Nonlinear Quantum Mechanics and Locality*

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Abstract

It is shown that, in order to avoid unacceptable nonlocal effects, the free parameters of the general Doebner-Goldin equation have to be chosen such that this nonlinear Schrödinger equation becomes Galilean covariant.

1 Introduction

Usually linear equations in physics have the status of useful approximations to actually nonlinear laws of nature. Therefore many authors asked whether the fundamental linearity of quantum mechanics in the form of the ‘superposition principle’ plays a similar role. Also in view of the persisting difficulties to combine the fundamental principles of quantum mechanics with those of relativity into a rigorous theory with nontrivial interaction it seems worthwhile to test nonlinear modifications of ordinary quantum mechanics. For such reasons, many authors suggested the addition of nonlinear terms to the linear Schrödinger equation while maintaining the usual statistical interpretation concerning the localization of the physical system (see, e.g., [Mic74, BBM76, HB78, Wei89, DG94]).

Unfortunately, the general interest in such theories was strongly diminished by N. Gisin’s claim that every (‘deterministic’) nonlinear Schrödinger equation leads to nonlocality of unacceptable type [Gis91, Gis95]. However, his reasoning relies on the tacit assumption – not justified at all [Luc95] – that the theory of measurements developed for the linear theory may be applied to the nonlinear case, too. Therefore the question whether nonlinear modifications of ordinary quantum mechanics may be physically consistent deserves further investigation. Actually, the question is whether, for a 2-particle system with fixed initial conditions, the nonlinearity allows

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to influence the position probability of particle 1 by acting on particle 2 if there is no explicit interaction between the particles. In the following we will show that this possibility really exists for some cases of the general Doebner-Goldin equation, at least.

This contribution is organized as follows. In Section 2 we will specify the type of nonlinear quantum mechanics we are going to analyze. In Section 3 the central problem will be posed and recent results by R. Werner related to this will be reported. In Section 4, finally, Werner’s conjecture concerning nonlocality of the general Doebner-Goldin equation will be confirmed by simple explicit calculations. We conclude with a short summary and further perspectives.

2 Nonlinear Quantum Mechanics

Let us consider a typical nonlinear Schrödinger equation

\[ i \partial_t \Psi_t(x) = H \Psi_t(x) + F_t(\Psi_t)(x) \] (1)

which is formally local in the sense that the nonlinearity \( F_t \) added to the usual Schrödinger equation with Hamiltonian \( H \) is some local (non-linear) functional \( F_t \),

\[ F_t[\Psi](x) = F_t[\Phi](x) \quad \forall x \notin \text{supp} (\Psi - \Phi), \]

that should be ‘sufficiently small’ in the sense that it does not introduce too strong deviations from the predictions of the linear theory. The question is whether the usual (nonrelativistic) quantum mechanical interpretation

\[ |\Psi_t(x)|^2 = \left\{ \begin{array}{ll} \text{probability density for the system to be localized around } x \text{ at time } t \\ \text{localization around } x \text{ at time } t \end{array} \right. \] (2)

may still be physically acceptable. Of course, (2) requires the norm of solutions \( \Psi_t \) of (1) to be \( t \)-independent. This is automatically fulfilled if we restrict to nonlinearities of the form

\[ F[\Psi] = R[\Psi] \Psi, \quad R[\Psi] = \overline{R[\Psi]}, \] (3)

since then (1) implies the ordinary continuity equation. In view of the mentioned locality problem let us concentrate on the case of two noninteracting particles of different type in individual external potentials \( V_1, V_2 \):

\[ i \partial_t \Psi_t^V(x_1, x_2) = \left( -\frac{1}{2m_1} \Delta x_1 + V_1(x_1, t) - \frac{1}{2m_2} \Delta x_2 + V_2(x_2, t) \right) \Psi_t^V(x_1, x_2) \]

\[ + F \left( \Psi_t^V(x_1, x_2) \right) \] (4)

