On connectedness of non-klt loci of singularities of pairs

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Abstract. We study the non-klt locus of singularities of pairs. We show that given a pair \((X, B)\) and a projective morphism \(X \to Z\) with connected fibres such that \(- (K_X + B)\) is nef over \(Z\), the non-klt locus of \((X, B)\) has at most two connected components near each fibre of \(X \to Z\). This was conjectured by Hacon and Han.

In a different direction we answer a question of Mark Gross on connectedness of the non-klt loci of certain pairs. This is motivated by constructions in Mirror Symmetry.

1. Introduction

We work over an algebraically closed field of characteristic zero.

Pairs and their singularities play a fundamental role in higher dimensional algebraic geometry. Let’s consider the simplest kind of pair, that is, a projective log smooth pair \((X, B)\) where \(X\) is a smooth projective variety and \(B = \sum b_iB_i\) is a divisor with simple normal crossing singularities and real coefficients \(b_i \geq 0\); here \(B_i\) are distinct prime divisors. In this setting, the non-klt locus \(\text{Nklt}(X, B)\) of \((X, B)\) is the union of the \(B_i\) with \(b_i \geq 1\). In general, \(\text{Nklt}(X, B)\) can have any number of connected components as a topological space with the Zariski topology. But in special situations the non-klt locus exhibits interesting behavior. For example, Shokurov [32, Connectedness Lemma 5.7] proved in the early 1990’s that if \(X\) is a surface and \(- (K_X + B)\) is ample, then \(\text{Nklt}(X, B)\) is connected. This was generalised to higher dimensions by Kollár [23, Theorem 17.4], which is known as the connectedness lemma or connectedness principle. The same holds if we only assume \(- (K_X + B)\) to be nef and big, and it also holds in the relative setting when \(X\) is defined over a base \(Z\) in which case connectedness holds near each fibre of \(X \to Z\) assuming \(X \to Z\) has connected fibres.

The connectedness principle plays an important role in higher dimensional algebraic geometry. For example, it is used in the proof of existence of flips [32][31], in the proof of inversion of adjunction [23, Theorem 17.6], in the proofs of boundedness of complements and boundedness of Fano varieties [6, Proposition 5.1, Proposition 6.7], in the proof of the Jordan property of birational automorphism groups of rationally connected varieties [28], in birational rigidity of Fano varieties [29], in the study of adjunction for fibre spaces [30], etc.

A natural question is what happens if we only assume that \(- (K_X + B)\) is nef? In this case, \(\text{Nklt}(X, B)\) may not be connected. The easiest example is to take \(X = \mathbb{P}^1\), \(B = B_1 + B_2\) where \(B_i\) are distinct points. It turns out that when connectedness fails this simple example is in a sense the reason. When \(K_X + B \equiv 0\) and the coefficients of \(B\) are \(\leq 1\), Shokurov [32, Theorem 6.9] in dimension two and Kollár and Kovács [24][22, Theorem 4.40] in higher dimension showed that if connectedness fails, then \(X\)
Caucher Birkar

is birational to a (possibly singular) model \((X', B')\) admitting a contraction \(X' \to Y'\) where the general fibres are \(\mathbb{P}^1\) and \([B']\) has exactly two disjoint components both horizontal over \(Y'\) \((X'\) is obtained by running a minimal model program on \(-[B]\)).

Hacon and Han [19] investigated the above phenomenon more generally. They showed that if \(\dim X \leq 4\) and if \(- (K_X + B)\) is nef, then \(\text{Nklt}(X, B)\) has at most two connected components. They conjectured that this holds in every dimension and then showed that it follows from the termination of klt flips conjecture. One of our main results is to verify their conjecture without assuming termination of klt flips (see Theorem 1.2).

Another main result of this paper (see Theorem 1.4 and Corollaries 1.5, 1.6), in a different but somewhat related direction, is an answer to the following question of Mark Gross. The question is motivated by constructions in Mirror Symmetry. In fact this work started in response to this question which was communicated privately.

In [17], Gross and Siebert construct mirror families to pairs \((X, B)\) where \(X\) is a non-singular projective variety and \(B\) is a simple normal crossings divisor satisfying certain hypotheses, see [17, Assumptions 1.1]. It is not clear how strong these hypotheses are, and thus it is important to know which pairs \((X, B)\) might satisfy them. These hypotheses include conditions (ii) and (iii) of Question 1.1 as well as the desired connectedness in the question. The conditions (ii) and (iii) are essential for mirror symmetry: mirrors are not expected to exist without similar assumptions. On the other hand, the connectedness condition would appear to be a technical one. Thus Corollary 1.5 below gives a useful criterion for checking when this connectedness assumption may hold, and in particular when the mirror to the pair \((X, B)\) may exist.

**Question 1.1.** Consider the following setup:

(i) \((X, B)\) is a projective log smooth pair where \(B = \sum B_i\) is reduced,

(ii) \(K_X + B \equiv \sum a_i B_i\) with \(a_i \geq 0\) real numbers (we say \(B_i\) is good if \(a_i = 0\)),

(iii) \((X, C)\) has a zero-dimensional stratum \(x\) where \(C\) is the sum of the good divisors,

(iv) for each stratum \(V\) of \((X, C)\), define \(K_V + C_V = (K_X + C)|_V\) by adjunction.

Then under what conditions is \(C_V\) connected for every stratum \(V\) of dimension \(\geq 2\)?

Here by a stratum of \((X, C)\) we mean \(X\) itself or an irreducible component of the intersection of any subset of the irreducible components of \(C\).

Gross pointed out that if \((X, B)\) has a good log minimal model which is log Calabi-Yau, that is, a log minimal model \((X', B')\) with \(K_{X'} + B' \sim_\mathbb{Q} 0\), then the connectedness in the question holds by [22, Theorem 4.40] mentioned above. If \((X, B)\) has Kodaira dimension zero, then standard conjectures of the minimal model program, including the abundance conjecture, imply that \((X, B)\) has such a log minimal model but the current technology of the minimal model program is not enough to guarantee existence of good log minimal models in general in dimension \(\geq 4\). The problem is then to find other reasonable assumptions, instead of Kodaira dimension zero, so that a good log minimal model exists or at least that the desired connectedness holds.

It turns out that in the setting of the question it is enough to assume the following property which is of a numerical nature so probably easy to check in explicit examples. Let \(\phi: Y \to X\) be the blowup of \(X\) at the zero-dimensional stratum \(x \in X\) and let \(E\) be the exceptional divisor. Assume that \(\phi^*(K_X + B) - tE\) is not pseudo-effective for any real number \(t > 0\) (a divisor is pseudo-effective if it is numerically the limit
of effective divisors). Then \( C_Y \) is connected for every stratum \( V \) of dimension \( \geq 2 \). See Corollary 1.5 for a statement that works in a much more general setting.

The non-pseudo-effectivity assumption of the previous paragraph is not as restrictive as it may seem. In fact, it is conjecturally equivalent to \((X, B)\) having Kodaira dimension zero. Indeed, assuming standard conjectures of the minimal model program, \((X, B)\) has a log minimal model \((X', B')\) where \(K_{X'} + B'\) is semi-ample, that is, \(|m(K_{X'} + B')|\) is base point free for some sufficiently divisible \(m \in \mathbb{N}\). Thus there is a contraction \(g: X' \to Z'\) such that \(K_{X'} + B' \sim_{\mathbb{Q}} g^*H\) where \(H\) is an ample \(\mathbb{Q}\)-divisor. Now \(X \to X'\) is an isomorphism near \(x\) because \(x\) does not belong to \(\text{Supp} \sum a_iB_i\). If \(\dim Z' > 0\), then we can find \(0 \leq P' \sim_{\mathbb{Q}} K_{X'} + B'\) so that \(x' \in \text{Supp} P'\) where \(x'\) is the image of \(x\). This gives \(0 \leq P \sim_{\mathbb{Q}} K_X + B\) so that \(x \in \text{Supp} P\), so in this case

\[
\phi^*(K_X + B) - tE \sim_{\mathbb{Q}} \phi^*P - tE
\]

is pseudo-effective for some \(t > 0\), a contradiction. Therefore, \(\dim Z' = 0\) which exactly means that \((X, B)\) has Kodaira dimension zero. Conversely, assume \((X, B)\) has Kodaira dimension zero. Then \(\phi^*(K_X + B) - tE\) is not pseudo-effective for any \(t > 0\) otherwise one can check that \(\phi^*(K_{X'} + B') - tE'\) is pseudo-effective for some \(t > 0\) where \(\phi'\) is the blowup of \(X'\) at \(x'\) and \(E'\) is the exceptional divisor, which is not possible as \(\phi^*(K_{X'} + B') \sim_{\mathbb{Q}} 0\).

Since conditions similar to the above non-pseudo-effectivity condition appear again in this text we make a definition to ease notation. Given a projective variety \(X\), a pseudo-effective \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(L\) on \(X\), and a prime divisor \(S\) over \(X\) (that is on birational models of \(X\)) we define the threshold

\[
\tau_S(L) = \sup \{ t \in \mathbb{R}_{\geq 0} \mid \phi^*L - tS \text{ is pseudo-effective} \}
\]

where \(\phi: W \to X\) is any resolution on which \(S\) is a divisor. This is independent of the choice of the resolution, see Lemma 2.3. A relative version of the threshold can similarly be defined when \(X\) is projective over a base \(Z\), in which case we denote it by \(\tau_S(L/Z)\).

**Non-klt loci of anti-nef pairs.** Our first main result concerns the non-klt loci of pairs \((X, B)\) with \(-(K_X + B)\) nef over some base.

**Theorem 1.2.** Let \((X, B)\) be a pair and \(f: X \to Z\) be a contraction. Assume \(-(K_X + B)\) is nef over \(Z\). Assume that the fibre of \(\text{Nklt}(X, B) \to Z\) over some point \(z \in Z\) is not connected. Then we have:

1. \(\text{Nklt}(X, B) \to Z\) is surjective and its fibre over \(z\) has exactly two connected components,

2. the pair \((X, B)\) is lc over \(z\) and after base change to an étale neighbourhood of \(z\), there exist a resolution \(\phi: X' \to X\) and a contraction \(X' \to Y'\) over \(Z\) such that if

   \[K_{X'} + B' := \phi^*(K_X + B)\]

   and if \(F'\) is a general fibre of \(X' \to Y'\), then \((F', B'|_{F'})\) is isomorphic to \((\mathbb{P}^1, p_1 + p_2)\) for distinct points \(p_1, p_2\).

   Moreover, \([B']\) has two disjoint irreducible components \(S', T'\), both horizontal over \(Y'\), and the images of \(S', T'\) on \(X\) are the two connected components of \(\text{Nklt}(X, B)\).

As mentioned above, the theorem implies [19, Conjecture 1.1]. After completion of this work we learnt that Filipazzi and Svaldi [14, Theorem 1.1] have also proved this
result using different arguments. We also learnt from Shokurov that similar results appeared in his unpublished work.

Here is an example of \((X, B)\) as in the theorem on which the non-klt locus has two components, one zero-dimensional and the other one-dimensional. Let \(X = \mathbb{P}^2\), \(B_1\) be a line, \(x\) a closed point not contained in \(B_1\), and \(B_2, \ldots, B_5\) be distinct lines passing through \(x\). Letting

\[
B = B_1 + \frac{1}{2}(B_2 + \cdots + B_5)
\]

we can see that \(K_X + B \equiv 0\) and \((X, B)\) is lc and \(\text{Nklt}(X, B)\) has two components, one is \(x\) and the other is \(B_1\). In this example, \(X' = \mathbb{F}_1 \to X\) is the blowup at \(x\), \(X' \to Y' = \mathbb{P}^1\) is the corresponding \(\mathbb{P}^1\)-bundle, \(S'\) is the birational transform of \(B_1\), and \(T'\) is the exceptional divisor of \(X' \to X\).

The theorem was already known (in this or other forms)

- in dimension two [32, Theorem 6.9][27, §3.3],
- in case \((X, B)\) is dlt and \(K_X + B \equiv 0\) assuming termination of klt flips [15, Proposition 2.1] and without assuming this termination [24][22, Proposition 4.37], and
- in any dimension assuming termination of klt flips and in dimension \(\leq 4\) without assuming termination [19, Theorem 1.2].

Also if \(K_X + B \sim_{\mathbb{R}} 0\) and if \((X, B)\) is not lc, it was proved in [1, Theorem 6.3] that the non-klt locus, even the non-lc locus, of \((X, B)\) is connected.

In the opposite direction we have the following result.

**Theorem 1.3.** Let \((X, B)\) be a pair and \(f: X \to Z\) be a contraction. Assume \(-(K_X + B)\) is nef over \(Z\). Then the fibres of \(\text{Nklt}(X, B) \to Z\) are connected if any of the following conditions holds:

1. \(-(K_X + B)\) is big over \(Z\), or
2. \(\text{Nklt}(X, B) \to Z\) is not surjective, or
3. \(\tau_S(-(K_X + B)/Z) > 0\) for every non-klt place \(S\) of \((X, B)\).

Here by a non-klt place we mean a prime divisor \(S\) over \(X\) (that is, on birational models of \(X\)) such that the log discrepancy \(\alpha(S, X, B) \leq 0\). Note that in view of Theorem 1.2 we can add another case in which the theorem holds, that is, the fibre of \(\text{Nklt}(X, B) \to Z\) over a point \(z \in Z\) is connected if \((X, B)\) is not lc over \(z\).

Case (1) is essentially the connectedness principle mentioned above. Case (3) implies cases (1) and (2) but in practice we first prove cases (1),(2) and then derive case (3) from Theorem 1.2. There are situations where one can apply (3) but not (1) and (2). For example, consider the following: assume \(-(K_X + B)\) is semi-ample over \(Z\) defining a non-birational contraction \(X \to T/Z\); assume that \(\text{Nklt}(X, B) \to T\) is not surjective but \(\text{Nklt}(X, B) \to Z\) is surjective; then \(\tau_S(-(K_X + B)/Z) > 0\) for every non-klt place \(S\) of \((X, B)\), so we can apply (3).

See also [19, Corollary 1.3] for some special situations in dimension \(\leq 4\) or any dimension assuming termination of klt flips.

**Non-klt loci for Mirror Symmetry.** The following result is the main step towards answering Question 1.1 which works in a much more general setting.

**Theorem 1.4.** Assume that

1. \((X, B)\) is a projective \(\mathbb{Q}\)-factorial dlt pair where \(B\) is a \(\mathbb{Q}\)-divisor,
2. \(K_X + B\) is pseudo-effective,
On connectedness of non-klt loci of singularities of pairs

(3) $x \in X$ is a zero-dimensional non-klt centre of $(X, B)$,
(4) $x$ is not contained in the restricted base locus $B_-(K_X + B)$,
(5) if $\phi : Y \to X$ is the blowup at $x$ with exceptional divisor $E$, then $\tau_E(K_X + B) = 0$, i.e. $\phi^*(K_X + B) - tE$ is not pseudo-effective for any real number $t > 0$.

Then $(X, B)$ has a good log minimal model which is log Calabi–Yau. More precisely, we can run a minimal model program on $K_X + B$ ending with a log minimal model $(X', B')$ with $K_{X'} + B'$ $\sim_{\mathbb{Q}} 0$.

Note that in the theorem, $(X, B)$ is dlt and $x \in X$ is its zero-dimensional non-klt centre, so $x$ is a smooth point of $X$ and $\text{Supp} B$ is simple normal crossing near $x$.

Recall that for a $\mathbb{Q}$-divisor $L$ on a normal projective variety, the stable base locus is defined as

$$B(L) := \bigcap_m \text{Bs} (mL)$$

where $m$ runs over the natural numbers such that $mL$ is an integral divisor. The restricted base locus of $L$ is defined as

$$B_-(L) := \bigcup_{\epsilon \in \mathbb{Q}^+} B(L + \epsilon A)$$

where $A$ is any fixed ample divisor (the locus is independent of the choice of $A$).

**Corollary 1.5.** Under the assumptions of Theorem 1.4, suppose that no non-klt centre of $(X, B)$ is contained in $B_-(K_X + B)$. For $V = X$ or $V$ a non-klt centre of $(X, B)$, define $K_V + B_V = (K_X + B)|_V$. Then $\text{Nklt}(V, B_V)$ is connected when $\dim V \geq 2$.

