Inverse elastic scattering problems with phaseless far field data

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Abstract
This paper is concerned with uniqueness, phase retrieval and shape reconstruction methods for inverse elastic scattering problems with phaseless far field data. The phaseless far field data is closely related to the outward energy flux, which is easily measured in practice. Systematically, we study two basic models, i.e. inverse scattering of plane waves by rigid bodies and inverse scattering of sources with compact support. For both models, we show that the phaseless far field data is invariant under translation of the underlying scattering objects, which implies that the location of the objects can not be uniquely recovered by the data. To solve this problem, we consider simultaneously the incident point sources with one fixed source point and at most three scattering strengths. With this technique, we establish some uniqueness results for source scattering problem with multi-frequency phaseless far field data. Furthermore, a fast and stable phase retrieval approach is proposed based on a simple geometric result which provides a stable reconstruction of a point in the plane from three distances to given points. Difficulties arise for inverse scattering by rigid bodies due to the additional unknown far field pattern of the point sources. To overcome this difficulty, we introduce an artificial rigid body into the system and show that the underlying rigid bodies can be uniquely determined by the corresponding phaseless far field data at a fixed frequency. Noting that the far field pattern of the scattered field corresponding to point sources is very small if the source point is far away from the scatterers, we propose an appropriate phase retrieval method for obstacle scattering problems, without using the artificial rigid body. Finally, we propose several sampling methods for shape reconstruction with phaseless far field data directly. For inverse obstacle scattering problems, two different direct sampling methods are proposed with data at a fixed frequency. For inverse source scattering problems, we introduce two direct sampling methods.
for source supports with sparse multi-frequency data. The phase retrieval techniques are also combined with the classical sampling methods for the shape reconstructions. Extended numerical examples in two dimensions are conducted with noisy data, and the results further verify the effectiveness and robustness of the proposed phase retrieval techniques and sampling methods.

Keywords: elastic scattering, phaseless far field data, uniqueness, phase retrieval, direct sampling methods

(Some figures may appear in colour only in the online journal)

1. Introduction

The inverse wave scattering problems are to determine the unknown objects using the prescribed radiated wave fields. There are two basic models, i.e. inverse scattering of plane waves by obstacles and inverse scattering of sources with compact support. These problems are mainly motivated by a lot of practical applications such as nondestructive testing, seismic exploration, antenna synthesis and biomedical imaging [6, 9, 19]. Waves are transmitted through the earth to detect oil or gas and to study the earth’s geological structure. In many non-invasive neurophysiological techniques, one measure the electromagnetic fields generated by the neuronal activity of the brain to localize the sources of the activities within the brain and to provide the information about both the structure and function of the brain. A survey on the state of the art of the mathematical theory and numerical approaches for inverse time harmonic acoustic and electromagnetic scattering problems can be found in the standard monograph [13]. The inverse elastic wave scattering is more challenging due to the coupling of compressional waves and shear waves that propagate at different speeds. We refer to [18, 20, 21, 25] for the uniqueness and stability theory. Further, several numerical approaches have been developed for shape or source reconstruction [3–5, 7, 8, 10, 11, 15, 19, 22–25, 28, 38, 43]. For the readers interested in a more comprehensive treatment of the direct and inverse elastic scattering problems, we suggest consulting [9, 14, 35, 36] on this subject.

The two well known difficulties of the inverse scattering problems are nonlinearity and ill-posedness. Actually, in many cases of practical interest, the third difficulty is incomplete data, i.e. only partial information can be measured directly. There are two cases of incomplete data. The first one is the limited-aperture data, where the measurements are only available in limited directions or positions. Limited-aperture data can present a severe challenge for the reconstructions [28, 38]. Recently, some data recovery techniques have been developed to obtain the full-aperture data by combining data and models [28, 39]. The far field pattern is a complex valued function defined on the unit circle. In many applications we have only the modulus of the far field pattern, while the phase information is difficult or even impossible to be captured. The second incomplete data of particular interest is the phaseless data. Actually, the phaseless far field data is closely related to the outward energy flux. Many works with phased far field patterns rely heavily on the fact that, by Rellich’s lemma, the radiating waves and their far field patterns are one-to-one. Unfortunately, the corresponding result is not available for the modulus of the far field patterns. Actually, it is well known that the modulus of the far field pattern is invariant under the translation of underlying objects [33, 34]. Thus, the location of the underlying objects can not be uniquely determined. If it is known a priori that the scatterer is a sound-soft ball centered at the origin, uniqueness is established to determine the radius of the ball by a single phaseless far field datum in [40]. Rellich’s lemma is avoided
in this special case. Initial effort is focused on the shape reconstruction numerically. Indeed, many efficient numerical implementations \([2, 17, 26, 27, 33]\) imply that shape reconstruction from the phaseless far field pattern is possible. In recent years, considerable effort has been made to break the translation invariance. The first breakthrough is given in \([48]\), where the authors prove that the translation invariance property of the phaseless far field pattern can be broken if superpositions of two plane waves are used as the incident fields for all wave numbers in a finite interval. We refer to \([12, 17, 44–47, 49, 50]\) on recent progress on uniqueness and numerical methods in this direction.

In this work, we consider the elastic scattering problems with phaseless far field data. To our best knowledge, this is the first work on uniqueness, phase retrieval and sampling methods for inverse elastic scattering problem with phaseless far field data. We refer to a recent manuscript \([16]\) on iterative methods with a reference ball for determining a rigid body using phaseless far field data. By considering simultaneously the scattering of point sources, we introduce a fast and simple phase retrieval method, and then propose some direct sampling methods with phaseless far field data for shape reconstruction. This work is a nontrivial extension of the results in \([29, 30]\) for the inverse acoustic scattering problem of the Helmholtz equation to the inverse elastic scattering of the Navier equation. The elastic wave equation is more challenging because of the coexistence of compressional waves and shear waves propagating at different speeds. The phaseless acoustic far field data for point sources/scatterers is just the modulus of the strength, which is a constant \([29, 30]\). However, the phaseless elastic far field data for point sources/scatterers changes for different observation directions, and even vanishes at some special directions. Hence more sophisticated modification is required.

This paper is organized as follows. In the next section, we introduce the two basic scattering models, i.e. scattering of plane waves by rigid bodies and scattering of sources with compact support. We show that the phaseless far field data is invariant under translation of the underlying scattering objects, which implies that the location of the underlying scattering objects can not be uniquely determined by the phaseless far field data. To overcome this difficulty, we consider simultaneously the scattering of point sources. Section 3 is devoted to uniqueness results with phaseless far field data. Some uniqueness results for inverse source scattering problem have been established using multi-frequency phaseless far field data. For inverse scattering by rigid bodies, with the help of an artificial rigid body, we prove some uniqueness results using phaseless far field data at a fixed frequency. In section 4, we introduce a phase retrieval technique and propose some direct sampling methods using phaseless far field data. Only one kind of phaseless far field data is needed in the numerical schemes, and the artificial rigid body is avoided. Extended numerical simulations in two dimensions are presented in the last section to indicate the efficiency and robustness of the proposed methods.

In this paper, we aim at conciseness, so that we restrict ourselves to the two dimensional case. Three dimensional case can be dealt with similarly after some modifications. Our methods proposed are independent of physical properties of the scatterers. However, for simplicity, we restrict our presentation to the case where the scatterer is a rigid body in the obstacle scattering problem. The other cases such as scattering by a cavity or by a penetrable inhomogeneous medium of compact support can be treated analogously.

2. Elastic scattering problems

This section is devoted to address the elastic scattering problems. We begin with the notations used throughout this paper. All vectors will be denoted in bold script. For a vector \(\mathbf{x} := (x_1, x_2)^T \in \mathbb{R}^2\), we introduce the two unit vectors \(\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}|\) and \(\hat{\mathbf{x}}^\perp\) obtained by
rotating \( \hat{x} \) anticlockwise by \( \pi/2 \). For simplicity, we write \( \partial_i \) for the usual partial derivative \( \frac{\partial}{\partial x_i}, \ i = 1, 2 \). Then, in addition to the usual differential operators \( \text{grad} := (\partial_1, \partial_2)^T \) and \( \text{div} := (\partial_1, \partial_2) \), we define two auxiliary differential operators \( \text{grad}^\perp := (-\partial_2, \partial_1)^T \) and \( \text{div}^\perp := (-\partial_2, \partial_1) \), respectively. It is easy to deduce the differential identities \( \text{div}^\perp \text{grad} = \text{grad}^\perp \text{div} = 0 \). In general, we shall use the notations \( \text{grad}^\perp \) and \( \text{div}^\perp \) to denote gradient and divergence with respect to \( y \), and \( ds(y) \) and \( dy \) to remind the reader that the appropriate integrals are with respect to \( y \). Denote by \( S := \{ x \in \mathbb{R}^2 : |x| = 1 \} \) the unit circle in \( \mathbb{R}^2 \). Let \( \Omega \subset \mathbb{R}^2 \) be an open and bounded Lipschitz domain such that the exterior \( \mathbb{R}^2 \setminus \overline{\Omega} \) of \( \Omega \) is connected. Here and throughout the paper we denote \( \overline{\Omega} \) the closure of the set \( \Omega \). A confusion with the complex conjugate \( \bar{z} \) of \( z \in \mathbb{C} \) is not expected. Let \( \omega > 0 \) be the circular frequency, \( \lambda \) and \( \mu \) be Lamé constants satisfying \( \mu > 0, 2\mu + \lambda > 0 \). Furthermore, denote by

\[
    k_p := \omega/\sqrt{\lambda + 2\mu} \quad \text{and} \quad k_s := \omega/\sqrt{\mu}
\]

the compressional and shear wave number, respectively.

The two basic problems in classical scattering theory are the scattering of elastic waves by bounded scatterers and of external forces with compact support. In this section we will present the details on these two problems in sections 2.1 and 2.2, respectively. We show that the corresponding phaseless elastic far field data is invariant under translation of the sources/scatterers. Thus, location of the underlying objects can not be uniquely determined from the phaseless elastic far field data. To solve this problem, we consider simultaneously the scattering of point sources.

### 2.1. Scattering of plane waves by scatterers

The first problem is the scattering of elastic waves by a bounded scatterer \( \Omega \). As incident fields \( u^\text{in} \), plane waves are of special interest. The time harmonic elastic plane wave with incident direction \( d \in S \) is given by

\[
    u^\text{in} = a_p u_p^\text{in} + a_s u_s^\text{in}, \quad a_p, a_s \in \mathbb{C},
\]

where \( u_p^\text{in} := de^{i\omega \cdot x \cdot d} \) is a plane compressional wave and \( u_s^\text{in} := d^\perp e^{i\omega \cdot x \cdot d} \) is a plane shear wave, respectively. The two coefficients \( a_p, a_s \in \mathbb{C} \) are weights of the plane compressional wave and the plane shear wave, respectively. Due to the linearity of the direct scattering problems, we take \( (a_p, a_s) = (0, 1) \) or \( (a_p, a_s) = (1, 0) \) for simplicity.

The propagation of time-harmonic elastic wave equation in an isotropic homogeneous media outside \( \Omega \) is governed by the reduced Navier equation

\[
    \Delta^* u_{\Omega} + \omega^2 u_{\Omega} = 0 \quad \text{in} \ \mathbb{R}^2 \setminus \overline{\Omega}, \quad \Delta^* := \mu \Delta + (\lambda + \mu) \text{grad} \text{div},
\]

where \( u_{\Omega} \) denotes the total displacement field. For a rigid body \( \Omega \), the total displacement field \( u_{\Omega} \) satisfies the first (Dirichlet) boundary condition

\[
    u_{\Omega} = 0 \quad \text{on} \ \partial \Omega.
\]

Straightforward calculations show that the incident plane waves (2.1) is an entire solution of the Navier equation (2.2). The scatterer \( \Omega \) gives rise to a scattered field \( u_{\Omega}^s = u_{\Omega} - u^\text{in} \), which is also a solution of the Navier equation (2.2). In the sequel, for the scattering of an incident plane wave \( u_{in}^m, m = p, s \), by a rigid body \( \Omega \), we denote the corresponding scattered field by \( u_{\Omega,m}^s(x, d), m = p, s \). It is well known that the scattered field has a decomposition in the form
\[ u_{1,m}^{sc} = u_{1,m}^{sc,\text{mp}} + u_{1,m}^{sc,\text{ms}} \quad \text{in } \mathbb{R}^2 \setminus \Omega, \]

where

\[ u_{1,m}^{sc,\text{mp}} := -\frac{1}{k_\rho^2} \text{grad div } u_{1,m}^{sc} \quad \text{and} \quad u_{1,m}^{sc,\text{ms}} := -\frac{1}{k_\rho^2} \text{div } \nabla u_{1,m}^{sc,\text{mp}}, \quad m = p, s, \]

are known as the compressional (longitudinal) and shear (transversal) parts of \( u_{1,m}^{sc} \) respectively. It is clear that

\[ \Delta u_{1,m}^{sc,\text{mp}} + k_\rho^2 u_{1,m}^{sc,\text{mp}} = 0, \quad \text{div } \nabla u_{1,m}^{sc,\text{mp}} = 0 \quad \text{in } \mathbb{R}^2 \setminus \Omega, \quad m = p, s \]

and

\[ \Delta u_{1,m}^{sc,\text{ms}} + k_\rho^2 u_{1,m}^{sc,\text{ms}} = 0, \quad \text{div } u_{1,m}^{sc,\text{ms}} = 0 \quad \text{in } \mathbb{R}^2 \setminus \Omega, \quad m = p, s. \]

To characterize outgoing waves, the scattered fields are required to satisfy the Kupradze’s radiation conditions

\[ \frac{\partial u_{1,m}^{sc,\text{mp}}}{\partial r} - ik_\rho u_{1,m}^{sc} = o(r^{-\frac{1}{2}}), \quad \frac{\partial u_{1,m}^{sc,\text{ms}}}{\partial r} - ik_\rho u_{1,m}^{sc} = o(r^{-\frac{1}{2}}), \quad m = p, s \]

uniformly in all directions \( \hat{x} \in S \) as \( r := |x| \to \infty \). For the unique solvability of the scattering problems (2.2)–(2.4) in the space \( [H^1_{\text{loc}}(\mathbb{R}^2 \setminus \Omega)]^2 \) we refer to Kupradze [35] and Li et al. [38].

It is well known that every radiating solution \( u_{1,m}^{sc} \) to the Navier equation has an asymptotic behaviour of the form

\[ u_{1,m}^{sc}(x) = \frac{k_\rho}{\omega} \frac{e^{i\pi/4}}{\sqrt{8\pi \omega}} \frac{e^{ik_\rho |x|}}{|x|} u_{1,m}^{sc,\text{mp}}(\hat{x}) \hat{x} + O(|x|^{-3/2}), \quad |x| \to \infty, \]

\[ u_{1,m}^{sc}(x) = \frac{k_\rho}{\omega} \frac{e^{i\pi/4}}{\sqrt{8\pi \omega}} \frac{e^{ik_\rho |x|}}{|x|} u_{1,m}^{sc,\text{ms}}(\hat{x}) x^\perp + O(|x|^{-3/2}), \quad |x| \to \infty \]

uniformly in all direction \( \hat{x} \in S \). The functions \( u_{1,m}^{sc,\text{mp}} \) and \( u_{1,m}^{sc,\text{ms}} \), known as the compressional and shear far field pattern of \( u_{1,m}^{sc} \) respectively, are complex valued analytic functions on \( S \). The far field patterns admit the following representations (see e.g. (2.29) and (2.30) in [28])

\[ u_{\Omega,m}^{\infty}(\hat{x}, \mathbf{d}) = \int_{\partial \Omega} \left\{ u_{n}^{in}(y, -\hat{x}) \cdot \mathbb{T} \nu(y) u_{1,m}^{sc}(y, \mathbf{d}) - \left[ \mathbb{T} \nu(y) u_{n}^{in}(y, -\hat{x}) \right] \cdot u_{1,m}^{sc}(y, \mathbf{d}) \right\} ds(y), \quad \hat{x}, \mathbf{d} \in S, \quad m, n = p, s. \]

Here, for a curve \( \Gamma \), \( \mathbb{T} \nu \) is the surface traction operator defined by

\[ \mathbb{T} \nu := 2\mu \nu \cdot \text{grad} + \lambda \nu \text{div} - \mu \nu \text{div } \nabla \]

in terms of the exterior unit normal vector \( \nu \) on \( \Gamma \).

