Improved Lower Bounds for Truthful Scheduling

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Abstract

The problem of scheduling unrelated machines by a truthful mechanism to minimize the makespan was introduced in the seminal “Algorithmic Mechanism Design” paper by Nisan and Ronen. Nisan and Ronen showed that there is a truthful mechanism that provides an approximation ratio of min(m, n), where n is the number of machines and m is the number of jobs. They also proved that no truthful mechanism can provide an approximation ratio better than 2. Since then, the lower bound was improved to 1 + √2 ≈ 2.41 by Christodoulou, Kotsoupias, and Vidali, and then to 1 + φ ≈ 2.618 by Kotsoupias and Vidali. Very recently, the lower bound was improved to 2.755 by Giannakopoulos, Hammerl, and Pocas. In this paper we further improve the bound to 2.8019.

Note that a gap between the upper bound and the lower bounds exists even when the number of machines and jobs is very small. In particular, the known 1 + √2 lower bound requires at least 3 machines and 5 jobs. In contrast, we show a lower bound of 2.2055 that uses only 3 machines and 3 jobs and a lower bound of 1 + √2 that uses only 3 machines and 4 jobs. For the case of two machines and two jobs we show a lower bound of 2. Similar bounds for two machines and two jobs were known before but only via complex proofs that characterized all truthful mechanisms that provide a finite approximation ratio in this setting, whereas our new proof uses a simple and direct approach.

1 Introduction

We consider the problem of scheduling unrelated machines to minimize the makespan by a truthful mechanism. In this problem we have n machines and m jobs. Denote by \( t^j_i \) the time it takes machine i to process job j. The \( t^j_i \)'s are the private information of machine i. The goal is to minimize the makespan by a truthful mechanism, that is, find an allocation \( (x_1, \ldots, x_n) \) of all jobs such that \( \max_i \sum_{j \in x_i} t^j_i \) is minimized.

The problem was introduced by Nisan and Ronen in their seminal “Algorithmic Mechanism Design” paper [16]. Nisan and Ronen showed that the VCG mechanism provides an approximation ratio of min(m, n). They also proved a lower bound of 2 on the approximation ratio. Closing this gap is a major open question that has attracted much attention.

The first improvement over these bounds was obtained by Christodoulou, Kotsoupias, and Vidali [6] who improved the bound to 1 + √2 ≈ 2.41. Kotsoupias and Vidali [10] further improved the bound to 1 + φ ≈ 2.618. No improvement over this bound was obtained for more than a decade until very recently when Giannakopoulos, Hammerl, and Pocas [9] were able to improve the bound to 2.755.

The gap between the lower and upper bounds is obviously still very large. Evidence that the “correct” answer is a lower bound of n was provided by Ashlagi et al. [1] who showed that no truthful anonymous mechanism can guarantee an approximation ratio better than n.

Generalizations of this problem were also studied. In particular, when the valuations are submodular (and not just additive), Christodoulou, Kotsoupias, and Kovacs obtain a lower bound of \( \Omega(\sqrt{n}) \) [5].

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Many papers also considered randomized and fractional versions of the problem \cite{16, 15, 17, 14, 13, 12, 8, 11, 4}.

The main result of this paper is an improvement of the lower bound to 2.8019. We achieve this bound by carefully refining the constructions of \cite{10, 9}. This result is provided in Section \ref{sec:main_result}.

We then go on by considering instances with small numbers of jobs and machines. Note that the VCG mechanism and slight modifications of it are the only mechanisms that we know that provide a finite approximation ratio. In fact, mechanisms that achieve non-trivial approximation guarantees are unknown even for very small instances. For example, the $1 + \sqrt{2}$ lower bound of \cite{6} requires three machines and five jobs. Currently, we do not even know whether there are mechanisms that provide an approximation ratio better than 3 for instances with only three jobs and three machines.

In this paper we make progress in understanding the power of truthful mechanisms for small instances. We show a lower bound of 2.2055 for instances with only 3 machines and 3 jobs (Section \ref{sec:lower_bound}). In Section \ref{sec:lower_bound_small_instances} we also provide a lower bound of $1 + \sqrt{2}$ that uses only 3 machines and 4 jobs (this matches the lower bound of \cite{6} that achieves the same bound with 3 machines and 5 jobs). We also consider instances with two machines and two jobs and show a lower bound of 2 (Section \ref{sec:lower_bound_small_instances}). Similar bounds for two machines and two jobs were known before \cite{8, 7} but only via complex proofs that characterized all truthful mechanisms that provide a finite approximation ratio in this setting, whereas our new proof uses a simple and direct approach which is much more in line with all other lower bounds.

## 2 Preliminaries

There are $n$ machines, $m$ tasks. Denote by $t^j_i$ the time it takes machine $i$ to process job $j$. The cost of machine $i$, denoted by $c_i = (t^1_i, ...t^m_i)$, is its private information. A time-processing matrix $T \in \mathbb{R}^{n \times m}$ is a matrix where the $i$'th row is $t_i$. Let $x = (x_1, ..., x_n)$ denote an allocation of tasks to machines, where $x_i$ is the set of tasks allocated to machine $i$ and $x^j_i$ equal 1 if machine $i$ gets job $j$ in $x$ and 0 otherwise. A valid allocation allocates each task to exactly one machine. Let $A$ be the set of all valid allocations. We are interested in truthful mechanisms, which are mechanisms where the dominant strategy of each agent (machine) is to reveal his true type $t_i$. A truthful mechanism is a tuple $M = (f, P)$ that consists of an allocation function $f : \mathbb{R}^{n \times m} \rightarrow A$ and a payment scheme $p : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^n$. The objective is to minimize the makespan, which is given by $\max_{i \in [n]} \sum_{j \in x_i} t^j_i$. Each machine is controlled by a selfish agent whose goal is to maximize his utility function: $u_i(x, p) = p_i - \sum_{j \in x_i} t^j_i$ (the mechanism pays the agents in order to incentivize them to perform the tasks).

A known characterization of a truthful mechanism is that its social choice function is weakly-monotone \cite{2}. In the setting of unrelated machine scheduling a mechanism is truthful if and only if it has a weakly-monotone allocation algorithm. An allocation algorithm $f : \mathbb{R}^{n \times m} \rightarrow A$ is weakly monotone if for every agent $i$, every $t_{-i}$ and every $T = (t_i, t_{-i}), T' = (t'_i, t_{-i})$ it holds that:

$$\sum_{j \in [m]} (t^j_i - t'^j_i) \cdot (x^j_i - x'^j_i) \leq 0$$

where $x = f(T), x' = f(T')$.

We will use $\infty$ to denote very large values, and $\varepsilon$ to denote values that are as small as we wish. A dummy job for player $i$ is a job which takes player $i$ some finite time to execute, while for all the other players it takes infinite time. That is, every mechanism which achieves a finite approximation must allocate this job to player $i$.

Several properties which follow from the weak monotonicity characterization are given in the lemmas below. After each lemma we provide an example to illustrate it. Similar lemmas are standard in the related literature.
Lemma 2.1 Let $M$ be a truthful mechanism with an allocation function $f$, and let $T, T' \in \mathbb{R}^{n \times m}$ be time-processing matrices that differ only on player $i$, i.e., $T_{-i} = T'_{-i}$, and $x = f(T), x' = f(T')$. Let $B_1 = \{j \mid x_i^j = 1 \land t_i^j > t_i'^j\}$, $B_2 = \{k \mid x_i^k = 0 \land t_i^k < t_i'^k\}$, and $B_3 = [m] \setminus (B_1 \cup B_2)$. Suppose that for every $q \in B_3$, it holds that $t_i^q = t_i'^q$. Then, for every $r \in (B_1 \cup B_2)$, it holds that $x_i^r = x_i'^r$.

