Equivariant Sheaves

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In this article we review some recent developments in heterotic compactifications. In particular we review an “inherently toric” description of certain sheaves, called equivariant sheaves, that has recently been discussed in the physics literature. We outline calculations that can be performed with these objects, and also outline more general phenomena in moduli spaces of sheaves.

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\[1\text{Invited paper to appear in the special issue of the journal } \textit{Chaos, Solitons, and Fractals} \text{ on “Superstrings, M, F, S, ... Theory” (M.S. El Naschie and C. Castro, editors).}\]
1 Introduction

Twelve years ago, during the first superstring revolution, there was virtually no technology at all to describe compactifications. Since then, matters have improved greatly.

For some string theories, namely the type II theories and M and F theory, a compactification requires essentially specifying only a Calabi-Yau. These compactifications are now reasonably well understood. Not only do we have technology for analyzing many Calabi-Yaus, but we also have a basic understanding of quantum effects, both in $\alpha'$ (such as mirror symmetry) and in the string coupling constant (such as enhanced gauge symmetries arising from singular Calabi-Yaus).

There are additional string theories (the heterotic and type I theories) whose compactifications are unfortunately understood much more poorly. The complication in these cases is that to compactify one must specify not only a Calabi-Yau, but also at least one bundle (or, more generally, a torsion-free sheaf) over the Calabi-Yau. Although physicists now have a lot of technology to analyze Calabi-Yaus, there are relatively few ways to get any handle on bundles on Calabi-Yaus.

Until recently, there were only two known ways to describe sheaves on Calabi-Yaus. First, about a year ago a description of bundles on elliptic Calabi-Yaus was published \[28\]. Their description is quite beautiful, but unfortunately only describes bundles, not more general sheaves, and only on elliptic Calabi-Yaus, not more general Calabi-Yaus. The other description of sheaves on Calabi-Yaus was first published several years prior \[29\]. The Distler-Kachru models described therein give one excellent control over the physics of heterotic compactifications, but are extremely cumbersome to work with mathematically.

In this article we shall describe a third approach to the problem of describing sheaves on Calabi-Yaus. Specifically, we shall review a very convenient set of sheaves on toric varieties – “equivariant” sheaves – which are mathematically quite easy to work with. Most of the Calabi-Yaus studied by physicists are realized as hypersurfaces (or complete intersections) in toric varieties, so sheaves on Calabi-Yaus can be constructed by restriction of a sheaf on a toric variety to the Calabi-Yau hypersurface (or complete intersection). (Not all sheaves on Calabi-Yaus can be constructed this way, however we can obtain a large subfamily.) Unfortunately we will have nothing to say about worldsheet instanton corrections – our discussion will be purely classical in nature. Equivariant sheaves were recently discussed by the authors in \[1\], in work which built upon the prior work largely by A. A. Klyachko

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2 We are ignoring, for example, flat M theory 3-form potentials which can be turned on.

3 In a manner that gives one control over moduli. Toroidal orbifolds and WZW models, for example, can also be used to construct heterotic compactifications, but with no handle on continuous moduli. Studying an entire moduli space by studying a few points is much like studying an ocean by studying a few water molecules – one will learn much about water, but nothing about waves, fish, or most of the other things that make an ocean interesting.
Why is it important to understand heterotic compactifications? For physicists, there are several good reasons. First, even after the advent of string duality, heterotic compactifications remain the most technically efficient ways to get phenomenologically viable results. Secondly, via string duality a good understanding of quantum effects in heterotic theories would surely yield insight into compactifications of other string theories.

For mathematicians, heterotic compactifications are interesting because of the potential existence of a generalization of mirror symmetry known as (0,2) mirror symmetry, which we shall discuss at greater length in section 7.

In section 2 we shall begin with a brief overview of some relevant characteristics of toric varieties. In section 3 we shall describe equivariant bundles on toric varieties. In section 4 we shall describe more general equivariant torsion-free sheaves on toric varieties, and outline the origins of the description presented herein. In section 5 we shall discuss the construction of moduli spaces of equivariant sheaves, and some of their prominent characteristics. In section 6 we shall comment on more general moduli spaces of sheaves. Finally in section 7 we shall comment on (0,2) mirror symmetry.

2 Toric varieties

Why are physicists interested in toric varieties? Essentially because most of the Calabi-Yaus presently studied are realized as hypersurfaces (or complete intersections) in toric varieties. Toric varieties are reasonably well-understood, in the sense that most computations one would like to perform are relatively straightforward.

What is a “toric variety”? A toric variety is, for the purposes of this paper, a variety which is an at least partial compactification of an “algebraic torus” – a product of $\mathbb{C}^\times = \mathbb{C} - \{0\}$’s – such that the algebraic torus action extends continuously over the entire variety. (Each $\mathbb{C}^\times$ contains an $\mathbb{S}^1$, so an algebraic torus can be thought of as a sort of complexification of an ordinary torus, thus the name.) For example,

$$\mathbb{P}^1 = \mathbb{C}^\times \cup \{0\} \cup \{\infty\}$$

Another example is

$$\mathbb{P}^2 = (\mathbb{C}^\times)^2 \cup \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$$

4 Or, for that matter, type I compactifications. Type I compactifications have additional technical complications beyond those of heterotic compactifications, so for this article we shall only be concerned with heterotic compactifications.

