Verification of the numerical method of phase enlargement of non-regenerating systems on the example of modelling the process of functioning the controlled technical system

M V Zamoryonov, V Y Kopp and D V Zamoryonova
Sevastopol State University, 33, Universitetskaya Street, Sevastopol, 299053, Russia

E-mail: zamoryonoff@gmail.com

Abstract. The article verifies the numerical method of phase integration of non-regenerating systems with a common phase space of states by comparing the results of phase integration, made by the proposed method and the classical, known from the literature. The transition probabilities of the enlarged system and the distribution function of the system's residence times in the enlarged states are compared. The results of modeling the obtained expressions are given.

1. Introduction
In modern production, there is an acute problem of increasing the productivity of products while ensuring a given level of its quality, as well as reducing the scrap rate. In this regard, the equipment is introduced into the production systems that control the manufactured products.
There are a large number of different types of controls [1–3]. In this case, one of them is considered, namely “with the work item disconnected for the period of the inspection”. This method allows identifying hidden failures [4], significantly increasing the percentage of defective products.
The complexity of the processes in large systems is explained by the presence of stochasticity, which must be taken into account when modeling production systems. One of the most common methods for modeling such systems is a method that uses the apparatus of semi-Markov processes [5–10].
The greatest difficulty is the construction of models of semi-Markov systems with a common phase space of states. Partial simplification of the simulation problem allows the phase integration algorithm proposed in the research work [5]. However, it should be noted that in order to apply this algorithm, it is necessary to determine the stationary distribution of the embedded Markov chain, which is a very complicated task associated with the need to solve systems of integral equations containing functions of sums and differences of variables.
The authors proposed a method that allows phase enlargement of semi-Markov systems without determining the stationary distribution of the embedded Markov chain (EMC) for systems with a common phase space of states, without resorting.

2. Problem formulation
The purpose of this article is to verify the numerical method of non-regenerating systems with a common phase space of states proposed in the research work [11] using the example of constructing a semi-Markov model of a technical system operation process with the work item disconnected for the period of monitoring.
The proposed method is based on the proven Lemma and Theorem of the distribution functions of
the residence time in a subset of continuous States taking into account the difficult recovery process
[11].

**Lemma.** The distribution function $F_{\gamma}(t)$ of the time difference of two CB ($\alpha - \beta$) with the
distribution functions $F_1(t)$ and $F_2(t)$, accordingly ($\alpha > \beta$), provided that determined by the
expression.

$$F_{\gamma}(t) = \int_0^\infty \frac{F_1(t+y)f_2(y)dy - \int_0^\infty F_1(y)f_2(y)dy}{\int_0^\infty F_1(y)f_2(y)dy},$$
given that

$$F_2(t) = 1_{x}(t) = \begin{cases} 1, & t \geq x; \\ 0, & t < x, \end{cases}$$
is the following

$$F_{\gamma}(t, x) = \frac{F_1(t+x) - F_1(x)}{F_1(x)},$$
where the parameter value $x$ is chosen from the equality

$$F_{\gamma}(y) = \int_0^\infty \frac{[F_1(t+y)-F_1(y)]f_2(y)dy}{\int_0^\infty F_1(y)f_2(y)dy} = \frac{F_1(t+x) - F_1(x)}{F_1(x)},$$

The theorem of the distribution functions of the residence time in a subset of continuous states taking
into account the difficult recovery process

DF of the difference between RV $\alpha$ and RV $\beta_\gamma = \gamma + \sum_{k=0}^\infty \beta_k$ - the recovery time in the same difficult stream generated by the DF $F_2(t)$ of RV $\beta$ and DF $F_{\gamma}(t)$ of the RV $\gamma$, provided that $\alpha > \beta_\gamma$, has the form:

$$F_{\alpha-\beta_\gamma}(t) = P[|\alpha - \beta_\gamma| \leq t] = \int_0^\infty \frac{[F_1(t+y)-F_1(y)]h_\gamma^f(y)dy}{\int_0^\infty [1-F_1(y)]h_\gamma^f(y)dy} = \frac{F_1(t+x) - F_1(x)}{F_1(x)},$$

where $f_\gamma(y)$ and $h_\gamma^f(y)$ - distribution density of RV $\gamma$ and density of recovery of the flow of restorations generated by RV $\beta$ and $\gamma$ equal to $h_\gamma^f(y) = \sum_{k=0}^\infty f_\gamma^f(y) * f_2^i(y)k(y),$
and $x$ is chosen numerically from the ratio (1).

It should be noted that in order to verify the method, it is necessary to obtain the characteristics of the
integrated system and compare them with the corresponding characteristics determined with the help of
the PEA [12].

