Recent Progress in Kähler Geometry

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Abstract

In recent years, there are many progress made in Kähler geometry. In particular, the topics related to the problems of the existence and uniqueness of extremal Kähler metrics, as well as obstructions to the existence of such metrics in general Kähler manifold. In this talk, we will report some recent developments in this direction. In particular, we will discuss the progress recently obtained in understanding the metric structure of the infinite dimensional space of Kähler potentials, and their applications to the problems mentioned above. We also will discuss some recent on Kähler Ricci flow.

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In the last few years, we have witnessed a rapid progress in Kähler geometry. In particular, the topic related to the existence, to the uniqueness of extremal Kähler metrics, and to obstructions to the existence of such metrics. In this talk, we will give a brief survey of these exciting progress made in this direction.

0.1. Some background

Let \((M, \omega)\) be a polarized \(n\)-dimensional compact Kähler manifold, where \(\omega\) is a Kähler form on \(M\). In local coordinates \(z_1, \cdots, z_n\), we have

\[
\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{ij} \, dz^i \wedge d\bar{z}^j > 0,
\]

where \(\{g_{ij}\}\) is a positive definite Hermitian matrix function. The Kähler condition requires that \(\omega\) is a closed positive \((1,1)\)-form. The Kähler metric corresponding to \(\omega\) is given by

\[
g_\omega = \sum_{\alpha, \beta=1}^{n} g_{\alpha \beta} \, dz^\alpha \otimes d\bar{z}^\beta.
\]

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For simplicity, in the following, we will often denote by $\omega$ the corresponding Kähler metric. The Kähler class of $\omega$ is its cohomology class $[\omega]$ in $H^2(M, \mathbb{R})$. It follows from the Hodge-Dolbeault theorem that any other Kähler metric in the same Kähler class is of the form

$$\omega_\varphi = \omega + \sqrt{-1} \sum_{i,j=1}^n \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} > 0$$

for some real valued function $\varphi$ on $M$.

Given a Kähler metric $\omega$, its volume form is

$$\frac{1}{n!} \omega^n = (\sqrt{-1})^n \det (g_{i\bar{j}}) \, dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n.$$

Its Ricci (curvature) form is:

$$\text{Ric}(\omega) = \sqrt{-1} R_{i\bar{j}} \, dw_i \, d\bar{w}_j = -\sqrt{-1} \partial \bar{\partial} \log \det \omega^n.$$

Note also that $R(\omega) = g^{\bar{l}j} R_{i\bar{j}}$ corresponds to one half times the scalar curvature as it is usually defines in Riemannian geometry. We say that the first Chern class of $M$ is positive of negative definite, if there exists a real valued function $\psi$ on $M$ such that $R_{i\bar{j}} + \frac{\partial^2 \psi}{\partial w_i \partial \bar{w}_j}$ is, respectively, positive of negative definite. A Kähler metric is Kähler-Einstein, if the Ricci form is proportional to the Kähler form by a constant factor. A Kähler metric is called extremal in the sense of E. Calabi [3], if it is a critical point of the functional $\int_M |\text{Ric}(\omega)|^2 \omega^n$, or, equivalently, if the complex gradient vector field of the scalar curvature function

$$g^{\alpha\bar{\beta}}(\omega) \frac{\partial R(\omega)}{\partial x^\alpha} \frac{\partial}{\partial x^\beta}$$

is a holomorphic vector field.

### 0.2. Existence of extremal Kähler metrics

It is well known that a Kähler-Einstein metric satisfies a Monge-Ampere equation

$$\log \det \frac{\omega_\varphi^n}{\omega^n} = -\lambda \varphi + h_\omega$$

where $[\text{Ric}(\omega)] = \lambda [\omega]$ and

$$\text{Ric}(\omega) - \lambda \omega = i \partial \bar{\partial} h_\omega.$$

In Calabi’s work in the 1950s, he made conjectures about the existence of Kähler-Einstein metrics on compact Kähler manifolds with definite first Chern class. In 1976, Aubin and Yau independently obtained existence when the first Chern class is negative. Around the same time, Yau proved also the existence of a Kähler-Einstein metric when the first Chern class vanishes. This is a celebrated work; and any Kähler manifold admit such a metric is called “Calabi-Yau” manifold. The positive case remains open, but significant progress has been made in the last two decades. G. Tian proved in [29] the existence of Kähler-Einstein metrics on any complex surface with positive first Chern class and reductive automorphism group.
In 1997, Tian \[30\] proved that existence of Kähler-Einstein metrics with positive scalar curvature is equivalent to an analytic stability. It remains open how this analytic stability follows from certain algebraic stability in geometric invariant theory.

