Cauchy Horizon Endpoints and Differentiability *

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Abstract

Cauchy horizons are shown to be differentiable at endpoints where only a single null generator leaves the horizon. A Cauchy horizon fails to have any null generator endpoints on a given open subset iff it is differentiable on the open subset and also iff the horizon is (at least) of class $C^1$ on the open subset. Given the null convergence condition, a compact horizon which is of class $C^2$ almost everywhere has no endpoints and is (at least) of class $C^1$ at all points.

I. Introduction

Cauchy horizons and black hole event horizons have been extensively studied and used in relativity [2 – 6, 10 – 14, 16, 17]. For general spacetimes, horizons may fail to be stable under small metric perturbations, however, some sufficiency conditions for various stability questions have been obtained [1], [7]. In the present paper, we will consider some differentiability questions for Cauchy horizons.

Let $(M, g)$ be a spacetime with a partial Cauchy surface $S$. The future Cauchy development $D^+(S)$ is the set of points of the spacetime where, in theory, one may calculate everything in terms of initial data on $S$. The future Cauchy horizon $H^+(S)$ is the future boundary of $S$. We state our results in terms of the future horizon $H^+(S)$, but similar results hold for any past Cauchy horizon $H^-(S)$.

Cauchy horizons are achronal (i.e., no two points on the horizon may be joined by a timelike curve) and this implies that Cauchy horizons (locally) satisfy a Lipschitz condition. This, in turn, implies that Cauchy horizons are differentiable almost everywhere. Because they are differentiable except for a set of (3 dimensional)

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measure zero, it seems that they have often been assumed to be smooth except for a set which may be more or less neglected. However, one must remember in the above that (1) differentiable only refers to being differentiable at a single point and (2) sets of measure zero may be quite widely distributed. In fact, Chruściel and Galloway have constructed examples of both a Cauchy horizon and a black hole event horizon where the horizons fail to be differentiable on dense subsets. Thus, they have constructed examples of horizons which contain no open subsets on which they are differentiable at all points. They point out that these examples raise definite questions concerning some major arguments that have been given in the past where smoothness assumptions have been implicitly assumed, compare \[6\]. In the light of these new examples, it is clear that there is a real need for a deeper understanding of the differentiability properties of horizons.

Each point $p$ of a Cauchy horizon $H^+(S)$ lies on at least one null generator. Since partial Cauchy surfaces are edgeless, a null generator continues to lie on the horizon when it is extended in the past. However, null generators may or may not stay on the horizon when they are extended in the future direction. If a null generator leaves the horizon, then there is a last point where it remains on the horizon. This last point is said to be an endpoint of the horizon.

Endpoints where two or more null generators leave the horizon are points where the horizon must fail to be differentiable. In addition, Chruściel and Galloway have shown that Cauchy horizons are differentiable at points which are not endpoints. Furthermore, Chruściel and Galloway pointed out the need to resolve the differentiability issue for endpoints where only one generator leaves the horizon. In the present paper, we show that Cauchy horizons are differentiable at these endpoints. This completes the classification of (pointwise) differentiability for Cauchy horizons in terms of null generators and their endpoints. However, this result raises the following question. Is it always true that the entire set of endpoints of a Cauchy horizon will have measure zero? We conjecture that the answer should be affirmative based on known examples. Also, some support for our conjecture is given by our proof that the set of endpoints with only one generator is in the closure of the set of endpoints with more than one generator.

Restricting our attention to an open subset $W$ of the Cauchy horizon $H^+(S)$ and assuming that the horizon has no endpoints on the open set $W$, we find the horizon must be differentiable at each point of $W$ and, in fact, that the horizon must be at least of class $C^1$ on $W$. Conversely, we find differentiability on an open set $W$ implies there are no endpoints on $W$. In general, (pointwise) differentiability on an open set yields class $C^1$, but not necessarily class $C^2$. We give an example to demonstrate that one may not conclude that either differentiability or lack of endpoints on an open set $W$ imply class $C^2$ on $W$. We also include a very simple example of a Cauchy horizon with an endpoint where only one null generator leaves
Hawking \cite{1} has argued that under certain conditions there should be no compactly generated Cauchy horizons. These are horizons with a compact set $K$ such that all null generators eventually enter and remain in this set $K$ when extended in the past direction. Hawking was interested in establishing arguments against the possibility of having closed timelike curves in physical spacetimes and, as he mentions, most of his arguments apply to the special case of compact horizons. A key part of Hawking’s paper involves flowing the horizon back along null generators to get a contradiction. We use his notation and technique to find sufficient conditions for a compact horizon to have no endpoints and to be at least of class $C^1$ at all points. More precisely, we show that if one has the null convergence condition and a compact horizon which is of class $C^2$ on an open set $G$ with complement of measure zero, then the horizon has no endpoints and is (at least) of class $C^1$ at all points.

2. Preliminaries

Let $(M, g)$ be a spacetime. Although our results hold for n-dimensional spacetimes, we will only give the proofs in the four dimensional case since similar proofs hold in other dimensions. Thus, we will take $M$ to be a smooth, connected, four dimensional, Hausdorff manifold with a Lorentzian metric, a countable basis, and a time orientation. The Lorentzian metric $g$ will have signature $(-,+,+,+)$. A partial Cauchy surface $S$ will be a connected, acausal, edgeless three dimensional submanifold of $(M, g)$, compare \cite{12, 11, 3}. The future Cauchy development $D^+(S)$ consists of all points $p \in M$ such that each past endless and past directed causal curve from $p$ intersects the set $S$. The future Cauchy horizon is $H^+(S) = (D^+(S)) - I^- (D^+(S))$.

