Thermalization and Black Holes

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Abstract. In the framework of AdS/CFT correspondence, we discuss how the thermalization in a Yang-Mills plasma is described by the exact solution of the time dependent black hole in the dual gravity side. We describe the prescription to match generic initial thermal states from the both sides. We also discuss implications of our solution on general non-equilibrium phenomena. The thermalization involves a loss of information leading to an increase of entropy, which is only possible in the thermodynamic limit of field theories. We argue that the large $N$ limit plays a role of this thermodynamic limit in our AdS/CFT correspondence.

1. Introduction

In the advent of the AdS/CFT correspondence or more generally gauge/gravity duality, there has been great progress in our understanding of the nature of gravity and dual field theories in a strongly coupled regime[1]. The conformal gauge theories are well defined quantum mechanically and this can be used to define quantum gravity using the correspondence. One of the main trouble in this definition of quantum gravity is that one needs a detailed dictionary of correspondence beyond the current level of our understanding.

A well known example of such correspondence is the AdS$_5$/CFT$_4$ correspondence. The $\mathcal{N} = 4$ SU(N) super-Yang-Mills (SYM) theory is dual to the type IIB superstring theory on AdS$_5 \times S^5$ background[1]. The $\mathcal{N} = 4$ SYM theory is parametrized by the number of colour $N$ and the gauge coupling $g_{YM}$ and its large $N$ planar limit is defined by sending $N \to \infty$ and $g_{YM} \to 0$ while holding the ’t Hooft coupling $\lambda = g_{YM}^2 N$ fixed. In this limit, the string coupling $g_s = g_{YM}^2 \sqrt{\frac{N}{4\pi}}$ responsible for the joining and splitting of strings becomes zero and thus the type IIB supergravity description becomes precise. If the AdS radius $L = \lambda^{1/4} t_s$ is large with $t_s$ denoting the string length scale, the supergravity is weakly coupled whereas the corresponding SYM theory becomes strongly coupled. Below we shall work in this limited framework of the correspondence where the classical type IIB supergravity is dual to the planar $\mathcal{N} = 4$ SYM theory in the strongly coupled regime of large $\lambda$.

The main topic of this note is how the thermal nature of gravity arises in the context of the gauge/gravity correspondence. We shall first present a detailed example of the time dependent black hole solution[2, 3] describing thermalization of a finite-size perturbation above an equilibrium SYM state. In the gauge theory side, what is expected is quite clear. The perturbations decay away in time with characteristic time scales called relaxation time scales. Of course the local perturbation is also spatially decaying, which is due to the thermal screening effects. Among the spatial scales, the true mass gap and the Debye screening mass are two important characteristic scales of plasma at finite temperature[4, 5]. The relaxation in time
eventually leads to a new equilibrium with an increased entropy. The total energy of the system is conserved in this process. In fact the relevant $N = 4$ SYM theory is defined over a compact space $\Sigma$ which will be specified below. Thus the volume of the system is fixed in time. Then the first law of thermodynamics dictates that there is no flow of heat into the system. Except the strongly coupled aspect, the above is typical and generic in any thermal system in the thermodynamic limit.

The time dependent black hole is precisely describing the same physics of thermalization from the gravity side. At an initial time $t = 0$, the perturbation is turned on deriving the time dependence of the horizon area caused by the (scalar) matter flux through the horizon. The entropy changes as the horizon area changes while the initial perturbation is relaxing away. Since there is no energy flux through the boundary, the energy and the volume of our system do not change through the processes which is simply following from the energy momentum conservation of gravity theories.

In fact one may be more specific about the correspondence. Namely using the generalized prescription of matching the initial states of the two sides based on the thermofield dynamics[6, 7, 8, 9], we shall show that the one-point function $\langle \mathcal{L}(t) \rangle$ of the field theory Lagrange density is decaying exponentially in time. The functional behaviours computed from both sides agree with each other precisely.

In this note we shall extend our discussion about detailed structure of the geometry. This will include the structure of the Euclidean geometry, from which one can generalize our prescription of matching initial states from both sides. We shall also discuss physical implications of the above correspondence. The first one is the thermalization time scale in the strong coupling limit. At high enough temperature, the thermalization happens very fast, which has a potential application to the understanding of the fast thermalization observed in the RHIC experiment. The second is how the information loss is occurring in the planar limit of the SYM theory via the change of the entropy. Our exact solution is particularly suited for the discussion of these aspects. We shall study the first law and its implication on what the gravity theory is describing. How to probe the time dependent entropy using the method of entanglement entropy will be discussed. We shall finish this note with a few final comments.

