Geometrically nonlinear buckling analysis of truss under mechanical and thermal load based on mixed finite element formulation

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Abstract. This paper is concerned with the numerical method for the geometrically nonlinear buckling analysis of truss under mechanical load and thermal load due to constant temperature change. The solution of a geometrically nonlinear problem of the system under thermal load based on a displacement-based finite formulation requires the implementation of thermal deformation constraint depending on the incremental element length. The additional constraint of thermal expansion considerably increases the difficulty in finite element formulation and solving method of the nonlinear buckling problem. This research proposes a novel approach to formulate the nonlinear buckling problem of truss under thermal load based on mixed finite formulation. The mixed balanced equation of thermal loading truss considering large displacement is formulated using the principle of virtual work. Based on the formulated equation, the research develops an incremental-iterative algorithm and calculation program for solving the nonlinear buckling problem of thermal loading truss using the arc length technique. The numerical test is presented to investigate the effect of thermal load to buckling behaviour of plan truss.

1. Introduction
Problems of thermal loading arise in many practical design of structural system including trusses. The temperature variation applied on the truss system tends to make the truss element physically expand or contract. The thermal deformation of truss element will significantly affect to the behavior of the structure system especially in buckling and post-buckling analysis. In geometrical linear finite element analysis the thermal loads are usually calculated using the restraining method suggested by J. M. C. Duhamel (1838) by adding the thermal loads to the nodal external force vector as equivalent loads [1-3]. In geometrical nonlinear finite element analysis, the thermal deformation of truss element is dependent on the incremental element length. Therefore, the solving a geometrically nonlinear problem of system under thermal loading using a displacement-based finite element requires the implementation of additional thermal deformation constraint. The implementation of additional constraints is required to produce modified system of equations by using mathematical methods for constrained optimization such as penalty augmentation method or Lagrange multiplier adjunction method [4-7]. The operation of imposing thermal deformation constraint considerably increases the
difficulty in the finite element formulation and solving nonlinear buckling problem. This research introduces novel approach to formulate the geometrical nonlinear buckling problem of truss under thermal loading to escape difficulties imposing the thermal deformation constraint based on mixed finite formulation. Based on mixed variational formulation the mixed finite truss element, including length-dependent thermal deformation due to constant temperature change, is constructed. The research formulates a mixed balanced equation of thermal loading truss considering large displacement using the principle of virtual work. Based on the formulated equation, the research develops an incremental-iterative algorithm and calculation program for solving the nonlinear buckling problem of thermal loading truss using the arc length technique. For investigating the effect of thermal loading to buckling behaviour of plan truss presented numerical test in different cases of temperature change

2. Mixed finite element formulation of truss problem under thermal load considering large displacement

2.1. Balanced equation of thermal loading truss element

Consider the homogeneous truss bar subjects to uniform thermal loads due to a constant temperature change $\Delta T$ shown in Figure 1. Designate the following

$L_0$ and $L$ - distances between two end nodes of the truss element before and after deformation;

$u_1, u_2, u_3, u_4$ and $P_1, P_2, P_3, P_4$ - nodal displacements and forces in a global coordinate system;

$P_e$ - resultant external force at the $i$th cross section after deformation;

$N$ - internal normal force of truss element;

$A$ - cross-sectional area of truss element; $E$ - modulus of elasticity of the material.

$\alpha$ - linear thermal expansion coefficient; $\Delta L_T = \alpha \cdot \Delta T \cdot L_0$ - the change in length due to thermal expansion [8] in case of temperature change $\Delta T$;

![Figure 1. Mixed thermal loading truss element considering large displacement.](image)

Using the relationship in geometry, the deformed length of the truss element is given by

$$L = \sqrt{(L_0 \sin \alpha_0 + u_4 - u_2)^2 + (L_0 \cos \alpha_0 + u_3 - u_1)^2}$$

The axial deformation of truss element is computed by

$$\Delta L = L - L_T = L - L_0 - \Delta L_T$$

Work done by internal forces is calculated as follows

\[ \delta V = - \int_0^L \sigma, \delta e, dV = - \int_0^L \sigma, dA \int_0^L \delta e, dx = -N \int_0^L \delta (\Delta x) \, dx = -N \int_0^L \delta \left( \frac{\Delta L}{\Delta x} \right) \, dx = \left[ N \delta \left( \frac{\Delta L}{\Delta x} \right) \right]_0^L \]

The virtual work due to external forces is computed by

\[ \delta V = P_i \delta u_i + P_2 \delta u_2 + P_3 \delta u_3 + P_4 \delta u_4 + P_e \delta \Delta L_e = \sum_{i=1}^4 P_i \delta u_i - P_e \delta \Delta L_e \]

