Generalised canonical systems related to matrix string equations: corresponding structured operators and high-energy asymptotics of the Weyl functions

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Abstract

We obtain high energy asymptotics of Titchmarsh-Weyl functions of the generalised canonical systems generalising in this way a seminal Gesztesy-Simon result. The matrix valued analog of the amplitude function satisfies in this case an interesting new identity. The corresponding structured operators are studied as well. Application to a procedure of solving an important inverse problem is presented in Appendix.

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1 Introduction

Canonical systems have the form

\[ w'(x, \lambda) = i\lambda JH(x)w(x, \lambda), \quad J := \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix} \quad \left( w' := \frac{d}{dx} w \right), \quad (1.1) \]
where \( i \) is the imaginary unit \( (i^2 = -1) \), \( \lambda \) is the so called spectral parameter, \( I_p \) is the \( p \times p \) \((p \in \mathbb{N})\) identity matrix, \( \mathbb{N} \) stands for the set of positive integer numbers, \( H(x) \) is a \( 2p \times 2p \) matrix function (matrix valued function), and \( H(x) \geq 0 \) (that is, the matrices \( H(x) \) are self-adjoint and the eigenvalues of \( H(x) \) are nonnegative). Canonical systems are important objects of analysis, being perhaps the most important class of the one-dimensional Hamiltonian systems and including (as subclasses) several classical equations. They have been actively studied in many already classical as well as in various recent works (see, e.g., \([1,5,7,13,14,19,21,23–25,31,32,34,38–40,43,44]\) and numerous references therein).

In most works on canonical systems, a somewhat simpler case of \( 2 \times 2 \) Hamiltonians \( H(x) \) (i.e, the case \( p = 1 \)) is dealt with. In particular, the trace normalisation \( \text{tr } H(x) \equiv 1 \) may be successfully used in the case of \( p = 1 \). The cases with other values of \( p \) \((p > 1)\) are equally important but more complicated and less studied.

**Remark 1.1** The fundamental results by L. de Branges and by M.G. Krein on canonical systems, where the Hamiltonians are \( 2 \times 2 \) trace-normalised matrix functions with real-valued entries, are well presented (and usefully reformulated sometimes) in the recent book \([22]\). In particular, the basic de Branges result on the one to one correspondence between generalized Herglotz functions and the above-mentioned canonical systems is presented there. An interesting research on the absolutely continuous spectrum by the author (C. Remling) himself is also contained in \([22]\). However, the procedure for solving inverse problem is missing in this excellent book.

In our paper, we consider generalised canonical systems

\[
 w'(x, \lambda) = i \lambda j x H(x) w(x, \lambda), \quad H(x) \geq 0, \quad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix} \quad (m_1 + m_2 =: m),
\]

\[
 (1.2)
\]

where \( x \geq 0, \ m_1, m_2 \in \mathbb{N}, \) and \( H \) is an \( m \times m \) locally integrable matrix function. More precisely, we consider mostly the case of the Hamiltonians \( H(x) \) of the form

\[
 H(x) = \beta(x)^* \beta(x),
\]

\[
 (1.3)
\]
where \( \beta \) are \( m_2 \times m \) matrix functions and \( \beta(x)^* \) is the matrix adjoint to \( \beta(x) \). Systems (1.2), (1.3) are studied on \([0, r]\) or \([0, \infty)\). We assume that

\[
\beta(x) \in \mathcal{U}_{m_2 \times m}[0, r],
\]

where

\[
\mathcal{U}^{p \times q}[0, r] = \{ \mathcal{G} : \mathcal{G}'(x) \equiv \mathcal{G}'(0) + \int_0^x \mathcal{G}''(t)dt, \quad \mathcal{G}'' \in L_2^{p \times q}(0, r) \}, \quad (1.4)
\]

\( L_2^{p \times q}(0, r) \) stands for the class of \( p \times q \) matrix functions with square integrable entries (i.e. the entries from \( L_2(0, r) \)) and \( \mathcal{G}' \) is the standard derivative of \( \mathcal{G} \). We say that \( \mathcal{G} \) in (1.4) is two times differentiable and that \( \mathcal{G}'' \) satisfying \( \mathcal{G}'(x) \equiv \mathcal{G}'(0) + \int_0^x \mathcal{G}''(t)dt \) is the second derivative of \( \mathcal{G} \).

**Remark 1.2** Clearly, \( \mathcal{U}^{p \times q}[0, r] \) is the class of \( p \times q \) matrix functions, the entries of which belong to the Sobolev class \( W^{2,2} \) on \([0, r]\) (although Sobolev norms are not used in this work).

When

\[
\beta(x) \in \mathcal{U}_{m_2 \times m}[0, r] \quad (1.5)
\]

for all \( r > 0 \), we write \( \beta(x) \in \mathcal{U}_{m_2 \times m}[0, \infty) \). We also assume that

\[
\beta(x) j \beta(x)^* \equiv 0, \quad \beta'(x) j \beta(x)^* \equiv iI_{m_2}. \quad (1.6)
\]

When \( m_1 = m_2 = p \), system (1.2) is equivalent to the canonical system (1.1) with a slightly different Hamiltonian (see Section 6). Moreover, system (1.2)–(1.6) (in this case and under some minor additional conditions) may be transformed into the matrix string equation [32, Appendix B]. The case of matrix Schrödinger equations is included in this class (see [32, §2 of Appendix B]). Systems (1.2)–(1.6) present a class of canonical systems complimentary to the canonical systems corresponding to Dirac systems. (On the canonical systems corresponding to Dirac systems see, e.g., [30, 34] and references therein.)

This paper is an important continuation and development of our article [32]. Here, we study the high energy asymptotics of Weyl (Titchmarsh–Weyl) functions of the systems (1.2)–(1.6). Asymptotics of the Weyl functions is an important topic with interesting results obtained, in particular, in [2, 8]. Some
fundamental results on Weyl functions followed (for the scalar Schrödinger equation) in the seminal papers [42] by B. Simon and [11] by F. Gesztesy and B. Simon (see further discussion in [9]). Closely related results in terms of spectral functions have been stated in the pioneering note [16] by M.G. Krein (unfortunately without proofs). It was later shown by H. Langer [19] that the assertions from [16] yield high energy asymptotics for Weyl functions as well (at least for the case treated in [42]). The corresponding assertions from [16] (for the special case of the orthogonal spectral functions) were also proved in [19]. Similar relations for Weyl functions corresponding to string equations one can find in [27] (see formula (45) and Statement 7 there based on the work [38] on S-colligations). The case of canonical systems, such that the Hamiltonians $H(x)$ are $2 \times 2$ matrix functions with real-valued entries and $\text{tr} \, H(x) \equiv 1$, was treated in the interesting papers [18–20].

Related results one can find in [17] and in [27, (32)] for self-adjoint Dirac systems, and in [28, p. 319] for skew-self-adjoint Dirac systems, with various important developments in [3, 4, 29]. The inverse approach to Dirac systems

$$y'(x, \lambda) = i(\lambda j + jV(x))y(x, \lambda), \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix},$$

(with $m_1 \times m_2$ potentials $v$) based on the A-function concept was studied in [10].

The main result of the present paper is Theorem 5.1. This theorem generalises (for the case of the canonical systems (1.2)–(1.6)) important Theorem 1.1 from [11], which constitutes, as also mentioned by the authors of [11], one of the main results of their seminal work. The result is new even for the $2 \times 2$ Hamiltonians (in particular, we do not require that the entries of $H$ are real-valued). An interesting new identity (5.10) for the analog of the so called A-amplitude appears here. We also give an analogue of Theorem 5.1 for the case of $m_1 = m_2$ and Weyl matrix functions belonging to Herglotz class (see Theorem 6.3).

**Remark 1.3** The results on the so called high energy asymptotics of Weyl functions (of the (5.9) and (6.20) type) have various applications including applications to the local uniqueness and other problems in the inverse spectral theory [11,12,42] (see also [10,19,26,27] and references therein). The work on
the applications of (5.9) and (6.20) is in progress. However, the uniqueness and procedure of solving inverse problem for the important case $m_1 = m_2$ is already derived from the results of this paper [33].

The next Section 2 is called “Preliminaries”. We generalise their some basic results from [32] and present an interesting Example 2.3. Section 3 is dedicated to the fundamental solutions of the canonical systems. The corresponding operator identities and structured operators are also of independent interest. They are studied in Section 4. The obtained results are summed up in Section 5 in the form of Theorem 5.1. An analogue of Theorem 5.1 for the case $m_1 = m_2$ and Weyl matrix functions belonging to Herglotz class as well as two useful examples are given in Section 6.

Some linear similarity problems are discussed in Appendix A.

Notations. Some notations were already introduced in the introduction above. As usual, $\mathbb{R}$ stands for the real axis, $\mathbb{C}$ stands for the complex plane, the open upper half-plane is denoted by $\mathbb{C}_+$, and $\overline{a}$ means the complex conjugate of $a$. The notation $\Re(a)$ stands for the real part of $a$, and $\Im(a)$ denotes the imaginary part of $a$. For a matrix $K$, $\overline{K}$ is the matrix such that all its entries are complex conjugates of the corresponding entries of $K$. For a matrix or operator $V$, $V^*$ stands for the adjoint matrix or operator.

We set $L_2^{p \times 1} = L_2^p$, $L_2^1 = L_2$ and $U^{p \times 1} = U^p$. ($L_2^p(0, r)$ and $L_2(0, r)$ stand also for the corresponding Hilbert spaces of square summable functions.) The notation $I$ stands for the identity operator. The norm $\|A\|$ of the $n \times n$ matrix $A$ means the norm of $A$ acting in the space $\ell_2^n$ of the sequences of length $n$. The class of bounded operators acting from the Hilbert space $\mathcal{H}_1$ into Hilbert space $\mathcal{H}_2$ is denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, and we set $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$. If $R \in \mathcal{B}(L_2^p(0, r))$ and $\Phi(x) \in L_2^{p \times q}(0, r)$, then $R\Phi(x) \in L_2^{p \times q}(0, r)$, that is, the operators ($R$ here) are applied to matrix functions columnwise.

2 Preliminaries

This article may be considered as an important development of our paper [32], where fundamental solutions of the class of canonical systems corresponding to matrix string equations were studied. Simple generalisations of some basic
results, which were obtained in [32] for the case $m_1 = m_2 = p$, are presented in this section.

1. Weyl functions of systems (1.2) (where $m_1 = m_2 = p$) were considered in [32, Appendix A]. The definitions, results and all proofs in [32, Appendix A] remain valid after we switch from the case $m_1 = m_2 = p$ to the general case of $j$ given in (1.2). (Note that we also switch from $I_p$ to $I_{m_1}$.) More precisely, we have the following relations, definitions and statements.

The $m \times m$ fundamental solution $W(x, \lambda)$ of (1.2) is normalised by the condition

$$W(0, \lambda) = I_m. \quad (2.1)$$

It is easy to see that

$$\frac{d}{dx} \left( W(x, \overline{\mu})^* j W(x, \lambda) \right) = i(\lambda - \mu)W(x, \overline{\mu})^* H(x)W(x, \lambda). \quad (2.2)$$

In particular, we have the equalities

$$W(r, \overline{\lambda})^* j W(r, \lambda) \equiv j \equiv W(r, \lambda) j W(r, \overline{\lambda})^*, \quad (2.3)$$

and we set

$$W(r, \lambda) = \{W_{ik}(r, \lambda)\}_{i,k=1}^2 := jW(r, \overline{\lambda})^* j = W(r, \lambda)^{-1} \quad (r \geq 0), \quad (2.4)$$

where the blocks $W_{ik}$ have the same dimensions as the corresponding blocks of $j$ in (1.2).

Pairs of meromorphic in $\mathbb{C}_+, m_k \times m_1$ matrix functions $P_k(\lambda)$ ($k = 1, 2$) such that

$$P_1(\lambda)^* P_1(\lambda) + P_2(\lambda)^* P_2(\lambda) > 0, \quad [P_1(\lambda)^* P_2(\lambda)^*]_j \begin{bmatrix} P_1(\lambda) \\ P_2(\lambda) \end{bmatrix} \geq 0 \quad (2.5)$$

(where the first inequality holds in one point (at least) of $\mathbb{C}_+$ and the second inequality holds in all the points of analyticity of $P_1$ and $P_2$), are called nonsingular, with property-$j$.

