Rational endomorphisms of plane preserving a rational volume form

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Abstract. Let \( \varphi \) be a rational map \( \mathbb{P}^2 \to \mathbb{P}^2 \) which preserves the rational volume form \( \frac{dx}{x} \wedge \frac{dy}{y} \). Sergey Galkin conjectured that in this case \( \varphi \) is necessarily birational. We show that such a map preserves the element \( \{x, y\} \) of the second K-group \( K_2(k(x, y)) \) up to multiplication by a constant, and restate this condition explicitly in terms of mutual intersections of the divisors of coordinates of \( \varphi \) in a way suitable for computations.

1. Introduction

Diller and Lin in their paper [2] showed that if a general rational self-map \( \varphi : \mathbb{P}^2 \to \mathbb{P}^2 \) preserves a rational two-form \( \omega \) on the complex plane, then up to a birational change of coordinates one of the following holds:
1) \(-\text{div}(\omega)\) is a smooth cubic curve
2) \(\omega = \frac{dx}{x} \wedge \frac{dy}{y}\)
3) \(\omega = dx \wedge dy\).

In this work we consider the second case of this classification. Let us call a rational map \( \varphi : \mathbb{P}^2 \to \mathbb{P}^2 \) which preserves the rational volume form \( \frac{dx}{x} \wedge \frac{dy}{y} \) a symplectic rational map. Sergey Galkin conjectured that any symplectic rational map is birational.

We make some preliminary steps towards this conjecture, namely:
1) We show that this conjecture is equivalent to another one, involving \( K_2(k(\mathbb{P}^2)) \)—see Theorem [1].
2) We utilize a description of \( K_2 \) in terms of algebraic cycles (Proposition [2]). This allows us to write down the condition of preserving the rational volume form explicitly in terms of the divisors of coordinates of \( \varphi \), in a way suitable for computations. (see Section [5])
3) Notice that for a birational map \( \varphi : \mathbb{P}^2 \to \mathbb{P}^2 \) given by its affine coordinates \( \varphi = (f, g) \), if \( \text{Bs}(f) \cap \text{Bs}(g) = \emptyset \), then \( \varphi \) is biregular. Using the condition of (2), we prove that the same holds for a symplectic rational map (under a certain condition, see Corollary [1]).

It may be interesting to compare our conditions with the generators and relations for the group of birational transformations of \( \mathbb{P}^2 \) preserving \( \frac{dx}{x} \wedge \frac{dy}{y} \), which were obtained by Jérémy Blanc [1].

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2. Statement of the conjecture

Let \( k \) be a field of characteristic 0. A dominant rational map \( \varphi : X \to Y \) of irreducible varieties over \( k \) induces an inverse image on the spaces of rational differential forms over \( k \), denoted by \( \varphi^* : \Omega^2_{k(Y)} \to \Omega^2_{k(X)} \).
Conjecture. Let \( \varphi : \mathbb{A}^2 \rightarrow \mathbb{A}^2 \) be a dominant rational map such that \( \varphi^* \left( \frac{dx}{x} \wedge \frac{dy}{y} \right) = \frac{dx}{x} \wedge \frac{dy}{y} \in \Omega^2_{k(x,y)} \). Then \( \varphi \) is birational.

Remark. This conjecture is related to the following famous question.

Jacobian conjecture. Let \( f : \mathbb{A}^2 \rightarrow \mathbb{A}^2 \) be a regular map such that \( f^* (dx \wedge dy) = dx \wedge dy \). Is it true that \( f \) is biregular?

One of the possible obstructions to proving the Jacobian conjecture via methods of birational geometry is the unconstrained growth of singularities of the pullback of \( dx \wedge dy \) under birational transformations. These complications do not arise when dealing with the logarithmic form \( \frac{dx}{x} \wedge \frac{dy}{y} \). There is another version of this conjecture, with Milnor K-groups in place of logarithmic forms. Let us introduce the notation first.

**Definition.** (see [3] Ch. III §7) For a field \( F \), let the graded ring \( K^M_n(F) \) be the quotient of the tensor algebra of \( F^* \) by the two-sided ideal generated by the elements \( \{ f \otimes (1 - f) \mid 1 \neq f \in F^* \} \). The \( n \)-th Milnor K-group \( K^M_n(F) \) is the \( n \)-th graded component of \( K^M_\bullet(F) \). We will write \( \{f_1, \ldots, f_n\} \) for the class of \( f_1 \otimes \cdots \otimes f_n \) in \( K^M_n(F) \).

