A note on composition operators between weighted spaces of smooth functions

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Abstract For certain weighted locally convex spaces $X$ and $Y$ of one real variable smooth functions, we characterize the smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$ for which the composition operator $C_\varphi : X \to Y$, $f \mapsto f \circ \varphi$ is well-defined and continuous. This problem has been recently considered for $X = Y$ being the space $\mathcal{S}$ of rapidly decreasing smooth functions [1] and the space $\mathcal{O}_M$ of slowly increasing smooth functions [2]. In particular, we recover both these results as well as obtain a characterization for $X = Y$ being the space $\mathcal{O}_C$ of very slowly increasing smooth functions.

1 Introduction

One of the most fundamental questions in the study of composition operators is to characterize when such an operator is well-defined and continuous in terms of its symbol. The goal of this article is to consider this question for weighted locally convex spaces of one real variable smooth functions.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be smooth. In [1] Galbis and Jordá showed that the composition operator $C_\varphi : \mathcal{S} \to \mathcal{S}$, $f \mapsto f \circ \varphi$, with $\mathcal{S}$ the space of rapidly decreasing smooth functions [3], is well-defined (continuous) if and only if

$$\exists N \in \mathbb{Z}_+ : \sup_{x \in \mathbb{R}} \frac{1 + |x|}{(1 + |\varphi(x)|)^N} < \infty$$

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and
\[ \forall p \in \mathbb{Z}_+ \exists N \in \mathbb{N} : \sup_{x \in \mathbb{R}} \frac{|\varphi^{(p)}(x)|}{1 + |\varphi(x)|^N} < \infty. \]

Albanese et al. [2] proved that the composition operator \( C_\varphi : \mathcal{O}_M \to \mathcal{O}_M \), with \( \mathcal{O}_M \) the space of slowly increasing smooth functions [3], is well-defined (continuous) if and only if \( \varphi \in \mathcal{O}_M \). In [2] Remark 2.6 they also pointed out that the corresponding result for the space \( \mathcal{O}_C \) of very slowly increasing smooth functions [3] is false, namely, they showed that \( \sin(x^2) \notin \mathcal{O}_C \), while, obviously, \( \sin x, x^2 \in \mathcal{O}_C \).

Inspired by these results, we study in this article the following general question: Given two weighted locally convex spaces \( X \) and \( Y \) of smooth functions, when is the composition operator \( C_\varphi : X \to Y \) well-defined (continuous)? We shall consider this problem for \( X \) and \( Y \) both being Fréchet spaces, \( (LF) \)-spaces, or \( (PLB) \)-spaces.

We now state a particular instance of our main result that covers many well-known spaces. We need some preparation. Given a positive continuous function \( \nu \) on \( \mathbb{R} \), we write \( \mathcal{B}_\nu^n, n \in \mathbb{N} \), for the Banach space consisting of all \( f \in C^n(\mathbb{R}) \) such that
\[ \|f\|_{\nu,n} = \max_{p \leq n} \sup_{x \in \mathbb{R}} \frac{|f^{(p)}(x)|}{\nu(x)} < \infty. \]

For \( \nu \geq 1 \) we consider the following three weighted spaces of smooth functions
\[ \mathcal{K}_\nu = \lim_{N \to \infty} \mathcal{B}_\nu^N, \]
\[ \mathcal{O}_{C,\nu} = \lim_{N \to \infty} \lim_{n \to \infty} \mathcal{B}_\nu^n, \]
\[ \mathcal{O}_{M,\nu} = \lim_{N \to \infty} \lim_{n \to \infty} \mathcal{B}_\nu^n. \]

Theorem below implies the following result:

**Theorem 1** Let \( v, w : \mathbb{R} \to [1, \infty) \) be continuous functions such that
\[ \sup_{x,t \in \mathbb{R}, |t| \leq 1} \frac{v(x+t)}{v(x)} < \infty \quad \text{and} \quad \sup_{x,t \in \mathbb{R}, |t| \leq 1} \frac{w(x+t)}{w(x)} < \infty, \]
for some \( \lambda, \mu > 0 \). Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be smooth. Then,

(1) The following statements are equivalent:

(i) \( C_\varphi(\mathcal{K}_\nu) \subseteq \mathcal{K}_\mu \).

(ii) \( C_\varphi : \mathcal{K}_\nu \to \mathcal{K}_\mu \) is continuous.