A system $S$ consisting of a single component that performs certain functions and equipment
monitoring its performance is considered. The system operates as follows. At the initial moment of time,
the component started to work, control is turned on. The uptime component is RV with the distribution
function (DF) $F(t) = P[\alpha \leq t]$ and distribution density (DD) $f(t)$. The control is carried out at random
time $\delta$ with DF $R(t) = P[\delta \leq t]$ and DD $r(t)$. The failure of a component is detected only as a result of monitoring (hidden failure), while the monitoring is in progress, the component is suspended. The duration of the control of RV $\gamma$ with DV $V(t) = P[\gamma \leq t]$ and DD $v(t)$. The recovery time (RT) of the component after the detection of the failure of the RV $\beta$ with DF $G(t) = P[\beta \leq t]$ and DD $g(t)$. For the period of restoration, control is suspended, after restoration all the properties of the component are updated. It is assumed that RV $\alpha, \beta, \delta, \gamma$ are independent and have finite mathematical expectations.

The functioning of the system will be described by a semi-Markov process (SMP) $\zeta(t)$ with a discrete-continuous phase space of states [8–12]. The authors introduce the following set $M$ of semi-Markov system states:

$$M = \{111, 212, 211, 101, 202, 220\}.$$

The authors decipher the meaningful sense of state codes:

- $111$ - component started to work, control is on;
- $212$ - control started, the component is operational and disconnected, there is still time left before the onset of failure (without taking into account the time of the control);
- $211$ - the control is over, the component continues to work, there is time $x > 0$ left until the failure occurs;
- $101$ - there was a refusal, there is time $x > 0$ left before the start of control;
- $202$ - monitoring started, the component in the failure is disabled;
- $220$ - control ended, failure detected, component recovery started, control suspended.

The transition graph of the system is shown in Figure 1, respectively.

![Figure 1. System transition graph](image)

### 3. Problem Solution

The time the system $\theta_{211}$ is in a state of $211$ is determined by two factors: the remaining time $x$ before the occurrence of latent failure and the time $\delta$ that determines the frequency of control. Therefore, $\theta_{211} = \delta \wedge x$ where $\wedge$ is the minimum sign. Similarly, the residence times in other states are determined:

$$\theta_{111} = \alpha \wedge \delta , \theta_{212} = \gamma , \theta_{101} = x , \theta_{202} = \gamma , \theta_{220} = \beta .$$

The authors define the distribution functions of the residence time in the system:

$$F_{n_{111}}(t) = 1 - F(t) \cdot R(t);$$
$$F_{n_{212}}(t) = V(t); F_{n_{202}}(t) = V(t); F_{n_{220}}(t) = G(t);$$
$$F_{n_{211}}(t) = 1 - F_{n}(t) \cdot R(t);$$
\[ F_{n_{101}}(t) = R_y(t), \]

where

\[ F_s(t) = \frac{F(t + x) + F(x)}{F(x)}, R_s(t) = \frac{R(t + y) + R(y)}{R(y)}, \]

\( x \) and \( y \) are defined from the expression (1).

In the research work [12] when using the PEA [5] were obtained the following probabilities of transitions of the embedded Markov chain and DF of times of stay of the system in integrated states \( \hat{S}_{101} \) and \( \hat{S}_{211} \):

\[
\begin{align*}
\hat{F}_{211}^{101} &= \frac{\int f(x)R(x)dx}{\int F_1(t)h_1(t)dt}, \\
\hat{F}_{211}^{212} &= 1 - \frac{\int f(x)R(x)dx}{\int F_1(t)h_1(t)dt}, \\
\hat{F}_{211}(t) &= 1 - \int_0^\infty F(t + z)h_1(z)dz, \\
\hat{F}_{101}(t) &= \int_0^\infty f(z)V_1(z,t)dz.
\end{align*}
\]

where \( V_1(z,x) = R(z + x) - \int_0^x R(z + x - s)h_1(s)ds \) – distribution function of direct residual time for the same recovery process.

The authors define these characteristics of the proposed method.

In this non-integrated system (Figure 1) there are two continuous components. The authors write two additional conditions necessary for their definition. Each condition, as indicated above, is associated with an unchanging parameter, which in this particular case is the mathematical expectation of the time between failures of the system \( m_\rho \) and the mathematical expectation of the operating time for control \( m_k \).

These values are:

\[
m_k = \frac{r_{on_{111}}}{r_{on_{111}} + r_{on_{211}}} (p_{n_{111}}^{101} (mn_{111} + mn_{101}) + p_{n_{211}}^{212} mn_{111}) + \frac{r_{on_{111}}}{r_{on_{111}} + r_{on_{211}}} (p_{n_{211}}^{101} (mn_{211} + mn_{101}) + p_{n_{211}}^{212} mn_{211}),
\]

\[
m_\rho = p_{n_{111}}^{212} \left( mn_{211} + \frac{mn_{101}}{p_{n_{211}}^{101}} \right) + p_{n_{111}}^{101} mn_{111},
\]

where \( mn_i \) – mathematical expectation of the time of stay in the state \( S_i \),

