EXTENSIONS OF GROUPS BY BRAIDED 2-GROUPS

EVAN JENKINS

Abstract. We classify extensions of a group $G$ by a braided 2-group $B$ as defined by Drinfeld, Gelaki, Nikshych, and Ostrik. We describe such extensions as homotopy classes of maps from the classifying space of $G$ to the classifying space of the 3-group of braided $B$-itororsors. The Postnikov system of the latter space contains a generalization of the classical Pontryagin square to the setting of local coefficients, which has been previously discussed by Baues and more recently, in a setting close to ours, by Etingof, Nikshych, and Ostrik. We give an explicit cochain-level description of this Pontryagin square for group cohomology.

1. Introduction

A classical problem in homological algebra is to classify extensions of a group $G$ by another group $K$, i.e., groups $E$ equipped with a surjection $\partial : E \to G$ and an identification $K \cong \partial^{-1}(e)$. When $K = A$ is an abelian group, the action of an element of $E$ on an element of $A$ by conjugation depends only on its image in $G$, so extensions of $G$ by $A$ include the data of an action of $G$ as automorphisms of $A$. If we fix such an action, then extensions with that action are classified by $H^2(G,A)$ (see, for example, [Wei95, 6.6]).

In [DGNO10, Appendix E], a categorification of the notion of an extension of a group by an abelian group is defined, in which the abelian group $A$ is replaced by a braided 2-group $B$ (see [BL04] for basic results about 2-groups, which have also been studied under the names gr-category and categorical group, and [JS93] for basic results about braided monoidal categories). Given such an extension one has an underlying action of $G$ as braided autoequivalences of the braided 2-group $B$, and also an underlying extension of $G$ by the abelian group $A = \pi_0(B)$ of isomorphism classes of objects of $B$. It is natural to ask under what circumstances these two pieces of data will determine an extension of $G$ by $B$ and how unique such an extension is.

In this paper, we show that we can lift these data to an extension if and only if a certain cohomology class in $H^4(G,H)$ vanishes, where $H = \pi_1(B)$ is the abelian group of isomorphisms of the unit object in $B$, and the action of $G$ on $H$ is induced from the action of $G$ on $B$. This cohomology class comes from a function $H^2(G,A) \to H^4(G,H)$ (determined by a fixed braided 2-group $B$ and an action of $G$ on $B$) that generalizes the classical Pontryagin square, which can be recovered in our language by taking the trivial action of $G$ on $B$. If this obstruction vanishes, then extensions form a torsor for $H^3(G,H)$.

These results are parallel to the results of [ENO10] on the related problem of classifying braided $G$-crossed fusion categories. In that paper, a braided $G$-crossed fusion category $\mathcal{C}$ is viewed as a $G$-indexed family of invertible bimodules over the neutral component. Analogously, we will view extensions of $G$ by a braided 2-group
We then use obstruction theory to obtain our classification. Sections 2 through 4 set up the machinery of braided bitorsors. Section 5 relates extensions and bitorsors by a construction of Grothendieck. Section 6 contains our classification result. Section 7 contains an explicit cochain-level description of the Pontryagin square defined in Section 6.

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2. TORSORS FOR 2-GROUPS

We can define torsors for 2-groups much the same as we do for groups. We first review the notion of modules for monoidal categories.

A monoidal category $C$ can be viewed as a one-object bicategory, which we will denote by $C[1]$. This can be viewed as the “delooping” of $C$, a notion we will revisit in Section 6. For now, we note only that strong monoidal functors between monoidal categories correspond to pseudofunctors between their deloopings, and that $C[1]^{\text{op}} = (C^{\text{rev}})[1]$, where $C^{\text{rev}}$ denotes the category $C$ with reversed tensor product.

Definition 2.1. Let $C$ be a monoidal category. A left (resp. right) module (or module category) over $C$ is a pseudofunctor $X : C[1] \to \text{Cat}$ (resp. $X : C[1]^{\text{op}} \to \text{Cat}$). By abuse of notation, we denote the image of the unique object of $C[1]$ by $X$. We denote by $C\text{-Mod}$ and $\text{Mod}_C$ the 2-categories of left and right $C$-modules, respectively.

We will write a left action of an element $c \in C$ on an element $x \in X$ by $c \triangleright x$, and a right action as $x \triangleleft c$. For the rest of this subsection, we will only deal with left $C$-modules. Completely analogous definitions and proofs work with “left” everywhere replaced by “right.”

Lemma 2.2. Let $F : X \to Y$ be a morphism of left $C$-modules, and suppose further that $F$ is an equivalence of categories. Then $F$ is an equivalence of $C$-modules.

Proof. The forgetful 2-functor $C\text{-Mod} \to \text{Cat}$ is monadic, and hence it reflects adjoint equivalences (see [CMV02]). □

Recall that a 2-group is a monoidal category in which all objects and morphisms are invertible.

Definition 2.3. Let $G$ be a 2-group. A left (resp. right) module $\mathcal{X}$ is a left $G$-torsor if $\mathcal{X}$ is nonempty and the characteristic map $\chi = (a, \pi_2) : \mathcal{G} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is an equivalence.

Definition 2.4. The trivial left $G$-module, denoted by $G$, is $G$ equipped with the action of left multiplication.

Definition 2.5. The action of $G$ on $\mathcal{X}$ is essentially simply transitive if, for every $x \in \mathcal{X}$, the map $G \to \mathcal{X}, g \mapsto g \triangleright x$ is an equivalence of categories.

Theorem 2.6. Let $G$ be a 2-group, $\mathcal{X}$ a left $G$-module. The following are equivalent.

1. $\mathcal{X}$ is a left $G$-torsor.
(2) $\mathcal{X}$ is nonempty, and the action of $\mathcal{G}$ on $\mathcal{X}$ is essentially simply transitive.

(3) $\mathcal{X}$ is equivalent to the trivial left $\mathcal{G}$-module.

**Proof.** (1) $\Rightarrow$ (2): Fix $x \in \mathcal{X}$. We will show that the map $F : g \mapsto g \triangleright x$ is full, faithful, and essentially surjective.

