Avoiding Chaos in Wonderland

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Abstract

Wonderland, a compact, integrated economic, demographic and environmental model is investigated using methods developed for studying critical phenomena. Simulation results show the parameter space separates into two phases, one of which contains the property of long term, sustainable development. By employing information contain in the phase diagram, an optimal strategy involving pollution taxes is developed as a means of moving a system initially in a unsustainable region of the phase diagram into a region of sustainability while ensuring minimal regret with respect to long term economic growth.

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1 Introduction

Long-term planning to ensure a sustainable world is a difficult problem for many reasons, not the least of which is the lack of certainty surrounding mankind's interactions with his environment. Given a host of models with varying levels of complexity [1]; it is difficult to develop a unified methodology for understanding the intricate dynamics contained within these models. A common approach to problem of long-term planning is the Scenario method [2, 3, 4]; whereby, the goal is to create a small set of reasonable scenarios extrapolating from the present into possible futures. These scenarios are often supported by quite large computer simulations of models describing the different sectors of an integrated world. Generally, these models include economic, demographic and environmental sectors [2], though they may also include political [3] and security [4] components as well. Basically, the idea behind the scenario approach is to selects a reasonable set of parameters for the model in question, then integrate the model forward over the planning time frame in order to ascertain what type of world might evolve.

While the scenario approach yields readily understandable projections based upon the current state and justifiable assumptions; its basic weakness is the complexity of the models themselves. The values of some of the parameters which enter the models are not know with any accuracy and can only be inferred from the current state of the system; at the same time some of the equations describing the system are chaotic. This combination of vaguely known parameters and chaotic equations of motion cast doubt on the reliability of the forecasts [5], leading critics to charge the authors with cherry picking, i.e., selecting sets of parameters to fit the authors preconceived notions of how the world should evolve.

Recently, Bankes, et al. [6, 7] suggested a more fundamental approach based upon large scale computer simulations, specifically aimed at avoiding the charges of cherry picking. By using Monte Carlo techniques to create a large ensemble of scenarios, they propose to avoid bias towards any one scenario. Projections used for long-term planning are then based upon the probability of any given scenario occurring in the ensemble.

In this paper the ensemble approach is extended to a systematic exploration of the parameter space, whereby the equations of motion are treated as equations of state, leading to a multi-dimensional phase diagram. The phase diagram allows a clean separation of the parameter space into regions where sustainable development is possible and where it is not. This removes the guess work from the construction of plausible scenarios and brings more focus to the ensemble method.

For discussing the current proposal in more depth, the Wonderland model [8, 9, 7] has been chosen as an example because of its relative simplicity and tractability. Although it lacks the details of many other models, it still captures enough general behavior of more sophisticated models to make its study worth-
while.

The next section describes the Wonderland model in more detail, after which a detailed discussion of Wonderland’s behavior and phase diagram is undertaken. Based upon these findings, section 4 presents a strategy for long term planning aimed at ensuring sustainable development.

2 The Wonderland Model

Sanderson’s Wonderland model [8, 9] describes in an integrative framework the economic, demographic and environmental sectors of an idealized world. The model is characterized by four state variables, \( \{x(t), y(t), z(t), p(t)\} \), representing the population, per capita output, stock of natural capital and pollution flow per unit of output respectively. These four state variables evolve according to the following set of non-linear, difference equations:

\[
\begin{align*}
  x(t+1) &= x(t) \left[ 1 + n(y(t), z(t)) \right], \\
  y(t+1) &= y(t) \left( 1 + \gamma - (\gamma + \eta) \left[ 1 - z(t) \right]^\lambda \right), \\
  z(t+1) &= \frac{g(x(t), y(t), z(t), p(t))}{1 + g(x(t), y(t), z(t), p(t))}, \\
  p(t+1) &= p(t)(1 - \chi).
\end{align*}
\]

The state variables for population \( (x) \) and per capita output \( (y) \) can assume all non-negative real values \( (x, y \in [0, \infty)) \), while the stock of natural capital \( (z) \) and the pollution per unit of output \( (p) \) are confined to the unit interval \( (z, p \in [0, 1]) \). A value of \( z = 1 \) represents a full stock of unpolluted natural capital and \( z = 0 \) represents the fully polluted state. \( p = 1 \) on the other hand represents maximal pollution per unit of output and \( p = 0 \) implies no pollution per unit of output.

