A note on $G_q$-summability of formal solutions of some linear $q$-difference-differential equations

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Abstract

Let $q > 1$ and $\delta > 0$. For a function $f(t, z)$, the $q$-shift operator $\sigma_q$ in $t$ is defined by $\sigma_q(f)(t, z) = f(qt, z)$. This article discusses a linear $q$-difference-differential equation $\sum_{j+\delta|\alpha| \leq m} a_{j,\alpha}(t, z)(\sigma_q)^j \partial_{\alpha}^X = F(t, z)$ in the complex domain, and shows a result on the $G_q$-summability of formal solutions (which may be divergent) in the framework of $q$-Laplace and $q$-Borel transforms by Ramis-Zhang.

Key words and phrases: $q$-difference-differential equations, summability, formal power series solutions, $q$-Gevrey asymptotic expansions.

2010 Mathematics Subject Classification Numbers: Primary 35C20; Secondary 35A01, 39A13.

1 Introduction

Let $(t, z)$ be the variable in $\mathbb{C}_t \times \mathbb{C}_z^d$. Let $q > 1$. For a function $f(t, z)$ we define a $q$-shift operator $\sigma_q$ in $t$ by $\sigma_q(f)(t, z) = f(qt, z)$.

In this note, we consider a linear $q$-difference-differential equation

\begin{equation}
\sum_{j+\delta|\alpha| \leq m} a_{j,\alpha}(t, z)(\sigma_q)^j \partial_{\alpha}^X = F(t, z)
\end{equation}

under the following assumptions:

(1) $q > 1, \delta > 0$ and $m \in \mathbb{N}^* (= \{1, 2, \ldots\});$

(2) $a_{j,\alpha}(t, z)$ ($j + \delta|\alpha| \leq m$) and $F(t, z)$ are holomorphic functions in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_z^d$;

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The first author is supported by JSPS KAKENHI Grant Number JP15K04966.
(3) \((1.1)\) has a formal power series solution

\[
X(t, z) = \sum_{n \geq 0} X_n(z) t^n \in \mathcal{O}_R[[t]]
\]

where \(\mathcal{O}_R\) denotes the set of all holomorphic functions on \(D_R = \{z \in \mathbb{C}^d; |z_i| < R \ (i = 1, \ldots, d)\}\).

Our basic problem is:

**Problem 1.1.** Under what condition can we get a true solution \(W(t, z)\) of \((1.1)\) which admits \(\hat{X}(t, z)\) as a \(q\)-Gevrey asymptotic expansion of order 1 (in the sense of Definition 1.2 given below)?

For \(\lambda \in \mathbb{C} \setminus \{0\}\) and \(\epsilon > 0\) we set

\[
\mathcal{Z}_\lambda = \{-\lambda q^m \in \mathbb{C}; m \in \mathbb{Z}\},
\]

\[
\mathcal{Z}_{\lambda, \epsilon} = \bigcup_{m \in \mathbb{Z}} \{t \in \mathbb{C} \setminus \{0\}; |1 + \lambda q^m/t| \leq \epsilon\}.
\]

It is easy to see that if \(\epsilon > 0\) is sufficiently small the set \(\mathcal{Z}_{\lambda, \epsilon}\) is a disjoint union of closed disks. For \(r > 0\) we write \(D_r^* = \{t \in \mathbb{C}; 0 < |t| < r\}\). The following definition is due to Ramis-Zhang [8].

**Definition 1.2.** (1) Let \(\hat{X}(t, z) = \sum_{n \geq 0} X_n(z) t^n \in \mathcal{O}_R[[t]]\) and let \(W(t, z)\) be a holomorphic function on \((D_r^* \setminus \mathcal{Z}_\lambda) \times D_R\) for some \(r > 0\). We say that \(W(t, z)\) admits \(\hat{X}(t, z)\) as a \(q\)-Gevrey asymptotic expansion of order 1, if there are \(M > 0\) and \(H > 0\) such that

\[
\left| W(t, z) - \sum_{n=0}^{N-1} X_n(z) t^n \right| \leq \frac{MH^N}{\epsilon} q^{N(N-1)/2} |t|^N
\]

holds on \((D_r^* \setminus \mathcal{Z}_{\lambda, \epsilon}) \times D_R\) for any \(N = 0, 1, 2, \ldots\) and any sufficiently small \(\epsilon > 0\).

