On the Unconditional Validity of J. von Neumann’s Proof of the Impossibility of Hidden Variables in Quantum Mechanics

C. S. Unnikrishnan

Tata Institute of Fundamental Research,
Homi Bhabha Road, Mumbai 400005
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Abstract

The impossibility of theories with hidden variables as an alternative and replacement for quantum mechanics was discussed by J. von Neumann in 1932. His proof was criticized as being logically circular, by Grete Hermann soon after, and as fundamentally flawed, by John Bell in 1964. Bell’s severe criticism of Neumann’s proof and the explicit (counter) example of a hidden variable model for the measurement of a quantum spin are considered by most researchers, though not all, as the definitive demonstration that Neumann’s proof is inadequate. Despite being an argument of mathematical physics, an ambiguity of decision remains to this day. I show that Neumann’s assumption of the linear additivity of the expectation values, even for incompatible (noncommuting) observables, is a necessary constraint related to the nature of observable physical variables and to the conservation laws. Therefore, any theory should necessarily obey it to qualify as a physically valid theory. Then, obviously, the hidden variable theories with dispersion-free ensembles that violate this assumption are ruled out. I show that it is Bell’s counter-example that is fundamentally flawed, being inconsistent with the factual mechanics. Further, it is shown that the local hidden variable theories, for which the Bell’s inequalities were derived, are grossly incompatible with the fundamental conservation laws. I identify the intrinsic uncertainty in the action as the reason for the irreducible dispersion, which implies that there are no dispersion-free ensembles at any scale of mechanics. With the unconditional validity of its central assumption shown, Neumann’s proof is fully resurrected.

Note: A French translation of the title, abstract and a slightly condensed version of the introductory section with a summary of the main results are included in the manuscript (pp 3-6), between the table of contents and the main text in English.
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Sur la validité inconditionnelle de la preuve de J. von Neumann de l'impossibilité de mécanique quantique à variables cachées

C. S. Unnikrishnan
Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005

Résumé

L'impossibilité de théories à variables cachées (variables supplémentaires) comme alternative et remplacement de la mécanique quantique fut débattue par J. von Neumann en 1932. Sa preuve fut critiquée peu après par Grete Hermann, qui la qualifia de raisonnement circulaire, puis par John Bell qui la déclara fondamentalement défectueuse en 1964. La critique sévère de Bell contre la preuve de Neumann, et le contre-exemple explicite d'un modèle à variables cachées pour la mesure d'un spin quantique, sont considérés par la plupart des chercheurs, mais non la totalité, comme la démonstration que la preuve de Neumann est inadéquate. Bien que ce soit un argument en physique mathématique, une ambiguïté de décision persiste à ce jour. Je démontre que l'hypothèse de Neumann sur la linéarité additive des valeurs moyennes, même pour des observables incompatibles (qui ne commutent pas), est une contrainte nécessaire relative à la nature des variables physiques observables et aux lois de conservation. Par conséquent, toute théorie devrait obligatoirement lui obéir pour être qualifiée de théorie physiquement valable. Ainsi, de toute évidence, les théories à variables cachées avec des ensembles de dispersion-libre (dispersion-nulle) qui violent cette hypothèse sont éliminées. Je démontre que c'est le contre-exemple de Bell qui est fondamentalement défectueux, étant incohérent avec la mécanique factuelle. De plus, il est démontré que les théories locales à variables cachées, pour lesquelles Bell déduisit les inégalités, sont incompatibles de façon flagrante avec les lois fondamentales de conservation. J'identifie l'incertitude intrinsèque dans l'action comme la raison de la dispersion irréductible, ce qui implique qu'il n'y a pas d'ensembles de dispersion-libre à aucun niveau de la mécanique. Avec la validité inconditionnelle de son hypothèse centrale démontrée, la preuve de Neumann est totalement ressuscitée.
Introduction des principaux résultats

« Une seule exigence de (Leonard) Nelson reçut toute mon approbation. C'était l'exigence de ne pas s'empêcher de répondre par peur de se couvrir de honte. » – Grete Hermann.

Le sujet principal de cet article est la preuve que l'hypothèse centrale de l'additivité linéaire des valeurs moyennes (valeurs attendues), dans la preuve de J. von Neumann [1], sur l'impossibilité de toute description de mécanique quantique à variables cachées (variables supplémentaires), est inconditionnellement valable parce qu'il existe une contrainte physique cruciale. Cet article restitue la validité générale de la preuve de Neumann, qui fut mise en doute par de nombreux chercheurs depuis sa présentation en 1932. Plusieurs autres résultats essentiellement liés au sujet principal sont aussi systématiquement examinés. Le résultat principal est la preuve que l'hypothèse de Neumann sur l'additivité linéaire des valeurs moyennes, $\text{Exp}(aR + bS) = a\text{Exp}(R) + b\text{Exp}(S)$, est effectivement une relation à laquelle tous les ensembles physiques obéissent, et qui s'applique dans toutes les théories physiques valables, indépendamment du fait que $R$ et $S$ sont, ou non, des quantités physiques mesurables simultanément, ou des observables qui commutent dans la terminologie de la mécanique quantique. Ceci ressuscite la preuve de Neumann de l'impossibilité de toute description de la mécanique quantique à variables cachées. Les failles des diverses critiques de la preuve de Neumann par Grete Hermann et John Bell, sont ensuite examinées en détail.

Le deuxième résultat est l'échec du célèbre contre-exemple de Bell du théorème de Neumann, un modèle à variables cachées des mesures de la projection spin d'une particule spin-1/2. Ceci est démontré de deux façons différentes, exposant clairement la nature dénue de signification physique et l'inconsistance de ce contre-exemple.

Le troisième résultat est une preuve clarifiant que « l'incomplétude de causalité », qui motiva le travail sur les théories à variables cachées, est très différente de « l'incomplétude selon le raisonnement EPR dans la mécanique quantique » débattue par Einstein, Podolsky et Rosen, en 1935. Une analyse claire de cette distinction était nécessaire car la preuve de Neumann concerne l'incomplétude causale qui est enracinée dans l'indéterminisme, alors que la discussion EPR se rapporte à un aspect entièrement différent de la mécanique quantique qui ne concerne pas l'indéterminisme et les variables cachées, bien qu'il y ait communément confusion avec la première notion.

Le quatrième résultat identifie que la relation entre la dispersion quantique et la fonction d'action est la raison exacte de l'impossibilité d'ensembles à dispersion-libre (dispersion-nulle). Ce résultat anticipe toute idée fausse qu'une description déterministe de la mécanique serait un jour possible à un quelconque degré, exceptée en tant qu'approximation.

Le cinquième résultat est que les théories locales à variables cachées, pour lesquelles Bell déduisit les inégalités, sont, dans les faits, incompatibles avec les lois fondamentales de conservation, et par conséquent ce sont des théories non physiques dans leur formulation même.

Une brève analyse de la réinterprétation que fit D. Bohm [21] de la mécanique quantique de Schrödinger en tant que théorie non locale « à variables cachées » est inclue pour que cet article soit complet, montrant certaines de ses sérieuses insuffisances.

En tant que résultat de la physique mathématique, on s'attend à ce que la preuve de Neumann soit par déduction une suite des hypothèses utilisées dans la preuve. Par conséquent, le seul moyen de mettre la preuve en doute est de questionner la validité des hypothèses de départ. C'est en fait ce que firent à la fois Grete Hermann en 1933-35 [2, 3] et John Bell en 1964-66 [4], quand ils critiquèrent l'hypothèse centrale de Neumann,
Selon laquelle la fonction de valeur moyenne $Exp(O)$ obéit à la linéarité additive dans toutes les théories, même pour les observables de type $O = aR + bS$, où $R$ et $S$ ne commutent pas en mécanique quantique. La fonction de «valeur moyenne» (valeur attendue) est $Exp(O) = \langle \psi | \hat{O} | \psi \rangle$ en mécanique quantique, où $\hat{O}$ est l’opérateur hermitique correspondant à l’observable $O$. Dans une théorie générale, et aussi de manière empirique, $Exp(O)$ est une moyenne (valeur moyenne) des valeurs possibles de la quantité $O$ dans un ensemble statistique. Chaque quantité physique $O$ a une valeur moyenne $Exp(O)$ et une variance $Var(O) = Exp(O - \bar{O})^2 = Exp(O^2) - \bar{O}^2$. Un ensemble pour lequel $Var(O) = 0$ pour toutes les observables est défini comme un ensemble de «dispersion-libre» [1].

La relation de linéarité additive

$$Exp(aR + bS) = aExp(R) + bExp(S)$$

est bien entendu valable pour des ensembles classiques, où la quantité $(aR + bS)$ se compose des mesures séparées de $R$ et $S$. Cela est également valable en mécanique quantique, même pour les observables qui ne commutent pour lesquelles $(aR + bS)$ n’est défini qu’implicitement. C’est à dire qu’une valeur pour $O = (aR + bS)$ ne peut pas se composer des mesures de $R$ et $S$ si les quantités ne sont pas simultanément mesurables. Pourtant, on qualifie $(aR + bS)$ ainsi que la quantité plus générale $(a f(R) + b g(S))$ de «d’observables» en mécanique quantique. Un exemple pratique est $L_n$, la composante du moment angulaire (à deux dimensions) dans la direction $n = a \hat{x} + b \hat{y}$. En mécanique quantique, l’opérateur pour cette composante est $\hat{L}_n = a L_x + b L_y$, mais la valeur de $L_n$ ne peut pas être obtenue des mesures de $L_x$ et $L_y$.

Pourtant, les valeurs moyennes des trois quantités obéissent au principe de linéarité additive. En fait, Neumann prit soin de prouver l’additivité linéaire en mécanique quantique de manière explicite, car sa validité n’était pas évidente [1]. Comme les opérateurs hermitiques de la mécanique quantique obéissent au principe de linéarité additive, $Exp(O) \equiv Exp(aR + bS) = \langle \psi | a \hat{R} + b \hat{S} | \psi \rangle = \langle \psi | a \hat{R} | \psi \rangle + \langle \psi | b \hat{S} | \psi \rangle = aExp(R) + bExp(S)$

(2)

Ici, l’opérateur $a \hat{R} + b \hat{S}$ représente une seule observable (quantité physique) comme $\hat{L}_n$, devant être mesurée par un dispositif adapté, qui en général est différent de ce qui est approprié pour mesurer $R$ ou $S$. Le reproche formulé dans les critiques était que Neumann n’avait pas raison d’assumer que la validité de la linéarité additive s’appliquait aussi dans les théories à variables cachées, car de telles théories implémentent la distribution de résultats physiques sous des prémisses très différentes, en comparaison avec la mécanique quantique. Selon les termes de Bell [3],

...L’additivité des valeurs attendues... est une propriété très particulière des états de la mécanique quantique, que l’on n’attend pas a priori. Il n’y a aucune raison de l’exiger individuellement des états hypothétiques de dispersion-libre, dont la fonction est de reproduire les particularités mesurables de la mécanique quantique quand la moyenne est faite.

