THE REPRESENTATIONS OF CYCLOTONIC BMW ALGEBRAS, II

HEBING RUI AND MEI SI

Abstract. In this paper, we go on Rui-Xu's work on cyclotomic Birman-Wenzl algebras $B_{r,n}$ in [19]. In particular, we use the representation theory of cellular algebras in [11] to classify the irreducible $B_{r,n}$-modules for all positive integers $r$ and $n$. By constructing cell filtrations for all cell modules of $B_{r,n}$, we compute the discriminants associated to all cell modules for $B_{r,n}$. Via such discriminants together with induction and restriction functors given in section 5, we determine explicitly when $B_{r,n}$ is semisimple over a field. This generalizes our previous result on Birman-Wenzl algebras in [17].

1. Introduction

Let $B_{r,n}$ be the cyclotomic Birman-Wenzl algebras defined in [12]. Motivated by Ariki, Mathas and Rui’s work on cyclotomic Nazarov-Wenzl algebras [4], Rui and Xu [19] proved that $B_{r,n}$ is cellular over $R$ for all positive odd integers $r$ under the so-called $u$-admissible conditions (see the assumption 2.2). Moreover, they have classified the irreducible $B_{r,n}$-modules.

In this paper, we will prove that $B_{r,n}$ is cellular over $R$ for all positive integers $r$ under the $u$-admissible conditions. By using arguments in [19], we classify the irreducible $B_{r,n}$-modules over an arbitrary field. This completes the classification of irreducible $B_{r,n}$-modules over a field. We remark that Yu [20] first proved that $B_{r,n}$ is cellular over $R$ under the similar conditions. However, she did assume that the parameter $\omega_0$, which is given in Definition 2.1, is invertible when she proved that $B_{r,n}$ is cellular.

Given a cell module $M$ of $B_{r,n}$. Following [17], we construct a $B_{r,n-1}$-filtration for $M$. Via it, we construct an $R$-basis for $M$, called JM-basis in the sense of [15]. This enables us to use standard arguments in [15] to construct an orthogonal basis for $M$ under so called separate condition in the sense of [15]. The key is that the Gram determinants associated to $M$ which are defined by the JM-basis and the previous orthogonal basis are the same. We will give a recursive formula to compute the later determinant.

Motivated by [9], we construct restriction functor $F$ and induction functor $G$ which set up a relationship between the category of $B_{r,n}$-modules and the category of $B_{r,n-2}$-modules. Via $F$ and $G$ together with certain explicit formulae on Gram determinants, we determine explicitly when $B_{r,n}$ is semisimple over a field.

We organize the paper as follows. In Section 2, we prove that $B_{r,n}$ is cellular over $R$ for all positive integers $r$ and $n$. We also classify the irreducible $B_{r,n}$-modules. In section 3, we construct the JM-basis and an orthogonal basis for each cell module of $B_{r,n}$. In section 4, we compute the discriminants associated to all cell modules of $B_{r,n}$. Restriction functor $F$ and induction functor $G$ will be constructed in section 5. In section 6, we determine explicitly when $B_{r,n}$ is semisimple over an arbitrary field.

Date: July 25, 2008.

The first author is supported in part by NSFC and NCET-05-0423.
2. The cyclotomic Birman-Wenzl algebras

Throughout the paper, we fix two positive integers \( r \) and \( n \). Let \( R \) be a commutative ring which contains the identity 1 and invertible elements \( q^{\pm 1}, u_1^{\pm 1}, \ldots, u_r^{\pm 1}, q^{\pm 1}, \delta^{\pm 1} \) such that \( \delta = q - q^{-1} \) and \( \omega_0 = 1 - \delta^{-1}(q - q^{-1}) \).

**Definition 2.1.** [12] The cyclotomic Birman-Wenzl algebra \( \mathcal{B}_{r,n} \) is the unital associative \( R \)-algebra generated by \( \{ T_i, E_i, X_j^{\pm 1} \mid 1 \leq i < n \text{ and } 1 \leq j \leq n \} \) subject to the following relations:

a) \( X_i X_i^{-1} = X_i^{-1} X_i = 1 \) for \( 1 \leq i \leq n \).

b) (Kauffman skein relation) \( 1 = T_i^2 - \delta T_i + \delta q E_i \), for \( 1 \leq i < n \).

c) (braid relations)
   
   i) \( T_i T_j = T_j T_i \) if \( |i - j| > 1 \),
   
   ii) \( T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \), for \( 1 \leq i < n - 1 \),
   
   iii) \( T_i X_j = X_j T_i \) if \( j \neq i, i + 1 \).

d) (Idempotent relations) \( E_i^2 = \omega_0 E_i \), for \( 1 \leq i < n \).

e) (Commutation relations) \( X_i X_j = X_j X_i \), for \( 1 \leq i, j \leq n \).

f) (Skein relations)
   
   i) \( T_i X_i - X_{i+1} T_i = \delta X_{i+1} (E_i - 1) \), for \( 1 \leq i < n \),
   
   ii) \( X_i T_i - T_{i+1} X_i = \delta (E_i - 1) X_{i+1} \), for \( 1 \leq i < n \).

g) (Anti–symmetry relations) \( E_i X_1 X_{j+1} = E_i X_{j+1} X_1 \), for \( 1 \leq i < n \).

h) (Untwisting relations)
   
   i) \( E_{i+1} E_i E_{i+1} = E_{i+1} \), for \( 1 \leq i < n - 2 \),
   
   ii) \( E_i E_{i+1} E_i = E_i \), for \( 1 \leq i < n - 2 \).

i) (Untwisting relations)
   
   i) \( E_{i+1} E_i E_{i+1} = E_{i+1} \), for \( 1 \leq i < n - 2 \),
   
   ii) \( E_i E_{i+1} E_i = E_i \), for \( 1 \leq i < n - 2 \).

j) (Cyclotomic relation) \( (X_1 - u_1)(X_1 - u_2) \cdots (X_1 - u_r) = 0 \)

For each \( x \in R \), let

\[
\gamma_r(x) = \begin{cases} 
1, & \text{if } 2 \nmid r, \\
-x, & \text{if } 2 \mid r.
\end{cases}
\]

In the remainder of this paper, We use \( u \) (resp. \( \Omega \)) to denote \( \{u_1, u_2, \ldots, u_r\} \) (resp. \( \{\omega_a \mid a \in \mathbb{Z}\} \)). In order to show that \( \mathcal{B}_{r,n} \) is free over \( R \), Rui and Xu introduced the \( u \)-admissible conditions in [19] 3.15 as follows.

**Assumption 2.2.** \( \Omega \cup \{q\} \) is called **\( u \)-admissible** if

\[
q^{-1} = \alpha \prod_{\ell=1}^r u_\ell, \quad \text{and} \quad \omega_a = \sum_{j=1}^r u_j^a \gamma_j, \forall a \in \mathbb{Z}
\]

where

1. \( \gamma_i = \gamma_r(u_i) + \delta^{-1} q(u_i^2 - 1) \prod_{j \neq i \neq i} u_j \prod_{j \neq i} \frac{u_i u_j - u_i}{u_i - u_j} \).
2. \( \alpha \in \{1, -1\} \) if \( 2 \nmid r \) and \( \alpha \in \{q^{-1}, -q\} \), otherwise.
3. \( \omega_0 = \delta^{-1} q(\prod_{\ell=1}^r u_\ell^2 - 1) + 1 - \frac{(-1)^{r+1}}{2} \alpha^{-1} q^{-1} \).

Note that there are infinite equalities in the definition of \( u \)-admissible conditions in Assumption 2.2. It has been proved in [19] 3.17 that \( \omega_j, \forall j \in \mathbb{Z} \), are determined by \( \omega_i \), \( 0 \leq i \leq r - 1 \). Furthermore, all \( \omega_i \) are elements in \( \mathbb{Z}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}, q^{\pm 1}, \delta^{\pm 1}] \) [19] 3.11. Therefore, \( \omega_i \in R \) for all \( i \in \mathbb{Z} \) if they are given in the Assumption 2.2.

In the remainder of this paper, unless otherwise stated, we always keep the Assumption 2.2 when we discuss \( \mathcal{B}_{r,n} \) over \( R \).
It has been proved in [19] that $\mathcal{B}_{r,n}$ is a free $R$-module with rank $r^n(2f-1)$! when $r$ is odd. We will prove that $\mathcal{B}_{r,n}$ is cellular over $R$ with rank $r^n(2f-1)$! when $r$ is even. We start by recalling the definition of Ariki-Koike algebras in [2].

The Ariki-Koike algebra $\mathcal{H}_{r,n}(\mathfrak{u}) := \mathbb{H}_{r,n}$ is the unital associative $R$-algebra generated by $y_1, \ldots, y_n$ and $g_1, g_2, \ldots, g_{n-1}$ subject to the following relations:

\begin{enumerate}
  \item \((g_i - q)(g_i + q^{-1}) = 0\), if \(1 \leq i \leq n - 1\),
  \item \(g_i g_j = g_j g_i\), if \(|i - j| > 1\),
  \item \(g_i g_{i+1} g_i = g_{i+1} g_i g_i\), for \(1 \leq i < n - 1\),
  \item \(g_i y_j = y_j g_i\), if \(j \neq i, i + 1\),
  \item \(y_i y_j = y_j y_i\), for \(1 \leq i, j \leq n\),
  \item \((y_i - u_1)(y_i - u_2) \cdots (y_i - u_r) = 0\).
\end{enumerate}

Let $\mathcal{E}_n = \mathcal{B}_{r,n}E_1\mathcal{B}_{r,n}$ be the two-sided ideal of $\mathcal{B}_{r,n}$ generated by $E_1$. It is proved in [19, 5.2] that $\mathcal{H}_{r,n} \cong \mathcal{B}_{r,n}/\mathcal{E}_n$. The corresponding $R$-algebraic isomorphism is determined by

\[ \varepsilon_n : y_i \mapsto T_i + \mathcal{E}_n, \quad y_j \mapsto X_j + \mathcal{E}_n, \]

for \(1 \leq i < n\) and \(1 \leq j \leq n\).

Let $\mathfrak{S}_n$ be the symmetric group on \(\{1, 2, \ldots, n\}\). Then $\mathfrak{S}_n$ is generated by $s_i = (i, i+1)$, \(1 \leq i \leq n - 1\). If $w = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_n$ is a reduced expression of $w$, then we write $T_w = T_{i_1}T_{i_2} \cdots T_{i_k} \in \mathcal{B}_{r,n}$. It has been pointed out in [19] that $T_w$ is independent of a reduced expression of $w$. We denote by

\[(2.3) \quad \mathbb{N}_r = \left\{ i \in \mathbb{Z} \mid -\left\lfloor \frac{r}{2} \right\rfloor + \frac{1}{2}(1 + (-1)^r) \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor \right\}.
\]

Given a non-negative integer $f$ with $f \leq \lfloor n/2 \rfloor$. Following [19, 5.5], we define

\[(2.4) \quad D_{f,n} = \left\{ s_{n-2f+1,i} s_{n-2f+2,j} \cdots s_{n-1,i} s_{n,j} \mid 1 \leq i < j \leq n, 1 \leq i_1 < j_1 \leq n-2k+2, 1 \leq k \leq f \right\},
\]

where

\[s_{i,j} = \begin{cases} s_{i-1}s_{i-2} \cdots s_j, & \text{if } i > j, \\ s_is_{i+1} \cdots s_{j-1}, & \text{if } i < j, \\ 1, & \text{if } i = j.\end{cases}\]

Let $\mathfrak{B}_f \subset \mathfrak{S}_n$ be the subgroup generated by $s_{n-2i+2}s_{n-2i+1}s_{n-2i+3}s_{n-2i+2}$, \(2 \leq i \leq f\), and $s_{n-1}$. Then $D_{f,n}$ is a right coset representatives for $\mathfrak{S}_{n-2f} \times \mathfrak{B}_f$ in $\mathfrak{S}_n$ (see e.g. [19]).

For each $d = s_{n-2f+1,i} s_{n-2f+2,j} \cdots s_{n-1,i} s_{n,j} \in D_{f,n}$, let $\kappa_d$ be the $n$-tuple $(k_1, \ldots, k_n)$ such that $k_i \in \mathbb{N}_r$ and $k_i \neq 0$ only for $i = i_1, i_2, \ldots, i_f$. Note that $\kappa_d$ may be equal to $\kappa_e$ although $e \neq d$ for $e, d \in D_{f,n}$. We set $X^{\kappa_d} = \prod_{i=1}^{f} X_i^{\kappa_i}$. By Definition 2.1

\[(2.5) \quad T_{d}X^{\kappa_d} = T_{n-2f+1,i} s_{n-2f+2,j} \cdots T_{n-1,i} X_{i,j},
\]

where $T_{i,j} = T_{s_{i,j}}$. For convenience, let

\[(2.6) \quad \mathbb{N}_r^{f,n} = \{ \kappa_d \mid d \in D_{f,n} \}.
\]

Recall that a composition $\lambda$ of $m$ is a sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots)$ such that $|\lambda| := \lambda_1 + \lambda_2 + \cdots = m$. $\lambda$ is called a partition if $\lambda_i \geq \lambda_{i+1}$ for all positive integers $i$. Similarly, an $r$-partition (resp. $r$-composition) of $m$ is an ordered $r$-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of partitions (resp. compositions) $\lambda^{(s)}$, $1 \leq s \leq r$, such that $|\lambda| := |\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = m$. In the remainder of this paper, we use multipartitions (resp. multicompositions) instead of $r$–partitions (resp. $r$-compositions). Let $\Lambda_+^r(m)$ (resp. $\Lambda_r(m)$) be the set of all multipartitions (resp. multicompositions) of $m$. 

It is known that both $\Lambda^+_r(m)$ and $\Lambda_r(m)$ are posets with the dominance order $\geq$ defined on them. We have $\lambda \leq \mu$ if
\[
\sum_{j=1}^{i-1} |\lambda^{(j)}| + \sum_{k=1}^{l} \lambda_k^{(j)} \leq \sum_{j=1}^{i-1} |\mu^{(j)}| + \sum_{k=1}^{l} \mu_k^{(j)}
\]
for $1 \leq i \leq r$ and $l \geq 0$. We write $\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$. Let
\[
\Lambda^+_{r,n} = \{ (k, \lambda) \mid 0 \leq k \leq \lfloor n/2 \rfloor, \lambda \in \Lambda^+_r(n-2k) \}.
\]
Then $\Lambda^+_{r,n}$ is a poset with $\geq$ as the partial order on it. More explicitly, $(k, \lambda) \geq (\ell, \mu)$ for $(k, \lambda), (\ell, \mu) \in \Lambda^+_{r,n}$ if either $k > \ell$ in the usual sense or $k = \ell$ and $\lambda \geq \mu$. Here $\geq$ is the dominance order defined on $\Lambda^+_r(n-2k)$.

The Young diagram $Y(\lambda)$ of a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a collection of boxes arranged in left-justified rows with $\lambda_i$ boxes in the $i$-th row of $Y(\lambda)$. A $\lambda$-tableau $t$ is obtained from $Y(\lambda)$ by inserting $\{1, \ldots, n\}$ into each box of $Y(\lambda)$ without repetition. If the entries in $t$ increase from left to right in each row and from top to bottom in each column, then $t$ is called a standard $\lambda$-tableau.

If $\lambda = (\lambda(1), \ldots, \lambda(n)) \in \Lambda^+_r(n)$, then the Young diagram $Y(\lambda)$ is an ordered Young diagrams $Y(\lambda(1)), \ldots, Y(\lambda(n))$. In this case, a $\lambda$-tableau $t$ is $(t_1, \ldots, t_r)$ where each $t_i, 1 \leq i \leq r$ is a $\lambda(i)$-tableau. If the entries in each $t_i$ increase from left to right in each row and from top to bottom in each column, then $t$ is called standard. Let $\mathcal{F}^{std}(\lambda)$ be the set of all standard $\lambda$-tableaux.

Suppose $\lambda \in \Lambda^+_r(n)$. It is well-known that $\mathcal{F}^{std}(\lambda)$ is a poset with dominance order $\geq$ on it. For each $s \in \mathcal{F}^{std}(\lambda)$ and a positive integer $i \leq n$, let $s_{\downarrow i}$ be obtained from $s$ by deleting all entries in $s$ greater than $i$. Let $s_i$ be the multipartition of $i$ such that $s_{\downarrow i}$ is the $s_i$-tableau. Then $s \geq t$ if and only if $s_i \geq t_i$ for all $i, 1 \leq i \leq n$. Write $s \succ t$ if $s \geq t$ and $s \neq t$.

