Classification of strongly asymptotically log del Pezzo flags and surfaces

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Abstract

We introduce the notion of strongly asymptotically log del Pezzo flags, and classify such flags under the assumption that their zero-dimensional part lies in the boundary. We use this result to give a new and conceptual proof of the classification of strongly asymptotically log del Pezzo surfaces, originally due to Cheltsov and the author.

1 Introduction

The classification problem for smooth Fano manifolds of low dimension has been a fundamental problem in algebraic geometry starting with the work of the Italian School in the second half of the 19th century. In an attempt to generalize the problem to pairs, Maeda introduced the following notion [10,11].

Definition 1.1. Let $X$ be a smooth variety and let $D$ be a simple normal crossing divisor in $X$. The pair $(X,D = \sum D_i)$ is called log Fano if $-K_X - D$ is ample.

Maeda posed the problem of classifying log Fano pairs and gave a complete classification up to dimension 3 (see also Loginov [9]), which we will come back to shortly.

Motivated by the study of Kähler–Einstein edge metrics, Cheltsov and the author introduced the following generalization of Maeda’s notion, that allows for the coefficients of the $D_i$ to be slightly less than 1 [2, Definition 1.1]:

Definition 1.2. We say that a pair $(X,D)$ consisting of a smooth complex variety $X$ and a divisor $D = \sum_{i=1}^r D_i$ with simple normal crossings on $X$ is strongly asymptotically log Fano if there exists $\epsilon > 0$ such that $-K_X - \sum_{i=1}^r (1 - \beta_i)D_i$ is ample for all $(\beta_1,\ldots,\beta_r) \in (0,\epsilon]^r$.

In fact, Maeda’s notion corresponds to the case $\beta_1 = \cdots = \beta_r = 0$, which by openness of ampleness [8, Example 1.3.14], implies ampleness for small $\beta_i$. In other words, every log Fano is strongly asymptotically log Fano. As we will see below the converse is far from true. For a survey on Maeda’s and Cheltsov–Rubinstein’s work we refer to [15, §8].

The differential geometric interpretation of positive $\beta_i$’s is as follows. By the resolution of the Calabi–Tian conjecture [7, Theorem 2], [5,14] $(X,D)$ is strongly asymptotically log Fano

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if and only if it admits Kähler metrics with positive Ricci curvature on $X \setminus D$ and with edge singularities of angle $2\pi \beta_i$ along $D_i$ for all small $\beta_i$. Edge singularities are, roughly, conic singularities transverse to the ‘edges’ $D_i$. We refer to \cite{15} §4 for a detailed survey.

As customary, from now and on, when discussing dimension 2 we will use ‘del Pezzo’ instead of ‘Fano’, and replace $(X, D)$ by $(S, C)$. Already in dimension 2, Maeda’s classification shows that it is rare for a pair to be log del Pezzo \cite{11} §3 (see \cite{1}, Proposition 4.1) for an expository proof:

**Proposition 1.3.** Log del Pezzo pairs $(S, C)$ are classified as follows:

(i) $S = \mathbb{P}^2$, and $C$ is a line,

(ii) $S = \mathbb{P}^2$, and $C = C_1 + C_2$, where each $C_i$ is a line,

(iii) $S = \mathbb{P}^2$, and $C$ is a smooth conic,

(iv) $S = \mathbb{F}_n$ for some $n \in \mathbb{N} \cup \{0\}$, and $C$ is a $-n$-curve,

(v) $S \cong \mathbb{F}_n$ for any $n \geq 0$, and $C = C_1 + C_2$ where $C_1$ is a $-n$-curve and $C_2$ is a 0curve,

(vi) $S = \mathbb{F}_1$, and $C$ is a smooth 1-curve,

(vii) $S = \mathbb{P}^1 \times \mathbb{P}^1$, and $C$ is a smooth 2-curve.

In particular, the rank of the Picard group is at most 2 which is extremely restrictive. What is more, the notion of log del Pezzo pairs does not even recover the classical notion of del Pezzo surfaces. As it turns out, strongly asymptotically log del Pezzos are rather richer in structure and the rank of their Picard group has no upper bound. From a geometric point of view this is interesting since it provides a host of manifolds on which to construct canonical Kähler edge metrics as well as metrics of positive Ricci curvature away from $C$. In addition, one easily realizes the classical del Pezzo surfaces in this picture by considering $(S, C)$ with $C$ a smooth anticanonical curve on a del Pezzo surface $S$ as then $-K_S - (1 - \beta)C \sim -\beta K_S > 0$.

Cheltsov and the author gave a classification of strongly asymptotically log del Pezzo pairs based on two main steps \cite{2} Theorems 2.1,3.1. First, an induction on the rank of the Picard group and the key observation that every strongly asymptotically log del Pezzo pair can be obtained from a pair $(s, c)$ with Picard group at most 2 via blow-ups along smooth points of the boundary $c$ and replacing the original boundary with its proper transform $\tilde{c}$. Second, an ad hoc verification of the generality conditions on the blow-ups allowed for each of the resulting pairs $(s, c)$.

Here, we would like to present another proof of the classification of strongly asymptotically log del Pezzo pairs, but our main goal is to actually introduce a different point of view of independent interest and with a number of applications. The first, necessary, part of the proof is quite similar to the one in \cite{2} though we give a more conceptual/pedagogical argument (see the flowchart, Figure 1 in the proof of Proposition 3.2) that has the advantage of generalizing to the setting of asymptotically log del Pezzo. For the second, sufficient, part of the proof, we present a new approach via the notion of SALdP flags that we introduce here (Definition 2.2). We show that this notion gives a new characterization of asymptotically log del Pezzo pairs that uses in a precise way blow-downs of the pair. This characterization is the first main result of this article (Theorem 2.3). The classification of all such flags (Theorem 4.1) is the second main result of the present note. We apply this result to give a slightly cleaner picture of the generality conditions on the blown-up points on the boundary in the classification of strongly asymptotically log del Pezzo pairs (Theorem 5.1). We believe this notion should also be important for the classification of the much larger class of asymptotically log del Pezzo pairs \cite{2, 15} and to the structure of the body of ample angles \cite{16}, as we hope to discuss elsewhere \cite{17}.
1.1 Organization

Section 2 starts by defining the three types of blow-ups that arise in this study. Then, strongly asymptotically log del Pezzo flags are introduced (Definition 2.2) and we state the main result on how such flags characterize strong asymptotic log positivity (Theorem 2.3). To prove this result we derive a characterization of flags containing a non-boundary component (Lemma 2.5) and those containing a boundary component (Lemma 2.7). Both of these lemmas are of independent interest and will serve us repeatedly in the classification of strongly asymptotically log del Pezzo flags (Theorem 4.1). In §3 we show any strongly asymptotically log del Pezzo pair can be described as a smooth boundary blow-up of another strongly asymptotically log del Pezzo pair with a smaller Picard group. The key result here is Proposition 3.2 which is more conceptual approach than that given in [2, Theorems 2.1, 3.1] and which generalizes to the setting of asymptotically log del Pezzo pairs [1]. In §5 we turn to the main application, the classification of strongly asymptotically log del Pezzo pairs (Theorem 5.1). In an Appendix we present an auxiliary classification result (Proposition 6.3) under a small Picard rank assumption.

2 Strongly asymptotically log del Pezzo flags

The classification of del Pezzo surfaces is one of the most classical results in algebraic geometry. It hinges on blowing down $-1$-curves and keeping track of intersection numbers in the process, and in the final step reverse engineering to determine the possible location (points) of the blow-downs (blow-ups).

It is therefore natural that when classifying strongly asymptotically log del Pezzo we similarly have to study carefully the $-1$-curves, this time on the pair, i.e., the added layer of difficulty is to understand not just the $-1$-curves on the surface but also their relative position to the boundary curve.

Thus, a key step in the proofs of Proposition 3.2 and Theorem 5.1 involves distinguishing between three different types of birational operations on pairs that we introduce in Definition 2.1.

First, let us establish some basic notation. Given a blow-down map

$$\pi_P : S \to s$$

of a reduced zero-dimensional locus (i.e., a collection of distinct points)

$$P \subset s$$

and an irreducible curve

$$\Sigma \subset s,$$

denote by

$$\hat{\Sigma} \subset S$$

the $\pi_P$-proper transform of $\Sigma$. For each $p \in P$, denote the exceptional curve by

$$E_p := \pi^{-1}(p).$$

All log pairs considered will be of the form $(s, c = \sum_{i=1}^r c_i)$ a log pair with each $c_i$ a smooth irreducible curve in $s$. We will also consider coefficient vectors $\bar{\beta} = (\beta_1, \ldots, \beta_r) \in (0, 1)^r$ with the convention

$$c_0 := c_r, \quad \beta_0 := \beta_r.$$
Definition 2.1. Let $P, \pi_P$ be as above.

- A pair $(S, C = \sum_{i=1}^r C_i)$ is called a smooth boundary blow-up of a pair $(s, c = \sum_{i=1}^r c_i)$ if $P$ is contained in the smooth locus of $c$ (i.e., $P \cap c_j \cap c_k = \emptyset$ for all $j \neq k \in \{1, \ldots, r\}$) and $C = \tilde{c}$, where $\tilde{c}$ is the $\pi_P$-proper transform of $c$. We say $(s, c)$ is the smooth boundary blow-down of $(S, C)$.

- A pair $(S, C = \sum_{i=1}^r C_i)$ is called an away blow-up of a pair $(s, c = \sum_{i=1}^r c_i)$ if $p \in s \setminus c$ and $C = \pi^{-1}(c) = \tilde{c}$. We say $(s, c)$ is the away blow-down of $(S, C)$. We use the same terminology if $\pi$ is the blow-up of a collection of distinct points in $s \setminus c$.

- A pair $(S, C = \sum_{i=1}^r C_i)$ is called a tail blow-up of a pair $(s, c = \sum_{i=1}^{r-1} c_i)$ if $p \in c_{r-1}$, $c_{r-1} = 1$ for precisely one $i \in \{1, \ldots, r-2\}$ and zero otherwise, and $C = \pi^{-1}(c)$ (i.e., $C_s = \pi^{-1}(p)$). We say $(s, c)$ is the tail blow-down of $(S, C)$. We use the same terminology if $\pi$ is the blow-up of a collection of distinct points in the smooth locus of $c$, each on a different tail component.

For the purpose of classification of strongly asymptotically log del Pezzos it turns out that it suffices to consider just the first type of blow-ups. This is the content of Proposition 3.2 and Theorem 5.1.

