Perfect Graeco-Latin balanced incomplete block designs
and related designs

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Abstract: Main effect plans orthogonal through the block factor (POTB) have been defined and a few series of them have been constructed in Bagchi (2010). These plans are very closely related to the ‘mutually orthogonal balanced nested row-column designs’ of Morgan and Uddin (1996) and many other combinatorial designs in the literature with different names like ‘BIBDs for two sets of treatment’, ‘Graeco-Latin designs’ and ‘PERGOLAs’. In fact all of them may be viewed as POTBs satisfying one or more additional conditions, making them ‘optimal’. However, the PERGOLAs are defined to satisfy an additional property, without which also it is optimal. Interestingly, this additional property is satisfied by all the hitherto known examples of POTBs, even when their definitions do not demand it.

In this paper we present direct and recursive constructions of POTBs. In the process we have constructed one design which seems to be the first example of an ‘optimal’ two-factor POTB which is not a PERGOLA (see Theorem 3.1).

1 Introduction.

Preece (1966) constructed ‘BIBDs for two sets of treatments’. Subsequently several authors constructed similar combinatorial objects. Among these, the ones relevant to the present paper are ‘balanced Graeco-Latin block designs’ of Seberry (1979), ‘Graeco-Latin designs of type 1’ of Street (1981) and ‘Perfect Graeco-Latin balanced incomplete block designs (PERGOLAs)’ of Rees and Preece (1999).

Morgan and Uddin (1996) considered main effects plans (MEPs) on a nested row-column set up and proved the optimality of ‘mutually orthogonal balanced nested row-column designs’. They also discussed the constructional aspects of such designs. Unfortunately, the relevance of these results in the context of blocked MEPs was overlooked by later authors studying blocked MEPs like Mukerjee, Dey and Chatterjee (2001) and Bagchi (2010). Optimal Main effects plans for three or more factors on non-orthogonal blocks of small size were obtained in Mukerjee, Dey and Chatterjee (2001). Bagchi (2010) defined and studied main effect plans orthogonal ‘through the block factor’ (POTB).

In the present paper we first note the relation between POTBs with optimality property (termed balanced POTB) and the combinatorial objects considered by earlier authors mentioned above. Next we construct a few series of POTBs. We also present a recursive construction by which the number of factors is multiplied, keeping the block size unchanged, thus yielding multi-factor POTBs from PERGOLAs and other similar two-factor designs.

We note that a PERGOLA is a balanced POTB with an additional condition. It is interesting that all the balanced POTBs available in the literature (with different names) do satisfy this condition. (Table 1 of Rees and Preece (1999) shows that such designs are plentiful). One would, therefore, suspect that this condition is implicit in the definition. We have, however, found a balanced POTB which does not satisfy this condition. [See Theorem 3.1].
2 Preliminaries

In this section we present the definition of a balanced POTB. We also list related combinatorial objects existing in the literature with various names.

Definition 2.1 By a block design with \( v \) treatments and \( b \) blocks of size \( k \) each we mean an incidence structure represented by a \( v \times b \) matrix \( N \) having constant column sum \( k \).

With any such block design \( D \) one associates the graph \( G(D) \) with the treatments of \( D \) as vertices, two treatments being adjacent in \( G(D) \) if there is a block containing both the treatments. One says that \( D \) is connected if the graph \( G(D) \) is connected in the usual sense.

\( D(b,k,v) \) will denote the class of all connected block designs with \( v \) treatments on \( b \) blocks of size \( k \) each.

Definition 2.2 Consider an \((m+1) \times n\) array \( A \) such that the entries of the \( i \)th row are elements of a set \( S_i \) of size \( s_i \), \( i = 0,1,\ldots,m \). This is said to be a main effect plan (MEP) for \( m+1 \) factors, say, \( F_0, F_1, \ldots, F_m \) on \( n \) runs. The \( i \)th row corresponds to the factor \( F_i \) and we say that \( F_i \) has \( s_i \) levels.

For \( 0 \leq i, j \leq m \), let \( M_{ij} \) be the \( s_i \times s_j \) matrix such that the rows and columns of \( M_{ij} \) are indexed by \( S_i \) and \( S_j \) respectively and the \((p,q)\)th entry of \( M_{ij} \) is the number of columns of \( A \) in which the \( i \)th and \( j \)th entries are \( p \) and \( q \) respectively, \( p \in S_i \), \( q \in S_j \). \( M_{ij} \) is said to be the \( F_i \) versus \( F_j \) incidence matrix.

