Codes arising from incidence matrices of points and hyperplanes in $PG(n, q)^*$

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Abstract

In this paper we completely characterize the words with second minimum weight in the $p$–ary linear code generated by the rows of the incidence matrix of points and hyperplanes of $PG(n, q)$, with $q = p^h$ and $p$ prime, proving that they are the scalar multiples of the difference of the incidence vectors of two distinct hyperplanes of $PG(n, q)$.

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1 Introduction

Consider the projective space $PG(n, q)$, with $q = p^h$, $h \geq 1$ and $p$ prime. We define the incidence matrix $A = (a_{i,j})$ of points and hyperplanes in $PG(n, q)$ as the matrix whose rows are indexed by the $\theta_n$ hyperplanes of $PG(n, q)$ and whose columns are indexed by the $\theta_n$ points of $PG(n, q)$, and with entry

$$a_{i,j} = \begin{cases} 1 & \text{if } P_j \in H_i, \\ 0 & \text{otherwise} \end{cases}.$$

The $p$-ary linear code of points and hyperplanes of $PG(n, q)$, which we denote by $C(n, q)$, is the code generated over $\mathbb{F}_p$ by the rows of the matrix $A$. These codes belong to a more general class of codes, the Reed-Muller codes. For comprehensive references see e.g. [1], [8] and [14].

The interest of these codes was born after the works of E. Prange [15] and L. D. Rudolph [16], which showed that projective planes can be used to define error-correcting codes.

Let $c$ be a codeword of $C(n, q)$, the subset of $\{1, \ldots, \theta_n\}$ which corresponds to nonzero components of $c$ is said to be the support of $c$, and it will be denoted by $\text{supp}(c)$. We identify this set with the corresponding set of points of $PG(n, q)$. The size of $\text{supp}(c)$ is said to be the weight of $c$ and we will denote it by $\text{wt}(c)$. Let $X$ be a subset of $PG(n, q)$, with $v^X$ we will denote the incidence vector of $X$. Note that $\text{wt}(v^X) = |X|$. Moreover, let $c_1 = (a_1, \ldots, a_{\theta_n})$, $c_2 = (b_1, \ldots, b_{\theta_n}) \in \mathbb{F}_p^{\theta_n}$, the standard inner product is $(c_1, c_2) = \sum_{i=1}^{\theta_n} a_i b_i$. The orthogonal code is denoted by $C(n, q)^\perp$ and is given by

$$C(n, q)^\perp = \{v \in \mathbb{F}_p^{\theta_n} : (v, c) = 0 \ \forall \ c \in C(n, q)\}.$$

The hull of $C(n, q)$ is defined as $C(n, q) \cap C(n, q)^\perp$. 
The fundamental parameters of these codes are well known (cfr. [1] and [14]):
the length is equal to the number of points of $PG(n, q)$, that is $\theta_n$; the dimension
is the p-rank of $A$, that is $(p^{n+1}-1)^h + 1$; the minimum distance is the number of
points of a hyperplane, that is $\theta_{n-1}$. Also,

**Theorem 1.1.** [1][14] The codewords of $C(n, q)$ with minimum weight are the
scalar multiples of the incidence vectors of hyperplanes.

In [5] the following result has been proved.

**Theorem 1.2.** [5]

1. In the $p$–ary linear code arising from $PG(2, p)$, $p$ prime, there are no code-
words with weight in $\lfloor p + 1, 2p \rfloor$.

2. The codewords of weight $2p$ in the $p$–ary linear code arising from $PG(2, p)$,
$p$ prime, are the scalar multiples of the differences of the incidence vectors
of two distinct lines of $PG(2, p)$.

In [10] the authors generalize 1. of Theorem 1.2 to codes generated by the
rows of the incidence matrix of points and hyperplanes in $PG(n, q)$. In particular,
they obtain the following result.

**Theorem 1.3.** [10, Corollary 19] There are no codewords with weight in the open
interval $\lfloor \theta_{n-1}, 2q^{n-1} \rfloor$ in the code $C(n, q)$, $q = p^h$, $p$ prime, $p > 5$.

In [19], the authors characterized small weight planar codewords of $C(2, q)$
improving Theorem 1.3 but no proof has been published yet.

