A Lower Bound on the Quantum Capacity of Channels with Correlated Errors

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The highest fidelity of quantum error-correcting codes of length $n$ and rate $R$ is proven to be lower bounded by $1 - \exp[-nE(R) + o(n)]$ for some function $E(R)$ on noisy quantum channels that are subject to not necessarily independent errors. The $E(R)$ is positive below some threshold $R_0$, which implies $R_0$ is a lower bound on the quantum capacity. This work is an extension of the author’s previous works [M. Hamada, Phys. Rev. A, 65, 052305 (2002), e-Print quant-ph/0109114, LANL, 2001, and M. Hamada, e-Print quant-ph/0112103, LANL, 2001], which presented the bound for channels subject to independent errors, or channels modeled as tensor products of copies of a completely positive linear map. The relation of the channel class treated in this paper to those in the previous works are similar to that of Markov chains to sequences of independent identically distributed random variables.

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I. INTRODUCTION

Quantum error-correcting codes (simply called quantum codes or codes in this work) were discovered by Shor\textsuperscript{1} and Steane\textsuperscript{2} as schemes that protect quantum states from decoherence during quantum computation. Shor\textsuperscript{1} not only gave the first quantum code but also posed a problem of determining the quantum analog of Shannon’s channel capacity. In classical information theory, channels with independent errors are called memoryless channels and channels with correlated errors are called channels with memory\textsuperscript{3}, which will be applied to quantum channels as well in the present work. On quantum memoryless channels, several bounds on the quantum capacity have been known\textsuperscript{1,4,5,6,7} and also exponential convergence of fidelity of codes was recently proved by the present author\textsuperscript{8,9}. It is natural to ask whether such bounds and exponential convergence hold true or not on channels with memory, which will be answered affirmatively in this work.

While one of the greatest incentives to investigate quantum codes is need in quantum computing, we are not sure which devices to use for this purpose currently. Hence, we do not know which channel models are appropriate, so that treating general channels may be among what we can proceed to now. Thus, this paper analyzes the code performance on a class of quantum channels that is much wider than was treated in the literature.

In the proof of the main result below, the method of types, which is a powerful tool from classical information theory, plays an important role\textsuperscript{10,11}. This method was exploited by the Hungarian mathematician (information theorist) Csiszár and coworkers around 1980 to present the strongest coding theorems such as the one showing the existence of universal channel codes asymptotically as good as any codes\textsuperscript{10,11}. It has often produced results in elementary enumerative manners, which is also the case in this paper.

II. MAIN RESULT FOR SIMPLE CASE

As usual, all quantum channels and decoding (state-recovery) operations in coding systems are described in terms of trace-preserving completely positive (TPCP) linear maps\textsuperscript{4,12,13,14}. Given a Hilbert space $H$ of finite dimension, let $L(H)$ denote the set of linear operators on $H$. In general, every CP linear map $\mathcal{M} : L(H) \to L(H)$ has an operator-sum representation $\mathcal{M}(\rho) = \sum_{i \in \mathcal{I}} M_i \rho M_i^\dagger$ for some $M_i \in L(H)$, $i \in \mathcal{I}$\textsuperscript{4,12,13,14}. When $\mathcal{M}$ is specified by a set of operators $\{M_i\}_{i \in \mathcal{I}}$ in this way, we write $\mathcal{M} \sim \{M_i\}_{i \in \mathcal{I}}$.

Hereafter, $H$ denotes an arbitrarily fixed Hilbert space of dimension $d$, which is a prime number. A quantum channel is a sequence of TPCP linear maps $\{A_n : L(H^\otimes n) \to L(H^\otimes n)\}$. We want a large subspace $C_n \subseteq H^\otimes n$ every state vector in which remains almost unchanged after the effect of a channel followed by some suitable recovery operation $R_n : L(H^\otimes n) \to L(H^\otimes n)$. A pair $(C_n, R_n)$ consisting of such a subspace $C_n$ and a TPCP map $R_n$ is called a code and its performance is evaluated in terms of minimum fidelity\textsuperscript{6,7,15}

$$F(C_n, R_n A_n) = \min_{|\psi\rangle \in C_n} \langle \psi | R_n A_n (|\psi\rangle \langle \psi|) |\psi\rangle,$$

where $R_n A_n$ denotes the composition of $A_n$ and $R_n$. Throughout, bras $\langle \cdot |$ and kets $|\cdot\rangle$ are assumed normalized. A subspace $C_n$ alone is also called a code assuming implicitly some recovery operator.

