Eigenvalues and triangles in graphs

Huiqiu Lin\textsuperscript{a} \quad Bo Ning\textsuperscript{b}\textsuperscript{†} \quad and Baoyindureng Wu\textsuperscript{c}\textsuperscript{‡}

\textbf{Abstract:} Bollobás and Nikiforov (J. Combin. Theory, Ser. B. 97 (2007) 859–865) conjectured the following. If $G$ is a $K_{r+1}$-free graph of order at least $r+1$ with $m$ edges, then $\lambda_1^2(G) + \lambda_2^2(G) \leq \frac{r-1}{r^2} m$, where $\lambda_1(G)$ and $\lambda_2(G)$ are the largest and the second largest eigenvalues of the adjacency matrix $A(G)$, respectively. In this paper, we confirm the conjecture in the case $r = 2$, by using tools from doubly stochastic matrix theory, and also characterize all families of extremal graphs. Motivated by classical theorems due to Erdős and Nosal, respectively, we prove that every non-bipartite graph $G$ of order $n$ and size $m$ contains a triangle, if one of the following is true: (1) $\lambda_1(G) \geq \sqrt{m - 1}$ and $G \neq C_5$; and (2) $\lambda_1(G) \geq \lambda_1(S(K_{\left\lfloor \frac{n-1}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil}))$ and $G \neq S(K_{\left\lfloor \frac{n-1}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil})$, where $S(K_{\left\lfloor \frac{n-1}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil})$ is obtained from $K_{\left\lfloor \frac{n-1}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil}$ by subdividing an edge. Both conditions are best possible. We conclude this paper with some open problems.

Keywords: Bollobás-Nikiforov conjecture; doubly stochastic matrix; the largest eigenvalue; the second largest eigenvalue; triangle

\section{Introduction}

It is well known that spectra of graphs can be used to describe structural properties and parameters of graphs, including cycle structures \cite{16}, maximum cuts \cite{2}, matchings and factors \cite{8}, regularity \cite{29}, diameter \cite{12} and expander properties \cite{1}, etc. On recent developments and trend, there has been extensive research in the literature (see \cite{11, 10, 24, 21, 33, 15, 22}). Referring to spectral extremal graph theory, a bulk of related work was included in the detailed survey \cite{29}, see references therein.

In this paper, we focus on spectral extremal graph theory and mainly investigate the relationship between eigenvalues of adjacency matrix of graphs and triangles. Throughout
this paper, let $G$ be a graph with order $v(G) := n$, size $e(G) := m$ and clique number $\omega(G) := \omega$. Let $A(G)$ be its adjacency matrix. The eigenvalues $\lambda_1(G) := \lambda_1 \geq \lambda_2(G) := \lambda_2 \geq \cdots \geq \lambda_n(G)$ of $A(G)$ are called the eigenvalues of $G$.

The study of bounding spectral radius in terms of some parameters of graphs has a rich history. Starting from 1985, Brualdi and Hoffman [9] proved that $\lambda_1 \leq k - 1$ if $m \leq \binom{k}{2}$ for some integer $k \geq 1$. This result was extended by Stanley [32] who showed that $\lambda_1 \leq \frac{1}{\sqrt{2}}(\sqrt{8m + 1} - 1)$. The bound is best possible for complete graphs (possibly with isolated vertices), but still can be improved for special classes of graphs, such as triangle-free graphs (see Nosal [20]). For further generalizations and related extensions of Stanley’s result, see Hong [19], Hong et al. [20], Nikiforov [26] and Zhou et al. [37]. Specific to bounding spectral radius of a graph in terms of clique number, Wilf [34] showed that $\lambda_1 \leq \omega - 1$. A better inequality $\lambda_1 \leq \sqrt{\frac{2(\omega - 1)m}{\omega}}$, implicitly conjectured by Edwards and Elphik [13], was confirmed by Nikiforov in [26], with the help of Motzkin–Straus technique [25]. Later, the extremal graphs when the equality holds were characterized in [27]. By the inequality $\lambda_1 \geq \frac{2m}{\omega}$, one can easily deduce the concise form of Turán’s theorem that $m \leq \frac{\omega - 1}{2\omega}n^2$ from Nikiforov’s inequality. So, Nikiforov’s inequality sometimes is called spectral Turán theorem.

In 2007, Bollobás and Nikiforov [6] posed the following conjecture. This nice conjecture is the original motivation of our article.

