Control of synchronization in delay-coupled neural networks of heterogeneous nodes

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We study synchronization in delay-coupled neural networks of heterogeneous nodes. It is well known that heterogeneities in the nodes hinder synchronization when becoming too large. We show that an adaptive tuning of the overall coupling strength can be used to counteract the effect of the heterogeneity. Our adaptive controller is demonstrated on ring networks of FitzHugh-Nagumo systems which are paradigmatic for excitable dynamics but can also – depending on the system parameters – exhibit self-sustained periodic firing. We show that the adaptively tuned time-delayed coupling enables synchronization even if parameter heterogeneities are so large that excitable nodes coexist with oscillatory ones.

The control of nonlinear systems is a central topic in dynamical system theory, with a diverse range of applications. Adaptive control schemes have emerged as a new type of control methods that enable control in situations where parameters are unknown or drift in time. They allow for automatically tuning the parameters to appropriate values and are, therefore, of particular interest for experiments and technological applications. Besides drifting or unknown parameters, heterogeneous parameters are a further challenge in the application of control methods. Here, we develop adaptive control schemes which are able to deal with heterogeneities in the node parameters of a delay-coupled network where the parameters of each node are randomly chosen from a distribution. Our goal is, hereby, to achieve synchronization of all nodes. In the presence of heterogeneities perfect synchronization is unfeasible, while synchronization in a state where the node’s trajectories are close though not identical is reachable with our adaptive method if the parameter heterogeneities are not too large.

I. INTRODUCTION

The ability to control nonlinear dynamical systems has brought up a wide interdisciplinary area of research that has evolved rapidly in the past decades. Besides the control of isolated systems, control of dynamics in spatiotemporal systems and on networks has recently gained much interest. Adaptive control methods are of particular interest in situations where parameters drift or are uncertain and have been successfully applied in the control of network dynamics. Here, we show that they also can be used to counteract the effect of heterogeneous nodes in the synchronization of delay-coupled networks.

Synchronization in neural networks has gained a lot of attention lately since it is involved in processes as diverse as learning and visual perception on the one hand and the occurrence of Parkinson’s disease and epilepsy on the other hand. Control of synchronization has so far focused on networks of identical nodes. However, in realistic networks the nodes will always be characterized by some diversity meaning that the parameters of the different nodes are not identical but drawn from a distribution. It is well known that such heterogeneities in the nodes can hinder or prevent synchronization and that the coupling strength is a crucial parameter in this context. Here, we develop a method to adaptively control synchronization in networks of heterogeneous nodes.

Our method is based on the speed-gradient (SG) method, which was previously used in the control of delay-coupled networks, however, not in the presence of node heterogeneities. In order to apply the SG method, we suggest a goal function which characterizes the quality of synchronization. Based on this measure an adaptive controller is developed which ensures synchronization even if the parameter heterogeneities become such large that some nodes – if uncoupled – undergo a Hopf bifurcation and behave distinctly different from the other nodes in the network. We demonstrate our algorithm on the FitzHugh-Nagumo (FHN) system, a generic model for neural dynamics.

The paper is organized as follows: Section I is a recapitulation of the SG method, while Sec. II introduces the model. Section III discusses two delay-coupled FHN systems: The possible bifurcation scenarios are investigated and the adaptive control algorithm is developed. In Sec. IV the method is generalized to larger ring networks. Finally, we conclude with Sec. V.
II. SPEED-GRADIENT METHOD

In this section, we briefly review the speed-gradient (SG) method\textsuperscript{23}. Consider a general nonlinear dynamical system

\[ \dot{x} = F(x, g, t) \]

with state vector \( x \in \mathbb{C}^n \), input (control) variables \( g \in \mathbb{C}^n \), and nonlinear function \( F \). Define a control goal

\[ Q(x(t), t) \leq \Delta, \]

for \( t \geq t^* \), where \( Q(x(t), t) \geq 0 \) is a smooth scalar goal function and \( \Delta \) is the desired level of precision. For example, if we want to force the trajectory of system \( \dot{x} = F(x, g, t) \) to a desired location \( x^\ast \), we use a goal function in the form \( Q(x(t), t) = \|x(t) - x^\ast(t)\|^2 \).

