Anomaly Cancellation in Seven-Dimensional Supergravity with a Boundary

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Abstract

We construct a seven-dimensional brane world in a slice of $\text{AdS}_7$, where the boundary matter content is fixed by the cancellation of anomalies. The seven-dimensional minimal $\mathcal{N} = 2$ gauged supergravity is compactified on the orbifold $S^1/\mathbb{Z}_2$, and the supersymmetric bulk-boundary Lagrangian is consistently derived for boundary vector and hypermultiplets up to fermionic bilinear terms. Anomaly cancellation then fixes the boundary gauge coupling in terms of the seven-dimensional Planck mass, and a topological mass parameter of the Chern-Simons term. In addition for gauge groups containing the standard model, anomaly cancellation restricts the gauge groups on the six-dimensional boundaries to be only one of the exceptional groups. There are also special values of the separation of the two boundaries, where the boundary couplings become singular, and lead to a possible phase transition in the boundary theory. Furthermore, by the AdS/CFT correspondence our brane world is dual to a six-dimensional conformal field theory, suggesting that our bulk theory describes the strong coupling dynamics of six-dimensional theories.
1 Introduction

It is an indelible fact that the particle content of the low-energy world is anomaly free. The cancellation of anomalies is a crucial guiding principle, especially in theories beyond the standard model. We are accustomed to the cancellation of anomalies in four dimensions, but as string theory has taught us the cancellation of anomalies in higher dimensions also leads to powerful constraints. Recently, the idea that we live on a brane in a higher-dimensional spacetime has led to new possibilities for physics beyond the standard model. In the brane world, where the geometry of extra dimensions can naturally account for hierarchies \[1, 2\], one would expect that anomaly cancellation can further lead to constraints on the matter content.

In this regard, the archetypal model is due to Horava and Witten \[3\], where it was shown that eleven-dimensional (11D) supergravity compactified on the orbifold \(S^1/Z_2\), uniquely fixes the gauge group on the ten-dimensional (10D) boundaries. This restriction arises due to the cancellation of the ten-dimensional anomalies. This is unlike the brane worlds constructed in five dimensions where the boundary gauge group is not restricted by any local anomaly, although global anomalies may impose some constraints \[4\]. However, gravitational anomalies also exist in six dimensions \[5\], and this places nontrivial constraints on six-dimensional (6D) theories \[6, 7, 8, 9\]. In this paper we will show that in seven-dimensional (7D) brane worlds the gauge group structure and matter content on the boundaries will be similarly restricted \[10\]. Of course, the dimensional reduction of the Horava-Witten (HW) model automatically gives rise to brane worlds in seven dimensions, that are anomaly free. But our analysis will be general, and we will construct seven-dimensional brane worlds which satisfy all the anomaly constraints and do not necessarily arise from the dimensional reduction of the HW model \[11, 12\].

Our starting point will be the minimal \(\mathcal{N} = 2\) 7D gauged supergravity. The ungauged theory is obtained from the compactification of M-theory on \(K3\) or, equivalently, from the compactification of strongly coupled heterotic theory on \(T^3\) \[13\]. The compactification produces twenty two vectors resulting from expanding the eleven-dimensional three-form on the \(b_2 = 22\) two-cycles of the \(K3\). Three of these vectors are members of the gravity multiplet, whereas the remaining nineteen fill vector multiplets of the \(\mathcal{N} = 2\) 7D theory. Each vector multiplet also contains three scalars, and the 57 total scalars parametrize the coset space \(SO(19, 3)/SO(19) \times SO(3)\), for which an \(SO(3) \times H\) or \(SO(3, 1) \times H\) subgroup of \(SO(19, 3)\) can be gauged. The corresponding gauged supergravity has been constructed in Ref. \[14\], and it is interesting that this theory admits a one-parameter extension which contains a topological mass term for the three-form. A supersymmetric gauged theory can be obtained after introducing an appropriate potential for the scalar field (corresponding to the \(K3\) volume). The scalar potential has two extrema, leading to either a supersymmetric or non-supersymmetric vacuum \[15\]. The supersymmetric vacuum has a negative cosmological constant implying that the vacuum in the gauged theory is not Minkowski
spacetime but rather anti-de Sitter, AdS. The AdS vacuum with $\mathcal{N} = 2$ supersymmetry has been considered in the context of the AdS/CFT correspondence [16], and was shown to be the supergravity dual of the 6D $\mathcal{N} = (0,1)$ SCFT [17].

The minimal $\mathcal{N} = 2$ 7D gauged supergravity may be compactified down to six dimensions on $S^1$, even in the presence of a cosmological constant as was shown in Ref. [18], where the non-chiral $\mathcal{N} = (1,1)$ 6D theory was obtained. However, we are interested in the chiral $\mathcal{N} = (0,1)$ 6D theory because in this case vector, tensor and hypermultiplets can couple to gravity in a way that is restricted by anomalies. In particular, the possibility of vector multiplets on the 6D boundaries allows one to construct theories which contain the standard model gauge group. Thus, we need to find a way to obtain the chiral 6D theory from the 7D gauged supergravity.

An immediate way to obtain the 6D chiral theory is to compactify on an orbifold $S^1/\mathbb{Z}_2$. This is similar to what happens in the HW model except that in our 7D scenario the vacuum is AdS, and not Minkowski. This difference in vacua means that the boundary branes must now have a tension. In fact, compactifying the AdS vacuum on an orbifold is analogous to similar compactifications of the supersymmetric Randall-Sundrum model in a slice of AdS5 [19, 20]. By adding suitable boundary potential terms, which at the AdS minimum become the brane tensions, we will see that the vacuum of our seven-dimensional brane world becomes a slice of AdS7. Besides the localized gravity multiplet there will also be a localized tensor and hypermultiplet in the resulting 6D $\mathcal{N} = (0,1)$ chiral theory.

However, unlike the five-dimensional case, the resulting spectrum in a slice of AdS7 is anomalous because in six dimensions there are gravitational anomalies, like in ten (and two) dimensions. In order to cancel these anomalies we are then forced to introduce boundary fields such as vector, tensor and hypermultiplets [10]. This leads to a restriction of the possible boundary gauge groups and matter content on the boundaries. In particular, for the case of one tensor multiplet, we will see that for gauge groups containing the standard model, only exceptional groups are allowed with a restriction on the number of generations transforming in the fundamental representation. This is one of the main results of our paper.

Furthermore, the locally supersymmetric bulk-boundary couplings are derived for the case of boundary vector and neutral hypermultiplets. In the HW scenario the Bianchi identity for the four-form field strength had to be modified in order to obtain a consistent coupling between the boundary gauge couplings and the bulk. In our scenario a similar modification for the Bianchi identity will be needed as well. In addition the anomaly cancellation conditions fixes a dimensionless ratio, $\eta$, as in the HW scenario, except that $\eta$ now relates the 6D gauge coupling, the 7D gravitational constant and a topological mass parameter of the Chern-Simon term. For the neutral hypermultiplet in the 6D theory, whose analogous multiplet does not exist in the 11D HW theory, we also construct the locally supersymmetric bulk-boundary Lagrangian. In particular we will need to modify the Bianchi identity of the two-form field strength resulting from the bulk gauge field corresponding to the
gauged R-symmetry. This modification is crucial in showing that the scalar manifold of the boundary hypermultiplets is indeed quaternionic, as expected for a locally supersymmetric theory. The derivation of these bulk-boundary couplings comprises the second main result of our paper. The case of boundary tensor multiplets is more complicated and their couplings will be presented elsewhere.

There is also an interesting novel phenomenon in our 7D brane world scenario, that is not found in the HW model. In the cancellation of the mixed anomaly terms by the Green-Schwarz mechanism, the coefficient of the boundary kinetic terms of the gauge couplings is related to the separation between the UV and IR brane. In certain instances, there is a critical separation for which the boundary kinetic term vanishes. This signifies that the gauge theory on the boundary becomes infinitely strongly coupled and suggests that a phase transition occurs which may be related to tensionless strings. Moreover, just as there is a dual description of the HW theory, where the strongly coupled 10D $E_8 \times E_8$ heterotic string theory is described by the 11D HW theory, we also have a similar dual picture in our model. By the AdS/CFT correspondence \cite{16}, suitably modified for the addition of boundaries \cite{21, 22, 23}, our seven-dimensional supergravity model is dual to a strongly coupled conformal field theory (CFT) in six dimensions \cite{17}, where the gauge and matter fields on the UV brane are fundamental fields added to the CFT, while the gauge and matter fields on the IR brane are bound states of the CFT. It remains an intriguing question to further understand these hybrid six-dimensional conformal field theories.

The outline of this paper is as follows. In Section 2 we compactify the 7D supergravity Lagrangian on the orbifold $S^1/\mathbb{Z}_2$, and identify the surviving 6D supermultiplets. By adding boundary potential terms we construct a 7D brane world in a slice of $AdS_7$, with localized gravity. In general this supersymmetric brane world is anomalous, and in Section 3 we derive the constraints needed for the cancellation of the gauge and gravitational anomalies. In particular we will see that the anomalies must be cancelled locally at the orbifold fixed points, as well as globally. Explicit examples of the anomaly cancellation constraints are given in Section 4. In Section 5 we derive the consistent locally supersymmetric bulk-boundary Lagrangian up to bilinear fermionic terms. We consider boundary vector multiplets and neutral boundary hypermultiplets. In the case of boundary vector multiplets the cancellation of the anomalies fixes the boundary gauge coupling in terms of the 7D gravitational constant, and a topological mass parameter. For the neutral boundary hypermultiplets we will show how the modified Bianchi identity is crucial in order to obtain the quaternionic structure of the scalar manifold. In Section 6 we mention the possibility of a phase transition when the two boundaries reach a critical separation, and comment on the dual correspondence of our model. Finally our conclusions are presented in Section 7. Note that our conventions and notations are summarized in Appendix A and details of the gravitational Chern-Simons term are given in Appendix B. In Appendix C all possible solutions satisfying the anomaly constraints with one tensor multiplet are tabulated.
2 The 7D supergravity Lagrangian on $S^1/\mathbb{Z}_2$

The minimal $\mathcal{N} = 2$ 7D supersymmetry has an $SU(2)$ R-symmetry group and the supersymmetry multiplets are

$$(A_M, A^i, \psi^i), \ M, N = 0, \ldots, 6, \ i, j = 1, 2, \ \text{vector multiplet}$$

$$(g_{MN}, A^i_{MN}, A^i_{M}, \phi, \psi^i_M, \chi^i), \ \text{gravity multiplet}$$

The vector multiplet contains a vector $A_M$, an $SU(2)$ triplet of scalars $A^i$, and an $SU(2)$ pseudo-Majorana spinor $\psi^i$, whereas the gravity multiplet contains the graviton $g_{MN}$, an antisymmetric three-form $A^i_{MN}$, an $SU(2)$ triplet of vectors $A^i_{M}$, a scalar $\phi$, and the $SU(2)$ pseudo-Majorana gravitinos $\psi^i_M$ and spinors $\chi^i$. The $SU(2)$ R-symmetry can be gauged and the resulting $\mathcal{N} = 2$ 7D gauged supergravity with an antisymmetric two-form $B_{MN}$ (the dual of the three-form $A^i_{MN}$) has been constructed in Refs. [24, 25], while the one with the three-form in Refs. [14, 15, 18]. The coupling of $n$ vector multiplets to the 7D $\mathcal{N} = 2$ supergravity leads to the irreducible multiplet

$$(g_{MN}, B_{MN}, A^I_M, \phi^a, \psi^a, \chi^a, \psi^i_M), \ I = 1, \ldots, n + 3, \ a = 1, \ldots, n, \ \alpha = 1, \ldots, 3n,$$

where the scalars parametrize the coset $SO(n,3)/SO(n) \times SO(3)$ as discussed in Ref. [26].

