Loops in inflationary correlation functions

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Abstract

We review the recent progress regarding the loop corrections to the correlation functions in the inflationary universe. A naive perturbation theory predicts that the loop corrections generated during inflation suffer from various infrared (IR) pathologies. Introducing an IR cutoff by hand is neither satisfactory nor enough to fix the problem of a secular growth, which may ruin the predictive power of inflation models if the inflation lasts sufficiently long. We discuss the origin of the IR divergences and explore the regularity conditions of the loop corrections for the adiabatic perturbation, the iso-curvature perturbation, and the tensor perturbation, in turn. These three kinds of perturbations have qualitative differences, but in discussing the IR regularity there is a feature common to all cases, which is the importance of the proper identification of observable quantities. Genuinely, observable quantities should respect the gauge invariance from the view point of a local observer. Interestingly, we find that the requirement of the IR regularity restricts the allowed quantum states.

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(Some figures may appear in colour only in the online journal)

1. Motivations and overview

The Planck satellite has provided the most precise map of perturbation at around the recombination epoch of the universe through the measurement of the cosmic microwave background (CMB). The observed spectrum of the CMB strongly suggests that there was an inflationary phase in the very early stage of the universe, which relaxes the fine tuning of the initial condition of the Big Bang universe. When we postulate the presence of a light scalar field, the inflaton, its quantum fluctuation is amplified to the observable level due to the rapid expansion during inflation, and it provides the source of the large scale structure of the universe. Thus generated primordial perturbation becomes almost scale-invariant, which is consistent with the measurement of the CMB.
Since the inflation paradigm is getting a more and more convincing scenario of the early universe, it will be high time to re-examine all the predictions based on this paradigm more carefully. In general relativity, nonlinearity necessarily enters into the evolution of the primordial perturbation. Therefore, to compute the primordial perturbation, we need to understand the interacting quantum fields in the inflationary universe. The nonlinear quantum evolution was initially discussed by considering the interacting scalar fields on a fixed background neglecting the gravitational fluctuation [1]. The systematic study of the quantum evolution of perturbation including the gravitational nonlinearity in the context of realistic inflation models was initiated by Maldacena in [2] more than 20 years after Bardeen’s gauge-invariant linear perturbation theory [3]. As a consequence of the general covariance, the description of gravitating system has the freedom in choosing the coordinates. To provide a theoretical prediction of the observable fluctuation, we need to identify the gauge-invariant degrees of freedom, excluding the gauge ambiguity. Using the Arnowitt–Deser–Misner (ADM) formalism, Maldacena derived the third-order action expressed only in terms of the so-called gauge-invariant variable. Using this nonlinear action, he has also computed the bi-spectrum of the primordial fluctuation at the tree level. The non-Gaussianity is now within the reach of observations. The constraint on the non-Gaussianities given by the Planck satellite has already excluded a number of inflation models, which highlights the importance of studying the nonlinear evolution of the primordial fluctuation.

The nonlinear evolution also generates the loop corrections. The amplitude of the loop corrections is typically suppressed by an extra power of the amplitude of the power spectrum \( \sim (H/M_{pl})^2 \), where \( H \) denotes the Hubble parameter during inflation and \( M_{pl} \) is the reduced Planck mass defined by \( M_{pl}^{-2} = 8\pi G \). However, the suppression by the factor \( (H/M_{pl})^2 \) might be compensated by the accumulation of the infrared (IR) contributions. When we assume the scale-invariant spectrum in the IR limit, a naive loop integral results in a factor \( \propto \int \frac{d^3 k}{k^3} \), which leads to the IR divergence (IRdiv). Even if we introduce an IR cutoff, say at the Hubble scale at a time \( t = t_0 \), integrating the IR modes leads to the logarithmic secular growth as

\[
\int_{a(t_0)H(t_0)}^{a(t)H(t)} \frac{dk}{k} \sim \ln \frac{a(t)}{a(t_0)},
\]

where \( a(t) \) denotes the scale factor. Therefore, the loop corrections may dominate in case inflation continues for a sufficiently long period, leading to the breakdown of perturbative expansion.

This IR singular behavior due to the accumulated IR contributions associated with a massless scalar field has long been known in the de Sitter space [4–6]. The study on the implication of this singular behavior to the interacting fields dates back to [7–9] and afterward many works such as [10–19] followed. A similar divergence has been reported for various fields [20–23]. The accumulated effect of the IR contributions has been also explored in the presence of the gravitational fluctuations [24–39]. The IR contributions for the multi scalar fields also have been studied [40–42]. (A thorough overview of the historical progress regarding the IR issues is nicely summarized in the review paper by Seery in [43].) The accumulation of the IR modes may cause the secular modification of the effective theory. Tsamis and Woodard claimed that the IR gravitons may screen the cosmological constant, which may explain the unnaturally small observed cosmological constant (the cosmological constant problem) [24]. The possible secular effects have been further examined by Woodard et al [10–12] and more recently by Polyakov et al [44–46], by Alvarez and Vidal [47], and also by Kitamoto and Kitazawa [34, 35, 48–50]. (See also the lecture note [51] and the references therein.) With these observations, one may worry if we cannot provide the sound theoretical predictions based on the inflationary universe paradigm. However, it is not manifest whether the reported IR
secular evolution is a truly physical effect or not. In particular, once we include the fluctuation of the gravitational field, we need to be careful in the discrimination between gauge-invariant physical degrees of freedom and spurious gauge artifacts.

Given that the inflationary universe includes the multi scalar fields, the physical degrees of freedom therein are decomposed into the three categories: the adiabatic perturbation, the iso-curvature perturbation and the tensor perturbation. The adiabatic perturbation describes the fluctuation of the inflaton and the iso-curvature perturbation describes the remaining scalar degrees of freedom. Note that an appropriate description of the adiabatic perturbation and the tensor perturbation requires taking into account the gravitational perturbation. While the iso-curvature perturbation is primarily well-approximated by a test field in a fixed background.

In this review, our main focus is on the adiabatic perturbation, but we also discuss the other types of perturbation briefly.

The IR issues have been addressed based on various approaches and approximations. The methodological variation may have made the mutual relation among them obscure. In this review paper, we aim at giving a consistent explanation more than giving an exhaustive review on this subject. The outline of this paper is as follows. In section 2, after we overview Maldacena’s method to compute the nonlinear contributions; in section 2.3, we will classify various divergences which appear in computing the loop corrections. The appearance of IRdivs is not peculiar to the cosmological perturbation theory. In section 2.4, we briefly overview the IR divergences in QED and QCD and compare them to those in the inflationary universe.

In section 3, we will show that the singular behavior of the IR contributions of the curvature perturbation $\zeta$ is deeply related to the influence from the outside of the observable universe, which can be rephrased as the gauge degrees of freedom in the local observable universe. In section 4, we will present how to introduce observable quantities that preserve the gauge invariance. Afterward, we will also show that requesting the regularity of the IR contributions is equivalent to requesting the gauge invariance in the local universe. In section 5, we show that the Euclidean vacuum preserves the gauge invariance in the local universe and guarantees the regularity of the loop corrections for the curvature perturbation $\zeta$. In section 6, we will summarize the recent progress regarding the IR issues for a test field in a fixed inflationary background, which is supposed to give a good approximation to the iso-curvature perturbation. In section 7, we will briefly discuss the IR issues of the graviton loops. Finally, in section 8, we will summarize the current status of this subject and will discuss the future issues.

In this review, we will introduce the following abbreviations:

- superH : super Hubble
- subH : sub Hubble
- IR : infrared
- UV : ultraviolet
- tIR : transient IR (section 2.3.2)
- IRdiv : IR divergence (section 2.3.1)
- IRsec : IR secular growth (section 2.3.2)
- SG : secular growth due to temporal integral (section 2.3.4)
- gauge : coordinate choice in the local universe (section 3.2)

where, in parentheses, we described the section in which the abbreviation is introduced.

2. Issues in calculating loop corrections

In this review, we mainly focus on a single scalar field model. For simplicity, we consider a scalar field with the canonical kinetic term, whose action takes the following form:

$$S = \frac{1}{2} \int \sqrt{-g} \left[ M_p^2 R - g^{\mu\nu} \Phi,_{\mu} \Phi,_{\nu} - 2U(\Phi) \right] d^4 x.$$  \hspace{1cm} (2)

An extension to a non-canonical kinetic term is straightforward. In this section, we explain our main concern in calculating the loop corrections to the correlation functions in this simple
model. For later use, we rescale the variables as
\[ \phi \equiv \Phi/M_{\text{Pl}}, \quad V(\phi) \equiv U(\Phi)/M_{\text{Pl}}^2, \] (3)
so that \( M_{\text{Pl}}^2 \) is factored out from the action as
\[ S = \frac{M_{\text{Pl}}^2}{2} \int \sqrt{-h} \left[ R - g^{ij} \phi_i \phi_j - 2V(\phi) \right] d^4x. \] (4)
Then, the equations of motion and the constraint equations do not depend on the Planck mass.

2.1. The action with non-locality

The ADM formalism is well suited for the nonlinear analysis of the Einstein gravity [2, 52]. The ADM line element is expressed as
\[ ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N_i dt)(dx^j + N_j dt), \] (5)
where we introduced the lapse function \( N \), the shift vector \( N_i \) and the purely spatial metric \( h_{ij} \). Using the metric form (5), we can express the action as
\[ S = \frac{M_{\text{Pl}}^2}{2} \int \sqrt{h} \left[ N^2 R - 2NV(\phi) + N(\kappa_{ij} \kappa^{ij} - \kappa^2) + \frac{1}{N} (\dot{\phi} - N^{i} \partial_i \phi)^2 - Nh_{ij} \partial_i \phi \partial_j \phi \right] d^4x, \] (6)
where \( R \) is the three-dimensional Ricci scalar, and \( \kappa_{ij} \) and \( \kappa \) are the extrinsic curvature and its trace defined by
\[ \kappa_{ij} = \frac{1}{2N} (\dot{h}_{ij} - D_i N_j - D_j N_i), \quad \kappa = h^{ij} \kappa_{ij}. \] (7)
The spatial indices \( i, j, \ldots \) are raised or lowered by the spatial metric \( h_{ij} \) and \( D_i \) which denotes the covariant differentiation associated with \( h_{ij} \). Since \( N \) and \( N^i \) are the Lagrange multipliers, varying the action with respect to them yields
\[ \dot{R} - 2V - (\kappa^{ij} \kappa_{ij} - \kappa^2) - N^{-2}(\dot{\phi} - N^{i} \partial_i \phi)^2 - h^{ij} \partial_i \phi \partial_j \phi = 0, \] (8)
\[ D_j [N(\kappa^i_j - \delta_i^j \kappa)] - N^{-1} \partial_i \phi (\dot{\phi} - N^{j} \partial_j \phi) = 0, \] (9)
which are called as the Hamiltonian and momentum constraint equations, respectively. As we will show below more explicitly, we eliminate the Lagrange multipliers \( N \) and \( N_i \) from the action, solving these constraint equations.

As for the gauge conditions, we fix the time slicing by adopting the uniform field gauge
\[ \dot{\phi} = 0. \] (10)
To impose the spatial gauge conditions, we decompose the spatial metric \( h_{ij} \) as
\[ h_{ij} = e^{2(\rho + \zeta)} [e^{\delta \gamma} h_{ij}], \] (11)
where \( \rho \equiv e^{\phi}(t) \) is the background scale factor, \( \zeta \) is the so-called curvature perturbation and \( \delta \gamma_{ij} \) is traceless, i.e., \( \delta \gamma^i_i = 0 \). As the spatial gauge condition, we impose the transverse conditions on \( \delta \gamma_{ij} \):
\[ \partial_i \delta \gamma^i_j = 0. \] (12)
We defer discussing the contribution from the tensor perturbation to section 7.

Next, we eliminate the Lagrange multipliers \( N \) and \( N_i \) to derive the action written in terms of the dynamical field \( \zeta \) alone. In the above choice of the gauge, the constraint equations are given by
\[ \dot{R} - 2V - (\kappa_{ij} \kappa^{ij} - \kappa^2) - N^{-2} \dot{\phi}^2 = 0, \] (13)
\[ D_j [N(\kappa^i_j - \delta_i^j \kappa)] = 0. \] (14)
By expanding the metric perturbations as
\[ \zeta = \zeta_I + \zeta_2 + \cdots, \]
\[ N = 1 + N_1 + N_2 + \cdots, \]
\[ N_i = N_{i,1} + N_{i,2} + \cdots, \tag{15} \]
where \( \zeta_I \) is the interaction picture field of \( \zeta \) and the subscripts indicate the order of perturbation; the \( n \)th order Hamiltonian and momentum constraints are expressed in the following form:
\[ VN_n - 3 \dot{\rho} \zeta_n + e^{-2\rho} \partial^2 \zeta_n + \dot{\rho} e^{-2\rho} \partial^i N_{i,n} = H_n, \tag{16} \]
\[ 4 \partial_i (\dot{\rho} N_n - \zeta_n) - e^{-2\rho} \partial^2 N_{i,n} + e^{-2\rho} \partial i \partial j N_{j,n} = M_{i,n}, \tag{17} \]
where \( \partial^2 \) denotes the spatial Laplacian. \( H_1 \) and \( M_{i,1} \) are 0 and \( H_n \) and \( M_{i,n} \) with \( n \geq 2 \) are the functions which consist of \( n \) interaction picture fields \( \zeta_I \). To obtain the \( n \)th order action, we need to solve the constraint equations up to \([n/2]\)th order, where the square brackets mean Gauss's floor function.

Since the constraint equations (16) and (17) are elliptic-type equations, we need to employ (spatial) boundary conditions to fix a solution for \( N_n \) and \( N_{i,n} \). As it was shown in appendix of [74], equations (16) and (17) give
\[ N_n = \frac{1}{\dot{\rho}} \zeta_n + \frac{V}{4\dot{\rho}} e^{-2\rho} (e^{2\rho} \partial^{-2} \partial^i M_{i,n} - G_n), \tag{18} \]
\[ N_{i,n} = \partial_i \partial^{-2} \left[ \frac{\dot{\rho}^2}{2\rho^2} e^{2\rho} \zeta_n - \frac{1}{\rho} \partial^2 \zeta_n + \frac{e^{2\rho}}{\dot{\rho}} H_n - \frac{V}{4\rho^2} \{ e^{2\rho} \partial^{-2} \partial^i M_{i,n} - G_n \} \right] - \left( \delta^i_j - \partial_i \partial^{-2} \partial^j \right) \left[ e^{2\rho} \partial^{-2} \left( M_{j,n} - \frac{4\dot{\rho}}{V} \partial j H_n \right) - G_{j,n} \right], \tag{19} \]
where the degrees of freedom for the boundary conditions are manifestly expressed by adding homogeneous solutions \( G_n(x) \) and \( G_{i,n}(x) \) that satisfy
\[ \partial^2 G_n(x) = 0, \]
\[ \partial^2 G_{i,n}(x) = 0. \tag{20} \]
Since the function \( G_{i,n}(x) \) contributes only through its transverse part, we see that the number of introduced independent functions is 3. By employing the appropriate boundary conditions at the spatial infinity, the degrees of freedom of the boundary conditions for these elliptic-type equations will be uniquely fixed. Substituting thus the obtained expressions for \( N_n \) and \( N_{i,n} \), the action \( S = \int d^4x L[\zeta, N, N_i] \) can be, in principle, expressed only in terms of the dynamical field \( \zeta \). Then, the evolution of \( \zeta \) is governed by a non-local action which contains the inverse Laplacian.

