Connectivity of direct products of graphs

Wei Wang *  Ni-Ni Xue
College of Information Engineering, Tarim University,
Alar, Xinjiang, 843300, P.R.China

Abstract

Let $\kappa(G)$ be the connectivity of $G$ and $G \times H$ the direct product
of $G$ and $H$. We prove that for any graphs $G$ and $K_n$ with $n \geq 3$,

$$\kappa(G \times K_n) = \min\{n\kappa(G), (n-1)\delta(G)\},$$

which was conjectured by Guji and Vumar.

Keywords: Connectivity; Direct product; Minimum degree

1 Introduction

Throughout this paper we consider only finite undirected graphs without
loops and multiple edges.

Let $G = (V(G), E(G))$ be a graph. The connectivity of $G$ is the number,
denoted as $\kappa(G)$, equal to the fewest number of vertices whose removal from
$G$ results in a disconnected or trivial graph. The direct (or Kronecker)
product $G \times H$ of graph $G$ and $H$ has vertex set $V(G \times H) = V(G) \times V(H)$
and edge set $E(G \times H) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G) \text{ and } v_1v_2 \in E(H)\}$.

The connectivity of direct products of graphs has been studied recently.
Unlike the case of Cartesian products where the general formula was ob-
tained [4, 5], results for direct products have been given only in special
cases. Mamut and Vumar [3] considered product of two complete graphs
and proved for any $K_m$ and $K_n$ with $n \geq m \geq 2$ and $n \geq 3$,

$$\kappa(K_m \times K_n) = (m-1)(n-1).$$

Later, Guji and Vumar [2] proved for any bipartite graph $G$ and $K_n$ with
$n \geq 3$,

$$\kappa(G \times K_n) = \min\{n\kappa(G), (n-1)\delta(G)\},$$

*Corresponding author: wangwei.math@gmail.com
where $\delta(G)$ denoted the minimum degree of $G$. In the same paper, Guji and Vumar conjectured \(^2\) holds even without the assumption of bipartiteness of $G$.

In the next section we shall prove the conjecture.

# The result

**Theorem 1.** $\kappa(G \times K_n) = \min\{n\kappa(G), (n - 1)\delta(G)\}$ for $n \geq 3$.

The proof of the theorem will be postponed to the end of this section. We first give some properties on direct products of graphs \(^1\).

**Lemma 1.** (1) The direct product of nontrivial graphs $G$ and $H$ is connected if and only if both factors are connected and at least one factor contains an odd cycle.

(2) $\delta(G \times H) = \delta(G)\delta(H)$, and in particular, $\delta(G \times K_n) = (n - 1)\delta(G)$.

We shall always label $V(G) = \{u_1, \ldots, u_m\}, V(K_n) = \{v_1, \ldots, v_n\}$ and set $S_i = \{u_i\} \times V(K_n)$. Let $S \subseteq V(G \times K_n)$ satisfy the following two conditions:

(1). $|S| < \min\{n\kappa(G), (n - 1)\delta(G)\}$, and

(2). $S'_i := S_i - S \neq \emptyset$, for $i = 1, 2, \ldots, m$.

Associated with $G, K_n$ and $S$, we define a new graph $G^*$ as follows:

(1). $V(G^*) = \{S'_1, S'_2, \ldots, S'_m\}$, and

(2). $E(G^*) = \{S'_i, S'_j \in E(S'_i, S'_j), \emptyset\}$, where $E(S'_i, S'_j)$ denotes the collection of all edges in $(G \times K_n - S)$ with one end in $S'_i$ and the other in $S'_j$.

Notice $G^*$ can be defined only if $\kappa(G) > 0$ since otherwise condition (1) is meaningless.

**Lemma 2.** If $G$ is connected then $G^*$ is connected.

**Proof.** Suppose $G^*$ is not connected. Then the vertices of $G^*$ can be partitioned into two parts, $X^*$ and $Y^*$, such that there are no edges joining a vertex in $X^*$ and a vertex in $Y^*$. Let $r = |X^*|$. Without loss of generality, we may assume $X^* = \{S'_1, \ldots, S'_r\}$ and $Y^* = \{S'_{r+1}, \ldots, S'_m\}$.

Let $X = \{u_1, \ldots, u_r\}$ and $Y = \{v_{r+1}, \ldots, v_m\}$. Since $G$ is connected, there is at least one edge joining a vertex in $X$ and a vertex in $Y$. Let $Z$ be the collection of ends of all edges in $E(X, Y)$.

