A generalized analytic Fourier-Feynman transform on the product function space $C^{2}_{a,b}[0, T]$ and related topics

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Abstract: In this paper we obtain various results involving the generalized analytic Fourier-Feynman transform and the first variation of functionals in a Fresnel type class defined on the product function space $C^{2}_{a,b}[0, T]$.

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1 Introduction

Let $H$ be a separable Hilbert space and let $\mathcal{M}(H)$ be the space of all complex-valued Borel measures on $H$. The Fourier transform of $\sigma$ in $\mathcal{M}(H)$ is defined by

$$f(\sigma)(h') \equiv \hat{\sigma}(h') = \int_H \exp\{i\langle h, h' \rangle\}d\sigma(h), \quad h' \in H. \quad (1.1)$$

The set of all functions of the form (1.1) is denoted by $\mathcal{F}(H)$ and is called the Fresnel class of $H$. Let $(H,B,\nu)$ be an abstract Wiener space. It is known [25, 26] that each functional of the form (1.1) can be extended to $B$ uniquely by

$$\tilde{\sigma}(x) = \int_H \exp\{i(h,x)\}d\sigma(h), \quad x \in B \quad (1.2)$$

where $\langle \cdot, \cdot \rangle$ is a stochastic inner product between $H$ and $B$. The Fresnel class $\mathcal{F}(B)$ of $B$ is the space of (equivalence classes of) all functionals of the form (1.2).

There has been a tremendous amount of papers and books in the literature on the Fresnel integral theory and Fresnel classes $\mathcal{F}(B)$ and $\mathcal{F}(H)$ on abstract Wiener and Hilbert spaces. For an elementary introduction, see [24 Chapter 20].

Furthermore, in [25], Kallianpur and Bromley introduced a larger class $\mathcal{F}_{A_1,A_2}$ than the Fresnel class $\mathcal{F}(B)$ and showed the existence of the analytic Feynman integral of functionals in $\mathcal{F}_{A_1,A_2}$ for a successful treatment of certain physical problems by means of a Feynman integral. The Fresnel class $\mathcal{F}_{A_1,A_2}$ of $B^2$ is the space of (equivalence classes of) all functionals on $B^2$ of the form

$$F(x_1,x_2) = \int_{H^2} \exp\left\{ \sum_{j=1}^{2} i(A_j^{1/2}h,x_j)\right\}d\sigma(h)$$

where $A_1$ and $A_2$ are bounded, nonnegative and self-adjoint operators on $H$ and $\sigma \in \mathcal{M}(H)$.

Let $A$ be a nonnegative self-adjoint operator on $H$ and let $\sigma$ be any complex Borel measure on $H$. Then the functional

$$F(x) = \int_{H} \exp\{i(A^{1/2}h,x)\}d\sigma(h) \quad (1.3)$$
belongs to the Fresnel class $\mathcal{F}(B)$ on $B$ because it can be rewritten as
\[
\int_H \exp\{i(h, x)\} d\sigma_A(h)
\]
for $\sigma_A = \sigma \circ (A^{1/2})^{-1}$. For the functional $F$ given by equation (1.3), the analytic
Feynman integral $\int_B^{\text{anf}_q} F(x) d\nu(x)$ with parameter $q = 1$ (based on the connection
with the Fresnel integral of $F$ in $\mathcal{F}(H)$ by Albeverio and Høegh-Krohn, the most
important value of the parameter $q$ is $q = 1$) on $B$ exists and is given by
\[
\int_B^{\text{anf}_q} F(x) d\nu(x) = \int_H \exp\left\{ -\frac{i}{2} \langle Ah, h \rangle \right\} d\sigma(h).
\]
If we choose $A$ to be the identity operator on $H$, then equation (1.4) is equal to ‘the Fresnel integral $\mathcal{F}(f)$’ of $f(\sigma)$, studied by Albeverio and Høegh-Krohn in [1].

The concept of the Fresnel integral is not based on the technique of analytic con-
tinuation, but appears as solutions of important problems in quantum mechanics
and in quantum field theory.

Let $A$ be a bounded self-adjoint operator on $H$. Then we may write
\[
A = A_+ - A_-
\]
where $A_+$ and $A_-$ are each bounded, nonnegative and self-adjoint. Take $A_1 = A_+$
and $A_2 = A_-$ in the definition of $\mathcal{F}_{A_1, A_2}$ above. For any $F$ in $\mathcal{F}_{A_1, A_2}$, the analytic
Feynman integral of $F$ with parameter $(1, -1)$ is given by
\[
\int_B^{\text{anf}_{(1,-1)}} F(x_1, x_2) d(\nu \times \nu)(x_1, x_2) = \int_H \exp\left\{ -\frac{i}{2} \langle Ah, h \rangle \right\} d\sigma(h).
\]

Kallianpur and Bromley, using this idea, studied and investigated relationships
between the Albeverio and Høegh-Krohn’s Fresnel integral with respect to a sym-
metric bilinear form $\Delta$ on $H$ (see [1, Chapter 4]) and the analytic Feynman integral
given by equation (1.5).

In this paper we extend the ideas in [25] from the functionals on $B^2$ to the
functionals on the product function space $C^2_{\alpha, \beta}[0, T]$. The function space $C_{\alpha, \beta}[0, T]$, induced by generalized Brownian motion, was introduced by J. Yeh [28, 29] and
was used extensively by Chang and Chung [11], Chang and Skoug [13], and Chang,
Chung and Skoug [19]. In this paper we also construct a concrete theory of the
generalized analytic Fourier-Feynman transform(GFFT) and the first variation of
functionals in a generalized Fresnel type class defined on $C^2_{\alpha, \beta}[0, T]$. Other work
involving GFFT theories on $C_{\alpha, \beta}[0, T]$ include [14, 17, 20].

The Wiener process used in [2, 3, 4, 5, 6, 7, 9, 10, 12, 22, 23, 25, 26, 30] is
stationary in time and is free of drift while the stochastic process used in this
paper, as well as in [11, 13, 14, 15, 16, 17, 18, 20, 28], is nonstationary in time, is
subject to a drift $a(t)$, and can be used to explain the position of the Ornstein-
Uhlenbeck process in an external force field [27]. It turns out, as noted in Remark
3.5 below, that including a drift term $a(t)$ makes establishing the existence of the
GFFT of functionals on $C^2_{\alpha, \beta}[0, T]$ very difficult. However, when $a(t) \equiv 0$
and $b(t) = t$ on $[0, T]$, the general function space $C_{\alpha, \beta}[0, T]$ reduces to the Wiener space
$C_0[0, T]$. 

3
2 Definitions and preliminaries

Let $D = [0, T]$ and let $(\Omega, \mathcal{F}, P)$ be a probability measure space. A real-valued stochastic process $Y$ on $(\Omega, \mathcal{F}, P)$ and $D$ is called a generalized Brownian motion process if $Y(0, \omega) = 0$ almost everywhere and for $0 = t_1 < t_2 < \cdots < t_n \leq T$, the $n$-dimensional random vector $(Y(t_1, \omega), \ldots, Y(t_n, \omega))$ is normally distributed with density function

$$K_n(t, \bar{u}) = \left( \prod_{j=1}^{n} 2\pi (b(t_j) - b(t_{j-1})) \right)^{-1/2} \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{[(u_j - a(t_j)) - (u_{j-1} - a(t_{j-1}))]^2}{b(t_j) - b(t_{j-1})} \right\}$$

where $\bar{u} = (u_1, \ldots, u_n)$, $u_0 = 0$, $t = (t_1, \ldots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0, T]$ with $a(0) = 0$, $a'(t) \in L^2[0, T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0, T]$. As explained in [29, p.18–20], $Y$ induces a probability measure $\mu$ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real-valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest $\sigma$-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_i(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process $Y$ determined by $a(\cdot)$ and $b(\cdot)$.

In [29], Yeh shows that the generalized Brownian motion process $Y$ determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s, t) = \min\{b(s), b(t)\}$, and that the probability measure $\mu$ induced by $Y$, taking a separable version, is supported by $C_{a,b}[0, T]$ (which is equivalent to the Banach space of continuous functions $x$ on $[0, T]$ with $x(0) = 0$ under the sup norm). Hence, $(C_{a,b}[0, T], B(C_{a,b}[0, T]), \mu)$ is the function space induced by $Y$ where $B(C_{a,b}[0, T])$ is the Borel $\sigma$-algebra of $C_{a,b}[0, T]$. We then complete this function space to obtain $(C_{a,b}[0, T], W(C_{a,b}[0, T]), \mu)$ where $W(C_{a,b}[0, T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0, T]$.

A subset $B$ of $C_{a,b}[0, T]$ is said to be scale-invariant measurable provided $\rho B$ is $W(C_{a,b}[0, T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set $N$ is said to be a scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional $F$ is said to be scale-invariant measurable provided $F$ is defined on a scale-invariant measurable set and $F(\cdot)$ is $W(C_{a,b}[0, T])$-measurable for every $\rho > 0$. If two functionals $F$ and $G$ defined on $C_{a,b}[0, T]$ are equal s-a.e., we write $F \approx G$.

Let $L^2_{a,b}[0, T]$ be the set of functions on $[0, T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0, T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0, T] = \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\}$$
where \(|a(\cdot)|\) is the total variation function of \(a(\cdot)\). Then \(L_{0, b}^2[0, T]\) is a separable
Hilbert space with inner product defined by
\[
(u, v)_{a, b} = \int_0^T u(t)v(t)d|b(t)| + |a|(t).
\]
In particular, note that \(\|u\|_{a, b} \equiv \sqrt{(u, u)_{a, b}} = 0\) if and only if \(u(t) = 0\) a.e. on
\([0, T]\).

Let \(\{\phi_j\}_{j=1}^\infty\) be a complete orthonormal set of real-valued functions of bounded
variation on \([0, T]\) such that
\[
(\phi_j, \phi_k)_{a, b} = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}
\]
Then for each \(v \in L_{a, b}^2[0, T]\), the Paley-Wiener-Zygmund (PWZ) stochastic integral
\(\langle v, x \rangle\) is defined by the formula
\[
\langle v, x \rangle = \lim_{n \to \infty} \int_0^T \sum_{j=1}^n (v, \phi_j)_{a, b}\phi_j(t)dx(t)
\]
for all \(x \in C_{a, b}[0, T]\) for which the limit exists; one can show that for each \(v \in L_{a, b}^2[0, T]\), the PWZ stochastic integral \(\langle v, x \rangle\) exists for \(\mu\)-a.e. \(x \in C_{a, b}[0, T]\) and if \(v\) is of bounded variation on \([0, T]\), then the PWZ stochastic integral \(\langle v, x \rangle\) equals
the Riemann-Stieltjes integral \(\int_0^T v(t)dx(t)\) for \(s\)-a.e. \(x \in C_{a, b}[0, T]\). For more
details, see [13].

**Remark 2.1.** For each \(v \in L_{a, b}^2[0, T]\), the PWZ stochastic integral \(\langle v, x \rangle\) is
a Gaussian random variable on \(C_{a, b}[0, T]\) with mean \(\int_0^T v(s)da(s)\) and variance
\(\int_0^T v^2(s)db(s)\). Note that for all \(u, v \in L_{a, b}^2[0, T]\),
\[
\int_{C_{a, b}[0, T]} (u, x)\langle v, x \rangle d\mu(x) = \int_0^T u(s)v(s)da(s) + \int_0^T u(s)da(s)\int_0^T v(s)da(s).
\]
Hence we see that for all \(u, v \in L_{a, b}^2[0, T]\), \(\int_0^T u(s)v(s)db(s) = 0\) if and only if
\(\langle u, x \rangle\) and \(\langle v, x \rangle\) are independent random variables.

**Remark 2.2.** Recall that above, as well as in papers [13, 14, 15, 17], we require that \(a : [0, T] \to \mathbb{R}\) is an absolutely continuous function with \(a(0) = 0\) and with
\(\int_0^T |a'(t)|^2 dt < \infty\). Now throughout this paper we add the requirement
\[
\int_0^T |a'(t)|^2 d|a|(t) < \infty. \tag{2.1}
\]

**Remark 2.3.** Note that the function \(a(t) = t^{2/3}, 0 \leq t \leq T\) doesn’t satisfy
condition (2.1) even though its derivative is an element of \(L^2[0, T]\).

**Remark 2.4.** The function \(a : [0, T] \to \mathbb{R}\) satisfies the requirements in Remark
2.2 if and only if the function \(a'\) is an element of \(L_{a, b}^2[0, T]\).
The following Cameron-Martin like subspace of $C_{a,b}[0,T]$ plays an important role throughout this paper.

Let

$$C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0,T] \right\}.$$  

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D_t : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$D_t w = z(t) = \frac{w'(t)}{b'(t)}.$$  

(2.2)

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1,w_2)_{C'_{a,b}} = \int_0^T D_tw_1D_tw_2db(t) = \int_0^T z_1(t)z_2(t)db(t)$$

is a separable Hilbert space.

Note that the linear operator given by equation (2.2) is a homeomorphism. In fact, the inverse operator $D_t^{-1} : L^2_{a,b}[0,T] \to C'_{a,b}[0,T]$ is given by

$$D_t^{-1}z = \int_0^t z(s)db(s).$$

It is easy to show that $D_t^{-1}$ is a bounded operator since

$$\|D_t^{-1}z\|_{C'_{a,b}} = \left\| \int_0^t z(s)db(s) \right\|_{C'_{a,b}} = \left( \int_0^T z^2(t)db(t) \right)^{\frac{1}{2}} \leq \left( \int_0^T z^2(t)d[b(t)] + |a(t)| \right)^{\frac{1}{2}} = \|z\|_{a,b}.$$  

Applying the open mapping theorem, we see that $D_t$ is also bounded and there exist positive real numbers $\alpha$ and $\beta$ such that $\alpha \|w\|_{C'_{a,b}} \leq \|D_tw\|_{a,b} \leq \beta \|w\|_{C'_{a,b}}$ for all $w \in C'_{a,b}[0,T]$. Hence we see that the Borel $\sigma$-algebra on $(C'_{a,b}[0,T], \| \cdot \|_{C'_{a,b}})$ is given by

$$\mathcal{B}(C'_{a,b}[0,T]) = \{ D_t^{-1}(E) : E \in \mathcal{B}(L^2_{a,b}[0,T]) \}.$$  

**Remark 2.5.** Our conditions on $b : [0,T] \to \mathbb{R}$ imply that $0 < \delta < b'(t) < M$ for some positive real numbers $\delta$ and $M$, and all $t \in [0,T]$.

The following lemma follows quite easily from Remarks 2.2, 2.4 and 2.5 above and the fact that $a(t) = \int_0^t \frac{a'(s)}{\nu(s)}db(s)$ on $[0,T]$.

**Lemma 2.6.** The function $a : [0,T] \to \mathbb{R}$ satisfies the conditions in Remark 2.2 if and only if $a$ is an element of $C'_{a,b}[0,T]$.

Throughout this paper for $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, we will use the notation $(w,x)^\sim$ instead of $\langle z,x \rangle = \langle D_tw,x \rangle$. Then we have the following assertions.
(i) For each \( w \in C'_{a,b}[0,T] \), the random variable \( x \mapsto (w, x) \sim \) is Gaussian with mean \( (w, a)_{C'_{a,b}} \) and variance \( \| w \|^2_{C_{a,b}} \).

(ii) \((w, \alpha x) \sim = (\alpha w, x) \sim = \alpha (w, x) \sim \) for any real number \( \alpha \), \( w \in C'_{a,b}[0,T] \) and \( x \in C_{a,b}[0,T] \).

(iii) If \( \{w_1, \ldots, w_n\} \) is an orthonormal set in \( C'_{a,b}[0,T] \), then the random variables \((w_i, x) \sim \)'s are independent.

