1. Introduction

Motivated by the classical paper [6] by Koch and Tataru, we study the generalized Navier-Stokes equation in $\mathbb{R}^n$

$$
\begin{cases}
    u_t + (u \cdot \nabla)u + (-\Delta)^\alpha u + \nabla p = 0 \\
    \nabla \cdot u = 0 \\
    u(x, 0) = u_0(x),
\end{cases}
$$

where $\alpha \in (\frac{1}{2}, 1)$ is a real parameter. In [6], they study the case $\alpha = 1$. For the case $\alpha \in (\frac{1}{2}, 1)$, heuristically speaking, the effect of the dissipative term is still stronger than the transport term.

Note that $(-\Delta)^\alpha$ is the Fourier multiplier defined by the obvious way

$$
(-\Delta)^\alpha u(\xi) = |\xi|^{2\alpha} \hat{u}(\xi).
$$

We use the Leray projection to rewrite the equation as

$$
u_t + (-\Delta)^\alpha u = -\nabla \Pi Nu, \quad N(u) = u \otimes u.
$$

Now we denote the fractional heat semigroup by $S(t) = e^{-t(-\Delta)^\alpha}$ without specifying the domain, where

$$
S(s)f(t) = \mathcal{F}^{-1}\left(\int e^{-s|\xi|^{2\alpha}} \hat{f}(\xi, t) \, d\xi \right).
$$

Date: today.
Let $\Phi(x) = \mathcal{F}^{-1}\left(e^{-|\xi|^{2\alpha}}\right)$, and $\Phi_t(x) = \frac{1}{t^n}\Phi\left(\frac{x}{t}\right)$. Then the kernel of the fractional heat semigroup $S(t)$ can be denoted by

$$
\Phi_t(x) = \left(t^{\frac{1}{2\alpha}}\right)^{-n}\Phi(t^{-\frac{1}{2\alpha}}x) := \Phi(t, x).
$$

And the kernel satisfies the estimate below

$$
|\Phi(x, t)| \lesssim \frac{t}{(t^{1/\alpha} + |x|^2)^{(n+2\alpha)/2}},
$$

which can be found in [2]. In particular, $\Phi(t, x)$ is integrable.

From now on, we only consider the case $\frac{1}{2} < \alpha < 1$ unless other specified. Let now $Q(x, R) = B(0, R) \times (0, R^{2\alpha})$.

**Definition 1.1.** Now we define the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ with the following norm

$$
\|v\|_E = \sup_{x, R > 0} \left( R^{-(n+2-2\alpha)} \int_{Q(x, R)} |S(t)v(y)|^2 \, dt \, dy \right)^{\frac{1}{2}}
$$

is finite. And the space is denoted by $E$.

The above definition is motivated by some scaling invariance property and inspired from [6]. Furthermore, when $\alpha \in \left(\frac{1}{2}, 1\right)$, we could expect more properties than the case $\alpha = 1$. We use the fractional heat kernel to present an equivalent definition of Besov space:

**Lemma 1.2.** The norm of the homogeneous Besov space $\dot{B}^{1-2\alpha, \infty}_\infty$ is

$$
\|v\|_{\dot{B}^{1-2\alpha, \infty}_\infty} \sim \sup_{t > 0} t^{1-\frac{1}{2\alpha}} \|S(t)v(x)\|_{L^\infty(\mathbb{R}^n)}
$$

when $\alpha \in \left(\frac{1}{2}, 1\right)$.

Note that the equivalence to the usual the definition of Besov space can be proved following an analogous proof of [7, Theorem 5.3]. We defer the proof to next section.

The above discussion motivates the introduction of the spaces $GX, GY$ of functions in $\mathbb{R}^n \times \mathbb{R}_+$ with norms

$$
\|u\|_{GX} = \sup_{t > 0} t^{1-\frac{1}{2\alpha}} \|u(t)\|_{L^\infty(\mathbb{R}^n)},
$$

$$
\|f\|_{GY} = \sup_{t > 0} t^{2-\frac{1}{\alpha}} \|f(t)\|_{L^\infty(\mathbb{R}^n)}.
$$

Actually, some of these function spaces are equivalent.

**Theorem 1.3.** For $h \in \mathcal{S}'(\mathbb{R}^n)$, the following statements are equivalent:

1. $h(x) \in \dot{B}^{1/\alpha-2, \infty}_\infty$.
2. $h(x) \in E$.
3. $S(t)h(x) \in GX$.

Moreover,

$$
\|h(x)\|_{\dot{B}^{1/\alpha-2, \infty}_\infty} \sim \|h(x)\|_E \sim \|S(t)h(x)\|_{GX}.
$$

And for simplicity, we denote $\dot{B}^{1/\alpha-2, \infty}_\infty, E$ by $B$.

We leave the proof to the following section.

Now we could state our main result for the wellposedness of generalized Navier-Stokes:
Theorem 1.4. The generalized Navier-Stokes equations
\[
\begin{cases}
  u_t + (u \cdot \nabla)u + (-\Delta)\alpha u + \nabla p = 0 \\
  \nabla \cdot u = 0 \\
  u(0) = u_0 
\end{cases}
\] (1.2)

have a unique small global solution in $B$ for all initial data $u_0$ with $\nabla \cdot u_0 = 0$ which are small in $G_X$, where $\alpha \in \left(\frac{1}{2}, 1\right)$.

Corollary 1.5. The global solution constructed in Theorem 1.4 satisfies that
\[u(x,t_0) \in B\] for each fixed $t_0 > 0$.

Note that this corollary is not true for $\alpha = 1$. In fact, the solution map is not continuous in $\dot{B}^{-1,\infty}_\infty$. See [3] for further discussion.

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2. Proof of Theorem 1.3

Thanks to scaling invariance, we present an important lemma at first.

Lemma 2.1.
\[|S(t)u_0(x)| \leq Ct^{-(1-\frac{1}{2\alpha})}\|u_0\|_\varepsilon\]

Proof. First, we show some scaling property. Note that $\mathcal{F}(u_0(\lambda x)) = \lambda^{-n} \hat{u}_0(\frac{\xi}{\lambda})$, then
\[
(S(t)u_0)(x) = \int e^{ix \cdot \xi} e^{-t|\xi|^{2\alpha}} \hat{u}_0(\xi) d\xi = \left(\lambda \frac{x}{\lambda}\right)^n \int e^{i\lambda \frac{x}{\lambda} \cdot \xi} e^{-t|\xi|^{2\alpha}} \hat{v}_0(\frac{\xi}{\lambda}) d\xi = \int e^{i\lambda \frac{x}{\lambda} \cdot \xi} e^{-t|\xi|^{2\alpha}} \hat{v}_0(\xi) d\xi = (S(t\lambda)v_0)(\lambda \frac{x}{\lambda}),
\] (2.1)

where $v_0(y) = u_0(y/\lambda^{\frac{1}{2\alpha}})$ and $\lambda > 0$. And from (2.1), we have
\[
\left(\int_{Q(x,R)} |S(t)u_0(y)|^2 dy \right)^{\frac{1}{2}} = \left(\int_{B(x,R)} S(t\lambda)v_0(\lambda \frac{x}{\lambda} y)^2 dy dt \right)^{\frac{1}{2}} = \lambda^{-(n+2-2\alpha)} \left(\lambda \frac{x}{\lambda} R\right)^{-(n+2-2\alpha)} \int_{Q(x,\lambda \frac{x}{\lambda} R)} |S(t)v_0(y)|^2 dy dt
\] (2.2)

where the last equality follows from a change of variable. Then taking supremum implies
\[\|u_0\|_\varepsilon = \lambda^{-(1-\frac{1}{2\alpha})}\|v_0\|_\varepsilon.\]

And note that $\|u_0\|_\varepsilon$ is translational invariant, thus it suffices to show
\[|S(1)u_0(0)| \leq C\|u_0\|_\varepsilon.\]
Since
\[ |S(1)u_0(0)|^2 = \left| \int_{\mathbb{R}^n} \Phi(1-s,-y)(S(s)u_0)(y) \, dy \right|^2 \lesssim \int_{\mathbb{R}^n} \Phi(1-s,-y)|(S(s)u_0)(y)|^2 \, dy , \]
then replace the right hand side by its time average, the inequality still holds that
\[ |S(1)u_0(0)|^2 \lesssim 2^{2\alpha} \int_0^{2-2\alpha} \int_{\mathbb{R}^n} \Phi(1-s,-y)|(S(s)u_0)(y)|^2 \, dy \, ds \]
\[ \lesssim \sum_{k \in \mathbb{Z}^n} \int_0^{2-2\alpha} \int_{B(\frac{1}{2^n} k, 4)} \Phi(1-s,-y)|(S(s)u_0)(y)|^2 \, dy \, ds \]
\[ \lesssim \| u_0 \|_E \left( \sum_{k \in \mathbb{Z}^n} \sup_{s \in [0,2-2\alpha]} \sup_{y \in B(\frac{1}{2^n} k, \frac{1}{2})} |\Phi(1-s,-y)| \right) \lesssim \| u_0 \|_E \]
where the last inequality comes from the kernel estimate
\[ |\Phi(x,t)| \lesssim \frac{t}{(t^{1/\alpha} + |x|^2)^{(n+2\alpha)/2}} \]
which can be found in [2]. Then the result follows.