In the setting of Question 1.1, conditions (1)-(4) of the theorem are automatically satisfied: indeed, $(X, B)$ is log smooth so we have (1); $K_X + B = \sum_i a_i B_i$ with $a_i \geq 0$, so $K_X + B$ is pseudo-effective which is (2); $x$ is a zero-dimensional stratum of $(X, C)$, so it is a non-klt centre of both $(X, C)$ and $(X, B)$, so we have (3); $x$ is contained only in the good components of $B$, so $x$ is not contained in $\text{Supp} \sum a_i B_i$, hence $x$ is not contained in $B_-(K_X + B) \subseteq \text{Supp} \sum a_i B_i$ so we have (4).

Applying Theorem 1.4 and Corollary 1.5, we will prove the following answer to Question 1.1.

**Corollary 1.6.** Assume that

(1) $(X, B)$ is a projective log smooth pair where $B = \sum B_i$ is reduced,
(2) $K_X + B = \sum a_i B_i$ with $a_i \geq 0$ real numbers (we say $B_i$ is good if $a_i = 0$),
(3) $(X, C)$ has a zero-dimensional stratum $x$ where $C$ is the sum of the good divisors,
(4) for each stratum $V$ of $(X, C)$, define $K_V + C_V = (K_X + C)|_V$ by adjunction,
(5) if $\phi : Y \to X$ is the blowup at $x$ with exceptional divisor $E$, then $\tau_E(K_X + B) = 0$.

Then $(X, B)$ has a good log minimal model which is log Calabi–Yau, and $C_V$ is connected for every stratum $V$ of dimension $\geq 2$.

**Plan of the paper.** We will prove Theorems 1.2 and 1.3 in Section 3 and Theorem 1.4 and Corollaries 1.5 and 1.6 in Section 4. We will actually prove more general forms of these results in the setting of generalised pairs. Generalised pairs play a key role in the proofs.
2. Preliminaries

All varieties in this paper are quasi-projective over an algebraically closed field of characteristic zero unless otherwise stated.

2.1. Contractions. A contraction is a projective morphism \( f : X \to Y \) of varieties such that \( f_\ast \mathcal{O}_X = \mathcal{O}_Y \) (\( f \) is not necessarily birational). In particular, \( f \) has connected fibres and if \( X \to Z \to Y \) is the Stein factorisation of \( f \), then \( Z \to Y \) is an isomorphism.

2.2. Pseudo-effective thresholds. Given a projective morphism \( X \to Z \) of varieties, a pseudo-effective \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor \( L \) on \( X \), and a prime divisor \( S \) over \( X \) (that is on birational models of \( X \)) we define the pseudo-effective threshold \( \tau_S(L/Z) \) as follows. Pick a birational contraction \( \phi : Y \to X \) from a normal variety so that \( S \) is a \( \mathbb{Q} \)-Cartier divisor on \( Y \), e.g. a resolution of singularities of \( X \). Then define

\[
\tau_S(L/Z) = \sup\{ t \in \mathbb{R}_{\geq 0} \mid \phi^\ast L - tS \text{ is pseudo-effective}/Z \}.
\]

The next lemma shows that this is well-defined. When \( Z \) is a point, we will simply denote the threshold by \( \tau_S(L) \).

**Lemma 2.3.** The threshold \( \tau_S(L/Z) \) is independent of the choice of \( \phi : Y \to X \).

**Proof.** Let \( \phi : Y \to X \) and \( \phi' : Y' \to X \) be birational morphisms from normal varieties so that \( S, S' \) are \( \mathbb{Q} \)-Cartier divisors on \( Y, Y' \), respectively, where \( S, S' \) represent the same divisor over \( X \), that is, \( S' \) is the birational transform of \( S \). We denote the corresponding thresholds by \( \tau^\phi_S(L/Z) \) and \( \tau^\phi_{S'}(L/Z) \). It is enough to show that \( \tau^\phi_S(L/Z) = \tau^\phi_{S'}(L/Z) \) when the induced map \( \psi : Y' \to Y \) is a morphism because in the general case we can use a common resolution of \( Y', Y \).

If \( \phi^\ast L - t S' \) is pseudo-effective\( /Z \), then obviously the pushdown

\[
\psi_\ast(\phi^\ast L - t S') = \phi^\ast L - t S
\]

is pseudo-effective\( /Z \). Thus \( \tau^\phi_{S'}(L/Z) \leq \tau^\phi_S(L/Z) \). Conversely, if \( \phi^\ast L - tS \) is pseudo-effective\( /Z \), then \( \psi^\ast(\phi^\ast L - tS) \) is pseudo-effective\( /Z \). But

\[
\psi^\ast(\phi^\ast L - tS) = \phi'^\ast L - t\psi^\ast S \leq \phi'^\ast L - t S',
\]

as \( \psi^\ast S \geq S' \), hence \( \phi'^\ast L - tS' \) is pseudo-effective\( /Z \) (this is where we use the \( \mathbb{Q} \)-Cartier condition of \( S \) so that we can take the pullback \( \psi^\ast S \) ). Thus \( \tau^\phi_{S'}(L/Z) \geq \tau^\phi_S(L/Z) \) which in turn implies the equality \( \tau^\phi_{S'}(L/Z) = \tau^\phi_S(L/Z) \).

\[ \square \]
2.4. Pairs. A sub-pair \((X,B)\) consists of a normal quasi-projective variety \(X\) and an \(\mathbb{R}\)-divisor \(B\) such that \(K_X + B\) is \(\mathbb{R}\)-Cartier. If the coefficients of \(B\) are at most 1 we say \(B\) is a sub-boundary, and if in addition \(B \geq 0\), we say \(B\) is a boundary. A sub-pair \((X,B)\) is called a pair if \(B \geq 0\).

Let \(\phi: W \to X\) be a log resolution of a sub-pair \((X,B)\). Let \(K_W + B_W\) be the pullback of \(K_X + B\). The log discrepancy of a prime divisor \(D\) on \(W\) with respect to \((X,B)\) is \(1 - \mu_D B_W\) and it is denoted by \(a(D, X, B)\). We say \((X,B)\) is sub-lc (resp. sub-\(c\)-lc) if \(a(D, X, B)\) is \(\geq 0\) (resp. \(> 0\)) for every \(D\). When \((X,B)\) is a pair we remove the sub and say the pair is lc, etc. Note that if \((X,B)\) is an lc pair, then the coefficients of \(B\) necessarily belong to \([0,1]\).

Let \((X,B)\) be a sub-pair. A non-klt place of \((X,B)\) is a prime divisor \(D\) on birational models of \(X\) such that \(a(D, X, B) \leq 0\). A non-klt centre is the image on \(X\) of a non-klt place. When \((X,B)\) is lc, a non-klt centre is also called an lc centre. The non-klt locus \(\text{Nklt}(X,B)\) of a sub-pair \((X,B)\) is the union of the non-klt centres with reduced structure.

A sub-pair \((X,B)\) is log smooth if \(X\) is smooth and \(\text{Supp} B\) has simple normal crossing singularities.

2.5. Minimal model program (MMP). We will use standard results of the minimal model program (cf. [25][11]). Assume \((X,B)\) is a pair, \(X \to Z\) is a projective morphism, \(H\) is an ample//\(\mathbb{R}\)-divisor, and that \(K_X + B + H\) is \(\text{nef}/Z\). First suppose \((X,B)\) is \(\mathbb{Q}\)-factorial dlt. Then we can run an MMP//\(\mathbb{Q}\)(Z) on \(K_X + B\) with scaling of \(H\) (cf. [10, §3]). In general we do not know whether the MMP terminates but we know that in some step of the MMP we reach a model \(Y\) on which \(K_Y + B_Y\), the pushdown of \(K_X + B\), is a limit of movable/\(\mathbb{Z}\)(\(\mathbb{R}\)-divisors: indeed, if the MMP terminates, then the claim is obvious; otherwise the MMP produces an infinite sequence \(X_i \to X_{i+1}\) of flips and a decreasing sequence \(\lambda_i\) of scaling numbers in \((0,1]\) such that \(K_{X_i} + B_i + \lambda_i H_i\) is \(\text{nef}/Z\); by [11][8, Theorem 1.9], \(\lim \lambda_i = 0\); in particular, if \(Y := X_1\), then \(K_Y + B_Y\) is the limit of the movable/\(\mathbb{Z}\)(\(\mathbb{R}\)-divisors \(K_Y + B_Y + \lambda_i H_Y\).

Now assume \((X,B)\) is klt with either \(K_X + B\) or \(B\) \(\text{big}/Z\) (but \(X\) not necessarily \(\mathbb{Q}\)-factorial). Then again we can run an MMP//\(\mathbb{Q}\)(Z) on \(K_X + B\) with scaling of \(H\) and this time the MMP terminates. We explain how this works.

Let \(X_1 = X, B_1 = B, H_1 = H\) and let \(\lambda_1\) be the smallest number such that \(K_{X_1} + B_1 + \lambda_1 H_1\) is \(\text{nef}/Z\). We can assume \(\lambda_1 > 0\) otherwise the MMP ends on \(X\). Moreover, we can choose \(H_1 \geq 0\) in its \(\mathbb{R}\)-linear equivalence class so that \((X_1, B_1 + \lambda_1 H_1)\) is klt. Then \(K_{X_1} + B_1 + \lambda_1 H_1\) is semi-ampleness over \(Z\) by the base point free theorem, hence it defines a contraction \(X_1 \to Z_1/Z\). Now \(X_1\) has only finitely many extremal rays over \(Z_1\) by the cone theorem [25, Theorem 3.25] as \(H_1\) is \(\text{big}/Z\) and \(K_{X_1} + B_1 + \lambda_1 H_1 \equiv Z_1\) (here \(X_1\) is of Fano type over \(Z_1\); see below for Fano type varieties). Since \(\lambda_1 > 0\), there is an extremal ray \(R_1\) over \(Z_1\) such that \((K_{X_1} + B_1) \cdot R_1 < 0\). By the cone theorem, \(R_1\) can be contracted via an extremal contraction \(X_1 \to T_1/Z_1\). If this contraction is not birational, then we get a Mori fibre space and the MMP ends here. Otherwise, \((X_1,B_1)\) has an lc model over \(T_1\), say \((X_2,B_2)\) [11, Theorem 1.2]. Let \(H_2\) be the pushdown of \(H_1\) and let \(\lambda_2\) be the smallest number such that \(K_{X_2} + B_2 + \lambda_2 H_2\) is \(\text{nef}/Z\). Now we repeat the same process on \(X_2\) and so on to get the desired MMP. Note that \(H_i\) may not be ample for \(i > 1\) but this is not a problem as we do not need ampleness beyond \(X_1\).

We argue that the MMP terminates. We can assume that after finitely many steps the extremal contractions \(X_i \to T_i\) are all flipping contractions. Pick \(i \geq 0\) and take
a small \(\mathbb{Q}\)-factorialisation of \(X_i\), say \(X_i' \to X_i\). Let \(B_i', H_i'\) be the birational transforms of \(B_i, H_i\). Running an MMP on \(K_{X_i'} + B_i'\) over \(T_i\) with scaling of \(\lambda_i H_i'\) lifts \(X_i \to X_{i+1}\) to a sequence of flips \(X_i' \to X_{i+1}'\) in the \(\mathbb{Q}\)-factorial case where \(X_{i+1}' \to X_{i+1}\) is a small \(\mathbb{Q}\)-factorialisation. Putting all these together gives a sequence of flips for an MMP with scaling (of a big divisor) in the \(\mathbb{Q}\)-factorial case which terminates by [11, Corollary 1.4.2].

Similar remarks apply to the MMP for generalised pairs defined below.

2.6. Fano pairs. Let \((X, B)\) be a pair and \(X \to Z\) a contraction. We say \((X, B)\) is log Fano over \(Z\) if it is lc and \(-(K_X + B)\) is ample over \(Z\); if \(B = 0\) we just say \(X\) is Fano over \(Z\). We say \(X\) is of Fano type over \(Z\) if \((X, B)\) is klt log Fano over \(Z\) for some choice of \(B\); it is easy to see this is equivalent to existence of a big/\(\mathbb{Q}\) \(\mathbb{Q}\)-boundary (resp. \(\mathbb{R}\)-boundary) \(\Gamma\) so that \((X, \Gamma)\) is klt and \(K_X + \Gamma \sim_{\mathbb{Q},Z} 0\) (resp. \(\sim_{\mathbb{R},Z}\) instead of \(\sim_{\mathbb{Q},Z}\)).

Assume \(X\) is of Fano type over \(Z\). Then we can run an MMP over \(Z\) on any \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(D\) on \(X\) which ends with some model \(Y\): the MMP is just an MMP on \(K_X + \Gamma + tD\) with scaling of some ample divisor for some small \(t > 0\), as in 2.5. If \(D_Y\) is nef over \(Z\), we call \(Y\) a minimal model over \(Z\) for \(D\). If \(D_Y\) is not nef/\(Z\), then there is a \(D_Y\)\(\)-negative extremal contraction \(Y \to T/Z\) with \(\dim Y > \dim T\) and we call \(Y\) a Mori fibre space over \(Z\) for \(D\).

If \(X\) is of Fano type over \(Z\) and \(D\) is a nef/\(\mathbb{R}\) \(\mathbb{R}\)-divisor on \(X\), then \(D\) is semi-ample over \(Z\): there is a boundary \(\Gamma\) so that \((X, \Gamma)\) is klt and \(K_X + \Gamma \sim_{\mathbb{R},Z} 0\); so choosing a small \(t > 0\), \((X, \Gamma + tD)\) is klt and \(tD \sim_{\mathbb{R},Z} K_X + \Gamma + tD\), hence \(D\) is semi-ample over \(Z\) because \(K_X + \Gamma + tD\) is semi-ample over \(Z\) by the base point free theorem.

2.7. b-divisors. We recall some definitions regarding b-divisors. Let \(X\) be a normal variety. A b-divisor \(M\) over \(X\) is a collection of \(\mathbb{R}\)-divisors \(M_Y\) on \(Y\) for each birational contraction \(Y \to X\) from a normal variety that are compatible with respect to pushdown, that is, if \(Y' \to X\) is another birational contraction and \(\psi: Y' \to Y\) is a morphism, then \(\psi_* M_{Y'} = M_Y\).

A b-divisor \(M\) is \(b\)-\(\mathbb{R}\)-Cartier if there is a birational contraction \(Y \to X\) such that \(M_Y\) is \(\mathbb{R}\)-Cartier and such that for any birational contraction \(\psi: Y' \to Y/X\) we have \(M_{Y'} = \psi^* M_Y\). In other words, a \(b\)-\(\mathbb{R}\)-Cartier b-divisor over \(X\) is determined by the choice of a birational contraction \(Y \to X\) and an \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(M\) on \(Y\). But this choice is not unique, that is, another birational contraction \(Y' \to X\) and an \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(M'\) on \(Y'\) defines the same \(b\)-\(\mathbb{R}\)-Cartier b-divisor if there is a common resolution \(W \to Y\) and \(W \to Y'\) on which the pullbacks of \(M\) and \(M'\) coincide.

A \(b\)-\(\mathbb{R}\)-Cartier b-divisor represented by some \(Y \to X\) and \(M\) is \(b\)-Cartier if \(M\) is \(b\)-Cartier, i.e. its pullback to some resolution is Cartier.

2.8. Generalised pairs. For the basic theory of generalised polarised pairs see [12, Section 4]. Below we recall some of the main notions and discuss some basic properties.

1. A generalised pair consists of
   - a normal variety \(X\) equipped with a projective morphism \(X \to Z\),
   - an \(\mathbb{R}\)-divisor \(B \geq 0\) on \(X\), and
   - a \(b\)-\(\mathbb{R}\)-Cartier b-divisor over \(X\) represented by some projective birational morphism \(X' \to X\) and \(\mathbb{R}\)-Cartier divisor \(M'\) on \(X'\).
such that $M'$ is nef$/Z$ and $K_X + B + M$ is $R$-Cartier, where $M := \phi_* M'$.

We usually refer to the pair by saying $(X, B + M)$ is a generalised pair with data $X' \xrightarrow{\phi} X \to Z$ and $M'$. Since a $b$-$R$-Cartier $b$-divisor is defined birationally, in practice we will often replace $X'$ with a resolution and replace $M'$ with its pullback. When $Z$ is not relevant we usually drop it and do not mention it: in this case one can just assume $X \to Z$ is the identity. When $Z$ is a point we also drop it but say the pair is projective.