We denote the function space integral of a \( W(C_{a,b}[0,T]) \)-measurable functional \( F \) by

\[
E[F] = E_x[F(x)] = \int_{C_{a,b}[0,T]} F(x) d\mu(x)
\]

whenever the integral exists.

**Remark 2.7.** For each \( t \in [0,T] \), let

\[
\beta_t(s) = \int_0^s \chi_{[0,t]}(\tau) d\tau = \begin{cases} b(s), & 0 \leq s \leq t \\ b(t), & t \leq s \leq T \end{cases}.
\]

Then the family of functions \( \{\beta_t : 0 \leq t \leq T\} \) from \( C'_{a,b}[0,T] \) has the reproducing property

\[
(w, \beta_t)_{C'_{a,b}} = w(t)
\]

for all \( w \in C'_{a,b}[0,T] \). Note that \( \beta_t(s) = \min\{b(s), b(t)\} \), the covariance function associated with the generalized Brownian motion \( Y \) used in this paper. We also note that for each \( x \in C_{a,b}[0,T] \),

\[
x(t) = \int_0^T \chi_{[0,t]}'(s) dx(s) = (\beta_t, x) \sim .
\]

**Remark 2.8.** Let \( A : C'_{a,b}[0,T] \rightarrow C'_{a,b}[0,T] \) be a bounded linear operator with adjoint \( A^* \). Let \( w \) and \( g \) be elements of \( C_{a,b}[0,T] \) with \( g(t) = \int_0^t z(s) db(s) \) for some \( z \in L^2_{a,b}[0,T] \). Then

\[
E_x[(Aw, x) \sim] = E_x[(D_t Aw, x)] = \int_0^T D_t Aw da(t) = \int_0^T D_t Aw \frac{a'(t)}{b'(t)} db(t)
\]

\[
= \int_0^T D_t Aw D_t adb(t) = (Aw, a)_{C'_{a,b}} = (w, A^* a)_{C_{a,b}}
\]

and

\[
(Aw, g) \sim = \langle D_t Aw, g \rangle = \int_0^T D_t Aw dg(t) = \int_0^T D_t Aw \frac{g'(t)}{b'(t)} db(t)
\]

\[
= \int_0^T D_t Aw D_t g(t) db(t) = (Aw, g)_{C'_{a,b}} = (w, A^* g)_{C_{a,b}}.
\]

Next, letting \( A \) be the identity operator, yields the formulas

\[
E_x[(w, x) \sim] = \int_0^T D_t da(t) = (g, a)_{C_{a,b}} \text{ and } (w, g) \sim = (w, g)_{C_{a,b}}.
\]
In this paper, as possible, we adopt the definitions and notations of [13, 15, 20] for the definitions of the generalized analytic Feynman integral and the GFFT of functionals on \( C_{a,b}[0, T] \).

The following integration formula is used several times in this paper:

\[
\int_{\mathbb{R}} \exp\{-\alpha u^2 + \beta u\} \, du = \sqrt{\frac{\pi}{\alpha}} \exp\left\{ \frac{\beta^2}{4\alpha} \right\}
\]

for complex numbers \( \alpha \) and \( \beta \) with \( \Re(\alpha) > 0 \).

3 The GFFT of functionals in a Banach algebra \( \mathcal{F}_{A_1, A_2}^{a,b} \)

Let \( \mathcal{M}(C'_{a,b}[0, T]) \) be the space of complex-valued, countably additive (and hence finite) Borel measures on \( C'_{a,b}[0, T] \). \( \mathcal{M}(C'_{a,b}[0, T]) \) is a Banach algebra under the total variation norm and with convolution as multiplication.

We define the Fresnel type class \( \mathcal{F}(C_{a,b}[0, T]) \) of functionals on \( C_{a,b}[0, T] \) as the space of all stochastic Fourier transforms of elements of \( \mathcal{M}(C'_{a,b}[0, T]) \); that is, \( F \in \mathcal{F}(C_{a,b}[0, T]) \) if and only if there exists a measure \( f \) in \( \mathcal{M}(C'_{a,b}[0, T]) \) such that

\[
F(x) = \int_{C'_{a,b}[0, T]} \exp\{iw, x\} \, df(w)
\]

for \( \text{s-a.e. } x \in C_{a,b}[0, T] \). More precisely, since we shall identify functionals which coincide \( \text{s-a.e.} \) on \( C_{a,b}[0, T] \), \( \mathcal{F}(C_{a,b}[0, T]) \) can be regarded as the space of all \( \text{s-equivalence classes of functionals of the form (3.1)} \).

The Fresnel type class \( \mathcal{F}(C_{a,b}[0, T]) \) is a Banach algebra with norm \( \|F\| = \|f\| = \int_{C'_{a,b}[0, T]} df(w) \).

In fact, the correspondence \( f \mapsto F \) is injective, carries convolution into pointwise multiplication and is a Banach algebra isomorphism where \( f \) and \( F \) are related by (3.1).

**Remark 3.1.** (1) The Banach algebra \( \mathcal{F}(C_{a,b}[0, T]) \) contains several interesting functions which arise naturally in quantum mechanics: Let \( \mathcal{M}(\mathbb{R}) \) be the class of \( \mathcal{C}- \)valued countably additive measures on \( \mathcal{B}(\mathbb{R}) \), the Borel class of \( \mathbb{R} \). For \( \nu \in \mathcal{M}(\mathbb{R}) \), the Fourier transform \( \widehat{\nu} \) of \( \nu \) is a complex-valued function defined on \( \mathbb{R} \) by the formula

\[
\widehat{\nu}(u) = \int_{\mathbb{R}} \exp\{iuv\} \, d\nu(v).
\]

Let \( \mathcal{G} \) be the set of all complex-valued functions on \( [0, T] \times \mathbb{R} \) of the form \( \theta(s, u) = \widehat{\sigma_s}(u) \) where \( \{\sigma_s : 0 \leq s \leq T\} \) is a family from \( \mathcal{M}(\mathbb{R}) \) satisfying the following two conditions:

(i) For every \( E \in \mathcal{B}(\mathbb{R}) \), \( \sigma_s(E) \) is Borel measurable in \( s \),
(ii) $\int_0^T \| \sigma_s \| dB(s) < +\infty$.

Let $\theta \in \mathcal{G}$ and let $H$ be given by

$$H(x) = \exp \left\{ \int_0^T \theta(t, x(t)) \, dt \right\}$$

for $s$-a.e. $x \in C_{a,b}[0,T]$. It was shown in [13] that the function $\theta(t,u)$ is Borel-measurable and that $\theta(t, x(t))$, $\int_0^T \theta(t, x(t)) \, dt$ and $H(x)$ are elements of $F(C_{a,b}[0,T])$. This fact is relevant to quantum mechanics where exponential functions play a prominent role. For more details, see [18].

(2) In [13] Chang and Skoug introduced a Banach algebra $S(L^2_{a,b}[0,T])$ of functionals on $C_{a,b}[0,T]$ given by

$$S(L^2_{a,b}[0,T]) = \left\{ F : F(x) \approx \int_{L^2_{a,b}[0,T]} \exp\{i(v, x)\} \, d\sigma(v), \sigma \in \mathcal{M}(L^2_{a,b}[0,T]) \right\},$$

and then showed that the generalized analytic Feynman integral and the GFFT exist for $F \in S(L^2_{a,b}[0,T])$ under appropriate conditions. If

$$F(x) \approx \int_{L^2_{a,b}[0,T]} \exp\{i(v, x)\} \, d\sigma(v)$$

for some $\sigma \in \mathcal{M}(L^2_{a,b}[0,T])$, then we have

$$F(x) \approx \int_{C'_{a,b}[0,T]} \exp\{i(w, x)\} \, d(\sigma \circ D_t)(w)$$

where $D_t : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ is given by equation (2.2) above. Conversely, if

$$F(x) \approx \int_{C'_{a,b}[0,T]} \exp\{i(w, x)\} \, df(w)$$

for some $f \in \mathcal{M}(C'_{a,b}[0,T])$, then we have

$$F(x) \approx \int_{L^2_{a,b}[0,T]} \exp\{i(v, x)\} \, d(f \circ D_t^{-1})(v).$$

Thus we have that $F \in S(L^2_{a,b}[0,T])$ if and only if $F \in F(C_{a,b}[0,T])$.

When $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, $S(L^2_{a,b}[0,T])$ reduces to the Banach algebra $S$ introduced by Cameron and Storvick [3]. In [24], pp.609–629, Johnson and Lapidus give a very complete summary of various relationships which exist among the Banach algebras $S$, $F(H)$ and $F(B)$.

Let $A$ be a nonnegative self-adjoint operator on $C'_{a,b}[0,T]$ and $f$ any complex measure on $C'_{a,b}[0,T]$. Then the functional

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\{i(A^{1/2}w, x)\} \, df(w)$$

(3.2)
belong to $\mathcal{F}(C_{a,b}[0,T])$ because it can be rewritten as $\int_{C_{a,b}[0,T]} \exp\{i(w, x)\} df_A(w)$ for $f_A = f \circ (A^{1/2})^{-1}$. Let $A$ be self-adjoint but not nonnegative. Then $A$ has the form
\begin{equation}
A = A^+ - A^-
\end{equation}
where both $A^+$ and $A^-$ are bounded, nonnegative self-adjoint operators.

In this section we will extend the ideas of [25] to obtain expressions of the generalized analytic Feynman integral and the GFFT of functionals of the form (3.2) when $A$ is no longer required to be nonnegative. To do this, we will introduce definitions and notations analogous to those in [13, 15, 20].

Let $\mathcal{W}(C_{a,b}^2[0,T])$ denote the class of all Wiener measurable subsets of the product function space $C_{a,b}[0,T] \times C_{a,b}[0,T] = C_{a,b}^2[0,T]$. A subset $B$ of $C_{a,b}^2[0,T]$ is said to be scale-invariant measurable provided $\{(\rho_1 x_1, \rho_2 x_2) : (x_1, x_2) \in B\}$ is $\mathcal{W}(C_{a,b}^2[0,T])$-measurable for every $\rho_1 > 0$ and $\rho_2 > 0$, and a scale-invariant measurable subset $N$ of $C_{a,b}^2[0,T]$ is said to be scale-invariant null provided $\{\mu \times \mu\}((\rho_1 x_1, \rho_2 x_2) : (x_1, x_2) \in N) = 0$ for every $\rho_1 > 0$ and $\rho_2 > 0$. A property that holds except on a scale-invariant null set is said to hold s-a.e. on $C_{a,b}^2[0,T]$. A functional $F$ on $C_{a,b}^2[0,T]$ is said to be scale-invariant measurable provided $F$ is defined on a scale-invariant measurable set and $F(\rho_1 \cdot, \rho_2 \cdot)$ is $\mathcal{W}(C_{a,b}^2[0,T])$-measurable for every $\rho_1 > 0$ and $\rho_2 > 0$. If two functionals $F$ and $G$ defined on $C_{a,b}^2[0,T]$ are equal s-a.e., then we write $F \approx G$. For more details, see [9, 25].

We denote the product function space integral of a $\mathcal{W}(C_{a,b}^2[0,T])$-measurable functional $F$ by
\begin{equation}
E[F] \equiv E_a[F(x_1, x_2)] = \int_{C_{a,b}^2[0,T]} F(x_1, x_2) d(\mu \times \mu)(x_1, x_2)
\end{equation}
whenever the integral exists.

Throughout this paper, let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+$ denote the complex numbers, the complex numbers with positive real part and the nonzero complex numbers with nonnegative real part, respectively. Furthermore, for all $\lambda \in \mathbb{C}_+$, $\lambda^{-1/2}$ (or $\lambda^{1/2}$) is always chosen to have positive real part. We also assume that every functional $F$ on $C_{a,b}^2[0,T]$ we consider is s-a.e. defined and is scale-invariant measurable.

**Definition 3.2.** Let $\mathbb{C}_+^2 = \{\bar{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \text{Re}(\lambda_j) > 0 \text{ for } j = 1, 2\}$ and let $\mathbb{C}_+^2 = \{\bar{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_j \neq 0 \text{ and } \text{Re}(\lambda_j) \geq 0 \text{ for } j = 1, 2\}$. Let $F : C_{a,b}^2[0,T] \to \mathbb{C}$ be such that for each $\lambda_1 > 0$ and $\lambda_2 > 0$, the function space integral
\begin{equation}
J(\lambda_1, \lambda_2) = \int_{C_{a,b}^2[0,T]} F(x_1^{-1/2}, x_2^{-1/2}) d(\mu \times \mu)(x_1, x_2)
\end{equation}
exists. If there exists a function $J^*(\lambda_1, \lambda_2)$ analytic in $\mathbb{C}_+^2$ such that $J^*(\lambda_1, \lambda_2) = J(\lambda_1, \lambda_2)$ for all $\lambda_1 > 0$ and $\lambda_2 > 0$, then $J^*(\lambda_1, \lambda_2)$ is defined to be the analytic function space integral of $F$ over $C_{a,b}^2[0,T]$ with parameter $\bar{\lambda} = (\lambda_1, \lambda_2)$, and for $\bar{\lambda} \in \mathbb{C}_+^2$ we write
\begin{equation}
E^{an}_{\bar{\lambda}}[F] \equiv E^{an}_{\bar{\lambda}}[F(x_1, x_2)] \equiv E_{x_1, x_2}^{an}(\lambda_1, \lambda_2)[F(x_1, x_2)] = J^*(\lambda_1, \lambda_2).
\end{equation}
Let $q_1$ and $q_2$ be nonzero real numbers. Let $F$ be a functional such that $E^{an\lambda}_q[F]$ exists for all $\lambda \in \mathbb{C}_+^2$. If the following limit exists, we call it the generalized analytic Feynman integral of $F$ with parameter $q = (q_1, q_2)$ and we write

$$E^{an\lambda}_q[F] = E^{an\lambda}_{x_1}(F(x_1, x_2)) = E^{an\lambda}_{x_2}(F(x_1, x_2)) = \lim_{\lambda \to -iq} E^{an\lambda}_q[F]$$

where $\lambda = (\lambda_1, \lambda_2) \to -iq = (-iq_1, -iq_2)$ through values in $\mathbb{C}_+^2$.

**Definition 3.3.** Let $q_1$ and $q_2$ be nonzero real numbers. For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}_+^2$ and $(y_1, y_2) \in C^2_{a,b}[0, T]$, let

$$T_{\lambda}(F)(y_1, y_2) = T_{(\lambda_1, \lambda_2)}(F)(y_1, y_2) = E^{an\lambda}_{\lambda}(F(y_1 + x_1, y_2 + x_2)).$$

For $p \in (1, 2]$, we define the $L_p$ analytic GFFT, $T_{q}^{(p)}(F)$ of $F$, by the formula ($\lambda \in \mathbb{C}_+^2$)

$$T_{q}^{(p)}(F)(y_1, y_2) = T_{(q_1, q_2)}^{(p)}(F)(y_1, y_2) = \frac{1}{i} \int \lambda \to -iq \int_{C_{a,b}[0, T]} [T_{\lambda}(F)(\rho_1 y_1, \rho_2 y_2) - T_{q}^{(p)}(F)(\rho_1 y_1, \rho_2 y_2)]^p d(\mu \times \mu)(y_1, y_2) = 0$$

where $1/p + 1/p' = 1$. We define the $L_1$ analytic GFFT, $T_{q}^{(1)}(F)$ of $F$, by the formula ($\lambda \in \mathbb{C}_+^2$)

$$T_{q}^{(1)}(F)(y_1, y_2) = \lim_{\lambda \to -iq} T_{\lambda}(F)(y_1, y_2)$$

if it exists.

We note that for $1 \leq p \leq 2$, $T_{q}^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_{q}^{(p)}(F)$ exists and if $F \approx G$, then $T_{q}^{(p)}(G)$ exists and $T_{q}^{(p)}(G) \approx T_{q}^{(p)}(F)$. Moreover, from Definition 3.3 we see that for $q_1, q_2 \in \mathbb{R} \setminus \{0\}$,

$$E^{an\lambda}_q[F(x_1, x_2)] = T_{q}^{(1)}(F)(0, 0).$$

Next we give the definition of the generalized Fresnel type class $F^{a,b}_{A_1, A_2}$.

