NONEXISTENCE RESULTS FOR A FULLY NONLINEAR EVOLUTION INEQUALITY

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Abstract. In this paper, a Liouville type theorem is proved for some global fully nonlinear evolution inequality via suitable choices of test functions and the argument of integration by parts.

1. Introduction

In the papers [7, 8], Phuc and Verbitsky deduced nonexistence results for the following Hessian inequality:

$$\sigma_k(-D^2u) \geq u^\alpha \quad \text{in } \mathbb{R}^n,$$

for $$\alpha \in (k, \frac{nk}{n-2k}]$$, where $$u \in C^2(\mathbb{R}^n)$$, $$u$$ is $$k$$-convex, and $$\sigma_k(-D^2u)$$ are the $$k$$-Hessian of $$(-D^2u)$$ as usual, i.e., the sum of all the $$k$$-th principle minors of $$(-D^2u)$$. They employed the potential theory developed by Trudinger-Wang [9, 10, 11] and Labutin [4], and they also showed that the power $$\alpha = \frac{nk}{n-2k}$$ is sharp. The nonexistence results were deduced in [7, 8] from sharp pointwise estimates of solutions in terms of Wolff potentials. Later in [6], the author reproved some of Phuc-Verbitsky’s results by a very different method – by using the argument of integration by parts via careful choices of the test functions.

In this paper, we will use the same method as in [6] and extend the results to the evolution case, namely, for the following fully nonlinear inequality:

$$u_t + \sigma_k(-D^2u) \geq u^\alpha \quad \text{in } (x,t) \in \mathbb{R}^n \times (0, +\infty),$$

with $$u \in C^2(\mathbb{R}^n)$$, $$u$$ is $$k$$-convex, and $$u(x,t) > 0 \ \forall \ (x,t) \in \mathbb{R}^n \times (0, +\infty)$$, and $$u_0 = u(x,0) \geq 0 \ \forall \ x \in \mathbb{R}^n$$. Denote $$k_* := k + \frac{2k}{n}$$. We will deduce a Liouville type theorem as follows:

**Theorem 1.1.** If $$2k < n$$, then (1.2) has no positive $$k$$-admissible solution $$u \in C^2(\mathbb{R}^n)$$ for any $$\alpha \in (k, k_*]$$.

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Note that according to Caffarelli-Nirenberg-Spruck [1], we say that $u$ is $k$-admissible (or $k$-convex) with respect to $\sigma_k(-D^2u)$ if $u \in \Gamma_k$, where $\Gamma_k$ is defined by

$$\Gamma_k = \{ u \in C^2(\mathbb{R}^n) : \sigma_s(-D^2u) \geq 0, s = 1, 2, \ldots, k \}.$$  

Similar nonexistence results for some quasilinear evolution inequalities were proved by Mitidieri-Pohozaev [5]. In particular, in the case of equality for $k = 1$, the results of our Theorem 1.1 were proved by Fujita [2] and Hayakawa [3]. Therefore, Theorem 1.1 can be viewed as a generalization of their results to the fully nonlinear case.

2. Proof of Theorem 1.1

Assume that $u > 0$ is a $k$-admissible solution of (1.2). In the following, we write $\sigma_k(-D^2u)$ simply as $\sigma_k$.

First, we will construct a suitable test function.

Denote by $D$ the gradient operator in the space directions only. Let $\varphi(x), \psi_0(s)$ be two $C^2$ cut-off function satisfying

$$\begin{cases}
\varphi \equiv 1 & \text{in } B_R, \\
0 \leq \varphi \leq 1 & \text{in } B_{2R}, \\
\varphi \equiv 0 & \text{in } \mathbb{R}^n \setminus B_{2R}, \\
|D\varphi| \lesssim \frac{1}{R} & \text{in } \mathbb{R}^n,
\end{cases} \quad (2.1)$$

and

$$\begin{cases}
\psi_0 \equiv 1 & \text{for } -1 \leq s \leq 1, \\
0 \leq \psi_0 \leq 1 & \text{for } 1 < |s| < 2, \\
\psi_0 \equiv 0 & \text{for } |s| \geq 2, \\
|\psi'_0| \lesssim 1 & \text{in } \mathbb{R}.
\end{cases} \quad (2.2)$$

Here and in the rest of the paper, $B_R$ denotes a ball in $\mathbb{R}^n$ centered at the origin with radius $R$, and we use “$\lesssim$”, “$\gtrsim$”, etc., to drop out some positive constants independent of $R$ and $u$.

Take $\psi(t) = \psi_0\left(\frac{1}{R^{\rho+\tau}} \right)$ and let $\eta(x,t) = \varphi(x)\psi(t)$. Denote, as in [6], for $s = 1, \ldots, k$,

$$b_s = \frac{k + s}{s!2^s} \delta(\delta + 1) \cdots (\delta + s - 1),$$

$$B_s = \int_{\mathbb{R}^n} \sigma_{k-s}Du^{2s}u^{-\delta-s}\varphi^\theta dx,$$

where $\rho$, $\delta$, $\theta$ are constants to be determined.

