The interaction between graviton spin and gravitomagnetic fields

J.Q. Shen
Zhejiang Institute of Modern Physics and Department of Physics, Zhejiang University, Hangzhou 310027, P.R. China
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This note is devoted to the detailed mathematical treatment of the coupling of graviton spin to gravitomagnetic fields. The expression \( (i.e., \sim g_{mn}g_{0n}(\partial_mg_{0n} - \partial_ng_{0m})) \) for the graviton spin-gravitomagnetic (S-G) coupling in the Lagrangian/Hamiltonian density of the weak gravitational fields is presented in this note.

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The interaction of graviton spin with gravitomagnetic fields under consideration is the extension of Mashhoon’s spin-rotation coupling \[1\], which is the interaction between the gravitomagnetic moment of a spinning particle and the noninertial frame of reference. To the best of our knowledge, the spin-rotation coupling of photon, electron and neutron has been taken into account in the literature \[1,2\]. However, the gravitational coupling of graviton spin to the gravitomagnetic fields, which may be of physical interest, has so far never yet been considered. In this note, we will extend Mashhoon’s spin-rotation coupling to a purely gravitational case, where the graviton spin will be coupled to gravitomagnetic fields. Here we are concerned mainly with the detailed mathematical treatment of the coupling of graviton spin to gravitomagnetic fields.

First let us consider the Christoffel symbol of the weak gravitational field, the first- and second-order terms of which may be expressed as follows \[3\]

\[
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} K \left( h^\alpha_{\beta,\gamma} + h^\alpha_{\gamma,\beta} - h^\alpha_{\beta\gamma} - \frac{1}{2} \delta^\alpha_{\beta} h^\lambda_{\gamma,\beta} - \frac{1}{2} \delta^\alpha_{\gamma} h^\lambda_{\beta,\gamma} + \frac{1}{2} \eta_{\beta\gamma} h^\alpha_{\lambda,\lambda} \right) \\
+ \frac{1}{2} K^2 \left( \frac{1}{2} \delta^\alpha_{\beta} h^\lambda_{\gamma,\gamma} + h^\alpha_{\gamma,\gamma} h^\lambda_{\beta,\alpha} + h^\alpha_{\gamma,\gamma} h^\lambda_{\beta,\alpha} - h^\alpha_{\beta,\gamma} h^\lambda_{\gamma,\alpha} + h^\alpha_{\beta,\gamma} h^\lambda_{\gamma,\alpha} \right) \\
+ \frac{1}{2} K^2 \left( -\frac{1}{2} \delta^\alpha_{\beta} h_{\lambda\gamma}^\gamma - \frac{1}{2} h^\alpha_{\gamma,\gamma} h^\lambda_{\beta,\gamma} - \frac{1}{2} h^\alpha_{\beta,\gamma} h^\lambda_{\gamma,\alpha} + \frac{1}{2} \eta_{\beta\gamma} h^\alpha_{\lambda,\lambda} \right) + O(h^3),
\]

(1)

where \( K \) and \( h^{\mu\nu} \) are so defined that \( \sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + Kh^{\mu\nu} \) is satisfied. Since there exists an exact analogy between general relativity and electrodynamics for weak gravitational fields \[4,5\], one may think of the expression associated with \( g_{mn}g_{0n}(\partial_mg_{0n} - \partial_ng_{0m}) \) in the gravitational Hamiltonian/Lagrangian density as the interaction term of the graviton spin with the gravitomagnetic fields, where \( \dot{g} \) denotes the time derivative of \( g_{0n} \). If the Lagrangian density \( L_{-g} \) contains the expression \( \sim g_{0n}g_{0n}(\partial_mg_{0n} - \partial_ng_{0m}) \) (here the both indices \( m \) and \( n \) take 1, 2, 3, i.e., summation is carried out over 1, 2, 3), then the Hamiltonian density of the weak gravitational field may be of the form \( T^{00} = 2 \frac{\partial L_{-g}}{\partial g_{0n}} \dot{g}_{0n} - L_{-g} = L_{-g} \). It should be noted that here the factor 2 results from the fact that \( \frac{\partial L_{-g}}{\partial g_{0n}} \dot{g}_{0n} \) includes \( \frac{\partial L_{-g}}{\partial g_{0n}} \dot{g}_{0n} + \frac{\partial L_{-g}}{\partial g_{0n}} \dot{g}_{0n} \). So, in order to find the interaction term between the graviton spin and the gravitomagnetic fields (in what follows it will be referred to as the graviton S-G coupling term) in the Hamiltonian density of the weak gravitational field, we should only take into consideration the graviton S-G coupling term in the Lagrangian density \( i.e., \sqrt{-g}g^{\mu\nu}R_{\mu\nu} \). Because of \( g^{\mu\nu}R_{\mu\nu} \) contains \( g^0mR_{0m} + g^m0R_{m0} \) \( (i.e., 2g^{0m}R_{0m}) \), we will analyze the \( 0m \) component of the Ricci tensor \( R_{\mu\nu} \) in the following. Note that the expression for \( R_{\mu\nu} \) is written as \( R_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\nu,\rho} + \Gamma^\rho_{\nu\sigma,\rho} + \Gamma^\rho_{\sigma\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu} \). Now we will extract the expression \( \sim g_{0n}(\partial_mg_{0n} - \partial_ng_{0m}) \) from \( R_{\mu\nu} \).