Proof: By weak monotonicity:

$$0 \geq \sum_{p \in [m]} (t_i^p - t_i'^p) \cdot (x_i^p - x_i'^p) = \sum_{j \in B_1} (t_i^j - t_i'^j) \cdot (x_i^j - x_i'^j) + \sum_{k \in B_2} (t_i^k - t_i'^k) \cdot (x_i^k - x_i'^k) = \sum_{j \in B_1} (t_i^j - t_i'^j) \cdot (1 - x_i^j) + \sum_{k \in B_2} (t_i^k - t_i'^k) \cdot (0 - x_i^k).$$

In order for the weak monotonicity inequality to hold, each term in the summation must be equal to 0, i.e., for every job $r$ such that $r \in (B_1 \cup B_2)$, it holds that $x_i^r = x_i'^r$, and for any other job $k \in B_3$, $x_i^k = x_i'^k = 0$.

When using Lemma 2.1, if not stated otherwise, $B_1$ will be the set of all the jobs that the $i$'th player gets in $x$ and $B_2$ will be the of all the jobs he does not get in $x$.

Example 2.2 (an example of Lemma 2.1) Consider the instances $G_1$ with the allocation indicated by stars and the instance $G_2$ (given below). Applying Lemma 2.1 on this instances where $T = G_1$, $T' = G_2$, $i = 2$, $B_1 = \{3\}$, $B_2 = \{1\}$ and $B_3 = \{2\}$ gives us that in the instance $G_2$ the second player gets the third job but not the first one.

$$\begin{pmatrix} 1^* & 2 & 3 \\ 2 & 1^* & 3^* \end{pmatrix} = G_1 \quad \Rightarrow \quad \begin{pmatrix} 1^* & 2 & 3 \\ 3 & 1 & 2^* \end{pmatrix} = G_2$$

Lemma 2.3 Let $M$ be a truthful mechanism with a social choice function $f$, let $T, T' \in \mathbb{R}^{n \times m}$ be time-processing matrices that differ only on player $i$, i.e., $T_{-i} = T'_{-i}$, and let $x = f(T), x' = f(T')$. Let $j$ be a job that player $i$ gets in $x$ where $t_i^j > t_i'^j$, and let $k$ be another job such that $t_i^k > t_i'^k$. Suppose that for any other job $l$, it holds that $t_i^l = t_i'^l$. Then, it follows that either $x_i^j = 1$ or $x_i^k = 1$ or both.

Proof: By weak monotonicity:

$$0 \geq \sum_{r \in [m]} (t_i^r - t_i'^r) \cdot (x_i^r - x_i'^r) = (t_i^j - t_i'^j) \cdot (x_i^j - x_i'^j) + (t_i^k - t_i'^k) \cdot (x_i^k - x_i'^k) = (t_i^j - t_i'^j) \cdot (1 - x_i^j) + (t_i^k - t_i'^k) \cdot (0 - x_i^k).$$

Suppose not. Then, $x_i^j = x_i^k = 0$, and we have that:

$$\sum_{r \in [m]} (t_i^r - t_i'^r) \cdot (x_i^r - x_i'^r) = (t_i^j - t_i'^j) + (t_i^k - t_i'^k) \cdot (x_i^k) \quad \text{given below.}$$

and so the inequality does not hold, therefore either $x_i^j = 1$ or $x_i^k = 1$. 

Example 2.4 (an example of Lemma 2.3) Consider the instances $H_1$ with the allocation indicated by stars and the instance $H_2$ (given below). Applying Lemma 2.3 on this instances where $T = H_1, T' = H_2, i = 2, j = 2$ and $k = 1$ gives us that in the instance $H_2$ the second player gets at least one of the first two jobs.

$$
\begin{pmatrix}
1^* & 2 & 3 \\
2 & 1^* & 3^*
\end{pmatrix}_{H_1} \Rightarrow
\begin{pmatrix}
1 & 2 & 3 \\
1^* & 0 & 3
\end{pmatrix}_{H_2}
$$

Lemma 2.5 Let $M$ be a truthful mechanism with a finite approximation ratio, its social choice function is denoted by $f$. Let $i$ be a player with a dummy job $j$, let $T, T' \in \mathbb{R}^{n \times m}$ be time-processing matrices that differ only on player $i$, i.e., $T_{-i} = T'_{-i}$, and let $x = f(T)$. Let $B_1 = \{ r \neq j \mid x_i^r = 1 \land t_i^r > t_i'^r \}$, $B_2 = \{ k \mid x_i^k = 0 \land t_i^k = t_i'^k \}$, and $B_3 = [m] \setminus (B_1 \cup B_2 \cup \{ j \})$. Suppose that for every $q \in B_3$, it holds that $t_i^q = t_i'^q$. Then for $p \in (B_1 \cup B_2 \cup \{ j \})$, it holds that $x_i^p = x_i'^p$, where $x' = f(T')$.

Proof: Observe that if the mechanism has a finite approximation ratio then it must allocate job $j$ to player $i$. Thus, we have that $x_i^j = 1$ and $x_i'^j = 1$. By weak monotonicity:

$$
0 \geq \sum_{s \in [m]} (t_i^s - t_i'^s) \cdot (x_i^s - x_i'^s)
= \sum_{r \in B_1} (t_i^r - t_i'^r) \cdot (x_i^r - x_i'^r) + \sum_{k \in B_2} (t_i^k - t_i'^k) \cdot (x_i^k - x_i'^k) + (t_i^j - t_i'^j) \cdot (x_i^j - x_i'^j) = 0
$$

and for the inequality to hold it must be that for every $r \in B_1$, it holds that $x_i^r = x_i'^r = 1$ and for every $k \in B_2$, it holds that $x_i^k = x_i'^k = 0$. As explained before, since the mechanism has a finite approximation ratio it must be that $x_i^j = x_i'^j = 1$.

When using Lemma 2.5, if not stated otherwise, $B_1$ will be the set of all the jobs that the $i$'th player gets in $x$ and $B_2$ will be the of all the jobs he does not get in $x$.

Example 2.6 (an example of Lemma 2.5) Consider the instance $I_1$ with the allocation indicated by stars and the instance $I_2$ (given below). Applying Lemma 2.5 on this instances where $T = I_1, T' = I_2, i = 2, j = 4, B_1 = \{ 3 \}, B_2 = \{ 1 \}$ and $B_3 = \{ 2 \}$ gives us that in instance $I_2$ the second player gets the third and fourth jobs but not the first job.

$$
\begin{pmatrix}
1^* & 1^* & 2 & \infty \\
1 & 2 & 1^* & 1^*
\end{pmatrix}_{I_1} \Rightarrow
\begin{pmatrix}
1^* & 1^* & 2 & \infty \\
2 & 2 & 1^* & 3^*
\end{pmatrix}_{I_2}
$$

Lemma 2.7 Let $M$ be a truthful mechanism with a social choice function $f$, let $T, T' \in \mathbb{R}^{n \times m}$ be time-processing matrices that differ only on player $i$, i.e., $T_{-i} = T'_{-i}$, and let $x = f(T), x' = f(T')$. Let $j_1, j_2$ be two jobs such that $x_i^{j_1} = x_i^{j_2} = 1$, $t_i^{j_1} > t_i^{j_1}$ and $t_i^{j_2} < t_i^{j_2}$. Suppose that for any other job $j \neq j_1, j_2$, it holds that $t_i^j = t_i'^j$. Then, if $x_i^{j_2} = 1$, then also $x_i^{j_1} = 1$. 

