Classification of gradient steady Ricci solitons with linear curvature decay

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Abstract In this paper, we study steady Ricci solitons with a linear decay of sectional curvature. In particular, we give a complete classification of 3-dimensional steady Ricci solitons and 4-dimensional \(\kappa\)-noncollapsed steady Ricci solitons with non-negative sectional curvature under the linear curvature decay.

Keywords Ricci flow, steady Ricci solitons, rotational symmetry

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1 Introduction

In his celebrated paper [19], Perelman conjectured that any 3-dimensional \(\kappa\)-noncollapsed and non-flat steady (gradient) Ricci soliton must be rotationally symmetric. The conjecture was solved by Brendle [1] in 2012. For higher dimensional steady Ricci solitons with positive sectional curvature, Brendle [2] also proved that they must be rotationally symmetric if they are asymptotically cylindrical. By verifying the asymptotically cylindrical property, Deng and Zhu [9] recently showed that any higher dimensional \(\kappa\)-noncollapsed steady Ricci solitons with the non-negative curvature operator must be rotationally symmetric, if its scalar curvature \(R(x)\) satisfies

\[
R(x) \leq \frac{C}{\rho(x)}, \quad \forall \rho(x) \geq r_0
\]

(1.1)

for some \(r_0\), where \(\rho(x)\) denotes the distance from a fixed point \(x_0\).

In the present paper, we study steady Ricci solitons with non-negative sectional curvature under the assumption (1.1) even without the \(\kappa\)-noncollapsed condition. By using the Ricci flow method, we prove the following theorem.

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Theorem 1.1. Let \((M, g, f)\) be an \(n\)-dimensional steady Ricci soliton with non-negative sectional curvature and satisfying (1.1). Then the universal cover \((\tilde{M}, \tilde{g}, \tilde{f})\) of \((M, g, f)\) is one of the following:

(i) \((\tilde{M}, \tilde{g})\) is the Euclidean space \((\mathbb{R}^n, g_{\text{Euclid}})\);
(ii) \((\tilde{M}, \tilde{g}) = (\mathbb{R}^2, g_{\text{cigar}}) \times (\mathbb{R}^{n-2}, g_{\text{Euclid}})\), where \(g_{\text{cigar}}\) is the cigar soliton on \(\mathbb{R}^2\) and \(g_{\text{Euclid}}\) is the Euclidean metric on \(\mathbb{R}^{n-2}\);
(iii) \((\tilde{M}, \tilde{g}) = (N^k, g_N, f_N) \times (\mathbb{R}^{n-k}, g_{\text{Euclid}})\) \((k > 2)\), where \((N, g_N)\) is a \(k\)-dimensional steady Ricci soliton with non-negative sectional curvature and positive Ricci curvature. Moreover, the scalar curvature \(R_N(\cdot)\) of \((N^k, g_N)\) satisfies

\[
\frac{C_1^{-1}}{\rho_N(x)} \leq R_N(x) \leq \frac{C_1}{\rho_N(x)}, \quad \forall \rho_N(x) > 1.
\]

Corollary 1.2. Let \((M, g, f)\) be a 3-dimensional steady Ricci soliton which satisfies (1.1). Then the universal cover \((\tilde{M}, \tilde{g}, \tilde{f})\) of \((M, g, f)\) is one of the following:

(i) \((\tilde{M}, \tilde{g})\) is the Euclidean space \((\mathbb{R}^3, g_{\text{Euclid}})\);
(ii) \((\tilde{M}, \tilde{g}) = (\mathbb{R}^2, g_{\text{cigar}}) \times \mathbb{R}\);
(iii) \((\tilde{M}, \tilde{g})\) is the rotationally symmetric Bryant soliton and \((M, g) = (\tilde{M}, \tilde{g})\).

Corollary 1.2 has certain implication on a conjecture of Hamilton. Hamilton conjectured that there should exist a family of collapsed 3-dimensional complete gradient steady Ricci solitons with positive curvature and \(S^1\)-symmetry (see [5]). Our result shows that the curvature of Hamilton’s conjectured examples could not have a linear decay.

In general, it is hard to classify steady Ricci solitons with a linear decay of sectional curvature in higher dimensions. For example, besides Bryant’s Ricci solitons as the rotationally symmetric solutions, Cao [3] constructed a family of \(U(n)\)-invariant steady Ricci solitons with positive sectional curvature and linear curvature decay. Dancer and Wang [8] also constructed many \(\kappa\)-noncollapsed steady Ricci solitons with Ricci curvature \(\text{Ric}(\cdot)\) and curvature tensor \(\text{Rm}(\cdot)\) satisfying

\[
\text{Ric}(x) \geq 0 \quad \text{and} \quad |\text{Rm}(x)| \leq \frac{C}{\rho(x)}, \quad \forall \rho(x) \geq r_0
\]

for some \(r_0 > 0\). However, by following the argument in the proof of Theorem 1.1, we further prove the following theorem.

Theorem 1.3. Let \((M, g, f)\) be an \(n\)-dimensional non-flat steady Ricci soliton with non-negative sectional curvature and normalized scalar curvature

\[
\sup_{x \in M} R = 1.
\]

Then, the universal cover of \((M, g, f)\) is \((\mathbb{R}^2, g_{\text{cigar}}) \times (\mathbb{R}^{n-2}, g_{\text{Euclid}})\) if

\[
R(x) \rho(x) \leq \varepsilon(n), \quad \forall \rho(x) \geq r_0,
\]

where \(r_0\) is a constant and \(\varepsilon(n)\) is a constant depending only on \(n\).

Theorem 1.3 is an improvement of Munteanu et al. [16, Corollary 5.5], where they proved that the universal cover of non-flat steady Ricci solitons with non-negative sectional curvature is \((\mathbb{R}^2, g_{\text{cigar}}) \times (\mathbb{R}^{n-2}, g_{\text{Euclid}})\) if it satisfies

\[
R(x) \rho(x) \to 0, \quad \text{as} \quad \rho(x) \to \infty.
\]

As another application of Theorem 1.1, we can classify non-flat 4-dimensional \(\kappa\)-noncollapsed steady Ricci solitons with the linear decay of non-negative sectional curvature.
Corollary 1.4. Let $(M, g)$ be a non-flat 4-dimensional $\kappa$-noncollapsed steady Ricci soliton with non-negative sectional curvature which satisfies (1.1). Then $(M, g)$ must be rotationally symmetric.

The proof of Corollary 1.4 is reduced to use a result on 4-dimensional steady Ricci solitons with non-negative sectional curvature in [9], where the asymptotically cylindrical property is verified in terms of Brendle [2] under the condition (1.2). In fact, for any $n$-dimensional $\kappa$-noncollapsed steady Ricci solitons, we can weaken the assumptions in Theorem 1.1 to (1.3) to study their asymptotic behavior of level sets $\Sigma_r = \{ x \in M | f(x) = r \}$ (see Theorem 6.1). In particular, when $n = 4$, we can also derive the asymptotically cylindrical property as follows.

Theorem 1.5. Let $(M, g)$ be a 4-dimensional $\kappa$-noncollapsed steady Ricci soliton with positive Ricci curvature. Suppose that there exists $r_0 > 0$ such that

$$|Rm(x)| \leq \frac{C}{\rho(x)}, \quad \forall \rho(x) \geq r_0.$$ 

Then, $(M, g)$ is asymptotically cylindrical. Moreover, the sectional curvature is positive away from a compact set $K$ in $M$.

In view of Corollary 1.4, we guess that the 4-dimensional steady Ricci soliton in Theorem 1.5 is actually a Bryant soliton.

At last, we give the main idea in the proof of Theorem 1.1. By a splitting result (see [12, Theorem 1.1]), we only need to deal with the case (iii) in [9, Lemma 5.1] (also see Lemma 2.1), and to prove the split-off steady Ricci soliton $(N^k, g_N)$ ($k \geq 3$) with non-negative sectional curvature and positive Ricci curvature satisfies (1.2). First, we show that each level set $\Sigma_r$ ($r \gg 1$) of $(N, g_N)$ has a uniform bounded diameter after the scale $r^{-1}$ (see Section 3). The method is to estimate the distance function along $\Sigma_r$ as done in [9]. Next, we show that the rescaled $(\Sigma_{r_i}, r_i^{-1}g_N)$ is an almost flat manifold for any $i \geq i_0$ if there is a sequence of $p_i \in N$ such that $\lim_{i \to \infty} R_N(p_i)\rho_N(p_i) = 0$, where $f_N(p_i) = r_i$ (see Section 4). Then by the famous Gromov’s theorem on the almost flat manifolds (see Theorem 4.1 or a new proof in [20]), we will derive a contradiction since $\Sigma_r$ is diffeomorphic to a sphere [9, Lemma 2.1].

The rest of this paper is organized as follows. In Section 2, we recall previous results for steady Ricci solitons with non-negative sectional curvature in [9] and give a curvature estimate (see Lemma 2.4). In Section 3, we give a diameter estimate of level sets (see Proposition 3.3). Theorems 1.1, 1.3 and 6.1 are proved in Sections 4–6, respectively.

2 Previous results in [9]

$(M, g, f)$ is called a gradient steady Ricci soliton if the Ricci curvature $\text{Ric}(\cdot)$ of $g$ on $M$ satisfies

$$\text{Ric} = \text{Hess}f,$$  \hspace{1cm} (2.1)

for some smooth function $f \in C^\infty(M)$.

In this section, we always assume the steady Ricci soliton $(M, g, f)$ has non-negative Ricci curvature and $M$ admits an equilibrium point $o$ such that $\nabla f(o) = 0$. Note that $R$ is non-negative by Chen’s result in [6]. Then by the identity

$$|
abla f|^2 + R \equiv \text{const},$$ \hspace{1cm} (2.2)

it is easy to see that

$$R(o) = R_{\text{max}} = \max_{x \in M} R(x).$$

The following lemma is due to [9, Lemma 5.1].