Now, suppose \( n = bk \), where \( b \) and \( k \) are integers. A blocked MEP (laid out on \( b \) blocks of size \( k \) each) is an \((m+1) \times bk\) array in which \( S_0 \) is the set of integers \( \{1,2,\ldots,b\} \) and each integer \( j \in S_0 \) appears exactly \( k \) times in the last row of \( A \). The 0-th row is said to correspond to the “block factor”, which is represented by \( B \) (and not \( F_0 \)). In this case, the incidence matrix of the \( i \)th row versus the last row is denoted by \( M_{iB} \), \( 1 \leq i \leq m \). A blocked MEP is said to be ‘connected’ (borrowing a term from the theory of block designs) if each \( M_{iB} \) is the incidence matrix of a connected block design. It is said to be symmetric if \( s_1 = s_2 = \ldots = s_m \). A symmetric MEP with \( s_i = s \) for every \( i \) is also referred to as an MEP for an \( s^m \) experiment.

In the applications, there are \( n \) experimental units, which are classified into homogeneous classes or blocks. These units are used to study the effects of \( m \) factors, the \( i \)th one having \( s_i \) ‘levels’. Typically, an experimental unit, say in the \( j \)th block, receives a ‘level combination’ say \( x = (x_1, \ldots, x_m) \), i.e. the level \( x_i \) of \( F_i \) is applied on that unit, \( i = 1, 2, \ldots, m \). This information is stored in the column vector \((j,x_1,\ldots,x_m)'\).

The array \( A \) consists of all such column vectors.

Definition 2.3 [Bagchi(2010)] The \( i \)th and \( j \)th factors of a blocked MEP \( \rho \) are said to be orthogonal through the block factor (OTB) if

\[
M_{iB}(M_{jB})' = kM_{ij}.
\]

(2.1)

If every pair of factors of a plan \( \rho \) is orthogonal to each other through the block factor, then \( \rho \) is said to be a plan orthogonal through the block factor (POTB).

Remark 2.1: What is the utility of condition (2.1)? This condition guaranties that for the inference on a factor \( F_i \) of a POTB one has to look at only its incidence with the block factor (i.e. \( M_{iB} \)) and forget all other treatment factors. Thus, the performance of a POTB \( \rho \) regarding the inference on the \( i \)th treatment factor depends only on the incidence matrix \( M_{iB} \). We present a more precise statement in the next theorem. We omit the proof which can be obtained by going along the same lines as in the proofs of Lemma 1 and Theorem 1 of Mukerjee, Dey and Chatterjee (2001). [See Shah and Sinha (1989) for definitions, results and other details about optimality].
Theorem 2.1 Suppose a connected POTB $\rho^*$ satisfies the following condition. For some non-increasing optimality criterion $\phi$, $M_{iB}$ is the incidence matrix of a block design $d$ which is $\phi$-optimal in the class of all connected block designs with $s_i$ treatments and $b$ blocks of size $k$ each. Then, $\rho^*$ is $\phi$-optimal in the class of all connected $m$-factor MEP in the same set-up for the inference on the $i$th factor.

In particular, using the well-known optimality property of a BIBD we get the following result.

Corollary 2.1 Suppose $\rho^*$ is a connected POTB. Suppose further that $M_{iB}$ is the incidence matrix of a BIBD. Then, $\rho^*$ is universally optimal in the class of all $m$-factor connected MEP in the same set up, for the inference on the $i$th factor.

In view of the above result, we introduce the following term.

Definition 2.4 A connected POTB is said to be balanced if each of its factors form a BIBD with the block factor, that is $M_{iB}$ is the incidence matrix of a BIBD for each factor $F_i$.

We now present a small example of a balanced POTB. This has two factors, each with four levels $0, 1, 2, 3$ on six blocks of size two each.

| Blocks | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ |
|--------|-------|-------|-------|-------|-------|-------|
| $F_1$  | 0 2   | 1 3   | 0 3   | 1 2   | 0 1   | 3 2   |
| $F_2$  | 1 3   | 0 2   | 2 1   | 3 0   | 3 2   | 0 1   |

Next we list a few combinatorial designs and note their relation with balanced POTBs.

