Abstract. In this article, we define Weyl transform on second countable type I locally compact group $G$, and as an operator on $L^2(G)$, we prove that the Weyl transform is compact when the symbol lies in $L^p(G \times \hat{G})$ with $1 \leq p \leq 2$. Further, for the Euclidean motion group and Heisenberg motion group, we prove that the Weyl transform can not be extended as a bounded operator for the symbol belongs to $L^p(G \times \hat{G})$ with $2 < p < \infty$. To carry out this, we construct positive, square integrable and compactly supported function, on the respective groups, such that $L^{p'}$ norm of its Fourier transform is infinite, where $p'$ is the conjugate index of $p$.

1. Introduction

In [15], Hermann Weyl studied the quantization problem in quantum mechanics and introduced a type of pseudo-differential operators. These operators are useful in physics and mathematics, especially in PDE, harmonic analysis, time frequency analysis. In [16], Wong called these operators as Weyl transform. Further in [16], for the symbol in $L^p(\mathbb{R}^{2n})$ with $1 \leq p \leq 2$, compactness of the Weyl transform, as an operator on $L^2(\mathbb{R}^n)$, is studied. Moreover in [12], Simon proved that the operator is not even bounded when the symbol is in $L^p(\mathbb{R}^{2n})$ with $2 < p < \infty$.

In general, for a locally compact group, Fourier transform is an operator valued function. For the Heisenberg group, in [5], Weyl transform is defined for operator valued symbol and proved its boundedness (even compactness) when the symbol is in the corresponding $L^p$ spaces with $1 \leq p \leq 2$, while unboundedness for $2 < p < \infty$. Further, in [3] Weyl transform on the upper half plane, and in [3] Weyl transform on the quaternion Heisenberg group, are studied.

In this article, we consider second countable type I locally compact group $G$ and define Weyl transform in view of its inversion formula. We prove that the Weyl transform, as an operator on $L^2(G)$, is compact, when the symbol is in $L^p(G \times \hat{G})$ with $1 \leq p \leq 2$, where $\hat{G}$ is the dual of $G$. Further, to prove that the Weyl transform can not be extended as a bounded operator for the symbol is in $L^p(G \times \hat{G})$ with $2 < p < \infty$, it is enough to construct a positive, square integrable and compactly supported function on $G$ such that $L^{p'}$ norm of its Fourier transform is infinite, where $\frac{1}{p} + \frac{1}{p'} = 1$. In this perspective, we construct...
such type of functions for the Euclidean motion group and Heisenberg motion group.

In addition, we want to mention that these examples will not follow directly from the Euclidean space and Heisenberg group, respectively, for the following reasons. The Fourier transform on the Euclidean motion group is operator valued, whereas it is just a function for the Euclidean space. Secondly, the case becomes more difficult for the Heisenberg motion group due to the presence of metaplectic representation whose implicitness may not be visible at the first instance.

2. Weyl transform on locally compact groups

In this section, we recall some harmonic analysis results, namely Fourier inversion, Plancherel formula, and Hausdorff-Young inequality on certain locally compact groups. Then in terms of Wigner transform, we define Weyl transform. After that, we prove the compactness of the Weyl transform for the symbol in corresponding $L^p$ spaces with $1 \leq p \leq 2$. This section concludes with a sufficient condition for the unboundedness of the Weyl transform for $2 < p < \infty$.

Let $G$ be a second countable locally compact group with type I left regular representation, and $A(G)$, $\hat{G}$ denote the Fourier algebra and dual of $G$, respectively. Then there is a standard measure $\mu$ on $\hat{G}$, called the Plancherel measure, a $\mu$-measurable field $(\pi, \mathcal{H}_\pi)$ of representation and a measurable field $K = (K_\pi)$ of nonzero positive self-adjoint operators such that $K_\pi$ is semi-invariant with weight $\triangle^{-1}$, for almost all $\pi \in \hat{G}$, where $\triangle$ be the modular function of $G$ and $A(G)$. If $G$ is unimodular, then $K_\pi = I_{\mathcal{H}_\pi}$, the identity operator on $\mathcal{H}_\pi$. For $f \in L^1(G) \cap A(G)$, the group Fourier transform

$$\hat{\pi}(f) = \int_G f(x)\pi(x)d\nu(x)$$

is a bounded operator on $\mathcal{H}_\pi$, where $d\nu$ is a left Haar measure on $G$. Further, $f$ can be recovered by the inversion formula.

**Theorem 2.1.** ([7]) (Inversion theorem) Let $f \in L^1(G) \cap A(G)$. Then

$$f(x) = \int_G \text{tr}(\pi(x)^{-1}\pi(f)K_\pi)d\mu(\pi).$$

Next, we discuss the Plancherel formula for the Fourier transform, and for this, we start with Schatten class operators. Let $S_p$ be the space of all $p$-Schatten class operators, which is a Banach space with the norm $\|T\|_{S_p} = \text{tr}(T^*T)^{p/2}$ for $1 \leq p < \infty$ and for $p = \infty$, the norm is the usual operator norm.

**Theorem 2.2.** ([4]) (Plancherel formula) Let $f \in L^1 \cap L^2(G)$. Then

$$\int_G |f(x)|^2d\nu(x) = \int_G \|\pi(f)K_\pi^{1/2}\|_{S_2}^2 d\mu(\pi).$$

This map $L^1 \cap L^2(G) \to L^2(\hat{G})$ can be extended uniquely to a unitary map from $L^2(G)$ onto $L^2(\hat{G})$. 
Let \( f \in L^1 \cap L^p(G) \), where \( 1 \leq p \leq \infty \). Define the \( L^p \) Fourier transform of \( f \) by \( \mathcal{F}_p(f)\pi = \pi(f)K_\pi^{1/p'} \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then the following Hausdorff-Young inequality holds.

**Theorem 2.3.** If \( f \in L^1 \cap L^p(G) \), where \( 1 < p < 2 \), then \( \mathcal{F}_p(f) \in L^{p'}(\hat{G}) \) and this map \( f \mapsto \mathcal{F}_p(f) \) extends uniquely to a bounded linear map from \( L^p(G) \) into \( L^{p'}(\hat{G}) \) with norm less than or equal to 1.

To study the Weyl transform, we need to define Wigner transform on \( G \) and investigate its boundedness properties. For this, we first describe the following product spaces and the left translation operator.

Consider the measure \( d\nu \otimes d\mu \) on \( G \times \hat{G} \) and for \( 1 \leq p \leq \infty \), let \( L^p(G \times \hat{G}, S_p, d\nu \otimes d\mu) \) be the space of \( S_p \) valued functions satisfying

\[
\|f\|_{p,\nu\otimes\mu}^p = \int_{G \times \hat{G}} \|f(x,\pi)K_\pi^{1/p}\|_{S_p}^p \, d\nu \, d\mu(x) < \infty, \quad 1 \leq p < \infty,
\]

\[
\|f\|_{\infty,\nu\otimes\mu} = \text{ess sup}_{(x,\pi) \in G \times \hat{G}} \|f(x,\pi)\|_{S_\infty} < \infty.
\]

The left translation operator is defined by \( \tau_x f(x) = f(x^{-1}x) \) for \( f \in L^p(G) \). Further, \( C_c(G) \) denotes the space of all compactly supported continuous functions on \( G \). Throughout this section, \( p, p' \) are the conjugate indices.