Throughout this paper, we will denote the pair of far field patterns

\[ [u_{1,m}^{\infty}(\hat{x}, \mathbf{d}); u_{1,m}^{sc}(\hat{x}, \mathbf{d})] \]

by \( u_{1,m}^{\infty}(\hat{x}, \mathbf{d}), \quad m = p, s \), indicating the dependence on the observation direction \( \hat{x} \in S \) and the incident direction \( \mathbf{d} \in S \). In our recent work [28], we have shown that the far field patterns satisfy the reciprocity relation
The inverse classical elastic scattering problem is to determine the scatterer \( \Omega \) from partial or full far field data \( \mathbf{u}_{1,\Omega,n}^\infty(\hat{s}, \hat{d}) \), \( \hat{s}, \hat{d} \in \mathbb{S} \), \( m = p, s \). A wealth of theory and numerical results for such an inverse problem is now available. We refer to \([18, 20, 25]\) for the uniqueness results. The numerical methods can be found in \([3, 4, 25, 28, 43]\).

In many applications we have only the modulus of the far field data, while the phase information is difficult to be captured. Actually, the phaseless far field data is closely related to the outward energy flux. In the case of time harmonic case, the outward energy flux \( J_{mr}, m = p, s \), through a circle \( \partial B_r \) with radius \( r \) centered at the origin is given by

\[
J_{mr} := -4\omega^2 \int_{\partial B_r} \mathbf{u}_{1,\Omega,m}^\infty \cdot \mathbf{n} \mathbf{u}_{1,\Omega,m}^\infty \, ds, \quad m = p, s.
\]

Here, we take \( r \) large enough such that \( \Omega \) is contained in the interior of \( \partial B_r \). We refer to the corresponding concept for acoustic wave in \([32]\).

**Theorem 2.1.**

\[
J_{mr} = \int_{\mathbb{S}} \{ k_p |\mathbf{u}_{1,\Omega,m}^\infty|^2 + k_s |\mathbf{u}_{1,\Omega,m}^\infty|^2 \} \, ds, \quad m = p, s.
\]

**Proof.** Let \( B_R \) be a disk with radius \( R > r \) centered at the origin. We now apply Betti’s formula \([4]\) in \( B_R \setminus B_r \), with the help of the Navier equation \((2.2)\), for \( m = p, s \), to obtain

\[
\int_{\partial B_R} \mathbf{u}_{1,\Omega,m}^\infty \cdot \mathbf{n} \mathbf{u}_{1,\Omega,m}^\infty \, ds = \int_{\partial B_r} \mathbf{u}_{1,\Omega,m}^\infty \cdot \mathbf{n} \mathbf{u}_{1,\Omega,m}^\infty \, ds + \int_{B_R \setminus B_r} \left\{ \mathbf{u}_{1,\Omega,m}^\infty \cdot \Delta \mathbf{u}_{1,\Omega,m}^\infty + \mathcal{E}(\mathbf{u}_{1,\Omega,m}^\infty, \mathbf{u}_{1,\Omega,m}^\infty) \right\} \, dx
\]

\[
= \int_{\partial B_R} \mathbf{u}_{1,\Omega,m}^\infty \cdot \mathbf{n} \mathbf{u}_{1,\Omega,m}^\infty \, ds + \int_{B_R \setminus B_r} \left\{ \omega^2 \mathbf{u}_{1,\Omega,m}^\infty \cdot \mathbf{u}_{1,\Omega,m}^\infty - \mathcal{E}(\mathbf{u}_{1,\Omega,m}^\infty, \mathbf{u}_{1,\Omega,m}^\infty) \right\} \, dx. \tag{2.8}
\]

Here, for any two smooth vector functions \( \mathbf{v} = (v_1, v_2)^T \) and \( \mathbf{w} = (w_1, w_2)^T \),

\[
\mathcal{E}(\mathbf{v}, \mathbf{w}) := \lambda \text{div} \mathbf{v} \text{div} \mathbf{w} - \mu \text{div} \mathbf{v} \text{div} \mathbf{w} + 2\mu \sum_{i,j=1}^2 \partial_i v_j \partial_i w_j.
\]

Taking the imaginary part of the last equation \((2.8)\) yields

\[
J_{mr} = -4\omega^2 \int_{\partial B_R} \mathbf{u}_{1,\Omega,m}^\infty \cdot \mathbf{n} \mathbf{u}_{1,\Omega,m}^\infty \, ds, \quad m = p, s.
\]

By straightforward calculations, we have the asymptotic behaviour \([4]\) for \( m = p, s \)

\[
\mathbf{T}(\mathbf{u}_{1,\Omega,m}^\infty)(\mathbf{x}) = i\omega e^{i\pi/4} \frac{e^{i\omega|x|}}{\sqrt{8\pi\omega}} \mathbf{u}_{1,\Omega,mp}^\infty(\mathbf{x}) \mathbf{x}
\]

\[
+ i\omega \frac{e^{i\pi/4}}{\sqrt{8\pi\omega}} \frac{e^{i\omega|x|}}{\sqrt{|x|}} \mathbf{u}_{1,\Omega,ms}^\infty(\mathbf{x}) \mathbf{x}^\perp + O(|x|^{-3/2}), \quad x \to \infty. \tag{2.9}
\]

The proof is now completed by letting \( R \to \infty \) and using \((2.5)\) and \((2.9)\). \( \square \)
We are interested in the inverse problems with phaseless data $|u_{\Omega,m}^\infty|, m,n = p,s$. Unfortunately, the following translation invariance property for the phaseless far field data implies that it is impossible to determine the location of the underlying obstacle, even when multiple directions and multiple frequencies are considered.

**Theorem 2.2.** Let $\Omega_h := \{ y \in \mathbb{R}^2 : y = z + h : z \in \Omega \}$ be the shifted rigid body with a fixed vector $h \in \mathbb{R}^2$. Then, for any fixed circular frequency $\omega > 0$,

$$
|u_{\Omega_h,m}^\infty(\hat{x},d)| = |u_{\Omega,m}^\infty(\hat{x},d)|, \quad \hat{x},d \in \mathbb{S}, m,n = p,s.
$$

**(Proof.)** We firstly consider the case with $m=p$ and $n=p$, i.e. the compressional far field pattern corresponding to incident plane compressional wave. For the shifted rigid body $\Omega_h$, we have

$$
u_h \in \mathbb{S} \quad \text{implies that it is impossible to determine the location of the underlying obstacle, even when multiple directions and multiple frequencies are considered.}
$$

Finally, (2.10) follows by taking modulus on both sides of the above four equalities (2.13)–(2.16). □
Recently, a simple phase retrieval technique has been proposed in [29] for acoustic obstacle scattering problems. The basic idea is to add three point like scatterers located at \( \mathbf{z} \in \mathbb{R}^3 \setminus \overline{\Omega} \) with different scattering strengths \( \tau_j \in \mathbb{C}, j = 1, 2, 3 \), into the scattering system. The corresponding far field patterns are given by
\[
u_{ij}^\infty = \tau_j e^{ik\hat{d} \cdot (\hat{d} - \hat{q})}, \quad j = 1, 2, 3,
\]
where \( k \) is the wave number. A key step of the proposed phase retrieval technique is to choose three different far field data \( \nu_{ij}^\infty \). This is possible by choosing \( \tau_j, j = 1, 2, 3 \), such that they are not collinear in the complex plane. Similarly, for the elastic scattering of plane compressional wave \( \nu_{kj}^\infty \) by point like scatterer located at \( \mathbf{z} \in \mathbb{R}^3 \setminus \overline{\Omega} \) with strengths \( \tau_j \in \mathbb{C}, j = 1, 2, 3 \), the corresponding compressional far field pattern is given by
\[
u_{kj}^\infty = \tau_j e^{ik\hat{d} \cdot (\hat{d} - \hat{q})}, \quad j = 1, 2, 3.
\]
(2.17)

Clearly, difficulties arise due to the term \( \hat{d} \cdot \hat{q} \). In particular, the three elastic compressional far field patterns \( \nu_{kj}^\infty, j = 1, 2, 3 \), vanish if \( \hat{d} \cdot \hat{q} = 0 \). In practice, \( \hat{d} \cdot \hat{q} \) is very small for many cases, which makes the phase retrieval quite unstable. However, to the best of our knowledge, there is no rigorous mathematical theory on why the compressional far field pattern take the form (2.17). Besides, multiple scattering between the point-like scatterer and the unknown object \( D \) makes the problem more complicated.

In this work, instead of adding point like scatterers, we introduce point sources into the scattering system. In other words, we also consider the scattering of point sources. Due to the linearity of the direct scattering problems, the corresponding far field data to the point sources are independent of the incident plane waves. This makes the subsequent phase retrieval scheme more flexible. Denote by \( \mathbf{z} \in \mathbb{R}^3 \setminus \overline{\Omega}, \tau \in \mathbb{C} \) and \( \mathbf{q} \in \mathbb{S} \) the location, scattering strength and polarization of the point source, respectively. The incident point source \( \nu^\text{in} \) is given by
\[
\nu^\text{in}(\mathbf{x}, \mathbf{z}, \mathbf{q}, \tau) := \tau \Phi(\mathbf{x}, \mathbf{z})\mathbf{q},
\]
where \( \Phi \) is the Green’s tensor [4] of the Navier equation in \( \mathbb{R}^2 \) given by
\[
\Phi(\mathbf{x}, \mathbf{y}) := \frac{i}{4\mu} H_0^{(1)}(k_\| \mathbf{x} - \mathbf{y})I + \frac{i}{4\omega^2} \text{grad}_\mathbf{x} \text{grad}_\mathbf{y}^T (H_0^{(1)}(k_\| \mathbf{x} - \mathbf{y})) - H_0^{(1)}(k_\| \mathbf{x} - \mathbf{y})), \quad \mathbf{x} \neq \mathbf{y}
\]
in terms of the identity matrix \( I \) and the Hankel function \( H_0^{(1)} \) of the first kind of order zero. Note that \( \Phi(\mathbf{x}, \mathbf{y})\mathbf{q} \) is a radiating elastic scattered field and the corresponding far field patterns are given by [28]
\[
\Phi^\infty_p(\mathbf{\hat{x}}, \mathbf{\hat{y}}, \mathbf{q}) = e^{-ik\hat{x} \cdot \hat{y}} (\mathbf{q} \cdot \hat{y}), \quad \mathbf{\hat{x}}, \mathbf{\hat{y}} \in \mathbb{S}, \mathbf{q} \in \mathbb{R}^2,
\]
(2.18)
\[
\Phi^\infty_s(\mathbf{\hat{x}}, \mathbf{\hat{y}}, \mathbf{q}) = e^{-ik\hat{x} \cdot \hat{y}} (\mathbf{q} \cdot \hat{y}^\perp), \quad \mathbf{\hat{x}}, \mathbf{\hat{y}} \in \mathbb{S}, \mathbf{q} \in \mathbb{R}^2.
\]
(2.19)

Denote by \( \nu^\infty_p(\mathbf{x}, \mathbf{z}, \mathbf{q}, \tau) \) and \( \nu^\infty_s(\mathbf{\hat{x}}, \mathbf{\hat{y}}, \mathbf{q}, \tau) \) the scattered field and far field pattern, respectively, due to the scattering of point source \( \nu^\text{in}(\mathbf{x}, \mathbf{z}, \mathbf{q}, \tau) \) by the rigid body \( \Omega \).

For all \( \mathbf{z} \in \mathbb{R}^2 \setminus \overline{\Omega} \cup \{ \mathbf{z} \} \), \( \mathbf{d}, \mathbf{q} \in \mathbb{S}, \tau \in \mathbb{C} \), define
\[
\nu^\infty(\mathbf{z}, \mathbf{d}, \mathbf{q}, \tau) := \nu^\infty_p(\mathbf{x}, \mathbf{d}) + \nu^\infty_s(\mathbf{x}, \mathbf{z}, \mathbf{q}, \tau) + \nu^\text{in}(\mathbf{x}, \mathbf{z}, \mathbf{q}, \tau).
\]
Then \( w_{\Omega,i}^m(z) \) is a radiating solution to
\[
\Delta w_{\Omega,i}^m(z) + \omega^2 w_{\Omega,i}^m(z) = -\tau \delta_z q \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{\Omega} \cup \{ z \},
\]
where \( \delta_z \) is the Dirac delta function at the point \( z \), and
\[
w_{\Omega,i}^m(z) = -u^m \quad \text{on} \quad \partial \Omega.
\]
Then, by linearity, for all \( \hat{x}, d, q \in \mathbb{S}, \tau \in \mathbb{C} \), denote respectively by
\[
w_{\Omega,i}^m(\hat{x}, d, q, \tau) = u_{\Omega,i}^m(\hat{x}, d) + \nu_{\Omega,i}^m(\hat{x}, z, q, \tau) + \tau e^{-i k z} (q \cdot \hat{x})
\]
the compressional far field pattern corresponding to the incident field \( u^m \), \( m = p, s \), and
\[
w_{\Omega,i}^m(\hat{x}, d, q, \tau) = u_{\Omega,i}^m(\hat{x}, d) + \nu_{\Omega,i}^m(\hat{x}, z, q, \tau) + \tau e^{-i k z} (q \cdot \hat{x}^\perp)
\]
the shear far field pattern corresponding to \( u^m \), \( m = p, s \).

The following theorem 2.3 implies that the far field pattern \( \nu_{i}^m \) corresponding to the point source \( v^m \) is quite small if the distance \( \rho := \text{dist}(z, \Omega) \) is large enough. To prove this, we recall the elastic single-layer boundary operator \( S : [H^{1/2}(\partial \Omega)]^2 \to [H^{1/2}(\partial \Omega)]^2 \), given by
\[
(S\psi)(x) = \int_{\partial \Omega} \Phi(x, y)\psi(y) d\sigma(y), \quad x \in \partial \Omega
\]
and the elastic double-layer boundary operator \( K : [H^{1/2}(\partial \Omega)]^2 \to [H^{1/2}(\partial \Omega)]^2 \) by
\[
(K\psi)(x) = \int_{\partial \Omega} (T_{\nu}(x, y) \Phi(x, y))^T \psi(y) d\sigma(y), \quad x \in \partial \Omega.
\]
The mapping properties of \( S \) and \( K \) are intensively studied in [20, 41].

**Theorem 2.3.** Let \( z \) be a point outside \( \Omega \) such that the distance \( \rho := \text{dist}(z, \Omega) \) is large enough. Then we have
\[
v_{\Omega,i}^m(\hat{x}, z, q, \tau) = O\left( \frac{1}{\sqrt{\rho}} \right), \quad \rho \to \infty, \ \hat{x}, q \in \mathbb{S}, \ \tau \in \mathbb{C}, \ m = p, s.
\]

**Proof.** We seek a solution in the form
\[
v_{\Omega,i}^m(x, z, q, \tau) = \int_{\partial \Omega} (|T_{\nu}(y) \Phi(x, y)|^T + i \Phi(x, y))\psi(y, x, z, q, \tau) d\sigma(y)
\]
for all \( x \in \mathbb{R}^2 \setminus \overline{\Omega} \cup \{ z \}, q \in \mathbb{S}, \tau \in \mathbb{C} \) with a density \( \psi \in [H^{1/2}(\partial \Omega)]^2 \). From the jump relation of the double layer potential [20, 41], we see that the representation \( v_{\Omega,i}^m \) given in (2.25) solves the exterior Dirichlet boundary problem provided the density is a solution of the integral equation
\[
(I/2 + K + iS)\psi = -v^m = -\tau \Phi(\cdot, z \cdot q) \quad \text{on} \quad \partial \Omega.
\]
Note that \( I/2 + K + iS \) is bijective and the inverse \( (I/2 + K + iS)^{-1} : [H^{1/2}(\partial \Omega)]^2 \to [H^{1/2}(\partial \Omega)]^2 \) is bounded [20]. Therefore
\[
\psi = -(I/2 + K + iS)^{-1}(\tau \Phi(\cdot, z \cdot q),
\]

(2.26)
A straightforward calculation shows that the Green tensor $\Phi$ satisfies Sommerfeld’s finiteness condition

$$\Phi(x, y)q = O\left(\frac{1}{|x - y|^{\frac{1}{2}}}\right), \quad |x - y| \to \infty.$$ 

This implies that

$$\Phi(y, z, q, \tau) = O\left(\frac{1}{\sqrt{\rho}}\right), \quad \rho \to \infty$$

for all $y \in \partial \Omega, q \in S$. Inserting this into (2.25), we find that

$$v^{\infty}(x, z, q, \tau) = O\left(\frac{1}{\sqrt{\rho}}\right), \quad \rho \to \infty$$

for all $x \in \mathbb{R}^2 \setminus \Omega \cup \{z\}, q \in S, \tau \in C$. The proof is completed by letting $|x| \to \infty$ in (2.25) and using (2.18) and (2.19).