In order to design a control algorithm, the scalar function \( Q = \omega(x, g, t) \) is calculated, that is, the speed (rate) at which \( Q(x(t), t) \) is changing along the trajectories of Eq. \( \dot{x} = F(x, g, t) \):

\[ \omega(x, g, t) = \frac{\partial Q(x, t)}{\partial t} + [\nabla_x Q(x, t)]^\top F(x, g, t). \]

Then the gradient of \( \omega(x, g, t) \) with respect to the input variables is evaluated as

\[ \nabla_g \omega(x, g, t) = \nabla_g [\nabla_x Q(x, t)]^\top F(x, g, t). \]

Finally, we obtain the control function \( g \) from

\[ g(t) = g^0 - \psi(x, g, t), \]

where the vector function \( \psi(x, g, t) = \gamma \nabla_g \omega(x, g, t) \) with some adaptation gain \( \gamma > 0 \) and \( g^0 = \text{const} \) is an initial (reference) control value (often \( g^0 = 0 \) is assumed). The algorithm \textsuperscript{23} is called speed-gradient (SG) algorithm in finite form since it suggests to change \( g \) proportionally to the gradient of the speed of changing \( Q \). For the speed-gradient (SG) in its differential form see Ref. \textsuperscript{23}.

Several analytic conditions exist guaranteeing that the control goal \textsuperscript{23} can be achieved in system \textsuperscript{1} and \textsuperscript{3}. The main condition is an existence of a constant value of the parameter \( g^\ast \), ensuring attainability of the goal in the system \( \dot{x} = F(x, g^\ast, t) \). Details can be found in the control-related literature\textsuperscript{23,24,25}.

The idea of this algorithm is the following: The term \(-\nabla_g \omega(x, g, t)\) points to the direction in which the value of \( Q \) decreases with the highest speed. Therefore, if one forces the control signal to "follow" this direction, the value of \( Q \) will decrease and finally be negative. When \( Q < 0 \), then \( Q \) will decrease and, eventually, will tend to zero.

\[ \dot{x} = F(x, g^\ast, t) \]

for excitable dynamics close to a Hopf bifurcation\textsuperscript{23}, which is not only characteristic for neurons but also occurs in the context of other systems ranging from electronic circuit\textsuperscript{23,24} to cardiovascular tissues and the climate system\textsuperscript{25,28}. Each node of the network is described as follows:

\[ \varepsilon u_i = u_i - \frac{u_i^3}{3} - v_i + C \sum_{j=1}^{N} G_{ij} [u_j(t - \tau) - u_i(t)], \]

\[ v_i = u_i + a_i, \quad i = 1, \ldots, N, \]

where \( u_i \) and \( v_i \) denote the activator and inhibitor variable of the nodes \( i = 1, \ldots, N \), respectively. \( \tau \) is the delay, i.e., the time the signal needs to propagate between nodes \( i \) and \( j \) (here we will use \( \tau = 1.5 \)). \( \varepsilon \) is a time-scale parameter and typically small (here we will use \( \varepsilon = 0.1 \)), i.e., \( u_i \) is a fast variable, while \( v_i \) changes slowly. The coupling matrix \( G = \{G_{ij}\} \) defines which nodes are connected to each other. We construct the matrix \( G \) by setting the entry \( G_{ij} \) equal to 1 (or 0) if the \( j \)th node couples (or does not couple) into the \( i \)th node. After repeating this procedure for all entries \( G_{ij} \), we normalize each row to unity. The overall coupling strength is given by \( C \).