Let $p$ be a point of the Cauchy horizon. It is well known, that there is at least one null generator of $H^+(S)$ containing $p$. Each null generator is at least part of a null geodesic of $M$. When a null generator of $H^+(S)$ is extended into the past, it continues to lie on the horizon, compare \cite[p. 203]{12}. However, if a null generator is extended into the future it may have a last point on the horizon which then said to be an endpoint the horizon. We will define the multiplicity of a point $p$ in $H^+(S)$ to be the number of null generators containing $p$. Points of the horizon which are not endpoints must have multiplicity one. The multiplicity of an endpoint may be any positive integer or infinite. We will call the set of endpoints of multiplicity two or higher the crease set, compare \cite{6}.

Consider any fixed point $p$ of the Cauchy horizon $H^+(S)$ and let $x^0, x^1, x^2, x^3$ be local coordinates defined on an open set about $p = (p^0, p^1, p^2, p^3)$. Let $H^+(S)$ be given near $p$ by an equation of the form

$$x^0 = f_H(x^1, x^2, x^3)$$
The horizon $H^+(S)$ is differentiable at the point $p$ iff the function $f_H$ is differentiable at the point $(p^1, p^2, p^3)$ using the advanced calculus definition of differentiability [18, p. 212]. In particular, if $p = (0, 0, 0, 0)$ corresponds to the origin in the given local coordinates and if

$$\Delta x = (x^1, x^2, x^3)$$

represents a small displacement from $p$ in the $x^0 = 0$ plane, then $H^+(S)$ is differentiable at $p$ iff one has

$$f_H(\Delta x) = f_H(0) + \sum a_i x^i + R_H(\Delta x) = 0 + \sum a_i x^i + R_H(\Delta x)$$

where the ratio $R_H(\Delta x)/|\Delta x|$ converges to zero as $|\Delta x|$ goes to zero. Here we use

$$|\Delta x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}.$$

Note that even though $|\Delta x|$ represents a coordinate dependent length, the Cauchy surface $H^+(S)$ will be differentiable or not differentiable at $p$ independent of the particular choice of local coordinates. Of course, when $H^+(S)$ is differentiable at $p$, one has $a_i = \partial f_H/\partial x^i$ evaluated at the origin for $i = 1, 2, 3$. The surface $H^+(S)$ is of class $C^r$ on an open neighborhood of $p$ iff $f_H$ is of class $C^r$ on an open neighborhood of the origin.

If $H^+(S)$ is differentiable at the point $p$, then there is a well defined 3-dimensional linear subspace $N_0$ in the tangent space $T_p(M)$ such that $N_0$ is tangent to the 3-dimensional surface $H^+(S)$ at $p$. In the above notation a basis for $N_0$ is given by $\{a_i \partial/\partial x^0 + \partial/\partial x^i \mid i = 1, 2, 3\}$. It is clear that if local coordinates $x^0, x^1, x^2, x^3$ have been chosen as above, then the partials $a_i = \partial f_H/\partial x^i$ at $p$ are determined by $N_0$ assuming, of course, that $H^+(S)$ is differentiable at $p$ and is given by $x^0 = f_H(\Delta x)$ near $p$. Notice that the given tangent plane $N_0$ cannot be spacelike since there is at least one null generator containing $p$. Also, the tangent plane $N_0$ cannot be timelike since otherwise $p$ would be in chronological past of some points of $H^+(S)$. Thus, $N_0$ is null (i.e., of signature $(0, +, +)$) and it follows that there is a one dimensional linear subspace $L_0$ in $N_0$ such that $L_0$ is null. In fact, $N_0 = L_0^\perp$ is the orthogonal space to $L_0$ and is thus uniquely determined by $L_0$. Given a tangent vector $X$ at $p$ with $X \in N_0$, then $X$ belongs to $L_0$ iff $g(X, Y) = 0$ for all $Y \in N_0$.

At a point $p$ where $H^+(S)$ is differentiable there can only be one null generator though $p$ since two different null generators though $p$ would yield a two dimensional
timelike plane lying in the (3-dimensional) tangent plane to the horizon and this would imply the existence of a timelike plane tangent to $H^+(S)$ at $p$ in contradiction to the above. Thus at points with two or more null generators of $H^+(S)$, the horizon $H^+(S)$ cannot be differentiable, compare [6]. In fact, if $p$ is a point of the horizon having two or more null generators, then all null generators through $p$ must leave the horizon at $p$ when traversed in the future direction. This follows since if $\gamma_1(t)$ and $\gamma_2(t)$ are future directed null generators with $\gamma_1(0) = \gamma_2(0) = p$ and if $\gamma_1(t)$ remains on $H^+(S)$ for some $\gamma_1(t_1)$ with $t_1 > 0$, one has that all points of $\gamma_2(t)$ with $t < 0$ are in the chronological past of $\gamma_1(t_1)$ since one can traverse $\gamma_1(t)$ backward to $p$, make a corner at $p$ to head in the past direction of the null geodesic $\gamma_2(t)$ and then traverse this null geodesic backward. This would imply the points $\gamma_2(t)$ for $t < 0$ are in the set $D^+(S)$ and not on the horizon, in contradiction.

We now give an example of an endpoint $p$ of multiplicity one.

