On Algebraic Multi-Group Spaces

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Abstract: A Smarandache multi-space is a union of $n$ spaces $A_1, A_2, \cdots, A_n$ with some additional conditions holding. Combining classical of a group with Smarandache multi-spaces, the conception of a multi-group space is introduced in this paper, which is a generalization of the classical algebraic structures, such as the group, filed, body, $\cdots$, etc.. Similar to groups, some characteristics of a multi-group space are obtained in this paper.

Key words: multi-space, group, multi-group space, theorem.

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1. Introduction

The notion of multi-spaces is introduced by Smarandache in [5] under his idea of hybrid mathematics: combining different fields into a unifying field([6]). Today, this idea is widely accepted by the world of sciences. For mathematics, definite or exact solution under a given condition is not the only object for mathematician. New creation power has emerged and new era for the mathematics has come now.

A Smarandache multi-space is defined by

Definition 1.1. For any integer $i, 1 \leq i \leq n$ let $A_i$ be a set with ensemble of law $L_i$, and the intersection of $k$ sets $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ of them constrains the law $I(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$. Then the union of $A_i, 1 \leq i \leq n$

$$\tilde{A} = \bigcup_{i=1}^{n} A_i$$

is called a multi-space.

The conception of multi-group space is a generalization of the classical algebraic structures, such as the group, filed, body, $\cdots$, etc., which is defined as follows.

Definition 1.2 Let $\tilde{G} = \bigcup_{i=1}^{n} G_i$ be a complete multi-space with a binary operation set $O(\tilde{G}) = \{\times_i, 1 \leq i \leq n\}$. If for any integer $i, 1 \leq i \leq n$, $(G_i; \times_i)$ is a group and for $\forall x, y, z \in \tilde{G}$ and any two binary operations $\times$ and $\circ$, $\times \neq \circ$, there is one operation,
for example the operation $\times$ satisfying the distribution law to the operation $\circ$ if their
operation results exist, i.e.,

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$

$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\tilde{G}$ is called a multi-group space.

**Remark:** The following special cases convince us that multi-group spaces are
generalization of group, field and body, · · ·, etc..

(i) If $n = 1$, then $\tilde{G} = (G_1; \times_1)$ is just a group.

(ii) If $n = 2$, $G_1 = G_2 = \tilde{G}$, Then $\tilde{G}$ is a body. If $(G_1; \times_1)$ and $(G_2; \times_2)$ are
commutative groups, then $\tilde{G}$ is a field.

Notice that in [7][8] various bispaces, such as bigroup, bisemigroup, biquasigroup,
biloop, bigroupoid, biring, bisemiring, bivector, bisemivector, binary-ring, · · ·, etc.,
consider two operation on two different sets are introduced.

2. Characteristics of multi-group spaces

For a multi-group space $\tilde{G}$ and a subset $\tilde{G}_1 \subset \tilde{G}$, if $\tilde{G}_1$ is also a multi-group
space under a subset $O(\tilde{G}_1), O(\tilde{G}_1) \subset O(\tilde{G})$, then $\tilde{G}$ is called a multi-group subspace,
denoted by $\tilde{G}_1 \preceq \tilde{G}$. We have the following criterion for the multi-group subspaces.

**Theorem 2.1** For a multi-group space $\tilde{G} = \bigcup_{i=1}^{n} G_i$ with an operation set $O(\tilde{G}) = \{\times_i | 1 \leq i \leq n\}$, a subset $\tilde{G}_1 \subset \tilde{G}$ is a multi-group subspace if and only if for any
integer $k, 1 \leq k \leq n$, $(\tilde{G}_1 \cap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$ or $\tilde{G}_1 \cap G_k = \emptyset$.

**Proof** If $\tilde{G}_1$ is a multi-group space with the operation set $O(\tilde{G}_1) = \{\times_{i_j} | 1 \leq j \leq s\} \subset O(\tilde{G})$, then

$$\tilde{G}_1 = \bigcup_{i=1}^{n} (\tilde{G}_1 \cap G_i) = \bigcup_{j=1}^{s} G'_{i_j}$$

where $G'_{i_j} \preceq G_{i_j}$ and $(G_{i_j}; \times_{i_j})$ is a group. Whence, if $\tilde{G}_1 \cap G_k \neq \emptyset$, then there exist
an integer $l, k = i_l$ such that $\tilde{G}_1 \cap G_k = G'_{i_l}$, i.e., $(\tilde{G}_1 \cap G_k; \times_k)$ is a subgroup of
$(G_k; \times_k)$.

