VIRTUAL ELEMENT METHODS ON MESHES WITH SMALL EDGES OR FACES

SUSANNE C. BRENNER AND LI-YENG SUNG

Abstract. We consider a model Poisson problem in \( \mathbb{R}^d \) \( (d = 2, 3) \) and establish error estimates for virtual element methods on polygonal or polyhedral meshes that can contain small edges \( (d = 2) \) or small faces \( (d = 3) \).

1. Introduction

Let \( \Omega \subset \mathbb{R}^d \) \( (d = 2, 3) \) be a bounded polygonal/polyhedral domain and \( f \in L^2(\Omega) \). The Poisson problem with the homogeneous Dirichlet boundary condition is to find \( u \in H^1_0(\Omega) \) such that

\[
(1.1) \quad a(u, v) = (f, v) \quad \forall v \in H^1_0(\Omega),
\]

where

\[
(1.2) \quad a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx
\]

and \( \langle \cdot, \cdot \rangle \) is the inner product for \( L^2(\Omega) \). Here and throughout the paper we follow standard notation for differential operators, function spaces and norms that can be found for example in [10, 1, 9].

Problem (1.1) can be solved by virtual element methods [5, 2] on polygonal or polyhedral meshes. It has been observed in numerical experiments that the convergence rates for the virtual element methods do not deteriorate noticeably even in the presence of small edges or faces (cf. [2, 4, 3]). Our goal is to establish error estimates that justify these numerical results for the virtual element methods introduced in [2].

We will develop error estimates that are based on general shape regularity assumptions on the subdomains in the polygonal or polyhedral meshes. For the two dimensional problem, we assume that (i) each polygonal subdomain is star-shaped with respect to a disc whose diameter is comparable to the diameter of the subdomain and (ii) the number of edges of the subdomains is uniformly bounded. For the three dimensional problem, we assume that (i) each polyhedral subdomain is star-shaped with respect to a ball whose diameter is comparable to the diameter of the subdomain; (ii) the number of faces of the subdomains is uniformly bounded; and (iii) the faces of the subdomains satisfy the two dimensional shape regularity assumptions. Our error estimates are optimal up to at most a logarithmic factor that involves the ratio of the lengths of the longest edge and the shortest edge of the subdomains.
each subdomain in two dimensions and a similar ratio over the edges of the faces on the subdomains in three dimensions.

The rest of the paper is organized as follows. We begin with a star-shaped condition in Section 2. Then we treat the two dimensional case in Section 3 and Section 4, where the analysis benefits from the techniques developed in [4] and [8]. The extension to the three dimensional Poisson problem is presented in Section 5. We end with some concluding remarks in Section 6.

In order to avoid the proliferation of constants, we will often use the notation \( A \lesssim B \) to represent the statement that \( A \leq (\text{constant})B \), where the positive constant is independent of mesh sizes. The notation \( A \approx B \) is equivalent to \( A \lesssim B \) and \( B \lesssim A \). The precise dependence of the hidden constants will be declared in the text.

To minimize the technicalities, we also assume that \( \Omega \) is convex so that the solution of (1.1) belongs to \( H^2(\Omega) \) by elliptic regularity [14, 11].

2. A Star-Shaped Condition

Let \( D \) be a bounded open polygon \((d = 2)\) or a bounded open polyhedron \((d = 3)\), and \( h_D \) be the diameter of \( D \).

We assume that

\[
D \text{ is star-shaped with respect to a disc/ball } \mathfrak{B}_D \subset D \text{ with radius } \rho_D h_D. \tag{2.1}
\]

We will denote by \( \mathfrak{B}_D \) the disc/ball concentric with \( \mathfrak{B}_D \) whose radius is \( h_D \). It is clear that

\[
\mathfrak{B}_D \subset D \subset \mathfrak{B}_D. \tag{2.2}
\]

Below are some consequences of the star-shaped condition (2.1). The hidden constants in Section 2.1–Section 2.4 only depend on \( \rho_D \), while those in Section 2.8 also depend on \( k \).

2.1. Sobolev Inequalities. It follows from (2.1) that

\[
\|\zeta\|_{L^\infty(D)} \lesssim h_D^{-(d/2)}\|\zeta\|_{L^2(D)} + h_D^{1-(d/2)}|\zeta|_{H^1(D)} + h_D^{2-(d/2)}|\zeta|_{H^2(D)} \quad \forall \zeta \in H^2(D), \tag{2.3}
\]

and in the case where \( d = 2 \),

\[
\|\zeta\|_{L^\infty(D)} \lesssim h_D^{-1}\|\zeta\|_{L^2(D)} + |\zeta|_{H^1(D)} + h_D^{1/2}|\zeta|_{H^{3/2}(D)} \quad \forall \zeta \in H^{3/2}(D). \tag{2.4}
\]

Details can be found in [9, Lemma 4.3.4] and [12, Section 6].

2.2. Bramble-Hilbert Estimates. Condition (2.1) also implies the following Bramble-Hilbert estimates [7]:

\[
\inf_{q \in \mathbb{P}_\ell} |\zeta - q|_{H^m(D)} \lesssim h_D^{\ell + 1 - m}|\zeta|_{H^{\ell+1}(D)} \quad \forall \zeta \in H^{\ell+1}(D), \ell = 0, \ldots, k, \text{ and } m \leq \ell, \tag{2.5}
\]

\[
\inf_{q \in \mathbb{P}_\ell} |\zeta - q|_{H^m(D)} \lesssim h_D^{\ell + 1/2 - m}|\zeta|_{H^{\ell+1/2}(D)} \quad \forall \zeta \in H^{\ell+1/2}(D), \ell = 0, \ldots, k, \text{ and } m \leq \ell. \tag{2.6}
\]

Details can be found in [9, Lemma 4.3.8] and [12, Section 6].
2.3. A Lipschitz Isomorphism between $D$ and $\mathcal{B}_D$. In view of the star-shaped condition (2.1), there exists a Lipschitz isomorphism $\Phi : \mathcal{B}_D \to D$ such that both $|\Phi|_{W^{1,\infty}(\mathcal{B}_D)}$ and $|\Phi^{-1}|_{W^{1,\infty}(D)}$ are bounded by a constant that only depends on $\rho_D$ (cf. [16, Section 1.1.8]).

It follows that

\begin{align}
|D| & \approx h_d^d, \\
|\partial D| & \approx h_d^{d-1},
\end{align}

where $|D|$ is the area of $D$ ($d = 2$) or the volume of $D$ ($d = 3$), and $|\partial D|$ is the arclength of $\partial D$ ($d = 2$) or the surface area of $D$ ($d = 3$).

Moreover we have (cf. [17, Theorem 4.1])

\begin{align}
|\zeta|_{L^2(\partial D)} & \approx |\zeta \circ \Phi|_{L^2(\partial \mathcal{B}_D)} \quad \forall \zeta \in L^2(\partial D), \\
|\zeta|_{L^2(D)} & \approx |\zeta \circ \Phi|_{L^2(\mathcal{B}_D)} \quad \forall \zeta \in L^2(D), \\
|\zeta|_{H^1(\partial D)} & \approx |\zeta \circ \Phi|_{H^1(\partial \mathcal{B}_D)} \quad \forall \zeta \in H^1(\partial D), \\
|\zeta|_{H^1(D)} & \approx |\zeta \circ \Phi|_{H^1(\mathcal{B}_D)} \quad \forall \zeta \in H^1(D), \\
|\zeta|_{H^{1/2}(\partial D)} & \approx |\zeta \circ \Phi|_{H^{1/2}(\partial \mathcal{B}_D)} \quad \forall \zeta \in H^{1/2}(\partial D).
\end{align}

2.4. Poincaré-Friedrichs Inequalities. The Bramble-Hilbert estimate (2.5) and the geometric estimates (2.7)–(2.8) imply the following Poincaré-Friedrichs inequalities:

\begin{align}
&h_d^{-(d/2)} |\zeta|_{L^2(\partial D)} \lesssim h_d^{-d} \int_D |\zeta|_{L^2(D)} dx + h_d^{1-(d/2)} |\zeta|_{H^1(D)} \quad \forall \zeta \in H^1(D), \\
&h_d^{-(d/2)} |\zeta|_{L^2(D)} \lesssim h_d^{-(d-1)} \int_{\partial D} |\zeta|_{\partial D} ds + h_d^{1-(d/2)} |\zeta|_{H^1(D)} \quad \forall \zeta \in H^1(D).
\end{align}

2.5. Estimates for $|\cdot|_{H^{1/2}(\partial D)}$. On the circle $\partial \mathcal{B}_D$, we have a standard estimate

\[ |\zeta|_{H^{1/2}(\partial \mathcal{B}_D)} \lesssim h_d^{-1/2} |\zeta|_{L^2(\partial \mathcal{B}_D)} + h_d^{1/2} |\zeta|_{H^1(\partial \mathcal{B}_D)} \quad \forall \zeta \in H^1(\partial \mathcal{B}_D). \]

It then follows from a Poincaré-Friedrichs inequality for a circle that

\[ |\zeta|_{H^{1/2}(\partial \mathcal{B}_D)} = |\zeta - \bar{\zeta}|_{H^{1/2}(\partial \mathcal{B}_D)} \lesssim h_d^{1/2} |\zeta - \bar{\zeta}|_{L^2(\partial \mathcal{B}_D)} + h_d^{1/2} |\zeta|_{H^1(\partial \mathcal{B}_D)} \lesssim h_d^{1/2} |\zeta|_{H^1(\partial \mathcal{B}_D)}, \]

where $\bar{\zeta}$ is the mean of $\zeta$ over $\partial \mathcal{B}_D$. Therefore, in view of (2.11) and (2.13), we have

\begin{align}
|\zeta|_{H^{1/2}(\partial D)} & \lesssim h_d^{1/2} |\zeta|_{H^1(\partial D)} \quad \forall \zeta \in H^1(\partial D).
\end{align}

Similarly, it follows from (2.12), (2.13) and the trace theorem for $H^1(\mathcal{B}_D)$ that

\begin{align}
|\zeta|_{H^{1/2}(\partial D)} & \lesssim |\zeta|_{H^1(D)} \quad \forall \zeta \in H^1(D).
\end{align}

2.6. Trace Inequalities. It follows from (2.9), (2.10), (2.12) and standard (scaled) trace inequalities for $H^1(\mathcal{B}_D)$ that

\begin{align}
|\zeta|_{L^2(\partial D)} & \lesssim h_d^{-1} |\zeta|_{L^2(D)} + h_d |\zeta|_{H^1(D)} \quad \forall \zeta \in H^1(D).
\end{align}

We also have trace inequalities for the $H^1$ norm on $\partial D$ that require a different derivation.

**Lemma 2.1.** Let $e$ be an edge of $D \subset \mathbb{R}^2$. We have

\[ h_d |\zeta|_{H^1(e)}^2 \lesssim |\zeta|_{H^1(D)}^2 + h_d |\zeta|_{H^{3/2}(D)}^2 \quad \forall \zeta \in H^{3/2}(D). \]
Proof. By scaling we can assume $h_D = 1$. Without loss of generality we may also assume that

\[ \int_D \zeta \, dx = 0. \tag{2.19} \]

The existence of the Lipschitz isomorphism $\Phi : D \rightarrow \mathfrak{B}_D$ implies that the domain $D$ satisfies a uniform cone condition \[1, \text{Section 4.8}\], with one reference cone and a finite cover of $\bar{D}$ that contains a fixed number of congruent open discs. Furthermore, the angle and the height of the reference cone and the radius of the open discs only depend on $\rho_D$. It follows that there exists a Calderon-Zygmund extension operator $E : H^1(D) \rightarrow H^1(\mathbb{R}^2)$ (cf. \[1, \text{Theorem 5.28}\]) such that $E$ maps $H^2(D)$ into $H^2(\mathbb{R}^2)$ and the restriction of $E\zeta$ to $D$ equals $\zeta$. Moreover we have

\[ \|E\zeta\|_{H^1(\mathbb{R}^2)} \lesssim \|\zeta\|_{H^1(D)} \quad \forall \zeta \in H^1(D) \quad \text{and} \quad \|E\zeta\|_{H^3(\mathbb{R}^2)} \lesssim \|\zeta\|_{H^3(D)} \quad \forall \zeta \in H^3(D). \]

It follows from the interpolation of Sobolev spaces \[1, \text{Chapter 7}\] that

\[ \|E\zeta\|_{H^{3/2}(\mathbb{R}^2)} \lesssim \|\zeta\|_{H^{3/2}(D)} \lesssim |\zeta|_{H^1(D)} + |\zeta|_{H^{3/2}(D)} \quad \forall \zeta \in H^{3/2}(D), \tag{2.20} \]

where we have used (2.15) and (2.19).

Let $e$ be an edge of $D$, $\bar{e}$ be the infinite line that contains $e$ and $G$ be a half-plane that borders $\bar{e}$. Then we have, by (2.20) and the trace theorem \[17, \text{Theorem 8.1}\],

\[ |\zeta|_{H^1(e)} = |E\zeta|_{H^1(e)} \leq |E\zeta|_{H^1(\bar{e})} \lesssim \|E\zeta\|_{H^{3/2}(G)} \lesssim |\zeta|_{H^1(D)} + |\zeta|_{H^{3/2}(D)}. \]

The proof of the following result is similar.

**Lemma 2.2.** Let $F$ be a face of $D \subset \mathbb{R}^3$. We have

\[ h_D|\zeta|_{H^1(F)}^2 \lesssim |\zeta|_{H^1(D)}^2 + h_D^2|\zeta|_{H^2(D)}^2 \quad \forall \zeta \in H^2(D). \]

2.7. **A Lifting Operator.** It follows from (2.10), (2.12), (2.13) and the inverse trace theorem for $H^1(\mathfrak{B}_D)$ (cf. \[17, \text{Theorem 8.8}\]) that there exists a linear operator $\text{Tr}^\dagger : H^{1/2}(\partial D) \rightarrow H^1(D)$ such that

\[ \text{Tr}^\dagger \zeta = \zeta \text{ on } \partial D \quad \text{and} \quad h_D^{-1}\|\text{Tr}^\dagger \zeta\|_{L^2(D)} + \|\text{Tr}^\dagger \zeta\|_{H^1(D)} \leq C|\zeta|_{H^{1/2}(\partial D)}, \]

where the constant $C$ depends only on $\rho_D$.

2.8. **Some Estimates for Polynomials.** Let $\mathbb{P}_k$ be the space of polynomials of total degree $\leq k$ in $d$ variables. We obtain the following estimates by using the equivalence of norms on finite dimensional vector spaces and scaling.

**Lemma 2.3.** We have

\[ \|p\|_{L^2(\partial D)}^2 \lesssim h_D^{-1}\|p\|_{L^2(D)}^2 \quad \forall p \in \mathbb{P}_k, \tag{2.21} \]
\[ |p|_{H^1(D)} \lesssim h_D^{-1}\|p\|_{L^2(D)} \quad \forall p \in \mathbb{P}_k, \tag{2.22} \]
\[ |p|_{H^1(\partial D)} \lesssim \bar{p}_{\partial D} + h_D^{1-(d/2)}|p|_{H^1(D)} \quad \forall p \in \mathbb{P}_k, \tag{2.23} \]
\[ |p|_{L^\infty(D)} \lesssim \bar{p}_{D} + h_D^{1-(d/2)}|p|_{H^1(D)} \quad \forall p \in \mathbb{P}_k, \tag{2.24} \]

where $\bar{p}_{\partial D}$ is the mean of $p$ over $\partial D$ and $\bar{p}_D$ is the mean of $p$ over $D$. 

Proof. In view of (2.2), we have

\[(2.25)\hspace{1cm} \|p\|_{L^\infty(D)} \leq \|p\|_{L^\infty(\mathcal{B}_D)} \lesssim \|p\|_{L^\infty(\mathcal{B}_D)} \lesssim (\text{diam } \mathcal{B}_D)^{-d/2}\|p\|_{L^2(\mathcal{B}_D)} \leq (\text{diam } \mathcal{B}_D)^{-d/2}\|p\|_{L^2(D)}.
\]

The estimate (2.21) then follows from (2.8) and (2.25):

\[
\|p\|_{L^2(\partial D)}^2 \lesssim h_D^{d-1}\|p\|_{L^\infty(D)}^2 \lesssim h_D^{-1}\|p\|_{L^2(D)}^2.
\]

Similarly, we have

\[
|p|_{H^1(D)} \lesssim |p|_{H^1(\mathcal{B}_D)} \lesssim (\text{diam } \mathcal{B}_D)^{-1}\|p\|_{L^2(\mathcal{B}_D)} \leq (\text{diam } \mathcal{B}_D)^{-1}\|p\|_{L^2(D)},
\]

which together with (2.1) implies (2.22).

The estimates (2.23) and (2.24) follow immediately from (2.7)--(2.8), (2.14)--(2.15) and (2.25).

Lemma 2.4. Given any \( p \in \mathbb{P}_{k-2} \) \((k \geq 2)\), there exists \( q \in \mathbb{P}_k \) such that \( \Delta q = p \) and

\[
\|\nabla q\|_{L^2(B)} \leq C(\text{diam } B)\|p\|_{L^2(B)},
\]

where \( B \subset \mathbb{R}^d \) is any ball and the positive constant \( C \) depends only on \( k \).