**Example 2.1.** Let $M_0 = L^3$ be three dimensional Minkowski spacetime with coordinates $(t, x, y)$ and $g = -dt^2 + dx^2 + dy^2$. Remove the parabola $y = x^2$ from the $t = 0$ plane to obtain the spacetime $M$. Let the partial Cauchy surface $S$ be the portion of the x-y plane which is interior to the parabola $y = x^2$ (i.e., $S = \{(0, x, y) \mid y > x^2\}$). The curvature $k$ of the curve $y = x^2$ has a maximum value of 2 at $x = 0$ and is always less than 2 for points on the parabola other than the origin. The osculating circle at the origin to the curve $y = x^2$ is a circle in the x-y plane of radius $1/k = 1/2$ and center at $y = 1/2$ on the y-axis. The chronological past of the point $(1/2, 0, 1/2)$ intersected with the x-y plane is the interior of this osculating circle and lies in the set $S$. It follows that the null geodesic $x = 0, y = t$ contains a null generator of $H^+(S)$. In particular, the curve $c(t) = (t, 0, t)$ for $0 < t < \infty$ is a null geodesic which lies on the horizon for $0 < t \leq 1/2$. Notice that if one intersects the x-y plane with the chronological past of $(t, 0, t)$ for $t > 1/2$, then one obtains some points outside of the above partial Cauchy surface $S$. Thus, the null generator $c(t) = (t, 0, t)$ must leave the horizon at $t = 1/2$ and one finds that $(1/2, 0, 1/2)$ is an endpoint of the horizon. However, the endpoint $p = (1/2, 0, 1/2)$ has no other null generators since all other null geodesics through this point intersect the set $S$. Thus, $p$ is an endpoint of multiplicity one as desired. In this example one may directly check that $H^+(S)$ is differentiable at the point $p$. The crease set is a curve which lies above the points on the y-axis for which $1/2 < y < \infty$.

In the next example we obtain a horizon which has an open set $W$ where the horizon is of class $C^1$, but where there are some points in this open set $W$ where the horizon fails to be of class $C^2$. In particular, there is a certain null generator and the horizon fails to be of class $C^2$ along this null generator.
Example 2.2. Let $M_0 = L^3$. The spacetime $M$ will be obtained by removing two half lines and one fourth of a circle. From $M_0$ remove the set in the $t = 0$ plane given by $y = -1$ and $0 \leq x$. Then remove the part of the circle $x^2 + y^2 = 1$ for which $x < 0$ and $y < 0$. Finally, remove the half line in the $t = 0$ plane given by $x = -1$ and $0 \leq y$. The partial Cauchy surface $S$ will be the open set in the x-y plane corresponding to the interior of the convex hull of the deleted set. The null generators corresponding to the quarter circle focus at the point $(1,0,0)$. The null generators corresponding to points on the line $y = -1$ for sufficiently small values of $t$ lie on lines of the form $c(t) = (t, x, t - 1)$ where $0 \leq x$. It follows that for values of $x$ and $y$ near $x = 0$ and $y = -1/2$ the height $t$ of the Cauchy horizon is given by $t = f_H(x,y)$ where $f_H(x,y) = 1 + y$ for $0 \leq x$ and also $f_H(x,y) = 1 - (x^2 + y^2)^{1/2}$ for $x < 0$. It is easily checked that $f_H$ is of class $C^1$ near $(0,-1/2)$, but that $f_H$ fails to be of class $C^2$ along the y-axis near $(0,-1/2)$. It follows that there is an open set $W$ in $H^+(S)$ about the point $(1/2,0,-1/2)$ where the horizon is of class $C^1$, but not of class $C^2$. In this example there are no endpoints of multiplicity one.

Both Examples 2.1 and 2.2 may be changed to four dimensional examples using a cartesian product with a positive definite $R^1$ corresponding to the z-axis. In particular, to modify Example 2.1, one may take $M_0 = L^4$, delete the two dimensional set $\{(0,x,y,z) \mid y = x^2\}$, and use the three dimensional partial Cauchy surface $S = \{(0,x,y,z) \mid y > x^2\}$. This generates a horizon $H^+(S)$ with an endpoint $p = (1/2,0,1/2,0)$ of multiplicity one. To modify Example 2.2, one may take $M_0 = L^4$ and delete the set formed by the cartesian product of the originally deleted set with the z-axis. Of course, in this four dimensional example, one takes the new partial Cauchy surface to be the cartesian product of the original Cauchy surface with the z-axis.

3. Differentiability of Cauchy Horizons

One may always represent tangent null directions using a normalization based on an auxiliary positive definite metric $g_{pos}$. Given a fixed point $p$ on a null geodesic $\gamma$, chose the uniquely defined tangent vector $V$ to $\gamma$ at $p$ which is future pointing and satisfies $g_{pos}(V,V) = 1$. With this normalization, it is clear that the null directions at $p$ form a compact set homeomorphic to $S^2$. Of course, we are identifying the null direction given by a null vector $V$ as the same as that given by $\alpha V$ for nonzero
values of $\alpha$. Thus, in particular, the null direction represented by $V$ is the same as the null direction represented by $-V$. Note that if one is given any compact subset $K$ of $M$, the set of null directions attached at points of $K$ forms a compact set.

Let $\gamma_n$ be a sequence of null generators of $H^+(S)$ and let $p_n \in \gamma_n$ for each $n$. Assume the sequence $\{p_n\}$ converges to $p$ and let $\gamma$ be a null generator at $p$. We will say the sequence $\gamma_n$ converges to $\gamma$ if there is a sequence of null vectors $\{V_n\}$ converging to the null vector $V$ such that each $V_n$ is tangent to the corresponding generator $\gamma_n$ at $p_n$ and $V$ is tangent to $\gamma$ at $p$. Notice that in this definition we do not assume that the points $\{p_n\}$ have unique null generators and we do not assume that the point $p$ has a unique null generator. In particular, we allow for the possibility that some (or all) of $\{p_n\}$ and $p$ may be endpoints.