**Definition 3.4.** Let $A_1$ and $A_2$ be bounded, nonnegative self-adjoint operators on $C^2_{a,b}[0, T]$. The generalized Fresnel type class $F^{a,b}_{A_1, A_2}$ of functionals on $C^2_{a,b}[0, T]$ is defined as the space of all functionals $F$ on $C^2_{a,b}[0, T]$ of the form

$$F(x_1, x_2) = \int_{C^2_{a,b}[0, T]} \exp \left\{ \sum_{j=1}^{2} i(A_1^{1/2} w, x_j) \right\} df(w)$$

for some $f \in \mathcal{M}(C^2_{a,b}[0, T])$. More precisely, since we identify functionals which coincide s-a.e. on $C^2_{a,b}[0, T]$, $F^{a,b}_{A_1, A_2}$ can be regarded as the space of all s-equivalence classes of functionals of the form (3.3).
Remark 3.5. (1) In Definition 3.4 above, let $A_1$ be the identity operator on $C^1_{a,b}[0, T]$ and $A_2 \equiv 0$. Then $\mathcal{F}_{A_1, A_2}$ is essentially the Fresnel type class $\mathcal{F}(C_{a,b}[0, T])$ and for $p \in [1, 2]$ and nonzero real numbers $q_1$ and $q_2$,

$$T^{(p)}_{(q_1, q_2)}(F)(y_1, y_2) = T^{(p)}_{q_1}(F_0)(y_1),$$

if it exists, where $F_0(x_1) = F(x_1, x_2)$ for all $(x_1, x_2) \in C^2_{a,b}[0, T]$ and $T^{(p)}_{q_1}(F_0)(y)$ means the $L_p$ analytic GFFT on $C_{a,b}[0, T]$, see [13, 14].

(2) The map $f \mapsto F$ defined by (3.5) sets up an algebra isomorphism between $\mathcal{M}(C_{a,b}[0, T])$ and $\mathcal{F}_{A_1, A_2}$ if $\text{Ran}(A_1 + A_2)$ is dense in $C_{a,b}[0, T]$ where $\text{Ran}$ indicates the range of an operator. In this case $\mathcal{F}_{A_1, A_2}$ becomes a Banach algebra under the norm $\|F\| = \|f\|$. For more details, see [22].

Remark 3.6. Let $F$ be given by equation (3.5). In evaluating $E_\mathbb{F}[F(\lambda_1^{-1/2} x_1, \lambda_2^{-1/2} x_2)]$ and $T_{(\lambda_1, \lambda_2)}(F)(y_1, y_2) = E_\mathbb{F}[F(y_1 + \lambda_1^{-1/2} x_1, y_2 + \lambda_2^{-1/2} x_2)]$ for $\lambda_1 > 0$ and $\lambda_2 > 0$, the expression

$$\psi(\vec{x}; \vec{A}; w) \equiv \psi(\lambda_1, \lambda_2; A_1, A_2; w)$$

$$= \exp \left\{ \sum_{j=1}^{2} \left[ - \frac{(A_j w, w)_{C_{a,b}^\prime}}{2\lambda_j} + i\lambda_j^{-1/2} (A_j^{1/2} w, a)_{C_{a,b}^\prime} \right] \right\} \quad (3.6)$$

occurs. Clearly, for $\lambda_j > 0$, $j \in \{1, 2\}$, $|\psi(\vec{x}; \vec{A}; w)| \leq 1$ for all $w \in C^2_{a,b}[0, T]$. But for $\vec{x} \in \mathbb{C}^2_+$, $|\psi(\vec{x}; \vec{A}; w)|$ is not necessarily bounded by 1.

Note that for each $\lambda_j \in \mathbb{C}_+$ with $\lambda_j = \alpha_j + i\beta_j$, $j \in \{1, 2\},$

$$\lambda_j^{1/2} = \sqrt{\frac{\alpha_j^2 + \beta_j^2 + \alpha_j}{2}} + i\beta_j, \quad \frac{\beta_j}{|\beta_j|}, \frac{\alpha_j}{|\alpha_j|} \sqrt{\frac{\alpha_j^2 + \beta_j^2 - \alpha_j}{2}}.$$  

and

$$\lambda_j^{-1/2} = \sqrt{\frac{\alpha_j^2 + \beta_j^2 + \alpha_j}{2(\alpha_j^2 + \beta_j^2)}} - i\beta_j, \quad \frac{\beta_j}{|\beta_j|}, \frac{\alpha_j}{|\alpha_j|} \sqrt{\frac{\alpha_j^2 + \beta_j^2 - \alpha_j}{2(\alpha_j^2 + \beta_j^2)}}.$$  

Hence, for $\lambda_j \in \mathbb{C}_+$ with $\lambda_j = \alpha_j + i\beta_j$, $j \in \{1, 2\},$

$$|\psi(\vec{x}; \vec{A}; w)|$$

$$= \left| \exp \left\{ \sum_{j=1}^{2} \left[ - \frac{1}{2\lambda_j} (A_j w, w)_{C_{a,b}^\prime} + i\lambda_j^{-1/2} (A_j^{1/2} w, a)_{C_{a,b}^\prime} \right] \right\} \right| \quad (3.7)$$

$$= \left| \exp \left\{ \sum_{j=1}^{2} \left[ - \frac{1}{2} \left( \frac{\alpha_j}{\alpha_j^2 + \beta_j^2} - i\frac{\beta_j}{\alpha_j^2 + \beta_j^2} \right) (A_j w, w)_{C_{a,b}^\prime} \right. \right. \right.$$  

$$+ i \left. \left( \frac{\sqrt{\alpha_j^2 + \beta_j^2 + \alpha_j}}{2(\alpha_j^2 + \beta_j^2)} - \frac{\beta_j}{\alpha_j^2 + \beta_j^2} \sqrt{\frac{\alpha_j^2 + \beta_j^2 - \alpha_j}{2}} \right) (A_j^{1/2} w, a)_{C_{a,b}^\prime} \right\} \right| \quad (3.7)$$

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\[
\begin{align*}
&= \exp \left\{ \sum_{j=1}^{2} \left[ -\frac{\alpha_j}{2(\alpha_j^2 + \beta_j^2)} (A_j w, w)_{C_{a,b}'} \\
&\quad + \frac{\beta_j}{|\beta_j|} \sqrt{\frac{\alpha_j^2 + \beta_j^2 - \alpha_j}{2(\alpha_j^2 + \beta_j^2)}} (A_j^{1/2} w, a)_{C_{a,b}'} \right] \right\}.
\end{align*}
\]

The last expression of (3.7) is an unbounded function of \( w \) for \( w \in C_{a,b}[0,T] \). Thus \( E_{\text{an}}[F] \), \( E_{\text{an}}^\xi[F] \), \( T_{\text{F}}[F] \) and \( T_{\text{qj}}[F] \) might not exist. Throughout this paper we thus will need to put additional restrictions on the complex measure \( F \) corresponding to \( F \) in order to obtain our results for the GFFT and the generalized analytic Feynman integral of \( F \).

In view of Remark 3.3 above, we clearly need to impose additional restrictions on the functionals \( F \) in \( \mathcal{F}_{a,b} \).

For a positive real number \( q_0 \), let
\[
\Gamma_{q_0} = \left\{ \tilde{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}_+^2 \mid \lambda_j = \alpha_j + i\beta_j, \\
\text{Im}(\lambda_j^{-1/2}) = \sqrt{\frac{\alpha_j^2 + \beta_j^2 - \alpha_j}{2(\alpha_j^2 + \beta_j^2)}} < \frac{1}{\sqrt{2q_0}}, \ j = 1, 2 \right\} \quad (3.8)
\]
and let
\[
k(q_0; \tilde{A}; w) \equiv k(q_0; A_1, A_2; w) \\
= \exp \left\{ \sum_{j=1}^{2} \frac{(2q_0)^{-1/2}}{2} \| A_j^{1/2} \|_a \| w \|_{C_{a,b}'} \| a \|_{C_{a,b}'} \right\} \quad (3.9)
\]
where \( \| A_j^{1/2} \|_a \) means the operator norm of \( A_j^{1/2} \) for \( j \in \{1, 2\} \). Then for all \( \tilde{\lambda} = (\lambda_1, \lambda_2) \in \Gamma_{q_0} \),
\[
|\psi(\tilde{\lambda}; \tilde{A}; w)| = \exp \left\{ \sum_{j=1}^{2} \left[ -\frac{\alpha_j}{2(\alpha_j^2 + \beta_j^2)} \| A_j^{1/2} w \|_{C_{a,b}'}^2 \\
+ \frac{\beta_j}{|\beta_j|} \sqrt{\frac{\alpha_j^2 + \beta_j^2 - \alpha_j}{2(\alpha_j^2 + \beta_j^2)}} (A_j^{1/2} w, a)_{C_{a,b}'} \right] \right\} \quad (3.10)
\]
\[
\leq \exp \left\{ \sum_{j=1}^{2} \sqrt{\frac{\alpha_j^2 + \beta_j^2 - \alpha_j}{2(\alpha_j^2 + \beta_j^2)}} |(A_j^{1/2} w, a)_{C_{a,b}'}| \right\}
\]
\[
\leq \exp \left\{ \sum_{j=1}^{2} \sqrt{\frac{\alpha_j^2 + \beta_j^2 - \alpha_j}{2(\alpha_j^2 + \beta_j^2)}} \| A_j^{1/2} w \|_{C_{a,b}'} \| a \|_{C_{a,b}'} \right\}
\]
\[
< k(q_0; \tilde{A}; w).
\]

We note that for all real \( q_j \) with \( |q_j| > q_0, \ j \in \{1, 2\} \),
\[
(-iq_j)^{-1/2} = \frac{1}{\sqrt{|2q_j|}} + \text{sign}(q_j) \frac{i}{\sqrt{|2q_j|}}
\]

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and $(-iq_1, -iq_2) \in \Gamma_{q_0}$.

For the existence of the GFFT of $F$, we define a subclass $\mathcal{F}_{A_1, A_2}^{q_0}$ of $\mathcal{F}_{A_1, A_2}^a$ by

$$
\int_{C_{a,b}^0[0,T]} k(q_0; \vec{A}; w) df(w) < +\infty
$$

(3.11)

where $f$ and $F$ are related by equation (3.5) and $k$ is given by equation (3.9).

**Remark 3.7.** Note that in case $a(t) \equiv 0$ and $b(t) = t$ on $[0, T]$, the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$ and $(w,a)_{C_{a,b}} = 0$ for all $w \in C_{a,b}'[0,T] = C_0'[0,T]$. Hence for all $\vec{x} \in \mathbb{C}_a^2$, $|\psi(\vec{x}, \vec{A}; w)| \leq 1$ and for any positive real number $q_0$, $\mathcal{F}_{A_1, A_2}^{q_0} = \mathcal{F}_{A_1, A_2}$, the Kallianpur and Bromley’s class introduced in Section 4

**Theorem 3.8.** Let $q_0$ be a positive real number and let $F$ be an element of $\mathcal{F}_{A_1, A_2}^{q_0}$. Then for any nonzero real numbers $q_1$ and $q_2$ with $|q_j| > q_0$, $j \in \{1, 2\}$, the $L_1$ analytic GFFT of $F$, $T^{(1)}_{q_1}(F)$ exists and is given by the formula

$$
T^{(1)}_{q_1}(F)(y_1, y_2) = \int_{C_{a,b}^0[0,T]} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2} w, y_j)^- \right\} \psi(i\vec{q}; \vec{A}; w) df(w)
$$

(3.12)

for s.a.e. $(y_1, y_2) \in C_{a,b}^2[0,T]$, where $\psi$ is given by equation (3.6).

**Proof.** We first note that for $j \in \{1, 2\}$, the PWZ stochastic integral $(A_j^{1/2} w, x)^-$ is a Gaussian random variable with mean $(A_j^{1/2} w, a)_{C_{a,b}}^c$ and variance $\|A_j^{1/2} w\|^2_{C_{a,b}} = \|A_j w, w\|_{C_{a,b}}^c$. Hence, using equation (3.5), the Fubini theorem, the change of variables theorem and equation (2.21), we have that for all $\lambda_1 > 0$ and $\lambda_2 > 0,$

$$
J(y_1, y_2; \lambda_1, \lambda_2)
$$

$$
= E_F[F(y_1 + A_1^{1/2} x_1, y_2 + A_2^{1/2} x_2)]
$$

$$
= \int_{C_{a,b}^0[0,T]} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2} w, y_j)^- \right\} \left( \prod_{j=1}^{2} E_{x_j} \left[ \exp \left\{ i\lambda_j^{-1/2}(A_j^{1/2} w, x_j)^- \right\} \right] \right) df(w)
$$

$$
= \int_{C_{a,b}^0[0,T]} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2} w, y_j)^- \right\} \left[ \prod_{j=1}^{2} \left( 2\pi(A_j w, w)_{C_{a,b}}^{-1} \right)^{1/2} \right]
$$

$$
\times \int_{\mathbb{R}} \exp \left\{ i\lambda_j^{-1/2} u_j - \frac{[u_j - (A_j^{1/2} w, a)_{C_{a,b}}^c]^2}{2(A_j w, w)_{C_{a,b}}^c} \right\} du_j df(w)
$$

$$
= \int_{C_{a,b}^0[0,T]} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2} w, y_j)^- \right\}
$$

$$
\times \left[ \prod_{j=1}^{2} \exp \left\{ \frac{(A_j w, w)_{C_{a,b}}^c}{2} i\lambda_j^{-1/2} + \frac{(A_j^{1/2} w, a)_{C_{a,b}}^c}{(A_j w, w)_{C_{a,b}}^c} \right\} - \frac{(A_j^{1/2} w, a)_{C_{a,b}}^c}{2(A_j w, w)_{C_{a,b}}^c} \right] df(w)
$$

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In view of Definition 3.3, it will suffice to show that for each \( \rho \in (3.8) \) that

\[
\text{Proof.}
\]

Let \( \Gamma \) be as in Theorem 3.8. Then for all \( \lambda \in \text{Int}(\Gamma_q) \),

\[
\left| T_{\lambda}(F)(y_1, y_2) \right| \leq \int_{C_{a,b}[0,T]} k(q_0; \tilde{A}; w)df(w) < +\infty.
\]

Using this fact and the dominated convergence theorem, we see that \( T_{\lambda}(F)(y_1, y_2) \) is a continuous function of \( \lambda = (\lambda_1, \lambda_2) \) on \( \text{Int}(\Gamma_q) \). For each \( w \in C_{a,b}[0,T] \), \( \psi(\lambda; \tilde{A}; w) \) is an analytic function of \( \lambda \) throughout the domain \( \text{Int}(\Gamma_q) \) so that

\[
\int_{\Delta} \psi(\lambda; \tilde{A}; w)d\lambda = 0 \quad \text{for every rectifiable simple closed curve} \quad \Delta \quad \text{in} \quad \text{Int}(\Gamma_q).
\]

By equation (3.13), the Fubini theorem and the Morera theorem, we see that \( T_{\lambda}(F)(y_1, y_2) \) is an analytic function of \( \lambda \) throughout the domain \( \text{Int}(\Gamma_q) \). Finally, by the dominated convergence theorem, it follows that

\[
T_{\lambda}^{(1)}(F)(y_1, y_2) = \lim_{\lambda \to -iq} T_{\lambda}(F)(y_1, y_2)
\]

\[
= \int_{C_{a,b}[0,T]} \lim_{\lambda \to -iq} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2}w, y_j) \right\} \psi(\lambda; \tilde{A}; w)df(w)
\]

\[
= \int_{C_{a,b}[0,T]} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2}w, y_j) \right\} \psi(-iq; \tilde{A}; w)df(w).
\]

\[
\square
\]

**Theorem 3.9.** Let \( q_0 \) and \( F \) be as in Theorem 3.8. Then for all \( p \in (1, 2] \) and all nonzero real numbers \( q_1 \) and \( q_2 \) with \( |q_j| > q_0 \), \( j \in \{1, 2\} \), the \( L_p \) analytic GFFT of \( F \), \( T_{q}^{(p)}(F) \) exists and is given by the right hand side of equation (5.12) for s.a.e. \( (y_1, y_2) \in C_{a,b}[0,T] \).