(2) If there is a \(W(t, z)\) as above, we say that the formal solution \(\hat{X}(t, z)\) is \(G_q\)-summable in the direction \(\lambda\).

A partial answer to Problem 1.1 was given in Tahara-Yamazawa [11]: in this paper, we will give an improvement of the result in [11]. As in [11], we will use the framework of \(q\)-Laplace and \(q\)-Borel transforms via Jacobi theta function, developed by Ramis-Zhang [8] and Zhang [10].

Similar problems are discussed by Zhang [9], Marotte-Zhang [5] and Ramis-Sauloy-Zhang [7] in the \(q\)-difference equations, and by Malek [3, 4], Lastra-Malek [1] and Lastra-Malek-Sanz [2] in the case of \(q\)-difference-differential equations. But, their equations are different from ours.
Main results

For a holomorphic function $f(t, z)$ in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_z^d$, we define the order of the zeros of the function $f(t, z)$ at $t = 0$ (we denote this by $\text{ord}_t(f)$) by

$$\text{ord}_t(f) = \min \{ k \in \mathbb{N} ; (\partial^k_t f)(0, z) \not\equiv 0 \text{ near } z = 0 \}$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

For $(a, b) \in \mathbb{R}^2$ we set $C(a, b) = \{(x, y) \in \mathbb{R}^2 ; x \leq a, y \geq b\}$. We define the $t$-Newton polygon $N_t(1.1)$ of equation (1.1) by

$$N_t(1.1) = \text{the convex hull of } \bigcup_{j + \delta|\alpha| \leq m} C(j, \text{ord}_t(a_{j, \alpha})),$$

and

In this note, we will consider the equation (1.1) under the following conditions (A1) and (A2):

(A1) There is an integer $m_0$ such that $0 \leq m_0 < m$ and

$$N_t(1.1) = \{(x, y) \in \mathbb{R}^2 ; x \leq m, y \geq \max\{0, x - m_0\}\}.$$

(A2) Moreover, we have

$$|\alpha| > 0 \implies (j, \text{ord}_t(a_{j, \alpha})) \in \text{int}(N_t(1.1)),$$

where $\text{int}(N_t(1.1))$ denotes the interior of the set $N_t(1.1)$ in $\mathbb{R}^2$.

The figure of $N_t(1.1)$ is as in Figure 1. In Figure 1 the boundary of $N_t(1.1)$ consists of a horizontal half-line $\Gamma_0$, a segment $\Gamma_1$ and a vertical half-line $\Gamma_2$, and $k_i$ is the slope of $\Gamma_i$ for $i = 0, 1, 2$.

Lemma 2.1. If (A1) and (A2) are satisfied, we have

(2.1) $\text{ord}_t(a_{j, \alpha}) \geq \begin{cases} \max\{0, j - m_0\}, & \text{if } |\alpha| = 0, \\ \max\{1, j - m_0 + 1\}, & \text{if } |\alpha| > 0. \end{cases}$

By the condition (2.1), we have the expression

(2.2) $a_{j, 0}(t, z) = t^{j-m_0}b_{j, 0}(t, z)$ for $m_0 < j \leq m$

for some holomorphic functions $b_{j, 0}(t, z) (m_0 < j \leq m)$ in a neighborhood of $(0, 0) \in \mathbb{C} \times \mathbb{C}_z^d$. We suppose:

(2.3) $a_{m_0, 0}(0, 0) \neq 0$ and $b_{m, 0}(0, 0) \neq 0$.

We set

(2.4) $P(\tau, z) = \sum_{m_0 < j \leq m} \frac{b_{j, 0}(0, z)}{\tau^{j-1}/2} \tau^{j-m_0} + \frac{a_{m_0, 0}(0, z)}{q^{m_0(m_0-1)/2}}$
and denote by \( \tau_1, \ldots, \tau_{m-m_0} \) the roots of \( P(\tau, 0) = 0 \). By (2.3) we have \( \tau_i \neq 0 \) for all \( i = 1, 2, \ldots, m - m_0 \). The set \( S \) of singular directions at \( z = 0 \) is defined by

\[
S = \bigcup_{i=1}^{m-m_0} \{ t = \tau_i \eta; \eta > 0 \}.
\]

In [11], we have shown the following result.

