Reservoir cross-over in entanglement dynamics

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We study the effects of spontaneous emission on the entanglement dynamics of two qubits interacting with a common Lorentzian structured reservoir. We assume that the qubits are initially prepared in a Bell-like state. We focus on the strong coupling regime and study the entanglement dynamics for different regions of the spontaneous emission decay parameter. This investigation allows us to explore the cross-over between common and independent reservoirs in entanglement dynamics.

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I. INTRODUCTION

Since its discovery by Yu and Eberly in 2004, considerable interest has been devoted to the phenomenon of early-stage disentanglement, also called entanglement sudden death (ESD) [1]. Indeed not only is this phenomenon important from a fundamental point of view, but it is likely to play a role in many applications of quantum information theory and technology [2,3]. For example, ESD might represent a threat to building a quantum computer. So far ESD has been predicted in many different theoretical systems such as pairs of qubits [4,5,6,7], continuous variable system [8], subset of multiple qubits and spin chains. It has also been observed experimentally in two different contexts [9,10].

In general, a key factor determining the dynamics is the type of environment in which the system of interest is immersed. With just a pair of qubits, differences appear in the entanglement evolution, depending on the properties of the environment. For example, when a bipartite system interacts with one reservoir (i.e. the two parts interact with the same reservoir), correlations between the two parts are created because of the reservoir-mediated interaction. These correlations cannot arise if each of the parts talks to only its own environment, and if there is no direct coupling present between them. On the other hand, if the correlation time of the bath is long, the environment keeps track of the dynamics of the small system, and revivals of entanglement may appear. These are the so-called memory effects, typical of non-Markovian dynamics, appearing not only in common, but also in independent reservoirs.

In Ref. [7] we have studied the exact dynamics of two qubits interacting with a common Lorentzian-structured, non-Markovian reservoir. Here we generalize these results and take into account the spontaneous emission from the two qubits. This model describes, for example, the situation of two atoms strongly coupled to the same one-dimensional high-Q cavity, in which both the atoms can also independently emit a photon in any direction outside the cavity. The uni-dimensional high-Q cavity constitutes a common non-Markovian reservoir for the qubits, while the independent spontaneous emission of a photon in a flat continuum of modes can be seen as a consequence of the interaction with two independent Markovian reservoirs.

The introduction of spontaneous emission for the two atoms allows us to study a rather realistic situation, characterized by different regimes of entanglement dynamics. In particular, we want to explore the cross-over between common and independent reservoir dynamics.

In Ref. [11] the generation of entanglement for two trapped ions coupled to a high-finesse cavity has been studied. In that case, not only the cavity losses, but also the entire atomic level structure (the ions are coupled to the cavity mode via a Raman scheme in a Λ-configuration) is taken into account. The aim of that work is to study realistic experimental conditions under which the collective Dicke model can be implemented in an ion-cavity QED context, so only states with one excitation in the atomic system are considered. Here we study entanglement dynamics when the atomic system is prepared in Bell-like states with one and two excitations.

In the following sections we first introduce the master equation describing the system of interest (section III), then we study the entanglement dynamics for initial Bell-like states for different regions of parameters (section III), and finally we interpret our results and make some conclusive remarks in section IV.

II. THE MODEL

We consider two two-level systems (qubits) interacting with the same leaky cavity in rotating-wave approximation, and emitting independently outside the cavity due to spontaneous emission (see figure I). The qubits have the same transition frequency, and they are equally and resonantly coupled with the leaky cavity. In Ref. [7] we have studied the exact dynamics of two qubits interacting with the same Lorentzian-structured reservoir, i.e. a lossy resonator. In that work we found that, for a certain class of initial states, the entanglement dynamics exhibits regions of sudden death and resurrections. We have interpreted these results as a consequence of the memory effects of the non-Markovian environment.
the interaction picture is
\[ \frac{\partial \hat{\rho}}{\partial t} = -i[H, \hat{\rho}] - \frac{\Gamma}{2}(a^\dagger a \hat{\rho} + \hat{\rho} a^\dagger a - 2a \hat{\rho} a^\dagger) \]
\[ -\frac{\gamma_A}{2}(\sigma_+^A \sigma_-^A \hat{\rho} + \hat{\rho} \sigma_+^A \sigma_-^A - 2\sigma_+^A \rho \sigma_-^A) \]
\[ -\frac{\gamma_B}{2}(\sigma_+^B \sigma_-^B \hat{\rho} + \hat{\rho} \sigma_+^B \sigma_-^B - 2\sigma_+^B \rho \sigma_-^B), \]
\)

