Zitterbewegung and the Magnetic Moment of the Electron

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Abstract

Zitterbewegung of a Dirac electron is an oscillation between positive and negative energy states, and is thus distinct from the analogous phenomena exhibited by spin half charged particles in electric and magnetic fields. Quantum field theory offers an insight into the velocity operator and provides an interpretation of zitterbewegung. Applying stationary perturbation theory to these results the electron $g$ factor is obtained analytically up to the Schwinger correction ($g = 2 + \alpha/\pi$).

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I. INTRODUCTION

The study of single quantum systems has gained importance over the past four decades. Single quantum systems have been studied using the method of quantum simulation [1], an approach of great significance for quantum computation [2]. A subject that has received much recent attention in the context of the simulation of single quantum systems is *zitterbewegung*, a rapid oscillation of the electron between positive and negative energy states with a frequency of $\frac{2E}{\hbar}$, where $E$ is the energy of the electron, so that the lowest possible zitterbewegung frequency of the electron is of the order of $10^{21}$ sec$^{-1}$ [3, 4]. This very high frequency has been an obstacle to a direct observation of zitterbewegung, but zitterbewegung has been simulated using particles that obey equations mathematically similar to the Dirac equation.

The oscillatory motion due to the force exerted on an electron moving in an electric field and the Larmor precession of an electron in a magnetic field are both described as zitterbewegung [5]. Studies on spintronics apply this term to the coupling between the components of the eigenstates of a system, and so this manifestation of zitterbewegung is not necessarily a relativistic effect [6]. Trapped ions are also candidates for simulating zitterbewegung [1, 7, 8]. Such particles obey Hamiltonians mathematically similar to that of a Dirac electron, such as the Dresselhaus and the Rashba Hamiltonians [9, 10]. But because all these phenomena are qualitatively distinct from the zitterbewegung of the Dirac-Schrödinger theory, authors qualify their statements with disclaimers such as “simulation of the Dirac equation” [7], “zitterbewegung-like phenomena” [11], “analogy” [12], “reminiscent of zitterbewegung” or “zitterbewegung effect” [13], etc. The essential difference between the simulated zitterbewegung observed in spintronics and the zitterbewegung of the Dirac electron - henceforth abbreviated as ZB - is that the former is encoded in the spin degree of freedom [14], whereas the latter involves an oscillation between positive and negative energy states.

With this resurgence of interest in ZB it is appropriate to take a fresh look at the fundamental physics of ZB. In this context a couple of issues have emerged which we will address in this paper.

I. In deriving the equations of ZB Dirac defined a velocity operator $c\alpha$. Each component of this operator has eigenvalues $\pm c$. Since the measured velocity of an electron can never equal the speed of light, this result appears to violate the principles of relativity [15].
II. Because the simulations of ZB are quite distinct from the original ZB it needs to be determined whether the ZB predicted by Schrödinger and Dirac is a real physical observable for a free electron [16, 17].

We will show that the resolution of these issues leads to a deeper insight into ZB, and it will become evident that ZB is indeed a physical observable. The electron magnetic moment has been theoretically predicted and experimentally measured to a very high order of precision [18–20], a fact that offers striking proof for the validity of quantum electrodynamics. We will show that the $g$ factor of the electron is a manifestation of ZB, and thereby establish the physical reality of ZB.

We shall use quantum field theory to provide an insight into the physics of ZB. Armed with this insight we shall work within the Dirac theory of positive and negative energy electron states, and apply perturbation theory to derive the magnetic moment of the electron up to the Schwinger correction. We will not use Feynman graphs, but it will be evident that our result is in agreement with the one obtained by the standard methods [21–26].

In this article we shall establish these results:

1. The eigenfunctions of the velocity operator of the Dirac electron are linear combinations of equal amplitudes of positive and negative energy states, and hence ZB states.

2. A field theoretical analysis of the Dirac equation shows that the eigenvalues ($\pm c$) of the velocity operator apply to the electromagnetic field, and thus represent the velocity of the electromagnetic field, and not of the real electron.

3. The application of perturbation theory to the positive and negative energy states offers an alternate method for obtaining the magnetic moment of an electron in a weak magnetic field, and enables us to calculate the electron $g$ factor to the first order ($g = 2$). The ZB of the electron modifies the magnetic field, and incorporating this modification into the perturbation calculation yields the Schwinger correction $\alpha/\pi$ to the $g$ factor.