In order to prove such a result, we establish a characterization of such blow-ups that preserve strong asymptotic log positivity (Theorem 2.3). This leads to an additional key new notion we introduce in this article:

Definition 2.2. Let $P, \pi_P, \Sigma$ be as above. We say $P \subset \Sigma \subset s$ is a strongly asymptotically log del Pezzo flag (SALdP flag) for $(s, c = \sum_{i=1}^r c_i)$ if there exists a sequence of vectors $\tilde{\beta}(j) = (\beta_1(j), \ldots, \beta_r(j)) \in (0, 1)^r$ tending to the origin such that

$$
\left(K_S + \sum_{i=1}^r (1 - \beta_i(j))\tilde{c}_i\right)\Sigma \geq 0, \quad \text{for all } j \in \mathbb{N}.
$$

(2.3)

The point of this definition is the following useful characterization of strong asymptotic log positivity of a pair in terms of certain SALdP flags.

Theorem 2.3. Let $(s, c)$ be a strongly asymptotically log del Pezzo pair and let $(S, C = \tilde{c})$ be the smooth boundary blow-up at $P = \{p_1, \ldots, p_m\} \subset c$ (Definition 2.1). Then $(S, C)$ is strongly asymptotically log del Pezzo if and only if there are no SALdP flags for $(s, c)$ of the form

$$
\{p_{i_1}, \ldots, p_{i_t}\} \subset \Sigma \subset s,
$$

(2.4)

with $\{i_1, \ldots, i_t\} \subset \{1, \ldots, m\}$.

Remark 2.4. Note that Theorem 2.3 characterizes a positivity property of $(S, C)$ in terms of a ‘downstairs’ pair $(s, c)$ as well as curves that live on intermediate pairs $(S', C')$ that are blow-ups of $(s, c)$ but blow-downs of $(S, C)$! See Remark 2.11. Still, after some thought the content of Theorem 2.3 might seem intuitive, however, as we will see below there are a few pitfalls to an ‘easy’ proof.

2.1 Flags with boundary points and no boundary curve

In order to prove Theorem 2.3 we will need to develop a basic understanding of SALdP flags whose zero-dimensional locus lies in the boundary. The next lemma characterizes such flags under the additional assumption that their one-dimensional locus is not a boundary component.
Lemma 2.5. Suppose that $\Sigma$ is not a component of $c$ and let $P$ be contained in the smooth locus of $c$. Then $c \supset P = \{p_1, \ldots, p_m\} \subset \Sigma \subset s$ is a SALdP flags for $(s, c)$ if and only if

$$0 \geq -(K_s + c).\Sigma + \text{sign}(\tilde{c}.\tilde{\Sigma}), \quad (2.5)$$

if and only if

$$\tilde{c}.\tilde{\Sigma} = c.\Sigma - \sum_{k=1}^{m} \text{mult}_{p_k} \Sigma = 0 = (K_s + c).\Sigma.$$

Proof. First (recall (2.1)),

$$\tilde{\Sigma} \sim \pi^* \Sigma - \sum_{k=1}^{m} \text{mult}_{p_k} \Sigma E_{p_k}. \quad (2.6)$$

Second, as $c \neq 0$ and $\{p_k\}_{k=1}^{m}$ are smooth points of $c$,

$$\tilde{c} \sim \pi^* c - \sum_{k=1}^{m} \text{mult}_{p_k} c E_{p_k} = \pi^* c - \sum_{k=1}^{m} E_{p_k}. \quad (2.7)$$

In addition,

$$K_S \sim \pi^* K_s + \sum_{k=1}^{m} E_{p_k}, \quad (2.8)$$

so

$$K_S + \tilde{c} \sim \pi^* (K_s + c). \quad (2.9)$$

The inequalities (2.3), $j \in \mathbb{N}$, i.e., $(K_S + \tilde{c}).\tilde{\Sigma} \geq \tilde{\Sigma} . \sum_{i=1}^{r} \beta_i(j)\tilde{c}_i$, become

$$0 \geq -(K_s + c).\Sigma + \sum_{\alpha=1}^{r} \beta_\alpha(j)\left(c_\alpha . \Sigma - \sum_{k=1}^{m} \text{mult}_{p_k} \Sigma \text{mult}_{p_k} c_\alpha\right), \quad \forall j \in \mathbb{N}. \quad (2.10)$$

We claim that

$$\sum_{i=1}^{m} \text{mult}_{p_i} \Sigma \text{mult}_{p_i} c_\alpha \leq \Sigma . c_\alpha, \quad \forall \alpha \in \{1, \ldots, r\}. \quad (2.11)$$

Indeed, $\Sigma \neq c_\alpha$ and thus also $\tilde{\Sigma} \neq \tilde{c}_\alpha$, so since the $p_k$ are smooth points of $c$, by Definition [2.1] and (2.6)-(2.7),

$$0 \leq \tilde{\Sigma}.\tilde{c}_\alpha = \Sigma . c_\alpha - \sum_{k=1}^{m} \text{mult}_{p_k} \Sigma \text{mult}_{p_k} c_\alpha, \quad (2.12)$$

as claimed. Next, by (2.6)-(2.7),

$$\tilde{c}.\tilde{\Sigma} = c.\Sigma - \sum_{k=1}^{m} \text{mult}_{p_k} \Sigma = \sum_{\alpha=1}^{r} \left(c_\alpha . \Sigma - \sum_{k=1}^{m} \text{mult}_{p_k} \Sigma \text{mult}_{p_k} c_\alpha\right)$$

with each summand nonnegative, the sign will equal the sign of the maximal summand. Finally, both terms in (2.10) are nonnegative; the first since $-K_S - C$ is nef for every strongly asymptotically log del Pezzo pair (as a limit of ample classes by Definition [1.2]), while the second by (2.12). Thus both must vanish in order for (2.10) to hold: the first term is independent of $j$ while the second is either zero or small and positive. Conversely, if both terms vanish than (2.10) holds (with equality) and so does (2.3), for each $j \in \mathbb{N}$. This concludes the proof of the equivalence of Lemma 2.5. \qed
Remark 2.6. The crucial consequence of Lemma 2.5 is that being a SALdP flag with zero-dimensional locus in the boundary and one-dimensional locus off the boundary does not affect the computation. We will use this crucially in the proof below. This will be generalized and used crucially also in [17].

2.2 Flags with a boundary curve

The next lemma characterizes flags whose one-dimensional locus is a component of the boundary.

Lemma 2.7. Suppose that \(s, c = \sum_{\alpha=1}^r c_\alpha\) is strongly asymptotically log del Pezzo. Then \(\{p_1, \ldots, p_m\} \subset c_i \subset s\) is a SALdP flag for \((s, c)\) if and only if:

- \(m \geq K_s^2\) if \(r = 1\) and \(c_1 \sim -K_s\),
- \(m > c_1^2\) if \(r \geq 2\) and \(c \sim -K_s\),
- \(m > c_i^2\) if \(r \geq 3\), \(c \not\sim -K_s\), and \(c_i\) intersects exactly two other \(c_j\)’s.

Proof. Suppose first that \(c_1\) is a smooth elliptic curve so \(\text{mult}_{p_k} c_1 = 1\) for all \(k \in \{1, \ldots, m\}\), \(c_1 \sim -K_s\), and \(r = 1\) [2, Lemma 2.2]. Plugging-in \(K_s + c \sim 0\) in [2.3] gives,

\[
0 \geq -(K_s + c).\Sigma + \beta_1(c_1^2 - \sum_{k=1}^m \text{mult}_{p_k} c_1) = \beta_1(c_1^2 - m),
\]

i.e., \(m \geq c_1^2 = K_s^2\).

If \(c\) has more than one component then each component of \(c\) (including \(c_i\)) must be a smooth rational curve [2, §3]. Let \(\epsilon \in \{0, 1, 2\}\) be the number of components of \(c\) (excluding \(c_i\) itself) that \(c_i\) intersects, counted with multiplicity. By [2.3],

\[
0 \geq -(K_s + c).c_i + \sum_{\alpha=1}^r \beta_\alpha(c_\alpha.c_i - \sum_{k=1}^m \text{mult}_{p_k} c_i \text{mult}_{p_k} c_\alpha)
= K_s.c_i - c_i^2 - \epsilon + \sum_{\alpha=1}^r \beta_\alpha(c_\alpha.c_i - \sum_{k=1}^m \text{mult}_{p_k} c_i \text{mult}_{p_k} c_\alpha)
\geq 2 - \epsilon + \sum_{\alpha=1}^r \beta_\alpha(c_\alpha.c_i - \sum_{k=1}^m \text{mult}_{p_k} c_i \text{mult}_{p_k} c_\alpha),
\]

so it follows that we must have \(\epsilon = 2\) and, using [2, Lemma 3.5], either (i) \(r = 2\), and \(c_i = c_2\) with \(c_1.c_2 = 2\) and \(c_1 + c_2 \sim -K_s\), or (ii) \(r \geq 3\), and \(c_i\) is a ‘middle’ component of \(c\), i.e., with \(c_i.c_\alpha = 1\) for \(\alpha = i \pm 1 \mod r\) (recall [2.2]) and \(c_i.c_\alpha = 0\) for \(\alpha \not\in \{i - 1, i, i + 1 \mod r\}\) (note that in theory \(c\) could have several connected components if \(c \not\sim -K_s\) [2, Lemma 3.5], but this does not affect the computation).

In case (i), [2.14] becomes,

\[
0 \geq \beta_1(2 - \sum_{k=1}^m \text{mult}_{p_k} c_2 \text{mult}_{p_k} c_1) + \beta_2(c_2^2 - \sum_{k=1}^m (\text{mult}_{p_k} c_2)^2)
= 2\beta_1 + \beta_2(c_2^2 - m)
\]

since \(\text{mult}_{p_k} c_2 \text{mult}_{p_k} c_1 = 0\),
as otherwise \( p_k \in c_1 \cap c_2 \) contrary to Definition 2.1. Thus, \( c_2 \) is part of a SALdP flag if and only if \( c_2^2 < m \), equivalently, \( \tilde{c}_2^2 < 0 \).

