Concr. Oper. 2021; 8:114–124

Research Article

Ashish Pathak* and Shrish Pandey

Besov-type spaces for the $\kappa$-Hankel wavelet transform on the real line

https://doi.org/10.1515/conop-2020-0117
Received March 1, 2021; accepted July 9, 2021

Abstract: In this paper, we shall introduce functions spaces as subspaces of $L^p_\kappa(\mathbb{R})$ that we call Besov-$\kappa$-Hankel spaces and extend the concept of $\kappa$-Hankel wavelet transform in $L^p_\kappa(\mathbb{R})$ space. Subsequently we will characterize the Besov-$\kappa$-Hankel space by using $\kappa$-Hankel wavelet coefficients.

Keywords: Besov $\kappa$-Hankel space, Continuous $\kappa$-Hankel wavelet transform, $\kappa$-Hankel transform, $\kappa$-Hankel convolution

MSC: 33A40, 44A05, 42C40

Dedicated to the memory of Prof. R. S. Pathak

1 Introduction

Besov spaces $B^{\alpha,q}_p(\mathbb{R})$ are subspaces of $L^p(\mathbb{R})$, having functions of smoothness $\alpha$ and $q$ gives a finer graduation to the smoothness. It is extension of classical Sobolev and Hölder spaces. It is also expressed as interpolation space lies in between two Sobolev spaces $H^x_p$ and $H^y_p$ ($1 \leq p, q \leq \infty$) with $\alpha = (1 - \beta)x + \beta y$; $\alpha, \beta, y \in \mathbb{R}$, $\beta \in (0, 1)$.

The Besov spaces $B^{\alpha,q}_p$ ($\alpha \in \mathbb{R}$ and $1 \leq p, q \leq \infty$) were recognized in about sixtieth decade of nineteenth century [3, 4]. They were generalized in mid of seventies by various authors in different directions with different ideas. The classical definition of Besov spaces depends on the modulus of smoothness [11, 13]. The Littlewood-Paley theory interlink Besov spaces with the fourier transform. Michael Frazier and Björn Jawerth [6] characterise Besov spaces with the help of Calderon’s formula while Dang Vu Giang and Ferenc Moricz [7] characterise Besov spaces in terms of it’s Riesz mean and Dirichlet integral. In 1996, Valerie Perrier and Claude Badea [10] characterise Besov spaces by the behavior of the continuous wavelet coefficients for $\alpha \in \mathbb{R}^+$, $\alpha \notin \mathbb{Z}^+$.

Jorge J. Betancor and L. Rodríguez-Mesa [5] pave the way for exploration of Besov-Hankel spaces and characterised by mean of the Bochner-Riesz mean and the partial Hankel integrals. Recently Salem Ben Saïd, Mohamed Amine Boubatra, Mohamed Sifi [2] come out with deformed Besov-Hankel spaces and characterised it in terms of the deformed Bochner-Riesz means and the deformed partial Hankel integral.

Hatem Mejjaoli, Khalifa Trimène [8] presented $\kappa$-Hankel wavelet transform in the year 2020. Using the approach of Betancor et al. [5] and Ben Saïd et al. [2], we define Besov $\kappa$-Hankel space and by exploiting the technique of Perrier et al. [10] we characterise Besov $\kappa$-Hankel space with the help of continuous $\kappa$-Hankel wavelet transform.

Present paper is organized in following manner: section 1 is introductory, in which we define the development from Besov space to Besov $\kappa$-Hankel space with the help of $\kappa$-Hankel wavelet and it’s characterisation...
with time period. Section 2 is preliminary, in which we recall some properties of $\kappa$-Hankel transform, Besov $\kappa$-Hankel space and continuous $\kappa$-Hankel wavelet transform. Section 3 is related the continuous $\kappa$-Hankel wavelet transform in $L_p^\kappa(\mathbb{R})$. In the section 4, we characterize Besov $\kappa$-Hankel norms in terms of continuous $\kappa$-Hankel wavelet transform.

## 2 Preliminary

In this paper, we denotes the weighted $L_p^\kappa(\mathbb{R})$ space as

$$\|f\|_{L_p^\kappa} = \|f\|_p = \left( \int_{\mathbb{R}} |f(x)|^p d\sigma_\kappa(x) \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty),$$

where $d\sigma_\kappa(x) = |x|^{2\kappa-1} dx$, $\kappa \geq \frac{1}{2}$.

$$\|f\|_{L_\infty^\kappa} = \|f\|_\infty = \text{esssup} |f(x)|.$$

The $\kappa$-Hankel transformation of the function $f \in L_1^\kappa(\mathbb{R})$ for order $\kappa \geq \frac{1}{2}$ is defined as see [8]

$$\mathcal{H}_\kappa(f)(\mu) = \tilde{f}(\mu) := \frac{1}{\rho_\kappa} \int_{\mathbb{R}} B_\kappa(\mu, x)f(x)d\sigma_\kappa(x), \quad x \in \mathbb{R},$$

where

$$\rho_\kappa = \int_{\mathbb{R}} e^{-|x|} d\sigma_\kappa(x) = 2\Gamma(2\kappa)$$

and $B_\kappa(\mu, x)$ is the $\kappa$-Hankel kernel given as

$$B_\kappa(\mu, x) = j_{2\kappa-1}\left(2\sqrt{|\mu x|}\right) - \frac{\mu x}{2\kappa(2\kappa + 1)} j_{2\kappa+1}\left(2\sqrt{|\mu x|}\right).$$