\(^1\text{We use natural units, therefore } \hbar = 1.\)
(here $\vec{x} = (\vec{x}_1, \vec{x}_2)$ and to nonlinearities of the Bialynicki-Birula–Mycielski type
\[ F_{BB}(\Psi) = -\ln \rho \Psi \]  
(5)
or the Doebner-Goldin type\footnote{The general Doebner-Goldin equation \cite{DG92, DG94} arises from this family by nonlinear gauge transformations that do not change $\rho(\vec{x}, t)$ \cite{Nat93, Nat94}.}
\[ F_{DG}(\Psi) = \left( c_1 \frac{\nabla \cdot \vec{J}}{\rho} + c_2 \Delta \rho + c_3 \frac{\vec{J}^2}{\rho^2} + c_4 \frac{\vec{J} \cdot \nabla \rho}{\rho^2} + c_5 \frac{(\nabla \rho)^2}{\rho^2} \right) \Psi , \]  
(6)
where we use the notation
\[ \rho = |\Psi|^2 , \quad \vec{J} = \frac{1}{2i} \left( \nabla \bar{\nabla} \Psi - \Psi \bar{\nabla} \right) , \]
We assume that there are sufficiently many solutions of (4) for which the formal singularities, introduced especially by (6), do not cause any problems (see \cite{CH80} and \cite{T97}, in this connection). Interaction between the particles and the case of identical particles will be discussed later.

3 The Locality Problem

In both cases, (5) and (6), $F$ is of the form (3) with
\[ R(\phi_1 \otimes \phi_2) = R(\phi_1) + R(\phi_2) , \]
and therefore
\[ \Psi^V_1(\vec{x}_1, \vec{x}_2) = \phi^V_1(\vec{x}_1) \phi^V_2(\vec{x}_2) \]
is a solution of (4) whenever the $\phi^V_i$ are solutions of the corresponding 1-particle equations
\[ i\partial_t \phi^V_i = \left( H_j + R(\phi^V_i) \right) \phi^V_j , \]
where
\[ H_j \overset{\text{def}}{=} -\frac{1}{2m_j} \Delta \vec{x} + V_j(\vec{x}, t) . \]
This ensures that we cannot influence particle 1 by action on particle 2 by change of $V_2$ \textbf{if} the fixed initial conditions are factorized. However, most interesting features of quantum mechanics are connected with entangled states (nonfactorized initial conditions). For linear $F$, since the particles do not interact with each other, we even have \textbf{full separability} \footnote{Actually, one should allow for magnetic fields.}

For arbitrarily fixed initial conditions, the partial state of particle 1 does not depend on $V_2$.\footnote{The general Doebner-Goldin equation \cite{DG92, DG94} arises from this family by nonlinear gauge transformations that do not change $\rho(\vec{x}, t)$ \cite{Nat93, Nat94}.}
In other words:

$$\langle \Psi_t \mid A \otimes 1 \mid \Psi_t \rangle$$

does not depend on $V_2$ for any self-adjoint operator in $L^2(\mathbb{R}^3)$.

That the latter statement is no longer true for nonlinear $F$ (irrelevant Gisin effect \cite{Gis95}) does not mean that the former statement is wrong for nonlinear $F$, too \cite{Luc95}. However, full separability should be equivalent to $V_2$-independence of

$$\rho_{1,V}(\vec{x}_1, t) \overset{\text{def}}{=} \int \left| \Psi^V_t(\vec{x}_1, \vec{x}_2) \right|^2 d\vec{x}_2.$$ (7)

If $\rho_{1,V}(\vec{x}_1, t)$ changes with localized (in space and time) variations of $V_2$ then we have a relevant Gisin effect (unacceptable nonlocality):

An arbitrarily small localized variation of $V_2$ may influence particle 1 at any distance by the same amount (just translate $V_1$ and the initial condition w.r.t. $\vec{x}_1$).

Unfortunately, we do not have sufficient control on solutions of (4). Therefore the only possibility to uncover relevant Gisin effects, for the time being, is to determine $((\partial_t)^{\nu} \rho_{1,V})_{|t=0}$ for fixed (entangled) initial conditions and see whether this depends on $V_2$ for sufficiently large $\nu$. Very recently Reinhard Werner (Technische Universität Braunschweig) performed a computer algebraic test of this sort for oscillator potentials $V_j(\vec{x}_j) = \kappa_j \| \vec{x}_j \|^2$ making the Ansatz

$$\Psi^V_t(\vec{x}_1, \vec{x}_2) = \exp (-Q_t(\vec{x}_1, \vec{x}_2)) ,$$

where $Q_t$ is a time-dependent 2nd order polynomial with positive real part initial value $Q_0$ such that $\Psi^V_0(\vec{x}_1, \vec{x}_2)$ is not factorized. Werner found that $((\partial_t)^{3} \rho_{1,V})_{|t=0}$ depends on $\kappa_2$ unless

$$c_3 = c_1 + c_4 = 0$$ (8)