If $x' \in \mathcal{X}$, essential surjectivity of the characteristic map implies that there exists $g \in \mathcal{G}$, $x'' \cong x$ such that $g \triangleright x'' \cong x'$. It follows that $F$ is essentially surjective.

If $g, g' \in \mathcal{G}$, $\phi_1, \phi_2 : g \rightarrow g'$, such that $\phi_1 \triangleright x = \phi_2 \triangleright x : g \triangleright x \rightarrow g' \triangleright x$, then $\phi_1 \triangleright (x, x) = (\phi_2 \triangleright x, x)$, so faithfulness of the characteristic map implies that $\phi_1 = \phi_2$. Thus, $F$ is faithful.

Finally, if $g, g' \in \mathcal{G}$, and $f : g \triangleright x \rightarrow g' \triangleright x$, then fullness of the characteristic map gives a map $(\phi, \text{id}) : (g, x) \rightarrow (g', x)$ such that $\phi \triangleright \text{id} : g \triangleright x \rightarrow g' \triangleright x$ is $f$. Thus, $F$ is full.

(2) $\Rightarrow$ (3): The map $g \mapsto g \triangleright x$ is a map $\mathcal{G} \rightarrow \mathcal{X}$ of $\mathcal{G}$-modules and an equivalence of categories, so by Lemma 2.2 it is an equivalence of $\mathcal{G}$-modules.

(3) $\Rightarrow$ (1): It suffices to show that the trivial left $\mathcal{G}$-module is a left $\mathcal{G}$-torsor. The map $(a, \tau_2) : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$ has quasi-inverse $(a \circ (-1), \tau_2)$, so $\mathcal{G}$ is a left $\mathcal{G}$-torsor. □

**Corollary 2.7.** Any morphism $F : \mathcal{X} \rightarrow \mathcal{Y}$ of left $\mathcal{G}$-torsors is an equivalence.

**Proof.** Fix $x \in \mathcal{X}$. Then the following diagram commutes up to isomorphism.

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{g \mapsto g \triangleright x} & \mathcal{X} \\
\downarrow & & \downarrow F \\
\mathcal{G} & \xrightarrow{g \mapsto g \triangleright F(x)} & \mathcal{Y}
\end{array}
\]

By Theorem 2.6 the top two arrows are equivalences, so $F$ is an equivalence. □

**Proposition 2.8.** The autoequivalence 2-group of a left $\mathcal{G}$-torsor is equivalent to $\mathcal{G}$.

**Proof.** By Theorem 2.6 every $\mathcal{G}$-torsor is equivalent to the trivial left $\mathcal{G}$-torsor, so it suffices to show this for the trivial left $\mathcal{G}$-torsor $\mathcal{G}$. We get a monoidal functor $\mathcal{G} \rightarrow \text{Aut}_\mathcal{G}(\mathcal{G})$ by sending $g \in \mathcal{G}$ to $x \mapsto xg^{-1}$. We get a monoidal functor $\text{Aut}_\mathcal{G}(\mathcal{G}) \rightarrow \mathcal{G}$ by sending $f \in \text{Aut}_\mathcal{G}(\mathcal{G})$ to $f(e)^{-1}$. The composition $\mathcal{G} \rightarrow \text{Aut}_\mathcal{G}(\mathcal{G}) \rightarrow \mathcal{G}$ is isomorphic to the identity; for the other direction, we note that if $f(e)^{-1} \cong g$, then $f(x) \cong xx^{-1}f(x) \cong xf(e) \cong xg^{-1}$ is an isomorphism of $\mathcal{G}$-module morphisms. □

### 3. Bitorsors for 2-groups

We can also define bitorsors for 2-groups much the same as we do for groups. We begin here with the notion of bimodules for monoidal categories.

**Definition 3.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A $(\mathcal{C}, \mathcal{D})$-bimodule is a $(\mathcal{D}^{op} \times \mathcal{C})$-module. We think of $\mathcal{C}$ as acting on the left and $\mathcal{D}$ as acting on the right.

We denote by $\mathcal{C} \text{ Mod}_\mathcal{D}$ the 2-category of $(\mathcal{C}, \mathcal{D})$-bimodules.
Definition 3.2. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be 2-groups, and let $\mathcal{X}$ and $\mathcal{Y}$ be $(\mathcal{C}, \mathcal{D})$- and $(\mathcal{D}, \mathcal{E})$-bimodules, respectively. The tensor product $\mathcal{X} \times \mathcal{D} \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$ is defined to be the bicolimit of the diagram

$$\mathcal{X} \times \mathcal{D} \times \mathcal{D} \times \mathcal{Y} \rightrightarrows \mathcal{X} \times \mathcal{D} \times \mathcal{Y} \Rightarrow \mathcal{X} \times \mathcal{Y},$$

where $\mathcal{C}$ acts on the left on each left factor, and $\mathcal{E}$ acts on the right on each right factor. Here, the arrows correspond to the product of $\mathcal{D}$ and the various actions of $\mathcal{D}$ on $\mathcal{X}$ and $\mathcal{Y}$.

In other words, a $(\mathcal{C}, \mathcal{E})$-bimodule map $\mathcal{X} \times \mathcal{D} \mathcal{Y} \rightarrow \mathcal{Z}$ is a map $\mathcal{F}: \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ equipped with a family of isomorphisms $\phi_d: \mathcal{F}(x \triangleright d, y) \cong \mathcal{F}(x, d \triangleright y)$ natural in $d$ and compatible with the product in $\mathcal{D}$.

The following important "folklore" result is entirely formal, although the author does not believe a reference exists.

Theorem 3.3. Let $\mathcal{C}$ be a monoidal category. Then $\mathcal{C}$-bimodules form a monoidal 2-category.

Proof. We give only the vaguest outline of a proof. We will identify $\mathcal{C}$-bimodules with endo-pseudofunctors of the presheaf 2-category $[(\mathcal{C}[1])^{\text{op}}, \text{Cat}]$ that preserve all weighted bicolimits. For the relevant bicategorical definitions, see [Str80] and [Str87].