Let $F^*_{n,k}(A_n)$ denote the supremum of $F(C_n, R_n A_n)$ such that there exists a code $(C_n, R_n)$ with $\log_d \dim C_n \geq k$, where $n$ is a positive integer and $k$ is a nonnegative real number. Our
goal is to estimate \( F_{n,k}^*(\mathcal{A}_n) \) as precisely as possible.

First, we state the main result for an easy case, and give a more general statement later. Fix an orthonormal basis \( \{|0\}, \ldots, |d-1\rangle \) of \( \mathcal{H} \). Put \( \mathcal{X} = \{0, \ldots, d-1\}^2 \) and \( N_{(i,j)} = X^i Z^j \) for \((i,j) \in \mathcal{X}\). Here, \( X, Z \in \mathcal{L}(\mathcal{H}) \) are Weyl’s unitaries, which could be viewed as generalized Pauli operators, and are defined by

\[
X|j\rangle = |(j - 1) \mod d\rangle, \quad Z|j\rangle = \omega^j|j\rangle,
\]

where \( \omega \) is a primitive \( d \)-th root of unity. From the \( \mathcal{L}(\mathcal{H}) \) basis \( \{N_{(i,j)}\} \), we obtain a basis \( \mathbf{N}_n = \{N_x \mid x \in \mathcal{X}^n\} \) of \( \mathcal{L}(\mathcal{H}^{\otimes n}) \), where \( N_x = N_{x_1} \otimes \ldots \otimes N_{x_n} \) for \( x = (x_1, \ldots, x_n) \in \mathcal{X}^n \). The first channel class to be considered here consists of those \( \{A_n\} \) such that \( \mathcal{A}_n \sim \{\sqrt{P_n(x)}N_x\}_{x \in \mathcal{X}^n} \), where we assume that \( P_n \) are the probability distributions of a (first-order) homogeneous Markov chain, i.e., that \( P_n \) has the form

\[
P_n(x_1, \ldots, x_n) = p(x_1) \prod_{j=1}^{n-1} P(x_{j+1}|x_j)
\]

with some transition probabilities \( P(v|u), u, v \in \mathcal{X} \), and some initial distribution \( p \). These are generalizations of the so-called depolarizing channel; see Ruskai et al.\cite{Ruskai1} for a thorough analysis of memoryless channels with \( d = 2 \).

Given a probability distribution \( Q \) on \( \mathcal{X}^2 \), we let \( \overline{Q} \) and \( \overline{Q} \) denote the two marginal distributions:

\[
\overline{Q}(u) = \sum_{v \in \mathcal{X}} Q(u, v), \quad \overline{Q}(u) = \sum_{u \in \mathcal{X}} Q(v, u), \quad u \in \mathcal{X}.
\]

The classical (conditional) Kullback-Leibler information (informational divergence or relative entropy) is denoted by \( D \) and entropy by \( H \)\cite{Cover}. Specifically, for a probability distribution \( Q \) on \( \mathcal{X}^2 \), transition (or conditional) probabilities \( P(v|u), u, v \in \mathcal{X} \), and a probability distribution \( p \) on \( \mathcal{X} \), we define \( \overline{Q}(\cdot|\cdot) \) by \( \overline{Q}(v|u) = Q(u, v)/\overline{Q}(u) \) for \( \overline{Q}(u) > 0 \), \( D(Q||P) \) by

\[
D(Q||P) = \sum_{u \in \mathcal{X}}: \overline{Q}(u) > 0 \sum_{v \in \mathcal{X}} Q(u, v) \log_d \frac{\overline{Q}(v|u)}{P(v|u)},
\]

and \( H(P|p) \) by

\[
H(P|p) = -\sum_{u \in \mathcal{X}}: p(u) > 0 \sum_{v \in \mathcal{X}} p(u)P(v|u) \log_d P(v|u),
\]

which is called the entropy of \( P(\cdot|\cdot) \) conditional on \( p \). We remark that \( D(Q||P) \) is a conditional Kullback-Leibler information, so that in a more consistent notation\cite{Cover}, it would be denoted by \( D(\overline{Q}||P|\overline{Q}) \).