Proof. By scaling it suffices to treat the case where \( B \) is a unit ball. Since \( \Delta \) maps \( \mathbb{P}_k \) onto \( \mathbb{P}_{k-2} \), there exists an operator \( \Delta^\dagger : \mathbb{P}_{k-2} \rightarrow \mathbb{P}_k \) such that \( \Delta \Delta^\dagger \) is the identity operator on \( \mathbb{P}_{k-2} \), and we can take \( q = \Delta^\dagger p \). The lemma follows from the observation that both \( \|p\|_{L^2(B)} \) and \( \|\Delta p\|_{L^2(B)} \) are norms on \( \mathbb{P}_{k-2} \) together with a standard inverse estimate [10, 9].

3. Local Virtual Element Spaces in Two Dimensions

In this section we obtain properties of the local virtual element spaces that will be used in the stability and error analyses in Section 4.

Let the space \( \mathbb{P}_k(D) \) be the restriction of \( \mathbb{P}_k \) to \( D \) and the space of \( \mathbb{P}_k(\partial D) \) be defined by

\[
\mathbb{P}_k(\partial D) = \{v \in C(\partial D) : v|_e \in \mathbb{P}_k(e) \text{ for all } e \in \mathcal{E}_D\},
\]

where \( C(\partial D) \) is the space of continuous functions on \( \partial D \), \( \mathbb{P}_k(e) \) is the restriction of \( \mathbb{P}_k \) to the edges \( e \), and \( \mathcal{E}_D \) is the set of the edges of \( D \). The length of an edge \( e \) is denoted by \( h_e \).

3.1. The Projection \( \Pi_{k,D}^\nabla \) and the Space \( \mathcal{Q}_k(D) \). The Sobolev space \( H^1(D) \) is a Hilbert space under the inner product

\[
\langle \zeta, \eta \rangle = (\nabla \zeta, \nabla \eta) + \left( \int_{\partial D} \zeta \eta ds \right) \left( \int_{\partial D} \zeta \eta ds \right).
\]

The projection operator from \( H^1(D) \) onto \( \mathbb{P}_k(D) \) with respect to \( \langle \cdot, \cdot \rangle \) is denoted by \( \Pi_{k,D}^\nabla \), i.e., \( \Pi_{k,D}^\nabla \zeta \in \mathbb{P}_k(D) \) satisfies

\[
\langle \Pi_{k,D}^\nabla \zeta, q \rangle = \langle \zeta, q \rangle \quad \forall q \in \mathbb{P}_k(D).
\]

In particular we have

\[
\Pi_{k,D}^\nabla p = p \quad \forall p \in \mathbb{P}_k(D).
\]
It is straightforward to check that (3.2) is equivalent to
\[
\int_D \nabla (\Pi_{k,D}^v \zeta) \cdot \nabla q \, dx = \int_D \nabla \zeta \cdot \nabla q \, dx \\
= \int_{\partial D} \zeta (n \cdot \nabla q) \, ds - \int_D \zeta (\Delta q) \, dx \quad \forall q \in \mathbb{P}_k(D),
\]
and
\[
\int_{\partial D} \Pi_{k,D}^v \zeta \, ds = \int_{\partial D} \zeta \, ds.
\]

For \( k \geq 1 \), the virtual element space \( \mathbb{Q}^k(D) \subset H^1(D) \) is defined by the following conditions:

\[
v \in H^1(D) \ \text{belongs to} \ \mathbb{Q}^k(D) \ \text{if and only if} \ \text{(i) the trace of} \ v \ \text{on} \ \partial D \ \text{belongs to} \ \mathbb{P}_k(\partial D), \ \text{(ii) the distribution} \ -\Delta v \ \text{belongs to} \ \mathbb{P}_k(D), \ \text{and (iii) we have} \nabla
\]

\[
\Pi_{k,D}^0 v - \Pi_{k,D} v \in \mathbb{P}_{k-2}(D),
\]

where \( \Pi_{k,D}^0 \) is the projection from \( L^2(D) \) onto \( \mathbb{P}_k(D) \) and \( \mathbb{P}_{-1}(D) = \{0\} \).

**Remark 3.1.** It follows from elliptic regularity for bounded Lipschitz domains [15, Section 1.2] and conditions (i) and (ii) in the definition of \( \mathbb{Q}^k(D) \) that \( \mathbb{Q}^k(D) \subset C(\bar{D}) \).

**Remark 3.2.** The dimension of \( \mathbb{Q}^k(D) \) is the sum of the dimension of \( \mathbb{P}_k(\partial D) \) and the dimension of \( \mathbb{P}_{k-1}(D) \) (cf. [2]). The degrees of freedom consist of (i) the values of \( v \) at the vertices of \( D \) and at the points in the interior of the edges that together determine \( \mathbb{P}_k(\partial D) \), and (ii) the moments of \( \Pi_{k,D}^0 v \). The set of the nodes in (i) will be denoted by \( \mathcal{N}_{\partial D} \).

**Remark 3.3.** It follows from (3.4) and (3.5) that the polynomial \( \Pi_{k,D}^v v \) can be computed in terms of the degrees of freedom of \( v \in \mathbb{Q}^k(D) \). Moreover, the polynomial \( \Pi_{k,D}^0 v \) can also be computed through (3.6).

### 3.2. A Minimum Energy Principle.

The following minimum energy principle is useful for bounding the \( H^1 \) norm of a virtual element function.

**Lemma 3.4.** The inequality
\[
|v|_{H^1(D)} \leq |\zeta|_{H^1(D)}
\]
holds for any \( v \in \mathbb{Q}^k(D) \) and \( \zeta \in H^1(D) \) such that \( \zeta - v = 0 \) on \( \partial D \) and \( \Pi_{k,D}^0 (\zeta - v) = 0 \).

**Proof.** It follows from condition (ii) in the definition of \( \mathbb{Q}^k(D) \) that
\[
\int_D \nabla v \cdot \nabla (\zeta - v) \, dx = \int_D (-\Delta v)(\zeta - v) \, dx = 0
\]
and hence \( |\zeta|_{H^1(D)}^2 = |\zeta - v|_{H^1(D)}^2 + |v|_{H^1(D)}^2 \). \( \square \)

### 3.3. A Maximum Principle.

We begin with a result from [4, Lemma 3.3].

**Lemma 3.5.** There exists a positive constant \( C \), depending only on \( \rho_D \) and \( k \), such that
\[
\|
\Delta v\|_{L^2(D)} \leq C h_D^{-1} \|
\nabla v\|_{L^2(D)} \quad \forall v \in \mathbb{Q}^k(D).
\]
Proof. By scaling we may assume \( h_D = 1 \).

Let \( \phi \geq 0 \) be a smooth (bump) function supported on the disc \( \mathfrak{B}_D \) with radius \( \rho_D \) such that

\[
\int_D \phi \, dx = 1, \quad |\phi| \lesssim 1 \quad \text{and} \quad |\nabla \phi(x)| \lesssim 1.
\]

We have, by the equivalence of norms on finite dimensional vector spaces, scaling and (2.2),

\[
\|p\|^2_{L^2(D)} \lesssim \|p\|_{L^2(\mathfrak{B}_D)}^2 \lesssim \int_{\mathfrak{B}_D} p^2 \phi \, dx \quad \forall p \in \mathbb{P}_k.
\]

Since \( \Delta v \in \mathbb{P}_k \), it follows from (2.22) (with \( h_D = 1 \)), (3.9) and integration by parts that

\[
\|\Delta v\|^2_{L^2(D)} \lesssim \int_{\mathfrak{B}_D} (\Delta v)^2 \phi \, dx
\]

\[
= - \int_{\mathfrak{B}_D} \nabla v \cdot (\phi \nabla (\Delta v) + (\Delta v) \nabla \phi) \, dx
\]

\[
\lesssim \|\nabla v\|_{L^2(D)} \left( \|\nabla (\Delta v)\|_{L^2(D)} + \|\Delta v\|_{L^2(D)} \right) \lesssim \|\nabla v\|_{L^2(D)} \left( \|\Delta v\|_{L^2(D)} \right),
\]

which implies (3.7) (with \( h_D = 1 \)).

The following maximum principle will be used in the analysis of the interpolation operator in Section 3.8 (cf. Lemma 3.21), and in the stability and error analyses for virtual element methods in three dimensions (cf. (5.14) and Lemma 5.7).

Lemma 3.6. There exists a positive constant \( C \), depending only on \( \rho_D \) and \( k \), such that

\[
\|v\|_{L^\infty(D)} \leq C \left[ \|v\|_{L^\infty(\partial D)} + |v|_{H^1(D)} \right] \quad \forall v \in \mathcal{Q}_k(D).
\]

Proof. There exists \( q \in \mathbb{P}_{k+2} \) such that

\[
\Delta q = \Delta v \quad \text{and} \quad \|\nabla q\|_{L^2(\mathfrak{B}_D)} \lesssim h_D \|\Delta v\|_{L^2(\mathfrak{B}_D)}
\]

by Lemma 2.4 (with \( p = \Delta v \in \mathbb{P}_k \)).

Without loss of generality we may assume that the mean of \( q \) over \( D \) is zero. Therefore we have

\[
\|q\|_{L^\infty(D)} \lesssim \|q\|_{H^1(D)} \lesssim \|q\|_{H^1(\mathfrak{B}_D)} \lesssim \|q\|_{H^1(\mathfrak{B}_D)} \lesssim h_D \|\Delta v\|_{L^2(\mathfrak{B}_D)} \lesssim |v|_{H^1(D)}
\]

by (2.2), (2.24), Lemma 3.5, (3.10) and scaling.

It then follows from the maximum principle for the harmonic function \( v - q \) (cf. [13]) that

\[
\|v\|_{L^\infty(D)} \leq \|v-q\|_{L^\infty(D)} + \|q\|_{L^\infty(D)}
\]

\[
\leq \|v-q\|_{L^\infty(\partial D)} + \|q\|_{L^\infty(D)} \lesssim \|v\|_{L^\infty(\partial D)} + |v|_{H^1(D)}.
\]

\[\square\]
3.4. The Semi-norm $\| \cdot \|_{k,D}$.

The semi-norm $\| \cdot \|_{k,D}$ on $H^1(D)$ is defined by

$$
\| \zeta \|_{k,D}^2 = \| \Pi_{k-2,D}^0 \zeta \|_{L_2(D)}^2 + h_D \sum_{e \in E_D} \| \Pi_{k-1,e}^0 \zeta \|_{L_2(e)}^2,
$$

where $\Pi_{k-1,e}^0$ is the orthogonal projection from $L_2(e)$ onto $P_{k-1}(e)$.

It follows from (2.18) and (3.11) that

$$
\| \zeta \|_{k,D} \leq C \left( \| \zeta \|_{L_2(D)} + h_D | \zeta |_{H^1(D)} \right) \quad \forall \zeta \in H^1(D),
$$

where the positive constant $C$ depends only on $\rho_D$ and $k$.

We also have, by (2.8) and a standard estimate for polynomials in one variable,

$$
\| v \|_{k,D} \lesssim h_D \| v \|_{L_\infty(\partial D)} + \| \Pi_{k-2,D}^0 v \|_{L_2(D)}
$$

$$
\lesssim h_D \left( \sum_{p \in N_0_D} v^2(p) \right)^{\frac{1}{2}} + \| \Pi_{k-2,D}^0 v \|_{L_2(D)} \quad \forall v \in Q^k(D),
$$

where the hidden constant depends only on $\rho_D$ and $k$.

3.5. Estimates for $\Pi_{k,D}^\nu$. All the hidden constants in this subsection depend only on $\rho_D$ and $k$. Besides the obvious stability estimate

$$
| \Pi_{k,D}^\nu \zeta |_{H^1(D)} \leq | \zeta |_{H^1(D)} \quad \forall \zeta \in H^1(D)
$$

that follows from (3.4), we also have a stability estimate for $\Pi_{k,D}^\nu$ in terms of $\| \cdot \|_{L_2(D)}$ and the semi-norm $\| \cdot \|_{k,D}$.

Lemma 3.7. We have

$$
\| \Pi_{k,D}^\nu \zeta \|_{L_2(D)} \lesssim \| \zeta \|_{k,D} \quad \forall \zeta \in H^1(D).
$$

Proof. It follows from (3.4) that

$$
\int_D \nabla (\Pi_{k,D}^\nu \zeta) \cdot \nabla (\Pi_{k,D}^\nu \zeta) \, dx = \int_{\partial D} \zeta n \cdot \nabla (\Pi_{k,D}^\nu \zeta) \, ds - \int_D \zeta \Delta (\Pi_{k,D}^\nu \zeta) \, dx
$$

$$
\leq \left( \sum_{e \in E_D} \| \Pi_{k-1,e}^0 \zeta \|_{L_2(e)}^2 \right)^{\frac{1}{2}} \left( \sum_{e \in E_D} \| \nabla \Pi_{k,D}^\nu \zeta \|_{L_2(e)}^2 \right)^{\frac{1}{2}} + \| \Pi_{k-2,D}^\nu \zeta \|_{L_2(D)} \| \Delta (\Pi_{k,D}^\nu \zeta) \|_{L_2(D)},
$$

and we have, by (2.21) and (2.22),

$$
\sum_{e \in E_D} \| \nabla \Pi_{k,D}^\nu \zeta \|_{L_2(e)}^2 \lesssim h_D^{-1} \| \nabla \Pi_{k,D}^\nu \zeta \|_{L_2(D)}^2 \quad \text{and} \quad \| \Delta (\Pi_{k,D}^\nu \zeta) \|_{L_2(D)} \lesssim h_D^{-1} \| \nabla \Pi_{k,D}^\nu \zeta \|_{L_2(D)}.
$$

It follows that

$$
\| \nabla \Pi_{k,D}^\nu \zeta \|_{L_2(D)} \leq h_D^{-1} \left( h_D \sum_{e \in E_D} \| \Pi_{k-1,e}^0 \zeta \|_{L_2(e)}^2 + \| \Pi_{k-2,D}^\nu \zeta \|_{L_2(D)} \right)^{\frac{1}{2}} = h_D^{-1} \| \zeta \|_{k,D}.
$$

Moreover (2.8), (3.5) and (3.11), imply

$$
\left| \int_{\partial D} \Pi_{k,D}^\nu \zeta \, ds \right| = \left| \sum_{e \in E_D} \int_e \Pi_{0,e}^0 \zeta \, ds \right| \leq \sum_{e \in E_D} \sqrt{h_e} \| \Pi_{k-1,e}^0 \zeta \|_{L_2(e)}.$$
Lemma 3.8. We have

\[ \| \Pi_{k,D}^\nu \|_{L^2(D)} \leq \sqrt{h_D \left( \sum_{e \in \Xi_D} \| \Pi_{k-1,e}^\nu \|_{L^2(e)}^2 \right)^{\frac{1}{2}}} \leq \| \zeta \|_{k,D}. \]

Finally we have, by (2.15),

\[ \| \Pi_{k,D}^\nu \zeta \|_{L^2(D)} \leq \left| \int_{\partial D} \Pi_{k,D}^\nu \zeta \, ds \right| + h_D \| \nabla \Pi_{k,D}^\nu \zeta \|_{L^2(D)} \lesssim \| \zeta \|_{k,D}. \]

We can now establish error estimates for \( \Pi_{k,D}^\nu \).

Lemma 3.8. We have

\[ \| \zeta - \Pi_{k,D}^\nu \zeta \|_{L^2(D)} \lesssim h_D^{\ell+1} | \zeta |_{H^{\ell+1}(D)} \quad \forall \zeta \in H^{\ell+1}(D), \ 0 \leq \ell \leq k, \]

(3.16) \[ | \zeta - \Pi_{k,D}^\nu \zeta |_{H^1(D)} \lesssim h_D^{\ell} | \zeta |_{H^{\ell+1}(D)} \quad \forall \zeta \in H^{\ell+1}(D), \ 1 \leq \ell \leq k, \]

(3.17) \[ | \zeta - \Pi_{k,D}^\nu \zeta |_{H^2(D)} \lesssim h_D^{\ell-1} | \zeta |_{H^{\ell+1}(D)} \quad \forall \zeta \in H^{\ell+1}(D), \ 1 \leq \ell \leq k. \]

Proof. The estimate (3.16) follows immediately from (2.5), (3.3) and (3.14).

In view of (2.22) and (3.14), we have

\[ | \Pi_{k,D}^\nu \zeta |_{H^2(D)} \lesssim h_D^{-1} | \Pi_{k,D}^\nu \zeta |_{H^1(D)} \leq h_D^{-1} | \zeta |_{H^1(D)}, \]

which together with (2.5) and (3.3) implies (3.17).

Similarly we have, by (3.12) and Lemma 3.7

\[ | \Pi_{k,D}^\nu \zeta |_{L^2(D)} \lesssim \| \zeta \|_{k,D} \lesssim \| \zeta \|_{L^2(D)} + h_D \| \zeta \|_{H^1(D)}, \]

which together with (2.5) and (3.3) implies (3.15). \( \square \)

3.6. Estimates for \( \Pi_{k,D}^0 \). All the hidden constants in this subsection only depend on \( \rho_D \) and \( k \). We have an obvious stability estimate

(3.18) \[ | \Pi_{k,D}^0 \zeta |_{L^2(D)} \leq \| \zeta \|_{L^2(D)} \quad \forall \zeta \in L^2(D) \]

and an obvious relation

(3.19) \[ \Pi_{k,D}^0 q = q \quad \forall q \in \mathbb{P}_k(D). \]

It follows from (2.5), (3.18) and (3.19) that

(3.20) \[ | \zeta - \Pi_{k,D}^0 \zeta |_{L^2(D)} \lesssim h_D^{\ell+1} | \zeta |_{H^{\ell+1}(D)} \quad \forall \zeta \in H^{\ell+1}(D), \ 0 \leq \ell \leq k. \]

We also have a stability estimate for \( \Pi_{k,D}^0 \) in \( | \cdot |_{H^1(D)} \).