**Notation 2.1** The notation $\mathcal{N}(r)$ stands for the set of matrix functions of the form

$$\phi(r, \lambda) = (W_{21}(r, \lambda) P_1(\lambda) + W_{22}(r, \lambda) P_2(\lambda)) \times (W_{11}(r, \lambda) P_1(\lambda) + W_{12}(r, \lambda) P_2(\lambda))^{-1}, \quad (2.6)$$

where the pairs $\{P_1, P_2\}$ are nonsingular, with property-$j$. 

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If $\phi(\lambda) \in \mathcal{N}(r)$ we have

$$
[I_{m_1} \quad \phi(\lambda)^*] \mathfrak{A}(r, \lambda) \begin{bmatrix} I_{m_1} \\ \phi(\lambda) \end{bmatrix} \succeq 0 \quad \text{for} \quad \mathfrak{A}(r, \lambda) := W(r, \lambda)^* jW(r, \lambda). \quad (2.7)
$$

The matrix functions $\mathcal{W}_{11}(r, \lambda)\mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r, \lambda)\mathcal{P}_2(\lambda)$ in (2.6) are invertible (excluding, possibly, isolated points $\lambda \in \mathbb{C}_+$), and the functions $\phi(r, \lambda)$ are holomorphic and contractive in $\mathbb{C}_+$. That is, only removable singularities of $\phi(r, \lambda)$ are possible in $\mathbb{C}_+$.

**Definition 2.2** Matrix functions $\phi(\lambda) \in \mathcal{N}(r)$ are called Weyl (Titchmarsh–Weyl) functions of the generalised canonical system (1.2) on $[0, r]$, where $0 < r < \infty$. Matrix functions $\varphi(\lambda)$ such that

$$
\varphi(\lambda) \in \bigcap_{r>0} \mathcal{N}(r) \quad (2.8)
$$

are called Weyl functions of the system (1.2) on $[0, \infty)$.

**Example 2.3** Consider canonical system for the case

$$
\beta(x) = \begin{bmatrix} e^{icx} \alpha & e^{-icx}I_{m_2} \end{bmatrix}, \quad \alpha\alpha^* = I_{m_2} \quad (m_1 \geq m_2), \quad c = 1/2, \quad (2.9)
$$

where $\alpha$ is an $m_2 \times m_1$ matrix. Clearly, (1.6) is valid for this $\beta$. It follows from (2.9) that

$$
H(x) = \beta(x)^* \beta(x) = e^{-icx} \mathcal{K} e^{icx}, \quad \mathcal{K} := \begin{bmatrix} \alpha^* \alpha^* & \alpha^* \\ \alpha & I_{m_2} \end{bmatrix}. \quad (2.10)
$$

One easily checks directly that $W(x, \lambda) = e^{-icx} \mathcal{K} e^{icx}$, that is,

$$
\mathcal{W}(x, \lambda) = e^{-icx} \mathcal{K} e^{icx} \quad (2.11)
$$

Further in this example, we assume that

$$
m_1 = m_2 = p. \quad (2.12)
$$

It is easily checked that under assumptions (2.9) and (2.12) we have

$$
\lambda j\mathcal{K} + cj = \Lambda \begin{bmatrix} \zeta_1I_p & 0 \\ 0 & \zeta_2I_p \end{bmatrix} \Lambda^{-1}, \quad \Lambda := \begin{bmatrix} \alpha^* & \alpha^* \\ \zeta_1I_p & \zeta_2I_p \end{bmatrix}. \quad (2.13)
$$
where \( \zeta_1 \) and \( \zeta_2 \) are the roots of the quadratic equation

\[
\zeta^2 + \left( \frac{1}{\lambda} + 2 \right) \zeta + 1 = 0, \quad \text{that is,}
\]

\[
\zeta_1(\lambda) = -1 - \frac{1}{2\lambda} + \frac{1}{\lambda} \sqrt{\lambda + \frac{1}{4}}, \quad \zeta_2(\lambda) = -1 - \frac{1}{2\lambda} - \frac{1}{\lambda} \sqrt{\lambda + \frac{1}{4}},
\]

and the branch of the square root in (2.15) belongs to the quadrant \( \Re(z) > 0 \), \( \Im(z) > 0 \). Relations (2.11) and (2.13) yield:

\[
\mathcal{W}(r, \lambda) = \Lambda \left[ \begin{array}{cc}
e^{-i\zeta_1 r}I_p & 0 \\
0 & e^{-i\zeta_2 r}I_p \end{array} \right] \Lambda^{-1} e^{i\varepsilon r j}.
\]

Note that in the case (2.12), the considerations (of this example) above are similar to the considerations in [32, Section 5]. Below, we calculate a Weyl function for our example.

In view of (2.14), we obtain

\[
\zeta_1(\lambda)\zeta_2(\lambda) = 1, \quad \zeta_1(\lambda) + \zeta_2(\lambda) = -2 - \frac{1}{\lambda},
\]

Simple calculations using (2.15) show that \( |z_2(\lambda)| > |z_1(\lambda)| \) for \( \lambda = \varepsilon i - \frac{1}{4} \), where \( \varepsilon \) is small and positive (\( \lambda \in \mathbb{C}_+ \)). Since \( \zeta_1\zeta_2 = 1 \), the equality

\[
\zeta_1(\lambda) = \overline{\zeta_2(\lambda)}
\]

is valid in the points \( \lambda \in \mathbb{C} \), where \( |\zeta_1(\lambda)| = 1 \) or, equivalently, \( |\zeta_2(\lambda)| = 1 \). Hence, in these points \( \zeta_1 + \zeta_2 \in \mathbb{R} \). Now, the relation \( \zeta_1 + \zeta_2 = -2 - \frac{1}{\lambda} \) implies that \( |\zeta_1(\lambda)| \neq 1 \) for \( \lambda \in \mathbb{C}_+ \). These considerations show that \( |\zeta_1(\lambda)| < 1 \) for \( \lambda \in \mathbb{C}_+ \). Therefore, the pair

\[
\mathcal{P}_1 \equiv e^{-i\varepsilon r}I_p, \quad \mathcal{P}_2 = e^{i\varepsilon r} \zeta_1(\lambda) \alpha
\]

is nonsingular, with property-\( j \). From the second equality in (2.13) and relations (2.16) and (2.18), we derive

\[
\mathcal{W}(r, \lambda) \left[ \begin{array}{c}
\mathcal{P}_1(r, \lambda) \\
\mathcal{P}_2(r, \lambda) \end{array} \right] = e^{-i\zeta_1(\lambda) r} \left[ \begin{array}{c}
I_p \\
\zeta_1(\lambda) \alpha \end{array} \right].
\]
According to (2.6) and (2.19), we have $\zeta_1(\lambda)\alpha \in \mathcal{N}(r)$ for any $r > 0$, that is,
\[
\varphi(\lambda) = \zeta_1(\lambda)\alpha \in \bigcap_{r>0} \mathcal{N}(r). \tag{2.20}
\]
Taking into account Definition 2.2, we see that $\zeta_1(\lambda)\alpha$ is a Weyl function of the canonical system (1.2), (1.3) (in the case (2.9), (2.12)) on $[0, \infty)$.

A simpler example is considered in the present paper as Example 6.5.

Finally, we have the following proposition.

**Proposition 2.4** Let generalised canonical system (1.2) be given on $[0, \infty)$. Then, the sets $\mathcal{N}(r)$ are nested (i.e., $\mathcal{N}(r_2) \subseteq \mathcal{N}(r_1)$ for $0 \leq r_1 < r_2$). Moreover, there is a matrix function $\varphi(\lambda)$ belonging to the intersection of all $\mathcal{N}(r)$, that is, the set of Weyl functions of the system (1.2) on $[0, \infty)$ (the set of $\varphi$ satisfying (2.8)) is nonempty.

If (2.8) holds, then the inequality
\[
\int_0^\infty \left[ I_{m_1} \varphi(\lambda)^* \right] W(x, \lambda)^* H(x) W(x, \lambda) \left[ I_{m_1} \varphi(\lambda) \right] dx < \infty \quad (\lambda \in \mathbb{C}_+) \tag{2.21}
\]
is valid.

We note that (2.21) may be used as an alternative definition of the Weyl function (see, e.g., [32]).

2. Similar to [32, Appendix A], the formulas, statements and proofs in [32, Appendix C] remain valid after we switch to the case corresponding to a more general form of $j$ (more precisely, when we switch to $m_2 \times m$ matrix functions $\beta(x)$). The corresponding results are given below.

**Theorem 2.5** Let the $m_2 \times m$ matrix function $\beta(x)$ belong to the class $\mathcal{U}^{m_2 \times m}[0, r]$ (defined in (1.4)) and let (1.6) hold. Introduce operators $A$ and $K$ acting in $L^2_{m_2}(0, r)$ by the equalities
\[
A := \int_0^x (t - x) \cdot dt, \quad K := i\beta(x)j \int_0^x \beta(t)^* \cdot dt. \tag{2.22}
\]
Then, $K$ is linear similar to $A$:
\[
K = VAV^{-1}, \quad V = u(x) \left( I_{m_2} + \int_0^x \mathcal{V}(x, t) \cdot dt \right), \tag{2.23}
\]
where \( u \in U^{m_2 \times m_2}[0, r] \), \( u^* = u^{-1} \), and
\[
\sup \|V(x, t)\| < \infty \quad (0 \leq t \leq x \leq r).
\]

Remark 2.6 It is important for the study of the generalised canonical systems (1.2)–(1.6) on the semi-axis \([0, \infty)\) that the matrix function \( V(x, t) \) in the domain \( 0 \leq t \leq x \leq \ell \) is uniquely determined by \( \beta(x) \) on \([0, \ell]\) (and does not depend on the choice of \( \beta(x) \) for \( \ell < x < r \) and the choice of \( r \geq \ell \)).

3 Fundamental solution

The following considerations are similar to the considerations of [32, Section 2] although we consider a more general case of \( j \) and normalised transformation operators \( E \) given by (A.41) instead of the transformation operators \( V \). Recalling definition (2.22), it is easy to see that
\[
K - K^* = i\beta(x) j \int_0^\ell \beta(t)^* \cdot dt.
\]

If \( \beta''(x) \in L^{m_2 \times m}(0, r) \), we have (according to Proposition A.4) \( K = EA E^{-1} \), which we substitute into (3.1). Multiplying both parts of the derived equality by \( E^{-1} \) from the left and by \((E^*)^{-1}\) from the right, we obtain the operator identity
\[
AS - SA^* = i\Pi j \Pi^*,
\]
where
\[
S = E^{-1}(E^*)^{-1} > 0, \quad \Pi h = \Pi(x) h, \quad \Pi(x) := (E^{-1}\beta)(x),
\]
\[
\Pi \in \mathcal{B}(\mathbb{C}^m, L^1_2(0, r)), \quad \Pi(x) \in U^{m_2 \times m}[0, r], \quad h \in \mathbb{C}^m.
\]

Note that \( \Pi \) above is the operator of multiplication by the matrix function \( \Pi(x) \) and the operator \( E^{-1} \) is applied to \( \beta \) (in the expression \( E^{-1}\beta \)) columnwise. The transfer matrix function corresponding to the so called S-node (i.e., to the triple \( \{A, S, \Pi\} \) satisfying (3.2)) has the form
\[
w_A(\lambda) = w_A(r, \lambda) = I_m - ij\Pi^* S^{-1}(A - \lambda I)^{-1} \Pi,
\]
and was first introduced and studied in [36]. We introduce the projectors $P_\ell \in \mathcal{B}(L^2_2(0, r), L^2_2(0, \ell))$:

\[(P_\ell f)(x) = f(x) \quad (0 < x < \ell, \quad \ell \leq r). \tag{3.6}\]

Now, we set

\[
S_\ell := P_\ell SP_\ell^*, \quad E_\ell := P_\ell EP_\ell^*, \quad A_\ell := P_\ell AP_\ell^*, \quad \Pi_\ell := P_\ell \Pi, \tag{3.7}
\]

\[
w_A(\ell, \lambda) = I_m - i j \Pi_\ell S_\ell^{-1}(A_\ell - \lambda I)^{-1} \Pi_\ell. \tag{3.8}
\]

Since $E$ is a triangular operator, $E^{-1}$ is triangular as well (see, e.g., the formula (A.49) and its proof), and we have $P_\ell E^{-1} = P_\ell E^{-1}P_\ell^*P_\ell$. Hence, taking into account (3.3) and (3.7) we derive

\[
P_\ell E^{-1}P_\ell^*E_\ell = P_\ell E^{-1}P_\ell^*P_\ell EP_\ell^* = P_\ell E^{-1}EP_\ell^* = I, \tag{3.9}
\]

\[
S_\ell = P_\ell E^{-1}(E^*)^{-1}P_\ell^* = P_\ell E^{-1}P_\ell^*P_\ell(E^*)^{-1}P_\ell^*. \tag{3.10}
\]

It follows that

\[
E_\ell^{-1} = P_\ell E^{-1}P_\ell^*, \quad S_\ell = E_\ell^{-1}(E^*)^{-1}. \tag{3.11}
\]

We also have $P_\ell A = P_\ell AP_\ell^*P_\ell$. Thus, multiplying both parts of (3.2) by $P_\ell$ from the left and by $P_\ell^*$ from the right (and using (3.7), (3.11), and the last equality in (3.3)) we obtain

\[
A_\ell S_\ell - S_\ell A_\ell^* = i j \Pi_\ell j \Pi_\ell^*, \quad \Pi_\ell(x) = (E_\ell^{-1}\beta)(x) \quad (0 < x < \ell). \tag{3.12}
\]

Clearly $w_A(\ell, \lambda)$ coincides with $w_A(r, \lambda)$ when $\ell = r$.