It is well-known that $\mathfrak{S}_n$ acts on a $\lambda$-tableau by permuting its entries. Let $t^\lambda$ be the $\lambda$-tableau obtained from $Y(\lambda)$ by adding $1, 2, \cdots, n$ from left to right along the rows of $Y(\lambda(1)), Y(\lambda(2))$, etc. For example, if $\lambda = ((3, 2), (2, 1), (1, 1)) \in \Lambda^+_3(10)$, then
\[
t^\lambda = \begin{array}{ccccccc}
1 & 2 & 3 & & & & \\
4 & 5 & & & & & \\
6 & 7 & & & & & \\
8 & & & & & & \\
9 & & & & & & \\
10 & & & & & & \\
\end{array}
\]

Let $\mathfrak{S}_\lambda$ be the Young subgroup associated to the multipartition $\lambda$. Then $\mathfrak{S}_\lambda$ is the row stabilizer of $t^\lambda$. Let $a_i = \sum_{j=1}^{i} |\lambda^{(j)}|$, $1 \leq i \leq r$ and $a_0 = 0$. For each $\lambda$-tableau $t$, there is a unique element, say $d(t) \in \mathfrak{S}_n$, such that $t = t^\lambda d(t)$. Suppose that $s, t \in \mathcal{F}^{std}(\lambda)$ where $\lambda \in \Lambda^+_r(n-2f)$ for some non-negative integer $f \leq \lfloor n/2 \rfloor$. It is defined in [19, 5.7] that
\[
M_{st} = T^r_{d(s)} \cdot \prod_{s=2}^{r} \prod_{i=1}^{a_{s-1}} (X_i - u_s) \sum_{w \in \mathfrak{S}_\lambda} q^{|w|} T_w \cdot T_d(t),
\]
where $*$ is the $R$-linear anti-involution on $\mathcal{B}_{r,n}$, which fixes $T_i$ and $X_j$, $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Note that
\[
m_{st} := \varepsilon_{n-2f}^{-1}(M_{st} + E_n)
\]
is the Murphy basis element for Ariki-Koike algebra $\mathcal{H}_{r,n-2f}$ in [2].

We define $M_{\lambda} = M_{t^\lambda t^\lambda}$ and $E^{f,n} = E_{n-1}E_{n-3} \cdots E_{n-2f+1}$ and $\mathcal{B}_{r,n} = \mathcal{B}_{r,n} E^{f,n} \mathcal{B}_{r,n}$ for each non-negative integer $f \leq \lfloor n/2 \rfloor$. Therefore, there is a filtration of two-sided ideals of $\mathcal{B}_{r,n}$ as follows:
\[
\mathcal{B}_{r,n} = \mathcal{B}_{r,n}^0 \supset \mathcal{B}_{r,n}^1 \supset \cdots \supset \mathcal{B}_{r,n}^{\lfloor n/2 \rfloor} \supset 0.
\]
Definition 2.10. Suppose that $0 \leq f \leq \left\lfloor \frac{n}{r} \right\rfloor$ and $\lambda \in \Lambda^+_{r}(n-2f)$. Define $\mathcal{B}_{r,n}^{(f,\lambda)}$ to be the two-sided ideal of $\mathcal{B}_{r,n}$ generated by $\mathcal{B}_{r,n}^{f+1}$ and $S$ where

$$S = \{ E_{f,n}^{s,t} | s, t \in \mathcal{F}^{std}(\mu) \text{ and } \mu \in \Lambda^+_{r}(n-2f) \text{ with } \mu \geq \lambda \}. $$

We also define $\mathcal{B}_{r,n}^{(f,\lambda)} = \sum_{\mu \gg \lambda} \mathcal{B}_{r,n}^{(f,\mu)}$, where in the sum $\mu \in \Lambda^+_{r}(n-2f)$.

By Definition 2.11 there is a natural homomorphism from $\mathcal{B}_{r,m}$ to $\mathcal{B}_{r,n}$ for positive integers $m \leq n$. Let $\mathcal{B}_{r,m}$ be the image of $\mathcal{B}_{r,m}$ in $\mathcal{B}_{r,n}$. The following result, which plays the key role, has been proved by Yu without assuming that $\omega_0$ is invertible [20].

Lemma 2.11. $N$ is a right $\mathcal{B}_{r,n}$-module if $N$ is the $R$-submodule generated by $\mathcal{B}_{r,n-2f}^{f,n} E_{f,n}^{s,t} X^{\kappa_d}$, for all $d \in D_{f,n}$ and $\kappa_d \in \mathbb{N}_{r,n}$. 

Proposition 2.12. (cf. [19, 5.10]) Suppose that $s \in \mathcal{F}^{std}(\lambda)$. We define $\Delta_s(f,\lambda)$ to be the $R$-submodule of $\mathcal{B}_{r,n}^{(f,\lambda)} / \mathcal{B}_{r,n}^{(f,\lambda)}$ spanned by

$$\{ E_{f,n}^{s,t} M_{st} D_{f,n} X^{\kappa_d} + \mathcal{B}_{r,n}^{(f,\lambda)} | (t,d,\kappa_d) \in \delta(f,\lambda) \}, $$

where $\delta(f,\lambda) = \{(t,d,\kappa_d) | t \in \mathcal{F}^{std}(\lambda), d \in D_{f,n} \text{ and } \kappa_d \in \mathbb{N}_{r,n}^f \}$. Then $\Delta_s(f,\lambda)$ is a right $\mathcal{B}_{r,n}$-module.

Proof. By Lemma 2.11 $E_{f,n}^{s,t} M_{st} D_{f,n} X^{\kappa_d} + \mathcal{B}_{r,n}^{(f,\lambda)}$ can be written as an $R$-linear combination of elements $M_{st} \mathcal{B}_{r,n-2f}^{f,n} E_{f,n}^{s,t} X^{\kappa_d} + \mathcal{B}_{r,n}^{(f,\lambda)}$ for $e \in D_{f,n}$ and $\kappa_e \in \mathbb{N}_{r,n}^f$. By [19, 5.8d],

$$M_{st} \mathcal{B}_{r,n-2f}^{f,n} E_{f,n}^{s,t} X^{\kappa_e} + \mathcal{B}_{r,n}^{(f,\lambda)} = \Delta_s(f,\lambda), $$

where $\mathcal{B}_{r,n-2f}^{f,n}$ is given in (2.8). Finally, using Dipper-James-Mathas’s result on Murphy basis for Ariki-Koike algebras in [7] yields

$$M_{st} \mathcal{B}_{r,n-2f}^{f,n} E_{f,n}^{s,t} X^{\kappa_e} + \mathcal{B}_{r,n}^{(f,\lambda)} = \Delta_s(f,\lambda). $$

So, $\Delta_s(f,\lambda)$ is a right $\mathcal{B}_{r,n}$-module.

We recall the definition of cellular algebras in [11].

Definition 2.13. [11] Let $R$ be a commutative ring and $A$ an $R$-algebra. Fix a partially ordered set $\Lambda = (\Lambda, \geq)$ and for each $\lambda \in \Lambda$ let $T(\lambda)$ be a finite set. Finally, fix $C^\lambda_{st} \in A$ for all $\lambda \in \Lambda$ and $s, t \in T(\lambda)$.

Then the triple $(\Lambda, T, C)$ is a cell datum for $A$ if:

1. $\{ C^\lambda_{st} | \lambda \in \Lambda \text{ and } s, t \in T(\lambda) \}$ is an $R$-basis for $A$;
2. the $R$-linear map $*: A \rightarrow A$ determined by $(C^\lambda_{st})^* = C^\lambda_{ts}$, for all $\lambda \in \Lambda$ and all $s, t \in T(\lambda)$ is an anti-isomorphism of $A$;
3. for all $\lambda \in \Lambda$, $s \in T(\lambda)$ and $a \in A$ there exist scalars $r_{tu}(a) \in R$ such that

$$C^\lambda_{st} a = \sum_{u \in T(\lambda)} r_{tu}(a) C^\lambda_{su} \quad (mod \; A^{>\lambda}), $$

where $A^{>\lambda} = R$-span $\{ C^\mu_{uv} | \mu \gg \lambda \text{ and } u, v \in T(\mu) \}$. Furthermore, each scalar $r_{tu}(a)$ is independent of $s$. An algebra $A$ is a cell algebra if it has a cell datum and in this case we call $\{ C^\lambda_{st} | s, t \in T(\lambda), \lambda \in \Lambda \}$ a cellular basis of $A$.

Theorem 2.14. Let $\mathcal{B}_{r,n}$ be the cyclotomic Birman–Wenzl algebras over $R$. Then

$$\mathcal{C} = \bigcup_{(f,\lambda) \in \Lambda^+_{r,n}} \{ C^{(f,\lambda)}_{(s,e,\kappa_e)}(t,d,\kappa_d) | (s,e,\kappa_e), (t,d,\kappa_d) \in \delta(f,\lambda) \}$$
is a cellular basis of $\mathcal{B}_{r,n}$ where
$$C^{(f,\lambda)}_{(s,\kappa_\alpha)(t,d,s_\beta)} = X^{s_\beta}T_x^*E^{f,n}M_{st}T_dX^{s_\alpha}.$$ The $R$-linear map $*$, which fixes $T_i, X_j, 1 \leq i \leq n-1$ and $1 \leq j \leq n$ is the required anti-involution. In particular, the rank of $\mathcal{B}_{r,n}$ is $r^n(2n-1))!!$.

**Proof.** This result can be proved by arguments in the proof of [19, 5.41]. We leave the details to the reader. The only difference is that we have to use Proposition 2.12 instead of [19, 5.10]. Finally, we remark that we use seminormal representations for $\mathcal{B}_{r,n}$ in the proof of [19, 5.41]. Such representations have been constructed in [19, 4.19] for all positive integers $r$.

**Remark 2.15.** Yu [20] has proved that $\mathcal{B}_{r,n}$ is cellular under the assumption that $\omega_0$ is invertible. Finally, we remark that Theorem 2.14 for all odd positive integers $r$ has been proved in [19, 5.41].

Let $F$ be an arbitrary field, which contains the non-zero parameters $q, u_1, \ldots, u_r$ and $q - q^{-1}$. Assume that $\Omega \cup \{q\} \subset F$ is $u$-admissible in the sense of the Assumption 2.2. We always keep this assumption when we consider $\mathcal{B}_{r,n}$ over $F$ later on. Let $\mathcal{B}_{r,n,F}$ be the cyclotomic Birman–Wenzl algebra over $F$. By standard arguments, we have
$$\mathcal{B}_{r,n,F} \cong \mathcal{B}_{r,n} \otimes_R F.$$ In the remainder of this paper, we use $\mathcal{B}_{r,n}$ instead of $\mathcal{B}_{r,n,F}$ if there is no confusion.

By using Dipper-Mathas’s Morita equivalent theorem for Ariki-Koike algebras [8], we can assume $u_i = q^{k_i}, k_i \in \mathbb{Z}$ in the following theorem without loss of generality. See the remark in [4, p130].

**Theorem 2.16.** Let $\mathcal{B}_{r,n}$ be the cyclotomic Birman–Wenzl algebra over $F$.

(a) If $n$ is odd, then the non-isomorphic irreducible $\mathcal{B}_{r,n}$-modules are indexed by $(f, \lambda)$ where $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$ and $\lambda$ are $u$-Kleshchev multipartitions of $n - 2f$ in the sense of [3].

(b) Suppose that $n$ is an even number.

(i) If $\omega_i \neq 0$ for some non-negative integers $i \leq r - 1$, then the non-isomorphic irreducible $\mathcal{B}_{r,n}$-modules are indexed by $(f, \lambda)$ where $0 \leq f \leq \frac{n}{2}$ and $\lambda$ are $u$-Kleshchev multipartitions of $n - 2f$.

(ii) If $\omega_i = 0$ for all non-negative integers $i \leq r - 1$, then the set of all pair-wise non-isomorphic irreducible $\mathcal{B}_{r,n}$-modules are indexed by $(f, \lambda)$ where $0 \leq f < \frac{n}{2}$ and $\lambda$ are $u$-Kleshchev multipartitions of $n - 2f$.

**Proof.** When $r$ is odd, this is [19, 6.3]. In general, the result still follows from the arguments in [19, 6.6]. The reason why Rui and Xu had to assume that $2 \nmid r$ in [19, 6.6] is that they did not have Proposition 2.12 for $2 \nmid r$ in [19]. We leave the details to the reader.

We close this section by giving a criterion on $\mathcal{B}_{r,n}$ being quasi-hereditary in the sense of [6].

**Corollary 2.17.** Suppose that $\mathcal{B}_{r,n}$ is defined over the field $F$.

(a) Suppose that $\omega_i \neq 0$ for some $i, 0 \leq i \leq r - 1$. Then $\mathcal{B}_{r,n}$ is quasi-hereditary if and only if $o(q^2) > n$ and $|d| \geq n$ whenever $u_i u_j^{-1} - q^{2d} = 0$ and $d \in \mathbb{Z}$ with $1 \leq i \neq j \leq r$.

(b) Suppose that $\omega_i = 0$ for all $i, 0 \leq i \leq r - 1$. Then $\mathcal{B}_{r,n}$ is quasi-hereditary if and only if $n$ is odd and $o(q^2) > n$ and $|d| \geq n$ whenever $u_i u_j^{-1} - q^{2d} = 0$ and $d \in \mathbb{Z}$ with $1 \leq i \neq j \leq r$.

**Proof.** Note that $\mathcal{B}_{r,n}$ is cellular. By [11, 3.10], $\mathcal{B}_{r,n}$ is quasi-hereditary if and only if the non-isomorphic irreducible $\mathcal{B}_{r,n}$-modules are indexed by $\Lambda_{r,n}$. So, the result follows from Theorem 2.16. In this case, the Ariki-Koike algebras $\mathcal{H}_{r,n-2f}$, $0 \leq f \leq \lfloor n/2 \rfloor$ are semisimple.
3. The JM-basis of \( \Delta(f, \lambda) \)

Throughout this section, we assume that \( \mathcal{B}_{n} \) is defined over a commutative \( R \). The main purpose of this section is to construct the JM-basis for \( \mathcal{B}_{n} \).

**Lemma 3.1.** Suppose that \( n \geq 2 \). We have \( E_{n-1}\mathcal{B}_{n}E_{n-1} = E_{n-1}\mathcal{B}_{n-2} \).

**Proof.** Since \( \mathcal{B}_{n-2}E_{n-1} = E_{n-1}\mathcal{B}_{n-2}E_{n-2}E_{n-1} \subset E_{n-1}\mathcal{B}_{n}E_{n-1} \), we need only to show the inverse inclusion.

By Lemma \( 2.11 \) for \( f = 1 \), we need only prove that \( E_{n-1}hE_{n-1} \in \mathcal{B}_{n-2}E_{n-1} \) for \( h = T_{d}X^{ne}d \in D_{1,n} \). By Definition \( 2.11(b)(c) \), we can assume \( X^{ne} = X^{k} \) for some \( k \in \mathbb{Z} \) without loss of generality.

Note that the Birman-Wenzl algebra \( \mathcal{B}_{1,n} \) is a subalgebra of \( \mathcal{B}_{n} \). The result for \( k = 0 \) follows from the corresponding result for \( \mathcal{B}_{1,n} \) in [5]. Assume that \( k \neq 0 \). We have \( i_{1} = n - 1 \) and \( j_{1} = n \) if \( d = s_{n-1,i_{1}}, s_{n,j_{1}} \). So, \( d = 1 \). By [19] 4.21, \( E_{n-1}X^{k}_{n-1}E_{n-1} = \omega^{(k)}_{n-1} \) for some \( \omega^{(k)}_{n-1} \in \mathcal{B}_{n-2} \). So, \( E_{n-1}\mathcal{B}_{n}E_{n-1} \subset E_{n-1}\mathcal{B}_{n-2} \). 

Using Lemma 3.1 repeatedly yields the following result.

**Corollary 3.2.** \( E^{f,n}\mathcal{B}_{n}E^{f,n} = \mathcal{B}_{n-2f}E^{f,n} \), for all positive integers \( f \leq \left\lceil \frac{n}{2} \right\rceil \).

