Completely bounded maps into certain Hilbertian operator spaces

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Abstract
We prove a factorization of completely bounded maps from a $C^*$-algebra $A$ (or an exact operator space $E \subset A$) to $\ell_2$ equipped with the operator space structure of $(C,R)_\theta$ $(0 < \theta < 1)$ obtained by complex interpolation between the column and row Hilbert spaces. More precisely, if $F$ denotes $\ell_2$ equipped with the operator space structure of $(C,R)_\theta$, then $u : A \to F$ is completely bounded iff there are states $f,g$ on $A$ and $C > 0$ such that

$$\forall a \in A \; \|ua\|^2 \leq Cf(a^*a)^{1-\theta}g(aa^*)^\theta.$$  

This extends the case $\theta = 1/2$ treated in a recent paper with Shlyakhtenko [25]. The constants we obtain tend to 1 when $\theta \to 0$ or $\theta \to 1$, so that we recover, when $\theta = 0$ (or $\theta = 1$), the case of mappings into $C$ (or into $R$), due to Effros and Ruan. We use analogues of “free Gaussian” families in non semifinite von Neumann algebras. As an application, we obtain that, if $0 < \theta < 1$, $(C,R)_\theta$ does not embed completely isomorphically into the predual of a semifinite von Neumann algebra. Moreover, we characterize the subspaces $S \subset R \oplus C$ such that the dual operator space $S^*$ embeds (completely isomorphically) into $M_\infty$ for some semifinite von neumann algebra $M$: the only possibilities are $S = R$, $S = C$, $S = R \cap C$ and direct sums built out of these three spaces. We also discuss when $S \subset R \oplus C$ is injective, and give a simpler proof of a result due to Oikhberg on this question. In the appendix, we present a proof of Junge’s theorem that $O\!H$ embeds completely isomorphically into a non-commutative $L_1$-space. The main idea is similar to Junge’s, but we base the argument on complex interpolation and Shlyakhtenko’s generalized circular systems (or “generalized free Gaussian”), that somewhat unifies Junge’s ideas with those of our work with Shlyakhtenko [25].

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Introduction

An operator space is a Banach space given together with an isometric embedding into the space $B(H)$ of all bounded operators on a Hilbert space $H$. Like the previous papers [25] and [19] (to which this one is a natural sequel) this paper mainly studies questions about Hilbertian operator spaces. As is well known, the Hilbert space $\ell_2$ can be equipped with many different “operator space structures”, i.e. there are many inequivalent ways to embed $\ell_2$ into the space $B(H)$ of all bounded operators on a Hilbert space $H$. Here the “inequivalence” is with respect to the operator space theory where the relevant notion of morphism and isomorphism are those of “completely bounded map” and “complete isomorphism”. The basic theory of operator spaces and their duality was developed by Effros–Ruan and Blecher–Paulsen (cf. [5, 22]).

The two basic ways to realize $\ell_2$ as an operator space are the “row” and “column” ways that are defined respectively as follows. Let

$$ R = \operatorname{span}\{e_{1j} \mid j \geq 1\} \subset B(\ell_2) $$

$$ C = \operatorname{span}\{e_{ij} \mid i \geq 1\} \subset B(\ell_2). $$

Then $R$ and $C$ are isometric to $\ell_2$ but are not completely isomorphic (see e.g. [22], p. 21]).

In [21], the author introduced another isometric embedding of $\ell_2$ into $B(H)$ and denoted by $OH$ the space $\ell_2$ equipped with this operator space structure. Being self-dual, the space $OH$ appears as the natural analogue of $\ell_2$ in operator space theory. It also appears as a “midpoint” between $R$ and $C$ in the sense of interpolation theory. More precisely, we have $OH = (C, R)\downarrow$. The latter formula roughly means that $OH$ is obtained by some sort of “geometric mean” out of the pair $(C, R)$.

In [21], the author introduced more generally a “natural” operator space structure on any $L_p$-space $(1 \leq p \leq \infty)$. When $p = 2$, one recovers the space $OH$ (up to cardinality considerations). This uses the theory developed by Effros–Ruan and Blecher–Paulsen. Roughly, one starts with the obvious (by Gelfand theory) operator space structure on von Neumann or $C^*$-algebras, then by operator space duality one passes to their duals and preduals and finally by interpolation one obtains the “natural” operator space structure on non-commutative $L_p$-spaces. In Banach space theory, the fact that $\ell_2$ embeds in $L_p = L_p([0,1], dt)$ for any $0 < p < \infty$ plays a very important role. The case $p = 1$ is probably the most useful one; it is closely related to Grothendieck’s theorem. More precisely, define $J: \ell_2 \to L_p$ by setting $Jx = \Sigma x_n g_n$ where $(g_n)$ is an independent identically distributed (i.i.d. in short) sequence of standard (complex valued) Gaussian random variables. If we normalize $g_n$ so that $\|g_1\|_p = 1$ (note $\|g_n\|_p = \|g_1\|_p$), then $J$ is an isometric embedding. If we replace $(g_n)$ by the Rademacher functions $(r_n)$, then by Khintchine’s inequalities, $J$ becomes an isomorphic embedding for any $0 < p < \infty$.

In operator space theory, the question whether $OH$ analogously embeds completely isomorphically (or even hopefully completely isometrically) into a non-commutative $L_p$-space arose immediately after [21]. Curiously, the above embedding $J$ produces a different operator space structure than that of $OH$. The latter structure actually depends on $p$ (with a sharp distinction between the cases $p < 2$ and $p > 2$) and gives $OH$ only when $p = 2$. Moreover, if we replace $(g_n)$ or $(r_n)$ by their free analogue in Voiculescu’s sense (see [22]), the resulting operator space remains the same (completely isomorphically) for each $p$ as long as $p < \infty$. These facts are closely related to Lust–Piquard’s non-commutative Khintchine inequalities (see [21], pp. 104–112 and pp. 115–121 for full details). Passing to the Fermionic analogue of Gaussian variables (see [9]) also produces the same result; thus neither does this yield a realization of $OH$ in non-commutative $L_p$. Actually, Junge had observed early on that (by the Lust-Piquard inequalities) such a realization is impossible for $2 < p < \infty$ ([9] and see also the final remark below), but the case $1 < p < 2$ and especially the crucial case $p = 1$ resisted all efforts until recently when Junge proved that $OH$ does embed completely isomorphically into $M_n$ for some non-semifinite von Neumann algebra $M$. Shortly before Junge’s paper, Shlyakhtenko and the
Main Results

Let $E \subset A$ be an operator space, given as a closed subspace of a $C^*$-algebra $A$. Let $u: E \rightarrow \ell_2$ be a linear map. We will identify $\ell_2$ successively with $R, C$, and other operator spaces isometric to $\ell_2$.

By [9], we know that $\|u\|: E \rightarrow R \|_{cb} \leq 1$ if and only if there is a state $f$ on $A$ such that

$$\forall x \in E \quad \|ux\|^2 \leq f(xx^*).$$

The letters WEP stand for Lance’s “weak expectation property”: a $C^*$-algebra $A$ has the WEP if the inclusion $A \rightarrow A^{**}$ factors completely contractively through $B(H)$ (see e.g. [22] for examples of its use in operator space theory).
Equivalently, by \[4\] this holds iff for any finite sequence \(x_1, \ldots, x_n\) in \(E\) we have

\[
\sum ||ux_i||^2 \leq \left\| \sum x_ix_i^* \right\|.
\]

Similarly, \(\|u\| : E \rightarrow C\) \(cb \leq 1\) iff there is a state \(g\) on \(A\) such that

\[
\forall x \in E \quad \|ux\|^2 \leq f(x^*x).
\]

Moreover, this holds iff for any finite sequence \(x_1, \ldots, x_n\) in \(E\) we have

\[
\sum ||ux_i||^2 \leq \left\| \sum x_i^*x_i \right\|.
\]

In \[20\], the author introduced a different operator space structure on \(\ell_2\), namely the space \(\text{OH}\), an operator space isometric to \(\ell_2\) and uniquely characterized among operator spaces by the property that it is (canonically) completely isometric to its anti-dual.

For any \(0 < \theta < 1\), one can extend (see \[20\]) complex interpolation to the operator space context. Applied to the interpolation pair \((C, R)\) (using the transposition map \(x \rightarrow {}^tx\) to define “compatibility” in the interpolation sense) this method produces new operator spaces, denoted by \((C, R)_\theta\), that are each isometric to \(\ell_2\). For \(\theta = 1/2\) we recover the space \(\text{OH}\). To abbreviate we will denote simply \(R[\theta] = (C, R)_\theta\). With this notation we have \(R[1/2] = (C, R)_{1/2} = (R, C)_{1/2} = \text{OH}\), and also \(R[\theta]^* = R[1-\theta]\) completely isometrically. By convention, we set \(R[0] = C\) and \(R[1] = R\).

The operator space structure on \(R[\theta]\) can be described more explicitly as follows. Let us denote by \(\{e_i(\theta) \mid i = 1, 2, \ldots\}\) an orthonormal basis in \(R[\theta]\) (recall \(R[\theta] \simeq \ell_2\) as Banach space). Then for any finite sequence \((a_i)\) in \(B(\ell_2)\) we have

\[
||\sum a_i \otimes e_i(\theta)||_{B(\ell_2) \otimes_{\text{min}} R[\theta]} = \sup \{\left(\sum |s^\theta a_it^{1-\theta}\right)^{1/2} \mid s \geq 0, t \geq 0, \|s\|_2 \leq 1, \|t\|_2 \leq 1\}
\]

where \(\| \|_{2}\) denotes the Hilbert–Schmidt norm. Equivalently, let \(p = (1-\theta)^{-1}\) and \(p' = \theta^{-1}\); let \(S_p\) and \(S_{p'}\) denote the corresponding Schatten classes. Then the left side is equal to the norm of the mapping \(x \rightarrow \sum a_i^*xa_i\) on \(S_{p'}\) and also equal to that of the mapping \(y \rightarrow \sum a_iya_i^*\) on \(S_p\).

In the extreme case \(\theta = 0\) (resp. \(\theta = 1\)) we recover the space \(C\) (resp. \(R\)) and the above supremum is equal to \(||\sum a_i^*a_i||^{1/2}\) (resp. \(||\sum a_i^*a_i||^{1/2}\)). The space \(R[\theta]\) can also be described as the space of “row matrices” inside the Schatten class \(S_p\) with \(p = 1/(1-\theta)\), when the latter is equipped with its “natural” operator space structure defined (by interpolation) in \[20\]. Similarly \(R[\theta]\) can be described as the space of “column matrices” inside the Schatten class \(S_{p'}\).

In \[25\] it is proved that, if \(E\) is exact, then \(||u\| : E \rightarrow \text{OH}\) \(cb < \infty\) iff there is a constant \(C\) and states \(f, g\) such that for all \(x \in E\) we have

\[
||ux||^2 \leq Cg(x^*x)^{1/2}f(x^*x)^{1/2}.
\]

The first goal of this note is to prove this result with \(R[\theta]\) instead of \(\text{OH}\). Although the ingredients are the same as in \[25\], our proof is somewhat more direct. Moreover, we are able to recover the extreme cases \(\theta = 0\) and \(\theta = 1\) described above (due to Effros and Ruan \[4\]). Note that no assumption on \(E\) is needed in the latter extreme cases, but some assumption (such as exactness) is definitely needed when \(0 < \theta < 1\) (see the remark p. 210 in \[25\]).

In an appendix to this note, we present a simpler proof of Junge’s recent remarkable embedding theorem of \(\text{OH}\) (or \(R[\theta]\) for \(0 < \theta < 1\)) into the predual of a von Neumann algebra \(M\). Combined with the results of the present note, the argument of \[19\] shows that such an embedding is impossible, for any \(0 < \theta < 1\), if \(M\) is semifinite.

