}{2}} \langle \xi_3 \rangle^{\frac{1}{2}} \mathcal{F} u_3(\tau_3, \xi_3)$$

With the Fourier multiplier $M$ defined in Lemma 4.1, we rewrite the left hand side as

$$\|u_1 u_2 \partial_x u_3\|_{X_{\frac{1}{2}, \frac{1}{2}}} = \left\| \int \sum_{\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3} \prod_{j=1}^{3} f_j(\tau_j, \xi_j) \right\|_{\mathcal{L}_2^2}$$

By an application of the triangle inequality we may assume $f_j, \mathcal{F} u_j \geq 0$ and $\|u_j\|_{X_{\frac{1}{2}, \frac{1}{2}}} = \|\mathcal{F} u_j\|_{X_{\frac{1}{2}, \frac{1}{2}}}$. By the point-wise bound (22) on $|M|$ the left hand side is bounded by the sum over the corresponding terms with $M$ replaced by $M_0, M_1, M_2, M_3$ and $N$, respectively.

#### Estimate for $M_0$:

$$\left\| \int \sum_{\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3} \prod_{j=1}^{3} f_j(\tau_j, \xi_j) \right\|_{\mathcal{L}_2^2} = \|u_1 u_2 \mathcal{J}^\frac{1}{2} u_3\|_{\mathcal{L}_2^2} =: m_0$$

Using Hölder, we get

$$m_0 \leq \|u_1\|_{\mathcal{L}_2^2} \|u_2\|_{\mathcal{L}_2^2} \|\mathcal{J}^\frac{1}{2} u_3\|_{\mathcal{L}_2^2} \leq c \|u_1\|_{X_{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X_{\frac{1}{2}, \frac{1}{2}}}$$

where we used Sobolev’s inequality w.r.t. space and time variables on $u_1, u_2$ as well as the $L^4$ Strichartz inequality on $\mathcal{J}^\frac{1}{2} u_3$. By the localization in time, see Lemma 4.3

$$m_0 \leq c T^\varepsilon \|u_1\|_{X_{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X_{\frac{1}{2}, \frac{1}{2}}}$$

#### Estimate for $M_1$:

$$\left\| \int \sum_{\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3} \prod_{j=1}^{3} f_j(\tau_j, \xi_j) \right\|_{\mathcal{L}_2^2} = \|\mathcal{J}^{-\frac{1}{2}} \mathcal{J}^{-1} u_1 u_2 \mathcal{J}^{\frac{1}{2}} u_3\|_{X_{0, \frac{1}{2}}} \leq \|\mathcal{J}^{-\frac{1}{2}} \mathcal{J}^{-1} f_1 u_2 \mathcal{J}^{\frac{1}{2}} u_3\|_{X_{0, \frac{1}{2}}} =: m_1 \quad (26)$$
Then, by Sobolev in time
\[
m_1 \leq c\|J^{-\frac{1}{2}} \mathcal{F}^{-1} f_1 u_2 J^{\frac{1}{2}} u_3\|_{L^{6/7}_x L^6_t}
\leq c\|J^{-\frac{1}{2}} \mathcal{F}^{-1} f_1\|_{L_x^6 L^6_t} \|u_2 J^{\frac{1}{2}} u_3\|_{L^{6/3}_x L^{6/3}_t}
\leq c\|J^{-\frac{1}{2}} \mathcal{F}^{-1} f_1\|_{L_x^6 L^6_t} \|u_2\|_{L^6_x L^6_t} \|J^{\frac{1}{2}} u_3\|_{L^{6}_x L^6_t}
\]

Now we use the Sobolev inequality on the first two factors as well as the \(L^4\) Strichartz inequality on \(J^{\frac{1}{2}} u_3\) and obtain
\[
m_1 \leq cT^c \|u_1\|_{X^{\frac{1}{2}, \frac{3}{2}}} \|u_2\|_{X^{\frac{1}{2}, \frac{3}{2}}} \|u_3\|_{X^{\frac{1}{2}, \frac{3}{2}}}
\]

**Estimate for \(M_2\):**
\[
\left\| \sum_j M_2(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^3 f_j(\tau_j, \xi_j) \right\|_{L^2_x L^2_t}
\leq \|u_1 J^{-\frac{1}{2}} \mathcal{F}^{-1} f_2 J^{\frac{1}{2}} u_3\|_{X^{0, -\frac{3}{2}}} \leq \|u_1 J^{-\frac{1}{2}} \mathcal{F}^{-1} f_2 J^{\frac{1}{2}} u_3\|_{X^{0, -\frac{3}{2}}} =: m_2
\]

As for \(m_1\), by exchanging the roles of the first two factors we obtain
\[
m_2 \leq cT^c \|u_1\|_{X^{\frac{1}{2}, \frac{3}{2}}} \|u_2\|_{X^{\frac{1}{2}, \frac{3}{2}}} \|u_3\|_{X^{\frac{1}{2}, \frac{3}{2}}}
\]

**Estimate for \(M_3\):**
\[
\left\| \sum_j M_3(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^3 f_j(\tau_j, \xi_j) \right\|_{L^2_x L^2_t}
\leq \|u_1 u_2 \mathcal{F}^{-1} f_3\|_{X^{0, -\frac{2}{3}}} \leq \|u_1 u_2 \mathcal{F}^{-1} f_3\|_{X^{0, -\frac{2}{3}}} =: m_3
\]

We apply dual Strichartz\(^{18}\), Hölder’s and Sobolev’s inequality to conclude
\[
m_3 \leq c\|u_1 u_2 \mathcal{F}^{-1} f_3\|_{L^{4/3}_x L^{4/3}_t}
\leq c\|u_1\|_{L^6_x L^6_t} \|u_2\|_{L^6_x L^6_t} \|f_3\|_{L^{2}_x L^{2}_t}
\leq cT^c \|u_1\|_{X^{\frac{1}{2}, \frac{3}{2}}} \|u_2\|_{X^{\frac{1}{2}, \frac{3}{2}}} \|u_3\|_{X^{\frac{1}{2}, \frac{3}{2}}}
\]

**Estimate for \(N\):**
\[
\left\| \sum_j N(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^3 f_j(\tau_j, \xi_j) \right\|_{L^2_x L^2_t}
\leq \|J^{\frac{1}{2}} u_1 J^{\frac{1}{2}} u_2 J^{\frac{1}{2}} u_3\|_{X^{0, -\frac{1}{2}}} \leq \|J^{\frac{1}{2}} u_1 J^{\frac{1}{2}} u_2 J^{\frac{1}{2}} u_3\|_{X^{0, -\frac{1}{2}}} =: n
\]

