Distribution function of electrons and phonons in semiconductors and semimetals in high electric and quantizing magnetic fields

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Abstract

The distribution function of electrons and phonons interacting with electrons in semiconductors and semimetals in high electric and quantizing magnetic fields as a result of the solution of the coupled system of equations for the density matrixes of electrons and phonons is obtained. The effects of heating of electrons and phonons and their arbitrary mutual drag are taken into account. The dispersion relation of energy of electrons is assumed to be arbitrarily spherically symmetric one. The spectrum of phonons is assumed to be isotropic.

The distribution function of electrons, phonons, the amplification coefficient of phonons and the dependence of chemical potential on $E$, $H$, $n$ and $T_e$ are obtained. The distribution function of phonons is obtained for arbitrary drift velocities of phonons.

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1 Introduction

In the present paper, the behavior of semiconductors and semimetals in crossed high external electric $E$ ($E = E_x$) and magnetic $H$ ($H = H_z$) fields is considered. The magnetic field is assumed to be high such that the cyclotron frequency $\Omega$ is much greater than the momentum relaxation frequency of electrons $\nu$, i.e. $\Omega \ll \nu$. It is known that an electron has stationary states at high electric and magnetic fields. In Landau representation, the stationary states of electrons are characterized by the magnetic quantum number $N$, the projection of momentum on the magnetic field direction $p_z$ and the rotation center of electrons $X$. 

If the spectrum of electrons is assumed to be isotropic and quadratic, then the eigenvalues of energy have the form:

$$\varepsilon_\alpha = \varepsilon_{N,p_z,X} = \hbar \Omega \left( N + \frac{1}{2} \right) + \frac{p_{2\alpha}^2}{2m_n} - eE X_\alpha + \frac{m_n v_{yo}^2}{2}.$$  \hspace{1cm} (1)

The stationary state of electrons with energy given by Eq. (1) is characterized with Hall drift along the $y$ direction with velocity $v_{yo} \equiv V_H = \frac{eE}{\hbar H}$. In the absence of scattering, the average velocity of electrons along the $x$ direction is equal to zero. The inclusion of scattering leads to the appearance of the conductance current. The motion of electrons along the $x$ direction changes the equilibrium position of the center of oscillation $X_0$, and, as a result, changes the $p_y$ component of the electron momentum connected with them $X_0 = - \frac{c p_y}{eH}$. In the case of scattering of electrons by phonons, the motion of electrons along the $x$ direction is connected with the transfer of the $y$ component of the momentum of electron $p_y$ to phonons, and leads to the stream of phonons along the $x$ axis, i.e., it leads to the mutual drag of electrons and phonons. During the motion along the $x$ axis, the electron gains energy from the field, $eE(X_\alpha - X_\beta) = eE X_{\alpha\beta} \equiv \frac{eER^2 q_y}{\hbar}$, and makes a transition from state $\alpha$ to state $\beta$. If this energy is more than the emitted phonon energy $\hbar \omega_q$, electrons and phonons are heated. Thus, the presence of external electric field leads to the mutual drag of electrons and phonons. If the electric field is high, then we have heating of electrons and phonons and their mutual drag.

In the quantized magnetic field the scattering frequencies of both electrons by phonons $\nu_p$ and phonons by electrons $\beta_e$ are increased, ($\nu_p, \beta_e \sim H^2$). On the other hand, the inter-electronic collision frequency $\nu_{ee}$ is decreased sharply. Because, for nondegenerate statistics of electrons only the lowest level of energy is fully occupied, and the number of electrons in higher energy levels is exponentially small. In this case the collision frequency between electrons in different Landau levels becomes exponentially small. The collisions between electrons in the lowest Landau level becomes elastic as a result of the one dimensionality of the motion, and do not contribute to the inter-electron relaxation. Then, we have $\frac{\nu_{ee}}{\nu_p} \sim \left( \frac{T_e}{\hbar \Omega} \right)^2 \exp \left( - \frac{\hbar \Omega}{T_e} \right)$. The decrease of inter-electronic collisions in quantized magnetic fields leads to a sharp decrease of the efficiency of redistribution of electrons in quantum states. Therefore, at high $E \perp H$ fields under the reasonable concentrations of electrons, the approximation of “effective electron temperature” is not satisfied. That is a reason
of why for the definition of distribution functions of electrons and phonons it is necessary to solve the coupled system of equations for the density matrix of electrons and phonons directly.

Such a problem was formulated and solved for the first time in [2] in the equilibrium state of phonons at lattice temperature \( T \). This situation can be realized at high lattice temperature \( T \), when the phonon–phonon collision frequency \( \beta_p \) is much higher than the collision frequency of phonons by electrons \( \beta_e \). However, for the quantization of orbital motion it is necessary to carry out the experiments at low temperatures (liquid Helium, or lower) of lattice when \( \beta_p \to 0 \). As it is shown experimentally [3]–[6], under these conditions it is necessary to take into account the heating and mutual drag of electrons and phonons, and the generation of phonons by hot electrons. In the experimental conditions the results of [2] satisfy for drift velocities \( V \ll s \). Nevertheless, as it follows from the result of [2], under the condition \( V \ll s \), the heating of electrons is negligible. Moreover, in [2] the change of chemical potential of electrons at high electric and magnetic fields is neglected. In fact, as it is shown in the present paper, for the nondegenerate statistics of electrons, the presence of high electric field leads to the renormalization of chemical potential.

