Numerical algorithm for two-dimensional time-fractional wave equation of distributed-order with a nonlinear source term

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Abstract

In this paper, an alternating direction implicit (ADI) difference scheme for two-dimensional time-fractional wave equation of distributed-order with a nonlinear source term is presented. The unique solvability of the difference solution is discussed, and the unconditional stability and convergence order of the numerical scheme are analysed. Finally, numerical experiments are carried out to verify the effectiveness and accuracy of the algorithm.

Keywords: Two-dimensional time-fractional wave equation of distributed-order, ADI scheme, Nonlinear source term, Stability, Convergence

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1. Introduction

The idea of distributed-order differential equation was first introduced by Caputo in his work for modeling the stress-strain behavior of an anelastic medium in 1960s \cite{1}. Being different from the differential equations with the single-order fractional derivative and the ones with sums of fractional derivatives, i.e., multi-term fractional differential equations (FDEs), the distributed-order differential equations are derived by integrating the order of differentiation over a

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certain range. It can be regarded as a generalization of the aforementioned two classes of FDEs. A typical application of this kind of FDEs is in the retarding sub-diffusion process, where a plume of particles spreads at a logarithmic rate, which leads to ultraslow diffusion (see [2][3][4]). Another example is the fractional Langevin equation of distributed-order, which was proposed to model the kinetics of retarding sub-diffusion whose scaling exponent decreases in time, and then was applied to simulate the strongly anomalous ultraslow diffusion with the mean square displacement growing as a power of logarithm of time [5]. The distributed-order FDEs were also found playing important role in other various research fields, such as control and signal processing [6], modelling dielectric induction and diffusion [7], identification of systems [8], and so on.

Till now, there have been many important progresses for the research on analytical solutions of distributed-order FDEs. For the kinetic description of anomalous diffusion and relaxation phenomena, A. V. Chechkin et al. presented the diffusion-like equation with time fractional derivative of distributed-order in [9], where the positivity of the solutions of the proposed equation was proved and the relation to the continuous-time random walk theory was established. T. M. Atanackovic et al. analysed a Cauchy problem for a time distributed-order diffusion-wave equation by means of the theory of an abstract Volterra equation [10]. In [11], for the one-dimensional distributed-order diffusion-wave equation, R. Gorenflo et al. gave the interpretation of the fundamental solution of the Cauchy problem as a probability density function of the space variable $x$ evolving in time $t$ in the transform domain by employing the technique of the Fourier and Laplace transforms. Using the Laplace transform method, Z. Li et al investigated the asymptotic behavior of solutions to the initial-boundary-value problem for the distributed-order time-fractional diffusion equations [12].

In most instances, the analytical solutions of distributed-order differential equations are not easy to available, thus it stimulates researchers to develop numerical algorithms for approximate solutions. To our knowledge, the research on numerically solving the distributed-order differential equations are still in its infancy. The literatures [13][14][15] concerned on developing numerical meth-
ods for solving distributed-order ordinary differential equations. In terms of the distributed-order partial differential equations, most of the work are about the one-dimensional time distributed-order differential equations, and the integrating range of the order of time derivative is the interval $[0, 1]$, which is named as time distributed-order diffusion equation. N. J. Ford et al. developed an implicit finite difference method for the solution of the diffusion equation with distributed order in time $[16]$. By using the Grünwald-Letnikov formula, Gao et al. proposed two difference schemes to solve the one-dimensional distributed-order differential equations, and the extrapolation method was applied to improve the approximate accuracy $[17]$. In $[18]$, the authors handled the same distributed-order differential equations by employing a weighted and shifted Grünwald-Letnikov formula to derive several second-order convergent difference schemes. When the order of the time derivative is distributed over the interval $[1, 2]$, it is called the time distributed-order wave equation. The study of the numerical solution of this kind of equation is rather more limited. Ye et al. derived and analysed a compact difference scheme for a distributed-order time-fractional wave equation in $[19]$.

When considering the high-dimensional models, Gao et al. investigated ADI schemes for two-dimensional distributed-order diffusion equations $[20][21]$, and they also developed two ADI difference schemes for solving the two-dimensional time distributed-order wave equations $[22]$. Due to the widespread use of the nonlinear models $[23][24]$, M. L. Morgado et al. developed an implicit difference scheme for one-dimensional time distributed-order diffusion equation with a nonlinear source term $[25]$. For further discussion on the numerical approaches for solving the high-dimensional distributed-order partial differential equations, this paper is devoted to develop effective numerical algorithm for two-dimensional
time-fractional wave equation of distributed-order with a nonlinear source term

\[
\int_{1}^{2} p(\beta) C_{0}^{\beta}D_{t}^{\beta} u(x, y, t)d\beta = \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial x^2} + f(x, y, t, u(x, y, t)),
\]

\((x, y) \in \Omega, \quad t \in (0, T],\)

\((1.1)\)

\[
u(x, y, t) = \phi(x, y, t), \quad (x, y) \in \partial \Omega, \quad 0 \leq t < T,
\]

\((1.2)\)

\[
u(x, y, 0) = \psi_1(x, y), \quad u_t(x, y, 0) = \psi_2(x, y), \quad (x, y) \in \Omega,
\]

\((1.3)\)

where \(\Omega = (0, L_1) \times (0, L_2),\) and \(\partial \Omega\) is the boundary of \(\Omega.\) The fractional derivative \(C_{0}^{\beta}D_{t}^{\beta} v(t)\) in \((1.1)\) is given in the Caputo sense

\[
C_{0}^{\beta}D_{t}^{\beta} v(t) = \begin{cases}
\frac{\partial v(t)}{\partial t} - \frac{\partial v(0)}{\partial t}, & \beta = 1, \\
\frac{1}{\Gamma(2 - \beta)} \int_{0}^{t} (t - \xi)^{1 - \beta} \frac{\partial^2 v(\xi)}{\partial \xi^2} d\xi, & 1 < \beta < 2, \\
\frac{\partial^2 v(t)}{\partial t^2}, & \beta = 2,
\end{cases}
\]

and the function \(p(\beta)\) is served as weight for the order of differentiation such that \(p(\beta) > 0\) and \(\int_{1}^{2} p(\beta) d\beta = c_0 > 0.\) We assume that \(p(\beta), \phi(x, y, t), \psi_1(x, y), \psi_2(x, y)\) and \(f(x, y, t, u)\) are continuous, and the nonlinear source term \(f\) satisfies a Lipschitz condition of the form

\[
|f(x, y, t, u_1) - f(x, y, t, u_2)| \leq L_f |u_1 - u_2|,
\]

\((1.4)\)

where \(L_f\) is a positive constant.

The main procedure of developing numerical scheme for solving problem \((1.1)-(1.3)\) is as follows. Firstly a suitable numerical quadrature formula is adopted to discrete the integral in \((1.1),\) and a multi-term time fractional wave equation is left whereafter. Then we develop an ADI finite difference scheme which is uniquely solvable for the multi-term time fractional wave equation. By using the discrete energy method, we prove the derived numerical scheme is unconditionally stable and convergent.