Now we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Multiplying both sides of (1.2) by $u^{-\delta}\eta^\theta$ and integrating over $\mathbb{R}^n \times (0, +\infty)$, we have

$$\int_0^\infty \int_{\mathbb{R}^n} u^{-\delta}\eta^\theta dxdt \leq \int_0^\infty \int_{\mathbb{R}^n} u_t u^{-\delta}\eta^\theta dxdt + \int_0^\infty \int_{\mathbb{R}^n} \sigma_k u^{-\delta}\eta^\theta dxdt. \quad (2.3)$$

Consider the first integral on the right hand side of (2.3). Integrating by parts once, we get

$$\int_0^\infty \int_{\mathbb{R}^n} u_t u^{-\delta}\eta^\theta dxdt = \frac{1}{1-\delta} \int_0^\infty \int_{\mathbb{R}^n} \partial_t(u^{1-\delta})\eta^\theta dxdt$$

$$= -\frac{1}{1-\delta} \int_{\mathbb{R}^n} u_0^{1-\delta}\varphi^\theta dx - \frac{\theta}{1-\delta} \int_0^\infty \int_{\mathbb{R}^n} u^{1-\delta}\eta^{\theta-1}\eta_t dxdt. \quad (2.4)$$
By the choice of the test function, we have $|\eta| \lesssim R^{-\rho}$. Then inserting (2.4) into (2.3) we have
\[
\int_0^\infty \int_{\mathbb{R}^n} u^{\alpha-\delta} \eta^\theta dx dt + \frac{1}{1-\delta} \int_{\mathbb{R}^n} u^{1-\delta} \phi^\theta dx \lesssim R^{-\rho} \int_{\mathbb{R}^n} u^{1-\delta} \phi^\theta dx + \int_0^{\infty} \int_{\mathbb{R}^n} \sigma_i u^{-\delta} \eta^\theta dx dt.
\] (2.5)
Now, for $\alpha \in (k, k_*)$ we split into two cases with suitable choice of $\delta$ respectively:
(i) Let $0 < \delta < \min\{1, \frac{\alpha-2k}{k_*-k}\}$ for $\alpha \in (k, k_*)$.
(ii) Let $\delta = 0$ first and then $0 < \delta < 1$ for $\alpha = k_*$. 
In case (i), the following integral estimate had been deduced by the author (see (3.10) in [6]):
\[
\int_{\mathbb{R}^n} \sigma_k u^{-\delta} \varphi^\theta dx + \sum_{s=1}^k (b_s - \varepsilon) B_s \lesssim \frac{1}{R^{2k}} \int_{\mathbb{R}^n} u^{-\delta+k} \varphi^{-2k} dx,
\] (2.6)
where $\varepsilon$ is a small positive constant which comes from Young’s inequality.
By the choice of $\delta$, we see $b_s > 0$ for $s = 1, \ldots, k$. Taking $\varepsilon$ small enough, we have
\[
k \int_{\mathbb{R}^n} \sigma_k u^{-\delta} \varphi^\theta dx \lesssim \frac{1}{R^{2k}} \int_{\mathbb{R}^n} u^{-\delta+k} \varphi^{-2k} dx.
\] (2.7)
Inserting this into (2.5), we get
\[
\int_0^\infty \int_{\mathbb{R}^n} u^{\alpha-\delta} \eta^\theta dx dt + \frac{1}{1-\delta} \int_{\mathbb{R}^n} u^{1-\delta} \phi^\theta dx \lesssim R^{-\rho} \int_{\mathbb{R}^n} u^{1-\delta} \phi^\theta dx + \frac{1}{R^{2k}} \int_0^{\infty} \int_{\mathbb{R}^n} u^{-\delta+k} \eta^\theta dx dt.
\] (2.8)
Since $\frac{\alpha-\delta}{-\delta+k} > 1$, using Young’s inequality with exponent pair $(\frac{\alpha-\delta}{-\delta+k}, \frac{\alpha-\delta}{k-\alpha})$ in the last term in (2.8) we get
\[
\frac{1}{R^{2k}} \int_0^{\infty} \int_{\mathbb{R}^n} u^{-\delta+k} \eta^\theta dx dt \lesssim \varepsilon \int_0^{\infty} \int_{\mathbb{R}^n} u^{\alpha-\delta} \eta^\theta dx dt + \frac{2k(\alpha-\delta)}{\alpha-k} \int_0^{\infty} \int_{\mathbb{R}^n} \eta^\theta dx dt.
\] (2.9)
Then, by choosing $\theta$ large enough, it follows that
\[
\int_0^{\infty} \int_{\mathbb{R}^n} \eta^\theta dx dt \lesssim R^{\alpha+\rho},
\] (2.10)
and then
\[
\frac{1}{R^{2k}} \int_0^{\infty} \int_{\mathbb{R}^n} u^{-\delta+k} \eta^\theta dx dt \lesssim \varepsilon \int_0^{\infty} \int_{\mathbb{R}^n} u^{\alpha-\delta} \eta^\theta dx dt + R^{\alpha+\rho - \frac{2k(\alpha-\delta)}{\alpha-k}}.
\] (2.11)
Similarly, for the second-to-last term in (2.8), we have
\[
R^{-\rho} \int_{\mathbb{R}^n} u^{\alpha-\delta} \eta^\theta dx dt \lesssim \varepsilon \int_0^{\infty} \int_{\mathbb{R}^n} u^{\alpha-\delta} \eta^\theta dx dt + R^{\alpha+\rho - \frac{2k(\alpha-\delta)}{\alpha-k}}.
\] (2.12)
Combining (2.8) with (2.11) and (2.12), we get
\[
\int_0^{\infty} \int_{\mathbb{R}^n} u^{\alpha-\delta} \eta^\theta dx dt + \frac{1}{1-\delta} \int_{\mathbb{R}^n} u^{1-\delta} \phi^\theta dx \lesssim R^{\alpha+\rho} \frac{\alpha^\rho}{\alpha-k} + R^{\alpha+\rho - \frac{2k(\alpha-\delta)}{\alpha-k}}.
\] (2.13)
Since $0 < \delta < \min\{1, \frac{n-2k}{2k}(k_\ast - \alpha)\}$ and $\alpha \in (k, k_\ast)$, we see
\[
\frac{n(\alpha - 1)}{1 - \delta} < \frac{2k(\alpha - \delta)}{\alpha - k} - n.
\]
So we can take $\rho$ such that
\[
n(\alpha - 1) \frac{1}{1 - \delta} < \rho < \frac{2k(\alpha - \delta)}{\alpha - k} - n,
\]
and hence $n + \rho - \frac{\rho \alpha}{\alpha - 1} < 0$, $n + \rho - \frac{2k(\alpha - \delta)}{\alpha - k} < 0$. Letting $R \to +\infty$ in (2.13), we deduce
\[
\int_0^\infty \int_{\mathbb{R}^n} u^{\alpha - \delta} \eta^\theta \, dx \, dt + \frac{1}{1 - \delta} \int_{\mathbb{R}^n} u_0^{1 - \delta} \varphi^\theta \, dx \leq 0. \tag{2.14}
\]
This is a contradiction, since $u > 0$ and $u_0 \geq 0$.