Now, a variation of $\kappa_2$ means a nonlocalized variation of $V_2$. But a global variation of $V_2$ may be approximated by a local variation. Thus Werner concluded that violation of (8) implies relevant Gisin effects.\footnote{Note that condition (8) is equivalent to Galilei covariance of (4) for $F = F_{DG}$ \cite{Nat93}!}

The only objection against the physical relevance of Werner’s result could be that in order to influence the position of particle 1 one might need local variations of $V_2$ of such strength that the nonrelativistic equation (4), designed for sufficiently low energies, is no longer applicable, anyway. Moreover, Werner himself admitted that Gaussian solutions might be too special and, therefore, (8) might not guarantee absence of relevant Gisin effects. Therefore it is desirable to determine the $V_2$-dependent part of $((\partial_t)^{\nu} \rho_{1,V})_{|t=0}$ for essentially arbitrary initial conditions and potentials. This will be done in the next Section for $\nu = 3$, as a first step.

\footnote{Anyway, by (3), full separability implies this condition.}

\footnote{Investigating $((\partial_t)^{\nu} \rho_{1,V})_{|t=0}$ also for $\nu = 4, \ldots, 8$ Werner did not find anything more.}
4 Confirmation of Werner’s Results

Obviously, as a consequence of the continuity equation

\[
\partial_t \left| \Psi_t^V(x_1, x_2) \right|^2 + \nabla_{x_1} \cdot j_{1,V}(x_1, x_2) + \nabla_{x_2} \cdot j_{2,V}(x_1, x_2) = 0, \tag{9}
\]

where

\[
\bar{j}_{1,V}(x_1, x_2) = \Re \left( \Psi_t^V(x_1, x_2) \right) \frac{1}{\imath m_1} \nabla_{x_1} \Psi_t^V(x_1, x_2)
\]

(and similarly for \( \bar{j}_{2,V} \)) a relevant Gisin effect is equivalent to nontrivial \( V_2 \)-dependence of

\[
\partial_t \rho_{1,V}(x_1, t) = - \int \left( \nabla_{x_1} \cdot \bar{j}_{1,V} \right)(x_1, x_2, t) \, d\bar{x}_2. \tag{10}
\]

So the crucial question is whether

\[
\partial_t \rho_{1,V}(x_1, t) \sim Q_\Phi^V(x_1, t) \triangleq \Im \int \left( H_1 \Phi + F(\Phi) \right) \Delta_{x_1} \Phi - \Phi \Delta_{x_1} \left( H_1 \Phi + F(\Phi) \right) d\bar{x}_2
\]

does not depend on \( V_2 \) and the only part of the r.h.s. of

\[
- \bar{Q}_\Phi^V(x_1, 0) = \Im \int \left( H_1 \Phi + F(\Phi) \right) \Delta_{x_1} \Phi - \Phi \Delta_{x_1} \left( H_1 \Phi + F(\Phi) \right) \, d\bar{x}_2,
\]

where

\[
\dot{F}(\Phi)(x_1, x_2) \triangleq \left. \partial_t F(\Psi_t^V)(x_1, x_2) \right|_{t=0}.
\]

By (3) this gives

\[
T_\Phi^F(x_1) = \Re \int \left( \Phi \bar{R} \Delta_1 \Phi - \Phi \Delta_1 \left( \Phi \dot{R} \right) \right) \, d\bar{x}_2, \tag{11}
\]

where

\[
\dot{R}(\Phi)(x_1, x_2) \triangleq \left. \partial_t R(\Psi_t^V)(x_1, x_2) \right|_{t=0}.
\]
For $R(\Psi) = G(\rho)$, therefore, (11) cannot depend on $V_2$ since neither $|\Psi_0^V|^2$ nor $(\partial_t \Psi_t^V)_{|t=0}$ does. In other words:

For arbitrary initial conditions $((\partial_t \rho_{1,V})_{|t=0})$ does not depend on $V_2$ if the nonlinear functional $R$ is a real linear combination of the three functionals

$$R_{BB}(\Psi) \equiv \ln \rho, \quad R_2(\Psi) \equiv \frac{\Delta \rho}{\rho}, \quad R_5(\Psi) \equiv \left(\frac{\nabla \rho}{\rho}\right)^2.$$

