LAGRANGIAN NON-INTERSECTIONS

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1. INTRODUCTION AND MAIN RESULTS

The present paper is devoted to Lagrangian submanifolds of symplectic manifolds and their intersection patterns.

One of the cornerstones of symplectic topology is the rigidity of symplectic structures reflected in the behavior of Lagrangian submanifolds and their intersections. The first non-trivial restrictions on Lagrangian embeddings go back to Gromov’s work of 1985 [21]. One of the many results of that paper is the following fundamental theorem: In \( \mathbb{R}^{2n} \) there are no closed exact Lagrangian submanifolds, in particular there are no closed Lagrangian submanifolds \( L \subset \mathbb{R}^{2n} \) with \( H^1(L; \mathbb{R}) = 0 \).

Since then, Gromov’s techniques have been extended and, in combination with other methods, have led to more restrictions on the topology of Lagrangian submanifolds, most of the results being for Lagrangians submanifolds of \( \mathbb{R}^{2n} \) and of cotangent bundles (see e.g. [2, 10, 16, 24, 30, 32, 33, 34, 35, 39, 40] for a partial list of older and newer results).

Only relatively recently first results on the topology of Lagrangians in closed manifolds have been obtained by Seidel [37] and later on by Biran and Cieliebak [9]. Note that when studying Lagrangians in an arbitrary manifold one encounters all Lagrangian submanifolds of \( \mathbb{R}^{2n} \). This is due to the fact that every symplectic manifold \( M^{2n} \) is locally modeled on \( \mathbb{R}^{2n} \), hence every Lagrangian submanifold \( L \subset \mathbb{R}^{2n} \) can also be Lagrangianly embedded into \( M^{2n} \). Thus, Lagrangian embeddings into \( \mathbb{R}^{2n} \) should, in a sense, be regarded as the local case. However, our understanding of Lagrangian submanifolds of \( \mathbb{R}^{2n} \) is still quite limited, in particular also that of “local” Lagrangian submanifolds in any symplectic manifold \( M^{2n} \). (Thus, in this case, “local” turns out to be difficult.)

In this paper we concentrate on “global” Lagrangian submanifolds. One way to “mod out” the local Lagrangians is to assume for example that the first homology of the Lagrangians is either zero or torsion. In view of the preceding theorem of Gromov such Lagrangian submanifolds must be global in the sense that they cannot lie entirely in a Darboux chart.

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One of the phenomena arising from our results below is that in certain symplectic manifolds the topology of Lagrangians with small first homology is extremely restricted. It turns out that in some cases (e.g. $M = \mathbb{C}P^n$) certain assumptions on $H_1(L; \mathbb{Z})$ of a Lagrangian $L$ completely determine the entire homology of $L$. This phenomenon is illustrated in Theorems A-C of Section 1.1 below. Note that very recently examples of this phenomenon have been discovered also for cotangent bundles of spheres by Buhovski [10] and by Seidel [35].

The second phenomenon presented in this paper belongs to the framework of Lagrangian intersections. Our results show that certain symplectic manifolds $M$ contain some kind of “Lagrangian core” $\Lambda \subset M$ which dominates intersections, in the sense that many global Lagrangians $L \subset M$ must intersect $\Lambda$, the intersection points being irremovable by symplectic diffeomorphisms. Results in this direction are presented in Section 1.3.

1.1. Homological uniqueness of Lagrangian submanifolds. Here and in what follows all Lagrangian submanifolds are assumed to be compact and without boundary, unless otherwise explicitly stated.

Lagrangian submanifolds of $\mathbb{C}P^n$. Let $\mathbb{C}P^n$ be the complex projective space, endowed with its standard Kähler structure. It is well known that $\mathbb{C}P^n$ has no Lagrangian submanifolds $L$ with $H_1(L; \mathbb{Z}) = 0$ (see [37], see also [9]). Note however that there do exist Lagrangians $L \subset \mathbb{C}P^n$ with $H_1(L; \mathbb{Z})$ torsion. For example, the real projective space $\mathbb{R}P^n \approx \{[z_0 : \ldots : z_n] \in \mathbb{C}P^n | z_i \in \mathbb{R} \forall i\} \subset \mathbb{C}P^n$ is such a Lagrangian submanifold (for $n \geq 2$). In fact Seidel proved in [37] that every Lagrangian submanifold $L \subset \mathbb{C}P^n$ with $H^1(L; \mathbb{Z}_{2n+2}) = \mathbb{Z}_2$ must satisfy $H^*(L; \mathbb{Z}_2) \cong H^*(\mathbb{R}P^n; \mathbb{Z}_2)$ as graded vector spaces. Below we shall prove a stronger statement which gives information also on the cohomology ring of $L$. Henceforth we say that an abelian group $H$ is $q$-torsion if for every $\alpha \in H$ we have $q\alpha = 0$. (This, by our conventions, includes the case when $H$ is the trivial group.) Our first result is:

**Theorem A.** Let $L \subset \mathbb{C}P^n$ be a Lagrangian submanifold with $H_1(L; \mathbb{Z})$ 2-torsion. Then:

1. There exists an isomorphism of graded vector spaces $H^*(L; \mathbb{Z}_2) \cong H^*(\mathbb{R}P^n; \mathbb{Z}_2)$. Moreover, if $a \in H^2(\mathbb{C}P^n; \mathbb{Z}_2)$ is the generator then $a|_L \in H^2(L; \mathbb{Z}_2)$ generates the subalgebra $H^{\text{even}}(L; \mathbb{Z}_2)$, and $H^{\text{odd}}(L; \mathbb{Z}_2) = H^1(L; \mathbb{Z}_2) \cup H^{\text{even}}(L; \mathbb{Z}_2)$.

2. When $n$ is even, the isomorphism $H^*(L; \mathbb{Z}_2) \cong H^*(\mathbb{R}P^n; \mathbb{Z}_2)$ is in fact an isomorphism of graded algebras.

Other than $\mathbb{R}P^n \subset \mathbb{C}P^{n+1}$ we are not aware of any example of Lagrangians $L$ with 2-torsion $H_1(L; \mathbb{Z})$. However, Chiang [11] constructed an example of a Lagrangian in
Let $L \subset \mathbb{C}P^3$ with $H_i(L; \mathbb{Z}_2) = \mathbb{Z}_2$ for every $i$. This Lagrangian is a quotient of $\mathbb{R}P^3$ by the dihedral group $D_3$. It satisfies $H_1(L; \mathbb{Z}) = \mathbb{Z}_4$. See [11] for the details as well as other interesting examples of Lagrangian submanifolds of complex projective spaces of various dimensions.

**Lagrangian submanifolds of $\mathbb{C}P^n \times X$.** It has been proved in [9] that if $X$ is a closed symplectic manifold with $\pi_2(X) = 0$ then for $n \geq \dim \mathbb{C}X$, $\mathbb{C}P^n \times X$ has no simply connected Lagrangian submanifolds. On the other hand, if $n < \dim \mathbb{C}X$, then $\mathbb{C}P^n \times X$ may have such Lagrangians. Indeed, for any symplectic manifold $X$ of $\dim \mathbb{C}X = n + 1$, $\mathbb{C}P^n \times X$ has a Lagrangian sphere, after a possible rescaling of the symplectic form on the $X$ factor (see [9] for details). The following theorem shows that homologically this is the only example.

**Theorem B.** Let $X$ be a symplectic manifold of $\dim \mathbb{C}X = n + 1$ with $\pi_2(X) = 0$ and which is either closed or has a symplectically convex end. If $L^{2n+1} \subset \mathbb{C}P^n \times X$, $n \geq 1$, is a simply connected Lagrangian submanifold then $H^*(L^{2n+1}; \mathbb{Z}_2) \cong H^*(S^{2n+1}; \mathbb{Z}_2)$.

In contrast to the result of [9] mentioned above, if one drops the condition “$\pi_2(X) = 0$”, then $\mathbb{C}P^n \times X$ may have simply connected Lagrangian submanifolds even if $n \geq \dim \mathbb{C}X$. For example take $X$ to be $\mathbb{C}P^n$ and consider $\mathbb{C}P^n \times \mathbb{C}P^n$ endowed with the equally weighted split standard symplectic structure. Then $L = \mathbb{C}P^n$ embeds Lagrangianly as the “anti-diagonal” in $\mathbb{C}P^n \times \mathbb{C}P^n$, namely

$$\mathbb{C}P^n \ni [z_0 : \ldots : z_n] \mapsto ([z_0 : \ldots : z_n], [\bar{z}_0 : \ldots : \bar{z}_n]) \in \mathbb{C}P^n \times \mathbb{C}P^n.$$ 

The following theorem shows that, again, homologically this is the only example.

**Theorem C.** Let $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$ be a Lagrangian submanifold with $H_1(L; \mathbb{Z}) = 0$. Then $H^*(L; \mathbb{Z}_2) \cong H^*(\mathbb{C}P^n; \mathbb{Z}_2)$ as graded algebras. Moreover, if $a \in H^2(\mathbb{C}P^n \times \mathbb{C}P^n; \mathbb{Z}_2)$ is the generator of $H^2$ of either factor of $\mathbb{C}P^n \times \mathbb{C}P^n$ then $a|_L \in H^2(L; \mathbb{Z}_2)$ generates $H^*(L; \mathbb{Z}_2)$ as an algebra.

Again, besides $L = \mathbb{C}P^n$ we are not aware of any other examples of Lagrangian submanifolds in $\mathbb{C}P^n \times \mathbb{C}P^n$ with $H_1(L; \mathbb{Z}) = 0$.

### 1.2. Lagrangian spheres

Here we present new restrictions on Lagrangian embeddings of spheres. Let us mention that Lagrangian spheres appear in various mathematical contexts other than symplectic geometry (e.g. singularity theory [4]) and thus deserve special attention beyond the scope of symplectic geometry.

In view of the fact that manifolds of the type $\mathbb{C}P^n \times X$ with $\dim \mathbb{C}X = n + 1$ have Lagrangian spheres it makes sense to ask what happens for $X$ of other dimensions. The next theorem gives a partial answer to this question.
Theorem D. Let $X$ be a symplectic manifold with $\pi_2(X) = 0$ which is either closed or has a symplectically convex end. If $\mathbb{C}P^n \times X$ has a Lagrangian sphere (where $n, \dim X > 0$) then $\dim_{\mathbb{C}} X \equiv n + 1 \pmod{2n + 2}$.

Recall that a symplectic manifold $(M, \omega)$ is called spherically monotone if $[\omega]|_{\pi_2(M)} \neq 0$ and there exists $\lambda > 0$ such that $[\omega] = \lambda c_1^M$ on $\pi_2(M)$. Here, and in what follows, $c_1^M$ stands for the first Chern class of the tangent bundle of $M$, viewed (in a canonical way) as a complex vector bundle. We denote by $N_M \in \mathbb{Z}_+$ the minimal Chern number, namely $N_M = \min\{c_1^M(A) \mid A \in \pi_2(M), c_1^M(A) > 0\}$.

Theorem E. Let $X$ be a symplectic manifold that can be covered by a symplectic manifold which is symplectomorphic to a domain in $\mathbb{C}^m$. Let $M$ be a spherically monotone closed symplectic manifold. Assume $\dim X, \dim M > 0$. If $M \times X$ has a Lagrangian sphere then $2N_M | \dim_{\mathbb{C}} M + \dim_{\mathbb{C}} X + 1$.

Examples of manifolds $X$ satisfying the conditions of Theorem E include symplectic tori, and ball quotients, both endowed with Kähler symplectic structures. Theorem E is in fact a special case of the more general Theorem F which will be proved in Section 7.

Theorem F. Let $(M, \omega)$ be a spherically monotone closed symplectic manifold with $[\omega] = c_1^M$ on $\pi_2(M)$ and denote $m = \dim_{\mathbb{C}} M$. Let $\mathbb{C}P^n \times M$ be endowed with the symplectic form $\Omega = (n+1)\sigma \oplus \omega$, where $\sigma$ is the standard symplectic Kähler form of $\mathbb{C}P^n$ normalized so that the area of a projective line is 1. Suppose that $(\mathbb{C}P^n \times M, \Omega)$ has a Lagrangian sphere, where $n + m \geq 3$. Then $2 \gcd (n + 1, N_M) | n + m + 1$.

Example. Theorem F implies that if $(\mathbb{C}P^n \times \mathbb{C}P^m, (n+1)\sigma \oplus (m+1)\sigma)$ has a Lagrangian sphere for $n + m \geq 3$ then $\gcd (n + 1, m + 1) = 1$ and $n + m$ is odd.

Remark. Embarrassingly, the only example known to us of Lagrangian spheres in manifolds of the types appearing in Theorems D,E,F are all in manifolds of the type $\mathbb{C}P^n \times X$ where $\dim_{\mathbb{C}} X = n + 1$. It would be interesting to figure out for example whether or not $\mathbb{C}P^n \times X$ admits a Lagrangian sphere (or even a homology sphere) when $\dim_{\mathbb{C}} X = (2k + 1)(n + 1)$, $k \geq 1$.

1.3. Lagrangian intersections. Here we present new results on Lagrangian intersections. The pattern that stands out in all the examples below is the existence of a “core” $\Lambda$ consisting of a finite union of (possibly) immersed Lagrangian spheres with the property that every Lagrangian submanifold with prescribed topological properties (e.g. simply connected) must intersect $\Lambda$. 
**Intersections in the quadric.** Let $Q^n \subset \mathbb{C}P^{n+1}$ be the complex $n$-dimensional smooth quadric defined by the equation:

$$Q^n = \left\{ [z_0 : \ldots : z_{n+1}] \in \mathbb{C}P^{n+1} \mid z_0^2 + \cdots + z_{n+1}^2 = 0 \right\},$$

endowed with the symplectic structure induced from $\mathbb{C}P^{n+1}$. Let $\Lambda_Q \subset Q^n$ be the corresponding “real” quadric, namely:

$$\Lambda_Q = \left\{ [z_0 : \ldots : z_{n+1}] \in Q^n \mid z_0 \in \mathbb{R}, z_1, \ldots, z_{n+1} \in i\mathbb{R} \right\}.$$

It is not hard to see that $\Lambda_Q \subset Q^n$ is in fact a Lagrangian sphere.