**Proof.** Let \( \Gamma_q \) be given by equation (5.8). It was shown in the proof of Theorem 3.8 that \( T_{\lambda}(F)(y_1, y_2) \) is an analytic function of \( \lambda \) throughout the domain \( \text{Int}(\Gamma_q) \). In view of Definition 3.3, it will suffice to show that for each \( \rho_1 > 0 \) and \( \rho_2 > 0 \),

\[
\lim_{\lambda \to -iq} \int_{C_{a,b}[0,T]} \left| T_{\lambda}(F)(\rho_1 y_1, \rho_2 y_2) - T_{q}^{(p)}(F)(\rho_1 y_1, \rho_2 y_2) \right|^p d(\mu \times \mu)(y_1, y_2) = 0.
\]

15
Fixing $p \in (1, 2]$ and using the inequalities (3.10) and (3.11), we obtain that for all $\rho_j > 0$, $j \in \{1, 2\}$ and all $\tilde{\lambda} \in \Gamma_{q_0}$,

$$\left| T_{\tilde{\lambda}}(F)(\rho_1 y_1, \rho_2 y_2) - T_{\tilde{q}}^{(p)}(F)(\rho_1 y_1, \rho_2 y_2) \right|^p \leq \int_{C_{a,b}[0,T]} \exp \left\{ \sum_{j=1}^{2} i \rho_j (A_j^{1/2}w, \rho_j y_j) \right\} \left| \psi(\tilde{\lambda}; \tilde{A}; w) - \psi(-i\tilde{q}; \tilde{A}; w) \right| df(w) \right|^p \leq \left( \int_{C_{a,b}[0,T]} \left[ \left| \psi(\tilde{\lambda}; \tilde{A}; w) \right| + \left| \psi(-i\tilde{q}; \tilde{A}; w) \right| \right] df(w) \right)^p \leq \left( 2 \int_{C_{a,b}[0,T]} \tilde{k}(q_0; \tilde{A}; w) df(w) \right)^p < +\infty.$$ 

Hence by the dominated convergence theorem, we see that for each $p \in (1, 2]$ and each $\rho_1 > 0$ and $\rho_2 > 0$,

$$\lim_{\tilde{\lambda} \to -i\tilde{q}} \int_{C_{a,b}[0,T]} \left| T_{\tilde{\lambda}}(F)(\rho_1 y_1, \rho_2 y_2) - T_{\tilde{q}}^{(p)}(F)(\rho_1 y_1, \rho_2 y_2) \right|^p d(\mu \times \mu)(y_1, y_2) = \lim_{\tilde{\lambda} \to -i\tilde{q}} \int_{C_{a,b}[0,T]} \int_{C_{a,b}[0,T]} \exp \left\{ \sum_{j=1}^{2} i (A_j^{1/2}w, \rho_j y_j) \right\} \psi(\tilde{\lambda}; \tilde{A}; w) df(w) \leq \int_{C_{a,b}[0,T]} \int_{C_{a,b}[0,T]} \exp \left\{ \sum_{j=1}^{2} i (A_j^{1/2}w, \rho_j y_j) \right\} \psi(-i\tilde{q}; \tilde{A}; w) df(w) d(\mu \times \mu)(y_1, y_2) \times \lim_{\tilde{\lambda} \to -i\tilde{q}} \left[ \psi(\tilde{\lambda}; \tilde{A}; w) - \psi(-i\tilde{q}; \tilde{A}; w) \right] df(w) \right|^p $$

$$= 0$$

which concludes the proof of Theorem (3.9).

**Remark 3.10.** (1) In view of Theorems (3.8) and (3.9), we see that for each $p \in [1, 2]$, the $L_p$ analytic GFFT of $F$, $T_{\tilde{q}}^{(p)}(F)$ is given by the right hand side of equation (3.12) for $q_0$, $\eta_1$, $\eta_2$ and $F$ as in Theorem (3.8) and for s.a.e. $(y_1, y_2) \in C_{a,b}^2(0, T)$,

$$T_{\tilde{q}}^{(p)}(F)(y_1, y_2) = E_{\tilde{x}}^{\text{an}}[F(y_1 + x_1, y_2 + x_2)], \quad p \in [1, 2].$$

In particular, using this fact and equation (3.4), we have that for all $p \in [1, 2]$

$$T_{\tilde{q}}^{(p)}(F)(0, 0) = E_{\tilde{x}}^{\text{an}}[F(x_1, x_2)].$$

(2) For nonzero real numbers $q_1$ and $q_2$ with $|q_j| > q_0$, $j \in \{1, 2\}$, define a set function $f_{\tilde{q}}^{\tilde{A}} : \mathcal{B}(C_{a,b}^\prime[0, T]) \to \mathbb{C}$ by

$$f_{\tilde{q}}^{\tilde{A}}(B) = \int_{B} \psi(-i\tilde{q}; \tilde{A}; w) df(w), \quad B \in \mathcal{B}(C_{a,b}^\prime[0, T]),$$
where \( f \) and \( F \) are related by equation (3.5) and \( \mathcal{B}(C'_{a,b}[0,T]) \) is the Borel \( \sigma \)-algebra of \( C'_{a,b}[0,T] \). Then it is obvious that \( f \hat{\mathcal{A}} \) belongs to \( \mathcal{M}(C'_{a,b}[0,T]) \) and for all \( p \in [1,2] \), \( T^{(p)}_{\hat{q}}(F) \) can be expressed as

\[
T^{(p)}_{\hat{q}}(F)(y_1, y_2) = \int_{C'_{a,b}[0,T]} \exp \left\{ \sum_{j=1}^{2} i(A_{1/2}^y, y_j) \right\} d\hat{A}_q(w)
\]

for s.a.e. \( (y_1, y_2) \in C^2_{a,b}[0,T] \). Hence \( T^{(p)}_{\hat{q}}(F) \) belongs to \( \mathcal{F}^{a,b}_{A_1,A_2} \) for all \( p \in [1,2] \).

(3) Let \( A \) be a bounded self-adjoint operator on \( C'_{a,b}[0,T] \). Then \( A \) has the form (3.3). Take \( A_1 = A_+ \) and \( A_2 = A_- \) in Definition 3.4 above. Then for \( F \in \mathcal{F}^{q_0}_{A_+,A_-} \) and for real \( q \) with \( |q| > q_0 \), equations (3.4) and (3.12) with \( \hat{q} = (q_1, q_2) \) replaced with \( \hat{q} = (q, -q) \) becomes

\[
E_{\hat{q}}^{\text{anf}}[F(x_1, x_2)] = T^{(p)}_{(q, -q)}(F)(0,0)
\]

\[
= \int_{C'_{a,b}[0,T]} \exp \left\{ -\frac{i}{2q} (Aw, w)_{C'_{a,b}} \right\} d\hat{A}_q(w).
\]

The following corollary follows from equations (3.4) and (3.12).

**Corollary 3.11.** Let \( q_0 \) and \( F \) be as in Theorem 3.8. Then for all real numbers \( q_1 \) and \( q_2 \) with \( |q_j| > q_0 \), \( j \in \{1,2\} \), the generalized analytic Feynman integral \( E_{\hat{q}}^{\text{anf}}[F] \) of \( F \) exists and is given by the formula

\[
E_{\hat{q}}^{\text{anf}}[F] = \int_{C'_{a,b}[0,T]} \psi(-i\hat{q}; \hat{A}; w) df(w)
\]

where \( \psi \) is given by equation (3.6).

In the proof of Theorem 3.8 we showed that \( T_{\hat{q}}(F) \) is an analytic function of \( \hat{q} \) throughout the domain \( \text{Int}(\Gamma_{q_0}) \). Thus we have the following corollary.

**Corollary 3.12.** Let \( q_0 \) and \( F \) be as in Theorem 3.8 and let \( \Gamma_{q_0} \) be given by (3.8). Then for each \( \hat{q} \in \text{Int}(\Gamma_{q_0}) \),

\[
E_{\hat{q}}^{\text{anf}}[F] = \int_{C'_{a,b}[0,T]} \psi(\hat{q}; \hat{A}; w) df(w)
\]

where \( \psi \) is given by equation (3.6).

### 4 Relationships between the GFFT and the function space integral of functionals in \( \mathcal{F}^{a,b}_{A_1,A_2} \)

In this section we establish a relationship between the GFFT and the function space integral of functionals in the Fresnel type class \( \mathcal{F}^{a,b}_{A_1,A_2} \).
Throughout this section, for convenience, we use the following notation: for given $\lambda \in \mathbb{C}_+$ and $n = 1, 2, \ldots$, let

$$G_n(\lambda, x) = \exp \left\{ \frac{1 - \lambda}{2} \sum_{k=1}^{n} [(e_k, x)^2 + (\lambda^{1/2} - 1) \sum_{k=1}^{n} (e_k, a)_{C'_{a,b}} (e_k, x)^2 \right\} \quad (4.1)$$

where $\{e_n\}_{n=1}^\infty$ is a complete orthonormal set in $C'_{a,b}[0, T]$.

To obtain our main results, Theorems 4.3 and 4.6 below, we state a fundamental integration formula for the function space $C_{a,b}[0, T]$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in $(C'_{a,b}[0, T], \| \cdot \|_{C'_{a,b}})$, let $k : \mathbb{R}^n \to \mathbb{C}$ be a Borel measurable function and let $K : C_{a,b}[0, T] \to \mathbb{C}$ be given by equation $K(x) = k((e_1, x)^-, \ldots, (e_n, x)^-)$.

Then

$$E[K] = \int_{C_{a,b}[0, T]} k((e_1, x)^-, \ldots, (e_n, x)^-)d\mu(x)$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} k(u_1, \ldots, u_n)$$

$$\times \exp \left\{ -\sum_{j=1}^{n} \frac{\|u_j - (e_j, a)_{C'_{a,b}}\|^2}{2} \right\} \, du_1 \cdots du_n \quad (4.2)$$

in the sense that if either side of equation (4.2) exists, both sides exist and equality holds.

We also need the following lemma to obtain our main theorem in this section.

**Lemma 4.1.** Let $\{e_1, \ldots, e_n\}$ be an orthonormal subset of $C'_{a,b}[0, T]$ and let $w \in C_{a,b}[0, T]$. Then for each $\lambda \in \mathbb{C}_+$, the function space integral

$$E_x[G_n(\lambda, x) \exp\{iw(x)^-\}]$$

exists and is given by the formula

$$E_x[G_n(\lambda, x) \exp\{iw(x)^-\}] = \lambda^{-n/2} \exp \left\{ \left[ \frac{\lambda - 1}{2\lambda} \sum_{k=1}^{n} (e_k, w)^2_{C'_{a,b}} - \frac{1}{2} \|w\|^2_{C'_{a,b}} \right. \right.$$  

$$+ i\lambda^{-1/2} \sum_{k=1}^{n} (e_k, a)_{C'_{a,b}} (e_k, w)_{C'_{a,b}}$$

$$+ i(e_{n+1}, a)_{C'_{a,b}} \left[ \|w\|^2_{C'_{a,b}} - \sum_{k=1}^{n} (e_k, w)^2_{C'_{a,b}} \right]^{1/2} \right\} \quad (4.3)$$

where $G_n$ is given by equation (4.1) above.

**Proof.** (Outline) Using the Gram-Schmidt process, for any $w \in C'_{a,b}[0, T]$ we can write $w = \sum_{k=1}^{n+1} c_k e_k$ where $\{e_1, \ldots, e_n, e_{n+1}\}$ is an orthonormal set in $C'_{a,b}[0, T]$ and

$$c_k = \begin{cases} (e_k, w)_{C'_{a,b}} & , \quad k = 1, \ldots, n \\ [\|w\|^2_{C'_{a,b}} - \sum_{j=1}^{n} (e_j, w)^2_{C'_{a,b}}]^{1/2} & , \quad k = n + 1 \end{cases}$$
Then using (4.1), (4.2), the Fubini theorem and (2.3), it follows that equation (4.3) holds for all \( \lambda > 0 \). Finally (4.3) holds for all \( \lambda \in \mathbb{C}_+ \) by analytic continuation.

The following remark will be very useful in the proof of our main theorem in this section.

**Remark 4.2.** Let \( q_0 \) be a positive real number and let \( \Gamma_{q_0} \) be given by equation (3.8). For real numbers \( q_1 \) and \( q_2 \) with \( |q_j| > q_0 \), \( j \in \{1, 2\} \), let \( \{\lambda_n\}_{n=1}^{\infty} = \{(\lambda_{1,n}, \lambda_{2,n})\}_{n=1}^{\infty} \) be a sequence in \( \mathbb{C}_+^2 \) such that

\[
\lambda_n = (\lambda_{1,n}, \lambda_{2,n}) \to -iq = (-iq_1, -iq_2).
\]

Let \( \lambda_{j,n} = \alpha_{j,n} + i\beta_{j,n} \) for \( j \in \{1, 2\} \) and \( n \in \mathbb{N} \). Then for \( j \in \{1, 2\} \), \( \Re(\lambda_{j,n}) = \alpha_{j,n} > 0 \) and

\[
\lambda_{j,n}^{-1} = (\alpha_{j,n} + i\beta_{j,n})^{-1} = \frac{\alpha_{j,n} - i\beta_{j,n}}{\alpha_{j,n}^2 + \beta_{j,n}^2}
\]

for each \( n \in \mathbb{N} \). Since \( |\Im((-iq_j)^{-1/2})| = 1/\sqrt{2|q_j|} < 1/\sqrt{2q_0} \) for \( j \in \{1, 2\} \), there exists a sufficiently large \( L \in \mathbb{N} \) such that for any \( n \geq L \), \( \lambda_{1,n} \) and \( \lambda_{2,n} \) are in \( \text{Int}(\Gamma_{q_0}) \) and

\[
\delta(q_1, q_2) \equiv \sup \left( \{|\Im(\lambda_{1,n}^{-1/2})| : n \geq L\} \right.
\]

\[
\cup \left\{ |\Im(\lambda_{2,n}^{-1/2})| : n \geq L \right\}
\]

\[
\cup \left\{ |\Im((-iq_1)^{-1/2})|, |\Im((-iq_2)^{-1/2})| \right\} < \frac{1}{\sqrt{2q_0}}.
\]

Thus there exists a positive real number \( \varepsilon > 1 \) such that

\[
\delta(q_1, q_2) < \frac{1}{\varepsilon} \sqrt{2q_0}.
\]

Let \( \{e_n\}_{n=1}^{\infty} \) be a complete orthonormal set in \( C_{a,b}[0,T] \). Using Parseval’s identity, it follows that

\[
(g_1, g_2)_{C_{a,b}} = \sum_{n=1}^{\infty} (e_n, g_1)_{C_{a,b}} (e_n, g_2)_{C_{a,b}}
\]

for all \( g_1, g_2 \in C_{a,b}[0,T] \). In addition for each \( g \in C_{a,b}[0,T] \),

\[
\|g\|_{C_{a,b}}^2 - \sum_{k=1}^{n} (e_k, g)_{C_{a,b}}^2 = \sum_{k=n+1}^{\infty} (e_k, g)_{C_{a,b}}^2 \geq 0
\]

for every \( n \in \mathbb{N} \). Note that for \( g \in C_{a,b}[0,T] \), \( (g, a)_{C_{a,b}} \) may be positive, negative or zero. Since

\[
(g, a)_{C_{a,b}} = \sum_{n=1}^{\infty} (e_n, g)_{C_{a,b}} (e_n, a)_{C_{a,b}}
\]
Theorem 4.3.

