HAMILTONIAN CIRCLE ACTIONS WITH MINIMAL FIXED SETS

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Abstract. Consider an effective Hamiltonian circle action on a compact symplectic 2n-dimensional manifold \((M, \omega)\). Assume that the fixed set \(M^{S^1}\) is minimal, in two senses: it has exactly two components, \(X\) and \(Y\), and \(\dim(X) + \dim(Y) = \dim(M) - 2\).

We prove that the integral cohomology ring and Chern classes of \(M\) are isomorphic to either those of \(\mathbb{C}P^n\) or to those of \(G_2(\mathbb{R}^{n+2})\), where \(G_2(\mathbb{R}^{n+2})\) denotes the Grassmannian of oriented two-planes in \(\mathbb{R}^{n+2}\). In particular, this implies that \(H'(M; \mathbb{Z}) = H'(\mathbb{C}P^n; \mathbb{Z})\) for all \(i\), and that the Chern classes of \(M\) are determined by the integral cohomology ring. We also prove that the fixed set data agrees exactly with one of the two standard examples. In particular, there are no points with stabilizer \(\mathbb{Z}_k\) for any \(k > 2\).

We also show that the same conclusions hold when \(M^{S^1}\) has exactly two components and the even Betti numbers of \(M\) are minimal, that is, \(b_{2i}(M) = 1\) for all \(i \in \{0, \ldots, \frac{1}{2}\dim(M)\}\). This provides additional evidence that very few symplectic manifolds with minimal even Betti numbers admit Hamiltonian actions.

1. Introduction

Let the circle \(S^1\) act in a Hamiltonian fashion\(^1\) on a compact symplectic manifold \((M, \omega)\). The moment map \(\phi: M \to \mathbb{R}\) is a Morse-Bott function whose critical set is exactly the fixed point set \(M^{S^1}\). Moreover, since \(M\) is a symplectic manifold, \(H^{2i}(M; \mathbb{R}) \neq 0\) for all \(i\) such that \(0 \leq 2i \leq \dim(M)\). Therefore, we can immediately conclude that

\[
\sum_{F \subseteq M^{S^1}} (\dim(F) + 2) \geq \dim(M) + 2,
\]

where the sum is over all fixed components.

In this paper, we consider the case that the fixed set has the smallest possible number of components, and these components have the smallest possible dimension. More concretely, we assume that \(M^{S^1}\) has exactly two components.

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\(^1\)Unless we specify otherwise, we shall always assume that our actions are non-trivial.
components, $X$ and $Y$, and that $\dim(X) + \dim(Y) = \dim(M) - 2$. It is easy to see that $\mathbb{CP}^n$, the complex projective space of lines in $\mathbb{C}^{n+1}$, admits a Hamiltonian circle action satisfying these assumptions; if $n$ is odd then $\tilde{G}_2(\mathbb{R}^{n+2})$, the Grassmannian of oriented two-planes in $\mathbb{R}^{n+2}$, does as well. (See Examples 1.4 and 1.5.)

Our first main theorem in this paper is that, under the above assumptions, the cohomology ring $H^*(M; \mathbb{Z})$ and Chern classes $^2 c(M)$ of $M$ are identical to those of one of these two spaces.

**Theorem 1.** Let the circle act in a Hamiltonian fashion on a compact $2n$-dimensional symplectic manifold $(M, \omega)$. Assume that $M$ has exactly two components, $X$ and $Y$, and that $\dim(X) + \dim(Y) = \dim(M) - 2$. Then one of the following is true:

(A) $H^*(M; \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$ and $c(M) = (1 + x)^{n+1}$; or

(B) $n$ is odd, $H^*(M; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^{\frac{1}{2}(n+1)} - 2y, y^2)$ and $c(M) = \frac{(1+x)^{n+2}}{1+2x}$.

In both cases, $x$ has degree 2; in case (B), $y$ has degree $n + 1$.

More generally, let $(M, \omega)$ be any compact symplectic manifold with a Hamiltonian circle action. Assume that $M$ has minimal even Betti numbers, that is,

$$b_{2i}(M) = 1 \quad \forall \ i \in \{0, \ldots, \frac{1}{2} \dim(M)\},$$

where $b_j(M) = \dim(H^j(M; \mathbb{R}))$ for all $j$. It is easy to check that this implies that the fixed components have minimal dimension, that is,

$$\sum_{F \subseteq M^{S^1}} (\dim(F) + 2) = \dim(M) + 2.$$  

(See Lemma 4.1.) Theorem 1 shows that if $M^{S^1}$ has exactly two components, then the converse also holds. In fact, we are able to show more generally that if (1.2) holds, then $H^j(M; \mathbb{R}) = H^j(\mathbb{CP}^{\frac{1}{2} \dim(M)}; \mathbb{R})$ for all $j \in \{0, \ldots, \dim(X) + 2\}$, where $X$ is the minimal fixed component; see Proposition 4.2. However, as we explain in Remark 4.3, (1.2) does not imply (1.1) in general.

Moreover, the second author considers the following symplectic generalization of the classical Petrie conjecture in [5]:

**Question 1.** Consider a Hamiltonian circle action on a compact symplectic manifold $(M, \omega)$ which satisfies $b_{2i}(M) = 1$ for all $i \in \{0, \ldots, \frac{1}{2} \dim(M)\}$.

(i) Is $H^j(M; \mathbb{Z}) = H^j(\mathbb{CP}^n; \mathbb{Z})$ for all $j$?

(ii) Are the Chern classes $c_i(M)$ determined by the cohomology ring $H^*(M; \mathbb{Z})$?

In the case that $M$ is 6-dimensional, both questions are answered affirmatively in [5]. By the theorem above, the same claim holds if $M^{S^1}$ has exactly

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$^2$See [2] for the definition of the Chern class of a symplectic manifold.
two components. Moreover, in both cases, every such manifold “looks like” some well-known Kähler example with additional symmetries; see [5] and [4] for the case when \( M \) is 6-dimensional. This provides additional evidence that very few symplectic manifolds with minimal even Betti numbers admit Hamiltonian actions.

Our second main theorem is that there are no exotic actions, that is, the fixed set data agrees exactly with one of the two standard examples. Here, the fixed set data is the cohomology ring of each fixed component \( X \), the set of weights for the isotropy representation on the normal bundle \( N_X \) to each fixed component \( X \), and – for each such weight \( k \) – the Chern classes of each subbundle \( V_k \subset N_X \) on which \( S^1 \) acts with weight \( k \). This theorem is an immediate consequence of Propositions 6.1 and 8.1.

**Theorem 2.** Let the circle act effectively on a compact symplectic manifold \( (M, \omega) \) with moment map \( \phi: M \to \mathbb{R} \). Assume that \( M^{S^1} \) has exactly two components, \( X \) and \( Y \), and that \( \dim(X) + \dim(Y) = \dim(M) - 2 \). Then

\[
\begin{align*}
H^*(X; \mathbb{Z}) &= \mathbb{Z}[u]/u^{i+1} \quad \text{and} \quad c(X) = (1 + u)^{i+1}, \quad \text{where} \ \dim(X) = 2i; \\
H^*(Y; \mathbb{Z}) &= \mathbb{Z}[v]/v^{j+1} \quad \text{and} \quad c(Y) = (1 + v)^{j+1}, \quad \text{where} \ \dim(Y) = 2j.
\end{align*}
\]

Moreover, one of the following is true:

(A) The action is semifree,

\[
c(N_X) = (1 + u)^{j+1}, \quad \text{and} \quad c(N_Y) = (1 + v)^{i+1},
\]

where \( N_X \) and \( N_Y \) are the normal bundles to \( X \) and \( Y \), respectively.

(B) The action is not semifree, but no point has stabilizer \( \mathbb{Z}_k \) for any \( k > 2 \); moreover,

\[
\begin{align*}
\dim(X) &= \dim(Y) \quad \text{and} \quad \dim(M^{\mathbb{Z}_2}) = \dim(M) - 2; \\
c(N_{M^{\mathbb{Z}_2}})|_X &= 1 + u \quad \text{and} \quad c(N_{M^{\mathbb{Z}_2}})|_Y = 1 + v; \\
c(N_{M^{\mathbb{Z}_2}})|_X &= \frac{(1 + u)^{i+1}}{1 + 2u} \quad \text{and} \quad c(N_{M^{\mathbb{Z}_2}})|_Y = \frac{(1 + v)^{j+1}}{1 + 2v},
\end{align*}
\]

where \( N_{M^{\mathbb{Z}_2}} \) denotes the normal bundle of \( M^{\mathbb{Z}_2} \) in \( M \), and, \( N_{X}^{M^{\mathbb{Z}_2}} \) and \( N_{Y}^{M^{\mathbb{Z}_2}} \) denote the normal bundles of \( X \) and of \( Y \), respectively, in \( M^{\mathbb{Z}_2} \).

The cohomology ring and Chern classes of \( \mathbb{C}P^n \) and \( \tilde{G}_2(\mathbb{R}^{n+2}) \) are exactly those described in Theorem [1]. Moreover, we can transform any non-trivial circle action into an effective circle action by quotienting out by the subgroup which acts trivially. Therefore, by the proposition below, which combines Corollary 3.16 and Remark 3.18 in [5], Theorem [1] is an immediate consequence of Theorem [2].

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3 A group \( G \) acts **effectively** on \( M \) if for every non-trivial \( g \in G \), there exists \( m \in M \) so that \( g \cdot m \neq m \).

4 A circle action is **semifree** if there are no points with stabilizer \( \mathbb{Z}_k \) for any \( k \geq 2 \).
Proposition 1.3 (Tolman). Let the circle act on compact symplectic manifolds \((M, \omega)\) and \((\tilde{M}, \tilde{\omega})\) with moment maps \(\phi: M \to \mathbb{R}\) and \(\tilde{\phi}: \tilde{M} \to \mathbb{R}\), respectively; assume that \(H^j(\tilde{M}; \mathbb{Z}) = H^j(\mathbb{C}P^n; \mathbb{Z})\) for all \(j\). Also assume that there is a bijection from the fixed components \(F_1, \ldots, F_k\) of \(M\) to the fixed components \(\tilde{F}_1, \ldots, \tilde{F}_k\) of \(\tilde{M}\) so that there exists an isomorphism \(f^*: H^*_{S^1}(\tilde{F}_i; \mathbb{Z}) \to H^*_{S^1}(F_i; \mathbb{Z})\) such that \(f^*(c^{S^1}(\tilde{M})|_{\tilde{F}_i}) = c^{S^1}(M)|_{F_i}\) for all \(i\). Then there is an isomorphism \(f^2: H^*(\tilde{M}; \mathbb{Z}) \to H^*(M; \mathbb{Z})\) so that \(f^2(c(\tilde{M})) = c(M)\).

Alternatively, we can use Theorem [2] to directly calculate the equivariant cohomology of \(M\) and to prove Theorem [1] see Section [3].

Note that, in the case that one of the fixed components is a point, Theorems [1] and [2] are an immediate consequence of Delzant’s theorem [1]. He classified (up to equivariant symplectomorphism) Hamiltonian circle actions on compact symplectic manifolds \((M, \omega)\) with one isolated fixed point and one other fixed component. Additionally, Haussmann and Holm are studying Hamiltonian circle actions with two fixed components, but from a very different perspective [2]. In particular, they are not focussing on the case that we consider here.

Example 1.4. Given \(n \geq 1\), let \(\mathbb{C}P^n\) denote the complex projective space. Since this \(2n\)-dimensional manifold naturally arises as a coadjoint orbit of \(SU(n + 1)\), it inherits a symplectic form \(\omega\) and a Hamiltonian \(SU(n + 1)\) action. Thus, for any \(j \in \{0, \ldots, n - 1\}\), there is a Hamiltonian circle action given by
\[
\lambda \cdot [z_0, z_1, \ldots, z_n] = [\lambda z_0, \lambda z_1, \ldots, \lambda z_j, z_{j+1}, \ldots, z_n].
\]
The fixed set \((\mathbb{C}P^n)^{S^1}\) consists of two components:
\[
\{[z] \in \mathbb{C}P^n \mid z_k = 0 \forall k \leq j\} \simeq \mathbb{C}P^{n-j-1} \quad \text{and}
\{[z] \in \mathbb{C}P^n \mid z_k = 0 \forall k > j\} \simeq \mathbb{C}P^j.
\]
Note that \(2(n - j - 1) + 2j = 2n - 2\), as required.

Example 1.5. Given \(n \geq 3\), let \(\tilde{G}_2(\mathbb{R}^{n+2})\) denote the Grassmannian of oriented two-planes in \(\mathbb{R}^{n+2}\). Since this \(2n\)-dimensional manifold naturally arises as a coadjoint orbit of \(SO(n + 2)\), it inherits a symplectic form \(\omega\) and a Hamiltonian \(SO(n + 2)\) action. Thus, if \(n\) is odd, there is a Hamiltonian circle action on \(\tilde{G}_2(\mathbb{R}^{n+2})\) induced by the action on \(\mathbb{R}^{n+2} \simeq \mathbb{R} \times \mathbb{C}P^{(n+1)}\) given by
\[
\lambda \cdot (t, z_1, \ldots, z_{\frac{n}{2}(n+1)}) = (t, \lambda z_1, \ldots, \lambda z_{\frac{n}{2}(n+1)}).
\]
The fixed set consists of two components, corresponding to the two orientations on the real two-planes in \(\mathbb{P}(\{0\} \times \mathbb{C}P^{(n+1)}) \simeq \mathbb{C}P^{\frac{n}{2}(n-1)}\). Note that \(2\left(\frac{n}{2}(n - 1)\right) + 2\left(\frac{1}{2}(n - 1)\right) = 2n - 2\), as required.
The techniques in this paper are new, and are quite different from those used in [5]. In that paper, it was nearly always sufficient to examine Chern classes on the fixed components, using the Atiyah-Bott-Berline-Vergne localization formula. In this paper, we work much more directly with the cohomology ring and Chern classes of the reduced space itself. We believe that this more direct approach will be vital for further progress.

The outline of this paper is fairly straightforward. We describe properties of moment maps in §2; this is mostly review. In §3, we use Theorem 2 to calculate the possible cohomology rings (ordinary and equivariant) and Chern classes of $M$. The rest of the paper is dedicated to proving Theorem 2. We consider the implications of our two main restriction – that the fixed components have minimal dimension, and that their are only two fixed components, in §4 and §5, respectively. In the next section, we bring these arguments together to complete the proof of Theorem 2 in the semifree case. Finally, in §7 we study isotropy submanifolds of actions with only two fixed components, and in §8 we use this to complete the proof of our main theorem.

1.1. Open Questions.

The results in this paper suggest some interesting open questions. First, although the assumption that the fixed components have minimal dimension does not in general imply that $b_{2i}(M) = 1$ for all $i \in \{0, \ldots, \frac{1}{2} \dim(M)\}$, we may consider extending Question 1 to this case. (See Remark 4.3.)

**Question 2.** Consider a Hamiltonian circle action on a compact symplectic manifold $(M, \omega)$ which satisfies $\sum_{F \subset M^{\#}} (\dim(F) + 2) = \dim(M) + 2$.

1. Is $b_{2i+1}(M) = 0$ for all $i$? Is $H^\ast(M; \mathbb{Z})$ torsion-free?
2. Are the Chern classes $c_i(M)$ determined by the cohomology ring $H^\ast(M; \mathbb{Z})$?

The first interesting case which we hope to consider is that of an 8-dimensional manifold $M$ with an isolated fixed minimum, an isolated fixed maximum, and a 4-dimensional fixed component of index 2.