With the previous analysis, we are now well prepared for studying the following elastic inverse obstacle scattering problem.

**Phaseless-IP1**: Determine the location and shape of the scatterer $\Omega$ from the following phaseless far field data

$$\left|w_{\Omega \cup \{z\}, m, n}(\hat{x}, d, q, \tau)\right|, \quad \hat{x}, d, q \in S, \tau \in C, m, n = p, s.$$ 

### 2.2. Scattering of sources

In this subsection, we study the scattering of sources with compact support. Denote by $F = F(x) \in [L^2(\mathbb{R}^2)]^2$ the external force with compact support $\Omega$. For the source scattering problem, we assume that

$$\omega \in \mathcal{W} := (\omega_{\min}, \omega_{\max}),$$

where $\omega_{\min} < \omega_{\max}$ are two fixed circular frequencies. For the scattering of a source $F$ we denote the scattered field by $u^{sc}_F(x)$, and the corresponding far field pattern by $u^{\infty}_F(\hat{x}, \omega) = [u^{\infty}_F(\hat{x}, \omega), q^{\infty}_F(\hat{x}, \omega)]$.

The propagation of time-harmonic elastic wave in an isotropic homogeneous media is governed by the reduced Navier equation

$$\Delta u^{sc}_F + \omega^2 u^{sc}_F = F \quad \text{in} \; \mathbb{R}^2.$$ 

Again, the scattered field $u^{sc}_F$ admits the decomposition

$$u^{sc}_{F,t} := -\frac{1}{k_x^2} \text{grad} \, \text{div} \, u^{sc}_F - \frac{1}{k_y^2} \text{grad} \, \text{div} \, u^{sc}_F \quad \text{in} \; \mathbb{R}^2 \setminus \Omega$$

and satisfies the Kupradze’s radiation conditions

$$\frac{\partial u^{sc}_{F,t}}{\partial r} - ik_x u^{sc}_{F,\rho} = o(r^{-1}), \quad \frac{\partial u^{sc}_{F,s}}{\partial r} - ik_s u^{sc}_{F,\rho} = o(r^{-1}),$$ 

uniformly in all directions $\hat{x} \in S$ as $r := |x| \to \infty$. 

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The radiating solution \( u^\infty_F \) to the scattering problem (2.27) and (2.28) takes the form
\[
u^\infty_F(x, \omega) = \int_{\Omega^2} \Phi(x, y) F(y) dy, \quad x \in \mathbb{R}^2, \ \omega \in \mathbb{W}. \tag{2.29}\]

By (2.18) and (2.19), the corresponding compressional and shear far field patterns of \( u^\infty_F \) are given by
\[
u^\infty_{F, p}(\hat{x}, \omega) = \int_{\mathbb{R}^2} e^{-ik_h y \hat{x} \cdot \hat{y}} F(y) dy, \quad \hat{x} \in \mathbb{S}, \ \omega \in \mathbb{W} \tag{2.30}\]
and
\[
u^\infty_{F, s}(\hat{x}, \omega) = \int_{\mathbb{R}^2} e^{-ik_s y \hat{x} \cdot \hat{y}} F(y) dy, \quad \hat{x} \in \mathbb{S}, \ \omega \in \mathbb{W}, \tag{2.31}\]
respectively.

The analogous result of theorem 2.2 is formulated in the following theorem.

**Theorem 2.4.** Let \( F_h(y) := F(y + h), \ y \in \mathbb{R}^2 \) be the shifted source with a fixed vector \( h \in \mathbb{R}^2 \). For the scattering of the source \( F_h \), we denote the scattered field by \( u^\infty_{F_h} \) and its far field pattern by \( (u^\infty_{F, p, m}, u^\infty_{F, s, m}) \). Then we have the following translation invariance property
\[
u^\infty_{F_{h, m}}(\hat{x}, \omega) = |\nu^\infty_{F_{h}}(\hat{x}, \omega)|, \quad \hat{x} \in \mathbb{S}, \ \omega \in \mathbb{W}, \ m = p, s. \tag{2.32}\]

**Proof.** By the far field representations (2.30) and (2.31), we find
\[
u^\infty_{F_{h, p}}(\hat{x}, \omega) = e^{ik_h y \hat{x} \cdot \hat{y}} u^\infty_{F, p}(\hat{x}, \omega), \quad \hat{x} \in \mathbb{S}, \ \omega \in \mathbb{W} \tag{2.33}\]
and
\[
u^\infty_{F_{h, s}}(\hat{x}, \omega) = e^{ik_h y \hat{x} \cdot \hat{y}} u^\infty_{F, s}(\hat{x}, \omega), \quad \hat{x} \in \mathbb{S}, \ \omega \in \mathbb{W}. \tag{2.34}\]

Then the statement follows by taking the modulus on both sides of (2.33) and (2.34). \( \square \)

Theorem 2.4 implies that the phaseless far field data are invariant under the translation of the source. Thus the location of the source can be not uniquely determined. Clearly, from (2.30) and (2.31), we have
\[
u^\infty_{F_{h, m}}(\hat{x}, \omega) = |\nu^\infty_{F_{h}}(\hat{x}, \omega)|, \quad \hat{x} \in \mathbb{S}, \ \omega \in \mathbb{W}, \ m = p, s
\]
for any constant \( c \in \mathbb{C} \) with \(|c| = 1\). Therefore, the source can not be uniquely determined from the phaseless far field data \( |\nu^\infty_{F_{h, p}}(\hat{x}, \omega)| \) and \( |\nu^\infty_{F_{h, s}}(\hat{x}, \omega)| \), \( \hat{x} \in \mathbb{S}, \ \omega \in \mathbb{W} \), even the location \( \Omega \) of the source is known in advance.

To solve the above mentioned problems, we introduce again point sources into the scattering system. Let \( z \in \mathbb{R}^3 \setminus \Omega, \ \tau \in \mathbb{C} \) and \( q \in \mathbb{S} \) be the location, scattering strength and polarization of the point source, respectively. For the scattering of combined sources, the scattered field \( u^\infty_{F_U(z)} \) is given by
\[
u^\infty_{F_U(z)}(x, q, \omega, \tau) = u^\infty_F(x, \omega) + \tau \Phi(x, z) q, \quad x \in \mathbb{R}^2, \ q \in \mathbb{S}, \ \omega \in \mathbb{W}, \ \tau \in \mathbb{C} \tag{2.35}\]
and the corresponding compressional and shear far field patterns are given by
\[
u^\infty_{F_{U, p}(z, \phi)}(\hat{x}, q, \omega, \tau) = u^\infty_{F_{p, m}}(\hat{x}, \omega) + \tau e^{-ik_h y \hat{x} \cdot \hat{y}} q \cdot \hat{x}, \quad \hat{x}, q \in \mathbb{S}, \ \omega \in \mathbb{W}, \ \tau \in \mathbb{C} \tag{2.36}\]
and
\[ u_{\infty}^\infty(\hat{x}, q, \omega, \tau) = u_{\infty}^\infty(\hat{x}, \omega) + \tau e^{-ik\hat{x} \cdot \hat{\mathbf{r}}} \cdot \hat{x}, \hat{x}, q \in S, \omega \in \mathbb{W}, \tau \in \mathbb{C}, \]
respectively.

Finally, the corresponding inverse source problem of our interest is as follows.

**Phaseless-IP2:** Determine the source \( \mathbf{F} \) from the following phaseless far field data
\[ |u_{\infty}^\infty(\hat{x}, q, \omega, \tau)|, \hat{x}, q \in S, \omega \in \mathbb{W}, \tau \in \mathbb{C}, m = p, s. \]

### 3. Uniqueness

Using the results of the previous section, this section is devoted to study uniqueness for the inverse elastic scattering problems with phaseless far field data.

#### 3.1. Uniqueness for sources

Firstly, we establish the uniqueness results for **Phaseless-IP2**, i.e. the inverse source scattering problems. Let \( B_R \) be a disk with radius \( R \) large enough such that \( \Omega \) is contained inside. In a recent paper [6], the authors show that the source \( \mathbf{F} \in H^3(B_R) \) can be uniquely determined by the scattered field \( u_{\infty}^\infty(\hat{x}, \omega) \), \( \hat{x} \in \partial B_R, \omega \in \mathbb{W} \). By Rellich’s lemma [13], the compressional scattered field \( u_{\infty}^p \) and the shear scattered field \( u_{\infty}^s \) are uniquely determined by the corresponding compressional far field \( u_{\infty}^\infty(\hat{x}, \omega) \), and shear far field \( u_{\infty}^\infty \), respectively. Therefore, we immediately deduce the following uniqueness result with phased far field data.

**Theorem 3.1.** The source \( \mathbf{F} \) is uniquely determined by the multi-frequency far field data sets
\[ A_p := \{ u_{\infty}^\infty(\hat{x}, \omega) : \hat{x} \in S, \omega \in \mathbb{W} \} \quad \text{(3.1)} \]
and
\[ A_s := \{ u_{\infty}^\infty(\hat{x}, \omega) : \hat{x} \in S, \omega \in \mathbb{W} \}. \quad \text{(3.2)} \]

We want to remark that by analyticity the far field pattern can be determined on the whole unit circle \( S \) using partial value on any circular arc of \( S \). Thus the previous theorem also holds if the far field data are given on any arc. In the sequel we set
\[ S_q := \{ \hat{x} \in S | q \cdot \hat{x} \geq 1/2 \} \quad \text{and} \quad S_q^\perp := \{ \hat{x} \in S | q \cdot \hat{x} \perp \geq 1/2 \} \quad \text{(3.3)} \]
for some fixed \( q \in S \). Define
\[ Q := \{ q_1, q_2, q_3 \}, \]
where
\[ q_1 := \left( \cos \frac{\pi}{4}, \sin \frac{\pi}{4} \right)^T, \quad q_2 := \left( \cos \frac{11\pi}{12}, \sin \frac{11\pi}{12} \right)^T, \quad q_3 := \left( \cos \frac{19\pi}{12}, \sin \frac{19\pi}{12} \right)^T. \]

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It follows that

\[ S = \bigcup_{q \in \Omega} S_q = \bigcup_{q \in \Omega} S_q^\perp. \]

**Theorem 3.2.** Define \( \tau_1 := |\tau_1|^\alpha \in \mathbb{C} \setminus \{0\} \) with the principal argument \( \alpha \in [0, 2\pi) \), and let \( q_0 \) and \( z_1 \) be two different points in \( \mathbb{R}^2 \setminus \bar{\Omega} \). For fixed polarization \( q \in S \) of the point sources, define two circular arcs \( S_q \) and \( S_q^\perp \) as in (3.3). Then the source \( F \) is uniquely determined by the two phaseless far field data sets

\[
B_p := \{ |u_{F_p,(z)\rho}(\hat{x}, q, \omega, \tau)| : \hat{x} \in S_q, \omega \in \mathbb{W}, \tau \in \{0, \tau_1\}, z \in \{z_0, z_1\} \}
\]

and

\[
B_s := \{ |u_{F_s,(z)\rho}(\hat{x}, q, \omega, \tau)| : \hat{x} \in S_q^\perp, \omega \in \mathbb{W}, \tau \in \{0, \tau_1\}, z \in \{z_0, z_1\} \}.
\]

**Proof.** We first show that the compressional far field data set \( A_p \) given in (3.1) is uniquely determined by the phaseless far field data set \( B_p \). To accomplish this, we observe that from the representation (2.36) it follows that

\[
|u_{F_p,(z)\rho}(\hat{x}, q, \omega, \tau_1)|^2 = |u_{F_p}(\hat{x}, \omega) + \tau_1 e^{-ikp \hat{x} \cdot z}|^2 = |u_{F_p}(\hat{x}, \omega)|^2 + 2q \cdot \hat{x} \mathcal{R}(u_{F_p}(\hat{x}, \omega)\tau_1 e^{-ikp \hat{x} \cdot z}) + |\tau_1|^2, \quad \hat{x} \in S_q, \omega \in \mathbb{W}, \ z \in \{z_0, z_1\}.
\]

From this, since \( q \cdot \hat{x} \geq 1/2 \) for \( \hat{x} \in S_q \), we deduce that \( \mathcal{R}(u_{F_p}(\hat{x}, \omega)\tau_1 e^{-ikp \hat{x} \cdot z}) \) can be uniquely determined for all \( \hat{x} \in S_q, \omega \in \mathbb{W}, z \in \{z_0, z_1\} \). We rewrite \( u_{F_p}(\hat{x}, \omega) \) and \( \tau_1 e^{-ikp \hat{x} \cdot z} \) in the form

\[
u_{F_p}(\hat{x}, \omega) = u_{\hat{x}}(\hat{x}, \omega)|\Omega(\hat{x}, \omega) \quad \text{and} \quad \tau_1 e^{-ikp \hat{x} \cdot z} = |\tau_1|\Omega(\hat{x}, \omega) \quad \text{for} \quad \hat{x} \in S_q, \omega \in \mathbb{W}, z \in \{z_0, z_1\}.
\]

respectively, for all \( \hat{x} \in S_q, \omega \in \mathbb{W}, z \in \{z_0, z_1\} \). Here, \( \phi(\hat{x}, \omega) \in [0, 2\pi) \) is the principal argument of \( u_{F_p}(\hat{x}, \omega) \). We claim that the principal argument \( \phi(\hat{x}, \omega) \) is uniquely determined. Indeed, assume that there are two arguments \( \phi_1(\hat{x}, \omega) \) and \( \phi_2(\hat{x}, \omega) \). Define

\[
S_{q,0}(z_0, z_1, \omega) := \{ \hat{x} \in S_q : u_{\hat{x}}(\hat{x}, \omega) \neq 0 \quad \text{and} \quad \hat{x} \cdot (z_0 - z_1) \neq 0 \}.
\]

It is easy to verify that \( S_{q,0}(z_0, z_1, \omega) \) has Lebesgue measure zero. Since \( |u_{\hat{x}}(\hat{x}, \omega)| \) is given in \( B_p \) and \( \mathcal{R}(u_{F_p}(\hat{x}, \omega)\tau_1 e^{-ikp \hat{x} \cdot z}) \) can be uniquely determined for all \( \hat{x} \in S_q, \omega \in \mathbb{W}, z \in \{z_0, z_1\} \), we have that for all \( \hat{x} \in S_{q,0}(z_0, z_1, \omega) \), \( \omega \in \mathbb{W}, z \in \{z_0, z_1\}, \)

\[
\cos[\phi_1(\hat{x}, \omega) - (\alpha - k_p \hat{x} \cdot z)] = \cos[\phi_2(\hat{x}, \omega) - (\alpha - k_p \hat{x} \cdot z)],
\]

and furthermore, we conclude that

\[
\phi_1(\hat{x}, \omega) - (\alpha - k_p \hat{x} \cdot z) = \phi_2(\hat{x}, \omega) - (\alpha - k_p \hat{x} \cdot z) + 2l\pi, \quad \text{for some} \ l \in \mathbb{Z},
\]

or

\[
\phi_1(\hat{x}, \omega) - (\alpha - k_p \hat{x} \cdot z) = -[\phi_2(\hat{x}, \omega) - (\alpha - k_p \hat{x} \cdot z)] + 2l'\pi, \quad \text{for some} \ l' \in \mathbb{Z}.
\]
We now show that the case (3.7) does not hold. Actually, (3.7) implies that
\[ \phi_1(\hat{x}, \omega) + \phi_2(\hat{x}, \omega) - 2l'\pi = 2[\alpha - k_p \hat{x} \cdot z], \quad z \in \{z_0, z_1\}, \text{ for some } l' \in \mathbb{Z}. \] 
(3.8)

The left hand side of (3.8) is independent of \( z \). However, the right hand side of (3.8) changes for different \( z \in \{z_0, z_1\} \). This leads to a contradiction, and thus (3.7) does not hold.