Furthermore, it is straightforward that
\[ \left( R^{-(n+2-2\alpha)} \int_{Q(x,R)} |S(t)u_0(y)|^2 \, dt \, dy \right)^{\frac{1}{2}} \]
\[ \leq \left( \frac{1}{R^{2-2\alpha}} \int_0^{R^{2\alpha}} t^{-(2-\frac{1}{4})} \, dt \right) \sup_{t>0} t^{1-\frac{1}{2\alpha}} \| S(t)u_0(y) \|_{L^\infty} \lesssim_{\alpha} \sup_{t>0} t^{(1-\frac{1}{2\alpha})} \| S(t)u_0(y) \|_{L^\infty} , \]
where the last equality follows from the assumption that \( \alpha \in (\frac{1}{2},1) \). Thus we know
\[ \| v \|_E \sim \sup_{t>0} t^{1-\frac{1}{2\alpha}} \| S(t)v(x) \|_{L^\infty(\mathbb{R}^n)} \quad (2.3) \]
are equivalent norms.

Now we prove Lemma [1,2]. We state a more precise version of the lemma: Let \( \sigma < 0 \), \( f \in \mathcal{S}'_h(\mathbb{R}^n) \), \( S(t) = e^{-t(-\Delta)\sigma} \) is the fractional heat kernel, (the definition of \( \mathcal{S}'_h \) can be found in [1] Definiton 1.26]), then we have
\[ \sup_{t>0} t^{-\frac{\sigma}{2\alpha}} \| S(t)f \|_{L^\infty} \lesssim \| f \|_{\mathcal{B}^\sigma_{2,\infty}} \lesssim \sup_{t>0} t^{-\frac{\sigma}{2\alpha}} \| S(t)f \|_{L^\infty} . \quad (2.4) \]

Proof. The first inequality: By the Littlewood Paley decomposition characterization of Besov spaces, (c.f. [1]) \( f = \sum_{j \in \mathbb{Z}} P_j f \) is true for \( f \in \mathcal{S}'_h \), with \( P_j f \in L^\infty(\mathbb{R}^n) \) for \( j \in \mathbb{Z} \) and \( \| P_j f \|_{L^\infty(\mathbb{R}^n)} = 2^{-j\sigma} \epsilon_j \) where \( (\epsilon_j)_{j \in \mathbb{Z}} \in \ell^\infty \). We estimate the norm \( t^{-\frac{\sigma}{2\alpha}} \| S(t)f \|_{L^\infty} \) by
\[ t^{-\frac{\sigma}{2\alpha}} \| S(t)P_j f \|_{L^\infty} \leq t^{-\frac{\sigma}{2\alpha}} \| P_j f \|_{L^\infty} . \]
Furthermore, we write
\[ S(t)P_j f = t^{-\frac{N}{4}} (t(-\Delta)^{\sigma})\frac{N}{4} S(t)(-\Delta)^{-\frac{N}{2}} \tilde{P}_j P_j f , \]
then since the kernel of \((t(-\Delta)^\alpha)^{\frac{N}{2}} S(t)\) (Lemma 3.4) and \((-\Delta)^{\frac{N}{2}} \widetilde{P}_0 = 2^{jN} (-\Delta)^{\frac{N}{2}} \widetilde{P}_j\) are integrable, respectively, therefore we have

\[
t^{-\frac{\alpha}{2N}}\|S(t)P_j f\|_{L^\infty} \lesssim t^{-\frac{jN}{2N}}\|(-\Delta)^{-\frac{N}{2}} \widetilde{P}_j f\|_{L^\infty} \lesssim t^{-\frac{jN}{2N}} 2^{-jN}\|P_j f\|_{L^\infty}
\]

for \(N \geq 0\). Hence, we know

\[
S(t)f \in L^\infty(\mathbb{R}^n), \quad \forall t > 0
\]

from the estimate above and the summability of \(\sum_{j \geq 0} 2^{-jN} < \infty\).

Now we choose \(j_0\) such that \(4^{-(j_0+1)\alpha} < t \leq 4^{-j_0\alpha}\), and we choose \(N > -\sigma\), then for \(j_0 > 0\),

\[
t^{-\frac{\alpha}{2N}}\|S(t)f\|_{L^\infty} \leq C \sum_{j \leq j_0} 2^{(j_0-j)\alpha} \epsilon_j + C_N \sum_{j > j_0} 2^{(j_0+1-j)(\sigma+N)} \epsilon_j \leq \widetilde{C}_N\|\epsilon_j\|_{\ell^\infty(\mathbb{Z})}.
\]

Thus

\[
\sup_{t > 0} t^{-\frac{\alpha}{2N}}\|S(t)f\|_{L^\infty} \lesssim \|f\|_{\dot{B}^{\alpha,\infty}_\infty}.
\]

**The second inequality:** For each \(j\), take some \(t\) such that \(4^{-(j+1)\alpha} < t \leq 4^{-j\alpha}\),

\[
2^{j\sigma} \|P_j f\|_{L^\infty} = 2^{j\sigma} \|t^{\frac{\alpha}{2N}} S(-t) P_j t^{-\frac{\alpha}{2N}} S(t) f\|_{L^\infty} \leq 2^{-\sigma} C \sup_{t > 0} \|t^{-\frac{\alpha}{2N}} S(t) f\|_{L^\infty},
\]

where the \(L^1\) norm for the kernel of \(S(-t) P_j\), \(\forall t \in (4^{-(j+1)\alpha}, 4^{-j\alpha}]\) are bounded by \(C\) uniformly.

And note that \(\sup_{t > 0} t^{1 - \frac{\alpha}{4\alpha}} \|S(t)u\|_{L^\infty} < \infty\) provided that \(\|u\|_\mathcal{E} < \infty\) by (2.3), this implicitly implies \(\tilde{u}(\xi)\) in frequency domain has some decay properties near 0, that is, \(\lim_{t \to \infty} \|S(t)u\|_{L^\infty} \to 0\). For \(\theta \in C^\infty_0\) such that \(\theta(\xi)\) behave like \(|\xi|^{4\alpha}\) near 0, then we choose some constant \(\delta < \frac{1}{2} - \frac{1}{4\alpha}\) and compute as follows that

\[
\|\theta(tD)u\|_{L^\infty} \lesssim \|\int \theta(t\xi)e^{i\xi|\xi|^{2\alpha}} e^{ix\cdot \xi} d\xi\|_{L^1_x} \|S(t) u\|_{L^\infty}
\]

\[
\lesssim \|\int \theta(t\xi)e^{i\xi|\xi|^{2\alpha}} (1 + |x|^2)^{\frac{\alpha}{2} + \delta} e^{ix\cdot \xi} d\xi\|_{L^\infty_x} t^{\frac{1}{2\alpha} - 1 - n} \|\int \theta(\xi)e^{t^{2\alpha}|\xi|^{2\alpha}} (1 + |tx|^2)^{\frac{\alpha}{2} + \delta} e^{ix\cdot \xi} d\xi\|_{L^\infty_x}
\]

\[
\lesssim t^{\frac{1}{2\alpha} - 1 - 2\delta} \|x|^{n+2\delta} \|\theta(\xi)e^{t^{2\alpha}|\xi|^{2\alpha}} e^{ix\cdot \xi} d\xi\|_{L^\infty_x}
\]

\[
\lesssim t^{\frac{1}{2\alpha} - 1 + 2\delta} \|\int (1 - \Delta\xi)^{\frac{\alpha}{2} + \delta} (\theta(\xi)e^{t^{2\alpha}|\xi|^{2\alpha}}) d\xi\|_{L^\infty_x} \lesssim t^{\frac{1}{2\alpha} - 1 + 2\delta} \to 0
\]

where the last inequality we use the fact \(t > 1\) and \(1 - 2\alpha < 0\). Then we see that \(u \in \mathcal{S}'_h\) from the definition in [11] Definition 1.26, Page 22.