Strichartz inequalities\(^{17}\) and \(18\) yield
\[
n \leq c\|J^{\frac{1}{2}} u_1 J^{\frac{1}{2}} u_2 J^{\frac{1}{2}} u_3\|_{L^{4/3}_x L^{4/3}_t}
\leq c\|J^{\frac{1}{2}} u_1\|_{L^4_x L^4_t} \|J^{\frac{1}{2}} u_2\|_{L^4_x L^4_t} \|J^{\frac{1}{2}} u_3\|_{L^4_x L^4_t}
\leq cT^c \|u_1\|_{X^{\frac{1}{2}, \frac{3}{2}}} \|u_2\|_{X^{\frac{1}{2}, \frac{3}{2}}} \|u_3\|_{X^{\frac{1}{2}, \frac{3}{2}}}
\]

and the proof is complete. □

**Lemma 4.2.** We use the notation from \textbf{Lemma 4.1} and define
\[
\tilde{M}(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) := \frac{M(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3)}{(\tau + \xi^2)^{\frac{1}{2}}}
\]
Then,
\[ |\hat{M}| \leq 128 \left( \sum_{j=0}^{3} \hat{M}_j + \hat{N} \right) \]  
(34)
where for \( \delta \in (0, \frac{1}{6}) \)
\[ \hat{M}_0 = \frac{\chi_{A_0}}{\langle \xi \rangle^{\frac{1}{2} - 3\delta} \langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2} - 3\delta} \langle \tau_1 + \xi_1^3 \rangle^{\frac{1}{2} + \delta} \langle \tau_2 + \xi_2^3 \rangle^{\frac{1}{2} + \delta} \langle \tau_3 - \xi_3^3 \rangle^{\frac{1}{2} + \delta}} \]
and
\[ \hat{M}_j = \langle \tau + \xi^2 \rangle^{-\frac{1}{2}} M_j, \ j \in \{1, 2, 3\}, \ \hat{N} := \langle \tau + \xi^2 \rangle^{-\frac{1}{2}} N \]

Proof. By Lemma 4.1 it suffices to consider the region \( A_0 \) and to show that
\[ \langle \tau + \xi^2 \rangle^{-\frac{1}{2}} M_0 \leq 8 \hat{M}_0 + 4 \hat{N} \]
1. \( |\xi| > 2|\xi_1| \) and \( |\xi| > 2|\xi_2| \): In this case \( |\xi_3| \leq 2|\xi| \). In \( A_0 \) we have \( 16\langle \tau + \xi^2 \rangle \geq \langle \xi \rangle \), since \( \langle \tau + \xi^2 \rangle \geq \langle \tau_1 + \xi_1^3 \rangle, \langle \tau_2 + \xi_2^3 \rangle, \langle \tau_3 - \xi_3^3 \rangle \) which implies
\[ 8\langle \tau + \xi^2 \rangle^{\frac{1}{2}} \geq \langle \tau_1 + \xi_1^3 \rangle^{\delta} \langle \tau_2 + \xi_2^3 \rangle^{\delta} \langle \tau_3 - \xi_3^3 \rangle^{\delta} \langle \xi \rangle^{\frac{1}{2} - 3\delta} \]
which is bounded by \( 32\langle \tau + \xi^2 \rangle \), since we are in region \( A_0 \). Then,
\[ 8\langle \tau + \xi^2 \rangle^{\frac{1}{2}} \geq \langle \xi \rangle^{\frac{1}{2} - 3\delta} \langle \xi_3 \rangle^{\frac{1}{2} - 3\delta} \langle \tau_1 + \xi_1^3 \rangle^{\delta} \langle \tau_2 + \xi_2^3 \rangle^{\delta} \langle \tau_3 - \xi_3^3 \rangle^{\delta} \]
which proves
\[ \langle \tau + \xi^2 \rangle^{-\frac{1}{2}} M_0 \leq 8 \hat{M}_0 \]
and therefore
\[ \langle \xi \rangle \langle \xi_3 \rangle \leq 2|\xi| |\xi_3| \leq 4|\xi| |\xi - \xi_2| \leq 8((\xi - \xi_1)(\xi - \xi_2)) \]
which is bounded by \( 32\langle \tau + \xi^2 \rangle \), since we are in region \( A_0 \). Then,
\[ 8\langle \tau + \xi^2 \rangle^{\frac{1}{2}} \geq \langle \xi \rangle^{\frac{1}{2} - 3\delta} \langle \xi_3 \rangle^{\frac{1}{2} - 3\delta} \langle \tau_1 + \xi_1^3 \rangle^{\delta} \langle \tau_2 + \xi_2^3 \rangle^{\delta} \langle \tau_3 - \xi_3^3 \rangle^{\delta} \]
which proves
\[ \langle \tau + \xi^2 \rangle^{-\frac{1}{2}} M_0 \leq 8 \hat{M}_0 \]
In the subregion where \( |\xi_1| > |\xi - \xi_2| \) we have \( |\xi_3| \leq 2|\xi_1| \) and we arrive at \( M \leq 4 \hat{N} \).
4. \( |\xi| \leq 2|\xi| \) and \( |\xi| > 2|\xi_2| \): By the symmetry of \( \hat{M} \) in \( \xi_1, \xi_2 \) we find the same estimate as in case 3.

**Theorem 4.2.** There exists \( c, \varepsilon > 0 \), such that for \( T \in (0, 1] \) and \( u_j \in \mathcal{S}_{\text{per}} \) with \( \text{supp}(u_j) \subset \{(t, x) \mid |t| \leq T\}, j = 1, 2, 3 \), we have
\[ \|u_1 u_2 \partial_x u_3\|_{Y_{\frac{1}{2}, -1}} \leq cT^\varepsilon \|u_1\|_{X_{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X_{\frac{1}{2}, -\frac{1}{2}}} \]  
(35)

Proof. We use the notation from the proof of Theorem 4.1. With the Fourier multiplier \( \hat{M} \) defined in Lemma 4.1, we rewrite the left hand side as
\[ \|u_1 u_2 \partial_x u_3\|_{Y_{\frac{1}{2}, -1}} = \left\| \int \sum_{\tau} \hat{M}(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^{3} f_j(\tau_j, \xi_j) \right\| \]
By the estimate (34) we successively replace $\hat{M}$ by $\hat{M}_0$, $\hat{M}_1$, $\hat{M}_2$, $\hat{M}_3$ and $\tilde{N}$.