C’est le point central sur lequel je me focalise et je vais prouver que l’additivité des valeurs moyennes est effectivement une propriété essentielle qui est exigée a priori, et qui est universellement nécessaire, dans toutes les théories physiques. Etant donné qu’un ensemble de dispersion-libre ne peut pas répondre à cette contrainte, de tels ensembles n’existent pas dans le monde physique. Ceci exclut définitivement toute description de la mécanique quantique à variables cachées et justifie pleinement la preuve de Neumann.

Les théories à variables cachées complètent l’état pur en mécanique quantique (la fonction-ψ) avec des variables cachées qui déterminent une relation causale entre les valeurs des variables et la valeur d’une quantité physique observée dans les faits. Le but
est d’obtenir les valeurs quantiques aléatoires, apparemment acasuelles, comme le résultat déterministe de l’état quantique et des valeurs stochastiques des variables cachées. Bien sûr, il est empiriquement connu et vérifié que la relation d’additivité est valable lorsque les résultats de l’ensemble complet des mesures sont pris en considération pour calculer la moyenne. Cependant, a-t-on raison de supposer sa validité également pour chaque sous-ensemble ? Cette question se pose car la relation fonctionnelle allant des valeurs des variables cachées à la valeur observée d’une quantité physique pourrait être non-linaire. Alors que G. Hermann jugeait la preuve de Neumann d’être «logiquement circulaire», J. S. Bell alla jusqu’à qualifier la preuve (dont l’abrégéation est JvNP) de «idiote», «absurde» et «folle» [5, 6]. Pour défendre son point de vue, Bell mit au point un contre-exemple à la JvNP, un modèle à variables cachées des mesures d’un seul spin quantisé avec comme résultat la valeur moyenne correcte [4, 7]. Comme on le sait bien, il alla plus loin pour démontrer qu’un tel modèle n’était pas possible pour la corrélation générale de deux spins, dans l’hypothèse de la localité.

Aujourd’hui, dans la communauté de physiciens la plupart se fient à la critique sévère de Bell, essentiellement parce qu’il fit une démonstration explicite contre la preuve de Neumann, d’un modèle à variables cachées des mesures d’un seul spin. Pourtant, il y eut de sérieux débats et désaccords [6, 8, 10]. Les énormes efforts et ressources qui furent investis pour tester les théories locales à variables cachées (LHVT) et les inégalités de Bell provenaient du fait que Bell réfuta la JvNP, et par ailleurs du résultat qu’il n’y a aucune différence distincte entre les fonctions de corrélation à deux particules et la mécanique quantique. D’un autre côté, si la JvNP avait toujours été valable et que Bell se soit trompé dans ses critiques et contre-exemple, nous serions alors face à une situation de recherche et tests expérimentaux malavisés depuis des décennies.

Nous allons voir que la linéarité additive de la fonction $Exp(O)$ découle de la nécessité que les observables physiques obéissent aux lois fondamentales de conservation dans tout ensemble. Il s’ensuit alors la preuve qu’il n’y a pas d’ensembles de dispersion-libre. Ainsi, la partie manquante de la JvNP est en place, découlant de considérations de physique fondamentale. Par conséquent, la critique de raisonnement circulaire de Hermann et la critique d’absurdité de Bell ne sont pas valables. Avec la JvNP démontrée comme généralement valable, il est possible d’identifier le point exact où Bell se trompe dans sa critique. De plus, je vais montrer explicitement que le contre-exemple de Bell du modèle à variables cachées de la mesure du spin est fondamentalement défectueux, et qu’il est incohérent avec la mécanique quantique simple du spin. Dans toutes les analyses précédentes du problème, la raison principale de l’impossibilité de tout ensemble à dispersion-libre ne fut pas explorée. Je fais remonter cette raison à la relation profonde entre la fonction d’action d’Hamilton et la dispersion irréductible des variables dynamiques en mécanique.

Je montre que la dispersion irréductible n’est pas dans les observables individuelles comme $p$, $q$, $L$, etc., mais dans la fonction d’action même, qui est une combinaison de deux observables conjuguées (incompatibles). Comme l’action $S$ est la même pour n’importe quel sous-ensemble d’un ensemble parent défini par une fonction-$\psi$, la preuve de l’impossibilité de théories à variables cachées avec des ensembles à dispersion-libre est immédiate.
I. INTRODUCTION TO THE MAIN RESULTS

"Only one demand that (Leonard) Nelson made got my total approval. It was the demand not to let oneself be prevented from answering for fear of disgrace" – Grete Hermann.

The main theme of this paper is the proof that the central assumption of the linear additivity of expectation values, in J. von Neumann’s proof of the impossibility of any hidden variable description of quantum mechanics, is unconditionally valid because of a crucial physical constraint. This restores the general validity of Neumann’s proof that has been questioned by many researchers after its presentation in 1932. Several supplementary results that are vitally related to the main theme are also discussed systematically. The central result is the proof that Neumann’s assumption of the linear additivity of the expectation values, \( \text{Exp}(aR+bS) = a\text{Exp}(R) + b\text{Exp}(S) \), is indeed a relation obeyed by all physical ensembles, and applicable in all physically valid theories, irrespective of whether or not \( R \) and \( S \) are physical quantities that are simultaneously measurable, or commuting observables in the terminology of quantum mechanics. This resurrects Neumann’s proof of the impossibility of any hidden variable description of quantum mechanics (QM). The fault lines of the different criticisms of Neumann’s proof, by Grete Hermann and John Bell, are then shown in detail.

The second result is the failure of Bell’s well known counter-example to Neumann’s theorem, a hidden variable model of the measurements of the spin projection of a spin-1/2 particle. This is demonstrated in two different ways, exposing clearly the unphysical nature and inconsistency of that counter-example.

The third result is a clarificatory proof that the ‘causality incompleteness’ that motivated the work on hidden variable theories is very different from the ‘EPR-incompleteness’ of QM, discussed by Einstein, Podolsky, and Rosen, in 1935. This was necessary because Neumann’s proof concerns the ‘causal incompleteness’ that is rooted in indeterminism, whereas the EPR discussion is on an entirely different aspect of QM that does not concern indeterminism and hidden variables, though commonly confused with the former notion.

The fourth result identifies the exact reason for the impossibility of dispersion-free ensembles, in the relation between the quantum dispersion and the action function. This result pre-empt any misconception that a deterministic description of mechanics will ever be possible at any scale, except as an approximation.

The fifth result is that the local hidden variable theories for which the Bell’s inequalities were derived are factually incompatible with the fundamental conservation laws, and hence they are unphysical theories in their very formulation.

A brief discussion of D. Bohm’s reinterpretation of the Schrödinger quantum mechanics as a nonlocal theory, with the position coordinate as a ‘hidden variable’, is included for completeness, pointing out some of its serious inadequacies.

Seen as a result of mathematical physics, one expects that Neumann’s proof deductively follows from the assumptions used in the proof. Therefore, the only way the proof could be put in doubt is by questioning the validity of the starting assumptions. In fact, this is what was done by both Grete Hermann in 1933-35 and John Bell in 1964-66, when they criticized Neumann’s central assumption that the expectation value function, \( \text{Exp}(O) \), obeys linear additivity in all theories, even for the observables of the form \( O = aR + bS \), where \( R \) and \( S \) are noncommuting in quantum mechanics. The function ‘expectation value’ is \( \text{Exp}(O) = \langle \psi | \hat{O} | \psi \rangle \) in quantum mechanics, where \( \hat{O} \) is the Hermitian operator corresponding to the observable \( O \). In a general theory, and also empirically, \( \text{Exp}(O) \) is an average (mean value) of the possible values of the quantity \( O \) over a statistical ensemble. Every physical quantity \( O \) has an expectation value \( \text{Exp}(O) \) and a variance \( \text{Var}(O) = \text{Exp}(O - \bar{O})^2 = \text{Exp}(O^2) - \bar{O}^2 \). An ensemble for which \( \text{Var}(O) = 0 \) for all
observables is defined as a ‘dispersion-free’ ensemble [1].

The linear additivity relation
\[
\text{Exp}(aR + bS) = a\text{Exp}(R) + b\text{Exp}(S)
\]
(3)
is of course valid for classical ensembles, where the quantity \((aR + bS)\) is assembled from separate measurements of \(R\) and \(S\). It is also valid in quantum mechanics (QM), even for noncommuting observables for which \((aR + bS)\) is defined only implicitly. That is, a value for \(O \equiv (aR + bS)\) cannot be formed from the measurements of \(R\) and \(S\) if the quantities are not simultaneously measurable. Yet, \((aR + bS)\) as well as the more general \((af(R) + bg(S))\) qualify as ‘observables’ in quantum mechanics. One example that Neumann himself discussed is the energy observable, \(E = (p^2/2m) + V(x)\). Another common and convenient example is \(L_n\), the component of the (two dimensional) angular momentum along the direction \(\hat{n} = a\hat{x} + b\hat{y}\). In QM, the operator for this component is \(\hat{L}_n = a\hat{L}_x + b\hat{L}_y\), but the value of \(L_n\) cannot be obtained from the measurements of \(L_x\) and \(L_y\). Yet, the expectation values of the three quantities obey the principle of linear additivity. In fact, Neumann took care to prove the linear additivity in quantum mechanics explicitly, because its validity was not obvious [1]. The proof within the mathematical structure of QM is easy because the Hermitian operators of QM obey the principle of linear additivity,
\[
\text{Exp}(O) \equiv \text{Exp}(aR + bS) = \langle \psi | a\hat{R} + b\hat{S} | \psi \rangle = \langle \psi | a\hat{R} | \psi \rangle + \langle \psi | b\hat{S} | \psi \rangle = a\text{Exp}(R) + b\text{Exp}(S)
\]
(4)

Here, the operator \(a\hat{R} + b\hat{S}\) represents a single observable (physical quantity) like \(\hat{L}_n\), to be measured by a suitable apparatus that is in general different from what is appropriate for measuring \(R\) or \(S\).

The complaint of the critics was that Neumann was not justified in assuming the validity of the linear additivity in the hidden variable theories as well, because such theories implement the distribution of observable physical results under very different physical premises, compared to QM. In Bell’s words [4],

...the additivity of expectation values.... is a quite peculiar property of quantum mechanical states, not to be expected a priori. There is no reason to demand it individually of the hypothetical dispersion-free states, whose function is to reproduce the measurable peculiarities of quantum mechanics when averaged over.

This is the central point that I focus on, and I will prove that the additivity of the expectation values is indeed an essential property that is demanded a priori, and universally necessary, in all physical theories. Since a dispersion-free ensemble cannot fulfill this constraint, such ensembles do not exist in the physical world. This definitely rules out any hidden variable description of quantum mechanics and fully justifies Neumann’s proof.