Therefore, the previously defined space \(\mathcal{E}\) is indeed the Besov space \(\dot{B}^{1-2\alpha,\infty}_\infty\), which completes the proof of Theorem 1.3.

**Remark 2.2.** A well known result by Bourgain-Pavlovic [3] had shown the illposedness of Navier-Stokes when initial data is in \(\dot{B}^{1-\infty}_\infty\). Indeed, this does not contradict to our result
here since (2) is false for $\alpha = 1$. Hence we could not expect the wellposedness to be true for $\dot{B}^{-1,\infty}_\infty$.

In the following sections, we denote both $\dot{B}^{1/\alpha-2,\infty}_\infty$, $\mathcal{E}$ by $\mathcal{B}$ thanks to Lemma 3.3.

3. Preliminaries for some kernel estimates

As defined in previous section, the kernel of $S(t)$ is $\Phi(t, x) = (t^{\frac{1}{2\alpha}})^{-n} \Phi(t^{-\frac{1}{2\alpha}} x)$. The operator $V$ is the parametrix for the inhomogeneous fractional heat equation with 0 Cauchy data, that is, $u = Vf$ if and only if
\[
 u_t + (-\Delta)^\alpha u = f, \quad u(0) = 0.
\]
Then
\[
 (Vf)(t) = \int_0^t S(t-s)f(s) \, ds.
\]
Hence the solutions to the fractional heat equation
\[
 u_t + (-\Delta)^\alpha u = f, \quad u(x, 0) = u_0(x)
\]
are given by
\[
 u(x, t) = (S(t)u_0)(x) + (Vf)(x,t).
\]
Then we intend to discuss the mild solution of the fixed point form generalized Navier-Stokes:
\[
 u(x, t) = S(t)u_0(x) - (V\nabla \Pi N(u))(x, t).
\]
Now we consider the symbol $m$ corresponding to the projection operator $\Pi$ to the divergence free vector fields, which is defined by its matrix valued Fourier multiplier $m(D_x)$, where
\[
 \Pi u(x) = m(D_x) u = \mathcal{F}^{-1}(m(\eta) \hat{u}(\eta))
\]
and its symbol $m$ is given by
\[
 (m(\eta))_{ij} = \delta_{ij} - \frac{\eta_i \eta_j}{|\eta|^2}.
\]
Then the symbol $m$ satisfies the Mihlin-Hormander condition
\[
 \sup_{\eta \neq 0} |\eta|^{|\alpha|} |\partial_\eta^\alpha m(\eta)| \leq C_\alpha
\]
for all multiindices $\alpha$; hence, $m(D_x)$ is a singular integral operator.

Lemma 3.1. Let $\Phi(x, t)$ as defined before in previous section, then combined with the Mihlin-Hormander condition (3.1), we have the bound that
\[
 |(\Pi \Phi)(x)| \leq c(1 + |x|)^{-n}.
\]
Proof. By definition, $m(D_x)\Phi(x) = K \ast \Phi$, where $K = (m)^\vee$. We use the Littlewood-Paley decomposition to manipulate:
\[
 m(D_x)\Phi = \sum_{k \in \mathbb{Z}} m(D_x)P_k \Phi = \sum_{k \in \mathbb{Z}} K_k \ast \Phi,
\]
where $K_k = (m(\eta)P_k(\eta))^\vee$ (an abuse of notation here, in $K_k$, $P_k(\eta)$ denotes the Fourier transform of the kernel of $P_k$), and here $P_k(\eta)$ is a cut-off (bump) function cutting off to $|\eta| \sim 2^k$. Since $P_k f = f \ast \phi_{2^{-k}}$, where $\phi_{2^{-k}}(x) = 2^{kn}\phi(2^k x)$, thus $P_k(\xi) = \hat{\phi}_{2^{-k}}(\xi) = \hat{\phi}(2^{-k}\xi) = P_0(2^{-k}\xi)$. 


Now we let \( \tilde{m}_k(\eta') = m(2^k\eta')P_k(2^k\eta') \), then \( |\eta'| \sim 1 \) and we can observe that \( \tilde{m}_k(\eta') = m(\eta') P_0(\eta') = \tilde{m}_0(\eta') \), thus \( \tilde{m}_k(\eta') \) also satisfies (3.1) with the same constant \( C_n \). Note that

\[
|m(D_x)\Phi(x)| \leq \sum_k \left| \left( m(\xi)P_k(\xi)e^{-|\xi|^{2\alpha}} \right)^\vee(x) \right|.
\]

Let \( \Phi_k(x) = \left( m(\xi)P_k(\xi)e^{-|\xi|^{2\alpha}} \right)^\vee(x) \), then

\[
\Phi_k(x) = \frac{1}{(2\pi)^n} \int e^{i\xi \cdot m(\xi)P_k(\xi)e^{-|\xi|^{2\alpha}}} d\xi = \frac{2^{kn}}{(2\pi)^n} \int e^{i(2^k x) \cdot \eta \tilde{m}_k(\eta)e^{-4k\alpha|\eta|^{2\alpha}}} d\eta
\]

Moreover, with \( N \) (assuming \( N \) is even) times integration by parts and applying the Mihlin-Hormander condition (3.1), we have

\[
\frac{e^{-4k\alpha|\eta|^{2\alpha}} d\eta}{(2\pi)^n} \leq C_N(1 + |x|)^{-N} \sum_{|\gamma|+|\beta|} \int |\partial^\gamma \tilde{m}_0(\eta)| |\partial^\alpha e^{-c|\eta|^{2\alpha}}| d\eta
\]

\[
\leq C_N(1 + c)^N(1 + |x|)^{-N} \sum_{|\gamma|+|\beta|} \int |\eta|^{-|\beta|} (1 + |\eta|)^{|\gamma|} e^{-e|\eta|^{2\alpha}} d\eta
\]

Then use the estimate above, we could give a bound for \( \Phi_k(x) \) in (3.2):

\[
|\Phi_k(x)| \leq C_N 2^{kn}(1 + |2^k x|)^{-N} \frac{(1 + 4^{k\alpha})^N}{e^{4^{k\alpha}}},
\]

Then for \( \forall k \), there exists some constant such that

\[
|\Phi_k(x)| \leq C_N 2^{kn}(1 + |2^k x|)^{-N}
\]

and \( \exists K_N \) sufficient large, \( \forall k > K_N \),

\[
|\Phi_k(x)| \leq C_N 2^{kn}(1 + |2^k x|)^{-N} 4^{-kn} \leq C_N 2^{-kn},
\]

since \( \frac{(1+x)^N e^{x}}{e^x} \to 0 \) as \( x \to +\infty \).