For $R_4(\Psi) = \vec{J} \cdot \vec{\nabla} \rho$, however, we have

$$\text{ess} \left( \partial_t R_4(\Psi_t^V) \right)_{|t=0} = \frac{\vec{\nabla} |\Phi|^2}{|\Phi|^4} \cdot \text{ess} \left( \partial_t \frac{\vec{\nabla} \Psi_t^V \vec{\nabla} \Psi_t^V - \Psi_t^V \vec{\nabla} \Psi_t^V}{2i} \right)_{|t=0}$$

$$= 2 \frac{\vec{\nabla} |\Phi|}{|\Phi|^3} \cdot \Re \left( \nabla_2 \Phi \vec{\nabla} \Phi - \Phi \vec{\nabla} \Phi_2 \right)$$

$$= -2 \frac{\vec{\nabla} |\Phi|}{|\Phi|^3} \cdot \Re \left( \Phi_2 \nabla_2 \Phi - \Phi \nabla \Phi_2 \right) = -2 \frac{\vec{\nabla} |\Phi|}{|\Phi|^3} \cdot \vec{\nabla} \Phi_2,$$

where 'ess' means 'V2-dependent part of' and $\Phi = \Psi_0^V$. Therefore the term

$$\text{ess} \left( \Phi \partial_t R_4(\Psi_t^V) \right)_{|t=0} = -2 \frac{\Phi}{|\Phi|} \left( \vec{\nabla} \Phi_2 \right) \cdot \vec{\nabla} \Phi_2$$

to be inserted in the integral defining $T_{F_4}^\Phi$ is sufficiently well behaved in order to take the limit of compactly supported $\Phi$. Thus we may simplify our check by considering localized $V_2$ fulfilling

$$r(\vec{x}_1, \vec{x}_1) \neq 0 \implies V_2(\vec{x}_2, 0) = g(x_2) \quad (12)$$

and initial conditions of the form

$$\Phi(\vec{x}_1, \vec{x}_2) = e^{is(\vec{x}_1, \vec{x}_2)} r(\vec{x}_1, \vec{x}_2), \quad r, s \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R}). \quad (13)$$

Then

$$\text{ess} \left( \Psi_t^V \partial_t R_4(\Psi_t^V) \right)_{|t=0} = \text{ess} \left( -2 \frac{\Phi \vec{\nabla} \Phi_2}{|\Phi|} \cdot \vec{\nabla} \Phi_2 \right) = -2g \frac{\Phi}{|\Phi|} \partial_{x_2} |\Phi|$$. 

and, consequently, the essential part of $T^\Phi_{F_4}$ is
\[
\text{ess } (T^\Phi_{F_4}) = -2g \Re \int \left( \frac{\Phi \partial_{x_2} |\Phi| \Delta_1 \Phi - \Phi \Delta_1 \Phi \partial_{x_2} |\Phi|}{|\Phi|} \right) \, d\vec{x}_2 \\
= -2g \Re \int \left( e^{-is} (\partial_{x_2} r) \Delta_{\vec{x}_1} (e^{is} r) - e^{-is} r \Delta_{\vec{x}_1} (e^{is} (\partial_{x_2} r)) \right) \, d\vec{x}_2.
\]

To simplify things further, let us assume that\(^7\)
\[s(\vec{x}_1, \vec{x}_2) = x_1 x_2. \tag{14}\]

Then
\[
\text{ess } (T^\Phi_{F_1}) = -2g \Re \int \left( e^{-is} (\partial_{x_2} r) \Delta_{\vec{x}_1} (e^{is} r) - e^{-is} r \Delta_{\vec{x}_1} (e^{is} (\partial_{x_2} r)) \right) \, d\vec{x}_2 \\
= -2g \int \left( (\partial_{x_2} r) \left( -x_2^2 r + \Delta_{\vec{x}_1} r \right) + r x_2^2 \partial_{x_2} r - r \Delta_{\vec{x}_1} \partial_{x_2} r \right) \, d\vec{x}_2 \\
= -2g \int \left( (\partial_{x_2} r) \Delta_{\vec{x}_1} r - r \Delta_{\vec{x}_1} \partial_{x_2} r \right) \, d\vec{x}_2 \\
= 4g \int r \Delta_{\vec{x}_1} \partial_{x_2} r \, d\vec{x}_2 \\
\neq 0 \text{ in general}. \tag{15}\]

We may conclude:

If $F(\Psi) = \vec{J} \cdot \vec{\nabla} \rho \Psi$ then there are allowed initial conditions $\Phi$ for which $((\partial_t)^3 \rho_{1,V})_{t=0}$ may be changed by localized variations of $V_2$.