4
Proof: Assume that \( x_{i}^{j_{2}} = 1 \). By weak monotonicity:

\[
0 \geq \sum_{s \in [m]} (t_{i}^{s} - t_{i}^{s}) \cdot (x_{i}^{s} - x_{i}^{s})
\]

\[
= (t_{i}^{j_{1}} - t_{i}^{j_{1}}) \cdot (x_{i}^{j_{1}} - x_{i}^{j_{1}}) + (t_{i}^{j_{2}} - t_{i}^{j_{2}}) \cdot (x_{i}^{j_{2}} - x_{i}^{j_{2}})
\]

\[
= (t_{i}^{j_{1}} - t_{i}^{j_{1}}) \cdot (1 - x_{i}^{j_{1}}) + (t_{i}^{j_{2}} - t_{i}^{j_{2}}) \cdot (1 - 1)
\]

\[
= (t_{i}^{j_{1}} - t_{i}^{j_{1}}) \cdot (1 - x_{i}^{j_{1}})
\]

In order for the inequality to hold it must be the case that \( x_{i}^{j_{1}} = 1 \).

Example 2.8 (an example of Lemma 2.7) Consider the instances \( K_{1} \) with the allocation indicated by stars and the instance \( I_{2} \) (given below). Applying Lemma 2.7 on this instances where \( T = K_{1}, T' = K_{2}, i = 2, j_{1} = 1 \) and \( j_{2} = 2 \) gives us that in the instance \( K_{2} \) if the second player gets the second job then he must also get the first job.

\[
\begin{pmatrix}
1 & 2 & 3^* \\
1^* & 0^* & 3
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 2 & 3 \\
3 & 1 & 3
\end{pmatrix}
= K_{1} \rightarrow K_{2}
\]

When applying lemmas 2.1, 2.3, 2.5, 2.7 we will often increase or decrease only some of the costs, and not all of them, in order to make the proof clearer. The convention will be that the other values are increased or decreased by a small amount, as was done in [10].

When visualizing instances using matrices, a blue star (*) will be used to indicate the allocation of the mechanism \( M \) in this instance, and a red at (@) will be used to indicate an optimal allocation in this instance. Moreover, note that sometimes the allocation of the mechanism \( M \) will be partial (namely, some jobs will not be visually assigned a player, since which player gets the job is irrelevant).

3 A Lower Bound of \( \approx 2.8019 \)

Theorem 3.1 There is a sequence of numbers \( \{a_{i}\}_{i=1}^{\infty} \), where \( a_{n} \rightarrow 1.8019 \) and for each \( i, a_{i} > 1 \), such that for every \( n \), every truthful mechanism for \( n + 2 \) machines and \( 2 \cdot (n + 2) \) jobs guarantees an approximation ratio no better than \( 1 + a_{n} \).

In the proof of the theorem we will write \( a \) instead of \( a_{n} \) to simplify notation. For every \( n \), let \( A_{n} \) be the following instance:

\[
A_{n} = \begin{pmatrix}
\infty & 1 & 1 & \frac{1}{a} & \frac{1}{a} & \ldots & \frac{1}{a^{\infty}} & 0 & \infty & \ldots & \ldots & \infty \\
1 & \varepsilon & \infty & \infty & \infty & \ldots & \infty & \infty & \infty & \ldots & \ldots & \ldots \\
1 & \infty & \varepsilon & \infty & \infty & \ldots & \infty & \infty & \infty & \ldots & \ldots & \ldots \\
\infty & \infty & \infty & \frac{1}{a} & \infty & \ldots & \infty & \infty & \infty & \ldots & \ldots & \ldots \\
\infty & \infty & \infty & \infty & \frac{1}{a} & \ldots & \infty & \infty & \infty & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\infty & \infty & \infty & \infty & \infty & \ldots & \frac{1}{a^{a-1}} & \infty & \infty & \ldots & \ldots & \infty \\
\end{pmatrix}
\]
Fix a mechanism $M$ with an approximation ratio better than $1 + a$. In the instance $A_n$, $M$ must allocate the first job to either the second player or the third. We analyze the case where $M$ allocates the first job to the second player. The analysis is similar if the job is instead allocated to the third player.

**Lemma 3.2** In the instance $A_n$, $M$ allocates the first two jobs to the second player.

**Proof:** Suppose that the second player is assigned the first job but not the second one. Consider the second player. Decrease his cost for the first job to 0, and increase his cost for the second job by $\varepsilon$. By Lemma 2.1, the second player will not get the second job. Thus, the first player will (otherwise, if some other player gets the second job, the approximation ratio will be infinite). Increase the first player’s cost for his dummy job to $\frac{1}{a}$. By Lemma 2.5 we have that the first player keeps both the second job and his dummy job (as shown in Figure 1). In this case the makespan of the mechanism $M$ is $1 + \frac{1}{a}$ whereas the optimal makespan is $\frac{1}{a}$. This results in an approximation ratio of $1 + a$.

By Lemma 3.2, we can assume that the second player gets the second job in addition to the first job in $A_n$. Increase the cost of the second player for the second job from $\varepsilon$ to 1 and decrease his cost for the first job by $\varepsilon$. Denote this instance by $B_n$.

We divide the analysis to different cases based on the player that gets the second job in $B_n$. There are two possible cases. In the first case, $M$ allocates the second job to the second player. In the second case, $M$ allocates the second job to the first player. Observe that if $M$ allocates the second job to any other player then it will have an approximation ratio worse than $1 + a$ (the cost of the other players for the second job is $\infty$).


**Case 1: The Second Player Keeps Job 2**

In this case, $M$ allocates the second job to the second player in $B_n$. Recall that in the instance $A_n$, $M$ allocates the first two jobs to the second player. By Lemma 2.7, for $i = 2$, $j_1 = 1$, $j_2 = 2$, we have that the second player gets the first job in addition to our assumption that he gets the second job in $B_n$. We increase the second player’s cost for his dummy job to 1, and by applying Lemma 2.5 we have that the second player keeps the first two jobs, and his dummy job. This results in $M$ having a makespan of 3 whereas the optimal makespan is 1 (see Figure 2). The optimal makespan can be achieved when the third player gets the first and third jobs, the first player gets the second job, the $i$’th player for $4 \leq i \leq n + 2$ gets the $i$’th job and every player gets his dummy job. Thus, the mechanism has an approximation ratio of 3.

\[
\begin{pmatrix}
\infty & 1^@ & 1 & \frac{1}{a^2} & \frac{1}{a^3} & \ldots & \frac{1}{a^n} & 0^@ & \infty & \ldots & \ldots & \infty \\
1^* & 1^* & \infty & \infty & \infty & \ldots & \infty & \infty & 1^* & \infty & \ldots & \ldots \\
1^@ & \infty & \varepsilon & \infty & \infty & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots \\
\infty & \infty & \infty & \frac{1}{a^0} & \infty & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots \\
\infty & \infty & \infty & \frac{1}{a^2} & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\infty & \infty & \infty & \infty & \ldots & \frac{1}{a^n} & \infty & \ldots & \ldots & \infty & 0^@ \\
\end{pmatrix}
\]

Figure 2: In Case 1 we get this instance by increasing the second player’s dummy job’s cost to 1.