**Estimate for $\hat{M}_0$.** We observe that by the Cauchy-Schwarz inequality we have for fixed $\xi$

$$\left\| (\tau + \xi^2)^{-\frac{1}{4} - \delta'} \phi(\tau, \xi) \right\|_{L^1_\tau} \leq \left( \int (\tau)^{-1-2\delta'} d\tau \right)^{\frac{1}{2}} \| \phi(\cdot, \xi) \|_{L^2_\tau}$$

(36)

for $\delta' > 0$. Now, for fixed $\xi_1, \xi_2, \xi_3$ and $\xi = \xi_1 + \xi_2 + \xi_3$

$$\left\| \int \hat{M}_0(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^3 f_j(\tau_j, \xi_j) \right\|_{L^1_\tau}$$

$$= \langle \xi \rangle^{-\frac{1}{4} + 3\delta} \left\| \int_{\tau=\tau_1+\tau_2+\tau_3} \prod_{j=1}^2 \frac{f_j(\tau_j, \xi_j)}{\langle \xi_j \rangle^\frac{1}{2} (\tau_j + \xi_j)^{\delta/2}} \frac{f_3(\tau_3, \xi_3)}{\langle \xi_3 \rangle^\frac{1}{2} (\tau_3 - \xi_3)^{\delta/2}} \right\|_{L^1_\tau}$$

$$\leq c \langle \xi \rangle^{-\frac{1}{4} + 3\delta} \prod_{j=1}^2 \left\| \frac{f_j(\tau_j, \xi_j)}{\langle \xi_j \rangle^\frac{1}{2} (\tau_j + \xi_j)^{\delta/2}} \right\|_{L^2_\tau} \left\| \frac{f_3(\tau_3, \xi_3)}{\langle \xi_3 \rangle^\frac{1}{2} (\tau_3 - \xi_3)^{\delta/2}} \right\|_{L^2_\tau}$$

by Young’s inequality and (36) with $\delta' = \delta/2$. With $g_j(\tau_j, \xi_j) = f_j(\tau_j, \xi_j)(\tau_j + \xi_j)^{-\delta/2}$ and $g_3(\tau_3, \xi_3) = f_3(\tau_3, \xi_3)(\tau_3 - \xi_3)^{-\delta/2}$ we have

$$\left\| \int \sum \hat{M}_0(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^3 g_j(\tau_j, \xi_j) \right\|_{L^1_\tau}$$

$$\leq c \langle \xi \rangle^{-\frac{1}{4} + 3\delta} \sum_{\xi = \xi_1 + \xi_2 + \xi_3} \left\| \langle \xi_1 \rangle^{-\frac{1}{4}} \langle \xi_2 \rangle^{-\frac{1}{4}} \langle \xi_3 \rangle^{-\frac{1}{4}} \prod_{j=1}^3 g_j(\cdot, \xi_j) \right\|_{L^2_\tau}$$

An application of Hölder’s and Young’s inequalities, choosing $\delta = 1/24$, gives the upper bound

$$c \left\| \sum_{\xi = \xi_1 + \xi_2 + \xi_3} \langle \xi_1 \rangle^{-\frac{1}{4}} \langle \xi_2 \rangle^{-\frac{1}{4}} \langle \xi_3 \rangle^{-\frac{1}{4}} \prod_{j=1}^3 g_j(\cdot, \xi_j) \right\|_{L^2_\tau}$$

$$\leq c \prod_{j=1}^3 \left\| \langle \xi_j \rangle^{-\frac{1}{4}} \| g_j(\cdot, \xi_j) \|_{L^2_\tau} \right\|_{L^{4/3}_\tau} \leq c \prod_{j=1}^3 \| g_j \|_{L^2_\tau}$$

$$\leq c \| u_1 \|_{X^{\frac{1}{2} - \frac{1}{24}}} \| u_2 \|_{X^{\frac{1}{2} - \frac{1}{24}}} \| u_3 \|_{X^{\frac{1}{2} - \frac{1}{24}}}$$

which finally proves that

$$\left\| \int \sum \hat{M}_0(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^3 f_j(\tau_j, \xi_j) \right\|_{L^1_\tau} \leq c T^c \| u_1 \|_{X^{\frac{1}{2} - \frac{1}{24}}} \| u_2 \|_{X^{\frac{1}{2} - \frac{1}{24}}} \| u_3 \|_{X^{\frac{1}{2} - \frac{1}{24}}}$$

**Estimate for $\hat{M}_1, \hat{M}_2, \hat{M}_3$ and $\tilde{N}$.** We show that the estimates from the proof of Theorem 4.1 are strong enough to treat these terms, too. Indeed, an application of (36) implies

$$\left\| \int \sum \hat{M}_1(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^3 f_j(\tau_j, \xi_j) \right\|_{L^1_\tau}$$

$$\leq c \left\| (\tau + \xi^2)^{\frac{1}{4}} \int \sum M_1(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^3 f_j(\tau_j, \xi_j) \right\|_{L^2_\tau} \leq c m_1$$
where \( m_1 \) is defined in (26) and is bounded according to (27). The same reasoning applies to \( \tilde{M}_2, \tilde{M}_3 \) and \( \tilde{N} \), where we use the bounds established in (29), (31) and (33). □

The next Lemma contains an auxiliary estimate, which will be used for polynomial terms in the nonlinearity. This suffices for our purposes, but it is far from optimal, see [2].

Lemma 4.3. For \( \delta > 0 \) there exists \( c, \varepsilon > 0 \), such that for \( T \in (0,1] \) and \( u_j \in \mathcal{S}_{\text{per}} \) with \( \text{supp}(u_j) \subset \{(t,x) \mid |t| \leq T \}, j = 1, \ldots, 5 \), we have

\[
\left\| \sum_{j=1}^{5} u_j \right\|_{X_{\frac{1}{2}, -\frac{3}{8} - \delta}} \leq c T^\varepsilon \left\| u_1 \right\|_{X_{\frac{1}{8}}} \left\| u_2 \right\|_{X_{\frac{3}{4}} - \frac{3}{8}} \prod_{k=3}^{5} \left\| u_j \right\|_{X_{\frac{1}{2} + \delta}}
\]

and

\[
\left\| \sum_{j=1}^{3} u_j \right\|_{X_{\frac{1}{2}, -\frac{3}{8} - \delta}} \leq c T^\varepsilon \left\| u_1 \right\|_{X_{\frac{1}{8}}} \left\| u_2 \right\|_{X_{\frac{3}{4} + \delta}} \left\| u_3 \right\|_{X_{\frac{1}{2} + \delta}}
\]

