Hybridized Summation-by-Parts Finite Difference Methods

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Abstract
We present a hybridization technique for summation-by-parts finite difference methods with weak enforcement of interface and boundary conditions for second order, linear elliptic partial differential equations. The method is based on techniques from the hybridized discontinuous Galerkin literature where local and global problems are defined for the volume and trace grid points, respectively. By using a Schur complement technique the volume points can be eliminated, which drastically reduces the system size. We derive both the local and global problems, and show that the resulting linear systems are symmetric positive definite. The theoretical stability results are confirmed with numerical experiments as is the accuracy of the method.

Keywords Summation-by-parts finite difference methods · Elliptic equations · Hybridization · Weak enforcement of boundary conditions · High-order finite difference methods · Numerical stability

1 Introduction

High-order finite difference methods have a long and rich history for solving second order, elliptic partial differential equations (PDEs); see for instance the short historical review of
Thomée [29]. When complex geometries are involved, finite difference methods are similar to finite element methods in that unstructured meshes and coordinate transforms can be used to handle complex geometries [25]. Summation-by-parts (SBP) finite difference methods [16, 17, 19, 22, 28] have been particularly effective for such problems, since inter-block coupling conditions can be handled weakly using the simultaneous approximation term (SAT) method [2, 3].

The combined SBP-SAT approach has been used extensively for problems that arise in the natural sciences where physical interfaces are ubiquitous, for example in earthquake problems where faults separate continental and oceanic crustal blocks or in multiphase fluids with discontinuous properties [8, 13, 14, 18]. The present work is particularly motivated by models of earthquake nucleation and rupture propagation over many thousands of years, where the slow, quiescent periods between earthquakes represent quasi-steady state problems [9]. In the steady-state regime, an elliptic PDE must be repeatedly solved, which results in large linear systems of equations for complex problems.

In this work we propose a hybridization technique for SBP-SAT methods in order to reduce the size of the linear systems. The inspiration for this is static condensation and hybridization for finite element methods [6, 12]. These techniques reduce system size by writing the numerical method in a way that allows the Schur complement to be used to eliminate degrees of freedom from within the element leaving only degrees of freedom on element boundaries. SBP-SAT methods have a similar discrete structure to discontinuous Galerkin methods, with the penalty terms in SBP-SAT methods being analogous to the numerical fluxes in discontinuous Galerkin methods.

Here we introduce independent trace variables along the faces of the blocks, and the inter-block coupling penalty terms are only a function of the trace variables. Thus, the solution in each block is uniquely determined by the trace variables which are applied as Dirichlet boundary data. The problem is broken into two pieces, a local problem, which is the solution within the block given the trace data, and a global problem, which is the value of the trace variable given the block data. Using a Schur complement technique either set of variables can be eliminated. When the trace variables are eliminated the scheme is similar to existing SBP-SAT schemes, for instance the method of Virta and Mattsson [30]. If on the other hand the volume variables are eliminated and the trace variables are retained, the system size is drastically reduced since the system only involves the unknowns along the block faces. That said, the cost of forming this later Schur complement system arises from the need to invert each finite difference block (though we note that each inverse is independent, involving only the block local degrees of freedom).

The developed method is symmetric positive definite for the monolithic system (trace and volume variables) as are the two Schur complement systems. Thus, the elliptic discretization is stable. Importantly, these properties are shown to hold even if the elliptic problem is variable coefficient or involves curvilinear blocks. Since the discretization is based on the hybridized interior penalty method [6, IP-H], there is a (spatially varying) penalty parameter that must be sufficiently large for stability and a bound for this penalty is given. It is also shown that the penalty parameter can be determined purely from the local problem, independent of the neighboring blocks.

The paper is organized as follows: in Sect. 2 we detail the block decomposition and SBP operators. Section 3 describes the model problem, an elliptic PDE, along with boundary and interface conditions which allow for jump discontinuities and material contrasts. Section 4 details the hybridized scheme, including the local and global problems. Proofs of positive-definiteness of both systems are provided; these results are confirmed with numer-
ical experiments in Sect. 5. Section 5 also provides results from convergence tests using an exact solution, and we conclude with a summary in Sect. 6.

2 Domain Decomposition and SBP Operators

As noted above, we apply the class of high-order accurate SBP finite difference methods which were introduced for first derivatives in Kreiss and Scherer [16,17]; Strand [28], and for second derivatives by Mattsson and Nordström [22], with the variable coefficients treated in Mattsson [19]. In addition to high-order accuracy, SBP methods can be combined with various boundary treatments so that the resulting linear PDE discretization is provably stable. In Sect. 4 we use weak enforcement of boundary and interface conditions with the Simultaneous-Approximation-Term (SAT) method. Here we introduce notation related to the decomposition of the computational domain into blocks as well as one-dimensional and two-dimensional SBP operators for first and second derivatives.

2.1 Domain Decomposition

We let the computational domain be $\Omega \subset \mathbb{R}^2$ which is partitioned into $N_b$ non-overlapping curved quadrilateral blocks; the partitioning is denoted $B(\Omega)$. For each block $B \in B(\Omega)$ we assume that there exists a diffeomorphic mapping from the reference block $\hat{B} = [0, 1] \times [0, 1]$ to $B$. The mapping $(x^B(r,s), y^B(r,s))$ goes from the reference block to the physical block and $(r^B(x,y), s^B(x,y))$ is the inverse mapping. An example of this is shown in Fig. 1; the figure also shows the face numbering for the reference block.

As will be seen in Sect. 3, the transformation to the reference block requires metric relations that relate the physical and reference derivatives. Four relations that are particularly useful are

$$
J \frac{\partial r}{\partial x} = \frac{\partial y}{\partial s}, \quad J \frac{\partial s}{\partial y} = \frac{\partial x}{\partial r}, \quad J \frac{\partial s}{\partial x} = -\frac{\partial y}{\partial r}, \quad J \frac{\partial r}{\partial y} = -\frac{\partial x}{\partial s},
$$

with $J$ being the Jacobian determinant for block $B$,

$$
J = \frac{\partial x}{\partial r} \frac{\partial y}{\partial s} - \frac{\partial x}{\partial s} \frac{\partial y}{\partial r};
$$

for simplicity of notation, unless required we suppress the block $B$ superscript and the relations should be understood as applying to a single block. For face $k$ of a block, the surface Jacobian is

Fig. 1 (left) Block decomposition of a disk with a single curved block highlighted along with its grid lines in physical space. (right) Mapping of the highlighted block to the reference domain; shown in the figure is the convention used to number the faces of the reference block.
\[ S_{j,k} = \begin{cases} \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}, & \text{if } k = 1, 2, \\ \sqrt{\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2}, & \text{if } k = 3, 4, \end{cases} \]

and the outward unit normal vectors are

\[ S_{j,1} \hat{n}_1 = \begin{bmatrix} -\frac{\partial y}{\partial x} \\ \frac{\partial x}{\partial x} \end{bmatrix}, \quad S_{j,2} \hat{n}_2 = \begin{bmatrix} \frac{\partial y}{\partial x} \\ -\frac{\partial x}{\partial x} \end{bmatrix}, \]

\[ S_{j,3} \hat{n}_3 = \begin{bmatrix} -\frac{\partial y}{\partial r} \\ \frac{\partial r}{\partial r} \end{bmatrix}, \quad S_{j,4} \hat{n}_4 = \begin{bmatrix} \frac{\partial y}{\partial r} \\ -\frac{\partial r}{\partial r} \end{bmatrix}. \]

### 2.2 One-Dimensional SBP Operators

Let the domain \(0 \leq r \leq 1\) be discretized with \(N + 1\) evenly spaced grid points \(r_i = i h, i = 0, \ldots, N\) and spacing \(h = 1/N\). The projection of a function \(u\) onto the computational grid is taken to be \(u = [u_0, u_1, \ldots, u_N]^T\); if \(u\) is known then \(u\) is often taken to be the interpolant at the grid points. The grid basis vector \(e_j\) is 1 at grid point \(j\) and zero at all other grid points and \(u_j = e_j^T u\).

**Definition 1** (First derivative) A matrix \(D_r\) is called an SBP approximation to \(\partial u/\partial r\) if it can be decomposed as \(HD_r = Q\) with \(H\) being symmetric positive definite and \(Q\) being such that \(u^T(Q + Q^T)v = u_Nv_N - u_0v_0\).

In this work we only consider diagonal-norm SBP, i.e., finite difference operators where \(H\) is a diagonal matrix and \(D_r\) is the standard central finite difference matrix in the interior which transitions to one-sided at the boundaries. The condition on \(Q\) can also be written as \(Q + Q^T = e_N e_N^T - e_0 e_0^T\).

The operator \(D_r\) is called SBP because the integration-by-parts property

\[ \int_0^1 u \frac{\partial v}{\partial r} + \int_0^1 \frac{\partial u}{\partial r} v = uv \bigg|_0^1, \]

is mimicked discretely by

\[ u^T HD_r v + u^T D_r^T H v = u^T \left( Q + Q^T \right) v = u_N v_N - u_0 v_0. \]

**Definition 2** (Second derivative) A matrix \(D_{rr}^{(c)}\) is called an SBP approximation to \(\partial^2 u/\partial r^2\) if it can be decomposed as \(HD_{rr}^{(c)} = -A^{(c)} + c_N e_N d_N + c_0 e_0 d_0^T\) where \(A^{(c)}\) is symmetric positive definite and \(d_0^T u\) and \(d_N^T u\) are approximations of the first derivative of \(u\) at the boundaries.

The operator \(D_{rr}^{(c)}\) is called SBP because the integration-by-parts equality

\[ \int_0^1 u \frac{\partial}{\partial r} \left( c \frac{\partial v}{\partial r} \right) + \int_0^1 \frac{\partial u}{\partial r} c \frac{\partial v}{\partial r} = uc \frac{\partial v}{\partial r} \bigg|_0^1, \]

is mimicked discretely by

\[ u^T HD_{rr}^{(c)} v + u^T A^{(c)} v = c_N u_N d_N^T v - c_0 u_0 d_0^T v. \]
Definition 3 (Compatibility) Matrices $D_r$ and $D_{rr}^{(c)}$ are called compatible SBP operators if they use the same matrix $H$ and the remainder matrix $R^{(c)} = A^{(c)} - D_r^T CHD_r$ is symmetric positive definite with $C = \text{diag}(c)$ being a diagonal matrix constructed from the grid interpolant of $c$.

It is important to note that compatibility does not assume that $d_0^T$ and $d_N^T$ are the first and last rows of $D_r$. When this is the case the operators are called fully-compatible [23] and such operators are not used in this work.