The construction of complete non-compact Calabi-Yau manifolds has also enjoyed a good deal of success through the work of Calabi, Tian and Yau, Anderson, Kronheimer, LeBrun, Joyce and many others. These non-compact metrics are related to manifolds with $G_2$ and $Spin(7)$ holonomy, which are important in M-theory.

A lot of effort has also gone into constructing special or explicit examples of Kähler-Einstein metrics and extremal Kähler metrics. The same is true for hyperkähler metrics as well. Counter examples to the existence of extremal metrics have given by Levine, Burns-De Bartolomeis, and LeBrun.

There has not been much progress made on the existence of extremal metrics in general. One of the possible reasons is the lack of maximum principle for nonlinear equations of 4th order. A general existence result, even in complex surfaces, will be highly interesting.

### 0.3. Obstructions

In 1983, A. Futaki \[19\] introduced a complex character $\mathcal{F}(X, [\omega])$ on the complex Lie algebra of all holomorphic vector fields $X$ in $M$, depending only on the Kähler class $[\omega]$, and show that its vanishing is a necessary condition for the existence of a Kähler-Einstein metric on the manifold. In 1985, E. Calabi\[4\] generalized Futaki’s result to cover the more general case of any extremal Kähler metric: the generalized Futaki invariant of a given Kähler class is zero or not, according to whether any extremal metric in that class has constant scalar curvature or not. S. Bando also obtained some generalizations of the Futaki invariant. More recently, a finite family of obstructions was introduced in \[14\]. For any holomorphic vector field $X$ inducing the trivial translation on the Albanese torus there exists a complex valued potential function $\theta_{X, \omega}$, uniquely determined up to additive constants, defined by the equation: $L_X \omega = \sqrt{-1} \partial \bar{\partial} \theta_{X}$ (Here $L_X$ denote the Lie derivative along vector field $X$). Now, for each $k = 0, 1, \cdots, n$, define the functional $\mathcal{I}_k(X, \omega)$ by\[1\]

$$
\mathcal{I}_k(X, \omega) = (n - k) \int_M \theta_{X, \omega} \omega^n + \int_M \left( (k + 1) \Delta_{\omega} \operatorname{Ric}(\omega)^k \wedge \omega^{n-k} - (n - k) \theta_{X} \operatorname{Ric}(\omega)^{k+1} \wedge \omega^{n-k-1} \right).
$$

Here and elsewhere, $\Delta_{\omega}$ denotes the one half times the Laplacian-Beltrami operator of the induced Riemannian structure $\omega$.

The next theorem assures that the above integral gives rise to a holomorphic invariant.

\[1\] This is a formula for canonical Kähler class. For general Kähler class, see \[14\].
Theorem 0.1. [14] The integral $\mathcal{I}_k(X, \omega)$ is independent of choices of Kähler metrics in the Kähler class $[\omega]$, that is, $\mathcal{I}_k(X, \omega) = \mathcal{I}_k(X, \omega')$ so long as the Kähler forms $\omega$ and $\omega'$ represent the same Kähler class. Hence, the integral $\mathcal{I}_k(X, \omega)$ is a holomorphic invariant, which will be denoted by $\mathcal{I}_k(X, [\omega])$. Note that $\mathcal{I}_0$ is the usual Futaki invariant.