Alternatively, one could attempt to classify Hamiltonian circle actions with two fixed components, $X$ and $Y$. In this case, in light of our results here and Remark 5.3, the interesting cases to consider are when $\dim(X) + \dim(Y) \geq \dim(M)$ and when $X$ and $Y$ have codimension at least four. The first such case would be an 8-dimensional manifold $M$ with two 4-dimensional fixed components.

More generally, we prove in the appendix that for any effective Hamiltonian circle action with two fixed components, no point has stabilizer $\mathbb{Z}_k$ for any $k > 6$. However, we do not know of any example with stabilizer $\mathbb{Z}_k$ for any $k > 2$. (See also Proposition 7.13.) This raises another question.
Question 3. Does there exist an effective Hamiltonian circle action on a compact symplectic manifold \((M, \omega)\) such that \(M^{S^1}\) has exactly two components and there exists a point \(x \in M\) with stabilizer \(Z_k\), for \(k > 2\)?

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2. Background

The main goal of this section is to introduce some background material and establish our notation. However, in a few cases we will need to slightly extend known results.

Let the circle act (possibly trivially) on a space \(X\). The equivariant cohomology of \(X\) is 
\[ H^*_S(X) = H^*(X \times S^1, S^\infty). \]

For example, if \(p\) is a point then 
\[ H^*_S(p; \mathbb{Z}) = H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[t]. \]

More generally, if \(F \subset X^{S^1}\) is a fixed component, then 
\[ H^*_S(F) \text{ is naturally isomorphic to } H^*(F) \otimes H^*(\mathbb{C}P^\infty) = H^*(F)[t]. \]

In contrast, if the stabilizer of every point \(x \in X\) is finite, then 
\[ H^*_S(X; \mathbb{R}) \text{ is naturally isomorphic to } H^*(X/S^1; \mathbb{R}). \]

Note that the projection map \(X \times S^1, S^\infty \to \mathbb{C}P^\infty\) induces a pull-back map
\[ (2.1) \quad \pi^* : H^*(\mathbb{C}P^\infty) \to H^*_S(X); \]

hence, \(H^*_S(X)\) is a \(H^*(\mathbb{C}P^\infty)\) module.

Let \(M\) be a compact manifold. A symplectic form on \(M\) is a closed, non-degenerate two-form \(\omega \in \Omega^2(M)\). A circle action on \(M\) is symplectic if it preserves \(\omega\). A symplectic circle action is Hamiltonian if there exists a moment map, that is, a map \(\phi : M \to \mathbb{R}\) such that 
\[ -d\phi = \iota_{\xi_M} \omega, \]

where \(\xi_M\) is the vector field on \(M\) induced by the circle action. Since \(\iota_{\xi_M} \omega\) is closed, every symplectic action is Hamiltonian if \(H^1(M; \mathbb{R}) = 0\).

As we mentioned in the introduction, the moment map \(\phi : M \to \mathbb{R}\) is a Morse-Bott function whose critical set is exactly the fixed point set \(M^{S^1}\). Therefore, if \(c \in \mathbb{R}\) is a regular value of \(\phi\), then every point in the level set \(\phi^{-1}(c)\) has finite stabilizer. Since \(\phi^{-1}(c)\) is a manifold, this implies that the symplectic reduction of \(M\) at \(c\),
\[ M_c := \phi^{-1}(c)/S^1, \]

is an orbifold. Since \(\omega|_{\phi^{-1}(c)}\) is a basic two-form, there exists a symplectic form \(\omega_c \in \Omega^2(M_c)\) such that \(\rho^*(\omega_c) = \omega|_{\phi^{-1}(c)}\), where \(\rho : \phi^{-1}(c) \to M_c\) is the quotient map. Let
\[ \kappa : H^*_S(M; \mathbb{R}) \to H^*(M_c; \mathbb{R}) \]
be the composition of the restriction map from $H^*_S(M; \mathbb{R})$ to $H^*_S(\phi^{-1}(c); \mathbb{R})$ and the isomorphism from $H^*_S(\phi^{-1}(c); \mathbb{R})$ to $H^*(M; \mathbb{R})$; this is called the Kirwan map.

Given a symplectic manifold $(M, \omega)$, there is an almost complex structure $J: T(M) \to T(M)$ which is compatible with $\omega$, that is, $\omega(J(\cdot), \cdot)$ is a Riemannian metric. Moreover, the set of such structures is connected, and so there is a well defined total Chern class $c(M) \in H^*(M; \mathbb{Z})$. Similarly, given a circle action on $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$, there is a well-defined multiset of integers, called weights, associated to each fixed point $p$. In fact, for any fixed component $F$, the tangent bundle $T(M)|_F$ naturally splits into subbundles – one corresponding to each weight.

The negative normal bundle $N_F^-$ at a fixed component $F$ is the sum of the subbundles of $T(M)|_F$ with negative weights. In particular, if $\lambda_F$ is the number of negative weights in $T_p M$ for any $p \in F$ (counted with multiplicity), then the index of $\phi$ as a Morse-Bott function at $F$ is $2\lambda_F$. Under the identification $H^*_S(F) = H^*(F)[t]$, the equivariant Euler class $e^S(N_F^-)$ is a polynomial in $t$ whose highest degree term is $\Lambda_F t^\lambda_F$, where $\Lambda_F \in \mathbb{Z} \setminus \{0\}$ is the product of the negative (integer) weights at $F$. As Atiyah and Bott pointed out, this fact implies that $e^S(N_F^-)$ is not a zero divisor in $H^*_S(F; \mathbb{R})$. Kirwan proved that this fact has remarkable consequences for Hamiltonian actions [3]; we explain some of these consequences below.

Let $R$ be a commutative ring (with unit), for example, $\mathbb{R}$, $\mathbb{Z}$, or $\mathbb{Z}_p$. If $R = \mathbb{R}$, or if the action is semifree, or if $H^{2\lambda_F}(F; \mathbb{Z})$ is torsion-free and $R = \mathbb{Z}$, then multiplication by $e^S(N_F^-)$ induces an injection from $H^S_{2i}(F; R)$ to $H^S_{2i}(F; R)$. (See [6] for comments on the case $R \neq \mathbb{R}$.) Let $M^\pm = \phi^{-1}(-\infty, \phi(F) \pm \epsilon)$, where $\epsilon > 0$ is sufficiently small. For simplicity, assume that $F$ is the only fixed set in $M^+ \setminus M^-$. By the previous paragraph, if $R = \mathbb{R}$, or if the action is semifree, or if $H^{2\lambda_F}(F; \mathbb{Z})$ and $H^{2\lambda_F+1}(F; \mathbb{Z})$ are torsion-free and $R = \mathbb{Z}$, then the map from $H^S_{2i}(M^+, M^-; R) = H^S_{2i}(M^+, M^-; R)$ to $H^S_{2i}(F; R)$ is injective for $* = j$ and $* = j + 1$; therefore, the long exact sequence in equivariant cohomology for the pair $(M^+, M^-)$ breaks into a short exact sequence

$$0 \to H^S_{2i}(M^+, M^-; R) \to H^S_{2i}(M^+; R) \to H^S_{2i}(M^-; R) \to H^S_{2i}(F; R) \to 0.$$  \hspace{1cm} (2.2)

In particular, if $i \leq 2\lambda_F$ and $R = \mathbb{Z}$, then (2.2) is exact because $H^i(F; \mathbb{Z})$ is torsion-free for all $i \leq 1$. (Note that, if $i \leq 2\lambda_F - 2$ then $H^S_{2i}(M^+, M^-; R) = H^S_{2i+1}(M^+, M^-; R) = 0$ and so $H^S_{2i}(M^+; R) = H^S_{2i}(M^-; R)$ for any commutative ring $R$.)

Additionally, restriction induces a natural map of exact sequences

$$\ldots \longrightarrow H^S_{2i}(M^+, M^-; R) \longrightarrow H^S_{2i}(M^+; R) \longrightarrow H^S_{2i}(M^-; R) \longrightarrow \ldots$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\ldots \longrightarrow H^S_{2i+1}(M^+, M^-; R) \longrightarrow H^S_{2i+1}(M^+; R) \longrightarrow H^S_{2i+1}(M^-; R) \longrightarrow \ldots$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\ldots \longrightarrow H^S_{2i+2}(M^+, M^-; R) \longrightarrow H^S_{2i+2}(M^+; R) \longrightarrow H^S_{2i+2}(M^-; R) \longrightarrow \ldots$$
The restriction map from $H^j_{S^1}(M^+, M^-; R)$ to $H^j(M^+, M^-; R)$ is surjective because $H^j_{S^1}(F; R) = H^j(F; R)[t]$. Hence, by an easy diagram chase, if (2.2) is exact and if the restriction map from $H^j_{S^1}(M^-; R)$ to $H^j(M^-; R)$ is surjective, then the restriction map from $H^j_{S^1}(M^+; R)$ to $H^j(M^+; R)$ is also surjective. If, additionally, the restriction map from $H^{j-1}_{S^1}(M^-; R)$ to $H^{j-1}(M^-; R)$ is surjective, then
\[(2.3) \quad 0 \to H^j(M^+, M^-; R) \to H^j(M^+; R) \to H^j(M^-; R) \to 0\]
is a short exact sequence.

Note that, if $j = 2$ and $R = \mathbb{Z}$, then (2.2) is exact for any fixed component $F$. This is because either $\lambda_F = 0$, in which case $e^{S^1}(N^-_F) = 1$, or $\lambda_F \geq 1$, in which case $H^{j-2\lambda_F}(F; \mathbb{Z})$ and $H^{j-2\lambda_F+1}(F; \mathbb{Z})$ are torsion-free. Therefore, the proposition below follows easily by induction and the paragraph above.

**Proposition 2.4.** Let the circle act on a compact symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. The natural restriction $H^*_{S^1}(M; \mathbb{Z}) \to H^2(M; \mathbb{Z})$ is onto.

More generally, as we saw above, if the action is semifree, if $R = \mathbb{R}$ or if $H^*(M^{S^1}; \mathbb{Z})$ is torsion-free and $R = \mathbb{Z}$, then (2.2) is exact for every $j$. By induction, this implies that (2.3) is exact for all $F$, that the restriction map $H^j_{S^1}(M; R) \to H^j(M; R)$ is a surjection, and that the restriction map $\iota^*: H^*_{S^1}(M; R) \to H^*_{S^1}(M^{S^1}; R)$ is an injection.

If $H^*(M^{S^1}; \mathbb{Z})$ is torsion-free and $R = \mathbb{Z}$, or if $R = \mathbb{R}$, then (2.2) and (2.3) imply that
\[
H^j_{S^1}(M; R) = \bigoplus_{F \subset M^{S^1}} H^{j-2\lambda_F}(F; R), \quad \text{and} \quad H^j(M; R) = \bigoplus_{F \subset M^{S^1}} H^{j-2\lambda_F}(F; R) \quad \forall j,
\]
where the sum is over all fixed components. In particular, $\phi$ is **perfect** and **equivariantly perfect**, that is, these equations hold for $R = \mathbb{R}$. Moreover, by Leray-Hirsch, the fact that $H^j(M^{S^1}; \mathbb{Z})$ is torsion-free and that the restriction map from $H^j_{S^1}(M; R)$ to $H^j(M; R)$ is surjective implies that the kernel of this map is the image $\pi^*(H^*(\mathbb{CP}^\infty; R))$, which is the ideal generated by $\pi^*(t)$, where $t \in H^2(\mathbb{CP}^\infty; R)$ is the generator. (See 2.1.) Hence, if we want to compute the (ordinary) cohomology of $M$, it is enough to determine the equivariant cohomology of $M$;
\[(2.5) \quad H^*(M; R) = H^*_{S^1}(M; R)/(t).
\]

We can use a similar argument as in [5] to prove the following claim.

**Proposition 2.6.** Let the circle act on a compact symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Consider $\beta \in H^*_{S^1}(M; \mathbb{R})$ so that $\beta|_{F^\omega} = 0$
for all fixed components \( F' \) such that \( \phi(F') < \phi(F) \), where \( F \) is a fixed component. Then \( \beta|_F \) is a multiple of \( e^{S^1}(N_F^*) \).

Given any manifold \( M \), there is a natural map from \( H^*(M; \mathbb{Z}) \) to \( H^*(M; \mathbb{R}) \). The image of this map is a lattice in \( H^*(M; \mathbb{R}) \). We shall say that a class is integral if it lies in the image of this map and is primitive if, in addition, it is not a positive integer multiple of any other integral class.

**Lemma 2.7.** Let the circle act on a compact symplectic manifold \((M, \omega)\) with moment map \( \phi: M \to \mathbb{R} \). Let \( F \) be a fixed component.

- There exists \( \tilde{u} \in H^2_{S^1}(M; \mathbb{R}) \) so that,
  \[
  \tilde{u}|_{F'} = [\omega|_{F'}] + t (\phi(F) - \phi(F')) \quad \text{and} \quad \kappa_c(\tilde{u} - t(\phi(F) - c)) = \omega_c
  \]
  for all fixed components \( F' \) and all regular values \( c \in \mathbb{R} \). Here, \( \kappa_c: H^2_{S^1}(M; \mathbb{R}) \to H^*(M_c) \) is the Kirwan map and \((M_c, \omega_c)\) is the symplectic reduction of \( M \) at \( c \).

- If \([\omega]\) is integral, then \( \tilde{u} \) is integral.

**Proof.** To prove the first claim, let \( \tilde{u} = [\omega - t\phi + t\phi(F)] \) in the De Rham model of equivariant cohomology. If \( c \) is a regular value, then \( \tilde{u} - t\phi(F) + tc = [\omega - t\phi + tc] \), and so \( (\tilde{u} - t\phi(F) + tc)|_{\phi^{-1}(c)} = [\omega|_{\phi^{-1}(c)}] \), which maps to \( \omega_c \) under the natural isomorphism \( H^2_{S^1}(\phi^{-1}(c); \mathbb{R}) \simeq H^*(M_c) \).

If \([\omega] \in H^2(M; \mathbb{R})\) is integral, then by Proposition 2.4 there exists an integral class \( \tilde{\alpha} \in H^2_{S^1}(M; \mathbb{R}) \) which maps to \([\omega]\) under the natural restriction \( H^2_{S^1}(M; \mathbb{R}) \to H^2(M; \mathbb{R}) \). Then the restriction of \( \tilde{\alpha} - \tilde{u} \) under the same map is zero. Since \( H^*(\mathbb{C}P^\infty; \mathbb{R}) = \mathbb{R}[t] \), this implies that \( \tilde{\alpha} - \tilde{u} = \lambda t \) for some constant \( \lambda \in \mathbb{R} \). Finally, since \( \tilde{\alpha} \) is integral, \( \tilde{\alpha}|_F = \tilde{u}|_F + \lambda t = [\omega|_F] + \lambda t \) is integral; hence \( \lambda \in \mathbb{Z} \).

**Lemma 2.8.** Let the circle act on a complex vector bundle \( E \) of rank \( d \) over a compact manifold \( X \) so that \( E^{S^1} = X \). Assume that there exists a non-zero \( \lambda \in \mathbb{Z} \) so that the circle acts on \( E \) with weight \( \lambda \). Then there exists \( c_i \in H^{2i}(X; \mathbb{Z}) \) for all \( i \in \{0, \ldots, \frac{1}{2}\dim(M)\} \) such that

\[
\begin{align*}
  c(E) &= 1 + c_1 + \cdots + c_{d-1} + c_d, \\
  c^{S^1}(E) &= (1 + \lambda t)^d + c_1(1 + \lambda t)^{d-1} + \cdots + c_{d-1}(1 + \lambda t) + c_d, \quad \text{and} \\
  e^{S^1}(E) &= (\lambda t)^d + c_1(\lambda t)^{d-1} + \cdots + c_{d-1}(\lambda t) + c_d.
\end{align*}
\]

Here, \( c(E) \), \( c^{S^1}(E) \), and \( e^{S^1}(E) \) are the total Chern class of \( E \), the total equivariant Chern class of \( E \), and the equivariant Euler class of \( E \), respectively.