Instead of checking the case
\[R(\Psi) = R_1(\Psi) \overset{\text{def}}{=} \frac{\vec{\nabla} \cdot \vec{J}}{\rho}\]
it is more convenient to consider
\[R(\Psi) = R_{1-4}(\Psi) \overset{\text{def}}{=} \frac{1}{2i} \Delta \ln \left( \frac{\Psi}{\overline{\Psi}} \right) = R_1(\Psi) - R_4(\Psi)\]
where we have
\[
\text{ess } \left( \partial_t R_{1-4}(\Psi^V) \right)_{t=0} = -\frac{1}{2} \Delta \text{ess } \left( \left( \overline{\Psi_t^V} / \Psi_t^V \right) i \partial_t \left( \Psi_t^V / \overline{\Psi_t^V} \right) \right)_{t=0} \\
= -\frac{1}{2} \Delta \text{ess } \left( \left( i \partial_t \Psi_t^V \right) / \Psi_t^V + (i \partial_t \overline{\Psi_t^V}) / \overline{\Psi_t^V} \right)_{t=0} \\
= -\Delta_2 V_2.
\]

\(^7\)In fact, (14) does not contribute to ess $(T^\Phi_{F_4})$, but will be needed later.

\(^8\)Note that
\[4g \int r \Delta_{\vec{x}_1} \partial_{x_2} r \, d\vec{x}_2 = \mp 8g (x_1 r_1 \partial_{x_1} r_1) \int r_2^2 \, d\vec{x}_2. \]

for
\[r(\vec{x}_1, \vec{x}_2) = (x_1 \pm x_2) r_1(\vec{x}_1) r_2(\vec{x}_2). \tag{15}\]
and therefore
\[
\text{ess} \left( T_{F_{3-4}}^\Phi \right) = \text{ess} \left( \Re \int \left( \Phi \bar{R}_{1-4} \Delta_1 \Phi - \bar{\Phi} \Delta_1 \left( \Phi \bar{R}_{1-4} \right) \right) \, d\vec{x}_2 \right) \\
= -\text{ess} \left( \Re \int \left( \left( \bar{\Phi} \Delta_2 V_2 \right) \Delta_1 \Phi - \bar{\Phi} \Delta_1 \left( \Phi \Delta_2 V_2 \right) \right) \, d\vec{x}_2 \right) \\
= 0,
\]
i.e.:

For arbitrary initial conditions, \( ((\partial_t)^3 \rho_1 V) \bigg|_{t=0} \) does not depend on \( V_2 \) if \( R(\Psi) = R_1(\Psi) - R_4(\Psi) \).

For
\[
R_3(\Psi) = \left( \frac{\vec{J}}{\rho} \right)^2,
\]
finally, we have
\[
\text{ess} \left( \partial_t R_3(\Psi^V) \right) \bigg|_{t=0} = \frac{2\vec{J}}{\rho^2} \cdot \text{ess} \left( \partial_t \vec{J} \right) \\
= -\frac{\vec{J}}{\rho^2} \left( \bar{\Phi} \left[ \nabla, V_2 \right] - \Phi \left[ \nabla, V_2 \right] - \bar{\Phi} \right) \\
\tag{12}
= i \text{g} \frac{\bar{\Phi} \partial \Phi - \Phi \partial \bar{\Phi}}{\rho}.
\]

