Distance Labelings on Random Power Law Graphs

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Abstract

A Distance Labeling scheme is a data structure that can answer shortest path queries on a graph. Experiment results from several recent studies (Akiba et al.’13, Delling et al.’14) found very efficient and very accurate labeling schemes, which scale to social and information networks with tens of millions of vertices and edges. Such a finding is not expected in the worst case, since even for graphs with maximum degree 3, it is known that any distance labeling requires $\Omega(n^{3/2})$ space (Gavoille et al.’03). On the other hand, social and information networks have a heavy-tailed degree distribution and small average distance, which are not captured in the worst case.

In this paper, we fill in the gap between empirical and worst case results. We consider distance labeling schemes on random graph models with a power law degree distribution. For such graphs, we show that simple breadth-first-search based algorithm can find near optimal labeling schemes. The intuition behind our proof reveals that the distances between different pairs of vertices are almost independent, even for polynomially many pairs.
1 Overview

Distance Labeling refers to a family of data structures for answering distance queries [28]. Each vertex is assigned a “labeling”; To answer a query between the distance of two vertices $x$ and $y$, one is only allowed to use the labels of $x$ and $y$ to compute $\text{dist}(x,y)$. Labeling schemes are designed to speed up distance queries on large graphs, where computing shortest path from scratch is expensive. In this paper, we study distance labeling schemes on random graph models for social and information networks.

A simple yet commonly used scheme is landmark based labelings (also known as 2-hop covers [19], hub labeling [1]). The idea is to find central landmarks that lie on the shortest paths of many sources and destinations. Meanwhile, every vertex stores a set of landmarks as well as its distance to each landmark. To answer a distance query $\text{dist}(x,y)$, we simply find a common landmark $z$ in the landmark sets of $x$ and $y$ to minimize the sum of distances $\text{dist}(x,z) + \text{dist}(z,y)$.

By cleverly finding local and global landmarks, Akiba et al. [4] and Delling et al. [22] found that only a few hundred landmarks per vertex suffices to recover all-pairs distances exactly, in a collection of social, Web, and computer networks with tens of millions of edges. Such a finding does not hold in worst case, since no distance labelings can always recover the exact distance while using sub-quadratic amount of space. Even for graphs with maximum degree 3, it is known that any distance labeling scheme requires $\Omega(n^{1.5})$ space [28].

Existing models for social and information networks build on random graphs with a fixed degree distribution [24, 15, 42]. Informally, we assume that the degree sequence of our graph is given, and then we draw a “uniform” sample from graphs that have the same or very similar degree sequences. Random graphs capture the small world phenomenon [15], because the average distance grows logarithmically in the number of vertices. They serve as a basic block to richer models with more realistic features, e.g. community structures [31], shrinking diameters in temporal graphs [34]. In this work, we fill in the gap between empirical and worst case results, by studying distance labelings on random graphs:

**Given a random graph from the Chung-Lu model with a power law degree distribution of exponent $\beta$, how much storage does a distance labeling scheme require overall, in order to answer distance queries with no distortion?**

In the Chung-Lu model [14], each vertex $x$ has a weight (expected degree) $p_x$. For every pair of vertices $x$ and $y$, there is an undirected and unweighted edge between them with probability proportional to $p_x \cdot p_y$, independent of other edges. Hence, Chung-Lu model generalizes Erdős-Rényi graph to arbitrary degree distributions. Later on, we will discuss the implication of our results for configuration model and directed Chung-Lu model as well. In the rest of the paper, we use the term “random power law graph” to refer to a graph that is sampled from the Chung-Lu model, where the weight of each vertex is independently drawn from a power law distribution with mean value $\nu > 1$ and exponent $\beta$. We are interested in the regime when $\beta > 2$ — this covers most of the empirical power law degree distributions that people have observed on social and information networks [16].

1.1 Results

When the degree distribution has finite variance ($\beta > 3$), we show that breadth first search produces a 2-hop cover which only requires each vertex to store $\tilde{O}(\sqrt{n})$ landmarks. As a complement, the total length of any distance labeling schemes that answer distance queries exactly is almost surely
\(\Omega(n^{1.5})\). The same conclusion also applies to Erdős-Rényi graphs \(G(n, \frac{c}{n})\) when \(c > 1\), or when \(c = (1 + \varepsilon) \log n\).

**Theorem 1.** Let \(\mathcal{G}^n(p)\) be a random power law graph model with average degree \(\nu > 1\) and exponent \(\beta > 3\). For a random graph \(G = (V, E)\) drawn from \(\mathcal{G}^n(p)\), we have that:

- Almost surely there exists a 2-hop cover \(F\) such that \(|F(x)| \leq O(\sqrt{n \log n})\) for all \(x \in V\).
- Almost surely any distance labeling scheme will output a labeling whose total length is \(\tilde{\Omega}(n^{3/2})\).

We then present an algorithm such that on a random power law graph when \(2 < \beta < 3\) (infinite variance of the degree distribution), it generates at most \(O(n^{\beta-2}/(\beta-1))\) landmarks per vertex when \(\beta \geq 2.5\); and \(\tilde{O}(n^{(3-\beta)/(4-\beta)})\) landmarks per vertex when \(2 < \beta < 2.5\) (See Figure 1 for an illustration). We also show that when \(2 < \beta < 3\), any distance labeling scheme will generate labels of total size \(n^{5-\beta}/2 - o(1)\) almost surely.

**Theorem 2.** Let \(\mathcal{G}^n(p)\) be a random power law graph model with average degree \(\nu > 1\) and exponent \(2 < \beta \leq 3\). For a random graph \(G = (V, E)\) drawn from \(\mathcal{G}^n(p)\), we have that:

- Almost surely there exists a 2-hop cover \(F\) such that \(|F(x)| \sim O(n^{1-\min(\frac{1}{\beta-1}, \frac{1}{\beta})} \cdot \log^3 n)\) for all \(x \in V\).
- For any distance labeling scheme, almost surely it will output a labeling whose total length is \(n^{\frac{5-\beta}{2}-o(1)}\).

Our algorithm starts from the following observation: do a breadth-first search from each vertex \(x\) until either the entire connected component has been explored or \(\sqrt{n}\) vertices has been traversed — let \(F(x)\) denote the set of vertices we discovered. For any two vertices \(x\) and \(y\) in the same component, \(F(x)\) and \(F(y)\) have a nonempty intersection, thus ensuring that there is a common landmark on the shortest path between \((x,y)\). This simple procedure generates a 2-hop cover where each vertex stores \(\tilde{O}(\sqrt{n})\) landmarks.

When \(2 < \beta < 3\), instead of stopping when \(\sqrt{n}\) vertices are explored, we stop before reaching the layer containing the maximum weighted vertex. The vertices that we already discovered together with the maximum weighted vertex is essentially a \((+1)\)-stretch scheme with at most \(\tilde{O}(n^{(\beta-2)/(\beta-1)})\) landmarks per vertex. However, the size of the boundary, which contains the maximum weighted vertex, can be much larger than \(\sqrt{n}\). This is because the branching process grows doubly exponentially near the boundary \([13]\). Therefore, we preprocess a set of high degree vertices first, then carefully add vertices on the boundary to resolve the \((+1)\)-stretch. It is also not hard to obtain

![Figure 1](image-url)

**Figure 1:** An illustration of the results: The \(x\)-axis is the exponent of the power law degree distribution and for each value in the \(y\)-axis it means that the amount of storage is \(\tilde{O}(n^y)\).
a (+2)-stretch scheme with $O(n^{\beta/2-1})$ landmarks per vertex — when $\beta < 2.5$, this improves the 3-approximate distance oracle of Chen et al. [13], which requires $O(n^{(\beta-2)/(2\beta-3)})$ space per vertex.

Now we describe the intuition behind our lower bound for the $\beta > 3$ case. Consider two randomly chosen vertices $x, y$ from $V$. We already know that $\text{dist}(x, y)$ is close to the average distance (on the order of $O(\log n)$). Let $d$ be slightly smaller than the average distance. While $\text{dist}(x, y)$ will be at least $2d + 1$ with high probability, it is crucial that $\Pr[\text{dist}(x, y) = 2d + 1]$ is already non-negligible (e.g. $\Omega(1/\log n)$ if we set $d$ appropriately). Hence the information of whether $\text{dist}(x, y)$ is equal to $2d + 1$ or not, is worth $\Omega(1/\log n)$ bits. If we can construct $n^{1.5}$ pairs of such $(x, y)$, then we could obtain the desired lower bound. Towards this goal, we note that even for $\sqrt{n}$ vertices, we could still maintain the neighborhood growth “almost” independently, up to distance $d$. In Section 3 we implement this idea via a martingale argument and an entropy argument, which yields a lower bound of $\tilde{\Omega}(n^{1.5})$.

To prove the lower bound when $2 < \beta < 3$, we adopt the high level plan described above. However, the neighborhood growth has very high variance, because there are lots of high degree vertices. To overcome the barrier, we use a technique from Van Der Hofstad (Chapter 3 [42]). We carefully construct a set of “good” path, so that with high probability, a vertex will follow our good path during the neighborhood growth. Our lower bound is nearly tight when $\beta$ is close to 2, and has a small gap when $\beta < 2.5$. It would be interesting to close the gap when $2.5 < \beta < 3$. We believe that the right answer should be $\Omega(n^{1.5})$ when $\beta$ is close to 3.

In Section 6 we test our algorithm on real world graphs. We found that our algorithm achieves fairly accurate results — the 80%-percentile multiplicative error is less than 0.25 in our experiment. In addition, the algorithm is scalable to preprocess graphs with millions of edges in several minutes.

One limitation is that our algorithm is designed for graphs with small average distance and a heavy tailed distribution. The second limitation is that we only derived results for Chung-Lu model. However, our technical proofs only rely on upper and lower bounding the growth rate of neighborhood growth. And it is well-known that configuration model have the same neighborhood growth rate [42], hence we expect our high level intuition to hold for the configuration model [24]. Finally, our lower bound technique only applies to labeling schemes. We refer the interested reader to Section 7 for more discussion about future work.

1.2 Related work

Landmark based Labelings The problem of computing the optimal landmark based labelings can be formulated as an integer program, similar to the set cover problem [19]. It is NP-hard to compute the optimal landmark labelings (or 2-hop cover), and a log $n$-approximation can be obtained via a greedy algorithm [19]. See also the references [29, 23, 8, 7] for a line of followup work. Another closely related line of work is approximate distance oracle. We refer the reader to the excellent survey of Sommer [39] on this topic, as well as the classic work [6, 17, 41], and recent developments [20, 36, 8, 37, 43, 38, 5, 12] for further reading. On the practical side, numerous studies have demenstated the effectiveness of landmark based labelings on large graphs [21, 18, 4, 22, 11]. Notably, Bahmani and Goel [9] implemented the distance sketch of Das Sarma et al. [21] in a distributed setting.

Power Law Graphs It has been empirically observed that many social and information networks have a heavy-tailed degree distribution [16] — concretely, the number of vertices whose degree is $x$, is proportional to $x^{-\beta}$. For real world graphs, the coefficient $\beta$ vary from 2 to 10, and are often greater than 3 [25].
Previous work of Chen et al.\textsuperscript{13} proved that the space complexity of the classic Thorup and Zwick distance oracle\textsuperscript{41} can be reduced on random graphs with a power law degree distribution, while achieving stretch 3, when $2 < \beta < 3$. They left open the question of finding distance oracles with better stretch and less space. Gavoille et al.\textsuperscript{27} reported a stretch 5 compact routing scheme (distance oracles for distributed settings) that also requires much less space on random power law graphs than in worst case. Enachescu et al.\textsuperscript{26} presented a compact routing scheme that achieves stretch 2 using space $O(n^{1.75})$ on Erdős-Rényi graphs. Existing mathematical models on special families of graphs related to distance queries include road networks\textsuperscript{2}, planar graphs\textsuperscript{35} and graphs with doubling dimension\textsuperscript{30}. However none of them can capture the expansion properties that have been observed on sub-networks of real-world social networks\textsuperscript{34}. Apart from the Chung-Lu model and the configuration model that we have mentioned, the preferential attachment graph is also well-understood\textsuperscript{24}. It would be interesting to see if our results extend to preferential attachment graphs as well. The Kronecker model\textsuperscript{32} allows a richer set of features by extending previous random graph models, however its mathematical properties are not as well-understood as the other three models.

**Organization:** The rest of the paper is organized as follows. In Section 2 we define the random graph model. In Section 3 we present our technical results along with the proof sketches. In Section 4 and 5 we fill in the missing proofs from Section 3. Section 6 describes our experiments. Finally, we discuss future extensions of our results in Section 7.

## 2 Preliminaries

In this section, we introduce notations and define graph models. Consider an undirected graph $G=(V,E)$. Let $n=|V|$ be the number of vertices. For any vertex $x\in V$, Let $d_x$ denote the degree of $x$. For a set of vertices $S$, let $d_S = \sum_{x\in S} d_x$ denote the sum of their degrees. We use the notation $x \sim y$ to indicate that $(x,y)\in E$. For two disjoint sets $S$ and $T$, $S \sim T$ means there exists an edge between $S$ and $T$; $S \sim T$ means there does not exist any edge between $S$ and $T$. Let $\text{dist}_G(x,y)$ denote the distance of $x$ and $y$ in $G$ (we drop the subscript $G$ if there is no ambiguity). For any integer $1 \leq k \leq n-1$, let $\Gamma_k(x) = \{y\in V: \text{dist}(x,y) = k\}$ denote the set of vertices whose distance from $x$ is equal to $i$. And let $N_i(x) = \{y\in V: \text{dist}(x,y) \leq i\}$ denote the set of vertices whose distance from $x$ is at most $i$.

### 2.1 Distance Labelings

A **distance labeling scheme** consists of two algorithms\textsuperscript{28,39}:

- **Preprocessing:** given a graph $G=(V,E)$, output a vertex-labeling $\mathcal{L}: V \rightarrow \{0,1\}^*$;
- **Query:** given input $\mathcal{L}(x)$ and $\mathcal{L}(y)$, compute $\text{dist}(x,y)$ without accessing any other information.

For a set of vertices $S$, the total label size of $S$ is defined as $\sum_{x\in S} |\mathcal{L}(x)|$. The total label size of $\mathcal{L}$ is given by $|\mathcal{L}(V)|$. The maximum size of $\mathcal{L}$ over all vertices is given by $\max_{x\in V} |\mathcal{L}(x)|$.

In a landmark based labeling scheme, the labeling of each vertex $x$ consists of a subset of vertices and their distances to $x$. A function $F: V \rightarrow 2^V$ is said to be a **2-hop cover**\textsuperscript{19}, if for every pair of vertices $x$ and $y$ in the same connected component of $G$, there exists a vertex $z \in F(x) \cap F(y)$ such that $\text{dist}(x,y) = \text{dist}(x,z) + \text{dist}(z,y)$. When $x$ and $y$ are not reachable from each other, then the intersection of $F(x)$ and $F(y)$ should be empty. Apart from $F$, a 2-hop cover also stores $\text{dist}(x,y)$ for every $y \in F(x)$.
The query algorithm for dist\((x, y)\) is given by
\[
\min_{z \in F(x) \cap F(y)} \text{dist}(x, z) + \text{dist}(z, y).
\]
If no common landmark in \(F(x)\) and \(F(y)\) is found, we return disconnected. Each query takes no more than \(O(|F(x)| + |F(y)|)\) time.

