Recursive overbetting of a satellite investment account:

enjoyable generalizations of the Kelly criterion, by

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Abstract

This paper builds a core-satellite model of semi-static Kelly betting and log-optimal investment. We study the problem of a saver whose core portfolio consists in unlevered (lx) retirement plans with no access to margin debt. However, the agent has a satellite investment account with recourse to significant, but not unlimited, leverage; accordingly, we study optimal controllers for the satellite gearing ratio. On a very short time horizon, the best policy is to overbet the satellite, whereby the overriding objective is to raise the aggregate beta toward a growth-optimal level. On an infinite horizon, by contrast, the correct behavior is to blithely ignore the core and optimize the exponential growth rate of the satellite, which will anyways come to dominate the entire bankroll in the limit. For time horizons strictly between zero and infinity, the optimal strategy is not so simple: there is a key trade-off between the instantaneous growth rate of the composite bankroll, and that of the satellite itself, which suffers ongoing volatility drag from the overbetting. Thus, a very perspicacious policy is called for, since any losses in the satellite will constrain the agent's access to leverage in the continuation problem. We characterize the optimal feedback controller, and compute it in earnest by solving the corresponding HJB equation recursively and backward in time. This solution is then compared to the best open-loop controller, which, in spite of its relative simplicity, is expected to perform similarly in practical situations.

Keywords Stochastic control · Kelly criterion · Asymptotic capital growth · Log-optimal portfolios · Geared investment · Leveraged ETFs · Controlled diffusion processes · Markov decision processes · Lattice-based models · Lifecycle investing · Leverage constraints

JEL classification C44 · C61 · D14 · D15 · D81 · G00 · G10 · G11

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"We take a buck, we shoot it full of steroids, and we call it leverage. I call it steroid banking."

—Gordon Gekko, Wall Street 2: Money Never Sleeps

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1 Introduction

1.1 The paper in a nutshell

My paper below studies and solves a semi-static optimal control problem in the theory of Kelly 1956 betting, or asymptotic capital growth (cf. with Thorp 2018; Poundstone 2010), also known as log-optimal portfolio theory.

I build a classical decision-theoretic model of a leverage-constrained investor, the bulk of whose funds are deposited into a “core” portfolio that wholly consists of unlevered (1x) retirement plans, wherein the participant has every last dollar invested in the tangency portfolio, or market index. At the same time, the agent controls a “satellite” investment account with substantial access to leverage, subject to constraints.¹ Not for no reason did the author ask himself the following economic question: what is the optimal control law for the satellite leverage ratio? This treatise contains a definitive answer.

For agents who are interested in compounding and long-term capital management, there is a natural temptation to operate the satellite independently of the core, using a special constant leverage ratio known as the Kelly bet, a parameter which we regard as a characteristic feature of the total stock market. On an infinite horizon, this simple mode of operation proves to be the best possible; but in the long run, we are all dead.

Given a medium-term horizon, then, the penny drops that the day-to-day fluctuations of the satellite are mostly irrelevant, especially if it comprises a small fraction of

¹Specifically, the leverage ratio of the auxiliary account will not be allowed to exceed some exogenously given multiplier, say, 3x or 4x, etc. However, the dollar amount of the margin loan can grow without bound as the account accumulates more and more collateral over time.
the aggregate bankroll. Watching the random gyrations, one comes to realize intuitively that it is the overall gearing ratio that actually rules the day; that is, if the current satellite share is small, then one can justify very high satellite gearing ratios (i.e., exceeding the Kelly bet) because the saver's composite leverage (or overall market beta) will in any case be fairly tame, and far below the Kelly criterion.

Thus, we have already hit upon what will become the pre- eminent tradeoff in the model: up to a point, it becomes worthwhile to sacrifice some of the welfare of the satellite (by “overbetting” it on a fixed horizon) in exchange for a higher expected CAGR of the aggregate bankroll. On a shorter planning horizon, this kind of exchange becomes relatively more attractive: we can bet the satellite like there is no tomorrow, because there isn’t one. Say, for a log-optimal investor whose time horizon is just a few days or weeks, the overriding objective is to get the aggregate gearing ratio as close to the Kelly bet as possible, and right quick.

On a longer time frame, by contrast, and especially in view of everyday leverage constraints, overbetting the satellite is much more reckless, since it endangers the Kelly bettor's long-term access to margin debt. Thus, on a finite horizon, the optimal satellite bet ought to lie somewhere between the two extremes, viz., between the Kelly bet and the idiosyncratic (very high) gearing ratio that makes the overall leverage equal to the Kelly multiple.

Based on this brief symposium, it has become clear that the optimal controller will condition its behavior not just on the time, but also on the precise amounts of capital that reside in the core and in the satellite, respectively; in fact, it will be shown below that the payoff-relevant state consists in the dynamic percentage of total capital that lives in the satellite, which we refer to as the “satellite share,” or the “satellite weight,” in earnest.

By continuity considerations, the satellite gearing policy must converge to the Kelly bet as the satellite weight tends to 100%, and on the other end, it ought to saturate the leverage constraint when the state variable approaches 0%. Expressed in terms of per-
formance values, if the satellite weight is close to 100% then the expected CAGR to the end of the horizon (the “CAGR to go”) should be close to the Kelly optimum growth rate (cf. with Cover and Thomas 2005); if the satellite share is practically nil, then the CAGR to go will be scantly more than that of the market index. As such, these are the boundary values that will be used to construct numerical solutions of the dynamic program that appears in the sequel.

As far as the qualitative behavior of the optimal feedback controller, let us observe that our problem admits a most mechanical and clear-cut notion of the substitutability of satellite gearing for satellite wealth. That is, if the auxiliary account is relatively starved for funding, then a $1 margin loan may substitute for a dollar of equity capital in a patently obvious way. Thus, in typical practice we will expect the optimal satellite bet to be decreasing in the state variable, e.g., as the satellite gradually eats the overall portfolio, the satellite gearing policy will take a “glide path” down to the Kelly bet. After crunching the relevant numbers, our analysis will show that the investor's overall beta undergoes a commensurate monotonic increase toward the Kelly rule.

1.2 Road map

On the strength of these mystical intuitions, we proceed to develop a purely scientific economic model that gives a precise quantitative answer as to what is the best satellite gearing ratio in all possible situations.

Section 2 starts with a simplified algebraic analysis on the Cox, Ross, and Rubinstein 1979 binomial lattice; in spite of this simplicity, the ensuing model can be flexibly calibrated to match the log-normal diffusion processes of financial mathematics. We study the fundamental state transition (6) in a lattice-based framework, and then we formulate the appropriate Bellman equation (17) and develop its properties. A proper numerical solution of the model requires some care, since we must iterate over finite grids for the time, the state, and the control; the dynamic objective (17) is tractable in so far as it leads
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to a concave program over a compact interval. The latter part of subsection 2.2 gives the exact conditions under which it is possible to enforce a given aggregate gearing ratio to the end of the horizon, regardless of the performance of the underlier; this phenomenon is a special artifact of the binomial lattice dynamics, and does not obtain when trading is continuous in time.

Section 3 develops the corresponding stochastic control model in continuous time; here, the key dynamics are expressed by the stochastic differential equation (39) and the HJB equation (42), which is solved by a backward Euler finite-difference scheme. The continuous time model turns out to be more computationally tractable, because there is now an explicit formula (46) for the policy function in terms of the value function. This being done, subsection 3.2 studies open-loop controllers, which, once initiated, decline to take any additional input from the path of the state variable. Finally, 3.3 analyzes more conservative preferences and control objectives: specifically, we argue in favor of optimizing the mean terminal satellite share, a process which takes a less cavalier attitude toward leveraging the auxiliary account. This generates an interesting rule of thumb, which we refer to as “half-Kelly and a half”: in the absence of gearing restrictions, the expected percentage of final wealth that resides in the satellite is maximized by setting the aggregate gearing ratio equal to the midpoint of the Kelly rule and 100%.

Section 4 concludes the paper, and the Appendix gives a full software implementation of our main controllers, along with Monte Carlo simulations of their sample behavior.

2 Lattice-based model

2.1 Basic definitions and notation

We assume that the time $t$ operates in discrete steps of the Cox, Ross, and Rubinstein 1979 binomial lattice, that is, $t \in \{0, 1, 2, ..., T\}$, where $T \in \mathbb{N}$ is the termination date and $t = 0$ is right now. $S_t$ denotes the price of the underlier (say, the stock market index)
at date $t$, where the initial asset price $S_0$ is exogenously given. We let the price relative $G_t := S_{t+1}/S_t$ denote the (random) gross return of a $1$ investment in the underlier over the time interval $[t, t+1]$, so that $G_t - 1 = \Delta S_t/S_t$ is the percentage change in the underlier between date $t$ and date $t + 1$, where $\Delta S_t = S_{t+1} - S_t$ is the forward difference of the asset price sequence. Thus, the Markov process $(S_t)_{t=0}^\infty$ is assumed to evolve according to a multiplicative binomial random walk (Cox, Ross, and Rubinstein 1979; Rendleman and Bartter 1979):

$$G_t = \frac{S_{t+1}}{S_t} = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } 1 - p, \end{cases}$$

(1)

where $p \in (0, 1)$ is the stationary probability of an uptick, $u > 1$ is the periodic gross return when the underlier goes up, and $d < 1$ is the periodic gross return when the underlier goes down. Thus, the gross returns $(G_t)_{t=0}^\infty$ make up an iid process, whereby $G_t$ is independent of the contemporaneous price history $S' := (S_0, S_1, \ldots, S_t)$. We let $R := 1 + r$ denote the gross return on cash over $[t, t + 1]$, where $r$ is the risk-free rate of periodic interest. Thus, in order to prevent arbitrage opportunities, we must have $u > R > d$.

As far as stochastic control, we let $x_t$ and $y_t$ denote the dollar values of the core and the satellite accounts, respectively, at time $t$ (in advance of the fluctuation $G_t$), where the initial values $x_0 > 0$ and $y_0 > 0$ are fixed parameters; $w_t := x_t + y_t$ is the investor’s aggregate wealth at date $t$. We will stipulate that the (unlevered) core is 100% invested in the market index at all times, so that the dynamics of $(x_t)_{t=0}^\infty$ are given by the transition law $x_{t+1} = x_t G_t = x_t S_{t+1}/S_t$, whence $x_t = x_0 \prod_{s=0}^{t-1} G_s$. As to the (leveraged) satellite, we let $b_t$ denote the investor’s chosen gearing ratio (or leverage ratio) over the time step $[t, t + 1]$, where $b_t = b_t(x_t, y_t)$ is a feedback control policy $b_t(\bullet, \bullet)$ whose value will generally depend on the time $t$ and the respective account balances $(x, y)$. Accordingly, the law of motion of the satellite account $y_t$ is

$$\frac{y_{t+1}}{y_t} = b_t G_t + (1 - b_t)R = R + b_t(G_t - R),$$

(2)
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whereby the gambler has bet the fraction $b_t$ of the account on the underlier and the fraction $1 - b_t$ on the risk-free bond over the time interval $[t, t + 1]$. Solving the stochastic difference equation (2), we obtain the formula

$$y_t = y_0 \prod_{s=0}^{t-1} (R + b_s(x_s, y_s)(G_s - R)).$$  \hspace{1cm} (3)

For the sake of convenience, we let $\lambda_t := y_t/(x_t + y_t)$ denote the percentage of the investor’s wealth that is contained in the satellite at time $t$; thus, we have the aggregate wealth dynamics

$$\frac{w_{t+1}}{w_t} = \lambda_t \frac{y_{t+1}}{y_t} + (1 - \lambda_t) \frac{x_{t+1}}{x_t} = \lambda_t[R + b_t(G_t - R)] + (1 - \lambda_t)G_t$$

$$= R + [1 + \lambda_t(b_t - 1)](G_t - R) = R + \hat{b}_t(G_t - R),$$  \hspace{1cm} (4)

where $\hat{b}_t := 1 + \lambda_t(b_t - 1)$ denotes the investor’s overall gearing ratio at date $t$. The percentage of the investor’s wealth that resides in the satellite evolves according to

$$\lambda_{t+1} = \frac{y_{t+1}}{x_{t+1} + y_{t+1}} = \frac{y_t[R + b_t(G_t - R)]}{x_tG_t + y_t[R + b_t(G_t - R)]} = \frac{R + b_t(G_t - R)}{R + b_t(G_t - R) + (\lambda_t^{-1} - 1)G_t}$$

$$= \left(1 + \frac{(\lambda_t^{-1} - 1)}{R + b_t(G_t - R)}\right)^{-1}. \hspace{1cm} (5)$$

Thus, we have the following definition, which formalizes our basic law of motion:

**Definition 1** (Transition function). We will write

$$\lambda^G = \lambda^G(\lambda, b) := \frac{R + b(G - R)}{R + b(G - R) + (\lambda^{-1} - 1)G}.$$  \hspace{1cm} (6)

for the (random) subsequent state that obtains from the (state, control) pair $(\lambda, b)$; the realized values in the “up” and “down” states will be denoted $\lambda^u$ and $\lambda^d$, respectively.