Now we define generalised singularities. Replacing $X'$ we can assume $\phi$ is a log resolution of $(X, B)$. We can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

for some uniquely determined $B'$. For a prime divisor $D$ on $X'$ the \emph{generalised log discrepancy} $a(D, X, B + M)$ is defined to be $1 - \mu_D B'$.

We say $(X, B + M)$ is generalised lc (resp. generalised klt)(resp. generalised $\epsilon$-lc) if for each $D$ the generalised log discrepancy $a(D, X, B + M)$ is $\geq 0$ (resp. $> 0$)(resp. $\geq \epsilon$). A \emph{generalised non-klt place} of $(X, B + M)$ is a prime divisor $D$ on birational models of $X$ with $a(D, X, B + M) \leq 0$, and a \emph{generalised non-klt centre} of $(X, B + M)$ is the image of a generalised non-klt place. The \emph{generalised non-klt locus} $Nklt(X, B + M)$ of the generalised pair is the union of all the generalised non-klt centres.

We will also use similar definitions when $B$ is not necessarily effective in which case we have a generalised sub-pair.

(2) Let $(X, B + M)$ be a generalised pair as in (1). We say $(X, B + M)$ is \emph{generalised dlt} if it is generalised lc and if $\eta$ is the generic point of any generalised non-klt centre of $(X, B + M)$, then $(X, B)$ is log smooth near $\eta$ and $M' = \phi^* M$ holds over a neighbourhood of $\eta$. Note that when $M' = 0$, then $(X, B)$ is generalised dlt if it is dlt in the usual sense.

The generalised dlt property is preserved under the MMP. Indeed, assume $(X, B + M)$ is generalised dlt and that $X \dashrightarrow X''/Z$ is a divisorial contraction or a flip with respect to $K_X + B + M$. Replacing $\phi$ we can assume $X' \dashrightarrow X''$ is a morphism. Let $B'', M''$ be the pushdowns of $B, M$ and consider $(X'', B'' + M'')$ as a generalised pair with data $X' \to X'' \to Z$ and $M'$. Then $(X'', B'' + M'')$ is also generalised dlt because it is generalised lc and because $X \dashrightarrow X''$ is an isomorphism over the generic point of any generalised non-klt center of $(X'', B'' + M'')$.

(3) Let $(X, B + M)$ be a generalised pair as in (1) and let $\psi: X'' \to X$ be a projective birational morphism from a normal variety. Replacing $\phi$ we can assume $\phi$ factors through $\psi$. We then let $B''$ and $M''$ be the pushdowns of $B'$ and $M'$ on $X''$ respectively. In particular,

$$K_{X''} + B'' + M'' = \psi^*(K_X + B + M).$$

If $B'' \geq 0$, then $(X'', B'' + M'')$ is also a generalised pair with data $X' \to X'' \to Z$ and $M'$.

Assume that we can write $B'' = \Delta'' + G''$ where $(X'', \Delta'' + M'')$ is $\mathbb{Q}$-factorial generalised dlt, $G'' \geq 0$ is supported in $\lfloor \Delta'' \rfloor$, and every exceptional prime divisor of $\psi$ is a component of $\lfloor \Delta'' \rfloor$. Then we say $(X'', \Delta'' + M'')$ is a \emph{Q-factorial generalised dlt model} of $(X, B + M)$. Such models exist by the next lemma (also see [20, Proposition 3.3.1] and [12, Lemma 4.5]). If $(X, B + M)$ is generalised lc, then $G'' = 0$.

**Lemma 2.9.** Let $(X, B + M)$ be a generalised pair with data $\phi: X' \to X$ and $M'$. Then the pair has a $\mathbb{Q}$-factorial generalised dlt model.
Proof. Replacing $\phi$ we can assume it is a log resolution. Write
$$K_{X'} + B' + M' = \phi^*(K_X + B + M).$$
Write $B = \Delta + G$ where $\Delta$ is obtained from $B$ by replacing each coefficient $> 1$ with $1$. So $G \geq 0$ is supported in $[\Delta]$. Let $\Delta'$ be the sum of the birational transform of $\Delta$ and the reduced exceptional divisor of $\phi$. Let $G' := B' - \Delta'$.

Run an MMP on $K_{X'} + \Delta' + M'$ over $\mathcal{X}$ with scaling of an ample divisor. We reach a model $\mathcal{X}'$ on which $K_{\mathcal{X}''} + \Delta'' + M''$ is a limit of movable $\mathcal{X}$-divisors. By construction,
$$K_{\mathcal{X}''} + \Delta'' + M'' + G'' \equiv_{\mathcal{X}} 0.$$ 
So for any exceptional $\mathcal{X}$ prime divisor $S''$ on $\mathcal{X}''$, $-G''|_{S''}$ is pseudo-effective over the image of $S''$ in $\mathcal{X}$. Therefore, by the general negativity lemma [8, Lemma 3.3], $G'' \geq 0$ (note that [8, Lemma 3.3] implicitly assumes the base field is uncountable; if in our case the base field is countable, then we do a base change and then apply the lemma as in that case for the very general curves $C''$ of $S''$ contracted over $\mathcal{X}$, we have $-G'' \cdot C \geq 0$).

By definition of $\Delta'$, each exceptional prime divisor of $\mathcal{X}'' \to \mathcal{X}$ is a component of $[\Delta'']$. Moreover, each component of $G''$ is either exceptional in which case it is a component of $[\Delta'']$, or non-exceptional in which case it is the birational transform of a component of $G$ hence again a component of $[\Delta'']$. Therefore, $(\mathcal{X}'', \Delta'' + M'')$ is a $\mathbb{Q}$-factorial generalised dlt model of $(\mathcal{X}, B + M)$.

(4) Next we prove two lemmas about generalised pairs that will be used later in Section 4.

Lemma 2.10. Let $d, p$ be natural numbers and $\Phi$ be a DCC set of non-negative real numbers. Then there is a positive real number $t > 0$ depending only on $d, p, \Phi$ satisfying the following. Assume that
- $(\mathcal{X}, B + M)$ is a $\mathbb{Q}$-factorial generalised pair of dimension $d$ with data $X' \to X \to Z$ and $M'$,
- $D \geq 0$ is an $\mathbb{R}$-divisor,
- the coefficients of $B, D$ are in $\Phi$ and $pM'$ is Cartier.

Then we have:
1. if $(\mathcal{X}, B + (1-t)D + M)$ is generalised lc, then $(\mathcal{X}, B + D + M)$ is generalised lc, and
2. if $(\mathcal{X}, B + (1-t)D + M)$ is generalised lc, $X \to Z$ is a Mori fibre space for $(\mathcal{X}, B + (1-t)D + M)$, and $K_X + B + D + M$ is nef over $Z$, then $K_X + B + D + M \equiv_Z 0$.

Proof. (1) If this is not true, then there exist a strictly decreasing sequence $t_i$ of real numbers approaching 0 and a sequence $(X_i, B_i + M_i), D_i$ of pairs and divisors as in the lemma such that
$$(X_i, B_i + (1-t_i)D_i + M_i)$$
is generalised lc but $(X_i, B_i + D_i + M_i)$ is not generalised lc. Let $u_i$ be the generalised lc threshold of $D_i$ with respect to $(X_i, B_i + M_i)$. Then $u_i$ belongs to an ACC set depending only on $d, p, \Phi$, by [12, Theorem 1.5]. On the other hand, $1 > u_i \geq 1 - t_i$, so the $u_i$ form a sequence approaching 1. This contradicts the ACC property.
On connectedness of non-klt loci of singularities of pairs

(2) Now we prove the second claim. Again if it is not true, then there exist a strictly decreasing sequence \( t_i \) of real numbers approaching 0 and a sequence \((X_i, B_i + M_i), D_i\) of pairs and divisors as in the lemma such that

\[
(X_i, B_i + (1 - t_i)D_i + M_i)
\]

is generalised lc, \( X_i \to Z_i \) a Mori fibre space structure, and with \( K_{X_i} + B_i + D_i + M_i \) nef over \( Z_i \) but such that

\[
K_{X_i} + B_i + D_i + M_i \not\equiv_{Z_i} 0.
\]

Let \( v_i \in (1 - t_i, 1) \) be the number such that

\[
K_{X_i} + B_i + v_iD_i + M_i \equiv_{Z_i} 0.
\]

By (1), \((X_i, B_i + D_i + M_i)\) is generalised lc. Let \( F_i \) be a general fibre of \( X_i \to Z_i \). Restricting to \( F_i \), we get

\[
K_{F_i} + B_{F_i} + v_iD_{F_i} + M_{F_i} \equiv 0
\]

where \((F_i, B_{F_i} + v_iD_{F_i} + M_{F_i})\) naturally inherits the structure of a generalised pair, induced by \((X_i, B_i + v_iD_i + M_i)\), with nef part \( M_{F_i} = M_i\big|_{F_i} \) where \( F_i \) is the fibre of \( X_i \to Z_i \) corresponding to \( F_i \). Now the coefficients of \( B_{F_i}, D_{F_i} \) belong to \( \Phi \) and \( pM_{F_i} \) is Cartier. Then we get a contradiction, by the global ACC [12, Theorem 1.6] as the coefficients of \( B_{F_i} + v_iD_{F_i} \) are in a DCC but not finite set.

\[\square\]

Lemma 2.11. Assume that

- \((X, B + M)\) is a generalised lc pair with data \( X' \to X \) and \( M' \),
- \((X, C + N)\) is generalised klt with data \( X' \to X \) and \( N' \), and
- \( S \) is a prime divisor over \( X \) with
  
  \[ a(S, X, B + M) < 1. \]

Then there is a birational contraction \( Y \to X \) from a normal variety such that \( S \) is a divisor on \( Y \) and \(-S\) is ample over \( X \).

Proof. Take a small rational number \( u > 0 \) and consider the generalised pair

\[
(X, uC + (1 - u)B + uN + (1 - u)M)
\]

with nef part \( uN' + (1 - u)M' \). The pair is generalised klt and

\[
a(S, X, uC + (1 - u)B + uN + (1 - u)M) < 1.
\]

Thus replacing \((X, B + M)\) with the pair above, we can assume that \((X, B + M)\) is generalised klt.

There exist an ample divisor \( A' \) and an effective divisor \( G' \) on \( X' \) such that \( A' + G' \sim_{\mathbb{Q}, X} 0 \). Take a small number \( t > 0 \) and general element \( 0 \leq L' \sim_{\mathbb{R}} M' + tA' \). Letting \( L \) be the pushdown of \( L' \) we see that \((X, B + tG + L)\) is klt and

\[
a(S, X, B + tG + L) < 1.
\]

Thus replacing \((X, B + M)\) with \((X, B + tG + L)\) we can assume \( M' = 0 \) and that \((X, B)\) is klt.

Now by [11, Corollary 1.4.3], there is a crepant \( \mathbb{Q} \)-factorial terminal model \((V, B_V)\) of \((X, B)\). Then \( S \) is a divisor on \( V \) and the coefficient of \( S \) in \( B_V \) is positive. Since

\[
K_V + B_V - cS \sim_{\mathbb{R}, X} -cS,
\]
where $c > 0$ is sufficiently small, we see that $-S$ has an ample model $Y$ over $X$. By the negativity lemma, $S$ is not contracted over $Y$. Abusing notation we denote the birational transform of $S$ on $Y$ again by $S$. Then on $Y$, $-S$ is ample over $X$, so we get the desired model.

\[ \square \]

2.12. **Generalised adjunction for fibrations.** Consider the following set-up. Assume that

- $(X, B + M)$ is a generalised sub-pair with data $X' \to X \to Z$ and $M'$,
- $f : X \to Z$ is a contraction with $\dim Z > 0$,
- $(X, B + M)$ is generalised sub-lc over the generic point of $Z$, and
- $K_X + B + M \sim_{\mathbb{R}, Z} 0$.

We define the discriminant divisor $B_Z$ for the above setting. Let $D$ be a prime divisor on $Z$. Let $t$ be the largest real number such that $(X, B + tf^*D + M)$ is generalised sub-lc over the generic point of $D$. This makes sense even if $D$ is not $\mathbb{Q}$-Cartier because we only need the pullback $f^*D$ over the generic point of $D$ where $Z$ is smooth. We then put the coefficient of $D$ in $B_Z$ to be $1 - t$. Note that since $(X, B + M)$ is generalised sub-lc over the generic point of $Z$, $t$ is a real number, that is, it is not $-\infty$ or $+\infty$.

Having defined $B_Z$, we can find $M_Z$ giving

$$K_X + B + M \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

where $M_Z$ is determined up to $\mathbb{R}$-linear equivalence. We call $B_Z$ the **discriminant divisor of adjunction** for $(X, B + M)$ over $Z$. If $B, M'$ are $\mathbb{Q}$-divisors and $K_X + B + M \sim_{\mathbb{Q}, Z} 0$, then $B_Z$ is a $\mathbb{Q}$-divisor and we can choose $M_Z$ also to be a $\mathbb{Q}$-divisor. For any birational morphism $Z' \to Z$ from a normal variety, we can similarly define $B_{Z'}$ and $M_{Z'}$. This gives the discriminant and moduli b-divisors.

For more details about generalised adjunction for fibrations, we refer to [13] and §6.1 of [4].

**Theorem 2.13.** Under the above notation, assume that $X$ is projective, $(X, B + M)$ is generalised lc over the generic point of $Z$, and $B, M'$ are $\mathbb{Q}$-divisors, and $M'$ is globally nef. Then the moduli divisor of adjunction is nef $b$-$\mathbb{Q}$-Cartier, that is, there is a resolution $Z' \to Z$ such that $M_{Z'}$ is nef $\mathbb{Q}$-Cartier and for any birational contraction $Z'' \to Z'$, $M_{Z''}$ is the pullback of $M_{Z'}$.

In particular, we can regard $(Z, B_Z + M_Z)$ as a generalised sub-pair with nef part $M_{Z'}$. The theorem is proved in [3, §3] (based on [21]) when $M' = 0$, and in [13, Theorem 1.4] in general. We will use the theorem in the proof of Theorem 1.4.

2.14. **Connected components and étale neighbourhoods.** Let $Z$ be a variety and $g : N \to Z$ be a projective morphism where $N$ is a scheme. In the discussion below keep in mind that the topology of the fibre of $g$ over a point $z \in Z$ is the same as the subset topology on $g^{-1}\{z\}$ induced by the topology on $N$ where $g^{-1}\{z\}$ means the set-theoretic inverse image (a similar fact holds in general for any morphism of schemes).

(1) We show that the fibres of $g$ are connected iff its fibres over closed points are connected. We actually prove a stronger statement. Let $z \in Z$ and let $R$ be the closure of $z$ in $Z$. We claim that for any closed point $y$ in some non-empty open subset of $R$, the number of connected components of $g^{-1}\{y\}$ is at least the number of connected components of $g^{-1}\{z\}$.
To prove the claim, we can replace $Z$ with $R$ and replace $N$ with the scheme-theoretic inverse image of $R$, hence assume $z$ is the generic point of $Z$. Let $C_1,\ldots,C_r$ be the connected components of $g^{-1}\{z\}$, and let $\overline{C_i}$ be the closure of $C_i$ in $N$. Then the fibre of $\overline{C_i} \to Z$ over $z$ is just $C_i$. Therefore, $\overline{C_i} \cap \overline{C_j}$ does not surject onto $Z$ for any $i \neq j$ otherwise $C_i$ and $C_j$ would intersect, so shrinking $Z$ we can assume that $\overline{C_i} \cap \overline{C_j} = \emptyset$ for $i \neq j$. But then since $\overline{C_i}$ surjects onto $Z$ for each $i$, the number of connected components of $g^{-1}\{y\}$ is at least $r$, for any closed point $y \in Z$. This proves the claim.

(2) For each closed point $z \in Z$, there is an étale neighbourhood $\tilde{Z} \to Z$ with a closed point $\tilde{z}$ mapping to $z$ such that there is a 1-1 correspondence between the connected components of $\tilde{N} := N \times_Z \tilde{Z}$, and the connected components of the fibre of $g$ over $z$ [22, claim 4.38.1]. If the ground field is $\mathbb{C}$, then this essentially says that the connected components of $N$ over a small analytic neighbourhood of $z$ correspond to the connected components of the fibre $N \to Z$ over $z$.

### 3. Non-klt loci of anti-nef pairs

In this section we prove Theorems 1.2 and 1.3 in the more general framework of generalised pairs. Using generalised pairs is important for the proofs even if one is only interested in usual pairs.