Lemma 3.9. We have

(3.21) \[ | \Pi_{k,D}^0 \zeta |_{H^1(D)} \lesssim | \zeta |_{H^1(D)} \quad \forall \zeta \in H^1(D). \]

Proof. This is a consequence of (2.22), (3.14), (3.15) and (3.20):

\[ | \Pi_{k,D}^0 \zeta |_{H^1(D)} \leq | \Pi_{k,D}^0 \zeta - \Pi_{k,D}^\nu \zeta |_{H^1(D)} + | \Pi_{k,D}^\nu \zeta |_{H^1(D)} \]
\[ \lesssim h_D^{-1} | \Pi_{k,D}^0 \zeta - \Pi_{k,D}^\nu \zeta |_{L^2(D)} + | \zeta |_{H^1(D)} \]
\[ \lesssim h_D^{-1} ( | \Pi_{k,D}^0 \zeta |_{L^2(D)} + | \zeta - \Pi_{k,D}^\nu \zeta |_{L^2(D)} ) + | \zeta |_{H^1(D)} \lesssim | \zeta |_{H^1(D)}. \]

\( \square \)
We can then derive error estimates for $\Pi_{k,D}^v$ by combining the Bramble-Hilbert estimates and Lemma 3.9.

**Lemma 3.10.** We have

\begin{align}
|\zeta - \Pi_{k,D}^0 \zeta|_{H^1(D)} &\lesssim h^{-1}_D |\zeta|_{H^{1+1}(D)} & \forall \zeta \in H^{\ell+1}(D), 1 \leq \ell \leq k; \\
|\zeta - \Pi_{k,D}^0 \zeta|_{H^1(D)} &\lesssim h^{-1}_D |\zeta|_{H^{1+1}(D)} & \forall \zeta \in H^{\ell+1}(D), 1 \leq \ell \leq k.
\end{align}

**Proof.** In view of (3.19), the estimate (3.22) follows from (2.5) and (3.21).

Similarly the estimate (3.23) follows from (2.5), (3.19) and the inequality

\[ |\Pi_{k,D}^0 \zeta|_{H^2(D)} \lesssim h^{-1}_D |\Pi_{k,D}^0 \zeta|_{H^1(D)} \lesssim h^{-1}_D |\zeta|_{H^1(D)} \]

obtained from (2.22) and (3.21). \hfill \Box

The following is another useful estimate.

**Lemma 3.11.** We have

\[ \| \Pi_{k,D}^0 v \|_{L^2(D)} \lesssim C \| v \|_{k,D} \quad \forall v \in Q^k(D). \]

**Proof.** Let $v \in Q^k(D)$ be arbitrary. It follows from (3.6) that

\[ \| \Pi_{k,D}^0 v \|_{L^2(D)}^2 = \| \Pi_{k-2,D}^0 v \|_{L^2(D)}^2 + \| (\Pi_{k,D}^0 - \Pi_{k-2,D}^0) v \|_{L^2(D)}^2 \\
= \| \Pi_{k-2,D}^0 v \|_{L^2(D)}^2 + \| (\Pi_{k,D}^0 - \Pi_{k-2,D}^0) \Pi_{k,D}^0 v \|_{L^2(D)}^2 \\
\leq \| \Pi_{k-2,D}^0 v \|_{L^2(D)}^2 + \| \Pi_{k,D}^0 v \|_{L^2(D)}^2, \]

which together with (3.11) and Lemma 3.7 completes the proof. \hfill \Box

### 3.7. Inverse Estimates.

These are estimates that bound the norm $|v|_{H^1(D)}$ of a virtual element function $v \in Q^k(D)$ in terms of $\| \Pi_{k-2,D}^0 v \|_{L^2(D)}$ and norms that only involve the boundary data of $v$. They are crucial for the stability analysis of virtual element methods in Section 4.2.

We begin with a key lemma.

**Lemma 3.12.** There exists a positive constant $C$ depending only on $\rho_D$ and $k$, such that

\[ |v|_{H^1(D)} \leq C [h^{-1}_D \| v \|_{k,D} + |v|_{H^{1/2}(\partial D)}]. \]

**Proof.** By scaling we may assume $h_D = 1$.

Let $\text{Tr}^\dagger$ be the lifting operator from Section 2.7. The function $w = \text{Tr}^\dagger v \in H^1(D)$ satisfies $w = v$ on $\partial D$ and

\[ |w|_{H^1(D)} \lesssim |v|_{H^{1/2}(D)}. \]

Let $\phi$ be the same (bump) function in the proof of Lemma 3.6 and $\zeta = w + p\phi$, where the polynomial $p \in \mathbb{P}_k(D)$ is determined by

\[ \int_D (\zeta - v)q \, dx = 0 \quad \forall q \in \mathbb{P}_k(D), \]

or equivalently

\[ \int_D pq\phi \, dx = \int_D (v - w)q \, dx = \int_D (\Pi_{k,D}^0 v - w)q \, dx \quad \forall q \in \mathbb{P}_k(D). \]
Then we have
\[(3.27) \quad |v|_{H^1(D)} \leq |\zeta|_{H^1(D)} \]
by Lemma 3.4 and, in view of (2.22), (3.8) and (3.25),
\[(3.28) \quad |\zeta|_{H^1(D)} \leq |w|_{H^1(D)} + |p\phi|_{H^1(D)} \lesssim |w|_{H^1(D)} + \|P\|_{L_2(D)} \lesssim |v|_{H^{1/2}(\partial D)} + \|P\|_{L_2(D)}. \]
Note that (3.9) and (3.26) imply
\[(3.29) \quad \|P\|_{L_2(D)} \lesssim \|\Pi^0_{k,D} v - w\|_{L_2(D)} \]
and hence
\[(3.30) \quad \|P\|_{L_2(D)} \lesssim \|\Pi^0_{k,D} v\|_{L_2(D)} + \|w\|_{L_2(D)} \lesssim \|v\|_{k,D} + |v|_{H^{1/2}(\partial D)} \]
by Lemma 3.11 and (3.25).

The estimate (3.24) (with \(h_D = 1\)) follows from (3.27)–(3.29). \(\Box\)

**Lemma 3.13.** There exists a positive constant \(C\), depending only \(\rho_D\) and \(k\), such that
\[(3.30) \quad |v|_{H^1(D)} \leq C \left[ h_D^{-1} \|v\|_{k,D} + h_D^{1/2} \|\partial v/\partial s\|_{L_2(\partial D)} \right] \quad \forall v \in Q^k(D), \]
where \(\partial v/\partial s\) is a tangential derivative of \(v\).

**Proof.** The estimate (3.30) follows immediately from (2.16) and Lemma 3.12 . \(\Box\)

**Lemma 3.14.** There exists a positive constant \(C\), depending only on \(\rho_D\), \(|E_D|\) and \(k\), such that
\[(3.31) \quad |v|_{H^1(D)} \leq C \left[ h_D^{-1} \|v\|_{k,D} + \sqrt{\ln(1 + \tau_D)} \|v\|_{L_\infty(\partial D)} \right] \quad \forall v \in Q^k(D), \]
where
\[(3.32) \quad \tau_D = \max_{e \in E_D} \frac{h_e}{\min_{e \in E_D} h_e}. \]

**Proof.** According to [4, Lemma 5.1], we have
\[(3.33) \quad |v|_{H^{1/2}(\partial D)} \leq C \sqrt{\ln(1 + \tau_D)} \|v\|_{L_\infty(\partial D)} \quad \forall v \in Q^k(D), \]
where the positive constant \(C\) only depends on \(\rho_D\), \(|E_D|\) and \(k\).

The estimate (3.31) follows from Lemma 3.12 and (3.33). \(\Box\)

Combining (3.13) and (3.31), we have the following corollary.

**Corollary 3.15.** There exists a positive constant \(C\), depending only on \(\rho_D\), \(|E_D|\) and \(k\), such that
\[(3.34) \quad |v|_{H^1(D)} \leq C \left[ h_D^{-1} \|\Pi^0_{k-2,D} v\|_{L_2(D)} + \sqrt{\ln(1 + \tau_D)} \|v\|_{L_\infty(\partial D)} \right] \quad \forall v \in Q^k(D). \]
3.8. The Interpolation Operator. For \( s > 1 \) the interpolation operator \( I_{k,D} : H^s(D) \rightarrow Q^k(D) \) is defined by the condition that \( \zeta \) and \( I_{k,D}\zeta \) share the same degrees of freedom, i.e., \( I_{k,D}\zeta \) agrees with \( \zeta \) at the nodes in \( \mathcal{N}_{\partial D} \) and

\[
(3.34) \quad \Pi_{k-2,D}^0 I_{k,D}\zeta = \Pi_{k-2,D}^0 \zeta.
\]

It is clear that

\[
(3.35) \quad I_{k,D}q = q \quad \forall q \in \mathbb{P}_k(D),
\]

and by a standard estimate for polynomials in one variable,

\[
(3.36) \quad \|I_{k,D}\zeta\|_{L^\infty(\partial D)} \leq C \max_{p \in \mathcal{N}_{\partial D}} |\zeta(p)| \leq \|\zeta\|_{L^\infty(\partial D)} \quad \forall \zeta \in H^s(D) \text{ and } s > 1,
\]

where the positive constant \( C \) only depends on \( k \).

For the three dimensional Poisson problem, if the solution belongs to \( H^{\ell+1}(\Omega) \), then its restriction to a face \( F \) of a polyhedral subdomain belongs to \( H^{\ell+\frac{1}{2}}(F) \). Therefore below we also consider the interpolants of functions in \( H^{\ell+\frac{1}{2}}(D) \).

We begin with several stability estimates for the interpolation operator.

**Lemma 3.16.** There exists a positive constant \( C \), depending only on \( \rho_D \) and \( k \), such that

\[
(3.37) \quad \|I_{k,D}\zeta\|_{k,D} \leq C \left[ \|\zeta\|_{L^2(D)} + h_D \|\zeta\|_{H^1(D)} + h_D^2 \|\zeta\|_{H^2(D)} \right] \quad \forall \zeta \in H^2(D),
\]

\[
(3.38) \quad \|I_{k,D}\zeta\|_{k,D} \leq C \left[ \|\zeta\|_{L^2(D)} + h_D \|\zeta\|_{H^1(D)} + h_D^{3/2} \|\zeta\|_{H^{3/2}(D)} \right] \quad \forall \zeta \in H^{3/2}(D).
\]

**Proof.** Let \( \zeta \in H^2(D) \) (resp., \( H^{3/2}(D) \)) be arbitrary. From (2.8), (3.13), (3.34) and (3.36), we have

\[
\|I_{k,D}\zeta\|_{k,D} \lesssim h_D \|I_{k,D}\zeta\|_{L^\infty(\partial D)} + \|\Pi_{k-2,D}^0 I_{k,D}\zeta\|_{L^2(D)} \\
\lesssim h_D \|\zeta\|_{L^\infty(\partial D)} + \|\Pi_{k-2,D}^0 \zeta\|_{L^2(D)} \lesssim h_D \|\zeta\|_{L^\infty(D)},
\]

which together with (2.3) (resp., (2.4)) implies (3.37) (resp., (3.38)). \( \square \)

**Lemma 3.17.** We have

\[
(3.39) \quad |I_{k,D}\zeta|_{H^1(D)} \lesssim |\zeta|_{H^1(D)} + h_D |\zeta|_{H^2(D)}
\]

for all \( \zeta \in H^2(D) \), and

\[
(3.40) \quad |I_{k,D}\zeta|_{H^1(D)} \lesssim |\zeta|_{H^1(D)} + h_D^{1/2} |\zeta|_{H^{3/2}(D)}
\]

for all \( \zeta \in H^{3/2}(D) \), where the hidden constants only depend on \( \rho_D \), \( |\mathcal{E}_D| \) and \( k \).

**Proof.** Let \( \zeta \in H^2(D) \) be arbitrary and \( \bar{\zeta}_D \) be the mean of \( \zeta \) over \( D \). Since \( I_{k,D}\bar{\zeta}_D = \bar{\zeta}_D \), it follows from (3.30) that

\[
|I_{k,D}\zeta|_{H^1(D)}^2 = |I_{k,D}(\zeta - \bar{\zeta}_D)|_{H^1(D)}^2 \\
\lesssim h_D^{-2} \|I_{k,D}(\zeta - \bar{\zeta}_D)\|_{k,D}^2 + h_D \|\partial(I_{k,D}\zeta) / \partial s\|_{L^2(\partial D)}^2.
\]
We have, by a standard interpolation estimate in one dimension and (2.18) (applied to the first order derivatives of $\zeta$),
\[
h_D \| \partial (I_{k,D} \zeta) / \partial s \|_{L^2(\partial D)}^2 \lesssim \sum_{e \in \mathcal{E}_D} h_D \| \partial \zeta / \partial s \|_{L^2(e)}^2
\]
\[
\lesssim \sum_{e \in \mathcal{E}_D} [\| \zeta \|_{H^1(D)}^2 + h_D^2 \| \zeta \|_{H^2(D)}^2] \leq \| \zeta \|_{H^1(D)}^2 + h_D^2 \| \zeta \|_{H^2(D)}^2.
\]

These two estimates together with (2.15) and (3.37) imply (3.39).

Similarly we obtain (3.40) by replacing (2.18) with the estimate in Lemma 2.1. \hfill \square

**Lemma 3.18.** We have
\begin{equation}
\| I_{k,D} \zeta \|_{L^2(D)} \lesssim \| \zeta \|_{L^2(D)} + h_D \| \zeta \|_{H^1(D)} + h_D^2 \| \zeta \|_{H^2(D)}
\end{equation}
for all $\zeta \in H^2(D)$, and
\begin{equation}
\| I_{k,D} \zeta \|_{L^2(D)} \lesssim \| \zeta \|_{L^2(D)} + h_D\| \zeta \|_{H^1(D)} + h_D^{3/2} \| \zeta \|_{H^{3/2}(D)}
\end{equation}
for all $\zeta \in H^{3/2}(D)$, where the hidden constants only depend on $\rho_D$, $|\mathcal{E}_D|$ and $k$.

**Proof.** From Lemma 3.7 and (3.15) we have
\begin{equation}
\| I_{k,D} \zeta \|_{L^2(D)} \lesssim \| I_{k,D} \zeta - \Pi_{k,D}^\Sigma I_{k,D} \zeta \|_{L^2(D)} + \| \Pi_{k,D}^\Sigma I_{k,D} \zeta \|_{L^2(D)}
\end{equation}
The estimates (3.41) and (3.42) follow from (3.43), Lemma 3.16 and Lemma 3.17. \hfill \square

We can now derive error estimates for the interpolation operator.

**Lemma 3.19.** We have, for $1 \leq \ell \leq k$,
\begin{align}
\| \zeta - I_{k,D} \zeta \|_{L^2(D)} + \| \zeta - \Pi_{k,D}^\Sigma I_{k,D} \zeta \|_{L^2(D)} & \lesssim h_D^{\ell+1} \| \zeta \|_{H^{\ell+1}(D)} \quad \forall \zeta \in H^{\ell+1}(D), \quad \tag{3.44} \\
\| \zeta - I_{k,D} \zeta \|_{H^{\ell}(D)} + \| \zeta - \Pi_{k,D}^\Sigma I_{k,D} \zeta \|_{H^{\ell}(D)} & \lesssim h_D^{\ell} \| \zeta \|_{H^{\ell+1}(D)} \quad \forall \zeta \in H^{\ell+1}(D), \quad \tag{3.45} \\
\| \zeta - \Pi_{k,D}^\Sigma I_{k,D} \zeta \|_{H^{\ell}(D)} & \lesssim h_D^{\ell-\frac{1}{2}} \| \zeta \|_{H^{\ell+\frac{1}{2}}(D)} \quad \forall \zeta \in H^{\ell+\frac{1}{2}}(D), \quad \tag{3.46}
\end{align}
and
\begin{align}
\| \zeta - I_{k,D} \zeta \|_{L^2(D)} & \lesssim h_D^{\ell+\frac{1}{2}} \| \zeta \|_{H^{\ell+\frac{1}{2}}(D)} \quad \forall \zeta \in H^{\ell+\frac{1}{2}}(D), \quad \tag{3.47} \\
\| \zeta - I_{k,D} \zeta \|_{H^{\ell}(D)} & \lesssim h_D^{\ell-\frac{1}{2}} \| \zeta \|_{H^{\ell+\frac{1}{2}}(D)} \quad \forall \zeta \in H^{\ell+\frac{1}{2}}(D), \quad \tag{3.48}
\end{align}
where the hidden constants only depend on $\rho_D$, $|\mathcal{E}_D|$ and $k$.