In order to avoid trouble vis-à-vis our leveraged investments on the binomial lattice,
we must constrain $b$ in order to guarantee that the numerator and denominator of (6) are positive, e.g., so that the gambler can never lose more than his initial investment. Hence, we have Definition 2:

**Definition 2** (Strong admissibility of satellite gearing ratios). A gearing ratio $b$ is called “admissible” (or “strongly admissible”) for the binomial lattice with parameters $(u, d, R)$ if and only if it guarantees a gross return that is non-negative (viz., if it guarantees that the investor’s periodic percentage return is $\geq -100\%$). Thus, an admissible gearing ratio can never cause the investor to become bankrupt as a consequence of an individual fluctuation $S_{t+1}/S_t$ of the underlier.

The next proposition codifies the set of admissible gearing ratios.

**Proposition 1** (Characterization of strong admissibility). The gearing ratio $b$ is admissible for the lattice parameters $(u, d, R)$ if and only if

$$b \in \mathbb{B} := \left[ -\frac{R}{u-R}, \frac{R}{R-d} \right].$$

**Proof.** $b$ is admissible if and only if we have the simultaneous inequalities $R+b(u-R) \geq 0$ and $R+b(d-R) \geq 0$, which is equivalent to the quoted relation (7).

In the sequel, our essential interest will be to fine-tune the lattice (1) so as to obtain a discretization of exponential Brownian motion; in all such calibrations, the periodic capital growth factors $(u, d, R)$ will approach unity and the Bernoulli parameter $p$ will converge to 50%. In this connection, Proposition 2 says that any particular leverage ratio will be admissible for a sufficiently fine mesh.

**Proposition 2** (Admissible leverage for a calibrated lattice). If an $n$-step binomial lattice $\mathcal{L}_n := (u_n, d_n, R_n)$ is calibrated in such a way that:

1. For $n$ sufficiently large, there are no arbitrage opportunities on $\mathcal{L}_n$; and
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Then every numerical gearing ratio \( b \in (-\infty, \infty) \) is admissible in the satellite, for \( n \) sufficiently large. If the number \( n \) of revisions is sufficiently great, then the investor’s overall gearing ratio can be made as large as desired, in spite of the lack of access to leverage in the core.

**Proof.** In order to (eventually) avoid arbitrage opportunities on \( L_n \), we must have \( d_n < R_n < u_n \) for all sufficiently large \( n \), so that \( \lim_{n \to \infty} R_n = 1 \) by the squeezing process. Hence, we have the relations \( \lim_{n \to \infty} R_n/(R_n - d_n) = \infty \) and \( \lim_{n \to \infty} -R_n/(u_n - R_n) = -\infty \), so that, for all sufficiently large \( n \), \( b \) will lie in the interval \((7)\), as promised.

Now, taking the standpoint of time \( t = 0 \) in an \( n \)-step lattice, the relation \( \hat{b}_0 = 1 + \lambda_0(b_0 - 1) \) implies that, subject to admissibility in the satellite, our overall gearing ratio is constrained by the fact that

\[
\hat{b}_0 \in \left[ 1 - \lambda_0 \left( \frac{R_n}{u_n - R_n} - 1 \right), 1 + \lambda_0 \left( \frac{R_n}{R_n - d_n} - 1 \right) \right].
\]

Given our assumption that \( \lambda_0 > 0 \), the right- and left-endpoints of the interval \((8)\) converge to \( \pm \infty \), respectively, as \( n \to \infty \), so that the gambler’s overall gearing ratio can be made arbitrarily large, using only admissible leverage in the core.

As an illustration of Proposition 2, we have

**Example 1** (Unbounded gearing for log-normal diffusions). If we divide a \( \tau \)-year time span into \( n \) binomial steps of duration \( \Delta t := \tau/n \) each, and we make the substitutions \( R_n := e^{\rho \Delta t}, u_n := e^{\sigma \sqrt{\Delta t}}, \) and \( d_n := e^{-\sigma \sqrt{\Delta t}} \), where \( \rho \) is the continuously-compounded annual interest rate and \( \sigma > 0 \) is the annual log-volatility of the underlying (cf. with Luenberger 1997), then unlimited leverage is permitted on a step-by-step basis, for a large enough number of subdivisions.

In the context of Example 1, the no-arbitrage condition \( R_n < u_n \) is equivalent to
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Figure 1: Monotonic expansion of the set of admissible gearing ratios on an $n$-step binomial lattice ($1 \leq n \leq 15$) calibrated to match the moments of a geometric Brownian motion process with 40% annual volatility (interest rate = 3% a year, horizon = 1 year), as in Example 1.

$\rho \sqrt{\Delta t} < \sigma$, which is eventually true in the limit as $n \to \infty$ and $\Delta t \to 0$. Thus, in the Black and Scholes 1973 market whereby the underlying follows a log-normal diffusion process (cf. with Wilmott 2007), leveraged ETFs of arbitrary scale can be admissibly used over the differential time step $dt$ (Garivaltis 2022). Under this calibration, the interval (7) of admissible gearing ratios expands monotonically with the number of binomial steps; this is illustrated in Figure 1.

The next definition generalizes the concept of admissibility to allow for the investor to pay off his margin loan by liquidating some of the assets that reside in the core:

**Definition 3** (Weak admissibility of satellite gearing ratios). A satellite gearing ratio $b$ is called “weakly admissible” iff it guarantees that the investor’s overall periodic gross return $w_{t+1}/w_t$ is always non-negative, assuming that the core portfolio can be pledged as collateral for the bets made in the satellite.

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2The lower (e.g., short-selling) bound $-R_n/(u_n-R_n)$ is decreasing in $n$ if $n \geq \tau(2\rho/\sigma)^2$; the upper bound $R_n/(R_n-d_n)$, which applies to levered long positions, is monotonically increasing for all parameter values.
In accordance with this definition, Proposition 3 specifies the segment of weakly admissible gearing ratios:

**Proposition 3** (Characterization of weak admissibility). If a satellite gearing ratio \( b_t \) is admissible, then it is weakly admissible. In fact, if the investor currently has the fraction \( \lambda_t > 0 \) of his wealth in the satellite, then the set of weakly admissible satellite bets consists in the segment

\[
\left[ 1 - \frac{u}{\lambda_t(u - R)}, 1 + \frac{d}{\lambda_t(R - d)} \right].
\]

(9)

Proof. If \( b_t \) is admissible, then, since it guarantees to never bankrupt the satellite, *a fortiori*, it guarantees to never bankrupt the investor’s overall net worth. Now, according to (4), the relation \( w_{t+1}/w_t \geq 0 \) means that \( R + \hat{b}_t(G_t - R) \geq 0 \), where \( \hat{b}_t = 1 + \lambda_t(b_t - 1) \) is the agent’s overall leverage ratio. Solving the inequalities

\[
- \frac{R}{u - R} \leq 1 + \lambda_t(b_t - 1) \leq \frac{d}{R - d}
\]

for \( b_t \), we obtain the stated interval (9).

\[
\boxed{\text{2.2 Restrictions on satellite gearing}}
\]

According to Proposition 2 and Example 1, if the investor can revise his positions with a great enough frequency, then he can *de facto* arrange any desired level of overall exposure \( \hat{b}_t \) for his aggregate net worth, regardless of the dynamic percentage \( \lambda_t \) of his total wealth that exists in the satellite. In the case of a Kelly 1956 bettor (e.g., a geometric mean maximizer), such as we are concerned with here, this amounts to the (unique) fixed optimal fraction (or “log-optimal portfolio”, cf. with Cover and Thomas 2005)

\[
b^* = \arg \max_{-R/(u - R) < b \leq R/(-R - d)} \mathbb{E}[\log(R + b(G_t - R))],
\]

(11)
Weakly admissible gearing bounds on an up-down lattice

![Diagram showing weakly admissible and strongly admissible satellite bets.]

Figure 2: The interval of weakly admissible satellite bets $b_t$, for different percentages $\lambda_t$ of wealth that are contained in the satellite. Here, we have used the binomial lattice parameters $u = 1.07$, $d = 1/u = 0.935$, and $R = 1.03$. The special case $\lambda_t = 1$ corresponds to strong admissibility.

where the strictly concave function

$$f(b) := \mathbb{E}_t [\log(R + b(G_t - R))] = p \log(R + b(u - R)) + (1 - p) \log(R - b(R - d))$$

(12)

gives the expected continuously-compounded capital growth rate from step $t$ to step $t+1$ (cf. with MacLean, Thorp, and Ziemba 2011). Notice that

$$\lim_{b \downarrow -R/(u-R)} f(b) = \lim_{b \uparrow R/(d-R)} f(b) = -\infty$$

(13)

if $p \in (0, 1)$. The first-order condition for the Kelly criterion amounts to

$$f'(b) = \mathbb{E}_t \left[ \left( b + \frac{R}{G_t - R} \right)^{-1} \right] = p \left( b + \frac{R}{u - R} \right)^{-1} + (1 - p) \left( b - \frac{R}{R - d} \right)^{-1} = 0,$$

(14)

or equivalently,

$$b^* = p \frac{R}{R - d} + (1 - p) \frac{R}{u - R}$$

(15)
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Exponential capital growth on a binomial lattice

Figure 3: The mean, standard deviation, and mean absolute deviation (MAD) of the continuously-compounded per-bet capital growth rate ($\log(R + b(G - R))$) on a binomial lattice with parameters $(u, d, R, p) := (1.08, 1/1.08, 1.025, 75\%)$, where $G \in \{u, d\}$. The set of admissible gearing ratios is $\mathbb{B} = [-18.63, 10.34]$, the Kelly bet is $b^* = 3.1$, and the Kelly growth rate is $f(b^*) = 5.1\%$ per step. The Kelly volatility is 22.1\% per step (MAD = 17.8\%).

is expressed uniquely as a convex combination of the endpoints of the admissible interval, $\mathbb{B}$ (cf. with Thorp 2008). Naturally, we have the second-order condition $f''(b) = -\mathbb{E}_t \left[ (b + R/(G_t - R))^{-2} \right] < 0$.

We proceed to take up the situation whereby the satellite account is leverage constrained, and thereby subject to the restriction $b \in [1, \overline{b}] \subset \mathbb{B}$. In general, we will be dealing with levered long positions ($b_t > 1$); this means that the investor’s access to leverage (i.e., $\lambda_t$) increases when the underlying goes up (since the satellite outperforms the core) and decreases when the underlier goes down (e.g., when the satellite underperforms the core). In the context of continuous betting on the price changes $dS_t$ of a geometric Brownian motion process, the satellite gearing ratio $b_t$ can be revised upward at will in response to a decline in the underlier; now that such is not the case ($b_t(x_t, y_t) \in [1, \overline{b}]$),
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we have a much richer problem:

\[ V_t(x_t, y_t) := \sup_{1 \leq b_t(\cdot, \cdot) \leq \bar{b}} \mathbb{E}_t \left[ \log \left( (x_t + y_t) \prod_{s=t}^{T-1} (G_s + \lambda_s (b_s - 1)(G_s - R)) \right) \right] \]

\[ = \log(x_t + y_t) + \sup_{1 \leq b_t(\cdot, \cdot) \leq \bar{b}} \sum_{s=t}^{T-1} \mathbb{E}_t [\log(R + \hat{b}_s (G_s - R))], \quad (16) \]

where \( x_{s+1}/x_s = G_s, \ y_{s+1}/y_s = R + b_s(x_s, y_s)(G_s - R) \), and \( V_t(\cdot, \cdot) \) is the maximum value function (Bellman 2003), that is, the optimized geometric mean terminal wealth (cf. with Samuelson 1969) in the continuation problem that starts with $x$ in the core and $y$ in the satellite at date $t$. In accordance with the principle of optimality, and the time-separable form of (16), we have the Bellman equation:

\[ V_t(x_t, y_t) = \max_{1 \leq \hat{b}_t \leq \bar{b}} \mathbb{E}_t [\log(G_t + \lambda_t (b_t - 1)(G_t - R) + V_{t+1}(x_{t+1}, y_t (R + b_t(G_t - R))))]
\]

\[ = \max_{1 \leq \hat{b}_t \leq 1 + \lambda_t(\bar{b} - 1)} f(\hat{b}_t) + \mathbb{E}_t \left[ V_{t+1}(x_t G_t, y_t \left( G_t + \frac{\hat{b}_t - 1}{\lambda_t} (G_t - R) \right)) \right], \quad (17) \]

where the first term in side the expectation in (17) is the investor’s expected log-return over the immediate lattice step ($t$ to $t + 1$), and the continuation value amounts to his expected log-return from date $t + 1$ through date $T$. Here, we have \( x_{t+1} = x_t G_t \) and \( \lambda_t = y_t/(x_t + y_t) \); the boundary condition is \( V_T(x, y) = \log(x + y) \). Note well that \( e^{V_t(x, y)} \) is the optimized geometric mean terminal wealth in the time-$t$ continuation problem from state \((x, y)\). The last equality in (17) expresses the investor’s dynamic optimization problem in terms of choosing a feasible aggregate gearing ratio, \( \hat{b}_t \), rather than a feasible satellite gearing ratio, \( b_t \).