0.4. Uniqueness of extremal Kähler metrics

We now turn to the uniqueness of extremal metrics. In the 1950s, Calabi used the maximum principle to prove the uniqueness of Kähler-Einstein metrics when the first Chern class is non-positive. In 1987, Mabuchi introduced the “K-energy”, which is essentially a potential function for the constant scalar curvature metric equation. Using the K-energy, he and Bando [2] proved that the uniqueness of Kähler-Einstein metric up to holomorphic transformations when the first Chern class is positive. Recently, Tian and X. H. Zhu proved that the uniqueness of Kähler-Ricci Soliton on any Kähler manifolds with positive first Chern class.

Theorem 0.2. [31], [32] The Kähler Ricci soliton of a Kähler manifold $M$ is unique modulo the automorphism subgroup $\text{Aut}_r(M)$; more precisely, if $\omega_1, \omega_2$ are two Kähler Ricci solitons with respect to a holomorphic vector field $X$, i.e., they satisfy

\[ \text{Ric} (\omega_i) - \omega_i = \mathcal{L}_X (\omega_i), \]  

where $i = 1, 2$. (0.1)

Then there are automorphism $\sigma$ in $\text{Aut}^a(M)$ and $\tau$ in $\text{Aut}_r(M)$ such that $\sigma^{-1}X \in \eta_r(M)$ and $\sigma^*\omega_2 = \tau^*\sigma^*\omega_1$, where $\eta_r(M)$ denotes the Lie algebra of $\text{Aut}_r(M)$. In fact, $\sigma^{-1}X$ lies in the center of $\eta_r(M)$. Moreover, this vector field $X$ is unique up to conjugations.

Following a program of Donaldson (which will be explained in Subsection 0.7), we proved in 1998 [10] that the uniqueness for constant scalar curvature metric in any Kähler class when $C_1 < 0$ along with some other interesting results:

Theorem 0.3. [10] If the first Chern class is strictly negative, then the extremal Kähler metric is unique in each Kähler class. Moreover, the K energy must have a uniform lower bound if there exists an extremal Kähler metric in that Kähler class.

Very recently, Donaldson proved a beautiful theorem which states

Theorem 0.4. [18] For algebraic Kähler class with no non-trivial holomorphic vector field, the constant scalar curvature metric is unique.

The two theorems overlaps in a lot cases, but mutually non-inclusive.

0.5. Lower bound of the K energy

According to T. Mabuchi and S. Bando [2], the existence of a lower bound of the K energy is a necessary condition for the existence of Kähler-Einstein metrics in the first Chern class. Tian [30] showed that in a Kähler manifold with positive first Chern class and no non-trivial holomorphic fields, the Kähler-Einstein metric

...
exists if and only if the Mabuchi functional is proper. When the first Chern class is negative, making use of Tian’s explicit formulation [30], a simple idea in [9] reduces a lower bound of the K energy to the existence of critical point for the following convex functional:

\[ J(\varphi) = -\sum_{p=0}^{n-1} \frac{1}{(p+1)!(n-p-1)!} \int_V \varphi \operatorname{Ricci}(\omega_0) \wedge \omega_0^{n-p-1} (\partial \overline{\partial} \varphi)^p, \]

where \( \operatorname{Ricci}(\omega_0) < 0 \). In complex surfaces, we solve this existence problem completely, which leads to the following interesting result:

**Theorem 0.5.** [9] Suppose \( \dim V = 2 \) and \( C_1(V) < 0 \). For any Kähler class \([\omega_0]\), if \( 2 \cdot [C_1(V)] \cdot [\omega_0] + |C_1(V)| > 0 \), then the K energy has a lower bound in this Kähler class.

It will be very interesting to generalize this result to higher dimensional Kähler manifold.