**Case 2: The First Player Takes Job 2**

In this case, in the instance $B_n$ the second job was allocated to the first player. Reduce the first player’s cost for the first two jobs to $\frac{1}{a}$ and denote the resulting instance by $C_n$.

\[
C_n = \begin{pmatrix}
\frac{1}{a} & \frac{1}{a} & 1 & \frac{1}{a^2} & \frac{1}{a^3} & \ldots & \frac{1}{a^n} & 0 & \infty & \ldots & \ldots & \infty \\
1 & 1 & \infty & \infty & \infty & \ldots & \infty & \infty & \ldots & \ldots & \ldots \\
1 & \infty & \varepsilon & \infty & \infty & \ldots & \infty & \ldots & \ldots & \ldots & \ldots \\
\infty & \infty & \infty & \frac{1}{a} & \infty & \ldots & \infty & \ldots & \ldots & \ldots & \ldots \\
\infty & \infty & \infty & \frac{1}{a^2} & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\infty & \infty & \infty & \infty & \ldots & \frac{1}{a^n} & \infty & \ldots & \ldots & \infty & 0 \\
\end{pmatrix}
\]

By Lemma 2.3 for $j = 2$, $k = 1$, we have that the first player gets at least one of the first two jobs in $C_n$. We analyze the case that the first player gets the first job (and possibly others). If the first player gets the second job the analysis is similar.

Let $C = \{1, 2, 4, \ldots, n + 2\}$ be the set of the first $n + 2$ tasks, except the third one. We divide the analysis to three cases based on the allocation of the mechanism $M$ in the instance $C_n$. In the first case, the first player gets the first job in $C_n$ but not the second job. In the second case, the first player gets the first two jobs, but not some other job $j$ for $4 \geq j \in C$. In the third case, the first player gets all the jobs in $C$. In each case we will show that $M$ has an approximation ratio of at least $1 + a$.

The analysis of all these cases is similar to the previous proofs of Koutsoupias and Vidali [10] and Giannakopoulos et al. [9].
Case 2.1

In this case, $M$ allocates the first job to the second player but not the second job. We decrease the first player’s cost for the first job to 0. By Lemma 2.1, the first player will still not get the second job, and since $M$ has a finite approximation ratio, the second player will. Increase the second player’s cost for his dummy job to $\frac{1}{a}$. By Lemma 2.5 the second player will get his dummy job in addition to the second job (see Figure 3). The makespan of this allocation is $1 + \frac{1}{a}$, whereas the optimal makespan is $\frac{1}{a}$ (this can be achieved when the first player gets the first two jobs, the third player gets the third job, the $i$’th player $i$ for $4 \leq i \leq n + 2$ gets the $i$’th job, and each player gets his dummy job). Thus, the mechanism has an approximation ratio of $1 + a$.

$$\begin{pmatrix} 0^\alpha & 1^\alpha & 1 & \frac{1}{a^2} & \frac{1}{a^3} & \ldots & \frac{1}{a^n} & 0^\alpha & \infty & \ldots & \ldots & \ldots & \ldots & \infty \\ 1 & 1^* & \infty & \infty & \infty & \ldots & \infty & \infty & \frac{1}{a^2} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \infty \\ 1 & \infty & \varepsilon^\alpha & \infty & \infty & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \infty & \infty & \infty & \frac{1}{a^2} & \infty & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \infty & \infty & \infty & \infty & \frac{1}{a^2} & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \infty & \infty & \infty & \infty & \infty & \ldots & \frac{1}{a^n} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \end{pmatrix}$$

Figure 3: This instance is the result of the transitions in Lemma Case 2.1.

Case 2.2

In this case the first player gets the first two jobs but not some other job $j$ for $4 \geq j \in C$. Denote by $i \geq 4$ the smallest index of a job from $C$ that the first player does not get. Reduce the first player’s cost for all jobs in $C$ that he does get to 0 and increase by $\varepsilon$ its cost for job $i$. By Lemma 2.1 we have that the first player still doesn’t get the $i$’th job, and because $M$ has an approximation ratio better than $1 + a$, the $i$’th player gets the $i$’th job. Increase the $i$’th player’s cost for his dummy job to $\frac{1}{a^{i-2}}$. By Lemma 2.5 we have that the $i$’th player will keep the $i$’th job and his dummy job (see Figure 4). Now, the mechanism’s makespan will be $\frac{1}{a^{i-2}} + \frac{1}{a^{i-3}}$ whereas the optimal makespan is $\frac{1}{a^{i-2}}$. The optimal makespan can be achieved when allocating the third job to the third player, the $k$’th job for $i \geq k \in C$ to the first player, the $r$’th job for $i < r \in C$ to the $r$’th player, and each dummy job to its respective player. Hence, the mechanism has an approximation ratio of $1 + a$.

$$\begin{pmatrix} 0^\alpha & 0^\alpha & 1 & 0^\alpha & \ldots & \frac{1}{a^{i-2}} & \ldots & \frac{1}{a^n} & 0^\alpha & \infty & \ldots & \ldots & \ldots & \ldots & \infty \\ 1 & 1 & \infty & \infty & \ldots & \infty & \ldots & \infty & \infty & 0^\alpha & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \infty & \varepsilon^\alpha & \infty & \infty & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \infty & \infty & \infty & \frac{1}{a} & \ldots & \infty & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \infty & \infty & \infty & \infty & \frac{1}{a} & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \infty & \infty & \infty & \infty & \infty & \ldots & \frac{1}{a^n} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \end{pmatrix}$$

Figure 4: This instance is the result of the transitions in Case 2.2.
Case 2.3

In this case, the first player gets all the jobs in $C$. We will increase the first player’s dummy job’s cost to 1 and by Lemma 2.5 we will get that the first player keeps all of his jobs (see Figure 5). The optimal makespan is 1 which can be achieved in the case where each player gets his dummy job, the second player gets the second job, the third player gets the first and third jobs, and the $j$'th player for $4 \leq j \leq n + 2$ gets the $j$'th job. Thus, the mechanism will have an approximation ratio of $1 + \frac{2}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^n}$. 

$$
\begin{pmatrix}
\frac{1}{a^*} & \frac{1}{a^*} & \frac{1}{a^*} & \ldots & \frac{1}{a^*} & \frac{1}{a^*} & 1^{*}_{a_1} & \ldots & \ldots & \ldots & \ldots & 1^{*}_{a_n} \\
1 & 1^{*}_{a} & \infty & \infty & \ldots & \infty & \infty & \infty & \infty & 0^{*}_{a_1} & \ldots & \ldots & \ldots & \ldots \\
1^{*}_{a} & \infty & \varepsilon & \varepsilon & \ldots & \infty & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\infty & \infty & \varepsilon & \varepsilon & \ldots & \infty & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\infty & \infty & \infty & \varepsilon & \ldots & \infty & \ldots & \infty & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\infty & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
$$

Figure 5: This is the instance that results from the transition described in Case 2.3.

Concluding the Proof

We finish the proof of Theorem 1 by choosing $a$ such that $1 + \frac{2}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^n} = 1 + \frac{1}{a} + \frac{2(a+1) - 1}{a-1} = 1 + a$. When the number of players is $n + 2$ for $n \geq 1$ we have that the approximation ratio is $1 + a$ where $a$ satisfies the equation $\frac{1}{a} + \frac{2(a+1) - 1}{a-1} = a$. For example, when the number of players is 5 we get a lower bound of 2.7106, and when the numbers of players is 7 we get a lower bound of 2.7704. When $n$ approaches infinity we have that $\frac{1}{a} + \frac{2(a+1) - 1}{a-1} = \frac{1}{a} + \frac{1}{a-1}$, and so $a = \frac{1}{a} + \frac{1}{a-1}$. That is, $a \approx 1.8019$ and the approximation ratio is 2.8019 (see Table 1).

| $n$ | 3 | 4 | ... | 7 | 8 | ... | $\infty$ |
|-----|---|---|-----|---|---|-----|--------|
| approximation ratio | $1 + \sqrt{2}$ | $1 + \phi$ | $2.770$ | $2.788$ | $2.8019$ |

Table 1: The lower bound of the approximation ratio by the number of players $n$.