Given a pseudofunctor $\hat{\mathcal{X}}: [(\mathcal{C}[1])^{\text{op}}, \text{Cat}] \rightarrow [(\mathcal{C}[1])^{\text{op}}, \text{Cat}]$ preserving weighted bicolimits, we restrict via the bicategorical Yoneda embedding to get a pseudo-functor $\mathcal{X}': \mathcal{C}[1] \rightarrow [(\mathcal{C}[1])^{\text{op}}, \text{Cat}]$, which corresponds under adjunction to a $\mathcal{C}$-bimodule $\mathcal{X}: (\mathcal{C}[1])^{\text{op}} \times \mathcal{C}[1] \rightarrow \text{Cat}$. In the other direction, given a module $\mathcal{X}: (\mathcal{C}[1])^{\text{op}} \times \mathcal{C}[1] \rightarrow \text{Cat}$, we construct a weighted bicolimit-preserving pseudofunctor $\hat{\mathcal{X}}: [(\mathcal{C}[1])^{\text{op}}, \text{Cat}] \rightarrow [(\mathcal{C}[1])^{\text{op}}, \text{Cat}]$ by

$$\hat{\mathcal{X}}(\mathcal{Y})(*) = \text{bicolim}(\mathcal{Y}(\cdot), \mathcal{X}(\cdot, *)) .$$

These two constructions give quasi-inverse equivalences of 2-categories, and take the tensor product of bimodules to the composition of pseudofunctors. Since pseudofunctors form a (strict) monoidal 2-category, we may transport this structure to the 2-category of $\mathcal{C}$-bimodules to get a monoidal 2-category. \qed

Next, we prove a coherence result for bimodules.

Definition 3.4. Let $\mathcal{C}$ be a strict monoidal category. A strict $\mathcal{C}$-bimodule is a category $\mathcal{X}$ equipped with a strict 2-functor $\mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \text{Aut}(\mathcal{X})$.

Definition 3.5. Let $\mathcal{C}$ be a strict monoidal category, and let $\mathcal{X}$ and $\mathcal{Y}$ be strict $\mathcal{C}$-bimodules. A strict morphism from $\mathcal{X}$ to $\mathcal{Y}$ is a strict 2-natural transformation $\mathcal{X} \Rightarrow \mathcal{Y}$.

Theorem 3.6. Let $\mathcal{C}$ be a monoidal category, and let $\mathcal{X}$ be a $\mathcal{C}$-bimodule. Then there is a strict $\mathcal{C}$-bimodule $\text{st}\mathcal{X}$ over $\text{st}\mathcal{C}$ and an equivalence $\mathcal{X} \rightarrow \text{st}\mathcal{X}$ that is equivariant with respect to the strictification maps $\mathcal{C} \rightarrow \text{st}\mathcal{C}$.

Proof. We consider the bicategory $\mathcal{C}_\mathcal{X}$ defined as follows.

- $\mathcal{C}_\mathcal{X}$ has two objects, which we will denote $\mathcal{A}$ and $\mathcal{B}$.
- $\text{End}(\mathcal{A}) = \text{End}(\mathcal{B}) = \mathcal{C}$.
- $\text{Hom}(\mathcal{A}, \mathcal{B}) = \mathcal{X}$, while $\text{Hom}(\mathcal{B}, \mathcal{A}) = \emptyset$.
- Composition is given by the left and right actions of $\mathcal{C}$ on $\mathcal{X}$.
We now appeal to the coherence theorem for bicategories (see, for example, [MLP85, Section 2]) to get a strict 2-category \( \mathcal{C}_X \) with the same object set as \( \mathcal{C}_X \) and a biequivalence \( F : \mathcal{C}_X \to \text{st}\mathcal{C}_X \). The biequivalence \( F \) induces monoidal equivalences \( \text{End}(\mathcal{A}) \to \text{st}\text{End}(\mathcal{A}) = \mathcal{C}, \text{End}(\mathcal{B}) \to \text{st}\text{End}(\mathcal{B}) = \mathcal{C}, \) and \( \mathcal{X} \to \text{st}\mathcal{X} \) compatible with the left and right actions. □

**Definition 3.7.** Let \( \mathcal{G} \) be a (strict) 2-group. A \( \mathcal{G} \)-bimodule \( \mathcal{X} \) is a \( \mathcal{G} \)-bitorsor if \( \mathcal{X} \) is a torsor separately with respect to both the left action and the right action.

**Proposition 3.8.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be \( \mathcal{G} \)-bitorsors. Then \( \mathcal{X} \times \mathcal{G} \mathcal{Y} \) is a \( \mathcal{G} \)-bitorsor.

**Proof.** By Theorem 2.6 \( \mathcal{Y} \) is equivalent to \( \mathcal{G} \) as a left \( \mathcal{G} \)-torsor. It follows that as a left \( \mathcal{G} \)-module, \( \mathcal{X} \times \mathcal{G} \mathcal{Y} \cong \mathcal{X} \times \mathcal{G} \mathcal{G} \cong \mathcal{X} \cong \mathcal{G} \), so \( \mathcal{X} \times \mathcal{G} \mathcal{Y} \) is a left \( \mathcal{G} \)-torsor. An identical argument shows that it is a right \( \mathcal{G} \)-torsor. □

We can construct nontrivial (strict) \( \mathcal{G} \)-bitorsors as follows. Given \( \Phi \in \text{Aut}(\mathcal{G}) \), we define a bitorsor \( \mathcal{G}_\Phi \) to be \( \mathcal{G} \) as a left \( \mathcal{G} \)-torsor. The right action is given by \( \mathcal{G}^\text{op} \overset{\Phi}{\to} \mathcal{G} \overset{\cong}{\to} \text{Aut}(\mathcal{G}) \), where the last arrow comes from Proposition 2.6. Explicitly, \( (g, g') \in \mathcal{G} \times \mathcal{G}^\text{op} \) by \( x \mapsto gx\Phi(g') \).

Our next goal is to show that every \( \mathcal{G} \)-bitorsor is equivalent to one of the above form. To do this, we introduce the adjoint autoequivalence of \( \mathcal{X} \).