Let $Z^* = \{S'_j : j \in \{1, \ldots, m\}$ and $|S'_1| = 1\}$. For each $u_i \in Z$, by the definition of $Z$, there is an edge $u_i u_j \in E(X, Y)$. It follows that both $S'_1$ and $S'_j$ contains exactly one element since otherwise $E(S'_i, S'_j)$ contains at least one edge by the definition of $G \times K_n$. Therefore $S'_1 \in Z^*$ and we have $|Z| \leq |Z^*|$. We need to consider two cases:
Case 1: Either \( X \subseteq Z \) or \( Y \subseteq Z \). We may assume \( X \subseteq Z \), then the degree of any vertex \( u_i \in X \) can not exceed \(|Z| - 1\). Therefore \( \delta(G) \leq |Z| - 1 \).

By a simple calculation, we have

\[
|S| \geq (n - 1)|Z^*| \geq (n - 1)|Z| > (n - 1)\delta(G) \geq \min\{n\kappa(G), (n - 1)\delta(G)\},
\]

a contradiction.

Case 2: \( X \not\subseteq Z \) and \( Y \not\subseteq Z \). Either of \( X \cap Z \) and \( Y \cap Z \) is a separating set. Therefore, \( \kappa(G) \leq \min\{|X \cap Z|, |Y \cap Z|\} \leq |Z|/2 \). Similarly, we have

\[
|S| \geq (n - 1)|Z^*| \geq (n - 1)|Z| > |Z|n/2 \geq n\kappa(G) \geq \min\{n\kappa(G), (n - 1)\delta(G)\},
\]

again a contradiction. \( \square \)

While the above lemma tells us that the new graph \( G^* \) is connected, what we most concern is the connectedness of \( G \times K_n - S \). We need the following lemma.

Lemma 3. Any vertex of \( G^* \), \( S'_i \), as a subset of \( V(G \times K_n - S) \), is contained in the vertex set of some component of \( G \times K_n - S \).

Proof. It suffices to prove the lemma for \( i = 1 \).

If \( |S'_i| = 1 \), then the assertion holds trivially. We need to consider two cases:

Case 1: \( |S'_i| \geq 3 \). Assume \( S'_i \) is not contained in any component of \( G \times K_n - S \). Then there must exist a component \( C \) such that \( 0 < |S'_i \cap V(C)| \leq |S'_i|/2 < |S'_i| - 1 \). Let \((u_1, v_u) \in S'_i \cap V(C)\) be any vertex. Since \( |S| < (n - 1)\delta(G) \), it follows by (2) of Lemma 2 that \((u_1, v_u)\) has at least one adjacent vertex in \( G \times K_n - S \). Let \((u_j, v_p)\) be an adjacent vertex of \((u_1, v_u)\).

Clearly, \((u_j, v_p) \in V(C)\) and \( S'_i - \{(u_1, v_u)\} \subseteq V(C) \) since every vertex in \( S'_i - \{(u_1, v_u)\} \) is adjacent to \((u_j, v_p)\). It follows \( |S'_i \cap V(C)| \geq |S'_i| - 1 \), a contradiction.

Case 2: \( |S'_i| = 2 \). Let \( Z^* = \{S'_j : j \in \{1, \ldots, m\} \text{ and } |S'_j| = 1 \} \) and \( C^* \) be a component of \( G^* - Z^* \) containing \( S'_i \). Let \( r \) be the order of \( C^* \).

Without loss of generality, we may assume \( V(C^*) = \{S'_1, \ldots, S'_r\} \).

Since each \( S'_j \in V(C^*) \) contains at least two elements, any edge \( S'_j \) in \( C^* \) implies every vertex in \( S'_k \) has at least one adjacent vertex in \( S'_j \). Therefore, if there is a vertex \( S'_j \) in \( C^* \) contained in the vertex set of some component \( C \) of \( G \times K_n - S \), then every \( S'_k \) is contained in \( V(C) \) provided \( S'_k \in E(C^*) \). It follows by the connectedness of \( C^* \) that \( \bigcup_{i=1}^{r} S'_i \subseteq V(C) \) and hence \( S'_i \subset V(C) \).

By case 1, we may assume each \( S'_j \in V(C^*) \) contains exactly two elements. Let \( S'_j = \{u_j\} \times F_j, j = 1, \ldots, r \).
Subcase 2.1: There exists an edge \( S'_i S'_k \) in \( C^* \) with \( F_j \neq F_k \). One easily verify that \( S'_i \cup S'_k \) induces a connected subgraph of \( G \times K_n - S \). The lemma follows.