Proposition 4 below describes the qualitative behavior of the maximum value function (16).

**Proposition 4** (Basic properties of the value function). The value function \( V_t(x, y) \) has the property that, for any \( \theta > 0, V_t(\theta x, \theta y) = \log \theta + V_t(x, y) \); the policy function \( b_t(x, y) \)
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Numerical solution of the Bellman equation

(a) feedback controller, conditional mean log final aggregate capital

(b) open-loop controller, expected per-bet continuously-compounded growth rate to the end of the horizon

Figure 4: Illustration of the value function $V_t(\lambda)$ for $\lambda \in [0, 1]$ and all horizons $\tau := T - t$ up to 100 lattice steps, using the parameters $(u, d, R, p, \bar{b}) := (1.07, 1/1.07, 1.025, 75\%, 4)$. The Kelly growth rate for this market is 4.1% per step; the growth rate of the underlier is 3.4% per step.

only depends on the percentage $\lambda$ of wealth that is held in the satellite, and, accordingly, we will abuse notation and write $b_t(\lambda)$ and $V_t(\lambda)$ for the policy function and the value function, respectively. The value function is concavely increasing in $\lambda$.

In view of Proposition 4, when we regard the percentage $\lambda$ of wealth that is held in the satellite as the payoff-relevant state, the Bellman equation (17) becomes (for $0 \leq t \leq T - 1$):

$$V_t(\lambda) = \max_{1 \leq b \leq \bar{b}} f(1 + \lambda(b - 1)) + \mathbb{E}_t \left[ V_{t+1} \left( \lambda^G(\lambda, b) \right) \right].$$

(18)

The boundary conditions for the value function are $V_T \equiv 0$, $V_t(1) = f(b^*)(T - t)$, and $V_t(0) = (p \log u + (1 - p) \log d)(T - t)$. The corresponding boundary values for the policy function are $b_{T-1}(\lambda) = \min(\bar{b}, 1 + (b^* - 1)/\lambda)$, $b_t(1) \equiv b^*$, and $b_t(0) \equiv \bar{b}$, where $b^*$ is the Kelly bet for this market. Note well that $b_{T-1}(\bullet)$ is the “greedy” (horizon-1) policy that acts so as to make the aggregate gearing ratio $\hat{b}_{T-1}(\bullet)$ as close as possible to the Kelly
Figure 5: Plot of the horizon-10 policy function \((T - t := 10)\), for the parameter values \((u, d, R, p, \overline{b}) := (1.07, 1/1.07, 1.025, 75\%, 4)\). If the satellite comprises a sufficiently small percentage of the investor’s net worth, then he will elect to saturate the leverage constraint in order to push his aggregate gearing ratio \(\hat{b}\) up to an acceptable level. As \(\lambda\) increases, and the satellite comes to dominate the overall portfolio, \(\hat{b}\) increases monotonically to the Kelly 1956 fraction, \(b^* = 2.81\).

bet; in this case, there is no longer any continuation problem to fret over, and we have

\[
V_{T-1}(\lambda) = f(1 + \lambda(b_{T-1}(\lambda) - 1)) = f(\min(b^*, 1 + \lambda(\overline{b} - 1))).
\]

For an interior optimum \(b_t(\lambda) \in (1, \overline{b})\), we have the first-order condition

\[
\lambda f'(\hat{b}) + E_t \left[V_{t+1}^* \left(\lambda^G(\lambda, b)\right) \frac{\partial \lambda^G}{\partial b}\right] = 0,
\]

where \(= 0\) is replaced by \(\geq 0\) for the corner solution \(b_t(\lambda) \equiv \overline{b}\), which saturates the leverage that is available in the satellite; we have the expression

\[
\frac{\partial \lambda^G}{\partial b} = \left(\frac{\lambda^G}{R + \hat{b}(G - R)}\right)^2 \left(\lambda^{-1} - 1\right) G(G - R) = \frac{(\lambda^{-1} - 1) G(G - R)}{(R + \hat{b}(G - R) + (\lambda^{-1} - 1) G)^2}.
\]
Figure 6: The horizon-100 gearing policy \((T - t := 100)\) for the parameters \((u, d, R, p, \bar{b}) := (1.07, 1/1.07, 1.025, 75\%, 4)\). Note well that the horizon-100 policy is significantly less aggressive than the horizon-10 policy; the substitutability of satellite wealth for satellite gearing is manifest in the fact that the optimal satellite bet \(b_t(\lambda)\) is monotonically decreasing in \(\lambda\).

In the same vein, there lies the envelope condition

\[
V_t'(\lambda) = (b - 1)f'(\hat{b}) + \mathbb{E}_t \left[ V_{t+1}' \left( \lambda^G(\lambda, b) \right) \frac{\partial \lambda^G}{\partial \lambda} \right],
\]

where

\[
\frac{\partial \lambda^G}{\partial \lambda} = \left( \frac{\lambda^G}{\hat{\lambda}} \right)^2 \frac{G}{R + b(G - R)} > 0.
\]

A special feature of the lattice-based model, which does not obtain in the continuous time case, is that there are always situations (which depend on the satellite share and the number of steps remaining) in which it is possible to enforce a given composite gearing ratio to the end of the horizon, regardless of what happens to the underlier. Such is the content of Definition 4.

**Definition 4** \((\tau\text{-achievability})\). A fixed aggregate gearing ratio \(\hat{b}\) is said to be “\(\tau\text{-achievable}\)” iff the investor can guarantee (using the policy \(b(\lambda) = 1 + (\hat{b} - 1)/\lambda\)) to achieve \(\hat{b}\) through-
Figure 7: Location of the kink in the policy function, for different binomial lattice horizons \(= T - t\), assuming the parameters \((u, d, R, p, \bar{b}) := (1.07, 1/1.07, 1.025, 75\%, 4)\). In the blue region, the percentage of wealth in the satellite is sufficiently low (for the given horizon) that the investor leverages to the fullest extent possible; here, the gearing constraint causes a loss of welfare. In the orange region, the gearing limit is irrelevant. On a short horizon, the Kelly bettor must hurry to exploit the leverage that is available in the satellite; on longer horizons, a more conservative approach is warranted.

**out a \(\tau\)-step horizon without violating the gearing constraint, even if the underlying goes down at every step.**

In the next theorem, we develop and solve a linear recurrence relation that gives a blueprint for the interval of enforceable gearing ratios.

**Theorem 1** (Characterization of \(\tau\)-achievability). In a given state \(\lambda\), the aggregate leverage \(\hat{b}\) is \(\tau\)-achievable if and only if

\[
\lambda \geq 1 - \frac{\bar{b} - \hat{b}}{\bar{b} - 1} \left(1 - (\hat{b} - 1) \left(\frac{R}{d} - 1\right)\right)^{\tau-1}.
\]

**Proof.** We let \(\Lambda_\tau\) denote the lowest value of \(\lambda\) for which \(\hat{b}\) is \(\tau\)-achievable; we proceed to derive a recurrence relation for the sequence \((\Lambda_\tau)_{\tau=1}^\infty\). For \(\tau := 1\) we must have \(1 + (\hat{b} - 1)/\lambda \leq \bar{b}\), so that \(\Lambda_1 = (\hat{b} - 1)/(\bar{b} - 1)\). Now, assuming that \(\hat{b}\) is \(\tau\)-achievable, then, even
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Figure 8: Plot of the policy function $b_t(\lambda)$ for $\lambda \in [0, 1]$ and various integer horizons $\tau := T - t$, using the parameters $(u, d, R, p, \hat{b}) := (1.07, 1/1.07, 1.025, 75\%, 4)$. The Kelly bet for this market is $b^* = 2.81$. The level curves $b(t, \lambda) \equiv \beta$ are indicated below each surface, in the $(t, \lambda)$ plane. Note well that the open-loop controller outputs a fixed satellite gearing ratio (based on $(t, \lambda_t)$) that is meant to be held constant over the remaining horizon, regardless of the eventual state path; the future output of the optimal controller $(b(s, \lambda_s))_{s=t}^T$ will fluctuate alongside $\lambda_s$.

when the underlier goes down, the subsequent state $\lambda^d$ must be such that $\hat{b}$ is $(\tau - 1)$-achievable:

$$\lambda^d = \left(1 + \frac{(\lambda^{-1} - 1) d}{R + (1 + (\hat{b} - 1)/\lambda)(d - R)}\right)^{-1} \geq \Lambda_{\tau - 1}. \quad (24)$$

Solving the inequality (24) for $\lambda$, we obtain

$$\lambda \geq \Lambda_{\tau - 1} + (1 - \Lambda_{\tau - 1})(\hat{b} - 1) \left(\frac{R}{d} - 1\right), \quad (25)$$

where the right-hand side of (25) is the lowest possible value of $\lambda$ that makes $\hat{b}$ $\tau$-achievable ($= \Lambda_\tau$). Hence, we have the first-order linear difference equation

$$\Lambda_\tau = \theta + (1 - \theta)\Lambda_{\tau - 1}, \quad (26)$$
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where \( \theta := (\hat{b} - 1)(R/d - 1)^3 \) and the initial value \( \Lambda_1 \) was just given above. This is readily solved (cf. with Chiang 1984) to obtain the expression\(^4\)

\[
\Lambda_\tau = 1 - \frac{\overline{b} - \hat{b}}{b - 1}(1 - \theta)^{\tau - 1},
\]

which is the desired result.

In particular, Corollary 1 gives the precise conditions under which we can enforce the aggregate Kelly bet to the end of the horizon.

**Corollary 1** (Enforcement of the aggregate Kelly bet). *If the remaining horizon is \( \tau = T - t \) steps, then over the interval

\[
\lambda \in [\Lambda_\tau(b^*), 1],
\]

where \( \Lambda_\tau(b^*) \) denotes the lowest state at which the aggregate Kelly bet is \( \tau \)-achievable, the value function is given by \( V_t(\lambda) = f(b^*)(T - t) \) and the policy function amounts to

\[
b_t(\lambda) = 1 + (b^* - 1)/\lambda,
\]

which is to say, the investor will expect to achieve the Kelly optimum growth rate over the remaining horizon.

In analogous fashion, Corollary 2 below uses the best \( \tau \)-achievable composite gearing ratio to generate a lower bound for the value of the optimal policy in (16):

**Corollary 2** (Open-loop aggregate gearing policy). *More generally, on a given horizon \( \tau \), if we take the greatest \( \tau \)-achievable aggregate gearing ratio \( \hat{b} \) that is \( \leq b^* \), that is, if

\[
\hat{b} := \min \left\{ b^*, \arg \text{zero } (\Lambda_\tau(z) - \lambda) \right\},
\]

then we have the minorant \( V_t(\lambda) \geq f(\hat{b})(T - t) \), which is the value of adhering to a fixed overall gearing ratio of \( \hat{b} \) over the remaining horizon.

\(^3\)Note that \( \theta \) will lie in \([0, 1]\) if \( \hat{b} \) is admissible and \( \hat{b} \geq 1 \).

\(^4\)(1 - \( \theta \))\(^{\tau - 1} \) solves the associated homogeneous equation, and \( \Lambda_\tau \equiv 1 \) is a particular solution, etc.
To close this section, we elucidate the manner in which the interval of achievability disappears in the continuous time limit; thus, on account of the gearing constraint, we can never guarantee to avoid a “leverage trap” when the trading is continuous in time.

**Example 2.** Consider an \( n \)-step lattice \( L_n := (u_n, d_n, R_n, p_n) \) that is calibrated to match a geometric Brownian motion with a \( (T - t) \)-year horizon: we put \( R_n := e^{r \Delta t} \) and \( d_n := e^{-v \Delta t - \sigma \sqrt{\Delta t}} \), where \( v \) is the geometric drift, \( \sigma \) is the volatility, and \( \Delta t := (T - t) / n \). For this binomial tree, we have

\[
\Lambda_n(\hat{b}) = 1 - \frac{\bar{b} - \hat{b}}{b - 1} \left( \hat{b} - (\hat{b} - 1)e^{-(v-r)\Delta t - \sigma \sqrt{\Delta t}} \right)^{n-1}.
\]

As \( n \to \infty \), we have two competing tendencies vis-à-vis the \( n \)-achievability of \( \hat{b} \). On the one hand we must survive a greater number of downticks without hitting the gearing constraint; on the other, the down factor \( d_n \) is converging to 1. Using L'Hôpital’s rule to resolve the indeterminacy \( 1^\infty \), we have \( \lim_{n \to \infty} \Lambda_n(\hat{b}) = 1 \). Thus, in the continuous time Black and Scholes 1973 market, there is no aggregate gearing ratio \( \hat{b} > 1 \) that can be reliably enforced to the end of the horizon, unless \( \lambda_t = 1 \).
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“Begin with the end in mind. Start with the end outcome and work backwards to make your dream possible.”