Proof. As in the previous proofs it suffices to consider \( \mathcal{F} u_j \geq 0 \). For \( \xi = \sum_{k=1}^{5} \xi_k \) we have

\[
\langle \xi \rangle^\frac{1}{2} \leq c \sum_{k=1}^{5} \langle \xi_k \rangle^\frac{1}{2}
\]

which implies

\[
\left\| \sum_{j=1}^{5} u_j \right\|_{X_{\frac{1}{2}, -\frac{3}{8} - \delta}} \leq c \sum_{k=1}^{5} \left\| J_1^\frac{1}{2} u_k \right\|_{X_{\frac{1}{2}, -\frac{3}{8} - \delta}}
\]

Each of the five terms can be estimated, using the dual Strichartz estimate [18] as follows

\[
\left\| J_1^\frac{1}{2} u_k \right\|_{X_{\frac{1}{2}, -\frac{3}{8} - \delta}} \leq c \left\| J_1^\frac{1}{2} u_k \right\|_{L_t^2 L_x^\frac{4}{3}} \left\| u_j \right\|_{L_t^4 L_x^{16}}
\]

\[
\leq c \left\| J_1^\frac{1}{2} u_k \right\|_{L_t^2 L_x^\frac{4}{3}} \prod_{j=1}^{5} \left\| u_j \right\|_{L_t^4 L_x^{16}}
\]

where in the last step we used the Sobolev embedding in space and time. The second claim follows in the same way, using the \( L_t^8 L_x^8 \) norm instead of the \( L_t^{16} L_x^{16} \) norm on the factors without derivatives. □

We put these estimates in a slightly more general form.

Corollary 4.1. Let \( s \geq \frac{1}{2} \) and \( \delta > 0 \). There exists \( c, \varepsilon > 0 \), such that for \( T \in (0,1] \) and \( u_j \in \mathcal{S}_{\text{per}} \) with \( \text{supp}(u_j) \subset \{(t,x) \mid |t| \leq T \}, j = 1, \ldots, 5 \), we have

\[
\left\| u_1 u_2 \partial_x u_3 \right\|_{Y_{s, -\frac{3}{4} - \delta}} \leq c T^\varepsilon \sum_{k=1}^{3} \left\| u_k \right\|_{X_{\frac{1}{8}}} \prod_{j=1}^{3} \left\| u_j \right\|_{X_{\frac{1}{2} + \delta}}
\]

\[
\left\| u_1 u_2 \prod_{j=3}^{5} u_j \right\|_{X_{s, -\frac{3}{8} - \delta}} \leq c T^\varepsilon \sum_{k=1}^{5} \left\| u_k \right\|_{X_{\frac{1}{8}}} \prod_{j=1}^{5} \left\| u_j \right\|_{X_{\frac{1}{2} + \delta}}
\]

\[
\left\| \mu(u_1) - \mu(u_2) \right\|_{Y_3 u_4 u_5} \right\|_{X_{s, -\frac{3}{4} - \delta}} \leq c T^\varepsilon \left\| u_1 - u_2 \right\|_{Z_0} \left\| u_1 \right\|_{Z_0} + \left\| u_2 \right\|_{Z_0} \sum_{k=3}^{5} \left\| u_k \right\|_{X_{\frac{1}{8}}} \prod_{j=3}^{5} \left\| u_j \right\|_{X_{\frac{1}{2} + \delta}}
\]

(41)
and
\[ \left\| (\psi(u_1) - \psi(u_2))u_3 \right\|_{X_{s,0}} \leq cT^s(1 + \|u_1\|_{X^{1+s}} \cap Z_0 + \|u_2\|_{X^{1+s}} \cap Z_0)^3 \|u_1 - u_2\|_{X^{1+s}} \cap Z_0} \|u_3\|_{X^{1+s}} \]

(42)

Proof. We observe that \( |\Pi|_{X_{s,b}} = \|u\|_{X_{s,b}} \) and
\[ \langle \xi \rangle^s \leq c \sum_{k=1}^{l} \langle \xi_k \rangle^s \text{, for } \xi = \sum_{k=1}^{l} \xi_k \text{ and } s \geq 0 \]

Furthermore, by the embedding \( Z_0 \hookrightarrow C(\mathbb{R}, L^2(T)) \)
\[ |\mu(u_1) - \mu(u_2)| \leq c\|u_1 - u_2\|_{Z_0}(\|u_1\|_{Z_0} + \|u_2\|_{Z_0}) \]
and by (3)
\[ \|\psi(u) - \psi(v)\|_{L^2} \leq cT^s(1 + \|u_1\|_{X^{1+s}} \cap Z_0 + \|u_2\|_{X^{1+s}} \cap Z_0)^3 \|u_1 - u_2\|_{X^{1+s}} \cap Z_0} \]

Using this, the corollary follows from (25, 35, 37) and (38).

5. The gauge equivalent Cauchy problem

Theorem 5.1. Let \( s \geq \frac{1}{2} \). There exists a non-increasing function \( T : (0, \infty) \rightarrow (0, \infty) \), such that for \( v_0 \in H^s(T) \) and \( T = T(\|v_0\|_{H^s(T)}) \) there exists a solution
\[ v \in Z^T_s \subset C([-T, T], H^s(T)) \]

of the Cauchy problem
\[ \partial_t v - i\partial_{xx}^2 v = -v^2 \partial_x \pi + \frac{i}{4}|v|^4 v - i\mu(v)|v|^2 v + i\psi(v)v \quad \text{in } (-T, T) \times \mathbb{T} \]
\[ v(0) = v_0 \]

where \( \mu(v) = \frac{1}{2\pi}\|v(0)\|_{L^2(\mathbb{T})}^2 \) and
\[ \psi(v)(t) = \frac{1}{2\pi} \int_0^{2\pi} 2 \text{Im}(\pi_x v)(t, \theta) - \frac{1}{2}|v|^2(t, \theta) \ d\theta + \mu(v)^2 \]

This solution is unique in \( Z^T_s \). Moreover, for any \( r > 0 \) there exists \( T = T(r) \), such that with
\[ B_r = \{ v_0 \in H^s(\mathbb{T}) | \|v_0\|_{H^s(\mathbb{T})} < r \} \]

the flow map
\[ \tilde{F} : H^s(\mathbb{T}) \ni B_r \rightarrow C([-T, T], H^s(\mathbb{T})) \quad , \quad v_0 \mapsto v \]
is Lipschitz continuous.

Remark 3. We remark that Theorem 5.1 extends to nonlinear terms of the type \( \pi^k \partial_x \pi \) by Grünrock’s result [10]. On the other hand, Christ [5] proved a strong ill-posedness result for the nonlinearities \( u^k \partial_x u \), for every \( k \in \mathbb{N} \).

