Decentralized Convergence to Nash Equilibria in Constrained Mean Field Control

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Abstract

This paper considers decentralized control and optimization methodologies for large populations of systems, consisting of several agents with different individual behaviors, constraints and interests, and affected by the aggregate behavior of the overall population. For such large-scale systems, the theory of “mean field” games and control has been successfully applied in various scientific disciplines. While the existing mean field control literature is limited to unconstrained problems, we formulate mean field problems in the presence of heterogeneous convex constraints at the level of individual agents, for instance arising from agents with linear dynamics subject to convex state and control constraints. We propose several iterative solution methods and show that, even in the presence of constraints, the mean field solution gets arbitrarily close to a mean field Nash equilibrium as the population size grows. We apply our methods to the constrained linear quadratic mean field control problem and to the constrained mean field charging control problem for large populations of plug-in electric vehicles.

I. INTRODUCTION

Control and optimization of large populations of systems are of interest to various scientific disciplines, such as engineering, mathematics, social sciences, system biology and economics. A population of systems comprises a large number of interacting heterogeneous agents, each with its own individual dynamic behavior and interest. Such interactions have been studied in dynamic noncooperative game theory [1].

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Since for large populations of systems the analytic solution of the game equations becomes intractable, Mean Field (MF) games [2] have emerged as a methodology to study large-scale control and optimization problems in which each agent is affected by the population distribution. Under an underlying rationality assumption, each agent in a MF setting responds optimally to the overall population behavior, which in turn is determined by the aggregation of the individual responses. As the number of agents tends to infinity, these coupled interactions can be studied mathematically via a system of two coupled Partial Differential Equations (PDEs), the Hamilton–Jacobi–Bellman (HJB) PDE for the optimal response of each individual agent and the Fokker–Planck–Kolmogorov (FPK) PDE for the dynamical evolution of the population distribution [2].

For large but finite population sizes, MF control theory [3], [4] considers the population behavior as the aggregate (for instance, average) behavior among the agents, also called mass effect [4, Section IV.A]. Due to the structure of the MF problem and the large population size, the aggregate population behavior affects the individual agents as a nearly deterministic quantity [4, Section I]. In the setting of [3], [4], each agent has linear dynamics and solves a classical linear quadratic optimal tracking problem, where the reference signal represents the population behavior, that corresponds to the average among all the optimal state trajectories.

The success of MF games and control lies not only in its theoretical foundations, but also in the applicability to non-trivial large-scale control and optimization problems. Applications in fact include synchronization among populations of coupled oscillators [5], charging control for large populations of Plug-in Electric Vehicles (PEVs) [6], cyber security in wireless networks [7] and demand-side management of aggregated loads [8].

In this paper, we adopt the MF control theory approach [3], [4], [9], including large populations of agents with discrete-time linear dynamics and quadratic cost, the latter coupling the individual agent to the overall population. Compared to the existing MF control literature, we take the step of formulating and solving MF problems in which individual agents are subject to heterogeneous convex constraints, for instance arising from different linear dynamics, convex state and input constraints. Our motivation comes from the fact that constrained systems arise in almost all engineering applications, playing an active role in the system dynamics and hence in the agent behavior.

In the presence of constraints, the optimal response of each agent is in general not known in closed form. To overcome this difficulty, we build on mathematical definitions and tools from
convex analysis and operator theory [10], [11], establishing useful regularity properties of the aggregation mapping. We solve the constrained MF control problem via several specific fixed point iterations and show convergence to a MF equilibrium in a decentralized fashion, making our methods scalable as the population size increases. Analogously to [4], we seek convergence to a MF Nash equilibrium, that is, we focus on equilibria in which each agent has no interest to change its strategy, given the aggregate strategy of the others.

The contributions of the paper are hence the following:

- We extend the mean field control setting to populations of agents with heterogeneous convex constraints.
- We show that a fixed point of the aggregation mapping gets arbitrarily close to a mean field Nash equilibrium as the population size grows.
- We show several regularity properties of the mappings arising in constrained mean field control problems.
- We show that specific fixed point iterations are suited to solve constrained mean field control problems.
- We apply our results to the general constrained linear quadratic mean field control problem and to the constrained mean field charging control problem for large populations of plug-in electric vehicles, showing extensions to literature results.

The paper is structured as follows. Section II presents the MF control problem and the technical result about the approximation of a MF Nash equilibrium with a fixed point of the aggregation mapping. Section III contains the main results, regarding some regularity properties of parametric convex programs arising in MF problems and the decentralized convergence to a MF Nash equilibrium of specific fixed point iterations. Section IV presents two applications of our technical results: the MF control problem for a population of agents with heterogeneous linear dynamics, convex state and input constraints; the constrained MF charging problem for a population of heterogeneous PEVs. Section V concludes the paper and highlights several possible extensions and applications. Appendix A presents some background definitions and results from operator theory; Appendix B justifies the use of finite-horizon formulations to approximate infinite-horizon discounted-cost ones; all the proofs of the main paper are given in Appendix C.
Notation

$\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$ respectively denote the set of real, positive real, non-negative real numbers; $\mathbb{Z}$ denotes the set of integer numbers; for $a, b \in \mathbb{Z}$, $a \leq b$, $\mathbb{Z}[a, b]$ denotes the integer interval $\{a, a+1, \ldots, b\}$, and $\mathbb{Z}[a, \infty)$ denotes the infinite integer interval $\{a, a+1, a+2, \ldots\}$. $A^\top \in \mathbb{R}^{m \times n}$ denotes the transpose of $A \in \mathbb{R}^{n \times m}$. For given matrices $A_1, \ldots, A_M$, $\text{diag}(A_1, \ldots, A_M)$ denotes the block diagonal matrix with $A_1, \ldots, A_M$ in block diagonal positions. For a given $Q \in \mathbb{R}^{n \times n}$, $Q \succ 0$, we denote by $\mathcal{H}_Q$ the Hilbert space $\mathbb{R}^n$ with inner product $\langle \cdot, \cdot \rangle_Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as $\langle x, y \rangle_Q := x^\top Q y$, and induced norm $\|\cdot\|_Q : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined as $\|x\|_Q := \sqrt{x^\top Q x}$. A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz in $\mathcal{H}_Q$ if there exists $L > 0$ such that $\|f(x) - f(y)\|_Q \leq L \|x - y\|_Q$ for all $x, y \in \mathbb{R}^n$. $\text{Id} : \mathbb{R}^n \to \mathbb{R}^n$ denotes the identity operator, $\text{Id}(x) := x$ for all $x \in \mathbb{R}^n$.

Every mentioned set $S \subseteq \mathbb{R}^n$ is meant to be nonempty, unless explicitly stated. The projection operator in $\mathcal{H}_Q$, $\text{Proj}^Q : \mathbb{R}^n \to C \subseteq \mathbb{R}^n$, is defined as $\text{Proj}^Q(x) := \arg\min_{y \in C} \|x - y\|_Q = \arg\min_{y \in \mathbb{R}^n} \|x - y\|_Q^2$. $I_n$ denotes the $n$-dimensional identity matrix; $\mathbb{1}$ denotes a matrix of all 1s; $\mathbb{0}$ denotes a matrix/vector of all 0s; $\mathbb{1}$ denotes a vector of all 1s. $A \otimes B$ denotes the Kronecker product between matrices $A$ and $B$. Given $S \subseteq \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, $A S + b$ denotes the set $\{y \in \mathbb{R}^n \mid y = Ax + b, \; x \in S\}$; hence given $S^1, \ldots, S^N \subseteq \mathbb{R}^n$ and $a_1, \ldots, a_N \in \mathbb{R}$, $\frac{1}{N} \left( \sum_{i=1}^N a_i S^i \right) := \left\{ y \in \mathbb{R}^n \mid y = \frac{1}{N} \sum_{i=1}^N a_i x^i, \; x^i \in S^i \right\} \forall i \in \mathbb{Z}[1, N]$.

II. MEAN FIELD CONTROL PROBLEM WITH CONVEX CONSTRAINTS

A. General mean field control problem

We consider a large population of $N$ heterogeneous agents, where each agent $i \in \mathbb{Z}[1, N]$ controls its decision variable $x^i$, taking values in the set $\mathcal{X}^i \subset \mathbb{R}^n$. The aim of agent $i$ is to minimize its individual cost $J(x^i, z)$, which depends on the variable $z \in \mathbb{R}^n$. In the MF setting, $z$ represents the aggregate of actions of all the agents, that hence affects their behaviors.

Formally, let $x^i(*) := \arg\min_{x \in \mathcal{X}^i} J(x, z)$ be the optimal response of agent $i$, given a signal $z$. Let the population state be summarized in the aggregate (e.g., average) behavior $\mathcal{A}(\cdot) := \frac{1}{N} \sum_{i=1}^N a_i x^i(*)$, for appropriate aggregation parameters $a_1, \ldots, a_N \geq 0$.

We consider a MF setting where the $N$ agents communicate to a central coordinator, called “virtual agent” in [4, Section IV.B], in a decentralized iterative fashion. Namely, for a given reference $z_k$ at iteration $k$, each agent computes its optimal response $x^i(*)(z_k)$ based only on its
own constraint set $X^i$, that is, based on its own “private information”. The central coordinator then aggregates all the individual responses in $A(z_k)$, computes an updated reference $z_{k+1} = \Phi_k (z_k, A(z_k))$, broadcasts it to the whole population, and the process is repeated.

Given the cost function $J$, the agents’ constraint sets $\{X^i\}_{i=1}^N$ and the aggregation parameters $\{a_i\}_{i=1}^N$, the MF control problem consists in designing the feedback iteration $z^+ = \Phi_k (z, A(z))$, that is, selecting the mappings $\{\Phi_k : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n\}_{k=1}^\infty$, such that, for any initial condition $z_0 \in \mathbb{R}^n$, the variable $z$ converges to some $\bar{z}$, generating a set of strategies $\{x^i(\bar{z})\}_{i=1}^N$ with desirable properties. Along the lines of [3], [4], we are interested in a feedback mapping $\Phi_k$ for which the mentioned iteration generates a MF (almost-) Nash equilibrium, according to the following definition.

**Definition 1 (Mean field Nash equilibrium):** Given a cost function $J : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and aggregation parameters $a_1, \ldots, a_N > 0$, a set of strategies $\{\bar{x}^i\}_{i=1}^N \in \mathbb{R}^{nN}$ is a MF $\varepsilon$-Nash equilibrium, with $\varepsilon > 0$, if for all $i \in \mathbb{Z}[1, N]$ we have

$$J \left( \bar{x}^i, \frac{1}{N} \sum_{j=1}^N a_j \bar{x}^j \right) \leq \min_{y \in X^i} J \left( y, \frac{1}{N} \left( a_i y + \sum_{j \neq i}^N a_j \bar{x}^j \right) \right) + \varepsilon. \quad (1)$$

It is a MF Nash equilibrium if (1) holds with $\varepsilon = 0$. □

In classical game theory, a population is at a Nash equilibrium $\{\bar{x}^i\}_{j=1}^N$, if each agent $i$ has no individual benefit in changing its strategy $\bar{x}^i$, given the strategies of the others $\{\bar{x}^j\}_{j \neq i}$. In the MF case, the concept is similar: if the population is at a MF $\varepsilon$-Nash equilibrium, then each agent has no more than $\varepsilon$ individual benefit to change its strategy, given the aggregation among the strategies of the others.