Conjecture 1. ([6, Conjecture 1]) Let $G$ be a $K_{r+1}$-free graph of order at least $r + 1$ with $m$ edges. Then

$$\lambda_1^2 + \lambda_2^2 \leq \frac{r - 1}{r}2m.$$

To the knowledge of us, the conjecture is still open now. In this paper, we make the first progress on this conjecture. We solve the case $r = 2$ by using tools from doubly stochastic matrix theory, and also characterize all extremal graphs.

Let $G$ be a graph. A “blow-up” of $G$ is defined to be a new graph obtained by replacing each vertex $x \in V(G)$ by an independent set $I_x$, in which for any two vertices $x, y \in V(G)$, we add all edges between $I_x$ and $I_y$ if $xy \in E(G)$.

Theorem 1.1. Let $G$ be a triangle-free graph of order at least 3 with $m$ edges. Then

$$\lambda_1^2 + \lambda_2^2 \leq m,$$

where the equality holds if and only if $G$ is a blow-up of some member of $G$, in which $G = \{ P_2 \cup K_1, 2P_2 \cup K_1, P_4 \cup K_1, P_5 \cup K_1 \}$.

Recall that a quintessential result in extremal graph theory is Mantel’s theorem, which maximizes the number of edges over all triangle-free graphs. This important result was
improved by Erdős [7, Ex.12.2.7] in the following form: Every non-bipartite triangle-free graph of order \( n \) satisfies that \( m \leq \frac{(n-1)^2}{4} + 1 \). Notice that a subdivision of \( K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil} \) on one edge shows the bound above is tight. As a counterpart of Mantel’s theorem, it was shown by Nosal [30] that every triangle-free graph \( G \) on \( m \) edges satisfies that \( \lambda_1(G) \leq \sqrt{m} \).

In this direction, we shall prove two spectral versions of Erdős’ theorem.

Our two results are the following.

**Theorem 1.2.** Let \( G \) be a non-bipartite graph with size \( m \). If \( \lambda_1 \geq \sqrt{m-1} \), then \( G \) contains a triangle unless \( G \cong C_5 \).

**Theorem 1.3.** Let \( G \) be a non-bipartite graph with order \( n \). If \( \lambda_1 \geq \lambda_1(S(K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil})) \) where \( S(K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}) \) denotes a subdivision of \( K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil} \) on one edge, then \( G \) contains a triangle unless \( G \cong S(K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}) \).

### 2 The Bollobás-Nikiforov conjecture for triangle-free graphs

In Section 2, we introduce necessary preliminaries for doubly stochastic matrix theory, and then prove Theorem 1.1. For more details on the related theory, we refer the reader to Zhan [36].

A square nonnegative matrix is called **doubly stochastic** if the sum of the entries in every row and every column is 1. A nonnegative square matrix is called **doubly substochastic** if the sum of the entries in every row and every column is less than or equal to 1. A square matrix is called a **weak permutation matrix** if every row and every column has at most one nonzero entry and all the nonzero entries (if existing) are 1.

We will use the definition of “a vector is weakly majorized by the other one”. We rearrange the components of \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) in decreasing order as \( x[1] \geq x[2] \geq \cdots \geq x[n] \).

**Definition.** Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \). If

\[
\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i], \quad k = 1, 2, \ldots, n,
\]

then we say that \( x \) is **weakly majorized by** \( y \) and denote it by \( x \prec_w y \). If \( x \prec_w y \) and \( \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \), then we say that \( x \) is **majorized by** \( y \) and denote it by \( x \prec y \).

**Lemma 1.** ([36 Lemma 3.24]) Let \( x, y \in \mathbb{R}_+^n = \{x \in \mathbb{R}^n | x = (x_1, \ldots, x_n), x_i \geq 0, 1 \leq i \leq n \} \). Then \( x \prec_w y \) if and only if there exists a doubly substochastic matrix \( A \) such that \( x = Ay \).
One of the main ingredients in our proof is investigating the relationship between doubly (sub)stochastic matrix and (weak)-permutation matrix.

**Lemma 2.** ([33] Theorem 3.22) Every doubly substochastic matrix is a convex combination of weak-permutation matrices.

The following theorem will play an essential role in our proof of Theorem 1.1.

**Theorem 2.1.** Let \( x = (x_1, x_2, \ldots, x_n)^T \) and \( y = (y_1, y_2, \ldots, y_n)^T \) be two \( n \)-vectors with \( x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \) and \( y_1 \geq y_2 \geq \cdots \geq y_n \geq 0 \). If \( y \prec_w x \), then \( \|x\|_p \geq \|y\|_p \) for every real number \( p > 1 \) with equality holding if and only if \( x = y \).