### 0.6. Donaldson’s program

Mabuchi defined in [25] a Weil-Petersson type metric on the space of Kähler potentials in a fixed Kähler class. Consider the space of Kähler potentials

\[ \mathcal{H} = \{ \varphi \mid \omega_\varphi = \omega + \overline{\partial} \varphi > 0, \text{ on } M \}. \]

A tangent vector in \( \mathcal{H} \) is just a real valued function in \( M \). For any vector \( \psi \in T_\varphi \mathcal{H} \), we define the length of this vector as:

\[ \|\psi\|_\varphi^2 = \int_V \psi^2 \, d\mu_\varphi. \]

It is easy to see that the geodesic equation for this metric is

\[ \varphi''(t) - g_{\alpha\overline{\beta}} \frac{\partial \varphi'}{\partial w_\alpha} \frac{\partial \varphi'}{\partial w_\overline{\beta}} = 0, \]

where \( g_{\alpha\overline{\beta}} = g_{\alpha\overline{\beta}} + \frac{\partial^2 \varphi}{\partial w_\alpha \partial w_\overline{\beta}} > 0 \). It is first observed (cf. Semmes S. [27]) that one can complexified the \( t \) variable, denoted it by \( w_{n+1} \). Then, the geodesic equation becomes a homogenous complex Monge-Ampere equation:

\[ \det \left( g_{\alpha\overline{\beta}} + \frac{\partial^2 \varphi}{\partial w_\alpha \partial w_\overline{\beta}} \right)_{(n+1)(n+1)} = 0, \quad \text{on } \Sigma \times M. \]  

(0.2)

Here \( \Sigma = [0, 1] \times S^1 \). It turns out that we don’t need to restrict to this special case. For any Riemann surface \( \Sigma \) with boundary, and for any \( C^\infty \) map \( \varphi_0 \) from \( \partial \Sigma \) to \( \mathcal{H} \), one can always ask the following existence problem:
Question 0.6. (Donaldson) For any smooth map \( \varphi_0 : \partial \Sigma \to H \), does there exists a smooth map \( \varphi : \Sigma \to H \) which satisfies the Homogenous Monge-Ampere equation such that \( \varphi = \varphi_0 \) in \( \partial \Sigma \)?

Theorem 0.7. (X. Chen) For any smooth map \( \varphi_0 : \partial \Sigma \to H \), there always exists a \( C^{1,1} \) map \( \varphi : \Sigma \to H \) which solves the Homogenous Monge-Ampere equation such that \( \varphi = \varphi_0 \) in \( \partial \Sigma \).

An important conjecture by Donaldson in was that the space of Kähler potentials is a metric space which is path-connected with respect to this Weil-Petersson metric. This conjecture was complete verified here.

Theorem 0.8. The space \( H \) is a genuine metric space: the minimal distance between any two Kähler metrics is realized by the unique \( C^{1,1} \) geodesic; and the length of this geodesic is positive.

Collaborating with E. Calabi, we proved the following

Theorem 0.9. The space \( H \) in a fixed Kähler class is a non-positively curved space in the sense of Alexandrov: Suppose \( A, B, C \) are three smooth points in \( H \) and \( P_\lambda \) is a geodesic interpolation point for \( 0 \leq \lambda \leq 1 \): the distance from \( P_\lambda \) to \( B \) and \( C \) are respectively \( \lambda d(B,C) \) and \( (1-\lambda)d(B,C)^2 \). Then the following inequality holds:

\[
d(A, P_\lambda)^2 \leq (1-\lambda)d(A,B)^2 + \lambda d(A,C)^2 - \lambda \cdot (1-\lambda)d(B,C)^2.
\]

Theorem 0.10. Given any two Kähler potentials \( \varphi_1 \) and \( \varphi_2 \) in \( H \) and a smooth curve \( \varphi(t), 0 \leq t \leq 1 \) which connects them in \( H \). Suppose \( \varphi(s,t) \) are the family of curves under the Calabi flow and suppose that \( L(s) \) is the length of this curve at time \( s \). Then

\[
\frac{dL}{ds} = - \int_0^1 \left( \int_M |D\frac{\partial \varphi}{\partial t}|^2 \omega_{\varphi(s,t)}^{n} \cdot \left( \int_M |\frac{\partial \varphi}{\partial t}|^2 \omega_{\varphi(s,t)}^{n} \right)^{\frac{1}{2}} \right) dt,
\]

where \( D \) is the 2nd order Lichernowicz operator. For any smooth function \( f \) in \( V \), \( D(f) = \sum_{\alpha,\beta=1}^{n} f_{,\alpha\beta}dz^\alpha \otimes dz^\beta \) where \( f_{,\alpha\beta} \) is the second covariant derivatives of \( f \).