Let $A^n_Q = \oplus_{i=0}^n A^n_Q$ be the following graded vector space over $\mathbb{Z}_2$:

$$\begin{align*}
A^n_Q &= \mathbb{Z}_2, 
A^n_Q &= \mathbb{Z}_2, 
A^n_i &= 0 \quad \text{for every } 1 < i < n - 1, 
A^n_{n-1} &= \mathbb{Z}_2, 
A^n_n &= \mathbb{Z}_2.
\end{align*}$$

**Theorem G.** Let $L \subset Q^n, n \geq 3$, be a Lagrangian submanifold such that $H_1(L; \mathbb{Z})$ is a 2-torsion group. If $H^*(L; \mathbb{Z}_2)$ is not isomorphic to $A^n_Q$ then $L \cap \Lambda_Q \neq \emptyset$. In particular for every Lagrangian submanifold $L \subset Q^n$ with $H_1(L; \mathbb{Z}) = 0$, we have $L \cap \Lambda_Q \neq \emptyset$.

**Remarks.**

1. Note that the intersection between $L$ and $\Lambda_Q$ is in general not due to topological reasons but rather to symplectic ones. For example when $n$ is odd, every Lagrangian sphere (in particular $\Lambda_Q$) can be displaced from itself via an arbitrary small (non symplectic) diffeomorphism.

2. The quadric $Q^n$ has many different Lagrangian submanifolds with $H_1(L; \mathbb{Z})$ either zero or 2-torsion. First of all it has simply connected Lagrangians (e.g. $\Lambda_Q$ itself). Next, for every $0 \leq r \leq n$ consider

$$L_r = \left\{ [z_0 : \ldots : z_{n+1}] \in Q^n \mid z_0, \ldots, z_r \in \mathbb{R}, \ z_{r+1}, \ldots, z_{n+1} \in i\mathbb{R} \right\}.$$

It is not hard to see that all the $L_r$’s are Lagrangian submanifolds of $Q^n$ and that $L_r$ is diffeomorphic to $(S^r \times S^{n-r})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on both factors by the antipode map. It easily follows that $L_0$ and $L_n$ are spheres, and a simple computation shows that for every $1 < r < n - 1$, we have $H_1(L_r; \mathbb{Z}) = \mathbb{Z}_2$ but $H^2(L_r; \mathbb{Z}_2) \neq 0$. Therefore when $n > 3$, $H^*(L_r; \mathbb{Z}_2)$ is not isomorphic to $A^n_Q$. It follows from Theorem [G] that for $n > 3$ any Lagrangian submanifold $L \subset Q^n$ diffeomorphic to one of the $L_r$’s must intersect $\Lambda_Q$. In particular the intersection $L_r \cap \Lambda_Q \neq \emptyset$ cannot be removed via a symplectic isotopy.

3. We do not know of any examples of Lagrangians $L$ with 2-torsion $H_1(L; \mathbb{Z})$ lying in the complement of $\Lambda_Q$. 
The statement of Theorem G remains true if instead of assuming $H_1(L;\mathbb{Z})$ is 2-torsion one assumes that $L$ is monotone with minimal Maslov number $N_L = n$. (See the second remark at the end of the proof of the Theorem in Section 6.) It would be interesting to figure out if at least there are any monotone Lagrangians $L \subset Q^n$ (not necessarily with 2-torsion $H_1$) with $N_L = n$ that lie in complement of $\Lambda_Q$.

**Questions.**

1. Theorem G implies that (for $n \geq 3$) every Lagrangian sphere $L \subset Q^n$, must intersect $\Lambda_Q$. Is it true that every two Lagrangian spheres $L_1, L_2 \subset Q^n$ must intersect each other? Note that when $n = \text{even}$ this easily follows by computing the intersection number $\langle L_1 \rangle \cdot \langle L_2 \rangle$, but for $n = \text{odd}$ this does not seem to follow from purely topological reasons. An affirmative answer would have the following consequences in algebraic geometry: the complex quadric (of dimension $\geq 3$) cannot be degenerated to a variety having two or more isolated singularities. See [6, 7, 11] for more details and precise statements.

2. Note that the cohomology of the Lagrangian $L \subset Q^n$ (taken from (2) above with $r = 1$) is precisely $A^\ast Q^n$. Can $L_1$ be Hamiltonianly isotoped to lie in the complement of $\Lambda_Q$? It is known that $L_1$ can also be embedded as a monotone Lagrangian in $\mathbb{C}^n$ with minimal Maslov number = $n$ (see [34]). Can $L_1$ be embedded as a monotone Lagrangian in $Q^n \setminus \Lambda_Q$?

3. Theorem G will not be proved by showing that the Floer homology $HF(L, \Lambda_Q)$ is not zero. It would be interesting to figure out whether this is indeed so.

**Intersections in hypersurfaces of $\mathbb{C}P^{n+1}$.** Theorem G can be generalized as follows. Let $\Sigma_d^n \subset \mathbb{C}P^{n+1}$ be a smooth complex hypersurface of degree $d$, viewed as a symplectic manifold endowed with the symplectic structure induced from $\mathbb{C}P^{n+1}$.

**Theorem H.** For every $d > 2$ there exist $d^{n+1}$ (possibly) immersed Lagrangian spheres $S_1, \ldots, S_{d^{n+1}} \subset \Sigma_d^n$ such that their union $\Lambda_d = S_1 \cup \ldots \cup S_{d^{n+1}}$ has the following properties:

1. When $d \leq \frac{n+1}{2}$ or $d \geq \frac{3}{2}(n + 1)$, $n \geq 3$, every Lagrangian submanifold $L \subset \Sigma_d^n$ with $H_1(L;\mathbb{Z}) = 0$ must satisfy $L \cap \Lambda_d \neq \emptyset$.

2. When $d \leq \frac{n+1}{2}$ and $n = \text{even}$, for every Lagrangian submanifold $L \subset \Sigma_d^n$ with $H_1(L;\mathbb{Z})$ 2-torsion either $L \cap \Lambda_d \neq \emptyset$ or $L$ has the following properties:
   (a) $H^d(L;\mathbb{Z}_2) = \ldots = H^{n-d}(L;\mathbb{Z}_2) = 0$.
   (b) $\beta_i(L) = \beta_{d-1-i}(L) = \beta_{i+1+n-d}(L) = \beta_{n-i}(L)$ for every $0 \leq i \leq d-1$, where $\beta_j(L)$ stands for the $j$'th $\mathbb{Z}_2$-Betti number of $L$.

3. When $d \leq \frac{n+1}{2}$ and $d = \text{odd}$, for every Lagrangian submanifold $L \subset \Sigma_d^n$ with $H_1(L;\mathbb{Z})$ 2-torsion either $L \cap \Lambda_d \neq \emptyset$ or $L$ has the following properties:
in Section 3.6.4 below we show that

\[ S \]

Questions.

(1) Theorem \text{H} implies in particular that (for some values of \( d \) is a Lagrangian sphere. When \( n \) namely \( \Sigma \)

\[ \text{a degeneration with isolated singularities, hence by the Lagrangian v anishing cycle con-} \]

(2) Under the conditions of statement (2) of Theorem \text{H}, Lagrang ians

\[ L \]

submanifold with \( H \) come from Picard-Lefschetz theory. Indeed \( \Sigma \)

\[ \text{Examples of Lagrangian submanifolds} \]

When \[ (4) \]

\[ \text{Let} \]

\[ \tau \]

\[ \text{We have not been able to explicitly compute the Lagrangian spheres in } \Lambda \]

\[ \text{of the type} \]

\[ L \}

Here, \( w \in H^2(L; \mathbb{Z}_2) \) is the restriction of the generator \( a \in H^2(\mathbb{C}P^{n+1}; \mathbb{Z}_2) \) to

\[ L \subset \Sigma^d_n \subset \mathbb{C}P^{n+1}. \]

(4) When \( 2 < d \leq n + 1, n \geq 3 \) and \( 2(n + 2 - d) \nmid n + 1, \) every Lagrangian sphere

\[ L \subset \Sigma^d_n \] must satisfy \( L \cap \Lambda_d \neq \emptyset. \)

(5) Let \( d, t \geq 2, n \geq 3 \) be such that \( d \geq \frac{(t(n-1)}{2} + n + 2. \) Then every Lagrangian

\[ \text{submanifold with } H_1(L; \mathbb{Z}) \text{ t-torsion must satisfy } L \cap \Lambda_d \neq \emptyset. \]

We have not been able to explicitly compute the Lagrangian spheres in \( \Lambda_d, \) however in Section 3.6.4 below we show that \( S_2, \ldots, S_{d^n+1} \) are all obtained from \( S_1 \) by applying suitable automorphisms of \( \Sigma^d_n. \)

Examples of Lagrangian submanifolds \( L \subset \Sigma^d_n \) that satisfy the conditions of the Theorem \text{H} come from Picard-Lefschetz theory. Indeed \( \Sigma^d_n, d \geq 2, \) can be included as a fibre in a degeneration with isolated singularities, hence by the Lagrangian vanishing cycle construction it must contain Lagrangian spheres (see \[ 11, 13, 22, 33, 36, \text{ see also}\[ 14]. \) Here are more explicit examples of Lagrangians in \( \Sigma^d_n: \) write \( \Sigma^d_n \) as \{ \( z_0^d + \cdots + z_{n+1}^d = 0 \} \subset \mathbb{C}P^{n+1}. \)

\[ \text{Let } \tau \in \mathbb{C} \text{ be a root of } -1 \text{ of order } d. \text{ Then it is easy to see that when } d = \text{ even} \]

\[ \left\{ [z_0: \ldots : z_{n+1}] \in \Sigma^d_n \left| z_0 \in \tau \mathbb{R}, z_j \in \mathbb{R} \text{ for every } 1 \leq j \leq n+1 \right. \right\} \]

is a Lagrangian sphere. When \( d = \text{ odd} \) we also have Lagrangians homeomorphic to \( \mathbb{R}P^n, \)

\[ \text{namely } \Sigma^d_n \cap \mathbb{R}P^{n+1}. \] (See Appendix A of \[ 25 \] for an explicit homeomorphism.)

Questions. (1) Theorem \text{H} implies in particular that (for some values of \( d \) it is impos-

\[ \text{sible to embed a Lagrangian sphere which is disjoint from the spheres } S_1, \ldots, S_{d^n+1}. \]

In view of this one is led to speculate that the maximal number of mutually dis-

\[ \text{joint Lagrangian spheres in } \Sigma^d_n \text{ is finite, or even that this number is not bigger} \]

\[ \text{than } d^{n+1} \text{ (c.f. the first question after Theorem \text{G}.)} \]

(2) Under the conditions of statement (2) of Theorem \text{H} Lagrangians \( L \subset \Sigma^d_n \setminus \Lambda_d \) with

\[ \text{2-torsion } H_1(L; \mathbb{Z}) \text{ must have the same } \mathbb{Z}_2-\text{coefficients cohomology as a manifold} \]

\[ \text{of the type } L_0 \times S^{n+1-d}, \text{ where } L_0 \text{ is a } (d - 1)-\text{dimensional manifold. Can such} \]

\[ \text{manifolds be Lagrangianly embedded in } \Sigma^d_n? \text{ In } \Sigma^d_n \setminus \Lambda_d? \]

(3) Consider the manifold \( L = (S^1 \times S^{n+1-d})/\mathbb{Z}_2 \times S^{d-2}, \text{ where } \mathbb{Z}_2 \text{ acts on both factors} \)

\[ \text{of } S^1 \times S^{n+1-d} \text{ by the antipode map. Note that } L \text{ also satisfies the cohomological restrictions predicted by statement (2) of Theorem \text{H}.} \]

It is known that \( L \) admits a monotone Lagrangian embedding into \( \mathbb{C}^n \) with minimal Maslov number = \( n + 2 - d \) (see \[ 31 \]). Does \( L \) admit a \text{monotone} Lagrangian embedding into \( \Sigma^d_n? \) Into \( \Sigma^d_n \setminus \Lambda_d? \)
Intersections in a hypersurface of $\mathbb{CP}^n \times \mathbb{CP}^n$. Consider $\mathbb{CP}^n \times \mathbb{CP}^n$ endowed with the standard symplectic form $\sigma_{\text{std}} \oplus \sigma_{\text{std}}$ and let $\Sigma^{2n-1} \subset \mathbb{CP}^n \times \mathbb{CP}^n$ be the complex hypersurface defined by the equation

$$\Sigma^{2n-1} = \left\{ \sum_{j=0}^{n-1} z_j w_j = z_n w_n \right\},$$

where $[z_0 : \ldots : z_n], [w_0 : \ldots : w_n]$ are homogeneous coordinates on $\mathbb{CP}^n \times \mathbb{CP}^n$. We endow $\Sigma^{2n-1}$ with the symplectic structure induced from $\mathbb{CP}^n \times \mathbb{CP}^n$. Put

$$\Lambda_\Sigma = \left\{ ([z_0 : \ldots : z_n], [w_0 : \ldots : w_n]) \in \Sigma^{2n-1} \mid w_j z_n = w_n z_j \forall 0 \leq j \leq n-1 \right\}.$$

A simple computation shows that $\Lambda_\Sigma \subset \Sigma^{2n-1}$ is a Lagrangian sphere.

**Theorem I.** Let $L^{2n-1} \subset \Sigma^{2n-1}$, $n \geq 2$, be a Lagrangian submanifold with $H_1(L; \mathbb{Z}) = 0$. Then either $H^*(L; \mathbb{Z}_2) \cong H^*(S^{2n-1}; \mathbb{Z}_2)$ or $L \cap \Lambda_\Sigma \neq \emptyset$.

1.4. **Discussion.** The phenomenon arising in the results of Section 1.3 above is the existence of certain “Lagrangian subsets” that dominate intersections in the sense that all Lagrangian submanifolds with specified topology must intersect them. A similar phenomenon is already known for cotangent bundles, where a result due to Gromov \[21\] implies that every Lagrangian submanifold of a cotangent bundle $L \subset T^*(X)$ with $H^1(L; \mathbb{R}) = 0$ must intersect the zero section. Our results show that such a phenomenon holds also in closed manifolds. It is interesting to note that, similarly to our case, also for cotangent bundles it is currently unknown whether or not the Floer homology $HF(L, O_X)$ of $L$ and the zero section $O_X$ is non-trivial. (Compare Question (2) after Theorem G.)

Finally, let us remark that the above intersection phenomena, in general, seize to hold in the $C^\infty$-category. Indeed in each of the Theorems G-I one can usually remove all the intersection points by a smooth diffeomorphism.

1.5. **Methods and ideas.** The methods and tools used in this paper consist of two main ingredients. The first one is Floer theory for Lagrangian submanifolds. In particular we use the extension of Floer homology to monotone Lagrangian submanifolds due to Oh \[31\ \ 33\ \ 32\] which gives rise to an algebraic approach for computing Floer homology in terms of a spectral sequence.