2.2. The non-interacting theory and the scale-invariant spectrum

In this subsection, we consider the linear theory for the curvature perturbation, which describes the evolution of the interaction picture field \( \zeta_I \). For brevity, we introduce the horizon flow functions as
\[ \epsilon_1 \equiv -\frac{1}{\dot{\rho}} \frac{d}{d\rho} \dot{\rho}, \quad \epsilon_n \equiv \frac{1}{\epsilon_{n-1}} \frac{d}{d\rho} \epsilon_{n-1}, \tag{21} \]
with \( n \geq 2 \), but we do not assume that these functions are small. The quadratic action is given by
\[ S_0 = M_p^2 \int d\tau \int d^3x e^{3\rho} \epsilon_1 [\zeta^2 - e^{-2\rho} (\partial_i \zeta)^2], \tag{22} \]
and the equation of motion for \( \zeta_I \) is given by
\[ \left[ \partial^2 + (3 + \epsilon_2) \dot{\rho} \partial_i - e^{-2\rho} \partial^2 \right] \zeta_I(x) = 0. \tag{23} \]
For quantization, we expand $\zeta(t)$ as

$$\zeta(t) = \int \frac{d^3k}{(2\pi)^3/2} a_k v_k(t) e^{i k \cdot x} + (\text{h.c.}),$$

with the mode function $v_k$ that satisfies

$$\left[ \partial_t^2 + (3 + e_2) \dot{\rho} \partial_t + e^{-2e_2 k^2} \right] v_k(t) = 0.$$

The mode function is normalized as

$$(v_k e^{i k \cdot x}, v_p e^{i p \cdot x}) = (2\pi)^3 \delta^{(3)}(k - p),$$

where the Klein–Gordon inner product is defined by

$$(f_1, f_2) = -2iM_{pl}^2 \int d^3x \, e^{i p \cdot x} f_1 \bar{f}_2 - (\partial_x f_1) \bar{f}_2.$$

With this normalization, the commutation relations for $\zeta$ and its conjugate momentum yield those for the creation and annihilation operators as

$$[a_k, a_p^\dagger] = \delta^{(3)}(k - p), \quad [a_k, a_p] = 0.$$

Using equation (24), we obtain the Wightman function of $\zeta$ for the vacuum defined by $a_k|0\rangle = 0$ as

$$G^\tau(x_1, x_2) = \langle 0|\zeta(x_1)\zeta(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot (x_1 - x_2)} v_k(t_1) v_k^*(t_2).$$

A couple of comments are in order regarding the mode function $v_k(t)$. As an implementation of the three-dimensional general covariance, the curvature perturbation is ensured to be massless, which can also be confirmed in equation (25). Because of that, when the physical wavelength $\lambda_{\text{phys}} \sim e^{\rho/k}$ becomes much longer than the Hubble scale, i.e., $\lambda_{\text{phys}} \rho \sim e^{\rho/k} \gg 1$, the growing mode of the mode equation (25) rapidly approaches a constant as

$$\frac{1}{\rho} \partial_t v_k(t) = O \left( (k/e^{\rho})^2 \right) v_k(t).$$

By choosing a solution of the mode equation, we select the vacuum state for the system with the interaction turned off. In the inflationary universe, the physical wavelength should be much shorter than the Hubble scale in the distant past. In this limit, the mode function approximately behaves as that of a harmonic oscillator with a constant frequency with respect to the conformal time,

$$\eta(t) \equiv \int^t dt' \frac{d\rho}{e^{\rho(t')}}.$$

Then, we can solve equation (25) using the WKB approximation with the asymptotic boundary condition

$$v_k(t) \approx \frac{1}{M_{pl} \sqrt{2e_3}} \sqrt{\frac{\rho}{2k}} e^{-ik\eta(t)} \frac{1}{\sqrt{2k}} e^{-ik\eta(t)}.$$

The vacuum state defined by this WKB solution is called the adiabatic vacuum. When the background spacetime is approximated by the de Sitter space, which is the case for $k/(e^{\rho/2}) \gtrsim 1$ in the slow roll inflation, the mode function for the adiabatic vacuum is reduced to

$$v_k(t) \approx \frac{i}{\sqrt{2k^3} \sqrt{2e_3}} \left( \frac{\rho}{M_{pl}} \right) \left( 1 + ik\eta(t) \right) e^{-ik\eta(t)}.$$

In this case, the power spectrum becomes almost scale-invariant in the IR limit as

$$P(k) \equiv |v_k(t)|^2 = \frac{1}{4k^3} \frac{1}{e_3(t_k)} \left( \frac{\dot{\rho}(t_k)}{M_{pl}} \right)^2 \left( 1 + O((k\eta)^2) \right),$$

where we evaluated $v_k(t)$ at the Hubble crossing time $t = t_h$ with $k = e^{\rho(t_h)} \dot{\rho}(t_h)$, since the curvature perturbation gets frozen rapidly after the time $t_h$.  

6
2.3. Various types of divergences

Now, we consider the $n$-point functions of the curvature perturbation $\zeta$, turning on the interaction. Using the in–in (or equivalently the closed time path) formalism [53], the $n$-point function for $\zeta$ is calculated as

$$\langle \zeta(t_1, x_1) \cdots \zeta(t_n, x_n) \rangle = \langle U_I(t_1, t_2) \zeta_I(t_1, x_1) \cdots \zeta_I(t_n, x_n) U_I(t_2, t_1) \rangle,$$

(35)

where $t_i$ is an initial time and

$$U_I(t_1, t_2) = T \exp \left[ -i \int_{t_1}^{t_2} ds \int d^3 x H_I(s, x) \right],$$

(36)

is the unitary operator with the interaction Hamiltonian density $H_I(x)$, which consists of the interaction picture field $\zeta_I$. Using equation (35), we can expand the $n$-point functions for $\zeta$ in terms of the Wightman propagator $G^{\pm}(x_1, x_2)$.

A naive computation of the $n$-point functions tells us that the loop integrals of perturbations in an inflationary spacetime apparently have various kinds of unsuppressed contribution from the deep IR modes. In this subsection, we illustrate and classify the potential origins of such pathological behaviors. The first three (discussed in section 2.3.1–2.3.3) are related to the momentum integrals, while the last (discussed in section 2.3.4) is originating from the time integral. Here, our illustration is focusing on the curvature perturbation in the single field models, but almost the same arguments will also follow for the tensor perturbation.

2.3.1. The IR divergence. When we assume that the corresponding free theory has an almost scale-invariant spectrum in the IR limit, a naive consideration can easily lead to the IRdiv due to the loop corrections. To explain this, we pick up the following quartic interaction vertex:

$$H_I(x) \ni \left\{ \zeta_I(x) \right\}^2 \left\{ \frac{\partial_i e^\rho}{e^\rho} \dot{\zeta}(x) \right\}^2$$

(37)

from the interaction Hamiltonian density $H_I(t)$, where we abbreviated the unimportant time-dependent coefficients. Using the in–in formalism, we find that the one-loop diagram depicted in figure 1 obtained from the contraction between the two $\zeta_I$s in the interaction vertex (37) yields the factor

$$G^+(x, x) = \langle (\zeta_I(x))^2 \rangle = \int \frac{d^3 k}{(2\pi)^3} |v_k(t)|^2.$$

(38)

We can easily understand that this momentum integral logarithmically diverges in the IR as $\int d^3 k / k^3$ for the scale-invariant spectrum. Even if the spectrum is not completely scale-invariant as given in equation (34); deep IR modes contribute to $\langle \zeta_I^2 \rangle$ significantly. We refer to the appearance of such an unsuppressed momentum integral for small $k$ as IRdiv, even though the integral does not diverge for the blue spectrum. Note that we encounter the same IRdiv also in a free theory, when we evaluate the spectrum in the position space.
2.3.2. The IR secular growth. One may think of regularizing the IRdiv by introducing an IR cutoff. When we introduce the IR cutoff, say at the Hubble scale for the initial time $t_i$, the variance of the super Hubble (superH) modes

$$\langle \{\zeta_I(x)\}^2\rangle_{\text{superH}} \propto \int \frac{dk}{k} \ln \left( \frac{\rho(t_0)}{\rho(t)} \right)$$

shows the secular growth which is logarithmic in the scale factor $a = e^\rho$. Then, the loop corrections, which are suppressed by an extra power of the amplitude of the power spectrum $(\dot{\rho}/M_{pl})^2$, may dominate in case inflation continues sufficiently long, leading to the breakdown of perturbation. We refer to the modes with $e^{\rho(t_i)}\dot{\rho}(t_i) \lesssim k \lesssim e^{\rho(t)}\dot{\rho}(t)$ as the transient IR (tIR) modes and refer to the enhancement of the loop contributions due to the tIR modes as the IR secular growth (IRsec), discriminating it from IRdiv. To be precise, we define the tIR modes as such that were in the sub Hubble (subH) range at the initial time $t_i$, but were transmitted into superH ones by the time $t$ as shown in figure 2. As inflation proceeds, the range of the tIR modes increases, which leads to the IRsec. Equation (39) shows that the introduction of an artificial comoving IR cutoff eliminates IRdiv, but it does not cure IRsec.

Note that the IRsec manifestly depends on the value of the IR cutoff. When we denote the comoving wavenumber for the IR cutoff as $k_{IR}$, the secular growth factor is given by $\ln(e^{\rho(t_0)}/k_{IR})$. In [54], Lyth discussed the cutoff-dependence of the correlation functions and proposed to set the IR cutoff length scale to a slightly larger scale than the observable universe, i.e., $k_{IR} \sim e^{\rho_0}\rho_0$, where the subscript 0 indicates the quantities evaluated at the present epoch (see also [55]). This cutoff-dependence was studied to the two-loop order by Bartolo et al in [56, 57]. More recently, introducing an IR cutoff $k_{IR}$, Byrnes et al examined the influence of the IRsec on the non-Gaussian parameters in [58–60]. Although introducing an IR cutoff might give an almost correct practical prescription, it should be verified to be a proper way to compute observable quantities. As Enqvist et al pointed out, the present Hubble scale is not a critical scale of the theory beyond which the presence of fluctuations is prohibited and hence the introduction of the IR cutoff at $k_{IR} \sim e^{\rho_0}\rho_0$ is ad hoc [61].
The potential IRsec has also been addressed from the point of the gradual change of the effective coupling constants. A particular interest is in the screening of the cosmological constant. As an example of the secular change of an effective coupling constant, we consider how the cosmological constant is affected by the loop correction due to a massless scalar field with a quartic potential $V(\phi) = \frac{\lambda \phi^4}{4!}$ in a fixed quasi-de Sitter background. By picking up only the tIR modes, one may naively expect that the variance of the free massless field would be given by

$$\langle \phi^2 \rangle_{\text{tIR}} \approx \left( \frac{\dot{\rho}}{M_{\text{pl}}} \right)^2 \ln \left( \frac{e^{\rho(t)}}{e^{\rho(t_i)}} \right).$$

If we could simply trust this expression, the expectation value of the potential term in the energy–momentum tensor would be evaluated as

$$\langle T_{\mu\nu} \rangle \cong -g_{\mu\nu} \langle V(\phi) \rangle \approx -\lambda g_{\mu\nu} \left( \frac{\dot{\rho}}{M_{\text{pl}}} \right)^2 \ln \left( \frac{e^{\rho(t)}}{e^{\rho(t_i)}} \right),$$

signaling the time-dependence of the cosmological constant. In this simple example, the cosmological constant increases but the screening may happen when we consider different field contents [10, 24, 25]. (For the detail of computation, see [62] and references therein.)

Of course, the secular change of the coupling constants due to the superH modes should be examined more carefully. The evolution of the superH modes can be naively understood based on the stochastic approach, which was initiated by Starobinsky [63], while the quantum loop effect is not essential there. As we shall discuss in more detail in section 6.1, in this approach the evolution of the field value averaged over the Hubble scale, $\bar{\phi}$, is described as a stochastic motion caused by the successive addition of modes transmitted from the subH modes to the tIR modes. This stochastic diffusion balances with the deterministic force pushing the average value toward the bottom of the potential in the end. As a result, the variance of the massless scalar field with the quartic potential approaches

$$\langle \bar{\phi}^2 \rangle \to \frac{\dot{\rho}^2}{(\sqrt{\lambda}M_{\text{pl}})^2}$$

after a sufficiently long time [64]. If we start with $\bar{\phi} = 0$, the stochastic diffusion dominates and the $\bar{\phi}$ deviates from the bottom of the potential. One can understand that $\langle \phi^2 \rangle_{\text{tIR}}$ increases in time because of this effect. Then, the energy–momentum tensor in each horizon patch will naturally have the value corresponding to $V(\bar{\phi})$, whose ensemble average will give the result in equation (40). However, the local physics in each Hubble patch is still described by the original $\lambda \phi^4$ model with the stochastic background value of $\bar{\phi}$. This stochastic interpretation mentioned above suggests that the secular change of the coupling constants obtained by explicit calculations does not necessarily mean that the accumulated IR modes can modify the local physics law.

In the above discussion, the metric perturbations have been neglected. Once we include them, we also need to pay attention to the gauge issue, which is our main focus of this review.

2.3.3. The inverse Laplacian. Another complication may arise from the inverse Laplacian operators, $\partial^{-2}$, contained in the expression for the lapse function $N$ and the shift vector $N_i$, in (18) and (19). Using these expressions for $N$ and $N_i$, we see that the interaction Lagrangian written in terms of the curvature perturbation $\xi$ contains $\partial^{-2}$. These inverse Laplacian operators are always associated with at least two derivative operators in the action, because $\partial^{-2}$ in $N$ explicitly accompanies the two spatial derivative operators, while $\partial^{-2}$ in $N_i$ accompanies at least one and $N_i$ is always multiplied by at least one spatial derivative operator in the action. For example, besides the unimportant time-dependence, the interaction Hamiltonian contains a term like

$$\mathcal{H}_I (x) \cong \frac{1}{\alpha^2} \hat{\xi}_I (x) \partial^i \xi_i (x) \partial^{-2} \left( \xi_I (x) \partial_i \xi_i (x) \right).$$

(41)
Using this vertex, we can consider the one-loop diagram as shown in the left-hand panel of figure 1, which yields the factor in the Fourier components

\[ \int d^3k_1 \frac{k \cdot k_1}{|k_1 - k_1|^2} \partial_t G_{k_1}(t, t') \bigg|_{t' \to t}. \] 

(42)

This factor is very pathological. For any value of \( k_1 \), the integrand is divergent. A different kind of pathology may appear from the two-loop diagram shown in the right-hand panel of figure 1. If we use the interaction vertex in equation (41) twice, the diagram yields the factor in Fourier components

\[ \int d^3k_1 d^3k_2 \frac{(k_1 \cdot k_2)^2}{|k - k_1|^2} G_{k_2}^+(t, t') G_{k_1}^+(t', t') \partial_t \partial_{t'} G_{k_1-k_2}(t, t'). \] 

(43)

Unless some non-trivial cancellation occurs, the contribution at around \( k_1 = k \) of this diagram diverges. We need to make sure that the inverse Laplacian operator does not give rise to a singular pole in the momentum integral.

2.3.4. The secular growth due to temporal integral. The remaining issue is the possible secular growth (SG) due to the accumulated contribution from the temporal integral. If the contribution to some observable quantity from the interaction vertex in the far past remains unsuppressed, it will diverge when we send the initial time to the infinite past. We think that the discrimination of this effect from the previously introduced IRsec is important. The main difference is in that the IRsec can be discussed without taking into account subH modes, specified by \( k \gtrsim e^{\rho(t)} \dot{\rho}(t) \), while the SG, in general, can be caused by the contribution from the vertex composed of the subH modes. In [65, 66], Weinberg investigated the SG from the time integration performing the time integral with the momenta of the propagators fixed. He assumed that the mode function in the limit \( k \gg e^{\rho(t)} \dot{\rho}(t) \) oscillates very rapidly and hence the subH modes \( k \gtrsim e^{\rho(t)} \dot{\rho}(t) \) give only a little contribution. This assumption will not be verified for an arbitrary initial state. Actually, in general, a time integration includes a mixture of the positive and negative frequency mode functions, which yields the phase in the UV limit \( e^{\eta(t)}(k_1-k_2+k_3-\cdots) \). Then, the phase does not necessarily exhibit the rapid oscillation even for the modes with \( -k_\infty \eta(t) \gg 1 \). In section 5, we will show that when we fix an initial state by employing the so-called \( i\epsilon \) prescription, the assumption of the rapid oscillations is satisfied.

The subH modes also include ultraviolet (UV) modes with \( k \gg e^{\rho(t)} \dot{\rho}(t) \). We refer to the divergence due to the UV modes as the UV divergence. In [65, 67], the UV divergence has been identified by using the dimensional regularization. Initially, it was pointed out that the integral over the subH modes can also contribute to the SG, but this SG is shown to be an artifact by means of a consistent dimensional regularization [67]. In this review, therefore, we will not provide a rigorous argument about the UV regularization.

2.4. The IR divergences in QED and QCD

It is widely known that the IRdiv also appears in QED or non-Abelian gauge theories. A frequently asked question is whether the IRdivs in these gauge theories in the flat spacetime has something to do with the IR pathologies discussed in the previous subsection. In the case of the IRdiv in gauge theories, we can discretize the singular pole by using the dimensional regularization, because changing the spacetime dimension from \( D = 4 \) to \( D = 4 - \delta \) with a negative \( \delta \) reduces the power in the IR, relaxing the singular behaviors in the IR limit. In contrast, in the cosmological setup, changing the spacetime dimensions does not change the behavior in the IR. For example, in the \( D \)-dimensional de Sitter space, the power
spectrum of a massless scalar field is given by \( P(k) \propto \frac{1}{k^{D-1}} \), and hence the logarithmic divergence remains as [68]

\[
\int \frac{d^{D-1}k}{(2\pi)^{D-1}} P(k) \propto \int \frac{dk}{k}
\]

indicating that the dimensional regularization cannot regularize the logarithmic divergence associated with the IR contributions.