Let us now consider the pure $\mathcal{N} = 2$ 7D gauged supergravity with no vector multiplets. This theory is described by the Lagrangian [14, 15, 18]

$$\kappa^2 e^{-1} \mathcal{L} = \frac{1}{2} R - \frac{\sigma^{-4}}{48} F^2_{MNPQ} - \frac{\sigma^2}{4} F_{MN} \nabla_i F^M_{\ n} \nabla^i - \frac{1}{2} \bar{\psi}^i \Gamma^M D_M \chi_i - \frac{1}{2} \bar{\psi}^i \Gamma^MNK D_N \psi_K - \frac{i \sigma}{2 \sqrt{2}} \left( \frac{1}{12} \bar{\psi}^i \Gamma^{KMN} \psi_R \bar{\psi}^j + \bar{\psi}^i \psi^N \psi^j \right) F_{MNI}$$

$$- \frac{\sigma^{-2}}{8 \sqrt{2}} \left( \frac{1}{12} \bar{\psi}^i \Gamma^{KMN}PQR \psi_R + \bar{\psi}^i \psi^M \psi^P \psi^Q \right) F_{MNPQ}$$

$$- \frac{\sigma}{2 \sqrt{10}} \bar{\psi}^i \Gamma^M \Gamma^{NK} \psi_M \psi_N + \frac{\sigma^{-2}}{24 \sqrt{10}} \bar{\psi}^i \Gamma^L \Gamma^{MN} \psi_M \psi^L$$

$$+ \frac{\sigma^{-2}}{160 \sqrt{2}} \chi^i \Gamma^{MN} \chi^i F_{MNPQ} - \frac{3 i \sigma}{20 \sqrt{2}} \chi^i \Gamma^M \chi^i F_{MNPQ}$$

$$+ \frac{i}{48 \sqrt{2}} \bar{F}^{MNPQ} \left( F^{KLI} \psi_R^j - \frac{2 i g}{3} \text{tr}(A_K A_L A_R) \right) e^{MNPQKLR}$$

$$+ 60 (m - \frac{2}{5} h \sigma^4)^2 - 10 (m + \frac{8}{5} h \sigma^4)^2 + \left( \frac{5}{2} m - h \sigma^4 \right) \bar{\psi}^i \Gamma^M \psi_N$$

$$+ \sqrt{5} \left( m + \frac{8}{5} h \sigma^4 \right) \bar{\psi}^i \Gamma^M \chi^i + \frac{3}{2} m + \frac{27}{5} h \sigma^4 \chi^i \chi^i + \frac{i}{2} \bar{\chi}^i \Gamma^M \Gamma^N \partial_N \phi \psi_M$$

$$+ \frac{h}{36} \bar{F}^{KLMN} PQR F_{KLMN} A_{PQR},$$

(2.1)
where \( F_{KLMN} = 4 \partial_{[K} A_{LMN]} \) and \( \kappa^2 \) is the 7D Newton’s constant. The local supersymmetry transformation rules are

\[
\delta e^A_M = \frac{1}{2} \epsilon^i \Gamma^A \psi_{Mi}, \tag{2.2}
\]

\[
\delta \psi_{Mi} = D_M e_i + \frac{\sigma^2}{80 \sqrt{2}} \left( \Gamma_{MKPQ}^N - \frac{8}{3} \delta_M^N \Gamma_{KPQ} \right) F_{NKPQ} e_i
+ \frac{i \sigma}{10 \sqrt{2}} \left( \Gamma_{NK}^M - 8 \delta_M^N \Gamma_K \right) F_{NKPQ} \epsilon_j
+ \left( m - \frac{2}{5} h \sigma^4 \right) \Gamma M e_i, \tag{2.3}
\]

\[
\delta A_{MNK} = \frac{3 \sigma^2}{2 \sqrt{2}} \bar{\psi}_{i [M} \Gamma_{NK]} e_i + \frac{\sigma^2}{\sqrt{10}} \bar{\chi} \Gamma_{MNK} e_i, \tag{2.4}
\]

\[
\delta A_{i j} = \frac{i \sigma^{-1}}{\sqrt{2}} \left( \bar{\psi} M e_i - \frac{1}{2} \delta_k^i \bar{\psi}_{M} e_k \right)
- \frac{i \sigma^{-1}}{\sqrt{10}} \left( \bar{\chi} j \Gamma_{M} e_i - \frac{1}{2} \delta_i^j \bar{\chi} \Gamma_{M} e_k \right), \tag{2.5}
\]

\[
\delta \chi_i = \frac{1}{2} \Gamma^M \partial_M \phi e_i - \frac{i \sigma}{2 \sqrt{10}} \Gamma_{MN} F_{MN} e_j + \frac{\sigma^2}{24 \sqrt{10}} \Gamma_{MNPQ} F_{MNPQ} e_i
- \sqrt{5} \left( m + \frac{8}{5} h \sigma^4 \right) e_i, \tag{2.6}
\]

\[
\delta \phi = \frac{1}{2} \epsilon^i \chi_i. \tag{2.7}
\]

The notation here is

\[
D_M \chi_i = \partial_M \chi_i + \frac{1}{4} \omega_{MAB} \Gamma^{AB} \chi_i + ig A_{Mi j} \chi_j, \quad m = - \frac{g \sigma^{-1}}{5 \sqrt{2}}, \tag{2.8}
\]

\[
F_{MN i j} = \partial_{M} A_{Ni j} + ig A_{Mi k} A_{Nk j} - M \leftrightarrow N, \quad \sigma = \exp \left( - \frac{\phi}{\sqrt{5}} \right), \tag{2.9}
\]

where \( g \) is the SU(2) coupling. The potential of the scalar \( \phi \) is

\[
V(\phi) = 16 h^2 \sigma^8 + 80 h m^4 - 50 m^2, \tag{2.10}
\]

which has (for \( h/g > 0 \)) two extrema, a non-supersymmetric local minimum and a supersymmetric local maximum [15]. The latter is the supersymmetric \( AdS_7 \) background, and so the \( \mathcal{N} = 2 \) 7D supergravity theory does not possess a Minkowski vacuum. Although there exists no stable Minkowski vacuum, one can still perform a dimensional reduction of the theory (even in the presence of a cosmological constant). This is done by writing the 7D metric in the standard Kaluza-Klein reduction form

\[
ds_7^2 = g_{MN} dx^M dx^N
= e^{-\xi/\sqrt{5}} g_{\mu \nu} (x^\mu) dx^\mu dx^\nu + e^{4 \xi/\sqrt{5}} \left( dx^7 + A_\mu dx^\mu \right)^2, \tag{2.11}
\]

where the theory is reduced along \( x^7 \). For the theory (2.1), this has been performed in [18], and the resulting dimensional reduction produces the \( \mathcal{N} = (1,1) \) 6D supergravity theory. The 6D spectrum obtained after appropriate rescaling, and redefinition of the various fields is

\[
(g_{\mu \nu}, \xi, A_\mu, A_{\mu \nu}, A_{\mu \nu}, A_{\mu i}^j, A_i^j, \phi, \psi_i^j, \psi_{ij}^i),
\]
Before the redefinitions the 7D graviton \( g_{MN} \) gives rise to a 6D graviton \( g_{\mu \nu} \), a scalar \( g_{77} = \xi \), and a vector (graviphoton) \( g_{\mu 7} = A_{\mu} \). From the three-form \( A_{MNP} \) we get a 6D three-form \( A_{\mu \nu \rho} \) and a two-form \( A_{\mu \nu} = A_{\mu \nu} \), from the \( SU(2) \) vector \( A_{M i} \) we get a 6D \( SU(2) \) vector \( A_{\mu i} \) and \( A_{\mu i} = A_{\mu} \), from the 7D gravitino \( \psi^i_M \) we get a 6D gravitino \( \psi^i \) and a spinor \( \psi^i = \psi^i \), while the 7D spinor \( \chi^i \) gives rise to a 6D one \( \chi^i \).

Dualizing the three-form \( A_{\mu \nu \rho} \) into a vector \( B_{\mu} \) we have the \( N = (1, 1) \) 6D massless spectrum

\[
(g_{\mu \nu}, A_{\mu}, A_{\mu \nu}, A_{\mu}^i, \phi, \psi^i, \chi^i, \), \quad \text{gravity multiplet}
\]
\[
(B_{\mu}, A^i, \xi, \psi^i), \quad \text{vector multiplet}
\]

It should be noted that the Poincaré (ungauged) theory is obtained by dimensional reduction of 11D supergravity on a \( K3 \) surface. In this picture, the Chern-Simons term in (2.1) results from the corresponding term in 11D where the parameter \( h \) is proportional to the \( F_4 \) “flux” through \( K3 \). The 11D supergravity Lagrangian is invariant under \( x^{11} \rightarrow -x^{11} \) provided that the three-form of the 11D gravity multiplet transforms as \( A_3 \rightarrow -A_3 \). Similarly, the 7D supergravity Lagrangian, (2.1) is invariant under \( x^7 \rightarrow -x^7 \) provided we have

\[
A_{MNP} \rightarrow -A_{MNP}, \quad A_{M i}^j \rightarrow -A_{M i}^j, \quad h \rightarrow -h, \quad m \rightarrow -m,
\]

which is actually the transformations inherited from the 11D parent theory. As the \( \mathbb{Z}_2 \) transformation \( x^7 \rightarrow -x^7 \) is a symmetry of the theory, we can mod it out, by considering the compactification on \( S^1/\mathbb{Z}_2 \). Thus the only fields which survive at the orbifold fixed points are the \( \mathbb{Z}_2 \) singlets. It is not difficult to see that the \( \mathbb{Z}_2 \) parity assignments consistent also with the supersymmetry transformations in Eqs. (2.2)–(2.7) are

\[
g_{\mu \nu}, A_{\mu \nu}, \phi, \xi, A^i, \psi^{-i}, \chi^i, \psi^i \quad \text{even parity}
\]
\[
A_{\mu}, B_{\mu}, A^i, \psi^i, \chi^i, \psi^{-i} \quad \text{odd parity}
\]

where the \( \pm \) indices refer to the chirality of the 6D reduced spinors. In addition for the supersymmetry parameters we have that \( \epsilon_- \) is even whereas \( \epsilon_+ \) is odd. The odd-parity fields are projected out while the even-parity fields survive the orbifold projection and are organized in 6D \( \mathcal{N} = (0, 1) \) representations. At this point let us recall that the massless representations of the \( (0, 1) \) supersymmetry in 6D, labeled by their \( SU(2) \times SU(2) \) representations are

\[
\begin{align*}
\text{(i) gravity} & : \quad (1, 1) + 2(\frac{1}{2}, 1) + (0, 1), \\
\text{(ii) tensor} & : \quad (1, 0) + 2(\frac{1}{2}, 0) + (0, 0), \\
\text{(iii) vector} & : \quad (\frac{1}{2}, 1) + 2(0, \frac{1}{2}), \\
\text{(iv) hyper} & : \quad 2(\frac{1}{2}, 0) + 4(0, 0).
\end{align*}
\]

Consequently, the gravity multiplet contains the graviton, a self-dual two-form field and a gravitino, the tensor multiplet contains an anti-self dual two-form field, a
scalar and a spinor (tensorino), the vector contains a vector, and a gaugino, while
the hypermultiplet consists of four scalars and a spinor (dilatino). It should be noted
that all spinors are symplectic-Majorana, and that the gaugino and gravitino have the
same chirality (left-handed), opposite to the tensorino and dilatino (right-handed).
Thus, the surviving fields on the orbifold $S^1/\mathbb{Z}_2$ are arranged into the following 6D
multiplets

$$\begin{align*}
&(g_{\mu\nu}, A^+_{\mu\nu}, \psi^i_\mu), \quad \text{gravity} \\
&(A^-_{\mu\nu}, \phi, \chi^i), \quad \text{tensor} \\
&(A^i_{\mu}, \xi, \psi^i), \quad \text{hypermultiplet}
\end{align*}$$

where $\psi^i_\mu = \psi^-^i_\mu$ are left-handed symplectic Majorana-Weyl fermions while $\chi^i = \chi^i_+$
and $\psi^i = \psi^i_+$ are right-handed.

It is also interesting to point out that the 6D multiplets (2.13)-(2.15), only exist
due to the orbifold compactification. Instead if we had compactified on $S^1$ then
for nonzero $h$, the two-form $A_{\mu\nu}$ becomes massive by eating the Nambu-Goldstone
bosons, $B_\mu$ in a generalized Higgs mechanism [18]. However by compactifying on the
orbifold, the $B_\mu$ fields are projected out and the two-form $A_{\mu\nu}$ remains massless.

Clearly, the 6D spectrum (2.13)-(2.15) at the orbifold fixed points is anomalous
and the only way to make sense of such a theory is to introduce extra vector, hyper,
and tensor multiplets at the fixed points in such a way as to cancel any anomalous
contribution.

## 2.1 7D Randall-Sundrum vacuum

The compactification of the 7D solution on an orbifold results in an $\mathcal{N} = (0, 1)$ 6D
theory with the massless spectrum (2.13)-(2.15), provided that under $x_7 \to -x_7$
Eq. (2.12) is satisfied. However, these relations do not necessarily respect the su-
persymmetry transformations (2.2)-(2.7) at the boundaries. For example, since the
parameters $h$ and $m$ are odd at the orbifold fixed points, the variation of the kinetic
energy terms will produce $\delta$-function terms. In order to make the truncated theory
on the orbifold supersymmetric we must introduce six-branes at the orbifold fixed
points with specific boundary potentials. This is very similar to the five-dimensional
supersymmetric Randall-Sundrum model [19][20], where supersymmetry requires the
introduction of brane tensions.

In the Lagrangian (2.1) and supersymmetry transformation rules (2.2)-(2.7) let
us make the replacement (with $y = x_7$)

$$h \to h [\epsilon(y) - \epsilon(y - \pi R)] ,$$

and similarly for $m$, where $\epsilon(y) = 1(-1)$ for $y > 0(y < 0)$. If we introduce the
boundary potential term

$$S_0 = \int d^6x \int dy \sqrt{-g} 20(m - \frac{2}{5}h\sigma^4) [\delta(y) - \delta(y - \pi R)] ,$$

where $\sigma^4$ is the volume of the 4D hypersphere at the orbifold fixed points. 