5 For more information on toric varieties see [8, 9, 10, 11]. Not all compactifications of algebraic tori are toric varieties – toric varieties have additional nice properties – but the distinctions will not be relevant for the purposes of this article.
where $x$, $y$, and $z$ are homogeneous coordinates defining the toric variety.

Toric varieties often have a description in terms of homogeneous coordinates \([2]\). How can the algebraic torus be seen in such a description? The algebraic torus is simply all possible $\mathbb{C}^\times$ rescalings of the individual homogeneous coordinates, modulo the $\mathbb{C}^\times$'s one mods out to form the toric variety.

The codimension one subvarieties that compactify the algebraic torus are called “toric divisors.” In the $\mathbb{P}^2$ example, the sets $\{x = 0\}$, $\{y = 0\}$, and $\{z = 0\}$ are the toric divisors.

In general, if we know everything about how the toric divisors are attached, then we know almost everything about the toric variety. Loosely speaking, given knowledge of the toric divisors we can use the underlying algebraic torus $(\mathbb{C}^\times)^n$ to sweep out the rest of the toric variety.

### 3 Equivariant bundles

Given that all toric varieties are a compactification of an algebraic torus $(\mathbb{C}^\times)^n$, what can we say about bundles on toric varieties?

Let $t \in (\mathbb{C}^\times)^n$, so $t$ has a natural action on the toric variety $- t$ simply rotates the underlying algebraic torus. (On $\mathbb{P}^1$, for example, this would correspond to rotations about an axis plus dilations that leave two poles fixed.)

Now, given any bundle $\mathcal{E}$, we can form the bundle $t^* \mathcal{E}$—we drag $\mathcal{E}$ back along the action of $t$. In general, $\mathcal{E} \not\cong t^* \mathcal{E}$.

In the special case that $\mathcal{E} \cong t^* \mathcal{E}$ for all $t$, we say that $\mathcal{E}$ is an equivariant bundle. (Equivariant with respect to the underlying algebraic torus.)

It is equivariant bundles (and more generally, equivariant sheaves) for which there exists a nice description.

What are some examples of equivariant bundles on smooth compact toric varieties? First, line bundles are equivariant. It turns out that all smooth compact toric varieties are simply connected, so line bundles have no moduli. Since line bundles cannot be deformed at all, they certainly cannot be deformed by the algebraic torus—thus, line bundles are equivariant. Similarly, direct sums of line bundles are equivariant. The tangent and cotangent bundles of

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6In the mathematics literature, by an equivariant bundle one would typically mean not only that $\mathcal{E} \cong t^* \mathcal{E}$, but one would have in mind a fixed choice of isomorphisms—an “equivariant structure.” The sheaves we describe in this paper all are implicitly associated with a specific choice of equivariant structure, a fact we will return to later. In this article we shall be somewhat loose and often ignore the equivariant structure.
a toric variety are equivariant. Examples of such bundles are not uncommon, and typically come in continuous families – they can certainly have moduli.

Now, suppose $E$ is an equivariant bundle. It turns out that to reconstruct $E$ it suffices to know its behavior in neighborhoods of the toric divisors. Given knowledge of $E$ near the toric divisors, we can then (loosely speaking) use the underlying algebraic torus to rotate that information around and recreate $E$ on the rest of the toric variety. So, precisely what information must we associate to each toric divisor to specify an equivariant bundle?

It turns out that an equivariant bundle $E$ can be specified by associating a “filtration” of a vector space to each toric divisor. Recall that a filtration of a vector space $E$ is simply

$$E \supseteq \cdots \supseteq E^n(i) \supseteq E^n(i + 1) \supseteq E^n(i + 2) \supseteq \cdots \supseteq 0$$

The vector space we filter is precisely the fiber of the vector bundle.

A random set of filtrations does not necessarily define a bundle – they must satisfy a compatibility condition. On a smooth toric variety this compatibility condition says that in any cone of the fan defining the toric variety, all the elements of the filtrations associated to toric divisors in the cone must be coordinate subspaces of the vector space $E$, with respect to some basis of the vector space. This compatibility condition is trivial for two dimensional varieties.

Two sets of compatible filtrations define the same bundle precisely when they differ by an automorphism of the vector space $E$.

Before we can describe some examples, we must first clear up some loose ends. The filtration description of equivariant sheaves given above hinges on a choice of “equivariant structure” of the bundle. What is an equivariant structure? We have mentioned that a bundle $E$ is equivariant precisely when it is isomorphic to $t^*E$ for all $t \in (\mathbb{C}^*)^n$; an equivariant structure is simply a precise choice of isomorphism for each $t$.