In case (ii), first we have, by taking $\delta = 0$ in (2.5),
\[
\int_0^\infty \int_{\mathbb{R}^n} u^{\alpha - \delta} \eta^\theta \, dx \, dt + \int_{\mathbb{R}^n} u_0 \varphi^\theta \, dx \lesssim R^{-\rho} \int_{\mathbb{R}^n} \sigma_k \eta^\theta \, dx + \int_0^\infty \int_{\mathbb{R}^n} \varphi^\theta \, dx. \tag{2.15}
\]
To deal with the last term in (2.15), we denote $U = \text{supp } |D\varphi| = B_{2R} \setminus B_R$ and
\[
V_s = R^{\frac{n-2k}{2} \delta - 2s} \int_U \sigma_{k-s} u^{(2s-1)\delta + s} \varphi^{-2s} \, dx
\]
and
\[
W_s = R^{\frac{n-2k}{2} \delta - 2s} \int_U \sigma_{k-s} u^s \varphi^{-2s} \, dx,
\]
where $0 < \delta < 1$ is fixed small enough. Then similar to lemma 3.2, 3.3 in [6], we have

**Lemma 2.1.** For $s = 1, \cdots, k - 1$,
\[
V_s \lesssim B_{s+1} + V_{s+1} + W_{s+1}, \tag{2.16}
\]
and
\[
W_s \lesssim B_{s+1} + V_{s+1} + W_{s+1}. \tag{2.17}
\]

By these, it is not difficult to deduce
\[
\int_{\mathbb{R}^n} \sigma_k \varphi^\theta \, dx \lesssim R^{-2k} \int_U u^k \varphi^{-2k} \, dx
\]
\[
+ R^{-\frac{n-2k}{2} \delta - 2k} \int_U u^{-\delta + k} \varphi^{-2k} \, dx + R^{(2k-1)\frac{n-2k}{2} \delta - 2k} \int_U u^{(2k-1)\delta + k} \varphi^{-2k} \, dx. \tag{2.18}
\]