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on the six-branes located at the orbifold fixed points $y^*$, then the complete action will be supersymmetric. The supersymmetric vacuum is the one in which the Killing equations

$$\delta\psi_{Mi} = \delta\chi_i = 0,$$

are satisfied. Assuming that all bulk fields are zero except for the scalar $\phi$ we find from Eq. (2.18) that

$$\langle \sigma \rangle = \left( \frac{g}{8\sqrt{2h}} \right)^{\frac{1}{5}}.$$

Substituting this vacuum expectation value back into the bulk Lagrangian (2.1) and boundary action (2.17) we obtain the action

$$S = S_7 + S_{(0)} + S_{(\pi R)},$$

$$S_7 = \int d^6x \int dy \sqrt{-g} \left[ \frac{1}{2} M^5 R - \Lambda_7 \right],$$

$$S_{(y^*)} = \int d^6x \sqrt{-g_6} \left[ L_{(y^*)} - \Lambda_{(y^*)} \right],$$

where $g_6$ is the induced metric on the six-brane located at $y^*$, and $M = \kappa^{-2/5}$ is the 7D Planck mass. The cosmological constants are given by

$$\Lambda_7 = -15 M^5 k^2; \quad \Lambda_{(0)} = -\Lambda_{(\pi R)} = 10 M^5 k,$$

where

$$k = \left( \frac{hg^4}{16} \right)^{\frac{1}{5}}.$$

The Einstein equations for the combined bulk and boundary action (2.20) can be solved to obtain a seven-dimensional Randall-Sundrum solution

$$ds^2 = e^{-2k|y|} dx_6^2 + dy^2,$$

where $0 \leq y \leq \pi R$ and $k$ is the AdS curvature scale which is given by (2.24). Note that supersymmetry automatically guarantees the fine-tuning conditions (2.23) required to obtain the Randall-Sundrum solution. This leads to a slice of $AdS_7$, where the 6D gravity multiplet is localized on the UV brane at $y^* = 0$. The localization of the gravitino on the UV brane follows from the fact that in the $AdS_7$ vacuum, the gravitino has a mass term

$$m_{3/2} = \frac{5}{2} k \left[ \epsilon(y) - \epsilon(y - \pi R) \right].$$
which leads to the zero mode wave function $\psi^{(0)}_{\mu} \propto e^{-\frac{1}{2}k|y|}$ for the left-handed $\psi^{(0)}_{\mu}$.

On the other hand the tensor and hypermultiplets are localized on the IR brane. For the tensor multiplet the simplest way to see the localization on the IR brane is to note that in the $AdS_7$ vacuum, the scalar in the tensor multiplet has a bulk mass term

$$m^2_\phi = -8k^2 + 8k [\delta(y) - \delta(y - \pi R)] ,$$

which leads to a zero mode wavefunction $\phi^{(0)} \propto e^{4k|y|}$. Similarly one can check that the right-handed tensorino, $\chi$ obtains a wavefunction $\chi^{(0)} \propto e^{\frac{3}{2}k|y|}$, which is consistent with supersymmetry. This follows from the fact that in the $AdS_7$ vacuum the tensorino has a bulk mass term

$$m_\chi = -\frac{3}{2}k [\epsilon(y) - \epsilon(y - \pi R)] .$$

Thus by supersymmetry the tensor field in the tensor multiplet must be localized on the IR brane.

For the hypermultiplet we notice that the scalar is identified as the radion and from an analysis similar to that studied in 5d we find that the radion is localized on the IR brane [27]. Thus by supersymmetry we expect the remainder of the component fields in the hypermultiplet to be localized on the IR brane.

### 3 Anomaly cancellation with a boundary

So far the compactification of the 7D solution has resulted in a theory with a tensor and a hypermultiplet coupled to gravity, which are localized at one of the two boundaries. However, this 6D theory containing only a gravity, hyper, and tensor multiplet is anomalous. The only way to make a consistent $S^1/Z_2$ compactification of the 7D $\mathcal{N} = 2$ theory is to introduce matter on the boundaries such that the complete theory, bulk plus boundary, is anomaly free. Unlike the five-dimensional case where there is no anomaly constraint, and arbitrary matter can be added to the boundaries [19], in our seven-dimensional slice of AdS, we will see that anomaly cancellation restricts the boundary matter content. In particular, the anomaly must be cancelled both locally on the boundaries of the 7D orbifold as well as globally by a Green-Schwarz (GS) mechanism [28]. Local cancellation is necessary because otherwise the boundary theory would be anomalous, while global cancellation is required since the 7D theory when reduced to 6D gives rise to massless fields which contribute to the anomaly.

#### 3.1 Local Green-Schwarz cancellation

For a Green-Schwarz mechanism to take place, a necessary condition is that the irreducible part of the anomaly cancels. Here we will examine the cancellation of
the irreducible part $\text{tr} R^4$ of the gravitational anomaly such that it cancels locally on each boundary. There are four contributions to the gravitational anomaly on each boundary coming from the gravity, vector, tensor, and hypermultiplets. The total gravitational anomaly from the bulk fields (the gravity, tensor and hypermultiplet) is

$$\mathcal{A}_{\text{grav}}^{\text{bulk}} = \frac{1}{5760} \left[ 243 \, \text{tr} R^4 - \frac{225}{4} (\text{tr} R^2)^2 \right], \quad (3.1)$$

where $\mathcal{A} = (2\pi)^4 I_8$, with $I_8$ the anomaly eight-form. This anomaly is distributed evenly [29] between the two boundaries at $x_7 = 0, \pi R$ as

$$\mathcal{A}_{\text{grav}}^{\text{bulk}} = \frac{1}{2 \cdot 5760} \left[ 243 \, \text{tr} R^4 - \frac{225}{4} (\text{tr} R^2)^2 \right] \delta(x_7)$$

$$+ \frac{1}{2 \cdot 5760} \left[ 243 \, \text{tr} R^4 - \frac{225}{4} (\text{tr} R^2)^2 \right] \delta(x_7 - \pi R). \quad (3.2)$$

To cancel the anomaly locally, an appropriate amount of matter must be added to each boundary. If there are $N_V$ vectors, $N_H$ hypers and $N_T$ tensors at $x_7 = 0$, and $\tilde{N}_V$ vectors, $\tilde{N}_H$ hypers and $\tilde{N}_T$ tensors at $x_7 = \pi R$, then their anomaly contribution is

$$\mathcal{A}_{\text{bound}}^{\text{grav}} = \frac{1}{5760} \left[ (N_V - N_H - 29 N_T) \, \text{tr} R^4 + \frac{5}{4} (N_V - N_H + 7 N_T)(\text{tr} R^2)^2 \right] \delta(x_7)$$

$$+ \frac{1}{5760} \left[ (\tilde{N}_V - \tilde{N}_H - 29 \tilde{N}_T) \, \text{tr} R^4 + \frac{5}{4} (\tilde{N}_V - \tilde{N}_H + 7 \tilde{N}_T)(\text{tr} R^2)^2 \right] \delta(x_7 - \pi R). \quad (3.3)$$

Cancellation of the irreducible part of the anomaly in (3.2), then leads to the conditions

$$N_H + 29 N_T - N_V = \frac{243}{2} - 15n, \quad (3.4)$$

$$\tilde{N}_H + 29 \tilde{N}_T - \tilde{N}_V = \frac{243}{2} + 15n, \quad (3.5)$$

where we have included a bulk contribution arising from a gravitational Chern-Simons term, that only gives a meaningful solution for half-integral values of $n$ (see Appendix B). This can also be thought of as the branes having a “magnetic” charge [30]. Thus we obtain for $N = N_H + 29 N_T - N_V$, and $\tilde{N} = \tilde{N}_H + 29 \tilde{N}_T - \tilde{N}_V$ the constraint $N + \tilde{N} = 243$. Then, the remaining reducible part of the total gravitational anomaly is

$$\mathcal{A}^{\text{grav}} = \frac{1}{128} \left( N_T - 4 + \frac{n}{2} \right) (\text{tr} R^2)^2 \delta(x_7) + \frac{1}{128} \left( \tilde{N}_T - 4 - \frac{n}{2} \right) (\text{tr} R^2)^2 \delta(x_7 - \pi R). \quad (3.6)$$
Let us now consider the gauge and mixed anomalies. The requirement of local anomaly cancellation for two six-branes located at the fixed points $x_7 = 0, \pi R$ of the $\mathbb{Z}_2$ orbifold means that the gauge group should be $G_1 \times G_2$, where each $G_i$ group is localized on one of the two fixed points (for simplicity we will only consider semisimple $G_i$). The pure six-dimensional anomaly is formally described by an anomaly polynomial eight-form, $I_8$. Thus, mathematically we require that the anomaly eight-form, $I_8$ should satisfy

$$\frac{\partial^2 I_8}{\partial \text{tr} F_1^2 \partial \text{tr} F_2^2} = 0 ,$$

(3.7)

where $F_1, F_2$ are the gauge field strengths of the $G_1 \times G_2$ gauge group. On the 7D orbifold this condition simply means that there is no matter charged under both the gauge groups located at the fixed points and the only interaction between the two six-branes is purely gravitational. Consequently, the eight-form $I_8$ is written as

$$I_8 = I_8^{(1)} + I_8^{(2)} ,$$

(3.8)

where $I_8^{(1)}, I_8^{(2)}$ are the anomaly polynomial for the boundary theories at $0, \pi R$, respectively. Then, by Eq.(3.7), $I_8^{(1)}$ and $I_8^{(2)}$, appropriately normalized, are explicitly written as

$$I_8^{(1)} = \left[ \frac{1}{2} \left( 1 - \frac{N_T}{4} + \frac{n}{8} \right) (\text{tr} R^2)^2 + \frac{1}{6} \text{tr} R^2 X_1^{(2)} - \frac{2}{3} X_1^{(4)} \right] \delta(x_7) ,$$

$$I_8^{(2)} = \left[ \frac{1}{2} \left( 1 - \frac{\tilde{N}_T}{4} - \frac{n}{8} \right) (\text{tr} R^2)^2 + \frac{1}{6} \text{tr} R^2 X_2^{(2)} - \frac{2}{3} X_2^{(4)} \right] \delta(x_7 - \pi R) ,$$

(3.9)

where $X_i^{(n)}$ are defined as [31]

$$X_i^{(n)} = \text{Tr} F_i^n - \sum n_i \text{tr}_i F_i^n , \quad i = 1, 2 .$$

(3.10)

As usual, Tr denotes the trace in the adjoint representation, tr$_i$ the trace in the representation $R_i$ of the group $G_i$, and $n_i$ is the number of hypermultiplets in the representation $R_i$. For the GS mechanism to work in this case, we demand the local factorization (omitting $\delta$-functions)

$$I_8^{(i)} = I_4^{(i)} \tilde{I}_4^{(i)} ,$$

(3.11)

where $i = 1, 2$ and,

$$I_4^{(i)} = c_i \text{tr} R^2 + a_i \text{tr} F_i^2 , \quad \tilde{I}_4^{(i)} = \text{tr} R^2 + b_i \text{tr} F_i^2 ,$$

(3.12)

and $c_1, c_2$ are $\frac{1}{2}(1 - N_T/4 + n/8), \frac{1}{2}(1 - \tilde{N}_T/4 - n/8)$, respectively. Then, the anomalies $I_8^{(1)}, I_8^{(2)}$ vanish by a local GS mechanism at $x_7 = 0$ and $x_7 = \pi R$, respectively.
A simple inspection of Eq. (3.9) reveals that such a factorization may be problematic due to the $X^{(4)}$ term. Indeed,

$$X^{(4)}_i = \text{Tr} F^4_i - \sum_i n_i \text{tr} F^4_i ,$$  \hspace{1cm} (3.13)

is the pure gauge anomaly and can be written as \[32, 33\],

$$X^{(4)}_i = \alpha_i \text{tr} F^4_i + \gamma_i (\text{tr} F^2_i)^2 .$$  \hspace{1cm} (3.14)

Similarly, we may write

$$X^{(2)}_i = \beta_i \text{tr} F^2_i .$$  \hspace{1cm} (3.15)

Thus, for each term (3.11) we have using (3.14) and (3.15)

$$I^{(i)}_8 = c_i (\text{tr} R^2)^2 + \frac{\beta_i}{6} \text{tr} R^2 \text{tr} F^2_i - \frac{2\alpha_i}{3} \text{tr} F^4_i - \frac{2}{3} \gamma_i (\text{tr} F^2_i)^2 .$$  \hspace{1cm} (3.16)

Then it is clear that the factorization (3.12) is possible as long as

$$\alpha_i \text{tr} F^4_i = 0 .$$  \hspace{1cm} (3.17)

There are two solutions to the above equation: i) either there is no fourth-order Casimir, or ii) $\alpha_i = 0$. The first possibility is satisfied for all the irreps of $E_8, E_7, E_6, F_4, G_2, SU(3), SU(2), U(1)$, for the 28 of Sp(4) and SU(8) and all the irreps of SO(2n) with highest weight $(f_1, f_2, f_1, -f_2, 0, ..., 0)$ in the Gel’fand-Zetlin basis \[7\]. The case ii) is model dependent and should be solved in each case.