Lemma 2.1. Let $(M, g, f)$ be a non-flat steady Ricci soliton with non-negative sectional curvature. Let $S = \{ p \in M \mid \nabla f(p) = 0 \}$ be equilibrium point set of $(M, g, f)$. Suppose that scalar curvature $R$ of $g$ decays uniformly. Then the following statements are true:
(1) \((S,g_S)\) is a compact flat manifold, where \(g_S\) is the induced metric of the metric \(g\).

(2) Let \(o \in S\). Then level set \(\Sigma_r = \{x \in M \mid f(x) = r\}\) is a compact hypersurface of \(M\). Moreover, each \(\Sigma_r\) is diffeomorphic to the other whenever \(r > f(o)\).

(3) \(\{x \in M \mid f(x) \leq r\}\) is compact for any \(r > f(o)\).

(4) \(f\) satisfies
\[c_1\rho(x) \leq f(x) \leq c_2\rho(x), \quad \forall \rho(x) \geq r_0.\] (2.3)

The constants \(c_1\) and \(c_2\) in (2.3) can be estimated more precisely.

**Lemma 2.2.** Let \((M, g, f)\) be a steady Ricci soliton with non-negative Ricci curvature. Suppose that \(f\) satisfies (2.3) and the scalar curvature \(R\) decays uniformly. Then the following statements are true:

(1) \(f(x)\) satisfies
\[\frac{f(x)}{\rho(x)} \to \sqrt{R_{\text{max}}}, \quad \text{as} \quad \rho(x) \to \infty.\] (2.4)

(2) If \(R(x) \leq \frac{C}{\rho(x)}\), then there exist constants \(C_1\) and \(C_2\) such that
\[-C_1\sqrt{\rho(x)} + \sqrt{R_{\text{max}}\rho(x)} \leq f(x) \leq \sqrt{R_{\text{max}}\rho(x)} + C_2, \quad \forall \rho \geq r_0.\] (2.5)

**Proof.** (1) Since \(f\) satisfies (2.3), \(R\) is not identically zero and \(R_{\text{max}} > 0\). Note that \(R\) decays uniformly. Then
\[|\nabla f|(x) > \frac{\sqrt{R_{\text{max}}}}{2}, \quad \forall f(x) \geq r_0.\] (2.6)

We can also choose \(r_1 > 0\) such that \(\Sigma_{r_0} \subseteq B(o, r)\) for all \(r > r_1\) and some fixed point \(o \in M\). Let \(\phi_t\) be generated by \(-\nabla f\). Thus for any \(x \in M \setminus B(o, r)\), there exists \(r_x > 0\) such that \(\phi_{r_x}(x) \in \Sigma_{r_0}\) by (2.6). It follows that there is a \(t_x > 0\) such that \(\phi_{t_x}(x) \in \partial B(o, r)\). By integrating along the curve \(\phi_s(x)\), we have
\[f(x) - f(\phi_{t_x}(x)) = \int_0^{t_x} |\nabla f|^2(\phi_s(x))ds,
\]
and
\[d(x, \phi_{t_x}(x)) \leq \text{Length}(\phi_s(x) |_{[0, t_x]}), g) = \int_0^{t_x} |\nabla f|(\phi_s(x))ds.\]

Since
\[\frac{d|\nabla f|^2}{dt}(\phi_t(x)) = -2\text{Ric}(\nabla f, \nabla f) \leq 0, \quad \forall t \in \mathbb{R},\]
we have
\[|\nabla f|(\phi_s(x)) \geq |\nabla f|(\phi_{t_x}(x)), \quad \forall s \in [0, t_x].\]

Consequently,
\[f(x) - f(\phi_{t_x}(x)) = \int_0^{t_x} |\nabla f|^2(\phi_s(x))ds \geq |\nabla f|(\phi_{t_x}(x)) \int_0^{t_x} |\nabla f|(\phi_s(x))ds \geq |\nabla f|(\phi_{t_x}(x))d(x, \phi_{t_x}(x)).\]

Note that \(\phi_{t_x}(x) \in \partial B(o, r)\). We derive
\[d(x, \phi_{t_x}(x)) \geq \rho(x) - \rho(\phi_{t_x}(x)) = \rho(x) - r.\]
and

\[ |\nabla f|(\phi_t(x)) = \sqrt{R_{\text{max}} - R(\phi_t(x))} \geq \sqrt{R_{\text{max}} - R}, \]

where \( R = \sup_{y \in \partial B(o,r)} R(y) \). Therefore, we obtain

\[ f(x) \geq \sqrt{R_{\text{max}} - R}(\rho(x) - r) + f(r), \tag{2.7} \]

where \( f(r) = \inf_{y \in \partial B(o,r)} f(y) \).

For \( \rho(x) \gg 1 \), we take \( r = \sqrt{\rho(x)} \). Then, by (2.3),

\[ f(x) \geq \sqrt{R_{\text{max}} - R}(\rho(x) - \sqrt{\rho(x)}) + C\sqrt{\rho(x)}. \tag{2.8} \]

On the other hand, by integrating \( f \) along the minimal geodesic \( \gamma(s) \) connecting \( x \) and \( o \), we have

\[ f(x) - f(o) \leq \sqrt{R_{\text{max}} \rho(x)}. \tag{2.9} \]

Hence, we get (2.4).

(2) Note that

\[ R \sqrt{\rho(x)} \leq \frac{C}{\sqrt{\rho(x)}}. \]

Then, by (2.8) and (2.9), we get (2.5).

The following lemma is due to [9, Lemma 2.2].

**Lemma 2.3.** Let \( o \in M \) be any fixed point of steady Ricci soliton \( (M, g, f) \). Then for any \( p \in M \) and number \( k > 0 \) with \( f(p) - k\sqrt{f(p)} > f(o) \), it holds

\[ B\left(p, \frac{k}{\sqrt{R_{\text{max}}}}, f^{-1}(p)g\right) \subset M_{p,k}, \tag{2.10} \]

where \( M_{p,k} \) is a set defined by

\[ M_{p,k} = \{ x \in M | f(p) - k\sqrt{f(p)} \leq f(x) \leq f(p) + k\sqrt{f(p)} \}. \]

**Proof.** For any \( q \in M \), let \( \gamma(s) \) be a minimal geodesic connecting \( p \) and \( q \) such that \( \gamma(s_1) = q \) and \( \gamma(s_2) = p \). Since

\[ |\nabla f|^2(x) = R_{\text{max}} - R(x) \leq R_{\text{max}}, \quad \forall x \in M, \]

we have

\[ |f(p) - f(q)| = \left| \int_{s_1}^{s_2} \langle \gamma'(s), \nabla f \rangle ds \right| \leq \int_{s_1}^{s_2} |\nabla f| ds \leq \sqrt{R_{\text{max}}}d(q,p). \]

In particular, for \( q \in M \setminus M_{p,k} \), we get

\[ d(q,p) \geq k\sqrt{f(p)} \cdot \frac{1}{\sqrt{R_{\text{max}}}}. \]

Hence

\[ B_{g_p}\left(p, \frac{k}{\sqrt{R_{\text{max}}}}\right) \subset M_{p,k}. \tag{2.11} \]

This completes the proof.

By Lemma 2.3, we prove the following lemma.
Lemma 2.4. Let \((M, g, f)\) be a steady Ricci soliton with Ricci curvature and sectional curvature satisfying (1.3). Suppose \(f\) satisfies (2.3). Then there exists a constant \(C(k)\) for each \(k \in \mathbb{N}\) such that
\[
|\nabla^k \text{Rm}(p)| \cdot f^{k+2} \leq C(k), \quad \forall p \in M.
\]

Proof. Fix any \(p \in M\) with \(f(p) \geq 2r_0 \gg 1\). Then
\[
|f(x) - f(p)| \leq \sqrt{f(p)}, \quad \forall x \in M_{p, 1}.
\]
By Lemma 2.2, we may assume that
\[
\frac{\sqrt{R_{\text{max}}}}{2} \rho(x) \leq f(x) \leq \sqrt{R_{\text{max}}} \rho(x), \quad \forall \rho(x) \geq r_0.
\]
Thus, for \(x \in M_{p, 1}\),
\[
|\text{Rm}(x)| \cdot f(p) \leq C \sqrt{R_{\text{max}}} \leq C \sqrt{R_{\text{max}}} f(p) \leq \frac{C \sqrt{R_{\text{max}}} f(p)}{f(p) - \sqrt{f(p)}} \leq 2C \sqrt{R_{\text{max}}}.
\]
By (2.10), we get
\[
|\text{Rm}(x)| \cdot f(p) \leq C', \quad \forall x \in B(p, \frac{1}{\sqrt{R_{\text{max}}}}; g_p) \subseteq M_{p, 1}, \quad (2.12)
\]
where \(g_p = f^{-1}(p)g\).

Let \(\phi_t\) be a one-parameter diffeomorphism group generated by \(-\nabla f\) on \(M\). Then
\[
\frac{\partial}{\partial t} f(\phi_t(p)) = -|\nabla f|^2(x, t) \leq 0.
\]
It follows that
\[
f(\phi_t(x)) \geq f(x) \geq r_0, \quad \forall t \in [0, \infty), \quad x \in B(p, \frac{1}{\sqrt{R_{\text{max}}}}; g_p).
\]
Combining this fact with (1.3), we obtain from (2.12),
\[
|\text{Rm}_{g_p(t)}(x)| \leq \frac{C \sqrt{R_{\text{max}}}}{f(x)} \leq \frac{C'}{f(p)} \quad \forall x \in B(p, \frac{1}{\sqrt{R_{\text{max}}}}; g_p(0)), \quad t \in [-1, 0],
\]
where \(\text{Rm}_{g_p(t)}\) is the sectional curvature of rescaled flow \(g_p(t) = f^{-1}(p)g(f(pt))\).