**Proof.** Since \( y \prec_w x \), there exists a doubly substochastic matrix \( A \) such that \( y = Ax \) by Lemma 1. By Lemma 2, there are weak-permutation matrices \( P_i \) for \( i = 1, \ldots, n \), such that \( A = \sum_{i=1}^n a_i P_i \) with \( \sum_{i=1}^n a_i = 1 \). Thus, we have \( y = Ax = (\sum_{i=1}^n a_i P_i)x = \sum_{i=1}^n a_i(P_ix) \).

So, \( \|y\|_p = \|\sum_{i=1}^n a_i(P_ix)\|_p \leq \sum_{i=1}^n a_i\|P_ix\|_p \leq \sum_{i=1}^n a_i\|x\|_p = (\sum_{i=1}^n a_i)\|x\|_p = \|x\|_p \).

If \( \|x\|_p = \|y\|_p \), then \( \|\sum_{i=1}^n a_i(P_ix)\|_p = \sum_{i=1}^n a_i\|P_ix\|_p = \sum_{i=1}^n a_i\|x\|_p \). The first equality holding implies that for any pair of integers \( i, j \) with \( i \neq j \), there exists a real number \( \alpha \) (related to \( i, j \)), such that \( P_ix = \alpha P_jx \). By the second equality, we have \( \|P_ix\|_p = \|P_jx\|_p \). Then \( \alpha = 1 \) and \( P_ix = P_jx \). Thus, \( P_ix = P_1x \) for any integer \( i \in [1, n] \), and it follows that \( y = \sum_{i=1}^n a_i(P_ix) = P_1x \).

Since \( y = P_1x \), \( P_1 \) is a weak-permutation matrix and \( \|x\|_p = \|y\|_p \). We can see that the number of 0-entries in \( x \) equals to those of \( y \). Furthermore, since both \( (x_1, x_2, \ldots, x_n) \) and \( (y_1, y_2, \ldots, y_n) \) are non-increase sequences, we have \( y = x \). The proof is complete. \( \square \)

For a graph \( G \), define the rank of \( G \) to be the rank of \( A(G) \), and denote it by \( \text{rank}(G) \).

We need Theorem 3.3 and Theorem 4.3 in [31] to characterize the extremal graphs in Theorem 1.1 and list them as a lemma below.

**Lemma 3.** ([31]) Let \( G \) be a graph with order \( n \). Then we have the following statements.

(I) If \( \text{rank}(G) = 2 \), then \( G \) is a blow-up of \( P_2 \cup K_1 \).

(II) If \( G \) is a bipartite graph with \( \text{rank}(G) = 4 \), then \( G \) is a blow-up of \( 2P_2 \cup K_1 \) or \( P_4 \cup K_1 \) or \( P_5 \cup K_1 \).

We shall give a proof of Theorem 1.1. We introduce the definition of “the inertia of a graph” as the ordered triple \((n^+, n^-, n^0)\), where \( n^+ \), \( n^- \) and \( n^0 \) are the numbers (counting multiplicities) of positive, negative and zero eigenvalues of the adjacency matrix \( A(G) \) respectively.

**Proof of Theorem 1.1.** Let \( n \) be the order of \( G \) and \((n^+, n^-, n^0)\) be the inertia of \( G \). Set \( s^+ = \lambda_1^2 + \cdots + \lambda_n^2 \) and \( s^- = \lambda_1^2 - n^+ + \cdots + \lambda_n^2 \). Since \( G \) is triangle-free, we have \( G \not\cong K_n \), and so \( \lambda_2(G) \geq 0 \) (see Lemma 5 in [18]).
Suppose that $\lambda_1^2 + \lambda_2^2 > m$. Since $s^+ + s^- = 2m$ and $\lambda_1^2 + \lambda_2^2 > \frac{s^+ + s^-}{2}$, we have $\lambda_1^2 + \lambda_2^2 \geq 2(\lambda_1^2 + \lambda_2^2) - s^+ > s^- \geq 0$. Now, we construct two $n$-vectors $x$ and $y$ such that $x = (\lambda_1^2, \lambda_2^2, 0, \ldots, 0)^T$ and $y = (\lambda_n^2, \lambda_{n-1}^2, \ldots, \lambda_{n-n-1}^2)^T$. Since $\lambda_1^2 + \lambda_2^2 > s^-$, we have $y \succeq_w x$ and $x \neq y$. Set $p = \frac{\lambda_1^2}{\lambda_2^2}$. By Theorem 2.1, we have $\|x\|\frac{3}{2} > \|y\|\frac{3}{2}$, that is, $\lambda_1^2 + \lambda_2^2 > |\lambda_n|^3 + |\lambda_{n-1}|^3 + \cdots + |\lambda_{n-n-1}|^3$. It implies that $t(G) = \frac{\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2}{6} > \frac{\lambda_1^2 + \lambda_2^2 + \lambda_{n-n-1}^2 + \cdots + \lambda_n^2}{6} > 0$.