For the case when (3.6) holds, we have
\[ \phi_1(\hat{x}, \omega) - \phi_2(\hat{x}, \omega) = 2l\pi, \quad \text{for some } l \in \mathbb{Z}. \]

Noting that \( \phi_1(\hat{x}, \omega), \phi_2(\hat{x}, \omega) \in [0, 2\pi) \), we have \( \phi_1(\hat{x}, \omega) - \phi_2(\hat{x}, \omega) \in (-2\pi, 2\pi) \), and thus \( l = 0 \), i.e.
\[ \phi_1(\hat{x}, \omega) = \phi_2(\hat{x}, \omega), \quad \forall \hat{x} \in S_{q,0}(z_0, z_1, \omega), \omega \in \mathbb{W}. \]

This further implies that \( u^\infty_{F,\rho}(\hat{x}, \omega) \) is uniquely determined for all \( \hat{x} \in S_{q,0}(z_0, z_1, \omega), \omega \in \mathbb{W} \) and also for \( \hat{x} \in \mathbb{S}, \omega \in \mathbb{W} \) by analytic continuation.

Similarly, the shear far field data set \( A_r \) given in (3.2) is uniquely determined by the set \( B_r \).
The conclusion of the theorem now follows from theorem 3.1.

In theorem 3.2, the phaseless data set includes the data with scattering strength \( \tau = 0 \), which means there are no additional scattering of point sources. For diversity, we now prove a uniqueness theorem with three different scattering strengths. Define
\[ T := \{\tau_1, \tau_2, \tau_3\}, \]
where \( \tau_1, \tau_2, \tau_3 \in \mathbb{C} \) are three different scattering strengths such that \( \tau_2 - \tau_1 \) and \( \tau_3 - \tau_1 \) are linearly independent.

**Theorem 3.3.** Fix \( z \in \mathbb{R}^2 \setminus \overline{\Omega} \) and \( q \in \mathbb{S} \). Define again two circular arcs \( S_q \) and \( S_q^\perp \) as in (3.3). Then the source \( F \) is uniquely determined by the phaseless data set
\[ D_p := \{ |u^\infty_{F,\rho}(\hat{x}, q, \omega, \tau)| : \hat{x} \in S_q, \omega \in \mathbb{W}, \tau \in T \} \]
and
\[ D_s := \{ |u^\infty_{F,\rho}(\hat{x}, q, \omega, \tau)| : \hat{x} \in S_q^\perp, \omega \in \mathbb{W}, \tau \in T \}. \]

**Proof.** Recalling the representation (2.36) and noting that \( \hat{x} \cdot q \geq 1/2 \) for \( \hat{x} \in S_q \), we deduce that
\[
\frac{1}{\hat{x} \cdot q} |u^\infty_{F,\rho}(\hat{x}, q, \omega, \tau)| \\
= \frac{1}{\hat{x} \cdot q} \left| u^\infty_{F,\rho}(\hat{x}, \omega) + \tau e^{-ik_p z \hat{x} \cdot q} \right| \\
= \left| \frac{u^\infty_{F,\rho}(\hat{x}, \omega)}{\hat{x} \cdot q} e^{ik_p z \hat{x} \cdot q} + \tau \right|, \quad \hat{x} \in S_q, \omega \in \mathbb{W}, \tau \in T.
\]
Define $z_j := -\tau_j$, $j = 1, 2, 3$. Using lemma 4.1 we find that $rac{u_{F_q}(\hat{k}\omega)e^{ik\hat{x} \cdot \hat{q}}}{\hat{q}}$ is uniquely determined for all $\hat{k} \in \mathbb{S}_{\hat{q}}$, $\omega \in \mathbb{W}$. By analyticity, we also deduce that $u_{F_q}(\hat{k}, \omega)$ is uniquely determined for all $\hat{k} \in \mathbb{S}$, $\omega \in \mathbb{W}$.

Similarly, the shear far field pattern $u_{F_s}(\hat{k}, \omega)$ is uniquely determined for all $\hat{k} \in \mathbb{S}$, $\omega \in \mathbb{W}$. By analyticity, we also deduce that $u_{F_s}(\hat{x}, \omega)$ is uniquely determined for all $\hat{x} \in \mathbb{S}$, $\omega \in \mathbb{W}$ from the data set $D_s$. Then the proof is finished by theorem 3.1.

In the previous uniqueness results, we determine the source from both the compressional and shear far field data. A challenging open problem is whether one of the compressional and shear far field data completely determines the source. We are also interested in broadband sparse measurements, i.e. the data can only be measured in finitely many observation directions

$$\{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_M\} =: \Theta \subset \mathbb{S}.$$ 

In particular, we are interested in what information can be obtained using multi-frequency data with a single observation direction. For a bounded domain $\Omega$, the $\hat{x}$-strip hull of $\Omega$ for a single observation direction $\hat{x} \in \Theta$ is defined by

$$S_{\Omega}(\hat{x}) := \{y \in \mathbb{R}^2 | \inf_{z \in \Omega} z \cdot \hat{x} \leq y \cdot \hat{x} \leq \sup_{z \in \Omega} z \cdot \hat{x}\},$$

which is the smallest strip (region between two parallel lines) with normals in the directions $\pm \hat{x}$ that contains $\Omega$. Let

$$\Pi_\alpha := \{y \in \mathbb{R}^2 | y \cdot \hat{x} + \alpha = 0\}, \quad \alpha \in \mathbb{R}$$

be a line with normal $\hat{x}$. Define

$$f_\alpha(\hat{x}) := \int_{\Pi_\alpha} \hat{x} \cdot F(y) dy, \quad \alpha \in \mathbb{R}.$$ 

(3.9)

The following theorem gives a uniqueness result on determine the strip by using only compressional far field data or shear far field data at a fixed observation direction. The proof follows the related results for inverse acoustic source scattering problems [1].

**Theorem 3.4.** For any fixed $\hat{x}_0 \in \mathbb{S}$, choose a polarization $q_0 \in \mathbb{S}$ such that $q_0 \cdot \hat{x}_0 \neq 0$. If the set

$$\{\alpha \in \mathbb{R} | \Pi_\alpha \subset S_{\Omega}(\hat{x}_0), f_\alpha(\hat{x}_0) = 0\}$$ 

has Lebesgue measure zero, then the strip $S_{\Omega}(\hat{x}_0)$ of the source support $\Omega$ can be uniquely determined by the following phaseless data set

$$\mathcal{E}_p := \{u_{F_q}(\hat{k}, \omega, \hat{x}_0, \omega) : \omega \in \mathbb{W}, \tau \in \mathcal{T}\}.$$ 

**Proof.** Essentially, the same arguments as in the proof of theorem 3.3 show that the compressional far field pattern $u_{F_q}(\hat{x}_0, \omega)$ is uniquely determined by $\mathcal{E}_p$ for all $\omega \in \mathbb{W}$ and for also all $\omega \in \mathbb{R}$ by analyticity. Recall the compressional far field pattern representation (2.30), we have
Clearly, the compressional far field pattern \( u_{F,p}^{∞}(\hat{S}_0, ω) \) is just the inverse Fourier transform of \( f_κ \). This implies that \( f_κ \) is uniquely determined by \( u_{F,p}^{∞}(\hat{S}_0, ω) \), \( ω \in \mathbb{W} \). Under the assumption that the set in (3.10) has Lebesgue measure zero, it is seen that

\[
S_Ω(\hat{κ}) = \bigcup_{α ∈ \mathbb{R}} \{ Π_α[f_κ(α)] \neq 0 \}
\]

which implies that the strip \( S_Ω(\hat{κ}) \) is uniquely determined by \( f_κ \), and also by \( u_{F,p}^{∞}(\hat{S}_0, ω) \), \( ω \in \mathbb{W} \). The proof is complete.

We end this subsection by the following remarks:

- **Unique determination of the strip can also be established by using the phaseless shear far field data**

  \[
  \mathcal{E}_1 := \{ u_{F,p}^{∞}(\hat{S}_0, q_0, ω, τ) : ω ∈ \mathbb{W}, τ ∈ \mathcal{T} \},
  \]

  where the polarization \( q_0 \) should be chosen such that \( \hat{S}_0 \cdot q_0 \neq 0 \).

- **Theorem 3.4 is not true in general if the set in (3.10) has positive Lebesgue measure. For example, for \( y = (y_1, y_2)^T ∈ \mathbb{R}^2 \), we consider**

  \[
  F_1(y) = \begin{cases} 
  (1,0)^T, & y_1 ∈ (-1,1), y_2 ∈ [1,2); \\
  (y_1,0)^T, & y_1 ∈ (-1,1), y_2 ∈ (-1,1); \\
  (1,0)^T, & y_1 ∈ (-1,1), y_2 ∈ (-2,-1]; \\
  (0,0)^T, & \text{otherwise.}
  \end{cases}
  \]

  and

  \[
  F_2(y) = \begin{cases} 
  (1,0)^T, & y_1 ∈ (-1,1), y_2 ∈ [1,2); \\
  (1,0)^T, & y_1 ∈ (-1,1), y_2 ∈ (-2,-1]; \\
  (0,0)^T, & \text{otherwise.}
  \end{cases}
  \]

Then \( F_1 ∈ [L^2(\mathbb{R}^2)]^2 \) has compact support in \( D_1 = [-1,1] × [-2,2] \) and \( F_2 ∈ [L^2(\mathbb{R}^2)]^2 \) has compact support in \( D_2 := D_2^{(1)} \cup D_2^{(2)} \), where \( D_2^{(1)} = [-1,1] × [-2,-1] \) and \( D_2^{(2)} = [-1,1] × [1,2] \). Straightforward calculations show that the corresponding compressional far field patterns corresponding to these two different sources coincide for all frequencies at the fixed observation direction \( \hat{κ} = (1,0)^T \).

- **A by-product of theorem 3.4 is that one may determine a convex hull of the source support \( Ω \) by using two or more observation directions, which is verified in section 5 numerically.**

In summary, the external source \( F \) can be uniquely determined by one of the following data set:

\[
B_p \cup B_t; \quad \text{or} \quad D_p \cup D_t.
\]
If we can only take the measurements at a fixed observation direction, we show in theorem 3.4 that the strip $S_0(\mathbf{\hat{x}}_0)$ of the source support $\Omega$ can be uniquely determined by the phaseless data set $E_p$.

### 3.2. Uniqueness for obstacles

Difficulties arise for the uniqueness results for Phaseless-IP1 because of the additional far field pattern $v^{\infty}_{\Omega}$ due to point sources, i.e. scattering effects of point sources can not be ignored, even if they are very weak if the source is far away from the obstacle (see theorem 2.3). Inspired by the recent works [45, 47], we introduce an artificial rigid body $D \subset \mathbb{R}^2 \backslash (\Omega \cup \{z\})$ into the scattering system. The reference ball technique dates back to [37], where such a technique is used to avoid eigenvalues and choose a cut-off value for the linear sampling method with phased far field patterns. We also refer to [31] and [42] for application of the reference ball techniques in phased inverse scattering problems. We guess that such a body $D$ can be removed. However, this requires other techniques. We hope to report this elsewhere in the near future.

To simplify notations, for fixed $z$ and $q$, we set

\[
\begin{align*}
\nu^{\infty}_{D;\Omega,0}(\mathbf{\hat{x}}) & := \nu^\infty_{D;\Omega,0}(\mathbf{\hat{x}}, z, q, 1) + e^{-ik_0 \cdot \mathbf{\hat{x}}}(q \cdot \mathbf{\hat{x}}), \quad \mathbf{\hat{x}} \in S_q, \\
\nu^{\infty}_{D;\Omega,1}(\mathbf{\hat{x}}) & := \nu^\infty_{D;\Omega,1}(\mathbf{\hat{x}}, z, q, 1) + e^{-ik_0 \cdot \mathbf{\hat{x}}}(q \cdot \mathbf{\hat{x}}^\perp), \quad \mathbf{\hat{x}} \in S_q^\perp.
\end{align*}
\]

**Theorem 3.5.** For any fixed frequency $\omega > 0$, source polarization $q \in Q$ and source point $z \in \mathbb{R}^2 \backslash (\mathcal{D} \cup \Omega)$, the scatterer $\Omega$ is uniquely determined by the following phaseless data

\[
\begin{align*}
\nu^{\infty}_{D;\Omega,0}(\mathbf{\hat{x}}, \mathbf{d}, q, \tau), & \quad \mathbf{\hat{x}} \in S_q, \quad \mathbf{d} \in S, \quad \tau \in \mathcal{T}, \quad m = p, s, \quad (3.13) \\
\nu^{\infty}_{D;\Omega,0}(\mathbf{\hat{x}}, \mathbf{d}, q, \tau), & \quad \mathbf{\hat{x}} \in S_q^\perp, \quad \mathbf{d} \in S, \quad \tau \in \mathcal{T}, \quad m = p, s, \quad (3.14) \\
\nu^{\infty}_{D;\Omega,0}(\mathbf{\hat{x}}), & \quad \mathbf{\hat{x}} \in S_q, \quad (3.15) \\
\nu^{\infty}_{D;\Omega,0}(\mathbf{\hat{x}}), & \quad \mathbf{\hat{x}} \in S_q^\perp. \quad (3.16)
\end{align*}
\]

**Proof.** Define two sets

\[
S_0 := \left\{ \mathbf{\hat{x}} \in S_q : \nu^{\infty}_{D;\Omega,0}(\mathbf{\hat{x}}) = 0 \right\} \quad \text{and} \quad S_0^\perp := \left\{ \mathbf{\hat{x}} \in S_q^\perp : \nu^{\infty}_{D;\Omega,0}(\mathbf{\hat{x}}) = 0 \right\}.
\]

We then claim that these two sets $S_0$ and $S_0^\perp$ have Lebesgue measure zero. For the case when the distance $\rho := \text{dist}(z, D \cup \Omega)$ is large enough, this is obvious with theorem 2.3. For the general case, we show only that $S_0$ has Lebesgue measure zero, the other one can be proved analogously. On the contrary, assume that $S_0$ has positive Lebesgue measure. By the analyticity of the far field pattern, we conclude that $S_0 = S_q$. This implies that $\nu^{\infty}_{D;\Omega,0}(\mathbf{\hat{x}}, z, q, 1) = -e^{-ik_0 \cdot \mathbf{\hat{x}}}(q \cdot \mathbf{\hat{x}})$ for $\mathbf{\hat{x}} \in S_q$, and also for all $\mathbf{\hat{x}} \in S$ by analyticity again. Note that by (2.18), $e^{-ik_0 \cdot \mathbf{\hat{x}}}(q \cdot \mathbf{\hat{x}})$ is just the compressional far field pattern of the scattered field $\Phi(\cdot, z)q$. We conclude by Rellich’s lemma [13] and analytic continuation that
\[
\mathbf{v}^\infty_{D\cup\Omega, p}(\mathbf{x}, \mathbf{z}, q, l) = \frac{1}{k^p} \text{grad div} [\Phi(\mathbf{x}, \mathbf{z})q], \quad \mathbf{x} \in \mathbb{R}^2 \setminus D \cup \Omega \cup \{\mathbf{z}\}.
\] (3.17)

This contradicts the fact that the scattered field \(\mathbf{v}^\infty_{D\cup\Omega, p}(\cdot, \mathbf{z}, q, l) \) is analytic in \(\mathbb{R}^2 \setminus D \cup \Omega\) since the right hand side of (3.17) is singular at \(\mathbf{x} = \mathbf{z}\).