2. Time dependent black hole and correspondence

In this section we review the properties of time dependent black hole and the correspondence to the SYM system. We start with a simple scalar Einstein gravity system,

$$ I = \frac{1}{16\pi G} \int d^d x \sqrt{g} \left( R - \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi + (d - 1)(d - 2) \right), $$

where the spacetime dimension $d$ is greater than or equals to three. Any solution of the above can be embedded into the type IIB supergravity for $d = 5$ and $d = 3$. The $d = 5$ ($d = 3$) solution describes the deformation of AdS$_5 \times S^5$ (AdS$_3 \times S^3$) geometry[10, 3]. Note that in five dimensions, $\phi$ is the IIB dilaton whereas $\phi/\sqrt{2}$ corresponds to the dilaton for $d = 3$. The AdS radius denoted by $L$ is set to be unity and we shall focus on the 5d case though all the results are extended to the three dimensions.

The metric for the time dependent black hole is given by

$$ ds^2 = f(\mu)(d\mu^2 - d\tau^2 + \cos^2 \tau ds^2_{\Sigma}), $$

where $ds^2_{\Sigma}$ is describing the compact, smooth, finite volume Einstein space metric in $(d - 2)$ dimensions satisfying $R_{\Sigma} = -(d - 3)g_{\Sigma}$. The coordinate $\tau$ is ranged over $[-\pi/2, \pi/2]$. For $d = 3$
case, the $\Sigma$ space corresponds to a circle $S^1$ and, for higher dimensions, the space can be given by the quotient of the hyperbolic space $H_d-2$ by a discrete subgroup of the hyperbolic symmetry group, $SO(1,d-2)$. The function $f(\mu)$ is given by the integral\[10, 2\]

$$
\mu_0 \pm \mu = \int_{-\infty}^{f} \frac{dx}{2\sqrt{x^3 - x^2 + \frac{\gamma^2}{(d-1)(d-2)} x^{4-d}}},
$$

where $\mu_0$ is chosen such that $\mu = 0$ at the turning point. Then $\mu \in [-\mu_0, \mu_0]$ with $\mu_0 \geq \pi/2$.

The dilaton field is given implicitly by the integral

$$
\phi(\mu) = \int_{-\mu_0}^{\mu} dx \sqrt{\frac{\gamma}{f^2(x)}} + \phi_+.
$$

and we introduce $\phi_\pm$ by $\phi_\pm = \phi(\pm \mu_0)$. This time dependent black hole solution is closely related to Janus solution dual to the interface conformal theory\[10, 11\]. If the deformation parameter $\gamma$ vanishes, the solution corresponds to the BTZ black hole in 3d\[12\] and the topological black holes in higher dimensions\[13, 14, 15, 16\].

![Penrose diagram for the time dependent black hole.](image)

**Figure 1.** Penrose diagram for the time dependent black hole. The $\tau \in [-\pi/2, \pi/2]$ coordinate runs vertically upward and $\mu \in [-\mu_0, \mu_0]$ to the right horizontally.

The above form of the solution is particularly suited for drawing the Penrose diagram representing the global structure of the spacetime. As in Figure 1, the Penrose diagram is elongated horizontally. The vertical coordinate $\tau$ runs from $-\pi/2$ to $\pi/2$ and the horizontal coordinate $\mu$ from $-\mu_0$ to $+\mu_0$ where $\mu_0$ is greater than or equal to $\pi/2$. In fact $\mu_0$ increases monotonically as the integration constant $\gamma$ is increasing. As $\gamma$ approaching $\gamma_c$ from below with

$$
\gamma_c^2 = (d-2) \left( \frac{d-2}{d-1} \right)^{d-2},
$$

the $\mu_0$ becomes infinitely large. Hence the horizontal size becomes larger and larger as $\gamma$ is approaching $\gamma_c$ from below. The black hole solution can have a physical meaning beyond the critical value of $\gamma$ but the spacetime no longer involve two timelike boundaries. We shall restrict our discussion for $\gamma < \gamma_c$ below. The dilaton runs from $\phi_-$ to $\phi_+$ from left boundary to the right boundary. Hence for $\gamma < \gamma_c$ there are two boundaries as the usual black hole in the AdS spacetime but the boundary coupling becomes different as a result of the deformation with the deformation parameter $\gamma$. There are two future and two past horizons as drawn in Figure 1 by 45 degree lines from the future and past infinities. For the future horizon, the horizon area starts from zero at $\tau = -\pi/2$ and monotonically increases reaching an equilibrium value