The second ingredient is a geometric technique, developed by the author in \[5\] and in this paper, by which it is sometimes possible to compute Floer homology in a geometric way. Our techniques enable to perform certain transformations to a given Lagrangian submanifold resulting in a new Lagrangian that can be Hamiltonianly displaced. In particular, we obtain vanishing of Floer homology. This vanishing combined with the algebraic computations mentioned above is the key point behind most of our results.
Let us now describe in some more detail the main ideas of the paper. Let \( \Sigma^{2n} \) be a closed symplectic manifold. We shall concentrate on the case when \( \Sigma \) can be symplectically embedded as a hyperplane section of a higher dimensional symplectic manifold, say \( M^{2n+2} \). The idea is that in view of the decomposition technique developed by the author in [5], the symplectic properties of \( M \setminus \Sigma \) can be used to study the symplectic topology of \( \Sigma \) itself. Consider a tubular neighbourhood \( U \) of \( \Sigma \) in \( M \). Its boundary \( \partial U \) is a circle bundle \( \pi : \partial U \to \Sigma \) over \( \Sigma \). Now let \( L \subset \Sigma \) be a Lagrangian submanifold. As we shall see in Section 4.1 below, if we chose \( U \) carefully the restriction of this circle bundle to \( L \), is a Lagrangian submanifold \( \Gamma_L = \pi^{-1}(L) \) lying in \( M \setminus \Sigma \). Note that \( M \setminus \Sigma \) is a Stein manifold.

The symplectic theory of Stein manifolds now comes into play. Recall that Stein manifolds are divided into subcritical and critical ones (see Section 2 for the precise definitions). Subcritical Stein manifolds have the feature that (after a suitable completion) all compact subsets can be Hamiltonianly displaced. In particular the Floer homology of Lagrangian submanifolds must vanish.

Returning to our case, if \( M \setminus \Sigma \) turns out to be subcritical, the Floer homology \( HF(\Gamma_L, \Gamma_L) \) vanishes. Thus, starting with a Lagrangian submanifold \( L \subset \Sigma \), we have transformed it into a new Lagrangian \( \Gamma_L \subset M \setminus \Sigma \) whose Floer homology vanishes due to geometric reasons. Note that in contrast to \( \Gamma_L \), in general \( L \) itself cannot be Hamiltonianly displaced and in fact its Floer homology might not vanish.

We now turn to algebraic computations in Floer homology. The idea is to perform the computation of \( HF(\Gamma_L, \Gamma_L) \) in an alternative way. The main tool for this end is a spectral sequence based on the theory developed by Oh [33, 32]. The second step of this spectral sequence is the singular cohomology of a Lagrangian, and the sequence converges to its Floer homology. Comparing this computation (performed on \( \Gamma_L \)) with the vanishing of \( HF(\Gamma_L, \Gamma_L) \) we obtain restrictions on the cohomology of \( \Gamma_L \). In some cases we are even able to reproduce the entire cohomology of \( \Gamma_L \). Having done this, we derive information on the cohomology of \( L \) itself (recall that \( \Gamma_L \to L \) is a circle bundle). We give a rather detailed construction of this spectral sequence in Section 5 since our approach is somewhat different than Oh’s original work [33, 32]. Detailed computations using the spectral sequence appear throughout the proofs of the main theorems in Section 6.

Let us turn now to the case when \( M \setminus \Sigma \) is a critical Stein manifold. In this case it is no longer true that all Lagrangian submanifolds in \( M \setminus \Sigma \) are displaceable. In Section 3.5 and 4.2 we introduce a kind of geometric obstruction for displaceability in \( M \setminus \Sigma \). This obstruction, which we call the Lagrangian trace, is a union of (possibly) immersed Lagrangian spheres \( \Lambda \subset \Sigma \). It has the property that for every Lagrangian \( L \subset \Sigma \) with \( L \cap \Lambda = \emptyset \), the circle bundle \( \Gamma_L \subset M \setminus \Sigma \) can be Hamiltonianly displaced, in particular
$HF(\Gamma_L, \Gamma_L) = 0$. Recomputing the vanishing of $HF(\Gamma_L, \Gamma_L)$ using the spectral sequence we deduce that Lagrangian submanifolds $L \subset \Sigma$ with certain topological properties must intersect $\Lambda$.

**The choice of the coefficients.** Our cohomological restrictions are with $\mathbb{Z}_2$-coefficients only. This has to do with technical reasons coming from Floer homology theory which is one of the main tools used in this paper. However, recent developments in Floer theory, due to Fukaya, Oh, Ohta and Ono [20] make it possible to define, in some cases, Floer homology with coefficients in $\mathbb{Q}$ or even $\mathbb{Z}$. It seems very likely that several of our results above continue to hold for cohomology with $\mathbb{Z}$-coefficients as well.

**The title of the paper.** “Lagrangian non-intersections” is derived from the following idea motivated in this paper: whenever the principle of Lagrangian intersections fails (in the sense that a Lagrangian can be Hamiltonianly displaced) we obtain restrictions on the topology of the Lagrangian via computations in Floer homology.

**Organization of the paper.** The rest of the paper is organized as follows. In Section 2 we collect some important facts from the symplectic theory of Stein manifolds that will be used in the sequel. We also develop in this section methods to displace Lagrangian submanifolds in both subcritical and critical Stein manifolds.

In Sections 3 and 4 we discuss symplectic manifolds $\Sigma$ that appear as hyperplane sections in other manifolds $M$. For Lagrangian submanifolds $L \subset \Sigma$ we introduce the Lagrangian circle bundle construction giving rise to a new Lagrangian submanifold $\Gamma_L \subset M \setminus \Sigma$. We then study the possibilities to displace $\Gamma_L$ in $M \setminus \Sigma$ and introduce the Lagrangian trace $\Lambda \subset \Sigma$ which is a kind of obstruction for displacing $\Gamma_L$. In Section 3.6 we present explicit calculations of $\Lambda$ for various examples of $\Sigma$. Section 5 is devoted to computations in Floer homology. In Section 6 we give the proofs of the main theorems. Finally, in Section 7 we present some generalizations of the theorems of Section 1.

### 2. SYMPLECTIC GEOMETRY OF STEIN MANIFOLDS

Here we briefly recall some basic facts on Stein manifolds from the symplectic viewpoint. The reader is referred to [14] [15] for the foundations of the symplectic theory of Stein manifolds. Apart from the contents of Subsection 2.2, most of the material below can be found in [16] [14] [15] [8].

A **Stein manifold** is a triple $(V, J, \varphi)$ where $(V, J)$ is an open complex manifold and $\varphi : V \to \mathbb{R}$ is a smooth exhausting plurisubharmonic function. The term “exhausting” means that $\varphi$ is proper and bounded from below. “Plurisubharmonic” means that the 2-form $\omega_\varphi = -dd^c\varphi$ is a $J$-positive symplectic form, i.e. $-dd^c\varphi(v, Jv) > 0$ for every $0 \neq v \in T_{xy}V$. The **complexification** of $V$ is $V^{1,0} = \{ (v, w) \mid v \in V, w \in J_0V \}$, where $J_0V$ is a 0-dimensional complex vector subspace of $V$.

The **holomorphic tangent bundle** $T^cV$ is the orthogonal complement of $T_{xy}V$ with respect to $\omega_\varphi$; it is a complex vector bundle on $V$. The **holomorphic$\times$Hamiltonian form** $\omega^h$ is the pull-back of the holomorphic$\times$Hamiltonian form on $V^{1,0}$ by the inclusion $T_{xy}V \subset T^cV$. The **first Chern class** $c_1(V)$ of $V$ is the obstruction to the trivialization of $T^cV$. The **Kähler form** $\omega$ is a non-degenerate $(1,1)$-form on $V$ which is positive everywhere except on the fibers of $T^cV$. The **Kähler class** $[\omega]$ is the cohomology class of $\omega$.

A **Lagrangian submanifold** $L$ is a submanifold of $V$ with $\omega|_L = 0$. A **Lagrangian submanifold** $L$ is **holomorphic** if $J_0L \subset L$. A **Lagrangian** $L$ is **displaceable** if there is a Hamiltonian $H$ on $V$ with $H|_L < H|_{L'}$ for some $L'$ near $L$. The **Lagrangian circle bundle** $\Gamma_L$ is a Lagrangian submanifold in $M \setminus \Sigma$. The **Lagrangian trace** $\Lambda$ is a kind of obstruction for displacing $\Gamma_L$. In Section 5 we present explicit calculations of $\Lambda$ for various examples of $\Sigma$. Section 6 is devoted to computations in Floer homology. In Section 7 we give the proofs of the main theorems. Finally, in Section 8 we present some generalizations of the theorems of Section 1.
Given a Stein manifold \((V, J, \varphi)\) we have the gradient vector field \(X_\varphi = \text{grad}_{g_\varphi} \varphi\) of \(\varphi\) with respect to the metric \(g_\varphi\). A simple computation shows that \(L_{X_\varphi} \omega_\varphi = \omega_\varphi\), hence the flow \(X_t^\varphi\) of \(X_\varphi\) is conformally symplectic, \((X_t^\varphi)^* \omega_\varphi = e^t \omega_\varphi\). We remark that in contrast to other texts (e.g. [17]), we do not assume that the flow of \(X_\varphi\) is complete, unless explicitly stated. Note however that since \(\varphi\) is exhausting the flow \(X_t^\varphi\) does exist for all negative times.

2.1. Canonical symplectic structures on Stein manifolds. Given a Stein manifold \((V, J, \varphi)\) and \(R \in \mathbb{R}\), we denote by \(V_{\varphi \leq R}\) the sublevel set \(\varphi^{-1}((-\infty, R])\). We write \(\text{Crit}(\varphi)\) for the set of critical points of the function \(\varphi\).

Following [17] we say that a Stein manifold \((V, J, \varphi)\) is complete if the flow of gradient vector field \(X_\varphi = \text{grad}_{g_\varphi} \varphi\) exists for all positive times.

**Lemma 2.1.A** (See [17], [8]). Let \((V, J, \varphi)\) be a Stein manifold. Then for every \(R \in \mathbb{R}\) there exists an exhausting plurisubharmonic function \(\varphi_R : V \to \mathbb{R}\) with the following properties:

1. \(\varphi_R = \varphi\) on \(V_{\varphi \leq R}\).
2. \((V, J, \varphi_R)\) is a complete Stein manifold.
3. \(\text{Crit}(\varphi_R) = \text{Crit}(\varphi)\) and for every \(p \in \text{Crit}(\varphi_R)\), \(\text{ind}_p(\varphi_R) = \text{ind}_p(\varphi)\).

In particular, the inclusion \((V_{\varphi \leq R}, \omega_\varphi) \subset (V, \omega_{\varphi_R})\) is a symplectic embedding.

The next lemma shows that the symplectic structure of a complete Stein manifold is unique up to symplectomorphism.

**Lemma 2.1.B** (See [17], compare [8]). Let \((V, J)\) be a complex manifold endowed with two exhausting plurisubharmonic functions \(\varphi_1, \varphi_2\) such that both Stein manifolds \((V, J, \varphi_1)\) and \((V, J, \varphi_2)\) are complete. Then the symplectic manifolds \((V, \omega_{\varphi_1})\) and \((V, \omega_{\varphi_2})\) are symplectomorphic.

**Remark 2.1.C.** In view of this lemma, we shall sometimes denote the symplectic manifold associated to the completion of a Stein manifold \((V, J, \varphi)\) by \((V, \hat{\omega})\) (since it does not depend on the choice of the plurisubharmonic function). Note that for every open subset \(W \subset V\) with compact closure we have a symplectic embedding \((W, \omega_\varphi) \hookrightarrow (V, \hat{\omega})\).
2.2. The skeleton of a Stein manifold $(V, J, \varphi)$. By this we mean the subset $\Delta_\varphi \subset V$ which is formed from the union of the stable submanifolds of the flow $X^t_\varphi$, namely:

$$\Delta_\varphi = \bigcup_{p \in \text{Crit}(\varphi)} W^s_p(X_\varphi) = \left\{ x \in V \mid \lim_{t \to \infty} X^t_\varphi(x) \in \text{Crit}(\varphi) \right\}.$$ 

**Lemma 2.2.A.** Let $(V, J, \varphi)$ be a Stein manifold and assume that $\varphi$ is a Morse-Bott function. Then for every critical submanifold $C \subset V$ of $\varphi$ we have:

1. $C$ is isotropic with respect to $\omega_\varphi$.
2. $\text{ind}_C(\varphi) + \dim C \leq \frac{1}{2} \dim \mathbb{R} V$.
3. For every $p \in C$ the stable submanifold $W^s_p(X_\varphi)$ is isotropic with respect to $\omega_\varphi$.
4. For every $p \in C$ the unstable submanifold $W^u_p(X_\varphi)$ is coisotropic with respect to $\omega_\varphi$.

*Proof.* For $\varphi$ being Morse a proof can be found in [15] (see also [17], and see [5] Section 8). The case of Morse-Bott $\varphi$ is an obvious extension of the “Morse case”. □

**Lemma 2.2.B** (See [5]). Let $(V, J, \varphi)$ be a Stein manifold, and assume that all the critical points of $\varphi$ lie in the subset $\{ \varphi < R \}$ for some $R \in \mathbb{R}$. Then arbitrarily close to $\varphi'$ in the $C^2$-topology there exists an exhausting plurisubharmonic function $\varphi' : V \to \mathbb{R}$ such that:

1. $\varphi' = \varphi$ on $\{ \varphi \geq R \}$.
2. $\varphi'$ is Morse.
3. The flow of $X_{\varphi'}$ is Morse-Smale. In particular all trajectories of $X_{\varphi'}$ go from critical points of $\varphi'$ to either critical points of strictly higher index or to “infinity” (i.e. do not go to any other critical point). Moreover the skeleton $\Delta_{\varphi'}$ is an isotropic CW-complex (see below).

Let $(Y, \omega)$ be a symplectic manifold and $\Delta \subset Y$ a subset. We call $\Delta$ an isotropic CW-complex if there exists an abstract CW-complex $K$ and a homeomorphism $i : K \to \Delta \subset Y$ such that for every cell $C \subset K$ the restriction $i|_{	ext{Int} C \cap \text{Int} (D^{\dim C})} : \text{Int} C \to (Y, \omega)$ is an isotropic embedding. We refer the reader to [5] for more details on this notion.