The hidden variable theories supplement the pure quantum mechanical state (the \(\psi\)-function) with a set of hidden variables that determine a causal relation between the values of the variables and the actually observed value of a physical quantity. The aim is to get the random, apparently acausal, quantum values as a deterministic outcome from the quantum state and the hidden stochastic values of the hidden variables. Of course, it is empirically known and verified that the additivity relation is valid when the entire ensemble of measurement results are considered to calculate the average. However, is one justified in assuming its validity for every subensemble as well? This question arises because the functional connection from the values of the hidden variables to the observed
value could be nonlinear. While G. Hermann judged Neumann’s proof as “logically circular”, J. S. Bell went to the extent of calling the proof (abbreviated as JvNP) as “silly”, “absurd”, and “foolish” [3, 6]. To prove his point, Bell constructed a counter-example to the JvNP, a hidden variable model of the measurements of a single quantized spin that resulted in the correct expectation value [4, 7]. As it is well known, he went further to show that such a model was not possible for the general correlation of two spins, under the assumption of locality. Recent discussions that review the situation differ in their judgement of the JvNP and Bell’s criticism. More balanced views describe the JvNP as of limited validity, applicable to only a certain class of hidden variable theories [8].

It is curious to note that E. P. Wigner’s account [9] of the hidden variable question judged that Bell had arrived at the same conclusion as Neumann’s, though with the aid of a different argument! This statement is correct in a sense, as far as the final results are concerned. The article, written in 1969, seems to fully accept Neumann’s reasoning, as well as Bell’s, and does not express any doubt about the integrity of Neumann’s proof. However, Wigner mentions that Neumann was convinced against hidden variables primarily because of physical reasons associated with multiple (sequential) quantum measurements. This is an important point, because the repeated measurements of a pair of incompatible observables do show the difficulty in proposing the existence of ensembles without dispersion (variance) in the physical quantities. However, in this paper we are concerned with Neumann’s formal proof and the major criticisms that were mentioned.

In the history of the discussions of the JvNP, there is another curious fact, that Hermann’s criticism did not reach Neumann, even though she sought comments from researchers like Bohr, Heisenberg and Dirac [2]. Bell’s criticisms were expressed after Neumann’s departure. Therefore, we do not know what justification Neumann would have given for taking the linear additivity of the function $\text{Exp}(O)$ as generally applicable for all statistical ensembles. Though Neumann discusses the issues in great detail in his monograph, he is terse at times, and seems to avoid explanatory comments on the mathematical steps that are perhaps evident to him. In any case, instead of trying to interpret Neumann’s reasons, I will present a strengthened proof, without the alleged ambiguities. At present, most people in the physics community trust Bell’s harsh criticism, primarily because of his explicit demonstration countering Neumann’s proof, of a hidden variable model of the measurement on a single spin. However, there has been some serious debates and disagreements, involving J. Bub, D. Dieks and D. Mermin, regarding Hermann’s and Bell’s criticisms about Neumann’s proof [6, 8, 10]. The enormous efforts and resources that had been invested in the tests of local hidden variable theories (LHVT) and the Bell’s inequalities were due to Bell’s refutation of the JvNP, along with the result that there was a distinct difference between the measurable two-particle correlation functions in the LHVT and QM. On the other hand, if the JvNP was always valid, and Bell was wrong in his criticisms and counter-example, then we are faced with a situation of decades’ long misguided research and experimental tests.

I will show that the linear additivity of the function $\text{Exp}(O)$ holds for all physical observables, in any physically valid theory, because every physical observable should be consistent with the space-time symmetries and should obey the fundamental conservation laws on the average. Then the proof that there are no dispersion-free ensembles follows. Thus, the missing part in the JvNP is in place, derived from fundamental physical considerations. Consequently, Hermann’s criticism of logical circularity and Bell’s criticism of absurdity are not valid. With the JvNP shown as generally valid, one is able to show where exactly Bell’s criticism is mistaken. Further, I will explicitly show that Bell’s counter-example of the hidden variable model of spin measurements was fundamentally flawed, and that it is inconsistent with the simple quantum mechanics of the spin.
In all previous discussions of the problem, the core reason for the impossibility of any dispersion-free ensemble was not explored. However, if one can prove such an impossibility from a general consideration obeyed by all valid theories, then there should be a deep and fundamental reason that could be stated simply. I trace this reason to the relation between Hamilton’s action function and the irreducible dispersion of dynamical variables in mechanics. I show that the irreducible dispersion is not in the individual observables like p, q, L etc., but in the action function itself, which is a combination of two conjugate (incompatible) observables. Since the action S is the same for any of the subensembles of a parent ensemble defined by a ψ-function, the proof of the impossibility of hidden variable theories with dispersion-free ensembles is immediate.

II. THE ONE CRUCIAL DIFFERENCE

I want to state very clearly the one crucial difference between quantum mechanics and a hidden variable theory, both of which rely on the ψ-function for the representation of a physical state. In QM, the ψ-function is the sole representation of the physical state, from which all observable results follow. The result of any single observation is not predictable even though the possible values and their statistical distribution are calculated. In contrast, in a hypothetical hidden variable theory, the ψ-function as well as many hidden variables together determine causally the precise observable values of the physical quantities, in each observation. The unpredictability arises merely from the unknown values taken by the hidden variables. The implication is that the physical nature is fundamentally deterministic and completely causal, and that at least a theoretical picture of that hypothetical deterministic world can be constructed, in a hidden variable theory. Stated this way, the drastic nature of Neumann’s mathematical assertion, that this idealistic expectation of the physical world is impossible, is striking.

The physical state in QM is completely specified by the ψ-function, in the sense that there is a one-to-one correspondence between the physical states and their ψ-function representations. But, the same state can be expressed in different basis states. When the two sets of basis states correspond to (eigenstates of) noncommuting observables, then each eigenstate in one basis is a superposition of eigenstates of the other basis. Denoting the two basis sets as ψ and φ, a pure state ψi is in general

$$\psi_i = \sum_j c_j \phi_j$$

(5)

This defines the dispersion in the ψ-state, of the values represented by the φj states. Clearly, every particle in the ψi state belongs to the single ‘homogeneous’ ensemble; ψ is not a ‘mixture’ of ensembles of different φj states. Similarly, each φi state is a linear superposition of the different ψj states. The ensemble specified by the ψ-function (or the φ-function) is dispersive in general; Var(O) ≠ 0 for some observables and hence, there are no dispersion-free ensembles in quantum mechanics.

In a hidden variable theory, the dispersion of a physical quantity in the ψ-ensemble is seen as due to the mixing of several pure (dispersion-free) φ-ensembles. Similarly, the dispersion in a pure φ-ensemble results from a mixing of different ψ-ensembles. Hence, the hidden variable theories assert the notional division of the entire ensemble into dispersion-free ensembles, (ψi, φj), each of which has no dispersion in any of the observables. This also means that the ψi state is not a superposition of the φj states. (I use the term ‘superposition’ exclusively for expressions of the kind in equation 5 and the term ‘mixture’ to refer to the composition of ensembles. Neumann has used the word ‘superposition’ for
both, with its contextual meaning, which I avoid). This absence of superposition in a hidden variable theory, and hence the absence of the concept of wave-particle duality that defines quantum mechanics, is the crucial conceptual difference to be noted. A hidden variable theory tries to match the measured dispersion in the given state $\psi$ to what is predicted by quantum mechanics, by distributing the measured values among the different dispersion-free sets. Each measurement results in a specific value among the many possible values randomly because of the hidden variables taking a random (unknown) realization, which then picks one among the many dispersion-free ensembles. I want to emphasize the point that the characteristic feature of a hidden variable theory is the existence of dispersion-free ensembles, and not merely the labelling of some unmeasured physical observables as hidden variables.

Both in QM and in any hidden variable extension, the following is true: A single measurement of a physical quantity $O$ in a general state specified by a $\psi$-function returns one of the possible values $a_i$ in the spectrum of that quantity, realized unpredictably. In QM this is one of the eigenvalues of the operator $\hat{O}$ corresponding to the observable $O$. Then the expectation value in $n$ measurements is $\text{Exp}(O) = \frac{1}{n} \sum a_i$. If the physical state is specified by a $\psi$-function that is an eigenstate $\psi_i$ of an observable $O$, with the eigenvalue $\alpha$, then the measurable as well as the measured values of $O$ are all $\alpha$, with $\text{Exp}(O) = \alpha$, irrespective of the values of the hidden variables. This will be important when we discuss Neumann’s proof as well as Bell’s counter-example.

Another relevant fact that one should keep in mind is that Neumann was asking the question whether quantum mechanics could be supplemented by additional variables to make it causal. Hence, he assumed that all the mathematical truths and the statistical predictions from the Schrödinger equation within the theory of quantum mechanics remain true. For example, the statement that every observable can be represented by a Hermitian operator must remain true in a hidden variable extension of QM. Then it is also true that if $\hat{R}$ and $\hat{S}$ are Hermitian operators, so is their sum $\hat{R} + \hat{S}$. However, we accept the general relation $\text{Exp}(a\hat{R} + b\hat{S}) = a\text{Exp}(\hat{R}) + b\text{Exp}(\hat{S})$ only for the full ensemble a priori (which agrees with the empirical fact), and not for the hypothetical subensembles which might be dispersion-free. It might be relevant to state that the ‘causal extension’ by supplementing QM with hidden variables is very different from a ‘total replacement’ by a classical hidden variable theory, without a $\psi$-function description of the physical state.

### III. JOHN VON NEUMANN’S IMPOSSIBILITY PROOF

Neumann’s proof of the impossibility of a hidden variable description of quantum mechanics has been discussed extensively. The question addressed by Neumann was whether a causally deterministic description of quantum mechanics was possible, by adding hidden supplementary variables to the QM description of the state specified by the $\psi$-function. In a causal and fully deterministic mechanics, one can form subensembles in which the values of the physical quantities are without any variance. In such an ensemble, the observables have one of their characteristic quantized values or eigenvalues (in the vocabulary of quantum mechanics). So, the possibility of dispersion-free ensembles characterize the hidden variable theories.

Neumann starts with the central assumption, provable as valid in quantum mechanics, of the general (unconditional) linear additivity of expectation values of the physical quantities $R, S$, etc.,

$$\text{Exp}(aR + bS) = a\text{Exp}(R) + b\text{Exp}(S)$$  \hspace{1cm} (6)

Also, it is assumed that if the observable values of $O$ are non-negative, then $\text{Exp}(O) \geq 0$. 

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Then he argued that there are no dispersion-free ensembles 1. First he showed that the function $\text{Exp}(O)$ can be expressed in general as the trace of the product of a Hermitian density matrix $D$ and another Hermitian matrix $O$ representing the observable, $\text{Exp}(O) = \text{Tr}(DO)$. A valid density matrix obeys $\text{Tr}(D) = 1$.