As noted above, we only consider diagonal-norm SBP finite difference operators. In the interior the operators use the minimum bandwidth central difference stencil and transition to one-sided near the boundary in a manner that maintains the SBP property. If the interior operator has accuracy $2p$, then the interior stencil bandwidth is $2p + 1$ and the boundary operator has accuracy $p$. The first and second derivative operators used are those given in Strand [28] and [19], respectively. In Sect. 5 we will use operators with interior accuracy $2p = 2, 4, 6$. The expected global order of accuracy is the minimum of $2p$ and $p + 2$ as evidenced experimentally [21,30] and proved rigorously for the Schrödinger equation [24]. In Sect. 5 we verify this result for the hybridized scheme through convergence tests.

Remark 1 If the second derivative finite difference operator is defined by repeated applications of the first derivatives operator, e.g. $D_{rr}^{(c)} = D_{r} C D_{r}$, then the operator is fully compatible with $R^{(c)}$ being the zero matrix but the operator does not have minimal bandwidth.

2.3 Two-Dimensional SBP Operators

Two-dimensional SBP operators can be developed for rectangular domains by applying the one-dimensional operators in a tensor product fashion (i.e., dimension-by-dimension application of the one-dimensional operators). Here we describe the operators for the reference block $\hat{B} = [0, 1] \times [0, 1]$ We assume that the domain is discretized using an $(N+1) \times (N+1)$ grid of points where grid point $(i,j)$ is at $(r_i, s_j) = (ih, jh)$ for $0 \leq i, j \leq N$ with $h = 1/N$; the generalization to different numbers of grid points in each dimension complicates the notation but does not impact the construction of the method and is discussed later.

A 2D grid function $\tilde{u}$ is taken to be a stacked vector of vectors with $\tilde{u} = [u^T_0, u^T_1, \ldots, u^T_N]^T$ and $u^T_j = [u^{j0}, u^{j1}, \ldots, u^{jN}]^T$ where $u^{ji} \approx u(r_j, s_i)$.

Derivative approximations are taken to be of the form

$$\frac{\partial}{\partial r} \begin{pmatrix} c_{rr} & \partial u \\ \partial r \end{pmatrix} \approx \tilde{D}_{rr}^{(c_{rr})} \tilde{u}, \quad \frac{\partial}{\partial s} \begin{pmatrix} c_{ss} & \partial u \\ \partial s \end{pmatrix} \approx \tilde{D}_{ss}^{(c_{ss})} \tilde{u},$$

$$\frac{\partial}{\partial r} \begin{pmatrix} c_{rs} & \partial u \\ \partial s \end{pmatrix} \approx \tilde{D}_{rs}^{(c_{rs})} \tilde{u}, \quad \frac{\partial}{\partial s} \begin{pmatrix} c_{sr} & \partial u \\ \partial s \end{pmatrix} \approx \tilde{D}_{sr}^{(c_{sr})} \tilde{u}.$$  

(1)

To explicitly define the derivative operators, we first let $\tilde{c}_{rr}$ be the grid interpolant of the weighting function $c_{rr}$ and define $\tilde{C}_{rr} = \text{diag}(\tilde{c}_{rr})$. Additionally, the diagonal matrices of the coefficient vectors along each of the grid lines are

$$C_{rr}^i = \text{diag}(0, c_{rr}, \ldots, c_{rr})^N, \quad C_{rr}^j = \text{diag}(c_{rr}, c_{rr}, \ldots, c_{rr})^N.)$$
Similar matrices are constructed for $c_{ss}$, $c_{rs}$, and $c_{sr}$. With this, the derivative operators in (1) are
\[
(H \otimes H) \hat{D}^{(crs)}_{rr} = \tilde{A}^{(crs)}_{rr} \quad \text{and} \quad (H \otimes H) \hat{D}^{(crs)}_{ss} = \tilde{A}^{(crs)}_{ss} + \left( e_N^T \otimes H C_{ss}^N \right),
\]
\[
(H \otimes H) \hat{D}^{(crs)}_{rs} = (I \otimes Q) \tilde{C}_{rs} (Q \otimes I)
\]
\[
(H \otimes H) \hat{D}^{(crs)}_{sr} = (Q \otimes I) \tilde{C}_{sr} (I \otimes Q)
\]
where $\otimes$ denotes the Kronecker product of two matrices. Here, the matrices $\tilde{A}^{(crs)}_{rr}$, $\tilde{A}^{(crs)}_{ss}$, $\tilde{A}^{(crs)}_{rs}$, and $\tilde{A}^{(crs)}_{sr}$ are
\[
\tilde{A}^{(crs)}_{rr} = (H \otimes I) \left[ \sum_{j=0}^{N} (e_j \otimes I) A^{(C_N^{ij})} (e_j^T \otimes I) \right],
\]
\[
\tilde{A}^{(crs)}_{ss} = (I \otimes H) \left[ \sum_{i=0}^{N} (I \otimes e_i) A^{(C_N^{ij})} (I \otimes e_i^T) \right],
\]
\[
\tilde{A}^{(crs)}_{rs} = (I \otimes Q^T) \tilde{C}_{rs} (Q \otimes I),
\]
\[
\tilde{A}^{(crs)}_{sr} = (Q^T \otimes I) \tilde{C}_{sr} (I \otimes Q),
\]
and can be viewed as approximations of the following integrals:
\[
\int_B \frac{\partial u}{\partial r} c_{rr} \frac{\partial v}{\partial r} \approx \tilde{u}^T \tilde{A}^{(crs)}_{rr} \tilde{v}, \quad \int_B \frac{\partial u}{\partial s} c_{rs} \frac{\partial v}{\partial r} \approx \tilde{u}^T \tilde{A}^{(crs)}_{rs} \tilde{v}, \quad \int_B \frac{\partial u}{\partial s} c_{sr} \frac{\partial v}{\partial s} \approx \tilde{u}^T \tilde{A}^{(crs)}_{sr} \tilde{v},
\]
The following equality will be useful later which splits the volume and surface contributions:
\[
(H \otimes H) \left[ -\tilde{D}^{(crs)}_{rr} - \tilde{D}^{(crs)}_{rs} - \tilde{D}^{(crs)}_{sr} - \tilde{D}^{(crs)}_{ss} \right]
\]
\[
= \tilde{A}^{(crs)}_{rr} + \tilde{A}^{(crs)}_{rs} + \tilde{A}^{(crs)}_{sr} + \tilde{A}^{(crs)}_{ss} - L_1^T G_1 - L_2^T G_2 - L_3^T G_3 - L_4^T G_4.
\]
Here the face point extraction operators are defined as
\[
L_1 = I \otimes e_0^T, \quad L_2 = I \otimes e_N^T, \quad L_3 = e_0^T \otimes I, \quad L_4 = e_N^T \otimes I.
\]

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and the matrices which compute the weighted boundary derivatives are

\[ G_1 = - \left( H C_{rr}^0 \otimes d_0^T \right) - \left( C_{rs}^0 Q \otimes e_0^T \right), \]
\[ G_2 = \left( H C_{rr}^N \otimes d_N^T \right) + \left( C_{rs}^N Q \otimes e_N^T \right), \]
\[ G_3 = - \left( d_0^T \otimes H C_{ss}^0 \right) - \left( e_0^T \otimes C_{sr}^0 Q \right), \]
\[ G_4 = \left( d_N^T \otimes H C_{ss}^N \right) + \left( e_N^T \otimes C_{sr}^N Q \right). \]  

(4)

The matrix \( G_f \) should be thought of as approximating the integral of the boundary derivative, for example

\[ v^T L_1^T G_1 u \approx - \int_0^1 \left( v \left( c_{rr} \frac{\partial u}{\partial r} + c_{rs} \frac{\partial u}{\partial s} \right) \right) \bigg|_{r=1}. \]

**Remark 2** As noted above, for simplicity of notation we have assumed that the grid dimension is the same in both directions. This can be relaxed by letting the first argument in the Kronecker products be with respect to the \( s \)-direction and the second with respect to the \( r \)-direction. If the grid were different in each direction then, for example, \( (H \otimes H) \) would be replaced by \( (H_s \otimes H_r) \) where \( H_r \) and \( H_s \) are the one-dimensional SBP norm matrices based on grids of size \( N_r + 1 \) and \( N_s + 1 \), respectively.

### 3 Model Problem

As a model problem we consider the following scalar, anisotropic elliptic equation in two spatial dimensions for the field \( u \):

\[- \nabla \cdot (b \nabla u) = f, \quad \text{on } \Omega, \]  
\[ u = g_D, \quad \text{on } \partial \Omega_D, \]  
\[ n \cdot b \nabla u = g_N, \quad \text{on } \partial \Omega_N, \]  
\[ \{ n \cdot b \nabla u \} = 0, \quad \text{on } \Gamma_I, \]  
\[ \{ u \} = \delta, \quad \text{on } \Gamma_I. \]  

Here \( b(x, y) \) is a matrix valued function that is symmetric positive definite and the scalar function \( f(x, y) \) is a source function. The boundary of the domain has been partitioned into Dirichlet and Neumann segments, i.e., \( \partial \Omega = \partial \Omega_D \cup \partial \Omega_N \) and \( \partial \Omega_D \cap \partial \Omega_N = \emptyset \). In the Neumann boundary conditions, the vector \( n \) is the outward pointing normal. The functions \( g_D \) and \( g_N \) are given data at the boundaries. An internal interface \( \Gamma_I \) has also been introduced. Along this interface the \( b \)-weighted normal derivative is taken to be continuous, with jumps allowed in the scalar field \( u \); this allowance is made so that the scheme can be used for the earthquake problems that motivate the work. Here \( \{ w \} = w^+ + w^- \) denotes the sum of the scalar quantity on both sides of the interface and \( \{ w \} = w^+ - w^- \) is the difference across the interface; the side defined as the plus- and minus-side are arbitrary though the choice affects the sign of the jump data \( \delta \).