By Theorem 2.14 \( \mathcal{B}_{n} \) is cellular over the poset \( \Lambda_{n}^{+} \) in the sense of [11]. For each \( (f, \lambda) \in \Lambda_{n}^{+} \), we have the cell module \( \Delta(f, \lambda) \) with respect to the cellular basis of \( \mathcal{B}_{n} \) given in Theorem 2.14. By definition, it is a right \( \mathcal{B}_{n} \)-module which is isomorphic to \( \Delta_{n}(f, \lambda) \) defined in Proposition 2.12. Later on, we will identify \( \Delta(f, \lambda) \) with \( \Delta_{n}(f, \lambda) \) for \( s = t^{\lambda} \). We are going to construct a \( \mathcal{B}_{n} \)-filtration of \( \Delta(f, \lambda) \) by using arguments in [18].

Let \( \sigma_{f} : \mathcal{B}_{n-2f} \longrightarrow \mathcal{B}^{f,n}_{n}/\mathcal{B}^{f+1,n}_{n} \) be the \( R \)-linear map defined by

\[
\sigma_{f}(h) = E^{f,n}_{n-2f}(h) + \mathcal{B}^{f+1,n}_{n},
\]

for all \( h \in \mathcal{B}_{n-2f}, 1 \leq f \leq \left\lceil \frac{n}{2} \right\rceil \). Here \( E^{f,n}_{n-2f} : \mathcal{B}_{n-2f} \rightarrow \mathcal{B}_{n-2f}/\mathcal{B}_{n-2f} \) is the algebraic isomorphism mentioned in section 2.

Given \( \lambda \in \Lambda_{n}^{+}(n) \) and \( \mu \in \Lambda_{r}(n) \). A \( \lambda \)-tableau \( S \) is of type \( \mu \) if it is obtained from \( Y(\lambda) \) by inserting the entries \( (k, i) \) with \( i \geq 1 \) and \( 1 \leq k \leq r \) such that the number of the entries in \( S \) which are equal to \( (k, i) \) is \( \mu_{i}^{(k)} \).

For any \( s \in \mathcal{T}^{std}(\lambda) \), let \( \mu(s) \) be obtained from \( s \) by replacing each entry \( m \) in \( s \) by \( (k, i) \) if \( m \) is in row \( i \) of the \( k \)-th component of \( t^{\mu} \). Then \( \mu(s) \) is a \( \lambda \)-tableau of type \( \mu \).

Given \( (k, i) \) and \( (\ell, j) \) in \( \{ 1, 2, \ldots, r \} \times \mathbb{N} \), we say that \( (k, i) < (\ell, j) \) if either \( k < \ell \) or \( k = \ell \) and \( i < j \). In other words, \( < \) is the lexicographic order on \( \{ 1, 2, \ldots, r \} \times \mathbb{N} \).

Following [7], we say that \( S = (S^{(1)}, S^{(2)}, \ldots, S^{(f)}) \), a \( \lambda \)-tableau of type \( \mu \), is semi-standard if

\( a) \) the entries in each row of each component \( S^{(k)} \) of \( S \) increase weakly,

\( b) \) the entries in each column of each component \( S^{(k)} \) of \( S \) increase strictly,

\( c) \) for each positive integer \( k \leq r \) no entry in \( S^{(k)} \) is of form \( (f, i) \) with \( \ell < k \).

Let \( \mathcal{T}^{ss}(\lambda, \mu) \) be the set of all semi-standard \( \lambda \)-tableaux of type \( \mu \). Given \( S \in \mathcal{T}^{ss}(\lambda, \mu) \) and \( t \in \mathcal{T}^{std}(\lambda) \). Motivated by [7], write

\[
M_{St} = \sum_{s \in \mathcal{T}^{ss}(\lambda)} M_{st}.
\]

**Lemma 3.5.** (cf. [18] 4.8, 4.11-4.13)
a) For any \( h \in \mathcal{B}_{r,n} \), we have
\[
E^{f,n}h \equiv \sum_{h_1 \in \mathcal{K}_{r,n-2f}} \sum_{d \in \mathcal{D}_{f,n}} \sum_{\kappa_d \in \mathcal{B}_{r,n}^{f,n}} \sigma_f(h_1)T_dX^{\kappa_d} \pmod{\mathcal{B}_{r,n}^{f+1,n}}.
\]

b) For each \( \mu \in \Lambda^+_r(n-2f) \), let \( L^\mu \) be the right \( \mathcal{B}_{r,n} \)-submodule of \( \mathcal{B}_{r,n}^{f,n}/\mathcal{B}_{r,n}^{f+1,n} \) generated by \( E^{f,n}M_\mu \pmod{\mathcal{B}_{r,n}^{f+1,n}} \). Then \( L^\mu \) is the free \( R \)-module generated by \( Y = \{ E^{f,n}M_{\mu}T_dX^{\kappa_d} \pmod{\mathcal{B}_{r,n}^{f+1,n}} \mid S \in \mathcal{Y}_{f}\rangle(\lambda, \mu), (t, d, \kappa_d) \in \delta(f, \lambda), \lambda \in \Lambda^+_r(n-2f) \} \).

c) Suppose that \( (f, \lambda) \in \Lambda^+_r(n-2f) \) with \( f > 0 \). If \( s \in \mathcal{Y}_{f}^{\text{std}}(\mu) \) such that \( \mu \in \Lambda^+_r(n-2f+1) \) and \( \tau = s_{n-2f} \models \lambda \), then \( E^{f,n}T_{n-1,n-2f+1}T_{d(s)}M_\mu \in \mathcal{B}_{r,n}^{\text{p}(f, \lambda)} \).

d) Suppose that \( (f, \lambda) \in \Lambda^+_r(n-2f) \) with \( f > 0 \) and \( h \in E^{f-1,n-1}M_\lambda \mathcal{B}_{r,n-1} \cap \mathcal{B}_{r,n-1}^{f+1,n} \).

Then
\[
E_{n-1}T_{n-1,n-2f+1}h \equiv \sum_{h_1 \in \mathcal{K}_{r,n-2f}} \sum_{d \in \mathcal{D}_{f,n-1}} \sum_{\kappa_d \in \mathcal{B}_{r,n}^{f,n-1}} E^{f,n}M_\lambda\varepsilon_{n-2f}(h_1)T_{n-2f,n}T_dX^{\kappa_d} \pmod{\mathcal{B}_{r,n}^{f+1,n}}.
\]

Proof. One can use arguments in the proof of [18, 4.8] together with Corollary 3.2 to verify (a). (b)-(d) can be proved by arguments in the proof of [18, 4.11-4.13].

Given two multipartitions \( \lambda \) and \( \mu \). We say that \( \mu \) is obtained from \( \lambda \) by adding a box (or node) and write \( \lambda \rightarrow \mu \) if there exists a pair \((s, i)\) such that \( \mu^{(s)}_i = \lambda^{(s)}_i + 1 \) and \( \mu^{(i)}_j = \lambda^{(i)}_j \) for \((t, j)\) \(\neq (s, i)\). In this case, we will also say that \( \lambda \) is obtained from \( \mu \) by removing a box (or node).

**Definition 3.6.** Suppose \( \lambda \in \Lambda^+_r(n-2f) \) with \( s \) removable nodes \( p_1, p_2, \ldots, p_s \) and \( m-s \) addable nodes \( p_{s+1}, p_{s+2}, \ldots, p_m \).

- Let \( \mu^{(s)}_i \in \Lambda^+_r(n-2f-1) \) be obtained from \( \lambda \) by removing the box \( p_i \) for \( 1 \leq i \leq s \).
- Let \( \mu^{(i)}_j \in \Lambda^+_r(n-2f+1) \) be obtained from \( \lambda \) by adding the box \( p_j \) for \( s+1 \leq j \leq m \).

We identify \( \mu^{(s)}_i \) with \( (f, \mu^{(s)}_i) \in \Lambda^+_{r,n-1} \) (resp. \((f-1, \mu^{(s)}_i) \in \Lambda^+_{r,n-1} \)) for \( 1 \leq i \leq s \) (resp. \( s+1 \leq i \leq m \)). So, \( \mu^{(s)}_i \triangleright \mu^{(j)}_j \) for all \( i, j \) with \( 1 \leq i \leq s \) and \( s+1 \leq j \leq m \). We arrange the nodes \( p_i, 1 \leq i \leq m \) such that
\[
\mu^{(s)}_i \triangleright \mu^{(j)}_j \quad \text{for all } i, 1 \leq i \leq m-1
\]
with respect to the partial order \( \trianglelefteq \) on \( \Lambda^+_{r,n-1} \).

For each \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}) \in \Lambda^+_r(n) \), let \( [\lambda] = [a_1, a_2, \ldots, a_r] \) such that \( a_i = \sum_{j=1}^{i} |\lambda^{(j)}|, 1 \leq i \leq r \). Write \( [\mu^{(s)}_i] = [b_1, b_2, \ldots, b_r] \) for \( s+1 \leq i \leq m \). We remark that \( (t, k, l) \in Y(\lambda) \) is in the \( k \)-th row, \( t \)-th column of the \( t \)-th component of \( Y(\lambda) \). When \( 1 \leq i \leq s \), \( \mu^{(s)}_i \) is obtained from \( \lambda \) by removing the box, say \( p_i = (t, k, \lambda^{(t)}_k) \). We define
\[
\left\{
\begin{array}{ll}
\alpha_{p_i} = a_{t-1} + \sum_{j=1}^{k} \lambda^{(j)}_j, & \text{if } 1 \leq i \leq s,
\beta_{p_i} = b_{t-1} + \sum_{j=1}^{s} \mu^{(s)}_i(i)_j, & \text{if } s+1 \leq i \leq m,
\end{array}
\right.
\]
and
\[
\nu^{(s)}_i = \left\{
\begin{array}{ll}
E^{f,n}M_{\lambda}T_{a_{p_i},n}, & \text{if } 1 \leq i \leq s,
E_{n-1}T_{n-1,b_{p_i}}E^{f-1,n-1}M_{\mu^{(s)}_i(i)}, & \text{if } s+1 \leq i \leq m.
\end{array}
\right.
\]
For each positive integer \(i \leq m\), define
\[
(3.10) \quad \delta(\lambda, i) = \{ (s, d, \kappa_d) \mid s \in \mathcal{F}^{sd}(\mu_\lambda(i)), d \in \mathcal{D}_{\ell,n-1}, \text{ and } \kappa_d \in \mathbb{N}_{r}^{\ell,n-1} \},
\]
where \(\ell = f\) (resp. \(\ell = f - 1\)) if \(1 \leq i \leq s\) (resp. \(s + 1 \leq i \leq m\)).

In the remainder of this section, we will keep our previous notation \(\mu_\lambda(i)\). In other words, \(\mu_\lambda(i)\) is obtained from \(\lambda\) by removing (resp. adding) the node \(p_i\) for \(1 \leq i \leq s\) (resp. \(s + 1 \leq i \leq m\)).

**Theorem 3.11.** For any \((f, \lambda) \in \Lambda^{+}_{r,n}\) with \(f \geq 0\), let \(S^{\varphi_{\mu}(i)}\) be the \(R\)-submodule of \(\Delta(f, \lambda)\) generated by \(\{ y_{\mu_{\lambda}(i)} T_{d(t)} T_{e} X^{e} \mid (t, d, \kappa_d) \in \delta(\lambda, i), 1 \leq j \leq i \}\). Then
\[
(0) \subseteq S^{\varphi_{\mu}(1)} \subseteq \cdots \subseteq S^{\varphi_{\mu}(m)} = \Delta(f, \lambda)
\]
is a \(S_{r,n-1}\) filtration of \(\Delta(f, \lambda)\). Further, we have the following \(S_{r,n-1}\)-isomorphism:
\[
\Delta(\ell, \mu_\lambda(i)) \cong S^{\varphi_{\mu}(i)} / S^{\varphi_{\mu}(i-1)}, \quad 1 \leq i \leq m.
\]

**Proof.** When \(f = 0\), each cell module \(\Delta(0, \lambda)\) can be considered as a cell module for \(S_{r,n}\). The result for \(f = 0\) has been given in [3]. In the remainder of the proof, we assume \(f > 0\).

Using arguments in the proof of [18] 4.9, 4.14], we can prove that all \(S^{\varphi_{\mu}(i)}, 1 \leq i \leq m\), are \(S_{r,n-1}\)-modules. Of course, we have to use Lemma 3.3 instead of [18] 4.8, 4.11-4.13]. So, \((0) \subseteq S^{\varphi_{\mu}(1)} \subseteq \cdots \subseteq S^{\varphi_{\mu}(m)}\) is a filtration of \(S_{r,n-1}\)-modules.

Let \(\phi_i : \Delta(\ell, \mu_\lambda(i)) \to S^{\varphi_{\mu}(i)} / S^{\varphi_{\mu}(i-1)}\) be the \(R\)-linear map sending
\[
E^{f,n} M_\lambda = E_{n-1} T_{n-1,n-2f+1} E^{f-1,n-1} M_\lambda T_{n-1,n-2f+1}^* = y^{\lambda}_{\varphi_{\mu}(m)} T_{n-1,n-2f+1}^* \in S^{\varphi_{\mu}(m)}.
\]
Since \(S^{\varphi_{\mu}(m)}\) is a right \(S_{r,n-1}\)-module, \(\Delta(f, \lambda) \subseteq S^{\varphi_{\mu}(m)}\). The inverse inclusion is trivial. This proves our claim. Counting the rank of \(\Delta(f, \lambda)\) forces each \(\phi_i\) to be an \(R\)-linear isomorphism. \(\square\)

We are going to recall the notion of \(n\)-updown tableaux in [4] in order to construct the JM-basis of \(S_{r,n}\).

Fix \((f, \lambda) \in \Lambda^{+}_{r,n}\). An \(n\)-updown \(\lambda\)-tableau, or more simply an updown \(\lambda\)-tableau, is a sequence \(\mathbf{t} = (t_0, t_1, t_2, \ldots, t_n)\) of multipartitions such that \(t_0 = 0\), \(t_n = \lambda\) and \(t_i\) is obtained from \(t_{i-1}\) by either adding or removing a box, for \(i = 1, \ldots, n\). Let \(\mathcal{F}^{ud}(\lambda)\) be the set of all \(n\)-updown \(\lambda\)-tableaux.

Given \(t \in \mathcal{F}^{ud}(\lambda)\) with \((f, \lambda) \in \Lambda^{+}_{r,n}\), define \(f_j \in \mathbb{N}\) by declaring that \(t_j \in \Lambda^{+}_{r,n}(j - 2f_j)\). So, \(0 \leq f_j \leq \frac{j}{2}\).

Motivated by [18], we define \(m_i = m_{\mathbf{t}_i} \in S_{r,n}\) inductively by declaring that \(m_0 = 1\) and
\begin{itemize}
    \item[a)] \(m_t = \sum_{a,s,k} a^{s,k} q^{a_s,k-1}T_{j,a} \prod_{j=1}^{a_s-k} (X_{j+1} - u_{j+1}) T_{a_j,a_{j+1}} T_{a_{s-k}} m_{t_{i-1}} \) if \(t_i = t_{i-1} \cup p\) with \(p = (s, k, \mu_k^{(s)})\) and \(a_{s-k} = a_{s-1} + \sum_{j=1}^{k} \mu_j^{(s)}\).
    \item[b)] \(m_{t_i} = E^{f,n} T_{n-1,b_i} m_{t_{i-1}} \) if \(t_i = t_{i-1} \cup p\) with \(p = (s, k, \mu_k^{(s)})\) and \(E^{f,n} T_{n-1,b_i} m_{t_{i-1}} = b_{s-1} + \sum_{j=1}^{k} \mu_j^{(s)}\).
\end{itemize}
where \( \mu = t_i \) and \( \nu = t_{i-1} \) with \( [\mu] = [a_1, a_2, \ldots, a_r] \) and \( [\nu] = [b_1, b_2, \ldots, b_r] \).