**Theorem 2.2** (Theorem 2.3 in [11]).

1. Suppose the conditions (A\(_1\)), (A\(_2\)) and (2.3). Then, if equation (1.1) has a formal solution \( \hat{X}(t, z) = \sum_{n \geq 0} X_n(z)t^n \in O_R[[t]] \), we can find \( A > 0, h > 0 \) and \( 0 < R_1 < R \) such that \( |X_n(z)| \leq Ah^nq^{(n-1)/2} \) on \( D_{R_1} \) for any \( n = 0, 1, 2, \ldots \).

2. In addition, if the condition

\[ \text{ord}_t(a_{j, \alpha}) \geq j - m_0 + 2, \quad \text{if } |\alpha| > 0 \text{ and } m_0 \leq j < m \]

is satisfied, for any \( \lambda \in \mathbb{C} \setminus (\{0\} \cup S) \) the formal solution \( \hat{X}(t, z) \) is \( G_q \)-summable in the direction \( \lambda \). In other words, there are \( r > 0, R_1 > 0 \) and a holomorphic solution \( W(t, z) \) of (1.1) on \( (D^*_r \setminus \mathcal{Z}_\lambda) \times D_{R_1} \) such that \( W(t, z) \) admits \( \hat{X}(t, z) \) as a \( q \)-Gevrey asymptotic expansion of order 1.

In this paper, we remove the additional condition (2.5) from the part (2) of Theorem 2.2. We have

**Theorem 2.3.** Suppose the conditions (A\(_1\)), (A\(_2\)) and (2.3). Then, for any \( \lambda \in \mathbb{C} \setminus (\{0\} \cup S) \) the formal solution \( \hat{X}(t, z) \) (in (1.2)) is \( G_q \)-summable in the direction \( \lambda \).
To prove this, we use the framework of \(q\)-Laplace and \(q\)-Borel transforms developed by Rramis-Zhang [8]. By (1) of Theorem 2.2 we know that the formal \(q\)-Borel transform of \(\hat{X}(t, z)\) in \(t\)

\[
(2.6) \quad u(\xi, z) = \sum_{k \geq 0} \frac{X_k(z)}{q^{k(k-1)/2}} \xi^k
\]

is convergent in a neighborhood of \((0, 0) \in \mathbb{C}_\xi \times \mathbb{C}_z\). For \(\lambda \in \mathbb{C} \setminus \{0\}\) and \(\theta > 0\) we write \(S_\theta(\lambda) = \{\xi \in \mathbb{C} \setminus \{0\}; |\arg \xi - \arg \lambda| < \theta\}\). Then, to show Theorem 2.3 it is enough to prove the following result.

**Proposition 2.4.** For any \(\lambda \in \mathbb{C} \setminus \{(0) \cup S\}\) there are \(\theta > 0\), \(R_1 > 0\), \(C > 0\) and \(H > 0\) such that \(u(\xi, z)\) has an analytic extension \(u^*(\xi, z)\) to the domain \(S_\theta(\lambda) \times D_{R_1}\) satisfying the following condition:

\[
(2.7) \quad |u^*(\lambda q^m, z)| \leq CH^m q^{m^2/2} \quad \text{on } D_{R_1}, \quad m = 0, 1, 2, \ldots.
\]

### 3 Some lemmas

Before the proof of Proposition 2.4, let us give some lemmas which are needed in the proof of Proposition 2.4.

**Lemma 3.1.** Let \(q > 1\). Let \(f(t, z)\) be a function in \((t, z)\).

1. We have \(\sigma_q(f)(t, z) = (\sigma\sqrt{q})^2(f)(t, z)\).
2. We set \(F(t, z) = f(t^2, z)\); then we have \(\sigma_q(f)(t^2, z) = \sigma\sqrt{q}(F)(t, z)\). Similarly, we have \((\sigma_q)^m(f)(t^2, z) = (\sigma\sqrt{q})^m(F)(t, z)\) for any \(m = 1, 2, \ldots\).

**Proof.** (1) is clear. (2) is verified as follows: \(\sigma_q(f)(t^2, z) = f(qt^2, z) = f((\sqrt{q}t)^2, z) = F(\sqrt{q}t, z) = \sigma\sqrt{q}(F)(t, z)\). The equality \((\sigma_q)^m(f)(t^2, z) = (\sigma\sqrt{q})^m(F)(t, z)\) can be proved in the same way.