For $\varepsilon > 1$,

$$-\varepsilon \|g\|_{C_{a,b}} \|a\|_{C_{a,b}} < -\|g\|_{C_{a,b}} \|a\|_{C_{a,b}}$$

$$\leq (g, a)_{C_{a,b}}$$

$$\leq \|g\|_{C_{a,b}} \|a\|_{C_{a,b}} < \varepsilon \|g\|_{C_{a,b}} \|a\|_{C_{a,b}},$$

there exists a sufficiently large $K_j \in \mathbb{N}$ such that for any $n \geq K_j$,

$$\left| \sum_{k=1}^{n} (e_k, A_j^{1/2} w)_{C_{a,b}} (e_k, a)_{C_{a,b}} \right| < \varepsilon \|A_j^{1/2} w\|_{C_{a,b}} \|a\|_{C_{a,b}}$$

$$\leq \varepsilon \|A_j^{1/2} w\|_o \|a\|_{C_{a,b}}$$

for $j \in \{1, 2\}$.

Using these and a long and tedious calculation we can show that for every $n \geq \max\{L, K_1, K_2\}$,

$$\left| \exp \left\{ \sum_{j=1}^{2} \left( \frac{\lambda_{j,n} - 1}{2\lambda_{j,n}} \sum_{k=1}^{n} (e_k, A_j^{1/2} w)_{C_{a,b}}^2 - \frac{1}{2} \|A_j^{1/2} w\|_{C_{a,b}}^2 \right) + i\lambda_{j,n}^{-1/2} \sum_{k=1}^{n} (e_k, A_j^{1/2} w)_{C_{a,b}} (e_k, a)_{C_{a,b}} \right. \right.$$  

$$+ \left. i(e_{n+1}, a)_{C_{a,b}} \left[ \|A_j^{1/2} w\|_{C_{a,b}}^2 - \sum_{k=1}^{n} (e_k, A_j^{1/2} w)_{C_{a,b}}^2 \right] \right)^{1/2} \right| < k(q_0; \tilde{A}; w)$$

where $k(q_0; \tilde{A}; w)$ is given by (3.39).

In our next theorem, for $F \in \mathcal{F}_{A_1, A_2}$, we express the GFFT of $F$ as the limit of a sequence of function space integrals on $C_{a,b}^2[0, T]$.

**Theorem 4.3.** Let $q_0$ and $F$ be as in Theorem 3.3. Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $C_{a,b}^2[0, T]$ and let $\{\lambda_{1,n}, \lambda_{2,n}\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}_+^2$ such that $\lambda_{j,n} \to -iq_j$ where $q_j$ is a real number with $|q_j| > q_0$, $j \in \{1, 2\}$. Then for $p \in [1, 2]$ and for s.a.e. $(y_1, y_2) \in C_{a,b}^2[0, T]$,

$$T_{q}^{(p)}(F)(y_1, y_2)$$

$$= \lim_{n \to \infty} \lambda_{1,n}^{n/2} \lambda_{2,n}^{n/2} E_{q}^{G_n(\lambda_{1,n}, x_1)G_n(\lambda_{2,n}, x_2)F(y_1 + x_1, y_2 + x_2)}$$

where $G_n$ is given by equation (3.11) above.

**Proof.** By Theorems 3.8 and 3.9 above, we know that for each $p \in [1, 2]$, the $L_p$ analytic GFFT of $F$, $T_{q}^{(p)}(F)$ exists and is given by the right hand side of equation (3.12). Thus it suffices to show that

$$T_{q}^{(1)}(F)(y_1, y_2) = E_{q}^{\text{anf}}[F(y_1 + x_1, y_2 + x_2)]$$

$$= \lim_{n \to \infty} \lambda_{1,n}^{n/2} \lambda_{2,n}^{n/2} E_{q}^{G_n(\lambda_{1,n}, x_1)G_n(\lambda_{2,n}, x_2)F(y_1 + x_1, y_2 + x_2)}.$$
Using equation (3.5), the Fubini theorem and equation (4.3) with \( \lambda \) and \( w \) replaced with \( \lambda_{j,n} \) and \( A_j^{1/2} w \), \( j \in \{1, 2\} \), respectively, we see that

\[
\lambda_{1,n}^{n/2} \lambda_{2,n}^{n/2} E_{q}(G_n(\lambda_{1,n}, x_1)G_n(\lambda_{2,n}, x_2)F(y_1 + x_1, y_2 + x_2))
\]

\[
= \lambda_{1,n}^{n/2} \lambda_{2,n}^{n/2} \int_{C_{a,b}[0,T]} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2} w, y_j)^{\sim} \right\} \times \left( \prod_{j=1}^{2} E_{q}[G_n(\lambda_{j,n}^{1/2}, x_j) \exp \{ i(A_j^{1/2} w, x_j)^{\sim} \}] \right) \, df(w)
\]

\[
= \int_{C_{a,b}[0,T]} \lim_{n \to \infty} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2} w, y_j)^{\sim} + \left[ \frac{\lambda_{j,n} - 1}{2 \lambda_{j,n}} \right] \sum_{k=1}^{n} (e_k, A_j^{1/2} w)^2 C_{a,b}^{\prime} \right\}
\]

\[
- \frac{1}{2} \| A_j^{1/2} w \|^2 C_{a,b}^{\prime} + i \lambda_{j,n}^{-1/2} \sum_{k=1}^{n} (e_k, A_j^{1/2} w)^2 C_{a,b}^{\prime}
\]

\[
+ i(e_{n+1}, A) C_{a,b}^{\prime} \left[ \| A_j^{1/2} w \|^2 C_{a,b}^{\prime} - \sum_{k=1}^{n} (e_k, A_j^{1/2} w)^2 C_{a,b}^{\prime} \right]^{1/2} \}
\right) \, df(w)
\]

But, by Remark 4.2, we see that the last expression of (4.3) is dominated by (3.11) on the region \( \Gamma_{q_0} \), given by equation (3.8) for all but a finite number of values of \( n \). Next using the dominated convergence theorem, Parseval’s relation and equation (3.12), it follows that for s-a.e. \( (y_1, y_2) \in C_{a,b}^{2}[0,T] \),

\[
\lim_{n \to \infty} \lambda_{1,n}^{n/2} \lambda_{2,n}^{n/2} E_{q}(G_n(\lambda_{1,n}, x_1)G_n(\lambda_{2,n}, x_2)F(y_1 + x_1, y_2 + x_2))
\]

\[
= \int_{C_{a,b}[0,T]} \lim_{n \to \infty} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2} w, y_j)^{\sim} + \left[ \frac{\lambda_{j,n} - 1}{2 \lambda_{j,n}} \right] \sum_{k=1}^{n} (e_k, A_j^{1/2} w)^2 C_{a,b}^{\prime} \right\}
\]

\[
- \frac{1}{2} \| A_j^{1/2} w \|^2 C_{a,b}^{\prime} + i \lambda_{j,n}^{-1/2} \sum_{k=1}^{n} (e_k, A_j^{1/2} w)^2 C_{a,b}^{\prime}
\]

\[
+ i(e_{n+1}, A) C_{a,b}^{\prime} \left[ \| A_j^{1/2} w \|^2 C_{a,b}^{\prime} - \sum_{k=1}^{n} (e_k, A_j^{1/2} w)^2 C_{a,b}^{\prime} \right]^{1/2} \}
\right) \, df(w)
\]

\[
= \int_{C_{a,b}[0,T]} \lim_{n \to \infty} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2} w, y_j)^{\sim} \right\} \psi(-i\hat{q}; A) \, df(w)
\]

\[
= T_{q_{(1)}}^{(1)}(F)(y_1, y_2)
\]

which concludes the proof of Theorem 4.3.

Corollary 4.4. Let \( q_0, F, \{e_n\}_{n=1}^{\infty}, \{(\lambda_{1,n}, \lambda_{2,n})\}_{n=1}^{\infty} \) and \( (q_1, q_2) \) be as in Theorem 4.3. Then

\[
E_{q_{(1)}}^{\text{antif}}[F(x_1, x_2)] = \lim_{n \to \infty} \lambda_{1,n}^{n/2} \lambda_{2,n}^{n/2} E_{q}(G_n(\lambda_{1,n}, x_1)G_n(\lambda_{2,n}, x_2)F(x_1, x_2))
\]

where \( G_n \) is given by equation (4.1) above.
Corollary 4.5. Let \( q_0, F \) and \( \{e_n\}_{n=1}^\infty \) be as in Theorem 4.3 and let \( \Gamma_{q_0} \) be given by (3.8). Let \( \lambda = (\lambda_1, \lambda_2) \in \text{Int}(\Gamma_{q_0}) \) and let \( \{ (\lambda_{1,n}, \lambda_{2,n}) \}_{n=1}^\infty \) be a sequence in \( \mathbb{C}^2_+ \) such that \( \lambda_{j,n} \to \lambda_j, \ j \in \{1, 2\} \). Then

\[
E_{\lambda, n}^x[F(x_1, x_2)] = \lim_{n \to \infty} \lambda_{1,n}^{n/2} \lambda_{2,n}^{n/2} E^x[G_n(\lambda_{1,n}, x_1)G_n(\lambda_{2,n}, x_2)F(x_1, x_2)] \tag{4.5}
\]

where \( G_n \) is given by equation (4.1) above.

Our another result, namely a change of scale formula for function space integrals now follows from Corollary 4.5 above.

**Theorem 4.6.** Let \( F \in \mathcal{F}_{A_1, A_2}^{a,b} \) and let \( \{e_n\}_{n=1}^\infty \) be a complete orthonormal set in \( C_{a,b}^1[0,T] \). Then for any \( \rho_1 > 0 \) and \( \rho_2 > 0 \),

\[
E^x[F(\rho_1 x_1, \rho_2 x_2)] = \lim_{n \to \infty} \rho_1^{-n} \rho_2^{-n} E^x[G_n(\rho_1^{-2}, x_1)G_n(\rho_2^{-2}, x_2)F(x_1, x_2)]
\]

where \( G_n \) is given by equation (4.1) above.

**Proof.** Simply choose \( \lambda_j = \rho_j^{-2} \) for \( j \in \{1, 2\} \) and \( \lambda_{j,n} = \rho_j^{-2} \) for \( j \in \{1, 2\} \) and \( n \in \mathbb{N} \) in equation (4.1).

**Remark 4.7.** Of course, if we choose \( a(t) \equiv 0, b(t) = t, A_1 = I \) (identity operator) and \( A_2 = 0 \) (zero operator), then the function space \( C_{a,b}^2[0,T] \) reduces to the classical Wiener space \( C_0[0,T] \) and the generalized Fresnel type class \( \mathcal{F}_{A_1, A_2}^{a,b} \) reduces to the Fresnel class \( \mathcal{F}(C_0[0,T]) \). It is known that \( \mathcal{F}(C_0[0,T]) \) forms a Banach algebra over the complex field and that \( \mathcal{F}(C_0[0,T]) \) and \( \mathcal{S} \) are isometrically isomorphic. See [22]. In this case, we have the relationships between the analytic Feynman integral and the Wiener integral on classical Wiener space as discussed in [2] and [6].

## 5 The first variation of functionals in \( \mathcal{F}_{A_1, A_2}^{a,b} \)

In this section, we first give the definition of the first variation of a functional \( F \) on \( C_{a,b}^2[0,T] \). The following definition of the first variation on product space is due to Yoo and Kim [30].

**Definition 5.1.** Let \( F \) be a functional on \( C_{a,b}^2[0,T] \) and let \( g_1 \) and \( g_2 \) be elements of \( C_{a,b}[0,T] \). Then

\[
\delta F(x_1, x_2|g_1, g_2) = \frac{\partial}{\partial h} F(x_1 + hg_1, x_2) \bigg|_{h=0} + \frac{\partial}{\partial h} F(x_1, x_2 + hg_2) \bigg|_{h=0} \tag{5.1}
\]

(if it exists) is called the first variation of \( F \) in the direction of \( (g_1, g_2) \).

Throughout this section, when working with \( \delta F(x_1, x_2|g_1, g_2) \), we will always require \( g_1 \) and \( g_2 \) to be elements of \( C_{a,b}^2[0,T] \).

For \( j \in \{1, 2\} \), let \( g_j \in C_{a,b}^{\alpha_j}[0,T] \) and let \( F \) be an element of \( \mathcal{F}_{A_1, A_2}^{a,b} \) whose associated measure \( f \), see equation (3.5), satisfies the inequality

\[
\int_{C_{a,b}^{\alpha_j}[0,T]} \|w\|_{C_{a,b}^{\alpha_j}} df(w) < +\infty. \tag{5.2}
\]
Then using equation (5.1), we obtain that
\[
\delta F(x_1, x_2 | g_1, g_2) = \sum_{k=1}^{2} \left[ \frac{\partial}{\partial h} \left( \int_{C_{a,b}^{'},[0,T]} \exp \left\{ 2 \sum_{j=1}^{2} i(A_j^{1/2}w, x_j)^{\sim} + ih(A_k^{1/2}w, g_k)^{\sim} \right\} df(w) \right) \right]_{h=0}
\]
(5.3)
where the complex measure \( \sigma^{\vec{A}, \vec{g}} \) is defined by
\[
\sigma^{\vec{A}, \vec{g}}(B) = \int_B \left[ \sum_{k=1}^{2} i(A_k^{1/2}w, g_k)C_{a,b}^{'},\sim \right] df(w), \quad B \in B(C_{a,b}^{'},[0,T]).
\]
The second equality of (5.3) follows from (5.2) and Theorem 2.27 in [21]. Also, \( \delta F(x_1, x_2 | g_1, g_2) \) is an element of \( F_{A_1, A_2}^{a,b} \) as a functional of \( (x_1, x_2) \), since by the Cauchy-Schwartz inequality and (5.2),
\[
\| \sigma^{\vec{A}, \vec{g}} \| \leq \int_{C_{a,b}^{'},[0,T]} \sum_{j=1}^{2} \| i(A_j^{1/2}w, g_j)C_{a,b}^{'},\sim \| df(w)
\]
(5.3)
where \( \| A_j^{1/2} \|_o \) is the operator norm of \( A_j^{1/2} \).

For given positive real number \( q_0 \), we define a subclass \( \mathcal{G}_{A_1, A_2,q}^{q_0} \) of \( F_{A_1, A_2}^{a,b} \) by \( F \in \mathcal{G}_{A_1, A_2,q}^{q_0} \) if and only if
\[
\int_{C_{a,b}^{'},[0,T]} \| w \|\| C_{a,b}^{'},\sim \| k(q_0; \vec{A}; w) df(w) < +\infty
\]
where \( f \), the associated measure of \( F \), and \( F \) are related by equation (3.3) and \( k(q_0; \vec{A}; w) \) is given by equation (3.9).

Our next two theorems follows quite readily from the techniques developed in Sections [3] and [4] above.

