Tree-level gluon amplitudes on the celestial sphere

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Abstract:
Pasterski, Shao and Strominger have recently proposed that massless scattering amplitudes can be mapped to correlators on the celestial sphere at infinity via a Mellin transform. We apply this prescription to arbitrary $n$-point tree-level gluon amplitudes. The Mellin transforms of MHV amplitudes are given by generalized hypergeometric functions on the Grassmannian $Gr(4,n)$, while generic non-MHV amplitudes are given by more complicated Gelfand $A$-hypergeometric functions.
1 Introduction

The holographic description of bulk physics in terms of a theory living on the boundary has been concretely realised by the AdS/CFT correspondence for spacetimes with global negative curvature. It remains an important outstanding problem to understand suitable formulations of holography for flat spacetime, a goal that has elicited a considerable amount of work from several complementary approaches [1].

Recently, Pasterski, Shao and Strominger [2] studied the scattering of particles in four-dimensional Minkowski space and formulated a prescription that maps these amplitudes to the celestial sphere at infinity. The Lorentz symmetry of four-dimensional Minkowski space acts as the conformal group $SL(2, \mathbb{C})$ on the celestial sphere. It has been shown explicitly that the near-extremal three-point amplitude in massive cubic scalar field theory has the correct structure to be identified as a three-point correlation function of a conformal field theory living on the celestial sphere [2]. The factorization singularities of more general scattering amplitudes in this CFT perspective have been further studied in [3, 4]. The map uses conformal primary wave functions which have been constructed for various fields in arbitrary dimensions in [5]. In [6] it was shown that the change of basis from plane waves to the conformal primary wave functions is implemented by a Mellin transform, which was computed explicitly for three and four-point tree-level gluon amplitudes. The optical theorem in the conformal basis and scattering in three dimensions were studied in [7]. One-loop and two-loop four-point amplitudes have also been considered in [8].

In this note we use the prescription [6] to investigate the structure of CFT correlators corresponding to arbitrary $n$-point gluon tree-level scattering amplitudes, thus generalizing
their three- and four-point MHV results. Gluon amplitudes can be represented in many different ways that exhibit different, complementary aspects of their rich mathematical structure. It is natural to suspect that they may also take a particularly interesting form when written as correlators on the celestial sphere. We find that Mellin transforms of \( n \)-point MHV gluon amplitudes are given by Aomoto-Gelfand generalized hypergeometric functions on the Grassmannian \( Gr(4, n) \) \((3.19)\). For non-MHV amplitudes the analytic structure of the resulting functions is more complicated, and they are given by Gelfand \( A \)-hypergeometric functions \((4.6)\) and its generalizations. It will be very interesting to explore further the structure of these functions, and possibly make connections to other representations of tree-level amplitudes \([9]\) which we leave for future work.

## 2 Gluon amplitudes on the celestial sphere

We work with tree-level \( n \)-point scattering amplitudes of massless particles \( \mathcal{A}_{\ell_1, \ldots, \ell_n}(k^\mu_j) \) which are functions of external momenta \( k^\mu_j \) and helicities \( \ell_j = \pm 1 \), where \( j = 1, \ldots, n \). We want to map these scattering amplitudes to the celestial sphere. To that end we can parametrize the massless external momenta \( k^\mu_j \) as

\[
k^\mu_j = \epsilon_j \omega_j q^\mu_j \equiv \epsilon_j \omega_j \left(1 + |z_j|^2, z_j + \bar{z}_j, -i(z_j - \bar{z}_j), 1 - |z_j|^2\right),
\]

where \( z_j, \bar{z}_j \) are the usual complex coordinates on the celestial sphere, \( \epsilon_j \) encodes a particle as incoming \((\epsilon_j = -1)\) or outgoing \((\epsilon_j = +1)\), and \( \omega_j \) is the angular frequency associated with the energy of the particle \([6]\). Therefore, the amplitude \( \mathcal{A}_{\ell_1, \ldots, \ell_n}(\omega_j, z_j, \bar{z}_j) \) is a function of \( \omega_j, z_j \) and \( \bar{z}_j \) under the parametrization \((2.1)\).

Usually, we write any massless scattering amplitude in terms of spinor-helicity angle-and square-brackets representing Weyl-spinors (see \([10]\) for a review). The spinor-helicity variables are related to external momenta \( k^\mu_j \), so that in turn we can express them in terms of variables on the celestial sphere via \([6]\):

\[
[ij] = 2\sqrt{\omega_j \omega_j} z_{ij}, \quad \langle ij \rangle = -2\epsilon_i \epsilon_j \sqrt{\omega_j \omega_j} \bar{z}_{ij},
\]

where \( z_{ij} = z_i - z_j \) and \( \bar{z}_{ij} = \bar{z}_i - \bar{z}_j \).