In case (ii), (2.14) becomes,

\[
0 \geq \beta_{i-1}\left(1 - \sum_{k=1}^{m} \text{mult}_{p_k} c_i \text{mult}_{p_k} c_{i-1}\right) + \beta_i\left(c_i^2 - \sum_{k=1}^{m} (\text{mult}_{p_k} c_i)^2\right)
+ \beta_{i+1}\left(1 - \sum_{k=1}^{m} \text{mult}_{p_k} c_i \text{mult}_{p_k} c_{i+1}\right)
= \beta_{i-1} + \beta_i(c_i^2 - m) + \beta_{i+1}
\]

since

\[
\text{mult}_{p_k} c_i \text{mult}_{p_k} c_{i+1} \mod r = 0
\]

(recall (2.2)), as otherwise \( p_k \in c_i \cap c_i \pm 1 \mod r \) contrary to Definition 2.1. Thus, \( c_i \) is part of a SALdP flag if and only if \( c_i^2 < \sum_{k=1}^{m} \text{mult}_{p_k} c_i = m \), i.e., \( c_i^2 < 0 \).}

2.3 Degree under smooth boundary blow-ups

**Lemma 2.8.** Let \((s,c)\) be strongly asymptotically log del Pezzo and let \((S,C)\) be obtained from \((s,c)\) via smooth boundary blow-up. Then \((K_S + \sum_{i=1}^{r}(1 - \beta_i)C_i)^2 > 0\) for all sufficiently small \(\beta_i\) if and only if one of the following three mutually exclusive conditions holds:

- \( m < K_s^2 \) if \( c_1 \sim -K_s \),
- at most \( c_i^2 \) points are blown on each \( c_i \) if \( c \sim -K_s \) and \( r > 1 \),
- \( c \not\sim -K_s \).

(2.15)

**Remark 2.9.** For the second condition note that \( c_i^2 \geq 0 \) in this case [2, Lemmas 3.6, 3.5].

**Proof.** Compute using (2.9),

\[
(K_S + \sum_{i=1}^{r}(1 - \beta_i)C_i)^2 = (K_S + C)^2 + \left(\sum_{i=1}^{r} \beta_i C_i\right)^2 - 2 \sum_{i=1}^{r} \beta_i C_i (K_S + C)
= (\pi^* K_s + \pi^* c)^2 + \left(\sum_{i=1}^{r} \beta_i C_i\right)^2 - 2 \sum_{i=1}^{r} \beta_i C_i (K_s + c)
= (K_s + c)^2 + \left(\sum_{i=1}^{r} \beta_i c_i\right)^2 - 2 \sum_{i=1}^{r} \beta_i c_i (K_s + c).
\]

(2.16)

There are two possibilities: \( c \) is a union of disjoint chains of smooth rational curves or else \( c \sim -K_s \) and it is a single cycle [2, Lemma 3.5]. Observe that \( C \) has the same structure as \( c \) by Definition 2.1.

In the latter case (2.16) reduces to

\[
\left(\sum_{i=1}^{r} \beta_i c_i\right)^2 > 0.
\]

We treat a few sub-cases: if \( r = 1 \) then (2.16) reduces to \( c^2 > 0 \). If \( r = 2 \), then \( c_1.c_2 = 2 \) [2, Lemma 3.5] so (2.16) reduces to

\[
\beta_1^2 c_1^2 + \beta_2^2 c_2^2 + 4 \beta_1 \beta_2 > 0.
\]
which holds for all small $\beta_i > 0$ if and only if $c_i^2, c_i^2 \geq 0$. If $r \geq 3$ then $c_i, c_{i-1} \mod r = c_i, c_{i+1} \mod r = 1$ and otherwise $c_i, c_j = 0$ for $i \neq j$ (recall (2.2)). Thus, (2.16) reduces to

$$\sum_{i=1}^{r} \beta_i c_i^2 + 2 \sum_{i=1}^{r} \beta_i \beta_{i+1} \mod r > 0,$$

which holds for all small $\beta_i > 0$ if and only if $\tilde{c}_i^2 \geq 0$ for each $i \in \{1, \ldots, r\}$.

In the former case, each $c_i$ is smooth and rational hence $K_s . c_i + c_i^2 = -2$. We can assume without loss of generality that $c$ is connected since the computations of the linear terms below are done for each connected component. So when $r = 1$ (2.16) reduces to

$$(K_s + c)^2 + \beta_1 c_1^2 + 4 \beta_1;$$

when $r = 2$ we obtain

$$(K_s + c)^2 + \left( \sum_{i=1}^{2} \beta_i c_i \right)^2 + 2 \beta_1 + 2 \beta_2;$$

when $r > 2$ we obtain

$$(K_s + c)^2 + \left( \sum_{i=1}^{r} \beta_i c_i \right)^2 + 2 \beta_1 + 2 \beta_r;$$

all three of these expressions are positive for all small $\beta_i > 0$ as the linear terms have positive coefficients (and the constant term is nonnegative since $-K_s - c$ is nef (recall the end of the proof of Lemma 2.5)). Here we used that $c_1 . c_2 = c_i, c_{i+1} = \ldots = c_{r-1}, c_r = 1$ for all $i = 1, \ldots, r - 1$ with all other $c_i, c_j = 0$ for $i \neq j$.

\section*{2.4 Proof of Theorem 2.3}

Let us first assume that $(S, \tilde{c})$ is strongly asymptotically log del Pezzo. If there exists a SALdP flag of the form (2.4) then (2.3) and the Nakai–Moishezon criterion [8, Theorem 1.2.23] imply that $(S, \tilde{c})$ is not strongly asymptotically log del Pezzo by Definition 1.2, a contradiction.

Suppose now that no SALdP flags of the form (2.4) exist. If $(S, \tilde{c})$ is not strongly asymptotically log del Pezzo there exists a sequence $\{\beta(j)\}_{j \in \mathbb{N}}$ tending to the origin in $\mathbb{R}^r$ such that the class $(K_S + \sum_{i=1}^{r} (1 - \beta_i(j)) \tilde{c}_i)$ is not negative for each $j$. By Nakai–Moishezon this means that either

$$\left( K_S + \sum_{i=1}^{r} (1 - \beta_i(j)) \tilde{c}_i \right)^2 \leq 0, \quad \text{for all } j \in \mathbb{N}, \quad (2.17)$$

and/or there exists a sequence of irreducible curves $0 \neq D_j \subset S$ such that

$$\left( K_S + \sum_{i=1}^{r} (1 - \beta_i(j)) \tilde{c}_i \right). D_j \geq 0, \quad \text{for all } j \in \mathbb{N}. \quad (2.18)$$

The first possibility (2.17) cannot hold by Lemma 2.8 if we can verify the conditions (2.15). This is the first subtle point in the proof of Theorem 2.3. Fortunately, our assumption that no SALdP flags of the form (2.4) exist precisely verifies these conditions: Lemma 2.7 shows that if any of these conditions is not verified then one of the boundary components of $c$ would be the one-dimensional strata of a SALdP flag of the form (2.4) (this is only needed in the case $c \sim -K_s$!)

Thus, only the second possibility, i.e., (2.18), can hold. A subtle, but important, point is that we can assume (2.18) holds with a fixed $D = D_j$:
Claim 2.10. There exists a fixed irreducible curve $D \subset S$ such that

$$
\left( K_S + \sum_{i=1}^{r} (1 - \beta_i(j)) \tilde{c}_i \right) . D \geq 0, \quad \text{for all } j \in J \subset \mathbb{N} \text{ with } J \text{ an infinite set.}
$$

Proof. The proof is based on Remark 2.6, but requires a bit more.

Let $\mathcal{D} = \{ D_j : j \in \mathbb{N} \}$ be the collection of divisors satisfying (2.18). If infinitely-many of the $D_j$ are boundary components, then there exists an unbounded subsequence $J \subset \mathbb{N}$ with $D_j = \tilde{c}_\alpha = C_\alpha$ for some fixed $\alpha$ and for all $j \in J$, and we are done. Otherwise, let $J := \{ j \in \mathbb{N} : D_j \not\subset C \}$. By Remark 2.6 we may take $D = D_j$ for an arbitrary (fixed) $j \in J$.

We return to the proof of Theorem 2.3. Let $D$ be the curve furnished by Claim 2.10. Write

$$
D = \pi^* \pi(D) - \sum_{k=1}^{m} \text{mult}_{p_k} \pi(D) E_{p_k},
$$

so using (2.7), (2.9), and Claim 2.10 there exists an unbounded subsequence $J \subset \mathbb{N}$ such that

$$
0 \leq \left( K_S + \sum_{i=1}^{r} (1 - \beta_i(j)) \tilde{c}_i \right) . D = \left( K_s + \sum_{i=1}^{r} (1 - \beta_i(j)) c_i \right) . \pi(D) + \sum_{i=1}^{r} \beta_i(j) \sum_{k=1}^{m} \text{mult}_{p_k} \pi(D) \text{mult}_{p_k} c_i.
$$

Since $(s,c)$ is strongly asymptotically log del Pezzo the first term in the last line is negative for all sufficiently large $j \in J$. By assumption, each $p_k$ is a smooth point of $c$, i.e., $\text{mult}_{p_k} c_i = 1$ for some (actually, exactly one) $i \in \{1, \ldots, r\}$ it follows that $\text{mult}_{p_k} \pi(D) > 0$ for at least one $k \in \{1, \ldots, m\}$. Thus, by Definition 2.2

$$
\emptyset \neq \{ p_k : \text{mult}_{p_k} \pi(D) > 0 \} \subset \pi(D) \subset s
$$

is a SALdP flag for $(s,c)$ of the form (2.4), contradicting our assumption. Thus, $(S,\tilde{c})$ is strongly asymptotically log del Pezzo, concluding the proof of Theorem 2.3.

Remark 2.11. Let $\pi' : S' \to s$ be the blow-up of $s$ at the subset of points $\{ p_k : \text{mult}_{p_k} \pi(D) > 0 \} \subset \pi(D) \subset s$. Then $D' := \pi'(D)$ lives on the ‘intermediate’ surface $S'$ ‘between’ $s$ and $S$ (unless of course $\pi(D)$ passes through all $p_1, \ldots, p_m$). A moment’s thought shows that $D'$ satisfies the same exact inequality (2.19) $D$ satisfied on $S$, but on $S'$ instead. Thus, we do get a SALdP flag for $(s,c)$ as claimed.