**Notation.** Let \( F \) be a field extension over \( k \); the group \( K^M_2(F) / \langle \{F^*, k^*\} \rangle \) will be denoted \( K_2(F)/\text{Const} \). For brevity, we will not distinguish between the elements of \( K^M_2(F) \) and their images in \( K_2(F)/\text{Const} \).

A dominant rational map \( \varphi : X \rightarrow Y \) of irreducible varieties over \( k \) induces an inverse image \( \varphi^* : K^M_\bullet(k(Y)) \rightarrow K^M_\bullet(k(X)) \) on K-groups of their function fields. In particular, there is an inverse image on \( K_2 \) and \( K_2/\text{Const} \) groups.

**Conjecture.** Let \( \varphi : \mathbb{A}^2 \rightarrow \mathbb{A}^2 \) be a dominant rational map such that \( \varphi^* \{x, y\} = \{x, y\} \in K_2/\text{Const} \). Then \( \varphi \) is birational.

We will show that the two conjectures are in fact equivalent, specifically:

**Theorem 1.** A dominant rational map \( \varphi : \mathbb{A}^2 \rightarrow \mathbb{A}^2 \) preserves the form \( \frac{dx}{x} \wedge \frac{dy}{y} \in \Omega^2_{k(x,y)} \) if and only if it preserves the element \( \{x, y\} \in K_2/\text{Const} \).

3. **\( K_2 \) in terms of algebraic cycles**

For a scheme \( X \), let \( \mathcal{K}_m^X \) be the Zariski sheaf associated to the presheaf \( U \mapsto K_m(U) \).

**Theorem.** (Quillen; see [3] ch.V Prop 9.8.1) Let \( X \) be a regular quasi-projective scheme over the field \( k \); then for each \( m \geq 0 \) the complex

\[
0 \rightarrow \mathcal{K}_m^X \overset{d_0}{\rightarrow} \bigoplus_{p \in X^{(0)}} i_{p,*}K_m(k(p)) \overset{d_1}{\rightarrow} \bigoplus_{p \in X^{(1)}} i_{p,*}K_{m-1}(k(p)) \overset{d_2}{\rightarrow} \cdots \overset{d_m}{\rightarrow} \bigoplus_{p \in X^{(m)}} i_{p,*}\mathbb{Z} \rightarrow 0,
\]

where \( \text{Spec}(k(p)) \xrightarrow{i_p} X \) denotes the natural inclusion map, provides a flabby resolution for the \( \mathcal{K}_m \)-sheaf of \( X \).
Let $X = \mathbb{A}^n_k$, and fix an identification $F := k(x_1, ..., x_n) \simeq k(X)$. Applying Gersten’s construction we obtain the following flabby resolution for the sheaf $K^X_2$ on $X$:

$$G := \left[ i_{0, *}K_2(F) \to \bigoplus_{p \in X^{(1)}} i_{p, *}K(p)^* \to \bigoplus_{p \in X^{(2)}} i_{p, *}\mathbb{Z} \to 0 \right].$$

We can compute the cohomology of $K^X_2$, and therefore of $G$, using the $\mathbb{A}^1$-invariance for the cohomology of the $\mathcal{K}_m$-sheaf (see [3]).

$$H^i(X, G) = H^i(\mathbb{A}^n_k, K_2) = \begin{cases} K_2(k), & i = 0 \\ 0, & i \neq 0 \end{cases}$$

Therefore

(2) \hspace{1cm} 0 \to K_2(k) \xrightarrow{d_0} K_2(F) \xrightarrow{d_1} \bigoplus_{p \in X^{(1)}} k(D_p)^* \xrightarrow{d_2} \bigoplus_{p \in X^{(2)}} \mathbb{Z} \to 0

is an exact sequence of abelian groups; here $D_p$ stands for the irreducible divisor $V(p) \subset X$ defined by the prime ideal $p \in X^{(1)}$. The group $\bigoplus_{p \in X^{(2)}} \mathbb{Z}$ is none other than the group $Z^2(X)$ of zero-cycles. The differentials of this complex are given by direct sums of residue maps along all the irreducible divisors; $d_1$ is called the tame symbol, computed (see [3] Ch. III Lemma 6.3) as

(3) \hspace{1cm} \text{Tame}\{x, y\} = \bigoplus_{p \in X^{(1)}} (-1)^{\nu_p(x)\nu_p(y)} \frac{y^{\nu_p(x)}}{x^{\nu_p(y)}} (\text{mod } p).