In \([5, 6]\) it was proposed that any massless scattering amplitude is mapped to the celestial sphere via a Mellin transform:

\[
\tilde{\mathcal{A}}_{J_1, \ldots, J_n}(\lambda_j, z_j, \bar{z}_j) = \prod_{j=1}^n \int_0^\infty d\omega_j \omega_j^{i\lambda_j} \mathcal{A}_{\ell_1, \ldots, \ell_n}(\omega_j, z_j, \bar{z}_j).
\]

The Mellin transform maps a plane wave solution for a helicity \( \ell_j \) field in momentum space to a corresponding conformal primary wave function on the boundary with spin \( J_j \), where helicity \( \ell_j \) and spin \( J_j \) are mapped onto each other, and the operator dimension takes values in the principal continuous series representation \( \Delta_j = 1 + i\lambda_j \) \([5]\). Therefore, \( \tilde{\mathcal{A}}_{J_1, \ldots, J_n}(\lambda_j, z_j, \bar{z}_j) \) has the structure of a conformal correlator on the celestial sphere, where the symmetry group of diffeomorphisms is the conformal group \( SL(2, \mathbb{C}) \).
Explicitly, under conformal transformations, we have the following behavior:

\[ \omega_j \to \omega'_j = |cz_j + d|^2 \omega_j, \quad z_j \to z'_j = \frac{a z_j + b}{c z_j + d}, \quad \bar{z}_j \to \bar{z}'_j = \frac{\bar{a} \bar{z}_j + \bar{b}}{\bar{c} \bar{z}_j + \bar{d}}, \quad (2.4) \]

where \( a, b, c, d \in \mathbb{C} \) and \( ad - bc = 1 \). The transformation for \( z_j, \bar{z}_j \) is familiar from the usual action of \( SL(2, \mathbb{C}) \) on the complex coordinates on a sphere. Concerning \( \omega_j \), recall that \( q_j^\mu \) transforms as \( q_j^\mu \to |cz_j + d|^2 \Lambda^\mu \nu q_j^\nu [5] \), where \( \Lambda^\mu \nu \) is a Lorentz transformation in Minkowski space corresponding to the celestial sphere conformal transformation. Thus, \( \omega_j \) must transform as in \( (2.4) \) to ensure that \( k_j^\mu \) transforms as a Lorentz vector: \( k_j^\mu \to \Lambda^\mu_\nu k_j^\nu \).

The conformal covariance of \( \tilde{A}_{J_1...J_n}(\lambda_j, z_j, \bar{z}_j) \) on the celestial sphere demands:

\[ \tilde{A}_{J_1...J_n} \left( \lambda_j, \frac{a z_j + b}{c z_j + d}, \frac{\bar{a} \bar{z}_j + \bar{b}}{\bar{c} \bar{z}_j + \bar{d}} \right) = \prod_{j=1}^n \left[ (cz_j + d)^{\Delta_j} + (\bar{c} \bar{z}_j + \bar{d})^{\Delta_j} \right] \tilde{A}_{J_1...J_n}(\lambda_j, z_j, \bar{z}_j), \quad (2.5) \]

as expected for a correlator of operators with weights \( \text{Delta}_j \) and spins \( J_j \).

## 3 \ n\text{-point} \ MHV

The cases of 3- and 4-point gluon amplitudes have been considered in [6]. Here we will map \( n \geq 5 \)-point MHV gluon amplitudes to the celestial sphere.

### 3.1 Integrating out one \( \omega_i \)

Starting from \( (2.3) \), we can anchor the integration to one of our variables \( \omega_i \) by making a change of variables for all \( l \neq i \)

\[ \omega_l \to \frac{\omega_l}{s_i} \omega_l, \quad (3.1) \]

where \( s_i \) is a constant factor that cancels the conformal scaling of \( \omega_i \) in \( (2.4) \), so that the ratio \( \frac{\omega_i}{s_i} \) is conformally invariant. One choice which is always possible in Minkowski signature is

\[ s_i = \frac{|z_{i-1}+1|}{|z_{i-1}| \cdot |z_{i+1}|}. \quad (3.2) \]

Since gluon scattering amplitudes scale homogeneously under uniform rescalings, collecting all the factors in front, we have

\[ \tilde{A}_{J_1...J_n}(\lambda_j, z_j, \bar{z}_j) = \int_0^\infty d\omega_i \left( \frac{\omega_i}{s_i} \right)^{\sum_{j=1}^n i \lambda_j} \left( \prod_{a=1}^n \int_0^\infty d\omega_a \omega_a^{i \lambda_a} \right) \tilde{A}_{\ell_1...\ell_n}(s_i, \omega_i, z_j, \bar{z}_j), \quad (3.3) \]

where we used that the scaling power of dressed gluon amplitudes is \( A_n(\Lambda \omega_i) \to \Lambda^{-n} A_n(\omega_i) \).

We recognize that the integral over \( \omega_i \) is the Mellin transform of 1, which is given by

\[ \int_0^\infty \frac{d\omega_i}{\omega_i} \left( \frac{\omega_i}{s_i} \right)^iz = 2\pi \delta(z). \quad (3.4) \]
With this we simplify the transformation prescription (2.3) to
\[
\tilde{A}_{\lambda_1...\lambda_n}(\lambda_j, z_j, \bar{z}_j) = 2\pi \delta \left( \sum_{j=1}^{n} \lambda_j \right) s_i^{1+i\lambda_i} \left( \prod_{a=i}^{n} \int_0^{\infty} d\omega_a \omega_a^{i\lambda_a} \right) A_{\lambda_1...\lambda_n}(s_i, \omega_i, z_j\bar{z}_j). \tag{3.5}
\]

