Two antisymmetric hypermultiplets in 
$\mathcal{N}=2$ SU($N$) gauge theory:
Seiberg-Witten curve and M-theory interpretation

Isabel P. Ennes
Martin Fisher School of Physics
Brandeis University, Waltham, MA 02454

Stephen G. Naculich
Department of Physics
Bowdoin College, Brunswick, ME 04011

Henric Rhedin
Department of Engineering Sciences, Physics and Mathematics
Karlstad University, S-651 88 Karlstad, Sweden

Howard J. Schnitzer
Martin Fisher School of Physics
Brandeis University, Waltham, MA 02454
and
Lyman Laboratory of Physics
Harvard University, Cambridge, MA 02138

Abstract

The one-instanton contribution to the prepotential for $\mathcal{N}=2$ supersymmetric gauge theories with classical groups exhibits a universality of form. We extrapolate the observed regularity to SU($N$) gauge theory with two antisymmetric hypermultiplets and $N_f \leq 3$ hypermultiplets in the defining representation. Using methods developed for the instanton expansion of non-hyperelliptic curves, we construct an effective quartic Seiberg-Witten curve that generates this one-instanton prepotential. We then interpret this curve in terms of an M-theoretic picture involving NS 5-branes, D4-branes, D6-branes, and orientifold sixplanes, and show that for consistency, an infinite chain of 5-branes and orientifold sixplanes is required, corresponding to a curve of infinite order.

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3Research supported in part by the DOE under grant DE-FG02-92ER40706.
4Permanent address.
naculich@bowdoin.edu; henric.rhedin@kau.se; ennes,schnitzer@binah.cc.brandeis.edu
1. Introduction

The program of Seiberg and Witten [1] allows one to extract the exact low-energy behavior of four-dimensional \( \mathcal{N} = 2 \) supersymmetric gauge theories from the following data: a Riemann surface or algebraic curve specific to the group and representation content of the underlying Lagrangian, and a preferred meromorphic 1-form, the Seiberg-Witten (SW) differential. From this information, one may (in principle) reconstruct the prepotential of the Coulomb branch of the theory in the low-energy limit from the period integrals of the SW differential. In practice, technical difficulties make the construction of the prepotential a challenging problem.

The Riemann surface associated with \( \mathcal{N} = 2 \) supersymmetric gauge theories based on the classical groups, either without matter hypermultiplets or with hypermultiplets in the defining representation, is hyperelliptic [2]. In this case systematic methods are available for extracting the relevant physical information [3]–[6].

For other \( \mathcal{N} = 2 \) supersymmetric gauge theories, however, the associated Riemann surface, obtained via geometric engineering [7, 8] or M-theory [9]–[11], is not hyperelliptic, and in fact one may encounter varieties that are not Riemann surfaces at all. For SU(\( N \)) gauge theories with one hypermultiplet in the symmetric (or antisymmetric) representation (with or without additional hypermultiplets in the defining representation), the Riemann surface is described by a cubic (non-hyperelliptic) curve [11]. For a gauge theory based on a product of \( m \) factors of SU(\( N \)) with hypermultiplets in bifundamental representations, the Riemann surface is described by an \(( m + 1)\)th order curve [10, 8]. In a series of papers [12]–[15], we have developed a systematic approximation scheme to compute the instanton expansion of the prepotential for non-hyperelliptic curves of cubic and higher order. This allows one to test the predictions of M-theory and geometric engineering for field theory, thereby increasing our confidence in the validity of these string-theoretic methods.

In this paper we will discuss the SW problem for SU(\( N \)) gauge theory with two matter
hypermultiplets in the antisymmetric representation and up to 3 additional hypermultiplets in
the defining representation. Since there is no existing M-theoretic or geometric engineering
prediction for the curve for this theory, our methods will differ from previous work on this
subject. We begin by predicting the form of $F_{1-\text{inst}}$ from the observed regularities of known
prepotentials in section 2.

In section 3, we then “reverse engineer” a Seiberg-Witten curve for this theory from the
prepotential, using methods we have developed \cite{12-15} for computing the instanton expansion
for non-hyperelliptic curves. The quartic curve that we derive has the correct limiting behavior
as the mass of either of the antisymmetric hypermultiplets goes to infinity.

We then attempt an M-theory interpretation\footnote{We describe the brane structure in terms of type IIA string theory, which then lifts to M-theory \cite{10}.} of the result in section 4. The quartic curve
corresponds to a picture containing four parallel NS 5-branes, with each adjacent pair linked by
$N$ D4-branes, and four orientifold 6-planes, one on each 5-brane. Hypermultiplets in the defining
representation correspond to additional D6-branes. The reflection symmetries of the orientifold
6-planes, however, imply an expanded M-theory picture with an infinite chain of equally spaced
parallel NS 5-branes, and an infinite set of orientifold 6-planes, one lying on each of the NS
5-branes. Thus, the effective quartic curve derived in section 3 is only a truncation of a curve
of infinite order. To calculate the prepotential to any given order in the instanton expansion,
however, the curve corresponding to only a finite subset of 5-branes is needed. For example, $2d$
NS 5-branes (corresponding to a curve of order $2d$) are necessary to compute the prepotential
to $\frac{1}{2}d(d-1)$-instanton accuracy.

In section 5 we sum the infinite series representing the leading order coefficients in the curve
for certain special cases, and are able to represent the curve in terms of theta functions. This
leads us to speculate that our curve is related to a “decompactification” of the elliptic model
described in M-theory by Uranga \cite{16}, who considered the scale invariant case of SU($N$) with
two hypermultiplets in the antisymmetric representation and four hypermultiplets in the defining
representation (although he does not specify a curve for this theory). The link to our work would
be if one (or more) of the defining hypermultiplets had their mass(es) sent to infinity.

Section 6 summarizes our results, and points to issues for further study.

2. The Prepotential

The Lagrangian for an $\mathcal{N} = 2$ gauge theory to lowest order in the momentum expansion is