Each component of $d_2$ maps a function to the corresponding principal divisor:

$$k(D)^* \xrightarrow{\text{div}} Z^2(D) \subset Z^2(X).$$

We will be interested in describing the group $K_2(F)/\text{Const}$; let us note that

(4) \hspace{1cm} d_0(K_2(k)) \subset \{F^*, k^*\} \subset K_2(F).

Thus the exact sequence [2] yields

$$K_2(F)/\langle\{F^*, k^*\}\rangle \simeq \ker \left( \bigoplus_{D \subset \mathbb{A}^n_k} k(D)^* \xrightarrow{\text{div}} Z^2(\mathbb{A}^n) \right)/\text{Tame}\langle\{F^*, k^*\}\rangle.$$

The image of $\text{Const}$ is just the sums of constant functions, as shown in the following proposition.

**Proposition 1.** \hspace{1cm} Tame$\langle\{F^*, k^*\}\rangle = \bigoplus_{p \in X^{(1)}} k^* \to \bigoplus_{p \in X^{(1)}} k(D_p)^*$ (included as sums of constant functions.)

**Proof.** We have

$$\text{Tame}\{f, c\} = \bigoplus_p \left( \frac{\epsilon^{\nu_p(f)}}{f^0} \right) \text{mod } p = \bigoplus_p \epsilon^{\nu_p(f)}.$$

Since the divisor class group is trivial for $X = \mathbb{A}^n$, for any $p \in X^{(1)}$ we can choose an equation $F_p$ defining the corresponding divisor $D_p$. Then for any finite collection of $p_i$ and $c_i$ we can take $\psi = \sum \{F_{p_i}, c_i\} \in K_2(F)$ so that $\text{Tame}(\psi) = \bigoplus_{p_i} c_i$; this shows that any finite collection of constant functions is contained in $\text{Tame}\langle\{F^*, k^*\}\rangle$. \hfill $\Box$
This gives our final presentation for $K_2(F)/\text{Const}$ which concludes this section.

**Proposition 2.** $K_2(F)/\text{Const} \simeq \ker \left( \bigoplus_{D \subset \mathbb{A}^n} k(D)^* / k^* \xrightarrow{\text{div}} Z^2(\mathbb{A}^n) \right)$.

4. **Logarithmic differential and proof of Theorem 1**

We will need a notion of rational logarithmic form (see [4], §2 for a thorough exposition).

**Definition.** Let $F$ be an extension of $k$ of finite transcendence degree. A rational differential form $\omega \in \Omega^n_{F/k}$ is called logarithmic if there exists a smooth projective variety $V/k$ with function field $F$ and a simple normal crossing divisor $D \subset V$ such that $\omega$ is locally logarithmic along $D$, and regular elsewhere on $V$. We will write $\Omega^n_{F, \log}$ for the subgroup of $\Omega^n_F$ generated by logarithmic forms.

There is a map $K_2^M(F) \xrightarrow{\text{dlog}} \Omega^2_F$ taking $\{f_1, \ldots, f_n\}$ to the form $\frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n}$. Such a form is indeed logarithmic.

**Theorem 2.** The map $K_2(F)/\text{Const} \xrightarrow{\text{dlog}} \Omega^2_{F, \log}$ is injective.

**Proof.** The following diagram is commutative:

$$
\begin{array}{ccc}
K_2(F)/\text{Const} & \xrightarrow{\text{Tame}} & \bigoplus_{D \subset X} k(D)^*/k^* \\
\downarrow \hspace{1cm} \text{dlog} & & \downarrow \hspace{1cm} \text{dlog} \\
\Omega^2_{F, \log} & \xrightarrow{\text{Res}} & \bigoplus_{D \subset \mathbb{A}} \Omega^1_{k(D), \log}
\end{array}
$$

(one can check this fact by applying $\text{dlog}$ to the formula (3) for Tame.)

The top map Tame is injective by Proposition 2.

The right map dlog is injective, as dlog of a function vanishes if and only if the function is constant. Therefore the left map is also injective. \qed

Recall that our goal was to prove that a rational map $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ preserves $\{x, y\} \in K_2/\text{Const}$ whenever it preserves the logarithmic volume form $\frac{dx}{x} \wedge \frac{dy}{y} \in \Omega^2_{k(x,y), \log}$ (Theorem 1).