4 Lower Bounds For Small Instances

Throughout this section we denote by $\varepsilon^+, \varepsilon^-, \varepsilon^{--}, \varepsilon^{---}$ values that are as small as we wish, such that $\varepsilon^+ >> \varepsilon^- >> \varepsilon^{--} >> \varepsilon^{---}$. Thus, we will treat $\frac{\varepsilon^+}{\varepsilon^-}, \frac{\varepsilon^-}{\varepsilon^{--}}, \frac{\varepsilon^{--}}{\varepsilon^{---}}$ as $\infty$.

4.1 2 Players and 2 Jobs

Theorem 4.1 Every truthful mechanism for the unrelated machine scheduling problem with two machines and two jobs has an approximation ratio of at least 2.
Let $M$ be a truthful mechanism for the $2 \times 2$ case of the unrelated machine scheduling problem with an approximation ratio better than 2. Consider the instance:

$$D = \begin{pmatrix} 1 & \epsilon^- \\
1 & \epsilon^+ \end{pmatrix}$$

**Lemma 4.2** If $M$ has a finite approximation ratio then it must allocate the second job to the first player.

**Proof:** Assume by contradiction that $M$ has a finite approximation ratio and it allocates the second job to the second player. Then, there are two possible cases. In the first case, $M$ allocates the first job to the first player, and in the second case, $M$ allocates the first job to the second player in addition to the second job.

In the first case, reduce the first player’s cost for the first job to 0. By Lemma 2.1, $M$ allocates the same jobs to the first player i.e., the first player will get job 1 and player 2 will get job 2 (see Figure 6). In this case the mechanism’s makespan is $\epsilon^+$ whereas the optimal makespan is $\epsilon^-$ (this can be achieved when the first player gets both jobs). Thus, $M$ has an approximation ratio of $\frac{\epsilon^+}{\epsilon^-}$ which is as large as we want.

$$\begin{pmatrix} 0^\alpha & \epsilon^-^\alpha \\
1 & \epsilon^+_\alpha \end{pmatrix}$$

Figure 6: This is the instance that result from the transition described in Lemma 4.2 Case 1.

In the second case, $M$ allocates both jobs to the second player. Reduce the cost of the second player for the first job to 0. By Lemma 2.1 we have that the second player keeps both jobs. Thus, $M$ has a makespan of $\epsilon^+$ whereas the optimal makespan is $\epsilon^-$ (see Figure 7). In this case, $M$ has an approximation ratio which can be made arbitrarily large.

$$\begin{pmatrix} 1 & \epsilon^-^\alpha \\
0^\alpha & \epsilon^+_\alpha \end{pmatrix}$$

Figure 7: This is the instance that result from the transition described in Lemma 4.2 Case 2.

By Lemma 4.2, we have that $M$ allocates the second job to the first player. There are two possible cases: the first is that the first player gets the first job and the second is that the second player gets the first job.

**Case 1: $M$ Allocates Job 1 to Player 1**

In this case, in the instance $D$, $M$ allocates both jobs to the first player. Increase the second player’s cost for the second job to $\infty$. This effectively makes the second job a dummy job for player 1. By Lemma 2.1 the second player’s allocation does not change. Thus, the first player keeps both jobs. Increase the cost of the first player for the second job to 1. By Lemma 2.3 the first player keeps both jobs and $M$ has a makespan of 2 whereas the optimal makespan is 1 (this can be achieved when the first player gets the second job and the second player gets the first job) (see Figure 8).

**Case 2: $M$ Allocates Job 1 to the Second Player**

In this case we reduce the second player’s cost for the second job to $\epsilon^{--}$ and reduce its cost for the first job by $\epsilon' > \epsilon^+ - \epsilon^{--}$. Denote this instance by $D_1$ (see Figure 9).
\[
\begin{pmatrix}
1 & \varepsilon^- \\
\varepsilon^+ & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & \varepsilon^- \\
1 & \infty
\end{pmatrix} \rightarrow
\begin{pmatrix}
1^* & 1^* \\
1^0 & 1^0
\end{pmatrix}
\]

Figure 8: The transitions described in the analysis of the first case of the proof of theorem 2.

\[
\begin{pmatrix}
1 & \varepsilon^-
\end{pmatrix}
\]

Figure 9: The instance \( D_1 \)

Lemma 4.3 In the instance \( D_1 \), \( M \) must allocate the first job to the second player.

Proof: Consider the weak monotonicity inequality applied on the second player where \( x' \) is the allocation of \( M \) in the instance \( D_1 \) and assume by contradiction that \( x'_2 = 0 \):

\[
(1 - (1 - \varepsilon')) \cdot (1 - x'_2) + (\varepsilon^+ - \varepsilon^-) \cdot (0 - x_2^2) \\
= (\varepsilon') + (\varepsilon^+ - \varepsilon^-) \cdot (0 - x_2^2) \\
\leq 0
\]

In this case the inequality does not hold since \( \varepsilon' > \varepsilon^+ - \varepsilon^- \).

By Lemma 4.3 we have that in \( D_1 \) the second player gets the first job. Similarly to Lemma 4.2 we can show that \( M \) allocates the second job to the second player in \( D_1 \) (otherwise we can decrease the second player’s cost for the first job to 0. Using Lemma 2.1 we get that \( M \) has an approximation ratio of \( \frac{1}{\varepsilon'} \) which is as large as we want).

It remains to show that in \( D_1 \), if the second player gets both jobs then \( M \) has an approximation ratio of 2. In this case, increase the first player’s cost for the second job to \( \infty \). By Lemma 2.1 the first player will not get either job. Thus, the second player will keep both jobs and the makespan of \( M \) will be \( 2 - \varepsilon' \), whereas the optimal makespan is 1 (this can be achieved when the first player gets the first job and the second player gets the second job). See Figure 10.

\[
\begin{pmatrix}
1 & \varepsilon^- \\
(1 - \varepsilon') & \varepsilon^-
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & \infty \\
(1 - \varepsilon') & \infty
\end{pmatrix} \rightarrow
\begin{pmatrix}
1^0 & \infty \\
1^* & \varepsilon^-
\end{pmatrix}
\]

Figure 10: The transitions described in the case that in \( D_1 \) the second player gets both jobs (as analyzed in the second case of the proof of theorem 2).

4.2 3 Players and 3 Jobs

Theorem 4.4 Every truthful mechanism for the unrelated machine scheduling with three machines and three jobs has an approximation ratio of at least 2.2055.

Let \( M \) be a truthful mechanism for the \( 3 \times 3 \) case of the unrelated machine scheduling problem with an approximation ratio better than 2.2055. Consider the instance:

\[
E = \begin{pmatrix}
\infty & c & \varepsilon^+ \\
b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
\]

We will show that \( M \) has an approximation ratio of at least \( \min\{1 + \frac{\varepsilon^+ + \varepsilon^-}{c}, \frac{a + b + c}{\varepsilon^+} \} \) where \( a < b < c \). We can then prove Theorem 4.4 by choosing \( a = 1, b \approx 2.2055 \) and \( c \approx 2.6589 \).
We divide the analysis to four cases based on the allocation of $M$ in $E$. In the first case, $M$ allocates the second job to the first player. In the second, $M$ allocates the first job to the second player and the second job to the third player. In the third case, $M$ allocates the first three jobs to the third player. In the forth case, $M$ allocates the first two jobs to the third player and the third job to the first player.

**Case 1: $M$ Allocates the Second Job to the First Player**

There are two possible cases for the allocation of the third job in $E$. In the first case, $M$ allocates the third job to the first player and in the second case, $M$ allocates it to the third player (otherwise the approximation ratio can be made arbitrarily large).