First we consider the terms \( \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\nu,\rho} \) in \( R_{\mu\nu} \).

A. \( \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\nu,\rho} \)

The second-order term (the coefficient of which is \( \frac{1}{2} K^2 \)) in \( \Gamma^\rho_{\mu\nu} \) reads

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*This paper presenting the trial and lengthy calculations only serves as the supplement to a paper entitled “The purely gravitational generalization of spin-motion couplings” (by J.Q. Shen).
Further analysis shows that the terms which truly give contribution to the graviton S-G coupling term, in the derivative of $\Gamma^\rho_{\mu\nu}$ with respect to $x^\nu$, i.e., $\Gamma^\rho_{\mu\rho,\nu}$, are given as follows

$$h_{\mu\lambda,\nu}h_{\gamma\rho} + h_{\rho\lambda,\nu}h_{\mu\gamma} + h_{\rho\gamma,\nu}h_{\mu\lambda} - h_{\mu\lambda,\nu}h_{\rho\gamma} - h_{\rho\lambda,\nu}h_{\mu\gamma} - h_{\rho\gamma,\nu}h_{\mu\lambda} - \frac{5}{2}h_{\lambda\tau,\nu}h_{\mu\gamma} - \frac{1}{2}h_{\mu\rho,\nu}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\nu}h_{\mu\gamma} + \frac{1}{2}h_{\mu\rho,\nu}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\nu}h_{\mu\gamma}.$$  

(2)

Thus the terms, which will probably contribute to the graviton S-G coupling term, in the derivative of $\Gamma^\rho_{\mu\nu}$, with respect to $x^\nu$, i.e., $\Gamma^\rho_{\mu\rho,\nu}$, are given as follows

$$h_{\mu\lambda,\nu}h_{\gamma\rho} + h_{\rho\lambda,\nu}h_{\mu\gamma} + h_{\rho\gamma,\nu}h_{\mu\lambda} - h_{\mu\lambda,\nu}h_{\rho\gamma} - h_{\rho\lambda,\nu}h_{\mu\gamma} - h_{\rho\gamma,\nu}h_{\mu\lambda} - \frac{5}{2}h_{\lambda\tau,\nu}h_{\mu\gamma} - \frac{1}{2}h_{\mu\rho,\nu}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\nu}h_{\mu\gamma} + \frac{1}{2}h_{\mu\rho,\nu}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\nu}h_{\mu\gamma}.$$  

(3)

But via the detailed analysis we find that the terms that truly give contribution to the graviton S-G coupling are $-h_{\rho\lambda,\nu}h_{\mu\gamma}$ and $h_{\rho\lambda,\nu}h_{\mu\gamma}$ (i.e., $-2\sqrt{-\eta}g_{\mu\nu} h_{\rho\lambda,\nu} h_{\mu\gamma}$), which contains

$$-2\sqrt{-\eta}g_{\mu\nu} h_{\rho\lambda,\nu} h_{\mu\gamma} - 2\sqrt{-\eta}h_{\rho\lambda,\nu} h_{\mu\gamma}.$$  

(4)

Here $h_{0\mu} = g^n$. The flat Minkowski metric $\eta_{\mu\nu} = \text{diag}[+1,-1,-1,-1]$. Clearly, the relation between the metric $g_{\mu\nu}$ (with $m = 1, 2, 3$) and the gravitomagnetic potential $g^m$ is that $\sqrt{-\eta}g_{\mu\nu} = K h_{0\mu} = Kg^m$.