II. DIRAC’S RELATIVISTIC WAVE EQUATION AND ZB

From Dirac’s equation we may write the Hamiltonian for a free electron as

$$H = c\mathbf{\alpha} \cdot \mathbf{p} + \beta mc^2$$

(1)
where $mc^2$ is the electron rest energy and $\alpha$ and $\beta$ are $4 \times 4$ matrices

$$
\alpha^i = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} \quad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}
$$

(2)

that operate on the $4 \times 1$ Dirac spinors. Here the $\sigma^i$ are the familiar $2 \times 2$ Pauli matrices and $I$ is a $2 \times 2$ identity matrix.

The velocity operator $\dot{x}_i$ may be defined by the relation $i\hbar\dot{x}_i = [x_i, H]$ where $H$ is given by Eq.(1). It is a straightforward step to show that

$$
\dot{x}_i = -\frac{i}{\hbar}[x_i, H] = c\alpha_i. \quad (3)
$$

Equivalently, $\dot{x}_i = \frac{\partial H}{\partial p_i} = c\alpha_i$. The eigenvalues of $\alpha_i$ are equal to $\pm 1$ for all three values of $i = 1, 2, 3$ so that each component of the velocity operator has the two eigenvalues $\pm c$.

At face value it appears that this result violates the principles of special relativity according to which a particle with non-zero rest mass cannot be observed traveling at the speed of light. Dirac offered an explanation of this result, but he used a classical definition of velocity, that the velocity is obtained by measuring the position of the particle at two different times:

"The great accuracy with which the position of the electron is known during the time-interval must give rise, according to the principle of uncertainty, to an almost complete indeterminacy in its momentum. This means that almost all values of the momentum are equally probable, so that the momentum is almost certain to be infinite. An infinite value for a component of momentum corresponds to the value of $\pm c$ for the corresponding component of velocity (p 262)."

We can add rigor to Dirac’s explanation by setting up a definition for velocity as a quantum mechanical observable. The kinematic relation

$$
p_i = \frac{mv_i}{\sqrt{1 - v^2/c^2}}
$$

permits us to define the “velocity squared” operator as

$$
v^2 = \frac{p^2}{m^2 + p^2/c^2}
$$

(5)

Since the momentum is totally indeterminate, all possible values of $(p_x, p_y, p_z)$ can occur with equal probability. So the expected value of $v^2$ is

$$
\langle v^2 \rangle = \frac{\int_0^\infty v^2d^3p}{\int_0^\infty d^3p} = \frac{4\pi \int_0^\infty v^2p^2dp}{4\pi \int_0^\infty p^2dp}
$$

(6)
The ratio of these diverging integrals may be evaluated by taking the limit \( k \to \infty \) of

\[
\langle v^2 \rangle_k = \frac{\int_0^k v^2 p^2 dp}{\int_0^k p^2 dp}
\]  

(7)

Using Eq.(5), we obtain

\[
\langle v^2 \rangle_k = \frac{\int_0^k \frac{p^4}{(m^2+p^2/c^2)} dp}{\int_0^k p^2 dp} = c^2 + O \left( \frac{m^2}{k^2} \right).
\]

(8)

So \( \langle v \rangle \equiv \sqrt{\langle v^2 \rangle} = c \).

So the expected velocity of any particle, be it luminal or subluminal, equals the speed of light. While this is an interesting result, it does not address our problem. The above derivation of the expected electron velocity did not use Dirac’s equation. Specifically, the matrix structure of the wave equation and the spinor wave functions did not feature in our calculation. So the proof we have given above does not offer a physical insight into the Dirac electron velocity operators having the eigenvalues \( \pm c \). We take up this issue below.

Upon diagonalizing the \( \alpha_i \) matrices, the eigenstates corresponding to the positive eigenvalue \( +c \) of \( c\alpha_x, c\alpha_y \) and \( c\alpha_z \) are, in that order:

\[
\text{Eigenstates of } c\alpha_x, \ c\alpha_y \text{ and } c\alpha_z = \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \quad \begin{bmatrix} a \\ b \\ -ib \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a \\ b \\ -a \end{bmatrix}
\]

(9)

and a similar set for the negative eigenvalue \( -c \). Now \( a \) and \( b \) are independent within normalization constraints. So by setting each of them separately to 0 it can be seen that each solution contains equal proportions of positive and negative energy states. Thus they are not eigenstates of the Hamiltonian. Conversely, a pure positive or negative state is not an eigenfunction of the velocity operator \( c\alpha_i \). A superposition of positive and negative energy states in equal proportions describes zitterbewegung (ZB), in which the electron oscillates between positive and negative energy states with frequency proportional to the energy difference or \( 2E \). We shall next show that ZB also involves a spatial oscillation with the same frequency.