Now we take a fixed \( N > n \) and let \( k_0 \in \mathbb{Z}^- \) such that \( 2^{-k_0-1} \leq |x| \leq 2^{-k_0} \), then

\[
|m(D_x)\Phi(x)| \leq C_N \sum_{k \in \mathbb{Z}} 2^{kn}(1 + |2^k x|)^{-N} \lesssim N \sum_{k \leq k_0} 2^{kn} + |x|^{-N} \sum_{k > k_0} 2^{k(n-N)}
\]

\[
\lesssim N 2^{kn} + 2^{k_0(n-N)} |x|^{-N} \lesssim_N |x|^{-n}
\]

when \( |x| > 1 \). And for \( |x| \leq 1 \), we have

\[
|m(D_x)\Phi(x)| \leq C_N \left( \sum_{k > K_N} 2^{-kn} + \sum_{k \leq K_N} 2^{kn} \right) \lesssim_{N,n} 1.
\]
Combining the cases for $|x| > 1$ and for $|x| \leq 1$, we have

$$|m(D_x)\Phi(x)| \lesssim (1 + |x|)^{-n}.$$  
\[ \square \]

**Lemma 3.2.** From Lemma 3.1, scaling shows that the kernel function $k_t(x) = \Pi \Phi_{\frac{1}{t^\alpha}}$ of $\Pi S(t)$ satisfies

$$|k_t(x)| \leq C \left(t^{\frac{\alpha}{n}} + |x|\right)^{-n}.$$  

**Lemma 3.3.** Similarly, we have bounds for the kernel of $\Pi \nabla S(t)$ that

$$|m(D_x)\nabla \Phi_{\frac{1}{t^\alpha}}(x)| \leq C \left(t^{\frac{\alpha}{n}} + |x|\right)^{-n-1}.$$  

**Lemma 3.4.** The kernel $k(t, x)$ of $(t(-\Delta)\frac{N}{n^2}) S(t)$ is integrable and its $L^1$ norm is uniformly bounded.

**Proof.** By scaling property, it suffices to show $\|k(1, x)\|_{L^1} < \infty$. Let

$$K_j(x) = (-\Delta)^{N/2}P_j\Phi(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi}|\xi|^NP_j(\xi)e^{-|\xi|^2\alpha}d\xi$$

$$= \frac{2^{j(\alpha+N)}}{(2\pi)^n} \int e^{i(2^jx)\cdot\xi}|\xi|^NP_0(\xi)e^{-4^j|\xi|^2\alpha}d\xi.$$  

Since

$$\left|\int e^{ix\cdot\eta}|\eta|^NP_0(\eta)e^{-c|\eta|^2\alpha}d\eta\right| \leq C_M(1 + |x|)^{-M} \int \left|(1 - \Delta_\eta)^{M/2}\left(|\eta|^NP_0(\eta)e^{-c|\eta|^2\alpha}\right)\right|d\eta$$

$$\lesssim C_M(1 + |x|)^{-M} \sum_{|\delta| \leq M} \int_{|\eta| \sim 1} \left|\partial^\delta e^{-c|\eta|^2\alpha}\right|d\eta \lesssim C_M(1 + c)^M(1 + |x|)^{-M}e^{-c},$$

we have

$$|K_j(x)| \leq C_M 2^{j(n+N)}(1 + |2^jx|)^{-M}(1 + 4^j)^M e^{-4^j\alpha}. $$

Then for $|x| \sim 2^{-j_0}$, $M > n + N$, we have

$$|k(1, x)| = |(-\Delta)^{N/2}\Phi(x)| \leq \sum_{j \in \mathbb{Z}} |K_j(x)| \leq C_M \sum_{j \leq j_0} 2^{j(n+N)} + \sum_{j > j_0} 2^{j(n+N-M)}|x|^{-M} \lesssim |x|^{-n-N}.$$  

And there exists $J$ such that $\forall j > J$, such that $|K_j(x)| \leq C_M 2^{-j}(1 + |2^jx|)^{-M}$ then

$$|k(1, x)| \leq \sum_{j \in \mathbb{Z}} |K_j(x)| \leq C_M \sum_{j \leq J} 2^{j(n+N)} + \sum_{j > J} 2^{-j} \lesssim 1.$$  

Hence $|k(1, x)| \lesssim (1 + |x|)^{-n-N}$. In particular, $k(1, x)$ is integrable.  
\[ \square \]

Now we devote a full section to the proof of Theorem 1.4.
4. Proof of Theorem 1.4 and Corollary 1.5

Let

\[
\Psi(u) = S(t)u_0(x) - (V \nabla \Pi N(u))(x, t),
\]

then we solve fixed point form of generalized Navier-Stokes:

\[
\Psi u = u. \tag{4.1}
\]

For small initial data we want to solve this in \( G_X \) using a fixed point argument. Since \( N \) is quadratic, the small Lipschitz constant follows for small initial data if the nonlinearity has the correct mapping properties. Hence the result is a consequence of the following two lemmas:

**Lemma 4.1.** \( N \) maps \( G_X \) into \( G_Y \).

**Lemma 4.2.** \( V \nabla \Pi \) maps \( G_Y \) into \( G_X \).

Since \( V \nabla \Pi \) is linear, and from Lemma 4.2, \( \| V \nabla \Pi f \|_{G_X} \leq C \| f \|_{G_Y} \). We assume \( \| u_0 \|_B \leq \epsilon \).

Then for \( u, v \in B_{G_X}(0, 2\epsilon C) \),

\[
\| \Psi(u) \|_{G_X} \leq \| S(t)u_0(x) \|_{G_X} + \| V \nabla \Pi N(u) \|_{G_X} \leq \| u_0 \|_B + C \| u \|_{G_X}^2 \leq \epsilon + C \| u \|_B^2.
\]

And

\[
\| \Psi(u) - \Psi(v) \|_{G_X} \leq C \| N(u) - N(v) \|_{G_Y} \leq C \| u - v \|_{G_X} (\| u \|_{G_X} + \| v \|_{G_X}) \leq 2\epsilon C \| u - v \|_{G_X}.
\]

Thus for \( \epsilon \) sufficiently small, namely \( \epsilon_n \leq \frac{1}{4Cn+1} \), \( \Psi \) maps to \( B_{G_X}(0, 2\epsilon C) \) to itself and is a contraction mapping. Therefore, we apply the fixed point theorem and this completes the proof of Theorem 1.4.

Now, we turn to the proof of these two lemmas. The proof of Lemma 4.1 is obvious and straightforward. Then for Lemma 4.2, we prove as follows.

**Proof of Lemma 4.2.** It suffices to prove the pointwise estimate

\[
| V \nabla \Pi f(x, t) | \leq ct^{-(1-\frac{1}{2\alpha})} \| f \|_{G_Y}. \tag{4.2}
\]

**Step 1. Scaling.** We claim that this estimate is scale invariant and translation invariant, which is the motivation of the definition of function spaces. Now we check this property: For (4.2), suppose we have proved the result for \( t = 1, x = 0 \), then apply the result to \( g(x, t) = f(\lambda \frac{1}{2\alpha} x, \lambda t) \), by a similar argument to (2.2), we have \( \| g \|_{G_Y} = \lambda^{-(2-1/\alpha)} \| f \|_{G_Y} \) and

\[
V \nabla \Pi g(0, 1) = \int_0^1 \int_{\mathbb{R}^n} \Phi(s, y) \nabla \Pi g(-y, 1-s) \, dy \, ds
\]

\[
= \lambda^{\frac{1}{2\alpha}} \int_0^1 \int_{\mathbb{R}^n} \Phi(s, y) (\nabla \Pi f)(\lambda^{\frac{1}{2\alpha}} (-y), \lambda (1 - s)) \, dy \, ds
\]

\[
= \lambda^{\frac{1}{2\alpha}} \lambda^{\frac{n}{2\alpha}} \int_0^\lambda \int_{\mathbb{R}^n} \Phi(s, y) (\nabla \Pi f)((-y), \lambda - s) \, d(\lambda^{\frac{1}{2\alpha}} y) \, d\left(\frac{s}{\lambda}\right)
\]

\[
= \lambda^{\frac{1}{2\alpha} - 1} \int_0^\lambda \int_{\mathbb{R}^n} \Phi(s, y) (\nabla \Pi f)((-y), \lambda - s) \, dy \, ds.
\]

Thus take \( \lambda = t \), then the estimate for \( (0, t) \) follows from the above scale invariance property. And the translation invariance is obvious.
Thus it suffices to prove
\[ |V \nabla \Pi f(0, 1)| \leq c\|f\|_{GY}. \] (4.3)