In QED and QCD, the IRdivs from the vertex corrections are canceled in the cross section by the ones from the soft photon and gluon radiations, respectively, (see [69] and references therein). This is an implementation of the Kinoshita–Lee–Nauenberg theorem, which states that in a theory with massless fields, the soft divergence should be canceled in the transition rates, if we sum over the initial and final degenerate states. Here, the degeneracy means that an electron accompanied by an arbitrary number of soft photons cannot be distinguished from a single electron in an experiment. The roll played by the soft photons might be attributed to the IR modes in the perturbation of the inflationary universe. It is analogous to the property of soft photons that the IR modes are hardly detected by the local observers. In fact, Seery suggested an analogy between the Fokker–Planck equation in the stochastic approach [63] and the equation which describes the evolution of the parton distribution functions in [70]. Hence, a similarity between these two cases might be worthy of examination.

However, the origin of divergence in QED is the neglection of the states with soft photons. In contrast, in the calculation of cosmological perturbation, what we calculate as observable quantities are the correlators in a particular quantum state. In this case, the degeneracy due to the IR modes in the final state is not neglected, because all the possible final states are automatically summed up in the in–in formalism. On the other hand, as for the initial state, it might be still suggestive to claim by analogy that all the IR fluctuations must be added to the initial state in a proper way to avoid the IRdiv and IRsec, but the precise meaning of this speculative statement is not so clear.

### 2.5. The dilatation symmetry

The regularization of the IRdiv and IRsec, which are both caused by the superH modes, has been attempted, based on various methods [71–85]. In this subsection, we give a brief review on the previous works, focusing on the dilatation symmetry that must be fully taken into account in proving the absence of the IRdiv and IRsec.

As it is expected from the fact that the spatial metric is given in the form \( e^{2(\rho + \zeta)} dx^2 \), a constant shift of the dynamical variable \( \zeta \) can be absorbed by the overall rescaling of the spatial coordinates. Hence, the action for \( \zeta \) preserves the dilatation symmetry

\[
x^i \rightarrow e^{-s} x^i, \quad \zeta(t, x) \rightarrow \zeta(t, e^{-s} x) - s,
\]

where \( s \) is a constant parameter. (In the literature this dilatation symmetry has been addressed many times. See, for instance, [86, 87] and the references therein.) One may naively expect that we can remove the divergent IR contribution in \( \zeta \) using this constant shift. In fact, if we set the parameter \( s \) to \( \zeta(t_i) \), the averaged value of \( \zeta \) over the Hubble patch at \( t_i \), the logarithmically divergent Wightman function would be regularized. For instance, its coincidence limit, \( \langle \{ \zeta(t_i, x) \}^2 \rangle \), would be replaced with \( \langle \{ \zeta(t_i, x) - \tilde{\zeta}(t_i) \}^2 \rangle \), whose superH modes give

\[
\langle \{ \zeta(t_i, x) - \tilde{\zeta}(t_i) \}^2 \rangle_{\text{superH}} \propto \int e^{(\rho(t) + \tilde{\rho}(t_i) - 2\tilde{\rho}(t_i))} \frac{dk}{k},
\]

where
where the comoving radius of the Hubble patch is given by $1/(\rho(t_i)\dot{\rho}(t_i))$. As we discussed in equation (39), although the introduction of the comoving IR cutoff eliminates the IRdiv, it does not eliminate the IRsec.

One may think that if the system can be described in such a way that the symmetry under the time-dependent dilatation transformation is manifest, the logarithmic growth of $\langle \{ \zeta(t, x) - \bar{\zeta}(t_i) \}^2 \rangle$ might be eliminated by setting $s(t)$ to the time-dependent spatial average in the Hubble patch. However, the reduced action written in terms of $\zeta$ does not preserve the invariance under the dilatation transformation with $s(t)$ being time-dependent. For example, in [87], the authors showed that when we consider the whole universe with the infinite spatial volume, the dilatation transformation should be time-independent to preserve the action invariant. In addition, the $n$-point functions with the inverse Laplacian $\partial^2$ in the interaction vertexes do not seem to be regularized merely by considering the dilatation symmetry. This quick consideration indicates that the presence of the dilatation symmetry may play an important role to show the absence of the IRdiv and IRsec, but it is not enough to resolve these pathologies.

3. The causality and the gauge invariance

Our goal is to judge whether or not the various divergences mentioned in the preceding section are just artifacts. In this review, we will show that the actual observable quantities are not spoiled by these divergences. In this section, we will provide the basic ingredients in the discussion. We will also clarify that a short proof based on the naive arguments is quite unsatisfactory.

3.1. Influence from the causally disconnected region

First, we define the observable region as the region causally connected to us. We denote the observable region on the time slicing at the end of inflation $t_f$ and its comoving radius as $O_{t_f}$ and $L_{t_f}$, respectively. The causality requires that $L_{t_f}$ should satisfy $L_{t_f} \lesssim \int_t^{t_f} dt/e^\rho(t)$, where $t_0$ is the present time. What we will detect through the cosmological observations will be the $n$-point functions of the fluctuation with the arguments $(t_f, x)$ contained in the observable region $O_{t_f}$. For later use, we refer to the causal past of $O_{t_f}$ as the observable region $O$ and refer to the intersection between $O$ and a $t$-constant slicing $\Sigma_t$ as $O_t$ (see figure 3). We approximate the comoving radius of the region $O_t$ as

$$L_t \equiv L_{t_f} + \int_t^{t_f} dt'/e^\rho(t') \simeq L_{t_f} + \frac{1}{e^\rho(t_0)\dot{\rho}(t)} + \frac{1}{e^\rho(t)\dot{\rho}(t)},$$

which approaches the comoving Hubble radius, $1/e^\rho(t)\dot{\rho}(t)$, in the distant past.
One can argue that the effects of the superH modes with $k \lesssim e^{\rho(t)}\dot{\rho}(t)$ are the influence from the outside of the observable region $O$. These modes potentially affect the fluctuations in $O_{tf}$ by two ways. One is due to the non-local interaction through the inverse Laplacian $\partial^{-2}$, while the other is through the Wightman function $G^+(x_1, x_2)$. Even if the spatial distance $|x_1 - x_2|$ is bounded from above by confining $x_1$ and $x_2$ within the observable region, the contribution to $G^+(x_1, x_2)$ from the IR modes with $k \leq |x_1 - x_2|^{-1}$ are not suppressed. These modes make $G^+(x_1, x_2)$ divergent for scale-invariant or red-tilted spectrum. To regularize the contribution from the superH modes, we need to prove the suppression of their effects.

3.2. The residual coordinates degrees of freedom in the local universe

In the previous subsection, we introduced the observable region $O$, which is a limited portion of the whole universe. We claim that the observable fluctuation must be composed of fluctuations in $O$. Furthermore, since the information that we can access is limited to within $O$, there is no reason to request the regularity at the spatial infinity in solving the elliptic constraint equations (16) and (17), at least, at the level of Heisenberg equations of motion. Then, there arise degrees of freedom in choosing the boundary conditions, which appear as the arbitrary homogeneous solutions of the Laplace equation, $G_n(x)$ and $G_{i,n}(x)$ in equations (18) and (19). These arbitrary functions in $N$ and $N_i$ can be understood as the degrees of freedom in choosing the spatial coordinates. Since the time slicing is fixed by the gauge condition (10), the residual gauge degrees of freedom only reside in the spatial coordinates $x^i$.

As we have shown in [71, 72], these residual coordinate transformations associated with $G_n(x)$ and $G_{i,n}(x)$ are expressed as

$$x^i \rightarrow x^i - \sum_{m=1}^{\infty} s^i_{j_1 \ldots j_m}(t) x^{j_1} \ldots x^{j_m} + \ldots,$$

(47)

where $s^i_{j_1 \ldots j_m}(t)$ are the symmetric traceless tensors, which satisfy $\delta^{ij} s^i_{j_1 \ldots j_m}(t) = 0$. Here, we abbreviated the nonlinear terms in equation (47). These transformations diverge at the spatial infinity, no matter how small the coefficients are. In contrast, restricted to the local region, the magnitude of the coordinate transformations (47) is kept perturbatively small. Since the transformations (47) are nothing but coordinate transformations, the Heisenberg equations of motion for the diffeomorphism invariant theory remains unchanged under these transformations. Note that these coordinate transformations include the dilatation transformation with the time-dependent function $s(t)$.

We should note that, once we substitute the expressions for $N$ and $N_i$ to obtain the equation of motion solely written in terms of the curvature perturbation $\zeta$, the symmetry under the residual coordinate transformations is lost, because $N$ and $N_i$ depend on the specified boundary conditions. Although in this sense, the coordinate transformations (47) are to be distinguished from the usual gauge transformation that leaves the overall action invariant, we are accustomed to call infinitesimal coordinate transformations gauge transformation. To avoid confusion, we distinguish the coordinate transformations (47) as the gauge transformation by using the italic font.

3.3. The IR issues and changing the local average

In the previous subsection, we pointed out the presence of residual gauge degrees of freedom from the point-of-view restricted to the local observable universe. Among them, here we focus on the dilatation transformation $x^i \rightarrow e^{-s(t)}x^i$, denoting the trace part of $s^i_j(t)$ as $s(t)$. As we mentioned earlier, this dilatation transformation shifts the spatial curvature perturbation as
\[ \zeta(t, x) \rightarrow \zeta(t, e^{-\delta t} x) - s(t), \] and hence, it can be understood as the subtraction of the local average in the observable region \( \mathcal{O} \) from \( \zeta \). (As shown in [71, 72], one of the residual gauge transformations (47) can absorb the local average of the tensor perturbation as well.) The gauge invariance will imply that the quantities that we can observe in actual measurements should be insensitive to this change of the local average of \( \zeta \).

As far as we know, computing the local observable quantity was first emphasized in the discussion of the long wavelength fluctuations by Unruh [88]. Afterward, Geshnizjani and Brandenberger examined the behavior of the long wavelength fluctuations by considering a local quantity [89]. They computed the local expansion rate \( \Theta \equiv u_{\mu} \), where \( u_{\mu} = \partial_{\mu} \phi / \sqrt{\partial_{\nu} \phi \partial^\nu \phi} \) in a single clock inflation. They showed that when the local expansion rate \( \Theta \) is evaluated as a function of the clock field \( \phi \), \( \Theta \) is not affected by the long wavelength modes, basically staying at the background value, \( \Theta \simeq \sqrt{V(\phi)}/3 \). In contrast, \( \Theta \) as a function of the cosmological time \( t \) suffers from the logarithmic secular growth discussed in section 2.3. Their analysis is totally classical, but their result suggests that the accumulation of the superH modes may disappear, if we evaluate genuinely gauge invariant quantities. (See also their discussion in two field models [90].)

In [77–79], the leading IR logarithms of the curvature perturbation are discussed focusing on the dilatation transformation, which introduces the shift of \( \zeta \). In these references, the time-independent dilatation is addressed, but here we extend it to the time-dependent one to consider both the IRdiv and IRsec. In [77, 78], the authors introduced the spatial average of the curvature perturbation in the Hubble patch with the size \( L_r \sim 1/(e^\rho \dot{\rho}) \), which is roughly expressed as \( \bar{\zeta}(t) \sim \int |k| < e^\rho \dot{\rho} d^3k \zeta_k(t) \) in terms of the Fourier components \( \zeta_k \). As inflation proceeds, the number of the modes that contribute to \( \bar{\zeta}(t) \) increases, leading to the secular growth of \( \bar{\zeta}(t) \). Because of the contribution of \( \bar{\zeta}(t) \), the physical meaning of the comoving coordinates \( x \) is effectively modified as \( e^{\bar{\zeta}(t)} x \). The IRdiv and IRsec computed in the \( \delta N \) formalism are shown to agree with the divergent contributions which appear from the above modification of the physical distance due to \( \bar{\zeta}(t) \). In [79], a similar argument is provided based on a semi-classical approach.

3.4. How to fix the residual gauge degrees of freedom

In the previous subsection, we claimed that the IRdiv and IRsec are deeply related to the presence of the residual gauge degrees of freedom. In this subsection, we discuss several attempts to show the absence of the IRdiv and IRsec by fixing the residual gauge degrees of freedom. The discussions in this subsection will not complete a rigorous proof of the IR regularity, but giving an overview of these attempts will be instructive to capture a key aspect that should be taken into account in proving the absence of the IRdiv and IRsec.

3.4.1. Absorbing the IR divergence by gauge fixing. One way to preserve the invariance under the gauge transformation is fixing the gauge conditions completely. The residual gauge degrees of freedom explained above can also be removed by employing the additional gauge conditions that fix the boundary conditions for \( N \) and \( N_i \) at the boundary of the local region \( \mathcal{O} \). Then, the IR regularity may be explicitly shown by performing the quantization in this local region, since the wavelengths that fit within this local region \( \mathcal{O} \) are bounded by the size of the region \( \mathcal{O} \). Although the quantization in the local region might be an interesting idea, it is not clear how to select a natural initial quantum state for the system after the removal of the residual gauge degrees of freedom. Even the spatial translation symmetry of the quantum state...
cannot be easily guaranteed in this approach, because it is manifestly broken by the boundary conditions imposed at a finite distance.

In [75], to preserve the global translation symmetry in the spatial directions manifestly, the initial state is set at an initial time \( t = t_i \) without fixing the residual gauge degrees of freedom. Then, a shift of the spatial coordinates \( x \rightarrow x + a \) is simply absorbed by multiplying the overall phase factor \( e^{i\phi(a)} \) to each Fourier mode. After setting the initial quantum state in this way, the residual gauge transformation (47) is performed to absorb the IR contributions. In this approach, one can show the absence of the IRdiv and IRsec, if they are absent at the initial time, which however, is not guaranteed. Below, we briefly summarize the discussion in [75].

As usual, in quantizing the curvature perturbation \( \zeta(x) \), we first consider the whole universe. The initial conditions for \( \zeta(x) \) and the conjugate momentum \( \pi(x) \) are set by

\[
\zeta(t_i, x) = \zeta_i(t_i, x), \quad \pi(t_i, x) = \pi_i(t_i, x),
\]

with the corresponding interaction picture fields \( \zeta_I(x) \) and \( \pi_I(x) \). The mode expansion of \( \zeta_I(x) \) is given in equation (24). Then, we perform the dilatation transformation \( x' \rightarrow e^{-i\phi(x')} \), which is one of the residual gauge transformation, and the curvature perturbation transforms as

\[
\zeta(t, x) \rightarrow \zeta'(t, x) = \zeta(t, e^{-i\phi}x) - s(t).
\]

We fixed the time-dependent parameter \( s(t) \), requesting

\[
\tilde{\zeta}'(t) = 0,
\]

where \( \tilde{\zeta}'(t) \) is the local spatial average in the observable region, defined by

\[
\tilde{\zeta}'(t) \equiv \frac{\int d^3x W_t(x)\zeta'(t, x)}{\int d^3x W_t(x)},
\]

with a window function \( W_t(x) \) which is non-vanishing only in the local region \( O_t \). Then, \( \zeta'(x) \) becomes the curvature perturbation in the gauge more relevant to the local observable universe. Here, we recast the discussion given in the flat gauge in [75] into the one in the uniform field gauge.

The issue of the inverse Laplacian addressed in section 2.3.3, can also be solved by using the remaining residual gauge degrees of freedom. If we choose the boundary conditions for \( \partial^{-2} \) in \( N_t \) and \( N_t \) appropriately, \( N_t \) and \( N_t \) in the region \( O_t \) can be specified by the fluctuations only within \( O_t \). In the general solutions of \( N_t \) and \( N_t \) given in equations (18) and (19), the residual gauge degrees of freedom are expressed by the arbitrary homogeneous solutions of the Laplace equation, \( G_n(x) \) and \( \left( \delta A - \partial^{-2} \partial A \right) G_{I,n}(x) \). We fix the homogeneous solution \( G_n(x) \), requesting that \( \partial^{-2} \partial A G_{I,n}(x) \) in \( N_t \) should satisfy

\[
-\frac{1}{4\pi} \int \frac{d^3y}{|x-y|} W_t(y) \partial^2 M_{I,n}(t, y) = \partial^{-2} \partial^2 M_{I,n}(x) - e^{-2\phi} G_n(x),
\]

in the observable region \( O_t \). Similarly, using the transverse part of \( G_{I,n}(x) \), we can fix the boundary conditions for the remaining \( \partial^{-2} \) so as to eliminate the influence from the region far outside of \( O_t \). (For a detailed explanation, see appendix of [74].) Then, all the interaction vertices are confined to the neighborhood of \( O_t \).