The choice of equivariant structure is not unique, and different choices yield distinct filtrations, but for all that the choice is relatively harmless – it adds no continuous moduli, and is well understood.

Let us consider an example – line bundles on $\mathbb{P}^2$. Let $x$, $y$, and $z$ denote homogeneous coordinates defining the toric variety, let $D_x$ denote $\{x = 0\}$, and so forth. For readers not acquainted with the notation, $\mathcal{O}(a)$ denotes a line bundle of $c_1 = a$.

In this context, consider the line bundles $\mathcal{O}(D_x)$, $\mathcal{O}(D_y)$, $\mathcal{O}(3D_z - 2D_y)$, and $\mathcal{O}(6D_x + 7D_y - 12D_z)$. These line bundles are all isomorphic as line bundles to $\mathcal{O}(1)$, however they...
all have distinct equivariant structures\[. More generally, the equivariant structure of a line bundle is given by a specific choice of torus-invariant divisor.

How can we describe a line bundle with filtrations? Consider the example $\mathcal{O}(nD_x)$ on $\mathbb{P}^2$. This line bundle is specified by the filtrations

$$E^x(i) = \begin{cases} \mathbb{C} & i \leq n \\ 0 & i > n \end{cases}$$

$$E^y(i) = E^z(i) = \begin{cases} \mathbb{C} & i \leq 0 \\ 0 & i > 0 \end{cases}$$

Since $\mathcal{O}(nD_x)$ is a line bundle, its fiber is $\mathbb{C}$, so the top vector space in each filtration is $\mathbb{C}$. The only complex vector subspace of $\mathbb{C}$ is 0, so each filtration necessarily looks like a string of $\mathbb{C}$’s followed by a string of 0’s. All information is contained in the precise value of $i$ at which the filtration changes dimension. Clearly the filtration description is overkill for line bundles, but for higher rank bundles it is quite useful.

It turns out that Chern classes and sheaf cohomology groups of equivariant bundles are quite straightforward to calculate. We shall not work through the details here, but shall merely outline the highlights. For example, if we define

$$E^{[\alpha]}(i) = \frac{E^\alpha(i)}{E^\alpha(i + 1)}$$

then it can be shown that for any bundle $\mathcal{E}$,

$$c_1(\mathcal{E}) = \sum_{\alpha, i} i \dim E^{[\alpha]}(i) D_\alpha$$

Sheaf cohomology groups of equivariant bundles have a natural decomposition, known as an “isotypic decomposition”, into subgroups each of which is associated with an element of the weight lattice of the algebraic torus:

$$H^p(\mathcal{E}) = \bigoplus_\chi H^p(\mathcal{E})_\chi$$

Sheaf cohomology of equivariant bundles on smooth toric varieties can be calculated as Čech cohomology on a natural Leray cover, a straightforward exercise.

\section{Equivariant sheaves}

The rather compact description of equivariant bundles given above can be generalized to equivariant sheaves. Before we do so, however, we shall review some basic definitions.

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\[ Note that if we worked with Chern classes in equivariant cohomology rather than in singular cohomology, we would be able to distinguish the Chern classes of line bundles with distinct equivariant structures.
A locally free sheaf is precisely the sheaf of sections of some vector bundle. Each stalk of the sheaf is a freely generated module, thus the nomenclature. In this article we shall fail to distinguish “bundle” from “locally free sheaf.”

A reflexive sheaf is a sheaf \( \mathcal{E} \) such that \( \mathcal{E} \cong \mathcal{E}^{\vee \vee} \), where \( \mathcal{E}^{\vee} \) is the dual sheaf: \( \mathcal{E}^{\vee} = \text{Hom}(\mathcal{E}, \mathcal{O}) \). For example, all bundles are reflexive sheaves. On a smooth variety, reflexive sheaves are locally free up to codimension three. Also, on a smooth variety all reflexive rank 1 sheaves are locally free.

A torsion-free sheaf is a sheaf such that each stalk is a torsion-free module. On a smooth variety, torsion-free sheaves are locally free up to codimension two. Physicists may think (rather loosely) of torsion-free sheaves as being bundles with possible small instanton singularities. In general, reflexive sheaves are special cases of torsion-free sheaves.

Any equivariant torsion-free sheaf looks like a trivial vector bundle over the open torus orbit.

It turns out that an equivariant reflexive sheaf can be specified by associating a filtration\(^9\) to each toric divisor. The difference between an equivariant reflexive sheaf and an equivariant bundle is that for an equivariant reflexive sheaf, the filtrations are not required to satisfy a compatibility condition.

How is this description derived, and how can we describe more general equivariant torsion-free sheaves? To explain these matters, we must make a very short digression into modern algebraic geometry.