2.2 Chung-Lu Model

In Chung-Lu model, each vertex \(x \in V\) has a weight \(p_x > 0\), which corresponds to \(x\)'s expected degree. Given weight vector \(\mathbf{p}\) over \(V\), the Chung-Lu model defines a probability distribution over the set of all graphs \(\mathcal{G}^n\). Let \(\text{vol}(S) = \sum_{x \in S} p_x\) denote the volume of \(S\). And let \(\text{vol}^2(S) := \sum_{x \in S} p_x^2\) denote the second moment of \(S\). Each edge \((x, y)\) is chosen independently with probability
\[
\Pr[x \sim y] = \min \left\{ \frac{p_x \cdot p_y}{\text{vol}(V)}, 1 \right\}.
\]
Thus, \(p_x\) is approximately the expected degree of \(x\), and \(\text{vol}(V)\) is approximately the expected number of edges. Let \(\mathcal{G}^n(\mathbf{p})\) denote such a probability distribution over \(\mathcal{G}^n\). We use the notation \(G \in \mathcal{G}^n(\mathbf{p})\) to refer to a graph drawn from the distribution defined by \(\mathcal{G}^n(\mathbf{p})\). We have the following convenient Proposition for bounding the probability of whether two sets connect or not.

**Proposition 3.** Let \(G = (V, E) \in \mathcal{G}^n(\mathbf{p})\) be a random graph. For any two disjoint set of vertices \(S\) and \(T\),
\[
\Pr[S \sim T] \leq \frac{\text{vol}(S) \text{vol}(T)}{\text{vol}(V)}, \quad \text{and}
\]
\[
\Pr[S \nsim T] \leq \exp \left( -\frac{\text{vol}(S) \text{vol}(T)}{\text{vol}(V)} \right).
\]

**Proof.** For the first bound, we have:
\[
\Pr[S \sim T] = 1 - \prod_{x \in S} \prod_{y \in T} (1 - \min(\frac{p_x p_y}{\text{vol}(V)}, 1))
\]
\[
\leq 1 - \left( 1 - \sum_{x \in S} \sum_{y \in T} \min(\frac{p_x p_y}{\text{vol}(V)}, 1) \right)
\]
\[
\leq 1 - \left( 1 - \sum_{x \in S} \sum_{y \in T} \frac{p_x p_y}{\text{vol}(V)} = \frac{\text{vol}(S) \text{vol}(T)}{\text{vol}(V)} \right)
\]
and
\[
\Pr[S \nsim T] = \prod_{x \in S} \prod_{y \in T} (1 - \min(\frac{p_x p_y}{\text{vol}(V)}, 1))
\]
\[
\leq \exp \left( -\sum_{x \in S} \sum_{y \in T} \min(\frac{p_x p_y}{\text{vol}(V)}, 1) \right)
\]
\[
\leq \exp(-\frac{\text{vol}(S) \text{vol}(T)}{\text{vol}(V)}).
\]
2.3 Power Law Distribution

Let \( f : [x_{\text{min}}, \infty) \to \mathbb{R} \) denote the probability density function of a power law distribution \[ \text{with exponent } \beta > 1, \text{ i.e. } f(x) = Z x^{-\beta}, \] where \( Z = (\beta - 1) \cdot x_{\text{min}}^{\beta-1} \). The expectation of \( f(\cdot) \) exists when \( \beta > 2 \):

\[
\nu = x_{\text{min}} \cdot \frac{\beta - 1}{\beta - 2}
\]

The second moment is finite, only when \( \beta > 3 \), which is equal to

\[
\omega = x_{\text{min}}^2 \cdot \frac{\beta - 1}{\beta - 3}.
\]

In the random power law graph model, the weight of each vertex \( x \) is drawn independently from a power law distribution (with the same mean \( \nu \) and exponent \( \beta \)). Given the weight vector \( p \), we then sample a random graph according to the Chung-Lu model \( G^n(p) \).

It is shown in Chung and Lu [15] that if \( \nu > 1 \), then almost surely a graph \( G \in G^n(p) \) has a unique giant component. If \( \nu < 1 \), almost surely all connected components have at most \( O(\log n) \) vertices. In this paper, we are interested in cases when the average degree \( \nu \) is a constant greater than 1. In this case, it is also known that the diameter of \( G \) is \( O(\log n) \) with high probability.

3 Proof Overview

In this section, we state our results formally along with a sketch for the proof. First, we consider the case when the degree distribution has finite variance.

**Theorem 1** (restate). Let \( G^n(p) \) be a random power law graph model with average degree \( \nu > 1 \) and exponent \( \beta > 3 \). For a random graph \( G = (V, E) \) drawn from \( G^n(p) \), we have that:

- Almost surely there exists a 2-hop cover \( F \) such that \( |F(x)| \leq O(\sqrt{n \log n}) \) for all \( x \in V \).
- Almost surely any distance labeling scheme will output a labeling whose total length is \( \tilde{\Omega}(n^{3/2}) \).

For the first part, we simply observe that for each vertex, if we add the closest \( \sqrt{n} \) vertices to the landmark set of every vertex, then the landmark sets of every pair of vertices will intersect with high probability, i.e. we have obtained a 2-hop cover. The proof can be found in Appendix A.

For the lower bound, we divide the set of vertices whose weight is at most \( 2\nu \) to groups of size \( \sqrt{n} \). The following Proposition shows that for each group, with high probability the total label size of \( \tilde{\Omega}(\sqrt{n}) \). This reduction will simplify the proof because the joint neighborhood growth of \( \sqrt{n} \) vertices is almost independent in early stages.

**Proposition 4.** Let \( S \) be a set of \( \sqrt{n} \) vertices, where every vertex has weight no more than \( 2\nu \). The total label size of \( S \) is at least \( \Omega\left(n \cdot \log^{-O(1+1/\gamma)} n \right) \) with high probability, where \( \gamma = (\beta - 3)/2. \)

Let us first show that the above proposition directly implies the lower bound in Theorem 1.

**Proof of Theorem 1.** We know that at least \( n/2 \) vertices have weight at most \( 2\nu \), by an averaging argument. Divide them into groups of size \( \sqrt{n} \), and apply Proposition 4 on each group. Clearly, there are at least \( \Theta(\sqrt{n}) \) disjoint groups. In expectation, except for \( o(\sqrt{n}) \) groups, most groups will have label size at least \( \Omega\left(n \cdot \log^{-O(1+1/\gamma)} n \right) \). Hence Markov’s inequality, at least \( \Omega(\sqrt{n}) \) of the groups have total labeling size at least \( \Omega\left(n \cdot \log^{-O(1+1/\gamma)} n \right) \) with probability \( 1 - o(1) \), thus they have a total labeling size of \( \tilde{\Omega}(n^{1.5}) \). This proves the theorem.

\[ \square \]
To build up intuition towards the proof of Proposition 4, we consider Erdős-Rényi graph \( G = G(n, p) \) where \( p = c/n \), for a constant \( c > 0 \). Our lower bound is derived via an entropy argument. We will carefully exploit the information given by the pairwise distances of \( S \). Note that the average distance of \( G \) is roughly \( \frac{\log n}{2\log c} \) (see e.g. [10]). We will consider distances slightly smaller than the average distance. Let \( d = \frac{\log n}{2\log c} - O(\log \log n) \). We observe the following facts.

- For every \( x \in S \), with constant probability we have that \( |\Gamma_d(x)| = \Theta(c^d) \).
- For every pair of vertices \( x, y \in S \), the probability that \( \text{dist}(x, y) \leq 2d + 1 \) is at most \( O\left(\frac{c^2 d n}{\log \log(n)}\right) = O\left(\frac{1}{\text{polylog}(n)}\right)\).
- If \( |\Gamma_d(x)| \) and \( |\Gamma_d(y)| \) are both on the order of \( \Theta(c^d) \) and \( \text{dist}(x, y) > 2d \), the probability that \( \text{dist}(x, y) = 2d + 1 \) is \( \Theta\left(\frac{c^2 d n}{\log \log(n)}\right) = \Theta\left(\frac{1}{\text{polylog}(n)}\right)\).

We know that the distance labeling of \( S \) determine the pairwise distances of \( S \). In particular, they determine whether each pair \( x, y \) has distance exactly \( 2d + 1 \) or more, if \( \text{dist}(x, y) \) is not yet revealed by \( \Gamma_d(x) \) and \( \Gamma_d(y) \). This is worth roughly \( \frac{1}{\text{polylog}(n)} \) bit of information. In expectation, we expect to have \( \Theta(n) \) such pairs from \( S \), which implies our lower bound on the labeling size of \( S \).

To implement the above plan, we need to deal two more issues. First, we need to argue that there are \( \Theta(n) \) pairs with high probability. Secondly, for general degree distributions, there exists high degree vertices which introduces high variance to neighborhood growth. To resolve the first issue, we construct a martingale to grow the neighborhood of each vertex in \( S \). To resolve the second issue, we note the second moment of the degree distribution is finite. During the martingale process, we carefully bound the second moment of degree sequence. We leave the full proof to Section 4.

Now we consider the case when the degree distribution has infinite variance.

**Theorem 2 (restate).** Let \( \mathcal{G}^n(p) \) be a random power law graph model with average degree \( \nu > 1 \) and exponent \( 2 < \beta \leq 3 \). For a random graph \( G = (V, E) \) drawn from \( \mathcal{G}^n(p) \), we have that:

- Almost surely there exists a 2-hop cover \( F \) such that \( |F(x)| \sim O\left(n^{1-\min\left(\frac{1}{\nu-1}, \frac{1}{3-\beta}\right)} \cdot \log^3 n\right) \) for all \( x \in V \).
- For any distance labeling scheme, almost surely it will output a labeling whose total length is \( n^{\frac{5-\beta}{2} - o(1)} \).

Our upper bound uses the fact that \( G \) contains a heavy vertex whose weight is approximately \( n^{\frac{1}{\nu-1}} \). We first add all high degree vertices to the landmark set of every vertex. Then we do a breadth-first search, but stop right before the boundary size exceeding \( \tilde{\Theta}(n^{\frac{\beta-2}{\beta-1}}) \), and put all vertices that we have explored in the landmark set. We claim that this gives up a (+1)-sketch labeling.

To see this, for two vertices \( x, y \), if their landmark sets intersect with each other, then we can already compute their distances correctly. Otherwise, the bottom layer of \( x \) and \( y \) have distance at most two (through the heavy vertex) with high probability. Therefore, we only have to check whether they have distance one, i.e., there is an edge between them.

To resolve the (+1)-stretch, for each vertex on the boundary, we add all of its neighbors with a higher degree to the landmark set. Clearly, this fixes the (+1)-stretch, if there is an edge connecting
Algorithm 1 ALGSKWEDEGREE

Input: An undirected graph $G = (V,E)$; Parameters $\delta$ and $K$.

1: Let $H = \{x \in V : d_x \geq K\}$
2: for $x \in V$ do
3: \hspace{1em} $(F(x), l(x)) = \text{ALGBFS}(x, \delta n^{\frac{3-\beta}{2}})$ \hspace{1em} (add closest $\delta n^{\frac{3-\beta}{2}}$ vertices)
4: \hspace{1em} $F(x) = F(x) \cup H$ \hspace{1em} (add all high degree vertices)
5: for $y \in \Gamma_{l(x)-1}(x)$ do
6: \hspace{2em} if $d_y \leq K$ then
7: \hspace{3em} $F(x) = F(x) \cup \{z \in N(y) : d_y \leq d_z \leq K\}$ \hspace{1em} (add all neighbors of $y$ with a higher degree)
8: \hspace{2em} end if
9: end for
10: end for

the two boundaries. To bound the landmark size, we refer the reader to Appendix A. A complete description of the algorithm is shown below.

For each vertex $x$ and $0 \leq k \leq n-1$, $\alpha_k(x)$ is the number of edges between $\Gamma_k(x)$ and $V \setminus \Gamma_{k-1}(x)$. $l(x)$ is the first non-negative integer that satisfies: $\alpha_{l(x)-1}(x) > \delta n^{(\beta-2)/(\beta-1)}$. It is clear that $l(x)$ always exists. The procedure $\text{ALGBFS}(x,T)$ expands from $x$ until reaching a level set whose volume is at least $T$; And the output will be all the vertices visited so far — the precise definition can be found in Appendix A. Set $\delta = 4\nu \log^2 n$ and

$$K = \begin{cases} \sqrt{n} & \text{if } 2.5 \leq \beta \leq 3 \\ n^{\frac{3-\beta}{1-\beta}} & \text{if } 2 < \beta < 2.5. \end{cases}$$

Remark One can also obtain a $(+2)$-stretch labeling by setting $K = \tilde{O}(n^{\beta/2-1})$; The maximum labeling size will be $K$. We omit the proof.

For the lower bound, we introduce a reduction to groups of size $n^{\frac{3-\beta}{2}}$.

**Proposition 5.** Let $S$ be a set of $n^{\frac{3-\beta}{2}}$ vertices with weights between $[a,b]$ such that $a,b = n^{\Theta(1/\log \log n)}$ and $b > 2a$. Then any labeling scheme must generate a total label size of at least $n^{3-\beta - O(1/\log \log n)}$ for $S$ with high probability.

We first show how to reduce the lower bound part of Theorem 2 to the above proposition.

**Proof of Theorem 2.** Consider the set of vertices with weights between $a$ and $b$, and divide them into groups of size $n^{\frac{3-\beta}{2}}$. The number of groups is $n^{\frac{3-\beta}{2}-o(1)}$ with high probability, because there are $n^{1-o(1)}$ such vertices. For each group, by Proposition 5 the total label size is at least $n^{3-\beta - O(1/\log \log n)}$ except with $o(1)$ probability. By Markov’s inequality, with probability $1-o(1)$, at least a constant fraction (i.e. $n^{\frac{3-\beta}{2}-o(1)}$) of the groups have label sizes at least $n^{3-\beta - O(1/\log \log n)}$. Hence, we conclude that the total label size is at least $n^{\frac{3-\beta}{2}-O(1/\log \log n)} = n^{\frac{3-\beta}{2}-o(1)}$ with $1-o(1)$ probability. \qed

To prove the proposition, we perform a breadth first search from $S$ to distance $d$ such that each vertex is “expected” to have a neighborhood of volume roughly $n^{\beta/2-1}$ at distance $d$. We show:

1. There exists a subset of vertices $S' \subseteq S$, such that: i) $|S'| = \Theta(n^{\frac{3-\beta}{2}})$; ii) for each vertex $x \in S'$, $\text{vol}(\Gamma_d(x)) = \Theta(n^{\frac{3-\beta}{2}-1})$; iii) for any two vertices $x,y \in S'$, their “neighborhoods” are disjoint.
2. For a constant fraction of vertices \( x \) in \( S' \), \( \Gamma_d(x) \) is connected to a vertex \( h(x) \) whose weight is about \( O(\sqrt{n}) \), and their pairwise distance is at least \( 2d + 4 \).

3. Distance between \( x \) and \( y \) being at least \( 2d + 4 \) implies that \( h(x) \) and \( h(y) \) are not connected, which happens with roughly constant probability. However, there are \( n^{3-\beta} \) such pairs, and thus every label of \( S \) which implies such a distance function could only occur with \( \exp(-n^{3-\beta}) \) probability. The total label size of \( S \) is at least \( n^{3-\beta} \) with high probability.

The detailed proof can be found in Section 5.

4 Proof of Proposition 4

Let \( r = \frac{\text{vol}_2(V)}{\text{vol}(V)} \) and \( d = \log_r \sqrt{n} - c \), where \( c \leq O(\log \log n) \) will be determined later. We describe an iterative process to grow the neighborhood of \( S \) up to distance \( d \). Let \( S = \{x_1, x_2, \ldots \} \) be any ordering of its vertices. Denote by \( G_1 = (V_1, E_1) \), where \( V_1 = V \) and \( E_1 = E \). For any \( i \geq 1 \), define \( T(x_i) \) to be the set of of vertices in \( G_i \) whose distance is at most \( d \) from \( x_i \). Define \( L(x_i) \) to be the set of vertices in \( G_i \) whose distance is equal to \( d \) from \( x_i \). More formally,

\[
T(x_i) := \begin{cases} 
\{ y : d_{G_i}(x_i, y) \leq d \}, & \text{if } x_i \in V_i \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

\[
L(x_i) := \{ y \in T(x_i) : d_{G_i}(x_i, y) = d \}
\]

We then define \( F_i = F_{i-1} \cup T(x_i) \) (\( F_0 := \emptyset \) by default). Denote by \( G_{i+1} \) to be the induced subgraph of \( G_i \) on the remaining vertices \( V_{i+1} = V \setminus F_i \).