This gives us a contradiction. Thus, we proved $\lambda_1^2 + \lambda_2^2 \leq m$.

If $\lambda_1^2 + \lambda_2^2 = m$, then $\lambda_1^2 + \lambda_2^2 \geq s^- \geq 0$ and $\|x\|\frac{3}{2} = \|y\|\frac{3}{2}$. By Theorem 2.1, $x = y$. It follows that $\lambda_1^2 = \lambda_2^2$ and $\lambda_n^2 = \lambda_{n-1}^2$, and the other eigenvalues are all 0. Note that $\lambda_1 + \lambda_2 + \lambda_{n-1} + \lambda_n = 0$. Then $\lambda_1 = -\lambda_n$, which implies that $G$ is bipartite and $\lambda_2 = -\lambda_{n-1}$.

If $\lambda_2 = 0$, then $\text{rank}(G) = 2$. By Lemma 3 (I), $G$ is a blow-up of $P_2 \cup K_1$. If $\lambda_2 \neq 0$, then $\text{rank}(G) = 4$. By Lemma 3 (II), $G$ is a blow-up of $2P_2 \cup K_1$ or $P_4 \cup K_1$ or $P_5 \cup K_1$. The proof is complete.

By the proof of Theorem 1.1, we can deduce the following as a corollary.

**Theorem 2.2.** ([27] Theorem 2(i)] Let $G$ be a graph of size $m$. If $\lambda_1^2 \geq m$, then $G$ contains a triangle, unless $G$ is a blow up of $P_2 \cup K_1$.

![Figure 1: The graphs $H_1$, $H_2$ and $H_3$.](image)

### 3 Proofs of Theorems 1.2 and 1.3

A walk $v_1v_2\cdots v_k$ ($k \geq 2$) in a graph $G$ is called an internal path, if these $k$ vertices are distinct (except possibly $v_1 = v_k$), $d_G(v_1) > 2$, $d_G(v_k) > 2$ and $d_G(v_2) = \cdots = d_G(v_k-1) = 2$ (unless $k = 2$). We denote by $G_{uv}$ the graph obtained from $G$ by subdividing the edge $uv$, that is, introducing a new vertex on the edge $uv$. Let $Y_n$ be the graph obtained from a path $v_1v_2\cdots v_{n-4}$ by attaching two pendant vertices to $v_1$ and another two to $v_{n-4}$. Hoffman and Smith [17] proved the following result, which is used towards the structure of extremal graphs in Theorem 1.2.

**Lemma 4.** ([17]) Let $G$ be a connected graph with $uv \in E(G)$. If $uv$ belongs to an internal path of $G$ and $G \not\cong Y_n$, then $\lambda_1(G_{uv}) < \lambda_1(G)$. 

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Now we shall prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. By contradiction. Suppose $G$ contains no triangle. Assume that $\lambda_2 \geq 1$. Then $\lambda_1^2 + \lambda_2^2 \geq m$. Since $G$ is non-bipartite, by Theorem 1.1 $G$ contains a triangle, a contradiction. Now assume that $\lambda_2 < 1$. Let the length of a smallest odd cycle be $s \geq 5$. Note that $\lambda_2(C_s) = 2 \cos \frac{\pi}{s}$. By Cauchy’s interlacing theorem, $G$ does not contain an induced odd cycle with length greater than 5, and so $s = 5$. Let $S \subseteq V(G)$ with $G[S] = u_1u_2u_3u_4u_5u_1$. If $n = 5$, then $G \cong C_5$, thus we are done. Let $V_1 = V(G) \setminus S$ and $T = N(S) \setminus S$.

We shall use the property that: $G$ contains no $H_i$ as an induced subgraph since $\lambda_2(H_i) = 1 > \lambda_2(G)$, where $i = 1, 2, 3$ (recall $H_i$ in Fig. 1). In the following, we say that $G$ is $H$-free if it contains no $H$ as an induced subgraph.