Here

$$j_\lambda(\omega) = \Gamma(\lambda+1) \left(\frac{\omega}{2}\right)^{-\lambda}J_\lambda(\omega) = \Gamma(\lambda+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\lambda+n+1)} \left(\frac{\omega}{2}\right)^{2n}$$

denote the normalized Bessel function of index $\lambda$.

If $\tilde{f} \in L_1^\kappa(\mathbb{R})$, then the inverse of $\kappa$-Hankel transformations is given by

$$f(x) := \frac{1}{\rho_\kappa} \int_{\mathbb{R}} B_\kappa(\mu, x)\tilde{f}(\mu)d\sigma_\kappa(\mu), \quad x \in \mathbb{R}. \quad (1)$$

Also, Parseval’s formula of the $\kappa$-Hankel transformation for $f, g \in L_1^\kappa(\mathbb{R}) \cap L_2^\kappa(\mathbb{R})$ is given by

$$\int_{\mathbb{R}} \tilde{f}(\mu)\overline{\tilde{g}(\mu)}d\sigma_\kappa(\mu) = \int_{\mathbb{R}} f(x)\overline{g(x)}d\sigma_\kappa(x).$$

By denseness and continuity the Parseval’s formula can be extended to all $f, g \in L_2^\kappa(\mathbb{R})$. Hence $\mathcal{H}_\kappa$ is isometry on $L_2^\kappa(\mathbb{R})$.

If $f, g \in L_2^\kappa(\mathbb{R})$, then the convolution associated with the $\kappa$-Hankel transform is defined as see [1]

$$(f \#_\kappa g)(x) = \frac{1}{\rho_\kappa} \int_{\mathbb{R}} f(y)r_{\kappa}^\ast g(y)d\sigma_\kappa(y), \quad (2)$$
where the operator \( \tau^\kappa_x \) is \( \kappa \)-Hankel translation is given by

\[
f^\kappa(x, y) = \tau^\kappa_x f(y) = \int_{\mathbb{R}} f(z) \mathcal{K}_x(x, y, z) d\sigma_x(z)
\]

and

\[
\int_{\mathbb{R}} \mathcal{B}_x(\mu, z) \mathcal{K}_x(x, y, z) d\sigma_x(z) = \mathcal{B}_x(\mu, y) \mathcal{B}_x(\mu, x). \tag{3}
\]

From (1) and (3), we have

\[
\mathcal{K}_x(x, y, z) = \frac{1}{\partial \kappa^2} \int_{\mathbb{R}} \mathcal{B}_x(\mu, x) \mathcal{B}_x(\mu, y) \mathcal{B}_x(\mu, z) d\sigma_x(\mu)
\]

and moreover,

\[
\int_{\mathbb{R}} \mathcal{K}_x(x, y, z) d\sigma_x(z) = 1
\]

\[
\int_{\mathbb{R}} |\mathcal{K}_x(x, y, z)| d\sigma_x(z) \leq M_x
\]

where \( M_x \) is independent of \( x \) and \( y \) such that \( M_x \xrightarrow{\kappa} 2 \) as \( \kappa \to \infty \) whenever \( xy < 0 \) and \( M_x \xrightarrow{\kappa} 3 \) as \( k \to \infty \) elsewhere.

\[
(f \hat{\#}_x g)(x) = \hat{f}(x) \hat{g}(x).
\]

Now, we recall some properties of \( \kappa \)-Hankel convolution [1] which are useful throughout the paper.

**Lemma 2.1.** Let \( f \in L^p_t(\mathbb{R}), \ 1 \leq p \leq \infty \). Then we have

\[
||\tau^\kappa_x f(y)||_p \leq M_x ||f||_p.
\]

**Lemma 2.2.** Let \( f \in L^p_t(\mathbb{R}) \) and \( g \in L^q_t(\mathbb{R}), \ \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{\gamma} \). Then we have

\[
||f \hat{\#}_x g||_r \leq M_x ||f||_p ||g||_q.
\]

**Definition 2.3.** (Besov \( \kappa \)-Hankel Space): Let measurable function \( \phi \) defined on \( \mathbb{R} \) belongs to \( \mathcal{B}_{p,q}^{\kappa,\alpha} \) if \( \phi \in L^p_t(\mathbb{R}) \) and

\[
\int_0^\infty (h^{-\alpha}w_{p,\kappa}(\phi)(h))^q \frac{dh}{h} < \infty \quad \text{for}\quad 1 \leq p, q < \infty,
\]

\[
\text{esssup}_{h>0} (h^{-\alpha}w_{p,\kappa}(\phi)(h)) < \infty \quad \text{for}\quad q = \infty,
\]

where \( w_{p,\kappa}(\phi)(h) = ||\tau^\kappa_x \phi - \phi||_{L^p_t}, \ h \in \mathbb{R}^+ \) and \( 0 < \alpha < 1 \).