In eq. 1, the endogenous population growth rate, \( n(y, z) \), can be written as the difference between the crude birth rate, \( b(y, z) \) (number of births per 1000 population per time at a given time step) and the crude death rate, \( d(y, z) \) (number of deaths per 1000 population per time at a given time step):

\[
\begin{align*}
  n(y, z) &= b(y, z) - d(y, z) \\
  b(y, z) &= \beta_0 \left[ \beta_1 - \left( \frac{e^{\beta y}}{1 + e^{\beta y}} \right) \right],
\end{align*}
\]

\( p(t+1) = p(t)(1 - \chi) \).
\[ d(y, z) = \alpha_0 \left[ \alpha_1 - \left( \frac{e^{\alpha y}}{1 + e^{\alpha y}} \right) \right] \left[ 1 + \alpha_2 (1 - z)^\theta \right], \]  

(7)

The parameters \( \beta, \beta_0 \) and \( \beta_1 \) govern the birth rate, while the parameters \( \alpha, \alpha_0, \alpha_1, \alpha_2 \) and \( \theta \) govern the death rate. From eqs. 6 and 7 it can be seen how both the birth rate and death rate decrease with increases in the per capita output, \( y \). Furthermore, in eq. 7, the death rate is seen to increase when the environment deteriorates, i.e., when \( z \) decreases. These effects are in line with recent studies relating population growth with industrial output [11, 12].

The function \( g(x, y, z, p) \) which determines the time evolution of the natural capital is given by:

\[ g(x, y, z, p) = \frac{z}{1 - z} e^{\delta \rho - \omega f(x, y, p)}, \]  

(8)

where, \( f(x, y, p) \), is the pollution flow:

\[ f(x, y, p) = xy p. \]  

(9)

The parameters \( \delta, \rho, \) and \( \omega \) determine the pollution flow at which the economic, environmental and demographic sectors are in balance. This critical pollution flow, \( \delta \rho / \omega \), determines the rate at which the natural capital is able to ameliorate the pollution flow, \( f \). As can be seen from eqs. 3 and 8, when \( f = \delta \rho / \omega \) the level of natural capital remains constant and the economic sector is in balance with the environmental sector.

The form of \( f \) originates from the I-PAT hypothesis [13]. In its original form, the I-PAT hypothesis states: a population’s impact on its environment is equal to its size multiplied by its per capita output and its level of technology. In wonderland, the impact is the pollution flow and the level of technology is represented by the pollution per unit of output; hence, \( f = xy p \).

Wonderland’s economic sector is characterized by the parameters: \( \gamma, \eta, \lambda \) and \( \chi \). \( \gamma \) is the exogenous economic growth rate and determines how fast the economy could grow if its capital stock were fully intact (see eq. 2). \( \eta \) and \( \lambda \) determine how rapidly the economy deteriorates (recovers) when the capital stock declines (increases). The parameter \( \chi \) governs the economic decoupling rate, i.e., the rate at which technological innovations reduce the pollution flow per unit of output.

Some variants of the Wonderland model include a term in eq. 10 governing pollution control expenditures [14, 15, 9, 16]. Since pollution control is essentially a policy decision and not an intrinsic part of the model, in this paper, we follow [7] and introduce pollution control later in section 4 where long term planning
is discussed in more detail. Such an approach allows us to cleanly separate the intrinsic dynamics of the model as given above, and perturbations of the dynamics due to policy decisions.

As can be seen, equations 1 to 10 depend upon 15 positive parameters which govern the overall behavior of model; whereby eight of them ($\alpha, \alpha_0, \alpha_1, \alpha_2, \beta, \beta_0, \beta_1$ and $\theta$) determine the population growth, four ($\delta, \rho, \omega$ and $\chi$) determine the state of the environment and three ($\gamma, \eta$ and $\lambda$) determine the health of the economy. In the next section we take some slices through this 15 dimensional parameter space to learn more about the models behavior.

3 Phase Diagram

As noted in the introduction, previous work on this model has mostly followed a scenario based approach [8, 14, 6], with the two most often investigated scenarios being the so-called “Dream” scenario and the “Horror” scenario. The Dream scenario earns its name because it holds out the possibility of continued economic growth combined with a stable population and a healthy environment; whereas the Horror scenario depicts environmental collapse followed by an economic collapse and a declining population. Table 1 contains the parameters for the Dream scenario. In the horror scenario, the only parameter whose value is changed, is the decoupling rate, $\chi$, which takes on the new value $\chi = 0.01$.