We first observe that $d_S(v) = 2$ for each $v \in T$ and $V(G) = S \cup T$, since $G$ is triangle-free and $H_1$-free. Recall that $n \geq 6$ and $G$ is triangle-free. Choose $v \in T$. Without loss of generality, assume $N_S(v) = \{u_1, u_3\}$. If $n = 6$, then by a simple calculation, $\lambda_1(G) = 2.3914 < 2.4 < \sqrt{6}$, a contradiction. So $n \geq 7$. Let $w \in T \setminus \{v\}$. If $N_S(w) = N_S(v)$ then $uv \notin E(G)$; if $N_S(w) \neq N_S(v)$, then $N_S(w) \cap N_S(v) = \emptyset$ since $G$ is $H_2$-free and triangle-free. Thus, $N_S(w) = \{u_1, u_3\}$, $N_S(w) = \{u_2, u_4\}$, or $N_S(w) = \{u_2, u_3\}$ and $uv \in E(G)$ since $G$ is $H_2$-free. Indeed, if $N_S(w) = \{u_2, u_4\}$, then every vertex in $T$ is adjacent to $u_1, u_3$, or to $u_2, u_4$. Without loss of generality, we assume $N_S(w) = \{u_2, u_4\}$. Let $T_1 = N_G(\{u_1, u_3\})$ and $T_2 = N_G(\{u_2, u_4\})$. Obviously, both $T_1$ and $T_2$ are independent sets, and $xy \in E(T_1, T_2)$ for each $x \in T_1$ and $y \in T_2$. Let $|T_1| = a$ and $|T_2| = b$. Then $m = ab + a + b + 2 = (a + 1)(b + 1) + 1$, and $G$ is a subdivision of $K_{a+1,b+1}$ on some edge. By Lemma 1, $\lambda_1(G) < \lambda_1(K_{a+1,b+1}) = \sqrt{(a + 1)(b + 1)} = \sqrt{e(G)} - 1$, a contradiction.

The proof is complete. □

Proof of Theorem 1.3. Suppose that $G$ is a non-bipartite triangle-free graph of order $n$ with the maximal spectral radius. In the following, we shall show $G = ST_2(n)$. First we claim that $G$ is connected; since otherwise we can add a new edge between a component with the maximal spectral radius and any other component to get a new graph with larger spectral radius. We also observe that adding any new edge gives us a triangle.

Let $x = (x_1, \ldots, x_n)^t$ be the Perron vector of $G$ and $u$ be a vertex of $G$ with $x_u = \max\{x_i| i = 1, \ldots, n\}$. Let $C = u_1u_2\cdots u_ku_1$ be a shortest odd cycle of $G$ with $k \geq 5$. We have the following claims.

Claim 1. For each vertex $w \in V(G) \setminus (N(u) \cup V(C))$, $N(w) = N(u)$.

Proof. If $V(G) \setminus (N(u) \cup V(C)) = \emptyset$, then there is nothing to prove. So we assume $V(G) \setminus (N(u) \cup V(C)) \neq \emptyset$. Suppose Claim 1 is false. Let $w \in V(G) \setminus (N(u) \cup V(C))$
Proof. Suppose that there exist two nonadjacent vertices \( x \) in \( G \) and \( y \) in \( d \) such that \( xy \) is a path of length 2. Then \( N(x) \cap N(y) \neq \emptyset \), and hence \( \lambda_1(G) < \lambda_1(G') \), a contradiction. This proves the claim.

Claim 2. For any \( s, t \in N(u) \setminus V(C) \), we have \( N(s) \cap V(C) = N(t) \cap V(C) \).

Proof. Let \( X = \{ v \mid N(v) = N(u), v \in V(G - C) \} \). Let \( p \in N(u) \setminus V(C) \) such that \( x_p = \max \{ x_v \mid v \in N(u) \} \). Note that \( N(u) \) is an independent set. Then there holds \( \lambda_1(G)x_i = \sum_{v \in X} x_v + \sum_{v \in N(i) \cap V(C)} x_v \) for any \( i \in N(u) \). This implies that \( \sum_{v \in N(p) \cap V(C)} x_v \geq \sum_{v \in N(i) \cap V(C)} x_v \) for any \( i \in N(u) \). If there exists a vertex \( j \in N(u) \setminus V(C) \) such that \( N(p) \cap V(C) \neq N(j) \cap V(C) \). Let \( G' = G - \{ j \} \cap V(C) \} \). Then \( \lambda_1(G') - \lambda_1(G) \geq x_1(A(G') - A(G))x \geq 2x_j \sum_{v \in N(p) \cap V(C)} x_v - \sum_{v \in N(j) \cap V(C)} x_v \geq 0. \)

Obviously, \( G' \) is connected, non-bipartite and triangle-free. Similar to the proof of Claim 1, we have \( \lambda_1(G') > \lambda_1(G) \), a contradiction. Thus, for any \( i \in N(u) \setminus V(C) \), \( N(i) \cap V(C) = N(p) \cap V(C) \). This proves Claim 2.