**Definition 2.4.** Let \( f, g \in C_c(G) \) and \( (x, \pi) \in G \times \hat{G} \). Then Wigner transform associated with \( f, g \) is defined by

\[
V(f, g)(x, \pi) = \int_G f(x') \tau_{x'} g(x) \pi(x') d\nu(x').
\]

That is, \( V(f, g)(x, \pi) = \pi(f \cdot \tau g(x)) \).

The following result proves that the Wigner transforms are in certain \( L^p \) spaces.

**Proposition 2.5.** Let \( f, g \in C_c(G) \). Then \( V(f, g) \in L^{p'}(G \times \hat{G}, S_{p'}, d\nu \otimes d\mu) \) and

\[
\|V(f, g)\|_{p',\nu\otimes\mu} \leq \|f\|_2 \|g\|_2
\]

for \( p' \in [2, \infty] \). Thus \( V : C_c(G) \times C_c(G) \to L^{p'}(G \times \hat{G}, S_{p'}, d\nu \otimes d\mu) \) can be extended uniquely to a bilinear operator \( V : L^2(G) \times L^2(G) \to L^{p'}(G \times \hat{G}, S_{p'}, d\nu \otimes d\mu) \) with

\[
\|V(f, g)\|_{p',\nu\otimes\mu} \leq \|f\|_2 \|g\|_2.
\]

**Proof.** Let \( p' = \infty \). Then

\[
\|V(f, g)\|_{\infty,\nu\otimes\mu} = \text{ess sup}_{(x,\pi) \in G \times \hat{G}} \|V(f, g)(x, \pi)\|_{S_\infty}
\]

\[
= \text{ess sup}_{(x,\pi) \in G \times \hat{G}} \int_G \|f(x') \tau_{x'} g(x) \pi(x') d\nu(x')\|_{S_\infty}
\]

\[
\leq \text{ess sup}_{x \in G} \int_G |f(x') \tau_{x'} g(x)| d\nu(x') \leq \|f\|_2 \|g\|_2.
\]
For $p' = 2$, Plancherel formula, Theorem 2.2 gives
\[ \|V(f, g)\|_{2,\nu\otimes\mu}^2 = \int_G \int_G \|\pi(f\cdot\tau g(x)) K_{\pi}^{1/2} \|_{L_2}^2 d\mu(\pi) d\nu(x) \]
\[ = \int_G \int_G |f(x')\tau_x g(x)|^2 d\nu(x') d\nu(x) = \|f\|_2^2 \|g\|_2^2. \]
Then Riesz-Thorin interpolation theorem completes the proof. □

Proposition 2.6. Let $f, g \in L^1 \cap L^2(G)$ and $C = \int_G g(x) d\nu(x) \neq 0$. Then
\[ \pi(f) = C^{-1} \int_G V(f, g)(x, \pi) d\nu(x) \text{ for } \pi \in \hat{G}. \]
Proof.
\[ \int_G V(f, g)(x, \pi) d\nu(x) = \int_G \int_G f(x') g(x'^{-1} x) \pi(x') d\nu(x') d\nu(x') \]
\[ = \left( \int_G g(x) d\nu(x) \right) \left( \int_G f(x') \pi(x') d\nu(x') \right) = C \pi(f). \]
□

In view of Proposition 2.6, the Inversion formula 2.1 can be reformulated.

Corollary 2.7. Let $f \in L^1(G) \cap A(G)$ and $g \in L^1 \cap L^2(G)$ with $C = \int_G g(x) d\nu(x) \neq 0$. Then
\[ f(x) = C^{-1} \int_G \text{tr} \left( (\pi(x)^{-1} \left( \int_G V(f, g)(x', \pi) d\nu(x') \right) K_{\pi} \right) d\mu(\pi). \]

Definition 2.8. Let $\varsigma \in L^p(G \times \hat{G}, S_p, d\nu \otimes d\mu)$, where $1 \leq p \leq 2$. Then corresponding to $\varsigma$, Weyl transform $W_\varsigma : L^2(G) \to L^2(G)$ is defined by
\[ \langle W_\varsigma f, \hat{g} \rangle = \langle V(f, g), \varsigma \rangle_{\nu\otimes\mu} = \int_G \int_G \text{tr} \left( \varsigma^*(x, \pi) V(f, g)(x, \pi) K_{\pi} \right) d\mu(\pi) d\nu(x), \]
where $f, g \in L^2(G)$. Here, by the abuse of notation $\langle \cdot, \cdot \rangle_{\nu\otimes\mu}$ is used.

After a bit of calculation, we can conclude that
\[ (2.2) \quad W_\varsigma f(x) = \int_G \int_G \text{tr} \left( \varsigma^*(x'x, \pi) \pi(x') K_{\pi} \right) f(x') d\nu(x') d\mu(\pi). \]
Thus $W_\varsigma : L^2(G) \to L^2(G)$ is an integral operator with kernel
\[ K(x, x') = \int_G \text{tr} \left( \varsigma^*(x'x, \pi) \pi(x') K_{\pi} \right) d\mu(\pi). \]

In the following proposition, we investigate the boundedness of the Weyl transform.

Proposition 2.9. Let $\varsigma \in L^p(G \times \hat{G}, S_p, d\nu \otimes d\mu)$, where $1 \leq p \leq 2$. Then
\[ \|W_\varsigma\| \leq \|\varsigma\|_{p,\nu\otimes\mu}. \]
Proof. Since \(|\text{tr}(A^*B)| \leq \|A\|_{S_p}\|B\|_{S_{p'}}\), for \(f, g \in L^2(G)\) we have

\[
|\langle W_\zeta f, g \rangle| \leq \int_G \int_G |\text{tr}(\zeta^*(x, \pi)V(f, g)(x, \pi)K_\pi)|d\mu(\pi)d\nu(x)
\leq \int_G \int_G \|\zeta(x, \pi)(K^{1/p}_\pi)^*\|_{S_p}\|V(f, g)(x, \pi)K^{1/p'}_\pi\|_{S_{p'}}d\mu(\pi)d\nu(x).
\]

By Hölder’s inequality, the above integral is less than or equal to

\[
\left( \int_G \int_G \|\zeta(x, \pi)K^{1/p}_\pi\|_{S_p}^p d\mu(\pi)d\nu(x) \right)^{\frac{1}{p}} \left( \int_G \int_G \|V(f, g)(x, \pi)K^{1/p'}_\pi\|_{S_{p'}}^p d\mu(\pi)d\nu(x) \right)^{\frac{1}{p'}}
= \|\zeta\|_{\mathcal{L}^{p,\nu\otimes\mu}} \|V(f, g)\|_{\mathcal{L}^{p',\nu\otimes\mu}} \leq \|\zeta\|_{\mathcal{L}^{p,\nu\otimes\mu}} \|f\|_2 \|g\|_2.
\]

Further for \(p = 2\), \(W_\zeta\) is a Hilbert-Schmidt operator, and for \(p = 1\), \(W_\zeta\) is a trace class operator.

**Proposition 2.10.** If \(\zeta \in L^2(G \times \hat{G}, S_2, d\nu \otimes d\mu)\), then \(W_\zeta\) is a Hilbert-Schmidt operator.