Governing Eq. (5) are not solved directly on \( \Omega \). Instead, the equations are solved over each \( B \in B(\Omega) \), where along each edge of \( B \) either continuity of the solution and the \( b \)-weighted normal derivative are enforced, or the appropriate boundary (or interface) condition. Additionally, we do not solve directly on \( B \) but instead transform to the reference block \( \hat{B} \).
With this, (5a) becomes for each \( B \in \mathcal{B} \):

\[-\hat{\nabla} \cdot (c \hat{\nabla} u) = Jf,\]  

(6a)

where \( \hat{\nabla} = [\frac{\partial u}{\partial r}, \frac{\partial u}{\partial s}]^T \), i.e., the \( \hat{\nabla} \) is the del operator with respect to \((r, s)\), and the matrix valued coefficient function \( c(r, s) \) has entries

\[c_{rr} = J \left( b_{xx} \frac{\partial r}{\partial x} \frac{\partial r}{\partial x} + 2b_{xy} \frac{\partial r}{\partial x} \frac{\partial y}{\partial y} + b_{yy} \frac{\partial y}{\partial y} \frac{\partial y}{\partial y} \right),\]  

(6b)

\[c_{ss} = J \left( b_{xx} \frac{\partial s}{\partial x} \frac{\partial s}{\partial x} + 2b_{xy} \frac{\partial s}{\partial x} \frac{\partial y}{\partial y} + b_{yy} \frac{\partial y}{\partial y} \frac{\partial y}{\partial y} \right),\]  

(6c)

\[c_{rs} = c_{sr} = J \left( b_{xx} \frac{\partial r}{\partial x} \frac{\partial s}{\partial x} + b_{xy} \left( \frac{\partial r}{\partial x} \frac{\partial s}{\partial y} + \frac{\partial r}{\partial y} \frac{\partial s}{\partial x} \right) + b_{yy} \frac{\partial r}{\partial y} \frac{\partial s}{\partial y} \right),\]  

(6d)

where \( b_{xx}, b_{yy}, \) and \( b_{xy} = b_{yx} \) are the four components of \( b \). For simplicity of notation we have suppressed the subscript \( B \) on terms in (6) and following. If \( J > 0 \) then the matrix formed by \( c_{rr}, c_{ss}, \) and \( c_{rs} = c_{sr} \) is symmetric positive definite and (6a) is of the same form as (5a) except on the unit square domain \( \hat{B} \).

The boundary conditions (5b)-(5c) and interface conditions (5d) are similarly transformed. Namely, letting \( \hat{\partial} \hat{B}_k \) for \( k = 1, 2, 3, 4 \) be the faces of \( \hat{B} \), we then require that for each \( k \):

\[u = g_D, \quad \text{if } \hat{\partial} \hat{B}_k \cap \partial \Omega_D \neq \emptyset,\]  

(6e)

\[\hat{n}_k \cdot c \hat{\nabla} u = S_{J,k} g_N, \quad \text{if } \hat{\partial} \hat{B}_k \cap \partial \Omega_N \neq \emptyset,\]  

(6f)

\[\{\{\hat{n}_k \cdot c \nabla u\}\} = 0, \quad \text{if } \hat{\partial} \hat{B}_k \cap \Gamma_I \neq \emptyset,\]  

(6g)

\[\{\{u\}\} = \delta, \quad \text{otherwise.}\]  

(6h)

Here \( \hat{n}_k \) is the outward pointing normal to face \( \hat{\partial} \hat{B}_k \) in the reference space (not the physical space) and \( S_{J,k} \) is the surface Jacobian which arises due to the fact that \( c \) includes metric terms. Condition (6h) is the same as (6g) if \( \delta \) is defined to be 0 on these faces.

## 4 Hybridized SBP Scheme

In the finite element literature, a hybrid method has one unknown function on element interiors and a second unknown function on element traces [5, p. 421]. For SBP methods, the big idea is to write the method in terms of local problems and a global problem. In the local problems, for each \( B \in \mathcal{B} \) the trace of the solution (i.e., the boundary and interface data) is assumed and the transformed Eq. (6) is solved locally over \( B \). In the global problem the solution traces for each \( B \in \mathcal{B} \) are coupled. As will be shown, this technique will result in a linear system of the form

\[
\begin{bmatrix}
\bar{M} & \bar{F} \\
\bar{F}^T & \bar{D}
\end{bmatrix}
\begin{bmatrix}
\bar{u} \\
\bar{\lambda}
\end{bmatrix}
=
\begin{bmatrix}
\bar{g} \\
\bar{g}_s
\end{bmatrix}.
\]

(7)

Here \( \bar{u} \) is the approximate solution to (6) at all the grid points and \( \bar{\lambda} \) are the trace variables along internal interfaces; trace variables related to boundary conditions can be eliminated.
The matrix $\tilde{M}$ is block diagonal with one symmetric positive definite block for each $B \in B$. $D$ is diagonal, and the matrix $\tilde{F}$ is sparse and incorporates the coupling conditions. The right-hand side vector $\tilde{g}$ incorporates the boundary data ($g_D, g_N$) and source terms whereas $\tilde{g}_\delta$ incorporates the interface data $\delta$.

Using the Schur complement we can transform (7) to

$$\left( D - \tilde{F}^T \tilde{M}^{-1} \tilde{F} \right) \lambda = \tilde{g}_\delta - \tilde{F}^T \tilde{M}^{-1} \tilde{g},$$

resulting in a substantially reduced problem size since the number of trace variables is significantly smaller than the number of solution variables. Since $\tilde{M}$ is block diagonal, the inverse can be applied in a decoupled manner for each $B \in B$. Thus there is a trade-off between the number of blocks and the size of system (8), since for a fixed resolution increasing the number of blocks means that $\tilde{M}$ will be more efficiently factored but the size of (8) will increase through the introduction of additional trace variables.

Now that the big picture is laid, we proceed to introduce the local problem (thus defining $\tilde{F}$ and $\tilde{D}$).

### 4.1 The Local Problems

For each $B \in B$ we solve (6a) with boundary conditions

$$u = \lambda_k \text{ on } \partial \tilde{B}_k \text{ for } k = 1, 2, 3, 4,$$

where for now we assume that the trace functions $\lambda_k$ are known; later these will be defined in terms of the boundary and coupling conditions. Using the SBP operators defined in Sect. 2.3 a discretization of (6a) is

$$-D_{rr}^{(C_{rr})} \hat{u} - D_{rs}^{(C_{rs})} \hat{u} - D_{sr}^{(C_{sr})} \hat{u} - D_{ss}^{(C_{ss})} \hat{u} = \tilde{J} \tilde{f} + \tilde{b}_1 + \tilde{b}_2 + \tilde{b}_3 + \tilde{b}_4.$$

Here $\hat{u}$ is the vector solution and $\tilde{J} \tilde{f}$ is the grid approximation of $Jf$. The terms $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3$, and $\tilde{b}_4$ are the penalty terms which incorporate local boundary conditions (9); this is known as the SAT method and is equivalent to the numerical flux in discontinuous Galerkin formulations [2,10]. The penalty terms are taken to be of the form

$$(H \otimes H) \hat{b}_1 = G_1^T \left[ L_1 \hat{u} - \lambda_1 \right] + L_1^T \left[ H\hat{\sigma}_1 - G_1 \hat{u} \right],$$

$$(H \otimes H) \hat{b}_2 = G_2^T \left[ L_2 \hat{u} - \lambda_2 \right] + L_2^T \left[ H\hat{\sigma}_2 - G_2 \hat{u} \right],$$

$$(H \otimes H) \hat{b}_3 = G_3^T \left[ L_3 \hat{u} - \lambda_3 \right] + L_3^T \left[ H\hat{\sigma}_3 - G_3 \hat{u} \right],$$

$$(H \otimes H) \hat{b}_4 = G_4^T \left[ L_4 \hat{u} - \lambda_4 \right] + L_4^T \left[ H\hat{\sigma}_4 - G_4 \hat{u} \right],$$

where $\lambda_1, \lambda_2, \lambda_3,$ and $\lambda_4$ are the grid values of $\lambda$ along each of the four faces. The yet-to-be-defined vectors $H\hat{\sigma}_1, H\hat{\sigma}_2, H\hat{\sigma}_3,$ and $H\hat{\sigma}_4$ are (within the HDG literature) known as the numerical fluxes and will be linear functions of the solution vector $\hat{u}$ and trace variables $\lambda_1, \lambda_2, \lambda_3,$ and $\lambda_4$. We have scaled $\hat{\sigma}_k$ by the matrix $H$ to highlight that these would be integrated flux terms in the HDG literature and $\hat{\sigma}_k$ can be thought of as an approximation of $\hat{n}_k \cdot \hat{c} \hat{\nabla} u$.

Motivated by the hybridized symmetric interior penalty (IP-H) method [6], we take the penalty fluxes to be of the form

$$H\hat{\sigma}_k = G_k \hat{u} - H\tau_k \left( L_k \hat{u} - \lambda_k \right);$$

thus $H\hat{\sigma}_k$ includes the norm-weighted boundary derivative $G_k$ (4) and penalties related to the trace function $\lambda_k$. Here $\tau_k$ is a positive, diagonal matrix of penalty parameters, which
as we will see below, is required to be sufficiently large for the local problem to be positive definite.

Multiplying (10) by $H \otimes H$, using the structure of the derivative matrices (3), and collecting all terms involving $\tilde{u}$ on the left-hand side gives a system of the form

$$
(\tilde{\mathbf{A}} + \tilde{\mathbf{C}}_1 + \tilde{\mathbf{C}}_2 + \tilde{\mathbf{C}}_3 + \tilde{\mathbf{C}}_4) \tilde{u} = \tilde{\mathbf{M}} \tilde{u} = \tilde{\mathbf{q}}.
$$

(12a)

Here the left-hand side matrices are

$$
\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(cr)}_{rr} + \tilde{\mathbf{A}}^{(cs)}_{ss} + \tilde{\mathbf{A}}^{(cr)}_{rs} + \tilde{\mathbf{A}}^{(cr)}_{sr},
$$

(12b)

and the right-hand side vector is

$$
\tilde{\mathbf{q}} = (H \otimes H) \tilde{\mathbf{J}} \tilde{\mathbf{f}} - \sum_{k=1}^{4} F_k \lambda_k,
$$

(12d)

with the face matrix $F_k$ being defined as

$$
F_k = G_k^T - L_k^T H \tau_k,
$$

(12e)

The following theorem characterizes the structure of $\tilde{\mathbf{M}}$.

**Theorem 1** The local problem matrix $\tilde{\mathbf{M}}$ is symmetric positive definite if the components of the diagonal penalty matrices $\tau_k$ for $k = 1, 2, 3, 4$ are sufficiently large.

**Proof** See “Proof of Theorem 1 (Symmetric Positive Definiteness of the Local Problem)” of appendix.

**Remark 3** Explicit bounds for the penalty terms are given in the proof of Theorem 1 given in Proof of Theorem 1 (Symmetric Positive Definiteness of the Local Problem)” of the appendix: see (35). Since they are fairly complicated to state, we have chosen to omit them from the statement of the theorem.

**Corollary 1** The local solution $\tilde{u}$ is uniquely determined by $\tilde{\mathbf{f}}$, $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$.

**Proof** Follows directly from Theorem 1 since $\tilde{\mathbf{f}}$, $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$ determine the right-hand side vector $\tilde{\mathbf{q}}$.