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[ \int d^4\theta \frac{\partial F(A)}{\partial A^i} \bar{A}^i + \frac{1}{2} \int d^2\theta \frac{\partial^2 F(A)}{\partial A^i \partial A^j} W^i W^j \right], \quad (2.1)$$

where $A^i$ are $\mathcal{N} = 1$ chiral superfields. The prepotential in the Coulomb phase has the form

$$F(A) = F_{\text{cl}}(A) + F_{1-\text{loop}}(A) + \sum_{d=1}^{\infty} \Lambda^{I(G)-I(R)}d F_{d-\text{inst}}(A), \quad (2.2)$$

where $I(G)$ [$I(R)$] is the Dynkin index of the adjoint (matter) representation. The prepotential
(2.2) may be obtained from the Seiberg-Witten data by first computing the renormalized order
parameters and their duals

$$2\pi i a_k = \oint_{A_k} \lambda \quad \text{and} \quad 2\pi i a_{D,k} = \oint_{B_k} \lambda, \quad (2.3)$$

where $\lambda$ is the Seiberg-Witten differential and $A_k$ and $B_k$ are a canonical basis of homology
cycles for the Riemann surface, and then integrating

$$a_{D,k} = \frac{\partial F}{\partial a_k}. \quad (2.4)$$

The one-instanton contribution to the prepotential, $F_{1-\text{inst}}$, for the classical groups exhibits
a remarkable universality of form when expressed in terms of the renormalized order parameters.
In particular, for $\text{SU}(N)$, the one-instanton prepotential has the form $\mathbb{3}$ [13 $\mathbb{14}$]

$$8\pi i F_{1-\text{inst}} = \sum_{k=1}^{N} S_k (a_k), \quad (2.5)$$
for (a) $N_f$ hypermultiplets in the defining representation, or (b) one hypermultiplet in the symmetric representation and $N_f$ hypermultiplets in the defining representation, while for (c) one hypermultiplet in the antisymmetric representation and $N_f$ hypermultiplets in the defining representation, it is given by \cite{12, 14}

$$8\pi i \mathcal{F}_{1-\text{inst}} = \sum_{k=1}^{N} S_k (a_k) - 2 S_m (-m).$$

(2.6)

The expressions $S_k(a_k)$ and $S_m(-m)$ in eqs. (2.5) and (2.6) are the coefficients of second order poles at $x = a_k$ and $x = -m$, respectively, of a function $S(x)$,

$$S(x) = \frac{S_k(x)}{(x - a_k)^2} = \frac{S_m(x)}{(x + m)^2},$$

(2.7)

which can be obtained from the hyperelliptic approximation to the curve, if it is known. The explicit form of $S(x)$ for each of the above theories is specified in Table 1. (The relative coefficient between the terms in (2.6) guarantees the absence of a pole in $\mathcal{F}_{1-\text{inst}}$ at $a_k = -m$.)

One observes from the table the following regularities for $S(x)$, which is a product of factors:

1. a factor

$$\frac{1}{\prod_{i=1}^{N} (x - a_i)^2},$$

(2.8)

for the pure gauge multiplet,

2. a factor

$$(x + M_j),$$

(2.9)

for each hypermultiplet of mass $M_j$ in the defining representation,

3. a factor

$$(-1)^N (x + m)^2 \prod_{i=1}^{N} (x + a_i + 2m),$$

(2.10)

for a hypermultiplet of mass $2m$ in the symmetric representation, and
| Hypermultiplet Representations | $S(x)$ |
|-------------------------------|-------|
| $N_f$ defining                | $4 \prod_{j=1}^{N_f} \frac{(x+M_j)}{\prod_{i=1}^{N_f} (x-a_i)^2}$ |
| (ref. [3])                    |       |
| 1 symmetric                   | $4(-1)^N (x+m)^2 \prod_{i=1}^{N} (x+a_i+2m) \prod_{j=1}^{N_f} (x+M_j) \prod_{i=1}^{N} (x-a_i)^2$ |
| + $N_f$ defining              |       |
| (ref. [13, 14])               |       |
| 1 antisymmetric               | $4(-1)^N \prod_{i=1}^{N} (x+a_i+2m) \prod_{j=1}^{N_f} (x+M_j) (x+m)^2 \prod_{i=1}^{N} (x-a_i)^2$ |
| + $N_f$ defining              |       |
| (ref. [12, 14])               |       |
| 2 antisymmetric               | $4 \prod_{i=1}^{N} (x+a_i+2m_1) \prod_{i=1}^{N} (x+a_i+2m_2) \prod_{j=1}^{N_f} (x+M_j) (x+m_1)^2 (x+m_2)^2 \prod_{i=1}^{N} (x-a_i)^2$ |
| + $N_f$ defining              |       |
| (This paper.)                 |       |

Table 1: The function $S(x)$ for SU($N$) gauge theory, with various matter contents. The hypermultiplets in the defining representation have masses $M_j$. The symmetric or antisymmetric representation has mass $2m$. If there are two antisymmetric representations, their masses are $2m_1$ and $2m_2$.

(4) a factor

$$\frac{(-1)^N}{(x+m)^2} \prod_{i=1}^{N} (x+a_i+2m),$$

(2.11)

for a hypermultiplet of mass $2m$ in the antisymmetric representation.

The first three entries of Table 1 almost exhaust the (generic) cases for the Coulomb phase of $\mathcal{N} = 2$ supersymmetric SU($N$) gauge theories. The remaining (generic) case of two antisymmetric hypermultiplets (with up to 3 additional hypermultiplets in the defining representation) has
not been treated to date. Since we know of no M-theoretic or geometric engineering description of this case, we begin instead by predicting $F_{1-\text{inst}}$, and then reverse engineer the SW curve. Based on the regularities described above, we postulate in the last row of Table 1 the form of $S(x)$ for two antisymmetric hypermultiplets with masses $2m_1$ and $2m_2$ and $0 \leq N_f \leq 3$ hypermultiplets in the defining representation with masses $M_j$. We then predict the one-instanton prepotential to have the form

$$8\pi i F_{1-\text{inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_{m_1}(-m_1) - 2S_{m_2}(-m_2) + C,$$

(2.12)

where, as before,

$$S(x) = \frac{S_k(x)}{(x-a_k)^2} = \frac{S_{m_1}(x)}{(x+m_1)^2} = \frac{S_{m_2}(x)}{(x+m_2)^2}.$$

(2.13)

Explicitly,

$$S_k(a_k) = \frac{4\prod_{i=1}^{N}(a_k + a_i + 2m_1)(a_k + a_i + 2m_2)\prod_{j=1}^{N_f}(a_k + M_j)}{(a_k + m_1)^2(a_k + m_2)^2\prod_{i\neq k}(a_k - a_i)^2},$$

(2.14)

$$S_{m_1}(-m_1) = \frac{4\prod_{i=1}^{N}(a_i + 2m_2 - m_1)\prod_{j=1}^{N_f}(M_j - m_1)}{(m_2 - m_1)^2\prod_{i=1}^{N}(a_i + m_1)},$$

(2.14)

$$S_{m_2}(-m_2) = \frac{4\prod_{i=1}^{N}(a_i + 2m_1 - m_2)\prod_{j=1}^{N_f}(M_j - m_2)}{(m_1 - m_2)^2\prod_{i=1}^{N}(a_i + m_2)}.$$

As before, the relative coefficients in the prepotential (2.12) guarantee the absence of poles at $a_k = -m_1$ and $a_k = -m_2$. The inclusion of the constant

$$C = \frac{16}{(m_2 - m_1)^2}\prod_{j=1}^{N_f}\left(M_j - \frac{1}{2}[m_1 + m_2]\right)$$

(2.15)

in eq. (2.12), although irrelevant to the computation of the dual order parameters, renders the prepotential finite in the limit $m_2 \to m_1$.