—Steven R. Covey, The 7 Habits of Highly Effective People

3 Continuous time model

3.1 Diffusive dynamics

We now presume that the time $t \in [0, T]$ is continuous, with $T$ again denoting the length of the planning horizon, in years. As before, we let $(S_t)_{t \geq 0}$ denote the (diffusion) process that is followed by the price of the underlier; the Markov processes $(x_t)_{t \geq 0}$ and $(y_t)_{t \geq 0}$ will represent the dollar values of the investor’s core and satellite accounts, respectively, at instant $t$, with $M_t$ denoting the aggregate wealth. Thus, we have the relations $M_t = x_t + y_t$ and $d x_t / x_t = d S_t / S_t$, where the underlying index is assumed to follow a process of geometric Brownian motion (cf. with Wilmott 1998):

$$\frac{d S_t}{S_t} = \mu \, dt + \sigma \, d W_t.$$  \hspace{1cm} (31)

Here, $\mu = \mathbb{E}[d S_t / S_t] / dt$ is the expected annual (arithmetic) drift, $\sigma > 0$ is the log-volatility of the market index (i.e., the tangency portfolio), and $d W_t = \epsilon \sqrt{dt}$ is the instantaneous change in a unit Wiener process, which is notated $(W_t)_{t \geq 0}$. Continuing our prior notation, we let $\lambda_t := y_t / (x_t + y_t)$ denote the percentage of the investor’s wealth that is (happily) contained in the satellite, the remaining fraction $1 - \lambda_t$ being marooned in the core. Our feedback control policy is again denoted $b_t = b_t(\lambda_t)$.

If $r$ now denotes the annual rate of continuously-compounded interest in the Black and Scholes 1973 market, then the transition law for the satellite (cf. with Garivaltis 2019;
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Ordentlich and Cover (1998) is

\[ \frac{dy_t}{y_t} = b_t \frac{dS_t}{S_t} + (1 - b_t) r \ dt = (r + (\mu - r)b_t) dt + b_t \sigma \ dW_t, \quad (32) \]

so that, by Itô’s Lemma (cf. with Wilmott, Howison, and Dewynne 1995), we have

\[ d \log y_t = \left( r + (\mu - r)b_t \right) dt + \sigma b_t dW_t. \]

Continuing with the notation \( \hat{b}_t = 1 + \lambda_t (b_t - 1) \) for the agent’s overall gearing ratio at instant \( t \), we have the aggregate capital dynamics

\[ \frac{dM_t}{M_t} = \left( r + (\mu - r)\hat{b}_t \right) dt + \hat{b}_t \sigma dW_t. \quad (34) \]

Thus, the logarithm of the gambler’s fortune has the fundamental decomposition:

\[ \log M_T = \log M_t + r(T - t) + \sigma^2 \int_t^T \left\{ b^* (1 + \lambda_s (b_s - 1)) - \frac{1}{2} (1 + \lambda_s (b_s - 1))^2 \right\} d\lambda_s + \sigma \int_t^T \lambda_s dW_s, \quad (35) \]

where \( b^* := (\mu - r)/\sigma^2 \) is the continuous time Kelly bet (cf. with Garivaltis 2018). Taking the conditional mean from the standpoint of time \( t \), and ignoring the constant terms, we obtain the Kelly bettor’s problem:

\[ J(t, \lambda_t) := \sup_{1 \leq b_t(\bullet) \leq \hat{b}} \mathbb{E}_t \int_t^T \left\{ b^* (1 + \lambda_s (b_s (\lambda_s) - 1)) - \frac{1}{2} (1 + \lambda_s (b_s (\lambda_s) - 1))^2 \right\} d\lambda_s, \quad (36) \]

where \( J(\bullet, \bullet) \) is the maximum value function, and the control policy \( b(s, \lambda) \) is given in feedback form. Note well that that the stochastic process \( (\lambda_t)_{t \geq 0} \) is the only payoff-relevant
state that appears in (36). In passing, we remark that our instantaneous reward function (cf. with Kamien and Schwartz 2012; Kao 2019) in (36) is given by

\[ R(t, \lambda, b) := (\mu - r)\hat{b} - \frac{\sigma^2}{2}\hat{b}^2, \]  

where \( \hat{b} = 1 + \lambda(b - 1) \) is the composite gearing ratio that the investor plans to use in state \((t, \lambda)\). Note well that a “greedy” satellite gearing policy \( b(\lambda) \), which seeks to continuously maximize the instantaneous reward, would amount to using

\[ \arg \max_{1 \leq b \leq \hat{b}} R(t, \lambda, b) = \min \left\{ 1 + \frac{b^* - 1}{\lambda}, \frac{\hat{b}}{b} \right\}. \]  

(38)

The next theorem, which is basic for the entire analysis, gives our fundamental transition law.

**Theorem 2** (Dynamics of the state variable). *The payoff-relevant state \((\lambda_t)_{t \geq 0}\) is a controlled diffusion process that evolves according to the stochastic differential equation

\[ \frac{d\lambda_t}{(b_t - \hat{b}_t)\lambda_t} = \sigma^2(b^* - \hat{b}_t)dt + \sigma dW_t; \]  

the percentage of the investor’s wealth that is held in the satellite is a submartingale (continuously expected to increase) if and only if the aggregate gearing ratio \( \hat{b}_t = 1 + \lambda(b_t - 1) \) is less than or equal to the continuous time Kelly bet.

The drift of \( \log \lambda_t \) is maximized when the satellite bet \( b_t \) is equal to \( 1 + (b^* - 1)/(1 + \lambda) \); the drift of \( \lambda_t \) proper is maximized when \( \hat{b}_t = (1 + (\mu - r)/\sigma^2)/2 \), e.g., when the aggregate gearing ratio is the midpoint of 1 and the Kelly criterion.\(^5\)

**Proof.** Starting from the fact that \( d(\log \lambda_t) = d(\log y_t) - d(\log M_t) \), we substitute the re-

\(^5\) Naturally, the constant gearing ratio \((\mu - r)/(2\sigma^2) = b^*/2\) is referred to as “half-Kelly” (cf. with MacLean, Thorp, and Ziemba 2010). Here, if one seeks to maximize \( \mathbb{E}_t[\lambda_t] \), then he is led to a different paradigm, an aggregate gearing ratio of “half-Kelly and a half”.
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spective expressions for \(d(\log y_t)\) and \(d(\log M_t)\), and simplify, to obtain the relation

\[
d(\log \lambda_t) = (b_t - \hat{b}_t) \left( \mu - r - \frac{\sigma^2}{2} (b_t + \hat{b}_t) \right) dt + \sigma (b_t - \hat{b}_t) dW_t.
\] (40)

Now, since we require the dynamics of the transformed variable \(\lambda_t = e^{\log \lambda_t}\), Itô’s Lemma (viz., the change-of-variables formula) in conjunction with (40) says that

\[
\frac{d\lambda_t}{\lambda_t} = d(\log \lambda_t) + \frac{\sigma^2}{2} (b_t - \hat{b}_t)^2 dt = \sigma^2 (b^* - \hat{b}_t)(b_t - \hat{b}_t) dt + \sigma (b_t - \hat{b}_t) dW_t,
\] (41)

which is the desired result. Note that, since \(b_t\) is always \(\geq \hat{b}_t\), the drift in (41) is positive if and only if the composite gearing ratio is less than the Kelly bet. Finally, maximizing the drift in (40) amounts to the policy \(\arg \max_{b \in \mathbb{R}} \{-1 + \lambda \} = 1 + (b^* - 1)/(1 + \lambda)\); optimizing the drift in (41) gives us \(\arg \max_{b \in \mathbb{R}} \{-\lambda (b - 1)^2 + (b^* - 1)(b - 1)\} = 1 + (b^* - 1)/(2\lambda)\), Q.E.D.

As a natural consequence of the state equation (39), we have our fundamental dynamic program in continuous time:

**Corollary 3** (Kelly bettor’s HJB equation). The maximum value function \(J(t, \lambda)\) in the program (36), which represents the investor’s optimized mean log aggregate final wealth (less \(\log M_t + r(T - t)\)) in the continuation from state \((t, \lambda)\), satisfies the Hamilton-Jacobi-Bellman PDE (cf. with Kamien and Schwartz 2012):

\[
-\frac{\partial J}{\partial t} = \sigma^2 \max_{1 \leq b \leq \bar{b}} \left\{ b^* \hat{b} - \frac{\hat{b}^2}{2} + \lambda (b - \hat{b})(b^* - \hat{b}) \frac{\partial J}{\partial \lambda} + \frac{\lambda^2}{2} (b - \hat{b})^2 \frac{\partial^2 J}{\partial \lambda^2} \right\},
\] (42)

where \(b^* \equiv (\mu - r)/\sigma^2\). The boundary conditions are \(J(T, \lambda) \equiv 0\), \(J(t, 1) = \sigma^2 b^* (T - t)/2\), and \(J(t, 0) = \sigma^2 (b^* - 1/2)(T - t)\); the corresponding boundary conditions for the policy function are \(b(T, \lambda) = \min(1 + (b^* - 1)/\lambda, \bar{b})\), \(b(t, 1) \equiv b^*\), and \(b(t, 0) \equiv \bar{b}\).

The following proposition establishes the basic bounds and monotonicity properties
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Figure 9: The instantaneous drift and diffusivity rates of the percentage of wealth (= $\lambda_t$) held in the satellite, for different (state, control) pairs $(\lambda, b) \in [0, 1] \times [1, \bar{b}]$. This example uses the parameters $(\sigma, b^*, \bar{b}) := (20\% \text{ a year}, 2.75, 4)$. In a given state $\lambda$, the local drift is maximized by using $b(\lambda) = \min(1 + (b^* - 1)/(2\lambda), \bar{b})$, where $b^* = (\mu - r)/\sigma^2$ is the continuous time Kelly bet. This makes the aggregate gearing ratio $\hat{b}$ as close to $(1 + b^*)/2$ as possible; the satellite constraint $\bar{b}$ will interfere if $\lambda$ is sufficiently low.

that obtain for the solution of the HJB equation:

**Proposition 5** (Bounds for the value function). The continuation value $J(t, \lambda)$ is is increasing in both $\lambda$ and in the horizon $\tau = T - t$, and we have the inequalities $\partial J/\partial \tau \geq \sigma^2 (b^* - 1/2)$ as well as

$$\lambda \frac{b^*^2}{2} + (1 - \lambda)(b^* - 1/2) \leq J(t, \lambda) \leq \frac{b^*^2}{2}. \quad (43)$$

**Proof.** In the HJB equation (42), the maximized value of the optimand is greater than or equal to its particular value when set $b = 1$; in this happenstance, we have $\hat{b} = b = 1$, which gives us the bound $\partial J/\partial \tau \geq \sigma^2 (b^* - 1/2)$, so that $J$ is increasing in the horizon $\tau$.

As far as the state variable $\lambda$, let us assume that at time $t$ we are given a special (one-
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off) privilege of transferring some percentage \( \delta > 0 \) of wealth from the core portfolio to the satellite. Such a transfer (from the constrained to the unconstrained account) can never leave the agent worse off, for he has the option, if he so chooses, to replicate the old outcome by earmarking the transferred funds for (1x) buy-and-hold. Say, if we perform a transfer-in-kind of \( \Delta = \delta M_t / S_t \) shares from the core to the satellite, then we have the option to pretend that the satellite wealth at time \( t \) is the counterfactual \( \tilde{y}_t := y_t - \Delta S_t \), that the core wealth is \( \tilde{x}_t := x_t + \Delta S_t \), and that the satellite weight is actually

\[
\tilde{\lambda}_t := \frac{y_t - \Delta S_t}{x_t + y_t}.
\]

(44)

Accordingly, we could (sub-optimally) use the counterfactual gearing ratio \( b(t, \tilde{\lambda}_t) \) and get exactly the same wealth dynamics \( d(x_t + y_t) = d(\tilde{x}_t + \tilde{y}_t) \) that we had before. When we max over all feedback-control strategies as in (36), we will do even better. These considerations prove that \( \partial J / \partial \lambda \geq 0 \).