**Proof.** Since \(W_\zeta\) is an integral operator, we have

\[
\|W_\zeta\|_{S_2}^2 = \int_G \int_G \int_G \int_G |K(x, x')|^2 d\nu(x)d\nu(x')
\]

\[
= \int_G \int_G \int_G \int_G \text{tr}(\zeta^*(x', \pi)\pi(x')K_\pi)d\mu(\pi)|^2 d\nu(x)d\nu(x')
\]

\[
= \int_G \int_G \int_G \int_G \text{tr}(\zeta^*(x, \pi)\pi(x)K_\pi)d\mu(\pi)|^2 d\nu(x)d\nu(x'),
\]

where last equality is ensured by applying the change of variable \(x = x'x\), as \(d\nu\) is a left invariant Haar measure. Since \(\text{tr}(T^*) = \text{tr}(T)\) and \(K_\pi^* = K_\pi\), the above integral becomes

\[
\int_G \int_G \left| \int_G \text{tr}(\zeta(x, \pi)K_\pi^{-1})(x, \pi)\right|^2 d\nu(x)d\nu(x').
\]

Applying the Fourier inversion and Plancherel formula, we get

\[
\|W_\zeta\|_{S_2}^2 = \int_G \int_G |\mathcal{F}^{-1}(\zeta)(x', \cdot)(x')|^2 d\nu(x')d\nu(x)
\]

\[
= \int_G \int_G \|\zeta(x, \pi)K^{1/2}_\pi\|_{S_2}^2 d\mu(\pi)d\nu(x) = \|\zeta\|_{\mathcal{L}^{2,\nu\otimes\mu}}^2.
\]

**Proposition 2.11.** If \(\zeta \in L^1(G \times \hat{G}, S_1, d\nu \otimes d\mu)\), then \(W_\zeta\) is a trace class operator.
Proof. Since $W_\zeta$ is an integral operator
\[
\|W_\zeta\|_{S_1} = \int_G |K(x, x)|d\nu(x)
\]
\[
= \int_G \left| \int_G \text{tr} \left( \zeta^*(xx, \pi)xK_\pi \right) d\mu(x) \right| d\nu(x)
\]
\[
\leq \int_G \int_G |\text{tr} \left( \pi(x)K_\pi \zeta^*(xx, \pi) \right)| d\mu(x) d\nu(x)
\]
\[
\leq \int_G \int_G \text{tr} \left( |\zeta^*(xx, \pi)K_\pi| \right) d\mu(x) d\nu(x) = \|\zeta\|_{1, \nu \otimes \mu},
\]
where the second last inequality true because of $|\text{tr}(T)| \leq \text{tr}(T^*T)^{1/2} = \text{tr}(|T|)$. \hfill \Box

Theorem 2.12. If $\zeta \in L^p(G \times \hat{G}, S_p, d\nu \otimes d\mu)$, where $1 \leq p \leq 2$, then $W_\zeta$ is a compact operator.

Proof. Let $\zeta \in L^p(G \times \hat{G}, S_p, d\nu \otimes d\mu)$ and $(\zeta_n)$ be a sequence in $C_c(G \times \hat{G})$ such that $\|\zeta_n - \zeta\|_{p, \nu \otimes \mu} \to 0$. From Proposition 2.4, it follows that $W_{\zeta_n}$ converges to $W_\zeta$ in operator norm. But in view of Proposition 2.11, $W_{\zeta_n}$ is a compact operator for each $n$. Hence $W_\zeta$ is a compact operator. \hfill \Box

Now, we shall see whether the Weyl transform defined in (2.2) can be extended as a bounded operator for $p > 2$. Next result gives a necessary and sufficient condition for boundedness of the Weyl transform in terms of the Wigner transform.

Proposition 2.13. Let $2 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then for all $\zeta \in L^p(G \times \hat{G}, S_p, d\nu \otimes d\mu)$, the Weyl transform $W_\zeta$ is a bounded operator on $L^2(G)$ if and only if there exists a constant $C$ such that $\|V(f, g)\|_{p', \nu \otimes \mu} \leq C\|f\|_2\|g\|_2$ for all $f, g \in L^2(G)$.

Proof. A similar argument as in Proposition 2.9 proves that $W_\zeta$ is bounded for all $\zeta \in L^p(G \times \hat{G}, S_p, d\nu \otimes d\mu)$, whenever there exists a constant $C$ such that $\|V(f, g)\|_{p', \nu \otimes \mu} \leq C\|f\|_2\|g\|_2$ for all $f, g \in L^2(G)$.

Conversely, assume that $W_\zeta$ is bounded for each $\zeta \in L^p(G \times \hat{G}, S_p, d\nu \otimes d\mu)$. Then there exists a constant $C_\zeta$ such that $\|W_\zeta f\|_2 \leq C_\zeta\|f\|_2$ for all $f \in L^2(G)$. For $f, g \in C_c(G)$ with $\|f\|_2 = \|g\|_2 = 1$, define the bounded linear functional on $L^p(G \times \hat{G}, S_p, d\nu \otimes d\mu)$ by $Q_{f, g}(\zeta) = \langle W_\zeta f, g \rangle$. Then $\sup |Q_{f, g}(\zeta)| \leq C_\zeta$, where the supremum is over all $f, g \in C_c(G)$ with $\|f\|_2 = \|g\|_2 = 1$. Therefore, by the uniform boundedness principle, there exists a constant $C$ such that $\|Q_{f, g}\| \leq C$ for all $f, g \in C_c(G)$ with $\|f\|_2 = \|g\|_2 = 1$. Hence $\|W_\zeta f, g\| \leq C\|\zeta\|_{p, \nu \otimes \mu}\|f\|_2\|g\|_2$, that is, $\|W_\zeta\| \leq C\|\zeta\|_{p, \nu \otimes \mu}$. Thus for $f, g \in C_c(G)$,
\[
\|V(f, g)\|_{p', \nu \otimes \mu} = \sup_{\|\zeta\|_{p, \nu \otimes \mu}=1} |\langle V(f, g), \zeta \rangle_{\nu \otimes \mu}| = \sup_{\|\zeta\|_{p, \nu \otimes \mu}=1} |\langle W_\zeta f, g \rangle| \leq C\|f\|_2\|g\|_2.
\]
The density argument completes the proof. \hfill \Box

A sufficient condition for unboundedness of the Weyl transform is obtained in the following result.
Proposition 2.14. Let \( 2 < p < \infty \) and \( f \) be a square integrable, compactly supported function on \( G \) with \( \int_G f(x) d\nu(x) \neq 0 \). If \( W_\zeta \) is a bounded operator on \( L^2(G) \) for all \( \zeta \in L^p(G \times \hat{G}, d\nu \otimes d\mu) \), then \( \int_G \| \mathcal{F}_p(f) \pi \|_{S^p_{\rho'}}^p d\mu(\pi) < \infty \).