**Proof.** By the splitting principle, there exists a space \( Y \) and a map \( p: Y \to X \) such that \( p^*: H^*(X; \mathbb{Z}) \to H^*(Y; \mathbb{Z}) \) is injective and the pullback bundle \( p^*(E) \) breaks up as the direct sum of line bundles. Therefore, without loss
of generality we may assume that the bundle is a direct sum of line bundles with first Chern class $\alpha_1, \ldots, \alpha_d$ respectively. Then
\[
c(E) = \prod_{i=1}^{d} (1 + \alpha_i) \quad \text{and} \quad c^{S^1}(E) = \prod_{i=1}^{d} (1 + \lambda t + \alpha_i).
\]
The claim follows immediately. $\square$

3. USING THEOREM 2 TO CALCULATE THE COHOMOLOGY RING OF $M$

In this section, we use Theorem 2 to compute the possible cohomology rings (ordinary and equivariant) and Chern classes of $M$. In particular, we give an alternative proof of Theorem 1 and prove the proposition below.

**Proposition 3.1.** Let the circle act effectively on a compact symplectic $2n$-dimensional manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume that $M^{S^1}$ has exactly two components, $X$ and $Y$, and that $\dim(X) + \dim(Y) = \dim(M) - 2$. Then
\[
H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{i+1} \quad \text{and} \quad H^*(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{j+1},
\]
where $\dim(X) = 2i$ and $\dim(Y) = 2j$. If $\phi(X) < \phi(Y)$, then one of the following is true:

(A) The action is semifree,
\[
H_{S^1}^*(M; \mathbb{Z}) = \mathbb{Z}[\bar{x}, t]/(\bar{x}^{j+1}(\bar{x} + t)^{j+1}) \quad \text{and} \quad c^{S^1}(M) = (1 + \bar{x})^{i+1}(1 + \bar{x} + t)^{j+1},
\]
where $\bar{x}|_X = u$, and $\bar{x}|_Y = v - t$.

(B) The action is not semifree, $\dim(X) = \dim(Y)$,
\[
H_{S^1}^*(M; \mathbb{Z}) = \mathbb{Z}[\bar{x}, \bar{y}, t]/\left((\bar{x}^{i+1} - 2\bar{y}, \bar{y}(\bar{y} + \frac{1}{2}(\bar{x} + 2t)^{i+1} - \bar{x}^{i+1}))\right) \quad \text{and} \quad c^{S^1}(M) = \frac{(1 + \bar{x})^{j+1}(1 + \bar{x} + 2t)^{i+1}(1 + \bar{x} + t)}{1 + 2\bar{x} + 2t},
\]
where $\bar{x}|_X = u$, $\bar{x}|_Y = v - 2t$, $\bar{y}|_X = 0$, and $\bar{y}|_Y = \frac{1}{2}(v - 2t)^{i+1}$.

In both cases, $t$ generates $\pi^1(H^2(\mathbb{C}P^\infty; \mathbb{Z})) \subset H_{S^1}^2(M; \mathbb{Z})$ and $\bar{x}$ has degree $2$; in case (B), $\bar{y}$ has degree $n+1$.

**Proof.** By Theorem 2 $H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{i+1}$ and $H^*(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{j+1}$. In particular, $H^*(M^{S^1}; \mathbb{Z})$ is torsion-free. As we showed in §2 this implies that the restriction map $H_{S^1}^*(M; \mathbb{Z}) \to H_{S^1}^*(M^{S^1}; \mathbb{Z})$ is injective and that
\[
H^k(M; \mathbb{Z}) = \bigoplus_{F \subset M^{S^1}} H^{k-2\lambda_F}(F; \mathbb{Z}) = H^k(\mathbb{C}P^n; \mathbb{Z}) \quad \forall k.
\]
Hence, Proposition 3.9 of [5] states that whenever $H^k(M; \mathbb{Z}) = H^k(\mathbb{C}P^n; \mathbb{Z})$ for all $k$, the classes $1, \alpha_1, \ldots, \alpha_n$ defined by

\begin{equation}
\alpha_i = \frac{\Lambda_{F_i}}{m_i} \left( c_1^{S_i}(M) - \Gamma_{F_i} t \right)^{i - \lambda_{F_i}} \prod_{\lambda_{F'} < \lambda_{F_i}} \left( \frac{c_1^{S_i}(M) - \Gamma_{F'} t}{\Gamma_{F_i} - \Gamma_{F'}} \right)^{\frac{1}{2} \dim(F') + 1}
\end{equation}

form a basis for $H^*_S(M; \mathbb{Z})$ as a $H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[t]$ module. Here, $F_i$ is the unique fixed component so that $H^{2i-2\lambda_{F_i}}(F_i; \mathbb{Z}) = \mathbb{Z}$, and $m_i \in \mathbb{Z}$ is chosen so that $\frac{1}{m_i} c_1(M)^{i - \lambda_{F_i}} |_{F_i}$ generates $H^{2i-2\lambda_{F_i}}(F_i; \mathbb{Z})$ for each integer $i$ such that $0 \leq 2i \leq 2n$. Moreover, $\Lambda_{F_i}$ is the product of the negative weights at $F_i$ and $\Gamma_{F_i}$ is the sum of the weights at $F_i$ for each fixed component $F_i$. Finally, the product is over all fixed components $F'$ such that $\lambda_{F'} < \lambda_{F_i}$.

Assume first that the action is semifree. By part (A) of Theorem 2 and Lemma 2.8,

\begin{equation}
\frac{1}{(n+1)^c} c_1^k(M) |_X = (n+1)u + (j+1)t \quad \text{and} \quad \frac{1}{(n+1)^c} c_1^k(M) |_Y = (n+1)v - (i+1)t.
\end{equation}

Hence, $\frac{1}{(n+1)^c} c_1^k(M) |_X$ generates $H^{2k}(X; \mathbb{Z})$ for all $k \in \{0, \ldots, i\}$; similarly, $\frac{1}{(n+1)^c} c_1^k(M) |_Y$ generates $H^{2k}(Y; \mathbb{Z})$ for all $k \in \{0, \ldots, j\}$. Additionally, $\Gamma_X = (j + 1)$, $\Gamma_Y = -(i + 1)$, $\Lambda_X^{-1} = 1$, and $\Lambda_Y^{-1} = (-1)^{i+1}$. Therefore, by (3.2),

\begin{equation}
\alpha_k = \begin{cases}
\frac{1}{(n+1)^c} c_1^k(M) - (j + 1)t \quad &0 \leq k \leq i \\
\frac{1}{(n+1)^c} c_1^k(M) - (j + 1)t \quad &i < k \leq n.
\end{cases}
\end{equation}

In particular, (3.3) implies that $\alpha_1 |_X = u$ and $\alpha_1 |_Y = v - t$. Hence, if we let $\bar{x} = \alpha_1$, then $\bar{x}^{j+1} + (\bar{x} + t)^{j+1} = 0$ and part (A) of Theorem 2 implies that

\begin{align*}
\alpha_k &= \left\{ \begin{array}{ll}
\frac{1}{n^c} (c_1^k(M) - nt)^k & 0 \leq k \leq i \\
\frac{1}{2n^c} (c_1^k(M) - nt)^{i+1} (c_1^k(M) + nt)^{k-i-1} & i < k \leq n.
\end{array} \right.
\end{align*}

Since the restriction map $H^*_S(M; \mathbb{Z}) \rightarrow H^*_S(M^{S_i}; \mathbb{Z})$ is injective, claim (A) follows easily.

Now assume that the action is not semifree. By part (B) of Theorem 2 and Lemma 2.8, $\dim(X) = \dim(Y)$ and so $i = \frac{1}{2} (n - 1)$; moreover,

\begin{equation}
\frac{1}{n^c} c_1^k(M) |_X = nu + nt \quad \text{and} \quad \frac{1}{n^c} c_1^k(M) |_Y = nv - nt.
\end{equation}

Hence, $\frac{1}{n^c} c_1^k(M) |_X$ generates $H^{2k}(X; \mathbb{Z})$ and $\frac{1}{n^c} c_1^k(M) |_Y$ generates $H^{2k}(Y; \mathbb{Z})$ for all $k \in \{0, \ldots, i\}$. Additionally, $\Gamma_X = n$, $\Gamma_Y = -n$, $\Lambda_X^{-1} = 1$, and $\Lambda_Y^{-1} = 2^j(-1)^{i+1}$. Therefore, by (3.2),

\begin{equation}
\alpha_k = \begin{cases}
\frac{1}{n^c} (c_1^k(M) - nt)^k & 0 \leq k \leq i \\
\frac{1}{2n^c} (c_1^k(M) - nt)^{i+1} (c_1^k(M) + nt)^{k-i-1} & i < k \leq n.
\end{cases}
\end{equation}

In particular, (3.4) implies that $\alpha_1 |_X = u$, $\alpha_1 |_Y = v - 2t$, $\alpha_{i+1} |_X = 0$, and $\alpha_{i+1} |_Y = \frac{1}{2} (v - 2t)^{i+1}$. Hence, if we let $\bar{x} = \alpha_1$ and $\bar{y} = \alpha_{i+1}$, then
\[
\tilde{x}^{i+1} - 2\tilde{y} = 0 \quad \text{and} \quad \tilde{y}(\tilde{y} + \frac{1}{2}(\tilde{x} + 2t)^{i+1} - \tilde{x}^{i+1})) = 0. \quad \text{(Note that the latter expression does lie in } \mathbb{Z}[\tilde{x}, \tilde{y}, t], \text{ while the expression } \frac{1}{2}\tilde{y}(\tilde{x} + 2t)^{i+1} \text{ does not.)}
\]
Moreover, part (B) of Theorem 2 implies that
\[
c^S(M)|_X = \frac{(1 + \tilde{x})^{i+1}(1 + \tilde{x} + 2t)^{i+1}(1 + \tilde{x} + t)}{1 + 2\tilde{x} + 2t} |_X \quad \text{and}
\]
\[
c^S(M)|_Y = \frac{(1 + \tilde{x})^{i+1}(1 + \tilde{x} + 2t)^{i+1}(1 + \tilde{x} + t)}{1 + 2\tilde{x} + 2t} |_Y.
\]
Since the restriction map \(H^*_S(M; \mathbb{Z}) \to H^*_S(M^S; \mathbb{Z})\) is injective, claim (B) follows easily.

Finally, as we showed in \([2]\) the fact that \(H^*(M^S; \mathbb{Z})\) is torsion-free implies that \(H^*(M; \mathbb{Z}) = H^*_S(M; \mathbb{Z})/(t)\), where \(t\) generates \(\pi^*(H^2(\mathbb{CP}^\infty; \mathbb{Z}))\). (See \((2.5)\).) Therefore, Theorem 4 follows immediately from the proposition above.

4. **The Case that the Fixed Components Have Minimal Dimension**

Consider a Hamiltonian circle action on a compact symplectic manifold \((M, \omega)\). Because \(M\) is symplectic, \(H^{2i}(M; \mathbb{R}) \neq 0\) for all \(i \in \{0, \ldots, \frac{1}{2}\dim(M)\}\). As we saw in the introduction, since the moment map is a Morse-Bott function this implies that
\[
\sum_{F \subseteq M^S} (\dim(F) + 2) \geq \dim(M) + 2,
\]
where the sum is over fixed components. In this section, we consider the case that the fixed components have minimal dimension, that is, \([1,2]\) holds. This assumption is closely related to the assumption that the even Betti numbers are minimal, that is, \([1,1]\) holds. For example, \([1,1]\) implies \([1,2]\).

**Lemma 4.1.** Let the circle act on a compact symplectic manifold \((M, \omega)\) with moment map \(\phi: M \to \mathbb{R}\). Assume that \(b_{2i}(M) = 1\) for all \(i \in \{0, \ldots, \frac{1}{2}\dim(M)\}\). Then \(\sum_F (\dim(F) + 2) = \dim(M) + 2\).

**Proof.** This claim is an immediate consequence of Lemma 3.3 in \([5]\), which states that, for each \(i \in \{0, \ldots, \frac{1}{2}\dim(M)\}\), there exists a unique fixed component \(F\) such that \(0 \leq 2i - 2\lambda_F \leq \dim(F)\). (Lemma 3.3 itself follows from the facts that \(\phi\) is a perfect Morse-Bott function and that \(H^{2i}(F; \mathbb{R}) \neq 0\) for all fixed components \(F\) and \(i \in \{0, \ldots, \frac{1}{2}\dim(F)\}\).)

The following proposition – which is the main result in this section – gives a partial converse.

**Proposition 4.2.** Let the circle act on a compact symplectic manifold \((M, \omega)\) with moment map \(\phi: M \to \mathbb{R}\). Assume that \(\sum_{F \subseteq M^S} (\dim(F) + 2) = \dim(M) + 2\). Let \(X\) be the minimal fixed component.

1. \(H^i(M; \mathbb{R}) = H^i(\mathbb{CP}^a; \mathbb{R})\) for all \(i \in \{0, \ldots, \dim(X) + 2\}\).
We now show that (1.1) holds if $M$ is a compact symplectic manifold.

**Theorem.** Let $M$ be a compact symplectic manifold. Then (1.1) holds.

**Proof.** We will prove the theorem by induction on the dimension of $M$. If $M$ is one-dimensional, then (1.1) holds trivially. Assume that (1.1) holds for all compact symplectic manifolds of dimension less than $n$. Let $M$ be a compact symplectic manifold of dimension $n$. If $M$ is a point, then (1.1) holds trivially. Assume that $M$ is not a point. Then there exists a Hamiltonian circle action on $M$.

Let $F$ be a fixed component of $M$. Then $H^0(F; \mathbb{R}) = \mathbb{R}$ and $H^1(F; \mathbb{R}) = 0$. Moreover, $H^i(F; \mathbb{R}) = 0$ for all $i > 1$. Therefore, (1.1) holds for all fixed components of $M$.

Consider a Hamiltonian circle action on a compact symplectic manifold $(M, \omega)$ which satisfies (1.2). Clearly, $H^0(F; \mathbb{R}) = H^{\dim(F)}(F; \mathbb{R}) = \mathbb{R}$ for every fixed component $F$. Therefore, if every fixed component has dimension 0 or 2, then (1.1) follows from Lemma 4.3 below.

In the remaining cases considered above, (1.1) follows from Lemma 4.4 and Poincaré duality on $M$ and $F$.

To see the last claim, note that for any $n > 2$ there is a Hamiltonian circle action on $\tilde{G}_2(\mathbb{R}^{n+2})$ induced by the action on $\mathbb{R}^{n+2} \cong \mathbb{C} \times \mathbb{R}^n$ given by

$$\lambda \cdot (z, x_1, \ldots, x_n) = (\lambda z, x_1, \ldots, x_n).$$

(See Example 1.5.) The fixed set has three components. Two are isolated fixed points which correspond to the orientations on the real two-plane $\mathbb{C} \times \{0\}$. The third component has dimension $2n - 4$ and corresponds to the set of oriented real two-planes in $\{0\} \times \mathbb{R}^n$. Hence, $(0 + 2) + (0 + 2) + (2n - 4 + 2) = 2n + 2$, as required by (1.2). However, if $n$ is even, then $H^n(\tilde{G}_2(\mathbb{R}^{n+2}); \mathbb{R}) = \mathbb{R}^2$.

To prove Proposition 4.2 we will need the following analog of Lemma 3.3 in [5].