In this paper we extend Results 1. and 2. of Chouinard (Theorem 1.2) in
$C(n, q)$ for each prime power $q$. More precisely, in Section 3 we prove the follow-
ing.
Theorem 1.4. Let $q = p^h$ with $p$ prime.

1. There are no codewords of $C(n, q)$ with weight in the interval $[\theta_{n-1}, 2q^{n-1}]$.

2. The codewords of weight $2q^{n-1}$ in $C(n, q)$ are the scalar multiples of the differences of the incidence vectors of two distinct hyperplanes of $PG(n, q)$.

2 Preliminaries

2.1 Blocking sets

Let $p$ a prime and $q = p^h$, with $h$ a positive integer. A subset $B$ of $PG(n, q)$ is a $k-$blocking set (or blocking set with respect to $(n - k)$-subspaces) of $PG(n, q)$, with $1 \leq k \leq n - 1$, if each $(n - k)$-subspace intersects $B$ in at least one point.

If $k = 1$, we simply say that $B$ is a blocking set of $PG(n, q)$. A $k-$blocking set is called trivial if it contains a $k-$subspace. An $(n - k)-$subspace which contains exactly one point of the $k-$blocking set $B$ is called $(n - k)-$tangent space of $B$ and such a point is called essential point. We say $B$ minimal if each point of $B$ is an essential point for $B$.

An $(n-1)-$blocking set of $PG(n, q)$ small enough can be reduced in a unique way to a minimal blocking set, as proved in [11] by using Lemma 2.11 of [7]. More precisely,

Theorem 2.1. [11] Corollary 1] Every $(n - 1)-$blocking set in $PG(n, q)$, of size smaller than $q^{n-1} + \theta_{n-1}$, can be uniquely reduced to a minimal $(n - 1)-$blocking set.

2.2 Blocking sets and codewords of small weight in $C(n, q)$

The following properties of the code $C(n, q)$ can be easily verified.
Property 2.2. (II Lemmas 1, 2 and 3)

1. If $U_1$ and $U_2$ are subspaces of dimension at least 1 in $PG(n, q)$, then $v^{U_1} - v^{U_2} \in C(n, q)^\perp$.

2. The scalar product $(c, v^U)$, with $c \in C(n, q)$ and $U$ an arbitrary subspace of dimension at least 1, is a constant.

3. A codeword $c$ is in $C(n, q) \cap C(n, q)^\perp$ if and only if $(c, v^U) = 0$ for all subspaces $U$ with $\dim(U) \geq 1$.

Codewords of small weight in $C(n, q)$ are related to $(n - 1)$–blocking sets. Indeed in (II), generalizing Lemma 23 of [5], the authors prove the following.

Theorem 2.3. (II Lemma 6) If $c \in C(n, q)$, $c \neq 0$, with weight less than $2q^{n-1}$, then

1. $\text{supp}(c)$ is a minimal $(n - 1)$–blocking set of $PG(n, q)$;

2. $c$, up to scalar, is the incidence vector of its support;

3. $\text{supp}(c)$ intersects every line in $1 \pmod{p}$ points.

The next Theorem, due to A. Blokhuis, A. E. Brouwer and H. Wilbrink in [3], gives geometric information on codewords of $C(2, q)$ with components 0 and 1.

Theorem 2.4. [3 Proposition] Let $X$ be a subset of $PG(2, q)$ such that $v^X \in C(2, q)$ and let $Q$ be a point of $PG(2, q)$ such that $Q \notin X$. Then the points $P \in X$ for which the line $PQ$ is tangent to $X$ are collinear.
3 The second minimum weight of $C(n, q)$ and the characterisation of the codeword of weight $2q^{n-1}$

In this section we prove Theorem 1.4.