The present paper is devoted to solving the coupled system of equations for the diagonal parts of electrons and phonons density matrices at high external crossed \( E \) and \( H \) fields with taking into account the heating of electrons and phonons, and their mutual drag. The problem is solved for arbitrary spherically symmetric spectrum of electrons. The spectrum of phonons is assumed to be isotropic.

## 2 The spectrum of electrons in high electric and quantizing magnetic fields

Let us assume that in the absence of electric field, the dispersion relation of electrons is

\[
B(\varepsilon_{\alpha 0}) = \frac{p_{z}^{2}}{2m_{n}} + \varepsilon_{N}, \quad \varepsilon_{N} = \hbar \Omega \left( N + \frac{1}{2} \right). \tag{2}
\]

The energy of electrons \( \varepsilon_{\alpha} \) at \( E \perp H \) fields may also be written as

\[
\varepsilon_{\alpha} = \varepsilon_{\alpha 0} - eEX_{\alpha} + \frac{m(\varepsilon_{\alpha})v_{y\alpha}^{2}}{2} = \varepsilon_{\alpha 0} - eEX_{\alpha} + \frac{m(\varepsilon_{\alpha})m_{n}e^{2}E^{2}}{2m_{n}H^{2}}, \tag{3}
\]

where \( m_{n} \) is the effective mass of electrons at the bottom of the conduction band, and \( m(\varepsilon_{\alpha}) = m_{n} \left( \frac{\partial B(\varepsilon_{\alpha})}{\partial \varepsilon_{\alpha}} \right) \) is the effective mass of electron. In the absence of the electric field,

\[
X_{\alpha} = -\frac{p_{y\alpha}}{m_{n}\Omega}. \tag{4}
\]

For the determination of the dispersion relation of electrons in \( E \perp H \) fields, we make a transition to the reference frame which drifts together with electrons with a velocity of \( V = \frac{eE}{H} \). It is obvious that in such reference frame, the dispersion relation of electrons must have the form of Eq. (2), and all properties of the system must be preserved if we substitute
Then, by using Eq. (5) in Eq. (4), we may obtain $X_\alpha$ as

$$X_\alpha = -\frac{p_{y\alpha}}{m_n \Omega} - \frac{m(\varepsilon_\alpha) V}{m_n \Omega} = -\frac{p_{y\alpha}}{m_n \Omega} - \frac{e E}{m_n \Omega^2} \left( \frac{\partial B(\varepsilon_\alpha)}{\partial \varepsilon_\alpha} \right).$$  

If we define $\varepsilon_{\alpha 0}$ as in Eq. (3) and substitute it into Eq. (2), we may obtain the dispersion relation of electrons in $\mathbf{E} \perp \mathbf{H}$ fields as

$$B(\varepsilon_\alpha^*) \equiv B\left(\varepsilon_\alpha + eEX_\alpha - \frac{m(\varepsilon_\alpha) c^2 E^2}{2 \Omega H^2}\right) = \frac{p_{z\alpha}^2}{2m_n} + \hbar \Omega \left(N + \frac{1}{2}\right),$$

or,

$$\frac{p_{z\alpha}^2}{2m_n} = B(\varepsilon_\alpha^*) - \varepsilon_N. \tag{8}$$

$X_\alpha$ in Eq. (7) is determined from Eq. (6).

If the increasing of energy of electrons in electric field \(eEX_\alpha - \frac{m(\varepsilon_\alpha) c^2 E^2}{2 \Omega H^2}\), is much less than $\varepsilon_\alpha$, by expanding $B(\varepsilon_\alpha^*)$ into a series around this small parameter, we find

$$\frac{p_{z\alpha}^2}{2m_n} = B(\varepsilon_\alpha) - \varepsilon_N + \left[ eEX_\alpha - \frac{m_n c^2}{2} \left( \frac{\partial B(\varepsilon)}{\partial \varepsilon_\alpha} \right) \right] \left( \frac{\partial B(\varepsilon_\alpha)}{\partial \varepsilon_\alpha} \right), \tag{9}$$

or,

$$\frac{p_{z\alpha}^2}{2m_n} = B(\varepsilon_\alpha) - \varepsilon_N + \frac{m(\varepsilon_\alpha)}{m_n} eEX_\alpha - \frac{m(\varepsilon_\alpha) m_n c^2 E^2}{2 \Omega H^2}. \tag{10}$$

In the condition $V = \frac{cE}{H} \ll v_{cr} = s$, Eq. (10) reduces to the relations obtained earlier in [7] and [8] as a result of the solution of Schrodinger’s equation for the Kane spectrum of electrons. In the present paper, Eq. (8) is obtained for arbitrary magnitude of the external electric field without any limit, and that is why this expression describes a more general case than the expressions given in [7] and [8].

### 3 The main equations and their solutions

At high magnetic field ($\Omega \gg \nu$) in the Landau representation, the diagonal elements of the density matrix of electrons $f_\alpha$ is larger than the nondiagonal elements $\left(\frac{\Omega}{\nu}\right)$ times, and that is why it is enough to write and solve the equation for the diagonal elements of the density matrix of electrons.