We now show that if $p$ is a point of multiplicity one, then any point on the horizon sufficiently close to $p$ must have all of its null generators close to the unique null generator containing $p$.

**Lemma 3.1.** Let $p$ be a point of multiplicity one on the horizon $H^+(S)$. If $\{p_n\}$ is a sequence of points on the horizon converging to $p$ and if for each $n$ a null generator $\gamma_n$ containing the point $p_n$ has been chosen, then the sequence $\gamma_n$ converges to the null generator $\gamma$ containing $p$.

**Proof.** Assume by way of contradiction that the sequence $\{p_n\}$ converges to $p$, but that the null directions defined by the generators $\gamma_n$ do not converge to the null direction defined by $\gamma$. Using compactness, it follows that there must be subsequence $\{j\}$ of the sequence $\{n\}$ with the $\gamma_j$ directions converging to a null direction at $p$ different from that defined by $\gamma$. This yields a second null generator at $p$, in contradiction to the assumption that $p$ was of multiplicity one. $\square$

Clearly, Lemma 3.1 implies that if $W$ is an open subset of the horizon and if each point of $W$ has multiplicity one, then the null generators move in a continuous fashion on $W$. More precisely, we have the following result.

**Lemma 3.2.** Let $W$ be an open subset of the future Cauchy horizon $H^+(S)$. If each point of $W$ has multiplicity one, then there is a (nonvanishing) null vector field $V$ tangent to null generators and defined on $W$ such that $V$ is continuous. Furthermore, the vector field $V$ may be taken to be future pointing.

We are now able to show that for open subsets of the horizon, differentiability implies (at least) class $C^1$. Example 2.2 shows that differentiability on an open set
does not, in general, imply class $C^2$ and hence the following is the best possible result one may obtain without additional assumptions. Recall that if a surface is of class $C^r$ for $r > 1$, then it is also of class $C^1$.

**Proposition 3.3.** If $W$ is an open subset of $H^+(S)$ and if $H^+(S)$ is differentiable at all points of $W$, then $H^+(S)$ is of class $C^r$ on $W$ for some $r \geq 1$.

**Proof.** A point where the horizon is differentiable must be a point of multiplicity one. Hence Lemma 3.2 yields the existence of a null vector field $V$ on $W$ which is continuous, tangent to $H^+(S)$ and serves to define the generator directions on $W$. Let $p \in W$ and, as in Section 2, introduce local coordinates $x^0, x^1, x^2, x^3$ near $p$ with $\partial / \partial x^0$ future timelike. Let $H^+(S)$ be given locally by $x^0 = f_H(\Delta x)$. The partial derivatives of $f_H$ exist near $p$ since $H^+(S)$ is differentiable near $p$. Notice that the null vector field $V$ determines the orthogonal space $V^\perp$ which is the null tangent plane to $H^+(S)$ at each point near $p$. Furthermore, the tangent plane $V^\perp$ determines the first order partial derivatives of $f_H$. The continuity of $V$ yields the continuity of $V^\perp$ and hence the continuity of the first order partial derivatives of $f_H$. Thus, $f_H$ is at least of class $C^1$ which is equivalent to $H^+(S)$ being at least of class $C^1$, as desired. $\Box$

Fix a point $p$ on $H^+(S)$ of multiplicity one. Choose normal coordinates $x^0, x^1, x^2, x^3$ with $p$ corresponding to the origin, with $\partial / \partial x^0$ future timelike at $p$, and with the future direction of the unique null generator at $p$ corresponding to the direction of $\partial / \partial x^0 + \partial / \partial x^1$. Assume also that the natural basis is orthonormal at $p$. As before, let $H^+(S)$ be given near $p$ by $x^0 = f_H(\Delta x)$. If $H^+(S)$ is differentiable at $p$ one has $\partial f_H / \partial x^1 = 1$ and $\partial f_H / \partial x^2 = \partial f_H / \partial x^3 = 0$ at $p$. Assuming that $f_H$ is of class $C^r$ for $r \geq 1$ near $p$, define new coordinates $z^0, z^1, z^2, z^3$ by $z^0 = x^0 - f_H(\Delta x)$, and $z^i = x^i$ for $i = 1, 2, 3$. Locally $H^+(S)$ corresponds to the set $z^0 = 0$. Note that $z^0, z^1, z^2, z^3$ are local coordinates of class $C^r$ in terms of the original $x$ coordinates.

**Proposition 3.4.** If a Cauchy horizon $H^+(S)$ is differentiable on an open subset $W$, then the horizon has no endpoints on $W$.

**Proof.** The horizon must be (at least) of class $C^1$ on $W$ by Proposition 3.3. Fixing $p \in W$, we may use the above local coordinates $z^0, z^1, z^2, z^3$ which are at least of class $C^1$ and the horizon is given near $p$ by $z^0 = 0$. From Lemma 3.2 it follows that there is a continuous future directed null vector field $V$ tangent to $H^+(S)$ near $p$. Then $V = \sum y^i(z^1, z^2, z^3) \partial / \partial z^i$ where $y^i = y^i(z^1, z^2, z^3)$ are continuous functions for $i = 1, 2, 3$. Consider the system