(a) Balanced Graco-Latin block design defined and constructed in Seberry (1979) are balanced POTBs with two factors.

(b) Graco-Latin block design of type 1 of Street (1981) are also two-factor balanced POTBs having

\[ M_{12} = J. \]

(c) Perfect Graeco-Latin balanced incomplete block designs (PERGOLAs) defined and discussed extensively in Rees and Preece (1999) are two-factor balanced POTBs having

(i) $s_1 = s_2 = s$, say and

\[ M_{12}M_{12}' = M_{12}'M_{12} = fI_s + gJ_s, \quad f, g \text{ are integers.} \quad (2.2) \]

(d) Mutually orthogonal BIBDs defined and constructed by Morgan and Uddin (1996) are multi-factor balanced POTBs.

Here $I_n$ is the identity matrix and $J_n$ is the all-one matrix of order $n$.

Remark 2.2: The definition of neither balanced Graco-Latin block designs nor of mutually orthogonal BIBDs include condition (2.2). However, it is interesting to note that all these designs constructed so far do satisfy this condition. [See theorem 3.3].

3 Constructions for symmetric POTBs

Now we present a few constructions of POTB's. Each of these constructions is in terms of some group $G$ of order $g$ (which is the additive group of the field $V$ in Theorem 3.3).

A block will consist of $k$ plots or runs represented by columns. By adding an element $u \in G$ to a block we mean adding $u$ to the level of every factor in every run of the block. By developing an initial block we mean generating $g$ blocks by adding distinct elements of $G$ to the initial block.

Let $N$ denote the set of integers modulo $n$ and $N^+$ denote $N \cup \{\infty\}$. 

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Theorem 3.1 Let $n$ be a positive integer $\geq 5$.
(a) Then there exists a POTB with three factors $F_0, F_1, F_2$ each having $n+1$ levels on $b = 6n$ blocks of size two.
(b) In the case $n = 5$, we get a Balanced POTB, which is not a PERGOLA.

Proof: (a) Let $N^+$ be the set of levels for each factor. The initial blocks $B_{ij}$, $i = 1, 2$, $j = 0, 1, 2$ are as follows. Here addition in the suffix of $F$ is modulo 3.

\[
\begin{array}{c|c|c|c}
F_{0+j} & B_{1j} & B_{2j} \\
\infty & 0 & \infty \\
F_{1+j} & 0 & 1 & 0 & 2 \\
F_{2+j} & -1 & 1 & 1 & 2 \\
\end{array}
\]

That the design satisfies the required property follows by straightforward verification.

(b) Let $n = 5$. One can verify that the incidence matrices satisfy the following.

\[
M_{ij} = \begin{bmatrix}
0 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 1 & 1 \\
2 & 2 & 2 & 2 & 1 \\
2 & 1 & 2 & 2 & 1 \\
2 & 1 & 1 & 2 & 2 \\
2 & 2 & 1 & 1 & 2
\end{bmatrix}, \quad i, j = 0, 1, 2 \quad (3.3)
\]

\[
M_{iB}(M_{iB})' = 8I_6 + 2J_6, \quad i = 1, 2, 3. \quad (3.4)
\]

We see that each $M_{iB}$ is the incidence matrix of a BIBD. Thus, by definition 2.4 it is a balanced POTB. But neither of $M_{ij}s$ satisfy (2.2), as is clear from (3.3). Thus, the two-factor balanced POTB obtained by ignoring any one of the factors is not a PERGOLA. □

Theorem 3.2 Suppose $n$ is an integer $\geq 5$. Then there exists
(a) a POTB with two $n$−level factors on $2n$ blocks and
(b) a POTB with four $n$−level factors on $4n$ blocks of size 2 each.
(c) We get a balanced POTB in the case $n = 5$ in series (a). Further, in the case $n = 10$ in series (b) we get a POTB which is E-optimal for the inference on each factor.

Proof: The set of levels of each factor is $N$. Let $a, b \in N$.
(a) We present initial blocks $B_1$ and $B_2$ below.

\[
\begin{array}{c|c|c|c}
F_0 & B_1 & B_2 \\
a & -a & b & -b \\
b & -b & -a & a \\
\end{array}
\]

(b) We present the initial blocks $B_l$, $l = 1, \cdots 4$ below.

\[
\begin{array}{c|c|c|c|c}
Blocks & B_1 & B_2 & B_3 & B_4 \\
F_1 & 0 & a & a & -a & 0 & b & -b & b \\
F_2 & a & -a & 0 & -a & -b & b & 0 & b \\
F_3 & 0 & b & b & -b & -a & 0 & a & -a \\
F_4 & b & -b & 0 & -b & a & -a & a & 0 \\
\end{array}
\]

That these initial blocks generate POTBs can be verified by straightforward computation.