Subcase 2.2: There exists no edge \( S'_i S'_k \) in \( C^* \) with \( F_j \neq F_k \). By the connectedness of \( C^* \), all \( F_j \) in \( C^* \) are equal. Notice that \( C^* \) and the subgraph induced by \( \cup_{i=1}^r S'_i \) are isomorphic to \( G[u_1, \ldots, u_r] \) and \( G[u_1, \ldots, u_r] \times K_2 \), respectively. We claim \( G[u_1, \ldots, u_r] \) must contain an odd cycle, which will finish our proof by (1) of lemma 1.

Suppose \( G[u_1, \ldots, u_r] \) does not contain an odd cycle. Then either \( r = 1 \) or \( G[u_1, \ldots, u_r] \) is bipartite. Either of the two cases implies \( \delta(G[u_1, \ldots, u_r]) \leq r/2 \). Let \( j \in \{1, \ldots, r\} \) such that \( \delta(G[u_1, \ldots, u_r]) \leq \delta(G[u_1, \ldots, u_r]) \).

Let \( u_k \) be any adjacent vertex of \( u_j \) in \( G \), then either \( S'_i \in Z^* \), or \( S'_i \) is an adjacent vertex of \( S'_k \) in \( C^* \). Therefore,

\[
\delta(G) \leq \deg_G(u_j) = \deg_{C^*}(S'_j) + |Z^*| = \deg_{G[u_1, \ldots, u_r]}(u_j) + |Z^*| \leq r/2 + |Z^*|.
\]

By a simple calculation,

\[
|S| \geq (n - 2)r + (n - 1)|Z^*| \geq (n - 1)\left(\frac{r}{2} + |Z^*|\right).
\]

(3)

From (3) and (4), we obtain

\[
|S| \geq (n - 1)\delta(G),
\]

(5)
a contradiction.

\[\square\]

Lemma 4. Let \( m = |G| \geq 2 \) and \( u_i \) be any vertex of \( G \). Then

(1). \( \delta(G - u_i) \geq \delta(G) - 1 \), and

(2). \( \kappa(G - u_i) \geq \kappa(G) - 1 \).

Proof of Theorem 1 We apply induction on \( m = |G| \). It trivially holds when \( m = 1 \). We therefore assume \( m \geq 2 \) and that the result holds for all graphs of order \( m - 1 \).

It is clear \( \kappa(G \times K_n) \leq \min\{nk(G), (n - 1)\delta(G)\} \) by lemma 1. The nontrivial part of the proof is hence to show the other inequality. We may assume \( \kappa(G) > 0 \). Let \( S \subseteq V(G \times K_n) \) satisfy condition (1), i.e.,

\[
|S| < \min\{nk(G), (n - 1)\delta(G)\}.
\]

Case 1: \( S \) satisfies condition (2). It follows by lemma 2 and lemma 3 that \( (G \times K_n - S) \) is connected.

Case 2: \( S \) does not satisfy condition (2). Then there exists an \( S_i \) contained in \( S \). Therefore, \( S - S_i \subseteq V((G - u_i) \times K_n) \) and
\[ |S - S_i| = |S| - n \]
\[ < \min\{n\kappa(G), (n - 1)\delta(G)\} - n \]
\[ \leq \min\{n(\kappa(G) - 1), (n - 1)(\delta(G) - 1)\} \]
\[ \leq \min\{n\kappa(G - u_i), (n - 1)\delta(G - u_i)\}. \]

the last inequality above follows from lemma. By the induction assumption,
\[ \kappa((G - u_i) \times K_n) = \min\{n\kappa(G - u_i), (n - 1)\delta(G - u_i)\}. \]

Hence, \((G - u_i) \times K_n - (S - S_i)\) is connected. It follows by isomorphism that \(G \times K_n - S\) is connected.

Either of the two cases implies \((G \times K_n - S)\) is connected. Thus,
\[ \kappa(G \times K_n) \geq \min\{n\kappa(G), (n - 1)\delta(G)\}. \]

The proof of the theorem is completed by induction.

Acknowledgement
The authors are much indebted to the referee for his/her valuable suggestions and corrections that improved the initial version of this paper.

References
[1] A. Bottreou, Y. Métivier, Some remarks on the Kronecker product of graph, Inform. Process. Lett. 68 (1998) 55-61.
[2] R. Guji and E. Vumar, A note on the connectivity of Kronecker products of graphs, Appl. Math. Lett. 22 (2009) 1360-1363.
[3] A. Mamut, E. Vumar, Vertex vulnerability parameters of Kronecker product of complete graphs, Inform. Process. Lett. 106 (2008) 258-262.
[4] S. Spacapan, Connectivity of Cartesian products of graphs, Appl. Math. Lett. 21 (2008) 682-685.
[5] J.-M Xu, C. Yang, Connectivity and super-connectivity of Cartesian product graphs, Ars Comb. 95 (2010) 235-245.