In the uncoupled system \( (C = 0) \), a\textsubscript{i} is a threshold parameter: For \( a_i > 1 \) the \( i \)th node of the system is excitable, while for \( a_i < 1 \) it exhibits self-sustained periodic firing. This is due to a supercritical Hopf bifurcation at \( a_i = 1 \) with a locally stable equilibrium point for \( a_i > 1 \) and a stable limit cycle for \( a_i < 1 \). In previous publications, networks of homogeneous FHN systems were considered, i.e., \( a_1 = a_2 = \ldots = a_N = \text{const} \). In particular, it was shown that for excitable systems, i.e., \( a > 1 \) and coupling matrices with positive entries zero-lag synchronization is always a stable solution independently of the coupling strength and delay time (as long as both are large enough to induce any spiking at all).

Here, we investigate the case of heterogeneous nodes. In this case, perfect synchronization, i.e., \( (u_1, v_1) = \ldots = (u_N, v_N) = (u_s, v_s) \), is no longer a solution of Eq. \textsuperscript{6} which can easily be seen by plugging \( (u_1, v_1) = \ldots = (u_N, v_N) = (u_s, v_s) \) into Eq. \textsuperscript{6}. The node dynamics is then described by

\[ \varepsilon \dot{u}_i = u_i - \frac{u_i^3}{3} - v_i + C \sum_{j=1}^{N} G_{ij} [u_j(t - \tau) - u_i(t)], \]

\[ \dot{v}_i = u_i + a_i, \quad i = 1, \ldots, N, \]

which is obviously not independent of \( i \). This means that a perfectly synchronous solution does not exist in system \textsuperscript{6} because the prerequisite for the existence of such a solution is that each node receives the same input if all nodes are in synchrony. However, solutions close to the synchronous solution might exist where the nodes spike at the same (or almost the same) time but with slightly different amplitudes. As we show, these solutions can be reached and stabilized by an adaptive tuning of the coupling strength.

III. MODEL EQUATION

The local dynamics of each node in the network is modeled by the FitzHugh-Nagumo (FHN) differential equation\textsuperscript{23,28}. The FHN model is paradigmatic...
IV. TWO DELAY-COUPLED FITZHUGH-NAGUMO SYSTEMS

This Section studies the most basic network motif consisting of two coupled systems without self-feedback. Before deriving the adaptive controller, we perform a linear stability analysis of the equilibrium point to get insight in the possible bifurcations.

A. Linear stability of the equilibrium point

The linear stability analysis follows the approach suggested in Ref. [31]. Consider two coupled FHN-systems with heterogeneous threshold parameters and bidirectional coupling

\[ \begin{align*}
\varepsilon u_1 &= u_1 - \frac{u_1^3}{3} - v_1 + C[u_2(t - \tau) - u_1(t)], \\
v_1 &= u_1 + a_1,
\end{align*} \]

\[ \begin{align*}
\varepsilon u_2 &= u_2 - \frac{u_2^3}{3} - v_2 + C[u_1(t - \tau) - u_2(t)], \\
v_2 &= u_2 + a_2.
\end{align*} \]

Equation (8)

The unique equilibrium point of the system (8) is given by \( \mathbf{x}^* = (u_1^*, v_1^*, u_2^*, v_2^*)^T \), where \( u_1^* = -a_1, v_1^* = -a_2, u_2^* = -a_2 + a_1^2/8 + C(a_2 - a_1) \) and \( v_2^* = -a_2 + a_1^2/8 + C(a_2 - a_1) \). Linearizing Eq. (8) around the equilibrium point \( \mathbf{x}^* \) by setting \( \mathbf{x}(t) = [u_1(t), v_1(t), u_2(t), v_2(t)]^T = \mathbf{x}^* + \delta \mathbf{x}(t) \), we obtain