Instead of working with topological spaces directly, algebraic geometers work with rings of functions on spaces. More precisely, given any (commutative) ring (with identity), say \( A \), there is a map \((\text{Spec})\) that associates an affine space to \( A \):

\[
\text{Spec} : \text{Rings} \rightarrow \text{Affine spaces}
\]

(One can then build up a compact space by working on coordinate patches.) For example,

\[
\text{Spec } C[x_1, \cdots, x_n] = C^n
\]

\[
\text{Spec } C[x_1, \cdots, x_n]/(p) = \text{the hypersurface } \{p = 0\} \subset C^n
\]

\[
\text{Spec } C = \text{a single point}
\]

Coherent sheaves over an affine space are described in terms of modules of sections of the sheaf – to each ring \( A \), we associate an \( A \)-module \( M \). Put another way, we can either

\(^9\)For example, the dual of a line bundle \( \mathcal{O}(D) \) is \( \mathcal{O}(-D) \). In the physics literature it is traditional to use * rather than \( \vee \) to denote duals; here we follow the notation of algebraic geometers.

\(^{10}\)A filtration of the fiber of the trivial vector bundle over the open torus orbit.
speak of pairs (ring $A$, $A$-module $M$) or of pairs (affine space $U$, sheaf on $U$) = (Spec $A$, $\tilde{M}$). These two descriptions are equivalent!

For example, consider $\mathbb{C}^2$ and sheaves on $\mathbb{C}^2$. The affine space $\mathbb{C}^2$ is associated with the polynomial ring $\mathbb{C}[x, y]$, i.e.,

$$\text{Spec } \mathbb{C}[x, y] = \mathbb{C}^2$$

Sheaves on $\mathbb{C}^2$ are then associated to $\mathbb{C}[x, y]$-modules. For example, the trivial rank $r$ vector bundle on $\mathbb{C}^2$ is associated to the $\mathbb{C}[x, y]$ module

$$\bigoplus_r \mathbb{C}[x, y]$$

Note this module is freely generated – that is why the corresponding sheaf is locally free.

The description of equivariant torsion-free sheaves can be derived by thinking along these lines. First, we shall set up the sheaf. To each maximal cone $\sigma$ of the fan defining the toric variety, associated a $\mathbb{C}[\sigma^\vee]$-module, call it $E^\sigma$. If $\tau$ is a subcone of $\sigma$, then $E^\tau$ is defined to be the restriction of the module $E^\sigma$ to the open subset $\text{Spec } \mathbb{C}[\tau^\vee] \hookrightarrow \text{Spec } \mathbb{C}[\sigma^\vee]$. For consistency, if $\tau$ is a subcone of $\sigma_1$, $\sigma_2$, then the restrictions of $E^{\sigma_1}$, $E^{\sigma_2}$ must agree. (This is how modules over overlapping open sets are glued together.)

So far all we have done is define a sheaf (or, rather, a presheaf) by associating modules of sections to open sets. To recover the description of equivariant reflexive sheaves given earlier, there are two steps. First, one shows that for a reflexive sheaf, the module $E^\sigma$ associated to any cone $\sigma$ is completely determined by the modules $E^\alpha$ associated to toric divisors $\alpha$ in $\sigma$. (Thus, to specify a reflexive sheaf, it suffices to know the modules associated to the one-dimensional edges of the fan.) Second, one shows that the modules associated to one-dimensional edges of the fan are all completely determined by filtrations. Thus, equivariant reflexive sheaves are specified by associating a filtration to each toric divisor.

This result – that equivariant reflexive sheaves are specified by associating a filtration to each toric divisor – is sufficiently important to warrant repetition. The point of interest is that codimension one behavior is enough to nail down reflexive sheaves; we need not go to higher codimension.

Now, we shall go over a few details behind these statements. The modules one sees in studying equivariant sheaves all have what is essentially an isotypic decomposition under the action of the algebraic torus. This means that we can specify a module by associating a vector space to each element of the weight lattice of the algebraic torus. The vector spaces are all subspaces of one fixed vector space – the fiber of the trivial vector bundle over the open torus orbit.

For example, consider $\mathbb{C}[x, y]$ as a $\mathbb{C}[x, y]$-module. Here we associate a vector subspace of $\mathbb{C}$ to each element of the weight lattice of $(\mathbb{C}^*)^2$, as follows:
In general, for any torsion-free $\mathbf{C}[\sigma^\vee]$-module, multiplying by an element of $\mathbf{C}[\sigma^\vee]$ induces inclusions. In the example above, if we let $E(i_1, i_2)$ denote the vector space associated with monomial $x^{i_1}y^{i_2}$, then we have inclusions

$$E(i_1, i_2) \hookrightarrow E(i_1 + 1, i_2)$$
$$\hookrightarrow E(i_1, i_2 + 1)$$