**Case 1.1: $M$ Allocates the Third Job to the First Player**

In this case, in the instance $E$, $M$ allocates the second and third jobs to the first player. Increase the third player’s cost for the third job to $\infty$ which makes the third job a dummy job for the first player. By Lemma 2.1, we have that the third player does not get these jobs. Thus, since $M$ has a finite approximation ratio, the first player keeps these jobs. Increase the cost of the first player for his dummy job to $b$. By Lemma 2.5, the first player gets the second and third jobs which results in an approximation ratio of $\frac{c^* + b}{b}$, as is visualized below.

\[
\begin{pmatrix}
\infty & c^* & \varepsilon^+ \\
 b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & c^* & \varepsilon^+ \\
b & \infty & \infty \\
a & b & \infty
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & c^* & b^* \\
b & \infty & \infty \\
a & b & \infty
\end{pmatrix}
\]

**Case 1.2: $M$ does not Allocate the Third Job to the First Player**

In this case, in $E$, $M$ allocates the second job but not the third job to the first player. Reduce the first player’s cost for the third job to $\varepsilon^-$ and reduce his cost for the second job by $\varepsilon' > \varepsilon^+ - \varepsilon^-$. Denote this instance by $E_1$ (as is visualized below).

\[
\begin{pmatrix}
\infty & c^* & \varepsilon^+ \\
 b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & c^* & \varepsilon^- \\
b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
\]

**Lemma 4.5** Suppose that in $E$, $M$ allocates the second job but not the third job, to the first player. Then, in $E_1$, $M$ allocates the second job to the first player.

**Proof:** Consider the weak monotonicity inequality applied on the first player where $x'$ is the allocation of $M$ in $E_1$ and $x$ is the allocation of $M$ in $E$. Assume by contradiction that $x_1^2 = 0$:

\[
(\infty - \infty) \cdot (0 - x_1^1) + (c - (c - \varepsilon')) \cdot (1 - x_1^2) + (\varepsilon^+ - \varepsilon^-) \cdot (0 - x_1^3) \\
= (\varepsilon') + (\varepsilon^+ - \varepsilon^-) \cdot (0 - x_1^3) \\
\leq 0
\]

and we arrive at a contradiction, since $\varepsilon' > \varepsilon^+ - \varepsilon^-$. 

By Lemma 4.5, in $E_1$, $M$ allocates the second job to the first player. There are two possible cases in the instance $E_1$: the first is that $M$ allocates the third job to the third player. The second case is that $M$ allocates the third job to the first player.
Case 1.2.1

In this case, in $E_1$, $M$ allocates the third job to the third player and the second job to the first player. Reduce the cost of the first player for the second job to 0, and by Lemma 2.1 the first player still gets only the second job (as visualized below).

$$
\begin{pmatrix}
\infty & c - \varepsilon^* & \varepsilon^- \\
b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & 0^* & \varepsilon^- \\
b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
$$

By Lemma 2.1

There are again two possible cases. In the first, the second player is assigned the first job, which results in an approximation ratio of $\frac{b}{a}$ (the makespan of $M$ is $b$ whereas the optimal one is $a$ which can be achieved, for example, when the first player takes the second and third jobs and the third player takes the first job).

$$
\begin{pmatrix}
\infty & 0^* & \varepsilon^- \\
b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
$$

In the second case, the third player takes the first and third jobs and the first player takes the second job. Reduce the third player’s cost for the third job to $\infty$, which makes the third job a dummy job for the first player. By Lemma 2.1, we have that the third player does not get the second or the third jobs. Thus, since $M$ has a makespan of $\varepsilon^-$ whereas the optimal one is $\varepsilon^-$ (this can be achieved when the first player gets the second and third jobs and the third player gets the first job) which results in an approximation ratio of $\frac{\varepsilon^-}{\varepsilon^-}$, which can be made arbitrarily large.

$$
\begin{pmatrix}
\infty & 0^* & \varepsilon^- \\
b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & 0^* & \varepsilon^- \\
b & \infty & \infty \\
0^* & b & \varepsilon^-
\end{pmatrix}
$$

Case 1.2.2

In this case, in $E_1$, $M$ allocates the second and third jobs to the first player.

$$
\begin{pmatrix}
\infty & c - \varepsilon^* & \varepsilon^-
\\b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & c - \varepsilon^* & \varepsilon^-
\\b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & c - \varepsilon^* & b^0 \\
b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
$$

The analysis is very similar to Case 1.1. Increase the third player’s cost for the third job to $\infty$, which makes the third job a dummy job for the first player. By Lemma 2.1 we have that the third player does not get the second or the third jobs. Thus, since $M$ has a finite approximation ratio the first player keeps these jobs. Increase the first player’s dummy job’s cost to $b$. By Lemma 2.5 the first player gets the second and third jobs which results in an approximation ratio of $\frac{c+b}{b}$. This is visualized below.

$$
\begin{pmatrix}
\infty & c - \varepsilon^* & \varepsilon^-
\\b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & c - \varepsilon^* & \varepsilon^-
\\b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & c - \varepsilon^* & b^0 \\
b & \infty & \infty \\
a & b & \varepsilon^-
\end{pmatrix}
$$

Case 2: $M$ Allocates the First Job to the Second Player and the Second Job to the Third Player

In this case, $M$ allocates the first job to the second player and the second job to the third player in $E$. Reduce the cost of the third player for the second job to 0. By Lemma 2.1 the third player will not
get the first job and since $M$ has a finite approximation ratio, the second player will get it. Then, $M$ will have an approximation ratio of $\frac{b}{a}$. This is visualized below.

$$\begin{pmatrix} \infty & c & \varepsilon^+ \\ b^* & \infty & \infty \\ a & b^* & \varepsilon^- \end{pmatrix} \xrightarrow{\text{Lemma 2.1}} \begin{pmatrix} \infty & c & \varepsilon^+ \\ b^* & \infty & \infty \\ a^{\oplus} & 0^{\oplus} & \varepsilon^-^{\oplus} \end{pmatrix}$$

Case 3: $M$ Allocates All Three Jobs to the Third Player

In this case, $M$ allocates all three jobs to the third player. Increase the cost of the first player for the third job to $\infty$. This makes the third job a dummy job for the first player. By Lemma 2.1, the first player will not get any job. Denote the resulting instance by $E_2$.

$$\begin{pmatrix} \infty & c & \varepsilon^+ \\ b & \infty & \infty \\ a^* & b^* & \varepsilon^-^{*} \end{pmatrix} \xrightarrow{\text{Lemma 2.1}} \begin{pmatrix} \infty & c & \infty \\ b & \infty & \infty \\ a^{\oplus} & 0^{\oplus} & \varepsilon^-^{\oplus} \end{pmatrix}$$

There are two possible cases based on the allocation of $M$ in the instance $E_2$. In the first case, $M$ allocates the first job to the second player and in the second case, $M$ allocates the first job to the third player.