(c) For $n = 5$, taking $a = 1, b = 2$ we get a balanced POTB.
For $n = 10$, we take $a = 1$ and $b = 3$. Then for every $i = 1, \cdots 4$, $M_{iB}$ is the incidence matrix of a group divisible design with five groups, the $jth$ group being the pair of levels $\{j, j+5\}$, $j = 0, \cdots 4$, satisfying
\( \lambda_0 = 0 \) and \( \lambda_1 = 1 \). This plan is, therefore, E-optimal for the inference on all the four factors by Takeuchi (1961).

**Theorem 3.3**  
(a) There exists a symmetric POTB with four \( n \)-level factors on \( 4n \) blocks of size 2 each, whenever \( n \geq 9 \). We get a balanced POTB in the case \( n = 9 \).

(b) There exists a symmetric POTB with four factors each having \( n + 1 \) levels on \( 6n \) blocks of size 2 each, whenever \( n \geq 7 \).

**Proof:** (a) The set of levels for each factor is \( N \). Let \( a, b, c, d \in N \). The initial blocks \( B_l, l = 1, \cdots, 4 \) are as follows.

| Blocks | \( B_1 \) | \( B_2 \) | \( B_3 \) | \( B_4 \) |
|--------|----------|----------|----------|----------|
| \( F_1 \) | \( a \)  | \( -a \)  | \( b \)  | \( -b \)  |
| \( F_2 \) | \( b \)  | \( -b \)  | \( -c \)  | \( c \)  |
| \( F_3 \) | \( c \)  | \( -c \)  | \( d \)  | \( -d \)  |
| \( F_4 \) | \( d \)  | \( -d \)  | \( -c \)  | \( c \)  |

One can easily verify that these initial blocks generate a symmetric POTB with the given parameters. By taking \( a = 1, b = 2, c = 3 \) and \( d = 4 \) in the case \( n = 9 \), we get a balanced POTB.

(b) The set of levels for each factor is \( N^+ \). Let \( a, b, c \in N \). The initial blocks \( B_l, l = 1, \cdots, 6 \) are as follows.

| Blocks | \( B_1 \) | \( B_2 \) | \( B_3 \) | \( B_4 \) | \( B_5 \) | \( B_6 \) |
|--------|----------|----------|----------|----------|----------|----------|
| \( F_1 \) | 0  | \( \infty \)  | \( a \)  | \( -a \)  | \( b \)  | \( -b \)  | \( c \)  | \( -c \)  | \( a \)  | \( -a \)  | \( \infty \)  |
| \( F_2 \) | \( a \)  | \( -a \)  | 0  | \( \infty \)  | \( c \)  | \( -c \)  | \( b \)  | \( -b \)  | \( a \)  | \( -a \)  | \( \infty \)  |
| \( F_3 \) | \( b \)  | \( -b \)  | \( c \)  | \( -c \)  | 0  | \( \infty \)  | \( a \)  | \( -a \)  | \( \infty \)  | \( c \)  | \( -c \)  | \( \infty \)  |
| \( F_4 \) | \( c \)  | \( -c \)  | \( b \)  | \( -b \)  | \( a \)  | \( -a \)  | 0  | \( \infty \)  | \( \infty \)  | \( c \)  | \( -c \)  | \( \infty \)  |

That the design satisfies the required property follows by straightforward verification.

Next we construct a series of balanced POTBs using finite fields. We first introduce the following notation.

**Notation 3.1**  
(i) Let \( \biguplus \) denote an union counting multiplicity.

(ii) For a set \( A \) and an integer \( n \), let \( nA \) denotes a multiset in which every member of \( A \) occurs \( n \) times.

(iii) For subsets \( A \) and \( B \) of a group \((G, +)\),

\[ A - B = \{ a - b : a \in A, b \in B \}. \]

**Notation 3.2**  
(i) \( v \) denotes an odd prime or a prime power, written as \( v = mf + 1 \). \( V \) denotes the Galois field of order \( v \). Further, \( V^* = V \setminus \{0\} \) and \( V^+ = V \cup \{\infty\} \).

