ON SMALL MODULES FOR QUANTUM GROUPS AT ROOTS OF UNITY

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Abstract. A conjecture of De Concini Kac and Procesi provides a bound on the minimal possible dimension of an irreducible module for quantized enveloping algebras at an odd root of unity. We pose the problem of the existence of modules whose dimension equals this bound. We show that this question cannot have a positive answer in full generality and discuss variants of this question.

1. Introduction

Let $\ell$ be an odd positive integer. A construction due to De Concini Kac and Procesi [11] associates with an irreducible representation $V$ of a quantized enveloping algebra at an $\ell$-th root of unity a conjugacy class $O_V$ in a corresponding algebraic group $G$. They conjectured that the dimension of each irreducible module $V$ is divisible by $\ell^{\frac{1}{2}\dim O_V}$. The De Concini-Kac-Procesi (DCKP for short) conjecture was confirmed in [13] for regular conjugacy classes, in [4] for subregular unipotent classes in type $A_n$, in [2] for all conjugacy classes in type $A_n$ when $\ell$ is a prime, in [3] and [5] for spherical conjugacy classes for arbitrary $G$. An approach to settle the conjecture for unipotent conjugacy classes was given in [18]. In [24, 25], which at the time of writing are under refereeing process, Sevostyanov proposes a strategy for the proof of the DCKP conjecture for simply-connected quantized enveloping algebras, with some restrictions on $\ell$.

We are interested in the assumptions under which the bound in the DCKP conjecture is attained, i.e., in conditions for the existence of a module $V$ of dimension $\ell^{\frac{1}{2}\dim O_V}$. We call such a module $V$ a small module. In the case of modular Lie algebras, the analogue problem was formulated in [17, §8] and is usually referred to as Humphreys conjecture. For an account on the development of this conjecture, see [23].

One of the big differences with the modular case is that for each reductive Lie algebra $\mathfrak{g}$, there are several quantized enveloping algebras, each corresponding to a lattice $M$ between the root lattice and the weight lattice of $\mathfrak{g}$. We say that $M$ is the isogeny type of the algebra. The theory is well developed for the simply connected case (i.e., when $M$ is the weight lattice), but does not seem to allow for standard inductive arguments as in the case of modular Lie algebras, as quantized Levi subalgebras are not always simply connected.

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This motivates our analysis of different isogeny types both with respect to the DCKP-conjecture and to the question of the existence of small modules.

The paper is structured as follows: we recall in Section 3 the DCKP construction, in particular we see that every irreducible module for a quantized enveloping algebra is a representation of some finite-dimensional quotient $U^M_\eta(g)$, corresponding to an $\ell$-character $\eta$. Each $\ell$-character is associated with a conjugacy class in $G$. In Section 4 we relate the finite-dimensional quotients $U^M_\eta(g)$ and $U^N_\eta(g)$ corresponding to different isogeny types in Theorem 4.3: in particular we provide necessary and sufficient conditions on $\ell$ ensuring that $U^M_\eta(g)$ is independent of the isogeny type. As a consequence, under these assumptions on $\ell$, in order to confirm the DCKP conjecture for all isogeny types it is enough to prove it for one type (Corollary 4.4). In Section 5 we give an inductive argument using parabolic induction which reduces the quest for small modules to rigid conjugacy classes and settles their existence for $\mathfrak{sl}_{n+1}$ when $(\ell,(n+1)!) = 1$. The results we get are in analogy with the ones in [15]. The last Section is devoted to the cases in which the assumption on $\ell$ is not verified. We show that in this case small modules may fail to exist for $U^N_\eta(g)$ even if they exist for $U^M_\eta(g)$. A complete answer to the question of existence of small modules for central elements in $G$ is given in Proposition 6.1. As a conclusion, we formulate Conjecture 2 which is a quantum analogue of Humphreys conjecture.

2. Notation

Let $\varepsilon$ be a primitive $\ell$-th root of 1. Throughout $g$ denotes a fixed semisimple Lie algebra of rank $n$ with root lattice $Q$ and weight lattice $\Lambda$. We assume that $\ell$ is odd and coprime with 3 if $g$ has a component of type $G_2$.

For a lattice $M$ such that $Q \subseteq M \subseteq \Lambda$ we denote by $G_M$ the complex semisimple algebraic group with $\text{Lie}(G_M) = g$ and of isogeny type determined by $M$. We let $T_M \subseteq G_M$ be a maximal torus and $B_M \subseteq G_M$ a Borel subgroup containing $T_M$. The opposite Borel subgroup is denoted by $B^-_M$. The unipotent radical of $B_M$ does not depend on $M$ and we denote it by $U$. Similarly, the radical of $B^-_M$ is denoted by $U^\circ$.

The set of roots w.r.t. $T_M$ is denoted by $\Phi$, the set of positive roots w.r.t. $B_M \subseteq G_M$ a Borel subgroup containing $T_M$. The opposite Borel subgroup is denoted by $B^-_M$. The unipotent radical of $B_M$ does not depend on $M$ and we denote it by $U$ so $B_M = T_M \ltimes U$. Similarly, the radical of $B^-_M$ is denoted by $U^\circ$.

For the lattice $M$ as above, we denote by $U^M_\varepsilon(g)$ the De Concini-Kac quantized enveloping algebra at the root $\varepsilon$ of one. It is defined by generators $E_\alpha, F_\alpha (\alpha \in \Delta)$ and $K_\mu (\mu \in M)$.
subject to the relations

(2.1) \[ K_0 = 1, \quad K_{\mu}K_{\nu} = K_{\mu + \nu} \]

(2.2) \[ K_{\mu}E_\alpha = \varepsilon^{(1,\alpha)}E_\alpha K_{\mu} \]

(2.3) \[ K_{\mu}F_\alpha = \varepsilon^{-1(\mu,\alpha)}F_\alpha K_{\mu} \]

(2.4) \[ E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha,\beta} \frac{K_{\alpha} - K_{-\alpha}}{\varepsilon_\alpha - \varepsilon_\beta} \]

(2.5) \[ \sum_{a,b \geq 0} (-1)^a \left[ \frac{a+b}{a} \right] \varepsilon^{a} E_\alpha E_\beta E_\alpha^b = 0 \]

(2.6) \[ \sum_{a,b \geq 0} (-1)^a \left[ \frac{a+b}{a} \right] \varepsilon^{a} F_\alpha F_\beta F_\alpha^b = 0 \]

where \[ \left[ \frac{c}{d} \right]_e = \frac{|c|! |e|!}{|d|! (c-d)!}, \quad [c]_e! = [c]_e [c-1]_e \cdots [1]_e, \quad [c]_e = \frac{e^c - e^{-c}}{e^c - e^{-1}}, \quad \varepsilon(a) = \varepsilon \frac{a}{2} \]

If \( \mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r \) is the decomposition of \( \mathfrak{g} \) into simple factors, \( M = M_1 \oplus \cdots \oplus M_r \), we have the isomorphism \( U^M_\varepsilon(\mathfrak{g}) \simeq \bigotimes_{i=1}^r U^M_i(\mathfrak{g}_i) \).

For every choice of the lattice \( \mathbb{M} \) and every reduced decomposition \( w_0 \), we have root vectors \( E_\gamma, F_\gamma \) for \( \gamma \in \Phi^+ \) \[ 19 \]. The set of root vectors of \( U^M_\varepsilon(\mathfrak{g}) \) is the union of the sets of root vectors for each factor \( U^M_i(\mathfrak{g}_i) \).