Finally, as far as bounding the continuation value \( J(t, \lambda) \), we will get a majorant if we dispense with the constraint \( 1 \leq b_t(\lambda) \leq \bar{b} \) in (36) and take an unconstrained maximum. In this case, the optimal control law is \( \hat{b}(\lambda) \equiv b^* \), which leaves us with an optimized value of \( \sigma^2 b^* \tau / 2 \).

To derive the minorant in (43), let us consider the following (sub-optimal) control policy from the standpoint of the state \( (t, \lambda) \): we will simply use the Kelly bet \( b(s, \lambda_s) \equiv b^* \) for the satellite, which will achieve the Kelly growth rate (the aggregate gearing ratio \( \hat{b}_t \) will float stochastically according to the performance of the underlier). This policy, if adhered to over \([t, T]\), will leaves us with a terminal expected payoff in (36) of

\[
\mathbb{E}_t \left[ \log \left( \lambda_t \frac{y_T}{y_t} + (1 - \lambda_t) \frac{S_T}{S_t} \right) \right] - r \geq \lambda_t \mathbb{E}_t \left[ \log \left( \frac{y_T}{y_t} \right) \right] + (1 - \lambda_t) \mathbb{E}_t \left[ \log \left( \frac{S_T}{S_t} \right) \right] - r = \lambda_t \cdot \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 + (1 - \lambda_t) \left( \mu - r - \frac{\sigma^2}{2} \right) \tau = \sigma^2 \tau \left( \lambda_t \frac{b^*^2}{2} + (1 - \lambda_t)(b^* - 1/2) \right),
\]

(45)
where we have used Jensen's inequality. Note that, in keeping with the definition of the objective (36), the initial wealth $M_t$ has been normalized to $1$ and the risk-free rate $r$ has been subtracted from the expected logarithmic return. *A fortiori*, in so far as the best gearing strategy in (36) outperforms all the others, including $b(s, \lambda) \equiv b^*$, the stated lower bound (43) obtains.

On the strength of the HJB equation (42), we can derive a serviceable formula for the policy function in terms of the value function; this relation forms the basis of our policy iteration in numerical software.

**Theorem 3** (Optimal control law). *At an interior solution, the aggregate policy function $\hat{b}(t, \lambda)$ corresponding to (42) is given by*

$$
\frac{\hat{b}(t, \lambda) - 1}{b^* - 1} = \frac{1 + (1 - \lambda)J'_\lambda(t, \lambda)}{1 + 2(1 - \lambda)J'_\lambda(t, \lambda) - (1 - \lambda)^2J''_{\lambda\lambda}(t, \lambda)},
$$

*where $J'_\lambda$ and $J''_{\lambda\lambda}$ are the first two partial derivatives with respect to $\lambda$; and, for such pairs $(t, \lambda)$, the HJB equation (42) simplifies to*

$$
\sigma^{-2} \frac{\partial J}{\partial \tau} = b^* - 1/2 + \frac{b^* - 1}{2}(1 + (1 - \lambda)J'_\lambda(t, \lambda)) \left(\hat{b}(t, \lambda) - 1\right).
$$

*The general policy function is gotten by projecting the expression (46) into the interval $[1, 1 + \lambda(\hat{b} - 1)]$, e.g., we overwrite $\hat{b}(t, \lambda) \leftarrow \min(\hat{b}, 1 + \lambda(\hat{b} - 1))$.

Proof. The optimand in (42), which is quadratic in $(b - 1)$, consists in the expression

$$
b^*(1 + \lambda(b - 1)) - 1/2(1 + \lambda(b - 1))^2 + \lambda(1 - \lambda)(b - 1)(b^* - 1 - \lambda(b - 1))J'_\lambda + \frac{\lambda^2}{2}(1 - \lambda)^2(b - 1)^2J''_{\lambda\lambda};
$$

the coefficient of $(b - 1)^2$ is $\alpha := \lambda^2 \left[(1 - \lambda)^2J''_{\lambda\lambda}/2 - (1 - \lambda)J'_\lambda - 1/2\right]$, and the coefficient of $(b - 1)$ is $\beta := \lambda(b^* - 1) \left[1 + (1 - \lambda)J'_\lambda\right]$. The constant term in the maximand is $\gamma :=$...
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Finite difference solution of the HJB equation

Figure 10: Numerical solution of the HJB equation (42) together with the boundary condition \( J(T, \lambda) \equiv 0 \), for the model parameters \((\sigma, b^*, b, T) := (20\% \text{ a year}, 2.75, 4, 2 \text{ years})\). The number \( \exp(r(T-t) + J(t, \lambda)) \) represents the maximized geometric mean aggregate terminal capital in the continuation problem from state \((t, \lambda)\), assuming an initial wealth of $1.

\( b^* - \frac{1}{2} \). Accordingly, at an interior solution, we have \( b(t, \lambda) = 1 - \beta/(2\alpha) \) and an optimized value of \( \gamma - \beta^2/(4\alpha) = \gamma + (b(t, \lambda) - 1)\beta/2 \), which correspond to formulas (46) and (47), respectively.

The next proposition gives a rigorous confirmation of our inaugural intuition from the dawn of the paper, viz., that without gearing restrictions, the growth-optimal policy is to equalize the composite gearing ratio to Kelly bet. Note that the closed-form solution for the unrestricted case corresponds exactly to the majorant for \( J(\bullet, \bullet) \) that was stated in Proposition 5 above.

**Proposition 6 (Unconstrained solution).** In the absence of a leverage constraint on the satellite (i.e., if \( \overline{b} := \infty \) in (42)), the HJB equation (42) is solved by putting \( b(t, \lambda) = 1 + (b^* - 1)/\lambda \) and

\[
J(t, \lambda) = \frac{\sigma^2 b^*^2}{2} (T-t),
\]

(49)
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The optimal feedback controller in continuous time

![Diagram](image)

(a) feedback controller

(b) open-loop controller

Figure 11: Plot of the optimal policy $b(t, \lambda)$ in (42), for the model parameters $(\sigma, b^*, \tilde{b}) := (20\% \text{ a year}, 2.75, 4)$. At a given concentration of satellite wealth, the correct behavior is to bet more aggressively on shorter horizons; for a given time horizon, one should bet bigger when the satellite contains a lower percentage of the aggregate bankroll. The best terminal policy $b(T, \lambda)$ is the greedy algorithm, whereby we make the composite gearing ratio $\hat{b}(T, \lambda)$ as close to the Kelly rule as possible, subject to the satellite constraint.

which corresponds to a constant aggregate gearing ratio of $\hat{b}(t, \lambda) \equiv b^*$.

Proof. By direct substitution of the purported solution (49) into (42), ignoring the leverage constraint, and simplifying, we get the assertion

$$\frac{\sigma^2 b^*^2}{2} = \sigma^2 \max_{\hat{b} \in (-\infty, \infty)} \left\{ b^* \hat{b} - \frac{\hat{b}^2}{2} \right\},$$

where the aggregate gearing ratio $\hat{b} = 1 + \lambda(b - 1)$ may be chosen freely, since there are supposed to be no restrictions on the value of $b$. The maximum in (50) is attained for $\hat{b} = b^*$, whence the solution is verified.

Somewhat more systematically, we may consider the ansatz $J(t, \lambda) = \phi(t) + \psi(\lambda)$; according to the boundary condition $J(T, \lambda) \equiv 0$, implies that $\psi(\lambda) \equiv -\phi(T)$ for all $\lambda$, so that $\psi'(\lambda) \equiv 0$, so that $J(\bullet)$ must be a function of time alone. Thus, the HJB equation
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reduces to

$$\frac{dJ}{d\tau} = \sigma^2 \max_{b \in (-\infty, \infty)} \left( b^* \hat{b} - \frac{\hat{b}^2}{2} \right) = \frac{\sigma^2 b^2}{2},$$

(51)

so that $J(\tau) = \sigma^2 b^2 \tau / 2$, as promised. The fact that $\hat{b} = b^*$ means that the optimal control policy is $b(t, \lambda) = 1 + (b^* - 1) / \lambda$, thanks to the possibility of unlimited leverage in the satellite.

Remark 1 (Leverage trap). According to Proposition 6, in the absence of leverage restrictions, the growth-optimal policy will increase the satellite gearing ratio in response to a decline in the underlier, and it will decrease leverage when the underlier goes up. The availability of potentially unlimited gearing in the satellite ensures that the overall gearing ratio can always be made equal to the Kelly fraction, regardless of any declines that may occur in the underlying index.

By contrast, a leverage constraint $\overline{b}$ imposes a discipline of careful, conservative, and perspicacious planning: if the investor bets the satellite too recklessly, then he may eventually get trapped in a situation (the saturation boundary $b(t, \lambda) = \overline{b}$) whereby he is not permitted to increase $b$ in response to a downturn in the underlier. These considerations render the HJB equation without any closed form solution, forcing us to solve (42) numerically by working backward in time, just as we did for the Bellman equation (18) on the binomial lattice.

The following proposition confirms our initial disposition in the introduction, which is to say, under no circumstances will an optimal policy prescribe an aggregate gearing ratio that exceeds the Kelly bet.

Proposition 7 (Sanity of the overall gearing level). The value function is concave in the state variable, $\lambda$; consequently, the optimum satellite gearing ratio cannot exceed the prescription of the greedy algorithm, viz., $b(t, \lambda) \leq 1 + (b^* - 1) / \lambda$. The corresponding aggregate gearing ratio $\hat{b}$ will never exceed the Kelly bet. The second-order condition for the dynamic program that occurs in the HJB equation (42) is automatically satisfied.
Proof. From the standpoint of time $t$, let us consider two different satellite weights $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$, and form the convex combination $\eta_t := \alpha \lambda_t^{(1)} + (1 - \alpha) \lambda_t^{(2)}$, for $\alpha \in [0, 1]$. Starting from an initial satellite weight of $\eta_t$, the maximized terminal log-wealth is, by definition, $J(t, \eta_t) + r(T - t)$; this utility will be minorized by that of a specific sub-optimal policy, as follows.

We will imagine that we deposit the fraction $\alpha$ of our wealth into a core-satellite nexus whose satellite weight is $\lambda_t^{(1)}$; the remaining percentage $1 - \alpha$ will be deposited into a different core-satellite portfolio whose state parameter is $\lambda_t^{(2)}$. Now, consider a policy that separately optimizes the geometric mean terminal capital of each distinct core-satellite model. To this end, we let $M_t^{(j)} / M_t^{(1)}$ represent the capital growth factor that model $j$ (for $j \in \{1, 2\}$) achieves over the time interval $[t, T]$, where $M_t^{(j)}$ denotes the total bankroll of model $j$ at time $t$. Such a program would generate an expected utility of

$$
\mathbb{E}_t \left[ \log \left( \alpha \frac{M_t^{(1)}}{M_t^{(1)}} + (1 - \alpha) \frac{M_t^{(2)}}{M_t^{(2)}} \right) \right] \geq \alpha \mathbb{E}_t \left[ \log \left( \frac{M_T^{(1)}}{M_t^{(1)}} \right) \right] + (1 - \alpha) \mathbb{E}_t \left[ \log \left( \frac{M_T^{(2)}}{M_t^{(2)}} \right) \right]
= \alpha J(t, \lambda_t^{(1)}) + (1 - \alpha) J(t, \lambda_t^{(2)}) + r(T - t),
$$

so that $J(t, \eta_t) \geq \alpha J(t, \lambda_t^{(1)}) + (1 - \alpha) J(t, \lambda_t^{(2)})$, and the concavity is proved.\footnote{In a nutshell: the best monolithic controller outperforms any policy that operates two separate core-satellite pools, in particular, the policy that optimizes the geometric mean of each individual pool.} Thus, referring to the policy formula (46), insofar as $\partial J / \partial \lambda \geq 0$ and $\partial^2 J / \partial \lambda^2 \leq 0$, we have $b(t, \lambda) \leq 1 + (b^* - 1) / \lambda$, so that $b(t, \lambda) = 1 + \lambda (b(t, \lambda) - 1) \leq b^*$, as promised. Finally, the second-order condition in the HJB equation (42) is equivalent to the relation $1 + 2(1 - \lambda) J'_\lambda(1 - \lambda)^2 J''_{\lambda\lambda} > 0$, which is rendered true by the concavity of $J(t, \lambda)$.

\[ \square \]

3.2 Open-loop policies

We proceed to consider, from the standpoint of time $t$, open-loop control policies $(b_s)_{s \in [t, T]}$ for the satellite gearing ratio, which policies are deterministic functions of time; the time
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path \( b(s) \) is decided upon at \( t \) and then strictly adhered to over \([t, T]\), without any feedback from the state variable, \( \lambda_s \). This will entail a slight degradation in performance versus the optimal feedback controller (42), and yet, as the next proposition shows, the optimum open-loop controller gains in elegance, tractability, and simplicity.