Let \(0 < \theta < 1\). Following \[20\] \(\S 2.7\), we can view \(R[\theta]\) as an operator space such that we have (isometrically)

\[
M_n(R[\theta]) = (M_n(C), M_n(R))_{\theta}
\]

for any \(n \geq 1\). We set

\[
c(\theta) = (\theta^\theta(1-\theta)^{1-\theta})^{-1}.
\]
**Theorem 1.** Let $A$ be a $C^*$-algebra. Then for any complete contraction $u$: $A \to R[\theta]$, there are states $f, g$ on $A$ such that

$$
\forall \, a \in E \quad \|ua\| \leq c(\theta) f(a^* a)^{\frac{1 - \theta}{2}} g(aa^*)^{\frac{\theta}{2}}.
$$

We will use the following known fact (see [20] and [23] for related results)

**Lemma 2.** For any $C^*$-algebra $B$ with the WEP and $0 < \theta < 1$, we have

$$
B \otimes_{\min} R[\theta] = (B \otimes_{\min} C, B \otimes_{\min} R)^\theta.
$$

with equal norms.

**Proof.** Indeed, this is equivalent to the validity of the following isometric identities for any $n \geq 1$

$$
\forall \, a \in E \quad \|ua\| \leq c(\theta) f(a^* a)^{\frac{1 - \theta}{2}} g(aa^*)^{\frac{\theta}{2}}.
$$

To verify we first observe that the case $B = K(H)$ follows from the definition of interpolation. Then taking the bidual of both sides of (1) (still restricted to the case $B = K(H)$), we obtain the case $B = B(H)$. Finally, if $B$ is WEP the inclusion $B \to B^{**}$ factors completely contractively through $B(H)$, so that (using [20] for $B = B(H)$) we have a complete contraction

$$
B \otimes_{\min} (C_n, R_n)^\theta \to (B^{**} \otimes_{\min} C_n, B^{**} \otimes_{\min} R_n)^\theta.
$$

But $B^{**} \otimes_{\min} C_n = (B \otimes_{\min} C_n)^{**}$ and $B^{**} \otimes_{\min} R_n = (B \otimes_{\min} R_n)^{**}$, and the norm induced on $B \otimes (C_n, R_n)^\theta$ by the space

$$
((B \otimes_{\min} C_n)^{**}, (B \otimes_{\min} R_n)^{**})^\theta
$$

coincides with the norm of $(B \otimes_{\min} C_n, B \otimes_{\min} R_n)^\theta$. Indeed (see e.g. [22] p. 57 for more details), since the sets appearing on both sides are obviously identical (and each identical to $(B^*)^n$), we have isometrically

$$
(B \otimes_{\min} C_n, B \otimes_{\min} R_n)^\theta = ((B \otimes_{\min} C_n)^*, (B \otimes_{\min} R_n)^*)^\theta,
$$

and hence repeating the same argument for the duals, we have isometrically $(B \otimes_{\min} C_n, B \otimes_{\min} R_n)^{\theta^*} = ((B \otimes_{\min} C_n)^{**}, (B \otimes_{\min} R_n)^{**})^\theta$. Therefore, we find a completely contractive inclusion

$$
B \otimes_{\min} (C_n, R_n)^\theta \to (B \otimes_{\min} C_n, B \otimes_{\min} R_n)^\theta.
$$

On the other hand, the fact that the converse inclusion is completely contractive is obviously true in general (without any assumption on $B$), as follows easily by considering an embedding of $B$ onto $B(H)$.

**Proof of Theorem 1.** We will use the following formula valid for any pair $\alpha_0, \alpha_1$ of positive numbers:

$$
\alpha_0^{1 - \theta} \alpha_1^\theta = \inf_{\lambda > 0} \{(1 - \theta) \lambda^\theta \alpha_0 + \theta \lambda^{-(1 - \theta)} \alpha_1\}.
$$

Using this, (6) can be rewritten as

$$
\forall \, \lambda > 0 \quad \forall \, a \in E \quad \|ua\|^2 \leq (c(\theta))^2 \{(1 - \theta) \lambda^\theta f(a^* a) + \theta \lambda^{-(1 - \theta)} g(aa^*)\}.
$$

By the Hahn–Banach theorem (cf. e.g. [6] Lemma 3.4) for details), it suffices to show that for all finite sequences $(a_i)$ in $E$ and all numbers $\lambda_i > 0$, we have

$$
\Sigma \|ua_i\|^2 \leq (c(\theta))^2 \{(1 - \theta) \|\Sigma \lambda_i^\theta a_i a_i^*\| + \theta \|\Sigma \lambda_i^{-(1 - \theta)} a_i a_i^*\|\}.
$$

5
We will use the “generalized circular elements” introduced in [30], following Voiculescu’s work. Since we follow closely the ideas in [22], we will be brief. Let \( H \) be a Hilbert space. We assume given a set \( I \) such that \( H \) has an orthonormal basis formed of the disjoint union
\[
\{e_i | i \in I\} \cup \{e'_i | i \in I\}.
\]
Let \( \mathcal{F} \) be the Full Fock space over \( H \), i.e.
\[
\mathcal{F} = \mathbb{C} \otimes H \otimes H^{\otimes 2} \otimes \cdots.
\]
Let \( \Omega \) be the unit of \( \mathbb{C} \), viewed as an element in \( \mathcal{F} \). For any \( h \) in \( H \), we denote by \( \ell(h) \) (resp. \( r(h) \)) the left (resp. right) creation operator on \( \mathcal{F} \) i.e. \( x \to h \otimes x \) (resp. \( x \to x \otimes h \)). Moreover, we set \( \ell_i = \ell(e_i) \) (resp. \( r_i = r(e_i) \)) and \( \ell'_i = \ell(e'_i) \) (resp. \( r'_i = r(e'_i) \)). We define
\[
x_i = (1 - \theta) \lambda_i^{\theta/2} \ell_i + \theta \lambda_i^{-(1-\theta)/2} \ell'_i
\]
and
\[
y_i = (1 - \theta) \lambda_i^{(1-\theta)/2} r_i + \theta \lambda_i^{\theta/2} r'_i.
\]
Let \( \mathcal{L} \) (resp. \( \mathcal{R} \)) be the von Neumann algebra generated by \( (x_i)_{i \in I} \) (resp. \( (y_i)_{i \in I} \)) in \( B(\mathcal{F}) \). Note that \( \mathcal{L} \) and \( \mathcal{R} \) commute with each other. Let \( (a_i) \) be a finite sequence in \( A \). By a well known argument (as in [25, p. 202]) we have
\[
\|\Sigma x_i \otimes a_i\|_{\min} \leq (1 - \theta) \|\Sigma \lambda_i^{\theta/2} \ell_i \otimes a_i\|_{\min} + \theta \|\Sigma \lambda_i^{-(1-\theta)/2} \ell'_i \otimes a_i\|_{\min}
\]
and hence by Cauchy–Schwarz
\[
\leq (1 - \theta) \|\Sigma a_i^* a_i \lambda_i^\theta\|^{1/2} + \theta \|\Sigma a_i^* a_i \lambda_i^{\theta-1}\|^{1/2}.
\]
Therefore the proof of (8) (and thus of Theorem 1) is reduced to that of the following sublemma. \( \square \)

**Sublemma.**

(10)
\[
(\Sigma \|u a_i\|^2)^{1/2} \leq c(\theta) \|\Sigma x_i \otimes a_i\|_{\min}.
\]

**Proof.** By [22] we know that \( \mathcal{L} \) is QWEP, i.e. there is a \( C^* \)-algebra \( B \) with the weak expectation property (WEP in short) and an ideal \( \mathcal{I} \subset B \) such that \( \mathcal{L} \simeq B/\mathcal{I} \). Since \( \|u\|_{cb} \leq 1, I \otimes u \)
defines a contraction from \( B \otimes_{\min} A \) to \( B \otimes_{\min} R[\theta] \). Since \( R[\theta] \) has the completely contractive approximation property, we may clearly assume that \( u \) has finite rank. In that case, if we denote by \( q : B \otimes_{\min} A \to (B/\mathcal{I}) \otimes_{\min} A \) the canonical map, we must have \( (I \otimes u) \ker(q) \subset \mathcal{I} \otimes_{\min} R[\theta] \) (see e.g. [22, Th. 15.11] for details). Therefore, \( I \otimes u \) defines a contractive map from \( (B \otimes_{\min} A)/\ker(q) \) to \( (B \otimes_{\min} R[\theta])/\mathcal{I} \otimes_{\min} R[\theta] \). Thus
\[
\|I \otimes u : \mathcal{L} \otimes_{\min} A \to (B \otimes_{\min} R[\theta])/(\mathcal{I} \otimes_{\min} R[\theta])\| \leq 1.
\]
But now since \( B \) has the WEP, by Lemma 2 the following isometric identity holds:
\[
B \otimes_{\min} R[\theta] = (B \otimes_{\min} C, B \otimes_{\min} R)[\theta].
\]
Hence we obtain a natural contractive map
\[
(B \otimes R[\theta])/(\mathcal{I} \otimes_{\min} R[\theta]) \to (\mathcal{L} \otimes_{\min} C, \mathcal{L} \otimes_{\min} R)[\theta].
\]
Thus to conclude it suffices to prove the following.
Claim. Let \( X_\theta = (\mathcal{L} \otimes_{\min} C, \mathcal{L} \otimes_{\min} R)_\theta \). Then we have

\[
(\Sigma \|u_i\|^2) \leq c(\theta) \|\Sigma x_i \otimes u_i\|_{X_\theta}.
\]

Let \((z_i)_{i \leq n}\) be a finite sequence in \((C,R)_\theta = R[\theta]\). We will show more generally that

\[
(\Sigma \|z_i\|^2)^{1/2} \leq c(\theta) \|\Sigma x_i \otimes z_i\|_{X_\theta}.
\]

Let us denote by \(L^2_\ell\) (resp. \(L^2_r\)) the completion of \(\mathcal{L}\) for the norm \(x \rightarrow \|x\Omega\|\) (resp. \(x \rightarrow \|x^{*}\Omega\|\)). Similarly we denote by \(R^2_\ell\) (resp. \(R^2_r\)) the completion of \(\mathcal{R}\) for the norm \(x \rightarrow \|x\Omega\|\) (resp. \(x \rightarrow \|x^{*}\Omega\|\)).

Note that actually \(L^2_\ell\) (resp. \(L^2_r\)) is clearly isometric to \(H\) and the map \(x \rightarrow x\Omega\) injects \(\mathcal{L}\) (resp. \(\mathcal{R}\)) into \(H\). We will denote

\[
\mathcal{L}_\theta = (L^2_\ell, L^2_r)_\theta \text{ and } \mathcal{R}_\theta = (R^2_\ell, R^2_r)_\theta.
\]

Clearly, for any \((b_1, \ldots, b_n)\) in \(\mathcal{L}\) we have

\[
(\Sigma \|b_k\|^2)^{1/2} \leq \|\Sigma b_k \otimes e_k\|^{1/2} = \|\Sigma b_k \otimes e_{k1}\|
\]

and

\[
(\Sigma \|b_k\|^2)^{1/2} \leq \|\Sigma b_k \otimes e_{k1}\|^{1/2} = \|\Sigma b_k \otimes e_{k1}\|
\]

hence we have a contractive inclusion

from \((\mathcal{L} \otimes_{\min} C, \mathcal{L} \otimes_{\min} R)_\theta\) to \((\ell^2_\ell(L^2_\ell), \ell^2_\ell(L^2_r))_\theta\)

and the latter space can be classically identified with \(\ell^2(L^2_\ell, L^2_r)_\theta = \ell^2(\mathcal{L}_\theta)\).