\textbf{Step 2: Localization.} Let \( \chi \) be the characteristic function of \( B(0, 2) \times [0, 1] \). Then \( f = \chi f + (1 - \chi) f \). Clearly both components are still in \( GY \). Since for \( g \in GY \),
\[ \Pi_x g(x - y) = \mathcal{F}^{-1}_x \left( \left( \delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2} \right) \widehat{g}(\xi) e^{-iy \cdot \xi} \right), \]
then we calculate by Parseval identity,
\[ \int_{\mathbb{R}^n} h(y) \Pi_x g(x - y) \, dy = \int_{\mathbb{R}^n} \widehat{h}(\xi) \left( \delta_{ij} - \frac{(-\xi_i)(-\xi_j)}{|\xi|^2} \right) \widehat{g}(-\xi) e^{ix \cdot (-\xi)} \, d\xi \]
\[ = \int_{\mathbb{R}^n} \left( \delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2} \widehat{h}(\xi) \right) \widehat{g}(-\xi) e^{ix \cdot \xi} \, d\xi = \int_{\mathbb{R}^n} (\Pi_y h(y)) g(x - y) \, dy. \] (4.4)

Then we know from integration by parts and applying the equation above to \( \nabla \Phi \) and \( g \),
\[ V \nabla \Pi g(x, t) = \int_{\mathbb{R}^n} \int_0^t \Phi(s, y) \nabla \Pi g(x - y, t - s) \, ds \, dy = \int_{\mathbb{R}^n} \int_0^t \Pi \nabla \Phi(s, y) g(x - y, t - s) \, ds \, dy. \]
Thus the kernel \( K \) of \( V \nabla \Pi \) is \( K = \Pi \nabla \Phi \), then from Lemma 3.3, we know that
\[ |K(x, t)| \leq C(t^{\frac{1}{n}} + |x|)^{-n-1}. \] (4.5)

Then we claim that
\[ \|V \nabla \Pi(1 - \chi)f\|_{L^\infty(Q(0, 1))} \leq C \sup_{x \in \mathbb{R}^n} \int_{Q(x, 1)} |f| \, dy \, dt. \] (4.6)

Take \((x_0, t_0) \in Q(0, 1)\), then
\[ |V \nabla \Pi(1 - \chi)f(x_0, t_0)| \]
\[ \leq C \int_{\mathbb{R}^n \setminus B(0, 2)} \int_0^1 \left( (t_0 - s)^{\frac{1}{2n}} + |x_0 - y|^n \right)^{-(n+1)} |f(y, s)| \, ds \, dy \]
\[ \leq C \sum_{k \in \{1, \ldots, n\}, |k| > 2} \int_{B(k, 1)} \int_0^1 \frac{1}{|x_0 - y|^{n+1}} |f(y, s)| \, ds \, dy \]
\[ \leq C \sup_{x \in \mathbb{R}^n} \int_{Q(x, 1)} |f| \, dy \, dt \sum_{k \in \{1, \ldots, n\}, |k| > 2} \frac{1}{(|k| - 2)^{n+1}} \]
\[ = C \sup_{x \in \mathbb{R}^n} \int_{Q(x, 1)} |f| \, dy \, dt \left( \tilde{C} + \sum_{k \in \{1, \ldots, n\}, |k| > 4} \frac{1}{(|k| - 2)^{n+1}} \right) \]
\[ \leq C \sup_{x \in \mathbb{R}^n} \int_{Q(x, 1)} |f| \, dy \, dt \left( \tilde{C} + \int_{|z| > 3} \frac{1}{|z|^{n+1}} \, dz \right) = C' \sup_{x \in \mathbb{R}^n} \int_{Q(x, 1)} |f| \, dy \, dt. \]

Thus we have verified the claim (4.6). Actually, the claim (4.6) is much stronger than (4.3) under the assumption that \( f \) supported outside \( B(0, 2) \times [0, 1] \) thanks to the following
For the part of \( f \):

\[
R^{-(n+2-2\alpha)} \int_{Q(x,R)} |f(y,t)| \, dt \, dy \lesssim \frac{1}{R^{2-2\alpha}} \int_0^{R^{2\alpha}} t^{-(2-\frac{4}{n})} \, dt \sup_{t>0} t^{\frac{2}{n}} \|f(\cdot,t)\|_{L^\infty}
\]

\[
= \sup_{t>0} t^{\frac{2}{n}} \|f(\cdot,t)\|_{L^\infty} = \|f\|_{G^Y}.
\]

Hence, it suffices to check the integrability of \( f \) combined with the integrability of the kernel at 0, that is,

\[
\sup_{t>0} t^{\frac{2}{n}} \|f(\cdot,t)\|_{L^\infty} = \|f\|_{G^Y}.
\]

**Step 3: The pointwise estimate.** We intend to show the pointwise estimate \( (4.3) \) when \( f \) is supported in \( B(0,2) \times [0,1] \). Actually, it follows easily from the kernel bound \( (4.5) \). Since

\[
|V \nabla \Pi f(0,1)| \lesssim \int_0^{\frac{1}{2}} \left( 1 - t \right)^{\frac{1}{2n}} |y|^{-(n+1)} |f(y,t)| \, dt \, dy
\]

\[
+ \int_{\frac{1}{2}}^1 \left( 1 - t \right)^{\frac{1}{2n}} |y|^{-(n+1)} |f(y,t)| \, dt \, dy.
\]

For the part of \( f \) in \( B(0,2) \times [0,\frac{1}{2}] \) we can use the \( L^1 \) bound on \( f \) combined with the boundedness of the kernel away from 0, that is,

\[
\left| \int_{B(0,2)} \int_0^{\frac{1}{2}} \left( 1 - t \right)^{\frac{1}{2n}} |y|^{-(n+1)} f(y,t) \, dt \, dy \right| \lesssim \int_0^{\frac{1}{2}} \|f(y,t)\|_{L^\infty} \, dt \, dy \lesssim \|f\|_{G^Y},
\]

where we use the observation \( (4.7) \) in Step 2 here. For the part of \( f \) in \( B(0,2) \times [\frac{1}{2},1] \) we can use the \( L^\infty \) bound on \( f \) combined with the integrability of the kernel at 0, that is,

\[
\left| \int_{\mathbb{R}^n} \int_{[\frac{1}{2},1]} \left( 1 - t \right)^{\frac{1}{2n}} |y|^{-(n+1)} f(y,t) \, dt \, dy \right| \lesssim \int_{\frac{1}{2}}^1 \|f(y,t)\|_{L^\infty} \, dt \, dy
\]

Then it suffices to check the integrability of \( \left( 1 - t \right)^{\frac{1}{2n}} |y|^{-(n+1)} \) near 0:

\[
\int_0^1 \left( 1 - t \right)^{\frac{1}{2n}} |y|^{-(n+1)} \, dt = 2\alpha \int_0^1 \frac{s^{2\alpha-1}}{(s+|y|)^{n+1}} \, ds
\]

\[
= 2\alpha |y|^{-(n+1-2\alpha)} \int_0^{|y|} \frac{s^{2\alpha-1}}{(s+|y|)^{n+1}} \, ds \leq 2\alpha B(2\alpha, n + 1 - 2\alpha) |y|^{-(n+1-2\alpha)},
\]

which is integrable near \( y = 0 \in \mathbb{R}^n \) when \( \alpha \in (\frac{1}{2}, 1) \).