By convention, we assume \( \log(a/0) = \infty \) for \( a > 0 \), \( 0 \log 0 = 0 \log(0/0) = 0 \). The first form of this work’s main result is the next one.

**Theorem 1** Let a channel \( \mathcal{A}_n \sim \{\sqrt{P_n(x)}N_x\}_{x \in \mathcal{X}^n}, n = 1, 2, \ldots, \) be specified by \( \mathcal{P} \) with some \( P(\cdot|\cdot) \) and \( p \). Then, for \( 0 \leq R \leq 1 \), we have

\[
\liminf_{n \to \infty} \frac{1}{n} \log_d [1 - F_{n,Rn}^*(\mathcal{A}_n)] \geq E(R, P),
\]

(3)
where

$$E(R, P) = \min_{Q, \bar{Q}} |D(Q||P)| + |1 - H(\bar{Q} | \bar{Q}) - R|^+,$$

$$|x|^+ = \max\{x, 0\},$$

and the minimization with respect to $Q$ is taken over all probability distributions on $\mathcal{X}^2$ with $\bar{Q} = \bar{Q}$.

Remarks. Roughly speaking, the theorem says $F_{n,Rn}^*(A_n) \gtrsim 1 - \exp_d[-nE(R, P)]$. An immediate consequence of the theorem is that when the Markov chain is irreducible, the quantum capacity of $\{A_n\}$ is lower bounded by $1 - H(P|q)$, where $q$ is the unique stationary (steady state, or equilibrium) distribution of the Markov chain. To see this, observe that $E(R, P)$ is positive for $R < 1 - H(P|q)$ due to an easily established inequality $D(Q||P) \geq 0$ where equality occurs if and only if $Q(u, v) = q(u)P(v|u)$ for all $u, v \in \mathcal{X}$ under the constraint $\bar{Q} = \bar{Q}$.

Example. Let us assume $d = 2$, rename the elements $(0, 0), (1, 0), (0, 1), (1, 1)$ in $\mathcal{X}$ as $0, 1, 2, 3$, and define $P(v|u), u, v \in \mathcal{X}$, by

$$P(v|u) = \begin{cases} 1 - \varepsilon & \text{if } u = 0 \text{ and } v = 0, \\ \varepsilon/3 & \text{if } u = 0 \text{ and } v \neq 0, \\ 1 - \gamma & \text{if } u \neq 0 \text{ and } v = 0, \\ \gamma/3 & \text{if } u \neq 0 \text{ and } v \neq 0. \end{cases}$$

In this case, $\{A_n\}$ is analogous to the channel with memory discussed by Gilbert in the context of classical channel coding (see also Gallager, Sec. 4.6). If we brought Gilbert’s idea into our quantum case innocently, we might assume $0 < \varepsilon \leq \gamma < 1$ and interpret $0$ as ‘good state,’ $1, 2, 3$ as ‘bad ones,’ where a state means that of the Markov chain, not a quantum state, and $\varepsilon$ (resp., $\gamma$) as the probability of going into a ‘bad state’ provided the current state be ‘good (resp., bad).’ For the above quantum channel, the lower bound $1 - H(P|q)$ becomes

$$1 - \frac{(1 - \gamma)[h(\varepsilon) + \varepsilon \log_2 3] + \varepsilon[h(\gamma) + \gamma \log_2 3]}{1 - \gamma + \varepsilon},$$

where $h$ is the binary entropy function $h(z) = -z \log_2 z - (1-z) \log_2 (1-z)$. Note that when $\varepsilon = \gamma$, the channel becomes the depolarizing channel and the lower bound on the capacity becomes the known one.

