Turbulence modeling using test modes

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Abstract. We present an analytical self-consistent approach, which is able to analyze the processes that appear in the nonlinear stage of turbulence. The results concern drift turbulence in confined plasmas. The method is based on a combined study of test particle and test modes in turbulent plasmas. We show that the main cause of the nonlinear processes that appear beyond the quasilinear stage of turbulence is trajectory trapping or eddying. Trapping introduces quasi-coherent aspects in test trajectory statistics, which lead to large scale correlations, nonlinear damping of the drift modes and generation of the zonal flow modes.

1. Introduction
Turbulence is considered to be the last great problem of classical physics that is still unsolved. It is of fundamental importance in plasma physics, astrophysics, fluid mechanics and atmospheric and oceanic sciences. Direct numerical simulations, which have obtained important results in the last decades, largely dominate the actual research in this field. The analytical advance based on first principle description and mathematically justifiable approximations is very small. The reason is the stochastic advection process, which is the dominant nonlinearity in all turbulent system: the advected field depends on the velocity in plasmas or it is just the velocity field in fluids. Moreover, even the most simplified models that deal with passive fields have not been solved analytically because the basic problem of particle trajectories in stochastic velocity fields is strongly nonlinear.

We present an analytical modeling of the evolution of drift turbulence in magnetically confined plasmas, which consists of a coupled study of the test modes on turbulent plasmas and of the statistics of the particle trajectories. It is similar to the Lagrangian approach initiated by Dupree [1] and developed in the 70’s. Their assumption of random trajectories with Gaussian distribution has limited the application of this method to the quasilinear regime. We introduce a realistic description of the rather complex trajectories that are determined by the ExB drift in turbulence, which enables the description of the processes that appear in the nonlinear regime. Particle motion is not only random but it includes coherent aspects. A typical trajectory is a random sequence of trapping or eddying events and large jumps. We have developed semi-analytical statistical methods (the decorrelation trajectory method DTM [2] and the nested subensemble approach NSA [3]), which are able to describe the process of trapping and the related coherent aspects of trajectory statistics.

We show that trajectory trapping is the main cause of the nonstandard (anomalous) transport regimes and of the strong nonlinear effects in the evolution of drift turbulence (generation of large scale...
correlations and of the zonal flow modes). This is the first analytical result on this complex problem that is in agreement with numerical simulations.

2. Test modes on turbulent plasmas

Test modes on turbulent plasmas were studied for drift turbulence in constant magnetic field [4] starting from the basic description provided by the drift kinetic equations. Analytical expressions are derived, which approximate the growth rates $\gamma$ and the frequencies $\omega$ of the test modes as functions of the characteristics of the background turbulence. They provide the tendencies in turbulence evolution. We consider the (universal) drift instability described by the drift kinetic equations in the collisionless limit. Electron kinetic effects produce the dissipation mechanism to release the energy and, combined with the ion polarization drift, make drift waves unstable. Beside this, the polarization drift

$$u_p = \frac{m_i}{eB^2} \partial_x E_{\perp}$$

has a more complex influence determined by its nonzero divergence, which produces compressibility effects in the background turbulence.

The dispersion relation of the test modes in turbulent plasma is shown to be formally the same as in quiescent plasma, except for a time dependent function $M(\tau)$, which embeds all the effects of the background turbulence:

$$-i\left(k_y V_{se} - \omega \rho_s k_{\perp}^2 \right) \int_{-\infty}^{\infty} d\tau \ M(\tau) \exp\left[-i\omega(\tau - t)\right] = 1 + i \sqrt{\frac{\pi}{2}} \frac{\omega - k_y V_{se}}{|k_{\perp} V_{le}|}$$

$$M(\tau) \equiv \left\{ \exp\left[i\mathbf{k} \cdot (\mathbf{x}(\tau) - \mathbf{x}) - \int_{\tau}^{t} d\tau' \mathbf{v} \cdot \mathbf{u}_p(\mathbf{x}(\tau')) \right] \right\}.$$  (3)

$V_{se} = T_e (eBL_n) = \rho_s c_s / L_n$ is the diamagnetic velocity produced by the density gradient (taken along the x direction), $\rho_i = c_i / \Omega_i$, $c_s = \sqrt{T_e / m_i}$, $T_e$ is the electron temperature, $m_i$ is the ion mass, $e$ is the absolute value of electron and ion charge, $\Omega_i = eB / m_i$ is the cyclotron frequency and $k_{\perp} = \sqrt{k_x^2 + k_y^2}$ is the perpendicular wave number.

