Graph MBO as a semi-discrete implicit Euler scheme for graph Allen–Cahn

Jeremy Budd, Yves van Gennip

Delft Institute of Applied Mathematics (DIAM),
Technische Universiteit Delft,
Delft, The Netherlands

Abstract

In recent years there has been an emerging interest in PDE-like flows defined on finite graphs, with applications in clustering and image segmentation. In particular for image segmentation and semi-supervised learning Bertozzi and Flenner (2012) developed an algorithm based on the Allen–Cahn gradient flow of a graph Ginzburg–Landau functional, and Merkurjev, Kostić and Bertozzi (2013) devised a variant algorithm based instead on graph Merriman–Bence–Osher (MBO) dynamics.

This work offers rigorous justification for this use of MBO in place of Allen–Cahn. First, we choose the double-obstacle potential for the Ginzburg–Landau functional, and derive existence, uniqueness and regularity results for the resulting graph Allen–Cahn flow. Next, we exhibit a “semi-discrete” time-discretisation scheme for Allen–Cahn of which MBO is a special case. We investigate the long-time behaviour of this scheme, and prove its convergence to the Allen–Cahn trajectory as the time-step vanishes. Finally, following a question raised in Van Gennip, Guillen, Osting and Bertozzi (2014), we exhibit results towards proving a link between double-obstacle Allen–Cahn and mean curvature flow on graphs. We show some promising Γ-convergence results, and translate to the graph setting two comparison principles used in Chen and Elliott (1994) to prove the analogous link in the continuum.

Keywords: Allen–Cahn equation, Ginzburg–Landau functional, Merriman–Bence–Osher algorithm, double-obstacle potential, mean curvature flow, Γ-convergence, graph dynamics.

1 Introduction

In this paper, we derive a link between graph formulations of the Merriman–Bence–Osher (MBO) algorithm for diffusion generated motion and the Allen–Cahn gradient flow of the Ginzburg–Landau functional. We go on to observe some promising results towards a link between these flows and a graph formulation of mean curvature flow.
The core background for this work is the paper of Van Gennip, Guillen, Osting and Bertozzi [1] in which a framework for graph-based analysis is defined, and within this framework described graph variants of MBO, Allen–Cahn and mean curvature flow. The work in this paper follows on from that work, seeking to elaborate more exactly on the links between the flows, especially in light of their interrelated use in image-processing algorithms developed by Bertozzi et al. [6, 7] inspired by the connections between mean curvature flow and the method of Chan–Vese (see e.g. [20]).

The central result of this paper is that taking as our potential in the Ginzburg–Landau functional the “double-obstacle” potential (see Blowey and Elliott [2, 3, 4] for detail in the continuum context, and recent work by Bosch, Klamt and Stoll [5] in the graph context) we can derive MBO exactly as a “semi-discrete” numerical scheme for the Allen–Cahn PDE, for a particular choice of time-step. We will explore the properties of our semi-discrete scheme (and thus in particular MBO) and how it relates to the continuous-time Allen–Cahn flow with this potential. Furthermore, we prove existence, uniqueness and regularity for this Allen–Cahn flow. Finally, we follow [1] in investigating links between MBO, Allen–Cahn and a graph formulation of mean curvature flow. We present encouraging Γ-convergence results, and prove a pair of comparison principles that are graph analogues of comparison principles used by Chen and Elliott [32] to prove convergence of continuum Allen–Cahn (with double-obstacle potential) to mean curvature flow.

1.1 Background

In the continuum, it is well-known that these three flows share important interrelations. The MBO algorithm was developed in [8] as a means of approximating motion according to mean curvature flow by iterative diffusion and thresholding of a set. The paper gave a formal analysis showing that diffusion of a set locally corresponded to motion with curvature dependent velocity, suggesting a convergence as the MBO time-step went to zero. This formal analysis was then supported by rigorous convergence proofs by Evans [9] and Barles and Georgelin [10]. Recently, Swartz and Kwan Yip [11] have presented an elementary proof of the convergence making use of the weak formulation of mean curvature flow in [12]. The connections between Ginzburg–Landau dynamics and mean curvature flow have been extensively studied, dating back at least to a formal analysis by Allen and Cahn in [13]. The basic convergence result, see for example [14], [15] and [16], is that as \( \varepsilon \to 0 \) the Allen–Cahn solution tends to a phase-separation with the interface evolving by mean curvature flow. Thus a method of approximating mean curvature flow is as a singular limit of “phase fields” evolving under the Allen–Cahn equation.

Mean curvature flow also arises in a discrete context in applications such as image segmentation. A major technique in this area is to use a variational approach involving minimising the Mumford–Shah functional [18]. As this functional is quite intensive to

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1See [14], [15] and [17] for detail on this method.
minimise in full generality, Chan and Vese [19] introduced a method of using a level-set approach in the simplified case of a piecewise-constant image. In [20] Esedoḡlu and Tsai considered in particular the case where the image $u$ takes just two values $c_1$ and $c_2$ on regions $\Sigma$ and $\Omega \setminus \Sigma$ respectively. The Euler–Lagrange level-set equations devised to minimise the Chan–Vese functional in this case closely resemble those associated to mean curvature flow of the boundary $\partial \Sigma$. Motivated by this and the continuum convergence results above, Esedoḡlu and Tsai devised a variant of the MBO algorithm to minimise the functional. Interpreting MBO as an approximation to a time-splitting of the Allen–Cahn equation, an idea we will later return to in our analysis in this paper, they consider a modified Mumford–Shah energy using the Ginzburg–Landau function in place of total variation and devise an MBO-like algorithm based on a modified diffusion followed by a thresholding to minimise this energy.

Inspired by these techniques, in [6] Bertozzi and Flenner embraced the discrete nature of an image and devised a discrete graph-based method for image segmentation (and related topics, such as semi-supervised learning) using the graph Ginzburg–Landau functional (with symmetric normalised Laplacian). In [7] Merkurjev, Kostić and Bertozzi developed a faster variant of this method by employing the MBO algorithm on a graph, motivated by the link between continuum Ginzburg–Landau and MBO through their common association with mean curvature flow. They also extended this method to apply it to non-local image inpainting. An example of an application of these techniques is recent work by Calatroni, Van Gennip, Schönlieb, Rowland and Flenner [21].

The use of these methods implicitly assumes that the continuum connections between these processes in general extend to their graph-based counterparts. An important challenge to this assumption is that graphs need not resemble the continuum objects to which the above convergence results apply. For example there has been some interest, though not to the authors’ knowledge any published work, in applying graph-based segmentation methods to social networks—which can be very different in structure to a mesh on a continuum manifold. Following work in [1] we seek to investigate rigorously the validity of this assumption.

1.2 Groundwork

The framework for analysis on graphs is presented in [1], here we reproduce those aspects needed for our discussion. We define $G = (V, E)$ a finite, undirected, weighted graph with vertex set $V$, edge set $E \subseteq V^2$ and positive weights $\{\omega_{ij}\}_{ij \in E}$ with $\omega_{ij} = \omega_{ji}$. We extend $\omega_{ij}$ to be zero when $ij \notin E$. We shall assume $G$ is simple and connected. Upon this graph we define the function spaces (where $X \subseteq \mathbb{R}$):

$$
\mathcal{V} := \{u : V \to \mathbb{R}\}, \quad \mathcal{V}_X := \{u : V \to X\}, \quad \mathcal{E} := \{\varphi : E \to \mathbb{R}|\varphi_{ij} = -\varphi_{ji}\}.
$$

In particular, a common approach leads to motion of $\partial \Sigma$ with normal velocity $\kappa - \lambda (c_1 - f)^2 + \lambda (c_2 - f)^2$ where $\kappa$ is the mean curvature, $f$ the reference and $\lambda$ the strength of fidelity to the reference.
Since $V$ is finite, elements of these spaces may be viewed as real vectors. We shall use these two interpretations interchangeably. Furthermore, we define the spaces of time-dependent vertex functions (where $T \subseteq \mathbb{R}$ an interval)

$$\mathcal{V}_{t \in T} := \{ u : T \to V \}, \quad \mathcal{V}_{X, t \in T} := \{ u : T \to \mathcal{V}_X \}.$$ 

For a parameter $r \in [0, 1]$, and denoting $d_i := \sum_j \omega_{ij}$, which we refer to as the degree of vertex $i$, we define the following inner products on $\mathcal{V}$ and $\mathcal{E}^2$:

$$\langle u, v \rangle_V := \sum_{i \in V} u_i v_i d_i, \quad \langle \varphi, \phi \rangle_E := \frac{1}{2} \sum_{i,j \in V} \varphi_{ij} \phi_{ij} \omega_{ij}$$

and define the inner product on $\mathcal{V}_{t \in T}$ (or $\mathcal{V}_{X, t \in T}$)

$$(u, v)_{t \in T} := \int_T \langle u(t), v(t) \rangle_V \, dt = \sum_{i \in V} d_i^r (u_i, v_i)_{L^2(T; \mathbb{R})}.$$ 

These induce norms $\| \cdot \|_V$, $\| \cdot \|_E$ and $\| \cdot \|_{t \in T}$ in the usual way. We also define for $u \in \mathcal{V}$ the norm $\| u \|_\infty := \max_{i \in V} |u_i|$. We furthermore define the space:

$$L^2(T; \mathcal{V}) := \{ u \in \mathcal{V}_{t \in T} \mid \| u \|_{t \in T} < \infty \}.$$ 

Finally, for $T$ an open interval, we define the Sobolev space $H^1(T; \mathcal{V})$ as the set of $u \in L^2(T; \mathcal{V})$ with generalised time derivative $du/dt \in L^2(T; \mathcal{V})$ such that

$$\forall \varphi \in C_c^\infty(T; \mathcal{V}) \quad \left( u, \frac{d\varphi}{dt} \right)_{t \in T} = - \left( \frac{du}{dt}, \varphi \right)_{t \in T}$$

where $C_c^\infty(T; \mathcal{V})$ denotes the set of elements of $\mathcal{V}_{t \in T}$ that are infinitely differentiable with respect to time and compactly supported in $T$. We link this to the familiar continuum setting:

**Proposition 1.** $u \in H^1(T; \mathcal{V})$ if and only if $u_i \in H^1(T; \mathbb{R})$ for each $i \in V$.

**Proof.** Note that $(du/dt)_i = du_i/dt$, so $u$ and $du/dt \in L^2(T; \mathcal{V})$ if and only if $\forall i \in V, u_i$ and $du_i/dt \in L^2(T; \mathbb{R})$. Next, $(u, d\varphi/dt)_{t \in T} = -(du/dt, \varphi)_{t \in T}$ if and only if

$$\sum_{i \in V} d_i^r (u_i, d\varphi_i/dt)_{L^2(T; \mathbb{R})} = - \sum_{i \in V} d_i^r (du_i/dt, \varphi_i)_{L^2(T; \mathbb{R})}.$$ 

It follows that $\forall \varphi \in C_c^\infty(T; \mathcal{V})$ $(u, d\varphi/dt)_{t \in T} = -(du/dt, \varphi)_{t \in T}$ if and only if

$$\forall i \in V \forall \phi \in C_c^\infty(T; \mathbb{R}) \quad (u_i, d\phi/dt)_{L^2(T; \mathbb{R})} = -(du_i/dt, \phi)_{L^2(T; \mathbb{R})}$$

and therefore $\forall i \in V \; u_i \in H^1(T; \mathbb{R})$.  

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We here take, so as to preserve the $\Gamma$-convergence result in [29], $q = 1$ in the definitions in [1].
We define the following inner product on $H^1(T; V)$:

$$(u, v)_{H^1(T; V)} := (u, v)_{t \in T} + \left( \frac{du}{dt}, \frac{dv}{dt} \right)_{t \in T} = \sum_{i \in V} d_t^T (u_i, v_i)_{H^1(T; \mathbb{R})}.$$ 

Since for $T$ unbounded we have the undesirable feature that non-zero constant functions are not of bounded $||\cdot||_{t \in T}$, but still have meaningful time derivative, we also define $H^1_{loc}(T; V) := \{ u \in V_{t \in T} \mid \forall a, b \in T, u \in H^1((a, b); V) \}$ and we likewise define $L^2_{loc}(T; V)$.

Next, we introduce the graph variants of familiar vector calculus operators of gradient and Laplacian:

$$(\nabla u)_{ij} := \begin{cases} u_j - u_i, & ij \in E \\ 0, & \text{otherwise} \end{cases} \quad (\Delta u)_i := d_t^{-r} \sum_{j \in V} \omega_{ij} (u_i - u_j)$$

where the graph Laplacian $\Delta$ is positive semi-definite, unlike the negative semi-definite continuum Laplacian. From this we define the graph diffusion operator

$$e^{-t\Delta} u := \sum_{n \geq 0} \frac{(-1)^n t^n}{n!} \Delta^n u$$

where $v(t) = e^{-t\Delta} u$ is the unique solution to the diffusion equation

$$\frac{dv}{dt} = -\Delta v, \quad v(0) = u.$$ 

We recall the familiar functional analysis notation, for some $F : \mathcal{V} \rightarrow \mathcal{V}$, of

$$\rho(F) := \max \{ |\lambda| : \lambda \text{ an eigenvalue of } F \}$$

$$||F|| := \sup_{||u||_{\mathcal{V}} = 1} ||Fu||_{\mathcal{V}}$$

and recall the standard result that if $F$ self-adjoint then $||F|| = \rho(F)$.

**Proposition 2.** If $u \in H^1(T; V)$ and $T$ bounded below, then $e^{-t\Delta} u \in H^1(T; V)$ with

$$\frac{d}{dt} (e^{-t\Delta} u) = e^{-t\Delta} \frac{du}{dt} - e^{-t\Delta} \Delta u.$$ 

**Proof.** Let $T = (a, b)$, $a > -\infty$. Now $e^{-t\Delta}$ has eigenvalues $e^{-\lambda_k t}$, where $\lambda_k \geq 0$ are the eigenvalues of $\Delta$, and $e^{-t\Delta}$ is self-adjoint so $||e^{-t\Delta}|| = \rho(e^{-t\Delta}) \leq \max \{ 1, e^{-a||\Delta||} \}$ for

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*Our choice of $r$ dictates which graph Laplacian we use. For $r = 0$ we have $\Delta = D - A$ the standard unnormalised Laplacian. For $r = 1$ we have $\Delta = I - D^{-1} A$ the random walk Laplacian. Note that the symmetric normalised Laplacian $I - D^{-1/2} AD^{-1/2}$ used in [6, 7] is not covered by our scheme.*
$t \in T$. So $e^{-t\Delta}$ a uniformly bounded operator for $t \in T$ and therefore $\|e^{-t\Delta u}\|_{t \in T} < \infty$ and $\|e^{-t\Delta \frac{du}{dt}} - e^{-t\Delta} \Delta u\|_{t \in T} < \infty$. Next, note that for $\varphi \in C_c(\mathcal{T}; \mathcal{V})$

$\left( e^{-t\Delta \frac{du}{dt}} - e^{-t\Delta} \Delta u, \varphi \right)_{t \in T} = \left( \frac{du}{dt} e^{-t\Delta} \varphi \right)_{t \in T} - \left( u, e^{-t\Delta} \Delta \varphi \right)_{t \in T}$

$= - \left( u, \frac{d}{dt} (e^{-t\Delta} \varphi) + e^{-t\Delta} \Delta \varphi \right)_{t \in T}$

$= - \left( u, e^{-t\Delta} \frac{d\varphi}{dt} \right)_{t \in T}$

(by the product rule)

$= - \left( e^{-t\Delta} u, \frac{d\varphi}{dt} \right)_{t \in T}$

so $e^{-t\Delta} u$ has the desired generalised derivative. \hfill \Box

Finally, when considering variational problems of the form

$$\arg\min_x f(x)$$

we write $f \simeq g$ and say the functionals are equivalent when $g(x) = af(x) + b$ for $a, b$ independent of $x$ and $a > 0$. This ensures that $f$ and $g$ have the same minimisers.

Our first process is the graph Merriman–Bence–Osher algorithm (MBO). This algorithm creates a series of vertex sets (or equivalently, binary elements of $\mathcal{V}$) by first diffusing the characteristic function $\chi_{S_n}$ of the set $S_n \subseteq V$ for a time $\tau$ to form a function $v = e^{-\tau\Delta} \chi_{S_n}$, and then thresholding to define $S_{n+1} = \{i \in V | v_i \geq 1/2\}$. In Proposition 4.6 it was shown that this algorithm can be expressed variationally, where $u_n = \chi_{S_n}$, by

$$u_{n+1} \in \arg\min_{u \in V_{[0,1]}} \left( 1 - 2e^{-\tau\Delta} u_n, u \right)_{\mathcal{V}}$$

(where $1$ is the vector of ones) which we can rewrite with the equivalent functional:

$$u_{n+1} \in \arg\min_{u \in V_{[0,1]}} \frac{1}{2\tau} (1 - u, u)_{\mathcal{V}} + \frac{||u - e^{-\tau\Delta} u_n||^2_{\mathcal{V}}}{2\tau}.$$  

This second formulation has a form resembling a discrete solution \cite[Definition 2.0.2]{22} (cf. the study of minimising movements) of a gradient flow. Importantly, this means that it resembles a sequence arising from an Euler discretisation of the gradient flow of some functional, which shall be important for motivating the link to Allen–Cahn.