Now if (3.17) is satisfied, then the anomaly can be locally factorized as in (3.12) with

$$a_i + b_i c_i = \frac{\beta_i}{6} , \quad a_i b_i = -\frac{2}{3} \gamma_i .$$  \hspace{1cm} (3.18)

Thus, $a_i, b_i c_i$ are the roots of the equation

$$x^2 - \frac{\beta_i}{6} x - \frac{2}{3} \gamma_i c_i = 0 .$$  \hspace{1cm} (3.19)

This equation has real solutions (so that the anomaly can always be factorized) for

$$\beta_i^2 + 96 c_i \gamma_i > 0 .$$  \hspace{1cm} (3.20)

Thus, for

$$\gamma_i > -\frac{\beta_i^2}{96c_i} ,$$  \hspace{1cm} (3.21)

the anomaly can be cancelled by a GS mechanism (provided Eq. (3.17) is satisfied).
3.2 Global Green-Schwarz cancellation

The locally factorized reducible part of the anomaly must also cancel globally. This follows from the fact that the 7D orbifold theory reduces in six dimensions to a Kaluza-Klein sum of massive modes, which do not contribute to the anomaly, and the massless 6D fields which do give an anomaly. This means that the reducible gravitational and gauge anomalies should be cancelled by a global GS mechanism. The irreducible part of the gravitational anomaly of an $\mathcal{N} = (0,1)$ 6D supergravity theory with $n_V$ vector multiplets, $n_T$ tensor multiplets, and $n_H$ hypermultiplets is cancelled when

$$n_V - n_H - 29n_T + 273 = 0.$$  \hspace{1cm} (3.22)

In the case of $n_T = 1$, global cancellation of the anomalies leads to the global GS condition

$$I_8 = \left( \text{tr} R^2 + \sum_i u_i \text{tr} F_i^2 \right) \left( \text{tr} R^2 + \sum_i v_i \text{tr} F_i^2 \right),$$  \hspace{1cm} (3.23)

where $u_i, v_i$ are constants. We will see that the combination of satisfying (3.11), (3.12), and (3.23) leads to very stringent possibilities for the boundary matter.

For the case of $n_T > 1$ one does not require the factorization (3.23) because the presence of additional tensor fields allows the reducible part of the anomaly to cancel via a generalized GS mechanism [32].

4 Gauge group analysis

As we have seen previously, the dimensional reduction of the bulk gravity multiplet gives rise to the gravity multiplet, one tensor multiplet, and one hypermultiplet of the $\mathcal{N} = (0,1)$ 6D theory. However, this theory by itself is anomalous since Eq. (3.22) is not satisfied. We are then forced to introduce boundary fields such that in the resulting 6D theory the anomaly can be cancelled by a GS mechanism. The fields which can be introduced on the 6D boundaries are vector, tensor and hypermultiplets and we will discuss next the various possibilities for the boundary theory.

4.1 $n_T = 1$

In the case where there is only one tensor multiplet in the 6D theory, arising from the dimensional reduction of the bulk theory (so that $N_T = \tilde{N}_T = 0$), we are led to the constraint

$$n_H = n_V + 244.$$  \hspace{1cm} (4.1)
This means that we can add vector multiplets and hypermultiplets on the boundaries. As discussed earlier we will assume that on each boundary there is a gauge group $G_i$. For the hypermultiplets charged under the gauge group $G_i$, we will assume that under $G_1 \times G_2$ the total number of hypermultiplets consist of the following representations

$$n_1 (d_{F_1}, 1) + n_2 (1, d_{F_2}) + (n_S + 1)(1, 1),$$

where $d_{F_i}$ is the dimension of the fundamental representation of the group $G_i$, and $n_{1,2}, n_S$ are constants representing the number of each representation. Note that we have automatically included the extra singlet hypermultiplet (or radion multiplet) arising from the dimensionally reduced bulk theory. Thus, assuming that the constraint (4.1) is satisfied, and simultaneously solving (3.12) and (3.23) we find the following solutions:

| $G_1 \times G_2$ | $n_1 + n_2$ | $n_S$ |
|------------------|-------------|-------|
| $G_2 \times G_2$ | 20          | 131   |
| $F_4 \times F_4$ | 10          | 87    |
| $E_6 \times E_6$ | 12          | 75    |
| $E_7 \times E_7$ | 8           | 61    |

We see from the table that the distribution of the 6D anomaly on the two boundaries constrains the number of generations of fundamental matter that can be added. In particular consider the $E_6 \times E_6$ solution. In this case we have

$$I_8 = \left[ \text{tr} R^2 - \frac{1}{3}(\text{tr} F_1^2 + \text{tr} F_2^2) \right] \left[ \text{tr} R^2 + (1 - \frac{n_1}{6})(\text{tr} F_1^2 - \text{tr} F_2^2) \right]$$

$$= \left( c_1 \text{tr} R^2 + a_1 \text{tr} F_1^2 \right) \left( \text{tr} R^2 + b_1 \text{tr} F_1^2 \right) + \left( c_2 \text{tr} R^2 + a_2 \text{tr} F_2^2 \right) \left( \text{tr} R^2 + b_2 \text{tr} F_2^2 \right),$$

where $n_1 + n_2 = 12$, $c_1 = 1/2 + n/16$, $c_2 = 1/2 - n/16$ and $a_i = \xi_\pm$, $b_i c_i = \xi_\mp$ with

$$\xi_\pm = \frac{1}{12} \left( 4 - n_i \pm \sqrt{(4 - n_i)^2 + 8c_i(6 - n_i)} \right).$$

In particular if one boundary contains 3 generations of the fundamental 27 then the other boundary must have 9 generations. There are also $(n_1, n_2)$ solutions $(2, 7)$ and $(5, 10)$, and similar exceptions exist for the other gauge groups. It is also possible to have two different gauge groups distributed between the fixed points. The complete solutions for exceptional groups can be found in Appendix C. It should be noted that in general, these solutions are not obtained from compactifications of the weakly coupled heterotic $E_8 \times E_8$ theory [34] or from compactifications of the HW theory because the latter involves matter charged under both local gauge groups [11, 12].

The simultaneous constraint of satisfying (3.12) and (3.23) has restricted the possible gauge group structure on the boundaries. In particular notice that it is not
possible to have $SU(n), (n > 3)$, $SO(n)$, and $Sp(n)$ on the boundaries because in order to cancel the fourth order Casimir one needs specific numbers of fundamentals which are incompatible with (3.23). For the $SO(n)$ groups $(n > 6)$ one can also try to add matter in the spinorial representations. However, in this case the fourth order Casimir cannot be cancelled as required by (3.17).

Finally note that the six-dimensional theory may still be ill defined due to non-perturbative anomalies [35, 36, 7]. Global anomalies exist as long as $\pi_6(G)$ is non-trivial. In our case only the gauge group $G_2$ may be plagued by global anomalies since $\pi_6(G_2) = \mathbb{Z}_3$. In particular, with $n_F$ fundamentals of $G_2$, the condition for the absence of global anomalies is $n_F = 1 \mod 3$ [37]. Thus, for the examples tabulated in Appendix C containing the gauge group $G_2$, the absence of non-perturbative anomalies will further restrict the values of $(n_1, n_2)$.

4.2 $n_T > 1$

More generally we can consider the addition of extra tensor multiplets, on the boundaries in addition to the one arising from dimensional reduction. This means that we must now satisfy the irreducible anomaly constraint (3.22). Once this is done the remaining part of the anomaly can only be cancelled by invoking the generalized GS mechanism where the extra tensor multiplets are used to cancel part of the anomaly [32].

This can best be illustrated by an example. Consider the product gauge group $SO(n_1) \times SO(n_2)$, where the hypermultiplets are in the representation

$$(n_1 - 8)(d_F, 1) + (n_2 - 8)(1, d_F) + (n_S + 1)(1, 1) .$$

Then by adding 3 tensor multiplets on each six-brane, the anomaly polynomial can be factorized in the form

$$I_8 = \frac{1}{4} \left\{ [trR^2 + 2(trF_1^2 + trF_2^2)]^2 - 2 (2trF_1^2 + 2trF_2^2)^2 - 4 (trF_1^2 - trF_2^2)^2 \right\}$$

$$= (c_1 trR^2 + a_1 trF_1^2) (trR^2 + b_1 trF_1^2) - 3(trF_1^2)^2$$

$$+ (c_2 trR^2 + a_2 trF_1^2) (trR^2 + b_2 trF_1^2) - 3(trF_1^2)^2 ,$$

where $c_1 = 1/8 + n/16$, $c_2 = 1/8 - n/16$ and $a_i = \xi$, $b_i c_i = \xi$ with

$$\xi = \frac{1}{2} \left( 1 \pm \sqrt{1 - 4c_i} \right) .$$

As we can see by allowing more tensor multiplets on the branes, we obtain solutions which can involve gauge groups other than the exceptional groups. This implies that the class of anomaly free 7D brane worlds is much bigger than those arising from using the usual GS mechanism if one invokes the generalized GS mechanism. These solutions that require a multiple number of tensor multiplets on the branes are interesting because they are not derivable from ten-dimensional heterotic string compactifications.
5 Constructing the bulk-boundary Lagrangian

We have seen that a consistent 7D orbifold theory can be obtained by adding boundary fields to cancel all the anomalies. Next we construct the locally supersymmetric bulk-boundary Lagrangian. The basic idea in theories with boundaries [3] is to introduce a globally supersymmetric theory at the boundary which is coupled to the bulk. The combined theory is then clearly invariant under local supersymmetry transformations in the bulk, and global supersymmetry transformations on the boundary. To construct a complete locally supersymmetric theory, bulk-boundary interactions must be added (if possible). In the HW setup, the only theory which can live on the boundary is a super Yang-Mills theory since in ten dimensions this is the only supermultiplet of the $\mathcal{N} = 1$ theory (besides the gravity multiplet, which in any case exists due to the $S^1/\mathbb{Z}_2$ compactification). Moreover, in the same framework, since there are no bulk or boundary scalars to organize a perturbative expansion, everything is given in terms of the 11D Newton’s constant and a dimensionless number ($\eta$) which controls the boundary-bulk coupling.

In our case, where the 7D supergravity Lagrangian is compactified on $S^1/\mathbb{Z}_2$ with two boundaries the situation is more involved because of basically two reasons. First, the 6D $(0,1)$ theory has not only vector multiplets but also tensor and hypermultiplets. The second reason is that there are scalars in the bulk in addition to the scalars in the boundary theory. In particular, the couplings of the bulk scalars to the boundary theory are not a priori known. One way to construct these couplings is to use supersymmetry because the final theory should have local $(0,1)$ supersymmetry on the boundary (after dimensional reduction). The $(0,1)$ 6D pure supergravity theory was first considered in Ref. [38]. The coupling of an arbitrary number of tensor multiplets to lowest order in the fermionic fields was considered in [39], while the coupling of a single tensor multiplet and an arbitrary number of hypers was studied in [40]. Following this work, it was shown in Ref. [32] how the model of [39] can be coupled to vector multiplets by employing gauge and gravitational anomaly arguments. Furthermore, this was shown to be related to the supersymmetry anomaly in Ref. [41]. The complete $(0,1)$ supergravity coupled to vectors and tensors has been constructed in [42], and the inclusion of hypermultiplets has partially been obtained in [43]. More recently, the most general up to date supergravity theory coupled to vectors, tensors and hypermultiplets has been given in [44].

Let us consider the most general globally supersymmetric theory on the boundary coupled to gravity, which contains $n_V$ vectors, $n_T$ tensors, and $n_H$ hypermultiplets. Then, the $n_T$ scalars of the tensor multiplet parametrize the coset $SO(1,n_T)/SO(n_T)$ [39], whereas the $4n_H$ scalars of the hypermultiplets parametrize a quaternionic manifold [45]. In the rigid (global) supersymmetric case, the scalar manifold of the $n_T$ scalars turns out to be flat, while the scalars in the hypermultiplets now parametrize a hyperkähler manifold (see Table 1). Thus we see that the coupling of the 7D bulk scalars to the boundary theory should be such that, in the reduced 6D theory the
original hyperkähler scalar manifold of the hypermultiplets should transform into a quaternionic manifold, while the original flat $\mathbb{E}^n_T$ scalar manifold for the scalars in the tensor multiplets should turn into the coset $SO(1,n_T)/SO(n_T)$.