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\[ dz^i / dt = y^i(z^1, z^2, z^3) \]

for \( i = 1, 2, 3 \). This is a system which we may regard as being defined on the coordinate plane \( z^0 = 0 \) (i.e., on the horizon) and solving the system corresponds to finding integral curves of the vector field \( V \). However, the above system does not necessarily satisfy a Lipschitz condition and thus, in principle, might fail to have unique solutions. Nevertheless, because of the continuity of the functions \( y^i \), one may apply the Cauchy-Peano Existence Theorem [8, p. 6] to obtain a class \( C^1 \) solution \( c(t) \) with \( c(0) = p \). Notice that \( c(t) \) is a null curve always tangent to \( H^+(S) \) because \( c'(t) \) is always equal to the vector field \( V \) at the corresponding point \( c(t) \). Moving along the curve \( c \) a little to the past, one obtains a point \( r = c(t_1) \in H^+(S) \) with \( t_1 < 0 \). Similarly, moving along \( c \) a little to the future one obtains \( q = c(t_2) \in H^+(S) \) with \( t_2 > 0 \). Traversing the curve \( c \) backward from \( q \) to \( r \) one finds it must be a null geodesic lying on the horizon since if it were not, then \( r \) would be in the chronological past of \( q \) and this would yield a contradiction to the achronality of \( H^+(S) \). Since there is a unique null direction tangent to \( H^+(S) \) at \( p \), the curve \( c \) (at least) restricted to the domain \([t_1, t_2]\) must lie on the generator containing \( p \) and it follows that \( p \) is not an endpoint, as desired. \( \square \)

We note in passing that the proof of Proposition 3.4 shows that, in fact, the system \( dz^i / dt = y^i(z^1, z^2, z^3) \) has unique solutions given an initial point \( p \in W \) and that each solution \( c(t) \) lies on (part of) a null generator.

Fix a point \( p \) of multiplicity one on \( H^+(S) \) with null direction \( L_0 \) tangent to the unique null generator at \( p \). Let \( Y_0, Y_1, Y_2, Y_3 \) be an orthonormal basis at \( p \) with \( Y_0 \) future pointing timelike and with \( Y_0 + Y_1 \) future pointing null in the direction of \( L_0 \). Then \( Y_2 \) and \( Y_3 \) lie in the orthogonal space to \( L_0 \). Let \( X_0, X_1, X_2, X_3 \) be a new basis of \( T_pM \) given by \( X_0 = Y_0, X_1 = Y_0 + Y_1, X_2 = Y_2, \) and \( X_3 = Y_3 \). Take normal coordinates \( x^0, x^1, x^2, x^3 \) centered at \( p \) determined by the basis \( X_0, X_1, X_2, X_3 \). Then \( \partial / \partial x^0 \) is timelike future pointing, \( \partial / \partial x^1 \) future pointing in the null direction \( L_0 \) and both \( \partial / \partial x^2 \) and \( \partial / \partial x^3 \) are in the orthogonal space to \( L_0 \). Of course, these normal coordinates fail to be orthonormal at \( p \) since, in particular, \( \partial / \partial x^1 \) is null. Notice that the metric tensor \( g \) has components (in the \( x \) coordinates) which at \( p \) have all zero values except for \( g_{00}(p) = g_{01}(p) = g_{10}(p) = -1 \) and \( g_{22}(p) = g_{33}(p) = 1 \). The \( x^1 \) axis is a geodesic which corresponds to a null generator of \( H^+(S) \) and (at least) the origin and negative \( x^1 \) axis lie on the horizon. The point \( p \) is an endpoint iff points on the positive \( x^1 \) axis do not lie on the horizon. In the following we allow \( p \) to be either an endpoint or a nonendpoint. Fix a point \( r = (0, t_1, 0, 0) \) on the negative \( x^1 \) axis and lying in a convex normal neighborhood of \( p \). Thus, \( t_1 < 0 \). Since we have used normal coordinates, we have \( p = exp_r(V_0) \) where \( V_0 = |t_1| \partial / \partial x^1 \).
and $V_0$ is a null vector attached at $r$. If one considers the exponential map at $r$ restricted to all null vectors at $r$ in some small neighborhood of $V_0$, then one obtains a smooth null surface which passes through $p = (0, 0, 0, 0)$. Since $\partial/\partial x^1$ at the origin is a null vector tangent to this null surface, it follows that the tangent plane to this null surface at the point $p$ is the orthogonal plane to $\partial/\partial x^1$ which is also the tangent plane to the coordinate plane $x^0 = 0$ at $p = (0, 0, 0, 0)$. Let $N(V_0)$ be a sufficiently small neighborhood of $V_0$ in the tangent bundle $TM$ and let $U(r)$ be a (small) 3 dimensional neighborhood of $r$ in the coordinate plane $x^1 = t_1$. Then points of $U(r)$ are of the form $u = (u^0, t_1, u^2, u^3)$ where the values of of $u^0, u^2$ and $u^3$ are all close to zero. For each fixed $u \in U(r)$ one generates a smooth null surface $S(u^0, u^2, u^3)$ defined near the origin by using the exponential map at $u$ and restricting $exp_u$ to null vectors in the set $N(V_0)$ attached at $u$. An element $V \in N(V_0)$ attached at $u \in U(r)$ may be represented in local coordinates as $V = V_0 \partial/\partial x^0 + \sum V^i \partial/\partial x^i$. Thus, using $g(V, V) = 0$ to determine the $V^0$ component in terms of the $V^1, V^2, V^3$ components, one may parametrize our six dimensional domain space by $u^0, u^2, u^3, V^1, V^2, V^3$. Here the values of $u^0, u^2, u^3, V^2, V^3$ are close to zero and the value of $V^1$ is close to $|t_1|$. Note that tangent vectors to this six dimensional space may be expressed in terms of a basis of the form $\partial/\partial u^0, \partial/\partial u^2, \partial/\partial u^3, \partial/\partial V^1, \partial/\partial V^2, \partial/\partial V^3$. Consider the map