2.3. Subcritical Stein manifolds. Let $(V, J, \varphi)$ be a Stein manifold. It is well known [15] that if $\varphi$ is Morse then for every critical point $p$ we have $\text{ind}_p \varphi \leq \frac{1}{2} \dim \mathbb{R} V$. We say that $(V, J, \varphi)$ is subcritical if $\varphi$ is Morse with finite number of critical points and for every $p \in \text{Crit} \varphi$, $\text{ind}_p \varphi < \frac{1}{2} \dim \mathbb{R} V$. Note that in this case $\dim \Delta_\varphi < \frac{1}{2} \dim \mathbb{R} V$ (hence the skeleton does not contain Lagrangian cells).

The following lemma shows that in subcritical Stein manifolds any compact subset can be Hamiltonianly displaced. See [8] for the proof.
Lemma 2.3.A. Let \((V, J, \varphi)\) be a complete subcritical Stein manifold. Then for every compact subset \(A \subset V\) there exists a compactly supported Hamiltonian diffeomorphism \(h : (V, \omega_\varphi) \to (V, \omega_\varphi)\) such that \(h(A) \cap A = \emptyset\).

2.4. The critical coskeleton. Let \((V, J, \varphi)\) be a Stein manifold, and suppose that \(\varphi\) is a Morse-Bott function with finitely many critical submanifolds. We denote by \(p_1, \ldots, p_N\) the isolated critical points of \(\varphi\) (if there are any). We say that \(\varphi\) has property \((S_0)\) if one of the following two conditions is satisfied:

1. For every positive dimensional critical submanifold \(S\) of \(\varphi\), \(\text{ind}_S(\varphi) + \dim S < \frac{1}{2} \dim \mathbb{R} V\).
2. \(\varphi\) has no isolated critical points and only one positive dimensional critical manifold \(S\), with \(\dim S = \frac{1}{2} \dim \mathbb{R} V\).

In the first case denote by \(\{p'_1, \ldots, p'_r\} \subset \{p_1, \ldots, p_N\}\) those critical points with \(\text{ind}_{p'_i}(\varphi) = \frac{1}{2} \dim \mathbb{R} V\) (again, it may happen that \(r = 0\)). In the second case pick a point \(p'_1 \in S\) and put \(r = 1\). We define the critical coskeleton \(\nabla^\text{crit}_\varphi \subset V\) to be the union of the unstable submanifolds of the \(p'_i\)'s, namely:

\[
\nabla^\text{crit}_\varphi = \bigcup_{i=1}^r W^u_{p'_i}(X_\varphi) = \left\{ x \in V \mid \lim_{t \to -\infty} X_\varphi(x) \in \{p'_1, \ldots, p'_r\} \right\}.
\]

Remark. Property \((S_0)\) is purely technical and may look somewhat artificial. Note that if \(\varphi\) is Morse then it automatically has property \((S_0)\) (since all its critical points are isolated). Property \((S_0)\) was created to accommodate a slightly more general situation than that. It covers two different (and unrelated) possibilities. The first possibility means that among the unstable submanifolds, those that are Lagrangian (i.e. have minimal dimension) all come from isolated critical points. The second possibility, roughly speaking, means that \((V, \omega_\varphi)\) looks like a neighbourhood of the zero section in \(T^*(S)\).

Examples. (1) If \((V, J, \varphi)\) is a subcritical Stein manifold then clearly \(r = 0\), hence \(\nabla^\text{crit}_\varphi = \emptyset\).

2) Let \(M\) be a closed manifold and \(V = T^*(M)\) be its cotangent bundle. Denote by \(q \in M\) local coordinates along \(X\) and by \(p \in T^*_q(M)\) the dual coordinates along the cotangent fibres. It is well known (see [15]) that \(V\) can be endowed with the structure of a Stein manifold with \(\varphi(q, p) = |p|^2\) and \(X_\varphi = p \frac{\partial}{\partial p}\). Here \(|\cdot|\) is a norm along the cotangent fibers (coming from a Riemannian metric on \(M\)). In this case the only critical submanifold is the zero section, and the critical coskeleton is just one fibre, \(\nabla^\text{crit}_\varphi = T^*_q(M)\).
2.4.1. Property (S). Let \((V, J, \varphi)\) be a Stein manifold. We say that \(\varphi\) has property \((S)\) if it has property \((S_0)\) above and in addition for every \(1 \leq i \leq r\) all gradient trajectories of \(X_\varphi\) emanating from the points \(p'_i\) go to "infinity". Note that this definition does not depend on the choice of the point \(p'_1\) in the case when \(\varphi\) has only one critical submanifold \(S\) with \(\dim S = \frac{1}{2} \dim_{\mathbb{R}} V\). Indeed, in that case \(S\) is the minimum of \(\varphi\), hence all gradient trajectories emanating from points of \(S\) go to "infinity".

**Lemma 2.4.A.** Let \((V, J, \varphi)\) be a complete Stein manifold and assume that \(\varphi\) has property \((S)\). Let \(A \subset V\) be a compact subset with \(A \cap \nabla_{\varphi}^{\text{crit}} = \emptyset\). Then there exists a compactly supported Hamiltonian diffeomorphism \(h : (V, \omega_\varphi) \to (V, \omega_\varphi)\) such that \(h(A) \cap A = \emptyset\).

**Examples.**

1. If \((V, J, \varphi)\) is subcritical then the lemma reduces to Lemma 2.3.A since in this case \(\nabla_{\varphi}^{\text{crit}} = \emptyset\), hence any compact subset can be Hamiltonianly displaced.

2. Let \(V = T^* M\). As we have just seen above \(\nabla_{\varphi}^{\text{crit}} = T^*_q M\) for some \(q \in M\). Hence we recover a statement due to Lalonde and Sikorav [24] that any compact subset of \(T^* M\) lying in the complement of a fibre \(T^*_q M\) can be Hamiltonianly displaced.

Before we prove Lemma 2.4.A we shall need some preparations. Given a Morse function \(\varphi : V \to \mathbb{R}\), denote by \(\text{Crit}_{\leq k} (\varphi)\) the set of critical points of \(\varphi\) of index \(\leq k\). Denote by \(\Delta^k_{\varphi}\) the subskeleton

\[
\Delta^k_{\varphi} = \bigcup_{p \in \text{Crit}_{\leq k}(\varphi)} W^s_{p}(X_\varphi) = \left\{ x \in V \mid \lim_{t \to \infty} X^t_\varphi(x) \in \text{Crit}(\varphi) \right\}.
\]

We shall need the following Proposition for the proof of Lemma 2.4.A

**Proposition 2.4.B.** Let \((V, J, \varphi)\) be a Stein manifold. Fix an integer \(0 \leq k \leq \text{dim}_{\mathbb{C}} V\). Assume that:

1. \(\varphi\) is a Morse function with finitely many critical points \(x_1, \ldots, x_\nu \in V\), arranged so that \(\text{Crit}_{\leq k}(\varphi) = \{x_1, \ldots, x_l\}, l \leq \nu\).
2. There are no trajectories of \(X_\varphi\) that connect any of the critical points \(x_{l+1}, \ldots, x_\nu\) with one of the critical points \(x_1, \ldots, x_l\).

Fix mutually disjoint neighbourhood \(U_1, \ldots, U_\nu\) of \(x_1, \ldots, x_\nu\) respectively. Then arbitrarily close to \(\varphi\) in the \(C^2\)-topology there exists an exhausting plurisubharmonic function \(\varphi'\) with the following properties:

1. \(\varphi = \varphi'\) on \(V \setminus (U_1 \cup \ldots \cup U_\nu)\).
2. \(\varphi'\) is Morse, \(\text{Crit}(\varphi') = \text{Crit}(\varphi)\) and for every \(1 \leq i \leq \nu\), \(\text{ind}_{x_i}(\varphi') = \text{ind}_{x_i}(\varphi)\).
Finally, it is a straightforward computation to check that

\( k \) submanifolds. Put

we have:

\[ \emptyset \]

compactly supported in an arbitrarily small neighbourhood of \( \Delta \)

\( k \) is a Hamiltonian isotopy (see [8], Lemma 3.2).

Proof of Lemma 2.4.A. The proof generalizes ideas from [8] (see Lemma 3.2 there).

Assume first that \( \varphi \) is a Morse function, hence it has no positive dimensional critical submanifolds. Put \( k = \frac{1}{2} \dim \mathbb{R} V - 1 \). Applying Proposition 2.4.B we obtain a new plurisubharmonic function \( \varphi' \) for which \( \Delta^k \) is a CW-complex. Note that if we choose the neighborhoods \( U_1, \ldots, U_l \) of the points in \( \text{Crit}_{\leq k}(\varphi) \) to be small enough we can arrange that \( \nabla^\varphi' = \nabla^\varphi \).

By Moser argument there is a symplectomorphism \( f : (V, \omega) \rightarrow (V, \omega) \) which is supported in \( U_1 \cup \ldots \cup U_l \). In particular \( f(\nabla^\varphi) = \nabla^{\varphi'} \), and \( f(A) \cap \nabla^\varphi = \emptyset \). Thus by replacing \( \varphi \) by \( \varphi' \) we may assume without loss of generality that \( \Delta^k \) is a CW-complex.

Since \( \dim \Delta^k < \frac{1}{2} \dim \mathbb{R} V \) there exists a Hamiltonian isotopy \( g_t : (V, \omega) \rightarrow (V, \omega) \), compactly supported in an arbitrarily small neighbourhood of \( \Delta^k \), such that \( g_1(\Delta^k) \cap \Delta^k = \emptyset \). As \( \Delta^k \) is compact there exists a small neighbourhood \( W \) of \( \Delta^k \) so that \( g_1(W) \cap W = \emptyset \).

Since \( A \cap \nabla^\varphi = \emptyset \), for large enough \( T > 0 \) we have \( X^{-T}(A) \subset W \). As \( g_1 \) displaces \( W \) we have:

\[ X^T \circ g_1 \circ X^{-T}(A) \cap A = \emptyset. \]

Finally, it is a straightforward computation to check that

\[ h_t = X^T \circ g_t \circ X^{-T} \]

is a Hamiltonian isotopy (see [8], Lemma 3.2).

Assume now that \( \varphi \) has also positive dimensional submanifolds say \( S_1, \ldots, S_q \) with \( \text{ind}_{S_i}(\varphi) + \dim S_i < \frac{1}{2} \dim \mathbb{R} V \) for every \( i \). Pick for every \( i \) a generic Morse function \( f_i : S_i \rightarrow \mathbb{R} \) and a cut off function \( \rho_i \) which is identically 1 near \( S_i \) and identically 0 outside a small neighbourhood \( W_i \) of \( S_i \). Consider now the function \( \varphi_\epsilon = \varphi + \epsilon \sum_{i=1}^q \rho_i f_i \). Clearly for small \( \epsilon \), \( \varphi_\epsilon \) is plurisubharmonic. Moreover \( \varphi_\epsilon \) is Morse and its critical points consists of the isolated critical points of \( \varphi \) and the critical points of the Morse functions \( f_1, \ldots, f_q \). Moreover, for every critical point \( p \in \text{Crit}(f_i) \), we have \( \text{ind}_p(\varphi_\epsilon) = \text{ind}_p(f_i) + \text{ind}_{S_i}(\varphi) \).

From property \((S_0)\) we get \( \text{ind}_p(\varphi_\epsilon) < \frac{1}{2} \dim \mathbb{R} V \). Therefore the critical points of \( \varphi_\epsilon \) of index \( \frac{1}{2} \dim \mathbb{R} V \) are exactly the same as those of \( \varphi \). Moreover \( \varphi_\epsilon = \varphi \) near these points. Next, note that if the perturbation above is in small enough neighborhoods \( W_i \) of the \( S_i \)'s then due to assumption \((S)\) we have \( \nabla^\varphi_\epsilon = \nabla^\varphi \).
Now, by Moser argument there is a symplectomorphism \( f : (V, \omega_\varphi) \to (V, \omega_{\varphi_0}) \) which is supported in \( W_1 \cup \ldots \cup W_q \). And again, \( f(\nabla_{\varphi_0} \text{crit}) = \nabla_{\varphi_0} \text{crit} \), and \( f(A) \cap \nabla_{\varphi_0} \text{crit} = \emptyset \). Replacing \( \omega_\varphi \) by \( \omega_{\varphi_0} \) and \( A \) by \( f(A) \) we arrive to the case from the beginning of the proof.

Finally assume that the only critical points of \( \varphi \) consist of one critical submanifolds \( S \) of dimension \( \frac{1}{2} \dim \mathbb{R} V \). Consider the map \( F : V \to S \) defined by

\[
F(x) = \lim_{t \to -\infty} X_t^\varphi(x).
\]

Since \( \varphi \) is Morse-Bott this map is a locally trivial fibration in a neighbourhood of \( S \) (see \[3\]). Note that \( \nabla_{\varphi_0} \text{crit} \) is just the preimage under \( F \) of a point \( p \in S \). Therefore since \( A \cap \nabla_{\varphi_0} \text{crit} = \emptyset \) we have \( F(A) \subset S \setminus \{p\} \).

Since \( S \) is Lagrangian a small neighbourhood of \( S \) can be identified with a neighbourhood of \( T^*(S) \). Pick a Morse function \( G \) on \( S \) having all its critical points in \( S \setminus F(A) \). Clearly the Hamiltonian isotopy \( g_t \) of \( G \) (viewed as a Hamiltonian in \( T^*(S) \)) will displace \( F(A) \) away of \( S \) within arbitrary small time, say \( t = \epsilon \) (compare \[24\]). The result now follows in the same way as in the beginning of the proof. Namely for large enough \( T > 0 \) the Hamiltonian diffeomorphism

\[
X_T^\varphi \circ g_\epsilon \circ X_{-T}^\varphi(A) \cap A = \emptyset
\]

will displace \( A \). \( \square \)

3. Polarizations and decompositions of \( \text{Kähler manifolds} \)

A basic tool that we shall use throughout this work is a decomposition technique for \( \text{Kähler manifolds} \) that was developed in \[5\]. In Subsections 3.1-3.4 we briefly summarize the necessary facts from \[5\] where more details can be found. In Subsection 3.5 we introduce the concept of Lagrangian trace and in 3.6 we compute some examples.

3.1. Polarized \( \text{Kähler manifolds} \). Throughout this paper, by a \( \text{Kähler manifold} \) we mean a triple \((M, \omega, J)\) where \((M, \omega)\) is a closed symplectic manifold and \( J \) is an (integrable) complex structure compatible with \( \omega \).