For the hypermultiplets, let us recall the corresponding situation in the $\mathcal{N} = 2$ 4D theory. There, the scalar manifold for the hypermultiplets is hyperkähler in the globally supersymmetric case and becomes quaternionic in the local case. This is achieved by introducing gravitinos which are coupled to the $Sp(1)$-connection of the hyperkähler manifold. Supersymmetry requires that this connection is no longer flat (as was the case for global supersymmetry). The hyperkähler manifold is then replaced by a quaternionic one and this is how, technically, the quaternionic structure arises in the local $\mathcal{N} = 2$ 4D theory. In our case, the scalars in the hypermultiplets that are added to the boundary also parametrize hyperkähler manifolds, but there are no boundary gravitinos which have to be added since the gravitinos simply emerge from the dimensional reduction. Thus, it appears that there is no obvious way to be consistent with 6D local supersymmetry, since the latter demands that the hyperkähler manifold must be quaternionic.

However, recall that there is an $SU(2)$ bulk gauge field which couples to the 7D gravitons. This plays the role of the $SU(2)(= Sp(1))$ connection and as we will show, local supersymmetry in the $S^1/Z_2$ compactification of the 7D theory demands that the $SU(2)$ bulk gauge field is related to the $Sp(1)$ connection of the scalar manifold. This boundary condition then determines the scalar manifold to be quaternionic. On the other hand, it is easy to see that with $n_T$ tensors on the boundary, there exists a coupling which is consistent with 6D local supersymmetry because in the reduced theory the scalar manifold for the scalars in the tensors changes from a flat manifold to the coset $SO(1,n_T)/SO(n_T)$.

### 5.1 Bulk-boundary action

The most general boundary theory contains vector, hyper and tensor multiplets. The boundary action may collectively be written as

$$S_{\text{boundary}} = S_0 + S_{YM} + S_H + S_T,$$

where $S_0$ is given in (2.17) and $S_{YM}$, $S_H$, $S_T$ are the actions for the vector, hyper, and tensor multiplets, respectively. By demanding that both the bulk and boundary is locally supersymmetric, we will determine the action for vectors and neutral hyper-
multiplets. The case of gauged hypermultiplets can be generalized from the neutral hypermultiplet case, and the tensor multiplet case will be presented elsewhere.

5.1.1 Boundary vector multiplets

Let us start by considering the 6D globally supersymmetric action for vector multiplets

$$S^{(0)}_{YM} = -\frac{1}{\lambda^2} \int d^6 x \sqrt{-g} \left( \frac{1}{4} \sigma^{-2} F_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} \bar{\lambda}^a \Gamma^\mu D_\mu \lambda^a \right),$$  \hspace{1cm} (5.2)$$

where $F_{\mu\nu}$ is the gauge field strength of the gauge fields propagating on the six-brane, and $\sigma$ is defined in Eq. (2.9). The supersymmetry transformations are

$$\delta A^a_\mu = \frac{1}{2} \sigma \epsilon \Gamma_\mu \lambda^a, \hspace{1cm} (5.3)$$

$$\delta \lambda^a = -\frac{1}{4} \sigma^{-1} \Gamma^{\mu\nu} F_{\mu\nu} \epsilon. \hspace{1cm} (5.4)$$

In order to make the combined action $S_{bulk} + S_{boundary}$ locally supersymmetric, where the bulk Lagrangian is defined by (2.1), we need to add an interaction $\bar{\psi} S_{YM}$, where $S_{YM}$ is the supercurrent of the vector supermultiplet. This interaction is

$$S^{(1)}_{YM} = -\frac{1}{16\lambda^2} \int d^6 x \sqrt{-g} \sigma^{-2} \bar{\psi}_i \Gamma_{\mu\rho} \Gamma^\mu \lambda^a F^a_{\mu\rho},$$  \hspace{1cm} (5.5)$$

and its variation cancels terms in $L_{YM}$ of the form $D\epsilon \lambda F$ and $\epsilon \psi \lambda D\lambda$, whereas the uncancelled part is

$$\Delta^{(1)} = \frac{1}{16\lambda^2} \int d^6 x \sqrt{-g} \sigma^{-2} \bar{\psi}_i \Gamma_{\mu\rho\sigma\tau} F^a_{\mu\rho} F^a_{\sigma\tau} \epsilon_i.$$  \hspace{1cm} (5.6)$$

Analogous to the 11D HW theory we can cancel this contribution from the variation of the bulk term, $\bar{\psi}_A \Gamma^{ABCDEF} \psi_F F_{BCDE}$ by modifying the Bianchi identity as

$$dF_{7\mu\nu\rho\sigma} = -3\sqrt{2} \frac{\kappa^2}{\lambda^2} \delta(x_7) F_{[\mu\nu} F^{a\rho\sigma]}.$$  \hspace{1cm} (5.7)$$

There is one more term in the variation of $L_{YM}$ which remains to be cancelled. This term comes from $\delta A^a_\mu$ and $\delta \lambda$, proportional to $\sigma$ and $\sigma^{-1}$, respectively, and it is

$$\delta L^{(0)}_{YM} = \frac{1}{4\sqrt{3}\lambda^2} \int d^6 x \sqrt{-g} \sigma^{-1} \bar{\lambda}^a F_{\mu\nu} \Gamma^\mu \Gamma^\nu \epsilon \partial_\kappa \phi.$$  \hspace{1cm} (5.8)$$

It can be cancelled by adding to the boundary action the term

$$S^{(2)}_{YM} = -\frac{1}{2\sqrt{3}\lambda^2} \int d^6 x \sqrt{-g} \sigma^{-1} \bar{\lambda}^a F_{\mu\nu} \Gamma^\mu \Gamma^\nu \chi,$$  \hspace{1cm} (5.9)$$
where $\chi$ is the partner of the bulk $SU(2)$ vector multiplet. In the variation of $S_{YM}^{(2)}$, terms of the form $\epsilon F^2 \chi$ are cancelled from the variation of $\sigma^{-2}$ in $L_{YM}$. The only uncanceled variation of $S_{YM}^{(2)}$ is

$$\Delta^{(2)} = -\frac{1}{8\sqrt{5}\lambda^2} \int d^6x \sqrt{-g} \sigma^{-2} \epsilon \Gamma^{\mu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \chi .$$

(5.10)

This term should also cancel from a bulk contribution. Indeed, in the bulk action (2.1) there exists the term $\chi \psi_L F_{MNPQ}$, and its variation will contain the term $\chi D_L \epsilon F_{MNPQ}$ which after partial integration gives $\chi \epsilon dF$. In the usual theory without boundaries, this contribution vanishes due to the Bianchi identity. However, in the presence of the boundary, there is a contribution

$$-\frac{1}{24\sqrt{10}\kappa^2} \sigma^{-2} \chi \Gamma^{KLMNP} \epsilon \partial_K F_{LMNP} = \frac{1}{8\sqrt{5}\lambda^2} \sigma^{-2} \Gamma^{\mu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \chi ,$$

(5.11)

as a result of (5.7) and it is this bulk contribution which exactly cancels $\Delta^{(2)}$. It should be noted that there are further terms in the variation of $S_{YM}^{(1)}$ and $S_{YM}^{(2)}$ which have to be checked. They arise from variations $\psi_\mu$ and $\chi$ proportional to $F_{\mu\nu\rho}^7$ and $F_{\mu\rho}^i j$, and give terms of the form $\epsilon FF_{\mu\nu\rho}^7$ and $\epsilon FF_{\mu\rho}^i j$. In particular, the only uncanceled variation of $S_{YM}^{(1)}$ and $S_{YM}^{(2)}$ so far is

$$\Delta^{(3)} = \frac{1}{48\sqrt{2}\lambda^2} \int d^6x \sqrt{-g} \sigma^{-3} \epsilon \left( \Gamma^{\mu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \right) \lambda_i^j \chi .$$

(5.12)

There is an additional contribution from the variation of the bulk $F_{\mu\nu\rho}^7 F_{\mu\nu\rho}^7$ term as in the HW case. The total variation can be cancelled by adding the term

$$S_{YM}^{(3)} = \frac{1}{24\sqrt{2}\lambda^2} \int d^6x \sqrt{-g} \sigma^{-2} \chi \Gamma^{KLMNP} \epsilon \partial_K F_{LMNP} .$$

(5.13)

Finally, the uncanceled variation proportional to $\epsilon FF_{\mu\rho}^i j$ is

$$\Delta^{(4)} = \frac{i}{4\sqrt{2}\lambda^2} \int d^6x \sqrt{-g} \sigma \chi \Gamma^{\mu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \chi .$$

(5.14)

which can be eliminated by adding the extra part

$$S_{YM}^{(4)} = -\frac{i}{2\sqrt{2}\lambda^2} \int d^6x \sqrt{-g} \epsilon \Gamma^{\mu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \chi .$$

(5.15)

Thus, the bulk and boundary actions (2.1) and (5.1) together with

$$S_{YM} = -\frac{1}{\chi^2} \int d^6x \sqrt{-g} \left[ \frac{1}{4} \sigma^{-2} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} \chi^a \Gamma^\mu F_{\mu\nu}^a \right]$$

$$+ \frac{1}{4} \sigma^{-1} \chi^a \Gamma^{\mu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a + \frac{1}{2\sqrt{3}} \sigma^{-1} \chi^a \Gamma^{\mu\rho\sigma} F_{\mu\rho\sigma}^a$$

$$- \frac{1}{24\sqrt{2}} \sigma^{-2} \chi^a \Gamma^{\mu\rho\sigma} F_{\mu\rho\sigma}^a + \frac{i}{2\sqrt{2}} \sigma \chi^a \Gamma^\mu F_{\mu\nu}^a F_{\nu\rho}^a,$$

(5.16)
are invariant under the supersymmetry transformations (2.2)-(2.7) and (5.3)-(5.4) up to fermionic bilinear terms.

Having obtained the locally supersymmetric action we can now obtain a relation between the boundary gauge coupling, the 7D Planck mass, and the mass parameter of the Chern-Simons term, by explicitly cancelling all the anomalies via the GS mechanism. First we must solve the modified Bianchi identity (5.7), and as in [3] we introduce

\[
\omega_{\mu\nu\rho} = \text{tr} \left( A_\mu F_{\nu\rho} - \frac{1}{3} A_\mu [A_\nu, A_\rho] + \text{cyclic perm.} \right), \tag{5.17}
\]

which satisfies

\[
\partial_\lambda \omega_{\mu\nu\rho} + \text{cyclic perm.} = 6 \text{tr} F_{[\lambda\mu} F_{\nu\rho]} . \tag{5.18}
\]

Then, (5.7) is satisfied if \( F_{7\mu\nu\rho} \) is defined as

\[
F_{7\mu\nu\rho} = 4 \partial_{[7} A_{\mu\nu\rho]} + \frac{\kappa^2}{\sqrt{2} \lambda^2} \delta(\tau) \omega_{\mu\nu\rho} . \tag{5.19}
\]

Since under infinitesimal gauge transformations, \( \delta A_\mu^a = -D_\mu \epsilon^a \), \( \omega \) transforms as

\[
\delta \omega_{\mu\nu\rho} = 3 \partial_\mu \text{tr}(\epsilon F_{\nu\rho}) , \tag{5.20}
\]
gauge invariance of \( F_{7\mu\nu\rho} \) is achieved if \( A_{7\mu\nu} \) transforms as

\[
\delta A_{7\mu\nu} = \frac{\kappa^2}{\sqrt{2} \lambda^2} \delta(\tau) \text{tr}(\epsilon F_{\mu\nu}) . \tag{5.21}
\]

Solving the the Bianchi identity in the “upstairs” approach, or specifying the boundary behaviour of \( F_4 \) in the “downstairs” version 1 we find, as in the HW case, that \( F_4 \) has a jump at \( x_7 = 0 \) given by

\[
F_{\mu\nu\kappa\lambda} = -\frac{3}{\sqrt{2} \lambda^2} \epsilon(x_7) \text{tr} F_{[\mu\nu} F_{\kappa\lambda]} , \tag{5.22}
\]

and similarly at \( x_7 = \pi R \). At the classical level, the transformation of the three-form under gauge transformations (3.21) makes the Chern-Simons term in the 7D Lagrangian (2.1) not gauge invariant, but we will see that at the quantum level an anomalous fermion contribution will cancel the non-gauge invariant term and restore gauge invariance, much like in the 11D HW theory. We will next consider the anomaly cancellation for both the gravitational and mixed anomalies. In order to do this, it is more convenient to use the downstairs approach in form notation.