The rest of this paper is organized in the following way. In Section 2, the ADI finite difference scheme is constructed and described detailedly. In Section 3, we give analysis on solvability, stability and convergence for the derived difference
scheme. Numerical results are illustrated in Section 4 to confirm the effectiveness and accuracy of our method, and some conclusions are drawn in the last section.

2. The derivation of the ADI scheme

This section focuses on deriving the ADI scheme for the problems (1.1)–(1.3).

Let \( M_1, M_2 \) and \( N \) be positive integers, and \( h_1 = L_1/M_1, h_2 = L_2/M_2 \) and \( \tau = T/N \) be the uniform sizes of spatial grid and time step, respectively. Then a spatial and temporal partition can be defined as \( x_i = ih_1 \) for \( i = 0, 1, \cdots, M_1 \), \( y_j = jh_2 \) for \( j = 0, 1, \cdots, M_2 \) and \( t_n = n\tau \) for \( n = 0, 1, \cdots, N \). Denote \( \Omega_h = \{(x_i, y_j) \mid 0 \leq i \leq M_1, 0 \leq j \leq M_2\} \) and \( \Omega_\tau = \{t_n \mid t_n = n\tau, 0 \leq n \leq N\} \), then the domain \( \Omega \times [0, T] \) is covered by \( \Omega_h \times \Omega_\tau \). Let \( u = \{u_{ij}^n \mid 0 \leq i \leq M_1, 0 \leq j \leq M_2, 0 \leq n \leq N\} \) be a grid function on \( \Omega_h \times \Omega_\tau \). We introduce the following notations:

\[
\begin{align*}
\Delta_h u_{ij} &= \delta_x u_{ij}^n + \delta_y u_{ij}^n, \\
\delta_x u_{ij}^n &= \frac{1}{h_1} (u_{ij}^n - u_{i-1,j}^n), \\
\delta_y u_{ij}^n &= \frac{1}{h_2} (u_{ij}^n - u_{i,j-1}^n), \\
\delta_x u_{ij} &= \frac{1}{h_1} (u_{ij+1} - u_{ij-1}), \\
\delta_y u_{ij} &= \frac{1}{h_2} (u_{i+1,j} - u_{i-1,j}).
\end{align*}
\]

Consider Eq. (1.1) at the point \((x_i, y_j, t_n)\), and we write it as

\[
\int_1^2 p(\beta) C_0 \psi_x^2 u(x_i, y_j, t_n) d\beta = \frac{\partial^2 u(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 u(x_i, y_j, t_n)}{\partial y^2} + f(x_i, y_j, t_n, u(x_i, y_j, t_n)).
\]

(2.1)

Take an average of Eq. (2.1) on time level \( t = t_n \) and \( t = t_{n-1} \), then we have

\[
\begin{align*}
\frac{1}{2} \left[ \int_1^2 p(\beta) C_0 \psi_x^2 u(x_i, y_j, t_n) d\beta + \int_1^2 p(\beta) C_0 \psi_x^2 u(x_i, y_j, t_{n-1}) d\beta \right] \\
= \frac{1}{2} \left[ \frac{\partial^2 u(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 u(x_i, y_j, t_{n-1})}{\partial x^2} \right] + \frac{1}{2} \left[ \frac{\partial^2 u(x_i, y_j, t_n)}{\partial y^2} + \frac{\partial^2 u(x_i, y_j, t_{n-1})}{\partial y^2} \right] \\
+ \frac{1}{2} \left[ f(x_i, y_j, t_n, u(x_i, y_j, t_n)) + f(x_i, y_j, t_{n-1}, u(x_i, y_j, t_{n-1})) \right].
\end{align*}
\]

(2.2)
Denote by \( U^n_{ij} = u(x_i, y_j, t_n) \) the grid functions on \( \tilde{\Omega}_h \times \Omega_{r, t} \) with \( 0 \leq i \leq M_1, 0 \leq j \leq M_2, 0 \leq n \leq N \). Eq. (2.2) can be expressed as

\[
\int_1^2 p(\beta)^C_0 \beta^\frac{n-\frac{1}{2}}{D^\frac{n-\frac{1}{2}}_t} U^n_{ij} \, d\beta = \frac{\partial^2}{\partial x^2} U^n_{ij} + \frac{\partial^2}{\partial y^2} U^n_{ij} + \frac{1}{2} \left[ f(x_i, y_j, t_n, U^n_{ij}) + f(x_i, y_j, t_{n-1}, U^{n-1}_{ij}) \right] \tag{2.3}
\]

Firstly we discretize the integral term in (2.3). Suppose \( p(\beta) \in C^2[1, 2], \beta_0^C D^\beta_0 u(x, y, t) \mid_{t=t_n-1} \text{ and } \beta_0^C D^\beta_0 u(x, y, t) \mid_{t=t_n} \in C^2[1, 2] \). Let \( K \) be a positive integer, and \( \Delta \beta = 1/K \) be the uniform step size. Take \( \beta_l = 1 + \frac{l-1}{2} \Delta \beta \), \( 1 \leq l \leq K \), then the mid-point quadrature rule is used for approximating the integral in (2.3)

\[
\Delta \beta \sum_{l=1}^{K} p(\beta_l)^C_0 \beta^\frac{n-\frac{1}{2}}{D^\frac{n-\frac{1}{2}}_t} U^n_{ij} + R_1 = \frac{\partial^2}{\partial x^2} U^n_{ij} + \frac{\partial^2}{\partial y^2} U^n_{ij}
\]

\[
+ \frac{1}{2} \left[ f(x_i, y_j, t_n, U(x_i, y_j, t_n)) + f(x_i, y_j, t_{n-1}, U(x_i, y_j, t_{n-1})) \right],
\]

where \( R_1 = \mathcal{O}(\Delta \beta^2) \).

Next, we solve the multi-term time fractional wave equation (2.4) with the initial and boundary conditions (1.3) and (1.2). Suppose \( u(x, y, t) \in C^{4,4,3}[\tilde{\Omega} \times [0, T]) \). According to Theorem 8.2.5 in [20], the Caputo derivative \( \beta_0^C D^\beta_0 u^n_{ij} \), \( 1 < \beta_l < 2 \) have the fully discrete difference scheme

\[
\beta_0^C D^\beta_0 u^n_{ij} = \frac{\tau^{1-\beta_l}}{\Gamma(3-\beta_l)} \left[ a_{0}^{(\beta_l)} \delta_t U^n_{ij}^{\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\beta_l)} - a_{n-k}^{(\beta_l)}) \delta_t U^n_{ij}^{\frac{1}{2}} - a_{n-1}^{(\beta_l)} \psi_2(x_i, y_j) \right] + R^l_2,
\]

where \( a_{k}^{(\beta_l)} = (k+1)^{2-\beta_l} - k^{2-\beta_l} \), \( k = 0, 1, 2, \ldots \), and

\[
| R^l_2 | \leq \frac{1}{\Gamma(3-\beta_l)} \left[ \frac{2-\beta_l}{12} + \frac{2^{3-\beta_l}}{3-\beta_l} - (1 + 2^{1-\beta_l}) + \frac{1}{12} \right],
\]

\[
\max_{0 \leq l \leq t_n} \left| \frac{\partial^3 u(x_i, y_j, t)}{\partial t^3} \right| \tau^{3-\beta_l}, \quad l = 1, 2, \ldots, K.
\]
Substituting (2.8) in (2.7), we are left with the following manner to avoid a system of nonlinear equations when computing:

\[
\frac{\partial^2 g(x_i)}{\partial x^2} = \frac{g(x_{i+1}) - 2g(x_i) + g(x_{i-1})}{(\Delta x)^2} - \frac{(\Delta x)^2 \partial^4 g(\xi_i)}{12} \frac{\partial^4 g(\xi_i)}{\partial x^4}, \quad \xi_i \in (x_{i-1}, x_{i+1})
\]

to approximate the second order derivatives in (2.4), it is obtained

\[
\Delta \beta \sum_{l=1}^{K} p(\beta_l) \frac{\Gamma(1-\beta_l)}{\Gamma(3-\beta_l)} \left[ a_0^{(\beta_l)} \delta_t U_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k}^{(\beta_l)} - a_{n-k-1}^{(\beta_l)}) \delta_t U_{ij}^{k-\frac{1}{2}} \right.
\]
\[
- a_{n-1}^{(\beta_l)} \psi_2(x_i, y_j) \left. + \sum_{i=1}^{K} \Delta \beta p(\beta_l) R_{ij}^l + R_i \right]
\]
\[
= \delta_x^2 U_{ij}^{n-\frac{1}{2}} + \delta_y^2 U_{ij}^{n-\frac{1}{2}} + \frac{1}{2} \left( f(x_i, y_j, t_{n-1}, U_{ij}^{n-1}) + f(x_i, y_j, t_n, U_{ij}^n) \right) + R_i,
\]

(2.7)

where \( R_i = O(h^2 + h_2^2) \). Subsequently, the nonlinear source term is dealt with in the following manner to avoid a system of nonlinear equations when computing:

\[
f(x_i, y_j, t_n, U_{ij}^n) = f(x_i, y_j, t_{n-1}, U_{ij}^{n-1}) + O(\tau).
\]

(2.8)

Substituting (2.8) in (2.7), we are left with

\[
\Delta \beta \sum_{l=1}^{K} p(\beta_l) \frac{\Gamma(1-\beta_l)}{\Gamma(3-\beta_l)} \left[ a_0^{(\beta_l)} \delta_t U_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k}^{(\beta_l)} - a_{n-k-1}^{(\beta_l)}) \delta_t U_{ij}^{k-\frac{1}{2}} - a_{n-1}^{(\beta_l)} \psi_2(x_i, y_j) \right]
\]
\[
= \delta_x^2 U_{ij}^{n-\frac{1}{2}} + \delta_y^2 U_{ij}^{n-\frac{1}{2}} + f(x_i, y_j, t_{n-1}, U_{ij}^{n-1}) + R_{ij}^{n-\frac{1}{2}} + \bar{R}_{ij}^{n-\frac{1}{2}},
\]

(2.9)

where

\[
R_{ij}^{n-\frac{1}{2}} = - \sum_{l=1}^{K} \Delta \beta p(\beta_l) R_{ij}^l + O(h^2 + h_2^2) + O(\Delta \beta^2)
\]

and

\[
\bar{R}_{ij}^{n-\frac{1}{2}} = O(\tau).
\]

From (2.6), we can deduce that there exists a positive constant \( C_1 \) such that

\[
\left| - \sum_{l=1}^{K} \Delta \beta p(\beta_l) R_{ij}^l \right| \leq C_1 \tau^{1+\frac{1}{2}} \Delta \beta \sum_{l=1}^{K} \Delta \beta p(\beta_l).
\]

Since

\[
\sum_{l=1}^{K} \Delta \beta p(\beta_l) \sim \int_{1}^{2} p(\beta) d\beta = c_0,
\]

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we get
\[ \sum_{l=1}^{K} \Delta \beta p(\beta_l) \leq C_2, \]
where \( C_2 \) is a positive constant. Thus there exists a positive constant \( C_3 \) such that
\[ \left| R_{ij}^{n-\frac{1}{2}} \right| \leq C_3 \left( \tau^{1+\frac{1}{2}} \Delta \beta + h_1^2 + h_2^2 + \Delta \beta^2 \right). \]
Besides, it is obvious that
\[ \left| \tilde{R}_{ij}^{n-\frac{1}{2}} \right| \leq C_4 \tau, \]
where \( C_4 \) is a positive constant.

Denote
\[ \mu = \Delta \beta \sum_{l=1}^{K} p(\beta_l) \frac{1}{\tau^{\beta_l} \Gamma(3 - \beta_l)}. \]
Since
\begin{align*}
\Delta \beta \sum_{l=1}^{K} p(\beta_l) \frac{1}{\tau^{\beta_l} \Gamma(3 - \beta_l)} & \sim \int_{1}^{2} p(\beta) \frac{1}{\tau^\beta \Gamma(3 - \beta)} d\beta \\
& = \frac{p(\beta^*)}{\Gamma(3 - \beta^*)} \int_{1}^{2} \frac{1}{\tau^\beta} d\beta \\
& = \frac{p(\beta^*)}{\Gamma(3 - \beta^*)} \frac{1 - \tau}{\tau^2 |\ln \tau|}.
\end{align*}

it can be concluded that
\[ \mu = \frac{1}{O(\tau^2 |\ln \tau|)}. \]
In addition, \( |\ln \tau| \leq C \tau^{-\epsilon} \) for any positive and small \( \epsilon \) when \( \tau \) is sufficiently small, thus the term \( O(\tau^2 |\ln \tau|) \) is almost the same as \( O(\tau^2) \) when \( \tau \) is sufficiently small. Adding the high order term
\[ \frac{\tau}{4\mu \delta_x^2 \delta_y^2} \frac{U_{ij}^n - U_{ij}^{n-1}}{\tau}, \]
on both sides of (2.9), we derive

\[
\Delta \beta \sum_{l=1}^{K} p(\beta_l) \frac{\tau^{1-\beta_l}}{\Gamma(3-\beta_l)} \left[ a_0^{(\beta_l)} \delta_t U_{i,j}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\beta_l)}) \delta_t U_{i,j}^{n-k-\frac{1}{2}} - a_{n-1}^{(\beta_l)} \psi_2(x_i, y_j) \right] \\
+ \frac{\tau}{4\mu} \delta_x^2 \delta_y^2 \frac{U_{i,j}^n - U_{i,j}^{n-1}}{\tau}
\]

\[
= \delta_x^2 U_{i,j}^{n-\frac{1}{2}} + \delta_y^2 U_{i,j}^{n-\frac{1}{2}} + f(x_i, y_j, t_{n-1}, U_{i,j}^{n-1}) + R_{i,j}^{n-\frac{1}{2}} + R_{i,j}^{n-\frac{1}{2}},
\]

(2.10)

where

\[
\hat{R}_{i,j}^{n-\frac{1}{2}} = \frac{\tau}{4\mu} \delta_x^2 \delta_y^2 \frac{U_{i,j}^n - U_{i,j}^{n-1}}{\tau},
\]

and it is clear that

\[
|\hat{R}_{i,j}^{n-\frac{1}{2}}| \leq C_5 \tau^2 |\ln \tau|.
\]

Also, for the initial and boundary value conditions, we have

\[
U_{i,j}^0 = \psi_1(x_i, y_j), \ (x_i, y_j) \in \Omega,
\]