Our second process is graph Allen–Cahn evolution (ACE). This evolution is the $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ gradient flow of the graph Ginzburg–Landau functional, which in slight variation from \cite{1} we shall define as:

$$\text{GL}_\varepsilon(u) := \frac{1}{2} \|\nabla u\|_\varepsilon^2 + \frac{1}{\varepsilon} \langle W \circ u, 1 \rangle_{\mathcal{V}}$$

\footnote{One can check that $\langle 1 - 2e^{-\tau\Delta} u_n, u \rangle_{\mathcal{V}} = \langle u, 1 - u \rangle_{\mathcal{V}} + \langle u - e^{-\tau\Delta} u_n, u - e^{-\tau\Delta} u_n \rangle_{\mathcal{V}} - \langle e^{-\tau\Delta} u_n, e^{-\tau\Delta} u_n \rangle_{\mathcal{V}}$. Then suppress the constant (in $u$) term $\langle e^{-\tau\Delta} u_n, e^{-\tau\Delta} u_n \rangle_{\mathcal{V}}$ and divide by $2\tau$.}
where $W$ is a double-well potential. This differs from the form in [1] only in that we have replaced their $\sum_i W(u_i)$ with $\langle W \circ u, 1 \rangle_V$, a change which we have found plays better with the Hilbert space structure and which facilitates the link we derive with MBO. ACE is then given, for $W$ differentiable, by the ODE:

$$\frac{du}{dt} = -\Delta u - \frac{1}{\varepsilon} W' \circ u = -\nabla_V G L_{\varepsilon}(u) \tag{1.4}$$

where $\nabla_V$ is the Hilbert space gradient on $V$. For the purposes of linking ACE to MBO, we must discretise it in time. Note that MBO, although discrete in time, thresholds after a continuous-time diffusion. To capture this behaviour, we introduce what we term a semi-discrete implicit Euler scheme:

$$u_{n+1} = e^{-\tau \Delta u_n} - \frac{\tau}{\varepsilon} W' \circ u_{n+1}. \tag{1.5}$$

We link this to our variational form of MBO by writing it as a discrete solution. We can rewrite (1.5) using the Hilbert space gradient as

$$0 = \nabla_V|_{u=u_{n+1}} \left( \frac{1}{\varepsilon} \langle W \circ u, 1 \rangle_V + \frac{||u - e^{-\tau \Delta u_n}||^2_V}{2\tau} \right)$$

suggesting that solutions to our semi-discrete scheme obey the variational equation

$$u_{n+1} \in \arg\min_{u \in V} \frac{1}{\varepsilon} \langle W \circ u, 1 \rangle_V + \frac{||u - e^{-\tau \Delta u_n}||^2_V}{2\tau}. \tag{1.6}$$

We thus note that for $\varepsilon = \tau$ there is a striking similarity between the functionals in (1.2) and (1.6), so long as we choose a suitable $W$. The form of (1.2) suggests that we take $W(x) = \frac{1}{2} x (1 - x)$. This however has the issue of not having two wells (or indeed minima). We further wish for our $W$ to have a form that forces minimisers of (1.6) to lie in $V_{[0,1]}$ as in (1.2). A potential that satisfies these demands is the well-known double-obstacle potential studied extensively by Blowey and Elliott [2, 3, 4]:

$$W(x) := \begin{cases} \frac{1}{2} x (1 - x), & \text{for } 0 \leq x \leq 1, \\ \infty, & \text{otherwise}. \end{cases} \tag{1.7}$$

In the remainder of this paper we shall use $W$ to denote this double-obstacle potential. This paper will prove and explore the following theorem:

**Theorem 3.** Taking $\varepsilon = \tau$ and choosing $W$ as in (1.7), we get that solutions to our semi-discrete scheme (1.5), which is (1.4) refined to apply to a non-differentiable $W$, obey the variational equation

$$u_{n+1} \in \arg\min_{u \in V_{[0,1]}} \langle u, 1 - u \rangle_V + ||u - e^{-\tau \Delta u_n}||^2_V \tag{1.8}$$

and thus the solutions correspond exactly to MBO trajectories.
Allen–Cahn evolution with a double-obstacle potential

Our definition of ACE in (1.4) assumed that $W$ was differentiable, which of course the double-obstacle potential is not at 0 and 1. Towards extending our definition, write

$$W(x) = \frac{1}{2} x(1-x) + I_{[0,1]}(x)$$

where $I_{[0,1]}$ is the indicator function taking value 0 on $[0,1]$ and $\infty$ elsewhere. Now following [4] we seek $H^1_{loc}$ solutions $u$ to (1.4) rewritten using the subdifferential as:

$$-\frac{du}{dt} - \Delta u \in \frac{1}{\varepsilon} \partial W(u).$$

(2.1)

That is, for almost every $t$ in some chosen interval $T$, and every $i \in V$,

$$\varepsilon \frac{du_i}{dt} + \varepsilon (\Delta u(t))_i + \frac{1}{2} - u_i(t) = \beta_i(t) \in -\partial I_{[0,1]}(u_i(t))$$

(2.2)

that is, $\beta$ obeys

$$\beta_i(t) \in \begin{cases} 
\{\infty\}, & u_i < 0, \\
[0, \infty), & u_i(t) = 0, \\
\{0\}, & 0 < u_i(t) < 1, \\
(-\infty, 0], & u_i(t) = 1, \\
\{-\infty\}, & u_i > 1.
\end{cases}$$

(2.4)

Notice that this expression only makes sense for trajectories such that $u(t) \in V_{[0,1]}$ at a.e. $t$, as $\partial I_{[0,1]}(x)$ has no real values for $x \notin [0,1]$. For tidyness of notation, we define

$$B(u) := \{\alpha \in V | \forall i \in V : \alpha_i \in -\partial I_{[0,1]}(u_i)\}$$

(2.3)

which is non-empty if and only if $u \in V_{[0,1]}$.

However not all values for $\beta$ in the subdifferential are attained in valid trajectories. To characterise the validly attained values, we first note a standard fact about continuous representatives of $H^1$ functions:

**Lemma 4.** If $u \in H^1_{loc}(T; V) \cap C^0(T; V)$, then $u$ is locally absolutely continous on $T$. It follows that $u$ is differentiable a.e. in $T$, and the weak derivative equals the classical derivative a.e. in $T$.

**Proof.** By Proposition [1] $u \in H^1_{loc}(T; V) \cap C^0(T; V)$ if and only if $\forall i \in V, u_i \in H^1_{loc}(T; \mathbb{R}) \cap C^0(T; \mathbb{R})$. The result then follows from standard results, cf. [23] Theorem 7.13. \qed

**Theorem 5.** Let $(u, \beta)$ obey (2.2) at a.e. $t \in T$, with $u \in H^1_{loc}(T; V) \cap C^0(T; V) \cap V_{[0,1], t \in T}$. Then for all $i \in V$ and a.e. $t \in T$,

$$\beta_i(t) = \begin{cases} 
\frac{1}{2} + \varepsilon (\Delta u(t))_i, & u_i(t) = 0, \\
0, & u_i(t) \in (0, 1), \\
-\frac{1}{2} + \varepsilon (\Delta u(t))_i, & u_i(t) = 1.
\end{cases}$$

(2.4)
Proof. Since \( \beta(t) \in B(u(t)) \) for a.e. \( t \in T \), (2.4) holds at a.e. \( t \in T \) and \( i \in V \) such that \( u_i(t) \in (0, 1) \). Let \( \tilde{T} \subseteq T \) denote the times when \( u \) is differentiable and has classical derivative equal to its weak derivative. Since \( u_i(t) \in [0, 1] \) at all times, when \( t \in \tilde{T} \) and \( u_i(t) \in \{0, 1\} \) we have \( du_i/dt = 0 \). Consider first \( u_i(t) = 0 \). Then for a.e. such \( t \in \tilde{T} \)

\[
0 = \frac{du_i}{dt} = -\left(\Delta u(t)\right)_i + \frac{1}{\varepsilon} \left(\beta_i(t) - \frac{1}{2}\right)
\]

and therefore

\[
\beta_i(t) = \frac{1}{2} + \varepsilon(\Delta u(t))_i
\]

and likewise for \( u_i(t) = 1 \). Thus (2.4) holds at a.e. \( t \in \tilde{T} \). By Lemma 1, \( T \setminus \tilde{T} \) has measure zero, so (2.4) holds at a.e. \( t \in T \).

Note. From (2.4) and the sign properties of \((\Delta u(t))_i\) at \( u_i(t) \in \{0, 1\} \), it follows that \( \beta(t) \in V_{-1/2, 1/2} \). This corresponds to the explicit restriction on the subdifferential imposed in the definition of \( \beta \) in 3.

Tying this all together, we define ACE solutions as follows:

Definition 6 (Double-obstacle ACE). Let \( T \) be an interval. Then a pair \((u, \beta) \in \mathcal{V}_{[0,1], t \in T} \times \mathcal{V}_{i \in T}\) is a solution to double-obstacle ACE on \( T \) when \( u \in H^1_{\text{loc}}(T; \mathcal{V}) \cap C^0(T; \mathcal{V}) \) and for almost every \( t \in T \),

\[
\varepsilon \frac{du}{dt} + \varepsilon \Delta u(t) + \frac{1}{2} - u(t) = \beta(t), \quad \beta(t) \in B(u(t)). \tag{2.5}
\]

Note that we will often for conciseness refer to just \( u \) as a solution to (2.5), since \( \beta \) is a.e. uniquely determined as a function of \( u \) by (2.4).

Note. The condition that \( u \) is continuous is not as such a further condition on \( u \) beyond it being \( H^1_{\text{loc}} \), but rather is to emphasise that we are taking the continuous representative of \( u \). Recall since \( T \) is one-dimensional we have by Sobolev embedding that any \( u \in H^1(T; \mathcal{V}) \) has a representative \( \tilde{u} \in C^{0,1/2}(T; \mathcal{V}) \) such that \( u(t) = \tilde{u}(t) \) for a.e. \( t \in T \).

We now investigate important properties of this equation. Firstly, we note the following existence and uniqueness theory, which we shall prove over the course of this paper:

Theorem 7. Let \( T = [0, \infty) \). Then for all \( u_0 \in \mathcal{V}_{[0,1], t \in T} \) there exists \((u, \beta) \in \mathcal{V}_{[0,1], t \in T} \times \mathcal{V}_{i \in T}\) satisfying (2.5) with \( u \in H^1_{\text{loc}}(T; \mathcal{V}) \cap C^{0,1}(T; \mathcal{V}) \) and with initial condition \( u(0) = u_0 \).

Proof. We construct a solution by approximating double-obstacle ACE by solutions to ACE with a \( C^1 \) approximation of the double-obstacle potential, and taking the limit as the approximations become more accurate. For detail, refer to Appendix A.

We also prove this as Theorem 21 by taking a limit of the semi-discrete approximations as defined in (2.8). \( \square \)

Theorem 8. Let \( T = [0, T_0] \) or \([0, \infty)\), and let \((u, \beta), (v, \gamma) \) be solutions to (2.5) on \( T \) with \( u(0) = v(0) \). Then for all \( t \in T \), \( u(t) = v(t) \), and for a.e. \( t \in T \), \( \beta(t) = \gamma(t) \).
Proof. We will prove this for finite and infinite intervals as Corollaries 29 and 30 as a consequence of a comparison principle. The proofs of these corollaries do not depend on results of the rest of this paper beyond Theorem 5, justifying us to use uniqueness language for ACE solutions (with given initial state) for the remainder.

Secondly, to meaningfully call this an ACE flow, we must verify that it monotonically decreases the Ginzburg–Landau functional:

**Proposition 9.** Let \((u, \beta)\) solve (2.5) on an interval \(T\). Then for a.e. \(t \in T\),

\[
\frac{d}{dt} GL_\varepsilon(u(t)) \leq 0.
\]

Proof. Define

\[
G_\varepsilon(u) := \frac{1}{2} \| \nabla u(t) \|^2 + \frac{1}{2\varepsilon} \langle u(t), 1 - u(t) \rangle_V
\]

then by (1.3) we have, for all \(t \in T\),

\[
GL_\varepsilon(u(t)) = G_\varepsilon(u(t)) + \frac{1}{\varepsilon} \langle I_{[0,1]} \circ u(t), 1 \rangle_V = G_\varepsilon(u(t))
\]

since \(u(t) \in V_{[0,1]}\) for all \(t \in T\). Hence, since \(\nabla_V G_\varepsilon(u) = \Delta u + \frac{1}{\varepsilon} (\frac{1}{2} \mathbf{1} - u)\), we note that

\[
\varepsilon^2 \frac{d}{dt} GL_\varepsilon(u(t)) = \varepsilon^2 \frac{dG_\varepsilon(u(t))}{dt} = \left\langle \varepsilon \frac{du}{dt}, \varepsilon \Delta u(t) + \frac{1}{2} \mathbf{1} - u(t) \right\rangle_V
\]

\[
= \left\langle \beta(t) - \varepsilon \Delta u(t) - \frac{1}{2} \mathbf{1} + u(t), \varepsilon \Delta u(t) + \frac{1}{2} \mathbf{1} - u(t) \right\rangle_V
\]

and so we seek to prove that for almost every \(t \in T\)

\[
\left\langle \beta(t) - \varepsilon \Delta u(t) - \frac{1}{2} \mathbf{1} + u(t), \varepsilon \Delta u(t) + \frac{1}{2} \mathbf{1} - u(t) \right\rangle_V \leq 0.
\]

By Theorem 5 at a.e. \(t \in T\) and all \(i \in V\) such that \(u_i(t) \in \{0,1\}\) we have that \(\beta_i(t) - \varepsilon (\Delta u(t))_i - \frac{1}{2} + u_i(t) = 0\), and at a.e. \(t \in T\) and all \(i \in V\) such that \(u_i(t) \in (0,1)\), we have \(\beta_i(t) = 0\). Therefore for a.e. \(t \in T\),

\[
\left\langle \beta(t) - \varepsilon \Delta u(t) - \frac{1}{2} \mathbf{1} + u(t), \varepsilon \Delta u(t) + \frac{1}{2} \mathbf{1} - u(t) \right\rangle_V = - \sum_{i \in V} d_i^r \left( \varepsilon (\Delta u(t))_i + \frac{1}{2} - u_i(t) \right)^2 \leq 0
\]

as desired.

Thirdly, following [4] we derive a weak formulation of the ACE flow:

**Proposition 10.** A continuous function \(u \in V_{[0,1], t \in T} \cap H^1_{loc}(T; V)\) (and associated \(\beta(t) = \varepsilon \frac{du}{dt} + \varepsilon \Delta u(t) - u(t) + \frac{1}{2} \mathbf{1}\) a.e.) is a solution to (2.5) if and only if for almost every \(t \in T\)

\[
\forall \eta \in V_{[0,1]}, \left\langle \varepsilon \frac{du}{dt} - u(t) + \frac{1}{2} \mathbf{1}, \eta - u(t) \right\rangle_V + \varepsilon \langle \nabla u(t), \nabla \eta - \nabla u(t) \rangle_\varepsilon \geq 0. \tag{2.6}
\]
Proof. Let \( u \) solve (2.5). Then for a.e. \( t \in T, \beta(t) \in B(u(t)) \) and in particular, \( \beta_i(t) \geq 0 \) and \( \beta_i(t) \leq 0 \) at a.e. \( t \in T \) for which \( u_i(t) \) is 0 and 1 respectively. So, at each such \( t \in T \), for any \( \eta \in V_{[0,1]} \)

\[
\text{LHS } (2.6) = (-\varepsilon \Delta u(t) + \beta(t), \eta - u(t))_\mathcal{V} + \varepsilon \langle \nabla u(t), \nabla \eta - \nabla u(t) \rangle_\mathcal{E} = \langle \beta(t), \eta - u(t) \rangle_\mathcal{V}
\]

\[
= \sum_{\{i:u_i(t)=0\}} d_i^\varepsilon \beta_i(t) \eta_i + \sum_{\{i:u_i(t)=1\}} d_i^\varepsilon \beta_i(t) (\eta_i - 1) \geq 0.
\]

Now let \( u \in V_{[0,1],t \in T} \cap H^1_{\text{loc}}(T; \mathcal{V}) \) satisfy (2.6) for almost every \( t \in T \). So for all such \( t \)

\[
\forall \eta \in V_{[0,1]}, \langle \beta(t), \eta - u(t) \rangle_\mathcal{V} \geq 0.
\]

Let \( \eta_j = u_j(t) \) for \( j \neq i \) and \( \eta_i = 0 \), and \( \eta'_j = u_j(t) \) for \( j \neq i \) and \( \eta'_i = 1 \). Substituting \( \eta \) and \( \eta' \) into the above we have \( \beta_i(t)u_i(t) \leq 0 \) and \( \beta_i(t)(1-u_i(t)) \geq 0 \). Therefore

\[
\beta_i(t) = \begin{cases} 
0, & u_i(t) \in (0,1) \\
\leq 0, & u_i(t) = 1 \\
\geq 0, & u_i(t) = 0 
\end{cases}
\]

so \( \beta(t) \in B(u(t)) \), and thus \( (u, \beta) \) solves (2.5). \( \square \)

Finally, we give an explicit integral form for the solution:

\[
u(t) = \frac{1}{2} + e^{t/\varepsilon} e^{-t\Delta} \left( u(0) - \frac{1}{2} \right) + \frac{1}{\varepsilon} \int_0^t e^{(t-s)/\varepsilon} e^{-(t-s)\Delta} \beta(s) \, ds. \tag{2.7}
\]

Note that (2.7) shows that \( \beta(s) \neq 0 \) a positive measure subset of the time, so \( u \) must remain at obstacles non-instantaneously rather than “bouncing” off them. Furthermore, this formula yields a regularity estimate:

**Theorem 11.** ACE solutions are globally Lipschitz, i.e. \( u \in C^{0,1}([0, \infty); \mathcal{V}) \).

Proof. By (2.7) we have for \( t_1 < t_2 \) and \( A := \varepsilon^{-1} I - \Delta \)

\[
u(t_2) - \nu(t_1) = (e^{t_2 - t_1}A - e^{t_1}A) \left( u(0) - \frac{1}{2} \right) + \frac{1}{\varepsilon} \int_0^{t_1} \left( e^{(t_2-s)A} - e^{(t_1-s)A} \right) \beta(s) \, ds + \frac{1}{\varepsilon} \int_{t_1}^{t_2} e^{(t_2-s)A} \beta(s) \, ds
\]

\[
= \left( e^{(t_2-t_1)A} - I \right) \left( e^{t_1}A \left( u(0) - \frac{1}{2} \right) + \frac{1}{\varepsilon} \int_0^{t_1} e^{(t_1-s)A} \beta(s) \, ds \right) + \frac{1}{\varepsilon} \int_{t_1}^{t_2} e^{sA} \beta(t_2-s) \, ds
\]

\[
= \left( e^{(t_2-t_1)A} - I \right) \left( u(t_1) - \frac{1}{2} \right) + \frac{1}{\varepsilon} \int_0^{t_2-t_1} e^{sA} \beta(t_2-s) \, ds.
\]