1In the downstairs approach, the theory is defined on \( M^6 \times S^1 / \mathbb{Z}_2 \), whereas in the upstairs version it is defined on \( M^6 \times S^1 \) with \( \mathbb{Z}_2 \)-symmetric fields. The volume integrals in the former case is half the integrals in the latter one [3], which means that \( \kappa^2 \) is replaced by \( \kappa^2 / 2 \) in (2.1).
The Chern-Simons term in the “downstairs” approach on the seven-dimensional manifold $M^7$ is

$$S_{CS} = \frac{2}{\kappa^2} \int_{M^7} h \ F_4 \wedge A_3 ,$$  \hspace{1cm} (5.23)

where $A_3$ is the three-form gauge field of 7D supergravity and $F_4 = dA_3$. On each component of the boundary, $\partial M^7$, we will have

$$F_4|_{\partial M^7} = \frac{\kappa^2}{\sqrt{2}\lambda^2} Q_4 ,$$  \hspace{1cm} (5.24)

where the four-form $Q_4$ is defined as

$$Q_4 = \xi_{CS} \text{tr} R^2 - \text{tr} F^2 ,$$  \hspace{1cm} (5.25)

and $\xi_{CS}$ is a numerical constant. We may now define $Q_3 = \xi_{CS} \omega_{3L} - \omega_{3Y}$ where as usual $\omega_{3Y,L}$ are the Yang-Mills and Lorentz Chern-Simons terms

$$\omega_{3Y} = \text{tr} \left( AF - \frac{1}{3} A^3 \right) ,$$  \hspace{1cm} (5.26)

$$\omega_{3L} = \text{tr} \left( \omega R - \frac{1}{3} \omega^3 \right) .$$  \hspace{1cm} (5.27)

Then, we have the descent equations

$$Q_4 = dQ_3 , \hspace{1cm} \delta Q_3 = dQ_2^1 ,$$  \hspace{1cm} (5.28)

for $\delta$ gauge and Lorentz transformations which follows from

$$d\omega_{3L} = \text{tr} R^2 , \hspace{1cm} \delta \omega_{3L} = d\omega_{2L}^1 ,$$
$$d\omega_{3Y} = \text{tr} F^2 , \hspace{1cm} \delta \omega_{3Y} = d\omega_{2Y}^1 .$$  \hspace{1cm} (5.29)

In the following we will not need the explicit forms of $\omega_3, \omega_2^1$. Then, following [46, 47, 48] we have

$$A_3|_{\partial M^7} = \frac{\kappa^2}{\sqrt{2}\lambda^2} Q_3 ,$$  \hspace{1cm} (5.30)

so that

$$\delta A_3|_{\partial M^7} = \frac{\kappa^2}{\sqrt{2}\lambda^2} dQ_2^1 .$$  \hspace{1cm} (5.31)

This variation is extendable to the bulk by writing

$$\delta A_3|_{\partial M^7} = d\Lambda , \hspace{1cm} \Lambda|_{\partial M^7} = \frac{\kappa^2}{\sqrt{2}\lambda^2} Q_2^1 .$$  \hspace{1cm} (5.32)
Then, the anomalous variation of the bulk action is

$$\delta S_{CS} = -\frac{\kappa^2 h}{\lambda^4} \int_{M^6} Q_2^1 \wedge Q_4 ,$$  \hfill (5.33)

where $M^6$ is the boundary at $x_7 = 0$, and it should be compensated by the anomaly of the boundary theory. Thus, the bulk anomaly eight-form for the Chern-Simons term $S_{CS}$ is

$$I_{CS} = -\frac{\kappa^2 h}{2\pi \lambda^4} Q_4 \wedge Q_4 .$$  \hfill (5.34)

However, these are not the only sources which contribute to the anomaly. In particular, we expect a term (see Appendix B)

$$S_R = -\xi_R \int_{M^7} A_3 \wedge \text{tr} R^2 ,$$  \hfill (5.35)

in the 7D action where $\xi_R$ is a dimensionful constant. As explained in Appendix B, such a term exists in the gauged 7D $\mathcal{N} = 2$ supergravity theory resulting from the $K3$ compactification of the 11D five-brane anomaly term, and is expected to survive after gauging. The anomalous variation of the term (5.35) is

$$\delta S_R = -\frac{\kappa^2 \xi_R}{\sqrt{2}\lambda^2} \int_{M^6} Q_2^1 \wedge \text{tr} R^2 ,$$  \hfill (5.36)

so that the corresponding anomaly eight-form becomes

$$I_R = -\frac{\kappa^2 \xi_R}{2\pi \sqrt{2}\lambda^2} Q_4 \wedge \text{tr} R^2 .$$  \hfill (5.37)

In addition, as follows from Section 3.1, the appropriately normalized anomaly eight-form for the boundary theory is

$$I_{bdy} = \frac{1}{(2\pi)^4} \left[ \frac{1}{4608} \left( N_V - N_H + 7N_T \right) (\text{tr} R^2)^2 - \frac{\beta}{96} \text{tr} R^2 \text{tr} F^2 + \frac{\gamma}{24} (\text{tr} F^2)^2 \right].$$  \hfill (5.38)

The reducible part of the anomaly eight-form from the bulk fields (3.2), is evenly distributed between the two fixed points and contributes a term

$$I_{bulk} = \frac{1}{(2\pi)^4} \cdot \frac{-1}{2\cdot 5760} \cdot \frac{225}{4} (\text{tr} R^2)^2 ,$$  \hfill (5.39)

at each fixed point. Finally, there exists a contribution to the anomaly arising from the gravitational Chern-Simons term (7.11). The irreducible $\text{tr} R^4$ part in Eq. (7.14) has been cancelled against the bulk and boundary irreducible parts of the anomaly.
The remaining contribution of the gravitational Chern-Simons term to the anomaly is then
\[ I_{GCS} = \frac{1}{(2\pi)^4} \frac{n}{8 \cdot 192} (\text{tr} R^2)^2. \] (5.40)

The total anomaly eight-form coming from the bulk, the boundary theory and the Chern-Simons terms is
\[ I_{\text{total}} = I_{\text{bulk}} + I_{\text{bdy}} + I_{CS} + I_R + I_{GCS}. \] (5.41)

It is a polynomial in \((\text{tr} R^2)^2, \text{tr} R^2 \text{tr} F^2\) and \((\text{tr} F^2)^2\) and the vanishing of the total anomaly is equivalent to the vanishing of the coefficients of these terms. In particular, the vanishing of the \((\text{tr} R^2)^2\) and \(\text{tr} R^2 \text{tr} F^2\) terms gives the conditions
\[ 320\gamma \xi_{CS}^2 - 80\beta \xi_{CS} + 2(N_V - N_H + N_T) + 3 = 0, \] (5.42)
\[ 384\sqrt{2}\pi^3 \xi_{R}^2 \kappa^2 = \beta - 8\gamma \xi_{CS}, \] (5.43)
respectively, whereas the vanishing of the \((\text{tr} F^2)^2\) specifies the dimensionless ratio, \(\eta\), as
\[ \eta \equiv \frac{h\kappa^2}{\lambda^4} = \frac{\gamma}{3(4\pi)^3}. \] (5.44)

This relation fixes the gauge coupling, \(\lambda\), in terms of the gravitational coupling, \(\kappa\), and the topological mass parameter, \(h\), of the Chern-Simons term. This is similar to the relation obtained in the HW theory except for the presence of the extra parameter, \(h\). The difference is due to the fact that in the 11D HW theory the Chern-Simons term is fixed by supersymmetry, whereas in seven dimensions the theory is supersymmetric up to an arbitrary topological mass parameter, \(h\).

Note that similar conditions to (5.42), (5.43), and (5.44) exist at the second fixed point \(x^7 = \pi R\). When the boundary theories at the two fixed points are the same, the anomaly cancellation conditions at \(x^7 = \pi R\) are identical to that at \(x^7 = 0\). However, if the boundary theory at \(x^7 = \pi R\) is different from the boundary theory at \(x^7 = 0\), then the anomaly cancellation conditions must be solved for another set of \(\xi_{CS}, \lambda, \beta,\) and \(\gamma\). In this case a different value of \(\xi_R\) is also needed at the second fixed point.

This requires more general compactifications of the 11D theory.

The effective gauge coupling on the boundaries, \(\lambda_{\text{eff}}^2 \equiv \lambda^2 \langle \sigma^2 \rangle\) can be written in terms of the geometric parameters as
\[ \lambda_{\text{eff}}^2 = 2\kappa^2 \sqrt{\frac{6\pi^3 \Lambda_{(0)}}{5\gamma}}. \] (5.45)

We see then that \(\gamma\) must be necessarily positive in order to have no ghosts. In this case, decoupling gravity leads to an anomaly free theory as pointed out in [33].
5.1.2 Boundary hypermultiplets

In the previous section we only considered boundary vector multiplets and their bulk couplings. We will now introduce hypermultiplets on the boundary and determine their supersymmetric interactions with bulk fields. Let us begin with the globally supersymmetric action for the hypermultiplet \((\phi^\alpha, \zeta^Y)\)

\[
S_H^{(0)} = \int d^6x \sqrt{-g} \left( -\frac{1}{2} g_{\alpha\beta}(\phi) \partial_\mu \phi^\alpha \partial^\mu \phi^\beta - \frac{1}{2} \bar{\zeta}^Y \Gamma^\mu D_\mu \zeta_Y \right). \tag{5.46}
\]

Here, \(\phi^\alpha (\alpha, \ldots = 1, \ldots, 4n_H)\), \(\zeta^Y (Y, \ldots = 1, \ldots, 2n_H)\) are, respectively, the scalar and fermion components of the \(n_H\) hypermultiplets on the boundary, and \(g_{\alpha\beta}\) is the metric of the scalar manifold. The action (5.46) is the standard \(\mathcal{N} = (0, 1)\) globally supersymmetric action where the \(4n_H\) scalars parametrize a space with \(Sp(n_H)\) holonomy group, i.e. a hyperkähler manifold. The covariant derivative in (5.46) is defined as

\[
D_\mu \zeta^Y = \partial_\mu \zeta^Y + \Gamma^Y_{\alphaX} \partial_\mu \phi^\alpha \zeta^X, \tag{5.47}
\]

where \(\Gamma^Y_{\alphaX}\) is the \(Sp(n_H)\) connection. We will demand invariance of the action (5.46) under the supersymmetry transformations

\[
\delta \phi^\alpha = \frac{1}{2} f_{\phi} V^\alpha_{iY} \epsilon^i \zeta^Y, \tag{5.48}
\]

\[
\delta \zeta^Y = \frac{1}{2} f_{\zeta} V^Y_{\alphai} \Gamma^\mu \partial_\mu \phi^\alpha \epsilon^i. \tag{5.49}
\]

where \(f_{\phi}, f_{\zeta}\) are also functions of the bulk fields, and \(V^Y_{\alphai}\) is the vielbein of the scalar manifold. The case \(f_{\phi} = f_{\zeta} = 1\) corresponds to the \(\mathcal{N} = (0, 1)\) globally supersymmetric theory. By simple inspection of the transformations (5.48)-(5.49), we find that \(f_{\phi} = f_{\zeta}^{-1}\) in order that two consecutive supersymmetry transformations on \(\phi^\alpha\) produce a correctly normalized translation. Checking the supersymmetry of the action (5.46), we find that \(V^Y_{\alphai}\) is covariantly constant so that the vielbein satisfies the relations

\[
g_{\alpha\beta} V^\alpha_{iY} V^\beta_{jZ} = \epsilon_{ij} \epsilon_{YZ}, \tag{5.50}
\]

\[
V^\alpha_{iY} V^\beta_{jY} + V^\beta_{iY} V^\alpha_{jY} = g_{\alpha\beta} \delta^i_j, \tag{5.51}
\]

\[
V^\alpha_{iY} V^\beta_{iZ} + V^\beta_{iY} V^\alpha_{iZ} = \frac{1}{n_H} g_{\alpha\beta} \delta^Y_Z, \tag{5.52}
\]

where \(\epsilon_{ij}\) and \(\epsilon_{YZ}\) are the \(Sp(n_1)\) and \(Sp(n_H)\) invariant antisymmetric tensors, respectively. We can now define a triplet of complex structures \(J_{\alpha\beta i}^j\) as

\[
J_{\alpha\beta i}^j = V^\alpha_{iY} V^\beta_{jY} - V^\beta_{iY} V^\alpha_{jY}, \tag{5.53}
\]

which obey the \(SU(2)\) algebra, and are covariantly constant with respect to the \(Sp(1)\) connection \(\omega_{ij}^k = \omega_{\alpha i}^j d\phi^\alpha\), namely

\[
\nabla J^i_j = dJ^i_j + \omega_{kj}^i J_k^j - J^k_i \omega^k_j = 0. \tag{5.54}
\]

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The $Sp(1)$ curvature two-form $\Omega^i_j = \frac{1}{2} \Omega_{\alpha\beta}^i d\varphi^\alpha \wedge d\varphi^\beta$ is
\[ \Omega^i_j = d\omega^i_j + \omega^k_i \wedge \omega^j_k, \tag{5.55} \]
and the manifold is then quaternionic if
\[ \Omega^i_j = \mu J^i_j, \tag{5.56} \]
for some constant $\mu \neq 0$, whereas it is hyperkähler if $\mu = 0$, i.e. a hyperkähler manifold has vanishing $Sp(1)$ curvature. So far we have seen that the holonomy group of the scalar manifold should be contained in $Sp(n_H) \times SU(2)$ as follows from the covariantly constant vielbein $V_{\alpha i} Y$. However, after imposing the latter condition, there still exists an uncancelled term in the variation of (5.46) under the transformations (5.48)-(5.49). This term is explicitly written as
\[ \Delta^{(0)}_H = \frac{1}{2} \int d^6 x \sqrt{-g} \epsilon^I \Gamma^\mu \Gamma^\nu \partial^\alpha \varphi^\alpha V_{\alpha i} Y \zeta^I \partial^I, \tag{5.57} \]
and we will return to this term shortly.