The function $M(\tau)$ depends on the background potential through the statistical average on the trajectories denoted by $\langle \cdot \rangle$, and also through the compressibility term determined by the polarization drift. The trajectories (the characteristics of the drift kinetic equation) are solutions of

$$\frac{d\mathbf{x}(\tau)}{d\tau} = -\frac{\nabla \phi(\mathbf{x} - V_{se} \mathbf{t}) \times \mathbf{e}_z}{B},$$  (4)

calculated backwards in time with the condition at $\tau = t$, $\mathbf{x}(t) = \mathbf{x}$. In the case of quiescent plasmas $M = 1$. The function $M(\tau)$ is estimated using the results of the test particle study.

3. Test particle statistics in stochastic velocity fields

The origin of trapping is the Hamiltonian structure of the equation of motion (4). The components x and y of particle position are conjugate variables and the potential $\phi(\mathbf{x}, t)$ is the Hamiltonian function. The trajectories are periodic and they wind on the contour lines of $\phi$ when the potential is time independent. This defines the state of permanent trapping. If the motion is weakly perturbed (by time variation of the potential, particle collisions, etc.), the trapping is temporary and alternates with large jumps.
Original semi-analytical methods for determining the statistics of stochastically advected particles were developed (the DTM [2] and the NSA [3]), and they have been used in a series of studies of the transport in magnetically confined plasmas (see [5]-[7] and the references there in). This represents the first analytical approach that is able to go beyond the quasi-linear regime that corresponds to quasi-Gaussian transport. The general conclusion of these studies is that the existence of space correlations of the stochastic velocity can generate trajectory trapping or eddying, which leads to nonstandard statistics of trajectories: non-Gaussian distributions, long time Lagrangian correlations (memory), strongly modified transport coefficients and an increased degree of coherence. We have also shown that the trapped trajectories form quasi-coherent structures similar to fluid vortices. These effects appear in two-dimensional, zero divergence velocity fields as in Eq. (4).

Essentially, the DTM and the NSA reduce the problem of determining the statistics of the stochastic trajectories to the calculation of weighted averages of some smooth, deterministic trajectories determined from the Eulerian correlation (EC) of the stochastic potential. These methods are in agreement with the statistical consequences of the invariance of the potential so that they are able to describe trajectory trapping.

The distribution of displacements strongly depends on the ordering of the main characteristic times of the stochastic process: the diamagnetic time \( \tau_y = \lambda_y / V \), defined by the motion of the potential along \( y \) with the diamagnetic velocity, and the time of flight (or eddying time) \( \tau_\beta = \lambda_y / V_\beta \). The decorrelation process is determined by two mechanisms: the time variation of the stochastic potential and the parallel decorrelation due to ion motion along the magnetic field. The correlation time of the potential is larger than the parallel time \( \tau_\parallel = \lambda_y / V_\parallel \), and thus the decorrelation parameter (Kubo number) is \( K_d = \tau_\parallel / \tau_\beta \). Trapping is statistically relevant for drift turbulence if the time of flight is the smallest characteristic time. This condition corresponds to \( K_d > 1 \) and \( K_y > 1 \), where \( K_y = \tau_y / \tau_\beta = V_y / V_\parallel \) is the diamagnetic parameter. The fraction of trapped trajectories \( n_t \) and the maximum size of trajectory structures are increasing functions of \( K_y \) and \( K_d \) (see Figure 1).