\( \varepsilon \)Note we can rewrite the ODE in (2.6) as \( \equiv \left( e^{t/\varepsilon} e^{t\Delta} (u - \frac{1}{2} \mathbf{1}) \right) = \varepsilon^{-1} e^{-t/\varepsilon} e^{t\Delta} \beta \). Then (2.7) follows by the ‘fundamental theorem of calculus’ on \( H^1 \) [24] Theorem 8.2] since \( e^{-s/\varepsilon} e^{s\Delta} (u - \frac{1}{2} \mathbf{1}) \in H^1((0, t); \mathcal{V}) \).
Now recall by the conditions on an ACE solution that \( u(t) \in \mathcal{V}_{[0,1]} \) for all \( t \geq 0 \), and by Theorem \([5]\) \( \beta(t) \in \mathcal{V}_{[-1/2,1/2]} \) for a.e. \( t \geq 0 \). Writing \( B_{\delta t} := (e^{\delta t A} - I)/\delta t \), notice that \( A \) has largest eigenvalue \( 1/\varepsilon \) and hence since \( B_{\delta t} \) is self-adjoint
\[
\| B_{\delta t} \| = \frac{e^{\delta t/\varepsilon} - 1}{\delta t}
\]
and note this is monotonically increasing in \( \delta t \) for \( \delta t > 0 \). We thus have for \( 0 < t_2 - t_1 < 1 \)
\[
\frac{\| u(t_2) - u(t_1) \|_V}{t_2 - t_1} \leq \| B_{t_2 - t_1} \| \cdot \frac{1}{2} \| 1 \|_V + \frac{1}{\varepsilon} \sup_{s \in [0,t_2-t_1]} \| e^{sA} \beta(t_2 - s) \|_V \\
\leq \frac{e^{(t_2-t_1)/\varepsilon} - 1}{t_2 - t_1} \cdot \frac{1}{2} \| 1 \|_V + \frac{1}{\varepsilon} \sup_{s \in [0,t_2-t_1]} \| e^{sA} \| \cdot \frac{1}{2} \| 1 \|_V \\
\leq \frac{e^{(t_2-t_1)/\varepsilon} - 1}{t_2 - t_1} \cdot \frac{1}{2} \| 1 \|_V + \frac{1}{\varepsilon} e^{(t_2-t_1)/\varepsilon} \cdot \frac{1}{2} \| 1 \|_V \\
\leq \frac{1}{2} \| 1 \|_V \left( e^{1/\varepsilon} - 1 + \frac{1}{\varepsilon} e^{1/\varepsilon} \right)
\]
and for \( t_2 - t_1 \geq 1 \) we have the simpler estimate
\[
\frac{\| u(t_2) - u(t_1) \|_V}{t_2 - t_1} \leq \| u(t_2) - u(t_1) \|_V \leq \| 1 \|_V
\]
completing the proof. \( \Box \)

**Note.** This regularity estimate is relatively optimal, i.e. \( u \) is not in general \( C^1 \). For example, suppose \( u(0) = \alpha 1 \) for \( \alpha \in (0,1/2) \), and take as an ansatz for \([2.5]\): \( u(t) = f(t)1 \) and \( \beta(t) = \gamma(t)1 \). Plugging this into \([2.5]\) we get
\[
\varepsilon \frac{df}{dt} + \frac{1}{2} - f(t) = \gamma(t).
\]
Then for \( f(t) > 0 \) this has solution \( f(t) = \frac{1}{2} + (\alpha - \frac{1}{2}) e^{t/\varepsilon} \). One can therefore check that the following (uniquely) solves \([2.5]\):
\[
u(t) = \begin{cases} \frac{1}{2} 1 + (\alpha - \frac{1}{2}) e^{t/\varepsilon} 1, & 0 \leq t < -\varepsilon \log(1 - 2\alpha) \\ 0, & t \geq -\varepsilon \log(1 - 2\alpha) \end{cases}
\]
\[
\beta(t) = \begin{cases} 0, & 0 \leq t < -\varepsilon \log(1 - 2\alpha) \\ \frac{1}{2} 1, & t \geq -\varepsilon \log(1 - 2\alpha) \end{cases}
\]
and note that this has a discontinuity in \( du/dt \) at \( t = -\varepsilon \log(1 - 2\alpha) \).

### 2.1 Semi-discrete scheme

In a similar vein, we extend \([1.5]\) to the non-differentiable case of the double-obstacle potential. We thus write our semi-discrete scheme, where \( \lambda := \tau/\varepsilon \), as
\[
\lambda^{-1} (u_{n+1} - e^{-\tau A} u_n) \in -\partial W(u_{n+1}), \text{ i.e.} \quad (2.8a)
\]
\[
(1 - \lambda) u_{n+1} - e^{-\tau A} u_n + \frac{\lambda}{2} 1 = \lambda \beta_{n+1} \quad (2.8b)
\]
for some \( \beta_{n+1} \in \mathcal{B}(u_{n+1}) \) defined as in \([2.3]\). We express this variationally.
Theorem 12. If $0 \leq \tau \leq \varepsilon$ then $(u_{n+1}, \beta_{n+1})$ is a solution to the semi-discrete scheme \ref{2.8} for some $\beta_{n+1} \in B(u_{n+1})$ if and only if $u_{n+1}$ solves the variational equation:

$$u_{n+1} \in \arg\min_{u \in \mathcal{V}_{[0,1]}} \lambda \langle u, 1-u \rangle_{\mathcal{V}} + \| u - e^{-\tau \Delta} u_n \|^2_{\mathcal{V}}, \quad (2.9)$$

Proof. Let $(u_{n+1}, \beta_{n+1})$ solve \ref{2.8}. First, note that $B(u_{n+1})$ is non-empty and so $u_{n+1} \in \mathcal{V}_{[0,1]}$. We seek to prove that for $0 \leq \lambda \leq 1$ and $\forall \eta \in \mathcal{V}_{[0,1]}:

\lambda(u_{n+1}, 1-u_{n+1})\eta + \langle u_{n+1} - e^{-\tau \Delta} u_n, u_{n+1} - e^{-\tau \Delta} u_n \rangle_{\mathcal{V}} \leq \lambda(\eta, 1-\eta)\eta + \langle \eta - e^{-\tau \Delta} u_n, \eta - e^{-\tau \Delta} u_n \rangle_{\mathcal{V}}

By rearranging and cancelling this is equivalent to

$$0 \leq \langle \eta - u_{n+1}, \lambda - 2e^{-\tau \Delta} u_n \rangle_{\mathcal{V}} + (1-\lambda)(\langle \eta, \eta \rangle_{\mathcal{V}} - \langle u_{n+1}, u_{n+1} \rangle_{\mathcal{V}})

= \langle \eta - u_{n+1}, \lambda - 2e^{-\tau \Delta} u_n + (1-\lambda)(\eta + u_{n+1}) \rangle_{\mathcal{V}}

= \langle \eta - u_{n+1}, 2\lambda\beta_{n+1} + (1-\lambda)(\eta - u_{n+1}) \rangle_{\mathcal{V}}

= 2\lambda(\eta - u_{n+1}, \beta_{n+1})\eta + (1-\lambda)\| \eta - u_{n+1} \|^2_{\mathcal{V}}$$

but it is easy to check from \ref{2.3} that $(\beta_{n+1})_i$ is either zero, when $(u_{n+1})_i \in (0,1)$, or has the same sign as $\eta_i - (u_{n+1})_i$, so $\langle \eta - u_{n+1}, \beta_{n+1} \rangle \geq 0$.

Now let $u$ solve \ref{2.9}. The functional in \ref{2.9} can be written

$$\lambda \langle u, 1-u \rangle_{\mathcal{V}} + \| u - e^{-\tau \Delta} u_n \|^2_{\mathcal{V}} = \sum_{i \in \mathcal{V}} d_i f_i(u_i)$$

where

$$f_i(x) := \lambda x(1-x) + (x - (e^{-\tau \Delta} u_n)_i)^2$$

so we can reduce \ref{2.9} to the system of 1-dimensional problems

$$(u_{n+1})_i \in \arg\min_{x \in [0,1]} f_i(x).$$

Differentiating, we get that for $0 \leq \lambda < 1$ $f_i$ is minimised at

$$x = \frac{(e^{-\tau \Delta} u_n)_i - \lambda/2}{1-\lambda} = \frac{1}{2} + \frac{(e^{-\tau \Delta} u_n)_i - 1/2}{1-\lambda}.$$

Therefore for $0 \leq \lambda < 1$ the solution $u$ is given by

$$u_i = \begin{cases} 0, & \text{if } (e^{-\tau \Delta} u_n)_i < \frac{1}{2} \lambda \\ \frac{1}{2} + \frac{(e^{-\tau \Delta} u_n)_i - 1/2}{1-\lambda}, & \text{if } \frac{1}{2} \lambda \leq (e^{-\tau \Delta} u_n)_i < 1 - \frac{1}{2} \lambda \\ 1, & \text{if } (e^{-\tau \Delta} u_n)_i \geq 1 - \frac{1}{2} \lambda \end{cases}$$

and hence

$$\lambda^{-1} \left( (1-\lambda)u_i - (e^{-\tau \Delta} u_n)_i + \frac{\lambda}{2} \right) = \begin{cases} \frac{1}{2} - \lambda^{-1}(e^{-\tau \Delta} u_n)_i, & \text{if } (e^{-\tau \Delta} u_n)_i < \frac{1}{2} \lambda \\ 0, & \text{if } \frac{1}{2} \lambda \leq (e^{-\tau \Delta} u_n)_i < 1 - \frac{1}{2} \lambda \\ -\frac{1}{2} + \lambda^{-1}(1 - (e^{-\tau \Delta} u_n)_i), & \text{if } (e^{-\tau \Delta} u_n)_i \geq 1 - \frac{1}{2} \lambda \end{cases}$$
Thus $\beta = \lambda^{-1} \left( (1 - \lambda) u - e^{-\tau \Delta} u_n + \frac{1}{2} \mathbf{1} \right) \in \mathcal{B}(u)$, so $u$ solves (2.8).

If $\lambda = 1$ then examine the functional in (2.9) for $\lambda = 1$:

$$
\langle u, 1 - u \rangle_V + \| u - e^{-\tau \Delta} u_n \|_V^2
= \langle u, 1 - u \rangle_V + \langle u - e^{-\tau \Delta} u_n, u - e^{-\tau \Delta} u_n \rangle_V
= \langle u, 1 \rangle_V - \langle u, u \rangle_V + \langle u, e^{-\tau \Delta} u_n \rangle_V + \langle e^{-\tau \Delta} u_n, e^{-\tau \Delta} u_n \rangle_V
\approx \langle u, 1 - 2e^{-\tau \Delta} u_n \rangle_V,
$$

and therefore $u$ as a minimiser must obey

$$
u_i \in \begin{cases} 
\{1\}, & (e^{-\tau \Delta} u_n)_i > 1/2, \\
[0, 1], & (e^{-\tau \Delta} u_n)_i = 1/2, \\
\{0\}, & (e^{-\tau \Delta} u_n)_i < 1/2.
\end{cases}
$$

Hence $\beta \in \mathcal{B}(u)$ if and only if for each $i \in V$

$$
\beta_i \in \begin{cases} 
[0, \infty), & (e^{-\tau \Delta} u_n)_i \leq 1/2 \\
\{0\}, & (e^{-\tau \Delta} u_n)_i = 1/2, u_i \in (0, 1) \\
(-\infty, 0], & (e^{-\tau \Delta} u_n)_i \geq 1/2
\end{cases}
$$

and thus $\frac{1}{2} \mathbf{1} - e^{-\tau \Delta} u_n \in \mathcal{B}(u)$, so $u$ solves (2.8).

**Proof of Theorem 3.** Follows directly from (2.10): for $\lambda = 1$ the functional in (2.9) is the functional describing MBO.

**Note.** The $0 \leq \lambda < 1$ semi-discrete solution given by

$$
(u_{n+1})_i = \begin{cases} 
0, & \text{if } (e^{-\tau \Delta} u_n)_i < \frac{1}{2} \lambda \\
\frac{1}{2} + \frac{(e^{-\tau \Delta} u_n)_i - 1/2}{1 - \lambda}, & \text{if } \frac{1}{2} \lambda \leq (e^{-\tau \Delta} u_n)_i < 1 - \frac{1}{2} \lambda \\
1, & \text{if } (e^{-\tau \Delta} u_n)_i \geq 1 - \frac{1}{2} \lambda
\end{cases} \quad (2.11)
$$

can be seen to approach MBO behaviour as $\lambda \uparrow 1$. Indeed, it can be seen (see Fig. 1) to be a relaxation of the MBO thresholding.
Figure 1: Plot of the semi-discrete updates $u_{n+1}$ (blue, left axis, see (2.11)) and $\beta_{n+1}$ (red, right axis, see (2.14)) at vertex $i$ for $0 \leq \lambda < 1$ as a function of the diffused value at $i$. Observe that the semi-discrete solution is a piecewise linear relaxation of the MBO step-function thresholding.

In [1, Theorem 4.2], it was proved that for $\tau$ below certain thresholds and $u_n = \chi_S$, the indicator function of $S$ a subset of $V$, MBO exhibits pinning. That is, $u_{n+1} = \chi_S$. We have an analogous result for our semi-discrete scheme.

**Theorem 13** (Cf. [1, Theorem 4.2]). If $u_n = \chi_S$ and $0 \leq \lambda < 1$, then the semi-discrete scheme pins, i.e. $u_{n+1} = \chi_S$, if

$$\tau \leq \|\Delta\|^{-1} \log \left( 1 + \frac{\lambda}{2} \sqrt{\min_{i \in V} d_i^r} \right), \quad \text{or}$$

$$\tau \leq \frac{\lambda}{2\|\Delta \chi_S\|_{\infty}}. \quad (2.12)$$

**Proof.** The proof of [1, Theorem 4.2] followed from observing that for MBO, $u_{n+1} = u_n$ if $\|e^{-\tau\Delta}u_n - u_n\|_{\infty} < 1/2$. Inspecting (2.11) we see that, for $0 \leq \lambda < 1$, we likewise have for the semi-discrete scheme that if $u_n = \chi_S$, then $u_{n+1} = \chi_S$ if and only if $\|e^{-\tau\Delta} \chi_S - \chi_S\|_{\infty} \leq \lambda/2$. Hence, the theorem is proved by modifying appropriately the proof of [1, Theorem 4.2] to derive sufficient conditions for this inequality to hold.

**Note.** Note that for $\lambda = 1$, the result of Theorem 13 still holds if we turn (2.12) into a strict inequality, as per [1, Theorem 4.2].

**Note.** Note that (2.13) is equivalent to the condition that $\varepsilon \leq \frac{1}{2} \|\Delta \chi_S\|_{\infty}^{-1}$. Since, as we shall prove in Theorem 21, the semi-discrete iterates converge to the continuous-time ACE solution as $\tau \downarrow 0$ for $\varepsilon$ fixed, it follows that if $\varepsilon \leq \frac{1}{2} \|\Delta \chi_S\|_{\infty}^{-1}$ and $u(0) = \chi_S$ then ACE also pins, i.e. $u(t) = \chi_S$ for all $t \geq 0$. This can be checked to be sharp directly from (2.6) by considering $u(t) := \chi_S$ for all $t \geq 0$ and noting that, since $du/dt \equiv 0$, this
is a valid trajectory (i.e. has a corresponding $\beta(t) \in B(\chi_S)$ for a.e. $t$) if and only if $\varepsilon \| \Delta \chi_S \|_{\infty} \leq \frac{1}{2}$, by the characterisation of $\beta$ in Theorem 2.

We round out this section by noting some trivia.

**Note.** Notice the similarity between (2.4) and the expression for the $\beta_{n+1}$ term derived in the proof of Theorem 12.

$$
\lambda < 1, \quad (\beta_{n+1})_i = \begin{cases} 
\frac{1}{2} - \lambda^{-1}(e^{-\tau \Delta} u_n)_i, & \text{if } (e^{-\tau \Delta} u_n)_i < \frac{1}{2} \lambda, \\
0, & \text{if } \frac{1}{2} \lambda \leq (e^{-\tau \Delta} u_n)_i < 1 - \frac{1}{2} \lambda, \\
- \frac{1}{2} + \lambda^{-1}(1 - (e^{-\tau \Delta} u_n)_i), & \text{if } (e^{-\tau \Delta} u_n)_i \geq 1 - \frac{1}{2} \lambda.
\end{cases} \quad (2.14a)
$$

$$
\lambda = 1, \quad \beta_{n+1} = \frac{1}{2} \mathbf{1} - e^{-\tau \Delta} u_n.
$$

Indeed, we can exaggerate this similarity further by noting that $e^{-\tau \Delta} = I - \tau \Delta + O(\tau^2)$ and recalling $\lambda := \tau / \varepsilon$. Therefore for $\tau \ll \varepsilon$,

$$
(\beta_{n+1})_i \approx \begin{cases} 
\frac{1}{2} + \varepsilon(\Delta u_n)_i - \lambda^{-1}(u_n)_i, & \text{if } (e^{-\tau \Delta} u_n)_i < \frac{1}{2} \lambda, \\
0, & \text{if } \frac{1}{2} \lambda \leq (e^{-\tau \Delta} u_n)_i < 1 - \frac{1}{2} \lambda, \\
- \frac{1}{2} + \varepsilon(\Delta u_n)_i + \lambda^{-1}(1 - (u_n)_i), & \text{if } (e^{-\tau \Delta} u_n)_i \geq 1 - \frac{1}{2} \lambda
\end{cases}
\tag{2.14b}
$$

with $O(\tau)$ error.

**Note.** Let us explore the cases when $\lambda \notin [0, 1]$. If $\lambda \geq 1$ then the function $f_i(x) := \lambda x(1 - x) + (x - (e^{-\tau \Delta} u_n)_i)^2$ is concave, so is minimised on the boundary. Thus the $\lambda(u, 1 - u)_\nu$ term in (2.4) equals zero, and minimising the norm term gives

$$
(u_{n+1})_i = \begin{cases} 
0, & \text{if } (e^{-\tau \Delta} u_n)_i < \frac{1}{2} \\
1, & \text{if } (e^{-\tau \Delta} u_n)_i > \frac{1}{2}
\end{cases} \quad (2.15)
$$

and underdetermined when $(e^{-\tau \Delta} u_n)_i = 1/2$. Therefore the variational problem yields the MBO solution even for $\lambda > 1$.