**Proof.** In view of (3.13), Lemma 3.7, (3.37) and (3.41), we have, for any $\zeta \in H^2(D)$,
\[
\| I_{k,D} \zeta \|_{L^2(D)} + \| \Pi_{k,D}^\Sigma I_{k,D} \zeta \|_{L^2(D)} \lesssim \| I_{k,D} \zeta \|_{L^2(D)} + \| I_{k,D} \zeta \|_{H^2(D)}
\]
\[
\lesssim \| \zeta \|_{L^2(D)} + h_D \| \zeta \|_{H^1(D)} + h_D^2 \| \zeta \|_{H^2(D)},
\]
which together with (2.5), (3.3) and (3.35) implies (3.44).

From (3.14) and (3.39) we also have, for any $\zeta \in H^2(D)$,
\[
| I_{k,D} \zeta \|_{H^1(D)} \leq 2 | I_{k,D} \zeta \|_{H^1(D)} \lesssim h_D^{-1} \| \zeta \|_{L^2(D)} + \| \zeta \|_{H^1(D)} + h_D \| \zeta \|_{H^2(D)},
\]
Lemma 3.20. There exists a positive constant $C$, depending only on $\rho_D$, $|E_D|$ and $k$, such that
\[
\| I_{k,D}^0 \zeta - \Pi_{1,D}^0 I_{k,D} \zeta \|_{L^2(D)} \leq C h_D^2 \| \zeta \|_{H^2(D)} \quad \forall \zeta \in H^2(D).
\]

We also have interpolation error estimates in the $L_\infty$ norm.

Lemma 3.21. There exists a positive constant $C$, depending only on $\rho_D$, $N$ and $k$, such that
\begin{align}
| \Pi_{k,D}^\nu I_{k,D} \zeta - \Pi_{1,D}^\nu I_{k,D} \zeta |_{H^2(D)} &\leq C h_D^\ell \| \zeta \|_{H^{\ell+1}(D)} \quad \forall \zeta \in H^{\ell+1}(D) \text{ and } 1 \leq \ell \leq k, \quad (3.49) \\
| \Pi_{k,D}^\nu I_{k,D} \zeta - \Pi_{1,D}^\nu I_{k,D} \zeta |_{L^\infty(D)} &\leq C h_D^{\ell-\frac{1}{2}} \| \zeta \|_{H^{\ell+\frac{1}{2}}(D)} \quad \forall \zeta \in H^{\ell+\frac{1}{2}}(D) \text{ and } 1 \leq \ell \leq k. \quad (3.50)
\end{align}

Proof. It follows from (2.3), Lemma 3.6, (3.36) and (3.39) that, for any $\zeta \in H^2(D)$,
\[
\| I_{k,D} \zeta \|_{L^\infty(D)} \leq \| I_{k,D} \zeta \|_{L^\infty(\partial D)} + |I_{k,D} \zeta|_{H^1(D)} \leq h_D^{-1} \| \zeta \|_{L^2(D)} + \| \zeta \|_{H^1(D)} + h_D \| \zeta \|_{H^2(D)},
\]
which together with (2.5) and (3.35) imply (3.49).

The proof for (3.50) is similar, but with (2.3) (resp., (2.5) and (3.39)) replaced by (2.4) (resp., (2.6) and (3.40)). \qed

3.9. The Null Space of $\Pi_{k,D}^\nu$. Let $\mathcal{N}(\Pi_{k,D}^\nu) = \{ v \in \mathcal{Q}^k(D) : \Pi_{k,D}^\nu v = 0 \}$ be the null space of the projection $\Pi_{k,D}^\nu$. The inverse estimates in Section 3.7 can be simplified for functions in $\mathcal{N}(\Pi_{k,D}^\nu)$.

Lemma 3.22. There exists a positive constant $C$, depending only on $\rho_D$ and $k$, such that
\[
\| v \|_{k,D} \leq C h_D \sum_{e \in E_D} \| \Pi_{k-1,e}^0 v \|_{L^2(e)}^2 \quad \forall v \in \mathcal{N}(\Pi_{k,D}^\nu).
\]

Proof. We can assume $k \geq 2$ since $\Pi_{k-2,D} v = 0$ for $k = 1$.

Let $v \in \mathcal{N}(\Pi_{k,D}^\nu)$ be arbitrary. It follows from (3.4) that
\[
\int_D v(\Delta q) \, dx = \int_{\partial D} v(n \cdot \nabla q) \, ds \quad \forall q \in \mathbb{P}_k(D).
\]

According to (2.2) and Lemma 2.4, given any $p \in \mathbb{P}_{k-2}$, there exists $q \in \mathbb{P}_k$ such that $\Delta q = p$ and
\[
\| \nabla q \|_{L^2(D)} \leq \| \nabla q \|_{L^2(\mathbb{S}_D)} \leq h_D \| p \|_{L^2(\mathbb{S}_D)} \leq h_D \| p \|_{L^2(\mathbb{S}_D)} \leq h_D \| p \|_{L^2(D)}.
\]
It follows from (2.21), (3.52) and (3.53) that
\[
\int_D v p \, dx \leq \left( \sum_{e \in \mathcal{E}_D} \| \Pi_{k-1,e}^0 v \|_{L^2(e)}^2 \right)^{\frac{1}{2}} \left( \sum_{e \in \mathcal{E}_D} \| \nabla q \|_{L^2(e)}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \sum_{e \in \mathcal{E}_D} \| \Pi_{k-1,e}^0 v \|_{L^2(e)}^2 \right)^{\frac{1}{2}} h_D^{-1/2} \| \nabla q \|_{L^2(D)}
\]
\[
\lesssim \left( \sum_{e \in \mathcal{E}_D} \| \Pi_{k-1,e}^0 v \|_{L^2(e)}^2 \right)^{\frac{1}{2}} h_D^{1/2} \| p \|_{L^2(D)}
\]
and hence
\[
(3.54) \quad \| \Pi_{k-2,D}^0 v \|_{L^2(D)} = \max_{p \in \mathbb{P}_k} \int_D v p \, dx / \| p \|_{L^2(D)} \lesssim \left( h_D \sum_{e \in \mathcal{E}_D} \| \Pi_{k-1,e}^0 v \|_{L^2(e)}^2 \right)^{\frac{1}{2}}.
\]

The estimate (3.51) follows from (3.11) and (3.54). \qed

**Lemma 3.23.** For any \( v \in \mathbb{P}_k(\partial D) \) that vanishes at some point on \( \partial D \), we have
\[
\| v \|_{L^2(\partial D)} \leq C h_D \| \partial v / \partial s \|_{L^2(\partial D)},
\]
where \( \partial v / \partial s \) denotes a tangential derivative of \( v \) along \( \partial D \) and the positive constant \( C \) only depends on \( k \).

**Proof.** It follows from a Poincaré-Friedrichs inequality on the circle \( \partial \mathcal{B}_D \) that
\[
\| \zeta \|_{L^2(\partial \mathcal{B}_D)} \lesssim h_D \| \zeta \|_{H^1(\partial \mathcal{B}_D)}
\]
for any \( \zeta \in H^1(\partial \mathcal{B}_D) \) that vanishes at some point on \( \partial \mathcal{B}_D \). The lemma then follows from (2.9) and (2.11). \qed

Note that every \( v \in \mathcal{N}(\Pi_{k,D}^\nabla) \) must vanish at some point on \( \partial \Omega \) because \( \int_{\partial D} v \, ds = 0 \) by (3.5). Therefore, in view of Lemma 3.22 and Lemma 3.23, the inverse estimates (3.30) and (3.31) can be simplified to
\[
|v|_{H^1(D)} \lesssim h_D^{1/2} \| \partial v / \partial s \|_{L^2(\partial D)} \quad \forall \, v \in \mathcal{N}(\Pi_{k,D}^\nabla)
\]
with a hidden constant depending only on \( \rho_D \) and \( k \), and
\[
|v|_{H^1(D)} \lesssim \sqrt{\ln(1 + \tau_D)} \| v \|_{L^\infty(\partial D)} \quad \forall \, v \in \mathcal{N}(\Pi_{k,D}^\nabla)
\]
with a hidden constant that also depends on \( |\mathcal{E}_D| \).

Hence we have
\[
(3.55) \quad |v|_{H^1(D)}^2 = |\Pi_{k,D}^\nabla v|_{H^1(D)}^2 + |v - \Pi_{k,D}^\nabla v|_{H^1(D)}^2
\]
\[
\lesssim \| \Pi_{k,D}^\nabla v \|_{H^1(D)}^2 + h_D \| \partial(v - \Pi_{k,D}^\nabla v) / \partial s \|_{L^2(\partial D)}^2 \quad \forall \, v \in \mathcal{Q}_k(D),
\]
with a hidden constant depending only on \( \rho_D \) and \( k \), and also
\[
(3.56) \quad |v|_{H^1(D)}^2 \lesssim |\Pi_{k,D}^\nabla v|_{H^1(D)}^2 + \ln(1 + \tau_D) \| v - \Pi_{k,D}^\nabla v \|_{L^\infty(\partial D)}^2 \quad \forall \, v \in \mathcal{Q}_k(D),
\]
with a hidden constant that also depends on \( |\mathcal{E}_D| \).
4. The Poisson Problem in Two Dimensions

In this section we consider virtual element methods for the Poisson problem (1.1) in two dimensions and establish error estimates under two global shape regularity assumptions.

Let $\mathcal{T}_h$ be a triangulation of the convex polygon $\Omega \subset \mathbb{R}^2$ by polygonal subdomains, where $h = \max_{D \in \mathcal{T}_h} h_D$ is the mesh parameter. The global virtual finite element space $Q_h^k$ is given by

$$ Q_h^k = \{ v \in H^1_0(\Omega) : v|_D \in Q^k(D) \quad \forall D \in \mathcal{T}_h \}. $$

The space of (discontinuous) piecewise polynomials of degree $\leq k$ with respect to $\mathcal{T}_h$ is denoted by $P_h^k$.

The operators $\Pi_{k,h}^\nabla : H^1(\Omega) \to P_h^k$, $\Pi_{k,h}^0 : L^2(\Omega) \to P_h^k$ and $I_{k,h} : H^2(\Omega) \cap H^1_0(\Omega) \to Q_h^k$ are defined in terms of their local counterparts, i.e.,

$$ (\Pi_{k,h}^\nabla \zeta)|_D = \Pi_{k,D}^\nabla (\zeta|_D), \quad (\Pi_{k,h}^0 \zeta)|_D = \Pi_{k,D}^0 (\zeta|_D) \quad \text{and} \quad (I_{k,h} \zeta)|_D = I_{k,D} (\zeta|_D). $$

The piecewise $H^1$ norm with respect to $\mathcal{T}_h$ is given by

$$ |v|_{h,1} = \left( \sum_{D \in \mathcal{T}_h} |v|_{H^1(D)}^2 \right)^{\frac{1}{2}}. $$

4.1. Global Shape Regularity Assumptions. We assume that the local shape regularity assumption (2.1) is satisfied by all $D \in \mathcal{T}_h$ and impose the following global regularity assumptions.

Assumption 1 There exists a positive number $\rho \in (0,1)$, independent of $h$, such that

$$ \rho_D \geq \rho \quad \forall D \in \mathcal{T}_h. $$

Assumption 2 There exists a positive integer $N$, independent of $h$, such that

$$ |E_D| \leq N \quad \forall D \in \mathcal{T}_h. $$

The hidden constants in the rest of Section 4 will only depend on $\rho$, $N$ and $k$.

4.2. The Discrete Problem. Let the local stabilizing bilinear form $S_1^D(\cdot,\cdot)$ and $S_2^D(\cdot,\cdot)$ be defined by

$$ S_1^D(w,v) = \sum_{N \neq D} w(p) v(p), $$

$$ S_2^D(w,v) = h_D (\partial w/\partial s, \partial v/\partial s)_{L^2(\partial D)}, $$

where $\partial v/\partial s$ denotes a tangential derivative of $v$ along $\partial D$.

Remark 4.1. The local stability bilinear form $S_1^D(\cdot,\cdot)$ is the boundary part of the local stability bilinear form in [5]. The bilinear form $S_2^D(\cdot,\cdot)$ was introduced in [18].

Remark 4.2. We can also use the bilinear form $\tilde{S}_2^D(\cdot,\cdot)$ defined by

$$ \tilde{S}_2^D(w,v) = \sum_{e \in E_D} h_e (\partial w/\partial s, \partial v/\partial s)_{L^2(e)}. $$
Remark 4.3. By the equivalence of norms on finite dimensional vector spaces, we have
\[ \sum_{p \in \mathcal{N}_e} v^2(p) \approx \sum_{e \in \mathcal{E}_D} \|v\|_{L_\infty(e)}^2 \approx \|v\|_{L_\infty(\partial D)}^2 \quad \forall v \in \mathcal{P}_k(\partial D). \]

The discrete problem for (1.1) is to find \( u_h \in \mathcal{Q}_h^k \) such that
\[ a_h(u_h, v) = (f, \Xi_h v) \quad \forall v \in \mathcal{Q}_h^k, \]
where
\[ a_h(w, v) = \sum_{D \in \mathcal{T}_h} \left[ a_D(\Pi_{k,D}^v w, \Pi_{k,D}^v v) + S_D(w - \Pi_{k,D}^v w, v - \Pi_{k,D}^v v) \right], \]
\[ a_D(w, v) = \int_D \nabla w \cdot \nabla v \, dx, \]
\( S_D(\cdot, \cdot) \) is either \( S_P^1(\cdot, \cdot) \) or \( S_P^2(\cdot, \cdot) \), and \( \Xi_h : \mathcal{Q}_h^k \rightarrow \mathcal{P}_h^k \) is given by
\[ \Xi_h = \begin{cases} \Pi_{1,h}^0 & \text{if } k = 1, 2, \\ \Pi_{k-2,h}^0 & \text{if } k \geq 3. \end{cases} \]

It follows from (3.55) and (3.56) that
\[ |v|^2_{H^1(\Omega)} \lesssim \alpha_h a_h(v, v) \quad \forall v \in \mathcal{Q}_h^k, \]
where
\[ \alpha_h = \begin{cases} \ln(1 + \max_{D \in \mathcal{T}_h} \tau_D) & \text{if } S_D(\cdot, \cdot) = S_P^1(\cdot, \cdot), \\ 1 & \text{if } S_D(\cdot, \cdot) = S_P^2(\cdot, \cdot). \end{cases} \]
The well-posedness of the discrete problem follows from the stability estimate (4.10).

We will use the following properties of \( \Xi_h \) in the error analysis.

Lemma 4.4. We have, for \( 1 \leq \ell \leq k \),
\[ (f, w - \Xi_h w) \lesssim h^\ell |f|_{H^{\ell-1}(\Omega)} |w|_{H^1(\Omega)} \quad \forall f \in H^{\ell-1}(\Omega), \ w \in \mathcal{Q}_h^k, \]
\[ (f, I_{k,h} \zeta - \Xi_h I_{k,h} \zeta) \lesssim h^{\ell+1} |f|_{H^{\ell-1}(\Omega)} \|\zeta\|_{H^2(\Omega)} \quad \forall f \in H^{\ell-1}(\Omega), \ \zeta \in H^2(\Omega). \]

Proof. In view of the relation
\[ (f, w - \Xi_h w) = (f, \Pi_{k-2,h}^0 f, w - \Xi_h w) \leq \|f - \Pi_{k-2,h}^0 f\|_{L_2(\Omega)} \|w - \Xi_h w\|_{L_2(\Omega)} \leq \|f - \Pi_{k-2,h}^0 f\|_{L_2(\Omega)} \|w - \Pi_{0,h}^0 w\|_{L_2(\Omega)}, \]
the estimate (4.12) follows from (3.20).

Similarly we have
\[ (f, I_{k,h} \zeta - \Xi_h I_{k,h} \zeta) = (f, \Pi_{k-2,h}^0 f, I_{k,h} \zeta - \Xi_h I_{k,h} \zeta) \leq \|f - \Pi_{k-2,h}^0 f\|_{L_2(\Omega)} \|I_{k,h} \zeta - \Xi_h I_{k,h} \zeta\|_{L_2(\Omega)} \leq \|f - \Pi_{k-2,h}^0 f\|_{L_2(\Omega)} \|I_{k,h} \zeta - \Pi_{1,h}^0 I_{k,h} \zeta\|_{L_2(\Omega)}, \]
and the estimate (4.13) follows from (3.20) and Lemma 3.20. Note that this is the reason why \( \Xi_h \) is chosen to be \( \Pi_{1,h}^0 \) for \( k = 2 \) instead of \( \Pi_{k-2,h}^0 = \Pi_{0,h}^0 \). \( \square \)
4.3. An Abstract Error Estimate in the Energy Norm. Let \( \| \cdot \|_h = \sqrt{a_h(\cdot, \cdot)} \) be the mesh-dependent energy norm. Note that (4.10) implies
\[
|v|_{H^1(\Omega)} \lesssim \sqrt{a_h} \| v \|_h \quad \forall v \in \mathcal{Q}_h^k.
\]

The discrete problem (4.6) is defined in terms of a non-inherited symmetric positive definite bilinear form. We have a standard error estimate (cf. [9, Lemma 10.1.7] and [6])
\[
\| u - u_h \|_h \leq \inf_{v \in \mathcal{Q}_h^k} \| u - v \|_h + \sup_{v \in \mathcal{Q}_h^k} \frac{a_h(u, v) - (f, \Xi_h v)}{\| v \|_h}.
\]

The key is to control the numerator on the right-hand side of (4.15).