where \(\hat{\rho}\) is the density matrix of the qubits plus the cavity mode and
\[ H = \Omega[(\sigma_+^A + \sigma_+^B)a + (\sigma_-^A + \sigma_-^B)a^\dagger]. \]

Here, \(\sigma_+^A\) and \(\sigma_-^B\) are, respectively, the Pauli raising and lowering operators for the atoms A and B, \(a\) and \(a^\dagger\) are the annihilation and creation operators for the cavity mode, \(\Omega\) is the cavity-qubits coupling constant, \(\Gamma\) is the cavity decay rate, and \(\gamma_{A/B}\) are the spontaneous emission rates for atoms A and B. For simplicity we consider \(\gamma_A = \gamma_B = \gamma_S\) and the coupling \(\Omega\) to be the same for both qubits.

We assume that the environment is at zero temperature, so there are at most two excitations in the total system. The basis we use to write the density matrix elements of \(\hat{\rho}\) is \(\{|000\}, \{|001\}, \{|010\}, \{|011\}, \{|100\}, \{|101\}, \{|110\}, \{|111\}\}\), where the first and second digit indicate ground (0) and excited (1) state of qubit A and B respectively, and the third one indicates the number of excitations inside the cavity.

We solve numerically the 64 differential equations for the density matrix elements, and then, to find the dynamics of the two-qubit system only, we trace out the cavity mode degree of freedom to find the reduced density matrix \(\rho\).

Once we have the reduced qubits density matrix we can derive the entanglement dynamics. To quantify entanglement we use the Wootters concurrence \(\mathcal{W}\), defined as
\[ \mathcal{W}(t) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \]
where \(\lambda_i\) are the eigenvalues of the matrix \(R = \rho(\sigma_y^A \otimes \sigma_y^B)\rho^* (\sigma_y^A \otimes \sigma_y^B),\) with \(\rho^*\) denoting the complex conjugate of \(\rho\) and \(\sigma_y^A/B\) are the Pauli matrices for atoms A and B. This quantity attains its maximum value of 1 for maximally entangled states and vanishes for separable states.

We assume the qubits are prepared in Bell-like states. The evolution then drives such initial pure states into a mixed state having an “X” form
\[ \rho(t) = \begin{pmatrix} a(t) & 0 & 0 & w(t) \\ 0 & b(t) & z(t) & 0 \\ 0 & z^*(t) & c(t) & 0 \\ w^*(t) & 0 & 0 & d(t) \end{pmatrix}. \]

For this class of states the concurrence assumes a simple analytic expression
\[ C(t) = \max\{0, C_1(t), C_2(t)\}, \]
where
\[ C_1(t) = 2|w(t)| - 2\sqrt{b(t)c(t)}, \]
\[ C_2(t) = 2|z(t)| - 2\sqrt{a(t)d(t)}. \]
where coherences give a positive contribution to $C_1(t)$ and $C_2(t)$ and hence to the concurrence, while the negative parts involve populations only. Of course, any initial state in the “X” form of Eq. (3), pure, or mixed, would lead to the same expressions for the concurrence. For states not in this form we would have to perform a numerical evaluation of the concurrence.

III. ENTANGLEMENT DYNAMICS

We investigate entanglement dynamics for the following initial Bell-like states of the qubits:

$$|\Phi\rangle = \alpha|10\rangle + e^{i\theta}(1 - \alpha^2)^{1/2}|01\rangle,$$

and

$$|\Psi\rangle = \alpha|00\rangle + e^{i\theta}(1 - \alpha^2)^{1/2}|11\rangle.$$

With appropriate choices of $\theta$ and $\alpha$ these states include the usual four Bell states.