The notation for the main objects that we consider is as follows:

- \( \mathcal{O}_q^H \) denotes the conjugacy class of \( q \) in the group \( H \), if it is clear what \( H \) is we write \( \mathcal{O}_q \)
- \( C_H(S) \) for \( H \) a group and \( S \subseteq H \) is the centralizer of \( S \) in \( H \)
- \( H^0 \) denotes the identity component of the algebraic group \( H \)
- \( \text{Rep}(A) \) is the set of modules of the algebra \( A \) up to isomorphism
- \( \text{Spec}(A) \) is the set of simple modules of the algebra \( A \) up to isomorphism
- \( U^M_\varepsilon(\mathfrak{h}) \) is the subalgebra of \( U^M_\varepsilon(\mathfrak{g}) \) generated by the \( K_\lambda \) for \( \lambda \in \mathbb{M} \)
- \( Z^M_\varepsilon(\mathfrak{g}) \) is the center of \( U^M_\varepsilon(\mathfrak{g}) \)
- \( Z^\ell_\varepsilon(\mathfrak{g}) \) is the \( \ell \)-center of \( U^M_\varepsilon(\mathfrak{g}) \), i.e. the central subalgebra generated by \( E^\ell_\alpha, F^\ell_\alpha \) for \( \alpha \in \Phi^+ \) and \( K^\ell_\mu \) for \( \mu \in \mathbb{M} \)
- \( U^M_\varepsilon(\mathfrak{g}) = U^M_\varepsilon(\mathfrak{g})/(\ker \eta) \)
- \( Z^M_\eta(\mathfrak{h}) \) is the subalgebra of \( U^M_\eta(\mathfrak{g}) \) generated by \( K_\lambda \) for \( \lambda \in \mathbb{M} \)

3. Preliminaries

3.1. For any reduced decomposition of \( w_0 \), we have an ordering of the positive roots and a choice of root vectors. Ordered monomials in the root vectors and the \( K_{\mu_j} \), for a \( \mathbb{Z} \)-basis \( \mu_1, \ldots, \mu_n \) of \( \mathbb{M} \), form a PBW basis for \( U^M_\varepsilon(\mathfrak{g}) \). This is proved in \[ 19 \], Proposition 4.2 for \( M = Q \) and the construction works for general \( M \).

In addition, \( U^M_\varepsilon(\mathfrak{g}) \) is a Hopf algebra, \[ 12 \] §9.1. Its \( \ell \)-center is the central Hopf subalgebra \( Z^M_\varepsilon(\mathfrak{g}) \) \[ 11 \] §5.6 Proposition (d) generated by \( \ell \)-th powers of the root vectors and of the \( K_\mu \) \[ 10 \] §4. Considering the same ordering of positive in order to describe \( U \) as
a product of root subgroups, the group \( U \) is identified with \( \text{Spec}(\mathbb{C}[E_{\alpha}, \alpha \in \Phi^+]) \); similarly, \( U^{-} \simeq \text{Spec}(\mathbb{C}[F_{\alpha}, \alpha \in \Phi^+]) \) and \( \text{Spec}(\mathbb{C}[K_{\mu_j}^{\pm \ell}, j = 1, \ldots, n]) \simeq \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \simeq T_M \). Combining these identifications we get isomorphisms of varieties

\[
\text{Spec}(\mathbb{C}[E_{\alpha}, \alpha \in \Phi^+]) \simeq U^{-} \times T_M \times U \simeq \{ (t^{-1}u, tv) \mid t \in T_M, u \in U^{-}, v \in U \}.
\]

In addition, \( Z_0^M(\mathfrak{g}) \) is a finitely generated commutative Poisson Hopf algebra [11, §7.3] and the composition of the above isomorphisms is an isomorphism of Poisson algebraic groups [11, §7.6].

If \( M \subset N \), the natural inclusion \( \iota : U_\varepsilon^M(\mathfrak{g}) \subset U_\varepsilon^N(\mathfrak{g}) \) induces an inclusion \( Z_0^M(\mathfrak{g}) \to Z_0^N(\mathfrak{g}) \) which we denote by the same symbol.

3.2. Considering the PBW basis, \( U_\varepsilon^M(\mathfrak{g}) \) is a free \( Z_0^M(\mathfrak{g}) \)-module with basis given by the subset of the PBW monomials of \( U_\varepsilon^M(\mathfrak{g}) \) with all exponents in \{0, \ldots, \ell - 1\}. As a consequence, \( U_\varepsilon^M(\mathfrak{g}) \) is finite over its center and every irreducible representation of \( U_\varepsilon^M(\mathfrak{g}) \) is finite-dimensional [8, §11.1A]. By Schur’s Lemma we have the natural restriction map \( \xi_M : \text{Spec}(U_\varepsilon^M(\mathfrak{g})) \to \text{Spec}(Z_0^M(\mathfrak{g})) \). Let \( \pi_M : \text{Spec}(Z_0^M(\mathfrak{g})) \to B_M^{-1}B_M \) be given by \((x, y) \mapsto x^{-1}y\), where we use the identification in [3,1]. By the formulas in [11, 0.5] the map \( \pi_M \) is the product of the \( \pi_{M_i} \) for each simple component of \( \mathfrak{g} \), since \( Z_0^M(\mathfrak{g}) = \bigotimes_{i=1}^r Z_0^{M_i}(\mathfrak{g}_i) \) (notation as in [2]).

If \( M \subset N \), we have an isogeny \( \phi_{NM} : G_N \to G_M \). Again using [3,1] \( \iota^\ast \) is the restriction of \( \phi_{NM} \times \phi_{NM} \) to \( \{(t^{-1}u, tv) \mid t \in T_N, u \in U^{-}, v \in U \} \leq G_N \times G_N \). We have surjective horizontal maps in the commutative diagram below [11].

\[
\begin{array}{ccc}
\text{Spec}(U_\varepsilon^M(\mathfrak{g})) & \xrightarrow{\xi_M} & \text{Spec}(Z_0^M(\mathfrak{g})) \\
\downarrow & & \downarrow \phi_{NM} \\
\text{Spec}(U_\varepsilon^N(\mathfrak{g})) & \xrightarrow{\xi_N} & \text{Spec}(Z_0^N(\mathfrak{g})) \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Spec}(U_\varepsilon^M(\mathfrak{g})) & \xrightarrow{\pi_M} & B_M^{-1}B_M \subseteq G_M \\
\downarrow & & \downarrow & \\
\text{Spec}(Z_0^N(\mathfrak{g})) & \xrightarrow{\pi_N} & B_N^{-1}B_N \subseteq G_N \\
\end{array}
\]

Let \( \eta \in \text{Spec}(Z_0^M(\mathfrak{g})) \) and let \( V \in \text{Spec}(U_\varepsilon^M(\mathfrak{g})) \). We say that \( V \) has \( \ell \)-character \( \eta \) if \( \xi_M(V) = \eta \). Since \( \xi_M \) is surjective and \( \iota^\ast \) is induced from \( \phi_{NM} \times \phi_{NM} \), any \( \ell \)-character of \( U_\varepsilon^M(\mathfrak{g}) \) lifts to exactly \([N/M]\) \( \ell \)-characters of \( U_\varepsilon^N(\mathfrak{g}) \). Moreover, considering the rows of the diagram, every \( V \in \text{Spec}(U_\varepsilon^M(\mathfrak{g})) \) is an irreducible module over the algebra

\[
U_{\eta}^M(\mathfrak{g}) := U_\varepsilon^M(\mathfrak{g})/(\text{Ker} \eta)U_\varepsilon^M(\mathfrak{g})
\]

for \( \eta = \xi_M(V) \). We call such quotients reduced quantized enveloping algebra.

By abuse of notation, we denote the image of the generators of \( U_\varepsilon^M(\mathfrak{g}) \) in \( U_{\eta}^M(\mathfrak{g}) \) with the same symbols. For any \( \eta \in \text{Spec}(Z_0^M(\mathfrak{g})) \), the image of any \( Z_0^M(\mathfrak{g}) \)-basis of \( U_\varepsilon^M(\mathfrak{g}) \) through the canonical projection \( U_\varepsilon^M(\mathfrak{g}) \to U_{\eta}^M(\mathfrak{g}) \) is a linear basis of \( U_{\eta}^M(\mathfrak{g}) \). In particular
dim \( U_\eta^M(g) \) = \( \ell \dim \mathcal{O}_g \) and fixing a basis \( X = \{ \mu_1, \ldots, \mu_n \} \) of \( M \), we may choose the basis to consist of monomials of the form

\[
F^A K^B \mathcal{E}^C := F_{\alpha_N} \cdots F_{\alpha_1} K_{\mu_1}^{b_1} \cdots K_{\mu_n}^{b_n} E_{\alpha_1}^{c_1} \cdots E_{\alpha_N}^{c_N}
\]

where \( N = |\Phi^+| \), \( B = (b_1, \ldots, b_n) \), \( A = (a_N, \ldots, a_1) \) and \( C = (c_1, \ldots, c_N) \) are tuples of elements in \( \{0, \ldots, \ell - 1\} \).

3.3. Let \( (\sigma_1, \ldots, \sigma_n) \in \{ \pm 1 \}^n \). For a given \( \mathbb{Z} \)-basis \( \mu_1, \ldots, \mu_n \) of the lattice \( M \), we consider the algebra automorphism \( \sigma \) of \( U^M_\varepsilon(h) \) given by \( \sigma(K_{\mu_i}) = \sigma_i K_{\mu_i} \). For every \( j = 1, \ldots, n \) we have \( \sigma(K_{\mu_j}) = \tau_j K_{\alpha_j} \) for some \( \tau_j \in \{ \pm 1 \} \). Then, \( \sigma \) extends to an algebra automorphism of \( U^M_\varepsilon(g) \) by setting \( \sigma(F_{\alpha_j}) = F_{\alpha_j} \) and \( \sigma(E_{\alpha_j}) = \tau_j E_{\alpha_j} \).