In view of (1.1), (1.2) and (3.4) we can write, for any \( v \in \mathcal{Q}_h^k \),
\[
a_h(u, v) = \sum_{D \in \mathcal{T}_h} \left[ a^D(\Pi_{k,D}^v u, \Pi_{k,D}^v v) + S^D(u - \Pi_{k,D}^v u, v - \Pi_{k,D}^v v) \right]
= \sum_{D \in \mathcal{T}_h} \left[ a^D(\Pi_{k,D}^v u, v) + S^D(u - \Pi_{k,D}^v u, v - \Pi_{k,D}^v v) \right]
= \sum_{D \in \mathcal{T}_h} \left[ a^D(\Pi_{k,D}^v u - u, v) + S^D(u - \Pi_{k,D}^v u, v - \Pi_{k,D}^v v) \right] + (f, v)
= \sum_{D \in \mathcal{T}_h} \left[ a^D(\Pi_{k,D}^v u - u, v - \Pi_{k,D}^v v) + S^D(u - \Pi_{k,D}^v u, v - \Pi_{k,D}^v v) \right] + (f, v),
\]
and hence, by (3.14), (4.7) and (4.14),
\[
a_h(u, v) - (f, \Xi_h v) = \sum_{D \in \mathcal{T}_h} \left[ a^D(\Pi_{k,D}^v u - u, v - \Pi_{k,D}^v v) + S^D(u - \Pi_{k,D}^v u, v - \Pi_{k,D}^v v) \right]
+ (f, v - \Xi_h v),
\]
\[
\lesssim \left( \sum_{D \in \mathcal{T}_h} |\Pi_{k,D}^v u - u|_{H^1(D)} |v - \Pi_{k,D}^v v|_{H^1(D)} \right) + \| u - \Pi_{k,h}^v u \|_h \| v \|_h
+ \left( \sup_{w \in \mathcal{Q}_h^k} \frac{(f, w - \Xi_h w)}{|w|_{H^1(\Omega)}} \right) \sqrt{a_h} \| v \|_h
\lesssim \| u - \Pi_{k,h}^v u \|_h \| v \|_h + \left( |u - \Pi_{k,h}^v u|_{h,1} + \sup_{w \in \mathcal{Q}_h^k} \frac{(f, w - \Xi_h w)}{|w|_{H^1(\Omega)}} \right) \sqrt{a_h} \| v \|_h.
\]

Putting (4.15) and (4.16) together we arrive at the estimate
\[
\| u - u_h \|_h \lesssim \| u - I_{k,h} u \|_h + \| u - \Pi_{k,h}^v u \|_h + \sqrt{a_h} \left( \| u - \Pi_{k,h}^v u|_{h,1} + \sup_{w \in \mathcal{Q}_h^k} \frac{(f, w - \Xi_h w)}{|w|_{H^1(\Omega)}} \right).
\]

Below we will derive concrete error estimates under the assumption that the solution \( u \) of (1.1) belongs to \( H^{\ell+1}(\Omega) \) for \( 1 \leq \ell \leq k \).
4.4. **Concrete Error Estimates in the Energy Norm.** The terms on the right-hand side of (4.17) are estimated as follows.

**Estimate for the Term Involving \( f \)**

Since \( u \in H^{\ell+1}(\Omega) \) and \( f = -\Delta u \), we have, by (4.12),

\[
(4.18) \quad \sup_{w \in \mathcal{Q}_h^k} \frac{(f, w - \Xi_h w)}{|w|_{H^1(\Omega)}} \lesssim h^\ell |u|_{H^{\ell+1}(\Omega)}.
\]

**Estimate for \( |u - \Pi_{k,h}^v u|_{h,1} \)**

It follows directly from (3.16) that

\[
|u - \Pi_{k,h}^v u|_{h,1} = \left( \sum_{D \in \mathcal{T}_h} |u - \Pi_{k,D}^v u|_{H^1(D)}^2 \right)^{\frac{1}{2}} \lesssim h^\ell |u|_{H^{\ell+1}(\Omega)}.
\]

**Estimate for \( \|u - I_{k,h} u\|_h \)**

We will establish the estimate

\[
(4.20) \quad \|u - I_{k,h} u\|_h \lesssim h^\ell |u|_{H^{\ell+1}(\Omega)}
\]

for both choices of \( S^D(\cdot, \cdot) \).

In the case where \( S^D(\cdot, \cdot) = S_2^D(\cdot, \cdot) \), it follows from (3.14), (4.4) and (4.7) that

\[
\|u - I_{k,h} u\|_h^2 \lesssim \sum_{D \in \mathcal{T}_h} |\Pi_{k,D}^v (u - I_{k,D} u)|_{H^1(D)}^2 + \sum_{D \in \mathcal{T}_h} \| (u - I_{k,D} u) - \Pi_{k,D}^v (u - I_{k,D} u) \|_{L^\infty(\partial D)}^2
\]

\[
(4.21) \lesssim \sum_{D \in \mathcal{T}_h} \left( |u - I_{k,D} u|_{H^1(D)}^2 + \| u - I_{k,D} u \|_{L^\infty(\partial D)}^2 \right)
+ \sum_{D \in \mathcal{T}_h} |\Pi_{k,D}^v (u - I_{k,D} u)|_{L^\infty(D)}^2,
\]

and we have

\[
\sum_{D \in \mathcal{T}_h} |\Pi_{k,D}^v (u - I_{k,D} u)|_{L^\infty(D)}^2 \lesssim \sum_{D \in \mathcal{T}_h} \left( \left( \frac{|\Pi_{k,D}^v (u - I_{k,D} u)|_{\partial D}}{|\Pi_{k,D}^v (u - I_{k,D} u)|_{H^1(D)}} \right)^2 + |\Pi_{k,D}^v (u - I_{k,D} u)|_{H^1(D)}^2 \right)
\]

\[
(4.22) \leq \sum_{D \in \mathcal{T}_h} \left( \left( \frac{|u - I_{k,D} u|_{\partial D}}{|u - I_{k,D} u|_{H^1(D)}} \right)^2 + |u - I_{k,D} u|_{H^1(D)}^2 \right)
\]

\[
\lesssim \sum_{D \in \mathcal{T}_h} \left( |u - I_{k,D} u|_{L^\infty(\partial D)}^2 + |u - I_{k,D} u|_{H^1(D)}^2 \right),
\]

by (2.23), (3.6) and (3.14).

The estimate (4.20) now follows from (3.45), (3.49), (4.21) and (4.22).

In the case where \( S^D(\cdot, \cdot) = S_2^D(\cdot, \cdot) \), it follows from (4.5) and (4.7) that

\[
\|u - I_{k,h} u\|_h^2 \lesssim \sum_{D \in \mathcal{T}_h} |\Pi_{k,D}^v (u - I_{k,D} u)|_{H^1(D)}^2 + \sum_{D \in \mathcal{T}_h} h_D \| \partial [\Pi_{k,D}^v (u - I_{k,D} u)] / \partial s \|_{L^2(\partial D)}^2
\]

\[
+ \sum_{D \in \mathcal{T}_h} h_D \| \partial (u - I_{k,D} u) / \partial s \|_{L^2(\partial D)}^2.
\]
We have
\[
\sum_{D \in T_h} |\Pi_{k,D}^v(u - I_{k,D}u)|_{H^1(D)}^2 \leq \sum_{D \in T_h} |u - I_{k,D}u|_{H^1(D)}^2 \lesssim h^{2\ell}|u|_{H^{\ell+1}(\Omega)}^2
\]
by (3.14) and (3.45), and hence, in view of (2.21) (applied to the first order derivatives of the polynomial \( \Pi_{k,D}^v(u - I_{k,D}u) \)),
\[
\sum_{D \in T_h} h_D \|\partial (\Pi_{k,D}^v(u - I_{k,D}u))/\partial s\|_{L_2(\partial D)}^2 \lesssim \sum_{D \in T_h} \|\Pi_{k,D}^v(u - I_{k,D}u)|_{H^1(D)}^2 \lesssim h^{2\ell}|u|_{H^{\ell+1}(\Omega)}^2.
\]
Finally it follows from a standard interpolation error estimate in one variable and (2.17) (applied to the \( \ell \)-th order derivatives of \( u \)) that
\[
\sum_{D \in T_h} h_D \|\partial (u - I_{k,D}u)/\partial s\|_{L_2(\partial D)}^2 \lesssim \sum_{D \in T_h} h_D \sum_{e \in \mathcal{E}_h} h_{e^{-2\ell-1}} |\partial^\ell u/\partial s|^2_{H^{1/2}(e)} \lesssim h^{2\ell}|u|_{H^{\ell+1}(\Omega)}^2.
\]
Together these estimates imply (4.20).

Estimate for \( \|u - \Pi_{k,h}^v u\|_h \)

We will show that the estimate
\[
(4.23) \quad \|u - \Pi_{k,h}^v u\|_h \lesssim h^{\ell}|u|_{H^{\ell+1}(\Omega)}
\]
holds for both choices of \( S^D(\cdot, \cdot) \).

In the case where \( S^D(\cdot, \cdot) = S_1^D(\cdot, \cdot) \), it follows from (2.3), Lemma 3.8, (4.4) and (4.7) that
\[
\|u - \Pi_{k,h}^v u\|_h^2 \lesssim \sum_{D \in T_h} \|u - \Pi_{k,h}^v u\|_{L_\infty(\partial D)}^2 \lesssim \sum_{D \in T_h} \left[ h_{D}^{-2}\|u - \Pi_{k,D}^v u\|_{L_2(D)}^2 + |u - \Pi_{k,D}^v u|_{H^1(D)}^2 + h_D^2\|u - \Pi_{k,D}^v u\|_{H^2(D)}^2 \right] \lesssim h^{2\ell}|u|_{H^{\ell+1}(\Omega)}^2.
\]
In the case where \( S^D(\cdot, \cdot) = S_2^D(\cdot, \cdot) \), we have
\[
\|u - \Pi_{k,h}^v u\|_h^2 = \sum_{D \in T_h} h_D \|\partial (u - \Pi_{k,D}^v u)/\partial s\|_{L_2(D)}^2 \lesssim \sum_{D \in T_h} \left[ |u - \Pi_{k,D}^v u|_{H^1(D)}^2 + h_D^2\|u - \Pi_{k,D}^v u\|_{H^2(D)}^2 \right] \lesssim h^{2\ell}|u|_{H^{\ell+1}(\Omega)}^2
\]
by (2.18) (applied to the first order derivatives of \( u - \Pi_{k,D}^v u \)) and Lemma 3.8.

Putting (4.17)–(4.20) and (4.23) together, we arrive at the following result.

**Theorem 4.5.** Assuming the solution \( u \) of (1.1) belongs to \( H^{\ell+1}(\Omega) \) for \( \ell \) between 1 and \( k \), we have
\[
\|u - u_h\|_h \leq C \sqrt{\alpha_h} h^{\ell}|u|_{H^{\ell+1}(\Omega)},
\]
where \( \alpha_h \) is defined in (4.11) and the positive constant \( C \) only depends on \( \rho \), \( k \) and \( N \).

We have similar estimates for the computable approximate solutions \( \Pi_{k,h}^v u_h \) and \( \Pi_{k,h}^0 u_h \).
Theorem 4.6. Assuming the solution \( u \) of (1.1) belongs to \( H^{\ell+1}(\Omega) \) for some \( \ell \) between 1 and 1, we have
\[
|u - u_h|_{H^{\ell+1}(\Omega)} + \sqrt{\alpha_h} [u - \Pi_{k,h}^v u]|_{h,1} + |u - \Pi_{k,h}^0 u|_{h,1} \leq C \alpha_h h^\ell |u|_{H^{\ell+1}(\Omega)},
\]
where \( \alpha_h \) is defined in (4.11) and the positive constant \( C \) only depends on \( \rho, N \) and \( k \).

Proof. In view of (4.7) and Theorem 4.5, we have
\[
|\Pi_{k,h}^v (u - u_h)|_{h,1} \leq \| u - u_h \|_h \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)},
\]
which together with (4.19) implies
\[
|u - \Pi_{k,h}^v u_h|_{h,1} \leq |u - \Pi_{k,h}^v u|_{h,1} + |\Pi_{k,h}^v (u - u_h)|_{h,1} \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}.
\]
It follows from this estimate and (3.21) that
\[
|u - \Pi_{k,h}^0 u_h|_{h,1} \leq |u - \Pi_{k,h}^v u|_{h,1} + |\Pi_{k,h}^v (u - u_h)|_{h,1} \lesssim |u - \Pi_{k,h}^v u|_{h,1} \lesssim \sqrt{\alpha_h} h^\ell |u|_{H^{\ell+1}(\Omega)}.
\]
Finally we have, by (3.45), (4.14), (4.20) and Theorem 4.5,
\[
|u - u_h|_{H^1(\Omega)} \leq |u - I_{k,h} u|_{H^1(\Omega)} + |I_{k,h} u - u_h|_{H^1(\Omega)} \\
\lesssim |u - I_{k,h} u|_{H^1(\Omega)} + \sqrt{\alpha_h} \| I_{k,h} u - u_h \|_h \\
\leq |u - I_{k,h} u|_{H^1(\Omega)} + \sqrt{\alpha_h} (\| u - I_{k,h} u \|_h + \| u - u_h \|_h) \lesssim \alpha_h h^\ell |u|_{H^{\ell+1}(\Omega)}.
\]
\[\square\]

Remark 4.7. In the case where \( S^D(\cdot, \cdot) = S_1^D(\cdot, \cdot) \), the estimates for \( |u - \Pi_{k,h}^v u_h|_{h,1} \) and \( |u - \Pi_{k,h}^0 u_h|_{h,1} \) are better than the estimate for \( |u - u_h|_{H^1(\Omega)} \).

4.5. Error Estimates in the \( L_2 \) Norm. We begin with two lemmas involving \( S^D(\cdot, \cdot) \).

Lemma 4.8. We have
\[
\sum_{D \in T_h} S_2^D (\zeta - \Pi_{k,D}^v I_{k,D} \zeta, \zeta - \Pi_{k,D}^v I_{k,D} \zeta) \lesssim h^2 |\zeta|_{H^2(\Omega)}^2, \quad \forall \zeta \in H^2(\Omega) \cap H_0^1(\Omega).
\]

Proof. It follows from (2.18) (applied to the first order partial derivatives of \( \zeta - \Pi_{k,D}^v I_{k,D} \zeta \)), Lemma 3.19 and (4.5) that
\[
\sum_{D \in T_h} S_2^D (\zeta - \Pi_{k,D}^v I_{k,D} \zeta, \zeta - \Pi_{k,D}^v I_{k,D} \zeta) = \sum_{D \in T_h} h_D \| \partial(\zeta - \Pi_{k,D}^v I_{k,D} \zeta) / \partial s \|_{L_2(\partial D)}^2 \\
\lesssim \sum_{D \in T_h} \| \zeta - \Pi_{k,D}^v I_{k,D} \zeta \|_{H^1(D)}^2 + h_D^2 \| \zeta - \Pi_{k,D}^v I_{k,D} \zeta \|_{H^2(D)}^2 \lesssim h_D^2 |\zeta|_{H^2(\Omega)}^2,
\]
and we have
\[
\sum_{D \in T_h} S_1^D (\zeta - \Pi_{k,D}^v I_{k,D} \zeta, \zeta - \Pi_{k,D}^v I_{k,D} \zeta) \lesssim \sum_{D \in T_h} \| \zeta - \Pi_{k,D}^v I_{k,D} \zeta \|_{L_\infty(\partial D)}^2 \\
\lesssim \sum_{D \in T_h} \| \zeta - \Pi_{k,D}^v I_{k,D} \zeta \|_{L_2(D)}^2 + \| \zeta - \Pi_{k,D}^v I_{k,D} \zeta \|_{H^1(D)}^2 + h_D^2 \| \zeta - \Pi_{k,D}^v I_{k,D} \zeta \|_{H^2(D)}^2 \\
\lesssim h_D^2 |\zeta|_{H^1(\Omega)}^2,
\]
by (2.3), Lemma 3.19 and (4.4). \[\square\]
Lemma 4.9. Assuming that $u \in H^{\ell+1}(\Omega)$ for some $\ell$ between 1 and $k$, we have
\begin{equation}
\sum_{D \in T_h} S^D(u_h - \Pi^\nu_{k,D} u_h, u_h - \Pi^\nu_{k,D} u_h) \lesssim \alpha_h^2 h^{2\ell+2}|u|_{H^{\ell+1}(\Omega)}^2.
\end{equation}

Proof. This is a consequence of (4.7), (4.23), Theorem 4.5 and Theorem 4.6:
\[\sum_{D \in T_h} S^D(u_h - \Pi^\nu_{k,D} u_h, u_h - \Pi^\nu_{k,D} u_h) = \|u_h - \Pi^\nu_{k,h} u_h\|_h^2\]
\[\lesssim \|u_h - u\|_h^2 + \|u - \Pi^\nu_{k,h} u\|_h^2 + \|\Pi^\nu_{k,h}(u - u_h)\|_h^2\]
\[= \|u_h - u\|_h^2 + \|u - \Pi^\nu_{k,h} u\|_h^2 + \|u - u_h\|_{H^1(\Omega)}^2 \lesssim \alpha_h^2 h^{2\ell+1}|u|_{H^{\ell+1}(\Omega)}^2.\]

We can now prove a consistency estimate.