**Remark 3.1** Relations (3.8), (3.11) and (3.12) show that $S_\ell$ and $w_A(\ell, \lambda)$ may be defined via $E_\ell$ (and $\beta(x)$ given on $[0, \ell]$) precisely in the same way as $w_A(r, \lambda)$ is constructed via $E$ (and $\beta(x)$ given on $[0, r]$). Moreover, according to Remark A.5, $E_\ell$ may be constructed in the same way as $E$, and so $w_A(\ell, \lambda)$ does not depend on the choice of $\beta(x)$ for $\ell < x < r$ and the choice of $r \geq \ell$. In particular, $w_A(\ell, \lambda)$ is uniquely defined on the semi-axis $0 < \ell < \infty$ for $\beta(x)$ considered on the semi-axis $0 \leq x < \infty$. 

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The fundamental solution of the canonical system (1.2)–(1.6) may be expressed via the transfer functions \( w_A(\ell, \lambda) \) using continuous factorisation theorem [40, p. 40] (see also [34, Theorem 1.20] as a more convenient for our purposes presentation).

**Theorem 3.2** Let the Hamiltonian of the generalised canonical system (1.2) have the form (1.3). Assume that \( \beta(x) \) in (1.3) belongs \( U^{m_2 \times m}[0, r] \) and satisfies (1.6). Then, the fundamental solution \( W(\ell, \lambda) \) of the generalised canonical system normalised by (2.1) admits representation

\[
W(\ell, \lambda) = w_A \left( \ell, \frac{1}{\lambda} \right)
\]

for \( 0 < \ell \leq r \). If theorem’s conditions hold for each \( 0 < r < \infty \), then (3.13) is valid for each \( \ell \) on the semi-axis \((0, \infty)\).

The proof of Theorem 3.2 coincides with the proof of Theorem 2.2 in [32].

**Remark 3.3** Using (3.3), (3.4) and the last equality in (A.48), we partition \( \Pi \) and \( \Pi(x) \) into two blocks and derive:

\[
\Pi = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}, \quad \Phi_k h_k = \Phi_k(x) h_k \quad (k = 1, 2);
\]

\[
\Pi(x) = \begin{bmatrix} \Phi_1(x) & \Phi_2(x) \end{bmatrix}, \quad \Phi_2(x) \equiv I_{m_2}.
\]

After partitioning \( \beta \) into blocks \( \beta_k \in U^{m_2 \times m_k}[0, r] \quad (k = 1, 2) \), relations (3.3) and (3.15) yield

\[
\Phi_1(x) = \left( E^{-1} \beta_1 \right)(x), \quad \beta(x) =: \left[ \beta_1(x) \beta_2(x) \right].
\]

### 4 Structured operators \( S \)

The study of the structured operators in direct and inverse spectral problems goes back to the classical works [16, 17] by M.G. Krein. See further developments, discussions and references in [10, 29, 34, 38–40]. The case of \( S \) satisfying operator identity (or equation) (3.2), where \( A \) is given in (2.22), that is,

\[
A = A^2, \quad A = i \int_0^x dt \in \mathcal{B}(L^{m_2}_2(0, r)),
\]

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was studied only in [6]. However, related Theorems 1.1 and 1.2 in [6] contain mistakes (since the functions in (1.4) and (1.18) in [6] are integrated over domains, where they are not defined). Hence, we will consider (3.2) here. In particular, we will study S satisfying an important operator identity

\[ ASA^* + S A^* = R, \quad R := \int_0^t R(x, t) \cdot dt \in \mathcal{B}(L_2^{m_2}(0, r)). \]  

Contrary to the identity \( ASA - SA^* = iR \), there is no literature on (4.18) as far as we know.

Since \( A = A^2 \), the operator \( ASA - SA^* \) may be rewritten in the form

\[ ASA - SA^* = A(ASA + SA^*) - (ASA + SA^*)A^*, \]  

that is, (3.2) may be rewritten as

\[ AZ - ZA^* = i\Pi j \Pi^*, \quad ASA + SA^* = Z. \]  

Thus, operators S satisfying (3.2) may be found in two steps, which we discuss below. First, we study Z such that

\[ AZ - ZA^* = i\Pi j \Pi^*. \]  

The operators Z satisfying (4.21) as well as more general identity

\[ AZ - ZA^* = iR, \quad R := \int_0^r R(x, t) \cdot dt \in \mathcal{B}(L_2^{m_2}(0, r)) \]  

were thoroughly studied in [37,41] (see also [15]). According to the resulting Theorem D.1 [34], there is a unique solution \( Z \in \mathcal{B}(L_2^{m_2}(0, r)) \) of the identity (4.21) where \( \Pi \) has the form (3.14), (3.15) (such that \( \Phi_1(x) \) has a bounded derivative):

\[ (Zf)(x) = (\Phi_1(0)\Phi_1(0)^* - I_{m_2})f(x) + \int_0^r Z(x, t)f(t)dt, \]  

\[ Z(x, t) := \begin{cases} 
\int_{\min(x, t)}^{x} \Phi_1'(x - \zeta)\Phi_1'(t - \zeta) d\zeta + \begin{cases} 
\Phi_1'(x - t)\Phi_1(0)^*, & x > t; \\
\Phi_1(0)\Phi_1'(t - x)^*, & t > x.
\end{cases} 
\end{cases} \]
Remark 4.1 According to [41, Corollary 1.1.7] the bounded operator $Z$, which satisfies (4.22) (that is, a more general identity than (4.21)), admits representation

$$Z = \frac{d}{dx} \int_0^r \left( \frac{\partial}{\partial t} \Psi(x,t) \right) \cdot dt,$$

(4.25)

$$\Psi(x,t) = \frac{1}{2} \int_{|x-t|}^{x+t} R \left( \frac{s+x-t}{2}, \frac{s-x+t}{2} \right) ds.$$

(4.26)

In our case, (3.16) and Proposition A.4 imply that $\Phi_1(x) \in U^{m_2 \times m_1}[0, r]$, and so the relations (4.23) and (4.24) hold. Moreover, taking into account (3.16), (A.9) and (A.42) we obtain

$$\Phi_1(0) = \beta_2(0)^{-1} \beta_1(0),$$

(4.27)

where $\beta_2(0)$ is invertible. In view of the first equality in (1.6) at $x = 0$, formula (4.27) yields

$$\Phi_1(0) \Phi_1(0)^* = I_{m_2}.$$

(4.28)

Therefore, we rewrite (4.23) in a simpler form

$$Z = \int_0^r Z(x,t) \cdot dt \in B(L_{m_2}^2(0, r)).$$

(4.29)

Next, we consider a similar to (4.18) identity

$$A S + S A^* = Z.$$

(4.30)

In order to study the operator identity (4.30), we start with the closely related identity

$$T A + A^* T = \tilde{Z}, \quad \tilde{Z} = \int_0^r \tilde{Z}(x,t) \cdot dt \in B(L_{m_2}^2(0, r)),$$

(4.31)

and modify the proof of [41, Theorem 1.1.6] (i.e., of Theorem 1.3 in [37, Chapter 1]).
Theorem 4.2 Let operator $T$ belong to the class $\mathcal{B}(L^2_0(0, r))$ of bounded operators and satisfy operator identity (4.31). Then, $T$ admits representation

$$T = \frac{d}{dx} \int_0^r \frac{\partial \tilde{Y}}{\partial t} (x, t) \cdot dt,$$

where $\tilde{Y}(x, t)$ has the form

$$\tilde{Y}(x, t) = \tilde{v}(x + t) + \frac{1}{2} \int_{\min\{x+t, 2r-x-t\}}^{x-t} \tilde{Z} \left( \frac{x + t + s}{2}, \frac{x + t - s}{2} \right) ds,$$

(4.33)

$$\frac{\partial}{\partial t} \tilde{Y}(x, t) \in L^2_0(0, r) \quad \text{for} \quad x \in [0, r],$$

(4.34)

$$\tilde{v}(x) \in L^2(0, r), \quad \tilde{v}(x) = 0 \quad \text{for} \quad r \leq x \leq 2r.$$  

(4.35)

Moreover, in this case the kernel $\tilde{Z}(x, t)$ satisfies the condition

$$\int_{\xi-r}^{\xi} \tilde{Z}(\xi - t, t) dt = 0 \quad \text{for} \quad r \leq \xi \leq 2r.$$  

(4.36)

Proof. Representation (4.32) of the bounded operators $T$ follows from [41, Theorem 1.1.1]. Clearly, we may assume additionally that

$$\tilde{Y}(r, t) = 0.$$  

(4.37)

It remains to prove relations (4.33), (4.35) and (4.36). For this purpose, one can use the first part of the proof of [41, Theorem 1.1.6], where the signs of the second terms on the left-hand sides of (1.1.25), (1.1.28) and (1.1.29) in [41] should be (evidently) changed. We also change some notations in [41]: $Q_2$ into $Z_2$, $iQ$ into $\tilde{Z}$ and $\omega$ into $r$. Then, in view of [41, (1.1.22)] and of the second formula in [41, (1.1.32)] we have

$$F_1(x, t) := - \int_x^r \int_t^r \frac{\partial \tilde{Y}}{\partial \zeta} (y, \zeta) d\zeta dy, \quad Z_2(x, t) := -i \int_x^r \tilde{Z}(y, t) dy.$$  

(4.38)

Taking into account the abovementioned changes, instead of [41, (1.1.30)] we obtain the following partial differential equation for $F_1$:

$$\frac{\partial F_1}{\partial t}(x, t) - \frac{\partial F_1}{\partial x}(x, t) = Z_2(x, t).$$  

(4.39)
Equation (4.39) may be rewritten in the form
\[-2 \frac{\partial F_2}{\partial \eta}(\xi, \eta) = \mathcal{Z}_2 \left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2} \right), \quad F_2(\xi, \eta) := F_1 \left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2} \right).\] (4.40)

According to (4.38), we have
\[F_1(r, t) = F_1(x, r) = 0,\] (4.41)
which yields
\[F_2(\xi, \pm(2r - \xi)) = 0 \quad \text{for} \quad r \leq \xi \leq 2r.\] (4.42)

Introduce the boundary value matrix function
\[
\vartheta(\xi) = \begin{cases} 
F_2(\xi, -\xi) = F_1(0, \xi) & \text{for} \quad 0 \leq \xi < r, \\
0 & \text{for} \quad r \leq \xi \leq 2r. 
\end{cases}
\] (4.43)

Using (4.40), (4.42) and (4.43), we derive
\[F_2(\xi, \eta) = \vartheta(\xi) - \frac{1}{2} \int_{-\xi}^{\eta} \mathcal{Z}_2 \left( \frac{\xi + y}{2}, \frac{\xi - y}{2} \right) dy \quad \text{for} \quad 0 \leq \xi < r,\] (4.44)
\[F_2(\xi, \eta) = \vartheta(\xi) - \frac{1}{2} \int_{\xi-2r}^{\eta} \mathcal{Z}_2 \left( \frac{\xi + y}{2}, \frac{\xi - y}{2} \right) dy \quad \text{for} \quad r \leq \xi \leq 2r.\] (4.45)