After the gauge fixing, the Heisenberg equation for \( \zeta'(x) \), which is perturbatively expanded as \( \zeta'(x) = \zeta_n'(x) + \zeta_2'(x) + \cdots \), can be iteratively solved as

\[
\zeta_n'(x) = \zeta_n(t, x) - \frac{\int d^3x W_t(x)\zeta_n(t, x)}{\int d^3x W_t(x)}.
\]

where \( \zeta_n(x) \) is given by

\[
\zeta_n(x) = -2M_p^2 \int dt' \int dx' \zeta_1(t') e^{3i\phi(t')} G_R(x, x') \Gamma_{n}(x'),
\]
with the retarded Green function that satisfies
\[ \left[ \partial_t^2 + (3 + \varepsilon_2) \partial_t - e^{-2\varepsilon_2} \partial^2 \right] G_R(x, x') = -\frac{1}{2M_{pl}^2} \frac{1}{\varepsilon_1 e^{\beta\varepsilon}} \delta^{(4)}(x - x'). \] (55)

Here, the interaction picture fields \( \zeta'(t, x) \) and its conjugate momentum \( \pi'(t, x) \) satisfy the initial conditions
\[ \zeta'(t_i, x) = \zeta(t_i, x) - \bar{\zeta}(t_i), \quad \pi'(t_i, x) = \pi(t_i, x), \] (56)
and \( \Gamma_n(x) \) denotes the nonlinear interaction terms that include \( n \zeta'(x) \)s. After we fixed the boundary condition of the inverse Laplacian as above, \( \Gamma_n(x) \) becomes a local function which is not affected by the far outside of \( \mathcal{O} \). Reflecting the fact that the retarded Green function \( G_R(x, x') \) takes a non-vanishing value only if the two points \( x \) and \( x' \) are causally connected, the above expression manifestly preserves the (approximate) causality. Note that the local average of \( \zeta'(x) \), given by operating \( \int d^3x W_k(x) \) on equation (53), vanishes as it is requested from the gauge condition (50).

Using equations (53) and (54), we can expand the correlation functions for the curvature perturbation \( \zeta(x) \) in terms of the retarded Green function and the correlation functions for \( \zeta' \). Since the integration region of the vertex integrals are restricted to the local observable region \( \mathcal{O} \), if the correlation functions for \( \zeta' \) were all finite, those for \( \zeta(x) \) would be finite as well. Here, however, we should note that \( \zeta' \) is not linear in the original interaction picture field \( \zeta(t) \).

The gauge condition (50) implies that the part of \( \zeta'(x) \) which is linear in \( \zeta(t) \) should be given by
\[ \zeta'(t, x) = \zeta(t, x) - \bar{\zeta}(t), \] (57)
where \( \bar{\zeta}(t) \) denotes the local average of \( \zeta(t) \), given by
\[ \bar{\zeta}(t) \equiv \frac{\int d^3x W_k(x) \zeta(t, x)}{\int d^3x W_k(x)}. \] (58)

Inserting the mode expansion of \( \zeta(t, x) \), given in equation (24) into equation (57), the two-point function for \( \zeta'(x) \) is calculated as
\[ \langle \zeta'(x_1) \zeta'(x_2) \rangle = \int \frac{d^3k}{(2\pi)^3} \left( e^{ikx_1} - \frac{\hat{W}_k(-k)}{W_k(0)} \right) \left( e^{-ikx_2} - \frac{\hat{W}_k(k)}{W_k(0)} \right) v_k(t_1) v_k^*(t_2) \] (59)
where \( \hat{W}_k(k) \) denotes the Fourier mode of the window function \( W_k(x) \). Since the contributions of the IR and tIR modes to \( \zeta'(x) \) are canceled by the second term in equation (57), the IR suppression factor
\[ \left| e^{ikx} - \frac{\hat{W}_k(-k)}{W_k(0)} \right| \lesssim \mathcal{O}(kL) \] (60)
as arises. As a result, the momentum integration in the two-point function for \( \zeta'(x) \) is regularized in the IR. To obtain the inequality (60), we used the facts that the spatial coordinates \( x \) satisfy \( |x| \lesssim L \) and that \( \hat{W}_k(k) \) can be expanded as \( \hat{W}_k(k)/\hat{W}_k(0) = 1 + \mathcal{O}(kL) \).

However, the regularity of \( \langle \zeta'(x_1) \zeta'(x_2) \rangle \) does not imply the regularity of the correlation functions for \( \zeta'(x) \), because \( \zeta'(x) \) also contains nonlinear terms of \( \zeta(t) \). Inserting equations (48) and (56) into equation (49), we find that \( \zeta'(x) \) and \( \zeta'(x) \) are related as
\[ \zeta'(t, x) = \zeta'(t, x) - \bar{\zeta}(t)(x - \partial_t \zeta(t, x) \bar{\zeta}(t)) + \mathcal{O}(\zeta^3), \] (61)
where \( x \cdot \partial_t \bar{\zeta}(t) \) denotes the local average of \( x \cdot \partial_t \bar{\zeta}(t) \). Since the nonlinear terms in equation (61) include \( \bar{\zeta}(t) \) which diverges due to the IR modes, they can make the correlation
functions for $\zeta'(x)$ divergent. The lesson is that, in general, it is not straightforward to absorb all the IR divergent contributions by performing the residual gauge transformation.

We should also note that, if we can eliminate the nonlinear terms in equation (61), $\zeta'_f$ agrees with $\zeta'_t$ at the initial time. Both variables satisfy the same equation of motion, i.e., the linearized equation of motion with the subtraction of the local average as given in equation (53). Therefore, in this case the correlation functions of $\zeta'_f$ can be replaced by the products of the two-point function of $\zeta'_t$, $\langle \zeta'_t(x_1)\zeta'_t(x_2) \rangle$, which is shown above to be regular in IR. Hence, the regularity of the correlation functions for $\zeta'(x)$ is as follows. We will see that the IR regularity conditions, which will be derived in section 4.3, agree with the conditions which request the nonlinear terms in equation (61) should vanish.

In [82], to absorb the IR modes, Senatore and Zaldarriaga introduced a physical distance measure, considering the map between a scale in comoving coordinates $L$ and the Hubble crossing time of the corresponding scale $t_{hc}$, which satisfies $\dot{\rho}(t_{hc})^{-3} = e^{3[\rho(t_{hc})+\zeta(t_{hc})]L^3}$, where $\zeta(t_{hc})$ denotes the spatial average of the curvature perturbation in the region with the size $L$. They computed the physical volume at the reheating $t_{rh}$, eliminating the influence of the IR modes which resides in the comoving coordinates as

$$V_{rh} = e^{3\rho(t_{hc})} \int d^3x e^{3\zeta(t_{hc},x)} = e^{3[\rho(t_{hc})+\zeta(t_{hc})]} \dot{\rho}(t_{hc})^{-3} \int d^3x 3 \frac{\dot{\zeta}(t_{hc},x)}{L^3} e^{3[\zeta(t_{hc},x)-\zeta(t_{hc})]}.$$  

If we could replace all $\zeta$ s in $V_{rh}$ with the interaction picture field $\zeta_I$, we would see that the IR modes $kL \ll 1$ in $\zeta_I(\mathbf{t}_{rh}, \mathbf{x})$ is canceled by these modes in $\zeta_I(t_{hc})$. In a nonlinear computation, the discussion will become more complicated because choosing the proper measure does not necessarily guarantee that all the interaction picture fields $\zeta_I$ appear in the combination $\zeta_I - \zeta_I$, but their argument still suggests that the choice of the proper measure is one of the crucial ingredients for the regularization of the IR contributions. A related aspect was focused also in [61, 91]. In [61], focusing on the fact that the IRdiv is canceled in the difference between the free propagators $\langle \zeta_I(t, x_1)\zeta_I(t, x_2) \rangle - \langle \zeta_I(t, x_1)\zeta_I(t, x_2) \rangle$, Enqvist et al computed a nonlinear quantity which partially includes the one-loop contributions but whose IRdiv is canceled. We also comment on the paper by Primentel et al [83]. They discussed fixing the residual gauge degrees of freedom in a different way from the one described above. The argument was extended to higher order loops in [84]. They performed the residual gauge transformation $x' \rightarrow M'_f(t)x' + C'(t)$, requesting that $N_i$ and $\partial_j N_i$ after the gauge transformation should vanish in the local region at the leading order in $kL$, where $k$ is the wavenumber assigned to these variables. If this requirement can be fulfilled, the Fourier modes of these variables $N_{i,k}$ and $k_i N_{i,k}$ will obtain an additional suppression in the IR by $kL$. In [83, 84], they gave a restricted analysis on the gauge fixing, picking up the term with $\partial_i \partial^{-2}\zeta_{n}$ in $N_{i,n}$ as an example.

### 3.4.2. Sending the initial time to past infinity.

The appearance of the IRdiv due to the residual gauge transformation mentioned in section 3.4.1 might be evaded by sending the initial time $t_i$ to the past infinity. This is because in this limit the size of the observable region $O_i$ in comoving coordinates, $L_i$, becomes infinitely large. As we get close to this limit, the discrepancy between the average in the local region and that in the whole universe becomes smaller and smaller. Then, the residual gauge transformation at the initial time might be unnecessary. Also, it is quite natural to eliminate the presence of a special time in setting the initial quantum state.

We should note that when we send the initial time to the past infinity, it is too naive to neglect the subH modes with $k \gtrsim e^{\rho(t)} \dot{\rho}(t)$ at the distant past. The discussion in section 3.4.1 suggests that if the nonlinear correlations are regular at a finite reference time, they will be also kept regular for the later times. However, the nonlinear correlations at a finite reference
time cannot be specified without knowing the evolution in the subH regime up to that time. This aspect makes the IR issues very complicated.

To obtain an intuition, let us consider the conservation of \( \zeta_k \) in the limit \( k/(e^\rho \dot{\rho}) \ll 1 \) as an example. This conservation is a well-known fact in the long-wavelength approximation. (For a linear analysis, see [92] and for a nonlinear extension, see [93] and references therein.) However, once we include the nonlinear contributions from the subH modes, the conservation does not hold any more. If we consider a vertex integral confined to the region \( \mathcal{O} \) as in section 3.4.1, the superH modes will be suppressed but the subH modes can still contribute. Since the domain of the time integration is infinite, it is easy to understand that there is a possible origin of the SG due to the subH modes, and the effects may diverge in the limit \( t_i \to -\infty \).

In [84, 94], the absence of the SG was claimed relying on the conservation of the curvature perturbation, but the aspect mentioned above has not been discussed. In addition, even if the conservation of \( \zeta_k \) in the limit \( k/(e^\rho \dot{\rho}) \ll 1 \) is proved, the logarithmic enhancement in the form \( (k/e^\rho \dot{\rho})^2 \ln(k/e^\rho \dot{\rho}(t_i)) \) may give rise. The factor \( \ln(k/e^\rho \dot{\rho}(t_i)) \) can become large to overcome the suppression by \( (k/e^\rho \dot{\rho}) \) when we send the initial time to the past infinity.

3.5. Constructing a gauge invariant operator

The observable fluctuations should be free from the residual gauge degrees of freedom, which were introduced in section 3.2. In this subsection, following [71, 72], we construct an operator which is invariant under the residual gauge transformations. We call such an operator a genuinely gauge invariant operator.

3.5.1. The definition. Since the time slicing is uniquely specified by the gauge condition (10), a quantity which is invariant under the transformation of spatial coordinates will be genuinely gauge invariant. To construct a genuinely gauge invariant operator, we propose to calculate \( n \)-point functions for the scalar curvature of the induced metric on a \( \phi = \text{constant} \) hypersurface, \( sR \). Although \( sR \) itself transforms as a scalar quantity, the \( n \)-point functions of \( sR \) with its \( n \) arguments specified in a coordinate-independent manner will be gauge invariant. The distances of spatial geodesics that connect pairs of \( n \) points characterize the configuration in a coordinate-independent manner. Based on this idea, we specify the \( n \) spatial points in terms of the geodesic distances and the directional cosines, measured from a reference point. Although the reference point and frame depend on the coordinates, this would not matter as long as we choose a quantum state that respects the spatial homogeneity and isotropy of the universe.

The geodesic normal coordinates on each time slice are introduced by solving the spatial geodesic equation

\[
\frac{d^2 x_{gl}^i}{d\lambda^2} + \Gamma^i_{jk} \frac{dx_{gl}^j}{d\lambda} \frac{dx_{gl}^k}{d\lambda} = 0, \tag{62}
\]

where \( \Gamma^i_{jk} \) is the Christoffel symbol with respect to the three-dimensional spatial metric and \( \lambda \) is the affine parameter. The affine parameter ranges from \( \lambda = 0 \) to 1, and the initial ‘velocity’ is given by

\[
\frac{dx_{gl}^i(x, \lambda)}{d\lambda} \bigg|_{\lambda=0} = e^{-\zeta(\lambda=0)} x^i. \tag{63}
\]

Here we associate the subscript \( gl \) with the global coordinates, reserving the simple notation \( x \) for the geodesic normal coordinates. A point \( x \) in the geodesic normal coordinates is mapped...
to the end point of the geodesic $x_{\ell}^i(x, \lambda = 1)$ in the original coordinates. We perturbatively expand $x_{\ell}^i$ in terms of $x^i$ as

$$x_{\ell}^i = x^i + \delta x^i(x).$$

With the aid of the geodesic normal coordinates, we can construct a genuinely gauge invariant variable as

$$\mathcal{R}(x) \equiv \mathcal{R}(t, x_{\ell}^i(x)) = \mathcal{R}(t, x^i + \delta x^i(x)). \quad (64)$$

Since the gauge invariant variable $\mathcal{R}$ does not include the conjugate momentum of $\zeta$, we can consider products of $\mathcal{R}$ at an equal time without any ambiguity of operator ordering. To calculate the $n$-point functions of $\mathcal{R}$, we need to specify the quantum state as well. One may think that the quantum state should be selected so that it preserves the invariance under the residual gauge transformations. However, we cannot directly discuss this invariance as a condition for allowed quantum states in this approach, because the residual gauge degrees of freedom are absent when we quantize fields in the whole universe.

Here, we note that even though the operator $\mathcal{R}$ is not affected by the residual gauge degrees of freedom, this does not imply that the $n$-point functions of $\mathcal{R}$ are uncorrelated to the fields in the causally disconnected region. In section 3.1, we discussed two ways in which the variables in the observable region $O$ are contaminated by the influence from the outside of $O$. Since changing the boundary condition for the inverse Laplacian $\partial^2$ can be thought of as a residual gauge transformation, selecting $\partial^2$ whose integration region is restricted to the region $O$ as in equation (52) does not affect the $n$-point functions of $\mathcal{R}$. Therefore, as long as we consider a genuinely gauge invariant operator, the inverse Laplacian $\partial^{-2}$ never gives the conjunction between the inside and the outside of $O$. On the other hand, the long-range correlation through the Wightman function can stay even in the genuinely gauge invariant variables, providing a possible origin of the IRdiv and IRsec. In section 4, we will show that requesting the absence of the IRdiv or IRsec, in fact, constrains the quantum state of the inflationary universe. This can be interpreted as that the IR regularity of observable fluctuations can be achieved only when the quantum state is selected so that the long-range correlation is suppressed for genuinely gauge invariant variables.

3.5.2. The coarse grained distance. Tsamis and Woodard [95] showed that using the geodesic normal coordinates can introduce an additional origin of UV divergence, which may not be renormalized by local counter terms [96]. This is expected because specifying the spatial distance precisely in the presence of the gravitational perturbation requires taking account of all the short wavelength modes of the gravitational perturbation. In any realistic observations, what we observe are smeared fields with a finite resolution. However, it is not so trivial how to introduce a realistic smearing in a gauge invariant manner. Here, just to keep the UV contributions under control, we replace the geodesic normal coordinates with approximate ones removing the UV contributions. Originally the geodesic normal coordinates are related to $x_{\ell}^i$ as

$$x_{\ell}^i = e^{-\xi(t, e^{-\zeta(x)})} x^i + \cdots, \quad (65)$$

where the ellipsis means the terms suppressed in the IR limit, which vanish when $\xi(x)$ is spatially homogeneous. We replace the above relation with

$$x_{\ell}^i = e^{-\tilde{\xi}(t)} x^i, \quad (66)$$

where we introduced the smeared curvature perturbation

$$\tilde{\xi}(t) \equiv \frac{\int d^3x W_t(x) \xi(t, e^{-\zeta(x)})}{\int d^3x W_t(x)}. \quad (67)$$
Although $\xi$ appears on the right-hand side of equation (67), this expression defines $\xi$ iteratively at each order of the perturbation. We calculate the $n$-point functions of $\mathcal{R}_x \xi(t, x)$, instead of $\mathcal{R}$, with
\[ \mathcal{R}_x \xi(t, x) = \zeta(t, e^{-\xi(t)} x). \] (68)

Here, $\mathcal{R}_x$ represents such operators, \( \partial_t \dot{\rho}, \partial_x e^{\rho(t)} \dot{\rho}(t), \left(1 - \int d^3x W_t(x) \int d^3y W_t(y) \right), \ldots \) (69) that manifestly suppress the IR contributions by acting on the field $\zeta(t, x)$.