Claim 3. For any two vertices \( x, y \in V(G) \), the distance between \( x \) and \( y \) in \( G \), denoted by \( d_G(x, y) \), satisfies that \( d_G(x, y) \leq 2 \).

Proof. Suppose that there exist two nonadjacent vertices \( x, y \in V(G) \) such that \( xy \notin E(G) \) and \( d_G(x, y) \geq 3 \). Let \( P = v_0v_1v_2 \ldots v_l \) be a shortest \( (x, y) \)-path in \( G \), where \( v_0 = x \) and \( v_l = y \). Thus, \( l \geq 3 \). Since \( G + xy \) is not bipartite and \( \lambda_1(G + xy) > \lambda_1(G) \), there is a triangle passing through the edge \( xy \) in \( G + xy \). That is, there is an \( (x, y) \)-path of length 2 in \( G \), a contradiction. This proves the claim.

Claim 4. \( k = 5 \).

Proof. Suppose to the contrary that \( k \geq 7 \). Since \( C \) is chordless, \( u_1u_4 \notin E(G) \). By Claim 3 \( d_G(u_1, u_4) = 2 \). This means that there exists a vertex outside \( C \), say \( u \), such that \( u_1v_4 \) is a path of length 2. Then \( u_1v_4u_3u_2u_1 \) is a cycle of length 5, a contradiction. This proves the claim.

By Claims 1 and 2 we can determine the structure of \( G - C \). Indeed, if \( V(G) \setminus (N(u) \cup V(C)) \neq \emptyset \), then \( G - C = B(X, Y) \cong K_{s,t} \), where \( |X| = s, |Y| = t \) and \( t \geq s \geq 1 \). We can see any two vertices in \( X \) have the same neighbor(neighbors) of \( C \), and any two vertices in \( Y \) also have the same neighbor(neighbors) of \( C \).
Claim 5. By symmetry, $N_C(X) = \{u_i, u_{i+2}\}$, $N_C(Y) = \{u_j, u_{j+2}\}$ and $N_C(X) \cap N_C(Y) = \emptyset$.

Proof. We first observe that for any vertex $v \in V(G \setminus C)$, $|N(v) \cap V(C)| \leq 2$, since $G$ is triangle-free. Next, we shall show that $d_C(X) = d_C(Y) = 2$. Since $G$ is connected, we have $d_C(X) \geq 1$ or $d_C(Y) \geq 1$. Assume that $d_C(X) = 1$ and set $N_C(X) = \{u_i\}$. If either $u_{i+2}$ or $u_{i-2}$ does not belong to $N_C(Y)$, then we connect such a vertex to all vertices in $X$, and create a new graph $G'$. Note that $G'$ is non-bipartite and triangle-free but with larger spectral radius, a contradiction. Hence $u_{i+2}, u_{i-2} \in N_C(Y)$. Recall that $C$ is a 5-cycle. So there is a triangle in $G$, a contradiction. So, $d_C(X) = 2$, and by symmetry, $N_C(X) = \{u_i, u_{i+2}\}$. The other assertion can be proved similarly. □

Without loss of generality, by Claim 5 and the symmetry, we can assume that $N_C(X) = \{u_1, u_3\}$ and $N_C(Y) = \{u_2, u_4\}$. By Claim 2, $C$ is an induced 5-cycle. By Claims 1 and 2, all vertices of $X$ are adjacent to $\{u_1, u_3\}$, and all vertices of $Y$ are adjacent to $\{u_2, u_4\}$. Furthermore, $u_5$ is a vertex of degree 2 in $G$. Observe that $G = S(K_{s+2, t+2})$.

By Proposition 2 and the choice of $G$, $G = S(K_{\lceil \frac{n+1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor})$.