Proof. Let \( f \) be supported on a compact set \( K \). Then \( \hat{K} = KK = \{ xy : x, y \in K \} \) is compact and \( V(f, f) \) is supported on \( \hat{K} \times \hat{G} \). In view of Proposition 2.6 instead of \( \int_G \| \mathcal{F}_p(f) \pi \|_{S^p_{\rho'}}^p d\mu(\pi) \), it is enough to consider the following integral

\[
\int_G \left\| \int_G V(f, f)(x, \pi) K^{1/p'}_\pi d\nu(x) \right\|_{S^p_{\rho'}}^p d\mu(\pi).
\]

Applying Minkowski’s integral inequality and Hölder’s inequality, the above integral is less than or equal to

\[
\left( \int_K \left( \int_G \| V(f, f)(x, \pi) K^{1/p'}_\pi \|_{S^p_{\rho'}} d\mu(\pi) \right)^{p'} d\nu(x) \right)^{1/p'} \leq \left( \int_K \int_G \| V(f, f)(x, \pi) K^{1/p'}_\pi \|_{S^p_{\rho'}} d\mu(\pi) d\nu(x) \right)^{1/p}. 
\]

Hence by Proposition 2.13 it completes the proof. \( \square \)

To prove that the Weyl transform \( W_\zeta \) can not be extended as a bounded operator for \( \zeta \in L^p(G \times \hat{G}, d\nu \otimes d\mu) \), where \( 2 < p < \infty \), it is enough to consider the following problem.

Question: For \( p \in (2, \infty) \), does there exists a square integrable, compactly supported function \( f \) on \( G \) with \( \int_G f(x) d\nu(x) \neq 0 \), such that \( \int_G \| \mathcal{F}_p(f) \pi \|_{S^p_{\rho'}} d\mu(\pi) = \infty \)?

In \([12]\), Simon gave the example of such functions for \( \mathbb{R}^n \), and later on, it is considered for the Heisenberg group \([8]\), quaternion Heisenberg group \([3]\) and upper half plane \([9]\). In this article, we define such functions for the Euclidean motion group and Heisenberg motion group.

3. Euclidean motion group

In this section, we briefly discuss the Fourier analysis on the Euclidean motion group. Thereafter, we shall precisely write down the formula for the Weyl transform in this setup and prove that it can not be extended as a bounded operator for the symbol in the corresponding \( L^p \) spaces with \( 2 < p < \infty \).

Let \( n \in \mathbb{N} \setminus \{1\} \) and \( SO(n) \) be the special orthogonal group of order \( n \). Then Euclidean motion group \( M(n) \) is the semidirect product of \( \mathbb{R}^n \) with \( K = SO(n) \). The group law on \( M(n) \) can be expressed as

\[
(x_1, k_1)(x_2, k_2) = (x_1 + k_1 \cdot x_2, k_1k_2),
\]

where \( x_1, x_2 \in \mathbb{R}^n \) and \( k_1, k_2 \in K \). The Haar measure on \( M(n) \) can be written as \( d\nu(x, k) = dxdk \), where \( dx \) and \( dk \) are the Haar measures on \( \mathbb{R}^n \) and \( K \) respectively. The Fourier analysis on \( M(n) \) can be discussed in the following two cases. For details, see \([6, 10, 13]\).
For \( n = 2 \): An arbitrary element of \( M(2) \) can be written as \((z, e^{i\varphi})\), where \( z \in \mathbb{C}, \varphi \in \mathbb{R} \). All the, upto unitarily equivalent, infinite dimensional irreducible unitary representations of \( M(2) \) are parametrized by \( a > 0 \). Moreover, for each \( a > 0 \), the representation \( \pi_a \), realized on \( L^2(S^1) \), is given by

\[
\pi_a(z, e^{i\varphi})g(\theta) = e^{i\text{Re}(ae^{i\theta}z)}g(\theta - \varphi),
\]

where \( g \in L^2(S^1) \). The Plancherel measure on \((0, \infty)\) is \( d\mu(a) = ada \), where \( da \) is the Lebesgue measures on \((0, \infty)\). Further, for \( f \in L^1(M(2)) \) the group Fourier transform, defined by

\[
\hat{f}(a) = \int_{M(2)} f(z, e^{i\varphi})\pi_a(z, e^{i\varphi})dzd\varphi,
\]

is a bounded operator on \( L^2(S^1) \). For \( \varsigma \in L^p(M(2) \times (0, \infty), S_p, d\nu \odot d\mu) \) and \( f \in L^2(M(2)) \), the Weyl transform \( W_\varsigma \) takes the form

\[
W_\varsigma f(z, e^{i\varphi}) = \int_0^\infty \int_{M(2)} \text{tr} \left( \varsigma^* (w, e^{i\psi}) (z, e^{i\varphi}) \right) \pi_a(w, e^{i\psi}) f(w, e^{i\psi}) dwd\psi da.
\]

For \( n \geq 3 \): Let \( M = SO(n-1) \) be the subgroup of \( K \) that fixes the point \( e_1 = (1, 0, \ldots, 0) \). Let \( \hat{M} \) be the unitary dual group of \( M \). Given a unitary irreducible representation \( \sigma \in \hat{M} \), realized on the Hilbert space \( \mathcal{H}_\sigma \) of dimension \( d_\sigma \), consider the space \( L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma}) \) consisting of \( d_\sigma \times d_\sigma \) complex matrices valued functions \( \varphi \) on \( K \) such that \( \varphi(uk) = \sigma(u)\varphi(k) \), where \( u \in M, k \in K \), and satisfying

\[
\int_K \text{tr} (\varphi(k)^*\varphi(k))dk < \infty.
\]

For \((a, \sigma) \in (0, \infty) \times \hat{M} \), a unitary representation \( \pi_{a,\sigma} \) of \( M(n) \) is defined by

\[
\pi_{a,\sigma}(x,k)(\varphi)(s) = e^{ia(s^{-1}e_1,x)}\varphi(sk),
\]

where \( \varphi \in L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma}) \). These are all, up to unitary equivalent, infinite dimensional, unitary, irreducible representations that appear in the Plancherel formula. The group Fourier transform of a function \( f \in L^1(M(n)) \) is defined by

\[
\hat{f}(a, \sigma) = \int_{M(n)} f(x,k)\pi_{a,\sigma}(x,k)dxdk.
\]

Further, for \( f \in L^1 \cap L^2(M(n)) \), Plancherel formula holds

\[
\int_{M(n)} |f(x,k)|^2dxdk = c_n \int_0^\infty \left( \sum_{\sigma \in \hat{M}} d_\sigma \|\hat{f}(a, \sigma)\|_{HS}^2 \right) a^{n-1}da,
\]

for some constant \( c_n \).

Next, we construct the required square integrable and compactly supported functions for \( M(n) \). To do so, we need to discuss some properties of Bessel functions.
For \( v \in \mathbb{R} \), first kind Bessel function \( J_v \) of order \( v \) is defined by

\[
J_v(t) = \sum_{l=0}^{\infty} \frac{(-1)^l (\frac{x}{2})^{vl+2l}}{l! \Gamma(l + v + 1)},
\]

where \( t \) is a non-negative real number. Then for \( n \geq 2 \), \( a \in (0, \infty) \) and \( x \in \mathbb{R}^n \), the following relation holds

\[
\int_{S^{n-1}} e^{i\alpha \omega} d\omega = c_n (a|x|)^{1-\frac{\alpha}{2}} J_{\frac{\alpha}{2}-1}(a|x|),
\]

where \( c_n > 0 \) is a constant depending only on \( n \) and \( d\omega \) is the surface measure on \( S^{n-1} \).