**B. Parametric convex programs arising in mean field control problems with quadratic cost**

In the sequel, we consider MF control problems with quadratic cost, that is, we assume that each agent $i \in \mathbb{Z}[1, N]$ responds to the common signal $z \in \mathbb{R}^n$ through the mapping $x^i : \mathbb{R}^n \rightarrow X^i$, defined as

$$x^i(z) := \arg \min_{x \in X^i} J(x, z) = \arg \min_{x \in X^i} x^\top Q x + (x - z)^\top \Delta (x - z) + 2 (C z + c)^\top x \quad (2)$$

where $X^i \subset \mathbb{R}^n$ is compact and convex, $Q, \Delta \succeq 0$, $Q + \Delta \succeq 0$, $C \in \mathbb{R}^{n \times n}$ and $c \in \mathbb{R}^n$.

The three terms in (2) emphasize the contribution of three different cost terms: a quadratic cost $x^\top Q x$, typical of Linear Quadratic (LQ) MF control [4], [9], a quadratic cost $(x - z)^\top \Delta (x - z)$
on the deviations from the reference signal $z$ \cite{4,6}, and a linear cost $2(Cz + c)^T x$ \cite{4 Section II.A, 6}. Note that each agent has its own individual constraint set $X^i$, which in the context of LQ MF control discussed in Section IV-A models the fact that each agent has its own linear dynamics and its own state and input constraints.

Let us start from the characterization of the optimal solution of (2).

**Lemma 1 (Parametric Optimizer):** The unconstrained optimizer of (2) is

$$\hat{x}^*(z) := \operatorname{arg min}_{x \in \mathbb{R}^n} J(z, x) = (Q + \Delta)^{-1} ((\Delta - C)z - c).$$

(3)

The (constrained) optimizer of (2) is

$$x^i*(z) = \operatorname{arg min}_{x \in X^i} J(x, z) = \operatorname{Proj}_{X^i}^{Q+\Delta}(\hat{x}^*(z)).$$

(4)

\hfill \square

**Remark 1:** Since the mapping $\hat{x}^*$ in (3) is affine and hence Lipschitz, and the projection operator $\operatorname{Proj}_{X^i}$ has Lipschitz constant $1$ \cite[Proposition 4.8]{10}, both mappings $\hat{x}^*$ and $x^i*(\cdot) = \operatorname{Proj}_{X^i}(\hat{x}^*(\cdot))$ in (4) are Lipschitz with the same constant, that is, for every norm $\|\cdot\|$, there exists $L > 0$ such that $\|\hat{x}^*(v) - \hat{x}^*(w)\| \leq L \|v - w\|$ and $\|x^i*(v) - x^i*(w)\| \leq L \|v - w\|$ for all $v, w \in \mathbb{R}^n$.

\hfill \square

**C. Approximating a mean field Nash equilibrium in the limit of infinite population size**

We now come back to the aggregation mapping $\mathcal{A} : \mathbb{R}^n \to \left( \frac{1}{N} \sum_{i=1}^N a_i X^i \right) \subset \mathbb{R}^n$, defined as

$$\mathcal{A}(z) := \frac{1}{N} \sum_{i=1}^N a_i x^i*(z)$$

(5)

where $a_1, \ldots, a_N \geq 0$ are aggregation parameters.

Since the objective of our MF control problem is to find a MF Nash equilibrium for large population size, we exploit the following observation \cite[Section IV.A]{4}. For any agent $i$, the aggregate behavior of the others, i.e., $\frac{1}{\sum_{j \neq i} a_j} \sum_{j \neq i} a_j x^j*$, can be approximated with the aggregate behavior of the whole population, i.e., $\frac{1}{\sum_{j=1}^N a_j} \sum_{j=1}^N a_j x^j*$, for large population size $N$. Therefore, if $\bar{z} = \frac{1}{N} \sum_{j=1}^N a_j x^j*(\bar{z}) = \mathcal{A}(\bar{z})$, then $x^i*(\bar{z})$ approximates the optimal response $\operatorname{arg min}_{y \in \mathcal{X}^i} J(y, \frac{1}{N} \left( a_i y + \sum_{j \neq i} a_j x^j*(\bar{z}) \right))$ for large $N$. Formally, under the assumption that the aggregation parameters and the individual constraint sets are uniformly bounded in the limit...
of infinite population size, the following result shows that a fixed point of the aggregation mapping corresponds to a MF Nash equilibrium in the limit of infinite population size.

**Assumption 1 (Compactness):** There exists \( \bar{a} > 0 \) and a compact set \( \mathcal{X} \subseteq \mathbb{R}^n \) such that \( 0 \leq a_i \leq \bar{a} \) for all \( i \in \mathbb{Z}[1, N] \), \( \sum_{i=1}^{N} a_i = N \), and \( \mathcal{X} \supseteq \bigcup_{i=1}^{N} \mathcal{X}_i \) hold for all population sizes \( N \).

**Theorem 1 (Infinite population limit):** If Assumption 1 holds, then for all \( \varepsilon > 0 \), there exists a population size \( \bar{N} \) such that, for all \( N \geq \bar{N} \), if \( \bar{z} \) is a fixed point of \( \mathcal{A} \) in (5), that is, \( \bar{z} = \frac{1}{N} \sum_{i=1}^{N} a_i x^{i*}(\bar{z}) \), then the set of strategies \( \{x^{i*}(\bar{z})\}_{i=1}^{N} \), with \( x^{i*} \) as in (2) for all \( i \in \mathbb{Z}[1, N] \), is a MF \( \varepsilon \)-Nash equilibrium.

**Remark 2:** It follows from the proof of Theorem 1 given in Appendix C that a fixed point of \( \mathcal{A} \) in (5) with population size \( N \) is a MF \( \varepsilon_N \)-Nash equilibrium with \( \varepsilon_N \sim O\left(\frac{1}{N}\right) \). Note that having a uniform upper bound \( \bar{a} \) on the aggregation parameters \( \{a_i\}_{i=1}^{N} \) means that no single agent has a disproportionate influence on the population aggregation for large population size, which is a typical feature of MF settings.

Theorem 1 suggests that one should design the feedback mapping \( \Phi_k \) to steer the iterative game towards a fixed point of the aggregation mapping \( \mathcal{A} \), as this is an approximate solution of the MF control problem for large population size.

### III. THE QUEST FOR A FIXED POINT OF THE AGGREGATION MAPPING

#### A. Mathematical tools

In this section we present the mathematical definitions needed for the technical results in Section III-B regarding appropriate fixed point iterations relative to the aggregation mapping. For ease of notation, the statements of this section are formulated in an arbitrary finite-dimensional Hilbert space \( \mathcal{H} \), that is, in terms of an arbitrary norm \( \|\cdot\| \) on \( \mathbb{R}^n \), but in general hold for infinite-dimensional metric spaces.

We start from the property of contractiveness [11, Definition 1.6], exploited in most of the MF control literature [4, 9, 6] to show, under appropriate technical assumptions, convergence to a fixed point of the aggregation mapping.
**Definition 2 (Contraction mapping):** A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction (CON) if there exists $\epsilon \in (0, 1]$ such that
\[
\|f(x) - f(y)\| \leq (1 - \epsilon) \|x - y\| \tag{6}
\]
for all $x, y \in \mathbb{R}^n$.

If a mapping $f$ is CON, then the Picard–Banach iteration, $k = 0, 1, 2, \ldots$,
\[
z_{k+1} = f(z_k) =: \Phi^{P-B}(z_k, f(z_k)) \tag{7}
\]
converges, for any initial point $z_0 \in \mathbb{R}^n$, to its unique fixed point [11, Theorem 2.1].

Though commonly used in the MF control literature, contractiveness is a quite restrictive property. In this paper we actually exploit less restrictive properties than contractiveness, starting with nonexpansiveness [10, Definition 4.1 (ii)].

**Definition 3 (NonExpansive mapping):** A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive (NE) if
\[
\|f(x) - f(y)\| \leq \|x - y\| \tag{8}
\]
for all $x, y \in \mathbb{R}^n$.

Clearly, a CON mapping is also NE, while the converse does not necessarily hold. Note that, unlike CON mappings, NE mappings, e.g., the identity mapping, may have more than one fixed point. Among NE mappings, let us refer to firmly nonexpansive mappings [10, Definition 4.1 (i)].

**Definition 4 (Firmly NonExpansive mapping):** A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is firmly nonexpansive (FNE) if
\[
\|f(x) - f(y)\|^2 \leq \|x - y\|^2 - \|f(x) - f(y) - (x - y)\|^2 \tag{9}
\]
for all $x, y \in \mathbb{R}^n$.

An example of FNE mapping is the metric projection onto a closed convex set $\text{Proj}_C : \mathbb{R}^n \to C \subseteq \mathbb{R}^n$ [10, Proposition 4.8].

The FNE condition is sufficient for the Picard–Banach in (7) iteration to converge to a fixed point [12, Section 1, p. 522]. This is not the case for NE mappings; for example, $z \mapsto f(z) := -z$ is NE, but not CON, and the Picard–Banach iteration $z_{k+1} = f(z_k)$ oscillates indefinitively.
between $z_0$ and $-z_0$. If a mapping $f : C \to C$ is NE, with $C \subset \mathbb{R}^n$ compact and convex, then the Krasnoselskij iteration

$$z_{k+1} = (1 - \lambda)z_k + \lambda f(z_k) =: \Phi^K(z_k, f(z_k))$$  \hspace{1cm} (10)

where $\lambda \in (0, 1)$, converges, for any initial point $z_0 \in C$, to a fixed point of $f$ [11, Theorem 3.2].

Finally, we consider the even weaker regularity property of strict pseudocontractiveness [11, Remark 4, pp. 12–13].

**Definition 5 (Strictly PseudoContractive mapping):** A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is strictly pseudocontractive (SPC) if there exists $\rho < 1$ such that

$$\|f(x) - f(y)\|^2 \leq \|x - y\|^2 + \rho \|f(x) - f(y) - (x - y)\|^2$$  \hspace{1cm} (11)

for all $x, y \in \mathbb{R}^n$. $\square$

If a mapping $f : C \to C$ is SPC with $C \subset \mathbb{R}^n$ compact and convex, then the Mann iteration

$$z_{k+1} = (1 - \alpha_k)z_k + \alpha_k f(z_k) =: \Phi^M(z_k, f(z_k))$$  \hspace{1cm} (12)

where $\{\alpha_k\}_{k=0}^{\infty}$ is such that $\alpha_k \in (0, 1) \forall k \geq 0$, $\lim_{k \to \infty} \alpha_k = 0$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$, converges, for any initial point $z_0 \in C$, to a fixed point of $f$ [11, Fact 4.9, p. 112], [13, Theorem R, Section I].

It follows from Definitions 2–5 that $f\text{ FNE} \Rightarrow f\text{ NE}$, $f\text{ CON} \Rightarrow f\text{ NE} \Rightarrow f\text{ SPC}$. Therefore, the Mann iteration in (12) ensures convergence to a fixed point for CON, FNE, NE and SPC mappings; the Krasnoselskij iteration in (10) ensures convergence for CON, FNE and NE mappings; the Picard–Banach iteration in (7) for CON and FNE mappings.