One can test the postulated form of the prepotential (2.12) by considering the special cases:

(a) $N = 2$, which is equivalent to SU(2) gauge theory with $N_f \leq 3$ defining hypermultiplets.
(b) $N = 3$, which is equivalent to SU(3) gauge theory with 2 anti-defining and $N_f \leq 3$ defining hypermultiplets, or equivalently, SU(3) with $2 \leq N_f \leq 5$ defining hypermultiplets.

(c) The limit $m_1$ or $m_2 \to \infty$, which removes one of the antisymmetric hypermultiplets from the theory, in which case eq. (2.12) should reduce to eq. (2.6).

In each of these cases, there is complete agreement (up to an irrelevant constant). Therefore, for the remainder of the paper we take eqs. (2.12) and (2.14) to correctly describe the one-instanton prepotential for SU($N$) gauge theory with two antisymmetric and $N_f \leq 3$ defining hypermultiplets.

Finally, we differentiate the prepotential to obtain the dual order parameter $a_{D,k}$ for this theory. Using eq. (2.12) together with the one-loop contribution to the prepotential, given by perturbation theory,

$$
F_{1\text{-loop}} = \frac{i}{8\pi} \left[ \sum_{i,j=1}^{N} (a_i - a_j)^2 \log \frac{(a_i - a_j)^2}{\Lambda^2} - \sum_{j=1}^{N_f} \sum_{i=1}^{N} (a_i + M_j)^2 \log \frac{(a_i + M_j)^2}{\Lambda^2} \right. \\
- \left. \sum_{\ell=1}^{2} \sum_{i<j}^{N} (a_i + a_j + 2m_\ell)^2 \log \frac{(a_i + a_j + 2m_\ell)^2}{\Lambda^2} \right],
$$

we use (2.2) and (2.4) to find

$$
2\pi i a_{D,k} = \left[ \text{const} a_k - 2 \sum_{i \neq k}^{N} (a_k - a_i) \log (a_k - a_i) + \sum_{j=1}^{N_f} (a_k + M_j) \log (a_k + M_j) \right. \\
+ \left. \sum_{\ell=1}^{2} \left[ \sum_{i}^{N} (a_k + a_i + 2m_\ell) \log (a_k + a_i + 2m_\ell) - 2(a_k + m_\ell) \log (a_k + m_\ell) \right] \right]
$$

$$
+ \frac{1}{2} \Lambda^{4-N_f} \left[ \frac{1}{2} \frac{\partial S_k}{\partial x} (a_k) - \sum_{i \neq k}^{N} S_i(a_i) \frac{a_k - a_i}{a_k} + \frac{1}{2} \sum_{\ell=1}^{2} \sum_{i=1}^{N} \frac{S_i(a_i)}{a_k + a_i + 2m_\ell} \right] \\
+ \frac{S_{m_1}(-m_1)}{a_k + m_1} - \frac{S_{m_1}(-m_1)}{a_k + 2m_2 - m_1} + \frac{S_{m_2}(-m_2)}{a_k + m_2} - \frac{S_{m_2}(-m_2)}{a_k + 2m_1 - m_2} + O(\Lambda^{4-2N_f}),
$$

accurate to one-instanton order. In the next section, we will reproduce this expression from the period integrals of a Riemann surface.
3. The Curve

Beginning from the one-instanton prepotential (2.12) for SU($N$) gauge theory with two antisymmetric and $N_f \leq 3$ defining hypermultiplets postulated in the last section, we will now “reverse engineer” an effective SW curve that can generate this prepotential.

The SW curve associated with SU($N$) gauge theory with matter hypermultiplets in the defining representation is quadratic [1, 2] while for the theory with one hypermultiplet in the symmetric or antisymmetric representations, the curve is cubic [11]. We expect at least a cubic curve for two antisymmetric hypermultiplets. The curve for a theory with a product of $m$ factors of SU($N$) with matter in bifundamental representations [10, 8] is of order $m + 1$, but we found [15] that to compute the dual order parameters for any of the factor groups to one-instanton accuracy, it suffices to use a quartic approximation to the full curve. (In all cases, the quadratic approximation is sufficient to compute the one-loop prepotential.)

For these reasons, we postulate a quartic curve for the theory with two hypermultiplets in the antisymmetric representation and $N_f \leq 3$, which takes the general form of the curve of ref. [10],

$$L^4 j_1(x)P_2(x) t^2 + L P_1(x) t + P_0(x)$$
$$+ Lj_0(x)P_{-1}(x) t^{-1} + L^4 j_0^2(x)j_{-1}(x) P_{-2}(x) t^{-2} = 0,$$

where $L^2 = \Lambda^{4-N_f}$ and the coefficient functions $P_n(x)$ and $j_n(x)$ are to be determined below [3]. (The $j_n(x)$ are written separately from the $P_n(x)$ to represent the contribution of the $N_f$ hypermultiplets in the defining representation.) The coefficient functions are chosen to satisfy

$$P_n(x; m_1, m_2) = P_{-n}(x; m_2, m_1);$$

$$j_n(x; m_1, m_2) = j_{-n}(x; m_2, m_1).$$

Alternatively, the factors of $j_0(x)$ can be associated with the positive powers of $t$ through the change of variables $t \rightarrow t_0(x)$, or more controversially, distributed symmetrically between positive and negative powers of $t$ through $t \rightarrow t_0^{1/2}(x)$. 

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2 Alternatively, the factors of $j_0(x)$ can be associated with the positive powers of $t$ through the change of variables $t \rightarrow t_0(x)$, or more controversially, distributed symmetrically between positive and negative powers of $t$ through $t \rightarrow t_0^{1/2}(x)$. 

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so that the curve is invariant under an involution that exchanges the two antisymmetric hyper-
multiplets

\[ t \rightarrow \frac{j_0(x)}{t}; \quad m_1 \leftrightarrow m_2. \quad (3.3) \]

We begin by changing variables

\[ t = \frac{y}{LP_1(x)}, \quad (3.4) \]

to recast the curve into a form suitable for the hyperelliptic expansion \[12\]-\[15\]

\[
\frac{L^2 j_1(x) P_2(x)}{P_1^2(x)} y^4 + y^3 + P_0(x) y^2 \\
+ L^2 j_0(x) P_1(x) P_{-1}(x) y + L^6 j_0^2(x) j_{-1}(x) P_1^2(x) P_{-2}(x) = 0. \quad (3.5)
\]

The first approximation to eq. \((3.5)\) in an instanton expansion is the hyperelliptic curve

\[ y^2 + 2A(x)y + B(x) = 0, \quad (3.6) \]

where

\[ A(x) = \frac{1}{2} P_0(x) \quad \text{and} \quad B(x) = L^2 j_0(x) P_1(x) P_{-1}(x). \quad (3.7) \]

