Maximal depth implies \( su(3) + su(2) + u(1) \)

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Abstract  
Hence it excludes proton decay and supersymmetry. The main predictions of a gauge model based on the exceptional simple Lie superalgebra \( \mathfrak{mb}(3|8) \) (a localized version of \( su(3) + su(2) + u(1) \)) are reviewed.

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Simple infinite-dimensional Lie superalgebras of polynomial vector fields have recently been classified \[7, 19\]. The exception \(\mathfrak{mb}(3|8)\) (a.k.a. \(E(3|8)\)) \[3, 13, 18\] is singled out as the unique algebra of maximal depth 3 in its consistent \(\mathbb{Z}\)-gradation. Its representation theory was recently worked out \[6, 11, 14\]. This superalgebra, and its close relative \(\mathfrak{vle}(3|6) = E(3|6)\) of depth 2, are Cartan prolongs \((g_-, g_0)\) where \(g_-\) is nilpotent and \(g_0 = su(3) + su(2) + u(1)\). This means that there is a 1-1 correspondence between \(\mathfrak{mb}(3|8)\) (and \(\mathfrak{vle}(3|6)\)) irreps and \(su(3) + su(2) + u(1)\) irreps, suggesting that an \(\mathfrak{mb}(3|8)\) symmetry might be mistaken experimentally for the symmetries of the standard model (SM) in particle physics.

With this in mind, I recently constructed a gauge model with \(\mathfrak{mb}(3|8)\) symmetry \[14\]. It was called the “second-gauged standard model” (SGSM), since the \(su(3) + su(2) + u(1)\) symmetry is made local not only in spacetime (first gauging), but also in internal space (second gauging). However, the details of this model are not necessary to extract the main experimental consequences, which differ sharply from predictions of popular theories such as supersymmetry (SUSY) and grand unified theories (GUTs). It is the purpose of this note to list and motivate these predictions, which will be tested at the LHC and at SuperKamiokande over the next decade. My hope is also that model builders will be attracted by this substantially new local mathematics, which immediately and unambiguously leads to the correct SM symmetries. Technical details are given in the appendix.

The main observation is that the correct symmetry of the SM is not \(su(3) + su(2) + u(1)\) itself, but rather its associated current algebra \(\text{map}(4, su(3) + su(2) + u(1))\) of maps from four-dimensional spacetime to the gauge algebra. Since the bosons in the SM are gauge bosons, i.e. current algebra connections, the bosons in the SGSM must be connections of \(\text{map}(4, \mathfrak{mb}(3|8))\); this point was missed in \[8\]. Just as there is a 1-1 correspondence between \(\mathfrak{mb}(3|8)\) and \(su(3) + su(2) + u(1)\) irreps, there is also a 1-1 correspondence between connections in the associated current algebras.

The SGSM thus predicts absence of new gauge bosons, and consequently no proton decay, except possibly by instanton effects. In contrast, all GUTs, and theories which can be approximated by GUTs at low energies, have interactions that do not conserve baryon number; in fact, the minimal \(SU(5)\) model was slain long ago precisely because it predicted too rapid proton decay. Also low-energy SUSY models, both the MSSM and the ESSM, predict proton decay at rates which are straining the bounds from SuperKamiokande \[16\]. Within a decade, proton decay will either be discovered, or the bounds will be stringent enough to rule out GUTs and SUSY.

The main reason why SUSY became popular is that it is the only (known)
way to circumvent the Coleman-Mandula theorem and combine internal and spacetime symmetries in a non-trivial way \[20\]. However, there is no good experimental reason why such unification would be desirable. On the contrary, if we elevate maximal depth to a guiding principle (and at least the \( su(3) + su(2) + u(1) \) part is strongly supported by experiments), internal/spacetime unification is ruled out. Although there certainly are simple Lie superalgebras containing both \( \mathfrak{m} \mathfrak{b}(3|8) \) and the Poincaré algebra as subalgebras (\( \mathfrak{vect}(7|8) \) is an obvious example), any such superalgebra is necessarily of less than maximal depth. Hence maximal depth requires that the \( \mathfrak{map}(4, \mathfrak{m} \mathfrak{b}(3|8)) \) symmetry be combined with spacetime symmetries in the trivial way, i.e. as a semi-direct product. GUTs are excluded for essentially the same reason. Any finite-dimensional Lie algebra \( \mathfrak{g} \supset \mathfrak{su}(3)+\mathfrak{su}(2)+\mathfrak{u}(1) \) is of less depth, as are all prolongs based on \( \mathfrak{g} \).