Now if for any integer $k$, $(\tilde{G}_1 \cap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$ or $\tilde{G}_1 \cap G_k = \emptyset$,
let $N$ denote the index set $k$ with $\tilde{G}_1 \cap G_k \neq \emptyset$. Then

$$\tilde{G}_1 = \bigcup_{j \in N} (\tilde{G}_1 \cap G_j).$$
and \((\tilde{G}_1 \cap G_j, \times_j)\) is a group. Since \(\tilde{G}_1 \subset \tilde{G}\) and \(O(\tilde{G}_1) \subset O(\tilde{G})\), the associative law and distribute law are true for the \(\tilde{G}_1\). Therefore, \(\tilde{G}_1\) is a multi-group subspace of \(\tilde{G}\). \(\blacksquare\)

For a finite multi-group subspace, we have the following criterion.

**Theorem 2.2** Let \(\tilde{G}\) be a finite multi-group space with an operation set \(O(\tilde{G}) = \{\times_i | 1 \leq i \leq n\}\). A subset \(\tilde{G}_1\) of \(\tilde{G}\) is a multi-group subspace under an operation subset \(O(\tilde{G}_1) \subset O(\tilde{G})\) if and only if for each operation \(\times\) in \(O(\tilde{G}_1)\), \((\tilde{G}_1; \times)\) is complete.

**Proof** Notice that for a multi-group space \(\tilde{G}\), its each multi-group subspace \(\tilde{G}_1\) is complete.

Now if \(\tilde{G}_1\) is a complete set under each operation \(\times_i\) in \(O(\tilde{G}_1)\), we know that \((\tilde{G}_1 \cap G_i; \times_i)\) is a group (see also [9]) or an empty set. Whence, we get that

\[
\tilde{G}_1 = \bigcup_{i=1}^{n} (\tilde{G}_1 \cap G_i).
\]

Therefore, \(\tilde{G}_1\) is a multi-group subspace of \(\tilde{G}\) under the operation set \(O(\tilde{G}_1)\). \(\blacksquare\)

For a multi-group subspace \(\tilde{H}\) of the multi-group space \(\tilde{G}\), \(g \in \tilde{G}\), define

\[
g \tilde{H} = \{g \times h | h \in \tilde{H}, \times \in O(\tilde{H})\}.
\]

Then for \(\forall x, y \in \tilde{G}\),

\[
x \tilde{H} \cap y \tilde{H} = \emptyset \text{ or } x \tilde{H} = y \tilde{H}.
\]

In fact, if \(x \tilde{H} \cap y \tilde{H} \neq \emptyset\), let \(z \in x \tilde{H} \cap y \tilde{H}\), then there exist elements \(h_1, h_2 \in \tilde{H}\) and operations \(\times_i\) and \(\times_j\) such that

\[
z = x \times_i h_1 = y \times_j h_2.
\]

Since \(\tilde{H}\) is a multi-group subspace, \((\tilde{H} \cap G_i; \times_i)\) is a subgroup. Whence, there exists an inverse element \(h_1^{-1}\) in \((\tilde{H} \cap G_i; \times_i)\). We have that

\[
x \times_i h_1 \times_i h_1^{-1} = y \times_j h_2 \times_i h_1^{-1}.
\]

That is,

\[
x = y \times_j h_2 \times_i h_1^{-1}.
\]

Whence,

\[
x \tilde{H} \subseteq y \tilde{H}.
\]

Similarly, we can also get that
Therefore, we get that

\[ x\tilde{H} \supseteq y\tilde{H}. \]

Denote the union of two set \( A \) and \( B \) by \( A \oplus B \) if \( A \cap B = \emptyset \). Then we get the following result by the previous proof.

**Theorem 2.3** For any multi-group subspace \( \tilde{H} \) of a multi-group space \( \tilde{G} \), there is a representation set \( T, T \subset \tilde{G} \), such that

\[ \tilde{G} = \bigoplus_{x \in T} x\tilde{H}. \]

For the case of finite groups, since there is only one binary operation \( \times \) and \( |x\tilde{H}| = |y\tilde{H}| \) for any \( x, y \in \tilde{G} \), We get the following corollary, which is just Lagrange theorem for finite groups.

**Corollary 2.1** (Lagrange theorem) For any finite group \( G \), if \( H \) is a subgroup of \( G \), then \( |H| \) is a divisor of \( |G| \).

For a multi-group space \( \tilde{G} \) and \( g \in \tilde{G} \), denote by \( \overline{O(g)} \) all the binary operations associative with \( g \) and by \( \tilde{G}(\times) \) the elements associative with the binary operation \( \times \). For a multi-group subspace \( \tilde{H} \) of \( \tilde{G} \), \( \times \in \overline{O(\tilde{H})} \) and \( \forall g \in \tilde{G}(\times) \), if \( \forall h \in \tilde{H} \),

\[ g \times h \times g^{-1} \in \tilde{H}, \]

then call \( \tilde{H} \) a normal multi-group subspace of \( \tilde{G} \), denoted by \( \tilde{H} \triangleleft \tilde{G} \). If \( \tilde{H} \) is a normal multi-group subspace of \( \tilde{G} \), similar to the normal subgroup of a group, it can be shown that \( g \times \tilde{H} = \tilde{H} \times g \), where \( g \in \tilde{G}(\times) \). We have the following result.