The function \( M(\tau) \) that accounts for the background turbulence is evaluated using simplified analytical expressions for the probability of displacements with the characteristics of the results of the DTM.

![Figure 1](image.png)
4. The evolution of the drift turbulence
The self-consistent modeling of the evolution of drift turbulence is obtained from these coupled
studies of test modes and test particles. Although the input for both studies is the EC of the stochastic
potential, it is possible to determine the evolution of this function. The difference between the
characteristic time of turbulence evolution and the decorrelation time of ion trajectories (the parallel
time) enables the evaluation of the EC evolution from the tendencies provided by the growth rates and
the frequencies of the test modes.

Starting from an initial condition with a large spectrum of very small amplitude, the EC of the
potential is determined (by the Fourier transform), and then the statistics of trajectories and the
function \( M(\tau) \) are calculated for this EC using the DTM. The dispersion relation (2) is solved and
eventually the short time evolution of the test modes is determined from
\[
\left\langle \left| \Phi^2(k, t+\tau) \right| \right\rangle = \left\langle \left| \Phi^2(k, t) \right| \right\rangle \exp(2\gamma(k; t)\tau) \tag{5}
\]
for times \( \tau \) of the order of the correlation time of the turbulence. We have taken the initial condition
\[
\left\langle \left| \Phi^2(k, 0) \right| \right\rangle = \Phi_0^2 \exp(-k^2\lambda_0^2/2), \tag{6}
\]
where the initial amplitude \( \Phi_0 \) is small and \( \lambda_0 \in (0.4, 0.8) \). The EC of the potential at time \( t+\tau \) is
the Fourier transform
\[
E(x, y', t+\tau) \equiv \left\langle \phi(x_i, y_i, t+\tau) \phi(x_i + x, y_i + y, t+\tau) \right\rangle = \int d^2k \left\langle \left| \Phi^2(k, t+\tau) \right| \right\rangle \exp\left(ik_x x + ik_y y\right). \tag{7}
\]
This determines the statistics of the trajectories at time \( t+\tau \), and the evolution of the EC is obtained
by repeating these calculations.

4.1. The quasilinear regime
For small amplitude of the background potential trapping does not appear and the transport is
Gaussian. One obtains
\[
M(\tau) = \exp \left[-k^2 D_1(t-\tau) - 2ik_y \frac{V^2}{\Omega V_y}\right] \tag{8}
\]
and the solution of the dispersion relation is
\[
\omega = \frac{k_y V_y}{1 + k^2}, \tag{9}
\]
\[
\gamma = \gamma_0 \omega \left(k_y V_y - \omega\right) - k^2 D_1. \tag{10}
\]
The effect of the background turbulence appears in the growth rate, while the frequency is the same as
in quiescent plasma. It determines the diffusive damping (second term in Eq. (10)). The damping is
produced by ion diffusion and it is important for the modes with large wave numbers.

The positive growth rate determines the increase of the amplitude of the modes accompanied by a
strong modification of the shape of the EC (see Figure 2, dashed lines). The latter is determined by the
fact that the growth rate is zero for \( k_y = 0 \), which leads to \( \left\langle \left| \Phi^2(k, k_y = 0, t) \right| \right\rangle = 0 \) and to
\[
\int dy E(x, y, t, \tau) = 0. \tag{11}
\]
This change of the shape of the EC determines a strong decrease of \( D_y \), which enables the development of the drift turbulence. The correlation lengths \( \lambda_i \) are of the order of \( \rho_y \).
4.2. The weakly nonlinear regime

When the amplitude reaches values that make \( K_s > 1 \), ion trajectory trapping appears and generates vortical structures of trapped ions. The fraction of trapped trajectories is small in this regime. The distribution of displacements is non-Gaussian, with an invariant maximum that is produced by trajectory structures and an expanding part that corresponds to the free ions and determines the diffusion. The effect of the background turbulence appears in the frequency

\[
\omega = \frac{k \gamma F V_s}{1 + F k^2}, \quad F = \exp \left( -\frac{1}{2} k^2 S_i^2 \right),
\]

where \( S_i \) are the average sizes of the trajectory structures. Trajectory structures determine the decrease of the frequencies, which leads to the displacement of the unstable range of wave numbers to small values of the order \( k_s \approx 1/S_i \). The solution for the growth rate remains as in the quasilinear regime (Eq. (10)), but \( \gamma \) decreases due to the decrease of the frequency.