Finally, we consider the case of $V(G) = N(u) \cup V(C)$. Thus, $u \in V(C)$. Set $X = N(u) - C$ and $u = u_1$. If $X = \emptyset$, then $G$ is an induced 5-cycle and it is true. Thus, assume that $X \neq \emptyset$ and $|X| = n - 5 \geq 1$. This implies $n \geq 6$. Similarly, we can find $\lambda_1(G) \leq \lambda_1(S(K_{n-3,2}))$, where the equality holds if and only if $G = K_{n-3,2}$. By Proposition 2 (see Appendix), $\lambda_1(S(K_{n-3,2})) \leq \lambda_1(S(K_{\lceil \frac{n+1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}))$ where the equality holds if and only if $n = 6$.

The proof is complete. □

4 Concluding remarks

In this paper, we consider the Bollobás-Nikiforov conjecture on the largest eigenvalue, the second largest eigenvalue and size of a graph, and settle the conjecture for triangle-free graphs, improving the spectral version of Mantel’s theorem. We also prove two spectral analogs of Erdős’ theorem. Many intriguing problems with respect to this topic remain open.

- One can prove the following extension of Erdős’ theorem: Let $G$ be a graph with order $n$ and the length of odd girth at least $2k + 3$. If $G$ is non-bipartite, then $\epsilon(G) \leq \left(\frac{n-(2k-1)}{2}\right)^2 + 2k - 1$. The following question naturally arises: Which class of graphs can attain the maximum spectral radius among the class of graphs above?
From Theorem 1.3, we can see $S(K_{\frac{n-1}{2}, \frac{n-1}{2}})$ is the answer for this problem when $k = 1$.

- Nosal [30] proved that every graph $G$ of size $m$ satisfying $\lambda_1(G) > \sqrt{m}$ contains a triangle. Nikiforov [28, Theorem 2] proved that every graph $G$ of size $m \geq 9$ contains a $C_4$ if $\lambda_1(G) > \sqrt{m}$. Let $k$ and $m$ be two integers such that $k|m$ and $k$ is odd. Let $S_{\frac{m}{k} + \frac{k+1}{2}, k}$ be the graph obtained by joining each vertex of $K_k$ to $\frac{m}{k} - \frac{k-1}{2}$ isolated vertices. Zhai, Lin and Shu conjectured a more general one.

**Conjecture 2** ([35]). Let $G$ be a graph of sufficiently large size $m$ without isolated vertices and $k \geq 1$ be an integer. If $\lambda_1(G) \geq \lambda_1(S_{\frac{m}{k} + \frac{k+1}{2}, k})$, then $G$ contains $C_t$ for every $t \leq 2k + 2$, unless $G = S_{\frac{m}{k} + \frac{k+1}{2}, k}$.

When $k = 1$, it includes Nosal’s theorem [30] and Nikiforov’s theorem [28, Theorem 2] as two special cases.

- By replacing $\lambda_1$ by $s^+$, Elphick et al. refined and generalized many classical results on spectral graph theory. For example, Hong’s famous theorem states that $\lambda_1 \leq \sqrt{2m - n + 1}$ holds for all connected graphs. (It also holds for all graphs without isolated vertices). Elphick, Farber, Goldberg and Wocjan conjectured that

**Conjecture 3** ([14]). Let $G$ be a connected graph of order $n$. Then $\min\{s^+, s^-\} \geq n - 1$.

Motivated by Conjecture 3, we can reconsider Conjecture 1 in a general form.

**Problem 1.** Let $H$ be a given graph and $G$ be an $H$-free graph of size $m$. How to estimate the upper bound of $s^+$ in terms of $H$ and $m$?

This problem includes Conjecture 1 as a special case when $H = K_{r+1}$. For graphs with given chromatic number, Ando and Lin studied a related problem in [3].

5  **Appendix**

In this section, we shall prove some proposition which is used in the proof of Theorem 1.3. Since the proof is by pure computation, we write it here.

**Proposition 1.** If $t \geq s \geq 1$, then $\lambda_1(S(K_{s+2, t+2})) > \lambda_1(S(K_{s+1, t+3}))$.

**Proof.** Set $G_1 = S(K_{s+2, t+2})$ and $G_2 = S(K_{s+1, t+3})$. Since both $G_1$ and $G_2$ contain $C_5$ as a proper subgraph, we have $\lambda_1(G_i) > 2$ for $i = 1, 2$. The characteristic polynomial of $G_1$
is $P_{G_1}(x) = x^{n+4} (x^{5} - (2s + 2t + st + 5)x^{3} + (4s + 4t + 3st + 5)x - 2s - 2t - 2st - 2)$, where $n = s + t + 5$. Let $f(x, s, t) = x^{5} - (2s + 2t + st + 5)x^{3} + (4s + 4t + 3st + 5)x - 2s - 2t - 2st - 2$. Then $\lambda_1(G_1)$ is the largest root of $f(x, s, t) = 0$. Note that $f(x, s - 1, t + 1) - f(x, s, t) = (x - 1)^2(x + 2)(t - s + 1)$. Since $t \geq s$, we have $f(x, s - 1, t + 1) - f(x, s, t) > 0$ when $x > 1$. Moreover, since $\lambda_1(G_2)$ is the largest root of $f(x, s - 1, t + 1) = 0$, it follows that $\lambda_1(G_1) > \lambda_1(G_2)$.