0.7. The Calabi flow and the Kähler Ricci flow

In a sequence of papers and , we develop some new techniques in attacking the convergence problems for the geometric flow, in particular, the Calabi flow and the Kähler Ricci flow. The main ideas are to find a set of new functionals which will be preserved (or decreased) under the flow with a uniform lower bound, then using the principle of concentration to attack the compactness/convergence problem. Following our work , M. Struwe gave a more concise proof on Ricci flow and Calabi flow in Riemann surface. This simple idea of using integral estimates in the flow should be able to be applied in other geometric flows.

\(^2\)In affine space, this means \( P_\lambda = \lambda B + (1-\lambda) C \).
0.7.1. The Calabi flow on Riemann surfaces

The Calabi flow is the gradient flow of the K energy and it is a 4th order parabolic equation, proposed by E. Calabi in 1982. Namely, for a given Kähler manifold \((M, [\omega])\), the Calabi flow was defined by

\[
\frac{\partial \phi(t)}{\partial t} = R(\omega_\phi) - \frac{1}{\text{vol}(M)} \int_M R(\omega) \omega^n.
\]

The local existence for this flow is known, while very little is known for its long term existence since this is a 4th order flow. The only known result is in Riemann surface where Chrusciel proved that the flow converges exponentially fast to a unique constant scalar curvature metric. In 1998 [11], we gave a new proof based on some geometrical integral estimate and concentration compactness principle. Now the challenging question is:

**Question 0.11.** Does the Calabi flow exists globally for any smooth initial metric?

0.7.2. The Kähler Ricci flow

A Kähler Ricci flow is defined by

\[
\frac{\partial }{\partial t} \omega_\phi = \omega_\phi - \text{Ric}(\omega_\phi).
\]

This flow was first studied by H. D. Cao, following the work of R. Hamilton on the Ricci flow\(^3\). Cao [6] proved that the flow always exists for all the time along with some other interesting results. It was proved by S. Bando [1] for 3-dimensional Kähler manifolds and by N. Mok [26] for higher dimensional Kähler manifolds that the positivity of bisectional curvature is preserved under the Kähler Ricci flow. The main issue here is the global convergence on manifold with positive bisectional curvature. In the work with Tian, we found a set of new functionals \(\{E_k\}_{k=0}^n\) on curvature tensors such that the Ricci flow is the gradient like flow of these functionals. On Kähler-Einstein manifold with positive scalar curvature, if the initial metric has positive bisectional curvature, we can prove that these functionals have a uniform lower bound, via the effective use of Tian’s inequality. Consequently, we are able to prove the following theorem:

**Theorem 0.12.** [12, 13] Let \(M\) be a Kähler-Einstein manifold with positive scalar curvature. If the initial metric has nonnegative bisectional curvature and positive at least at one point, then the Kähler Ricci flow will converge exponentially fast to a Kähler-Einstein metric with constant bisectional curvature.

The above theorem in complex dimension 1 was proved first by Hamilton [21]. B. Chow [13] later showed that the assumption that the initial metric has positive

\(^3\)The Ricci flow was introduced by R. Hamilton [20] in 1982. There are extensive study in this subject (cf. [22]) since his famous work in 3-dimensional manifold with positive Ricci curvature (cf. [22] for further references). Another important geometric flow is the so called “mean curvature flow.” The codimension 1 case was studied extensively by G. Huisken and many others. Recently, there are some interesting progress made in codimension 2 case (cf. [7] [24] for further references).
Corollary 0.13. The space of Kähler metrics with non-negative bisectional curvature is path-connected.