**Theorem 3.1.** Let $(X, B + M)$ be a generalised pair with data $X' \to X \to Z$ and $M'$ where $f: X \to Z$ is a contraction. Assume $-(K_X + B + M)$ is nef over $Z$. Then the fibres of

$$Nklt(X, B + M) \to Z$$

are connected if any of the following conditions holds:

1. $-(K_X + B + M)$ is big over $Z$, or
2. $Nklt(X, B + M) \to Z$ is not surjective, or
3. $\tau_S(-(K_X+B+M)/Z) > 0$ for every generalised non-klt place $S$ of $(X, B+M)$.

We prove cases (1) and (2) first and then prove case (3) towards the end of this section. Note that in view of Theorem 3.5 below, we can add another case in which the theorem holds, that is, the fibre of $Nklt(X, B + M) \to Z$ over a point $z \in Z$ is connected if $(X, B + M)$ is not generalised lc over $z$.

**Remark 3.2.** In view of 2.14, to prove the theorem, it is enough to prove the weaker statement that $Nklt(X, B + M)$ is connected near each fibre of $X \to Z$ as all the conditions (1)-(3) are preserved after étale base change. We explain this point in detail. By 2.14(1), it is enough to show that the fibres of

$$N := Nklt(X, B + M) \to Z$$

over closed points are connected. And by 2.14(2), for each closed point $z \in Z$, there is an étale neighbourhood $\tilde{Z} \to Z$ with a closed point $\tilde{z}$ mapping to $z$ such that there is a 1-1 correspondence between the connected components of $\tilde{N} := N \times_Z \tilde{Z}$, and the connected components of the fibre of $N \to Z$ over $z$. Fibre product with $\tilde{Z}$ induces a generalised pair $(\tilde{X}, \tilde{B} + \tilde{M})$ with data $\tilde{X}' \to \tilde{X} \to \tilde{Z}$ and $\tilde{M}'$ where $\tilde{N} = Nklt(\tilde{X}, \tilde{B} + \tilde{M})$. It is enough to show that $Nklt(\tilde{X}, \tilde{B} + \tilde{M})$ is connected near the fibre of $\tilde{X} \to \tilde{Z}$ over $\tilde{z}$, hence replacing $(X, B + M)$, $X' \to X \to Z$ and $M'$ with $(\tilde{X}, \tilde{B} + \tilde{M})$, $\tilde{X}' \to \tilde{X} \to \tilde{Z}$ and $\tilde{M}'$, respectively, it is enough to show that $Nklt(X, B + M)$ is connected near each fibre of $X \to Z$. 


Lemma 3.3. **Theorem 3.1 (1) holds.**

*Proof.* As pointed out above it is enough to show that $\text{Nklt}(X, B + M)$ is connected near each fibre of $X \to Z$. We can then use [6, Lemma 2.14] which reduces the statement to the connectedness principle for usual pairs.

\[\square\]

Lemma 3.4. **Theorem 3.1 (2) holds.**

*Proof.* Step 1. In this step we reduce the statement to the case $K_X + B + M \equiv_Z 0$. By Remark 3.2, it is enough to show that $\text{Nklt}(X, B + M)$ is connected near each fibre of $X \to Z$ over closed points. Assume that $\text{Nklt}(X, B + M)$ is not connected near the fibre of $X \to Z$ over some closed point $z$. Shrinking $Z$ around $z$, we can assume that $\text{Nklt}(X, B + M)$ is not connected globally. Extending the ground field we can assume it is not countable.

Let

$$L := -(K_X + B + M)$$

and let $L'$ be the pullback of $L$ on $X'$. Since $L$ is nef over $Z$, $L'$ is nef over $Z$, hence $M' + L'$ is nef over $Z$. Consider the generalised pair $(X, B + M + L)$ with data $X' \to X \to Z$ and $M' + L'$. The generalised non-klt locus of $(X, B + M + L)$ coincides with that of $(X, B + M)$ because $L$ being nef over $Z$ means that for any prime divisor $S$ over $X$ we have

$$a(S, X, B + M + L) = a(S, X, B + M).$$

Thus replacing $M'$ with $M' + L'$ we can assume that $K_X + B + M \equiv_Z 0$.

Step 2. In this step we modify $(X, B + M)$ so that $B = \Delta + G$ where $B, G \geq 0$, $G$ is supported in $|\Delta|$, and $(X, \Delta - t|\Delta| + M)$ is $\mathbb{Q}$-factorial generalised klt for some $t \in (0, 1)$. Let $(X'', \Delta'' + M'')$ be a $\mathbb{Q}$-factorial generalised dlt model of $(X, B + M)$ which exists by Lemma 2.9. By definition, $(X'', \Delta'' + M'')$ is $\mathbb{Q}$-factorial generalised dlt. Denoting $X'' \to X$ by $\psi$, we have

$$K_{X''} + \Delta'' + G'' + M'' = \psi^*(K_X + B + M)$$

where $G'' \geq 0$ is supported in $|\Delta''|$. We can assume $X'' \dasharrow X''$ is a morphism, so we can consider $(X'', \Delta'' + G'' + M'')$ as a generalised pair with nef part $M'$. Since

$$\text{Nklt}(X, B + M) = \psi(\text{Nklt}(X'', \Delta'' + G'' + M'')),$$

we deduce that

$$\text{Nklt}(X'', \Delta'' + G'' + M'')$$

is not connected over $z$. Thus we can replace $(X, B + M)$ with $(X'', \Delta'' + G'' + M'')$, hence we can assume that the following condition holds:

$$(*) \quad B = \Delta + G \text{ where } G \geq 0 \text{ is supported in } |\Delta| \text{ and } (X, \Delta - t|\Delta| + M) \text{ is } \mathbb{Q}\text{-factorial generalised klt for some } t \in (0, 1).$$

The condition implies that any generalised non-klt centre of $(X, B + M)$ is contained in the support of $t|\Delta| + G$, so we have

$$|\Delta| \subseteq \text{Nklt}(X, B + M) \subseteq \text{Supp}(t|\Delta| + G) = |\Delta|,$$

giving

$$\text{Nklt}(X, B + M) = \text{Supp}(t|\Delta| + G) = |\Delta|.$$
By assumption, \( \text{Nklt}(X, B + M) \) is vertical over \( Z \), so \( [\Delta] \) is vertical over \( Z \). Thus \( t \cdot [\Delta] + G \) is also vertical over \( Z \), so

\[
K_X + \Delta - t \cdot [\Delta] + M \equiv 0
\]

over some non-empty open subset of \( Z \), hence \( K_X + \Delta - t \cdot [\Delta] + M \) is pseudo-effective over \( Z \).

**Step 3.** In this step we run an MMP on \( K_X + \Delta - t \cdot [\Delta] + M \) and see that the condition \( \text{Nklt}(X, B + M) = [\Delta] \) is preserved by the MMP. Indeed, we can run an MMP on \( K_X + \Delta - t \cdot [\Delta] + M \) over \( Z \) with scaling of some ample divisor \( H \) [12, Lemma 4.4] but we do not claim that the MMP terminates. However, if \( \lambda_i \) are the scaling numbers that appear in the MMP, then \( \lim \lambda_i = 0 \), by [12, Lemma 4.4]. Since

\[
K_X + \Delta + G + M \equiv_Z 0,
\]

the divisor \( t \cdot [\Delta] + G \) is numerically positive on the extremal ray of each step of the MMP. Clearly the condition \((*)\) is preserved by the MMP, so the property

\[
\text{Nklt}(X, B + M) = \text{Supp}(t \cdot [\Delta] + G) = [\Delta]
\]

is also preserved. Also note that since \( K_X + \Delta - t \cdot [\Delta] + M \) is pseudo-effective over \( Z \), every step of the MMP is a divisorial contraction or a flip.

**Step 4.** In this step we show that \( \text{Nklt}(X, B + M) \) remains disconnected near the fibre over \( z \) during the MMP. More precisely, we show that there is a 1-1 correspondence (given by divisorial pushdown) between the connected components of \( \text{Nklt}(X, B + M) \) and of \( \text{Nklt}(X'', B'' + M'') \) for any model \( X'' \) appearing in the MMP. By Step 3, \( \text{Nklt}(X, B + M) \) coincides with \( \text{Supp}(t \cdot [\Delta] + G) \), and \( \text{Nklt}(X'', B'' + M'') \) coincides with \( \text{Supp}(t \cdot [\Delta]' + G'') \), so it is enough to prove the assertion for connected components of \( \text{Supp}(t \cdot [\Delta] + G) \) and of \( \text{Supp}(t \cdot [\Delta]' + G'') \).

Say \( X \to X'' \) is the first step of the MMP which is either a divisorial contraction or a flip. First assume \( X \to X'' \) is a flip and let \( X \to V \) be the corresponding flipping contraction. Each connected component of \( \text{Supp}(t \cdot [\Delta]' + G'') \) away from the flipped locus is just the birational transform of a connected component of \( \text{Supp}(t \cdot [\Delta] + G) \) away from the flipping locus. On the other hand, since \( t \cdot [\Delta] + G \) is ample over \( V \), there is a connected component \( C \) of \( \text{Supp}(t \cdot [\Delta] + G) \) intersecting every positive-dimensional fibre of \( X \to V \). By Step 2 and Lemma 3.3, \( \text{Supp}(t \cdot [\Delta] + G) \) is connected near any fibre of \( X \to V \) as \( -(K_X + B + M) \) is nef and big over \( V \). Therefore, no other connected component of \( \text{Supp}(t \cdot [\Delta] + G) \) intersects the exceptional locus of \( X \to V \). It is enough to show that the birational transform \( C'' \) of \( C \) is connected.

If a connected component of \( C'' \) does not intersect the flipped locus, then it is the birational transform of a connected component of \( C \), hence it coincides with the whole \( C'' \) (as \( C \) is connected), which is not possible because \( C'' \) intersects the flipped locus. Thus every connected component of \( C'' \) intersects the flipped locus. On the other hand, the birational transform \( t \cdot [\Delta]' + G'' \) of \( t \cdot [\Delta] + G \) is anti-ample over \( V \), so at least one of its irreducible components, say \( D'' \), contains the flipped locus. Thus \( D'' \) intersects every connected component of \( \text{Supp}(t \cdot [\Delta]' + G'') \) near the flipped locus. Therefore, \( \text{Supp}(t \cdot [\Delta]' + G'') \) has only one connected component near the flipped locus, and this connected component is exactly \( C'' \). The claim is then proved in the flip case.
A similar argument shows that if $X \to X''$ is a divisorial contraction, then exactly one connected component $C$ of $\text{Supp}(t \lfloor \Delta \rfloor + G)$ intersects the exceptional divisor. Moreover, in this case $C$ is not contracted by $X \to X''$, that is, $C$ is not equal to the exceptional divisor, by the negativity lemma, because $t \lfloor \Delta \rfloor + G$ is ample over $X''$. Thus each connected component of $\text{Supp}(t \lfloor \Delta \rfloor'' + G'')$ is just the divisorial pushdown of exactly one connected component of $\text{Supp}(t \lfloor \Delta \rfloor + G)$. This proves the claim in the divisorial contraction case.

Since the claim holds in each step of the MMP, it holds on any model appearing in the MMP.

**Step 5.** In this step we show that after finitely many steps of the MMP, the irreducible components of the fibre of $X \to Z$ over $z$ not contained in the non-klt locus $\text{Nklt}(X, B + M)$ stabilise. Let $F_1, \ldots, F_r$ be the irreducible components of the fibre of $X \to Z$ over $z$. We claim that there is $i$ such that $F_i$ is not contained in $\text{Nklt}(X, B + M)$ and that this is preserved by the MMP, that is, the birational transform of $F_i$ is not contained in the non-klt locus during the MMP.

By assumption, $f: X \to Z$ is a contraction, so its fibre over $z$ is connected which means the set-theoretic inverse image $f^{-1}\{z\}$ is connected. On the other hand, we assumed that $\text{Nklt}(X, B + M)$ is not connected near the fibre of $X \to Z$ over $z$, so

$$f^{-1}\{z\} \not\subseteq \text{Nklt}(X, B + M) = \text{Supp}(t \lfloor \Delta \rfloor + G)$$

as $f^{-1}\{z\}$ intersects every connected component of $\text{Nklt}(X, B + M)$ over $z$. Thus some of the $F_i$ are not contained in $\text{Nklt}(X, B + M)$; rearranging the indices we can assume that $F_1, \ldots, F_s$ are not contained in $\text{Nklt}(X, B + M)$ but $F_{s+1}, \ldots, F_r$ are contained.

Let $X \to X''$ be the first step of the MMP. Recall that $t \lfloor \Delta \rfloor + G$ is positive on the extremal ray of each step of the MMP. Thus if $X \to X''$ is a divisorial contraction, then $\text{Nklt}(X'', B'' + M'')$ contains the image of the exceptional divisor, and if $X \to X''$ is a flip, then similarly $\text{Nklt}(X'', B'' + M'')$ contains the flipped locus, that is, the exceptional locus of $X'' \to V$ where $X \to V$ is the corresponding flipping contraction. This implies that any irreducible component of the fibre of $X'' \to Z$ over $z$ which is not contained in $\text{Nklt}(X'', B'' + M'')$ is the birational transform of one of the $F_1, \ldots, F_s$. Therefore, after finitely many steps, the irreducible components of the fibre over $z$ not contained in the non-klt locus stabilise, that is, replacing $X$ and possibly decreasing $s$ we can assume that on each model appearing in the MMP, the birational transforms of the $F_1, \ldots, F_s$ are exactly the irreducible components of the fibre over $z$ not contained in the non-klt locus.

**Step 6.** In this step we show that

$$(K_X + \Delta - t \lfloor \Delta \rfloor + M)|_{F_j}$$

is pseudo-effective for each $F_j$, $1 \leq j \leq s$, where $F_1, \ldots, F_s$ are as in the previous step. Since $F_j$ is not contained in $\text{Supp}(t \lfloor \Delta \rfloor + G)$ in the course of the MMP, the map $X \to X''$ is an isomorphism near the generic point of $F_j$ for any model $X''$ that appears in a step of the MMP. Since the MMP is an MMP on $K_X + \Delta - t \lfloor \Delta \rfloor + M$ over $Z$ with scaling of a big divisor, we deduce that $F_j$ is not contained in the relative stable base locus

$$\mathcal{B}(K_X + \Delta - t \lfloor \Delta \rfloor + M + \lambda_i H/Z)$$
On connectedness of non-klt loci of singularities of pairs

for any $i$ where $\lambda_i$ are the scaling numbers in the MMP. In particular, this implies that

$$(K_X + \Delta - t |\Delta| + M + \lambda_i H)|_{F_j}$$

is pseudo-effective for every $i$. But then since $\lim \lambda_i = 0$,

$$(K_X + \Delta - t |\Delta| + M)|_{F_j}$$

is pseudo-effective. Therefore,

$$(K_X + \Delta - t |\Delta| + M) \cdot C \geq 0$$

for any curve $C \subset F_j$ outside some countable union of subvarieties of $F_j$. Since the ground field is not countable, this countable union is not equal to $F_j$.

Step 7. In this step we finish the proof. Since the fibre of $X \to Z$ over $z$ is connected, there is $1 \leq j \leq s$ such that $F_j$ intersects $\text{Nklt}(X,B + M)$. By the previous step, we can find a curve $C \subset F_j$ intersecting $\text{Nklt}(X,B + M)$ but not contained in it such that

$$(K_X + \Delta - t |\Delta| + M) \cdot C > 0$$

Then $C$ intersects $\text{Supp}(t |\Delta| + G)$ but is not contained in it. Therefore,

$$(t |\Delta| + G) \cdot C > 0$$

which contradicts

$$(K_X + \Delta - t |\Delta| + M + t |\Delta| + G) \cdot C = 0.$$

□

Next we treat a generalised version of Theorem 1.2.