Note that Ricci curvature is non-negative when \(f(x) \geq r_0\). Since \(g(t) = \phi_1^* g\) satisfies the Ricci flow equation,
\[
\frac{\partial g(t)}{\partial t} = -2\text{Ric}(t), \quad (2.13)
\]
by Lemma 2.3, the flow \(g_p(t)\) is shrinking on \(B(p, \frac{1}{\sqrt{R_{\text{max}}}}; g_p(0))\). Thus
\[
B(p, \frac{1}{\sqrt{R_{\text{max}}}}; g_p(t)) \subseteq B(p, \frac{1}{\sqrt{R_{\text{max}}}}; g_p(0)), \quad \forall t \leq 0.
\]
By Shi’s higher order estimates [21], we have
\[
|\nabla^k \text{Rm}_{g_p(t)}(x)| \leq C'(k), \quad \forall x \in B(p, \frac{1}{2 \sqrt{R_{\text{max}}}}; g_p(-1)), \quad t \in \left[-\frac{1}{2}, 0\right].
\]
It follows that
\[
|\nabla^k \text{Rm}(x)| \cdot f^{k+2} \leq C'(k), \quad \forall x \in B(p, \frac{1}{2 \sqrt{R_{\text{max}}}}; g_p(-1)), \quad (2.14)
\]
In particular, we have
\[
|\nabla^k \text{Rm}(p)| \cdot f^{k+2} \leq C'(k), \quad \text{as} \quad f(p) \geq 2r_0.
\]
The lemma is proved. □
Remark 2.5. When $R_{\text{max}} = 1$, by the proof of Lemma 2.4, there exists a constant $C(k)$ depending only on $k$ and $C$ for each $k \in \mathbb{N}$ such that
\[
|\nabla^k Rm|(p) \cdot f^{\frac{k+2}{2}}(p) \leq C(k), \quad \forall f \geq r_0,
\]
where $C$ is the constant in (1.3).

3 The diameter estimate

In this section, we follow the argument in [9] to give a diameter estimate for steady Ricci solitons with Ricci curvature and sectional curvature satisfying (1.3).

Let $\phi_t$ be the one-parameter diffeomorphism group generated by $-\nabla f$. Since $g(t) = \phi_t^*(g)$ is a solution of Ricci flow (2.13), the Ricci curvature $R_{ij}(\cdot, t)$ of $g(t)$ satisfies,
\[
\frac{\partial R_{ij}}{\partial t} = \Delta R_{ij} + 2R_{kijl}R_{kl} - 2R_{ik}R_{jk}.
\] (3.1)

Note that $L_X R_{ij} = -\frac{\partial R_{ij}}{\partial t}$ and $L_X R_{ij} - \nabla_X R_{ij} = \text{Ric}(\nabla e_i X, e_j) + \text{Ric}(e_i, \nabla e_j X) = 2R_{ik}R_{jk},$

where $X = \nabla f$. Then,
\[
\nabla_X R_{ij} + \Delta R_{ij} + 2R_{kijl}R_{kl} = 0.
\]

On the other hand, by the soliton equation, we have
\[
\nabla_i R_{jk} - \nabla_j R_{ik} = \nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = -R_{ijkl} \nabla_l f.
\] (3.2)

Hence,
\[
\text{Rm}(X, e_j, e_k, X) = (\nabla \text{Ric})(e_j, X, e_k) - (\nabla \text{Ric})(X, e_j, e_k).
\] (3.3)

By taking trace of (3.2) and the contracted Bianchi identity, we also have
\[
\text{Ric}(X, e_i) = -\frac{1}{2} \nabla e_i R.
\]

Then
\[
(\nabla \text{Ric})(e_j, X, e_k) = \frac{\partial(\text{Ric}(X, e_k))}{\partial x_j} - \text{Ric}(X, \nabla e_j e_k) - \text{Ric}(\nabla e_j X, e_k)
\]
\[
= -\frac{1}{2} \frac{\partial^2 R}{\partial x_j \partial x_k} + \frac{1}{2} \langle \nabla e_j e_k, \nabla R \rangle - R_{ij} R_{ik}
\]
\[
= -\frac{1}{2} (\text{Hess} R)_{jk} - R_{ji} R_{kl}.
\] (3.4)

Therefore, combining (3.3) and (3.4), we get the following lemma.

Lemma 3.1. It holds that
\[
\text{Rm}(X, e_j, e_k, X) = -\frac{1}{2} (\text{Hess} R)_{jk} - R_{ji} R_{kl} + \Delta R_{jk} + 2R_{ijl} R_{kl}.
\] (3.5)
When \( f(x) \) satisfies (2.3) and Ricci curvature and curvature tensor satisfy (1.3), it is easy to check that \( S = \{ x \in M \mid \nabla f(x) = 0 \} \) is non-empty and compact. Then, we may assume that \( |\nabla f|^2(x) = R_{\text{max}} - R(x) > 0, \forall f(x) > r_0 \). Thus, every \( \Sigma_r \) is diffeomorphic to the same compact manifold \( \Sigma \) for \( f(x) \geq r_0 \). As in [9, Lemma 3.1], we introduce one parameter group of diffeomorphisms \( F_r : \Sigma \to \Sigma_r \subseteq M (r \geq r_0), \) which is generated by flow

\[
\frac{\partial F_r}{\partial r} = \frac{\nabla f}{|\nabla f|^2}.
\]

Let \( h_r = F^*_r(g) \) and \( e_i = F_*(\bar{e}_i), e_j = F_*(\bar{e}_j) \), where \( \bar{e}_i, \bar{e}_j \in T\Sigma \). Then, a direct computation shows that

\[
\frac{\partial h_r}{\partial r}(\bar{e}_i, \bar{e}_j) = 2|\nabla f|^2 \text{Ric}(e_i, e_j). \quad (3.6)
\]

**Lemma 3.2.** Let \((M, g, f)\) be a steady Ricci soliton with Ricci curvature and curvature tensor satisfying (1.3). Suppose \( f \) satisfies (2.3). Then for \( x_1, x_2 \in \Sigma \) with \( d_r(x_1, x_2) \geq 2\tau_0 \), we have

\[
\frac{d}{dr} d_r(x_1, x_2) \leq C \left( \frac{\tau_0}{r} + \frac{1}{\tau_0} + \frac{r}{\tau_0^2} \right), \quad \forall r \geq r_0,
\]

where \( d_r(\cdot, \cdot) \) is the distance function of \((\Sigma, h_r)\).

**Proof.** The proof follows from [9, Lemma 3.1]. Let \( \gamma \) be a normalized minimal geodesic from \( x_1 \) to \( x_2 \) with velocity field \( X(s) = \frac{d\gamma}{ds} \) and \( V \) be any piecewise smooth normal vector field along \( \gamma \) which vanishes at the endpoints. By the second variation formula, we have

\[
\int_0^{d_r(x_1, x_2)} (|\nabla X|^2 + \langle \bar{R}(V, X)V, X \rangle) ds \geq 0.
\]

Let \( \{e_i(s)\}_{i=1}^{n} \) be a parallel orthonormal frame along \( \gamma \) that is perpendicular to \( X \). Put \( V_i(s) = f(s)e_i(s) \), where \( f(s) \) is defined as

\[
\begin{align*}
  f(s) &= \frac{s}{\tau_0}, \quad \text{if} \quad 0 \leq s \leq \tau_0, \\
  f(s) &= 1, \quad \text{if} \quad \tau_0 \leq s \leq d_r(x_1, x_2) - \tau_0, \\
  f(s) &= \frac{d_r(x_1, x_2) - s}{\tau_0}, \quad \text{if} \quad d_r(x_1, x_2) - \tau_0 \leq s \leq d_r(x_1, x_2).
\end{align*}
\]

Hence, by a direct computation, we have

\[
0 \leq \sum_{i=1}^{n-1} \int_0^{d_r(x_1, x_2)} (|\nabla X|^2 + \langle \bar{R}(V_i, X)V_i, X \rangle) ds \\
= \frac{2(n-1)}{\tau_0} - \int_0^{d_r(x_1, x_2)} \bar{\text{Ric}}(X, X) ds + \int_0^{\tau_0} \left( 1 - \frac{s^2}{\tau_0^2} \right) \bar{\text{Ric}}(X, X) ds \\
+ \int_{d_r(x_1, x_2) - \tau_0}^{d_r(x_1, x_2)} \left( 1 - \frac{(d_r(x_1, x_2) - s)^2}{\tau_0^2} \right) \bar{\text{Ric}}(X, X) ds. \quad (3.8)
\]

We claim

\[
\bar{\text{Ric}}(X, X) \leq \frac{C}{r} \tilde{g}(X, X), \quad \forall x \in \Sigma_r. \quad (3.9)
\]

By the Gauss formula, we have

\[
\begin{align*}
\text{Rm}(X, Y, Z, W) &= \bar{\text{Rm}}(X, Y, Z, W) \\
&\quad + \frac{1}{|\nabla f|^2} (\bar{\text{Ric}}(X, Z)\bar{\text{Ric}}(Y, W) - \bar{\text{Ric}}(X, W)\bar{\text{Ric}}(Y, Z)). \quad (3.10)
\end{align*}
\]
and
\[ R_{ij} = R_{ij} + R\left( \frac{\nabla f}{|\nabla f|}, e_i, e_j, \frac{\nabla f}{|\nabla f|} \right) - \frac{1}{|\nabla f|^2} \sum_k (R_{ij} R_{kk} - R_{ik} R_{kj}), \]
where indices \( i, j \) and \( k \) are corresponding to vector fields on \( T\Sigma_r \). Thus for a unit vector \( Y \), we derive
\[
(Ric - \overline{Ric})(Y, Y) = R\left( \frac{\nabla f}{|\nabla f|}, Y, Y, \frac{\nabla f}{|\nabla f|} \right) - \frac{1}{|\nabla f|^2} \sum_{i=1}^{n-1} [\text{Ric}(Y, Y)\text{Ric}(e_i, e_i) - \text{Ric}^2(Y, e_i)].
\]
(3.11)