The time dependent velocity operator of an electron is given by [4, 24, 28, 29]:

\[
c\alpha(t) = c[\alpha(0) - c\mathbf{p}H^{-1}]e^{-2iHt/\hbar} + c^2\mathbf{p}H^{-1}
\]

(10)
where $p$ is the momentum and $H$ the Hamiltonian of the electron. The last term on the right side of the equation is the “traditional” velocity of the electron, equal to its momentum divided by its relativistic mass. The other terms are ZB terms, oscillating with frequency $\frac{2E}{\hbar}$. The first term within brackets is an operator with eigenvalues $\pm c$. This term can be resolved into a longitudinal component in the direction of the electron momentum and a transverse component perpendicular to the momentum. The transverse component

$$c\alpha(t) = c\alpha(0)e^{-iHt/\hbar} \quad (11)$$

may be formally integrated to provide the displacement matrix operator with $x$ component

$$x(t) = \frac{i\hbar}{2H}c\alpha_x(0)e^{-iHt/\hbar} + \text{constant} \quad (12)$$

where the constant may be set to zero without loss of generality for displacement. Defining the operator $x^2 = x^\dagger x$ we obtain

$$x^2 = \frac{c^2\hbar^2}{4H^2} \quad (13)$$

A ZB state, which is a linear combination of positive and negative energy states, is an eigenstate of the $H^2$ operator, though not of the $H$ operator. So a ZB state is also an eigenfunction of the $x^2$ operator. Thus, for an electron executing ZB the eigenvalue of the $x^2$ operator is

$$\frac{c^2\hbar^2}{4E^2} = \frac{\hbar^2}{4m^2c^2} \quad \text{for an electron at rest.}$$

This suggests that the only measurable values of $x$ are $\frac{\hbar}{2mc}$ and $-\frac{\hbar}{2mc}$. And since the measured velocity is $\pm c$, the conclusion appears to be that the electron jumps discretely from one value of $x$ to the other at the speed of light. This apparently unphysical result will be interpreted below using quantum field theory.

### III. INTERACTION OF A FREE ELECTRON WITH THE ZERO POINT ENERGY

ZB persists when the Dirac field is quantized [30], showing that it is not an artifact of the single particle theory. Upon quantization $\psi$ is reinterpreted as the field operator $\hat{\psi}$ in Fock space, and the operator property of $\hat{\psi}$ is borne by the creation and annihilation operators. Now the electron current operator $\int \hat{\psi}^\dagger c\alpha \hat{\psi} d^3x$ is expandable into corresponding time independent and time dependent terms, and these latter are responsible for the ZB.
There are two such terms, and each of them is expressible as a current equation [30]. One of them represents the transverse ZB current:

\[
\hat{Z}_\perp = \sum_p c(\sqrt{2}\eta_\perp [\hat{c}(p,2)\hat{d}(-p,1) \exp(i2Et/\hbar) - \hat{c}(-p,1)\hat{d}(p,2) \exp(-i2Et/\hbar)] + h.c.) \tag{14}
\]

The other represents the longitudinal ZB current:

\[
\hat{Z}_\parallel = \sum_p c((mc^2/E)\eta_\parallel ([\hat{c}(p,1)\hat{d}(-p,1) - \hat{c}(p,2)\hat{d}(-p,2)] \exp(i2Et/\hbar) + h.c.) \tag{15}
\]

Here the \(\eta\) are unit vectors perpendicular or parallel to the momentum of the electron and \(h.c.\) stands for Hermitian conjugate. The creation and annihilation operators indicate that virtual electron-positron pairs are continuously created and annihilated around a real electron which may be in a bound or a free state, and the original electron annihilates with the positron, leaving the (newly created) electron as the real particle [30, 31]. Thus Sakurai’s conjecture has been placed on a rigorous footing, that ZB arises from the influence of virtual electron-positron pairs (or vacuum fluctuations) on the electron [16, 24]. Thus quantum field theory offers an intuitive understanding of ZB.