III. PROOF OF THEOREM 1

A. Codes Based on Symplectic Geometry

The codes to be proven to have the desired performance are symplectic (stabilizer, or additive) codes. Let us recall first the basics of symplectic codes. We can regard the index of $N_{(i,j)} = X^i Z^j$, $(i,j) \in \mathcal{X}$, as a pair of elements from the field $\mathbb{F} = \mathbb{F}_d = \mathbb{Z}/d\mathbb{Z}$, the finite field consisting of $d$ elements. Recall we put $N_x = N_{x_1} \otimes \ldots \otimes N_{x_n}$ for $x = (x_1, \ldots, x_n) \in (\mathbb{F}^2)^n$. We write $N_J$ for $\{N_x \in N_n | x \in J\}$ where $J \subseteq (\mathbb{F}^2)^n$. The index
\((u_1, v_1), \ldots, (u_n, v_n)\) \(\in (\mathbb{F}^2)^n\) of a basis element can be regarded as the plain 2\(n\)-dimensional vector
\[
x = (u_1, v_1, \ldots, u_n, v_n) \in \mathbb{F}^{2n}.
\]
We can equip the vector space \(\mathbb{F}^{2n}\) over \(\mathbb{F}\) with a symplectic bilinear form (symplectic paring), which is defined by
\[
(x, y)_{sp} = \sum_{i=1}^{n} u_i v'_i - v_i u'_i
\]
for the above \(x\) and \(y = (u'_1, v'_1, \ldots, u'_n, v'_n) \in \mathbb{F}^{2n}\). Given a subspace \(L \subseteq \mathbb{F}^{2n}\), let
\[
L^\perp = \{ x \in \mathbb{F}^{2n} | \forall y \in L, (x, y)_{sp} = 0 \}.
\]

**Lemma 1** \(^{25,27}\) Let a subspace \(L \subseteq \mathbb{F}^{2n}\) satisfy
\[
L \subseteq L^\perp \quad \text{and} \quad \dim L = n - k.
\]
In addition, let \(J_0 \subseteq \mathbb{F}^{2n}\) be a set satisfying
\[
\forall x, y \in J_0, \ [y - x \in L^\perp \Rightarrow x = y].
\]
(4)

Then, there exist \(d^k\)-dimensional \(N_{J_0}\)-correcting codes.

In fact, given a subspace \(L\) as above, there are \(d^k\) subspaces of the form
\[
\{ \psi \in \mathbb{H}^\otimes n | \forall M \in N_L, \ M\psi = \tau(M)\psi \},
\]
with some scalars \(\tau(M)\) (eigenvalues of \(M \in N_L\), and each of them, together with a suitable recovery operator, serves as an \(N_{J_0}\)-correcting quantum code of dimension \(d^k\). Note that the direct sum of these subspaces is the whole space \(\mathbb{H}^\otimes n\). The precise meaning of \(N_{J_0}\)-correcting can be found, e.g., in Knill and Laflamme\(^{15}\). Originally, Lemma \(^1\) was claimed for the case where \(d = 2\), and has been generalized to the case where \(d\) is a general prime \(^{18,19,20,31}\).

By definition, for an \(N_{J_0}\)-correcting code \((\mathcal{C}_n, \mathcal{R}_n)\) and the channel \(\{\mathcal{A}_n\}\) in the theorem, it holds
\[
1 - F(\mathcal{C}) \leq \sum_{x \notin J_0} P_n(x),
\]
(5)
where \(F(\mathcal{C}) = F(\mathcal{C}, \mathcal{R}_n, \mathcal{A}_n)\). We remark that, as is usually done in the literature, it is assumed in this paper that when we speak of an \(N_{J_0}\)-correcting code \((\mathcal{C}_n, \mathcal{R}_n)\), the \(\mathcal{R}_n\) indicates the one constructed by Knill and Laflamme\(^{14}\). Note that \(\mathcal{R}_n\) is determined from \(J_0\) and \(\mathcal{C}\). The premise \(^4\) of Lemma \(^1\) is restated as that \(J_0\) is a set of representatives of cosets of \(L^\perp\) in \(\mathbb{F}^{2n}\). A natural choice for \(J_0\) would be a set consisting of representatives each of which maximizes the probability \(P_n(x)\) in its coset\(^7\) since it is analogous to maximum likelihood decoding, which is an optimum strategy for classical coding (see Slepian\(^32\) or any textbook of information theory). In the proof below, we choose another set of representatives, the classical counterpart of which (minimum entropy decoding) asymptotically yields the same performance as maximum likelihood decoding\(^{10,33}\).
B. The Method of Types

The theorem can be proved along the lines of Ref. 8, which employed the method of types. In the present case, second-order (Markov) types rather than the usual types are used. Needed technical tools from the method of types in the Markov case can be found in Csiszar et al. and papers cited therein. We collect here a few basic facts on this method to be used below.