If internal and spacetime symmetries are not combined non-trivially, gravity is not unified with other interactions. Since the main argument for string theory is that it allegedly contains gravity \[17\], maximal depth and string theory are mutually exclusive. \( \mathfrak{m} \mathfrak{b}(3|8) \) is proposed as a generalization of the SM symmetries, whereas gravity must be dealt with separately. It can be remarked that all projective lowest-energy Fock representations of the spacetime diffeomorphism algebra, which is the correct symmetry of classical gravity, naturally involve worldlines \[12\]. This indicates that the correct treatment of quantum gravity conceptually involves field theory, or at least a theory with particle-like quanta.

The predictions so far were negative – proton decay and supersymmetric particles are not to occur. Mainly negative predictions are in a sense expected, since the SM is so successful experimentally, but one would nevertheless like to have some way to discriminate between \( \mathfrak{m} \mathfrak{b}(3|8) \) and \( su(3) + su(2) + u(1) \). In fact, there are two generic effects that point to \( \mathfrak{m} \mathfrak{b}(3|8) \) and that have already been experimentally confirmed: CP violation and several generations.

CP violation can be accounted for already within the SM by several mechanisms (theta angle, neutrino mass matrix, CKM matrix) \[1\], but it is one of the least understood aspects, as witnessed by expressions like “strong CP problem” and “flavour problem”. Also, CP violation in the SM is too weak to explain certain cosmological data, in particular the observed baryon-to-photon ratio. Almost any extension of the SM provides new sources of CP violation \[13\], but \( \mathfrak{m} \mathfrak{b}(3|8) \) has a mechanism which makes CP violation manifest already at the symmetry level: conjugate representations are inequivalent.

In fact, every infinite-dimensional algebra of vector fields has this property. E.g., in \( \mathfrak{gl}(n) \) the distinction between conjugate representations is purely conventional, because this algebra is generated by vector fields of the form \( x^\mu \partial_\nu \), where \( [\partial_\nu, x^\mu] = \delta^\mu_\nu \). One usually considers \( x^\mu \) as the coordinate and \( \partial_\nu = \partial/\partial x^\nu \) as the derivative, but the opposite interpretation is
equally valid. In contrast, in the full algebra \( \text{vect}(n) \) of vector fields (which is related to \( gl(n) \) as \( \mathfrak{mb}(3|8) \) is related to \( su(3) + su(2) + u(1) \)), the distinction between co- and contra- is substantial. A general vector field is of the form \( \xi = \xi^\mu(x) \partial_\mu \), and thus the symmetry between \( x^\mu \) and \( \partial_\nu \) is lost. On the representation level, this is manifested in the existence of the exterior derivative, which acts on covariant skew tensor fields only.

Let us now turn to the fermions. Because gauge bosons must be identified with connections of the associated current algebra, they are not unified into a single multiplet. Analogously, there is no compelling reason why different species of quarks and leptons should be unified. Note in passing that all applications of conformal field theory to statistical physics contain several Virasoro irreps \( [4] \). This is the reason why \( \mathfrak{mb}(3|8) \), despite being a superalgebra, does not predict unobserved SUSY partners. Note also that \( \mathfrak{mb}(3|8) \) is not technically a SUSY, because it does not contain a Poincaré subalgebra.

However, any \( \mathfrak{mb}(3|8) \) irrep decomposes into several \( su(3) + su(2) + u(1) \) irreps under restriction, which can account for several generations. I thus suggest “vertical” unification between generations, but not “horizontal” unification within a single generation. Denote the basis of a \( \text{map}(4, \mathfrak{mb}(3|8)) \) module by \( \psi(x, \theta, u, \vartheta) \), where \( x \) is the spacetime coordinate and \( (\theta, u, \vartheta) \) are coordinates in internal 3|8-dimensional space (see Appendix). Upon restriction to \( \text{map}(4, su(3) + su(2) + u(1)) \), this module contains three natural sub-modules, defined by the conditions \( \int d^6 \theta \psi(x, \theta, u, \vartheta) = 0 \), \( \int d^3 u \psi(x, \theta, u, \vartheta) = 0 \), and \( \int d^2 \vartheta \psi(x, \theta, u, \vartheta) = 0 \), respectively. As a very speculative note, this might suggest that the three SM generations are related to depth 3.