The second-order term (the coefficient of which is $\frac{1}{2}K^2$) in $\Gamma^\rho_{\mu\nu}$ reads

$$\Gamma^\rho_{\mu\nu} \left( \frac{1}{2}K^2 \right) = h_{\mu\lambda,\nu}h_{\gamma\rho} + h_{\rho\lambda,\nu}h_{\mu\gamma} + h_{\rho\gamma,\nu}h_{\mu\lambda} - h_{\mu\lambda,\nu}h_{\rho\gamma} - h_{\rho\lambda,\nu}h_{\mu\gamma} - h_{\rho\gamma,\nu}h_{\mu\lambda} - \frac{5}{2}h_{\lambda\tau,\nu}h_{\mu\gamma} - \frac{1}{2}h_{\mu\rho,\nu}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\nu}h_{\mu\gamma} + \frac{1}{2}h_{\mu\rho,\nu}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\nu}h_{\mu\gamma}.$$  

(5)

Those terms in $\Gamma^\rho_{\mu\rho,\nu}$ which may have effect on the graviton S-G coupling are

$$h_{\mu\lambda,\rho}h_{\gamma\nu} + h_{\nu\lambda,\rho}h_{\mu\gamma} + h_{\nu\gamma,\rho}h_{\mu\lambda} - h_{\mu\lambda,\rho}h_{\nu\gamma} - h_{\nu\lambda,\rho}h_{\mu\gamma} - h_{\nu\gamma,\rho}h_{\mu\lambda} - \frac{5}{2}h_{\lambda\tau,\rho}h_{\mu\gamma} - \frac{1}{2}h_{\mu\sigma,\rho}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\rho}h_{\mu\gamma} + \frac{1}{2}h_{\mu\sigma,\rho}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\rho}h_{\mu\gamma}.$$  

(6)

Further analysis shows that the terms which truly give contribution to the graviton S-G coupling are the following four terms:

(i) $\sqrt{-\eta}g^m (h_{\mu\lambda,\rho}h_{\gamma\nu} + h_{\nu\lambda,\rho}h_{\mu\gamma} + h_{\nu\gamma,\rho}h_{\mu\lambda} - h_{\mu\lambda,\rho}h_{\nu\gamma} - h_{\nu\lambda,\rho}h_{\mu\gamma} - h_{\nu\gamma,\rho}h_{\mu\lambda} - \frac{5}{2}h_{\lambda\tau,\rho}h_{\mu\gamma} - \frac{1}{2}h_{\mu\sigma,\rho}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\rho}h_{\mu\gamma} + \frac{1}{2}h_{\mu\sigma,\rho}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\rho}h_{\mu\gamma})$ contains

$$\sqrt{-\eta}g^m \left( h_{0\lambda,\rho}h_{\gamma\mu} + h_{0\gamma,\rho}h_{\lambda\mu} + h_{0\lambda,\rho}h_{\gamma\mu} + h_{0\gamma,\rho}h_{\lambda\mu} \right) = 2\sqrt{-\eta}g^m h_{0\lambda,\rho}h_{\gamma\mu},$$  

(7)

which includes the following terms

$$2\sqrt{-\eta}g^m \left( h_{0\lambda,0}h_{\gamma\mu} + h_{0\gamma,0}h_{\lambda\mu} \right) = -2\sqrt{-\eta}g^m g_n (\partial_m g_n + \partial_n g_m).$$  

(8)