(2.11)

\[
U_{i,j}^n = \phi(x_i, y_j, t_n), \ (x_i, y_j) \in \partial \Omega, \ 0 \leq n \leq N.
\]

(2.12)

Let \( u_{i,j}^n \) be the numerical approximation to \( u(x_i, y_j, t_n) \). Neglecting the small term \( R_{i,j}^{n-\frac{1}{2}}, \ R_{i,j}^{n-\frac{1}{2}} \) and \( \hat{R}_{i,j}^{n-\frac{1}{2}} \) in (2.10), and using \( u_{i,j}^n \) instead of \( U_{i,j}^n \) in (2.10)–(2.12), we construct the difference scheme for (1.1)–(1.3) as follows:

\[
\Delta \beta \sum_{l=1}^{K} p(\beta_l) \frac{\tau^{1-\beta_l}}{\Gamma(3-\beta_l)} \left[ a_0^{(\beta_l)} \delta_t u_{i,j}^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\beta_l)}) \delta_t u_{i,j}^{n-k-\frac{1}{2}} \\
- a_{n-1}^{(\beta_l)} (\psi_2)_{i,j} \right] \\
+ \frac{\tau}{4\mu} \delta_x^2 \delta_y^2 \frac{u_{i,j}^n - u_{i,j}^{n-1}}{\tau}
\]

\[
= \delta_x^2 u_{i,j}^{n-\frac{1}{2}} + \delta_y^2 u_{i,j}^{n-\frac{1}{2}} + f(x_i, y_j, t_{n-1}, u_{i,j}^{n-1}),
\]

1 \leq i \leq M_1 - 1, \ 1 \leq j \leq M_2 - 1, \ 1 \leq n \leq N,

(2.13)

\[
u_{i,j}^0 = (\psi_1)_{i,j}, \ 1 \leq i \leq M_1 - 1, \ 1 \leq j \leq M_2 - 1,
\]

(2.14)

\[
u_{i,j}^n = \phi_{i,j}^n, \ (i, j) \in \gamma = \{(i,j) \mid (x_i, y_j) \in \partial \Omega \}, \ 0 \leq n \leq N,
\]

(2.15)

where

\[
(\psi_1)_{i,j} = \psi_1(x_i, y_j), \ (\psi_2)_{i,j} = \psi_2(x_i, y_j), \ 1 \leq i \leq M_1 - 1, \ 1 \leq j \leq M_2 - 1,
\]
and
\[ \phi^n_i = \phi(x_i, y_j, t_n), \quad (i, j) \in \gamma, \quad 0 \leq n \leq N. \]

Notice \( a_0^{(\beta_i)} = 1 \), then Eq. (2.13) can be rewritten as:
\[
\Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{1}{\tau^{\beta_i} \Gamma(3 - \beta_i)} u_{ij}^n - \frac{1}{2} \delta_x u_{ij}^n - \frac{1}{2} \delta_y u_{ij}^n + \frac{1}{4\mu} \delta_x^2 \delta_y u_{ij}^n
\]
\[
= \Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{1}{\tau^{\beta_i} \Gamma(3 - \beta_i)} \left[ u_{ij}^{n-1} + \sum_{k=1}^{n-1} (a_{n-k-1}^{(\beta_i)} - a_{n-k}^{(\beta_i)}) (u_{ij}^k - u_{ij}^{k-1}) + \tau a_{n-1}^{(\beta_i)}(\psi_2)_{ij} \right] + f(x_i, y_j, t_{n-1}, u_{ij}^{n-1}),
\]
or
\[
\left( \sqrt{\mu I} - \frac{1}{2\sqrt{\mu}} \delta_x^2 \right) \left( \sqrt{\mu I} - \frac{1}{2\sqrt{\mu}} \delta_y^2 \right) u_{ij}^n
\]
\[
= \left( \sqrt{\mu I} + \frac{1}{2\sqrt{\mu}} \delta_x^2 \right) \left( \sqrt{\mu I} + \frac{1}{2\sqrt{\mu}} \delta_y^2 \right) u_{ij}^{n-1} + \Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{1}{\tau^{\beta_i} \Gamma(3 - \beta_i)} \left[ \sum_{k=1}^{n-1} (a_{n-k-1}^{(\beta_i)} - a_{n-k}^{(\beta_i)}) (u_{ij}^k - u_{ij}^{k-1}) + \tau a_{n-1}^{(\beta_i)}(\psi_2)_{ij} \right] + f(x_i, y_j, t_{n-1}, u_{ij}^{n-1}),
\]
where \( I \) denotes the identity operator.

Let
\[ u_{ij}^* = \left( \sqrt{\mu I} - \frac{1}{2\sqrt{\mu}} \delta_y^2 \right) u_{ij}^n. \]

Together with (2.14) and (2.15) the ADI difference scheme is derived, and the procedure can be executed as follows:

On each time level \( t = t_n \) \((1 \leq n \leq N)\), firstly, for all fixed \( y = y_j \) \((1 \leq j \leq M_2 - 1)\), solving a set of \( M_1 - 1 \) equations at the mesh points \( x_i \) \((1 \leq i \leq M_1 - 1)\) to get the intermediate solution \( u_{ij}^* \):
\[
\left\{ \begin{array}{l}
\left( \sqrt{\mu I} - \frac{1}{2\sqrt{\mu}} \delta_x^2 \right) u_{ij}^* = \left( \sqrt{\mu I} + \frac{1}{2\sqrt{\mu}} \delta_x^2 \right) \left( \sqrt{\mu I} + \frac{1}{2\sqrt{\mu}} \delta_y^2 \right) u_{ij}^{n-1} \\
+ \Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{1}{\tau^{\beta_i} \Gamma(3 - \beta_i)} \left[ \sum_{k=1}^{n-1} (a_{n-k-1}^{(\beta_i)} - a_{n-k}^{(\beta_i)}) (u_{ij}^k - u_{ij}^{k-1}) + \tau a_{n-1}^{(\beta_i)}(\psi_2)_{ij} \right] + f(x_i, y_j, t_{n-1}, u_{ij}^{n-1}), \quad 1 \leq i \leq M_1 - 1,
\end{array} \right.
\]
\[
\left. \begin{array}{l}
u_{ij}^* = \left( \sqrt{\mu I} - \frac{1}{2\sqrt{\mu}} \delta_y^2 \right) u_{ij}^*, \quad u_{ij}^* = \left( \sqrt{\mu I} - \frac{1}{2\sqrt{\mu}} \delta_y^2 \right) u_{ij}^n;
\end{array} \right. \quad (2.16)
\]
afterwards, for all fixed \( x = x_i \) (1 \( \leq i \leq M_1 - 1 \)), by computing a set of \( M_2 - 1 \) equations at the mesh points \( y_j \) (1 \( \leq j \leq M_2 - 1 \)), the solution \( u^*_{ij} \) can be obtained:

\[
\begin{cases}
(\sqrt{\mu} I - \frac{1}{2\sqrt{\mu}} \delta_y^2) u^*_{ij} = u^*_{ij}, & 1 \leq j \leq M_2 - 1, \\
u^*_{i0} = \phi(x_i, y_0, t_n), & u^*_{iM_2} = \phi(x_i, y_{M_2}, t_n).
\end{cases}
\]  

(2.17)

3. Analysis of the ADI difference scheme

3.1. Solvability

It is clear that the ADI scheme (2.16)–(2.17) is a linear tridiagonal system in unknowns, and the coefficient matrices are strictly diagonally dominant. Thus the scheme (2.16)–(2.17) has a unique solution. This result can be written as following.