Here we complete the proof of \( (4.3) \). And this completes the proof of Theorem \( 1.4 \). \( \square \)

To conclude, we prove the Corollary \( 1.5 \) at the end.
Proof of Corollary \[ \text{Suppose } u(x, t) \text{ is the unique solution constructed above for small initial data } u_0(x) \in B. \]

For \( t \leq t_0 \), we have

\[
\sup_{0 < t \leq t_0} t^{1 - \frac{1}{2\alpha}} \| S(t)u(\cdot, t_0) \|_{L^\infty} \lesssim t_0^{1 - \frac{1}{2\alpha}} \| u(\cdot, t_0) \|_{L^\infty} \leq \| u \|_{G^X},
\]

where we use the integrability of the kernel of \( S(t) \). For \( t > t_0 \), since

\[
u(x, t) = S(t-t_0)u(x, t_0) - \int_{0}^{t} S(t-s)(\nabla I^N(u))(x, s) ds,
\]

which is equivalent to

\[
S(t)u(x, t_0) = S(t_0)u(x, t) + S(t_0) \int_{0}^{t} S(t-s)(\nabla I^N u)(x, s) ds - S(t) \int_{0}^{t_0} S(t_0-s)(\nabla I^N u)(x, s) ds
\]

\[
= S(t_0)u(x, t) + S(t_0)(V\nabla I^N u)(x, t) - S(t)(V\nabla I^N u)(x, t_0).
\]

(4.8)

From Theorem 1.4 and Lemma 4.2 we know \( u(x, t) \in G^X \), \((V\nabla I^N u)(x, t) \in G^X\), respectively. Then applying (4.8) and the fact that the kernel \( \Phi(t_0, x) \) of \( S(t_0) \) is integrable, we have

\[
\sup_{t > t_0} t^{1 - \frac{1}{2\alpha}} \| S(t)u(\cdot, t_0) \|_{L^\infty} \lesssim \| u \|_{G^X} + \| V\nabla I^N u \|_{G^X} + \sup_{t > t_0} t^{1 - \frac{1}{2\alpha}} \| S(t)(V\nabla I^N u)(\cdot, t_0) \|_{L^\infty}.
\]

(4.9)

Then we compute

\[
t^{1 - \frac{1}{2\alpha}} \| S(t)(V\nabla I^N u)(\cdot, t_0) \|_{L^\infty}
\]

\[
\leq t^{1 - \frac{1}{2\alpha}} \| \int_{0}^{t_0} (\Pi \nabla S(t + t_0 - s))(u(s) \otimes u(s)) ds \|_{L^\infty}
\]

\[
\leq t^{1 - \frac{1}{2\alpha}} \left\| \int_{0}^{t_0} \int_{\mathbb{R}^n} \frac{|u(y, s)|^2}{(t + t_0 - s)^{\frac{2}{\alpha}} + |x - y|} \frac{dy ds}{n+1} \right\|_{L^\infty}
\]

\[
\leq t^{1 - \frac{1}{2\alpha}} \left\| \int_{0}^{t_0} \int_{\mathbb{R}^n} \frac{1}{s^{2 - \frac{1}{\alpha}}(t + t_0 - s)^{\frac{2}{\alpha}} + |x - y|} \frac{dy ds}{n+1} \right\|_{L^\infty}
\]

\[
\leq t^{1 - \frac{1}{2\alpha}} \left( \int_{0}^{t_0} \int_{\mathbb{R}^n} \frac{1}{s^{2 - \frac{1}{\alpha}}(t + t_0 - s)^{\frac{2}{\alpha}} + |y|} \frac{dy ds}{n+1} \right) \| u \|_{G^X}^2
\]

\[
\lesssim t^{1 - \frac{1}{2\alpha}} \left( \int_{\mathbb{R}^n} \frac{1}{(t^{2/\alpha} + |y|)^{n+1}} \frac{dy}{12} \right) \| u \|_{G^X}^2 \lesssim \| u \|_{G^X}^2,
\]
where the first inequality is the similar fact to (4.4), the second is by Lemma 3.3, the fifth and last inequality follow from the choice of $\alpha \in (\frac{1}{2}, 1)$ and $t > t_0$. Then combined this result with (4.9), we have
\[
\sup_{t > t_0} t^{1 - \frac{1}{2\alpha}} \| S(t) u(\cdot, t_0) \|_{L^\infty} \lesssim \| u \|_{G^X} + \| V \nabla \Pi Nu \|_{G^X} + \| u \|_{G^X}^2 \lesssim \| u \|_{G^X} + \| u \|_{G^X}^2.
\]
Combining the two cases above, and we know
\[
\sup_{t > t_0} t^{1 - \frac{1}{2\alpha}} \| S(t) u(\cdot, t_0) \|_{L^\infty} \lesssim \| u \|_{G^X} + \| u \|_{G^X}^2,
\]
which completes the proof.

\[\square\]

5. Further results about regularities

Motivated by [4], we study the regularity about the solution constructed by Theorem 1.4. To state the results, we introduce some new notations.

\begin{definition}
For any nonnegative integer $k$, we introduce the space $G^X_k$ which equipped with the norm
\[
\| u \|_{G^X_k} = \sup_{\alpha_1 + \cdots + \alpha_n = k} \sup_{t > 0} t^{(1 - \frac{1}{2\alpha}) + \frac{k}{2\alpha}} \| \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(\cdot, t) \|_{L^\infty}.
\]
And note that $G^X_0$ is the same as the definition for $G^X$ above. For simplicity, we denote
\[
\nabla^k u = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u, \quad \alpha_1 + \cdots + \alpha_n = k.
\]
Then
\[
\| u \|_{G^X_k} = \sup_{t > 0} t^{(1 - \frac{1}{2\alpha}) + \frac{k}{2\alpha}} \| \nabla^k u \|_{L^\infty}.
\]
\end{definition}

\begin{theorem}
For small initial data $\| u_0 \|_B < \epsilon$, the solution constructed in Theorem 1.4 satisfies
\[
i^{(1 - \frac{1}{2\alpha}) + \frac{k}{2\alpha}} \nabla^k u \in G^X_0
\]
for any $k \geq 0$. And this is equivalent to $u \in G^X_k$.
\end{theorem}

Similar to Lemma 3.3, we have the general version that

\begin{lemma}
Similarly, we have bounds for the kernel of $\Pi \nabla^{k+1} S(t)$ that
\[
| \nabla^{k+1} \Pi \Phi_{\frac{t}{2^\alpha}}(x) | \leq C \left( t^{\frac{1}{2\alpha}} + |x| \right)^{-(n+k+1)}.
\]
\end{lemma}

In fact, we need a more precise estimate for the proof of the analyticity result:

\begin{lemma}
The kernel of $\Pi \nabla^{k+1} S(t)$ satisfies that
\[
| \nabla^{k+1} \Pi \Phi_{\frac{t}{2^\alpha}}(x) | \leq C^k k^{\frac{k}{2\alpha}} \left( \frac{t}{2^\alpha} \right)^{\frac{1}{2\alpha}} \left( \frac{t}{k} \right)^{\frac{1}{2\alpha}} + |x| \right)^{-(n+1)}
\]
for all $t > 0$, $x \in \mathbb{R}^n$, $\nabla^{k+1}$ denote $\partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} \nabla$ where $\beta_1 + \cdots + \beta_n = k$.
\end{lemma}

Before stating the proof, we need the following fact cited from [4].
Lemma 5.5. There exists some constant $C$ such that
\[ \int_{\mathbb{R}^n} (a + |x - y|)^{-n-1} (b + |y|)^{-n-1} \, dy \leq C a^{-1} (a + |x|)^{-n-1} \]
for all $x \in \mathbb{R}^n$ and $0 < a < b$.