Remark 3.1. [11, Proof of Theorem 5] Note that the restriction of a codeword to a subspace is a codeword of the code associated with the fixed subspace. Indeed, if $c \in C(n, q)$ then there exist $\alpha_1, \ldots, \alpha_{\theta_n} \in \mathbb{F}_p$ such that $c = \alpha_1 v^{H_1} + \cdots + \alpha_{\theta_n} v^{H_{\theta_n}}$, where $H_1, \ldots, H_{\theta_n}$ are the hyperplanes of $PG(n, q)$. Let $S$ be a subspace of $PG(n, q)$ of dimension at least 2 and let $C(S)$ be the linear code of points and hyperplanes of $S$. Then the restriction of $c$ to $S$ is defined as

$$c|_S = \alpha_1 v^{H_1 \cap S} + \cdots + \alpha_{\theta_n} v^{H_{\theta_n} \cap S}.$$ 

Note that $S \cap H_i$ is either a hyperplane of $S$ or is equal to $S$, for each $i$, so $c|_S \in C(S)$, since $c|_S$ is the sum of a linear combination of incidence vectors of hyperplanes of $S$ and of an $\mathbb{F}_p-$proportional of the all-one vector $j$ belonging to $C(S)$. Also, $supp(c|_S) = supp(c) \cap S$. Furthermore, if $c = v^X$ for some subset $X \subseteq PG(n, q)$, then its restriction to a subspace $S$ is the incidence vector of $X \cap S$, that is $c|_S = v^{X \cap S}$.

Using Theorem 2.4 we prove the first point of Theorem 1.4 in the planar case for any prime $p$.

Theorem 3.2. There are no codewords of $C(2, q)$ with weight in the interval $]q + 1, 2q[\) for any prime $p$.

Proof. Let $c \in C(2, q)$ with weight in $]q + 1, 2q[$. By Theorem 2.3, $supp(c)$ defines a minimal blocking set $B$ of the plane $PG(2, q)$, which intersects every line in $1(mod p)$ points and the nonzero components of $c$ are equal to some $a \in \mathbb{F}_p^*$. Dividing by $a$ the codeword $c$, we obtain another codeword $c'$ of $C(2, q)$. So we
may assume that the nonzero components of $c$ are 1 and hence $c = v^B$. If $q + k + 1$ is the cardinality of $B$, by 3. of Theorem 2.4, we have $|B| \equiv 1 \pmod{p}$. Consider a point $P \in B$. Let $t$ be a tangent line through $P$ and let $Q$ be in $t \setminus B$. Since every secant line to $B$ has at least $1 + p$ points of $B$, the number of tangent lines to $B$ through $Q$ is at least $q - \frac{k}{p} + 1$. By Theorem 2.4 the points of $B$ which belong to tangent lines through $Q$ are collinear. Therefore there exists at least one secant line $l$ to $B$ through $P$ containing at least $q - \frac{k}{p} + 1$ points of $B$. Since $k < q - 1$, we have that

$$|B \cap l| \geq q - \frac{k}{p} + 1 > q - \frac{q}{p} + \frac{1}{p} + 1,$$

and hence $|B \cap l| \geq q + 1$, if $q = p$ and

$$|B \cap l| \geq q - \frac{q}{p} + p + 1,$$

(3.1) if $q > p$. In the first case we have $l \subseteq B$ and, by the minimality of $B$, we get $B = l$, a contradiction. Hence let $q > p$. So we have that for each point $P \in B$ there exists a line $l$ through $P$ containing at least $q - \frac{q}{p} + p + 1$ points of $B$. Since $B$ is not a line and cannot be contained in the union of two lines, there exist at least 3 lines $l_1, l_2$ and $l_3$ satisfying [3.1] So,

$$|B| \geq 3 \left( q - \frac{q}{p} + p - 1 \right) + 3 = 3q - 3\frac{q}{p} + 3p,$$

hence

$$|B| \geq 3\frac{q - 1}{p} + 3p + 1,$$

and this is not possible if $p \geq 3$. Finally, let $p = 2$ and note that there cannot exist another line, different from $l_1, l_2$ and $l_3$ intersecting $B$ in at least $\frac{q}{2} + 3$ points, otherwise $|B| \geq 4 \cdot \frac{q}{2} + 4$. In this way, we have shown that $B = (B \cap l_1) \cup (B \cap l_2) \cup (B \cap l_3)$ and hence, since $B$ is a blocking set and $|B| < 2q$, $l_i \cap l_j \in B$ for each $i, j \in \{1, 2, 3\}$ with $i \neq j$. Let $P$ be the intersection point of $l_1$ and $l_2$ and let $Q$ be a point of $l_3 \cap B$ different from $l_1 \cap l_3$ and $l_2 \cap l_3$, then $|PQ \cap B| = 2$ and this is not possible by point 3. of Theorem 2.3. \qed
Now, we are able to prove 1. of Theorem 1.4 in the general case.