### 2.1 \( \kappa \)-Hankel Wavelet

Using the properites of \( \kappa \)-Hankel transform see [8] define the \( \kappa \)-Hankel wavelet for \( \varphi \in L^p_t(\mathbb{R}), 1 \leq p < \infty, b \in \mathbb{R} \) and \( a > 0 \) as

\[
\varphi_{b,a}^\kappa(x) : = D_\alpha \tau^\kappa_x \varphi(x) = D_\alpha \varphi^\kappa(x, b) = a^{-2\kappa} \varphi^\kappa(\frac{x}{a}, \frac{b}{a}) = a^{-2\kappa} \int_{\mathbb{R}} \varphi(z) \mathcal{K}_x(\frac{b}{a}, \frac{x}{a}, z) d\sigma_x(z)
\]
where $D_a$ denote the dilation operator such that
\[ D_a \varphi(x, y) = a^{-2x} \varphi\left( \frac{x}{a}, \frac{y}{a} \right). \]

The continuous $\kappa$-Hankel wavelet transform of $f \in L^2_\kappa(\mathbb{R})$ with respect to a wavelet $\varphi \in L^1_\kappa(\mathbb{R})$ is defined as [8]
\[ (\mathcal{H}_\kappa^f)(b, a) := \frac{1}{\rho_\kappa} \int_{\mathbb{R}} f(x) \overline{\varphi_{b,a}(x)} d\sigma_a(x) \]
\[ = a^{-2x} \rho_\kappa \int_{\mathbb{R}} f(x) \overline{\varphi(z)} \mathcal{K}_\kappa \left( \frac{b}{a}, \frac{x}{a}, z \right) d\sigma_a(z) d\sigma_a(x). \]

Moreover, using (2), we have
\[ (\mathcal{H}_\kappa^f)(b, a) = (f \# \varphi^\kappa_b)(b) \]
where $\varphi^\kappa_b(t) = a^{-2x}(t/a)$
for more about $\kappa$-Hankel wavelet see [8].

### 3 The Continuous $\kappa$-Hankel Wavelet Transform in $L^p_\kappa(\mathbb{R})$

In this section we extend the concept of $\kappa$-Hankel wavelet transform on $L^p_\kappa(\mathbb{R})$.

**Theorem 3.1.** Suppose that a function $\psi \in L^2_\kappa(\mathbb{R})$ satisfies the admissibility condition
\[ C_{\kappa, \psi} = \int_0^\infty \omega^{-1} |\hat{\psi}(\omega)|^2 d\omega < \infty, \quad (4) \]
where $\hat{\psi}$ denote the $\kappa$-Hankel transform of $\psi$ then continuous $\kappa$-Hankel wavelet transform is a bounded linear operator
\[ L^p_\kappa(\mathbb{R}) \rightarrow L^2_\kappa(\mathbb{R}^+, \frac{d\sigma_a(a)}{a^{2\kappa}}) \times L^p_\kappa(\mathbb{R}), \]
moreover, for any $f \in L^p_\kappa(\mathbb{R})$, $1 < p < \infty$
\[ \|f\|_{L^p_\kappa(\mathbb{R})} \approx \left( \int_0^\infty \left( \int_0^\infty |(\mathcal{H}_\kappa^f)(b, a)|^2 \frac{d\sigma_a(a)}{a^{2\kappa}} \right)^{\frac{p}{2}} d\sigma_a(b) \right)^{\frac{1}{p}}. \quad (5) \]

**Proof.** Let $S_p$ denote the space $L^2_\kappa(\mathbb{R}^+, \frac{d\sigma_a(a)}{a^{2\kappa}}) \times L^p_\kappa(\mathbb{R})$ associated to the norm
\[ \|f\|_{S_p} = \left\{ \int_0^\infty \left( \int_0^\infty |f(b, a)|^2 \frac{d\sigma_a(a)}{a^{2\kappa}} \right)^{\frac{p}{2}} d\sigma_a(b) \right\}^{\frac{1}{2}}. \]
If we take $p = 2$, then from Plancherel’s theorem [8]:
\[ \|(\mathcal{H}_\kappa^f)\|_{S_2} = \left\{ \int_0^\infty \left( \int_0^\infty |(\mathcal{H}_\kappa^f)(b, a)|^2 \frac{d\sigma_a(a)}{a^{2\kappa}} \right) d\sigma_a(b) \right\}^{\frac{1}{2}} \]
\[ = \sqrt{C_{\kappa, \psi}} \|f\|_{L^2(\mathbb{R})}. \]
where \( C_{k,\psi} = \int_{0}^{\infty} \omega^{-1} |\psi(\omega)|^2 \, d\omega < \infty \), if \( \psi \) is real. From singular integral theorem, the operators on \( L^2(\mathbb{R}^+, \frac{d\sigma(\omega)}{\omega}) \) holds inequality:

\[
\|\mathcal{H}_{k} f\|_{L^{p}(\mathbb{R})} \leq C_{p} \|f\|_{L^{p}(\mathbb{R})} \quad \text{for} \quad 1 < p \leq 2,
\]

where the constant \( C_{p} \) depends only on \( p \) and \( \psi \)(see [12]). Due to duality the inequality is also valid for \( 1 < p < \infty \). It follows that

\[
\left\{ \frac{1}{C_{k,\psi}} \int_{-\infty}^{\infty} \left( \int_{0}^{\infty} |(\mathcal{H}_{k} f)(b, a)|^2 \frac{d\sigma_k(a)}{a^{2k}} \right)^{\frac{q}{2}} \, d\sigma_k(b) \right\}^{\frac{1}{q}} \leq C_{p} \|f\|_{L^{p}(\mathbb{R})}.
\] (6)

Conversely suppose that \( f \in L^2_{\psi}(\mathbb{R}) \cap L^p_{\psi}(\mathbb{R}) \). Since continuous \( \kappa \)-Hankel wavelet transform is isometry for every \( g \in L^2_{\psi}(\mathbb{R}) \cap L^p_{\psi}(\mathbb{R}) \), we can write

\[
\frac{1}{C_{k,\psi}} \int_{-\infty}^{\infty} \left( \int_{0}^{\infty} |(\mathcal{H}_{k} f)(b, a)|^2 \frac{d\sigma_k(a)}{a^{2k}} \right)^{\frac{q}{2}} \, d\sigma_k(b) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{1}{C_{k,\psi}} \left| \frac{\mathcal{H}_{k} f(b, a)}{g(b, a)} \right|^2 \, d\sigma_k(a) \, d\sigma_k(b)
\]

Now,

\[
| \int_{-\infty}^{\infty} f(x) g(x) d\sigma_k(x) | = \frac{1}{C_{k,\psi}} | \int_{-\infty}^{\infty} \left( \int_{0}^{\infty} |(\mathcal{H}_{k} f)(b, a)|^2 \frac{d\sigma_k(a)}{a^{2k}} \right)^{\frac{q}{2}} \, d\sigma_k(b) |
\]

using Schwarz inequality and then Holder’s inequality, we have

\[
\leq \frac{1}{C_{k,\psi}} \left( \int_{-\infty}^{\infty} \left( \int_{0}^{\infty} |(\mathcal{H}_{k} f)(b, a)|^2 \, d\sigma_k(a) \right)^{\frac{q}{2}} \, d\sigma_k(b) \right)^{\frac{1}{q}}
\]

\[
\times \left( \int_{-\infty}^{\infty} \left( \int_{0}^{\infty} |(\mathcal{H}_{k} g)(b, a)|^2 \, d\sigma_k(a) \right)^{\frac{q}{2}} \, d\sigma_k(b) \right)^{\frac{1}{q}},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

From equation (6), we get

\[
\|f\|_{L^{p}(\mathbb{R})} \leq \frac{C_{q}}{C_{k,\psi}} \left( \int_{-\infty}^{\infty} \left( \int_{0}^{\infty} |(\mathcal{H}_{k} f)(b, a)|^2 \, d\sigma_k(a) \right)^{\frac{q}{2}} \, d\sigma_k(b) \right)^{\frac{1}{q}} \|g\|_{L^{q}(\mathbb{R})}.
\]

By Density theorem

\[
\|f\|_{L^{p}(\mathbb{R})} \leq A \left( \int_{-\infty}^{\infty} \left( \int_{0}^{\infty} |(\mathcal{H}_{k} f)(b, a)|^2 \, d\sigma_k(a) \right)^{\frac{q}{2}} \, d\sigma_k(b) \right)^{\frac{1}{q}},
\]

where \( A = \frac{C_{q}}{C_{k,\psi}} \).
Theorem 3.2. **Parseval’s formula** Let us assume \( \phi_1 \in L_p^\kappa(\mathbb{R}) \), \( \phi_2 \in L_q^\kappa(\mathbb{R}) \) with \( 1 < p, q < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( \psi \) is a real wavelet then

\[
\frac{1}{C_{\kappa, \psi}} \int_{-\infty}^{\infty} (\mathfrak{C}_\kappa \phi_1)(b, a) \overline{(\mathfrak{C}_\kappa \phi_2)(b, a)} e^{-2\pi i a x} d\sigma_a(b) \approx \int_{-\infty}^{\infty} \phi_1(x) \overline{\phi_2(x)} d\sigma_x(x),
\]

where \( C_{\kappa, \psi} = \int_{0}^{\infty} \omega^{-1} |\hat{\psi}(\omega)|^2 d\omega < \infty \) and \( \hat{\psi} \) denote the \( \kappa \)-Hankel transform.