If \( \xi, \zeta \) and \( \eta \) are complex parameters with \( \eta \neq 0, -1, \ldots \), then the complex power series

\[
\sum_{l=0}^{\infty} \frac{(\xi)_l (\zeta)_l}{(\eta)_l} \frac{z^l}{l!},
\]

converges for \( |z| < 1 \), where \( (\xi)_0 = 1 \) and \( (\xi)_l = \xi(\xi+1) \cdots (\xi+l-1) \) for \( l \geq 1 \). The sum of the above series is denoted by \( \Gamma F_1(\xi, \zeta; \eta; z) \) and called it hypergeometric function. Moreover, if \( \text{Re}(\eta - \xi - \zeta) > 0 \), then the series \( \sum_{l=0}^{\infty} \frac{(\xi)_l (\zeta)_l}{(\eta)_l} \frac{z^l}{l!} \) also converges for \( |z| = 1 \) and

\[
\Gamma F_1(\xi, \zeta; \eta; 1) = \frac{\Gamma(\eta) \Gamma(\eta - \xi - \zeta)}{\Gamma(\eta - \xi) \Gamma(\eta - \zeta)}.
\]

For more details, see [1], page no. 62-66.

The following property of Bessel functions hold, see [14], page no. 385.

**Lemma 3.1.** [14] Let \( u, \alpha, v \in \mathbb{R} \) be such that \( u, \alpha > 0 \) and \( v \geq 0 \). Then

\[
\int_0^\infty e^{-ut} t^{\alpha-1} J_v(t) dt = \frac{(\frac{1}{2})^\alpha \Gamma(\alpha + v)}{(1 + u^2)^{\alpha/2} \Gamma(v + 1)} 2 \Gamma F_1 \left( \frac{\alpha + v}{2}, \frac{1 - \alpha + v}{2}; 1 + v; \frac{1}{1 + u^2} \right),
\]

where \( 2 \Gamma F_1 \) is the hypergeometric function as defined by the series \( \sum_{l=0}^{\infty} \frac{(\zeta)_l (\zeta)_l}{(\eta)_l} \frac{z^l}{l!} \).

Next lemma is needful to prove the main result in this section.

**Lemma 3.2.** Let \( n \in \mathbb{N} \setminus \{1\} \) and \( \beta \in (\frac{\alpha}{2}, \frac{\alpha}{2}) \). Then there exists \( C > 0 \) and \( a_0 > 0 \), both independent of \( \beta \), such that

\[
\int_0^a t^{n-\beta} J_{\frac{\alpha}{2}-1}(t) dt \geq C
\]

for all \( a \geq a_0 \).

**Proof.** Let \( \frac{\alpha}{3} < \beta < \frac{\alpha}{2} \) and \( \alpha = \frac{\alpha}{2} - \beta + 1 \), then \( 1 < \alpha < \frac{\alpha}{6} + 1 \). Again for \( 0 < u < 1 \), we have \( \frac{1}{2} < \frac{1}{1+u^2} < 1 \). Now, we shall find an independent of \( u \) and positive lower bound of the integral define in Lemma 3.1 when \( \alpha = \frac{\alpha}{3} - \beta + 1 \) and \( v = \frac{\alpha}{2} - 1 \). For that, consider the following two cases.

**For** \( n = 2 \): Since \( 1 < \alpha < \frac{\alpha}{3} \), except for the first term, all other terms in the series expansion of \( 2 \Gamma F_1 \left( \frac{1}{2}, \frac{1-\alpha}{2}; 1; \frac{1}{1+u^2} \right) \) are negative. Further, as \( 1 - \frac{\alpha}{2} - \frac{1-\alpha}{2} > 0 \),
From Lemma 3.1, using (3.4 - 3.5), we have
\[ 2F_1 \left( \frac{\alpha}{2}, 1 - \frac{\alpha}{2}; 1; \frac{1}{1 + u^2} \right) = 1 + \sum_{l=1}^{\infty} \frac{(\frac{\alpha}{2})_l (1 - \frac{\alpha}{2})_l}{(1)_l l!} \frac{1}{(1 + u^2)^l} \]
(3.5)

From Lemma 3.1 using (3.4 - 3.5), we have
\[
\int_0^\infty e^{-ut} t^{1-\beta} J_0(t) dt = \int_0^\infty e^{-ut} t^{\alpha-1} J_0(t) dt \\
= \frac{\Gamma(\alpha)}{(1 + u^2)^{\alpha/2}} 2F_1 \left( \frac{\alpha}{2}, 1 - \frac{\alpha}{2}; 1; \frac{1}{1 + u^2} \right) \\
\geq \frac{\Gamma(2)}{2} \frac{\Gamma(\frac{\beta}{2})}{\Gamma\left(1 - \frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha + 1}{2}\right)} > 0, 
\]
(3.6)
where the last inequality true as \( 1 \leq \frac{1}{1 + u^2} < 2 \) for \( 1 < \alpha < \frac{4}{3} \).

For \( n \geq 3 \): Since \( \frac{\alpha}{2} < \beta < \frac{\alpha}{2} \), all the terms in the series expansion of \( 2F_1 \left( \frac{n-\beta}{2}, \frac{\beta-1}{2}; \frac{n}{2}; \frac{1}{1+u^2} \right) \) are positive. Hence \( 2F_1 \left( \frac{n-\beta}{2}, \frac{\beta-1}{2}; \frac{n}{2}; \frac{1}{1+u^2} \right) \geq 1 \). Thus from Lemma 3.1, we have
\[
\int_0^\infty e^{-ut} t^{\frac{n}{2}-\beta} J_{\frac{n}{2}-1}(t) dt = \frac{(\frac{\alpha}{2})_l \Gamma(n - \beta)}{(1 + u^2)^{\alpha/2} \Gamma\left(\frac{n}{2}\right)} 2F_1 \left( \frac{n - \beta}{2}, \frac{\beta - 1}{2}; \frac{n}{2}; \frac{1}{1 + u^2} \right) \\
\geq \frac{(\frac{\alpha}{2})_l \Gamma(n - \beta)}{2^\frac{n-\beta}{2} \Gamma\left(\frac{n}{2}\right)} > 0. 
\]
(3.7)

Consider arbitrary \( n \geq 2 \). Since for \( x > 0 \), \( \Gamma(x) \) is continuous, the lower bounds in (3.6) and (3.7) are independent of \( \alpha, \beta \). Hence for \( 0 < u < 1 \) and \( \frac{n}{3} < \beta < \frac{n}{2} \), there exists \( a_0, C > 0 \), independent of \( u, \beta \), such that for all \( a \geq a_0 \),
\[
\int_0^a e^{-ut} t^{\frac{n}{2}-\beta} J_{\frac{n}{2}-1}(t) dt \geq C. 
\]
Therefore, \( \lim_{u \to 0^+} \int_0^a e^{-ut} t^{\frac{n}{2}-\beta} J_{\frac{n}{2}-1}(t) dt \geq C \) for all \( a \geq a_0 \). Thus
\[
\int_0^a t^{\frac{n}{2}-\beta} J_{\frac{n}{2}-1}(t) dt \geq C 
\]
for all \( a \geq a_0 \) and \( \frac{n}{3} < \beta < \frac{n}{2} \). \( \square \)

Now we prove the main results of the section.