**Theorem 5.2.** Let \( q_0 \) be a positive real number and let \( g_1 \) and \( g_2 \) be elements of \( C_{a,b}^{'},[0,T] \). Let \( F \) be an element of \( \mathcal{G}_{A_1, A_2,q}^{q_0} \) and let \( \Gamma_{q_0} \) be given by (3.8). Then:
Also for each

\[ y_1,y_2 \in C_{a,b}^2[0,T], \]

where \( \psi \) is given by equation (3.6); and

(2) for all real numbers \( q_1 \) and \( q_2 \) with \( |q_j| > q_0, j \in \{1,2\} \), the generalized analytic Feynman integral of \( \delta F(\cdot, g_2) \) exists and is given by the formula

\[
E^{\text{an}F}_2[\delta F(x_1,x_2|g_1,g_2)] = \int_{C_{a,b}^2[0,T]} \left[ \sum_{j=1}^{2} i(A_j^{1/2} w,g_j)\mathcal{C}_{a,b} \right] \psi(-i\tilde{g}; \tilde{A}; w)df(w).
\]

In addition, for each \( \tilde{\lambda} \in \text{Int}(\Gamma_{\varrho_0}) \),

\[
E^{\text{an}x}_2[\delta F(x_1,x_2|g_1,g_2)] = \int_{C_{a,b}^2[0,T]} \left[ \sum_{j=1}^{2} i(A_j^{1/2} w,g_j)\mathcal{C}_{a,b} \right] \psi(\tilde{\lambda}; \tilde{A}; w)df(w).
\]

**Theorem 5.3.** Let \( q_0, \{\epsilon_n\}_{n=1}^\infty, \{\lambda_{1,n},\lambda_{2,n}\}_{n=1}^\infty \) and \( (q_1,q_2) \) be as in Theorem 4.3 above, and let \( g_1 \) and \( g_2 \) be elements of \( C_{a,b}^2[0,T] \). Let \( F \) be an element of \( \mathcal{G}_{A_1,A_2}^\varrho_0 \) and let \( \Gamma_{\varrho_0} \) be given by (3.8). Then:

(1) for all \( p \in [1,2] \),

\[
T^{(p)}_q[\delta F(\cdot, g_2)](y_1,y_2) = \lim_{n \to \infty} \lambda_1^{n/2} \lambda_{2,n}^{n/2} \mathcal{E}[G_n(\lambda_{1,n},x_1)G_n(\lambda_{2,n},x_2)\delta F(y_1 + x_1 + y_2 + x_2|g_1,g_2)]
\]

for s.a.e. \( (y_1,y_2) \in C_{a,b}^2[0,T] \), where \( G_n \) is given by equation (4.1) above;

(2) the generalized analytic Feynman integral \( E^{\text{an}F}_2[\delta F(\cdot, g_2)] \) of \( \delta F(\cdot, g_2) \)

is expressed as follows:

\[
E^{\text{an}F}_2[\delta F(x_1,x_2|g_1,g_2)] = \lim_{n \to \infty} \lambda_1^{n/2} \lambda_{2,n}^{n/2} \mathcal{E}_x[G_n(\lambda_{1,n},x_1)G_n(\lambda_{2,n},x_2)\delta F(x_1 + x_2|g_1,g_2)]
\]

Also for each \( \tilde{\lambda} \in \text{Int}(\Gamma_{\varrho_0}) \) and all sequence \( \{\lambda_{1,n},\lambda_{2,n}\}_{n=1}^\infty \) in \( C_{a,b}^2 \) which converges to \( \tilde{\lambda} \),

\[
E^{\text{an}x}_2[\delta F(x_1,x_2|g_1,g_2)] = \lim_{n \to \infty} \lambda_1^{n/2} \lambda_{2,n}^{n/2} \mathcal{E}_{\tilde{x}}[G_n(\lambda_{1,n},x_1)G_n(\lambda_{2,n},x_2)\delta F(x_1 + x_2|g_1,g_2)]
\]

and

(3) for any \( \rho_1 > 0 \) and \( \rho_2 > 0 \),

\[
E^{\text{an}}_{\tilde{x}}[\delta F(\rho_1 x_1,\rho_2 x_2|g_1,g_2)] = \lim_{n \to \infty} \rho_1^{-n} \rho_2^{-n} E_{\tilde{x}}[G_n(\rho_1^{-2},x_1)G_n(\rho_2^{-2},x_2)F(x_1,x_2)].
\]
6 Functionals in \( F^\alpha_{A_1, A_2} \)

In this section, we first prove a theorem ensuring that various functionals are in \( F^\alpha_{A_1, A_2} \).

**Theorem 6.1.** Let \( A_1 \) and \( A_2 \) be bounded, nonnegative, self-adjoint operators on \( C^*_a[0,T] \). Let \( (Y, \mathcal{Y}, \gamma) \) be a \( \sigma \)-finite measure space and let \( \varphi_l : Y \to C^*_a[0,T] \) be \( \mathcal{Y} \)-\( \mathcal{B}(C^*_a[0,T]) \) measurable for \( l \in \{1, \ldots, d\} \). Let \( \theta : Y \times \mathbb{R}^d \to \mathbb{C} \) be given by \( \theta(\eta; \cdot) = \tilde{\nu}_\eta(\cdot) \) where \( \nu_\eta \in \mathcal{M}(\mathbb{R}^d) \) for every \( \eta \in Y \) and where the family \( \{\nu_\eta : \eta \in Y\} \) satisfies:

(i) \( \nu_\eta(E) \) is a \( \mathcal{Y} \)-measurable function of \( \eta \) for every \( E \in \mathcal{B}(\mathbb{R}^d) \); and

(ii) \( ||\nu_\eta|| \in L^1(Y, \mathcal{Y}, \gamma) \).

Under these hypothesis, the functional \( F : C^2_{a,b}[0,T] \to \mathbb{C} \) given by

\[
F(x_1, x_2) = \int_Y \theta(\eta; \sum_{j=1}^2 (A_j^{1/2} \varphi_1(\eta), x_j)^\sim, \ldots, \sum_{j=1}^2 (A_j^{1/2} \varphi_d(\eta), x_j)^\sim) d\gamma(\eta) \tag{6.1}
\]

belongs to \( F^\alpha_{A_1, A_2} \) and satisfies the inequality

\[
||F|| \leq \int_Y ||\nu_\eta|| d\gamma(\eta).
\]

**Proof.** Using the techniques similar to those used in [3], we can show that \( ||\nu_\eta|| \) is measurable as a function of \( \eta \), that \( \theta \) is \( \mathcal{Y} \)-measurable, and that the integrand in equation (6.1) is a measurable function of \( \eta \) for every \( (x_1, x_2) \in C^2_{a,b}[0,T] \).

We define a measure \( \tau \) on \( \mathcal{Y} \times \mathcal{B}(\mathbb{R}^d) \) by

\[
\tau(E) = \int_Y \nu_\eta(E(\eta)) d\gamma(\eta), \quad \text{for } E \in \mathcal{Y} \times \mathcal{B}(\mathbb{R}^d).
\]

Then by the first assertion of Theorem 3.1 in [23], \( \tau \) satisfies \( ||\tau|| \leq \int_Y ||\nu_\eta|| d\gamma(\eta) \).

Now let \( \Phi : Y \times \mathbb{R}^d \to C^*_a[0,T] \) be defined by \( \Phi(\eta; v_1, \ldots, v_d) = \sum_{j=1}^d v_j \varphi_j(\eta) \).

Then \( \Phi \) is \( \mathcal{Y} \times \mathcal{B}(\mathbb{R}^d) \)-\( \mathcal{B}(C^*_a[0,T]) \)-measurable on the hypothesis for \( \varphi_l \), \( l \in \{1, \ldots, d\} \). Let \( \sigma = \tau \circ \Phi^{-1} \). Then clearly \( \sigma \in \mathcal{M}(C^*_a[0,T]) \) and satisfies \( ||\sigma|| \leq ||\tau|| \).

From the change of variables theorem and the second assertion of Theorem 3.1 in [23], it follows that for a.e. \( (x_1, x_2) \in C^2_{a,b}[0,T] \) and for every \( \rho_1 > 0 \) and \( \rho_2 > 0 \),

\[
F(\rho_1 x_1, \rho_2 x_2) = \int_Y \tilde{\nu}_\eta \left( \sum_{j=1}^2 (A_j^{1/2} \varphi_1(\eta), \rho_j x_j)^\sim, \ldots, \sum_{j=1}^2 (A_j^{1/2} \varphi_d(\eta), \rho_j x_j)^\sim \right) d\gamma(\eta)
\]

\[
= \int Y \left[ \int_{\mathbb{R}^d} \exp \left\{ \sum_{l=1}^d v_l \left[ \sum_{j=1}^2 (A_j^{1/2} \varphi_l(\eta), \rho_j x_j)^\sim \right] \right\} d\nu_\eta(v_1, \ldots, v_d) \right] d\gamma(\eta)
\]

\[
= \int_{Y \times \mathbb{R}^d} \exp \left\{ \sum_{l=1}^d v_l \left[ \sum_{j=1}^2 (A_j^{1/2} \varphi_l(\eta), \rho_j x_j)^\sim \right] \right\} d\tau(\eta; v_1, \ldots, v_d)
\]
Thus the functional $F$ given by equation (6.1) belongs to $F_{A_1,A_2}^{a,b}$ and satisfies the inequality 

$$\|F\| = \|\sigma\| \leq \|\tau\| \leq \int Y \|\nu\|d\gamma(\eta).$$

As mentioned in (2) of Remark 3.5, $F_{A_1,A_2}^{a,b}$ is a Banach algebra if $\text{Ran}(A_1 + A_2)$ is dense in $C_{a,b}[0,T]$. In this case, many analytic functionals of $F$ can be formed. The following corollary is relevant to Feynman integration theories and quantum mechanics where exponential functions play an important role.

**Corollary 6.2.** Let $A_1$ and $A_2$ be bounded, nonnegative and self-adjoint operators on $C_{a,b}'[0,T]$ such that $\text{Ran}(A_1 + A_2)$ is dense in $C_{a,b}[0,T]$. Let $F$ be given by equation (6.1) with $\theta$ as in Theorem 6.1, and let $\beta : \mathbb{C} \to \mathbb{C}$ be an entire function. Then $(\beta \circ F)(x_1, x_2)$ is in $F_{A_1,A_2}^{a,b}$. In particular, $\exp\{F(x_1, x_2)\} \in F_{A_1,A_2}^{a,b}$.

**Corollary 6.3.** Let $A_1$ and $A_2$ be bounded, nonnegative, self-adjoint operators on $C_{a,b}'[0,T]$, and let $\{g_1, \ldots, g_d\}$ be a finite subset of $C_{a,b}[0,T]$. Given $\beta = \tilde{\nu}$ where $\nu \in M(\mathbb{R}^d)$, define $F : C_{a,b}^2[0,T] \to \mathbb{C}$ by

$$F(x_1, x_2) = \beta\left(\sum_{j=1}^2 (A_j^{1/2}g_1, x_j)^\sim, \ldots, \sum_{j=1}^2 (A_j^{1/2}g_d, x_j)^\sim\right).$$

Then $F$ is an element of $F_{A_1,A_2}^{a,b}$.

**Proof.** Let $(Y, \mathcal{Y}, \gamma)$ be a probability space and for $l \in \{1, \ldots, d\}$, let $\varphi_l(\eta) \equiv g_l$. Take $\theta(\eta; \cdot) = \beta(\cdot) = \tilde{\nu}(\cdot)$. Then for all $\rho_1 > 0$ and $\rho_2 > 0$ and for a.e. $(x_1, x_2) \in C_{a,b}^2[0,T]$,

$$\int_Y \theta(\eta) \sum_{j=1}^2 (A_j^{1/2}\varphi_1(\eta), x_j)^\sim, \ldots, \sum_{j=1}^2 (A_j^{1/2}\varphi_d(\eta), x_j)^\sim) d\gamma(\eta)$$

$$= \int_Y \beta\left(\sum_{j=1}^2 (A_j^{1/2}g_1, x_j)^\sim, \ldots, \sum_{j=1}^2 (A_j^{1/2}g_d, x_j)^\sim\right) d\gamma(\eta)$$

$$= \beta\left(\sum_{j=1}^2 (A_j^{1/2}g_1, x_j)^\sim, \ldots, \sum_{j=1}^2 (A_j^{1/2}g_d, x_j)^\sim\right)$$

$$= F(\rho_1 x_1, \rho_2 x_2).$$

Hence $F \in F_{A_1,A_2}^{a,b}$. \qed

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Remark 6.4. Let \( d = 1 \) and let \( (Y, \mathcal{Y}, \gamma) = ([0, T], B([0, T]), m_L) \) in Theorem 6.1 where \( m_L \) denotes Lebesgue measure on \([0, T]\). Then Theorems 4.6, 4.7 and 4.9 in [18] follows from the results in this section by letting \( A_1 \) be the identity operator and letting \( A_2 \equiv 0 \) on \( C'_{a,b}[0, T] \). The function \( \theta \) studied in [18] (and mentioned in (1) of Remark 7.1 above) is interpreted as the potential energy in quantum mechanics.

7 A translation theorem for the generalized analytic Feynman integral of functionals in \( \mathcal{F}^{a,b}_{A_1,A_2} \)

In [4], Cameron and Storvick derived a translation theorem for the analytic Feynman integral of functionals in the Banach algebra \( S \) on classical Wiener space and in [11], Chang and Chung derived a translation theorem for function space integral of functionals on \( C_{a,b}[0, T] \). The translation theorem in [11], using the notation of this paper, states that if \( x_0 \in C'_{a,b}[0, T] \) and if \( G \) is a \( \mu \)-integrable function on \( C_{a,b}[0, T] \), then

\[
E[G(x + x_0)] = \exp \left\{ -\frac{1}{2}\|x_0\|^2_{C_{a,b}} - (x_0, a)_{C'_{a,b}} \right\} E[G(x) \exp\{(x_0, x)^\sim\}] \tag{7.1}
\]

In this section, we will present a generalized analytic Feynman integral version of the translation theorem for functionals in \( \mathcal{F}^{a,b}_{A_1,A_2} \).

Theorem 7.1. Let \( q_0 \) and \( F \) be as in Theorem 3.8. Let \( g_1 \) and \( g_2 \) be elements of \( C_{a,b}'[0, T] \). Then for all \( p \in [1, 2] \), all real numbers \( q_1 \) and \( q_2 \) with \( |q_j| > q_0 \), \( j \in \{1, 2\} \) and for s-a.e. \((y_1, y_2) \in C_{a,b}'[0, T]\),

\[
T_q^{(p)}(F)(y_1 + A_1^{1/2}g_1, y_2 + A_2^{1/2}g_2) = \exp \left\{ \sum_{j=1}^{2} \left[ \frac{iq_j}{2}(A_j g_j, g_j)_{C_{a,b}'} - (\frac{1}{2}i)(A_j^{1/2}g_j, a)_{C_{a,b}'} \right] \right\} \times \exp \left\{ \sum_{j=1}^{2} iq_j(A_j^{1/2}g_j, y_j)^\sim \right\} T_q^{(p)}(F^*)(y_1, y_2) \tag{7.2}
\]

where

\[
F^*(y_1, y_2) = F(y_1, y_2) \exp \left\{ \sum_{j=1}^{2} \left[ -iq_j(A_j^{1/2}g_j, y_j)^\sim \right] \right\}.
\]

Proof. By Theorems 3.8 and 3.9, the \( L_p \) analytic GFFT \( T_q^{(p)}(F) \) of \( F \) exists for all \( p \in [1, 2] \) and is given by the right hand side of equation (3.12). Thus we only need to verify the equality in equation (7.2). We will give the proof for the case \( p = 1 \). The case \( p = 2 \) is similar, but somewhat easier.