(ii) \( \alpha \) denotes a primitive element of \( V \).

(iii) \( \beta = \alpha^m \) is a generator of the subgroup \( C_0 \) of order \( f \) of \((V^*, .)\).

(iv) \( C_0, C_1, \cdots, C_{m-1} \) are the cosets of \( C_0 \) in \((V^*, .)\).

(v) \( (i, j) = \) the number of ordered pairs of integers \((s, t)\) such that the following equation is satisfied in \( V \). [This notation is borrowed from the theory of cyclotomy]

\[ 1 + \alpha^s = \alpha^t, \ s \equiv i, t \equiv j \pmod{m}. \]

We need the following well-known result. [See Hall (1986), for instance].

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Lemma 3.1 Suppose \( m = 2 \). Then the following hold.

(a) \(-1 \in C_0 \) (respectively \( C_1 \)) if \( f \) is even (respectively odd).

(b) If \( f \) is even, then \( \alpha - 1 \in C_i \Rightarrow \alpha^{-1} - 1 \in C_{i+1} \).

(c) If \( f \) is odd, then \( \alpha - 1 \in C_i \Rightarrow 1 - \alpha^{-1} \in C_{i+1} \).

Here + in the suffix is modulo 2.

We present the following well-known results for ready reference. [See equations (11.6.30), (11.6.40) and (11.6.43) of Hall (1986)].

Lemma 3.2 The differences between the cosets \( C_j \)’s of \( V^* \) can be expressed in terms of the cyclotomy numbers \((p,q)’s\) as follows.

\[
C_i - C_j = \begin{cases} 
\bigcup_{k=0}^{m-1} (k-j,i-j)C_k & \text{if } j \neq i \\
\{0\} \cup \bigcup_{k=0}^{m-1} (k-j,0)C_k & \text{if } j = i 
\end{cases}
\]

The following cyclotomy numbers are known for the case \( m = 2 \).

Case 1: \( f \) odd. \((0,0) = (1,1) = (1,0) = (f-1)/2, (0,1) = (f+1)/2\).

Case 2: \( f \) even. \((0,0) = f/2 - 1, (0,1) = (1,0) = (1,1) = f/2\).

A series of two-factor balanced POTBs:

Theorem 3.4 Suppose \( v \) is an odd prime or a prime power. Then there exists a balanced POTB for a \((v + 1)^2\) experiment on \( b = 2v \) blocks of size \((v + 1)/2\).

Proof: We write \( v = 2f + 1 \). The set of levels of each factor is \( V \cup \{\infty\} \). The plan is obtained by developing the following initial blocks \( B_0 \) and \( B_1 \) presented below.

Case 1: \( f \) is even.

\[
B_0 = \begin{bmatrix} \infty & 1 & \beta & \cdots & \beta^{f-1} \\
0 & \alpha & \alpha \beta & \cdots & \alpha \beta^{f-1} \end{bmatrix}
\]

and \( B_1 = \begin{bmatrix} 0 & 1 & \beta & \cdots & \beta^{f-1} \\
\infty & \alpha^{-1} & \alpha^{-1} \beta & \cdots & \alpha^{-1} \beta^{f-1} \end{bmatrix} \).

Case 2: \( f \) is odd.

Block \( B_0 \) as in case 1, while Block \( B_1 \) is as follows.

\[
B_1 = \begin{bmatrix} 0 & \alpha^{-1} & \alpha^{-1} \beta & \cdots & \alpha^{-1} \beta^{f-1} \\
\infty & 1 & \beta & \cdots & \beta^{f-1} \end{bmatrix}.
\]

Clearly block size is \( f + 1 = (v + 1)/2 \). To show that the plan satisfies the required property, we have to show that

(a) the plan is a POTB and (b) each factor forms a BIBD with the block factor.

Condition (b) follows from Lemma 3.2. So, we prove (a) .

Let us use the following simplified notation \( M = (m_{ij}) \) for \( M_{12} \) and \( A \) for \( M_{1B}(M_{2B})' \). We note that \( m_{ij} \) is the total number of plots (runs) in which the level combination \((i,j)\) appears, while \( a_{ij} \) is the number of blocks in which \( F_1 \) is at level \( i \) and \( F_2 \) at level \( j \), (in same or different plots).