We note that in the above iterative process, the neighborhood growth of \( x_i \) only depends on the degree sequence of \( V_i \). In order to bound rate of growth, we introduce the following Lemma. The result is standard (see e.g. [15]) – we leave the proof to Appendix C.1.

**Lemma 6.** Let \( \mathcal{G}^n(p) \) be a random graph model with weight sequence \( p \) satisfying the following properties:

1. \( \text{vol}(V) = (1 + o(1)) \nu \cdot n \) for some constant \( \nu \);
2. \( \text{vol}_2(V) = (1 + o(1)) \omega \cdot n \) for some constant \( \omega \);
3. \( \text{vol}_{2+\gamma}(V) = \tau \cdot n \) for some positive constant \( \gamma < 1/2 \) and \( \tau \), where \( \text{vol}_{2+\gamma}(S) := \sum_{x \in S} p_x^{2+\gamma} \);  
4. The growth rate \( r = \frac{\text{vol}_2(V)}{\text{vol}(V)} \) is bounded away from 1 \( (\nu > \omega) \).

Then for any vertex \( x \) with a constant weight, the set of vertices \( \Gamma_k(x) \) at distance exactly \( k \) from \( x \) has:

1. \( \mathbb{E}[\text{vol}(\Gamma_k(x))] = O\left( r^k \right) \) for every \( k \leq \log_r n \);
2. \( \Pr[\text{vol}(\Gamma_k(x)) \geq \Omega\left( r^k \right)] \geq \Omega(1) \) for every \( k \leq \frac{1}{2} \log_r n \).

As a corollary, we have that \( \Pr[\text{dist}(x, y) \leq k + 1] \leq O(r^k/n) \) for every \( k \leq \frac{1}{2} \log_r n \), where \( y \) is any vertex with constant weight.

We now claim that with high probability, at least \( c_1 \sqrt{n} \) vertices \( x \in S \) have that \( \text{vol}(L(x)) \geq \Omega(\sqrt{n} \cdot r^{-c}) \), for some absolute constant \( c_1 \) and \( c = (3 + 1/\gamma) \log \log n \); The proof consists of two parts:
a) For every $1 \leq i \leq \sqrt{n}$, conditional on $x_i \in V_i$ and $\text{vol}(F_{i-1}) \leq n/\log^{2+\gamma} n$, we have that $\text{vol}(L(x_i)) \geq \Omega(\sqrt{n} \cdot r^{-c})$ with constant probability.

b) By constructing a martingale and then apply Azuma-Hoeffding inequality, at least $c_1 \sqrt{n}$ vertices $x \in S$ have that $\text{vol}(L(x)) \geq \Omega(\sqrt{n} \cdot r^{-c})$.

To prove Claim a), we note that the subgraph $G_i$ on $V_i$ is also a random graph sampled from Chung-Lu model. The vertices at distance $d$ from $x_i$ in this subgraph is exactly $L_i$. As the total weight of the subgraph is smaller than the $\text{vol}(V)$, the normalizer for the probability of an edge changes if restricted to $V_i$. To adjust for this change, we set

$$p_y^{(i)} = p_y \cdot \left(1 - \frac{\text{vol}(F_{i-1})}{\text{vol}(V)}\right), \quad \forall y \in V_i.$$ 

Then we have

$$\Pr[y \sim z] = \frac{p_y^{(i)} \cdot p_z^{(i)}}{\sum_{x \in V_i} p_x^{(i)}} = \frac{p_y \cdot p_z}{\text{vol}(V)}.$$

Hence we see that $G_i$ is equivalent to a random graph drawn from degree sequence $p^i$. The growth rate is $r_i := \frac{\text{vol}(V)}{\text{vol}(V)}$. When $\text{vol}(F_{i-1}) \leq n/\log^{2+\gamma} n$, by Hölder’s inequality,

$$\text{vol}(F_{i-1}) \leq \Omega(n^{1+\gamma}) \leq o(n/\log n).$$

Thus, the growth rate $r_i > 1$. By our growth Lemma 6 with constant probability,

$$\text{vol}(L_i) \geq \Omega\left(r_i^{d \cdot \frac{\text{vol}(V)}{\text{vol}(V)}}\right) = \Omega(r_i^{d}) = \Omega(\sqrt{n} \cdot r^{-c}).$$

Hence we proved Claim a).

To prove Claim b), we consider the following random variable, for any $1 \leq i \leq \sqrt{n}$.

$$X_i := \begin{cases} 1 & \text{if } x_i \notin V_i, \text{ or } \text{vol}(F_{i-1}) > n/\log^{2+\gamma} n, \\ & \text{or } \text{vol}(L_i) \geq \Omega(\sqrt{n} \cdot r^{-c}) \\ 0 & \text{otherwise.} \end{cases}$$

We have $\Pr[X_i = 1 \mid X_1, \ldots, X_{i-1}] \geq \Omega(1)$ by Claim a). Thus by Azuma-Hoeffding inequality, $\sum_{i=1}^{S} X_i \geq \Omega(\sqrt{n})$ with probability $1 - o(1)$. We shall prove below that the contributions to $\sum_{i=1}^{S} X_i$ from the first two predicates is $o(\sqrt{n})$. Hence by taking union bound, we obtain Claim b).

- First, the number of $x_i$ such that $x_i \notin V_i$ is $o(\sqrt{n})$ with high probability. Note that $x_i \notin V_i$ implies that there exists some vertex $x_j \in F_{i-1}$ such that $\text{dist}(x_i, x_j) \leq d$. On the other hand, for any two vertices $x, y \in S$, $\Pr[\text{dist}(x, y) \leq d] \leq O(r^d/n)$, by Lemma 6. Hence, the expected number of vertex pairs in $S$ whose distance is at least $d$, is $O(r^d/n) \leq o(1/\sqrt{n})$. By Markov’s inequality, with probability $1 - o(1)$, only $o(\sqrt{n})$ vertex pairs have distance at most $d$ in $S$. Hence for at most $o(\sqrt{n})$ $i$’s we have that $x_i \notin V_i$.

- Secondly for all $1 \leq i \leq \sqrt{n}$, $\text{vol}(F_i) \leq n/\log^{2+\gamma} n$ with high probability. This is because the set of vertices $F_i$ is a subset of $N_d(x_i)$, the vertices within distance $d$ to $x_i$. Thus, by Lemma 6 we have

$$\mathbb{E}[\text{vol}(T_i)] \leq \mathbb{E}[\text{vol}(N_d(x_i))] \leq O(r^d).$$
Thus, the expected volume of $F_i$ is at most
\[ O(i \cdot r^d) = O(\sqrt{n} \cdot r^d) = O(\log^{-1} n) \times n / \log^{2+1/\gamma}, \]

because $d = \log_r \sqrt{n} - c$ and $c \geq (3 + 1/\gamma) \log_r \log n$. Hence by Markov’s inequality, the probability that $\text{vol}(F_{|S|}) > n / \log^{2+1/\gamma} n$ is at most $O(\log^{-1} n)$.

Now we are ready to prove Proposition 4. Given the labelings of $S$, we can recover the pairwise distances for all vertex pairs in $S$. Let $\text{dist}_S : S \times S \to \mathbb{N}$ denote the distance function restricted to all pairs in $S$. We show that with high probability, the total labeling size of $S$ is at least $\Omega(n \cdot r^{-2c}) = \Omega(n \cdot \log^{-O(1+1/\gamma)} n)$ (recall that $c = (3 + 1/\gamma) \log_r \log n$). Consider the following two cases:

1. \( \exists c_1^2 \cdot n/4 \) pairs \((x_i, x_j)\) such that $\text{dist}_S(x_i, x_j) \leq 2d + 1$. By Lemma 6, we know that $\Pr[\text{dist}(x_i, x_j) \leq 2d + 1] = O(r^{2d}/n)$, for any $x_i, x_j \in S$. Hence the expected number of pairs with distance at most $2d + 1$ in $S$, is at most $O(r^{2d}/n)$. By Markov’s inequality, the probability that a random graph induces any such distance function is $O(r^{-2c}) = o(1)$.

2. The number of pairs such that $\text{dist}_S(x_i, x_j) \leq 2d + 1$ is at most $c_1^2 \cdot n/4$ in $S$. Let $A = \{(x,y) \in S \times S \mid \text{dist}(x,y) > 2d + 1, \text{and vol}(L(x)), \text{vol}(L(y)) > \Omega(\sqrt{n} \cdot r^{-c})\}$. By Claim b), the size of $A$ is at least $c_1 n (c_1 n - 1)/2 - c_2^2 n/4 \geq c_2^2 n/5$. For any $(x,y) \in A$, $L(x)$ and $L(y)$ are clearly disjoint. Conditional on \( \{T(x)\} \) for all $x \in S$, the probability of the existences of edges between $L_i$ and $L_j$ are unaffected.

\[
\Pr \left[ \text{dist}_S(x,y) > 2d + 1, \forall (x,y) \in A \mid \{T_i\}_{i=1}^{|S|} \right] \\
\leq \prod_{(x,y) \in A} \Pr[L(x) \neq L(y) \mid L(x) \cap L(y) = \emptyset] \\
\text{and vol}(L(x)), \text{vol}(L(y)) \geq \Omega(\sqrt{n} \cdot r^{-c}) \\
\leq \prod_{(x,y) \in A} \exp \left( -\frac{\text{vol}(L(x)) \text{vol}(L(y))}{\text{vol}(V)} \right) \\
\leq \exp \left( -\Omega(r^{-2c}) \right) c_2^2 n/5 \leq \exp(-\Omega(n \cdot r^{-2c})).
\]

Now let $c_2$ be a sufficiently small value (e.g. $1/\log \log n$ suffices). The number of labelings of size less than $c_2 n \cdot r^{-2c}$ is at most $2^{c_2 n \cdot r^{-2c}}$. The probability that the total label size of $|S|$ is at most $c_2 n \cdot r^{-2c}$, is at most $2^{c_2 n \cdot r^{-2c}} \times \exp(-\Omega(n \cdot r^{-2c})) = o(1)$ by union bound.

By taking a union bound on the two cases, we prove that with probability $1 - o(1)$, the total label size of $S$ has to be at least $\Omega(n \cdot r^{-2c}) = \Omega(n \cdot \log^{-O(1+1/\gamma)} n)$.

5 Proof of Proposition 5

We divide the proof into three parts. Let $d = \log \log \log n / \log \varepsilon^{-1/2}$. In part 1, we specify the set of good path. We argue that the growth of $S$ to the $d$-th level follows our good path with high probability. In part 2, we connect to vertices with weight $\sqrt{n}$ in the $(d+1)$-th level. In part 3, we use the entropy argument to show that with high probability, the label size of $S$ is at least $n^3 - \beta - O(\varepsilon)$, where $\varepsilon = 1/\log \log n$. 

Part I: neighborhood growth  For all $0 \leq i \leq d + 1$, let
\[
\mu_i := n^{(\beta/2 - 1 - \varepsilon)(\beta - 2)^{d-i}}, \quad \text{and}
\sigma_i := w^{1/(\beta - 2)^{2i}(3 - \beta)},
\]
where $w = \log \log n$. Denote by $a_i = \mu_i/\sigma_i$ and $b_i = \mu_i \sigma_i$. $\mu_i$ can be thought of as the “expected” volume at distance $i$ from a vertex $x \in S$. If the volume at distance $i$ from $x$ always stays inside $[a_i, b_i]$, then we think of $x$ as a “good” vertex.

It is easy to verify that both $a_0$ and $b_0$ are $n^{\Theta(1/\log \log n)}$ and $b_0 > 2a_0$. Let $S = \{x_1, x_2, \ldots\}$ be an arbitrary vertex set of size $n^{\frac{3-\beta}{2}}$ such that all $x_i$ have weights between $a_0$ and $b_0$. Clearly, for any $i \neq j$, the neighborhood growth of $i$ and $j$ are correlated. However, one would expect that the correlation is small, so long as the volume of the neighborhood has not reached $O(\sqrt{n})$. We leverage the observation by exploring the neighborhood of $S$ one vertex at a time. Denote by $G_1 = (V_1, E_1)$ where $V_1 = V$ and $E_1 = E$. For $1 \leq i \leq |S|$, we consider the following inductive process:

1. If $x_i \in V_i$, let $1 \leq \lambda_i \leq d$ be the maximum $k$ that still satisfy $\text{vol}(\Gamma_k(x_i)) \in [a_k, b_k]$ in graph $G_i$ (Recall that $\Gamma_k(x_i)$ is the set of vertices at distance exactly $k$ from $x_i$ in $G_i$);

2. Denote by $T_i$ the set of vertices within distance $\min\{d, \lambda_i + 1\}$ from $x_i$ in $G_i$;

3. If $\lambda_i = d$, let $L(x_i) = \Gamma_d(x_i)$; otherwise, let $L(x_i) = \emptyset$;

We then define $F_i = F_{i-1} \cup T_i$ ($F_0 = \emptyset$ by default). Let $G_{i+1}$ be the subgraph of $G_i$ on remaining vertices $V_{i+1} = V_i \setminus T_i$. In the above process, we keep expanding the neighborhood of $x_i$ until we reach distance $d$ or we find a distance $\lambda_i + 1$ such that the volume of vertices at distance $\lambda_i + 1$ from $x_i$ is not in $[a_{\lambda_i + 1}, b_{\lambda_i + 1}]$. If we reach distance $d$, then $L(x_i)$ is the set of vertices at distance $d$ from $x_i$.

In order to bound the growth rate of $x_i$ on graph $G_i$. We introduce the following Lemma. This result follows standard arguments (see e.g. [42]) – a proof can be found in Appendix C.2 for completeness.

**Lemma 7.** Let $c_1, c_2, c_3 > 0$ be absolute constants. Let $S$ be a set of fixed values within $[1, n^{\beta/2 - 1}]$ whose size is at most $2d$. Let $\mathcal{G}^n(p)$ be a random graph with weight sequence $p$ satisfying $\text{vol}(V) = \Theta(n)$, and for any $t \in S$,

\[
\sum_{g, p_g \geq t} p_g \geq c_1 \times nt^{2-\beta}
\]
\[
\sum_{g, p_g \leq t} p_g^2 \leq c_2 \times nt^{3-\beta},
\]
\[
\sum_{g, p_g \geq t} p_g \leq c_3 \times nt^{2-\beta}.
\]

Then we have the following facts regarding neighborhood growth:

a) **Following a good path:** Let $x$ be a fixed vertex and $1 \leq k \leq d + 1$. Suppose that $\text{vol}(\Gamma_k(x)) \in [a_k, b_k]$ for any $1 \leq i < k$, then $\text{vol}(\Gamma_k(x)) \in [a_k, b_k]$, with probability at least $1 - O(1/w^{\beta - 2})$, where $w = \log \log n$;

b) **Average distance:** let $x, y$ be two vertices such that $p_x, p_y \in [a_0, b_0]$, then $\text{Pr}[\text{dist}(x, y) \leq 2d + 3] = o(1)$. 
We make the following crucial claim.

**Claim 8.** With probability $1 - o(1)$, at least $\Theta(n^{3/2 - \beta})$ vertices $x_i$ in $S$ have $\text{vol}(L(x_i)) \in [a_d, b_d]$.