### 4.2 Global Problem

We now turn to the global problem, namely the system that determines the trace vector $\tilde{\lambda}$. To do this we let $\mathcal{F}$ be the set of all block faces with $\mathcal{F}_D$ and $\mathcal{F}_N$ being those faces that occur on the Dirichlet and Neumann boundaries, respectively, and $\mathcal{F}_I$ being the interior faces; internal faces that both have a jump and those that do not are included in $\mathcal{F}_I$ with the latter having $\delta := 0$. For each face $f \in \mathcal{F}_D \cup \mathcal{F}_N$ we let the corresponding block and block face be $B_f \in \mathcal{B}$ and $k_f$, respectively. For each face $f \in \mathcal{F}_I$ we let $B_f^\pm \in \mathcal{B}$ be the blocks connected to the two sides of the interface and let $k_f^\pm$ be the connected sides of the blocks; for the jump interfaces the plus- and minus-sides should correspond to those in (6g). In what follows the subscript $f$ is dropped when only one face $f \in \mathcal{F}$ is being considered. Finally, for each $B \in \mathcal{B}$ we let $\lambda_k = P_{B,k} \tilde{\lambda}$, where $P_{B,k}$ selects the values out of the global vector of trace variables $\tilde{\lambda}$ that correspond to face $k$ and block $B$.
Dirichlet Boundary Conditions Consider face $f \in \mathcal{F}_D$ which corresponds to face $k$ of block $B \in \mathcal{B}$. In this case we set $\lambda_k$ in (11) to be
\[
\lambda_k = g_{D,f},
\]
where $g_{D,f}$ denotes the projection of $g_D$ to face $f$. With this the penalty term $\tilde{b}_k$ becomes
\[
(H \otimes H) \tilde{b}_k = F_k \left( L_k \tilde{u} - g_{D,k} \right),
\]
which is penalization of the grid function along interface $k$ to the Dirichlet boundary data. Since $\lambda_k$ is determined independently of $\tilde{u}$ and the structure of the matrix $\tilde{M}$ remains unchanged.

Neumann Boundary Condition Consider face $f \in \mathcal{F}_N$ which corresponds to face $k$ of block $B \in \mathcal{B}$. In this case we require that $\lambda_k$ in (11) satisfies
\[
H \hat{\sigma}_k = H S_{J,k} g_{N,f},
\]
where $g_{N,f}$ denotes the projection of $g_N$ to face $f$ and $S_{J,k}$ is a diagonal matrix of surface Jacobians along block face $k$. As with the Dirichlet boundary condition, the variable $\lambda_k$ can be found uniquely in terms of the boundary data:
\[
\lambda_k = L_k \tilde{u} + \tau_k^{-1} \left( S_{J,k} g_{N,k} - H^{-1} G_k \tilde{u} \right),
\]
which represents penalization of the boundary derivative towards the Neumann boundary data. If $\lambda_k$ is eliminated in this fashion from the scheme, then $\tilde{M}$ is modified as
\[
\tilde{M} := \tilde{M} - F_k H^{-1} \tau_k^{-1} F_k^T.
\]

**Theorem 2** The modified local problem matrix $\tilde{M}$ in (16) is symmetric positive definite if the components of the diagonal penalty matrices $\tau_k$ for $k = 1, 2, 3, 4$ are sufficiently large and at least one face of the local block $B \in \mathcal{B}$ is a Dirichlet boundary or interior interface.

**Proof** See “Proof of Theorem 2 (Positive Definiteness of the Local Problem with Neumann Boundary Conditions)” of the appendix.

Interfaces We now consider an $f \in \mathcal{F}_I$ which is connected to face $k^\pm$ of blocks $B^\pm \in \mathcal{B}$; below a subscript $B^\pm$ is added to denoted terms associated with each block and a subscript $f, B^\pm$ for terms associated with the respective faces of the blocks. Continuity of the solution and the $b$-weighted normal derivative are enforced by requiring
\[
H \hat{\sigma}_{f,B^+} + H \hat{\sigma}_{f,B^-} = 0;
\]
since $\hat{\sigma}_{f,B^\pm}$ includes the outward pointing normal to the blocks, condition (17) implies that the terms are equal in magnitude but opposite in sign. Using penalty formulation (11) in (17) with $\lambda_k$ replaced with $\lambda_f \mp \delta_f/2$ gives
\[
0 = \left( G_{f,B^+} \tilde{u}_{B^+} + G_{f,B^-} \tilde{u}_{B^-} \right) - H \tau_{f,B^+} \left( L_{f,B^+} \tilde{u}_{B^+} - \left( \lambda_f - \frac{1}{2} \delta_f \right) \right) - H \tau_{f,B^-} \left( L_{f,B^-} \tilde{u}_{B^-} - \left( \lambda_f + \frac{1}{2} \delta_f \right) \right).
\]
where the first term represents penalization of the face normal derivative on the two sides to
the common value and the second two terms the penalization of the $u_{B^\pm}$ to $\lambda_f \mp \delta_f / 2$. By
grouping terms, the above equation can be rewritten as

$$F_{f,B^+}^T \tilde{u}_{B^+} + F_{f,B^-}^T \tilde{u}_{B^-} + D_f \lambda_f = \frac{1}{2} H \left( \tau_{f,B^+} - \tau_{f,B^-} \right) \delta_f. \quad (18)$$

Here the matrices $F_{f,B^\pm}$ are defined by (12e) and the diagonal matrix $D_f$ is

$$D_f = H \left( \tau_{f,B^+} + \tau_{f,B^-} \right).$$

With this, all the terms in linear system (7) can be defined. The solution vector and trace
vectors are

$$\tilde{u} = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_{N_f} \end{bmatrix}, \quad \tilde{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{N_I} \end{bmatrix},$$

with $N_f$ being the number of interfaces. Multiplying out the terms in (7) gives

$$\tilde{M} \tilde{u} + \tilde{F} \tilde{\lambda} = \tilde{g},$$

$$\tilde{F}^T \tilde{u} + \tilde{D} \tilde{\lambda} = \tilde{g}_\delta.$$  

This form, along with the definition of the local problem (12) and the coupling equation (18),
implies that the matrices $\tilde{M}$ and $\tilde{D}$ are

$$\tilde{M} = \begin{bmatrix} \tilde{M}_1 & \tilde{M}_2 & \cdots \\ & \tilde{M}_2 \\ & & \ddots \\ & & & \tilde{M}_{N_f} \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} \tilde{D}_1 & \tilde{D}_2 & \cdots \\ & \tilde{D}_2 \\ & & \ddots \\ & & & \tilde{D}_{N_I} \end{bmatrix}.$$ 

Furthermore, since each matrix $\tilde{D}_f$ is diagonal, the matrix $\tilde{D}$ is also diagonal. To write down
the form of $\tilde{F}$ it is convenient to think of it as a block matrix with sub-matrix $fB$ being the
columns associated with interface $f$ and rows associated with block $B$. Thus, block $\tilde{F}_{fB}$ is
zero unless block $B$ is connected to interface $f$ through local face $k_f$ in which case

$$\tilde{F}_{fB} = F_{k_f,B}.$$ 

The right-hand side vector $\tilde{g}$ is defined from the boundary data using (14) and (15), and
similarly $\tilde{g}_\delta$ is defined from the right-hand side of (18).

In order to prove the positive definiteness of the coupled system, we first note that $\tilde{M}$
and $\tilde{D}$ are symmetric positive definite since they are block diagonal matrices formed from
symmetric positive definite matrices. If the trace variables $\tilde{\lambda}$ are eliminated using the Schur
complement of the $\tilde{D}$ block the system for $\tilde{u}$, the resulting system is

$$\left( \tilde{M} - \tilde{F} \tilde{D}^{-1} \tilde{F}^T \right) \tilde{u} = \tilde{g} - \tilde{F} \tilde{D}^{-1} \tilde{g}_\delta.$$ \quad (19)

This corresponds to the elimination of the trace variables by solving the coupling relation (18)
for $\lambda_f$ and substituting in the local problem (12) for each block. The matrix on the left-hand
side of (19) is characterized by the following theorem which says that if the individual local
problems are symmetric positive definite, then the coupled problem is symmetric positive
definite.
Theorem 3  The matrix $\bar{M} - \bar{F}D^{-1}\bar{F}^T$ is symmetric positive definite as long as the penalty matrices $\tau_{k,B}$ for $k = 1, 2, 3, 4$ and $B \in B$ are sufficiently large that each $\bar{M}_B$ is positive definite.

Proof  See “Proof of Theorem 3 and Corollary 2 (Positive Definiteness of the Global Problem)” of the appendix.

The following corollary characterizes the global system and the Schur complement of the $\bar{M}$ block of the global system.

Corollary 2  The global system matrix

$$
\begin{bmatrix}
\bar{M} & \bar{F} \\
\bar{F}^T & D
\end{bmatrix}
$$

(20)

and the Schur complement of $\bar{M}$ block, $\bar{D} - \bar{F}^T\bar{M}^{-1}\bar{F}$, are symmetric positive definite.

Proof  See “Proof of Theorem 3 and Corollary 2 (Positive Definiteness of the Global Problem)” of the appendix.

5 Numerical Results

We now confirm the theoretical results concerning the positive definiteness of the system, the bounds on the penalty parameters, and numerically investigate accuracy of the hybridized technique. All of the solves in this section are done using direct solves. The Julia [1] codes used to generate the numerical results are available at https://doi.org/10.5281/zenodo.4716826.

5.1 Positive Definiteness of the Local and Global Problems

We begin by confirming that the local problem with both Dirichlet and Neumann boundary conditions is symmetric positive definite. To do this, we consider a single block, and assign a pseudo-random generated symmetric positive definite coefficient matrix $c$ at each grid point. The blocks are taken to use grids of size $N \times N = (3p + 2) \times (3p + 2)$ where $2p$ is the interior order of the SBP operator.

To confirm that the operator is positive definite we compute 1000 realizations of the pseudo-random coefficients and numerically compute the minimum eigenvalue with the penalty parameter defined by the equality version of (35). Two sets of boundary conditions are considered: (1) when all four faces of the block are Dirichlet and (2) when three faces are Neumann and one face is Dirichlet. The result of these calculations are shown in Fig. 2. From this we see that the system is positive definite. One thing of note is that the local system with Neumann boundary conditions has a minimum eigenvalue which is an order of magnitude lower than the purely Dirichlet case. Though not shown, when all four boundaries are Neumann the minimum computed eigenvalue is $\sim 10^{-16} - 10^{-14}$. This conforms with the theory since in this case the system should be singular. An important implication of Fig. 2 is that the bound on the penalty parameter given in (35) is not tight for all cases.

Another question to consider is how the penalty parameter affects the spectral radius of the operator. In Fig. 3 we plot the minimum and maximum eigenvalues versus increasing $\tau_s$;

\[ ^2 \text{Simulations run with Julia 1.5.3.} \]
here $\tau_s$ is a scaling of the penalty parameter so that the actual penalty parameter at each grid point is $\tau_s$ times the equality version of (35). From Fig. 3a it is seen that once $\tau_s$ is large enough, the minimum eigenvalue remains roughly constant. From Fig. 3b we see that the maximum eigenvalue increases linearly with $\tau_s$ in all cases, and that the slope of the line depends on the order of the operators; note that in this figure a log-log axis has been used so the higher the line the larger the slope.