Let \((e_k)\) be the canonical basis of \(R[\theta]\) (corresponding to \((e_{k1})\) or \((e_{k1})\)). Let \(z_i = \Sigma z_i(k)e_k\), and let

\[
x(k) = \Sigma_i z_i(k)x_i.
\]

By the preceding discussion, to prove our claim it suffices to show

\[
\Sigma \|z_i\|^2 \leq c(\theta)^2 \Sigma \|x(k)\|^2_{\ell_2^\theta}.
\]

Actually we will show that there is equality in the above. To verify this, we now introduce

\[
y(k) = \Sigma_i z_i(k) y_i
\]

so that

\[
\Sigma \langle y(k)x(k)\Omega, \Omega \rangle = \Sigma k \Sigma_i |z_i(k)|^2 \theta(1-\theta) = \theta(1-\theta) \Sigma \|z_i\|^2.
\]

We will use the fact that for any \((X_k)\) in \(\mathcal{L}\) and any \((Y_k)\) in \(\mathcal{R}\) we have

\[
|\Sigma \langle Y_k X_k \Omega, \Omega \rangle| \leq (\Sigma \|X_k\|^2_{\ell_2^\theta})^{1/2}(\Sigma \|Y_k\|^2_{R^2_{1-\theta}})^{1/2}.
\]

Indeed, we have

\[
|\Sigma \langle Y_k X_k \Omega, \Omega \rangle| \leq (\Sigma \|X_k \Omega\|^2)^{1/2}(\Sigma \|Y_k \Omega\|^2)^{1/2}
\]

\[
= (\Sigma \|X_k\|^2_{\ell_2^\theta})^{1/2}(\Sigma \|Y_k\|^2_{R^2_{1-\theta}})^{1/2}
\]

and also since \(Y_k X_k = X_k Y_k\)

\[
|\Sigma \langle Y_k X_k \Omega, \Omega \rangle| \leq (\Sigma \|X_k\|^2_{\ell_2^\theta})^{1/2}(\Sigma \|Y_k\|^2_{R^2_{1-\theta}})^{1/2},
\]

hence by the bilinear interpolation theorem follows (since \((R^2_\ell, R^2_r)_{1-\theta} = (R^2_{\ell}, R^2_r)_{1-\theta} = R_{1-\theta})\).
We will show that

\[(\Sigma \|y(k)\|^2_{L^2_{k\rightarrow 0}})^{1/2} \leq (1-\theta)^\theta\theta^{1-\theta}(\Sigma \|z_i\|^2)^{1/2}.\]

Let \( f_i \) be the \( R \)-valued analytic function defined on \( C \) by

\[f_i(z) = (1-\theta)^{1+z}\theta^{z} - ((1-\theta)^{1-\theta} - 1)y_i.\]

Note that \( f_i(1-\theta) = (1-\theta)^{\theta-1}(1-\theta)^{-1}y_i. \) Moreover, for any \( (\alpha_i) \in \ell_2(I) \), we have

\[
\begin{align*}
\|\Sigma \alpha_i f_i(z)\|_{R^2} &= (\Sigma |\alpha_i|^2)^{1/2} & \text{if} \ Re(z) = 0 \\
\|\Sigma \alpha_i f_i(z)\|_{R^2} &= (\Sigma |\alpha_i|^2)^{1/2} & \text{if} \ Re(z) = 1.
\end{align*}
\]

Hence

\[(1-\theta)\Sigma \|z_i\|^2 = \Sigma \langle y(k)x(k)\rangle \Omega, \Omega \]

\[
\leq (\Sigma \|x(k)\|_{L^2_0}^2)^{1/2}(\Sigma \|y(k)\|_{L^2_{k\rightarrow 0}}^2)^{1/2}
\]

\[
\leq (\Sigma \|x(k)\|_{L^2_0}^2)^{1/2}(1-\theta)^\theta\theta^{1-\theta}(\Sigma \|z_i\|^2)^{1/2}
\]

which, after a suitable division, yields \((12)\). This completes the proof of \((12)\), of the above claim, and of the sublemma. Note that an obvious modification of the proof of \((15)\) shows that the converse of \((12)\) also holds so \((12)\) is indeed an equality.

Remark. Arguing as in [25], it is easy to check that Theorem [4] remains valid when \( A \) is replaced by an exact subspace \( E \subset A \) with exactness constant \( \leq c \), provided the constant \( c(\theta) \) in \((9)\) is replaced by \( c(\theta)c \).

Remark. Note that \( c(\theta) \rightarrow 1 \) when either \( \theta \rightarrow 0 \) or \( \theta \rightarrow 1 \). In the cases \( \theta = 0 \) and \( \theta = 1 \), Theorem [4] is well known (cf. [9]). In that case, Theorem [4] still holds when \( A \) is replaced by an arbitrary subspace \( E \subset A \). However, when \( 0 < \theta < 1 \), some extra assumption (such as exactness) is necessary. Indeed, if we take \( \theta = 1/2 \), let \( (a_i) \) be the orthonormal basis of \( OH = (C, R)_{1/2} \), and let \( u \) be the identity map, we have \( \left\| \sum_{i=1}^n a_i^*a_i \right\|^{1/2} = \left\| \sum_{i=1}^n a_i^*a_i \right\|^{1/2} = n^{1/4} \)

but \( \sum_{i=1}^n \|a_i\|^2 = \sum_{i=1}^n \|a_i\|^2 = n \), which shows that \((12)\) fails. Similarly, the extension property valid when either \( \theta = 0 \) or \( \theta = 1 \) is no longer true in general, indeed this is closely related to the fact that \( R \) or \( C \) are injective operator spaces, while \( R[\theta] \) is not when \( 0 < \theta < 1 \).

Remark. In the preceding argument, the only delicate point is \((12)\). Note that actually, it is easy to show that equality holds in \((12)\). We chose to base the above proof of \((12)\) solely on complex interpolation to make it accessible to a reader unfamiliar with the Tomita–Takesaki theory. However, if one uses the latter theory, in the form made explicit by Shlyakhtenko in [30], it is very easy to explain why \((12)\) should be true. We now review this alternate approach.

Alternate proof of \((12)\). Let \( \varphi \) be the vacuum state, defined on \( B(\mathcal{F}) \) by \( \varphi(T) = \langle T\Omega, \Omega \rangle \). Let \( \xi_i = (1-\theta)^{-1/2}\lambda_i^{-1/2} \). Note that

\[(18) \quad x_i = (1-\theta)\lambda_i^{1/2}(\xi_i + \xi_i^*) ;\]
where \( z \) is the obvious analytic extension of \( t \to \sigma_t \). As is well known (cf. e.g. \([26, 27]\))

we have in this situation

\[
\|x\|_{\mathcal{L}_b} = \|x\|_{(L^2_b, L^2)} = \|\sigma_{-i\theta/2}(x)\|_{L^2} = \varphi(\sigma_{-i\theta/2}(x)\sigma_{-i\theta/2}(x))^{1/2}.
\]

Hence, we can write since \( \xi_j \lambda_j^{1/2} = \theta(1 - \theta)^{-1} \)

\[
\Sigma\|x(k)\|_{\mathcal{L}_b}^2 = \Sigma \varphi(\sigma_{-i\theta/2}(x(k))^{*}\sigma_{-i\theta/2}(x(k))) = \Sigma \|\sigma_{-i\theta/2}(x(k))\Omega\|^2
\]

\[
= \sum_k \left( \sum_j z_j(k)\sigma_{-i\theta/2}(x_j)\Omega \right)^2
\]

\[
= \sum_k \left( \sum_j z_j(k)\xi_j^{\theta} x_j \Omega \right)^2
\]

\[
= \sum_k \left( \sum_j z_j(k)\xi_j^{\theta}(1 - \theta)\lambda_j^{\theta/2} e_j \right)^2
\]

\[
= \sum_j \|z_j\|^2(\varphi(\theta))^{-2}.
\]

Hence, we obtain the announced equality

\[
(\Sigma\|z_j\|^2)^{1/2} = \varphi(\theta)(\Sigma\|x(k)\|_{\mathcal{L}_b}^2)^{1/2}.
\]

\[\Box\]

The converse of Theorem \([\ref{cbs}1]\) also holds, as follows. In the case \( \theta = 1/2 \), this was proved in \([\ref{cbs}25]\). We give a more direct argument.

**Proposition 3.** Let \( E \subset A \) be an operator space embedded in a \( C^* \)-algebra \( A \). Then any linear map \( u : E \to \ell_2 \) for which there are states \( f, g \) on \( A \) and a constant \( C \) such that

\[
\forall a \in E \quad \|ua\| \leq Cf(a^*a)^{(1-\theta)/2}g(aa^*)^{\theta/2}
\]

is completely bounded, with \( \|u\|_{cb} \leq C \), as a mapping into \( \ell_2 \) equipped with the operator space structure of \( (C, R)_\theta \).
Proof. Let $F = (C,R)_\theta$ (recall this is isometric to $\ell_2$). We will show that for any $a = (a_{ij})$ in the unit ball of $M_n(E)$ we have $\|u(a_{ij})\|_{M_n(F)} \leq C$. Let $(T_k)$ be an orthonormal basis of $F = (C,R)_\theta$. Let $u_k : E \to \mathbb{C}$ be defined by $u(a) = \Sigma u_k(a)T_k$. Let us denote $\gamma_k = \sum_{ij} u_k(a_{ij})e_{ij} \in M_n$. Using $M_n(F) \simeq M_n \otimes F$, the matrix $(u(a_{ij}))$ can then be rewritten as

$$\Sigma e_{ij} \otimes u(a_{ij}) = \sum_k \gamma_k \otimes T_k.$$ 

Let $p, p'$ be defined by $1 - \theta = 1/p$ and $\theta = 1/p'$. By definition of $(C,R)_\theta$, we have (see the identity (8.5), p. 83 in [20], but note that our space $R[\theta]$ corresponds to the space denoted by $R(1 - \theta)$ in [20]):

$$(20) \quad \|(u(a_{ij}))\|_{M_n(F)} = \sup \left\{ \left( \sum_k \|s\gamma_k t\|_2^2 \right)^{1/2} \right\}$$

where $\| \cdot \|_2$ denotes the Hilbert–Schmidt norm on $M_n$ and where the supremum runs over all pairs $(s,t)$ in $(M_n)_+ \times (M_n)_+$ such that $\text{tr} s^{2p'} \leq 1$ and $\text{tr} t^{2p} \leq 1$.

Let $x_{ij}$ denote the $(ij)$-entry of $s\gamma t$ (so that $x_{ij} = \sum_k s_{ik}a_{kl}t_{lj}$). We have

$$\sum_k \|s\gamma_k t\|_2^2 = \sum_{ij} \|u(x_{ij})\|_2^2.$$ 

We claim that (19) implies

$$(21) \quad \sum_{ij} \|u(x_{ij})\|_2^2 \leq C^2.$$ 

By (20), this claim implies that $\|u\|_{cb} \leq C$, thus completing the proof. To show this claim, we may as well assume (replacing $a$ by $v_1av_2$ for suitable unitaries $v_1, v_2$ in $M_n$, and using (20) again) that $s$ and $t$ are diagonal matrices. We then have

$$\sum_{ij} \|x_{ij}\|^2 = \sum_{ij} s_{ii}^2 \|u(a_{ij})\|^2 t_{jj}^2$$

hence, since $\Sigma s_{ii}^{2p'} \leq 1$ and $\Sigma t_{jj}^{2p} \leq 1$, we have using (19)

$$\sum_{ij} \|x_{ij}\|^2 \leq C^2 \|\Sigma f(a_{ij}^*a_{ij})^{1-\theta} g(a_{ij}a_{ij}^*)^\theta e_{ij}\|_{B(\ell_p^2, \ell_p^2)}.$$ 

Thus, the above claim follows from the next lemma.