As usual for local supersymmetry, we should add to the action (5.46) the standard $\bar{\psi}_\mu S_H$ interaction where $S_H$ is the supercurrent of the hypermultiplet. In particular, the term which has to be added is
\[ S^{(1)}_H = \frac{1}{2} \int d^6 x \sqrt{-g} f_{\zeta} \bar{\psi}_\mu^i \Gamma^\nu \Gamma^\mu \partial^\nu \varphi^\alpha V_{\alpha i} Y \zeta^I Y. \tag{5.58} \]
However, due to the variation of $\zeta^I$ in $S^{(1)}_H$, we get an uncancelled term which is
\[ \Delta^{(1)}_H = -\frac{1}{8} \int d^6 x \sqrt{-g} f_{\zeta}^2 \bar{\psi}_\mu^i \Gamma^{\mu\nu\rho} \partial^\nu \varphi^\alpha \partial^\rho \varphi^\beta \left( V_{\alpha i} Y V_{\beta j} Y - V_{\alpha j} Y V_{\beta i} Y \right) \epsilon^I. \tag{5.59} \]
The only way to cancel $\Delta^{(1)}_H$ is to find an opposite contribution from the bulk. As in the vector case, there exists a bulk term $\bar{\psi}_K^j \Gamma^{KMN} \bar{\psi}_M^i F_{MN}^j$, which can be shown to be supersymmetric by invoking a Bianchi identity. Using this term we may cancel $\Delta^{(1)}_H$ in the downstairs approach by modifying the Bianchi identity to be
\[ DF^i_{\mu\nu} = \frac{iK^2}{2\sqrt{2}} \delta(x_7) \partial^\alpha \varphi^\alpha \partial^\beta \varphi^\beta J_{\alpha\beta}^i, \tag{5.60} \]
with $f_{\zeta} = \sigma^{1/2}$. The latter is also needed for cancelling (5.57) as we will see below. There are also other terms that need to be cancelled in $S^{(1)}_H$, which arise from the variation of $\bar{\psi}_\mu$, and are of the form $F_{\mu\nu\rho \sigma} \partial^\sigma \varphi$ and $F_{\mu i}^j \partial^I \varphi$. The former can be cancelled by adding
\[ S^{(2)}_H = \frac{1}{2\sqrt{5}} \int d^6 x \sqrt{-g} \sigma^{1/2} V_{\alpha i} Y \zeta^I \Gamma^\mu \partial^\mu \varphi^\alpha \chi^i, \tag{5.61} \]
\[ S^{(3)}_H = \frac{1}{24\sqrt{2}} \int d^6 x \sqrt{-g} \sigma^{1/2} \zeta^Y \Gamma^{\mu\nu\rho} \zeta_Y F_{\mu\nu\rho} . \tag{5.62} \]
In addition, the variation of \( S^{(2)} \) with \( \delta \chi \sim \partial \varphi \epsilon \) exactly cancels (5.57) for \( f_\zeta = \sigma^{1/2} \) as promised.

Finally there remains the cancellation of terms of the form \( \zeta F_{\mu_7}^i \) which arise from variations of \( \psi_\mu \) and \( \chi \) proportional to \( F_{\mu_7}^i \) in \( S^{(1)}_H \) and \( S^{(2)}_H \), respectively. The uncanceled terms are of the form \( V_{\alpha i} \epsilon^j \Gamma^\mu \zeta Y \partial_\mu \varphi^\alpha F_{\tau j}^i \) and \( V_{\alpha i} \epsilon^j \zeta Y \partial_\mu \varphi^\alpha F_{\mu_7}^i \). The former cancels exactly while the latter does not and so we have to look for a possible bulk contribution. We need to implement a correction to the supersymmetry variation of \( F_{\mu_7}^i \) which then produces a contribution from the kinetic term of the two-form, just as there was in the vector multiplet case from the kinetic term of the four-form.

The correction to the supersymmetry variation of \( F_{\mu_7}^i \) is

\[
\tilde{\delta} F_{\mu_7}^i = \frac{k^2}{5\sqrt{2}} \sigma^{-1/2} \partial_\mu \varphi^\alpha \left( V_{\alpha i} \epsilon^j - \frac{1}{2} \delta^j_i \right) V_{\alpha k} \epsilon^k \zeta Y . \hspace{1cm} (5.63)
\]

Summarizing, the supersymmetric boundary action for neutral hypermultiplets is

\[
S_H = \int d^6x \sqrt{-g} \left[ -\frac{1}{2} \partial_\alpha \varphi^\beta \partial^\beta \varphi^\alpha - \frac{1}{2} \zeta Y \Gamma^\mu \partial_\mu \varphi^\alpha \right] + \frac{1}{2} \sigma^{1/2} \delta^i_j \Gamma^\nu \partial_\nu \varphi^\alpha V_{\alpha i} \epsilon^j \zeta Y + \frac{1}{2} \sigma^{1/2} V_{\alpha i} \zeta Y \Gamma^\mu \partial_\mu \varphi^\alpha \chi^i + \frac{1}{24\sqrt{2}} \zeta Y \Gamma^\mu \nu \zeta Y F_{\tau \mu \rho} \right] , \hspace{1cm} (5.64)
\]

and it is invariant under the supersymmetry transformations

\[
\delta \varphi^\alpha = \frac{1}{2} \sigma^{-1/2} V_{\alpha i} \epsilon^j \zeta Y , \hspace{1cm} (5.65)
\]
\[
\delta \zeta Y = \frac{1}{2} \sigma^{1/2} V_{\alpha i} \epsilon^j \Gamma^\mu \partial_\mu \varphi^\alpha \epsilon^i . \hspace{1cm} (5.66)
\]

Let us now return to the scalar manifold where the only constraint we have so far is that the holonomy group is in \( Sp(n_H) \times Sp(1) \). In 6D (as well as \( \mathcal{N} = 2 \) in 4D), \( \mathcal{N} = (0, 1) \) supersymmetry requires that the scalar manifold be hyperkähler in rigid supersymmetry and quaternionic for local supersymmetry. It is interesting to see how the quaternionic structure arises. In the local case, the supersymmetry parameters are charged (with minimal coupling) under \( SU(2) \), and after a supersymmetry transformation a term proportional to the \( SU(2) \) curvature is generated from the gravitino kinetic term leading to a quaternionic scalar manifold. In our case, the gravitino is not minimally coupled to the \( SU(2) \) connection of the boundary scalar manifold, and it is not clear how to obtain a quaternionic structure as required by \( \mathcal{N} = (0, 1) \) 6D supersymmetry.

Although the 7D gravitino kinetic term does not contribute to the \( Sp(1) \) curvature of the scalar manifold, there is a contribution from the Bianchi identity (5.60). Indeed,
either from the Bianchi identity or in the downstairs approach, we find that the boundary value of the $SU(2)$ bulk gauge field is

$$F_{\mu\nu}^i = \frac{i}{4\sqrt{2}} \kappa^2 \partial_\mu \varphi^\alpha \partial_\nu \varphi^\beta J_{\alpha\beta}^i .$$  \hspace{1cm} (5.67)$$

It is more convenient to use form notation so that the boundary value of the $SU(2)$ field strength can be rewritten in the form

$$F_i^j |_{\partial M^7} = \frac{i}{4\sqrt{2}} \kappa^2 (\varphi^* J)_i^j ,$$  \hspace{1cm} (5.68)$$

where, as usual, $(\varphi^* J)_i^j$ is the pullback of $J$ on $M^7$ defined as

$$(\varphi^* J)_i^j = \frac{1}{2} J_{\alpha\beta}^i \partial_\mu \varphi^\alpha \partial_\nu \varphi^\beta \, dx^\mu \wedge dx^\nu ,$$  \hspace{1cm} (5.69)$$

and similarly for higher-order forms. Since we have that $D F_i^j = \nabla_J_i^j = 0$, we obtain

$$A_i^j |_{\partial M^7} = - \frac{i}{g} (\varphi^* \omega)_i^j ,$$  \hspace{1cm} (5.70)$$

where $(\varphi^* \omega)_i^j$ is the pullback of the $Sp(1)$ connection in the scalar manifold. As a result, the boundary value $F_i^j = d A_i^j + ig A_i^k \wedge A_k^j$ is proportional to the pullback of the $Sp(1)$ curvature two-form of the scalar manifold

$$F_i^j |_{\partial M^7} = - \frac{i}{g} (\varphi^* \Omega)_i^j .$$  \hspace{1cm} (5.71)$$

Then, using Eq.(5.68) we obtain

$$\Omega_i^j = - \frac{g \kappa^2}{4\sqrt{2}} J_i^j ,$$  \hspace{1cm} (5.72)$$

so that, the scalar manifold is indeed quaternionic as required by $\mathcal{N} = (0, 1)$ 6D local supersymmetry.

### 6 Phases of the Boundary Theory

We have seen that the supersymmetric vacua of the $\mathcal{N} = 2$ 7D theory are pure $AdS_7$, and the two-brane Randall-Sundrum configuration (RS1). There also exists the one-brane Randall-Sundrum vacuum (RS2), where a single six-brane is sitting at $x_7 = 0$. It is obtained by omitting the last $\epsilon$ and $\delta$ functions in eqs.(2.16) and (2.17), respectively. The metric is still given by $\epsilon\delta$, except that now $0 \leq y < \infty$, and the singularity at $y = 0$ is resolved by placing a positive-tension brane there. In this case,
only the graviton is localized on the brane so that there exists an $\mathcal{N} = (0, 1)$ 6D theory on the brane. Again, the 6D theory is anomalous (since Eq. (3.22) is violated) and we should again include matter fields on the brane to cancel the anomaly by a local GS mechanism at the single point $y = 0$ as we did before. The only difference with the two-brane RS1 vacuum is that here we have only local cancellation of the anomaly since global and local cancellations coincide (only one brane). Thus, the irreducible part of the anomaly cancels if Eq. (3.22) is satisfied and then the rest of the anomaly should be cancelled by a 6D GS mechanism. As a result, any anomaly free 6D theory can be put on the brane leading to a consistent $\mathbb{R}/\mathbb{Z}_2$ "compactification" of the 7D $\mathcal{N} = 2$ theory. There are many 6D theories which can exist on the boundary. In the case of one tensor multiplet, some of them can be obtained from heterotic string compactifications like $E_8 \times E_7$, and others like $SU(n) \times SU(n)$ with matter in the representation $(\mathbf{n}, \mathbf{\bar{n}}) + (\mathbf{\bar{n}}, \mathbf{n}) + 242(1, 1)$ described in [31] do not arise from any known compactifications of string theory. In the case of more than one tensor multiplets, a generalized GS mechanism may operate as in [32]. As a result, any anomaly free $\mathcal{N} = (0, 1)$ 6D theory can be lifted to the boundary of a 7D $\mathcal{N} = 2$ one-brane vacuum. The boundary 6D theory can be consistently coupled to the 7D bulk in a similar fashion as considered in the previous section.

In the two-brane scenario, it is also interesting to consider the two possible limits $R \to 0$, $R \to \infty$ in the $n_T = 1$ case. These limits correspond to coincident branes and a single brane, respectively. As we have discussed earlier, the anomaly must be factored both globally and locally. In particular, we have seen that, due to Eq. (3.18) the local anomaly (3.11) can be written in the form

$$I_8^{(i)} = (c_i \text{tr} R^2 + |a_i| \text{tr} F_i^2) \left( \text{tr} R^2 - |b_i| \text{tr} F_i^2 \right),$$

(6.1)

where $|a_i| |b_i| = \frac{2}{3} \gamma_i > 0$. Then it is possible that a phase transition on the boundary may occur. This is because the factorization (6.1) is correlated with the gauge kinetic terms on the boundary via [32]

$$- \left( |a_i| \sigma^{-2} - |b_i| \sigma^2 \right) \text{tr} F_i^2,$$

(6.2)

and the gauge coupling on the boundaries always becomes infinite at the value

$$\sigma^4 = \frac{|a_i|}{|b_i|}.$$

(6.3)

As the radius of the $S^1/\mathbb{Z}_2$ compactification is $R \sim \sigma^{-2}$, there is always a value of $R$ where the gauge coupling blows up and a phase transition takes place. Thus, one of the limits $R \to 0$ or $R \to \infty$ drives the gauge theory to a phase transition where the number of massless degrees of freedom rearrange themselves such that the 6D theory remains anomaly free. One of the possibilities for this singularity is that it could signal the appearance of tensionless strings [49, 50]. This behaviour is novel and it does not occur in the HW scenario.
Although the coincident limit of the two branes is unique, the limit \( R \to \infty \) can be taken in two ways. We can either send the UV or the IR brane to infinity. When the IR brane is sent to infinity the localized theory on the UV-brane is simply the single brane vacuum considered above, and the 6D boundary theory is anomaly free.