However, solutions to (2.8b) are no longer necessarily solutions to (2.4) for $\lambda > 1$. Let $\lambda = 1 + \delta$, then (2.8b) becomes

$$
- \delta(u_{n+1})_i - (e^{-\tau \Delta} u_n)_i + \frac{1}{2} + \frac{1}{2} \delta = (1 + \delta)(\beta_{n+1})_i. \quad (2.16)
$$

This has admisible solution (i.e. there is a corresponding $\beta_{n+1} \in B(u_{n+1})$) with

$$
(u_{n+1})_i = 0 \quad \text{if and only if} \quad (e^{-\tau \Delta} u_n)_i \in \left[ 0, \frac{1}{2}(1 + \delta) \right],
$$

$$
(u_{n+1})_i = \frac{1}{2} + \frac{1}{2} \delta^{-1} - \delta^{-1}(e^{-\tau \Delta} u_n)_i \quad \text{if and only if} \quad (e^{-\tau \Delta} u_n)_i \in \left[ \frac{1}{2}(1 - \delta), \frac{1}{2}(1 + \delta) \right],
$$

$$
(u_{n+1})_i = 1 \quad \text{if and only if} \quad (e^{-\tau \Delta} u_n)_i \in \left[ \frac{1}{2}(1 - \delta), 1 \right].
$$

16
Hence for $\lambda > 1$ when $(e^{-\tau \Delta_u} u)_i \in \left[ \frac{1}{2} (1 - \delta), \frac{1}{2} (1 + \delta) \right] \setminus \left\{ \frac{1}{2} \right\}$ for any $i \in V$ then (2.8b) has solutions that are not solutions to (2.9). However, the solutions to (2.9) remain solutions to (2.8b).

Finally we consider $\lambda < 0$, though this regime has less obvious meaning. By the same argument as for $0 \leq \lambda < 1$ we get that (2.9) has unique solution

$$u := \left( 1 - \frac{\lambda}{2} \right) - \frac{\lambda}{1 - \lambda} (e^{-\tau \Delta_u} u_n - \frac{\lambda}{2}) \in V_{(0,1)}$$

and so

$$\beta = \lambda^{-1} \left( (1 - \lambda)u - e^{-\tau \Delta_u} u_n + \frac{\lambda}{2} \right) = 0 \in B(u)$$

since $u \in V_{(0,1)}$, so $u$ solves (2.8b). Now, if $v$ solves (2.8b) then

$$v_i \in (0,1) \Rightarrow \beta_i = 0 \Rightarrow v_i = \frac{(e^{-\tau \Delta_u} u_n)_i - \lambda/2}{1 - \lambda},$$

$$v_i = 0 \Rightarrow \beta_i = \frac{1}{2} - \frac{1}{\lambda^{-1}} (e^{-\tau \Delta_u} u_n)_i \geq \frac{1}{2},$$

$$v_i = 1 \Rightarrow \beta_i = \frac{1}{2} + \lambda^{-1} (1 - (e^{-\tau \Delta_u} u_n)_i) \leq -\frac{1}{2}.$$ 

Hence (2.8b) has as solution any $v \in V$ obeying $v_i \in \{0, u_i, 1\}$ for all $i \in V$.

### 2.2 A Lyapunov functional for the semi-discrete flow

In [1, Proposition 4.6] it was proved that the strictly concave functional

$$J(u) := \langle 1 - u, e^{-\tau \Delta_u} u \rangle_V$$

has first variation at $u$

$$L_u(v) := \langle v, 1 - 2e^{-\tau \Delta_u} u \rangle_V$$

and hence is monotonically decreasing along MBO trajectories. Performing a calculation similar to that of (2.10), we find the functional in (2.9) is equivalent to

$$F_{u_n}(u) := L_{u_n}(u) + (\lambda - 1)\langle u, 1 - u \rangle_V.$$ 

We can therefore deduce a Lyapunov functional for the semi-discrete flow. Using a similar approach to Bertozzi and Luo’s analysis of semi-implicit graph Allen–Cahn in [25] we furthermore prove results about long-time behaviour of the semi-discrete flow.

**Theorem 14.** When $0 \leq \lambda \leq 1$ the functional (on $V_{[0,1]}$)

$$H(u) := J(u) + (\lambda - 1)\langle u, 1 - u \rangle_V = \lambda \langle u, 1 - u \rangle_V + \langle u, (I - e^{-\tau \Delta}) u \rangle_V$$

(2.17)
is non-negative, and furthermore the functional is a Lyapunov functional for \( (2.8b) \), i.e. \( H(u_{n+1}) \leq H(u_n) \) with equality if and only if \( u_{n+1} = u_n \) for \( u_{n+1} \) defined by \( (2.8b) \). In particular, we have that

\[
H(u_n) - H(u_{n+1}) \geq (1 - \lambda) \|u_{n+1} - u_n\|_V^2 .
\] (2.18)

Proof. Note that \( I - e^{-\tau \Delta} \) has eigenvalues \( 1 - e^{-\tau \lambda_k} \geq 0 \), since \( \lambda_k \) the eigenvalues of \( \Delta \) are non-negative, and so \( \langle u, (I - e^{-\tau \Delta}) u \rangle_V \geq 0 \). Since \( u \in V_{[0,1]} \) it follows that \( H(u) \geq 0 \).

Next by the concavity of \( J \) and linearity of \( L_{u_n} \) we have:

\[
H(u_n) - H(u_{n+1}) = J(u_n) - J(u_{n+1}) + (1 - \lambda) \langle u_{n+1}, 1 - u_{n+1} \rangle_V - (1 - \lambda) \langle u_n, 1 - u_n \rangle_V
\geq L_{u_n}(u_n - u_{n+1}) + (1 - \lambda) \langle u_{n+1}, 1 - u_{n+1} \rangle_V - (1 - \lambda) \langle u_n, 1 - u_n \rangle_V (*)
\]

with equality if and only if \( u_{n+1} = u_n \) as the concavity of \( J \) is strict. Finally, we can continue the above calculation

\[
(*) = \langle u_n - u_{n+1}, 1 - 2e^{-\tau \Delta} u_n \rangle_V + (1 - \lambda) \langle u_{n+1}, 1 - u_{n+1} \rangle_V - (1 - \lambda) \langle u_n, 1 - u_n \rangle_V
= \langle u_n - u_{n+1}, 1 - 2e^{-\tau \Delta} u_n \rangle_V + (1 - \lambda)(\langle u_{n+1} - u_n, 1 \rangle_V + \langle u_n, u_n \rangle_V - \langle u_{n+1}, u_{n+1} \rangle_V)
= \langle u_n - u_{n+1}, \lambda - 2e^{-\tau \Delta} u_n + (1 - \lambda)u_{n+1} + (1 - \lambda)u_n \rangle_V
= \langle u_n - u_{n+1}, 2\lambda \beta_{n+1} + (1 - \lambda)(u_n - u_{n+1}) \rangle_V \quad \text{by \( (2.8b) \)}, \quad \text{recall } \beta_{n+1} \in B(u_{n+1})
\geq (1 - \lambda) \|u_{n+1} - u_n\|_V^2
\]

where the final line follows from \( (2.3) \) as in the proof of Theorem 12. \( \square \)

Corollary 15. For \( 0 \leq \lambda \leq 1 \), we have that for the sequence \( u_n \) given by \( (2.8b) \)

\[
\sum_{n=0}^\infty \|u_{n+1} - u_n\|_V^2 < \infty
\]

and therefore in particular

\[
\lim_{n \to \infty} \|u_{n+1} - u_n\|_V = 0.
\]

Proof. If \( \lambda = 1 \) the result follows directly from Theorem 3 and the fact that MBO trajectories are eventually constant [1, Proposition 4.6]. If \( 0 \leq \lambda < 1 \) then by the non-negativity of \( H \) and \( (2.18) \) we have

\[
(1 - \lambda) \sum_{n=0}^N \|u_{n+1} - u_n\|_V^2 \leq H(u_0) - H(u_{N+1}) \leq H(u_0)
\]

so result follows by taking \( N \to \infty \). \( \square \)

\^Since \( e^{-\tau \Delta} \) is self-adjoint and \( e^{-\tau \Delta} 1 = 1 \), \( J(u) = \langle 1, u \rangle_V - \langle u, e^{-\tau \Delta} u \rangle_V \) and so it follows that \( J(u) + (\lambda - 1)\langle u, 1 - u \rangle_V = \langle 1, u \rangle_V - \langle u, e^{-\tau \Delta} u \rangle_V + \lambda \langle u, 1 - u \rangle_V - \langle 1, u \rangle_V + \langle u, u \rangle_V \).
Proposition 16. The Lyapunov functional has Hilbert space gradient (for $u \in V_{(0,1)}$)

$$\nabla_V H(u) = \lambda 1 - 2e^{-\tau \Delta} u + 2(1-\lambda)u \quad (2.19)$$

and therefore:

i. For the sequence $u_n \in V_{(0,1)}$ given by (2.8b)

$$\nabla_V H(u_n) = 2\lambda \beta_{n+1} + 2(1-\lambda) (u_n - u_{n+1}). \quad (2.20)$$

ii. Let $E$ denote the eigenspace of $\Delta$ with eigenvalue $-\tau^{-1} \log(1-\lambda)$ (i.e. the eigenspace of $e^{-\tau \Delta}$ with eigenvalue $1 - \lambda$) or $\{0\}$ if there is no such eigenvalue. Then if $u \in V_{(0,1)}$, it follows that $\nabla_V H(u) = 0$ (i.e. $u$ is a critical point of $H$) if and only if $u \in \left(\frac{1}{2}1 + E\right) \cap V_{(0,1)}$.

Proof. It is straightforward to check that

$$\langle \nabla_V H(u), v \rangle _V := \lim_{t \to 0} \frac{H(u + tv) - H(u)}{t} = \langle 1 - 2e^{-\tau \Delta} u, v \rangle _V + (\lambda - 1) \langle 1 - 2u, v \rangle _V$$

and therefore

$$\nabla_V H(u) = 1 - 2e^{-\tau \Delta} u + (\lambda - 1)(1 - 2u) = \lambda 1 - 2e^{-\tau \Delta} u + 2(1-\lambda)u.$$ 

i. From (2.8b) we have

$$\lambda 1 - 2e^{-\tau \Delta} u_n = 2\lambda \beta_{n+1} - 2(1-\lambda)u_{n+1}$$

so (2.20) follows by substituting into (2.19).

ii. Let $A := 2e^{-\tau \Delta} + 2(\lambda - 1)I$. Then by (2.19), $u \in V_{(0,1)}$ satisfies $\nabla_V H(u) = 0$ if and only if $Au = \lambda 1$. Note that $\frac{1}{2}A1 = 1 + \lambda 1 - 1 = \lambda 1$. Therefore $Au = \lambda 1$ if and only if $u \in \frac{1}{2}1 + \ker A$. But $E$ is by definition the kernel of $A$. Therefore $u \in V_{(0,1)}$ satisfies $\nabla_V H(u) = 0$ if and only if $u \in \left(\frac{1}{2}1 + E\right) \cap V_{(0,1)}$.

Note. Considering the quadratic terms one can observe that

$$H\left(\frac{1}{2}1 + \eta\right) = H\left(\frac{1}{2}1\right) - \langle \eta, e^{-\tau \Delta} \eta \rangle _V - (1 - \lambda) \langle \eta, \eta \rangle _V$$

so, for $0 \leq \lambda \leq 1$, $u = \frac{1}{2}1$ is a global maximiser of $H$ if and only if

$$P := e^{-\tau \Delta} - (1 - \lambda)I$$

is positive semi-definite. For $\lambda_k$ the eigenvalues of $\Delta$, we desire $P$ have corresponding $k$-th eigenvalues:

$$e^{-\tau \lambda_k} - (1 - \lambda) \geq 0$$

i.e.

$$\tau \varepsilon^{-1} \geq 1 - e^{-\tau \lambda_k}.$$
Therefore we have $\lambda \leq 1$ and $u = \frac{1}{2}1$ as a global maximiser of $H$ if and only if

$$\varepsilon \in \left[ \tau, \frac{\tau}{1 - e^{-\tau||\Delta||}} \right].$$

Furthermore, for $\xi \in \mathcal{E}$ we have $e^{-\tau\Delta}\xi = (1 - \lambda)\xi$, and so

$$H\left(\frac{1}{2}1 + \xi\right) = H\left(\frac{1}{2}1\right) - (\langle \xi, e^{-\tau\Delta}\xi \rangle_{\mathcal{V}} - (1 - \lambda)(\xi, \xi)_{\mathcal{V}}) = H\left(\frac{1}{2}1\right).$$

Therefore $\left(\frac{1}{2}1 + \mathcal{E}\right) \cap \mathcal{V}_{(0,1)}$ are all global maxima in this case.

Since $H(u_n)$ is monotonically decreasing and bounded below, it follows that $H(u_n) \downarrow H_\infty$ for some $H_\infty \geq 0$. Furthermore, since the sequence $u_n \in \mathcal{V}_{[0,1]}$ is compact, there exist subsequences $u_{n_k} \to u^* \in \mathcal{V}_{[0,1]}$ with $H(u^*) = H_\infty$. Unfortunately, just like for ACE with the standard quartic potential, we are unable to infer convergence of the whole sequence from these facts. However, by the same argument as in [25], if the sequence $u_n$ has finitely many accumulation points then in fact the whole sequence converges. Notably, if $u^* \in \mathcal{V}_{(0,1)}$ is an accumulation point of $u_n$ then by Corollary 15 and (2.20) we have $\nabla_Y H(u^*) = 0$, and so $H(u^*) = H(\frac{1}{2}1)$. Hence if $H(u_0) < H(\frac{1}{2}1)$, then no accumulation points of $u_n$ lie in $\mathcal{V}_{(0,1)}$.

We can suggestively generalise the Lyapunov functional to all of $\mathcal{V}$ by defining

$$H(u) := \langle u, u - e^{-\tau\Delta}u \rangle_{\mathcal{V}} + 2\lambda\langle W \circ u, 1 \rangle_{\mathcal{V}} \geq 0 \quad (2.21)$$

which we can rewrite as:

$$H(u) = \tau\langle u, \Delta u \rangle_{\mathcal{V}} + 2\lambda\langle W \circ u, 1 \rangle_{\mathcal{V}} - \tau^2\langle u, Q_\tau u \rangle_{\mathcal{V}}$$

where $Q_\tau$ satisfies $e^{-\tau\Delta} = I - \tau\Delta + \tau^2Q_\tau$ and therefore

$$\frac{1}{2\tau}H(u) = GL(u) - \frac{1}{2}\tau\langle u, Q_\tau u \rangle_{\mathcal{V}}. \quad (2.22)$$

### 2.3 The semi-discrete flow and a time-splitting of the ACE flow

Theorem 3 shows that MBO is Allen–Cahn approximated via our semi-discrete scheme (2.8). This scheme can also be related to the following two-step time-splitting of the continuous ACE flow. We fix $\tau > 0$ and take $\tilde{u}_0 \in \mathcal{V}_{[0,1]}$, then iteratively apply the steps:

1. Take $\tilde{u}_n$ from the previous iteration.

2. **(Diffusion step)** Define $v := e^{-\tau\Delta}\tilde{u}_n$ the heat equation solution with $v(0) = \tilde{u}_n$ and define $v_n := v(\tau)$.

\[8\] The link between MBO and this time-splitting was also noted in the continuum case in [20].
3. (Reaction step) Define \( U_n \in H^1((0, \tau); \mathcal{V}) \cap C^0([0, \tau]; \mathcal{V}) \cap \mathcal{V}_{[0, \tau]} \) obeying
\[
\frac{d(U_n)_i}{dt} = \varepsilon^{-1} \left( (U_n(t))_i - \frac{1}{2} \right) + \varepsilon^{-1} \beta(t) , \quad U_n(0) = v_n = e^{-\tau \Delta} \tilde{u}_n \tag{2.23}
\]
where \( \beta \in \mathcal{B}(U_n) \) as defined in (2.23).

4. Finally, define \( \bar{u}_{n+1} := U_n(\tau) \) and define \( U(t) := U_n(t-n\tau) \) for \( t \in (n\tau, (n+1)\tau] \).

**Note.** If \( u_n = \bar{u}_n \) we can therefore describe the semi-discrete update \( u_{n+1} \approx \bar{u}_{n+1} \) as the implicit Euler approximation:
\[
\frac{(u_{n+1})_i - (v_n)_i}{\tau} \in -\frac{1}{\varepsilon} \partial W((u_{n+1})_i).
\]
That is, we get the semi-discrete update by dissecting the flow in (2.23) into a diffusion for time \( \tau \) followed by a gradient flow of \( W \), again for time \( \tau \). Then, we use the exact solution for the former and approximate the latter by an implicit Euler scheme.

Consider the ODE in (2.23). We prove that minimisers of \( W \) are stationary for this gradient flow of \( W \).

**Proposition 17.** Let \( x : [0, T] \to [0, 1] \) with \( x \in H^1((0, T); \mathbb{R}) \cap C^0([0, T]; \mathbb{R}) \) solve
\[
\varepsilon \frac{dx}{dt} = x(t) - \frac{1}{2} + \beta(t)
\]
with \( x(0) \in \{0, 1\} \) and \( \beta(t) \in -\partial I_{[0,1]}(x(t)) \). Then \( x(t) = x(0) \) for all \( t \in [0, T] \).

**Proof.** By symmetry, it suffices to prove the case when \( x(0) = 0 \). Let \( T_0 \) equal the first time at which \( x(t) = 1/2 \), or let \( T_0 = T \) if there is no such time. Therefore for \( t \in [0, T_0] \) we have \( 0 \leq x(t) \leq 1/2 \). Furthermore, note that for any \( t \in [0, T] \), \( x(t) \beta(t) \leq 0 \). Hence, for all \( t \in [0, T_0] \),
\[
\frac{1}{2} \varepsilon \frac{d}{dt} (x(t)^2) = \varepsilon x(t) \frac{dx}{dt} = x(t) \left( x(t) - \frac{1}{2} \right) + \beta(t) x(t) \leq 0
\]
and thus \( x(t)^2 \leq x(0)^2 = 0 \). Therefore for all \( t \in [0, T_0] \), \( x(t) = 0 \). Since \( x(T_0) \neq 1/2 \), we must have \( T_0 = T \), completing the proof. \( \square \)

We therefore solve (2.23). Let \( T_i \) denote the first time \((U_n(t))_i \in \{0, 1\} \). Then for \( t < T_i \), \( \beta_i(t) = 0 \), so by separation of variables we have
\[
(U_n(t))_i - \frac{1}{2} = \left((v_n)_i - \frac{1}{2}\right) e^{t/\varepsilon}
\]
and for \( t \geq T_i \), \( (U_n(t))_i = (U_n(T_i))_i \) by the above proposition. Thus
\[
(\bar{u}_{n+1})_i = \begin{cases} \frac{1}{2} + e^\lambda ((v_n)_i - \frac{1}{2}), & \text{if } e^\lambda |(v_n)_i - \frac{1}{2}| < \frac{1}{2} \\ \Theta ((v_n)_i - \frac{1}{2}), & \text{otherwise} \end{cases}
\]
where $\Theta$ is the Heaviside step function. We compare to the semi-discrete update $u_{n+1}$. If $u_n = \tilde{u}_n$ then by (2.21) this obeys:

$$(u_{n+1})_i = \begin{cases} \frac{1}{2} + \frac{1}{1-\lambda} ((v_n)_i - \frac{1}{2}), & \text{if } \lambda < 1 \text{ and } \frac{1}{1-\lambda} |(v_n)_i - \frac{1}{2}| < \frac{1}{2}, \\ \Theta ((v_n)_i - \frac{1}{2}), & \text{otherwise.} \end{cases}$$

Note a basic result: for $0 \leq \lambda \leq 1$, $e^\lambda \leq (1-\lambda)^{-1}$. Thus we have that $|(\tilde{u}_{n+1})_i - 1/2| \leq |(u_{n+1})_i - 1/2|$, and so after one step the semi-discrete approximation is always at least as binary as the time-splitting approximation.