**Theorem 3.1.** The ADI difference scheme (2.16)–(2.17) is uniquely solvable.

3.2. Stability

In this subsection we prove the unconditional stability and the convergence of the difference scheme (2.16)–(2.17). We start with some auxiliary definitions and useful results.

Denote the space of grid functions on \( \Omega_h \)

\[ \mathcal{V}_h = \{ v \mid v = \{ v_{ij} \mid (x_i, y_j) \in \Omega_h \} \text{ and } v_{ij} = 0 \text{ if } (x_i, y_j) \in \partial \Omega_h \}. \]

For any grid function \( v \in \mathcal{V}_h \), the following discrete norms and Sobolev seminorm are introduced:

\[
\|v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} |v_{ij}|^2}, \quad \|\delta_x \delta_y v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} |\delta_x \delta_y v_{i-\frac{1}{2},j-\frac{1}{2}}|^2},
\]

\[
\|\delta_x v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2-1} |\delta_x v_{i-\frac{1}{2},j}|^2}, \quad \|\delta_y v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2} |\delta_y v_{i,j-\frac{1}{2}}|^2},
\]

\[
\|\Delta_h v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} |\Delta_h v_{ij}|^2}, \quad \|v\|_1 = \sqrt{\|\delta_x v\|^2 + \|\delta_y v\|^2}.
\]
Lemma 3.1. [27] For any grid function \( v \in \mathcal{V}_h \), \( \|v\| \leq \frac{1}{\sqrt{3}} |v|_1 \).

Lemma 3.2. [26] For any grid function \( v \in \mathcal{V}_h \), \( |v|_1 \leq \frac{1}{\sqrt{3}} \|\Delta_h v\| \).

Lemma 3.3. [26] For any \( G = \{G_1, G_2, G_3, \ldots\} \) and \( q \), we have
\[
\sum_{n=1}^{m} \left[ b_0 G_n - \sum_{k=1}^{n-1} (b_{n-k-1} - b_{n-k}) G_k - b_{n-1} q \right] G_n \\
\geq \frac{t_m^{1-\alpha}}{2} \tau \sum_{n=1}^{m} G_n^2 - \frac{t_m^{2-\alpha}}{2(2-\alpha)} q^2, \quad m = 1, 2, 3, \ldots,
\]
where
\[
b_l = \frac{\tau^{2-\alpha}}{2-\alpha} [(l + 1)^{2-\alpha} - l^{2-\alpha}], \quad l = 0, 1, 2, \ldots.
\]

The discrete Gronwall’s inequality is also introduced below since it is necessary to prove the stability and convergence of the proposed method.

Lemma 3.4. [28] Assume that \( k_n \) and \( p_n \) are nonnegative sequences, and the sequence \( \Phi_n \) satisfies
\[
\Phi_0 \leq g_0, \quad \Phi_n \leq g_0 + \sum_{l=0}^{n-1} p_l + \sum_{l=0}^{n-1} k_l \Phi_l, \quad n \geq 1,
\]
where \( g_0 \geq 0 \). Then the sequence \( \Phi_n \) satisfies
\[
\Phi_n \leq \left( g_0 + \sum_{l=0}^{n-1} p_l \right) \exp \left( \sum_{l=0}^{n-1} k_l \right), \quad n \geq 1.
\]

Since the ADI difference scheme (2.16)–(2.17) is equivalent to (2.13)–(2.15) if the intermediate variable \( u^* \) is eliminated, we analyze the stability and convergence by employing the difference scheme (2.13)–(2.15).

Assume that \( \bar{u}^n_{ij} \) is the approximate solution of \( u^n_{ij} \), which is the exact solution of the scheme (2.13)–(2.15). Denote \( \varepsilon^n_{ij} = u^n_{ij} - \bar{u}^n_{ij}, 0 \leq i \leq M_1, 0 \leq j \leq M_2 \).
$M_2$, $0 \leq n \leq N$, then we have the perturbation error equations

$$
\Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{\tau^{1-\beta_i}}{\Gamma(2-\beta_i)} \left[ a^{(\beta_i)}_0 \delta_t \varepsilon_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a^{(\beta_i)}_{n-k-1} - a^{(\beta_i)}_{n-k}) \delta_t \varepsilon_i^{k-\frac{1}{2}} - a^{(\beta_i)}_{n-1} (\psi^*_2)_{ij} \right] \\
+ \frac{\tau}{4\mu} \delta_x^2 \delta_y^2 \frac{\varepsilon_i^n - \varepsilon_i^{n-1}}{\tau} \\
= \delta_x^2 \varepsilon_i^{n-\frac{1}{2}} + \delta_y^2 \varepsilon_i^{n-\frac{1}{2}} + f(x_i, y_j, t_{n-1}, u_i^{n-1}) - f(x_i, y_j, t_{n-1}, \tilde{u}_i^{n-1}),
$$

$1 \leq i \leq M_1 - 1, 1 \leq j \leq M_2 - 1, 1 \leq n \leq N, 

$$
\varepsilon_{ij}^{n} = 0, (i, j) \in \gamma, 0 \leq n \leq N,
$$

where

$$
(\psi^*_{2})_{ij} = (\psi_2)_{ij} - (\tilde{\psi}_2)_{ij}.
$$

**Theorem 3.2.** Asssume that the condition (1.4) is satisfied, then the difference scheme (2.16)-(2.17) is unconditionally stable.

**Proof.** Let

$$
b^{(\beta_i)}_k = \frac{\tau^{2-\beta_i}}{2 - \beta_i} a^{(\beta_i)}_k, 1 \leq l \leq K,
$$

then Eq. (3.1) is equivalent to

$$
\Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{1}{\Gamma(2-\beta_i)} \left[ b^{(\beta_i)}_0 \delta_t \varepsilon_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (b^{(\beta_i)}_{n-k-1} - b^{(\beta_i)}_{n-k}) \delta_t \varepsilon_i^{k-\frac{1}{2}} - b^{(\beta_i)}_{n-1} (\psi^*_2)_{ij} \right] \\
+ \frac{\tau}{4\mu} \delta_x^2 \delta_y^2 \frac{\varepsilon_i^n - \varepsilon_i^{n-1}}{\tau} \\
= \delta_x^2 \varepsilon_i^{n-\frac{1}{2}} + \delta_y^2 \varepsilon_i^{n-\frac{1}{2}} + f(x_i, y_j, t_{n-1}, u_i^{n-1}) - f(x_i, y_j, t_{n-1}, \tilde{u}_i^{n-1}),
$$

$1 \leq i \leq M_1 - 1, 1 \leq j \leq M_2 - 1, 1 \leq n \leq N.