Taking into account the second equality in (4.40) and setting \(\xi = x + t, \ \eta = x - t\), we rewrite (4.44) and (4.45) as
\[F_1(x, t) = \vartheta(x + t) - \frac{1}{2} \int_{-\min(x+t, 2r-x-t)}^{x-t} \mathcal{Z}_2 \left( \frac{x + t + y}{2}, \frac{x - t - y}{2} \right) dy.\] (4.46)

Setting \(\zeta = (x + t - y)/2\) and using the second equality in (4.38), we rewrite (4.46) in the form
\[F_1(x, t) = \vartheta(x + t) + i \int_t^{\min\{x+t, r\}} \int_{x+t-\zeta}^r \tilde{\mathcal{Z}}(s, \zeta) ds \, d\zeta.\] (4.47)
Hence, we have
\[
\frac{\partial}{\partial x} F_1(x, t) = -\tilde{v}(x + t) - i \int_t^{\min\{x+t, r\}} \tilde{Z}(x + t - \zeta, \zeta) \, d\zeta, \quad (4.48)
\]
\[
\tilde{v}(x) = -\vartheta'(x) - i \int_0^r \tilde{Z}(s, x) \, ds \text{ for } x < r, \quad \tilde{v}(x) = 0 \text{ for } r \leq x \leq 2r. \quad (4.49)
\]

It follows from (4.37), (4.38) and (4.48) that
\[
\tilde{\Upsilon}(x, t) = -\frac{\partial F_1}{\partial x}(x, t) = \tilde{v}(x + t) + i \int_t^{\min\{x+t, r\}} \tilde{Z}(x + t - \zeta, \zeta) \, d\zeta
\]
\[
= \tilde{v}(x + t) + \frac{i}{2} \int_{-\min\{x+t, 2r-x-t\}}^{x-t} \tilde{Z}\left(\frac{x + t + s}{2}, \frac{x + t - s}{2}\right) \, ds. \quad (4.50)
\]

According to (4.43), (4.49) and (4.50), relations (4.33) and (4.35) are valid.

Finally, in order to prove (4.36) we note that the first equality in (4.40) and relations (4.42) imply that
\[
\int_{\xi-2r}^{2r-\xi} \int_{\frac{\xi+\eta}{2}}^{r} \tilde{Z}\left(\frac{x, \xi - \eta}{2}\right) \, dx \, d\eta = 0. \quad (4.51)
\]
Setting \( t = \frac{\xi - r}{2} \) in (4.51), we rewrite (4.51) in the form
\[
\int_{\xi-r}^{r} \int_{\xi-t}^{r} \tilde{Z}(x, t) \, dx \, dt = 0 \quad \text{for} \quad r \leq \xi \leq 2r. \quad (4.52)
\]

Since the right-hand side of (4.52) equals zero at \( \xi = r \), by differentiating (4.52) with respect to \( \xi \) we obtain an equivalent equality, that is, (4.36) follows. \[\blacksquare\]

Introduce operators \( U \) such that
\[
(U f)(x) = \overline{f(r - x)} \quad (f \in L^2_{mp}(0, r)). \quad (4.53)
\]

It is immediate that
\[
UAU = A^*, \quad U^2 = I. \quad (4.54)
\]
Hence, the identity (4.30) is equivalent to the identity
\[ \mathcal{A} U SU + \mathcal{A}^* USU = U Z U \]  
(4.55)
and Theorem 4.2 yields the following corollary.

**Corollary 4.3** Let operator $S$ belong to the class $\mathcal{B}(L_{2}^{m_2}(0, r))$ of bounded operators and satisfy operator identity (4.30) where $Z$ has the form (4.29). Then, $S$ admits representation
\[ S = \frac{d}{dx} \int_{0}^{r} \left( \frac{\partial}{\partial t} \Upsilon(x, t) \right) \cdot dt, \]
(4.56)
where $\Upsilon(x, t)$ has the form
\[ \Upsilon(x, t) = v(x + t) - \frac{i}{2} \int_{x-t}^{\min\{x+t, 2r-x-t\}} Z \left( \frac{x + t + s}{2}, \frac{x + t - s}{2} \right) ds, \]
(4.57)
\[ \frac{\partial}{\partial t} \Upsilon(x, t) \in L_{2}^{m_2 \times m_2}(0, r) \quad \text{for} \quad x \in [0, r], \]
(4.58)
\[ v(x) = 0 \quad \text{for} \quad 0 \leq x \leq r, \quad v(x) \in L_{2}^{m_2 \times m_2}(r, 2r). \]
(4.59)
Moreover, in this case the kernel $Z(x, t)$ satisfies the condition
\[ \int_{0}^{\xi} Z(\xi - t, t) dt = 0 \quad \text{for} \quad 0 \leq \xi \leq r. \]
(4.60)

**Proof.** Setting
\[ S = UTU, \quad Z = U \tilde{Z} U, \]
(4.61)
where $T$, $\tilde{Z}$ satisfy (4.31), and using (4.29) and (4.32) we derive (4.56), where
\[ \Upsilon(x, t) = \overline{\Upsilon(r - x, r - t)}, \quad \tilde{Z}(x, t) = Z(r - x, r - t). \]
(4.62)
According to (4.33) and (4.62) we have
\[
\begin{align*}
\Upsilon(x, t) &= \overline{v(2r - x - t)} \\
&= \frac{i}{2} \int_{\min\{2r-x-t, x+t\}}^{t-x} \tilde{Z} \left( \frac{2r - x - t + s}{2}, \frac{2r - x - t - s}{2} \right) ds \\
&= \overline{v(2r - x - t)} \\
&- \frac{i}{2} \int_{\min\{2r-x-t, x+t\}}^{t-x} \tilde{Z} \left( \frac{x + t - s}{2}, \frac{x + t + s}{2} \right) ds.
\end{align*}
\]
(4.63)
Thus, we obtain (4.57) and (4.59) after we switch from $s$ to $-s$ on the right-hand side of (4.63). Finally, in view of the second equality in (4.62), condition (4.36) may be rewritten in the form

$$
\int_{\xi-r}^{r} Z(r - \xi + t, r - t) dt = \int_{0}^{2r - \xi} Z(2r - \xi - s, s) ds, \quad r \leq \xi \leq 2r.
$$

(4.64)

Substituting in (4.64) $\xi$ instead of $2r - \xi$, we derive (4.60). \[\blacksquare\]

**Remark 4.4** Since operators $S_\ell$ in (3.8) (and, correspondingly, in the representation (3.13) of the fundamental solution) satisfy operator identities (3.12), they have the form (4.56)–(4.59), where $r$ is substituted by $\ell$ and $Z$ is given by (4.24).

In order to consider our conditions on $\Phi_1$, it is convenient to write down the right-hand side of (4.19) in another form:

$$
AS - SA^* = A(AS - SA^*) + (AS - SA^*)A^*.
$$

(4.65)

Then, taking into account (3.2) and (3.15), we rewrite condition (4.60) in the form

$$
\int_{0}^{\xi} \Phi_1(\xi - t)\Phi_1(t)^* dt = \xi I_{m_2}.
$$

(4.66)

**Corollary 4.5** If the conditions of Theorem 3.2 hold on $[0, r]$ (on $[0, \infty)$), then (4.66) is valid for all $\xi \in [0, r]$ ($\xi \in [0, \infty)$) as well.

Finally, let us consider some examples.

**Example 4.6** Let $Z$ be a bounded operator, which satisfies (4.22), where

$$
\mathcal{R}(x, t) = \mathcal{R}_0(x + t), \quad \mathcal{R}_0'(s) \in L^{m_2 \times m_2}_2(0, 2r).
$$

(4.67)

Then, (4.26) yields

$$
\Psi(x, t) = \frac{1}{2} \int_{|x-t|}^{x+t} \mathcal{R}_0(s) ds,
$$

(4.68)

and explicit representation of $Z$ follows from (4.25), (4.68):

$$
(Zf)(x) = \mathcal{R}_0(0)f(x) + \frac{1}{2} \int_{0}^{r} \left( \mathcal{R}_0'(x + t) + \mathcal{R}_0'(|x - t|) \right) f(t) dt.
$$

(4.69)
It is also easily checked that each $Z$ of the form (4.69) (where $\mathcal{R}_0$ is differentiable and $\mathcal{R}_0'(s) \in L^{m_2 \times m_2}(0, 2r)$) satisfies the identity (4.22) with

$$R = \int_0^r \mathcal{R}_0(x + t) \cdot dt. \quad (4.70)$$

Indeed, we set

$$Z = Z_1 + Z_2 + Z_3, \quad Z_1 f = \mathcal{R}_0(0)f, \quad Z_2 f = \frac{1}{2} \int_0^r \mathcal{R}_0'(x + t)f(t)dt,$$

$$Z_3 f = \frac{1}{2} \int_0^r \mathcal{R}_0'(|x - t|)f(t)dt.$$

Clearly,

$$(A\mathcal{Z}_1 - \mathcal{Z}_1A^*)f = i \int_0^r f(t)dt. \quad (4.71)$$

Simple calculations similar to (4.77) and (4.78) below show that

$$(A\mathcal{Z}_2 - \mathcal{Z}_2A^*)f = \frac{i}{2} \int_0^r \left(2\mathcal{R}_0(x + t) - \mathcal{R}_0(t) - \mathcal{R}_0(x)\right)f(t)dt. \quad (4.72)$$

Finally, changing the order of integration, we obtain

$$A\mathcal{Z}_3 f = \frac{i}{2} \left( \int_0^x \left( \mathcal{R}_0(x - t) + \mathcal{R}_0(t) - 2\mathcal{R}_0(0) \right)f(t)dt 
+ \int_x^r \left( \mathcal{R}_0(t) - \mathcal{R}_0(t - x) \right)f(t)dt \right). \quad (4.73)$$

$$Z_3A^* = -\frac{i}{2} \left( \int_0^x \left( \mathcal{R}_0(x) - \mathcal{R}_0(x - t) \right)f(t)dt 
+ \int_x^r \left( \mathcal{R}_0(x) + \mathcal{R}_0(t - x) - 2\mathcal{R}_0(0) \right) \right). \quad (4.74)$$

Equalities (4.71)–(4.74) imply the identity (4.22) where $R$ has the form (4.70).

**Example 4.7** The operator

$$S_0 = \int_0^r v_0(x + t) \cdot dt \quad (v_0 \in L^{m_2 \times m_2}(0, 2r)) \quad (4.75)$$

satisfies the operator identity

$$AS_0 + S_0A^* = i \int_0^r \int_t^x v_0(s)ds \cdot dt. \quad (4.76)$$
Indeed, we have

\[(A_S f)(x) = i \int_0^x \int_0^r v_0(t + s) f(s) ds dt = i \int_0^x \int_0^r v_0(t + s) dt f(s) ds, \tag{4.77}\]

\[(S_0 A^* f)(x) = -i \int_0^r v_0(x + t) \int_t^r f(s) ds dt = -i \int_0^r \int_0^x v_0(x + t) dt f(s) ds, \tag{4.78}\]

and (4.76) follows.

According to (4.57), (4.59) and (4.76), \(\Upsilon(x, t)\) is given in this case by the equalities

\[
\Upsilon(x, t) = \frac{1}{2} \int_{x-t}^{x+t} \int_{(x-t-s)/2}^{(x+s)/2} v_0(\xi) d\xi ds \quad (x + t \leq r), \tag{4.79}\]

\[
\Upsilon(x, t) = v(x + t) + \frac{1}{2} \int_{x-t}^{x+t} \int_{(x-s)/2}^{(x+s)/2} v_0(\xi) d\xi ds \quad (x + t \geq r). \tag{4.80}\]

Hence, we derive

\[
\frac{\partial}{\partial t} \Upsilon(x, t) = \int_t^{x+t} v_0(\xi) d\xi \quad (x + t \leq r), \tag{4.81}\]

\[
\frac{\partial}{\partial t} \Upsilon(x, t) = v'(x + t) - \int_{x+t-r}^t v_0(\xi) d\xi \quad (x + t \geq r). \tag{4.82}\]

Comparing (4.56) and (4.75) and taking into account (4.81) and (4.82), we see that

\[
v''(\xi) = v_0(\xi) - v_0(\xi - r) \quad \text{for} \quad \xi > r \tag{4.83}\]

in our example. The condition (4.60) takes the form

\[
i \int_0^\xi \int_t^{\xi-t} v_0(s) ds dt = 0 \quad (0 \leq \xi \leq r), \tag{4.84}\]

which is equivalent to

\[
\left( \int_0^r \int_t^{x-t} v_0(s) ds dt \right)' = 0 \quad (0 \leq \xi \leq r). \tag{4.85}\]

Clearly, (4.85) is always valid.