**Proof.** Let us define bilinear transform \( T : L_p^\kappa(\mathbb{R}) \times L_q^\kappa(\mathbb{R}) \to \mathbb{R} \) by

\[
T(\phi_1, \phi_2) = \langle (\mathfrak{C}_\kappa \phi_1)(b, a), (\mathfrak{C}_\kappa \phi_2)(b, a) \rangle_{\frac{d\sigma_a}{d\sigma_b}},
\]

Now, applying Hölder's inequality two times we obtain

\[
|T(\phi_1, \phi_2)| \leq \left( \int_{-\infty}^{\infty} \left[ \int_{0}^{\infty} (\mathfrak{C}_\kappa \phi_1)(b, a) \frac{d\sigma_a}{a^{2\kappa}} \right]^2 d\sigma_b(b) \right)^{\frac{1}{2}} \left( \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} (\mathfrak{C}_\kappa \phi_2)(b, a) \frac{d\sigma_a}{a^{2\kappa}} \right)^{\frac{q}{2}} d\sigma_b(b) \right)^{\frac{1}{2}} 
\]

using Theorem 3.1. we have

\[
|T(\phi_1, \phi_2)| \leq A \|\phi_1\|_{L_p^\kappa(\mathbb{R})} \|\phi_2\|_{L_q^\kappa(\mathbb{R})}. \tag{7}
\]

Moreover for all \( \phi_1 \in L_p^\kappa(\mathbb{R}) \cap L_q^\kappa(\mathbb{R}) \) and \( \phi_2 \in L_p^\kappa(\mathbb{R}) \cap L_q^\kappa(\mathbb{R}) \) we get

\[
T(\phi_1, \phi_2) = \langle (\mathfrak{C}_\kappa \phi_1)(b, a), (\mathfrak{C}_\kappa \phi_2)(b, a) \rangle_{\frac{d\sigma_a}{d\sigma_b}} = C_{\kappa, \psi} \langle \phi_1, \phi_2 \rangle. \tag{8}
\]

From equations (7), (8) and density of spaces \( L_p^\kappa(\mathbb{R}) \cap L_q^\kappa(\mathbb{R}) \) in \( L_p^\kappa(\mathbb{R}) \) gives the result. \( \square \)

### 3.1 An inversion formula

**Theorem 3.3.** Let us consider \( \phi \in L_p^\kappa(\mathbb{R}) \) with \( 1 < p < \infty \) and \( \psi \) is a real wavelet. Then

\[
\phi(x) = \frac{1}{C_{\kappa, \psi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mathfrak{C}_\kappa \phi)(b, a) \psi_{b,a}(x) \frac{d\sigma_a}{a^{2\kappa}} d\sigma_b(b).
\]

The equality holds in \( L_p^\kappa(\mathbb{R}) \) sense and the integral of right hand side have to be taken in the sense of distributions.

**Proof.** The proof followed from theorem 3.2. \( \square \)

### 4 Characterization of Besov \( \kappa \)-Hankel Norms

In present section, By using the above results, we characterize the Besov \( \kappa \)-Hankel norms associated with the \( \kappa \)-Hankel wavelet transform.
Theorem 4.1. Let $f \in B_{p,q}^{q,a}(\mathbb{R})(p, q \geq 1, 0 < a < 1)$ and $\psi, z^a \psi \in L^p_1(\mathbb{R})$, then the wavelet transform of function $f$ holds following conditions:

- If $q < \infty$, \[ \int_0^\infty \left[ a^{-a} \| (\mathcal{H}^k f)(b,a) \|_{L^p_1} \right]^q \frac{da}{a} < \infty \]
- If $q = \infty$, $a \rightarrow a^{-a} \| (\mathcal{H}^k f)(b,a) \|_{L^p_1} \in L^\infty_p(\mathbb{R}^+)$. 

Moreover the function $a \rightarrow a^{-a} \| (\mathcal{H}^k f)(b,a) \|_{L^p_1} \in L^p_1(\mathbb{R}^+, \frac{da}{a})$.

Proof. By the definition of continuous $\kappa$-Hankel wavelet transform and equation (4), we have

\[
(\mathcal{H}^k \psi f)(b,a) = \frac{1}{\rho_k} \int f(x) \overline{\psi_{b,a}(x)} d\sigma_x(x)
\]

\[
= \frac{1}{\rho_k} \int f(x) \left( \int \overline{a^{-2\kappa} \mathcal{K}_b \left( \frac{b}{a}, \frac{x}{a} \right)} \overline{\psi(z)} d\sigma_x(z) \right) d\sigma_x(x)
\]

\[
= \frac{1}{\rho_k} \int \overline{\psi(z)} \left( \int a^{-2\kappa} \mathcal{K}_b \left( \frac{b}{a}, \frac{x}{a} \right) f(x) d\sigma_x(x) \right) d\sigma_x(z)
\]

\[
= \frac{1}{\rho_k} \int \overline{\psi(z)} \left( \int \mathcal{K}_b \left( b, x, az \right) f(x) d\sigma_x(x) \right) d\sigma_x(z)
\]

\[
= \frac{1}{\rho_k} \int \overline{\psi(z)} \left( (\tau_{ax}^z f)(b) - f(b) \right) d\sigma_x(z).
\]