On the other hand, by Lemma 3.1, we have
\[
|\nabla f|^2 \cdot Rm \left( \frac{\nabla f}{|\nabla f|}, Y, Y, \frac{\nabla f}{|\nabla f|} \right) = -\frac{1}{2} (\text{Hess}R)(Y, Y) - \text{Ric}(Y, e_i)\text{Ric}(Y, e_i)
\]
\[ + (\Delta \text{Ric})(e_j, e_k) + 2\text{Rm}(e_i, Y, Y, e_j)\text{Ric}(e_i, e_j). \]

Note that by Lemma 2.4,
\[
|\text{Hess}R| + |\Delta \text{Ric}| + |\text{Ric}|^2 + |\text{Rm}| \cdot |\text{Ric}| \leq \frac{C_1}{f^2}.
\]
Then
\[
\left| \text{Rm} \left( \frac{\nabla f}{|\nabla f|}, Y, Y, \frac{\nabla f}{|\nabla f|} \right) \right| \leq \frac{C_1}{|\nabla f|^2 \cdot f^2} \leq \frac{C}{2r^2}.
\]

Also, we have
\[
\frac{n-1}{|\nabla f|^2} \sum_{i=1}^{n-1} [\text{Ric}(Y, Y)\text{Ric}(e_i, e_i) - \text{Ric}^2(Y, e_i)] \leq \frac{R^2 + |\text{Ric}|^2}{|\nabla f|^2} \leq \frac{C}{2r^2}.
\]

Hence, we get from (3.11),
\[
|(Ric - \overline{Ric})(Y, Y)| \leq \frac{C}{r^2}, \quad r \geq r_0.
\]
(3.12)

In particular,
\[
\text{Ric}(Y, Y) \leq \frac{C_1}{r}, \quad r \geq r_0.
\]

This proves (3.9).

By (3.8) and (3.9), it is easy to see
\[
\int_0^{d_r(x_1, x_2)} \overline{\text{Ric}}(Y, Y) ds \leq \frac{2(n-1)}{\tau_0} + \frac{4C \tau_0}{3r}.
\]
(3.13)

Also by (3.12), we see
\[
\int_0^{d_r(x_1, x_2)} (\text{Ric} - \overline{\text{Ric}})(Y, Y) ds \leq \frac{C}{r^2} d_r(x_1, x_2).
\]
(3.14)

On the other hand, if we let \( Y(s) = (F_r)_*(X(s)) \) with \( |Y(s)|_{(\Sigma_r, g)} = 1 \), then by (3.6), we have
\[
\frac{d}{dr} d_r(x_1, x_2)
\]
\[
\int_0^d r(x_1, x_2) \left( L_{\nabla f} - g \right)(Y, Y) ds
= \frac{1}{|\nabla f|^2} \int_0^d r(x_1, x_2) \frac{1}{Ric(Y, Y)} ds + \frac{1}{|\nabla f|^2} \int_0^d r(x_1, x_2) \left( Ric - \bar{Ric} \right)(Y, Y) ds.
\]

Thus inserting (3.13) and (3.14) into the above relation, we obtain

\[
\frac{d}{dr} d_r(x_1, x_2) \leq C \left( \frac{\tau_0}{r} + \frac{1}{\tau_0} + \frac{d_r(x_1, x_2)}{r^2} \right).
\]

This completes the proof.

By Lemma 3.2, we get the following diameter estimate for \((\Sigma_r, \bar{g})\) (see [9, Proposition 3.3]).

**Proposition 3.3.** Let \((M, g, f)\) be a steady Ricci soliton as in Lemma 3.2. Then there exists a constant \(C\) independent of \(r\) such that

\[
\text{diam}(\Sigma_r, g) \leq C \sqrt{r}, \quad \forall r \geq r_0.
\]

**Remark 3.4.** When \(R_{\text{max}} = 1\), there exist constants \(C_1\) and \(C_2\) independent of \(r\) such that

\[
\text{diam}(\Sigma_r, g) \leq C_1 \sqrt{r} + C_2, \quad \forall r \geq r_0.
\]

Moreover, \(C_1\) only depends on \(n\) and \(C\), where \(C\) is the constant in (1.3).

As a corollary, we have the following.

**Corollary 3.5.** Under the condition of Proposition 3.3, there exists a uniform constant \(C_0 > 0\) such that the following is true: for any \(k \in \mathbb{N}\), there exists \(\tilde{r}_0 = \tilde{r}_0(k)\) such that

\[
M_{p, k} \subset B \left( p, C_0 + \frac{2k}{\sqrt{R_{\text{max}}}}; f^{-1}(p)g \right), \quad \forall \rho(p) \geq \tilde{r}_0.
\]

The proof is the same as the proof of Corollary 3.4 in [9]. We only need to replace \(R(p)\) in Corollary 3.4 in [9] by \(f^{-1}(p)\) to prove Corollary 3.5. We skip over the proof.

### 4 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following famous result of Gromov [11].

**Theorem 4.1.** If \(V\) is \(\varepsilon(n)\)-flat, then its universal cover is diffeomorphic to \(\mathbb{R}^n\), where \(\varepsilon(n)\) is a constant depending only on \(n\).

The definition of almost flat manifolds is as follows.

**Definition 4.2.** Let \((V, g)\) be a connected \(n\)-dimensional complete Riemannian manifold. Let \(d(V)\) be the diameter of \((V, g)\). Let \(c^+(V)\) and \(c^-(V)\) be the upper and lower bounds of the sectional curvature of \((V, g)\), respectively. We set \(c(V) = \max\{|c^+(V)|, |c^-(V)|\}\). We say \(V\) is \(\varepsilon\)-flat for \(\varepsilon \geq 0\) if \(c(V)d^2(V) \leq \varepsilon\).

With the help of Theorem 4.1, we prove the following proposition.

**Proposition 4.3.** Let \((M, g, f)\) be an \(n\) \((\geq 3)\)-dimensional steady Ricci soliton with non-negative sectional curvature and positive Ricci curvature. Suppose that (1.1) holds. Then there is a constant \(c_0 > 0\) such that

\[
R(x) \geq \frac{c_0}{\rho(x)}.
\]

We first prove two lemmas below.
Lemma 4.4. Let \((M, g, f)\) be an \(n\)-dimensional steady Ricci soliton with non-negative sectional curvature, positive Ricci curvature and linear scalar curvature decay \((1.1)\). If there exists a sequence of points \(p_i\) such that \(f(p_i) \to 0\) and \(R(p_i)f(p_i) \to 0\) as \(i \to \infty\), then there exists a constant \(\varepsilon > 0\) such that
\[
R(x_i)f(x_i) \to 0, \quad \text{as } i \to \infty, \quad \forall x_i \in B(p_i, \varepsilon; f^{-1}(p_i)g). \tag{4.1}
\]
Moreover, \(\varepsilon\) only depends on the constant \(C\) in \((1.1)\) and \(n\).

**Proof.** Let \(g_i(t) = f(p_i)^{-1}g(f(p_i)t)\). We consider the rescaled sequence of Ricci flows \((M, g_i(t), p_i)\). By Lemma 2.3, we see that
\[
B(p_i, 1; f(p_i)^{-1}g) \subseteq M_{p_i, \sqrt{R_{\text{max}}}}. \tag{4.2}
\]
Then, for any \(q_i \in B(p_i, 1; f(p_i)^{-1}g)\),
\[
\frac{f(q_i)}{f(p_i)} \to 1, \quad \text{as } i \to \infty. \tag{4.3}
\]
Note that \((2.3)\) holds by Lemma 2.1. By the curvature decay \((1.1)\), it follows that
\[
R(q_i)f(p_i) \leq 2C, \quad \text{as } i \to \infty. \tag{4.4}
\]
Hence, the sectional curvature of \((M, f(p_i)^{-1}g)\) is uniformly bounded by \(C(n)C\) on \((p_i, 1; f(p_i)^{-1}g)\) when \(i\) is large enough.

Let \(\iota_i : B(0, r_0; f(p_i)^{-1}g) \to M\) be an exponential map such that \(\iota_i(0) = p_i\). Then there exists an \(r_0 \leq 1\) (depending only on \(C\) and \(n\)) such that \(\iota_i\) is smooth, \(\iota_i'(f(p_i)^{-1}g)(0) = g_e(0)\) and \(C_0^{-1}g_e \leq \iota_i'(f(p_i)^{-1}g)(x) \leq C_0 g_e\) for \(x \in B(0, r_0)\), where \(g_e\) is the Euclidean metric and \(C_0\) is a constant independent of \(i\). Let \(\tilde{g}_i(t) = \iota_i^*(g_i(t))\). By Lemma 2.4 and \((4.3)\), it follows that
\[
|\nabla^k \tilde{R}_{\tilde{g}_i(0)}|_{\tilde{g}_i(0)}(x) = \frac{|\nabla^k Rm|_{\iota_i(x)}(x)}{f(\iota_i(x))^{-\frac{k+2}{2}}} \cdot \frac{f(p_i)^{\frac{k+2}{2}}}{f(\iota_i(x))^{\frac{k+2}{2}}} \leq C_1, \quad \forall x \in \tilde{B}(0, r_0).
\]
Note that
\[
f(\phi_i(p)) - f(p) = \int_0^t |\nabla f|^2(\phi_s(p))ds \geq 0, \quad \forall t \leq 0, \quad p \in M. \tag{4.5}
\]
Then, for \(t \leq 0\) and \(x \in \tilde{B}(0, r_0)\), we also get
\[
|\nabla^k \tilde{R}_{\tilde{g}_i(t)}|_{\tilde{g}_i(t)}(x) = \frac{|\nabla^k Rm|_{\iota_i(t,f(p_i)t)}(x)}{f(\phi_{f(p_i)}t(x_i))^{-\frac{k+2}{2}}} \cdot \frac{f(p_i)^{\frac{k+2}{2}}}{f(\phi_{f(p_i)}t(x_i))^{\frac{k+2}{2}}} \leq C_1 \cdot \frac{f(p_i)^{\frac{k+2}{2}}}{f(x_i)^{\frac{k+2}{2}}} \leq 2C_1.
\]
Here \(x_i = \iota_i(x)\). Hence, \((\tilde{B}(0, r_0), \tilde{g}_i(t), 0)\) converges to a limit \((\tilde{B}(0, r_0), \tilde{g}_\infty(t), 0)\) in the \(C^\infty_{\text{loc}}\) sense.