$\mathcal{R}_x \xi(t, x)$ is not invariant under general residual gauge transformations but still is invariant under the dilatation transformation, which can absorb the dominant IR contributions. In fact, since the genuinely gauge invariant variable $\mathcal{R}(x)$ should be invariant under the dilatation transformation, $\xi(x)$ appears only in the form of $\mathcal{R}_x \xi(x)$, when we express $\mathcal{R}(x)$ in terms of $\xi(x)$. As we can compute $\mathcal{R}(x)$ from $\mathcal{R}_x \xi(x)$, the $n$-point functions of $\mathcal{R}$ should be regular if those of $\mathcal{R}_x \xi$ are regular. Therefore, we compute the latter.

Note that the genuinely gauge invariant quantity $\mathcal{R}(x)$ should be composed of $\mathcal{R}_x \xi(x)$, but this does not imply that all the interaction field operators in $\mathcal{R}(x)$ are associated with IR suppressing operators $\mathcal{R}_x \xi_I(x)$. Therefore, as we mentioned at the end of section 3.5.1, the long-range correlation in the Wightman function $G^+(x, x')$ is not always suppressed. In section 5, we will show that only for restricted quantum states all the Wightman functions contained in the expression for the $n$-point functions of $\mathcal{R}$ are accompanied with the IR suppressing operators.

4. Restricting initial states from the gauge invariance and the regularity

In the previous section, we revealed that the IRdiv and IRsec originate from the influence from the outside of the observable region, and this influence is mediated by the residual gauge degrees of freedom from the view point of the local observable universe. To remove the influence of the residual gauge degrees of freedom, it is essentially important to focus on the correlation functions of genuinely gauge invariant operators. However, evaluating genuinely gauge invariant operators is not sufficient. Another important aspect is to choose the quantum state which is not affected by the residual gauge degrees of freedom. In other words, we need to choose a quantum state whose correlation with the outside of the observable region is suppressed. In this section, we will show that the conditions for the absence of the IRdiv and IRsec yield a non-trivial restriction on the quantum state. Then, we will show that this condition can be interpreted as the condition for the invariance of the quantum state under the dilatation transformation. In this section, for an illustrative purpose, we employ a simple assumption that the interaction is turned on at a finite initial time $t_i$.

4.1. Restricting initial states from the absence of IRdiv and IRsec

In this subsection, we compute the two-point function of $\mathcal{R}_x \xi(x)$ up to the one-loop order, and derive the condition on the initial state for the absence of the IRdiv and IRsec. Assuming that the interaction is turned on at the initial time $t_i$, we set
\[ \zeta(t_i, x) = \zeta_I(t_i, x), \quad \pi(t_i, x) = \pi_I(t_i, x), \] (70)
where $\pi_I$ is the conjugate momentum of the interaction picture field $\zeta_I$. Here, we compute the two-point function by solving the Heisenberg equation of motion for the curvature perturbation
operator \( \zeta \). Using the retarded Green function \( G_{R}(x, x') \), we obtain the solution of \( \zeta \) that satisfies the initial condition (63) as

\[
\zeta(x) = \zeta_{I}(x) + L_{R}^{-1}S_{NL}(x)
\]  

(71)

with

\[
L_{R}^{-1}S_{NL}(t, x) \equiv -2M_{pl}^{2} \int d^{4}x' \epsilon_{1}(t') e^{3\rho(t')} G_{R}(x, x') S_{NL}(x'),
\]

(72)

where the explicit form of the nonlinear source term \( S_{NL}(x) \) will be given later. Evaluating equation (72) iteratively, we can obtain an expression for the curvature perturbation. Inserting thus obtained solution \( \zeta \) into equation (67), we can perturbatively compute \( \tilde{\zeta}(x) \) as

\[
\tilde{\zeta}(x) = \zeta_{I}(x) + \tilde{\zeta}_{2}(x) + \tilde{\zeta}_{3}(x) + \cdots,
\]

(73)

where \( \tilde{\zeta}_{n}(x) \) represents the term that consists of \( n \) interaction picture fields \( \zeta_{I} \). Expanding the interaction picture field \( \zeta_{I} \) as equation (24), the initial vacuum state is defined by

\[
a_{0}(0) = 0.
\]

(74)

The \( n \)-point functions computed by taking the expectation values of products of thus obtained \( \tilde{\zeta}(x) \) can be formally shown to agree with those calculated in the in–in formalism (see, for example, appendix of [73]).

Using equation (73), the one-loop contributions to the two-point function of \( R_{\nu} \tilde{\zeta}(x) \) are given by

\[
\langle R_{\nu} \tilde{\zeta}(x_{1}) R_{\nu} \tilde{\zeta}(x_{2}) \rangle_{\text{1-loop}} = \langle R_{\nu} \tilde{\zeta}(x_{1}) R_{\nu} \tilde{\zeta}(x_{2}) \rangle + \langle R_{\nu} \zeta_{I}(x_{1}) R_{\nu} \tilde{\zeta}(x_{2}) \rangle + \langle R_{\nu} \tilde{\zeta}(x_{1}) R_{\nu} \zeta_{I}(x_{2}) \rangle.
\]

(75)

After we choose the boundary conditions for \( \partial^{-2} \) as given in equation (52), the inverse Laplacian does not enhance the singular behavior of the superH modes, and hence, the IRdiv and IRsec can appear only from the variance

\[
\langle \tilde{\zeta}_{1}^{2}(t) \rangle \simeq \int_{k_{\perp} \leq k_{\perp}^{*}} \frac{d^{3}k}{(2\pi)^{3}} P(k).
\]

If \( \tilde{\zeta}_{2} \) includes \( \tilde{\zeta}_{1} \), the first term on the right-hand side in equation (75) can give \( \langle \tilde{\zeta}_{2}^{2} \rangle \). Similarly, if \( \tilde{\zeta}_{3} \) includes \( \tilde{\zeta}_{2} \), the second and third terms can give \( \langle \tilde{\zeta}_{3}^{2} \rangle \). To make our discussion compact and transparent, here, we pick up only the potentially divergent contributions, which yield \( \langle \tilde{\zeta}_{n}^{2} \rangle \). We introduce the symbol ‘\( \approx^{\text{IR}} \)’ to denote the approximate equality neglecting the terms which do not yield \( \langle \tilde{\zeta}_{n}^{2} \rangle \) at the one-loop level [71, 72].

We can easily derive the nonlinear action which is relevant for yielding \( \langle \tilde{\zeta}_{n}^{2} \rangle \) as

\[
S \approx M_{pl}^{2} \int d^{4}x e^{3\rho(\zeta)} \epsilon_{1}[(\delta_{i}(\zeta))^{2} - e^{-2(\rho+\zeta)}(\partial_{i}(\zeta))^{2}],
\]

(76)

where the terms with more than two fields with differentiations, which do not give \( \langle \tilde{\zeta}_{n}^{2} \rangle \), are abbreviated. This approximate expression for the action also preserves the dilatation symmetry (44) [73].

The variation of the above action gives the equation of motion as

\[
\left[ \partial_{i}^{2} + (3 + \epsilon_{2}) \partial_{i} - e^{-2\rho} \partial_{i}^{2} \right] \zeta(x) = S_{NL}(x)
\]

(77)

with

\[
S_{NL}(x) \approx e^{-2\rho}(e^{-2\zeta} - 1) \partial_{i}^{2} \zeta(x) - \delta(t - t_{i})(e^{3\zeta} - 1) \delta_{i} \zeta(x),
\]

(78)
where the last term is added to satisfy the second condition in equations (63) [73]. By inserting equation (78) into equation (71), the solution that satisfies equation (63) is obtained. Then, we can express \( \tilde{g}(x) \) as

\[
\begin{align*}
\xi_2(x) &\approx -\tilde{\xi} D_x \xi_1, \\
\xi_3(x) &\approx \frac{1}{2} \tilde{\xi}^2 D_x^2 \xi_1,
\end{align*}
\]

(79)

with

\[
D_x \equiv 2L_R^{-1} e^{-2\rho} \delta^2 + 3L_R^{-1} \delta(t - t_i) \partial_x + \partial_x^2,
\]

(80)

where we used the factors

\[
L_R^{-1} \xi_1 \approx \tilde{\xi} L_R^{-1} \xi_1,
\]

(81)

and \( L_R^{-1} f(x) \approx 0 \) for \( f(x) \approx 0 \) [73]. In the above expression (79) \( \tilde{\xi} \) appears in the combination of \( \tilde{\xi} D_x \). To be more precise, the terms with the delta function \( \delta(t - t_i) \) in \( D_x \) are multiplied by \( \delta(t_i) \), while the remaining terms are multiplied by \( \tilde{\xi}(t) \). Therefore, the former terms contribute only to the IRdiv, while the latter terms contribute to both the IRdiv and IRsec. First, we focus on the IRdiv, neglecting the IRsec for a while. Inserting equations (79) into equation (75), we obtain the one loop correction with the factor \( \langle \tilde{\xi}^2(t_i) \rangle \) as

\[
\langle R_{\xi_1} \xi(x_1) \rangle _{\text{loop}} \approx \frac{\langle \tilde{\xi}^2(t_i) \rangle}{2} R_{\xi_1} R_{\xi_2} \left( 2D_x \xi_1(x_1) D_x \xi_1(x_2) + D_x^2 \xi_1(x_1) \xi_2(x_2) + \xi_1(x_1) D_x^2 \xi_1(x_2) \right).
\]

(82)

In general, the above expression is not free from the IRdiv and IRsec. One may think that the absence of the IRdiv requests \( D_x \xi_1(x) = 0 \). However, this condition immediately contradicts, because an operation of \( x \cdot \partial_x \) on a Fourier mode \( e^{ik \cdot x} \) yields the term with \( (x \cdot k) e^{ik \cdot x} \), which cannot be canceled by the remaining two terms with the retarded integral \( L_R^{-1} \) [73].

A simple alternative way we can think of is to impose

\[
D_x \xi_1(x) = \int \frac{d^3k}{(2\pi)^3/2} \left( \partial_k D e^{ik \cdot x} v_k + \text{h.c.} \right)
\]

(83)

where

\[
D \equiv k^{-3/2} e^{-i\phi(k)} k \cdot \partial_k k^{3/2} e^{i\phi(k)},
\]

(84)

with an arbitrary phase function \( \phi(k) \). (The choice of the \( k \)-dependent phase in the mode functions is irrelevant from the beginning as usual.) With these conditions, the terms with \( \langle \tilde{\xi}^2(t_i) \rangle \) in equation (82) can be summarized in the total derivative form as

\[
\langle R_{\xi_1} \xi(x_1) \rangle _{\text{loop}} \approx \frac{\langle \tilde{\xi}^2(t_i) \rangle}{2} R_{\xi_1} R_{\xi_2} \int \frac{d\ln k}{(2\pi)^3/2} \partial^2_{\ln k} \left[ k^2 v_k^2 e^{ik \cdot (x_i - x_j)} \right],
\]

where \( \int \partial \Omega_k \) denotes the integration over the angular directions of \( k \). Since the integral of a total derivative vanishes, the IRdiv is eliminated. The condition (83) can be rewritten as a condition on mode functions

\[
L_R^{-1} \left( -2(k e^{-\rho})^2 + 3 \delta(t - t_i) \partial_x \right) v_k = D v_k,
\]

(85)

where \( L_R^{-1} \) is the Fourier mode of \( L_R^{-1} \). We request the mode functions \( v_k \) to satisfy equation (85) and its time derivative just after the initial time \( t = t_i \). Then, the condition (85) continues to hold also for \( t > t_i \), because this condition vanishes under the operation of the second-order differential operator \( L \).
Similarly, we can also discuss the absence of the IRsec, which requests the terms with
\[
\langle \hat{\xi}^2_f (t) \rangle - \langle \hat{\xi}^2_f (t) \rangle \simeq \int_{1/L_e \leq k \leq 1/L_e} \frac{d^3k}{(2\pi)^3} P(k)
\]
should vanish. Recalling that \( \tilde{\xi} \) associated with \( \delta (t - t_i) \) is to be interpreted as \( \tilde{\xi} (t_i) \), we find that the terms which contain \( \langle \hat{\xi}^2_f (t) \rangle - \langle \hat{\xi}^2_f (t) \rangle \) vanish as an integral of a total derivative, if
\[
-2L_e^{-1} (ke^{s})^2 v_k = D v_k
\]
is satisfied.

In this subsection, we have observed that requesting the absence of the IRdiv and IRsec restricts the initial states. In the succeeding subsection, we will show that the same conditions as equations (85) and (86) are derived from the requirement that the quantum state is invariant under the dilatation transformation, which will clarify the physical meaning of these conditions.

### 4.2. The canonical systems connected by the dilatation

To discuss the physical meaning of the IR regularity conditions, we introduce another set of the canonical variables. As long as we consider a theory which preserves the three-dimensional diffeomorphism invariance, the dilatation symmetry, \( x \to e^{-\bar{x}} \) with a constant parameter \( s \), is preserved as a part of spatial coordinate transformations. This implies that the action for \( \zeta \) should be invariant under the change of the variable \( \zeta (t, x) \to \zeta (t, e^{-\bar{x}} - s) \), i.e.,
\[
S = \int dt \int d^3x L [\zeta (x)] = \int dt \int d^3x L [\zeta (t, e^{-\bar{x}} - s)].
\]

We introduce another set of canonical variables than \( \zeta (x) \) and its conjugate momentum \( \pi (x) \) by
\[
\tilde{\zeta} (x) \equiv \zeta (t, e^{-\bar{x}}), \quad \tilde{\pi} (x) \equiv e^{-3s} \pi (t, e^{-\bar{x}}).
\]

One can show that, using the commutation relations for \( \zeta (x) \) and \( \pi (x) \), these new variables also satisfy the canonical commutation relations
\[
[\tilde{\zeta} (t, x), \tilde{\pi} (t, y)] = [\zeta (t, e^{-\bar{x}}), \pi (t, e^{-\bar{x}})] = i\delta (3)(x - y),
\]
and
\[
[\tilde{\zeta} (t, x), \tilde{\zeta} (t, y)] = [\tilde{\pi} (t, x), \tilde{\pi} (t, y)] = 0.
\]

Using equation (87), we can show that the Hamiltonian densities expressed in terms of these two sets of the canonical variables are related to each other as
\[
\int d^3x \mathcal{H}[\tilde{\zeta} (x), \tilde{\pi} (x)] = \int d^3x \left\{ \pi (x) \tilde{\zeta} (x) - \mathcal{L}[\zeta (x)] \right\}
= \int d^3x \mathcal{H}[\tilde{\zeta} (x) - s, \tilde{\pi} (x)]
= \int d^3x \tilde{\mathcal{H}}[\tilde{\zeta} (x), \tilde{\pi} (x)],
\]
where we changed the spatial coordinates as \( x \to e^{-\bar{x}} \) on the second equality. We find that the Hamiltonian density for the system \{\tilde{\zeta}, \tilde{\pi}\} is given by the same functional as the one for the system \{\zeta, \pi\} with \( \tilde{\zeta} \) shifted by \( -s \).