(ii) $\sqrt{-\eta}g^m (h_{\mu\lambda,\rho}h_{\nu\gamma} + h_{\nu\lambda,\rho}h_{\mu\gamma} + h_{\nu\gamma,\rho}h_{\mu\lambda} - h_{\mu\lambda,\rho}h_{\nu\gamma} - h_{\nu\lambda,\rho}h_{\mu\gamma} - h_{\nu\gamma,\rho}h_{\mu\lambda} - \frac{5}{2}h_{\lambda\tau,\rho}h_{\mu\gamma} - \frac{1}{2}h_{\mu\sigma,\rho}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\rho}h_{\mu\gamma} + \frac{1}{2}h_{\mu\sigma,\rho}h_{\lambda\gamma} + \frac{1}{2}h_{\lambda\tau,\rho}h_{\mu\gamma})$ contains

$$\sqrt{-\eta}g^m \left( h_{0\lambda,\rho}h_{\nu\mu} + h_{0\nu,\rho}h_{\lambda\mu} + h_{0\lambda,\rho}h_{\nu\mu} + h_{0\nu,\rho}h_{\lambda\mu} \right) = -2\sqrt{-\eta}g^m h_{0\lambda,\rho}h_{\nu\mu},$$  

(9)

which will give no contribution to the graviton S-G coupling. So here we will not further consider it.

(iii) $\sqrt{-\eta}g^m h_{\mu\nu,\rho} h_{\mu\lambda,\rho}$ contains

$$\sqrt{-\eta}g^m \left( h_{\nu\lambda,\rho} h_{0\mu,\rho} + h_{\nu\rho,\mu} h_{0\lambda,\rho} \right) = 2\sqrt{-\eta}g^m h_{\nu\lambda,\rho} h_{0\mu,\rho}.$$  

(10)

which includes

$$2\sqrt{-\eta}g^m h_{0\mu,0} h_{0\lambda,\rho} = 2\sqrt{-\eta}g^m g_n \partial_n g_m,$$  

(11)

(iv) $\frac{1}{2}\sqrt{-\eta}g^m (h_{\lambda\tau,\rho}h_{\mu\gamma} + h_{\lambda\tau,\rho}h_{\mu\gamma})$ (i.e., $-\sqrt{-\eta}g^m h_{\lambda\tau,\rho}h_{\mu\gamma}$) includes
\[ \sqrt{-g} g^{0m} (-h_{\lambda\tau,0} h_{m}^{\lambda\tau} - h_{\lambda\tau,m} h_{0}^{\lambda\tau}) = -2 \sqrt{-g} g^{0m} h_{\lambda\tau,0} h_{m}^{\lambda\tau}, \] (12)

which contains
\[ \sqrt{-g} g^{0m} (2h_{0,0} h_{m}^{0m} - 2h_{n,0} h_{m}^{n0}) = -4 \sqrt{-g} g^{0m} h_{0,0} h_{m}^{0m} = 4 \sqrt{-g} g^{0m} g_{n} \partial_{m} g_{n}. \] (13)

Thus it follows from Eq.(8), (11) and (13) that the terms in \( \sqrt{-g} g^{\mu\nu} \Gamma_{\mu\nu}^{\rho} \), which contribute to the graviton S-G coupling, are given as follows
\[ \sqrt{-g} g^{0m} [-2g_{n} (\partial_{m} g_{n} + \partial_{n} g_{m}) - 2g_{n} \partial_{n} g_{m} + 4g_{n} \partial_{m} g_{n}] = \sqrt{-g} g^{0m} [2g_{n} \partial_{m} g_{n} - 4g_{n} \partial_{n} g_{m}] . \] (14)

**THE RESULT:**
Hence it follows from (4) and (14) that the terms in \( \sqrt{-g} g^{\mu\nu} (\Gamma_{\rho\mu\nu}^{\sigma} - \Gamma_{\mu\nu}^{\rho}) \) which contribute to the graviton S-G coupling are written in the form
\[ \sqrt{-g} g^{0m} [4g_{n} \partial_{m} g_{n} - (2g_{n} \partial_{m} g_{n} - 4g_{n} \partial_{n} g_{m})] = \sqrt{-g} g^{0m} (2g_{n} \partial_{m} g_{n} + 4g_{n} \partial_{n} g_{m}) . \] (15)