**Lemma 4.4.** Let the circle act on a compact symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume that $\sum_{F \subset M} (\dim(F) + 2) = \dim(M) + 2$.

- For each $i \in \{0, \ldots, \frac{1}{2} \dim(M)\}$, there exists a unique fixed component $F$ such that $0 \leq 2i - 2\lambda_F \leq \dim(F)$.
- In particular, if $X$ is the minimal fixed component, then $\dim(X) \leq 2\lambda_F - 2$ for all other fixed components $F$.

**Proof.** Since $M$ is symplectic $H^{2i}(M; \mathbb{R}) \neq 0$ for all $i \in \{0, \ldots, \frac{1}{2} \dim(M)\}$. Since $\phi$ is a Morse-Bott function, there is at least one fixed component $F$ such that $0 \leq 2i - 2\lambda_F \leq \dim(F)$. Since $\sum_{F \subset M} (\dim(F) + 2) = \dim(M) + 2$, this proves the claim.

**Remark 4.5.** Consider a Hamiltonian circle action on a compact symplectic manifold $(M, \omega)$; assume that (1.2) holds. Although we will not use them in this paper, several of the results in §3 of [5] still work in this context if we

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5 A set of integers is relatively prime if their greatest common divisor is 1.
use Lemma 4.4 above instead of Lemma 3.3 in [5]. For example, the proof of Proposition 3.4 and Lemma 3.7 in [5] otherwise go through without any changes. Therefore, for all fixed components \( F \) and \( F' \),
\[
\phi(F') < \phi(F) \quad \text{exactly if} \quad \lambda_{F'} < \lambda_F; \quad \text{moreover}
\]
\[
H^j(M; \mathbb{Z}) = \bigoplus_{F' \subseteq M} H^{j−2\lambda_F}(F; \mathbb{Z}) \quad \forall \ j,
\]
where the sum is over all fixed components.

**Lemma 4.6.** Let the circle act on a compact symplectic manifold \((M, \omega)\) with moment map \( \phi: M \to \mathbb{R} \). Let \( X \) be the minimal fixed component and let \( \mathbb{F} \) be a field. Assume that \( \dim(X) \leq 2\lambda_F - 2 \) for all other fixed components \( F \). Assume also that there exist classes \( \tilde{u} \in H^2_{S^1}(M; \mathbb{F}) \) and \( u \in H^2(X; \mathbb{F}) \), such that \( \tilde{u}|_X = u \), and a fixed point \( y \) such that \( \tilde{u}|_y \neq 0 \). Then
\[
H^*(X; \mathbb{F}) = \mathbb{F}[u]/u^2 \dim(X)+1.
\]

**Proof.** Assume that, on the contrary, \( H^*(X; \mathbb{F}) \neq \mathbb{F}[u]/u^2 \dim(X)+1 \).

First, we claim that there exist \( \alpha \in H^j(X; \mathbb{F}) \) and \( \tilde{\alpha} \in H^j_{S^1}(M; \mathbb{F}) \) such that \( \alpha \neq 0, \tilde{\alpha}|_X = \alpha, \) and \( \tilde{\alpha}|_y = 0 \). To see this, note that at least one of the following is true:

(a) there exists a non-zero class \( \alpha \in H^{2i+1}(X; \mathbb{F}) \) for some \( i \); or

(b) there exists a class \( \alpha \in H^{2i}(X; \mathbb{F}) \) which is not a multiple of \( u^i \) for some \( i \). (Since \( X \) is symplectic; \( H^2(X; \mathbb{F}) \neq 0 \) for all \( i \in \{0, \ldots, \frac{1}{2} \dim(X)\} \).)

If (a) is true, then since \( \dim(X) < 2\lambda_F \) for all other fixed components \( F \), there exists a class \( \tilde{\alpha} \in H^{2i+1}_{S^1}(M; \mathbb{F}) \) such that \( \tilde{\alpha}|_X = \alpha \). (See (2.2).) Since \( H^{2i+1}(\mathbb{C}P^\infty; \mathbb{F}) = 0, \tilde{\alpha}|_y = 0 \). Similarly, if (b) is true then there exists \( \tilde{\alpha} \in H^{2i}_{S^1}(M; \mathbb{F}) \) such that \( \tilde{\alpha}|_X = \alpha \). Since \( \tilde{u}|_y \neq 0 \), we can define \( \lambda = \frac{\tilde{\alpha}|_y}{u|_y} \)
and then replace \( \alpha \) by \( \alpha - \lambda u^i \) and \( \tilde{\alpha} \) by \( \tilde{\alpha} - \lambda \tilde{u}^i \).

Since \( \mathbb{F} \) is a field, Poincaré duality implies that there exists a class \( \beta \in H^{\dim(X)−j}(X; \mathbb{F}) \) such that \( \alpha \cup \beta = u^{\frac{1}{2} \dim(X)} \). As before, there exists \( \tilde{\beta} \in H^{\dim(X)−j}_{S^1}(M; \mathbb{F}) \) such that \( \tilde{\beta}|_X = \beta \). Since \( \tilde{u}^{\frac{1}{2} \dim(X)}|_X = \left( \tilde{\alpha} \cup \tilde{\beta} \right)|_X, \) and since \( \dim(X) \leq 2\lambda_F - 2 \) for all other fixed components \( F \), we can conclude that \( \tilde{u}^{\frac{1}{2} \dim(X)} = \tilde{\alpha} \cup \tilde{\beta} \). But \( (\tilde{\alpha} \cup \tilde{\beta})|_y = \tilde{\alpha}|_y \cup \tilde{\beta}|_y = 0, \) while \( \tilde{u}^{\frac{1}{2} \dim(X)}|_y = (\tilde{u}|_y)^{\frac{1}{2} \dim(X)} \neq 0 \). This gives a contradiction. \( \square \)

We are now ready to prove our main result.

**Proof of Proposition 4.2.** By Lemma 4.4, \( \dim(X) \leq 2\lambda_F - 2 \) for every other fixed component \( F \). Moreover, there is exactly one fixed component \( F \) with \( 2\lambda_F = \dim(X) + 2 \).

By Lemma 2.7, there exists \( \tilde{u} \in H^2_{S^1}(M; \mathbb{R}) \) such that \( \tilde{u}|_X = [\omega|_X] \) and \( \tilde{u}|_y = t(\phi(X) - \phi(y)) \neq 0 \) for all fixed points \( y \notin X \). Since \( \phi \) is a perfect
If $[\omega]$ is integral, then by Lemma 4.6 there exists $\tilde{u} \in H^2_{S^1}(M; \mathbb{Z})$ and $u \in H^2(X; \mathbb{Z})$ so that $\tilde{u}|_X = u$, $\tilde{u}|_y = t(\phi(X) - \phi(y))$ for all fixed points $y$, and $u$ maps to $[\omega|_X] \in H^2(X; \mathbb{R})$. If the integers $\{\phi(X) - \phi(F)\}_{F \subset (M \setminus X)^{S^1}}$ are relatively prime, then for any prime $p$ there exists a fixed point $y$ so that $\phi(X) - \phi(y) \neq 0 \mod p$. Therefore, by Lemma 4.6, $H^*(X; \mathbb{Z}_p) = \mathbb{Z}_p[u]/u^{\frac{1}{2}\dim(X)+1}$. On the one hand, by Lemma 4.7 below, this implies that $H^*(X; \mathbb{Z})$ is torsion-free. On the other hand, it implies that $u^i$ is primitive for all $i \in \{0, \ldots, \frac{1}{2}\dim(X)\}$. Claim (3) follows immediately.

Lemma 4.7. Let $X$ be a compact manifold. Assume that $H^{2i+1}(X; \mathbb{Z}_p) = 0$ for all $i$ and all primes $p$. Then $H^*(X; \mathbb{Z})$ is torsion free.

Proof. Since $X$ is compact, the homology ring of $X$ is finitely generated. Moreover, $\text{Hom}(\mathbb{Z}, \mathbb{Z}_p) = \mathbb{Z}_p$ for all primes $p$, while $\text{Hom}(\mathbb{Z}_q, \mathbb{Z}_p) = \mathbb{Z}_p$ and $\text{Ext}(\mathbb{Z}_q, \mathbb{Z}_p) = \mathbb{Z}_p$ if $p$ divides $q$. Therefore, the claim follows immediately from the universal coefficient theorem. □

Lemma 4.8. Let the circle act on a compact connected symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Let $X$ be the minimal fixed component. Let $e^{S^1}(N_X) \in H_{S^1}^{\dim(M)-\dim(X)}(X; \mathbb{R})$ be the equivariant Euler class of the normal bundle of $X$, and let $\Lambda_X$ be the product of the weights (with multiplicity) on the normal bundle of $X$.

1. If $\sum_{F \subset (M \setminus X)^{S^1}} (\dim(F) + 2) = \dim(M) + 2$, then

$$e^{S^1}(N_X) = \Lambda_X \prod_{F \subset (M \setminus X)^{S^1}} \left( t + \frac{[\omega|_X]}{\phi(F) - \phi(X)} \right)^{\frac{1}{2}\dim(F)+1},$$

where the product is over all fixed components except $X$.

(2) More generally, $e^{S^1}(N_X)$ is a non-zero multiple of

$$\prod_{F \subset (M \setminus X)^{S^1}} \left( t + \frac{[\omega|_X]}{\phi(F) - \phi(X)} \right)^{\frac{1}{2}\dim(F)+1}.$$

Proof. By Lemma 2.7 there exists $\tilde{u} \in H^2_{S^1}(M; \mathbb{R})$ so that $\tilde{u}|_F = [\omega|_F] + t(\phi(X) - \phi(F))$ for each fixed component $F$. Hence

$$\left(\tilde{u}|_F + t(\phi(F) - \phi(X))\right)^{\frac{1}{2}\dim(F)+1} = 0$$

for all $F$, and so the class

$$\prod_{F \subset (M \setminus X)^{S^1}} \left( \tilde{u} + t(\phi(F) - \phi(X)) \right)^{\frac{1}{2}\dim(F)+1}$$
vanishes when restricted to any fixed component other than \(X\). Therefore, Proposition 2.6 (applied to \(−φ\)) implies that

\[
(4.9) \prod_{F \subset (M \setminus X)} (\omega|_F + t(\phi(F) - \phi(X)))^{\frac{1}{2}\dim(F) + 1} = \lambda e^{S^1}(N_X)
\]

for some \(\lambda \in H^*(\mathbb{C}P^\infty; \mathbb{R})\). This proves (2).

Finally, if \(\sum_{F \subset M} (\dim(F) + 2) = \dim(M) + 2\), then

\[
\sum_{F \subset (M \setminus X)} (\dim(F) + 2) = \dim(M) - \dim(X).
\]

Therefore, \(\lambda \in \mathbb{R}\), and by comparing the coefficients of \(t^{\frac{1}{2}\dim(M) - \frac{1}{2}\dim(X)}\) on both sides of (4.9), we see that

\[
\prod_{F \subset (M \setminus X)} (\phi(F) - \phi(X))^{\frac{1}{2}\dim(F) + 1} = \lambda \Lambda_X.
\]
Proof. By Lemma 2.7 there exists \( \tilde{u} \in H^2_{S^1}(M; \mathbb{R}) \) such that \( \tilde{u}|_F = [\omega|_F] + t(\phi(X) - \phi(F)) \) for each fixed component \( F \). Since \( \phi \) is a Morse-Bott function and \( \dim(X) < 2\lambda_F \) for all fixed components \( F \) other than \( X \), the natural restriction map from \( H^2_{S^1}(M; \mathbb{Z}) \) to \( H^2_{S^1}(X; \mathbb{Z}) \) is an isomorphism for all \( i \in \{0, \ldots, \frac{1}{2}\dim(X)\} \). Therefore, if \( \lambda \tilde{u}|_X = \lambda[\omega|_X]^j \) is an integral class, then so is \( \lambda \tilde{u}^j \). Therefore, for any \( y \) in a fixed component \( F \), \( \lambda \tilde{u}|_y^j = \lambda(\phi(X) - \phi(F))^j t^j \) is integral.

\[ \square \]

5. THE CASE THAT THERE ARE ONLY TWO FIXED COMPONENTS

In this section, we turn to considering the implications of our main restriction – the assumption that there are only two fixed components, \( X \) and \( Y \). The key idea is to exploit the fact that each (nonempty regular) reduced space is a bundle over \( X \) and a bundle over \( Y \); more specifically, it is the projectivization of the normal bundle to \( X \) and of the normal bundle to \( Y \).

Proposition 5.1. Let the circle act on a connected compact symplectic manifold \((M, \omega)\) with moment map \( \phi: M \to \mathbb{R} \). Let \( X \) be the maximal fixed component and fix \( i \in \mathbb{N} \). If the action is semifree, or if \( H^*(X; \mathbb{Z}) \) is torsion-free, or if \( i \leq \dim(M) - \dim(X) \), let \( R = \mathbb{Z} \); otherwise, let \( R = \mathbb{R} \). Given a regular value \( c \in \mathbb{R} \) so that \( M^{S^1} \cap \phi^{-1}(c, +\infty) = X \), there is an isomorphism

\[ \kappa_{X, c}: H^i_{S^1}(X; R)/e^{S^1}(N_X) \xrightarrow{\cong} H^i_{S^1}(\phi^{-1}(c); R) \quad \text{such that} \]

\[ \kappa_{X, c}(\tilde{\alpha}|_X) = \tilde{\alpha}|_{\phi^{-1}(c)} \quad \forall \tilde{\alpha} \in H^i_{S^1}(M; R). \]

Here, \( e^{S^1}(N_X) \) is the equivariant Euler class of the normal bundle to \( X \).

Proof. Assume first that the action is semifree, or \( H^*(X; \mathbb{Z}) \) is torsion-free, or \( R = \mathbb{R} \). Then this claim is a special case of the theorem on the cohomology of reduced spaces proved in [6]; see Theorem 3 and Propositions 6.4 and 6.7.

Alternatively, as we showed in [2] if any of these criteria holds or if \( i \leq 2\lambda_X \) and \( R = \mathbb{Z} \), the long exact sequence in equivariant cohomology for the pair \( (N_X, N_X \setminus X) \) breaks into a short exact sequence:

\[ 0 \to H^i_{S^1}(N_X, N_X \setminus X; R) \to H^i_{S^1}(N_X; R) \to H^i_{S^1}(N_X \setminus X; R) \to 0. \]

Since \( N_X \sim X \) and \( N_X \setminus X \sim \phi^{-1}(c) \), by the Thom isomorphism theorem we can rewrite this short exact sequence as follows:

\[ 0 \to H^{i-\dim(M)+\dim(X)}_{S^1}(X; R) \to H^i_{S^1}(X; R) \to H^i_{S^1}(\phi^{-1}(c); R) \to 0, \]

where the second arrow is multiplication by \( e^{S^1}(N_X) \).

\[ \square \]

If there are exactly two fixed sets, this has the following corollary:

Corollary 5.2. Let the circle act on a compact symplectic manifold \((M, \omega)\) with moment map \( \phi: M \to \mathbb{R} \). Assume that \( M \) has exactly two fixed components, \( X \) and \( Y \). Fix \( i \in \mathbb{N} \). If the action is semifree, or if \( i \leq \frac{1}{2}\dim(M) \), \( \kappa_{X, c}: H^i_{S^1}(X; R)/e^{S^1}(N_X) \xrightarrow{\cong} H^i_{S^1}(\phi^{-1}(c); R) \), then so is \( \lambda \tilde{u}^j \). Therefore, for any \( y \) in a fixed component \( F \), \( \lambda \tilde{u}|_y^j = \lambda(\phi(X) - \phi(F))^j t^j \) is integral.
There is an isomorphism
\[ f: H^i_{S^1}(X; R)/e^{S^1}(N_X) \xrightarrow{\simeq} H^i_{S^1}(Y; R)/e^{S^1}(N_Y) \]
such that
\[ f(\bar{\alpha}|_X) = \bar{\alpha}|_Y \quad \forall \bar{\alpha} \in H^i_{S^1}(M; R). \]
Moreover,
\[ f([\omega|_X]) = [\omega|_Y] + t(\phi(X) - \phi(Y)) \quad \text{and} \]
\[ s f([\omega|_X]) + (1-s)[\omega|_Y] \neq 0 \quad \forall \ s \in (0,1). \]

Proof. For simplicity, we may assume that \( \phi(X) < \phi(Y) \).