We now confirm the positive definiteness of the global problem by considering two blocks coupled along a single locked interface with Dirichlet boundaries. Each of the blocks has grids of size $N \times N = (3p - 1) \times (3p - 1)$ where $2p$ is the interior order of the SBP operator. As before, the coefficient matrix $c$ at each grid point is generated using pseudo-random numbers with the penalty parameters set to the equality version of (35). In Fig. 4 the minimum eigenvalue for 1000 realizations of the material properties is shown. Eigenvalues

---

**Fig. 2** Plot of the minimum eigenvalue of the local operator for 1000 psuedo-randomly assigned sets of coefficient matrix values for SBP operators with interior orders 2 (blue line), 4 (red line), and 6 (brown line) (Color figure online)

**Fig. 3** Plot of the minimum and maximum eigenvalue for increasing $\tau_s$ for a single psuedo-random parameter realization when all four faces are Dirichlet for SBP operators with interior orders 2 (blue line), 4 (red line), and 6 (brown line) (Color figure online)
from three different systems are shown: the full system (7) and the two Schur complement systems (8) and (19). In all cases it is seen that the minimum eigenvalue is positive, confirming that the systems are positive definite.

5.2 Numerical Accuracy and Convergence

Next we explore the accuracy of the method by applying the method of manufactured solutions (MMS), see for example Roache [26]. In the MMS technique an analytic solution is assumed, and compatible boundary and source data derived. The domain is taken to be the square $\Omega = \{(x, y) | -2 \leq x, y \leq 2\}$. We partition $\Omega$ into the closed unit disk $\Omega_1 = \{(x, y) | x^2 + y^2 \leq 1\}$ and $\Omega_2 = \text{cl}(\Omega \setminus \Omega_1)$, and define the unit circle $\Gamma_I = \{(x, y) | x^2 + y^2 = 1\}$ to be the interface between $\Omega_1$ and $\Omega_2$. The domain can be seen in Fig. 5. The material properties are taken to be $b = I_2$; the metric terms will cause the transformed material properties $c$ to be spatially variable. The right and left boundaries of $\Omega$ are taken to the Dirichlet, the top and bottom boundaries Neumann, and the interface $\Gamma_I$ will have a jump in the solution.
Fig. 5 Domain used for MMS solution (21). The thick red line is the interface between the two subdomains $\Omega_1$ and $\Omega_2$. The thin black lines show the finite difference block interfaces.

Table 1 Error and convergence rates using the method of manufactured solutions

| $N$         | 2nd order |       | 4th order |       | 6th order |       |
|-------------|-----------|-------|-----------|-------|-----------|-------|
|             | Error$_N$ | Rate  | Error$_N$ | Rate  | Error$_N$ | Rate  |
| $17 \times 2^0$ | $2.90 \times 10^{-4}$ |       | $1.81 \times 10^{-6}$ |       | $3.02 \times 10^{-7}$ |       |
| $17 \times 2^1$ | $7.23 \times 10^{-5}$  | 2.01  | $1.25 \times 10^{-7}$  | 3.86  | $1.10 \times 10^{-8}$  | 4.66  |
| $17 \times 2^2$ | $1.80 \times 10^{-5}$  | 2.00  | $8.32 \times 10^{-9}$  | 3.91  | $4.26 \times 10^{-10}$ | 4.81  |
| $17 \times 2^3$ | $4.51 \times 10^{-6}$  | 2.00  | $5.45 \times 10^{-10}$ | 3.93  | $1.42 \times 10^{-11}$ | 4.90  |

The manufactured solution is taken to be

$$u(x, y) = \begin{cases} \frac{e}{1+e} \left( 2 - e^{-r^2} \right) r \sin(\theta), & (x, y) \in \Omega_1, \\ (r-1)^2 \cos(\theta) + (r-1) \sin(\theta), & (x, y) \in \Omega_2, \end{cases}$$

where $r = \sqrt{x^2 + y^2}$ and $-\pi \leq \theta = \tan^{-1}(y/x) < \pi$. This solution has the property that along $\Gamma_I$ the solution $u$ is discontinuous but the weighted normal derivative $n \cdot \nabla u$ is continuous. The boundary, jump, and forcing data are found by using (21) in governing Eq. (5).

The test is run with the block decomposition shown in Fig. 5. Each block uses an $(N + 1) \times (N + 1)$ grid of points where $N$ will be increased with grid refinement. The error is measured using the discrete norm

$$\text{error}_N = \sqrt{\sum_{b=1}^{N_b} \tilde{\Delta}_b^T \tilde{J}_b (H \otimes H) \tilde{\Delta}_b},$$

where $\tilde{J}_b$ is the diagonal matrix of Jacobian determinants for block $b$ and $u(\tilde{x}_b, \tilde{y}_b)$ is the exact solution (21) evaluated at the grid points of block $b$. Table 1 shows the error and convergence rate estimates with increasing $N$ for $2p = 2, 4, 6$, and reflect global convergence rates of 2, 4, and 5, respectively.

As a final numerical result, the accuracy of the weighted normal derivative along the interface is considered. Using the same problem setup as above, the weighted interface derivative are taken to be the penalty term $\tilde{\sigma}$ computed using (11); by construction (17) implies that the normal derivative is equal and magnitude and opposite in sign across the interface. The error in the normal derivative is defined to be

$$\tilde{\sigma}$$ Springer
Table 2  Error and convergence rates using the method of manufactured solutions for the interface normal derivative

| $N$     | 2nd order | 4th order | 6th order |
|---------|-----------|-----------|-----------|
|         | Interface error | Rate | Interface error | Rate | Interface error | Rate |
| $17 \times 2^0$ | $4.93 \times 10^{-3}$ | | $1.35 \times 10^{-4}$ | | $2.39 \times 10^{-5}$ | |
| $17 \times 2^1$ | $1.83 \times 10^{-3}$ | $1.43$ | $2.69 \times 10^{-5}$ | $2.33$ | $2.53 \times 10^{-6}$ | $3.24$ |
| $17 \times 2^2$ | $6.66 \times 10^{-4}$ | $1.46$ | $5.03 \times 10^{-6}$ | $2.42$ | $2.46 \times 10^{-7}$ | $3.37$ |
| $17 \times 2^3$ | $2.39 \times 10^{-4}$ | $1.48$ | $9.16 \times 10^{-7}$ | $2.46$ | $2.28 \times 10^{-8}$ | $3.43$ |

Fig. 6  Spy plots showing the sparsity pattern for various systems related to the mesh in Fig. 5 with SBP order 6 and $N = 17 \times 2^2$. The red lines in Subfigures (a) and (c) denote the diagonal submatrices associated with a single finite difference block and for the monolithic system the connections between the trace and volume variables, $\bar{F}$ in (7).
Table 3 Comparison of the number of volume and trace points for the mesh shown in Fig. 5 with the mesh sizes of Table 1

| $N$    | $N_p^{(\text{vol})}$ | $N_p^{(\text{tr})}$ | $N_p^{(\text{vol})} / N_p^{(\text{tr})}$ |
|--------|----------------------|---------------------|-----------------------------------------|
| $17 \times 2^0$ | 18,144               | 1728                | 10.5                                     |
| $17 \times 2^1$ | 68,600               | 3360                | 20.4                                     |
| $17 \times 2^2$ | 266,616              | 6624                | 40.3                                     |
| $17 \times 2^3$ | 1,051,064            | 13,152              | 79.9                                     |

interface error $N_p = \sqrt{\sum_{f \in F_I} \Delta_f^T S_{f,f} H \Delta_f}$,

$\tilde{\Delta}_f = \hat{\sigma}_f - \sigma (x_f, y_f)$.

The results of this are show in Table 2. As can be seen, the interface derivative converges at a rate of the $p + 1/2$. Since the boundary derivative operators only have accuracy of $p$ a reduced convergence rate is expected.

To highlight the sparsity and reduction of system size we consider spy plots in Fig. 6 for the following matrices: the four matrices in the monolithic system with both volume and trace variables (7), a single finite difference block from the monolithic system, the Schur complement matrix obtained by removing the volume variables (8), and the Schur complement matrix obtained by removing the trace variables (20). Additionally, Table 3 gives the number of volume and trace points for each $N$ are given. If $N_b$ is the number of blocks and $N_I$ the number of internal interfaces, the number of volume and trace points are

$N_p^{(\text{vol})} = (N + 1)^2 N_b$,

$N_p^{(\text{tr})} = (N + 1) N_I$,

respectively; the mesh in Fig. 5 has $N_b = 56$ blocks and $N_I = 96$ internal interfaces.

6 Conclusions

We have developed a hybridized, summation-by-parts finite difference method for elliptic PDEs, where boundary and interface conditions are enforced weakly through the simultaneous-approximation-term method. The hybridization defines a global and local problem, which through the Schur complement, results in a linear system with reduced size. We proved positive-definiteness of both the local and global problems with arbitrarily heterogeneous material properties. The theoretical results were corroborated through numerical experiments and showed convergence to an exact solution at the expected rate.

All of the results in Sect. 5 used sparse direct solves; the sparse Cholesky factorization within Julia is CHOLMOD [4]. Even though the trace variable system is smaller and has fewer non-zero entries, it is still possible that this system may be harder to solve in general than either the volume variable system or the monolithic system. If direct methods are used, comparisons with other direct methods, such as nested dissection, e.g., [7,11], should be considered. Similarly, if iterative methods are to be used, there is the possibility of using direct methods for the local problems and iterative methods for the trace system. To do this efficiently will require the development of preconditioners for the trace system. Since the discrete scheme closely resembles the hybridized discontinuous Galerkin interior penalty method [6, IP-H], problems which can be handled with IP-H maybe amenable to the presented...
hybridized SBP scheme. Iterative methods could also be explored for the local problem, such as SBP-SAT specific geometric multigrid techniques [27].

Another avenue for future work is construction of hybridized SBP schemes that more closely resemble other hybridizable discontinuous Galerkin methods. The scheme may also be applicable to nonlinear elliptic problems, the challenge here would be efficient methods for solving the local problems, as in the non-linear context direct solves may be inefficient. In recent years non-conforming coupling of SBP schemes has been of particular interest [15,20,31], and could be explored in the hybridized context. One challenge which may arise is the need for a single trace variable, which maybe challenging for most of the present non-conforming SBP formulations.

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**Data availability** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**Code availability** The Julia based computer codes used that support the findings of this study are available in the github repository https://github.com/bfam/HybridSBP.

**A Proofs of Key Results**

To simplify the presentation of the results, the proofs of the key results in the paper are given here in the appendix.

**A.1 Proof of Theorem 1 (Symmetric Positive Definiteness of the Local Problem)**

Here we provide conditions that ensure that the local problem is symmetric positive definite. To do this we need a few auxiliary lemmas.