We compare to the ACE flow. Note the simplest case, when $u$ remains in $V_{(0,1)}$ so $\beta$ is constantly 0. Then substituting into (2.27):

$$u(t) = \frac{1}{2} 1 + e^{t/\varepsilon} \left( e^{-t\Delta} u(0) - \frac{1}{2} 1 \right).$$

(2.25)

Note that in this case we also have for $t \in [0, \tau]$ and with $\bar{u}_0 = u(0)$,

$$U_0(t) = \frac{1}{2} 1 + e^{t/\varepsilon} \left( e^{-\tau\Delta} u(0) - \frac{1}{2} 1 \right)$$

so $\bar{u}_1 = u(\tau)$, and so by induction $\bar{u}_n = u(n\tau)$, i.e. the time-splitting agrees with ACE after each time-step of $\tau$. Note also that by [1] Lemma 2.6(d), for $t \in (n\tau, (n+1)\tau)$, we have $\|e^{-(t-n\tau)\Delta} u(n\tau) - \frac{1}{2} 1\|_\infty \geq ||e^{-t\Delta} u(n\tau) - \frac{1}{2} 1||_\infty$, and so

$$\left\|u(t) - \frac{1}{2} 1\right\|_\infty \geq \left\|U(t) - \frac{1}{2} 1\right\|_\infty$$

(2.26)

with strict inequality when $u(n\tau)$ is not a constant vector and $t \in (n\tau, (n+1)\tau)$.

However in general $u(t)$ will eventually have a vertex take value 0 or 1, and indeed if $\langle u(0), 1 \rangle_V \neq \frac{1}{2} \langle 1, 1 \rangle_V$ then (2.26) yields an upper bound on how long until this occurs. Furthermore, and importantly, by (2.20) it will do so before this happens for $U(t)$.

### 2.4 Convergence of the semi-discrete scheme to continuous-time ACE

We consider the behaviour of the semi-discrete solution as $\tau, \lambda \downarrow 0$ ($\varepsilon$ fixed). Drawing inspiration from the form of the ACE solution in (2.7), we show that the semi-discrete solutions converge pointwise (in a sense we make explicit) to the ACE solution.

Recall the semi-discrete scheme is defined as

$$(1 - \lambda)u_{n+1} = e^{-\tau\Delta} u_n - \frac{\lambda}{2} 1 + \lambda \beta_{n+1}$$

and note that $\beta_{n+1} \in V_{[-1/2,1/2]}$ by (2.14). Iterating, we get the general term:

By [1] Lemma 2.6(c)], for all $i \in V$ $(e^{-\tau\Delta} u(0))_i - \frac{1}{2} \rightarrow \langle u(0) - \frac{1}{2} 1, 1 \rangle_V (1, 1)_V^{-1} =: \alpha$ as $t \rightarrow \infty$. From that lemma, we have an explicit $t_0$ such that for all $t > t_0$, $|(e^{-\tau\Delta} u(0))_i - \frac{1}{2} | > \frac{1}{2} |\alpha|$. Then by (2.20) for $t > \max\{t_0, -\varepsilon \log |\alpha|\}$ we have $u_i(t) \notin (0,1)$.
Proposition 18. For $\lambda := \tau/\varepsilon \in [0, 1)$, the semi-discrete solution has $n$-th iterate

$$u_n = \frac{1}{2} + (1 - \lambda)^{-n} e^{-n\tau\Delta} \left( u_0 - \frac{1}{2} \right) + \frac{\lambda}{1 - \lambda} \sum_{k=1}^{n} (1 - \lambda)^{-(n-k)} e^{-(n-k)\tau\Delta} \beta_k. \quad (2.27)$$

Understanding $O$ to refer to the limit of $\tau \downarrow 0$ and $n \to \infty$ with $n\tau - t \in [0, \tau)$ for some fixed $t \geq 0$ and for fixed $\varepsilon > 0$, we therefore have

$$u_n = \frac{1}{2} e^{n\lambda} e^{-n\tau\Delta} \left( u_0 - \frac{1}{2} \right) + \frac{\lambda}{1 - \lambda} \sum_{k=1}^{n} (1 - \lambda)^{-(n-k)} e^{-(n-k)\tau\Delta} \beta_k + O(\tau). \quad (2.28)$$

Proof. We proceed by induction. The $n = 0$ base case is trivial. Then inducting:

$$u_{n+1} = (1 - \lambda)^{-1} e^{-\tau\Delta} u_n - \frac{1}{2} \frac{\lambda}{1 - \lambda} 1 + \frac{\lambda}{1 - \lambda} \beta_{n+1}$$

$$= (1 - \lambda)^{-1} e^{-\tau\Delta} \left[ \frac{1}{2} + (1 - \lambda)^{-n} e^{-n\tau\Delta} \left( u_0 - \frac{1}{2} \right) + \frac{\lambda}{1 - \lambda} \sum_{k=1}^{n} (1 - \lambda)^{-(n-k)} e^{-(n-k)\tau\Delta} \beta_k \right]$$

$$- \frac{1}{2} \frac{\lambda}{1 - \lambda} 1 + \frac{\lambda}{1 - \lambda} \beta_{n+1}$$

$$= \frac{1}{2} \left( \frac{1}{1 - \lambda} - \frac{\lambda}{1 - \lambda} \right) 1 + (1 - \lambda)^{-(n+1)} e^{-(n+1)\tau\Delta} \left( u_0 - \frac{1}{2} \right)$$

$$+ \frac{\lambda}{1 - \lambda} \sum_{k=1}^{n} (1 - \lambda)^{-(n-k+1)} e^{-(n-k+1)\tau\Delta} \beta_k + \frac{\lambda}{1 - \lambda} \beta_{n+1}$$

$$= \frac{1}{2} 1 + (1 - \lambda)^{-(n+1)} e^{-(n+1)\tau\Delta} \left( u_0 - \frac{1}{2} \right) + \frac{\lambda}{1 - \lambda} \sum_{k=1}^{n+1} (1 - \lambda)^{-(n-k+1)} e^{-(n-k+1)\tau\Delta} \beta_k$$

completing the induction.

Next, consider (with $n\tau = t + O(\tau)$ and $n\lambda = t/\varepsilon + O(\tau)$ for fixed $t$) the difference:
For any sequence $\tau_n \downarrow 0$, $n \to \infty$ with $n \tau \to t$ for some fixed $t$ and $\tau < \varepsilon$. As a prelude to this, for reasons that will soon become clear, we define the piecewise constant function $z_\tau : [0, \infty) \to V$

$$z_\tau(s) := \begin{cases} e^{-s/\varepsilon} e^{-s\Delta \beta_1}, & 0 \leq s \leq \tau \\ e^{-k\tau/\varepsilon} e^{k\tau \Delta \beta_k}, & (k - 1)\tau < s \leq k\tau \text{ for } k \in \mathbb{N} \end{cases}$$

and the function

$$\gamma_\tau(s) := e^{s/\varepsilon} e^{-s\Delta z_\tau(s)} = \begin{cases} e^{-(\tau-s)/\varepsilon} e^{(\tau-s)\Delta \beta_1}, & 0 \leq s \leq \tau \\ e^{-(k\tau-s)/\varepsilon} e^{(k\tau-s)\Delta \beta_k}, & (k - 1)\tau < s \leq k\tau \text{ for } k \in \mathbb{N} \end{cases}$$

(note that for bookkeeping we introduce the superscript $[\tau]$ to keep track of the time-step governing a particular sequence of $u_n$ and $\beta_n$). We note an important convergence:

**Proposition 19.** For any sequence $\tau_n^{(0)} \downarrow 0$ with $\tau_n^{(0)} < \varepsilon$ for all $n$, there exists a $z : [0, \infty) \to V$ and a subsequence $\tau_n$ such that $z_{\tau_n}$ converges weakly to $z$ in $L^2_{\text{loc}}$.\footnote{Follows from triangle inequality since $\lambda(1-\lambda)^{-r} - \lambda e^{\varepsilon} \geq 0$ as $e^{-\lambda r/(r+1)} \geq 1 - \lambda r/(r+1) \geq 1 - \lambda.$}
Proof. For \( N \in \mathbb{N} \), consider \( z_{\tau}|_{[0,N]} \). As the \( \beta_k^{[\tau]} \in \mathcal{V}_{-1/2,1/2} \) for all \( k \) and \( \tau \), we have for all \( s \in [0,N] \) and \( 0 \leq \tau < \varepsilon \):

\[
||z_{\tau}(s)||_V \leq \sup_{s' \in [0,N+\varepsilon]} \left| e^{-s'(\frac{1}{2}\Delta)} \right| \cdot \frac{1}{2} ||1||_V \leq \max\left\{ 1, e^{(N+\varepsilon)(||\Delta||_{-1}^{-1})} \right\} \cdot \frac{1}{2} ||1||_V. \]

Therefore the \( z_{\tau}|_{[0,N]} \) are uniformly bounded for \( 0 \leq \tau < \varepsilon \), and hence they lie in a closed ball in \( L^2(\mathcal{V}) \). By the Banach–Alaoglu theorem on a Hilbert space, this ball is weak-compact. We proceed by a “local-to-global” diagonal argument: choose any \( \tau_n^{(0)} \rightarrow 0 \) with \( \tau_n^{(0)} < \varepsilon \) for all \( n \). Then, for \( N = 1 \): by compactness we may choose a subsequence \( \tau^{(1)} \) of \( \tau^{(0)} \) such that \( z_{\tau_n^{(1)}} \) converges weakly to \( z \) on \( [0,1] \). From \( N \) to \( N + 1 \): by compactness we may choose a subsequence \( \tau^{(N+1)} \) of \( \tau^{(N)} \) such that \( z_{\tau_n^{(N+1)}} \) converges weakly to \( z \) on \( [0,N+1] \). Finally, define \( \tau_n := \tau_n^{(n)} \). Then for all bounded \( T \subseteq [0,\infty) \), we have \( T \subseteq [0,M] \) for some \( M \in \mathbb{N} \) and hence \( z_{\tau_n}|_{T} \) is eventually a subsequence of \( z_{\tau_n}|_{T} \), so converges weakly to \( z|_{T} \). \( \square \)

**Corollary 20.** From \( z_{\tau_n} \rightarrow z \) in \( L^2_{loc} \) we infer:

A. \( \gamma_{\tau_n} \rightarrow \gamma \) (in \( L^2_{loc} \)) where \( \gamma(s) := e^{s/\varepsilon}e^{-s\Delta}z \).

B. For all \( t \geq 0 \),

\[
\int_0^t z_{\tau_n}(s) \, ds \rightarrow \int_0^t z(s) \, ds.
\]

C. Replacing \( \tau_n \) by a subsequence, we have strong convergence of the Cesàro sums, i.e. for all bounded \( T \subseteq [0,\infty) \)

\[
\frac{1}{N} \sum_{n=1}^{N} z_{\tau_n} \rightarrow z \quad \frac{1}{N} \sum_{n=1}^{N} \gamma_{\tau_n} \rightarrow \gamma \quad \text{in } L^2(T;\mathcal{V})
\]

as \( N \rightarrow \infty \).

**Proof.** (A) follows since \( f \mapsto e^{s/\varepsilon}e^{-s\Delta}f \) is a continuous self-adjoint map on \( L^2(T;\mathcal{V}) \) for \( T \) bounded. Hence for all \( f \in L^2(T;\mathcal{V}) \),

\[
(\gamma_{\tau_n}, f)_{s \in T} = (z_{\tau_n}, e^{s/\varepsilon}e^{-s\Delta}f)_{s \in T} \rightarrow (z, e^{s/\varepsilon}e^{-s\Delta}f)_{s \in T} = (\gamma, f)_{s \in T}.
\]

(B) is a direct consequence of weak convergence. (C) follows by the Banach–Saks theorem and a “local-to-global” diagonal argument as in the above proof. \( \square \)

**Note.** Since \( L^2(T;\mathcal{V}) \) is separable, the theorem of Banach–Alaoglu can be stated constructively, as can the theorem of Banach–Saks. Hence these proofs can be made constructive to yield an explicit \( \tau_n \). We omit the details of this construction.\footnote{Here we have used that for \( s \leq N \) the corresponding \( k\tau \) in the exponent of \( z_{\tau}(s) \) is less than \( N + \tau \), and that \( ||e^{-s'(\frac{1}{2}\Delta)}|| = e^{s'(|\Delta|^{-1})} \) is maximised at the endpoints of \( [0,N+\varepsilon] \).}
We now return to the question of convergence of the semi-disc rete iterates. Taking $\tau$ to zero along the sequence $\tau_n$, we define for all $t \geq 0$ the continuous-time function

$$\hat{u}(t) := \lim_{n \to \infty, m = \lceil t/\tau_n \rceil} u_{m}^{[\tau_n]}.$$  \hfill (2.29)

Therefore by (2.28)

$$\hat{u}(t) = \frac{1}{2} 1 + \lim_{n \to \infty} e^{m\tau_n/\varepsilon} e^{-m\tau_n \Delta} \left( u_0 - \frac{1}{2} 1 \right) + \frac{1}{\varepsilon} e^{m\tau_n/\varepsilon} e^{-m\tau_n \Delta} \sum_{k=1}^{m} e^{-k\tau_n/\varepsilon} e^{k\tau_n \Delta} \beta_{k}^{[\tau_n]}$$

and by rewriting the sum term via the definition of $z_{\tau_n}$:

$$\hat{u}(t) = \frac{1}{2} 1 + \lim_{n \to \infty} e^{m\tau_n/\varepsilon} e^{-m\tau_n \Delta} \left( u_0 - \frac{1}{2} 1 \right) + \frac{1}{\varepsilon} e^{m\tau_n/\varepsilon} e^{-m\tau_n \Delta} \int_{0}^{m\tau_n} z_{\tau_n}(s) \, ds.$$  \hfill (2.30)

Then to prove global convergence we must show the following desiderata:

i. $\hat{u}(t)$ exists for all $t \geq 0$,

ii. $\hat{u}(t) \in V_{[0,1]}$ for all $t \geq 0$,

iii. $\hat{u}$ is continuous and $H^1_{loc}$,

iv. $\hat{u}(t)$ is a solution to ACE.

Towards (i), let $A := \varepsilon^{-1} I - \Delta$ and $e_n := m\tau_n - t \in [0, \tau_n)$. Then

$$e^{m\tau_n/\varepsilon} e^{-m\tau_n \Delta} = e^{(t+e_n)A} = e^{tA} (I + O(e_n)) = e^{tA} + O(\tau_n)$$

and so

$$\hat{u}(t) = \frac{1}{2} 1 + \lim_{n \to \infty} e^{tA} \left( u_0 - \frac{1}{2} 1 \right) + \frac{1}{\varepsilon} (e^{tA} + O(\tau_n)) \left( \int_{0}^{t} z_{\tau_n}(s) \, ds + \int_{t}^{t+e_n} z_{\tau_n}(s) \, ds \right).$$

Hence since the $z_{\tau_n}$ are uniformly bounded on $[0, t+\max_{n} e_n]$ and by Corollary [20]B

$$\hat{u}(t) = \frac{1}{2} 1 + e^{t/\varepsilon} e^{-t\Delta} \left( u_0 - \frac{1}{2} 1 \right) + \frac{1}{\varepsilon} e^{t/\varepsilon} e^{-t\Delta} \int_{0}^{t} z(s) \, ds.$$  \hfill (2.30)

To show (ii), we note simply that $\hat{u}(t)$ is a limit of semi-discrete iterates, each of which we know lies in $V_{[0,1]}$.

To show (iii), we note that $\int_{0}^{t} z(s) \, ds$ is continuous since $z$ is a weak limit of locally bounded functions, so is locally bounded, and hence $\hat{u}$ is continuous by (2.24). Next, by (ii) $\hat{u}$ is bounded so is locally $L^2$. It is easy to check that $\hat{u}$ has weak derivative

$$\frac{d\hat{u}}{dt} = \left( \frac{1}{\varepsilon} I - \Delta \right) \left( u_0 - \frac{1}{2} 1 \right) + \frac{1}{\varepsilon} e^{t/\varepsilon} e^{-t\Delta} \left( z(t) + \left( \frac{1}{\varepsilon} I - \Delta \right) \int_{0}^{t} z(s) \, ds \right)$$

26
which is locally $L^2$ since $z$ is a weak limit of locally $L^2$ functions (so is locally $L^2$) and $\int_0^t z(s) \, ds$ is a pointwise limit of locally bounded functions, so is locally bounded.