A polarized \( \text{Kähler manifold} \) \( \mathcal{P} = (M^{2n}, \omega, J; \Sigma) \) is a \( \text{Kähler manifold} \) \((M, \omega, J)\) with \([\omega] \in H^2(M; \mathbb{Z})\) together with a smooth and reduced complex hypersurface \( \Sigma \subset M \) whose homology class \([\Sigma] \in H_{2n-2}(M)\) is the Poincaré dual to \( k[\omega] \in H^2(M) \) for some \( k \in \mathbb{N} \). The number \( k \) will be called the degree of the polarization \( \mathcal{P} \) and denoted by \( k_{\mathcal{P}} \). Note that our notion of polarized \( \text{Kähler manifolds} \) is slightly different from the one common in algebraic geometry.
3.2. **Additional structures associated with a polarization.** We shall now define a distinguished plurisubharmonic function $\varphi_P : M \setminus \Sigma \to \mathbb{R}$ which is canonically associated with the polarization $\mathcal{P}$. For this purpose let $\mathcal{L} = \mathcal{O}_M(\Sigma)$ be the holomorphic line bundle defined by the divisor $\Sigma$. Denote by $s : M \to \mathcal{L}$ the (unique up to a constant factor) holomorphic section whose zero set $\{s = 0\}$ is $\Sigma$. Choose a hermitian metric $\| \cdot \|$ on $\mathcal{L}$, and a compatible connection $\nabla$ with curvature $R^\nabla = 2\pi ik_P \omega$. Finally, define $\varphi_P : M \setminus \Sigma \to \mathbb{R}$ to be

$$\varphi_P(x) = -\frac{1}{4\pi k_P} \log \|s(x)\|^2.$$ 

Put $V = M \setminus \Sigma$. A simple computation shows that $-dd^c \varphi_P = \omega$, hence $\varphi_P$ is plurisubharmonic. Moreover, it is not hard to see that $\varphi_P$ is exhausting and that it has no critical points outside some compact subset of $V$ (see [5]).

It is important to remark that the function $\varphi_P$ is canonically determined by the polarization $\mathcal{P}$ up to an additive constant and does not depend on any of the choices made for $\| \cdot \|$, $s$ or $\nabla$. This is due to the requirement on the curvature $R^\nabla$ and the fact that $J$ is integrable (see [5] for more details). Next, let $g_{\omega,J} = \omega(\cdot, J \cdot)$ be the Kähler Riemannian metric associated with the pair $(\omega, J)$. Finally denote by $X^t_P$ the gradient flow of $\varphi_P$ with respect to $g_{\omega,J}$. (Note that $X^t_P$ is not complete for $t > 0$, since $(V, \omega)$ has finite volume.)

Consider the Stein manifold $(V = M \setminus \Sigma, J, \varphi_P)$. We denote by $\Delta_P \subset V$ its skeleton (see 2.2 above). Note that $\Delta_P \subset M \setminus \Sigma$ is compact since the flow $X^t_P$ is complete at $-\infty$ and $\text{Crit}(\varphi_P)$ is a compact subset of $M \setminus \Sigma$. We remark that $\Delta_P$ is completely determined by the polarization $\mathcal{P}$ without any further choices since the function $\varphi_P$ is determined (up to an additive constant) by $\mathcal{P}$. We shall therefore call $\Delta_P$ *the skeleton* associated with the polarization $\mathcal{P}$.

3.3. **The decomposition associated to a polarization.** In this section we explain how to decompose a Kähler manifold into two basic building blocks. The first piece is a standard symplectic disc bundle over a complex hypersurface. The second piece is the isotropic skeleton of the Stein manifold which is the complement of this hypersurface.

3.3.1. **Standard symplectic disc bundles.** Let $\mathcal{P} = (M, \omega, J; \Sigma)$ be a polarization of degree $k_P$ of a Kähler manifold.

Put $\omega_\Sigma = \omega|_{T(\Sigma)}$ and let $\pi : N_\Sigma \to \Sigma$ be the (complex) normal line bundle of $\Sigma$ in $M$ with first Chern class $c_1^{N_\Sigma} = k_P[\omega_\Sigma] \in H^2(\Sigma)$. Let $\| \cdot \|$ be any hermitian metric on $N_\Sigma$ and denote by $E_\Sigma = \{v \in N_\Sigma \mid \|v\| < 1\}$ the open unit disc bundle of $N_\Sigma$. Choose a connection $\nabla$ on $N_\Sigma$ with curvature $R^\nabla = 2\pi ik_P \omega_\Sigma$ and denote by $\alpha^\nabla$ the associated *transgression 1-form* on $N_\Sigma \setminus 0$ defined by:

- $\alpha^\nabla(u) = 0$, $\alpha^\nabla(iu) = \frac{1}{2\pi}$ for every $u \in N_\Sigma \setminus 0$. 

\( \alpha^\nabla |_{H^\nabla} = 0 \), where \( H^\nabla \) is the horizontal distribution of \( \nabla \).

With this normalization of \( \alpha^\nabla \) we have \( d\alpha^\nabla = -\pi^*(k_P \omega_\Sigma) \). Define now the following symplectic form \( \omega_{\text{can}} \) on \( E_\Sigma \):

\[
\omega_{\text{can}} = k_P \pi^* \omega_\Sigma + d(r^2 \alpha^\nabla),
\]

where \( r \) is the radial coordinate along the fibres induced by \( \| \cdot \| \). It is easy to check that \( \omega_{\text{can}} \) is well defined, that it is symplectic, and has the following three properties:

1. All fibres of \( \pi : E_\Sigma \to \Sigma \) are symplectic with respect to \( \omega_{\text{can}} \) and have area 1.
2. The restriction of \( \omega_{\text{can}} \) to the zero section \( \Sigma \subset E_\Sigma \) equals \( k_P \omega_\Sigma \).
3. \( \omega_{\text{can}} \) is \( S^1 \)-invariant with respect to the obvious circle action on \( E_\Sigma \).

Although \( \omega_{\text{can}} \) a priori depends on \( \| \cdot \| \) and \( \nabla \), different choices of these structures in fact lead to symplectically equivalent results (see [27], see also [5, 9]). We shall henceforth call \( (E_\Sigma, \omega_{\text{can}}) \) the standard symplectic disc bundle over \((\Sigma, \omega_\Sigma)\) modeled on \( N_\Sigma \). Often we shall multiply \( \omega_{\text{can}} \) by a positive number \( c > 0 \) (usually by \( c = \frac{1}{k_P} \) and refer to \((E_\Sigma, c\omega_{\text{can}})\) as the standard symplectic disc bundle with fibres of area \( c \). (Note that now the restriction of this symplectic form to \( \Sigma \subset E_\Sigma \) equals \( c k_P \omega_\Sigma \), not \( k_P \omega_\Sigma \).)

3.3.2. The decomposition. Let \( \mathcal{P} = (M, \omega, J; \Sigma) \) be a polarized Kähler manifold. Denote by \( \rho_\mathcal{P} : M \to \mathbb{R} \) the function \( \rho_\mathcal{P}(x) = \|s(x)\|^2 \), and let \( Z_\mathcal{P} \) be the gradient vector field of \( -\rho_\mathcal{P} \). Note that since \( \rho_\mathcal{P} = e^{-4\pi k_P \varphi_\mathcal{P}} \) the vector fields \( Z_\mathcal{P} \) and \( X_\mathcal{P} \) are positively proportional on \( M \setminus \Delta_\mathcal{P} \).

Theorem 3.3.A (See [5]). Let \( \mathcal{P} = (M, \omega, J; \Sigma) \) be a polarized Kähler manifold. Then, the complement of the skeleton \((M \setminus \Delta_\mathcal{P}, \omega)\) is symplectomorphic to the following standard symplectic disc bundle over \( \Sigma \)

\[
(E_\Sigma; \frac{1}{k_P} \omega_{\text{can}}) \to (\Sigma, k_P \omega_\Sigma)
\]

which is modeled on the normal bundle \( N_\Sigma \), and with fibres of area \( \frac{1}{k_P} \). In fact, there exists a canonical symplectomorphism \( F_\mathcal{P} \), which depends only on \( \mathcal{P} \), such that the following diagram commutes:

\[
\begin{array}{ccc}
(E_\Sigma; \frac{1}{k_P} \omega_{\text{can}}) & \xrightarrow{F_\mathcal{P}} & (M \setminus \Delta_\mathcal{P}, \omega) \\
0 \text{-section} & & \text{inclusion} \\
(\Sigma, \omega_\Sigma) & \xrightarrow{\text{inclusion}} & (\Sigma, \omega_\Sigma)
\end{array}
\]

Moreover, \( F_\mathcal{P} \) sends the flow lines of \( Z_\mathcal{P} \) to the lines of the negative radial flow on \( E_\Sigma \), namely \( DF_\mathcal{P}(Z_\mathcal{P}) \) is negatively proportional to the radial vector field \( r \frac{\partial}{\partial r} \) on \( E_\Sigma \).

The proof of this theorem appears in [5]. The “Moreover” statement, is not stated explicitly in [5] as a theorem but is proved there (see proof of Proposition 7.B in [5]).
3.4. Subcritical polarizations

A polarization $\mathcal{P} = (M, \omega, J; \Sigma)$ is called subcritical if there exists a plurisubharmonic function $\varphi : (V = M \setminus \Sigma, J) \to \mathbb{R}$ such that $(V, J, \varphi)$ is a subcritical Stein manifold (namely $\varphi$ is Morse and for every $p \in \text{Crit} (\varphi)$, $\text{ind}_p (\varphi) < \frac{1}{2} \dim \mathbb{R} V$). Note that we do not assume $\varphi$ to be the canonical function $\varphi_p$. We refer the reader to [9] for more information on subcritical polarizations, examples and criteria for identifying them.

3.5. The Lagrangian trace. Let $\mathcal{P} = (M, \omega, J; \Sigma)$ be a polarized Kähler manifold. We say that the polarization $\mathcal{P}$ has property $(\mathcal{S})$ if $\varphi_p : M \setminus \Sigma \to \mathbb{R}$ has property $(\mathcal{S})$ of Section 2.4.1. We denote by $\nabla^\text{crit}_\mathcal{P} \subset M \setminus \Sigma$ the critical coskeleton of $\varphi_p$.

Let $\rho_p : M \to \mathbb{R}$ be the function defined by $\rho_p (x) = \|s(x)\|^2$. Note that the critical points of $\rho_p$ consist of those of $\varphi_p$ and $\Sigma$ which is a non-degenerate critical submanifold (of index 0). On $M \setminus \Sigma$ the gradients of $\varphi_p$ and of $-\rho_p$ have the same (oriented) flow lines. Denote by $Z^t_p : M \to M$ the gradient flow of $-\rho_p$.

Assume now that $\mathcal{P}$ has property $(\mathcal{S})$ and let $\nabla^\text{crit}_\mathcal{P} \subset M \setminus \Sigma$ be its critical coskeleton. Denote by $\Lambda_\mathcal{P} \subset \Sigma$ the subset obtained from $\nabla^\text{crit}_\mathcal{P}$ by “projecting” it using $\lim_{t \to \infty} Z^t_p$ to $\Sigma$, namely

$$\Lambda_\mathcal{P} = \left\{ x \in \Sigma \mid x = \lim_{t \to \infty} Z^t_p (p), \text{for some } p \in \nabla^\text{crit}_\mathcal{P} \right\}.$$  

In case $\nabla^\text{crit}_\mathcal{P} = \emptyset$ (e.g. if $\varphi_p$ is subcritical) we put $\Lambda_\mathcal{P} = \emptyset$. We call $\Lambda_\mathcal{P} \subset \Sigma$ the Lagrangian trace of the polarization $\mathcal{P}$. This term is justified by the following proposition.

**Proposition 3.5.A.** Let $\mathcal{P} = (M, \omega, J; \Sigma)$ be a polarized Kähler manifold with $\varphi_p$ having property $(\mathcal{S})$. Let $p_1', \ldots, p_r'$ be those critical points of $\varphi_p$ as chosen in 2.4 above. Then the corresponding Lagrangian trace $\Lambda_\mathcal{P} \subset \Sigma$ consists of a union of $r$ immersed (but possibly embedded) Lagrangian spheres one for each of the points $p_1', \ldots, p_r'$.

**Proof.** Denote by $G : M \setminus \Delta_\mathcal{P} \to \Sigma$ the end point map of the flow $Z^t_p$; namely $G(x) = \lim_{t \to \infty} Z^t_p (x)$. Clearly

$$\Lambda_\mathcal{P} = \bigcup_{i=1}^r G(W^u_{p_i'}),$$

where $W^u_{p_i'} = W^u_{p_i'} (X_{p_i}) \setminus \{ p_i' \} = W^u_{p_i'} (Z_{p_i}) \setminus \{ p_i' \}$ are the unstable submanifolds of $X_{p_i}$ at $p_i'$. We shall prove that for every $i$, $G(W^u_{p_i'})$ is an immersed Lagrangian sphere in $\Sigma$.

Denote by $\tau : E_\Sigma \to \Sigma$ the standard symplectic disc bundle, endowed with the symplectic structure $\frac{1}{k_p} \omega_{\text{can}}$. Using Theorem 3.3.A we may identify $(E_\Sigma, \frac{1}{k_p} \omega_{\text{can}})$ with $(M \setminus \Delta_\mathcal{P}, \omega)$. We shall view from now on $W^u_{p_i'}$ as a submanifold of $E_\Sigma$ and $G$ as a map $G : E_\Sigma \to \Sigma$. It follows from Theorem 3.3.A that the map $G$ coincides with projection map $\pi : E_\Sigma \to \Sigma$.

Denote $P_\epsilon = \{ v \in E_\Sigma \mid \|v\| = \epsilon \}$. Note that $W^u_{p_i'}$ intersects $P_\epsilon$ transversely because the vector field $\frac{\partial}{\partial r}$ is tangent to $W^u_{p_i'}$. Put $L_t = W^u_{p_i'} \cap P_\epsilon$. We claim that $L_t$ is diffeomorphic to a sphere. Indeed, pick a small ball $B_t \subset W^u_{p_i'}$ centered around $p_i'$ whose boundary $\partial B_t$ is
transverse to \( Z_p \). Taking \( B_i \) to be small enough we may assume that every flow line of \( Z_p \) intersects \( \partial B_i \) exactly once. Denote by \( \nu : E_\Sigma \setminus \Sigma \to P_t \) the map \( \nu(v) = \epsilon \frac{v}{\|v\|} \). By our choice of \( B_i \) we have that \( \nu \) sends the sphere \( \partial B_i \) diffeomorphically onto \( L_i \).