Lemma 4. Let $f, g$ be states on a $C^*$-algebra $A$. We have then for any $n \geq 1$ and any $a$ in $M_n(A)$

$$\left\| \sum_j f(a_{ij}^*a_{ij})^{1-\theta} g(a_{ij}a_{ij}^*)^\theta e_{ij} \right\|_{B(\ell_p^2, \ell_p^2)} \leq \|a\|_{M_n(A)}.$$ 

Proof. To prove Lemma 4, we first check that, if $\|a\|_{M_n(A)} \leq 1$, then $\alpha_0(i,j) = f(a_{ij}^*a_{ij})$ and $\alpha_1(i,j) = g(a_{ij}a_{ij}^*)$ satisfy

$$(22) \quad \sup_j \Sigma_i \alpha_0(i,j) \leq 1 \quad \text{and} \quad \sup_i \Sigma_j \alpha_1(i,j) \leq 1.$$ 

Indeed, we have for any fixed $j$

$$\sum_i f(a_{ij}^*a_{ij}) \leq \sum_i a_{ij}^*a_{ij} = \left\| \sum_i a_{ij} \otimes e_{ij} \right\| \leq \|a\|_{M_n(A)},$$
and similarly for the other sum. Now let \( \alpha_0(i, j), \alpha_1(i, j) \) be any \( n \times n \) matrices with nonnegative entries satisfying (22). Then it is well known and easy to check by interpolation (see e.g. [24] for more on this topic) that

\[
\left\| \sum_{ij} \alpha_0(i, j)^{1-\theta} \alpha_1(i, j)^{\theta} e_{ij} \right\|_{B(t_0^p, t_0^p)} \leq 1,
\]

or equivalently for any \( s_i, t_j \geq 0 \) with \( \Sigma s_i \leq 1, \Sigma t_j \leq 1 \) we have

\[
\sum_{ij} \alpha_0(i, j)^{1-\theta} \alpha_1(i, j)^{\theta} s_i^{1/p} t_j^{1/p} \leq 1.
\]

Indeed, by Hölder’s inequality (recall \( 1/p = 1 - \theta \) and \( 1/p' = \theta \)) this is

\[
\leq \left( \sum_{ij} \alpha_0(i, j) t_j \right)^{1-\theta} \left( \sum_{ij} \alpha_1(i, j) s_i \right)^\theta \leq \left( \sum_j t_j \right)^{1-\theta} \left( \sum_i s_i \right)^\theta \leq 1.
\]

Remark. When \( \lambda_i = 1 \) for all \( i \) and \( \theta = 1/2 \), we have \( x_i = (1/2)x'_i \) where \( x'_i = \ell_i + \ell'_i \).

Then \( (x'_i) \) is a free circular (i.e. free analogue of complex Gaussian) family in Voiculescu’s sense (cf. [33]). It is easy to see in this case that for any finite sequence \( (a_i) \) in \( B(\ell_2) \) we have

\[
(1/2) \left\| \sum a_i \otimes x'_i \right\| \leq \max \left\{ \left\| \sum a_i^* a_i \right\|^{1/2}, \left\| \sum a_i a_i^* \right\|^{1/2} \right\} \leq \left\| \sum a_i \otimes x'_i \right\|.
\]

Thus \( \text{span}[x'_i] \) is completely isomorphic to the space \( R \cap C \) studied in [14] (see also [22, p. 209]). The notation \( R \cap C \) comes from the fact that if we consider \( \delta_i = e_{1i} \oplus e_{i1} \) in \( R \oplus C \) then we have for \( (a_i) \) as above

\[
\left\| \sum a_i \otimes \delta_i \right\| = \max \left\{ \left\| \sum a_i a_i^* \right\|^{1/2}, \left\| \sum a_i^* a_i \right\|^{1/2} \right\};
\]

so that

\[
\text{span}[\delta_i] = \{(x, t^i x) \mid x \in R\} \subset R \oplus C
\]

appears as the diagonal in \( R \oplus C \). Let \( \mathcal{L} \) be again the von Neumann algebra generated by \( \{x'_i\} \).

We claim (see [14], see also [22, p. 209]) that there is a normal c.b. projection \( P : \mathcal{L} \to \text{span}[x'_i] \) with \( \|P\|_{cb} \leq 2 \). Indeed, let \( Q : \mathcal{F} \to \mathcal{F} \) (resp. \( Q' : \mathcal{F} \to \mathcal{F} \)) be the orthogonal projection onto \( \text{span}[e_i] \mid i \in I \) (resp. \( \text{span}[e'_i] \mid i \in I \)) viewed as a subspace of \( H \), itself embedded into \( \mathcal{F} \) via tensors of degree 1. Then the map \( P \) defined by

\[
(23) \quad \forall T \in \mathcal{L} \quad P(T) = \ell(Q(T\Omega)) + \ell(Q'(T^*\Omega))^*
\]

is the announced projection (see more generally Lemma [4] below). Therefore \( (R \cap C)^* \) embeds completely isomorphically into \( \mathcal{L}_* \) and in the present special case \( \mathcal{L} \) is semifinite (and actually finite).
Consider now a family $\xi = (\xi_i)$ with $\xi_i > 0$. Let $\delta_i^x = e_{1i} \oplus \xi_i e_{11} \in R \oplus C$. Thus if $(a_i)$ is as before, we have
\[
\left\| \sum a_i \otimes \delta_i^x \right\| = \max \left\{ \left\| \sum a_i a_i^* \right\|^{1/2}, \left\| \sum \xi_i^2 a_i a_i^* \right\|^{1/2} \right\},
\]
and also
\[
(1/2) \left\| \sum a_i \otimes (\ell_i + \xi_i \ell'_i) \right\| \leq \left\| \sum a_i \otimes \delta_i^x \right\| \leq \left\| \sum a_i \otimes (\ell_i + \xi_i \ell'_i) \right\|.
\]
Thus we have completely isomorphically
\[
\text{span}[\ell_i + \xi_i \ell'_i] \simeq \text{span}[\delta_i^x].
\]
In particular, if $(\xi_i)$ is as in (18) then $\text{span}[x_i] \simeq \text{span}[\delta_i^x]$ (completely isomorphically). Note that $\text{span}[\delta_i^x]$ can also be viewed as the graph of the unbounded operator $\Lambda : R \rightarrow C$ taking $e_{11}$ to $\xi_i e_{11}$, with Dom$(\Lambda) = \{ x = \Sigma x_i e_{11} \in R \mid \Sigma |\xi_i x_i|^2 < \infty \}$. More precisely, if we denote
\[
G(\Lambda) = \{ (x, Ax) \mid x \in \text{Dom}(\Lambda) \},
\]
then we have $\text{span} \delta_i^x \simeq G(\Lambda)$ completely isometrically. (Note: In analogy with $R \cap C$, it would be natural to denote $G(\Lambda)$ by $R \cap \Lambda^{-1}(C)$ but we prefer not to use this notation.)

The next result (extending the case $\xi_i = 1 \forall i$) is easy to deduce from Shlyakhtenko’s [30].

**Lemma 5.** Let $\mathcal{L}$ be the von Neumann algebra generated by the family $(x_i)$ defined by (18). Then the mapping $P : \mathcal{L} \rightarrow \mathcal{L}$ defined by (20) is a normal c.b. projection from $\mathcal{L}$ onto $\text{span}[x_i]$ with $\|P\|_{cb} \leq 2$. In particular $\text{span}[x_i]$ embeds completely isometrically into $\mathcal{L}_\ast$.

**Proof.** We first claim that $T \rightarrow P(T)$ is c.b. on $B(\mathcal{F})$ with cb-norm $\leq 2$. This is easy to see. Indeed, consider $P_1 : B(\mathcal{F}) \rightarrow B(\mathcal{F})$ and $P_2 : B(\mathcal{F}) \rightarrow B(\mathcal{F})$ defined by
\[
P_1(T) = \ell(Q(T\Omega)) \quad \text{and} \quad P_2(T) = \ell(Q'(T^*\Omega))^\ast
\]
so that $P(T) = P_1(T) + P_2(T)$. We will show that $\|P_1\|_{cb} \leq 1$ and $\|P_2\|_{cb} \leq 1$. Indeed, the ranges of $P_1$ and $P_2$ are respectively $\text{span}[\ell_i]$ and $\text{span}[\ell'_i]$. Assuming $I = \mathbb{N}$ for simplicity, we have (see e.g. [22] p. 176-177)
\[
\text{span}[\ell_i] \simeq C \quad \text{and} \quad \text{span}[\ell'_i] \simeq R.
\]
Note that we have obviously (recall $\varphi$ is the vacuum state)
\[
\|P_1(T)\| = \|Q(T\Omega)\| \leq \|T\Omega\| = \langle T^*T\Omega, \Omega \rangle^{1/2} = \varphi(T^*T)^{1/2}
\]
\[
\|P_2(T)\| = \|Q'(T^*\Omega)\| \leq \|T^*\Omega\| = \langle TT^*\Omega, \Omega \rangle^{1/2} = \varphi(TT^*)^{1/2}
\]
Hence, $\|P_1\|_{cb} \leq 1$ and $\|P_2\|_{cb} \leq 1$ (and a fortiori $\|P\|_{cb} \leq 1$) follow using (1) and (2). Thus it suffices to prove that the restriction of $P$ to $\mathcal{L}$ is a projection onto $\text{span}[x_i]$. We know (see [30]) that the map $T \rightarrow T\Omega$ is faithful (i.e. injective) on $\mathcal{L}$. Let $T$ be a polynomial in $x_i, x'_i$ ($i \in I$). We can write a priori
\[
T\Omega = \sum t_i e_i + \sum t'_i e'_i + r
\]
where $r$ is a sum of tensors of degree $> 1$. By [30] Lemma 3.2], we know that the (antilinear) map $S$ taking $T\Omega$ to $T^*\Omega$ takes $r$ to another sum $r'$ of tensors of degree $> 1$ in $\mathcal{F}$. Moreover, since $(\ell_i + \xi_i \ell'_i)\Omega = e_i$ and $(\ell_i + \xi_i \ell'_i)^\ast\Omega = \xi_i e'_i$, we have
\[
Se_i = \xi_i e'_i \quad \text{and} \quad Se'_i = \xi_i^{-1} e_i,
\]
and hence
\[
T^*\Omega = \sum \xi_i e'_i + \sum \xi_i^{-1} e_i + r'.
\]
Therefore we have $Q(T\Omega) = \sum t_i e_i$, $Q'(T^*\Omega) = \sum \xi_i \ell_i e'_i$, and we finally obtain
\[
P(T) = \sum t_i \ell_i(e_i) + \sum t_i \xi_i \ell_i(e'_i)^* = \sum t_i(\ell_i + \xi_i \ell_i^*) \in \text{span}[x_i].
\]
In particular, we find $P(\ell_i + \xi_i \ell_i^*) = \ell_i + \xi_i \ell_i^*$ and hence $P(x_i) = x_i$ for all $i$. This proves that $P|_\mathcal{L}$ is a projection from $\mathcal{L}$ onto $\text{span}[x_i]$.