**Theorem 4** (Constancy of the best open-loop controller). A log-optimal open-loop policy \((b(s))_{s \in [t, T]} \) (that maximizes the geometric mean of time-\(T\) aggregate wealth) is necessarily a constant function \( b(s) \equiv b \).

**Proof.** Integrating the satellite dynamics (32), the expected final log-wealth that accrues to \((b_s)_{s \in [t, T]}\) consists in the expression \( \mathbb{E}_t [\log ((1 - \lambda_t)S_T/S_t + \lambda_t y_T/y_t)] \), where

\[
\frac{y_T}{y_t} = \exp \left( r(T-t) + (\mu - r) \int_t^T b_s ds - \frac{\sigma^2}{2} \int_t^T b_s^2 ds + \sigma \int_t^T b_s dW_s \right) . \tag{53}
\]

Now, the noise term \( \int_t^T b_s dW_s \) is normally distributed with variance \( \int_t^T b_s^2 ds \) (cf. with Björk 2019); thus, putting \( A := \int_t^T b_s ds \) and \( B := \sqrt{\int_t^T b_s^2 ds} \), the exponent in (53) is given by \( r(T-t) + (\mu - r)A - \sigma^2 B^2 / 2 + \sigma B Z \), where \( Z \) is a unit normal deviate. Note that the possible values of \( A \) and \( B \) are constrained by the relation \( B^2 (T-t) \geq A^2 \), e.g., the sample variance of the family of gearing ratios \((b_s)_{s \in [t, T]}\) is non-negative. Hence, since \( A \) contributes positively to the deterministic term of the exponent (53), \( y_T/y_t \) is optimized (in the sense of state-by-state dominance) by using the highest value of \( A \) that is possible for a given \( B \), so that \( A = B \sqrt{T-t} \). This implies that the sample variance of any optimal gearing path must be zero, so that \( b(s) \) is a constant function (equal to its average value \( A/(T-t) \)), Q.E.D.

According to Theorem 4, then, the optimum open-loop controller emerges as the

---

\( ^7 \)To re-iterate, we are dealing with a constant time path \( b : [t, T] \rightarrow \mathbb{R} \), a path that is conditioned on the particular numerical value of \( \lambda_t \); the controller will ignore the future state values \( \lambda_s \), for \( s > t \).
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fixed satellite gearing ratio

\[ b := \arg \max_{1 \leq b \leq \bar{b}} \mathbb{E}_t \left[ \log(\omega_t + \phi(b; Z)) \right], \quad (54) \]

where \( \omega_t := (1 - \lambda_t)/\lambda_t \) is the odds ratio, \( Z \) is a standard normal,

\[ \phi(b; Z) := \exp \left( \sigma^2 \tau \left( b^* (b - 1) - \frac{b^2 + 1}{2} \right) + \sigma \sqrt{\tau} (b - 1) Z \right), \quad (55) \]

and \( \tau := T - t \) is the remaining horizon. The first-order condition that corresponds to (54) can be expressed as

\[ \omega \mathbb{E}_t \left[ \frac{Z}{\omega_t + \phi(b; Z)} \right] = \sigma \sqrt{\tau} (b^* - b) \mathbb{E}_t \left[ \frac{1}{1 + \omega_t \phi(b; Z)^{-1}} \right], \quad (56) \]

where the integration in (56) is against the Gaussian density \( e^{-z^2/2}/\sqrt{2\pi} \).

The following corollary establishes a general theoretical basis for the “overbetting” mentioned in the title of the paper. Such overbetting is proved to be universally valid for all optimum open-loop policies, as distinct from optimal feedback controllers, which can be seen to underbet the satellite in certain situations, albeit for absurd values of \( b^* \) and \( \sigma \).

**Corollary 4 (Overbetting theorem).** Assuming that \( b^* < \bar{b} \), the growth-optimal open-loop policy \( \mathbb{b} \) is always greater than the Kelly bet; in fact, the geometric mean (54) is strictly increasing in \( b \) for \( b \in (-\infty, b^*] \).

**Proof.** In the first-order condition (56), the right-hand side is non-negative if \( b \leq b^* \); we proceed to show that the left-hand side is negative for all \( b \in \mathbb{R} \). To this end, note that

\[ \int_{-\infty}^{\infty} \frac{ze^{-z^2/2}}{\omega_t + \phi(b, z)} \, dz = \int_0^{\infty} \frac{e^{-\xi}}{\omega_t + \phi(b, \sqrt{2} \xi)} \, d\xi - \int_0^{\infty} \frac{e^{-\xi}}{\omega_t + \phi(b, -\sqrt{2} \xi)} \, d\xi, \quad (57) \]

\( ^8 \)The odds that a dollar of the investor’s capital lies in the core portfolio.
Figure 12: An essential counterexample, based on the parameters $(\sigma, b^*, \overline{b}) := (150% \text{ a year}, 3, 4)$, depicting both 10- and 30-year horizons. The (closed-loop) satellite gearing ratio need not be decreasing in $\lambda$, and it can lie below the Kelly bet if the underlier is especially volatile. However, the aggregate gearing ratio never exceeds the Kelly fraction, as we proved above.

where we have made the substitution $\xi := z^2/2$. Since $\phi(b, z)$ is increasing in $z$, the subtrahend in (57) has a smaller denominator than the minuend, hence is bigger, so that the difference is negative, Q.E.D.

### 3.3 More conservative objectives

On account of the fact that $\log M_T = \log x_T - \log(1 - \lambda_T)$, the semi-static Kelly betting problem can be reformulated as a terminal control problem, or a *problem of Mayer* (cf. with Chiang 1992; Bliss 1963), whereby the instantaneous reward is zero and we seek to optimize the pure bequest:

$$\mathfrak{Y}(t, \lambda_t) := \sup_{1 \leq b_t(\bullet) \leq \overline{b}} \mathbb{E}_t[-\log(1 - \lambda_T)|\lambda_t].$$

(58)
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Problem of Mayer: risk preferences over the final capital share

Figure 13: Kelly bettor’s risk preferences vis-à-vis the terminal satellite weight, $\lambda_T$. The equivalent bequest function ($\lambda \mapsto -\log(1 - \lambda)$) is risk-loving, albeit in service of risk aversion over the aggregate bankroll; the implied negative risk aversion gets amplified as $\lambda_T$ approaches 100%.

Now, once can raise the objection that the Kelly bettor (who is risk-averse with respect to his total capital) happens to have risk-loving preferences over the distribution of the terminal satellite share, $\lambda_T$. This consideration leads us to the more conservative objective of maximizing the mean terminal share:

$$K(t, \lambda_t) := \sup_{1 \leq b_1(\bullet) \leq \bar{b}} \mathbb{E}_t [\lambda_T | \lambda_t].$$  \hspace{1cm} (59)

The corresponding HJB equation consists in (cf. with Kamien and Schwartz 2012)

$$-\frac{\partial K}{\partial t} = \sigma^2 (1 - \lambda) \max_{1 \leq b \leq \bar{b}} \left\{ (b - 1)(b^* - 1 - \lambda(b - 1)) \frac{\partial K}{\partial \lambda} + \frac{\lambda}{2} (1 - \lambda)(b - 1)^2 \frac{\partial^2 K}{\partial \lambda^2} \right\},$$  \hspace{1cm} (60)

whereby the boundary values are $K(T, \lambda) = \lambda$, $K(t, 0) = 0$, and $K(t, 1) = 1$. Note that we have $K(t, \lambda) \geq \lambda$, since the policy $b(s, \lambda) \equiv 1$ guarantees that $\lambda_T = \lambda_t$ with certainty.

Based on the same considerations that were used above, we have the fact that $K(t, \bullet)$
is concavely increasing in the state variable. *Mutatis mutandis*, any transference of capital from the core to the satellite (viz., an increase from $\lambda_t^{(1)}$ to $\lambda_t^{(2)}$) could always be treated as a transfer-in-kind of a certain number of shares of the underlier, and we have the option to earmark these shares for buy-and-hold, exactly replicating the counterfactual outcome, for an expected terminal share of $K(t, \lambda_t^{(1)})$. Inasmuch as $K(t, \lambda_t^{(2)})$ is the best possible, the monotonicity of $K(t, \bullet)$ follows.

As far as the concavity, if $\eta_t := \alpha \lambda_t^{(1)} + (1 - \alpha) \lambda_t^{(2)}$ is the current satellite weight, then we have the option to behave as if the fraction $\alpha$ of our capital is invested in a core-satellite nexus with satellite share $\lambda_t^{(1)}$, and the remaining $1 - \alpha$ is invested in a model with state $\lambda_t^{(2)}$. If we wanted to, we could separately optimize the mean terminal share in each model $j \in \{1, 2\}$, achieving an expected overall share of

$$E_t[\eta_T] = \alpha E_t[\lambda_T^{(1)}] + (1 - \alpha) E_t[\lambda_T^{(2)}] = \alpha K(t, \lambda_t^{(1)}) + (1 - \alpha) K(t, \lambda_t^{(2)}) \leq K(t, \eta_t), \quad (61)$$

hence the concavity, since, by definition, $K(t, \eta_t)$ is the maximum achievable value of the expected terminal state. Thus, the optimand in (60) is concave quadratic in $(b - 1)$, with leading coefficient $\lambda ((1 - \lambda)K''/2 - K') < 0$, so that the second-order condition is always fulfilled.

Our working formula for an interior policy is now

$$\hat{b}(t, \lambda) = 1 + \frac{b^* - 1}{2 - (1 - \lambda)K''/K'} \leq \frac{1 + b^*}{2}, \quad (62)$$

so that the aggregate gearing ratio will never exceed the midpoint of one and the Kelly rule (“half-Kelly and a half”); this corresponds to the relation $b(t, \lambda) \leq 1 + (b^* - 1)/(2\lambda)$. Substitution of the optimal controller (62) into the HJB equation gives us

$$\frac{\partial K}{\partial \tau} = \frac{\sigma^2}{2} (1 - \lambda)(b^* - 1)(\hat{b} - 1) \frac{\partial K}{\partial \lambda}; \quad (63)$$
Exact HJB solution without gearing constraints

Figure 14: Illustration of Theorem 5 for the parameters \((\sigma, b^*) := (20\%, 2.75)\). Without gearing constraints, the expected final satellite share is saturated by maintaining an aggregate gearing ratio that is the midpoint of 1 and the Kelly rule (i.e., “half-Kelly and a half”). The minorant \(K(t, \lambda; \bar{b}) \geq \lambda\) obtains because the policy \(b(s, \lambda) \equiv 1\) guarantees that \(\lambda_T = \lambda_t\).

Note that the terminal value is increasing in the horizon \((\partial K/\partial \tau \geq 0)\) since, in response to any increase \(\Delta T\), we have the option to preserve the terminal state by setting \(b(t, \lambda) \equiv 1\) for all \(t \in (T, T + \Delta T]\).

In the theorem given below, we derive a closed-form solution of the modified HJB equation (60) when it is free of gearing constraints; this spectacle is the raison d'être for our safer “Half-Kelly and a half” rule of thumb.

**Theorem 5** (Half-Kelly and a half). In the absence of a gearing constraint \((\bar{b} := \infty)\), the dynamic program (60) is solved by putting \(b(t, \lambda) = 1 + (b^* - 1)/(2\lambda), \bar{b} \equiv (1 + b^*)/2, \) and \(K(t, \lambda) = 1 - (1 - \lambda) \exp\left(-\sigma^2(b^* - 1)^2 \tau /4\right),\) which majorizes the value function (59) for all \(\bar{b}\).

**Proof.** The stated formulas can be verified by direct substitution into (60); a more sys-
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A systematic approach starts with the ansatz \( \hat{b} = (1 + b^*)/2 \). Juxtaposing this with the interior policy (62) implies that \( K''_{\lambda} \equiv 0 \), so that \( K(t, \lambda) = A(t) + B(t)\lambda \), where \( A(\cdot) \) and \( B(\cdot) \) are arbitrary time functions; on account of the boundary condition \( K(T, \lambda) = \lambda \), the corresponding function values at the terminus must be \( A(T) = 0 \) and \( B(T) = 1 \), respectively. Now, substituting the linear form \( K(t, \lambda) = A(t) + B(t)\lambda \) into (63), we get a pleasant system of ODEs in \( A(t) \) and \( B(t) \), whereby we integrate the relation \( dB/dt = -\sigma^2 (b^* - 1)^2 B(t)/4 \), and the result follows.

As a corollary to Theorem 5, we get a practical bound on the chances of ever reaching a prespecified satellite weight.