Now, we define \( b_i \) inductively such that

\[
m_1 \equiv E^{f,n}_b M_\lambda b_{n-1} \pmod {R_n(f,\lambda)}.
\]

We write \( m_\lambda \equiv E^{f,n}_b M_\lambda \). Suppose that \( t_{n-1} = \mu \), \( [\lambda] = [a_1, a_2, \ldots, a_r] \) and \( [\mu] = [b_1, b_2, \ldots, b_r] \). We have \( b_{0_0} = 1 \) and

\[
(3.12) \quad b_n = \begin{cases} 
T_{a_k, b_{n-1}} & \text{if } t_n = t_{n-1} \cup \{(t, k, \lambda^{(i)})\}, \\
T_{n-1, b_{n-1} b_{n-1}} & \text{if } t_n = t_{n-1} \cup \{(t, k, \mu^{(i)})\}, \\
T_{n-1, b_{n-1}} \sum_{j=b_{n-1}+1}^{b_{n-1}+1} q^{b_{n-1}+j} T_{b_{n-1} b_{n-1}} & \text{if } t_n = t_{n-1} \cup \{(t, k, \mu^{(i)})\},
\end{cases}
\]

where \( s \neq r \) and

\[
h = \prod_{j=r}^{a_2+2} ((X_{b_j} - u_j) T_{b_{j-1}, b_{j-2}}) \times (X_{b_j} - u_j) T_{b_{j-1}, b_{j-2}} \sum_{j=b_{n-1}+1}^{b_{n-1}+1} q^{b_{n-1}+j} T_{b_{n-1} b_{n-1}}.
\]

We also use \( b_i \) instead of \( b_{n-1} \).

For any \( s, t \in T_n^{ud}(\lambda) \), we identify \( s_i \) (resp. \( t_i \)) with \( (f_i, s_i) \) (resp. \( (f_i, t_i) \)) where \( (f_i, s_i), (f_i, t_i) \in \Lambda_{r,n}, \) \( i = 1, 2, \ldots, r, \lambda \). We write \( s \triangleright t \) if \( s_j > t_j \) and \( s_i = t_i \) for \( j + 1 \leq l \leq n \) and \( j \geq k \). We write \( s \triangleright t \) if there is a positive integer \( k \leq n - 1 \) such that \( s > t \). In \[18\], we have verified that \( s \triangleright v \) if \( s \triangleright t \) and \( t \triangleright v \). So, \( \triangleright \) can be refined to be a linear order on \( T_n^{ud}(\lambda) \).

There is a partial order \( \triangleright \) on \( T_n^{ud}(\lambda) \). More explicitly, we have \( s \triangleright t \) if \( s_i \triangleright t_i \), \( 1 \leq i \leq n \). We write \( s \triangleright t \) if \( s \triangleright t \) and \( s \neq t \).

There is a unique element, say \( t^k_\lambda \in T_n^{ud}(\lambda) \), which is maximal with respect to \( \triangleright \). More explicitly, we have \( t^k_\lambda = (1, 2, \ldots, 1) \) for \( 1 \leq i \leq f \) and \( t^k_\lambda = (1, 2, \ldots, 1) \) for \( 1 \leq i \leq f \) and \( j = 2t - 1 \).

Let \( E_i = E_{i+1} \cdots E_{j-1} \) for \( i < j \). If \( i = j \), we set \( E_i = 1 \). When \( i > j \), we define \( E_i = E_{i-1} E_{i-2} \cdots E_j \). So,

\[
(3.13) \quad m_\lambda = E^{f,n}_b M_\lambda \prod_{i=1}^f E_{n-2(f-i)} (F_{i-1})^{-1} \prod_{j=2}^r \prod_{k=1}^f (X_{2k-1} - u_j).
\]

Suppose \( t \in T_n^{ud}(\lambda) \) with \( (f, \lambda) \in \Lambda_{r,n}^+ \). Let

\[
(3.14) \quad c_t(k) = \begin{cases} 
 u_s q^{2(j-i)} & \text{if } t_k = (s, i, j), \\
u_s^{-1} q^{-2(j-i)} & \text{if } t_{k-1} = (s, i, j),
\end{cases}
\]

and

\[
(3.15) \quad c_\lambda(p) = \begin{cases} 
 u_s q^{2(j-i)} & \text{if } p = (s, i, j) \text{ is an addable node of } \lambda, \\
u_s^{-1} q^{-2(j-i)} & \text{if } p = (s, i, j) \text{ is a removable node of } \lambda.
\end{cases}
\]

In the remainder of this paper, unless otherwise stated, we always use \( m_t \) instead of \( m_t + R_n^{\triangleright}(f, \lambda) \in \Delta(f, \lambda) \).

\begin{proposition} \[3.16\]
\begin{enumerate}
\item \( \{ m_t \mid t \in T_n^{ud}(\lambda) \} \) is an \( R \)-basis of \( \Delta(f, \lambda) \) for any \((f, \lambda) \in \Lambda_{r,n}^+ \).
\item \( m_t (\prod_{i=1}^r X_i) = \prod_{k=1}^r c_t(k) m_t, \forall t \in T_n^{ud}(\lambda). \)
\end{enumerate}
\end{proposition}

\begin{proof}
\begin{enumerate}
\item \( m_t \) follows immediately from Theorem 3.11. 
\item We consider \( R_n \) over the field of fraction of \( R_0 \) where \( R_0 = \mathbb{Z}[u_1^\pm, u_2^\pm, \ldots, u_r^\pm, q^\pm, (q - q^{-1})^{-1}] \). Note that we are assuming that \( u_1, u_2, \ldots, u_r, q \) are indeterminates. 
\end{enumerate}
\end{proof}
\[ \prod_{i=1}^{n} X_i \text{ acts on } \Delta(f, \lambda) \text{ as a scalar. This enables us to consider the special case } t = t^\lambda \text{ without loss of generality. By direct computation,} \]
\[ m_{\lambda^i} X_i = \begin{cases} u^{(i-1)\lambda^i} m_{\lambda^i}, & \text{if } 1 \leq i \leq 2f, \\ c_{\lambda^i}(i)m_{\lambda^i}, & \text{if } 2f + 1 \leq i \leq n. \end{cases} \]

So, \( m_{\lambda}(\prod_{i=1}^{n} X_i) = \prod_{k=1}^{n} c_{\lambda^i}(k)m_{\lambda^i} \). By (a), \( m_{\lambda} \) is an \( R_0 \)-basis. So (b) holds over \( R_0 \). Finally, we use standard arguments on base change to get (b) over a commutative ring \( R \). \( \square \)

**Theorem 3.17.** (cf. [18, 5.12]) Let \( t \in \mathcal{T}_n^u(\lambda) \) with \((f, \lambda) \in \Lambda_{\pm}^+ \). For any \( k \), \( 1 \leq k \leq n \), there are some \( u \in \mathcal{T}_n^u(\lambda) \) and \( a_u \in R \) such that
\[ m_{\lambda}X_k = c_{\lambda}(k)m_t + \sum_{u \succ_t} a_u m_u. \]

**Proof.** Note that \( \prod_{k=1}^{n} c_{\lambda^i}(k) = \prod_{k=1}^{n} c_{\lambda^i}(k) \) for any \( t \in \mathcal{T}_n^u(\lambda) \). By Lemma 3.16(b), \( m_{\lambda} \prod_{k=1}^{n} X_k = \prod_{k=1}^{n} c_{\lambda^i}(k)m_{\lambda^i} \). We consider the action of \( \prod_{k=1}^{n} X_i \) on \( m_{\lambda} \). We use the \( R_{r,n-1} \)-filtration of \( \Delta(f, \lambda) \) in Theorem 3.11. By Lemma 3.16(b),
\[ m_{\lambda} \prod_{j=1}^{n-1} X_j - \prod_{j=1}^{n-1} c_{\lambda^i}(j)m_{\lambda^i} \in S_{\pm}^{\lambda(i-1)} \]
where \( \mu_{\lambda}(j), 1 \leq j \leq m \) are defined in Theorem 3.11 with \( \mu_{\lambda}(i) = t_{n-1} \). Since \( S_{\pm}^{\lambda(i-1)} \) is a right \( R_{r,n-1} \)-module,
\[ m_{\lambda}X_{\mu_{\lambda}(n)}^{-1} - m_{\lambda} = m_{\lambda} \prod_{j=1}^{n-1} c_{\lambda}^{-1}(j) \prod_{j=1}^{n-1} X_{j}^{-1} - m_{\lambda} \in S_{\pm}^{\lambda(i-1)}. \]

So, Theorem 3.17 holds for \( k = n \). When we deal with the case \( k = n - 1 \), we consider the filtration of \( R_{r,n-2} \)-submodules of \( S_{\pm}^{\lambda(i)/\lambda(i-1)} \). Note that \( S_{\pm}^{\lambda(i)/\lambda(i-1)} \cong \Delta(\ell, \mu_{\lambda}(i)) \) where \( \Delta(\ell, \mu_{\lambda}(i)) \) is the cell module for \( R_{r,n-1} \) with respect to \( (\ell, \mu_{\lambda}(i)) \in \Lambda_{\pm}^+(n-1) \). By similar arguments as above we can verify the result for \( k = n - 2 \). Using these arguments repeatedly yields the required formula for general \( k \). \( \square \)

Standard arguments prove the following result (cf. [18, 2.7]).

**Theorem 3.19.** For each \( t, s \in \mathcal{T}_n^u(\lambda) \) with \((f, \lambda) \in \Lambda_{\pm}^+ \), let \( m_{st} = b_s^t m_{b_1} \), where \( * : R_{r,n} \to R_{r,n} \) is the \( R \)-linear anti-involution which fixes the generators \( T_i, X_j \) for \( 1 \leq i \leq n-1 \) and \( 1 \leq j \leq n \).

a) \( \mathcal{M} = \{ m_{st} | s, t \in \mathcal{T}_n^u(\lambda), (f, \lambda) \in \Lambda_{\pm}^+ \} \) is a cellular basis of \( R_{r,n} \) over \( R \).

b) \( m_{st}X_k = c_{\lambda}(k)m_{st} + \sum_{u \succ_t} a_u m_{su} \pmod{R_{r,n}^{\lambda}} \).

**Remark 3.20.** Note that \( \succ \) is a linear order on \( \mathcal{T}_n(\lambda) \). So, \( \mathcal{M} \) is a JM-basis and \( \{X_1, \ldots, X_n\} \) is a family of JM-element in the sense of [18, 2.4].

Given two partitions \( \lambda, \mu \), write \( \lambda \ominus \mu \) if either \( \lambda \subset \mu \) and \( \mu \setminus \lambda = p \) for some removable node \( p \) of \( \mu \) or \( \lambda \supset \mu \) and \( \lambda \setminus \mu = p \) for some removable node \( p \) of \( \lambda \).

Given an \( s \in \mathcal{T}_n^u(\lambda) \) and a positive integer \( k < n \), if \( s_k \ominus s_{k+1} \) and \( s_{k+1} \ominus s_k \) are in different rows and in different columns then we define, following [18], \( s_{sk} \) to be the updown \( \lambda \)-tableau
\[ s_{sk} = (s_1, \cdots, s_{k-1}, t_k, s_{k+1}, \cdots, s_n) \]
where \( t_k \) is the multipartition which is uniquely determined by the conditions \( t_k \ominus s_{k+1} = t_{k-1} \ominus s_k \) and \( s_{k-1} \ominus t_k = s_k \ominus s_{k+1} \). If the nodes \( s_k \ominus s_{k-1} \) and \( s_{k+1} \ominus s_k \) are both in the same row, or both in the same column, then \( s_{sk} \) is not defined.
Lemma 3.21. (cf. [13, 5.13]) Suppose that $t \in \mathcal{F}_n^{\text{ud}}(\lambda)$ with $t_{i-2} \neq t_i$ and $t_{s_i-1} < t_i$.

a) If $t_{i-2} \subset t_{i-1} \subset t_i$, then $m_t T_{i-1} = m_{t_{s_i-1}} + \sum_{u \geq t_{s_i-1}} a_u m_u$ for some scalars $a_u \in R$.

b) If $t_{i-2} \supset t_{i-1} \subset t_i$ such that $(\tilde{p}, \ell) > (p, k)$ where $t_{i-2} \setminus t_{i-1} = (p, k, v^{(p)}_k)$, $t_i \setminus t_{i-1} = (\tilde{p}, \ell, \mu^{(\tilde{p})}_\ell)$, $t_{i-2} = \nu$ and $t_i = \mu$, then $m_{t_{i-1}} T_{i-1} = m_{t_{s_{i-1}}} + \sum_{u \geq t_{s_{i-1}}} a_u m_u$ for some scalars $a_u \in R$.

Proof. First, we assume $i = n$. One can prove (a) by verifying $m_{T_{n-1}} = m_{t_{s_{n-1}}}$ via (3.12). We leave the details to the reader.

In order to prove (b), write $t_{n-2} \setminus t_{n-1} = (p, k, v^{(p)}_k)$ and $t_n \setminus t_{n-1} = (\tilde{p}, \ell, \mu^{(\tilde{p})}_\ell)$. Let $a = a_{p-1} + \sum_{i=1}^{\ell} \lambda_i^{(\tilde{p})}$, $c = c_{p+1} + \sum_{i=1}^{k} v_i^{(p)}$. Since either $\tilde{p} > p$ or $\tilde{p} = p$ and $\ell > k$, we have $a \geq c$.

First, we assume $p < r$, then

$$m_1 = E^{f,n} M_{1,n} T_{n-2,c_{r-1}} A b_{n-2} + \mathcal{B}^{p,(f,\lambda)}_{r,n}$$

where

$$A = \prod_{j=r}^{p+2} (X_{c_{j-1}} - u_j) T_{c_{j-1},c_{j-2}} \times (X_{c_{p}} - u_{p+1}) T_{c_{p},c} \sum_{c=p,k-1} c q^{-j} T_{c_{j}}. \tag{3.22}$$

We prove (b) by induction on $\tilde{p}$.

If $\tilde{p} = r$, then $a \geq c_{r-1}$. It is routine to verify $m_{T_{n-1}} = m_{t_{s_{n-1}}}$.

If $\tilde{p} = r - 1$, then $c_{r-2} \leq a \leq c_{r-1}$. We have

$$m_{T_{n-1}} = E^{f,n} M_{\lambda} T_{n-1,c_{r-1}} + T_{a,c_{r-1}} \{ (X_{c_{r-1}+1} - u_r) T_{c_{r-1}+1,c_{r-2}} + \delta X_{c_{r-1}+1} E_{c_{r-1}} T_{c_{r-1},c_{r-2}} - \delta X_{c_{r-1}+1} E_{c_{r-1}+1} T_{c_{r-1},c_{r-2}} \} A \times T_{c_{r-1}+1,n-1} b_{n-2} \text{ (mod } \mathcal{B}^{p,(f,\lambda)}_{r,n}) \tag{3.23}$$

Since $T_{n-1,c_{r-1}+1} X_{c_{r-1}+1} T_{c_{r-1}+1,n-1} = X_{n-1}$, the third term on the right hand side of (3.23) is equal to

$$h := \delta \sum_{j=a_{p-1}+1}^{a} q^{-j} T_{j,a} T_{a,c} E^{f,n} M_{\lambda} b_{n-2} X_{n-1}$$

with $\nu = n-2$. Since we are assuming that $\nu \succ \lambda$, $h \in \mathcal{B}^{p,(f,\lambda)}_{r,n}$.

The first term on the right hand side of the above equality is equal to $m_{t_{s_{n-1}}}$.

One can verify it by arguments in the proof of [13, 5.13]. We leave the details to the reader.

Finally we consider the second term $h_1$ on the right hand side of (3.23).