(iii) \( \varphi \) satisfies the following two properties

(a) \( \exists \lambda > 0 : \sup_{x \in \mathbb{R}} \frac{w(x)}{v^\lambda(\varphi(x))} < \infty \).

(b) \( \forall p \in \mathbb{Z}_+ \exists \lambda > 0 : \sup_{x \in \mathbb{R}} \frac{|\varphi^{(p)}(x)|}{v^\lambda(\varphi(x))} < \infty \).

(II) The following statements are equivalent:
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(i) $C_{\varphi}(\mathcal{O}_{C,w}) \subseteq \mathcal{O}_{C,w}$.
(ii) $C_{\varphi} : \mathcal{O}_{C,v} \to \mathcal{O}_{C,w}$ is continuous.
(iii) $\varphi$ satisfies the following two properties

(a) $\exists \mu > 0 : \sup_{x \in \mathbb{R}} \frac{v(\varphi(x))}{w^\mu(x)} < \infty$.
(b) $\forall p, k \in \mathbb{Z}_+ : \sup_{x \in \mathbb{R}} \frac{|\varphi^{(p)}(x)|}{w^{1/k}(x)} < \infty$.

(III) The following statements are equivalent:

(i) $C_{\varphi}(\mathcal{O}_{M,v}) \subseteq \mathcal{O}_{M,w}$.
(ii) $C_{\varphi} : \mathcal{O}_{M,v} \to \mathcal{O}_{M,w}$ is continuous.
(iii) $\varphi$ satisfies the following two properties

(a) $\exists \mu > 0 : \sup_{x \in \mathbb{R}} \frac{v(\varphi(x))}{w^\mu(x)} < \infty$.
(b) $\forall p \in \mathbb{Z}_+ \exists \mu > 0 : \sup_{x \in \mathbb{R}} \frac{|\varphi^{(p)}(x)|}{w^\mu(x)} < \infty$.

By setting $v(x) = w(x) = 1 + |x|$ in Theorem 1, we recover the above results about $\mathcal{O}$ and $\mathcal{O}_{M}$ from [1, 2] as well as the following characterization for the space $\mathcal{O}_{C}$ of very slowly increasing smooth functions: $C_{\varphi} : \mathcal{O}_{C} \to \mathcal{O}_{C}$ is well defined (continuous) if and only if

$$\exists N \in \mathbb{N} : \sup_{x \in \mathbb{R}} \frac{|\varphi(x)|}{(1 + |x|)^N} < \infty \quad \text{and} \quad \forall p, k \in \mathbb{Z}_+ : \sup_{x \in \mathbb{R}} \frac{|\varphi^{(p)}(x)|}{(1 + |x|)^{1/k}} < \infty.$$  

For $v = w = 1$, Theorem 1 gives the following result for the Fréchet space $\mathcal{B}$ of smooth functions that are bounded together with all their derivatives [3]: $C_{\varphi} : \mathcal{B} \to \mathcal{B}$ is well defined (continuous) if and only if $\varphi' \in \mathcal{B}$. Another interesting choice is $v(x) = w(x) = e^{x}$, for which Theorem 2 characterizes composition operators on spaces of exponentially decreasing/increasing smooth functions [4, 5]. We leave it to the reader to explicitly formulate this and other examples.

2 Statement of the main result

A pointwise non-decreasing sequence $V = (v_N)_{N \in \mathbb{N}}$ of positive continuous functions on $\mathbb{R}$ is called a weight system if $v_0 \geq 1$ and

$$\forall N \exists M \geq N : \sup_{x,t \in \mathbb{R}, |t| \leq 1} \frac{v_N(x+t)}{v_M(x)} < \infty.$$  

We shall also make use of the following condition on a weight system $V = (v_N)_{N \in \mathbb{N}}$:

$$\forall N, M \exists K \geq N, M : \sup_{x \in \mathbb{R}} \frac{v_N(x)v_M(x)}{v_K(x)} < \infty. \quad (1)$$
Example 1 Let \( v : \mathbb{R} \to [1, \infty) \) be a continuous function satisfying
\[
\sup_{x, t \in \mathbb{R}, |t| \leq 1} \frac{v(x + t)}{v^N(x)} < \infty,
\]
for some \( N \in \mathbb{N} \) (cf. Theorem 1). Then,
\[
V_v = (v^N)_{N \in \mathbb{N}}
\]
is a weight system satisfying (1). \( \square \)