### 3.2 Integrating out momentum conservation $\delta$-functions

For simplicity, we choose the anchor variable above to be $\omega_1$ and use $\omega_{n-3}, \ldots, \omega_n$ to localize the momentum conservation $\delta$-functions in the amplitude. These $\delta$-functions can then be equivalently rewritten as follows, compensating the transformation by a Jacobian:
\[
\delta^4(\epsilon_{1}s_1q_1 + \sum_{i=2}^{n} \epsilon_{i}\omega_iq_i) = \frac{4}{U} \prod_{j=n-3}^{n} s_j \delta(\omega_j - \omega^*_j) 1_{\omega_0}(\omega^*_j), \tag{3.6}
\]
where $\omega^*_j$ are solutions to the initial set of linear equations:
\[
\omega^*_j = -s_j \left( \frac{U_{i,j}}{U} + \sum_{i=2}^{n-4} \omega_i \frac{U_{i,j}}{s_i} \right). \tag{3.7}
\]
The $U_{i,j}$ and $U$ are minor determinants by Cramer’s rule:
\[
U_{i,j} = \det(M^{(n-3,...,i,...,n)}), \quad U = \det(M^{(n-3,...,n)}), \tag{3.8}
\]
where $j \to i$ means that index $j$ is replaced by index $i$. $M^{(a,b,c,d)}$ denotes the $4 \times 4$ matrix
\[
M^{(a,b,c,d)} = (p_a p_b p_c p_d). \tag{3.9}
\]
For the purpose of determinant conformally invariant calculation, the column vectors $p_i^\mu = \epsilon_i s_i q_i^\mu$ can be written in a manifestly conformally invariant form:
\[
p_i^\mu(z, \bar{z}) = \epsilon_1(1, 0, 0, -1), \quad p_2^\mu(z, \bar{z}) = \epsilon_2(1, 0, 0, 1), \quad p_3^\mu(z, \bar{z}) = \epsilon_3(2, 2, 0, 0),
\]
\[
p_i^\mu(z, \bar{z}) = \frac{1}{|u_i|^2} (1 + |u_i|^2, u_i + \bar{u}_i, -i(u_i - \bar{u}_i), 1 - |u_i|^2) \quad \text{for} \quad i = 4, 5, \ldots, n, \tag{3.10}
\]
in terms of conformal invariant cross-ratios
\[
u_i = \frac{z_{31}z_{22}}{z_{32}z_{11}} \quad \text{and} \quad \bar{u}_i = \frac{\bar{z}_{31}\bar{z}_{22}}{\bar{z}_{32}\bar{z}_{11}} \quad \text{for} \quad i = 4, 5, \ldots, n, \tag{3.11}
\]
but if, and only if, we also specify the explicit choice
\[
s_1 = \frac{|z_{3,2}|}{|z_{3,1}| |z_{1,2}|}, \quad s_2 = \frac{|z_{3,1}|}{|z_{3,2}| |z_{2,1}|}, \quad \text{and} \quad s_i = \frac{|z_{1,2}|}{|z_{1,1}| |z_{2,2}|} \quad \text{for} \quad i = 3, \ldots, n. \tag{3.12}
\]
The indicator functions $\prod_{i=n-3}^{n} 1_{\omega_0}(\omega^*_i)$ appear due to the integration range in all $\omega$ being along the positive real line, such that the $\delta$-functions can only be localized in this region.

Furthermore, in order for all the remaining integration variables $\omega_j$ with $j = 2, \ldots, n-4$ to be defined on the whole integration range, the indicator functions $\prod_{i=n-3}^{n} 1_{\omega_0}(\omega^*_i)$ have to demand $\frac{U_{i,j}}{U} < 0$ for all $i = 1, \ldots, n-4$ and $j = n-3, \ldots, n$, so that we can write them as $\prod_{i,j} 1_{\omega_0}(\frac{U_{i,j}}{U})$. 

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3.3 Integrating the remaining $\omega_i$

In this section we apply (3.5) to the usual $n$-point MHV Parke-Taylor amplitude [11] in spinor-helicity formalism for $n \geq 5$ rewritten via (2.2):

$$\mathcal{A}_{-+...+}(s_1, \omega_j, z_j, \bar{z}_j) = \frac{z_1^2 s_1^2\omega_2 \delta^4(\epsilon_1 s_1 q_1 + \sum_{i=2}^{n} \epsilon_i \omega_i q_i)}{(-2)^{n-4}23z_{34}...z_{n1}\omega_3\omega_4...\omega_n}. \tag{3.13}$$

Making use of the solutions (3.6) and performing four of the integrations in (3.5), we have:

$$\tilde{\mathcal{A}}_{-+...+}(\lambda_i, z_i, \bar{z}_i) = 2\pi \frac{\delta(\sum_{j=1}^{n} \lambda_j) z_{12} s_1 \lambda_1 + 2 s_2 \lambda_2 + \sum_{j=3}^{n} s_j \lambda_j}{23z_{34}...z_{n1}U_{i1,n}} \prod_{a=2}^{n-4} \int_{0}^{\infty} d\omega_a \frac{\omega_a}{\omega_3\omega_4...\omega_n} \prod_{i,j} 1_{c0}(\frac{U_{i,j}}{U}). \tag{3.14}$$

For convenience, we transform the remaining integration variables as:

$$\omega_i = \frac{U_{i1,n}}{U_{i,n}} \frac{u_{i-1}}{1 - \sum_{j=1}^{n} u_j}, \quad i = 2, 3, ..., n - 4, \tag{3.15}$$

which leads to

$$\tilde{\mathcal{A}}_{-+...+}(\lambda_i, z_i, \bar{z}_i) \sim \frac{z_1^2 s_1 \lambda_1 + 2 s_2 \lambda_2 + \sum_{j=3}^{n} s_j \lambda_j}{23z_{34}...z_{n1}U_{i1,n}} \delta(\sum_{j=1}^{n} \lambda_j) \varphi(\{\alpha\}, x) \prod_{i,j} 1_{c0}(\frac{U_{i,j}}{U}). \tag{3.16}$$

Note that the overall factor in (3.16) accounts for proper transformation weight of the resulting correlator under conformal transformations (2.5).