The known upper bounds on the convergence rates suggest that a simpler iteration has faster convergence in general. The convergence rate for the Picard–Banach iteration is linear, that is $\|z_{k+1} - \bar{z}\| / \|z_k - \bar{z}\| \leq 1 - \epsilon$ [11, Chapter 1]. Instead, the convergence rate for the Mann iteration is sublinear, specifically $\|z_{k+1} - \bar{z}\| / \|z_k - \bar{z}\| \leq 1 - \epsilon \alpha_k$ [11, Chapter 4], for some $\epsilon > 0$.

Note that CON mappings have a unique fixed point [11, Theorem 1.1], whereas FNE, NE, SPC mappings may have multiple fixed points. In our context, this implies that there could exist multiple MF Nash equilibria, unless the aggregation mapping is CON.
B. Main results

Using the definitions and properties of the previous section, we can now state our technical result about the regularity of the optimal solution \( x^{i*} \) in (4) of the parametric convex program in (2).

**Theorem 2 (Regularity of the optimizer):** The mapping \( x^{i*} \) in (4) is:

CON in \( \mathcal{H}_{Q+\Delta} \) if there exists \( \epsilon > 0 \) such that

\[
\begin{bmatrix}
Q + \Delta & \Delta - C \\
(\Delta - C)^\top & Q + \Delta
\end{bmatrix} \succeq \epsilon I;
\]

(13)

NE in \( \mathcal{H}_{Q+\Delta} \) if (13) holds with \( \epsilon \geq 0 \);

FNE in \( \mathcal{H}_{\Delta - C} \) if \( -Q \preceq C = C^\top \prec \Delta \);

SPC in \( \mathcal{H}_{C - \Delta} \) if \( \Delta \prec C = C^\top \).

Remark 3: The condition \( -Q \preceq C = C^\top \prec \Delta \) in Theorem 2 implies (13) with \( \epsilon = 0 \), in fact

\[
\begin{bmatrix}
Q + \Delta & \Delta - C \\
(\Delta - C)^\top & Q + \Delta
\end{bmatrix} = I_2 \otimes (Q + C) + 1 \otimes (\Delta - C) \succeq 1 \otimes (\Delta - C) \succeq 0,
\]

where the last matrix inequality holds true because the eigenvalues of \( 1 \otimes (\Delta - C) \) equal the product of the eigenvalues of \( \Delta - C \), which are positive as \( \Delta - C \succ 0 \), and the eigenvalues of \( 1 = [1 \ 1] \), which are non-negative (0 and 2).

We can now exploit the structure of the aggregation mapping to establish our main result about its regularity. Specifically, under the conditions of Theorem 2, the aggregation mapping inherits the same regularity properties of the individual optimizer mappings.

**Theorem 3 (Regularity of the mean of the optimizers):** For all \( i \in \mathbb{Z}[1, N] \), let \( x^{i*} \) be defined as in (2). The mapping \( A \) in (5) is Lipschitz and:

CON in \( \mathcal{H}_{Q+\Delta} \) if (13) holds with \( \epsilon > 0 \);

NE in \( \mathcal{H}_{Q+\Delta} \) if (13) holds with \( \epsilon \geq 0 \);

FNE in \( \mathcal{H}_{\Delta - C} \) if \( -Q \preceq C = C^\top \prec \Delta \);

SPC in \( \mathcal{H}_{C - \Delta} \) if \( \Delta \prec C = C^\top \).

Theorem 3 directly leads to iterative methods for finding a fixed point of the aggregation mapping, that is a solution of the MF control problem in the limit of infinite population size.
Corollary 1 (Convergence of fixed point iterations): The following iterations and conditions guarantee convergence, from any initial point, to a fixed point of $A$ in (5), where $x^{i*}$ is as in (2) for all $i \in \mathbb{Z}[1,N]$:

1. Picard–Banach (7) if (13) holds ($\epsilon > 0$) or $-Q \preceq C = C^T \prec \Delta$;
2. Krasnoselskij (10) if (13) holds ($\epsilon \geq 0$);
3. Mann (12) if (13) holds ($\epsilon \geq 0$) or $\Delta \prec C = C^T$.

Note that convergence is ensured in different norms, namely $\| \cdot \|_{Q+\Delta}$, $\| \cdot \|_{\Delta-C}$ if $\Delta-C \succ 0$ or $\| \cdot \|_{C-\Delta}$ if $C-\Delta \succ 0$; this is not a limitation since all norms are equivalent in finite-dimensional Euclidean spaces.

We emphasize that each iterative method presented in Corollary 1 has its specific range of applicability depending on the specific MF problem. This allows us to select one or more fixed point iterations from the specific knowledge of the regularity property at hand. An important advantage of Corollary 1 is that decentralized convergence is guaranteed under conditions independent of the individual constraints $\{X_i\}^N_{i=1}$, but only on the population-level cost function $J$ in (2). Therefore, the results and methods apply naturally to populations of heterogeneous agents.

Let us summarize in Algorithm 1 our proposed decentralized procedure to compute a fixed point of the aggregation mapping $A$, where the feedback mapping $\Phi_k \in \{\Phi^{P-B}, \Phi^K, \Phi^M\}$ is chosen in view of Corollary 1.

**Algorithm 1:** Decentralized iterations towards a fixed point of the aggregation mapping.

**Initialization:** $z \leftarrow z_0$, $k \leftarrow 1$.

**Iterate until convergence:**

$x^{i*}(z) \leftarrow \arg \min_{x \in X_i} J(x, z)$, $i = 1, 2, \ldots, N$;

$A(z) \leftarrow \frac{1}{N} \sum_{i=1}^{N} a_i x^{i*}(z)$;

$z \leftarrow \Phi_k (z, A(z))$;

$k \leftarrow k + 1$. 

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IV. MEAN FIELD CONTROL APPLICATIONS

A. Discrete-time constrained linear quadratic mean field control

We now address a discrete-time constrained LQ MF control problem, that is the discrete-time constrained counterpart of the MF control problem in [4]. We consider a population of $N$ agents, in which each agent $i \in \mathbb{Z}[1, N]$ has the discrete-time, $t = 0, 1, 2, \ldots$, linear dynamics

$$x_{t+1}^i = A_i x_t^i + B_i u_t^i$$  \hspace{1cm} (14)

where $x_t^i \in \mathbb{R}^n$ is the state variable, $u_t^i \in \mathbb{R}^m$ is the control input, and $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$. For each agent $i \in \mathbb{Z}[1, N]$, we allow for convex time-varying, $t = 0, 1, 2, \ldots$, state and input constraints

$$x_{t+1}^i \in X_{t+1}^i, \quad u_t^i \in U_t^i$$  \hspace{1cm} (15)

where $X_{t+1}^i \subset \mathbb{R}^n$ and $U_t^i \subset \mathbb{R}^m$ are compact and convex sets.

Let us consider that all agents seek a dynamical evolution minimizing the finite-horizon cost function $J : \mathbb{R}^{nT} \times \mathbb{R}^{mT} \times \mathbb{R}^{nT} \rightarrow \mathbb{R}_{\geq 0}$ defined as

$$J(x, u, z) := \sum_{t=0}^{T-1} \|x_{t+1} - (z_{t+1} + \eta)\|_{Q_t}^2 + \|u_t\|_{R_t}^2$$  \hspace{1cm} (16)

where $Q_t, R_t > 0$ for all $t \in \mathbb{Z}[0, T - 1]$, $x = (x_1^T, \ldots, x_T^T) \in \mathbb{R}^{nT}$, $u = (u_0^T, \ldots, u_{T-1}^T)^T \in \mathbb{R}^{mT}$, $z = (z_1^T, \ldots, z_T^T)^T \in \mathbb{R}^{nT}$ and $\eta \in \mathbb{R}^n$. Formally, for given parameter $z \in \mathbb{R}^{nT}$, each agent $i \in \mathbb{Z}[1, N]$ solves the finite horizon optimal tracking problem

$$(x^i(\cdot), u^i(\cdot)) := \arg \min_{(x, u) \in \mathbb{R}^{(n+m)T}} \quad J(x, u, z)$$

subject to: $x_{t+1} = A_i x_t + B_i u_t$ \hspace{1cm} (17)

$x_t \in X_t^i$ \hspace{1cm} $\forall t \in \mathbb{Z}[0, T - 1]$

$u_t \in U_t^i$ \hspace{1cm} $\forall t \in \mathbb{Z}[0, T - 1]$.

We assume that the optimization problem (17) is feasible, that is, given the initial state $x_0 \in \mathbb{R}^n$, we assume that there exists a control input sequence $\{u_t\}_{t=0}^{T-1}$ such that the sets $\{X_t^i\}_{t=1}^T$ are reachable at time steps $t = 1, \ldots, T$, respectively [14, Chapter 6]. This assumption can be checked by solving a convex feasibility problem; furthermore, the set of initial states $x_0$ such that (17) is solvable can be computed by solving the feasibility problem parametrically in $x_0$. 

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We refer to [4, Section III] for the continuous-time infinite-horizon *unconstrained* counterpart of (16)–(17). Our motivations for studying the discrete-time finite-horizon formulation in (16)–(17) are mainly twofold. First, from the technical point of view, we show in Appendix B that in the presence of an exponential cost-discount factor as in [4, Equation 2.2], [9, Equation 2], e.g., $Q_t = \beta^t Q$ and $R_t = \beta^t R$, for some $\beta \in (0, 1)$ and $Q, R \succ 0$, the optimal value of the corresponding finite-horizon problem is arbitrarily close to the infinite-horizon one, if the finite horizon $T$ is chosen large enough. Second, from the computational point of view, our discrete-time finite-horizon formulation allows us to efficiently address state and input constraints, in the sense that we can embed them in finite-dimensional convex optimization problems (e.g., quadratic programs (QPs) if state and input constraints are linear) that are known to be efficiently solvable numerically.

Along the lines of [4, Section IV], and in view of Theorem 1, our discrete-time, finite-horizon, constrained LQ MF control problem for large population size consists of finding a fixed point of the average mapping, that is, $z \in \mathbb{R}^{nT}$ such that

$$z = \frac{1}{N} \sum_{i=1}^{N} x^{i*}(z) =: A(z), \quad (18)$$

where $x^{i*}$ is defined in (17). In (18), we average the optimal tracking trajectories $\{x^{i*}(z)\}_{i=1}^{N}$ among the whole population (that is, we take $a_1 = \cdots a_N = 1$ in (5), so that Assumption 1 is satisfied with $\bar{a} = 1$) and we require the reference trajectory $z$ to equal such average [4, Section IV.A]. For large populations, the interpretation is that each agent $i$ responds optimally with state and control trajectory $x^{i*}(z), u^{i*}(z)$, to the mass influence $z = A(z)$ [4, Section I, p. 1560].