For a somewhat less trivial example, consider the ideal generated by $(x, xy)$ in $\mathbf{C}[x, xy, xy^2]$. As a $\mathbf{C}[x, xy, xy^2]$-module, it has an isotypic decomposition

$$\begin{array}{cccc}
2 & 0 & \mathbf{C} & \mathbf{C} \\
1 & 0 & \mathbf{C} & \mathbf{C} \\
0 & \cdots & 0 & \mathbf{C} & \mathbf{C} & \cdots \\
-1 & 0 & 0 & 0 & 0 \\
\end{array}$$

With notation as before, it is easy to check one has inclusions

$$E(i_1, i_2) \hookrightarrow E(i_1 + 1, i_2)$$
$$\hookrightarrow E(i_1 + 1, i_2 + 1)$$
$$\hookrightarrow E(i_1 + 1, i_2 + 2)$$

As a $\mathbf{C}[x, xy, xy^2]$-module, it has two generators (located at $x$, $xy$) and one relation.

What is the geometry behind the example above? Spec $\mathbf{C}[x, xy, xy^2] = \mathbf{C}^2/\mathbf{Z}_2$, and so it turns out this module defines a reflexive rank 1 sheaf on $\mathbf{C}^2/\mathbf{Z}_2$. In particular, as the module is not freely generated, this is an example of a reflexive rank 1 sheaf which is not a line bundle.
So far we have told you about a particularly convenient (isotypic) decomposition of the modules appearing in equivariant sheaves. It turns out that when the module $E^{\sigma}$ associated to cone $\sigma$ is reflexive, it can be specified in terms of modules associated to one-dimensional fan edges as

$$E^{\sigma}(\chi) = \bigcap_{\alpha \in |\sigma|} E^{\alpha}(\chi)$$

We shall not derive this relation here\(^\text{11}\), but see instead [1].

So far we have told you that modules defining reflexive sheaves are completely determined by modules associated to one-dimensional edges of the fan; we still need to demonstrate that modules associated to one-dimensional edges of the fan are completely determined by a filtration.

Why should a module associated to a neighborhood of a toric divisor be equivalent to a filtration? A toric neighborhood of any toric divisor is simply $\mathbb{C}^\times \times (\mathbb{C}^\times)^n$, which is associated to a ring, say $A$,

$$A = \mathbb{C}[x_1, x_2, x_2^{-1}, x_3, x_3^{-1}, \ldots, x_n, x_n^{-1}]$$

Consider the inclusions generated in any associated torsion-free module:

$$E(i_1, \ldots, i_n) \hookrightarrow E(i_1, i_2 + 1, \ldots, i_n)$$

$$E(i_1, i_2 - 1, \ldots, i_n)$$

Clearly, $E(i_1, \ldots, i_n)$ is independent of $i_2, \ldots, i_n$. The only nontrivial inclusion is simply

$$E(i_1, \ldots, i_n) \hookrightarrow E(i_1 + 1, i_2, \ldots, i_n)$$

Thus, this module is equivalent to a filtration.

Let us consider a simple example to help clarify matters. Consider the trivial rank 1 line bundle $\mathcal{O}$ (known more formally as the structure sheaf) on $\mathbb{P}^2$. Let $x, y, z$ be homogeneous coordinates defining $\mathbb{P}^2$. A fan defining $\mathbb{P}^2$ as a toric variety is shown in figure 1. The structure sheaf of a variety is defined by associating to each neighborhood $\text{Spec } A$, a module that is precisely the ring $A$.

First, the toric neighborhood of $D_x = \{x = 0\}$ is $\text{Spec } \mathbb{C}[x, y, y^{-1}]$, and the associated module describing the structure sheaf is $\mathbb{C}[x, y, y^{-1}]$:

\(^{11}\)We should mention, however, that it is formally similar to a standard result on reflexive modules over noetherian integrally closed domains [15, chapter 7.4], which says that if $M$ is a reflexive module over such a domain, then

$$M = \bigcap_p M_p$$

where the intersection is over all prime ideals of height 1.
Figure 1: A fan defining $\mathbb{P}^2$ as a toric variety

The toric neighborhood of $D_y = \{ y = 0 \}$ is $\text{Spec } \mathbb{C}[x, x^{-1}, y]$, and the associated module describing the structure sheaf is $\mathbb{C}[x, x^{-1}, y]$: 

\[
\begin{array}{c|ccc}
 & -1 & 0 & 1 \\
\hline
\vdots & & & \\
1 & 0 & C & C \\
0 & \cdots & 0 & C & C & \cdots \\
-1 & 0 & C & C \\
\vdots & & & \\
\end{array}
\]

The toric neighborhood of $D_z = \{ z = 0 \}$ is $\text{Spec } \mathbb{C}[x^{-1}y, xy^{-1}, x^{-1}y^{-1}]$, and the associated module describing the structure sheaf is $\mathbb{C}[x^{-1}y, xy^{-1}, x^{-1}y^{-1}]$: 

\[
\begin{array}{c|ccc}
 & -1 & 0 & 1 \\
\hline
\vdots & & & \\
1 & C & C & C \\
0 & \cdots & C & C & C & \cdots \\
-1 & 0 & 0 & 0 \\
\vdots & & & \\
\end{array}
\]
For a reflexive sheaf $E$ (of which the structure sheaf is a trivial example), if we denote the module associated to divisor $\alpha$ by $E^\alpha$, then the sections of $E$ are given by

$$H^0(E)_\chi = \bigcap_\alpha E^\alpha(\chi)$$

and in particular in the case at hand it is trivial to compute

$$H^0(P^2, \mathcal{O})_\chi = \begin{cases} C & \chi = 0 \\ 0 & \text{otherwise} \end{cases}$$

so $h^0(P^2, \mathcal{O}) = 1$, as is well known.