On the other hand,
\[
\tilde{R}_\infty(0, 0) = \lim_{i \to \infty} R(p_i)f(p_i) = 0. \tag{4.6}
\]
This means that \(\tilde{R}_\infty(x, t)\) attains its minimum 0 at the interior point \(0 \in \tilde{B}(0, r_0)\), since \(\tilde{g}_i(t)\) has non-negative sectional curvature. By the maximum principle, \((\tilde{B}(0, r_0), \tilde{g}_\infty(t))\) has constant scalar curvature 0 (see [15, Theorem 4.18]). Hence, \((\tilde{B}(0, \varepsilon), \tilde{g}_\infty(t))\) has constant sectional curvature 0 by taking \(\varepsilon = r_0\). Therefore, by the convergence of \(\tilde{g}_i(t)\), we prove \((4.1)\). \(\square\)
Lemma 4.5. Let \((M, g, f)\) be a steady Ricci soliton and \(\{p_i\}\) a sequence such that \(f(p_i) \to \infty\) and \(R(p_i)f(p_i) \to 0\) as \(i \to \infty\) as in Lemma 4.4. Then by taking a substitution,

\[
R(x_i)f(x_i) \to 0, \quad \text{as} \quad i \to \infty, \quad \forall x_i \in \Sigma_f(p_i).
\]  

(4.7)

Proof. We prove by contradiction. Suppose the lemma is not true. Then, there exist a sequence of points \(q_i \to \infty\) and a constant \(c' \geq 0\) such that

\[
R(q_i)f(q_i) \geq c'.
\]  

(4.8)

We consider the sequence of manifolds \((\Sigma_f(p_i), f(p_i)^{-1}\bar{g})\), where \(\bar{g}\) is the induced metric on \(\Sigma_f(p_i)\) as the submanifold of \((M, g)\). By the Gauss formula (3.10) together with the curvature estimate in Lemma 2.4, we see that \((\Sigma_f(p_i), f(p_i)^{-1}\bar{g})\) has uniform bounded curvature. Moreover, by Proposition 3.3, it has a uniform bounded diameter estimate. Thus \((\Sigma_f(p_i), f(p_i)^{-1}\bar{g})\) converges to a length space \((\Sigma_\infty, \bar{g}_\infty)\) in the Gromov-Hausdorff topology. Without loss of generality, we may assume that \(p_i \to p_\infty\) and \(q_i \to q_\infty\). Then \(d_\infty(p_\infty, q_\infty) = L < \infty\).

Let \(N = \lceil \frac{2k}{\varepsilon} \rceil\), where \(\varepsilon\) is chosen as in Lemma 4.4. Then we can choose \(N\) points \(p^1_\infty, \ldots, p^N_\infty\) such that

\[
\begin{align*}
d_\infty(p_\infty, p^1_\infty) &= \frac{\varepsilon}{2}, \\
d_\infty(p^k_\infty, p^{k+1}_\infty) &= \frac{\varepsilon}{2}, \quad 1 \leq k \leq N - 1, \\
d_\infty(q_\infty, p^N_\infty) &\leq \frac{\varepsilon}{2}.
\end{align*}
\]

By the convergence of \((\Sigma_f(p_i), f(p_i)^{-1}\bar{g})\), there exists sequences \(p^k_i, 1 \leq k \leq N - 1\) such that \(p^k_i \to p^k_\infty\). Note that \(R(p_i)f(p_i) \to 0\) and \(p^1_i \in B(p_i, \varepsilon; f^{-1}(p_i)\bar{g})\). Thus by Lemma 4.4, we see that

\[
R(p^k_i)f(p^k_i) \to 0, \quad \text{as} \quad i \to \infty.
\]  

(4.9)

Since the constant \(\varepsilon\) is independent of the choice of the sequence of base points in Lemma 4.4, by induction on \(k\), we conclude from Lemma 4.4 that

\[
R(p^k_i)f(p^k_i) \to 0, \quad \text{as} \quad i \to \infty, \quad 1 \leq k \leq N + 1.
\]  

(4.10)

In particular, for \(k = N + 1\), we get

\[
R(q_i)f(q_i) \to 0, \quad \text{as} \quad i \to \infty,
\]

which is a contradiction to (4.8). The lemma is proved. \(\square\)

Proof of Proposition 4.3. We prove by contradiction. If the proposition is not true, there exists a sequence of points \(p_i \to \infty\) such that \(R(p_i)f(p_i) \to 0\). Then by Lemma 4.5,

\[
R(x_i)f(x_i) \to 0, \quad \text{as} \quad i \to \infty, \quad \forall x_i \in \Sigma_f(p_i).
\]

Since the sectional curvature is non-negative,

\[
Rm(x_i)f(x_i) \to 0, \quad \text{as} \quad i \to \infty, \quad \forall x_i \in \Sigma_f(p_i).
\]

By (3.10), it follows that

\[
|\overline{Rm}(x_i)|_{f(p_i)^{-1}\bar{g}} \to 0, \quad \text{as} \quad i \to \infty, \quad \forall x_i \in \Sigma_f(p_i).
\]

On the other hand, by Proposition 3.3,

\[
diam(\Sigma_f(p_i), f(p_i)^{-1}\bar{g}) \leq C.
\]

Hence, we see that \((\Sigma_f(p_i), f(p_i)^{-1}\bar{g})\) is a sequence of \(\bar{Z}(n)\)-flat manifolds when \(i\) is large enough. By Theorem 4.1, the universal cover of \((\Sigma_f(p_i), f(p_i)^{-1}\bar{g})\) is diffeomorphic to \(\mathbb{R}^{n-1}\). However, \(\Sigma_f(p_i)\) is diffeomorphic to \(S^{n-1}\) (see [9, Lemma 2.1]). Therefore, there is a contradiction. The proposition is proved. \(\square\)
By Proposition 4.3, we can finish the proof of Theorem 1.1.

Proof of Theorem 1.1. We may suppose that \((M, g, f)\) is non-flat. Let \((\tilde{M}, \tilde{g})\) be the universal cover of \((M, g)\) and \(\pi : \tilde{M} \to M\) be the covering map. Let \(f = f \circ \pi\). Then, \((\tilde{M}, \tilde{g}, \tilde{f})\) is also a steady Ricci soliton.

By the proof of Lemma 5.1 in [9], \((\tilde{M}, \tilde{g})\) splits as \((N, h) \times \mathbb{R}^{n-k}\) and \(f\) is a constant when restricted on \(\mathbb{R}^{n-k}\), where \(k \geq 2\). Moreover, \((N, h, f_N)\) is a \(k\)-dimensional steady Ricci soliton with non-negative sectional curvature and positive Ricci curvature, where \(f_N(q) = \tilde{f}(q, \cdot), \forall q \in N\). By [9, Lemma 5.1(4)] and (1.1), we see that

\[
R(x)f(x) \leq c, \quad \forall f(x) \geq r_0, \tag{4.11}
\]

and consequently,

\[
R_N(x)f_N(x) \leq c, \quad \forall f(x) \geq r_0. \tag{4.12}
\]

When \(k = 2\), \((N, h)\) is a two-dimensional steady Ricci soliton, which is a cigar soliton [7, 14]. When \(k > 2\), by (4.12), we can apply Proposition 4.3 to the steady Ricci soliton \((N, h)\) and conclude that \(R_N(x)\) satisfies (1.2). The proof is completed.

Corollary 1.2 follows from Theorem 1.1 and [9, Theorem 1.5] directly. We need to consider Theorem 1.1(iii). In this case, we can show that the scalar curvature \(R(x)\) of \((M, g)\) satisfies (4.13). However, it is proved in [9] that any 4-dimensional \(\kappa\)-noncollapsed steady Ricci soliton with scalar curvature decay (4.13) must be rotationally symmetric.

Proof of Corollary 1.4. By Theorem 1.1, the universal cover \((\tilde{M}, \tilde{g}, \tilde{f})\) of \((M, g, f)\) is either \((\mathbb{R}^2, g_{\text{cigar}}) \times (\mathbb{R}^2, g_{\text{Euclid}})\), or a 4-dimensional steady Ricci soliton \((N^k, g_N, f_N) \times (\mathbb{R}^{n-k}, g_{\text{Euclid}})\) \((k > 2)\) with the scalar curvature \(R_N(\cdot)\) of \((N^k, g_N)\) satisfying (1.2). Since \((M, g)\) is \(\kappa\)-noncollapsed, \((\tilde{M}, \tilde{g})\) is also \(\kappa\)-noncollapsed, and so it cannot be \((\mathbb{R}^2, g_{\text{cigar}}) \times (\mathbb{R}^2, g_{\text{Euclid}})\).