3.4. For a lattice \( M \), let \( M' = \Lambda \cap \frac{1}{2} M \), and the corresponding central isogeny \( \phi_{M', M} : G_{M'} \to G_M \). By [11, p. 176], the map \( \pi_M \) factors through the big cell of \( G_{M'} \), so we have the following composition of maps:

\[
\text{Spec}(U^M_\varepsilon(g)) \xrightarrow{\xi_M} \text{Spec}(Z_0^M(g)) \xrightarrow{\pi} B^{-}_{M'}B_{M'} \subseteq G_{M'} \xrightarrow{\phi_{M', M}^{-1}} B^{-}_{M}B_{M} \subseteq G_{M}
\]

By [11] §6.6 Theorem, if \( g = \pi'(\eta) \in G_{M'} \setminus Z(G_{M'}) \), then for every \( \tau \in (\pi')^{-1}(\mathcal{O}_g) \) we have \( U^M_\eta(h) \simeq U^M_\eta(h) \). Now, \( \mathcal{O}_g \) has a representative in \( B^{-}_{M} \), so we may assume \( g = \pi'(\eta) \in B^{-}_{M} \). Therefore \( \pi_M(\eta) \in B^{-}_{M} \). Thus, by definition of the map \( \pi_M \) in [3.2] and the identification in [3.1] we have \( \eta(E_{\alpha}) = 0 \) for every \( \alpha \in \Phi^+ \). If \( h = \phi_{M', M}(g) \in B^{-}_{M} \) then \( \pi^{-1}_M(\mathcal{O}_h) \subseteq \bigcup_{z \in \text{Ker} \phi_{M', M}} (\pi')^{-1}(\mathcal{O}_z) \). For any \( z \in \text{Ker} \phi_{M', M} \), we choose \( \delta \in (\pi')^{-1}(z) \). Since \( \text{Ker} \phi_{M', M} \subseteq \{ z \in Z(G_{M'}) \mid z^2 = 1 \} \), we have \( \eta(E_{\alpha}) = \delta(E_{\alpha}) = 0, \eta(F_{\alpha}) = \delta(F_{\alpha}) \) and \( \eta(K_{\mu_j}) = \sigma_j \delta(K_{\mu_j}) \), for \( \sigma_j \in \{ \pm 1 \} \). The automorphism \( \sigma \) from paragraph 3.3 induces an isomorphism \( U^M_\eta(g) \simeq U^M_\delta(g) \). In other words, if \( h = \pi_M(\eta) \in G_M \), then for every \( \delta \in (\pi_M)^{-1}(\mathcal{O}_h) \) we have \( U^M_\eta(g) \simeq U^M_\delta(g) \).

3.5. In 1992, De Concini, Kac, Procesi formulated the following conjecture:

**Conjecture 1.** If \( V \) is an irreducible \( U^M_\eta(g) \)-module with \( \pi_M(\eta) = g \) then \( \ell^{\frac{1}{2} \dim \mathcal{O}_g} \) \mid \dim V \).

Since the map \( \pi_M \) is compatible with the tensor product decomposition of \( U^M_\varepsilon(h) \), verification of the conjecture can be reduced to the case of \( g \) simple.

Note that, since the diagram in [3.2] is commutative and since \( \phi_{N,M} \) is a central isogeny, \( \dim \mathcal{O}_{\pi_M(\eta)} = \dim \mathcal{O}_{\pi_M(\eta)} \) for any \( \eta \in \text{Spec}(Z_0^M(g)) \).

The following questions arise:

1. Let \( M \subset N \). Under which assumptions Conjecture [11] for the lattice \( M \) follows from or implies Conjecture [11] for the lattice \( N \)?
2. Let \( \eta \in \text{Spec}(Z_0^M(g)) \). Does there exist a \( U^M_\eta(g) \)-module of dimension \( \ell^{\frac{1}{2} \dim \mathcal{O}_{\pi_M(\eta)}} \) ? Is it irreducible?
3. Let $M \subset N$. Under which assumptions can we deduce an answer to Question (2) for the lattice $M$ from the case of the lattice $N$ or vice versa?

Question (2) is the quantum analogue of a problem posed by Humphreys, on representations of restricted Lie algebras. We call a module $V$ whose dimension satisfies an equality as in Question (2) a small module for $U^M_\eta(g)$. If the DCKP holds for $\pi^M(M)$, then any corresponding small module is irreducible. We show in §6 that Question (2) does not always have an affirmative answer and we formulate necessary conditions under which an affirmative answer can be expected.

3.6. We close this section by noticing that if $(\ell, |N/M|) = 1$, the center $Z^M_\varepsilon(g)$ of $U^M_\varepsilon(g)$ behaves well with respect to inclusion. This fact will not be needed in the sequel.

**Lemma 3.1.** Let $M \subset N$ be such that $(\ell, |N/M|) = 1$. Then, $Z^M_\varepsilon(g) \subseteq Z^N_\varepsilon(g)$. In particular, this holds for every $M, N$ whenever $(\ell, |A/Q|) = 1$.

**Proof.** The algebra $U^N_\varepsilon(g)$ is generated by $U^M_\varepsilon(g)$ and the elements $K_\omega$ with $\omega$ running through a set $N_M$ of representatives of $N/\ell M$ in $N$. Therefore $z \in Z^M_\varepsilon(g)$ lies in $Z^N_\varepsilon(g)$ if and only if $[K_\omega, z] = 0$ for every $\omega \in N_M$. Let $z = \sum z_i$ be the expression of $z$ with respect to the PBW basis. By construction of the PBW basis, $K_\omega z_i = c_i \omega z_i K_\omega$ for $c_i \omega$ some power of $\varepsilon$. By linear independence of the $z_i$, for every $k \geq 0$ and every $\omega$ we have $[K^k_\omega, z] = 0$ if and only if $[K^k_\omega, z_i] = 0$ for every $i$, which happens if and only if $c_i^{\ell \omega} = 1$ for every $i$. Since $K^{|N/M|}_\omega \in U^M_\varepsilon(g)$, for every $\omega$ and every $i$ we have $c_i^{\ell \omega} = 1$, which under our hypotheses forces $c_i^{\ell} = 1$, whence the statement. □

The map $\xi_\ell$ in (3.2) is the composition of the (surjective) restriction maps $\chi_\ell$ to $Z^M_\varepsilon(g)$ and $\tau_\ell$ to $Z^N_\varepsilon(g)$, so in the the diagram

$$
\begin{array}{ccc}
\text{Spec}(U^M_\varepsilon(g)) & \xrightarrow{\chi_\ell} & \text{Spec}(Z^M_\varepsilon(g)) \\
& \downarrow{t^*} & \downarrow{t^*} \\
\text{Spec}(U^N_\varepsilon(g)) & \xrightarrow{\chi_N} & \text{Spec}(Z^N_\varepsilon(g)) \\
\end{array}
\begin{array}{ccc}
\text{Spec}(Z^M_\varepsilon(g)) & \xrightarrow{\tau_\ell} & \text{Spec}(Z^N_\varepsilon(g)) \\
\end{array}
$$

the first vertical arrow is surjective if $(\ell, |N/M|) = 1$.

4. **Reduced quantized enveloping algebras**

In order to deal with the questions from §3.5, we compare reduced algebras corresponding to different lattices $M \subset N$.

4.1. Let $\eta_\ell \in \text{Spec}(Z^M_0(g))$ and $\eta_N \in \text{Spec}(Z^N_0(g))$ such that $t^* \eta_N = \eta_\ell$. It follows from the inclusion $Z^M_0(g) \subseteq Z^N_0(g)$ that there is a natural algebra morphism $f_{MN} : U^M_{\eta_M}(g) \rightarrow U^N_{\eta_N}(g)$. Indeed, we always have

$$(\ker \eta_M)U^M_\varepsilon(g) \subseteq (\ker \eta_N \cap Z^M_0(g))U^M_\varepsilon(g) \subseteq (\ker \eta_N)U^N_\varepsilon(g).$$
Note that, with respect to PBW bases corresponding to the same reduced decomposition of $w_\alpha$, we have $f_{MN}(F^A K_\lambda E^C) = F^A f_{MN}(K_\lambda) E^C$ for any $\lambda \in M$. Hence the dimension of the image equals $\ell^{2N} \dim f_{MN}(U^M_\eta(h))$.