3 Reduction to smooth boundary blow-ups

3.1 Trichotomy of $-1$-curves

By a $-1$-curve we will always mean an irreducible smooth rational curve of self-intersection $-1$ (and genus $g$ zero). Let $e$ be a $-1$-curve on an asymptotically log del Pezzo pair $(s,c = \sum_{i=1}^{r} c_i)$. By adjunction [4],

$$
K_s.e = 2g(e) - 2 - e^2 = -1,
$$

(3.1)
0 > (K_s + \sum_{i=1}^{r} (1 - \beta_i) c_i) \cdot e = -1 + e \cdot e + O(\beta)

i.e., $e \cdot e < 2$. If $e$ is not contained in $c$, i.e., $e \neq c_i$ for any $i$, then also $e \cdot e \geq 0$. Thus, on any asymptotically log del Pezzo pair $(s, c)$ there are three disjoint families of $-1$-curves:

- $E_{\perp C}(s, c) := \{ E : E \not\subset C \text{ is a } -1\text{-curve with } E \cdot c = 1 \}$
- $E_{\setminus C}(s, c) := \{ E : E \not\subset c \text{ is a } -1\text{-curve with } E \cdot c = 0 \}$
- $E_C(s, c) := \{ E : E \subset c \text{ is a } -1\text{-curve} \}$

and the disjoint union

$$\mathcal{E}(s) := E_{\perp C}(s, c) \cup E_{\setminus C}(s, c) \cup E_C(s, c)$$

consists of all $-1$-curves on $s$ \cite[Lemma 3.3]{2}. What is more, by Remark 3.1 below, this trichotomy precisely corresponds to the three birational operations in Definition 2.1.

**Remark 3.1.** In fact, $E_C(s, c)$ consists exclusively of tail $-1$-components of the boundary \cite[Lemma 3.6]{2}.

### 3.2 Smooth boundary blow-ups suffice

The next result is key to the proof of Theorem 5.1. The proof is modelled on the one in \cite{2} but is more conceptual. It shows that the first type of blow-ups in Definition 2.1 suffice as far as the classification of strongly asymptotically log del Pezzos is concerned.

**Proposition 3.2.** Let $(S, C)$ be a strongly asymptotically log del Pezzo pair. Let $(s, c)$ be the smooth boundary blow-down of $(S, C)$ given by the contraction of $E_{\perp C}(S, C)$. Then $(s, c)$ is a strongly asymptotically log del Pezzo pair with $rk \text{Pic}(s) \leq 2$.

**Proof.** A flowchart for the proof is provided in Figure 1. Note that $S$ and hence $s$ are rational surfaces \cite[p. 1253]{2}. We will use repeatedly the following consequences of the classification of rational surfaces \cite[p. 520]{4}:

- every rational surface with $rk \text{Pic}(s) > 2$ contains a $-1$-curve,
- a rational surface with $rk \text{Pic}(s) \leq 2$ is either $\mathbb{P}^2$ or $\mathbb{F}_n$, $n \geq 0$.

**Step 1: the case $C \sim -K_S$.** Suppose first that $C \sim -K_S$. Then $E_C(S, C) = \emptyset$ \cite[Lemma 3.6]{2} and $E_{\setminus C}(S, C) = \emptyset$ since $-K_S \cdot E = 1$ by \cite[3.1]{2}, and this equals $C \cdot E$. So by (3.2), $\mathcal{E}(S) = E_{\perp C}(S, C)$ and contracting all of these curves is a smooth boundary blow-down that yields a pair $(s, c)$ with no $-1$-curves. By (3.3) then $rk \text{Pic}(s) \leq 2$ and, finally, the pair $(s, c)$ is strongly asymptotically log del Pezzo \cite[Lemma 3.4]{2}, as desired.

**Step 2: contracting $E_{\perp C}(S, C)$.** From now on, suppose $C \not\sim -K_S$. Let

$$\pi_1 : S \to S(1)$$

be the contraction of $E_{\perp C}(S, C)$ and let

$$(S(1), C(1))$$

be the smooth boundary blow-down of $(S, C)$. The pair $(S(1), C(1))$ is strongly asymptotically log del Pezzo as $(S, C)$ is by assumption \cite[Lemma 3.4]{2}.
contract $\mathcal{E}_\perp C(S, C)$ to obtain $(S(1), C(1))$

$C \sim -K_S$?

rk Pic$(S(1)) \leq 2$

no

contract $\mathcal{E}_\perp C(S, C)$ to obtain $(S(2), C(2))$

$\mathcal{E}_C(S(1), C(1))$ empty?

no

$r = 1$?

yes

$\mathcal{E}_\perp C(S(1), C(1))$ nonempty, contradiction

no

contract a subset of $\mathcal{E}_C(S(2), C(2))$ to obtain $(S(3), C(3))$ with rk Pic$(S(3)) = 2$

$\mathcal{E}_\perp C(S(2), C(2))$ nonempty, contradiction

yes

Figure 1: A flowchart for the proof of Proposition 3.2
**Claim 3.3.** Assume $C \neq -K_S$ and let $(S(1), C(1))$ be the smooth boundary blow-down of $(S, C)$. Then,
\[ \mathcal{E}_{\perp C}(S(1), C(1)) = \emptyset. \] (3.5)

**Proof.** Denote $\{E_1, \ldots, E_m\} = \mathcal{E}_{\perp C}(S, C)$ and $q_i := \pi_1(E_i) \in C(1)$ (the last inclusion is since otherwise $E_i$ would not intersect $C$ by assumption $E_i, C = 1$). Let $e \in \mathcal{E}_{\perp C}(S(1), C(1))$. Then $\tilde{e} \subset S$ satisfies $\tilde{e}^2 = (\pi_1^* e - \sum_{i=1}^m \mult_q e_i)^2 \leq e^2 = -1$. Equality is not possible since then $\tilde{e} \in \mathcal{E}_{\perp C}(S, C)$ and would be contracted by $\pi_1$ to a point and not to a curve $e$. So $\tilde{e}^2 \leq -2$.

On the other hand, $e \not\subset C(1)$ so $\tilde{e} \not\subset C$ and (compare to the proof of the case $r = 1$ \cite{2} Lemma 2.5)
\[
\tilde{e}^2 = 2 h^1(O_{\tilde{e}}) - 2 - K_S. \tilde{e} \\
\geq -2 - (K_S + \sum_{i=1}^r (1 - \beta_i) C_i). \tilde{e} + \sum_{i=1}^r (1 - \beta_i) C_i. \tilde{e} \\
\geq -2 - (K_S + \sum_{i=1}^r (1 - \beta_i) C_i). \tilde{e} > -2.
\] (3.6)

Therefore, $\tilde{e}$ cannot exist on $S$, so neither can $e$ on $S(1)$. \hfill \Box

**Step 3: away blow-down of $(S(1), C(1))$.** Let
\[
\pi_2 : S(1) \to S(2)
\] (3.7)
be the blow-down map of $\mathcal{E}_{\perp C}(S(1), C(1))$ and denote by
\[
(S(2), C(2))
\]
the away blow-down of $(S(1), C(1))$. The pair $(S(2), C(2))$ is strongly asymptotically log del Pezzo as $(S(1), C(1))$ is (the latter was noted right before Claim \cite{3.3} \cite{2} Lemma 3.4). Observe that
\[
\mathcal{E}_{\perp C}(S(2), C(2)) = \emptyset.
\] (3.8)
Indeed, by the same reasoning as in the proof of Claim 3.3 (see (3.6)), if there were a $-1$-curve disjoint from $C(2)$ then its $\pi_2$-proper transform would be a curve disjoint from $C(1)$ and of self-intersection at most, and hence exactly, $-1$, but then it would be contracted by $\pi_2$. Next, observe that by (3.5) and Claim 3.5 below also
\[
\mathcal{E}_{\perp C}(S(2), C(2)) = \emptyset.
\] (3.9)
As $\pi_2$ is an isomorphism on $C(2)$, i.e., $C(1)$ and $C(2)$ are isomorphic and $C(1) \cong C(2)$, for each $i \in \{1, \ldots, r\}$, the self-intersection numbers of the boundary components are unchanged under $\pi_2$: $C(1)_i^2 = C(2)_i^2$. Thus,
\[
\mathcal{E}_C(S(2), C(2)) \cong \mathcal{E}_C(S(1), C(1)).
\] (3.10)

**Step 4: the case $C(1)$ contains no $-1$-curves.**

**Claim 3.4.** Assume $C \neq -K_S$ and let $(S(1), C(1))$ be the smooth boundary blow-down of $(S, C)$. If $\mathcal{E}_C(S(1), C(1)) = \emptyset$ then either $\mathcal{E}_C(S(1), C(1)) = \emptyset$ or $(S(1), C(1)) \in \{(I.3A), (I.3B), (II.3)\}$. In particular, $\text{rk Pic}(S(1)) \leq 2$. 
Proof. Suppose that $E_C(S(1), C(1)) = \emptyset$. By (3.10) also $E_C(S(2), C(2)) = \emptyset$. So by (3.8)–(3.9) and (3.2), $E(S(2)) = \emptyset$ and by (3.3) then $\text{rk Pic}(S(2)) \leq 2$. Additionally, $C(2) \not\sim -K_{S(2)}$: by (2.9),

$$-K_{S(1)} - C(1) \sim -\pi^*(K_{S(2)} + C(2)) - \sum_{i=1}^{k} E_i,$$

and the left hand side is a limit of ample classes, while if $C(2) \sim -K_{S(2)}$, the right hand side would have negative square. In sum:

- $E(S(2)) = \emptyset$,
- $\text{rk Pic}(S(2)) \leq 2$,
- $C(2) \not\sim -K_{S(2)}$.
- $(S(2), C(2))$ is strongly asymptotically log del Pezzo.

By Proposition 6.3, $(S(2), C(2))$ must be one of the following 11 pairs:

(I.1B), (I.1C), (II.1B),
(I.2.n), (I.4B), (I.4C), (II.2A.n), (II.2B.n), (II.2C.n), (II.4B), or (III.3.n).

There are two cases to consider.

First, each of the last 8 surfaces is ruled and contains a fiber through every point. If exists $T \in E_C(S(1), C(1))$, let $F$ be the fiber (i.e., $F^2 = 0$) through the point $\pi_2(T) \not\in C(2)$. In each of the above cases $F.C(2) \geq 1$. More precisely, one can arrange (i.e., in the case (I.4B) we may choose $F$ to belong to either fibration resulting in different intersection numbers):

$$F.C(2) = \begin{cases} 1, & (S(2), C(2)) \in \{(I.2.n), (I.4B), (I.4C), (II.2C.n)\}, \\ 2 & (S(2), C(2)) \in \{(I.4B), (II.2A.n), (II.2B.n), (II.4B), (III.3.n)\}. \end{cases}$$

The $\pi_2$-proper transforms of $F$ and of $C(2)$ are curves on $S(1)$. Note that $\tilde{C}(2) = \pi^*C(2) = C(1)$. Since $(S(1), C(1))$ is strongly asymptotically log del Pezzo, the curve $\tilde{F}$ on $S(1)$ must satisfy $\tilde{F}^2 \geq -1$ by the argument of (3.6) (cf. [2, Lemma 2.5] for the case $r = 1$). So while by choice $\pi_2(T) \in F$, no other blow-up point of $\pi_2$ can contained in $F$. Thus, $\tilde{F} = \pi_2^*F - T$, and $\tilde{F}^2 = -1$. Moreover, since by assumption $T.C(1) = 0$,

$$F.C(2) = (\pi_2^*F - T).C(2) = \tilde{F}.C(2) = \tilde{F}.C(1).$$

Thus, the possibility $F.C(2) = 1$ is impossible as then $\tilde{F} \in E_{\perp C}(S(1), C(1))$ contradicting (3.5). The possibility $F.C(2) = 2$ is also impossible as then $\tilde{F}$ is a $-1$-curve intersecting $C(1)$ at 2 points, contradicting the fact that $(S(1), C(1))$ is strongly asymptotically log del Pezzo (recall, once again, (3.6)). Altogether, we have shown that if $(S(2), C(2))$ is one of the last 8 pairs in (3.11) then $T$ cannot exist, i.e., $E_{\perp C}(S(1), C(1)) = \emptyset$.