To summarize, we have argued that for a reflexive sheaf, the module associated to any cone of the fan is completely determined by modules associated to one-dimensional fan edges, and modules associated to one-dimensional fan edges are equivalent to filtrations. Thus, to specify an equivariant reflexive sheaf, we associate a filtration to each toric divisor.

When do two sets of filtrations define the same equivariant reflexive sheaf? When they differ by an automorphism of the topmost vector space. To construct a moduli space of equivariant reflexive sheaves, we would have to mod out a space of all filtrations by automorphisms of the topmost vector space. In the next section we shall study such constructions in detail.

5 Moduli spaces of equivariant sheaves

In this section we will outline how to construct moduli spaces of equivariant reflexive sheaves via GIT quotients.

First, what is a GIT quotient? GIT (Geometric Invariant Theory) quotients are closely related to symplectic quotients, and can be loosely thought of as a holomorphic way to compute a symplectic quotient [33, section 8]. Suppose we have a complex algebraic variety $T$ with an action of a reductive algebraic group $G$ and an ample line bundle $L$ on $T$. (The analogous data for a symplectic quotient would be a symplectic manifold $\mathcal{T}$ with the action of a compact Lie group $G$ and a specific choice of symplectic form $\omega$ on $\mathcal{T}$.) Points on $\mathcal{T}$
are classified as stable, semistable, or unstable, depending upon the action of $G$ and the behavior of $L$. (For example, a point is not stable if the dimension of its stabilizer in $G$ is greater than zero.)

Denote a GIT quotient by $\mathcal{T}//G$, then technically

$$\mathcal{T}//G = \text{Proj} \bigoplus_n H^0(\mathcal{T}, \mathcal{L}^n)^G$$

though for the purposes of this article it will suffice to say, loosely,

$$\mathcal{T}//G = (\mathcal{T} - \mathcal{T}^{us})/G$$

where $\mathcal{T}^{us}$ denotes the unstable points of $\mathcal{T}$.

For example, projective spaces can be realized as very elementary examples of GIT quotients:

$$\mathbb{P}^n = \mathbb{C}^{n+1}//\mathbb{C}^\times = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times$$

What is the relevant notion of stability for sheaves on a Kähler variety? The relevant notion is called “Mumford-Takemoto stability.” (This necessary but not sufficient condition for a consistent heterotic compactification arises, for example, as a $D$-term constraint in compactifications to $N = 1$ in $3 + 1$ dimensions.) For a torsion-free sheaf $\mathcal{E}$ of rank $r$, define the slope of $\mathcal{E}$ to be

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cup \omega^{n-1}}{r}$$

where $\omega$ is the Kähler form and $n$ is the dimension of the underlying variety. With this definition, we say $\mathcal{E}$ is Mumford-Takemoto (semi)stable if for all proper coherent subsheaves $\mathcal{F} \subset \mathcal{E}$ such that $0 < \text{rank } \mathcal{F} < \text{rank } \mathcal{E}$ and $\mathcal{E}/\mathcal{F}$ is torsion-free,

$$\mu(\mathcal{F}) (\leq) < \mu(\mathcal{E})$$

For equivariant reflexive sheaves, the notion of Mumford-Takemoto stability simplifies. First, in general for a reflexive sheaf on any variety it suffices to check only reflexive subsheaves to determine Mumford-Takemoto stability. Second, for any equivariant sheaf it suffices to check only equivariant subsheaves to determine Mumford-Takemoto stability. Thus, for equivariant reflexive sheaves, we need only test equivariant reflexive subsheaves.

Before we can finally construct moduli spaces, we need a little more information. Instead of specifying filtrations, we can specify parabolic subgroup of $G = GL(n, \mathbb{C})$, as

$$P^\alpha = \{ g \in GL(n, \mathbb{C}) \mid gE^\alpha(i) = E^\alpha(i) \forall i \}$$

\footnote{12A parabolic subgroup of $GL(n, \mathbb{C})$ is conjugate to a subgroup consisting of upper-block-triangular matrices.}
Specifying a parabolic subgroup $P^\alpha$ does not uniquely identify a filtration $\{E^\alpha(i)\}$ – it does not say at which values of $i$ the filtration changes dimension. That additional information is given by specifying an ample line bundle on $G/P^\alpha$. Denote this ample line bundle by $L_\alpha$.