For the purpose of analyzing $f_{MN}$, note that we are not bound to considering a specific basis $X$ of $M$. For each $\mu \in X$, $\eta(K_\mu) \neq 0$ since $K_\mu$ is invertible. Fix $c_\mu^n$ an $\ell$-th root of $\eta(K_\mu)$ and denote $K_\mu/c_\mu^n$ by $K_\mu^n$. Clearly, replacing the $K_\mu$ in the monomials of the basis (paragraph 3.2) with $K_\mu^n$ we still have a basis of $U^M_\eta(h)$. The latter is isomorphic to the group algebra $\mathbb{C}[M/\ell M]$. Let $f_N^\prime : \mathbb{C}[N/\ell N] \to \mathbb{C}[N/\ell N] \simeq U^N_\eta(h)$ be the invertible linear map defined by $f_N^\prime(K_\mu^n) = (c_\mu^{nM}/c_\mu^{nN})K_\mu^{nN}$. Then $k_{MN} = f_N^\prime \circ f_{MN}$ restricts to the canonical group homomorphism $M/\ell M \to N/\ell N$.

**Lemma 4.1.** Let $N$ be a lattice and $M \subseteq N$ a sublattice of finite index. The natural group homomorphism $k_{MN} : M/\ell M \to N/\ell N$ is an isomorphism if and only if $(|N/M|, \ell) = 1$.

**Proof.** Note that $\ker k_{MN} = (\ell N \cap M)/\ell M$. Assume that $(|N/M|, \ell) = 1$. Since $k_{MN}$ is an endomorphism of the finite group $Z^{\text{rank } N}$ it is enough to show that $k_{MN}$ is surjective. By assumption, there are $a, b \in \mathbb{Z}$ such that $a \ell + b |N/M| = 1$ so, for any $x \in N$ we have $x = a\ell x + b |N/M|x \in \ell N$. If instead there is a prime $p$ dividing $(|N/M|, \ell)$, then, there is $\mu \in \Lambda$ such that $\mu \not\in M$ and $p\mu \in M$. Then, $\ell \mu = \frac{\ell}{p} (p\mu) \in \ell N \cap M$ and $\ell \mu \not\in \ell M$ hence $k_{MN}$ is not injective.

For our purposes we will have to consider lattices $M \subseteq N$ for which $(|N/M|, \ell) \neq 1$. As $\ell$ is odd, they occur only for Lie algebras with components of type $A_n$ or $E_6$. Recall that the simple roots are in Bourbaki ordering.

**Lemma 4.2.** If $\mathfrak{g}$ is simple of type $A_n$ or $E_6$, then there is $\lambda_\Lambda \in \Lambda$ such that $\lambda_\Lambda, \alpha_1, \ldots, \alpha_{n-1}$ is a basis for $\Lambda$ and any lattice $Q \subseteq M \subseteq \Lambda$ is generated by $|\Lambda/M|\lambda_\Lambda, \alpha_1, \ldots, \alpha_{n-1}$.

**Proof.** We use [16, §13.2]. For $A_n$ we have $(n + 1)\lambda_1 = \sum_{i=1}^n (n - i + 1)\alpha_i$, so $\Lambda = \langle \lambda_1, Q \rangle$ and since the coefficient of $\alpha_n$ is 1, we have $\Lambda = (\lambda_1, \alpha_1, \ldots, \alpha_{n-1})$ as claimed with $\lambda_\Lambda = \lambda_1$. For $E_6$ we choose $\lambda_\Lambda := \lambda_3 - \lambda_5 = \frac{1}{4}(\alpha_1 + 2\alpha_3 - 2\alpha_5 - \alpha_6)$. Since $\lambda_\Lambda \in \Lambda \setminus Q$ and $\Lambda/Q = \mathbb{Z}_3$ we have $\Lambda = \langle \lambda_\Lambda, Q \rangle$. As $-3\lambda_\Lambda \in \lambda_\Lambda + \langle \alpha_1, \ldots, \alpha_{n-1} \rangle$ the claim follows also in this case. The last claim follows from the fact that $\Lambda/Q$ is cyclic.

The following theorem relates different isogeny types for reduced quantized enveloping algebras. A result comparing different isogeny types for the infinite-dimensional algebras $U^M_\ell(\mathfrak{g})$ is described in [14, §5].

**Theorem 4.3.** Let $Q \subseteq M \subseteq N \subseteq \Lambda$ with $M_i \subseteq N_i$ corresponding to the simple factors of $\mathfrak{g}$. Then $U^M_\eta(\mathfrak{g})$ is a free $f_{MN}(U^M_\ell(\mathfrak{g}))$-module of rank $\prod_i (\ell, |N_i/M_i|)$. In particular $f_{MN}$ is an algebra isomorphism if and only if $(|N/M|, \ell) = 1$.

**Proof.** It is enough to prove the statement for $\mathfrak{g}$ simple of rank $n$. Consider the group homomorphism $k_{MN} : M/\ell M \to N/\ell N$ and note that $\dim k_{MN}(\mathbb{C}[M/\ell M])$ equals the order of the image of $N/\ell N \to M/\ell M$. By Lemma 4.1 when $k_{MN}$ is not an isomorphism, then $d := (\ell, |N/M|) \neq 1$ and $\mathfrak{g}$ is of type $A_n$ or $E_6$. Moreover in this case, by Lemma
4.2 $k_{MN}$ is the endomorphism $Z^\ell_n \to Z^\ell_n$ which restricts to the identity on the last $n - 1$ $Z^\ell$-factors and on the first factor restricts to $x \mapsto |N/M| x$. Then, a basis of $f_{MN}(U^M_{\nu'(\eta)}(g))$ is given by $F^A K^{db} \left| A/M \right| \lambda_\alpha K^{b_1} \cdots K^{b_{n-1}} C^C$, where $b_0, \ldots, \frac{\ell}{d} - 1$, $b_1, \ldots, \ell - 1$, and $A,C$ are as in \cite{14}.

Theorem 4.3 and the discussion in \cite{3.2} give the following answer to Questions (1) and (3).

**Corollary 4.4.** Let $Q \subseteq M \subseteq N \subseteq \Lambda$ and let $\eta \in \Spec(Z^N_0(g))$. If $(\ell, |N/M|) = 1$ then

(a) the DCKP conjecture holds for $U^N_{\eta}(g)$ if and only if it holds for $U^M_{\nu'(\eta)}(g)$;

(b) there exists a small $U^N_{\eta}(g)$-module if and only if there exists a small $U^M_{\nu'(\eta)}(g)$-module.

Let $b(g)$ be the maximum between the largest bad prime for $g$ and the maximum $m$ such that $\Delta$ contains a subset of type $A_{m-1}$. The values of $b(g)$ for $g$ simple are listed in Table I.

Table I

| $A_n$ | $B_n$ | $C_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $n+1$ | $n$   | $n$   | $n$   | $6$   | $7$   | $8$   | $3$   | $3$   |

**Remark 4.5.** The strategy proposed in \cite{21}, \cite{25} aims at settling the DCKP-conjecture for $(\ell, n!) = 1$ if $g$ is of type $A_n$ and $(\ell, b(g)!) = 1$ otherwise, in the case of $M = \Lambda$. So, if $(\ell, b(g)!) = 1$, Corollary 4.4 together with this result would imply the DCKP conjecture for every lattice $M$.

5. SOME POSITIVE ANSWERS TO QUESTION 2

In this section we apply an inductive argument on the rank of $g$ in order to give affirmative answers to Question 2, under certain coprimality assumptions on $\ell$. With notation explained in the sequel, there are two main parts in the argument: a reduction to $U^N_{\eta}(l)$ for some Levi subalgebra $l$ of $g$ as in \cite{6} and a further reduction to a subalgebra determined by $[l, l]$ for which the coprimality condition is needed. By parabolic induction, the problem of determining the existence of small modules is reduced to rigid orbits, see Remark 5.8, hence the existence of small modules is settled for $sl_{n+1}$ when $(\ell, (n+1)!) = 1$.