We develop a systematic expansion about the hyperelliptic approximation \((3.6)\), where on one
of the sheets of the Riemann surface \((3.5)\),

\[ y = y_I + y_{II} + \cdots, \quad (3.8) \]

with

\[ y_I = -A - r \quad \text{and} \quad r = \sqrt{A^2 - B}, \quad (3.9) \]

the solution to the hyperelliptic approximation \((3.6)\), and \[13\]

\[
y_{II} = -\frac{(A + r)^3}{2r} \frac{L^2 j_1(x) P_2(x)}{P_1^2(x)} - \frac{1}{2r(A + r)} L^6 j_0^2(x) j_{-1}(x) P_1^2(x) P_{-2}(x), \quad (3.10)
\]
the first correction to this. This induces a comparable expansion of the SW differential,
\[ \lambda = x \frac{dy}{y} = \lambda_I + \lambda_{II} + \ldots, \quad (3.11) \]
where
\[ \lambda_I = \frac{x \left( A' - \frac{B'}{2x} \right)}{\sqrt{1 - \frac{B}{A}}} dx, \quad (3.12) \]
and
\[ \lambda_{II} = \frac{-L^2 \left[ \frac{j_1(x) P_0(x) P_2(x)}{P_1^2(x)} + \frac{j_{-1}(x) P_0(x) P_{-2}(x)}{P_{-1}^2(x)} \right]}{dx}. \quad (3.13) \]

Following the Seiberg-Witten approach, we will now use the curve \((3.5)\) together with the SW differential \((3.11)-(3.13)\) to compute the renormalized order parameters \(a_k\) and their duals \(a_{D,k}\) using eq. \((2.3)\). Our goal will be to choose \(P_n(x)\) and \(j_n(x)\) so that \(a_{D,k}\) computed from the curve agrees with eq. \((2.17)\).

The hyperelliptic curve \((3.6)\) has two sheets connected by \(N\) branch cuts extending from \(x^-_k\) to \(x^+_k\) and centered about \(x = e_k\), the zeros of \(P_0(x)\). We choose the canonical homology basis as follows: the cycle \(A_k\) is a simple contour enclosing the branch cut centered about \(e_k\); the cycle \(B_k\) goes from \(x^-_{k-1}\) to \(x^-_k\) on one sheet and from \(x^+_k\) to \(x^-_1\) on the other. The order parameter \(2\pi i a_k = \oint_{A_k} \lambda\) is calculated as in ref. \[3\] \(a_k = e_k + O(L^2). \quad (3.14)\]

From Sec. 5 of ref. \[2\], one of the contributions to the first approximation to the dual order parameter \((2\pi i a_{D,k})_I = \oint_{B_k} \lambda_I\) is
\[ \frac{1}{2} \int_{x^-_1}^{x^-_k} dx \frac{B}{A^2} = 2L^2 \int_{x^-_1}^{x^-_k} dx \frac{j_0(x) P_1(x) P_{-1}(x)}{P_0^2(x)}. \quad (3.15) \]
The integrand has second-order poles at \(x = e_k\) from the factor \(1/P_0^2(x)\). The coefficients of these poles are chosen to be \(S_k(a_k)\), in analogy with the first three entries of Table 1, \(i.e.,\)
\[ \frac{4L^2 j_0(x) P_{-1}(x) P_1(x)}{P_0^2(x)} = L^2 \sum_k \frac{S_k(a_k)}{(x - e_k)^2} + \cdots. \quad (3.16) \]
As a result, eq. (3.15) will produce the term \(-\frac{1}{2}L^2 \sum_{i \neq k} S_i(a_i)/(a_k - a_i)\) in eq. (2.17). Equation (3.16) is attained by setting

\[ j_0(x) \frac{P_{-1}(x)P_1(x)}{P_0^2(x)} = \frac{1}{4} S(x) + O(L^2), \]

where (from the last entry of Table 1)

\[ S(x) = \frac{4 \prod_{i=1}^{N}(x + a_i + 2m_1) \prod_{i=1}^{N}(x + a_i + 2m_2) \prod_{j=1}^{N}(x + M_j)}{(x + m_1)^2 (x + m_2)^2 \prod_i(x - a_i)^2}. \]

(3.18)

The correction to the dual order parameter \[ 5 \]

\[ (2\pi a_{D,k})_{II} = 2 \int_{x_1}^{x_k} \lambda_{II} = -L^2 \int_{x_1}^{x_k} dx \left[ \frac{j_1(x)P_0(x)P_2(x)}{P_1^2(x)} + \frac{j_{-1}(x)P_0(x)P_{-2}(x)}{P_{-1}^2(x)} \right], \]

(3.19)

evaluated using Sec. 5(b) of ref. \[ 2 \], gives rise to the term \( \frac{1}{4}L^2 \sum_{\ell=1}^N S_i(a_i)/(a_k + a_i + 2m_k) \) in eq. (2.17) if we choose

\[ j_1(x) \frac{P_0(x)P_2(x)}{P_1^2(x)} = \frac{1}{4} S(-x - 2m_2) + O(L^2), \]

(3.20)

and

\[ j_{-1}(x) \frac{P_{-2}(x)P_0(x)}{P_{-1}^2(x)} = \frac{1}{4} S(-x - 2m_1) + O(L^2). \]

(3.21)

The remaining terms of eq. (2.17) arise from subleading (in \( L \)) terms of the coefficient functions, discussed later in this section.

Observe from the right-hand side of (3.17), (3.20) and (3.21) that the ratio

\[ j_n(x) \frac{P_{n-1}(x)P_{n+1}(x)}{P_n^2(x)} \]

is invariant, up to a predictable reflection and shift in the argument of \( S(x) \). This fact will be useful in understanding the M-theory interpretation of our results.

Equations (3.17), (3.20), and (3.21) suffice to determine the leading order terms of \( P_n(x) \) and \( j_n(x) \). The general solution to these equations is

\[ P_2(x) = F(x) G(x)^2 (x + m_2)^{-6} (x + 2m_2 - m_1)^{-2} \prod_{i=1}^{N}(x - a_i + 2m_2 - 2m_1) + O(L^2), \]
\[
P_1(x) = F(x)G(x)(x + m_2)^{-2}(-1)^N \prod_{i=1}^N (x + a_i + 2m_2) + O(L^2),
\]
\[
P_0(x) = F(x) \prod_{i=1}^N (x - a_i) + O(L^2),
\]
\[
P_{-1}(x) = F(x)G(x)^{-1}(x + m_1)^{-2}(-1)^N \prod_{i=1}^N (x + a_i + 2m_1) + O(L^2),
\]
\[
P_{-2}(x) = F(x)G(x)^{-2}(x + m_1)^{-6}(x + 2m_1 - m_2)^{-2} \prod_{i=1}^N (x - a_i + 2m_1 - 2m_2) + O(L^2),
\]
\[
j_1(x) = (-1)^{N_f} \prod_{j=1}^{N_f} (x + 2m_2 - M_j),
\]
\[
j_0(x) = \prod_{j=1}^{N_f} (x + M_j),
\]
\[
j_{-1}(x) = (-1)^{N_f} \prod_{j=1}^{N_f} (x + 2m_1 - M_j),
\]

where \(F(x)\) and \(G(x)\) are arbitrary functions. The function \(F(x)\) can be simply factored out of the curve, and \(G(x)\) eliminated by the change of variables \(t \rightarrow t/G(x)\).