Note that, by (24), if
\[
0 < \inf \xi_i \leq \sup \xi_i < \infty
\]
then $\text{span}[x_i]$ (or equivalently $G(\Lambda)$) is again completely isomorphic to $R \cap C$ and hence its dual embeds in $M_\ast$ for some semifinite $M$. We will now show that if either $\inf \xi_i = 0$ or $\sup \xi_i = \infty$, then such an embedding $G(\Lambda)^* \subset M_\ast$ with $M$ semifinite exists if and only if we have for some $\varepsilon > 0$
\[
\sum_{i: \xi_i < \varepsilon} \xi_i^2 + \sum_{i: \xi_i^{-1} > \varepsilon} \xi_i^{-2} < \infty.
\]

Let $M$ be a von Neumann algebra with predual $M_\ast$. As already mentioned at the end of [19], Theorem 11 and its converse (Proposition 3) admit the following corollary: for any $0 < \theta < 1$, the space $R[\theta]$ does not embed completely isomorphically into $M_\ast$ when $M$ is semifinite. It can be shown (see the appendix below) that $R[\theta]$ is completely isometric to a quotient of a subspace $S$ of $R \oplus C$. Thus $R[\theta]^*$ embeds in $S^\ast$, and hence to embed $R[\theta]^*$ into $M_\ast$ it suffices to embed $S^\ast$ into $M_\ast$. Indeed, Marius Junge announced that if $S$ is any subspace of $R \oplus C$ then $S^\ast$ embeds completely isomorphically into $M_\ast$ for some von Neumann algebra $M$. Let $S \subset R \oplus C$ be such a subspace. For convenience, let us assume that $S$ is not completely isomorphic to either $R \cap C$ or $C R \parallel C$. Then Q. Xu [22] observed the fact (presumably known to Junge) that $S$ can be rewritten (up to complete isomorphism) as a direct sum $H_r \parallel \tilde{S} \oplus K_c$ where $H_r, K_c$ are suitable Hilbert spaces equipped respectively with the row and column operator space structure, and where $\tilde{S} \subset R \parallel C$ is the (closed) graph of a (closed) densely defined operator $\Lambda: R \rightarrow C$, injective (on its domain) and with dense range. As explained in the appendix, the fact that $(\tilde{S})^\ast$ embeds into $M_\ast$ for some suitable $M$ can be deduced from the basic properties of Shylakhtenko’s generalized free circular elements, already used in [22]. The typical $M$ is then not semifinite. The next result shows that this cannot be avoided.

**Theorem 6.** Let $S \subset R \parallel C$ be an arbitrary infinite dimensional subspace. Then there is a semifinite von Neumann algebra $M$ such that $S^\ast$ embeds completely isomorphically into $M_\ast$ iff $S$ is completely isomorphic to one of the spaces
\[
R, C, R \parallel C, R \cap C, R \oplus (R \cap C), C \oplus (R \cap C), R \oplus (R \cap C) \parallel C.
\]

**Remark.** As is well known, we have $R^\ast \simeq C$ and $C^\ast \simeq R$, so that $R^\ast$ and $C^\ast$ embed (completely isometrically) in $K^\ast \simeq S_1$ (the trace class). Consequently $(R \parallel C)^\ast \simeq R \parallel C$ embeds completely isomorphically into $S_1 \parallel S_1 \simeq S_1$. The space $R \cap C$ is less trivial, but it was shown by Lust-Piquard and the author (see [10] or [22, p. 194]) that $(R \cap C)^\ast$ embeds completely isomorphically into the most classical $L_1$-space, namely $L_1([0, 1], dt)$. Therefore, we do have $S^\ast \subset M_\ast$ with $M$ semifinite for any of the 7 spaces in the above list.

Our task will now be to show that the latter list is complete.

**Remark.** Consider a (closed) subspace $S \subset R \parallel C$. As explained above, we can write
\[
S \simeq H_r \oplus G(\Lambda) \oplus K_c
\]
with
\[
G(\Lambda) = \{(x, \Lambda x) \mid x \in \text{Dom}(\Lambda)\} \subset R \parallel C
\]
where $\text{Dom}(\Lambda) \subset R$ is a dense subspace and $\Lambda: \text{Dom}(\Lambda) \rightarrow C$ is a closed unbounded operator with zero kernel and dense range.
By the polar decomposition of $\Lambda$ and the “homogeneity” of $R$ and $C$ (in the sense of [22, p. 172]), we may assume that $\Lambda > 0$. Using the spectral theory of Hermitian operators, we can then decompose $\Lambda$ as $\Lambda = \Lambda_1 + \Lambda_2$ with $0 < \Lambda_1 \leq 1$ and $\Lambda_2 \geq 1$, and consequently we may decompose

$$G(\Lambda) \simeq G(\Lambda_1) \oplus G(\Lambda_2)$$

where $\Lambda_1, \Lambda_2$ are unbounded self-adjoints of the same form as $\Lambda$ but in addition such that $\Lambda_1$ and $\Lambda_2^{-1}$ are bounded with norm $\leq 1$. The key to the preceding theorem then lies in the next statement.

**Lemma 7.** Consider $\Lambda > 0$ with $\|\Lambda\| \leq 1$ and $\Lambda^{-1}$ unbounded. Let $E(\varepsilon)$ be the spectral projection of $\Lambda$ relative to $(0, \varepsilon)$, so that $0 \neq \|\Lambda E(\varepsilon)\| \leq \varepsilon$ for any $\varepsilon > 0$. Assume that there is a semifinite $M$ such that $G(\Lambda)^* \Lambda G(\Lambda)$ embeds completely isomorphically into $M_*$. Then, for $\varepsilon > 0$ small enough, $\Lambda E(\varepsilon)$ must be Hilbert–Schmidt.

**Proof.** The basic idea is similar to the one in [19] but the details are more complicated. By assumption, we have an embedding $j: G(\Lambda)^* \subset M_*$. Let $u = j^* : M \to G(\Lambda)$. We may assume that $\|u\|_{cb} \leq 1$ and that there is a constant $c$ such that for any $n$ and any $a$ in $M_n(G(\Lambda))$ with $\|a\| < 1$, there is $\tilde{a}$ in $M_n(M)$ with

$$\|\tilde{a}\| < c$$

such that $(I \otimes u)(\tilde{a}) = a$. Note that $u$ is “normal”, i.e. is $(\sigma(M, M_*), \sigma(G(\Lambda), G(\Lambda)^*))$ continuous. The map $u$ can clearly be rewritten as $ux = (vx, \Lambda vx)$ with

$$\|v: M \to R\|_{cb} \leq 1 \quad \text{and} \quad \|\Lambda v: M \to C\|_{cb} \leq 1.$$  

Let $\tau$ be a semifinite faithful normal trace on $M$. Since $v$ and $\Lambda v$ are normal, arguing as in [19], we find normal states $f, g$ on $M$ such that

$$\|vx\| \leq f(xx^*)^{1/2}$$

$$\|\Lambda vx\| \leq g(x^*x)^{1/2}$$

for all $x$ in $M$.

We may view $f, g$, as elements of $L_1(\tau)$, i.e. positive unbounded operators affiliated to $M$ such that $\tau(f) = \tau(g) = 1$, and consequently we will write $f(\cdot) = \tau(f\cdot)$ and $g(\cdot) = \tau(g\cdot)$. Fix $\alpha > 0$. Let $p$ (resp. $q$) be the spectral projection of $f$ (resp. $g$) associated to $(\alpha^{-1}, \alpha]$, so that in $M_*$, we have $\alpha^{-1}p \leq pf \leq \alpha p$ and $\alpha^{-1}q \leq gq \leq \alpha q$. Choosing $\alpha = \alpha(\delta)$ large enough we can ensure that moreover

$$\|f(1 - p)\|_{M_*} < \delta^2 \quad \text{and} \quad \|g(1 - q)\|_{M_*} < \delta^2.$$  

Moreover, we have $\alpha^{-1}\tau(p) \leq \tau(f) = 1$ and $\alpha^{-1}\tau(q) \leq \tau(g) = 1$, and hence

$$\tau(p) \leq \alpha \quad \text{and} \quad \tau(q) \leq \alpha.$$  

We then define

$$v_\delta x = v(pqx).$$

Note that

$$\|v_\delta x\| \leq f(pqx^*)^{1/2} \leq \sqrt{\alpha} \tau(pqx^*)^{1/2} = \sqrt{\alpha} \tau(qx^*px)^{1/2} \leq \sqrt{\alpha} \tau(qx^*x)^{1/2},$$

where the last equality holds by the trace property, so that by [1]

$$\|v_\delta: M \to C\|_{cb} \leq (\alpha \tau(q))^{1/2} \leq \alpha.$$
On the other hand, we have
\[(31)\] \[vx - v_3x = v_1x + v_2x\]
with \(v_1x = v(x(1 - q))\) and \(v_2x = v((1 - p)xq)\). Note that
\[\|\Lambda v_1\| \leq g((1 - q)x^*x(1 - q))^{1/2} = \tau(g(1 - q)x^*x)^{1/2}\]
hence by (4)
\[(32)\] \[\|\Lambda v_1\| : M \rightarrow C\|_{cb} \leq \delta.\]
Similarly, we have
\[(33)\] \[\|v_2x\| \leq \tau(f(1 - p)x^*)^{1/2}\]
hence by (2)
\[\|v_2\| \leq \|v_2\| : M \rightarrow R\|_{cb} \leq \delta.\]
We now turn to the following

**Claim 1.** \(\varepsilon > 0\) can be chosen so that \(\|\Lambda E(\varepsilon)\|_2 \leq 1\).

For each integer \(n \geq 1\), let
\[\pi_n^2 = \sup \left\{ \left( \sum_{i=1}^{n} \|\Lambda e_i\|^2 \right)^{1/2} \right\}\]
where the supremum runs over all possible orthonormal \(n\)-tuples \((e_1, \ldots, e_n)\) in the range of the projector \(E(\varepsilon)\). Note that for any operator \(w: M \rightarrow E(\varepsilon)\) and for any \(a_1, \ldots, a_n\) in \(M\) we have
\[(34)\] \[\left( \sum_{i=1}^{n} \|\Lambda w a_i\|^2 \right)^{1/2} \leq \pi_n^2 \|w\| \|\sum a_i^* a_i\|^{1/2}.\]
Indeed, let \(T: [e_1, \ldots, e_n] \rightarrow M\) be the map defined by \(T e_i = a_i\). Note \(\|T\| \leq \|\Sigma a_i^* a_i\|^{1/2}\). Let \(F = \text{span}[wa_i]\)
\[(35)\] \[\sum \|\Lambda w a_i\|^2 = \sum_{i=1}^{n} \|\Lambda w T e_i\|^2 = \|\Lambda w T\|_2^2 \leq \|\Lambda_{1F}\|_2 \|w\|^2 \|T\|^2\]
and since \(\dim F \leq n\) we have \(\|\Lambda_{1F}\|_2 \leq \pi_n^2\) hence \(\Sigma \|\Lambda w a_i\|^2 \leq (\pi_n^2)^2 \|w\|^2 \|\Sigma a_i^* a_i\|\), which establishes (44).