\( \rho n_i \) – stationary distribution of the embedded Markov chain for the state \( S_i \),

\( p_{n_i}^{j} \) – transition probabilities from the state \( S_j \) to the state \( S_i \).
\[ \rho_{101} = \frac{1}{4 + 2 \cdot \frac{P_{111}^{212}}{P_{211}^{101}}} = \rho_{220} = \rho_{202} = \rho_{111}; \]

\[ \rho_{212} = \frac{P_{111}^{212}}{4 \cdot P_{211}^{101} + 2 \cdot P_{111}^{212}} = \rho_{211}; \]

\[ P_{111}^{101} = \int_{0}^{\infty} \bar{R}(t)f(t)dt; \]

\[ P_{111}^{212} = \int_{0}^{\infty} \bar{F}(t)r(t)dt; \]

\[ P_{111}^{212} = \int_{0}^{\infty} \bar{R}(t)f_{e}(t)dt; \]

\[ P_{211}^{212} = \int_{0}^{\infty} \bar{F}_{e}(t)r(t)dt. \]

The system, which must satisfy the desired parameters, has the form:

\[
\begin{align*}
\rho_{101} &= \frac{1}{4 + 2 \cdot \frac{P_{111}^{212}}{P_{211}^{101}}} \\
\rho_{220} &= \rho_{202} = \rho_{111} = \rho_{101} \\
\rho_{212} &= \frac{P_{111}^{212}}{4 \cdot P_{211}^{101} + 2 \cdot P_{111}^{212}} \\
\rho_{211} &= \rho_{212} \\
m_{k} &= m(\delta) \\
m_{o} &= m(\alpha)
\end{align*}
\]

(2)

The simulation results are as follows.

The initial data for the simulation are the DF \( F(t) \), \( R(t) \), \( G(t) \) and \( V(t) \), distributed according to the second order Erlang law with the parameters \( \nu, \lambda, \mu, \sigma \). And

\[ f(t) = \frac{\nu_{1}\nu_{2}(e^{-\nu_{1}t} - e^{-\nu_{2}t})}{\nu_{2} - \nu_{1}}, \]

where \( \nu_{1} = 0.0333 \) (h\(^{-1}\)); \( \nu_{2} = 0.1 \) (h\(^{-1}\)).

\[ r(t) = \frac{\lambda_{1}\lambda_{2}(e^{-\lambda_{1}t} - e^{-\lambda_{2}t})}{\lambda_{2} - \lambda_{1}}, \]

where \( \lambda_{1} = 0.0667 \) (h\(^{-1}\)); \( \lambda_{2} = 0.2 \) (h\(^{-1}\)).
\[ g(t) = \frac{\mu_1 \mu_2 (e^{-\mu_1 t} - e^{-\mu_2 t})}{\mu_2 - \mu_1}, \]

where \( \mu_1 = 0.2667 \ (h^{-1}); \mu_2 = 0.8 \ (h^{-1}). \)

\[ \nu(t) = \frac{\sigma_1 \sigma_2 (e^{-\sigma_1 t} - e^{-\sigma_2 t})}{\sigma_2 - \sigma_1}, \]

where \( \sigma_1 = 1.333 \ (h^{-1}); \sigma_2 = 4.0 \ (h^{-1}). \)

As well as the mathematical expectations of the times of maintenance of a unit of production, mean time to failure and recovery, which are respectively 0.5, 10 and 1.

By solving a system of equations (2) the authors get 

\[ x = 19.489244, \ y = 7.662384 \]

Find the transition probabilities:

\[ \hat{P}_{211}^{101} = 0.398095, \ P_{n211}^{101} = 0.398095 \]
\[ \hat{P}_{211}^{212} = 0.601905, \ P_{n211}^{212} = 0.601905 \]

where \( \hat{P}_{211}^{101}, \ P_{n211}^{101} \) - transition probabilities obtained by the classical method; \( P_{n211}^{101}, \ P_{n211}^{212} \) - transition probabilities obtained using the phase enlargement method.

The conditions of equality of the mathematical expectation of the time between failures of the system and the mathematical expectation of the time of the operating time for control are checked.

\[ m_k = 20 = m(\delta), \ m_o = 40 = m(\alpha). \]

The following (Figure 2) are the results of modeling the functions of the residence time in states 101 and 211.

![Figure 2](image-url)

**Figure 2.** Results of modeling functions of staying in states 101 (a) and 211 (b) using the classical and proposed method

As it can be seen from the graphs, the functions of the system in the states \( S_{101} \) and \( S_{211} \) those obtained using the classical and the proposed method are identical.
4. Conclusion
The complete coincidence of the results was shown, which proved the correctness of the developed method. The proposed method together with the paths method [12] developed by the authors and the trajectory method [13] allows, in further studies, proceeding to the task of automated construction of semi-Markov models.

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