Plots of the time evolution for the Dream and Horror scenarios are shown in Figure 1. Initially, both scenarios follow the same growth curves in terms of per capita output and population; however, after approximately 90 years, the world in the Horror scenario undergoes a spontaneous transition to a phase marked by a depleted stock of natural capital, an economic depression and population decline. The sudden collapse of the environment after 90 years of relative stability is indicative of a first order phase transition.

Until now, our analysis has concentrated upon the classical scenario approach. As a first step in going beyond this approach, we examine the behavior of the model when systematically moving through the parameter space along the line $\chi = 0.01$ to $\chi = 0.04$ while holding all the other parameters fixed. For this purpose, define an order parameter, $t_c$, as the number of years before the natural stock collapses. The justification of calling $t_c$ an order parameter [17] becomes apparent when examining Figure 2. As can be seen, $t_c$ follows the scaling behavior expected in the presence of a second order phase transition:

$$t_c \sim (\chi - \chi_c)^{\xi},$$  

(11)

Furthermore, $t_c$ is undefined for all $\chi \geq \chi_c$. From the data in Figure 2, the crit-
ical value of $\chi$ can be estimated: $\chi_c \approx 0.0385$, along with the critical exponent, $\zeta \approx 0.945$. The symbols in Figure 2, indicate different starting states. Depending upon the exact initial state, the collapse may take place sooner or later, but the scaling behavior is the same. From this data we can view $\chi_c$ as marking a phase boundary between the phase of unsustainable development, as epitomized by the Horror scenario, and the phase of sustainable development as epitomized by the Dream scenario. (The reader should not take this diagram to mean we are advocating a planning horizon extending to 100000 years! Rather, the diagram indicates how rapidly $t_c$ changes given small changes in $\chi$.)

In the Wonderland model, the parameters most responsible for the interactions of the economic and environmental sectors are $\gamma$ and $\chi$. Therefore, if we repeat the above analysis for the remainder of the $\gamma - \chi$ plane, we arrive at the phase diagram shown in Figure 3. The line of points delineates the parameter space into a phase of sustainable development and one of unsustainable development. In the phase of sustainable development the economic, demographic and environmental sectors of Wonderland are in equilibrium, while in the phase of unsustainable development, these sectors are out of equilibrium, eventually leading to a world wide collapse. As one approaches the phase boundary from below, the time before the collapse occurs increases as a power law of the distance from the boundary.

One can continue this exercise for the remaining parameters in the model; however, most of the other parameters effect the quantitative results but not the qualitative features described above. The parameters $\delta, \rho$ and $\omega$, for example, determine the exact value of the critical pollution flow above which the environment begins deteriorating, but not the existence of a critical pollution flow [14].

### 4 Policy Planning

In the previous section, the analysis focused on the intrinsic behavior of the Wonderland model; however, this is only part of the problem, the other more intricate question is whether or not it is possible to introduce control mechanisms in order to avoid the phase transition in the Horror scenario and the attendant catastrophic consequences. Note, the model as described by equations 1 through 10 has no steerable parameters. All of the 15 parameters entering into the Model are in principal measurable or can be estimated using available data [8, 7]; hence, once they are known one can look on the phase diagram of Figure 3 to determine whether the world of Wonderland is in a sustainable phase or an unsustainable phase approaching a potentially catastrophic collapse.

When faced with the question of how to handle sustainable development, policy makers can choose to either control the emission of pollutants at their source or to spend funds abating pollutants afterwards. Either of these approaches may be
effective or they may induce undesirable side affects. As a first step, we look at the pollution abatement approach.

4.1 Pollution Abatement

To abate the effects of pollution, funds must be drawn from other sources of income; furthermore, as the environmental degradation becomes more serious, more funds are required. A simple, non-linear model governing the expenditures for pollution abatement, \( c(y, z) \), has been proposed by previous authors [8, 9, 16]:

\[
  c(y, z) = \phi (1 - z)^\mu y. \tag{12}
\]

In this model, as natural capital deteriorates, the expenditures to abate pollution increase, whereby the rate at which expenditures increase are governed by the policy parameters \( \phi \) and \( \mu \). Withdrawing capital in this manner, decreases the per capita output available for other uses; hence, the per capita output, \( y \), in equations 6 and 7 must be replaced by \( y' = y - c \). (Eq. 2 does not change, because goods and services needed for pollution abatement are part of the overall output.) Furthermore, since the aim is to improve the state of the environment, the pollution flow is reduced by the effectiveness of the regulations, i.e., eq. 10, becomes [8, 9, 16]:

\[
  f(x, y, p) = xyp - \kappa \frac{e^{exc}}{1 + e^{exc}}. \tag{13}
\]

where, \( \kappa \) determines the effectiveness of the expenditures, \( c(y, z) \).