Moreover, we can carry over the proof of Theorem 0.12 to a more general case of Kähler orbifolds, for which we will not go into details here. Now the definition of these functionals $E_k = E_k^0 - J_k(k = 0, 1, \cdots, n)$:

Definition 0.14. For any $k = 0, 1, \cdots, n$, we define a functional $E_k^0$ on $\mathcal{H}$ by
\[
E_k^0, \omega (\varphi) = \frac{1}{\text{vol}(M)} \int_M \left( \log \frac{\omega_{\varphi}^n}{\omega^n} - h_{\omega} \right) \left( \sum_{i=0}^{k} \text{Ric}(\omega_{\varphi})^i \wedge \omega_{\varphi}^{k-i} \right) \wedge \omega_{\varphi}^{n-k} + c_k,
\]
where
\[
c_k = \frac{1}{\text{vol}(M)} \int_M h_{\omega} \left( \sum_{i=0}^{k} \text{Ric}(\omega)^i \wedge \omega^{k-i} \right) \wedge \omega^{n-k},
\]
and
\[
\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} h_{\omega}, \quad \text{and} \quad \int_M (e^{h_{\omega}} - 1) \omega^n = 0.
\]

Definition 0.15. For each $k = 0, 1, 2, \cdots, n-1$, we will define $J_{k, \omega}$ as follows: Let $\varphi(t) (t \in [0, 1])$ be a path from $0$ to $\varphi$ in $\mathcal{H}$, we define
\[
J_{k, \omega}(\varphi) = -\frac{n-k}{\text{vol}(M)} \int_0^1 \int_M \frac{\partial \varphi}{\partial t} \left( \omega_{\varphi}^{k+1} - \omega_{\varphi}^{k+1} \right) \wedge \omega_{\varphi}^{n-k-1} \wedge dt.
\]
Put $J_n = 0$ for convenience in notations.

Note that $E_0$ is the well known K energy function introduced by T. Mabuchi in 1987. Direct computations lead to

Theorem 0.16. For any $k = 0, 1, \cdots, n$, we have
\[
\frac{dE_k}{dt} = \frac{k+1}{\text{vol}(M)} \int_M \Delta \varphi \left( \frac{\partial \varphi}{\partial t} \right) \text{Ric}(\omega_{\varphi})^k \wedge \omega_{\varphi}^{n-k} - \frac{n-k}{\text{vol}(M)} \int_M \frac{\partial \varphi}{\partial t} \left( \text{Ric}(\omega_{\varphi})^{k+1} - \omega_{\varphi}^{k+1} \right) \wedge \omega_{\varphi}^{n-k-1}. \tag{0.3}
\]

Here $\{\varphi(t)\}$ is any path in $\mathcal{H}$.

Note that under the Kähler Ricci flow, these functionals essentially decreases! We then prove the derivative of these functionals along a curve of holomorphic automorphisms give rise to a set of holomorphic invariants $\mathcal{I}_k(k = 0, 1, \cdots, n)$ (cf. Theorem 0.1). In case of Kähler-Einstein manifolds, all these invariants vanishes. This give us freedom to re-adjust the flow so that the evolving Kähler potentials are perpendicular to the first eigenspace of a fixed Kähler-Einstein metric. Then we will be able to show that the evolved volume form has a uniform lower bound. From this point on, the boot-strapping process will give us necessary estimates to obtain global convergence.
0.8. Some new result with G. Tian

In 2001, Donaldson proved the following

**Theorem 0.17.** (Openness) For any smooth solution to the geodesic equation with a disc domain, there are always exists a smooth solution to the geodesic equation if we perturb the boundary data in a small open set (of the given boundary data).

This is somewhat surprising result since it is very hard to deform any solution of a homogenous Monge-Ampere equation even locally. However, Donaldson was able to make clever use of the Fredholm theory of holomorphic discs with totally real boundary in his proof. Then the problem of closed-ness becomes very important in light of this theorem. Tian and I are able to establish the closed-ness in this case.

**Theorem 0.18.** (Closure property) The defomation of geodesic solution in the preceding theorem is indeed closed, provided we allow solution to be smooth almost everywhere.

This is a deep theorem and we will not go into detail here due to the expository nature of this talk. However, this theorem, along with the ideas of proof, shall have implication in both geometry and other Monge-Ampere type equation in the future.

**References**

[1] S. Bando. On the three dimensional compact Kähler manifolds of nonnegative bisectional curvature. *J. D. G.*, 19:283–297, 1984.

[2] S. Bando and T. Mabuchi. Uniqueness of Einstein Kähler metrics modulo connected group actions. In *Algebraic Geometry*, Advanced Studies in Pure Math., 1987.