By Proposition 5.1, for any \( c \in (\phi(X), \phi(Y)) \)
\[ f = (\kappa_{Y,c})^{-1} \circ \kappa_{X,c}: H^i_{S^1}(X; R)/e^{S^1}(N_X) \to H^i_{S^1}(Y; R)/e^{S^1}(N_Y) \]
is an isomorphism such that \( f(\bar{\alpha}|_X) = \bar{\alpha}|_Y \) for all \( \bar{\alpha} \in H^i_{S^1}(M; R) \).

By Lemma 2.7, there exists \( \tilde{u} \in H^*_S(M; \mathbb{R}) \) such that \( \tilde{u}|_X = [\omega|_X] \) and \( \tilde{u}|_Y = [\omega|_Y] + t(\phi(X) - \phi(Y)) \). Therefore, \( f([\omega|_X]) = [\omega|_Y] + t(\phi(X) - \phi(Y)) \).

Finally, fix any \( s \in (0,1) \) and let
\[ c = s\phi(X) + (1-s)\phi(Y) \in (\phi(X), \phi(Y)). \]

Since \( c \) is a regular value, Lemma 2.7 implies that \( \kappa_c (\tilde{u} - t(\phi(X) - c)) = \omega_c \),
where \( \kappa_c \) is the Kirwan map and \((M_c, \omega_c)\) is the symplectic reduction of \( M \) at \( c \). Therefore, under the identification of \( H^*_S(\phi^{-1}(c); \mathbb{R}) \) and \( H^*(M_c; \mathbb{R}) \),
\[ \kappa_{Y,c}(sf([\omega|_X] + (1-s)[\omega|_Y])) = \kappa_{X,c}(s[\omega|_X] + (1-s)(([\omega|_X] - t(\phi(X) - \phi(Y)))) \]
\[ = \kappa_{X,c}([\omega|_X] - t(\phi(X) - c)) \]
\[ = (\tilde{u} - t(\phi(X) - c))|_{\phi^{-1}(c)} \]
\[ = \kappa_c(\tilde{u} - t(\phi(X) - c)) \]
\[ = \omega_c. \]

Since \( \omega_c \neq 0 \), the final claim follows immediately. \( \square \)

It is particularly easy to analyze the case where one of the two fixed components has codimension two.

Remark 5.3. Consider an effective Hamiltonian circle action on a compact symplectic manifold \((M, \omega)\). Assume that \( M^{S^1} \) has exactly two components, \( X \) and \( Y \), and that \( Y \) has codimension two. Then the fixed point data near \( Y \) is determined by the data near \( X \). More precisely, there is a natural isomorphism
\[ H^*(Y; \mathbb{Z}) = H^*_S(X; \mathbb{Z})/e^{S^1}(N_X); \]
under this identification,
\[ e(N_Y) = t \quad \text{and} \quad c(Y) = c(X)e^{S^1}(N_X). \]
To see this note that, since $\text{rank}_C(N_Y) = 1$, the action must be semifree. Moreover, since $e^{S^1}(N_Y) = -t + e(N_Y)$, the inclusion $H^*(Y; \mathbb{Z}) \to H^*_{S^1}(Y; \mathbb{Z})$ induces an isomorphism from $H^*(Y; \mathbb{Z})$ to $H^*_{S^1}(Y; \mathbb{Z})/e^{S^1}(N_Y)$. Finally,

$$c^{S^1}(M)|_X = c(X)e^{S^1}(N_X) \quad \text{and} \quad c^{S^1}(M)|_Y = c(Y) \left(1 + e^{S^1}(N_Y)\right).$$

Therefore, the claims follow immediately from Corollary 5.2.

On the other hand, if $(X, \omega)$ is any symplectic manifold and $V$ is any complex vector bundle over $X$, we can use symplectic cutting to construct a symplectic manifold $(M, \omega)$ which admits a Hamiltonian circle action with exactly two fixed components – one component is $X$ and has normal bundle $V$, and the other component has codimension two.

Additionally, when the two fixed components have minimal dimension, the Chern class of each component is determined by the Chern class of its normal bundle and the weights of the isotropy action on the other component.

**Lemma 5.4.** Let the circle act on a compact symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume that $M^{S^1}$ has exactly two components, $X$ and $Y$, where $\dim(X) + \dim(Y) = \dim(M) - 2$. Under the natural isomorphism

$$H^*(X) \cong H^*_{S^1}(X)/([\omega]|_X + t(\phi(Y) - \phi(X)))$$

the total Chern class of $X$ is

$$c(X) = \prod_\lambda \left(1 + \lambda t\right) \frac{e^{S^1}(N_X)}{e^{S^1}(N_X)}.$$

where the product is over the weights (counted with multiplicity) $\lambda$ in $N_Y$. Here, $N_X$ and $N_Y$ are the normal bundles to $X$ and $Y$, respectively.

**Proof.** By Corollary 5.2 there is an isomorphism

$$f: H^*_{S^1}(X; \mathbb{R})/e^{S^1}(N_X) \to H^*_{S^1}(Y; \mathbb{R})/e^{S^1}(N_Y)$$

such that

$$f(t) = t, \quad f\left(c^{S^1}(M)|_X\right) = c^{S^1}(M)|_Y, \quad \text{and} \quad f(u) = v - mt,$$

where $u = [\omega]|_X$, $v = [\omega]|_Y$, and $m = \phi(Y) - \phi(X)$.

Fix a point $y \in Y$. Since $f(u + mt)|_y = 0$, the composition of $f$ and the restriction map

$$t^*_y: H^*_{S^1}(Y; \mathbb{R})/e^{S^1}(N_Y) \to H^*_{S^1}(\{y\}; \mathbb{R})/t^{\frac{1}{2}\dim(M) - \frac{1}{2}\dim(Y)}$$

induces a map

$$g: H^*_{S^1}(X; \mathbb{R})/\left(u + mt, e^{S^1}(N_X)\right) \to H^*_{S^1}(\{y\}; \mathbb{R})/t^{\frac{1}{2}\dim(M) - \frac{1}{2}\dim(Y)}$$

so that

$$g(u) = -mt \quad \text{and} \quad g\left(c^{S^1}(M)|_X\right) = c^{S^1}(M)|_y = \prod_\lambda \left(1 + \lambda t\right),$$

where $\lambda$ is any weight in $N_Y$. Therefore, we have

$$c(X) = c(\{y\}) \frac{e^{S^1}(N_X)}{e^{S^1}(N_X)} = \prod_\lambda \left(1 + \lambda t\right) \frac{e^{S^1}(N_X)}{e^{S^1}(N_X)}.$$

which completes the proof.
where again the product is over all the weights \( \lambda \) in \( N_Y \). Moreover, since \( \dim(X) + \dim(Y) = \dim(M) - 2 \), Lemma 4.8 implies that
\[
e^{S^1}(N_X) = \Lambda_X \left( t + \frac{u}{m} \right)^{\frac{j}{2} \dim(Y) + 1};
\]
in particular, \( e^{S^1}(N_X) \) is a multiple of \( u + mt \). Therefore,
\[
H^*_{S^1}(X; \mathbb{R})/(u + mt, e^{S^1}(N_X)) = H^*_{S^1}(X; \mathbb{R})/(u + mt).
\]
Finally, Proposition 4.2 implies that
\[
H^*(X; \mathbb{R}) = \mathbb{R}[u]/u^{\frac{j}{2} \dim(X) + 1}.
\]
Therefore, \( g \) is an isomorphism. Since \( c^{S^1}(M)|_X = c(X)c^{S^1}(N_X) \), the claim follows immediately. \( \square \)

6. Proof of the main theorem for semifree actions

In this section, we prove the main theorem in the case when the circle action is semifree, i.e., the action is free outside fixed point set.

**Proposition 6.1.** Let the circle act on a compact symplectic manifold \((M, \omega)\) with moment map \( \phi: M \to \mathbb{R} \). Assume that \( M^{S^1} \) has exactly two components, \( X \) and \( Y \), and that \( \dim(X) + \dim(Y) = \dim(M) - 2 \). Also assume that the action is semifree. Then
\[
H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{\frac{j}{2} \dim(X) + 1} \quad \text{and} \quad H^*(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{\frac{i}{2} \dim(Y) + 1};
\]
\[
c(X) = (1 + u)^{\frac{j}{2} \dim(X) + 1} \quad \text{and} \quad c(Y) = (1 + v)^{\frac{i}{2} \dim(Y) + 1};
\]
\[
c(N_X) = (1 + u)^{\frac{j}{2} \dim(Y) + 1} \quad \text{and} \quad c(N_Y) = (1 + v)^{\frac{i}{2} \dim(X) + 1};
\]
where \( N_X \) and \( N_Y \) denote the normal bundles to \( X \) and \( Y \), respectively.

**Proof.** Clearly, the proposition holds if \( \dim(X) = \dim(Y) = 0 \). Without loss of generality, we assume that \( \phi(X) < \phi(Y) \) and that \( \dim(X) > 0 \). By assumption, there exist natural numbers \( i > 0 \) and \( j \) such that
\[
(6.2) \quad \dim(X) = 2i, \quad \dim(Y) = 2j, \quad \text{and} \quad \dim(M) = 2i + 2j + 2; \quad \text{hence}
(6.3) \quad \text{rank}_\mathbb{C}(N_X) = j + 1 \quad \text{and} \quad \text{rank}_\mathbb{C}(N_Y) = i + 1.
\]
By Proposition 4.2 and Lemma 4.8
\[
(6.4) \quad H^*(X; \mathbb{R}) = \mathbb{R}[u]/u^{i+1}, \quad \text{where} \ u = [\omega]|_X, \quad \text{and}
(6.5) \quad e^{S^1}(N_X) = \left( t + \frac{u}{m} \right)^{j+1}, \quad \text{where} \ m = \phi(Y) - \phi(X).
\]
Since the action is semifree, \( (6.5) \) and Lemma 2.8 imply that the total equivariant Chern class of \( N_X \) is
\[
(6.6) \quad c^{S^1}(N_X) = \left( 1 + t + \frac{u}{m} \right)^{j+1}.
\]
Similarly, (6.3), (6.6), and Lemma 5.4 imply that the total Chern class of $X$ is
\begin{equation}
(6.7) \quad c(X) = \left(1 + \frac{u}{m}\right)^{i+1} = 1 + (i+1)\frac{u}{m} + \cdots + (i+1)\frac{u^i}{m^i}.
\end{equation}

By (6.4) and (6.7), the Euler characteristic of $X$ is
\begin{equation}
(6.8) \quad i + 1 = \sum_k (-1)^k \dim(H^k(X; \mathbb{R})) = \int_X c_1(X) = (i+1)\int_X \frac{u^i}{m^i}.
\end{equation}

Therefore, $\frac{u^i}{m^i} \in H^{2i}(X; \mathbb{R})$ is a primitive integral class. By multiplying $[\omega]$ by a constant, we may assume that $[\omega]$ is also a primitive integral class. Hence, $u = [\omega|_X] \in H^2(X; \mathbb{R})$ is a primitive integral class by Lemma 4.11.

By Poincaré duality, these two facts imply that $\frac{u^{i-1}}{m^i} \in H^{2i-2}(X, \mathbb{R})$ is an integral class. By Lemma 4.12 this implies that $m^i$ divides $m^{i-1}$, that is,
\begin{equation}
(6.9) \quad m = 1.
\end{equation}

By Proposition 4.2 this implies that
\begin{equation}
(6.10) \quad H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^{i+1}.
\end{equation}

Since nearly identical arguments can be applied to $Y$, the claim now follows immediately from (6.2), (6.6), (6.7), (6.9), and (6.10).

\[\square\]

### 7. Isotropy submanifolds

Let the circle act effectively on a compact symplectic manifold $(M, \omega)$. If the action is not semifree, then the assumption that there are only two fixed components induces strong restrictions on $M$ itself and on its isotropy submanifolds, especially if the fixed components have relatively simple cohomology. Here, an **isotropy submanifold** is a symplectic submanifold $M^{2k} \subset M$ which is not fixed by the $S^1$ action, but is fixed by the $\mathbb{Z}_k$ action for some $k > 1$.

We begin with some results which do not depend on the cohomology of the fixed components.

**Lemma 7.1.** Let the circle act effectively on a connected compact symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume that $M$ has exactly two fixed components, $X$ and $Y$.

- If the action is not semifree, then $\dim(X) = \dim(Y)$.
- Given an isotropy submanifold $Q \subset M$, there exists a cohomology class $\tilde{\alpha} \in H_{S^1}^{\dim(Q) - \dim(X)}(M; \mathbb{Z})$ so that
  \[\tilde{\alpha}|_X = e^{S^1}(N^Q_X)\quad \text{and}\quad \tilde{\alpha}|_Y = \pm e^{S^1}(N^Q_Y),\]
  where $N^Q_X$ and $N^Q_Y$ are the normal bundles of $X$ and $Y$ in $Q$. 

Then \( \tilde{\iota} \) implies that \( \tilde{\iota} \) such that any cohomology class \( \mu \in H^\ast_{S^1}(M; \mathbb{R}) \).

By applying Corollary 5.2 to \( \mu|_Q \in H^\ast_{S^1}(Q; \mathbb{R}) \), we see that \( \mu|_X \) is a multiple of \( e^{S^1}(N^Q_X) \in H^\ast_{S^1}(X; \mathbb{R}) \) exactly if \( \mu|_Y \) is a multiple of \( e^{S^1}(N^Q_Y) \in H^\ast_{S^1}(Y; \mathbb{R}) \).

Since \( \phi \) is equivariantly perfect, there exists \( \tilde{\alpha} \in H^\ast_{S^1}(M; \mathbb{R}) \) such that \( \tilde{\alpha}|_X = e^{S^1}(N^Q_X) \). Similarly, there exists \( \tilde{\beta} \in H^\ast_{S^1}(Y; \mathbb{R}) \) such that \( \tilde{\beta}|_Y = e^{S^1}(N^Q_Y) \). By the first paragraph, \( \tilde{\alpha}|_Y = ae^{S^1}(N^Q_Y) \) for some \( a \in H^\ast_{S^1}(Y; \mathbb{R}) \) and \( \tilde{\beta}|_X = be^{S^1}(N^Q_Y) \) for some \( b \in H^\ast_{S^1}(X; \mathbb{R}) \). Then \( (\tilde{\beta} - b\tilde{\alpha})|_X = 0; \) hence the fact that

\[
\dim(Q) - \dim(Y) < \dim(M) - \dim(Y) = 2\lambda_Y
\]

implies that \( (\tilde{\beta} - b\tilde{\alpha})|_Y = 0 \). On the other hand, by a direct computation, \( (\tilde{\beta} - b\tilde{\alpha})|_Y = (1 - ab)e^{S^1}(N^Q_Y) \). This is only possible if \( ab = 1 \), which implies that \( a \) and \( b \) are constants. Therefore,

\[
\dim(X) = \dim(Y).
\]

Since \( \dim(Q) - \dim(X) = \dim(Q) - \dim(Y) \), Corollary 5.2 implies that for any cohomology class \( \mu \in H^\ast_{S^1}(M; \mathbb{Z}) \), \( \mu|_X \) is an integer multiple of \( e^{S^1}(N^Q_X) \in H^\ast_{S^1}(X; \mathbb{Z}) \) exactly if \( \mu|_Y \) is an integer multiple of \( e^{S^1}(N^Q_Y) \in H^\ast_{S^1}(Y; \mathbb{Z}) \). Moreover, since \( \dim(Q) - \dim(X) < \dim(M) - \dim(Y) = 2\lambda_Y \), there exists an integer class \( \tilde{\alpha} \in H^\ast_{S^1}(M; \mathbb{Z}) \) such that \( \tilde{\alpha}|_X = e^{S^1}(N^Q_X) \). Therefore, \( a \in \mathbb{Z} \). By a similar argument, \( b \in \mathbb{Z} \). Since \( ab = 1 \), this implies that \( a = b = \pm 1 \).