Proposition 1 implies that

**Proposition 2.** If $t \geq s \geq 1$ and $s + t = n - 5$, then $\lambda_1(S([\frac{n-1}{2}],[\frac{n-1}{2}])) \geq \lambda_1(S(K_{s+2,t+2}))$, where the equality holds if and only if $(s,t) = ([\frac{n-1}{2}],[\frac{n-1}{2}])$.

We also include a proof of the result mentioned in the last section here. The proof uses a result due to Andrásfai, Erdős and Sós to control the minimum degree.

**Lemma 5.** (III) Let $G$ be a graph with order $n$ and the length of odd girth at least $2k + 1$. If the minimum degree $\delta(G) > \frac{2n}{2k+1}$, then $G$ is bipartite.

**Theorem 5.1.** Let $G$ be a graph with order $n$ and the length of odd girth at least $2k + 3$. If $G$ is non-bipartite, then $e(G) \leq \left(\frac{n-(2k-1)}{2}\right)^2 + 2k - 1$.

**Proof of Theorem 5.1.** We prove Theorem 5.1 by induction on $n$. If $n = 2k + 3$, then the length of odd girth is $2k + 3$, and hence $G \cong C_{2k+3}$ and $e(G) = 2k + 3$. The result holds. Now we assume $n \geq 2k + 4$ and the result holds for graphs with order less than $n - 1$.

First suppose $\delta(G) > \frac{n-(2k-1)}{2} - \frac{1}{4}$, i.e., $\delta(G) \geq \frac{n-(2k-1)}{2}$. Since $n > 2k + 3$, we have $(2k - 1)n > (2k - 1)(2k + 3)$, which implies that $\frac{n-(2k-1)}{2} > \frac{2n}{2k+3}$. By Lemma 3, $G$ is bipartite, a contradiction. Thus, in the following we can assume $\delta(G) \leq \frac{n-(2k-1)}{2} - \frac{1}{4}$. Let $v$ be a vertex with $d(v) = \delta(G)$ and let $G' = G - \{v\}$.

If $G'$ is non-bipartite, then by the hypothesis, we have $e(G') \leq \left(\frac{n-1-(2k-1)}{2}\right)^2 + 2k - 1 = \left(\frac{n-(2k-1)}{2}\right)^2 + 2k - 1 - \frac{n-(2k-1)}{2} + \frac{1}{4}$. It follows $e(G) \leq \left(\frac{n-(2k-1)}{2}\right)^2 + 2k - 1$. Thus, $G'$ is bipartite. Then every odd cycle passes through $v$ in $G$. Choose $C$ as a shortest odd cycle of $G$, where $|C| \geq 2k + 3$. Let $G' = B(X, Y)$, where $(X, Y)$ is the bipartition of $G'$. Let
$X_0 = X \cap (V(C) - \{v\})$, $Y_0 = Y \cap (V(C) - \{v\})$, $X_1 = X - X_0$ and $Y_1 = Y - Y_0$. Then

$$e(G) = e(G[C]) + |E(C,G[X_1,Y_1])| + |E(G[X_1,Y_1])|$$

$$\leq |C| + 2|X_1| + 2|Y_1| + |X_1||Y_1|$$

$$= |C| + (|X_1| + 2)(|Y_1| + 2) - 4$$

$$\leq |C| + \left(\frac{|X_1| + 2 + |Y_1| + 2}{2}\right)^2 - 4$$

$$= |C| + \left(\frac{n - |C| + 4}{2}\right)^2 - 4.$$ 

Let $f(x) = x + (\frac{n-x+4}{2})^2 - 4$. Then $f'(x) = 1 - (n - x + 4) = x - (n + 3)$. Hence for $2k + 3 \leq x \leq n - 1$, we have $f(x) \leq f(2k + 3) = \left(\frac{n-(2k-1)}{2}\right)^2 + 2k - 1$. The proof is complete. 

\[\square\]

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