Next, we claim that \( \pi|_{L_i} : L_i \to \Sigma \) is an immersion. To prove this, note that for every \( x \in E_\Sigma \setminus \Sigma \), \( \ker D\pi_x = \mathbb{R} \frac{\partial}{\partial r} \oplus i\mathbb{R} \frac{\partial}{\partial r} \). Now let \( v = a \frac{\partial}{\partial r} + ib \frac{\partial}{\partial r} \in \ker D\pi_x \cap T_x(L_i) \). Since \( L_i \subset P_c \), we have \( a = 0 \). By Lemma 2.2.4 \( W'_i \) is Lagrangian (it is coisotropic and has half the dimension of \( M \)). As \( \frac{\partial}{\partial r} \) is tangent to \( W'_i \), we have \( \omega_{\text{can}}(\frac{\partial}{\partial r}, v) = 0 \). But \( \omega_{\text{can}}(\frac{\partial}{\partial r}, v) = b\omega_{\text{can}}(\frac{\partial}{\partial r}, i\frac{\partial}{\partial r}) \) which can vanish only if \( b = 0 \). Thus \( v = 0 \). This proves that \( \pi|_{L_i} \) is an immersion.

It remains to prove that \( \pi(L_i) \) is Lagrangian. Let \( \xi = \ker(\alpha^\nabla|_{(P_i)}) \) be the contact distribution on \( P_i \). Then \( T(P_i) = \xi \oplus i\mathbb{R} \frac{\partial}{\partial r} \). We first claim that \( T(L_i) \subset \xi \). Indeed, let \( v = u + ia \frac{\partial}{\partial r} \in T(L_i) \), where \( u \in \xi \), \( a \in \mathbb{R} \). As before \( \omega_{\text{can}}(v, \frac{\partial}{\partial r}) = 0 \), because \( W'_i \) is Lagrangian. But \( \omega_{\text{can}}(v, \frac{\partial}{\partial r}) = a\omega_{\text{can}}(i\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) \) which can vanish only if \( a = 0 \). Thus \( v = u \in \xi \). Finally note that

\[
\omega_{\text{can}} = k_p \pi^* \omega_\Sigma + d(r^2 \alpha^\nabla) = k_p(1 - r^2)\omega_\Sigma + 2rd\alpha \wedge \alpha^\nabla,
\]

hence \( D\pi \) sends \( (\xi, (1 - r^2)\omega_{\text{can}}|_\xi) \) isomorphically to \( (T(\Sigma), \omega_\Sigma) \). As \( T(L_i) \) is Lagrangian in \( (\xi, \omega_{\text{can}}|_\xi) \) we conclude that \( D\pi(T(L_i)) \) is Lagrangian in \( (T(\Sigma), \omega_\Sigma) \).

3.6. **Examples.** Let us present a few explicit examples of Lagrangian traces coming from various polarizations.

3.6.1. **Subcritical polarizations.** Let \( \mathcal{P} = (M, \omega, J; \Sigma) \) be a subcritical polarization (see 3.4). In this case \( \nabla^\text{crit} = 0 \), hence \( \Lambda_\mathcal{P} = 0 \).

The simplest example of a subcritical polarization consists of \( M = \mathbb{C}P^n \) endowed with its standard symplectic Kähler form \( \sigma \) and \( \Sigma \approx \mathbb{C}P^{n-1} \) being a linear hyperplane. The skeleton in this case is a point \( \Delta_\mathcal{P} = \text{pt} \) (see [5] for more details).

Another example of a subcritical polarization is \( M = \mathbb{C}P^n \times \mathbb{C}P^{n+r}, r \geq 1 \), endowed with the split symplectic structure \( \sigma \oplus \sigma \), and

\[
\Sigma = \left\{ ([z_0 : \ldots : z_n], [w_0 : \ldots : w_{n+r}]) \in \mathbb{C}P^n \times \mathbb{C}P^{n+r} \mid \sum_{i=0}^{n-1} z_i w_i = z_n \sum_{j=n}^{n+r} w_j \right\}.
\]

In this case the skeleton turns out to be an isotropic copy of \( \mathbb{C}P^n \) (see [5] for more details).

We refer the reader to [5] for more details and examples of subcritical polarizations.

3.6.2. **The quadric.** Consider the polarization \( \mathcal{P} = (M, \omega, J; \Sigma) \) with \( M = \mathbb{C}P^{n+1} \) and \( \Sigma \) the quadric:

\[
\Sigma = \{ z_0^2 + \cdots + z_{n+1}^2 = 0 \} \subset \mathbb{C}P^{n+1}.
\]
The function \( \varphi_p : \mathbb{C}P^{n+1} \setminus \Sigma \to \mathbb{R} \) is (up to a constant factor):

\[
\varphi_p([z_0 : \ldots : z_{n+1}]) = \log \frac{\sum_{j=0}^{n+1} z_j^2}{(\sum_{j=0}^{n+1} |z_j|^2)^2}.
\]

A straightforward computation shows that \( \varphi_p \) is Morse-Bott and \( \text{Crit}(\varphi_p) = \mathbb{R}P^{n+1} \), where \( \mathbb{R}P^{n+1} \) is embedded in \( \mathbb{C}P^{n+1} \) as

\[
\mathbb{R}P^{n+1} = \{[z_0 : \ldots : z_{n+1}] \in \mathbb{C}P^{n+1} \mid z_j \in \mathbb{R} \text{ for every } j \}.
\]

Note also that \( \varphi_p \) has property (S) (see 2.4.1). Let \( p = [1 : 0 : \ldots : 0] \in \mathbb{R}P^{n+1} \). A straightforward computation shows that

\[
\nabla_{\varphi}^{\text{crit}} = W_p^p(X_p) = \{[1 : ix_1 : \ldots : ix_{n+1}] \mid x_j \in \mathbb{R} \text{ for every } j \} \setminus \Sigma.
\]

Hence the Lagrangian trace \( \Lambda_p \subset \Sigma \) is the following Lagrangian sphere:

\[
\Lambda_p = \left\{ [1 : ix_1 : \ldots : ix_{n+1}] \mid x_j \in \mathbb{R} \text{ for every } j, \text{ and } \sum_{j=1}^{n+1} x_j^2 = 1 \right\}.
\]

3.6.3. Polarization of \( \mathbb{C}P^n \times \mathbb{C}P^n \). Let \( \mathcal{P} = (M = \mathbb{C}P^n \times \mathbb{C}P^n, \omega = \sigma \oplus \sigma, J; \Sigma) \) where

\[
\Sigma = \left\{ ([z_0 : \ldots : z_n], [w_0 : \ldots : w_n]) \in \mathbb{C}P^n \times \mathbb{C}P^n \mid \sum_{j=0}^{n-1} z_j w_j = z_n w_n \right\}.
\]

A simple computation of the function \( \varphi_p \) show that it is Morse-Bott. It has one critical submanifold which is a Lagrangian copy of \( \mathbb{C}P^n \):

\[
\Delta_p = \{([z_0 : \ldots : z_n], [\overline{z}_0 : \ldots : \overline{z}_{n-1} : -z_n]) \in \mathbb{C}P^n \times \mathbb{C}P^n \mid [z_0 : \ldots : z_n] \in \mathbb{C}P^n \}.
\]

Pick \( p = ([0 : \ldots : 0 : 1], [0 : \ldots : 0 : 1]) \in \Delta_p \). A straightforward computation shows that

\[
\nabla_{\varphi}^{\text{crit}} = W_p^p(X_p) = \{([z_0 : \ldots : z_n], [\overline{z}_0 : \ldots : \overline{z}_n]) \mid [z_0 : \ldots : z_n] \in \mathbb{C}P^n \text{ and } z_n \neq 0 \} \setminus \Sigma.
\]

Finally, the Lagrangian trace is the following Lagrangian sphere:

\[
\Lambda_p = \left\{ ([z_0 : \ldots : z_n], [\overline{z}_0 : \ldots : \overline{z}_n]) \in \Sigma \mid \sum_{j=0}^{n-1} |z_j|^2 = |z_n|^2 \right\}.
\]

3.6.4. Lagrangian trace of hypersurfaces in \( \mathbb{C}P^{n+1} \). Generalizing Example 3.6.2 above, consider \( \mathcal{P} = (M, \omega, J; \Sigma) \) with \( M = \mathbb{C}P^{n+1} \) and \( \Sigma \) the degree \( d > 2 \) hypersurface:

\[
\Sigma = \{ z_0^d + \cdots + z_{n+1}^d = 0 \} \subset \mathbb{C}P^{n+1}.
\]

The function \( \varphi_p : \mathbb{C}P^{n+1} \setminus \Sigma \to \mathbb{R} \) is (up to a constant factor):

\[
\varphi_p([z_0 : \ldots : z_{n+1}]) = \log \frac{\sum_{j=0}^{n+1} z_j^d}{(\sum_{j=0}^{n+1} |z_j|^2)^d}.
\]
When \( d > 2 \) all the critical points of \( \varphi_p \) are isolated. The critical points of index \( n + 1 \) are all the points \([1 : \xi_1 : \ldots : \xi_{n+1}]\) with \( \xi_i^d = 1 \) for every \( i \). Denote by \( W^u_{[1 : \ldots : 1]} = W^u_{[1 : \ldots : 1]}(X_p) \) the unstable submanifold corresponding to the critical point \([1 : \ldots : 1]\), and let

\[
\Lambda_{[1 : \ldots : 1]} = \left\{ \lim_{t \to \infty} Z^t_p (p) \mid p \in W^u_{[1 : \ldots : 1]} \setminus \{[1 : \ldots : 1]\} \right\}
\]

be the part of the trace corresponding to \([1 : \ldots : 1]\). Here \( Z^t_p \) is the flow defined in 3.5.

We have not managed to compute \( W^u_{[1 : \ldots : 1]} \) nor \( \Lambda_{[1 : \ldots : 1]} \) explicitly. However we do have the following information on \( \Lambda_P \). Let \( \xi \in \mathbb{C} \) be a primitive root of unity of degree \( d \).

For every multi-index \( \hat{i} = (i_1, \ldots, i_{n+1}) \) where \( i_1, \ldots, i_{n+1} \in \{0, \ldots, d-1\} \) denote by \( R_{\hat{i}} : \mathbb{C}P^{n+1} \to \mathbb{C}P^{n+1} \) the map:

\[
R_{\hat{i}}([z_0 : z_1 : \ldots : z_{n+1}]) = [z_0 : \xi^{i_1} z_1 : \ldots : \xi^{i_{n+1}} z_{n+1}].
\]

A simple computation shows that the vector field \( X_p \) is invariant under the action of each of the maps \( R_{\hat{i}} \). Therefore the unstable submanifold \( W^u_{[1 : \xi^{i_1} : \ldots : \xi^{i_{n+1}}]} \) coincides with \( R_{\hat{i}}(W^u_{[1 : \ldots : 1]}) \). We conclude that

\[
\nabla^\text{crit}_\varphi = \bigcup_{\hat{i} \in I} R_{\hat{i}}(W^u_{[1 : \ldots : 1]}),
\]

where \( I \) is the set of all multi-indices \( \hat{i} \in \{0, \ldots, d-1\}^{n+1} \). In particular

\[
\Lambda_P = \bigcup_{\hat{i} \in I} R_{\hat{i}}(\Lambda_{[1 : \ldots : 1]}),
\]

hence the Lagrangian trace \( \Lambda_P \) is a union of not more than \( d^{n+1} \) possibly immersed Lagrangian spheres.

4. LAGRANGIAN SUBMANIFOLDS AND POLARIZATIONS

In this section we consider Lagrangian submanifolds of manifolds \( \Sigma \) which appear as hyperplane sections in some polarization \( \mathcal{P} = (M, \omega, J; \Sigma) \). Given a Lagrangian \( L \subset \Sigma \) our strategy will be to go one dimension up and construct a new Lagrangian \( \Gamma_L \subset M \setminus \Sigma \). The advantage is that, sometimes, due to the ambient geometry of \( M \setminus \Sigma \) it is easier to compute symplectic invariants of \( \Gamma_L \) than those of \( L \). The basic construction is presented in detail in subsection 4.1 below. Before we continue we remark again that all Lagrangian submanifolds are assumed to be compact and without boundary, unless explicitly otherwise stated.

Let us briefly recall now the notions of monotone symplectic manifold and monotone Lagrangian. Given a symplectic manifold \( (X, \omega) \) we denote by \( c_1^X \in H^2(X) \) the first Chern class of its tangent bundle (viewed as a complex vector bundle). A symplectic manifold \( (X, \omega) \) is called \emph{spherically monotone} if the following two conditions are satisfied:
• $c_1^X$ does not vanish on $\pi_2(X)$.
• There exists $\lambda > 0$ such that for every $A \in \pi_2(X)$, $\omega(A) = \lambda c_1^X(A)$.

Denote by $N_X \in \mathbb{Z}_+$ the positive generator of the subgroup $c_1^X(\pi_2(X)) \subset \mathbb{Z}$. We call $N_X$ the minimal Chern number of $(X,\omega)$.

A Lagrangian submanifold $K \subset (X,\omega)$ is called monotone if there exists $\eta > 0$ such that the following two conditions are satisfied:

• The Maslov class of $K$, $\mu_K : \pi_2(X,K) \to \mathbb{Z}$ is not zero.
• There exists $\eta > 0$ such that for every $A \in \pi_2(X,K)$, $\omega(A) = \eta \mu_K(A)$.

Denote by $N_K \in \mathbb{Z}_+$ the positive generator of the subgroup $\mu_K(\pi_2(X,K)) \subset \mathbb{Z}$. We call $N_K$ the minimal Maslov number of $K$.

4.1. The Lagrangian circle bundle construction. Let $\mathcal{P} = (M,\omega, J; \Sigma)$ be a polarized Kähler manifold. Put $\omega_\Sigma = \omega|_{T(\Sigma)}$. Let $L \subset (\Sigma, \omega_\Sigma)$ be a Lagrangian submanifold. Consider the standard symplectic disc bundle $E_\Sigma \to \Sigma$ endowed with the symplectic structure $\frac{1}{k_{p}}\omega_{\text{can}}$ (see 3.3.1). By Theorem 3.3.A we can identify $(E_\Sigma, \frac{1}{k_{p}}\omega_{\text{can}})$ with $(M \setminus \Delta_{\mathcal{P}}, \omega)$. Pick $0 < \epsilon < 1$, and consider the circle bundle $P = \{ v \in E_\Sigma \mid \|v\| = \epsilon \}$ over $\Sigma$. Denote by $\pi_{\mathcal{P}} : P \to \Sigma$ the projection. Finally define

$$\Gamma_L = \pi_{\mathcal{P}}^{-1}(L) \subset P$$

to be the total space of the restriction of $P$ to $L$. As we shall see in a moment

$$\Gamma_L \subset (E_\Sigma \setminus \Sigma, \frac{1}{k_{p}}\omega_{\text{can}}) \hookrightarrow (M \setminus \Sigma, \omega)$$

is a Lagrangian submanifold. We call $\Gamma_L$ the Lagrangian circle bundle over $L$.