Using our results for Examples 4.6 and (4.7) we obtain one more example.
Example 4.8 Let $S = Z$ be given by (4.69) and let $A = A^2$ (i.e., let $A$ be given by (2.22)). Then, $S$ satisfies the operator identity

$$AS - SA^* = - \int_0^r \int_t^x \mathcal{R}_0(s) ds \cdot dt.$$  \hspace{1cm} (4.86)

Moreover, the rank of the right-hand side of (4.86) is no more than $2m_2$.

Indeed, for $S_0$ given (4.75), relations (4.22) and (4.70) yield

$$AS - SA^* = S_0, \text{ where } v_0(s) := i\mathcal{R}_0(s).$$ \hspace{1cm} (4.87)

Now, using equalities (4.76) and (4.87) we obtain

$$A(AS - SA^*) + (AS - SA^*)A^* = A S_0 + S_0 A^* = - \int_0^r \int_t^x \mathcal{R}_0(s) ds \cdot dt.$$ \hspace{1cm} (4.88)

Since the left-hand sides of (4.86) and (4.88) coincide, the operator identity (4.86) follows. The estimate of the rank of the right-hand side of (4.86) follows from the representation

$$\int_t^x \mathcal{R}_0(s) ds = \int_0^x \mathcal{R}_0(s) ds - \int_0^t \mathcal{R}_0(s) ds,$$

where the first and second terms on the right-hand side are $m_2 \times m_2$ matrix functions, the first one depending only on $x$ and the second one depending on $t$.

Operators $S$ of the form (4.69), some similar ones, and spectral theory of the corresponding string equations have been studied, for instance, in [16, 19] (in the scalar case), in [38, pp. 53-56] as well as in [27, §7].

In the spirit of the continuous factorisation theorem [34, 38, 40], formula (3.3) and Remark 3.1 yield $\Pi_r^* S_r^{-1} \Pi_r = \int_0^r \beta(x)^* \beta(x) dx$, that is,

$$\frac{d}{dr} \left( \Pi_r^* S_r^{-1} \Pi_r \right) = \beta(r)^* \beta(r) = H(r).$$ \hspace{1cm} (4.89)

The recovery of $\Phi_1$ and operators $S_r$, and so (in view of (4.89)) the solution of the inverse problem to recover a generalised canonical system, is the subject of our next paper.
5 High energy asymptotics of the Weyl functions

Relations (3.2), (3.5) and (3.13) yield (see [34, (1.88)]):

\[ W(r, \lambda)^* j W(r, \lambda) = j + i(\lambda - \overline{\lambda}) \Pi^*(I - \overline{\lambda}A^*)^{-1} S^{-1}(I - \lambda A)^{-1} \Pi, \quad (5.1) \]

where \( A = A_r, S = S_r, \Pi = \Pi_r \). Hence, in view of (2.7) we obtain

\[ I_{m_1} \geq i(\overline{\lambda} - \lambda) \left[ I_{m_1} \phi(\lambda)^* \right] \Pi^*(I - \overline{\lambda}A^*)^{-1} S^{-1}(I - \lambda A)^{-1} \Pi \left[ I_{m_1} \phi(\lambda) \right] \quad (5.2) \]

for \( \lambda \in \mathbb{C}_+ \) and \( \phi(\lambda) \in \mathcal{N}(r) \). Since the operator \( S \) is strictly positive, (5.2) implies that

\[ \left\| (I - \lambda A)^{-1} \Pi \left[ I_{m_1} \phi(\lambda) \right] \right\| \leq \frac{C}{\sqrt{\Im(\lambda)}} \text{ for } \lambda \in \mathbb{C}_+ \text{ and some } C = C(r) > 0. \quad (5.3) \]

It is easily checked directly and follows from the formula for the resolvent \( (I + zA)^{-1} = I - iz \int_0^x e^{iz(t-x)} \cdot dt \) (see, e.g., [34, (1.157)]) that

\[ (I - z^2 A)^{-1} = I + \frac{iz}{2} \int_0^x \left( e^{iz(x-t)} - e^{iz(t-x)} \right) \cdot dt. \quad (5.4) \]

In particular, for the operators \( \Phi_k \in \mathcal{B}((\mathbb{C}^{m_k}, L_{2m_2}(0, r)) \ (k = 1, 2), \) which are defined in (3.14) and (3.15), we have

\[ (I - z^2 A)^{-1} \Phi_2 = \frac{1}{2} \left( e^{inz} + e^{-inz} \right) \Phi_2, \quad (5.5) \]

\[ \Phi_2^*(I - z^2 A)^{-1} \Phi_2 = \frac{1}{2iz} \left( e^{izr} - e^{-izr} \right) I_{m_2}, \quad (5.6) \]

\[ \Phi_2^*(I - z^2 A)^{-1} \Phi_1 = \frac{1}{2} \int_0^r \left( e^{iz(r-t)} + e^{iz(t-r)} \right) \Phi_1(t) dt. \quad (5.7) \]

It is immediate from (5.6) that

\[ (\Phi_2^*(I - z^2 A)^{-1} \Phi_2)^{-1} = -2iz e^{izr} \left( 1 + O(e^{2izr}) \right) I_{m_2} \text{ for } \Im(z) \to \infty. \quad (5.8) \]
In turn, relations (5.3), (5.7) and (5.8) yield

$$\phi(z^2) = \left( o(1) + \frac{1}{\sqrt{\Im(z^2)}} \right) |z|O(e^{izr}) + iz \int_0^r e^{izt} \Phi_1(t)dt \quad (\Im(z) \to \infty).$$

(5.9)

Note that $O(e^{izr})$ in (5.9) characterises the growth of the corresponding matrix norm and $z$ is assumed to be situated in the first quadrant ($\Re(z) > 0$, $\Im(z) > 0$), that is, $\Im(z) > 0$ and $\Im(z^2) > 0$.

**Theorem 5.1** Let generalised canonical system (1.2), (1.3) be given, such that $\beta(x) \in U_{m^2 \times m}[0, r]$ and (1.6) holds. Then, the Weyl functions $\phi \in N(r)$ admit representation (5.9), where $\Phi_1(x)$ is given by (3.16). The corresponding matrix function $\Phi_1(x)$ is two times differentiable and belongs to $U_{m^2 \times m_1}[0, r]$. Moreover, $\Phi_1(x)$ satisfies the equality

$$\int_0^\xi \Phi_1(\xi - t)\Phi_1(t)^*dt = \xi I_{m_2}.$$  

(5.10)

The matrix function $\Phi_1(x)$ is uniquely determined on $[0, r]$ by the representation (5.9).

**Proof.** Formula (5.9) was proved above. The equality (3.16) and Proposition A.4 show that $\Phi_1(x) \in U_{m^2 \times m_1}[0, r]$. Condition (5.10) follows from Corollary 4.5.

Finally, suppose that there is a matrix function $\tilde{\Phi}(x) \in L^2_{m_2 \times m_1}(0, r)$ such that

$$\phi(z^2) = \left( o(1) + \frac{1}{\sqrt{\Im(z^2)}} \right) |z|O(e^{izr}) + iz \int_0^r e^{izt} \tilde{\Phi}(t)dt.$$  

(5.11)

The equalities (5.9) and (5.11) yield

$$\omega(z) := \int_0^r e^{iz(t-r)} (\Phi_1(t) - \tilde{\Phi}(t))dt = \left( o(1) + \frac{1}{\sqrt{\Im(z^2)}} \right) O(1).$$

(5.12)

Clearly, $||\omega(z)||$ is bounded in the domain $\{ z : \Im(z) \leq \varepsilon \}$ ($\varepsilon > 0$). According to (5.12), $||\omega(z)||$ is bounded in the domain $\{ z : \Re(z) \geq \varepsilon, \Im(z) \geq \varepsilon \}$.
Finally, by virtue of the Phragmen–Lindelöf theorem (see, e.g., [34, Corollary E.7] after using the change of variables $z = (\exp\{3i\pi/4\}) \tilde{z} + \varepsilon (1 + i)$ in order to come to the standard angle considered in the theorem), $\|\omega(z)\|$ is bounded in the domain $\{z : \Re(z) < \varepsilon, \Im(z) > \varepsilon\}$. Thus, $\|\omega(z)\|$ is bounded in $\mathbb{C}$ and $\omega(z)$ tends also to zero on some rays in $\mathbb{C}$. Hence, we obtain $\omega(z) \equiv 0$, and the identity $\hat{\Phi}(x) \equiv \Phi_1(x)$ follows. ■

Remark 5.2 Since $\Phi_1(x)$ is two times differentiable, the integral on the right-hand side of (5.9) can be integrated by parts, which will produce a slightly more precise asymptotics.

6 A special case $m_1 = m_2 = p$: Weyl functions from Herglotz class

1. In the important case $m_1 = m_2 = p$, together with system (1.2) studied above, we also consider the standard form (1.1) of the canonical system:

$$\hat{w}'(x, \lambda) = i\lambda J \hat{H}(x) \hat{w}(x, \lambda), \quad (6.1)$$

and assume that

$$\hat{H}(x) = \hat{\beta}(x)^* \hat{\beta}(x), \quad \hat{\beta}(x) J \hat{\beta}(x)^* \equiv 0, \quad \hat{\beta}'(x) J \hat{\beta}(x)^* \equiv iI_p, \quad (6.2)$$

where $J$ is given in (1.1) and the “widehat” is used (in the notations of this section) in order to show that the notations correspond to the system (1.1) (or, more precisely, (6.1)) instead of the system (1.2). We require that $\hat{\beta}(x) \in U^{p \times 2p}[0, r]$. According to [32, (1.3)], we have

$$J = \Theta j \Theta^*, \quad \Theta := \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & -I_p \\ I_p & I_p \end{bmatrix} \quad (\Theta \Theta^* = I_{2p}). \quad (6.3)$$

Thus, one may assume the following simple correspondence between the systems (1.2), (1.3), (1.6) (where $m_1 = m_2 = p$) and systems (6.1), (6.2):

$$\hat{\beta}(x) = \beta(x) \Theta^*, \quad \hat{H}(x) = \Theta H(x) \Theta^*, \quad \hat{W}(x, \lambda) = \Theta W(x, \lambda) \Theta^*. \quad (6.4)$$
Here, \( \hat{W}(x, \lambda) \) is the normalised fundamental solution of (6.1). Similar to \( \mathcal{N}(r) \), the set \( \hat{\mathcal{N}}(r) \) is introduced as the set of linear-fractional transformations \( \hat{\phi} \):

\[
\hat{\phi}(r, \lambda) = i (\hat{W}_{21}(r, \lambda) \hat{P}_1(\lambda) + \hat{W}_{22}(r, \lambda) \hat{P}_2(\lambda)) \\
\times (\hat{W}_{11}(r, \lambda) \hat{P}_1(\lambda) + \hat{W}_{12}(r, \lambda) \hat{P}_2(\lambda))^{-1},
\]

(6.5)

where the pairs \( \{\hat{P}_1, \hat{P}_2\} \) of the \( p \times p \) matrix functions are nonsingular, with property-\( J \), and

\[
\{\hat{W}_{ik}(r, \lambda)\}_{i,k=1}^{2} = \hat{W}(r, \lambda) = \hat{W}(r, \lambda)^{-1}.
\]

(6.6)

In view of (6.4)–(6.6), it is easy to see that the function \( \hat{\phi}(r, \lambda) \), generated by the pair \( \{\hat{P}_1, \hat{P}_2\} \) (with property-\( J \)) of the form

\[
\begin{bmatrix}
\hat{P}_1(\lambda) \\
\hat{P}_2(\lambda)
\end{bmatrix}
\begin{bmatrix}
\Theta \\
\mathcal{P}_1(\lambda) \\
\mathcal{P}_2(\lambda)
\end{bmatrix},
\]

is connected with the function \( \phi(r, \lambda) \) given by (2.6) by a simple linear-fractional transformation

\[
\hat{\phi}(r, \lambda) = i (I_p \pm \phi(r, \lambda)) (I_p \mp \phi(r, \lambda))^{-1}.
\]

(6.7)

Since \( \phi(r, \lambda) \) are contractions, the matrix functions \( \hat{\phi}(r, \lambda) \in \hat{\mathcal{N}}(r) \) belong to Herglotz class, that is, \( i(\hat{\phi}(r, \lambda)^* - \hat{\phi}(r, \lambda)) \geq 0 \). Hence, the matrix functions \( \hat{\phi}(r, \lambda) \) admit Herglotz representation

\[
\hat{\phi}(r, \lambda) = \mu \lambda + \nu + \int_{-\infty}^{\infty} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\tau(t), \quad \mu \geq 0, \quad \nu = \nu^*,
\]

(6.8)

where \( \tau(t) \) is a \( p \times p \) matrix function such that \( \tau(t_1) \geq \tau(t_2) \) for \( t_1 > t_2 \) (i.e., \( \tau \) is monotonically increasing) and

\[
\frac{d\tau(t)}{1 + t^2} < \infty.
\]

(6.9)

**Definition 6.1** The matrix functions \( \hat{\phi}(\lambda) \in \hat{\mathcal{N}}(r) \), that is, \( \hat{\phi}(\lambda) \) of the form (6.5) are called Weyl functions of the canonical system (6.1), (6.2) on \([0, r]\).