A check of these coefficient functions is obtained if one of the antisymmetric hypermultiplets is removed from the spectrum by letting its mass go to infinity. One may verify that in the limit \(m_2 \rightarrow \infty\), the quartic curve given by (3.5) and (3.23) reduces, after the redefinition
\[
L^2 \rightarrow m_2^2(-2m_2)^{-N}L^2,
\]
to the cubic curve [13, 14] for a single hypermultiplet in the antisymmetric representation and \(N_f\) hypermultiplets in the defining representation, for leading terms of the coefficient functions. The same result holds in the \(m_1 \rightarrow \infty\) limit, in light of the involution (3.3).

Consideration of the one-instanton contribution to the prepotential also allows us to place some contraints on (but not uniquely determine) the subleading (in \(L\)) terms of \(P_n(x)\). We postulate that the coefficient functions in eq. (3.1) have subleading terms of the form
\[
P_1(x) = (x + m_2)^{-2} \left[ (-1)^N \prod_{i=1}^N (x + a_i + 2m_2) + L^2Q(-x - 2m_2) + O(L^4) \right],
\]
\[ P_0(x) = \prod_{i=1}^{N} (x - a_i) + L^2 Q(x) + O(L^4), \]
\[ P_{-1}(x) = (x + m_1)^{-2} \left[ (-1)^N \prod_{i=1}^{N} (x + a_i + 2m_1) + L^2 Q(-x - 2m_1) + O(L^4) \right], \quad (3.25) \]

with
\[ Q(x) = \frac{3A^{(1)}}{(x + m_1)^2} + \frac{B^{(1)}}{(x + m_1)} + \frac{3A^{(2)}}{(x + m_2)^2} + \frac{B^{(2)}}{(x + m_2)} + \frac{3A^{(3)}}{(x + 2m_2 - m_1)^2} + \]
\[ \frac{B^{(3)}}{(x + 2m_2 - m_1)} + \frac{3A^{(4)}}{(x + 2m_1 - m_2)^2} + \frac{B^{(4)}}{(x + 2m_1 - m_2)} + \frac{3A^{(5)}}{(x + 3m_1 - 2m_2)^2} + \]
\[ \frac{B^{(5)}}{(x + 3m_1 - 2m_2)} + \frac{3A^{(6)}}{(x + 3m_2 - 2m_1)^2} + \frac{B^{(6)}}{(x + 3m_2 - 2m_1)} + \cdots \quad (3.26) \]

The expression (3.26) for \( Q(x) \) is motivated by the results of sec. 4 of this paper, in which an infinite number of orientifold sixplanes are required in order to satisfy the reflection symmetries of the curve. Eq. (3.26) reduces to
\[ Q(x) \rightarrow \frac{3A^{(1)}}{(x + m_1)^2} + \frac{B^{(1)}}{(x + m_1)} \quad (3.27) \]
in the limit \( m_2 \to \infty \), in agreement with the form of curve for one antisymmetric and \( N_f \) defining hypermultiplets [11, 14].

The constants \( A^{(1)}, B^{(1)}, \ldots \) in eq. (3.26) are constrained by:

(a) the involution symmetry (3.3),
(b) the absence of \( \log(a_k + m_1), \log(a_k + 2m_2 - m_1), \) etc. terms in \( a_{D,k} \),
(c) the absence of poles in \( (a_k + 3m_2 - 2m_1), (a_k + 4m_2 - 3m_1) \), etc. in \( a_{D,k} \),
(d) agreement of the coefficients of the simple poles at \( (a_k + m_1), (a_k + 2m_2 - m_1), (a_k + m_2), \) and \( (a_k + 2m_1 - m_2) \) in \( a_{D,k} \) calculated from the curve with those in the expression (2.17), and
(e) the correct limiting behavior of the subleading terms as \( m_1 \to \infty \) and \( m_2 \to \infty \).

These constraints give rise to a set of recursion relations that determine all but two of the constants. We are unaware of any additional field-theoretic constraints that would allow us to fix the subleading terms of the curve uniquely.
We mention that if one arbitrarily truncates the expression $Q(x)$ to a finite number of terms, the constants are uniquely determined by the constraints. The resulting curve, however, has an unphysical singular limit as $m_2 \to m_1$. We therefore conclude that such a truncation is inconsistent.

4. M-theory picture

In section 3, we showed that the quartic SW curve

$$L^4 j_1(x)P_2(x)t^2 + LP_1(x)t + P_0(x) + Lj_0(x)P_{-1}(x)t^{-1} + L^4 j_0(x)j_{-1}(x)P_{-2}(x)t^{-2} = 0,$$

(4.1)

with coefficient functions (given to leading order in $L$) given by eqs. (3.23) ff, as

$$P_2(x) = (x + m_2)^{-6} (x + 2m_2 - m_1)^{-2} \prod_{i=1}^{N}(x - a_i + 2m_2 - 2m_1) + O(L^2),$$

$$P_1(x) = (x + m_2)^{-2} \left[ (-1)^N \prod_{i=1}^{N}(x + a_i + 2m_2) + O(L^2) \right],$$

$$P_0(x) = \prod_{i=1}^{N}(x - a_i) + O(L^2),$$

$$P_{-1}(x) = (x + m_1)^{-2} \left[ (-1)^N \prod_{i=1}^{N}(x + a_i + 2m_1) + O(L^2) \right],$$

$$P_{-2}(x) = (x + m_1)^{-6} (x + 2m_1 - m_2)^{-2} \prod_{i=1}^{N}(x - a_i + 2m_1 - 2m_2) + O(L^2),$$

$$j_1(x) = (-1)^N f \prod_{j=1}^{N_f}(x + 2m_2 - M_j),$$

$$j_0(x) = \prod_{j=1}^{N_f}(x + M_j),$$

$$j_{-1}(x) = (-1)^N f \prod_{j=1}^{N_f}(x + 2m_1 - M_j),$$

(4.2)

gives rise to the one-instanton prepotential (2.12) postulated in section 2 for SU($N$) gauge theory with two antisymmetric hypermultiplets and $N_f$ defining hypermultiplets. In this section, we
interpret this curve in terms of M-theory \[3\]-\[11\]. We then present evidence that the curve (4.1), though sufficient to generate \( F_{1-{\text{inst}}} \), is only an effective curve, and is in fact embedded in an infinite power series in \( t \).

We begin by attempting to associate an M-theory picture with the quartic curve (4.1). Such a picture involves four parallel NS 5-branes, with each adjacent pair connected by \( N \) parallel D4-branes (see figure 1). The factors \( \prod_{i=1}^{N} (x + a_i + 2m_2) \), \( \prod_{i=1}^{N} (x - a_i) \), and \( \prod_{i=1}^{N} (x + a_i + 2m_1) \), in \( P_1, P_0 \), and \( P_{-1} \), respectively, determine the positions of the connecting D4-branes. There are also D6-branes between the NS 5-branes, which correspond to the factors of \( j_n(x) \) in eq. (4.1) representing the hypermultiplets in the defining representation \[10\].