For a given \( x \in \mathcal{X} \), we define a 2-group \( \Gamma_{\mathcal{G}, \mathcal{X}, x} \) consisting of triples \( (g, g', \phi) \), where \( g, g' \in \mathcal{G} \) and \( \phi : g \triangleright x \overset{\cong}{\to} x \triangleleft g' \). A morphism of triples \( (g_1, g_1', \phi_1) \to (g_2, g_2', \phi_2) \) is a pair of morphisms \( g_1 \to g_2, g_1' \to g_2' \) that intertwine \( \phi_1 \) and \( \phi_2 \). The tensor product on \( \Gamma_{\mathcal{G}, \mathcal{X}, x} \) is given by \( (g_1, g_1', \phi_1) \otimes (g_2, g_2', \phi_2) = (g_1 \otimes g_2, g_1' \otimes g_2', \phi_1 \otimes \phi_2) \), where \( \phi_1 \otimes \phi_2 = (\phi_1 g_2') \circ (g_1 \phi_2) \). Theorem 3.7 guarantees that this product is coherently associative.

There are two natural projections \( \pi_1, \pi_2 : \Gamma_{\mathcal{G}, \mathcal{X}, x} \to \mathcal{G} \). Each of these is an equivalence, as for any \( g \in \mathcal{G} \), there is a unique (up to unique isomorphism) \( g' \in \mathcal{G} \) with \( g \triangleright x \cong x \triangleleft g' \), and the freedom in choosing this isomorphism is precisely the automorphism group of \( g \) itself.

**Definition 3.9.** The adjoin autoequivalence of the pair \( (\mathcal{X}, x) \) is the autoequivalence of \( \mathcal{G} \) given by \( \text{Ad}_{\mathcal{X}, x} = \pi_1 \circ \pi_2^{-1} \).

**Proposition 3.10.** Let \( \mathcal{X} \) be a \( \mathcal{G} \)-bitorsor, \( x \in \mathcal{X} \). Then \( \mathcal{X} \cong \mathcal{G}_{\text{Ad}_{\mathcal{X}, x}} \).

**Proof.** We define a functor \( \mathcal{G}_{\text{Ad}_{\mathcal{X}, x}} \to \mathcal{X} \) by \( g \mapsto g \triangleright x \). This is a map of left \( \mathcal{G} \)-modules by definition. The compatibility with the right action is given by the isomorphism

\[
(g \otimes \text{Ad}_{\mathcal{X}, x}(g')) \triangleright x \overset{\cong}{\to} g \triangleright (\text{Ad}_{\mathcal{X}, x}(g') \triangleright x) \overset{\cong}{\to} g \triangleright (x \triangleleft g') \overset{\cong}{\to} (g \triangleright x) \triangleleft g'.
\]

This map of bitorsors is necessarily an equivalence by Corollary 2.7. □

**Proposition 3.11.** Let \( \Phi, \Psi \in \text{Aut}(\mathcal{G}) \). Then \( \mathcal{G}_\Phi \times \mathcal{G} \mathcal{G}_\Psi \cong \mathcal{G}_{\Phi \circ \Psi} \).

**Proof.** We define a map \( \mathcal{G} \times \mathcal{G} \to \mathcal{G} \) by \( (g, g') \mapsto g \otimes \Phi(g') \). This map extends to a \( \mathcal{G} \)-bimodule map \( \mathcal{G}_\Phi \times \mathcal{G} \mathcal{G}_\Psi \to \mathcal{G}_{\Phi \circ \Psi} \) and descends to a \( \mathcal{G} \)-bitorsor map \( \mathcal{G}_\Phi \times \mathcal{G} \mathcal{G}_\Psi \to \mathcal{G}_{\Phi \circ \Psi} \). This is an equivalence by Corollary 2.7. □
Definition 3.12. Let $\mathcal{X}$ be a $(\mathcal{G}, \mathcal{H})$-bimodule. The opposite bimodule $\mathcal{X}^{-1}$ is the $(\mathcal{H}, \mathcal{G})$-bimodule with the same objects as $\mathcal{X}$ but where the action factors through the isomorphism $(-)^{-1}: \mathcal{H} \times \mathcal{G}^{rev} \cong \mathcal{G} \times \mathcal{H}^{rev}$.

Proposition 3.13. If $\Phi \in \text{Aut}(\mathcal{G})$, then $(\mathcal{G}_\Phi)^{-1} \cong \mathcal{G}_{-\Phi}^{-1}$.

Proof. The map $\Phi: \mathcal{G} \to \mathcal{G}$ extends to an equivalence $\mathcal{G}_\Phi^{-1} \cong (\mathcal{G}_\Phi)^{-1}$ of $\mathcal{G}$-bitorsors. □

Proposition 3.14. Let $\mathcal{G}$ be a 2-group, and let $\mathcal{X}$ be a $\mathcal{G}$-bitorsor. Then $\mathcal{X} \times \mathcal{G} \mathcal{X}^{-1} \cong \mathcal{G} \cong \mathcal{X}^{-1} \times \mathcal{G} \mathcal{X}$.

Proof. This result follows immediately from Corollary 3.10, Proposition 3.11, and Proposition 3.13. □

Proposition 3.15. Let $\mathcal{G}$ be a 2-group, and let $\mathcal{X}$ be a $\mathcal{G}$-bitorsor. Then $\text{Ad}: \mathcal{X} \to \text{Aut}(\mathcal{G})$, $x \mapsto \text{Ad}_{\mathcal{X},x}$ is a morphism of $\mathcal{G}$-bimodules, where $\mathcal{G}$ acts on $\text{Aut}(\mathcal{G})$ via left and right multiplication by inner autoequivalences.

Proof. We must construct natural isomorphisms $\text{Ad}_{\mathcal{X},g \triangleright x}(h) \cong g^{-1} \text{Ad}_{\mathcal{X},x}(h)g$ and $\text{Ad}_{\mathcal{X},x \triangleright g}(h) \cong \text{Ad}_{\mathcal{X},x}(g^{-1}hg)$.