For $x = (x_1, \ldots, x_n) \in \mathcal{X}^n$, $n > 1$, define a probability distribution $M_x$ on $\mathcal{X}^2$ by

$$M_x(u, v) = \frac{\{i \mid 1 \leq i \leq n-1, (x_i, x_{i+1}) = (u, v)\}}{n-1}, \quad u \in \mathcal{X},$$

which is called the second-order type or Markov type of $x$. With $\mathcal{X}$ and an element $u \in \mathcal{X}$ fixed, the set of all possible Markov types of sequences $(x_1, \ldots, x_n)$ from $\mathcal{X}^n$ satisfying $x_1 = u$ is denoted by $Q_n(\mathcal{X}, u)$ simply by $Q_n(u)$, and $Q_n$ stands for $\bigcup_{u \in \mathcal{X}} Q_n(u)$. For a type $Q \in Q_n(u)$, $T_Q^n(u)$ is defined as $\{(x_1, \ldots, x_n) \in \mathcal{X}^n \mid x_1 = u$ and $M_x = Q\}$, and $T_Q^n$ denotes $\bigcup_{u \in \mathcal{X}} T_Q^n(u)$.

In what follows, we use

$$|T_Q^n(u)| \leq \exp_d[(n-1)H(\overline{Q} | Q)], \quad u \in \mathcal{X}. \quad (6)$$

Note that if $x = (x_1, \ldots, x_n) \in \mathcal{X}^n$ with $x_1 = u$ has type $Q$, then $P_n(x) = p(u) \prod_{(a,b) \in \mathcal{X}^2} P(b|a)^{(n-1)Q(a,b)} = p(u) \exp_{\overline{d}}\{-(n-1)[H(\overline{Q} | Q) + D(Q||P)]\}$ and hence, (6) is equivalent to the latter inequality in (39) of Csiszar et al., i.e.,

$$\Pr\{M_X = Q \mid X_1 = u\} \leq \exp_{\overline{d}}\{-(n-1)D(Q||P)\}, \quad (7)$$

where the sequence of random variables $X = (X_1, \ldots, X_n)$ represents the Markov chain in the theorem, i.e., $\Pr\{X_1 = x_1, \ldots, X_n = x_n\} = P_n(x_1, \ldots, x_n)$ with $P_n$ defined in (2). Eq. (4) or (7) is a consequence of Whittle's formula for $|T_Q^n(u)|$, a simple proof of which was given by Billingsley. The upper bound in (7) can be proved even easier with a simple way of enumeration (Davisson et al. or the paragraph containing (9) of Ref. 34).

C. Proof of Theorem 1

The case where $R = 1$ is trivial, so that we assume $R < 1$ from now on. Putting $k = \lceil Rn \rceil$, we apply Lemma 1, where we choose $J_0$ as follows. Assume $\dim L = n - k$. Then, $\dim L^\perp = n + k$. For notational simplicity, we write $H_c(Q)$ in place of $H(\overline{Q} | Q)$ for a probability distribution $Q$ on $\mathcal{X}^2$. From each of the $d^{n-k}$ cosets of $L^\perp$ in $\mathbb{F}^{2n}$, select a vector that minimizes $H_c(M_x)$, i.e., a vector $x$ satisfying $H_c(M_x) \leq H_c(M_y)$ for any $y$ in the coset. This selection uses the idea of the minimum entropy decoder known in the classical information theory literature.

Let $J_0(L)$ denote the set of the $d^{n-k}$ selected vectors, let

$$A = \{L \subseteq \mathbb{F}^{2n} \mid L \text{ linear, } L \subseteq L^\perp, \dim L = n - k\}$$

and for each $L \in A$, let $C(L)$ be an $N_{J_0(L)}$-correcting code existence of which is ensured by Lemma 1. Putting

$$\overline{F} = \frac{1}{|A|} \sum_{L \in A} F(C(L)),$$
we will show \( \lim \inf_n -n^{-1} \log_d (1 - \overline{F}) \geq E(R, P) \), which implies that, at least, one sequence of codes has fidelity as high as promised in the theorem. Such a method for a proof is referred to as random coding\(^{10,37}\).