5.1. For $\Pi \subseteq \Delta$, let $Q^\Pi = Z\Pi, \Phi^\Pi = Q^\Pi \cap \Phi, \Phi^+_\Pi = Q^\Pi \cap \Phi^+$. We denote the weight lattice of $\Phi^\Pi$ by $\Lambda^\Pi$ and the longest element of the corresponding Weyl group by $w^\Pi_0$. Let $p$ be the associated standard parabolic subalgebra of $g$ with standard Levi factor $l$. For a lattice $N$ between $Q$ and $\Lambda$, if $P$ and $L$ are the connected subgroups of $G_N$ with $\Lie(P) = p$ and $\Lie(L) = l$, then $P = LU_P$ for a connected unipotent subgroup $U_P \subseteq U$. We set $U_P := w_0 U_P w_0^{-1}$.

Let $U^N_{\xi}(p)$ be the subalgebra of $U^N_{\eta}(g)$ generated by $F_\alpha, K_\gamma, E_\beta$ for $\alpha \in \Pi, \beta \in \Delta, \gamma \in N$, let $U^N_{\xi}(l)$ be the subalgebra of $U^N_{\eta}(p)$ generated by $F_\alpha, K_\gamma, E_\beta, \alpha, \beta \in \Pi, \gamma \in N$.

When dealing with such subalgebras, we always assume that the reduced decomposition of $w_0$ is such that the first $|\Phi^+_\Pi|$ terms form a reduced decomposition of $w^\Pi_0$. This way, the root vectors corresponding to roots in $\Phi^+_\Pi$ will be contained in $U^N_{\xi}(l)$.

For $\eta \in \Spec(Z^N_0(g))$, let $U^N_{\eta}(p) = U^N_{\xi}(p)/(\Ker \eta) U^N_{\xi}(g) \cap U^N_{\xi}(p))$ and $U^N_{\eta}(l) = U^N_{\xi}(l)/(\Ker \eta) U^N_{\xi}(g) \cap U^N_{\xi}(l))$.  

By construction, $U^N_\eta(p)$ and $U^N_{\eta}(I)$ are subalgebras of $U^N_\eta(g)$ generated, respectively by $F_\alpha, K_\gamma, E_\beta$ for $\alpha \in \Pi$, $\beta \in \Delta$, $\gamma \in N$, and $F_\alpha, K_\gamma, E_\beta, \alpha, \beta \in \Pi, \gamma \in N$. Note that $U^N_{\eta}(I)$ depends only on the restriction of $\eta$ to the subalgebra of $Z^N_0(g)$ generated by $F_\alpha^\ell, K_\gamma^\ell, E_\beta^\ell$, $\alpha, \beta \in \Phi^\ell, \gamma \in N$.

5.2. If $\eta$ is such that $\eta(E'_\alpha) = 0$ for every $\alpha \in \Phi^+ \setminus \Phi^N_H$ then, extending trivially the action of $\{E_\alpha : \alpha \in \Phi^+ \setminus \Phi^N_H\}$ induces a natural map

\[(5.7) \text{Rep}(U^N_\eta(I)) \to \text{Rep}(U^N_\eta(p)).\]

If $V \in \text{Spec}(U^N_\eta(p))$, then $E_\alpha V = 0$ for every $\alpha \in \Phi^+ \setminus \Phi^N_H$. Indeed, if $I$ is the left ideal of $U^N_\eta(p)$ generated by $\{E_\alpha : \alpha \in \Phi^+ \setminus \Phi^N_H\}$ then $IV$ is a proper submodule of $V$. Therefore, the map $(5.7)$ restricts to a bijection $\text{Spec}(U^N_\eta(I)) \to \text{Spec}(U^N_\eta(p))$.

The composition of the map $(5.7)$ with extension of scalars to $U^N_\eta(g)$, i.e., with $V \mapsto U^N_{\eta}(g) \otimes_{U^N_\eta(p)} V$, is the induction map defined in [6] §2.1

\[
\text{Ind}^{\eta}_\eta : \text{Spec}(U^N_\eta(I)) \to \text{Rep}(U^N_\eta(g)).
\]

5.3. We make use of the generalization of Lusztig-Spaltenstein induction [21] to arbitrary elements described in [7]. If $\mathcal{O}_L^G$ is a conjugacy class in a Levi subgroup $L$ of a parabolic subgroup $P$, with decomposition $P = U^P_L$, then $\text{Ind}^G_P(\mathcal{O}_L^G)$ is the unique conjugacy class in $G$ intersecting $U^P_Lx$ in a dense subset. A conjugacy class which is not induced from a class in any proper Levi subgroup is called rigid. For $g \in G_N$ the semisimple and unipotent factors in the Jordan decomposition are denoted by $g_s$ and $g_u$ respectively.

For $\eta \in \text{Spec}(Z^N_0(g))$, let $\eta_\ell \in \text{Spec}(Z^N_0(g))$ be defined as $\eta_\ell(x) = \eta(x)$ for every $x \in Z^N_0(g) \cap U^N_\eta(I)$ and 0 elsewhere. By our choice of reduced decomposition of $w_0$, the ordering in $\Phi^+$ begins with the roots in $\Phi^+_H$ and the ordering in $-\Phi^+$ ends with the roots in $-\Phi^+_H$. By the identification in §3.1 and the definition of $\pi_N$ in §3.2 we see that $\pi_N(\eta_\ell) \in L$.

Lemma 5.1. For $\eta \in \text{Spec}(U^N_{\epsilon}(g))$ and $\Pi \subseteq \Delta$ let $L$ be the standard Levi subgroup corresponding to $\Pi$, $I = \text{Lie}(L)$. Assume $g = \pi_N(\eta) \in B^\epsilon_N$ and that $\mathcal{O}_g^G = \text{Ind}^G_N(\mathcal{O}_g^L)$ where $g' = \pi_N(\eta_\ell)$.

(a) If there exists an $U^N_\eta(I)$-module of dimension $\ell^{1/2} \dim \mathcal{O}_g^L$ then there exists a small module for $U^N_\eta(g)$.

(b) Assume in addition that $L = CG(N(C_G(g_s))^\circ)$. If there exists a $U^N_{\epsilon}(g)$-module of dimension $\ell^{1/2} \dim \mathcal{O}_g^G$ then there exists a $U^N_\eta(I)$-module of dimension $\ell^{1/2} \dim \mathcal{O}_g^L$ and one is irreducible if and only if the other is so.

Proof. Note that $U^N_\eta(I) \simeq U^N_{\eta}(I)$. If $V$ is an $U^N_{\eta}(I)$-module of dimension $\ell^{1/2} \dim \mathcal{O}_g^L$ then, by [6] (2.2)

\[
\dim \text{Ind}^{\eta}_\eta V = \ell^{1/2} |\Phi - \Phi_H^N| \dim V = \ell^{1/2} |\Phi - \Phi_H^N| + \dim \mathcal{O}_g^L.
\]

By [21] Theorem 1.3, [7] Proposition 4.6], $\dim \text{Ind}^{G}_N(\mathcal{O}_g^L) = |\Phi - \Phi_H^N| + \dim \mathcal{O}_g^L$ which proves (a). The main theorem in [10] §8 applies when $L = C_G(N(C_G(g_s))^\circ)$, the minimal Levi subgroup containing $C_G(g_s)^\circ$ [20] §3.1 and the claim (b) follows. \square
Remark 5.2. If the DCKP conjecture holds for the lattice $N$ and the integer $\ell$, the $U_\eta^N(\mathfrak{g})$-modules considered in the lemma are always irreducible.

5.4. The inductive argument that we are aiming for is for quantized enveloping algebras of semisimple Lie algebras. In order to make this possible we want to pass from $U_\eta^N(\mathfrak{l})$ to a product of quantized enveloping algebra corresponding to the simple factors of $[\mathfrak{l}, \mathfrak{l}]$, as suggested, in a special case, in [10, Remark 8.1].

Let $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_r$ be the decomposition of $[\mathfrak{l}, \mathfrak{l}]$ in simple factors, with $\mathfrak{P} = \mathfrak{P}_1 \sqcup \cdots \sqcup \mathfrak{P}_r$. We set $\varepsilon_i := \varepsilon_\alpha$ if $(\alpha|\alpha) > 2$ for all $\alpha \in \mathfrak{P}_i$ and $\varepsilon_i := \varepsilon$ otherwise. Then for the $r$-tuple $\varepsilon$ consisting of the $\varepsilon_i$’s, the subalgebra $U_\varepsilon^Q([\mathfrak{l}, \mathfrak{l}])$ of $U_\varepsilon^Q(\mathfrak{l})$ generated by $F_\alpha, K_\gamma, E_\beta$ for $\alpha, \beta, \pm \gamma \in \mathfrak{P}$ is isomorphic to $\bigotimes_{i=1}^r U_{\varepsilon_i}^Q(\mathfrak{l}_i)$.