To prove the claim, let us consider the following random variables. Define

$$X_i = \begin{cases} 1 & \text{vol}(L(x_i)) \in [a_d, b_d], \\ 0 & \text{otherwise}. \end{cases}$$

We show that $X_i = 1$ with high probability for all $1 \leq i \leq |S|$. We first verify that $\Pr[x_i \not\in V_i] = \Pr[x_i \in F_{i-1}] \leq o(1)$. Consider any vertex $z \in V$ and $1 \leq j \leq i - 1$, we have that

$$\Pr[z \in T_j] \leq \sum_{l=0}^{d-1} \frac{p_z \cdot b_l}{\text{vol}(V_j)} \leq p_z \cdot n^{(\beta/2-1-\varepsilon)(\beta-2)-1} \cdot O(\log^2 \log n).$$

Thus, by union bound from $1 \leq j \leq i - 1$, we have

$$\Pr[z \in F_{i-1}] \leq p_z \cdot n^{(\beta^2-5\beta+5)-\varepsilon(\beta-2)+o(\varepsilon)} = p_z \cdot n^\lambda,$$

i.e. denote the exponent by $\lambda$ above.

Next, we verify that the weight sequence of $G_i$ satisfies the premises of Lemma 7 with high probability. It suffices to verify the first premise – the second and the third hold because $V_i$ is a subset of $V$. It’s not hard to see that the initial weight sequence of $G_1 = G$ satisfies all the premises by Chernoff bound (details omitted). It suffices to show that $F_{i-1}$ has small volume, i.e., we only remove a small volume from $G$ in total. By Equation (1), we have:

$$E[\text{vol}(F_{i-1})] \leq \sum_{z \in V} p_z \cdot \min\{1, p_z \cdot n^\lambda\} \leq \sum_{z: p_z \geq n^{-\lambda}} p_z + \sum_{z: p_z < n^{-\lambda}} p_z^2 \cdot n^\lambda \leq O(n^{1-(\lambda(2-\beta))} + n^{1-(\lambda(3-\beta)+\lambda)}) = O(n^{1+\lambda(\beta-2)}).$$

The second inequality above is because of Lemma 7. It is not hard to verify that

$$1 + \lambda(\beta - 2) \leq 1 - \frac{1}{2} (\beta - 2)^2 - \Theta(\varepsilon).$$

Having bounded the expected volume of $F_{i-1}$, we obtain that with high probability only a total volume of $o(n^{1-(\beta-2)^2/2})$ is from $V$ in $V_i$. Thus, we obtain the first premise of Lemma 7 because $nt^{2-\beta} = \Omega(n^{1-(\beta-2)^2/2})$ for any $t \leq n^{\beta/2-1}$.

Now we can apply Lemma 7 to $G_i$ to obtain that $\Pr[X_i = 0] = o(1)$. Finally, by Markov’s inequality, $\sum_i (1 - X_i) \geq \frac{1}{2} \cdot n^{3/2 - \beta}$ with $o(1)$ probability. Hence at least $0.99n^{3/2 - \beta}$ vertices in $S$ have $\text{vol}(L(x_i)) \in [a_d, b_d]$ with high probability. Denote by $S_1 \subseteq S$ the set of such vertices.
Part II: connecting to heavy vertices

In this part, we show how $L(x_i)$ and certain set of high degree vertices are connected. Let $A = \{ x \in V : p_x \in [(a_d/w)^{1/(\beta-2)}, 2(a_d/w)^{1/(\beta-2)}] \setminus F_{[S]} \}$. We claim that there is a constant fraction of the vertices $x \in S_1$ such that each $L(x)$ is connected to a different vertex in $A$. More formally, we make the following claim.

Claim 9. With $1 - o(1)$ probability, there exists a set $S_2 \subseteq S_1$ and a function $h : S_2 \rightarrow A$ such that $|S_2| \geq \frac{1}{3} n^{\frac{3-\beta}{2}}$, for every $x \in S_2$, $h(x)$ connects to some vertex in $L(x)$ and $h$ is an injection.

We first show that $A$ has a large volume. By Equation (1), any vertex with weight at most $2(a_d/w)^{1/(\beta-2)} < n^{1/2}$ belongs to $F_{[S]}$ with probability at most $n^{\frac{1}{2}(\beta-2)/(\beta-3)} = o(1)$. Thus, the volume of $A$ is

$$\text{vol}(A) = \Theta(nw/a_d)$$

with $1 - o(1)$ probability. This implies that $|A| \geq \Theta(nw/(a_d(a_d/w)^{1/(\beta-2)})) \gg \omega(n^{(3-\beta)/2})$.

Now we construct the set $S_2$ and the function $h$ as follows:

1. Go over all vertices $x$ in $S_1$, if $L(x)$ has a neighbor $y$ in $A$ that is “unused”, add $x$ to $S_2$ and set $h(x)$ to $y$;
2. Mark $y$ as “used”.

This procedure will generate a set $S_2$ of size at least $\frac{1}{3} n^{\frac{3-\beta}{2}}$ with high probability, since

1. Only an $o(1)$ fraction of the vertices in $A$ are marked as “used”, as $|A| \gg n^{\frac{3-\beta}{2}}$;
2. $\text{vol}(L(x)) \geq a_d$ by Claim 8 and thus,

$$\Pr[\forall y \in A, \text{ s.t. } y \text{ "unused", } L(x) \sim y] \leq \exp(-\text{vol}(L(x)) \cdot \text{vol}(A)/\text{vol}(V)) \leq \exp(-a_d \cdot nw/a_dn) = o(1).$$

The claim follows by Markov’s inequality.

Part III: upper bounding the label size of $S$

Consider the distance function on $G$ restricted to all vertex pairs in $S$, $\text{dist}_S : S \times S \rightarrow \mathbb{N}$. Clearly, $\text{dist}_S$ can be determined from the labels of $S$. We consider three cases:

1. The random graph does not satisfy the bound on $|S_2|$ in Claim 9. By the above argument, such a graph can only be generated with $o(1)$ probability.
2. If there exists $0.01 \cdot n^{3-\beta}$ vertex pairs from $S$ whose distance is at most $2d + 3$, we claim that the probability that a random graph induces any such distance function is at most $o(1)$. By Lemma 8, the probability that two vertices have distance at most $2d + 3$ is $o(1)$, hence the expected number of vertex pairs in $S$ within distance $2d + 3$ is $o(n^{3-\beta})$. The claim then follows by Markov’s inequality.
3. If the number of vertex pairs from $S$ within distance $2d + 3$ is at most $0.01 \cdot n^{3-\beta}$, then we infer that there are $0.2n^{(3-\beta)/2}$ vertices in $S_2$ whose pairwise distance is at least $2d + 4$. Let $B \subseteq S_2 \times S_2$ be the set of all such vertex pairs. The size of $B$ is least $0.01n^{3-\beta}$. Note that the
distance function \( \text{dist} \) determines the set \( B \), and we have \( |B| \geq \Theta(n^{3-\beta}) \), for any \((x, y) \in B\), \( h(x) \) and \( h(y) \) must not be connected by an edge. Hence, for any such set \( B \),

\[
\Pr \left[ \text{dist}(x, y) \geq 2d + 4, \ \forall (x, y) \in B \mid |S_2| = \Theta(n^{3-\beta}) \right] \\
\leq \prod_{(x, y) \in B} \Pr[h(x) \neq h(y)] \\
\leq (1 - \Theta((a_d/w)^{2/(\beta-2)} / n))^{\Theta(n^{3-3})} \\
\leq \exp \left( -(a_d/w)^{2/(\beta-2)} \cdot n^{3-4} \right) \\
= \exp \left( -n^{-2e/(\beta-2)} \cdot w^{-2/(\beta-2)} \cdot \log^2 \log n \cdot n^{3-3} \right) \\
= \exp \left( -n^{3-3-O(\varepsilon)} \right).
\]

That is, each of such labels for \( S \) could only occur with probability \( \exp(-n^{3-3-O(\varepsilon)}) \). However, there are only \( O(2^n) \) different labels of total size at most \( s \). Then by union bound, the probability that a random graph induces labels for \( S \) of size at most \( n^{3-3-O(\varepsilon)} \) is \( o(1) \). This finishes the proof.

## 6 Experiment

In this section, we evaluate our algorithms on a collection of large networks. We compare with the algorithm of Akiba et al.’s [1] and the Thorup-Zwick distance oracle [41, 13]. The first algorithm produces an exact landmark labeling via recursively pruning during breadth first search over all vertices – we will refer to it as \text{PrunedLabel} later. The second algorithm adapts the 3-approximate distance oracle of Thorup and Zwick [41], via picking high degree vertices as global landmarks – we refer to it as \text{BallGrow}. In Table 1 we list the graphs used in our experiment. More details are available at Stanford Large Network Dataset Collection [33].

| graph       | # nodes | # edges | category | 90% effective diameter | average distance |
|-------------|---------|---------|----------|------------------------|-----------------|
| Twitter     | 81,306  | 1,768,149 | Social   | 4.5                    | 3.8             |
| Stanford    | 281,903 | 2,312,497 | Web      | 9.7                    | 5.2             |
| Google      | 875,713 | 5,105,039 | Web      | 8.1                    | 6.0             |
| BerkStan    | 685,230 | 7,600,595 | Web      | 9.9                    | 6.3             |

Table 1: Basic statistics of graphs in experiments.

**Implementation** We implemented all three algorithms in Scala. The graph library we used is available at [https://github.com/teapot-co/tempest](https://github.com/teapot-co/tempest). We run the experiments on Amazon EC2 m4.4xlarge instance, with 64GB of RAM and 16 Intel Xeon 2.3GHz CPUs. We used a variant of \text{AlgSkewDegree} in the experiments [6]. We hand tune the two parameters used in the algorithm. For \text{PrunedLabel}, a vertex ordering is required: we simply sort all vertices by indegree plus outdegree. For \text{BallGrow}, it is necessary to specify the number of global landmarks; we handtune this parameter and choose the number of high degree vertices as global landmarks accordingly.

We measure accuracy over 2000 randomly sampled pairs of source/destination vertices. We look at the 80 and 90-percentile multiplicative error (\(|\text{estimated-distance} / \text{true-distance} - 1|\) ).
Algorithm 2 A description of our algorithm in experiments.

Input: A directed graph $G = (V, E)$; Parameters $d$ and $K$.

1: $\sigma = \text{vertices ordered by (indegree + outdegree)}$
2: for $i \leq n$ do
3: if $i \leq K$ then
4: computeGlobalLm($\sigma_i$)
5: else
6: computeLocalLm($\sigma_i$)
7: end if
8: end for
9: procedure computeGlobalLm($x$)
10: $(y, \text{dist}(x, y)), \forall y \in V =$ Run a forward BFS
11: $(y, \text{dist}(y, x)), \forall y \in V =$ Run a backward BFS
12: end procedure
13: procedure computeLocalLm($x$)
14: Run a forward BFS from $x$ up to distance $d$, prune any node from $\{\sigma_i\}_{i=1}^K$
15: end procedure

Results Table 2 compares the landmark size and running time of the three tested algorithms. Table 3 compares the accuracy. Looking at accuracy, we found that both our algorithm and BallGrow are fairly accurate on the three Web graphs. However, our algorithm does slightly worse for the first test cases. From the performance comparison, we found that both our algorithm and BallGrow are more scalable compared to PrunedLabel. This is to be expected, since PrunedLabel is designed to guarantee exact distances. Our algorithm found smaller landmark sets compared to BallGrow and PrunedLabel in three out of four tests, and runs faster than PrunedLabel on the two largest instance.

|            | Landmark size per node | Running time (min) |
|------------|------------------------|--------------------|
|            | Ours  | PrunedLabel | BallGrow   | Ours  | PrunedLabel | BallGrow   |
| Twitter    | 227   | 261        | 637        | <1    | 10.5        | 1.8        |
| Stanford   | 82    | 95         | 367        | <1    | <1          | 2.0        |
| Google     | 215   | 285        | 276        | 7.2   | 84          | 8.2        |
| BerkStan   | 63    | 155        | 742        | 1.0   | 46.9        | 8.6        |

Table 2: Comparison of performances over our algorithm, PrunedLabel and BallGrow. The landmark size is equal to the total number of forward and backward landmarks stored, divided by the total number of vertices.

|            | 90% error | 80% error |
|------------|-----------|-----------|
|            | Ours | BallGrow | Ours | BallGrow |
| Twitter    | 0.5  | 0.0      | 0.25 | 0.125    |
| Stanford   | 0.07 | 0.08     | 0.0  | 0.0      |
| Google     | 0.0  | 0.0      | 0.0  | 0.0      |
| BerkStan   | 0.125| 0.1      | 0.0  | 0.0      |

Table 3: Comparison of accuracy. The accuracy of PrunedLabel is not listed because it is guaranteed to output exact distances.
7 Discussions

In this work, we studied distance labeling schemes on random graphs. We showed that simple breadth first search based algorithms are near optimal. Our experiments suggest that the algorithms we developed are effective on real world graphs that have small average distance and power law degree distribution. Apart from closing the gap between upper and lower bounds, we discuss about future work below.

Extensions. The Chung-Lu model has a natural extension to directed graphs. Consider two power law distributions $f^{\text{in}}(x)$ and $f^{\text{out}}(x)$ with mean value bigger than 1, representing the indegree and outdegree distributions, respectively. Each node $v$ is associated with two parameters $p^{\text{in}}_v \sim f^{\text{in}}(\cdot)$ and $p^{\text{out}}_v \sim f^{\text{out}}(\cdot)$, respectively. For any two nodes $u$ and $v$, there is a directed edge from $u$ to $v$ with probability $\frac{p^{\text{out}}_u \cdot p^{\text{in}}_v}{M}$, where $M$ is a normalization term. We sketch a heuristic argument which shows that $O(\sqrt{n})$ landmarks per node suffices to get a 2-hop cover: If we do a breadth-first search forward from every node $x$ to include $\tilde{O}(\sqrt{n})$ landmarks for $x$ as well as a backward BFS from $x$ to include $\tilde{O}(\sqrt{n})$ landmarks, then for every pair of nodes $x$ and $y$, the forward frontier of $x$ and the reverse frontier of $y$ will intersect with high probability. A second possible extension is to consider configuration models with a power law degree distribution. We believe all of our proofs can be extended to configuration models, since our technical tools only involve bounding the growth of branching processes from every node; We leave the details to future work.

Beyond worst case analysis. It would be interesting to consider deterministic characterizations that will ensure short distance labelings. Do constant expansion ensure the existence of sub-quadratic distance labeling schemes? The high level intuition behind our algorithmic result is that as long as the breadth-first search process grows neither too fast nor too last, but rather at a uniform rate, then it is possible to obtain a “short” labeling scheme. Here is a more concrete instantiation: consider any graph $G = (V, E)$ such that for all $x \in V$ and $0 \leq i \leq n$, either $|N_i(x)| \leq \sqrt{n}$ or $\frac{1}{|N_i(x)|} \in [l, c]$, where $c > l > 1$ are fixed values. Now for each $x \in V$, add all the vertices within distance $\log \frac{n}{2}$ plus the closest $\sqrt{n}$ vertices to the landmark set of $x$. It is not hard to see that this gives a 2-hop cover. The total size of the 2-hop cover is $O(l \log \frac{n}{2} + \sum_{i=1}^{\log \frac{n}{2}} H_i)$, where $H_i$ is the total number of length-$i$ path in $G$. For random power law graphs when $\beta > 3$, $l$ and $c$ are asymptotically equal to $r = \omega/\nu$, $E[H_i] = O(nr^i)$ and the above formula becomes $O(n^{3/2} \log n)$.

Distance oracles. For the finite variance case, the query complexity is $\Theta(\sqrt{n})$ for labeling schemes. However, if we want faster query schemes, the situation seems very mysterious. Can we even obtain sub-quadratic data structures with $o(\sqrt{n})$ query complexity? In another direction, we suspect that random graphs might provide a candidate hard instance for the set intersection conjecture [40].

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A Upper Bound of Theorem 1

In this section we consider the degree sequence \( p \) when it is drawn from a power law distribution with finite second moment.