Recall the representation (2.22), we have for all \(\hat{\mathbf{x}} \in S_q \setminus S_0, \mathbf{d} \in S, \tau \in T\),

\[
\begin{align*}
|w^\infty_{D\cup\Omega, \{\tau\}, mp}(\hat{\mathbf{x}}, \mathbf{d}, q, \tau)| &= \left|u^\infty_{D\cup\Omega, mp}(\hat{\mathbf{x}}, \mathbf{d}) + \tau v^\infty_{D\cup\Omega, p}(\hat{\mathbf{x}})\right| \\
&= \left|\frac{u^\infty_{D\cup\Omega, mp}(\hat{\mathbf{x}}, \mathbf{d})}{v^\infty_{D\cup\Omega, p}(\hat{\mathbf{x}})} + \tau\right|, \quad m, p, s.
\end{align*}
\]

By lemma 4.1, we deduce that

\[
\frac{u^\infty_{D\cup\Omega, mp}(\hat{\mathbf{x}}, \mathbf{d})}{v^\infty_{D\cup\Omega, p}(\hat{\mathbf{x}})} \quad \hat{\mathbf{x}} \in S_q \setminus S_0, \quad \mathbf{d} \in S, \quad m, p, s
\]

is uniquely determined by the phaseless data given in (3.13) and (3.15). Note that \(|v^\infty_{D\cup\Omega, p}(\hat{\mathbf{x}})|\) is given in (3.15), we thus further obtain the phaseless data \[u^\infty_{D\cup\Omega, mp}(\hat{\mathbf{x}}, \mathbf{d}), m, p, s \text{ uniquely for all } \hat{\mathbf{x}}, \mathbf{d} \in S \text{ by analyticity. Similarly,} \]

\[
|u^\infty_{D\cup\Omega, mp}(\hat{\mathbf{x}}, \mathbf{d})| \quad \text{and} \quad \frac{u^\infty_{D\cup\Omega, mp}(\hat{\mathbf{x}}, \mathbf{d})}{v^\infty_{D\cup\Omega, p}(\hat{\mathbf{x}})} \quad \hat{\mathbf{x}} \in S_q \setminus S_0, \quad \mathbf{d} \in S, \quad m, p, s
\]

are uniquely determined by the phaseless data given in (3.14) and (3.16), then \[u^\infty_{D\cup\Omega, mp}(\hat{\mathbf{x}}, \mathbf{d}) \text{, } m, p, s \] are obtained for all \(\hat{\mathbf{x}}, \mathbf{d} \in S \) uniquely. Writing

\[
\begin{align*}
&u_{D\cup\Omega, mn}(\hat{\mathbf{x}}, \mathbf{d}) := |u^\infty_{D\cup\Omega, mn}(\hat{\mathbf{x}}, \mathbf{d})| e^{i\alpha_{mn}(\hat{\mathbf{x}}, \mathbf{d})}, \quad \hat{\mathbf{x}}, \mathbf{d} \in S, \quad m, n, p, s, \\
v_{D\cup\Omega, n}(\hat{\mathbf{x}}) &= |v^\infty_{D\cup\Omega, p}(\hat{\mathbf{x}})| e^{i\beta_n(\hat{\mathbf{x}})}, \quad \hat{\mathbf{x}} \in S, \quad n, p, s,
\end{align*}
\]

where the phase functions \(\alpha_{mn} \in [0, 2\pi)\) and \(\beta_n \in [0, 2\pi)\) are analytic on \(S\), and from the previous analysis, we obtain that \(e^{i[\alpha_{mn}(\hat{\mathbf{x}}, \mathbf{d}) - \beta_n(\hat{\mathbf{x}})]}\) is uniquely determined for all \(\hat{\mathbf{x}} \in S_q \setminus S_0 \text{ or } \hat{\mathbf{x}} \in S_q \setminus S_0^2, \quad \mathbf{d} \in S\), and thus for all \(\hat{\mathbf{x}}, \mathbf{d} \in S \) by analyticity. We then claim that \(\alpha_{mn}\) and \(\beta_n\) are uniquely determined for all \(m, n, p, s\). To prove this, we assume that there are two sets of phase functions \(\alpha_{mn}^{(i)}\) and \(\beta_n^{(i)}, i = 1, 2\) and \(m, n, p, s\). Denote by \(\Omega_1\) and \(\Omega_2\) the corresponding rigid bodies. Then we obtain

\[
e^{i[\alpha_{mn}^{(1)}(\hat{\mathbf{x}}, \mathbf{d}) - \beta_n^{(1)}(\hat{\mathbf{x}})]} = e^{i[\alpha_{mn}^{(2)}(\hat{\mathbf{x}}, \mathbf{d}) - \beta_n^{(2)}(\hat{\mathbf{x}})]}, \quad \hat{\mathbf{x}}, \mathbf{d} \in S, \quad m, n, p, s
\]

that is,

\[
\alpha_{mn}^{(1)}(\hat{\mathbf{x}}, \mathbf{d}) - \beta_n^{(1)}(\hat{\mathbf{x}}) = \alpha_{mn}^{(2)}(\hat{\mathbf{x}}, \mathbf{d}) - \beta_n^{(2)}(\hat{\mathbf{x}}) + 2l_{mn}\pi, \quad \hat{\mathbf{x}}, \mathbf{d} \in S, \quad m, n, p, s
\] (3.18)

for some \(l_{mn} \in \{0, \pm 1\}\). Define

\[
\gamma_{mn}(\hat{\mathbf{x}}) := \beta_n^{(1)}(\hat{\mathbf{x}}) - \beta_n^{(2)}(\hat{\mathbf{x}}) + 2l_{mn}\pi, \quad \hat{\mathbf{x}} \in S, \quad m, n, p, s.
\] (3.19)
Then by (3.18) we have
\[
\gamma_{mn}(\hat{x}) = \alpha_{mn}^{(1)}(\hat{x}, d) - \alpha_{mn}^{(2)}(\hat{x}, \hat{d}), \quad \hat{x}, \hat{d} \in \mathbb{S}, \ m, n = p, s.
\]
From this, noting that \(|u_{D_1, \Omega_1}^\infty(\hat{x}, d)| = |u_{D_1, \Omega_2}^\infty(\hat{x}, d)|\) for \(\hat{x}, \hat{d} \in \mathbb{S}\), we observe that
\[
u_{D_1, \Omega_1, mn}(\hat{x}, d) = |u_{D_1, \Omega_1, mn}(\hat{x}, d)| e^{i\alpha_{mn}^{(1)}(\hat{x}, d)}
= |u_{D_1, \Omega_2, mn}(\hat{x}, d)| e^{i\alpha_{mn}^{(2)}(\hat{x}, d)} e^{i\gamma_{mn}(\hat{x})}
= u_{D_1, \Omega_2, mn}(\hat{x}, d) e^{i\gamma_{mn}(\hat{x})}, \quad \hat{x}, \hat{d} \in \mathbb{S}, \ m, n = p, s.
\]
(3.20)
Interchanging the roles of \(\hat{x}\) and \(-\hat{d}\) gives
\[
u_{D_1, \Omega_1, mn}(\hat{x}, d) = u_{D_1, \Omega_2, mn}(\hat{x}, d) e^{i\gamma_{mn}(\hat{x})}, \quad \hat{x}, \hat{d} \in \mathbb{S}, \ m, n = p, s.
\]
(3.21)
Using the reciprocity relation (2.7), from (3.20), we see that
\[
u_{D_1, \Omega_1, mn}(-\hat{d}, -\hat{x}) = u_{D_1, \Omega_2, mn}(-\hat{d}, -\hat{x}) e^{i\gamma_{mn}(\hat{x})}, \quad \hat{x}, \hat{d} \in \mathbb{S}, \ m, n = p, s.
\]
(3.22)
Comparing (3.21) and (3.22), we deduce that
\[
\gamma_{mn}(\hat{x}) = \gamma_{mn}(\hat{x}), \quad \hat{x}, \hat{d} \in \mathbb{S}, \ m, n = p, s.
\]
Thus, \(\gamma_{mn} = \gamma_{nm}\) are constants independent of the directions \(\hat{x}, \hat{d} \in \mathbb{S}\). Recall the definition (3.19) of \(\gamma_{mn}\), we find that \(\gamma_{pm} - \gamma_{nm} = 2(l_{pm} - l_{nm})\pi\) for two constants \(l_{pm}, l_{nm} \in \{0, \pm 1\}\), \(m = p, s\). Thus \(e^{i\gamma_{pm}} = e^{i\gamma_{nm}}, m, n = p, s\) for some fixed constant \(\gamma\).

Let \(G\) be the unbounded connected component of the complement of \(\Omega_1 \cup \Omega_2\). We apply Rellich lemma to conclude that
\[
u_{D_1, \Omega_1, mn}(\hat{x}, d) = \nu_{D_1, \Omega_2, mn}(\hat{x}, d) e^{i\gamma_{mn}}, \quad \hat{x} \in G \setminus \overline{\mathbb{D}}, \hat{d} \in \mathbb{S}, \ m, n = p, s.
\]
(3.23)
Since \(D \cap \Omega_i = \emptyset, i = 1, 2\), we have
\[
D \subset G \quad \text{or} \quad D \subset \mathbb{R}^2 \setminus \overline{\mathbb{C}}.
\]
If \(D \subset \mathbb{R}^2 \setminus \overline{\mathbb{C}}\), there exists a point \(y_0 \in \partial G\) such that \(y_0 \in \partial \Omega_1 \cap \partial \Omega_2\). For the case of \(D \subset G\), we choose any point \(y_1 \in \partial D\). Then we define
\[
y^* := \begin{cases} y_0, & D \subset \mathbb{R}^2 \setminus \overline{\mathbb{C}}; \\
y_1, & D \subset G.
\end{cases}
\]
Using the Dirichlet boundary conditions, we have
\[
\sum_{n \in \{p, s\}} u_{D_1, \Omega_1, mn}(y^*, \hat{x}) = -u_{D_1, \Omega_1, mn}(y^*, \hat{x})
= \sum_{n \in \{p, s\}} u_{D_1, \Omega_2, mn}(y^*, \hat{x}), \quad \hat{x} \in \mathbb{S}, \ m = p, s.
\]
(3.24)
Combining this with (3.23) we find
\[
u_{D_1, \Omega_1, mn}(y^*, \hat{x}) e^{i\gamma_{mn}} = 0, \quad \hat{x} \in \mathbb{S}, \ m = p, s.
\]
(3.25)
From this and \(|u_m^\infty(x^*, -\hat{x})| = 1\), it can be deduced that 
\(e^{i\gamma} = 1\).

Combining this with (3.20), we find that 
\[ u_{D,\Omega, mn}^\infty(\hat{x}, d) = u_{D,\Omega, mn}^\infty(\hat{x}, d), \quad \hat{x}, d \in \mathbb{S}, \ m, n = p, s. \]

The proof is completed by applying the classical uniqueness result with phased far field pattern [20].

**Remark 3.6.** For the case of \(D \subset \mathbb{R}^2 \setminus \Omega\), we have used the Dirichlet boundary conditions on \(\partial \Omega_i\) in the proof. Such a case can be avoided if we know a priori that the unknown scatterers \(\Omega_i, i = 1, 2\) are located in some big ball \(B_R\) with radius \(R\) centered at the origin, and we choose \(D\) outside of \(B_R\).

Essentially the same arguments as in the proof of theorem 3.5 show that the scatterer can be uniquely determined by the phaseless far field patterns with multiple frequencies and one fixed observation direction.

Using lemma 4.1 and following the first part of the previous proof of theorem 3.5, for any fixed polarization \(q \in Q\) and source point \(z \in \mathbb{R}^2 \setminus \Omega\), we can show that the scatterer can be uniquely determined by the following data 
\[
\begin{align*}
&|w_{\Omega \cup \{z\}, mn}(\hat{x}, d, q, \tau)|, \quad \hat{x} \in \mathbb{S}_q, \ d \in \mathbb{S}, \ \tau \in \Theta, \ m, n = p, s, \tag{3.26} \\
&|w_{\Omega \cup \{z\}, mn}(\hat{x}, d, q, \tau)|, \quad \hat{x} \in \mathbb{S}_q^\perp, \ d \in \mathbb{S}, \ \tau \in \Theta, \ m, n = p, s, \tag{3.27} \\
&v_{\Omega, 0}(\hat{x}, z, q, 1), \quad \hat{x} \in \mathbb{S}_q, \tag{3.28} \\
&v_{\Omega, 0}(\hat{x}, z, q, 1), \quad \hat{x} \in \mathbb{S}_q^\perp. \tag{3.29}
\end{align*}
\]

Here, we remove the artificial rigid body \(D\). However, there is a price to pay, that is, we need phased data (3.28) and (3.29).

Under more assumptions on the artificial rigid body \(D\), the following theorem gives a uniqueness result with less data.

**Theorem 3.7.** Assume that \(\Omega \subset B_R\) for some disk \(B_R\) with radius \(R\) centered at the origin. We choose a convex rigid body \(D \subset \mathbb{R}^2 \setminus \Omega\) such that \(\omega^s\) is not a Dirichlet eigenvalue of \(-\Delta^s\) in \(D\). Define \(\Theta_0 := \{0, \tau_1\}\), where \(\tau_1 \in \mathbb{R}\setminus\{0\}\). For the fixed frequency \(\omega > 0\), source polarization \(q \in Q\) and source point \(z \in \mathbb{R}^2 \setminus \Omega\), the scatterer \(\Omega\) is uniquely determined by the following phaseless data 
\[
\begin{align*}
&|w_{D,\Omega \cup \{z\}, mn}(\hat{x}, d, q, \tau)|, \quad \hat{x} \in \mathbb{S}_q, \ d \in \mathbb{S}, \ \tau \in \Theta_0, \ m, n = p, s, \tag{3.30} \\
&|w_{D,\Omega \cup \{z\}, mn}(\hat{x}, d, q, \tau)|, \quad \hat{x} \in \mathbb{S}_q^\perp, \ d \in \mathbb{S}, \ \tau \in \Theta_0, \ m, n = p, s, \tag{3.31} \\
&v_{D,\Omega, 0}(\hat{x}), \quad \hat{x} \in \mathbb{S}_q. \tag{3.32}
\end{align*}
\]
\begin{align}
\left| u_{D_{s},\Omega,2}^{\infty}(\xi) \right|, \quad \xi \in S_q^+. \tag{3.33}
\end{align}

**Proof.** Using (2.22), we have
\begin{align}
&w_{D_{s},\Omega,2}^{\infty}(\hat{\xi}, d, q, \tau) =
\left| u_{D_{s},\Omega,2}^{\infty}(\hat{\xi}, d) + \tau v_{D_{s},\Omega,2}^{\infty}(\hat{\xi}) \right|^2
\end{align}
Thus \( \Re \left[ u_{D_{s},\Omega,2}^{\infty}(\hat{\xi}, d) v_{D_{s},\Omega,2}^{\infty}(\hat{\xi}) \right], \hat{\xi} \in S_q, d \in \mathbb{S}, m = p, s \) is uniquely determined by the phaseless data (3.30) and (3.32). Similarly, \( \Re \left[ u_{D_{s},\Omega,2}^{\infty}(\hat{\xi}, d) v_{D_{s},\Omega,2}^{\infty}(\hat{\xi}) \right], \hat{\xi} \in S_q^+, d \in \mathbb{S}, m = p, s \) is uniquely determined by the phaseless data (3.31) and (3.33). Using analyticity, both of them are uniquely determined for \( \hat{\xi} \in \mathbb{S} \).