**Theorem 3.5.** Let $(X,B + M)$ be a generalised pair with data $X' \to X \to Z$ where $f : X \to Z$ is a contraction. Assume $-(K_X + B + M)$ is nef over $Z$ and that the fibre of

$$g : \text{Nklt}(X,B + M) \to Z$$

over some point $z \in Z$ is not connected. Then we have:

1. $g$ is surjective and its fibre over $z$ has exactly two connected components,
2. the pair $(X,B + M)$ is generalised lc over $z$ and after base change to an étale neighbourhood of $z$ and replacing $X'$ with a high resolution, there exist a contraction $X' \to Y'/Z$ such that if

   $$K_{X'} + B' + M' := \phi^*(K_X + B + M)$$

and if $F'$ is a general fibre of $X' \to Y'$, then $M'|_{F'} \equiv 0$ and $(F',B'|_{F'})$ is isomorphic to $(\mathbb{P}^1, p_1 + p_2)$ for distinct points $p_1, p_2$.

   Moreover, $|B'|$ has two disjoint irreducible components $S', T'$, both horizontal over $Y'$, and the images of $S', T'$ on $X$ are exactly the two connected components of $\text{Nklt}(X,B + M)$.

**Proof.** The proof is similar to the proof of [22, Proposition 4.37] but with some crucial differences.

Step 1. In this step we reduce the theorem to the case $K_X + B + M \equiv_Z 0$. By Lemma 3.4, Theorem 3.1(2) holds, so $g$ is surjective as we are assuming that the fibre of $g$ over $z$ is not connected (the corresponding argument in [22, Proposition 4.37]
instead uses torsion freeness of certain higher direct image sheaves when $M' = 0$ and $K_X + B \sim_{\mathbb{Q}, \mathbb{Z}} 0$.

Let

$$L := - (K_X + B + M)$$

and let $L'$ be the pullback of $L$ on $X'$. Consider the generalised pair $(X, B + M + L)$ with nef part $M' + L'$. Writing

$$K_{X'} + B' + M' := \phi^*(K_X + B + M)$$

we get

$$K_{X'} + B' + M' + L' := \phi^*(K_X + B + M + L)$$

meaning $B'$ is unchanged after adding $L$. The generalised log discrepancies of the two pairs $(X, B + M + L)$ and $(X, B + M)$ are equal, in particular, the non-klt locus of $(X, B + M + L)$ coincides with that of $(X, B + M)$. Thus replacing $M'$ with $M' + L'$ we can assume that $K_X + B + M \equiv_{\mathbb{Z}} 0$.

**Step 2.** In this step we argue that to prove the theorem we can replace $Z$ with an étale neighbourhood of $z$. Let $\tilde{Z}$ be an étale neighbourhood of $z$ with a point $\tilde{z}$ mapping to $z$. Fibre product with $\tilde{Z}$ gives a generalised pair $(\tilde{X}, \tilde{B} + \tilde{M})$ with data $\tilde{X}' \to \tilde{X} \to \tilde{Z}$ and $\tilde{M}'$, and $\Nklt(\tilde{X}, \tilde{B} + \tilde{M})$ is the inverse image of $\Nklt(X, B + M)$ under the morphism $\tilde{X} \to X$. If the fibre of

$$\Nklt(\tilde{X}, \tilde{B} + \tilde{M}) \to \tilde{Z}$$

over $\tilde{z}$ has exactly two connected components, then the fibre of $g$ over $z$ also has exactly two connected components because the former fibre maps surjectively onto the latter fibre (note we already know the latter fibre is not connected). Therefore, to prove (1) and (2) we are free to replace $Z$ with an étale neighbourhood of $z$.

**Step 3.** In this step we reduce the theorem to the case when $z$ is a closed point, and replace $Z$ with an étale neighbourhood of $z$. Let $R$ be the closure of $z$ in $Z$. By 2.14 (1), the number of connected components of $g^{-1}\{v\}$ is at least the number of connected components of $g^{-1}\{z\}$ for the closed points $v$ in some non-empty open subset of $R$. Then to show that $g^{-1}\{z\}$ has exactly two connected components, it is enough to show that $g^{-1}\{v\}$ has exactly two connected components for a general closed point $v \in R$.

On the other hand, by 2.14 (2), after base change to an étale neighbourhood of $v$ (which is automatically an étale neighbourhood of $z$), we can assume that $\Nklt(X, B + M)$ is not connected and that distinct connected components of $g^{-1}\{v\}$ are contained in distinct connected components of $\Nklt(X, B + M)$ (note that the number of connected components of $g^{-1}\{v\}$ is unchanged by the base change). We can then replace $z$ with $v$ and assume it is a closed point, and it is enough to prove (2) of the theorem without taking further étale base change.

**Step 4.** In this step we modify $(X, B + M)$ by taking a dlt model. Indeed, after taking a $\mathbb{Q}$-factorial generalised dlt model, as in Step 2 of the proof of Lemma 3.4, we can replace $X$ so that the following holds:

(*) $B = \Delta + G$ where $G \geq 0$ is supported in $\lfloor \Delta \rfloor$ and $(X, \Delta - t \lfloor \Delta \rfloor + M)$ is $\mathbb{Q}$-factorial generalised klt for some real number $t > 0$. 

In particular, we have
\[ \text{Nklt}(X, B + M) = \text{Supp}(t \lfloor \Delta \rfloor + G) = \lfloor \Delta \rfloor. \]
Moreover, since some component of Nklt$(X, B + M)$ is horizontal over $Z$, $t \lfloor \Delta \rfloor + G$ also dominates $Z$. Therefore, we see that
\[ K_X + \Delta - t \lfloor \Delta \rfloor + M \equiv_Z -(t \lfloor \Delta \rfloor + G) \]
is not pseudo-effective over $Z$.

**Step 5.** In this step we get a Mori fibre space for $(X, \Delta - t \lfloor \Delta \rfloor + M)$ and study its non-klt locus. We can run an MMP on $K_X + \Delta - t \lfloor \Delta \rfloor + M$ over $Z$ ending with a Mori fibre space $X'' \to Y'/Z$ [12, Lemma 4.4]. Since $t \lfloor \Delta \rfloor + G$ is positive on the extremal ray in each step of the MMP, arguing as in Step 4 of the proof of Lemma 3.4, we see that the number of the connected components of Nklt$(X, B + M)$ remains the same throughout the MMP. Moreover, $t \lfloor \Delta'' \rfloor + G''$ is ample over $Y'$. So at least one irreducible component $S''$ of $\lfloor \Delta'' \rfloor$ is ample over $Y'$ which implies that $S''$ intersects every irreducible component of each fibre of $X'' \to Y'$. In particular, if $T''$ is any vertical/Y' component of $\lfloor \Delta'' \rfloor$, then $S''$ intersects $T''$.

Since
\[ (X'', \Delta'' - t \lfloor \Delta'' \rfloor + M'') \]
is generalised klt,
\[ \text{Nklt}(X'', B'' + M'') = \text{Supp}(t \lfloor \Delta'' \rfloor + G'') = \lfloor \Delta'' \rfloor. \]
Thus $\lfloor \Delta'' \rfloor$ has at least two connected components, hence every irreducible component of any connected component of $\lfloor \Delta'' \rfloor$ is horizontal over $Y'$.

**Step 6.** In this step we show that the general fibres of $X'' \to Y'$ are isomorphic to $\mathbb{P}^1$ and further study Nklt$(X'', B'' + M'')$. Let $C''_1, C''_2$ be two connected components of Nklt$(X'', B'' + M'')$ where $S''$ of the previous step is an irreducible component of $C''_1$. Pick an irreducible component $T''$ of $C''_2$. Let $F''$ be a general fibre of $X'' \to Y'$. If $\dim T'' \cap F'' > 0$, then $S''$ intersects $T'' \cap F''$ as $S''$ is ample over $Y'$, which is not possible as $S'' \cap T'' = \emptyset$. Therefore, $\dim F'' = 1$, hence $F'' \cong \mathbb{P}^1$. Since
\[ (K_{X''} + \Delta'' + G'' + M'')|_{F''} \equiv 0 \]
and since at least two irreducible components $S'', T''$ of $\lfloor \Delta'' \rfloor$ intersect $F''$ we see that near $F''$ we have
\[ B'' = \Delta'' + G'' = S'' + T'', \]
and that $M''|_{F''} \equiv 0$. Therefore,
\[ (F'', B''|_{F''}) = (F'', (S'' + T'')|_{F''}) \cong (\mathbb{P}^1, p_1 + p_2) \]
where $p_1, p_2$ are two distinct points.

**Step 7.** In this step we show that $(X'', B'' + M'')$ is generalised plt whose only generalised non-klt places are $S'', T''$. By the previous two steps, $C''_1 = S'', C''_2 = T''$ are the only connected components of Nklt$(X'', B'' + M'')$ otherwise we would find a horizontal over $Y'$ component of $\lfloor \Delta'' \rfloor$ other than $S'', T''$, which is not possible. Assume that $(X'', B'' + M'')$ has another generalised non-klt place, say $R''$. Consider $(X'', sB'' + sM'')$ where $s$ is the biggest number so that the pair is generalised lc. Replacing $R''$ we can assume $R''$ is a generalised non-klt place of $(X'', sB'' + sM'')$. Then we can extract $R''$ say via an extremal contraction $X'' \to X''$. Note that $R''$ is
vertical over \( Y' \). Also \( R''' \) maps into one of \( S'', T'' \), say \( S'' \), so \( R''' \) does not intersect the birational transform \( T''' \) of \( T'' \).

Since \( X'' \) is of Fano type over \( Y' \), so is \( X''' \). Then we can run an MMP on \( -R''' \) over \( Y' \) ending with a good minimal model \( X''' \). Since \( R''' \) does not dominate \( Y' \), \( -R''' \) defines a contraction \( X''' \to V'/Y' \) where \( V' \to Y' \) is birational. Moreover, \( R''' \) and \( T''' \) are disjoint: indeed, if \( K_X'' + B'' + M'' \) is the pullback of \( K_X' + B' + M' \), then applying Theorem 3.5 (1) (through Lemma 3.3), we see that at most one connected component of \( \text{Nklt}(X'', B'' + M'') \) intersects the exceptional locus of each step of the MMP and that component is the one containing \( R''' \).

However, \( T''' \) is horizontal over \( V' \) as \( X''' \to V' \) and \( X'' \to Y' \) are the same over the generic point of \( Y' \). But then \( T''' \) intersects every fibre of \( X''' \to V' \), hence it also intersects \( R''' \) as \( R''' \) is the pullback of some effective divisor on \( V' \), a contradiction. Therefore, \( (X'', B'' + M'') \) is generalised plt.

**Step 8.** In this step we finish the proof. Replacing the given morphism \( \phi: X' \to X \) we can assume \( X' \) is a common resolution of \( X, X'' \). Recall

\[
K_{X'} + B' + M' := \phi^*(K_X + B + M)
\]

and let \( F' \) be a general fibre of \( X' \to Y' \). Since \( X'' \to Y' \) has relative dimension one and since the exceptional locus of \( X' \to X'' \) maps onto a closed subset of \( X'' \) of codimension \( \geq 2 \), we see that \( X' \to X'' \) is an isomorphism over the generic point of \( Y' \). Thus if \( F'' \) is the fibre of \( X' \to Y' \) corresponding to \( F' \), then \( (F', B'|_{F'}) \) is isomorphic to \( (F'', B''|_{F''}) \) which is in turn isomorphic to \( (\mathbb{P}^1, p_1 + p_2) \) for distinct points \( p_1, p_2 \). Moreover, if \( S', T' \) on \( X' \) are the birational transforms of \( S'', T'' \), then \([B'] = S' + T' \) as we showed that \( S'', T'' \) are the only generalised non-klt places of \((X'', B'' + M'') \). In addition, \( M'|_{F'} \equiv 0 \) as we already showed that \( M''|_{F''} \equiv 0 \).

It is then clear that the images of \( S', T' \) on \( X \) are the only generalised non-klt centres of \((X, B + M) \). Since \( \text{Nklt}(X, B + M) \) is not connected, the images of \( S', T' \) are disjoint and each gives a connected component of \( \text{Nklt}(X, B + M) \).

\[\square\]

**Proof.** (of Theorem 3.1) Cases (1) and (2) were proved in Lemmas 3.3.3, 3.4, respectively. We treat case (3). Assume that the fibre of

\[
\text{Nklt}(X, B + M) \to Z
\]

over \( z \) is not connected. Then by Theorem 3.5, after base change to an étale neighbourhood of \( z \), we can assume that \( (X, B + M) \) is generalised lc and that replacing \( X' \) with a high resolution, there is a contraction \( X' \to Y'/Z \) such that if

\[
K_{X'} + B' + M' := \phi^*(K_X + B + M)
\]

and if \( F' \) is a general fibre of \( X' \to Y' \), then \( M'|_{F'} \equiv 0 \) and \( (F', B'|_{F'}) \) is isomorphic to \( (\mathbb{P}^1, p_1 + p_2) \) for distinct points \( p_1, p_2 \). Moreover, \([B'] \) has exactly two disjoint components \( S', T' \), both horizontal over \( Y' \).

But then \(-(K_{X'} + B' + M') - tS' \) is not pseudo-effective over \( Y' \) for any real number \( t > 0 \) because

\[-(K_{X'} + B' + M') \cdot F' = -(K_{X'} + B') \cdot F' = 0\]

while \( S' \cdot F' > 0 \). Therefore, \(-(K_{X'} + B' + M') - tS' \) is not pseudo-effective over \( Z \) for any \( t > 0 \). This contradicts the assumption that

\[\tau_{S'}(-(K_X + B + M)/Z) > 0.\]
Proof. (of Theorem 1.2) This is a special case of Theorem 3.5.

Proof. (of Theorem 1.3) This is a special case of Theorem 3.1.

4. Non-klt loci for mirror symmetry

In this section we prove Theorem 1.4 and Corollary 1.6. First we prove a generalised version of Theorem 1.4 which occupies much of the section. Similar to the previous section, using generalised pairs is crucial for the proofs.

Theorem 4.1. Assume that

1. \((X, B + M)\) is a projective \(\mathbb{Q}\)-factorial generalised dlt pair with data \(X' \to X\) and \(M'\) where \(B, M'\) are \(\mathbb{Q}\)-divisors,
2. \(K_X + B + M\) is pseudoeffective,
3. \(x \in X\) is a zero-dimensional generalised non-klt centre of \((X, B + M)\),
4. \(x\) is not contained in the restricted base locus \(B - (K_X + B + M)\),
5. if \(\psi: Y \to X\) is the blowup at \(x\) with exceptional divisor \(E\), then we have \(\tau_E(K_X + B + M) = 0\), that is, \(\psi^*(K_X + B + M) - tE\) is not pseudoeffective for any real number \(t > 0\).

Then \((X, B + M)\) has a good minimal model which is generalised log Calabi–Yau. More precisely, we can run a minimal model program on \(K_X + B + M\) ending with a minimal model \((X'', B'' + M'')\) with \(K_{X''} + B'' + M'' \sim_\mathbb{Q} 0\).

Note that since \((X, B + M)\) is generalised dlt and \(x\) is a zero-dimensional non-klt centre, \(x\) is a smooth point of \(X\).

Before giving the proof of the theorem we prove several lemmas. We start with some basic properties of the pseudo-effective threshold \(\tau\).

4.2. Pseudo-effective thresholds.

Lemma 4.3. Assume that

- \(X\) is a normal projective variety,
- \(L\) is a pseudo-effective \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor on \(X\),
- \(\phi: Y \to X\) is a birational contraction from a normal variety and \(S\) is a prime divisor on \(Y\) such that \(-S\) is ample over \(X\),
- \(T\) is a prime divisor over \(X\) whose centre on \(X\) is contained in the centre of \(S\), and
- \(\tau_S(L) > 0\).

Then \(\tau_T(L) > 0\).

Proof. Take a resolution \(\alpha: V \to X\) on which \(T\) is a divisor and such that the induced map \(\beta: V \to Y\) is a morphism. Assume \(\tau_S(L) > 0\). Then \(\phi^*L - tS\) is pseudo-effective for some real number \(t > 0\). On the other hand, \(-S\) is ample over \(X\), hence \(\phi^{-1}\{x\} \subseteq S\) for every \(x \in \phi(S)\), so \(S = \phi^{-1}(\phi(S))\). Since the centre of \(T\) on \(X\) is contained in the centre of \(S\), we see that the centre of \(T\) on \(Y\) is contained in \(S\). Then the coefficient of \(T\) in \(\beta^*S\), say \(e\), is positive. But then

\[
\alpha^*L - teT = \beta^*(\phi^*L - tS) + \beta^*tS - teT
\]
is pseudo-effective as $\beta^*(\phi^*L - tS)$ is pseudo-effective and $t\beta^*S - teT \geq 0$. Therefore, $\tau_T(L) > 0$.