Since in the present paper we consider that the space is uniform and the temperature and concentration gradients are absent, it is enough to write the equations for the diagonal elements of density matrix $N(\mathbf{q}, t)$ for the phonon system.

After averaging over the electron states with fixed energies, the system of equations for the diagonal elements of the density matrixes of electrons $f_\alpha = f(\varepsilon_\alpha, t)$, and phonons $N_\mathbf{q}(t) \equiv N(\mathbf{q}, t)$ have the form[9]:

\begin{align*}
\end{align*}
\[
\frac{\partial f(\varepsilon, t)}{\partial t} = \frac{2\pi}{\hbar} \sum_{\alpha, \beta, \mathbf{q}} |C_{\mathbf{q}}|^2 \langle \alpha | \exp(-i\mathbf{q}\cdot\mathbf{r}) | \beta \rangle^2 \left\{ \delta(\varepsilon_\beta - \varepsilon_\alpha - \hbar\omega_\alpha^*)[(f_\beta - f_\alpha)N(\mathbf{q}, t) + f_\alpha(1 - f_\beta)] \right\} \delta(\varepsilon_\alpha - \varepsilon) + I_{ee}[f] + I_{ed}[f],
\]

where \( f(\varepsilon, t) = \sum_\alpha f(\varepsilon_\alpha, t)\delta(\varepsilon_\alpha - \varepsilon) \) is the average number of carriers with energy \( \varepsilon \), e.g., their distribution function, \( \hbar\omega_\alpha^* = \hbar\omega_\alpha - eE(X_\beta - X_\alpha) \), \( I_{ee}[f] \) and \( I_{ed}[f] \) are the inter-electronic and electron–defects collision integrals, respectively,

\[
I_{pp}[N_\mathbf{q}] = \beta(\mathbf{q})[N(\mathbf{q}, t) - N(\mathbf{q}, T_p)], \quad I_{pb}[N_\mathbf{q}] = \beta(\mathbf{q})[N(\mathbf{q}, t) - N(\mathbf{q}, T)],
\]

are the phonon–phonon and phonon–crystal boundries collision integrals, \( T_p \) is the temperature of the heated phonons, and \( T \) is the lattice temperature. Both phonon collision integrals are considered in the relaxation time approximation: \( \beta_\alpha^{-1} = \tau_\alpha(q), \beta_\alpha^{-1} = \tau_\alpha(q) \).

We consider the low electron concentration case \( n < n_{cr} \) when \( I_{ee} \ll I_{ep} \), where \( I_{ep} \) is the collisions integral of electrons with phonons. For simplicity we neglect the contribution of collisions of electrons with defects. Since in the Born approximation under quantizing magnetic field the collision frequency of electrons with neutral and ionized defects do not depend on the electron energy, the role of the scattering of electrons by defects may be easily taken into account in the final expressions.

Replacing \( \alpha \) by \( \beta \) in the second component of the expression under the sum in Eq. (11), we find

\[
\frac{\partial f(\varepsilon, t)}{\partial t} = \frac{2\pi}{\hbar} \sum_{\alpha, \beta, \mathbf{q}} |C_{\mathbf{q}}|^2 \langle \alpha | \exp(-i\mathbf{q}\cdot\mathbf{r}) | \beta \rangle^2 \left\{ \delta(\varepsilon_\beta - \varepsilon_\alpha - \hbar\omega_\alpha^*)[(f_\beta - f_\alpha)N(\mathbf{q}, t) + f_\alpha(1 - f_\beta)] \right\} \delta(\varepsilon_\alpha - \varepsilon) - \delta(\varepsilon_\beta - \varepsilon).
\]

We consider the case when the scattering of electrons by phonons is quasielastic and, therefore, changing of the energy of electrons \( \varepsilon_\alpha - \varepsilon_\beta = h\omega_\alpha - Vq_y = h\omega_\alpha^* \) is less than the energy scale of the changing electrons distribution function.

Then, expanding \( (f_\beta - f_\alpha) \) and \( [\delta(\varepsilon_\alpha - \varepsilon) - \delta(\varepsilon_\beta - \varepsilon)] \) into series, we find

\[
f_\beta - f_\alpha = f(\varepsilon_\alpha + h\omega_\alpha^*) - f(\varepsilon_\alpha) = h\omega_\alpha^* \left( \frac{\partial f(\varepsilon_\alpha)}{\partial \varepsilon_\alpha} \right) + \ldots,
\]