As noted in the previous section, the quartic curve (4.1) reduces in the limit \( m_2 \to \infty \) to a cubic curve describing SU(\( N \)) gauge theory with one antisymmetric hypermultiplet of mass \( 2m_1 \). In the M-theory picture (fig. 1), in this limit the D4-branes dependent on \( m_2 \) slide off to infinity, and the NS 5-brane denoted by \( \bullet \) becomes disconnected, leaving three parallel NS 5-branes connected by \( N \) parallel D4-branes. (In the limit \( m_1 \to \infty \), the NS 5-brane \( \bigcirc \) becomes disconnected instead.)

Recall that the M-theory picture for a single antisymmetric hypermultiplet involves a negative charge orientifold sixplane (O6−) on the central of three parallel NS 5-branes \[11\]. This pair of O6− planes must also be present before the limits \( m_1 \to \infty \) and \( m_2 \to \infty \) are taken, and we indicate their positions in fig. 1 by \( \otimes \)'s on \( \bigcirc \) and \( \bigcirc \). The factors (and exponents) of \( (x + m_1) \) and \( (x + m_2) \) in the coefficient functions (4.2) are exactly those expected for O6− planes in these locations. To see this, recall that the presence of an O6− plane at \( x = -m_1 \) implies that the geometry far from the orientifold is represented by the complex manifold \[11\]

\[
\hat{t} \hat{t} = \frac{L^2 j_0(x) j_{-1}(x)}{(x + m_1)^4} \tag{4.3}
\]

\(^3\)We describe the brane structure in terms of type IIA string theory, which then lifts to M-theory \[10\].
and that the curve should be invariant under the orientifold projection

\[ x \rightarrow -x - 2m_1 \quad ; \quad t \rightarrow \hat{t}. \]  

(4.4)

Imposing this invariance on the last four terms of the curve (1.1) (those that remain when \( m_2 \rightarrow \infty \)) yields the relations \( P_{-2}(x) = (x+m_1)^{-6}P_1(-x-2m_1) \) and \( P_{-1}(x) = (x+m_1)^{-2}P_0(-x-2m_1) \), in agreement with (4.2). A similar story holds for the factors of \( x + m_2 \). The fact that the full quartic curve (1.1) is not invariant under the projection (4.4) is the first indication that this curve is incomplete.

O6\textsuperscript{−} planes represent reflection symmetries in the M-theory picture, and the reflections of the two O6\textsuperscript{−} planes on \( \mathcal{O}_2 \) and \( \mathcal{O}_3 \) generate an infinite number of parallel NS 5-branes, with an O6\textsuperscript{−} plane on each of them. The factors of \( (x + 2m_2 - m_1) \) and \( (x + 2m_1 - m_2) \) in the coefficient functions (4.2) exactly correspond to O6\textsuperscript{−} planes on \( \mathcal{O}_1 \) and \( \mathcal{O}_4 \), and the factors of \( \prod_{i=1}^{N}(x - a_i + 2m_2 - 2m_1) \) and \( \prod_{i=1}^{N}(x - a_i + 2m_1 - 2m_2) \) in \( P_2 \) and \( P_{-2} \), respectively, represent D4-branes connecting \( \mathcal{O}_1 \) and \( \mathcal{O}_4 \) to the rest of the infinite chain of NS 5-branes.

The necessary presence of additional NS 5-branes may equivalently be seen by requiring the SW curve to possess the involution symmetries implied by the O6\textsuperscript{−} planes at \( x = -m_1 \) and \( x = -m_2 \). As we saw above, the last four terms of the curve (1.1) are invariant under the involution

\[ x \rightarrow -x - 2m_1 \quad ; \quad t \rightarrow \frac{L^2 j_0(x) j_{-1}(x)}{(x + m_1)^2 t}, \]  

(4.5)

but invariance of the full curve requires the presence of a \( t^{-3} \) term. Similarly, the first four terms of the curve (1.1) are invariant under the involution

\[ x \rightarrow -x - 2m_2 \quad ; \quad t \rightarrow \frac{(x + m_2)^4}{L^2 t}, \]  

(4.6)

but invariance of the full curve requires the presence of a \( t^3 \) term. Only a curve of infinite order can be simultaneously invariant under both involutions.
Figure 2 represents an expanded view of the brane configuration, involving six parallel NS 5-branes. The positions of the D4-branes, D6-branes, and O6$^-$ planes are completely dictated by reflection symmetries. The sextic curve associated with this truncation of the infinite chain of branes is

$$L^9 j_1^2(x) j_2(x) P_3(x) t^3 + L^4 j_1(x) P_2(x) t^2 + L P_1(x) t + P_0(x) + L j_0(x) P_{-1}(x) t^{-1}$$

$$+ L^4 j_0(x) j_{-1}(x) P_{-2}(x) t^{-2} + L^9 j_0^3(x) j_{-1}^2(x) j_{-2}(x) P_{-3}(x) t^{-3} = 0,$$  \hspace{1cm} (4.7)

and would be required to compute the prepotential to three-instanton accuracy. The involution symmetries (4.5) and (4.6) determine the new coefficients (to leading order in $L$) to be

$$P_3(x) = (x + m_2)^{-10} (x + 2m_2 - m_1)^{-6} (x + 3m_2 - 2m_1)^{-2} (-1)^N \prod_{i=1}^{N} (x + a_i + 4m_2 - 2m_1),$$

$$P_{-3}(x) = (x + m_1)^{-10} (x + 2m_1 - m_2)^{-6} (x + 3m_1 - 2m_2)^{-2} (-1)^N \prod_{i=1}^{N} (x + a_i + 4m_1 - 2m_2),$$

$$j_2(x) = \prod_{j=1}^{N_f} (x + 2m_2 - 2m_1 + M_j),$$

$$j_{-2}(x) = \prod_{j=1}^{N_f} (x + 2m_1 - 2m_2 + M_j).$$ \hspace{1cm} (4.8)

Observe from eqs. (4.2) and (4.8) that the ratio

$$j_n(x) \frac{P_{n-1}(x) P_{n+1}(x)}{P_n^2(x)}$$ \hspace{1cm} (4.9)

is identical to (3.17) up to a predictable reflection and shift in the argument of $S(x)$. One may take this as a general principle, which implies that one may take any pair of adjacent parallel 5-branes to define a hyperelliptic approximation for the instanton expansion. This observation leads to results equivalent to the successive imposition of involution symmetries such as (4.5) and (4.6) for all possible embedded quartic curves.