\[ \delta \dot{\mathbf{x}} = \frac{1}{\varepsilon} \begin{pmatrix}
\xi_1 & -1 & 0 & 0 \\
0 & \varepsilon & 0 & 0 \\
0 & 0 & \xi_2 & -1 \\
0 & 0 & \varepsilon & 0
\end{pmatrix} \delta \mathbf{x}(t) + \frac{1}{\varepsilon} \begin{pmatrix}
0 & 0 & C & 0 \\
0 & 0 & 0 & 0 \\
C & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \delta \mathbf{x}(t - \tau), \]

which can be rearranged to

\[ (\xi_1^2 \xi_2^2 - C^4)\omega^4 + (\xi_1^2 + \xi_2^2)\omega^2 + 1 = 0. \]

Equation (14) is biquadratic, i.e. quadratic in \( \omega^2 \). Therefore, we can use Vieta’s formulas to analyze whether it has non-negative roots. According to Vieta the following holds

\[ z_1 + z_2 = -\frac{\xi_1^2 + \xi_2^2}{\xi_1^2 \xi_2^2 - C^4}, \quad z_1 z_2 = \frac{1}{\xi_1^2 \xi_2^2 - C^4}, \]

where \( z_1 \) and \( z_2 \) are roots of the quadratic equation (14). For \( \xi_1^2 \xi_2^2 > C^4 \), \( z_1 < 0 \) and \( z_2 < 0 \) follows meaning that Eq. (14) has no real-valued solution \( \omega \), and, thus, no Hopf bifurcation will take place. Taking the square root of this inequality and resubstituting \( \xi_i = 1 - a_i^2 - C \) yields

\[ |(1 - C - a_1^2)(1 - C - a_2^2)| > C^2. \]

B. Adaptive control of two coupled FHN-systems

We now want to apply the SG method to system (8) with the goal to synchronize the two heterogeneous nodes. As discussed above perfect synchronization in the form \( (u_1, v_1) = (u_2, v_2) \) is not attainable in this case but the two systems will follow slightly different trajectories in the synchronized case. We, therefore, use as a goal function

\[ Q(\mathbf{x}(t), t) = \frac{1}{2} (u_1(t) - u_2(t) + a_1 - a_2)^2. \]
The choice [17] ensures that the system follows trajectories for which
\[
\begin{align*}
  u_1(t) - u_2(t) &\approx -a_1 + a_2, \\
  v_1(t) - v_2(t) &\approx c
\end{align*}
\] (18a) (18b)
holds for \( t \geq t^* \), where \( c \) is a constant. Approximation (18a) directly follows from the chosen goal function [17]; approximation (18b) is obtained by plugging (18a) into Eq. (8). Thus, the goal function [17] yields synchronization with a shift in the values of the inhibitors and activators of the two nodes.

From Eq. (3) with \( g = C \), system (5), goal function (17), and \( \psi(x, C, t) = \gamma \nabla_C \omega(x, C, t) \) an adaptive law is straightforwardly derived:
\[
C(t) = C_0 + \frac{\gamma}{\varepsilon} (u_1(t) - u_2(t) + a_1 - a_2) 
\times (u_1(t) - u_2(t) + u_1(t - \tau) - u_2(t - \tau)),
\] (19)
where \( \gamma > 0 \) is the gain and \( C_0 \) is the initial value of the control parameter. The appropriate value of \( \gamma \) has to be determined by numerical simulations. Note that a similar approach has been used to tune the coupling strength in a network of Rössler systems in Ref. [18].