**Theorem 3.3.** Consider the square integrable and compactly supported function \( f_\beta(z, e^{i\varphi}) = \chi_{B(0)}(z) \frac{1}{|z|^q} \), where \( \beta < 1 \), on \( M(2) \). Then for \( 1 < q < 2 \), there exist \( \beta \in (\frac{2}{q}, 1) \) such that \( \int_0^\infty \| f_\beta(a) \|_{S_q}^q ada = \infty. \)
Proof. Let \( e_0(\theta) = 1 \), then \( e_0 \in L^2(S^1) \). For \( a > 0 \),
\[
\langle \hat{f}_\beta(a)e_0, e_0 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}_\beta(a)e_0(\theta)e_0(\theta)d\theta
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1}{|z|^2} e^{\text{Re}(az\theta)} e_0(\theta - \varphi)dzd\varphi d\theta
= \int_{-\pi}^{\pi} \int_{0}^{1} \frac{1}{r^\beta} \left( \int_{-\pi}^{\pi} e^{iar\text{Re}(e^{i\varphi\psi})} d\theta \right) rdrd\psi.
\]
In view of (3.2), we can write
\[
(3.8) \quad \langle \hat{f}_\beta(a)e_0, e_0 \rangle = 2\pi c_2 \int_{0}^{1} \frac{1}{r^\beta-1} J_0(ar)dr = 2\pi c_2 a^{\beta-2} \int_{0}^{a} \frac{1}{r^\beta-1} J_0(r)dr.
\]
Hence by Lemma 3.2, if \( \beta \in (\frac{3}{2}, 1) \), then there exists \( C > 0 \) and \( a_0 > 0 \) such that
\[
\langle \hat{f}_\beta(a)e_0, e_0 \rangle \geq 2\pi c_2 C a^{\beta-2}
\]
for all \( a > a_0 \). Thus for \( 1 < q < 2 \),
\[
\int_{0}^{\infty} \| \hat{f}_\beta(a) \|_{S_q}^q ada \geq \int_{0}^{\infty} \| \hat{f}_\beta(a) \|_{S_q}^q ada \geq \int_{0}^{\infty} |\langle \hat{f}_\beta(a)e_0, e_0 \rangle|^q ada
\]
\[
\geq \int_{0}^{\infty} |\langle \hat{f}_\beta(a)e_0, e_0 \rangle|^q ada \geq \hat{C} \int_{0}^{\infty} a^{(\beta-2)q+1} ada.
\]
The integral \( \int_{0}^{\infty} a^{(\beta-2)q+1} da \) is finite if and only if \( (\beta - 2)q + 1 < -1 \), that is, \( \beta < 2 - \frac{2}{q} \). Since \( q < 2 \), we have \( 2 - \frac{2}{q} < 1 \). If we choose \( \beta \in \left( \frac{n}{3}, \frac{n}{2} \right) \), the corresponding function \( f_\beta \) will be the required function. \( \square \)

Theorem 3.4. Let \( n \geq 3 \). Consider the square integrable and compactly supported function \( f_\beta(x, k) = \chi_{B_1(0)}(x) \frac{1}{|z|^2} \), where \( \beta < \frac{n}{2} \), on \( M(n) \). Then for \( 1 < q < 2 \), there exists \( \beta \in \left( \frac{n}{3}, \frac{n}{2} \right) \) such that \( \int_{M} \sum_{\sigma \in \hat{M}} d_\sigma \| \hat{f}(a, \sigma) \|_{S_q}^q a^{n-1} da = \infty \).

Proof. Let \( \sigma_0 \in \hat{M} \), be the trivial representation. Consider the function \( \varphi_0 \) on \( K \) defined by \( \varphi_0(k) = 1 \) for all \( k \in K \). Since \( d_{\sigma_0} = 1 \), we have \( \varphi_0 \in L^2(K, \mathbb{C}^{d_{\sigma_0} \times d_{\sigma_0}}) \).

Now for \( a > 0 \),
\[
\langle \hat{f}_\beta(a, \sigma_0)\varphi_0, \varphi_0 \rangle = \int_{K} \int_{K} \int_{\mathbb{R}^n} f_\beta(x, k)e^{ia\langle s^{-1}e_1, x \rangle} \varphi_0(sk)\varphi_0(s) dxdks
\]
\[
= \int_{K} \int_{\mathbb{R}^n} \chi_{B_1(0)}(x) \frac{1}{|x|^\beta} e^{ia\langle s^{-1}e_1, x \rangle} dxd.
\]
Again \( S^{n-1} = \{ s^{-1} \cdot e_1 : s \in K \} \). Therefore, using (3.2) we get
\[
\langle \hat{f}_\beta(a, \sigma_0)\varphi_0, \varphi_0 \rangle = \int_{S^{n-1}} \int_{\mathbb{R}^n} \chi_{B_1(0)}(x) \frac{1}{|x|^\beta} e^{ia\langle \omega, x \rangle} dxd\omega
\]
\[
= c_n \int_{\mathbb{R}^n} \chi_{B_1(0)}(x) \frac{1}{|x|^\beta} (a|x|)^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(a|x|)dx
\]
\[
= c_n \int_{0}^{1} \int_{S^{n-1}} \frac{1}{r^\beta} (ar)^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(ar) r^{n-1} drd\omega
\]
\[
= c_n a^{\beta-n} \int_{0}^{a} r^{n-\beta-1} J_{\frac{n}{2}-1}(r) dr.
\]
Hence by Lemma 3.2 if $\beta \in \left(\frac{n}{q}, \frac{n}{2}\right)$, then there exists $C > 0$ and $a_0 > 0$ such that 
\[
\langle \hat{f}_\beta(a, \sigma_0)\varphi_0, \varphi_0 \rangle \geq C a \beta^{-n} \text{ for all } a \geq a_0.
\]
Thus for $1 < q < 2$,
\[
\int_0^\infty \sum_{\sigma \in \mathcal{M}} d_\sigma \|\hat{f}(a, \sigma)\|_q^q a^{n-1} da \geq \int_0^\infty \|\hat{f}(a, \sigma_0)\|_q^q a^{n-1} da \geq \int_0^\infty \langle \hat{f}_\beta(a, \sigma_0)\varphi_0, \varphi_0 \rangle |a|^{n-1} da \\
\geq \int_0^\infty \langle \hat{f}_\beta(a, \sigma_0)\varphi_0, \varphi_0 \rangle |a|^{n-1} da \geq C \int_0^\infty a^{(\beta-n)q+n-1} da.
\]
The integral $\int_0^\infty a^{(\beta-n)q+n-1} da$ is infinite if $(\beta-n)q+n-1 \geq -1$, that is, 
$\beta \geq n - \frac{n}{q}$. Since $q < 2$, we have $n - \frac{n}{q} < \frac{n}{2}$. If we choose $\beta \in \left(\max\left\{\frac{n}{3}, n - \frac{n}{q}\right\}, \frac{n}{2}\right)$, the corresponding function $f_\beta$ will be the required function.

4. **Heisenberg motion group**

The Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is a step two nilpotent Lie group having center $\mathbb{R}$ that equipped with the group law
\[
(z, t) \cdot (w, s) = \left( z + w, t + s + \frac{1}{2} \text{Im}(z \cdot \bar{w}) \right).
\]

By the Stone-von Neumann theorem, the infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ can be parameterized by $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. That is, each $\lambda \in \mathbb{R}^*$ defines a Schrödinger representation $\pi_\lambda$ of $\mathbb{H}^n$ via
\[
\pi_\lambda(z, t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{4} x \cdot y)} \varphi(\xi + y),
\]
where $z = x + iy$ and $\varphi \in L^2(\mathbb{R}^n)$.