Our aim is to understand the interplay between common non-Markovian reservoir and independent Markovian reservoirs in entanglement dynamics. To do so we consider a strong coupling between the qubits and the cavity, and then we change the spontaneous emission decay rate $\gamma_S$.

A Bell-like state with one excitation as in Eq. (7) never presents entanglement sudden death [1]. In the strong coupling regime and in absence of spontaneous emission the entanglement exhibits oscillations which are damped in time. The introduction of spontaneous emission from the two qubits leads to an additional damping in the oscillations. The result is that the larger the spontaneous emission rate is, the smaller the entanglement revivals are: for $\gamma_S \gtrsim 3\Omega$ the oscillations are completely killed and the entanglement does not revive anymore.

The dynamics of the Bell-like state involving two excitations, Eq. (8), is more interesting. First of all, let us recall the two limiting cases: no coupling with the cavity and no spontaneous emission. We obtain the Markovian independent reservoirs dynamics when setting the coupling between the qubits and the cavity to zero. In this case there are two different regions of the entanglement dynamics: for $\alpha^2 < 1/2$ entanglement dies suddenly, and for $\alpha^2 \geq 1/2$ entanglement decays exponentially. In the second limiting case, i.e. with cavity coupling and in absence of spontaneous emission, the qubits interact exclusively with a common non-Markovian reservoir. In this case entanglement presents a much richer dynamics with oscillations for every value of $\alpha^2$ and a series of dark periods (ESD regions) and resurrections of entanglement for $\alpha^2 \lesssim 1/4$, as shown in figure 2 [2] [3].

Here we explore entanglement dynamics for different values of the spontaneous emission rate $\gamma_S$, fixing the strong coupling between the qubits and the cavity. In particular we set the cavity parameters to $\Omega = 0.2\gamma_0$ and $\Gamma = \sqrt{0.05\gamma_0}$ where $\gamma_0$ is the decay parameter in the case of an infinitely broad cavity. These parameters correspond to experimentally feasible conditions in the context of, for example, trapped ions [15] or circuit-QED [16]. The results do not depend on the relative phase $\theta$.

In figure 3 we present the entanglement dynamics for two qubits prepared in the state (8) with a weaker spontaneous emission rate $\gamma_S = \Omega/10$. We notice that for short times the dynamics is not affected by spontaneous emission (compare with figure 2). As time passes the effects of spontaneous emission become prominent: i.e. the oscillations get smaller in amplitude, and the ESD region dramatically increases until, for a time around $t = 2/\gamma_S \approx 100\gamma_0$, entanglement is dead irrespective of the value of $\alpha^2$.

A clearer explanation of the dynamics can be found by looking at the evolution of the density matrix elements. For a two-photon Bell-like state the concurrence is given by $C_1(t)$, where the two-photon coherence gives a positive contribution, and the populations of the state with one excitation in one of the qubits give a negative contribution. Looking at the differential equations for the density matrix elements $\rho$, Eq. (1), we notice that the terms describing spontaneous emission cause a faster decay of two-photon coherence. On the other hand the population of the states $|10\rangle$ and $|01\rangle$ is less affected, since spontaneous emission not only adds a decay channel from $|10\rangle/\langle 01|$ to $|00\rangle$ but also one from $|11\rangle$ to $|10\rangle/\langle 01|$. As a consequence, at a certain time depending on the parameter $\alpha^2$, the two-photon coherence will be smaller than the one-excitation populations and entanglement will be definitely lost.

In the case of a strong spontaneous emission rate, $\gamma_S = 10\Omega$, the entanglement dynamics tends to be the same as the independent Markovian reservoirs case, as figure 4 shows. Even though the coupling with the cavity is strong, the entanglement does not revive and the
spontaneous emission damps down any entanglement oscillations completely. We also see that the entanglement vanishes exponentially for \( \alpha^2 \geq 1/2 \) and dies suddenly for \( \alpha^2 < 1/2 \).