**Corollary 7.2.** Let the circle act effectively on a compact symplectic manifold \((M, \omega)\) with moment map \( \phi: M \to \mathbb{R} \). Assume that \( M \) has exactly two fixed components, \( X \) and \( Y \), and that the action is not semifree. Then \( \Xi_X = -\Xi_Y \),

where \( \Xi_X \) and \( \Xi_Y \) denote the multisets of weights (counted with multiplicity) for the isotropy action on \( N_X \) and \( N_Y \), respectively.

**Proof.** Consider any \( k > 1 \). Since \( M^{S^1} \) has only two components, if there exists any points with stabilizer \( \mathbb{Z}_k \), then the isotropy submanifold \( M^{\mathbb{Z}_k} \) is connected and contains \( X \) and \( Y \). Moreover, since the action is not semifree, \( \dim(X) = \dim(Y) \) by Lemma 7.1. Therefore, \( k \) divides exactly the same number of weights in \( \Xi_X \) and \( \Xi_Y \).

**Lemma 7.3.** Let \( A \) be a set of relatively prime natural numbers \( a_1 < \cdots < a_N \). Assume that for each \( i \) and \( k \) in \( \{1, \ldots, N\} \), there exists \( j \in \{1, \ldots, N\} \) such that \( a_i + a_j = 0 \) mod \( a_k \). Then \( a_i = i \) for all \( i \).
Proposition 7.5. Let the circle act effectively on a compact symplectic manifold \( (\mathcal{M}, \omega) \) with moment map \( \phi: \mathcal{M} \to \mathbb{R} \). Assume that \( \mathcal{M}^{S^1} \) has exactly two components, \( X \) and \( Y \). Then there exists \( \mathcal{N} \subseteq \mathcal{M} \) such that \( \dim(\mathcal{Q}) = 2 \) implies that \( \dim(\mathcal{M}) - \dim(\mathcal{Q}) = \dim(\mathcal{M}) - \dim(\mathcal{Y}) > 2 \). Hence, \( H^2(\mathcal{M}; \mathbb{R}) = H^2(\mathcal{Y}; \mathbb{R}) = H^2(\mathcal{X}; \mathbb{R}) = \mathbb{R} \). In particular, after possibly multiplying \([\omega]\) by a constant, we may assume that \([\omega]\) is a Hamiltonian saddle point at \( q \in \mathcal{Q} \).

Proof. The claim is obvious if \( \mathcal{N} = 1 \). Assume that the claim holds for \( \mathcal{N} - 1 \).

Consider any \( i \in \{1, \ldots, \mathcal{N} - 1\} \). By assumption, there exists \( j \in \{1, \ldots, \mathcal{N}\} \) such that \( a_i + a_j = 0 \mod a_N \). Since \( a_i < a_N \) and \( a_j \leq a_N \), this implies that \( a_i + a_j = a_N \). Since \( a_1 < \cdots < a_{\mathcal{N} - 1} \), this immediately implies that

\[
a_i + a_{\mathcal{N} - i} = a_N \quad \forall i \in \{1, \ldots, \mathcal{N} - 1\}.
\]

Let \( A' = \{a_1, \ldots, a_{\mathcal{N} - 1}\} \). Since the elements of \( A \) are relatively prime, the equation above immediately implies that the elements of \( A' \) are relatively prime. Moreover, fix \( i \) and \( k \) in \( \{1, \ldots, \mathcal{N} - 1\} \). By assumption, there exists \( j \in \{1, \ldots, \mathcal{N}\} \) such that \( a_i + a_j = 0 \mod a_k \). Moreover, if \( j = \mathcal{N} \), then since \( a_k + a_{\mathcal{N} - k} = a_N \) this implies that \( a_i + a_k + a_{\mathcal{N} - k} = 0 \mod a_k \), and hence \( a_i + a_{\mathcal{N} - k} = 0 \mod a_k \). By the inductive hypothesis, this implies that \( A' = \{1, \ldots, \mathcal{N} - 1\} \). The result follows immediately. \( \square \)

Lemma 7.4. Let the circle act on a compact symplectic manifold \( (\mathcal{M}, \omega) \). Let \( p \) and \( q \) be fixed points which lie on the same component \( \mathcal{N} \) of \( \mathcal{M}^{S^1} \) for some \( k > 1 \). Then the weights of the action at \( p \) and at \( q \) are equal modulo \( k \).

For a proof of this lemma, see Lemma 2.6 in [5].

Proposition 7.5. Let the circle act effectively on a compact symplectic manifold \( (\mathcal{M}, \omega) \) with moment map \( \phi: \mathcal{M} \to \mathbb{R} \). Assume that \( \mathcal{M}^{S^1} \) has exactly two components, \( X \) and \( Y \). Then there exists \( \mathcal{N} \subseteq \mathcal{M} \) so that the set of distinct weights for the isotropy action on \( \mathcal{N}_X \) is \( \{1, \ldots, \mathcal{N}\} \).

Proof. Let \( A = \{a_1, \ldots, a_N\} \subset \mathbb{N} \) be the set of distinct weights for the isotropy action on \( \mathcal{N}_X \). By Corollary 7.2 the set of distinct weights for the isotropy action on \( \mathcal{N}_Y \) is \( \{-a_1, \ldots, -a_N\} \). Moreover, by Lemma 7.3 for each \( i \) and \( k \) in \( \{1, \ldots, \mathcal{N}\} \), there exists \( j \in \{1, \ldots, \mathcal{N}\} \) such that \( a_i = -a_j \mod a_k \). Finally, since the action is effective, \( a_1, \ldots, a_N \) are relatively prime. Therefore, Lemma 7.3 implies that \( A = \{1, 2, \cdots, \mathcal{N}\} \) for some \( \mathcal{N} \in \mathbb{N} \). \( \square \)

The remaining results depend on the cohomology of the fixed components.

Lemma 7.6. Let the circle act effectively on a compact symplectic manifold \( (\mathcal{M}, \omega) \) with moment map \( \phi: \mathcal{M} \to \mathbb{R} \). Assume that \( \mathcal{M}^{S^1} \) has exactly two components, \( X \) and \( Y \). Assume that \( b_2(X) = 1 \), and let \( \mathcal{Q} \subseteq \mathcal{M} \) be an isotropy submanifold such that \( \dim(\mathcal{Q}) = 2 \). Then

\[
c_1(N_X^Q) = 0,
\]

where \( N_X^Q \) denotes the normal bundle to \( X \) in \( \mathcal{Q} \).

Proof. By Lemma 7.1 \( \dim(X) = \dim(Y) \). Since \( \dim(M) > \dim(Q) \), the fact that \( \dim(\mathcal{Q}) = 2 \) implies that \( \dim(M) - \dim(\mathcal{Q}) = \dim(M) - \dim(Y) > 2 \). Hence, \( H^2(\mathcal{M}; \mathbb{R}) = H^2(\mathcal{Y}; \mathbb{R}) = H^2(\mathcal{X}; \mathbb{R}) = \mathbb{R} \). In particular, after possibly multiplying \([\omega]\) by a constant, we may assume that \([\omega]\) is a
primitive integral class. The induced $S^1/\mathbb{Z}_q$ action on $Q = M^{\mathbb{Z}_q}$ is semifree, and the moment map for this action is $\phi' = \frac{\phi}{q}$. Let $u = [\omega|_X]$, $v = [\omega|_Y]$, and $m = \phi'(Y) - \phi'(X)$.

Since $\dim(Q) - \dim(Y) = 2$, $e^{S^1}(N_Q^Y) = -t + e(N_Q^Y)$ and so $H^*(Y; \mathbb{Z}) \cong H^*_S(Y; \mathbb{Z})/e^{S^1}(N_Q^Y)$ (see Remark 5.3); similarly, $e^{S^1}(N_Q^X) = t + e(N_Q^Y)$ and $H^*(X; \mathbb{Z}) \cong H^*_S(X; \mathbb{Z})/e^{S^1}(N_Q^X)$. Therefore, by Corollary 5.2 (applied on $Q$), there exists an isomorphism $f: H^*(Y; \mathbb{Z}) \to H^*(X; \mathbb{Z})$ so that $f(v) = u - m e(N_Q^Y)$ and so that $sf(v) + (1 - s)u \neq 0$ for all $s \in (0, 1)$. On the one hand, by Lemma 4.11 both $u$ and $v$ are primitive integral classes. Since $f$ is an isomorphism, $f(v)$ is also primitive. Since $H^2(X; \mathbb{R}) = \mathbb{R}$, this implies that $f(v) = \pm u$. On the other hand, since $H^2(X; \mathbb{R}) = \mathbb{R}$, the fact that $sf(v) + (1 - s)u \neq 0$ for all $s \in (0, 1)$ implies that $f(v)$ is a positive multiple of $u$. Together, these two claims imply that $f(v) = u$. Since $f(v) = u - m e(N_Q^Y)$ and $m \neq 0$, this implies that $c_1(N_Q^Y) = e(N_Q^Y) = 0$. □

**Lemma 7.7.** Let the circle act on a compact symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume that $M^{S^1}$ has exactly two components, $X$ and $Y$. Assume that $b_2(X) = 1$, and let $Q \subset M$ be an isotropy submanifold such that $\dim(Q) - \dim(Y) > 2$. Then

$$c_1(N_Q|_X) = 2 \Gamma_Q \frac{u}{m},$$

where $N_Q$ is the normal bundle of $Q$ in $M$, $\Gamma_Q$ is the sum of the weights (counted with multiplicities) of the isotropy action on $N_Q|_X$, $m = \phi(Y) - \phi(X)$, and $u = [\omega|_X]$.

**Proof.** By Corollary 7.2, the sum of the weights (counted with multiplicity) of the $S^1$ action on $N_Q|_Y$ is $-\Gamma_Q$. Hence,

$$c^{S^1}_1(N_Q)|_x = \Gamma_Q t \quad \forall \, x \in X, \quad \text{and} \quad c^{S^1}_1(N_Q)|_y = -\Gamma_Q t \quad \forall \, y \in Y.$$

By Lemma 2.7 there exists $\tilde{u} \in H^2_S(Q; \mathbb{R})$ such that $\tilde{u}|_X = u$ and $\tilde{u}|_Y = v - mt$, where $v = [\omega|_Y]$. Since $H^2(X; \mathbb{R}) = \mathbb{R}$ and $\dim(Q) - \dim(Y) > 2$, there exists $a$ and $b$ in $\mathbb{R}$ such that

$$c^{S^1}_1(N_Q) = a\tilde{u} + b(\tilde{u} + mt).$$

Therefore,

$$c^{S^1}_1(N_Q)|_x = bmt \quad \forall \, x \in X, \quad \text{and} \quad c^{S^1}_1(N_Q)|_y = -amt \quad \forall \, y \in Y.$$

The claim follows immediately. □

**Lemma 7.8.** Let the circle act effectively on a compact symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume that $M^{S^1}$ has exactly two components, $X$ and $Y$. Assume that $b_2(X) = 1$ for all $i \in \{0, \ldots, \frac{1}{2}\dim(X)\}$. Finally, assume that the action is not semifree, and split $N_X = \bigoplus_k V_k$, where $N_X$ is the normal bundle to $X$ in $M$ and $V_k \subset N_X$ is
Here, $\Gamma_V$ is the sum of the weights (counted with multiplicities) of the $S^1$ action on $V$, $m = \phi(Y) - \phi(X)$ and $u = [\omega|_X]$.

Proof. Let $Q = M^2 \subset M$ be an isotropy submanifold. By Lemma 7.1, $\dim(X) = \dim(Y)$, and there exists $\tilde{\alpha} \in H^2_{s^1}(M; \mathbb{R})$ so that

$$\tilde{\alpha}|_X = e^{s^1}(N_X^Q)$$

and

$$\tilde{\alpha}|_Y = \pm e^{s^1}(N_Y^Q).$$

Here, $N_X^Q$ and $N_Y^Q$ denote the normal bundles of $X$ and $Y$, respectively, in $Q$, and $\dim(Q) - \dim(X) = \dim(Q) - \dim(Y) = 2r$.

Let $\Lambda_X^Q$ denote the product of the weights (counted with multiplicity) of the isotropy action on $N_X^Q$. By Corollary 7.2, the product of the weights of the isotropy action on $N_Y^Q$ is $(-1)^r \Lambda_X^Q$. Hence,

$$e^{s^1}(N_X^Q)|_x = \Lambda_X^Q t^r \forall x \in X, \quad \text{and} \quad e^{s^1}(N_Y^Q)|_y = (-1)^r \Lambda_X^Q t^r \forall y \in Y.$$ 

By Lemma 2.7, there exists $\tilde{u} \in H^2_{s^1}(M; \mathbb{R})$ such that $\tilde{u}|_X = u$ and $\tilde{u}|_Y = v - mt$, where $v = [\omega|_Y]$. Since $X$ is symplectic and $b_{2i}(X) = 1$ for all $i \in \{0, \ldots, \frac{1}{2} \dim(X)\}$,

$$H^{even}(X; \mathbb{R}) = \mathbb{R}[u]/u^{\frac{1}{2} \dim(X) + 1}.$$ 

Hence, since $\dim(M) - \dim(Y) > \dim(Q) - \dim(Y) = 2r$, we can write

$$\tilde{\alpha} = \sum a_i \left(\frac{\tilde{u}}{m}\right)^i \left(\frac{\tilde{u}}{m} + t\right)^{r-i},$$

where

$$\tilde{\alpha}|_x = a_0 t^r \forall x \in X, \quad \text{and} \quad \tilde{\alpha}|_y = a_r (-t)^r \forall y \in Y.$$ 

Combining equations (7.9), (7.10), and (7.12), we see that $a_0 = \pm a_r$. Therefore, (7.10) implies that

$$e^{s^1}(N_X^Q) = \sum a_i \left(\frac{u}{m}\right)^i \left(\frac{u}{m} + t\right)^{r-i}, \quad \text{where} \quad a_0 = \pm a_r.$$ 

On the other hand, split $N_X = \bigoplus_k V_k$, where $V_k \subset N_X$ is the subbundle on which $S^1$ acts with weight $k$. By Lemma 4.8, $\left(\frac{u}{m} + t\right)^{\frac{1}{2} \dim(X) + 1}$ is a multiple of $e^{s^1}(N_X)$. Since

$$e^{s^1}(N_X) = \prod e^{s^1}(V_k) \in H^{even}_{s^1}(X; \mathbb{R}) \simeq \mathbb{R}[u, t]/u^{\frac{1}{2} \dim(X) + 1},$$

this implies that the $e^{s^1}(V_k)$'s can be identified with polynomials in $\mathbb{C}[u, t]$ whose product divides $\left(\frac{u}{m} + t\right)^{\frac{1}{2} \dim(X) + 1} + (\lambda \frac{u}{m})^{\frac{1}{2} \dim(X) + 1}$ for some $\lambda \in \mathbb{C}$. Write

$$e^{s^1}(V_k) = k^r \sum \alpha_{k,i} \left(\frac{u}{m}\right)^i \left(\frac{u}{m} + t\right)^{r-k-i},$$

where

$$k^r \sum \alpha_{k,i} \left(\frac{u}{m}\right)^i \left(\frac{u}{m} + t\right)^{r-k-i}$$

is the sum of some subset of the $V_k$'s.
where $r_k = \text{rank}_{C} V_k$; note that $\alpha_{k,0} = 1$. Since

$$\left( \frac{u}{m} + t \right)^{\frac{1}{2} \dim(X) + 1} + \left( \lambda \frac{u}{m} \right)^{\frac{1}{2} \dim(X) + 1} = \prod_{i=0}^{\frac{1}{2} \dim(X)} \left( \frac{u}{m} + t + e^{\frac{i \pi}{\dim(X)+2}} \lambda \frac{u}{m} \right),$$

this implies that for all $k$,

$$|\alpha_{k,r_k}| = |\lambda|^{r_k} \quad \text{and} \quad |\alpha_{k,1}| \leq r_k |\lambda|.$$

Moreover, if $r_k > 1$ then $|\alpha_{k,1}| < r_k |\lambda|$, while if $r_k = r_{k'} = 1$, then $\alpha_{k,1} \neq \alpha_{k',1}$ unless $k = k'$. Since $N^Q_X = \bigoplus_n V_{nq}$, $N_Q|_X$ is the direct sum of the remaining $V_k$'s. Hence, the fact $a_0 = \pm a_r$ implies that $|\lambda| = 1$. Therefore, (since $e^{S_1}(V_k)$ is real)

$$e^{S_1}(V_k) = (kt)^{r_k} + \nu_k k r_k \frac{u}{m} (kt)^{r_k-1} + \text{lower order terms},$$

where $0 < \nu_k < 2$ for all $k$ except possibly:

- at most one $k$ such that $r_k = 1$ and $\nu_k = 0$; and
- at most one $k$ such that $r_k = 1$ and $\nu_k = 2$.