\[
\{\delta(\varepsilon_\alpha - \varepsilon) - \delta(\varepsilon_\beta - \varepsilon)\} = -h\omega_\alpha^* \frac{\partial}{\partial \varepsilon_\alpha} \delta(\varepsilon_\alpha - \varepsilon) + \ldots.
\]
By using the identity \( \frac{\partial}{\partial \varepsilon} \delta(\varepsilon_\alpha - \varepsilon) \equiv \frac{\partial}{\partial \varepsilon} \delta(\varepsilon_\alpha - \varepsilon) \), we obtain
\[
\frac{\partial f(\varepsilon, t)}{\partial t} = -\frac{\partial}{\partial \varepsilon} \left\{ A(\varepsilon) \frac{\partial f(\varepsilon, t)}{\partial \varepsilon} + D(\varepsilon) f(\varepsilon, t)[1 - f(\varepsilon, t)] \right\},
\] (17)
where,
\[
A(\varepsilon) = \frac{2\pi}{\hbar} \sum_{\alpha, \beta, \mathbf{q}} |C_{\mathbf{q}}|^2 \left| I_{\alpha, \beta} \right|^2 (\hbar \omega^*_\beta)^2 \delta(\varepsilon_\beta - \varepsilon_\alpha - \hbar \omega^*_\beta) N(\mathbf{q}, t) \delta(\varepsilon_\alpha - \varepsilon),
\] (18)
\[
D(\varepsilon) = \frac{2\pi}{\hbar} \sum_{\alpha, \beta, \mathbf{q}} |C_{\mathbf{q}}|^2 \left| I_{\alpha, \beta} \right|^2 \hbar \omega^*_\beta \delta(\varepsilon_\alpha - \varepsilon_\beta - \hbar \omega^*_\beta) \delta(\varepsilon_\alpha - \varepsilon).
\]
The stationary solution of Eq. (17) satisfying boundary condition \( \lim_{\varepsilon \to \infty} f(\varepsilon) \to 0 \) is
\[
f(\varepsilon) = \left\{ \text{const.} \frac{1}{1 + \frac{\partial}{\partial \varepsilon} \left\{ A(\varepsilon) \frac{\partial f(\varepsilon, t)}{\partial \varepsilon} + D(\varepsilon) f(\varepsilon, t)[1 - f(\varepsilon, t)] \right\}} \right\}^{-1},
\] (19)
where \( T_e(\varepsilon) = \frac{A(\varepsilon)}{D(\varepsilon)} \) is the temperature of electrons with energy \( \varepsilon \).

The solution of Eq. (12) is
\[
N(\mathbf{q}, t) = \left\{ N(\mathbf{q}, 0) + \beta \gamma_q^{-1} \widetilde{N}(\mathbf{q}) \right\} \exp(\gamma_q t) - \beta \gamma_q^{-1} \widetilde{N}(\mathbf{q}),
\] (20)
where \( \gamma_q = \beta \left( \frac{u \cdot \mathbf{q}}{\hbar \omega_q} - 1 \right) \) is the increment of the generation of phonons, \( \beta = \beta_e + \beta_p + \beta_b \) is the total collision frequencies of phonons by the scatterers and
\[
\beta_e = \frac{2\pi}{\hbar} \sum_{\alpha, \beta} |C_{\mathbf{q}}|^2 \left| I_{\alpha, \beta} \right|^2 (f_{\beta} - f_\alpha) \delta(\varepsilon_\beta - \varepsilon_\alpha - \hbar \omega^*_\beta)
\] (21)
\[
\approx \frac{2\pi}{\hbar} \sum_{\alpha, \beta, \mathbf{q}} |C_{\mathbf{q}}|^2 \left| I_{\alpha, \beta} \right|^2 (\varepsilon_\beta - \varepsilon_\alpha) \left( \frac{\partial f(\varepsilon_\alpha)}{\partial \varepsilon_\alpha} \right) \delta(\varepsilon_\beta - \varepsilon_\alpha - \hbar \omega^*_\beta).
\]

\( N(\mathbf{q}, 0) \equiv N(\mathbf{q}, T) \) is the initial distribution function of phonons at \( t = 0 \) in the absence of external fields,
\[
u = \left( 1 - \frac{\beta_p}{\beta} \right) V, \quad |I_{\alpha, \beta}|^2 = |\langle \alpha \mid \exp(-i\mathbf{q}\cdot\mathbf{r}) \mid \beta \rangle|^2.
\] (22)

As it follows from Eq. (20) in the \( \gamma_q > 0 \) case, i.e., when the drift velocity of phonons \( u \) is larger than the sound velocity \( s \), the distribution function of phonons \( N(\mathbf{q}, t) \) increases exponentially with time, whereas in the \( \gamma_q < 0 \) case the solution, Eq. (20), is stationary,
\[
N(\mathbf{q}) = \lim_{t \to \infty} N(\mathbf{q}, t) = -\beta \gamma_q^{-1} \widetilde{N}(\mathbf{q}), \quad \widetilde{N}(\mathbf{q}) = \gamma_e N(\mathbf{q}, T_e) + \gamma_p N(\mathbf{q}, T_p),
\] (23)
where \( \gamma_e = \frac{\beta_e}{\beta}, \gamma_p = \frac{\beta_p}{\beta}. \)
\[ N(q, T_e) = \sum_{\alpha,\beta} \frac{|I_{\alpha\beta}|^2 T_e(\varepsilon)}{\sum_{\alpha,\beta} |I_{\alpha\beta}|^2} \left[ \frac{\partial f(\varepsilon_\alpha) / \partial \varepsilon_\alpha}{\beta} \delta(\varepsilon_\beta - \varepsilon_\alpha - \hbar \omega_q^*) \right], \tag{24} \]

\[ T_e = \frac{\sum_{\alpha,\beta} |I_{\alpha\beta}|^2 T_e(\varepsilon) \left[ \frac{\partial f(\varepsilon_\alpha) / \partial \varepsilon_\alpha}{\beta} \delta(\varepsilon_\beta - \varepsilon_\alpha - \hbar \omega_q^*) \right]}{\sum_{\alpha,\beta} |I_{\alpha\beta}|^2 \left[ \frac{\partial f(\varepsilon_\alpha) / \partial \varepsilon_\alpha}{\beta} \delta(\varepsilon_\beta - \varepsilon_\alpha - \hbar \omega_q^*) \right]}, \tag{25} \]

where \( T_e \) is the effective temperature of electrons.