In the *unconstrained* linear quadratic setting, that is, $X_i^t = \mathbb{R}^n$ and $U_i^t = \mathbb{R}^m$ for all $i \in \mathbb{Z}[1, N]$ and $t \geq 0$, the mappings $x^{i*}$ and $u^{i*}$ in (17) are known in closed form, in both continuous- and discrete-time case, for both infinite and finite horizon [15, Chapter 11]. Using this knowledge, [4, Theorem 3.4] shows that if we replace $(z_{t+1} + \eta)$ in (16) by $\gamma(z_{t+1} + \eta)$, for $\gamma$ small enough, then the mapping $A$ in (18) is CON $^1$ and therefore the Picard–Banach iteration converges to the unique fixed point of $A$ [4, Proposition 3.4].

Unfortunately, it turns out that the mapping $A$ in (18) is not necessarily CON. We therefore apply the results in Section III-B to ensure convergence of suitable fixed point iterations.

---

$^1$The mapping $A$ in (18) is continuous, compact valued and constant, hence CON, if $\gamma = 0$. 

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Following [4 Equation 2.6], for given $\gamma \in \mathbb{R}$, let us consider

$$J_\gamma (x, u, z) := \sum_{t=0}^{T-1} \|x_{t+1} - \gamma(z_{t+1} + \eta)\|_{Q_t}^2 + \|u_t\|_{R_t}^2$$  \hspace{1cm} (19)$$

which coincides with $J$ in (16) when $\gamma = 1$. We notice that the cost function in both (16) and (19) can be rewritten as a particular case of the general cost function in (2) as follows:

$$\arg \min_{x,u} \sum_{t=0}^{T-1} \|x_{t+1} - \gamma(z_{t+1} + \eta)\|_{Q_t}^2 + \|u_t\|_{R_t}^2$$

$$= \arg \min_{x,u} \sum_{t=0}^{T-1} \|x_{t+1}\|_{Q_t}^2 - 2\gamma z_{t+1}^T Q_t x_{t+1} - 2\gamma \eta^T Q_t x_{t+1} + \|u_t\|_{R_t}^2$$

$$= \arg \min_{x,u} \sum_{t=0}^{T-1} \|x_{t+1} - z_{t+1}\|_{Q_t}^2 + 2z_{t+1}^T Q_t x_{t+1} - 2\gamma z_{t+1}^T Q_t x_{t+1} - 2\gamma \eta^T Q_t x_{t+1} + \|u_t\|_{R_t}^2$$

$$= \arg \min_{x,u} \sum_{t=0}^{T-1} \|u_t\|_{R_t}^2 + \|x_{t+1} - z_{t+1}\|_{Q_t}^2 + 2z_{t+1}^T (1 - \gamma) Q_t x_{t+1} - 2\gamma \eta^T Q_t x_{t+1}$$

$$= \arg \min_{\xi} \|\xi\|_Q^2 + \|\xi - \zeta\|_\Delta^2 + 2(C\zeta + c)^T \xi,$$  \hspace{1cm} (20)$$

where $\xi := (x, u) \in \mathbb{R}^{(n+m)T}$, $\zeta := (z^T, \tilde{z}^T)^T \in \mathbb{R}^{(n+m)T}$, for arbitrarily chosen $\tilde{z} \in \mathbb{R}^{mT}$, and

$$Q := \text{diag} (0, \text{diag} (R_1, \ldots, R_T)), \quad \Delta := \text{diag} (\text{diag} (Q_1, \ldots, Q_T) , 0), \quad C := (1 - \gamma) \Delta, \quad (21)$$

$$c := -\gamma \text{diag} (\text{diag} (Q_1, \ldots, Q_T) , 0) ((\frac{1}{0}) \otimes \eta).$$

Note that $(C\zeta)^T \xi = (1 - \gamma) \zeta \Delta^T \xi = (1 - \gamma) z^T \text{diag} (Q_1, \ldots, Q_T) x$, therefore the choice of $\tilde{z}$ does not affect the optimization problem in (20); we just need formally to consider a vector $\zeta$ of the same dimensions of $\xi$ in order to recover the same setting in Section III-B.

We can now show an extension of [4 Theorem 3.4], for the discrete-time finite-horizon constrained case, as corollary to our results in Section III-B.

**Corollary 2 (Fixed point iterations in LQ MF control):** The following iterations and conditions guarantee convergence, from any initial point, to a fixed point of $A$ in (18), where $x^i*$ is as in (17) for all $i \in \mathbb{Z}[1, N]$, with cost function $J_\gamma$ as in (19) in place of $J$:

1. Picard–Banach (7) if $-1 < \gamma < 1$;
2. Krasnoselskij (10) if $-1 \leq \gamma \leq 1$;
3. Mann (12) if $-1 \leq \gamma \leq 1$. 

\[\square\]
Let us illustrate the LQ MF setting with a production planning example inspired by [4, Section II.A]. We consider \( N \) firms supplying the same product to the market. Let \( x^i_t \geq 0 \) represent the production level of firm \( i \) at time \( t \). We assume that each firm can change its production according to the linear dynamics

\[
x^i_{t+1} = x^i_t + u^i_t,
\]

where both the states and inputs are subject to heterogeneous constraints of the form \( x^i_t \in [0, \bar{x}^i] \) and \( u^i_t \in [-\bar{u}^i, \bar{u}^i] \) for all \( t \). We assume that the price of the product reads as

\[
p = p_0 - \rho \left( \frac{1}{N} \sum_{i=1}^{N} x^i \right),
\]

for \( p_0, \rho > 0 \). Each firm seeks a production level \( x^i \) proportional to the product price \( p \), while facing the cost to change its production level (for example, for adding or removing production lines). We can then formulate the associated LQ MF finite horizon cost function as

\[
J(x, u, z) := \sum_{t=0}^{T-1} (x_{t+1} - \gamma (z_{t+1} + \eta))^2 + ru^2_t
\]

(22)

where \( \eta := -p_0/\rho, \gamma := -\rho, r > 0, x = (x_1, \ldots, x_T)^\top \in \mathbb{R}^T, u = (u_0, \ldots, u_{T-1})^\top \in \mathbb{R}^T \) and \( z = (z_1, \ldots, z_T)^\top \in \mathbb{R}^T \). Given a parameter \( z \in \mathbb{R}^T \), each agent, \( i = 1, \ldots, N \), solves a finite horizon optimal tracking problem as defined in (17). For illustration, we consider the case of a heterogeneous population of firms where we sample \( \bar{x}^i \) from a uniform distribution supported on \([0, 10]\) and \( \bar{u}^i \) from a uniform distribution supported on \([0, \bar{x}^i/5]\). We consider the parameters \( p_0 = 10, \rho = 1, T = 20 \), and hence \( \gamma = -1 \). The mapping \( A(\cdot) \) defined in (18) is then NE, thus the Krasnoleskij iteration does guarantee convergence to a fixed point, according to Corollary 2.

For different population sizes \( N \), we first numerically compute a fixed point \( \bar{z} \) of \( A(\cdot) \) using Krasnoleskij iteration with parameter \( \lambda = 0.5 \), and we hence compute the strategies \( \{x^i(\bar{z}), u^i(\bar{z})\}_{i=1}^{N} \). We then verify that this is an \( \varepsilon_N \)-Nash equilibrium: for each firm \( i \), we evaluate the individual cost \( \bar{J}^i := J(x^i(\bar{z}), u^i(\bar{z}), \bar{z}) \) and the actual optimal cost \( J^i \) under the knowledge of the production plan of the other firms at the fixed point \( \bar{z} \). In Figure 1 we plot the maximum benefit \( \varepsilon_N := \max_{i \in \mathbb{Z}[1, N]} |J^i - \bar{J}^i| \) that a firm could achieve by deviating from the solution computed via the fixed point iteration, normalized by the optimal cost in the homogeneous case with expected constraints (\( x^i \in [0, 5], u^i \in [-1, 1] \) \( \forall i \in \mathbb{Z}[1, N] \)). According to Theorem 1 such benefit vanishes as the population size increases.
charging rates, vehicle-to-grid operations. However, we prefer to keep the same setting of [6], [16] for simplicity.

We consider a dynamic pricing, where the electricity price function over the charging horizon. We aim to minimize its charging cost over the charging horizon, which represents the charging efficiency.

As second control application, we investigate the problem of coordinating the charging of a large population of PEVs, introduced in [6] and extended to the constrained case in [16]. For each PEV \( i \in \mathbb{Z}[1, N] \), we consider the discrete-time, linear dynamics

\[
x_{t+1}^i = x_t^i + b_i u_t^i
\]

where \( x_t^i \in [0, 1] \) is the state of charge, \( u_t^i \in [0, 1] \) is the charging control input and \( b_i > 0 \) represents the charging efficiency.

The objective of each PEV \( i \) is to acquire a charge amount \( \gamma_i \in [0, 1] \) within a finite charging horizon \( T \in \mathbb{Z}[1, \infty) \), hence to satisfy the charging constraint

\[
\sum_{t=0}^{T-1} u_t^i = 1^T u^i = \gamma_i,
\]

while minimizing its charging cost

\[
\sum_{t=0}^{T-1} p_t(\cdot) u_t^i = p(\cdot)^T u^i,
\]

where \( p(\cdot)^T = (p_0(\cdot), \ldots, p_{T-1}(\cdot))^T \) is the electricity price function over the charging horizon. We consider a dynamic pricing, where

\[\text{Fig. 1: As the population size } N \text{ increases, the maximum achievable individual cost improvement } \varepsilon_N, \text{ relative to the optimal cost in the homogeneous case with expected constraints } (x_t^i \in [0, 5], u_t^i \in [-1, 1] \ \forall i \in \mathbb{Z}[1, N]), \text{ decreases to zero. For all population sizes, } N \text{ agents are randomly selected.} \]

B. Decentralized constrained charging control of large populations of plug-in electric vehicles

As second control application, we investigate the problem of coordinating the charging of a large population of PEVs, introduced in [6] and extended to the constrained case in [16]. For each PEV \( i \in \mathbb{Z}[1, N] \), we consider the discrete-time, linear dynamics

\[
x_{t+1}^i = x_t^i + b_i u_t^i
\]

where \( x_t^i \in [0, 1] \) is the state of charge, \( u_t^i \in [0, 1] \) is the charging control input and \( b_i > 0 \) represents the charging efficiency.

The objective of each PEV \( i \) is to acquire a charge amount \( \gamma_i \in [0, 1] \) within a finite charging horizon \( T \in \mathbb{Z}[1, \infty) \), hence to satisfy the charging constraint

\[
\sum_{t=0}^{T-1} u_t^i = 1^T u^i = \gamma_i,
\]

while minimizing its charging cost

\[
\sum_{t=0}^{T-1} p_t(\cdot) u_t^i = p(\cdot)^T u^i,
\]

where \( p(\cdot)^T = (p_0(\cdot), \ldots, p_{T-1}(\cdot))^T \) is the electricity price function over the charging horizon. We consider a dynamic pricing, where

\[\text{We can also allow for more general convex constraints, for instance on the desired state of charge, multiple charging intervals, charging rates, vehicle-to-grid operations. However, we prefer to keep the same setting of } [6], [16] \text{ for simplicity.} \]
the price of electricity depends on the overall demand, namely the inflexible demand plus the aggregate PEV demand. In particular, in line with the (almost-affine) price function in \[6\], \[16\], we consider an affine price function \( p(z) := 2(a z + c) \), where \( a > 0 \) represents the inverse of the price elasticity of demand and \( c \geq 0 \) denotes the average inflexible demand. The interest of each agent is to minimize its own charging cost \( 2(a z + c)^\top u^i \), leading to a linear program with undesired discontinuous optimal solution. Therefore, following \[6\], \[16\], we also introduce a quadratic relaxation term as follows.