More delicate is the case of a comparable cavity-qubits coupling and spontaneous emission decay rate. Figure 5 shows the entanglement dynamics at different time-scales for \( \gamma_S = \Omega \). Compared to the case with no spontaneous emission, the dynamics is quite different from the very beginning. Indeed the region of ESD appears to be strongly increased. However, later on the coherent interaction with the cavity makes entanglement revive and, as time passes, revivals appear for a wider and wider range of the \( \alpha^2 \) parameter, as shown in figure 5 (a). For longer times (figure 5 (b)), the range of \( \alpha^2 \) over which we find multiple entanglement revivals shrinks, and then expands again and, in fact, we find that this pattern repeats many times. Clearly, as time passes the amplitude of the oscillations and revivals dramatically decreases, for example, for \( t \sim 150/\gamma_0 \) the concurrence is of the order of \( 10^{-6} \).

The details of the dynamics change a lot depending on the particular value of \( \gamma_S \). However, for \( \gamma_S \) of the order of \( \Omega \), some general features of the dynamics can be identified. For example, for \( \alpha^2 \to 0 \) and \( \alpha^2 \to 1 \) the dynamics of entanglement is mainly controlled by spontaneous emission, resembling the independent Markovian reservoirs case. For intermediate values of \( \alpha^2 \) a number of death and revival periods follow one after the other with decreasing intensity. This is a sign of the non-Markovian backaction of the cavity and of the cavity-mediated coupling between the qubits as well. In general the long-time picture looks like the superimposition of death and revivals on the independent Markovian reservoir dynamical pattern.

**IV. CONCLUSION**

We have studied the entanglement dynamics of two qubits interacting with a leaky cavity (shared non-Markovian reservoir) and emitting independently via spontaneous emission (independent Markovian reservoirs). Starting with the exact master equation of Ref. [7] describing the non-Markovian dynamics with the cavity, we have phenomenologically added two dissipative terms for the qubits. Depending on the ratio between the spontaneous emission parameter \( \gamma_S \) and cavity-qubit coupling constant \( \Omega \), different regimes in entanglement dynamics can be identified. When the spontaneous emission rate is small, the entanglement dynamics is not very much affected for short times, but, as time passes, the region of sudden death spreads out until for long times the entanglement is lost for every value of \( \alpha^2 \). When \( \gamma_S \) is much larger than \( \Omega \), the interaction with the cavity does not play any role. The two dissipative spontaneous emission terms in Eq. (1) control the dynamics, preventing any possibility of revivals in entanglement. The intermediate region when \( \gamma_S \) and \( \Omega \) are of the same order of magnitude shows a more complex behaviour exhibiting dynamical elements of both the two limiting cases.

We emphasize that the method presented here can be applied to any general initial state of the qubits, and not only to Bell-like states. In particular, it is interesting to see the effects of mixedness of the initial state on the entanglement dynamics given that the dynamics depends dramatically on the state of preparation of the qubits. Here we just mention the case of an extended Werner-like state with two excitations of the form \( \rho = r|\Psi\rangle\langle\Psi| + (1-r)/4 \). This state is characterized by an entangled part, formed from the Bell-like state of Eq. (8), and a maximally mixed part, identified with the unit matrix. Clearly, due to the mixedness of the state, the initial amount of entanglement is smaller than for the state \( |\Psi\rangle \) alone. The presence of the mixed part causes a faster death of entanglement. However, for small values of the \( \gamma_S \) parameter the entanglement exhibits oscillations and, as \( \gamma_S \) increases, features of the reservoir
cross-over appear.

In realistic experimental ion-cavity QED conditions it is not always possible to limit the losses of the system to cavity-losses, and often spontaneous emission needs to be taken into account. In our work we have demonstrated how the phenomenon of spontaneous emission comes into play in the entanglement dynamics of strongly interacting cavity QED systems.

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