**Theorem 3.3.** There are no codewords of \( C(n, q) \) with weight in the interval \( ]\theta_{n-1}, 2q^{n-1}[ \), where \( q = p^h, p \) prime.

**Proof.** We prove the theorem by induction on \( n \). The statement holds in the case \( n = 2 \) by Theorem 3.2. Now suppose \( n > 2 \) and that statement holds for each \( m \) less than \( n \). Let \( c \in C(n, q) \) with weight in \( ]\theta_{n-1}, 2q^{n-1}[ \). By Theorem 2.3, \( B = \text{supp}(c) \) is a minimal blocking set of \( PG(n, q) \) with respect to lines and \( B \) meets every line in \( 1 \) (mod \( p \)) points. The nonzero components of \( c \) are equal to some \( a \in \mathbb{F}_{p^h}^* \), so, up to a scalar, we may assume that the nonzero components of \( c \) are 1 and hence \( c = v_B \). Now, let \( P \) be a point of \( B \) and suppose that there is an integer \( m \) such that \( 1 \leq m \leq n - 2 \) and there exists an \( m-\)subspace \( S_m \) such that \( S_m \cap B \) is an \( (m-1)-\)subspace through \( P \). In this case there exists an \( (m+1)-\)subspace containing \( S_m \) such that \( S_{m+1} \cap B \) is an \( m-\)subspace. Indeed, if each \( (m+1)-\)subspace \( S_{m+1} \), which contains \( S_m \), intersects \( B \) in at least \( 2q^m \) points, then

\[
2q^{n-1} > |B| \geq \frac{\theta_n - \theta_m}{q^{m+1}}(2q^n - \theta_{m-1}) + \theta_{m-1},
\]

and we get

\[
(q^{n-m-1} - 1)[q^{m-1}(q - 2) + 1] < 0,
\]

a contradiction for every \( q \). So there exists an \( (m+1)-\)subspace \( S_{m+1} \) containing \( S_m \) such that \( 0 < |B \cap S_{m+1}| < 2q^m \). By Remark 3.1, the restriction of \( c \) to \( S_{m+1} \) is a codeword of \( C(S_{m+1}) = C(m+1, q) \) and its support is \( B \cap S_{m+1} \), i.e. \( c|_{S_{m+1}} = v_{B \cap S_{m+1}} \). So, by the induction hypothesis and by Theorem 1.1 we have that \( B \cap S_{m+1} \) is an \( m-\)subspace through \( P \). Now, since \( B \) is minimal, we know that for each point \( P \in B \) there exists a tangent line \( l \), then we can apply the previous considerations to obtain the existence of a hyperplane \( S_{n-1} \) through \( P \) such that \( S_{n-1} \cap B = S_{n-2} \). Let \( S \) be the set of all the \( (n-2)-\)subspaces \( S_{n-2} \).
of $PG(n, q)$ for which there exists a hyperplane $\tilde{S}_{n-1}$ such that $\tilde{S}_{n-1} \cap B = S_{n-2}$. Note that for each point of $B$ there exists an element of $S$ through it, and since $|B| > \theta_{n-1}$, it is clear that $S$ contains at least two elements. Let $S_{n-2}$ and $S'_{n-2}$ be two elements of $S$, then $S_{n-2} \cap S'_{n-2}$ is either an $(n-3)-$subspace or an $(n-4)-$subspace. In the latter case, each hyperplane through $S_{n-2}$ has to intersect $S'_{n-2} \setminus S_{n-2}$, but this is not possible for the hyperplane $\tilde{S}_{n-1}$, since $\tilde{S} \cap B = S_{n-2}$.