Writing
\begin{align}
u_{D_{s},\Omega,2}^{\infty}(\hat{\xi}, d) = |u_{D_{s},\Omega,2}^{\infty}(\hat{\xi}, d)| e^{i\alpha_{mn}(\hat{\xi}, d)}, \hat{\xi} \in \mathbb{S}, d \in \mathbb{S}, m = p, s
\end{align}
where the phase functions \( \alpha_{mn} \in [0, 2\pi) \) and \( \beta_n \in [0, 2\pi) \) are analytic on \( \mathbb{S} \). Then, by analyticity of \( \alpha_{mn} \) and \( \beta_n \), the previous analysis shows that
\begin{align}
\cos[\alpha_{mn}(\hat{\xi}, d) - \beta_n(\hat{\xi})], \quad \hat{\xi} \in \mathbb{S}, d \in \mathbb{S}, m = p, s
\end{align}
are uniquely determined. We claim that \( \alpha_{mn} \) and \( \beta_n \) are uniquely determined. Assume on the contrary that there are two sets of \( \alpha_{mn}^{(1)} \) and \( \beta_n^{(1)} \), and the corresponding scatterers are \( \Omega_i, i = 1, 2 \). Then we have either
\begin{align}
\alpha_{mn}^{(1)}(\hat{\xi}, d) - \beta_n^{(1)}(\hat{\xi}) = \alpha_{mn}^{(2)}(\hat{\xi}, d) - \beta_n^{(2)}(\hat{\xi}) + 2l_{mn}\pi, \quad \hat{\xi} \in \mathbb{S}, d \in \mathbb{S}, \tag{3.35}
\end{align}
or
\begin{align}
\alpha_{mn}^{(1)}(\hat{\xi}, d) - \beta_n^{(1)}(\hat{\xi}) = -\alpha_{mn}^{(2)}(\hat{\xi}, d) + \beta_n^{(2)}(\hat{\xi}) + 2l'_{mn}\pi, \quad \hat{\xi} \in \mathbb{S}, d \in \mathbb{S}, \tag{3.36}
\end{align}
for some \( l_{mn}, l'_{mn} \in \{0, \pm 1\} \), \( m = p, s \). Arguing similarly as in the proof of theorem 3.5, we can obtain from (3.35) and (3.36) that either
\begin{align}
u_{D_{s},\Omega,2}^{\infty}(\hat{\xi}, d) = e^{i\xi} u_{D_{s},\Omega,2}^{\infty}(\hat{\xi}, d), \quad \hat{\xi} \in \mathbb{S}, d \in \mathbb{S}, m = p, s \tag{3.37}
\end{align}
or
\begin{align}
u_{D_{s},\Omega,2}^{\infty}(\hat{\xi}, d) = e^{i\eta} u_{D_{s},\Omega,2}^{\infty}(\hat{\xi}, d), \quad \hat{\xi} \in \mathbb{S}, d \in \mathbb{S}, m = p, s \tag{3.38}
\end{align}
where \( \xi \) and \( \eta \) are two constants independent of \( \hat{\xi}, d \in \mathbb{S} \) and \( m, n \in \{p, s\} \).
Using further Rellich’s lemma, from (3.37), we deduce that
\[ \mathbf{u}_{D,\Omega_2,\infty}^n(x, d) = e^{i\xi} \mathbf{u}_{D,\Omega_2,\infty}^n(x, d), \quad x \in G, \ d \in S, \ m, n = p, s, \]
where \( G \) is the unbounded connected component of the complement of \( D \cup \Omega_1 \cup \Omega_2 \). The assumption on the artificial rigid body \( D \) implies that \( \partial D \subset \partial G \) and
\[ \sum_{n \in \{p,s\}} \mathbf{u}_{D,\Omega_1,\infty}^n(x, d) = -\mathbf{u}_{se}^n(x, d) = \sum_{n \in \{p,s\}} \mathbf{u}_{D,\Omega_2,\infty}^n(x, d), \quad x \in \partial D, \ d \in S, \ m = p, s, \]
Combining the previous two equalities implies that
\[ (e^{i\xi} - 1) \mathbf{u}_{se}^n(x, d) = 0, \quad x \in \partial D, \ d \in S, \ m = p, s. \]
Thus \( e^{i\xi} = 1 \) by noting the fact that \( |\mathbf{u}_{se}^n(x, d)| = 1 \) and consequently
\[ u_{D,\Omega_1,\infty}^x(x, d) = u_{D,\Omega_2,\infty}^x(x, d), \quad x, d \in S, \ m, n = p, s. \]
Then the statement of the theorem follows from the classical uniqueness result with phased far field patterns [20].

We finally show that (3.38) does not hold. Denote by
\[ \mathbf{u}_{D,\Omega_2,\infty}^n(x, d) := \sum_{n \in \{p,s\}} \mathbf{u}_{D,\Omega_2,\infty}^n(x, d), \quad x \in \mathbb{R}^2 \setminus (D \cup \Omega_1), \ d \in S \]
the scattered field due to scattering of plane wave \( \mathbf{u}_{se}^n \) by the scatterer \( D \cup \Omega_i, i = 1, 2 \). From (3.38), by (2.6), we have
\[
\begin{align*}
\mathbf{u}_{D,\Omega_1,\infty}^x(x, d) &= e^{in} \mathbf{u}_{D,\Omega_2,\infty}^x(x, d) \\
&= -e^{in} \int_{\partial D \cup \partial R_\infty} \left\{ \mathbf{u}_{se}^n(y, \hat{x}) \cdot \nabla_{y} \mathbf{u}_{D,\Omega_2,\infty}^n(y, d) \\
&- \left[ T_{y} \cdot y \mathbf{u}_{se}^n(y, \hat{x}) \right] \cdot \mathbf{u}_{D,\Omega_2,\infty}^n(y, d) \right\} ds(y) \\
&= e^{in} \int_{\partial D \cup \partial R_\infty} \left\{ \mathbf{u}_{se}^n(y, -\hat{x}) \cdot \nabla_{y} \mathbf{u}_{D,\Omega_2,\infty}^n(-y, d) \\
&- \left[ T_{y} \cdot y \mathbf{u}_{se}^n(y, -\hat{x}) \right] \cdot \mathbf{u}_{D,\Omega_2,\infty}^n(-y, d) \right\} ds(y), \quad \hat{x}, d \in S, \ m, n = p, s.
\end{align*}
\]
(3.39)
where \( \tilde{D} := \{ x \in \mathbb{R}^2 : -x \in D \} \). Since \( D \subset \mathbb{R}^2 \setminus \overline{R_\infty} \) is convex, we have \( \tilde{D} \subset \mathbb{R}^2 \setminus \overline{R_\infty} \) and \( D \cap \tilde{D} = \emptyset \). Define
\[ \mathbb{I}_n := \begin{cases} -\frac{1}{n} \text{grad div}, & n = p; \\
-\frac{1}{n} \text{grad}^\perp \text{div}^\perp, & n = s.
\end{cases} \]
Note that the right hand side of the above equality (3.39) is the far field pattern of the scattered field 

\[ g_{\text{sc}}^{\infty}(x, d) := e^{i \eta L n} \int_{\partial D \cup \partial B_R} \left\{ \Phi(x, y) \Phi^{(\nu)}(y) u_{D, \Omega_1, \alpha}^\infty(-y, d) \right. \\
- \left[ \Phi^{(\nu)}(y) \Phi(x, y) \right] u_{D, \Omega_1, \alpha}^\infty(-y, d) \} \text{ds}(y), \quad x \in \mathbb{R}^2 \setminus \overline{D \cup B_R}, \quad d \in \mathbb{S}, \quad m, n = p, s. \]

Rellich’s lemma gives that

\[ u_{D, \Omega_1, \alpha}^\infty(x, d) = g_{\text{sc}}^\infty(x, d), \quad x \in \mathbb{R}^2 \setminus \overline{D \cup D \cup B_R}, \quad d \in \mathbb{S}, \quad m, n = p, s. \]

By the analyticity of \( g_{\text{sc}}^\infty \) in \( \mathbb{R}^2 \setminus \overline{D \cup B_R} \), we find that the scattered field \( u_{D, \Omega_1, \alpha}^\infty, m, n = p, s \) can be analytically extended into \( D \) and satisfies the Navier equation in \( D \). Therefore, we deduce that the total field \( u_{D, \Omega_1, \alpha}^\infty := u_{D, \Omega_1, \alpha}^m + u_{D, \Omega_1, \alpha}^s \) satisfies

\[ \Delta^* u_{D, \Omega_1, \alpha}^\infty + \omega^2 u_{D, \Omega_1, \alpha}^\infty = 0 \text{ in } D \quad \text{and} \quad u_{D, \Omega_1, \alpha}^\infty = 0 \text{ on } \partial D, \quad m = p, s. \]

The assumption that \( \omega^2 \) is not a Dirichlet eigenvalue of \( -\Delta^* \) in \( D \) implies that \( u_{D, \Omega_1, \alpha}^\infty = 0 \) in \( D \). From this and the analyticity of \( u_{D, \Omega_1, \alpha}^m \) we finally obtain \( u_{D, \Omega_1, \alpha}^\infty = 0 \) in \( \mathbb{R}^2 \setminus \overline{B_R} \). However, this leads to a contradiction to \( |u_{D, \Omega_1, \alpha}^m(x, d)| = 1 \) for all \( x \in \mathbb{R}^2 \) and \( u_{D, \Omega_1, \alpha}^\infty(x, d) \to 0 \) as \( |x| \to \infty \).

4. Phase retrieval and shape reconstruction methods

This section devotes to the numerical schemes for shape reconstruction with phaseless far field data. First, we propose a fast and stable phase retrieval approach using a simple geometric structure which provides a stable reconstruction of a point in the plane from three given distances. Then, several sampling methods for shape reconstruction with phaseless far field data are given. For obstacle scattering problems, the shear far field pattern corresponding to incident plane shear wave is considered, two different direct sampling methods are proposed with data at a fixed frequency. For inverse source scattering problems, the shear far field pattern is considered, we introduce two direct sampling methods for source supports with sparse multi-frequency data. Other cases follow similarly. The phase retrieval techniques are also combined with the classical sampling methods for the shape reconstructions.

4.1. Phase retrieval

In this subsection, we introduce a phase retrieval method based on the following geometric result [29].

**Lemma 4.1.** Let \( z_j := x_j + iy_j, j = 1, 2, 3 \), be three different complex numbers such that they are not collinear. Then there is at most one complex number \( z \in \mathbb{C} \) with the distances \( r_j = |z - z_j|, j = 1, 2, 3 \). Let further \( \epsilon > 0 \) and assume that

\[ |r_j - r_i| \leq \epsilon, \quad j = 1, 2, 3. \]
Here, and throughout the paper, we use the subscript $\epsilon$ to denote the polluted data. Then there exists a constant $c > 0$ depending on $z_j, j = 1, 2, 3$, such that
\[ |z^\epsilon - z| \leq c. \]

Based on lemma 4.1, we have the following stable phase retrieval scheme which can be implemented easily.

**Phase retrieval scheme.**

1. Collect the distances $r_j := |z - z_j|$ with given complex numbers $z_j, j = 1, 2, 3$. If $r_j = 0$ for some $j \in \{1, 2, 3\}$, then $z = z_j$. Otherwise, go to next step.

2. Look for the point $M = (x_M, y_M)$. As shown in figure 1, $M$ is the intersection of circle centered at $z_2$ with radius $r_2$ and the ray $z_2z_1$ with initial point $z_2$. Denote by $d_{1,2} := |z_1 - z_2|$ the distance between $z_1$ and $z_2$, then
\[
\begin{align*}
x_M &= \frac{r_2}{d_{1,2}} x_1 + \frac{d_1 - r_2}{d_{1,2}} x_2, \\
y_M &= \frac{r_2}{d_{1,2}} y_1 + \frac{d_1 - r_2}{d_{1,2}} y_2,
\end{align*}
\]

3. Look for the points $z_A = (x_A, y_A)$ and $z_B = (x_B, y_B)$. Note that $z_A$ and $z_B$ are just two rotations of $M$ around the point $z_2$. Let $\alpha \in [0, \pi]$ be the angle between rays $z_2z_1$ and $z_2z_A$. Then, by the law of cosines, we have
\[
\cos \alpha = \frac{r_2^2 + d_{1,2}^2 - r_1^2}{2r_2d_{1,2}}.
\]

Note that $\alpha \in [0, \pi]$ and $\sin^2 \alpha + \cos^2 \alpha = 1$, we deduce that $\sin \alpha = \sqrt{1 - \cos^2 \alpha}$. Then
\[
\begin{align*}
x_A &= x_2 + \Re\{[x_M - x_2] + i(y_M - y_2)e^{-i\alpha}\}, \\
y_A &= y_2 + \Im\{[x_M - x_2] + i(y_M - y_2)e^{-i\alpha}\}, \\
x_B &= x_2 + \Re\{[x_M - x_2] + i(y_M - y_2)e^{i\alpha}\}, \\
y_B &= y_2 + \Im\{[x_M - x_2] + i(y_M - y_2)e^{i\alpha}\}.
\end{align*}
\]

4. Determine the point $z, z = z_A$ if the distance $|z_Az_3| = r_3$, or else $z = z_B$. 

![Figure 1. Sketch map for phase retrieval scheme.](image-url)
Lemma 4.1 can immediately be applied to the elastic source scattering problems. Indeed, we set $z_j = -\tau_j$, where $\tau_j \in \mathbb{C}, j = 1, 2, 3$ are three scattering strengths with different principle arguments. For any observation direction $\hat{x} \in \mathbb{S}$, there exists an arc $\bar{S}_{q_j}^+$ for some $q_j \in \mathbb{Q}$ such that $\hat{x} \in \bar{S}_{q_j}^+$, i.e. $\hat{x} \cdot q_j \geq 1/2$. Applying lemma 4.1 and following the proof in theorem 3.3, we obtain the approximate far field pattern $u_{\hat{x},\omega}^\infty(\hat{x}, \omega)$ from the perturbed phaseless far field data set $\{u_{F_j,(z_j)\cdot \hat{x}},(\hat{x}, \omega) \triangleright \tau \in \mathcal{T}\}$.

**Theorem 4.2.** Let $\tau_j \in \mathbb{C}, j = 1, 2, 3$ be three scattering strengths with different principle arguments. For fixed $z \in \mathbb{R}^2 \setminus \bar{\Omega}$, assume that

$$ \left| u_{F_j,(z_j)\cdot \hat{x}}^\infty(\hat{x}, \omega) - u_{F_j,(z_j)\cdot \hat{x}}^\infty(\hat{x}, \omega) \right| \leq \epsilon, \quad \hat{x} \in \bar{S}_{q_j}^+, q_j \in \mathbb{Q}, \tau \in \mathcal{T}, \omega \in \mathbb{W}. $$

Then we have

$$ \left| u_{F_j,(z_j)\cdot \hat{x}}^\infty(\hat{x}, \omega) - u_{F_j,(z_j)\cdot \hat{x}}^\infty(\hat{x}, \omega) \right| \leq c\epsilon, \quad \hat{x} \in \bar{S}_{q_j}^+, \omega \in \mathbb{W} $$

for some constant $c > 0$ depending only on $\tau_j, j = 1, 2, 3$.

Difficulties arise for the obstacle scattering problems because of the additional unknown far field pattern $v_{\hat{x},\omega}^\infty$ corresponding to the point sources. For a source point $z \in \mathbb{R}^2 \setminus \bar{\Omega}$, let $\rho := \text{dist}(z, \Omega)$ be the distance from $z$ to the unknown target $\Omega$. By theorem 2.3, $v_{\hat{x},\omega}^\infty$ is very weak if $\rho \to \infty$, and thus

$$ w_{\Omega,(z_j)\cdot \hat{x}}^\infty(\hat{x}, d, q, \tau) = u_{\Omega,(z_j)\cdot \hat{x}}^\infty(\hat{x}, d) + \tau e^{-i\hat{x} \cdot q_j} \cdot \hat{x} \perp \cdot O \left( \frac{1}{\sqrt{\rho}} \right), \quad \rho \to \infty $$

for all $\hat{x}, d, q, q_j \in \mathbb{S}, \tau \in \mathbb{C}$. Using the phase retrieval scheme, we wish to approximately reconstruct $w_{\Omega,(z_j)\cdot \hat{x}}^\infty(\hat{x}, d)$ from the knowledge of the perturbed phaseless data $u_{\Omega,(z_j)\cdot \hat{x}}^\infty(\hat{x}, d, q_j, \tau)$ with a known error level

$$ \left| w_{\Omega,(z_j)\cdot \hat{x}}^\infty(\hat{x}, d, q_j, \tau) - w_{\Omega,(z_j)\cdot \hat{x}}^\infty(\hat{x}, d, q_j, \tau) \right| \leq \epsilon, \quad \hat{x} \in \bar{S}_{q_j}^+, d \in \mathbb{S}, q_j \in \mathbb{Q}, \tau \in \mathcal{T} $$

uniformly with respect to $\rho > 0$.