**Claim 2.** If \(\pi_n^2 \leq 1\), then we have
\[(36)\] \[\pi_n^2 \leq (\varepsilon \alpha + \delta + \delta \pi_n^2)c.\]
To prove this, consider \((e_1, \ldots, e_n)\) in \(E(\varepsilon)\) and let
\[a = \sum_{i=1}^{n} e_i \otimes (e_i, \Lambda e_i) \in M_n(G(\Lambda)).\]
We have
\[\|a\| = \max \left\{ \left\| \sum e_i \otimes e_i \right\|_{C_n \otimes \min R}, \left\| \sum e_i \otimes \Lambda e_i \right\|_{C_n \otimes \min C} \right\}\]
hence (since \(\pi_n^2 \leq 1\))
\[\|a\| = \max \left\{ 1, \left( \sum \|\Lambda e_i\|^2 \right)^{1/2} \right\} \leq 1.\]
By (29), there is \( \tilde{a} \) in \( M_v(M) \) with \( \| \tilde{a} \| \leq c \), such that \( (I \otimes u)(\tilde{a}) = a \). Clearly we may assume \( \tilde{a} = \sum_{i} e_i \otimes a_i \) with \( a_i \in M \) such that \( e_i = v a_i \). Note that

\[
(37) \quad \left\| \sum a_i a_i^\| \right\| = \| \tilde{a} \| \leq c.
\]

Note that since \( v = v_3 + v_1 + v_2 \), we have \( e_i = v_3 a_i + v_1 a_i + v_2 a_i \), hence if we let \( \Lambda_\varepsilon = E(\varepsilon) \Lambda = \Lambda E(\varepsilon) \), we have

\[
(38) \quad \left( \sum \| \Lambda e_i \|^2 \right)^{1/2} \leq \left( \sum \| \Lambda_\varepsilon v_3 a_i \|^2 \right)^{1/2} + \left( \sum \| \Lambda_\varepsilon v_1 a_i \|^2 \right)^{1/2} + \left( \sum \| \Lambda_\varepsilon v_2 a_i \|^2 \right)^{1/2}.
\]

By (31), (37) and (39) we have

\[
\left( \sum \| \Lambda_\varepsilon v_3 a_i \|^2 \right)^{1/2} \leq \varepsilon \left( \sum \| v_3 a_i \|^2 \right)^{1/2} \leq \varepsilon (\alpha \tau(q))^{1/2} c
\]

and also by (38)

\[
\left( \sum \| \Lambda_\varepsilon v_1 a_i \|^2 \right)^{1/2} \leq \left( \sum \| \Lambda v_1 a_i \|^2 \right)^{1/2} \leq \delta c.
\]

Finally, by (38), (31) and (37) we have (recall \( \Lambda_\varepsilon = \Lambda E(\varepsilon) \))

\[
\left( \sum \| \Lambda_\varepsilon v_2 a_i \|^2 \right)^{1/2} \leq \pi_2^n \| E(\varepsilon) v_2 \| c \leq \delta c \pi_2^n.
\]

Recapitulating, we can now deduce (38) from (39), and Claim 2 follows.

We can now prove Claim 1.

We assume \( \varepsilon < 1/2 \). We will argue by contradiction. Assume that \( \sup_{m} \pi_2^n > 1 \). We will show that this is impossible. Let \( n + 1 \) be the smallest integer such that \( \pi_2^{n+1} > 1 \). Note that \( \pi_2^n \leq 1 \) and \( n \geq 1 \) (because \( \pi_2^1 \leq \varepsilon < 1 \)). Moreover, we have obviously \( \pi_2^{n+1} \leq \pi_2^n + \varepsilon \leq \pi_2^n + 1/2 \), hence \( \pi_2^n > 1/2 \). But now if we choose \( \delta \) so that \( \delta c < 1/2 \), (39) implies

\[
\pi_2^n \leq c(\varepsilon \alpha + \delta) + (1/2)\pi_2^n
\]

hence

\[
\pi_2^n \leq 2c(\varepsilon \alpha + \delta),
\]

so that since \( \pi_2^n > 1/2 \) we obtain

\[
1/2 \leq 2c(\varepsilon \alpha + \delta).
\]

But now if we choose \( \delta = 1/8 c \) this implies

\[
(39) \quad 1/4 \leq 2c \varepsilon \alpha,
\]

and here \( \alpha = \alpha(\delta) \) is determined by \( \delta \) but \( \varepsilon \) can still be made arbitrarily small. Thus we reach a contradiction, proving that \( \sup_{m} \pi_2^n \leq 1 \) for any \( \varepsilon < 1/2 \) for which (31) fails. This proves Claim 1 and completes the proof of Lemma 4.

**Proof of Theorem 6.** By Xu’s result [28], we are reduced to \( S = G(\Lambda) \) for \( \Lambda > 0 \) with dense range. By [28], we may assume that either \( \Lambda \) or \( \Lambda^{-1} \) has norm \( \leq 1 \). But observe that if \( \| \Lambda^{-1} \| \leq 1 \)

\[
G(\Lambda^{-1}) = \{(x, \Lambda^{-1} x) \mid x \in C \} = \{(\Lambda y, y) \mid y \in \text{Dom}(\Lambda) \} \subset C \oplus R
\]

hence \( G(\Lambda^{-1}) \cong G(\Lambda) \) since the first is obtained from the second via the mapping \( (x, y) \to (y, x) \) which is obviously a complete isometry from \( C \oplus R \) to \( R \subset C \). In particular, \( G(\Lambda^{-1}) \) embeds in \( M_v(M) \) iff \( G(\Lambda) \) embeds in \( M_v(M) \). Thus to conclude we may as well assume that \( \| \Lambda \| \leq 1 \). But then Lemma 4 shows that for \( \varepsilon \) small enough we have a decomposition \( R = H_c \oplus H_c^\perp \) and \( \Lambda = \Lambda_c \oplus \Lambda_c^\perp \) with \( \| \Lambda_c \| \leq 1 \). Clearly this implies \( G(\Lambda) \cong G(\Lambda_c) \oplus G(\Lambda_c^\perp) \) but since \( \| \Lambda_c \| \cong (H_c^\perp)_c \cap (H_c^\perp)_c \). This completes the proof of Theorem 6.
Remark. It may be worthwhile to point out that in Lemma 4 even if we know that $G(Λ)^*$ is completely c-isomorphic to a subspace of $M_c$ with a fixed $c$, the $ε$ given by Lemma 4 may be arbitrarily small, and this happens even for $M$ finite. Indeed, for the relevant examples, consider a free circular sequence $(x'_i)$ on $(M, τ)$ (with $τ$ a normalized trace) and a projection $p$ that is free from that family and such that $τ(p) = ε (K3)$. Then $\text{span}[px_i']$ provides the required phenomenon.

In the next statement, we observe that Xu’s decomposition for subspaces of $R ⊕ C$ leads to an easy proof of a result due to T. Oikhberg (with an improved bound), as follows.

**Theorem 8.** Let $S ⊂ R ⊕ C$ be a closed subspace. If there is a completely bounded projection $P: R ⊕ C → S$ then there are Hilbert spaces $H, K$ such that $S ≃ H_r ⊕ K_c$. Moreover there is a numerical constant $C$ such that $d_{cb}(S, H_r ⊕ K_c) ≤ C \|P\|_{cb}$.

**Proof.** By Xu’s decomposition and the above remarks, it suffices to prove this for $S = G(Λ)$ with $0 < Λ$ and $\|Λ\| ≤ 1$. Then the projection $P$ can be written as

$$∀(x, y) ∈ R ⊕ C \quad P(x, y) = (αx + βy, Λ(αx + βy))$$

where $α ∈ CB(R, R)$ and $β ∈ CB(C, R)$. By restricting $P$, we find

$$\max\{\|α\|_{CB(R, R)}, \|Λα\|_{CB(R, C)}\|β\|_{CB(C, R)}, \|Λβ\|_{CB(C, C)}\} ≤ \|P\|_{cb}.$$ (40)

Moreover since $P$ is a projection onto $G(Λ)$ we have for any $x$ in $R$

$$αx + βΛx = x$$

hence

$$Λα + ΛβΛ = Λ$$

which implies by (11) and (10) (since we assume $Λ ≤ 1$)

$$\|Λ\|_{CB(R, C)} = \|Λ\|_2 ≤ \|Λα\|_2 + \|ΛβΛ\|_2 = \|Λα\|_{CB(R, C)} + \|Λβ\|_{CB(C, R)} \leq 2 \|P\|_{cb}.$$ (10)

Thus we conclude

$$\|Λ\|_{CB(R, C)} ≤ 2 \|P\|_{cb}$$

and hence the map $u: x → (x, Λx)$ is a complete isomorphism between $R$ and $G(Λ)$ with

$$d_{cb}(R, G(Λ)) ≤ \|u\|_{cb} \|u^{-1}\|_{cb} ≤ 2 \|P\|_{cb}. \quad \Box$$

**Remark.** The preceding statement yields a rather satisfactory estimate in the following result from [25]: If an operator space $E$ is exact as well as its dual, then there are Hilbert spaces, $H, K$ such that $E ≃ H_r ⊕ K_c$ and moreover

$$d_{cb}(E, H_r ⊕ K_c) ≤ 2^{5/2}ex(E)ex(E^*)$$

where $ex(E)$ denotes the exactness constant of $E$. This seems rather sharp when $ex(E)ex(E^*)$ is large: Consider for instance the case $E = OH_{n1}$, we have then (cf. [19] p. 336) $ex(E) = ex(E^*) ≃ n^{1/4}$ but on the other hand it is easily checked that

$$d_{cb}(OH_{n1}, H_r ⊕ K_c) ≃ n^{1/2} ≃ ex(E)ex(E^*).$$

**Appendix**

In this appendix, we will reprove Junge’s result [10] that $OH$ embeds completely isomorphically into a non-commutative $L_1$-space. The main idea is the same as his, but our exposition is shorter and makes more transparent the relationship between the methods from [10] and [25].
We base the argument on the complex interpolation method instead of the Pusz–Woronowicz formula. Actually, there is nothing mysterious there: indeed the “purification of states” associated in [26] (see also [27, 34, 35, 36]) to a pair of faithful states (ϕ, ψ) on a C∗-algebra A is known to be very closely related to the complex interpolation space (A₀, A₁)₁/₂ where the Hilbert spaces A₀, A₁ are obtained by completing A for the norms

\[ \|x\|_{A₀} = (\varphi(x^*x))^{1/2}, \quad \|x\|_{A₁} = (\psi(xx^*))^{1/2}. \]

This close connection has been explored in depth notably by B. Simon, Uhlman, Peetre and probably others, besides Pusz and Woronowicz.

The proof rests on the following basic fact which had been known to the author (and probably others, besides Pusz and Woronowicz).

**Proposition A1.** \( OH \) is completely isometric to a quotient of a subspace of \( R \oplus C \).

**Proof.** Let \( \mu \) be the harmonic measure of the point \( z = 1/2 \) in the strip \( S = \{ z \in \mathbb{C} \mid 0 < \text{Re}(z) < 1 \} \). Recall that \( \mu \) is a probability measure on \( \partial S \) such that \( f(1/2) = \int f \, d\mu \) whenever \( f \) is a bounded harmonic function on \( S \) extended non-tangentially to \( \overline{S} \). Obviously \( \mu \) can be written as \( \mu = 2^{-1}(\mu₀ + \mu₁) \) where \( \mu₀ \) and \( \mu₁ \) are probability measures supported respectively by

\[ \partial₀ = \{ z \mid \text{Re}(z) = 0 \} \quad \text{and} \quad \partial₁ = \{ z \mid \text{Re}(z) = 1 \}. \]

Let \( (A₀, A₁) \) be a compatible pair of Banach spaces. We first need to describe \( (A₀, A₁)₁/₂ \) as a quotient of a subspace of \( L₂(\mu₀; A₀) \oplus L₂(\mu₁; A₁) \). The classical argument for this is as follows.

We denote by \( \mathcal{F}(E₀, E₁) \) the set of all bounded continuous functions \( f : \overline{S} \to E₀ + E₁ \) which are holomorphic on \( S \) and such that \( f|_{\partial₀} \) and \( f|_{\partial₁} \) are bounded continuous functions with values respectively in \( E₀ \) and \( E₁ \).