In the case of \( \gamma_q = 0 \) or \( \frac{u_q}{\hbar \omega_q} \approx 1 \) or \( \hbar \omega_q^* = 0 \), from Eq. (17) we have \( \frac{\partial f(\varepsilon, t)}{\partial t} = 0 \), or \( f(\varepsilon, t) = \text{const.} \). At this point the distribution function of electrons \( f(\varepsilon, t) \), the effective electron temperature \( T_e \) and the drift velocity of electrons are constant. Therefore, the state is nondissipative and the current is constant. Moreover, the distribution function of phonons is nonstationary and grows by time linearly, \( N(q, t) = N(q, T) + \beta t N(q, T_e) \). Indeed, \( u = V = s = \text{const.} \) and \( \frac{dN(q, t)}{dt} = \beta N(q, T_e) = \text{const.} \). Namely, \( P(T_e) = \sum_q \hbar \omega_q^* \left( \frac{dN(q, t)}{dt} \right) = 0 \), where \( P(T_e) \) is the power transferred by electrons to phonons. The point \( u = s \) is the acoustical instability threshold (AIT). At this point, the stimulated emission of phonons is equal to the stimulated absorption of phonons, and we have only spontaneous emission of phonons at high external electric and magnetic fields. At this point collisions of electrons with phonons are exactly elastic, i.e., state is nondissipative and dynamically stationary because of the power received from electric field emitted as phonons by the process of the stimulated emission. Moreover, at this point the current \( J = \text{const.} \) and \( u_d = V = \frac{cE}{H} = s \), or \( cE = sH \), i.e., \( E_{cr} = \frac{sH}{c} \) or \( H_{cr} = \frac{cE}{s} \).

Substituting \( \frac{\partial N(q, T_e)}{\partial t} = 0 \) in Eq. (12), we may directly solve this equation under the boundary conditions

\[ N(q, T_e) \big|_{t=0} = N(q, 0) \equiv N(q, T) = \left\{ \exp \left( \frac{\hbar \omega_q}{T} \right) - 1 \right\}^{-1}, \tag{26} \]

and we obtain Eq. (23).

Under the considered conditions \( \hbar \omega_q^* \ll T_e, T \) from Eq. (23), we find

\[ \tilde{N}(q) = \frac{\gamma_e T_e + \gamma_p T_p}{\hbar \omega_q} = \frac{\tilde{T}}{\hbar \omega_q} = \tilde{N}(q, \tilde{T}), \quad \tilde{T} = \gamma_e T_e + \gamma_p T_p, \tag{27} \]

where \( \tilde{T} \) is the temperature of coupled by the mutual drag system of electrons and phonons. Therefore, the stationary solution of Eq. (12) has the form \( \hbar \omega_q^* > 0 \):

\[ N(q) = \frac{\tilde{N}(q)}{1 - \frac{u_q}{\hbar \omega_q}} = -\beta \gamma_q^{-1} \tilde{N}(q) \approx \frac{\tilde{T}}{\hbar \omega_q^*}. \tag{28} \]
Let us substitute Eq. (28) in Eq. (18) and take into account the relations:

\[ |I_{\alpha\beta}|^2 = |I_{NN'}|^2 \delta_{p_{\alpha\beta},p_{\gamma}+q_y} \delta_{p_{\alpha\beta},p_{\lambda}+q_z}, \]

\[ |I_{NN'}|^2 = \left( \frac{N!}{N'}! \right)^{1/2} \exp \left( -\frac{q_{\perp}^2}{q_H^2} \right) \left( -\frac{q_{\perp}}{q_H} \right)^{N'-N} L_h^{N'-N} \frac{q_{\perp}^2}{q_H^2}. \]

Here \( q_H = \frac{1}{R} (m_n\varepsilon_N)^{1/2} \), \( q_{\perp}^2 = q_x^2 + q_y^2 \), \( L_h^{|m|} \) is the Laguerre’s polynomial normalized to unity, and \( |m| = |N - N'| \).

Choosing the cylindrical coordinate system with direction along the magnetic field

\[ dq = q_{\perp} \, dq_{\perp} \, dq_z \, d\varphi, \quad q_y = q_{\perp} \sin \varphi. \]

As a result of the integration, we have

\[ A(\varepsilon) = \bar{T} \Phi(u/s) D(\varepsilon), \quad \Phi(u/s) = \varphi - 2 \frac{V}{u} (\varphi_1 - 1) - \left( \frac{V}{u} \right)^2 (\varphi_1 - 1)^2, \]

\[ D(\varepsilon) = \frac{s\Omega}{\hbar(2\pi\hbar)^{3/2}} \sum_{N,N'} q_{\perp}^2 W(q_H) \int_0^\infty dx \frac{x^2 |I_{NN'}|^2 m^2(\varepsilon)}{[B(\varepsilon + \hbar \omega_q^*) - \varepsilon_N]^{1/2} [B(\varepsilon - \hbar \omega_q^*) - \varepsilon_N]^{1/2}}, \]

where \( x = q_{\perp}/q_H \), and \( W(q_H) \) is the constant part of the potential in mutual interaction of electrons by phonons.