In terms of parabolics, the constraint for a reflexive sheaf on a smooth toric variety to be a bundle is that for all cones $\sigma$,

$$\bigcap_{\alpha \in |\sigma|} P^\alpha \text{ contains a maximal torus of } G$$

(1)

Any pair of filtrations automatically satisfies this constraint, so on a smooth toric two-fold, all equivariant reflexive sheaves are bundles. (More generally, on a smooth two-dimensional variety, all reflexive sheaves are bundles – we have merely noted how this standard result can be rederived in the equivariant context.)

There is also a description of equivariant principal $G$-bundles on smooth varieties. We will not go into much detail, but the description simply associates a parabolic $P^\alpha \subset G$ and ample line bundle $L_\alpha$ on $G/P^\alpha$ to each toric divisor $\alpha$, satisfying constraint (1).

Now we are almost ready to form moduli spaces. What does the space $T$ of equivariant reflexive sheaves look like before performing a GIT quotient? Recall to each toric divisor we associated a filtration or equivalently a parabolic $P^\alpha$. The space of filtrations of the same form is simply $G/P^\alpha$. Thus, before quotienting, the space of reflexive sheaves is

$$\prod\alpha G/P^\alpha$$

Well, almost. For nongeneric flags the Chern classes can change, so truthfully

$$T \subset \prod\alpha G/P^\alpha$$

as we want the Chern classes constant on a component of a moduli space.

Finally we can define the relevant GIT quotient. Recall that two sets of filtrations define the same reflexive sheaf if they differ by an automorphism of the top vector space, meaning, if they differ by an element of $G = GL(n, \mathbb{C})$, therefore the moduli space we want is simply $T//G$, with $T$ constructed as above. To make sense out of this we must make a specific choice of ample line bundle on $T$. Let $\pi_\alpha : \prod_\beta G/P^\beta \to G/P^\alpha$ be the canonical projection, and let $n_\alpha = D_\alpha \cup \omega^{n-1}$ be an integer (for a dense subset of the Kähler cone, the $D_\alpha \cup \omega^{n-1}$ will all be proportional to an integer), then the ample line bundle on $T$ is simply

$$\otimes_\alpha \pi_\alpha^* L_\alpha^{n_\alpha}$$

In defining this GIT quotient we have implicitly defined some notion of stability of reflexive sheaves; how does that notion compare to Mumford-Takemoto stability, the notion
of stability relevant for physics? It turns out (see [1] for details) that the notion of stability implicit above precisely coincides with Mumford-Takemoto stability.

In general, GIT quotients of products of flag manifolds are a standard exercise in the mathematics literature, so in principle a great deal of information can be extracted from this description.

For example, we can make general remarks concerning singularities present in such moduli spaces. Singularities roughly fall into two classes.

First, there are singularities present for nongeneric Kähler forms. As the Kähler form is varied, sometimes semistable sheaves become unstable, or unstable sheaves become semistable. When this happens, the topology of the moduli space changes, and at the transition point there is a singularity. In extreme cases, such as rank two sheaves on surfaces, the Kähler cone splits into subcones, and one has a topologically distinct moduli space of sheaves associated to each subcone. Typically (but not always) these moduli spaces are birational to one another. For a review of this phenomenon and references in the mathematics literature, see [1]. In general, this sort of behavior of GIT quotients under change of ample line bundle is ubiquitous; see [16, 17] for recent expositions. We shall speak at greater length on this phenomenon in section 6.2.

Secondly, in moduli spaces of principal $G$-bundles (for $G$ other than $GL(n, C)$), there are orbifold singularities, present for generic Kähler forms.

6 More general moduli spaces

6.1 More uses of equivariant sheaves

So far we have only spoken about equivariant sheaves on toric varieties, though in principle information can be gained about more general sheaves on toric varieties.

In the mathematics literature, given an action of a group $G$ on some space, all one needs to know to essentially reconstruct the space is the fixed points of $G$ and its action on the normal bundle to the fixed points. In the present context, given knowledge of equivariant sheaves and the algebraic torus action on a normal bundle to equivariant sheaves, one can – in principle – reconstruct the rest of the moduli space.
6.2 Kähler cone substructure

As mentioned previously in section 5, a necessary but not sufficient condition for a consistent heterotic compactification is that the sheaf $E$ be Mumford-Takemoto semistable, as defined earlier. To review, this constraint is satisfied when for all reasonably well-behaved subsheaves $F \subset E$,

$$\frac{c_1(F) \cup \omega^{n-1}}{\text{rank } F} \leq \frac{c_1(E) \cup \omega^{n-1}}{\text{rank } E}$$

where $\omega$ is the Kähler form and $n$ is the dimension of the variety.

The relevant point concerning Mumford-Takemoto stability is that it explicitly depends upon the choice of Kähler form. In particular, as we move around in the Kähler cone, sheaves that are semistable with respect to some Kähler forms may become unstable with respect to others, and vice-versa.