First we assume that the operators \( \tilde{A}_{rr}^{(crr)} \) and \( \tilde{A}_{ss}^{(css)} \) are compatible with the first derivative (volume) operator \( D \) in the sense of Mattsson [19, Definition 2.4]:

**Assumption 1** *(Remainder assumption)* The matrices \( \tilde{A}_{rr}^{(crr)} \) and \( \tilde{A}_{ss}^{(css)} \) satisfy the following remainder equalities:

\[
\tilde{A}_{rr}^{(crr)} = \left( I \otimes D^T \right) \tilde{C}_{rr} (H \otimes H) (I \otimes D) + \tilde{R}_{rr}^{(crr)},
\]

\[
\tilde{A}_{ss}^{(css)} = \left( D^T \otimes I \right) \tilde{C}_{ss} (H \otimes H) (D \otimes I) + \tilde{R}_{ss}^{(css)},
\]

where \( \tilde{R}_{rr}^{(crr)} \) and \( \tilde{R}_{ss}^{(css)} \) are symmetric positive semidefinite matrices and that

\[
\tilde{1} \in \text{null} \left( \tilde{A}_{rr}^{(crr)} \right), \quad \tilde{1} \in \text{null} \left( \tilde{A}_{ss}^{(css)} \right).
\]
Table 4 Borrowing Lemma parameters $l$ and $\beta$ for operators used in this work [30, Table 1]

|       | 2nd order | 4th order | 6th order |
|-------|-----------|-----------|-----------|
| $l$   | 2         | 4         | 6         |
| $\beta$ | 0.363636363 | 0.2505765857 | 0.1878687080 |

The assumption on the nullspace was not a part of the original assumption of from Mattsson [19], but is reasonable for a consistent approximation of the second derivative. The operators used in Sect. 5 satisfy the Remainder Assumption [19].

We also utilize the following lemma from Virta and Mattsson [30, Lemma 3] which relates the $\tilde{A}_{rr}^{(crr)}$ and $\tilde{A}_{ss}^{(css)}$ to boundary derivative operators $d_0$ and $d_N$:

**Lemma 1** (Borrowing Lemma) The matrices $\tilde{A}_{rr}^{(crr)}$ and $\tilde{A}_{ss}^{(css)}$ satisfy the following borrowing equalities:

\[
\tilde{A}_{rr}^{(crr)} = h \beta \left( I \otimes d_0 \right) H C_{rr}^0 \left( I \otimes d_0^T \right) + h \beta \left( I \otimes d_N \right) H C_{rr}^N \left( I \otimes d_N^T \right) + \tilde{A}_{rr}^{(crr)} ,
\]

\[
\tilde{A}_{ss}^{(css)} = h \beta \left( d_0 \otimes I \right) H C_{ss}^0 \left( d_0^T \otimes I \right) + h \beta \left( d_N \otimes I \right) H C_{ss}^N \left( d_N^T \otimes I \right) + \tilde{A}_{ss}^{(css)} .
\]

Here $\tilde{A}_{rr}^{(crr)}$ and $\tilde{A}_{ss}^{(css)}$ are symmetric positive semidefinite matrices and the parameter $\beta$ depends on the order of the operators but is independent of $N$. The diagonal matrices $C_{rr}^0$, $C_{rr}^N$, $C_{ss}^0$, and $C_{ss}^N$ have nonzero elements:

\[
\left[ C_{rr}^0 \right]_{jj} = \min_{k=0,...,l} \{ c_{rr} \}_{kj} , \quad \left[ C_{rr}^N \right]_{jj} = \min_{k=N-l,...,N} \{ c_{rr} \}_{kj} ,
\]

\[
\left[ C_{ss}^0 \right]_{ii} = \min_{k=0,...,l} \{ c_{ss} \}_{ik} , \quad \left[ C_{ss}^N \right]_{ii} = \min_{k=N-l,...,N} \{ c_{ss} \}_{ik} ,
\]

where $l$ is a parameter that depends on the order of the scheme and the notation $\{ \cdot \}_{ij}$ denotes that the grid function inside the bracket is evaluated at grid point $i, j$.

The values of $\beta$ and $l$ used in the Borrowing Lemma (Lemma 1) for the operators used in this work are given in Table 4.

We additionally make the following linearity assumption (which the operators we use satisfy) concerning the operators’ dependence on the variable coefficients and an assumption concerning the symmetric positive definiteness of the variable coefficient matrix at each grid point.

**Assumption 2** The matrices $\tilde{A}_{rr}^{(crr)}$ and $\tilde{A}_{ss}^{(css)}$ depend linearly on the coefficient grid functions $c_{rr}$ and $c_{ss}$ so that they can be decomposed as

\[
\tilde{A}_{rr}^{(crr)} = \tilde{A}_{rr}^{(crr-\delta)} + \tilde{A}_{rr}^{(\delta)} ,
\]

\[
\tilde{A}_{ss}^{(css)} = \tilde{A}_{ss}^{(css-\delta)} + \tilde{A}_{ss}^{(\delta)} ,
\]

where $\delta$ is a grid function.

**Assumption 3** At every grid point the grid functions $c_{rr}$, $c_{ss}$, and $c_{rs} = c_{sr}$ satisfy

\[
c_{rr} > 0, \quad c_{ss} > 0, \quad c_{rr}c_{ss} > c_{rs}^2 .
\]
which implies that the matrix
\[ C = \begin{bmatrix} c_{rr} & c_{rs} \\ c_{rs} & c_{ss} \end{bmatrix} \]
is symmetric positive definite with eigenvalues
\[
\psi_{\text{max}} = \frac{1}{2} \left( c_{rr} + c_{ss} + \sqrt{(c_{rr} - c_{ss})^2 + 4c_{rs}^2} \right), \\
\psi_{\text{min}} = \frac{1}{2} \left( c_{rr} + c_{ss} - \sqrt{(c_{rr} - c_{ss})^2 + 4c_{rs}^2} \right). 
\]
(23)

We now state the following lemma which allows us to separate \( \tilde{A} \) into three symmetric positive definite matrices by peeling off \( \psi_{\text{min}} \) at every grid point.

**Lemma 2** The matrix \( \tilde{A} \), defined by (12b), can be written in the form
\[
\tilde{A} = \tilde{A}_{rr}^{(\psi_{\text{min}})} + \tilde{A}_{rs}^{(\psi_{\text{min}})} + \tilde{A}_{ss}^{(\psi_{\text{min}})},
\]
where \( \tilde{A}, \tilde{A}_{rr}^{(\psi_{\text{min}})}, \) and \( \tilde{A}_{ss}^{(\psi_{\text{min}})} \) are symmetric positive semidefinite matrices. Here \( \psi_{\text{min}} \) is the grid function defined by (23). Furthermore the nullspace of \( \hat{A} \) is \( \text{null}(\hat{A}) = \text{span}(\mathbf{1}) \), where \( \mathbf{1} \) is the vector of ones.

**Proof** By Assumption 2 we can write
\[
\tilde{A} = \tilde{A}_{rr}^{(c_{rr} - \psi_{\text{min}})} + \tilde{A}_{ss}^{(c_{ss} - \psi_{\text{min}})} + \tilde{A}_{rs}^{(c_{rs})} + \tilde{A}_{sr}^{(c_{rs})} + \tilde{A}_{rr}^{(\psi_{\text{min}})} + \tilde{A}_{ss}^{(\psi_{\text{min}})}.
\]
The matrix
\[
\tilde{A} = \tilde{A}_{rr}^{(c_{rr} - \psi_{\text{min}})} + \tilde{A}_{ss}^{(c_{ss} - \psi_{\text{min}})} + \tilde{A}_{rs}^{(c_{rs})} + \tilde{A}_{sr}^{(c_{sr})}
\]
is clearly symmetric by construction. To show that the matrix is positive semidefinite we note that
\[
\tilde{u}^T \tilde{A}_{rr}^{(c_{rr} - \psi_{\text{min}})} \tilde{u} \geq \tilde{u}^T (H \otimes H) (\tilde{C}_{rr} - \tilde{\psi}_{\text{min}}) \tilde{u}, 
\]
(24)
\[
\tilde{u}^T \tilde{A}_{ss}^{(c_{ss} - \psi_{\text{min}})} \tilde{u} \geq \tilde{u}^T (H \otimes H) (\tilde{C}_{ss} - \tilde{\psi}_{\text{min}}) \tilde{u}, 
\]
(25)
\[
\tilde{u}^T \tilde{A}_{rs}^{(c_{rs})} \tilde{u} = \tilde{u}^T (H \otimes H) \tilde{C}_{rs} \tilde{u}.
\]
(26)

Here we have defined the vectors \( \tilde{u}_r = (I \otimes D) \tilde{u} \) and \( \tilde{u}_s = (D \otimes I) \tilde{u} \). Inequalities (24) and (25) follow from the Remainder Assumption and equality (26) follows from (2) and the symmetry assumption \( (c_{rs} = c_{sr}) \). Using relationships (24)–(26) we have that
\[
\tilde{u}^T \tilde{A} \tilde{u} \geq \sum_{i=0}^{N} \sum_{j=0}^{N} \{(H \otimes H)\}_{ij} \begin{bmatrix} u_r^T & u_s^T \end{bmatrix} \begin{bmatrix} c_{rr} - \psi_{\text{min}} & c_{rs} \\ c_{rs} & c_{ss} - \psi_{\text{min}} \end{bmatrix} \begin{bmatrix} u_r \\ u_s \end{bmatrix}, 
\]
(27)

where the notation \( \{(\cdot)\}_{i,j} \) denotes that the grid function inside the brackets is evaluated at grid point \( i, j \). The \( 2 \times 2 \) matrix in (27) is the shift of the matrix \( C \) by its minimum eigenvalue, thus by Assumption 3 is symmetric positive semidefinite. It then follows that each term in the summation is non-negative and the matrix \( \tilde{A} \) is symmetric positive semidefinite.

The matrices \( \tilde{A}_{rr}^{(\psi_{\text{min}})} \) and \( \tilde{A}_{ss}^{(\psi_{\text{min}})} \) are clearly symmetric by construction, with positive semidefiniteness following from the positivity of \( \psi_{\text{min}} \) and the Remainder Assumption.
We now show that \( \text{null}(\hat{\mathcal{A}}) = \text{span}\{\mathbf{I}\} \). For the right-hand side of (27) to be zero it is required that \((u_r)_{i,j} = (u_s)_{i,j} = 0\) for all \(i, j\). The only way for this to happen is if \(\tilde{u} = \alpha \mathbf{I}\) for some constant \(\alpha\). Thus we have shown that \(\text{null}(\hat{\mathcal{A}}) \subseteq \text{span}\{\mathbf{I}\}\). To show equality we note that by Assumption 1 and the structure of \(\hat{\mathcal{A}}^{(C_s)}_{rs}\) and \(\hat{\mathcal{A}}^{(C_r)}_{sr}\) given in (2), the constant vector \(\mathbf{I} \in \text{null}(\hat{\mathcal{A}})\). Together the above two results imply that \(\text{null}(\hat{\mathcal{A}}) = \text{span}\{\mathbf{I}\}\). \(\square\)

We now state the following lemma concerning \(\hat{\mathcal{A}}^{(\psi_{\text{min}})}_{rr}\) and \(\hat{\mathcal{A}}^{(\psi_{\text{min}})}_{ss}\) which combine the Remainder Assumption and the Borrowing Lemma to provide terms that can be used to bound indefinite terms in the local operator \(\hat{\mathcal{M}}\).