**B. \( -\Gamma_{\mu\nu}^{\rho} \Gamma_{\rho\sigma}^{\sigma} \)**

The first-order terms (proportional to \( -\frac{1}{2} K \)) in \( \Gamma_{\mu\nu}^{\sigma} \) is of the form
\[ \Gamma_{\mu\nu}^{\sigma} \left( \propto - \frac{1}{2} K \right) = h_{\mu\nu}^{\sigma} + h_{\nu\mu}^{\sigma} - h_{\mu\sigma}^{\nu} - \frac{1}{2} \delta_{\nu}^{\sigma} h_{\lambda\nu}^{\lambda} - \frac{1}{2} \delta_{\mu}^{\sigma} h_{\lambda\mu}^{\lambda} + \frac{1}{2} \eta_{\mu\nu} h_{\lambda\nu}^{\lambda\sigma} , \]
which includes the valuable terms \( -\frac{1}{2} K (h_{\mu\nu}^{\sigma} + h_{\nu\mu}^{\sigma} - h_{\mu\sigma}^{\nu}) \). In the meanwhile, the terms proportional to \( -\frac{1}{2} K \) in \( \Gamma_{\rho\sigma}^{\rho} \) take the following form
\[ \Gamma_{\rho\sigma}^{\rho} \left( \propto - \frac{1}{2} K \right) = - \frac{1}{2} K (h_{\rho\sigma}^{\rho} + h_{\rho\sigma}^{\rho} - h_{\rho\sigma}^{\rho} + ...) = - \frac{1}{2} K h_{\rho\sigma}^{\rho} . \]

It is readily verified that the contribution of \( -\Gamma_{\mu\nu}^{\rho} \Gamma_{\rho\sigma}^{\sigma} \) to the graviton S-G coupling is vanishing. So, we will not consider it further in this note.

**C. \( \Gamma_{\rho\sigma}^{\sigma} \Gamma_{\mu\nu}^{\sigma} \)**

Apparently, \( \sqrt{-g} g^{\mu\nu} \Gamma_{\rho\sigma}^{\rho} \Gamma_{\mu\nu}^{0m} \) includes the following terms
\[ \sqrt{-g} g^{0m} (\Gamma_{\rho\sigma}^{0m} \Gamma_{\mu\nu}^{\sigma} + \Gamma_{\rho\sigma}^{\mu} \Gamma_{\mu\nu}^{\sigma} + \Gamma_{\rho\sigma}^{\sigma} \Gamma_{\mu\nu}^{\sigma}) = 2 \sqrt{-g} g^{0m} \Gamma_{\sigma\rho}^{0m} \Gamma_{\sigma\rho}^{\sigma} \] (18)

where
\[ \Gamma_{\rho\sigma}^{0m} \left( \propto - \frac{1}{2} K \right) = h_{\rho\sigma}^{0m} + h_{\rho\sigma}^{0m} - h_{\rho\sigma}^{0m} , \] (19)

and
\[ \Gamma_{\rho\sigma}^{m\rho} \left( \propto - \frac{1}{2} K \right) = h_{\rho\sigma}^{m\rho} + h_{\rho\sigma}^{m\rho} - h_{\rho\sigma}^{m\rho} . \] (20)

Thus, one can arrive at
\[ (h_{\rho\sigma}^{0m} + h_{\rho\sigma}^{0m} + h_{\rho\sigma}^{0m} - h_{\rho\sigma}^{0m}) (h_{m\rho}^{\sigma} + h_{m\rho}^{\sigma} - h_{m\rho}^{\sigma}) = [h_{\rho\sigma}^{0m} (h_{\rho\sigma}^{m\rho} - h_{\rho\sigma}^{m\rho})] + h_{\rho\sigma}^{0m} h_{m\rho}^{\sigma} + h_{\rho\sigma}^{0m} h_{m\rho}^{\sigma} + h_{\rho\sigma}^{0m} h_{m\rho}^{\sigma} . \] (21)

In the following discussions, for convenience, we classify the terms on the right-handed side of Eq.(21) into three categories:

(i) \( h_{\rho\sigma}^{0m} (h_{\rho\sigma}^{m\rho} - h_{\rho\sigma}^{m\rho}) \) contains
So, the terms in coupling as follows 

\[ h_{n,0}^0 (h_{0,m}^n - h_{m,0}^n) = -\dot{g}_n (\partial_m g_n - \partial_n g_m). \]  

(22)

(ii) \( h_{\sigma,0}^\sigma h_{\mu,\rho}^\rho \) contains 

\[ h_{0,0}^0 h_{m,n}^0 = -\dot{g}_n \partial_n g_m. \]  