We shall show that

\[
M = J - I \quad \text{and} \quad A = (f + 1)(J - I) \tag{3.5}
\]

We begin with \( M \). It is clear that \( m_{ii} = 0, i \in V^+ \) and \( m_{\infty,i} = m_{i,\infty} = 1, i \in V \).
We, therefore, assume \( i \neq j, \ i, j \in V \). Let \( u = j - i \). Then, \( m_{ij} \) is the number of times \( u \) appears in the multiset

\[
\begin{align*}
\{(\alpha - 1)C_0 \cup (\alpha^{-1} - 1)C_0 \} & \quad \text{if } f \text{ is even} \\
\{(\alpha - 1)C_0 \cup (1 - \alpha^{-1})C_0 \} & \quad \text{if } f \text{ is odd}
\end{align*}
\]

The relations above imply (3.5) in view of Lemma 3.1.

Next we consider \( i \neq j, \ i, j \in V \). Let \( u = j - i \). Then, \( a_{ij} \) is the number of times \( u \) appears in the multiset

\[
\tilde{S} = \begin{cases} 
(\{(0) \cup C_1) - C_0 \} \cup (C_1 - (0) \cup C_0)) & \text{if } f \text{ is even} \\
(\{(0) \cup C_1) - C_0 \} \cup (C_0 - (0) \cup C_1)) & \text{if } f \text{ is odd} 
\end{cases}
\]

These, together with Lemma 3.2 and (a) of Lemma 3.1 imply the equation next to (3.5). \( \square \)

Now we present the series of balanced POTBs available in the literature, together with two newly constructed balanced POTBs.

**Notation 3.3** \( v \) denotes an odd prime or a prime power of the form \( v = mf + 1 \).

### Table 3.1: Balanced POTBs

| No. | The expt. | \( m \) | \# of Blocks | Block size | Inc. matrix \((M_{ij})\) | Reference |
|-----|-----------|--------|--------------|------------|-----------------|-----------|
| 1.  | \((v + 1) \times v\) | 2 2 | \(2v\) | \(f = (v - 1)/2\) | \(J\) | Seberry (1979) |
| 2.  | \((v + 1)^2\) | 2 2 | \(2v\) | \(f + 1 = (v + 1)/2\) | \(J\) | Street (1981) |
| 3(a). | \(v^m\) | \(m\) | \(mv\) | \(f\) | \(J - I\) | Morgan and Uddin (1996) |
| 3(b). | \(v^t\) | \(tg\) | \(mv\) | \(hf, h \leq g\) | \(h(J - I)\) | Morgan and Uddin (1996) |
| 4.  | \(v^f\) | \(m\) | \(mv\) | \(1 + hf, h \leq m\) | \((m - h)I + hJ\) | Morgan and Uddin (1996) |
| 5.  | \((v + 1)^2\) | 2 | \(2v\) | \(f + 1\) | \(J - I\) | Theorem 3.4 |
| 6.  | \((5 + 1)^3\) | - 30 | 2 | given in \(3.5\) | Theorem 3.3 |
| 7.  | \(9^4\) | - 36 | 2 | \(J - I\) | Theorem 3.3 |

In 3(b) above \( t \) is a factor of \( m, g = m/t \) and \( h \) is an integer \( \leq g \).

Using the information in Table 3.1, one may verify the following result.

**Theorem 3.5** Every two-factor balanced POTB obtained from an existing multi-factor balanced POTB, except the one with \( n = 5 \), constructed in Theorem 3.1, is a PERGOLA.

One may also look at Table 1 of Rees and Preece (1999) for many more examples of PERGOLAS.

**A recursive construction**

**Notation 3.4** An orthogonal array with \( m \) rows, \( n \) columns, \( k \) symbols and strength 2, will be denoted by \( OA(n, m, k, 2) \).

**Theorem 3.6** Suppose there exists a balanced POTB with \( f \) factors on \( b \) blocks of size \( k \) each, with \( f \leq k \). If further an \( OA(k^2, m + 1, k, 2) \) exists, then a balanced POTB with \( mf \) factors on \( bk \) blocks of size \( k \) each also exists.
The proof of this theorem is based on the following lemma.