Here we recognize a hypergeometric function $\varphi(\{\alpha\}, x)$ of type $(n-4, n)$, as defined in section 3.8.1 of [12] and described in appendix A. In particular, here we have:

$$\varphi(\{\alpha\}, x) = \int_{\sum_{a=0}^{n=m-5} U_{a1,n}}^{n} \prod_{j=1}^{n} P_j(u)^{\alpha_j} d\varphi, \quad d\varphi = \frac{dP_2}{P_2} \wedge ... \wedge \frac{dP_{n-4}}{P_{n-4}}, \tag{3.17}$$

$$P_j(u) = x_{0j} + x_{1j} u_1 + ... + x_{n-5j} u_{n-5}, \quad 1 \leq j \leq n. \tag{3.18}$$

The parameters in (3.17) corresponding to (3.16) read:

$$\alpha_1 = 1, \quad \alpha_2 = 2 + i\lambda_2, \quad \alpha_3 = i\lambda_3, \quad ..., \quad \alpha_{n-4} = i\lambda_{n-4}, \quad \alpha_{n-3} = i\lambda_{n-3} - 1, \quad ..., \quad \alpha_{n-1} = i\lambda_{n-1} - 1, \quad \alpha_n = 1 + i\lambda_1, \quad x_{0i} = \frac{U_{i1,n}}{U_{1,n}}, \quad x_{j-1} = \frac{U_{j1,n}}{U_{1,n}} - \frac{U_{i1,n}}{U_{1,n}}, \quad x_{0n} = -\frac{U}{U_{1,n}}, \quad x_{j-1-n} = \frac{U_{j1,n}}{U_{1,n}}, \quad x_{01} = 1, \quad x_{j-1-j} = -\frac{U}{U_{j,n}}, \tag{3.18}$$

for $i = n-3, n-2, n-1$ and $j = 2, 3, ..., n-4$, and all other $x_{ab} = 0$.

These kinds of functions are also known as Aomoto-Gelfand hypergeometric functions on the Grassmannian $Gr(n-4, n)$.

Making use of eq. (3.24) and (3.25) from [12], we can write down a dual representation of the same function, which yields a hypergeometric function of type $(4, n)$:

$$\varphi(\{\alpha\}, x) = \frac{c_{2}}{c_1} \int_{\sum_{a=0}^{n=m-5} U_{a1,n}}^{n} \prod_{j=1}^{n} P_j(u)^{\alpha_j} d\varphi, \quad d\varphi = \frac{dP_{n-3}}{P_{n-3}} \wedge ... \wedge \frac{dP_{n-1}}{P_{n-1}}, \tag{3.19}$$

$$P_j(u) = x_{0j} + x_{1j} u_1 + x_{2j} u_2 + x_{3j} u_3, \quad 1 \leq j \leq n. \tag{3.19}$$

\footnote{For $n = 5$, the normally different cases $\alpha_2 = 2 + i\lambda_2$ and $\alpha_{n-3} = i\lambda_{n-3} - 1$ are reduced to a single $\alpha_2 = 1 + i\lambda_2$. In this case there also are no integrations so that the result becomes a simple product of factors.}
In this case, the parameters of (3.19) corresponding to (3.16) read:

\[\begin{align*}
\alpha_1 &= 1, \quad \alpha_2 = -2 - i \lambda_2, \quad \alpha_3 = -i \lambda_3, \ldots, \quad \alpha_{n-4} = -i \lambda_{n-4}, \quad \alpha_{n-3} = 1 - i \lambda_{n-3}, \ldots, \quad \alpha_{n-1} = 1 - i \lambda_{n-1}, \\
\alpha_n &= -i \lambda_n, \quad x_{0j} = \frac{U_{j,n}}{U_{1,n}}, \quad x_{ij} = \frac{U_{j,n-4+i}}{U_{1,n}}, \quad x_{0n} = -\frac{U}{U_{1,n}}, \quad x_{in} = \frac{U}{U_{1,n}}, \quad x_{01} = 1, \\
x_{1n-3} &= -\frac{U}{U_{1,n-3}}, \quad x_{2n-2} = -\frac{U}{U_{1,n-2}}, \quad x_{3n-1} = -\frac{U}{U_{1,n-1}}, \quad c_2 = \frac{\Gamma(2 + i \lambda_1) \Gamma(2 + i \lambda_2) \prod_{j=3}^{n-4} \Gamma(i \lambda_j)}{\Gamma(1 - i \lambda_1) \prod_{i=1}^{n-3} \Gamma(1 - i \lambda_{n-i})}.
\end{align*}\]

for \(i = 1, 2, 3\) and \(j = 2, 3, \ldots, n - 4\), and all other \(x_{ab} = 0\).