By Lemma 7.8,

$$c_1(V_k) = \nu_k k r_k \frac{u}{m}.$$

The claim follows immediately. \qed

**Proposition 7.13.** Let the circle act effectively on a compact symplectic manifold $(M, \omega)$ with moment map $\phi : M \to \mathbb{R}$. Assume that $M^{S_1}$ has exactly two components, $X$ and $Y$, and that $b_2(X) = 1$ for all $i \in \{0, \ldots, \frac{1}{2} \dim(X)\}$.

- No point has stabilizer $\mathbb{Z}_k$ for any $k > 2$.
- If the action is not semifree, then

  $$\dim(M^{\mathbb{Z}_2}) - \dim(Y) = 2 \quad \text{or} \quad \dim(M) - \dim(M^{\mathbb{Z}_2}) = 2 \quad \text{(or both)}.$$  

**Proof.** To begin, let $Q \subset M$ be an isotropy submanifold such that $\dim(Q) - \dim(Y) > 2$ and $\dim(M) - \dim(Q) > 2$. Let $N_Q$ be the normal bundle of $Q$ in $M$, $\Gamma_Q$ be the sum of the weights (counted with multiplicity) of the isotropy action on $N_Q|_X$, $m = \phi(Y) - \phi(X)$, and $u = [\omega|_X]$. By Lemma 7.8,

$$c_1(N_Q|_X) = \nu \Gamma_Q \frac{u}{m}, \quad \text{where} \quad \nu < 2.$$  

On the other hand, the fact that $\dim(M) - \dim(Q) > 2$ implies that $\dim(M) - \dim(Y) > 2$. Since $\phi$ is a perfect Morse-Bott function, $\dim(X) > 0$, and so $b_2(X) = 1$ by assumption. Hence, by Lemma 7.7

$$c_1(N_Q|_X) = 2 \Gamma_Q \frac{u}{m}.$$  

This gives a contradiction. Therefore, for any isotropy submanifold $Q \subset M$,

(7.14) \quad $\dim(Q) - \dim(Y) = 2$ \quad or \quad $\dim(M) - \dim(Q) = 2$ \quad (or both).
Let $N_X$ be the normal bundle to $X$. By Proposition 7.5, there exists $N \in \mathbb{N}$ so that the set of distinct weights for the isotropy action on $N_X$ is \{1, 2, \ldots, N\}. Split $N_X = \sum_{k=1}^{N} V_k$, where $V_k$ is the subbundle of $N_X$ on which $S^1$ acts with weight $k$.

Assume that $N > 2$. Then it is easy to check that $\dim(M) - \dim(M^Z_k) > 2$ for all $k \in \{2, \ldots, N\}$. By (7.14), this implies that $\dim(M^Z_k) - \dim(Y) = 2$ for all such $k$. Therefore, by Lemma 7.6, $c_1(V_{N-1}) = 0$ and $c_1(V_N) = 0$, and so $c_1(V_{N-1} \oplus V_N) = 0$. This contradicts Lemma 7.8, which implies that $c_1(V_{N-1} \oplus V_N) \neq 0$.

\section{Proof of the main theorem for actions which are not semifree}

In this section, we prove the main theorem in the case that the circle action is not semifree.

\begin{proposition}
Let the circle act effectively on a compact symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume that $M^{S^1}$ has exactly two components, $X$ and $Y$, and that $\dim(X) + \dim(Y) = \dim(M) - 2$. Also assume that the action is not semifree. Then

\begin{align*}
H^*(X; \mathbb{Z}) &= \mathbb{Z}[u]/u^{i+1} \text{ and } c(X) = (1 + u)^{i+1}; \\
H^*(Y; \mathbb{Z}) &= \mathbb{Z}[v]/v^{i+1} \text{ and } c(Y) = (1 + v)^{i+1};
\end{align*}

where $\dim(X) = \dim(Y) = 2i$.

Moreover, no point has stabilizer $\mathbb{Z}_k$ for any $k > 2$; $\dim(M^{\mathbb{Z}_2}) = \dim(M) - 2$;

\begin{align*}
c(N_{M^{\mathbb{Z}_2}})|_X &= 1 + u \quad \text{and} \quad c(N_{M^{\mathbb{Z}_2}})|_Y = 1 + v; \\
c(N_{M^{\mathbb{Z}_2}})|_X &= \frac{(1 + u)^{i+1}}{1 + 2u} \quad \text{and} \quad c(N_{M^{\mathbb{Z}_2}})|_Y = \frac{(1 + v)^{i+1}}{1 + 2v},
\end{align*}

where $N_{M^{\mathbb{Z}_2}}$ denotes the normal bundle of $M^{\mathbb{Z}_2}$ in $M$, and $N_{X}^{M^{\mathbb{Z}_2}}$ and $N_{Y}^{M^{\mathbb{Z}_2}}$ denote the normal bundles of $X$ and $Y$, respectively, in $M^{\mathbb{Z}_2}$.

\end{proposition}

\begin{proof}
This claim follows from Lemmas 8.2, 8.3, 8.15, and 8.34.
\end{proof}

To begin, note that the following lemma is an immediate consequence of Propositions 4.2 and 7.13.

\begin{lemma}
If the assumptions of Proposition 8.1 hold, then no point has stabilizer $\mathbb{Z}_k$ for any $k > 2$.
\end{lemma}
Lemma 8.3. If the assumptions of Proposition 8.1 hold and, additionally, \( \dim(M^{\mathbb{Z}_2}) - \dim(Y) = 2 \), then

\[
\dim(M) = 6 \quad \text{and} \quad \dim(X) = \dim(Y) = 2;
\]
\[
H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^2 \quad \text{and} \quad H^*(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^2;
\]
\[
c(X) = 1 + 2u \quad \text{and} \quad c(Y) = 1 + 2v;
\]
\[
c(N_M^{\mathbb{Z}_2})|_X = 1 + u \quad \text{and} \quad c(N_M^{\mathbb{Z}_2})|_Y = 1 + v;
\]
\[
c(N_M^{\mathbb{Z}_2})|_X = 1 \quad \text{and} \quad c(N_M^{\mathbb{Z}_2})|_Y = 1.
\]

Proof. Without loss of generality, we may assume that \( \phi(X) < \phi(Y) \). By Lemma 7.1, since the action is not semifree, \( \dim(X) = \dim(Y) \); hence there exists \( i \in \mathbb{N} \) such that

\[
\dim(X) = \dim(Y) = 2i \quad \text{and} \quad \dim(M) = 4i + 2.
\]

By assumption,

\[
\dim(M^{\mathbb{Z}_2}) - \dim(Y) = 2.
\]

By Proposition 4.2

\[
H^*(X; \mathbb{R}) = \mathbb{R}[u]/u^{i+1}, \quad \text{where} \ u = [\omega|_X].
\]

By (8.4), (8.5), and Lemma 4.8

\[
e^S_1(N_X) = 2 \left( t + \frac{u}{m} \right)^{i+1}, \quad \text{where} \ m = \phi(Y) - \phi(X).
\]

Here, \( N_X \) is the normal bundle to \( X \) in \( M \). Moreover, by Lemma 7.6 and (8.5),

\[
e^S_1(N_M^{\mathbb{Z}_2}) = 2t.
\]

Since \( e^S_1(N_X) = e^S_1(N_M^{\mathbb{Z}_2}) e^S_1(N_M^{\mathbb{Z}_2})|_X \), (8.7) and (8.8) imply that

\[
e^S_1(N_M^{\mathbb{Z}_2})|_X = \frac{1}{t} \left( t + \frac{u}{m} \right)^{i+1} = t^i + (i+1) \frac{u}{m} t^{i-1} + \cdots + (i+1) \frac{u}{m}.
\]

By (8.8), (8.9) and Lemma 2.8

\[
c^S_1(N_M^{\mathbb{Z}_2})|_X = \frac{1}{1+t} \left( 1 + t + \frac{u}{m} \right)^{i+1} \quad \text{and} \quad c^S_1(N_M^{\mathbb{Z}_2}) = 1 + 2t.
\]

Therefore, since \( e^S_1(N_X) = e^S_1(N_M^{\mathbb{Z}_2}) e^S_1(N_M^{\mathbb{Z}_2}) \),

\[
c^S_1(N_X) = \left( 1 + t + \frac{u}{m} \right)^{i+1} \frac{1 + 2t}{1+t}.
\]
By Lemma 5.4, (8.5) and (8.11) imply that,
\[ c(X) = \left(1 + \frac{u}{m}\right) \left(1 + 2 \frac{u}{m}\right) \left(1 - \frac{u}{m}\right) \frac{1}{1 - 2 \frac{u}{m}} \]
\[ = \frac{1 + \frac{u}{m}}{1 - 2 \frac{u}{m}} + \frac{u}{m} \left(1 + \frac{u}{m}\right)^i \]
\[ = 1 + (i + 3) \frac{u}{m} + \cdots + (3i + i) \frac{u^i}{m^i}. \]
(8.12)

By (8.6) and (8.12), the Euler characteristic of \( X \) is
\[ i + 1 = \sum_k (-1)^k \dim(H^k(X; \mathbb{R})) = \int_X c_t(X) = (3i + i) \int_X \frac{u^i}{m^i}. \]
Therefore, \( \frac{3i + i}{i + 1} \frac{u^i}{m} \in H^{2i}(X; \mathbb{R}) \) is a primitive integral class. On the other hand, since \( e^{S^1(N_M)} \) is an integral class, \( (8.9) \) implies that \( (i + 1) \frac{u}{m} \) is an integral class. Combined, these two facts imply that \( \frac{(i + 1)^2}{3i + i} \) is an integer. But this is impossible unless
(8.13)
\[ i = 1, \]
and so \( 2 \frac{u}{m} \in H^2(M; \mathbb{R}) \) is a primitive integral class. By multiplying \( \omega \) by a constant, we may also assume that \( [\omega] \) is a primitive integral class. Hence, \( u \in H^2(M; \mathbb{R}) \) is a primitive integral class by Lemma 4.11. Therefore,
(8.14)
\[ m = 2. \]
Since nearly identical arguments can be applied to \( Y \), the claims now follow from (8.4), (8.10), (8.12), (8.6), (8.13), and (8.14).

Lemma 8.15. If the assumptions of Proposition 8.1 hold and, additionally, \( \dim(M^{\mathbb{Z}_2}) - \dim(Y) > 2 \), then
\[ \dim(X) = \dim(Y) = 2i > 2 \quad \text{and} \quad \dim(M^{\mathbb{Z}_2}) = \dim(M) - 2; \]
\[ H^*(X; \mathbb{Z})/\text{torsion} = \mathbb{Z}[u]/u^{i+1} \quad \text{and} \quad H^*(Y; \mathbb{Z})/\text{torsion} = \mathbb{Z}[v]/v^{i+1}; \]
\[ c(X) = (1 + u)^{i+1} \quad \text{and} \quad c(Y) = (1 + v)^{i+1}; \]
\[ c(N_{M^{\mathbb{Z}_2}})|_X = 1 + u \quad \text{and} \quad c(N_{M^{\mathbb{Z}_2}})|_Y = 1 + v; \]
\[ c(N_X^{\mathbb{Z}_2}) = \frac{(1 + u)^{i+1}}{1 + 2u} \quad \text{and} \quad c(N_Y^{\mathbb{Z}_2}) = \frac{(1 + v)^{i+1}}{1 + 2v}, \]
where the last six equations are as elements of \( H^*(X; \mathbb{R}) \) or of \( H^*(Y; \mathbb{R}) \).
Moreover, if \( \phi(Y) > \phi(X) \) and \( [\omega] \) is a primitive integral class, then \( \phi(Y) - \phi(X) = 2 \).