Having chosen sublaplacian $\mathcal{L}$ of the Heisenberg group $\mathbb{H}^n$ and its geometry, there is a larger group of isometries that commute with $\mathcal{L}$, known as Heisenberg motion group. The Heisenberg motion group $G$ is the semidirect product of $\mathbb{H}^n$ with the unitary group $K = U(n)$. Since $K$ defines a group of automorphisms on $\mathbb{H}^n$, via $k \cdot (z, t) = (kz, t)$, the group law on $G$ can be expressed as
\[
(z, t, k_1) \cdot (w, s, k_2) = \left( z + k_1 w, t + s - \frac{1}{2} \text{Im}(k_1 w \cdot \bar{z}), k_1 k_2 \right).
\]

Since a right $K$-invariant function on $G$ can be thought as a function on $\mathbb{H}^n$, the Haar measure on $G$ is given by $dg = dz dt dk$, where $dz dt$ and $dk$ are the normalized Haar measure on $\mathbb{H}^n$ and $K$ respectively.

For $k \in K$ define another set of representations of the Heisenberg group $\mathbb{H}^n$ by $\pi_{\lambda, k}(z, t) = \pi_\lambda(kz, t)$. Since $\pi_{\lambda, k}$ agrees with $\pi_\lambda$ on the center of $\mathbb{H}^n$, it follows by the Stone-Von Neumann theorem for the Schrödinger representation that $\pi_{\lambda, k}$ is equivalent to $\pi_\lambda$. Hence there exists an intertwining operator $\mu_\lambda(k)$ satisfying
\[
\pi_\lambda(kz, t) = \mu_\lambda(k) \pi_\lambda(z, t) \mu_\lambda(k)^*.
\]

Then $\mu_\lambda$ can be thought of as a unitary representation of $K$ on $L^2(\mathbb{R}^n)$, called metaplectic representation. Let $(\sigma, \mathcal{H}_\sigma)$ be an irreducible unitary representation of $K$ and $\mathcal{H}_\sigma = \text{span}\{e^j, e^j_\sigma : 1 \leq j \leq d_\sigma\}$. For $k \in K$, the matrix coefficients of the representation $\sigma \in \mathcal{K}$ are given by
\[
\varphi^\sigma_{ij}(k) = \langle \sigma(k) e^j_\sigma, e^i_\sigma \rangle,
\]
where \( i, j = 1, \ldots, d_\sigma \).

Let \( \phi^\alpha_\sigma(x) = |\lambda|^{\frac{2\sigma}{n}} \phi_\alpha(\sqrt{\lambda}x) \); \( \alpha \in \mathbb{Z}_+^n \), where \( \phi_\alpha \)'s are the Hermite functions on \( \mathbb{R}^n \). Since for each \( \lambda \in \mathbb{R}^* \), the set \( \{ \phi^\alpha_\sigma : \alpha \in \mathbb{Z}_+^n \} \) forms an orthonormal basis for \( L^2(\mathbb{R}^n) \), letting \( P^\lambda_m = \text{span}\{ \phi^\alpha_\sigma : |\alpha| = m \} \), \( \mu_\lambda \) becomes an irreducible unitary representation of \( K \) on \( P^\lambda_m \). Hence, the action of \( \mu_\lambda \) can be realized on \( P^\lambda_m \) by

\[
\mu_\lambda(k) \phi^\alpha_\sigma = \sum_{|\alpha|=|\gamma|} \eta^\lambda_\gamma(k) \phi^\alpha_\sigma,
\]

where \( \eta^\lambda_\gamma \)'s are the matrix coefficients of \( \mu_\lambda(k) \). Define a bilinear form \( \phi^\alpha_\sigma \otimes e^\sigma_j \) on \( L^2(\mathbb{R}^n) \times \mathcal{H}_\sigma \) by \( \phi^\alpha_\sigma \otimes e^\sigma_j = \phi^\alpha_\sigma e^\sigma_j \). Then \( \{ \phi^\alpha_\sigma \otimes e^\sigma_j : \alpha \in \mathbb{Z}_+^n, 1 \leq j \leq d_\sigma \} \) forms an orthonormal basis for \( L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma \). Denote \( \mathcal{H}^2_\sigma = L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma \).

Define a representation \( \rho^\lambda_\sigma \) of \( G \) on the space \( \mathcal{H}^2_\sigma \) by

\[
\rho^\lambda_\sigma(z, t, k) = \pi_\lambda(z, t) \mu_\lambda(k) \otimes \sigma(k).
\]

In the article \([11]\), it is shown that \( \rho^\lambda_\sigma \) are all possible irreducible unitary representations of \( G \) that participate in the Plancherel formula. Thus, in view of the above discussion, we shall denote the partial dual of the group \( G \) by \( G' = \mathbb{R}^* \times \bar{K} \).

The Fourier transform of \( f \in L^1(G) \) defined by

\[
\hat{f}(\lambda, \sigma) = \int_K \int_{\mathbb{R}^*} \int_{\mathbb{C}^n} f(z, t, k) \rho^\lambda_\sigma(z, t, k) dz dt dk,
\]

is a bounded linear operator on \( \mathcal{H}^2_\sigma \). As the Plancherel formula

\[
\int_K \int_{\mathbb{R}^n} |f(z, t, k)|^2 dz dt dk = (2\pi)^{-n} \int_{\mathbb{R}\setminus\{0\}} \sum_{\sigma \in \bar{K}} d_\sigma \| \hat{f}(\lambda, \sigma) \|_{\mathcal{S}_2^2}^2 |\lambda|^n d\lambda
\]

holds for \( f \in L^2(G) \), it follows that \( \hat{f}(\lambda, \sigma) \) is a Hilbert-Schmidt operator on \( \mathcal{H}^2_\sigma \). For detailed Fourier analysis on the Heisenberg motion group, see \([2, 5, 11]\).

First, we recall the example of the previously mentioned required function for the Heisenberg group and show that the example can be extended to the Heisenberg motion group.

Let \( A = \{(x, y, t) : |x| \leq 1, |y| \leq 1, |t| \leq 1; l = 1, \ldots, n \} \) be a compact subset of \( \mathbb{R}^{2n} \times \mathbb{R} \) and

\[
ge_\xi(z, t) = g_\xi(x, y, t) = |t|^{\xi} \prod_{j=1}^n |x_j|^{\xi} \prod_{j=1}^n |y_j|^{\xi} \chi_A(z, t),
\]

where \( \xi > -\frac{1}{2} \). Then the following result holds.

**Theorem 4.1.** \([5]\) For \( 1 < q < 2 \), there exists \( \xi > -\frac{1}{2} \), such that

\[
\int_{\mathbb{R}\setminus\{0\}} |\langle g_\xi(\lambda) \phi^\lambda_0, \phi^\lambda_0 \rangle|^q |\lambda|^n d\lambda
\]

is infinite.

The following proposition gives the required function for the Heisenberg motion group.
Theorem 4.2. Consider the function $f_\xi(z, t, k) = g_\xi(z, t)$ on $G$, where $g_\xi$ is defined in (4.2). Then $f_\xi$ is square integrable and compactly supported. Further, for $1 < q < 2$, there exists $\xi > -\frac{1}{2}$, such that $\int_{\mathbb{R}\setminus\{0\}} \sum_{\sigma \in K} d_\sigma \|f_\xi(\lambda, \sigma)\|^q_{S_2} |\lambda|^n d\lambda$ is infinite.