The total work done by internal and external forces is obtained by summing Eq. (3) and Eq. (4),

\[ \delta V + \delta V = -N \left\{ \sum_{i=1}^4 \frac{\partial \Delta L}{\partial u_i} \delta u_i + \frac{\partial \Delta L}{\partial \Delta L_e} \delta \Delta L_e \right\} + \left\{ \sum_{i=1}^4 P_i \delta u_i - P_e \delta \Delta L_e \right\} = \sum_{i=1}^4 \left[ -N \frac{\partial \Delta L}{\partial u_i} + P_i \right] \delta u_i + \left\{ -N \frac{\partial \Delta L}{\partial \Delta L_e} - P_e \right\} \delta \Delta L_e = 0 \]

Using the principle of virtual work [3], it can be given

\[ \left\{ -N \frac{\partial \Delta L}{\partial u_i} + P_i = 0 \quad (i = 1, 2, 3, 4) \right\} \]

Replacing the axial deformation of truss in Eq. (6) with deformation in Eq. (2) to, then expressing axial force through deformation, getting

\[ \begin{align*}
\frac{E A}{L_0} \left( L - L_0 - \Delta L_e \right) \frac{\partial (L - L_0 - \Delta L_e)}{\partial u_i} &= P_i \quad (i = 1, 2, 3, 4) \\
-\frac{E A}{L_0} \left( L - L_0 - \Delta L_e \right) \frac{\partial (L - L_0 - \Delta L_e)}{\partial \Delta L_e} - P_e &= 0
\end{align*} \]

Combining Eq. (7) with Eq. (2), having

\[ \begin{align*}
\frac{E A}{L_0} \left( L - L_0 - \Delta L_e \right) \frac{\partial L}{\partial u_i} &= P_i \quad (i = 1, 2, 3, 4) \\
\frac{E A}{L_0} \left( L - L_0 - \Delta L_e \right) - P_e &= 0
\end{align*} \]

Designate

\[ \begin{align*}
q_i^{(e)}(u) &= \frac{E A}{L_0} \left( L - L_0 - \Delta L_e \right) \frac{\partial L}{\partial u_i} \quad (i = 1, 2, 3, 4) \\
q_5^{(e)}(u) &= \frac{E A}{L_0} \left( L - L_0 - \Delta L_e \right) - P_e
\end{align*} \]

The Eq. (8) can be compactly written as

\[ q_k^{(e)}(u) = P_k^{(e)} \quad k = 1, 2, \ldots, 5 \]
Where: \( u \equiv u = \{u_1, u_2, u_3, u_4, u_5\}^T \) is an unknown vector consisting of all nodal variables in the global coordinate system, including displacement unknowns and external force unknown at the \( i \)th cross section \( u_i \equiv P_i = N \).

Based on mixed finite element formulation, the balanced equation of element can be written as

\[
q^{(e)}(u) = P^{(e)} 
\]  

(10)

Designating: \( q^{(e)}(u) = \{q_1^{(e)}(u), q_2^{(e)}(u), q_3^{(e)}(u), q_4^{(e)}(u), q_5^{(e)}(u)\}^T; P^{(e)} = \{P_1^{(e)}, P_2^{(e)}, P_3^{(e)}, P_4^{(e)}, P_5^{(e)}\}^T \)

For nonlinear solution using the incremental loading into the of Eq. (10), getting

\[
q^{(e)}(u) + \frac{\partial q^{(e)}(u)}{\partial u} \delta u = P^{(e)} + \Delta P^{(e)} 
\]  

(11)

The second order infinitesimal of \( \Delta q^{(e)}(u) \) can be negligible, the Eq. (11) is becoming

\[
q^{(e)}(u) + \frac{\partial q^{(e)}(u)}{\partial u} \delta u = P^{(e)} + \Delta P^{(e)} 
\]  

(12)

Setting \( k^{(e)}(u) = \frac{\partial q^{(e)}(u)}{\partial u} \),

The Eq. (12) can be expressed in compact form as

\[
k^{(e)}(u) \delta u = (P^{(e)} + \Delta P^{(e)}) - q^{(e)}(u) 
\]  

(13)

Analysing the Eq. (13), recognizing \( k^{(e)}(u) \) - a mixed matrix of finite truss element considering length-dependent thermal deformation

\[
k^{(e)}(u) = \begin{bmatrix}
    k_{11} & k_{12} & \cdots & k_{15} \\
    k_{21} & k_{22} & \cdots & k_{25} \\
    \vdots & \vdots & \ddots & \vdots \\
    k_{51} & k_{52} & \cdots & k_{55}
\end{bmatrix}, k_i = \frac{\partial q^{(e)}(u)}{\partial u_j}, (i, j = 1, 2, \ldots, 5) 
\]  