The following result is proved in [Proposition 2.1 in 5]:

**Proposition 3.2.** Let \(\hat{f}(t) = \sum_{n \geq 0} a_n t^n \in \mathbb{C}[[t]]\). The following two conditions are equivalent:

1. There are \(A > 0\) and \(H > 0\) such that
   \[
   |a_n| \leq \frac{AH^n}{q^{n(n-1)/2}}, \quad n = 0, 1, 2, \ldots
   \]

2. \(\hat{f}(t)\) is the Taylor expansion at \(t = 0\) of an entire function \(f(t)\) satisfying the estimate
   \[
   |f(t)| \leq M \exp\left(\frac{(\log |t|)^2}{2\log q} + \alpha \log |t|\right) \quad \text{on } \mathbb{C} \setminus \{0\}
   \]

for some \(M > 0\) and \(\alpha \in \mathbb{R}\).
4 Proof of Proposition 2.4

We set \( q_1 = q^{1/4} \), replace \( t \) by \( t^2 \) in (1.1), and apply Lemma 3.1 to the equation (1.1): then (1.1) is rewritten into the form

\[
(4.1) \quad \sum_{j + \delta|\alpha| \leq m} A_{j,\alpha}(t, z)(\sigma_{q_1})^{2j} \partial_z^\alpha Y = G(t, z)
\]

where

\[
A_{j,\alpha}(t, z) = a_{j,\alpha}(t^2, z) \quad (j + \delta|\alpha| \leq m),
\]

\[
Y(t, z) = X(t^2, z) = \sum_{k \geq 0} X_k(z) t^{2k},
\]

\[
G(t, z) = F(t^2, z).
\]

We can regards (4.1) as a \( q_1 \)-difference-differential equation, and in this case, the order of the equation is \( 2m \) in \( t \). Therefore, the \( t \)-Newton polygon \( N_t(4.1) \) of (4.1) (as a \( q_1 \)-difference equation) is

\[
N_t(4.1) = \{(x, y) \in \mathbb{R}^2 ; x \leq 2m, y \geq \max\{0, x - 2m_0\}\}
\]

which is as in Figure 2.

Moreover, we have

\[
(4.2) \quad \text{ord}_t(A_{j,\alpha}) \geq \begin{cases} 
\max\{0, 2j - 2m_0\}, & \text{if } |\alpha| = 0, \\
\max\{2, 2j - 2m_0 + 2\}, & \text{if } |\alpha| > 0.
\end{cases}
\]

By (2.2), we have

\[
A_{j,0}(t, z) = t^{2j - 2m_0} B_{j,0}(t, z) \quad \text{for } m_0 < j \leq m
\]
for $B_{j,0}(t, z) = b_{j,0}(t^2, z)$ ($m_0 < j \leq m$). The set $S_1$ of singular directions of (4.1) is defined by using
\[ P_1(\rho, z) = \sum_{m_0 < j \leq m} \frac{B_{j,0}(0, z)}{q_1^{2j(2j-1)/2}} \rho^{2j-2m_0} + \frac{A_{m_0,0}(0, z)}{q_1^{2m_0(2m_0-1)/2}}. \]
Let $\rho_1, \ldots, \rho_{2m-2m_0}$ be the roots of $P_1(\rho, 0) = 0$: then $S_1$ is defined by
\[ S_1 = \bigcup_{i=1}^{2m-2m_0} \{ t = \rho_i \eta; \eta > 0 \}. \]
Let $u_1(\xi, x)$ be the $q_1$-formal Borel transform of $Y(t, x)$, that is,
\[ u_1(\xi, z) = \sum_{k \geq 0} \frac{X_k(z)}{q_1^{2k(2k-1)/2}} \xi^{2k}. \]
Since $q_1 = q^{1/4}$ we can easily see:
\begin{align*}
(4.3) \quad u_1(\xi, z) &= u(q^{-1/4}\xi^2, z), \\
(4.4) \quad P_1(\lambda, z) &= q^{-m_0/4}P(q^{-1/4}\lambda^2, z),
\end{align*}
where $u(\xi, z)$ and $P(\tau, z)$ are the ones in (2.0) and (2.3), respectively.

By (1.3) we see that $u_1(\xi, z)$ is convergent in a neighborhood of $(\xi, z) = (0, 0)$. The equality (4.4) implies that $\lambda \in \mathbb{C} \setminus (\{0\} \cup S_1)$ is equivalent to the condition $\lambda^2 \in \mathbb{C} \setminus (\{0\} \cup S)$.