For all \( i \in \mathbb{Z}[1, N] \), the optimal charging control \( u^{i*} \), given the price signal \( z = (z_0, \ldots, z_{T-1}) \in \mathbb{R}^T \), is defined as
\[
    u^{i*}(z) := \arg \min_{u \in \mathbb{R}^T} \delta \| u - z \|^2 + 2(a z + c)^\top u \text{ subject to: } 0 \leq u \leq U^i, \quad 1^\top u = \gamma_i,
\]
where \( \delta > 0 \) and \( U^i \in \mathbb{R}^T_{\geq 0} \) is a vector of desired upper bounds on the charging inputs.

The perturbation \( \delta > 0 \) should be chosen small to approximate the original linear cost \( 2(a z + c)^\top u^i \). We refer to \[16\] Section V] for a numerical evidence of the beneficial effect of choosing a small \( \delta > 0 \) for the perturbed cost in (23).

In view of Theorem 1, we seek a fixed point of the mapping
\[
    \mathcal{A}(z) := \frac{1}{N} \sum_{i=1}^N u^{i*}(z)
\]
which represents the average among the optimal charging control inputs \( \{u^{i*}(z)\}_{i=1}^N \).

Since the cost function in (23) is a particular case of the general cost function in (2), namely with \( Q = 0 \), \( \Delta = \delta I \), \( C = a I \), we can establish conditions on \( \delta > 0 \) under which a specific fixed point iteration converges to a MF almost-Nash solution of the constrained charging control problem. In particular, the Mann iteration always converges to a fixed point of the aggregation mapping and hence solves the constrained MF control problem for large population size.

**Corollary 3 (Fixed point iterations in MF PEVs charging):** The following iterations and conditions guarantee convergence, from any initial point, to a fixed point of \( \mathcal{A} \) in (24), where \( u^{i*} \) is as in (23) for all \( i \in \mathbb{Z}[1, N] \):

1. Picard–Banach \( (7) \) if \( \delta > a/2 \);
2. Krasnoselskij \( (10) \) if \( \delta \geq a/2 \);
3. Mann \( (12) \) if \( \delta > 0 \).

\[ \square \]
In [6], only the Picard–Banach iteration is considered, for some values of $\delta > a/2$. For small values of $\delta$, it is shown in both [6] and [16] that the Picard–Banach iteration causes permanent price oscillations. On the other hand, in [16] it is observed in simulation that the Mann iteration does converge. Corollary provides theoretical support for this observation.

Using the same numerical values as in [6], Figure 2 shows that, if we choose the parameter $\delta > 0$ small enough, we recover the valley-filling solution, known to be globally optimal in the case without charging upper bounds [6, Lemma 3.1]. For the same case, we show in Figure 3 that the Picard–Banach iteration oscillates indefinitely, while the Mann iteration converges.

![Fig. 2: Charging setting without upper bounds ($\delta = 10^{-4}$): the Mann iteration converges to the desired valley-filling solution.](image)

We refer to [16] for discussions and further numerical simulations. Application to realistic PEVs case studies is topic of current work.

V. CONCLUSION AND OUTLOOK

Conclusion

We have considered mean field control approaches for large populations of systems, consisting of several agents with different individual behaviors, constraints and interests, and affected by the aggregate behavior of the overall population. We have extended mean field control theory to problems with heterogeneous convex constraints, for instance arising from agents with linear
dynamics subject to convex state and control constraints. We have proposed several decentralized iterative methods for constrained mean field problems, as summarized in Table I converging to a mean field Nash equilibrium for large population size. We believe that our methods and results open several research directions in mean field control theory and inspire novel methods to various applications.

**Outlook on extensions and applications**

Most of the mathematical results from operator theory we adopted for finite-dimensional Euclidean spaces, also hold for infinite-dimensional Hilbert spaces. Therefore, our technical results can be adapted to infinite-horizon MF control problems, as addressed in [4]. For completeness, in Appendix A we present the most general known fixed point iteration, that is, the Ishikawa iteration in (26), which guarantees convergence to a fixed point of a (non-strictly) PseudoContractive (PC) mapping [11, Theorem 5.1]. In this paper we have considered MF problems in which the aggregation of the optimizers is at least SPC, so that the generally faster Mann iteration in (12) ensures convergence. An open question is whether there exist MF problems in which the average among optimizers is PC, but not SPC, so that the use of the Ishikawa iteration is actually necessary.
TABLE I: Conditions on the problem data, corresponding regularity properties of the aggregation mapping and iterations that ensure convergence to a fixed point of the aggregation mapping.

| Constrained MF control with quadratic cost (Sections II, III) | Fixed Point Iterations | Condition | Property | Picard–Banach | Krasnoselskij | Mann |
|-------------------------------------------------------------|-------------------------|-----------|----------|---------------|---------------|------|
|                                                             |                         | \[
\begin{bmatrix}
Q^+ + ∆ \\
(∆ - C)^	op
\end{bmatrix} \succ 0
\] | CON | ✓ | ✓ | ✓ |
|                                                             |                         | \[-Q \preceq C = C^\top \prec ∆\] | FNE | ✓ | ✓ | ✓ |
|                                                             |                         | \[
\begin{bmatrix}
Q^+ + ∆ \\
(∆ - C)^	op
\end{bmatrix} \succeq 0
\] | NE | ✓ | ✓ |  |
|                                                             |                         | ∆ \prec C = C^\top | SPC | ✓ |  |  |

| Discrete-time constrained LQ MF control (Section IV-A) | Fixed Point Iterations | Condition | Property | Picard–Banach | Krasnoselskij | Mann |
|-------------------------------------------------------|------------------------|-----------|----------|---------------|---------------|------|
|                                                       |                        | \[-1 < γ < 1\] | CON | ✓ | ✓ | ✓ |
|                                                       |                        | \[-1 \leq γ \leq 1\] | NE | ✓ | ✓ |  |

| Constrained MF PEV-charging control (Section IV-B) | Fixed Point Iterations | Condition | Property | Picard–Banach | Krasnoselskij | Mann |
|-----------------------------------------------------|------------------------|-----------|----------|---------------|---------------|------|
|                                                     |                        | δ > a/2   | CON | ✓ | ✓ | ✓ |
|                                                     |                        | δ \geq a/2 | NE | ✓ | ✓ |  |
|                                                     |                        | δ > 0     | SPC | ✓ |  |  |
We have considered agents with homogeneous cost functions, coupled via the aggregate population behavior. The cases of heterogeneous cost functions and couplings in the constraints are possible generalizations, motivated by settings where different agents may have different local interests and local mutual constraints.

Since we have addressed a deterministic setting, inspired by the deterministic agent dynamics in [6], a valuable extension would be a stochastic setting in presence of state and input constraints. For instance, the matrices $A_i$, $B_i$ of each agent $i$ can be thought as extracted from a probability distribution [4, Section V], and a zero-mean random input can enter linearly in the agent dynamics [4, Equation 2.1].

The concept of social global optimality [9] has not been considered in this paper. Following the lines of [9, Section IV], it would be valuable to use our mathematical tools to show, under suitable technical conditions, that the MF structure allows one to coordinate efficiently decentralized constrained optimization schemes.

Our constrained MF setting can be also extended in many transverse directions. For instance, the effect of local heterogeneous constraints can be studied in MF games with leader-follower (major-minor) agents [17] and in coalition formation MF games [18]. Furthermore, we believe that our constrained setting and methods can be also exploited in large-scale network problems [19].

Potential applications of our results include decentralized control and game-theoretic coordination of large-scale systems. Among others, an application field which is potentially suited for our constrained MF control approach is the efficient regulation of power grids and energy markets, indeed characterized by a large number of agents with heterogeneous behaviors and interests, for instance wishing to efficiently buy and/or sell services and energy [20]. Typical case studies, which can be further explored in view of our constrained MF problems, are demand-side management of aggregated loads [8], synchronization and frequency regulation among populations of coupled oscillators [5], [21].

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A. Further mathematical tools from operator theory

In this section, we present some useful definitions from operator theory, adapted to finite-dimensional Euclidean spaces from [10], [11]. We start from the most general fixed point iteration, the Ishikawa iteration, which guarantees convergence to a fixed point of pseudocontractive mappings, as formalized next [11, Remark 3, pp. 12–13].

**Definition 6 (PseudoContractive mapping):** A mapping \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is pseudocontractive (PC) in \( H_Q \) if

\[
\| f(x) - f(y) \|_Q^2 \leq \| x - y \|_Q^2 + \| f(x) - f(y) - (x - y) \|_Q^2
\]

for all \( x, y \in \mathbb{R}^n \). □

If a mapping \( f : \mathcal{C} \rightarrow \mathcal{C} \) is PC and Lipschitz, with \( \mathcal{C} \subseteq \mathbb{R}^n \) compact and convex, then the Ishikawa iteration

\[
z_{k+1} = (1 - \alpha_k)z_k + \alpha_k f ((1 - \beta_k)z_k + \beta_k f(z_k))
\]

where \( \{\alpha_k\}_{k=0}^\infty, \ \{\beta_k\}_{k=0}^\infty \) are such that \( 0 \leq \alpha_k \leq \beta_k \leq 1 \ \forall k \geq 0, \ \lim_{k \to \infty} \beta_k = 0 \) and \( \sum_{k=0}^\infty \alpha_k \beta_k = \infty \), converges, for any initial point \( z_0 \in \mathcal{C} \), to a fixed point of \( f \) [11, Theorem 5.1].

We notice that an SPC mapping is PC as well, therefore the Ishikawa iteration in (26) can be used in place of the Mann iteration in Corollary I. However, unlike the Mann iteration, in general there is no known convergence rate for the Ishikawa iteration, and in fact the convergence is usually much slower compared to the Mann iteration.

As exploited in the proofs of the main results, both SPC in Definition 5 and PC in Definition 6 can be characterized in terms of accretive and monotone mappings, according to the following definitions and results [11, Definition 1.14, p. 13], [10, Definition 20.1].

**Definition 7 (Strongly Accretive mapping):** A mapping \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is strongly accretive (SAC) in \( H_Q \) if there exists \( \epsilon > 0 \) such that

\[
(f(x) - f(y))^\top Q (x - y) \geq \epsilon \| x - y \|_Q^2
\]

for all \( x, y \in \mathbb{R}^n \). □
Definition 8 (Monotone mapping): A mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) is monotone (MON) in \( \mathcal{H}_Q \) if
\[
(f(x) - f(y))^\top Q (x - y) \geq 0
\]
for all \( x, y \in \mathbb{R}^n \).

Lemma 2: If \( f : \mathbb{R}^n \to \mathbb{R}^n \) is MON in \( \mathcal{H}_Q \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \) is SAC in \( \mathcal{H}_Q \), then \( f + g \) is SAC in \( \mathcal{H}_Q \).

Proof: It follows from Definitions [7, 8] that there exists \( \epsilon > 0 \) such that:
\[
(f(x) + g(x) - (f(y) + g(y)))^\top Q (x - y) = (f(x) - f(y))^\top Q (x - y) + (g(x) - g(y))^\top Q (x - y) \geq \epsilon \|x - y\|_Q^2
\]
for all \( x, y \in \mathbb{R}^n \).