More generally, let $\mathcal{P} = (M,\omega, J; \Sigma)$ be a polarized Kähler manifold and $(X,\omega_{\Sigma})$ be another symplectic manifold. Let $L \subset (\Sigma \times X, \omega_\Sigma \oplus \omega_X)$ be a Lagrangian submanifold. Pick $0 < \epsilon < 1$ and let $P = \{ v \in E_\Sigma \mid \|v\| = \epsilon \}$ be as before. Consider now the circle bundle $\pi_{\mathcal{P} \times X} : P \times X \to \Sigma \times X$. Define

$$\Gamma_L = \pi_{\mathcal{P} \times X}^{-1}(L) \subset P \times X.$$

We claim that

$$\Gamma_L \subset (E_\Sigma \setminus \Sigma) \times X, \frac{1}{k_{p}}\omega_{\text{can}} \oplus (1 - \epsilon^2)\omega_X) \hookrightarrow ((M \setminus \Sigma) \times X, \omega \oplus (1 - \epsilon^2)\omega_X)$$

is a Lagrangian submanifold. This follows immediately from the definition of $\omega_{\text{can}}$ (see 3.3.1) since

$$\left.\left(\frac{1}{k_{p}}\omega_{\text{can}} \oplus (1 - \epsilon^2)\omega_X\right)\right|_{T(P \times X)} = \left.(\pi_{\mathcal{P}}^*\omega_\Sigma + 2rdr \wedge \alpha \nabla + r^2d\alpha \nabla \oplus (1 - \epsilon^2)\omega_X)\right|_{T(P \times X)}$$

$$= \left.\left(\left(1 - \epsilon^2\right)\pi_{\mathcal{P}}^*\omega_\Sigma \oplus (1 - \epsilon^2)\omega_X\right)\right|_{T(P \times X)} = \left.(1 - \epsilon^2)\pi_{\mathcal{P} \times X}^*(\omega_\Sigma \oplus \omega_X)\right).$$
Note the \((1 - \epsilon^2)\) rescaling of the symplectic structure along the \(X\) factor. Note also that this construction coincides with the previous one when \(X\) is a point. We shall call this \(\Gamma_L\) too the Lagrangian circle bundle over \(L\). Note also that this construction coincides with the previous one when \(X\) is a point. We shall call this \(\Gamma_L\) too the Lagrangian circle bundle over \(L\). Note that \(\Gamma_L\) depends on the parameter \(\epsilon\) (since \(P\) does). In fact \(\Gamma_L\)'s corresponding to different choices of \(\epsilon\) are not Hamiltonianly isotopic (even when \(X = pt\)), however they are conformally symplectic in \((M \setminus \Sigma) \times X\). This dependence on \(\epsilon\) will not trouble us in the sequel as \(\Gamma_L\) will be used only as an auxiliary object for studying the topology of \(L\).

The following proposition compares the Maslov classes of \(L \subset \Sigma \times X\) and \(\Gamma_L \subset (M \setminus \Sigma) \times X\). We denote these classes by \(\mu_L : \pi_2(\Sigma \times X, L) \to \mathbb{Z}\), \(\mu_{\Gamma_L} : \pi_2((M \setminus \Sigma) \times X, \Gamma_L) \to \mathbb{Z}\).

**Proposition 4.1.A.** Let \(\mathcal{P} = (M, \omega, J; \Sigma)\) be a polarized Kähler manifold and \((X, \omega_X)\) another symplectic manifold. Let \(L \subset (\Sigma \times X, \omega_\Sigma \oplus \omega_X)\) be Lagrangian submanifold. Let \(\Gamma_L \subset ((M \setminus \Sigma) \times X, \omega \oplus (1 - \epsilon^2)\omega_X)\) be its corresponding circle bundle. Then for every \(A \in \pi_2(P \times X, \Gamma_L)\),

\[
\mu_{\Gamma_L}(i_* A) = \mu_L(\pi_{P \times X_\ast} A),
\]

where \(i : P \times X \to M \times X\) is the inclusion. Moreover, if \(L \subset (\Sigma \times X, \omega_\Sigma \oplus \omega_X)\) is monotone and \(\dim_{\mathbb{C}} \Sigma \geq 2\) then \(\Gamma_L \subset ((M \setminus \Sigma) \times X, \omega \oplus (1 - \epsilon^2)\omega_X)\) is also monotone and \(N_{\Gamma_L} = N_L\).

**Remarks.**

1. Note that although \(\Gamma_L \subset (M \setminus \Sigma) \times X\) is monotone (when \(L\) is), \(\Gamma_L\) is usually not monotone in \(M \times X\).

2. It can be easily seen from the proof below that statement on monotonicity in Proposition 4.1.A remains true when \(\dim_{\mathbb{C}} \Sigma = 1\) and \(\mathcal{P}\) is a subcritical polarization. Note however that the only subcritical polarization \(\mathcal{P} = (M, \omega, J; \Sigma)\) with \(\dim_{\mathbb{C}} \Sigma = 1\) is \(M = \mathbb{C}P^2, \Sigma = \mathbb{C}P^1\) where \(\Sigma\) is embedded as a projective line in \(\mathbb{C}P^2\) (see [9]).

**Proof.** To simplify notation we present the proof for the case \(X = pt\). The proof of the general case is very similar.

Let \(A \in \pi_2(P, \Gamma_L)\) be represented by \(\tilde{u} : (D, \partial D) \to (P, \Gamma_L)\). Put \(u = \pi_P \circ \tilde{u} : (D, \partial D) \to (\Sigma, L)\). We have to prove that \(\mu_{\Gamma_L}([\tilde{u}]) = \mu_L([u])\).

Put \(V = M \setminus \Sigma\) and let \(\xi = \ker(\alpha^\vee|_{T(P)})\) be the contact distribution on \(P\). Denote by \(N_{\Sigma} \to \Sigma\) the normal bundle of \(\Sigma\) in \(M\) (viewed as a complex line bundle). Throughout the proof we shall also use the following notation: given a symplectic vector bundle \((W, \Omega) \to X\) over a manifold \(X\) and a map \(v : Y \to X\) we shall write \((v^* W, v^* \Omega) \to Y\) for the pulled back symplectic vector bundle, namely for every \(y \in Y\), \((v^* W_y, v^* \Omega_y) = (W_{v(y)}, \Omega_{v(y)})\).
With these notation the symplectic vector bundle \((T(V)|_p, \omega)\) is isomorphic to \((\xi \oplus \pi^*_p N_{\Sigma}, \omega|_\xi \oplus \pi^*_p \sigma)\), where \(\sigma = \omega|_{N_{\Sigma}}\). Now,

\[
\begin{align*}
(\tilde{u}^*T(V), \tilde{u}^*\omega) &\cong \left(\tilde{u}^*\xi \oplus \tilde{u}^*\pi^*_p N_{\Sigma}, \tilde{u}^*\omega|_\xi \oplus \tilde{u}^*\pi^*_p \sigma\right) \\
&\cong \left(u^*T(\Sigma) \oplus \tilde{u}^*\pi^*_p N_{\Sigma}, (1 - \epsilon^2)u^*\omega_{\Sigma} \oplus \tilde{u}^*\pi^*_p \sigma\right),
\end{align*}
\]

where the last isomorphism follows from the fact that \((\xi, \omega|_\xi) \cong (\pi^*_p T(\Sigma), (1 - \epsilon^2)\pi^*_p \omega_{\Sigma})\) and \(\tilde{u}^*\pi^*_p = u^*\).

Consider now the loop of Lagrangian subspaces

\[\tilde{\lambda}(t) = T_{\tilde{u}(t)}(\Gamma_L) \subset T_{\tilde{u}(t)}(V), \quad t \in \partial \Delta.\]

In a compatible way to the symplectic isomorphism \((\ref{symplectic-isomorphism})\) we have the following isomorphism of Lagrangian subbundles over \(\partial \Delta\):

\[
\tilde{\lambda}(t) \cong T_{\tilde{u}(t)}(L) \oplus \tau_{\tilde{u}(t)},
\]

where \(\tau \subset T(P)\) is the subbundle \(\tau = i\mathbb{R} \frac{\partial}{\partial r}\). The Maslov index of the loop of Lagrangian subspaces \(\{\tau_{\tilde{u}(t)}\}_{t \in \partial \Delta} \subset \left(\tilde{u}^*\pi^*_p N_{\Sigma}, \tilde{u}^*\pi^*_p \sigma\right)\) is 0 because the bundle \(\tau\) is globally defined, hence this loop extends to the disc \(\Delta\). Thus by \((\ref{maslov-index})\), \(\mu_{\Gamma_L}([\tilde{u}])\) equals the Maslov index of the loop \(\{T_{\tilde{u}(t)}(L)\}_{t \in \partial \Delta} \subset (u^*T(\Sigma), u^*\omega_{\Sigma})\) which is exactly \(\mu_L([u])\). This proves the equality of the Maslov indices.

Now suppose that \(\dim_{\mathbb{C}} \Sigma \geq 2\). Put \(n = \dim_{\mathbb{C}} M\). As \(\Delta_p \subset M\) has dimension at most \(n\) and \(n > 2\) we have by a general position argument that

\[\pi_2(M \setminus \Sigma, \Gamma_L) \cong \pi_2((M \setminus \Sigma) \setminus \Delta_p, \Gamma_L) \cong \pi_2(P, \Gamma_L).\]

But \(\pi_p : \pi_2(P, \Gamma_L) \to \pi_2(\Sigma, L)\) is an isomorphism. By the equality of Maslov indices just proved it follows that \(\Gamma_L\) is monotone if and only if \(L\) is monotone and moreover that \(N_{\Gamma_L} = N_L\). \qedhere

4.2. Displaceable Lagrangian submanifolds. Let \((\Sigma, \omega_{\Sigma})\) be a closed symplectic manifold. We say that \((\Sigma, \omega_{\Sigma})\) participates in a polarization \(P\) if it can be embedded in a polarized Kähler manifold \(P = (M, \omega, J; \Sigma)\) in such a way that \(\omega_{\Sigma} = \omega|_{\Sigma}\).

Recall that given a polarization \(P = (M, \omega, J; \Sigma), (M \setminus \Sigma, J, \varphi_p)\) is a Stein manifold. Denote by \(\hat{\omega}\) the symplectic structure associated to the completion of this Stein manifold (see Section 2.1 and Remark 2.1.C). Given a Lagrangian submanifold \(K\) we denote by \(HF(K, K)\) the Floer homology of \(K\) with itself (see Section 5).

**Theorem 4.2.A.** Let \((\Sigma, \omega_{\Sigma})\) be a symplectic manifold that participates in a subcritical polarization \(P = (M, \omega, J; \Sigma)\). Let \((X, \omega_X)\) be another tame symplectic manifold. Let \(L \subset (\Sigma \times X, \omega_{\Sigma} \oplus \omega_X)\) be a Lagrangian submanifold. Then there exists a compactly supported \(h \in \text{Ham}((M \setminus \Sigma) \times X, \hat{\omega} \oplus \omega_X)\) such that \(h(\Gamma_L) \cap \Gamma_L = \emptyset\). In particular, if \(L\)}
is monotone with \( N_L \geq 2 \) then \( HF(\Gamma_L, \Gamma_L) = 0 \). Here the Floer homology is computed in \((M \setminus \Sigma) \times X\) (not in \(M \times X\)).

Remark. The Floer homology \( HF(\Gamma_L, \Gamma_L) \) when computed in \((M \setminus \Sigma) \times X\) with either of the symplectic structures \( \omega \oplus (1 - \epsilon^2)\omega_X \) or \( \hat{\omega} \oplus (1 - \epsilon^2)\omega_X \) is the same. This follows from convexity at infinity of \(M \setminus \Sigma\). See [S] for more details on this type of argument.

Proof of Theorem 4.2.4. Let \( V = M \setminus \Sigma \), and denote by \( \hat{\omega} \) the symplectic structure associated to the completion of \((V, J, \varphi_p)\). Since \( \mathcal{P} = (M, \omega, J; \Sigma) \) is subcritical there exists a subcritical exhausting plurisubharmonic function \( \varphi: V \to \mathbb{R} \) such that \((V, J, \varphi)\) is complete too (see Section 3.3 and Lemma 2.1.4). By Lemma 2.1.B \((V, \hat{\omega})\) and \((V, \omega_\varphi)\) are symplectomorphic.

Denote by \( pr_V : V \times X \to V \) the projection. It now follows from Lemma 2.3.A that there exists a compactly supported \( h' \in Ham(V, \hat{\omega}) \) such that \( h'(pr_V(\Gamma_L)) \cap pr_V(\Gamma_L) = \emptyset \). Let \( h = h' \times \mathbb{1}_X : V \times X \to V \times X \). Clearly \( h(\Gamma_L) \cap \Gamma_L = \emptyset \). If necessary one can Hamiltonianly cut off \( \mathbb{1}_X \), in case \( X \) is not compact, in order to obtain a compactly supported \( h \).

The statement on Floer homology follows immediately because by Proposition 4.1.A whenever \( L \) is monotone \( \Gamma_L \) is monotone too and \( N_{\Gamma_L} = N_L \). (Note the second remark after the statement of Proposition 4.1.A for the case when \( \dim \mathbb{C} \Sigma = 1 \)).

Theorem 4.2.B. Let \((\Sigma, \omega_\Sigma)\) be a symplectic manifold that participates in the polarization \( \mathcal{P} = (M, \omega, J; \Sigma) \) (see 3.2). Let \((X, \omega_X)\) be another tame symplectic manifold and \( L \subset (\Sigma \times X, \omega_\Sigma \oplus \omega_X) \) a Lagrangian submanifold. If \( L \cap (\Lambda_\mathcal{P} \times X) = \emptyset \), then there exists a compactly supported \( h \in Ham((M \setminus \Sigma) \times X, \hat{\omega} \oplus (1 - \epsilon^2)\omega_X) \) such that \( h(\Gamma_L) \cap \Gamma_L = \emptyset \). In particular, if in addition \( \dim \mathbb{C} \Sigma \geq 2 \) and \( L \) is monotone with \( N_L \geq 2 \) then \( HF(\Gamma_L, \Gamma_L) = 0 \). Here the Floer homology is computed in \((M \setminus \Sigma) \times X\) (not in \(M \times X\)).