**Theorem 2.4** Let \( \tilde{G} = \bigcup_{i=1}^{n} G_i \) be a multi-group space with an operation set \( \overline{O(\tilde{G})} = \{ \times_i | 1 \leq i \leq n \} \). Then a multi-group subspace \( \tilde{H} \) of \( \tilde{G} \) is normal if and only if for any integer \( i, 1 \leq i \leq n \), \( \tilde{H} \cap G_i; \times_i \) is a normal subgroup of \( (G_i; \times_i) \) or \( \tilde{H} \cap G_i = \emptyset \).

**Proof** We have known that

\[ \tilde{H} = \bigcup_{i=1}^{n} (\tilde{H} \cap G_i). \]

If for any integer \( i, 1 \leq i \leq n \), \( \tilde{H} \cap G_i; \times_i \) is a normal subgroup of \( (G_i; \times_i) \), then we know that for \( \forall g \in G_i, 1 \leq i \leq n \),

\[ g \times_i (\tilde{H} \cap G_i) \times_i g^{-1} = \tilde{H} \cap G_i. \]
Whence, for $\forall \circ \in O(\tilde{H})$ and $\forall g \in \overrightarrow{G(\circ)}$,

$$g \circ \tilde{H} \circ g^{-1} = \tilde{H}.$$ 

That is, $\tilde{H}$ is a normal multi-group subspace of $\tilde{G}$.

Now if $\tilde{H}$ is a normal multi-group subspace of $\tilde{G}$, then by definition, we know that for $\forall \circ \in O(\tilde{H})$ and $\forall g \in \overrightarrow{G(\circ)}$,

$$g \circ \tilde{H} \circ g^{-1} = \tilde{H}.$$ 

Not loss of generality, we assume that $\circ = \times_k$, then we get that

$$g \times_k (\tilde{H} \cap G_k) \times_k g^{-1} = \tilde{H} \cap G_k.$$ 

Therefore, $(\tilde{H} \cap G_k; \times_k)$ is a normal subgroup of $(G_k, \times_k)$. For operation $\circ$ is chosen arbitrarily, we know that for any integer $i$, $1 \leq i \leq n$, $(\tilde{H} \cap G_i; \times_i)$ is a normal subgroup of $(G_i; \times_i)$ or an empty set.

For a multi-group space $\tilde{G}$ with an operation set $O(\tilde{G}) = \{\times_i \mid 1 \leq i \leq n\}$, an order of operations in $O(\tilde{G})$ is said an oriented operation sequence, denoted by $\overrightarrow{O}(\tilde{G})$. For example, if $O(\tilde{G}) = \{\times_1, \times_2 \times_3\}$, then $\times_1 \succ \times_2 \succ \times_3$ is an oriented operation sequence and $\times_2 \succ \times_1 \succ \times_3$ is also an oriented operation sequence.

For an oriented operation sequence $\overrightarrow{O}(\tilde{G})$, we construct a series of normal multi-group subspaces

$$\tilde{G} \rhd \tilde{G}_1 \rhd \tilde{G}_2 \rhd \cdots \rhd \tilde{G}_m = \{1 \times_n\}$$

by the following programming.

**STEP 1:** Construct a series

$$\tilde{G} \rhd \tilde{G}_{11} \rhd \tilde{G}_{12} \rhd \cdots \rhd \tilde{G}_{1l_1}$$

under the operation $\times_1$.

**STEP 2:** If a series

$$\tilde{G}_{(k-1)l_1} \rhd \tilde{G}_{k1} \rhd \tilde{G}_{k2} \rhd \cdots \rhd \tilde{G}_{kl_k}$$

has be constructed under the operation $\times_k$ and $\tilde{G}_{kl_k} \neq \{1 \times_n\}$, then construct a series

$$\tilde{G}_{kl_1} \rhd \tilde{G}_{(k+1)1} \rhd \tilde{G}_{(k+1)2} \rhd \cdots \rhd \tilde{G}_{(k+1)l_{k+1}}$$

under the operation $\times_{k+1}$.