Taking $L^p_1 -$ norm of the wavelet coefficient

\[
\| (\mathcal{H}^k \psi f)(b,a) \|_{L^p_1} = \frac{1}{\rho_k} \left\{ \int \left( \int \overline{\psi(z)} \left( (\tau_{ax}^z f)(b) - f(b) \right) d\sigma_x(z) \right)^p d\sigma_k(b) \right\}^{\frac{1}{p}}.
\]

Using Minkowski inequality of integrability for $p \neq \infty$

\[
\| (\mathcal{H}^k \psi f)(b,a) \|_{L^p_1} \leq \frac{1}{\rho_k} \int \left\{ \int (\tau_{ax}^z f)(b) - f(b) )^p d\sigma_k(b) \right\}^{\frac{1}{p}} \overline{\psi(z)} d\sigma_x(z).
\]

Suppose that $q < \infty$ and integrating w.r.t. $a$, we get

\[
\int_0^\infty \left[ a^{-a} \| (\mathcal{H}^k f)(b,a) \|_{L^p_1} \right]^q \frac{da}{a} \leq \frac{1}{\rho_k} \int \left[ a^{-a} \int (\tau_{ax}^z f)(b) - f(b) )^p d\sigma_k(b) \right]^{\frac{1}{p}} \overline{\psi(z)} d\sigma_x(z).
\]

Again using Minkowski integrability inequality

\[
\int_0^\infty \left[ a^{-a} \| (\mathcal{H}^k f)(b,a) \|_{L^p_1} \right]^q \frac{da}{a} \leq \frac{1}{\rho_k} \int \left( \int (\tau_{ax}^z f)(b) - f(b) \right)^p d\sigma_k(b) \left\{ \int (a^{-a} \omega_{p,k}(f, az))^q \frac{da}{a} \right\}^{\frac{1}{p}} \overline{\psi(z)} d\sigma_x(z).
\]
Applying change of variable \( h = a z \)

\[
\begin{align*}
&= \frac{1}{p_x} \left[ \int_{-\infty}^{\infty} |\psi(z)| \, d\sigma_h(z) \left\{ \int_0^\infty (h^{-a} \omega_{p,k}(f, h))^q \, \frac{dh}{h} \right\} \right]^q \\
&\leq \frac{1}{p_x} \left[ \int_{-\infty}^{\infty} |z^a \psi(z)| \, d\sigma_h(z) \right]^q \times \left\{ \int_0^\infty (h^{-a} \omega_{p,k}(f, h))^q \, \frac{dh}{h} \right\} \\
&= \frac{1}{p_x} \left[ \int_{-\infty}^{\infty} |z^a \psi(z)| \, d\sigma_h(z) \right]^q \times \left\{ \int_0^\infty (h^{-a} \omega_{p,k}(f, h))^q \, \frac{dh}{h} \right\} \\
&< \infty.
\end{align*}
\]

If \( q = \infty \) the hypothesis on \( f \) says that \( h^{-a} \omega_{p,k}(f, h) \in L^\infty_k(\mathbb{R}^+) \), so

\[
\| (\mathcal{I}^{(\psi)} f)(b, a) \|_{L^p_k} \leq \frac{a^a}{p_x} \| h^{-a} \omega_{p,k}(f, h) \|_{L^\infty_k(\mathbb{R}^+)} \times \int_{-\infty}^{\infty} |z^a \psi(z)| \, d\sigma_h(z) \\
\leq \frac{a^a}{p_x} \| h^{-a} \omega_{p,k}(f, h) \|_{L^\infty_k(\mathbb{R}^+)} \times \| z^a \psi \|_{L^q_k(\mathbb{R}^+)}.
\]

Next theorem is the converse of the above theorem. The \( \kappa \)-Hankel wavelet coefficients is sufficient to characterize Besov \( \kappa \)-Hankel spaces.