Turbulence amplitude continues to increase in this stage, but with a smaller rate. The shape of the EC is not modified, but the correlation length increases in both directions (see Figure 2, the dashed-dotted line). An inverse cascade appears as the shift of the unstable wave-number range toward small wave numbers.

![Figure 2](image)

**Figure 2.** The Eulerian correlation of the potential normalized with the amplitude of the drift turbulence \( E(x,t)/E(0,t) \) as function of \( x \) (left panel) and of \( y \) (right panel). The initial condition (dotted lines), the shape of the EC at the end of the quasilinear regime (dashed lines), in the weakly nonlinear regime (dashed-dotted lines) and in the strongly nonlinear regime where zonal flow modes are generated (continuous lines).

4.3. The strongly nonlinear regime

When the fraction of trapped ions \( n_{tr} \) becomes comparable with the fraction of free ions \( n_f \), ion flows are generated by the trapping process. The trapped ions move with the potential, while the free ions move in the opposite direction with the velocity \( V_f = -V_s n_{tr} / n_f \), such that the total flux is zero. This determines the splitting of the distribution of ion displacement in two parts that depart from each other. A new term appears in the function \( M(\tau) \) and the dispersion relation is strongly modified. It has two solutions (see [4] for the calculations).
One of the solutions corresponds to the drift modes. It is strongly modified by the ion flows, which determine a damping effect. It begins with the damping of small $k_y$ modes and it extends to the entire spectrum as $n_w$ increases (the growth rates are negative for the entire range of $k$ at $n_w = 1/2$). The other solution corresponds to a new type of modes, the zonal flow modes with $k_y = 0$ and very small frequencies. They are produced by the combined action of the ion flows and of the compressibility due to the polarization drift in the background turbulence. The growth rate of the zonal flow modes is proportional to $n_w$ and it increases with the decrease of the correlation length of the drift turbulence. The shape of the total potential is strongly changed due to the zonal flow mode component (Figure 2, continuous line). This determined the decrease of $D_x$ and the strong increase of $D_y$.

The damping of the drift modes is not determined by the zonal flows mode. There is only an indirect contribution through the diffusive damping, which is increased by the zonal flow modes. The growth and the decay of turbulence are produced on different paths (hysteresis process). Large scales are generated at the increase of turbulence amplitude, when trapping is weak. Later in the nonlinear evolution, when trapping is stronger and produces ion flows, turbulence amplitude continues to increase, but it is accompanied by the decrease of the correlation length and by the generation of zonal flow modes. Then, the amplitude decays. A closed evolution curve in the $(\beta, \lambda)$ space is described by the turbulence, which remains in the nonlinear stage characterized by trapping and oscillates between weak and strong trapping. The characteristic time $\Delta t$ for turbulence and transport oscillations estimated as the inverse of the growth rates is in agreement with numerical simulations.

The predator-prey paradigm is not sustained by these results, although there is time correlation between the growth of zonal flow modes and the damping of the drift modes.

5. Conclusions

Ion trajectory trapping determines strong transitory changes in the turbulence. Drift turbulence does not saturate but has a complex oscillatory behavior. The trapped ions that form trajectory structures lead to longer correlations lengths. The potential motion combined with trapping stabilizes and damps the drift modes (first those with small $k$) and in the same time generates zonal flow modes, which eventually are damped when the fraction of trapped trajectories decreases.

In conclusion, this first principle semi-analytical approach is able to describe the complex evolution of drift turbulence and yields results in agreement with numerical simulations. A different perspective on important aspects of the physics of drift type turbulence in the strongly non-linear regime (as zonal flow mode generation) is deduced.

6. References

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