Suppose now that $(S(2), C(2))$ is one of the first 3 cases in (3.11), namely, $(S(2), C(2)) \in \{(I.1B), (I.1C), (II.1B)\}$, so $S(2) = \mathbb{P}^2$. If $E_{\perp C}(S(1), C(1)) \neq \emptyset$, denote

$$\pi_2(E_{\perp C}(S(1), C(1))) = \{q_1, \ldots, q_k\},$$

a nonempty collection of points in $S(2) = \mathbb{P}^2$ away from $C(2)$. Blowing up any one of these points, say $q_1$, we obtain a map $\tilde{\pi}_2 : \tilde{S}(2) \to S(2)$. Note that $\tilde{S}(2), C(2) = \tilde{\pi}_2^{-1}(C(2))$ is strongly asymptotically log del Pezzo as it is an away blow-down of $(S(1), C(1))$ at $\pi_2^{-1}(q_2), \ldots, \pi_2^{-1}(q_k)$ [2, Lemma 3.4]. If $k = 1$ then $S(1) = \tilde{S}(2) = F_1$ and $(S(1), C(1)) \in \{(I.3A), (I.3B), \ldots\}$.

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The cases $k \geq 2$ are not possible by the same analysis of the previous paragraph (i.e., there is a fiber on $S(2)$ whose proper transform on $S(1)$ would either be an element of $\mathcal{E}_{\perp C}(S(1), C(1))$ (contradicting (3.5)) or else a $-1$-curve intersecting the boundary more than once (contradicting the argument of (3.6)). Thus, we have shown that $(S(1), C(1)) \in \{\text{(I.3A), (I.3B), (II.3)}\}$.

In sum, either $\mathcal{E}_{\vert C}(S(1), C(1)) = \emptyset$ or $(S(1), C(1)) \in \{\text{(I.3A), (I.3B), (II.3)}\}$, and in both cases $\text{rk } \text{Pic}(S(1)) \leq 2$ (recall (3.3)), as stated in Claim 3.4.

**Claim 3.5.** Let $(M, A)$ be a strongly asymptotically log del Pezzo pair with $\mathcal{E}_{\perp C}(M, A) = \emptyset$ and let $(M', A')$ be the pair obtained by a composition of any number of away and/or tail blow-downs (Definition 2.1). Then $\mathcal{E}_{\perp C}(M', A') = \emptyset$.

**Proof.** By the same reasoning as in the proof of Claim 3.3, any blow-down map can only increase the self-intersection of curves. However, there are no $-n$-curves in $M$ intersecting $A$ transversally: $n = 1$ by assumption and $n \geq 2$ by [2, Lemma 2.5]. Thus, no new $-1$-curves intersecting $A'$ can appear downstairs.

**Step 5: the case $C(1)$ contains a $-1$-curve and $r = 1$.** Suppose that $C(1)$ contains a $-1$-curve and $r = 1$. Then $\mathcal{E}_{\vert C}(S(1), C(1)) = \{C(1)\}$. By (3.10), $\mathcal{E}_{\vert C}(S(2), C(2)) = \{C(2)\}$. Together with (3.8)–(3.9) and (3.2), $\mathcal{E}(S(2)) = \{C(2)\}$. By (3.3) then $\text{rk } \text{Pic}(S(2)) \leq 2$ and in fact, by Proposition 6.3, $(S(2), C(2))$ must be (I.2.1) (recall from the proof of Claim 3.4 that $(S(2), C(2))$ is strongly asymptotically log del Pezzo). In particular, $S(2) = F_1$ using the notation of the proof of Claim 3.4 the $\pi_2$-proper transform of a fiber through the point $q_1$ is an element of $\mathcal{E}_{\perp C}(S(1), C(1))$, contradicting (3.5).

**Step 6: the case $C(1)$ contains a $-1$-curve and $r > 1$.** Suppose that $C(1)$ contains a $-1$-curve and $r > 1$. If $\mathcal{E}_{\vert C}(S(1), C(1))$ is a singleton we are reduced to Step 5 above. So assume that $C(1)$, and hence also $C(2)$, contains at least two $-1$-curves. Together with (3.4) this implies that $\text{rk } \text{Pic}(S(2)) > 2$.

Note that by Remark 3.1 together with (3.8)–(3.9) and (3.2), all $-1$-curves on $S(2)$ are are mutually disjoint tail components of $C(2)$. Consider the tail blow-down

$$\pi_3 : S(2) \to S(3)$$

of a nonempty subset of $\mathcal{E}_{\vert C}(S(2), C(2)) = \mathcal{E}(S(2))$ so that $\text{rk } \text{Pic}(S(3)) = 2$. Denote by

$$(S(3), C(3))$$

the resulting tail blow-down of $(S(2), C(2))$. Note $(S(3), C(3))$ is strongly asymptotically log del Pezzo [2, Lemma 3.12]. By Proposition 6.3 $(S(3), C(3))$ is one of the following (recall that $-K_{S(3)} \not\sim C(3)$ as we showed earlier that $-K_{S(2)} \not\sim C(2)$): (I.2.n), (I.3A), (I.3B), (I.4B), (I.4C), (II.2A.n), (II.2B.n), (II.2C.n), (II.3), or (III.3.n). Each of these surfaces are ruled and contain a fiber through every point. Let $F$ be the fiber through the point $\pi_3(T) \in C(3)$ with $T$ in the exceptional locus of $\pi_3$ (and hence $T \in \mathcal{E}_{\vert C}(S(2), C(2))$). Observe that $\pi_3(T)$ is a smooth point of $C(3)$ so it cannot be the intersection point of the two components of (II.2C.n). The $\pi_3$-proper transform of $F$ is then a $-1$-curve in $S(2)$ not contained in $C(2)$ but intersecting $C(2)$ at $T \in C(2)$ transversally, i.e., $\tilde{F} \in \mathcal{E}_{\perp C}(S(2), C(2))$. This contradicts (3.9). In conclusion then $T$ cannot exist, i.e., $\mathcal{E}_{\vert C}(S(2), C(2)) = \mathcal{E}(S(2))$ must both be empty. This means that also that $\mathcal{E}_{\vert C}(S(1), C(1)) = \emptyset$, i.e., the hypothetical case of Step 6 is impossible.

Altogether, combining the 6 steps above, the proof of of Proposition 3.2 is complete.

\[\square\]
4 Classification of SALdP flags with zero-dimensional locus in the boundary

Motivated by Proposition 3.2, we restrict our attention in this article to smooth boundary blow-ups. The next result builds on the tools of 2 to completely classify SALdP flags arising from blow-ups of points on the boundary.

**Theorem 4.1.** Let \((s, c = \sum_{i=1}^{r} c_i)\) be strongly asymptotically log del Pezzo pair that is not the smooth boundary blow-up (recall Definition 2.1) of any other strongly asymptotically log del Pezzo pair. Suppose also that \(\text{rk}(\text{Pic}(s)) \leq 2\) so that \((s, c)\) is one of the pairs listed in Proposition 6.3. Then \(P_\pi \subset \Sigma \subset s\) is a SALdP flag with \(P_\pi \subset c\) if and only if it is one of the following (the numbering of the pairs matches the notation of Proposition 6.3):

(I.1A) \(c \cap \Sigma = \{p_1, \ldots, p_m\} \subset \Sigma \neq c\) with \(m > 0\), so that \(\sum_{i=1}^{m} \text{mult}_{p_i} \Sigma = \Sigma, c\), i.e., \(\tilde{\Sigma}.c = 0\),

\(\{p_1, \ldots, p_9, \ldots, p_m\} \subset \Sigma = c\) with \(m \geq 9\),

(I.3A) \(c \cap \Sigma = \{p_1, p_2\} \subset \Sigma = \text{fiber}\),

(I.4A) \(c \cap \Sigma = \{p_1, \ldots, p_m\} \subset \Sigma \neq c\) with \(m > 0\), so that \(\sum_{i=1}^{m} \text{mult}_{p_i} \Sigma = \Sigma, c\), i.e., \(\tilde{\Sigma}.c = 0\),

\(\{p_1, \ldots, p_8, \ldots, p_m\} \subset \Sigma = c\) with \(m \geq 8\),

(I.4B) \(c \cap \Sigma = \{p_1, p_2\} \subset \Sigma = (0,1)\text{-curve}\),

(II.1A) \(c \cap \Sigma = \{p_1, \ldots, p_m\} \subset \Sigma \neq c\) with \(m > 0\), so that \(\sum_{i=1}^{m} \text{mult}_{p_i} \Sigma = \Sigma, c\), i.e., \(\tilde{\Sigma}.c = 0\),

\(\{p_1, p_2, p_3, p_4, p_5, \ldots, p_m\} \subset c_1\) with \(m \geq 5\),

\(\{p_1, p_2, \ldots, p_m\} \subset c_2\) with \(m \geq 2\),

(II.2A) \(c \cap \Sigma = \{p_1, p_2\} \subset \Sigma = \text{fiber}\),

(II.2B) \(c \cap \Sigma = \{p_1, p_2\} \subset \Sigma = \text{fiber}\),

(II.3) \(c \cap \Sigma = \{p_1, p_2\} \subset \Sigma = \text{fiber}\),

(II.4A) \(c \cap \Sigma = \{p_1, \ldots, p_m\} \subset \Sigma \neq c_i\) with \(m > 0\), so that \(\sum_{i=1}^{m} \text{mult}_{p_i} \Sigma = \Sigma, c\), i.e., \(\tilde{\Sigma}.c = 0\),