As in the case of the real line, we show that below \( s = \frac{1}{2} \) it is not possible to prove similar estimates on the tri-linear term which contains the derivative.

Theorem 5.2. Let \( s < \frac{1}{2} \) and \( T > 0 \). There does not exist a normed space \( Z^T_s \hookrightarrow C([-T, T], H^s(T)) \), such that
\[ \|W(t)u_0\|_{Z^T_s} \leq c\|u_0\|_{H^s(\mathbb{T})} \]

and
\[ \left\| \int_0^t W(t - t') (u^2 \partial_x \pi) (t') \ dt' \right\|_{Z^T_s} \leq c\|u\|_{Z^T_s}^3 \]

hold.
We recall the definition of the space and we define which forces application of the contraction mapping principle, cp. [2, 7]. We define for since it only depends on \( \mu \) where \( \mu = \chi W(\cdot) v_0 \), see (9), (10) and (11). By Corollary 4.1, the embedding (16) and the linear estimate (20), we may extend \( \Phi_T \) uniquely to \( \Phi_T : Z_s \to Z_s \) for all \( s \geq \frac{1}{2} \). We also have \( \Phi_T|_{[-T,T]} : Z^T_s \to Z^T_s \) since it only depends on \( v|_{[-T,T]} \).

**Local Existence.** Our aim is to find a solution \( v \in Z_s \) of

\[
v = \chi W(\cdot) v_0 + \Phi_T(v)
\]

For \( v_0 \in H^s(\mathbb{T}) \) we use again the estimates from Corollary 4.1 and (16), (19) and (20) as well as Lemma 3.3 to show that there exists \( c, \varepsilon > 0 \), such that

\[
\|\chi W(\cdot) v_0 + \Phi_T(v)\|_{Z_s} \leq c\|v_0\|_{H^s} + cT^\varepsilon (1 + \|v\|_{Z_s})^3\|v\|_{Z_s}^2
\]

and

\[
\|\Phi_T(v_1) - \Phi_T(v_2)\|_{Z_s} \leq cT^\varepsilon (1 + \|v_1\|_{Z_s} + \|v_2\|_{Z_s})^3(\|v_1\|_{Z_s} + \|v_2\|_{Z_s})\|v_1 - v_2\|_{Z_s}
\]

Then, for all \( v_0 \in H^s \) with \( \|v_0\|_{H^s} \leq r \) and \( R = 2cr \) and \( T > 0 \) so small that \( T \leq (4c^2 r(1 + 4c r)^3)^{-\frac{1}{2}} \) we see that

\[
v \mapsto \chi W(\cdot) v_0 + \Phi_T(v)
\]

maps the closed ball \( B_R \subset Z_s \) to itself and is a strict contraction. This shows the existence of a solution \( v \in B_R \subset Z_s \). By restriction to the interval \( [-T,T] \) we found a solution \( v \in Z^T_s \subset C([-T,T], H^s(\mathbb{T})) \) of

\[
v(t) = W(t)v_0 + \Phi_T(v)(t) \quad t \in [-T,T]
\]

**Uniqueness.** Assume that \( v_1, v_2 \in Z^T_s \) are two solutions of (45), such that

\[
T' := \text{sup}\{ t \in [0,T] \mid v_1(t) = v_2(t) \} < T
\]

and we define \( w_j(t) = \hat{v}_j(T'+t), j = 1,2 \) for extensions \( \hat{v}_j \) of \( v_j \). By approximation we see

\[
w_1(t) - w_2(t) = \Phi_T(w_1)(t) - \Phi_T(w_2)(t) \quad -T' \leq t \leq T - T'
\]

Choosing \( \delta > 0 \) small enough, we arrive at

\[
\|\chi s(w_1 - w_2)\|_{Z^s} \leq c\delta^\varepsilon (1 + \|w_1\|_{Z^s} + \|w_2\|_{Z^s})^4\|\chi s(w_1 - w_2)\|_{Z^s}
\]

which forces \( w_1(t) = w_2(t) \) for \( |t| \leq \delta \) and therefore contradicts the definition of \( T' \). The same argument applies in the interval \( [-T,0] \).
Local Lipschitz continuity of the flow. Let $v_0, w_0 \in H^s(\mathbb{T})$ with $\|v_0\|_{H^s}, \|w_0\|_{H^s} \leq r$. Let $v, w \in Z^s_T$ be two solutions of \ref{4.5} with $v(0) = v_0$ and $w(0) = w_0$ with extensions $\tilde{v}, \tilde{w}$ constructed in part 1 of the proof. Then, $\|\tilde{v}\|_{Z_s}, \|\tilde{w}\|_{Z_s} \leq 2cr$ and

$$\|\tilde{v} - \tilde{w}\|_{Z_s} \leq c\|v_0 - w_0\|_{H^s} + cT^s4c^2(1 + 4cr)^3\|\tilde{v} - \tilde{w}\|_{Z_s}$$

and the choice of $T$ from part 1 guarantees

$$\|\tilde{v} - \tilde{w}\|_{Z_s} \leq 2c\|v_0 - w_0\|_{H^s}$$

and by restriction

$$\|v - w\|_{C([-T,T], H^s)} \leq 2c\|v_0 - w_0\|_{H^s}$$

$\square$

Time of existence. Finally, the standard iteration argument, using the estimates from Corollary \ref{4.4} shows that the maximal time of existence $T > 0$ depends only on $\|v_0\|^{\frac{1}{2}}_{H^s}$.

$\square$

Finally, we remark that the counterexamples from \ref{18} also show the optimality of our tri-linear estimate:

Proof of Theorem \ref{5.2} We follow the general idea from \ref{17}. Let $n \in \mathbb{N}$ and $u^{(n)}_0 := n^{-s}e^{inx}$. Then, $\|u^{(n)}_0\|_{H^s} = c$ and

$$\int_0^t W(-t') \left((W(t')u^{(n)}_0)^2 \partial_x W(t')u^{(n)}_0\right) dt' = -itn^{-3s}ne^{inx}$$

which shows that

$$\left\|\int_0^t W(t - t') \left((W(t')u^{(n)}_0)^2 \partial_x W(t')u^{(n)}_0\right) dt'\right\|_{H^s} \geq c|t|n^{1-2s}$$

If the linear and tri-linear estimates in a space $Z_T \hookrightarrow C([-T,T], H^s(\mathbb{T}))$ were true, there would exist $c > 0$ such that $|t|n^{1-2s} \leq c$ for all $n \in \mathbb{N}$, which is a contradiction for $s < \frac{1}{2}$.