Lemma 4.10. Assuming that $u \in H^{\ell+1}(\Omega)$ for some $\ell$ between 1 and $k$, we have
\[a(u - u_h, I_{k,h}\zeta) \leq C\alpha_h h^{\ell+1}|u|_{H^{\ell+1}(\Omega)}|\zeta|_{H^2(\Omega)} \quad \forall \zeta \in H^2(\Omega) \cap H^1_0(\Omega),\]
where $\alpha_h$ is defined in (4.11) and the positive constant $C$ only depends on $\rho$, $k$ and $N$.

Proof. We have, by (1.1), (1.2), (3.4) and (4.6)–(4.8),
\[a(u - u_h, I_{k,h}\zeta) = a(u, I_{k,h}\zeta) - \sum_{D \in T_h} a^D(u_h, I_{k,D}\zeta)\]
\[= (f, I_{k,h}\zeta) - \sum_{D \in T_h} a^D(u_h, \Pi^\nu_{k,D} I_{k,D}\zeta) - \sum_{D \in T_h} a^D(u_h, I_{k,D}\zeta - \Pi^\nu_{k,D} I_{k,D}\zeta)\]
\[= (f, I_{k,h}\zeta) - a_h(u_h, I_{k,D}\zeta) + \sum_{D \in T_h} S^D(u_h - \Pi^\nu_{k,D} u_h, I_{k,D}\zeta - \Pi^\nu_{k,D} I_{k,D}\zeta)\]
\[+ \sum_{D \in T_h} a^D(\Pi^\nu_{k,D} u_h - u_h, I_{k,D}\zeta - \Pi^\nu_{k,D} I_{k,D}\zeta)\]
\[= (f, I_{k,h}\zeta - \Xi_h I_{k,h}\zeta) + \sum_{D \in T_h} S^D(u_h - \Pi^\nu_{k,D} u_h, I_{k,D}\zeta - \Pi^\nu_{k,D} I_{k,D}\zeta)\]
\[+ \sum_{D \in T_h} a^D(\Pi^\nu_{k,D} u_h - u_h, I_{k,D}\zeta - \Pi^\nu_{k,D} I_{k,D}\zeta),\]
and the three terms on the right-hand side can be estimated as follows.
We have
\[|(f, I_{k,h}\zeta - \Xi_h I_{k,h}\zeta)| \lesssim h^{\ell+1}|u|_{H^{\ell+1}(\Omega)}|\zeta|_{H^2(\Omega)}\]
by (4.13),
\[\sum_{D \in T_h} S^D(u_h - \Pi^\nu_{k,D} u_h, I_{k,D}\zeta - \Pi^\nu_{k,D} I_{k,D}\zeta) \lesssim \alpha_h h^{\ell+1}|u|_{H^{\ell+1}(\Omega)}|\zeta|_{H^2(\Omega)}\]
by Lemma 4.8 and Lemma 4.9, and
\[\sum_{D \in T_h} a^D(\Pi^\nu_{k,D} u_h - u_h, I_{k,D}\zeta - \Pi^\nu_{k,D} I_{k,D}\zeta)\]
We have by Lemma 3.19, (4.19) and Theorem 4.6.

Theorem 4.11. Assuming \( u \in H^{\ell+1}(\Omega) \) for some \( \ell \) between 1 and \( k \), there exists a positive constant \( C \), depending only on \( \rho \), \( N \) and \( k \), such that

\[
\| u - u_h \|_{L_2(\Omega)} \leq C \alpha_h h^{\ell+1} |u|_{H^{\ell+1}(\Omega)},
\]

where \( \alpha_h \) is defined in (4.11).

Proof. Let \( \zeta \in H^1_0(\Omega) \) be defined by

\[
a(v, \zeta) = (v, u - u_h) \quad \forall v \in H^1_0(\Omega).
\]

We have

\[
\| u - u_h \|_{L_2(\Omega)}^2 = a(u - u_h, \zeta) = a(u - u_h, \zeta - I_{k,h} \zeta) + a(u - u_h, I_{k,h} \zeta),
\]

and since \( \Omega \) is convex,

\[
\| \zeta \|_{H^2(\Omega)} \leq C_\Omega \| u - u_h \|_{L_2(\Omega)}
\]

by elliptic regularity [14, 11].

The first term on the right-hand side of (4.28) satisfies

\[
a(u - u_h, \zeta - I_{k,h} \zeta) \leq |u - u_h|_{H^1(\Omega)} |\zeta - I_{k,h} \zeta|_{H^1(\Omega)} \leq h |u - u_h|_{H^1(\Omega)} |\zeta|_{H^2(\Omega)}
\]

by (3.45), and then (4.27) follows from Theorem 4.6, Lemma 4.10 and (4.28)–(4.30).

We have similar \( L_2 \) error estimates for the computable approximations \( \Pi_{k,h}^0 u_h \) and \( \Pi_{k,h}^\ast u_h \).

Theorem 4.12. Assuming \( u \in H^{\ell+1}(\Omega) \) for some \( \ell \) between 1 and \( k \), there exists a positive constant \( C \), depending only on \( \rho \), \( N \) and \( k \), such that

\[
\| u - \Pi_{k,h}^0 u_h \|_{L_2(\Omega)} + \| u - \Pi_{k,h}^\ast u_h \|_{L_2(\Omega)} \leq C \alpha_h h^{\ell+1} |u|_{H^{\ell+1}(\Omega)},
\]

where \( \alpha_h \) is defined in (4.11).

Proof. The estimate for \( \Pi_{k,h}^0 u_h \) follows from (3.20), Theorem 4.11 and the relation

\[
\| u - \Pi_{k,h}^0 u_h \|_{L_2(\Omega)} \leq \| u - \Pi_{k,h}^0 u_h \|_{L_2(\Omega)} + \| \Pi_{k,h}^0 (u - u_h) \|_{L_2(\Omega)} \leq \| u - \Pi_{k,h}^0 u_h \|_{L_2(\Omega)} + \| u - u_h \|_{L_2(\Omega)}.
\]

For the approximation \( \Pi_{k,h}^\ast u_h \), we have

\[
\| u - \Pi_{k,h}^\ast u_h \|_{L_2(\Omega)} \leq \| u - \Pi_{k,h}^\ast I_{k,h} u_h \|_{L_2(\Omega)} + \| \Pi_{k,h}^\ast (I_{k,h} u - u_h) \|_{L_2(\Omega)}
\]

and, in view of (2.18), (3.11), and Lemma 3.7,

\[
\| \Pi_{k,h}^\ast (I_{k,h} u - u_h) \|_{L_2(\Omega)}^2 \leq \sum_{D \in T_h} \| I_{k,D} u - u_h \|_{L_2(D)}^2
\]

(4.32)

\[
\leq \sum_{D \in T_h} (h_D \| I_{k,D} u - u_h \|_{L_2(\partial D)} + \| \Pi_{k-2,D}^0 (I_{k,D} u - u_h) \|_{L_2(D)}^2).
\]
\[ S \]

Assuming that the solution \( S \) follows from Lemma 3.19, Theorem 4.6, Theorem 4.11 and (4.31)–(4.32).

4.6. Error Estimates for \( u_h \) in the \( L_\infty \) Norm. Here we consider a \( L_\infty \) error estimate for \( u_h \) over the edges of \( T_h \), where \( u_h \) is computable. We will treat the two choices of \( S^D(\cdot, \cdot) \) separately. The set of all the edges in \( T_h \) will be denoted by \( E_h \).

4.6.1. The Case where \( S^D(\cdot, \cdot) = S^D_2(\cdot, \cdot) \). We have the following result for this choice of \( S^D(\cdot, \cdot) \).

**Theorem 4.13.** Assuming that the solution \( u \) of (1.1) belongs to \( H^{\ell+1}(\Omega) \) for some \( \ell \) between 1 and \( k \), we have

\[
\max_{e \in E_h} \| u - u_h \|_{L_\infty(\Omega)} \leq Ch^\ell |u|_{H^{\ell+1}(\Omega)},
\]

where the positive constant \( C \) only depends on \( \rho \), \( N \) and \( k \).

**Proof.** First we observe that, by (4.5), (4.7) and Theorem 4.5,

\[
(3.33) \quad \sum_{D \in T_h} \sum_{e \in E_D} h_D \| \partial((u - u_h) - \Pi_{k,D}^v (u - u_h))/\partial s \|_{L_2(e)}^2 \leq \| u - u_h \|_{L_\infty(\Omega)}^2 \lesssim h^{2\ell} |u|_{H^{\ell+1}(\Omega)}^2.
\]

We can connect any point in \( e \in E_h \) to \( \partial \Omega \), where \( u - u_h = 0 \), by a path along the edges in \( E_h \). Therefore it follows from a direct calculation (or a Sobolev inequality in one variable) that

\[
(3.34) \quad \| u - u_h \|_{L_\infty(e)}^2 \lesssim \sum_{D \in T_h} \sum_{e \in E_D} h_e \| \partial(u - u_h)/\partial s \|_{L_2(e)}^2
\]

\[
\lesssim \sum_{D \in T_h} \sum_{e \in E_D} h_D \| \partial((u - u_h) - \Pi_{k,D}^v (u - u_h))/\partial s \|_{L_2(e)}^2
\]

\[
+ \sum_{D \in T_h} \sum_{e \in E_D} h_D \| \partial[\Pi_{k,D}^v (u - u_h)]/\partial s \|_{L_2(e)}^2 \quad \forall e \in E_h,
\]

and we have, by (2.18), (2.22), (3.14) and Theorem 4.6,

\[
(3.35) \quad \sum_{D \in T_h} \sum_{e \in E_D} h_D \| \partial[\Pi_{k,D}^v (u - u_h)]/\partial s \|_{L_2(e)}^2 \lesssim \sum_{D \in T_h} \left( \| \Pi_{k,D}^v (u - u_h) \|_{H^1(D)}^2 + h_D^2 \| \Pi_{k,D}^v (u - u_h) \|_{H^2(D)}^2 \right)
\]

\[
\lesssim \sum_{D \in T_h} \| \Pi_{k,D}^v (u - u_h) \|_{H^1(D)}^2 \lesssim h^{2\ell} |u|_{H^{\ell+1}(\Omega)}^2.
\]
The estimates (4.33)–(4.35) together imply
\[ \|u - u_h\|_{L^\infty(e)} \leq h^\ell |u|_{H^{\ell+1}(\Omega)} \quad \forall e \in \mathcal{E}_h. \]

4.6.2. The Case where \( S^D(\cdot, \cdot) = S^D_1(\cdot, \cdot) \). We will establish an analog of Theorem 4.13 under the additional assumption that \( \mathcal{T}_h \) is quasi-uniform, i.e., there exists a positive constant \( \gamma \) independent of \( h \) such that
\[ h_D \geq \gamma h \quad \forall \gamma \in \mathcal{T}_h. \]

**Theorem 4.14.** Assuming \( \mathcal{T}_h \) is quasi-uniform and the solution \( u \) of (1.1) belongs to \( H^{\ell+1}(\Omega) \) for some \( \ell \) between 1 and \( k \), we have
\[ \max_{e \in \mathcal{E}_h} \|u - u_h\|_{L^\infty(e)} \leq C \ln(1 + \max_{D \in \mathcal{T}_h} \tau_D) h^\ell |u|_{H^{\ell+1}(\Omega)}, \]
where the positive constant \( C \) only depends on \( \rho, N, \gamma \) and \( k \).

**Proof.** Let \( D \in \mathcal{T}_h \) be arbitrary. First we observe that, by (4.4), Remark 4.3, (4.7), (4.20), Theorem 4.5 and Theorem 4.6,
\[ \| (I_k,Du - u_h) - \Pi_{k,D}^\gamma (I_k,Du - u_h) \|_{L^\infty(\partial D)} \]
\[ \lesssim \| I_k,Du - u_h \| \]
\[ \lesssim \| I_k,Du - u_h \| + \| u - u_h \| \lesssim [\ln(1 + \max_{D \in \mathcal{T}_h} \tau_D)]^{\frac{\ell}{2}} h^\ell |u|_{H^{\ell+1}(\Omega)}. \]

Furthermore, it follows from (2.8), (2.24), (3.14), (3.44), Theorem 4.5, Theorem 4.12 and (4.36) that
\[ \| \Pi_{k,D}^\gamma (I_k,Du - u_h) \|_{L^\infty(\partial D)} \lesssim h_D^{-1} \| \Pi_{k,D}^\gamma (I_k,Du - u_h) \|_{L^2(D)} + \| \Pi_{k,D}^\gamma (I_k,Du - u_h) \|_{H^1(D)} \]
\[ \lesssim h_D^{-1} \| I_k,Du - u_h \|_{L^2(D)} + \| u - \Pi_{k,D}^\gamma u_h \|_{L^2(D)} + \| I_k,Du - u_h \|_{H^1(D)} \]
\[ \lesssim \ln(1 + \max_{D \in \mathcal{T}_h} \tau_D) h^\ell |u|_{H^{\ell+1}(\Omega)}. \]

The theorem follows from (3.49), (4.37), (4.38) and the triangle inequality. \( \square \)

4.7. **Error Estimates for** \( \Pi_{k,h}^\gamma u_h \) **and** \( \Pi_{k,h}^0 u_h \) **in the** \( L^\infty \) **Norm.** Again we treat the two choices of \( S^D(\cdot, \cdot) \) separately.

4.7.1. The Case where \( S^D(\cdot, \cdot) = S^D_2(\cdot, \cdot) \). For this choice of \( S^D(\cdot, \cdot) \), we can establish the following result without assuming that \( \mathcal{T}_h \) is quasi-uniform.

**Theorem 4.15.** Assuming the solution \( u \) of (1.1) belongs to \( H^{\ell+1}(\Omega) \) for some \( \ell \) between 1 and \( k \), there exists a positive constant \( C \), depending only on \( \rho, N \) and \( k \), such that
\[ \|u - \Pi_{k,h}^\gamma u_h\|_{L^\infty(\Omega)} + \|u - \Pi_{k,h}^0 u_h\|_{L^\infty(\Omega)} \leq C h^\ell |u|_{H^{\ell+1}(\Omega)}. \]

**Proof.** For any \( D \in \mathcal{T}_h \), we have, by (2.3), (2.22) and (3.14),
\[ \|u - \Pi_{k,D}^\gamma u_h\|_{L^\infty(D)} \lesssim h_D^{-1} \|u - \Pi_{k,D}^\gamma u_h\|_{L^2(D)} + \|u - \Pi_{k,D}^\gamma u_h\|_{H^1(D)} + h_D |u - \Pi_{k,D}^\gamma u_h|_{H^2(D)} \]
\[ + h_D |\Pi_{k,D}^\gamma (u - u_h)|_{H^2(D)} \]
4.7.2. The Case where Theorem 4.6 and Theorem 4.13. □

Assuming by the same arguments but with Theorem 4.13 replaced by Theorem 4.14.

There exists a positive integer Assumption 2

Assumption 3 The shape regularity assumptions in Section 4.1 are satisfied by all the faces and F by T

Theorem 4.13 imply the estimate for u by (2.8), (2.15) and (3.5). These two estimates together with Lemma 3.8, Theorem 4.6 and 5.1. Shape Regularity Assumptions in Three Dimensions.

There exists Assumption 1

F by arguments. We will only provide details for estimates that require different derivations.

in two dimensions and many of the results in Section 3 and Section 4 carry over by identical

virtual element methods for the Poisson problem in three dimensions

follows from Lemma 3.10, Theorem 4.6 and Theorem 4.13. □

4.7.2. The Case where \( S^D(\cdot, \cdot) = S^D_1(\cdot, \cdot) \). The following analog of Theorem 4.15 is proved by the same arguments but with Theorem 4.13 replaced by Theorem 4.14.

**Theorem 4.16.** Assuming \( T_h \) is quasi-uniform and the solution u of (1.1) belongs to \( H^{\ell+1}(\Omega) \) for some \( \ell \) between 1 and k, there exists a positive constant C, depending only on \( \rho, N, \gamma \) and k, such that

\[
\|u - \Pi^v_{k,D} u_h\|_{L_2(\Omega)} + \|u - \Pi^0_{k,D} u_h\|_{L_\infty(\Omega)} \leq C \ln(1 + \max_{D \in T_h} \tau_D) h^{\ell} \|u\|_{H^{\ell+1}(\Omega)}.
\]

5. Virtual Element Methods for the Poisson Problem in Three Dimensions

The analysis of virtual element methods in three dimensions follows the same strategy as in two dimensions and many of the results in Section 3 and Section 4 carry over by identical arguments. We will only provide details for estimates that require different derivations.

Let \( T_h \) be a polyhedral mesh on \( \Omega \). The set of the faces of a subdomain \( D \in T_h \) is denoted by \( \mathcal{F}_D \) and the set of the edges of \( F \) is denoted by \( \mathcal{E}_F \). The set of all the faces of \( T_h \) is denoted by \( \mathcal{F}_h \) and the set of all the edges of \( T_h \) is denoted by \( \mathcal{E}_h \).

5.1. Shape Regularity Assumptions in Three Dimensions. We impose the following shape regularity assumptions on \( T_h \), where \( h_D \) is the diameter of \( D \).

**Assumption 1** There exists \( \rho \in (0, 1) \), independent of \( h \), such that every polyhedron \( D \in T_h \) is star-shaped with respect to a ball \( \mathcal{B}_D \) with radius \( \geq \rho h_D \).