\(\{p_1, p_2, p_3, \ldots, p_m\} \subset c_i, i \in \{1, 2\}\) with \(m \geq 3\),

(II.4B) \(c \cap \Sigma = \{p_1, p_2\} \subset \Sigma = (0,1)\text{-fiber}\),

\(\{p_1, p_2, p_3, p_4, p_5, \ldots, p_m\} \subset c_1, \text{ with } m \geq 5\),

\(\{p_1, \ldots, p_m\} \subset c_2\) with \(m \geq 1\),

(III.1) \(c \cap \Sigma = \{p_1, \ldots, p_m\} \subset \Sigma \neq c_i\) with \(m > 0\), so that \(\sum_{i=1}^{m} \text{mult}_{p_i} \Sigma = \Sigma, c\), i.e., \(\tilde{\Sigma}.c = 0\),

\(\{p_1, p_2, \ldots, p_m\} \subset c_i, i \in \{1, 2, 3\}\) with \(m \geq 2\),

(III.2) \(c \cap \Sigma \supset \{p_1, \ldots, p_m\} \subset \Sigma \neq c_i\) with \(m > 0\), so that \(\sum_{i=1}^{m} \text{mult}_{p_i} \Sigma = \Sigma, c\), i.e., \(\tilde{\Sigma}.c = 0\),

\(\{p_1, p_2, p_3, \ldots, p_m\} \subset c_1\) with \(m \geq 3\),

\(\{p_1, \ldots, p_m\} \subset c_i, i \in \{2, 3\}\) with \(m \geq 1\),

(III.3) \(c \cap \Sigma = \{p_1, p_2\} \subset \Sigma = \text{fiber} \neq c_2\),

\(\{p_1, \ldots, p_m\} \subset c_2\) with \(m \geq 1\).
(IV) \( c \cap \Sigma \supset \{p_1, \ldots, p_m\} \subset \Sigma \neq c \) with \( m > 0 \), so that \( \sum_{i=1}^{m} \text{mult}_{p_i} \Sigma = \Sigma.c \), i.e., \( \tilde{\Sigma}.\tilde{c} = 0 \), \( \{p_1, \ldots, p_m\} \subset c, i \in \{1, 2, 3, 4\} \), \( m \geq 1 \), so that
\[
\sum_{k=1}^{m} c \cap \Sigma \subset \Sigma \not\subset c
\]
Proof. Proposition 6.3 gives the list of strongly asymptotically log del Pezzo pairs \((s, c)\) with \( \text{rk}(\text{Pic}(s)) \leq 2 \) that are not the smooth boundary blow-up (Definition 2.1) of any other strongly asymptotically log del Pezzo pair. Applying to this list case-by-case Lemma 2.5 readily gives all SALdP flags \( P = c \cap \Sigma \subset \Sigma \), \( \tilde{\Sigma}.\tilde{c} = 0 \),
\[
\{p_1, \ldots, p_m\} \subset c, i \in \{1, 2, 3, 4\} \), \( m \geq 1 \),
so that \( \sum_{k=1}^{m} \text{mult}_{p_k} \Sigma = \Sigma.c \), i.e., \( \tilde{\Sigma}.\tilde{c} = 0 \),
\[
\{p_1, \ldots, p_m\} \subset c, i \in \{1, 2, 3, 4\} \), \( m \geq 1 \), so that
\[
\sum_{k=1}^{m} c \cap \Sigma \subset \Sigma \not\subset c
\]
5 Classification of strongly asymptotically log del Pezzo pairs

The following theorem describes all strongly asymptotically log del Pezzo pairs in terms of smooth boundary blow-ups (recall Definition 2.1). It is due to Cheltsov and the author [2, Theorems 2.1, 3.1].

Theorem 5.1. Let \( S \) be a smooth surface, let \( C_1, \ldots, C_r \) be distinct irreducible smooth curves on \( S \) such that \( \sum_{i=1}^{r} C_i \) is a divisor with simple normal crossings. Then \((S, \sum_{i=1}^{r} C_i)\) is strongly asymptotically log del Pezzo if and only if it is one of the following pairs:

(I.1A) \( S = \mathbb{P}^2 \), \( C_1 \) is a cubic,

(I.1B) \( S = \mathbb{P}^2 \), \( C_1 \) is a conic,

(I.1C) \( S = \mathbb{P}^2 \), \( C_1 \) is a line,

(I.2.A) \( S = \mathbb{F}_n \) for any \( n \geq 0 \), \( C_1 = Z_n \),

(I.3A) \( S = \mathbb{F}_1 \), \( C_1 \in |2(Z_1 + F)| \),

(I.3B) \( S = \mathbb{F}_1 \), \( C_1 \in |Z_1 + F| \),

(I.4A) \( S = \mathbb{P}^1 \times \mathbb{P}^1 \), \( C_1 \) is a \((2,2)\)-curve,

(I.4B) \( S = \mathbb{P}^1 \times \mathbb{P}^1 \), \( C_1 \) is a \((2,1)\)-curve

(I.4C) \( S = \mathbb{P}^1 \times \mathbb{P}^1 \), \( C_1 \) is a \((1,1)\)-curve

(I.5.m) \((S, C)\) is the smooth boundary blow-up of (I.1A) at \( 1 \leq m \leq 8 \) points with no three colinear, no six on a conic, and no eight on a cubic smooth away from its single double point with one of the points being that double point,

(I.6B.m) \((S, C)\) is the smooth boundary blow-up of (I.1B) at \( m \geq 1 \) points,

(I.6C.m) \((S, C)\) is the smooth boundary blow-up of (I.1C) at \( m \geq 1 \) points,

(I.7.n.m) \((S, C)\) is the smooth boundary blow-up of (I.2.n) at \( m \geq 1 \) points,

(I.8B.m) \((S, C)\) is the smooth boundary blow-up of (I.3B) at \( m \geq 1 \) points,

(I.9B.m) \((S, C)\) is the smooth boundary blow-up of (I.4B) at \( m \geq 1 \) points with no two on the same \((0,1)\)-curve,

(I.9C.m) \((S, C)\) is the smooth boundary blow-up of (I.4C) at \( m \geq 1 \) points,

(II.1A) \( S = \mathbb{P}^2 \), \( C_1 \) is a line, \( C_2 \) is a conic,
Remark 5.2. In Theorem [5.1] we fixed a few typos and minor omissions from [2] Theorems 2.1, 3.1:

- In case (I.7.n.m) we fixed the typo (I.2) [2] p. 1253, line 18 to (I.2.n).
- Case (II.8.m), \( m \in \{1,2,3,4\} \) in [2] Theorem 3.1, while correct (modulo the requirement that no two of the points be on the same (0,1)-fiber) coincides with (II.5A.m+1), \( m \in \{1,2,3,4\} \), so we removed the former.

Note that the generality condition implied by SALdP flags with curves not in the boundary (Lemma 2.5) require no two points on the same fiber and no four on the same (1,1)-curve (as well as further generality conditions involving more than four points). The SALdP flags with boundary curves (Lemma 2.7) require no points on the fiber \( \pi(C_2) \) and no five points on the (2,1)-curve \( \pi(C_2) \). Thus the latter flags already rule out the flag with the (1,1)-curve (or any higher bi-degree curves) as such a curve would have to intersect \( \pi(C_2) \) but do not rule out the flag \( \{p_1,p_2\} \subset \Sigma \) with \( \Sigma \) a (0,1)-fiber different from \( \pi(C_2) \).
• In case (II.5A.m) we fixed a typo where $c_1$ and $c_2$ were interchanged [2, p. 1260, line −3].

• Still in (II.5A.3), (II.5A.4), and (II.5A.5) we added the requirement that no three of the $m$ points be collinear (such a line would intersect the conic $c_1$ at two points and the line $c_2$ at one point, see Figure 2).

• In (III.4.3) we also added the requirement that the $m = 3$ points not be collinear (such a line would intersect each of the lines $c_i$ at one point, see Figure 3).

Remark 5.3. Since the statement of Theorem 5.1 stipulates the $C_i$ intersect simply and normally, the curves in (II.1A), (II.4A), (II.4B), (II.5A.m) intersect at two distinct points (i.e., we do not allow tangency).

The proof of Theorem 5.1 we give here contains the following main steps.

Step 1. We classify all strongly asymptotically log del Pezzo pairs $(s, c)$ with $\text{rk} \, \text{Pic}(s) \leq 2$. This is a straightforward computation involving ampleness conditions on Hirzebruch surfaces and is carried out in an Appendix (Proposition 6.1).

Step 2. We eliminate from the list of Step 1 all pairs that are obtained from another pair in the list via a smooth boundary blow-up (Definition 2.1). This is straightforward and is stated in Proposition 6.3.

Step 3. We introduced the notion of a SALdP flag in Definition 2.2. In this step (Theorem 4.1) we classify all SALdP flags whose zero-dimensional part lies in the boundary for the pairs in the
list of Step 1. This yields the precise generality conditions appearing in the final classification statement (Theorem 5.1) which slightly improves on the statement in 2.

**Step 4.** This step was carried out in Proposition 3.2. In sum, we prove, similarly to 2, but with different emphasis, that the list obtained in Step 3 contains all strongly asymptotically log del Pezzo pairs. The proof of this step is summarized in Figure 1.

**Proof of Theorem 5.1.** According to Proposition 3.2 every strongly asymptotically log del Pezzo pair can be expressed as the smooth boundary blow-ups of a strongly asymptotically log del Pezzo pairs \((s, c)\) with \(\text{rk}\ \text{Pic}(s) \leq 2\). According to Proposition 6.3, Theorem 4.1, and Theorem 2.3, the list in the statement of Theorem 5.1 consists precisely of all strongly asymptotically log del Pezzo pairs \((s, c)\) with \(\text{rk}\ \text{Pic}(s) \leq 2\) that are not the smooth boundary blow-up of any other strongly asymptotically log del Pezzo pair, together with all of their smooth boundary blow-ups that remain strongly asymptotically log del Pezzo.