(23)

(iii) \((h_{\sigma,\sigma}^\sigma - h_{\sigma,0}^\sigma) (h_{\mu,\rho}^\mu + h_{\rho,\mu}^\rho - h_{\mu,\mu}^\mu)\) contains the following six terms

\[ h_{0,0}^\sigma h_{m,n}^\rho = h_{0,0}^\sigma h_{m,n}^\rho + h_{0,0}^\rho h_{0,m}^\sigma \sim -\dot{g}_n \partial_n g_m, \]

\[ h_{0,0}^\rho h_{m,n}^\sigma = h_{0,0}^\rho h_{m,n}^\sigma + h_{0,0}^\sigma h_{0,m}^\rho \sim \dot{g}_n \partial_n g_m, \]

\[ -h_{0,0}^\sigma h_{m,n}^\sigma = \text{(giving no contribution to S-G coupling)}, \]

\[ -h_{0,0}^\rho h_{m,n}^\rho = \text{(giving no contribution to S-G coupling)}, \]

\[ -h_{0,0}^\rho h_{m,n}^\sigma = h_{0,0}^\rho h_{m,n}^\sigma + h_{0,0}^\sigma h_{0,m}^\rho \sim \dot{g}_n \partial_n g_m, \]

\[ (h_{0,0}^\rho) (h_{m,n}^\rho) = h_{0,0}^\rho h_{m,n}^\rho + h_{0,0}^\rho h_{0,m}^\rho \sim -\dot{g}_n \partial_n g_m. \]  

(24)

So, the terms in \((h_{0,0}^\rho) (h_{m,n}^\rho + h_{\rho,\mu}^\mu - h_{m,\mu}^\mu),\) which will have effect on the graviton S-G coupling, are

\[ -2\dot{g}_n \partial_n g_m. \]  

(25)

THE RESULT:

Thus it follows from (22), (23) and (25) that the total terms in \(\sqrt{-g}g^{\mu\nu}\Gamma^\sigma_{\nu\mu}\) which will give contribution to the graviton S-G coupling are expressed as follows

\[ 2\sqrt{-g}g^{0m} [-\dot{g}_n (\partial_m g_n - \partial_n g_m) - \dot{g}_n \partial_m g_n - 2\dot{g}_m \partial_n g_m] = 2\sqrt{-g}g^{0m} [-\dot{g}_n (\partial_m g_n - \partial_n g_m) - 3\dot{g}_n \partial_n g_m]. \]  

(26)

THE FINAL RESULT:

Hence, according to the expressions (15) and (26), one can finally obtain the total contribution to the graviton S-G coupling as follows

\[ \frac{1}{2} K^2 \sqrt{-g}g^{0m} (2\dot{g}_n \partial_m g_n + 4\dot{g}_m \partial_n g_m) + \frac{1}{4} K^2 : 2\sqrt{-g}g^{0m} [-\dot{g}_n (\partial_m g_n - \partial_n g_m) - 3\dot{g}_n \partial_n g_m] \]

\[ = \frac{1}{2} K^2 \sqrt{-g}g^{0m} [-\dot{g}_n (\partial_m g_n - \partial_n g_m) + 3\dot{g}_n \partial_m g_n]. \]  

(27)

Thus one can arrive at

\[ L_{s-g} = -K^2 \sqrt{-g}g^{0m} g_n (\partial_m g_n - \partial_n g_m) = K^3 g_m g_n (\partial_m g_n - \partial_n g_m), \]  

(28)

where \(\sqrt{-g}g^{0m} = K h^{0m} = K g^{0m} = -K g_m \) is applied, and the summation is carried out over the values 1, 2, 3 of the repeated indices. As stated previously, the Lagrangian density \(L_{s-g}\) is just equal to the Hamiltonian density describing the interaction of graviton spin with the gravitomagnetic fields. Hence this note that concerns ourselves with the detailed mathematical treatment of the coupling of graviton spin to gravitomagnetic fields may therefore provide us with a deep insight into the generalized (purely gravitational) version of Mashhoon’s spin-rotation couplings [1,4]. It may be reasonably believed that such a graviton S-G coupling deserves further detailed investigation.

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