(14)

2.2. Balanced equation of the truss system

Based on the balanced equation of element, obtaining the global balanced equation of truss system

\[
q(u) = P 
\]  

(15)

With

\[
\begin{align*}
    u \equiv u &= \{u_1, u_2, \ldots, u_n\}^T; q(u) = \{q_1(u), q_2(u), \ldots, q_5(u)\}^T; P = \{P_1, P_2, \ldots, P_5\}^T \\
 q_i(u) &= \sum_{i=1}^{n} q_i^{(e)}(u); P_i = \sum_{i=1}^{m} P_i^{(e)} (i = 1, 2, \ldots, n)
\end{align*}
\]

Where: “\( m \)” is a number of truss elements and “\( n \)” is number of unknowns;

The incremental loading form of system of nonlinear Eq. (15) is expressed as

\[
q(u) + \frac{\partial q(u)}{\partial u} \delta u = P + \Delta P 
\]  

(16)

Setting

\[
K(u) = \frac{\partial q(u)}{\partial u} 
\]  

(17)

The Eq. (16)&(17) can be compactly written as
\[ K(u)\delta u = (P + \Delta P) - q(u); \]  
\[ K_{i,j}(u) = \sum_{e=1}^{n} k_{i,j}^{e}(u), \ (i, j = 1, 2, ..., n) \]  
(18)  
(19)

With \( K(u) \) - the global mixed matrix of system, getting by assembling mixed matrix of truss elements

3. Test example

3.1. Example formulation

Investigate the truss system shown in Figure 2 subject to a thermal and mechanical loading due to constant temperature change. All the truss bars made of the same material and have the same cross-sectional area. The parameters are given

\[ E = 2.10^4 \text{kN/cm}^2, \quad A = 4 \text{cm}^2, \quad \alpha = 11.10^{-6} \text{ (°C)}^{-1}, \quad \Delta T = 0; 50; 100 \text{°C} \]

For solving the geometrically nonlinear buckling problem of investigated system, the research established the incremental-iterative algorithm based on the arc length method because of its efficiency in solving non-linear systems of equations when the problem under consideration exhibits one or more critical points [9-11]. Based on proposed algorithm, the calculation program for solving the geometrically nonlinear buckling problem of truss is written.

![Figure 2. Examined system, designating unknowns of the system.](image)

3.2. Numerical results

The results of solving process are load-displacement and load-internal force equilibrium path in different cases of temperature change. The most of important results is shown in Figures 3 & 4.

![Figure 3. Load-displacement equilibrium path \( P(u_2) \) in cases \( \Delta T = 0; 50; 100 \text{°C} \).](image)
3.3. Comments
The thermal load significantly effects to buckling and post buckling behaviour of truss structure. The influence of the temperature expansion can change the critical load value into both negative and positive sides.

4. Conclusions
The mixed model of the finite element formulation has a remarkable advantage in the analysis of problems with length-dependent thermal deformation. Using presented mixed formulation helps to escape difficulties of the mathematical treatment of additional constraints in nonlinear finite element model.

Figure 4. Load-internal force equilibrium path in case \( \Delta T = 100^\circ C \).

References
[1] Bathe KJ 2016 Finite Element Procedures (Prentice Hall)
[2] Zienkiewicz O C, Taylor R L 2014 The Finite Element Method for Solid and Structural Mechanics (7th edn) (Butterworth-Heinemann)
[3] Mircea Radeş 2006 Finite Element Analysis (Bucuresti: Printech) p 274
[4] Strodiot J J 2002 Numerical Methods in Optimization (Namur – Belgium)
[5] Sun W and YuanY -X 2006 Optimization Theory and Methods - Nonlinear Programming, (Springer)
[6] Bertsekas D P 1982 Constrained optimization and Lagrange multiplier method (Academic Press, Inc.)
[7] Oden J T 1981 Penalty-finite element methods for constrained problems in elasticity, Symposium on finite element method, Hefei politecnical university (China)
[8] Naotake Noda, Richard B Hetnarski, Yoshinobu Tanigawa 2003 Thermal stresses (Taylor & Francis Group)
[9] Crisfield M A 1997 Nonlinear Finite Element Analysis of Solids and Structures (John Wiley & Sons Ltd.)
[10] Ritto-Correia M. Camotim D 2008 On the Arc-Length and Other Quadratic Control Methods: Established, Less Known and New Implementation Procedures Comput. Struct. 86(11–12) 1353–1368
[11] Lam W F J, Morley C T 1992 Arc-Length Method for Passing Limit Points in Structural Calculation J. Struct. Eng. 118(1) 169–185