(3.2)

Multiplying (3.2) by $h_1 h_2 \tau \delta_t \varepsilon_i^{n-\frac{1}{2}}$, summing up for $i$ from 1 to $M_1 - 1$, for $j$ from 1 to $M_2 - 1$ and for $n$ from 1 to $m$, we analyze each term in the derived
equation. Firstly, by employing Lemma 3.3 we have

\[
\Delta \beta \sum_{l=1}^{K} p(\beta_l) \frac{1}{\Gamma(2 - \beta_l)} h_{1h_2} \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \left\{ \sum_{n=1}^{m} \left[ b_0(\beta_l) \delta_l \varepsilon_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} \left( b_{n-k-1}(\beta_l) \delta_l \varepsilon_{ij}^{k-\frac{1}{2}} - b_{n-k}(\psi_{ij}^{*}) \delta_l \varepsilon_{ij}^{n-\frac{1}{2}} \right) \right] \right\}
\]

\[
\geq \Delta \beta \sum_{l=1}^{K} p(\beta_l) \frac{1}{\Gamma(2 - \beta_l)} \left[ \frac{1}{2} t_m^{1-\beta_l} \sum_{n=1}^{m} \left\| \delta_l \varepsilon_{ij}^{n-\frac{1}{2}} \right\|^2 \right] \frac{t_m^{2-\beta_l}}{2(\beta_l)} h_{1h_2} \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \left( \psi_{ij}^{*} \right)^2,
\]

(3.3)

where

\[
K_m = \Delta \beta \sum_{l=1}^{K} p(\beta_l) \frac{t_m^{1-\beta_l}}{\Gamma(2 - \beta_l)} > 0.
\]

Whereafter using the discrete Green formula, we get

\[
h_{1h_2} \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \sum_{n=1}^{m} \frac{\tau}{4\mu} \delta_{x}^{2} \delta_{y}^{2} \frac{\varepsilon_{ij}^{n} - \varepsilon_{ij}^{n-1}}{\tau} \delta_l \varepsilon_{ij}^{n-\frac{1}{2}}
\]

\[
= \frac{1}{4\mu} \sum_{n=1}^{m} h_{1h_2} \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \left( \delta_x \delta_y \left( \varepsilon_{ij}^{n-\frac{1}{2},j-i} - \varepsilon_{ij}^{n-1-\frac{1}{2},j-i} \right) \right) \left( \delta_x \delta_y \left( \varepsilon_{ij}^{n-\frac{1}{2},j-i} - \varepsilon_{ij}^{n-1-\frac{1}{2},j-i} \right) \right)
\]

\[
= \frac{1}{4\mu} \sum_{n=1}^{m} \left\| \delta_x \delta_y \left( \varepsilon^n - \varepsilon^{n-1} \right) \right\|^2 \geq 0,
\]

(3.4)

and

\[
\tau \sum_{n=1}^{m} h_{1h_2} \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \left( \delta_l \varepsilon_{ij}^{n-\frac{1}{2}} \right) \left( \delta_x^{2} \varepsilon_{ij}^{n-\frac{1}{2}} \right)
\]

\[
= - \tau \sum_{n=1}^{m} \left[ h_{1h_2} \sum_{j=1}^{M_2-1} \sum_{i=1}^{M_1} \left( \delta_x \varepsilon_{ij}^{n-\frac{1}{2},j-i} \right) \left( \delta_l \delta_x \varepsilon_{ij}^{n-\frac{1}{2},j-i} \right) \right]
\]

\[
= - \tau \sum_{n=1}^{m} \left[ h_{1h_2} \sum_{j=1}^{M_2-1} \sum_{i=1}^{M_1} \left( \frac{\delta_x \varepsilon_{ij}^{n-\frac{1}{2},j-i} + \delta_x \varepsilon_{ij}^{n-1-\frac{1}{2},j-i}}{2} \right) \left( \frac{\delta_x \varepsilon_{ij}^{n-\frac{1}{2},j-i} - \delta_x \varepsilon_{ij}^{n-1-\frac{1}{2},j-i}}{\tau} \right) \right]
\]

\[
= - \frac{1}{2} \left\| \delta_x \varepsilon^n \right\|^2 - \left\| \delta_x \varepsilon^0 \right\|^2.
\]

(3.5)
Analogous to (3.5), it is also obtained
\[
\tau \sum_{n=1}^{m} \left[ h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \left( \delta \varepsilon_{ij}^{n-\frac{1}{2}} \right) \left( \delta \varepsilon_{ij}^{n-\frac{1}{2}} \right) \right] = -\frac{1}{2} \left[ \| \delta \varepsilon^{m} \|^{2} - \| \delta \varepsilon^{0} \|^{2} \right].
\]

(3.6)

On the basis of (1.4), there holds that
\[
\begin{align*}
&h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \left[ \tau \sum_{n=1}^{m} \left( \delta \varepsilon_{ij}^{n-\frac{1}{2}} \right) f \left( x_i, y_j, t_{n-1}, u_{ij}^{n-1} \right) - f \left( x_i, y_j, t_{n-1}, \bar{u}_{ij}^{n-1} \right) \right] \\
&\leq h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \left[ \tau \sum_{n=1}^{m} \left( \delta \varepsilon_{ij}^{n-\frac{1}{2}} \right) L_f \left| u_{ij}^{n-1} - \bar{u}_{ij}^{n-1} \right| \right] \\
&\leq L_f h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \left[ \tau \sum_{n=1}^{m} \left( \delta \varepsilon_{ij}^{n-\frac{1}{2}} \right) \varepsilon_{ij}^{n-1} \right] \\
&\leq L_f h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \tau \sum_{n=1}^{m} \left[ \frac{K_m}{2L_f} \left( \delta \varepsilon_{ij}^{n-\frac{1}{2}} \right)^2 + \frac{L_f}{2K_m} \left( \varepsilon_{ij}^{n-1} \right)^2 \right] \\
&= \frac{\tau K_m}{2} \sum_{n=1}^{m} \left\| \delta \varepsilon^{n-\frac{1}{2}} \right\|^{2} + \frac{\tau L_f^2}{2K_m} \sum_{n=1}^{m} \| \varepsilon^{n-1} \|^{2}.
\end{align*}
\]

(3.7)

From Equations (3.3)–(3.7), the inequality below is derived
\[
\begin{align*}
\left\| \delta \varepsilon^{m} \right\|^{2} + \left\| \delta \varepsilon^{m} \right\|^{2} &\leq \left\| \delta \varepsilon^{0} \right\|^{2} + \left\| \delta \varepsilon^{0} \right\|^{2} \\
&+ \Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{T^2-\beta_i}{(3-\beta_i)} \left\| \varepsilon_{i}^{n} \right\|^{2} + \frac{\tau L_f^2}{2K_m} \sum_{n=1}^{m} \| \varepsilon^{n-1} \|^{2}.
\end{align*}
\]

(3.8)

According to Lemma 3.1 and Lemma 3.2 we deduce from (3.8) that
\[
\left\| \varepsilon^{n} \right\|^{2} \leq \frac{1}{144} \left\| \Delta h \varepsilon^{0} \right\|^{2} + \frac{1}{12} \Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{T^2-\beta_i}{(3-\beta_i)} \left\| \varepsilon_{i}^{n} \right\|^{2} \\
&+ \frac{\tau L_f^2}{12 \Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{T^2-\beta_i}{(3-\beta_i)}} \sum_{k=1}^{n} \| \varepsilon^{k-1} \|^{2}, \quad 1 \leq n \leq N.
\]

(3.9)

Finally, taking Lemma 3.4 it follows that
\[
\left\| \varepsilon^{n} \right\|^{2} \leq \exp \left( - \frac{L_f}{12 \Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{T^2-\beta_i}{(3-\beta_i)}} \right).
\]

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This completes the proof.