**Proof of Theorem 1.** If $\varphi$ leaves $\{x, y\} \in K_2(k(x,y))/\text{Const}$ in place, then it clearly preserves $\frac{dx}{x} \wedge \frac{dy}{y} = \text{dlog} \{x, y\}$, as $\varphi^*$ commutes with dlog.

Suppose $\varphi$ preserves $\frac{dx}{x} \wedge \frac{dy}{y}$. Then it also preserves $\text{dlog}^{-1} \left( \frac{dx}{x} \wedge \frac{dy}{y} \right)$ as a set. But since we have observed that dlog is an inclusion, $\{x, y\}$ is the only element of $\text{dlog}^{-1} \left( \frac{dx}{x} \wedge \frac{dy}{y} \right)$; this finishes the proof. \qed

5. **Computing tame symbol**

Let us fix the projectivization $\mathbb{A}^2 = \mathbb{P}^2$. Given a divisor $D$ on $\mathbb{A}^2$, denote its closure in $\mathbb{P}^2$ by $\overline{D}$ for a moment. The divisorial map $k(D)^*/k^* \to \text{PDiv}($\overline{D}$)$ is an isomorphism, which implies the
following.

\[ K_2(\mathbf{k}(x, y))/\text{Const} \cong \ker \left( \bigoplus_{D \in \mathbb{A}^2} \operatorname{PDIV}(D) \xrightarrow{\text{div}} \mathbb{Z}^2(\mathbb{A}^2) \right) \]

The last proposition can be rephrased to obtain the following combinatorial description of the group on the left: an element of \( K_2(\mathbf{k}(x, y))/\text{Const} \) amounts to a finite collection of divisors \( D_i \) on \( \mathbb{A}^2 \), and a choice of a principal divisor \( \alpha_i \) on the projectivization \( \mathbf{D}_i \) of each of the \( D_i \), such that \( \sum \text{mult}_x(\alpha_i) = 0 \) for all closed points \( x \in \mathbb{A}^2 \). Notice that we do not impose any conditions on \( \alpha_i \) at the boundary \( \mathbb{P}^2 \setminus \mathbb{A}^2 \).

This description of elements of \( K_2(\mathbf{k}(x, y))/\text{Const} \) is suitable for explicit computations. Since it involves the closures of divisors, we will now switch to considering principal divisors on \( \mathbb{P}^2 \) in place of affine divisors for brevity.

**Notation.** 1) For a pair of irreducible divisors \( D, E \) on \( X \)

define \( D \cap E \in \mathbb{Z}^2(X) \), \( D \cap E := \begin{cases} D \cap E, & \text{if } D \neq E \\ 0, & \text{if } D = E \end{cases} \)

This operation extends linearly over the whole \( \operatorname{Div}(X) \).

2) There exist \( f_D, f_E \in \mathbf{k}(\mathbb{A}^2) \) such that \( D = (f_D), E = (f_E) \). Since these functions are determined uniquely up to a constant multiplier, \( \text{Tame}\{D, E\} := \text{Tame}\{f_D, f_E\} \in K_2(\mathbf{k}(x, y))/\text{Const} \) is well-defined.

Given a pair of principal divisors \( D, E \) on \( \mathbb{P}^2 \), let us express the components of \( \text{Tame}\{D, E\} \in K_2(\mathbf{k}(x, y))/\text{Const} \) in terms of the original divisors. If \( \text{Tame}\{D, E\} = \bigoplus_{l_\infty \neq C \subset \mathbb{P}^2} \alpha_C \), then \( \alpha_C \) can be computed as follows:

If \( C \not\in \text{Supp}(D) \cup \text{Supp}(E) \), then \( \alpha_C = 0 \).

Case I: if \( C \in \text{Supp}(D) \setminus \text{Supp}(E) \),

then \( \alpha_C = \nu_C(D) \cdot (C \cap E) \).

Case II: if \( C \in \text{Supp}(E) \setminus \text{Supp}(D) \),

then \( \alpha_C = -\nu_C(E) \cdot (C \cap D) \).

Case III: if \( C \in \text{Supp}(D) \cap \text{Supp}(E) \),

then \( \alpha_C = \nu_C(D) \cdot (C \cap E) - \nu_C(E) \cdot (C \cap D) \).