In case of the correspondence (6.3), the operators \( K = i \hat{\beta}(x) J \int_0^x \hat{\beta}(t)^* dt \) and \( V \) considered in this section coincide with the operators \( K \) and \( V \) from Section 2 and Appendix A. (Thus, we don’t write \( \hat{K} \) or \( \hat{V} \).) Under condition

\[
\det \hat{\beta}_2(0) \neq 0,
\]

(6.10)
the operators $\hat{E}$ and $\hat{V}_0$ (acting in $L^p_2(0,r)$) are introduced similar to $E$ and $V_0$, respectively:

$$\hat{E} = V\hat{V}_0, \quad \hat{V}_0 f = \hat{\beta}_2(0)f + \int_0^x \hat{V}_0(x-t)f(t)dt, \quad \hat{V}_0(x) = (V^{-1}\hat{\beta}_2)'(x).$$

(6.11)

Compare (6.11) with (A.41). Note that the complete analog of Lemma A.2 is valid in our case and its proof coincides with the proof of Lemma A.2. Hence, an analogue of Proposition A.4 is valid (see below).

**Proposition 6.2** The operator $\hat{E}$ given by (6.11) satisfies the equalities

$$K = \hat{E}A\hat{E}^{-1}, \quad \hat{E}^{-1}\beta_2 \equiv I_p.$$  \hspace{1cm} (6.12)

Moreover, the operators $\hat{E}$ and $\hat{E}^{-1}$ map $U^p[0,r]$ into $U^p[0,r]$.

Here, the proof of the second equality in (6.12) coincides with the proof of the second equality in (A.48).

Similar to the operators $S$ and $\Phi_1$, we introduce the operators $\hat{S}$ and $\hat{\Phi}_1$ ($\hat{\Phi}_1 \in B(\mathbb{C}^p, L^p_2(0,r))$):

$$\hat{S} = \hat{E}^{-1}(\hat{E}^*)^{-1}, \quad \hat{\Phi}_1 h = \hat{\Phi}_1(x)h, \quad \hat{\Phi}_1(x) = (\hat{E}^{-1}\hat{\beta}_1)(x).$$  \hspace{1cm} (6.13)

Relations (6.12) and (6.13) yield an analog of the operator identity (3.2):

$$A\hat{S} - \hat{S}A^* = i\hat{\Pi}J\hat{\Pi}^*, \quad \hat{\Pi} := \begin{bmatrix} \hat{\Phi}_1 & \Phi_2 \end{bmatrix}.$$  \hspace{1cm} (6.14)

The operator $\Phi_2$ in (6.14) coincides with the embedding $\Phi_2$ introduced in Section 3. The transfer matrix function $\hat{w}_A$ has the form

$$\hat{w}_A(\lambda) = \hat{w}_A(r,\lambda) = I_{2p} - iJ\hat{\Pi}^*\hat{S}^{-1}(A - \lambda I)^{-1}\hat{\Pi},$$  \hspace{1cm} (6.15)

and the equality

$$\hat{W}(r,\lambda) = \hat{w}_A(r,1/\lambda)$$  \hspace{1cm} (6.16)

follows similar to (3.13) from the continuous factorisation theorem [40, p. 40] (or from its particular case [34, Theorem 1.20]).
Now, we are ready to study the high energy asymptotics of the Weyl functions $\hat{\phi}(\lambda)$. Let $\hat{\phi}(\lambda) \in \hat{\mathcal{N}}(r)$. Then, the formulas (6.5) and (6.6) together with the property- $J$ of the pairs $\{\hat{P}_1, \hat{P}_2\}$ yield

$$\left[ I_p \ i \hat{\phi}(\lambda)^* \right] \hat{W}(r, \lambda)^* J \hat{W}(r, \lambda) \left[ I_p \ -i \hat{\phi}(\lambda) \right] \geq 0. \quad (6.17)$$

Taking into account (6.14)–(6.16) and using again [34, (1.88)], we obtain an analogue of (5.1):

$$\hat{W}(r, \lambda)^* J \hat{W}(r, \lambda) = J + i(\lambda - \lambda) \hat{\Pi}^* (I - \lambda A)^{-1} \hat{\Sigma}^{-1} (I - \lambda A)^{-1} \hat{\Pi}. \quad (6.18)$$

It easily follows from (6.17) and (6.18) that

$$\frac{\hat{\phi}(\lambda) - \hat{\phi}(\lambda)^*}{\lambda - \lambda} \geq \left[ I_p \ i \hat{\phi}(\lambda)^* \right] \hat{\Pi}^* (I - \lambda A)^{-1} \hat{\Sigma}^{-1} (I - \lambda A)^{-1} \hat{\Pi} \left[ I_p \ -i \hat{\phi}(\lambda) \right]. \quad (6.19)$$

Note that $\mu = 0$ in the Herglotz representation (6.8) of $\hat{\phi} \in \hat{\mathcal{N}}(r)$ (see [33, (4.1)]). Hence, (6.8) and (6.9) imply that the norm of the left-hand side of (6.19) tends to zero in any angle $\delta < \arg(\lambda) < \pi - \delta$ ($\delta > 0$) when $\lambda \to \infty$ in that angle. Therefore, taking into account (5.7) and (5.8), we similar to (5.9) derive

$$\hat{\phi}(z^2) = -z \int_0^r e^{izt} \hat{\Phi}_1(t) dt + o(z e^{izr}) \quad \text{for } |z| \to \infty, \quad (6.20)$$

where $\delta/2 < \arg(z) < (\pi - \delta)/2$.

**Theorem 6.3** Let canonical system (6.1), (6.2) be given, such that $\hat{\beta}(x) \in \mathcal{U}^{p \times 2p}[0, r]$ and (6.10) holds. Then, the Weyl functions $\hat{\phi} \in \hat{\mathcal{N}}(r)$ admit (for any $\delta > 0$) the representation (6.20), where $\hat{\Phi}_1(x) \in \mathcal{U}^{p \times p}[0, r]$ is given by (6.13). The matrix function $\hat{\Phi}_1(x)$ is uniquely determined on $[0, r]$ by the representation (6.20).

**Proof.** The remaining proof that $\hat{\Phi}_1(x)$ is uniquely determined on $[0, r]$ by $\hat{\phi}(\lambda)$ is similar to the proof of Theorem 5.1. Indeed, let another continuous matrix function $\hat{\Phi}(x)$ satisfy (6.20). Then, we have

$$\omega(z) := \int_0^r e^{iz(t-r)} (\hat{\Phi}_1(t) - \hat{\Phi}(t)) dt = o(1) \quad \text{for } |z| \to \infty, \quad (6.21)$$
where $\delta/2 < \arg(z) < (\pi - \delta)/2$. Clearly, $\omega(z)$ is bounded in the lower half-plane $\mathfrak{R}(z) \leq 0$. Hence, taking into account (6.21) and using Phragmen–Lindelöf theorem, we see that the entire function $\omega(z)$ is bounded in $\mathbb{C}$ and tends to zero on some rays. Therefore, $\omega(z) \equiv 0$, and so $\hat{\Phi}_1(x) \equiv \hat{\Phi}(x)$ on $[0,r]$. ■

Let us consider a simple example.

**Example 6.4** In the case $\hat{S} = I$, we have $A\hat{S} - \hat{S}A^* = \int_0^r (t - x) \cdot dt$. Thus, the operator identity (6.14) holds for

$$\hat{S} = I, \quad \hat{\Phi}_1(x) = ixI_p, \quad \Phi_2(x) \equiv I_p.$$  

(6.22)

Recall that $\hat{\Pi}$ in (6.14) is determined by the equality $\hat{\Pi} h = \begin{bmatrix} \hat{\Phi}_1(x) & \Phi_2(x) \end{bmatrix} h$. According to the continuous factorisation theorem [34, Theorem 1.20] (and to (6.22)), $\hat{W}(x, \lambda) = \hat{\omega}_A(x, 1/\lambda)$ is the normalised fundamental solution of the canonical system (6.1) where

$$\hat{H}(x) = \hat{\beta}(x)^*\hat{\beta}(x), \quad \hat{\beta}(x) = \begin{bmatrix} \hat{\Phi}_1(x) & \Phi_2(x) \end{bmatrix} = [ixI_p \quad I_p].$$  

(6.23)

In view of (6.23), relations (6.2) are satisfied. This example (for different purposes) was introduced in [40, p. 164]. Taking into account the equalities (6.15), (6.16) as well as (5.4), (5.5), (6.22), we calculate in a standard way the $p \times p$ blocks $\hat{W}_{ik}$ of $\hat{W}$:

$$\hat{W}_{11}(r, \lambda) = \frac{1}{2}(e^{ivXr} + e^{-ivXr})I_p, \quad \hat{W}_{12}(r, \lambda) = \frac{\sqrt{\lambda}}{2}(e^{ivXr} - e^{-ivXr})I_p,$$

$$\hat{W}_{21}(r, \lambda) = \frac{1}{2i}(r(e^{ivXr} + e^{-ivXr}) - \frac{1}{i\sqrt{\lambda}}(e^{ivXr} - e^{-ivXr}))I_p,$$

$$\hat{W}_{22}(r, \lambda) = \frac{1}{2}(e^{ivXr} + e^{-ivXr} - i\sqrt{\lambda}r(e^{ivXr} - e^{-ivXr}))I_p.$$

By virtue of (6.6) and (6.18) we have $\hat{W}(r, \lambda) = J\hat{W}(r, \lambda)^*J$, and so the expressions for $\hat{W}_{ik}$ above imply the following representation of $\hat{W}$:

$$\hat{W}(r, \lambda) = \frac{1}{2}e^{-izr} \left[ \begin{array}{c} (izr + 1)I_p \\ izr + 1 \\ -izr \end{array} \right] + O(2e^{izr}),$$

(6.24)
where \( \lambda \in \mathbb{C}_+ \), \( z = \sqrt[2]{\lambda} \) and the branch \( \sqrt[2]{\lambda} \) is chosen so that 
\( 0 < \arg(z) < \pi/2 \). We choose a simple nonsingular pair (with property-J):
\[
\hat{P}_1 \equiv I_p, \quad \hat{P}_2 \equiv I_p.
\] (6.25)

It easily follows from (6.5), (6.24) and (6.25) that in the angle 
\( \delta/2 < \arg(z) < (\pi - \delta)/2 \) (for any \( \delta > 0 \)) we have
\[
\hat{\phi}(z^2) = \frac{i}{z}I_p + O(e^{2izr}) \quad (|z| \to \infty).
\] (6.26)

On the other hand, substitution \( \hat{\Phi}_1(t) = itI_p \) in the expression on the right-hand side of (6.20) yields
\[
-z \int_0^r e^{izt}\hat{\Phi}_1(t)dt + o(z^{e^{izr}}) = \frac{i}{z}I_p + o(z^{e^{izr}}) \quad (|z| \to \infty).
\] (6.27)

Formula (6.20) for our example is immediate from (6.26) and (6.27). That is, \( \hat{\Phi}_1(x) = ixI_p \) is the matrix function given by the last equality in (6.13).