In the following we consider the convergence of the difference approximation. Noticing that \( U^n_{ij} \) is the exact solution of the system (1.1)–(1.3) and \( u^n_{ij} \) is the numerical solution of the difference scheme (2.13)–(2.15), we denote the error

\[
e^n_{ij} = U^n_{ij} - u^n_{ij}, \quad 0 \leq i \leq M_1, \quad 0 \leq j \leq M_2, \quad 0 \leq n \leq N.
\]

Substituting (2.13)–(2.15) from (2.10)–(2.12), we get the error equations

\[
\Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{\tau^{1-\beta_i}}{\Gamma(3-\beta_i)} \left[ e_n^{(\beta_i)} \delta_i e^{n-\frac{1}{2}}_{ij} - \sum_{k=1}^{n-1} (a^{(\beta_i)}_{n-k-1} - a^{(\beta_i)}_{n-k}) \delta_t e^{n-\frac{1}{2}}_{ij} \right] + \frac{\tau}{4\mu} \delta_x^2 \delta_y^2 e^n_{ij} - e^{n-1}_{ij} = 0
\]

\[
= \delta_x^2 e^n_{ij} - \delta_x e^{n-\frac{1}{2}}_{ij} + \delta_y^2 e^n_{ij} - \delta_y e^{n-\frac{1}{2}}_{ij} + f(x_i, y_j, t_{n-1}, U^n_{ij}) - f(x_i, y_j, t_{n-1}, u^n_{ij})
\]

\[
+ R^n_{ij} - \tilde{R}^n_{ij} + \hat{R}^{n-\frac{1}{2}}_{ij}, \quad 1 \leq i \leq M_1 - 1, \quad 1 \leq j \leq M_2 - 1, \quad 1 \leq n \leq N,
\]

(3.9)

\[
e^n_{ij} = 0, \quad 1 \leq i \leq M_1 - 1, \quad 1 \leq j \leq M_2 - 1,
\]

\[
e^n_{ij} = 0, \quad (i, j) \in \gamma, \quad 0 \leq n \leq N.
\]

**Theorem 3.3.** Suppose that the continuous problem (1.1)–(1.3) has solution \( u(x, y, t) \in C_{x,y,t}^{4,4,3}(\Omega \times [0, T]) \). Then there is a positive constant \( C \) such that

\[
\|e^n\| \leq C(\tau + h_1^2 + h_2^2 + \Delta \beta^2).
\]

**Proof.** The proof of convergence is similar to that of Theorem 3.2. Multiplying (3.9) by \( h_1 h_2 \tau \delta_t e_{ij}^{n-\frac{1}{2}} \), summing up for \( i \) from 1 to \( M_1 - 1 \), for \( j \) from 1 to \( M_2 - 1 \) and for \( n \) from 1 to \( m \), we estimate each term in the resulted equation.

By using analogous strategies as (3.3)–(3.7), we get (3.10)–(3.14) correspondingly.

\[
\Delta \beta \sum_{i=1}^{K} p(\beta_i) \frac{\tau^{1-\beta_i}}{\Gamma(3-\beta_i)} h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} \left\{ \sum_{k=1}^{n-1} (a^{(\beta_i)}_{n-k-1} - a^{(\beta_i)}_{n-k}) \delta_t e^{k-\frac{1}{2}}_{ij} \right\} + \frac{\tau}{4\mu} \delta_x^2 \delta_y^2 \sum_{i=1}^{m} \sum_{j=1}^{m} \delta_t e^{n-\frac{1}{2}}_{ij} = 0
\]

\[
\geq \frac{1}{2} \tau K_m \sum_{n=1}^{m} \|\delta_t e^{n-\frac{1}{2}}\|^2,
\]

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As for the remainder, it is deduced that

\[
\tau \sum_{n=1}^{m} \left[ \frac{M_1 - M_2 - 1}{4K} \left( \delta e_v^{(n)} \right)^2 \right] = \frac{1}{4K} \sum_{n=1}^{m} \left\| \delta e_v^{(n)} \right\|^2 = 0,
\]

(3.11)

and

\[
\tau \sum_{n=1}^{m} \left[ \frac{M_1 - M_2 - 1}{4K} \left( \delta e_v^{(n)} \right)^2 \right] = \frac{1}{2} \left\| \delta e_v^{(n)} \right\|^2 = 0,
\]

(3.12)

\[
\tau \sum_{n=1}^{m} \left[ \frac{M_1 - M_2 - 1}{4K} \left( \delta e_v^{(n)} \right)^2 \right] = \frac{1}{2} \left\| \delta e_v^{(n)} \right\|^2 = 0.
\]

(3.13)

As for the remainder, it is deduced that

\[
\sum_{n=1}^{m} \left[ \frac{M_1 - M_2 - 1}{4K} \left( \delta e_v^{(n)} \right)^2 \right] = \frac{1}{4K} \sum_{n=1}^{m} \left\| \delta e_v^{(n)} \right\|^2 = 0,
\]

(3.14)

\[
\sum_{n=1}^{m} \left[ \frac{M_1 - M_2 - 1}{4K} \left( \delta e_v^{(n)} \right)^2 \right] = \frac{1}{2} \left\| \delta e_v^{(n)} \right\|^2 = 0.
\]

(3.15)
From (3.10)–(3.15) it follows that
\[
\frac{1}{2} \left(\|\delta_x e^m\|^2 + \|\delta_y e^m\|^2\right) \leq \frac{TL_1 L_2}{K_m} \left[ (C_3 + C_4 + C_5) \left(\tau + h_1^2 + h_2^2 + \Delta^2\right) \right]^2 \\
+ \frac{\tau L_j^2}{K_m} \sum_{n=1}^m \|e^{n-1}\|^2,
\]
i.e.,
\[
\frac{1}{2} \|e^m\|^2 \leq \frac{TL_1 L_2}{6\Delta^2 \sum_{i=1}^K p(\beta_i) 1/(2-\beta_i) T_{1-\beta_i}} \left[ (C_3 + C_4 + C_5) \left(\tau + h_1^2 + h_2^2 + \Delta^2\right) \right]^2 \\
+ \frac{\tau L_j^2}{6K_n} \sum_{k=1}^n \|e^{k-1}\|^2.
\]
According to Lemma 3.1 we obtain
\[
\|e^n\|^2 \leq \frac{TL_1 L_2}{6\Delta^2 \sum_{i=1}^K p(\beta_i) 1/(2-\beta_i) T_{1-\beta_i}} \left[ (C_3 + C_4 + C_5) \left(\tau + h_1^2 + h_2^2 + \Delta^2\right) \right]^2 \\
+ \frac{\tau L_j^2}{6K_n} \sum_{k=1}^n \|e^{k-1}\|^2, \quad 0 \leq n \leq N.
\]
Therefore,
\[
\|e^n\|^2 \leq \frac{TL_1 L_2}{6\Delta^2 \sum_{i=1}^K p(\beta_i) 1/(2-\beta_i) T_{1-\beta_i}} \left[ (C_3 + C_4 + C_5) \left(\tau + h_1^2 + h_2^2 + \Delta^2\right) \right]^2 \\
\cdot \exp \left( \frac{L_j^2}{6\Delta^2 \sum_{i=1}^K p(\beta_i) 1/(2-\beta_i) T_{1-\beta_i}} \right),
\]
where Lemma 3.4 is applied. This completes the proof.