Finally to show (iv), we recall we can rearrange the ACE ODE into

$$\frac{d}{dt} \left( e^{-t/\varepsilon} e^{t\Delta} \left( u(t) - \frac{1}{2} \right) \right) = e^{-t/\varepsilon} e^{t\Delta} \beta(t).$$

Inspection of (2.30) shows that to prove that $\hat{u}$ solves this ODE a.e. it suffices to check that $\gamma(t) \in B(\hat{u}(t))$ for a.e. $t \geq 0$. By Corollary 20(C) we have that on each bounded $T \subseteq [0,\infty)$, $\gamma$ is the $L^2(T;\mathcal{V})$ limit of

$$S_N := \frac{1}{N} \sum_{n=1}^{N} \gamma_{\tau_n}$$

as $N \to \infty$. As $L^2$ convergence implies a.e. pointwise convergence along a subsequence, by a “local-to-global” diagonal argument we have $N_k \to \infty$ such that for a.e. $t \geq 0$

$$\gamma(t) = \lim_{k \to \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} \gamma_{\tau_n}(t).$$

Now recall $m := \lceil t/\tau_n \rceil$ and $e_n := m\tau_n - t \in [0,\tau_n)$. Then

$$\gamma_{\tau_n}(t) = e^{-e_n A} \beta_{m}[\tau_n] = \beta_{m}[\tau_n] + O(e_n) = \beta_{m}[\tau_n] + O(\tau_n)$$

as the $\beta$ are uniformly bounded. Therefore for a.e. $t \geq 0$,

$$\gamma(t) = \lim_{k \to \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} \beta_{m}[\tau_n].$$

Recall that $u_{m}[\tau_n] \to \hat{u}(t)$ and $\beta_{m}[\tau_n] \in B(u_{m}[\tau_n])$. Suppose first that $\hat{u}_i(t) \in (0,1)$. Then we have some $M$ such that for all $n > M$, $(u_{m}[\tau_n])_i \in (0,1)$ and so $(\beta_{m}[\tau_n])_i = 0$. Hence

$$\gamma_i(t) = \lim_{k \to \infty} \frac{1}{N_k} \left( \sum_{n=1}^{M} (\beta_{m}[\tau_n])_i + \sum_{n=M+1}^{N_k} 0 \right) = 0$$

as desired. Next suppose $\hat{u}_i(t) = 0$. Then we have some $M$ such that for all $n > M$, $(u_{m}[\tau_n])_i \in [0,1)$ and so $(\beta_{m}[\tau_n])_i \geq 0$. Hence

$$\gamma_i(t) \geq \lim_{k \to \infty} \frac{1}{N_k} \left( \sum_{n=1}^{M} (\beta_{m}[\tau_n])_i + \sum_{n=M+1}^{N_k} 0 \right) = 0$$

as desired. Likewise for $\hat{u}_i(t) = 1$, $\gamma_i(t) \leq 0$. Hence we have $\gamma(t) \in B(\hat{u}(t))$.

In summary, we have the convergence result:
Theorem 21. For any given $u_0 \in V_{[0,1]}$, $\varepsilon > 0$ and $\tau_n \downarrow 0$, there exists a subsequence $\tau'_n$ of $\tau_n$ with $\tau'_n < \varepsilon$ for all $n$, such that along this subsequence the semi-discrete iterates $(u_{m_{\tau'_n}}, \beta_{m_{\tau'_n}})$ given by (2.23) with initial state $u_0$ converge to the ACE solution with initial condition $u_0$. That is, for each $t \geq 0$, as $n \to \infty$ and $m = \lfloor t/\tau'_n \rfloor$, $u_{m_{\tau'_n}} \to \hat{u}(t)$, and there is a sequence $N_k \to \infty$ (independent of $t$) such that for almost every $t \geq 0$, 
\[
\frac{1}{N_k} \sum_{n=1}^{N_k} \beta_{m_{\tau'_n}} \to \gamma(t), \text{ where } (\hat{u}, \gamma) \text{ is the solution to (2.5) with } \hat{u}(0) = u_0.
\]

Note. This argument also yields a separate proof of Theorem 7, i.e. we show existence of a solution to ACE as a limit of semi-discrete approximations.

Note. Given the uniqueness of ACE trajectories given by Theorem 8 (i.e. Corollaries 28, 30), we can employ a trivial, if lesser known, fact about topological spaces towards removing the need to pass to a subsequence. That is, we note the standard result:

Fact. If $(X, \rho)$ is a topological space and $x_n, x \in X$ are such that every subsequence of $x_n$ has a further subsequence converging to $x$, then $x_n \to x$.

Let $\tau_n \downarrow 0$ with $\tau_n < \varepsilon$ for all $n$: define the sequence $x_n := t \mapsto u_{\lfloor t/\tau_n \rfloor} \in (V_{[0,\infty)}, \rho)$ with $\rho$ the topology of pointwise convergence. Then by the above method, every subsequence $x_{n_k}$ has a subsequence converging to an ACE solution with initial condition $u_0$. By uniqueness, there is only one such solution $\hat{u}$, which we can take as the “$x$” in the fact. We therefore have that $x_n \to \hat{u}$ pointwise, without passing to a subsequence.

Likewise, since the corresponding $\gamma$ is unique up to a.e. equivalence, we have that $z_{\tau_n} \to z$ and $\gamma_{\tau_n} \to \gamma$ in $L^2_{\text{loc}}$ without passing to a subsequence.

To round out this section, we verify that the solution obtained this way is a decreasing flow of $G_{\epsilon}$ by considering the Lyapunov functional $H_{\tau}$ for the semi-discrete scheme defined in (2.17), and in doing so obtain a control on the behaviour of $G_{\epsilon}(\hat{u}(t))$.

We wish to prove that for $t > s \geq 0$,
\[
G_{\epsilon}(\hat{u}(t)) \leq G_{\epsilon}(\hat{u}(s)).
\]

Recalling (2.22), for $u \in \mathcal{V}_{[0,1]}$ we consider a scaling of the Lyapunov functional
\[
H_{\tau}(u) := \frac{1}{2\tau} H(u) = G_{\epsilon}(u) - \frac{1}{2} \tau \langle u, Q_{\tau} u \rangle_{\mathcal{V}}
\]
where $\tau^2 Q_{\tau} := e^{-\tau \Delta} - I + \tau \Delta$. Note that $Q_{\tau}$ has eigenvalues $\tau^{-2}(e^{-\tau \lambda_k} - 1 + \tau \lambda_k) \leq \frac{1}{2} \lambda_k^2$, for $\lambda_k$ the eigenvalues of $\Delta$, and so $|\langle u, Q_{\tau} u \rangle_{\mathcal{V}}| \leq \frac{1}{2} \|\Delta\|_2^2 \|u\|_{\mathcal{V}}^2 \leq \frac{1}{2} \|\Delta\|_2^2 \|1\|_{\mathcal{V}}^2$.

It follows that as $\tau \to 0$, $H_{\tau} \to G_{\epsilon}$ uniformly on $\mathcal{V}_{[0,1]}$. Furthermore

Proposition 22. Let $u_{\tau}, u \in \mathcal{V}_{[0,1]}$ satisfy $\|u_{\tau} - u\|_{\mathcal{V}} \to 0$ as $\tau \to 0$. Then it follows that $H_{\tau}(u_{\tau}) \to G_{\epsilon}(u)$.

Suppose $x_n \to x$. Then there exists $U \in \rho$ such that $x \in U$ and infinitely many $x_n \notin U$. Choose $x_{n_k}$ such that for all $k$, $x_{n_k} \notin U$. This subsequence has no further subsequence converging to $x$. \hfill $\square$
Proof. We note that it suffices to show that \( H_r(u_r) - H_r(u) \to 0 \), since

\[
H_r(u_r) - \text{GL}_\varepsilon(u) = H_r(u_r) - H_r(u) + H_r(u) - \text{GL}_\varepsilon(u).
\]

Considering (2.17) (and recalling that \( \lambda := \tau/\varepsilon \)) we get that

\[
H_r(u_r) - H_r(u) = \frac{1}{2} \left( u_r - u, \frac{1}{\varepsilon} (1 - u_r - u) + (\Delta - \tau Q_r)(u_r + u) \right)_\mathcal{V} \to 0
\]
since the latter term in the inner product is bounded uniformly in \( \tau \).

\[\square\]

**Theorem 23.** The ACE trajectory \( \hat{u} \) defined by (2.23) has \( \text{GL}_\varepsilon(\hat{u}(t)) \) monotonically decreasing in \( t \). More precisely: for all \( t > s \geq 0 \),

\[
\text{GL}_\varepsilon(\hat{u}(s)) - \text{GL}_\varepsilon(\hat{u}(t)) \geq \frac{1}{2(t-s)} ||\hat{u}(s) - \hat{u}(t)||_\mathcal{V}^2. \tag{2.31}
\]

**Proof.** Let \( t > s \geq 0 \) and \( m := \lceil s/\tau_n \rceil \) and \( \ell := \lceil t/\tau_n \rceil \). We note a simple fact about inner product spaces\(^{13}\) for all sequences \( v_n \in \mathcal{V} \),

\[
\sum_{n=1}^{N} ||v_n||^2_\mathcal{V} = \frac{1}{N} \left| \sum_{n=1}^{N} v_n \right|_\mathcal{V}^2 + \frac{1}{N} \sum_{k<n} ||v_n - v_k||^2_\mathcal{V} \geq \frac{1}{N} \left| \sum_{n=1}^{N} v_n \right|_\mathcal{V}^2. \tag{2.32}
\]

Now by (2.29), we have \( u_n^{[\tau_n]} \to \hat{u}(s) \) and \( u_k^{[\tau_n]} \to \hat{u}(t) \). It follows that:

\[
\text{GL}_\varepsilon(\hat{u}(s)) - \text{GL}_\varepsilon(\hat{u}(t)) = \lim_{n \to \infty} H_{\tau_n} \left( u_n^{[\tau_n]} \right) - H_{\tau_n} \left( u_k^{[\tau_n]} \right) \quad \text{by Proposition 22}
\]

\[
\geq \lim_{n \to \infty} \frac{1}{2\tau_n} \left( 1 - \frac{\tau_n}{\varepsilon} \right) \sum_{k=m}^{\ell-1} ||u_k^{[\tau_n]} - u_k^{[\tau_n]}||^2_\mathcal{V} \quad \text{by (2.18)}
\]

\[
\geq \lim_{n \to \infty} \frac{1}{2\tau_n} \left( 1 - \frac{\tau_n}{\varepsilon} \right) \frac{1}{\ell - m} ||u_\ell^{[\tau_n]} - u_m^{[\tau_n]}||^2_\mathcal{V} \quad \text{by (2.32)}
\]

\[
= \frac{1}{2(t-s)} ||\hat{u}(s) - \hat{u}(t)||^2_\mathcal{V} \geq 0
\]
as desired. \[\square\]

**Note.** Since \( \text{GL}_\varepsilon(\hat{u}(s)) - \text{GL}_\varepsilon(\hat{u}(t)) \leq \text{GL}_\varepsilon(\hat{u}(s)) \leq \text{GL}_\varepsilon(\hat{u}(0)) \) it follows by (2.31) that

\[
||\hat{u}(s) - \hat{u}(t)||_\mathcal{V} \leq \sqrt{|t-s|} \sqrt{2 \text{GL}_\varepsilon(\hat{u}(0))}
\]

which is an explicit \( C^{0,1/2} \) condition for \( \hat{u} \).

\(^{13}\)To verify this, simply expand the \( || \cdot ||^2_\mathcal{V} \) terms as inner products and collect terms.
3 Towards a link to mean curvature flow

Following work in [1], we wish to explore the question of if the well-known continuum links between ACE, MBO and mean curvature flow (MCF) extend to graph ACE and MBO and a formulation of MCF on a graph. Towards this, in this section we first prove some relevant $\Gamma$-convergence results.

We then prove in the graph context a pair of comparison principles used by Chen and Elliott in [32] to prove convergence of continuum double-obstacle ACE to MCF.

3.1 $\Gamma$-convergence results

A positive answer to this question has been suggested by $\Gamma$-convergence results linking the associated energies of graph ACE [29] and MBO [30] to graph total variation

$$\text{TV}(u) := \frac{1}{2} \sum_{i,j \in V} \omega_{ij} |u_i - u_j|$$

of which graph MCF is a type of descending flow\footnote{For detail on $\Gamma$-convergence, see e.g. [27] and [28].} We here show analogous $\Gamma$-convergence results for the new functionals defined in this paper.

Define the function on $V_{[0,1]}$:

$$f_0(u) := \begin{cases} \frac{1}{2} \text{TV}(u), & u \in V_{[0,1]} \cap V_{[0,1]}, \\ \infty, & u \in V_{[0,1]} \setminus V_{[0,1]} \end{cases}$$

Then we have the following $\Gamma$-convergences:

**Theorem 24** (Cf. [29, Theorem 3.1]). The Ginzburg–Landau functional $\text{GL}_e$ with double-obstacle potential defined in (1.3) has $\Gamma$-limit in $V_{[0,1]}$:

$$\Gamma\lim_{\epsilon \downarrow 0} \text{GL}_e = f_0.$$

**Proof.** Let $u_\epsilon \to u$ for $u_\epsilon, u \in V_{[0,1]}$. Suppose $u_i \in (0,1)$ for some $i \in V$, then eventually $(u_\epsilon)_i \in (0,1)$ and $\text{GL}_e(u_\epsilon) \geq \frac{1}{2\epsilon} \sum_{i} d^r_i(u_\epsilon)_i(1 - (u_\epsilon)_i) \to \infty$, so $f_0(u) \leq \liminf_{\epsilon \to 0} \text{GL}_e(u_\epsilon)$. Now if $u \in V_{[0,1]}$ then $f_0(u) = \frac{1}{2} ||\nabla u||^2_V = \lim_{\epsilon \to 0} \frac{1}{2} ||\nabla u_\epsilon||^2_V \leq \liminf_{\epsilon \to 0} \text{GL}_e(u_\epsilon)$.

Now let $u \in V_{[0,1]}$ and choose the recovery sequence $\bar{u}_\epsilon \equiv u$. If $u_i \in (0,1)$ for some $i \in V$, then $\text{GL}_e(u) \geq \frac{1}{2\epsilon} \sum_{i} d^r_i(u_\epsilon)_i(1 - u_\epsilon)_i \to \infty$ so $f_0(u) = \liminf_{\epsilon \to 0} \text{GL}_e(u)$. If $u \in V_{[0,1]}$ then $\text{GL}_e(u) = \frac{1}{2} ||\nabla u||^2_V = f_0(u)$ so again $f_0(u) = \lim_{\epsilon \to 0} \text{GL}_e(u)$. \hfill $\square$

\footnote{Different authors present different definitions of graph MCF: for example in [1] it was defined as a discrete-time process inspired by the variational formulation of MCF in [33] (i.e. a generalised minimising movement of TV), whilst Elmoataz et al. (see e.g. [34]) have defined it as a continuous-time process inspired by the well-known level-set PDE for MCF. In both of these definitions however one can observe graph TV monotonically decreasing along trajectories.}
Corollary 25. The Lyapunov functional for the semi-discrete flow defined in (2.17) has $\Gamma$-limit in $\mathcal{V}_{[0,1]}$:  
\[ \Gamma\lim_{\varepsilon \downarrow 0, 0 < \tau \leq \varepsilon} \frac{1}{2\tau} H = f_0. \]

Proof. Recall from (2.22) that  
\[ \frac{1}{2\tau} H(u) = \mathrm{GL}_\varepsilon(u) - \frac{1}{2} \tau \langle u, Q_\tau u \rangle_{\mathcal{V}} \]
where $Q_\tau = \tau^{-2} (e^{-\tau \Delta} - I + \tau \Delta)$. Now $Q_\tau$ has eigenvalues $\tau^{-2} (e^{-\tau \lambda_k} - 1 + \tau \lambda_k) \leq \frac{1}{2} \lambda_k^2$, for $\lambda_k$ the eigenvalues of $\Delta$, and so $\langle u, Q_\tau u \rangle_{\mathcal{V}} \leq \frac{1}{2} \| \Delta \|^2 \| u \|^2_{\mathcal{V}}$ is bounded in $\mathcal{V}_{[0,1]}$ uniformly in $\tau$. It follows that if $\varepsilon_j \downarrow 0$, $0 < \tau_j \leq \varepsilon_j$ and $u_j \to u$ in $\mathcal{V}_{[0,1]}$ then $\tau_j \langle u_j, Q_\tau u_j \rangle_{\mathcal{V}} \to 0$.

We derive the $\Gamma$-convergence of $\frac{1}{2\tau} H$ from that of $\mathrm{GL}_\varepsilon$. Consider the lim-inf inequality:  
\[ f_0(u) \leq \liminf_j \mathrm{GL}_\varepsilon_j(u_j) \quad \text{(by the $\Gamma$-convergence of $\mathrm{GL}_\varepsilon$ to $f_0$)} \]
\[ = \liminf_j \mathrm{GL}_\varepsilon_j(u_j) + \liminf_j \left( -\frac{1}{2} \tau_j \langle u_j, Q_\tau u_j \rangle_{\mathcal{V}} \right) \leq \liminf_j \frac{1}{2\tau_j} H(u_j). \]

Next take $\bar{u}_j \to u$ a recovery sequence for $\mathrm{GL}_\varepsilon$ to $f_0$:  
\[ f_0(u) = \lim_{j \to \infty} \mathrm{GL}_\varepsilon_j(\bar{u}_j) = \lim_{j \to \infty} \left( \mathrm{GL}_\varepsilon_j(\bar{u}_j) - \frac{1}{2} \tau_j \langle \bar{u}_j, Q_\tau \bar{u}_j \rangle_{\mathcal{V}} \right) = \lim_{j \to \infty} \frac{1}{2\tau_j} H(\bar{u}_j) \]
and this proves the $\Gamma$-convergence of $\frac{1}{2\tau} H$ to $f_0$. \[ \square \]

Note. Taking $\tau = \varepsilon$ and considering $J(u) := (1 - u, e^{-\tau \Delta} u)_{\mathcal{V}}$, the Lyapunov functional for MBO (see [1, Proposition 4.6]), we have that $H = J$ and so we can immediately infer that in $\mathcal{V}_{[0,1]}$:  
\[ \Gamma\lim_{\tau \downarrow 0} \frac{1}{\tau} J|_{\mathcal{V}_{[0,1]}} = 2f_0. \]

This is a special case of the result of [30, Theorem 5.10].

3.2 Comparison principles for double-obstacle ACE

In [32], a pair of comparison principles were used to prove convergence of ACE with the double-obstacle potential to MCF in the continuum. We here follow their method to prove analogous principles. We also use the latter principle to prove uniqueness of solutions to ACE.