For the former, we construct an equivalence $\Gamma_{\mathcal{G},\mathcal{X},g \triangleright x} \to \Gamma_{\mathcal{G},\mathcal{X},x}$ by $(h, h', \phi) \mapsto (g^{-1}hg, h', g^{-1} \phi)$. This equivalence fits into the following commutative diagram, which gives the desired isomorphism of functors.

The latter isomorphism is constructed similarly. We construct an equivalence $\Gamma_{\mathcal{G},\mathcal{X},x \triangleright g} \to \Gamma_{\mathcal{G},\mathcal{X},x}$ by $(h, h', \phi) \mapsto (h, gh^{-1}g^{-1}, \phi g^{-1})$. This equivalence fits into the following commutative diagram, which gives the desired isomorphism of functors.

□

4. Bitorsors and Braided Bitorsors for Braided 2-Groups

For general groups $G$, the adjoint automorphism associated with a $G$-bitorsor $X$ is not unique; changing the basepoint $x \in X$ will modify the automorphism by an inner automorphism. If $G$ is abelian, however, there are no nontrivial inner autoequivalences, and the adjoint automorphism is unique. A similar phenomenon holds
in the 2-group setting; in particular, braided 2-groups are equipped with a canonical trivialization of any inner automorphism, and so the adjoint autoequivalence is unique.

**Proposition 4.1.** Let $\mathcal{B}$ be a braided 2-group, $\mathcal{X}$ a $\mathcal{B}$-bitorsor, and $x \in \mathcal{X}$. Then the autoequivalence $\text{Ad}_{\mathcal{X},x}$ is independent of the choice of $x$ up to unique isomorphism.

**Proof.** The braiding on $\mathcal{B}$ provides a trivialization of the right action of $\mathcal{B}$ on $\text{Aut}^\otimes(\mathcal{B})$ given in Proposition 3.15. Thus, the image of $x \in \mathcal{X}$ under $\text{Ad}$ is independent of $x$ up to unique isomorphism. □

We will abuse notation and denote by $\text{Ad}_{\mathcal{X}}$ the autoequivalence $\text{Ad}_{\mathcal{X},x}$. We denote by $\mathcal{B}^T \mathcal{B}$ the 3-group of $\mathcal{B}$-bitorsors, and by $\text{Aut}^\otimes(\mathcal{B})$ the 2-group of (not necessarily braided) autoequivalences of $\mathcal{B}$.

**Corollary 4.2.** The assignment $\mathcal{X} \mapsto \text{Ad}_{\mathcal{X}}$ defines a monoidal pseudofunctor $\text{Ad} : \mathcal{B}^T \mathcal{B} \rightarrow \text{Aut}^\otimes(\mathcal{B})$.

Let $A = \pi_0(\mathcal{B})$. Then there is a monoidal pseudofunctor $\pi_0 : \mathcal{B}^T \mathcal{B} \rightarrow A^T A$.

**Theorem 4.3.** The maps $\text{Ad} : \mathcal{B}^T \mathcal{B} \rightarrow \text{Aut}^\otimes(\mathcal{B})$ and $\pi_0 : \mathcal{B}^T \mathcal{B} \rightarrow A^T A$ identify $\text{Aut}^\otimes(\mathcal{B}) \times_{\text{Aut} A} A^T A$ with the monoidal 1-truncation $\mathcal{B}^T \mathcal{B}^{\leq 1}$ of $\mathcal{B}^T \mathcal{B}$.

**Proof.** It suffices to note that specifying an autoequivalence of $\mathcal{B}$ is determined, up to isomorphism, by a choice of $x \in \mathcal{B}$ (the image of the unit object $e$), or equivalently, an automorphism of the underlying $A$-bitorsor, and an automorphism of $\Phi$. □

**Definition 4.4.** A $\mathcal{B}$-bitorsor $\mathcal{X}$ is **braided** if the autoequivalence $\text{Ad}_{\mathcal{X}}$ is braided. We denote by $\mathcal{B}^T \mathcal{B}$ the full sub-3-group of the 3-group $\mathcal{B}^T \mathcal{B}$ consisting of braided bitorsors, and by $\text{Aut}^\otimes(\mathcal{B})$ the 2-group of braided autoequivalences of $\mathcal{B}$.

**Corollary 4.5.** The maps $\text{Ad} : \mathcal{B}^T \mathcal{B} \rightarrow \text{Aut}^\otimes(\mathcal{B})$ and $\pi_0 : \mathcal{B}^T \mathcal{B} \rightarrow A^T A$ identify $\text{Aut}^\otimes(\mathcal{B}) \times_{\text{Aut} A} A^T A$ with the monoidal 1-truncation $\mathcal{B}^T \mathcal{B}^{\leq 1}$ of $\mathcal{B}^T \mathcal{B}$.

**Proposition 4.6.** Let $\mathcal{B}$ be a braided 2-group with $\pi_1(\mathcal{B}) = H$. Then $\pi_2(\mathcal{B}^T \mathcal{B}) \cong H$, and the action of $\pi_0(\mathcal{B}^T \mathcal{B})$ on $\pi_2$ factors through the natural map $\pi_0(\mathcal{B}^T \mathcal{B}) \rightarrow \text{Aut}(H)$.

**Proof.** The first part follows from Proposition 2.8 and the existence of the braiding. The second part follows from Proposition 3.11. □

5. Extensions and bitorsors

Having developed the theory of braided bitorsors, we now turn to the problem of lifting group extensions to 2-group extensions. We start with a review of group extensions.

In [SGA72, Exposé VII], Grothendieck describes extensions of groups in terms of $G$-bitorsors. We summarize his results as follows.

**Theorem 5.1** (Grothendieck). Let $G$ and $K$ be groups. Isomorphism classes of extensions of $G$ by $K$ are in bijection with isomorphism classes of monoidal functors from the discrete monoidal category $G$ to the 2-group $K^T K$ of $K$-bitorsors. If $K = A$ is abelian, the action of $G$ on $A$ in such an extension is given by the composite map $G \rightarrow A^T A \xrightarrow{\text{Ad}} \text{Aut}(A)$. 