The hypergeometric functions \(\hat{\varphi}(\{\alpha\}, x)\) form a basis of solutions to a Pfaffian form equation which defines a Gauss-Manin connection as described in section 3.8 of [12]. This Pfaffian form equation can be interpreted as a generalized Knizhnik-Zamolodchikov equation satisfied by our correlators [13, 14]. Similar generalized hypergeometric functions appeared in [15] in the context of \(\mathcal{N} = 4\) Yang-Mills scattering amplitudes and the deformed Grassmannian.

### 3.4 6-point MHV

In the special case of six gluons there is only one integral in (3.17), such that the function reduces to the simpler case of Lauricella function \(\hat{\varphi}_D\):

\[
\hat{\varphi}_D(\{\alpha\}, x) = \left(\frac{-U}{U_{2,6}}\right)^{\lambda_1+1} \left(\frac{-U}{U_{1,6}}\right)^{\lambda_2+2} \left(\frac{U_{2,3}}{U_{2,6}}\right)^{\lambda_3-1} \left(\frac{U_{2,4}}{U_{2,6}}\right)^{\lambda_4-1} \left(\frac{U_{2,5}}{U_{2,6}}\right)^{\lambda_5-1} \times \int_0^1 dt t^{\gamma-1} (1-t)^{\alpha-1} \prod_{i=1}^3 (1-x_i t)^{-\beta_i},
\]

with parameters and arguments given by

\[
\alpha = 2 + i \lambda_2, \quad \gamma = 4 + i \lambda_1 + i \lambda_2, \quad \beta_i = 1 - i \lambda_{i+2}, \quad x_i = 1 - \frac{U_{1,i+2} U_{2,6}}{U_{1,6} U_{2,i+2}} \quad \mathrm{for} \quad i = 1, 2, 3. \quad (3.22)
\]

Note that \(x_{0j}\) arguments have been factored out of the integrand to achieve this form.

### 4 \(n\)-point NMHV

In this section we will map the \(n\)-point NMHV split helicity amplitude \(A_{++++\ldots+}\) to the celestial sphere via (3.5). The spinor-helicity expression for \(A_{++++\ldots+}\) can be found e.g. in [16]

\[
A_{++++\ldots+} = \frac{1}{F_{3,1}} \sum_{j=4}^{n-1} \frac{\langle P_{2,j} P_{j+1,2} \rangle^3}{P_{2,j}^2 P_{j+1,2}^2} \left[ \frac{j + 1}{2} \right] \langle j+1 \rangle \langle j \rangle \langle j \rangle \langle j+1 \rangle \langle j+1 \rangle \langle j+1 \rangle = \sum_{j=4}^{n-1} \{ M_j \} \quad (4.1)
\]

where \(F_{i,j} \equiv \langle i \ i+1 \rangle \langle i+1 \ i+2 \rangle \cdots \langle j-1 \ j \rangle\) and \(P_{x,y} \equiv \sum_{k=x}^{y} \langle k \rangle \langle k \rangle\) where \(x < y\) cyclically.
We will work with \{M_4\} for the purpose of our calculations. Using momentum conservation and writing \{M_4\} in terms of spinor-helicity variables, we find

\[
\{M_4\} = \frac{1}{((12)[24](43) + (13)[34](43))^3} \times \left(\frac{((23)[23] + (24)[24] + (34)[34])[34]}{(23)(35) + [24](45))(43)[32]}\right). 
\]

Writing this in terms of celestial sphere variables via (2.2), we find

\[
\{M_4\} = \frac{\omega_1 \omega_2 (\epsilon_2 \epsilon_1 \epsilon_5 \epsilon_6 \omega_2 + \epsilon_3 \epsilon_4 \epsilon_{35} \omega_3 \omega_4)}{2^{n-1} z_{01}^{\alpha_1} z_{12}^{\alpha_2} \epsilon_{a_1} z_{34}^{\alpha_3} \epsilon_{a_2} z_{56}^{\alpha_4} \epsilon_{a_3} \epsilon_{a_4} z_{35}^{\alpha_5} \epsilon_{a_5} z_{46}^{\alpha_6} z_{12}^{\alpha_7}} 
\]

The following map of the above formula to the celestial sphere will only be strictly valid for \(n \geq 8\). We will comment on changes at 6- and 7-points in the next section. We use the map (3.5), anchor the calculation about \(\omega_1\), make use of solutions (3.6) and perform a change of variables

\[
\omega_i = s_i \frac{u_{i-1}}{1 - \sum_{j=1}^{n-5} u_j}, \quad i = 2, \ldots, n - 4, 
\]

to find the resulting term in the \(n\)-point NMHV correlator

\[
\{\hat{M}_4\} \sim \delta \left(\sum_{j=1}^{n} \lambda_j\right) \frac{\prod_{j=1}^{n} s_i^{\lambda_i}}{z_{12}^{\epsilon_{a_1} z_{34}^{\epsilon_{a_2}}} \epsilon_{a_3} z_{35}^{\epsilon_{a_4} z_{46}^{\epsilon_{a_5} \epsilon_{a_6}} z_{12}^{\epsilon_{a_7}}} \hat{\mathcal{F}}(\alpha, x) \prod_{i,j} 1_{c_0} \left(\frac{U_{i,j} - U}{U}\right), 
\]

with the function \(\hat{\mathcal{F}}(\alpha, x)\) being a Gelfand A-hypergeometric function as defined in Appendix A. In this case it explicitly reads:

\[
\hat{\mathcal{F}}(\alpha, x) = \int_{u_{1,0}^{a_1}, \ldots, u_{n,0}^{a_n}} \prod_{a=1}^{n-5} u_{a}^{n-5} u_{j}^{\lambda_j} u_{3}^{n} (u_1 u_2 x_{10} + u_3 u_2 x_{20} + u_2 u_3 x_{30})^{-1} \times \prod_{i=1}^{7}(x_{i0} + u_{1} x_{i1} + \cdots + u_{n-5} x_{n-5,i})^{\alpha_i}, 
\]

where parameters are given by

\[
\alpha_1 = 3, \quad \alpha_2 = -1, \quad \alpha_3 = i \lambda_1 + 1, \quad \alpha_4 = i \lambda_{n-3} - 1, \quad \alpha_5 = i \lambda_{n-2} - 1, \quad \alpha_6 = i \lambda_{n-1} - 1, \quad \alpha_7 = i \lambda_n - 1, 
\]

and function arguments are given by

\[
x_{10} = \epsilon_2 \epsilon_3 |z_{23}|^2 s_2 s_3, \quad x_{20} = \epsilon_2 \epsilon_4 |z_{24}|^2 s_2 s_4, \quad x_{30} = \epsilon_3 \epsilon_4 |z_{34}|^2 s_3 s_4, \\
x_{11} = \epsilon_2 \epsilon_1 z_{24} s_2, \quad x_{21} = \epsilon_3 \epsilon_1 z_{34} s_3, \quad x_{22} = \epsilon_3 \epsilon_4 z_{35} s_3, \quad x_{32} = \epsilon_4 |z_{24}| z_{24} s_4, \\
x_{03} = 1, \quad x_{33} = -1, \quad x_{3j} = 1, \ldots, n - 5, \quad x_{04} = \frac{U_{1,n-3}}{U}, \quad x_{40} = \frac{U_{j,n-3} - U_{1,n-3}}{U}, \quad j = 1, \ldots, n - 5, \\
x_{05} = \frac{U_{1,n-2}}{U}, \quad x_{50} = \frac{U_{j,n-2} - U_{1,n-2}}{U}, \quad j = 1, \ldots, n - 5, \quad x_{06} = \frac{U_{1,n-1}}{U}, \quad x_{60} = \frac{U_{j,n-1} - U_{1,n-1}}{U}, \quad j = 1, \ldots, n - 5, \\
x_{07} = \frac{U_{1,n}}{U}, \quad x_{70} = \frac{U_{j,n} - U_{1,n}}{U}, \quad j = 1, \ldots, n - 5.
\]
Note that the first fraction in (4.5) accounts for the correct transformation weight of the correlator under conformal transformation (2.5).

6- and 7-point NMHV

In the cases of 6- and 7-point the results in the previous section change somewhat, due to the presence of $\omega_3$ and $\omega_4$ in the denominator of (4.3). These variables are fixed by momentum conservation $\delta$-functions in the lower point cases, such that the parameters and function arguments of the resulting Gelfand $A$-hypergeometric functions change.

For the 6-point case, we find that the resulting correlator part $\{\tilde{M}_4\}$ is proportional to a Gelfand $A$-hypergeometric function as defined in Appendix A:

$$\tilde{\mathcal{F}}(\{\alpha\}, x) = \int_{u_{12} \geq 0} \int_{1 - u_{12} \geq 0} \frac{du_1}{u_1} \frac{du_2}{u_2} \frac{u_1^{i\lambda_2} (x_{00} + u_1 x_{10} + u_1^2 x_{20})^{-1} (1 - u_1)^{i\lambda_1+1} \prod_{i=2}^{7} (x_{0i} + u_1 x_{1i})^{\alpha_i}}{(1 - u_1)^{i\lambda_1+1} \prod_{i=2}^{7} (x_{0i} + u_1 x_{1i})^{\alpha_i}} (4.9)$$

where parameters are given by

$$\alpha_2 = i\lambda_3 - 1, \quad \alpha_3 = i\lambda_4 + 1, \quad \alpha_4 = i\lambda_5 - 1, \quad \alpha_5 = i\lambda_6 - 1, \quad \alpha_6 = 3, \quad \alpha_7 = -1, \quad (4.10)$$

and function arguments $x_{ij}$ depend on $\epsilon_i, z_i, \bar{z}_i$ and $U_{ij}$. Performing a partial fraction decomposition on the quadratic denominator in (4.9), we can reduce the result to a sum of two Lauricella functions.

In the 7-point case, we find that the resulting correlator part $\{\tilde{M}_4\}$ is proportional to a Gelfand $A$-hypergeometric function as defined in Appendix A:

$$\tilde{\mathcal{F}}(\{\alpha\}, x) = \int_{u_{1236} \geq 0} \int_{1 - u_{1236} \geq 0} \frac{du_1}{u_1} \frac{du_2}{u_2} \frac{u_1^{i\lambda_2} u_2^{i\lambda_3} (u_1 x_{10} + u_2 x_{20} + u_1 u_2 x_{30} + u_1^2 x_{40} + u_2^2 x_{50})^{-1}}{(u_1 x_{10} + u_2 x_{20} + u_1 u_2 x_{30} + u_1^2 x_{40} + u_2^2 x_{50})^{-1}} \times \prod_{i=1}^{7} (x_{0i} + u_1 x_{1i} + u_2 x_{2i})^{\alpha_i}, \quad (4.11)$$

where parameters are given by

$$\alpha_1 = i\lambda_1 + 1, \quad \alpha_2 = i\lambda_4 + 1, \quad \alpha_3 = i\lambda_5 - 1, \quad \alpha_4 = i\lambda_6 - 1, \quad \alpha_5 = i\lambda_7 - 1, \quad \alpha_6 = 3, \quad \alpha_7 = -1, \quad (4.12)$$

and function arguments $x_{ij}$ again depend on $\epsilon_i, z_i, \bar{z}_i$ and $U_{ij}$.