Since $T_{a,c_{r-1}}, X_{c_{r-1}}^{-1} = X_{a}^{-1} T_{c_{r-1},a}$ and $E^{f,n} T_{n-1,c_{r-1}+1} E_{c_{r-1}} T_{c_{r-1},c_{r-1}+1,n-1} = E^{f,n} T_{c_{r-1},n} T_{n-2,c_{r-1}}, \delta^{-1} h_1$ is equal to

$$E^{f,n} M_{\lambda} X_{a}^{-1} T_{c_{r-1},a} T_{c_{r-1},n} T_{n-2,c_{r-1}} A b_{n-2} + \mathcal{B}^{p,(f,\lambda)}_{r,n}$$

$$= c_{\lambda}(a)^{-1} E^{f,n} M_{\lambda} \prod_{j=a_{r-1}}^{c_{r-1}-1} (T_j - \delta) T_{c_{r-1},n} T_{n-2,c_{r-1}} T_{c_{r-1},c_{r-2}} \times A b_{n-2} + \mathcal{B}^{p,(f,\lambda)}_{r,n}. \tag{3.24}$$

Note that $\prod_{j=a_{r-1}}^{c_{r-1}-1} (T_j - \delta) T_{c_{r-1},n}$ can be written as an $R$-linear combination of $T_{\ell,n} h$, with $a \leq \ell \leq c_{r-1}$ and $h \in \mathcal{B}_{r,\ell-1}$. So $\delta^{-1} c_{\lambda}(a) h_1$ can be written as an $R$-linear combination of the following elements

$$E^{f,n} M_{\lambda} T_{\ell,n} T_{n-2,c_{r-1}} T_{c_{r-1},c_{r-2}} A b_{n-2} + \mathcal{B}^{p,(f,\lambda)}_{r,n}. \tag{3.25}$$
Note that $M_{\lambda}T_w \equiv q^{\ell(w)}M_{\lambda} \pmod{(E_1)}$ if $w \in \mathfrak{S}_\lambda$. So, $M_{\lambda}T_{\ell,n-2f} \equiv q^kM_{\lambda}T_{b,n-2f} \pmod{(E_1)}$ for some integers $k,b$ such that $v = t^k s_{b,n-2f}$ is a row standard tableau. Furthermore, since $b \geq \ell \geq a$, $v_{n-2f-1} \geq t_{n-1}$. If $v$ is not standard, we use [14, 3.15] and [19, 5.8] to get

$$E^{f,n}M_{\lambda}T_{\ell,n-2f} = \sum_{s \in \mathcal{F}_{st}(\lambda), s \geq v} a_s E^{f,n}M_{\lambda}T_{d(s)} \pmod{\mathcal{B}_{r,n}^{f,(\alpha, \lambda)}}$$

for some scalars $a_s \in R$. We write $d(s) = sv_{n-2f}d(s')$ where $s'$ is obtained from $s$ by removing the entry $n - 2f$. Since $s \geq v$, $s' \in \mathcal{F}_{st}(\alpha)$ for $\alpha \in \Lambda_+^+(n - 2f - 1)$ with $\alpha \geq v_{n-2f-1} \geq t_{n-1} \geq (ts_{n-1})_{n-1}$. Therefore, $h_1$ can be written as an $R$-linear combination of the elements

$$E^{f,n}M_{\lambda}T_{v,n}T_{d(s')}T_{r-2,cr-1}T_{c_{r-1},cr-2}A b_{t_{n-2}}h \pmod{\mathcal{B}_{r,n}^{f,(\alpha, \lambda)}}$$

Note that $E^{f,n}M_{\lambda}T_{v,n} = y_\alpha$, and $\alpha = \mu_\lambda(i)$ for some $i, 1 \leq i \leq s$. So, the above element can be written as an $R$-linear combination of the elements in $\{m_s \mid s \in \mathcal{F}_{st}(\lambda), s_{n-1} \geq t_{n-1} \geq (ts_{n-1})_{n-1}\}$. In this case, $s \geq t_{s_{n-1}}$.

However, when $\tilde{p} < r - 1$, the first term is not equal to $m_{t_{s_{n-1}}}$. We will use it instead of $m_{t_{s_{n-1}}}T_{n-1}$ to get a similar equality for $i = c_{r-2}$. This will enable us to get three terms. If $\tilde{p} = r - 2$, we will be done since the first term must be $m_{t_{s_{n-1}}}$. The second and the third term can be written as an $R$-linear combination of $m_u$ with $u \geq t_{s_{n-1}}$. In general, we have to repeat the above procedure to get the required formula. This completes the proof of our result under the assumption $p < r$.

Let $p = r$. Note that $a \geq c$. It is routine to check that

$$m_{t_{n-1}}T_{n-1} = m_{t_{s_{n-1}}} \pmod{\mathcal{B}_{r,n}^{f,(\alpha, \lambda)}}.$$

This completes the proof of the result for $i = n$. In general, we use Theorem 3.11 and the definition of $\succ$ to reduce the result to the case for $i = n$. \qed

4. Recursive formulae for Gram determinants

In this section, we assume that $\mathcal{B}_{r,n}$ is defined over a field $F$ such that the following assumptions hold.

Assumption 4.1. Assume that $\mathfrak{u} = (u_1, u_2, \cdots, u_r) \in F^r$ is generic in the sense that $|d| \geq 2n$ whenever there exists $d \in \mathbb{Z}$ such that either $u_i u_j^{\pm 1} = q^d 1_F$ and $i \neq j$, or $u_i = \pm q^d \cdot 1_F$. We will also assume $o(q^d) > n$.

Suppose that $s, t \in \mathcal{F}_{st}(\lambda)$. Under the Assumption 4.1, Rui and Xu have proved that $s = t$ if and only if $c_s(k) = c_t(k), 1 \leq k \leq n$ [19, 4.5]. So, Assumption 4.1 is the separate condition in the sense of [15, 2.8]. This enables us to use standard arguments in [15] to construct an orthogonal basis for $\Delta(f, \lambda)$ as follows.

For each positive integer $k \leq n$, let

$$R(k) = \{c_t(k) \mid t \in \mathcal{F}_{st}(\lambda)\}.$$  

For $s, t \in \mathcal{F}_{st}(\lambda)$, let

a) $F_t = \prod_{k=1}^k F_{t,k}$,  
b) $f_{st} = F_t m_{st} F_i$,  
c) $f_s = m_s F_s \pmod{\mathcal{B}_{r,n}^{f,(\alpha, \lambda)}}$,

where

$$F_{t,k} = \prod_{r \in \mathcal{F}_{st}(\lambda)} \frac{X_k - r}{c_t(k) - r}.  \tag{4.2}$$

The following results hold for a general class of cellular algebras which have JM-bases such that the separate condition holds [15, §3].
Lemma 4.3. Suppose that $t \in \mathcal{T}_n^u(\lambda)$ with $(f, \lambda) \in \Lambda_{r,n}^+$. 

a) $f_t = m_t + \sum_{s \in \mathcal{T}_n^u(\lambda)} a_s m_s$, and $s \nrightarrow t$ if $a_s \neq 0$.
b) $m_t = f_t + \sum_{s \in \mathcal{T}_n^u(\lambda)} b_s f_s$, and $s \nrightarrow t$ if $b_s \neq 0$.
c) $f_t X_k = \alpha(k) f_t$, for any integer $k$, $1 \leq k \leq n$.
d) $f_t F_s = \delta_{s t} f_t$ for all $s \in \mathcal{T}_n^u(\mu)$ with $(\frac{n-|\mu|}{2}, \mu) \in \Lambda_{r,n}^+$.
e) $\{ f_t \mid t \in \mathcal{T}_n^u(\lambda) \}$ is a basis of $\Delta(f, \lambda)$.
f) The Gram determinants associated to $\Delta(f, \lambda)$ defined by $\{ f_t \mid t \in \mathcal{T}_n^u(\lambda) \}$ and the JM-basis in Proposition [4,10] are the same.
g) $\{ f_s \mid s, t \in \mathcal{T}_n^u(\lambda), (f, \lambda) \in \Lambda_{r,n}^+ \}$ is an F-basis of $\mathcal{B}_{r,n}$. Further, we have $f_s f_{st} = \delta_{us}(f_t, f_s) f_{st}$ where $s, t, u, v$ are updown tableaux and $(\ , \ )$ is the invariant bilinear form defined on the cell module $\Delta(f, \lambda)$.

By Lemma 4.3(f), we can compute the Gram determinant associated to $\Delta(f, \lambda)$ by computing each $(f_t, f_t)$, for $t \in \mathcal{T}_n^u(\lambda)$.

Given two $s, t \in \mathcal{T}_n^u(\lambda)$ and a positive integer $k \leq n - 1$. We write $s \stackrel{k}{\sim} t$ if $s_j = t_j$ for $1 \leq j \leq n$ and $j \neq k$.

Definition 4.4. For any $s, t \in \mathcal{T}_n^u(\lambda)$ and a positive integer $k \leq n - 1$, define $T_{ns}(k), E_{ns}(k) \in F$ by declaring that

$$f_t T_k = \sum_{s \in \mathcal{T}_n^u(\lambda)} T_{ns}(k) f_s, \quad f_t E_k = \sum_{s \in \mathcal{T}_n^u(\lambda)} E_{ns}(k) f_s.$$ 

Standard arguments prove the following result (cf. [14, 6.8–6.9]).

Lemma 4.5. Suppose $t \in \mathcal{T}_n^u(\lambda)$ and $(f, \lambda) \in \Lambda_{r,n}^+$.

a) $s \stackrel{k}{\sim} t$ if either $T_{ns}(k) \neq 0$ or $E_{ns}(k) \neq 0$.
b) $f_t E_k = 0$ if $t_{k-1} \neq t_{k+1}$ for any $1 \leq k \leq n - 1$.
c) Assume $t_{k-1} \neq t_{k+1}$.

(i) If $t_k \oplus t_{k-1}$ and $t_k \oplus t_{k+1}$ are in the same row of a component, then $f_t T_k = q f_t$.

(ii) If $t_k \oplus t_{k-1}$ and $t_k \oplus t_{k+1}$ are in the same column of a component, then $f_t T_k = -q^{-1} f_t$.
d) Assume $t_{k-1} = t_{k+1}$.

(i) $f_t E_k = \sum_{s \sim t} E_{ns}(k) f_s$. Furthermore, $(f_s, f_t) E_{ns}(k) = (f_t, f_s) E_{ts}(k)$.

(ii) $f_t T_k = \sum_{s \sim t} T_{ns}(k) f_s$. Furthermore, $T_{ns}(k) = \delta_{c_s(k),c_t(k)} - T_{ns}(k-1)$.

Lemma 4.6. Suppose that $t \in \mathcal{T}_n^u(\lambda)$ with $t_{k-1} \neq t_{k+1}$ and $ts_k \in \mathcal{T}_n^u(\lambda)$. Then $f_t T_k = T_{t^p}(k) f_t + T_{t^p t}(k) f_{ts_k}$, with $T_{t^p}(k) = \frac{c_t(k) - c_t(k+1)}{c_t(k+1) - c_t(k)}$. Suppose one of the following conditions holds:

1. $t_{k-1} \subset t_k \subset t_{k+1}$,
2. $t_{k-1} \supset t_k \subset t_{k+1}$ such that $(\tilde{p}, l) > (p, i)$ where $t_{k-1} \setminus t_k = (p, i, \nu_i^{(p)})$ if $t_{k-1} \setminus t_k = (\tilde{p}, \ell, \mu_{\ell}^{(p)})$, $t_{k-1} = \nu$ and $t_{k+1} = \mu$.

Then

$$T_{t^p t}(k) = \begin{cases} 1 - \frac{c_t(k)}{c_t(k+1)} T_{t^p}(k), & \text{if } ts_k \nrightarrow t, \\ 1, & \text{if } ts_k \prec t. \end{cases}$$

Proof. By defining relation [2,11(f)],

$$f_t T_k X_k - f_t X_{k+1} T_k = \delta_{t_1} X_{k+1} (E_k - 1).$$

Since we are assuming that $t_{k-1} \neq t_{k+1}$, $s \in \{ t, ts_k \}$ if $s \nrightarrow t$. Comparing the coefficients of $f_t$ on both sides of (4.7) and using Lemma 4.3(b) yields the formula for $T_{t^p}(k)$, as required.
First, we assume that \( t \triangleright ts_k \) and \( ts_{k-1} \subset t \subset tk+1 \). By Lemma 4.3(a),

\[
f_t = m_t + \sum_{u \triangleright t} a_u f_u
\]

for some scalars \( a_u \in F \).

By Lemma 3.21(a) and Lemma 4.3(b), \( m_tT_k = m_t + \sum_{u \triangleright ts_k} b_u f_u \) for some scalars \( b_u \in R \). We claim that \( f_{ts_k} \) cannot appear in the expressions of \( f_uT_k \) with non-zero coefficient. Otherwise, \( u \sim ts_k \), forcing \( u \in \{ t, ts_k \} \). This is a contradiction since \( ts_k \prec t \). By Lemma 4.3(b), the coefficient of \( f_{ts_k} \) in \( f_iT_k \) is 1.

Suppose that \( ts_{k-1} \supset t \subset tk+1 \). By Lemma 3.21(b),

\[
m_iT_k = m_{ts_k} + \sum_{u \triangleright ts_k} a_u m_u,
\]

for some scalars \( a_u \in F \). Using 2.1(b) to rewrite the above equality yields

\[
m_iT_k = m_{ts_k} + \sum_{u \triangleright ts_k} a_u m_u + \delta m_t - \delta m_t E_k.
\]

We use Lemma 4.3(b) to write the terms on the right hand side of the above equality as a linear combination of orthogonal basis elements. Since \( ts_k \prec t \), \( f_{ts_k} \) can not appear in the expression of \( \sum_{u \triangleright ts_k} a_u m_u + \delta m_t \).

We claim that \( f_{ts_k} \) can not appear in the expression of \( m_iE_k \). Otherwise, by Lemma 4.3(b), we write \( m_t = \sum_{u \triangleright t} a_u f_u \). Therefore, there is a \( v \) such that \( f_{ts_k} \) appears in the expression of \( f_vE_k \) with non-zero coefficient. So, \( v \sim ts_k \), forcing \( v \triangleright ts_{k-1} \neq v_{k+1} \). Thus \( f_vE_k = 0 \), a contradiction. This completes the proof of our claim. Therefore, the coefficient of \( f_{ts_k} \) in \( m_iT_k \) is 1.

Using Lemma 4.3(b) again, we write \( m_t = f_t + \sum_{u \triangleright t} a_u f_u \) for some scalars \( a_u \in F \). If \( f_{ts_k} \) appears in the expression of \( \sum_{u \triangleright t} a_u f_u T_k \), then \( f_{ts_k} \) must appear in the expression of \( f_u T_k \) for some \( u \). So, \( ts_k \sim u \), forcing \( u \in \{ t, ts_k \} \). This contradicts the fact \( u \triangleright t \). So, the coefficient of \( f_{ts_k} \) in \( f_iT_k \) is 1.

We have proved that

\[
(4.8) \quad f_iT_k = \frac{\delta c_t(k+1)}{c_t(k+1) - c_t(k)} f_t + f_{ts_k},
\]

if \( ts_k \prec t \) and one of conditions (1)-(2) holds. Multiplying \( T_k \) on both sides of (4.8) and using 2.1(b) yields

\[
(4.9) \quad f_{ts_k} T_k = f_t + \delta f_t T_k - \frac{\delta c_t(k+1)}{c_t(k+1) - c_t(k)} f_t T_k \delta f_i E_k.
\]

Note that \( ts_{k-1} \neq ts_k \). By Lemma 4.3(b), \( f_{ts_k} E_k = 0 \). Using (4.8) to simplify (4.9) and switching the role between \( ts_k \) and \( t \) yields the formula for \( f_{ts_k} (k \triangleright t) \) provided \( t_{s_k} > t \) together with one of conditions in (1)-(2) being true. \( \square \)

Note that \( \langle f_i T_k, f_{ts_k} \rangle = \langle f_t, f_{ts_k} T_k \rangle \). By Lemma 4.6 we have the following result immediately.

**Corollary 4.10.** Suppose \( t \in \mathcal{R}_n^u(\lambda) \) with \( (f, \lambda) \in \Lambda_n^+ \) and \( t_{k-1} \neq t_{k+1} \). If \( ts_k \in \mathcal{R}_n^u(\lambda) \), \( ts_k \prec t \) and one of the conditions (1)-(2) in Lemma 4.6 holds, then

\[
\langle f_{ts_k}, f_{ts_k} \rangle = (1 - \frac{\delta^2 c_t(k)c_t(k+1)}{(c_t(k+1) - c_t(k))^2})\langle f_t, f_t \rangle.
\]
Let $a$ be an integer. Let $[a]_q^2 = \frac{q^a - 1}{q^2 - 1}$. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, let $[\lambda]_q^2 = [\lambda_1]_q^2[\lambda_2]_q^2 \cdots [\lambda_k]_q^2$. If $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}) \in \Lambda^+_{r}(n)$, let $[\lambda]_{q^2} = [\lambda^{(1)}]_{q^2}[\lambda^{(2)}]_{q^2} \cdots [\lambda^{(r)}]_{q^2}$.\

**Lemma 4.11.** (cf. [18, 6.15]) Suppose that $(f, \lambda) \in \Lambda^+_{r,n_1}$ and $(f, \mu) \in \Lambda^+_{r,n_2}$. Let $[\lambda] = [a_1, a_2, \ldots, a_r]$ and $[\mu] = [b_1, b_2, \ldots, b_r]$. Then\n
\[
\langle f_\lambda, f_\lambda \rangle = [\lambda]_{q^2} \prod_{j=2}^{2} \prod_{k=1}^{a_{j-1}} (c_\lambda(k) - u_j),
\]
\[
\langle f_\mu, f_\mu \rangle = [\mu]_{q^2} \prod_{j=2}^{2} \prod_{k=1}^{b_{j-1}} (c_\mu(k) - u_j).
\]

Proof. This can be verified by arguments in the proof of [18, 6.11]. We leave the details to the reader. 