Case 3.1

In this case, $M$ allocates the first job to the second player and allocates the second and third jobs to the third player in the instance $E_2$. Reduce the third player cost for the second job to 0 and by Lemma 2.1 the third player will not get the first job. Thus, since $M$ has a finite approximation ratio, the second player will get the first job which results in an approximation ratio of $\frac{b}{a}$. This is shown below.

$$\begin{pmatrix} \infty & c & \infty \\ b^* & \infty & \infty \\ a & b^* & \varepsilon^-^{*} \end{pmatrix} \xrightarrow{\text{Lemma 2.1}} \begin{pmatrix} \infty & c & \infty \\ b^* & \infty & \infty \\ a^{\oplus} & 0^{\oplus} & \varepsilon^-^{\oplus} \end{pmatrix}$$

Case 3.2

In this case, $M$ allocates all three jobs to the third player in the instance $E_2$. Increase the cost of the third player for its dummy job (the third job) to $c$ and by Lemma 2.5 the third player will get all three jobs. In this case $M$ has an approximation ratio of $\frac{a+b+c}{c}$, as depicted below.

$$\begin{pmatrix} \infty & c & \infty \\ b & \infty & \infty \\ a^* & b^* & \varepsilon^-^{*} \end{pmatrix} \xrightarrow{\text{Lemma 2.5}} \begin{pmatrix} \infty & c^{\oplus} & \infty \\ b^* & \infty & \infty \\ a^{\oplus} & b^* & c^{\oplus} \end{pmatrix}$$

Case 4: $M$ Allocates the First Two Jobs to the Third Player and the Third Job to the First Player

Reduce the third player’s cost for the first two jobs to 0. By Lemma 2.1 the third player will not get the third job and since the approximation is finite the third player will. In this case, $M$ has a makespan of $\varepsilon^+$ whereas the optimal makespan is of $\varepsilon^-$ which results in an approximation ratio of $\frac{\varepsilon^+}{\varepsilon^-}$ which can be arbitrarily large.
4.3 3 Players and 4 Jobs

**Theorem 4.6** Every truthful mechanism for the unrelated machine scheduling with three machines and four jobs has an approximation ratio of at least \( 1 + \sqrt{2} \).

Let \( M \) be a truthful mechanism for the \( 3 \times 4 \) case of the unrelated machine scheduling problem with an approximation ratio better than \( 1 + \sqrt{2} \).

Consider the instance:

\[
F = \begin{pmatrix}
\infty & c & \epsilon^+ \\
1 & \epsilon^- & \epsilon^- \\
1 & \epsilon^+ & \epsilon^- \\
\end{pmatrix}
\]

We will prove that \( M \) has an approximation ratio of at least \( \min \left\{ 1 + x, \frac{2 + x}{x} \right\} \) where \( x > 1 \). This proves Theorem 4.6 when using \( x = \sqrt{2} \).

Consider the instance \( F \). We divide the analysis to cases based on the allocation of \( M \). In the first case, \( M \) allocates the second job to the first player. In the second case, \( M \) allocates the first two jobs to the second player. In the third case, \( M \) allocates the first job to the second player and the second job to the third player. In the fourth case, \( M \) allocates the first job to the third player and the second job to the second player. In the fifth case, \( M \) allocates the first two jobs to the third player.

**Case 1**

In this case, in the instance \( F \), \( M \) allocates the second job to the first player. Increase the first player’s dummy job’s cost to 1 and by Lemma 2.5 we have that the first player keeps the second and fourth jobs which results in a makespan of \( 1 + x \) whereas the optimal makespan is 1.

\[
\begin{pmatrix}
\infty & x^* & \infty & \epsilon^{---} \\
1 & \epsilon^- & \epsilon^- & \infty \\
1 & \epsilon^+ & \epsilon^- & \infty \\
\end{pmatrix} \xrightarrow{\text{By Lemma 2.5}} \begin{pmatrix}
\infty & x^* & \infty & 1^* \\
1 & \epsilon^- & \epsilon^- & \infty \\
1 & \epsilon^+ & \epsilon^- & \infty \\
\end{pmatrix}
\]

**Case 2**

In this case, in the instance \( F \), \( M \) allocates the first two jobs to the second player.

\[
\begin{pmatrix}
\infty & x^* & \infty & \epsilon^{---} \\
1^* & \epsilon^- & \epsilon^- & \infty \\
1 & \epsilon^+ & \epsilon^- & \infty \\
\end{pmatrix} \xrightarrow{\text{By Lemma 2.5}} \begin{pmatrix}
\infty & x^* & \infty & \epsilon^{---} \\
1^* & \epsilon^- & \epsilon^- & \infty \\
1 & \epsilon^+ & \epsilon^- & \infty \\
\end{pmatrix}
\]

By Lemma 2.1 the third player doesn’t get the first or the second jobs. In the case that the first player gets the second job we have an \( 1 + x \) approximation ratio as shown below:

\[
\begin{pmatrix}
\infty & x^* & \infty & \epsilon^{---} \\
1 & \epsilon^- & \epsilon^- & \infty \\
1 & \epsilon^- & \epsilon^- & \infty \\
\end{pmatrix} \xrightarrow{\text{By Lemma 2.5}} \begin{pmatrix}
\infty & x^* & \infty & 1^* \\
1 & \epsilon^- & \epsilon^- & \infty \\
1 & \epsilon^- & \epsilon^- & \infty \\
\end{pmatrix}
\]
Otherwise, the second player gets the first two jobs.

\[
\begin{pmatrix}
\infty & x & \infty & \varepsilon^- - \\
1^* & \varepsilon^- & \infty & \infty \\
1 & \infty & \varepsilon^-- & \infty
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & x & \infty & \varepsilon^- - \\
1 & 1 & \varepsilon^- & \infty \\
1 & \infty & \varepsilon^-- & \infty
\end{pmatrix}
= F_1
\]

There are two possible cases. In the first, the first player gets the second job in the instance \( F_1 \) (this is analyzed in Case 2.1). In the second case, the second player keeps the second job (this analyzed in Case 2.2).

**Case 2.1**

\[
\begin{pmatrix}
\infty & x^* & \infty & \varepsilon^- - \\
1 & 1 & \varepsilon^- & \infty \\
1 & \infty & \varepsilon^-- & \infty
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & x^* & \infty & 1^*0 \\
1 & 10 & \varepsilon^-0 & \infty \\
10 & \infty & \varepsilon^-- & \infty
\end{pmatrix}
\]

By Lemma 2.5

In this case, \( M \) has an approximation ratio of \( 1 + x \).

**Case 2.2**

In this case, in the instance \( F \), \( M \) allocates the first two jobs to the second player and in the instance \( F_1 \), \( M \) allocates the second job to the second player. By Lemma 2.7 for \( i = 2, j_1 = 1, j_2 = 2 \) we have that in the instance \( F_1 \) the second player gets the first two jobs.

There are two possible cases, in the first case the second player gets the first three jobs in \( F_1 \) (analyzed in Case 2.2.1). In the second case the second player gets the first two jobs and the third player gets the third job (analyzed in Case 2.2.2).

**Case 2.2.1**

\[
\begin{pmatrix}
\infty & x^* & \infty & \varepsilon^- - \\
1^* & 1^* & \varepsilon^- & \infty \\
1 & \infty & \varepsilon^-- & \infty
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & x^* & \infty & 1^*0 \\
0^*0 & 0^*0 & \varepsilon^-- & \infty \\
1 & \infty & \varepsilon^-- & \infty
\end{pmatrix}
\]

By Lemma 2.7

This results in an approximation ratio of \( \frac{\varepsilon^-}{\varepsilon^-} \) which can be made arbitrarily large.

**Case 2.2.2**

Let \( \varepsilon' > \varepsilon^- - \varepsilon^- - \).

\[
\begin{pmatrix}
\infty & x^* & \infty & \varepsilon^- - \\
1^* & 1^* & \varepsilon^- & \infty \\
1 & \infty & \varepsilon^-- & \infty
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & x & \infty & \varepsilon^- - \\
1 - \varepsilon' & 1 - \varepsilon' & \varepsilon^- - & \infty \\
1 & \infty & \varepsilon^-- & \infty
\end{pmatrix}
= F_2
\]

Similarly to Lemma 4.5, it can be shown that the second player keeps the first two jobs in the instance \( F_2 \). There are two possible cases. In the first, the second player gets the first three jobs in the instance \( F_2 \) (analyzed in Case 2.2.2.1). In the second case the second player gets the first two jobs and the third player gets the third job (analyzed in Case 2.2.2.2).
Case 2.2.2.1

\[
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{-}\varepsilon \varepsilon \\
1 - \varepsilon' & 1 - \varepsilon' & \varepsilon^{-}\varepsilon & \infty \\
1 & \infty & \varepsilon^{-} & \infty \\
\end{pmatrix} \rightarrow
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{-}\varepsilon \\
1 - \varepsilon' & 1 - \varepsilon' & \varepsilon^{-}\varepsilon & \infty \\
1 & \infty & \varepsilon^{-} & \infty \\
\end{pmatrix}
\]

By Lemma 2.1, the third player doesn’t get a job from the first three jobs. There are two possible cases, the first is that the first player takes the second job (Case 2.2.2.1.1) and the second is that the second player takes the second job (and jobs 1,3) (Case 2.2.2.1.2).