Close to the control goal, \( u_1(t) - u_2(t) \approx u_1(t - \tau) - u_2(t - \tau) \approx a_1 - a_2 \) holds. We, therefore, can simplify the adaptation law by substituting the delayed variables by their non-delayed versions and obtain
\[
C(t) = C_0 + \frac{\gamma}{\varepsilon} (u_1(t) - u_2(t) + a_1 - a_2)(u_1(t) - u_2(t)).
\] (20)

For constant coupling strength, i.e., \( \gamma = 0 \), the two coupled FHN systems do not synchronize in-phase, but approach an anti-phase synchronized state: Figure 2 shows in panels (a) and (b) the time series of the activators and the inhibitors, respectively, and in panel (d) the phase portrait. Though the first node is in the excitatory regime \( (a_1 = 1.1 > 1) \) both nodes oscillate due to the nonzero coupling strength \( C \). However, they do not synchronize as can clearly be seen in panel (c) which depicts the difference \( u_1 - u_2 \) between the activator values. Instead, they phase lock with a phase shift of approximately \( \pi \) which corresponds to an anti-phase synchronized state.

We now adapt the coupling strength according to Eq. (20) in order to synchronize the two systems where the result is shown in Fig. 3. After a transient time of approximately 15 units of time the two systems reach the desired synchronized state (see the time series of the activator in Fig. 3(a) and the difference between its values Fig. 3(c)). Thus, the control is successful.

If the gain \( \gamma \) is chosen too low the control fails: Figure 3 depicts the results of the adaptive control according to Eq. (20) for \( \gamma = 0.05 \). Clearly, the control does not succeed in synchronizing the two systems (see the time series of the activators in Fig. 3(a) and the difference between their values in (c)).

The adaptive controller (19) ensures synchronization of the activators with a shift given by \( a_2 - a_1 \) (see Eq. (18a)).

Furthermore, there is a finite, constant shift in the inhibitor values (see Eq. (18b)). The shift in the inhibitor values can be reduced to a value close to zero if we control the coupling strength of each node separately. The two FHN systems are then described by
\[
\begin{align*}
  \varepsilon u_i &= u_i - \frac{u_i^3}{3} - v_i + C_i(t)[u_{(i+1) \mod 2}(t - \tau) - u_i(t)], \\
  v_i &= u_i + a_i, \quad i = 1, 2
\end{align*}
\] (21)
where \( C_i(t) \) describes the strength of the coupling to node \( i \). From Eq. (5) with \( g = (C_1, C_2) \), system (21), goal function (17), and \( \psi(x, g, t) = \gamma \nabla_g \omega(x, g, t) \) an adaptive law is straightforwardly derived:
\[
C_i(t) = C_i^0 + \frac{\gamma}{\varepsilon} (u_i(t) - u_{(i+1) \mod 2}(t) + a_i - a_{(i+1) \mod 2}) 
\times (u_i(t) - u_{(i+1) \mod 2}(t - \tau)), \quad i = 1, 2
\] (22)
where \( \gamma > 0 \) is the gain and \( C_i^0 \) is the initial value of control parameter.

The results of the adaptation according to Eq. (22) are shown in Fig. 4. The two systems reach the desired synchronized state (see the time series of the activators in Fig. 4(a) and the difference between their values in Fig. 4(c)). Moreover, the difference between inhibitor values is close to zero (see the time series of the inhibitors in Fig. 4(b) and the difference between their values in Fig. 4(d)). Thus, the control is successful.
FIG. 3. Adaptive control of two coupled FitzHugh-Nagumo systems (Eq. (8)). (a) and (b): time series of the activator and the inhibitor, respectively; (c) and (d): differences $u_1 - u_2$ and $v_1 - v_2$ between the activator and the inhibitor values, respectively; (e): phase space, and (f): time series of the coupling strength adapted according to Eq. (20). Parameters: $\gamma = 3$, $C_0 = 0$. Other parameters and initial conditions as in Fig 2.

V. ADAPTIVE SYNCHRONIZATION IN RING NETWORKS

We now want to apply our method to larger networks. To this end, we consider a ring network of $N$ nodes where the coupling matrix $G$ has the following form

$$
G = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}.
$$  \hspace{1cm} (23)

Our approach aims to synchronize two nodes of the ring; in this case the other nodes will follow these two nodes and synchronize as well. The same idea has been used in the control of wave motion in a chain of pendula.\textsuperscript{133} However, there is a limitation to this approach: The nodes with the highest threshold and the lowest one have to be neighbors.