Remark 4: \( f \) FNE \( \implies f \) MON [10, Example 20.5]; \( f \) PC \( \iff \) Id \( - f \) MON [10, Example 20.8].

Lemma 3: For any \( f : \mathbb{R}^n \to \mathbb{R}^n \), the mapping Id \( - f \) is SPC in \( \mathcal{H}_Q \) if and only if there exists \( \epsilon > 0 \) such that \( (f(x) - f(y))^\top Q (x - y) \geq \epsilon \|f(x) - f(y)\|_Q^2 \) for all \( x, y \in \mathbb{R}^n \).

If \( f \) is Lipschitz and SAC in \( \mathcal{H}_Q \), then Id \( - f \) is SPC in \( \mathcal{H}_Q \).

Proof: By Definition [5] Id \( - f \) is SPC if there exists \( \rho < 1 \) such that
\[
\|f(x) - f(y) - (x - y)\|_Q^2 \leq \|x - y\|_Q^2 + \rho \|f(x) - f(y)\|_Q^2
\]
for all \( x, y \in \mathbb{R}^n \). Equivalently, since
\[
\|f(x) - f(y) - (x - y)\|_Q^2 = \|f(x) - f(y)\|_Q^2 + \|x - y\|_Q^2 - 2 (f(x) - f(y))^\top Q (x - y)
\]
we have
\[
\|f(x) - f(y)\|_Q^2 - 2 (f(x) - f(y))^\top Q (x - y) \leq \rho \|f(x) - f(y)\|_Q^2
\]
\[
\iff \frac{1 - \rho}{2} \|f(x) - f(y)\|_Q^2 \leq (f(x) - f(y))^\top Q (x - y)
\]
for all \( x, y \in \mathbb{R}^n \), which proofs the first statement with \( \epsilon = \frac{1 - \rho}{2} \).

If \( f \) is Lipschitz and SAC then there exist \( L, \epsilon > 0 \) such that \( \epsilon \|f(x) - f(y)\|_Q^2 \leq \epsilon L \|x - y\|_Q^2 \leq L (f(x) - f(y))^\top Q (x - y) \) for all \( x, y \in \mathbb{R}^n \). Therefore, we have \((f(x) - f(y))^\top Q (x - y) \geq \frac{\epsilon}{L} \|f(x) - f(y)\|_Q^2\), which implies that Id \( - f \) is SPC from the previous part of the proof.

Regularity of affine mappings

We next present necessary and sufficient conditions to characterize the regularity of affine mappings. Some of these equivalences are exploited in the proofs in Appendix [C]. The statements
could be further exploited to show which fixed point iteration solves the unconstrained LQ MF control problem introduced in Section IV-A.

**Lemma 4 (Regularity of affine mappings):** The following equivalencies hold true for any mapping \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined as \( f(x) := Ax + b \), for some \( A \in \mathbb{R}^{n \times n} \) and \( b \in \mathbb{R}^n \).

1. CON in \( \mathcal{H}_Q \) \( \iff \) \( A^\top QA - Q \prec 0 \)
2. NE in \( \mathcal{H}_Q \) \( \iff \) \( A^\top QA - Q \preceq 0 \)
3. FNE in \( \mathcal{H}_Q \) \( \iff \) \( 2A^\top QA \preceq A^\top Q + QA \)
4. SAC in \( \mathcal{H}_Q \) \( \iff \) \( A^\top Q + QA \succ 0 \)
5. MON in \( \mathcal{H}_Q \) \( \iff \) \( A^\top Q + QA \succeq 0 \)
6. PC in \( \mathcal{H}_Q \) \( \iff \) \( A^\top Q + QA \preceq 2Q \)

**Proof:** The mapping \( f \) is CON in \( \mathcal{H}_Q \) if and only if there exists \( \epsilon > 0 \) such that
\[
\|f(x) - f(y)\|_Q^2 = \|A(x - y)\|_Q^2 \leq (1 - \epsilon)^2 \|x - y\|_Q^2 \quad \text{for all } x, y \in \mathbb{R}^n; \text{ equivalently,}
\]
\[
(x - y)^\top A^\top QA (x - y) \leq (1 - \epsilon)^2 (x - y)^\top Q (x - y) \quad \text{for all } x, y \in \mathbb{R}^n, \text{ that is } A^\top QA \preceq (1 - \epsilon)^2 Q \Leftrightarrow A^\top QA - Q \preceq -(2\epsilon - \epsilon^2)Q.
\]
Since \( Q \succ 0 \), the existence of \( \epsilon > 0 \) such that the latter matrix inequality holds is equivalent to the existence of \( \epsilon > 0 \) such that \( A^\top QA - Q \preceq -\epsilon I \). An analogous proof with \( \epsilon = \epsilon = 0 \) shows that the mapping \( f \) is NE in \( \mathcal{H}_Q \) if and only if \( A^\top QA - Q \preceq 0 \).

The mapping \( f \) is FNE in \( \mathcal{H}_Q \) if and only if
\[
\|f(x) - f(y)\|_Q^2 = \|A(x - y)\|_Q^2 \leq \|x - y\|_Q^2 - \|f(x) - f(y) - (x - y)\|_Q^2 \quad \text{for all } x, y \in \mathbb{R}^n. \text{ Equivalently, we get}
\]
\[
(x - y)^\top A^\top QA (x - y) \leq (x - y)^\top Q (x - y) - (x - y)^\top (A - I)^\top Q (A - I) (x - y) \quad \text{for all } x, y \in \mathbb{R}^n, \text{ that is } A^\top QA \preceq Q - (A - I)^\top Q (A - I) = Q - A^\top QA + A^\top Q + QA - Q \Leftrightarrow 2A^\top QA \preceq A^\top Q + QA.
\]

The mapping \( f \) is SAC in \( \mathcal{H}_Q \) if and only if there exists \( \epsilon > 0 \) such that
\[
(f(x) - f(y))^\top Q (x - y) = (x - y)^\top A^\top Q (x - y) \geq \epsilon \|x - y\|_Q^2 = \epsilon (x - y)^\top Q (x - y) \quad \text{for all } x, y \in \mathbb{R}^n, \text{ that is equivalent to } \frac{1}{2} (A^\top Q + QA) \succ \epsilon Q. \text{ Since } Q \succ 0, \text{ the existence of } \epsilon > 0 \text{ such that the latter matrix inequality holds is equivalent to the existence of } \epsilon > 0 \text{ such that } A^\top Q + QA \succ \epsilon I. \text{ An analogous proof with } \epsilon = \epsilon = 0 \text{ shows that the mapping } f \text{ is MON in } \mathcal{H}_Q \text{ if and only if } A^\top Q + QA \succeq 0.
The mapping \( f \) is PC in \( \mathcal{H}_Q \) if and only if
\[
\|f(x) - f(y)\|_Q^2 = \|A(x - y)\|_Q^2 \leq \|x - y\|_Q^2 + \|f(x) - f(y) - (x - y)\|_Q^2 = \|x - y\|_Q^2 + \|(A - I)(x - y)\|_Q^2
\]
for all \( x, y \in \mathbb{R}^n \). Equivalently, we get
\[
(x - y)^\top A^\top Q A (x - y) \leq (x - y)^\top Q (x - y) + (I - A)^\top Q (I - A) (x - y)
\]
for all \( x, y \in \mathbb{R}^n \), that is \( A^\top Q A \preceq Q + (I - A)^\top Q (I - A) = 2Q - (A^\top Q + QA) + A^\top QA \) and hence \( A^\top Q + QA \preceq 2Q \).

\[
(x - y)^\top A^\top Q A (x - y) \leq (x - y)^\top Q (x - y) + (I - A)^\top Q (I - A) (x - y)
\]
for all \( x, y \in \mathbb{R}^n \), that is \( A^\top Q A \preceq Q + (I - A)^\top Q (I - A) = 2Q - (A^\top Q + QA) + A^\top QA \) and hence \( A^\top Q + QA \preceq 2Q \).

B. Finite-horizon approximation of infinite-horizon discounted-cost optimization problems

Let us consider continuous, uniformly bounded, convex functions \( \{\ell_t : \mathcal{X}_t \to \mathbb{R}_{\geq 0}\}_{t=1}^\infty \), where for all \( t \in \mathbb{Z}[1, \infty) \), \( \mathcal{X}_t \subset \mathcal{X} \subset \mathbb{R}^n \) is compact and convex, and \( \mathcal{X} \subset \mathbb{R}^n \) is compact and convex as well. Consider the infinite-dimensional set \( S := \times_{t=1}^\infty \mathcal{X}_t = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots, \beta \in (0, 1) \), and the function \( J_\infty : S \to \mathbb{R}_{\geq 0} \) defined as
\[
J_\infty (\{x_t\}_{t=1}^\infty) := \sum_{t=1}^\infty \beta^t \ell_t (\{x_h\}_{h=1}^t).
\]
(29)

Let us also define
\[
J^*_\infty := \inf_{y \in S} J_\infty (y), \quad x^*_\infty := \arg \min_{y \in S} J_\infty (y),
\]
(30)
where we assume that the infimum \( J^*_\infty \) is attained in a unique point \( x^*_\infty \in S \).

Analogously, let us define the finite-dimensional counterparts of the above quantities. We consider \( S_T := \times_{t=1}^T \mathcal{X}_t \subset \mathcal{X}^T \subset (\mathbb{R}^n)^T, J_T : S_T \to \mathbb{R}_{\geq 0} \) defined as
\[
J_T (\{x_t\}_{t=1}^T) := \sum_{t=1}^T \beta^t \ell_t (\{x_h\}_{h=1}^t),
\]
(31)
besides the optimal value \( J^*_T \) and optimizer \( x^*_T \), assumed to be unique:
\[
J^*_T := \min_{x \in S_T} J_T (x), \quad x^*_T := \arg \min_{x \in S_T} J_T (x).
\]
(32)

We next show that if \( T \) is chosen large enough, then \( J^*_T \) gets arbitrarily close to \( J^*_\infty \).

**Proposition 1 (Finite-horizon approximation):** Let \( J^*_\infty \) and \( J^*_K \) be defined respectively as in (29) and (31). Then \( \lim_{T \to \infty} |J^*_T - J^*_\infty| = 0 \).

\[
(x - y)^\top A^\top Q A (x - y) \leq (x - y)^\top Q (x - y) + (I - A)^\top Q (I - A) (x - y)
\]
for all \( x, y \in \mathbb{R}^n \), that is \( A^\top Q A \preceq Q + (I - A)^\top Q (I - A) = 2Q - (A^\top Q + QA) + A^\top QA \) and hence \( A^\top Q + QA \preceq 2Q \).
Proof: Let \((x^*_\infty)_t\) denote the component \(t\) of \(x^*_\infty\), which we rewrite as \(x^*_\infty = \{(x^*_\infty)_t\}_{t=1}^{\infty}\). We start from the following inequalities:

\[
J^*_T \leq J^*_T \left(\left(\left(x^*_\infty\right)_t\right)_t^T\right) \leq J^*_\infty = J^*_T \left(\left(\left(x^*_\infty\right)_t\right)_t^T\right) + \sum_{t=T+1}^{\infty} \beta^t \ell_t \left(\left\{\left(x^*_\infty\right)_\tau\right\}_{\tau=1}^{\infty}\right) \leq J^*_T + \sum_{t=T+1}^{\infty} \beta^t \ell_t \left(\left\{y^t_\tau\right\}_{\tau=1}^{\infty}\right)
\]

where, for all \(\tau \geq 1\), \(y^t_\tau := (x^*_\tau)^T \in X_\tau\) if \(\tau \in [1,T]\), \(y^t_\tau := (x^*_\infty)^T \in X_\tau\) if \(\tau \geq T + 1\).