Recall that for a positive continuous function \( v \) on \( \mathbb{R} \) and \( n \in \mathbb{N} \), we write \( \mathcal{B}_n^v \) for the Banach space consisting of all \( f \in C^n(\mathbb{R}) \) such that
\[
\| f \|_{v,n} = \max_{p \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{|f^{(p)}(x)|}{v(x)} < \infty.
\]

Let \( V = (v_N)_{N \in \mathbb{N}} \) be a weight system. We shall be concerned with the following weighted spaces of smooth functions
\[
\mathcal{K}_V = \lim_{N \in \mathbb{N}} \mathcal{B}_1^{v_N},
\]
\[
\mathcal{O}_{C,V} = \lim_{N \in \mathbb{N}} \mathcal{B}_n^{v_N},
\]
\[
\mathcal{O}_{M,V} = \lim_{n \in \mathbb{N}} \mathcal{B}_n^{v_N}.
\]

Note that \( \mathcal{K}_V \) is a Fréchet space, \( \mathcal{O}_{C,V} \) is an \((LF)\)-space, and \( \mathcal{O}_{M,V} \) is a \((PLB)\)-space. Furthermore, we have the following continuous inclusions
\[
\mathcal{D}(\mathbb{R}) \subset \mathcal{K}_V \subset \mathcal{O}_{C,V} \subset \mathcal{O}_{M,V} \subset C^\infty(\mathbb{R}),
\]
where \( \mathcal{D}(\mathbb{R}) \) denotes the space of compactly supported smooth functions. The spaces \( \mathcal{K}_V \) were introduced and studied by Gelfand and Shilov [6], while we refer to [7] for more information on the spaces \( \mathcal{O}_{C,V} \). For \( N, n \in \mathbb{N} \) fixed we will also need the following spaces
\[
\mathcal{B}_n^v = \lim_{N \in \mathbb{N}} \mathcal{B}_n^{v_N},
\]
\[
\mathcal{O}_{M,V} = \lim_{N \in \mathbb{N}} \mathcal{B}_n^{v_N}.
\]

The goal of this article is to show the following result.

Theorem 2 Let \( V = (v_N)_{N \in \mathbb{N}} \) and \( W = (w_M)_{M \in \mathbb{N}} \) be two weight systems and let \( \varphi : \mathbb{R} \to \mathbb{R} \) be smooth.

(1) Suppose that \( V \) satisfies (1). The following statements are equivalent:

(i) \( C_\varphi(\mathcal{K}_V) \subseteq \mathcal{K}_W \).
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(ii) \( C_\varphi : \mathcal{H}_V \to \mathcal{H}_W \) is continuous.

(iii) \( \varphi \) satisfies the following two properties

\[
\begin{align*}
& (a) \, \forall M \, \exists N : \sup_{x \in \mathbb{R}} \frac{w_M(x)}{v_N(\varphi(x))} < \infty, \\
& (b) \, \forall p \in \mathbb{Z}_+ \, \exists N : \sup_{x \in \mathbb{R}} \frac{|\varphi^{(p)}(x)|}{w_N(\varphi(x))} < \infty.
\end{align*}
\]

(II) Suppose that \( W \) satisfies (I). The following statements are equivalent:

(i) \( C_\varphi(\mathcal{O}_{C,V}) \subseteq \mathcal{O}_{C,W} \).

(ii) \( C_\varphi : \mathcal{O}_{C,V} \to \mathcal{O}_{C,W} \) is continuous.

(iii) \( \forall N \, \exists M \) such that \( C_\varphi : \mathcal{B}_N \to \mathcal{B}_M \) is continuous.

(iv) \( \varphi \) satisfies the following two properties

\[
\begin{align*}
& (a) \, \forall N \, \exists M : \sup_{x \in \mathbb{R}} \frac{v_N(\varphi(x))}{w_M(x)} < \infty, \\
& (b) \, \exists M, k \in \mathbb{Z}_+ : \sup_{x \in \mathbb{R}} \frac{|\varphi^{(p)}(x)|}{w_M^{1/k}(x)} < \infty.
\end{align*}
\]

(III) Suppose that \( W \) satisfies (I). The following statements are equivalent:

(i) \( C_\varphi(\mathcal{O}_{M,V}) \subseteq \mathcal{O}_{M,W} \).

(ii) \( C_\varphi : \mathcal{O}_{M,V} \to \mathcal{O}_{M,W} \) is continuous.