By the assumption of linear additivity

$$\text{Exp}(aR + bS) = a\text{Tr}(DR) + b\text{Tr}(DS)$$

(7)

For a dispersion-free ensemble $[\text{Tr}(DO)]^2 = \text{Tr}(DO^2)$. Now, take $O = P_\phi$, which is the projection operator onto the pure state $\phi$. Since $P_\phi = P_\phi^2$, $\text{Tr}(DP_\phi) = \langle \phi | D | \phi \rangle$ in the state $\phi$, and $\langle \phi | D | \phi \rangle = (\langle \phi | D | \phi \rangle)^2$, because the ensemble is dispersion-free. This implies that $\langle \phi | D | \phi \rangle$ is a constant. Since this should vary continuously when one goes continuously from a normalized state $\phi$ to another $\phi'$, $\langle \phi | D | \phi \rangle$ is a constant. The value $\langle \phi | D | \phi \rangle = 0$ for all states $\phi$ is physically irrelevant. Then we conclude that the matrix $D$ is the unit matrix, $D = 1$, for a dispersion-free ensemble. But this is not a valid (probability) density matrix. Hence, dispersion-free ensembles do not exist and the hidden variable theories are impossible.

It is clear that if Neumann’s impossibility proof is generally valid, the entire exercise of discussing and testing hidden variable theories during the past half century has been a colossal misadventure. Therefore, one should carefully scrutinize the central assumption and its criticisms. The use of the mathematical machinery of quantum mechanics in the proof can cause the suspicion that ‘peculiar’ factors that are legitimate in QM, but not justified in a hidden variable theory, might have entered the proof. Hence, I aim to recast Neumann’s proof in a way that does not use the mathematical tools of QM. In a way, I want to bypass the question whether Neumann was justified in making the assumption of linear additivity of the expectation values for hidden variable theories, by proving the unconditional validity of the contested assumption in all physical theories.

IV. A STRAIGHT PATH TO THE IMPOSSIBILITY PROOF

Whether or not the JvNP is general, the central assumption in the proof is valid as a fact in quantum mechanics, which is the only theory that is known to be consistent with the empirical facts. It is also valid in classical mechanics. How come the allegedly limited assumption made by Neumann is valid in the only physical theories that stand today?! It might be a fact that Neumann did not prove what he should have, but if the linear additivity is a physical constraint to be obeyed by any physical theory, then the JvNP is automatically valid for all theories that qualify as physical theories. I will now show that a transparent proof of the impossibility of hidden variable extensions of quantum mechanics can be built entirely on the necessary fundamental physical features of microscopic mechanics, without the aid of the mathematical features specific to quantum mechanics. There are certain physical principles that every valid theory should obey, irrespective of the structure and details of the theory. Among them are the fundamental conservation laws that are related to basic space-time symmetries. However, for microscopic physics, these can be valid only on the average and not for individual observations in an ensemble. Yet, the requirement that the expectation values in any ensemble obey the relations familiar in classical mechanics is a strict physical requirement, independent of any particular theory. Any theory where this is violated (for expectation values) does not respect the fundamental conservation laws and space-time symmetries, and does not qualify as a physical theory. The average value of a quantized observable in an ensemble corresponds to a fundamental physical quantity that is identical in meaning and scope to
the same quantity in classical mechanics. Such quantities have to necessarily obey the physical constraints that cannot be obeyed by single quantized values of the quantity; this is the key point. There are other contexts where such physical constraints are evident, like the Bohr correspondence principle and the Ehrenfest theorem. This physical requirement is the reason for the general validity of the linear additivity of the expectation values, irrespective of the nature of the theory and whether the physical quantities are measurable without disturbing the values of each other.

Consider the ensemble denoted by \( \psi_i \), which is an eigenstate of some observable \( S \) with the eigenvalue \( s_i \) in QM. This is ‘pure’ and dispersionless for \( S \), with \( Exp(S) = s_i \) but it is not dispersionless for another observable \( R \) that cannot be simultaneously measured with \( S \) (I use the term ‘dispersionless’ for denoting a single observable that has zero variance, and the more general term ‘dispersion-free’ to denote ensembles with zero variance in all observables). However, the hidden variable theories propose that \( \psi_i \) has zero variance, and the more general term ‘dispersion-free’ to denote ensembles with measured with \( S \). Now, the crucial point is that each of these symbols represents a physical quantity. Therefore, even though a single value \( s_i \) or a set of values \( (s_i, r_j, m_k) \) notionally associated with a single copy of the physical system (particle) is quantized, their average values in any ensemble should obey the constraints like the conservation laws, exactly as they do in classical mechanics. Whether or not the quantities are commuting is irrelevant for the conservation laws of the average quantity in every individual ensemble. Therefore, the functional relation between the quantities is maintained intact when averages over any ensemble are taken, even though a single set of quantized values cannot obey the relation.

This can be clarified with an example. If \( R \equiv L_x \) and \( S \equiv L_y \), and \( T \equiv L_z \), the components of the angular momentum in the directions \( x \) and \( y \), and \( z \), then

\[
\vec{L} \equiv \hat{n}L = \hat{x}L_x + \hat{y}L_y + \hat{z}L_z = \hat{x}\sin \theta \cos \varphi \hat{L} + \hat{y}\sin \theta \sin \varphi \hat{L} + \hat{z}\cos \theta \hat{L} \quad (8)
\]

which is an exact geometrical relation that reflects the physical constraint on the angular momentum. The operator relation in QM follows this universal physical constraint, and it is not merely ‘a peculiarity of quantum mechanics’. The measurable values of these components are quantized as \( \pm 1 \) for each of these quantities \( (L, L_x, L_y, L_z) \) in a single measurement. But, the total of these quantized values in any subensemble, like \( \sum l_i \), is the total angular momentum of that ensemble in a specific direction (figure 1). Then, it has to obey the physical law of the addition of the angular momentum and the conservation constraint, with \( \sum l_{xi} \) etc. as the component of the angular momentum along the different directions. Thus, every physically valid subensemble should unconditionally obey the relation \( \sum l_i = a \sum l_{xi} + b \sum l_{yi} + c \sum l_{zi} \), even though a set of single measurements obviously cannot obey \( l_i = a l_{xi} + b l_{yi} + c l_{zi} \). Therefore, the relation for the linear additivity of their expectation values \( \langle L \rangle = a \langle L_x \rangle + b \langle L_y \rangle + c \langle L_z \rangle \) follows.

In the particular example,

\[
\langle L \rangle \equiv \sin \theta \cos \varphi \langle L_x \rangle + \sin \theta \sin \varphi \langle L_y \rangle + \cos \theta \langle L_z \rangle \quad (9)
\]

Since \( \langle L_x \rangle = \sin \theta \cos \varphi \langle L \rangle \) etc., we have the identity

\[
\langle L \rangle = (\sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \theta) \langle L \rangle \quad (10)
\]

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Figure 1: The linear additivity of the expectation values is an unconditional physical requirement on all valid theories. This example illustrates this fact for the components of the quantized angular momentum in two dimensions. A) The ensemble notionally characterized by $L_x = \langle L_x \rangle = +1$ and $L_y = \langle L_y \rangle = +1$. The quantity $L = \cos \theta L_x + \sin \theta L_y$ also has the observable values $\pm 1$, with the maximum possible average value +1. But, $\langle L \rangle$ being an angular momentum it has to obey the linear additivity relation $\langle L \rangle = \cos \theta \langle L_x \rangle + \sin \theta \langle L_y \rangle \geq 1$. Therefore, this is an unphysical ensemble. B) A physical ensemble necessarily has the right dispersion to satisfy the linear additivity relation.

which confirms the general validity of the linear additivity of the expectation values of observables that are not simultaneously measurable. An ensemble that violates this cannot be a physically valid ensemble because it violates the addition and conservation of the angular momentum. We have proved the unconditional validity of Neumann’s central assumption, the linear additivity of the expectation values,

$$Exp(M) = Exp(aR + bS) = aExp(R) + bExp(S) \quad (11)$$

Then, it is obvious that dispersion-free ensembles are not possible. If an ensemble is dispersionless in the observables $R$ and $S$, it has to have a dispersion in $M$ such that the relation [11] is true on the average (figure 1). Therefore, dispersion-free ensembles are unphysical and they cannot exist in the physical world. This completes the proof of the impossibility of hidden variable extensions of quantum mechanics.

The impossibility of dispersion-free subensembles can be demonstrated in another way as well. We know that the subensembles can be physically separated on the basis of the distinct values of an observable $R$ (the different $\phi$-states), without any measurement of either $R$ or $S$. But, empirical facts demand that if the subensembles are physically separable, then each of the homogeneous ensembles specified by $\phi_j, \phi_k...$ etc. is not dispersion-free, which is also demanded by the relation $\phi_i = \sum c_n \psi_n$. Each $\phi_j$-subensemble shows dispersion in the quantity $S$, if measured. A hidden variable theory can accommodate this requirement only by postulating a redistribution of the original $\psi_i$-state into different $\psi_n$ states in the process of the separation into the pure $\phi_j$-subensembles. But that is not consistent with another physical fact that no dispersion will be seen in the quantity $S$ when the different $\phi_j$-subensembles are recombined before a measurement of $S$! Therefore, each state $\psi_i$ or $\phi_i$ represents a homogeneous ensemble, but it is not dispersion-free. This proves that dispersion-free ensembles do not exist.

An obvious example is when the pure state $\psi_i$ represents the $x$-component of spin, $S_x$. An observable that is not simultaneously measurable (noncommuting in QM) is the $z$-component of spin $S_z$. Considering only these two directions, the dispersion-free ensembles in a hidden variable theory are ($+x, +z$) and ($+x, -z$). Separating the $+x$ ensemble into these subensembles is easy, by passing through a $z$-directed magnetic field with a gradient. Now, there are two possibilities before any measurement verifies the predictions: 1) the act
of separation might have perturbed the $x$-component to randomize into two ensembles $+x$ and $-x$ in each $z$-subensembles, or 2) the $x$-component is unaffected. The second option is ruled out because it does not agree with the empirical fact that both the $+z$ ensemble and $-z$ ensemble have the maximal dispersion in the observable $S_z$. Therefore, a hidden variable theory demands the randomization of the $x$-component in the two subensembles that are dispersion-free in $S_z$. But, this also is in conflict with empirical facts, because recombining the two $z$-subensembles before any actual measurement of the $x$-component shows that the whole (combined) ensemble is in the state of pure $+x$ for $S_x$. Therefore, the states represented by $|x^+\rangle$, $|z^+\rangle$, and $|z^-\rangle$ are homogeneous (pure) ensembles, but they are not dispersion-free. The notional dispersion-free subensembles $(+x, +z)$ and $(+x, -z)$ cannot exist.