Proof of Lemma 5.4. By scaling property, it suffices to prove
\[ \| \nabla^{k+1} \Pi \Phi(x) \| \leq C k^{\frac{1}{2\alpha}} (k^{\frac{1}{2\alpha}} + |x|)^{-(n+1)} \]
Since
\[ \nabla^{k+1} \Pi \Phi(x) = \nabla \Pi \Phi \left( \frac{1}{2}, \cdot \right) * \nabla^{k} \Phi \left( \frac{1}{2}, \cdot \right), \]
where
\[ \| \nabla \Pi \Phi \left( \frac{1}{2}, x \right) \| \leq C (1 + |x|)^{n-1} \]
Then by Lemma 5.5, it suffices to prove
\[ \| \nabla \Phi \left( \frac{1}{2}, x \right) \| \leq C \left( 1 + |x| \right)^{-n-1} \]
From Lemma 3.4, then \[ \| \nabla \Phi \left( \frac{1}{2}, x \right) \| \leq C \left( 1 + |x| \right)^{n-1} \]
and then scaling implies
\[ \| \nabla \Phi \left( \frac{1}{2k}, x \right) \| \leq C k^{\frac{n+1}{2\alpha}} (1 + |x|)^{n-1} = C (k^{-\frac{1}{2\alpha}} + |x|)^{-n-1}. \] (5.1)
For fixed $k > 0$, we write
\[ \nabla^{l+1} \Phi \left( \frac{1}{2k}, x \right) = \nabla \Phi \left( \frac{1}{2k}, \cdot \right) * \nabla^{l} \Phi \left( \frac{1}{2k}, \cdot \right), \]
then we could use induction on $l \geq 0$ by applying Lemma 5.5 to get the estimate
\[ \| \nabla^{l+1} \Phi \left( \frac{1}{2k}, x \right) \| \leq C l^{\frac{n+1}{2\alpha}} (k^{-\frac{1}{2\alpha}} + |x|)^{-n-1}, \]
where (5.1) is our induction hypothesis. Let $l = k$, then we get our desired result. \qed

And from Lemma 3.4, we know that the kernel of $\nabla S(t)$ is integrable, thus we have the trivial estimate that
\[ \| \nabla S(t) u \|_{L^\infty} \leq C \frac{1}{t^\alpha} \| u \|_{L^\infty}. \] (5.2)

Now we state two lemmas, which are the constituents of the proof of Theorem 5.2. We first manipulate the linear terms.

Lemma 5.6. For $k \geq 0$, we have
\[ \| S(t) u \|_{G^k} \leq C_k \| u \|_{B}. \]

Proof. By the equivalence of norms that we have proved previously in the precise version (2.3) of Lemma 1.2, we know that
\[ t^{1-\frac{1}{2\alpha}+\frac{k}{2\alpha}} \| S(t) \nabla^k u \|_{L^\infty} = \| \nabla^k u \|_{\dot{B}^{-k-(2\alpha-1),\infty}_{\infty}}. \]
Then this lemma follows from the fact that $\nabla$ is bounded from $\dot{B}^{-l,\infty}_{\infty}$ to $\dot{B}^{-l-1,\infty}_{\infty}$ for $l \geq 0$. \qed

Then for the nonlinear terms, we have
Lemma 5.7. For $k \geq 1$, we have
\[
\|B(u, v)\|_{G^{k}} \leq C_0(k)\|u\|_{G^{0}}\|v\|_{G^{0}} + C_1\|u\|_{G^{k}}\|v\|_{G^{k}} + C_1\|u\|_{G^{k}}\|v\|_{G^{0}} + C_2 \sum_{l=1}^{k-1} \binom{k}{l} \|u\|_{G^{l}}\|v\|_{G^{k-l}}
\]
where
\[
B(u, v)(x, t) = V^{+} \nabla (u \otimes v) = \int_{0}^{t} S(t-s) \Pi \nabla (u(\cdot, s) \otimes v(\cdot, s)) \, ds.
\]
And for $k = 0$, we have proved that $\|B(u, v)\|_{G^{0}} \leq C\|u\|_{G^{0}}\|v\|_{G^{0}}$.

Proof. Fix some $m = m(k)$ which will be determined later. If $0 < s < t(1 - \frac{1}{m})$, we use Lemma 5.4 to obtain
\[
\int_{0}^{t(1 - \frac{1}{m})} |\nabla^k S(t-s)\Pi \nabla u(s) \otimes v(s)| \, ds
\]
\[
\leq \int_{0}^{t(1 - \frac{1}{m})} C^{k} k^{\frac{k}{2\alpha}} \int_{\mathbb{R}^n} \frac{|u(y, s)||v(y, s)|}{(t-s)^{\frac{k}{2\alpha}} \left( (\frac{t-s}{k})^{\frac{1}{2\alpha}} + |x-y| \right)^{n+1}} \, dy \, ds
\]
\[
\leq \int_{0}^{t(1 - \frac{1}{m})} C^{k} k^{\frac{k}{2\alpha}} \int_{\mathbb{R}^n} \frac{|u(y, s)||v(y, s)|}{(t-s)^{\frac{k}{2\alpha}} \left( (\frac{1}{k})^{\frac{1}{2\alpha}} + |x-y| \right)^{n+1}} \, dy \, ds
\]
\[
\leq C^{k} k^{\frac{k}{2\alpha}} \left( \frac{m}{t} \right)^{\frac{k+n+1}{2\alpha}} \int_{0}^{t(1 - \frac{1}{m})} \sum_{q \in \mathbb{Z}^d} \frac{1}{(\frac{1}{k} + |q|)^{n+1}} \int_{y \in x + B(t \frac{1}{m}, q \frac{1}{t})} |u(y, s)||v(y, s)| \, dy \, ds
\]
\[
\leq C^{k} k^{\frac{k+1}{2\alpha}} m \left( \frac{m}{t} \right)^{\frac{k+n+1}{2\alpha}} \frac{1}{t^{n+2-\frac{2}{\alpha}}} \sup_{q \in \mathbb{Z}^d} \int_{0}^{t(1 - \frac{1}{m})} \int_{y \in x + B(t \frac{1}{m}, q \frac{1}{t})} |u(y, s)||v(y, s)| \, dy \, ds
\]
\[
\leq C^{k} k^{\frac{k+1}{2\alpha}} m \left( \frac{m}{t} \right)^{\frac{k+n+1}{2\alpha}} \frac{1}{t^{n+2-\frac{2}{\alpha}}} \|u\|_{G^{k}}\|v\|_{G^{0}}.
\]
If $t(1 - \frac{1}{m}) \leq s < t$, we use (5.2) to obtain
\[
|\nabla^k S(t-s)\Pi \nabla (u \otimes v)| \leq \frac{1}{(t-s)^{\frac{k}{2\alpha}}} \sum_{l=0}^{k} \binom{k}{l} \|\nabla^l (u(\cdot, s))\|_{L^\infty} \|\nabla^{k-l} v(\cdot, s)\|_{L^\infty}
\]
\[
\leq \frac{1}{(t-s)^{\frac{k}{2\alpha}}} \sum_{l=0}^{k} s^{-\frac{k+2\alpha-2}{2\alpha}} \binom{k}{l} \|u\|_{G^{l}}\|v\|_{G^{k-l}}.
\]
Therefore,
\[
\left| \int_{t(1 - \frac{1}{m})}^{t} \nabla^k S(t-s)\Pi \nabla (u(s) \otimes v(s)) \, ds \right| \leq \int_{t(1 - \frac{1}{m})}^{t} \frac{1}{(t-s)^{\frac{k}{2\alpha}}} s^{-\frac{k+2\alpha-2}{2\alpha}} ds \sum_{l=0}^{k} \binom{k}{l} \|u\|_{G^{l}}\|v\|_{G^{k-l}}.
\]
Let 

\[ I(k,m,t) = \int_{t(1-\frac{1}{m})}^{t} \frac{1}{(t-s)^{\frac{k+4\alpha-2}{2\alpha}}} ds, \]

then

\[
I(k,m,t) = \frac{1}{t^{\frac{k+4\alpha-2}{2\alpha}}} \int_{1-\frac{1}{m}}^{1} \frac{1}{(1-z)^{\frac{k+4\alpha-2}{2\alpha}}} dz \leq t^{-\frac{k+4\alpha-2}{2\alpha}} \int_{1-\frac{1}{m}}^{1} (1-z)^{-\frac{1}{2\alpha}} dz
\]

where \( g(m) = \frac{1}{m^{\frac{k+4\alpha-2}{2\alpha}}} \). Take \( m = m(k) = k^{\frac{(2\alpha-1)k-(2\alpha+1)}{\alpha+k+4}} \), then \( g(m) \to 0 \) as \( k \to \infty \). Thus \( |g(m)| \leq C \) are uniformly bounded. Therefore, Lemma 5.7 follows from (5.3), (5.5), (5.6).