Now, consider $S, S' \in S$ and let $\overline{S}_{n-1} = S \cup S'$ and $\overline{S}_{n-3} = S \cap S'$. Since the intersection of two elements of $S$ is always an $(n-3)-$subspace, either $B \subseteq \overline{S}_{n-1}$ and this is not possible, or $B$ is a cone with vertex $\overline{S}_{n-3}$. In this case, consider a plane $\pi$ disjoint from $\overline{S}_{n-3}$ and note that each element of $S$ intersects $\pi$ in a point, hence, if $x$ is the size of $S$, then

$$|B| \geq xq^{n-2} + \theta_{n-3}.$$ 

Since $|B| < 2q^{n-1}$, we have $0 < x = |\pi \cap B| < 2q$. So, by Theorem 3.2 by Theorem 1.1 and by Remark 3.1 $\pi \cap B$ is a line $r$. Then $B \subseteq \langle r, \overline{S}_{n-3} \rangle = \overline{S}_{n-1}$, a contradiction.

Now, we characterize the codewords with weight $2q^{n-1}$ in $C(n, q)$.

In the planar case the following holds.

**Theorem 3.4.** [1, Corollary 6.4.4]

1. The minimum weight of $C(2, q) \cap C(2, q)^\perp$ is $2q$.

2. The codewords of $C(2, q) \cap C(2, q)^\perp$ with weight $2q$ are, up to scalar, the difference of incidence vectors of any two distinct lines.

M. Lavrauw, L. Storme and G. Van de Voorde in [11] generalize the first point of the previous result.

**Theorem 3.5.** [11, Theorem 5] The minimum weight of $C(n, q) \cap C(n, q)^\perp$ is $2q^{n-1}$. 

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Also, if \( q = p \) the words in \( C(n, p) \cap C(n, p)^\perp \) with weight \( 2p^n-1 \) are the scalar multiples of the difference of the incidence vectors of two hyperplanes of \( PG(n, p) \), see \([11, \text{Remark 2}]\) and \([12, \text{Theorem 12}]\).

In the next, we will prove that the second minimum weight of \( C(n, q) \) is \( 2q^n-1 \), \( q = p^h, \; h \geq 1 \) and that the words of \( C(n, q) \) with this weight are, up to scalar, the difference of the incidence vectors of two hyperplanes of \( PG(n, q) \) for each prime \( p \).

In the same way as it was done by the authors in \([11, \text{Lemma 6}]\), one can prove the following.

**Proposition 3.6.** There are no codewords in \( C(n, q) \setminus C(n, q)^\perp \) with weight \( 2q^n-1 \).

**Proof.** Suppose to the contrary that there exists \( c \) be in \( C(n, q) \setminus C(n, q)^\perp \) with weight \( 2q^n-1 \). Since \( c \notin C(n, q)^\perp \), by 2. and 3. of Property 2.2 for each subspace \( U \) with \( \dim U \geq 1 \) we have \( (c, v^U) = a \), for some \( a \in \mathbb{F}_p^* \), i.e. \( U \cap \text{supp}(c) \neq \emptyset \).

In particular, this holds for the lines, and so \( B = \text{supp}(c) \) is an \((n - 1)\)–blocking set in \( PG(n, q) \). Also, if \( R \) is an essential point of \( B \) and \( t \) is a tangent line to \( B \) through \( R \), then \( a \) is the component of \( c \) corresponding to \( R \). This implies that \( B \) is minimal, indeed if there exists a point \( P \in B \) which is not an essential point for \( B \), since \( |B| = 2q^n-1 \), there exists a line \( l \) through \( P \) intersecting \( B \) in exactly two points. If \( l \cap B = \{ P, Q \} \), then by Theorem 2.1 \( Q \) is an essential point of \( B \). So, the corresponding component of \( c \) is \( a \) and, denoted by \( x \) the component of \( c \) corresponding to \( P \), we get

\[
(c, v^l) = x + a = a,
\]

i.e. \( x = 0 \), a contradiction. So \( B \) is a minimal \((n - 1)\)–blocking set and hence for each point of \( B \) there exists a tangent line. This means that the nonzero components of \( c \) are equal to \( a \), that is \( c = av^B \). Since \( c \in C(n, q) \setminus C(n, q)^\perp \), we
have that $v^B \in C(n, q) \setminus C(n, q)\perp$. Then, each line of $PG(n, q)$ intersects $B$ in $1 \pmod{p}$ points, and hence $|B| \equiv 1 \pmod{p}$, but this is not possible since $|B| = 2q^{n-1}$. 