**Proposition 10.** Let \( f \) denote a power law distribution with mean value \( \nu > 1 \) and exponent \( \beta > 3 \). Let \( p \) denote \( n \) independent samples from \( f(\cdot) \) and \( 0 < \varepsilon < 1/2 \) be a fixed value. Let \( d \sim o(n^{3/5}) \) be a threshold value. The following holds almost surely:

i) The maximum weight \( \max p \leq o(n^{3/5}) \).

ii) The volume of \( V \) is \( \text{vol}(V) = \sum_{i=1}^{n} p_i = \nu n \pm O(n^{1/2+\varepsilon}) \).

iii) The second moment below \( d \) is

\[
\sum_{x \in V} p_x^2 1_{p_x \leq d} = \left( \omega - \frac{Zd^{3-\beta}}{\beta - 3} \right) n \pm O(n^{(1+\varepsilon)/2}d).
\]

iv) The second moment above \( d \) is

\[
\sum_{x \in V} p_x^2 1_{p_x > d} = \frac{Z}{\beta - 3} n d^{3-\beta} (1 + o(1)).
\]

v) The second moment \( \text{vol}_2(V) = \omega n \pm O(n^{3(\beta-2)+\varepsilon}) \).

vi) The first moment below \( d \) is \( \sum_{x \in V} p_x 1_{p_x \leq d} \leq (\nu - \frac{Zd^{3-\beta}}{\beta - 2}) n + o(n) \).

vii) The first moment above \( d \) is \( \sum_{x \in V} p_x 1_{p_x > d} \leq \frac{Zd^{3-\beta}}{\beta - 2} n (1 + o(1)) \).

The proof is via standard concentration inequality – we leave it to the reader.

We will assume that Proposition 10 holds for the degree sequence \( p \), and condition on \( p \) being fixed. Given a random graph \( G = (V, E) \), we present an algorithm (shown below) for finding a 2-hop cover of \( G \).

**Algorithm 3 ALGBoundedVar**

**Input:** An undirected graph \( G = (V, E) \); A parameter \( \delta \).

1. for \( x \in V \) do
2. \( (F(x), l(x)) = \text{ALGBfs}(x, \delta \sqrt{n}) \)
3. end for
4. procedure ALGBfs\((x, t)\)
5. \( S = \{x\} \)
6. \( \alpha_0(x) = d_x; k = 0 \)
7. while \( \alpha_k(x) \leq t \land |\Gamma_{k+1}(x)| > 0 \) do
8. \( S = S \cup \Gamma_k(x) \)
9. \( Y = \{(y,z) \in E : y \in \Gamma_k(x), z \in \Gamma_{k+1}(x)\} \)
10. \( k = k + 1 \)
11. \( \alpha_k(x) = \sum_{y \in \Gamma_k(x)} d_y - |Y| \) \hspace{1cm} (estimate the next boundary size)
12. end while
13. return \((S, k)\)
14. end procedure
Lemma 11. Algorithm A with parameter \( \delta \sqrt{n} \) where \( \delta = 5\sqrt{n \log n} \) finds a 2-hop cover \( F \) of \( G \) with high probability.

Proof. Consider a fixed vertex \( x \in V \), we first show that unless \( F(x) \) contains the entire connected component of \( x \), the Algorithm will stop with a boundary layer whose volume is \( \Omega(\sqrt{n \log n}) \) with probability at least \( 1 - n^{-2} \). If \( l(x) = 1 \), then either \( x \) is an isolated node, or \( d_x \geq \delta \sqrt{n} \). Since \( d_x \leq \sqrt{n} \), this happens with probability at most \( \exp(-\frac{(\delta-1)\sqrt{n}}{2}) \sim o(n^{-4}) \) by Proposition 21.

When \( l(x) = k + 1 \geq 2 \), we show that \( \text{vol}(\Gamma_k(x)) \geq \delta \sqrt{n}/3 \) with high probability. The termination condition implies that \( \alpha_k(x) \geq \delta \sqrt{n} \). Consider the process in which \( \alpha_k(x) \) is generated: we keep branching out from \( x \) until we reach the \( k \)-th level from \( x \), then we reveal the edges between \( \Gamma_k(x) \) and \( V\setminus N_{k-1}(x) \). Conditional on \( a = \text{vol}(\Gamma_k(x)) \leq \delta \sqrt{n}/3 \), \( \alpha_k(x) \) is the sum of independent 0-1-2 random variables (because the edges inside \( \Gamma_k(x) \) are counted twice), and

\[
\mathbb{E}[\alpha_k(x)] = \sum_{y \in \Gamma_k(x)} \sum_{z \notin N_{k-1}(x)} \frac{p_y p_z}{\text{vol}(V)} \leq a \sum_{z \in V} \frac{p_z}{\text{vol}(V)} = a \leq \frac{\delta \sqrt{n}}{3}
\]

Therefore by Chernoff bound,

\[
\Pr[\alpha_k(u) \geq \delta \sqrt{n} | \text{vol}(\Gamma_k(u)) \leq \delta \sqrt{n}/3] \leq \exp(-\frac{\delta \sqrt{n}}{18}) \sim o(n^{-4}).
\]

Secondly, we show that if two level sets have volume \( \Omega(\sqrt{n \log n}) \) but are disjoint, then the probability that there is no edge between them is very small. Let \( x \) and \( y \) be any two vertices. Let \( 1 \leq k_1 \leq n - 1 \) and \( 1 \leq k_2 \leq n - 1 \). Let \( \Omega_{k_1,k_2} \) denote the set of graphs satisfying \( \text{vol}(\Gamma_{k_1}(x)) \geq \delta \sqrt{n}/3 \), \( \text{vol}(\Gamma_{k_2}(y)) \geq \delta \sqrt{n}/3 \) and \( \Gamma_{k_1}(x) \cap \Gamma_{k_2}(y) = \emptyset \). Then by Proposition 3

\[
\Pr[\Gamma_{k_1}(x) \sim \Gamma_{k_2}(y) | \Omega_{k_1,k_2}] \leq \exp(-\frac{\delta^2 n}{9 \text{vol}(V)}) \sim o(n^{-6}).
\]

Now we are ready to bound the probability that \( F \) is not a 2-hop cover. Let \( \Omega_S \) denote the set of graphs such that

1. \( \Gamma_{l(x)}(x) = \emptyset \) or \( \text{vol}(\Gamma_{l(x)-1}(x)) \geq \delta \sqrt{n}/3 \), for all nodes \( x \);
2. if \( \Gamma_{l(x)}(x) \) and \( \Gamma_{l(y)}(y) \) are both non-empty, then \( F(x) \cap F(y) \) is non-empty, for all \( x, y \in V \), where \( x \neq y \).

It’s clear that if \( G \in \Omega_S \), then the algorithm successfully finds a 2-hop cover. The probability that Condition (1) or (2) does not hold is at most \( o(n^2) \) by taking union bound.

Lemma 12. Let \( x \) be a fixed node. Let \( 0 \leq k \leq O(\log n) \). Let \( \Omega_k \) denote the set of graphs such that

\[
\text{vol}(\Gamma_i(x)) \leq 4\delta \sqrt{n}, \quad \text{for any } 0 \leq i \leq k - 1,
\]

and

\[
\text{vol}(\Gamma_k(x)) > 4\delta \sqrt{n}.
\]

Then \( \Pr[\alpha_k(x) \leq \delta \sqrt{n} | \Omega_k] \leq n^{-2} \).
Proof. Let $a = \text{vol}(\Gamma_k(x))$ and $b = \text{vol}(N_{k-1}(x))$. Then conditional on $\Omega_k$, $a$ and $b$ satisfies

$$a > 4\delta \sqrt{n} \quad \text{and} \quad b \leq 4k\delta \sqrt{n}$$

And $\alpha_k$ is the sum of independent 0-1-2 random variables. Let $\mu$ denote its expected value, then

$$\mu = \sum_{y \in \Gamma_k(x)} \sum_{z \notin N_{k-1}(x)} \frac{p_y p_z}{\text{vol}(V)} = a(1 - \frac{b}{\text{vol}(V)})$$

Since $\text{vol}(V) \sim \Theta(n)$ by Proposition [10], $b/\text{vol}(V) \sim o(1)$. By Chernoff bound,

$$\Pr[\alpha_k \leq \delta \sqrt{n} \mid \Omega_k] \leq \exp(-\frac{\delta \sqrt{n}}{4}) \sim o(n^{-2}).$$

Lemma 13. Let $x$ be a fixed node. Let $0 \leq k \leq O(\log n)$. Denote by $\Omega^*_k$ the set of graphs such that

$$\alpha_i \leq \delta \sqrt{n}, \quad \text{for all } 0 \leq i \leq k$$

Then $\Pr[\text{vol}(\Gamma_k(x)) > 4\delta \sqrt{n}, \Omega^*_k] \leq (k+1)n^{-2}$.

Proof. For $k = 0$, the claim follows by applying Lemma [12]. For $k > 0$, we repeatedly apply Lemma [12] for all values of $i$ smaller than $k$. For $0 \leq i \leq k$, denote by $S_i \subset \Omega^*_k$ the set of graphs such that:

1. $\text{vol}(\Gamma_k(x)) > 4\delta \sqrt{n}$;
2. $\text{vol}(\Gamma_j(x)) \leq 4\delta \sqrt{n}$ for any $0 \leq j \leq i - 1$.

We claim that

$$\Pr[S_i] - \Pr[S_{i+1}] \leq n^{-2}, \quad \text{for } 0 \leq i \leq k - 1,$$

and $\Pr[S_k] \leq n^{-2}$. These two combined together would imply this conclusion.

To see this,

$$\Pr[S_i] - \Pr[S_{i+1}] = \Pr[\text{vol}(\Gamma_i(x)) > 4\delta \sqrt{n}, S_i]$$

The first inequality is because if a graph $G$ is in $S_i$ (in particular condition (2)) and the volume of $\Gamma_i(x)$ is bigger than $4\delta \sqrt{n}$, then $G \in \Omega_i$. And the condition on $S_i \subset \Omega^*_k$ implies $\alpha_i \leq \delta \sqrt{n}$. The second inequality is because of Lemma [12]. One can similarly argue about $\Pr[S_{k+1}]$ and we omit the details. \hfill \Box

Lemma 14. The following holds almost surely:

- $|F(x)| \sim O(\sqrt{n \log n})$ for all $x \in V$.
- The algorithm runs in time $O(\sqrt{n \log^2 n})$ for all $x \in V$.

Proof. We first bound the number of landmarks added before reaching the boundary layer. Since $\alpha_i(x) \leq \delta \sqrt{n}$, for $i = 0, \ldots, l(x) - 2$, and $l(x) \leq O(\log n)$, we conclude that there are at most $O(\sqrt{n \log^2 n})$ landmarks before layer $l(x) - 1$. The rest of the proof shows that $|\Gamma_{l(x) - 1}(x)| \sim O(\sqrt{n \log^2 n})$. Set $c = 8 \log n + 8\delta\omega/\nu$. When $l(x) = 1$, clearly $d_x \leq \delta \sqrt{n}$ by Proposition [10]. When $l(x) = k + 1 \geq 2$, we have that $G \in \Omega^*_k$ and hence $\text{vol}(\Gamma_{k-1}(x)) \leq 4\delta \sqrt{n}$ with probability at least $1 - kn^{-2}$, by Lemma [13]. We now argue that $\text{vol}(\Gamma_k(x)) \leq c\sqrt{n}$ with high probability.
Conditional on $a = \text{vol}(\Gamma_{k-1}(x)) \leq 4\delta \sqrt{n}$, $\text{vol}(\Gamma_k(x))$ is the sum of independent random variables that are all bounded in $[0, \sqrt{n}]$, with expected value $\mu$ as

$$\mu = \sum_{y \notin N_{k-1}(x)} \text{Pr}[z \sim \Gamma_{k-1}(y)] \times p_y \quad \text{(by Proposition 3)}$$

$$\leq \sum_{y \notin N_{k-1}(x)} \frac{p_y^2 a}{\text{vol}(V)} \leq \frac{\text{vol}_2(V)}{\text{vol}(V)} \times a \sim a \left( 2 - o(1) \right).$$

The last line is because $\text{vol}_2(V) = \omega n(1 + o(1))$ and $\text{vol}(V) = \nu n(1 + o(1))$, by Proposition 10. By Chernoff bound,

$$\text{Pr}[\text{vol}(\Gamma_k(x)) > c\sqrt{n} \mid \text{vol}(\Gamma_{k-1}(x)) \leq a] \leq \exp \left( -\frac{c\sqrt{n} - \mu}{2\sqrt{n}} \right) \leq \exp (-3 \log n) \sim o(n^{-2}),$$

because $\mu \leq \frac{\omega}{\nu} a(1 + o(1))$ and $a \leq 4\delta \sqrt{n}$.

For the last part, we argue that if $\text{vol}(\Gamma_k(x)) \leq c\sqrt{n}$, then $\alpha_k(x) \leq 3c\sqrt{n}$. This is because conditional on $a = \text{vol}(\Gamma_k(x)) \leq c\sqrt{n}$, the expected value of $\alpha_k(x)$ is at most $c\sqrt{n}$ (details omitted). Since $\alpha_k(x)$ is the sum of independent 0-1-2 random variables, by Chernoff bound

$$\text{Pr}[\alpha_k(x) > 3c\sqrt{n} \mid \text{vol}(\Gamma_k)(x) \leq c\sqrt{n}] \leq \exp \left( -\frac{c\sqrt{n}}{4} \right) \sim o(n^{-2}).$$

In summary,

$$\text{Pr}[l(x) = k + 1, \alpha_k(x) > 3c\sqrt{n}] \sim O(n^{-2}).$$

Taking a union bound over $k \leq O(\log n)$ and $x \in V$, we obtain the desired conclusion.

B Upper Bound of Theorem 2

In this section, we consider degree sequence $p$ whose second moment is infinite.

**Proposition 15.** Let $f$ denote the probability density function of a power law distribution with mean value $\nu > 1$ and exponent $2 < \beta \leq 3$. Let $p$ denote $n$ independent samples from $f(\cdot)$. Let $\log n \leq d \leq 2\sqrt{n}$ be any fixed value and let $\epsilon(n)$ be a function that goes to 0 when $n$ goes to infinity. Then almost surely the following holds:

i) The maximum weight $\max p \geq \epsilon(n)n^{\frac{1}{\beta - 2}}$.

ii) The sum of weights beyond $d$ is $\sum_{x \in V} p_x 1_{p_x \geq d} \sim o(n)$.

iii) The volume of $V$ is $\text{vol}(V) = \nu n \pm o(n)$. 
\textbf{iv) Let }\log n < K \leq 2\sqrt{n}\text{ be a fixed value. Set}
\[ c(K) = \begin{cases} 32^{5-2\beta} & \text{if } 2.5 \leq \beta \leq 3 \\ \frac{32}{5-2\beta} K^{5-2\beta} & \text{if } 2 < \beta < 2.5 \end{cases} \]

Then
\[ \sum_{x \in V} p_x^{4-\beta} 1_{p_x \leq K} \leq c(K)n. \]

\textbf{v) Let }c > 1\text{ denote a fixed constant value. For any vertex }x \in V,
\[ \sum_{y \in V} p_y 1_{\frac{\beta}{c} \leq p_x \leq 2\sqrt{n}} \leq 6 \max\left(\frac{c^{\beta-2} Z}{\beta - 2} np_x^{2-\beta}, \sqrt{n} \log n\right). \]

The proof is via standard concentration inequality (details omitted). In the following we assume that \( p \) satisfies all properties in Proposition 15. In the following, we prove that Algorithm 1 is correct in Lemma 16 and bound its output size in Lemma 19.

\textbf{Lemma 16.} Algorithm 1 with parameter \( \delta \) and \( K \) finds a 2-hop cover \( F \) with high probability.