In particular, for the case (I.5.m), there are two types of flags according to Theorem 4.1 (I.1A). The second type of flags already forces \(m \leq 8\). Feeding this information into the first type of flags precludes \(\Sigma\) from being a smooth curve of degree 3 or higher and by 2, Lemma 2.2, we know the resulting blow-up will be strongly asymptotically log del Pezzo if and only if \(S\) is del Pezzo. The del Pezzo–Manin–Hitchin’s classification of del Pezzo surfaces 3, 6, 12, 13 (see also 17 for an exposition in the language of flags close in spirit to this article) then shows that the only possibilities for \(\Sigma\) are a smooth line, a smooth conic, or a singular cubic with a single double point. The analysis for the cases (II.5A.m) and (III.4.m) is similar but simpler since the fact that no two points can be on a boundary line component (coming from the second type of flags in Theorem 4.1 (II.5A.m), (III.4.m)) precludes any flag with \(\Sigma \not\subset c\) of degree 2 or higher. To illustrate this, let us consider the case (II.5A.m). If \(\Sigma \neq \pi(C_2)\) is a conic in \(\mathbb{P}^2\) then either it intersects the line \(\pi(C_1)\) at two distinct points which is already taken care of by the first type of flags, or else it is tangent to \(\pi(C_1)\) at a single point, however that is also taken care of since we cannot blow-up infinitely near points on the smooth part of the boundary by 2, Lemma 2.5. Blow-ups of case (I.4A) are contained in the case (I.5.m) by the classification of del Pezzo surfaces, while blow-ups of (II.4B) are contained in case (II.5A.m) by Remark 5.2. For cases (I.9B.m), (II.6A.n.m), (II.6B.n.m), (II.7.m) the only obstruction are pairs of points on the same fiber according to Theorem 4.1 (I.4B), (II.2A.n), (II.2B.n), (II.3), and for the case (III.5.n.m) one has the previous obstruction as well as the obstruction of blowing-up any points on the fiber boundary component of (III.3.n) by Theorem 4.1 (III.3.n). For the cases (I.6B.m), (I.6C.m), (I.7.n.m), (I.8B.m), (I.9C.m), (II.5B.m) any smooth boundary blow-ups are allowed as by Theorem 4.1 there are no obstructing SALdP flags for (I.1B), (I.1C), (I.2.n), (I.3B), (I.4C), (II.1B).

6  **Appendix: Classification with small Picard group**

Let \(F_n\) be the \(n\)-th Hirzebruch surfaces, i.e., the unique rational surface with Picard group of rank 2 and a curve of self-intersection \(-n\), denoted \(Z_n \subset F_n\). We refer the reader to 2 §1.5 for our conventions and further background. Denote by \(F\) the class of fiber, i.e., an irreducible smooth rational curve such that \(F^2 = 0\) and \(F.Z_n = 1\). If \(n = 0\) when we refer to \(Z_0\) and \(F\) we intend that each is a fiber of a different projection to \(\mathbb{P}^1\). Hirzebruch surfaces are ruled toric surfaces and applying adjunction yields

\[
-K_S \sim 2Z_n + (n + 2)F.
\]  

(6.1)
Recall that every smooth irreducible curve in \(|Z_n + nF|\) (a ‘zero section’) intersects each fiber transversally at a single point and does not intersect the ‘infinity section’ \(Z_n\). Any curve \(C\) on \(\mathbb{F}_n\) satisfies

\[ C \sim aZ_n + bF, \tag{6.2} \]

with \(a, b \in \mathbb{N} \cup \{0\}\). Also,

\[ C \text{ is ample if and only if } a > 0 \text{ and } b > na, \tag{6.3} \]

and furthermore,

\[ C \text{ is an irreducible curve only if } C = Z_n \text{ or } b \geq na \geq 0, \tag{6.4} \]

and under such conditions the class \(6.2\) always contains an irreducible curve which in the latter case is nef.

**Proposition 6.1.** Let \(S\) be a smooth surface with \(\text{rk}(\text{Pic}(S)) \leq 2\), and let \(C_1, \ldots, C_r\) be distinct irreducible smooth curves on \(S\) such that \(C = \sum_{i=1}^r C_i\) is a divisor with simple normal crossings. Then \((S, C)\) is a strongly asymptotically log del Pezzo pair if and only if it is one of the following:

(I.1A) \(S = \mathbb{P}^2\), \(C_1\) is a cubic,

(I.1B) \(S = \mathbb{P}^2\), \(C_1\) is a conic,

(I.1C) \(S = \mathbb{P}^2\), \(C_1\) is a line,

(I.2n) \(S = \mathbb{F}_n\) for any \(n \geq 0\), \(C_1 = Z_n\),

(I.3A) \(S = \mathbb{F}_1\), \(C_1 \in |2(Z_1 + F)||,

(I.3B) \(S = \mathbb{F}_1\), \(C_1 \in |Z_1 + F|,

(I.4A) \(S = \mathbb{P}^1 \times \mathbb{P}^1\), \(C_1\) is a \((2, 2)\)-curve,

(I.4B) \(S = \mathbb{P}^1 \times \mathbb{P}^1\), \(C_1\) is a \((2, 1)\)-curve,

(I.4C) \(S = \mathbb{P}^1 \times \mathbb{P}^1\), \(C_1\) is a \((1, 1)\)-curve,

(I.5.1) \(S = \mathbb{F}_1\), \(C_1 \in |2Z_1 + 3F|,

(I.6B.1) \(S = \mathbb{F}_1\), \(C_1 \in |Z_1 + 2F|,

(I.6C.1) \(S = \mathbb{F}_1\), \(C_1 \in |F|,

(II.1A) \(S = \mathbb{P}^2\), \(C_1\) is a conic, \(C_2\) is a line,

(II.1B) \(S = \mathbb{P}^2\), \(C_1, C_2\) are lines,

(II.2A.n) \(S = \mathbb{F}_n\) for any \(n \geq 0\), \(C_1 = Z_n\), \(C_2 \in |Z_n + nF|,

(II.2B.n) \(S = \mathbb{F}_n\) for any \(n \geq 0\), \(C_1 = Z_n\), \(C_2 \in |Z_n + (n + 1)F|,

(II.2C.n) \(S = \mathbb{F}_n\) for any \(n \geq 0\), \(C_1 = Z_n\), \(C_2 \in |F|,

(II.3) \(S = \mathbb{F}_1\), \(C_1, C_2 \in |Z_1 + F|,

(II.4A) \(S = \mathbb{P}^1 \times \mathbb{P}^1\), \(C_1, C_2\) are \((1, 1)\)-curves,
(II.4B) \( S = \mathbb{P}^1 \times \mathbb{P}^1 \), \( C_1 \) is a \((2,1)\)-curve, \( C_2 \) is a \((0,1)\)-curve,

(II.5A.1(a)) \( S = F_1 \), \( C_1 \in |2Z_1 + 2F| \), \( C_2 \in |F| \),

(II.5A.1(b)) \( S = F_1 \), \( C_1 \in |Z_1 + 2F| \), \( C_2 \in |Z_1 + F| \),

(II.5B.1) \( S = F_1 \), \( C_1 \in |F| \), \( C_2 \in |Z_1 + F| \),

(III.1) \( S = \mathbb{P}^2 \), \( C_1, C_2, C_3 \) are lines,

(III.2) \( S = \mathbb{P}^1 \times \mathbb{P}^1 \), \( C_1, C_2, C_3 \) are \((1,1)\)-, \((0,1)\)- and \((1,0)\)-curves, respectively,

(III.3.n) \( S = F_n \) for any \( n \geq 0 \), \( C_1 = Z_n \), \( C_2 \in |F| \), \( C_3 \in |Z_n + nF| \),

(III.4.1) \( S = F_1 \), \( C_1 \in |F| \), \( C_2, C_3 \) are curves in \(|Z_1 + F|\),

(IV) \( S = \mathbb{P}^1 \times \mathbb{P}^1 \), \( C_1, C_2 \) are \((1,0)\)-curves, \( C_3, C_4 \) are \((0,1)\)-curves.

Remark 6.2. The curves in (II.1B), (II.4A), (II.4B), (II.5A.1(a)), (II.5A.1(b)) intersect at two distinct points, see Remark 5.3.

Proof. By Castelnuovo’s rationality criterion, \( S \) is rational, hence by the classification of rational surfaces with \( \text{rk}(\text{Pic}(S)) \leq 2 \) it is either \( \mathbb{P}^2 \) or \( F_n \) \( \text{[2–3]} \). When \( S = \mathbb{P}^2 \), \( \text{rk}(\text{Pic}(S)) = 1 \), \(-K_S \sim 3H\) and we see the possibilities are \((1.1A),(1.1B),(1.1C),(1.1A),(1.1B),(1.1B),(1.1B)\), (III.1). Assume now that \( S = F_n \). Denote

\[
C_i \in |a_i Z_n + b_i F|.
\]

Since \(-K_S - C\) is nef and using \( [6.4] \), we see that

\[
\sum_i a_i \in \{0, 1, 2\}, \quad \sum_i b_i \in \{0, \ldots , n + 2\}.
\]

Note that by \( [6.4] \) either \( b_i \geq na_i > 0 \) or else \((a_i, b_i) = (0, 1)\), i.e., \( C_i \) is a fiber. Thus, at most two components of \( C_i \) are not fibers. Also, if \( a_i = 2 \) for some \( i \) then \( n \leq 2 \) by \( n + 2 \geq b_i \geq 2n \). So we get the following possibilities when max, \( a_i = 2 \):

\[
[n, (a_1, b_1), \ldots , (a_r, b_r)] \in \{(2, (2, 4)], [1, (2, 3)], [1, (2, 2), (0, 1)]],
\]

\[
[1, (2, 2)], [0, (2, 2)], [0, (2, 1), (0, 1)], [0, (2, 1)]\}.
\]

(6.5)

When max, \( a_i = 1 \), we split into two subcases: when there are at least two pairs \((a_i, b_i)\) with all coefficients positive:

\[
[n, (a_1, b_1), \ldots , (a_r, b_r)] \in \{(2, (1, 2), (1, 2)], [1, (1, 2), (1, 1)],
\]

\[
[1, (1, 1), (1, 1), (0, 1)], [1, (1, 1), (1, 1), (0, 1)], (0, (1, 1), (1, 1))\}.
\]

(6.6)

and otherwise, still with max, \( a_i = 1 \), now for all \( n \) (so we omit the first index), and with max, \( b_i = n \),

\[
[(a_1, b_1), \ldots , (a_r, b_r)] \in \{(1, n), (1, 0), (0, 1)], [1, n), (1, 0), (0, 1)]],
\]

\[
[(1, n), (1, 0)], [(1, n), (0, 1), (0, 1)], [(1, n), (0, 1)], [(1, n)]\}.
\]

(6.7)

with max, \( b_i = n + 1 \),

\[
[(a_1, b_1), \ldots , (a_r, b_r)] \in \{(1, n + 1), (1, 0), (0, 1)],
\]

\[
[(1, n + 1), (1, 0)], [(1, n + 1), (0, 1)], [(1, n + 1)]\}.
\]

(6.8)
with \( \max_i b_i = n + 2 \),

\[
[(a_1, b_1), \ldots, (a_r, b_r)] \in \left\{ \left[(1, n + 2), (1, 0)\right], \left[(1, n + 2)\right] \right\}, \tag{6.9}
\]

when \( \max_i a_i = 1 \),

\[
[(a_1, b_1), \ldots, (a_r, b_r)] \in \left\{ \left[(1, 0)\right], \left[(1, 0), (0, 1)\right], \ldots, \left[(1, 0), (0, 1), \ldots, (0, 1)\right] \right\}, \tag{6.10}
\]

and finally when \( \max_i a_i = 0 \),

\[
[(a_1, b_1), \ldots, (a_r, b_r)] \in \left\{ \left[(0, 1)\right], \ldots, \left[(0, 1), \ldots, (0, 1)\right] \right\}. \tag{6.11}
\]

A few of these cases can be eliminated, though most of them actually occur. In [6.5], \([2,(2,4)]\) corresponds to a smooth anticanonical curve in \( \mathbb{P}_2 \), which is excluded as \( \mathbb{P}_2 \) is not del Pezzo. The remaining cases are: \([1,(2,3)] = (I.5.1)\), \([1,(2,2),(0,1)] = (II.5A.1)\) (a), \([1,(2,2)] = (I.3A)\), \([0,(2,2)] = (I.4A)\), \([0,(2,1),(0,1)] = (II.4B),(0,(2,1)] = (I.4B)\).