Let us consider canonical system of the form (1.2), (1.3) corresponding to system (6.1), (6.23), that is, canonical system (1.2), (1.3) where
\[
m_1 = m_2 = p, \quad \beta(x) = \hat{\beta}(x)\Theta = [ixI_p \quad I_p] \Theta.
\] (6.28)

Example 6.5 Relations (6.7) and (6.26) show that the Weyl function of the system (1.2), (1.3), (6.28) generated by the pair \( P_1 = I_p, \ P_2 = 0 \) (corresponding to the pair (6.25)) has the form
\[
\phi(z^2) = \frac{1 - z}{1 + z}I_p + O(e^{2izr}).
\] (6.29)

Thus, it is easily checked directly that
\[
\Phi_1(t) = (2e^{it} - 1)I_p
\] (6.30)
satisfies (5.9). (In other words, we uniquely recovered \( \Phi_1 \) from \( \phi \) in this example.)
A  Linear similarity transformation

1. In this appendix, we consider linear operators $V$ and $V^{-1}$, which appear in the similarity transformation (2.23), in greater detail. The operators $V$ may be constructed quite similar to the operators $V$ in [32, Appendix C], where some results of [35] are further developed. First, we note that (similar to the case $m_1 = m_2 = p$ in [32, Appendix C]) the $m_2 \times m_2$ integral kernel $\mathcal{V}(x, \zeta)$ of the operator $V$ constructed in the way of [32, Appendix C] admits representation

$$
\mathcal{V}(x, \zeta) := \sum_{k=1}^{\infty} \mathcal{V}_k(x, \zeta), \quad (A.1)
$$

where $\mathcal{V}_1(x, \zeta)$ has the form

$$
\mathcal{V}_1(x, \zeta) = \frac{1}{2} \left( \int_0^{(x+\zeta)/2} u_4(t)dt + \int_0^{(x-\zeta)/2} u_4(t)dt - \int_{(x+\zeta)/2}^{x} \tilde{F}(t, x - t + \zeta)dt 
- \int_{(x-\zeta)/2}^{x-\zeta} \tilde{F}(t, x - t - \zeta)dt - \int_{(x-\zeta)}^{x} \tilde{F}(t, \zeta + t - x)dt \right), \quad (A.2)
$$

and the matrix functions $\mathcal{V}_k(x, \zeta)$ (for $k > 1$) have the form

$$
2\mathcal{V}_k(x, \zeta) = \int_{(x-\zeta)}^{x} \int_{(x+\zeta)/2}^{x} \int_{x-t-\zeta}^{t} u_4(s)\mathcal{V}_{k-1}(s, \zeta + t - x)ds dt ds
+ \int_{(x+\zeta)/2}^{x} \int_{(x-\zeta)/2}^{x} \int_{x-t-\zeta}^{t} u_4(s)\mathcal{V}_{k-1}(s, \zeta + t - x)ds dt ds
+ \int_{(x-\zeta)/2}^{x} \int_{(x+\zeta)/2}^{x} \int_{x-t-\zeta}^{t} u_4(s)\mathcal{V}_{k-1}(s, \zeta + t - x)ds dt ds
- \int_{(x-\zeta)}^{x} \int_{x-t-\zeta}^{t} \tilde{F}(s, \eta)\mathcal{V}_{k-1}(\eta, \zeta + t - x)\eta ds dt ds
- \int_{(x+\zeta)/2}^{x} \int_{(x-\zeta)/2}^{x} \int_{x-t-\zeta}^{t} \tilde{F}(s, \eta)\mathcal{V}_{k-1}(\eta, \zeta + t - x)\eta ds dt ds
- \int_{(x-\zeta)/2}^{x} \int_{x-t-\zeta}^{t} \tilde{F}(s, \eta)\mathcal{V}_{k-1}(\eta, \zeta + t - x)\eta ds dt ds.
$$

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The $m_2 \times m_2$ matrix functions $u_4, F$ and $\tilde{F}$ in (A.2) and (A.3) satisfy relations

$$u_4(t) = u_4(t)^*, \quad u_4 \in L_{m_2}^{m_2}([0, r]), \quad F(s, \eta) = h_1(s)h_2(\eta),$$

$$h_1 \in L_{2}^{m_2 \times m_2}([0, r]), \quad h_2 \in L_{2}^{m_2 \times m_2}([0, r]); \quad \tilde{F}(t, \eta) := \int_{\eta}^{t} F(s, \eta)ds.$$  

Here, the notations $h_k$ slightly differ from the corresponding notations in [32].

For $C(r) = C > 0$ such that

$$\int_{0}^{r} \|h_k(t)\|dt \leq C \quad (k = 1, 2), \quad \int_{0}^{r} \|u_4(t)\|dt \leq C^2,$$

$$\sup_{0 \leq \xi \leq x \leq r} \|\mathcal{V}_1(x, \zeta)\| \leq C,$$

the following inequalities are valid:

$$\|\mathcal{V}_k(x, \zeta)\| \leq \frac{(3C^2)^{k-1}}{(k-1)!} Cx^{k-1} \quad (k \geq 1).$$

Similar to [32, (C.22)], we have the equality

$$u(0) = I_{m_2}.$$  

2. Using the above-mentioned estimates and equalities, we prove our next proposition, which completes Theorem 2.5.

**Proposition A.1** Let the conditions of Theorem 2.5 hold. Then, the similarity transformation operators $V$ and $V^{-1}$ in (2.23) (constructed by the procedure from [32, Appendix C]) map vector functions $f \in U_{m_2}^{m_2}[0, r]$ into $U_{m_2}^{m_2}[0, r]$ (where $U_{m_2}^{m_2}[0, r] = U_{m_2 \times 1}^{m_2}[0, r]$ is introduced in (1.4)).

**Proof.** Step 1. Since $A \in B(L_{2}^{m_2}(0, r))$ is the operator of the squared integration multiplied by $-1$, the relation $f \in U_{m_2}^{m_2}[0, r]$ is equivalent to the representation

$$f(x) = -Af'' + xf'(0) + f(0).$$

In the first step, we show that $\left( V(xf'(0)) \right) \in U_{m_2}^{m_2}[0, r]$, i.e.,

$$\left( V(xf'(0)) \right)'' \in L_{2}^{m_2}(0, r).$$  

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For this purpose, we rewrite the first equality in (2.23) as $KV = VA$, where $V$ is the integral operator given by the second equality in (2.23), and $K$ is given in (2.22). Moreover, from the relations (A.1)–(A.8) it is easy to see that $\mathcal{V}(x, \zeta)$ is continuous on the triangular $x \geq \zeta$. Hence, in terms of the integral kernels, the equality $KV = VA$ (after we change the order of integration) is equivalent to

$$i\beta(x) j(\beta(\zeta)^* u(\zeta) + \int_{\zeta}^{x} \beta(t)^* u(t) \mathcal{V}(t, \zeta) dt) = (\zeta - x) u(x) + u(x) \int_{\zeta}^{x} (\zeta - t) \times \mathcal{V}(x, t) dt.$$  \hspace{1cm} (A.12)

Sometimes, for convenience, we write $V g(t)$ (instead of $V g(x)$). Setting in (A.12) $\zeta = 0$, we derive

$$V(tI_{m_2}) = -i\beta(x) j(\beta(0)^* + \int_{0}^{x} \beta(t)^* u(t) \mathcal{V}(t, 0) dt),$$  \hspace{1cm} (A.13)

where $V$ is applied to $tI_{m_2}$ columnwise. We also note that according to (A.2), (A.4) and (A.5) we have

$$\mathcal{V}_1(x, 0) = \int_{0}^{x/2} u_4(t) dt - \int_{0}^{x/2} \left( \int_{y}^{x-y} h_1(s) ds \right) h_2(y) dy,$$

$$\frac{d}{dx} \mathcal{V}_1(x, 0) = \frac{1}{2} u_4(x) - \int_{0}^{x/2} h_1(x - y) h_2(y) dy \in L_{m_2 \times m_2}^{2}(0, r).$$  \hspace{1cm} (A.14)

From (A.3) and (A.4) we obtain the relations

$$\mathcal{V}_k(x, 0) = \int_{0}^{x/2} \int_{y}^{x-y} u_4(s) \mathcal{V}_{k-1}(s, y) ds dy$$

$$\quad - \int_{0}^{x/2} \int_{y}^{x-y} h_1(s) \int_{y}^{s} h_2(\eta) \mathcal{V}_{k-1}(\eta, y) d\eta dy,$$  \hspace{1cm} (A.16)

$$\frac{d}{dx} \mathcal{V}_k(x, 0) = \int_{0}^{x/2} u_4(x - y) \mathcal{V}_{k-1}(x - y, y) dy$$

$$\quad - \int_{0}^{x/2} h_1(x - y) \int_{y}^{x-y} h_2(\eta) \mathcal{V}_{k-1}(\eta, y) d\eta dy,$$  \hspace{1cm} (A.17)
for \( k > 1 \). Formulas (A.1), (A.8), (A.15), and (A.17) show that \( \mathcal{V}(x, 0) \) is differentiable and

\[
\frac{d}{dx} \mathcal{V}(x, 0) \in L^2_{m_2 \times m_2}(0, r). \quad (A.18)
\]

In view of (A.13) and (A.18), \( V(tI_{m_2}) \) is two times differentiable and

\[
\frac{d^2}{dx^2} V(tI_{m_2}) \in L^2_{m_2 \times m_2}(0, r). \quad (A.19)
\]

Thus, (A.11) holds.

Step 2. Next, we should show that \( \mathcal{V}(x, \zeta) \) is differentiable with respect to \( \zeta \) on \([0, x]\). Taking into account (A.2), (A.4) and (A.5), it is easy to see that

\[
\frac{\partial}{\partial \zeta} \mathcal{V}_1(x, \zeta) = \frac{1}{4} u_4 \left( \frac{x + \zeta}{2} \right) - \frac{1}{4} u_4 \left( \frac{x - \zeta}{2} \right) \in L^2_{m_2 \times m_2}(0, x). \quad (A.20)
\]

It follows from (A.3) and (A.4) that \( \mathcal{V}_2(x, \zeta) \) is differentiable and for some \( \hat{C} = \hat{C}(r) > 0 \) we have

\[
\sup_{0 \leq \zeta \leq x} \left\| \frac{\partial}{\partial \zeta} \mathcal{V}_2(x, \zeta) \right\| < \hat{C}. \quad (A.21)
\]

In view of (A.2) and (A.3), for \( \mathcal{V}_k(x, x) := \lim_{\zeta \to x-0} \mathcal{V}(x, \zeta) \) we also have

\[
\mathcal{V}_1(x, x) = \frac{1}{2} \int_0^x u_4(t) dt, \quad \mathcal{V}_k(x, x) = 0 \quad (k > 1). \quad (A.22)
\]

If \( k > 2 \) and \( \frac{\partial}{\partial \zeta} \mathcal{V}_{k-1}(x, \zeta) \) exists and is bounded, simple calculations using
(A.3), (A.4) and (A.22) show that

\[
2 \frac{\partial}{\partial \zeta} V_k(x, \zeta) = \int_{x-\zeta}^{x} \int_{\zeta+t-x}^{t} u_4(s) \frac{\partial}{\partial \zeta} V_{k-1}(s, \zeta + t - x) ds dt \\
+ \int_{(x+\zeta)/2}^{x} \int_{\zeta + x - t}^{t} u_4(s) \frac{\partial}{\partial \zeta} V_{k-1}(s, \zeta + x - t) ds dt \\
+ \int_{(x-\zeta)/2}^{x} \int_{x-t-\zeta}^{t} u_4(s) \frac{\partial}{\partial \zeta} V_{k-1}(s, x - t - \zeta) ds dt \\
- \int_{x-\zeta}^{x} \int_{\zeta + t - x}^{t} \int_{s}^{\zeta + t - x} F(s, \eta) \frac{\partial}{\partial \zeta} V_{k-1}(\eta, \zeta + t - x) d\eta ds dt \\
- \int_{(x+\zeta)/2}^{x} \int_{\zeta + x - t}^{t} \int_{s}^{\zeta + x - t} F(s, \eta) \frac{\partial}{\partial \zeta} V_{k-1}(\eta, \zeta + x - t) d\eta ds dt \\
- \int_{(x-\zeta)/2}^{x} \int_{x-t-\zeta}^{t} \int_{s}^{x-t-\zeta} F(s, \eta) \frac{\partial}{\partial \zeta} V_{k-1}(\eta, x - t - \zeta) d\eta ds dt.
\]  

(A.23)

Now, we choose \( \hat{C} \) in (A.21) such that \( \hat{C} > C \), and so (A.6) is valid for \( \hat{C} \). Let us show by induction that

\[
\left\| \frac{\partial}{\partial \zeta} V_k(x, \zeta) \right\| \leq \frac{(3\hat{C}^2)^{k-2}}{(k-2)!} \hat{C} x^{k-2} \quad (k \geq 2).
\]  

(A.24)

Indeed, (A.24) holds for \( k = 2 \). If (A.24) is valid for \( k - 1 \), we derive from (A.4), (A.23) and the inequality \( \hat{C} > C \) that (A.24) holds for \( k \).