As in the proof of Theorem 1 of Ref. 8, we have

\[
1 - \overline{F} \leq \sum_{x \in \mathbb{F}^{2n}} P_n(x) \frac{|B(x)|}{|A|},
\]

(8)

where

\[
B(x) = \{ L \in A \mid x \notin J_0(L) \}, \quad x \in \mathbb{F}^{2n}.
\]

The fraction \(|B(x)|/|A|\) is trivially bounded as

\[
\frac{|B(x)|}{|A|} \leq 1, \quad x \in \mathbb{F}^{2n}.
\]

(9)

We use the next inequality\(^{9}\). Let

\[
A(x) = \{ L \in A \mid x \in L^\perp \setminus \{0\} \}.
\]

Then, \(|A(0)| = 0\) and

\[
\frac{|A(x)|}{|A|} = \frac{d^{n+k} - 1}{d^{2n} - 1} \leq \frac{1}{d^{n-k}}, \quad x \in \mathbb{F}^{2n}, \ x \neq 0.
\]

(10)

This is a variant of the relation established by Calderbank \textit{et. al.}\(^{25}\), or its analog proved by Matsumoto and Uyematsu\(^{38}\) with an explicit use of the Witt lemma\(^{28,29}\) from the theory of bilinear forms.

Since \(B(x) \subseteq \{ L \in A \mid \exists y \in \mathbb{F}^{2n} : H_c(M_y) \leq H_c(M_x), y - x \in L^\perp \setminus \{0\} \}\) from the design of \(J_0(L)\) specified above (cf. Goppa\(^{37}\)), it follows that

\[
|B(x)| \leq \sum_{y \in \mathbb{F}^{2n} : H_c(M_y) \leq H_c(M_x), \ y \neq x} |A(y - x)|
\]

\[
\leq \sum_{y \in \mathbb{F}^{2n} : H_c(M_y) \leq H_c(M_x), \ y \neq x} |A| d^{-n+k},
\]

(11)

where we have used (10) for the latter inequality. Combining (8), (9) and (11), we obtain the following chain of inequalities with the aid of the basic inequalities in (6) and (7) as well.
as the inequality \( \min\{a + b, 1\} \leq \min\{a, 1\} + \min\{b, 1\} \) for \( a, b \geq 0 \):

\[
1 - \mathcal{F} \\
\leq \sum_{x \in \mathbb{F}^{2n}} P_n(x) \min \left\{ \sum_{y \in \mathbb{F}^{2n}: H_c(M_y) \leq H_c(M_x), y \neq x} d^{-(n-k)}, 1 \right\} \\
\leq \sum_{u \in \mathcal{X}} p(u) \sum_{Q \in \mathcal{Q}_n(u)} \Pr\{M_X = Q | X_1 = u\} \min \left\{ \sum_{Q' \in \mathcal{Q}_n: H_c(Q') \leq H_c(Q)} \frac{|\mathcal{T}_{Q'}|}{d^{n(1-R)-1}}, 1 \right\} \\
\leq d^3 \sum_{Q \in \mathcal{Q}_n} \exp_d\{-(n-1)D(Q || P)\} \max_{Q' \in \mathcal{Q}_n: H_c(Q') \leq H_c(Q)} \exp_d\{-(n-1)|1 - R - H_c(Q')|^+\} \\
\leq d^3 \sum_{Q \in \mathcal{Q}_n} \exp_d\{-(n-1)D(Q || P)\} |\mathcal{Q}_n| \exp_d\{-(n-1)|1 - R - H_c(Q)|^+\} \\
\leq d^3 |\mathcal{Q}_n|^2 \exp_d\{-(n-1) \min_{Q \in \mathcal{Q}_n} [D(Q || P) + |1 - R - H_c(Q)|^+]\}.
\]

Since \( |\mathcal{Q}_n| \) is polynomial in \( n \), the remaining task is to show that

\[
\liminf_{n \to \infty} \min_{Q \in \mathcal{Q}_n} [D(Q || P) + |1 - R - H_c(Q)|^+] \\
\text{is not less than} \\
\min_{Q: ||Q - \mathcal{Q}|| = 0} [D(Q || P) + |1 - R - H_c(Q)|^+],
\]

which is \( E(R, P) \). One sees this holds immediately noticing that any \( Q \in \mathcal{Q}_n \) satisfies \( ||Q - \mathcal{Q}|| \leq \frac{1}{n-1} \) for the norm \( ||(z_1, \ldots, z_\mathcal{X})|| = \max_i |z_i| \) the set of all probability distributions is compact, and \( D(Q) = D(Q || P) \) is continuous in its effective domain \( \{Q \mid D(Q) < \infty\} \) (cf., the proof of Lemma 2 in Csiszár et al.\(^4\)). This completes the proof.