The predictions so far were quite generic, and apply to \( \mathfrak{vle}(3|6) \) as well, but the fermion content is \( \mathfrak{mb}(3|8) \)-specific. To consider \( \mathfrak{mb}(3|8) \) would be overkill if we wanted to identify fermions with tensor modules, since all information about these is encoded already in the \( su(3) + su(2) + u(1) \) subalgebra. Instead, I suggested in [14] that fermions should be identified with closed form modules, i.e. the irreducible quotients \( \ker \nabla \), where \( \nabla \) is an invariant morphism (the analogue of the exterior derivative). \( \mathfrak{mb}(3|8) \) tensor modules are labelled by weights \( (p, q; r; Y) \), \( p, q, r \in \mathbb{N}, Y \in \mathbb{R} \), where \( p\pi_1 + q\pi_2 \) is an \( su(3) \) weight, \( r \) is an \( su(2) \) weight and \( Y \) is weak hypercharge.\(^1\) The tensor module \( T(p, q; r; Y) \) is irreducible except for four series \( [11, 14] \):

\[\begin{align*}
\Omega_A(p, r) &= T(p, 0; r; -\frac{4p}{3} + r - 2), \\
\Omega_B(p, r) &= T(p, 0; r; -\frac{4p}{3} - r - 4), \\
\Omega_C(q, r) &= T(0, q; r; \frac{4q}{3} + r + 2), \\
\Omega_D(q, r) &= T(0, q; r; \frac{4q}{3} - r).
\end{align*}\]

\(^2\)In [14], the abelian weight was defined with a factor 3 in order to work exclusively with integers.

\(^3\)Kac and Rudakov [11] obtain the contragredient result.

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The best assignment possible is described by the following table:

| weight     | fermion | form       | $Y$(SM) | $Y$(mb) |
|------------|---------|------------|---------|---------|
| $(0,1;1;\frac{1}{3})$ | $\begin{pmatrix} u_L \\ d_L \end{pmatrix}$ | $\Omega_D(1,1)$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $(0,1;0;\frac{4}{3})$ | $u_R$ | $\Omega_D(1,0)$ | $\frac{4}{3}$ | $\frac{4}{3}$ |
| $(0,1;0;\frac{-2}{3})$ | $d_R$ | $\Omega_C(1,0)$ | $\frac{-2}{3}$ | $\frac{10}{3}$ |
| $(0,0;1;1)$ | $\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$ | $\Omega_D(0,1)$ | $-1$ | $-1$ |
| $(0,0;0;2)$ | $e_R$ | $\Omega_C(0,0)$ | $-2$ | $2$ |

We see that the $su(3) + su(2)$ weights of the fundamental fermions can be fitted very snugly into the list of form modules, but that the hypercharge assignment in the C sector ($d_R$ and $e_R$) is off by four units. Also, the anti-fermions fit into the A and B sectors, but here we have an additional discrepancy $\Delta Y = 2$. Replacing $\mathfrak{mb}(3|8)$ by $\mathfrak{vle}(3|6)$ does not help; the reducibility conditions are different \cite{9}, but it is still impossible to fit all fermions to closed form modules.

Although the hypercharge problem is embarrassing, it can be circumvented in several ways. Maybe $d_R$ and $e_R$ (but not the fermions in the D sector) should be identified with irreducible tensor modules. Another possibility is that the restriction $\mathfrak{mb}(3|8) \rightarrow su(3) + su(2) + u(1)$, which is not understood, introduces extra hypercharge in the C sector. E.g., integration over the internal $3|8$-dimensional space contributes $\Delta Y = 2$. However, we can conclude that hypercharge is quantized in multiples of $1/3$.

To summarize, the main predictions from the assumption of maximal depth are:

- $su(3)+su(2)+u(1)$ symmetry; more precisely, a localized form thereof.
- No proton decay, since there are no new gauge bosons.
- No sparticles, since supersymmetry is incompatible with maximal depth.
- Manifest CP violation, since conjugate representations are inequivalent.
- Several generations, since the restriction of an $\mathfrak{mb}(3|8)$ irrep to $su(3) + su(2) + u(1)$ is reducible.
- Several fermion species in qualitative agreement with experiments, but the naïve hypercharge assignment in the C sector is off by four units.
- Charge quantization in units of $1/3$.  