Lemma 3 The matrices \(\hat{\mathcal{A}}^{(\psi_{\text{min}})}_{rr}\) and \(\hat{\mathcal{A}}^{(\psi_{\text{min}})}_{ss}\) satisfy the following inequalities:

\[
\begin{align*}
\tilde{u}^T \hat{\mathcal{A}}^{(\psi_{\text{min}})}_{rr} \tilde{u} \geq & \quad \frac{1}{2} \left[ h \beta (v^0_r)^T H \Psi_{\text{min}}^0 v^0_r + h \beta (v^N_r)^T H \Psi_{\text{min}}^N v^N_r \right] \\
+ & \quad \frac{1}{2} \left[ h \alpha (w^0_r)^T H \Psi_{\text{min}}^0 w^0_r + h \alpha (w^N_r)^T H \Psi_{\text{min}}^N w^N_r \right],
\end{align*}
\]

\[
\begin{align*}
\tilde{u}^T \hat{\mathcal{A}}^{(\psi_{\text{min}})}_{ss} \tilde{u} \geq & \quad \frac{1}{2} \left[ h \beta (v^0_s)^T H \Psi_{\text{min}}^0 v^0_s + h \beta (v^N_s)^T H \Psi_{\text{min}}^N v^N_s \right] \\
+ & \quad \frac{1}{2} \left[ h \alpha (w^0_s)^T H \Psi_{\text{min}}^0 w^0_s + h \alpha (w^N_s)^T H \Psi_{\text{min}}^N w^N_s \right],
\end{align*}
\]

with \(\alpha = \min \{H_{00}, H_{NN}\} / h\), i.e., the unscaled corner value in the \(H\)-matrix, and the (boundary) derivative vectors are defined as

\[
\begin{align*}
v^0_r & = (I \otimes d^T_0) \tilde{u}, & v^N_r & = (I \otimes d^T_N) \tilde{u}, \\
w^0_r & = (e^T_0 \otimes D) \tilde{u}, & w^N_r & = (e^T_N \otimes D) \tilde{u}, \\
w^0_s & = (D \otimes e^T_0) \tilde{u}, & w^N_s & = (D \otimes e^T_N) \tilde{u}, \\
v^0_s & = (d^T_0 \otimes I) \tilde{u}, & v^N_s & = (d^T_N \otimes I) \tilde{u}.
\end{align*}
\]

The diagonal matrices \(\tilde{\Psi}^{0}_{\text{min}}, \tilde{\Psi}^{N}_{\text{min}}, \tilde{\Psi}^{0}_{\text{min}}, \) and \(\tilde{\Psi}^{N}_{\text{min}}\) are defined by (22) using \(\psi_{\text{min}}\).

Proof We will prove the relationship for \(\hat{\mathcal{A}}^{(\psi_{\text{min}})}_{rr}\), and the proof \(\hat{\mathcal{A}}^{(\psi_{\text{min}})}_{ss}\) is analogous. First we note that by the Borrowing Lemma it immediately follows that

\[
\tilde{u}^T \hat{\mathcal{A}}^{(\psi_{\text{min}})}_{rr} \tilde{u} \geq h \beta (v^0_r)^T H \Psi_{\text{min}}^0 v^0_r + h \beta (v^N_r)^T H \Psi_{\text{min}}^N v^N_r.
\]  

(28)

Additionally by the Remainder Assumption it follows that

\[
\tilde{u}^T \hat{\mathcal{A}}^{(\psi_{\text{min}})}_{rr} \tilde{u} \geq \tilde{u}^T (I \otimes D^T) (H \otimes H) \tilde{\Psi}_{\text{min}} (I \otimes D) \tilde{u}
\]

\[
= \sum_{j=0}^{N} (H)_{jj} \tilde{u}^T (e_j \otimes D^T) H \tilde{\Psi}_{\text{min}}^j (e_j^T \otimes D) \tilde{u}
\]

(29)

\[
= \alpha h (w^0_r)^T H \tilde{\Psi}_{\text{min}}^0 (w^0_r)^T + \alpha h (w^N_r)^T H \tilde{\Psi}_{\text{min}}^N (w^N_r)^T;
\]

since each term of the summation is positive, the last inequality follows by dropping all but the \(j = 0\) and \(j = N\) terms of the summation. The result follows immediately by averaging (28) and (29). \(\square\)
We can now prove Theorem 1 on the symmetric positive definiteness of $\tilde{M}$ as defined by (12a).

**Proof** The structure of (12a) directly implies that $\tilde{M}$ is symmetric, in the remainder of the proof it is shown that $\tilde{M}$ is also positive definite.

We begin by recalling the definitions of $\tilde{C}_k$ and $F_k$ in (12) which allows us to write

$$\tilde{C}_k = F_kH^{-1}\tau_k^{-1}F_k^T - G_k^T H^{-1}\tau_k^{-1}G_k.$$  \hfill (30)

Now considering the $\tilde{M}$ weighted inner product we have that

$$\tilde{u}^T \tilde{M} \tilde{u} = \tilde{u}^T \left( \tilde{A} + \sum_{k=1}^{4} \tilde{C}_k \right) \tilde{u}$$

$$= \tilde{u}^T \left( \tilde{A} + \sum_{k=1}^{4} F_kH^{-1}\tau_k^{-1}F_k^T \right) \tilde{u}$$

$$+ \tilde{u}^T \left( \tilde{A}_{rr}^{(\psi_{\min})} + \tilde{A}_{ss}^{(\psi_{\min})} - \sum_{k=1}^{4} G_k^T H^{-1}\tau_k^{-1}G_k \right) \tilde{u}.$$  \hfill (31)

Here we have used Lemma 2 to split $\tilde{A}$.

If $\tau_k > 0$ then it follows for all $\tilde{u}$ that

$$\tilde{u}^T F_kH^{-1}\tau_k^{-1}F_k^T \tilde{u} \geq 0.$$  

Additionally, if $\tilde{u} = c\tilde{1}$ for some constant $c \neq 0$ then it is a strict inequality since

$$F_k^T \tilde{1} = -H \tau_k \tilde{1} \neq 0.$$  

Since by Lemma 2 the matrix $\tilde{A}$ is symmetric positive semidefinite with null($\tilde{A}$) = span($\tilde{1}$), this implies that the matrix

$$\tilde{A} + \sum_{k=1}^{4} F_kH^{-1}\tau_k^{-1}F_k^T > 0,$$

that is the matrix is positive definite. To complete the proof all that remains is to show the remaining matrix in (31) is positive semidefinite, namely

$$\tilde{A}_{rr}^{(\psi_{\min})} + \tilde{A}_{ss}^{(\psi_{\min})} - \sum_{k=1}^{4} G_k^T H^{-1}\tau_k^{-1}G_k \succeq 0.$$  

Considering the quantity $\tilde{u}^T \left( \tilde{A}_{rr}^{(\psi_{\min})} + \tilde{A}_{ss}^{(\psi_{\min})} \right) \tilde{u}$ and using Lemma 3 we can write:

$$\tilde{u}^T \left( \tilde{A}_{rr}^{(\psi_{\min})} + \tilde{A}_{ss}^{(\psi_{\min})} \right) \tilde{u}$$

$$\geq \frac{1}{2} \left( h \beta \left( v_r^0 \right)^T H \Psi_{\min}^0 v_r^0 + h \alpha \left( w_s^0 \right)^T H \Psi_{\min}^0 w_s^0 \right)$$

$$+ \frac{1}{2} \left( h \beta \left( v_r^N \right)^T H \Psi_{\min}^N v_r^N + h \alpha \left( w_s^N \right)^T H \Psi_{\min}^N w_s^N \right)$$

$$+ \frac{1}{2} \left( h \alpha \left( w_r^0 \right)^T H \Psi_{\min}^0 w_r^0 + h \beta \left( v_s^0 \right)^T H \Psi_{\min}^0 v_s^0 \right)$$

$$+ \frac{1}{2} \left( h \alpha \left( w_r^N \right)^T H \Psi_{\min}^N w_r^N + h \beta \left( v_s^N \right)^T H \Psi_{\min}^N v_s^N \right).$$  \hfill (32)
Now considering the $k = 1$ term of the last summation in (31) we have

$$
\tilde{u}^T G_1^T H^{-1} \tau_1^{-1} G_1 \tilde{u} = (C_{rr}^0 v_r^0 + C_{rs}^0 w_s^0)^T H \tau_1^{-1} (C_{rr}^0 v_r^0 + C_{rs}^0 w_s^0).
$$

We now need to use the positive term related to face 1 of (32) to bound the negative contribution from (33). Doing this subtraction for face 1 then gives:

$$
\frac{1}{2} \left( h \beta (v_r^0)^T H \Psi_{\min}^0 v_r^0 + h \alpha (w_s^0)^T H \Psi_{\min}^0 w_s^0 \right)
- (C_{rr}^0 v_r^0 + C_{rs}^0 w_s^0)^T H \tau_1^{-1} (C_{rr}^0 v_r^0 + C_{rs}^0 w_s^0)
= \left[ \begin{array}{c} v_r^0 \\ w_s^0 \end{array} \right]^T (I_{2 \times 2} \otimes H) \left[ \begin{array}{cccc} 1 & -\frac{(\hat{C}_{rr}^0)^2}{\tau_1^2} & \frac{-\hat{C}_{rr}^0 \hat{C}_{rs}^0}{\tau_1^2} \\ -\frac{\hat{C}_{rr}^0 \hat{C}_{rs}^0}{\tau_1^2} & 1 & -\frac{(\hat{C}_{rs}^0)^2}{\tau_1^2} \\ \frac{-\hat{C}_{rr}^0 \hat{C}_{rs}^0}{\tau_1^2} & -\frac{\hat{C}_{rr}^0 \hat{C}_{rs}^0}{\tau_1^2} & 1 \end{array} \right] \left[ \begin{array}{c} v_r^0 \\ w_s^0 \end{array} \right].
$$