Now define \(L := \sup_{t \in [1,\infty]} \sup_{x \in X_t} \ell_t(x)\), and notice that \(L < \infty\) as the functions \(\{\ell_t\}_{t \geq 1}\) are assumed uniformly bounded. We then have

\[
0 \leq J^*_\infty - J^*_T \leq L \sum_{t=T+1}^{\infty} \beta^t \leq \frac{L}{1-\beta} T^{\to \infty} \to 0,
\]

from which we conclude that \(\lim_{T \to \infty} |J^*_T - J^*_\infty| = 0\).

In presence of an exponential cost-discount factor in the cost as in [4, Equation 2.2], [9, Equation 2], Proposition 1 suggests the finite-horizon approach as a way to approximate MF \(\varepsilon\)-Nash equilibria relative to infinite-horizon formulations. The formalization of such claim, under suitable regularity conditions, goes beyond the purposes of this paper and hence it is left as future work.

C. Main proofs

Proof of Lemma 1

The expression of the (unique) unconstrained optimizer \(\hat{x}^*(z)\) directly follows from the equation

\[
0 = \frac{\partial}{\partial x} J(x, z) = \frac{\partial}{\partial x} \left(x^T Q x + (x - z)^T \Delta (x - z) + 2 (C z + c)^T x\right) = 2x^T Q + 2(x - z)^T \Delta + 2 (C z + c)^T \Delta.
\]

Then the following equalities hold:

\[
\text{Proj}_{X^i}^{Q+\Delta} (\hat{x}^*(z)) = \arg \min_{y \in X^i} \|y - \hat{x}^*(z)\|_{Q+\Delta}^2 = \arg \min_{y \in X^i} \left(y - (Q + \Delta)^{-1} ((\Delta - C) z - c)\right)^T (Q + \Delta) \left(y - (Q + \Delta)^{-1} ((\Delta - C) z - c)\right)
\]

\[
= \arg \min_{y \in X^i} y^T (Q + \Delta) y - 2y^T ((\Delta - C) z - c)
\]

\[
= \arg \min_{y \in X^i} y^T Q y + y^T \Delta y - 2y^T \Delta z + 2 (C z + c)^T y
\]

\[
= \arg \min_{y \in X^i} y^T Q y + (y - z)^T \Delta (y - z) + 2 (C z + c)^T y = x^i_\infty(z).
\]
Proof of Theorem 1

From Lemma 1, we have that \(x^i(\mathbf{z}) = \text{Proj}_{X_i}(\mathbf{z})\), that is the metric projection (in the Euclidean space \(H_{Q+\Delta}\)) onto the compact and convex set \(X^i\) of the affine mapping \(z \mapsto (Q + \Delta)^{-1}((\Delta - C)z - c)\). Therefore the mappings \(\{x^i\}_i=1\) are Lipschitz with the same constant, that is, there exists \(L > 0\) such that \(\|x^i(v) - x^i(w)\|_\infty \leq L\|v - w\|_\infty\) for all \(v, w \in \mathbb{R}^n\) and for all \(i \in \mathbb{Z}[1, N]\).

Now, \(J\) in (2) is a quadratic function and takes values on a compact subset of \(\mathbb{R}^n \times \mathbb{R}^n\), therefore it is Lipschitz, and hence there exists \(M > 0\) such that
\[
|J(v, z_1) - J(w, z_2)| \leq M (\|v - w\|_\infty + \|z_1 - z_2\|_\infty) \quad \text{for all } v, w \in \mathbb{R}^n, \quad z_1, z_2 \in \mathbb{R}^n.
\]
Let us also define \(D := \max_{v, w \in \mathcal{X}} \|v - w\|_\infty\), where \(\mathcal{X} \supseteq \bigcup_{N \geq 0} \bigcup_{i=1}^N X^i\) is compact from Assumption 1.

We now consider an arbitrary fixed point \(\bar{\mathbf{z}} = \frac{1}{N} \sum_{i=1}^N a_i \bar{x}^i(\bar{\mathbf{z}})\) and let \(\bar{x}^i := \arg \min_{y \in \mathcal{X}^i} J \left( y, \frac{1}{N} \left( a_i y + \sum_{j \neq i}^N a_j \bar{x}^j \right) \right)\). Let \(\bar{x}^i = \arg \min_{x \in \mathcal{X}^i} J \left( x, \frac{1}{N} \left( a_i x + \sum_{j \neq i}^N a_j \bar{x}^j \right) \right)\).

Let us also define the associated costs:
\[
\bar{J}^i = J \left( \bar{x}^i, \frac{1}{N} \left( a_i \bar{x}^i + \sum_{j \neq i}^N a_j \bar{x}^j \right) \right) = \min_{y \in \mathcal{X}} J \left( y, \frac{1}{N} \left( a_i y + \sum_{j \neq i}^N a_j \bar{x}^j \right) \right),
\]
\[
\tilde{J}^i = J \left( \tilde{x}^i, \frac{1}{N} \left( a_i \tilde{x}^i + \sum_{j \neq i}^N a_j \bar{x}^j \right) \right) = \min_{y \in \mathcal{X}} J \left( y, \frac{1}{N} \left( a_i y + \sum_{j \neq i}^N a_j \bar{x}^j \right) \right),
\]
\[
\tilde{J}^i = J \left( \tilde{x}^i, \frac{1}{N} \left( a_i \tilde{x}^i + \sum_{j \neq i}^N a_j \bar{x}^j \right) \right) = \min_{y \in \mathcal{X}} J \left( y, \frac{1}{N} \left( a_i y + \sum_{j \neq i}^N a_j \bar{x}^j \right) \right).
\]

Note that \(\tilde{J}^i \leq \tilde{J}^i \leq \tilde{J}^i\). Then we define \(\hat{\mathbf{z}} := \frac{1}{N} \left( a_i \tilde{x}^i + \sum_{j \neq i}^N a_j \bar{x}^j \right)\) and notice that \(\bar{x}^i = x^i(\bar{\mathbf{z}})\) and \(\tilde{x}^i = x^i(\tilde{\mathbf{z}})\). Therefore, the following inequalities hold true:
\[
0 \leq J^i - J^i \leq \tilde{J}^i - \tilde{J}^i \leq \left| J \left( \bar{x}^i, \bar{\mathbf{z}} \right) - J \left( \tilde{x}^i, \tilde{\mathbf{z}} \right) \right| \leq M \|\bar{x}^i - \tilde{x}^i\|_\infty + M \|\bar{\mathbf{z}} - \tilde{\mathbf{z}}\|_\infty =
\]
\[
M \|\bar{x}^i - \tilde{x}^i\|_\infty + \frac{M}{N} a_i \|\bar{x}^i - \tilde{x}^i\|_\infty = M \|\bar{x}^i(\bar{\mathbf{z}}) - \bar{x}^i(\tilde{\mathbf{z}})\|_\infty + \frac{M}{N} a_i \|\bar{x}^i - \tilde{x}^i\|_\infty \leq
\]
\[
M L \|\bar{\mathbf{z}} - \tilde{\mathbf{z}}\|_\infty + \frac{M a}{N} \|\bar{x}^i - \tilde{x}^i\|_\infty = \frac{\bar{a} M (L+1)}{N} \|\bar{x}^i - \tilde{x}^i\|_\infty \leq \frac{\bar{a} M D (L+1)}{\tilde{\varepsilon}} =: \varepsilon_N.
\]
This proves that for all \(\varepsilon > 0\) there exists \(N = N_\varepsilon := \frac{\bar{a} M D (L+1)}{\varepsilon}\) such that the cost \(J^i\) of any agent \(i\) at a fixed point \(\bar{\mathbf{z}}\) is \(\varepsilon\)-close to its true optimal cost \(\tilde{J}^i\). $$\blacktriangleleft$$
Proof of Theorem 2

It follows from the proof of Lemma 4 in Appendix A that the unconstrained optimizer $\hat{x}^*$ in (3) is CON in $\mathcal{H}_{Q+\Delta}$ if and only if there exist $\epsilon > 0$ such that $((Q + \Delta)^{-1}(\Delta - C))^\top (Q + \Delta) ((Q + \Delta)^{-1}(\Delta - C)) = (\Delta - C)^\top (Q + \Delta)^{-1}(\Delta - C) \preceq (1 - \epsilon)^2 (Q + \Delta)$ equivalent to $(\Delta - C)^\top ((1 - \epsilon)(Q + \Delta))^{-1}(\Delta - C) \preceq (1 - \epsilon)(Q + \Delta)$.

As $Q + \Delta > 0$, by Schur complement [22, Section A.5.5] the last inequality is equivalent to

$$
\begin{bmatrix}
(1 - \epsilon)(Q + \Delta) & \Delta - C \\
(\Delta - C)^\top & (1 - \epsilon)(Q + \Delta)
\end{bmatrix} \succeq 0 \iff
\begin{bmatrix}
Q + \Delta & \Delta - C \\
(\Delta - C)^\top & Q + \Delta
\end{bmatrix} \succeq \epsilon
\begin{bmatrix}
Q + \Delta & 0 \\
0 & Q + \Delta
\end{bmatrix} \iff
\begin{bmatrix}
Q + \Delta & \Delta - C \\
(\Delta - C)^\top & Q + \Delta
\end{bmatrix} \succeq \epsilon I_{2n}
$$

for some $\epsilon > 0$. The proof that $\hat{x}^*$ in (3) is NE in $\mathcal{H}_{Q+\Delta}$ if and only if (13) holds with $\epsilon > 0$ is analogous (with $\epsilon = \varepsilon = 0$).

Since $\text{Proj}_{Q+\Delta}^{x^i}$ is FNE [10, Proposition 4.8] and hence NE in $\mathcal{H}_{Q+\Delta}$, that is

$$\left\|\text{Proj}_{Q+\Delta}^{x^i} (x) - \text{Proj}_{Q+\Delta}^{x^i} (y)\right\|_{Q+\Delta} \leq \|x - y\|_{Q+\Delta} \text{ for all } x, y \in \mathbb{R}^n$$

it follows that the composition $x^i \ast (\cdot) = \text{Proj}_{Q+\Delta}^{x^i} (\hat{x}^i(\cdot))$ is CON in $\mathcal{H}_{Q+\Delta}$ if $\hat{x}^i$ is CON in $\mathcal{H}_{Q+\Delta}$, NE in $\mathcal{H}_{Q+\Delta}$ if $\hat{x}^*$ is NE in $\mathcal{H}_{Q+\Delta}$.

For the rest of the proof, we need the following side result, adapted from [23, Theorem 12.1 (d)].