Since $\text{ord}_t(A_{j,\alpha}) \geq 2j-2m_0+2$ holds for any $(j, \alpha)$ with $m_0 < j < m$ and $|\alpha| > 0$, the $q_1$-difference equation (4.1) satisfies the condition (2.5) (with $j, m_0, m$ replaced by $2j, 2m_0, 2m$, respectively). Therefore, we can apply (2) of Theorem 2.2 and its proof to the equation (4.1).

In particular, by the proof of [Proposition 5.6 in [11]] we have

**Proposition 4.1.** For any $\rho \in \mathbb{C} \setminus (\{0\} \cup S_1)$ we can find $\theta_1 > 0$ and $R_1 > 0$ which satisfy the following conditions (1) and (2):

(1) $u_1(\xi, z)$ has an analytic extension $u_1^*(\xi, z)$ to the domain $S_{\theta_1}(\rho) \times D_{R_1}$.

(2) There are $\mu > 0$ and holomorphic functions $w_n(\xi, z)$ ($n \geq \mu$) on $S_{\theta_1}(\rho) \times D_{R_1}$ which satisfy
\begin{equation}
(4.5) \quad u_1^*(\xi, z) = \sum_{n \geq 2\mu} w_n(\xi, z) + \sum_{0 \leq k < \mu} \frac{X_k(z)}{q_1^{2k(2k-1)/2}} \xi^{2k} \quad \text{on } S_{\theta_1}(\rho) \times D_{R_1},
\end{equation}
and
\[ |w_n(\xi, z)| \leq \frac{AH^n|\xi|^n}{q_1^{n(n-1)/2}} \quad \text{on } S_{\theta_1}(\rho) \times D_{R_1}, \quad n \geq 2\mu \]
for some $A > 0$ and $H > 0$. 7
Therefore, by applying Proposition 3.2 to (4.5) we have the estimate

\[(4.6) \quad |u^*_1(\xi, x)| \leq M \exp \left( \frac{(\log |\xi|)^2}{2 \log q_1} + \alpha \log |\xi| \right) \quad \text{on} \quad S_{\theta_1}(\rho) \times D_{R_1}\]

for some \(M > 0\) and \(\alpha \in \mathbb{R}\).

**Completion of the proof of Proposition 2.4.** Take any \(\lambda = re^{i\theta_1} \in \mathbb{C} \setminus \{0\} \cup S\). We set \(\rho = \sqrt{re^{i\theta_1}/2}\); then we have \(\rho \in \mathbb{C} \setminus \{0\} \cup S_1\). Therefore, by Proposition 4.1 we can get \(\theta_1 > 0, R_1 > 0, M > 0\) and \(\alpha \in \mathbb{R}\) such that \(u_1(\xi, z)\) has an analytic extension \(u^*_1(\xi, z)\) to the domain \(S_{\theta_1}(\rho) \times D_{R_1}\) satisfying the estimate (4.6) on \(S_{\theta_1}(\rho) \times D_{R_1}\).

Since \(u_1(\xi, z) = u(q^{-1/4}\xi^2, z)\) holds, this shows that \(u(\xi, z)\) has also an analytic continuation \(u^*(\xi, x)\) to the domain \(S_\theta(\lambda) \times D_{R_1}\) (with \(\theta = 2\theta_1\)), and we have \(u^*(\xi, z) = u^*_1(q^{1/8}\xi^{1/2}, z)\) on \(S_\theta(\lambda) \times D_{R_1}\). Therefore, by (4.6) we have the estimate

\[|u^*(\xi, x)| \leq M \exp \left( \frac{(\log(q^{1/8}|\xi|^{1/2}))^2}{2 \log q} + \alpha \log(q^{1/8}|\xi|^{1/2}) \right) \]

\[= M_1|\xi|^\beta \exp \left( \frac{(\log |\xi|)^2}{2 \log q} \right) \quad \text{on} \quad S_\theta(\lambda) \times D_{R_1}\]

(with \(M = M_1q^{1/32+\alpha/8}\) and \(\beta = 1/4 + \alpha/2\)).

Thus, by setting \(\xi = \lambda q^m\) we obtain

\[|u^*(\lambda q^m, x)| \leq M_1|\lambda q^m|^\beta \exp \left( \frac{(\log |\lambda q^m|)^2}{2 \log q} \right) \]

\[= M_1|\lambda|^\beta \exp \left( \frac{(\log |\lambda|)^2}{2 \log q} \right)(|\lambda|^q)^m q^{m^2/2}, \quad m = 0, 1, 2, \ldots \]

This proves (2.7). \(\square\)

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