Basically, policy makers have the parameters \( \phi \), \( \mu \) and \( \kappa \) for control purposes, though none of these parameters can be varied without limits. \( \phi \) and \( \mu \) determine how much output is diverted to pollution abatement once the stock of natural capital starts deteriorating. Since we require \( c < y \), this limits \( \phi < 1 \). \( \mu \) determines how quickly the inhabitants of Wonderland respond to early signs of environmental degradation. While these parameters cannot of themselves drive the system from the phase of unsustainable development to the phase of sustainable development, they do have an impact on the amount of chaos present in the unsustainable phase.

An intriguing new phenomena in this model is the environments recovery from collapse. Indeed, one can simulate the Horror scenario over several millennium, and observe how it undergoes repeating cycles of collapse and recovery. To gain insight into the underlying dynamics it is instructive to plot the orbits of the real growth rate as a function of the stock of natural capital. The normalized real grow rate can be defined as:
\[ r(t) = \frac{y(t) - y(t - 1)}{\gamma y(t - 1)}, \]  

where we have normalized the real growth rate by the exogenous growth rate, \( \gamma \). A plot of \( r(t) \) versus \( z(t) \) is shown in Figure 4. When the environment is deteriorating, the economy follows the upper curve and when the environment is improving, the economy follows the lower curve. Hysteresis curves of this type are expected in the presence of a first order phase transition. From this figure it can be seen that the momentary performance of the economy sheds little light on the overall health of Wonderland, since the growth rate at first decreases only slowly with the declining capital stock until the natural capital is nearly exhausted, at which point the growth rate rapidly turns from positive to negative. The exact shape of the orbits depend upon the parameters \( \eta \) and \( \lambda \) from eq. 2. This general picture is consistent with scenarios described by more detailed models such as World3 [2], thereby lending credence to the proposition of using Wonderland as a tractable model for in depth studies.

Notice how the trajectories of the Horror scenario appear to contract to an aperiodic recurrent attractor [5]. Using the techniques described in [18] we can estimated the Lyapunov exponent for this strange attractor: \( \lambda \approx 0.026 \). A positive value of the Lyapunov exponent is another indication of the problems facing the Horror scenario as past experience does not provide detailed guidance on future behavior. In the next cycle the economy may collapse more quickly or more slowly than it did in the previous cycle. This type of behavior is typical of a dynamical system operating in the chaotic regime [5].

While previous research has shown it is possible to avoid the chaos and collapse in the horror scenario by setting \( \kappa = 100 \) [14, 15, 9], such large values of \( \kappa \) seem unrealistic, because the first term in eq. 13 is \( O(1) \) initially.

Unfortunately, pollution abatement alone, does not move the system from the phase of unsustainable development to the phase of sustainable development, though it does help to recover from an environmental collapse; in fact, as long as the parameters remain within reasonable bounds they have no impact on the phase diagram shown in Figures 3.

### 4.2 Pollution Control

Pollution taxes have been introduced into the Wonderland model in previous studies [9, 16, 7]. The goal of a pollution tax is to increase the effective decoupling rate, \( \chi \), by making pollution unprofitable. In this paper, the pollution tax rate, \( \tau \), enters the model first via eq. 4, which changes to:
\[ p(t+1) = p(t) \left( 1 - \chi - \chi_0 \frac{\tau}{1 + \tau} \right). \]  \hspace{1cm} (15)

where, \( \chi_0 \) is the maximum additional decoupling rate for the assumed level of technology, i.e., \( \chi + \chi_0 \) is the maximum achievable decoupling rate for an assumed level of technology.