In the latter case, \(f_N(q) = \tilde{f}(q, \cdot), \forall q \in N\). Moreover, \((N^k, g_N, f_N)\) has positive Ricci curvature and it admits an equilibrium point \(o \in N\) by Lemma 2.1(2). Hence, \(f_N\) satisfies (2.3). Thus, we have

\[
\frac{C^{-1}}{f_N(x)} \leq R_N(x) \leq \frac{C}{f_N(x)}, \quad \tilde{f}_N(x) \geq r_0.
\]

Therefore,

\[
\frac{C^{-1}}{f(x)} \leq R_N(x) \leq \frac{C}{f(x)}, \quad \tilde{f}(x) \geq r_0.
\]

It follows

\[
\frac{C^{-1}}{f(x)} \leq R(x) \leq \frac{C}{f(x)}, \quad f(x) \geq r_0,
\]

and consequently, by Lemma 2.1(4),

\[
\frac{(C')^{-1}}{\rho(x)} \leq R(x) \leq \frac{C'}{\rho(x)}, \quad \rho(x) \geq r'_0. \tag{4.13}
\]

This means that \((M, g)\) is a 4-dimensional \(\kappa\)-noncollapsed steady Ricci soliton with non-negative sectional curvature, which satisfies (4.13). Hence, by [9, Theorem 1.5], it must be rotationally symmetric.

5 Proof of Theorem 1.3

In this section, we generalize the argument in the proof of Theorem 1.1 to prove Theorem 1.3. We have assumed that \(R_{\max} = 1\). Then, by Lemma 2.2 and (1.4),

\[
R(x)f(x) \leq 2\varepsilon(n), \quad \forall f(x) \geq r_0, \tag{5.1}
\]

when \(r_0\) is large enough.
Proof of Theorem 1.3. Let \((\tilde{M}, \tilde{g}, \tilde{f})\) be the covering steady Ricci soliton of \((M, g, f)\). Then, by Theorem 1.1, \((\tilde{M}, \tilde{g})\) is an \((\mathbb{R}^2, g_{\text{cigar}}) \times (\mathbb{R}^{n-2}, g_{\text{Euclid}})\), or \((\tilde{M}, \tilde{g}) = (N, g_N) \times (\mathbb{R}^{n-k}, g_{\text{Euclid}}), k > 2\), where \((N, g_N)\) is a \(k\)-dimensional steady Ricci soliton with non-negative sectional curvature and positive Ricci curvature. Moreover, there is a positive constant \(c_0\) such that the scalar curvature \(R_N\) of \((N, g_N)\) satisfies
\[
\frac{c_0}{f_N(x)} \leq R_N(x) \leq 2\varepsilon(n) \frac{f_N(x)}{f_N(x)}, \quad \forall f_N(x) \geq r_0, \tag{5.2}
\]
where \(f_N(q) = \tilde{f}(q, \cdot), \quad \forall q \in N\). Hence, to prove Theorem 1.3, it suffices to eliminate the second case \((N^k, g_N) \times (\mathbb{R}^{n-k}, g_{\text{Euclid}})\) by contradiction.

Let \(\Sigma_r = \{x \in N \mid f_N(x) = r\}\) be the level set of \(f_N\). By Proposition 3.3, the diameter \(D_r\) of \((\Sigma_r, r^{-1}h)\) satisfies
\[
D_r \leq C_1(n), \quad \forall r \geq r_1,
\]
where \(r_1\) is a large number and \(C_1(n)\) is a uniform constant depending only on the dimension \(n\). Thus, as in the proofs of Lemma 2.3 and Corollary 3.5, we see that
\[
\Sigma_r \subseteq B(x, 2C_1(n); r^{-1}h) \subseteq M_{x, 2C_1(n)}, \quad \forall x \in \Sigma_r.
\]

On the other hand, by (5.2) and the nonnegativity of sectional curvature,
\[
|R_{\text{rm}}|_h(x) \cdot r \leq C_2(n)\varepsilon(n), \quad \forall x \in \Sigma_r.
\]

Then by Lemmas 2.4 and 3.1 together with (3.10), we get
\[
|R_{\text{rm}}|_{r^{-1}h} = r \cdot |R_{\text{rm}}|_h \leq 2r \cdot |R_{\text{rm}}|_h(x) \leq 2C_2(n)\varepsilon(n), \quad \forall x \in \Sigma_r.
\]
It follows that
\[
|R_{\text{rm}}|_{r^{-1}h} \cdot D_r^2 \leq 2C_1(n)C_2^2(n)\varepsilon(n), \quad \forall x \in \Sigma_r, \quad r \geq r_1.
\]
When \(\varepsilon(n) \leq \frac{1}{2}C_1^{-1}(n)C_2^2(n)\varepsilon(n), (\Sigma_r, r^{-1}h)\) is an \(\varepsilon(n)\)-flat metric. Hence, by Gromov’s theorem, Theorem 4.1, the universal cover of \(\Sigma_r\) must be diffeomorphic to \(\mathbb{R}^{k-1}\). However, \(\Sigma_r\) is diffeomorphic to \(S^{k-1}\) (see [9, Lemma 2.1]) . Therefore, we get a contradiction. As a consequence, \((\tilde{M}, \tilde{g})\) must be \((\mathbb{R}^2, g_{\text{cigar}}) \times (\mathbb{R}^{n-2}, g_{\text{Euclid}})\). \(\square\)

6 A generalization of Proposition 4.3

In this section, we prove an analogy of Proposition 4.3 for \(\kappa\)-noncollapsed steady Ricci solitons satisfying (1.3). Namely, we show the following theorem.

Theorem 6.1. Let \((M^n, g, f)\) be a \(\kappa\)-noncollapsed steady Ricci soliton that satisfies (1.3) and \(f\) satisfies (2.3). Then \(\Sigma_r\) is diffeomorphic to a compact shrinking Ricci soliton with non-negative Ricci curvature for any \(r \geq r_1 > 0\). Moreover, there are \(c_0, r_0 > 0\) such that
\[
R(x) \leq \frac{c_0}{f(x)}, \quad \forall f(x) \geq r_0. \tag{6.1}
\]

Because of lack of positivity of sectional curvature, we could not use Gromov’s theorem, Theorem 4.1 to study the structure of level sets of steady Ricci solitons as in Section 4. Here, we first use the \(\kappa\)-noncollapsed condition to derive the convergence of rescaled flows \((M, f^{-1}(p_i)g(f(p_i)t), p_i)\) with the help of the diameter estimate established in Section 3.
Lemma 6.2. Let $(M, g, f)$ be a $\kappa$-noncollapsed steady Ricci soliton as in Theorem 6.1. Then, for any $p_i \to \infty$, rescaled flows $(M, f^{-1}(p_i)g(f(p_i)t), p_i)$ converge subsequently to $(\mathbb{R} \times \Sigma, ds^2 + g_\Sigma(t))$ $(t \in (-\infty, 0])$ in the Cheeger-Gromov topology, where $\Sigma$ is diffeomorphic to the level set $\Sigma_{r_0}$ of $f$ and $(\Sigma, g_\Sigma(t))$ is a $\kappa$-noncollapsed ancient Ricci flow with non-negative Ricci curvature. Moreover, the scalar curvature $R_\Sigma(t)$ satisfies
\[
\frac{\partial}{\partial t} R_\Sigma(x, t) \geq 0, \quad \forall x \in \Sigma, \quad t \leq 0 \tag{6.2}
\]
and the curvature tensor $\text{Rm}_\Sigma(x, t)$ of $(\Sigma, g_\Sigma(t))$ satisfies
\[
|\text{Rm}_\Sigma|(x, t) \leq \frac{C}{1 + |t|}, \quad \forall x \in \Sigma, \quad t \leq 0, \tag{6.3}
\]
where $C$ is a uniform constant.

Proof. The proof is a modification of [9, Theorem 1.4], where the sectional curvature is assumed non-negative. For any fixed $\bar{r} > 0$, by (2.10), we have
\[
B(p_i, \bar{r}; f^{-1}(p_i)g(0)) \subseteq M_{p_i, \bar{r} \sqrt{R_{\max}}}.
\]
Moreover, by the relation,
\[
\frac{d|\nabla f|^2(\phi_t(p))}{dt} = -2\text{Ric}(\nabla f, \nabla f)(\phi_t(p)) \leq 0, \quad \forall t \leq 0, \quad f(p) \geq r_0,
\]
one can show that for any $t \leq 0$,
\[
|\nabla f|^2(\phi_t(x)) \geq |\nabla f|^2(x) \geq \frac{R_{\max}}{2}, \quad \forall x \in B(p_i, \bar{r}; f^{-1}(p_i)g(0)). \tag{6.4}
\]
For $x \in B(p_i, \bar{r}; f^{-1}(p_i)g(0))$, by (1.3), it is easy to see that
\[
|Rm_{g_{p_i}}(x)|_{g_{p_i}} = |Rm(x, f(p_i)t)| \cdot f(p_i) \leq \frac{Cf(p_i)}{f(\phi_{f(p_i)t}(x))}.
\]
Moreover, by (6.4),
\[
f(x, f(p_i)t) - f(x) = \int_0^{f(p_i)t} |
\nabla f|^2(\phi_s(x)) ds \geq \frac{R_{\max}}{2} \cdot f(p_i)|t|.
\]
Thus
\[
|Rm_{g_{p_i}}(x)|_{g_{p_i}} \leq \frac{Cf(p_i)}{f(x) + \frac{R_{\max}}{2} \cdot f(p_i)|t|} \leq \frac{2C}{1 + R_{\max}|t|}, \quad \forall t \leq 0.
\]
As in the proof of Lemma 2.4, we further get
\[
|Rm_{g_{p_i}}(x)|_{g_{p_i}} \leq \frac{2C(k)}{1 + R_{\max}|t|}, \quad \forall t \leq 0, \quad x \in B(p_i, \bar{r}; f^{-1}(p_i)g(0)),
\]
where $C(k)$ is independent of $i$ and $\bar{r}$. Note that $(M, g_{p_i}(t))$ is $\kappa$-noncollapsed. Hence by Hamilton’s compactness theorem (see [13]), $(M, g_{p_i}(t), p_i)$ converge subsequently to a Ricci flow $(M_\infty, g_\infty(t), p_\infty)$ with non-negative Ricci curvature, which satisfies the curvature decay
\[
|Rm_{\infty}(x, t)| \leq \frac{C}{1 + |t|}, \quad \forall x \in M_\infty, \quad t \leq 0. \tag{6.5}
\]
On the other hand, for any $t \leq 0$ and $x \in B(p_i, \bar{r}; f^{-1}(p_i)g(0))$, it holds that
\[
\frac{\partial}{\partial t} R_{g_{p_i}}(x) = f(p_i)^2 \frac{\partial}{\partial t} f(x, f(p_i)t)
\]
Therefore, we obtain (6.2) from (6.5) immediately.