Let \( \tilde{G}X^k = \bigcap_{l=0}^{k} GX^l \) equipped with the norm \( \| \cdot \|_{GX^k} = \sum_{l=0}^{k} \| \cdot \|_{GX^l} \). Then Lemma 5.7 implies that

\[
\| B(u,v) \|_{GX^k} \leq C_0(k) \| u \|_{GX^0} \| v \|_{GX^0} + C_1 \| u \|_{GX^0} \| v \|_{GX^k} + C_1 \| u \|_{GX^k} \| v \|_{GX^0} + C(k) \| u \|_{GX^{k-1}} \| v \|_{GX^{k-1}}
\]

for any \( k \geq 1 \). And we already know \( \| B(u,v) \|_{GX^0} \leq C \| u \|_{GX^0} \| v \|_{GX^0} \).

Now we define an approximating sequence

\[
v^{-1} = 0, \quad v^0 = S(t)u_0, \]

\[
v^{j+1} = v^0 + B(v^j,v^j),
\]

then for any \( k \geq 0 \), \( v_j \) converges in \( \tilde{G}X^k \) provided that \( v^0 \) is small enough in \( \tilde{G}X^k \). Moreover, we need the following lemma to conclude the proof of Theorem 5.2.

**Lemma 5.8.** Let \( u_0 \) be small enough in \( B \), then for \( \forall k \geq 0 \), there exist constants \( D_k, E_k \) such that

\[
\| v^j \|_{GX^k} \leq D_k,
\]

\[
\| v^{j+1} - v^j \|_{GX^k} \leq E_k \frac{2}{3}^j.
\]

In particular, for any \( k \geq 0 \), \( v^j \) converges in \( \tilde{G}X^k \).

The proof is omitted, which is exactly the same with Lemma 4.3. in [4] with different formulation for \( B \). Then the lemma implies that \( v^j \) to \( v \) converges in \( \tilde{G}X^k \), where \( v \) is the solution in the sense of Theorem 1.4, therefore this completes the proof of Theorem 5.2.

As a straightforward corollary of the theorem, we get the decay of space derivatives:

**Corollary 5.9.** If the initial data \( \| u_0 \|_B \) is small enough, then the solution constructed in Theorem 1.4 satisfies

\[
\| \nabla^k u \|_{L^\infty} \leq C \frac{1}{t^{(1-\frac{1}{m})+\frac{k}{2\alpha}}}
\]

for any \( t \geq 0 \) and \( k > 0 \).
6. **Analyticity of the solution in space variable**

In this section, we prove the following result.

**Theorem 6.1.** If \( \|u_0\|_B \) is sufficiently small, then the solution constructed in Theorem 1.4 is analytic in space variable.

By applying the Stirling’s formula, it suffices to prove
\[
\|\nabla^k u\|_{L^\infty} \lesssim C^k k^{k-1} \frac{1}{t^{(1 - \frac{1}{2\alpha}) + \frac{k+2}{2\alpha}}},
\]
where \( C \) is a constant independent of \( k \). And by definition, we will establish the estimate that
\[
\|u\|_{G^k} \lesssim C^{k-1} k^{k-1},
\]
where \( k > K \) for some \( K \) sufficiently large.

**Lemma 6.2.**
\[
\|\nabla^k S(t)u_0\|_{L^\infty} \leq C^{k+2} k^{\frac{k+2}{2\alpha}} t^{-\frac{1}{2\alpha}} - \frac{1}{2\alpha} \|u_0\|_B.
\]

**Proof.** Take \( N \) such that \( 2^N \sim t^{-\frac{1}{2\alpha}} \), then
\[
\|\nabla^k S(t)u_0\|_{L^\infty} \leq \sum_{j \leq N} \|P_j(\nabla^k S(t)u_0)\|_{L^\infty} + \sum_{j > N} \|P_j(\nabla^k S(t)u_0)\|_{L^\infty}
\]
\[
\leq \sum_{j \leq N} \|P_j(\nabla^k S(t)u_0)\|_{L^\infty} + C \sum_{j > N} 2^{-2j} \|P_j(\nabla^{k+2} S(t)u_0)\|_{L^\infty}
\]
\[
\leq \sum_{j \leq N} \|\nabla^k S(t)P_j u_0\|_{L^\infty} + C \sum_{j > N} 2^{-2j} \|\nabla^{k+2} S(t)P_j u_0\|_{L^\infty}
\]
\[
\leq \sum_{j \leq N} \left( C \left( \frac{k}{t} \right)^{\frac{1}{2\alpha}} \right)^k \|P_j u_0\|_{L^\infty} + C \sum_{j > N} 2^{-2j} \left( C \left( \frac{k+2}{t} \right)^{\frac{1}{2\alpha}} \right)^{k+2} \|P_j u_0\|_{L^\infty}
\]
\[
\leq C^{k+2} \|u_0\|_B \left( \sum_{j \leq N} \left( \frac{k}{t} \right)^{\frac{k}{2\alpha}} 2^j (2^{\alpha - 1}) + C \sum_{j > N} \left( \frac{k+2}{t} \right)^{\frac{k+2}{2\alpha}} 2^{-2j_1} \right)
\]
\[
\leq C^{k+2} \|u_0\|_B \left( \left( \frac{k}{t} \right)^{\frac{k}{2\alpha}} 2^{N(2\alpha - 1)} + C \left( \frac{k+2}{t} \right)^{\frac{k+2}{2\alpha}} 2^{-2N + N(2\alpha - 1)} \right)
\]
\[
\leq C^{k+2} \|u_0\|_B k^{\frac{k+2}{2\alpha}} t^{-\frac{k+2-2(2\alpha-1)}{2\alpha}},
\]
where the fourth inequality follows from applying (5.2) \( k \) times, the fifth inequality follows from the definition of Besov space and Theorem 1.3.

Now we show the key result
\[
\|u\|_{G^k} \leq C^{k-1} k^{k-1}. \tag{6.1}
\]
by induction on \( k \). Recall a combinatorial result from [5].
**Lemma 6.3.** Let $\delta > \frac{1}{2}$, then there exists some constant $C = C(\delta) > 0$, such that

$$
\sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} |\gamma|^{\mid \gamma\mid - \delta} |\alpha - \gamma|^{\mid \alpha - \gamma\mid - \delta} \leq C |\alpha|^{\mid \alpha\mid - \delta}
$$

for all $\alpha \in \mathbb{N}_0^n$.

From Lemma 5.7 we know

$$
\|B(u, v)\|_{GX^k} \leq C_0(k)\|u\|_{GX^0}\|v\|_{GX^0} + C_1\|u\|_{GX^0}\|v\|_{GX^k} + C_1\|u\|_{GX^k}\|v\|_{GX^0} + C \sum_{l=1}^{k-1} \binom{k}{l} \|u\|_{GX^l}\|v\|_{GX^{k-l}}.
$$

Now let us assume that (6.1) is valid for $1, \cdots, k - 1$, then for $k$, using the induction hypothesis and applying Lemma 6.3 we have

$$
\sum_{l=1}^{k-1} \binom{k}{l} \|u\|_{GX^l}\|u\|_{GX^{k-l}} \leq \sum_{l=1}^{k-1} \binom{k}{l} C^{l-1}k^{l-1}C^{k-l-1}(k-l)^{k-l-1} \leq C^{k-2}k^{k-1}.
$$

Then apply Lemma 6.2 we have

$$
\|u\|_{GX^k} \leq \|S(t)u_0\|_{GX^k} + \|B(u, u)\|_{GX^k} \lesssim C^{k-1}k^{\frac{k+1}{2m}}\|u_0\|_B + C_0(k)\|u\|_{GX^0}^2 + 2C_1\|u\|_{GX^0}\|u\|_{GX^k} + C^{k-1}k^{k-1}.
$$

Theorem 1.4 tells us $\|u\|_{GX^0}$ is small, so that the term $2C_1\|u\|_{GX^0}\|u\|_{GX^k}$ can be incorporated into LHS. Note that $k^{\frac{k+1}{2m}} \leq k^{k-1}$ for all sufficiently large $k$. (For small $k$, we could choose $C$ large enough.) Thus the theorem follows from the construction of $C_0(k)$ in (5.3) and the choice of $m = m(k) = k^{\frac{2(2-k)}{2m}}m^{\frac{k+1}{2m}}$ so that $C_0(k) = Ck^k m^{\frac{k+1}{2m}}$ is of the form $C^{k-1}k^{k-1}$ from the proof of Theorem 5.2. Here we completes the proof.

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