We can consider an example that Neumann himself mentioned, of a physical system with its energy $E(x, p) = (p^2/2m) + V(x)$. For a harmonic oscillator in its ground state

$$E_0 = \frac{p^2}{2m} + \frac{m\omega^2x^2}{2} \quad (12)$$

If the hidden variable theories are possible, with dispersion-free ensembles, $(\delta(p), \delta(x))$ is one such, where the $\delta$-functions may be taken as very narrow distributions centred on zero (nearly dispersion-free ensemble). Though a single set of sequential measurements may return the value 0 for the quantities $E, p,$ and $x$, the average of $E$ over any ensemble must be the zero-point energy $\hbar\omega/2$, even without any measurement. An ensemble in which the average zero-point energy is zero ($\ll \hbar\omega/2$) is not a physical ensemble! Hence, the subensemble $(\delta(p), \delta(x))$ is unphysical and cannot exist.

Another striking example from ‘new physics’ is that of a stream of neutrinos, prepared as a pure ensemble of electron type. The three initial hypothetical dispersion-free subensembles of a hidden variable scenario are $(v_e, m_1), (v_e, m_2)$ and $(v_e, m_3)$. But, a subensemble tagged by the mass, say $m_2$, cannot be uniquely $(m_2, v_e)$! Instead, it has the dispersion spread in the values $(m_2, v_e), (m_2, v_\mu)$ and $(m_2, v_\tau)$. In a way, the quantum mechanical irreducible dispersion is displayed symbolically as $(v_e, m_2) \neq (m_2, v_e)$, reminding us of the noncommuting nature of observables.

Now that we are guaranteed about the impossibility of a hidden variable description of quantum mechanics, we can examine the influential criticisms that kept alive the thoroughly unphysical proposal. Our verification that the proof had complete general validity implies that the criticisms by Hermann and Bell (and many others) were not valid. Further, it also implies that Bell’s famous counter-example must be flawed in a serious way. I will show in detail that this is indeed the case.

V. GRETE HERMANN’S CRITICISM

Grete Hermann’s investigations in QM were motivated by the general philosophy of causality and determinism in physics. Her criticism of Neumann’s proof was actually a corollary of her analysis of the indeterminism in QM, as represented in the Dirac formalism. She examined the expression $\text{Exp}(O) = \langle \psi | \hat{O} | \psi \rangle$ with $\psi = \sum c_i \psi_i$, and focussed on the superposition $\sum c_i \psi_i$ as the source of indeterminism in the theory. She set up the task clearly in her paper in 1935 [3],

Dirac’s own presentation and interpretation of his formalism undeniably includes indeterminism. Hence the question is only whether it is a necessary component of his theory, such that in giving it up one loses essential elements
of the theory, or whether indeterminism can be detached from the theory
without thereby affecting its explanatory value.

Thus the investigation was whether the indeterminism could be attributed to some
elements other than the $\psi$-function. Her conclusion was that the possibility of additional
hidden traits in the physical state, which would eliminate indeterminism, was open and
not excluded by the physical requirements of Dirac’s theory.

Hermann then proceeded to analyze Neumann’s assumptions in his proof and wrote,

...the expectation value of a sum of physical quantities is equal to the sum of
the expectation values of the two quantities. Neumann’s proof stands or falls
with this assumption.

She noted that this is not shown as valid for the hidden variable theories. After agreeing
that this holds for the full ensemble characterized by the $\psi$-function, Hermann poses the
question whether this is true also for the subensembles of the ensemble, which are further
selected on the basis of some other distinguishing feature that remains hidden (or not
noticed) in the parent ensemble. If the possibility of such subensembles is kept open,
then one cannot assume that the expectation value of a sum of two physical quantities
is equal to the sum of the expectation values of the two quantities. Already in 1933 she
noted [2],

...for ensembles of physical systems agreeing with one another besides in the
wave function also in terms of such a newly discovered trait, it has not been
shown that the expectation value function has the form $\langle \psi | \hat{O} | \psi \rangle$ and is thus
an $\text{Exp}(O)$.

Therefore, she argued that not addressing this explicitly is equivalent to assuming the
validity of the linear additivity for the subensembles without any justification. In other
words, one is assuming that the complete description is in the $\psi$-function and there can
be no further distinguishing features. But, the proof is about the impossibility of such
distinguishing features and therefore taking it as an assumption before the proof makes
the proof logically circular. If one assumes at the start that there are no such features,
then the deduction that such features are impossible is a redundant fact. Hermann’s
conclusion was that one could hope for a causal description of microscopic physics, since
the impossibility proof was found inadequate.

Of course, this criticism is no more valid, now that we understand that the linear
additivity is applicable to all physical ensembles. Therefore, Neumann’s assumption did
not exclude any ensembles that are more general than ensembles specified by $\psi$-functions.
Neumann’s assumption excluded only unphysical ensembles, as it should.

VI. JOHN BELL’S CRITICISM AND HIS COUNTER-EXAMPLE

John Bell’s critique starts on the same point noticed by Hermann, that Neumann did
not justify his assumption of linear additivity for the hidden variable theories when the
observables were not simultaneously measurable (noncommuting in QM). To make sure
that we are discussing exactly what Bell meant, let me quote his clear statement [11],

That the statistical averages should then turn out to be additive is really
a quite remarkable feature of quantum mechanical states, which could not
be guessed a priori. It is by no means a ‘law of thought’ and there is no a priori reason to exclude the possibility of states for which it is false. It can be objected that although the additivity of expectation values is not a law of thought, it is after all experimentally true. Yes, but what we are now investigating is precisely the hypothesis that the states presented to us by nature are in fact mixtures of component states which we cannot (for the present) prepare individually. The component states need only have such properties that ensembles of them have the statistical properties of observed states.

Bell chose to be harsh and dismissive about Neumann’s proof. He had his reasons for this. Bell demonstrated a hidden variable model of the measurements on a spin-1/2 particle, explicitly countering Neumann’s proof that such theories are impossible. The model had the \( \psi \)-function description of the quantum state polarized along direction \( \hat{a} \), supplemented by a vector hidden variable that took random directions \( \hat{r} \) in each realization of the measurement. A simple rule \[4, 7, 10\] prescribed the predictable (causal and deterministic) discrete eigen results of \( \pm 1 \), when measured along a direction \( \hat{n} \), given \( \hat{a}, \hat{n} \) and \( \hat{r} \). When averaged over the hidden variable, the correct expectation value of \( \hat{a} \cdot \hat{n} = \cos \theta \) was reproduced. This was presented as a convincing proof that the JvNP was flawed.

Then Bell went on to examine the case of the singlet composite of two spin-1/2 particles and showed that there is a significant difference for the prediction of spin correlations in a hidden variable theory and in QM. Again, it has been taken for granted that the hidden variable theories discussed by Bell are otherwise physically valid and deserved consideration as viable physical theories. Because of this, people took up the task of testing whether the Bell’s inequalities on correlations follow the prediction of the hidden variable theories or of QM. Finally, the results disfavoured the hidden variable theories. The interesting point here is that if the JvNP had been convincing as generally valid, none of those tests would have been considered worthwhile or supported as a genuine activity in physics. However, Bell’s criticism and the successful demonstration of his counter-example eclipsed the JvNP, and created the impression of a legitimate undecided territory that required empirical scrutiny for the final judgement.

I have already shown that the linear additivity postulate is generally valid in all theories that qualify as physical theories. Thus, Bell’s assertion that it “need not be satisfied by the values of physical quantities of hidden-variables theories, even though the grand average over all the dispersion-free ensembles will satisfy the postulate” is not correct. Note that nobody explored the physical reason why the linear additivity is a “quite remarkable feature of quantum mechanical states”. Contrary to Bell’s statements, it should have been deduced a priori from very general physical requirements. It is indeed a “law of thought in physics”. How can that be, and also consistent with the counter-example that Bell demonstrated?! It turns out that Bell’s hidden variable model is fundamentally flawed, because it blatantly contradicts what is empirically true! Further, we will see that the class of hidden variable theories that Bell explored and many others tested are theories that do not respect the fundamental conservation laws of physics.

VII. THE INVALIDITY OF BELL’S CRITICISM OF NEUMANN’S PROOF

It is easy to show the invalidity of Bell’s criticisms of the JvNP because Bell’s examples were similar to the ones that we already considered while discussing the strengthened proof of Neumann’s theorem. Bell was barking up the wrong tree. Bell cited an example
involving the measurements of the components of a quantized spin in different directions. The expectation value of a physical quantity is equal to the eigenvalue in a dispersion-free ensemble. For a spin-1/2 particle, the eigenvalues of $S_x$ and $S_y$ are $\pm 1$. In quantum mechanics, these observables are noncommuting. If we consider the bisecting direction, the quantum mechanical operator $S_{\theta_{\pm}}$ involves the combination $(S_x + S_y)/\sqrt{2}$. Again the eigenvalues are $\pm 1$. However, for a dispersion-free ensemble, it cannot be true that $\text{Exp}(S_x + S_y) = (\text{Exp}(S_x) + \text{Exp}(S_y))$ because $\text{Exp}(S_x + S_y) = \pm \sqrt{2}$, and $\text{Exp}(S_x)$ and $\text{Exp}(S_y)$ are $\pm 1$.

Bell’s scathing criticism was that Neumann made the “silly” assumption that the linear additivity of the expectation values should be obeyed by the hidden variable (dispersion-free) ensembles as well, just like the quantum mechanical ensemble. However, here Bell made a fundamental mistake. I repeat the proof discussed earlier, transcribed to Bell’s example.

The quantity $\text{Exp}(S_x + S_y)/\sqrt{2}$ is the average spin angular momentum in the direction that bisects $\hat{x}$ and $\hat{y}$. It is a physical quantity, averaged over the ensemble. In any other direction, the components of the angular momentum should obey the law of vector projection, expected in any physical theory, without regard to the commutation rules, discreteness of spin projections etc. That is, though the spin projection for a single particle is $\pm 1$ in any direction, the average over any statistical ensemble is the same as the average angular momentum, which can take any value between $-1$ and $+1$. Then, necessarily, it is the vector sum of the average angular momentum in the directions $\hat{x}$ and $\hat{y}$, $\text{Exp}(S_x) + \text{Exp}(S_y)$. Thus,

$$\text{Exp}(aS_x + bS_y) = a\text{Exp}(S_x) + b\text{Exp}(S_y) \quad (13)$$

This is a physical truth, to be obeyed in any valid theory. Ensembles that do not obey this cannot exist in the physical world! Then, Neumann’s proof is resurrected without a blemish: dispersion-free ensembles are impossible because all physical ensembles obey the linear additivity of the expectation values, as Neumann assumed.