$$E(u, V) = exp_u(V)$$

taking elements of the domain space to points of $M$ near $p$. This map may be written as $E(u, V) = (E^0, E^1, E^2, E^3)$ where each component $E^\mu(u^0, u^2, u^3, V^1, V^2, V^3)$ is a real valued function of six variables. The reader will note a slight abuse of notation in the above. For example, we will use $u$ to denote both the point $(u^0, t_1, u^2, u^3)$ in $M$ and also the corresponding coordinates $u^0, u^2, u^3$ in our domain space. A similar comment holds for our use of $V$. Notice that if $u$ is chosen to be the point $r$, then the map $E(r, -)$ takes the null vectors attached at $r$ to the null surface $S(0, 0, 0)$ which we noted above is tangent to the coordinate plane $x^0 = 0$ at $p$. In our domain space the vector $V_0$ in $T_rM$ corresponds to $(0, 0, 0, |t_1|, 0, 0)$. Notice that the derivative of the exponential map for $u = r$ (i.e., $(exp_r)_*$) is nondegenerate at $V_0$ since $r$ was chosen close to $p$. Furthermore, $(exp_r)_*$ takes the null hyperplane at $V_0$ tangent to the null vectors at $r$ to the null plane tangent to $x^0 = 0$ at $p = (0, 0, 0, 0)$. Thus, for fixed $u = r$ and $V = V_0$, $E_*$ takes the three dimensional space with basis $\partial/\partial V^1, \partial/\partial V^2, \partial/\partial V^3$ corresponding to certain vectors tangent to our six dimensional domain space and attached to this domain space at $V_0 = (0, 0, 0, |t_1|, 0, 0)$ in a nonsingular fashion to the three dimensional space with basis $\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3$ of vectors tangent to $M$ and attached at $p$. Thus, using $x^i = E^i(u^0, u^2, u^3, V^1, V^2, V^3)$ one finds that the three by three matrix $[\partial x^i/\partial V^j]$ is nonsingular when evaluated at the point $(0, 0, 0, |t_1|, 0, 0)$ of our domain space.
the Implicit Function Theorem \[13\], we may thus solve for the \(V^1, V^2, V^3\) variables in terms of the \(u^0, u^2, u^3, x^1, x^2, x^3\) variables. Hence, one obtains three \(C^\infty\) functions \(V^i = V^i(u^0, u^2, u^3, x^1, x^2, x^3) = V^i(u, \Delta x)\). We will let \(F\) be the \(x^0\) component of \(E\) using the \(u^0, u^2, u^3, x^1, x^2, x^3\) variables. In other words, we define \(F\) to be the real valued function

\[
F(u^0, u^2, u^3, x^1, x^2, x^3) = E^0(u, V^1(u, \Delta x), V^2(u, \Delta x), V^3(u, \Delta x)).
\]

Then, for fixed \(u\), the surface \(S(u^0, u^2, u^3)\) is given near the origin by \(x^0 = F_u(x^1, x^2, x^3)\) where \(F_u(\Delta x) = F(u^0, u^2, u^3, x^1, x^2, x^3)\) is a smooth function of six variables. Still holding \(u \in U(r)\) fixed, select a point \(q = (q^0, q^1, q^2, q^3)\) near the origin on the surface \(S(u^0, u^2, u^3)\). Then one has \(q^0 = F_u(q^1, q^2, q^3) = F_u(\Delta q)\). Now expand \(F_u\) about this point (holding the \(u\) coordinates fixed) to obtain

\[
F_u(x^1, x^2, x^3) = q^0 + \sum (\partial F_u/\partial x^i)(x^i - q^i) + R(u, q, \Delta(x - q))
\]

where \(\Delta(x - q) = (x^1 - q^1, x^2 - q^2, x^3 - q^3)\) and the partials of \(F_u\) are evaluated at \(\Delta q\).

Using the smoothness of \(F\), it follows that the remainder term \(R(u, q, \Delta(x - q))\) may be bounded on some compact set in the domain space of six variables using the size of the second partial derivatives of \(F\) and the magnitude of \(|\Delta(x - q)|^2\). Of course the bound will depend on the mixed second partials of \(F\) involving the \(u^0, u^2, u^3\) variables as well as the \(x^1, x^2, x^3\) variables, compare \([9, p. 252]\). In particular, one may obtain an inequality of the form

\[
|R(u, q, \Delta(x - q))| < M|\Delta(x - q)|^2
\]

where \(M\) is a constant which holds for \(u\) near \(r\), \(q\) near \(p\), and all sufficiently small \(|\Delta(x - q)|\). We will use this bound on the size of the remainder term in the proof of Theorem 3.5 below. We remark in passing that points of any sufficiently small neighborhood \(W(p)\) of \(p = (0, 0, 0, 0)\) will lie on many of the null surfaces \(S(u^0, u^2, u^3)\). In fact, the null cone from each point of a sufficiently small \(W(p)\) will intersect the coordinate plane \(x^1 = t_1\) in a two dimensional surface and thus through a point of \(W(p)\) there will be a two parameter family of null surfaces in the collection \(S(u^0, u^1, u^2)\).

The next result answers a question raised by Chruściel and Galloway \[1\]. They proved that Cauchy horizons fail to be differentiable at endpoints with more than one null generator and that Cauchy horizons are differentiable at points which are not endpoints \[4\]. They mentioned that these results left open the question of the differentiability of Cauchy horizons at endpoints where there is only null generator (i.e., endpoints of multiplicity one in our terminology). As Chruściel and Galloway noted, it is of interest to resolve this remaining differentiability question. In Theorem
3.5, we obtain a positive answer to this question. We show that a Cauchy horizon is always differentiable at points of multiplicity one.

**Theorem 3.5.** A Cauchy horizon is differentiable at all points of multiplicity one. In particular, a Cauchy horizon is differentiable at an endpoint where only one null generator leaves the horizon.