From Proposition 10 we recall the weak formulation of double-obstacle ACE: if $u \in \mathcal{V}_{[0,1], t \in T}$ a solution to double-obstacle ACE (2.5) then for a.e. $t \in T$  
\[ \forall \eta \in \mathcal{V}_{[0,1]}, \left( \frac{du}{dt} - u(t) + \frac{1}{2\varepsilon} \eta - u(t) \right)_{\mathcal{V}} + \langle \nabla u(t), \nabla \eta - \nabla u(t) \rangle_{\mathcal{V}} \geq 0. \]

We first note a useful pair of facts:
Proposition 26. Let \( z \in \mathcal{V}_{i \in T} \) for \( T \) any interval and let \( z_+(t) \) be the positive part of \( z(t) \), i.e. \( (z_+)_i(t) := \max\{z_i(t), 0\} \). Then for all \( t \in T \),
\[
(\nabla z_+(t), \nabla z_+(t))_\varepsilon \leq (\nabla z(t), \nabla z_+(t))_\varepsilon.
\]
Also if \( z \in H^1(T; \mathcal{V}) \cap C^0(\bar{T}; \mathcal{V}) \) and \( T = (0, T^*) \) for \( T^* > 0 \) then,
\[
\frac{1}{2} ||z_+(T^*))_\varepsilon^2 - \frac{1}{2} ||z_+(0))_\varepsilon^2\right) = \int_0^{T^*} \left( \frac{dz}{dt}, z_+ \right)_\mathcal{V} dt.
\]
Proof. Consider \( (\nabla z - \nabla z_+, \nabla z)_\varepsilon \). By definition we have
\[
(\nabla z - \nabla z_+, \nabla z_+)_\varepsilon = \frac{1}{2} \sum_{i,j \in V} \omega_{ij} X_{ij}
\]
where \( X_{ij} := ((z_+)_i - (z_+)_j)(z_i - (z_+)_i - z_j + (z_+)_j) \). We claim that \( X_{ij} \geq 0 \). WLOG suppose \( (z_+)_i \geq (z_+)_j \). If \( (z_+)_i = (z_+)_j \) then \( X_{ij} = 0 \), and if \( z_i, z_j > 0 \) then \( X_{ij} = 0 \). Finally if \( z_i > 0, z_j \leq 0 \) then \( X_{ij} = -z_i z_j \geq 0 \). So \( (\nabla z - \nabla z_+, \nabla z)_\varepsilon \geq 0 \).
For the latter claim, note that it suffices to show that for each \( i \in V \)
\[
\frac{1}{2} (z_+(T^*))_i^2 - \frac{1}{2} (z_+(0))_i^2 = \int_0^{T^*} \frac{dz}{dt}_i (z_+)_i (t) dt
\]
and recall that \( z \in H^1(T; \mathcal{V}) \) if and only if \( z_i \in H^1(T; \mathcal{R}) \) for each \( i \in V \). The equation \( (3.2) \) then follows from \([31] \) Lemma 3.3. \( \square \)

We can now derive the following comparison principles:

Theorem 27 (Cf. \([32] \) Lemma 2.3). Let \( w \in \mathcal{V}_{(-\infty, 1], t \in (0, T)} \cap H^1((0, T); \mathcal{V}) \) be continuous and let \( u \in \mathcal{V}_{[0, 1], t \in (0, T)} \cap H^1((0, T); \mathcal{V}) \) be continuous and obey \( (3.1) \). Suppose that \( w_i(0) \leq u_i(0) \) and that there exists \( f \in \mathcal{V}_{t \in (0, T)} \) such that for all \( T^* \in [0, T] \)
\[
\int_0^{T^*} \langle f(t), (w - u)_+(t) \rangle_\mathcal{V} dt \leq \int_0^{T^*} \langle w(t), (w - u)_+(t) \rangle_\mathcal{V} dt \tag{3.3}
\]
and
\[
\forall \eta \in \mathcal{V}_{(0, \infty), t \in (0, T)} \int_0^T \left( \frac{d(u \cdot \eta)}{dt} + \frac{1}{2\varepsilon} 1, \eta \right)_\mathcal{V} + \langle \nabla w, \nabla \eta \rangle_\varepsilon - \frac{1}{\varepsilon} \langle f, \eta \rangle_\mathcal{V} dt \leq 0. \tag{3.4}
\]
Then it follows that \( \forall i \in V \forall t \in [0, T], \)
\[
w_i(t) \leq u_i(t).
\]
Proof. Let \( z := w - u \) and \( T^* \in [0, T] \). Taking \( \eta = z_+ + u \in \mathcal{V}_{[0,1], t \in (0,T]}^{16} \) and then integrating (3.1) gives
\[
\int_0^{T^*} \left( \frac{du}{dt} - \frac{1}{\varepsilon} u + \frac{1}{2\varepsilon} 1, z_+ \right)_\mathcal{V} + \langle \nabla u, \nabla z_+ \rangle_\mathcal{E} \ dt \geq 0
\]
and taking \( \eta(t) = z_+(t) \) for \( t < T^* \) and \( \eta(t) = 0 \) thereafter in (3.4) gives
\[
\int_0^{T^*} \left( \frac{dw}{dt} + \frac{1}{2\varepsilon} 1, z_+ \right)_\mathcal{V} + \langle \nabla w, \nabla z_+ \rangle_\mathcal{E} - \frac{1}{\varepsilon} \langle f, z_+ \rangle \ dt \leq 0.
\]
Thus combining these inequalities we get
\[
\int_0^{T^*} \left( \frac{dz}{dt}, z_+ \right)_\mathcal{V} + \langle \nabla z, \nabla z_+ \rangle_\mathcal{E} \ dt \leq \frac{1}{\varepsilon} \int_0^{T^*} \langle f - u, z_+ \rangle_\mathcal{V} \ dt. \tag{\ast}
\]
Now by subtracting \( \langle u, z_+ \rangle_\mathcal{V} \) from (3.3),
\[
RHS (\ast) \leq \frac{1}{\varepsilon} \int_0^{T^*} \langle z, z_+ \rangle_\mathcal{V} \ dt = \frac{1}{\varepsilon} \int_0^{T^*} \| z_+ \|^2_\mathcal{V} \ dt
\]
and by Proposition 26 we have that
\[
LHS (\ast) \geq \frac{1}{2} \| z_+(T^*) \|^2_\mathcal{V} - \frac{1}{2} \| z_+(0) \|^2_\mathcal{V} + \int_0^{T^*} \langle \nabla z_+, \nabla z_+ \rangle_\mathcal{E} \ dt.
\]
Thus let \( Z(t) := \| z_+(t) \|^2_\mathcal{V} \). Note that \( Z(0) = 0 \) and thus we have by the above that
\[
\frac{1}{2} Z(T^*) \leq \frac{1}{2} Z(T^*) \int_0^{T^*} \| \nabla z_+ \|^2_\mathcal{V} \ dt \leq LHS (\ast) \leq RHS (\ast) \leq \frac{1}{\varepsilon} \int_0^{T^*} Z(t) \ dt.
\]
Then by Grönwall’s integral inequality \( Z(T^*) \leq 0 \) for all \( T^* \in [0, T] \) and hence \( Z(t) = 0 \) in \( [0, T] \), thus for \( t \in [0, T] \) \( z_+(t) = 0 \) and so \( w(t) \leq u(t) \).

Theorem 28 (Cf. [32] Lemma 2.4). Let \( v \in H^1((0,T); \mathcal{V}) \cap C^0([0,T]; \mathcal{V}) \) and \( \gamma \in \mathcal{V}_{t \in [0,T]} \) be such that for all \( i \in V \), for all \( t \in (0,T) \) \( 0 \leq \gamma_i(t) \leq 1 \), and for a.e. \( t \in (0,T) \)
\[
\varepsilon \frac{dv_i}{dt} + \varepsilon (\Delta v(t))_i + \frac{1}{2} - \gamma_i(t), \quad \gamma(t) \in B(v(t)). \tag{3.5}
\]
Then if \( u \in H^1((0,T); \mathcal{V}) \cap C^0([0,T]; \mathcal{V}_{[0,1]}) \) is a solution to (2.3) and for all \( i \in V \), \( v_i(0) = u_i(0) \), it follows that for all \( t \in [0,T] \) and \( i \in V \), \( v_i(t) \leq u_i(t) \).

Proof. Subtracting (2.3) from (3.3), we get that for a.e. \( t \in (0,T) \) (understanding the inequality vertexwise)
\[
\varepsilon \frac{d}{dt} (v(t) - u(t)) + \varepsilon (\Delta (v(t) - u(t)) - (v(t) - u(t)) \leq \gamma(t) - \beta(t)
\]
\(\text{16\textsuperscript{Since either } } \eta_i(t) = u_i(t) \in [0, 1] \text{ or } \eta_i(t) = w_i(t) \text{ in the case when } 0 \leq u_i(t) \leq w_i(t) \leq 1.\)
where $\beta(t) \in B(u(t))$. Let $w := v - u$ and take the inner product with $w_+$

$$
\varepsilon \left\langle \frac{dw}{dt}, w_+(t) \right\rangle_{\mathcal{V}} + \varepsilon\langle \Delta w(t), w_+(t) \rangle_{\mathcal{V}} - \langle w(t), w_+(t) \rangle_{\mathcal{V}} \leq \langle \gamma(t) - \beta(t), w_+(t) \rangle_{\mathcal{V}}.
$$

Consider the RHS. If $v_i(t) \leq u_i(t)$ then $(w_+)_{Si}(\gamma_i(t) - \beta_i(t)) = 0$, and if $v_i(t) > u_i(t)$ a simple case check reveals that $\gamma_i(t) \leq \beta_i(t)$. Therefore $RHS \leq 0$. Hence we have that

$$
\langle w_+(t), w_+(t) \rangle_{\mathcal{V}} = \langle w(t), w_+(t) \rangle_{\mathcal{V}}
$$

for a.e. $t \in (0, T)$.

Note $w_+(0) = 0$. Integrating and applying the second part of Proposition 26, we have

$$
\frac{\varepsilon}{2} \left\| w_+(T) \right\|_{\mathcal{V}}^2 \leq \int_0^T \left\| w_+(t) \right\|_{\mathcal{V}}^2 \, dt
$$

so by Grönwall’s integral inequality $\left\| w_+(t) \right\|_{\mathcal{V}}^2 \leq 0$ for all $t \in [0, T]$, and hence $w_+(t) = 0$ in $[0, T]$. Therefore $v_i(t) \leq u_i(t)$ for all $i \in V$ and $t \in [0, T]$.

**Note.** The condition that $v_i(t) \geq 0$ can somewhat be relaxed. If $v_i(t) < 0$ then from the subdifferential $\gamma_i(t) = \infty$, in which case (3.5) is still meaningfully satisfied. The only hiccup this raises in the proof is when we consider the $(w_+)_{Si}(\gamma_i(t) - \beta_i(t))$ term in the $v_i(t) \leq u_i(t)$ case, which now becomes the undefined $0 \times \infty$. If however we consider $u, v, \beta$ and $\gamma$ to arise as in Appendix A from a limit of $C^1$ potentials approaching the double-obstacle potential, then for $v_i(t) < 0$ the corresponding limiting term

$$
(\nu_{\nu_n}(t))_i - (\nu_{v_n}(t))_i + \left( \frac{1}{2} - (\nu_{v_n}(t))_i - W_{\nu_n}'((\nu_{v_n}(t))_i) - \frac{1}{2} + (\nu_{v_n}(t))_i + W_{\nu_n}'((\nu_{v_n}(t))_i) \right)
$$

has eventually $(\nu_{v_n}(t))_i \leq -\nu_n \leq (\nu_{v_n}(t))_i$, so the term is eventually constantly zero. Hence we may take $(w_+(t))_{Si}(\gamma_i(t) - \beta_i(t)) = 0$ as desired.

As a notable aside, this comparison principle also yields uniqueness of ACE solutions:

**Corollary 29.** Let $u, v \in H^1((0, T); \mathcal{V}) \cap C^0([0, T]; \mathcal{V}_{[0,1]})$ and $\beta, \gamma \in \mathcal{V}_{\mathcal{F}[0, T]}$. Let $(u, \beta), (v, \gamma)$ be solutions to (2.3) on $[0, T]$ with $u(0) = v(0)$. Then for all $t \in [0, T]$, $u(t) = v(t)$, and for a.e. $t \in [0, T]$, $\beta(t) = \gamma(t)$.
We therefore rescale results quoted from [32] in order to preserve continuity with the rest of our notation. Furthermore, \( u(0) \leq v(0) \) vertexwise. Hence by the comparison principle, \( u(t) \leq v(t) \) vertexwise for all \( t \in [0, T] \). By symmetry, \( v(t) \leq u(t) \) vertexwise, and hence \( u(t) = v(t) \), for all \( t \in [0, T] \). Finally, by Theorem 35, \( \beta(t) \) and \( \gamma(t) \) are uniquely determined a.e. by \( u(t) \) and \( v(t) \), and therefore \( \beta(t) = \gamma(t) \) for a.e. \( t \in [0, T] \).

**Corollary 30.** Let \( u, v \in H^1((0, \infty); V) \cap C^0([0, \infty); V_{[0,1]} \) and \( \beta, \gamma \in V_{\epsilon[0,\infty)} \). Let \( (u, \beta), (v, \gamma) \) be solutions to (2.5) on \([0, \infty)\) with \( u(0) = v(0) \). Then for all \( t \in [0, \infty) \), \( u(t) = v(t) \), and for a.e. \( t \in [0, \infty) \), \( \beta(t) = \gamma(t) \).

**Proof.** For all \( T \in [0, \infty) \), \( (u|_{[0,T]}, \beta|_{[0,T]}), (v|_{[0,T]}, \gamma|_{[0,T]}) \) are solutions on \([0, T]\) with \( u(0) = v(0) \). Hence \( u|_{[0,T]} = v|_{[0,T]} \), and \( \beta|_{[0,T]} = \gamma|_{[0,T]} \) almost everywhere. Thus for all \( t \in [0, \infty) \), \( u(t) = v(t) \), and for a.e. \( t \in [0, \infty) \), \( \beta(t) = \gamma(t) \).

### 3.3 A note on the context of Theorem 27

Returning to Theorem 27, the reader may be forgiven for thinking the condition (3.4) looks a little obscure. To go some ways towards remedying this, we recap the context in which this principle is employed in [32]. Since [32] is concerned with ACE in the continuum context, we here depart from the graph setting of the rest of this paper.

Let \( \Gamma(t) \) for \( t \in [0, T] \) be a surface evolving by mean curvature flow, and let \( d(x,t) \) be the signed distance function

\[
d(x,t) := \begin{cases} 
-\text{dist}(x, \Gamma(t)), & x \text{ “inside” } \Gamma(t), \\
\text{dist}(x, \Gamma(t)), & x \text{ “outside” } \Gamma(t).
\end{cases}
\]

We suppose for some bounded domain \( \Omega \), \( \Gamma(t) \subset \Omega \) for all time and is bounded away from \( \partial \Omega \) by a distance \( \delta > 0 \). Suppose further that for some \( D_0 < \infty \),

\[
\sup_{t \in [0,T], |d(x,t)| \leq \delta} |\nabla (d_t(x,t) - \Delta d(x,t))| \leq D_0.
\]

Then one of the key results proved in [32] Theorem 3.1\(^{17}\) is that if double-obstacle ACE is initialised with

\[
u(x,0) = 1 \text{ when } d(x,0) > \gamma(\epsilon) \\
u(x,0) = 0 \text{ when } d(x,0) < -\gamma(\epsilon)
\]

that is, if the initial interfacial region is of width \( \leq 2\gamma(\epsilon)\)\(^{18}\) and contains \( \Gamma(0) \), the initial mean curvature flow surface, then the solution obeys

\[
u(x,t) = 1 \text{ when } d(x,t) > \gamma(\epsilon) \left(1 + 2e^{2D_0 t}\right) \\
u(x,t) = 0 \text{ when } d(x,t) < -\gamma(\epsilon) \left(1 + 2e^{2D_0 t}\right)
\]

\(^{17}\)Note that [32] uses a potential with obstacles at \( \pm 1 \) rather than our 0, 1 and uses a different \( \epsilon \) scaling.

\(^{18}\)We therefore rescale results quoted from [32] in order to preserve continuity with the rest of our notation.

\[\sqrt{\pi}/2 \leq \gamma(\epsilon) \leq (\delta - \sqrt{\pi}/2)e^{-2D_0 T}\].
i.e. the interfacial region at time $t$ is of width $\leq 2(1 + 2e^{2D_0t})\gamma(\varepsilon)$ and contains $\Gamma(t)$, the mean curvature flow surface at time $t$. This is proved via the comparison principle.

Define

$$z(x,t) := d(x,t) - 2e^{2D_0t}\gamma(\varepsilon).$$

We seek to prove that $u(x,t) = 1$ when

$$z(x,t) > \gamma(\varepsilon) \geq \frac{1}{2}\sqrt{\varepsilon\pi}.$$

To this end, we define

$$w(x,t) := \begin{cases} 1, & z(x,t) \geq \frac{1}{2}\sqrt{\varepsilon\pi} \\ \frac{1}{2} \left(1 + \sin \frac{z(x,t)}{\sqrt{\varepsilon}}\right), & z(x,t) \in \left(-\frac{1}{2}\sqrt{\varepsilon\pi}, \frac{1}{2}\sqrt{\varepsilon\pi}\right) \\ 0, & z(x,t) \leq -\frac{1}{2}\sqrt{\varepsilon\pi} \end{cases}$$

and if we can prove that $w(x,t) \leq u(x,t)$ we have the desired result. Now note that $w(x,0) > 0$ if and only if $d(x,0) > 2\gamma(\varepsilon) - \sqrt{\varepsilon\pi}/2 \geq \gamma(\varepsilon)$ in which case $u(x,0) = 1$. Thus $w(x,0) \leq u(x,0)$. We apply the comparison principle. Set the function $f$ to be

$$f(x,t) := \begin{cases} w(x,t), & w(x,t) > 0 \\ 1/2, & w(x,t) = 0 \end{cases}$$

We clearly have $f(x,t)(w(x,t) - u(x,t))_+ = w(x,t)(w(x,t) - u(x,t))_+$. Finally, Chen and Elliott check that for any function $\eta \geq 0$

$$\int_0^T \int_\Omega \frac{dw}{dt}\eta + \frac{1}{2\varepsilon}\eta + \nabla w\nabla \eta - \frac{1}{\varepsilon}f\eta \; dx \; dt \leq 0.$$

Thus $w(x,t) \leq u(x,t)$ and so $w(x,t) = 1 \Rightarrow u(x,t) = 1$, i.e. if $d(x,t) > \gamma(\varepsilon)(1 + 2e^{2D_0t})$ then $u(x,t) = 1$. The proof for $d(x,t) < -\gamma(\varepsilon)(1 + 2e^{2D_0t})$ is the same idea.

However, this proof does not obviously translate to the graph context. In particular, \cite{32} uses facts about the signed distance function such as

$$|\nabla d(x,t)| = 1 \text{ a.e.} \quad \text{ and } \quad d_t - \Delta d = 0 \text{ on } \Gamma(t),$$

which do not have obvious analogues in the graph setting. A topic for future research is whether a proof of a version of \cite{32, Theorem 3.1} can be developed in the graph setting using the comparison principles proved in this paper.