This programming is terminated until the series

$$\tilde{G}_{(n-1)l_1} \rhd \tilde{G}_{n1} \rhd \tilde{G}_{n2} \rhd \cdots \rhd \tilde{G}_{nl_n} = \{1 \times_n\}$$

has be constructed under the operation $\times_n$. 5
The number $m$ is called the length of the series of normal multi-group subspaces. For a series

$$\tilde{G} \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n = \{1 \times n\}$$

of normal multi-group subspaces, if for any integer $k, s, 1 \leq k \leq n, 1 \leq s \leq l_k$, there exists a normal multi-group subspace $\tilde{H}$ such that

$$\tilde{G}_{ks} \triangleright \tilde{H} \triangleright \tilde{G}_{k(s+1)},$$

then $\tilde{H} = \tilde{G}_{ks}$ or $\tilde{H} = \tilde{G}_{k(s+1)}$, we call this series is maximal. For a maximal series of finite normal multi-group subspaces, we have the following result.

**Theorem 2.5** For a finite multi-group space $\tilde{G} = \bigcup_{i=1}^{n} G_i$ and an oriented operation sequence $\tilde{O}(\tilde{G})$, the length of maximal series of normal multi-group subspaces is a constant, only dependent on $\tilde{G}$ itself.

**Proof** The proof is by induction on the integer $n$.

For $n = 1$, the maximal series of normal multi-group subspaces is just a composition series of a finite group. By Jordan-Hölder theorem (see [1] or [3]), we know the length of a composition series is a constant, only dependent on $\tilde{G}$. Whence, the assertion is true in the case of $n = 1$.

Assume the assertion is true for cases of $n \leq k$. We prove it is true in the case of $n = k + 1$. Not loss of generality, assume the order of binary operations in $\tilde{O}(\tilde{G})$ being $\times_1 \succ \times_2 \succ \cdots \succ \times_n$ and the composition series of the group $(G_1, \times_1)$ being

$$G_1 \triangleright G_2 \triangleright \cdots \triangleright G_s = \{1 \times 1\}.$$

By Jordan-Hölder theorem, we know the length of this composition series is a constant, dependent only on $(G_1; \times_1)$. According to Theorem 3.6, we know a maximal series of normal multi-group subspace of $\tilde{G}$ gotten by the STEP 1 under the operation $\times_1$ is

$$\tilde{G} \triangleright \tilde{G} \setminus (G_1 \setminus G_2) \triangleright \tilde{G} \setminus (G_1 \setminus G_3) \triangleright \cdots \triangleright \tilde{G} \setminus (G_1 \setminus \{1 \times 1\}).$$

Notice that $\tilde{G} \setminus (G_1 \setminus \{1 \times 1\})$ is still a multi-group space with less or equal to $k$ operations. By the induction assumption, we know the length of its maximal series of normal multi-group subspaces is only dependent on $\tilde{G} \setminus (G_1 \setminus \{1 \times 1\})$, is a constant. Therefore, the length of a maximal series of normal multi-group subspaces is also a constant, only dependent on $\tilde{G}$.

Applying the induction principle, we know that the length of a maximal series of normal multi-group subspaces of $\tilde{G}$ is a constant under an oriented operations $\tilde{O}(\tilde{G})$, only dependent on $\tilde{G}$ itself.

As a special case, we get the following corollary.
Corollary 2.2 (Jordan-Hölder theorem)  For a finite group $G$, the length of the composition series is a constant, only dependent on $G$.

3. Open Problems on Multi-group Spaces

**Problem 3.1** Establish a decomposition theory for multi-group spaces.

In group theory, we know the following decomposition results ([1][3]) for a group.

Let $G$ be a finite $\Omega$-group. Then $G$ can be uniquely decomposed as a direct product of finite non-decomposition $\Omega$-subgroups.

Each finite Abelian group is a direct product of its Sylow $p$-subgroups.

Then Problem 3.1 can be restated as follows.

Whether can we establish a decomposition theory for multi-group spaces similar to above two results in group theory, especially, for finite multi-group spaces?

**Problem 3.2** Define the conception of simple multi-group spaces for multi-group spaces. For finite multi-group spaces, whether can we find all simple multi-group spaces?

For finite groups, we know that there are four simple group classes ([9]):

- **Class** 1: the cyclic groups of prime order;
- **Class** 2: the alternating groups $A_n$, $n \geq 5$;
- **Class** 3: the 16 groups of Lie types;
- **Class** 4: the 26 sporadic simple groups.

**Problem 2.3** Determine the structure properties of a multi-group space generated by finite elements.

For a subset $A$ of a multi-group space $\bar{G}$, define its spanning set by

$$\langle A \rangle = \{a \circ b | a, b \in A \text{ and } \circ \in O(\bar{G})\}.$$  

If there exists a subset $A \subseteq \bar{G}$ such that $\bar{G} = \langle A \rangle$, then call $\bar{G}$ is generated by $A$. Call $\bar{G}$ is finitely generated if there exist a finite set $A$ such that $\bar{G} = \langle A \rangle$. Then Problem 2.3 can be restated by

Can we establish a finite generated multi-group theory similar to the finite generated group theory?

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