Figure 3 shows that in the $m_2 \to \infty$ limit, the M-theory picture of fig. 2 reduces to that of one antisymmetric representation with an O6$^-$ plane on NS 5-brane $\mathfrak{O}$. Had we taken $m_1 \to \infty$
instead, the analogue to figure 3 would have involved 5-branes $\circled{1}$, $\circled{2}$, and $\circled{3}$ connected by D4 branes with an O6$^-$ plane on NS 5-brane $\circled{2}$.

The full curve describing SU($N$) gauge theory with two antisymmetric and $N_f$ defining hypermultiplets, in which the quartic (4.10) and sextic (4.17) approximations are embedded, is

\[ \sum_{n=1}^{\infty} L_n^2 \prod_{s=1}^{n-1} j^{n-s}_s(x)P_n(x)t^n + P_0(x) + \sum_{n=1}^{\infty} L_n^2 j_0^n(x) \prod_{s=1}^{n-1} j^{n-s}_s(x)P_{-n}(x)t^{-n} = 0. \] (4.10)

The involution symmetries (4.5) and (4.6) imply the following recursion relations

\begin{align*}
P_{-n-1}(x) &= (x + m_1)^{-4n-2}P_n(-x - 2m_1), \\
P_{n+1}(x) &= (x + m_2)^{-4n-2}P_{-n}(-x - 2m_2), \\
j_{-n-1}(x) &= j_n(-x - 2m_1), \\
j_{n+1}(x) &= j_{-n}(-x - 2m_2),
\end{align*} (4.11)

which fully determine the coefficient functions of the curve (including subleading terms) in terms of $P_0(x)$. Note that the coefficients determined from the recursion relations (4.11) imply that the ratio (4.9) is invariant, up to a predictable reflection and shift in the argument of $S(x)$.

5. Summing the Series

In this section we sum the infinite series (4.10) for certain special cases, obtaining results in terms of theta functions. The solution to the recursion relations (4.11) allows one to write (4.10) explicitly, keeping the leading terms only,

\[ \sum_{n=-\infty}^{n=\infty} L_n^2 t^n (-1)^N n^{1/2} j_0^{n/2}(n)|n|^{n/2} \prod_{s=1}^{n-1} j^{n-s}_{s/|n|}(x) \left( x - (-1)^n a_i - n \Delta + \frac{1}{2}(1 - (-1)^n)(m_1 + m_2) \right) \]

\[ \times \prod_{i=1}^{N} \left[ x + (-1)^n m_1 - n \Delta + \frac{1}{2}(1 - (-1)^n)(m_1 + m_2) \right] \]

\[ \times \prod_{p=0}^{\left| n \right|-1} \left[ x + m_1 - n \Delta + \left( \frac{n - |n|}{2|n|} \right) \Delta + \frac{n}{|n|} p \Delta \right]^{-(4p+2)} = 0, \] (5.1)
where $\Delta = m_1 - m_2$, and

$$j_0(x) = \prod_{j=1}^{N_f} (x + M_j),$$

$$j_{ns/|n|}(x) = (-1)^{n|n|} \prod_{j=1}^{N_f} \left[ x - \frac{n}{|n|} \Delta + (-1)^{n} M_j + \frac{1}{2}(1 - (-1)^{s})(m_1 + m_2) \right]. \quad (5.2)$$

Rather than attempting to sum (5.1) for the general case, let us consider the special case $m_1 = m_2 = m$ (i.e., $\Delta = 0$), for which the curve simplifies to

$$\sum_{n=-\infty}^{\infty} \frac{L^{n^2} t^n}{(x + m)^{2n} j_0^{1/4} j_1^{1/4} (x)} \prod_{s=1}^{(|n|-1)/2} j_{ns/|n|}^{1/4} (x) = 0,$$  \quad (5.3)

where

$$\prod_{s=1}^{(|n|-1)/2} j_{ns/|n|}^{1/4} (x) = j_{0}^{(|n|-1)/2} j_{1}^{n-1/2} \quad \text{for } n \text{ even},$$

$$= j_{0}^{(|n|-1)/2} j_{1}^{n^2-1/2} \quad \text{for } n \text{ odd}, \quad (5.4)$$

with

$$j_0(x) = \prod_{j=1}^{N_f} (x + M_j),$$

$$j_1(x) = (-1)^{N_f} \prod_{j=1}^{N_f} (x + 2m - M_j). \quad (5.5)$$

Defining

$$H_0(x) = \prod_{i=1}^{N} (x - a_i) + O(L^2),$$

$$H_1(x) = (-1)^{N_f} \prod_{i=1}^{N_f} (x + a_i + 2m) + O(L^2), \quad (5.6)$$

we rewrite the curve (5.3) as

$$H_0(x) \sum_{n \text{ even}} \left[ \frac{L^{n^2} j_0^{1/4} j_1^{1/4} (x)}{(x + m)^n} \right] n^2 \left( t j_0^{-1/2} (x) \right)^n$$

$$+ H_1(x) \left( \frac{j_0(x)}{j_1(x)} \right)^{1/4} \sum_{n \text{ odd}} \left[ \frac{L^{n^2} j_0^{1/4} j_1^{1/4} (x)}{(x + m)^n} \right] n^2 \left( t j_0^{-1/2} (x) \right)^n = 0, \quad (5.7)$$

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which may be expressed as

\[
H_0(x) \sum_{n=-\infty}^{\infty} q(x)^{n^2} \tilde{t}^n + \left( \frac{j_0(x)}{j_1(x)} \right)^{1/4} H_1(x) \sum_{n=-\infty}^{\infty} q(x)^{(n+\frac{1}{2})^2} \tilde{t}^{(n+\frac{1}{2})} = 0, \quad (5.8)
\]

with

\[
q(x) = \frac{L^4 j_0(x) j_1(x)}{(x+m)^8} \quad \text{and} \quad \tilde{t} = t^2 j_0^{-1}(x). \quad (5.9)
\]

The curve (5.8) may now be recast in terms of theta functions as

\[
H_0(x) \theta_3(s|q(x)) + \left( \frac{j_0(x)}{j_1(x)} \right)^{1/4} H_1(x) \theta_2(s|q(x)) = 0,
\quad (5.10)
\]

where \( \tilde{t} = e^{2\pi i s} \) and

\[
\theta_3(s|q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi i n s},
\]

\[
\theta_2(s|q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{2\pi i (n+\frac{1}{2}) s}.
\quad (5.11)
\]

Since the theta functions (5.11) are only defined for \(|q(x)| < 1\), however, we observe from eq. (5.9) that the series (5.8) is not well-defined for \(x \to -m\), nor for large \(L\). This may be an indication that eq. (5.3) is inconsistent with the \(O(L^2)\) subleading terms omitted. It is possible that these subleading terms sum up to allow a continuation to regions where \(|q(x)| > 1\). As we saw in sec. 3, however, such subleading terms are not uniquely determined by the one-instanton prepotential. Nor do we have a prediction of these subleading terms from M-theory.