Grothendieck’s correspondence is given by assigning to an extension $1 \to K \to E \xrightarrow{\partial} G \to 1$ the functor $g \mapsto E_g = \partial^{-1}(g)$.

**Definition 5.2.** Let $G$ be a group, and let $K$ be a 2-group. An extension of $G$ by $K$ is a 2-group $E$ equipped with a surjective monoidal functor $\partial : E \to G$ and an identification of $K$ with $\partial^{-1}(e)$ as 2-groups.

We have the following straightforward analogue of Grothendieck’s theorem in this setting.

**Theorem 5.3.** Let $G$ be a group, and let $K$ be a 2-group. Equivalence classes of extensions of $G$ by $K$ are in bijection with equivalence classes of monoidal pseudofunctors from the discrete monoidal category $G$ to the 3-group $\kappa T_K$.

The analogue of an abelian group in our setting is a braided 2-group. The notion of an extension of a group by a braided 2-group was defined in [DGNO10] as follows.

**Definition 5.4.** Let $G$ be a group, and let $B$ be a braided 2-group. Suppose we are given an action $a : G \to \text{Aut}^\otimes(B)$ of $G$ as braided autoequivalences of $B$. An extension of $G$ by $B$ with action $a$ is an underlying 2-group extension $E$ equipped with an isomorphism between $a \circ \partial : E \to \text{Aut}^\otimes(B)$ and the adjoint action of $E$ on $B \cong \partial^{-1}(e)$ which restricts to the trivialization of the adjoint action of $B$ on itself given by the braiding.

We recall the following equivalent characterization of such extensions.

**Proposition 5.5 ([DGNO10 Proposition E.10]).** Let $1 \to B \to \mathcal{E} \to G \to 1$ be an ordinary extension of 2-groups with $B$ braided. This extension can be given the structure of a braided extension if and only if each $\text{Ad}_x : B \to B$ is braided for $x \in \mathcal{E}$, and such a structure is unique up to unique isomorphism.

This proposition implies that the fibers of a braided extension are precisely braided $B$-bitorsors, so we can characterize braided extensions as follows.

**Corollary 5.6.** Equivalence classes of braided extensions of $G$ by $B$ are in bijection with equivalence classes of monoidal pseudofunctors $G \to \mathcal{B}_B$.

This problem of classifying braided extensions can be naturally broken up into smaller problems. First, we can limit ourselves to classifying extensions with a fixed action $a : G \to \text{Aut}^\otimes(B)$. Secondly, since every extension of 2-groups has an underlying group extension, we can study those braided extensions living above a fixed group extension. Of course, these two restrictions must be compatible: both actions of $G$ on $B$ and extensions of $G$ by $A = \pi_0(B)$ have an underlying automorphism of $A$, and these must agree. Thus, specifying these two restrictions is the same as specifying a monoidal functor from $G$ to $\text{Aut}^\otimes(B) \times_{\text{Aut} A} A T_A$. By Corollary 4.5 this is the same as giving a monoidal functor from $G$ to $\mathcal{B}_{\mathcal{T}_B \leq 1}$. Lifting the data of a group extension and an action of $G$ on $B$ to an extension of $G$ by $B$ is thus equivalent to lifting a monoidal functor $G \to \mathcal{B}_{\mathcal{T}_B \leq 1}$ to a monoidal pseudofunctor $G \to \mathcal{B}_B$. We will describe a cohomological obstruction to such a lifting in the next section.
6. Obstructions to lifting extensions

Grothendieck’s homotopy hypothesis, first formulated in the manuscript [Gro83], argues that \( n \)-groupoids should be “the same as” homotopy \( n \)-types. One way to make this statement a bit more precise is to say that, for any reasonable notion of \( n \)-groupoid, there should be a simplicial nerve functor that takes \( n \)-groupoids to Kan complexes whose homotopy groups vanish above level \( n \) (which we will henceforth refer to simply as “\( n \)-types”), and this nerve functor should be quasi-inverse to the fundamental \( n \)-groupoid functor.

In our current setting, we can make this precise as follows. We may “deloop” all of our groups, 2-groups, and 3-groups and functors between them to obtain one-object groupoids, 2-groupoids, and 3-groupoids. We will again denote by \( C[1] \) the delooping of a group (or 2-group or 3-group) \( C \). It is shown in [Ber99] that there is a simplicial nerve functor for 3-groupoids. We will henceforth abuse notation and consider the deloopings \( C[1] \) themselves as \( n \)-groupoids and Kan complexes interchangeably. We note that \( \pi_i(C[1]) = \pi_{i-1}(C) \).

The nerve of a braided 2-group \( B \) is a 2-fold loop space, so we can talk not only about the delooping \( B[1] \) but also the double delooping \( B[2] \). Abelian groups, being infinite loop spaces, can be delooped arbitrarily many times; we will denote by \( A[n] \) the \( n \)th delooping of an abelian group \( A \) (i.e., the Eilenberg-Mac Lane space \( K(A,n) \)).

As a warm-up, we will give a cohomological interpretation of Grothendieck’s Theorem 5.1. Classifying extensions of \( G \) by \( A \) with specified action \( G[1] \to Aut(A)[1] \) is equivalent, by Grothendieck’s result, to classifying lifts as in the following diagram.

\[
\begin{array}{c}
A \mathcal{T}_A[1] \\
\downarrow \\
G[1] \\
\downarrow \\
\text{Aut}(A)[1]
\end{array}
\]

We note that \( A \mathcal{T}_A[1] \) is a very special space: it is a 2-type with \( \pi_1 = \text{Aut} A \) and \( \pi_2 = A \), and the action of \( \text{Aut} A \) on \( A \) is the usual one. Thus, it is the space \( \hat{K}(A,2) \) that classifies arbitrary (i.e., not necessarily simple) fibrations with fiber \( K(A,1) \). In particular, there is a universal fibration

\( K(A,1) \to P \to \hat{K}(A,2) \)

(which was constructed in [Rob72] and generalizes the path-space fibration over \( K(A,2) \)) from which every fibration with fiber \( K(A,1) \) is a pullback. Similarly, the spaces \( \hat{K}(A,n) \) admit universal fibrations with fiber \( K(A,n-1) \). In analogy with our notation of \( A[n] \) for \( K(A,n) \), we will write \( \hat{A}[n] \) for \( \hat{K}(A,n) \).