On the other hand when we send the UV brane to infinity there is no longer any localized graviton on the IR brane. This is due to the fact that the bulk theory is \( AdS_7 \). Thus there are no gravitational anomalies in the 6D boundary theory. However there are still vector and tensor multiplets on the boundary which give rise to gauge anomalies. Assuming that the hypermultiplet and tensor also decouple (since they arise from the 7D gravity multiplet), we are left with \( n_V \) vector multiplets, one tensor multiplet and one neutral hypermultiplet on the IR brane where the latter are localized in any case there. The theory is then anomaly free as long as \( \gamma_2 > 0 \) as discussed in [33].

There is also a dual description of our anomaly free 7D brane worlds. This is similar to the HW model where the dual description of the 11D supergravity is the ten-dimensional strongly coupled \( E_8 \times E_8 \) heterotic string theory. Since the vacua of the 7D brane worlds is \( AdS_7 \), we simply rely on AdS/CFT correspondence \([16, 21, 22, 23]\) to identify the dual description as a 6D strongly coupled CFT. These field theories are very interesting since by the AdS/CFT dictionary the fields on the UV brane correspond to fundamental fields added to the CFT, while fields on the IR brane are bound states of the CFT. For example, since the gauge group is in general a product group \( G_1 \times G_2 \), this means that the \( G_1 \) gauge fields are fundamental, while the \( G_2 \) gauge fields are composite in the dual description. In addition since gravity is localized on the UV brane, we must add a 4D gravity multiplet to the dual field theory. Unlike the HW case, where the dual theory is identified as the 10D heterotic string theory, not much is known about the dual 6D theories of our 7D brane worlds. These await further investigation.

7 Conclusions

We have constructed anomaly-free seven-dimensional brane worlds based on the minimal \( \mathcal{N} = 2 \) 7D gauged supergravity. Under an \( S^1/\mathbb{Z}_2 \) compactification, we showed that the resulting spectrum is the chiral \( \mathcal{N} = (0, 1) \) 6D supergravity. In order to maintain supersymmetry, we introduced two six-branes of opposite tension at the orbifold fixed points. Since there exists a negative bulk cosmological constant, this leads to a 7D vacuum solution which is a slice of \( AdS_7 \). However, unlike the celebrated 5D Randall-Sundrum solution, where there are no constraints on the boundary theory, in our case, anomaly cancellation puts stringent restrictions on the possible boundary matter content. This is because there are gravitational anomalies in six dimensions. This is analogous to the HW model, where the cancellation of 10D gravitational anomalies uniquely specifies the boundary theory to be \( E_8 \times E_8 \) with an \( E_8 \) factor.
localized on each boundary. In contrast, while there are many anomaly-free 6D theories, the boundary matter content of our 7D brane world is nevertheless constrained. In particular, for the case of one tensor multiplet and gauge groups containing the standard model, only exceptional groups are allowed on the boundaries with a further restriction on the number of fundamental generations. If one allows $n_T > 1$ then using the Sagnotti mechanism many more possibilities exist for the boundary theory.

We also explicitly constructed the locally supersymmetric bulk-boundary Lagrangian, up to fermionic bilinear terms, for a boundary vector multiplet and hypermultiplet. The case of the vector multiplet relies on the modification of the Bianchi identity of the four-form field strength. This is very similar to what happens in the HW model. Moreover, anomaly cancellation fixes a dimensionless ratio $\eta$, formed from the boundary gauge coupling, the 7D gravitational coupling, and the topological mass parameter of the Chern-Simons term. Again this is analogous to a similar relation in the HW theory, except that now there is a dependence on the extra parameter from the Chern-Simons term. However, the case of boundary hypermultiplets has no counterpart in the 11D HW model. For these boundary fields we had to modify the Bianchi identity of the bulk gauge field resulting from the gauged R-symmetry. This modification was also crucial in showing that the scalar manifold of the boundary hypermultiplets becomes quaternionic, as expected in the locally supersymmetric limit.

The brane worlds constructed in this paper are a first step in obtaining boundary theories which have the standard model matter content. For example, one can envisage compactifying the six-branes in our anomaly-free models on the sphere $S^2$ with a monopole background [51] to obtain an effective four-dimensional chiral theory. This would be the counterpart of the Calabi-Yau compactifications of M-theory, except that in our case the bulk space is AdS, and not Minkowski. In addition there is also the intriguing question of understanding the dual formulation of our models. By the AdS/CFT correspondence our 7D brane worlds are dual to a class of 6D strongly coupled conformal field theories. These 6D theories must necessarily include fundamental fields associated with the localized fields on the UV brane, such as the 4D gravity multiplet, and any gauge and matter fields, while for the fields localized on the IR brane, they will appear as bound states of the CFT. Our 7D AdS brane worlds suggest a way to study these mysterious hybrid 6D theories further.

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Appendix A

The $8 \times 8$ 6D gamma matrices satisfy the Clifford algebra
\[
\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}, \quad \alpha, \beta = 0, 1, \ldots, 5
\] (7.1)
with $\eta_{\alpha\beta} = \text{diag}(-, +, \ldots, +)$. The matrix $\gamma^7$ is defined as
\[
\gamma^7 = \gamma^0 \gamma^1 \ldots \gamma^5, \quad (\gamma^7)^2 = 1
\] (7.2)
and the $8 \times 8$ matrices $\Gamma^A = (\gamma^\alpha, \gamma^7)$ satisfy the 7D Clifford algebra
\[
\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}, \quad A, B = 0, 1, \ldots 6
\] (7.3)
All 7D spinors are symplectic-Majorana
\[
\chi^i = \epsilon^{ij} \chi^T_j, \quad \bar{\chi}_i = \chi^i \Gamma_0
\] (7.4)
and $SU(2)$ indices $i, j = 1, 2$ are raised and lowered as
\[
\chi^i = \epsilon^{ij} \chi_j, \quad \chi_i = \chi^j \epsilon_{ji}, \quad \epsilon^{ij} = \epsilon_{ij} = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)
\] (7.5)
A 7D spinor $\chi^i$ decomposes into $\chi^i = \chi^i_+ + \chi^i_-$, where $\chi^i_\pm$ are 6D symplectic Majorana-Weyl spinors satisfying $\gamma_7 \chi^i_\pm = \pm \chi^i_\pm$. Contraction with the $SU(2)$-invariant antisymmetric tensor is always understood in spinor inner-products, e.g. $\bar{\chi} \Gamma^{ABC} \psi = \bar{\chi} \Gamma^{ABC} \psi$. As a result we have
\[
\bar{\chi} \Gamma^{A_1 \ldots A_n} \psi = (-1)^n \bar{\psi} \Gamma^{A_n \ldots A_1} \chi.
\] (7.6)
The same conventions also hold for 6D spinors. Finally, the Riemann tensor is $R^{AB}_{\ MN} = \partial_M \omega_N^{\ AB} + \omega_M^{\ AC} \omega_N^{\ CB} - \omega^{\ AC} \omega_N^{\ B} - M \leftrightarrow N$, and the Ricci scalar $R = R^{AB}_{\ MN} e^M_A e^N_B$. 

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Appendix B

Let us now see how the dimensional reduction of the interaction term $F_4 \wedge X_7$ of 11D supergravity gives rise to a gravitational Chern-Simons term, $S_{GCS}$ and the term $S_R$ in (5.35). In 11D supergravity the correct normalization of the $F_4 \wedge X_7$ term is

$$S_{GCS} = \frac{1}{2} \left( \frac{4\pi^2}{3\kappa_{11}^2} \right)^{1/3} \int F_4 \wedge X_7 ,$$  \hspace{1cm} (7.7)

where $\kappa_{11}$ is the 11D Newton’s constant. Upon compactification on $K3$, the above interaction gives rise to two terms in the 7D effective supergravity action

$$S_{GCS} = \frac{1}{2} \left( \frac{4\pi^2}{3\kappa_{11}^2} \right)^{1/3} \left( \int_{K3} F_4 \right) \int X_7 \equiv \xi_G \int X_7 ,$$  \hspace{1cm} (7.8)

and

$$S_R = \frac{1}{2} \left( \frac{4\pi^2}{3\kappa_{11}^2} \right)^{1/3} \int F_4 \wedge \int_{K3} X_7 .$$  \hspace{1cm} (7.9)

According to [52, 53], the $F_4$-fluxes are quantized as

$$\int_{C_4} F_4 = (6\pi)^{1/3} \kappa_{11}^{2/3} n ,$$  \hspace{1cm} (7.10)

where $n$ is an integer or half-integer. If $C_4$ is the $K3$ surface, we obtain for $S_{GCS}$ the result

$$S_{GCS} = n\pi \int X_7 ,$$  \hspace{1cm} (7.11)

so that $\xi_G = n\pi$ and $X_7$ is a 7D Chern-Simons term satisfying

$$X_8 = dX_7 = \frac{1}{(2\pi)^4 192} \left[ \frac{1}{4} (\text{tr} R^2)^2 - \text{tr} R^4 \right] .$$  \hspace{1cm} (7.12)

Under diffeomorphisms, $X_7$ transforms as $X_7 \rightarrow X_7 + dX_6^{(0)}$. Thus, on a manifold with a boundary, the anomalous variation of $S_{GCS}$ is

$$\delta S_{GCS} = \xi_G \int X_6^{(0)} ,$$  \hspace{1cm} (7.13)

and the corresponding anomaly eight form is then

$$I_{GCS} = \frac{\xi_G}{2\pi} X_8 [\delta(y) - \delta(y - \pi R)] ,$$  \hspace{1cm} (7.14)
and similarly at $y = \pi R$.

Performing the integral in (7.9), it is not difficult to see that

$$X_3 = \int_{K_3} X_7 = \frac{1}{4} \cdot (2\pi)^2 \omega_{3L},$$

so that the term $S_R$ can be written as

$$S_R = -\frac{1}{8 \cdot (2\pi)^2} \left( \frac{4\pi^2}{3\kappa_{11}^2} \right)^{1/3} \int A_3 \wedge \text{tr} R^2.$$

Thus the coefficient $\xi_R$ in (5.35) is determined to be

$$\xi_R = \left( \frac{\pi^2}{48\kappa_{11}^2} \right)^{1/3} = \left( \frac{\pi^2}{48\kappa^2} \right)^{1/3} V_{K_3}^{-1/3},$$

where $V_{K_3}$ is the volume of $K_3$ and the relation $\kappa_{11}^2 = V_{K_3}\kappa^2$ between the 11D and 7D Newton constants $\kappa_{11}$ and $\kappa$, respectively, has been used.
### Appendix C

We tabulate here all the solutions which satisfy the anomaly constraint conditions for $n_T = 1$, and matter content (4.2).

| $G_1 \times G_2$ | $(n_1, n_2, n_S)$ |
|------------------|-------------------|
| $E_8 \times E_7$ | $(0,10,64), (1,5,96)$ |
| $E_8 \times E_6$ | $(0,18,83), (1,8,105)$ |
| $E_8 \times F_4$ | $(0,17,101), (1,7,113)$ |
| $E_8 \times G_2$ | $(0,11,428), (0,46,183), (1,16,145), (2,1,2)$ |
| $E_7 \times E_7$ | $(n_1, 8 - n_1, 61)$ |
| $E_7 \times E_6$ | $(n_1, 14 - 2n_1, 76 - 2n_1), (2,7,153)$ |
| $E_7 \times F_4$ | $(n_1, 13 - 2n_1, 90 - 4n_1), (2,6,160)$ |
| $E_7 \times G_2$ | $(n_1, 34 - 6n_1, 152 - 14n_1), (1,12,250),(2,13,187), (6,7,5)$ |
| $E_6 \times E_6$ | $(n_1, 12 - n_1, 75), (2,7,156)$ |
| $E_6 \times F_4$ | $(n_1, 11 - n_1, 87 - n_1), (2,6,163),(5,9,4), (7,1,158)$ |
| $E_6 \times G_2$ | $(n_1, 28 - 3n_1, 139 - 6n_1), (0,12,251), (2,13,190), (3,14,156), (5,22,46), (9,6,50), (10,7,16)$ |
| $F_4 \times F_4$ | $(n_1, 10 - n_1, 87), (1,6,165),(4,9,9)$ |
| $F_4 \times G_2$ | $(n_1, 25 - 3n_1, 134 - 5n_1), (1,13,192), (2,14,159), (4,22,51), (8,6,59), (9,7,26)$ |
| $G_2 \times G_2$ | $(n_1, 20 - n_1, 131), (1,14,166), (6,19,96), (7,22,68), (8,28,19)$ |
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