**Corollary 5** (Hitting a concentration target). *If the investor always keeps his aggregate gearing ratio below the Kelly bet, then, from the standpoint of time \( t \), the probability of breaching a given concentration target \( \lambda \) before the termination date is at most \( K(t, \lambda)/\lambda \), that is,

\[
\mathbb{P}\left\{ \max_{t \leq s \leq T} \lambda_s \geq \lambda \right\} \leq \frac{1 - (1 - \lambda) \exp (-\sigma^2 (b^* - 1)^2 (T - t)/4)}{\lambda}.
\]

(64)

*Proof.* Since the process \((\lambda_t)_{t \geq 0}\) is a submartingale, Doob's martingale inequality (cf. with Evans 2013) implies that

\[
\text{Prob}\left\{ \max_{t \leq s \leq T} \lambda_s \geq \lambda \right\} \leq \frac{\mathbb{E}_t [\lambda_T | \lambda_t]}{\lambda} \leq \frac{K(t, \lambda)}{\lambda},
\]

(65)

since, by definition, \( K(t, \lambda) \) is the greatest possible value of \( \mathbb{E}_t [\lambda_T | \lambda_t] \) over all control policies, in particular, those that render \((\lambda_t)_{t \geq 0}\) a submartingale.

As a practical use case of Corollary 5, we have the following illustrative example.

**Example 3.** For the parameters \((\lambda, T-t, \sigma, b^*, \lambda_t) := (75\%, 5 \text{ years}, 20\% \text{ a year}, 2.75, 30\%)\), the expected percentage of wealth in the satellite 5 years hence is at most 39.9%; the probability of the satellite concentration breaching 75% within 5 years is at most \( .399/.75 = 53.2\% \).
Figure 15: Garivaltis’ controllers for the satellite gearing ratio, illustrated for the sample parameters \((σ, b^*, \tilde{b}, τ) := (20\% \text{ a year}, 2.75, 4, 10 \text{ years})\). The open-loop expected share optimizer is stylistically the most careful: it is a reliable, path-independent technique for expanding the investor’s long-term access to leverage. Note that the geometric mean maximizing controls (both open- and closed-loop) are much more cavalier about what happens to the satellite, so long as the expected aggregate CAGR to the end of the horizon is as high as possible.

To close the paper, we present our main theorem on the stochastic ordering of open-loop leverage controllers, in the context of the Mayer problem (59). Among other things, Theorem 6 gives the clear-cut conditions under which a pair of open-loop policies admits a stochastic dominance relation vis-à-vis an agent who is risk-averse over the final satellite weight.

**Theorem 6** (Open-loop control of the terminal capital share). The **open-loop policy that maximizes the conditional median of \(λ_T\) is precisely the Kelly bet \(b(t, λ) \equiv b^*\). If \(F^b(l) := \text{Prob}(λ_T \leq l|λ_t; b)\) denotes the conditional CDF of the terminal satellite share under the open-loop policy \(b\), then no two distinct members of the family of distributions \(\left\{ F^b(\bullet) \right\}_{b \in [1, \infty)}\) are comparable via first-order stochastic dominance (cf. with Mas-Colell, Whinston, and Green 1995). If \(b\) is the open-loop policy that optimizes \(\mathbb{E}_t[λ_T|λ_t]\), then any riskier open-loop policy \(c > b\) is such that \(F^c(\bullet)\) is second-order stochastically dominated by \(F^b(\bullet)\).
STOCHASTIC ORDERING OF OPEN-LOOP POLICIES

(a) cdf of final satellite share

(b) area under cdf of satellite share

Figure 16: Illustration of the conditional CDF $F^b(l)$ (●) and the cumulative area, $\int_0^l F^b(l) \, dl$ of the end satellite share, for $b \in \{2, 3, 4\}$ and the example parameters $(\sigma, b^*, \tau, \lambda_t) := (20\% \text{ a year}, 2.75, 5 \text{ years}, 40\%)$. Because of the single-crossing property, no two such CDFs are comparable via first-order stochastic dominance (cf. with Rothschild and Stiglitz 1970); however, in keeping with Theorem 6, $F^4(●)$ is second-order stochastically dominated by the other two. Here, $E_t[\lambda_T | \lambda_t]$ is maximized by the open-loop policy $b = 2.81$. Note that the CDFs that correspond to lower (open-loop) gearing ratios always start lower and end higher.

Proof. We start with the condensed formula

$$\frac{1}{\lambda_T} = 1 + \omega_t \exp \left( \sigma \sqrt{\tau} (b - 1) \left( \sigma \sqrt{\tau} \left( \frac{b + 1}{2} - b^* \right) + Z \right) \right),$$

where $b$ is a constant satellite gearing ratio, $\omega_t := (1 - \lambda_t)/\lambda_t$ is the odds ratio, $\tau := T - t$ is the remaining horizon, and $Z$ is a standard normal deviate. If we take the left-tail event $\{\lambda_T \leq l\}$, and solve the inequality for $Z$ (bearing in mind that $b > 1$), we get the equivalent event

$$-Z \leq \frac{\log (\omega_t/(l^{-1} - 1))}{\sigma \sqrt{\tau} (b - 1)} + \sigma \sqrt{\tau} \left( \frac{b + 1}{2} - b^* \right),$$

so that

$$F^b(l) = N \left( \frac{\log (\omega_t/(l^{-1} - 1))}{\sigma \sqrt{\tau} (b - 1)} + \sigma \sqrt{\tau} \left( \frac{b + 1}{2} - b^* \right) \right),$$

where $N$ denotes the cumulative standard normal distribution.
where $N(\bullet)$ denotes the cumulative normal distribution function. Now, the conditional median $l_{\text{med}}$ of $\lambda_T$ is defined by the equation $F^b(l_{\text{med}}) = 50\%$; using (68) to solve for $l_{\text{med}}$, we obtain the expression\(^9\)

$$
\frac{1}{l_{\text{med}}} = 1 + \omega_1 \exp \left( \sigma^2 \tau (b - 1) \left( \frac{b + 1}{2} - b^* \right) \right).
$$

The optimizer of $l_{\text{med}}(b)$ is now seen to consist in

$$
\arg \min_{b \in [1, \infty)} \left( b^2 / 2 - b^* b + b^* - 1/2 \right) = b^*,
$$

as promised.

Next, we note that the family of CDFs $F^b(\bullet)_{b \in [1, \infty)}$ has a single-crossing property, viz., there is a unique solution $l = l_{\text{cross}}(b, c)$ to the equation $F^b(l) = F^c(l)$ whenever $b \neq c$. In fact, this solution is given by

$$
l_{\text{cross}}(b, c) = \frac{1}{1 + \omega_1 \exp (-\sigma^2 (b - 1)(c - 1) \tau / 2)} \in (0, 1),
$$

so that the CDFs $F^b(\bullet)$ and $F^c(\bullet)$ cannot be compared by first-order stochastic dominance. That is, the CDF corresponding to the lower gearing ratio starts lower and ends higher, i.e., if $b \leq c$ then $F^b(l) \leq F^c(l)$ if $l \leq 1/(1 + \omega_1)$. Thus, if $b < c$ then it is not possible for $c$ to second-order stochastically dominate $b$; $b$ will SOSD $c$ if and only if

$$
\int_0^1 F^b(l)dl \leq \int_0^1 F^c(l)dl,
$$

since

$$
\int_0^L (F^c(l) - F^b(l))dl = \int_0^{l_{\text{cross}}(b, c)} (F^c(l) - F^b(l))dl + \int_{l_{\text{cross}}(b, c)}^L (F^c(l) - F^b(l))dl \geq \int_0^1 (F^c(l) - F^b(l))dl
$$

for all $L \in [l_{\text{cross}}(b, c), 100\%]$.

\(^9\)The 50th percentile of $\lambda_T$ corresponds to putting $Z = 0$ in (66).
Hence, if the open-loop policy

\[ b := \arg \min_{1 \leq b \leq \bar{b}} \int_0^1 F^b(l)dl \]  \hspace{1cm} (72)

is such that the total area under \( F^b(\bullet) \) is minimized, then, \( a \text{ fortiori} \), any riskier open-loop policy \( \gamma > b \) will be second-order stochastically dominated by \( b \), since \( \int_0^1 (F^\gamma(l) - F^b(l))dl \geq 0 \). Finally, applying integration by parts to the minimand (72), we have \( \int_0^1 F^b(l)dl = 1 - \int_0^1 l \cdot dF^b(l) = 1 - \mathbb{E}_t[\lambda_T|\lambda_t] \), so that minimizing the area under \( F^b(\bullet) \) is equivalent to maximizing the conditional mean of the final satellite share, Q.E.D.

4 Epilogue

This article formulated, studied, and solved a practical, semi-static log-optimal investment problem that was found to be naturally occurring in actual life. We considered the control problem of an investor who has the bulk of his funds (the "core" portfolio) in unlevered, buy-and-hold (1x) retirement plans that have 100% of capital invested in the market index, with no access to margin loans or other sources of leverage. On the other hand, the agent has a satellite account that does have access to leverage, albeit a limited amount: specifically, there is a hard upper bound (which we call \( \bar{b} \)) on the satellite gearing ratio. Say, \( \bar{b} = 3 \) is a natural value when the satellite gearing is being provided by a 3x leveraged ETF; \( \bar{b} = 4 \) might obtain in a margin account that requires 25% maintenance margin.

In the backdrop of a geometric Brownian motion process that is followed by the market index (similarly, on a calibrated binomial lattice), if we are unhampered by borrowing constraints, then it becomes arithmetically possible to engineer whatever composite gearing ratio is desired (here denoted \( \hat{b} \)), by trading continuously in time and cranking up the satellite bet ("\( b \)"") as needed, thereby raising the satellite gearing ratio in commen-
surate response to every downtick of the underlier. In this connection, we showed that a Kelly bettor should dynamically adjust $b$ so as to equalize his composite gearing ratio with the Kelly 1956 bet, a numerical parameter $b^*$ which we regard as a parsimonious and characteristic feature of the tangency portfolio, along with its annual volatility, $\sigma$.

However, under realistic and practical conditions, whereby the satellite gearing ratio is band-limited, the most optimum, perfect, and perspicacious control strategy becomes kaleidoscopically more interesting in all of its byways and ramifications. Accordingly, we penetrated the general controlled diffusion problem by working backward from a fixed calendar date $T$, measured in years, the length of the remaining horizon (called $\tau$) being a critical factor in the optimal variations of countermove. We demonstrated carefully that the payoff-relevant state is uni-dimensional, viz., it consists in the percentage of the gambler's fortune that resides in the satellite, which we styled as “$\lambda$”. That is, if one's objective is to optimize the expected CAGR to the end of the horizon (the actual, realized CAGR being of course, random), then the numerical wealth level (“$M_t$”) is unimportant, once we know the internal capital distribution. Deducing the controlled dynamics of the state variable, we found that its drift parameter is a quadratic function of the state-control pair; its coefficient of diffusivity happens to be quartic in the state variable, and yet, it again presents itself as a tractable, second-degree polynomial in the satellite bet.

Thus, we hit upon the fundamental HJB equation (42) that expresses the general recursive optimization scheme of a Kelly bettor; we unraveled its qualitative properties and used a finite-difference scheme to give computational solutions in numerical software. We proved that the corresponding value function is concavely increasing in the state variable, a proposition which is model-independent: it holds good for completely arbitrary stochastic dynamics of the underlying index. As a result, in no circumstance will the prescribed aggregate gearing ratio of the optimal feedback controller ever exceed the Kelly bet. For typical parameter values, it will also transpire that the satellite bet exceeds the Kelly fraction (“overbetting”), and in fact, the policy function will be decreas-
ing in the state variable, giving an intuitive instance of substitutability between satellite leverage and satellite wealth. However, as we showed in a key counterexample above, such monotonicity is not universally valid over the whole parameter space: if the underlying volatility is (absurdly) high, the Kelly bet being held constant, it becomes possible for the policy function to increase in lockstep with the state variable, and it can accordingly lie below the stationary Kelly criterion, $b^*$.

Having constructed the optimal (geometric mean maximizing) feedback controller in earnest, we proceeded to build robustly simple open-loop controllers for the same task, strategies which fix in advance a certain time path for the satellite leverage ratio, and adhere to it over the remaining horizon, without any additional feedback or input from the state variable. On this score, we showed that the best possible open-loop controller must in fact correspond to a constant satellite gearing ratio, $b$, which policy is guaranteed to exceed the Kelly bet in all possible situations. Based on a slew of revealing computational examples, we concluded that the performance value of the best open-loop policy is a splendid approximation to that of the optimal feedback control law, in spite of its path-independence and its reduced algorithmic complexity.