These results may now be applied to the conclusions reached in Section II. The displacement from \(x = \frac{\hbar}{2mc}\) to \(x = -\frac{\hbar}{2mc}\) may be thought of as happening at the speed of light. An electron is annihilated at the first point and another created at the second point at a later time \(\Delta t\) such that \(2x = c\Delta t\). This is of course not the same as the electron itself traveling from \(+\frac{\hbar}{2mc}\) to \(-\frac{\hbar}{2mc}\) at the speed of light. The velocity jumps from \(c\) to \(-c\) and back again to \(c\) ad infinitum. The particle itself does not travel, only the field does. The Dirac velocity operator expresses the speed with which the electron is transported by the electromagnetic field. The situation is analogous to the classical case of electrons being transported along a wave guide at the speed of light. Here it is the electromagnetic wave that travels at the speed of light, not the individual electrons themselves. (For a discussion and references, cf. [32]).

One important effect of ZB is to generate a small increase of energy of the electron due to the electrostatic repulsion between the electron and its ZB twin. Because the time interval between the annihilation of the electron at a point \(A\) and the creation of another at \(A'\) is exactly the distance \(AA'\) divided by \(c\), each electron will experience the electrostatic force of its twin, since this force is mediated at the speed of light. This is therefore a relativistic
effect, and would vanish if the speed of light were infinity. Thus ZB will cause an increase in energy of the electron in the amount

\[ \frac{e^2}{4\pi\epsilon_0 R} \]

where \( R \) is the distance between the electron and its twin = \( \frac{h}{m_c} \). So the increase of energy = \( \frac{e^2 m_c}{4\pi\epsilon_0 h} = \alpha m_c^2 \) where \( \alpha = \frac{e^2}{4\pi\epsilon_0 h c} \) is the fine structure constant (expressed as \( \frac{e^2}{h c} \) in electrostatic cgs units).

Thus the rest energy of the electron is augmented to \( m_c^2 (1 + \alpha) \). And so ZB explains the “electromagnetic mass due to a weak interaction between matter and radiation”, which requires “a small correction (\( \sim (\frac{e^2}{h c})m_0 \)) to the mechanical mass \( m_0 \)” [21].

In the following section we shall obtain an expression for the magnetic moment of the electron.

**IV. MAGNETIC MOMENT OF AN ELECTRON**

In this section we shall use the mathematical formalism of the positive and negative energy states of the Dirac theory. We shall not use the results of field theory, but save those for the higher order correction to the magnetic moment that will be discussed in a later section.

We shall consider a free electron at rest in the positive energy state. We shall introduce a weak magnetic field as a small perturbation. It will be seen that because of this perturbation the electron will no longer be in a pure positive state, but will be in a mixture of positive and negative states. To obtain the magnetic moment, we calculate the change in energy due to the perturbation, divide this change of energy by the magnetic field strength, and take the limit as the magnetic field goes to zero.

Since we are applying stationary perturbation theory, the time dependence of the spinors may be ignored, as it will disappear in the matrix elements.

A free electron at rest is described by the Hamiltonian \( H_0 = \beta m_c^2 \). In the absence of external fields the positive and negative energy states can each exist separately as spin up or spin down states of positive or negative energy. We write four linearly independent eigenstates of \( H_0 \) as

\[ |+ \uparrow\rangle, |+ \downarrow\rangle, |- \uparrow\rangle, |- \downarrow\rangle \]
where the first and second states have positive energy and the other two have negative energy. The eigenvalues of this Hamiltonian are $E_+ = mc^2$ and $E_- = -mc^2$. The corresponding eigenstates listed above may also be written as $\begin{bmatrix} \phi \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \chi \end{bmatrix}$ where $\phi$ and $\chi$ can exist in the pure spin state forms $\phi_1$ or $\chi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\phi_2$ or $\chi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ depending on whether the spin is up or down along the $z$ axis. Let us consider an electron in the unperturbed state to be in a positive energy state with energy eigenvalue $mc^2$. The same energy eigenvalue describes both the spin up and the spin down states in the absence of a field. Likewise there is a two-fold degeneracy for the negative energy ($-mc^2$) eigenvalue. All four states are eigenstates of the unperturbed Hamiltonian $H_0 = \beta mc^2$. Let us consider the unperturbed electron to have spin up along the $z$ axis. Now we introduce a weak magnetic field $B$ which we take to be along the $z$ axis, so that the electron has spin parallel to the direction of the magnetic field. So there is a weak perturbation described by the interaction Hamiltonian $H_I = c\alpha \cdot (p - eA) \equiv c\alpha \cdot \pi$.