**Remark 2.2.** For the details of the proof of (2.18), we refer the readers to [6] (see (3.30) combining with (3.15) in [6]). But here we must point out that the term $R^{-\frac{n-2k}{2} \delta} V_k$ had been left out in the inequality (3.26) (and hence (3.30)) in [6]), although the final result is still valid.
For the last term in (2.19), using the Hölder inequality we have
\[
\int_0^\infty \int_R^n u^\alpha \eta^\theta dx dt + \int_R^n u_0^{1-\delta} \varphi^\delta dx 
\leq R^{-\rho} \int_{R^\rho} u \eta^{\theta - 1} dx dt + R^{-2k} \int_0^\infty \int_U U^k \eta^{\theta - 2k} dx dt 
+ R^{-\frac{n+\alpha \delta - 2k}{\alpha}} \int_0^\infty \int_U u^{-\delta + k} \eta^{\theta - 2k} dx dt 
+ R^{2(k-1) \frac{n+\alpha \delta - 2k}{\alpha}} \int_0^\infty \int_U (2k-1) \gamma + k \eta^{\theta - 2k} dx dt.
\] (2.19)

For the last term in (2.19), using the Hölder inequality we have
\[
R^{2(k-1) \frac{n+\alpha \delta - 2k}{\alpha}} \int_0^\infty \int_U (2k-1) \gamma + k \eta^{\theta - 2k} dx dt 
\leq R^{2(k-1) \frac{n+\alpha \delta - 2k}{\alpha}} \left( \int_0^\infty \int_U u \eta^{\theta} dx dt \right)^{(2k-1) \frac{\gamma + k}{\alpha}} 
\times \left( \int_0^\infty \int_U u \eta^{\theta} dx dt \right)^{-\frac{\gamma}{\alpha}}. \tag{2.20}
\]

Similarly, we have
\[
R^{-\rho} \int_{R^\rho} u \eta^{\theta - 1} dx dt \leq R^{\frac{n-k}{\alpha} (n+\rho - \frac{2k}{\alpha})} \left( \int_{R^\rho} u \eta^{\theta} dx dt \right)^{\frac{k}{n}}, \tag{2.21}
\]
\[
R^{-2k} \int_0^\infty \int_U u^k \eta^{\theta - 2k} dx dt \leq R^{\frac{n-k}{\alpha} (n+\rho - \frac{2k}{\alpha})} \left( \int_0^\infty \int_U u^\alpha \eta^\theta dx dt \right)^{\frac{k}{n}}, \tag{2.22}
\]
and
\[
R^{-\frac{n+\alpha \delta - 2k}{\alpha}} \int_0^\infty \int_U u^{-\delta + k} \eta^{\theta - 2k} dx dt \leq R^{\frac{n-k}{\alpha} (n+\rho - \frac{2k}{\alpha})} \left( \int_0^\infty \int_U u^\alpha \eta^\theta dx dt \right)^{\frac{k}{n}}. \tag{2.23}
\]

Notice that \( \alpha = k + \frac{2}{n} k \), if we take \( \rho = (n+2)k - n \), then \( n+\rho - \frac{2k}{\alpha} = n+\rho - \frac{\rho_0}{\alpha} = 0 \). Hence (2.19) combining with (2.20)–(2.23) implies
\[
\int_0^\infty \int_R^n u^\alpha \eta^\theta dx dt + \int_R^n u_0^{1-\delta} \varphi^\delta dx 
\leq \left( \int_{R^\rho} u^\alpha \eta^\theta dx dt \right)^{\frac{k}{n}} + \left( \int_0^\infty \int_U u^\alpha \eta^\theta dx dt \right)^{\frac{k}{n}} 
+ \left( \int_0^\infty \int_U u^\alpha \eta^\theta dx dt \right)^{-\frac{\gamma}{\alpha}} \left( \int_0^\infty \int_U u^\alpha \eta^\theta dx dt \right)^{(2k-1) \frac{\gamma + k}{\alpha}}. \tag{2.24}
\]

Since \( 0 < \frac{1}{\alpha}, \frac{k}{\alpha}, \frac{k-\delta}{\alpha}, \frac{(2k-1)\delta + k}{\alpha} \) < 1 (fixed \( \delta > 0 \) small enough), (2.24) shows that
\[
\int_0^\infty \int_R^n u^\alpha \eta^\theta dx dt \leq \text{constant} < \infty. \tag{2.25}
\]
This implies
\[
\int_{R^n} u^\alpha \eta^\beta dx \rightarrow 0, \quad \int_{U} u^\alpha \eta^\beta dx dt \rightarrow 0 \quad \text{as} \quad R \to +\infty. \quad (2.26)
\]
Returning to (2.24), we deduce
\[
\int_{U} u^\alpha \eta^\beta dx dt \rightarrow 0 \quad \text{as} \quad R \to +\infty. \quad (2.27)
\]
This is also a contradiction, and hence the proof of Theorem 1.1 is complete. \qed

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