Let \( X(t) = f(p_i)^2 \nabla f \). We have

\[
\sup_{B(p_i, \rho; g_{p_i})} |\nabla(X_{\bar{p}_i})|_{g_{p_i}} = \sup_{B(p_i, \rho; g_{p_i})} |\text{Ric}| \cdot \sqrt{f(p_i)} \leq C \sqrt{R(p_i)} \to 0.
\]

Then, by Lemma 2.4,

\[
\sup_{B(p_i, \rho; g_{p_i})} |\nabla^{k}(X_{\bar{p}_i})|_{g_{p_i}} \leq C(n) \sup_{B(p_i, \rho; g_{p_i})} |\nabla^{k-1}(\text{Ric})(g_{p_i})|_{g_{p_i}} \leq C_1.
\]

Hence, \( X(t) \) converges subsequently to a parallel vector field \( X(t) \) on \((M_\infty, g_\infty(0))\). Moreover,

\[
|X(t)|_{g_{p_i}}(x) = |\nabla f|(p_i) = \sqrt{R_{\max}} + o(1) > 0, \quad \forall x \in B(p_i, \rho; g_{p_i}),
\]

as long as \( f(p_i) \) is large enough. This implies that \( X(t) \) is non-trivial. As a consequence, \((M_\infty, g_\infty(t))\) locally splits off a line along \( X(t) \). By the same argument in the proof of Lemma 4.6 in [9], we can further show that \( X(t) \) generates a line through \( p_\infty \). Hence, \((M_\infty, g_\infty(0))\) splits off a line.

Now, we may assume that \((M_\infty, g_\infty(t)) = (\mathbb{R} \times \Sigma, ds^2 + g_\Sigma(t))\) for \( t \leq 0 \). Consequently, the curvature \( Rm_{\Sigma}(x, t) \) of \((\Sigma, g_\Sigma(t))\) satisfies (6.3). We are left to show that \( \Sigma \) is diffeomorphic to \( \Sigma_{\infty} \), when \( r \geq r_0 \). It suffices to prove the convergence of \((\Sigma_f(g_{p_i}), R(p_i))g_{p_i}\) as in [9]. In fact, [9, Section 4, Lemmas 4.2–4.4, Proposition 4.5 and Lemma 4.7] are valid, while the corresponding estimates have been obtained here in Lemmas 2.3 and 2.4, Proposition 3.3 and Corollary 3.5. Thus \((\Sigma_f(g_{p_i}), f(p_i)^{-1}g_{p_i})\) converge subsequently to a compact manifold \((\Sigma_\infty, g_\infty)\) in the Cheeger-Gromov sense. Moreover, \((\Sigma_\infty, g_\infty) \subseteq \Sigma \times \{p_\infty\}\). By the connectedness of \( \Sigma, \Sigma_\infty \) is diffeomorphic to \( \Sigma \), and so is each \( \Sigma_r \). The lemma is proved.

The following lemma seems well-known.

**Lemma 6.3.** The scalar curvature of a compact gradient shrinking Ricci soliton does not vanish.

**Proof.** Let \((\Sigma, g, f)\) be a compact gradient shrinking Ricci soliton, i.e.,

\[
\text{Ric} - \frac{g}{2} + \text{Hess} f = 0.
\]

By taking trace of (6.6), we get

\[
R + \Delta f = \frac{n}{2}.
\]

Note that \( \Sigma \) is compact. By (6.7), we have

\[
\frac{1}{\text{vol}(\Sigma, g)} \int_{\Sigma} R = \frac{n}{2}
\]

Hence, \( R \) cannot vanish.

**Proof of Theorem 6.1.** We first show that \( \Sigma \) in Lemma 6.2 is diffeomorphic to a compact shrinking Ricci soliton with non-negative Ricci curvature when \( r \geq r_0 \). Since \((\Sigma, g_\Sigma(t))\) is a \( \kappa \)-noncollapsed ancient Ricci flow with non-negative Ricci curvature satisfying (6.3), by [17, Theorem 3.1], we see that for any fixed \( x \in \Sigma \) and any sequence \( \{\tau_i\} \to \infty \), \((\Sigma, \tau_i^{-1}g_\Sigma, (\tau_i, x)\)

subsequently converges to a shrinking Ricci soliton \((\Sigma', g'(t), x')\) with non-negative Ricci curvature. On the other hand, by [18, Lemma 0.3], there exists a constant \( C_1 \) such that

\[
\text{diam}(\Sigma, g_\Sigma(t)) \leq C_1 \sqrt{|t| + 1}.
\]

In particular,

\[
\text{diam}(\Sigma, \tau_i^{-1}g_\Sigma(-\tau_i)) \leq C_1.
\]
Thus, $\Sigma'$ is diffeomorphic to $\Sigma$ by the Cheeger-Gromov compactness theorem. Consequently, each $\Sigma_r$ is diffeomorphic to $\Sigma'$ for any $r \geq r_1$.

Next, we prove (6.1). Suppose that (6.1) is not true. Then, there exists $p_i \to \infty$ such that

$$R(p_i)f(p_i) \to 0, \text{ as } i \to \infty.$$  \hspace{1cm} (6.10)

By Lemma 6.2, for any $p_i \to \infty$, rescaled flows $(M, f^{-1}(p_i)g(f(p_i)t), p_i)$ converge subsequently to $(\mathbb{R} \times \Sigma, ds^2 + g_\Sigma(t), p_\infty)$ ($t \in (-\infty, 0]$) in the Cheeger-Gromov topology. By (6.10), $R_\Sigma(p_\infty, 0) = 0$. Since $g_\Sigma(t)$ is ancient, $R_\Sigma(t)$ is non-negative by [6]. Combining this fact with (6.2), $R_\Sigma(p_\infty, t) = 0, \forall t \leq 0$. By the maximum principle, we see that

$$R_\Sigma(x, t) = 0, \forall t \leq 0, \ x \in \Sigma.$$  

Note that $(\Sigma, r_1^{-1}g_\Sigma(-r_1), p_\infty)$ subsequently converges to a shrinking Ricci soliton $(\Sigma', g'(-1), p')$ for some $\{r_1\} \to \infty$. Hence, we get a compact shrinking Ricci soliton $(\Sigma', g'(-1))$ whose scalar curvature is zero. This is impossible by Lemma 6.3. Hence, we complete the proof. 

\begin{flushright}
\Box
\end{flushright}

**6.1 Proof of Theorem 1.5**

According to [2], we have the following definition.

**Definition 6.4.** An $n$-dimensional steady Ricci soliton $(M, g, f)$ is called asymptotically cylindrical if the following holds:

(i) The scalar curvature $R(x)$ of $g$ satisfies

$$\frac{C_1}{\rho(x)} \leq R(x) \leq \frac{C_2}{\rho(x)}, \ \forall \rho(x) \geq r_0,$$

where $C_1$ and $C_2$ are two positive constants and $\rho(x)$ denotes the distance from a fixed point $x_0$.

(ii) Let $p_m$ be an arbitrary sequence of marked points going to infinity. Consider rescaled metrics $g_m(t) = r_m^{-1}\phi_m^*g$, where $r_mR(p_m) = \frac{n-1}{2} + o(1)$ and $\phi_t$ is a one-parameter subgroup generated by $X = -\nabla f$. As $m \to \infty$, the flows $(M, g_m(t), p_m)$ converge in the Cheeger-Gromov sense to a family of shrinking cylinders $(\mathbb{R} \times \mathbb{S}^{n-1}, \tilde{g}(t)), t \in (0, 1)$. The metric $\tilde{g}(t)$ is given by

$$\tilde{g}(t) = dt^2 + (n - 2)(2 - 2t)g_\mathbb{S}^{n-1}(1),$$

where $\mathbb{S}^{n-1}(1)$ is the unit sphere of the Euclidean space.

We need to describe some properties of asymptotically cylindrical geometry on the steady Ricci solitons.

**Lemma 6.5.** Let $(M, g, f)$ be a noncompact steady Ricci soliton which is asymptotically cylindrical. Then, we have

$$\frac{|\text{Hess}R| + |\Delta \text{Ric}|}{R^2} \to 0, \text{ as } \rho(x) \to \infty,$$  \hspace{1cm} (6.11)

$$\frac{\text{Ric}(e_i, e_j)}{R} \to \frac{\delta_{ij}}{n-1}, \text{ as } \rho(x) \to \infty,$$  \hspace{1cm} (6.12)

$$\frac{\text{Ric}(e_i, e_n)}{R} \to 0, \text{ as } \rho(x) \to \infty,$$  \hspace{1cm} (6.13)

$$\frac{\text{Rm}(e_j, e_k, e_k, e_l)}{R} \to \frac{(1 - \delta_{jk})\delta_{jl}}{(n-1)(n-2)}, \text{ as } \rho(x) \to \infty,$$  \hspace{1cm} (6.14)

$$\frac{\text{Rm}(e_n, e_i, e_j, e_k)}{R} \to 0, \text{ as } \rho(x) \to \infty,$$  \hspace{1cm} (6.15)

where $\{e_1, \ldots, e_n\}$ are the orthonormal basis of $T_xM$ with respect to the metric $g$ and $1 \leq i, j, k, l \leq n - 1$, $e_n = \frac{\nabla f}{|\nabla f|}$.  