Applying perturbation theory of degenerate states [33], we obtain the first order perturbation correction to the energy as the eigenvalues of the matrix whose elements are

$$\langle +\alpha|H_I| +\alpha' \rangle$$

where $\alpha$ and $\alpha'$ take on the two different values of spin (up or down). (These $\alpha$ indicate the spin direction and should not to be confused with the 4 × 4 matrix velocity operators.) Thus the matrix whose eigenvalues we seek is

$$\begin{bmatrix} \langle + \uparrow|H_I|+\uparrow \rangle & \langle + \uparrow|H_I|+\downarrow \rangle \\ \langle + \downarrow|H_I|+\uparrow \rangle & \langle + \downarrow|H_I|+\downarrow \rangle \end{bmatrix}$$

Now $|+\uparrow\rangle = \begin{bmatrix} \phi_1 \\ 0 \end{bmatrix}$ and $|+\downarrow\rangle = \begin{bmatrix} \phi_2 \\ 0 \end{bmatrix}$ while $H_I = c\begin{bmatrix} 0 & \sigma \cdot \pi \\ \sigma \cdot \pi & 0 \end{bmatrix}$

Now

$$\begin{bmatrix} \phi_i \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma \cdot \pi \\ \sigma \cdot \pi & 0 \end{bmatrix} \begin{bmatrix} \phi_j \\ 0 \end{bmatrix} = 0$$

(16)

for $i, j = 1, 2$. 
Thus we have the important result that

$$\langle +\alpha|H_I| + \alpha' \rangle = 0$$  \hspace{1cm} (17)

for all $\alpha, \alpha'$. The first order perturbation energy $\epsilon_1$ is zero.

The perturbation energy due to second order perturbation of degenerate states is given by

$$\epsilon^{(\alpha)}_2 = \sum_\beta \frac{|\langle -\beta|H_I| + \alpha \rangle|^2}{E_+^{(0)} - E_-^{(0)}} = \sum_\beta \frac{|\langle -\beta|H_I| + \alpha \rangle|^2}{2mc^2}$$  \hspace{1cm} (18)

where $\alpha$ expresses the spin quantum number of the unperturbed positive state, and $\beta$ the spin quantum numbers of the unperturbed negative energy states. $E_-^{(0)} = -mc^2$ and $E_+^{(0)} = +mc^2$, the energies of the unperturbed (zero magnetic field) negative and positive energy states respectively.

Using the result of Eq. (17) we may add a vanishing term to the numerator to obtain

$$\epsilon^{(\alpha)}_2 = \sum_\beta \frac{\langle +\alpha|H_I| - \beta \rangle \langle -\beta|H_I| + \alpha \rangle}{2mc^2}$$  \hspace{1cm} (19)

Using the identity $\sum_\beta (| - \beta\rangle \langle - \beta| + | + \beta\rangle \langle + \beta|) = I$ we get

$$\epsilon^{(\alpha)}_2 = \frac{\langle +\alpha|H_IH_I| + \alpha \rangle}{2mc^2} = \frac{\langle +\alpha|H_I^2 + \alpha \rangle}{2mc^2}$$  \hspace{1cm} (20)

Thus

$$\epsilon^{(\alpha)}_2 = \frac{\langle +\alpha|(|\sigma \cdot \pi|)^2 + \alpha \rangle}{2m} = \frac{\langle +\alpha|\left(\frac{\pi^2}{2m} - \frac{eh}{2m} \sigma \cdot B\right) + \alpha \rangle}{2m}$$  \hspace{1cm} (21)

where $B$ is the magnitude of the magnetic field, taken to be in the direction of the electron spin (the $z$ axis), and we have used the relation \[1, \text{ and } 27\]

$$(\sigma \cdot \pi)^2 = \pi^2 - eh\sigma \cdot B$$

In order to obtain the magnetic moment of the electron, we take the negative of the partial derivative of the energy with respect to the magnetic field strength $B$ and take the limit as $B \to 0$:

$$m = \lim_{B \to 0} - \frac{\partial \epsilon}{\partial B} = \frac{eh}{2m}$$  \hspace{1cm} (23)
An important observation can be made. $H_I$ couples energy states of opposite sign. So the magnetic moment of the electron arises from the coupling between positive and negative states. It is therefore an aspect of electronic ZB.