We can go through the exercise of dividing the ensemble corresponding to the state into dispersion-free ensembles. Let the state be $|n\rangle$, polarized along $\hat{n} = \hat{x} + \hat{y}$. Then the values along $\hat{n}$ are all $+1$. The average spin angular momentum of the state is $+1(\hbar/2)$ along $\hat{n}$. The dispersion-free ensembles of possible values for $S_x$ and $S_y$ are $(+1,+1), (+1,-1), (-1,+1), (-1,-1)$. The average angular momentum of these subensembles are $\sqrt{2}\hat{n}, \sqrt{2}\hat{n}', -\sqrt{2}\hat{n}'$ and $-\sqrt{2}\hat{n}$, where $\hat{n}'$ is the unit vector perpendicular to $\hat{n}$. None of these is compatible with the conservation of the angular momentum and the value of the spin angular momentum of the parent ensemble, which is $+1\hat{n}$. Therefore, such ensembles are unphysical and cannot exist, exactly as Neumann asserted.

I now proceed to show that Bell’s famous counter-example of a hidden variable model of spin measurements fails in a way that is quite independent of whether or not it obeys the assumption in Neumann’s Proof.

**VIII. THE INCONSISTENCY OF BELL’S HIDDEN VARIABLE MODEL**

As already mentioned, Bell’s hidden variable counter-example, meant to disprove Neumann’s assertion, was for the measurements of the component of a spin polarized along a direction $\hat{a}$, measured along the direction $\hat{n}$. In QM, this is the state $|a+\rangle$ and the results of the measurements are $S_n = \pm 1$ in any direction $\hat{n}$, with the expectation value $\langle a + | \hat{n} | a+ \rangle = \hat{a} \cdot \hat{n} = \cos \theta$. Bell constructed the model by supplementing the state polarized along $\hat{a}$ with a vector hidden variable $\hat{r}$ that took random directions in each
Figure 2: The failure of Bell’s hidden variable counter-example to Neumann’s impossibility assertion. A) The initial polarization of the spin-1/2 particles is in the +x direction. A small magnetic field with a constant gradient acts as an ensemble separator for spin projections +1 and −1 in the z-direction. QM predicts the detection of equal proportions of +x and −x polarization in each subensemble! B) A symmetric Stern-Gerlach interferometer between the state preparation and detection. The spin rotator (SR) in only the lower path rotates the spin by $2\pi$ about the x-axis. With the SR absent, there is no change in the spin state. With the SR, all the particles are detected with the spin state reversed, in the −x-direction. A hidden variable theory does not reproduce this result.

realization of the measurement. A simple nonlinear rule prescribed the predictable eigen results of ±1, when measured along $\hat{n}$ [4, 7, 10]. The randomness in the observable values arose from the randomness in the hidden variable. The average over the directions of the hidden variable reproduced the correct expectation value $\hat{a} \cdot \hat{n} = \cos \theta$. We take this apparently successful model and analyze the difference between QM and the hidden variable theories. Consider the simple case when $\hat{a} = \hat{n} = \hat{x}$. Then the result for each particle is $S_x = +1$, and the expectation value is $\text{Exp}(S_x) = +1$. In the hidden variable theory, each particle in the total ensemble has $S_x = +1$, with $S_y$ and $S_z$ distributed as ±1 equally. In contrast, in QM, each particle is in the state $|x\rangle$ with $S_x = +1$, which is also an equal superposition of the states $|y\rangle$ and $|y\rangle$, and also of $|z\rangle$ and $|z\rangle$. It is on this crucial difference that Bell’s famous model collapses. This can be demonstrated in many ways.

As indicated in the figure 2(A), we introduce an ensemble separator that divides the particles into a subensemble that would give $S_z = +1$, if measured, and another with $S_z = -1$. A spatially limited magnetic field $B$ in the z-direction, with a constant gradient $B'$ will do this job. In the hidden variable model, the $S_x = +1$ parent ensemble is divided further into two subensembles of $S_z = +1$ and $S_z = -1$, in equal proportions. The magnetic field would have perturbed the x-component to precess by some amount, which depends on the exact value of the field and the duration for which the particle is in the field. But in QM, each particle is in the $|x\rangle$ state (or the $|z\rangle$ state) is an equal superposition of $|x\rangle$ and $|z\rangle$, irrespective of the field used in the ensemble separator. Clearly, Bell’s hidden variable model fails completely now to reproduce the results of the measurements in the subensembles, in the x-direction.

Another dramatic demonstration of the inadequacy of Bell’s model is when an active version of the Stern-Gerlach (S-G) closed interferometer is inserted between the state preparation and the measurement apparatus (figure 2(B). There is a spin-rotator in one of the arms (say, lower arm), which rotates the spin by $2\pi$ about the x-axis. Hence, the x-polarization is not affected by the spin rotator. Both the y and z components rotate through full 360°. When the spin rotator is not operative, the particles polarized along $\hat{x}$ enter the symmetric device and come out without any change in the polarization state. This is certainly so in QM, and let us assume that similar results are obtained in Bell’s
model as well (though it is not the case in a detailed scrutiny). When the spin rotator is operative, nothing should change in the hidden variable model: the $x$-polarization is not touched and the $z$ and $y$ polarizations are restored back after the full $2\pi$ rotation. But, the QM result is very different. The state that enters the S-G device is

$$|x+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle)$$

(14)

The state that exits after the spin rotator acts on the $|z-\rangle$ state is

$$|s\rangle = \frac{1}{\sqrt{2}}\left(|z+\rangle + e^{-2\pi/2}|z-\rangle\right) = \frac{1}{\sqrt{2}}\left(|z+\rangle - |z-\rangle\right) = |x-\rangle$$

(15)

Then all the particles should be measured as $S_x = -1$, with the expectation value $\text{Exp}(S_x) = -1$, though all the particles entered with $S_x = +1$! The failure of the hidden variable counter-example is total. So, Bell’s hidden variable scheme for single spin fails, contrary to the widely held general impression. The quantized spin component is only one of the features of a ‘state’ in quantum mechanics; the evolution of the relative phase of the states, which remains alien to such hidden variable theories, determines the measured values and their distribution.

All these results are explicit demonstrations of the fact that a hidden variable description of even simplest quantum mechanical situations is inconsistent. The dispersion of quantum mechanics is due to some form of wave-particle duality that cannot be reduced to the trivial statistical dispersion of deterministic hidden variables. Having proved the unphysical nature of Bell’s hidden variable model for single spin, I now proceed to show the drastic unphysical nature of the local hidden variable theories (LHVT) of two-particle states, for which Bell derived the Bell’s inequality and prompted experimental tests [12, 13]. In fact, those hidden variable theories had no physical sanity to start with and it was easy to rule them out as unworthy of any experimental test. Surely, people who spent enormous efforts in testing the LHVT would not have done so if they had realized that they were testing thoroughly unphysical theories that grossly violate the conservation of the angular momentum. In that sense, this result is gloomy, but its presentation here is essential.

Before we discuss the proof of the unphysical nature of the LHVT, I note that Bell’s exercise was motivated by the desire to study whether it was possible to cure the ‘incompleteness’ of QM, pointed out in the Einstein-Podolsky-Rosen (EPR) paper in 1935 [14], using hidden variables. Bell had obviously thought that the ‘incompleteness’ mentioned in the EPR paper was the same as the ‘incompleteness’ discussed by Neumann, the lack of deterministic causality, for which the prescription was a local hidden variable theory. He stated as much in the opening paragraph of his 1964 paper. However, the ‘incompleteness’ in the EPR paper is very different from the ‘causal incompleteness’ addressed by Neumann, Hermann, and Bell! Since this is a crucial point for physics, I prove this point before showing the inherently unphysical nature of the LHVT.

IX. THE CONFUSION OF TWO KINDS OF ‘INCOMPLETENESS’

We expect that physicists are careful in distinguishing two entirely different conceptual points for which the same term has been used, because physics has the advantage of the availability of precise mathematical representations for the statement of each important concept. However, in the case of quantum mechanics an unfortunate ‘uncertainty’ has crept in. I am referring to the indiscriminate use of the term ‘incompleteness’ to designate
two different notions in the discussion of QM! It should be obvious to most that Neumann’s use of that term in 1932, in the context of causality, couldn’t be in the same sense as Einstein’s use in 1935 in the context of physical reality; otherwise Einstein would not have bothered to define explicitly that term in the EPR paper and beyond. To be (linguistically) precise, at least at this late stage in the situation, let me quote Neumann in detail, in his first use in the context of indeterminism in QM:

If we want to explain the non-causal character of the connection between $\psi$ and the values of physical quantities following the pattern of classical mechanics, then this interpretation is clearly the proper one: In reality, $\psi$ does not determine the state exactly. In order to know this state absolutely, additional numerical data are necessary. That is, the system has other characteristics or coordinates in addition to $\psi$. If we were to know all of these we could then give the values of all physical quantities exactly and with certainty... It is customary to call these hypothetical additional coordinates “hidden parameters” or “hidden coordinates”, since they must play a hidden role, in addition to the $\psi$ which alone have been uncovered by investigation thus far. Explanations by means of hidden parameters have (in classical mechanics) reduced many apparently statistical relations to the causal foundations of mechanics. An example of this is the kinetic theory of gases.

Whether or not an explanation of this type, by means of hidden parameters, is possible for quantum mechanics is a much discussed question. The view that it will sometime be answered in the affirmative has at present some prominent representatives. If it were correct, it would brand the present form of the theory provisional, since then the description of states would be essentially incomplete.

What I just quoted from Neumann’s treatise is the ‘determinism incompleteness’ or ‘causality incompleteness’ of QM, which was very much the part of debates right from Heisenberg’s discussion of the uncertainty relations in 1927. In fact, Hermann’s criticism of Neumann’s proof was in the context of her investigations of the question of causality.

Now, I quote from the EPR paper [14], their definition of “completeness” of a theory:

...every element of the physical reality must have a counterpart in the physical theory We shall call this the condition of completeness.

Thus, the term “completeness” in the sense of EPR demands that there is a one-to-one correspondence between the physical states and the QM states represented by the $\psi$-functions. The EPR paper does not mention the terms ‘indeterminism’ or ‘uncertainty’, or even ‘causality’, anywhere. Nor does it mention hidden variables and such. In fact, the EPR claim, quoting from the paper is

Previously we proved that either (1) the quantum-mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality.

Note carefully that the second option is the denial of hidden variable theories, which is the one EPR opted for! Continuing with their proof (italics are mine),

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Starting then with the assumption that the wave function does give a complete description of the physical reality, we arrived at the conclusion that two physical quantities, with noncommuting operators, can have simultaneous reality. Thus the negation of (1) leads to the negation of the only other alternative (2). We are thus forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete.

It is stated clearly that the authors (EPR) do not accept the negation of the option (2); they were ‘forced’ to accept the option (1) because of their conviction in the impossibility of simultaneous values for noncommuting observables. Thus, there is not even the slightest ambiguity that the EPR assertion of ‘incompleteness’ followed from denying dispersion-free ensembles and simultaneous values of noncommuting observables – a situation that accepts the acausal nature of QM. The ‘EPR-incompleteness’ means that there is no one-to-one correspondence between the physical states and their representation by the ψ-functions of QM. Therefore, it is amply clear that the ‘representation incompleteness’ in the EPR paper, which demanded an entirely different theory, is not at all the same as Neumann’s ‘determinism incompleteness’, proposed to be cured with a hidden variable theory.