[3] E. Calabi. Extremal Kähler metrics. In *Seminar on Differential Geometry*, volume 16 of *Annals of Mathematics*. New York University Press, 1982.

[4] E. Calabi. Extremal Kähler metrics, II. In *Differential geometry and Complex analysis*, 96–114. Springer, 1985.

[5] E. Calabi and X. X. Chen. Space of Kähler metrics (II), 1999. to appear in J.D.G.

[6] H. D. Cao. Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.*, 81:359–372, 1985.

[7] J.Y. Chen and J.Y. Li. mean curvature flow of surfaces in 4-manifolds, 2000. Adv. in Math. (to appear).

[8] J.Y. Chen and J.Y. Li. quaternionic maps between hyperkähler manifolds, 2000. J. D. G.

[9] X. X. Chen. On lower bound of the Mabuchi energy and its application. *International Mathematics Research Notices*, 12, 2000.

[10] X. X. Chen. Space of Kähler metrics. *Journal of Differential Geometry*, 56:189–234, 2000.

[11] X. X. Chen. Calabi flow in Riemann surface revisited: a new point of views. (6):276–297, 2001. “International Mathematics Research Notices”.
[12] X. X. Chen and G. Tian. Ricci flow on Kähler-Einstein manifolds, 2000. Submitted to Annals of Mathematics.

[13] X. X. Chen and G. Tian. Space of Kähler metrics (III), 2000. preprint.

[14] X. X. Chen and G. Tian. Ricci flow on complex surfaces, 2002. *Inventiones mathematicae*.

[15] B. Chow. The Ricci flow on the 2-sphere. *J. Diff. Geom.*, 33:325–334, 1991.

[16] S.K. Donaldson. Symmetric spaces, kähler geometry and Hamiltonian dynamics. *Amer. Math. Soc. Transl. Ser. 2*, 196, 13–33, 1999. Northern California Symplectic Geometry Seminar.

[17] S.K. Donaldson. Holomorphic Discs and the complex Monge-Ampère equation, 2001. to appear in Journal of Symplectic Geometry.

[18] S.K. Donaldson. Scalar curvature and projective embeddings, I, 2001. to appear in Journal of Differential Geometry.

[19] A. Futaki. An obstruction to the existence of Einstein Kähler metrics. *Inv. Math. Fasc.*, 73(3):437–443, 1983.

[20] R. Hamilton. Three-manifolds with positive Ricci curvature. *J. Diff. Geom.*, 17:255–306, 1982.

[21] R. Hamilton. The Ricci flow on surfaces. *Contemporary Mathematics*, 71:237–261, 1988.

[22] R. Hamilton. *The formation of singularities in the Ricci flow*, volume II. Internat. Press, 1993.

[23] J.Y. Li J.Y. Chen and G. Tian. two dimensional graphs moving by mean curvature flow, 2000. preprint.

[24] Wang M. T. Mean curvature flow of surfaces in einstein four-manifolds. *J. D. Geometry*, 57(2):301–338, 2001.

[25] T. Mabuchi. Some Symplectic geometry on compact kähler manifolds I. *Osaka, J. Math.*, 24:227–252, 1987.

[26] N. Mok. The uniformization theorem for compact Kähler manifolds of non-negative holomorphic bisectional curvature. *J. Differential Geom.*, 27:179–214, 1988.

[27] S. Semmes. Complex monge-ampere and sympletic manifolds. *Amer. J. Math.*, 114:495–550, 1992.

[28] M. S. Struwe. Curvature flows on surfaces, August 2000. preprint.

[29] G. Tian. On Calabi's conjecture for complex surfaces with positive first chern class. *Invent. Math.*, 101(1):101–172, 1990.

[30] G. Tian. Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.*, 130:1–39, 1997.

[31] G. Tian and X. H. Zhu. Uniqueness of kähler-Ricci Soliton, 1998. preprint.

[32] G. Tian and X. H. Zhu. A new holomorphic invariant and uniqueness of Kähler-Ricci Soliton, 2000. preprint.