For \( \lambda_j > 0 \), \( j \in \{1, 2\} \) and \( w \in C_{a,b}'[0, T] \), let \( G_j(w; \cdot) \) be a functional on \( C_{a,b}[0, T] \) given by

\[
G_j(w; x_j) = \exp \left\{ i(A_j^{1/2}w, \lambda_j^{1/2}x_j)^\sim \right\} \tag{7.3}
\]
Then for \( j \in \{1, 2\} \),

\[
\|x_{0,j}\|_{C^{2,2}_{a,b}}^2 = \lambda_j (A_j g_j, g_j) c_{a,b}^\prime \quad \text{and} \quad (x_{0,j}, a) c_{a,b}^\prime = \lambda_j^{1/2} (A_j^{1/2} g_j, a) c_{a,b}^\prime.
\] (7.5)

Using (6.5), the Fubini theorem, (7.4), (7.3), (7.1) and (7.5), we obtain that for \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \),

\[
T_x(F)(y_1 + A^{1/2}_1 g_1, y_2 + A^{1/2}_2 g_2)
\]

\[
= \int_{C^{\prime}_{a,b}[0,T]} \exp \left\{ \sum_{j=1}^{2} \left[ i(A_j^{1/2} w, y_j)^{\sim} \right] \right\} \times \left( \prod_{j=1}^{2} E_{x_j} \left[ \exp \left\{ i(A_j^{1/2} w, \lambda_j^{1/2} x_j + A_j^{1/2} g_j)^{\sim} \right\} \right] \right) \, df(w)
\]

\[
= \int_{C^{\prime}_{a,b}[0,T]} \exp \left\{ \sum_{j=1}^{2} \left[ i(A_j^{1/2} w, y_j)^{\sim} \right] \right\} \times \left( \prod_{j=1}^{2} E_{x_j} \left[ \exp \left\{ i(A_j^{1/2} w, \lambda_j^{1/2} x_j)^{\sim} + \lambda_j^{1/2} (A_j^{1/2} g_j, x_j)^{\sim} \right\} \right] \right) \, df(w)
\]

\[
= \exp \left\{ \sum_{j=1}^{2} \left[ -\frac{\lambda_j}{2} (A_j g_j, g_j) c_{a,b}^\prime - \lambda_j^{1/2} (A_j^{1/2} g_j, a) c_{a,b}^\prime \right] \right\} \times \left( \prod_{j=1}^{2} E_{x_j} \left[ \exp \left\{ i(A_j^{1/2} w, \lambda_j^{1/2} x_j)^{\sim} + \lambda_j^{1/2} (A_j^{1/2} g_j, x_j)^{\sim} \right\} \right] \right) \, df(w)
\]

\[
= \exp \left\{ \sum_{j=1}^{2} \left[ -\frac{\lambda_j}{2} (A_j g_j, g_j) c_{a,b}^\prime - \lambda_j^{1/2} (A_j^{1/2} g_j, a) c_{a,b}^\prime - \lambda_j (A_j^{1/2} g_j, y_j)^{\sim} \right] \right\} \times \left( \prod_{j=1}^{2} E_{x_j} \left[ \exp \left\{ \sum_{j=1}^{2} \left[ i(A_j^{1/2} w, y_j)^{\sim} + i(A_j^{1/2} w, \lambda_j^{1/2} x_j)^{\sim} \right] \right\} \right] \right) \, df(w)
\]

\[
= \exp \left\{ \sum_{j=1}^{2} \left[ -\frac{\lambda_j}{2} (A_j g_j, g_j) c_{a,b}^\prime - \lambda_j^{1/2} (A_j^{1/2} g_j, a) c_{a,b}^\prime - \lambda_j (A_j^{1/2} g_j, y_j)^{\sim} \right] \right\} \times \left( \prod_{j=1}^{2} E_{x_j} \left[ \exp \left\{ \sum_{j=1}^{2} \left[ i(A_j^{1/2} g_j, y_j)^{\sim} + i(A_j^{1/2} g_j, x_j)^{\sim} \right] \right\} \right] \right) \, df(w)
\]

\[= \exp \left\{ \sum_{j=1}^{2} \left[ -\frac{\lambda_j}{2} (A_j g_j, g_j) c_{a,b}^\prime - \lambda_j^{1/2} (A_j^{1/2} g_j, a) c_{a,b}^\prime - \lambda_j (A_j^{1/2} g_j, y_j)^{\sim} \right] \right\} \times \left( \prod_{j=1}^{2} E_{x_j} \left[ \exp \left\{ \sum_{j=1}^{2} \left[ i(A_j^{1/2} g_j, y_j)^{\sim} + i(A_j^{1/2} g_j, x_j)^{\sim} \right] \right\} \right] \right) \, df(w)
\]
Note that each factor in the last expression has a limit as \( \lambda \rightarrow 0 \). Next, using Hölder’s inequality with \( \lambda > 0 \), we see that

\[
T_\lambda(F^*)(y_1, y_2) = E_{\bar{F}}[F^*(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)]
\]

\[
= E_{\bar{F}}[F(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)
\times \exp \left\{ \sum_{j=1}^{2} \left[ -iy_j(A_j^{1/2}g_j, y_j) - \lambda_j(A_j^{1/2}g_j, y_j)^\sim \right] \right\}
\]

\[
= E_{\bar{F}}[F(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)
\times \exp \left\{ \sum_{j=1}^{2} \left[ -iy_j(A_j^{1/2}g_j, y_j) - \lambda_j(A_j^{1/2}g_j, y_j)^\sim - i\lambda_j(A_j^{1/2}g_j, x_j)^\sim \right] \right\}]
\]

Next, using Hölder’s inequality with \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), it follows that

\[
E_{\bar{F}}[F^*(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)
- \Phi_1(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)]
\]

\[
= E_{\bar{F}}[F(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)
\times \left \{ 1 - \exp \left \{ \sum_{j=1}^{2} \left[ (iq_j + \lambda_j(A_j^{1/2}g_j, y_j)^\sim
+ (iq_j\lambda_j^{-1/2} + \lambda_j^{1/2}(A_j^{1/2}g_j, x_j)^\sim \right] \right \} \right \}
\]

\[
\leq \left( E_{\bar{F}} \left[ F(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)^p \right] \right)^{1/p}
\times \left( E_{\bar{F}} \left[ 1 - \exp \left \{ \sum_{j=1}^{2} \left[ (iq_j + \lambda_j(A_j^{1/2}g_j, y_j)^\sim
+ (iq_j\lambda_j^{-1/2} + \lambda_j^{1/2}(A_j^{1/2}g_j, x_j)^\sim \right] \right \} \right \] \right)^{1/p'}
\]

Note that each factor in the last expression has a limit as \( \lambda = (\lambda_1, \lambda_2) \rightarrow -i\tilde{q} = 0 \).
Corollary 7.2. Let \( q_0, F, g_1 \) and \( g_2 \) be as in Theorem 7.1. Then for all real numbers \( q_1 \) and \( q_2 \) with \( |q_j| > q_0, j \in \{1, 2\} \),

\[
E_{\mathcal{F}}^{\text{anf}}[F(x_1 + A_1^{1/2} g_1, x_2 + A_2^{1/2} g_2)]
\]

\[
= \exp \left\{ \sum_{j=1}^{2} \frac{i |q_j|}{2} (A_j g_j, g_j) c_{\alpha,\beta}^\prime - (-i|q_j|)^{1/2} (A_j^{1/2} g_j, a) c_{\alpha,\beta}^\prime + i |q_j| (A_j^{1/2} g_j, y_j) \right\}
\]

\[
\times E_{\mathcal{F}}^{\text{anf}} \left[ F(x_1, x_2) \exp \left\{ \sum_{j=1}^{2} - i q_j (A_j^{1/2} g_j, x_j) \right\} \right].
\]

The following corollary follows from equation [5.3] above.

Corollary 7.2. Let \( q_0, F, g_1 \) and \( g_2 \) be as in Theorem 7.1. Then for all real numbers \( q_1 \) and \( q_2 \) with \( |q_j| > q_0, j \in \{1, 2\} \),

\[
\left( -i q_1, -i q_2 \right) \in \mathbb{C}_1^2, \text{ and that}
\]

\[
\left( E_{\mathcal{F}} \left[ 1 - \exp \left\{ \sum_{j=1}^{2} \frac{i |q_j|}{2} (A_j g_j, g_j) c_{\alpha,\beta}^\prime - (-i|q_j|)^{1/2} (A_j^{1/2} g_j, a) c_{\alpha,\beta}^\prime + i |q_j| (A_j^{1/2} g_j, y_j) \right\} \right] \right)^{1/p'} \to 0
\]

as \( \lambda = (\lambda_1, \lambda_2) \to -i \frac{|q_1|}{2} \). Hence we have that

\[
\frac{T_q^{(p)}(F)(y_1 + A_1^{1/2} g_1, y_2 + A_2^{1/2} g_2)}{E_{\mathcal{F}} \left[ \Phi_1(y_1 + \lambda_1^{1/2} x_1, y_2 + \lambda_2^{1/2} x_2) \right]}
\]

\[
= \exp \left\{ \sum_{j=1}^{2} \frac{i |q_j|}{2} (A_j g_j, g_j) c_{\alpha,\beta}^\prime - (-i|q_j|)^{1/2} (A_j^{1/2} g_j, a) c_{\alpha,\beta}^\prime + i |q_j| (A_j^{1/2} g_j, y_j) \right\}
\]

\[
\times E_{\mathcal{F}}^{\text{anf}} \left[ F(x_1, x_2) \exp \left\{ \sum_{j=1}^{2} - i q_j (A_j^{1/2} g_j, x_j) \right\} \right].
\]

The following corollary follows from equation [5.3] above.

Corollary 7.2. Let \( q_0, F, g_1 \) and \( g_2 \) be as in Theorem 7.1. Then for all real numbers \( q_1 \) and \( q_2 \) with \( |q_j| > q_0, j \in \{1, 2\} \),

\[
E_{\mathcal{F}}^{\text{anf}}[F(x_1 + A_1^{1/2} g_1, x_2 + A_2^{1/2} g_2)]
\]

\[
= \exp \left\{ \sum_{j=1}^{2} \frac{i |q_j|}{2} (A_j g_j, g_j) c_{\alpha,\beta}^\prime - (-i|q_j|)^{1/2} (A_j^{1/2} g_j, a) c_{\alpha,\beta}^\prime + i |q_j| (A_j^{1/2} g_j, y_j) \right\}
\]

\[
\times E_{\mathcal{F}}^{\text{anf}} \left[ F(x_1, x_2) \exp \left\{ \sum_{j=1}^{2} - i q_j (A_j^{1/2} g_j, x_j) \right\} \right].
\]

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By (1) of Remark 3.5 and Corollary 7.2 above, we have the following corollary.

**Corollary 7.3.** Let $q$ be a nonzero real number, let $g_0 \in C_{a,b}^0[0,T]$ and let $F$ be an element of $F(C_{a,b}[0,T])$ given by equation (3.1). Then

$$
\int_{C_{a,b}[0,T]} F(x + g_0) d\mu(x)
$$

$$
= \exp\left\{ \frac{i q}{2} \|g_0\|_{C_{a,b}'}^2 - (-iq)^{1/2} (g_0, a)_{C_{a,b}'} \right\} \times \int_{C_{a,b}[0,T]} F(x) \exp\left\{ -iq (g_0, x)^\sim \right\} d\mu(x)
$$

(7.6)

where $^\sim$ means that if either side exists, then both sides exist and equality holds.

**Remark 7.4.** In Corollary 7.3, taking $a(t) \equiv 0$ and $b(t) = t$, the general function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$. Also, we know that equation (7.6) becomes

$$
\int_{C_0[0,T]} F(x + g_0) dm_w(x)
$$

$$
= \exp\left\{ \frac{i q}{2} \|g_0\|_2^2 \right\} \int_{C_0[0,T]} F(x) \exp\left\{ -iq \int_0^T g_0'(t) dx(t) \right\} dm_w(x)
$$

where $\| \cdot \|_2$ is the norm on $L^2[0,T]$. This result subsume similar known result obtained by Cameron and Storvick [4].

8 A Cameron-Storvick type theorem on $C_{a,b}^2[0,T]$}

In [2], Cameron (see [7] Theorem A, p.145) expressed the Wiener integral of the first variation of a functional $F$ in terms of the Wiener integral of the product of $F$ by a linear functional, and in [7] Theorem 1, Cameron and Storvick obtained a similar result for the analytic Feynman integral on classical Wiener space. In [10] Theorem 2.4, p.491], Chang, Song and Yoo also obtained a Cameron-Storvick theorem on abstract Wiener space. In [13], Chang and Skoug obtained these results for functionals on the function space $C_{a,b}[0,T]$. Also see [12, 15] for related results involving conditional (generalized) Feynman integrals and Fourier-Feynman transforms.

In order to establish similar results for functionals in $F_{A_1}^{a,b}$ (see Theorem 8.3 below) we first obtain a Cameron-Storvick type theorem for the function space $C_{a,b}^2[0,T]$.

**Theorem 8.1.** Let $g_1$ and $g_2$ be elements of $C_{a,b}^0[0,T]$. Let $F(x_1, x_2)$ be $\mu \times \mu$-integrable over $C_{a,b}^2[0,T]$. Assume that $F$ has a first variation $\delta F(x_1, x_2 | g_1, g_2)$ for all $(x_1, x_2) \in C_{a,b}^2[0,T]$ such that for some $\gamma > 0$,

$$
\sup_{|h| \leq \gamma} |\delta F(x_1 + hg_1, x_2 + hg_2 | g_1, g_2)|
$$
is $\mu \times \mu$-integrable over $C^2_{a,b}[0,T]$ as a function of $(x_1, x_2) \in C^2_{a,b}[0,T]$. Then

$$
E_x[\delta F(x_1, x_2|g_1, g_2)] = E_x\left[F\left(x_1, x_2\right)\left\{(g_1, x_1)^{\sim} + (g_2, x_2)^{\sim}\right\}\right] - \left\{(g_1, a)_{C^2_{a,b}} + (g_2, a)_{C^2_{a,b}}\right\}E_x[F(x_1, x_2)].
$$

(8.1)

**Proof.** First note that

$$
\delta F(x_1 + h g_1, x_2 + h g_2|g_1, g_2) = \frac{\partial}{\partial \lambda} F(x_1 + h g_1 + \lambda g_1, x_2 + h g_2)\bigg|_{\lambda=0} + \frac{\partial}{\partial \lambda} F(x_1 + h g_1, x_2 + h g_2 + \lambda g_2)\bigg|_{\lambda=0} = \frac{\partial}{\partial \mu} F(x_1 + \mu g_1, x_2 + h g_2) + \frac{\partial}{\partial \mu} F(x_1 + h g_1, x_2 + \mu g_2)\bigg|_{\mu=h}
$$

$$
= 2 \frac{\partial}{\partial h} F(x_1 + h g_1, x_2 + h g_2).
$$

But since

$$
\sup_{|h| \leq \gamma} \left| \frac{\partial}{\partial h} F(x_1 + h g_1, x_2 + h g_2) \right|
$$

is $\mu \times \mu$-integrable,

$$
\frac{\partial}{\partial h} F(x_1 + h g_1, x_2 + h g_2)
$$

is $\mu \times \mu$-integrable for sufficiently small values of $h$. Hence by the Fubini theorem and equation (7.11), we see that

$$
E_x[\delta F(x_1, x_2|g_1, g_2)] = E_x\left[\frac{\partial}{\partial h} F(x_1, x_2)\bigg|_{h=0}\right] + E_x\left[\frac{\partial}{\partial h} F(x_1, x_2 + h g_2)\bigg|_{h=0}\right]
$$

$$
= E_{x_2}\left[\frac{\partial}{\partial h} E_{x_1}[F(x_1 + h g_1, x_2)]\bigg|_{h=0}\right] + E_{x_1}\left[\frac{\partial}{\partial h} E_{x_2}[F(x_1, x_2 + h g_2)]\bigg|_{h=0}\right]
$$

$$
= E_{x_2}\left[\frac{\partial}{\partial h} \left\{ \exp\left\{ -\frac{h^2}{2} \|g_1\|_{C^2_{a,b}}^2 - h(g_1, a)_{C^2_{a,b}} \right\} \times E_{x_1}[F(x_1, x_2) \exp\{ h(g_1, x_1)^{\sim}\}] \right\}\bigg|_{h=0}\right]
$$

$$
+ E_{x_1}\left[\frac{\partial}{\partial h} \left\{ \exp\left\{ -\frac{h^2}{2} \|g_2\|_{C^2_{a,b}}^2 - h(g_2, a)_{C^2_{a,b}} \right\} \times E_{x_2}[F(x_1, x_2) \exp\{ h(g_2, x_2)^{\sim}\}] \right\}\bigg|_{h=0}\right]
$$

$$
= E_x[F(x_1, x_2)\{ (g_1, x_1)^{\sim} + (g_2, x_2)^{\sim}\}]
$$

$$
- \left\{(g_1, a)_{C^2_{a,b}} + (g_2, a)_{C^2_{a,b}}\right\}E_x[F(x_1, x_2)].
$$