Lemma 4.4. Assume that

- $(X, B + M)$ is a projective generalised lc pair with data $\phi: X' \to X$ and $M'$,
- $(X, C)$ is klt for some boundary $C$,
- $K_X + B + M$ is pseudo-effective,
- $S$ is a prime divisor over $X$ with $a(S, X, B + M) < 1$,
- $T$ is a prime divisor over $X$ whose centre on $X$ is contained in the centre of $S$, and
- $\tau_S(K_X + B + M) > 0$.

Then $\tau_T(K_X + B + M) > 0$.

Proof. By Lemma 2.11, there is a birational contraction $\psi: Y \to X$ from a normal variety such that $S$ is a divisor on $Y$ and $-S$ is ample over $X$. Thus by Lemma 4.3, $\tau_S(K_X + B + M) > 0$ implies $\tau_T(K_X + B + M) > 0$.

Lemma 4.5. Assume that

- $(X, B + M)$ is a projective generalised lc pair with data $\phi: X' \to X$ and $M'$,
- $(X, C)$ is klt for some boundary $C$,
- $K_X + B + M$ is pseudo-effective,
- $S, T$ are prime divisors over $X$ with equal centre on $X$, and
- the generalised log discrepancies satisfy $a(S, X, B + M) = 0$ and $a(T, X, B + M) < 1$.

Then $\tau_S(K_X + B + M) > 0$ iff $\tau_T(K_X + B + M) > 0$.

Proof. This follows from Lemma 4.4.

Next we look at pseudo-effective thresholds when we modify our pair birationally in certain situations.

Lemma 4.6. Assume that

- $(X, B + M)$ is a projective generalised lc pair with data $\phi: X' \to X$ and $M'$,
- $(X, C)$ is klt for some boundary $C$,
- $K_X + B + M$ is pseudo-effective,
- $S$ is a prime divisor over $X$ with $a(S, X, B + M) = 0$,

and $\tau_S(K_X + B + M) = 0$.

Writing $K_{X'} + B' + M' = \phi^*(K_X + B + M)$,
let $\Delta' = B' + R' \geq 0$ where $R' \geq 0$ is exceptional over $X$, and assume $(X', \Delta' + M')$ is generalised lc. Then

$$\tau_S(K_{X'} + \Delta' + M') = 0.$$ 

**Proof.** By Lemma 2.11, there is a birational contraction $\psi: Y \to X$ from a normal variety such that $S$ is a divisor on $Y$ and $-S$ is ample over $X$. In particular, $S$ contains the exceptional locus of $\psi$, so $\psi$ does not contract any divisor except possibly $S$.

Assume

$$\tau_S(K_{X'} + \Delta' + M') > 0.$$ 

Let $\rho: W' \to X'$ be a resolution so that the induced map $\alpha: W' \to Y$ is a morphism. Let $S'$ on $W'$ be the birational transform of $S$. Then

$$\rho^*(K_{X'} + \Delta' + M') - tS'$$

is pseudo-effective for some real number $t > 0$. By assumption, $S'$ is a generalised non-klt place of $(X, B + M)$, so it is a generalised non-klt place of $(X', B' + M')$, hence $S'$ is not a component of $\rho^*R'$ otherwise $(X', \Delta' + M')$ would not be generalised lc.

By assumption, $R'$ is exceptional over $X$, hence $\rho^*R'$ is exceptional over $X$. Then since $S'$ is not a component of $\rho^*R'$ and since the only possible exceptional divisor of $Y \to X$ is $S$, we deduce that $\rho^*R'$ is exceptional over $Y$. Thus

$$\psi^*(K_X + B + M) - tS = \alpha\ast(\rho^*(\phi^*(K_X + B + M)) + \rho^*R' - tS')$$

$$= \alpha\ast(\rho^*(K_{X'} + B' + M') + \rho^*(\Delta' - B') - tS')$$

$$= \alpha\ast(\rho^*(K_{X'} + \Delta' + M') - tS')$$

is pseudo-effective. Therefore, we get

$$\tau_S(K_{X} + B + M) > 0,$$

a contradiction.

\[ \square \]

### 4.7. Lifting zero-dimensional non-klt centres.

**Lemma 4.8.** Assume that $(X, B + M)$ is a generalised dlt pair with data $\phi: X' \to X$ and $M'$, and $x \in X$ is a zero-dimensional generalised non-klt centre of $(X, B + M)$. Write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

and assume $(X', B')$ is log smooth. Then $(X', B' + M')$ has a zero-dimensional generalised non-klt centre $x'$ mapping to $x$.

**Proof.** Note that here we are considering $(X', B' + M')$ as a generalised sub-pair with data $X' \to X'$ and $M'$. The generalised non-klt centres of $(X', B' + M')$ are just the non-klt centres of $(X', B')$. So we want to show that $(X', B')$ has a zero-dimensional non-klt centre $x'$ mapping to $x$.

Since $(X, B + M)$ is generalised dlt and $x \in X$ is a generalised non-klt centre, by definition, $(X, B)$ is log smooth near $x$ and $M' = \phi^*M$ over a neighbourhood of $x$. Shrinking $X$ we can assume $(X, B)$ is log smooth and that $M' = \phi^*M$. Writing

$$K_{X'} + C' := \phi^*(K_X + B),$$

we see that $C' = B'$ because $M' = \phi^*M$, hence $x$ is a non-klt centre of $(X, B)$. Thus in this situation $M'$ is not relevant, so removing it from now on we can assume $M' = 0$. 

On connectedness of non-klt loci of singularities of pairs 23
We can assume $d := \dim X > 1$ as the lemma is obvious in dimension one. We use induction on dimension. Since $(X,B)$ is log smooth and $x$ is a zero-dimensional non-klt centre, $x$ is an intersection point of $d$ components of $[B]$ passing through $x$. Let $S$ be one such component and let $S'$ be its birational transform on $X'$. By adjunction, define $K_S + B_S = (K_X + B)|_S$ and $K'_{S'} + B_{S'} = (K_{X'} + B')|_{S'}$. Since $(X',B')$ is log smooth and $S'$ is a component of $B'$ with coefficient 1, $(S',B_{S'})$ is log smooth. Denoting the induced morphism $S' \to S$ by $\pi$, we have

$$K'_{S'} + B_{S'} = \pi^*(K_S + B_S).$$

Moreover, $(S,B_S)$ is log smooth and $x$ is a non-klt centre of $(S,B_S)$. Thus by induction, there is a zero-dimensional non-klt centre $x'$ of $(S',B_{S'})$ mapping onto $x$. But then $x'$ is also a non-klt centre of $(X',B')$.

$\square$

4.9. Ample models for certain generalised pairs. In this subsection we show that certain generalised pairs have ample models in the relative birational setting.

**Lemma 4.10.** Assume that

- $(X,B+M)$ is an lc generalised pair with data $X' \xrightarrow{\phi} X \xrightarrow{\delta} Y$ and $M'$,
- $B,M'$ are $\mathbb{Q}$-divisors and $g$ is a birational contraction,
- $(Y,By+M_Y)$ is generalised lc with nef part $M'$ where $K_Y + By + M_Y$ is the pushdown of $K_X + B + M$, and
- $(Y,C)$ is klt for some boundary $C$.

Then $(X,B+M)$ has an ample model over $Y$, i.e.

$$\bigoplus_{m \geq 0} g_*O_X(m(K_X+B+M))$$

is a finitely generated $O_Y$-module.

**Proof.** We can assume $\phi$ is a log resolution of $(X,B)$ and that $X' \to Y$ is a log resolution of $(Y,By)$. Write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

and let $\Delta'$ be obtained from $B'$ by increasing the coefficient of every exceptional $X$ prime divisor to 1. Since $(X',\Delta')$ is log smooth and log canonical, no divisor in $\Delta'$ with coefficient less than 1 can contain any stratum of intersections of divisors of coefficient 1. Thus replacing $(X,B+M)$ with $(X',\Delta'+M')$ we can assume that $(X,B+M)$ is $\mathbb{Q}$-factorial generalised dlt.

Run an MMP on $K_X + B + M$ over $Y$ with scaling of some ample divisor. We reach a model $V$ on which the pushdown $K_V + B_V + M_V$ is numerically a limit of movable $Y$ $\mathbb{R}$-divisors (see 2.5 or [9, Step 2 in the proof of Theorem 1.5] for relevant details). Replacing $X$ with $V$ we can assume that $K_X + B + M$ is numerically a limit of movable $Y$ $\mathbb{R}$-divisors.

Since $K_Y + By + M_Y$ is $\mathbb{R}$-Cartier, we can write

$$K_X + B + M + R = g^*(K_Y + By + M_Y)$$

where $R$ is exceptional over $Y$. Then $-R$ is numerically a limit of movable $Y$ $\mathbb{R}$-divisors, so for any exceptional $Y$ prime divisor $S \subset X$, the divisor $-R|_S$ is pseudoeffective over $g(S)$. Therefore, by the general negativity lemma (cf. [8, Lemma 3.3]), we have $R \geq 0$ (as in the proof of Lemma 2.9, to apply the negativity lemma we can first do a base change to an uncountable ground field if necessary).
Let $E$ be the sum of the prime exceptional divisors of $g$ that are not components of $B + R$. We do induction on the number of components of $E$. If $E = 0$, then $\text{Supp}(B + R)$ contains all the exceptional divisors of $g$, so $X$ is of Fano type over $Y$ because $(Y, C)$ is klt for some $C$ (by the relative version of [6, 2.13(7)] applied over $Y$), hence $(X, B + M)$ has an ample model over $Y$.

Now assume $E \neq 0$. By construction $(X, B + M)$ is $\mathbb{Q}$-factorial generalised dlt because this property is preserved by running MMP (see 2.8(2)). So since $E$ has no common component with $B$, $(X, B + tE + M)$ is generalised dlt for some small number $t > 0$ by definition of generalised dlt pairs. Thus running an MMP on

$$K_X + B + tE + M$$

over $Y$ with scaling of some ample divisor contracts $E$ because by the general negativity lemma the right hand side of the equation

$$K_X + B + tE + M \equiv_Y tE - R$$

eventually becomes anti-effective.

Let $W \to U$ be the step of the MMP where a component $E$ is contracted for the first time. Then $X \to W$ is an isomorphism in codimension one because the MMP cannot contract any divisor other than components of $E$. Moreover,

$$K_W + B_W + M_W \sim_{\mathbb{Q}, U} -R_W \sim_{\mathbb{Q}, U} 0$$

because $W \to U$ is extremal and because $-R_W$ is numerically a limit of movable $Y \mathbb{R}$-divisors and the divisor contracted by $W \to U$ is not a component of $R_W$. Therefore, if $K_U + B_U + M_U$ has an ample model over $Y$, then $K_W + B_W + M_W$ has an ample model over $Y$ which in turn implies that $K_X + B + M$ has an ample model over $Y$.

Note that $(U, B_U + M_U)$ is $\mathbb{Q}$-factorial generalised dlt because $X \to U$ is a partial MMP on

$$K_X + B + tE + M,$$

so

$$(U, B_U + tE_U + M_U)$$

is $\mathbb{Q}$-factorial generalised dlt. So we can replace $(X, B + M)$ with $(U, B_U + M_U)$. We are then done by induction as $E$ has one less component.

\[\square\]

4.11. Fibrations. In this subsection we prove a few results in order to treat Theorem 4.1 when the underlying space admits a suitable fibration.

Lemma 4.12. Let $(X, B + M)$ be as in Theorem 4.1. Assume that

- $f: X \to Z$ is a contraction where $\dim Z > 0$,
- $K_X + B + M \sim_{\mathbb{Q}, Z} 0$, and
- $X$ is of Fano type over $Z$.

Then there is a high resolution $Z' \to Z$ and a closed point $z' \in Z'$ such that

- $z'$ maps to $f(x)$,
- $z'$ is a generalised non-klt centre of $(Z', B_{Z'} + M_{Z'})$ where the latter is given by adjunction as in 2.12,
- and we have

$$\tau_{G'}(K_Z + B_Z + M_Z) = 0$$

where $G'$ is the exceptional divisor of the blowup of $Z'$ at $z'$.
Proof. By adjunction, for each birational contraction \( Z' \to Z \) where \( Z' \) is normal, we get \((Z', B_{Z'} + M_{Z'})\) as defined in 2.12. Assuming \( Z' \to Z \) is a high log resolution of \((Z, B_Z), M_{Z'} \) is nef. In particular, \((Z', B_{Z'} + M_{Z'})\) is a generalised sub-pair and \((Z, B_Z + M_Z)\) is a generalised pair with nef part \( M_{Z'} \). Replace the given morphism \( \phi: X' \to X \) with a high log resolution of \((X, B)\) so that the induced map \( f': X' \to Z' \) is a morphism. Since \( K_X + B + M \) is pseudo-effective and
\[
K_X + B + M \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z),
\]
\( K_Z + B_Z + M_Z \) is pseudo-effective.

Write
\[
K_{X'} + B' + M' = \phi^*(K_X + B + M)
\]
where \( M' \) is the nef part of \((X, B + M)\). By assumption, \((X, B + M)\) is \( \mathbb{Q} \)-factorial generalised dlt and \( x \) is a zero-dimensional generalised non-klt centre. Moreover, \((X', B')\) is log smooth as \( \phi \) is assumed to be a log resolution of \((X, B)\). Applying Lemma 4.8, there is a zero-dimensional generalised non-klt centre \( x' \) of \((X', B' + M')\) mapping to \( x \).

Let \( z' = f'(x') \). We will show \( z' \) is a generalised non-klt centre of \((Z', B_{Z'} + M_{Z'})\). Let \( E' \) be the exceptional divisor of the blowup of \( X' \) at \( x' \). Since \( x' \) is a generalised non-klt centre of \((X', B' + M')\), it is a non-klt centre of \((X', B')\), so there are \( d := \dim X \) components of \( [B'] \) intersecting transversally at \( x' \). So
\[
a(E', X', B' + M') = 0,
\]
that is, \( E' \) is a non-klt place of \((X', B' + M')\).

Now pick resolutions \( Z'' \to Z' \) and \( X'' \to X' \) such that \( f'': X'' \to Z'' \) is a morphism and if \( E'' \subset X'' \) is the birational transform of \( E' \), then \( H'' := f''(E'') \) is a divisor on \( Z'' \). Write \( K_{X''} + B'' + M'' \) for the pullback of \( K_{X'} + B' + M' \). Then \( E'' \) is a component of \([B'']\), hence \( H'' \) is a component of \( [B_{Z''}] \) because the generalised lc threshold of \( f''(x'') \) with respect to \((X'', B'' + M'')\) over the generic point of \( H'' \) is zero as \( E'' \leq f''(x'') \). This shows that \( z'' \) is a generalised non-klt centre of \((Z', B_{Z'} + M_{Z'})\) as claimed since \( H'' \) maps to \( z'' \) as \( E'' \) maps to \( x' \).