$\square$

6. PROOF OF THEOREM \ref{1.1} AND COROLLARY \ref{1.1}

In this section, we will use the solutions of \ref{4.5} constructed in the previous section to prove Theorem \ref{1.1} similar to \ref{18} \cite{11}.

Existence. We fix $s \geq \frac{1}{2}$ and let $u_0 \in H^s(\mathbb{T})$ with $\mu := \frac{1}{4\mu}\|u_0\|^2_{L^2}$. Then, we define $v_0 := \mathcal{G}(u_0) \in H^s(\mathbb{T})$, see Lemma \ref{2.1}. According to Theorem \ref{5.1} there exists a unique solution $v \in Z^s_T \subset C([-T,T], H^s(\mathbb{T}))$ of \ref{4.5}. Now, we claim that $u := \mathcal{G}^{-1}(v) \in X^s_T \subset C([-T,T], H^s(\mathbb{T}))$ solves

$$u(t) = W(t)u_0 + \int_0^t W(t - t')\partial_x(|u|^2u)(t') dt', \quad t \in (-T,T) \quad (46)$$

For smooth functions this follows from Lemma \ref{5.2}. Let $u^{(n)}_0 \in C^\infty$ with $u^{(n)}_0 \rightarrow u_0$ in $H^s$ and $\|u^{(n)}_0\|_{L^2} = \|u_0\|_{L^2}$. Moreover, let $v^{(n)} \in Z^s_T$ be the solution of \ref{4.5} with initial data $\mathcal{G}(u^{(n)}_0)$ and $u^{(n)} := \mathcal{G}^{-1}(v^{(n)})$. Then,

$$\sup_{t \in (-T,T)}\left\|\int_0^t W(t - t')\partial_x(|u|^2u - |u^{(n)}|^2u^{(n)})(t') dt'\right\|_{H^{-1}} \leq c(\|u\|^2_{L^\infty H^{\frac{1}{2}}} + \|u^{(n)}\|^2_{L^\infty H^{\frac{1}{2}}} )\|u - u^{(n)}\|_{L_T^1L^2}$$

Because $\mathcal{G}$ is continuous in $H^s$, $\mathcal{G}(u^{(n)}_0) \rightarrow v_0$ and due to the continuity of the flow map of \ref{4.5} we have $v^{(n)} \rightarrow v$ in $C([-T,T], H^s)$. Since also $\mathcal{G}^{-1}$ is continuous, the above term tends to zero. This shows that $u$ solves \ref{46} because obviously also the linear part converges in $H^s(\mathbb{T})$.

$\square$
**Uniqueness.** Let \( u_1, u_2 \in \mathcal{X}^T \) be two solutions of (45) with \( u_1(0) = u_2(0) \), such that \( \mathcal{G}(u_j) \in Z^T_2 \) solve (45) with the same initial datum. By the uniqueness of the solutions to (45) we have \( \mathcal{G}(u_1) = \mathcal{G}(u_2) \) and therefore \( u_1 = u_2 \).

We now prove that the hypothesis that \( \mathcal{G}(u_j) \) solve (45) is fulfilled if \( u_j \) are limits of smooth solutions in \( \mathcal{X}^T_2 \), say \( u_j^{(n)} \in C([-T,T], H^1(T)) \) such that \( \| \mathcal{G}(u_j^{(n)}) - \mathcal{G}(u_j) \|_{Z^T_2} \to 0 \). By Lemma 2.2 \( \mathcal{G}(u_j^{(n)}) \in Z^T_2 \) solve (45). Moreover, \( \mathcal{G}(u_j^{(n)}(0)) \to \mathcal{G}(u_j(0)) \in H^\frac{1}{2} \). There exists a unique solution \( v \in Z^T_2 \) to (45) with \( v(0) = \mathcal{G}(u_j(0)) \) and due to the continuity of the flow \( \mathcal{F} \) it follows \( \mathcal{G}(u_j^{(n)}) \to v \), which implies that \( \mathcal{G}(u_j) = v \) is a solution to (45).

**Continuity of the flow.** Since the flow map to (46) \( F : H^s(T) \to C([-T,T], H^s(T)) \) results from conjugating the flow map to (45) \( \mathcal{F} : H^s(T) \to C([-T,T], H^s(T)) \) with the gauge transformation \( \mathcal{G} \), i.e. \( F = \mathcal{G}^{-1} \circ \mathcal{F} \circ \mathcal{G} \), its continuity properties follow from the local Lipschitz continuity of \( \mathcal{F} \) and Lemma 2.1.

**Global existence.** It suffices to prove an a priori bound for smooth solutions. By Lemma B.2 and the Sobolev embedding we have

\[
\|\partial_x u(t)\|_{L^2(T)}^2 + \frac{3}{2} \int_0^{2\pi} |u|^2 u \partial_x \pi(t) \, dx + \frac{1}{2} \|u(t)\|_{L^6(T)}^6 \leq c(1 + \|u_0\|_{H^1(T)})^6
\]

(47)

Now, we use the Gagliardo-Nirenberg inequality

\[
\|u(t)\|_{L^6(T)} \leq \|u(t)\|_{L^2(T)} \left( \|\partial_x u(t)\|_{L^2(T)} + \frac{1}{2\pi} \|u(t)\|_{L^2(T)} \right)
\]

(48)

see Appendix C and estimate

\[
\frac{3}{2} \int_0^{2\pi} |u|^2 u \partial_x \pi(t) \, dx \geq -\frac{3}{2} \|u(t)\|_{L^2}^2 \left( \|\partial_x u(t)\|_{L^2} + \frac{1}{2\pi} \|u(t)\|_{L^2} \right) \|\partial_x u(t)\|_{L^2}
\]

Then, using this in (47) we have for \( \|u(t)\|_{L^2} \leq \delta \)

\[
(1 - \frac{3}{2} \delta^2) \|\partial_x u(t)\|_{L^2}^2 - \frac{3}{4\pi} \|\partial_x u(t)\|_{L^2} \leq c(1 + \|u_0\|_{H^1(T)})^6
\]

which shows for \( \delta < \sqrt{\frac{2}{3}} \) that there exists \( c(\delta) > 0 \) such that

\[
\|\partial_x u(t)\|_{L^2}^2 \leq c(\delta)(1 + \|u_0\|_{H^1(T)})^6
\]

This estimate, together with the \( L^2 \) conservation law from Lemma 5.1 shows that for \( \|u_0\|_{L^2} \leq \delta \) there exists \( C(\delta) > 0 \) such that

\[
\|u(t)\|_{H^1(T)} \leq C(\delta)(1 + \|u_0\|_{H^1(T)})^3
\]

\[\square\]