(iii) \( C_\varphi : \mathcal{O}_{M,V}^n \to \mathcal{O}_{M,W}^n \) is continuous for all \( n \in \mathbb{N} \).

(iv) \( \varphi \) satisfies the following two properties

\[
\begin{align*}
& (a) \, \forall N \, \exists M : \sup_{x \in \mathbb{R}} \frac{v_N(\varphi(x))}{w_M(x)} < \infty, \\
& (b) \, \forall p \in \mathbb{Z}_+ \, \exists M : \sup_{x \in \mathbb{R}} \frac{|\varphi^{(p)}(x)|}{w_M(x)} < \infty.
\end{align*}
\]

The proof of Theorem2 will be given in the next section. The spaces \( \mathcal{H}_V, \mathcal{O}_{C,V} \) and \( \mathcal{O}_{M,V} \) from the introduction can be written as

\[
\mathcal{H}_V = \mathcal{H}_{V_0}, \quad \mathcal{O}_{C,V} = \mathcal{O}_{C,V_0}, \quad \mathcal{O}_{M,V} = \mathcal{O}_{M,V_0},
\]

where \( V_0 = \{v^n\}_{n \in \mathbb{N}} \) is the weight system from Example1. Hence, Theorem1 is a direct consequence of Theorem2 with \( V = V_0 \) and \( W = V_0 \).

3 Proof of the main result

Throughout this section we fix a smooth symbol \( \varphi : \mathbb{R} \to \mathbb{R} \). We need two lemmas in preparation for the proof of Theorem2. For \( n \in \mathbb{N} \) we set

\[
\|f\|_n = \|f\|_{1,n} = \max_{p \leq n} \sup_{x \in \mathbb{R}} |f^{(p)}(x)|.
\]
Lemma 1 Let \(v, \tilde{v}, w\) be three positive continuous functions on \(\mathbb{R}\) such that
\[
C_0 = \sup_{x, t \in \mathbb{R}, |t| \leq 1} \frac{v(x + t)}{v(x)} < \infty.
\]

Let \(p, n \in \mathbb{N}\) be such that
\[
\|C_{\varphi}(f)\|_{w, p} \leq C_1 \|f\|_{\tilde{v}, n}, \quad \forall f \in \mathcal{D}(\mathbb{R}), \tag{2}
\]
for some \(C_1 > 0\). Then,
\[
\sup_{x \in \mathbb{R}} \frac{v(\varphi(x))}{w(x)} < \infty, \tag{3}
\]
and, if \(p \geq 1\), also
\[
\sup_{x \in \mathbb{R}} \frac{v(\varphi(x))|\varphi'(x)|^p}{w(x)} < \infty, \tag{4}
\]
and
\[
\sup_{x \in \mathbb{R}} \frac{v(\varphi(x))|\varphi^{(p)}(x)|}{w(x)} < \infty. \tag{5}
\]

Proof Given \(f \in \mathcal{D}(\mathbb{R})\) with \(\text{supp } f \subseteq [-1, 1]\), we set \(f_x = f(\cdot - \varphi(x))\) for \(x \in \mathbb{R}\). Note that
\[
\|f_x\|_{\tilde{v}, n} \leq C_0 \|f\|_n \frac{|\varphi(x)|}{v(\varphi(x))}, \quad x \in \mathbb{R}. \tag{6}
\]
We first show (3). Choose \(f \in \mathcal{D}(\mathbb{R})\) with \(\text{supp } f \subseteq [-1, 1]\) such that \(f(0) = 1\). For all \(x \in \mathbb{R}\) it holds that
\[
\|C_{\varphi}(f_x)\|_{w, p} \geq \frac{|\varphi(f_x)(x)|}{w(x)} = \frac{1}{w(x)}. \tag{7}
\]
Hence, by (2) and (6), we obtain that
\[
\frac{v(\varphi(x))}{w(x)} \leq C_0 C_1 \|f\|_n, \quad \forall x \in \mathbb{R}. \tag{8}
\]
Now assume that \(p \geq 1\). We prove (4). Choose \(f \in \mathcal{D}(\mathbb{R})\) with \(\text{supp } f \subseteq [-1, 1]\) such that \(f^{(j)}(0) = 0\) for \(j = 1, \ldots, p - 1\) and \(f^{(p)}(0) = 1\). Faà di Bruno’s formula implies that for all \(x \in \mathbb{R}\)
\[
\|C_{\varphi}(f_x)\|_{w, p} \geq \frac{|\varphi(f_x)^{(p)}(x)|}{w(x)} = \frac{|\varphi'(x)|^p}{w(x)}. \tag{9}
\]
Similarly as in the proof of (3), the result now follows from (2) and (6). Finally, we show (5). Choose \(f \in \mathcal{D}(\mathbb{R})\) with \(\text{supp } f \subseteq [-1, 1]\) such that \(f'(0) = 1\) and \(f^{(j)}(0) = 0\) for \(j = 2, \ldots, p\). Faà di Bruno’s formula implies that for all \(x \in \mathbb{R}\)
\[
\|C_{\varphi}(f_x)\|_{w, p} \geq \frac{|\varphi(f_x)^{(p)}(x)|}{w(x)} = \frac{|\varphi^{(p)}(x)|}{w(x)}. \tag{10}
\]
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As before, the result is now a consequence of \(\square\) and \(\square\).