So roughly speaking, \( A[2] \) is the classifying space for \( A \)-bitorsors, and we can view an extension of \( G \) by \( A \) as a family of \( A \)-bitorsors on \( G[1] \), which comes via pullback from \( \hat{A}[2] \). Given an action \( G[1] \to Aut(A)[1] \), homotopy classes of lifts to \( G[1] \to \hat{A}[2] \) are precisely elements of \( H^2(G,A) \) with the given action, so we recover the cohomological classification of group extensions.

We now return to the setting of braided 2-groups. We again let \( B \) be a braided 2-group with \( \pi_0(B) = A \) and \( \pi_1(B) = H \). In analogy with the case of groups, we denote by \( \hat{B}[2] \) the (delooping of) the 3-group of braided \( B \)-bitorsors. By Corollary \ref{cor:braided-classification} the following is a homotopy pullback square.
The data of an extension of $G$ by $A$ and a braided action of $G$ on $B$ with compatible underlying actions of $G$ on $A$ is thus given by a homotopy class of maps $G[1] \to \hat{B}[2] \leq 2$. We wish to know when this lifts to a map $G[1] \to \hat{A}[2]$. Since $\hat{B}[2]$ is a 3-type, the truncation map $\hat{B}[2] \to \hat{B}[2] \leq 2$ is a fibration with fiber $\pi_3(\hat{B}[2])[3] = H[3]$ by Proposition 4.6. This fibration is a pullback from the universal fibration over $\hat{H}[4]$. Thus, the obstruction to lifting an extension lies in $H^4(G, H)$, where the action of $G$ on $H$ is the specified one by Proposition 4.6.

We will call the map $H^2(G, A) \to H^4(G, H)$ that assigns to a class $[\omega] \in H^2(G, A)$ the corresponding obstruction in $H^4(G, H)$ the Pontryagin square, and we will denote it by $\text{Pontr}_{G,a}(\omega)$, where $a : G \to \text{Aut}(\hat{\otimes}(B))$ is the action of $G$ on $B$. We have thus proved the following classification result.

**Theorem 6.1.** Let $G$ be a group, and let $B$ be a braided 2-group. Suppose we are given an action $a : G \to \text{Aut}(\hat{\otimes}(B))$ and an extension $E$ of $G$ by $\pi_0(B) = A$ such that the underlying actions of $G$ on $A$ coming from $a$ and $E$ agree. This data lifts to an extension of $G$ by $B$ if and only if $\text{Pontr}_{G,a}(\omega) = 0$, and if this obstruction vanishes, such extensions form a torsor for $H^3(G, H)$.

When the action of $G$ on $B$ is trivial (corresponding to the case of central extensions), we have the following diagram, where all squares are (homotopy) pullbacks.

$$
\begin{array}{c}
\hat{B}[2] \leq 2 \ar[r] & \hat{A}[2] \\
\downarrow & \\
\text{Aut}(\hat{\otimes}(B))[1] \ar[r] & \text{Aut}(A)[1] \\
\end{array}
$$

From this diagram it follows that the fibration of $H[3]$ over $A[2]$ defining $B[2]$ pulls back from $H[4]$ via the map we have defined. Thus, our definition of Pontryagin square reduces to the classical Pontryagin square $A[2] \to H[4]$ corresponding to the quadratic map $A \to H$ coming from the braided 2-group $B$ (i.e., the Whitehead half-square map $\pi_2(B[2]) \to \pi_3(B[2])$).

**7. Cochain-level description of the Pontryagin square**

In the previous section, we defined the Pontryagin square as a “parametrized cohomology operation” $\text{Aut}(\hat{\otimes}(B))[1] \times_{\text{Aut}(A)[1]} \hat{A}[2] \to \hat{H}[4]$. In order to give an
explicit description of this operation in terms of group cochains, we need first to understand the structure of $\text{Aut}^\otimes(B)$.

We recall (see [JS86] or [JS93, Section 3] for definitions and details) that the data of a braided 2-group with $\pi_0 = A$ and $\pi_1 = H$ may be presented skeletally as an abelian 3-cocycle $(A, H, h, c)$; two such abelian 3-cocycles describe the same braided 2-group if and only if their difference is an abelian 3-coboundary.

**Proposition 7.1.** Let $B$ be the braided 2-group corresponding to the abelian 3-cocycle $(A, H, h, c)$. Then the 2-group $\text{Aut}^\otimes(B)$ can be described as follows.

- An object consists of a triple $(\phi, \psi, k)$, where $\phi \in \text{Aut}(A)$, $\psi \in \text{Aut}(H)$, and $k : A \times A \to H$ is a normalized 2-cochain such that
  \begin{align*}
  dk &= \psi \circ h - h \circ \phi^3, \\
  \psi \circ c + k &= k \circ \tau + c \circ \phi^2,
  \end{align*}

  where $\tau : A \times A \to A \times A$ switches the two factors.

- A morphism $(\phi, \psi, k) \to (\phi', \psi', k')$ consists of a normalized 1-cochain $\eta : A \to H$ such that $d\eta = k - k'$. Composition of morphisms is given by addition of 1-cochains.

- The tensor product on objects is given by
  $$(\phi, \psi, k) \circ (\phi', \psi', k') = (\phi \circ \phi', \psi \circ \psi', k \circ (\phi')^2 + \psi \circ k').$$

This product is strictly associative.

**Proof.** See [JS86, Proposition 14]. □

We can now describe actions of $G$ as braided autoequivalences of $B$ that lift given actions of $G$ as automorphisms of $A$ and $H$.

**Corollary 7.2.** Fix actions of $G$ on $A$ and $H$. An action of $G$ as braided autoequivalences of $B$ that induces these two actions is given by the following data.