Uranga [16] considers the scale invariant case of SU(\(N\)) with two antisymmetric hypermultiplets and four defining hypermultiplets. This corresponds to an elliptic model [17] with a brane configuration of two NS 5-branes, with an O6\(^-\) orientifold on each and two sets of \(N\) D4-branes connecting the two NS 5-branes pair-wise around a circle \(S^1\) in the \(t\)-direction (or the \(x_6\)-direction in the notation of ref. [10]), together with \(N_f = 4\) D6-branes. Our results for \(N_f \leq 3\) may be regarded as a decompactification of Uranga’s elliptic model, where \(t\) is now the covering space of

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the $S^1$ of the elliptic model. (For $\Delta = 0$, this is an untwisted elliptic model.) This interpretation is supported by the fact that the terms of the series (5.3) sum up to form theta functions.

Note also the work of Yokono [18], who considered softly broken $\mathcal{N} = 4$ USp($2N_c$) and SO($N_c$) gauge theory. He finds a brane picture analogous to ours (see for example his figs. 2 and 3). An infinite number of parallel 5-branes and O4$^-$-planes result from the decompactification of an elliptic model.

We return to the curve (5.1), taking $N_f = 0$ for simplicity. It is straightforward to expand the curve in powers of $\Delta = m_1 - m_2$, and express the result in terms of theta functions. To first order in $\Delta$, keeping only the leading terms of $P_n(x)$, we find

$$
(1 - \frac{2\Delta}{3(x + m_1)} \left[ 8(i \frac{\partial}{\partial \tilde{t}})(q(x)\frac{\partial}{\partial q(x)}) - 6q(x)\frac{\partial}{\partial q(x)} + i \frac{\partial}{\partial \tilde{t}} \right]) \left[ H_0(x)\theta_3(s|q(x)) + H_1(x)\theta_2(s|q(x)) \right] \\
+ 2\Delta \frac{\partial}{\partial z} H_0(x+z)|_{z=0} (\tilde{t} \frac{\partial}{\partial \tilde{t}} \theta_3(s|q(x))) + 2\Delta \frac{\partial}{\partial z} H_1(x+z)|_{z=0} (\tilde{t} \frac{\partial}{\partial \tilde{t}} - \frac{1}{2}) \theta_2(s|q(x)) = 0
$$

(5.12)

where eqs. (5.6), (5.9), and (5.11) have been used, and $\tilde{t} = e^{2\pi i s}$ as before. Again this is valid only for $|q(x)| < 1$. The derivation of (5.10) or the expected elliptic model from an integrable model remains a challenging problem for future work.

6. Summary

From our previous work, we are able to exhibit sufficient universality in the form of $\mathcal{F}_{1-\text{inst}}$ for SU($N$) with matter hypermultiplets to present an extremely plausible form (2.12) for $\mathcal{F}_{1-\text{inst}}$ for two antisymmetric and $0 \leq N_f \leq 3$ defining hypermultiplets. Using methods developed in refs. [12]–[15] for a systematic instanton expansion based on a perturbative expansion beginning with a hyperelliptic approximation to a SW curve, we were able to “reverse engineer” a quartic curve which reproduces $\mathcal{F}_{1-\text{inst}}$. The leading order terms in $L$ are unique, and there are strong constraints on the subleading terms. When the mass of either of the antisymmetric hypermultiplets goes to infinity, the curve reduces to that for one antisymmetric and $N_f$ defining
hypermultiplets.

The quartic curve constructed in this way led us to an M-theory picture containing four NS 5-branes, connected by D4-branes. However, since there are also O6− planes on each of the parallel 5-branes, we were forced to consider an infinite chain of 5-branes and O6− planes. A finite subset of 2d of these 5-branes yields an effective curve of order 2d, which is necessary to compute the prepotential to \( \frac{1}{2}d(d - 1) \)-instanton accuracy. Without requiring consistency with M-theory, one could have stopped with the quartic curve of sec. 3, if the only input were \( \mathcal{F}_{1-\text{inst}} \). A computation of \( \mathcal{F}_{3-\text{inst}} \) from an underlying Lagrangian, which could be compared with the 3-instanton prediction of the sextic curve\(^4\) of sec. 4, would therefore provide support for the M-theory picture we have developed.

It is interesting that the SW curve and M-theory picture for SU(\(N\)) gauge theory with two antisymmetric hypermultiplets differs so radically from that with only one antisymmetric hypermultiplet. It is not a “trivial” extension of known results, as we had originally anticipated, but is in fact much richer. Nevertheless, our curve and M-theory picture reduce to that of ref. \([11]\) in the large \(m_1\) or \(m_2\) limit.

The summation of the infinite series representing the curve allowed us to represent it in terms of theta functions in sec. 5. This suggests that our curve may be related to the decompactification of a scale-invariant elliptic model. Uranga \([16]\) has discussed an M-theory picture for SU(\(N\)) with two antisymmetric and four defining hypermultiplets, a scale-invariant case, but without specifying a curve. We speculate that, were the curve for this theory known, sending the mass of one or more of the defining hypermultiplets to infinity would be consistent with our analysis.

Softly broken \( \mathcal{N} = 4 \) SO(\(N_c\)) and USp(2\(N_c\)) gauge theories have been considered by Yokono \([18]\). His M-theory picture appears to be compatible with our analysis as well, in the sense that

\( ^4\)This prediction is somewhat ambiguous because the subleading coefficients of the curve are not known exactly.
there is a decompactification with an infinite number of equally spaced parallel 5-branes, and an infinite number of orientifold planes.

Finally, we remark that the observed universality of form for $\mathcal{F}_{1-\text{inst}}$ shown in Table 1 is still without a satisfactory derivation from first principles, particularly when the curve is not hyperelliptic. It is clear from this Table, however, that the renormalized order parameters are the natural variables for this problem, as emphasized in refs. \cite{3} and \cite{12–15}.

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Figure 1: Four parallel, equally-spaced (in $t$) NS 5-branes are indicated by solid vertical lines, numbered ① to ④ above. $N$ parallel D4-branes, linking each pair of adjacent 5-branes, are indicated by horizontal dashed lines, with the position of the $i$th 4-brane ($i = 1$ to $N$) shown in the figure. D6-branes are indicated by □ and O6$^{-}$ planes by ⊗, with their positions in the $x$-direction shown. All elements in the figure respect the (multiple) mirror symmetries of the O6$^{-}$ planes. The reflection symmetries imply additional 4-branes to the left and right of 5-branes ① and ④, respectively, and an infinite number of parallel 5-branes, etc. Our $x$ corresponds to the variable $v = x_4 + ix_5$ and $\ln t = -(x_6 + ix_{10})/R$ of ref. [10].
Figure 2: An expanded version of figure 1 with six NS 5-branes.
Figure 3: The $m_2 \to \infty$ limit of Figure 2. In this limit, only the 5-branes ②, ③, and ④ remain connected by 4-branes. The other 4-branes and O6− planes have “slid off” to $x \sim \infty$. 
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