**Theorem 4.2.** Suppose \( 0 < \alpha < 1 \), and a function \( \psi \) is a real \( C^1 \)-regular wavelet with first derivative rapidly decreasing. If \( f \in L^p_k(\mathbb{R}) \) \( (1 < p < \infty) \), and if \( a^{-\alpha} \| (\mathcal{I}^{(\psi)} f)(a, \cdot) \|_{L^k_\alpha} \in L^q_k(\mathbb{R}^+, \frac{da}{a}) \) \( (1 \leq q \leq \infty) \), then \( f \in \mathcal{B} \mathcal{I}^{(\psi)}_{\kappa, \kappa} \) and we have

\[
\| h^{-a} \omega_{p,k}(f, h) \|_{L^\infty_k(\mathbb{R}^+)} \leq \frac{M_k}{C_{x, \psi} p_x} \left( \frac{M_k + 1}{a} \| \psi \|_{L^1} + \frac{M_k}{1-\alpha} \| \psi' \|_{L^1} \right) \\
\times \| a^{-\alpha} \| (\mathcal{I}^{(\psi)} f)(a, \cdot) \|_{L^q_k(\mathbb{R}^+, \frac{da}{a})}.
\]

**Proof.** Let \( f \in L^p_k(\mathbb{R}) \). By inversion formula of \( \kappa \)-Hankel wavelet transform

\[
f(x) = \frac{1}{C_{x, \psi} p_x} \int_0^\infty \frac{d\sigma_h(a)}{a^{2\alpha}} \int_{-\infty}^{\infty} (\mathcal{I}^{(\psi)} f)(a, b) \psi^x_{a,b}(x) d\sigma_k(b)
\]

and

\[
\tau_h^x f(x) = \frac{1}{C_{x, \psi} p_x} \int_0^\infty \frac{d\sigma_h(a)}{a^{2\alpha}} \int_{-\infty}^{\infty} (\mathcal{I}^{(\psi)} f)(a, b) \tau^x_h \psi^x_{a,b}(x) d\sigma_k(b).
\]
Then
\[
\tau_h^\kappa f(x) - f(x) = \frac{1}{C_{\kappa,\phi\rho}} \int \frac{d\sigma_s(a)}{a^{2k}} \int (\mathcal{H}_{\kappa}(\psi) f)(a, b) \left\{ \tau_h^\kappa \psi_{a,b}^\kappa(x) - \psi_{a,b}^\kappa(x) \right\} d\sigma_s(b)
\]
\[
= \frac{1}{C_{\kappa,\phi\rho}} \int \frac{d\sigma_s(a)}{a^{2k}} \int (\mathcal{H}_{\kappa}(\psi) f)(a, b) a^{-2k} \left\{ \tau_h^\kappa \psi_{a,b}^\kappa(x) - \tau_h^\kappa \psi_{a,b}^\kappa(x) \right\} d\sigma_s(b)
\]
\[
= \frac{1}{C_{\kappa,\phi\rho}} \int \frac{d\sigma_s(a)}{a^{2k}} \int (\mathcal{H}_{\kappa}(\psi) f)(a, b) a^{-2k} \kappa \left( \frac{b - x}{a}, y \right) d\sigma_s(b)
\]
\[
\times \int \left\{ \tau_h^\kappa \psi(y) - \psi(y) \right\} d\sigma_s(y)
\]
\[
= \frac{1}{C_{\kappa,\phi\rho}} \int \frac{d\sigma_s(a)}{a^{2k}} \tau_h^\kappa (\mathcal{H}_{\kappa}(\psi) f)(a, x) \int \left\{ \tau_h^\kappa \psi(y) - \psi(y) \right\} d\sigma_s(y).
\]

Taking $L^p_k$-norm on both side and applying Minkowski's inequality, we have

\[
w_{p,k}(f, h) = \frac{1}{C_{\kappa,\phi\rho}} \left\{ \int \left| \int \frac{d\sigma_s(a)}{a^{2k}} \tau_h^\kappa (\mathcal{H}_{\kappa}(\psi) f)(a, x) \int \left\{ \tau_h^\kappa \psi(y) - \psi(y) \right\} d\sigma_s(y) \right|^p d\sigma_s(x) \right\}^{\frac{1}{p}}
\]
\[
\leq \frac{1}{C_{\kappa,\phi\rho}} \int \frac{d\sigma_s(a)}{a^{2k}} \left\{ \int \left| \tau_h^\kappa \psi(y) - \psi(y) \right| d\sigma_s(y) \right\}^{p} d\sigma_s(x)
\]
\[
\leq \frac{M_k}{C_{\kappa,\phi\rho}} \int \frac{d\sigma_s(t)}{t^{2k}} \left\{ \left\| (\mathcal{H}_{\kappa}(\psi) f) \left( \frac{h}{t}, \cdot \right) \right\|_{L^p_k} \right\}^{q/p} d\sigma_s(y)
\]