We can extend the dilatation transformation to a time-dependent one, \( s \to s(t) \) [74]. Again, we introduce another set of the canonical variables \( \tilde{\zeta} (x) \) and \( \tilde{\pi} (x) \), using \( \zeta \) and \( \pi \) evaluated at the transformed point \( e^{-s(t)} x \),
\[
\tilde{\zeta} (x) \equiv \zeta (t, e^{-s(t)} x), \quad \tilde{\pi} (x) \equiv e^{-3s(t)} \pi (t, e^{-s(t)} x),
\]

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which satisfy the canonical commutation relations. The Hamiltonian density for the variables \( \{ \zeta, \pi \} \) can be expressed in terms of the one for \( \{ \zeta , \pi \} \) as

\[
\hat{\mathcal{H}}[\{ \zeta (x), \pi (x) \}] = \mathcal{H}[\{ \zeta (x) - s(t), \pi (x) \}] - \dot{s}(t) \pi (x) \cdot \partial_x \zeta (x).
\]

(93)

Assuming that \( s(t) \) is as small as \( \zeta (x) \) and \( \pi (x) \), we decompose the Hamiltonian densities \( \mathcal{H} \) and \( \hat{\mathcal{H}} \) into the quadratic free parts and the higher order interaction parts as

\[
\hat{\mathcal{H}}[\{ \zeta (x), \pi (x) \}] = \mathcal{H}_0[\{ \zeta (x), \pi (x) \}] + \hat{\mathcal{H}}[\{ \zeta (x), \pi (x) \}],
\]

(94)

and

\[
\hat{\mathcal{H}}_I[\{ \zeta (x), \pi (x) \}] = \mathcal{H}_0[\{ \zeta (x), \pi (x) \}] + \hat{\mathcal{H}}_I[\{ \zeta (x), \pi (x) \}].
\]

(95)

Here, we note that \( \mathcal{H}_0[\{ \zeta (x) - s(t), \pi (x) \}] = \mathcal{H}_0[\{ \zeta (x), \pi (x) \}] \), since \( \zeta (x) \) always appears with spatial differentiations in \( \mathcal{H}_0[\{ \zeta (x), \pi (x) \}] \). Remarkably, the quadratic part of the Hamiltonian densities \( \mathcal{H} \) and \( \hat{\mathcal{H}} \) have the same functional form. Using equation (93), we find that the interaction Hamiltonian densities are related with each other as

\[
\hat{\mathcal{H}}_I[\{ \zeta (x), \pi (x) \}] \equiv \mathcal{H}_I[\{ \zeta (x) - s(t), \pi (x) \}] - \dot{s}(t) \pi (x) \cdot \partial_x \zeta (x). \]

(96)

Thus, we find that \( \zeta (x) \) in \( \hat{\mathcal{H}}_I \) only appears in the form of \( \zeta (x) - s(t) \) or with differentiations.

4.3. The gauge invariance and the IR regularity

Now, we are ready to give an alternative interpretation of the conditions (85) and (86). We adopted the initial condition (63) for the system \( \{ \zeta , \pi \} \), which identifies the Heisenberg fields with the corresponding interaction picture fields at the initial time and selected the vacuum state at the initial time by (74). These procedures specify a quantum state for the system \( \{ \zeta , \pi \} \). If we adopt the same scheme in the canonical system \( \{ \zeta, \pi \} \), it is not obvious whether physically the same vacuum state is picked up or not. In this subsection, we show that these two vacua are equivalent only when the conditions (85) and (86) are satisfied.

Adopting the same scheme to give a quantum state, both \( \zeta \) and \( \tilde{\zeta} \) are solved by using \( \mathcal{L}_R^{-1} \) with the conditions identifying the Heisenberg fields to the interaction picture fields at the initial time. We expand the respective interaction picture fields, \( \zeta_I \) and \( \tilde{\zeta}_I \), in terms of the same mode function \( v_k \) as

\[
\zeta_I(x) = \int \frac{d^3k}{(2\pi)^{3/2}} g_k v_k(t) e^{ikx} + (\text{h.c.}),
\]

(97)

\[
\tilde{\zeta}_I(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{g}_k v_k(t) e^{ikx} + (\text{h.c.}).
\]

(98)

Then, we choose the vacuum states, \( |0\rangle \) and \( |\tilde{0}\rangle \) erased by operations of \( a_k \) and \( \hat{a}_k \), respectively. Now we compare the two-point functions calculated in the respective systems to show that the requirement

\[
\langle 0 | \zeta (x_1) \zeta (x_2) | 0 \rangle = \langle \tilde{0} | \tilde{\zeta} (t, e^{it\xi_1}) \tilde{\zeta} (t, e^{it\xi_2}) | \tilde{0} \rangle
\]

(99)

yields the conditions (85) and (86). We expand \( \tilde{\zeta} (t, e^{it\xi}) \) as

\[
\tilde{\zeta} (t, e^{it\xi}) \equiv \tilde{\zeta} (t, e^{it\xi}) = \tilde{\zeta} (t, e^{it\xi}) + s(t) x \cdot \partial_x \tilde{\zeta} (x) + \frac{1}{2} s^2(t) (x \cdot \partial_x \tilde{\zeta} (x))^2 + \mathcal{O}(s^3) \]

\[
\mathcal{L}_R^{-1} \tilde{\zeta} (x) - \tilde{\zeta} (x) \mathcal{L}_R^{-1} \left( 2e^{-2\rho} \partial^2 + 3\delta (t - t_i) \partial_{t_i} \right) \tilde{\zeta} (x) + s(t) \partial_t \zeta_I (x) + \tilde{\zeta}_I (x)
\]

(100)

where ‘\( \cdots \)’ denotes higher order terms in perturbation. Note that the interaction Hamiltonian density for \( \{ \zeta, \pi \} \) is given by the same functional as \( \hat{\mathcal{H}}_I \) with the argument shifted by \( -s(t) \) and...
the second term in equation (93), which is higher order in perturbation. Then, the right-hand side of equation (99) gives

\[
\langle 0 | \zeta(t, e^x x_1) \zeta(t, e^x x_2) | \tilde{0} \rangle \overset{\text{IR}}{=} \langle 0 | \xi(x_1) \xi(x_2) | 0 \rangle + s(t) \langle 0 | D_{x_1} \tilde{\zeta}(x_1) \tilde{\zeta}(x_2) + \cdots | \tilde{0} \rangle + \mathcal{O}(s^2).
\]

(101)

Now, it is clear that equation (99) implies that the terms proportional to \(s(t)\) on the right-hand side in equation (101) should vanish.

Notice that, more precisely, \(s(t)\) multiplied by the term \(L_{\nu}^{-1} \delta(t - t_i) \cdots\) in \(\mathcal{D}\) should be replaced with \(s(t_i)\). Now we decompose \(s(t)\) in equation (101) into \(s(t_i)\) and \(s(t) - s(t_i)\). Then, we can see that if the condition for the absence of IRdiv (85) is valid, the terms multiplied by \(s(t_i)\) vanish. Whilst, if the condition for the absence of the IRsec (86) is valid, the terms multiplied by \(s(t) - s(t_i)\) vanish. Thus, the conditions for the invariance under the dilatation (99) gives the same conditions as requesting the absence of the IRdiv and IRsec [73].

### 4.4. Inconsistency in removing the IRdiv and IRsec

In this section, we assumed that the interaction is turned on at a finite initial time. However, we should also note that the conditions for the absence of the IRdiv and IRsec cannot be naturally satisfied in this setup. Regarding the condition for the absence of IRdiv (85), since the left-hand side of (85) vanishes at \(t = t_i\), the condition (85) yields

\[
Dv_k(t_i) = 0,
\]

(102)

These conditions are compatible with the normalization of the mode functions

\[
-2i\mathcal{M}_k^2 e^{3\nu} \varepsilon_1 (\nu \dot{u}_k^* - \dot{v}_k v_k^*) = 1.
\]

(103)

Operating the derivative \(\partial_{\nu n k}\) on it, we find that the both sides vanish. However, the first condition in (102) requests the scale-invariant spectrum with \(|v_k(t_i)| \propto 1/k^{3/2}\) for all wavenumbers, which is not compatible with the Hadamard condition in the UV limit. Therefore, it cannot be a physically natural quantum state.

If we introduce an IR cutoff by hand, the IRdiv can be eliminated. Even in this case, we cannot eliminate the IRsec. Since the left-hand side of (86) and its time derivative vanish at the initial time, the condition (86) requests

\[
Dv_k(t_i) = 0, \quad D\dot{v}_k(t_i) = 0,
\]

(104)

which are incompatible with the normalization condition (103). The right-hand side of (103) trivially vanishes after the operation of \(\partial_{\nu n k}\), while the left-hand side gives 3.

The intuitive reason why these conditions cannot be compatible with the initial condition (63) is as follows. When we abruptly switch on the interaction at \(t = t_i\), we introduce a particular time into the system. Then, the Hubble scale at the initial time is distinguished from other scales. Therefore, the invariance under the dilatation transformation is naturally broken. One possible way to avoid this symmetry breaking might be sending the initial time to the infinite past. In this limit, the IR regularity no longer requests the condition (85), because equation (81) does not hold.

### 5. IR regularity of the Euclidean vacuum in the inflationary universe

In the previous section we showed that the gauge invariance in the local observable universe is essential to remove the IRdiv and IRsec. However, we found it impossible to prepare a natural quantum state that maintains the gauge invariance, as long as we start with the vacuum state of
the free field, turning on the interaction at a finite time. In this section, we keep the interaction
turned on from the infinite past. We consider the Euclidean vacuum, which is specified by
the regularity at the infinite past with the time coordinate rotated towards the imaginary axis.
As will be explained, the Euclidean vacuum keeps the gauge invariance, and hence the loop
corrections become IR regular. We also show the absence of the SG without neglecting the
subH modes (to a certain order in the perturbative expansion).

5.1. The Euclidean vacuum
In the case of a massive scalar field in de Sitter spacetime, the boundary condition specified
by rotating the time path in the complex plane can be understood as requesting the regularity
of correlation functions on the Euclidean sphere which can be obtained by the analytic
continuation from those on de Sitter spacetime (see section 6.2). The vacuum state thus
defined is called the Euclidean vacuum state. Here, we also refer to the state which is specified
by a similar boundary condition in more general spacetime as the Euclidean vacuum.

To be more precise, we define the Euclidean vacuum, requesting the regularity of the
\( n \)-point functions,
\[
\langle T_c \zeta(x_1) \cdots \zeta(x_n) \rangle < \infty, \quad \text{for } \eta(t_a) \to -\infty(1 \pm i \epsilon),
\]
where \( a = 1, \ldots, n \) and \( T_c \) denotes the path ordering along the closed time path, \(-\infty(1 - i \epsilon) \to \eta(t_f) \to -\infty(1 + i \epsilon)\), in the conformal time defined in equation (31). For simplicity, here we
assume that \( e^{\rho(t)} \dot{\rho}(t) \) is rapidly increasing in time so that
\[
|\eta(t)| = O\left(1/e^{\rho(t)} \dot{\rho}(t)\right).
\]

The Euclidean vacuum is expected to possess the gauge invariance in the local universe,
especially the invariance under the dilatation transformation, since its conditions do not
introduce any artificial scale. In fact, the Euclidean vacuum is specified independently of
which canonical variables \( \{\zeta, \pi\} \) or \( \{\tilde{\zeta}, \tilde{\pi}\} \) we use. The boundary conditions of the Euclidean vacuum for the canonical variable \( \tilde{\zeta} \) request
\[
\langle T_c \tilde{\zeta}(x_1) \cdots \tilde{\zeta}(x_n) \rangle < \infty, \quad \text{for } \eta(t_a) \to -\infty(1 \pm i \epsilon),
\]
Then, we can show the equivalence
\[
\langle T_c \zeta(x_1) \cdots \zeta(x_n) \rangle = \langle T_c \tilde{\zeta}(t_1, e^{i\zeta x_1}) \cdots \tilde{\zeta}(t_n, e^{i\zeta x_n}) \rangle
\]
is satisfied. This is a generalization of the condition (99), which requests the invariance under
the dilatation transformation. A more detailed explanation regarding the uniqueness of the
Euclidean vacuum can be found in [74]. The distinctive property (108) will be the key to show
that the Euclidean vacuum is free from the IRdiv and IRsec.

Here we took the boundary conditions for the \( n \)-point functions as the definition of the
Euclidean vacuum state, assuming the existence of such a quantum state. In the in-in formalism,
the \( n \)-point functions are perturbatively expanded using the Wightman functions. At this point,
the vertex integrals along the closed time path start and end with \( \text{Re}[\eta] \to -\infty \). The infinitely
oscillating vertex integrals along this path can be made convergent by rotating the time path
toward the imaginary axis, which is nothing but the ordinary \( ie \) prescription. Thus obtained
\( n \)-point functions also satisfy the boundary conditions (105)/(107) (see section IV A of [74]).

5.2. The IR regular Hamiltonian
In this subsection, we discuss the quantization using the canonical variables \( \{\tilde{\zeta}, \tilde{\pi}\} \). When we
choose the Euclidean vacuum, the interaction Hamiltonian density for \( \{\zeta, \pi\} \), \( \tilde{\mathcal{H}}_I \) can be recast
into the form

\[ \tilde{\mathcal{H}}_{\xi}[\tilde{\xi}_{\xi}(x), \tilde{\pi}_{\xi}(x)] = M_{\text{pl}}^{3} \rho^{2} \tilde{\varepsilon}_{1}(t) \sum_{n=3}^{\infty} \lambda(t) \prod_{m=1}^{n} \mathcal{R}_{\xi}^{(m)} \tilde{\xi}_{\xi}(x), \tag{109} \]

where \( \lambda(t) \) represents an \( \mathcal{O}(1) \) dimensionless time-dependent function expressed in terms of the horizon flow functions. To discriminate different IR suppressing operators, we accompany \( \mathcal{R}_{\xi} \) with a superscript \((m)\). Although the interaction Hamiltonian for the curvature perturbation is very messy, what we need to verify equation (109) is only the formal expression given in equation (93), i.e., \( \tilde{\mathcal{H}}_{\xi} \) can be written down solely in terms of \( \tilde{\xi}_{\xi}(x) - s(t) \), \( \tilde{\xi}_{\xi}(x) \) with differentiation, and \( \tilde{s}(t) \) as a manifestation of the dilatation symmetry. We should note that the expression (93) does not immediately imply that \( \tilde{\mathcal{H}}_{\xi} \) is composed of IR irrelevant operators because of the terms with \( \tilde{\xi}_{\xi}(x) - s(t) \) and the inverse Laplacian \( \partial^{-2} \).

First, we consider the terms with \( \tilde{\xi}_{\xi}(x) - s(t) \). Owing to the uniqueness of the Euclidean vacuum discussed in the previous subsection, we can replace all \( \{\tilde{\xi}_{\xi}(x) - s(t)\} \)s in the interaction Hamiltonian with \( \tilde{\xi}_{\xi}(x) - \tilde{\xi}_{\xi}(t) \), which is in the form \( \mathcal{R}_{\xi} \tilde{\xi}_{\xi}(t) \) [74]. This replacement introduces additional terms, but these terms are shown to be products of \( \tilde{\xi}_{\xi}(x) \)s suppressed by \( \mathcal{R}_{\xi} \). Similarly, we can replace all \( s(t) \)s with the terms which are products of \( \mathcal{R}_{\xi} \tilde{\xi}_{\xi}(x) \). Thus, we can show that all the interaction picture fields \( \tilde{\xi}_{\xi}(x) \) are multiplied by the IR suppressing operator \( \mathcal{R}_{\xi} \).

Next, we consider the inverse Laplacian \( \partial^{-2} \), which appears in solving the constraint equations for the lapse function and the shift vector. If \( \partial^{-2} \) does not introduce additional inverse power of \( k \), equation (109) is verified. Repeating the discussion about the boundary condition of \( \partial^{-2} \) in section 3.4.1, we can restrict all the interaction vertices within the causally connected local region \( \mathcal{O} \), prohibiting the appearance of additional inverse power of \( k \). When we calculate \( n \)-point functions for the genuinely gauge invariant operator \( \mathcal{R} \) from those for \( \mathcal{R}_{\xi} \tilde{\xi}_{\xi}(x) \), the choice of the boundary conditions should not affect the results. In this way, we can express all the interaction Hamiltonian in the form of equation (109).

5.3. The regularized Wightman function

Since all \( \tilde{\xi}_{\xi}(x) \)s in the interaction Hamiltonian are multiplied by the IR suppressing operators \( \mathcal{R}_{\xi} \), the \( n \)-point function of \( \mathcal{R}_{\xi} \tilde{\xi}_{\xi}(x) \) can be expanded by the Wightman function \( \mathcal{R}_{\xi} \mathcal{R}_{\xi} G^{+}(x, x') \) and its complex conjugate \( \mathcal{R}_{\xi} \mathcal{R}_{\xi} G^{-}(x, x') \). In this subsection, we calculate these Wightman functions multiplied by the IR suppressing operators. We will find that the boundary condition of the Euclidean vacuum guarantees that the amplitude of \( \mathcal{R}_{\xi} \mathcal{R}_{\xi} G^{+}(x, x') \) is bounded from above for finite values of \( x \) and \( x' \), except for the coincidence limit.