It follows from the construction that for our choice of a reduced decomposition of $w_0$ and $w_0^{\Pi}$, the root vectors of each $U_{\varepsilon_i}^Q(\mathfrak{l}_i)$ are exactly the $E_\alpha, F_\beta$ for $\alpha, \beta \in \Phi^+_\Pi$. Thus, $Z_0^Q(\mathfrak{g}) \cap U_\varepsilon^Q([\mathfrak{l}, \mathfrak{l}])$ is the tensor product of the $\ell$-centers of the $U_{\varepsilon_i}^Q(\mathfrak{l}_i)$.

Now, let $U_{\eta}^Q([\mathfrak{l}, \mathfrak{l}]) := U_\varepsilon^Q([\mathfrak{l}, \mathfrak{l}])/((\text{Ker} \eta U_\varepsilon^Q(\mathfrak{g})) \cap U_\varepsilon^Q([\mathfrak{l}, \mathfrak{l}]))$. Then, for $\eta$, the restriction of $\eta$ to the $\ell$-center of $U_{\varepsilon_i}^Q(\mathfrak{l}_i)$, this algebra is the product of restricted quantized enveloping algebras, for possibly different primitive $\ell$-th roots of unity:

\begin{equation}
U_{\eta}^Q([\mathfrak{l}, \mathfrak{l}]) \simeq \bigotimes_{i=1}^r U_{\varepsilon_i}^Q(\mathfrak{l}_i)/(\text{ker} \eta)U_{\varepsilon_i}^Q(\mathfrak{l}_i).
\end{equation}

For the subset of simple roots $\mathfrak{P} \subset \Delta$ associated to $\mathfrak{l}$, let $N_\mathfrak{P}^\mathfrak{l} := N \cap \mathfrak{P}^{\perp}$, where $\mathfrak{P}^{\perp}$ is the orthogonal subgroup of $\mathfrak{P}$ in $\Lambda$ with respect to the natural pairing. Let $K_\mu^\mathfrak{P}$ be the central subalgebra of $U_\eta^N(\mathfrak{l})$ generated by $K_\mu$ for $\mu \in N_\mathfrak{P}^\mathfrak{l}$.

Identifying $\mathfrak{P}$ and $\Delta \setminus \mathfrak{P}$ with the set of corresponding indices parametrizing the simple roots, we see that a basis for $\Lambda_\mathfrak{P}^\mathfrak{l}$ is given by $\{\lambda_i, i \notin \mathfrak{P}\}$. In general, $N_\mathfrak{P}^\mathfrak{l}$ has rank $|\Delta - \mathfrak{P}|$ and $K_\mu^\mathfrak{P} \simeq \mathbb{C}[Z_\ell^{\Delta \setminus \mathfrak{P}}]$.

The following Lemma partially generalizes Theorem 4.3.

Lemma 5.3. Let $Q \subseteq N \subseteq \Lambda$ and let $\mathfrak{l}$ be a standard Levi subalgebra corresponding to $\mathfrak{P} \subseteq \Delta$. If $(\ell, |N/Q_\mathfrak{P} + N_\mathfrak{P}^\mathfrak{l}|) = 1$ then $U_\eta^N(\mathfrak{l}) \simeq U_\eta^Q([\mathfrak{l}, \mathfrak{l}]) \otimes K_\mathfrak{P}^\mathfrak{P}$.

Proof. Consider the product map $j: U_{\eta}^Q([\mathfrak{l}, \mathfrak{l}]) \otimes K_\mathfrak{P}^\mathfrak{P} \rightarrow U_\eta^N(\mathfrak{l})$. Let $M = Q_\mathfrak{P} + N_\mathfrak{P}^\mathfrak{l}$ and $m = |N/M|$. For any $\mu \in N$, $m\mu = x + y$ with $x \in Q_\mathfrak{P}$ and $y \in N_\mathfrak{P}^\mathfrak{l}$. If $(\ell, m) = 1$ then there are $b_1, b_2 \in \mathbb{Z}$ such that $\mu = (b_1\ell + b_2m)\mu = b_1\ell\mu + b_2x + b_2y$. Hence

$$K_\mu = \eta((K_{b_1\mu})^\ell)K_{b_2x}K_{b_2y} \in j(U_\eta^Q([\mathfrak{l}, \mathfrak{l}]) \otimes K_\mathfrak{P}^\mathfrak{P})$$

and $j$ is surjective. Using the PBW bases we see that $\dim U_\eta^Q([\mathfrak{l}, \mathfrak{l}]) \otimes K_\mathfrak{P}^\mathfrak{P} = \dim U_\eta^N(\mathfrak{l})$, whence the statement. \hfill \Box

We treat now the two special cases $N = \Lambda$ and $N = Q$.

Lemma 5.4. Let $\mathfrak{l}$ be a standard Levi subalgebra corresponding to $\mathfrak{P} \subseteq \Delta$, let $A_\mathfrak{P}$ be the Cartan matrix of $[\mathfrak{l}, \mathfrak{l}]$. Assume $(\ell, \det(A_\mathfrak{P})) = 1$. Then,

\begin{equation}
U_\eta^N(\mathfrak{l}) \simeq U_\eta^Q([\mathfrak{l}, \mathfrak{l}]) \otimes K_\Lambda^\mathfrak{P} \text{ and } U_\eta^Q(\mathfrak{l}) \simeq U_\eta^Q([\mathfrak{l}, \mathfrak{l}]) \otimes K_\mathfrak{Q}^\mathfrak{P}.
\end{equation}
Proof. Recall that \( \det(A_\Pi) = |A_\Pi/Q_\Pi| \). In order to show that \( j \) as in Lemma 5.3 is an isomorphism when \( N = \Lambda \), we observe that for two lattices \( M \subseteq N \) of rank \( n \) and \( A_{NM} \) the matrix expressing \( n \) generators of \( M \) in terms of \( n \) generators of \( N \) we have \( |N/M| = \det A_{NM} \). Let \( M = Q_\Pi \oplus \Lambda^\Pi \) and \( N = \Lambda \). Since a basis of \( \Lambda^\Pi \) is a subset of the fundamental weights, after a suitable reordering of the vectors, we may assume that the matrix \( A_{NM} \) is the block diagonal matrix \( \text{diag}(A_\Pi, I_{|\Delta|\Pi}) \). In particular \( \det A_{NM} = \det A_\Pi \).

We consider now the case \( N = Q \). Let \( V = \mathbb{Q} \Delta \). We rearrange the basis \( \Delta \) in such a way that the first \( \ell \) elements are the simple roots in \( \Pi \). Let \( DA \) be the symmetrized Cartan matrix of \( \Phi \), let \( D_\Pi \) be the square submatrix of \( D \) consisting of its first \( |\Pi| \) rows and columns, and let \( A' = (D_\Pi A_\Pi \; C) \) be the \( |\Pi| \times |\Delta| \) submatrix of \( DA \) consisting of the first \( |\Pi| \) rows. The orthogonal subspace \( V^\Pi \subset \Pi \) in \( V \) is given by those vectors whose coordinate columns \( X \) with respect to the reordered basis \( \Delta \) are solutions to \( A'X = 0 \). Multiplying by \( (D_\Pi A_\Pi)^{-1} \) on the left we have \( (I_{|\Pi|} - (D_\Pi A_\Pi)^{-1}C)X = 0 \). For \( B := (D_\Pi A_\Pi)^{-1}C \), a \( \mathbb{Q} \)-basis for \( V^\Pi \) is then given by the vectors of the form \( \beta_i := - \sum_{j=1}^{r} b_{ij} \alpha_j + \alpha_{|\Pi|+i} \) for \( i = 1, \ldots, |\Delta \setminus \Pi| \).

Let \( M := (\Pi, \beta_j, j = 1, \ldots, |\Delta \setminus \Pi|) \). Then, we have \( Q_\Pi \oplus Q^\Pi \subset Q \subset M \) and \( |Q/Q_\Pi \oplus Q^\Pi| \) divides \( |M/Q_\Pi \oplus Q^\Pi| \). Now, the exponent of \( M/Q_\Pi \oplus Q^\Pi \) divides \( \det(D_\Pi A_\Pi) \) because \( \det(D_\Pi A_\Pi) = \det(Q_\Pi \oplus Q^\Pi) \) for every \( j \). Since \( (\ell, \det(A_\Pi)) = 1 \) and \( (\ell, d_i) = 1 \), we have \( (\ell, |M/Q_\Pi \oplus Q^\Pi|) = 1 \), whence \( (\ell, |Q/Q_\Pi \oplus Q^\Pi|) = 1 \). Lemma 5.3 applies.