**IV. MAIN RESULT FOR GENERAL CASE**

Theorem\(^1\) actually holds for a wider class of channels. To evaluate the fidelity of codes on a more general channel \( \{\mathcal{A}_n\} \), we first associate a sequence of probability distributions \( \{P_{\mathcal{A}_n}\} \) with the channel \( \{\mathcal{A}_n\} \) as in Ref.\(^1\).

**Definition 1** For each \( n \), let \( \mathcal{A}_n \sim \{A_x^{(n)}\}_{x \in \mathcal{X}^n} \), expand \( A_x^{(n)} \) as \( A_x^{(n)} = \sum_{y \in \mathcal{X}^n} a_{xy} N_y, x \in \mathcal{X}^n \), and define a probability distribution \( P_{\mathcal{A}_n} \) on \( \mathcal{X}^n \) by

\[
P_{\mathcal{A}_n}(y) = \sum_x |a_{xy}|^2, \quad y \in \mathcal{X}^n.
\]

**Example.** Let \( \{\mathcal{A}_n\} \) be a memoryless channel \( \mathcal{A}_n = \mathcal{A}^{\otimes n}, n = 1, 2, \ldots \). It is easy to see that \( P_{\mathcal{A}_n}(y_1, \ldots, y_n) = \prod_{i=1}^n P_\mathcal{A}(y_i) \).

The case of memoryless channels as above was discussed in this author’s previous work\(^4\). This work claims the next.
Theorem 2 Consider a channel \( \{A_n\} \) whose \( \{P_n = P_{A_n}\} \) satisfies (2) with some \( P(\cdot|\cdot) \) and \( p \). Then, again, for \( 0 \leq R \leq 1 \), (3) in Theorem 1 holds.

The above theorem can be proved along the lines of this author’s previous work, which treated general memoryless quantum channels. Namely, Theorem 1 can be generalized to Theorem 2 in the same way as the result in Ref. 8 was strengthened in Ref. 9. Here it is briefly described how to prove Theorem 2. First, we evaluate the minimum average fidelity \( F_a(C) \), which is another performance measure for a code \( C \) introduced in Ref. 9, instead of the minimum fidelity \( F(C) \). Actually, we evaluate the average of \( F_a(C(L,i)) \) over the whole ensemble of quantum codes \( \{C(L,i) \mid L \in A, 0 \leq i < d_{n-k}\} \), where \( C(L,i) \), \( i = 0, \ldots, d_{n-k} - 1 \), are the \( d_{n-k} \) quantum codes associated with \( L \) as in Lemma 1; compare the proof of Theorem 1 above, where using an arbitrarily chosen code \( C(L,i) \) for each \( L \) was enough. The average of \( F_a(C(L,i)) \) turns out to be lower bounded by \( 1 - \exp[-nE(R,P) + o(n)] \). Then, at least, one code \( C(L,i) \) has this performance or higher. As proved in Ref. 9, if we have a code with \( 1 - F_a(C) \leq G \), we can choose a subcode \( C' \) of half the dimension with \( 1 - F(C') \leq 2G \), which implies Theorem 2.

The major difficulty in the analysis on general channels lay in the fact that (5) is no longer true in the general case; this was resolved in Ref. 9 by proving that (5) holds true if we replace \( F(C) = F(C(L)) \) by \( F_a(C(L,i)) \) averaged over \( 0 \leq i < d_{n-k} \).

We remark that the result of this paper readily extends to the case where \( P_n \) is the probability distributions of a higher-order Markov chain. For this extension, we have only to use higher-order types instead of second-order types.

V. CONCLUDING REMARKS

It should be remarked that the lower bound \( 1 - H(P|q) \) on the quantum capacity is not tight in general since there is an example of a code which slightly goes beyond the bound for some very noisy memoryless channels. This work, however, seems the first to demonstrate that standard error correction schemes work reliably even in the presence of correlated errors with positive information rate for all large enough code lengths. Moreover, the established convergence of the fidelity is exponential. Research in this direction is yet to be developed in quantum information theory, while exponent problems have already been central issues in other fields including large-deviation theory and classical information theory.

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