**Lemma 3.3** Consider a set of \( k \) runs of a plan for an experiment with \( f(\leq k) \) factors, such that no level of any factor is repeated. If an \( OA(k^2, m+1, k, 2) \) exists, then there exists an MEP with \( f \) classes of \( m \) \( k \)-level factors on \( k \) blocks of size \( k \) each with the following property. Every factor is orthogonal (w.r.t. the block factor) to every factor of a different class.

**Proof**: Let \( D \) denote the given set of runs. Let \( F = \{P, Q, \cdots\} \) denote the set of \( f \) factors of \( D \). For each \( P \in F \), let \( K_P \) denote the set of levels of \( P \) appearing in \( D \). Let \( K = \{1, \cdots k\} \) denote the set of symbols of the given OA (\( \hat{O} \)), say. For every \( P \in F \), let \( L_P \) denote the following one-one function from \( K \) to \( K_P \). [By assumption, size of \( K_P \) is \( k \) for each \( P \)].

\[
L_P(i) = j, \ i \in K, j \in K_P, \text{ if } P \text{ has level } j \text{ in the } i\text{th run of } D. \tag{3.7}
\]

Let us arrange the columns of the given OA (\( \hat{O} \)) as

\[
(\hat{O}) = \begin{bmatrix} \hat{A}_1 & \cdots & \hat{A}_k \end{bmatrix},
\]

such that the 1st row of \( \hat{A}_i \) consists of the symbol \( i \) repeated \( k \) times, \( 1 \leq i \leq k \). Let \( A_i \) denote the \( m \times k \) array obtained from \( A_i \) by deleting the 1st row. Thus, every member of \( K \) appear exactly once in every row of each \( A_i, i = 1, \cdots k \).

We now construct \( D^* \), the reqd MEP. For each factor \( P \) of \( D \), there will be \( m \) factors \( P_1, \cdots P_m \) in \( D^* \), each of which will have \( K_P \) as the set of levels.

For \( i \in K \), let us fix \( A_i \) and a factor, say \( P \) of \( D \). If the \( j \) th column of \( A_i \) is \( (s_1, \cdots s_m), s_u \in K \), then in the \( j \)th plot of the \( i \)th block of \( D^* \), the factor \( P \) will have level \( L_P(s_t), t = 1, 2, \cdots m \), where \( L_P \) is as in \( \text{(3.7)} \). Doing the same for all the factors and varying \( j \) over \( \{1, 2, \cdots k\} \) we get a block of \( D^* \). Finally varying \( i \) over \( K \) we we generate the \( k \) blocks of the reqd MEP.

We now show that the MEP \( D^* \) satisfies the required property. We fix two factors, say \( P_i \) and \( Q_j \), \( i \neq j \) and an ordered pair of levels, say \( (u, v), u \in K_P, v \in K_Q \). From the construction the following is clear. In every block there is a plot in which \( P_i \) is at level \( u \) and a plot where \( Q_j \) is at level \( v \). Moreover, there is exactly one block in which these factors are set at these levels in the same plot. Thus, the factors \( P_i \) and \( Q_j \) are mutually orthogonal through the block factor. We see that if \( P = Q \), then also the argument above holds. Thus \( P_i \) is orthogonal to \( P_j \), \( j \neq i \). However, \( P_i \) may not be orthogonal to \( Q_i \). We, therefore form the classes as \( C_i = \{P_i, Q_i, \cdots\}, P, Q \in F, i = 1, 2, \cdots m \). Now the factors satisfy the orthogonality condition of the hypothesis. □

**Proof of the theorem**: Let \( D \) denote the given POTB. Let \( A_i, i \in K \) be as in Lemma. For every block of \( D \) we construct a an MEP following the method described in the proof of the lemma above. Let the resultant MEP be named \( D^* \). By Lemma 3.3 every pair of factors belonging to different classes are orthogonal w.r.t. the block factor. Further, since the pair of factors \( P, Q \) are mutually orthogonal w.r.t. the block factor in \( D \), it follows that for every \( i \in K \), the factors \( P_i \) and \( Q_i \) are also mutually orthogonal w.r.t. the block factor in \( D^* \). □

**Remark 3.1**: If we look at the restriction of \( D^* \) to one factor, say \( P \), we see that it is nothing but \( k \) times repetition of each block of the block design obtained from the restriction of \( D \) to the factor \( P \).

## 4 References

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