This magnetic moment is identical with the one obtained from the non-relativistic Pauli equation which does not explicitly make use of positive and negative energy states \cite{4, 24, 27, 28}. However, it must be kept in mind that the expression for the kinetic energy of a non-relativistic particle is nothing but an approximation for the relativistic kinetic energy at small velocities

$$\frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 \rightarrow mv^2/2 = p^2/2m$$

It is evident that the exponent 2 in the numerator of the nonrelativistic kinetic energy expression came from the power of $v$ inside the radical, and the factor 2 in the denominator arises from the quadratic nature of the square root, which is the basis for the Dirac theory of negative energy states. We showed above that the denominator $2m$ in the magnetic moment arises from the energy difference between the positive and negative energy states of a stationary electron ($2mc^2$). We may therefore conclude that the magnetic moment of the electron is an integral feature of the Dirac theory of positive and negative energy states of a spin half charged particle. We shall next show that a higher order correction (the Schwinger correction) to the magnetic moment shows even more clearly the explicit link between ZB and the magnetic moment.

V. ANGULAR VELOCITY OPERATOR

It was shown in Section II that the ZB frequency of an electron is given by $\frac{2E}{\hbar}$ which is $\frac{2mc^2}{\hbar}$ for an electron at rest, and that the electron executes a spatial linear ZB motion of amplitude $\frac{\hbar}{2mc}$. A linear motion of a charged particle is converted to a circular motion in a plane perpendicular to an imposed magnetic field. Though there is no physical transportation of the ZB electron, yet there is a displacement of charge, and so it is a reasonable hypothesis that the effect of a magnetic field on this linear displacement is to convert it into a circular movement with angular velocity $\frac{2mc^2}{\hbar}$ and radius $\frac{\hbar}{2mc}$. Further support for this hypothesis will be provided through the use of the angular velocity operator, which we define as:

$$\dot{\phi} = \frac{i}{\hbar} [H, \phi]$$

(24)
It is evident that for the unperturbed Hamiltonian \( H = \beta mc^2 \) the angular velocity operator is zero. But if we were to introduce a uniform magnetic field we may write the total Hamiltonian as

\[
H = c\alpha \cdot \mathbf{p} - ec\alpha \cdot \mathbf{A} + \beta mc^2
\]  

(25)

If the magnetic field is along the \( z \) direction then the movement of the charge would be entirely in the \( xy \) plane, giving a Hamiltonian

\[
H = -i\hbar c \left( \alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} \right) - ec(\alpha_x A_x + \alpha_y A_y) + \beta mc^2
\]  

(26)

So

\[
\dot{\phi} = c \left[ \left( \alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} \right), \phi \right]
\]  

(27)

For motion in the \( xy \) plane \( \phi = \tan^{-1}(y/x) \) and this yields

\[
\dot{\phi} = \frac{ic}{r} \begin{bmatrix}
0 & 0 & 0 & -e^{-i\phi} \\
0 & 0 & e^{i\phi} & 0 \\
0 & -e^{-i\phi} & 0 & 0 \\
e^{i\phi} & 0 & 0 & 0
\end{bmatrix}
\]  

(28)

which is a traceless operator with eigenvalues \( \pm c/r \). So the magnitude of the angular velocity equals \( c/r \), which corresponds to a motion along a circle of radius \( r \) at a linear velocity \( c \).

This angular velocity operator is independent of the magnetic field strength, and so in the limit that the external field goes to zero, we would get the ZB of a free electron. A circular motion of radius \( r \) and angular velocity \( c/r \) can be resolved into two orthogonal harmonic vibrations of amplitude \( r \) and frequency \( c/r \). So letting \( r = \frac{\hbar}{2mc} \) it follows readily that \( \dot{\phi} \) has the eigenvalues \( \pm \frac{2mc^2}{\hbar} \), in agreement with the ZB results. We shall now show that the eigenfunctions of the angular velocity operator are also ZB eigenfunctions.