**Proposition 4.13.** Suppose that $\lambda \in \Lambda^+_{r}(n-2f)$. Following [18], we define $\mathcal{A}(\lambda)$ (resp. $\mathcal{R}(\lambda)$) to be the set of all admissible (resp. removable) nodes of $\lambda$. Given a removable (resp. an admissible) node $p = (s, k, \lambda_k)$ (resp. $(s, k, \lambda_k+1)$) of $\lambda$, define\n
a) $\mathcal{R}(\lambda)^{<p} = \{(h, l, \lambda_l) \in \mathcal{R}(\lambda) \mid (h, l) > (s, k)\}$,\n
b) $\mathcal{A}(\lambda)^{<p} = \{(h, l, \lambda_l + 1) \in \mathcal{A}(\lambda) \mid (h, l) > (s, k)\}$,

c) $\mathcal{A}(\lambda)^{\geq p} = \{(h, l, \lambda_l) \in \mathcal{A}(\lambda) \mid (h, l) \leq (s, k)\} \cup \{(h, l, \lambda_l + 1) \in \mathcal{A}(\lambda) \mid (h, l) \leq (s, k)\}$.

Following [16], let $\bar{t} = (t_0, t_1, t_2, \ldots, t_{n-1})$ and $\bar{t} = (s_0, s_1, s_2, \ldots, s_{n-1}, t_n)$ with $t_{n-1} = \mu$ and $(s_0, s_1, s_2, \ldots, s_{n-1}) = t^a$ for any $t = (t_0, t_1, t_2, \ldots, t_n) \in \mathcal{F}^{ud}(\lambda)$. Standard arguments prove the following result (cf. [18, 6.15]).

**Proposition 4.14.** Assume that $t \in \mathcal{F}^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda^+_{r,n}$. If $t_{n-1} = \mu$, then\n
\[
\langle f_t, f_t \rangle = \langle f_{t_1}, f_{t_1} \rangle \langle f_{t_2}, f_{t_2} \rangle.
\]

By Proposition 4.14 we can compute $\langle f_t, f_t \rangle$ recursively if we know how to compute $\langle f_{t_1}, f_{t_1} \rangle$. There are three cases which will be given in Propositions 4.15 and 4.18.

**Proposition 4.15.** Suppose that $t \in \mathcal{F}^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda^+_{r,n}$. If $\bar{t} = t^a$ with $t_n = t_{n-1} \cup \{p\}$ and $p = (m, k, \lambda_k^{(m)})$, then\n
\[
\langle f_t, f_t \rangle = \frac{(-1)^{r-m} q^{2k}}{u_m(1-q^2)} \prod_{a \in \mathcal{A}(\lambda)^{<p}} (c_\lambda(a) - c_\lambda(p)^{-1}) \prod_{a \in \mathcal{R}(\lambda)^{<p}} (c_\lambda(a)^{-1} - c_\lambda(p)^{-1}).
\]

Proof. Let $\lambda = [a_1, a_2, \ldots, a_r]$ and $t = t^k s_{a,n}$ where $a = 2f + a_{m-1} + \sum_{j=1}^{k} \lambda_j^{(m)}$. Note that $t < t_{s_{a-1}} < \cdots < t_{s_{a,n}} = t^k$, and $t_n \subset t_{a+1} \subset \cdots \subset t_n$. Applying Corollary 4.10 on the pairs $\{t_{a,s_{a,j}}, t_{a,s_{a,j+1}}\}$, $a \leq j \leq n-1$, we have\n
\[
\langle f_t, f_t \rangle = \langle f_{t_1}, f_{t_1} \rangle \prod_{j=a+1}^{n} (1 - \delta^2 \frac{c_\lambda(j)c_\lambda(a)}{(c_\lambda(j) - c_\lambda(a))^2}).
\]

Simplifying (4.17) via the definition of $c_\lambda(j) a \leq j \leq n$ together with (4.12) yields (4.16).

**Proposition 4.18.** Suppose that $t \in \mathcal{F}^{ud}(\lambda)$ with $\lambda = \Lambda^+_{r,n}$. Let $\bar{t} = (s, \mu_j^{(s)})$ such that $\mu^{(j)} = \emptyset$ for all integers $j$, $s < j \leq r$ and $l(\mu_j^{(s)}) = k$, then\n
\[
\langle f_t, f_t \rangle = [\mu_j^{(s)}]_{q^2} E_t(n-1) \prod_{j=s+1}^{r} (u_j q^{2(\mu_j^{(s)} - k)} - u_j).
\]
Proof. We have

\[ E^{f, n} T_{n-1, n-2f+1} X_n^{k} F_{n} F_{t, n-1} E_{n-1} \]

and

\[ E^{f, n} T_{n-1, n-2f+1} X_n^{k} F_{n} F_{t, n-1} E_{n-1} \]

By [19] 4.27a and Definition 2.1, we can write \( E_{n-1} T_{n-2} X_n^{k} F_{n} F_{t, n-1} E_{n-1} \) as an \( R \)-linear combination of elements \( E_{n-1} g(X_1^\pm, \ldots, X_{n-2}^\pm) \) where \( g(X_1^\pm, \ldots, X_{n-2}^\pm) \) is a polynomial in variables \( X_1^\pm, \ldots, X_{n-2}^\pm \), which is in the center of \( \mathcal{B}_{r, n-2} \). Therefore,

\[
E^{f-1, n} T_{n-2f, n-2} E_{n-1} X_n^{k} F_{n} F_{t, n-1} E_{n-1} \\
= E^{f, n} T_{n-2f, n-2} X_n^{k} F_{n} F_{t, n-1} E_{n-1} \\
= X_n^{k} E_{n-1} \sum_{i=f}^{2} E_{n-2i+2, n-2i} T_{n-2f, n-2f} E_{n-1} T_{n-2, n-2f} (X_1^\pm, \ldots, X_{n-2}^\pm) \\
= E^{f, n} X_n^{k} F_{n} F_{t, n-1} E_{n-1}.
\]

Note that \( f_t = m_t F_t \). Here we use \( m_t \) instead of \( m_t \) (mod \( \mathcal{B}_{r, n}(f, \lambda) \)). By (3.12),

\[
f_t E_{n-1} = E_{n-1} T_{n-1, n-2f+1} m_t T_{n-2f+1, n-1} b_{n-2} F_{n} E_{n-1} \\
= M_k E^{f, n} T_{n-1, n-2f+1} \prod_{j=1}^{r} (X_n^{2f+1} - u_j) \\
\times \sum_{i=a_{j, k-1}+1}^{n-2f+1} q^{n-2f+1-i} T_{n-2f+1, i} T_{n-2f+1, n-1} E_{n-1} b_{n-2} F_{i, k} \\
= M_k E^{f, n} T_{n-1, n-2f+1} \prod_{j=1}^{r} (X_n^{2f+1} - u_j) \\
(1 + T_{n-2f} \sum_{i=a_{j, k-1}+1}^{n-2f} q^{n-2f+1-i} T_{n-2f, i} T_{n-2f+1, n-1} E_{n-1} b_{n-2} F_{i, k}).
\]
By [10, 4.21] and our two equalities in the beginning of the proof, we can find $\Phi_t, \Psi_\ell \in F[X_1^\pm, X_2^\pm, \ldots, X_{n-2}^\pm] \cap Z(\mathcal{B}_{r,n-2})$, $\ell \in \mathbb{Z}$ such that

$$f_t E_{n-1} = E^{f,n} M_\lambda(\Phi_t + \sum_\ell X_\ell^{-1} \Psi_\ell \sum_{i=a_{s,k-1}+1}^{n-2} q^{n-2f+1-i} T_{n-2f,i}) b_{t,n-2} \prod_{k=1}^{n-2} F_{t,k}. $$

More explicitly, $\Phi_t$ is defined by (4.20) as follows:

$$E_{n-1} \prod_{j=s+1}^r (X_1^{-1} - u_j) F_{t,n-1} F_{n-1} = \Phi_t E_{n-1}. $$

By (4.21), we have $f_t E_{n-1} = (\Phi_{t,\lambda} + q [\lambda_k^{(s)}] q^2 \Psi_{t,\lambda}) m_u$ where

$$\Psi_{t,\lambda} \sum_\ell \Psi_{\ell,\lambda} c_\lambda(n - 2f) \Phi_t = \Phi_t E_{n-1}. $$

We compute $\Phi_{t,\lambda}$ and $\Psi_{t,\lambda}$ as follows. By (4.22),

$$\Phi_{t,\lambda} f_t E_{n-1} = f_t E_{n-1} \prod_{j=s+1}^r (X_1^{-1} - u_j) F_{t,n-1} F_{n-1} E_{n-1}$$

$$= E_{tt} (n-1) \prod_{j=s+1}^r (c_1^{-1}(n) - u_j) f_t E_{n-1}. $$

When we get the last equation, we use the fact that $f_s F_{t,n-1} F_{t,n} = 0$ for all $s \in \mathcal{T}_n^{ad}(\lambda)$ with $s \sim t$ and $s \neq t$, which follows from Lemma 4.3(d). So,

$$\Phi_{t,\lambda} = E_{tt} (n-1) \prod_{j=s+1}^r (c_1^{-1}(n) - u_j). $$

Similarly, we can verify

$$\Psi_{t,\lambda} = q E_{tt} (n-1) \prod_{j=s+1}^r (c_1^{-1}(n) - u_j). $$

By (4.22), (4.23),

$$E_{tt} (n-1) = (1 + q^2 [\lambda_k^{(s)}] q^2) E_{tt} (n-1) \prod_{j=s+1}^r (c_1^{-1}(n) - u_j). $$

On the other hand, by similar arguments for $f_{t,\lambda} f_{tt}^\lambda$ in [19, 6.22] for cyclotomic Nazarov-Wenzl algebra, we can verify

$$f_{t,\lambda} f_{tt}^\lambda \equiv E_{uu} (n-1) (f_u, f_v) f_{t,\lambda} \mod \mathcal{B}_{r,n}^{(f,\lambda)},$$

where $u = (u_1, u_2, \cdots, u_{n-2}) \in \mathcal{T}_n^{ad}(\lambda)$. So, $E_{uu} (n-1) (f_u, f_v)$ Note that
In [19, 4.7], Rui and Xu introduced rational functions $W_k(y, s)$ in variable $y$ for any $s \in \mathcal{B}_{n}^{ud}(\lambda)$ such that
\[ f_s E_k \frac{y}{y - X_k} E_k = E_k W_k(y, s). \]
Suppose that $s = t$. By comparing the coefficient of $f_u$ on both sides of the above equality, we have
\[ E_{tu}(n - 1) E_{ut}(n - 1) = E_{tt}(n - 1) E_{uu}(n - 1). \]
Note that $\lbrack \mu_k^{(s)} \rbrack_{q^2} = 1 + q^2 \lbrack \mu_k^{(s)} \rbrack_{q^2}$ and $c_t(n) = u_s^{-1} q^{2(k - \mu_k^{(s)})}$. Therefore,
\[ \frac{\langle f_t, f_t \rangle}{\langle f_{tu}, f_{tu} \rangle} = \frac{E_{tu}(n - 1)\langle f_u, f_u \rangle}{E_{tu}(n - 1)\langle f_{tu}, f_{tu} \rangle} = \frac{E_{tu}(n - 1)\langle f_u, f_u \rangle}{E_{uu}(n - 1)\langle f_{tu}, f_{tu} \rangle} = \frac{E_{uu}(n - 1)\langle f_{tu}, f_{tu} \rangle}{E_{uu}(n - 1)\langle f_{tu}, f_{tu} \rangle} = (1 + q^2 \lbrack \mu_k^{(s)} \rbrack_{q^2})^2 E_{tt}(n - 1) \prod_{j=s+1}^{r} (c_t^{-1}(n) - u_j)^2 \frac{\langle f_v, f_v \rangle}{\langle f_{tu}, f_{tu} \rangle}. \]
By Lemma 4.10, \( \frac{\langle f_s, f_s \rangle}{\langle f_{tu}, f_{tu} \rangle} = \lbrack \mu_k^{(s)} \rbrack_{q^2} \prod_{j=s+1}^{r} (u_s q^{2(\mu_k^{(s)} - k) - u_j})^{-1}. \) So,
\[ \frac{\langle f_t, f_t \rangle}{\langle f_{tu}, f_{tu} \rangle} = \lbrack \mu_k^{(s)} \rbrack_{q^2} E_{tu}(n - 1) \prod_{j=s+1}^{r} (u_s q^{2(\mu_k^{(s)} - k) - u_j}). \]

\[ \square \]

**Proposition 4.24.** Suppose that $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(s)}, \theta, \ldots, \theta) \in \Lambda^+(n - 2f)$ and $l(\lambda^{(s)}) = l$. Let $t \in \mathcal{B}_{n}^{ud}(\lambda)$ with $(f, \lambda) \in \Lambda^+_r$ such that $t = t^u$, and $t_{n-1} = t_n \cup \{ p \}$ with $p = (m, k, \mu^{(m)})$ and $(m, k) < (s, t)$. Let $\mu = [b_1, b_2, \ldots, b_r]$. We define $u = ts_{n, a+1}$ with $a = 2(f - 1) + b_{m-1} + \sum_{j=1}^{k} \mu_j^{(m)}$ and $v = (u_1, \ldots, u_{a+1})$. Then
\[ \frac{\langle f_t, f_t \rangle}{\langle f_{uu}, f_{uu} \rangle} = \lbrack \mu_k^{(m)} \rbrack_{q^2} E_{uv}(u_m q^{-2k} - u_m^{-1} q^{-2(\mu_k^{(m)} - k)})^{-1} A \]
where $A = \prod_{j=m+1}^{r} \frac{(u_m q^{2(\mu_k^{(m)} - k)} - u_j)}{(u_j - u_m^{-1} q^{-2(\mu_k^{(m)} - k)})} \prod_{b \in \pi(\mu) < e} (c_{a}(b) - c_{a}(p)).$

**Proof.** We have $t < ts_{n-1} < \cdots < ts_{n, a+1} = u$, and $v = (u_1, u_2, \ldots, u_{a+1})$. Using Corollary 4.10 repeatedly yields
\[ \frac{\langle f_t, f_t \rangle}{\langle f_{tu}, f_{tu} \rangle} = \lbrack \mu_k^{(s)} \rbrack_{q^2} \prod_{j=s+1}^{r} (u_m q^{2(\mu_k^{(m)} - k) - u_j}). \]
By Propositions 4.15 and 4.18 we have
\[ \frac{\langle f_u, f_u \rangle}{\langle f_{tu}, f_{tu} \rangle} = E_{uv}(u_m q^{-2k} - u_m^{-1} q^{-2(\mu_k^{(m)} - k)})^{-1} A \]
Simplifying (4.27) via the definition of $c_{a}(j)$, $a + 1 \leq j \leq n$ together with (4.27) yields (4.25), as required.

Assume that $(f, \lambda) \in \Lambda^+_r$ and $(l, \mu) \in \Lambda^-_{r,n-1}$. Write $(l, \mu) \to (f, \lambda)$ if either $l = f$ and $\mu$ is obtained from $\lambda$ by removing a removable node or $l = f - 1$ and $\mu$ is obtained from $\lambda$ by adding an addable node. Assume that $\mathcal{B}_{r,n}$ is semisimple. By Theorem 3.11
\[ \Delta(f, \lambda) \downarrow \equiv \bigoplus_{(l, \mu) \to (f, \lambda)} \Delta(l, \mu), \]
where $\Delta(f, \lambda) \downarrow$ is $\Delta(f, \lambda)$ considered as $\mathcal{B}_{r, n-1}$-module. We remark that (4.28) has been proved in [12] over $\mathbb{C}$.