5 n-point NkMHV

In this section we discuss the schematic structure of NkMHV amplitudes with higher $k$ under the Mellin transform (3.5).

N2MHV amplitude

In the 8-point N2MHV split helicity case, $\mathcal{A}_{+++++++}$, we consider one of the six terms of the amplitude found in e.g. [16] on page 6 as an example:

$$\frac{1}{F_{4,1} F_{2,3}} \frac{\{1\}_{P_{2,6}P_{7,2}P_{3,5}P_{6,3}}}{F_{2,6}^2 F_{7,2} P_{3,5}^2 P_{6,3}^2} \frac{\{76\}_{P_{2,6}P_{7,2}P_{3,5}P_{6,3}}}{\{23\}_{P_{2,6}P_{7,2}P_{3,5}P_{6,3}}^{3} \{65\}_{P_{2,6}P_{7,2}P_{3,5}P_{6,3}}^{3}} \quad (5.1)$$
where \( \hat{F}_{ij} \) is the complex conjugate of \( F_{ij} \). Performing the same sequence of steps as in the previous sections, we find a resulting Gelfand \( A \)-hypergeometric function of the form

\[
\hat{F}(\{\alpha\}, x) = \int_{u_{120}, u_{220}, u_{320}} \frac{du_1}{u_1} \frac{du_2}{u_2} \frac{du_3}{u_3} u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3} \mathcal{P}_3^{(4)} \prod_{i=1}^{13} (x_{0i} + u_1 x_{1i} + u_2 x_{2i} + u_3 x_{3i})^{\alpha_i} \tag{5.2}
\]

\[
\times \prod_{j=14}^{17} (x_{0j} + u_1 x_{1j} + u_2 x_{2j} + u_3 x_{3j} + u_1 u_3 x_{5j} + u_2 u_3 x_{6j} + u_2^2 x_{7j} + u_2^2 x_{8j} + u_3 x_{9j})^{\alpha_j},
\]

for some parameters \( \alpha_i \), where \( \mathcal{P}_3^{(4)} \) is a degree four polynomial in \( u_i \), and function arguments \( x_{ij} \) again depend on \( \epsilon_i, z_i, \bar{z}_i \) and \( U_{ij} \).

**\( N^k \)MHV amplitude**

More generally a split helicity \( N^k \)MHV amplitude \( A_{\ldots,\ldots,\ldots} \) involves a sum over the terms described in eq. (3.1), (3.2) of [16]. Terms corresponding in complexity to \( \{ \tilde{M}_4 \} \) discussed in the previous section are always present, with constant Laurent polynomial powers at any \( k \). However, for higher \( k \), the most complicated contributing summands result in hypergeometric integrals schematically given by

\[
\hat{F}(\{\alpha\}, x) = \int_{u_{1n}, \ldots, u_{n-4}, u_{n-1}, u_{n-2}, \ldots, u_{n-4}, u_{n-1}} \prod_{i=2}^{n-1} \frac{du_i}{u_i} u_i^{\alpha_i} \left( 1 - \sum_{j=2}^{n-1} u_j \right)^{\alpha_1} \mathcal{P}_3^{(d)} \left( \prod_{i=1}^{d} (\mathcal{P}_{\{1\}})^{\alpha_i} \right) \left( \prod_{j=2}^{d} (\mathcal{P}_{\{2\}})^{\alpha_j} \right) \tag{5.3}
\]

where \( \alpha_i \) are parameters and \( \mathcal{P}_{\{d\}} \) is a degree \( d \) polynomial in \( u_a \). Here we explicitly see an increase in power of the Laurent polynomials with increasing \( k \) in \( N^k \)MHV. The examples above feature the Gelfand \( A \)-hypergeometric function \( \hat{F} \). The increase in Laurent polynomial degree is traced back to the presence of Mandelstam invariants \( \mathcal{P}_{ij}^2 \) for degree two polynomials, as well as the factors \( \langle a|P_{ij} P_{kl} \ldots P_{rs}|b \rangle \) for higher degree polynomials. The length of chains of the \( P_{ij} \) depends on \( n \) and \( k \), such that multivariate Laurent polynomials of any positive degree are present at sufficiently high \( n, k \).

Similar generalized hypergeometric functions, or, equivalently, generalized Euler integrals are found in the case of string scattering amplitudes [17, 18]. It will be interesting to explore this connection further.