**Remark 4.** The proof shows that it suffices to choose \( \delta < \sqrt{\frac{2}{3}} \). By following the idea of Hayashi and Ozawa [14], using the gauge transform together with sharp versions of the Gagliardo-Nirenberg inequality we expect that this can be improved, but our aim here is to give a short proof of the qualitative result.
APPENDIX A. PROOF OF ESTIMATE \((\ref{eq:estimate})\)

We prove that for all \(s \geq 0\) there exists \(c > 0\), such that for \(f, g, h \in H^s(\mathbb{T})\) we have
\[
\left\| (e^{i \mathcal{I}(f)} - e^{i \mathcal{I}(g)} h) \right\|_{H^s} \leq c e^{c \|f\|_{H^s}^2 + c \|g\|_{H^s}^2} (\|f\|_{H^s} + \|g\|_{H^s}) \|f - g\|_{H^s} \|h\|_{H^s}
\]
To simplify the notation we only consider the plus sign since the same argument works with the minus sign. Moreover, it suffices to consider smooth \(f, g, h\) and we start with the case \(s > 0\). We will exploit the Sobolev multiplication law
\[
\|fg\|_{H^s} \leq c \|f\|_{H^s} \|g\|_{H^s}, \quad \beta = \begin{cases} \alpha + \varepsilon, & \alpha > \frac{1}{2} \\ \frac{1}{2} + \varepsilon, & \text{otherwise} \end{cases}
\]
We write
\[
(e^{i \mathcal{I}(f)} - e^{i \mathcal{I}(g)}) h = i h (\mathcal{I}(f) - \mathcal{I}(g)) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} (i \mathcal{I}(f))^j (i \mathcal{I}(g))^{k-j-1} \tag{49}
\]
Let \(s' = \max\{s, \frac{1}{2} + \varepsilon\}\) for some \(0 < \varepsilon < \frac{1}{2}\) to be chosen later. Then, the \(H^s\) norm of the expression \((49)\) is bounded by
\[
\|h\|_{H^s} \|\mathcal{I}(f) - \mathcal{I}(g)\|_{H^{s'}} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} (\|\mathcal{I}(f)\|_{H^{s'}})^j (\|\mathcal{I}(g)\|_{H^{s'}})^{k-1-j}
\]
Now, we observe that
\[
\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} (\|\mathcal{I}(f)\|_{H^{s'}})^j (\|\mathcal{I}(g)\|_{H^{s'}})^{k-1-j} \leq e^{c \|\mathcal{I}(f)\|_{H^{s'}} + c \|\mathcal{I}(g)\|_{H^{s'}}}
\]
Moreover,
\[
\|\mathcal{I}(f)\|_{H^{s'}} \leq \|f\|_{H^{s'}} + \|f\|_{L^2}^2
\]
In the case where \(s \geq \frac{1}{2} + \varepsilon\) it follows \(\|f\|_{H^{s'}}^2 \leq \|f\|_{H^s}^2 \leq c \|f\|_{H^s}^2\) and otherwise, with \(p = \frac{1}{1-\varepsilon}\)
\[
\|f\|_{H^{\frac{1}{2}+\varepsilon}}^2 \leq c \|f\|_{H^s}^2 \leq c \|f\|_{L^2}^2 \leq c \|f\|_{H^s}^2
\]
by Sobolev embeddings. Now, choosing \(\varepsilon \leq 2s\) we have
\[
\|\mathcal{I}(f)\|_{H^{s'}} \leq c \|f\|_{H^s}^2
\]
Similarly, we get
\[
\|\mathcal{I}(f) - \mathcal{I}(g)\|_{H^{s'}} \leq c \|f\|_{H^s} + \|g\|_{H^s} \|f - g\|_{H^s}
\]
and the claim follows for \(s > 0\). Finally, for \(s = 0\)
\[
\left\| (e^{i \mathcal{I}(f)} - e^{i \mathcal{I}(g)} h) \right\|_{L^2} \leq \left\| e^{i \mathcal{I}(f)} - e^{i \mathcal{I}(g)} \right\|_{L^\infty} \|h\|_{L^2}
\]
\[
\leq \|\mathcal{I}(f) - \mathcal{I}(g)\|_{L^\infty} \|h\|_{L^2}
\]
\[
\leq 2 \left( \|f\|_{L^2} + \|g\|_{L^2} \right) \|f - g\|_{L^2} \|h\|_{L^2}
\]

APPENDIX B. CONSERVATION LAWS

The results in this section are well-known in the case of the real line (cp. \((\ref{appendix:real_line})\), Proposition 6.1.1, appendix of \((\ref{appendix:real_line}), \text{or} \((\ref{appendix:real_line})\)) and formally everything transfers to the periodic setting. Nevertheless, we briefly repeat the main points for completeness of the paper.

**Lemma B.1.** If \(u \in C([-T, T], H^2(\mathbb{T})) \cap C^1([-T, T], L^2(\mathbb{T}))\) is a solution of \((\ref{eq:universal_equation}), \text{or} \((\ref{eq:universal_equation})\), we have for \(t \in (-T, T)\)
\[
\frac{d}{dt} \|u(t)\|_{L^2(\mathbb{T})} = 0
\]
Partial integration yields
\[ \frac{d}{dt} |u(t)|^2_{L^2(\mathbb{T})} = 2 \Re \int_0^{2\pi} \overline{N}(u(t)) \, dx \]
for \( N(u) = \partial_x (|u|^2 u) \) or \( N(u) = -u^2 \partial_x \overline{u} + \frac{i}{2} |u|^4 u - i\mu(u) |u|^2 u + i\psi(u) u \), respectively. Partial integration yields
\[ \Re \int_0^{2\pi} \overline{\mu} \partial_x (|u|^2 u) \, dx = 0 \]
and
\[ \Re \int_0^{2\pi} \overline{\mu} u^2 \partial_x \overline{u} \, dx = 0 \]
Obviously, all the other terms also vanish and the \( L^2 \) conservation law follows. \( \square \)