Lemma 2 Let \(v\) and \(w\) be positive continuous functions on \(\mathbb{R}\). Then,

(i) If
\[
\sup_{x \in \mathbb{R}} \frac{v(\varphi(x))}{w(x)} < \infty,
\]
then \(C_\varphi : \mathcal{B}_v^0 \rightarrow \mathcal{B}_w^0\) is well-defined and continuous.

(ii) Let \(n \in \mathbb{Z}_+\). If
\[
\sup_{x \in \mathbb{R}} \frac{v(\varphi(x))}{w(x)} \left( \prod_{p=1}^{n} \left| \varphi^{(p)}(x) \right|^{b_p} \right) < \infty
\]
for all \((k_1, \ldots, k_n) \in \mathbb{N}^n\) with \(\sum_{j=1}^{p} jk_j \leq p\) for all \(p = 1, \ldots, n\), then \(C_\varphi : \mathcal{B}_v^n \rightarrow \mathcal{B}_w^n\) is well-defined and continuous.

Proof (i) Obvious.

(ii) This is a direct consequence of (i) and Faà di Bruno’s formula. \(\square\)

Proof (of Theorem 2) (i) \(\Rightarrow\) (ii): Since \(C_\varphi : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})\) is continuous, this follows from the closed graph theorem for Fréchet spaces.

(ii) \(\Rightarrow\) (iii): For all \(p, M \in \mathbb{N}\) there are \(n, L \in \mathbb{N}\) such that
\[
\|C_\varphi(f)\|_{p,1/w_M} \leq C \|f\|_{n,1/v_L}, \quad \forall f \in \mathcal{K}_V.
\]

Choose \(N \geq L\) such that
\[
\sup_{x, t \in \mathbb{R}, |t| \leq 1} \frac{v_L(x+t)}{v_N(x)} = \sup_{x, t \in \mathbb{R}, |t| \leq 1} \frac{1}{v_N(x+t)} \frac{1/v_L(x)}{1/v_N(x)} < \infty.
\]

Lemma 1 with \(w = 1/w_M\), \(v = 1/v_N\) and \(\tilde{v} = 1/v_L\) yields that
\[
\sup_{x \in \mathbb{R}} \frac{w_M(x)}{v_N(\varphi(x))} < \infty
\]
and (recall that \(w_M \geq 1\))
\[
\sup_{x \in \mathbb{R}} \frac{\left| \varphi^{(p)}(x) \right|}{v_N(\varphi(x))} < \infty.
\]

(iii) \(\Rightarrow\) (i): As \(V\) satisfies [1], this follows from Lemma 2.

(II) (i) \(\Rightarrow\) (ii): Since \(C_\varphi : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})\) is continuous, this follows from De Wilde’s closed graph theorem.

(ii) \(\Rightarrow\) (iii): This is a consequence of Grothendieck’s factorization theorem.