4. Numerical results

In this section, a numerical example is tested to demonstrate the effectiveness of the proposed scheme, and verify the theoretical results including convergence orders and numerical stability. The discrete $L^2$ and $L^\infty$ norms are both taken to measure the numerical errors. Denote
\[
\|e^N\|_{L^2} := \left( \sum_{j=1}^{M_2-1} \sum_{i=1}^{M_1-1} \left| U_{ij}^N - u_{ij}^N \right|^2 h_1 h_2 \right)^{\frac{1}{2}},
\]
and
\[
\|e^N\|_{L^\infty} := \max_{1 \leq j \leq M_2-1, 1 \leq i \leq M_1-1} \left| U_{ij}^N - u_{ij}^N \right|.
\]
Example 4.1.

\[ \int_1^2 \Gamma(4 - \beta)_0^C D_t^\beta u(x, y, t) d\beta = \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} + \sin x \sin y \left[ 2(t^3 + 2t + 4) + \frac{6t^2 - 6t}{\ln t} \right] - (t^3 + 2t + 4)^2 \sin^2 x \sin^2 y + u^2, \]

\[ 0 < t < 1/2, \quad (x, y) \in \Omega = (0, \pi) \times (0, \pi), \]

\[ u(x, y, t) = 0, \quad (x, y) \in \partial \Omega, \quad 0 < t < 1/2, \]

\[ u(x, y, 0) = 4 \sin x \sin y, \quad u_t(x, y, 0) = 2 \sin x \sin y, \quad (x, y) \in \Omega, \]

whose analytical solution is known and is given by

\[ u(x, y, t) = (t^3 + 2t + 4) \sin x \sin y. \]

In Figure 1 we illustrate the relative error, which verifies the convergence of the algorithm we proposed.

In Figure 2 we present a comparison of the exact and numerical solutions. It can be seen that the numerical solution is in good agreement with the exact solution.

Figure 1: Relative error at \( T = 0.5 \), obtained by algorithm (2.16) \( - (2.17) \) with mesh \( h_1 = h_2 = \frac{\pi}{4 \tau}, \Delta \beta = \frac{1}{4 \tau}, \) and \( \tau = \frac{1}{100 \pi \tau} \).
Figure 2: Exact solution (a) and approximate solution (b) obtained by algorithm \((2.16)-(2.17)\) at \(T = 0.5\) with mesh \(h_1 = h_2 = \frac{\pi}{64}, \Delta \beta = \frac{1}{64}\), and \(\tau = \frac{1}{4096}\).

In Table 1, the numerical accuracy of difference scheme \((2.16)-(2.17)\) in time is recorded. Let the step sizes \(h_1, h_2\), and \(\Delta \beta\) be fixed and small enough such that the dominated error arise from the approximation of the time derivatives. Varying the step sizes in time, the numerical errors in discrete both \(L^\infty\) and \(L^2\) norms and the associated convergence orders are shown in this table respectively, which can be found in agreement with the theoretical analysis.

In Table 2, we take the fixed and small enough step sizes in space, and adopt an optimal step size ratio in time and distributed order. As \(\Delta \beta\) and \(\tau\) vary, we compute the errors and convergence orders listed in the table, which indicates that the convergence order in time and distributed order are about one and two, respectively.

Table 3 displays the computational results with an optimal step size ratio in time, space and distributed order. We can conclude from this table that the convergence orders with respect to time, space and distributed order are approximately one, two and two, respectively, which is in good agreement with our theoretical results analyzed in Section 3.
Table 1: Errors and convergence orders for Example 4.1 in temporal direction with $h_1 = h_2 = \frac{\pi}{500}$ and $\Delta \beta = \frac{1}{160}$.

| $\tau$ | $\|e^N\|_{L^\infty}$ | Order | $\|e^N\|_{L^2}$ | Order |
|--------|-----------------|-------|-----------------|-------|
| 1/10   | 0.0839          | -     | 0.1225          | -     |
| 1/20   | 0.0439          | 0.9344| 0.0634          | 0.9502|
| 1/40   | 0.0227          | 0.9515| 0.0326          | 0.9596|
| 1/80   | 0.0117          | 0.9526| 0.0167          | 0.9650|
| 1/160  | 0.0059          | 0.9877| 0.0085          | 0.9743|

Table 2: Errors and convergence orders for Example 4.1 with an optimal step size ratio for $\tau$ and $\Delta \beta$, and $h_1 = h_2 = \frac{\pi}{500}$.

| $\tau$ | $\Delta \beta$ | $\|e^N\|_{L^\infty}$ | Order | $\|e^N\|_{L^2}$ | Order |
|--------|----------------|-----------------|-------|-----------------|-------|
| 1/100  | 1/10           | 0.0093          | -     | 0.0133          | -     |
| 1/400  | 1/20           | 0.0024          | 1.9542| 0.0034          | 1.9678|
| 1/1600 | 1/40           | 6.0481e-04      | 1.9885| 8.6411e-04      | 1.9762|
| 1/6400 | 1/80           | 1.4751e-04      | 2.0357| 2.1076e-04      | 2.0365|

Table 3: Errors and convergence orders for Example 4.1 with an optimal step size ratio for $\tau$, $h_1$, $h_2$, and $\Delta \beta$.

| $\tau$ | $h_1 = h_2$ | $\Delta \beta$ | $\|e^N\|_{L^\infty}$ | Order | $\|e^N\|_{L^2}$ | Order |
|--------|-------------|----------------|-----------------|-------|-----------------|-------|
| 1/64   | $\pi/2$    | 1/8            | 0.4602          | -     | 0.7230          | -     |
| 1/256  | $\pi/4$    | 1/16           | 0.1195          | 1.9453| 0.1689          | 2.0978|
| 1/1024 | $\pi/8$    | 1/32           | 0.0301          | 1.9892| 0.0426          | 1.9872|
| 1/4096 | $\pi/16$   | 1/64           | 0.0075          | 2.0048| 0.0107          | 1.9932|
| 1/16384| $\pi/32$   | 1/128          | 0.0019          | 1.9809| 0.0027          | 1.9866|
| 1/65536| $\pi/64$   | 1/256          | 4.7098e-04      | 2.0123| 6.6801e-04      | 2.0150|
5. Conclusion

In this paper, we construct efficient numerical scheme for solving two-dimensional time-fractional wave equation of distributed-order with a nonlinear source term, and provide the theoretical analysis on stability and convergence by the discrete energy method. Numerical results are provided by figures and tables, which show the algorithm proposed in this work is effective and feasible. In the future work, the promotion of computational efficiency will be considered so that the more complicated problems can be handled.

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