**Theorem 4.3.** Let $\tau_j \in \mathbb{C}, j = 1, 2, 3$ be three scattering strengths with different principle arguments. Under the measurement error estimate (4.8), we have

$$ \left| u_{\Omega,(z_j)\cdot \hat{x}}^\infty(\hat{x}, d) - u_{\Omega,(z_j)\cdot \hat{x}}^\infty(\hat{x}, d) \right| \leq c\epsilon \cdot O \left( \frac{1}{\sqrt{\rho}} \right), \quad \rho \to \infty, \quad \hat{x} \in \mathbb{S} $$

for some constant $c > 0$ depending only on $\tau_j, j = 1, 2, 3$.

**Proof.** Define $r'_j := \left| u_{\Omega,(z_j)\cdot \hat{x}}^\infty(\hat{x}, d, q_j, \tau) \right|, j = 1, 2, 3$. By theorem 2.3, we have

$$ r'_j = \left| u_{\Omega,(z_j)\cdot \hat{x}}^\infty(\hat{x}, d) + \tau e^{-i\hat{x} \cdot q_j} \cdot \hat{x} \perp \cdot O \left( \frac{1}{\sqrt{\rho}} \right) \right| $$

$$ = q \cdot \hat{x} \perp \left| u_{\Omega,(z_j)\cdot \hat{x}}^\infty(\hat{x}, d) e^{i\hat{x} \cdot q_j} \cdot \hat{x} \perp \cdot O \left( \frac{1}{\sqrt{\rho}} \right) \right| + \tau + O \left( \frac{1}{\sqrt{\rho}} \right), \quad \rho \to \infty $$

(4.10)
for all $\hat{x} \in S_\perp^1, d \in S, q \in Q$. Here we have used the fact that $q \cdot \hat{x} \perp \geq 1/2$ for $\hat{x} \in S_\perp^1$. Let now

$$z_j = -\tau_j, \ j = 1, 2, 3.$$  

Then the assumption on the strengths implies that the three points $z_j, j = 1, 2, 3$ are not collinear. Applying lemma 4.1, we have

$$| u_{\Omega,d}^\infty(\hat{x},d) - u_{\Omega,d}^\infty(\hat{x}',d) | + O \left( \frac{1}{\sqrt{\rho}} \right) \leq c \tau_j,$$

for some constant $c > 0$ depending only on $\tau_j, j = 1, 2, 3$. The stability estimate (4.9) now follows by using the triangle inequality.

Finally, we want to remark that the same estimates in theorems 4.2 and 4.3 also hold for the other phaseless far field data with appropriate modification of the observation arcs.

4.2. Scatterer shape reconstruction

This subsection is devoted to introduce some direct sampling methods for reconstruction of $\Omega$ by using phaseless shear far field data $|w_{\Omega,d}(x,\lambda)|$. The direct sampling methods proposed do not need any a priori information of the obstacle. We want to remark that, in most theoretical analysis (except theorem 3.5), we need both compressional and shear waves. However, only one of the waves are involved in our numerical methods. This is in accordance with the direct sampling methods with phased data [28].

For some polarization direction $q$, we first introduce two auxiliary functions

$$G(p,d,q) := \int_{S} u_{\Omega,d}^\infty(\hat{x},d)e^{ikd\hat{x} \cdot p} q \cdot \hat{x} \perp ds(\hat{x}), \ p \in \mathbb{R}^2,$$

$$A(p,q) := \int_{S} G(p,d,q)e^{-ikd \cdot \hat{x}} p \cdot \hat{x} \perp ds(\hat{x}), \ p \in \mathbb{R}^2.$$  

(4.11)

By the well-known Riemann–Lebesgue lemma, both $G$ and $A$ tend to 0 as $|p| \to \infty$. In fact, due to the systematic analysis in [28], $G$ is a superposition of the Bessel functions. We thus expect that $G$ (and therefore $A$) decays like Bessel functions as the sampling points away from the boundary of the scatterer. Then one may look for the scatterers by using the following indicators [28] with phased far field patterns

$$I_2(p) = \bigcup_{q \in Q} | A(p,q) | \ \ \text{and} \ \ I_3(z,d) = \bigcup_{q \in Q} | G(p,d,q) |.$$  

(4.12)

In [28], it has been shown that the indicator $I_2$ has a positive lower bound for sampling points inside the scatterer, and decays like Bessel functions as the sampling points tend to infinity. If the size of the scatterer $\Omega$ is small enough (compared with the wavelength), $I_3$ takes its local maximum at the location of the scatterer.

Consider now the case of phaseless far field measurements. Using the notations in theorem 2.3, for all $\hat{x}, d \in S$, fixed $\tau_1 \in \mathbb{C}\{0\}$ and $z_0 \in \mathbb{R}^2\overline{\Omega}$, we have
Then we introduce the following two indicators

\[ I_{\alpha}(\mathbf{p}, \mathbf{d}) := \int_{\Omega} \mathcal{F}_{\alpha}(\hat{\mathbf{s}}, \mathbf{d}, \mathbf{q}, \tau) \cos[k\hat{\mathbf{s}} \cdot (\mathbf{p} - \mathbf{z}_0)] ds(\hat{\mathbf{s}}), \quad \mathbf{d} \in \mathbb{S}, \mathbf{p} \in \mathbb{R}^2, \]

and

\[ I_{\beta}(\mathbf{p}) := \int_{\mathbb{R}^2} I_{\alpha}(\mathbf{p}, \mathbf{d}) ds(\mathbf{d}), \quad \mathbf{p} \in \mathbb{R}^2. \]

Insert (4.13) into (4.14) and (4.15). Then a straightforward calculation shows that

\[ I_{\alpha}(\mathbf{p}, \mathbf{d}) = \sum_{\mathbf{q} \in \Omega} \left| V_{\alpha}(\mathbf{p}, \mathbf{d}, \mathbf{q}) + V_{\beta}(\mathbf{p}, \mathbf{d}, \mathbf{q}) \right|^2 + O\left(\rho^{-1/2}\right), \quad \mathbf{d} \in \mathbb{S}, \mathbf{p} \in \mathbb{R}^2, \]

\[ I_{\beta}(\mathbf{p}) = \int_{\mathbb{R}^2} \sum_{\mathbf{q} \in \Omega} \left| V_{\alpha}(\mathbf{p}, \mathbf{d}, \mathbf{q}) + V_{\beta}(\mathbf{p}, \mathbf{d}, \mathbf{q}) \right|^2 ds(\mathbf{d}) + O\left(\rho^{-1/2}\right), \quad \mathbf{p} \in \mathbb{R}^2 \]

with

\[ V_{\alpha}(\mathbf{p}, \mathbf{d}, \mathbf{q}) := \frac{1}{\pi} \left| G(z, \mathbf{d}, \mathbf{q}) + G(2z_0 - \mathbf{p}, \mathbf{d}, \mathbf{q}) \right|, \quad \mathbf{p} \in \mathbb{R}^2, \mathbf{d} \in \mathbb{S}, \mathbf{q} \in \Omega. \]

Let \( \Omega(\mathbf{z}_0) \) be the point symmetric domain of \( \Omega \) with respect to \( \mathbf{z}_0 \). If the size of the scatterer \( \Omega \) is small enough, from the properties of \( G \), we expect that the indicator \( I_{\alpha}(\mathbf{p}, \mathbf{d}) \) takes its local maximum on the locations of \( \Omega \) and \( \Omega(\mathbf{z}_0) \). For extended scatterer \( \Omega \), we expect that the indicator \( I_{\beta} \) takes its maximum on or near the boundary \( \partial \Omega \cup \partial \Omega(\mathbf{z}_0) \).

Note that the indicator \( I_{\alpha}(\mathbf{p}) \) (or \( I_{\beta}(\mathbf{p}, \mathbf{d}) \) the case of small scatterers) produces a false scatterer \( \Omega(\mathbf{z}_0) \). However, since we have the freedom to choose the point \( \mathbf{z}_0 \), we can always choose it such that the false domain \( \Omega(\mathbf{z}_0) \) located outside our searching domain of interest. One may also overcome this problem by considering another indicator \( I_{\alpha}(\mathbf{p}) \) (or \( I_{\beta}(\mathbf{p}, \mathbf{d}) \) the case of small scatterers) with \( \mathbf{z}_1 \in \mathbb{R}^2 \setminus \Omega \) and \( \mathbf{z}_1 \neq \mathbf{z}_0 \).

**Scatterer reconstruction scheme one.**

- **Collect the phaseless data set**

\[ \left\{ |w_{\Omega,\mathbf{z}_0,\mathbf{z}_1}(\hat{\mathbf{s}}, \mathbf{d}, \mathbf{q}, \tau)| : \hat{\mathbf{s}} \in \mathbb{S}^1, \mathbf{q} \in \Omega, \mathbf{d} \in \mathbb{S}, \tau \in \{0, \tau_1\} \right\}. \]

- **Select a sampling region in \( \mathbb{R}^2 \) with a fine mesh \( \mathcal{Z} \) containing the scatterer \( \Omega \).**

- **Compute the indicator functional \( I_{\alpha}(\mathbf{p}) \) (or \( I_{\beta}(\mathbf{p}, \mathbf{d}) \) with some fixed \( \mathbf{d} \in \mathbb{S} \) in the case of small scatterers) for all sampling points \( \mathbf{p} \in \mathcal{Z} \).**

- **Plot the indicator functional \( I_{\alpha}(\mathbf{p}) \) (or \( I_{\beta}(\mathbf{p}, \mathbf{d}) \) in the case of small scatterers).**

Using the Phase retrieval scheme proposed in the previous subsection, we can retrieve approximately the phased far field pattern \( u_{\Omega,\mathbf{z}_0,\mathbf{z}_1}^\infty \). Then we have the second scatterer reconstruction algorithm.
Scatterer reconstruction scheme two.

- Collect the phaseless data set 
  \[ \left\{ |w_{\infty,\Omega}^{\infty}(\hat{x},d,q,\tau)| : \hat{x} \in \mathbb{S}_q^\perp, q \in \mathbb{Q}, d \in \mathbb{S}, \tau \in T \right\}. \]

- Use the Phase retrieval scheme to retrieve approximately the phased far field patterns 
  \[ u_{\infty,\Omega}^{\infty}(\hat{x},d,q) \]
  for all \( \hat{x}, d \in \mathbb{S} \).

- Select a sampling region in \( \mathbb{R}^2 \) with a fine mesh \( Z \) containing \( \Omega \).

- Compute the indicator functional \( I_2(p) \) (or \( I_3(p,d) \) with fixed \( d \in \mathbb{S} \)) for all sampling points \( p \in Z \).

- Plot the indicator functional \( I_2(p) \) (or \( I_3(p,d) \) in the case of small scatterers).

4.3. Source support reconstruction

The uniqueness results discussed before ensure the possibility to reconstruct the unknown objects by stable algorithms. In this section, we investigate the numerical methods for support reconstruction of the source \( F \) using phaseless far field data 

[\[ u_{\infty,\Omega}^{\infty} \cup \{ z_0 \}, ss(\hat{x},d,q,\tau) \]].

Denote by \( \Theta \) a finite set with finitely many observation directions as elements. We first introduce an auxiliary function

\[ H(p,\hat{x}) := \int_{\mathbb{W}} u_{\infty,\Omega}^{\infty}(\hat{x},\omega)e^{ik\hat{x} \cdot p}d\omega, \quad p \in \mathbb{R}^2, \hat{x} \in \Theta. \] (4.16)

Clearly,

\[ H(p + \alpha \hat{x}^\perp,\hat{x}) = H(p,\hat{x}), \quad p \in \mathbb{R}^2, \alpha \in \mathbb{R}. \] (4.17)

This further implies that the functional \( H \) has the same value for sampling points in the hyperplane with normal direction \( \hat{x} \). By the well known Riemann–Lebesgue lemma, we obtain that \( H \) tends to 0 as \( |p| \to \infty \).

Recall from (2.31) that the far field pattern has the following representation

\[ u_{\infty,\Omega}^{\infty}(\hat{x},\omega) = \int_{\mathbb{R}^2} e^{-ik\hat{x} \cdot y} S_p(y,\hat{x})F(y)dy, \quad \hat{x} \in \mathbb{S}, \omega \in \mathbb{W}. \] (4.18)

Inserting it into the indicator \( H \) defined in (4.16), changing the order of integration, and integrating by parts, we have

\[ H(p,\hat{x}) = \int_\Omega \int_{\mathbb{W}} e^{ik\hat{x} \cdot (p-y)} S_p(y,\hat{x})F(y)dyd\omega \]

\[ = \int_\Omega \int \frac{S_p(y,\hat{x})}{ik \cdot (p - y)}dyd\omega \]

\[ = \int_\Omega S_p(y,\hat{x})dyd\omega \]

(4.19)

where \( S_p \in L^\infty(\Omega) \) is given by

\[ S_p(y,\hat{x}) := \hat{x}^\perp \cdot F(y)e^{ik\hat{x} \cdot (p - y)} \bigg|_{\omega_{\text{max}}}. \]

This implies that the functional \( H \) is a superposition of functions that decays as \( 1/|\hat{x} \cdot (p - y)| \) as the sampling point \( p \) goes away from the strip \( S_\Omega(\hat{x}) \).
For any \( \hat{x} \in S, \omega \in \mathbb{W}, \tau \in \mathbb{C} \), we define
\[
K_\omega (\hat{x}, q, \omega, \tau) := |u_{\omega, F}^\infty (\hat{x}, q, \omega, \tau) |^2 - |u_{\omega, F}^\infty (\hat{x}, \omega) |^2 - |q \cdot \hat{x}^\bot |^2 \\
= |u_{\omega, F}^\infty (\hat{x}, \omega) + \tau e^{-ik x^\perp q \cdot \hat{x}^\bot} |^2 - |u_{\omega, F}^\infty (\hat{x}, \omega) |^2 - |q \cdot \hat{x}^\bot |^2 \\
= \left( u_{\omega, F}^\infty (\hat{x}, \omega) \tau e^{ik x^\perp q \cdot \hat{x}^\perp} + u_{\omega, F}^\infty (\hat{x}, \omega) \tau e^{-ik x^\perp q \cdot \hat{x}^\perp} \right) q \cdot \hat{x}^\perp. \tag{4.20}
\]

Then, for any fixed \( \tau \in \mathbb{C} \setminus \{0\} \) and \( z_0 \in \mathbb{R}^2 \setminus \overline{\Omega} \), we introduce the following indicator
\[
I_{z_0,S}^\omega (p) := \sum_{x \in \Theta, q \in Q} \left| \int_{\mathbb{W}} K_{z_0} (\hat{x}, q, \omega, \tau) \cos[k_\omega (p - z_0)] d\omega \right|, \quad p \in \mathbb{R}^2. \tag{4.21}
\]

Inserting (4.20) into (4.21), straightforward calculations show that
\[
I_{z_0,S}^\omega (p) = \sum_{\hat{x} \in \Theta, q \in Q} \left| U_{z_0} (p, \hat{x}, q) + U_{z_0} (p, \hat{x}, q) \right|, \quad p \in \mathbb{R}^2
\]
with
\[
U_{z_0} (p, \hat{x}, q) := \frac{q \cdot \hat{x}^\perp}{2} \left[ H(p, \hat{x}) + H(2z_0 - p, \hat{x}) \right], \quad p \in \mathbb{R}^2, \hat{x} \in \Theta, q \in Q.
\]

Let \( \Omega(z_0) \) be again the point symmetric domain of \( \Omega \) with respect to \( z_0 \). We expect that the indicator \( I_{z_0,S}^\omega \) takes its maximum on the locations of \( \Omega \) and \( \Omega(z_0) \). Similarly to the phaseless obstacle scattering problem, we can choose \( z_0 \) such that the false domain \( \Omega(z_0) \) located outside our searching domain of interest or another point \( z_1 \in \mathbb{R}^2 \setminus \overline{\Omega} \) and \( z_1 \neq z_0 \).