(iii) \(\Rightarrow\) (iv): Fix an arbitrary \(N \in \mathbb{N}\). Choose \(L \geq N\) such that
\[
\sup_{x, t \in \mathbb{R}, |t| \leq 1} \frac{v_N(x+t)}{v_L(x)} < \infty.
\]
Choose $K \in \mathbb{N}$ such that $C_p : \mathcal{B}_{v_L} \to \mathcal{B}_{w_K}$ is continuous. For all $m \in \mathbb{Z}_+$ there are $n \in \mathbb{Z}_+$ and $C > 0$ such that

$$\|C_p(f)\|_{m, w_K} \leq C\|f\|_{n, v_L}, \quad \forall f \in \mathcal{B}_{v_L}.$$  

Lemma 1 with $w = w_K$, $v = v_N$ and $\tilde{v} = v_L$ yields that

$$\sup_{x \in \mathbb{R}} \frac{v_N(\varphi(x))}{w_K(x)} < \infty \quad \text{(7)}$$

and (recall that $v_N \geq 1$)

$$\sup_{x \in \mathbb{R}} \frac{|\varphi'(x)|}{w_{1/m}(x)} < \infty \quad \text{and} \quad \sup_{x \in \mathbb{R}} \frac{|\varphi^{(m)}(x)|}{w_K(x)} < \infty, \quad \text{(8)}$$

Equation (7) shows (a). We now prove (b). To this end, we will make use of the following Landau-Kolmogorov type inequality due to Gornyi [8]: For all $\|\cdot\|$ denotes the sup-norm on $[-1, 1]$. Choose $M \geq K$ such that

$$\sup_{x, t \in \mathbb{R}, \|t\| \leq 1} \frac{w_K(x + t)}{w_M(x)} < \infty.$$  

Let $p, k \in \mathbb{Z}_+$ and $x \in \mathbb{R}$ be arbitrary. Equation (8) yields that for all $m \in \mathbb{Z}_+$ there is $C > 0$ such that

$$\|\varphi'(x + \cdot)\| \leq C w_{1/m}(x) \quad \text{and} \quad \|\varphi^{(m)}(x + \cdot)\| \leq C w_M(x).$$

By applying (9) to $g = \varphi'(x + \cdot)$ and $m \geq p$ such that

$$\left(1 - \frac{p - 1}{m}\right) \frac{1}{m} + \frac{p - 1}{m} \leq \frac{1}{k}$$

we find that (recall that $w_M \geq 1$)

$$|\varphi^{(p)}(x)| \leq \|\varphi^{(p)}(x + \cdot)\|
\leq C\|\varphi'(x + \cdot)\|^{1-(p-1)/m}\left(\max\{\|\varphi'(\cdot)\|, \|\varphi^{(m+1)}(\cdot)\|\}\right)^{(p-1)/m}
\leq C' w_{1/k}(x).$$

(iv) $\Rightarrow$ (i): As $W$ satisfies (1), this follows from Lemma 2.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i): Obvious.

(i) $\Rightarrow$ (iv): Fix arbitrary $p \in \mathbb{Z}_+$ and $N \in \mathbb{N}$. Choose $L \geq N$ such that
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\[
\sup_{x,t \in \mathbb{R}, |\beta| \leq 1} \frac{v_N(x + t)}{v_L(x)} < \infty.
\]

Since \( \mathcal{B}_{\nu_L} \subset \mathcal{O}_{M, W} \) and \( \mathcal{O}^p_{M, W} \subset \mathcal{O}^p_{M, W} \), we obtain that \( C_{\varphi}(\mathcal{B}_{\nu_L}) \subset \mathcal{O}^p_{M, W} \). As \( C_{\varphi} : C^\infty(\mathbb{R}) \to \mathcal{C}^p(\mathbb{R}) \) is continuous, De Wilde’s closed graph theorem implies that \( C_{\varphi} : \mathcal{B}_{\nu_L} \to \mathcal{O}^p_{M, W} \) is continuous. Grothendieck’s factorization theorem yields that there is \( M \in \mathbb{N} \) such that \( C_{\varphi} : \mathcal{B}_{\nu_L} \to \mathcal{B}_{\nu_M}^p \) is well-defined and continuous, and thus that

\[
\|C_{\varphi}(f)\|_{p, \nu_M} \leq C\|f\|_{n, \nu_L}, \quad \forall f \in \mathcal{B}_{\nu_L},
\]

for some \( n \in \mathbb{N} \) and \( C > 0 \). Lemma 1 with \( w = w_M, v = v_N \) and \( \overline{v} = v_L \) yields that

\[
\sup_{x \in \mathbb{R}} \frac{v_N(\varphi(x))}{w_M(x)} < \infty
\]

and (recall that \( v_N \geq 1 \))

\[
\sup_{x \in \mathbb{R}} \frac{|\varphi(\beta)(x)|}{w_M(x)} < \infty.
\]

(iv) \( \Rightarrow (iii) \): As \( W \) satisfies (i), this follows from Lemma 2.

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