As mentioned above, the boundary conditions of the Euclidean vacuum (105)/(107) are equivalent to the \( ie \) prescription in the in-in formalism. We expand the curvature perturbation \( \tilde{\xi}_{\xi}(x) \) as in equation (98), using the mode function \( v_{k}(t) \). The boundary conditions (105)/(107) at the tree level imply that \( v_{k}(t) \) should be \( \propto e^{-ik\psi(t)} \) asymptotically. Factoring out this time dependence, we express \( v_{k}(t) \) as

\[ v_{k}(t) = \frac{A(t)}{k^{3/2}} f_{k}(t) e^{-ik\psi(t)}, \tag{110} \]

where \( A(t) \) is an approximate amplitude of the fluctuation defined by

\[ A(t) \equiv \frac{\dot{\rho}(t)}{\sqrt{\varepsilon_{1}(t)} m_{\text{pl}}}, \tag{111} \]
Using equation (110) and integrating over the angular part of the momentum, the Wightman function \( \mathcal{R}_x \mathcal{R}_x G^+(x, x') \) can be expressed as

\[
\mathcal{R}_x \mathcal{R}_x G^+(x, x') = \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} \mathcal{R}_x \mathcal{A}(t) f_k(t) A(t') f_k(t') \left[ e^{ik\sigma_+ (x, x')} - e^{ik\sigma_-(x, x')} \right],
\]

(112)

where we introduced \( \sigma_\pm (x, x') = \eta(t') - \eta(t) \pm |x - x'| \). The function \( f_k(t) \) satisfies a regular second order differential equation with the regular boundary condition \( f_k(t) \to k/(\sqrt{2} e^{\rho}) \) for \( -k\eta(t) \to \infty \). Since both the differential equation and the boundary condition for \( f_k(t) \) are analytic in \( k \), the resulting function \( f_k(t) \) should be analytic as well. Namely, \( f_k(t) \) does not have any singularity such as a pole on the complex \( k \)-plane.

Now we are ready to discuss the regularity of \( \mathcal{R}_x \mathcal{R}_x G^+(x, x') \), particularly the regularity of the \( k \) integration in equation (112). Since the function \( f_k(t) \) is not singular, the regularity can be verified if the integration converges both in the IR and UV limits. The regularity in the IR limit is guaranteed by the IR suppressing operators \( \mathcal{R}_x \), which add at least one extra factor of \( k|\eta(t)| \) or eliminate the leading \( k \)-independent term in the IR limit. In the UV limit, the contour of the \( k \)-integral in equation (112) should be appropriately modified at \( k \to \infty \) so that the integral becomes convergent, which is a part of the \( ie \) prescription. With this prescription, the integral is made finite for the UV contribution except for the case \( \sigma_\pm (x, x') = 0 \), where \( x \) and \( x' \) are mutually light-like. Since the expression of the Wightman function obtained after the \( k \) integration is independent of the value of \( \epsilon \), the regulator makes the UV contributions convergent even after \( \epsilon \) is sent to zero. For \( \sigma_\pm (x, x') = 0 \), the integral becomes divergent in the limit \( \epsilon \to 0 \), but this divergence is to be interpreted as the ordinary UV divergences, whose contribution to the vertex integrals must be renormalized by introducing local counter terms.

5.4. The secular growth (SG)

Since the amplitude of \( \mathcal{R}_x \mathcal{R}_x G^+(x, x') \) is shown to be finite, we can verify the regularity of the \( n \)-point functions if the non-vanishing support of the integrands of the vertex integrals is effectively restricted to a finite spacetime region. Since the interaction vertexes are restricted to the causal past with the appropriate choice of the residual gauge degrees of freedom, the question to address is whether the contribution from the vertexes in the distant past is effectively shut off or not. In this subsection, focusing on the long-term correlation, we will give an intuitive explanation why the vertex integrals of the \( n \)-point functions converge for the Euclidean vacuum (see [74] for details).

When we choose the Euclidean vacuum as the initial state, the deep IR modes \( k|\eta| \ll 1 \) are suppressed by the operation of \( \mathcal{R}_x \) and the UV modes \( k|\eta| \gg 1 \) are suppressed owing to the boundary condition of the \( ie \) prescription. Thus, only the modes around the Hubble scale, i.e., \( k|\eta| \simeq k e^{\rho} = O(1) \), remain to be relevant. Then, the Wightman function \( \mathcal{R}_x \mathcal{R}_x G^+(x, x') \) is necessarily suppressed when \( \eta(t)/\eta(t') \ll 1 \) \footnote{An explicit computation shows that in the limit \( |\eta(t)| \ll |\eta(t')| \), the Wightman function \( \mathcal{R}_x \mathcal{R}_x G^+(x, x') \) is suppressed as

\[
\mathcal{R}_x \mathcal{R}_x G^+(x, x') = \mathcal{A}(t) \mathcal{A}(t') O\left( \frac{|\eta(t)|}{|\eta(t')|} \right)^{n_a+1},
\]

(113)

where \( n_a \) is the spectral index.} because, if \( x \) and \( x' \) are largely separated in time, any Fourier mode in the Wightman function cannot be at the Hubble scale simultaneously at \( t \) and \( t' \). When we consider the contribution of vertexes located in the distant past, at least one Wightman function should satisfy \( \eta(t)/\eta(t') \ll 1 \), and therefore it is suppressed. When
all the time integrations converge, being dominated by the contributions at around \( t = t_f \), we have an estimate,

\[
\langle 0 | \mathcal{R}_{\xi} (t_f, x_1) \cdots \mathcal{R}_{\xi} (t_f, x_n) | 0 \rangle = \mathcal{O} (\lambda (t_f) |A (t_f)| N),
\]

where \( N \equiv N_f - 2N_e \) with \( N_f \) and \( N_e \) being the numbers of \( \tilde{\xi} \)'s and the vertices contained in the corresponding diagram, respectively.

When we consider a diagram for which a cluster of vertices in the distant past is connected to the vertices around the observation time by a single propagator, the IR suppression comes only from this propagator. If the past cluster of vertices includes a sufficiently large number of operators, the increase of the amplitude of fluctuation \( A (t) \) toward the past high energy regime may overtake the suppression due to this propagator. This happens only when \( N \) is extremely large such as \( 1/\epsilon^1 \simeq \mathcal{O} (10^2) \). We should also stress that the SG is totally suppressed in the slow roll limit.

In [84, 94], the absence of the SG is claimed by computing \( \tilde{\zeta}_k \) in the limit \( k/(e^\rho \dot{\rho}) \to 0 \). In these papers, the mode function in de Sitter spacetime, whose amplitude at large scales is given by a constant Hubble parameter, is used in proving the conservation of the curvature perturbation, while the time variation of the amplitude cannot be neglected namely for the tIR modes. This leads to the quantitative discrepancy in the evaluation of the SG from the one given above. For instance, in [84], ensuring that \( \tilde{\zeta}_I (x, t) \) given in equation (22) of the paper does not have long-term correlations is crucial in their proof. However, the locality is not necessarily valid, once we take into account the fact that in the chaotic inflation, the amplitude of the fluctuation becomes larger and larger in the distant past as \( \dot{\rho} \propto e^{-1/2m^2} \). When we neglect this effect by setting \( A \propto (\dot{\rho}/\sqrt{\epsilon}) \) to constant, the above discussion will also lead to the absence of the SG irrespective of the order of perturbation. Therefore the result here does not contradict the conservation of the curvature perturbation they claimed.

Here we also comment on the related works [77–79]. In these references, the authors showed that the two point function which contains the logarithmic IRdiv is related to the one which does not by the dilatation transformation (see section 3.3). In our terminology, the former is \( \langle \zeta (x) \zeta (x') \rangle \) and the later is \( \langle \tilde{\zeta} (x) \tilde{\zeta} (x') \rangle \). Note that \( \langle \tilde{\zeta} (x) \tilde{\zeta} (x') \rangle \) can still suffer from the SG, which can be eliminated only for a limited class of quantum state that is invariant under the dilatation transformation. In fact, explicit realizations of such quantum states that we know are limited to the Euclidean vacuum and its variants.

### 5.5. The summary of the proof

Now we conclude that, when we choose the Euclidean vacuum, the \( n \)-point functions for the genuinely gauge invariant curvature perturbation contain neither the IRdiv nor the IRsec. Furthermore, they do not suffer from the SG unless a very high order in the perturbative expansion is concerned. (The outline of the proof is depicted in figure 3 of [74].) We repeat two key points which ensure the absence of the IRdiv, IRsec, and SG:

- Evaluating a genuinely gauge invariant operator.
- Choosing the quantum state to be invariant under the dilatation transformation.

The genuinely gauge invariant operator should be entirely composed of the Heisenberg picture field \( \zeta (x) \) with \( \mathcal{R}_\xi \), but the operator is not necessarily expanded solely in terms of \( \mathcal{R}_\xi \). Therefore, even if we consider the correlators of genuinely gauge invariant operators, they can suffer from the IRdiv and IRsec. Then, the second point becomes important. By choosing the Euclidean vacuum, which is invariant under the dilatation transformation, the correlators of genuinely gauge invariant operators can be expanded only in terms of \( \mathcal{R}_\xi \).
and thus the \( \text{IRdiv} \) and \( \text{IRsec} \) are eliminated. As we described in section 2.3, all IR and \( \text{tIR} \) modes are initially sub\( H \) modes, and hence it is not satisfactory to neglect sub\( H \) modes from the beginning namely in examining the SG. When we choose the Euclidean vacuum, the UV modes much below the Hubble length scale are also suppressed, leaving aside the ordinary UV divergences to be renormalized by the local counter terms.

6. The IR issues in the absence of the gravitational fluctuation

So far, we discussed the fluctuation of the inflaton taking into account (the longitudinal mode of) the gravitational perturbation. Then, the modes far beyond the Hubble scale are almost indistinguishable from the residual gauge degrees of freedom, which is crucial in showing the absence of the \( \text{IRdiv} \), \( \text{IRsec} \) and SG. By contrast, the same argument does not apply to the IR issues of a test field in a fixed quasi de Sitter background spacetime, which is frequently discussed as an approximate toy model to discuss the iso-curvature perturbations. In this section, we briefly summarize the recent progress in this subject.

6.1. Resummation and the dynamical mass generation

As is described in section 2.3, the logarithmic \( \text{IRdiv} \) originates from the scale-invariant spectrum of a light field in an inflationary spacetime. It has been pointed out that resumming loop diagrams leads to a dynamical mass generation, which will remedy the singular behaviour in the IR. Here, we consider a scalar field \( \phi \) with the quartic coupling \( \lambda \phi^4 \) in a fixed background inflationary spacetime. In this section, we use the dimensionful scalar field \( \Phi \) instead of the dimensionless scalar field \( \phi \).

6.1.1. The stochastic approach. The stochastic approach, initiated by Starobinsky \[63\], describes the evolution of the super\( H \) modes, \( \Phi_{\text{sp}} \), defined by eliminating the contribution from the sub\( H \) modes as

\[
\Phi_{\text{sp}}(x) = \Phi(x) - \int \frac{d^3k}{(2\pi)^{3/2}} \delta(k - \epsilon \epsilon(t) \hat{\rho}(t)) \left[ a_k \Phi_k(t) e^{ik \cdot x} + (\text{h.c.)} \right],
\]

with a small positive parameter \( \epsilon \). The evolution equation for \( \Phi_{\text{sp}} \) is given by

\[
\dot{\Phi}_{\text{sp}}(x) = -\frac{1}{3\hat{\rho}} \frac{dV(\Phi_{\text{sp}})}{d\Phi_{\text{sp}}} + f, \tag{116}
\]

in the slow roll approximation, where \( f(x) \) denotes the stochastic noise due to the modes with \( k \approx \epsilon \epsilon(t) \hat{\rho}(t) \) whose variance is given by

\[
\langle f(x_1) f(x_2) \rangle = \frac{\hat{\rho}^3}{4\pi^2} \delta(t_1 - t_2) \left| x_1 - x_2 \right| \epsilon \epsilon(t) \hat{\rho}(t) \sin \frac{\epsilon \epsilon(t) \hat{\rho}(t)}{\epsilon \epsilon(t) \hat{\rho}(t)} \left| x_1 - x_2 \right|. \tag{117}
\]

The Fokker-Planck equation obtained from equations (116) and (117) gives the probability distribution function (PDF) for \( \Phi_{\text{sp}} \). In \[97\], the PDF for the curvature perturbation was discussed based on the stochastic approach.

Solving the Fokker-Planck equation, Starobinsky and Yokoyama \[64\] showed that the variance of a massless scalar field with a quartic potential \( \lambda \Phi^4 \) approaches a constant value

\[
\langle \Phi_{\text{sp}}^2 \rangle \rightarrow \frac{c}{\sqrt{\lambda}} \hat{\rho}^2, \tag{118}
\]

at late times, where \( c \) is an \( O(1) \) numerical factor. Note that the late time behaviour of the variance does not suffer from the logarithmic enhancement. This equilibrium state can be understood as the balance between the potential force (the first term in equation (116)) and the quantum fluctuation (the second term).
6.1.2. The two-particle irreducible formalism. Riotto and Sloth pointed out that the late time behaviour of \( \langle \Phi_1^2 \rangle \) signals the dynamical mass generation due to the higher loop contributions [98]. Using the two-particle irreducible (2PI) effective action, we can obtain the equation of motion for the resummed propagator \( G(x_1, x_2) \) (see [99] and the references therein). A local contribution in the self-energy \( \Sigma_1(x_1, x_2) \), which is proportional to the delta function \( \delta(x_1 - x_2) \), shifts the effective mass which appears in the equation of motion for \( G(x_1, x_2) \). (For instance, the left diagram of figure 1 gives a local contribution to the self-energy.) Namely, for a quartic interaction, the dynamically generated mass is given by \( M_{\text{dyn}}^2 \sim \lambda G(x, x) \) at the leading order of the 2PI expansion. Replacing \( G(x, x) \) with the variance (118), Riotto and Sloth claimed that resumming higher loops yields the effective mass of order of the Hubble scale as \( M_{\text{dyn}}^2 \sim \sqrt{\lambda} \dot{\rho}^2 \). Garbrecht and Rigopoulos [100], Serreau [101–103], and Arai [104–106] discussed the dynamical mass generation in more detail, using the 2PI formalism. In particular, in [102] and [105, 106], they elaborated on the UV contributions, identifying the necessary counter terms.

6.1.3. The dynamical renormalization group. The mass generation due to the resummation was reported also by Burges et al [107, 108], based on the dynamical renormalization group (RG) technique, which is useful to address the RG flow in the presence of the secular growth in time. They focused on the secular growth which appears from the naive estimation of the accumulated superH modes. By using the conventional cosmological perturbation theory, the power spectrum of the test field in the de Sitter space with the loop correction is given by

\[
P_\Phi(k, \eta) = \frac{\dot{\rho}^2}{2k^3} \left[ 1 + \delta \ln \left( \frac{k}{e^{e^\rho} \dot{\rho}} \right) \right] \tag{119}
\]

with

\[
\delta \equiv \frac{\lambda}{3} \frac{\langle \Phi_1^2 \rangle}{\dot{\rho}^2}. \tag{120}
\]

where the logarithmic term \( \ln(k/e^{e^\rho}) \) appeared by integrating the superH modes in time. Here we neglected the subH contribution. The dynamical RG technique suggests resumming the loop correction as

\[
P_\Phi(k, \eta) = \frac{\dot{\rho}^2}{2k^3} \left( \frac{k}{e^{e^\rho} \dot{\rho}} \right)^{\frac{\delta}{2}} [1 + \mathcal{O}(\delta^2)]. \tag{121}
\]

Since the power spectrum for the massive scalar field is given by equation (121) with \( \delta = 2M_{\text{dyn}}^2/(3\dot{\rho}^2) \), we see that the resummation generates the mass of order \( M_{\text{res}}^2 \sim \dot{\rho}^2 \delta \sim \lambda \langle \Phi_1^2 \rangle \), reproducing the result obtained by Riotto and Sloth [98]. (An impact of the resummation was explored based on different approaches in [109, 110].)

The dynamical mass generation, addressed in this subsection, can be interpreted as the thermalization process. Since the de Sitter space does not possess a global timelike Killing vector, there is no positive definite conserved energy, and the background spacetime plays the role of the heat bath with the de Sitter temperature \( \approx \dot{\rho} \). A related particle decay process in the de Sitter space was also discussed by Bros et al in [111–113].

6.2. The regularity for the Euclidean vacuum

For the non-interacting theory, a systematic study of the de Sitter invariant Euclidean vacuum was done by Mottola in [114] and Allen in [115]. Recently, Hollands [116] and Marolf and Morrison [117–119] systematically investigated the higher order loop corrections of a massive test scalar field in the exact de Sitter space. They showed the perturbative stability of the
Euclidean vacuum, which is identical to the so-called Bunch-Davies vacuum in the exact de Sitter case.