Now let \( \text{Tame}\{D, E\} = \text{Tame}\{x, y\} \). This is equivalent to \( \alpha_C \) being equal to the \( C \)-component of \( \text{Tame}\{x, y\} \) for all divisors \( l_\infty \neq C \subset \mathbb{P}^2 \).

\[ \text{Tame}\{x, y\} = (0 - \infty_h) \oplus (0 + \infty_v), \]

where \( l_h = (y), l_v = (x), 0 = [0 : 0 : 1], \infty_h = [1 : 0 : 0], \infty_v = [0 : 1 : 0] \).

All the components of \( \text{Tame}\{x, y\} \) except \( l_h \) and \( l_v \) are trivial, i.e. \( \forall C \neq l_h, l_v : \alpha_C = 0 \), which has the following implications for \( D \) and \( E \):

1) \( C \in \text{Supp}(D), C \not\in \text{Supp}(E) \cup \{l_h, l_v\} \),
then \( 0 = \alpha_C = \nu_C(E) \cdot (C \cap D) \Leftrightarrow \nu_C(D) = 1, \text{Div}^0(C) \ni C \cap E = 0 \).

2) \( C \in \text{Supp}(E), C \not\in \text{Supp}(D) \cup \{l_h, l_v\} \),
then \( 0 = \alpha_C = \nu_C(D) \cdot (C \cap E) \Leftrightarrow \nu_C(E) = 1, \text{Div}^0(C) \ni C \cap D = 0 \).
3) $C \in \text{Supp}(E) \cup \text{Supp}(D)$, $C \notin \{l_h, l_v\}$, then $0 = \alpha_C = \nu_C(D) \cdot (C \cap E) - \nu_C(E) \cdot (C \cap D)$

For $l_h$ and $l_v$, we have similar conditions depending on which of the divisors contain these lines as their components. From symmetry considerations there are four distinct cases:

**Case (I, I):**
\begin{align*}
\nu_h(D) &= \pm 1, \nu_v(D) = 1 \\
\nu_h(E) &= \nu_v(E) = 0 \\
l_h \cap E &= \pm (0 - \infty_h) \\
l_v \cap E &= -0 + \infty_v
\end{align*}

**Case (I, II):**
\begin{align*}
\nu_h(D) &= 1, \nu_v(D) = 0 \\
\nu_h(E) &= 0, \nu_v(E) = 1 \\
l_h \cap E &= 0 - \infty_h \\
l_v \cap D &= -0 + \infty_v
\end{align*}

**Case (I, III):**
\begin{align*}
|\nu_h(D)| &\geq 1, \nu_v(D) = 1 \\
\nu_h(E) &\geq 1, \nu_v(E) = 1 \\
0 - \infty_h &= \alpha_h = \nu_h(D) \cdot (l_h \cap E) - \nu_h(E) \cdot (l_h \cap D) \\
-0 + \infty_v &= \alpha_v = \nu_v(D) \cdot (l_v \cap E) - \nu_v(E) \cdot (l_v \cap D)
\end{align*}

**Case (III, III):**
\begin{align*}
|\nu_h(D)| &\geq 1, \nu_v(D) = 1 \\
\nu_h(E) &\geq 1, \nu_v(E) = 1 \\
0 - \infty_h &= \alpha_h = \nu_h(D) \cdot (l_h \cap E) - \nu_h(E) \cdot (l_h \cap D) \\
-0 + \infty_v &= \alpha_v = \nu_v(D) \cdot (l_v \cap E) - \nu_v(E) \cdot (l_v \cap D)
\end{align*}

### 6. An example of application

Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a rational map given by its affine coordinates $\varphi = (f, g)$. Denote by $D, E \in \text{PDiv}(\mathbb{P}^2)$ the corresponding principal divisors, and let $D = D^+ - D^-$, where $D^+$ and $D^-$ are effective divisors without common components. Define $E^+, E^-$ in the same way.

In what follows $\text{Bs}(f), \text{Bs}(g)$ will refer to the indeterminacy loci of $f, g$ considered as rational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. Note that $\text{Bs}(f) = |D^+| \cap |D^-|$, $\text{Bs}(g) = |E^+| \cap |E^-|$. For $C$ an irreducible divisor on $\mathbb{P}^2$, $\alpha_C$ stands for the $C$-component of Tame $\{D, E\}$, as in section 5.

**Remark.** Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map given by its affine coordinates $\varphi = (f, g)$. If $\text{Bs}(f) \cap \text{Bs}(g) = \emptyset$, then $\varphi$ is biregular.