We start by showing that for any \( x \in (A₀, A₁)₁/₂ \) we have

\[ \|x\|_{(A₀, A₁)₁/₂} = \inf \{ \max \{ \|f\|_{L₂(\mu₀; A₀)}, \|f\|_{L₂(\mu₁; A₁)} \} \mid f ∈ \mathcal{F}(A₀, A₁) \} \]

where the infimum runs over all \( f \) in \( \mathcal{F}(A₀, A₁) \) such that \( f(1/2) = x \). For a proof, see [15], p. 224. Let then \( E = L₂(\mu₀; A₀) ⊕_∞ L₂(\mu₁; A₁) \) and let \( G ⊆ E \) be the closure of the subspace \( \{ f|_{\partial₀} ⊕ f|_{\partial₁} \mid f ∈ \mathcal{F}(A₀, A₁) \} \). The preceding equality shows that the mapping \( f \to f(1/2) \) defines a metric surjection \( Q : G \to (A₀, A₁)₁/₂ \). We now consider the couple \( (A₀, A₁) = (R, C) \), where we think of \( R \) and \( C \) as operator space structures on the “same” underlying vector space, identified with \( ℓ₂ \). We introduce the operator space \( E = L₂(\mu₀; ℓ₂) ⊕ L₂(\mu₁; ℓ₂) \). Let \( G \) and \( Q : G \to ℓ₂ \) be the same as before. Note that \( G \) is nothing but the \( ℓ₂ \)-valued version of the Hardy space \( H² \) on the strip \( S \), so that if we assume \( f \) analytically extended inside \( S \), we have \( Q(f) = f(1/2) \).

We first claim that

\[ \|Q : G \to OH\|_{cb} ≤ 1. \]

To verify this, consider \( x \in Mₙ(G) \) with \( \|x\|_{Mₙ(G)} ≤ 1 \). We claim that \( \|x(1/2)\|_{Mₙ(OH)} ≤ 1 \). We may view \( x \) as a sequence \( (xₖ) \) of \( Mₙ \)-valued functions on \( \partial S \) extended analytically inside \( S \), so that

\[ \|x\|_{Mₙ(G)} = \max \left\{ \left\| \left( \int xₖxₖ^* \, d\mu₀ \right)^{1/2} \right\|_{Mₙ}, \left\| \left( \int xₖxₖ^* \, d\mu₁ \right)^{1/2} \right\|_{Mₙ} \right\}, \]

and by [22] (7.3)′

\[ \|x(1/2)\|_{Mₙ(OH)} = \left\| \sum xₖ(1/2) ⊗ xₖ(1/2) \right\|_{∞} = \sup \left\{ \left| \text{tr} \left( \sum xₖ(1/2)aₖxₖ(1/2)^*b \right) \right| \right\} \]

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where the supremum runs over all \(a,b \geq 0\) in \(M_n\) such that \(\text{tr}|a|^2 \leq 1\) and \(\text{tr}|b|^2 \leq 1\). Fix \(a,b\) satisfying these conditions. Consider then the analytic function

\[
F(z) = \text{tr}\left( \sum x_k(z) a x_k(\bar{z})^* b^{2(1-z)} \right),
\]
on \(S\). Note that

\[
F(1/2) = \text{tr}\left( \sum x_k(1/2) a x_k(1/2)^* b \right) = 2^{-1} \left( \int_{\partial_0} F \, d\mu_0 + \int_{\partial_1} F \, d\mu_1 \right).
\]

But for all \(z = it\) in \(\partial_0\) we have

\[
F(it) = \sum_k \text{tr}(b^{1-it} x_k(it) a^{2it} x_k(-it)^* b^{1-it})
\]
hence by Cauchy–Schwarz for any \(z\) in \(\partial_0\)

\[
|F(z)| \leq \left( \sum_k \text{tr}(b x_k(z) x_k(z)^* b) \right)^{1/2} \left( \sum_k \text{tr}(b x_k(z) x_k(z)^* b) \right)^{1/2}.
\]

A similar verification shows that for any \(z\) in \(\partial_1\) we have

\[
|F(z)| \leq \left( \sum_k \text{tr}(a x_k(z)^* x_k(z) a) \right)^{1/2} \left( \sum_k \text{tr}(a x_k(z)^* x_k(z) a) \right)^{1/2}.
\]

Thus we obtain by Cauchy–Schwarz

\[
|F(1/2)| = | \int F d\mu | \leq 2^{-1} \left( \int_{\partial_0} |F| \, d\mu_0 + \int_{\partial_1} |F| \, d\mu_1 \right)
\]

\[
\leq 2^{-1} \left\{ \text{tr} \left( b^2 \int \sum x_k x_k^* \, d\mu_0 \right) + \text{tr} \left( a^2 \int \sum x_k^* x_k \, d\mu_1 \right) \right\} \leq \|x\|_{M_n(G)} \leq 1,
\]
which proves our claim.

It is now easy to show that \(Q\) is actually a complete metric surjection, or equivalently, that \(I \otimes Q : M_n(G) \to M_n(OH)\) is a metric surjection for any \(n \geq 1\). Indeed, consider \(x \in M_n(OH)\) with \(\|x\|_{M_n(OH)} < 1\). Since \(M_n(OH) = (M_n(R), M_n(C))_{1/2}\) (isometrically) by [22 Corollary 5.9], there is a bounded continuous analytic function \(f\) on \(S\) with values in \(M_n(R) + M_n(C)\) such that

\[
\alpha_0 = \sup\{\|f(z)\|_{M_n(R)} \mid z \in \partial_0\} < 1, \quad \alpha_1 = \sup\{\|f(z)\|_{M_n(C)} \mid z \in \partial_1\} < 1 \quad \text{and} \quad f(1/2) = x.
\]

Let us write \(f(z) = (f_k(z))_k\) where \(f_k\) is an \(M_n\)-valued function on \(S\). We have trivially

\[
\left\| \left( \int \sum f_k(z) f_k(z)^* \, d\mu_0(z) \right)^{1/2} \right\|_{M_n} \leq \alpha_0 < 1
\]

and

\[
\left\| \left( \int \sum f_k(z)^* f_k(z) \, d\mu_1(z) \right)^{1/2} \right\|_{M_n} \leq \alpha_1 < 1
\]
hence \(\|f\|_{M_n(G)} < 1\). Since clearly \((I \otimes Q)(f) = x\), this shows that \(I \otimes Q : M_n(G) \to M_n(OH)\) is a metric surjection. Thus we have completely isometrically \(OH \simeq G/\ker(Q)\). Finally since \(G \subset R \oplus C\) this completes the proof.
Let $E = L_2(\mu_0; \ell_2)_r \oplus L_2(\mu_1; \ell_2)_c$. The preceding argument shows that

\[(41)\quad OH \simeq G/N \]

where $G \subset E$ is the subspace of boundary values of analytic functions on the strip $S = \{0 < \Re z < 1\}$, and where $N$ is the subspace of $G$ formed of all $f$ in $G$ such that $f(1/2) = 0$. Thus, $OH$ appears as a quotient, namely $G/N$, of a subspace, namely $G$, of $R \oplus C$ since obviously $E \simeq R \oplus C$. Moreover, the subspace $G \subset E$ is the graph of a (necessarily closed) unbounded operator $T$: $\text{Dom}(T) \to L_2(\mu_1; \ell_2)_c$ where $\text{Dom}(T) \subset L_2(\mu_0; \ell_2)_r$ is the dense subspace formed of all the restrictions $f|_{\partial_0}$ when $f$ runs over $G$. Since $G$ is formed of analytic functions, the restriction of $f$ to $\partial_0$ (or $\partial_1$) entirely determines $f$, therefore $f \in G \to f|_{\partial_0}$ and $f \in G \to f|_{\partial_1}$ are one to one, so that the definition of $T$ is clear: we simply have

\[T(f|_{\partial_0}) = f|_{\partial_1}.\]

Note that $T$ has dense range. By the polar decomposition of $T$ (cf. \cite[p. 1249]{2}) we have $T = U|T|$ where $U: L_2(\mu_0; \ell_2) \to L_2(\mu_1; \ell_2)$ is unitary and where $|T|: L_2(\mu_0; \ell_2) \to L_2(\mu_1; \ell_2)$ is an unbounded, $\geq 0$ and self-adjoint operator.

Clearly, since $L_2(\mu_0; \ell_2)_r \simeq R$ and $L_2(\mu_1; \ell_2)_c \simeq C$ are “homogeneous” operator spaces (i.e. for any $u: R \to R$ or $u: C \to C$ we have $\|u\|_{cb} = \|u\|$), $U$ (or its inverse) is completely isometric from $L_2(\mu_0; \ell_2)_c$ to $L_2(\mu_1; \ell_2)_c$, and hence $I \oplus U^{-1}$ is completely isometric on $L_2(\mu_0; \ell_2)_r \oplus L_2(\mu_1; \ell_2)_c$. Let $\Lambda: L_2(\mu_1; \ell_2)_r \to L_2(\mu_1; \ell_2)_c$ be the same map as $|T|$ but viewed as acting between the indicated operator spaces (so that $T = UA$).

Then we have trivially

\[G \simeq (I \oplus U^{-1})(G)\]

but

\[(I \oplus U^{-1})(G) = \{(x, \Lambda x) \mid x \in \text{Dom}(\Lambda)\}.\]

So we are reduced to the following.

**Proposition A2.** Let $\Lambda: R \to C$ be a closed self-adjoint densely defined unbounded operator with $\Lambda \geq 0$. Let

\[G(\Lambda) = \{(x, \Lambda x) \mid x \in \text{Dom}(\Lambda)\} \subset R \oplus C\]

be the graph of $\Lambda$. Then the dual $G(\Lambda)^*$ is completely 2-isomorphic to a subspace of a non-commutative $L_1$-space. In particular $OH$ embeds completely 2-isomorphically in $M_\ast$ for some von Neumann algebra $M$.

**Proof.** Let $\{E_\alpha\}$ be a net of finite dimensional subspaces of $\text{Dom}(\Lambda)$ directed by inclusion and such that $\cup E_\alpha = \text{Dom}(\Lambda)$. Let $G_\alpha = \{(x, \Lambda x) \mid x \in E_\alpha\}$. Then $G(\Lambda) = \cup G_\alpha$ (directed union) and hence for any c.b. map $u$: $G(\Lambda) \to M_n$ we have

\[\|u\|_{cb} = \lim \|u|_{G_\alpha}: G_\alpha \to M_n\|_{cb}.\]

It follows that $G(\Lambda)^*$ embeds completely isometrically into an ultraproduct of the spaces $G_\alpha^*$. Since by \cite{28}, ultraproducts preserve the class of subspaces of non-commutative $L_1$-spaces (the operator space version of this is easy to derive from \cite{28}) we are reduced to proving this with $G(\Lambda)$ replaced by $G_\alpha$. In that case we may as well replace $C$ by $C_n$ (where $= \dim(E_\alpha)$) and replace $R$ by $R_n$.

Thus we are reduced to proving the result for $G(\Lambda) \subset R_n \oplus C_n$ for some invertible operator $\Lambda \geq 0$ from $R_n$ to $C_n$. In that case, we may as well assume (by homogeneity) that $\Lambda e_i = \lambda_i e_i$ for some $\lambda_i > 0$. But then this follows from the next result which is somewhat implicit in Shlyakhtenko’s work \cite{20}, and in any case is included in the above Lemma \ref{lemma}.

**Proposition A3.** With the notation as in the first part. Let $I = \{1, \ldots, n\}$ and

\[a_i = \ell_i + \lambda_i \ell^*_i.\]

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Let $G_n = \text{span}[a_1, \ldots, a_n]$ and let $W_n$ be the von Neumann algebra generated by $G_n$. Then $G_n$ is completely 2-isomorphic to $G(\Lambda_n) \subset R_n \oplus C_n$ and $G_n$ is completely 2-complemented in $W_n$.