**Source support reconstruction scheme one.**

1. Collect the phaseless data set
   \[
   \{ |u_{\omega, F}^\infty (\hat{x}, q, \omega, \tau) | : \hat{x} \in \Theta \cap S_q^+, q \in Q, \omega \in \mathbb{W}, \tau \in \{0, \tau_1\} \}.
   \]
2. Select a sampling region in \( \mathbb{R}^2 \) with a fine mesh \( Z \) containing the source support \( \Omega \).
3. Compute the indicator functional \( I_{z_0,S}^\omega (p) \) for all sampling points \( p \in Z \).
4. Plot the indicator functional \( I_{z_0,S}^\omega (p) \).

Using the phase retrieval scheme proposed in the previous subsection, we obtain the phased far field pattern \( u_{\omega, F}^\infty \). Then we have the second scatterer reconstruction algorithm using the following indicator [1]
\[
I_{\Theta}^\omega (p) := \sum_{\hat{x} \in \Theta} |H(p, \hat{x})|, \quad p \in \mathbb{R}^2. \tag{4.22}
\]

**Source support reconstruction scheme two.**

1. Collect the phaseless data set
   \[
   \{ |u_{\omega, F}^\infty (\hat{x}, q, \tau, \omega) | : \hat{x} \in \Theta \cap S_q^+, q \in Q, \omega \in \mathbb{W}, \tau \in \mathbb{T} \}.
   \]
2. Use the phase retrieval scheme to obtain the phased far field patterns \( u_{\omega, F}^\infty \) for all \( \hat{x} \in \Theta, \omega \in \mathbb{W} \).
3. Select a sampling region in \( \mathbb{R}^2 \) with a fine mesh \( Z \) containing \( \Omega \).
4. Compute the indicator functional \( I_{\Theta}^\omega (p) \) for all sampling points \( p \in Z \).
5. Plot the indicator functional \( I_{\Theta}^\omega (p) \).
For each scattering model, we have two reconstruction schemes. The first scheme uses the phaseless data directly to reconstruct the objects, while the second one is divided into two steps: phase retrieval and shape reconstruction. The first scheme uses less data, but may produce a false domain. After obtaining the phased data in the second scheme, many existing numerical methods can be used to reconstruct the objects.

5. Numerical experiments

Now we present a variety of numerical examples in two dimensions to illustrate the applicability and effectiveness of our sampling methods. In the simulations, we use 0.05 as the sampling space. If not otherwise stated, we add 10% noise. We take $\tau = 1$ for the indicators $I_{z_0}$ and $I_{z_0, S}$, while $\tau = \pm 0.5, 0.5i$ are used in the Phase retrieval scheme.

5.1. Phaseless inverse scattering problems

There are totally four groups of numerical tests to be considered, and they are respectively referred to as $I_{z_0}(p)$-Big, $I_{z_0}(p, d)$-Small, phaseretrieval, $I_{z_0}$-Big + $I_{z_0}$-Small. The boundaries of the scatterers used in our numerical experiments are parameterized as follows.

\[ \text{Circle}: \quad x(t) = (a, b) + 0.1 (\cos t, \sin t), \quad 0 \leq t \leq 2\pi, \quad (5.1) \]

\[ \text{Kite}: \quad x(t) = \cos t + 0.65 \cos 2t - 0.65, 1.5 \sin t, \quad 0 \leq t \leq 2\pi, \quad (5.2) \]

with $(a, b)$ being the location.

The direct problem is solved by the boundary integral equation method [28]. Define $\theta_l := 2\pi l / N, l = 0, 1, \cdots, N - 1$, let $d_l = (\cos \theta_l, \sin \theta_l)^T$ and $\hat{x}_j = (\cos \theta_j, \sin \theta_j)^T$ for $j, l = 0, 1, \cdots, N - 1$. In our simulations, we compute the far field patterns $w_{\Omega, \{z_0\}, \sigma}^{\infty, \delta}(\hat{x}_j, d_l, q, \tau), q \in \mathbb{Q}, j, l = 0, 1, \cdots, N - 1$, for $N$ equidistantly distributed incident directions and $N$ observation directions. We further perturb this data by random noise

\[ \left| w_{\Omega, \{z_0\}, \sigma}^{\infty, \delta}(\hat{x}_j, d_l, q, \tau) \right| = \left| w_{\Omega, \{z_0\}, \sigma}^{\infty, \delta}(\hat{x}_j, d_l, q, \tau) \right| (1 + \delta * e_{rel}), \quad q \in \mathbb{Q}, j, l = 0, 1, \cdots, N - 1, \]

where $e_{rel}$ is a uniformly distributed random number in the open interval $(-1, 1)$. The value of $\delta$ used in our code is the relative error level. We also consider absolute error in example phaseretrieval. In this case, we perturb the phaseless data

\[ \left| w_{\Omega, \{z_0\}, \sigma}^{\infty, \delta}(\hat{x}_j, d_l, q, \tau) \right| = \max \left\{ 0, \left| w_{\Omega, \{z_0\}, \sigma}^{\infty, \delta}(\hat{x}_j, d_l, q, \tau) \right| + \delta * e_{abs} \right\}, \quad q \in \mathbb{Q}, j, l = 0, 1, \cdots, N - 1, \]

where $e_{abs}$ is again a uniformly distributed random number in the open interval $(-1, 1)$. Here, the value $\delta$ denotes the total error level in the measured data.

In the following experiments, we take $N = 512$, the Lamé constants $\lambda = 1$ and $\mu = 1$, and the circular frequency $\omega = 8\pi$.

Example $I_{z_0}(p)$-Big. This example checks the validity of our method for scatterers with different source points. We consider the benchmark example with a kited domain. Figure 2 shows the results with three source points $z_0 = (2, 4)^T, (4, 4)^T$ and $(12, 12)^T$. As expected, the indicator $I_{z_0}(p)$ takes a large value on $\partial \Omega \cup \partial \Omega(z_0)$, where $\Omega(z_0)$ is the symmetric domain of
with respect to the source point \(z_0\). The symmetric domain of \(z_0 = (12, 12)\) is outside of the sampling space. Note that \(\Omega(z_0)\) changes as the source point \(z_0\) changes, thus it is very easy to pick the correct domain \(\Omega\) by considering the indicator \(I_{z_0}\) with different source points, or we can just choose \(z_0\) far enough. As shown in figure 2, the left hand kite should be the one desired.

**Example** \(I_{z_0}(p, d)\)-Small. In this example, the scatterer is a combination of two mini disks with radius 0.1, one centered at \((a, b) = (3, 3)\) and the other at \((a, b) = (1, 1)\). Figure 3 shows the reconstructions using \(I_{z_0}(p, d)\), \(d = (1, 0)^T\) with the same source points as in the example \(I_{z_0}(p)\)-Big.

**Example phaseretrieval.** This example is designed to check the phase retrieval scheme proposed in section 4.1. The underlying scatterer is chosen to be a kite shaped domain. For comparison, we consider the real part of far field pattern at a fixed incident direction \(d = (1, 0)^T\). Figure 4 shows the results without measurement noise by using three different source points \((2, 4)^T, (4, 4)^T\) and \((12, 12)^T\). In particular, the source point \(2, 4)^T\) is very close to the kite shaped domain. However, Figure 4(a) shows that the multiple scattering is very week. Of course, Figures 4(b) and (c) show that the interaction between the source point and the kite shaped domain decreases as the source point away from the target. Figures 5 and 6 show the results with relative error and absolute error considered, respectively. We find that our phase retrieval scheme is quite robust with respect to noise. This also verifies the theory provided in theorem 4.3.

**Example** \(I_{z_0}(p)\)-Big + \(I_{z_0}(p, d)\)-Small. In figure 7, the scatterers are the same as the **Example** \(I_{z_0}(p)\)-Big and example \(I_{z_0}(p, d)\)-Small. We choose the same source point \(z_0 = (2, 4)^T\), no false domain appears in the reconstructions. Figures 7(a) uses all the incident directions, while figure 7(b) uses one incident direction \(d = (1, 0)^T\).

We summarize our schemes and the corresponding numerical results in the following table 1.

### 5.2. Phaseless inverse source problems

The forward problems are computed the same as in [1]. In all examples, for \(\hat{\mathbf{x}} \in \Theta\), we consider multiple frequency far field data \(u^{\infty}_{F,j}(\hat{\mathbf{x}}, k_j), j = 1, \cdots, N\), where \(N = 20, k_{\text{min}} = 0.5, k_{\text{max}} = 20\) such that \(k_j = (j - 0.5) \Delta k, \Delta k = \frac{k_{\text{max}}}{N}\).
Figure 3. Example $I_{E}(p,d)$-Small. Reconstruction of two small disks by using $I_{E}(p,d)$ with different source points. Here, $d = (1, 0)^T$. (a) $z_0 = (2, 4)^T$. (b) $z_0 = (4, 4)^T$. (c) $z_0 = (12, 12)^T$.

Figure 4. Example PhaseRetrieval. Phase retrieval for the real part of the far field pattern without error using different source points at a fixed direction $d = (1, 0)^T$. (a) $z_0 = (2, 4)^T$. (b) $z_0 = (4, 4)^T$. (c) $z_0 = (12, 12)^T$.

Figure 5. Example PhaseRetrieval. Phase retrieval for the real part of the far field pattern with different relative error and $z_0 = (2, 4)^T$ at a fixed direction $d = (1, 0)^T$. (a) 0 noise. (b) 10\% noise. (c) 30\% noise.

Figure 6. Example PhaseRetrieval. Phase retrieval for the real part of the far field pattern with different absolute error and $z_0 = (2, 4)^T$ at a fixed direction $d = (1, 0)^T$. (a) 0 noise. (b) 0.1 noise. (c) 0.3 noise.
Then the phaseless data are stored in the matrices $M_{Fz_0}$, $z_0 = (|u_{\infty}F| \cup \{z_0\}, s(\hat{x}, kj, \tau))$, $\hat{x} \in \Theta, j = 1, \cdots, N$. We further perturb $M_{Fz_0}$ by random relative error and absolute error as before. Three shapes are considered: a rectangle given by $(1, 2) \times (1, 1.6)$, a L-shaped domain given by $(0, 2) \times (0, 2) \setminus (1/16, 2) \times (1/16, 2)$ and an equilateral triangle with vertices $(-2, 0), (1, 0), (-1/2, 3/2\sqrt{3})$. Here, we use

**Figure 7.** $I_2$—Big $+ I_3$—Small Reconstruction of different shapes. (a) Kite. (b) Two circles.

**Table 1.** Schemes and the corresponding numerical results for the scatterer reconstructions.

| Schemes                                | Numerical results          |
|----------------------------------------|----------------------------|
| Scatterer reconstruction scheme one    | Figures 2 and 3            |
| Phase retrieval                        | Figures 4–6                |
| Scatterer reconstruction scheme two    | Figure 7                   |

**Figure 8.** $I^0_{\Theta z_0}$ with one observation direction for rectangle. (a) $z_0 = (4, 1.3)^T$. (b) $z_0 = (4, 4)^T$. (c) $z_0 = (12, 12)^T$.

Then the phaseless data are stored in the matrices $M_{Fz_0} = (|a_{Fz_0}(\hat{x}, s(\hat{x}, kj, \tau))|)$, $\hat{x} \in \Theta, j = 1, \cdots, N$. We further perturb $M_{Fz_0}$ by random relative error and absolute error as before. Three shapes are considered: a rectangle given by $(1, 2) \times (1, 1.6)$, a L-shaped domain given by $(0, 2) \times (0, 2) \setminus (1/16, 2) \times (1/16, 2)$ and an equilateral triangle with vertices $(-2, 0), (1, 0), (-1/2, 3/2\sqrt{3})$. Here, we use
F = (x^2 + y^2 + 5, x^2 - y^2 + 5)^T.

\( I_{\Theta z_0}^{z_0} \) with one and two observation directions. We first consider the case of one observation using different \( z_0 \). The support of \( F \) is the rectangle. In figure 8, we plot the indicators using \( \hat{x} = (0, 1)^T \) and three source points \( z_0 = (4, 1.3)^T, (4, 4)^T \) and \((12, 12)^T\). The picture clearly shows that the source and its the point symmetric domain (with respect to \( z_0 \)) lies in a strip.
which is perpendicular to the observation direction. Next we consider two observation directions \((1,0)^T\) and \((0,1)^T\), we plot the indicators in figure 9. Since the observation directions are perpendicular to each other, the strips are perpendicular to each other too.

**I** \( \Theta \) \( z_0 \), with multiple observation directions. We use the rectangle and the L-shaped domain in this example. Now we use 20 observation angles \( \theta_j, j = 1, \cdots, 20 \) such that \( \theta_j = -\pi/2 + j\pi/20 \). Note that \( \theta_j \in (-\pi/2, \pi/2) \). Figure 10 gives the results for rectangle with different \( z_0 \). Figure 11 gives the results for the L-shaped domain. The locations and sizes of support of \( F \) are reconstructed correctly.

**Figure 12.** Example PhaseRetrieval. Phase retrieval for the real part of the far field pattern with relative error at a fixed direction \( \hat{x} = (0,1)^T \). (a) 10% noise. (b) 30% noise. (c) 50% noise.

**Figure 13.** Example PhaseRetrieval. Phase retrieval for the real part of the far field pattern with absolute error at a fixed direction \( \hat{x} = (0,1)^T \). (a) 0.1 noise. (b) 0.3 noise. (c) 0.5 noise.

**Figure 14.** Example Extended Objects. Reconstructions the phase retrieval scheme using multiple directions for different domains with \( z_0 = (4,4)^T \). (a) L-shaped domain. (b) Triangle.
The validation the phase retrieval scheme. This example is designed to check the phase retrieval scheme proposed before. The underlying scatterer is the rectangle. In figures 12 and 13, we compare the phase retrieval data with the exact one, the real part of far field pattern at a fixed direction \( \hat{x} = (0, 1)^T \) is given. We observe that the phaseless data are reconstructed very well for small relative error level. These two figures also show that our phase retrieval scheme is robust to noise.

\( \text{I}_\Theta(p) \) for extended objects. In this example, we take \( z_0 = (4, 4)^T \). We show the reconstructions of extended objects with the same 20 observation directions. One is the L-shaped domain, the other is the triangle. Figure 14 give the reconstructions.

We summarize our schemes and the corresponding numerical results in the following table 2.

### Table 2. Schemes and the corresponding numerical results for source support reconstructions.

| Schemes                                | Numerical results |
|----------------------------------------|-------------------|
| Source support reconstruction scheme one | Figures 8-11      |
| Phase retrieval                        | Figures 12 and 13  |
| Source support reconstruction scheme two | Figure 14         |

6. Concluding remarks

In this paper, we study systematically the inverse elastic scattering problems with phaseless far field data. By considering simultaneously the scattering of point sources, we establish some uniqueness results with phaseless far field data, propose a simple and stable phase retrieval technique and some direct sampling methods for shape reconstructions. The theoretical investigations are then complemented by numerical examples which exploit generated synthetic far-field data for a variety of surfaces in two dimensions. The elaborated numerical reconstructions reveal that the phase retrieval technique is quite robust to noise and the proposed direct sampling methods are capable of identifying unknown objects effectively, even only spare data are used.

Numerically, we observe that if the indicator \( \text{I}_{\Theta}(p, d) \) given in (4.14) is replaced by

\[
\bar{I}_{\Theta}(p, d) := \sum_{q \in Q} \left| \int_S \mathcal{F}_{z_0}(\hat{x}, d, q, \tau_1) \cos[k_0 \hat{x} \cdot (p - z_0) - k_0 p \cdot d] ds(\hat{x}) \right|^2, \quad d \in \mathbb{S}, \quad p \in \mathbb{R}^2. \tag{6.1}
\]

Then the false domain \( \Omega(z_0) \) disappear surprisingly. Unfortunately, we are not clear of the theory basis for this fact.

We are also interested in the phaseless total fields taken on some measurement surface containing the unknown objects. For the source scattering problems, the phase retrieval technique is also applicable. However, since the point sources are also radiating solutions to the Navier equation, to retrieve the phased data stably, we have to choose the source points close to the measurement surface. We will address this problem in a forthcoming paper.
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