The side affect of taxing pollution is a reduction in the real per capita growth rate since resources are diverted into the pollution control; hence, eq. 2 becomes:

\[ y(t+1) = y(t) \left( 1 + \gamma - (\gamma + \eta) \left[ 1 - z(t) \right]^{\lambda} - \gamma_0 \frac{\tau}{1 - \tau} \right), \]  \hspace{1cm} (16)

where the parameter \( \gamma_0 \) determines the amount by which the pollution tax retards growth. Theoretically, a modest pollution tax does not diminish growth to the full extent of the tax, because the tax itself spurs investment in pollution reduction technologies which in turn increase growth; however, as the tax becomes larger, the numerator in equation 16 tends toward zero and the tax can become a considerable drag on the economy.

By varying the decoupling rate, \( \chi \) and the tax rate \( \tau \), while keeping the other parameters fixed to their values in Table 1 (with \( \gamma_0 = 0.5 \) and \( \chi_0 = \chi/2 \)), the phase diagram in Figure 5 can be constructed using the techniques discussed in the last section. As can be seen, with the help of a moderate pollution tax, the system can be moved from the phase of unsustainable development to the phase of sustainable development.

In today’s political climate many of society’s leaders do not dispute the ability of pollution taxes or other remedies to improve the state of the world’s natural capital, rather they claim the cost in terms of foregone economic development is too high. Indeed, many leaders would prefer to maintain high per capita growth now and deal with the aftermath of an environmental collapse later, in the same manner the stock market bubble was allowed to expand at the end of 1990s until it burst in 2000.

Concentrating on short term growth rates alone, however, can be misleading. As shown in Figure 6, the average growth rate is the same when the pollution tax rate is too low, or when the pollution tax is too high. Hence, the average growth rate alone is not definitive, policy makers must also be concerned about reducing economic volatility [19, 20], i.e., reducing fluctuations in the growth rate. Figure 6 also plots the variation in the normalized real growth rate as a function of the pollution tax rate. As can be seen, volatility drops to zero at the critical tax rate, which is also the point where the real growth rate is a maximum. This is to be expected, since in the phase of sustainable development, the Wonderland model contains no mechanism to create endogenous variability in the economic growth
rate.

As an additional check on the viability of differing long term strategies, the maximal regret\([21]\) for alternative strategies should be calculated and the strategy yielding the minimal of the maximal regret should be chosen. Towards this end, we define the critical regret for per capita output for different value of the tax rate, \(\tau\) as:

\[
R_c(t, \tau) = \frac{y_c(t) - y(t, \tau)}{y_c(t)}
\]

where \(y_c(t)\) is the per capita output at the critical value of \(\tau\), i.e., the value of \(\tau\) on the phase boundary in Figure 5 (all other parameters are assumed fixed). Figure 7 depicts the critical regret as a function of time for different tax rates when all other parameters are held fix to the values they assume in the Horror scenario. For tax rates lower than the critical tax rate, the short term regret is negative, meaning faster economic growth than that achievable at the critical tax rate; however, this faster growth is completely dissipated once the environment collapses, leaving the long term regret at its maximum possible value. For tax rates above the critical tax rate, economic growth is slower, leading to increased regret. Hence, the critical tax rate is the optimal tax rate in terms of Wonderland’s long term prospects.

The relative ineffectiveness of pollution abatement versus pollution control was noted previously by Leeves and Herbert\([16]\) who studied a modified version of the Wonderland model in which eq. 2 is replaced by a Cobb-Douglas production function. Their model showed a transient dynamics consisting of large volatility in the per capita output and the stock of natural capital. As in the present model, pollution abatement expenditures were ineffective at eliminating the unwanted behavior, whereas adequate levels of pollution control expenditures were effective in eliminating the volatility.

In summary, it seem reasonable to conclude that controlling pollution at the source is more import as far as sustainability is concerned, than attempting to abate its effects afterwards.

5 Conclusion

The advantages of basically treating eqs. 1-10 as dynamical equations of state have been demonstrated. The phase diagram depicted in Figure 3 contains an infinite number of scenarios, some of which have the property of sustainable development, others of which do not. In the lower right hand corner, for example, one can find scenarios of environmental collapse due to over production; while in the lower left hand corner are scenarios of collapse due to over population.