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The absence of proton decay and sparticles will be tested at the LHC and SuperKamiokande over the next decade. However, maximal depth excludes SUSY and GUTs in principle, not only at energy scales accessible to the next generation of experiments. The reason is simple: if nature reaches maximal depth at some scale, it would be unnatural to retract to less depth at higher energy. There is of course also the possibility that proton decay and sparticles will be seen in experiments soon. If so, the fact that maximal depth immediately leads to the correct SM symmetries will merely be a mathematical curiosity.

Appendix

Every algebra of polynomial vector fields $\mathfrak{g}$ admits a grading by finite-dimensional vector spaces of depth $d$ and height $h$:

$$\mathfrak{g} = \mathfrak{g}_{-d} + \ldots + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \ldots + \mathfrak{g}_h.$$  

Finite-dimensional Lie superalgebras have gradings with depth and height equal and at most 2, whereas infinite-dimensional algebras have infinite height and finite depth. Apart from an inconsistent regrading of $\mathfrak{esk}(5|10)$, $\mathfrak{mb}(3|8)$ is the unique simple Lie superalgebra of maximal depth 3 [7, 19].

The grading is said to be consistent if odd subspaces are purely fermionic and even subspaces purely bosonic. It is known that the only consistently graded simple algebras are the contact algebras $\mathfrak{k}(1|m)$ (a.k.a. the centerless $N = m$ superconformal algebra) and the four exceptions $\mathfrak{esk}(5|10)$, $\mathfrak{vle}(3|6)$, $\mathfrak{mb}(3|8)$ and $\mathfrak{esk}(1|6)$. These algebras were first discovered by Shchepochkina [8, 9] (the $\mathfrak{esk}(1|6)$ was independently found in [2]), and their consistent gradings first appeared in [3]. They were described explicitly in terms of generators and brackets in [4], and the preserved geometries were discovered in [13].

It is instructive to consider some examples.

The polynomial part of the Virasoro algebra $\mathfrak{vect}(1)$ has generators $L_m = t^{m+1} \frac{\partial}{\partial t}$, $m \geq -1$. Setting $\deg L_m = m$, we see that this algebra has depth 1; the grading is inconsistent since $L_{-1}$ is bosonic.

The polynomial part of the superconformal algebra $\mathfrak{f}(1|1)$ has generators $L_m = t^m (t \frac{\partial}{\partial t} + \frac{m+1}{2} \theta \frac{\partial}{\partial \theta})$, $m \geq -1$, and $G_r = t^{r+1/2} (\theta \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial t})$, $r \geq -1/2$. Setting $\deg L_m = 2m$ and $\deg G_r = 2r$ makes this into a consistently graded superalgebra of depth 2.

The Clifford algebra $\mathfrak{Cl}(m)$ has $m$ fermionic generators $D_i$ and a single bosonic generator $E$. The brackets read

$$\{D_i, D_j\} = 2g_{ij}E,$$

$$[D_i, E] = [E, E] = 0,$$

where $g_{ij}$ are symmetric structure constants. $\mathfrak{Cl}(m)$ can be realized as vector
fields acting on $\mathbb{C}^{1|m}$ as follows:

\[
D_i = \frac{\partial}{\partial \theta_i} + g_{ij} \theta^j \frac{\partial}{\partial t}, \\
E = \frac{\partial}{\partial t}.
\]

The contact superalgebra $\mathfrak{g}(1|m) \subset \text{vect}(1|m)$ is generated by vector fields $X$ which preserve the dual Pfaff equation $D_i = 0$. In other words, $X \in \mathfrak{g}(1|m)$ if

\[
[D_i, X] = f^j_i D_j,
\]

where $f^j_i$ are some functions depending on $X$. Note that a general vector field $X \in \text{vect}(1|m)$ satisfies

\[
[D_i, X] = f^j_i D_j + g_i E,
\]

so $\mathfrak{g}(1|m)$ is characterized by $g_i = 0$. Setting $\deg \theta^i = 1$, $\deg t = 2$ gives $\mathfrak{g}(1|m)$ a consistent grading of depth 2. $\mathfrak{g}(1|m)$ is known in physics as the centerless $N = m$ superconformal algebra, which is well known to have a central extension iff $m \leq 4$ [5]. In general, a Virasoro-like extension can be obtained by restriction from $\text{vect}(1|m)$; this cocycle is evidently non-central for $m \geq 5$.