**Lemma 5:** A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is FNE in $\mathcal{H}_P$, with $P > 0$, if and only if

$$
\|f(x) - f(y)\|_P \leq (x - y)^\top P (f(x) - f(y))
$$

for all $x, y \in \mathbb{R}^n$.

**Proof:** From Definition 4, we have $f$ FNE if and only if, for all $x, y \in \mathbb{R}^n$,

$$
\|f(x) - f(y)\|_P \leq \|x - y\|_P - \|f(x) - f(y)\|_P =
\|x - y\|_P - \Big(\|x - y\|_P^2 + \|f(x) - f(y)\|_P^2 - 2 (x - y)^\top P (f(x) - f(y)) \Big) =
- \|f(x) - f(y)\|_P^2 + 2 (x - y)^\top P (f(x) - f(y))
$$

therefore, equivalently, $\|f(x) - f(y)\|_P^2 \leq (x - y)^\top P (f(x) - f(y))$ for all $x, y \in \mathbb{R}^n$. \qed
From [10, Proposition 4.8] we have that \( \text{Proj}_C^P \) is FNE in \( \mathcal{H}_P \), hence by Lemma 5:

\[
\|\text{Proj}_C^P(\tilde{x}) - \text{Proj}_C^P(\tilde{y})\|_P^2 \leq (\tilde{x} - \tilde{y})^\top P (\text{Proj}_C^P(\tilde{x}) - \text{Proj}_C^P(\tilde{y}))
\]

(34)

for all \( \tilde{x}, \tilde{y} \in \mathbb{R}^n \). Therefore, with \( \tilde{x} := Ax + b \) and \( \tilde{y} := Ay + b \), the FNE condition (34) implies that:

\[
\|\text{Proj}_C^P(Ax + b) - \text{Proj}_C^P(Ay + b)\|_P^2 \leq (x - y)^\top A^\top P (\text{Proj}_C^P(Ax + b) - \text{Proj}_C^P(Ay + b))
\]

(35)

for all \( x, y \in \mathbb{R}^n \).

Now, since \( x^{i\ast}(z) = \text{Proj}_{\mathcal{X}^i}^{Q + \Delta}(\hat{x}(z)) = \text{Proj}_{\mathcal{X}^i}^{Q + \Delta}((Q + \Delta)^{-1}((\Delta - C)z - c)) \) from (4), let us consider (35) with \( Q + \Delta \) in place of \( P \), \( (Q + \Delta)^{-1}(\Delta - C) \) in place of \( A \), \( -(Q + \Delta)^{-1}c \) in place of \( b \), \( \mathcal{X}^i \) in place of \( C \), and \( v, w \) in place of \( x, y \). We hence obtain

\[
0 \leq \|x^{i\ast}(v) - x^{i\ast}(w)\|_{Q + \Delta}^2 \leq (v - w)^\top (\Delta - C)^\top (x^{i\ast}(v) - x^{i\ast}(w))
\]

(36)

for all \( v, w \in \mathbb{R}^n \).

If \( Q + \Delta \succ \Delta - C \succeq 0 \), i.e., \( -Q \preceq C = C^\top \prec \Delta \), then \( \|x^\ast(v) - x^\ast(w)\|_{\Delta - C}^2 \leq \|x^\ast(v) - x^\ast(w)\|_{Q + \Delta}^2 \) for all \( v, w \in \mathbb{R}^n \). Therefore, it follows from (36) that

\[
\|x^\ast(v) - x^\ast(w)\|_{\Delta - C}^2 \leq (v - w)^\top (\Delta - C) (x^\ast(v) - x^\ast(w))
\]

for all \( v, w \in \mathbb{R}^n \), which is equivalent to \( x^\ast(\cdot) \) being FNE in \( \mathcal{H}_{\Delta - C} \) by Lemma 5.

On the other hand, from (36) we get

\[
0 \leq (x^\ast(w) - x^\ast(v))^\top (C - \Delta) (v - w)
\]

for all \( v, w \), which, for \( C - \Delta \succ 0 \), is equivalent to \( -x^\ast(\cdot) \) being MON in \( \mathcal{H}_{C - \Delta} \) by Definition 8. We now notice that \( \text{Id}(\cdot) \) is a SAC mapping by Definition 7, therefore \( \text{Id} - x^\ast \), sum of SAC and MON mappings, is SAC in \( \mathcal{H}_{C - \Delta} \) by Lemma 2. It then follows from Lemma 3 in Appendix A that \( \text{Id} - x^\ast \) Lipschitz and SAC in \( \mathcal{H}_{C - \Delta} \) implies that \( \text{Id} - (\text{Id} - x^\ast) = x^\ast \) is SPC in \( \mathcal{H}_{C - \Delta} \).

Proof of Theorem 3

The mapping \( \mathcal{A} \) in (5) is a convex hull among the mappings \( \{x^{i\ast}\}_{i=1}^N \), that are uniformly Lipschitz in view of Remark [1] therefore \( \mathcal{A} \) is Lipschitz as well.
It follows from Theorem 2 that if $-Q \preceq C = C^T \preceq \Delta$ then, for all $i \in \mathbb{Z}[1, N]$, the mapping $x^i(\cdot)$ is FNE in $\mathcal{H}_{\Delta-C}$. Therefore, $A(\cdot) = \frac{1}{N} \sum_{i=1}^{N} a_i x^i(\cdot)$, convex combination of FNE mappings, is FNE as well [10, Example 4.31]. Analogously, the convex combination of CON (NE) mappings is CON (NE) as well.

For the SPC case, if $\Delta \prec C$ then it follows from the proof of Theorem 2 that, for all $i \in \mathbb{Z}[1, N]$, $\text{Id} - x^i(\cdot)$ is SAC in $\mathcal{H}_{C-\Delta}$, see Definition 7. Then it follows from Lemma 2 that

$$\frac{1}{N} \sum_{i=1}^{N} a_i (\text{Id}(\cdot) - x^i(\cdot))$$

is SAC as well, which implies that

$$\text{Id}(\cdot) - \frac{1}{N} \sum_{i=1}^{N} a_i x^i(\cdot) = \mathcal{A}(\cdot)$$

is SPC in view of Lemma 3.

Proof of Corollary 7

From Theorem 2, if (13) holds for some $\epsilon > 0$, then $A$ is CON and if $-Q \preceq C = C^T \preceq \Delta$, then $A$ is FNE. In both cases, the Picard–Banach iteration converges a fixed point of $A$ [11, Theorem 2.1], [12, Section 1, p. 522], which is unique if $A$ is CON.

For the other two fixed point iterations, we need to consider $A$ as a mapping from a compact convex set to itself. This can be assumed without loss of generality (that is, up to discarding the initial point $z_0$) since $A$ takes values in $\frac{1}{N} \sum_{i=1}^{N} a_i \mathcal{X}^i$, which is a linear transformation of the compact convex sets $\{\mathcal{X}^i\}_{i=1}^{N}$, as hence compact and convex as well [24, Section 3, Theorem 3.1]. If (13) holds for some $\epsilon \geq 0$ then $A$ is NE from Theorem 2 and the Krasnoselskij iteration converges to a fixed point of $A$ [11, Theorem 3.2].

Finally, if $\epsilon \geq 0$ in (13) or $\Delta \prec C$ hold true, then $A$ is SPC. Therefore the Mann iteration converges to a fixed point [11, Fact 4.9, p. 112], [13, Theorem R, Section I].

Proof of Corollary 2

It follows from (20) that the optimization problem in (17) with cost $J$ as in (19) can be rewritten in the same format of (2), where the optimization variable of agent $i$ is the vector $\xi^i = \left( x^T_1, \ldots, x^T_{T-1}, u^T_0, \ldots, u^T_{T-1} \right)^T \in \mathbb{R}^{(n+m)T}$. In particular, it follows from (21) that in the notation in (2) we get the block structured matrices

$$Q = \text{diag}(0, \tilde{R}), \quad \Delta = \text{diag}(Q, 0), \quad C = (1 - \gamma)\Delta,$$
where $\tilde{R} := \text{diag}(R_0, \ldots, R_{T-1}) \succ 0$ and $\tilde{Q} := \text{diag}(Q_1, \ldots, Q_T) \succ 0$. In order to exploit the first point in Corollary 1, we need to consider the matrix

$$
\begin{bmatrix}
Q + \Delta & \Delta - C \\
(\Delta - C)^\top & Q + \Delta
\end{bmatrix} =
\begin{bmatrix}
\tilde{Q} & \gamma \tilde{Q} \\
\gamma \tilde{Q} & \tilde{Q}
\end{bmatrix}
= \Pi^\top \text{diag} \left( \begin{bmatrix} 1 & \gamma \end{bmatrix} \otimes \tilde{Q}, I_2 \otimes \tilde{R} \right) \Pi,
$$

where $\Pi \in \mathbb{R}^{2(n+m)T \times 2(n+m)T}$ is an opportune permutation matrix (which swaps the second and third block columns). Since the eigenvalues of the Kronecker product of two matrices equal to the product of the eigenvalues of the two matrices, we have that $I_2 \otimes \tilde{R} \succ 0$ and that $\left[ \begin{bmatrix} 1 & \gamma \end{bmatrix} \otimes \tilde{Q} \right]$ is positive definite if $-1 < \gamma < 1$, positive semidefinite if $-1 \leq \gamma \leq 1$. Since $\Pi$ is invertible ($\Pi^\top \Pi = I$) and hence has no null eigenvalues, we conclude that $\Pi^\top \text{diag} \left( \begin{bmatrix} 1 & \gamma \end{bmatrix} \otimes \tilde{Q}, I_2 \otimes \tilde{R} \right) \Pi \succ 0$ ($\succeq 0$) if $-1 < \gamma < 1$ ($-1 \leq \gamma \leq 1$). The proof then follows from Corollary 1.

**Proof of Corollary 3**

We consider the matrix inequality (13) in Theorem 2 with $Q = 0$, $\Delta = \delta I$, $\delta > 0$, and $C = aI$, $a > 0$. The existence of $\epsilon > 0$ such that

$$
\begin{bmatrix}
\delta I & (\delta - a)I \\
(\delta - a)I & \delta I
\end{bmatrix} \succeq \epsilon I,
$$

is equivalent, by Schur complement [22, Section A.5.5], to $\delta - (\delta - a)\delta^{-1}(\delta - a) > 0 \iff \delta^2 - (\delta - a)^2 > 0 \iff \delta > a/2$. This implies that if $\delta > a/2$ then $A$ is CON in $\mathcal{H}_{\delta I}$ and, from Corollary 1, the Picard–Banach iteration in (7) converges to its unique fixed point. We now consider the case of $\delta = a/2$. The condition of Theorem 2 for $A$ being NE in $\mathcal{H}_{\delta I}$ reads as $\frac{a}{2} \left[ \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \right] = \frac{a}{2} \left[ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right] \otimes I \succeq 0$, which is satisfied as $a > 0$ and $\left[ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right] \otimes I$ has non-negative eigenvalues. The convergence of the Krasnoselskij iteration in (10) then follows from Corollary 1.

We finally consider the case $\delta \in (0, a/2)$. From the sufficient condition in Theorem 2, we get that $A$ is SPC in $\mathcal{H}_{(a-\delta)I}$ if $\delta \in (0, a)$. The convergence of the Mann iteration in (12) then follows from Corollary 1.

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