\textbf{Proof.} Consider the random variable \( l(x) \) that is computed in the Algorithm for each node \( x \). Let \( \Omega_S \) denote the set of graphs that satisfies
\[ \Gamma_{l(x)}(x) = \emptyset \text{ or } \dist(v^*, x) \leq l(x), \forall x \in V \]
where \( v^* \) is the node with the maximum weight. We argue that Algorithm 1 finds a 2-hop cover for any \( G \in \Omega_S \), and
\[ 1 - \Pr[\Omega_S] \leq \frac{2}{n} \]
This would imply that Algorithm 1 succeeds with probability at least \( 1 - 2/n \).

We first argue that Algorithm 1 is correct if \( G \in \Omega_S \). Let \( x \) and \( y \) be two different vertices in \( V \). If \( x \) and \( y \) are not reachable from each other, then clearly \( F(x) \cap F(y) = \emptyset \). If \( x \) and \( y \) are reachable from each other, consider their distance \( \dist(x, y) \). Note that when \( \Gamma_{l(x)}(x) \) (or \( \Gamma_{l(y)}(y) \)) is empty, then \( F(x) \) (or \( F(y) \)) includes the entire connected component that contains \( x \) (or \( y \)). Therefore, \( y \in F(x) \), vice versa. When none of them are empty, we know that \( \dist(x, v^*) \leq l(x) \) and \( \dist(y, v^*) \leq l(y) \) since \( G \in \Omega_S \). We consider three cases:

- If \( \dist(x, y) \leq l(x) + l(y) - 2 \), then there exists a node \( z \) such that \( \dist(x, z) \leq l(x) - 1 \) and \( \dist(y, z) \leq l(y) - 1 \). By our construction, \( z \) is in \( F(x) \) and \( F(y) \).

- If \( \dist(x, y) = l(x) + l(y) - 1 \), then consider the two nodes \( z \) and \( z' \) on one of the shortest path from \( x \) to \( y \), with \( \dist(x, z) = l(x) - 1 \) and \( \dist(y, z) = l(y) \). If either \( d_z \) or \( d_z' \) is at least \( K \), then they have been added as a landmark to every node in \( V \). Otherwise, assume without loss of generality that \( d_z \geq d_z' \). Then our construction adds \( z \) into \( F(y) \) and clearly \( z \) is also in \( F(x) \), hence \( z \) is a common landmark for \( x \) and \( y \).

- If \( \dist(x, y) = l(x) + l(y) \), then clearly \( v^* \) is a common landmark for \( x \) and \( y \).
We now bound \(1 - \Pr[\Omega_S]\). Clearly,
\[
1 - \Pr[\Omega_S] \\
\leq \sum_{x \in V} \Pr[\Gamma_l(x) \neq \emptyset, \text{dist}(v^*, x) > l(x)] \\
= \sum_{x \in V} \sum_{k=0}^{n-1} \Pr[l(x) = k + 1, \Gamma_{k+1}(x) \neq \emptyset, \text{dist}(v^*, x) > k + 1]
\]

Note that \(l(x) = k + 1\) and \(\Gamma_{k+1}(x) \neq \emptyset\) is the same as the event that:

- \(\alpha_i(x) \leq \delta n^{1 - \frac{1}{\beta-\tau}}\) for \(i = 0, \ldots, k - 1\);
- \(\alpha_k(x) > \delta n^{1 - \frac{1}{\beta-\tau}}\).

Hence,
\[
\Pr[l(x) = k + 1, \Gamma_{k+1}(x) \neq \emptyset, \text{dist}(v^*, x) > k + 1] \\
\leq \Pr[\alpha_k(x) > \delta n^{1 - \frac{1}{\beta-\tau}}, \text{dist}(v^*, x) > k + 1] \\
\leq \Pr[\alpha_k(x) > \delta n^{1 - \frac{1}{\beta-\tau}}, \text{vol}(\Gamma_k(x)) \leq \frac{\delta n^{1 - \frac{1}{\beta-\tau}}}{3}] \\
+ \Pr[\text{vol}(\Gamma_k(x)) > \frac{\delta n^{1 - \frac{1}{\beta-\tau}}}{3}, \text{dist}(v^*, u) > k + 1]
\]

For Equation (2), consider how \(\alpha_k(x)\) is discovered when we do the level set expansion from node \(x\). Conditioned on \(a = \text{vol}(\Gamma_k(x)) \leq \delta n^{1 - \frac{1}{\beta-\tau}}/3\), \(\alpha_k(x)\) is the sum of 0-1-2 independent random variables, with expected value less than \(\delta n^{1 - \frac{1}{\beta-\tau}}/3\). Hence by Chernoff bound, Equation (2) is at most \(\exp(-\delta n^{1 - \frac{1}{\beta-\tau}}/6) \sim o(n^{-3})\). For Equation (3), conditioned on \(\text{vol}(\Gamma_k(x)) \geq \delta n^{1 - \frac{1}{\beta-\tau}}/2\) and \(v^* \notin N_k(x)\),
\[
\Pr[v^* \sim \Gamma_k(x)] \leq \exp(-\frac{\delta n^{1 - \frac{1}{\beta-\tau}}p_{v^*}}{2\text{vol}(V)}) \sim o(n^{-3})
\]

The first inequality is because of Proposition 3. The second inequality is because \(\text{vol}(V) \sim \nu n \pm o(n)\) by Proposition 13. In summary, \(1 - \Pr[\Omega_S] \leq 2/n\). □

We now consider the size of our landmark scheme. There are three parts in each landmark set: (1) the heavy nodes whose degree is at least \(K\); (2) all the level sets before the last layer; (3) the last layer that we carefully constructed. It’s not hard to bound the first part, since the degree of a node is concentrated near its weight, and the number of nodes whose weight is \(\Omega(K)\) is \(O(nK^{1 - \beta})\).

The second part can be bounded by the maximum number of layers, hence the diameter of \(G\), which is \(O(\log n)\). For the third part, the idea is that before adding all the nodes on the boundary layer, we already have a (+1)-stretch scheme. Therefore, for a given vertex \(x\), it is enough if we only add neighbors whose degree is bigger than \(d_x\) — this reduces the amount of vertices from \(d_x\) to \(O(d_x^{\frac{1}{\beta}})\).

We first show that the volume of all the level sets is at most \(O(\delta n^{1 - \frac{1}{\beta-\tau}})\) before the boundary layer. For the rest of the section, let \(\alpha_k = \alpha_k(x)\) for any \(0 \leq k \leq n - 1\), unless there is any ambiguity on the vertex we are considering. Recall that \(\alpha_k\) denotes the number of edges between \(\Gamma_k(x)\) and \(V \setminus N_{k-1}(x)\).
Lemma 17. Let $x$ be a fixed node. Let $k$ be an integer less than $\leq O(\log n)$. Let $\Omega_k$ denote the set of graphs such that

$$\text{vol}(\Gamma_i(x)) < 4\delta n^{1 - \frac{1}{\beta - 1}}, \text{ for any } 0 \leq i \leq k - 1,$$

and

$$\text{vol}(\Gamma_k(x)) > 4\delta n^{1 - \frac{1}{\beta - 1}}.$$

Then $\Pr[\alpha_k \leq \delta n^{1 - \frac{1}{\beta - 1}} \mid \Omega_k] \leq n^{-2}$.

Proof. Let $a = \text{vol}(\Gamma_k(x))$ and $b = \text{vol}(N_{k-1}(x))$. Conditioned on $\Omega_k$,

$$a > 4\delta n^{1 - \frac{1}{\beta - 1}} \text{ and } b \leq 4k\delta n^{1 - \frac{1}{\beta - 1}}.$$

Clearly, the random variable $\alpha_k$ is the sum of independent 0-1 random variables. Let $\mu$ denote its expected value. For each $y \in \Gamma_k(x)$, we know that $p_y \leq a = O(\sqrt{n})$. Let $\mu_y$ denote the expected number of edges between $y$ and $V \setminus N_{k-1}(x)$, then

$$\mu_y = \sum_{y \in \Gamma_k(x)} \min(\frac{p_y p_z}{\text{vol}(V)}, 1) \mathbb{1}_{p_z \leq \sqrt{n}} \geq p_y (1 - \frac{b + \sum_{y \in V} p_z \mathbb{1}_{p_z \geq \sqrt{n}}}{\text{vol}(V)}) = p_y (1 - (n - o(1)))$$

because of Proposition [13]. And $\mu = \sum_{y \in \Gamma_k(x)} \mu_y = (1 - o(1))a$. Let $c = \frac{\mu}{\delta n^{1 - \frac{1}{\beta - 1}}} \geq 2 - o(1)$. By Chernoff bound,

$$\Pr[\alpha_k \leq \delta n^{1 - \frac{1}{\beta - 1}} \mid \Omega_k] \leq \exp(-\frac{(c - 1)^2\delta n^{1 - \frac{1}{\beta - 1}}}{4}) \sim o(n^{-2}).$$

Lemma 18. Let $x$ be a fixed vertex. Let $0 \leq k \leq O(\log n)$. Denote by $\Omega^*_k$ the set of graphs such that

$$\alpha_i \leq \delta n^{1 - \frac{1}{\beta - 1}}, \text{ for any } 0 \leq i \leq k$$

Then $\Pr[\text{vol}(\Gamma_k(x)) > 4\delta n^{1 - \frac{1}{\beta - 1}} \mid \Omega^*_k] \leq (k + 1)n^{-2}$.

Proof. When $k = 0$, the claim is proved by Lemma [17]. When $k \geq 1$, we will repeatedly apply Lemma [17] to prove the statement. For any values of $i$ smaller than or equal to $k$, let $S_i \subset \Omega^*_k$ denote the set of graphs that also satisfy: (1) $\text{vol}(\Gamma_j(x)) \leq 4\delta n^{1 - \frac{1}{\beta - 1}}$, for any $0 \leq j \leq i - 1$; (2) $\text{vol}(\Gamma_k(x)) > 4\delta n^{1 - \frac{1}{\beta - 1}}$. We show that $\Pr[S_i] - \Pr[S_{i+1}] \leq n^{-2}$ if $0 \leq i \leq k - 1$, and $\Pr[S_k] \leq n^{-2}$. Our Lemma follows from the two claims.

For the first part,

$$\Pr[S_i] - \Pr[S_{i+1}] = \Pr[\text{vol}(\Gamma_i(x)) > 4\delta n^{1 - \frac{1}{\beta - 1}}, S_i] \leq \Pr[\Omega_i, \alpha_i \leq \delta n^{1 - \frac{1}{\beta - 1}}] \leq n^{-2}.$$
Now we are ready to bound the size of our landmark scheme.

**Lemma 19.** The following holds almost surely

- \(|F(x)| \sim O(n^{1 - \min\left(\frac{1}{\beta - 1}, \frac{1}{4 - \beta}\right)} \cdot \log^3 n)\) for all \(x \in V\);
- The algorithm terminates in time \(O(n^{2 - \min\left(\frac{1}{\beta - 1}, \frac{1}{4 - \beta}\right)} \cdot \log^3 n)\).

**Remark** To implement Line 7, one can first sort \(N(x)\) for each \(x \in V\), in descending order on their degrees, and then create a separate list that truncates the nodes whose degree is at least \(K\). Given this list, one can find the set of neighbors of \(x\) whose degree is between \([d_x, K]\). The amount of time it takes to sort \(N(x)\) is \(O(d_x \log d_x) \sim O(d_x \log n)\). Hence the total amount of time it takes to sort all the adjacency lists is \(O(|E| \log n) = O(n \log n)\).

We will use the following lemma for technical reasons — the proof is deferred to the end of the section.

**Lemma 20.** Let \(x\) be a fixed node with weight \(p_x \leq 2K\). Denote by

\[
S_x = \{y \in N(x) : d_x \leq d_y \text{ and } d_y \leq K\}
\]

and let \(\hat{d}_x = |S_x|\). Then

\[
\Pr[\hat{d}_x \geq \max(c_1 p_x^{3 - \beta}, c_2 \log n)] \leq n^{-3}
\]

where \(c_1 = \frac{192Z}{\text{vol}(\Gamma) - 2}\) and \(c_2 = 130\).

**Proof of Lemma 20.** We first bound the number of nodes in \(H\). By Proposition 22 with probability \(1 - n^{-1}\)

\[
|H| = O(nK^{1 - \beta}) = O(n^{1 - \min\left(\frac{1}{\beta - 1}, \frac{1}{4 - \beta}\right)})
\]

Secondly, we bound the number of landmarks added before reaching the boundary layer. For any vertex \(x\), with \(i = 0, \ldots, l(x) - 2\), \(|\Gamma_{i+1}(x)| \leq \alpha_i(x) = O(\delta n^{1 - \frac{1}{\beta - 1}})\). Since \(l(x) \leq O(\log n)\), the total landmarks for these layers are at most \(O(n^{1 - \frac{1}{\beta - 1}} \log^3 n)\). The rest of the proof will bound the number of landmarks on the boundary layer with depth \(l(x) - 1\).

Denote by

\[
\pi_k(x) = \sum_{y \in \Gamma_k(x)} \hat{d}_y \mathbb{1}_{d_y \leq K} \quad \text{for } x \in V, 0 \leq k \leq n - 1
\]

Hence \(\pi_{\ell(x)-1}(x)\) gives the number of landmarks added on the boundary layer.

Set \(c_3 = \frac{3Z}{|\text{vol}(\Gamma)|} \max(x_{\min}^{5 - 2\beta}, 1), \psi = 12c_3 \delta n^{1 - \min\left(\frac{1}{\beta - 1}, \frac{1}{4 - \beta}\right)}\), and \(\Delta = \max(c_1 \psi, c_2 \delta n^{1 - \frac{1}{\beta - 1}} \log n)\), where \(c_1\) and \(c_2\) are defined in Lemma 20. We show that \(\pi_{\ell(x)-1}(x) \leq \Delta\) with probability \(1 - n^{-2}\) for the rest of the proof — our conclusion follows by taking union bound over \(x \in V\) and \(1 \leq l(x) \leq O(\log n)\).