Thus, in general, under the arbitrary degree of quantization, the ratio of \( A(\varepsilon) \) to \( D(\varepsilon) \) is \( \bar{T} \Phi(u/s) \); and it does not depend on the energy of electrons and the potential of mutual interactions with phonons. In other words, in a more general case of interaction of electrons by the acoustic and optical phonons the expression:

\[ T_{ef} = \frac{A(\varepsilon)}{D(\varepsilon)} = \bar{T} \Phi(u/s), \]

does not depend on the energy of electrons.

By substituting Eq. (32) into Eq. (19), we find

\[ f(\varepsilon) = \left\{ 1 + \exp \left( \frac{\varepsilon - \zeta(E,H)}{T_{ef}} \right) \right\}^{-1}. \]

In other words, at quantizing magnetic fields the distribution function of electrons is a Fermi one, and includes the effective temperature in the following form:

\[ T_{ef} = \bar{T} \left\{ 1 + \left( 1 - \frac{V}{u} \right)^2 (\varphi_1 - 1) \right\}, \quad \varphi_1 = \left( 1 - \frac{u^2}{s^2} \right)^{-1/2}. \]

In the classical region of strong magnetic fields (\( \Omega \gg \nu \))
\[ \Phi(u/s) = \left\{ 1 + \left(1 - \frac{V}{u}\right)(\varphi_2 - 1) + \frac{1}{3}\left(\frac{V}{s}\right)^2 \right\}, \quad \varphi_2 = \frac{s}{2u} \ln \left| \frac{s + u}{s - u} \right|. \] (35)

As it follows from Eq. (33), the distribution function of electrons is the Fermi one with effective temperature:

\[ T_{\text{ef}} = \tilde{T} \left\{ 1 + \left(1 - \frac{V}{u}\right)(\varphi_2 - 1) + \frac{1}{3}\left(\frac{V}{s}\right)^2 \right\}. \] (36)

The fact that the distribution function of electrons in both classical and quantum regions of magnetic fields are the Fermi ones with effective temperature is a result of the independency of drift velocity of electrons \( V = \frac{cE}{H} \) from energy of electrons \( \varepsilon \).

At small values of the drift velocity of electrons, \( V \ll s \), from Eq. (34) for the quantizing magnetic field, we have

\[ T_{\text{ef}} \approx \tilde{T} \left\{ 1 + \frac{1}{2}\left(\frac{V}{s}\right)^2 - \frac{uV}{s^2} \right\} = \tilde{T} \left\{ 1 + \left(\frac{\gamma_p - \frac{1}{2}}{cE/sH}\right)^2 \right\}. \] (37)

If \( \beta_e \gg \beta_p \), \( T_{\text{ef}} = \tilde{T} \left\{ 1 - \frac{1}{2}\left(\frac{cE}{sH}\right)^2 \right\} \); and if \( \beta_e \ll \beta_p \), \( T_{\text{ef}} = \tilde{T} \left\{ 1 + \frac{1}{2}\left(\frac{cE}{sH}\right)^2 \right\} \).

In the classical region of magnetic fields

\[ T_{\text{ef}} \approx \tilde{T} \left\{ 1 + \frac{1}{3}\left(\frac{V}{s}\right)^2 \right\} = \tilde{T} \left\{ 1 + \frac{1}{3}\left(\frac{cE}{sH}\right)^2 \right\}. \] (38)

If \( \beta_p \gg \beta_e \), phonons are not heated, \( \tilde{T} = T \) and \( V \ll s \). Under these conditions from Eqs. (37) and (38) we can obtain the results of [2].

4 The statistics of electrons at high magnetic fields

In crossed \( \mathbf{E} \perp \mathbf{H} \) field, the chemical potential of electrons may be obtained from the normalization condition of the distribution function as

\[ n_e = \int_{\infty}^{\infty} d\varepsilon \ g_H(\varepsilon) f(\varepsilon), \quad g_H(\varepsilon) = \frac{2m_n(eH/c)}{(2\pi\hbar)^2} \sum_N p_1^{-1}(\varepsilon). \] (39)

For arbitrary spherical symmetric spectrum of electrons

\[ n_e = \frac{2(2m_n)^{1/2}m_n\Omega}{(2\pi\hbar)^2} \sum_N \int_{\varepsilon_1}^{\infty} d\varepsilon^* \ [B(\varepsilon^*) - \varepsilon_N]^{1/2} \left\{ 1 + \exp \left( \frac{\varepsilon^* - \zeta^*(E, H)}{T_{\text{ef}}} \right) \right\}^{-1}, \] (40)

\[ \zeta^*(E, H) = \zeta(E, H) + eEX(\varepsilon) - \frac{m(\varepsilon)c^2}{2} \left( \frac{E^2}{H^2} \right). \] (41)
The $\varepsilon_1^*$ is determined as a solution

$$B(\varepsilon^*) - \varepsilon_N = 0. \quad (42)$$

By partial integration of Eq. (40), we find

$$n_e = \frac{4(2m_n)^{1/2}m_n\Omega}{(2\pi\hbar)^2} \sum_N \int_{\varepsilon_1^*}^\infty d\varepsilon^* \left[ B(\varepsilon^*) - \varepsilon_N \right]^{1/2} \left( - \frac{\partial f(\varepsilon^*)}{\partial \varepsilon^*} \right). \quad (43)$$