Yet, it is imperative that I explicitly prove that the ‘EPR-incompleteness’ is different from the ‘determinism incompleteness’ that motivated the hidden variable program. There are many reasons for this. One sufficient reason is the stark contrast and conflict between the statements quoted from the EPR paper and the opening sentence of Bell’s 1964 paper titled “On the Einstein-Podolsky-Rosen paradox”:

The paradox of Einstein, Podolsky and Rosen was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables.

Try as one might, no mention of what Bell says can be found in the EPR paper, that “QM should be supplemented by additional variables”, even as an implication! This point has been mildly raised by A. Fine earlier [15]. It is very plausible that Bell really did not consult the original EPR paper, but relied on a paper by D. Bohm and Y. Aharonov in 1957 that discussed the EPR situation in terms of the spin variables and nonlocal (superluminal) physical signals [16]. This is suggested by certain facts. Bohm and Aharonov made an error in the order of the authors in their citation to the original article, as Einstein-Rosen-Podolsky (ERP) instead of Einstein-Podolsky-Rosen (EPR). Then they called the EPR argument as the “ERP paradox”. Bell repeated this error of bibliographic citation and also the description of the EPR argument as a ‘paradox’, in his 1964 paper, which is a curious correlation. In any case, the proof that the ‘EPR-incompleteness’ is very different from the ‘causal incompleteness’ that Bell refers to is easy. Considering the importance of such a proof, let us go through the argument in the EPR scenario of two particles that had a prior interaction and then separated into two spatially distinct regions A and B.

Assume that the randomness in the observed values is only apparent and a more complete hidden variable description exists. As EPR and Bell did, we assume strict Einstein locality. Now consider the joint ψ-function for the two particles, Ψ_{12}. The hidden variable description consists of this function and a set of hidden variable realizations at

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1 In fact, the citation is unusually casual, without mentioning the initials of the three authors. The scrambled order can be traced back to Bohm’s textbook, “Quantum Theory”, published in 1951 [17].
the locations A and B. There is no causal indeterminism left in this hypothetical scenario because we have already admitted the hidden variables. Now I show that the ‘EPR-incompleteness’ is still present! A measurement at the location A results in a new localized \( \psi \)-function \( \psi_1 \) at A and another correlated function \( \psi_2 \) at the location B. Therefore, a measurement at A did change the \( \psi \)-function representation of the physical state at B, from \( \Psi_{12} \) to \( \psi_2 \), whether or not the hidden variables are operative. However, we have assumed strict locality, that the physically factual state at B cannot be changed by a measurement at A. Since \( \Psi_{12} \) and \( \psi_2 \) represent two different physical states, it is proved that there is no one-to-one correspondence between the \( \psi \)-function representations of the theory (supplemented by the hidden variables) and the factual physical states. We have proved the ‘EPR-incompleteness’ even for a hypothetical hidden variable deterministic extension of QM! This is a remarkable result that shows how far unrelated is the EPR argument from the hidden variable issue.

We are now in a quandary, because Bell and numerous others mixed up and equated the two unrelated notions of incompleteness, and proceeded with their assertions. Einstein warned about such a misunderstanding when he commented on M. Born’s (mis)understanding of the EPR argument [18]:

The whole thing is rather sloppily thought out, and for this I must respectfully clip your ear. I just want to explain what I mean when I say that we should try to hold on to physical reality... But whatever we regard as existing (real) should somehow be localised in time and space. That is, the real in part of space A should (in theory) somehow ‘exist’ independently of what is thought of as real in space B. When a system in physics extends over the parts of space A and B, then that which exists in B should somehow exist independently of that which exists in A. That which really exists in B should therefore not depend on what kind of measurement is carried out in part of space A; it should also be independent of whether or not any measurement at all is carried out in space A. If one adheres to this programme, one can hardly consider the quantum-theoretical description as a complete representation of the physically real.

Einstein was always referring to ‘reality’ in the sense of ‘objective physical existence’ and not in the sense of deterministic values for all physical observables. The notion of ‘local realism’ implied in the EPR paper is very different from the goal of ‘local determinism’ in a hidden variable theory. This has been confirmed by the ever-careful W. Pauli, in his revealing letters to M. Born in 1954 [18], when he assumed the role of an adjudicator in the friendly debate between Born and Einstein. Pauli wrote,

It seemed to me as if you had erected some dummy Einstein for yourself, which you then knocked down with great pomp. In particular, Einstein does not consider the concept of ‘determinism’ to be as fundamental as it is frequently held to be (as he told me emphatically many times)... In the same way, he disputes that he uses as criterion for the admissibility of a theory the question: ‘Is it rigorously deterministic?’ Einstein’s point of departure is ‘realistic’ rather than ‘deterministic’, which means that his philosophical prejudice is a different one... it seems to me misleading to bring the concept of determinism into the dispute with Einstein.

In a follow up letter he wrote to Born, “I have already tried in my last letter to explain Einstein’s point of view to you. It is exactly the same in Einstein’s printed work and
in what he said to me.” This shows that Pauli’s sharp and straight criticism applies to Bell and many others, who misread between the lines in ‘the printed work’, and failed to notice that Einstein’s point of departure was ‘realistic’ rather than ‘deterministic’.

Stated in the context of the often quoted Einstein-anecdote of the “reality of the moon when one is not looking”, the ‘EPR-completeness’ insists on the objective physical existence of the matter-moon, but does not demand anything about its trajectory that is more deterministic than what is specified with an uncertainty $\hbar$ of action. It is really an unfortunate turn of the course of physics that this kind of ‘EPR-incompleteness’ was widely confused as the ‘determinism incompleteness’ requiring hidden variables. The transference of a linguistic confusion to a lasting physical delusion with myriad irrational features like telepathic nonlocality is a serious issue. Einstein was vocal and explicit in his conviction that the $\psi$-function is already the representation of the statistical ensemble, and not of the single system. Then, it is obvious that supplementing $\psi$ with any number of hidden variables cannot restore determinism and causality. Einstein would have clipped our ears, and not so respectfully, for confusing his proof of ‘representation incompleteness’ as a demand for a quantum theory supplemented with the hidden variable dressing.

X. THE UNPHYSICAL NATURE OF THE LHVT

The spin correlation of a spin-singlet concerns the measurements on two (spin-1/2) particles with the total spin zero, and the possible results at locations A and B are $A_i = \pm 1$ and $B_i = \pm 1$. When both the measurements are in the same direction, the angle between the apparatus directions is zero, $\theta = 0$. Then, if $A_i = +1$, then $B_i = -1$ and vice versa; only then the total spin is zero, as demanded by the conservation of the angular momentum. To see what the expected correlation is for a general angle, dictated by just the conservation of the angular momentum, consider the measurements at A and B with the difference $\theta \neq 0$ in the angular settings. The correlation is the average of a large number of products $A_iB_i$.

Since the average of a sum of quantities is the same as the sum of the averages of two subsets that are half the size, we can make the subsets $A_i(+1)B_i$ and $A_i(-1)B_i$, where the first set has all $A_i = +1$ and the second has all $A_i = -1$. Then the correlation function is

$$C_e(\theta) = \frac{1}{N} \sum A_iB_i = \frac{1}{N/2} \sum A_i(+1)B_i + \frac{1}{N/2} \sum A_i(-1)B_i$$

This involves only a simple reordering of the pairs of data, keeping the pairwise data intact, which does not affect any average (we can do the same exercise with the B values as the anchor – the situation is symmetric). Now note that the first set has the average angular momentum at A as $+1$ and the second set as $-1$ (in units of $\hbar/2$). Then, the conservation of the angular momentum on the average dictates that the corresponding average value at B, at a relative angle $\theta$, should be the projection of the opposite angular momentum along the direction of B; in the first set it should be $-\cos \theta$ and in the second, $+\cos \theta$. Then the average for the whole set is just

$$C_e(\theta) = \frac{1}{2} [(+1 \times -\cos \theta) + (-1 \times +\cos \theta)] = -\cos \theta$$

This is predicted to be the experimentally observable singlet correlation, purely from the conservation of the angular momentum, independent of any theory. I have not even mentioned QM in this derivation. Sure, the prediction of QM agrees with this because it is a theory that respects the conservation laws for the average over the ensembles. A theory
that has a different prediction is then definitely not compatible with the conservation of the average angular momentum. Obviously, such theories are unphysical. Since the LHVT predict a very different correlation function that has a linear dependence on $\theta$ for small angles, the LHVT are in that unphysical class. They grossly violate the fundamental conservation laws. It is unlikely that anyone would have tested them, or even discussed them, if this result had been known. But this result can be seen as an extension of the physical factors that resurrect Neumann’s proof, which was always known.

XI. BOHM’S “HIDDEN VARIABLE” QUANTUM MECHANICS

When D. Bohm published his reinterpretation of the Schrödinger quantum mechanics [19], he preferred to call it a “hidden variable” construction. This was because the position variable in the $\psi$-function was treated as a hidden variable. However, it is very important to emphasize that the theory was based on just a reinterpretation of the Schrödinger equation of standard quantum mechanics, without any modification. All the peculiarities related to the $\psi$-function ($\psi(x,t) = A \exp(iS/h)$) were transferred to the nonlocal quantum potential, $(\nabla^2 A)/A$. Therefore, it was guaranteed to give the same statistical results as the Schrödinger quantum mechanics, while allowing notional trajectories, albeit in the simplest cases involving single particle space-time quantum mechanics. But, the theory has many unphysical features and inadequacies. As it is well known, the special status given to the position variable has not allowed any viable treatment of most of the standard quantum mechanical problems involving spin, atomic excitation and radiation, unstable particles, two-particle correlations etc. If a quantum theory cannot deal with atomic spectra and two-particle correlations, what relevance can it possibly have, except as a distracting example in a discussion of myriad features of the Schrödinger mechanics? In any case, classifying it as a hidden variable theory of the kind Neumann discarded with his proof is incorrect, because Neumann’s proof was for the impossibility of hidden variable theories that allowed dispersion-free ensembles. In Bohm’s theory, there are no physical ensembles that are dispersion-free in all possible observables, even with the ‘quantum potential’.