**Lemma 3.7.** Let $X \subseteq PG(n, q), n \geq 2$, with $|X| = 2q^{n-1}$ and such that for each $h$–dimensional subspace $S_h$, with $1 \leq h \leq n - 1$, one of the following occurs:

1. $X \cap S_h = \emptyset$;
2. $X \cap S_h$ is the symmetric difference of two hyperplanes of $S_h$ (if $h = 1$, $S_1$ is a 2–secant line to $X$);
3. $X \cap S_h = S_h \setminus S_{h-1}$, where $S_{h-1}$ is a hyperplane of $S_h$;

then $X$ is the symmetric difference of two hyperplanes of $PG(n, q)$.

**Proof.** Note that if $q = 2$, then points 2. and 3. describe the same set of points and hence for each hyperplane $S_{n-1}$ of $PG(n, 2)$ we have that either $S_{n-1} \cap X = \emptyset$ or $S_{n-1} \setminus S_{n-2} \subseteq X$ where $S_{n-2}$ is a hyperplane of $S_{n-1}$. Since $|X| = 2q^{n-1} = 2^n$ we easily get that $X$ is the symmetric difference of two hyperplanes of $PG(n, 2)$. So, let $q > 2$ and we prove that:

(a) For each $P \in X$ there exists a line $l$ through $P$ such that $l \setminus \{Q\} \subseteq X$, for some $Q \in l$;

(b) If $S_m$ is an $m$–subspace of $PG(n, q)$, with $0 \leq m \leq n - 2$, for which case 3. holds, then there exists an $(m + 1)$–subspace $S_{m+1}$ containing $S_m$ satisfying 3. .

Let $P$ be a point of $X$ and assume that (a) is not satisfied, then every line through $P$ is a 2–secant line to $X$. Hence,

$$|X| = \theta_{n-1} + 1,$$
and this is not possible since \( q > 2 \). So, (a) is proved. Now, let \( S_m \) be an \( m \)-subspace, with \( 0 \leq m \leq n-2 \), such that \( S_m \setminus S_{m-1} \subseteq X \), where \( S_{m-1} \) is a hyperplane of \( S_m \), and assume that every \( S_{m+1} \) containing \( S_m \) intersects \( X \) in the symmetric difference of two \( m \)-subspaces, one of which is \( S_m \), then

\[
|X| = \theta_{n-m-1}(2q^m - q^m) + q^m,
\]

and since \( m \leq n-2 \) this is possible only for \( q = 2 \). In this way we have proven (b). Now, from (a) and (b) we get that for each point \( P \in X \) there exists an \( S_{n-1} \) through \( P \) such that \( S_{n-1} \setminus S_{n-2} \subseteq X \). If there exists another hyperplane \( S_{n-1} \) containing \( S_{n-2} \) such that \( S_{n-1} \setminus S_{n-2} \subseteq X \), then \( X \) is the symmetric difference of two hyperplanes. Otherwise, denoted by \( x \) the number of the hyperplanes through \( S_{n-2} \) intersecting \( X \) in a symmetric difference of two of its hyperplanes, we get

\[
q^{n-1} = x 2q^{n-2},
\]

and so \( x = \frac{q}{2} \), and this is not possible if \( p \geq 3 \). Finally, let \( p = 2 \) and \( q > 2 \), i.e. \( q \geq 4 \). If there exist at least 3 hyperplanes satisfying 3., we obtain

\[
|X| \geq q^{n-1} + q^{n-1} - q^{n-2} + q^{n-1} - 2q^{n-2},
\]

which is not possible for \( q > 3 \). Then there exist two hyperplanes verifying 3., and since \( |X| = 2q^{n-1}, X \) is the symmetric difference of two hyperplanes of \( PG(n,q) \).

\[\square\]

**Theorem 3.8.** The codewords in \( C(n, q) \) of weight \( 2q^{n-1} \) are the scalar multiples of the difference of the incidence vectors of two distinct hyperplanes of \( PG(n,q) \).