**Proposition 3.** Let $(f, g) = \varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a rational map which preserves Tame $\{x, y\}$. Without loss of generality, suppose that $l_h \in \text{Supp}(D)$, $l_v \in \text{Supp}(E)$. Assume further that and $D$ and $E$ have no common components except $l_\infty$ (so that we are in the situation of Case (I, II).) If $\text{Bs}(f) \cap \text{Bs}(g) = \emptyset$, then $D = l_h - l_\infty$, $E = l_v - l_\infty$.

**Proof.** Assume Tame $\{D, E\} = \text{Tame}\{x, y\}$, but $|D^+| \cap |D^-| \cap |E^+| \cap |E^-| = \emptyset$. Recall that $\text{Supp}(\alpha_C) \subset \{0, \infty_h, \infty_v\} \subset Z^2(\mathbb{P}^2)$ for all $C$.

**Step 1.** We have $\text{Supp}(D \cap E) \subset \{0, \infty_h, \infty_v\}$.
Proof of Step 1. Choose an arbitrary component $C \subset D$, and let $p$ be any point of the intersection $C \cap E$. If $p$ is not contained in $|E^+|$ (resp. $|E^-|$), then up to a sign mult$_p(\alpha_C)$ equals $\nu_C(D)$ times the intersection multiplicity of $C$ and the effective divisor $E^-$ (resp. $E^+$), which is strictly bigger than zero, so mult$_p(\alpha_C) \neq 0$. If, on the other hand, $p \in |E^+| \cap |E^-|$, then $p \notin |D^+| \cap |D^-|$, and we can repeat the same argument, this time choosing a component of $E$ which passes through $p$, with the conclusion that mult$_p(\alpha_C) \neq 0$. But if mult$_p(\alpha_C) \neq 0$ then $p \in \{0, \infty_h, \infty_v\}$.

**Step 2.** If $C \in \text{Supp}(D) \cup \text{Supp}(E)$, $C \neq l_h, l_v$ then $0 \notin C$.

Proof of Step 2. If there exists such a component $C$ in either $D^-$ or $E^-$, then mult$_0(\alpha_C)$ is necessarily non-zero, which contradicts the assumption Tame $\{D, E\} = \text{Tame} \{x, y\}$.

Suppose that neither of the divisors $D^-$ and $E^-$ passes through $0$, and assume without loss of generality that $l_v \neq C \in \text{Supp}(E^+)$. Then mult$_0(\alpha_{l_h}) \geq \text{mult}_0(l_h \cap l_v) + \text{mult}_0(l_h \cap C) \geq 2$, which is again a contradiction.

**Step 3.** It follows that $D = l_h - l_{\infty}$, $E = l_v - l_{\infty}$.

Proof of Step 3. We have $\alpha_{l_h} = 0 - \infty_h \Rightarrow \infty_h \in |E^-| \Rightarrow \infty_v \in |E^-|$ by Bézout’s theorem.

Analogously $\alpha_{l_v} = -0 + \infty_v \Rightarrow \infty_v \in |D^-| \Rightarrow \infty_h \in |D^-|$. As $|D^+| \cap |D^-| \cap |E^+| \cap |E^-| = \emptyset$, this implies that $D^+ = l_h$, $E^+ = l_v$, and since $D$ and $E$ are both of degree $0$, the proposition follows.

**Corollary 1.** Let $(f, g) = \varphi : \mathbb{P}^2 \to \mathbb{P}^2$ be a rational map that preserves the form $\frac{dx}{x} \wedge \frac{dy}{y}$, and assume that the divisors $(f)$ and $(g)$ have no common components except possibly $l_{\infty}$. If $\text{Bs}(f) \cap \text{Bs}(g) = \emptyset$, then $\varphi$ is biregular.

**References**

[1] Jérémy Blanc: *Symplectic birational transformations of the plane*, [arXiv:1012.0706](http://arxiv.org/abs/1012.0706)

[2] Jeffrey Diller, Jan-Li Lin: *Rational surface maps with invariant meromorphic two forms*, [arXiv:1308.2567](http://arxiv.org/abs/1308.2567)

[3] Henri Gillet, *Riemann-Roch theorems for higher algebraic K-theory*, Bull. Amer. Math. Soc. (N.S.) Volume 3, Number 2 (1980), 849-852

[4] Sergey Gorchinskiy, Alexei Rosly: *A polar complex for locally free sheaves*, [arXiv:1101.5114](http://arxiv.org/abs/1101.5114)

[5] Charles A. Weibel: *The K-Book: An Introduction to Algebraic K-Theory*, Graduate Studies in Math. vol. 145, AMS, 2013