Let \( Q \subseteq N \subseteq \Lambda \) be such that \( (\ell, |N/Q_\Pi \oplus N^\Pi|) = 1 \) and let \( \eta \in \text{Spec}(Z_0^N(\mathfrak{g})) \). An immediate consequence of the above lemma is the following equality of sets

\[
\{ \dim V \mid V \in \text{Spec}(U^N_\eta([l, l])) \} = \{ \dim V \mid V \in \text{Spec}(U^Q_\eta([l, l])) \}.
\]

5.5. We are now in a position to prove the main statement. Recall that a unipotent conjugacy class \( \mathcal{O} \) is called Richardson if it is induced from the trivial class in some Levi subgroup of a parabolic subgroup. Note that for unipotent conjugacy classes this property does not depend on the isogeny type of \( G_N \). As \( U^- \) does not depend on the isogeny type, if \( \eta \) is such that \( \pi_N(\eta) \in U^- \), then \( \pi_M(\iota^* \eta) = \pi_N(\eta) \) for every \( M \subset N \) and we simply write \( \pi_M(\eta) \).

**Lemma 5.5.** Let \( Q \subseteq N \subseteq \Lambda, \eta \in \text{Spec}(Z_0^N(\mathfrak{g})) \) and \( g = \pi_N(\eta) \in B_N^- \). Assume that \( (\ell, |N/Q_\Pi \oplus N^\Pi|) = 1 \) and that \( L = C_{G_N}(g_\ell) \) is the standard Levi subgroup of a standard parabolic subgroup associated with \( \Pi \subset \Delta \). Let \( g_u \in [L, L] = L_1 \cdots L_r \) decompose as a product \( g_u = h_1 \cdots h_r \), so that \( \pi_{Q_\Pi^u}(\eta_h) = h_i \). If each \( U^Q_\eta^i(\mathfrak{l}_i)/(\ker \eta_i) U^Q_\eta^i(\mathfrak{l}_i) \) has a module of dimension \( \ell \dim \mathcal{O}_i^{N_u} \), then \( U^N_\eta(\mathfrak{g}) \) has a small module.

**Proof.** Under these assumptions, \( U^Q_\eta([l, l]) \) has a module of dimension \( \ell \dim \mathcal{O}_\eta^{L_u} \). Therefore, by Lemma 5.3, \( U^N_\eta([l, l]) \) has a module of the same dimension.Lemma 5.1 (b) applies.

**Lemma 5.6.** Let \( \eta \in \text{Spec}(Z_0^N(\mathfrak{g})) \) and \( g = \pi_N(\eta) \in U^- \). If \( \mathcal{O}_g^N \) is Richardson, then \( U^N_\eta(\mathfrak{g}) \) has a small module. In particular this hold if \( \mathfrak{g} = \mathfrak{sl}_{n+1} \) and \( \pi_N(\eta) \in U^- \).

**Proof.** By assumption we have \( g \in \text{Ind}^{G_N}_L(\mathcal{O}_L^1) \), for some standard Levi subgroup \( L \) of a parabolic subgroup of \( G \), associated with \( \Pi \). By Lemma 5.1 (a) it is enough to show that
$U_q^N(l) = U_1^N(l)$ has a 1-dimensional module. Being a Hopf algebra, the counit gives a small module. The last statement follows because all unipotent conjugacy classes in type $A_n$ are Richardson.

We are ready to state our result on small modules for $g = sl_{n+1}(C)$.

**Theorem 5.7.** If $(l, (n + 1)!)=1$ then $U_q^M(sl_{n+1})$ has a small module for every $\eta \in \text{Spec}(Z_0^M(sl_{n+1}))$ and every lattice $M$.

**Proof.** By Lemma 4.3 it suffices to consider the lattice $M = Q$. The connected centralizer $L$ of any semisimple element in $G = PSL_{n+1}(C)$ is a Levi subgroup of a parabolic subgroup of $G_N$ corresponding to some $\Pi \subseteq \Delta$ and every unipotent class in $[L, L]$ is Richardson. We apply Lemmas 5.5 and 5.6.

The condition on $l$ in the theorem is in accordance to the one given by Friedlander and Parshall in their proof of the existence of small modules for $sl_{n+1}$ in the modular case [15, Theorem 5.1].

**Remark 5.8.** Lemma 5.6 and Theorem 5.7 are special cases of the following pattern, which is similar to [15, Theorem 5.1]. By transitivity of induction, combining Lemma 5.3 with Lemma 5.4 (a) and the product decomposition in (5.8), we see that if there exists a small module for all rigid conjugacy classes in all simple factors of Levi subgroups of $G$, then there is a small module for all conjugacy classes in $G$.

### 6. The case $(l, |N/M|) \neq 1$

In this section we deal with Questions (1), (2) and (3) when $g$ is simple of type $A_n$ or $E_6$, $M \subset N$ and $d := (l, |N/M|) \neq 1$. With respect to the dimension of $g$, the smallest case to consider is type $A_2$ when $3 \mid l$. From a different perspective, with respect to $\dim \mathcal{O}_{\pi_N(\eta)}$ the smallest cases are the central $\ell$-characters. Type $A_2$ is considered in [6.3] and we describe precisely when there exists a small module for a central $\ell$-character and all $g$ in [6.3]. We observe that if $\pi_N(\eta) = 1$ then $U_1^N(g)$ is a Hopf algebra, so it always has a small module, namely the one given by the counit. For this reason we will deal only with $\eta$ such that $\pi_N(\eta) \neq 1$.

#### 6.1. An $\ell$-character $\eta \in Z_0^N(g)$ is called central if $\pi_N(\eta) \in Z(G_N)$. For such characters we have $\eta(E_\alpha^\ell) = \eta(F_\alpha^\ell) = 0$ for all $\alpha \in \Phi^+$. Moreover, $\pi_N(\eta) \in Z(G_N)$ if and only $\alpha(\pi_N(\eta)) = 1$ for every $\alpha \in \Delta$, i.e., if and only if $\eta(K_\lambda^\ell) = 1$ for all $\alpha \in Q$. In our setting, by Lemma 4.2 a central $\ell$-character $\eta$ is uniquely determined by $\eta(K_\lambda^\ell) = \omega$ a $|N/Q|$-th root of one. By (3.3) it suffices to treat the case $\eta(K_\lambda^\ell) = \omega$.

#### 6.2. If it exists, a small module $V$ for a central $\ell$-character $\eta$ has dimension 1. We recall from [12, §9.1] that in this case $E_\alpha V = F_\alpha V = 0$ for any $\alpha$ and

$$E_\alpha F_\alpha - F_\alpha E_\alpha = \frac{K_\alpha - K_{-\alpha}}{\varepsilon_\alpha - \varepsilon_{-\alpha}} \Rightarrow K_\alpha^2 = \text{id}_V.$$
6.3. We explore here the case where $E = \ell$. The example below shows that Question (2) from §6.3 has a negative answer in general.

By §6.2 the possible 1-dimensional representations of $U^\Lambda(\mathfrak{s}\mathfrak{l}_3)$ are in bijection with Hom($\Lambda/2Q, \mathbb{C}$). Let $V = \mathbb{C}$ be a 1-dimensional module affording the central $\ell$-character $\eta$. As $3\lambda_i \in Q$, we have $K_{2\lambda_i} = 1$ on $V$ and since $3 \mid \ell$, necessarily $K_{2\lambda_i} = 1$ on $V$. The image of $V$ through the map defined in [11, §4.3] is the element $t = \pi_{\Lambda}(\eta)$ in $T_\Lambda$ such that $\lambda_i(t) = \eta(K_{2\lambda_i})$ for every $i$. But $\eta(K_{2\lambda_i}) = 1$ on $V$ hence $t = \pi_{\Lambda}(\eta) = 1$. Therefore, if $\pi_{\Lambda}(\eta) \in Z(SL_3(\mathbb{C}))$ is of order 3, there does not exist a small module for $U^\Lambda(\mathfrak{s}\mathfrak{l}_3)$.

We show now that the minimal dimension of an irreducible $U^\Lambda(\mathfrak{s}\mathfrak{l}_3)$-module with central $\ell$-character $\eta$ with $\pi_{\Lambda}(\eta) \neq 1$ is at least 3. Let $w_0$ have reduced expression $s_\beta s_\alpha s_\beta$. From [19, §5], $E_{\beta + \alpha} = -E_\beta E_\alpha + \varepsilon^{-1}E_\alpha E_\beta$.