The analysis of the preceding two sections has yielded the information needed
to formulate an optimal strategy for ensuring the long term health of Wonderland. Such a strategy would consist of the following steps:

1. Determine the model parameters as accurately as possible.

2. Map out the phase diagram in that part of parameter space covered by the parameters measured in step 1.

3. Determine the approximate critical pollution tax required for sustainable development.

4. Periodically repeat steps 1-3 adjusting the pollution tax as necessary.

As demonstrated by Figure 7, this simple strategy is optimal in the sense of yielding the minimal regret in the long term, at the sacrifice of short term (and short lived) gains.

One may question whether it is necessary to aim for the phase boundary, or whether simply being “not too far” is sufficient, especially since the time to collapse increases exponentially as one approaches the phase boundary. The answer is given by the discussion in 3 and in particular Figures 1 and 4; namely, the system provides little forewarning of an impending collapse. Relative stability may prevail for decades, until the system suddenly undergoes a phase transition; hence, prudence would dictate avoiding this parameter regime if at all possible.

As noted above, the Wonderland model is a comparatively simple integrated model. For example, the maximum growth rate, $\gamma$, is treated in this model as an exogenous parameter; in reality the maximum growth rate is a complex function of the state of the economy, the environment and the population. When all three collapse as in Figure 1, $\gamma$ would also be expected to fall, leading to a faster collapse and longer recovery times. Hence, the phase diagram in Figure 3 should be taken as upper bound on the sustainability due to the use of a constant $\gamma$. An extension of the Wonderland model to include more realistic, endogenous growth through a Cobb-Douglas production function has been studied by Leeves and Herbert [16].

In summary, this paper has demonstrated how concepts originally developed for studying critical phenomena in physical systems can be successfully applied to problems in long-term, socio-economic planning. Of course, Wonderland is a toy model which captures the global features of the complete human-environment interactions, but not its details. Indeed, detailed models transferable to the real world, where the models can contain an order of magnitude more equations [1]; however, we believe the principals elucidated here can still apply. Furthermore, the principled approach developed here can augment the ensemble approach [7] by applying large scale computer simulations more systematically.
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Table 1: Parameter values for the Dream scenario.

|   | Economy |   | Environment |   | Population |
|---|---------|---|-------------|---|------------|
|   | name    | value | name        | value | name | value |
| γ | 0.04    |       | χ           | 0.04  | α   | 0.09  |
| η | 0.04    |       | δ           | 1.0   | α₀  | 10.0  |
| λ | 2.0     |       | ρ           | 0.2   | α₁  | 2.5   |
|   | ω       | 0.1   |             |       | α₂  | 2.0   |
| β |         |       |             |       | β₀  | 40.0  |
|   |         |       |             |       | β₁  | 1.375 |
| θ |         |       |             |       | θ   | 15.0  |
Figures

Figure 1 The basic scenarios Dream and Horror. (a) Semi-log plot of the per capita output as a function of time. (b) Plot of relative population as a function of time. (c) Plot of natural capital as a function of time.

Figure 2 Log-log plot of the time before the first collapse of natural capital as a function of the distance $\chi - \chi_c$. Different point types indicate different initial states of the system.

Figure 3 Phase diagram in the $\gamma - \chi$ plane. The line marks the boundary between the phases of sustainable, S, and unsustainable, U, development. H marks the location of the Horror scenario, while D marks the location of the Dream scenario.

Figure 4 Plot of the normalized, real growth rate as a function of the stock of natural capital over a 40,000 year time span.

Figure 5 Phase diagram in the $\tau - \chi$ plane when the other parameters are fixed to their values in Table 1 and $\gamma_0 = 0.5$.

Figure 6 The average, $\mu_r$, and deviation, $\sigma_r^2$ of the normalized real growth rate, $r(t)$ (see eq. 14) as a function of the tax rate, $\tau$, when the other parameters are the same as those used in the Horror scenario.

Figure 7 The critical regret as a function of time for different values of the tax rate, $\tau$, when the other parameters are the same as those used in the Horror scenario.
Figure 1:
Figure 2:
Figure 3:
Figure 4:
Figure 5:
Figure 6: Graph showing the relationship between \( \tau \) and functions \( \mu_r \) and \( \sigma^2_r \). The graph displays the change in these functions as \( \tau \) increases from 0 to 0.3.
Figure 7: Plots of $R_c(t, \tau)$ for different values of $\tau$: $\tau=0.0$, $\tau=0.012$, and $\tau=0.055$. The plots show the variation of $R_c(t, \tau)$ with respect to time $t$ (in years) for each of the given $\tau$ values.