Let \( G' \) be the exceptional divisor of the blowup of \( Z' \) at \( z' \). By assumption, \((Z', B_{Z'})\) is log smooth and \( M_{Z'} \) is the nef part of \((Z', B_{Z'} + M_{Z'})\). Since \( z' \) is a generalised non-klt centre of \((Z', B_{Z'} + M_{Z'})\),
\[
a(G', Z, B_Z + M_Z) = a(G', Z', B_{Z'} + M_{Z'}) = 0.
\]

On the other hand, by the previous paragraph,
\[
a(H'', Z, B_Z + M_Z) = a(H'', Z', B_{Z'} + M_{Z'}) = 0.
\]

Now since \( X \) is of Fano type over \( Z \), there is a boundary \( D \) on \( X \) such that \((X, D)\) is klt and \( K_X + D \sim_{\mathbb{Q}, Z} 0 \). Thus applying adjunction, we get \((Z, D_Z + N_Z)\) which is generalised klt. From this we get \( C \) so that \((Z, C)\) is klt (as in the proof of Lemma 2.11). Moreover, \( G', H'' \) both map to \( f(x) \). Thus, applying Lemma 4.5 to \((Z, B_Z + M_Z), G', H''\), we see that to prove that
\[
\tau_{G'}(K_Z + B_Z + M_Z) = 0
\]
it is enough to prove that
\[
\tau_{H''}(K_Z + B_Z + M_Z) = 0.
\]

By assumption,
\[
\tau_E(K_X' + B' + M') = \tau_E(K_X + B + M) = 0
\]
where $E$ is the exceptional divisor of the blowup of $X$ at $x$. Also
\[ a(E, X, B + M) = 0 \]
and
\[ a(E', X, B + M) = a(E', X', B' + M') = 0 \]
where $E'$ is the exceptional divisor of the blowup of $X'$ at $x'$. Then by Lemma 4.5,
\[ \tau_{E''}(K_X + B + M) = \tau_{E'}(K_X + B + M) = 0 \]
where recall that $E'' \subset X''$ is the birational transform of $E'$. Thus
\[ K_{X''} + B'' + M'' - tE'' \]
is not pseudo-effective for any real number $t > 0$. This in turn implies that
\[ K_{X''} + B'' + M'' - tf''^*H'' \]
is not pseudo-effective for any real number $t > 0$ because $E'' \leq f''^* H''$. But then
\[ K_{Z''} + B_{Z''} + M_{Z''} - tH'' \]
is not pseudo-effective as
\[ K_{X''} + B'' + M'' - t f''^* H'' \sim_Q f''^* (K_{Z''} + B_{Z''} + M_{Z''} - tH'') . \]
Therefore,
\[ \psi^*(K_Z + B_Z + M_Z) - tH'' \]
is not pseudo-effective for any $t > 0$ where $\psi$ denotes $Z'' \to Z$, hence
\[ \tau_{H''}(K_Z + B_Z + M_Z) = 0 \]
as required.

\begin{lemma}
Assume that
\begin{itemize}
  \item $(X, B + M)$ is a projective $\mathbb{Q}$-factorial generalised lc pair,
  \item $K_X + B + M$ is pseudo-effective,
  \item $x \in X$ is a point not contained in $B - (K_X + B + M)$,
  \item $f : X \to Z$ is a contraction with $\dim Z > 0$ and $K_X + B + M \sim_{\mathbb{R}} f^* L$ for some $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $L$, and
  \item $X$ is of Fano type over $Z$.
\end{itemize}
Then
\[ f(x) \notin B_-(L). \]
\end{lemma}

\begin{proof}
Pick an ample Cartier divisor $A_Z$ on $Z$ and let $A = f^* A_Z$. We want to show that
\[ z := f(x) \notin B(L + tA_Z) \]
for any $t \in \mathbb{R}^{>0}$. Assume otherwise, that is, assume that
\[ z \in B(L + tA_Z) \]
for some $t \in \mathbb{R}^{>0}$. Thus $z$ belongs to the support of every divisor $0 \leq R_Z \sim_{\mathbb{R}} L + tA_Z$.
Then $f^{-1}\{z\} \subset \text{Supp} R$ for every divisor
\[ 0 \leq R \sim_{\mathbb{R}} K_X + B + M + tA \]
because any such $R$ is the pullback of an $R_Z$ as above. Therefore,
\[ x \in B(K_X + B + M + tA). \]
Since $X$ is of Fano type over $\mathbb{Z}$, $B + M$ is big over $\mathbb{Z}$. Moreover, $B + M$ has a minimal model $Y$ over $\mathbb{Z}$ on which $B_Y + M_Y$ is semi-ample over $\mathbb{Z}$. Then $s(B_Y + M_Y) + tA_Y$ is semi-ample for every $0 < s \ll t$ (this can be seen by considering the ample model of $B_Y + M_Y$ over $\mathbb{Z}$). Pick one such number $s$ and pick a general

$$0 \leq D_Y \sim_{\mathbb{R}} s(B_Y + M_Y) + tA_Y.$$ 

Since $X$ is of Fano type over $\mathbb{Z}$, $Y$ is of Fano type over $\mathbb{Z}$, hence $Y$ has $\mathbb{Q}$-factorial klt singularities. Thus

$$(Y, (1 - s)B_Y + (1 - s)M_Y)$$

has generalised klt singularities with nef part $(1 - s)M'$ where replacing $X'$ we are assuming the induced map $X' \to Y$ is a morphism. Since $D_Y$ is general, we deduce that

$$(Y, (1 - s)B_Y + D_Y + (1 - s)M_Y)$$

is generalised klt with nef part $(1 - s)M'$. On the other hand, since $X \to Y$ is an MMP on $B + M$ over $\mathbb{Z}$, it is also an MMP on $s(B + M) + tA$. Thus $D_Y$ determines a unique divisor

$$0 \leq D \sim_{\mathbb{R}} s(B + M) + tA$$

whose pushdown to $Y$ is $D_Y$. Then the pair

$$(X, (1 - s)B + D + (1 - s)M)$$

is generalised klt with nef part $(1 - s)M'$ because

$$K_X + (1 - s)B + D + (1 - s)M \sim_{\mathbb{R}, \mathbb{Z}} K_X + (1 - s)B + sB + sM + tA + (1 - s)M$$

$$= K_X + B + M + tA \sim_{\mathbb{R}, \mathbb{Z}} 0$$

which ensures that the generalised log discrepancies of

$$(X, (1 - s)B + D + (1 - s)M)$$

coincide with those of

$$(Y, (1 - s)B_Y + D_Y + (1 - s)M_Y)$$

which is generalised klt with nef part $(1 - s)M'$. Moreover, $D$ is big.

Now, by [12, Lemma 4.4], we can run an MMP on

$$K_X + (1 - s)B + D + (1 - s)M$$

which ends with a good minimal model, say $V$. By the first paragraph,

$$x \in B(K_X + B + M + tA)$$

which means that $X \to V$ is not an isomorphism near $x$. That is, $x$ belongs to the exceptional locus of some step of the MMP. But then

$$x \in B(K_X + B + M + tA + uH)$$

where $H$ is an ample divisor and $u > 0$ is a sufficiently small real number. Therefore,

$$x \in B(K_X + B + M + uH)$$

as $A$ is semi-ample, hence

$$x \in B_+(K_X + B + M),$$

a contradiction.

\[ \square \]

**Lemma 4.14.** Assume that Theorem 4.1 holds in dimension $\leq d - 1$. Suppose
On connectedness of non-klt loci of singularities of pairs

• $(X, B + M)$ is as in Theorem 4.1 in dimension $d$ with data $X' \to X$ and $M'$,
• $X \to Y$ is a birational contraction and $Y \to Z$ is a non-birational contraction,
• $(Y, B_Y + M_Y)$ is generalised lc with nef part $M'$, where $K_Y + B_Y + M_Y$ denotes the pushdown of $K_X + B + M$,
• $Y$ is of Fano type over $Z$, and
• $K_Y + B_Y + M_Y \sim_{Q,Z} 0$.

Then

$$\kappa_s(K_X + B + M) = \kappa(K_X + B + M) = 0.$$  

Proof. Note that $\kappa_s$ is defined as in [26, Definition 6.2.7].

Step 1. In this step we consider an ample model of $(X, B + M)$ over $Y$ from which we derive a certain minimal model over $Z$. Since $Y$ is of Fano type over $Z$, $(Y, C)$ is klt and $K_Y + C \sim_{Q,Z} 0$ for some $C$. Thus by Lemma 4.10, $(X, B + M)$ has an ample model over $Y$, say $U$. Denoting the morphism $U \to Y$ by $\pi$, we can write

$$K_U + B_U + M_U + R_U = \pi^*(K_Y + B_Y + M_Y)$$

for some $R_U$ exceptional over $Y$. Then $-R_U$ is ample over $Y$, so $R_U \geq 0$ by the negativity lemma. Moreover, $\text{Supp} R_U$ contains every exceptional divisor of $\pi$. Therefore, $U$ is of Fano type over $Z$ as $Y$ is of Fano type over $Z$, by the relative version of [6, 2.13(7)]. In particular, $(U, B_U + M_U)$ has a minimal model over $Z$, say $V$.

Note that since $K_U + B_U + M_U$ is pseudo-effective and

$$R_U \text{ is vertical over } Z \text{ and}$$

$$K_U + B_U + M_U \sim_{Q} 0$$

over the generic point of $Z$.

Step 2. In this step, after replacing $X'$, we define a boundary on $X'$ and find a generalised non-klt centre mapping to $x$ and study its properties. Recall that $x \in X$ is a zero-dimensional generalised non-klt centre of $(X, B + M)$ as in the statement of Theorem 4.1. Replacing $X'$ we can assume that it is a log resolution of $(X, B)$ and that the induced map $\rho: X' \to V$ is a morphism and a log resolution of $(V, B_V)$. Write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M).$$

Let $\Delta'$ be the sum of the birational transform of $B_V$ and the reduced exceptional divisor of $\rho$. Then $R' := \Delta' - B'$ is exceptional over $V$ as $\rho_* \Delta' = B_V = \rho_* B'$. Moreover, $R' \geq 0$: indeed, for any prime divisor $D'$ exceptional over $V$, we have

$$\mu_{D'}(\Delta' - B') = 1 - \mu_{D'} B' \geq 0.$$  

By Lemma 4.8, there is a zero-dimensional generalised non-klt centre $x'$ of $(X', B' + M')$ mapping to the given point $x$. Then $x'$ is also a generalised non-klt centre of $(X', \Delta' + M')$, and $x' \notin \text{Supp } R'$ as $(X', \Delta' + M')$ is generalised lc. On the other hand, we claim that

$$x' \notin \mathcal{B}_-(K_{X'} + \Delta' + M').$$

This follows from

$$\mathcal{B}_-(K_{X'} + \Delta' + M') \subseteq \mathcal{B}_-(K_X + B + M) \cup \text{Supp } R' \subseteq \phi^{-1} \mathcal{B}_-(K_X + B + M) \cup \text{Supp } R'$$


and the assumption 
\[ x \notin \mathbf{B}_-(K_X + B + M) \]
and the fact \( x' \notin \text{Supp} R' \).

**Step 3.** In this step we construct a minimal model of \((X', \Delta' + M')\) over \(V\) and study its properties. By construction,
\[ K_{X'} + \Delta' + M' = \rho^*(K_V + B_V + M_V) + P' \]
where \(P' \geq 0\) is exceptional over \(V\). Thus running an MMP on \(K_{X'} + \Delta' + M'\) over \(V\) with scaling of some ample divisor contracts \(P'\) and ends with a minimal model \(W/V\) because \(P'\) cannot be a limit of movable \(V\) divisors as it is exceptional over \(V\).

In fact, \((W, \Delta_W + M_W)\) is a \(\mathbb{Q}\)-factorial generalised dlt model of \((V, B_V + M_V)\) where \(\Delta_W + M_W\) is the pushdown of \(\Delta' + M'\). The map \(X' \to W\) is an isomorphism near \(x'\) because \(x'\) is not contained in \(\mathbf{B}_-(K_{X'} + \Delta' + M')\).

Let \(E\) (resp. \(E'\)) be the exceptional divisor of the blowup of \(X\) at \(x\) (resp. of \(X'\) at \(x'\)). Then
\[ a(E, X, B + M) = 0 = a(E', X', B' + M') = a(E', X, B + M). \]
This implies
\[ a(E', V, B_V + M_V) = a(E', W, \Delta_W + M_W) = a(E', X', \Delta' + M') = 0 \]
by the previous paragraph.

Now from the assumption
\[ \tau_E(K_X + B + M) = 0 \]
we deduce that
\[ \tau_{E'}(K_X + B + M) = 0, \]
by Lemma 4.5. This in turn implies
\[ \tau_{E'}(K_V + B_V + M_V) = 0 \]
because by the construction of \(V\),
\[ \phi^*(K_X + B + M) - \rho^*(K_V + B_V + M_V) \geq 0. \]
Thus
\[ \tau_{E'}(K_W + \Delta_W + M_W) = 0. \]

**Step 4.** In this step we replace \(X, Y, Z\) so that we can assume that \(K_X + B + M \sim_{\mathbb{Q}, Z} 0\) and that \(X\) is of Fano type over \(Z\). By Step 1, \(K_V + B_V + M_V\) is nef over \(Z\) and \(V\) is of Fano type over \(Z\) as \(U\) is of Fano type over \(Z\). Then \(K_V + B_V + M_V\) is semi-ample over \(Z\) by the Fano type property (see 2.6). Let \(V \to T/Z\) be the contraction defined by \(K_V + B_V + M_V\). Since \(K_V + B_V + M_V \sim_{\mathbb{Q}} 0\) over the generic point of \(Z\) by Step 1, \(T \to Z\) is birational but \(V \to T\) is not birational as we assumed \(Y \to Z\) is not birational. By construction,
\[ \kappa_\sigma(K_X + B + M) = \kappa_\sigma(K_V + B_V + M_V) = \kappa_\sigma(K_W + \Delta_W + M_W) \]
and
\[ \kappa(K_X + B + M) = \kappa(K_V + B_V + M_V) = \kappa(K_W + \Delta_W + M_W). \]
Moreover, \(W\) is of Fano type over \(Z\) as \(V\) is of Fano type over \(Z\) (by the relative version of [6, 2.13(7)]), so \(W\) is also of Fano type over \(T\). Therefore, replacing \(Z\) with \(T\), replacing \((X, B + M)\) with \((W, \Delta_W + M_W)\), replacing \(x\) with the image of \(x'\) in \(W\),
On connectedness of non-klt loci of singularities of pairs

and replacing $Y$ with $V$, from now on we can assume that $K_X + B + M \sim_{\mathbb{Q}, Z} 0$ and that $X$ is of Fano type over $Z$. In particular, there is $Q$ such that $(X, Q)$ is klt and $K_X + Q \sim_{\mathbb{Q}, Z} 0$. From this we can get $Q_Z$ so that $(Z, Q_Z)$ is klt in case $\dim Z > 0$ (for this we can either apply [2, Theorem 0.2] or use adjunction to get a generalised klt structure on $Z$ from which we can derive a klt boundary).

**Step 5.** In this step we settle the case $\dim Z = 0$, and prepare for induction in the case $\dim Z > 0$. If $\dim Z = 0$, then $K_X + B + M \sim_{\mathbb{Q}} 0$ by the previous step, so the lemma holds in this case. From now on we assume that $\dim Z > 0$. Denote $X \to Z$ by $f$ and let $z := f(x)$. For each birational contraction $Z' \to Z$ from a normal variety, consider $(Z', B_{Z'} + M_{Z'})$ given by adjunction as in 2.12. Then by Lemma 4.12, there exist a high resolution $\psi: Z' \to Z$ and a closed point $z' \in Z'$ mapping to $z$ such that

- $z'$ is a generalised non-klt centre of $(Z', B_{Z'} + M_{Z'})$, so
  
  $$a(E', Z, B_Z + M_Z) = a(E', Z', B_{Z'} + M_{Z'}) = 0,$$

- and
  
  $$\tau_{G'}(K_Z + B_Z + M_Z) = 0$$

where $G'$ is the exceptional divisor of the blowup of $Z'$ at $z'$.

**Step 6.** In this step we finish the proof. Let $\Delta_{Z'} := B^{>0}_{Z'}$ and $P_{Z'} = \Delta_{Z'} - B_{Z'}$. Then $z'$ is a generalised non-klt centre of $(Z', \Delta_{Z'} + M_{Z'})$ and $z' \notin \text{Supp} P_{Z'}$ because $(Z', B_{Z'})$ is log smooth so only $\dim Z$ exceptional divisors can meet at $z'$, and all of these have coefficient 1 in $B_{Z'}$. Moreover, by Lemma 4.6 applied to $(Z, B_Z + M_Z)$ we have

$$\tau_{G'}(K_{Z'} + \Delta_{Z'} + M_{Z'}) = 0.$$  

Moreover, by Lemma 4.13,

$$z \notin B_-(K_Z + B_Z + M_Z),$$

hence

$$z' \notin B_-(K_{Z'} + \Delta_{Z'} + M_{Z'}) \subseteq \psi^{-1}B_-(K_Z + B_Z + M_Z) \cup \text{Supp} P_{Z'}.$$  

Thus $(Z', \Delta_{Z'} + M_{Z'})$ satisfies the properties listed in Theorem 4.1. Since we are assuming the theorem in dimension $\leq d - 1$, $(Z', \Delta_{Z'} + M_{Z'})$ has a minimal model which is generalised log Calabi–Yau. Therefore,

$$\kappa_\sigma(K_{Z'} + \Delta_{Z'} + M_{Z'}) = \kappa(K_{Z'} + \Delta_{Z'} + M_{Z'}) = 0,$$

so we have

$$\kappa_\sigma(K_Z + B_Z + M_Z) = \kappa(K_Z + B_Z + M_Z) = 0.$$  

These in turn imply that

$$\kappa_\sigma(K_X + B + M) = \kappa(K_X + B + M) = 0$$

as desired, by [26, Proposition 6.2.8].