Therefore the essential part of \( T_3^\Phi \) is
\[
-g \Im \int \left( \partial_2 \bar{\Phi} \Delta_{\vec{x}_1} \Phi - \bar{\Phi} \partial_2 \partial_1 \Phi + \bar{\Phi} \Delta_{\vec{x}_1} \partial_2 \Phi - \Phi \Delta_{\vec{x}_1} \left( \Phi \frac{\partial_2 \Phi}{\Phi} \right) \right) \, d\vec{x}_2 \\
= g \Im \int \bar{\Phi} \left( \partial_2 \bar{\Phi} \Delta_{\vec{x}_1} \partial_1 \Phi + \bar{\Phi} \Delta_{\vec{x}_1} \partial_2 \Phi - \Phi \Delta_{\vec{x}_1} \left( \Phi \frac{\partial_2 \Phi}{\Phi} \right) \right) \, d\vec{x}_2 \\
= g \Im \int \left( e^{-2is} \partial_2 \left( e^{is} \partial_1 \right) \Delta_{\vec{x}_1} \left( e^{is} \partial_1 \right) + e^{-is} \Delta_{\vec{x}_1} \left( e^{is} \partial_1 \partial_2 \partial_1 \Phi \frac{\partial_2 \Phi}{\Phi} \right) \right) \, d\vec{x}_2 \\
= g \Im \int \left( e^{-is} \left( \partial_2 \partial_1 \right) \Delta_{\vec{x}_1} \left( e^{is} \partial_1 \right) + e^{-is} \Delta_{\vec{x}_1} \left( e^{is} \partial_1 \partial_2 \partial_1 \Phi \frac{\partial_2 \Phi}{\Phi} \right) \right) \, d\vec{x}_2 \\
\tag{14}
= g \int \left( 2x_2 \left( \partial_2 \partial_1 \right) + 2x_2 \partial_1 \partial_2 \partial_1 \partial_2 r - 2r \partial_1 \partial_2 \partial_1 \partial_2 r \right) \, d\vec{x}_2,
\]
i.e.:
\[
\text{ess} \left( T_{F_3}^\Phi \right) = -4g \int r \partial_1 r \, d\vec{x}_2. \tag{16}
\]

Obviously, \( (16) \) is functionally independent of
\[
\text{ess} \left( T_{F_3}^\Phi \right) = 4g \int r \Delta_{\vec{x}_1} \partial_2 r \, d\vec{x}_2. \tag{17}
\]

\textit{For instance, \( (16) \) does not always vanish for factorized \( r \) whereas \( (17) \) does.}
Since, as shown in [Nat93], Werner’s condition (8) is equivalent to Galilei invariance of the general Doebner-Goldin equation (equation (1 with $F = F_{DG}$) we may conclude:

For solutions $\Psi^V_t$ of (1) with $F(\Psi) = \left( c_1 \frac{\vec{\nabla} \cdot \vec{J}}{\rho} + c_2 \frac{\Delta \rho}{\rho} + c_3 \left( \frac{\vec{J}}{\rho} \right)^2 + c_4 \frac{\vec{J} \cdot \vec{\nabla} \rho}{\rho^2} + c_5 \left( \frac{\vec{\nabla} \rho}{\rho} \right)^2 \right) \Psi$

$\left. (\partial_t^2 \rho_{1,V}(\vec{x}_1, t)) \right|_{t=0}$ cannot be changed by local $V_2$-variations (for arbitrarily fixed initial condition) if and only if the coefficients $c_\nu \in \mathbb{R}$ are chosen such that (1) is Galilei covariant.

5 Summary

We have seen that the general Doebner-Goldin equation has to be Galilei invariant in order to avoid unacceptable nonlocalities for noninteracting particles. Obviously, an interaction between the particles that vanishes for infinite separation of the particles would not have any influence on this conclusion. Similarly, since we considered local variations of $V_2$ and since (14) and (15) do not forbid any permutation symmetry of $\Phi$, the same conclusion applies to pairs of identical particles.

Whether Galilei invariance protects the general Doebner-Goldin equation against relevant Gisin effects is not yet clarified. It may well be that already $\left. (\partial_t^2 \rho_{1,V}(\vec{x}_1, t)) \right|_{t=0}$ depends on local $V_2$ variations even in the Galilei covariant case. The same, of course, applies to the Bialynicki-Birula–Mycielski equation (equation (1) with $F = F_{BB}$).

Let us finally remark that even ‘full separability’ would not yet be all one would like to have:

For every 2-particle initial wave function $\Phi$, $\rho_{1,V}$ should be the position probability density of a (possibly mixed) one-particle state, i.e. there should exist a sequence of families of (unnormalized) 1-particle solutions $\psi^V_{1,t}$ with

$$\sum_{\nu} \left| \psi^V_{1,t}(\vec{x}_1) \right|^2 = \int \left| \Psi^V_t(\vec{x}_1, \vec{x}_2) \right|^2 d\vec{x}_2 \quad \forall V = (V_1, V_2),$$

where $\Psi^V_t$ denotes the corresponding family of 2-particle solutions with $\Psi^V_0 = \Phi$.

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