**Proof.** The assert holds in the case \( n = 2 \) by Theorem 3.4 and by Proposition 3.6. Now, suppose the assert true in the code \( C(t, q) \), with \( 2 \leq t \leq n-1 \) and let \( c \in C(n, q) \) with \( wt(c) = 2q^{t-1} \). Then, by Proposition 3.6, \( c \in C(n, q) \cap C(n, q) \perp \). Denote by \( X \) the support of \( c \) and note that, by 3. of Property 2.2 \( (c, v^U) = 0 \).
for each subspace $U$ of $PG(n, q)$ of dimension $h$, with $1 \leq h \leq n - 1$, hence $X$ has no tangent space. Also, if $U$ is a subspace of dimension $h$ with $2 \leq h \leq n - 1$, then by Remark 3.1 $\text{supp}(c|_U) = \text{supp}(c) \cap U$ and $c|_U \in C(U) = \mathcal{C}(h, q)$, hence if $\text{supp}(c) \cap U \neq \emptyset$ by the Theorem 3.3 and by the induction hypothesis one of the following holds:

(a) $\text{supp}(c) \cap U$ is a hyperplane of $U$;

(b) $\text{supp}(c) \cap U$ is a symmetric difference of two hyperplanes of $U$;

(c) $|\text{supp}(c) \cap U| > 2q^{h-1}$.

Now, we are able to prove the following:

(*) If $S_m$ is an $m$–subspace, with $1 \leq m \leq n - 2$, such that $S_m \cap X$ is the symmetric difference of two hyperplanes of $S_m$ (if $m = 1$ $S_1$ is a 2–secant line to $X$), then each $(m + 1)$–subspace $S_{m+1}$ containing $S_m$ intersects $X$ in the symmetric difference of two $m$–subspaces.

Indeed, if $S_{m+1}$ is an $(m+1)$–subspace containing $S_m$, since $S_{m+1} \cap X \supseteq S_m \cap X$, Case (a) does not occur, hence either $S_{m+1} \cap X$ is the symmetric difference of two hyperplanes of $S_{m+1}$ or $|S_{m+1} \cap X| > 2q^m$. Let $x$ be the number of $S_{m+1}$ containing $S_m$ such that $S_{m+1} \cap X$ is the symmetric difference of two hyperplanes of $S_{m+1}$, then:

$$2q^{n-1} = |X| \geq x(2q^m - 2q^{m-1}) + (\theta_{n-m-1} - x)(2q^m - 2q^{m-1} + 1) + 2q^{m-1},$$

where $\theta_{n-m-1}$ is the number of the $(m+1)$–subspaces containing $S_m$. Then we get

$$x \geq \theta_{n-m-1},$$

and hence $x = \theta_{n-m-1}$, i.e. each $(m+1)$–subspace $S_{m+1}$ containing $S_m$ intersects $X$ in the symmetric difference of two $m$–subspaces. Since $|X| = 2q^{n-1}$ and there
are no tangent lines to $X$, for each point $P$ of $X$ there exists a 2–secant line, so applying (⋆), we get that for each point $P \in X$, each $h$–subspace, with $2 \leq h \leq n - 1$, through a 2–secant line containing $P$ intersects $X$ in the symmetric difference of two $(h - 1)$–subspaces, one of which contains $P$. As a consequence, we get that for a line $l$ of $PG(n, q)$ one of the following holds true:

(a’) $l$ is external to $X$;

(b’) $l$ is 2–secant to $X$;

(c’) $l \setminus \{Q\}$ is contained in $X$, where $Q \in l$.

Indeed, if $l$ is a line which contains more than two points and $R \in l \cap X$, through $R$ there exists at least one 2–secant line $l'$ to $X$. The plane $l \lor l'$ intersects $X$ in the symmetric difference of two lines, and one of these must be the line $l$.

By the previous considerations, moving forward by finite induction on $h$, we get that for each subspace $S_h$ of $PG(n, q)$ with $1 \leq h \leq n - 1$ one of the following occurs:

1. $S_h$ is external to $X$;

2. $X \cap S_h$ is the symmetric difference of two hyperplanes of $S_h$ (if $h = 1$, $S_1$ is a 2–secant line to $X$);

3. $S_h \setminus S_{h-1} \subseteq X$, where $S_{h-1} \subseteq S_h$;

then, by Lemma 3.7, $X$ is the symmetric difference of two distinct hyperplanes.

\[\Box\]

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