4 Conclusions

Following Blowey and Elliott \cite{2, 3, 4}, we have defined a graph Allen–Cahn equation using a double-obstacle potential, proved existence, uniqueness and Lipschitz regularity
of solutions, and demonstrated that graph MBO is a special case of a “semi-discrete”
time-discretisation of this equation. We exhibited a Lyapunov functional for this scheme,
and used this to perform an analysis of the long-time behaviour along the lines of Luo and
Bertozzi [25], yielding similar results. Furthermore, we proved that for any $\tau_n \downarrow 0$
there is a (constructible) subsequence along which the semi-discrete iterates $(u_m, \beta_m)$, with initial
state $u_0$ and parameter $\tau_n$, converge a.e. to the Allen–Cahn trajectory with $u(0) = u_0$,
and in particular the $u_m$ converge pointwise without passing to a subsequence.

Towards a link to mean curvature flow, we have proved some promising $\Gamma$-convergences,
and translated two comparison principles—used by Chen and Elliott to show convergence
of Allen–Cahn to mean curvature flow in [32]—into the graph setting.

In future work we will continue investigating links to mean curvature flow, in particular
seeking a representation of mean curvature flow on a graph that will allow us to employ
the above comparison principles to derive a concrete link between Allen–Cahn/MBO and
mean curvature flow on graphs. Currently ongoing work by the authors investigates the
properties of the semi-discrete link between Allen–Cahn and MBO in the presence of
further constraints, in particular [35] showing that the link continues to hold in the pres-
ce of a mass-conservation constraint. We furthermore suspect that the link between
MBO and the semi-discrete scheme for Allen–Cahn can also be made in the continuum,
which may prove an interesting topic for future research.

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A Proof of Theorem 7

Recall that given initial condition $u(0) = u_0 \in \mathcal{V}_{[0,1]}$, we seek to find $(u, \beta) \in \mathcal{V}_{[0,1], t \in [0, \infty)} \times
\mathcal{V}_{t \in [0, \infty)}$ with $u \in H^1_{\text{loc}}([0, \infty); \mathcal{V}) \cap C^0([0, \infty); \mathcal{V})$ that solves for a.e. $t \geq 0$

$$\frac{du(t)}{dt} = -\Delta u(t) + \frac{1}{\epsilon} \left( u(t) - \frac{1}{2} \right) + \frac{1}{\epsilon} \beta(t), \quad \beta(t) \in \mathcal{B}(u(t)). \quad (A.1)$$
Let $\nu > 0$. Define the following $C^1$ approximation to the double obstacle potential

$$W_\nu(x) = \begin{cases} \frac{1}{4\nu}x^2 + \frac{1}{2}x, & x < 0 \\ \frac{1}{2}x(1-x), & 0 \leq x \leq 1 \\ \frac{1}{4\nu}(x-1)^2 - \frac{1}{2}(x-1), & x > 1 \end{cases} \quad (A.2)$$

with derivative

$$W'_\nu(x) = \begin{cases} \frac{1}{4\nu}x + \frac{1}{2}, & x < 0, \\ \frac{3}{4} - x, & 0 \leq x \leq 1, \\ \frac{1}{4\nu}(x-1) - \frac{1}{2}, & x > 1. \end{cases} \quad (A.3)$$

Note that $W_\nu$ is a double-well potential with wells of depth $-\nu/4$ at $-\nu$ and $1+\nu$. Then we define the corresponding ACE

$$\frac{du_\nu(t)}{dt} = -\Delta u_\nu(t) - \frac{1}{\varepsilon} W'_\nu \circ u_\nu(t) \quad (A.4)$$

with $u_\nu(0) = u_0 \in \mathcal{V}_{[0,1]}$. We note the following facts.

**Proposition 31.** For $u_\nu$ solving (A.4) with $u_\nu(0) \in \mathcal{V}_{[0,1]}$:

1. $u_\nu \in C^1([0, \infty); \mathcal{V})$ exists.

2. $u_\nu \in \mathcal{V}_{[-\nu, 1+\nu], t \in [0, \infty)}$.

**Proof.** We employ standard arguments:

1. $W'_\nu$ is a piecewise linear function, and hence is Lipschitz. Therefore the right-hand side of (A.4) is a Lipschitz function of $u_\nu$, so existence of a $C^1$ solution on $[0, \infty)$ follows by the Picard–Lindelöf Theorem.

2. Note that $u_0 \in \mathcal{V}_{[-\nu, 1+\nu]}$ and $-\nu, 1+\nu$ are the locations of the wells of $W_\nu$. Suppose there were a $T > 0$ and a $k \in \mathcal{V}$ such that $(u_\nu(T))_k < -\nu$. Then on the interval $[0, T]$ each $(u_\nu)_i$ is continuous so attains its lower bound. As $\mathcal{V}$ is finite we may choose the $i \in \mathcal{V}$ with the lowest such bound, which by assumption must be less than $-\nu$, and let $t \in [0, T]$ be a time this bound is attained. Then we have that $(u_\nu(t))_i < -\nu$, and for all $j \in \mathcal{V}$, $(u_\nu(t))_j \geq (u_\nu(t))_i$. Then

$$\frac{d(u_\nu)_i}{dt}(t) = -\Delta u_\nu(t)_i - \frac{1}{\varepsilon} W'_\nu((u_\nu(t))_i) > -(\Delta u_\nu(t))_i \geq 0$$

with the final inequality following since $i$ is a minimiser of $u_\nu(t)$. Now if $t > 0$ then we must have some $0 < t' < t$ such that

$$\frac{(u_\nu(t))_i - (u_\nu(t'))_i}{t - t'} \geq \frac{1}{2} \frac{d(u_\nu)_i}{dt}(t) > 0$$

and hence $(u_\nu(t'))_i < (u_\nu(t))_i$ contradicting the minimality of $t$. But when $t = 0$, $(u_\nu(t))_i = (u_0)_i \geq -\nu$. So we attain a contradiction in either case. Likewise for the upper bound.
Lemma 32. If \( u \in C^{0,1}([0, \infty), V) \) then \( u \in H^1_{loc}([0, \infty); V) \).

Proof. As \( u \) is Lipschitz, it follows as a standard result that \( u \) is absolutely continuous and differentiable a.e., and hence there exists an integrable function \( g \) such that
\[
g = \frac{du}{dt} \text{ a.e.} \quad \text{and} \quad u(t_2) = u(t_1) + \int_{t_1}^{t_2} g(s) \, ds.
\]
Consider any open bounded interval \( T = (a, b) \subseteq [0, \infty) \). As \( u \) is Lipschitz it follows that \( g \) is bounded a.e. on \( T \), so \( g \in L^2(T; V) \). Finally for any \( \varphi \in C^\infty_c(T; V) \) we have
\[
\int_a^b \left< u(t), \frac{d\varphi(t)}{dt} \right>_V \, dt = \int_a^b \left< u(a) + \int_a^t g(s) \, ds, \frac{d\varphi(t)}{dt} \right>_V \, dt
\]
\[
= \left< u(a), \varphi(b) - \varphi(a) \right>_V + \int_a^b \int_a^t \left< g(s), \frac{d\varphi(t)}{dt} \right>_V \, ds \, dt
\]
\[
= \int_a^b \int_a^t \left< g(s), \frac{d\varphi(t)}{dt} \right>_V \, ds \, dt \text{ by Fubini's theorem}
\]
\[
= \int_a^b \left< g(s), \varphi(b) - \varphi(s) \right>_V \, ds
\]
\[
= - \int_a^b \left< g(s), \varphi(s) \right>_V \, ds
\]
so \( g \) is the desired weak derivative and \( u \in H^1(T; V) \). 

We next demonstrate the following convergences:

Lemma 33. For any sequence \( \hat{\nu}_n \downarrow 0 \) there exists a subsequence \( \nu_n \) and a Lipschitz function \( u \in V_{[0,1], t \in [0, \infty)} \cap H^1_{loc}([0, \infty); V) \) such that for all compact intervals \( T \subseteq [0, \infty) \), \( u_{\nu_n}|_T \to u|_T \) uniformly, where \( u_{\nu_n} \) are \( C^1 \) solutions to \( (\text{L.3}) \). Furthermore the derivatives \( \frac{du_{\nu}}{dt} \) converge weakly to \( \frac{du}{dt} \) in \( L^2_{loc} \), where \( \frac{du}{dt} \) is the weak derivative of \( u \).

Proof. We employ the Arzela–Ascoli theorem. WLOG we may take \( \hat{\nu}_n \leq 1 \) and so we have \( u_{\hat{\nu}_n} \in V_{[-1,2], t \in [0, \infty)} \), thus the \( u_{\hat{\nu}_n} \) are uniformly bounded in \( t \). Furthermore, \( \Delta \) is a bounded linear map so the \( \Delta u_{\hat{\nu}_n} \) are uniformly bounded in \( n \) and \( t \), and we furthermore have \( W^1_2 \circ u_{\hat{\nu}_n}(t) \in V_{[-1/2,1/2]} \). Therefore by \( (\text{L.3}) \) \( \frac{du_{\hat{\nu}_n}}{dt} \) is uniformly bounded in \( n \) and \( t \), and so for any \( t_1 \leq t_2 \),
\[
||u_{\hat{\nu}_n}(t_1) - u_{\hat{\nu}_n}(t_2)||_V \leq \int_{t_1}^{t_2} \left| \frac{du_{\hat{\nu}_n}}{dt}(t) \right|_V \, dt \leq C|t_1 - t_2| \quad (\text{A.5})
\]
where \( C \) is independent of \( n \), \( t_1 \) and \( t_2 \). Hence the \( u_{\hat{\nu}_n} \) are uniformly bounded and equicontinuous on every compact interval.
We now define the subsequence \( \nu_n \) by a diagonal argument. Take \( T = [0, 1] \), then by Arzela–Ascoli we have some \( \nu_n^{(1)} \), a subsequence of \( \hat{\nu}_n \), such that \( u_{\nu_n^{(1)}} \mid_{[0, 1]} \) converges uniformly. We then iterate as follows: take \( T = [0, k + 1] \), then by Arzela–Ascoli we have some \( \nu_n^{(k+1)} \), a subsequence of \( \nu_n^{(k)} \), such that \( u_{\nu_n^{(k+1)}} \mid_{[0, k+1]} \) converges uniformly. Finally, let \( \nu_n := \nu_n^{(n)} \). Then \( \nu_n \) is eventually a subsequence of each \( \nu_n^{(k)} \), so we have \( u : [0, \infty) \to V \) well-defined by: \( u_{[0,N]} := \) the uniform limit of \( u_{\nu_n} \mid_{[0,N]} \), for all \( N \in \mathbb{N} \). For all \( i \in V, t \in [0, \infty) \), we have \( \nu_{\nu_n(t)} \in [-\nu_n, 1 + \nu_n] \), and so taking \( n \to \infty \), \( u_i(t) \in [0, 1] \). Thus \( u \in V_{[0,1], t \in [0,\infty)} \). Taking limits as \( n \to \infty \) in (A.5):

\[
\forall t, s \geq 0 \quad ||u(t) - u(s)||_V \leq C |t - s|
\]

hence \( u \) is Lipschitz on \([0, \infty)\) and therefore is \( H^1_{loc}([0, \infty); V) \) by the above lemma.

Finally, we show weak convergence of \( \frac{du_n}{dt} \) up to a subsequence of \( \nu_n \). Let \( T \subseteq [0, \infty) \) be any bounded interval. It follows that since the \( \frac{du_n}{dt} \) are uniformly bounded in \( n \) and \( t \), they lie in a closed bounded ball in \( L^2(T; V) \), which by the Banach–Alaoglu theorem is weak-compact. Hence on \( T \) we have weak convergence of \( \frac{du_n}{dt} \) up to a subsequence of \( \nu_n \). Covering \([0, \infty)\) by a countable collection of such \( T \)s and running a diagonal argument as above yields weak convergence on all of \([0, \infty)\) up to a subsequence of \( \nu_n \). We relabel to denote this new subsequence \( \nu_n \).

Denote the weak limit of \( \frac{du_n}{dt} \) by \( g \). Then for any open bounded \( T \) and \( \varphi \in C_c^\infty(T; V) \)

\[
(g, \varphi)_{t \in T} = \lim_{n \to \infty} \left( \frac{du_n}{dt}, \varphi \right)_{t \in T} = - \lim_{n \to \infty} \left( u_{\nu_n}, \frac{d\varphi}{dt} \right)_{t \in T} = - \left( u, \frac{d\varphi}{dt} \right)_{t \in T}
\]

so \( g = \frac{du}{dt} \), where the first equality comes from weak convergence of \( \frac{du_n}{dt} \) and the final equality from uniform convergence of \( u_{\nu_n} \) to \( u \) on \( T \). \( \square \)

**Theorem 34.** For \( u \in C^{0,1}([0, \infty); V_{[0,1]} \cap H^1_{loc}([0, \infty); V)) \) as in the previous lemma and almost every \( t \in [0, \infty) \), there exists \( \beta(t) \in B(u(t)) \) such that

\[
\frac{du}{dt} = -\Delta u(t) + \frac{1}{\varepsilon} \left( u(t) - \frac{1}{2} \right) + \frac{1}{\varepsilon} \beta(t).
\]

**Proof.** Take \( \hat{\nu}_n \downarrow 0 \) with subsequence \( \nu_n \) and corresponding \( u_{\nu_n} \) as in the previous lemma. Define \( \beta_\nu := \frac{1}{2} - u_\nu - W'_\nu \circ u_\nu \). Then it is easy to check that

\[
(\beta_\nu(t)) = \begin{cases} -\left( 1 + \frac{1}{\varepsilon \nu_n} \right) (u_{\nu_n}(t))_i, & (u_{\nu_n}(t))_i \in [-\nu_n, 0] \\ 0, & (u_{\nu_n}(t))_i \in [0, 1] \\ \left( 1 + \frac{1}{\varepsilon \nu_n} \right) (1 - (u_{\nu_n}(t))_i), & (u_{\nu_n}(t))_i \in [1, 1 + \nu_n] \end{cases}
\]

and by (A.4),

\[
\frac{du_{\nu_n}}{dt} = -\Delta u_{\nu_n}(t) + \frac{1}{\varepsilon} \left( u_{\nu_n}(t) - \frac{1}{2} \right) + \frac{1}{\varepsilon} \beta_{\nu_n}(t).
\]

40
Since \( u_{\nu_n} \to u \) uniformly and \( \frac{du_{\nu_n}}{dt} \to \frac{du}{dt} \) (in \( L^2_{\text{loc}} \)), we have that
\[
\frac{1}{\varepsilon} \beta_{\nu_n} = \frac{du_{\nu_n}}{dt} + \Delta u_{\nu_n} - \frac{1}{\varepsilon} \left( u_{\nu_n} - \frac{1}{2} 1 \right) \to \frac{du}{dt} + \Delta u - \frac{1}{\varepsilon} \left( u - \frac{1}{2} 1 \right) =: \frac{1}{\varepsilon} \beta.
\]

It suffices to check that the function \( \beta \) defined in this way satisfies \( \beta(t) \in B(u(t)) \) for a.e. \( t \geq 0 \). By the Banach–Saks theorem [20], passing to a further subsequence of \( \nu_n \), we have that the Cesàro sums converge strongly, i.e. for all bounded intervals \( T \)
\[
\frac{1}{N} \sum_{n=1}^{N} \beta_{\nu_n} \to \beta \text{ in } L^2(T; \mathcal{V})
\]
as \( N \to \infty \). Recall that \( L^2 \) convergence implies a.e. pointwise convergence along a subsequence. We cover \([0, \infty)\) by a countable number of such \( T \) and extract by a diagonal argument (as in the previous lemma) a sequence \( N_k \to \infty \) such that for a.e. \( t \geq 0 \)
\[
\beta(t) = \lim_{k \to \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} \beta_{\nu_n}(t). \tag{A.6}
\]

Fix a \( t \) at which this convergence holds. First, suppose \( u_i(t) \in (0, 1) \). Then eventually \( (u_{\nu_n}(t))_i \in (0, 1) \) and so \( (\beta_{\nu_n}(t))_i = 0 \). Therefore by (A.6), \( \beta_i(t) = 0 \) as desired.

Next, suppose that \( u_i(t) = 0 \). Then eventually \( (u_{\nu_n}(t))_i \in [0, 1] \) and note that for \( (u_{\nu_n}(t))_i < 1 \) we have \( (\beta_{\nu_n}(t))_i \in [0, \frac{1}{2} + \nu_n] \). It follows by (A.6) that \( \beta_i(t) \geq 0 \), as desired. Likewise for \( u_i(t) = 1 \), \( \beta_i(t) \leq 0 \). Therefore \( \beta(t) \in B(u(t)) \), completing the proof. \( \square \)

**Note.** As an example, let \( u_0 = 0 \). One can check that (A.4) has unique solution:
\[
u_{\nu}(t) = -\nu \left(1 - e^{-\frac{t}{2\nu}}\right) 1.
\]

Hence we have the expression for \( \beta_{\nu} \)
\[
\beta_{\nu}(t) = \left(\nu + \frac{1}{2}\right) \left(1 - e^{-\frac{t}{2\nu}}\right) 1
\]
and taking \( \nu \downarrow 0 \) we therefore get the expected ACE solution:
\[
u(t) = 0, \quad \beta(t) = \begin{cases} 0, & t = 0, \\ \frac{1}{2} 1, & t > 0. \end{cases}
\]

**Note.** We can eliminate some of the reliance on passing to a subsequence using the uniqueness of ACE trajectories given by Theorem 8 and recalling the useful standard result: if \( (X, \rho) \) is a topological space and \( x_n, x \in X \) are such that every subsequence of \( x_n \) has a further subsequence converging to \( x \), then \( x_n \to x \).

\[19\]Suppose \( x_n \not\to x \). Then there exists \( U \in \rho \) such that \( x \in U \) and infinitely many \( x_n \notin U \). Choose \( x_{n_k} \) such that for all \( k, x_{n_k} \notin U \). This subsequence has no further subsequence converging to \( x \). \( \square \)
Let $\tilde{\nu}_n \downarrow 0$. We proved above that every subsequence $u_{\tilde{\nu}_n}$ of $u_{\tilde{\nu}_n}$ has a further subsequence $u_{\nu_n}$ uniformly converging (on every compact interval) to an ACE solution $u$ with initial condition $u_0$. By uniqueness, there is only one such solution $u$, so we take $x$ as this $u$ and $(X, \rho)$ as $V_{t \in [0, \infty)}$ with the topology of “uniform convergence on every compact interval” in the result. It follows that $u_{\tilde{\nu}_n} \to u$ uniformly on every compact interval.

Likewise, since the corresponding $du/dt$ and $\beta$ are unique up to a.e. equivalence, we have that $du_{\tilde{\nu}_n}/dt \rightharpoonup du/dt$ and $\beta_{\tilde{\nu}_n} \rightharpoonup \beta$ in $L^2_{\text{loc}}$ without passing to a subsequence.

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