When \(l(x) = 1\), \(\pi_0(x) = d_x \leq K \leq \Delta\). When \(l(x) = k + 1 \geq 2\), we know that \(G \in \Omega_{k-1}^\ast\). Hence by Lemma 18 \(\text{vol}(\Gamma_{k-1}(x)) \leq 4\delta n^{1 - \frac{1}{\beta - 1}}\) with high probability. More concretely,

\[
\Pr[l(x) = k + 1, \pi_k(x) \geq \Delta] \\
\leq (k + 1)n^{-2} + \Pr[l(x) = k + 1, \text{vol}(\Gamma_{k-1}(x)) \leq 4\delta n^{1 - \frac{1}{\beta - 1}}, \pi_k(x) \geq \Delta]
\]

Denote by

\[
w_k = \sum_{y \in \Gamma_k(x)} p_y^{3 - \beta} \mathbb{1}_{p_y \leq 2K}.
\]
Conditional on \( a = \text{vol}(\Gamma_{k-1}(x)) \leq 4\delta n^{1-\frac{1}{\sigma+1}} \), we show that \( w_k \leq \psi \) with high probability. Denote by \( \Omega_w \) the set of graphs satisfying \( a \leq 4\delta n^{1-\frac{1}{\sigma+1}} \). Conditioned on \( \Omega_w \), \( w_k \) is the sum of independent random variables that are all bounded in \([0, (2K)^{3-\beta}]\). Hence

\[
\mathbb{E}[w_k] = \sum_{y \not\in N_{k-1}(x)} \text{Pr}[y \sim N_{k-1}(x)] p_y^{3-\beta} 1_{p_y \leq 2K} \\
\leq \frac{a}{\text{vol}(V)} \left( \sum_{y \not\in N_{k-1}(x)} p_y^{4-\beta} 1_{p_y \leq 2K} \right) \quad \text{(by Proposition 3)} \\
\leq \frac{a}{\text{vol}(V)} \left( \sum_{y \in V} p_y^{4-\beta} 1_{p_y \leq 2K} \right) \\
\leq \frac{a \phi(K)n}{\text{vol}(V)} \quad \text{(by Proposition 15)} \\
\sim \frac{a \phi(K)}{\nu} \quad \text{(vol}(V) = \nu n \pm o(n) \text{ by Proposition 15)} \\
\leq \frac{\psi}{3},
\]

The last line follows by \( a \leq 4\delta n^{1-\frac{1}{\sigma+1}} \) and \( \phi(K)n^{1-\frac{1}{\sigma+1}} \leq c_2 n^{1-\min(\frac{1}{\sigma-1}, \frac{1}{\beta})} \). Now we apply Chernoff bound on \( w_k \),

\[
\text{Pr}[w_k > \psi \mid \Omega_w] \leq \exp(-\frac{\psi}{4(2K)^{3-\beta}}) \sim o(n^{-2})
\]

because when \( 2.5 \leq \beta \leq 3 \),

\[
\frac{\psi}{K^{3-\beta}} = \Theta(n^{1-\frac{1}{\sigma+1} - \frac{3-\beta}{2}}) = \Theta(n^{\frac{(\beta-1)^2}{2(\sigma-1)}})
\]

And when \( 2 < \beta < 2.5 \),

\[
\frac{\psi}{K^{3-\beta}} = \Theta(n^{\frac{(3-\beta)(\beta-2)}{4(\beta-1)}})
\]

Hence the second part in Equation (1) is bounded by \( o(n^{-2}) \) plus

\[
\text{Pr}[l(x) = k + 1, \text{vol}(\Gamma_{k-1}(x)) \leq 4\delta n^{1-\frac{1}{\sigma+1}}, w_k \leq \psi, \pi_k(x) \geq \Delta] \\
\leq \text{Pr}[w_k \leq \psi, \alpha_{k-1} \leq \delta n^{1-\frac{1}{\sigma+1}}, \pi_k(x) \geq \Delta] \\
\leq \text{Pr}[w_k \leq \psi, |\Gamma_k(x)| \leq \delta n^{1-\frac{1}{\sigma+1}}, \pi_k(x) \geq \Delta]
\]

In the reminder of the proof we show the above Equation is at most \( n^{-2} \). Denote by

\[
\pi'_k(x) = \sum_{y \in \Gamma_k(x)} \delta_y 1_{p_y \leq 2K}
\]

By Proposition 21 \( \text{Pr}[d_y \leq K \mid p_y > 2K] \leq \exp(-K/8) \sim o(n^{-3}) \) for any \( y \in V \). Hence \( \pi'_k(x) = \pi_k(x) \) with probability at least \( 1 - o(n^{-2}) \). Lastly, we have

\[
\text{Pr}[w_k \leq \psi, |\Gamma_k(x)| \leq \delta n^{1-\frac{1}{\sigma+1}}, \pi'_k(x) \geq \Delta] \leq n^{-2}
\]
Otherwise, there exists a vertex $y \in \Gamma_k(x)$ such that $p_y \leq 2K$ and $d_y \geq \max(c_1 p_y^{3-\beta}, c_2 \log n)$, because $\Delta \geq \max(c_1 \psi, c_2 \delta n^{1-\frac{\beta}{2}} \log n)$. This happens with probability at most $n^{-2}$, by taking union bound over every vertex with Lemma 20.

**Proof of Lemma 20** When $p_x \leq c_2 \log n/2$,

$$\Pr[d_x \geq c_2 \log n] \leq \Pr[d_x \geq c_2 \log n] \leq o(n^{-3})$$

Now suppose that $p_x > c_2 \log n/2$. Consider any vertex $y$ whose weight is at most $p_x/8$. Then

$$\Pr[d_y \geq d_x] \leq \Pr[d_y \geq p_x/4] + \Pr[d_x < p_x/4] \sim o(n^{-4})$$

The second inequality is because of Proposition 21. Hence $y$ is not in $S_x$.

Now if $p_y \geq 2K$, then $\Pr[d_y \leq K] \sim o(n^{-4})$. Hence $y$ is also not in $S_x$. Lastly, let $X$ denote the set of vertices whose weight is between $[p_x/8, 2K]$ and who is connected to $x$. We have

$$\mathbb{E}[X] = \sum_{y \in V \setminus \{x\}: p_x/8 \leq p_y \leq 2K} \frac{p_x p_y}{\text{vol}(V)}\leq \frac{4p_x}{\text{vol}(V)} \max\left(\frac{8Z}{\beta - 1}, \sqrt{n} \log n\right)\leq \max(c_1 p_x^{3-\beta}, c_2 \log n)/3$$

The first inequality is because of Proposition 15. The second inequality is because $\text{vol}(V) = \nu n + o(n)$ by Proposition 15 and $p_x \leq 2K \leq 2\sqrt{n}$. From here it is not hard to obtain that $\Pr[|X| \leq \max(c_1 p_x^{3-\beta}, c_2 \log n)] \sim o(n^{-3})$.

**C Random Graph Toolbox**

The following Lemma characterizes the probability that a vertex’s actual degree deviates from its weight.

**Proposition 21.** Let $G = (V, E) \in \mathcal{G}^n(p)$ be a random graph. Let $x$ be a fixed vertex with weight $p_x$ and degree $d_x$ in $G$. Then

1. If $c \geq 3$, then

$$\Pr[d_x \geq cp_x] \leq \exp\left(-\frac{(c - 1)p_x}{2}\right)$$

2. If $0 < c < 1$, then

$$\Pr[d_x \leq cp_x] \leq \exp\left(-\frac{(1 - c)^2 p_x}{8}\right)$$

**Proof.** Let $\mu = \mathbb{E}[d_x]$. First,

$$\mu = \sum_{y \in V \setminus \{x\}} \min\left(\frac{p_x p_y}{\text{vol}(V)}, 1\right)\leq p_x \sum_{y \in V} \frac{p_x p_y}{\text{vol}(V)} = p_x$$
By Chernoff bound, for any $c \geq 3$,
\[
\Pr[d_x \geq cp_x] \leq \exp(-\frac{cp_x - \mu}{2}) \\
\leq \exp\left(-\frac{(c-1)p_x}{2}\right)
\]
since $cp_x - \mu \geq 2\mu$.

On the other hand, let $t = \frac{\nu}{2\epsilon(n)} n^{1 - \frac{1}{\beta+1}}$, then for any $y \in V$ where $p_y \leq t$, we know that $p_xp_y \leq \text{vol}(V)$ by Proposition 15. Hence
\[
\mu \geq p_x(1 - \frac{p_x}{\text{vol}(V)} - \frac{\sum_{y \in V} p_y \mathbb{1}_{p_y \geq t}}{\text{vol}(V)})
\]

By Proposition 15,
\[
\sum_{y \in V} p_y \mathbb{1}_{p_y \geq t} \sim o(n)
\]

Since $p_x \sim o(n)$ and $\text{vol}(V) = \nu n + o(n)$, we conclude that $\mu = p_x(1 - o(1))$. By Chernoff bound, for any $0 < c < 1$,
\[
\Pr[d_x \leq cp_x] \leq \exp\left(-\frac{(cp_x - \mu)^2}{4\mu}\right) \leq \exp\left(-\frac{p_x(1-c)^2}{8}\right)
\]
for large enough $n$. \hfill \square

The following Proposition characterizes the number of nodes whose degree is at least $K$.

**Proposition 22.** Let $G = (V, E) \in \mathcal{G}^n(p)$ be a random graph. Let $8 \log n \leq K \leq \sqrt{n}$ denote a fixed value and $S = \{x \in V : d_x \geq K\}$. With probability at least $1 - n^{-1}$, $|S| \leq 3 \max(\frac{Z_{3,\beta-1}}{\beta-1} nK^{1-\beta}, \log n)$.

**Proof.** Let $Y_1 = \{x \in V : p_x \geq \frac{K}{3}\}$ and $Y_2 = \{x \in V : p_x < \frac{K}{3}\}$ and $K \leq d_x$. Clearly, $S \subset Y_1 \cup Y_2$. We first show that $Y_2$ is empty with probability at least $1 - n^{-1}$. Consider a fixed node $x \in V$ with weight $p_x \leq K/3$. By Proposition 21
\[
\Pr[d_x \geq K] \leq \exp(-K) \sim o(n^{-2})
\]
Hence $\Pr[Y_2 \neq \emptyset] = o(n^{-1})$ by union bound.

We then bound the size of $Y_1$. The expected value of $Y_1$ is $\frac{Z_{3,\beta-1}}{\beta-1} nK^{1-\beta}$. Then by Chernoff bound, it’s not hard to obtain the desired conclusion (details omitted).

### C.1 Proof of Growth Lemma 6: $\beta > 3$

We first show an upper bound for the expected volume for each level $\Gamma_k(x)$.

**The expected volume of** $\Gamma_k(x)$ **is** $O(r^k)$. Let us first fix $\Gamma_k(x)$, and consider the set $\Gamma_{k+1}(x)$. For a vertex $y$, the probability that it is in $\Gamma_{k+1}(x)$ is at most
\[
p_y \cdot \frac{\text{vol}(\Gamma_k(x))}{\text{vol}(V)}
\]
by Proposition 3. Thus, the expected volume of $\Gamma_{k+1}(x)$ conditioned on $\text{vol}(\Gamma_k(x))$ is at most

$$
\mathbb{E}[\text{vol}(\Gamma_{k+1}(x)) \mid \text{vol}(\Gamma_k(x))] \\
\leq \sum_{y \not\in N_k(x)} p_y^2 \frac{\text{vol}(\Gamma_k(x))}{\text{vol}(V)} \\
\leq \text{vol}(\Gamma_k(x)) \cdot \frac{\text{vol}_2(V)}{\text{vol}(V)} \\
= \text{vol}(\Gamma_k(x)) \cdot r.
$$

On the other hand, $\text{vol}(\Gamma_0(x)) = O(1)$. Thus, we have $\mathbb{E}[\text{vol}(\Gamma_k(x))] = O(r^k)$. This proves Item 1.

**Two fixed constant-weight vertices are close with very low probability**

Fix two vertices $x, y \in S$. By Item 1,

$$
\mathbb{E}[\text{vol}(N_k(x))] \leq O(r^k).
$$

However, for each $i$, the probability that $y$ is at distance $i$ from $x$ conditioned on $N_{i-1}(x)$ is at most

$$
\Pr[y \in \Gamma_i(x) \mid N_{i-1}(x)] \leq p_y \cdot \frac{\text{vol}(\Gamma_{i-1}(x))}{\text{vol}(V)}.
$$

The probability $y$ is within distance $k + 1$ from $x$ is at most

$$
\Pr[y \in N_{k+1}(x)] \leq \sum_{i=1}^{k+1} \Pr[y \in \Gamma_i(x)] \\
\leq \frac{p_y}{\text{vol}(V)} \cdot \sum_{i=1}^{k+1} \mathbb{E}[\text{vol}(\Gamma_{i-1}(x))] \\
= p_y \cdot \frac{N_k(x)}{\text{vol}(V)} \\
\leq O(r^k).
$$

**With large probability, $\Gamma_{k+1}(x)$ has volume not much smaller than $\Gamma_k(x) \cdot \frac{\text{vol}_2(V) \cdot N_k(x)}{\text{vol}(V)}$**

Conditioned on $\Gamma_k(x)$, the probability that a vertex $y \not\in N_k(x)$ is in $\Gamma_{k+1}(x)$ is at least

$$
1 - e^{-p_y \frac{\text{vol}(\Gamma_k(x))}{\text{vol}(V)}}
$$

by Proposition 3.

For any $T > 0$, we have

$$
\sum_{y: p_y > T} p_y^2 \leq T^{-\gamma} \cdot \sum_{y: p_y > T} p_y^{2+\gamma} \leq \tau \cdot n \cdot T^{-\gamma}.
$$

We also have

$$
\sum_{y: p_y \leq T} p_y^3 \leq T^{1-\gamma} \cdot \sum_{y: p_y \leq T} p_y^{2+\gamma} \leq \tau \cdot n \cdot T^{1-\gamma}.
$$

Let us focus on all $y$'s with weight at most $T$. By that fact that $1 - e^{-x} \geq x - x^2/2$ when $x \geq 0$, the expected volume of $\Gamma_{k+1}(x) \cap \{y : p_y \leq T\}$ conditioned on $N_k(x)$ is at least
\[
\sum_{y \notin N_k(x) : p_y \leq T} p_y \left( 1 - e^{-p_y \frac{\text{vol}(\Gamma_k(x))}{\text{vol}(V)}} \right)
\geq \sum_{y \notin N_k(x) : p_y \leq T} p_y^2 \cdot \frac{\text{vol}(\Gamma_k(x))}{\text{vol}(V)} - \frac{1}{2} \sum_{y : p_y \leq T} p_y^3 \cdot \left( \frac{\text{vol}(\Gamma_k(x))}{\text{vol}(V)} \right)^2
\geq (\text{vol}(V \setminus N_k(x))) \cdot \frac{\text{vol}(\Gamma_k(x))}{\text{vol}(V)} - \sum_{y : p_y \geq T} p_y^2 \cdot \frac{\text{vol}(\Gamma_k(x))}{\text{vol}(V)} - \frac{1}{2} \sum_{y : p_y \leq T} p_y^3 \cdot \left( \frac{\text{vol}(\Gamma_k(x))}{\text{vol}(V)} \right)^2
\geq \text{vol}(\Gamma_k(x)) \cdot \frac{\text{vol}(V \setminus N_k(x))}{\text{vol}(V)} - \tau \cdot \text{vol}(\Gamma_k(x))^{-\gamma/2} \cdot \left( T^{-\gamma} + T^{-1} \cdot \frac{\text{vol}(\Gamma_k(x))}{\text{vol}(V)} \right).
\]

Note that “\(y \in \Gamma_{k+1}(x)\)” are independent events conditioned on \(N_k(x)\) for different \(y \notin N_k(x)\). Now we apply Chernoff Bound to lower bound the probability that the volume of \(\Gamma_{k+1}(x)\) is too small. The above inequality holds for every \(T > 0\). In the following, we set \(T = \text{vol}(\Gamma_k(x))^{1/2}\).

When \(\text{vol}(\Gamma_k(x)) \leq \text{vol}(V)^{2/3}\), \(T^{-\gamma} \geq T^{-1} \cdot \frac{\text{vol}(\Gamma_k(x))}{\text{vol}(V)}\), the expected volume of \(\Gamma_{k+1}(x)\) conditioned on \(N_k(x)\) is at least:

\[
\mathbb{E}[\text{vol}(\Gamma_{k+1}(x) \cap \{y : p_y \leq T\}) \mid N_k(x)] \geq \text{vol}(\Gamma_k(x)) \cdot \left( \frac{\text{vol}(V \setminus N_k(x))}{\text{vol}(V)} - 2\tau \cdot \text{vol}(\Gamma_k(x))^{-\gamma/2} \right).
\]

Since each \(p_y \leq T = \text{vol}(\Gamma_k(x))^{1/2}\), by Chernoff bound, we have

\[
\Pr \left[ \text{vol}(\Gamma_{k+1}(x)) \leq \text{vol}(\Gamma_k(x)) \cdot \left( \frac{\text{vol}(V \setminus N_k(x))}{\text{vol}(V)} - \text{vol}(\Gamma_k(x))^{-\gamma/3} \right) \mid N_k(x) \right] \leq 2^{-\Theta(\text{vol}(\Gamma_k(x))^{1/2-2\gamma/3})},
\]

as long as \(\text{vol}(\Gamma_k(x)) = O(n^{2/3})\) and \(\text{vol}(\Gamma_k(x))\) sufficiently large.