Proof. Let $1 < q < 2$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Then

$$\sum_{\sigma \in K} d_\sigma \|f_\xi(\lambda, \sigma)\|^q_{S_2} \geq \sum_{\sigma \in K} \left( d^{1/q}_{\sigma} \|\hat{f}_\xi(\lambda, \sigma)\|_{S_2} \right)^q \geq \left( \sum_{\sigma \in K} d^{2/q}_{\sigma} \|\hat{f}_\xi(\lambda, \sigma)\|_{S_2}^2 \right)^{q/2} \geq \left( \sum_{\sigma \in K} d^{2/q}_{\sigma} \|\hat{f}_\xi(\lambda, \sigma)\|_{S_2}^2 \right)^{q/2}. \quad (4.3)$$

It is already discussed that $\{\phi^\lambda_\alpha \otimes e^\sigma_\beta : \alpha \in \mathbb{Z}^n_+, 1 \leq j \leq d_\sigma \}$ forms an orthonormal basis for $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$. Thus by \((4.1)\),

$$\langle \hat{f}_\xi(\lambda, \sigma)(\phi^\lambda_\alpha \otimes e^\sigma_\beta), (\phi^\lambda_\alpha \otimes e^\sigma_\beta) \rangle = \int_{\mathbb{R}^n} \int_K f_\xi(z, t, k) \sum_{|\gamma|=|\alpha|} \eta_{\lambda\gamma}(k) \Phi^{\lambda\gamma}_{\alpha\beta}(z, t) \phi^\sigma_{\beta}(k) dz dt dk,$$

where $\Phi^{\lambda\gamma}_{\alpha\beta}(z, t) = \langle \pi_\lambda(z, t) \phi^\lambda_\alpha, \phi^\gamma_\beta \rangle$. Therefore, $\|f_\xi(\lambda, \sigma)(\phi^\lambda_\alpha \otimes e^\sigma_\beta)\|^2_{S_2}$ is equal to

$$\sum_{\beta \in \mathbb{Z}^n_+} \sum_{1 \leq l \leq d_\sigma} \left| \int_{\mathbb{R}^n} \int_K f_\xi(z, t, k) \sum_{|\gamma|=|\alpha|} \eta_{\lambda\gamma}(k) \Phi^{\lambda\gamma}_{\alpha\beta}(z, t) \phi^\sigma_{\beta}(k) dz dt dk \right|^2$$

$$= \sum_{\beta \in \mathbb{Z}^n_+} \sum_{1 \leq l \leq d_\sigma} \left| \sum_{|\gamma|=|\alpha|} \langle \hat{f}_\xi(\lambda, \sigma)(\phi^\lambda_\alpha \otimes e^\sigma_\beta), \phi^\gamma_\beta \rangle \int_K \eta_{\lambda\gamma}(k) \phi^\sigma_{\beta}(k) dk \right|^2.$$

That is,

$$\|f_\xi(\lambda, \sigma)\|^2_{S_2} = \sum_{\alpha, \beta \in \mathbb{Z}^n_+} \sum_{1 \leq j \leq d_\sigma} \left| \sum_{|\gamma|=|\alpha|} \langle \hat{f}_\xi(\lambda, \sigma)(\phi^\lambda_\alpha \otimes e^\sigma_\beta), \phi^\gamma_\beta \rangle \int_K \eta_{\lambda\gamma}(k) \phi^\sigma_{\beta}(k) dk \right|^2.$$

Hence by the Peter-Weyl theorem (Plancherel) for compact groups, we get

$$\sum_{\sigma \in K} d_\sigma \|f_\xi(\lambda, \sigma)\|^2_{S_2} = \sum_{\alpha, \beta \in \mathbb{Z}^n_+} \int_K \left| \sum_{|\gamma|=|\alpha|} \langle \hat{f}_\xi(\lambda, \sigma)\phi^\lambda_\alpha, \phi^\gamma_\beta \rangle \eta_{\lambda\gamma}(k) \right|^2 dk$$

$$\geq \sum_{\alpha, \beta \in \mathbb{Z}^n_+} \int_K \left| \sum_{|\gamma|=|\alpha|} \langle \hat{f}_\xi(\lambda, \sigma)\phi^\lambda_\alpha, \phi^\gamma_\beta \rangle \eta_{\lambda\gamma}(k) \right|^2 dk$$

$$= \int_K \left| \langle \hat{f}_\xi(\lambda)\phi^\lambda_0, \phi^\lambda_0 \rangle \eta_{00}(k) \right|^2 dk = \left| \langle \hat{f}_\xi(\lambda)\phi^\lambda_0, \phi^\lambda_0 \rangle \right|^2.$$
Since $\mu_\lambda|_{\mathcal{P}_0}$ is irreducible, the last equality follows from Schur’s orthogonality relation. Therefore, from (4.3), we can conclude that

$$
\int_{\mathbb{R}\setminus\{0\}} \sum_{\sigma \in \mathcal{K}} d_\sigma \|\hat{f}_\xi(\lambda, \sigma)\|_2^q |\lambda|^n d\lambda \geq \int_{\mathbb{R}\setminus\{0\}} \left( \sum_{\sigma \in \mathcal{K}} d_\sigma \|\hat{f}_\xi(\lambda, \sigma)\|_2^q \right)^{q/2} |\lambda|^n d\lambda \\
\geq \int_{\mathbb{R}\setminus\{0\}} |\langle \hat{g}_\xi(\lambda)\phi_0^\lambda, \phi_0^\lambda \rangle|^q |\lambda|^n d\lambda.
$$

Thus Theorem 4.1 proves the result. \(\Box\)

**Remark 4.3.** As compare to motion groups, the result in product space is straightforward. Let $G$ be a second countable type $I$ locally compact unimodular group and $(\pi, \mathcal{H}_x)$ be its representation. Consider $G_P = \mathbb{R}^n \times G$. For $f \in L^1(G_P)$, Fourier transform defined by $\hat{f}(x, \pi) = \int_{\mathbb{R}^n} \int_{G} f(y, u)e^{-2\pi i x \cdot \pi(y)u}d\nu(u)dy$ is a bounded operator on $\mathcal{H}_x$. If we take $f(x, u) = f_1(x)f_2(u)$, then

(4.4) \[\int_{\mathbb{R}^n} \int_{G} \|\hat{f}(x, \pi)\|_q^2 d\mu(\pi)dy = \int_{\mathbb{R}^n} \|\hat{f}_1(y)\|_q^2 dy \int_{\mathcal{P}_0} \|\hat{f}_2(\pi)\|_q^2 d\mu(\pi).\]

It is known that, for $q \in (1, 2)$, there exists a positive, square integrable and compactly supported function $f_1$ on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |\hat{f}_1(y)|^q dy$ is infinite (see [12]). If we choose a square integrable and compactly supported function $f_2$ on $G$ such that $\int_{G} \|\hat{f}_2(\pi)\|^2 d\mu(\pi) \neq 0$, then by (4.4), $\int_{\mathbb{R}^n} \int_{G} \|\hat{f}(x, \pi)\|^2 d\mu(\pi)dy$ is infinite. Hence in view of Proposition 2.14, the Weyl transform $W_\zeta$ on $G_P$ is not bounded for $\zeta \in L^p(G_P \times \hat{G}_P)$ with $2 < p < \infty$.

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