FIG. 4. As in Fig. 3 but with smaller adaptation gain $\gamma = 0.025$. (a) and (b): time series of the activator and the inhibitor, respectively; (c) and (d): differences $u_1 - u_2$ and $v_1 - v_2$ between the activator and the inhibitor values, respectively.

FIG. 5. Adaptive control of two coupled FitzHugh-Nagumo systems (Eq. (21)) where the coupling strength $C_i$ of each node is adapted separately according to Eq. (22). (a) and (b): time series of the activator and the inhibitor, respectively; (c) and (d): differences $u_1 - u_2$ and $v_1 - v_2$ between the activator and the inhibitor values, respectively. Parameters: $\gamma = 2$, $C_0 = 0$, $i = 1, 2$. Other parameters and initial conditions as in Fig 2.

Let us assume that $k$ is the node with the highest threshold and $l$ is the one with the lowest threshold, i.e.,

$$
a_k = \max_{i=1,\ldots,N} a_i, \quad a_l = \min_{i=1,\ldots,N} a_i,
$$

and that $a_k$ and $a_l$ are neighbors, i.e., $k = (l + 1) \mod N$. 
FIG. 6. Adaptive control of synchronization of a ring of ten FitzHugh-Nagumo systems (according to Eq. (6) with coupling matrix (23)). (a) and (b): time series of the activator and the inhibitor, respectively; (c) and (d): differences between the activator and the inhibitor values, respectively; (e): phase space, (f): time series of the coupling strength adapted according to Eq. (24); (g) and (h): time series of the activators and the inhibitors of all nodes, respectively.

Parameters: $N = 10$, $\varepsilon = 0.1$, $\tau = 1.5$, $\gamma = 100$, $C_0 = 0$. Threshold parameters $a_i$ are chosen randomly from $[0.8, 1.1]$, while the highest threshold, here $a_1$, equals 1.1 and the lowest threshold, here $a_2$, equals 0.8. Initial conditions: $u_i(t) = 0$, $v_i(t) = 0$, $i = 1, \ldots, N$, for $t \in [-\tau, 0]$.

FIG. 7. As in Fig. 6 but with a bounded coupling strength $|C| \leq 5$. (a) and (b): time series of the activators and the inhibitors of all nodes, respectively.

FIG. 8. Adaptive control of synchronization of ten FitzHugh-Nagumo systems where the coupling strength of each node is adapted separately according to Eq. (29). (a) and (b): time series of the activator and the inhibitor of all nodes, respectively. Parameters: $C_{0i} = 0$, $i = 1, \ldots, N$, $\gamma = 1$. Other parameters and initial conditions as in Fig. 6.

or $k = (l - 1) \mod N$.

We now use the adaptation law (20) to synchronize these two nodes. With nodes $k$ and $l$ instead of nodes 1 and 2, Eq. (20) reads

$$C(t) = C_0 + \frac{\gamma}{\varepsilon} (u_k(t) - u_l(t) + a_k - a_l)(u_k(t) - u_l(t)), \quad (24)$$

where $\gamma$ is the gain and $C_0$ is an initial value of the coupling strength. As before, we aim to achieve the control
where we derive the following adaption law

\[
\begin{align*}
  u_i(t) - u_j(t) &\sim -a_i + a_j, \quad (25a) \\
v_i(t) - v_j(t) &\sim c_i, \quad (25b)
\end{align*}
\]

for \( t \geq t^* \), where \( c_i \) is constant and \( i, j = 1, \ldots, N \).