**Proof.** Let $p$ be a point of multiplicity one. We will use the above described coordinates $x^0, x^1, x^2, x^3$, neighborhoods $U(r), N(V_0), W(p)$, null surfaces $S(u^0, u^2, u^3)$, functions $F_k(x^1, x^2, x^3)$ and Inequality (3.1).

Assume that the Cauchy horizon $H^+(S)$ is given near $p$ by $x^0 = f_H(\Delta x)$. We will show that $f_H$ must be differentiable at the origin and have all of its first order partials equal to zero at the origin. This will prove that $H^+(S)$ is differentiable at the origin and that it has its tangent plane at the origin tangent to the coordinate plane $x^0 = 0$. To show the desired properties of $f_H$ are true, it is sufficient to show that $f_H(\Delta x)/|\Delta x|$ converges to zero as $|\Delta x|$ converges to zero. Hence, by way of contradiction, we assume that there is a sequence of points $\{q_k = (q^0_k, q^1_k, q^2_k, q^3_k)\}$ on $H^+(S)$ and converging to $p$ with

$$|f_H(\Delta q_k)|/|\Delta q_k| > c > 0$$

for all $k$. Here

$$\Delta q_k = (q^1_k, q^2_k, q^3_k) \quad \text{and} \quad |\Delta q_k| = \sqrt{(q^1_k)^2 + (q^2_k)^2 + (q^3_k)^2}.$$ 

The surface $S(0,0,0)$ is tangent at the origin to the coordinate plane $x^0 = 0$ and represents (part of) the null cone from the point $r = (0, \alpha, 0)$. Note that points in the chronological future of $r$ must lie above the Cauchy horizon and that $\partial/\partial x^0$ is future pointing timelike at the origin. It follows that points of the horizon lie on or below the surface $S(0,0,0)$ and hence

$$q^0_k = f_H(\Delta q_k) \leq F(0,0,0,q^1_k,q^2_k,q^3_k)$$

for all $k$. Expanding $F(0,0,0,x^1,x^2,x^3)$ about the origin in the $x$ variables, one has $F(0,0,0,q^1_k,q^2_k,q^3_k) = 0 + 0 + R(\Delta q_k)$ where $R(\Delta q_k)/|\Delta q_k|$ converges to zero as $k$ increases. Using $q^0_k = f_H(\Delta q_k) \leq F(0,0,0,\Delta q_k)$ and $|f_H(\Delta q_k)|/|\Delta q_k| > c$, one finds that for all large $k$ one must have $f_H(\Delta q_k) < 0$. Also, one obtains the inequality $f_H(\Delta q_k) < -c|\Delta q_k|$ for all large $k$. For each $k$, choose a null generator $\gamma_k$ containing $q_k$. Lemma 3.1 guarantees that the sequence $\{\gamma_k\}$ converges to the unique null generator $\gamma$ ($x^1$ axis) containing $p$. Thus, for sufficiently large $k$, each $\gamma_k$ contains
Recall that as $x$ to the null plane tangent to $\gamma$. Consequently, we find $p$ past of $k$ that for large $F$ $x$ to the origin, as desired.

We will show that for large $k$ one has $F_k(0,0,0) < 0$ and thus that the surface $S(u_k^0, u_k^2, u_k^3)$ cuts the $x^0$ axis below the origin. To this end, expand $F_k$ about the point $q_k$ in the $x$ variables to obtain

$$F_k(\Delta x) = q_k^0 + \sum a_{ki}(x^i - q_k^i) + R_k(\Delta(x - q_k))$$

where $\Delta(x-q_k) = (x^1-q_k^1, x^2-q_k^2, x^3-q_k^3)$, $R_k(\Delta(x-q_k)) = R(u_k, q_k, \Delta(x-q_k)) = R(u_k^0, u_k^2, u_k^3, q^1, q^2, q^3, \Delta(x-q_k))$ and $a_{ki}$ represents the partial of $F_k$ with respect to $x^i$ evaluated at the point $q_k$ for $i = 1, 2, 3$. Using $x^1 = x^2 = x^3 = 0$ and $q_k^0 = f_H(\Delta q_k) < -c|\Delta q_k|$, one obtains the following inequality for large $k$

$$F_k(0,0,0) < -c|\Delta q_k| + \sum a_{ki}(0 - q_k^i) + R_k(-q_k^1, -q_k^2, -q_k^3)$$

Recall that as $k$ increases, the tangent direction to $\gamma_k$ converges to the tangent direction to $\gamma$. Thus, the null tangent plane to $x^0 = F_k(\Delta x)$ at $q_k$ must converge to the null plane tangent to $x^0 = 0$ at the origin. It follows that the coefficients $a_{ki}$ converge to zero for $i = 1, 2, 3$ and thus for large $k$ one has an inequality of the form

$$|\sum a_{ki}(0 - q_k^i)| < (c/3)|\Delta q_k|.$$ 

Also, using Inequality (3.1) one has $|R_k(-q_k^1, -q_k^2, -q_k^3)| < M|\Delta q_k|^2$ which implies that for large $k$

$$|R_k(-q_k^1, -q_k^2, -q_k^3)| < (c/3)|\Delta q_k|.$$ 

Consequently, we find

$$F_k(0,0,0) < -c|\Delta q_k| + (c/3)|\Delta q_k| + (c/3)|\Delta q_k|$$

which yields $F_k(0,0,0) < 0$ for large $k$ since $|\Delta q_k| \neq 0$. Using the fact that the $x^0$ axis is timelike future directed, one finds that $F_k(0,0,0)$ is in the chronological past of $p = (0,0,0)$ for large $k$. Using the fact that $u_k$ is in the causal past of $F_k(0,0,0)$, one finds that $u_k$ must be in the chronological past of $p = (0,0,0)$ for large $k$. Since $p$ and $u_k$ must lie on $H^+(S)$, this contradicts the achronality of the horizon. We conclude that $f_H$ is differentiable at the origin and all of its partials with respect to $x^i$ are zero at the origin. It follows that $H^+(S)$ is differentiable at the origin, as desired. \(\square\)
Recall that the *crease set* is the set of points in the horizon having multiplicity at least two and that the horizon fails to be differentiable at any such point. Clearly, Theorem 3.5 is equivalent to the statement that *the crease set consists precisely of those points on the Cauchy horizon where the horizon fails to be differentiable.*

Combining Propositions 3.3 and 3.4 with Theorem 3.5, one obtains the following result.