This is an important fact which has so far been completely overlooked in the physics literature. At minimum, one can expect that at certain nongeneric points in the Kähler cone, a moduli space of sheaves will become singular. More extreme behavior is also possible.

In particular the case of rank 2 sheaves on complex surfaces has been thoroughly studied in the mathematics literature [18, 19, 20, 21, 22, 23, 24, 25], and is closely related to analogous phenomena occurring for continuous (rather than holomorphic) bundles [26, 27]. For the special case of rank 2 sheaves on complex surfaces, the Kähler cone actually splits into subcones (or “chambers”), with a topologically distinct moduli space associated to each chamber. (The precise decomposition depends upon the Chern classes of the sheaves appearing on the moduli space.) Typically (but not always) the moduli spaces associated to distinct chambers are birational to one another.

This fact was mentioned previously in section 3, in the context of equivariant sheaves, where this phenomenon can be seen explicitly. However this phenomenon occurs not only for equivariant sheaves but for general sheaves, and is sufficiently important to warrant repeating.

7 (0,2) mirror symmetry

There potentially exists a generalization of ordinary mirror symmetry, known as (0,2) mirror symmetry.

First, recall the definition of ordinary (so-called (2,2)) mirror symmetry. It says that there exist pairs of Calabi-Yaus, call them $X, Y$, both described by the same conformal field theory – a string cannot tell which of the two it is propagating on.
By contrast, (0,2) mirror symmetry exchanges\(^{13}\) pairs \((X,\mathcal{E}), (Y,\mathcal{F})\) where \(X, Y\) are Calabi-Yaus and \(\mathcal{E}, \mathcal{F}\) are torsion-free sheaves on \(X, Y\), respectively. (0,2) mirror symmetry then reduces to ordinary mirror symmetry in the special case that \(\mathcal{E} = TX\) and \(\mathcal{F} = TY\).

Ordinary mirror symmetry exchanges complex and Kähler moduli. By contrast, (0,2) mirror symmetry is believed to exchange complex, Kähler, and sheaf moduli as a unit: sheaf moduli may be mirror not only to other sheaf moduli but perhaps also to complex or Kähler moduli, for example.

In addition, (0,2) mirror symmetry presumably acts on charged matter as well as neutral moduli. Recall ordinary mirror symmetry exchanges Hodge numbers; for example, for threefolds,

\[
\begin{align*}
    h^{1,1}(X) &= h^{2,1}(Y) \\
    H^{2,1}(X) &= h^{1,1}(Y)
\end{align*}
\]

Analogously, at least in simple cases (0,2) mirror symmetry is believed to exchange global Ext groups

\[
\begin{align*}
    \text{Ext}^1_X(\mathcal{O}, \mathcal{E}) &\cong \text{Ext}^1_Y(\mathcal{F}, \mathcal{O}) \\
    \text{Ext}^1_X(\mathcal{E}, \mathcal{O}) &\cong \text{Ext}^1_Y(\mathcal{O}, \mathcal{F})
\end{align*}
\]

In particular, in heterotic compactifications massless modes are counted by Ext groups\(^{13,14}\), so if \(\mathcal{E}\) and \(\mathcal{F}\) are both rank 3, each embedded in an \(E_8\), then the congruence above simply says that \(27\)'s and \(\bar{27}\)'s of \(E_6\) are exchanged. Note that in the special case that \(\mathcal{E} = TX\) we have

\[
\begin{align*}
    \text{Ext}^1_X(\mathcal{O}, \mathcal{E}) &\cong H^{2,1}(X) \\
    \text{Ext}^1_Y(\mathcal{E}, \mathcal{O}) &\cong H^{1,1}(X)
\end{align*}
\]

and so we recover the analogous expressions for ordinary mirror symmetry.

Ordinary mirror symmetry is not deeply understood, but a lot of empirical facts about it are known. By contrast, very little is known about (0,2) mirror symmetry\(^{30,31,32}\). Existing work on the subject has attempted to construct (0,2) mirrors by orbifolds. Using such ideas, one can argue (somewhat weakly) that mirrors to (restrictions to Calabi-Yaus of) equivariant sheaves are other equivariant sheaves\(^1\). An attempt to get insight into how the monomial-divisor mirror map might be generalized has appeared in\(^{34}\).

\(^{13}\)In fact, more complicated examples than this are quite possible, but for the purposes of this article we shall not go into such details.
8 Conclusions

In this paper we have reviewed a new description of sheaves on Calabi-Yaus, explained in detail recently in [1] which builds upon work largely done by A. A. Klyachko. We have also commented on how this work is related to understanding moduli spaces of more general sheaves, and on a potential generalization of mirror symmetry.

One of the biggest outstanding problems in string compactifications is understanding quantum effects in heterotic compactifications; hopefully our work will be of use in studying this issue.

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