As a remark aside, the denial of clearance for Bohm to work in the Manhattan project and his eventual exile from the USA to Brazil in 1951 prevented any interaction between Neumann and Bohm. As for Neumann’s attitude towards Bohm’s claim of a hidden variable reinterpretation of QM, a letter from Bohm to Pauli in 1951 [20] has a relevant mention: “It appears that von Neumann has agreed that my interpretation is logically consistent and leads to all results of the usual interpretation. (This I am told by some people.) Also, he came to a talk of mine and did not raise any objections”. However, this much is clearly not an approval for a hidden variable theory of the kind Neumann analyzed, in which the characteristic feature is the existence of dispersion-free ensembles. It would have been ironical that the major work of a physicist who was denied access to the nuclear research in the USA apparently trashed the famous proof of the illustrious commissioner of the US Atomic Energy Commission! Though we do not have any record of what Neumann thought about Bohm’s hidden variable claim, we have Bohm’s view of Neumann’s proof, for example, in the same letter:

(Neumann’s) proof involves the demonstration that no “dispersionless” states can exist in the quantum theory, so that no single distribution of hidden parameters could possible determine the results of all experiments (including for example, the measurements of momentum and position). However, von
Neumann implicitly assumes that the hidden variables are only in the observed system and not in the measuring apparatus. On the other hand, in my interpretation, the hidden variables are in both the measuring apparatus and the observed system. Moreover, since different apparatus is needed to measure momentum and position, the actual results in each respective type of measurement are determined by different distributions of hidden parameters. Thus, von Neumann’s proof is irrelevant to my interpretation.

However, in his book “The Undivided Universe” [21], Bohm directly relies on Bell’s argument and treads tentatively in his interpretation of Neumann’s proof, almost cautioning us to the fact that the linear additivity is nevertheless true in quantum mechanics, and in the empirical data of measurements. Also, Bohm’s well-known text book on quantum mechanics [17], published first in 1951, has a section titled, “Proof that quantum theory is inconsistent with hidden variables”! The republished editions retain the section. There, mentioning the necessity of wave-particle duality as well as the analysis of the EPR argument, Bohm writes,

We conclude then that no theory of mechanically determined hidden variables can lead to all of the results of the quantum theory.

But, there is no mention of this in the sections discussing the proofs of the impossibility of hidden variables, in the later book, “The Undivided Universe”. One gets the impression that Bohm considered his theory as a ‘different kind of hidden variable theory’, while agreeing with the general view that the hidden variable theories cannot reproduce all the results of the quantum theory.

It is very easy to prove that Bohm’s theory with the spatial position as a hidden variable is no replacement for quantum mechanics. Consider a source of electron neutrinos, like the decay of neutrons. Every neutrino in this example is of the electron flavour, when it is emitted. We know that there is a finite probability for the detection of the neutrino as muon type. But, the initial position, which is Bohm’s hidden variable, or the guiding equation \( v = \nabla S/m \), is not relevant for the random detection of neutrinos in separate flavours. In other words, the much touted determinism in the theory is an illusion. So, Bohm’s theory is just a toy model still, and not useful as a general theory of physical phenomena. Those who are familiar with the damning criticisms with explicit examples, like that of Einstein’s in 1953 [22], would know how unphysical the theory can be. In Einstein’s example, a macroscopic particle in a box has the wavefunction \( \psi(x) = A \exp(iS/\hbar) \sim \exp(-ipx)+\exp(ipx) \) and velocity \( v = \nabla S/m = 0 \). So, the particle remains at rest, until observed, when it acquires the velocity \( \pm p/m \) within the duration of the observation, implying an arbitrary hidden acceleration. This example is seriously damaging for Bohm’s theory when reconsidered with photons. This kind of unavoidable physical inconsistency of Bohm’s theory is related to its nonlocal quantum potential. Einstein had perhaps thought through several aspects of such a theory because his own discarded attempt in 1927, at a causal interpretation of the Schrödinger equation, had many similarities to Bohm’s later attempt [23]. Further, as Bohm himself emphasized, the poly-dimensional spatial nonlocality, inherent in the \( 3n \)-dimensional \( n \)-particle \( \psi \)-function, is the hallmark of the theory. But, this unphysical aspect goes against the core tenet of relativity, as Bohm admitted in his paper on the ‘ERP-paradox’, “It must be admitted, however, that this quantum potential seems rather artificial in form, besides being subject to the criticism... that it implies instantaneous interactions between distant particles, so that it is not consistent with the theory of relativity.”
XII. WHAT PROHIBITS DISPERSION-FREE ENSEMBLES?

Since it has been shown in multiple ways, and by analyzing explicit examples, that hidden variable extensions to QM and dispersion-free ensembles cannot exist, I will now discuss the core physical reason for the general result that the dispersion in mutually incompatible observables is irreducible. Let an ensemble be defined through a $\psi$-function. All $\psi$-functions are of the form $\psi(x,t) = A \exp(iS/\hbar)$, where the function $S$ is the action. Usually, the QM dispersion is presented through the uncertainty principle involving two conjugate observables, which in turn finds a justification in the wave-particle duality. However, the real origin of the quantum uncertainty is the intrinsic uncertainty in the action, characterized by its scale $\hbar$; therefore the irreducible uncertainty should be correctly expressed as $\Delta S \geq \hbar$. Then, without any forced interpretation of ascribing an unrealistic wave nature to the matter particles themselves, we see that there would be the irreducible uncertainty relation between variables that define the action, which are by nature the conjugate variables of dynamics. The action is the product of a dynamical quantity (like the momentum $p$) and the corresponding coordinate (displacement $x$). Since the dispersion in the $\psi$-ensemble is in the action, $\Delta S \geq \hbar$, it cannot be split into definite values of both a dynamical variable and a conjugate coordinate. That is, a dynamical ensemble in mechanics is characterized by the action and its irreducible variance. Therefore, dispersion-free ensembles do not exist. In fact, this statement is more general, applicable to all mechanics, but I do not discuss the details here. A full description may be found in the reference [24].

The continuity equation for the classical probability density $\rho(x,t)$ of a statistical ensemble concerning the dynamics of a particle is

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot j \quad (17)$$

The probability current $j = \rho v = \rho \nabla S/m$ where $S(x,p,t)$ is the action and $m$ is the mass of the particle. Defining the positive quantity $\rho = \psi \psi^*$, where $\psi = A e^{i\phi}$, the continuity equation is

$$\frac{\partial (\psi \psi^*)}{\partial t} = \psi \frac{\partial \psi^*}{\partial t} + \psi^* \frac{\partial \psi}{\partial t} = -\nabla \cdot j = -\frac{1}{m} \nabla \cdot (\psi \psi^* v) \quad (18)$$

If the dummy angle $\phi$ is defined such that $\nabla \phi = v$, the equation can be written entirely in terms of $\psi, \psi^*$ and their spatial derivatives. We have $\nabla \psi = i\psi \nabla \phi + e^{i\phi} \nabla A$. Multiplying by $-i\psi^*$ gives the desired current term $\rho v$ and an additional term. Then

$$-i\psi^* \nabla \psi = \rho v - i\psi^* e^{i\phi} \nabla A = \rho v - iA \nabla A \quad (19)$$

Adding the complex conjugate eliminates the second term,

$$i (\psi \nabla \psi^* - \psi^* \nabla \psi) = 2 \rho v = 2j \quad (20)$$

Therefore, the continuity equation takes the form

$$\psi \frac{\partial \psi^*}{\partial t} + \psi^* \frac{\partial \psi}{\partial t} = \frac{i}{2} \nabla \cdot (\psi^* \nabla \psi - \psi \nabla \psi^*) = \frac{i}{2} \left( \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right) \quad (21)$$
But, $\partial \psi/\partial t = \left(\frac{i}{2}\right) \nabla^2 \psi$ is the free particle Schrödinger equation! With the identification of $\phi$ with the scaled action of dynamics, $S/\hbar$, the correspondence is complete. What is crucially important is that the $\psi$-function describes the statistical ensemble, and does not represent a matter-wave or the single particle dynamics, exactly as Einstein speculated.

The single particle dynamics is governed by another new dynamical equation for the $\zeta$-function of the action [24], defined as $\zeta(x,t) = \exp(iS/\hbar)$,

$$\frac{\partial \zeta(S)}{\partial t} = -\frac{i}{\hbar} H \zeta$$

This equation, universally applicable to the dynamics at all scales of masses and velocities, contains the standard Hamilton-Jacobi equation and a small additional term that reflects the intrinsic uncertainty of the action. Thus, the correct equation for the nonrelativistic dynamics of a single particle is a modified Hamilton’s equation [24],

$$\frac{\partial S}{\partial t} = -H + \frac{i\epsilon}{2m} \nabla^2 S$$

In the factual quantum dynamics, there are neither matter-waves, nor pilot waves with a quantum potential. The particle is separate from the ‘action-wave’ $\zeta(x,t) = \exp(iS/\hbar)$, and all the results of the single particle interference are reproduced without the collapse of the state. There is no measurement problem either, and full ontological consistency and locality are restored. The two-particle correlations follow from the local interference of the action-waves. The fundamental uncertainty $\Delta S \geq \hbar$ persists at all scales and there is no quantum-classical (micro-macro) divide. This theory verifiably solves all the foundational issues of QM in one stroke, while affirming that the irreducible stochastic aspect is universal, consistent with the validity of the principle of stationary action at all scales and situations of dynamics [24]. In a world in which the action principle is operative, the action-waves and the fundamental uncertainty $\Delta S \geq \hbar$ are inevitable; then, there cannot be any dispersion-free ensemble, even in the macroscopic world.

It is also clear now that misinterpreting the Schrödinger equation for the evolution of probability density as the dynamical equation for single particles is the reason for the unphysical features of Bohm’s theory as well. In effect, the probability density of the whole statistical ensemble was made to affect the dynamics of the single particle, through the fictitious and nonlocal quantum potential (without a source).

XIII. SUMMARY

In this paper I discussed several results that uphold the well-known assertion made by J. von Neumann on the impossibility of hidden variable descriptions of quantum mechanics, even at the level of single particle quantum mechanics. The central result is the proof that Neumann’s assumption of the linear additivity of the expectation values, $Exp(aR + bS) = aExp(R) + bExp(S)$, is indeed a relation of general validity, obeyed by all physical quantities. This resurrects Neumann’s famously contested proof of the impossibility of any hidden variable description of quantum mechanics. The much discussed criticisms of this assumption, by G. Hermann and J. S. Bell, were not valid. The second result is the failure of Bell’s counter-example to Neumann’s theorem, a hidden variable model of the measurements of the spin projection of a spin-1/2 particle. This was demonstrated in different ways. Another related result discussed is that the local hidden variable theories, for which Bell derived the inequalities, are unphysical theories because they are not compatible with the fundamental conservation laws. The discussion
of the correlation functions of such theories and the subsequent experimental tests of these intrinsically unphysical theories were prompted and supported by Bell’s critique of Neumann’s proof and his counter-example, both of which were incorrect. At this point, it was necessary to present a clarificatory proof that the ‘causality incompleteness’ that motivated the work on hidden variable theories is very different from the ‘EPR-incompleteness’ of QM, discussed by Einstein, Podolsky and Rosen in 1935. Finally, I identified the core reason for the impossibility of dispersion-free ensembles in the relation between the quantum dispersion and the action function. This result pre-empts any misconception that a deterministic description of mechanics will ever be possible at any scale, except as an approximation.

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