- For each $g \in G$, we have a normalized 2-cochain $k_g : A \times A \to H$ satisfying Equations (7.1.1) and (7.1.2).
- For each pair $g_1, g_2 \in G$, we have a normalized 1-cochain $\theta_{g_1, g_2} : A \to H$ such that
  \begin{align*}
  d\theta_{g_1, g_2} &= k_{g_1, g_2} - k_{g_1} \circ (g_2 \triangleright -) - g_1 \triangleright k_{g_2}, \\
  \theta_{g_1, g_2} \circ (g_3 \triangleright -) &= \theta_{g_1, g_3} + g_1 \triangleright \theta_{g_2, g_3}.
  \end{align*}

An isomorphism $(k, \theta) \overset{\sim}{\to} (k', \theta')$ of such actions is given by the data of, for each $g \in G$, a normalized 1-cochain $\eta_g : A \to H$ such that $d\eta_g = k'_g - k_g$, and such that for each pair $g_1, g_2 \in G$,

$$\theta'_{g_1, g_2} - \theta_{g_1, g_2} - \eta_{g_1, g_2} = \eta_{g_1, g_2} - \eta_{g_1} \circ (g_2 \triangleright -) - g_1 \triangleright \eta_{g_2}.$$

Since $\text{Aut}^\otimes(B)[1] \times_{\text{Aut}(A)[1]} \tilde{A}[2] \overset{\sim}{\to}$ is the 2-truncation of $\tilde{B}[2]$, maps from $G[1]$ to this 2-truncation can be presented as quintuples $(\phi, \psi, k, \theta, \omega)$, where $\phi$ and $\psi$ are the actions of $G$ on $A$ and $H$, $k$ and $\theta$ are the data described in Corollary 7.2, and $\omega : G \times G \to A$ is a 2-cocycle for $G$ with coefficients in $A$.

We wish to write down an obstruction cocycle in $Z^3(G, H)$ representing the Pontryagin square in terms of the data $(A, H, h, c)$ of the braided 2-group $B$ and $(\phi, \psi, k, \theta, \omega)$ of a map $G[1] \to \tilde{B}[2] \overset{\sim}{\to}$.
Proposition 7.3. Let $\mathcal{B}$ be a braided 2-group with data $(A, H, h, c)$, $a : G \to \text{Aut}^\widehat{\otimes}(\mathcal{B})$ an action with data $(\phi, \psi, k, \theta)$, and $[\omega] \in H^2(G, A)$. The Pontryagin square $\text{Pontr}_{G,a}([\omega])$ is represented by the 4-cocycle
\[
\pi(g_1, g_2, g_3, g_4) = \phi(g_1, g_2), g_1 g_2 \triangleright \psi(g_3, g_4)
\]
\[
+ h(g_1 g_2 \triangleright \psi(g_3, g_4), \psi(g_1, g_2), \psi(g_1 g_2, g_3 g_4))
\]
\[
- h(g_1 g_2 \triangleright \psi(g_3, g_4), g_1 \triangleright \psi(g_2, g_3 g_4), \psi(g_1, g_2 g_3 g_4))
\]
\[
+ h(g_1 \triangleright \psi(g_2, g_3), g_1 \triangleright \psi(g_2 g_3, g_4), \psi(g_1, g_2 g_3 g_4))
\]
\[
- h(g_1 \triangleright \psi(g_2, g_3), \psi(g_1, g_2), \psi(g_1 g_2 g_3, g_4))
\]
\[
+ h(\psi(g_1, g_2), \psi(g_1 g_2, g_3), \psi(g_1 g_2 g_3, g_4))
\]
\[
- h(\psi(g_1, g_2), g_1 g_2 \triangleright \psi(g_3, g_4), \psi(g_1, g_2 g_3 g_4))
\]
\[
- h(\psi(g_1, g_2), \psi(g_1, g_2), \psi(g_1 g_2 g_3, g_4))
\]
\[
+ h(g_1, g_2) \triangleright \psi(g_3, g_4), \psi(g_2, g_3 g_4))
\]
\[
+ h(g_1, g_2) \triangleright \psi(g_3, g_4), \psi(g_2, g_3 g_4)
\]
\[
+ h(g_1, g_2) \triangleright \psi(g_3, g_4)
\]
\[
+ h(g_1, g_2) \triangleright \psi(g_3, g_4)
\]

Proof. The cocycle $\omega : G \times G \to A$ describes an “extension with section”: an extension $0 \to A \to E \xrightarrow{\phi} G \to 1$ equipped with a section $s : G \to E$. The multiplication on $E$ is described by $\omega$: we have
\[
(a_1 + s(g_1)) \cdot (a_2 + s(g_2)) = a_1 + g_1 \triangleright a_2 + \omega(g_1, g_2) + s(g_1 g_2).
\]
The cocycle condition ensures that this multiplication is associative.

The cocycle $\omega$ lifts uniquely to $\Omega : G \times G \to \mathcal{B}$. We would like to understand when this cocycle describes an “extension with section” of $G$ by $\mathcal{B}$, but this requires additional data: a collection of isomorphisms
\[
\Upsilon_{g_1,g_2,g_3} : \Omega(g_1, g_2) \otimes \Omega(g_1 g_2, g_3) \Rightarrow g_1 \triangleright \Omega(g_1, g_3) \otimes \Omega(g_1),
\]
which corresponds to the associator. Such isomorphisms always exist because $\omega$ is a cocycle.

In order to be a true extension, this associator must satisfy the pentagon identity, which translates to the commutativity of the following diagram.

Translating this diagram into the skeletal setting yields the desired formula for $\pi$ (plus a coboundary corresponding to $\Upsilon$).

Remark 7.4. It is not a priori clear, without our earlier homotopy theoretic discussion, that the cochain $\pi$ is even a cocycle, much less that is well-defined (up to changing it by a coboundary) independently of the choice of rigidifying skeletal data. Indeed, direct computations of these facts at the level of cochains are extremely lengthy.
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Department of Mathematics, University of Chicago, Chicago, IL 60637

E-mail address: ejenkins@math.uchicago.edu