**Proposition 3.6.** Let \( W \) be an open subset of the Cauchy horizon \( H^+(S) \). Then the following are equivalent.

1. \( H^+(S) \) is differentiable on \( W \).
2. \( H^+(S) \) is of class \( C^r \) on \( W \) for some \( r \geq 1 \).
3. \( H^+(S) \) has no endpoints on \( W \).
4. All points of \( W \) have multiplicity one.

Note that the four parts of Proposition 3.6 are logically equivalent for an open set \( W \), but that, in general, they are not necessarily equivalent for sets which fail to be open. Using the equivalence of parts (1) and (3) of Proposition 3.6, it now follows that near each endpoint of multiplicity one there must be points where the horizon fails to be differentiable. Hence, each neighborhood of an endpoint of multiplicity one must contain endpoints of higher multiplicity. This yields the following corollary.

**Corollary 3.7.** If \( p \) is an endpoint of multiplicity one on a Cauchy horizon \( H^+(S) \), then each neighborhood \( W(p) \) of \( p \) on \( H^+(S) \) contains points where the horizon fails to be differentiable. Hence, the set of endpoints of multiplicity one is in the closure of the crease set.

We know that the nondifferentiable set (i.e., the crease set) has (3-dimensional) measure zero since the horizon satisfies a Lipschitz condition. On the other hand, since a Cauchy horizon is differentiable at an endpoint of multiplicity one, the fact that \( H^+(S) \) satisfies a Lipschitz condition does not give us any direct information on the measure of the set of endpoints of multiplicity one. Based on known examples it seems likely that this set should be a relatively small set. Thus, we conjecture that the set of endpoints of multiplicity one is of measure zero for all Cauchy horizons. Clearly, this is equivalent to the following endpoint conjecture.

**Conjecture.** The set of all endpoints of a Cauchy horizon must have measure zero.
An affirmative or negative answer to the above conjecture would certainly help in the understanding of Cauchy horizons and in future studies of these horizons.

4. Compact Horizons

In the previous sections we have not made any curvature assumptions. In this section we will assume $\text{Ric}(V,V) \geq 0$ for all null $V$ which is the null convergence condition. We will also assume that we have a Cauchy horizon $H^+(S)$ which is compact and of class $C^2$ on an open subset $G$ such that the complement of $G$ is of 3-dimensional measure zero.

**Theorem 4.1.** Let $(M,g)$ be a spacetime and assume that $(M,g)$ satisfies the null convergence condition (i.e., $\text{Ric}(V,V) \geq 0$ for all null $V$). Let $S$ be a partial Cauchy surface. Assume that the future horizon $H^+(S)$ is compact and contains an open set $G$ where it is $C^2$ and is such that the complement of $G$ in $H^+(S)$ has 3-dimensional measure zero. Then, $H^+(S)$ has no endpoints and hence is differentiable of (at least) class $C^1$ for all points.

**Proof.** The compactness of $H^+(S)$ yields the past completeness of all null generators [12, p. 295]. We use the notation of [11] to obtain a map $u_t: G \to G$, see [11, p. 606] and equation

$$d/dt \int_{u_t(G)} dA = 2 \int_{u_t(G)} \rho dA$$

(4.1)

Notice that since $H^+(S)$ is compact we find that $\int_{u_t(G)} dA$ is finite. However, the derivative of $\int_{u_t(G)} dA$ cannot be positive since the set $G$ is mapped into itself. Thus, the left hand side of Equation 4.1 is nonpositive. On the other hand, the right hand side of Equation 4.1 must be nonnegative since $\rho \geq 0$. Assume now that $H^+(S)$ has an endpoint $p$ of a null generator $\gamma$. Even if $\gamma$ does not lie in $G$, the horizon will be differentiable on the part of $\gamma$ in the past of $p$. Choose some $q$ on $\gamma$ in the past of $p$ and some $u$ on $\gamma$ in the future of $p$. Then for some small neighborhood $W(q)$ of $q$ on $H^+(S)$, all null generators though points $r$ of $W(q)$ will have directions close to the direction of $\gamma$ at $q$. Recall that given a compact domain set in the $t-$axis, geodesics with close initial conditions remain close on the compact domain set. Thus, by choosing $W(q)$ sufficiently small we may get all null generators though points of $W(q)$ come arbitrarily close to $u$. Thus for sufficiently small $W(q)$ all null generators intersecting $W(q)$ must leave $H^+(S)$ in the future. Since $W(q)$ is open in $H^+(S)$ it must have a nontrivial intersection with the set $G$. Thus, $u_t(G)$ cannot
be all of $G$ for some positive values of $t$ and this yields a negative for some values of $t$ on the left hand side of Equation 4.1, in contradiction. Thus, $H^+(S)$ has no endpoints and by Proposition 3.6 must be (at least) of class $C^1$ at all points. □

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