In the above calculation we have used the fact that $H$, $\tau_1$, $C_{rr}^0$, and $C_{rs}^0$ are diagonal as well as made the following definitions:

$$
\hat{v}_r^0 = v_r^0 \sqrt{\frac{1}{2} h \beta \Psi_{\min}^0}, \quad \hat{C}_{rr}^0 = C_{rr}^0 \sqrt{\frac{2}{h \beta \Psi_{\min}^0}},
$$

$$
\hat{w}_s^0 = w_s^0 \sqrt{\frac{1}{2} h \alpha \Psi_{\min}^0}, \quad \hat{C}_{rs}^0 = C_{rs}^0 \sqrt{\frac{2}{h \alpha \Psi_{\min}^0}}.
$$

The eigenvalues of the matrix in (34) are:

$$
\mu_1 = 1, \quad \mu_2 = 1 - \frac{(\hat{C}_{rr}^0)^2 + (\hat{C}_{rs}^0)^2}{\tau_1^2}.
$$

The first eigenvalue $\mu_1$ is clearly positive and $\mu_2$ will be positive if:

$$
\tau_1^j > (\hat{C}_{rr}^0)^2 + (\hat{C}_{rs}^0)^2 = \frac{2(C_{rr}^0)^2}{h \beta \Psi_{\min}^0} + \frac{2(C_{rs}^0)^2}{h \alpha \Psi_{\min}^0}.
$$

With such a definition of $\tau_1$ all the terms in (34) are positive and thus for face 1 the terms in (31) are positive. An identical argument holds for the other faces if:

$$
\tau_2^j > \frac{2(C_{rr}^{Nj})^2}{h \beta \Psi_{\min}^{Nj}} + \frac{2(C_{rs}^{Nj})^2}{h \alpha \Psi_{\min}^{Nj}},
$$

$$
\tau_3^j > \frac{2(C_{rr}^{0j})^2}{h \beta \Psi_{\min}^{0j}} + \frac{2(C_{rs}^{0j})^2}{h \alpha \Psi_{\min}^{0j}},
$$

$$
\tau_4^j > \frac{2(C_{rr}^{iN})^2}{h \beta \Psi_{\min}^{iN}} + \frac{2(C_{rs}^{iN})^2}{h \alpha \Psi_{\min}^{iN}}.
$$
and thus $\tilde{M}$ is positive definite since $\tilde{u}^T \tilde{M} \tilde{u} > 0$ for all $\tilde{u} \neq \tilde{0}$. □

A.2 Proof of Theorem 2 (Positive Definiteness of the Local Problem with Neumann Boundary Conditions)

Here we prove Theorem 2 on the symmetric positive definiteness of $\tilde{M}$ with Neumann boundary conditions.

Proof We begin by considering

$$\tilde{u}^T \tilde{M} \tilde{u} = \tilde{u}^T \left( \tilde{A} + \tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2 + \tilde{\mathcal{C}}_3 + \tilde{\mathcal{C}}_4 \right) \tilde{u},$$

where we define the modified surface matrices $\tilde{\mathcal{C}}_k$ to be

$$\tilde{\mathcal{C}}_k = \mathcal{C}_k - F_k H^{-1} \tau_k^{-1} F_k^T = - G_k^T H^{-1} \tau_k^{-1} G_k,$$

if face $k$ is a Neumann boundary and $\tilde{\mathcal{C}}_k = \mathcal{C}_k$ otherwise; see the definition of the modified $\tilde{M}$ with Neumann boundary conditions (16) and (30). In the proof of Theorem 1 it was shown that terms of the form of (36) combine with $A$ is a way that is non-negative if $\tau_k$ satisfy (35); see (32) and following. Thus $\tilde{u}^T \tilde{M} \tilde{u} \geq 0$ for all $\tilde{u}$. The inequality will be strict for $\tilde{u} \neq \tilde{0}$ as long as one face is Dirichlet; the argument is that same as that made in the proof of Theorem 1. □

A.3 Proof of Theorem 3 and Corollary 2 (Positive Definiteness of the Global Problem)

Proof of Theorem 3 Without loss of generality, we consider a two block mesh with Dirichlet boundary conditions with a single face $f \in \mathcal{F}_F$ and assume that it is connected to face $k^+$ of block $B^+$ and face $k^-$ of block $B^-$. Solving for $\lambda_f$ in the global coupling Eq. (18) in terms of $\tilde{u}_{B^+}$ and $\tilde{u}_{B^-}$ gives

$$\lambda_f = D_{f}\left( \frac{1}{2} H \left( \tau_{f,B^+} - \tau_{f,B^-} \right) \delta_f - F_{f,B^+}^T \tilde{u}_{B^+} - F_{f,B^-}^T \tilde{u}_{B^-} \right).$$

Plugging this expression into the local problem (12), gives

$$\begin{align*}
&\left( \tilde{A}_{B^+} - F_{f,B^+} D_f^{-1} F_{f,B^+} + \sum_{k=1}^4 \tilde{\mathcal{C}}_{k,B^+} \right) \tilde{u}_{B^+} \\
&- F_{f,B^+} D_f^{-1} F_{f,B^-} \tilde{u}_{B^-} = \tilde{q}_{B^+ \setminus f}, \\
&\left( \tilde{A}_{B^-} - F_{f,B^-} D_f^{-1} F_{f,B^-} + \sum_{k=1}^4 \tilde{\mathcal{C}}_{k,B^-} \right) \tilde{u}_{B^-} \\
&- F_{f,B^-} D_f^{-1} F_{f,B^+} \tilde{u}_{B^+} = \tilde{q}_{B^- \setminus f}.
\end{align*}$$

(37)

Here $\tilde{q}_{B^+ \setminus f}$ denotes $\tilde{q}_{B^+}$ (see (12d)) with the term dependent on $\tilde{u}$ associated with face $f$ removed. Using (30) which relates $\tilde{\mathcal{C}}_{f,B^\pm}$ to $F_{f,B^\pm}$ we have that

$$\begin{align*}
\tilde{\mathcal{C}}_{f,B^\pm} &= F_{f,B^\pm} D_f^{-1} F_{f,B^\pm} H^{-1} \left( \tau_{f,B^\pm}^{-1} - \left( \tau_{f,B^\pm} + \tau_{f,B^-} \right)^{-1} \right) F_{f,B^\pm}^T \\
&- G_{k,B^\pm}^T H^{-1} \tau_{f,B^-} G_{k,B^\pm}.
\end{align*}$$

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Plugging this into (37), and rewriting the two equations as single system gives:

\[
\begin{pmatrix}
A + F T T^T
\end{pmatrix}
\begin{pmatrix}
\hat{u}^+ \\
\hat{u}^-
\end{pmatrix}
= \begin{pmatrix}
\hat{q}^+ \backslash f \\
\hat{q}^- \backslash f
\end{pmatrix},
\]

where we have defined the following matrices:

\[
F = \begin{bmatrix}
H_1^{1/2} F_{f,B^+} & 0 \\
0 & H_1^{1/2} F_{f,B^-}
\end{bmatrix},
\]

\[
T = \begin{bmatrix}
\tau_{f,B^+}^{-1} - (\tau_{f,B^+} + \tau_{f,B^-})^{-1} & - (\tau_{f,B^+} + \tau_{f,B^-})^{-1} \\
-(\tau_{f,B^+} + \tau_{f,B^-})^{-1} & \tau_{f,B^-}^{-1} - (\tau_{f,B^+} + \tau_{f,B^-})^{-1}
\end{bmatrix},
\]

\[
\bar{A} = \begin{bmatrix}
\bar{A}^+ & 0 \\
0 & \bar{A}^-
\end{bmatrix},
\]

\[
\bar{A}^{\pm} = \bar{A}_{B^\pm} - G_{k,B^\pm}^T H_{k,B^\pm}^{-1} \tau_{f,B^\pm}^{-1} G_{k,B^\pm} + \sum_{k=1}^{4} \bar{C}_{k,B^\pm}.
\]

The matrix \(\bar{A}\) is block diagonal, and each of the blocks was shown in the proof of Theorem 1 to be symmetric positive semidefinite. Thus, if \(T\) is symmetric positive semidefinite, then the whole system is symmetric positive semidefinite. Since \(\tau_{f,B^\pm}\) are diagonal, the eigenvalues \(T\) are the same as the eigenvalues of the \(2 \times 2\) systems

\[
T^j = \begin{bmatrix}
\frac{1}{\tau_{f,B^+}^j} & \frac{1}{\tau_{f,B^-}^j} & 0 & 0 \\
0 & 0 & \frac{1}{\tau_{f,B^+}^j} & \frac{1}{\tau_{f,B^-}^j}
\end{bmatrix},
\]

for each \(j = 0\) to \(N_f\) (number of points on the face). The eigenvalues of \(T^j\) are

\[
\mu_1 = 0, \quad \mu_2 = \frac{\tau_{f,B^+}^2 + \tau_{f,B^-}^2}{\tau_{f,B^+} + \tau_{f,B^-}}.
\]

which shows that \(T^j\) and that \(T\) are positive semidefinite as long as \(\tau_{f,B^\pm} > 0\).

An identical argument holds for each interface \(f \in \mathcal{F}\), thus the interface treatment guarantees the global system of equations is symmetric positive semidefinite. Positive definiteness results as long as one of the faces of the mesh is a Dirichlet boundary since only the constant state over the entire domain is in the null(\(\bar{A}_{B}\)) for all \(B \in \mathcal{B}\) and this is removed as long as some face of the mesh has a Dirichlet boundary condition; see proof of Theorem 1.

**Proof of Corollary 2** Begin by noting that

\[
\begin{pmatrix}
\bar{M} & \bar{F} \\
\bar{F}^T & \bar{D}
\end{pmatrix}
= \begin{bmatrix}
\bar{I} & \bar{F} \bar{D}^{-1} \\
0 & \bar{I}
\end{bmatrix}
\begin{pmatrix}
\bar{M} - \bar{F} \bar{D}^{-1} \bar{F}^T & 0 \\
0 & \bar{D}
\end{pmatrix}
\begin{bmatrix}
\bar{I} & 0 \\
0 & \bar{I}
\end{bmatrix}.
\]

By Theorem 3 and structure of \(\bar{D}\) the block diagonal center matrix is symmetric positive definite. Since the outer two matrices are the transposes of one another, it immediately follows that the global system matrix is symmetric positive definite.
Since the global system matrix and $\bar{M}$ are symmetric positive definite, symmetric positive definiteness of the Schur complement of the $\bar{M}$ block follows directly from the decomposition

$$
\begin{bmatrix}
\bar{M} & \bar{F} \\
\bar{F}^T & D
\end{bmatrix} =
\begin{bmatrix}
\tilde{I} & 0 \\
0 & \bar{D}
\end{bmatrix}
\begin{bmatrix}
\bar{M} & 0 \\
0 & \bar{D} - \bar{F}^T \bar{M}^{-1} \bar{F}
\end{bmatrix}
\begin{bmatrix}
\tilde{I} & \bar{M}^{-1} \bar{F} \\
0 & \tilde{I}
\end{bmatrix}.
$$

\[\square\]

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