For the case of parabolic spectrum of electrons

$$\varepsilon_1^* = \varepsilon_N = \hbar \Omega \left( N + \frac{1}{2} \right), \quad (44)$$

and for the Kane spectrum of electrons:

$$\varepsilon_1^* = -\frac{\varepsilon_g}{2} \left[ 1 - \left( 1 + \frac{4\varepsilon_N}{\varepsilon_g} \right)^{1/2} \right]. \quad (45)$$

Let us consider the case of degenerate and non–degenerate statistics of electrons alone. For strong degenerate electrons

$$-\left( \frac{\partial f_0(\varepsilon^*)}{\partial \varepsilon^*} \right) = \delta(\varepsilon - \zeta^*). \quad (46)$$

With the help of this expression, we can integrate the Eq. (43) and get

$$n_e = \frac{4(2m_n)^{1/2}}{\hbar(2\pi R)^2} \sum_N \left[ B^*(\zeta^*) - \hbar \Omega \left( N + \frac{1}{2} \right) \right]^{1/2}, \quad (47)$$

For the parabolic spectrum of electrons

$$n_e = \frac{4(2m_n)^{1/2}}{\hbar(2\pi R)^2} \sum_N \left[ \zeta^* - \hbar \Omega \left( N + \frac{1}{2} \right) \right]^{1/2}. \quad (48)$$

For the Kane spectrum of electrons

$$n_e = \frac{4(2m_n)^{1/2}}{\hbar(2\pi R)^2} \sum_N \left[ \zeta^* \left( 1 + \frac{\zeta^*}{\varepsilon_g} \right) - \hbar \Omega \left( N + \frac{1}{2} \right) \right]^{1/2}. \quad (49)$$

For ultraquantum limits ($N = N' = 0$), we have

$$n_e = \frac{4(2m_n)^{1/2}}{\hbar(2\pi R)^2} \left[ B^*(\zeta^*) - \frac{\hbar \Omega}{2} \right]^{1/2} \quad (50)$$

$$= B(\zeta^*) - \frac{\hbar \Omega}{2} = \frac{\hbar^2(2\pi R)^4n_e^2}{16(2m_n)},$$

where $B(\zeta^*) = \frac{\hbar \Omega}{2} + \frac{\hbar^2(2\pi R)^4n_e^2}{16(2m_n)}$. From this relation for the parabolic spectrum of electrons we obtain the chemical potential as follows:
\[ \zeta^*(E, H) = \frac{\hbar \Omega}{2} + \frac{\pi^4 \hbar^4 n_e^2}{2m_n^2 \Omega^2}. \]  \hspace{1cm} (51)

This expression is the same of Eqs. (32) and (22) in [12].

In the case of Kane spectrum of electrons:

\[ \zeta^*(E, H) = -\frac{\varepsilon_g}{2} \left\{ 1 - \left( 1 + \frac{2\hbar \Omega}{\varepsilon_g} + \frac{2\pi^4 \hbar^2 R^4 n_e^2}{m_n \varepsilon_g} \right)^{1/2} \right\}. \]  \hspace{1cm} (52)

In the case of nondegenerate electrons by taking \([B(\varepsilon^*) - \varepsilon_N]\) out of the integral when \(\varepsilon^* = \varepsilon_1^*(N) + T_e\), we find

\[ \exp \left( \frac{\zeta^{**}(E, H)}{T_e} \right) = \frac{(2\pi R)^2 \hbar n_e}{4(2m_n)^{1/2}} \left\{ \sum_N \exp \left( -\frac{\varepsilon_1^*(N)}{T_e} \right) \right\}^{-1} \left[ B(\varepsilon_1^*(N) + T_e) - \varepsilon_N^{1/2} \right]. \]  \hspace{1cm} (53)

As before \(\varepsilon_1^*\) is obtained from Eqs. (44) and (45). If the condition \(\varepsilon_1^* \gg T_e\) is satisfied, then by expanding \(B(\varepsilon_1^* + T_e)\) in Eq. (53) into series and by taking into account Eq. (42), we have

\[ \exp \left( \frac{\zeta^{**}(E, H)}{T_e} \right) \simeq \frac{(2\pi R)^2 \hbar n_e}{4\sqrt{2}} T_e^{-1/2} \left[ \sum_N \exp \left( -\frac{\varepsilon_1^*(N)}{T_e} \right) m^{1/2}(\varepsilon_1^*) \right]^{-1}, \]  \hspace{1cm} (54)

where

\[ m(\varepsilon_1^*) = m_n \left( \frac{\partial B(\varepsilon^*)}{\partial \varepsilon^*} \right) \varepsilon^* = \varepsilon_1^*. \]  \hspace{1cm} (55)

Dividing Eq. (53) by itself for \(E = 0\), we get

\[ \zeta^{**}(E, H) = \zeta(H) \frac{T_e}{T} + T_e \ln \left( \frac{n_e}{n_0} \right) + T_e \ln \left[ \frac{F_N(T)}{F_N(T_e)} \right], \]  \hspace{1cm} (56)

where

\[ \zeta^{**}(E, H) = \zeta(E, H) - \frac{m}{m_n} \frac{m_n c^2 E^2}{2 H^2}; \]  \hspace{1cm} (57)

\[ F_N(T_e) = \sum_N \exp \left( -\frac{\varepsilon_1^*(N)}{T_e} \right) \left\{ B(\varepsilon_1^*(N) + T_e) - \varepsilon_N^{1/2} \right\}. \]  \hspace{1cm} (58)

\(F_N(T)\) may be obtained from Eq. (58) by replacing \(T_e\) with \(T\).