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A Generalized hypergeometric functions

The Aomoto-Gelfand hypergeometric functions of type \((n + 1, m + 1)\) relevant in this work can be defined as in section 3.5.1 of [12]:

\[
\hat{\varphi}(\{\alpha\}, x) \equiv \int_{\sum u_a \geq 0} \frac{m}{\prod_{j=0}^m P_j(u)^{\alpha_j}} d\varphi, \tag{A.1}
\]

\[
d\varphi = \frac{dP_{j_1}}{P_{j_1}} \wedge \ldots \wedge \frac{dP_{j_n}}{P_{j_n}}, \quad 0 \leq j_1 < \ldots < j_n \leq m, \tag{A.2}
\]

\[
P_j(u) = x_{0j} + x_{1j} u_1 + \ldots + x_{nj} u_n, \quad 1 \leq j \leq m, \tag{A.3}
\]

where here the parameters \(\alpha_i\) collectively describe all the powers for the factors in the integrand. When all \(\alpha_i\) are zero, the function reduces to the Aomoto polylogarithm.

The arguments \(x_{ij}\) of the hypergeometric function of type \((m + 1, n + 1)\) in (A.3) can be arranged in a matrix:

\[
\tilde{X} = \begin{pmatrix}
x_{00} & \cdots & x_{0m} \\
x_{10} & \cdots & x_{1m} \\
\vdots & \ddots & \vdots \\
x_{n0} & \cdots & x_{nm}
\end{pmatrix}. \tag{A.4}
\]

Each column in this matrix defines a hyperplane in \(\mathbb{C}^n\) that appears in the hypergeometric integral as \((x_{0j} + \sum_{i=1}^u x_{ij} u_i)^{\alpha_i}\). Furthermore, \((n + 1) \times (n + 1)\) minor determinants of the matrix can be regarded as Plücker coordinates on the Grassmannian \(Gr(n + 1, m + 1)\) over the space of arguments \(x_{ij}\).

Sometimes it is convenient to transform the argument arrangement (A.4) to the following gauge fixed form

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\
0 & 1 & \ldots & 0 & -x_{11} & \ldots & -x_{1m-n-1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 1 & -x_{n1} & \ldots & -x_{nm-n-1}
\end{pmatrix}. \tag{A.5}
\]

In this case the hypergeometric function can then be written in the following two equivalent ways, eq. (3.24) of [12]:

\[
F((\alpha_i), (\beta_j), \gamma; x) = c_1 \int_{\sum_{u_a \geq 0}} u_1^{\alpha_1-1} \ldots u_n^{\alpha_n-1} (1 - \sum_{i=1}^n u_i)^{\gamma - \sum_{i=1}^n \alpha_i} \prod_{j=1}^{m-n-1} \left(1 - \sum_{i=1}^n x_{ij} u_i\right)^{-\beta_j},
\]

\[
c_1 = \Gamma(\gamma) / \Gamma(\gamma - \sum_{i=1}^n \alpha_i) \cdot \prod_{i=1}^n \Gamma(\alpha_i), \tag{A.6}
\]

and the dual representation in eq. (3.25) of [12]:

\[
F((\alpha_i), (\beta_j), \gamma; x) = c_2 \int_{\sum_{u_a \geq 0}} u_1^{\beta_1-1} \ldots u_n^{\beta_n-1} (1 - \sum_{i=1}^n u_i)^{\gamma - \sum_{i=1}^n \beta_i} \prod_{j=1}^{m-n-1} \left(1 - \sum_{i=1}^n x_{ij} u_i\right)^{-\alpha_j},
\]

\[
c_2 = \Gamma(\gamma) / \Gamma(\gamma - \sum_{i=1}^{m-n-1} \beta_i) \cdot \prod_{i=1}^{m-n-1} \Gamma(\beta_i), \tag{A.7}
\]
where the parameters are assumed to satisfy the conditions

\[
\alpha_i \notin \mathbb{Z}, \quad 1 \leq i \leq n, \quad \beta_j \notin \mathbb{Z}, \quad 1 \leq j \leq m - n - 1,
\]

\[
\gamma - \sum_{i=1}^{n} \alpha_i \notin \mathbb{Z}, \quad \gamma - \sum_{j=1}^{m-n-1} \beta_j \notin \mathbb{Z}.
\]  

(A.8)

The hypergeometric functions (A.1) comprise a basis of solutions to the defining set of differential equations

\[
\begin{align*}
(1) \quad \sum_{i=0}^{n} x_{ij} \frac{\partial \hat{\phi}}{\partial x_{ij}} &= \alpha_j \hat{\phi}, \quad 0 \leq j \leq m, \\
(2) \quad \sum_{j=0}^{m} x_{ij} \frac{\partial \hat{\phi}}{\partial x_{ij}} &= -(1 + \alpha_i) \hat{\phi}, \quad 0 \leq i \leq n, \\
(3) \quad \frac{\partial^2 \hat{\phi}}{\partial x_{ij} \partial x_{pq}} &= \frac{\partial^2 \hat{\phi}}{\partial x_{iq} \partial x_{pj}}, \quad 0 \leq i,p \leq n, \quad 0 \leq j,q \leq m.
\end{align*}
\]  

(A.9)

In cases where factors of the integrand are non-linear in the integration variables, the functions can be generalized further to Gelfand A-hypergeometric functions [19, 20] defined as:

\[
\hat{F}(\{\alpha\}, x) = \int_{u_{1} \geq 0, \ldots, u_{k} \geq 0} \prod_{i} P_i(u_1, \ldots, u_k) \alpha_1^{a_1} \ldots \alpha_k^{a_k} du_1 \ldots du_k,
\]

(A.10)

where \(\alpha_i\) are complex parameters and \(P_i\) now are Laurent polynomials in \(u_1, \ldots, u_k\).

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