In [6.6] \([2,(1,2),(1,2)]\) is excluded as \( Z_2(Z_2+2F)=Z_2(2Z_2+4F)=Z_2(-K_{F_2})=0 \). The remaining cases are: \([1,(1,2),(1,1)] = (II.5A.1)\) (b);
\([1,(1,1),(1,1),(0,1)] = (III.4.1)\), \([1,(1,1),(1,1)] = (II.3)\), \([0,(1,1),(1,1)] = (II.4A)\).

In [6.7], \([1,n),(1,0),(0,1),(0,1)\) gives \(-K_{F_n}-(1-\beta_1)(Z_n+nF)-(1-\beta_2)Z_n-(1-\beta_3)Z_n-(1-\beta_4)Z_n+(n\beta_1+\beta_2+\beta_3+\beta_4)F\), that is ample if and only if \( n\beta_1+\beta_2+\beta_3 > n\beta_1+n\beta_2 \)
forcing \( n = 0 \) and this is (IV) \([1,n),(1,0),(0,1)] = (III.3)\). \([1,n),(1,0)\) \( = (II.2A.m)\). For \([1,n),(0,1),(0,1)\] consider \(-K_{F_n}-(1-\beta_1)(Z_n+nF)-(1-\beta_2)Z_n-(1-\beta_3)Z_n+(n\beta_1+\beta_2+\beta_3)F\), that is ample if and only if \( n\beta_1+\beta_2+\beta_3 > n+n\beta_1 \), i.e., \( n = 0 \), and this is (III.3.0).

For \([1,n),(0,1)\], consider \(-K_{F_n}-(1-\beta_1)(Z_n+nF)-(1-\beta_2)F \sim (1+\beta_1)Z_n+(n\beta_1+\beta_2)F\), that is ample if and only if \( 1+n\beta_1+\beta_2 > n+n\beta_1 \), i.e., \( n = 0,1 \), and these are (II.2C.0), (II.5B.1).

For \([n,1)\], \(-K_{F_n}-(1-\beta_1)(Z_n+nF)-(1-\beta_2)F \sim (1+\beta_1)Z_n+(n\beta_1+\beta_2+\beta_3)F\) implying \( n = 0,1,2 \) and these are (II.2.0), (I.3B) while the case \( n = 2 \) is excluded as in the first paragraph.

In [6.8] \([1,n+1),(1,0),(0,1)\] \( -K_{F_n}-(1-\beta_1)(Z_n+n+1F)-(1-\beta_2)Z_n-(1-\beta_3)F \sim (\beta_1+\beta_2)Z_n+(n+1\beta_1+\beta_3)F \) that is ample if and only if \( n+1\beta_1+\beta_3 > n(\beta_1+\beta_2) \), forcing \( n = 0 \) and this is (III.2) \([1,n+1),(1,0)] = (II.2B.n)\). \([1,n+1),(0,1)] \(-K_{F_n}-(1-\beta_1)(Z_n+n+1F)-(1-\beta_2)F \sim (1+\beta_1)Z_n+(1+\beta_1)Z_n+(n+1\beta_1+\beta_2)F\) that is ample if and only if \( n+1\beta_1+\beta_3 > n(n+1\beta_1+\beta_2) \), i.e., \( n = 0 \) and this is (II.2B.0) \([1,n+1)]\). \(-K_{F_n}-(1-\beta_1)(Z_n+n+1F)-(1+\beta_1)Z_n+(1+(n+1)\beta_1+\beta_3)F\) that is ample if and only if \( 1+(n+1)\beta_1 > n(1+\beta_1) \), i.e., \( n = 0,1 \), and these are (I.4C), (I.6B.1).

In [6.9] \([1,n+2),(1)] \(-K_{F_n}-(1-\beta_1)(Z_n+n+2F)-(1-\beta_2)Z_n+(n+2+\beta_1+\beta_3)F\) if \( 2\beta_1 > n\beta_2 \), so \( n = 0 \) and this is (II.4B) \([1,n+2)\). \(-K_{F_n}-(1-\beta_1)(Z_n+n+2F) \sim (1+\beta_1)Z_n+(n+2\beta_1+\beta_3+\beta_4)F\), i.e., \( n = 0 \) and this is (I.4B).

In [6.10] there is one \( Z_2 \) fibers, with \( k \geq 0 \): \(-K_{F_n}-(1-\beta_1)Z_n-(1-\beta_2)F-(1-\beta_3)F \sim (1+\beta_1)Z_n+(n+2\beta_1+\beta_3+\beta_4)F\), that is ample if and only if \( n(1+\beta_1) < n+2-k+\beta_2+\ldots+\beta_k+1 \), i.e., \( k = 0,1,2 \). When \( k = 0 \) this is (I.2.n), when \( k = 1 \) this is (II.2C.n), and when \( k = 2 \) this means \( n\beta_1 < (\beta_2+\beta_3) \), forcing \( n = 0 \) and this is (III.3.0).

Finally, in [6.11] there are \( k > 0 \) fibers, so \(-K_{F_n}-(1-\beta_1)F-(1-\beta_2)F-(1-\beta_3)F \sim 2Z_n+(n+2-k+\beta_1+\ldots+\beta_k)F\), that is ample if and only if \( 2n < n+2-k+\beta_1+\ldots+\beta_k \), i.e., \( n = k = 1 \) and (I.6C.1), or \( n = 0 \) and \( k = 1,2 \) and these are (I.2.0), (I.2A.0).

As a corollary, we obtain the following which is needed as a step in the proof of Theorem

[#]
Proposition 6.3. Let \((S, C = \sum_{i=1}^r C_i)\) be strongly asymptotically log del Pezzo pair with \(\text{rk} (\text{Pic} (S)) \leq 2\) that is not the smooth boundary blow-up (Definition 2.1) of any other strongly asymptotically log del Pezzo pair. Then \((S, C)\) is a strongly asymptotically log del Pezzo pair if and only if it is one of the following:

(I.1A) \(S = \mathbb{P}^2, C_1\) is a cubic,

(I.1B) \(S = \mathbb{P}^2, C_1\) is a conic,

(I.1C) \(S = \mathbb{P}^2, C_1\) is a line,

(I.2.n) \(S = F_n\) for any \(n \geq 0\), \(C_1 = Z_n\),

(I.3A) \(S = F_1, C_1 \in |2(Z_1 + F)|\),

(I.3B) \(S = F_1, C_1 \in |Z_1 + F|\),

(I.4A) \(S = \mathbb{P}^1 \times \mathbb{P}^1, C_1\) is a \((2, 2)\)-curve,

(I.4B) \(S = \mathbb{P}^1 \times \mathbb{P}^1, C_1\) is a \((2, 1)\)-curve,

(I.4C) \(S = \mathbb{P}^1 \times \mathbb{P}^1, C_1\) is a \((1, 1)\)-curve,

(II.1A) \(S = \mathbb{P}^2, C_1\) is a conic, \(C_2\) is a line,

(II.1B) \(S = \mathbb{P}^2, C_1, C_2\) are lines,

(II.2.n) \(S = F_n\) for any \(n \geq 0\), \(C_1 = Z_n, C_2 \in |Z_n + nF|\),

(II.2.n) \(S = F_n\) for any \(n \geq 0\), \(C_1 = Z_n, C_2 \in |Z_n + (n + 1)F|\),

(II.2.n) \(S = F_n\) for any \(n \geq 0\), \(C_1 = Z_n, C_2 \in |F|\),

(II.3) \(S = F_1, C_1, C_2 \in |Z_1 + F|\),

(II.4A) \(S = \mathbb{P}^1 \times \mathbb{P}^1, C_1, C_2\) are \((1, 1)\)-curves,

(II.4B) \(S = \mathbb{P}^1 \times \mathbb{P}^1, C_1\) is a \((2, 1)\)-curve, \(C_2\) is a \((0, 1)\)-curve,

(III.1) \(S = \mathbb{P}^2, C_1, C_2, C_3\) are lines,

(III.2) \(S = \mathbb{P}^1 \times \mathbb{P}^1, C_1\) is a \((1, 1)\)-curve, \(C_2\) is a \((0, 1)\)-curve, \(C_3\) is a \((1, 0)\)-curve,

(III.3.n) \(S = F_n\) for any \(n \geq 0\), \(C_1 = Z_n, C_2 \in |F|, C_3 \in |Z_n + nF|\),

(IV) \(S = \mathbb{P}^1 \times \mathbb{P}^1, C_1, C_2\) are \((1, 0)\)-curves, \(C_3, C_4\) are \((0, 1)\)-curves,

Proof. By Proposition 6.1 \((s, c)\) must be one of the pairs listed there. Of those listed there, \((I.5.1), (I.6B.1), (I.6C.1), (II.5A.1 (a)), (II.5A.1 (b)), (II.5B.1), (III.4.1)\) are manifestly obtained as the smooth boundary blow-up (Definition 2.1) of the pairs \((I.1A), (I.1B), (I.1C), (II.1A)\) (blow-up on the line in \((a)\), blow-up on the conic in \((b)\)), \((II.1B), (III.1)\), respectively. This completes the proof since the list of Proposition 6.3 coincides with that of Proposition 6.1 modulo those 7 cases.

\(\square\)
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