$$
S_{final} = \frac{\mathcal{V}_\Sigma}{4G}.
$$
By studying the near boundary behavior of the metric tensor, one can obtain the expectation value of the boundary energy momentum tensor:

\[
E = \frac{1}{8\pi G} \left( \frac{d-2}{d-1} \right) \left( \frac{d-3}{d-1} \right) \left( d-3 \right)^{d-3/2},
\]

where \(E\) is the energy density in the boundary field theory. The total energy of the system \(E = EV\) agrees with the equilibrium value. The result is consistent with the tracelessness condition of the energy momentum tensor, i.e. \(T^{\mu\nu} = 0\), which is due to the conformal symmetry.

We also note that the operator dual to the dilaton is the CFT Lagrange density \(L\). From the study of the boundary behaviours of the dilaton field, one finds

\[
\langle L \rangle = \frac{\gamma}{8\pi G} \frac{1}{\cosh^{d-1} t},
\]

where \(t\) is the boundary time related to the conformal time \(\tau\) by \(\tanh^{-1}(\sin \tau)\).

By choosing the conformal factor \(h^2 = \cos^2 \tau/f(\mu)\), the boundary metric for the CFT is given by

\[
d\Sigma^2_B = -dt^2 + ds_\Sigma^2.
\]

There are two separated boundaries at \(\mu = \pm \mu_0\) and the boundary time \(t \in (-\infty, \infty)\).

For \(d = 5\) case, the \(\mathcal{N} = 4\) SYM theory on the above boundary spacetime is the corresponding dual system. Since the values of the dilaton on the two boundaries are different from each other, the corresponding SYM theories of the two boundaries now become different as a result of the deformation. The number of colors \(N\) agrees with each other while the 't Hooft couplings \(\lambda_\pm = 4\pi e^{\phi/2} N\) become different as a result of deformation.

To talk about the correspondence further, one has to specify the initial density matrix of the boundary theory. Here we encounter a new situation where there is no standard prescription to find the initial density matrix. Our proposal is based on a generalization of the thermofield dynamics which is a mathematical technique to describe the real time thermodynamics.

We note first that the time dependent black hole solution allows an analytic continuation, \(\tau = -it_E\), leading to the Euclidean geometry,

\[
d\Sigma^2_E = f(\mu)(d\mu^2 + d\tau_E^2 + \cosh^2 \tau_E ds_\Sigma^2)\]

with the scalar field \(\phi(\mu)\) unchanged. The Euclidean coordinate \(\tau_E\) is ranged over \((-\infty, \infty)\). The Euclidean geometry is smooth everywhere and has a boundary. The conformal shape of the \((\mu, \tau_E)\) space is disk shaped. Below we shall provide the physical interpretation of the above Euclidean geometry in terms of thermofield dynamics[7, 8, 9].

We begin with the field theory instanton and its role as an initial state. For the 4d Poincaré-invariant theories, their instanton solution has \(O(4)\) invariance and let us take \(t_E = 0\) as the fixed point of the \(Z_2\) symmetry \(t_E \rightarrow -t_E\) where \(t_E\) is the Euclidean time. Thus, at \(t_E = 0\), the time derivative of fields vanish due to the \(Z_2\) symmetry. This \(t_E = 0\) field configuration can then be interpreted as an initial configuration from which the Lorentzian dynamics follows. The subsequent Lorentzian dynamics for \(t \geq 0\) can be obtained from the instanton solution by analytic continuation[17]. At \(t_E = t = 0\), the Lorentzian and the Euclidean configurations agree with each other and the time derivatives (velocities) of both fields vanish, which helps them join smoothly. Thus at least semi-classically we conclude that the Euclidean solution provides an initial state for the Lorentzian time evolution.
In case of geometry, this corresponds to the so called Hartle-Hawking construction of the wave function of universe[18]. In our problem we shall basically follow the proposal of Ref. [6] to construct the thermofield initial state. We patch the half of Euclidean geometry sliced at $\tau_E = 0$ to the upper half of the Lorentzian solution from $\tau = 0$. Since the geometry involves two boundaries, the corresponding Hilbert space consists of $\mathcal{H} = \mathcal{H}_+ \times \mathcal{H}_-$. Unlike the conventional thermofield formalism, the two Hamiltonians $H_+$ and $H_-$ differ from each other. Following the proposal in Ref. [6], the Euclidean geometry defines a boundary Hamiltonian and allows a Euclidean boundary time evolution, which will determine the initial thermofield state by