**Assumption 2** There exists a positive integer \( N \), independent of \( h \), such that \( |\mathcal{F}_D| \leq N \) for all \( D \in T_h \).

**Assumption 3** The shape regularity assumptions in Section 4.1 are satisfied by all the faces in \( \mathcal{F}_h \), with the same \( \rho \) from Assumption 1 and the same \( N \) from Assumption 2.
All the hidden constants below will only depend on $\rho$, $N$ and $k$.

Let $D$ be a polyhedron in $T_h$. We can define the inner product $\langle \cdot, \cdot \rangle$ by (3.1) where the infinitesimal arc-length $ds$ is replaced by the infinitesimal surface area $dS$. Then the projection operator $\Pi_{k,D}^v : H^1(D) \rightarrow \mathbb{P}_k(D)$ is defined by (3.2) and

$$\Pi_{k,D}^v \zeta|_{H^1(D)} \leq |\zeta|_{H^1(D)} \quad \forall \zeta \in H^1(D).$$

The projection from $L_2(D)$ to $\mathbb{P}_k(D)$ is again denoted by $\Pi_{k,D}^0$.

The results in Section 2 are valid for $D \in T_h$ under Assumption 1. Consequently the results in Section 3.5 and Section 3.6 are also valid provided the semi-norm $\| \cdot \|_{k,D}$ is defined by the following analog of (3.11):

$$\| \zeta \|_{k,D}^2 = \| \Pi_{k-2,D}^0 \zeta \|_{L_2(D)}^2 + h_D \sum_{F \in F_D} \| \Pi_{k-1,F}^0 \zeta \|_{L_2(F)}^2,$$

where $\Pi_{k-1,F}^0$ is the projection from $L_2(F)$ onto $\mathbb{P}_{k-1}(F)$.

We have the following estimates for $\Pi_{k,D}^v$ and $\Pi_{k,D}^0$:

$$\| \zeta - \Pi_{k,D}^v \zeta \|_{L_2(D)} + \| \zeta - \Pi_{k,D}^0 \zeta \|_{L_2(D)} \lesssim h_{\ell+1} \| \zeta \|_{H^{\ell+1}(D)} \quad \forall \zeta \in H^{\ell+1}(D), 0 \leq \ell \leq k,$$

and for $1 \leq \ell \leq k$,

$$\| \zeta - \Pi_{k,D}^0 \zeta \|_{H^1(D)} + \| \zeta - \Pi_{k,D}^0 \zeta \|_{H^1(D)} \lesssim h_{\ell+1} \| \zeta \|_{H^{\ell+1}(D)} \quad \forall \zeta \in H^{\ell+1}(D),$$

$$\| \zeta - \Pi_{k,D}^0 \zeta \|_{H^2(D)} + \| \zeta - \Pi_{k,D}^0 \zeta \|_{H^2(D)} \lesssim h_{\ell} \| \zeta \|_{H^{\ell+1}(D)} \quad \forall \zeta \in H^{\ell+1}(D).$$

The analogs of Lemma 3.7 and (5.2) lead to the estimate

$$\| \Pi_{k,D}^v \zeta \|_{L_2(D)}^2 \lesssim \| \zeta \|_{k,D}^2 \lesssim \| \zeta \|_{L_2(D)}^2 + h_D \| \zeta \|_{L_2(\partial D)}^2 \quad \forall \zeta \in H^1(D),$$

and we also have the following analog of (3.21):

$$\Pi_{k,D}^0 \zeta \|_{H^1(D)} \lesssim \zeta \|_{H^1(D)} \quad \forall \zeta \in H^1(D).$$

### 5.2. The Local Virtual Element Space $Q^k(D)$.

The space $Q^k(\partial D)$ of continuous piecewise (two dimensional) virtual element functions of order $\leq k$ on $\partial D$ is defined by

$$Q^k(\partial D) = \{ v \in C(\partial D) : \quad v|_F \in Q^k(F) \quad \forall F \in F_D \}.$$

For $k \geq 1$, the virtual element space $Q^k(D) \subset H^1(D)$ is defined by the following conditions: $v \in H^1(D)$ belongs to $Q^k(D)$ if and only if (i) the trace of $v$ on $\partial D$ belongs to $Q^k(\partial D)$, (ii) the distribution $-\Delta v$ belongs to $\mathbb{P}_k(D)$, and (iii) condition (3.6) is satisfied.

**Remark 5.1.** Since the restriction of $v \in Q^k(D)$ to $\partial D$ belongs to $C(\partial D)$ and $-\Delta v \in \mathbb{P}_k(D)$, the virtual element function $v$ is also continuous on $\bar{D}$ (cf. [15, Section 1.2]).

**Remark 5.2.** The degrees of freedom of $Q^k(D)$ (cf. [2]) consist of (i) the values of $v$ at the vertices of $D$ and nodes on the interior of each edge of $D$ that determine a polynomial of degree $k$ on each edge of $D$, (ii) the moments of $\Pi_{k-2,F}^0 v$ on each face $F$ of $D$, and (iii) the moments of $\Pi_{k-2,D}^0 v$ on $D$. 

Remark 5.3. For $v \in \mathcal{Q}^k(D)$ and $F \in \mathcal{F}_D$, the polynomial $\Pi^0_{k,F}v$ can be computed in terms of the degrees of freedom of $v|_F$ (cf. Remark 3.3). Therefore the polynomial $\Pi^\ast_{k,D}v$ can be computed in terms of the degrees of freedom of $v \in \mathcal{Q}^k(D)$ through (3.4) and (3.5). The polynomial $\Pi^\ast_{k,D}v$ can then be computed through (3.6).

Remark 5.4. Under Assumption 3 in Section 5.1, the results in Section 3 (with $D$ replaced by $F$) are valid for the restriction of $v \in \mathcal{Q}^k(D)$ to any face $F$ of $D$.

The three dimensional analogs of Lemma 3.4, Lemma 3.12 and Lemma 3.13 lead to the estimate

\begin{equation}
|v|^2_{H^1(D)} \lesssim h_D^{-2}\|\zeta\|^2_{k,D} + h_D \sum_{F \in \mathcal{F}_D} \|\nabla_F v\|^2_{L^2(F)} \quad \forall v \in \mathcal{Q}^k(D),
\end{equation}

where $\nabla_F$ is the two dimensional gradient operator on the face $F$, and we also have an analog of (3.4):

\begin{equation}
\|\Pi^0_{k-2,F}v\|^2_{L^2(D)} \lesssim h_D \sum_{F \in \mathcal{F}_D} \|\Pi^0_{k-1,F}v\|^2_{L^2(F)} \quad \forall v \in \mathcal{N}(\Pi^\ast_{k,F}).
\end{equation}

Hence we have, by (5.2), (5.9) and (5.10),

\begin{equation}
|v|^2_{H^1(D)} \lesssim h_D^{-2}\sum_{F \in \mathcal{F}_D} \|\Pi^0_{k-1,F}v\|^2_{L^2(F)} + h_D \sum_{F \in \mathcal{F}_D} \|\nabla_F v\|^2_{L^2(F)} \quad \forall v \in \mathcal{N}(\Pi^\ast_{k,F}).
\end{equation}

The interpolation operator $I_{k,D} : H^2(D) \rightarrow \mathcal{Q}^k(D)$ is defined by the condition that $I_{k,D}\zeta$ and $\zeta$ share the same degrees of freedom. In particular we have

\begin{equation}
I_{k,D}q = q \quad \forall q \in \mathcal{P}_k(D).
\end{equation}

Note that

\begin{equation}
I_{k,F}(\zeta|_F) = (I_{k,D}\zeta)|_F \quad \forall F \in \mathcal{F}_D,
\end{equation}

and hence, in view of (2.8), Lemma 3.6 and (5.2),

\begin{align}
\|I_{k,D}\zeta\|^2_{k,D} & = \|\Pi^0_{k-2,D}(I_{k,D}\zeta)\|^2_{L^2(D)} + h_D \sum_{F \in \mathcal{F}_D} \|\Pi^0_{k-1,F}(I_{k,D}\zeta)\|^2_{L^2(F)} \\
& = \|\Pi^0_{k-2,D}\zeta\|^2_{L^2(D)} + h_D \sum_{F \in \mathcal{F}_D} \|\Pi^0_{k-1,F}(I_{k,F}\zeta)\|^2_{L^2(F)} \\
& \lesssim \|\zeta\|^2_{L^2(D)} + h_D \sum_{F \in \mathcal{F}_D} h_F^2 \|I_{k,F}\zeta\|^2_{L^\infty(F)} \\
& \lesssim \|\zeta\|^2_{L^2(D)} + h_D \sum_{F \in \mathcal{F}_D} h_F^2 (\|I_{k,F}\zeta\|^2_{L^\infty(\partial F)} + \|\nabla_F(I_{k,F}\zeta)\|^2_{L^2(F)}).
\end{align}

The error estimates for $I_{k,D}$ rely on the following analog of (3.39), where $\tau_F$ is defined by replacing $D$ by $F$ in (3.32).

Lemma 5.5. We have

\begin{equation}
|I_{k,D}\zeta|_{H^1(D)} \lesssim h_D^{-1}\|\zeta\|_{L^2(D)} + |\zeta|_{H^1(D)} + h_D|\zeta|_{H^2(D)}
\end{equation}

for all $\zeta \in H^2(D)$.
Remark 5.6. Let $\zeta \in H^2(D)$ be arbitrary. It follows from (5.9) and (5.14) that
\[
|I_k,D\zeta|_{H^1(D)}^2 \lesssim h_D^{-2}||\zeta||_{L_2(D)}^2 + h_D \sum_{F \in F_D} ||I_k,F\zeta||_{L_\infty(\partial F)}^2 + h_D \sum_{F \in F_D} ||\nabla_F(I_k,F\zeta)||_{L_2(F)}^2.
\]
We have, by (2.3) and (3.36),
\[
(5.16) \quad h_D \sum_{F \in F_D} ||I_k,F\zeta||_{L_\infty(\partial F)}^2 \lesssim h_D \sum_{F \in F_D} ||\zeta||_{L_\infty(\partial F)}^2 \lesssim h_D^{-2}||\zeta||_{L_2(D)}^2 + ||\zeta||_{H^1(D)}^2 + h_D^2||\zeta||_{H^2(D)}^2;
\]
and by (2.17), Lemma 2.2 and (3.40),
\[
(5.17) \quad h_D \sum_{F \in F_D} ||\nabla_F(I_k,F\zeta)||_{L_2(F)}^2 \lesssim h_D \sum_{F \in F_D} (||\zeta||_{H^1(F)}^2 + h_F||\zeta||_{H^{3/2}(F)}^2) \lesssim \sum_{D \in T_h} (||\zeta||_{H^1(D)}^2 + h_D^2||\zeta||_{H^2(D)}^2).
\]
Note that (5.14), (5.16) and (5.17) imply
\[
(5.18) \quad ||I_k,D\zeta||_{k,D} \lesssim ||\zeta||_{L_2(D)} + h_D||\zeta||_{H^1(D)} + h_D^2||\zeta||_{H^2(D)} \quad \forall \zeta \in H^2(D),
\]
and hence we have, in view of (5.3), (5.6), (5.14) and (5.15),
\[
(5.19) \quad ||I_k,D\zeta||_{L_2(D)} \lesssim ||I_k,D\zeta - \Pi_{k,D}^{r}\zeta||_{L_2(D)} + ||\Pi_{k,D}^{r} I_k,D\zeta||_{L_2(D)} \lesssim h_D ||I_k,D\zeta||_{H^1(D)} + ||I_k,D\zeta||_{k,D} \lesssim \left(||\zeta||_{L_2(D)} + h_D||\zeta||_{H^1(D)} + h_D^2||\zeta||_{H^2(D)}\right)
\]
for all $\zeta \in H^2(D)$.

In view of (5.12), the following analogs of (3.45)–(3.46), where $\zeta \in H^{l+1}(D)$ and $1 \leq l \leq k$, can be obtained by combining the Bramble-Hilbert estimates (2.5) with the stability estimates (5.1), (5.6), (5.15), (5.18) and (5.19).
\[
(5.20) \quad ||\zeta - I_k,D\zeta||_{L_2(D)} + ||\zeta - \Pi_{k,D}^{r}\zeta||_{L_2(D)} \lesssim h_D^{l+1}||\zeta||_{H^{l+1}(D)}
\]
\[
(5.21) \quad ||\zeta - I_k,D\zeta||_{H^1(D)} + ||\zeta - \Pi_{k,D}^{r}\zeta||_{H^1(D)} \lesssim h_D^l||\zeta||_{H^{l+1}(D)}
\]
\[
(5.22) \quad ||\zeta - \Pi_{k,D}^{r}\zeta||_{H^1(D)} \lesssim h_D^{l-1}||\zeta||_{H^{l+1}(D)}
\]

Remark 5.6. We also have the following analog of (3.49):
\[
||\zeta - I_k,D\zeta||_{L_\infty(D)} \lesssim h_D^{l-\frac{1}{2}}||\zeta||_{H^{l+1}(D)}
\]
for all $\zeta \in H^{l+1}(D)$ and $1 \leq l \leq k$. The proof uses Lemma 3.6 (which is valid in three dimensions) and the arguments for (3.49). But we do not need this estimate in the error analysis.
5.3. The Discrete Problem. Let the global virtual element space $Q^k_h$ be defined by
$$Q^k_h = \{ v \in H^1_0(\Omega) : v|_D \in Q^k(D) \quad \forall D \in T_h \}.$$ 
The discrete problem for (1.1) is to find $u_h \in Q^k_h$ such that
$$a_h(u_h, v) = (f, \Xi_h v) \quad \forall v \in Q^k_h,$$
where $\Xi_h$ is defined as in (4.9),
$$a_h(w, v) = \sum_{D \in T_h} \left[ (\nabla (\Pi^v_{k,D} w), \nabla (\Pi^v_{k,D} v))_{L^2(D)} + S^D(w - \Pi^v_{k,D} w, v - \Pi^v_{k,D} v) \right],$$
and the local stabilizing bilinear form $S^D(\cdot, \cdot)$ is given by
$$S^D(w, v) = h_D \sum_{F \in F_D} \left( h_F^{-2} (\Pi^0_{k-2,F} w, \Pi^0_{k-2,F} v)_{L^2(F)} + \sum_{p \in N_{\partial F}} w(p) v(p) \right).$$
Here $N_{\partial F}$ is the set of the nodes along $\partial F$ associated with the degrees of freedom of a virtual element function.

**Lemma 5.7.** There exists a positive constant $C$, depending only on $\rho, N$ and $k$, such that
$$|v|^2_{H^1(D)} \leq C \left[ \ln(1 + \max_{F \in F_D} \tau_F) \right] S^D(v, v) \quad \forall v \in N(\Pi^v_{k,D}).$$

**Proof.** Let $v \in N(\Pi^v_{k,D})$ be arbitrary. We have, by (2.8), (2.15), Lemma 3.6, Corollary 3.15 and (5.11),
$$|v|^2_{H^1(D)} \lesssim h_D \sum_{F \in F_D} \left( h_D^{-2} \|v\|^2_{L^2(F)} + \|\nabla_F v\|^2_{L^2(F)} \right) \lesssim h_D \sum_{F \in F_D} \left( h_D^{-2} h_F^2 \|v\|^2_{L^\infty(\partial F)} + \|\nabla_F v\|^2_{L^2(F)} \right) \lesssim h_D \sum_{F \in F_D} \left( \|v\|^2_{L^\infty(\partial F)} + \|\nabla_F v\|^2_{L^2(F)} \right) \lesssim h_D \sum_{F \in F_D} \left( h_F^{-2} \|\Pi^0_{k-2,F} v\|^2_{L^2(F)} + \ln(1 + \tau_F) \|v\|^2_{L^\infty(\partial F)} \right),$$
which together with Remark 4.3 and (5.23) implies (5.24). \hfill \Box

It follows from (5.24) that we have an analog of (4.10):
$$|v|^2_{H^1(\Omega)} \leq 2 \sum_{D \in T_h} \left[ \|\Pi^v_{k,h} v\|^2_{H^1(\Omega)} + |v - \Pi^v_{k,D} v|^2_{H^1(D)} \right] \lesssim \beta_h a_h(v, v) \quad \forall v \in Q^k_h,$$
where
$$\beta_h = \ln(1 + \max_{F \in F_D} \tau_F).$$
Hence the discrete problem is well-posed.

**Remark 5.8.** The constants in the error estimates for the virtual element methods will only depend on $\rho, N, k$ and $\beta_h$. Therefore the existence of small faces in $T_h$ does not affect the performance of the method. It is only the relative sizes of the edges on each face that matter.
Note that the estimates in Lemma 4.4 are also valid for \( \Omega \subset \mathbb{R}^3 \).

### 5.4. Error Estimates in the Energy Norm.

The abstract error estimate

\[(5.27) \quad \|u - u_h\|_h \lesssim \|u - I_{k,h}u\|_h + \|u - \Pi_k^v u\|_h + \sqrt{\beta_h} \left( \|u - \Pi_{k,h} u\|_{h,1} + \sup_{w \in Q_k} \frac{(f, w - \Xi_h w)}{|w|_{H^1(\Omega)}} \right) \]

is obtained by the same arguments as in Section 4.3, where \(| \cdot |_{h,1}\) is defined in (4.1).