Proof. We only need to show that for any $p_i$ tending to infinity and for any sequence of orthonormal basis $\{e_1^{(i)}, \ldots, e_{n-1}^{(i)}\} \subseteq T_{p_i} \Sigma_{f(p_i)}$, by taking a subsequence, the following hold:

$$\frac{\|\text{Hess} R(p_i) + |\Delta \text{Ric}|(p_i)\|}{R^2(p_i)} \to 0, \quad \text{as } i \to \infty,$$

(6.16)

$$\frac{\text{Ric}(e_j^{(i)}, e_k^{(i)})}{R(p_i)} \to \frac{\delta_{jk}}{n-1}, \quad \text{as } i \to \infty,$$

(6.17)

$$\frac{\text{Ric}(e_j^{(i)}, e_n^{(i)})}{R(p_i)} \to 0, \quad \text{as } i \to \infty,$$

(6.18)

$$\frac{\text{Rm}(e_j^{(i)}, e_k^{(i)}, e_l^{(i)}, e_j^{(i)})}{R(p_i)} \to (1 - \delta_{jk})\delta_{jl} \left(\frac{n-1}{(n-1)(n-2)}\right), \quad \text{as } i \to \infty,$$

(6.19)

$$\frac{\text{Rm}(e_n^{(i)}, e_j^{(i)}, e_k^{(i)}, e_l^{(i)})}{R(p_i)} \to 0, \quad \text{as } i \to \infty.$$

(6.20)

For any $p_i$ tending to infinity, we consider the sequence of pointed manifolds $(M, g_{p_i}, p_i)$, where $g_{p_i} = R(p_i)g$. By the asymptotically cylindrical condition, by taking a subsequence, we have $(M, g_{p_i}, p_i)$ converge to $(\mathbb{R} \times S^{n-1}, g_\infty, p_\infty)$, where $g_\infty = ds^2 + (n-1)(n-2)g_{S^{n-1}}$. Then,

$$\lim_{i \to \infty} \frac{\|\text{Hess} R(p_i) + |\Delta \text{Ric}|(p_i)\|}{R^2(p_i)} = \lim_{i \to \infty} (\|\text{Hess} R_{g_{p_i}}|_{g_{p_i}}(p_i) + |\Delta_{g_{p_i}} \text{Ric}_{g_{p_i}}|(p_i)) = |\text{Hess} R_{g_\infty}|_{g_\infty}(p_\infty) + |\Delta_{g_\infty} \text{Ric}_{g_\infty}|(p_\infty) = 0.$$

(6.21)

For any fixed $\tau > 0$, we have

$$\lim_{i \to \infty} \frac{\|\text{Ric}\|}{R(p_i)} = \lim_{i \to \infty} \frac{\|\text{Ric}_{g_{p_i}}\|_{g_{p_i}}}{R(p_i)} = \frac{1}{\sqrt{n-1}}, \quad \forall x \in B(p_i, \tau; g_{p_i}).$$

(6.22)

Let $X_i = R(p_i)^{-\frac{1}{2}} \nabla f$. We have

$$\sup_{B(p_i, \tau; g_{p_i})} |\nabla (g_{p_i}) X_i|_{g_{p_i}} = \sup_{B(p_i, \tau; g_{p_i})} \frac{|\text{Ric}|}{\sqrt{R(p_i)}} \leq C \sqrt{R(p_i)} \to 0.$$

Note that

$$\sup_{B(p_i, \tau; g_{p_i})} |\nabla (g_{p_i}) X_i|_{g_{p_i}} \leq C(n) \sup_{B(p_i, \tau; g_{p_i})} |\nabla (g_{p_i})^{-1} \text{Ric}(g_{p_i})|_{g_{p_i}} \leq C_1.$$

Then $X_i$ converges to a parallel vector field $X_\infty$ on $(M_\infty, g_\infty)$. Moreover,

$$|X_i|_{g_{p_i}}(x) = |\nabla f|_{g_{p_i}} = \sqrt{R_{\max}} + o(1) > 0, \quad \forall x \in B(p_i, \tau; g_{p_i}).$$

This implies that $X_\infty$ is non-trivial. Hence, $X_\infty$ is tangent to $\mathbb{R} \times S^{n-1}$. Suppose $R(p_i)^{-\frac{1}{2}} e_j^{(i)} \to e_j^{(\infty)}$ for $1 \leq j \leq n-1$. Then,

$$e_n^{(\infty)} = \lim_{i \to \infty} \frac{X_i}{|X_i|_{g_{p_i}}} = \frac{X_\infty}{|X_\infty|_{g_\infty}},$$

(6.23)

$$\langle e_j^{(\infty)}, X_\infty \rangle_{g_{\infty}} = \lim_{i \to \infty} \langle e_j^{(i)}, X_i \rangle_{g_{p_i}} = 0,$$

(6.24)

$$|e_j^{(\infty)}|_{g_{\infty}} = \lim_{i \to \infty} R(p_i)^{-\frac{1}{2}} |e_j^{(i)}|_{g_{p_i}} = 1.$$

(6.25)
Then,
\[
\text{Ric}_{g_n}(e_j^{(\infty)}, e_k^{(\infty)}) = \frac{\delta_{jk}}{n-1}, \quad \forall 1 \leq j, k \leq n-1,
\]
\[
\text{Ric}_{g_n}(e_j^{(\infty)}, e_n^{(\infty)}) = 0, \quad \forall 1 \leq j \leq n-1,
\]
\[
\text{Rm}_{g_n}(e_j^{(\infty)}, e_k^{(\infty)}, e_n^{(\infty)}, e_l^{(\infty)}) = \frac{(1 - \delta_{jk})\delta_{jl}}{(n-1)(n-2)}, \quad \forall 1 \leq j, k, l \leq n-1,
\]
\[
\text{Rm}_{g_n}(e_n^{(\infty)}, e_n^{(\infty)}, e_n^{(\infty)}, e_l^{(\infty)}) = 0, \quad \forall 1 \leq j, k, l \leq n-1.
\]
By the convergence of \((M, g_{p_i}, p_i)\), it is easy to check (6.17)–(6.20).

**Lemma 6.6.** Let \((M, g, f)\) be a noncompact steady Ricci soliton with positive Ricci curvature, which is asymptotically cylindrical. Then, there exists a compact set \(K \subseteq M\) such that \((M, g)\) has positive sectional curvature on \(M \setminus K\).

**Proof.** For \(p \in M\), let \(\{e_1, \ldots, e_n\}\) be an orthonormal basis of \(T_pM\) with respect to metric \(g\) and \(e_n = \frac{\nabla f}{|\nabla f|}\). When \(\rho(p) \geq r_0\), we have
\[
\text{Rm}(e_j, e_k, e_k, e_j) \geq \frac{R(p)}{2(n-1)(n-2)} > 0, \quad \forall 1 \leq j < k \leq n-1.
\] (6.26)

By Lemmas 3.1 and 6.5, for any \(\epsilon > 0\), we have
\[
|\nabla f|^2(p) \cdot \text{Rm}(e_n, e_j, e_j, e_n)
= -\frac{1}{2} (\text{Hess} R)_{jj} - \sum_{i=1}^n R_{jl}R_{jl} + \Delta R_{jj} + 2 \sum_{i,j,l=1}^n R_{ijjl}R_{il} \\
\geq \left( \frac{1}{(n-1)^2} - \epsilon \right) R^2(p) > 0.
\] (6.27)

Hence, \((M, g)\) has positive sectional curvature on \(M \setminus B(p_0, r_0; g)\) for some \(p_0 \in M\) and \(r_0 > 0\). □

**Proof of Theorem 1.5.** Since \(R(x)\) decays uniformly, \(R(x)\) attains its maximum at some point \(o\). Hence
\[
\text{Ric}(\nabla f, \nabla f)(o) = -\frac{1}{2} (\nabla R, \nabla f)(o) = 0.
\] (6.28)

Since \(\text{Ric}\) is positive, we get \(\nabla f(o) = 0\). By [4], we know that \(f\) satisfies (2.3).

By Theorem 6.1, we have
\[
\frac{C}{f(x)} \geq |\text{Rm}(x)| \geq R(x) \geq \frac{c}{f(x)}, \quad \forall \rho(x) \geq r_0.
\] (6.29)

By Lemma 6.2, for any \(p_i \to \infty\), rescaled flows \((M, f^{-1}(p_i)g(f(p_i)t), p_i)\) converge subsequentially to \((\mathbb{R} \times \Sigma, ds^2 + g_\Sigma(t))\) \((t \in (-\infty, 0])\) in the Cheeger-Gromov topology. By the equivalence of \(f\) and \(R\) (see (6.29)), we may assume that \(f(p_i)R(p_i) \to C_0\) as \(i \to \infty\) by taking a subsequence. Then, the rescaled flows \((M, R(p_i)g(R^{-1}(p_i)t), p_i)\) converge subsequentially to \((\mathbb{R} \times \Sigma, ds^2 + g_\Sigma(t))\) \((t \in (-\infty, 0])\) in the Cheeger-Gromov topology, where \(g_\Sigma(t) = C_0g_\Sigma(C_0^{-1}t)\). Note that \((\Sigma, g_\Sigma(t))\) is a 3-dimensional \(\kappa\)-noncollapsed Ricci flow which satisfies (6.3). By [6], \((\Sigma, g_\Sigma(t))\) has non-negative sectional curvature. Since the Ricci curvature is positive, the level set of \(f\) is diffeomorphic to \(S^3\) by [9]. Hence, \(\Sigma\) is diffeomorphic to \(S^3\). By [18], \((\Sigma, g_\Sigma(t))\) is a group of shrinking spheres. Hence, we have shown that \((M, g, f)\) is asymptotically cylindrical.

By Lemma 6.6, \((M, g)\) has positive sectional curvature on \(M \setminus K\) for some compact \(K\). □

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