Motivated by [16], we define $\gamma_{\lambda/\mu} \in F$ to be the scalar given by

$$\gamma_{\lambda/\mu} = \frac{(f_1, f_1)}{(f_{\iota'}, f_{\iota'})}$$

(4.29)

where $\iota \in \mathcal{T}_n^d(\lambda)$ with $\iota = \iota' \in \mathcal{T}_n^d(\mu)$. By [18] 5.1,

$$\text{rank} \Delta(f, \lambda) = \frac{r^2 r! n!(2f-1)!!}{(2f)! \prod_{r=1}^r (a_i - a_{i-1})!} \prod_{r=1}^r a_i! \prod_{(k, \ell) \in \lambda} h_k^{(\ell)},$$

(4.30)

where $(f, \lambda) \in \Lambda^+_{r, n}$ and $[\lambda] = [a_1, a_2, \ldots, a_r]$ and $h_k^{(\ell)} = \lambda_k^{(i)} + \lambda_{k+1}^{(i)} - k - \ell + 1$ is the hook length of $(k, \ell)$ in $\lambda^{(i)}$.

Standard arguments prove the following result (cf. [18] 6.38).

**Theorem 4.31.** Let $\mathcal{B}_{r, n}$ be over $R$ where $R = \mathbb{Z}[u^\pm_1, \ldots, u^\pm_r, q^\pm, \delta^{-1}]$ satisfying the assumption [23]. Let $\det G_{f, \lambda}$ be the Gram determinant associated to the cell module $\Delta(f, \lambda)$ of $\mathcal{B}_{r, n}$. Then

$$\det G_{f, \lambda} = \prod_{(l, \mu) \in (f, \lambda)} \det G_{l, \mu} \cdot \gamma_{\lambda/\mu} \in R.$$  

(4.32)

Furthermore, $\text{rank} \Delta(l, \mu)$ is given by (4.30) and each scalar $\gamma_{\lambda/\mu}$ can be computed explicitly by Proposition 4.17, Proposition 4.18 and Proposition 4.22.

We compute $E_{ss}(k)$ for any $s \in \mathcal{T}_n^d(\lambda)$ and $1 \leq k \leq n$. In section 4 of [19], Rui and Xu have constructed the seminormal representations $\Delta(\lambda)$ for $\mathcal{B}_{r, n}$ where $\lambda \in \Lambda^+_{r, n-2f}$. More explicitly, $\Delta(\lambda)$ has a basis $v_s, s \in \mathcal{T}_n^d(\lambda)$. By standard arguments (cf. [16] 3.16), one can verify that $f_s$ constructed in the current section is equal to $v_s$ up to a scalar. Therefore, $E_{ss}(k)$ can be computed by [19] 4.12-4.13.

We list such formulae as follows. Let $\varepsilon \in \{1, -1\}$.

If $r$ is odd and $\vartheta^{-1} = \varepsilon \prod_{i=1}^r u_i$, then

$$E_{ss}(k) = \frac{1}{\vartheta c_s(k)} \left( \frac{c_s(k) - c_s(k)^{-1}}{\delta} + \varepsilon \prod_{\alpha} c_s(k) - c(\alpha)^{-1} \right),$$

where $\alpha$ run over all addable and removable nodes of $s_{k-1}$ with $\alpha \neq s_k \setminus s_{k-1}$.

If $r$ is even and $\vartheta^{-1} = -\varepsilon q^2 \prod_{i=1}^r u_i$, then

$$E_{ss}(k) = \frac{1}{\vartheta} \left( 1 - \frac{q^{-2e}}{c_s(k)^2} \right) \prod_{\alpha} c_s(k) - c(\alpha)^{-1},$$

where $\alpha$ run over all addable and removable nodes of $s_{k-1}$ with $\alpha \neq s_k \setminus s_{k-1}$.

By Propositions 4.14, 4.15, 4.18 and 4.22 together with (4.33)-(4.34), we have the following result immediately.

**Corollary 4.35.** Suppose that $(f, \lambda) \in \Lambda^+_{r, n}$. Let $[\lambda] = [a_1, a_2, \ldots, a_r]$ and $\varepsilon \in \{1, -1\}$. Then

$$\langle f_\lambda, f_\iota \rangle = \frac{[\lambda]!}{\vartheta \delta^2} A \prod_{j=2}^{a_j-1} \prod_{k=1}^{a_j} (c_{s_k}(k) - u_j) \prod_{j=2}^{r} (u_1 - u_j)^{\varepsilon} (u_1 - u_j)^{-1} \varepsilon,$$

where

$$A = \begin{cases} (u_1^{-1} + q^{-e})^{\varepsilon} (-u_1^{-1} + q^e)^{\varepsilon} & \text{if } 2 \nmid r \text{ and } \vartheta^{-1} = \varepsilon \prod_{i=1}^r u_i, \\ (u_1 + q^e)^{\varepsilon} (u_1 - q^e)^{\varepsilon} u_1^{-2} & \text{if } 2 \mid r \text{ and } \vartheta^{-1} = \varepsilon q^{-e} \prod_{i=1}^r u_i. \end{cases}$$
Given an multi-partition of $\lambda$. We denote $\mu$ by $\lambda \cup p$ (resp. $\lambda/p$) if $Y(\mu)$ is obtained from $\lambda$ by adding (resp. removing) the addable (resp. removable) node $p$. Let $p = (i, j, k)$ be the node which is in the $j$th-row, $k$th column of $i$th component of $Y(\lambda)$. We define $p^+ = (i, j, k + 1)$ and $p^- = (i, j + 1, k)$.

In the remainder of this section, we assume that

$$R = \mathbb{Z}[u_1^\pm, u_2^\pm, \ldots, u_r^\pm, q^{\pm1}, \delta^{-1}]$$

such that the assumption 2.2 holds. Let $R_1$ be the multiplicative sub-semigroup of $R$ generated by $1, u_i^\pm, q^\pm, \delta^\pm$ and $u_iu_j^{-1} - q^{2d}$ for integers $i, j, d$ with $|d| < n$ and $1 \leq i, j \leq r$. Let $F_1$ be the field of fraction of $R_1$.

**Theorem 4.36.** Suppose $\lambda \in \Lambda^+_1(n-2)$. Let $r_{\lambda,p,\tilde{p}} = \dim \Delta(0, \lambda \cup p \cup \tilde{p})$ if $\lambda \cup p \cup \tilde{p}$ is an multipartition. If $2 \nmid r$ and $q^{-1} = \varepsilon \prod_{i=1}^n u_i$, we define

$$B = \prod_{\lambda \cup p \cup p^+ \in \Lambda^+_1(n)} (c_\lambda(p) - \varepsilon q^{-1})^{r_{\lambda,p,p^+}} \prod_{\lambda \cup p \cup p^- \in \Lambda^+_1(n)} (c_\lambda(p) + \varepsilon q)^{r_{\lambda,p,p^-}}.$$

Otherwise, we define

$$B = \begin{cases} \prod_{\lambda \cup p \cup p^- \in \Lambda^+_1(n)} (c_\lambda(p)^2 - q^{-2})^{r_{\lambda,p,p^-}}, & \text{if } 2 \nmid r, q^{-1} = q^{-1} \prod_{i=1}^n u_i, \\ \prod_{\lambda \cup p \cup p^+ \in \Lambda^+_1(n)} (c_\lambda(p)^2 - q^{-2})^{r_{\lambda,p,p^+}}, & \text{if } 2 \mid r, q^{-1} = -q \prod_{i=1}^n u_i. \end{cases}$$

Then there is an $A \in R_1$ such that

$$\det G_{1,\lambda} = AB \prod_{\tilde{p} \in \mathcal{A}(\lambda)} (c_\lambda(p)c_\lambda(\tilde{p}) - 1)^{\dim \Delta(0, \lambda \cup p \cup \tilde{p})}. \quad (4.37)$$

**Proof.** Suppose that there are $s$ (resp. $m - s$) addable (resp. removable) nodes $p_1, p_2, \ldots, p_s$ (resp. $p_{s+1}, p_{s+2}, \ldots, p_m$) in $Y(\lambda)$. Let

$$\mu[i] = \begin{cases} \lambda \cup p_i, & \text{if } 1 \leq i \leq s, \\ \lambda/p_i, & \text{if } s + 1 \leq i \leq m. \end{cases}$$

We need (4.38)–(4.39) which can be verified directly. Suppose $s + 1 \leq k \leq m$.

$$(p, p) \in \mathcal{A}(\mu[k]), p \neq \tilde{p})$$

$$\{(p, p) \mid p, \tilde{p} \in \mathcal{A}(\mu[k]), p \neq \tilde{p}\} \cup \{(p, p) \mid p \in \mathcal{A}(\mu[k])\} \cup \{(p, p) \mid p \in \mathcal{A}(\mu[k])\}$$

and

$$(p, p) \in \mathcal{A}(\mu[k]), 1 \leq i \leq s$$

$$\cup \bigcup_{k=s+1}^{n} \{(p, p) \mid p \in \mathcal{A}(\mu[k])\} \cup \bigcup_{k=s+1}^{n} \{(p, p) \mid p \in \mathcal{A}(\mu[k])\}.$$ (4.39)

Now, we prove the result by induction on $n$. It is routine to check (4.37) for the case $n = 2$. Suppose $n \geq 3$. By Theorem 4.31

$$\det G_{1,\lambda} = \prod_{i=1}^n \det G_{0,\mu[i]} \cdot \gamma_{\lambda/\mu[i]}^{\dim \Delta(0, \mu[i])} \prod_{j=s+1}^m \det G_{1,\mu[j]} \cdot \gamma_{\lambda/\mu[j]}^{\dim \Delta(1, \mu[j])}. \quad (4.40)$$

By Proposition 4.15 $\det G_{0,\mu[i]} \in R_1$ and $\gamma_{\mu[i]} \in F_1$ for $1 \leq i \leq s$ and $s + 1 \leq j \leq m$. Suppose $1 \leq i \leq s$. By Propositions 4.18, 4.22

$$\gamma_{\lambda/\mu[i]} = CD \prod_{1 \leq j \neq i \leq s} (c_\lambda(p_i)c_\lambda(p_j) - 1) \prod_{s+1 \leq k \leq m} (c_\lambda(p_i) - c_\lambda(p_k)) \quad (4.41)$$

where $C \in F_1$ and

$$D = \begin{cases} (c_\lambda(p_i) + \varepsilon q)(c_\lambda(p_i) - \varepsilon q^{-1}), & \text{if } 2 \nmid r, q^{-1} = \varepsilon \prod_{i=1}^n u_i, \\ (c_\lambda(p_i)^2 - q^{-2}), & \text{if } 2 \mid r, q^{-1} = q^{-1} \prod_{i=1}^n u_i. \end{cases}$$

By induction assumption, $\det G_{1,\mu[j]}$ can be computed by (4.37) if $s + 1 \leq j \leq m$. We rewrite the terms on the right hand side of (4.40) so as to get $(c_\lambda(p)c_\lambda(\tilde{p}) - 1)^{r_{\lambda,p,\tilde{p}}}$.
5. Induction and Restriction

In this section, we consider \( B_{r,n} \) over a field \( F \). Let \( B_{r,n} \)-mod be the category of right \( B_{r,n} \)-modules. We define two functors

\[
F_n : B_{r,n} \text{-mod} \to B_{r,n-2} \text{-mod}, \quad G_{n-2} : B_{r,n-2} \text{-mod} \to B_{r,n} \text{-mod}
\]

such that

\[
F_n(M) = ME_{n-1} \text{ and } G_{n-2}(N) = N \otimes E_{n-1} B_{r,n},
\]

for all right \( B_{r,n} \)-modules \( M \) and right \( B_{r,n-2} \)-modules \( N \). By Lemma 5.1, \( F_n \) and \( G_{n-2} \) are well-defined. For the simplification of notation, we will omit the subscripts of \( F_n \) and \( G_{n-2} \) later on.

**Lemma 5.1.** Suppose that \( (f, \lambda) \in \Lambda_{r,n}^+ \) and \( (\ell, \mu) \in \Lambda_{r,n+2}^+ \).

1. \( FG = 1 \).
2. \( G(\Delta(f, \lambda)) = \Delta(f + 1, \lambda) \).
3. \( F(\Delta(f, \lambda)) = \Delta(f - 1, \lambda) \).
4. As right \( B_{r,n} \)-modules, \( \text{Hom}_{B_{r,n+2}}(E_{n+1} B_{r,n+2}, \Delta(\ell, \mu)) \cong \Delta(\ell, \mu) E_n + 1 \).
5. \( \text{Hom}_{B_{r,n+2}}(\Delta(f, \lambda), \Delta(\ell, \mu)) \cong \text{Hom}_{B_{r,n}}(\Delta(f, \lambda), F(\Delta(\ell, \mu))) \text{ as } F\text{-modules.} \)

**Proof.** (a) follows from Lemma 3.1 immediately. By standard arguments, we define \( \psi : \Delta(f, \lambda) \otimes E_{n+1} B_{r,n+2} \to \Delta(f + 1, \lambda) \) such that

\[
\psi(E^{f,n} M_{\Delta} + B_{r,n}^{(f,\lambda)} \otimes E_{n+1} h) = E^{f+1,n+2} M_{\lambda} h + B_{r,n+2}^{(f+1,\lambda)}
\]

for \( h \in B_{r,n+2} \). Since \( E^{f+1,n+2} M_{\lambda} \) generates \( \Delta(f + 1, \lambda) \) as \( B_{r,n+2} \)-module, \( \psi \) is an epimorphism. Note that \( E^{f,n} = E^{f,n} E^{f,n-1} E^{f,n} \). We have

\[
\Delta(f, \lambda) \otimes E_{n+1} B_{r,n+2} = (M_{\lambda} E^{f,n} E_{n+1} E^{f,n} + B_{r,n}^{(f,\lambda)}) \otimes E^{f+1,n+2} B_{r,n+2}.
\]

By Lemma 5.8, \( E^{f+1,n+2} B_{r,n+2} \) can be written as \( F \)-linear combination of elements in \( B_{r,n-2} E^{f+1,n+2} T_d X^{\kappa_d} \) where \( d \in \mathbb{D}_{f+1,n+2} \) and \( \kappa_d \in \mathbb{N}^{f+1,n+2} \). By [13]

\[
(M_{\lambda} E^{f,n} E_{n+1} E^{f,n} + B_{r,n}^{(f,\lambda)}) \otimes B_{r,n+2} E^{f+1,n+2} = (M_{\lambda} E^{f,n} B_{r,n-2} + B_{r,n}^{(f,\lambda)}) \otimes E_{n+1}.
\]

Therefore, \( \text{dim}_F(\Delta(f, \lambda) \otimes E_{n+1} B_{r,n+2}) \leq \text{dim}_F(\Delta(f + 1, \lambda)) \). So, \( \psi \) is injective. This completes the proof of (b). (c) follows from (a)-(b), immediately.

We define the \( F \)-linear map \( \phi : \text{Hom}_{B_{r,n-2}}(E_{n+1} B_{r,n+2}, \Delta(\ell, \mu)) \to \Delta(\ell, \mu) E_{n+1} \) such that \( \phi(f) = f(E_{n+1}) \), for \( f \in \text{Hom}_{B_{r,n-2}}(E_{n+1} B_{r,n+2}, \Delta(\ell, \mu)). \) Note that \( f(E_{n+1}) \in \Delta(\ell, \mu) E_{n+1} \). So, \( \phi \) is an epimorphism. Note that any \( f \in \text{Hom}_{B_{r,n-2}}(E_{n+1} B_{r,n+2}, \Delta(\ell, \mu)) \) is determined uniquely by \( f(E_{n+1}) \). So, \( \phi \) is injective. This proves (d). Finally, (e) follows from adjoint associativity and (d).

Given two \( B_{r,n} \)-modules \( M, N \). Let \( \langle M, N \rangle_n = \text{dim}_F \text{ Hom}_{B_{r,n}}(M, N) \). By Lemma 5.1(e), we have the following result immediately.

**Theorem 5.2.** Given \( (f, \lambda) \in \Lambda_{r,n+2}^+ \) and \( (\ell, \mu) \in \Lambda_{r,n+2}^+ \) with \( f \geq 1 \). Then

\[
\langle \Delta(f, \lambda), \Delta(\ell, \mu) \rangle_{n+2} = \langle \Delta(f - 1, \lambda), \Delta(\ell - 1, \mu) \rangle_n.
\]
6. A CRITERION ON $\mathcal{B}_{r,n}$ BEING SEMISIMPLE

In this section, we consider $\mathcal{B}_{r,n}$ over a field $F$. The main purpose of this section is to give a necessary and sufficient condition for $\mathcal{B}_{r,n}$ being semisimple over $F$.