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_{mn} \langle E^+_m|U|E^-_n \rangle |E^+_m\rangle \otimes |E^-_n\rangle$$  \hspace{1cm} (11)

where $Z$ is the normalization factor. The Euclidean evolution operator $U$ is given by

$$U = T \exp \left[ - \int_{t^E_-}^{t^E_+} dt^E H(t^E) \right]$$  \hspace{1cm} (12)

where $t^E$ is the boundary Euclidean time with $t^E_\pm$ denoting the two boundary times at $\tau_E = 0$. Since the two boundary Hamiltonians differ from each other, $H(t^E)$ becomes time dependent. In this respect, the above is a slight generalization of the Maldacena’s proposal but this naturally follows from the fact that our Euclidean boundary Hamiltonian is now time dependent. The conformal shape of $(\mu, \tau_E)$ space is depicted in Figure 2.

**Figure 2.** The conformal diagram of the Euclidean solution in $(\mu, \tau_E)$ space. The curves represent constant $\mu$ coordinate. On the curve, $\tau_E$ runs from $-\infty$ at the south pole to $+\infty$ at the north pole. The dotted line is the $\tau_E = 0$ line and the lower half is used to construct the thermofield initial state at $t = 0$.

This procedure can be extended to a more general case, once we restrict ourselves to a class of gravity solutions which possess a time reflection symmetry under $\tau \rightarrow -\tau$. This corresponds to the case where one has zero velocity at $\tau = 0$. By the same patching of the Euclidean solution, the thermofield state determined from the boundary evolution can serve as an initial thermofield state. Of course we assume that the resulting geometry remains asymptotically AdS at both boundaries.

The density matrix $\rho_+$ for the boundary system on the right hand side is given by

$$\rho_+(t) = \text{tr}_- |\Psi(t, t')\rangle \langle \Psi(t, t')|$$  \hspace{1cm} (13)
where \( \text{tr}_+ \) denotes the trace over the \( \mathcal{H}_+ \) Hilbert space. The time evolution for the thermofield state for the + and − boundary states are respectively by \( H_+ \) and \( H_- \) with time \( t \) and \( t' \). If \( O_+ \) is any operator defined in the \( \mathcal{H}_+ \) space, the thermal expectation values are defined by

\[
\langle O_+(t) \rangle = \text{tr}_+ \rho_+(t) O_+(t).
\] (14)

This description of the system by the density matrix provides us with the single boundary view.

The density matrix \( \rho_+ \) is no longer commuting with \( H_+ \) and, hence, time dependent. Then the expectation value \( \langle O_+(t) \rangle \) is in general time dependent, which is consistent with our result of the previous section for the operator \( \mathcal{L}(t) \).

Using this identification of the density matrix, one can show for example the expectation values of energy \( E \) is time independent, which agrees with the gravity computation. Furthermore, using the so called conformal perturbation theory, one may show that the gravity evaluation of \( \langle \mathcal{L}(t) \rangle \) can be recovered to the linear order in \( \gamma \), which supports our identification of the initial density matrix[3].

### 3. Thermalization

The exponential decay of the one point function describes a thermalization occurring in our boundary SYM system. The initial difference of the kinetic minus potential energy densities at \( t = 0 \) is the finite perturbation above an equilibrium state. As the time goes by, the initial difference relaxes away exponentially reaching a new equilibrium state at \( t = \infty \). Since there is no external perturbation, the total energy cannot be changing due to the energy conservation. This is a typical situation when one has a thermalization or a relaxation of finite-size perturbation. Namely, a system beginning with a non equilibrium state approaches a new equilibrium state as time goes by, whose time scale is called the relaxation time scale. From the general grounds, the relaxation time scale depends on operator causing the perturbation, the coupling and the final temperature \( T \) of the system. In conformal field theories, operators of definite scale dimension \( \Delta \) are relevant from general grounds.

Recovering the temperature, one finds the thermalization time scale is given by

\[
\tau_{\text{relax}} = 1/(2\pi T).
\] (15)

This agrees with the value occurring in other studies[19, 20, 21] where it played the role of a limiting value.