□
Lemma 8.2. Let $g_1$, $g_2$, and $F$ be as in Theorem 8.1. For each $\rho_1 > 0$ and $\rho_2 > 0$, assume that $F(\rho_1 x_1, \rho_2 x_2)$ is $\mu \times \mu$-integrable. Furthermore assume that $F(\rho_1 x_1, \rho_2 x_2)$ has a first variation $\delta F(\rho_1 x_1, \rho_2 x_2 | \rho_1 g_1, \rho_2 g_2)$ for all $(x_1, x_2) \in C^2_{a,b}[0,T]$ such that for some positive function $\gamma(\rho_1, \rho_2)$,

$$\sup_{|h| \leq \gamma(\rho_1, \rho_2)} |\delta F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2 + \rho_2 h g_2 | \rho_1 g_1, \rho_2 g_2)|$$

is $\mu \times \mu$-integrable over $C^2_{a,b}[0,T]$ as a function of $(x_1, x_2) \in C^2_{a,b}[0,T]$. Then

$$E_\mathbb{F}[\delta F(\rho_1 x_1, \rho_2 x_2 | \rho_1 g_1, \rho_2 g_2)]$$

$$= E_\mathbb{F}[F(\rho_1 x_1, \rho_2 x_2) \{ (g_1, x_1) - (g_2, x_2) \} - \{ (g_1, a) C_{a,b} + (g_2, a) C_{a,b} \} E_\mathbb{F}[F(\rho_1 x_1, \rho_2 x_2)].$$

Proof. Let $R(x_1, x_2) = F(\rho_1 x_1, \rho_2 x_2).$ Then we have that

$$R(x_1 + h g_1, x_2) = F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2)$$

and

$$R(x_1, x_2 + h g_2) = F(\rho_1 x_1, \rho_2 x_2 + \rho_2 h g_2)$$

and that

$$\left. \frac{\partial}{\partial h} R(x_1 + h g_1, x_2) \right|_{h=0} = \left. \frac{\partial}{\partial h} F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2) \right|_{h=0}$$

and

$$\left. \frac{\partial}{\partial h} R(x_1, x_2 + h g_2) \right|_{h=0} = \left. \frac{\partial}{\partial h} F(\rho_1 x_1, \rho_2 x_2 + \rho_2 h g_2) \right|_{h=0} .$$

Thus we have

$$\delta F(\rho_1 x_1, \rho_2 x_2 | \rho_1 g_1, \rho_2 g_2)$$

$$= \left. \frac{\partial}{\partial h} F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2) \right|_{h=0} + \left. \frac{\partial}{\partial h} F(\rho_1 x_1, \rho_2 x_2 + \rho_2 h g_2) \right|_{h=0}$$

$$= \left. \frac{\partial}{\partial h} R(x_1 + h g_1, x_2) \right|_{h=0} + \left. \frac{\partial}{\partial h} R(x_1, x_2 + h g_2) \right|_{h=0}$$

$$= \delta R(x_1, x_2 | g_1, g_2).$$

Hence by equation (8.1), we have

$$E_\mathbb{F}[\delta F(\rho_1 x_1, \rho_2 x_2 | \rho_1 g_1, \rho_2 g_2)]$$

$$= E_\mathbb{F}[\delta R(x_1, x_2 | g_1, g_2)]$$

$$= E_\mathbb{F}[R(x_1, x_2) \{ (g_1, x_1) + (g_2, x_2) \} - \{ (g_1, a) C_{a,b} + (g_2, a) C_{a,b} \} E_\mathbb{F}[R(x_1, x_2)]$$

$$= E_\mathbb{F}[F(\rho_1 x_1, \rho_2 x_2) \{ (g_1, x_1) + (g_2, x_2) \} - \{ (g_1, a) C_{a,b} + (g_2, a) C_{a,b} \} E_\mathbb{F}[F(\rho_1 x_1, \rho_2 x_2)]$$

which establishes equation (8.2).
Theorem 8.3. Let $g_1$, $g_2$, and $F$ be as in Lemma 8.2. Then if any two of the three generalized analytic Feynman integrals in the following equation exist, then the third one also exists, and equality holds:

$$E_{x}^{\text{ani} (g_1, g_2)}[\delta F(x_1, x_2|g_1, g_2)] = -i E_{x}^{\text{ani} (g_1, g_2)}[F(x_1, x_2)\{g_1(x_1)^{-} + g_2(x_2)^{-}\}]$$

$$= -i \left\{(-i q_1)^{1/2}(g_1, a)_{C_{a,b}} + (-i q_2)^{1/2}(g_2, a)_{C_{a,b}} \right\} E_{x}^{\text{ani} (g_1, g_2)}[F(x_1, x_2)].$$

**Proof.** Let $p_1 > 0$ and $p_2 > 0$ be given. Let $y_1 = \rho_1^{-1}g_1$ and $y_2 = \rho_2^{-1}g_2$. By equation (8.2),

$$E_{x}[\delta F(p_1 x_1, p_2 x_2|g_1, g_2)] = E_{x}[\delta F(p_1 x_1, p_2 x_2|p_1 y_1, p_2 y_2)]$$

$$= E_{x}[F(p_1 x_1, p_2 x_2)(\{y_1, x_1\}^{-} + (y_2, x_2)^{-})]$$

$$- \{(y_1, a)C_{a,b} + (y_2, a)C_{a,b} \} E_{x}[F(p_1 x_1, p_2 x_2)]$$

$$= E_{x}[F(p_1 x_1, p_2 x_2)(\rho_1^{-1}(g_1, p_1 x_1)^{-} + \rho_2^{-1}(g_2, p_2 x_2)^{-})]$$

$$- \{\rho_1^{-1}(g_1, a)C_{a,b} + \rho_2^{-1}(g_2, a)C_{a,b} \} E_{x}[F(p_1 x_1, p_2 x_2)].$$

Now let $p_1 = \lambda_1^{-1/2}$ and $p_2 = \lambda_2^{-1/2}$. Then equation (8.4) becomes

$$E_{x}[\delta F(\lambda_1^{1/2} x_1, \lambda_2^{-1/2} x_2|g_1, g_2)]$$

$$= E_{x}[F(\lambda_1^{1/2} x_1, \lambda_2^{-1/2} x_2) (\lambda_1 (g_1, \lambda_1^{1/2} x_1)^{-} + \lambda_2 (g_2, \lambda_2^{-1/2} x_2)^{-})]$$

$$- \{\lambda_1^{1/2}(g_1, a)C_{a,b} + \lambda_2^{1/2}(g_2, a)C_{a,b} \} E_{x}[F(\lambda_1^{1/2} x_1, \lambda_2^{-1/2} x_2)].$$

Since $p_1 > 0$ and $p_2 > 0$ were arbitrary, we have that equation (8.5) holds for all $\lambda_1 > 0$ and $\lambda_2 > 0$. We now use Definition 3.2 to obtain our desired conclusions.

\[\square\]

9 Applications of the Cameron-Storvick type theorem

In this section we consider functionals in the generalized Fresnel type class $F_{A}^{a,b} \equiv F_{A_{+}, A_{-}}$, where $A_{+}$ and $A_{-}$ are related by equation (3.3) above.

Let $\phi$ be a function of bounded variation on $[0, T]$. Define an operator $A : C'_{a,b}[0, T] \rightarrow C'_{a,b}[0, T]$ by

$$Aw(t) = \int_{0}^{t} \phi(s)D_{s}wdb(s) = \int_{0}^{t} \phi(s)\frac{w'(s)}{b(s)}db(s) = \int_{0}^{t} \phi(s)z(s)db(s)$$

for $w(t) = \int_{0}^{t} z(s)db(s)$. It is easily shown that $A$ is a self-adjoint operator. We also see that $A = A_{+} - A_{-}$ where

$$A_{+}w(t) = \int_{0}^{t} \phi^{+}(s)D_{s}wdb(s) \quad \text{and} \quad A_{-}w(t) = \int_{0}^{t} \phi^{-}(s)D_{s}wdb(s)$$

$$\int_{0}^{t} \phi(s)z(s)db(s).$$

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and $\phi^+$ and $\phi^-$ are the positive part and the negative part of $\phi$, respectively. Also, $A_{1/2}^+$ and $A_{1/2}^-$ are given by

$$A_{1/2}^+(t) = \int_0^t \sqrt{\phi^+(s)} D_s wdb(s) \quad \text{and} \quad A_{1/2}^-(t) = \int_0^t \sqrt{\phi^-(s)} D_s wdb(s),$$

respectively. For a more detailed study of this decomposition, see [24, pp.187-189].

For fixed $g \in C'_{a,b}[0,T]$, let $g_1 = A_{1/2}^+ g$ and $g_2 = A_{1/2}^- (-g)$. Then we see that for all $w(t) = \int_0^t z(s)db(s)$ in $C'_{a,b}[0,T]$,

$$\begin{align*}
(A_{1/2}^+, g_1)_{C'_{a,b}} + (A_{1/2}^-, g_2)_{C'_{a,b}} & = (A_{1/2}^+, A_{1/2}^+ g)_{C'_{a,b}} - (A_{1/2}^+, A_{1/2}^- g)_{C'_{a,b}} \\
& = (A^+ w, g)_{C'_{a,b}} - (A^- w, g)_{C'_{a,b}} \\
& = (A^+ w, g)_{C'_{a,b}}. \\
\int_0^T \frac{\partial}{\partial t} \phi(t) \sqrt{\phi^+(t)} (A^+ w) \, dt & = \int_0^T \frac{\partial}{\partial t} \phi(t) \sqrt{\phi^+(t)} (A^+ w) \, dt \\
& = \int_0^T \frac{\partial}{\partial t} \phi(t) \sqrt{\phi^+(t)} (A^+ w) \, dt \\
& = \int_0^T \frac{\partial}{\partial t} \phi(t) \sqrt{\phi^+(t)} (A^+ w) \, dt \\
& = \int_0^T \frac{\partial}{\partial t} \phi(t) \sqrt{\phi^+(t)} (A^+ w) \, dt.
\end{align*}$$

Using equation (2.2), we also see that

\begin{equation}
(A_{1/2}^+, A_{1/2}^+)_{C'_{a,b}} = (w, A_{1/2}^+ a)_{C'_{a,b}}
= \int_0^T D_t \sqrt{\phi^+(t)} (A^+ a) \, dt
= \int_0^T D_t \sqrt{\phi^+(t)} (A^+ a) \, dt.
\end{equation}

and

\begin{equation}
(A_{1/2}^-, A_{1/2}^-)_{C'_{a,b}} = \int_0^T D_t \sqrt{\phi^-(t)} (A^- a) \, dt
= \int_0^T D_t \sqrt{\phi^-(t)} (A^- a) \, dt.
\end{equation}

Assume that $F$ is an element of $\mathcal{F}_A^0 \cap \mathcal{G}_A^0$ for some $q_0 \in (0,1)$ where $\mathcal{F}_A^0 \equiv \mathcal{F}_{A_+ A_+}$ and $\mathcal{G}_A^0 \equiv \mathcal{G}_{A_+ A_+}$ (the classes $\mathcal{F}_{A_+ A_+}$ and $\mathcal{G}_{A_+ A_+}$ are defined in Sections 3 and 5, respectively).

Using (5.3) with $(g_1, g_2) = (1, -1)$, (9.1), (9.2) and (9.3), we obtain that

$$\begin{align*}
E_{g}^{\text{aff},(1,-1)} & \left[ \delta F(x_1, x_2 | A_{1/2}^+ g, -A_{1/2}^- g) \right] \\
= E_{g}^{\text{aff},(1,-1)} & \left[ \delta F(x_1, x_2 | g_1, g_2) \right] \\
= \int_{C'_{a,b}[0,T]} & \left[ i(A_{1/2}^+ w, A_{1/2}^+ g)_{C_{a,b}'} - i(A_{1/2}^- w, A_{1/2}^- g)_{C_{a,b}'} \right] \\
\times & \exp \left\{ -\frac{i}{2} ((A_+ - A_-)w, w)_{C_{a,b}'} \right\} \\
\times & \left\{ i \left[ (-i)^{-1/2}(A_{1/2}^+ w, a)_{C_{a,b}'} + (i)^{-1/2}(A_{1/2}^- w, a)_{C_{a,b}'} \right] \right\} df(w) \\
= \int_{C'_{a,b}[0,T]} & \left[ i(Aw, g)_{C_{a,b}'} \right] \left\{ -\frac{i}{2} (Aw, w)_{C_{a,b}'} \right\} \\
\times & \left\{ i \left[ (-i)^{-1/2}(D_t \sqrt{\phi^+}, a') + (i)^{-1/2}(D_t \sqrt{\phi^-}, a') \right] \right\} df(w).
\end{align*}$$
We also see that for all $\rho_1 > 0$, $\rho_2 > 0$ and $h \in \mathbb{R}$
\[
|\delta F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2 + \rho_2 h g_2|) + (\rho_1 g_1, \rho_2 g_2)\rangle | \\
\leq \int_{C_{a,b}[0,T]} \left| \left( (A_{1/2}^{1/2} w, A_{1/2}^{1/2} (\rho_1 g)) C_{a,b}[0,T] + (A_{1/2}^{1/2} w, A_{1/2}^{1/2} (-\rho_1 g)) C_{a,b}[0,T] \right) \right| df(w) \\
\leq \rho_1 \int_{C_{a,b}[0,T]} |(A_{+} w, g) C_{a,b}[0,T] + (A_{-} w, g) C_{a,b}[0,T]| df(w) \\
\leq (\rho_1 \|A_{+}\|_0 + \rho_2 \|A_{-}\|_0) g \|C_{a,b}[0,T] \| w \|C_{a,b}[0,T] \| df(w).
\]
But the last expression above is bounded and is independent of $(x_1, x_2) \in C_{a,b}[0,T]$. Hence $\delta F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2 + \rho_2 h g_2|)$ is $\mu \times \mu$-integrable in $(x_1, x_2) \in C_{a,b}[0,T]$ for every $\rho_1 > 0$ and $\rho_2 > 0$. Also by Theorem 5.4 and Corollary 3.11, the generalized analytic Feynman integrals $E_{\mathcal{F}}^{anf(1,-1)}(\delta F(x_1, x_2, A_{1/2}^2 g, -A_{1/2}^2 g))$ and $E_{\mathcal{F}}^{anf(1,-1)}(\delta F(x_1, x_2))$ exist. Thus by equation (8.3) together with equation (9.3) we have
\[
\int_{C_{a,b}[0,T]} i(Aw, g) C_{a,b}[0,T] \exp \left\{ -\frac{i}{2}(Aw, w) C_{a,b}[0,T] \right\} \\
\times \exp \left\{ i \left[ (-i)^{-1/2} (D_t w \sqrt{\phi^+}, a') + (i)^{-1/2} (D_t w \sqrt{\phi^-}, a') \right] \right\} df(w) \\
= -i E_{\mathcal{F}}^{anf(1,-1)}(F(x_1, x_2)) \left( \left( A_{1/2}^{1/2} g, x_1 \right)^2 + (A_{1/2}^{1/2} g, x_2)^2 \right) \\
- i \left\{ (-i)^{1/2} (D_t g \sqrt{\phi^+}, a') - (i)^{1/2} (D_t g \sqrt{\phi^-}, a') \right\} E_{\mathcal{F}}^{anf(1,-1)}(F(x_1, x_2)).
\]

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