By the discussion in §6.2 it suffices to show that if $W$ is a $U^\Lambda(\mathfrak{s}\mathfrak{l}_3)$-module of dimension 2 then $E_\alpha W = 0$ for some $\gamma \in \Phi^+$ because then $\eta(K_{2\alpha}^{2\gamma}) = 1$ for every $\alpha \in Q$ and the previous argument applies. There exists a basis $\{v, w\}$ of $W$ consisting of weight vectors for $U^\Lambda(\mathfrak{h})$. If $E_\alpha W \neq 0$ then we may assume that $w = E_\alpha v \neq 0$. Moreover, since $E_\alpha$ is nilpotent, $E_\alpha^2 W = 0$. If $K_\alpha v = \lambda v$, then $K_\alpha w = \varepsilon^2 \lambda w$. As $K_\alpha E_{\alpha + \beta} v = \varepsilon \lambda E_{\alpha + \beta} v$, we have $0 = E_{\alpha + \beta} v = (-E_\beta E_\alpha + \varepsilon^{-1}E_\alpha E_\beta) v$. Hence $E_\beta w = \varepsilon^{-1}E_\alpha E_\beta w$. Now $K_\alpha E_{\beta} w = \varepsilon \lambda E_\beta w$ so as before $E_{\beta} w = 0$ and $E_{\alpha + \beta} w = (-E_\beta E_\alpha + \varepsilon^{-1}E_\alpha E_\beta) w = -E_\beta E_\alpha w = 0$ where the last equality follows from $E_\alpha^2 v = E_\alpha w = 0$. Hence $E_{\gamma} W = 0$ for $\gamma = \alpha + \beta$.

6.3.1. We look at the case $\ell = 3$ and we show that there is an irreducible representation of $U^\Lambda(\mathfrak{s}\mathfrak{l}_3)$ with central $\ell$-character $\pi_{\Lambda}(\eta) \neq 1$ and dimension 3. By the above discussion, the minimal dimension of an irreducible $U^\Lambda(\mathfrak{s}\mathfrak{l}_3)$-module is therefore exactly 3.

Let $z \in \mathbb{C}$ be such that $z^3 = \varepsilon^2$. The map $\rho: U^\Lambda(\mathfrak{s}\mathfrak{l}_3) \to \text{Mat}_3(\mathbb{C})$ given on generators by

$$
E_\alpha \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_\beta \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad F_\alpha \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_\beta \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

and $K_{\lambda_1} \mapsto \text{diag}(z, z^{-2}, z), K_{\alpha} \mapsto \text{diag}(1, \varepsilon, \varepsilon^2)$ is a representation. It is irreducible because $\rho$ is surjective. Its $\ell$-character $\eta$ is central, with $\eta(K_{\lambda_1}^{2\gamma}) = \varepsilon$. Since $U^\Lambda(\mathfrak{s}\mathfrak{l}_3)$ is a Hopf algebra we can construct the dual representation. It is again irreducible and has central $\ell$-character $\eta^*$ with $\eta^*(K_{\lambda_1}^{2\gamma}) = \varepsilon^2$.

6.4. The observations on central $\ell$-characters in the case of $\mathfrak{s}\mathfrak{l}_3$ can be generalized to arbitrary simple $\mathfrak{g}$. 

In general, consider a finite-dimensional $U^M(\mathfrak{g})$-module $W$. If $E_\alpha W = 0$ or $F_\alpha W = 0$ for some $\alpha \in \Phi$, then $K^{2\alpha}_\alpha = \text{id}_W$. Conversely, if $K^{2\alpha}_\alpha = \text{id}_W$, then for any $\beta \in \Phi$ such $(\alpha|\beta) \neq 0$ mod $\ell$ we have $F_\beta W = E_\beta W = 0$. Indeed, as $K_\alpha$ is diagonalizable, it is enough to show that $E_\beta$ and $F_\beta$ act trivially on each of its eigenspaces. However, these operators map any $\pm 1$-eigenvector to the $\pm \varepsilon^{\pm (\alpha|\beta)}$-eigenspace. The latter is trivial because $\pm \varepsilon^{\pm (\alpha|\beta)} \neq 1$. 

Proposition 6.1. Let $\mathfrak{g}$ be a simple Lie algebra, let $M$ be a lattice with $m := |M/Q|$ and $d := (\ell, |M/Q|)$. Let $\eta \in \text{Spec}(Z_0^M(\mathfrak{g}))$ be such that $\pi_M(\eta) = z \in Z(G_M)$. Then, there exists a 1-dimensional representation of $U_\eta^M(\mathfrak{g})$ if and only if the order of $z$ divides $\frac{m}{d}$.

Proof. If $d = 1$, Theorem 4.3 yields $U_\eta^M(\mathfrak{g}) \simeq U_1^Q(\mathfrak{g})$ so we assume $d \neq 1$. Here $\mathfrak{g}$ is either of type $E_6$ or $A_n$. We set $\ell' := \frac{\ell}{d}$ and $m' := \frac{m}{d}$. From Lemma 4.2, $U_\ell^M(\mathfrak{h})$ has a $\mathbb{Z}$-basis given by $\lambda_M, \alpha_1, \ldots, \alpha_{n-1}$, where the order of $\lambda_M Q$ in $M/Q$ is exactly $m$, and $m\lambda_M, \alpha_1, \ldots, \alpha_{n-1}$ is a $\mathbb{Z}$-basis for $Q$. If $V$ is a 1-dimensional $U_\eta^M(\mathfrak{g})$-module, by Lemma 6.2 we have $K_{2\alpha_j} = \text{id}_V$ for every $j$ and $\lambda_M(z) = \eta(K^{2\ell'}_{\lambda_M})$ so $\lambda_M(z^{m'}) = \eta(K^{2\ell'}_{\lambda_M}) = 1$, whence $z^{m'} = 1$. Conversely, if $z^{m'} = 1$, we define the following action of $U_\ell^M(\mathfrak{h})$ on $V = \mathbb{C}v$: we set $K_{\alpha_j}.v = v$ for every $j \neq n$ and for $a, b \in \mathbb{Z}$ and $\xi \in \mathbb{C}$ satisfying $d = a\ell + bm$ and $\xi^{2\ell} = \lambda_M(z)^a$, we set $K_{\lambda_M}.v = \xi v$. Then, $K_{2\lambda_M}.v = \xi^{2m'}v = \lambda_M(z^{m'})^{2a}v = v$, so $K_{2\alpha_n}.v = v$. Thus, setting $E_a.v = F_a.v = 0$ for every $\alpha \in \Phi^+$ gives a well-defined representation of $U_\eta^M(\mathfrak{g})$. As $E_a = F_a = 0$ for every $\alpha$, we have $\chi_M(V) \in \text{Spec}(\mathbb{C}[K^{\pm\ell}_{\mu}, \mu \in M])$. Moreover, $K_{2\alpha_j} = 1$ for every $j$, so $\tau_M \chi_M(V) \in Z(G_M)$. Finally, $K_{2\lambda_M} = \xi^{2\ell} = \xi^{2d\ell} = \lambda_M(z)^{2d} = \lambda_M(z)^{2\ell} = \lambda_M(z)^{1-bm'} = \lambda_M(z)$. Hence, $V$ is a 1-dimensional representation of $U_\eta^M(\mathfrak{g})$. □

Motivated by the positive results from Section 5 and the above discussion, we formulate a quantum analogue of Humphreys conjecture.

Conjecture 2. Let $\mathfrak{g}$ be a Lie algebra with root system $\Phi$, let $M$ be a lattice satisfying $Q \subset M \subset \Lambda$, and let $\ell$ be such that $(\ell, \mathfrak{b}(\mathfrak{g})) = 1$. Then, for every $\eta \in \text{Spec}(Z_0^M(\mathfrak{g}))$ with $\tau_M(\eta) \in \mathcal{O}$ there exists an irreducible $U_\eta^M(\mathfrak{g})$-module $V$ such that $\dim V = \ell^{\frac{2}{d}} \dim \mathcal{O}$.

Conjecture 2 holds for $\mathfrak{sl}_{n+1}$, Theorem 5.7. Evidence for this conjecture is also given by Lemmas 5.3 and 5.6. Remark 5.8 shows that it is enough to consider rigid orbits. The bound $\mathfrak{b}(\mathfrak{g})$ could be relaxed for specific $\eta$’s, see Proposition 6.1 but seems to be necessary if we wish to have a general statement for $\mathfrak{g}$.

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