Proof. Without loss of generality, we may assume that \( \phi(X) < \phi(Y) \). By Lemma 7.4, since the action is not semifree, \( \dim(X) = \dim(Y) \); hence there exists \( i \in \mathbb{N} \) such that
(8.16) \[ \dim(X) = \dim(Y) = 2i \quad \text{and} \quad \dim(M) = 4i + 2. \]
Since \( \dim (M) - \dim (M^\mathbb{Z}_2) = 2 \), \( \text{Proposition 7.13} \) implies that

\[
\dim (M) - \dim (M^\mathbb{Z}_2) = 2.
\]

By \( \text{Proposition 4.2} \),

\[
H^\ast (X; \mathbb{R}) = \mathbb{R}[u]/u^{i+1}, \quad \text{where } u = [\omega|_X].
\]

By \( \text{(8.16)} \) and \( \text{(8.17)} \) and \( \text{Lemma 1.8} \),

\[
\text{e}^s (N_X) = 2^i \left( t + \frac{u}{m} \right)^{i+1}, \quad \text{where } m = \phi(Y) - \phi(X).
\]

Here, \( N_X \) is the normal bundle to \( X \) in \( M \). Moreover, since \( \dim (M^\mathbb{Z}_2) - \dim(Y) > 2 \), \( \text{Lemma 7.7} \) implies that

\[
\text{e}^s (N_{M^\mathbb{Z}_2})|_X = t + 2\frac{u}{m}.
\]

Since \( \text{e}^s (N_X) = \text{e}^s (N_{M^\mathbb{Z}_2}) \text{e}^s (N_{M^\mathbb{Z}_2})|_X \), \( \text{(8.19)} \) and \( \text{(8.20)} \) imply that

\[
\text{e}^s (N_{M^\mathbb{Z}_2}) = \frac{(2t + 2\frac{u}{m})^{i+1}}{2t + 4\frac{u}{m}}.
\]

By \( \text{Lemma 2.8} \), \( \text{(8.20)} \) and \( \text{(8.21)} \) imply that

\[
\text{e}^s (N_{M^\mathbb{Z}_2})|_X = 1 + t + 2\frac{u}{m} \quad \text{and} \quad \text{e}^s (N_{M^\mathbb{Z}_2}) = \frac{1 + 2t + 2\frac{u}{m})^{i+1}}{1 + 2t + 4\frac{u}{m}}.
\]

Therefore, since \( \text{e}^s (N_X) = \text{e}^s (N_{M^\mathbb{Z}_2}) \text{e}^s (N_{M^\mathbb{Z}_2})|_X \),

\[
\text{c}^s (N_{M^\mathbb{Z}_2}) = \frac{(1 + 2t + 2\frac{u}{m})^{i+1}}{1 + 2t + 4\frac{u}{m}}(1 + t + 2\frac{u}{m}).
\]

By \( \text{Lemma 5.4} \), \( \text{(8.16)} \), \( \text{(8.17)} \), and \( \text{(8.23)} \) imply that

\[
\text{c}(X) = \left( 1 + 2\frac{u}{m} \right)^{i+1} = 1 + (i + 1)\frac{2u}{m} + \cdots + (i + 1)\left( \frac{2u}{m} \right)^i.
\]

By \( \text{(8.18)} \) and \( \text{(8.24)} \) the Euler characteristic of \( X \) is

\[
(i + 1)^2 \int_X \frac{u^i}{m^i} = \int_X c_i(X) = \sum_k (-1)^k \dim H^k(X) = i + 1.
\]

So \( 2^i \frac{u^i}{m^i} \in H^{2i}(X, \mathbb{Z}) \) is a primitive integral class. By multiplying \( \omega \) by a constant, we may assume that \( [\omega] \) is a primitive integral class. Hence, \( u \) is a primitive integral class by \( \text{Lemma 4.11} \). On the one hand, since \( \text{c}^s (N_{M^\mathbb{Z}_2})|_X \) is an integral class, \( \text{(8.22)} \) implies that \( \frac{2}{m} \in \mathbb{Z} \). On the other hand, since \( u^i \) is an integral class and \( \frac{m^i}{2^i} \) is a primitive integral class, \( \frac{m^i}{2^i} \in \mathbb{Z} \). Together, these imply that

\[
m = 2 \quad \text{and} \quad u^i \text{ is a primitive integral class.}
\]

Since nearly identical arguments can be applied to \( Y \), claim now follows from \( \text{(8.18)}, \text{(8.22)}, \text{(8.24)}, \text{and (8.25)}. \)
Comparing the highest order terms of (8.28) and (8.31), we see that for all $\beta \in H^2_{S^1}(Y; \mathbb{F})$, where $N^Q_Y$ is the normal bundle to $Y$ in $Q$. Assume also that there exist classes $\bar{u}$ and $\bar{\mu}$ in $H^2_{S^1}(Q; \mathbb{F})$ such that

1. $\bar{u}_x = 0$ for all $x \in X$;
2. $\bar{u}_y \neq 0$ for all $y \in Y$;
3. $\bar{\mu}_Y = \mu$; and
4. $\bar{\mu}_x \neq -i\mu$ for all $x \in X$.

Then $H^{2k+1}(X; \mathbb{F}) = 0$ for all $k$.

**Proof.** Assume on the contrary that there exists a non-zero class $\alpha \in H^{2k+1}(X; \mathbb{F})$ for some $k$. By assumptions (1) and (2), there exist $u$ and $v$ in $H^2(X; \mathbb{F})$ and a non-zero $m \in \mathbb{F}$ such that

$$e^{s^1}(N^Q_Y) = \Lambda_Y (t^i + \mu t^{-1} + \text{lower order terms}) \in H^2_{S^1}(Y; \mathbb{F}),$$

where $\alpha = \bar{u}|_x$ and $\bar{\mu}$ in $H^2_{S^1}(Q; \mathbb{F})$ such that

1. $\bar{u}|_x = 0$ for all $x \in X$;
2. $\bar{u}_y \neq 0$ for all $y \in Y$;
3. $\bar{\mu}_Y = \mu$; and
4. $\bar{\mu}_x \neq -i\mu$ for all $x \in X$.

Since $\mathbb{F}$ is a field, Poincaré duality implies that there exists $\beta \in H^{2i-2k-1}(X; \mathbb{F})$ such that $\alpha \cup \beta = u^i$. Since $2k+1$ and $2i-2k-1$ are both smaller than $2\lambda_Y = \dim(Q) - \dim(Y)$, there exist classes $\tilde{\alpha} \in H^{2k+1}_{S^1}(Q; \mathbb{F})$ and $\tilde{\beta} \in H^{2i-2k-1}_{S^1}(Q; \mathbb{F})$ such that $\tilde{\alpha}|_Y = \alpha$ and $\tilde{\beta}|_X = \beta$.

Since $H^*_S(Y; \mathbb{F}) = H^*(Y; \mathbb{F})[t]$, we can write $\tilde{\alpha}|_Y = \sum a_{2j+1} t^j$ and $\tilde{\beta}|_Y = \sum b_{2j+1} t^{i-1-j}$, where $a_{2j+1}$ and $b_{2j+1}$ lie in $H^{2j+1}(Y; \mathbb{F})$ for all $j$. Moreover, since $1 < 2i$, there exist classes $\tilde{a}_1 \in H^1_{S^1}(Q; \mathbb{F})$ and $\tilde{b}_1 \in H^1_{S^1}(Q; \mathbb{F})$ such that $\tilde{a}_1|_Y = a_1$ and $\tilde{b}_1|_Y = b_1$. Finally, since $H^1(\mathbb{CP}^\infty; \mathbb{F}) = 0$ for all $x \in X$, $\tilde{a}_1|_x = \tilde{b}_1|_x = 0$. Therefore,

$$\tilde{\alpha} \cup \tilde{\beta}|_Y = (a_1 \cup b_1) t^{i-1} + \text{lower order terms},$$
$$\tilde{a}_1 \cup \tilde{b}_1|_Y = a_1 \cup b_1, \quad \text{and}$$
$$\tilde{a}_1 \cup \tilde{b}_1|_X = 0.$$

Since the action is semifree when $\mathbb{F} = \mathbb{Z}_p$, we have the short exact sequence (2.2) for both $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{Z}_p$. Since $\dim(Q) - \dim(Y) = 2i$ and $\tilde{u}|_X = (\tilde{a} \cup \tilde{b})|_X$, by (2.2), there exists $c \in \mathbb{F}$ such that

$$\tilde{\alpha} \cup \tilde{\beta}|_Y = \tilde{u}|_Y + ce^{s^1}(N^Q_Y)$$
$$= (m^i t^i + ivm^{i-1} t^{i-1}) + \Lambda_Y (ct^i + cmt^{i-1}) + \text{lower order terms}.$$
Hence, by comparing the next highest order terms, we see that
\[ a_1 \cup b_1 = m^{i-1} (iv - m\mu). \]
By (8.27), (8.29), and assumption (3), this implies that
\[ (\tilde{a}_1 \cup \tilde{b}_1) Y = m^{i-1} (i\tilde{u} - int - m\tilde{\mu}) Y. \]
Since \( 2 < \dim(Q) - \dim(X) = 2i \), this implies that
\[ (8.32) \quad \tilde{a}_1 \cup \tilde{b}_1 = m^{i-1} (i\tilde{u} - int - m\tilde{\mu}). \]
But by (8.27) and assumption (4),
\[ (8.33) \quad m^{i-1} (i\tilde{u} - int - m\tilde{\mu}) X \neq 0 \quad \forall x \in X. \]
Clearly, (8.30), (8.32), and (8.33) give a contradiction. \( \square \)

**Lemma 8.34.** If the assumptions of Proposition 8.1 hold, then \( H^*(M^{S^1}; \mathbb{Z}) \) is torsion-free.

**Proof.** By Lemma 8.3, no point in \( M \) has stabilizer \( \mathbb{Z}_k \) for any \( k > 2 \). By Lemma 8.3 the cohomology \( H^*(M^{S^1}; \mathbb{Z}) \) is torsion-free if \( \dim(M^{\mathbb{Z}_2}) - \dim(Y) = 2 \), and so we may assume that \( \dim(M^{\mathbb{Z}_2}) - \dim(Y) > 2 \). By Lemma 8.15 \( \dim(X) = \dim(Y) = 2i \) and \( \dim(M^{\mathbb{Z}_2}) = 4i \) for some \( i > 1 \).

By Lemma 8.15 \( H^2(M; \mathbb{R}) = \mathbb{R} \). Therefore, by multiplying \( \omega \) by a constant, we may assume that \( [\omega] \) is a primitive integral class. The induced effective \( S^1 = S^1/\mathbb{Z}_2 \) action on \( M^{\mathbb{Z}_2} \) is semifree, and the moment map for this action is \( \phi' = \phi/2 \). By Lemma 8.15 \( \phi(Y) - \phi(X) = 2 \), and so \( \phi'(Y) - \phi'(X) = 1 \). Hence, by Lemma 2.7 there exists an integral class \( \tilde{u} \in H^2_{S^1}(M^{\mathbb{Z}_2}; \mathbb{R}) \), such that
\[ (8.35) \quad \tilde{u}|X = [\omega|X] \in H^2_{S^1}(X; \mathbb{R}) \quad \text{and} \quad \tilde{u}|Y = [\omega|Y] - t \in H^2_{S^1}(Y; \mathbb{R}). \]
In particular, for any prime \( p \), there exists \( \tilde{u} \in H^2_{S^1}(M^{\mathbb{Z}_2}; \mathbb{Z}_p) \) such that \( \tilde{u}|x = 0 \) for all \( x \in X \) and \( \tilde{u}|y \neq 0 \) for all \( y \in Y \).

By Lemmas 8.15 and 2.8, the equivariant Euler class of the normal bundle of \( Y \) in \( M^{\mathbb{Z}_2} \) (for the semifree \( S^1/\mathbb{Z}_2 \) action on \( M^{\mathbb{Z}_2} \) ) is given by
\[ e^{S^1}(N^M_{Y^{M^{\mathbb{Z}_2}}}) = \frac{(-t + v)^{i+1}}{-t + 2v} \]
\[ = (-1)^i \frac{(t - v)^{i+1}}{t} \left( 1 + 2v t + \text{lower order terms} \right) \]
\[ = (-1)^i \left( t^i + (1 - i)vt^{i-1} + \text{lower order terms} \right) \in H^{2i}_{S^1}(M^{\mathbb{Z}_2}; \mathbb{R}), \]
where \( v = [\omega|Y] \in H^2(Y; \mathbb{R}) \). Moreover, by (8.35),
\[ (8.36) \quad (1 - i) (\tilde{u} + t)|Y = (1 - i)v, \quad \text{and} \]
\[ (8.37) \quad (1 - i) (\tilde{u} + t)|x = (1 - i)t \quad \forall x \in X. \]
Finally, fix any prime \( p \), and write
\[ e^{S^1}(N^M_{Y^{M^{\mathbb{Z}_2}}}) = (-1)^i \left( t^i + \mu t^{i-1} + \text{lower order terms} \right) \in H^{2i}_{S^1}(Y; \mathbb{Z}_p), \]
where \( \mu \in H^2(Y; \mathbb{Z}_p) \). Since \( 2 < \dim(M^{S^2}) - \dim(X) = 2i \), there exists a unique \( \tilde{\mu} \in H^2_{S^1}(M^{S^2}; \mathbb{Z}_p) \) such that \( \tilde{\mu}|_Y = \mu \). By the preceding paragraph, \( \tilde{\mu}|_x = (1 - i)t \neq -it \in H^2(\mathbb{C}P^\infty; \mathbb{Z}_p) \). By Lemma 8.20, this implies that \( H^{2k+1}(X; \mathbb{Z}_p) = 0 \) for all \( k \) and all primes \( p \). By Lemma 4.7, this proves the claim. \( \square \)

**Remark 8.38.** In fact, we can use Lemma 4.6 to give a simpler proof that \( H^{2k+1}(M^{S^1}; \mathbb{Z}_k) = 0 \) for all \( k > 2 \).

**Appendix A. Possible stabilizer subgroups**

The goal of this appendix is to prove the following proposition.

**Proposition A.1.** Let the circle act effectively on a compact symplectic manifold \((M, \omega)\) with moment map \( \phi: M \to \mathbb{R} \). Assume that \( M^{S^1} \) has exactly two components. Then no point has stabilizer \( \mathbb{Z}_k \) for any \( k > 6 \).

**Proof.** Let \( X \) and \( Y \) be the fixed components. Let \( \Xi_X \) denote the multiset of weights for the isotropy action on the normal bundle to \( X \). By Corollary 7.22, if the action is not semifree, the multiset of weights for the isotropy action on the normal bundle to \( Y \) is \(-\Xi_X\). By Lemma 7.4, \( \Xi_X = -\Xi_X \mod a \) for each \( a \in \Xi_X \). Finally, since the action is effective, the weights in \( \Xi_X \) are relatively prime. The result now follows immediately from Lemma A.2. \( \square \)

**Lemma A.2.** Let \( W \) be a multiset of natural numbers which are relatively prime. Assume that \( W \) contains exactly \( N \) distinct numbers \( a_1 < \cdots < a_N \) which have (non-zero) multiplicities \( m_1, \ldots, m_N \), respectively. Let \( -W \) be the multiset of negative integers which contains \(-a_1, \ldots, -a_N\) with the same multiplicity. Assume that \( W = -W \mod a_i \) for all \( i \in \{1, \ldots, N\} \). Then

1. \( a_i = i \) for all \( i \in \{1, \ldots, N\} \).
2. If \( N = 3 \), then \( m_2 = m_1 \).
3. If \( N = 4 \), then \( m_3 = m_1 \) and \( m_2 = m_1 + m_4 \).
4. If \( N = 5 \), then \( m_4 = m_1 = 2m_5 \) and \( m_3 = m_2 = 3m_5 \).
5. If \( N = 6 \), then \( m_2 = m_3 = m_4 = 2m_1 = 2m_5 = 2m_6 \).
6. \( N \leq 6 \).

**Proof.** The first claim follows immediately from Lemma 7.3. Now, the fact that \( W = -W \mod N \) implies that \( m_i = m_{N-i} \) for all \( i \in \{1, \ldots, N-1\} \). Moreover, if \( N > 3 \), the fact that \( W = -W \mod (N-1) \) implies \( m_1 + m_N = m_{N-2} \) and \( m_i = m_{N-i} \) for all \( i \in \{2, \ldots, N-3\} \). Therefore,

\[
\text{(A.3)} \quad m_1 = m_{N-1}, \quad \text{and} \\
\text{(A.4)} \quad m_2 = m_3 = \cdots = m_{N-2} = m_1 + m_N \quad \forall \ N > 3.
\]

Claim (2) follows immediately from (A.3), while claim (3) follows immediately from (A.3) and (A.4). If \( N = 5 \), then since \( W = -W \mod 3 \), \( m_1 + m_4 = m_2 + m_5 \). Claim (4) follows immediately from this fact and (A.3) and (A.4). Similarly, if \( N = 6 \), then since \( W = -W \mod 4 \), \( m_3 = m_1 + m_5 \); claim (5) follows easily. Finally, if \( N > 6 \), the fact that \( W = -W \)
mod \((N - 2)\) implies that \(m_2 + m_N = m_{N-4}\), which contradicts \([A.4]\). This proves the last claim.

\[
\square
\]

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