□
4.15. Proofs of Theorems 1.4 and 4.1.

Proof. (of Theorem 4.1) We will apply induction on dimension, so assume the theorem holds in lower dimension. The case dim = 1 is easy to verify.

Step 1. In this step we prove the theorem assuming that we have
\[(\ast) \quad \kappa_\sigma(K_X + B + M) = \kappa(K_X + B + M) = 0.\]
From this we get
\[K_X + B + M \equiv N_\sigma(K_X + B + M) \geq 0,\]
by \[26, Proposition 6.2.8\], where \(N_\sigma\) is defined as in \[26, Definition 2.1.8\]. Run an MMP on \(K_X + B + M\) with scaling of some ample divisor \(A\). We show that the MMP terminates (the case \(M = 0\) was established in \[16\] where more details can be found). Let \(\lambda_i\) be the scaling numbers and \(X_i \rightarrow X_{i+1}\) the steps of the MMP. Then \(\lim \lambda_i = 0\) by 2.5. Moreover,
\[K_{X_i} + B_i + M_i + \lambda_i A_i\]
is semi-ample, so any divisor not contracted by \(X_1 \rightarrow X_i\) is not a component of \(B(K_X + B + M + \lambda_i A).\)
Thus any prime divisor not contracted by the MMP is not a component of the restricted base locus \(B_-(K_X + B + M)\). On the other hand, by definition of \(N_\sigma\),
\[\text{Supp} \ N_\sigma(K_X + B + M)\]
is contained in \(B_-(K_X + B + M)\). Therefore, \(N_\sigma(K_X + B + M)\) is contracted by the MMP. This ensures
\[K_{X_i} + B_i + M_i \equiv 0\]
for \(i \gg 0\). Since
\[\kappa(K_X + B + M) = 0,\]
we deduce that
\[K_{X_i} + B_i + M_i \sim_0 0,\]
for \(i \gg 0\). Thus the MMP ends with a good minimal model as required. It is then enough to show that \((X, B + M)\) satisfies (\(\ast\)).

Step 2. In this step we modify \((X, B + M)\) so that we can assume
\[\tau_S(K_X + B + M) = 0\]
for some component \(S\) of \(|B|\). Let \(Y \rightarrow X\) be the blowup of \(X\) at \(x\) with exceptional divisor \(E\). We can assume that the induced map \(X' \rightarrow Y\) is a morphism and that \(\phi: X' \rightarrow X\) is a log resolution of \((X, B)\). Writing
\[K_{X'} + B' + M' = \phi^*(K_X + B + M),\]
let \(K_Y + B_Y + M_Y\) be the pushdown of \(K_{X'} + B' + M'\). We consider \((Y, B_Y + M_Y)\) as a generalised pair with data \(x' \rightarrow Y\) and \(M'\). There is a zero-dimensional generalised non-klt centre \(y\) of \((Y, B_Y + M_Y)\) mapping to \(x\) because there is a zero-dimensional generalised non-klt centre \(x'\) of \((X', B' + M')\) mapping to \(x\), by Lemma 4.8, and we can take \(y\) to be the image of \(x'\) on \(Y\); alternatively we can find \(y\) using the fact that \(Y \rightarrow X\) is the blowup of a smooth point \(x\) and that \((X, B)\) is log smooth near \(x\); this in particular shows that \((Y, B_Y + M_Y)\) is \(\mathbb{Q}\)-factorial generalised dlt, so \((Y, B_Y)\) is log smooth near \(y\).
Let $G$ be the exceptional divisor of the blowup of $Y$ at $y$. Then
$$a(E, X, B + M) = 0 = a(G, Y, B_Y + M_Y) = a(G, X, B + M).$$
Also by assumption,
$$\tau_E(K_Y + B_Y + M_Y) = \tau_E(K_X + B + M) = 0.$$
Applying Lemma 4.5, we see that
$$\tau_G(K_Y + B_Y + M_Y) = \tau_G(K_X + B + M) = 0.$$
Now $(Y, B_Y + M_Y), y$ satisfies the properties (1)-(3), (5) listed in Theorem 4.1. It also satisfies (4), that is,
$$y \notin B_-(K_Y + B_Y + M_Y)$$
as
$$B_-(K_Y + B_Y + M_Y) \subseteq \alpha^{-1}B_-(K_X + B + M)$$
where $\alpha$ denotes $Y \to X$. Therefore, replacing $(X, B + M), x$ with $(Y, B_Y + M_Y), y$, we can assume that
$$\tau_S(K_X + B + M) = 0$$
for some component $S$ of $\lfloor B \rfloor$.

**Step 3.** In this step we choose a number $t$ and find a Mori fibre space for $(X, B + M - t \lfloor B \rfloor)$. Let $\Phi$ be the set of the coefficients of $B$ and $p$ be a natural number so that $pM'$ is Cartier. Pick a small rational number $t > 0$ as in Lemma 2.10 for the data $\dim X, p, \Phi$. By the previous step,
$$K_X + B + M - t \lfloor B \rfloor$$
is not pseudo-effective. Thus we can run an MMP on
$$K_X + B + M - t \lfloor B \rfloor$$
ending with a Mori fibre space $V \to Z$. Since $K_X + B + M$ is pseudo-effective, $K_V + B_V + M_V$ is nef over $Z$.

Writing
$$B_V - t \lfloor B_V \rfloor + M_V = B_V - \lfloor B_V \rfloor + (1 - t) \lfloor B_V \rfloor + M_V,$$
and applying Lemma 2.10, we deduce that $(V, B_V + M_V)$ is generalised lc and
$$K_V + B_V + M_V = Z 0.$$

**Step 4.** In this step we introduce a boundary $\Delta'$ on $X'$ and study its properties. Replacing $X'$ we can assume that $\phi$ is a log resolution of $(X, B)$ and that the induced map $X' \to V$ is a morphism. Recall
$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$
from Step 2. Let $\Delta' := B'^{-0}$. Applying Lemma 4.8, we see that there is a zero-dimensional generalised non-klt centre $x'$ of $(X', B' + M')$ mapping to $x$. Note that $x'$ is also a generalised non-klt centre of $(X', \Delta' + M')$ and $x' \notin \text{Supp}(\Delta' - B')$. Moreover, if $G'$ is the exceptional divisor of the blowup of $X'$ at $x'$, then by Lemma 4.5,
$$\tau_{G'}(K_X + B + M) = 0,$$
so
$$\tau_{G'}(K_{X'} + \Delta' + M') = 0.$$
Caucher Birkar

by Lemma 4.6.

In addition,

\[ x' \notin B_-(K_{X'} + \Delta' + M') \]

because

\[ B_-(K_{X'} + \Delta' + M') \subseteq \phi^{-1}B_-(K_X + B + M) \cup \text{Supp}(\Delta' - B') \]

and \( x \notin B_-(K + B + M) \) and \( x' \notin \text{Supp}(\Delta' - B') \).

**Step 5.** In this step we finish the proof. By the previous step, we can replace \((X, B+M), x\) with \((X', \Delta'+M'), x'\), hence assume that there is a birational contraction \(X \to V\) and a non-birational contraction \(V \to Z\) such that

- \((V, B_V + M_V)\) is generalised lc with nef part \(M'\), where \(K_V + B_V + M_V\) denotes the pushdown of \(K_X + B + M\),
- \(V\) is of Fano type over \(Z\), and
- \(K_V + B_V + M_V \sim_{Q,Z} 0\).

Now since we are assuming Theorem 4.1 in lower dimension, applying Lemma 4.14, we get

\[ \kappa_\sigma(K_X + B + M) = \kappa(K_X + B + M) = 0 \]

as desired.

\[ \square \]

**Proof.** (of Theorem 1.4) This is a special case of Theorem 4.1.

\[ \square \]

4.16. **Proofs of Corollaries 1.5 and 1.6.** Before giving the proofs of the corollaries we make a bit of preparation.

**Lemma 4.17.** Let \((X, B)\) be a projective dlt pair with \(K_X + B \equiv 0\) having a zero-dimensional non-klt centre \(x\). Assume that either \(V = X\) or \(V\) is a non-klt centre of \((X, B)\), and that \(\dim V \geq 2\). Let \(K_V + B_V = (K_X + B)|_V\) be given by adjunction \((B_V = B \text{ when } V = X)\). Then the non-klt locus \(\text{Nklt}(V, B_V) = \lfloor B_V \rfloor\) is connected.

**Proof.** The equality \(\text{Nklt}(V, B_V) = \lfloor B_V \rfloor\) follows from the assumption that \((X, B)\) is dlt which implies that \((V, B_V)\) is dlt. Assume that \(\text{Nklt}(V, B_V)\) is not connected. By [22, Theorem 4.40], all the minimal non-klt centres of \((X, B)\) are birational to each other, in particular, they have the same dimension. Since we already have a zero-dimensional non-klt centre \(x\), all the minimal non-klt centres are zero-dimensional (we recall the proof of this fact below). Therefore, any minimal non-klt centre of \((V, B_V)\) is also zero-dimensional because any minimal non-klt centre of \((V, B_V)\) is a minimal non-klt centre of \((X, B)\).

Since we assumed that \(\text{Nklt}(V, B_V) = \lfloor B_V \rfloor\) is not connected, by Theorem 1.2, \((V, B_V)\) has exactly two disjoint non-klt centres. As \((V, B_V)\) is dlt, these centres are two disjoint irreducible components of \(\lfloor B_V \rfloor\). This contradicts the fact that \((V, B_V)\) has a zero-dimensional non-klt centre.

The lemma does not hold if we relax the dlt assumption to lc; see the example given after Theorem 1.2.

In the proof of the lemma we used the fact that all the minimal non-klt centres of \((X, B)\) are zero-dimensional. For convenience, we give the proof here essentially following [22, Theorem 4.40]. First applying Theorem 1.2, we see that \(\lfloor B \rfloor\) is connected.
because \((X, B)\) has a zero-dimensional non-klt centre \(x\). Now pick any minimal non-klt centre \(W\) of \((X, B)\). Then there exist components \(D_1, \ldots, D_r\) of \(|B|\) such that \(x \in D_1\), \(W \subset D_r\), \(D_i\) intersects \(D_{i+1}\) for each \(1 \leq i < r\), and that \(r\) is minimal with these properties. Then by induction on dimension, any minimal non-klt centre of \((D_1, B_{D_1})\) is zero-dimensional where \(K_{D_1} + B_{D_1} = (K_X + B)|_{D_1}\). In particular, \(D_1 \cap D_2\) contains a zero-dimensional non-klt centre \(x_2\) of \((D_1, B_{D_1})\) which is in turn a non-klt centre of \((X, B)\). Repeating this argument we find a zero-dimensional non-klt centre of \((x, B)\) in each \(D_i\), in particular, in \(D_r\). But \(W \subset D_r\), so applying induction to \((D_r, B_{D_r})\) implies that \(\dim W = 0\).

**Proof.** (of Corollary 1.5) By Theorem 1.4, we can run an MMP on \(K_X + B\) ending with a minimal model \(X'\) with \(K_{X'} + B' \sim_Q 0\). By assumption, no non-klt centre of \((X, B)\) is contained in \(B_-(K_X + B)\). Thus \(X \dashrightarrow X'\) is an isomorphism near the generic point of each non-klt centre. So each non-klt centre has a birational transform on \(X'\). On the other hand, if \(W'\) is any non-klt centre of \((X', B')\), then \(X \dashrightarrow X'\) is an isomorphism over the generic point of \(W'\) because \(X \dashrightarrow X'\) is an MMP on \(K_X + B\). Thus there is a 1-1 correspondence between the non-klt centres of \((X, B)\) and \((X', B')\) given by birational transform. In particular, the image \(x'\) on \(X'\) of the given non-klt centre \(x\) is a zero-dimensional non-klt centre of \((X', B')\).

Assume that either \(V = X\) or that \(V\) is a non-klt centre of \((X, B)\). Define \(K_V + B_V = (K_X + B)|_V\) by adjunction \((B_V = B\) if \(V = X)\). Since \((X, B)\) is dlt, \((V, V_B)\) is dlt, so

\[
\text{Nklt}(V, V_B) = |B_V|.
\]

Similarly, define \(K_{V'} + B_{V'} = (K_X + B)|_{V'}\) by adjunction. Then

\[
\text{Nklt}(V', V_B) = |B_{V'}|.
\]

Each non-klt centre of \((V, V_B)\) is a non-klt centre of \((X, B)\), and similarly, each non-klt centre of \((V', V_B)\) is a non-klt centre of \((X', B')\). Thus by the previous paragraph, \(|B_{V'}|\) is the birational transform of \(|B_V|\).

Now assume that \(\dim V \geq 2\). Since \((X', B')\) is dlt with \(K_{X'} + B' \sim_Q 0\) having a zero-dimensional non-klt centre \(x'\), applying Lemma 4.17 shows that \(\text{Nklt}(V', V_B) = |B_{V'}|\) is connected.

Assume that \(\text{Nklt}(V, V_B) = |B_V|\) is not connected. We will derive a contradiction. Since \(|B_V|\) is not connected but \(|B_{V'}|\) is connected, we can find disjoint components \(S, T\) of \(|B_V|\) such that their birational transforms \(S', T'\) on \(X'\) intersect. Let \(W'\) be an irreducible component of \(S' \cap T'\). Then \(S', T'\) are components of \(|B_{V'}|\), hence \(W'\) is a non-klt centre of \((V', V_B)\), so \(W'\) is also a non-klt centre of \((X', B')\). Thus \(X \dashrightarrow X'\) is an isomorphism over the generic point of \(W'\), hence \(W'\) is the birational transform of a non-klt centre \(W\) of \((X, B)\), and \(S, T\) both contain \(W\), a contradiction. Therefore, \(|B_{V'}|\) is connected.

**Proof.** (of Corollary 1.6) For each \(i\), let \(0 \leq u_i \leq a_i\) be a rational number such that \(u_i \leq 1\). Let \(\Delta = B - \sum u_i B_i\). Then

\[
K_X + \Delta = K_X + B - \sum u_i B_i = \sum (a_i - u_i) B_i \geq 0.
\]

In particular, \(K_X + \Delta\) is pseudo-effective and

\[
B_-(K_X + \Delta) \subseteq \text{Supp} \sum (a_i - u_i) B_i \subseteq B - C
\]
where $C$ is the sum of the good components of $B$. On the other hand, by assumption, $C$ and $B-C$ have no common components, and $x$ is a zero-dimensional non-klt centre of $(X,C)$. Thus $x$ is not contained in $B-C$ which implies that $x \notin \mathbb{B}_-(K_X + \Delta)$.

Moreover, $\Delta \geq C$, so $x$ is a zero-dimensional non-klt centre of $(X,\Delta)$. Also

$$0 \leq \tau_E(K_X + \Delta) \leq \tau_E(K_X + B) = 0$$

where $E$ is the exceptional divisor of the blow up of $X$ at $x$. Therefore, $(X,\Delta)$ satisfies all the assumptions of Theorem 1.4, so we can run an MMP on $K_X + \Delta$ which ends with a log minimal model $(X',\Delta')$ with $K_X + \Delta ' \sim_{\mathbb{Q}} 0$. In particular, taking $u_i = 0$ for every $i$, we have $\Delta = B$, so we get the first claim of the corollary.

Assume that $V$ is a stratum of $(X,C)$, that is, either $V = X$ or that $V$ is a non-klt centre of $(X,C)$. Assume dim $V \geq 2$. We want to show that $C_V$ is connected where $K_V + C_V = (K_X + C)|_V$.

From now on we assume that $u_i > 0$ when $a_i > 0$. Then $|\Delta| = C$, so the non-klt centres of $(X,\Delta)$ and $(X,C)$ are the same. So $C_V = |\Delta_V|$ where $K_V + \Delta_V = (K_X + \Delta)|_V$. Moreover, no non-klt centre of $(X,C)$ is contained in $B-C$, so no non-klt centre of $(X,\Delta)$ is contained in $B-C$, hence by the first paragraph of this proof, no non-klt centre of $(X,\Delta)$ is contained in $\mathbb{B}_-(K_X + \Delta)$. Therefore, by Corollary 1.5, $C_V = |\Delta_V|$ is connected as desired.

$$\square$$

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