In Propositions 6.1-6.5 we assume $a(q^2) > n$ and $|d| \geq n$ whenever $u_i u_j^{-1} - q^{2d} = 0$ and $d \in \mathbb{Z}$. So, $\mathcal{B}_{r,n}$ is semisimple over $F$. By Theorem 4.3, we describe explicitly when $\det G_{1,\lambda} \neq 0$ for all $\lambda \in \Lambda_n$ where $\Lambda_n$ is defined in Definition 6.4.

**Proposition 6.1.** $G_{1,\varnothing} \neq 0$ if and only if the following conditions hold:

- a) $u_i u_j - 1 \neq 0$ for all $1 \leq i \neq j \leq r$,
- b) $u_i \notin \{-\varepsilon q, \varepsilon q^{-1}\}$ if $2 \nmid r$ and $q^{-1} = \varepsilon \prod_{i=1}^{r} u_i$,
- c) $u_i \notin \{-q^r, q^r\}$ if $2 | r$ and $q^{-1} = \varepsilon q^{-e} \prod_{i=1}^{r} u_i$.

**Proposition 6.2.** Suppose that $n \geq 3$. Let $\lambda \in \Lambda_C^+(n-2)$ with $\lambda^{(m)} = (n-2)$ for some positive integer $m \leq r$. $\det G_{1,\lambda} \neq 0$ if and only if the following conditions hold:

- a) $u_m \notin \{q^{3-n}, -q^{3-n}\}$,
- b) $u_i u_m \notin \{q^{2n-4}, q^{-2}\}$, for all $1 \leq i \leq r$ and $i \neq m$,
- c) $u_i u_j \neq 1$ for all $m \notin \{i, j\}$ and $i \neq j$.
- d) $u_m \notin \{-\varepsilon q^3, q^{-3}, q\}$ and $u_i \notin \{-\varepsilon q, \varepsilon q^{-1}\}$ for all $i \neq m$ if $2 \nmid r$ and $q^{-1} = \varepsilon \prod_{j=1}^{r} u_j$,
- e) $u_m \notin \{-q^3, q^3\}$ and $u_i \notin \{q, -q\}$ for all $i \neq m$ if $2 | r$ and $q^{-1} = q^{-1} \prod_{j=1}^{r} u_j$,
- f) $u_m \notin \{-q^{3-2n}, -q^{3-2n}, -q\}$ and $u_i \notin \{q^{-1}, -q^{-1}\}$ if $2 | r$ and $q^{-1} = -q \prod_{j=1}^{r} u_j$.

**Lemma 6.3.** Suppose that $n \geq 3$. Let $\varepsilon = \pm 1$. Let $\lambda \in \Lambda_C^+(n-2)$ with $\lambda^{(m)} = (1^{n-2})$. $\det G_{1,\lambda} \neq 0$ if and only if the following conditions hold:

- a) $u_m \notin \{q^{n-3}, -q^{n-3}\}$,
- b) $u_i u_m \notin \{q^{2n-4}, q^{-2}\}$, for all $1 \leq i \leq r$ and $i \neq m$,
- c) $u_i u_j \neq 1$ for all $m \notin \{i, j\}$ and $i \neq j$.
- d) $u_m \notin \{-\varepsilon q^3, -q^{-3}, q^{3}, q^{-1}\}$ and $u_i \notin \{-\varepsilon q, \varepsilon q^{-1}\}$ for all $i \neq m$ if $2 \nmid r$ and $q^{-1} = \varepsilon \prod_{j=1}^{r} u_j$,
- e) $u_m \notin \{-q^{3}, q^{3}, q^{-3}, q^{-3}, -q^{-1}, -q^{-1}\}$ and $u_i \notin \{q, -q\}$ for all $i \neq m$ if $2 | r$ and $q^{-1} = q^{-1} \prod_{j=1}^{r} u_j$,
- f) $u_m \notin \{-q^3, q^{-3}\}$ and $u_i \notin \{q^{-1}, -q^{-1}\}$ if $2 | r$ and $q^{-1} = -q \prod_{j=1}^{r} u_j$.

**Definition 6.4.** Fix positive integers $r$ and $n$. Let

$$\Lambda_n = \bigcup_{k=2}^{n} \{\lambda \in \Lambda_C^+(k-2) \mid \lambda^{(i)} \in \{(k-2), (1^{k-2})\} \text{ for some } 1 \leq i \leq r\}$$

**Proposition 6.5.** Suppose that $r \geq 2$ and $n \geq 2$.

- a) Assume $\det G_{1,\varnothing} \neq 0$. Then $\Pi_{\lambda \in \Lambda_n} \det G_{1,\lambda} \neq 0$ if and only if $\mathcal{B}_{r,n}$ is (split) semisimple over $F$.
- b) $\mathcal{B}_{r,n}$ is not semisimple over $F$ if $\det G_{1,\varnothing} = 0$.

**Proof.** By Propositions 6.1-6.3, $\Pi_{\lambda \in \Lambda_n \setminus \Lambda_{n-1}} \det G_{1,\lambda} = 0$ if $\det G_{1,\varnothing} = 0$. This proves (b).

We are going to prove (a) by induction on $n$. When $n = 2$, there is nothing to prove. We assume $n \geq 3$ in the remainder of the proof.

In [11], Graham and Lehrer proved that a cellular algebra is (split) semisimple if and only if no Gram determinant associated to a cell module which is defined by a cellular basis is equal to zero. We use it frequently in the proof of this proposition.
(⇒) If \( \mathcal{R}_{r,n} \) is not semisimple, then \( \det G_{f, \lambda} = 0 \) for some \( (f, \lambda) \in \Lambda^+ \). Under our assumption, \( \mathcal{H}_{r,n} \) is semisimple. Since each cell module \( \Delta(0, \mu/p^\lambda) \) for \( \mathcal{R}_{r,n} \) can be considered as the cell module of \( \mathcal{H}_{r,n} \) with respect to \( \lambda \), so, \( \det G_{0, \lambda} \neq 0 \) for all \( \lambda \in \Lambda^+ \). Therefore, we can assume that \( f > 1 \).

Take an irreducible module \( D^{f, \mu} \subset \text{Rad} \Delta(f, \lambda) \). By general theory about cellular algebras, we know that \( \ell \leq f \). When \( \ell > 1 \), we use Theorem 5.2 to get a non-zero \( \mathcal{R}_{r,n-2} \)-homomorphism from \( \Delta(\ell - 1, \mu) \) to \( \Delta(f - 1, \lambda) \). So, \( \mathcal{R}_{r,n-2} \) is not semisimple. This contradicts to our assumption since \( \Lambda^+ \) is semisimple. By Theorem 3.11, there is a \( \mathcal{R}_{r,n-1} \)-homomorphism from \( \Delta(0, \mu/p^\lambda) \) to \( \Delta(f, \lambda) \) where \( p \) is a removable node of \( \mu \) and \( \mu/p^\lambda \) is obtained from \( \mu \) by removing the removable node \( p \). Here we use classical branching rule for \( \Delta(0, \mu/p^\lambda) \) to \( \Delta(f, \lambda) \) and \( \mu/p^\lambda \) is a composition factor of \( \Delta(f, \lambda) \). Since we are assuming that \( f > 1 \), \( k \geq f - 1 > 0 \). So, \( (0, \mu/p^\lambda) \neq (f, \lambda) \). Therefore, \( \mathcal{R}_{r,n-1} \) is not semisimple.

This contradicts our induction assumption again.

(⇐) By assumption, \( \det G_{1, \lambda} \neq 0 \) for all \( \lambda \in \Lambda_0 \setminus \Lambda_{n-1} \). Suppose that \( \det G_{1, \lambda} = 0 \) for \( \lambda \in \Lambda_{n-1} \). We can find an irreducible module \( D^{f, \mu} \subset \text{Rad} \Delta(1, \lambda) \). We have \( \ell = 0 \). Otherwise, since \( \ell \leq 1 \), we have \( \ell = 1 \). By Theorem 5.2, \( \lambda = \mu \), a contradiction.

If \( n - 2 - \lfloor \lambda \rfloor = 2a \) for some \( a \in \mathbb{N} \), we can use Theorem 5.2 to get a non-zero homomorphism from \( \Delta(a, \mu) \) to \( \Delta(1 + a, \lambda) \). So, \( \det G_{1+a, \lambda} = 0 \), forcing \( \mathcal{R}_{r,n} \) not being semisimple, a contradiction.

Suppose \( n - 2 - \lfloor \lambda \rfloor \) is odd. By Theorem 4.5, we can find a suitable multipartition, say \( \hat{\lambda} \) which is obtained from \( \lambda \) by adding an addable node, such that \( \det G_{1, \hat{\lambda}} = 0 \). First, we assume that \( \lambda \in \Lambda^+(k-2) \) with \( \lambda^{(m)} = k - 2 \) and \( k \leq n - 1 \) without loss of generality. By Proposition 6.2, either \( u_i \in \{ q^a, -q^b \} \) or \( u_i u_j = q^\ell \) for some \( 1 \leq i \neq j \leq r \) and some integers \( a, b, c \). In the first case, we add a box on \( \lambda^{(j)} \) with \( j \neq i \). In the remainder case, we define \( \lambda^{(m)} = (k - 2, 1) \) (resp. \( \lambda^{(m)} = (k - 1, 1) \)) if \( u_i u_m = q^{2k - 2} \) (resp. otherwise).

In each case, \( \hat{\lambda} \in \Lambda^+(k-2) \) and \( \det G_{1, \hat{\lambda}} = 0 \). Since \( n - 2 - \lfloor \hat{\lambda} \rfloor \) is a non-negative even number, we get a contradiction by our previous arguments.

By similar arguments, we get a contradiction if we assume \( \lambda \in \Lambda^+(k-2) \). We leave the details to the reader.

For convenience, we define

\[
Q_{r, \varepsilon} = \begin{cases} 
\{ -\varepsilon q, \varepsilon q^{-1} \}, & \text{if } 2 \nmid r, q^\varepsilon = \varepsilon \prod_{i=1}^r u_i, \\
\{ -q^\varepsilon, q^\varepsilon \}, & \text{if } 2 \mid r, q^\varepsilon = \varepsilon q^{-\varepsilon} \prod_{i=1}^r u_i, 
\end{cases}
\]

and

\[
S_{r, \varepsilon} = \begin{cases} 
\bigcup_{i=1}^n \{ \pm q^{3-k}, \pm q^{k-3}, \pm q^{3-2k}, \pm q^{2k-3} \}, & \text{if } 2 \nmid r, q^\varepsilon = \varepsilon \prod_{i=1}^r u_i, \\
\bigcup_{i=1}^n \{ \pm q^{3-k}, \pm q^{k-3}, \pm q^{3-2k}, \pm q^{2k-3} \}, & \text{if } 2 \mid r, q^\varepsilon = \varepsilon q^{-\varepsilon} \prod_{i=1}^r u_i. 
\end{cases}
\]

**Theorem 6.8.** Let \( n \geq 2 \) and \( r \geq 2 \). Let \( \mathcal{R}_{r,n} \) be defined over the field \( F \) which contains non-zero \( u_i \), \( 1 \leq i \leq r \), \( q, q - q^{-1} \) such that the assumption 2.2 holds.

a) If either \( u_i - u_j^{-1} = 0 \) for different positive integers \( i, j \leq r \) or \( u_i \in Q_{r, \varepsilon} \) for some positive integer \( i \leq r \), then \( \mathcal{R}_{r,n} \) is not semisimple.

b) Assume \( u_i - u_j^{-1} \neq 0 \) for all different positive integers \( i, j \leq r \) and \( u_i \notin Q_{r, \varepsilon} \) for all positive integers \( i \leq r \).
(1) $\mathcal{B}_{r,2}$ is semisimple if and only if $o(q^2) > 2$ and $|d| ≥ 2$ whenever $u_i u_j^{-1} = q^{2d}$ for any $1 ≤ i < j ≤ r$ and $d ∈ \mathbb{Z}$.

(2) Suppose $n ≥ 3$. Then $\mathcal{B}_{r,n}$ is semisimple if and only if
(a) $o(q^2) > n$,
(b) $|d| ≥ n$ whenever $u_i u_j^{-1} = q^{2d}$ for any $1 ≤ i < j ≤ r$ and $d ∈ \mathbb{Z}$,
(c) $u_i ∉ S_{r,g}$,
(d) $u_i u_j ∉ \bigcup_{k=3}^{n} \{q^{1-2k}, q^{2k-4}\}$ for all different positive integers $i, j ≤ r$.

Proof. Each cell module $\Delta(0, \lambda)$ for $\lambda ∈ \Lambda^+(n)$ can be considered as the cell module of $\mathcal{H}_{r,n}$. So, $\mathcal{B}_{r,n}$ is not semisimple over $F$ if $\mathcal{H}_{r,n}$ is not semisimple. Therefore, we can assume $\mathcal{H}_{r,n}$ is semisimple when we discuss the semisimplicity of $\mathcal{B}_{r,n}$. Now, the result follows from Ariki’s result on $\mathcal{H}_{r,n}$ being semisimple in [1] together with Propositions 6.1–6.5.

When $r = 1$, Theorem 6.8 has been proved in [17, 5.9]. We remark that the notation $r$ (resp. $ω$) in [17, 1.1] is the same as $r^{-1}$ (resp. $δ$) in the current paper.

References

[1] S. Ariki, “On the semi-simplicity of the Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$”, J. Algebra 169 (1994), 216–225.
[2] S. Ariki and K. Koike, “A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and construction of its irreducible representations”, Adv. Math. 106 (1994), 216–243.
[3] S. Ariki and A. Mathas, “The number of simple modules of the Hecke algebras of type $G(r, 1, n)$”, Math. Z. 233 (2000), 601–623.
[4] S. Ariki and A. Mathas, H. Rui, “Cyclotomic Nazarov-Wenzl algebras”, Nagoya Math. J., Special issue in honor of Prof. G. Lusztig’s sixty birthday. 182 (2006), 47–134.
[5] J. S. Birman and H. Wenzl, “Braids, link polynomials and a new algebra”, Trans. Amer. Math. Soc. 313 (1989), 249–273.
[6] E. Cline, B. Parshall and L.L. Scott, “Finite dimensional algebras and highest weight categories,”, J. Reine. Angew. Math. 391 (1988), 85-99.
[7] R. Dipper, G. James and A. Mathas, “Cyclotomic $q$-Schur algebras”, Math. Z. 229 (1999), 385–416.
[8] R. Dipper and A. Mathas, “Morita equivalences of Ariki–Koike algebras”, Math. Z. 240 (2002), 579–610.
[9] W. Doran IV, D. Wales and P. Hanlon, “On the semisimplicity of Brauer centralizer algebras”, J. Algebra 211 (1999), 647–685.
[10] F. M. Goodman and H. H. Mosley, “Cyclotomic Birman-Wenzl-Murakami Algebras, I: freeness and realization as tangle algebras”, arXiv:math.QA/0612064.
[11] J. J. Graham and G. I. Lehrer, “Cellular algebras”, Invent. Math. 123 (1996), 1–34.
[12] R. Häring-Oldenburg, “Cyclotomic Birman-Murakami-Wenzl algebras”, J. Pure Appl. Algebra 161 (2001), 113–144.
[13] G. James and A. Mathias, “The Jantzen sum formula for cyclotomic $q$-Schur algebras”, Trans. Amer. Math. Soc. 352 (2000), 5381–5404.
[14] A. Mathias, Hecke algebras and Schur algebras of the symmetric group, Univ. Lecture Notes, 15, Amer. Math. Soc., 1999.
[15] , “Seminormal forms and Gram determinants for cellular algebras”, J. Reine. Angew. Math., to appear.
[16] H. Rui and M. Si, “Discriminants of Brauer algebra”, Math. Zeit., 258 (2008), 925-944.
[17] , “Gram determinants and semisimple criteria for Birman-Murakami-Wenzl algebras”, J. Reine. Angew. Math., to appear.
[18] , “On the structure of cyclotomic Nazarov–Wenzl algebras”, J. Pure Appl. Algebra, 212, no. 10, (2008), 2209-2235.
[19] H. Rui and J. Xu, “The representations of cyclotomic BMW algebras”, Arxiv:0801.0465, 2007.
[20] Shona Yu, “The cyclotomic Birman-Murakami-Wenzl algebras”, Ph.D thesis, Sydney University, 2007.

H. Rui, Department of Mathematics, East China Normal University, 200241 Shanghai, P.R. China.
E-mail address: hbrui@math.ecnu.edu.cn

M. Si, Department of Mathematics, Shanghai Jiaotong University, 200240, Shanghai, P.R. China.
E-mail address: simeism@hotmail.com