The thermalization happens very fast at high enough temperatures. We are looking at the strongly coupled quark gluon plasma. So one can compare this time scale with those thermalization time scale of the plasma produced at the RHIC experiment. In a split second less than 0.6 fermi/c, the fireballs are thermalized and the hydrodynamic description works pretty well explaining the final experimental outcomes. Since the size of the fireball is about 5 fermi and the time scale of 0.6 fermi/c even does not allow any collisions in the average sense, such fast thermalization time scale has been quite puzzling. Note that the temperature keeps changing as the expansion goes on. We use the typical temperature \( T=300\text{MeV} \) to estimate the thermalization time scales, which is certainly an extreme temperature from any standard. This leads to the thermalization time scale 0.1 fermi/c or 0.5 fermi/c for the 99% thermalization, which explains the actual thermalization time scale pretty well.

### 4. Entropy and Information loss

In this section, we focus on the implication of the above result to the gravity theory and the strongly coupled SYM theory. We begin with the first law of the thermodynamics, which is given by

\[
dQ = dE + pdV_{\Sigma},
\] (16)
for any thermodynamic processes. This is basically a statement on the conservation of energy of system including the heat flow. For our case, we have shown $dE = 0$ and $dV = 0$ since the volume of the compact space of the boundary is fixed in time. In addition, there is no inflow of heat as we have explained before, i.e. $dQ = 0$. On the other hand, we know that the entropy changes in time along the thermalization process. Therefore,

$$S_{\text{final}} > S_{\text{initial}}.$$  \hfill (17)

This implies that the relation $dQ = TdS$ does not hold along our time dependent processes. This simply implies that our system is not in any quasi-equilibrium states during the relaxation process. This is in contrast with the claim that the classical gravity may follow from the consideration of entropic force[22]. Our solution and the correspondence clearly shows that the classical gravity includes thermalization effects for which the concept of entropic force is not applicable.

Let us now turn to the discussion of the problem of entropy and the information loss. Our result on the exponential decay of the one-point function implies that the corresponding information of initial perturbation is permanently lost at the future infinity. This loss of information is of course not consistent with the unitarity in the time evolution of the system. Without any truncation, the full SYM theory defined on a compact space is of course unitary. In our case, the system involves a truncation of the planar limit where one ignores any subleading contributions in $1/N$; This is precisely what the classical supergravity is describing. Therefore we conclude that the source of the information loss is basically from the large $N$ planar limit.

To talk about the information loss quantitatively, one needs to define entropy of the system. If it is not defined for all time, we need at least the definition of the initial entropy $S_{\text{initial}}$ and the final entropy $S_{\text{final}}$. This is not a simple matter. For instance one may try the following definition.

$$S(t) = -\text{tr}_+ \rho_+(t) \ln \rho_+(t).$$  \hfill (18)

But by a simple manipulation, one may show that $S$ is time independent since the time evolution of the density matrix is simply given by a similarity transformation of the form $\rho_+(t) = e^{-iH_+ t} \rho_+(0) e^{iH_+ t}$ and the similarity transformation inside the trace has no effect. Hence above definition is not working. We do not know how to proceed for the thermal field theory, but it appears that one needs a some properly defined coarse graining to see the effect of changing entropy.

In the gravity side, one does not have any proper definition for the time-dependent entropy. The changing area along the future event horizon could be one candidate for the entropy. But the geometry has the $Z_2$ reflection symmetry along $\tau = t = 0$. Now if one is following the future horizon in the backward direction starting from the future infinity, the area keeps decreasing beyond the symmetric $t = 0$ point. This is not satisfactory. For this time dependent case, the apparent horizon and the event horizon do not agree with each other. Then which horizon is to be used to measure the area is not clear. There is no well defined definition of the time-dependent temperature. The usual method of the Hartle-Hawking prescription is not working for our case.

We wonder if the study of the entanglement entropy[23] for our background may help our understanding of the time dependence of the entropy. The generalization of our solutions to various cases will be quite interesting in understanding nature of thermalization in strongly coupled gauge theories. In particular, our ansatz preserves locally $SO(d-2,2)$ conformal symmetries. One obvious generalization will be the case of $SO(d-2) \times SO(2)$ symmetry where $SO(2)$ corresponds to the time translation. This leads to the non linearly coupled partial differential equations of two variables. The treatment this coupled system may be complicated.
due to its nonlinear nature. However, expanding in powers of the radial coordinate of $S^{d-2}$, the equation for the remaining variable becomes linear differential equation except the leading part whose solution is already known. Therefore a systematic study in this expansion seems feasible, which may lead to rich solutions for time dependent black holes including formation of black hole. Further study in this direction is required.

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