Case 2.2.2.1.1

\[
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{-}\varepsilon \varepsilon \\
1 - \varepsilon' & 1 - \varepsilon' & \varepsilon^{-}\varepsilon & \infty \\
1 & \infty & \varepsilon^{-} & \infty \\
\end{pmatrix} \rightarrow
\begin{pmatrix}
\infty & x & \infty & 1^{*} \\
1 - \varepsilon' & 1 - \varepsilon' & \varepsilon^{-}\varepsilon & \infty \\
1 & \infty & \varepsilon^{-} & \infty \\
\end{pmatrix}
\]

In this case, \(M\) has an approximation ratio of \(1 + x\).

Case 2.2.2.1.2

\[
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{-}\varepsilon \varepsilon \\
1 - \varepsilon' & 1 - \varepsilon' & \varepsilon^{-}\varepsilon & \infty \\
1 & \infty & \varepsilon^{-} & \infty \\
\end{pmatrix} \rightarrow
\begin{pmatrix}
\infty & x & \infty & 1^{*} \\
1 - \varepsilon' & 1 - \varepsilon' & \varepsilon^{-}\varepsilon & \infty \\
1 & \infty & \varepsilon^{-} & \infty \\
\end{pmatrix}
\]

In this case, \(M\) has an approximation ratio of \(\frac{2 + x}{x}\).

Case 2.2.2.2

\[
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{-}\varepsilon \\
1 - \varepsilon' & 1 - \varepsilon' & \varepsilon^{-}\varepsilon & \infty \\
1 & \infty & \varepsilon^{-}\varepsilon & \infty \\
\end{pmatrix} \rightarrow
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{-}\varepsilon \\
0^{*} & 0 & \varepsilon^{-}\varepsilon & \infty \\
1 & \infty & \varepsilon^{-}\varepsilon & \infty \\
\end{pmatrix}
\]

By Lemma 2.1, the second player doesn’t get the third job and since \(M\) has a finite approximation ratio the third job goes to the third player. This results in an approximation ratio of \(\frac{\varepsilon^{-}\varepsilon}{\varepsilon}\) which can be made arbitrarily large.

Case 3

In this case, \(M\) has an approximation ratio of \(\frac{\varepsilon^{-}\varepsilon}{\varepsilon}\) which can be made arbitrarily large.

\[
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{-}\varepsilon \\
1 & \varepsilon - & \varepsilon & \infty \\
1 & \varepsilon' - & \varepsilon & \infty \\
\end{pmatrix} \rightarrow
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{-}\varepsilon \\
0^{*} & \varepsilon - & \varepsilon & \infty \\
1 & \varepsilon' - & \varepsilon & \infty \\
\end{pmatrix}
\]

Case 4

Let \(\varepsilon' > \varepsilon + \varepsilon^{-}\)

\[
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{-}\varepsilon \\
1 & \varepsilon' - & \varepsilon & \infty \\
1 & \varepsilon' - & \varepsilon & \infty \\
\end{pmatrix} \rightarrow
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{-}\varepsilon \\
1 & \varepsilon' - & \varepsilon & \infty \\
1 & \varepsilon' - & \varepsilon & \infty \\
\end{pmatrix}
\]

Similarly to Lemma 4.5, the third player gets the first job. There are three possible cases. In the first, the first player gets the second job (Case 4.1), in the second case, the second player gets the second job (Case 4.2) and in the third case the third player gets the second job (Case 4.3).
Case 4.1
In this case, $M$ has an approximation ratio of $1 + x$.

$$
\begin{pmatrix}
\infty & x^* & \infty & \varepsilon^{---*} \\
1 & \varepsilon^- & \varepsilon^- & \infty \\
1 - \varepsilon' & \varepsilon^- & \varepsilon^- & \infty \\
\end{pmatrix}
\xrightarrow{\text{By Lemma 2.5}}
\begin{pmatrix}
\infty & x^* & \infty & 1^{*0} \\
1 & \varepsilon^- & \varepsilon^- & \infty \\
1 - \varepsilon' & \varepsilon^- & \varepsilon^- & \infty \\
\end{pmatrix}
$$

Case 4.2
In this case, $M$ has an approximation ratio which can be made arbitrarily large.

$$
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{---*} \\
1 & \varepsilon^- & \varepsilon^- & \infty \\
1 - \varepsilon' & \varepsilon^- & \varepsilon^- & \infty \\
\end{pmatrix}
\xrightarrow{\text{By Lemma 2.1}}
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{---*}^{0} \\
1 & \varepsilon^- & \varepsilon^- & \infty \\
0 & \varepsilon^- & \varepsilon^- & \infty \\
\end{pmatrix}
$$

This results in an approximation ratio of $\frac{\varepsilon}{\varepsilon}$ or, $\frac{\varepsilon}{\varepsilon^*}$. The second approximation is achieved when the first player gets the second job and not the second player (after the above transition).

Case 4.3
In this case there are two possible cases, the first is that the third job is allocated to the second player (Case 4.3.1) and the second is that the third job is allocated to the third player (Case 4.3.2).

Case 4.3.1
In this case, $M$ has an approximation ratio of $\frac{\varepsilon}{\varepsilon}$ which can be made arbitrarily large.

Case 4.3.2
By Lemma 2.1, the second player does not get the first job. Thus, since $M$ has a finite approximation ratio, the third player gets it. There are two possible cases. The first is that the first player takes the second job (Case 4.3.2.1) and the second case, that the third player does (Case 4.3.2.2).

Case 4.3.2.1
In this case, $M$ has an approximation ratio of $1 + x$.

$$
\begin{pmatrix}
\infty & x^* & \infty & \varepsilon^{---*} \\
1 & \infty & \infty & \infty \\
1 - \varepsilon' & \varepsilon^- & \varepsilon^- & \infty \\
\end{pmatrix}
\xrightarrow{\text{By Lemma 2.5}}
\begin{pmatrix}
\infty & x^* & \infty & 1^{*0} \\
1 & \infty & \infty & \infty \\
1 - \varepsilon' & \varepsilon^- & \varepsilon^- & \infty \\
\end{pmatrix}
$$
Case 4.3.2.2

\[
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{---} \\
1 & \infty & \infty & \infty \\
1 - \varepsilon' & \varepsilon^{-*} & \varepsilon^{---} & \infty
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\infty & x & \infty & \varepsilon^{-*0} \\
1 & \infty & \infty & \infty \\
1 - \varepsilon' & 1^* & \varepsilon^{---} & \infty
\end{pmatrix}
\]

By Lemma 2.5

By Lemma 2.7, If the third player gets the second job, he also gets the first job, this case is analyzed in 4.3.2.2.1. Otherwise, the first player gets the second job, this is analyzed in 4.3.2.2.2.

Case 4.3.2.2.1

In this case, \( M \) has an approximation ratio of \( \frac{2 + x}{x} \).

Case 4.3.2.2.2

In this case, \( M \) has an approximation ratio of \( 1 + x \).

Case 5

In this case, \( M \) has an approximation ratio of \( \frac{\varepsilon^+}{\varepsilon} \) which can be made arbitrarily large.

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