One set of corresponding (not normalized) eigenfunctions of the \( \dot{\phi} \) operator:

Eigenvale \(-\frac{2mc^2}{\hbar} \): 
\[
\begin{bmatrix}
ie^{-i\phi/2} \\
0 \\
ie^{i\phi/2}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
0 \\
ie^{i\phi/2} \\
ie^{-i\phi/2}
\end{bmatrix}
\]

Eigenvale \(\frac{2mc^2}{\hbar} \): 
\[
\begin{bmatrix}
e^{-i\phi/2} \\
0 \\
i e^{i\phi/2}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
0 \\
ie^{i\phi/2} \\
e^{-i\phi/2}
\end{bmatrix}
\]

Each one of these four linearly independent eigenfunctions is a ZB state, having equal proportions of positive and negative energy states. Moreover, each of these eigenstates has...
VI. HIGHER ORDER CORRECTIONS

We shall next apply a correction due to the rotation of the ZB motion of the electron. We introduce a weak magnetic field of strength $B$ along the $z$ axis. In the absence of the field the ZB current is linear, according to Eq. (11). The effect of the magnetic field on the electronic ZB is to generate a rotation of the ZB in the $xy$ plane, as discussed in Section V. Fig. 1 shows the rotation of the ZB in the $xy$ plane. The magnetic field is directed into the page. The negative electron current has a clockwise flow according to the laws of electrodynamics. As in the case of linear ZB there is no mechanical flow of electrons, and so we may picture the electron leaping from point to point along the circle without mechanical momentum. However, because there is a net circular current flow, there is a momentum due to the electromagnetic field [27]:

$$\frac{d\pi}{dt} = e [c\alpha \times B]$$

(29)

Fig. 1 shows the electron $A$ and its ZB twin $A'$ navigating a circle of radius $\frac{\hbar}{2mc}$ in a clockwise direction. The situation is equivalently depicted in Fig. 2 where the electron orbits around its ZB twin along a circle of radius $\frac{\hbar}{mc}$. The actual electromagnetic effects felt by the electron are better expressed by Fig. 2. For the purpose of determining the magnetic moment it is only the magnetic effects that are of interest to us. We shall now obtain an
FIG. 2: ZB rotation of electron $A$ with respect to its twin $A'$.  

FIG. 3: Electron $A$ undergoing a small displacement from $A_1$ to $A_2$.  

expression for the magnetic field experienced by the electron at $A$ due to its twin at $A'$.  

Consider a small displacement $A_1 \rightarrow A_2$ in the path of the electron as in Fig. 3. From the perspective of electron $A$ the ZB twin $A'$ would appear to have undergone a displacement $A'_1 \rightarrow A'_2$ (Fig. 4). One could relate this to macroscopic experience by analogy with a train traveling along a curved track. As the train travels from $A_1$ to $A_2$ the passengers looking out of the window would see a stationary structure at $A'$ appear to move in the forward direction.

FIG. 4: From the perspective of electron $A$ its twin $A'$ undergoes a small displacement from $A'_1$ to $A'_2$.  

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direction from $A'_1$ to $A'_2$. In our case the infinitesimal angular widths of the arcs in the figures have been exaggerated for clarity. The isosceles triangles drawn within the figures merely serve to clarify the equivalence of the two figures. So $\Delta A_1PA'$ of Fig. 3 is identical with $\Delta APA'_1$ of Fig. 4 and $\Delta A_2PA'$ of Fig. 3 is identical with $\Delta APA'_2$ of Fig. 4. Thus the electron at $A$ will see a negative electron current flowing from $A'_1$ to $A'_2$, which means that the electron at $A$ will experience an additional magnetic field in the same direction as the external field. Let the strength of this additional ZB magnetic field be denoted by $\delta B$.

It must be borne in mind that at any given instant the electron is either at $A$ or at $A'$. The situation is symmetrical with respect to $A$ and $A'$. So in calculating the change in energy of the electron it suffices to consider the change in energy of the electron at $A$ only.

We will now obtain the change in energy due to the effect of the external magnetic field on the ZB motion of the electron.