The metric of the $D$-dimensional de Sitter space in global coordinates is given by

$$ds^2 = -\left(\frac{l}{t}\right)^2 \cosh^2 \frac{t}{l} \, d\Omega_{D-1}^2,$$

where $l \equiv 1/\dot{\rho}$ is the de Sitter length scale and $d\Omega_{D-1}^2$ is the metric on a unit $(D - 1)$-dimensional sphere. Performing the Wick rotation from $t$ to $\tau \equiv \pi/2 - i(t/l)$, the metric (122) is analytically continued to the metric on a $D$-dimensional sphere,

$$ds^2 = l^2 \left[ d\tau^2 + (\sin \tau)^2 d\Omega_{D-1}^2 \right].$$

(123)

It is obvious that the IRdiv is absent in the vertex integral over a compact manifold as long as the free propagator is well behaved there. Therefore, they first computed the $n$-point functions on the Euclidean sphere and then analytically continued the results to the Lorentzian domain.

The de Sitter space can be described as the $D$-dimensional hyperboloid with $\eta_{AB}X^A X^B = l^2$ embedded in the $(D + 1)$-dimensional Minkowski spacetime with the metric tensor $\eta_{AB}$. The free propagator $\langle \Phi(x_1) \Phi(x_2) \rangle$ for the Bunch-Davies vacuum is described only in terms of the invariant distance $Z_{12} \equiv \eta_{AB}X^A(x_1)X^B(x_2)/l^2$. In [116] and [119] the $n$-point functions in the limit where two arguments are largely separated as $|Z_{12}| \gg 1$ was examined, and they showed that the loop corrections decays with the power in $|Z_{12}|$ not slower than the free field two-point function. The equivalence between the correlators obtained after the analytic continuation and those computed in the Poicaré patch, i.e., the expanding cosmological patch, with the ie prescription is shown for interacting massive fields by Higuchi et al [120] (and also by Korai and Tanaka in a different way [121]).

By contrast, for a massless scalar field, the IR regularity has not been shown and the absence of the SG is unclear [48–50, 122]. Although the dynamical mass generation is one possible answer [123, 124], we can think of several possible situations in which mass generation is prohibited for the symmetry reason. The adiabatic curvature perturbation is a sort of massless field whose mass generation is not allowed because of the diffeomorphism invariance. In this case the IR suppressing operators $R_x$ are associated with the observable quantities by virtue of the residual gauge symmetry. Because of that, although its Wightman function $G^\tau(x, x')$ behaves similar to a massless scalar field, the singular behavior is cured for the Euclidean vacuum. In this sense, it would be intriguing to discuss a massless field with the exact shift symmetry in the de Sitter space (see also [91]).

6.3. The quantum decoherence and the IR problem

Even if the careful computations of correlation functions for observable quantities may give divergent results, this does not immediately indicate a pathology. This is because what we actually observe still can be different from what we compute based on the standard quantum field theory. The primordial perturbations are supposed to decohere through the cosmic expansion and/or through various interactions [63, 125, 126]. This decoherence process transmutes the quantum fluctuations at a long wavelength to a statistical ensemble. In the standard computation, however, the effect of this quantum decoherence is not taken into account, and hence the correlation functions that we calculate are the expectation values for a superposition of various wave packets which will never be observed simultaneously in reality.

Here, we focus on the spatial average of a test scalar field $\Phi$ given by

$$\overline{\Phi}(t) \equiv \frac{\int d^3x W_t(x) \Phi(t, x)}{\int d^3x W_t(x)}.$$
Figure 4. The left-hand panel shows the wave packets at the early stage of inflation, which are correlated with each other. At later times, the wave packets get decohered as depicted in the right-hand panel. At observation, only one of the decohered wave packets is picked up. To eliminate the influence from the irrelevant wave packets to us, we introduce the operator $P_{\alpha}$.

When we choose an initial state with the scale-invariant spectrum in the IR limit, the variance $\langle \bar{\Phi}_1(t) \rangle^2$ diverges. The unbounded variance implies that the wave function of $\bar{\Phi}_1$, $\Psi[\bar{\Phi}]$, does not have a sharp peak around a specific value but spreads infinitely. As is shown in figure 4, such a wave function can be decomposed into a superposition of wave packets. Even if the quantum state starts with a coherent superposition of wave packets, the quantum coherence is gradually lost through the time evolution. Thus, at a later observation time $t_{\text{obs}}$ the quantum coherence will be kept only among adjacent wave packets. Our universe will select a particular value $\bar{\Phi}(t_f) = \alpha$ once it is observed. This corresponds to picking up a single decohered wave packet peaked at $\bar{\Phi}(t_f) = \alpha$ from the superposition of wave packets. After the decoherence takes place, the other wave packets far from $\bar{\Phi}(t_f) \approx \alpha$ never contribute to observable quantities. Therefore it is more appropriate to remove the influence of these wave packets from the computation of observable quantities.

To take into account this selection effect, in [76], we proposed to insert an operator:

$$P_{\alpha} \equiv \exp \left[ \frac{-(\bar{\Phi}(t_f) - \alpha)^2}{2\sigma^2 M_{\text{pl}}^2} \right].$$

(124)

Here, $\sigma$ denotes the width of the projection window, which is supposed to satisfy $\dot{\rho}/M_{\text{pl}} < \sigma < 1$ (see [76], in which we considered a multi-field model of inflation, decomposing the fluctuations into the curvature perturbation and the iso-curvature perturbations. The same discussion for the iso-curvature perturbation applies to the test field discussed here). For instance, as the decohered $n$-point functions of $\mathcal{R}$, we proposed to compute

$$\frac{\langle P_{\alpha} \mathcal{R}(t_{\text{obs}}, x_1) \cdots \mathcal{R}(t_{\text{obs}}, x_n) \rangle}{\langle P_{\alpha} \rangle}.$$ 

(125)

The introduction of $P_{\alpha}$ removes the contamination from the wave packets (parallel worlds) that are not correlated with our wave packet at observation. In [76], we showed that after the introduction of the operator $P_{\alpha}$, the superH modes $k \lesssim e^{\rho(t_f)} \dot{\rho}(t_f)$ of the iso-curvature fields are suppressed. Although the IRdiv and IRsec can be removed by this prescription, the SG for the iso-curvature perturbation are still left as an open question.

The stochastic approach discussed briefly in section 6.1.1 assumes the decoherence when the scale exceeds the Hubble scale because of the large squeezing of the quantum state [63, 64, 125, 127, 128]. The decoherence process has been discussed by means of the coarse-graining of some degrees of freedom identified as environment. As a result, the reduced density matrix evolves from the initial pure state to a mixed state. (For instance, see [126, 129–131] and also...
This process is interpreted as the transition from the initial coherent superposition of many different worlds to the final statistical ensemble of them. We believe that the stochastic approach will provide a good approximation to the description of the decohered fluctuation. However, we also think that the stochastic approach is not sufficient to discuss the IR regularity issue, because the quantum nature of fluctuations of long wavelength modes is omitted by its assumption from the beginning.

7. The graviton loops

We have discussed the IR issues of the scalar perturbation so far, neglecting the tensor perturbation. In this section, we briefly discuss the IR issues related to the graviton and overview the recent progress.

7.1. The IR divergence and the secular growth from the graviton loops

The quadratic action for the tensor perturbation $\delta \gamma_{ij}$, which describes the evolution of the interaction picture field $\delta \gamma_{ij}^I$, is given by

$$S_{0, GW} = \frac{M_{pl}^2}{8} \int dt d^3x e^{\rho} \left[ e^{-\rho} \delta \gamma_{ij}^I \delta \gamma_{ij}^I - e^{-2\rho} \partial_t \delta \gamma_{ij}^I \delta \gamma_{ij}^I \right],$$  \hspace{1cm} (126)

and the equation of motion is given by

$$\left[ \partial_t^2 + 3 \dot{\rho} \partial_t - e^{-2\rho} \partial^2 \right] \delta \gamma_{ij} = 0.$$ \hspace{1cm} (127)

We quantize $\delta \gamma_{ij}^I$ as

$$\delta \gamma_{ij}^I(x) = \sum_{\lambda = \pm} \int \frac{d^3k}{(2\pi)^{3/2}} \delta \gamma_{ij}^{(\lambda)}(k) e^{i k \cdot x} a_{\lambda}^i(k) + \text{(h.c.)},$$ \hspace{1cm} (128)

where $\lambda$ is the helicity of the tensor perturbation, $e_{ij}^{(\lambda)}$ are the transverse and traceless polarization tensors normalized as $e_{ij}^{(\lambda)}(k) e^{(\lambda') ij}(k) = \delta_{\lambda \lambda'}$, and $a_{\lambda}^i(k)$ are the annihilation operators which satisfy

$$[a_{\lambda}^i(k), a_{\lambda'}^j(k')] = \delta_{\lambda \lambda'} \delta^{(3)}(k - k').$$ \hspace{1cm} (129)

Since the equation for $\delta \gamma_{ij}^{(\lambda)}$ is identical to the one for a massless scalar field, the graviton field in the adiabatic vacuum (see section 2.2) has the almost scale-invariant spectrum in the IR limit as

$$P_{GW}(k) = 2 \left| \delta \gamma_{ij}(t) \right|^2 = \frac{4}{k^3} \left( \frac{\dot{\rho}(t)}{M_{pl}} \right)^2 \left[ 1 + O((k\eta)^2) \right].$$ \hspace{1cm} (130)

Since the spectrum is isotropic, the amplitude of $\delta \gamma_{ij}$ does not depend on the helicity and hence, we doubled the amplitude. Using equation (128), the variance of the graviton (in the coincidence limit) is given by

$$\langle \delta \gamma_{ij}(x) \delta \gamma_{kl}(x) \rangle = \frac{1}{10} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right) \int \frac{d^3k}{(2\pi)^3} P_{GW}(k).$$ \hspace{1cm} (131)

We can see that similarly to the curvature perturbation, the superH modes in equation (131) yield the IRdiv and IRsec, respectively. As in the case of the curvature perturbation $\zeta$, one possible way to prove the IR regularity and the absence of the SG might be showing that the $n$-point functions are perturbatively expanded with respect to $\delta \gamma_{ij}$ associated with IR suppressing operators.

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Tsamis and Woodard claimed that the logarithmic secular growth due to the graviton loops can lead to the screening of the cosmological constant in [24]. More recently, Kitamoto and Kitazawa claimed that the secular growth from the graviton loops can screen the gauge coupling as well in [34, 35]. A related issue is discussed for the U(1) gauge field in [36, 37]. If the secular growth due to the graviton loops is really physical, it will provide an interesting phenomenological impact. However, we should also keep in mind the subtlety in the interpretation of the calculated results [136]. It was shown that the spatially averaged Hubble expansion computed by Tsamis and Woodard is not invariant under the change of the time slicing and hence the observed screening effect suffers from the gauge artifact [88, 136].

Focusing on the fact that a conformally coupled scalar field which interacts with a source which is homogeneously distributed in the spacetime measures the Hubble expansion rate \( \dot{\rho} \) as \( \dot{\rho}^2(t) \simeq R/12 \), where \( R \) is the four-dimensional Ricci scalar, the authors of [136] computed the expectation value of the smeared Ricci scalar in a local region as a local counterpart of the Hubble expansion rate. It was shown that the smeared Ricci scalar is time independent at least if an appropriate UV renormalization is assumed. This example tells us that once gravitational perturbations are concerned computing observable quantities unaffected by the gauge artifact is quite important [136] (see also [137] and [19, 39]). This is in harmony with our claim regarding the IR issues of the curvature perturbation [71, 72].

7.2. Fixing the residual gauge degrees of freedom

The study of graviton loops is still in progress and we need more elaborated discussions to provide a conclusive argument. However, we think that an important clue to solve this problem is that the homogeneous mode of the graviton \( \delta\gamma_{ij} \) is also indistinguishable from the residual gauge degrees of freedom [71, 72]. Among the residual gauge transformations, discussed in section 3.2, we focus on

\[
x^i \rightarrow e^{-i\alpha(t)} \left[e^{-S(t)/2}\right]_j x^j,
\]

where \( S_{ij}(t) \) is a time-dependent traceless tensor. As for the curvature perturbation, computing an invariant quantity under the dilatation transformation parametrized by \( s(t) \) was a key to show the regularity of the correlation functions. Intriguingly, at the linear level tensor perturbation is shifted as

\[
\delta\gamma_{ij}(x) \rightarrow \delta\gamma_{ij}(x) - S_{ij}(t),
\]

analogously to \( \zeta \). Although the nonlinear extension of the above transformation is more complicated than in the case of \( \zeta \), this observation suggests that the analogous proof of the IR regularity may work for the graviton loops.

The relation between the IRdiv due to graviton loops and the homogeneous shift (133) has been pointed out several times. Gerstenlauer \textit{et al} [78] and Giddings and Sloth [79] showed that the leading IRdiv of the graviton loops can be understood as the change of the spatial coordinates in the form (132) with \( s = 0 \) due to the accumulated effect of IR gravitons. In [71, 72], we found that the graviton one-loop in the calculation of \( \langle RR \rangle \) becomes regular without restricting the initial vacuum states. In this analysis, however, we heuristically adopted a particular solution of the Heisenberg equation. In our forthcoming publication [138], we will provide a comprehensive study on the regularity of graviton loops, focusing on the genuinely gauge invariant quantities.

The gravitons can exist also in the exact de Sitter background unlike the curvature perturbation. In the exact de Sitter background, the IR regularity might be rephrased as the existence of a regular quantum state that respects the de Sitter invariance. If the two-point function in the Euclidean vacuum for the free graviton field is regular, it will be extended to the
higher order perturbation straightforwardly without suffering from the IR div by performing the vertex integrations on the Euclidean sphere that is the analytic continuation of the de Sitter space. Even at the linear level, however, the regularity of the graviton two-point function is still under debate. In [139], Higuchi et al claimed the existence of a regular two-point function, while Miao et al objected against it in [140]. This issue was also discussed in [141–143]. At the moment we are writing this review article, there is no consensus regarding this issue.

8. Concluding remarks and future issues

We summarized the issues regarding the loop corrections of the three different types of perturbation in the inflationary universe, the adiabatic perturbation, the iso-curvature perturbation, and the tensor perturbation. Irrespective of the type of perturbations, what is crucially important for ensuring the IR regularity is to remove the unobservable effects. Namely, unless we concede the influence of the residual gauge modes, the adiabatic and tensor perturbations are free from the IR pathologies. The most intriguing result we have obtained will be that when we perform the quantization in the global universe, choosing an appropriate quantum state is required to regularize the IR contributions. Fortunately, the IR regularity is guaranteed if we choose the ordinary Euclidean vacuum.

In this section, we further address this point asking the question, ‘When we require the \( n \)-point functions to be finite and free from the SG, is the Euclidean vacuum the unique possible quantum state?’ If we inquire the regularity of the \( n \)-point functions on the real time axis all the way back to the distant past, we naively expect that the Euclidean vacuum is the unique possibility. This is because any excitations are blue-shifted toward the past, and hence any small deviation from the Euclidean vacuum will be enhanced to an infinite magnitude in the limit. On the other hand, if we require the regularity only in the future of a given initial hypersurface, we would be able to construct a variety of allowed quantum states. (See also the studies by Einhorn and Larsen in [144, 145] and by Marolf et al in [146].)

The residual gauge degrees of freedom can also affect the notion of the tree-level non-Gaussianity. In [147], we re-defined the primordial non-Gaussianity in the single field models, requesting the genuinely gauge invariance. We studied the local bi-spectrum, taking the squeezed limit, where one of \( \mathbf{k}_i \) is sent to 0. Namely, we revisited the so-called consistency relation, which relates the local non-Gaussianity with the power spectrum. As it is pointed out by Maldacena in [2], the leading contribution in the consistency relation stems from the effect of the IR mode \( \mathbf{k}_1 \), which shifts \( \mathbf{k}_j \) with \( j = 2, 3 \) as \( \mathbf{k}_j \rightarrow e^{-\zeta k_1} \mathbf{k}_j \). As it is expected from the fact that this dilatation transformation is one of the residual gauge transformations; the leading contribution in the consistency relation does not appear when we evaluate the genuinely gauge invariant three point function. This analysis is extended to the multi-field models of inflation in [85]. A related issue has been studied recently by Creminelli et al [148] and by Pajer et al [149]. These studies indicate that the gauge degrees of freedom should be carefully treated to provide a theoretical prediction to compare with observations even at the tree level.

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