We will derive concrete error estimates under the assumption that \(u\) belongs to \(H^{\ell+1}(\Omega)\) for \(1 \leq \ell \leq k\). Since the estimate

\[(5.28) \quad \|u - \Pi_{k,h} u\|_{h,1} + \sup_{w \in Q_k} \left( f, w - \Xi_h w \right) \lesssim h^\ell \|u\|_{H^{\ell+1}(\Omega)} \]

remains the same, we only need to estimate \(\|u - I_{k,h} u\|_h\) and \(\|u - \Pi_k^v u\|_h\).

It follows from (2.7), (5.1), (5.13) and (5.23) that

\[
\|u - I_{k,h} u\|_h^2 \lesssim \sum_{D \in T_h} |\Pi_{k,D}^v (u - I_{k,D} u)|^2_{H^1(D)} + \sum_{D \in T_h} h_D \sum_{F \in F_D} h_F^{-2} \|u - I_{k,D} u\|_{L_2(F)}^2 \\
+ \sum_{D \in T_h} h_D \sum_{F \in F_D} \|\Pi_{k,D}^v (u - I_{k,D} u)\|_{L_\infty(F)}^2
\]

\[(5.29) \quad \lesssim \sum_{D \in T_h} |u - I_{k,D} u|^2_{H^1(D)} + \sum_{D \in T_h} h_D \sum_{F \in F_D} h_F^{-2} \|u - I_{k,F} u\|_{L_2(F)}^2 \]

\[
+ \sum_{D \in T_h} h_D \|\Pi_{k,D}^v (u - I_{k,D} u)\|_{L_\infty(D)}^2,
\]

and we have, by (2.17) and (3.47),

\[(5.30) \quad \sum_{D \in T_h} h_D \sum_{F \in F_D} h_F^{-2} \|u - I_{k,F} u\|_{L_2(F)}^2 \lesssim \sum_{D \in T_h} h_D \sum_{F \in F_D} h_F^{-2} |u|^2_{H^{\ell+1/2}(\Omega)} \lesssim h^{2\ell} |u|_{H^{\ell+1}(\Omega)}^2.
\]

Moreover the estimates (2.7), (2.24) and (5.1) imply

\[
\sum_{D \in T_h} h_D \|\Pi_{k,D}^v (u - I_{k,D} u)\|_{L_\infty(D)}^2 \lesssim \sum_{D \in T_h} h_D \|\Pi_{k,D}^v (u - I_{k,D} u)\|_{L_2(D)}^2 + \sum_{D \in T_h} |\Pi_{k,D}^v (u - I_{k,D} u)|_{H^1(D)}^2 \]

\[
\lesssim \sum_{D \in T_h} h_D^{-2} \|\Pi_{k,D}^v (u - I_{k,D} u)\|_{L_2(D)}^2 + \sum_{D \in T_h} \|u - \Pi_{k,D}^v I_{k,D} u\|_{L_2(D)}^2 \]

\[
+ \sum_{D \in T_h} |u - I_{k,D} u|^2_{H^1(D)}.
\]

Combining (5.3), (5.20), (5.21) and (5.29)–(5.31), we obtain

\[(5.32) \quad \|u - I_{k,h} u\|_h^2 \lesssim h^{2\ell} |u|_{H^{\ell+1}(\Omega)}^2,
\]
which is the analog of (4.20).

From (2.3), (2.7), (5.3)–(5.5) and (5.23), we find

\[
\| u - \Pi_{k,h} u \|_h^2 \lesssim \sum_{D \in T_h} h_D \sum_{F \in \mathcal{F}_D} h_F^{-2} \| u - \Pi_{k,D}^V u \|_{L_2(F)}^2 + \sum_{D \in T_h} h_D \sum_{F \in \mathcal{F}_D} \| u - \Pi_{k,D}^V u \|_{L_\infty(\partial F)}^2
\]

(5.33)

\[
\lesssim \sum_{D \in T_h} h_D \| u - \Pi_{k,D}^V u \|_{L_\infty(D)}^2
\]

\[
\lesssim \sum_{D \in T_h} \left( h_D^{-2} \| u - \Pi_{k,D}^V u \|_{L_2(D)}^2 + | u - \Pi_{k,D}^V u |_{H^1(D)}^2 + h_D^2 | u - \Pi_{k,D}^V u |_{H^2(D)}^2 \right)
\]

\[
\lesssim h^2 | u |_{H^3(\Omega)}^2,
\]

which is the analog of (4.23).

The estimates (5.27), (5.28), (5.32) and (5.33) lead to the following analog of Theorem 4.5.

**Theorem 5.9.** Assuming the solution \( u \) of (1.1) belongs to \( H^{\ell+1}(\Omega) \) for \( \ell \) between 1 and \( k \), we have

\[
\| u - u_h \|_h \leq C \sqrt{\beta_h h^{\ell}} | u |_{H^{\ell+1}(\Omega)},
\]

where \( \beta_h \) is defined in (5.26) and the positive constant \( C \) depends only on \( \rho, N \) and \( k \).

The following analog of Theorem 4.6 on the computable approximate solutions \( \Pi_{k,h} u_h \) and \( \Pi_{k,h}^0 u_h \) is obtained by the same arguments.

**Theorem 5.10.** Assuming the solution \( u \) of (1.1) belongs to \( H^{\ell+1}(\Omega) \) for \( \ell \) between 1 and \( k \), there exists a positive constant \( C \), depending only on \( \rho, N \) and \( k \), such that

\[
| u - u_h |_{H^1(\Omega)} + \sqrt{\beta_h} | u - \Pi_{k,h}^V u_h |_{h,1} + | u - \Pi_{k,h}^0 u_h |_{h,1} \leq C \beta_h h^{\ell} | u |_{H^{\ell+1}(\Omega)},
\]

where \( \beta_h \) is defined in (5.26).

### 5.5. Error Estimates in the \( L_2 \) Norm

We begin with an analog of Lemma 4.8.

**Lemma 5.11.** We have

\[
\sum_{D \in T_h} S^D(\zeta - \Pi_{k,h}^V I_{k,h} \zeta, \zeta - \Pi_{k,h}^V I_{k,h} \zeta) \lesssim h^2 | \zeta |_{H^2(\Omega)}^2 \quad \forall \zeta \in H^2(\Omega) \cap H^1_0(\Omega).
\]

**Proof.** It follows from (2.3), (5.20)–(5.22) and (5.23) that

\[
\sum_{D \in T_h} S^D(\zeta - \Pi_{k,h}^V I_{k,h} \zeta, \zeta - \Pi_{k,h}^V I_{k,h} \zeta)
\]

\[
\lesssim \sum_{D \in T_h} h_D \sum_{F \in \mathcal{F}_D} (h_F^{-2} \| \zeta - \Pi_{k,D}^V I_{k,D} \zeta \|_{L_2(F)}^2 + \| \zeta - \Pi_{k,D}^V I_{k,D} \zeta \|_{L_\infty(\partial F)}^2)
\]

\[
\lesssim \sum_{D \in T_h} h_D \| \zeta - \Pi_{k,D}^V I_{k,D} \zeta \|_{L_\infty(\partial D)}^2
\]

\[
\lesssim \sum_{D \in T_h} \left( h_D^{-2} \| \zeta - \Pi_{k,D}^V I_{k,D} \zeta \|_{L_2(D)}^2 + | \zeta - \Pi_{k,D}^V I_{k,D} \zeta |_{H^1(D)}^2 + h_D^2 | \zeta - \Pi_{k,D}^V I_{k,D} \zeta |_{H^2(D)}^2 \right)
\]

\[
\lesssim h^2 | \zeta |_{H^2(\Omega)}^2.
\]
The same arguments as in the proof of Lemma 4.9 lead to the following result.

**Lemma 5.12.** We have

\[
\sum_{D \in T_h} S^D(u_h - \Pi_{k,h}^u u_h, u_h - \Pi_{k,h}^u u_h) \lesssim \beta_h^2 h^{2\ell} |u|_{H^{\ell+1}(\Omega)}^2.
\]

With Lemma 5.11 and Lemma 5.12 in hand, we obtain the following analog of Theorem 4.11 and Theorem 4.12 by identical arguments.

**Theorem 5.13.** Assuming \(u \in H^{\ell+1}(\Omega)\) for some \(\ell\) between 1 and \(k\), there exists a positive constant \(C\), depending only on \(N\), \(k\) and \(\rho\), such that

\[
\|u - u_h\|_{L_2(\Omega)} + \|u - \Pi_0^{k,h} u_h\|_{L_2(\Omega)} + \|u - \Pi_{k,h}^u u_h\|_{L_2(\Omega)} \leq C \beta_h h^{\ell+1} |u|_{H^{\ell+1}(\Omega)},
\]

where \(\beta_h\) is defined in (5.26).

**5.6. Error Estimate in the \(L_\infty\) Norm.** We will derive \(L_\infty\) error estimates under the additional assumption that \(T_h\) is quasi-uniform (cf. (4.36)). We begin with an analog of Theorem 4.14.

**Theorem 5.14.** Assuming \(T_h\) is quasi-uniform and the solution \(u\) of (1.1) belongs to \(H^{\ell+1}(\Omega)\) for \(\ell\) between 1 and \(k\), we have

\[
\max_{e \in \mathcal{E}_h} \|u - u_h\|_{L_\infty(e)} \leq C \beta_h h^{\ell+1} |u|_{H^{\ell+1}(\Omega)},
\]

where \(\beta_h\) is defined in (5.26) and the positive constant \(C\) only depends on \(\rho\), \(N\), \(\gamma\) and \(k\).

**Proof.** It follows from Remark 4.3 that

\[
\sum_{D \in T_h} h_D \sum_{F \in \mathcal{F}_D} \| (I_{k,D} u - u_h) - \Pi_{k,D}^u (I_{k,D} u - u_h) \|_{L_\infty(\partial F)}^2 \lesssim \| I_{k,D} u - u_h \|_h^2
\]

and hence, for any \(D \in T_h\) and \(F \in \mathcal{F}_D\),

\[
\| (I_{k,D} u - u_h) - \Pi_{k,D}^u (I_{k,D} u - u_h) \|_{L_\infty(\partial F)}^2 \lesssim \beta_h h^{2\ell-1} |u|_{H^{\ell+1}(\Omega)}^2
\]

by (5.32), Theorem 5.9 and the quasi-uniformity of \(T_h\).

For any \(D \in T_h\) and \(F \in \mathcal{F}_D\), we have, by (2.7), (2.24), (5.1), (5.20), (5.21), Theorem 5.13 and the quasi-uniformity of \(T_h\),

\[
\| \Pi_{k,D}^u (I_{k,D} u - u_h) \|_{L_\infty(F)}^2 \lesssim h_D^{-3} \| \Pi_{k,D}^u (I_{k,D} u - u_h) \|_{L_2(D)}^2 + h_D^{-1} \| \Pi_{k,D}^u (I_{k,D} u - u_h) \|_{H^1(D)}^2
\]

\[
\lesssim h_D^{-3} \left( \| \Pi_{k,D}^u I_{k,D} u - u \|_{L_2(D)}^2 + \| u - \Pi_{k,D}^u u_h \|_{L_2(D)}^2 \right) + h_D^{-1} \| I_{k,D} u - u_h \|_{H^1(D)}^2 \lesssim \beta_h h^{2\ell-1} |u|_{H^{\ell+1}(\Omega)}^2.
\]

Finally we have, for any \(D \in T_h\) and \(F \in \mathcal{F}_D\),

\[
\| u - I_{k,D} u \|_{L_\infty(F)}^2 = \| u - I_{k,F} u \|_{L_\infty(F)}^2 \lesssim h_D^{2\ell-1} |u|_{H^{\ell+1}(\Omega)}^2 \lesssim h_D^{2\ell-1} |u|_{H^{\ell+1}(D)}^2
\]

by (2.17), (3.50) and (5.13).

The estimate (5.36) then follows from (5.37)–(5.39) and the triangle inequality.
We also have estimates for the computable approximate solutions \( \Pi_{k,h}^\nabla u_h \) and \( \Pi_{k,h}^0 u_h \).

**Theorem 5.15.** Assuming \( \mathcal{T}_h \) is quasi-uniform and the solution \( u \) of (1.1) belongs to \( H^{\ell+1}(\Omega) \) for \( \ell \) between 1 and \( k \), there exists a positive constant \( C \), depending only on \( \rho \), \( N \), \( \gamma \) and \( k \), such that

\[
\| u - \Pi_{k,h}^\nabla u_h \|_{L_\infty(\Omega)} + \| u - \Pi_{k,h}^0 u_h \|_{L_\infty(\Omega)} \leq C \beta_h h^{\ell-1/2} |u|_{H^{\ell+1}(\Omega)},
\]

where \( \beta_h \) is defined in (5.26).

**Proof.** For any \( D \in \mathcal{T}_h \), we have, by (2.3), (2.22) and (5.1),

\[
\| u - \Pi_{k,D}^\nabla u_h \|_{L_\infty(D)} \leq h_D^{-3} \| u - \Pi_{k,D}^\nabla u_h \|_{L_2(D)} + h_D^{-1} \| u - \Pi_{k,D}^\nabla u_h \|_{H^1(D)} + h_D^{-1/2} |u - \Pi_{k,D}^\nabla u_h|_{H^2(D)} + h_D^{-1}\| \Pi_{k,D}^\nabla(u - u_h)\|_{H^2(D)}
\]

which together with (5.5), Theorem 5.10, Theorem 5.13 and the quasi-uniformity of \( \mathcal{T}_h \) implies the estimate for \( u - \Pi_{k,h}^\nabla u_h \).

Similarly we have, by (2.3), (2.22) and (5.7),

\[
\| u - \Pi_{k,D}^0 u_h \|_{L_\infty(D)} \leq h_D^{-3} \| u - \Pi_{k,D}^0 u_h \|_{L_2(D)} + h_D^{-1} \| u - \Pi_{k,D}^0 u_h \|_{H^1(D)} + h_D^{-1/2} |u - \Pi_{k,D}^0 u_h|_{H^2(D)} + h_D^{-1}\| \Pi_{k,D}^0(u - u_h)\|_{H^2(D)}.
\]

which together with (5.5), Theorem 5.10, Theorem 5.13 and the quasi-uniformity of \( \mathcal{T}_h \) implies the estimate for \( u - \Pi_{k,h}^0 u_h \). \( \square \)

### 6. Concluding Remarks

We have developed error estimates for virtual element methods for the model Poisson problem in two and three dimensions that provide justifications for existing numerical results for polygonal (or polyhedral) meshes with small edges (or faces).

For the two dimensional problem, the convergence of the virtual element method based on the stabilizing bilinear form \( S_2^D(\cdot,\cdot) \) is optimal under the shape regularity assumptions in Section 4.1. Under the additional assumption that the edges of any subdomain in a polygonal mesh are comparable to one another, convergence of the virtual element method based on the stabilizing bilinear form \( S_1^D(\cdot,\cdot) \) is also optimal.

For the three dimensional problem, the convergence of the virtual element method is optimal if, in addition to the assumptions in Section 5.1, we also assume that the edges of any face in the polyhedral mesh are comparable to one another.
The results in this paper can be extended to virtual element methods with the stabilizing bilinear form
\[ S_D(w, v) = (\Pi_{k-2,D}^0 w, \Pi_{k-2,D}^0 v) + \sum_{p \in N_{DD}} w(p)v(p) \]
in two dimensions, and the stabilizing bilinear form
\[ S_D(w, v) = (\Pi_{k-2,D}^0 w, \Pi_{k-2,D}^0 v) + h_D \sum_{F \in F_D} \left( h_F^{-2} (\Pi_{k-2,F}^0 w, \Pi_{k-2,F}^0 v)_{L^2(F)} + \sum_{p \in N_{F} \partial F} w(p)v(p) \right) \]
in three dimensions. The stability for these virtual element methods is automatic and the error analysis also does not pose any new difficulties.

The results in this paper can also be extended to virtual element methods \((k \geq 2)\) where the inner product \((3.1)\) is replaced by the inner product
\[ \langle (\zeta, \eta) \rangle = (\nabla \zeta, \nabla \eta) + \left( \int_D \zeta \, dx \right) \left( \int_D \eta \, dx \right). \]

We note that error estimates for the Poisson problem on general polygonal or polyhedral domains can also be obtained by the techniques developed in this paper.

Finally it would be interesting to construct a three dimensional analog of the stabilizing bilinear form \(S_D^3(\cdot, \cdot)\) defined in Section 4.2 so that the convergence of the virtual element methods is optimal for polyhedral meshes with arbitrarily small faces and edges, and \(L^\infty\) error estimates can be established without assuming the meshes are quasi-uniform. We conjecture that such a bilinear form can be defined by
\[
S_D^3(v, w) = \sum_{F \in F_D} h_F (\nabla_F \Pi_{k,F}^v v, \nabla_F \Pi_{k,F}^v w)_{L^2(F)} + \sum_{F \in F_D} h_F \sum_{e \in E_F} h_e \left( \partial(v - \Pi_{k,F}^v v) / \partial s, \partial(w - \Pi_{k,F}^v w) / \partial s \right)_{L^2(e)}.
\]

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Susanne C. Brenner, Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803

E-mail address: brenner@math.lsu.edu

Li-yeng Sung, Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803

E-mail address: sung@math.lsu.edu