All this being done, we found it illuminating to re-formulate our main programme (36) as a problem of Mayer (e.g., a terminal control problem), rather than our original setup, which constitutes a Lagrange problem. That is, our semi-static Kelly gambler can be taken to optimize a pure bequest in the amount of $-\log(1 - \lambda)$ utils, where $\lambda$ is the terminal value of the state variable. Thus, although the growth-optimal investor is (naturally) risk-averse with respect to his aggregate terminal capital, he is seen to be risk-loving with respect to the final satellite weight. This convex attitude obtains on account of the fact the log-optimal controller, by definition, cares only about the investor’s geometric mean terminal capital, properly averaged over the entire ensemble of sample paths; and there is a certain magnitudinal asymmetry between the “successful” paths (on which the satellite multiplies its grubstake many times over), and the unsuccessful
ones, on which the satellite may perform so poorly that it gets written off entirely.

Hence, since an actual investor must grapple with a single sample path for the entirety of his lived experience (rather than the ensemble mean CAGR that we have so strenuously optimized above), it is sensible to consider slightly more conservative objectives, which modified preferences ensure a more reliable outperformance of the satellite over the core. This insight leads us to adopt a risk-neutral stance vis-à-vis the ending satellite weight, the new maximum value function corresponding to the optimum conditionally expected terminal satellite share that can be achieved subsequent to a particular state and time. In this connection, the solution to the Hamilton-Jacobi-Bellman PDE is again found to be increasing and concave in the state variable; we proved that the optimal betting scheme will never allow the composite gearing ratio to exceed the midpoint the Kelly rule and 100%, a rule of thumb which we refer to as “half-Kelly and a half.” In the unrestricted case, then, we discovered that the HJB equation admits a gratifying closed-form solution, which dynamically operates the satellite so as to maintain a continuously fixed aggregate gearing multiplier in the amount of $(1 + b^*)/2$.

Finally, we considered open-loop controllers of the expected terminal share $(\lambda_T)$; this led to a family of cumulative distribution functions that happen to obey a certain single-crossing property, which property implies that no two of these probability laws (that are generated by open-loop policies) can be compared by the notion of first-order stochastic dominance. As far as the central tendency of the ending satellite weight, we derived the fact that the conditional median of $\lambda_T$ is continuously optimized by a constant satellite gearing policy that is equal to $b^*$. The last theorem of the paper says that the particular open-loop policy which maximizes the conditional mean of the final satellite weight is in fact the most aggressive open-loop controller whose terminal satellite share is not second-order stochastically dominated in the sense of Rothschild and Stiglitz 1970. The Kelly bettor is unfazed by all of this, of course, since he is consciously competing on the basis of the expected compound growth rate of the entire bankroll, regardless of what
happens to the individual components. Go big, or go home.

*Knee-deep in the inflation, Summer 2022.*

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A Software implementation

A.1 Closed-loop controller

The MATLAB® function file directHJB.m, supplied below, solves our main HJB equation \((42)\) using a backward Euler finite-difference scheme, and returns a pair of matrices \([J, pol]\) representing the value function \(J(\tau, \lambda)\) and the Kelly feedback controller \(b(\tau, \lambda)\) on a uniform grid with \(m\) subdivisions on the spatial axis and \(n\) subdivisions on the time axis. In order to get an accurate solution, \(n\) must be large in relation to \(m\).

To illustrate, the function call

\[
[J, pol] = \text{directHJB}(10, 100, 10000, .2, 2.75, 4);
\]

corresponds to a 10-year horizon, \(\Delta \lambda := 1\%\), \(\Delta t := 0.001\) years, 20\% annual volatility, a
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Kelly bet of 2.75, and a maximum satellite gearing ratio of 4x. The \((i, j)\)-entries of the returned matrices correspond to the grid points \((\lambda(i), t(j))\), where \(\lambda(i) := (i - 1)\Delta\lambda\) and \(t(j) := (j - 1)\Delta t\), for \(i = 1, 2, ..., m + 1\) and \(j = 1, 2, ..., n + 1\).

\[
\% \text{directHJB.m} \\
\% \text{------------------------------------------------------------} \\
\% \text{T is the horizon in years. There are m sub-divisions on the} \\
\% \text{lambda axis, and n sub-divisions on the time axis. sigma is} \\
\% \text{the annual volatility of the underlier, bstar is the Kelly} \\
\% \text{bet, and bmax is the satellite gearing constraint.} \\
\% \text{The function returns and plots (m+1)x(n+1) matrices} \\
\% \text{containing the value function J and the policy function pol,} \\
\% \text{respectively. n should be large in relation to m.} \\
\% \text{------------------------------------------------------------} \\
\]

\[
\text{function} \ [\text{J, pol}] = \text{directHJB(T, m, n, sigma, bstar, bmax)} \\
\]

\[
\% \text{precompute loop constants} \\
bsh = bstar -.5; \\
bs1 = bstar -1; \\
hbs1 = .5*bs1; \\
mm = m*m; \\
hm = .5*m; \\
dt = T/n; \\
sst = sigma^2*dt; \\
lambda = (1/m)*(0:m)'; \\
bsl = bsl./lambda; \\
L = 1-lambda; \\
LL = L.^2; \\
bm1 = bmax - 1; \\
lbh = lambda.*L*bm1; \\
bmh = 1+lambda*bm1; \\
A = bmh.*(bstar - .5*bmh); \\
B = lbh.*(bstar - bmh); \\
C = .5*lbh.^2; \\
\]

\[
\% \text{initialize boundary values of J and pol} \\
J = [sst*bsh*(n :-1: 0); zeros(m-1, n+1); .5*sst*bstar^2*(n :-1: 0)]; \\
pol = [bmax*ones(1, n+1); zeros(m-1, n+1); bstar*ones(1, n+1)]; \\
pol( :, n+1) = min(bmax, 1+bsl); \% \text{greedy policy (horizon-0)} \\
\]
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\% populate J and pol backward in time
\% j=n+1 is horizon-0, j=1 is horizon-T
\% i=1 is lambda=0, i=m+1 is lambda=1

for j=n:-1:1
    for i=2:m

        \% derivatives of the value function
        JP=hm*(J(i+1,j+1)-J(i-1,j+1)); \% J'
        LJP=L(i)*JP;
        JPP=mm*(J(i+1,j+1)+J(i-1,j+1)-2*J(i,j+1)); \% J''

        \% satellite gearing policy
        b=min(bmax,max(1,1+bsl(i)*(1+LJP)/(1+2*LJP-LL(i)*JPP)));
        pol(i,j)=b;

        if b==bmax
            \% corner solution
            slope=A(i)+B(i)*JP+C(i)*JPP;
        else
            \% interior solution
            slope=bsh+hbs1*(1+LJP)*lambda(i)*(b-1); \% simplest formula
        end
        \% latest boundary values of J
        J(i,j)=J(i,j+1)+slope*sst; \% backward Euler
    end
end

\% plot the value and policy functions
X=lambda*ones(1,n+1);
Y=ones(m+1,1)*(0:n)*dt;
figure
mesh(X,Y,J); \% plot value function
legend('value function')
figure
mesh(X,Y,pol); \% plot policy function
legend('policy function')
end
A.2 Open-loop controller

In a similar vein, the MATLAB® procedure garivaltisOpenLoop.m returns the best open-loop controller (54) for all state-horizon pairs \((\lambda(i), \tau(j))\) on an equally-spaced mesh with \(m\) points on the \(\lambda\)-axis and \(n\) points on the \(\tau\)-axis. Here, it is no longer essential that \(\Delta \tau\) be small in relation to \(\Delta \lambda\), since we are independently optimizing the \(m \times n\) utilities (54) for \(i = 1, 2, ..., m\) and \(j = 1, 2, ..., n\). For example, the function call

\[
b = \text{garivaltisOpenLoop}(10, 100, 100, .2, 2.75, 4);
\]

provides us proper coverage of the domain \([0, 1] \times [0, 10]\).

% garivaltisOpenLoop.m
% -------------------------------------------------------------
% T is the horizon in years. There are m points on the lambda
% axis, and n points on the time axis. sigma is the annual
% volatility of the underlier, bstar is the Kelly bet, and
% bmax is the satellite gearing constraint.
% The function returns and plots an mxn matrix containing the
% best open-loop policy for each given (state,horizon) pair.
% The accuracy of the computation does not require small
% subdivisions (viz., large values of m and n), as it does for
% the solution of the HJB equation, since we are solving
% optimization problems that are independent of each other.
% Accurate gearing ratios \(b(i,j)\) will be returned for the
% specified uniform grid.
% -------------------------------------------------------------

function b = garivaltisOpenLoop(T,m,n,sigma,bstar,bmax)

    % initialize axes
    lambda = linspace(0,1,m)';
    tau = linspace(0,T,n);

    % precompute utility parameters
    st = sigma*sqrt(tau);
    stau = sigma^2*tau;
    bsh = bstar -.5;

    ...
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% expected utility of a satellite gearing ratio
function u=util(g)
    C=(g-1)*(bsh-.5*g)*stau(j);
    D=(g-1)*st(j);
    % unit normal density, 4 s.d. in each direction
    u=-integral(@(Z)log(1+lambda(i)*(exp(C+D*Z)-1)).*exp(-.5*Z.^2),-4,4);
end

% initialize open-loop policy matrix
b=[bmax*ones(1,n);zeros(m-2,n);bstar*ones(1,n)];
b(:,1)=min(bmax,1+(bstar-1)./lambda);

% i=state index, j=horizon index
% entries of the policy matrix are filled independently
for i=2:m-1
    for j=2:n
        b(i,j)=fminbnd(@util,bstar,bmax); % max expected utility
        % optimum is known to be >= bstar
        display([num2str(floor(100*(i-1)/(m-2))) '% done']) % echo progress
    end
end
figure
mesh(lambda*ones(1,n),ones(m,1)*tau,b) % plot policy matrix
legend('open-loop policy')
end

A.3 Simulations

This final subsection provides two simulations of the behavior of the feedback controller
that we implemented above in directHJB.m. Figure 17 plots a sample path for the parameter vector
(σ, r, ν, b̄₀, λ₀, S₀) := (18% a year, 2.5% a year, 9% a year, 4, 50%, $100) over
a decade-long horizon, where ν = μ − σ²/2 is the asymptotic CAGR of the market index,
and b* = 2.51 is the associated Kelly bet.

In exhibit 17a, we have compared the controller’s aggregate bankroll to the performance
of the underlier itself (i.e., the accumulation of the core portfolio), and its geometric mean path g(t) := S₀e⁻vt. Panel 17b plots the evolution of the control law b(t, λₜ)
and the composite gearing ratio $\hat{b}(t, \lambda_t)$ for the same Monte Carlo trial. Along this (favorable) path of the underlier, the satellite eats the overall portfolio, and the aggregate gearing ratio converges rapidly to the Kelly criterion. Note how the satellite bet remains elevated for many years.

For the sake of contrast, Figure 18, which uses the same parameters, depicts a “failed” sample path whereby the underlying did not rise enough (and not in the right temporal order); as a result, although the market index nearly doubled, the satellite underperformed badly, achieving a CAGR of only 1.13% continuous, amid a violent storm of gut-wrenching volatility. Frame 18b gives us a vivid picture of the dynamically optimized CAGR expectations. Say, at the 5-year mark, the ensemble average CAGR to the end of the horizon was north of 12.6%, and practically equal to the Kelly growth rate; but there was no such luck for this particular outcome of the experiment.
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Simulation #1

(a) Sample path of \((S_t, M_t)_{t \in [0,T]}\). The geometric mean path of the underlier is \(g(t) := S_0 \exp(\nu t)\).

(b) The optimal control policies \(b(t, \lambda_t)\) and \(\hat{b}(t, \lambda_t)\) along the same sample path. The satellite constraint is \(\bar{b} := 4\).

Figure 17: Simulation of the model over a 10-year horizon for the parameters \((\sigma, r, \nu, \bar{b}, \lambda_0, S_0) := (18\% \text{ a year}, 2.5\% \text{ a year}, 9\% \text{ a year}, 4, 50\%, \$100)\). Here, \(\nu = \mu - \sigma^2 / 2\); the Kelly bet for this market is \(b^* = 2.51\).
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SIMULATION #2

(a) 10-year sample path of \((x_t, y_t)_{t \in [0, T]}\), where \(x_0 = y_0 = 50\). The two accounts start at parity, but the satellite gets whipsawed and realizes a continuous CAGR of only 1.13%.

(b) Evolution of the investor’s optimized expected aggregate CAGR to the end of the horizon, \( r + J(t, \lambda_t) / (T - t) \), where \( J(t, \lambda_t) \) is the value function (N.B. the actual CAGR is random).

Figure 18: A less favorable sample path of the same model: \((T, \sigma, r, v, \bar{b}, \lambda_0) := (10 \text{ years}, 18\% \text{ a year}, 2.5\% \text{ a year}, 9\% \text{ a year}, 4, 50\%)\). The Kelly growth rate for this market is 12.68%; such would be achievable under unlimited satellite gearing \((\bar{b} := \infty)\) or if \(\lambda_0 := 1\).