Finally, relations (A.1), (A.20) and (A.24) imply the differentiability of \( V(x, \zeta) \) with respect to \( \zeta \) and the equality

\[
\frac{\partial}{\partial \zeta} V(x, \zeta) = \sum_{k=1}^{\infty} \frac{\partial}{\partial \zeta} V_k(x, \zeta).
\]  

(A.25)

Moreover, the equalities (A.20) and (A.23) yield

\[
\frac{\partial}{\partial \zeta} V_k(x, \zeta) \bigg|_{\zeta = 0} \equiv 0
\]

(A.26)

for \( k = 1 \) and for \( k > 2 \). After slightly longer calculations, using (A.3) one obtains \( \frac{\partial}{\partial \zeta} V_2(x, \zeta) \bigg|_{\zeta = 0} \equiv 0 \). Thus, (A.26) holds for all \( k > 0 \). Therefore,
taking into account (A.25), we derive
\[ \frac{\partial}{\partial \zeta} V(x, \zeta) \bigg|_{\zeta=0} \equiv 0. \] (A.27)

Step 3. It follows from (A.1) and (A.22) that
\[ V(0, 0) = 0. \] (A.28)

Taking the derivatives (with respect to \( \zeta \)) of both parts of (A.12) at \( \zeta = 0 \) and using (A.27) and (A.28), we rewrite the result in the form
\[ V(I_{m_2}) = i \beta(x) j(\beta^* u)'(0), \text{ i.e., } V(f(0)) \in \mathcal{U}^{m_2}[0, r]. \] (A.29)

Moreover, using again the equality \( KV = VA \) we easily obtain
\[ \left(V A g\right) \in \mathcal{U}^{m_2}[0, r] \text{ for } g(x) \in L^{m_2}_2(0, r). \] (A.30)

Finally, relations (A.10), (A.11), (A.29) and (A.30) yield
\[ V f \in \mathcal{U}^{m_2}[0, r] \text{ for } f \in \mathcal{U}^{m_2}[0, r]. \] (A.31)

Step 4. Let us consider the operator \( V^{-1} \). For this purpose, we rewrite the first equality in (2.23) as \( V^{-1}K = AV^{-1} \). In view of the first equality in (1.6) and the second equality in (2.22), we obtain
\[ (Kg)(0) = 0, \quad (Kg)'(0) = 0 \text{ for } g \in L^{m_2}_2(0, r). \] (A.32)

Thus, presenting the function \( Kg \) in the form (A.10) and using (A.32), we derive \( Kg = A(-Kg)'' \) or, equivalently,
\[ K = A\tilde{K}, \quad \tilde{K} := I - i\beta''(x) j \int_0^x \beta(t)^* \cdot dt. \] (A.33)

Since \( \tilde{K} \) has a semi-separable kernel, it is invertible and \( \tilde{K}^{-1} \in \mathcal{B}(L^{m_2}_2(0, r)) \) (see explicit expressions [32, (C.5)–(C.7)] for the inverse operator \( \tilde{K}^{-1} \)). Hence, using (A.33) we can rewrite \( V^{-1}K = AV^{-1} \) in the form
\[ V^{-1}A = AV^{-1}\tilde{K}^{-1}. \] (A.34)
From (1.6) and (A.13) one can see that
\[
(V(xI_{m_2}))(0) = 0, \quad (V(xI_{m_2}))(0) = I_{m_2}.
\] (A.35)
Hence, representation (A.10) of \(V(xI_{m_2})\) has the form
\[
V(xI_{m_2}) = xI_{m_2} - A(V(xI_{m_2}))''.
\] (A.36)
Multiplying both parts of (A.36) by \(V^{-1}\), we easily rewrite the result in the form
\[
V^{-1}(xI_{m_2}) = xI_{m_2} + V^{-1}A(V(xI_{m_2}))''.
\] (A.37)
In the same way, the equality (A.29) (in view of (1.6), (A.9) and (A.10)) yields
\[
V I_{m_2} = I_{m_2} + x(V I_{m_2})'(0) - A(V I_{m_2})'',
\] (A.38)
\[
V^{-1} I_{m_2} = I_{m_2} - V^{-1}(x(V I_{m_2})'(0)) + V^{-1}A(V I_{m_2})''.
\] (A.39)
Clearly, the right-hand side of (A.34) (and so the left-hand side as well) maps functions from \(L^{m_2}_2(0, r)\) into \(U^{m_2}_2[0, r]\). Hence, (A.37) implies that \(V^{-1}(xI_{m_2}) \in U^{m_2}_2[0, r]\). Therefore, (A.39) implies that the matrix function \(V^{-1} I_{m_2}\) also belongs to \(U^{m_2}_2[0, r]\). Summing up, we have
\[
V^{-1}Af'', \quad V^{-1}(xf'(0)), \quad V^{-1}f(0) \in U^{m_2}_2[0, r]
\] (A.40)
for any \(f \in U^{m_2}_2[0, r]\). Thus, using (A.10) we see that the statement of the proposition regarding operator \(V^{-1}\) is valid. ■

3. Assuming that the conditions of Proposition A.1 hold, we introduce operators \(E\) and \(V_0\) by the equalities
\[
E := VV_0, \quad V_0f = \beta_2(0)f + \int_0^x V_0(x - t)f(t)dt, \quad V_0 := (V^{-1}\beta_2)',
\] (A.41)
where \(\beta_2\) is the \(m_2 \times m_2\) block of \(\beta\) (see (3.16)) and \(E\) is another triangular operator
\[
Ef = u(x)\beta_2(0)f + \int_0^x E(x, t)f(t)dt.
\] (A.42)
Lemma A.2 The operator $V_0$ is invertible and commutes with $A$:

$$V_0A = AV_0.$$  \hfill (A.43)

Moreover, the operators $V_0$ and $V_0^{-1}$ map $\mathcal{U}^{m_2}[0,r]$ into $\mathcal{U}^{m_2}[0,r]$.

Proof. Taking into account (1.6), it is easy to see that $\det \beta_2(0) \neq 0$. Indeed, if $\det \beta_2(0) = 0$ we have $\beta_2(0)^*g = 0$ for some vector $g \neq 0$. We also have $\beta_2(x)\beta_2(x)^* = \beta_1(x)\beta_1(x)^*$ (which is immediate from $\beta j \beta^* \equiv 0$), and so $\beta_2(0)^*g = 0$ implies $\beta_1(0)^*g = 0$ and $\beta(0)^*g = 0$. Since $g \neq 0$, the last equality contradicts the equality $\beta(0)'j\beta(0)^* = iI_{m_2}$. The invertibility of $V_0$ follows from the invertibility of $\beta_2(0)$ and from the boundedness of $V_0(x)$ in the matrix norm.

The commutation property (A.43) is immediate from the commutation property of the integral term of $V_0$ and $A$. In fact, there is a more general commutation property for triangular convolution operators. Indeed, for any operator $A = \int_0^x a(x-t)I_{m_2} \cdot dt$ (where $a$ is a scalar function and is assumed, for convenience, to be bounded) we have

\[
\begin{align*}
\int_0^x V_0(x-t) \int_0^t a(t-s)f(s)dsdt &= \int_0^x \int_s^x a(t-s)V_0(x-t)dtf(s)ds, \\
\int_0^x a(x-t) \int_0^t V_0(t-s)f(s)dsdt &= \int_0^x \int_s^x a(x-t)V_0(t-s)dtf(s)ds.
\end{align*}
\]  \hfill (A.44) \hfill (A.45)

Using the corresponding change of variables, it is easy to show that the integral kernels on the right-hand sides of (A.44) and (A.45) coincide:

\[
\int_s^x a(t-s)V_0(x-t)dt = \int_0^{x-s} a(y)V_0(x-y-s)dy = \int_s^x a(x-t)V_0(t-s)dt.
\]  \hfill (A.46)

Thus, the left-hand sides of (A.44) and (A.45) coincide as well (and (A.43) is proved).

According to Proposition A.1 and the last equality in (A.41), $V_0(x)$ is differentiable and its derivative belongs to $L_2^{m_2 \times m_2}(0,r)$. Hence, the direct differentiation shows that $V_0$ maps $\mathcal{U}^{m_2}[0,r]$ into the same class.
It is easy to see that

\[(V_0(xI_{m_2}))(0) = 0, \quad (V_0(xI_{m_2}))'(0) = (V_0I_{m_2})(0) = \beta_2(0). \quad (A.47)\]

Finally, the fact that \(V_0^{-1}\) maps \(U^{m_2}[0, r]\) into the same class follows from (A.43) (more precisely from the equivalent equality \(V_0^{-1}A = AV_0^{-1}\)) and from (A.47) similar to the case of \(V^{-1}\) in the proof of Proposition A.1.  

**Remark A.3** The proof of Lemma A.2 shows that a wide class of operators commutes with \(A\), and so the transformation operators \(\hat{V}\) such that \(K = \hat{V}A\hat{V}^{-1}\) are not uniquely defined. In this context, the normalised transformation operator \(E\) given by (A.41) is important.

**Proposition A.4** The operator \(E\) given by (A.41) satisfies the equalities

\[K = EAE^{-1}, \quad E^{-1}\beta_2 \equiv I_{m_2}. \quad (A.48)\]

Moreover, the operators \(E\) and \(E^{-1}\) map \(U^{m_2}[0, r]\) into \(U^{m_2}[0, r]\).

**Proof.** The first equality in (A.48) is immediate from the definition of \(E\) and formulas (2.23) and (A.43). In order to derive the second equality in (A.48), we note that the representation of \(V\) in (2.23) and the inequality (2.24) yield that

\[(V^{-1}f)(x) = u(x)^*f(x) + \int_0^x Q(x, \xi)f(\xi)d\xi, \quad \sup_{0 \leq \xi \leq x \leq r} \|Q(x, \xi)\| < \infty. \quad (A.49)\]

In particular, in view of (A.9) we have

\[(V^{-1}\beta_2)(0) = \beta_2(0). \quad (A.50)\]

Taking into account (A.50) and the last two equalities in (A.41), we derive

\[(V_0I_{m_2})(x) = \beta_2(0) + \int_0^x V_0(\xi)d\xi = (V^{-1}\beta_2)(x), \quad (A.51)\]

and the second equality in (A.48) follows.

Finally, the fact that the operators \(E\) and \(E^{-1}\) map \(U^{m_2}[0, r]\) into \(U^{m_2}[0, r]\) follows from the corresponding properties of \(V^{\pm 1}\) and \(V_0^{\pm 1}\) (see Proposition A.1 and Lemma A.2).  

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Remark A.5  It follows from the Remark 2.6 and formulas (A.41) and (A.42) that the integral kernel \( E(x,t) \) (of \( E \)) in the domain \( 0 \leq t \leq x \leq \ell < r \) is uniquely determined by \( \beta(x) \) on \([0, \ell]\) (and does not depend on the choice of \( \beta(x) \) for \( \ell < x < r \) and the choice of \( r \geq \ell \)).

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