More precisely, we have a surjective mapping $P_n : W_n \to G(\Lambda_n)$ with $\|P_n\|_{cb} \leq 1$ such that $G(\Lambda_n)$ is completely 2-isomorphic to the quotient $W_n/\ker(P_n)$. Therefore, $G(\Lambda_n)^* \text{ is completely } 2\text{-isomorphic to a subspace of a non-commutative } L_1\text{-space, namely the predual of } W_n.$

Proof. Let $P$ be as in the proof of Lemma [3]. Let $V : G_n \to G(\Lambda_n)$ be defined by $V(\ell_i + \lambda_i \ell'_i) = e_{i1} \oplus \lambda_i e_{11}$. Finally, let $P_n = VP : W_n \to G(\Lambda_n)$. The proof of Lemma [3] shows that $\|P_n\|_{cb} \leq 1$ and by the triangle inequality we have $\|V^{-1}\|_{cb} \leq 2$. Therefore, $G(\Lambda_n)$ is completely 2-isomorphic to $W_n/\ker(P_n)$. $\Box$

Note: In the above we used a discretization of $\Lambda$ to make the proof as elementary as possible, but this is not really necessary if one uses the general picture described in [30]. This alternate route is much more elegant but perhaps a bit more “abstract”. We will merely outline it. We consider the (complex) Hilbert space $H = L_2(\mu_0; \ell_2) \oplus L_2(\mu_1; \ell_2)$ equipped with the norm

$$\|(x_0, x_1)\| = \left(2^{-1}(\|x_0\|^2 + \|x_1\|^2)\right)^{1/2}.$$ 

As is classical, any $x = (x_0, x_1)$ admits (via Poisson integrals) a harmonic extension inside $S$, i.e. there is a harmonic function $\tilde{x} : S \to \ell_2$ such that $\|\tilde{x}\|^2$ admits a harmonic majorant $u$ on $S$ and admitting $x_0$ and $x_1$ as its nontangential boundary values respectively on $\partial_0$ and on $\partial_1$. Note that $\|x\| = \inf\{u(1/2)\}$ where the infimum runs over all such majorants and the Poisson integral of the function equal to $\|x_0\|^2$ on $\partial_0$ and $\|x_1\|^2$ on $\partial_1$ produces the minimal $u$.

We will denote by $h^2(\ell_2)$ the space of all harmonic functions $\tilde{x}$ obtained in this way. All such functions are implicitly extended nontangentially to the closure of $S$. Thus $h^2(\ell_2)$ can be identified with $H$. We denote by $H^2(\ell_2)$ the subspace of $h^2(\ell_2)$ formed of all the analytic functions. The spaces $h^2(\ell_2)$ and $H^2(\ell_2)$ may be viewed as conformally equivalent copies of the usual spaces on the unit disc.

For any $f = (f_k)$ in $h^2(\ell_2)$, we set $\tilde{f} = (\tilde{f}_k)$. We denote by $H_\mathbb{R}$ the real linear subspace of $H$ of all elements of the form $(\tilde{f}_0, \tilde{f}_1)$ when $f$ runs over all functions in $H^2(\ell_2)$. Note that the map

$$j : H^2(\ell_2) \to (\tilde{f}_0, \tilde{f}_1) \in H$$

is a real linear isometry, with range $H_\mathbb{R}$. It is easy to check that $H_\mathbb{R} \cap iH_\mathbb{R} = \{0\}$ (because an analytic function in $H^2(\ell_2)$ that vanishes on $\partial_1$ must vanish everywhere) and that $H_\mathbb{R} + iH_\mathbb{R}$ is dense in $H$ (because if an element of $h^2(\ell_2)$ is supported on $\partial_1$ or $\partial_0$ and is orthogonal to $H^2(\ell_2)$, it must be anti-analytic, and hence must vanish identically; therefore the restrictions $\{f|\partial_1 | f \in H^2(\ell_2)\}$ are dense in $L_2(\mu_0; \ell_2)$, and similarly for $\partial_0$).

As pointed out in [30] Remark 2.6, the basic construction of [30] can be carried out starting from the data of the embedding $j : H_\mathbb{R} \to H$, using [28] to obtain a group of orthogonal transformations of $H$ satisfying the KMS condition relative to this embedding. Let $\mathcal{F}$ be the full Fock space over $H$. We will identify $H$ with $L_2(\partial_0 \cup \partial_1; \mu)$. With the previous notation we set for any $f$ in $H^2(\ell_2)$

$$t(f) = (\ell(\tilde{f}_1))^* + \ell(\tilde{f}_1)$$

(42) Observe that $f \to t(f)$ is now a complex linear isomorphic embedding of $H^2(\ell_2)$ into $B(\mathcal{F})$. Note that this “quantization” of $H^2(\ell_2)$ seems to be of independent interest (even for scalar valued Hardy spaces, when $\ell_2$ is replaced by $\mathbb{C}$). More generally, [12] makes sense for any $f$ in $h^2(\ell_2)$; the resulting mapping is then a completely isomorphic embedding of $L_2(\mu_0; \ell_2) \oplus L_2(\mu_1; \ell_2)$ into $B(\mathcal{F})$.

Shlyakhtenko [30] made an extensive study of the von Neumann algebra $M$ generated by the operators $\{s(h) = \ell(h) + \ell(h)^*, \ h \in H_\mathbb{R}\}$. Since for any $f$ in $H^2(\ell_2)$,

$$2t(f) = s(j(f)) - is(j(if)),$$
in Proposition A.3, that \( G \) via a normal surjection \( t \) of a subspace of \( R(\mathcal{M}) \). Thus we can conclude, as in Proposition A.3 that \( G \) (and a fortiori \( OH \)) is completely 2-isomorphic to a quotient of \( M \), via a normal surjection \( M \to OH \). Thus, taking adjoints, we find that \( OH \) embeds completely isomorphically into \( M \).

**Remark.** The constant 2 appearing in Proposition A.3 is better than in Junge’s original work. This is not too significant but the question whether \( OH \) embeds completely isomorphically into \( M \), for some von Neumann algebra \( M \) remains open.

**Remark.** Our proof does not yield the fact announced by Junge (yet unpublished) that, in the above Proposition A.2 \( M \) can be chosen hyperfinite. Note however, that since by 25 we obtain an \( M \) that is a quotient of a \( C^* \)-algebra with the WEP (QWEP), one can deduce from the strong local reflexivity of non-commutative \( L_1 \)-spaces (see 3) Junge’s result that for each \( n \) and \( c > 2 \) there is an integer \( N \) and a subspace \( E_n \subset S_N^1 \) such that \( d_{cb}(E_n, OH_n) \leq c \).

**Remark.** The same proof suitably modified shows that \( OH \) embeds completely isomorphically in a non-commutative \( L_p \)-space for any \( p \) with \( 1 < p < 2 \). (The case \( p = 2 \) is of course trivial.) That result was known to Junge and Xu. Indeed, for any \( 0 < \theta < 1 \), we have by 20 \( OH = (R[\theta], R[1 - \theta]) \), hence (arguing as for Proposition A.1) we find that \( OH \) is a quotient of a subspace of \( R[\theta] \oplus R[1 - \theta] \). Now let \( p = (1 - \theta)^{-1} \) as before. In that case we claim that \( (R[\theta] \oplus R[1 - \theta])^* \) embeds completely isomorphically into \( S_p \). Indeed, as we already mentioned, \( R[\theta]^* = R[1 - \theta] \) (resp. \( R[1 - \theta]^* = R[\theta] \)) can be identified with the subspace of column (resp. row) matrices in \( S_p \). This proves our claim.

More generally, it follows from Xu’s results in 37 39 (see also 13 for related facts) that for any closed unbounded positive operator \( \Lambda : R[\theta] \to R[1 - \theta] \) with dense domain, dense range and zero kernel, the graph \( G(\Lambda) \subset R[\theta] \oplus R[1 - \theta] \) is such that \( G(\Lambda)^* \) embeds in a non-commutative \( L_p \)-space. Thus by the same principle as above for \( p = 1 \), we can show that \( OH \) embeds completely isomorphically in a non-commutative \( L_p \) for all \( 1 < p < 2 \). See 39 for more on this theme.

**Remark.** Junge observed already in 11 that \( OH \) does not embed completely isomorphically into a non-commutative \( L_q \)-space for \( 2 < q < \infty \). Actually, in that case it is even impossible to embed \( OH_n \) uniformly over \( n \) into such a space. For the reader’s convenience, we now sketch Junge’s argument for this fact. We will use the non-commutative Khintchine inequalities due to Lust–Piquard (cf. 22, p. 193). For our present purpose, it is convenient to state them using the “vector valued” version of the Schatten classes introduced in 21 and denoted by \( S_p[E] \) where \( 1 \leq p < \infty \) and \( E \) is an arbitrary operator space. Let \( (\varepsilon_k) \) denote the Rademacher functions on \( (\Omega, m) \) where (say) \( \Omega = [0, 1] \) and \( m \) is normalized Lebesgue measure. Then if the operator space \( E \) is assumed to be (completely isometrically) a subspace of a non-commutative \( L_q \)-space and if \( 2 \leq q < \infty \), then for any finite sequence \( a_1, \ldots, a_n \) in \( E \) we have

\[
\left( \int \left\| \sum_1^n \varepsilon_k a_k \right\|^q dm \right)^{1/q} \leq B_q \left( \left\| \sum_1^n e_{1k} \otimes a_k \right\|_{S_p[E]} + \left\| \sum_1^n e_{k1} \otimes a_k \right\|_{S_q[E]} \right)
\]

where \( B_q \) is a constant depending only on \( q \). See 13 for the extension of 23 to the case of general non-commutative \( L_q \)-spaces, including the non-semifinite case.

Now, let \( u : OH_n \to E \) be a linear isomorphism and let \((e_1, \ldots, e_n)\) denote an orthonormal basis in \( OH_n \). We have clearly

\[
\sqrt{n} \leq \left\| u^{-1} \right\| \left( \int \left\| \sum_1^n \varepsilon_kue_k \right\|^q dm \right)^{1/q}.
\]

On the other hand applying 23 to \( a_k = ue_k \) and using 21 Cor. 1.2 we find

\[
\left( \int \left\| \sum_1^n \varepsilon_kue_k \right\|^q dm \right)^{1/q} \leq \left\| u \right\|_{cb} B_q \left( \left\| \sum_1^n e_{1k} \otimes e_k \right\|_{S_q[OH_n]} + \left\| \sum_1^n e_{k1} \otimes e_k \right\|_{S_q[OH_n]} \right).
\]
Finally, by easy interpolation arguments (based on [21, Cor. 1.4]) we find
\[ \left\| \sum_{1}^{k} e_{1k} \otimes e_{k} \right\|_{S_{q}[OH_{n}]} \leq n^{\frac{1}{2n} + \frac{1}{4}} \]
and similarly for \( \left\| \sum_{1}^{k} e_{k1} \otimes e_{k} \right\|_{S_{q}[OH_{n}]} \). Thus we conclude
\[ n^{\frac{1}{2}} \leq \|u^{-1}\|_{cb} \|u\|_{cb} 2B_{q} n^{\frac{1}{2n} + \frac{1}{4}} \]
and a fortiori we find
\[ d_{cb}(E, OH_{n}) = \inf \|u\|_{cb} \|u^{-1}\|_{cb} \geq (2B_{q})^{-1} n^{\frac{1}{3} - \frac{\beta}{2}} . \]

A similar argument can be applied with \((C_{n}, R_{n})_{\theta}\) instead of \(OH_{n}\). The same calculations yield that for any \( q > \max\{p, p'\} \) with \( p = (1 - \theta)^{-1} \) and \( p' = \theta^{-1} \), we have
\[ d_{cb}(E, (C_{n}, R_{n})_{\theta}) \geq (2B_{q})^{-1} n^{\beta/2} \]
where \( \beta = \min\{p^{-1} - q^{-1}, p'^{-1} - q^{-1}\} \). Note however that if \( p = (1 - \theta)^{-1} \) then \((C_{n}, R_{n})_{\theta}\) embeds completely isometrically in both \( S_{p} \) and \( S_{p'} \); indeed it can be identified with \( \text{span}\{e_{1k} | 1 \leq k \leq n\} \) in \( S_{p} \) and with \( \text{span}\{e_{k1} | 1 \leq k \leq n\} \) in \( S_{p'} \).

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