**With constant probability, \(\Gamma_k(x)\) has volume at least \(\Omega(r^k)\)** Fix a sufficiently large constant \(C\), denote by \(\mathcal{E}_0\) the event that \(x\) has a neighborhood of volume at least \(C\). Then it is not hard to verify that for any constant \(C\), the probability of \(\mathcal{E}_0\) is at least a constant:

\[
\Pr[\text{vol}(\Gamma_1(x)) \geq C] \geq \Omega(1).
\]

Moreover, for \(i \geq 1\), denote by \(\mathcal{E}_i\) the event that either

\[
\text{vol}(\Gamma_{i+1}(x)) > \text{vol}(\Gamma_i(x)) \cdot \left( \frac{\text{vol}(V \setminus N_i(x))}{\text{vol}(V)} - \text{vol}(\Gamma_i(x))^{-\gamma/3} \right)
\]

or

\[
\text{vol}(\Gamma_i(x)) \geq n^{2/3}.
\]

By the argument above,

\[
\Pr[\mathcal{E}_i \mid N_i(x)] \leq 2^{-\Theta(\text{vol}(\Gamma_i(x))^{1/2-2\gamma/3})}.
\]

We claim that these events have the following properties.
Claim 23. When \( \mathcal{E}_i \) occurs for all \( 0 \leq i < k \), we must have either \( \text{vol}(\Gamma_k(x)) \geq \Omega(r^k) \) or \( \text{vol}(N_k(x)) \geq n^{2/3} \) for sufficiently large \( C \).

Claim 24. All events \( \mathcal{E}_i \)'s \( (0 \leq i < k) \) occur simultaneously with constant probability.

Before proving the two claims, let us first show that they together imply that \( \Pr[\text{vol}(\Gamma_k(x)) \geq \Omega(r^k)] \geq \Omega(1) \).

By Markov’s inequality, the first inequality in the lemma statement and \( k \leq \frac{1}{2}\log r n \), we have

\[
\Pr[\text{vol}(N_k(x)) \geq n^{2/3}] \leq O(\sqrt{n}/n^{2/3}) = o(1).
\]

Therefore, we have the lower bound

\[
\Pr[\text{vol}(\Gamma_k(x)) \geq \Omega(r^k)]
\geq \Pr[\mathcal{E}_0, \ldots, \mathcal{E}_{k-1}] \cdot \Pr[\text{vol}(\Gamma_k(x)) \geq \Omega(r^k) \mid \mathcal{E}_0, \ldots, \mathcal{E}_{k-1}]
\geq \Pr[\mathcal{E}_0, \ldots, \mathcal{E}_{k-1}] \cdot (1 - \Pr[\text{vol}(N_k(x)) \geq n^{2/3} \mid \mathcal{E}_0, \ldots, \mathcal{E}_{k-1}])
\geq \Pr[\mathcal{E}_0, \ldots, \mathcal{E}_{k-1}] \cdot (1 - \Pr[\text{vol}(N_k(x)) \geq n^{2/3}] / \Pr[\mathcal{E}_0, \ldots, \mathcal{E}_{k-1}])
\geq \Omega(1).
\]

This proves the lemma.

Proof of Claim 23. Assume \( \mathcal{E}_i \) occurs for all \( 0 \leq i < k \) and \( \text{vol}(N_k(x)) < n^{2/3} \). The goal is to show that in this case, we must have \( \text{vol}(\Gamma_k(x)) \geq \Omega(r^k) \).

In particular, \( \text{vol}(N_k(x)) < n^{2/3} \) implies that \( \text{vol}(N_i(x)) < n^{2/3} \) and \( \text{vol}(\Gamma_i(x)) \leq n^{2/3} \) for every \( i \leq k \). By Hölder’s inequality, we also have

\[
\text{vol}_2(N_i(x)) \leq \text{vol}_{2+\gamma}(N_i(x))^\frac{1}{1+\gamma} \cdot \text{vol}(N_i(x))^\frac{\gamma}{1+\gamma}
\leq \tau^{\frac{1}{1+\gamma}} \cdot n^{1-\frac{\gamma}{3(1+\gamma)}}.
\]

Thus, the event \( \mathcal{E}_i \) \( (i > 0) \) implies

\[
\text{vol}(\Gamma_{i+1}(x)) > \text{vol}(\Gamma_i(x)) \cdot \left( r - \tau^{\frac{1}{1+\gamma}} \cdot n^{-\frac{\gamma}{3(1+\gamma)}} - \text{vol}(\Gamma_i(x))^{-\gamma/3} \right).
\]

Let \( \hat{r} = r - \tau^{\frac{1}{1+\gamma}} \cdot n^{-\frac{\gamma}{3(1+\gamma)}} - C^{-\gamma/3} \). For sufficiently large \( C \), \( \hat{r} > \sqrt{r} > 1 \). First we can show \( \text{vol}(\Gamma_i(x)) \geq C \cdot \hat{r}^{i-1} \geq C \cdot r^{(i-1)/2} \) for \( i \geq 1 \) inductively:

- By the definition of \( \mathcal{E}_0 \), \( \text{vol}(\Gamma_1(x)) \geq C \);
- If \( \text{vol}(\Gamma_i(x)) \geq C \cdot \hat{r}^{i-1} \), then we have
  \[
  \text{vol}(\Gamma_{i+1}(x)) \geq \text{vol}(\Gamma_i(x)) \cdot \left( r - \tau^{\frac{1}{1+\gamma}} \cdot n^{-\frac{\gamma}{3(1+\gamma)}} - \text{vol}(\Gamma_i(x))^{-\gamma/3} \right)
  \geq \text{vol}(\Gamma_i(x)) \cdot \left( r - \tau^{\frac{1}{1+\gamma}} \cdot n^{-\frac{\gamma}{3(1+\gamma)}} - C^{-\gamma/3} \right)
  \geq \text{vol}(\Gamma_i(x)) \cdot \hat{r}.
  \]
Thus, we have \( \text{vol}(\Gamma_i(x)) \geq \Omega(\hat{r}^i) \geq \Omega(r^{i/2}) \). By Equation (6) again, we have

\[
\text{vol}(\Gamma_k(x)) > \text{vol}(\Gamma_{k-1}(x)) \cdot (r - \tau^{1/6} \cdot n^{-\gamma(1/6)} - r^{-(k-1)\gamma/6})
\]

\[
\geq \text{vol}(\Gamma_1(x)) \cdot \left(\prod_{i=1}^{k-1} (r - \tau^{1/6} \cdot n^{-\gamma(1/6)} - r^{-(i-1)\gamma/6})\right)
\]

\[
\geq r^{k-1} \cdot C \cdot \left(\prod_{i=1}^{k-1} (1 - \tau^{1/6} \cdot n^{-\gamma(1/6)} \cdot r^{-1} - r^{-(i-1)\gamma/6-1})\right)
\]

\[
= r^k \cdot \alpha_k,
\]

where \( \alpha_k \) is decreasing as \( k \) increases. \( \alpha \log_n n \) is lower bounded by a constant \( \alpha \). Thus, \( \text{vol}(\Gamma_k(x)) \geq \alpha \cdot r^k \geq \Omega(r^k) \). This proves the claim.  

**Proof of Claim 24:** By Lemma 23, conditioned on \( \mathcal{E}_0, \ldots, \mathcal{E}_{i-1} \), we have either

\[
\text{vol}(\Gamma_i(x)) \geq \alpha \cdot r^i
\]

or

\[
\text{vol}(N_i(x)) \geq n^{2/3}.
\]

Thus, by Equation (5), we have

\[
\Pr[\mathcal{E}_i | \mathcal{E}_0, \ldots, \mathcal{E}_{i-1}] \leq 2^{-\Theta(r^{\Omega(i)})}.
\]

Since \( \Pr[\mathcal{E}_0] = \Omega(1) \), we may lower bound the probability that all events happen simultaneously

\[
\Pr[\mathcal{E}_0, \ldots, \mathcal{E}_{k-1}] \geq \Omega\left(\prod_{i=0}^{k-1} \left(1 - 2^{-\Theta(r^{\Omega(i)})}\right)\right)
\]

\[
\geq \Omega(1).
\]

**C.2 Proof of Growth Lemma 7:** \( 2 < \beta < 3 \)

Let us first upper bound the volume of \( k \)-neighborhood of any vertex \( x \).

**Upper bounding \( \text{vol}(\Gamma_k(x)) \)** We will upper bound \( \text{vol}(\Gamma_k(x)) \) in terms of \( \text{vol}(\Gamma_{k-1}(x)) \). First, we have \( \text{vol}(\Gamma_0(x)) = \text{vol}(x) \) by definition. For any vertex \( y \), the probability that it is connected to some vertex in \( \Gamma_{k-1}(x) \) is at most

\[
\sum_{z \in \Gamma_{k-1}(x)} p_y \cdot \frac{p_z}{\text{vol}(V)} = p_y \cdot \frac{\text{vol}(\Gamma_{k-1}(x))}{\text{vol}(V)}.
\]

Thus, the probability that \( \Gamma_k(x) \) contains any vertex with very high weight is low:

\[
\Pr\left[ \exists y, p_y \geq (\text{vol}(\Gamma_{k-1}(x)))^{1/(\beta-2)} \cdot w, y \in \Gamma_k(x) \mid \text{vol}(\Gamma_{k-1}(x)) \right]
\]

\[
\leq \sum_{y, p_y \geq (\text{vol}(\Gamma_{k-1}(x)))^{1/(\beta-2)} \cdot w} p_y \cdot \frac{\text{vol}(\Gamma_{k-1}(x))}{\text{vol}(V)}
\]

\[
\leq O\left(n \cdot \frac{1}{\text{vol}(\Gamma_{k-1}(x))} \cdot w^{\beta-2} \cdot \frac{\text{vol}(\Gamma_{k-1}(x))}{\text{vol}(V)}\right)
\]

\[
= O(1/w^{\beta-2}).
\]
That is, the highest weight in $\Gamma_k(x)$ is at most
\[ w \cdot \left(\text{vol}(\Gamma_{k-1}(x))\right)^{1/(\beta-2)} \]
with probability at least $1 - O(1/w^{\beta-2})$. Denote this event by $E_k$. We have
\[
\mathbb{E}[\text{vol}(\Gamma_k(x)) \mid E_k, \text{vol}(\Gamma_{k-1}(x))] \leq \sum_{y, p_y \leq \left(\text{vol}(\Gamma_{k-1}(x))\right)^{1/(\beta-2)}.w} p_y^2 \cdot \frac{\text{vol}(\Gamma_{k-1}(x))}{\text{vol}(V)} \leq O\left(n \cdot \text{vol}(\Gamma_{k-1}(x))^{(3-\beta)/(\beta-2)} \cdot w^{3-\beta} \cdot \frac{\text{vol}(\Gamma_{k-1}(x))}{\text{vol}(V)}\right) \leq O\left(\text{vol}(\Gamma_{k-1}(x))^{1/(\beta-2)} \cdot w^{3-\beta}\right).
\]
By Markov’s inequality, we have
\[
\Pr[\text{vol}(\Gamma_k(x)) \geq \text{vol}(\Gamma_{k-1}(x))^{1/(\beta-2)} \cdot w \mid E_k] \leq O(1/w^{\beta-2}).
\]
Since $E_k$ occurs with high probability, by union bound, we have
\[
\Pr[\text{vol}(\Gamma_k(x)) \geq \text{vol}(\Gamma_{k-1}(x))^{1/(\beta-2)} \cdot w] \leq O(1/w^{\beta-2}).
\]
On the other hand, we have
\[
\text{vol}(\Gamma_{k-1}(x))^{1/(\beta-2)} \cdot w \leq b_{k-1}^{1/(\beta-2)} \cdot w
\]
\[
= (c_{k-1} \cdot w^{1/(\beta-2)2^{(k-1)}(3-\beta)})^{1/(\beta-2)} \cdot w
\]
\[
= c_k \cdot w^{1/(\beta-2)2^{(k-1)(3-\beta)+1}}
\]
\[
= c_k \cdot w^{1/(\beta-2)2^k(3-\beta)(\beta-2+3(3-\beta)(\beta-2)^{2^k})}
\]
\[
\leq b_k.
\]
Thus, we have $\Pr[\text{vol}(\Gamma_k(x)) > b_k] \leq O(1/w^{\beta-2})$.

**Lower Bounding $\text{vol}(\Gamma_k(x))$**  For any vertex $y$, if $p_y \cdot p_z \geq \text{vol}(V)$ for some $z \in \Gamma_{k-1}(x)$, then $y$ must be connected to $\Gamma_{k-1}(x)$, otherwise the probability that $y$ does not connect to $\Gamma_{k-1}(x)$ is at most
\[
\prod_{z \in \Gamma_{k-1}(x)} \left(1 - \frac{p_y \cdot p_z}{\text{vol}(V)}\right) \leq e^{-\sum_{z \in \Gamma_{k-1}(x)} \frac{p_y \cdot p_z}{\text{vol}(V)}}
\]
\[
= e^{-p_y \cdot \text{vol}(\Gamma_{k-1}(x)) / \text{vol}(V)}.
\]
That is, in either case, if $y \notin N_{k-1}(x)$, then $\Pr[y \notin \Gamma_k(x)] \leq e^{-p_y \cdot \text{vol}(\Gamma_{k-1}(x)) / \text{vol}(V)}$. When $n$ is large enough, we have $b_{k-1} < a_k$, and $N_{k-1}(x)$ does not contain high weight vertex by the premises of
the lemma. Therefore, the probability that $\Gamma_k(x)$ contains no high weight vertex is low:

$$\Pr[\forall y, \text{ s.t. } p_y \geq a_k, y \notin \Gamma_k(x)] \leq \prod_{y: p_y \geq a_k} e^{-p_y \cdot \frac{\text{vol}(\Gamma_{k-1}(x))}{\text{vol}(V)}}$$

$$= e^{-\sum_{y: p_y \geq a_k} p_y \cdot \frac{\text{vol}(\Gamma_{k-1}(x))}{\text{vol}(V)}}$$

$$\leq e^{-\Omega(a_k^{2-\beta} - a_{k-1})}$$

$$= e^{-\Omega(\sum_{y: p_y \geq a_k} p_y \cdot \text{vol}(\Gamma_{k-1}(x)) \text{vol}(V))}$$

$$= e^{-\Omega((\beta - 2)^{2k-1} - 1/(\beta - 2)^{2k-2}(3-\beta))}$$

$$\leq e^{-w} = 1/\log n.$$  

In particular, it implies that the probability that $\text{vol}(\Gamma_k(x)) < a_k$ is at most $1/\log n$.

Finally, by union bound, the probability that $\text{vol}(\Gamma_k(x)) \in [a_k, b_k]$ is at least $1 - O(1/\log n)$ as the lemma states.

**Lower Bounding dist(x, y)** By the above argument, we have

$$\Pr[\text{dist}(x, y) \leq 2d + 3]$$

$$\leq \Pr[\exists 0 \leq i, j \leq d + 1, \exists u \in \Gamma_i(x), v \in \Gamma_j(y), u \sim v]$$

$$\leq \sum_{i, j=0}^{d+1} \Pr[\exists u \in \Gamma_i(x), v \in \Gamma_j(y), u \sim v]$$

$$= \sum_{i, j=0}^{d+1} \mathbb{E}_{\Gamma_i(x), \Gamma_j(y)} \left[ \Pr[\exists u \in \Gamma_i(x), v \in \Gamma_j(y), u \sim v | \Gamma_i(x), \Gamma_j(y)] \right]$$

$$\leq \sum_{i, j=0}^{d+1} \left( \Pr[\exists u \in \Gamma_i(x), v \in \Gamma_j(y), u \sim v | \Gamma_i(x) \in [a_i, b_i], \Gamma_j(y) \in [a_j, b_j]] + \Pr[\Gamma_i(x) \notin [a_i, b_i]] + \Pr[\Gamma_j(y) \notin [a_j, b_j]] \right)$$

$$\leq \sum_{i, j=0}^{d+1} \left( b_i \cdot b_j / \text{vol}(V) + O(1/\log n) \right)$$

$$\leq O\left(b_{d+1}^2 / n + d^2 / \log n\right)$$

$$= O\left(n^{-\epsilon/(\beta - 2)} + \log^2 \log \log n / \log^2 \log n\right) = o(1).$$

This proves the lemma.