Let us now turn to the definition of $\mathfrak{m}b(3|8)$ [13]. Consider the following nilpotent Lie superalgebra $\tilde{g}_-$:

\[
\{D^i_a, D^j_b\} = -6 \epsilon^{ijk} \epsilon^{ab} E_k, \\
[D^i_a, E_j] = 2 \delta^{ja}_i F_b, \\
\{D^i_a, F_b\} = [E_i, E_j] = [E_i, F_a] = \{F_a, F_b\} = 0,
\]

where $i, j, k = 1, 2, 3$ are $\mathbb{C}^3$ indices, $a, b = 1, 2$ are $\mathbb{C}^2$ indices, and $\epsilon^{ijk}$ and $\epsilon^{ab}$ are the totally anti-symmetric constant tensors. Setting $\deg D^i_a = -1$, $\deg E_i = -2$, $\deg F_a = -3$ makes $\tilde{g}_-$ into a graded superalgebra; the grading is evidently consistent. Verification of the Jacobi identities is non-trivial; one needs that the three generators $F_a$, $F_b$, and $F_c$ can never be linearly independent.

Consider $\mathbb{C}^{3|8}$ with basis spanned by three even coordinates $u^i$, six odd coordinates $\theta^a$, and two more odd coordinates $\varphi^a$, where $\deg \theta^a = 1$, $\deg u^i = 2$ and $\deg \varphi^a = 3$. $\tilde{g}_-$ can be realized as a subalgebra of $\text{vect}(3|8)$ as follows:

\[
D^i_a = \frac{\partial}{\partial \theta_i} - 3 \epsilon^{ijk} \theta^a \frac{\partial}{\partial u_k} + \epsilon^{ijk} \theta^a \theta^b \frac{\partial}{\partial \varphi^c} - \epsilon^{ab} u^i \frac{\partial}{\partial \varphi^c}, \\
E_i = \frac{\partial}{\partial u^i} - \theta^a \frac{\partial}{\partial \varphi^a}, \\
F_a = \frac{\partial}{\partial \varphi^a},
\]
where $\theta_i^a = \epsilon^{ab} \theta_{bi}$. $\tilde{g}_-$ is naturally an $su(3) + su(2) + u(1)$ module; this is where contact with the SM symmetries is made. $su(3)$ acts on $\mathbb{C}^3$, $su(2)$ acts on $\mathbb{C}^2$ and $u(1)$ computes the grading; the grading operator is

$$Z = 3Y = 3\theta^a \frac{\partial}{\partial \theta^a} + 2u^i \frac{\partial}{\partial u_i} + \theta_{ia} \frac{\partial}{\partial \theta_{ia}},$$

where $Y$ is weak hypercharge.

$mb(3|8)$ is now defined as the subsuperalgebra of $\text{vect}(3|8)$ which preserves the dual Pfaff equation $D^{ia} = 0$. In other words, $X \in mb(3|8)$ iff

$$[D^{ia}, X] = f^{ia}_{jb} D^{jb},$$

for some functions $f^{ia}_{jb}$ (depending on $X$). An equivalent definition $mb(3|8)$ is as the Cartan prolong $(\tilde{g}_-, g_0)_*$, where $\tilde{g}_-$ is isomorphic to $\tilde{g}_-$ but commutes with it. This means that we start from the realizations of $\tilde{g}_-$ and $g_0$ as vector fields on $\mathbb{C}^{3|8}$, and define $\mathfrak{g}_k$, $k > 0$, recursively as the maximal subalgebra of $\text{vect}(3|8)$ such that $[\mathfrak{g}_{-1}, \mathfrak{g}_k] \subset \mathfrak{g}_{k-1}$.

$\mathfrak{ve}(3|6)$ is defined in a similar manner, except that we set $F_a = 0$ in $\tilde{g}_-$. Since the realization requires two less fermionic coordinates, this is a subalgebra of $\text{vect}(3|6)$.

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