The electron current at $A'$ will induce a magnetic field at $A$ of magnitude

$$\delta B = \frac{I}{2\pi \epsilon_0 c^2 R}$$  \hspace{1cm} (30)

where $I$ is the current and the distance between the two electrons $R = \frac{h}{mc}$. The electron of charge $e$ traverses a circle of radius $\frac{h}{mc}$ at the speed of light $c$. This last sentence may require some clarification. From the results of Section V the ZB electron and its diametrical twin orbit a common center at the speed of light $c$ as viewed in the reference frame of the center. By the laws of relativistic kinematics each particle will see the other traveling at the speed $c$. More accurately, each electron will see a charge being transported at speed $c$. So the current generated by $A'$ as seen by $A$ is given by

$$I = \frac{ec}{2\pi R} = \frac{emc^2}{2\pi \hbar}$$  \hspace{1cm} (31)

So

$$\delta B = \frac{em^2 c}{4\pi^2 \epsilon_0 \hbar^2}$$  \hspace{1cm} (32)

The total energy of the electron due to the sum of the external magnetic field and the field generated by ZB is

$$e'_2 = \frac{\pi^2}{2m} - \frac{eh}{2m} (B + \delta B)$$  \hspace{1cm} (33)

Since $\delta B$ is independent of $B$, it does not contribute to the magnetic moment of the electron which remains at $\frac{eh}{2m}$. So we need to consider a higher order correction to the perturbation energy.
VII. HIGHER ORDER PERTURBATION

We next consider higher order corrections to the perturbation energy due to a weak magnetic field. We saw that \( \epsilon_1 = 0 \). It turns out that all the odd corrections vanish, so \( \epsilon_3 = 0 \). The next higher non-vanishing correction to the energy is the fourth order:

\[ \epsilon_4 = -\frac{\epsilon_2^2}{2mc^2} \]

So

\[ \epsilon_4 = -\frac{1}{2mc^2} \left[ \frac{\pi^4}{4m^2} - 2\frac{\pi^2}{2m} \frac{e\hbar}{2m} \frac{e^2h^2}{2m} B + \frac{e^2h^2}{4m^2} B^2 \right] \] (34)

Here we replace \( B \) by the corrected magnetic field \( B + \delta B \). This yields

\[ \epsilon_4 = -\frac{1}{2mc^2} \left[ \frac{\pi^4}{4m^2} - 2\frac{\pi^2}{2m} \frac{e\hbar(B + \delta B)}{2m} + \frac{e^2h^2}{4m^2} (B + \delta B)^2 \right] \] (35)

The first term is fourth order in the field strength and the second term is third order. Only the third term makes a non-vanishing contribution to the magnetic moment. So the correction to the magnetic moment becomes

\[ \delta m = \lim_{B \to 0} -\frac{\partial \epsilon_4}{\partial B} = \frac{e^2h^2}{4m^3c^2} \delta B \] (36)

\[ \delta m = \frac{e^3}{16\pi^2\epsilon_0 mc} = \frac{e\hbar}{2m} \frac{e^2}{8\pi^2\epsilon_0 \hbar c} = \frac{e\hbar}{2m} \frac{\alpha}{2\pi} \] (37)

Hence the magnetic moment corrected for the ZB effect is given by

\[ m + \delta m = \frac{e\hbar}{2m} \left( 1 + \frac{\alpha}{2\pi} \right) \] (38)

Thus the \( g \) factor of the electron is corrected to \( 2(1 + \alpha/2\pi) \). The leading terms of the magnetic moment of the electron are therefore

\[ m = \frac{e\hbar}{2m} \quad \text{and} \quad \delta m = \frac{e^3}{16\pi^2m\epsilon_0 c}. \]

The first (Dirac) term was derived using relativistic quantum theory. It is a quantum effect since it is proportional to \( \hbar \). The second (Schwinger) term is independent of \( \hbar \) and vanishes as \( c \to \infty \) and so it appears as a relativistic correction.
VIII. CONCLUSION

We have established that the ZB predicted by Schrödinger and Dirac does indeed have a valid physical basis and that the magnetic moment of the electron can be understood as an observable manifestation of ZB. In this process we have provided justification for the field theoretic interpretation of ZB and offered an interpretation for the anomalous velocity of the Dirac electron.

This article is essentially a proof-of-principle study. It opens the doorway to further investigations on ZB related issues, such as higher order corrections to the magnetic moment of the electron, magnetic moments of other particles, the use of Feynman graphs in ZB calculations, etc. The close relationship between ZB and the vacuum indicated by quantum field theory invites further studies on the relevance of ZB theory for the Casimir effect and other vacuum related phenomena.

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