For the case of parabolic spectrum of electrons \(\varepsilon_1^* = \varepsilon_N = \hbar \Omega \left( N + \frac{1}{2} \right)\), and, we have

\[ \zeta^{**}(E, H) = \zeta(H) \frac{T_e}{T} + T_e \ln \left( \frac{n_e}{n_0} \right) - T_e \ln \left( \frac{T_e}{2} \right) + T_e \ln \left[ \frac{\sinh(\hbar \Omega / 2T)}{\sinh(\hbar \Omega / 2T_e)} \right]. \]  \hspace{1cm} (59)

Therefore, in the case of parabolic spectrum of electrons we finally find the chemical potential as
\[
\zeta(E, H) = \zeta(H) \frac{T_e}{T} - eEX + \frac{m_\text{e}c^2 E^2}{2H^2} + T_e \ln \left( \frac{n_\text{e}}{n_0} \right) + 
\]

\[
\frac{T_e}{2} \ln \left( \frac{T_e}{T} \right) + T_e \ln \left( \frac{\sinh(h\Omega/2T_e)}{\sinh(h\Omega/2T)} \right).
\]

As it follows from Eq. (60) in external electric field if the concentration of electrons is increased \((n_\text{e} > n_0)\), then the chemical potential of electrons must also increase.

If we have full ionization of small impurity centers, then \(n_\text{e} = n_0 = \text{const}\). Thus, from Eq. (60), we may obtain

\[
\zeta(E, H) - \frac{\hbar \Omega}{2} = \frac{T_e}{T} \left[ \zeta(H) - \frac{\hbar \Omega}{2} \right] + \frac{m_\text{e}c^2 E^2}{2H^2} - 
\frac{T_e}{2} \ln \left( \frac{T_e}{T} \right) + T_e \ln \left[ \frac{1 - \exp(h\Omega/2T_e)}{1 - \exp(h\Omega/2T)} \right].
\]

In the quantizing magnetic fields \(\hbar \Omega > T_e, T\) and, for this reason, the last term in Eq. (61) may be presented as \(\frac{T_e \hbar \Omega}{T} \). In this case

\[
\zeta(E, H) - \frac{\hbar \Omega}{2} = \frac{T_e}{T} \left[ \zeta(H) - \frac{\hbar \Omega}{2} \right] + \frac{m_\text{e}c^2 E^2}{2H^2} - 
\frac{T_e}{2} \ln \left( \frac{T_e}{T} \right).
\]

For the weak electric fields \(T_e = T\),

\[
\zeta(E, H) = \zeta(H) - \frac{\hbar \Omega}{2} + \frac{m_\text{e}c^2 E^2}{2H^2}.
\]

As it follows from Eq. (63), the expression for \(\zeta(E, H)\) differs from the expression given in \([13]\) by the factor \(\frac{m_\text{e}c^2 E^2}{2H^2}\), which is connected with the Hall drift of electrons. In the case of heating of electrons at external electric field the main contribution to the free energy is obtained by the expression \(\zeta(H) \frac{T_e}{T}\), i.e., in high external electric field the chemical potential of electrons in common case increases linearly by \(\frac{T_e}{T}\).

For the calculation of the statistical behavior of electrons in external fields such as magnetic susceptibility, heat capacity and \textit{etc}., it is necessary to know the dependence of the chemical potential on the intensity of external electric and magnetic fields.

The magnetic susceptibility of hot electrons in the case of high concentration of electrons was investigated earlier\([14]\).

5 Conclusion

In the present work, it is shown that under the conditions of arbitrary degree of quantization and for the interaction of electrons with both the acoustical and optical phonons,
the distribution function of electrons has the form of Fermi distribution function with effective electron temperature. This result is obtained in the case of small concentration of electrons $n \leq n_{cr}$ when the usual approximation of “effective temperature” for electrons is not satisfied, i.e., $\nu_{ee} \ll \nu_p$.

The distribution function of phonons interacting with electrons is obtained for arbitrary drift velocities of phonons. If the drift velocity of phonons $u$ is smaller than the sound velocity $s$ there is a stationary state and the distribution function of phonons is shifted Planck’s one with effective temperature of phonons. In the case of $u > s$, the distribution function of phonons grows with time exponentially, i.e., we have the effect of generation or amplification of phonons by electric field. The amplification coefficient of phonons $\gamma_q = \frac{\beta}{s} \left( \frac{u q}{\hbar \omega_q} - 1 \right)$ is obtained. It has been established that in the considered case the chemical potential of electrons is renormalised. The $E$, $H$, $n$ and $T_e$ dependences of the chemical potential are obtained.

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