**Lemma B.2.** If
\[ u \in C([-T, T], H^3(\mathbb{T})) \cap C^1([-T, T], H^1(\mathbb{T})) \]
is a solution of (1) with \( \lambda = 1 \), we have for \( t \in (-T, T) \)
\[ \frac{d}{dt} \| u_x(t) \|^2_{L^2(\mathbb{T})} + \frac{3}{2} \text{Im} \int_0^{2\pi} |u|^2 u \overline{u}_x(t) \, dx + \frac{1}{2} \| u(t) \|^6_{L^6(\mathbb{T})} = 0 \]
**Proof.** Firstly, using (1) we verify
\[ \frac{d}{dt} \| u_x \|^2_{L^2} = 2 \Re \int_0^{2\pi} (|u|^2 u_{xx} \overline{u}_x) \, dx \] (50)
Secondly, we again exploit (1) and carry out all the differentiations
\[ \frac{d}{dt} \text{Im} \int_0^{2\pi} |u|^2 u \overline{u}_x \, dx = \text{Im} \int_0^{2\pi} (|u|^2 u \overline{u}_x) \, dx + \text{Im} \int_0^{2\pi} |u|^2 u \overline{u}_xx \, dx \]
\[ = 4 \Re \int_0^{2\pi} |u|^2 u \overline{u}_xx \, dx + 4 \int_0^{2\pi} \overline{u}_x^2 |u|^2 \, dx \] (51)
Thirdly,
\[ \frac{d}{dt} \| u \|^6_{L^6} = -6 \text{Im} \int_0^{2\pi} |u|^4 \overline{u}_{xxx} \, dx + 6 \Re \int_0^{2\pi} |u|^4 \overline{u}_x (|u|^2 u)_x \, dx \]
and
\[ \Re \int_0^{2\pi} |u|^4 \overline{u}_x (|u|^2 u)_x \, dx = \frac{3}{8} \int_0^{2\pi} (|u|^8)_x \, dx = 0 \]
Moreover, we integrate by parts and obtain
\[ \frac{d}{dt} \| u \|^6_{L^6} = -12 \text{Im} \int_0^{2\pi} |u|^2 u^2 \overline{u}_x^2 \, dx \] (52)
Now, combining (50), (51) and (52) and integrating by parts we get
\[ \frac{d}{dt} \left( \| u \|^2_{L^2} + \frac{3}{2} \text{Im} \int_0^{2\pi} |u|^2 u \overline{u}_x(t) \, dx + \frac{1}{2} \| u(t) \|^6_{L^6(\mathbb{T})} \right) \]
\[ = 6 \Re \int_0^{2\pi} |u|^2 u \overline{u}_xx \, dx - 2 \Re \int_0^{2\pi} (|u|^2 u)_x \overline{u}_{xx} \, dx \]
\[ = 2 \Re \int_0^{2\pi} |u|^2 u \overline{u}_{xx} \, dx - 2 \Re \int_0^{2\pi} u^2 \overline{u}_x \overline{u}_{xx} \, dx = 0 \]
\( \square \)
APPENDIX C. PROOF OF THE ESTIMATE \(48\)

Let \(f, g\) be smooth and \(2\pi\)-periodic with \(g(0) = 0\). Then, \(g(x) = \int_0^x g'(y) \, dy\) and \(g(x) = -\int_x^{2\pi} g'(y) \, dy\) such that

\[
2\|g\|_{L^\infty(T)} \leq \|g'\|_{L^1(T)}
\]

and by Hölder’s inequality

\[
\|fg\|_{L^2(T)} \leq \frac{1}{2} \|f\|_{L^2(T)} \|g'\|_{L^1(T)}
\]

By a translation \(x \mapsto x + \xi\) we see that this holds for all \(g\) with \(g(\xi) = 0\) for some \(\xi \in [0, 2\pi]\). Now, let \(u\) be smooth and \(2\pi\)-periodic and set \(f = u\) and \(g = u^2 - \frac{1}{2\pi} \int_0^{2\pi} u^2(y) \, dy\).

Then,

\[
\left\| (u^2 - \frac{1}{2\pi} \int_0^{2\pi} u^2(y) \, dy) \right\|_{L^2(T)} \leq \|u\|_{L^2(T)} \|u'\|_{L^1(T)} \leq \|u\|_{L^2(T)}^2 \|u'\|_{L^2(T)}
\]

and the estimate \(48\) follows.

REFERENCES

[1] H. A. Biagioni and F. Linares. Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations. Trans. Amer. Math. Soc., 353(9):3649–3659 (electronic), 2001.
[2] J. Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Geom. Funct. Anal., 3(2/3):107–156/209–262, 1993.
[3] T. Cazenave and A. Haraux. An introduction to semilinear evolution equations. Oxford Lecture Series in Mathematics and its Applications. 13. Oxford: Clarendon Press., 1998.
[4] Y. Chen. The initial boundary value problem for a class of nonlinear Schrödinger equations. Acta Math. Sci., 6:405–418, 1986.
[5] M. Christ. Illposedness of a Schrödinger equation with derivative nonlinearity. Preprint, 2003, http://math.berkeley.edu/~mchrist/Papers/dnls.ps.
[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. A refined global well-posedness result for Schrödinger equations with derivative. SIAM J. Math. Anal., 34(1):64–86, 2002.
[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Sharp global well-posedness for KdV and modified KdV on \(\mathbb{R}\) and \(\mathbb{T}\). J. Amer. Math. Soc., 16(3):705–749 (electronic), 2003.
[8] J. Ginibre. Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d’espace [d’après Bourgain]. In Séminaire Bourbaki. Volume 1994/95. Exposés 790-804, pages 163–187. Société Mathématique de France. Paris: Astérisque, 1996.
[9] J. Ginibre, Y. Tsutsumi, and G. Velo. On the Cauchy problem for the Zakharov system. J. Funct. Anal., 152(2):384–436,1997.
[10] A. Grünrock. On the Cauchy- and periodic boundary value problem for a certain class of derivative nonlinear Schrödinger equations, 2000, arXiv:math.AP/0006195.
[11] A. Grünrock. Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS. Int. Math. Res. Not., 2005(41):2525–2558, 2005.
[12] N. Hayashi. The initial value problem for the derivative nonlinear Schrödinger equation in the energy space. Nonlinear Anal., 20(7):823–833, 1993.
[13] N. Hayashi and T. Ozawa. On the derivative nonlinear Schrödinger equation. Phys. D, 55(1-2):14–36, 1992.
[14] N. Hayashi and T. Ozawa. Finite energy solutions of nonlinear Schrödinger equations of derivative type. SIAM J. Math. Anal., 25(6):1488–1503, 1994.
[15] D.J. Kaup and A.C. Newell. An exact solution for a derivative nonlinear Schrödinger equation. J. Math. Phys., 19:798–810, 1978.
[16] T. Meškauskas. On well-posedness of the initial boundary-value problem for the derivative nonlinear Schrödinger equation. Lith. Math. J., 38(3):250–261, 1998.
[17] L. Molinet, J.C. Saut, and N. Tzvetkov. Ill-posedness issues for the Benjamin-Ono and related equations. SIAM J. Math. Anal., 33(4):982–988 (electronic), 2001.
[18] H. Takaoka. Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity. Adv. Differ. Equ., 4(4):561–580, 1999.

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