Now
\[
\left\{ \int \frac{dh}{h} h^{-aq} w_{p,k}(f, h)^q \right\}^{\frac{1}{q}} \leq \frac{M_k}{C_{\kappa,\phi\rho}} \int \frac{d\sigma_s(t)}{t^{2k}} \left\{ \int \left| \tau_h^\kappa \psi(y) - \psi(y) \right| d\sigma_s(y) \right\}^{p} d\sigma_s(x)
\]
\[
\times \left\{ \int \frac{dh}{h} h^{-aq} \left\| (\mathcal{H}_{\kappa}(\psi) f) \left( \frac{h}{t}, \cdot \right) \right\|_{L^p_k}^{q} \right\}^{\frac{1}{q}}
\]
\[
= \frac{M_k}{C_{\kappa,\phi\rho}} \int \frac{d\sigma_s(t)}{t^{2k+1}} \left\{ \int \left| \tau_h^\kappa \psi(y) - \psi(y) \right| d\sigma_s(y) \right\}^{p} d\sigma_s(x)
\]
\[
\times \left\{ \int \frac{da}{a} a^{-aq} \left\| (\mathcal{H}_{\kappa}(\psi) f)(a, \cdot) \right\|_{L^p_k}^{q} \right\}^{\frac{1}{q}}
\]
\[
= \frac{M_k}{C_{\kappa,\phi\rho}} \int \frac{dt}{t^{1+\alpha}} \left\{ \int \left| \tau_h^\kappa \psi(y) - \psi(y) \right| d\sigma_s(y) \right\}^{p} d\sigma_s(x)
\]
\[
\times \left\{ \int \frac{da}{a} a^{-aq} \left\| (\mathcal{H}_{\kappa}(\psi) f)(a, \cdot) \right\|_{L^p_k}^{q} \right\}^{\frac{1}{q}}
\]
Using Lemma 2.1, we obtain

\[
\left\{ \int_0^{\infty} \frac{dh}{h} h^{-aq} w_{p,k}(f, h)^q \right\}^{\frac{1}{q}} \leq \frac{M_k}{C_k \rho^k} \left( \frac{M_k + 1}{\alpha} \right) \left( \int_0^{\infty} \frac{dt}{t^{1/a}} \right)^{\frac{1}{q}} \left( \int_0^{\infty} \frac{dt}{t^{1/a}} \right)^{\frac{1}{q}} \times \left( \int_0^{\infty} \frac{da}{a} a^{-\alpha} \left\| (\mathcal{H}_p^k)(a, \cdot) \right\|_{L_d^c}^q \right)^{\frac{1}{q}}.
\]

Corollary 4.3. Let \( f \in \mathcal{B}_{\alpha, q, p}^c (\mathbb{R}) \) \( (p, q > 1, 0 < \alpha < 1) \), then

\[
\| f \|_{\mathcal{B}_{\alpha, q, p}^c (\mathbb{R})} = \| f \|_{L^q_d (\mathbb{R})} + | f |_{\mathcal{B}_{\alpha, q, p}^c (\mathbb{R})}
\]

where \( | f |_{\mathcal{B}_{\alpha, q, p}^c (\mathbb{R})} \) is equal to

\[
| f |_{\mathcal{B}_{\alpha, q, p}^c (\mathbb{R})} = \int_0^{\infty} \left( h^{-a} w_{p,k}(\phi)(h) \right)^q \frac{dh}{h} = \int_0^{\infty} \left[ a^{-a} \left\| (\mathcal{H}_p^k)(., a) \right\|_{L_d^c} \right]^q \frac{da}{a}.
\]

Acknowledgement: The research of the second author is supported by University Grants Commission (UGC), grant number: F.No. 16-9 (June 2019)/2019(NET/CSIR), New Delhi, India.

References

[1] S. Ben Saïd, A product formula and a convolution structure for a k-Hankel transform on R. J. Math. Anal. Appl. 463 , no. 2, (2018), 1132-1146.
[2] S. Ben Saïd, M.A. Boubatra, M. Sifi, On the deformed Besov-Hankel spaces. Opuscula Math. 40, no. 2,(2020), 171-207.
[3] O.V. Besov, On some families of functional spaces. Imbedding and extension theorems. (Russian) Dokl. Akad. Nauk SSSR 126,(1959),1163-1165.
[4] O.V. Besov, Investigation of a class of function spaces in connection with imbedding and extension theorems. (Russian) Trudy. Mat. Inst. Steklov. 60,(1961),42-81.
[5] J.J. Betancor, L. Rodríguez-Mesa, On the Besov-Hankel spaces. J. Math. Soc. Japan 50, no. 3,(1998),781-788.
[6] M. Frazier, B. Jawerth, Decomposition of Besov spaces, Indiana Univ. Math. J. 34, no. 4,(1985),777-799.
[7] D.V. Giang, F. Móricz, A new characterization of Besov spaces on the real line. J. Math. Anal. Appl. 189, no. 2,(1995),533-551.
[8] H. Mejjaoli, K. Trimèche, k-Hankel two-wavelet theory and localization operators. Integral Transforms Spec. Funct. 31, no. 8,(2020),620-644.
[9] Ashish Pathak, Dileep Kumar, Besov-Hankel norms in terms of the Continuous Bessel wavelet transform, arXiv:2012.01354.
[10] V. Perrier, C. Basdevant, Besov norms in terms of the continuous wavelet transform. Application to structure functions. Math. Models Methods Appl. Sci. 6, no. 5,(1996),649-664.
[11] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press,(1970).
[12] E.M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, (1993).
[13] A. Zygmund, Trigonometric series: Vols. I, II. Second edition, reprinted with corrections and some additions Cambridge University Press, London-New York, (1968).