Figure 6 presents the results of a simulation of ten FitzHugh-Nagumo systems coupled in a ring where the adaption law \((24)\) is applied with \( a_1 = a_k \) being the node with the highest threshold, and \( a_2 = a_l \) being the node with the lowest threshold. In Fig. 6(a) and 6(c), it is shown that activators \( u_1 \) and \( u_2 \) synchronize. Figure 6(e) shows the phase space of the two nodes. In Fig. 6(g) it can be seen that not only node one and two are synchronized but that all nodes follow these two nodes and synchronize after a transient time. Thus, the control goal is achieved.

So far we considered the case that the nodes with the highest and lowest coupling strength are neighbors. If this is not fulfilled, the adaptation of the overall coupling strength does not yield synchronization. However, if we control the coupling strength of each node separately the control goal can be reached. The ring network is then described by

\[
\begin{align*}
  \varepsilon u_i &= u_i - \frac{u_i^3}{3} - v_i + C_i(t)[u_{(i+1) \mod N}(t-\tau) - u_i(t)], \\
v_i &= u_i + a_i, \quad i = 1, \ldots, N,
\end{align*}
\]

where \( C_i(t) \) describes the strength of the coupling to node \( i \). As in the case of two nodes, from Eq. 5 with \( g = (C_1, \ldots, C_N) \), goal function

\[
Q(x(t), t) = \frac{1}{2} \sum_{i=1}^{N} \left( u_i(t) - u_{(i+1) \mod N}(t) \right)^2 + \left( a_i - a_{(i+1) \mod N}(t) \right)^2,
\]

and

\[
\psi(x, g, t) = \gamma \nabla g \omega(x, g, t),
\]

we derive the following adaption law

\[
C_i(t) = C_i^0 + \frac{\gamma}{\varepsilon} (u_i(t) - u_{(i+1) \mod N}(t - \tau)) \\
\times \left[ 2u_i(t) - u_{(i-1) \mod N}(t) - u_{(i+1) \mod N}(t) + 2a_i - a_{(i-1) \mod N}(t) - a_{(i+1) \mod N}(t) \right], \quad i = 1, \ldots, N,
\]

where \( \gamma \) is the gain and \( C_i^0 \) is the initial value of \( C_i \).

VI. CONCLUSION

We have proposed a novel adaptive method for controlling synchrony in heterogeneous networks. It is well known that networks with heterogeneous nodes are much less likely to synchronize than networks of identical nodes. Furthermore, synchrony will take place in a state where the trajectories of the different nodes are not identical but small deviations can be observed. We have suggested a goal function to characterize this type of synchrony. Based on this goal function and the speed-gradient (SG) method, we have derived an adaptive controller which tunes the overall coupling strength such that synchrony is stable despite the node heterogeneities.

We have demonstrated our method on networks of FitzHugh-Nagumo systems, a neural model which is considered to be generic for excitable systems close to a Hopf bifurcation. Before applying the adaptive control, we have studied the simple motif of two delay-coupled, heterogeneous nodes and have given analytic conditions for the occurrence of the Hopf bifurcations. We have then applied our adaptive method and discussed its dependencies on the node and control parameters. It has been shown that our method enables synchronization even if the node parameters are chosen such diverse that one of the systems would exhibit self-sustained oscillations without coupling, while the other one would remain in a stable equilibrium point, i.e., one of the uncoupled systems is above, and the other is below the Hopf bifurcation. Furthermore, we have generalized our method to larger networks and applied it to ring networks. As an alternative and complement to adapting the overall coupling strength, we have suggested adapting the coupling strength of each node separately. This allows for adaptively controlling the network in situations where the node with the highest threshold is not a direct neighbor of the node with the lowest threshold, in contrast to the restriction imposed by tuning only the overall coupling strength.

Given the paradigmatic nature of the FitzHugh-Nagumo system, we expect our method to be applicable in a wide range of excitable systems. Furthermore, the application of the SG method to the control of networks with heterogeneous nodes suggests that other adaptive controllers that are based on the SG method (see, for example, Refs. 7 8 18 and 34) are also robust towards heterogeneities.
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