Remark. Theorem 4.2.B generalizes Theorem 4.2.A since if \( \mathcal{P} = (M, \omega, J; \Sigma) \) is a subcritical polarization then \( \Lambda_\mathcal{P} = \emptyset \).

Proof of Theorem 4.2.B. The proof is similar to the one of Theorem 4.2.A. The only additional point is that, due to the assumption that \( L \cap (\Lambda_\mathcal{P} \times X) = \emptyset \), we have that \( \Gamma_L \) lies in the complement of \( \nabla^{\crit} \times X \). Indeed, due to Lemma 2.1.A the Stein manifold \((M \setminus \Sigma, \varphi_p)\) can be made complete in such a way that \( \varphi_p \) is not altered in a neighbourhood of \( pr_M(\Gamma_L) \), where \( pr_M : M \times X \to M \) is the projection on \( M \). Therefore, \( \Gamma_L \) continues to lie in the complement of \( \nabla^{\crit} \times X \) in \((M \setminus \Sigma) \times X, \hat{\omega} \oplus (1 - \epsilon^2)\omega_X) \). The Hamiltonian displacement along the \( M \setminus \Sigma \) factor follows now from Lemma 2.4.A.
5. Computations in Floer homology

In this section we summarize necessary facts from Floer theory for monotone Lagrangian submanifolds. This extension of Floer’s work was developed by Oh [31]. In section 5.2 we describe a spectral sequence which converges to the Floer homology of a Lagrangian. This spectral sequence is based on the theory developed by Oh [33, 32], however our construction is somewhat different. We refer the reader to [19, 18, 31, 33, 32, 37] for more details on Floer theory.

Let \((M, \omega)\) be a tame symplectic manifold (see [2], chapter 10), and let \(L \subset (M, \omega)\) be a monotone Lagrangian submanifold with \(N_L \geq 2\). In this situation one can define the Floer homology of \(L\) which is an invariant of the Hamiltonian isotopy class of \(L\). Let \(L' = \phi(L)\), \(\phi \in \text{Ham}(M, \omega)\), be a Hamiltonianly isotopic copy of \(L\), and assume that \(L \pitchfork L'\). Let

\[
CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x
\]

be the vector space over \(\mathbb{Z}_2\) spanned by the intersection points of \(L \cap L'\). One defines a differential \(d_J : CF(L, L') \to CF(L, L')\) by choosing an almost complex structure \(J\) and counting Floer trajectories (pseudo-holomorphic strips) connecting pairs of points of \(L \cap L'\). The homology of \(d_J\), denoted by \(HF(L, L'; J)\) is called the Floer homology of the pair \((L, L')\).

The Floer complex \(CF(L, L')\) has a (relative) \(\mathbb{Z}/N_L\) grading. This grading depends on a choice of a base intersection point \(x_0 \in L \cap L'\). Different choices of such a point yield a shift in the grading. Once \(x_0\) is fixed we denote by \(CF^\ast(\mod N_L)(L, L'; x_0)\) the \(i\)'th \((\mod N_L)\) component of \(CF\). An index computation (see [33]) shows that the differential \(d_J\) increases grading by 1, \(d_J : CF^\ast(\mod N_L)(L, L'; x_0) \to CF^{\ast+1}(\mod N_L)(L, L'; x_0)\). Thus the Floer homology

\[
HF(L, L'; J) = \bigoplus_{i=0}^{N_L-1} HF^i(\mod N_L)(L, L'; J, x_0)
\]

has a \(\mathbb{Z}/N_L\) grading. Again, this grading is relative as different choices of the base point \(x_0\) result in a shifted \(\mathbb{Z}/N_L\) grading. Note that there exists a more sophisticated approach to grading, due to Seidel [37], which overcomes the relativity problem.

The main feature of the Floer homology is its invariance under the choice of \(L'\) (and of \(J\)), namely, for every \(L'' = \psi(L)\), \(\psi \in \text{Ham}(M, \omega)\), intersecting \(L\) transversely and any generic almost complex structures \(J', J''\) there is an isomorphism \(HF(L, L'; J') \cong HF(L, L''; J'')\). Moreover, this isomorphism preserves the \(\mathbb{Z}/N_L\) grading up to a shift,
namely for given choices \( x_0' \in L \cap L' \) and \( x_0'' \in L \cap L'' \) there exists \( s \) such that
\[
HF^*(\mod N_L)(L, L'; J', x_0') \cong HF^{*+s(\mod N_L)}(L, L''; J'', x_0'').
\]

Finally, in case \( L, L' \) do not intersect transversely we define \( HF(L, L') = HF(L, L') \), where \( L_\epsilon \) is a small Hamiltonian perturbation of \( L' \) with \( L_\epsilon \cap L = \emptyset \).

5.1. The case of \( HF(L, L) \). Let \( L_\epsilon \) be a perturbation of \( L \) built in a Weinstein neighbourhood \( U \) of \( L \) using a \( C^2 \)-small Hamiltonian Morse function \( f : L \to \mathbb{R} \). Assume also that \( f \) has exactly one relative minimum \( x_0 \). Denote by \( C_f^\epsilon \) the Morse complex of \( f \). We shall use \( x_0 \) as a base intersection point for the Floer complex. From now on we shall drop \( J \) and \( x_0 \) from the notation of the Floer complex and Floer homology and simply write \( CF = CF(L, L_\epsilon) \), \( d = d_J \) and \( HF^* (\mod N_L) = HF^* (\mod N_L) (L, L; J, x_0) \). It is shown in [33] that
\[
CF^i(\mod N_L) = \bigoplus_{j=0}^i C^i_f.
\]
As \( d : CF^*(\mod N_L) \to CF^{*+1}(\mod N_L) \) we can write \( d = \sum_{j \in \mathbb{Z}} \partial_j \) where \( \partial_j \) is an operator \( \partial_j : C^*_f \to C^{*+1-j}_{f} \). An index computation shows that \( \partial_j = 0 \) for every \( j < 0 \) and, due to dimension reasons, \( \partial_j = 0 \) also for every \( j > \nu \), where \( \nu = \left\lfloor \frac{\dim L + 1}{N_L} \right\rfloor \). Thus
\[
d = \partial_0 + \cdots + \partial_\nu.
\]
Roughly speaking \( \partial_0 \) counts the Floer trajectories that lie in the small neighbourhood \( U \) of \( L \), while \( \partial_1, \ldots, \partial_\nu \) count the “fat” trajectories which go out of \( U \). Oh proves in [33] that (for suitable choices of \( J \) and Riemannian metric on \( L \)) the operator \( \partial_0 : C^*_f \to C^{*+1}_{f} \) can be identified with the Morse complex differential, hence \( H^*(C_f, \partial_0) \cong H^*(L; \mathbb{Z}_2) \). (Note however that \( \partial_j, j \geq 1 \), are not differentials in general, namely \( \partial_j \circ \partial_j \) may not be zero.)

5.2. A spectral sequence. We shall now present a spectral sequence which enables to calculate the Floer homology \( HF(L, L) \) using the operators \( \partial_1, \ldots, \partial_\nu \). We continue to use the shortened notation omitting \( J \) and \( x_0 \) in \( CF, d \) and \( HF \).

Let \( A = \mathbb{Z}_2 [T, T^{-1}] \) be the algebra of Laurent polynomials over \( \mathbb{Z}_2 \) in the variable \( T \). We define the degree of \( T \) to be \( N_L \). Thus
\[
A = \bigoplus_{i \in \mathbb{Z}} A^i, \quad \text{where} \quad A^i = \begin{cases} \mathbb{Z}_2 T^{i/N_L} & i \equiv 0(\mod N_L) \\ 0 & \text{otherwise} \end{cases}
\]
Set \( \tilde{C} = C_f \otimes A \), namely
\[
\tilde{C}^l = \bigoplus_{k \in \mathbb{Z}} C_f^{l-kN_L} \otimes A^{kN_L}, \quad \text{for every} \ l \in \mathbb{Z},
\]
and let $\tilde{d}: \tilde{C}^* \to \tilde{C}^{*+1}$ be $\tilde{d} = \partial_0 \otimes 1 + \partial_1 \otimes \tau + \cdots + \partial_p \otimes \tau^p$, where $\tau^i: A^* \to A^{*+iN_k}$ is multiplication by $T^i$. A simple algebraic computation shows that:

1. $\tilde{d} \circ \tilde{d} = 0$.
2. $H^l(\tilde{C}, \tilde{d}) \cong HF^l(\text{mod } N_k)$ for every $l \in \mathbb{Z}$.

Next we define a decreasing filtration $\cdots \subset F^{p+1}\tilde{C} \subset F^p\tilde{C} \subset F^{p-1}\tilde{C} \subset \cdots$ on $\tilde{C}$. For every $p \in \mathbb{Z}$ let $A_p = \bigoplus_{k \geq p} A_{kN}^l$ be the space of Laurent polynomials of the form $\sum_{k \geq p} \alpha_k T^k$. Define

$$F^p\tilde{C} = \tilde{C} \otimes A_p,$$

namely $F^p\tilde{C}^d = \bigoplus_{k \geq p} C_{kN}^{d-kN} \otimes A_{kN}^l$ for every $p, l \in \mathbb{Z}$.

Note that since $C^j_f = 0$ for every $j > \dim L$ and $j < 0$, the filtration $F^p\tilde{C}$ is bounded. Denote by $\{E^{p,a}_r, d_r\}$ the spectral sequence defined by this filtration.

**Theorem 5.2.A.** The spectral sequence $\{E^{p,a}_r, d_r\}$ has the following properties:

1. $E^{p,a}_0 = C^{p+q-pN_k}_f \otimes A_{pN_k}^l$, $d_0 = \partial_0 \otimes 1$.
2. $E^{p,a}_1 = H^{p+q-pN_k}(L; \mathbb{Z}_2) \otimes A_{pN_k}^l$, $d_1 = [\partial_1] \otimes \tau$, where
   $$[\partial_1]: H^{p+q-pN_k}(L; \mathbb{Z}_2) \to H^{p+1+q-(p+1)N_k}(L; \mathbb{Z}_2)$$
   is induced from $\partial_1$.
3. For every $r \geq 1$, $E^{p,a}_r$ has the form $E^{p,a}_r = V^{p,a}_r \otimes A_{pN_k}^l$ with $d_r = \delta_r \otimes \tau^r$, where $V^{p,a}_r$ are vector spaces over $\mathbb{Z}_2$ and $\delta_r$ are homomorphisms $\delta_r: V^{p,a}_r \to V^{p+r,a-r+1}_r$ defined for every $p, q$ and satisfy $\delta_r \circ \delta_r = 0$. Moreover
   $$V^{p,a}_{r+1} = \frac{\ker(\delta_r: V^{p,a}_r \to V^{p+r,a-r+1}_r)}{\text{image } (\delta_r: V^{p-r,a+r-1}_r \to V^{p,a}_r)}.$$
   (For $r = 0, 1$ we have $V^{p,a}_0 = C^{p+q-pN_k}_f$, $V^{p,a}_1 = H^{p+q-pN_k}(L; \mathbb{Z}_2)$.)
4. $\{E^{p,a}_r, d_r\}$ collapses at the $\nu + 1$ step, namely $d_r = 0$ for every $r \geq \nu + 1$ (and so $E^{p,a}_r = E^{p,a}_\infty$ for every $r \geq \nu + 1$). Moreover, the sequence converges to $HF$, i.e.
   $$\bigoplus_{p+q=l} E^{p,a}_\infty \cong HF^l(\text{mod } N_k)$$
   for every $l \in \mathbb{Z}$.
5. For every $p \in \mathbb{Z}$, $\bigoplus_{q \in \mathbb{Z}} E^{p,a}_\infty \cong HF$.

**Remarks.**

1. Our filtration $F^p\tilde{C}$ is different than the filtration used by Oh [33, 32]. The filtration used by Oh comes from the following filtration on the singular cohomology of $L$: $F^pH^*(L; \mathbb{Z}_2) = \bigoplus_{0 \leq j \leq n-p} H^j(L; \mathbb{Z}_2)$. Thus the spectral sequence in [33, 32] is different than ours.
(2) Semi-rigorous arguments suggest that the spectral sequence is multiplicative in the following sense. For every $r \geq 1$ we have a product $E^r_{p,q} \otimes E^r_{p',q'} \rightarrow E^r_{p+p',q+q'}$ and the differential $d_r$ satisfies Leibniz rule with respect to this product. Moreover the product on $E_1$ comes from the cup product on the cohomology $H^*(L; \mathbb{Z}_2)$.

Taking the product structure in consideration in combination with the other techniques of this paper should lead to new restrictions on the topology of Lagrangian submanifolds.

**Proof of Theorem 5.2.1.** Most of the proof is purely algebraic and follows from the basic construction of a spectral sequence associated to a filtered complex. For the readers who are not familiar with spectral sequences we shall now give a brief summary of this construction. More details can be found in any basic text on spectral sequences. Below we follow the conventions of [26].

Let $(\tilde{C}, \tilde{d})$ be a complex with a decreasing filtration $F^p\tilde{C}$, $p \in \mathbb{Z}$. (We write the $\sim$ over $C$ and $d$ to be consistent with the notation of our theorem.) Assume further that the filtration is bounded, namely for every $l \in \mathbb{Z}$, there exist $s = s(l)$, $t = t(l)$ such that $F^s\tilde{C}^l = 0$ and $F^t\tilde{C}^l = \tilde{C}^l$.

Define for every $p, q \in \mathbb{Z}$ and $r \geq -1$:

* $Z^p,q_r = \{ x \in F^p\tilde{C}^{p+q} \mid \tilde{d}x \in F^{p+r}\tilde{C}^{p+q+1} \}$,
* $B^p,q_r = \{ x \in F^p\tilde{C}^{p+q} \mid x = \tilde{d}y, \text{with } y \in F^{p-r}\tilde{C}^{p+q-1} \}$,
* $E^r_{p,q} = Z^p,q_r / (Z^p,q_{r+1} + B^p,q_r)$,
* $E^\infty_{p,q} = Z^p,q_{\infty} / (Z^p,q_{\infty+1} + B^p,q_{\infty})$.

A simple computation shows that $\tilde{d}$ maps $Z^p,q_r$ into $Z^{p+r,q-r+1}$ and moreover it descends to a homomorphism $d_r : E^r_{p,q} \rightarrow E^{r+r,q-r+1}_{p,q}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
Z^p,q_r & \xrightarrow{\tilde{d}} & Z^{p+r,q-r+1}
\end{array}
\]

\[
\begin{array}{ccc}
& \downarrow & \\
E^r_{p,q} & \xrightarrow{d_r} & E^{r+r,q-r+1}_{p,q}
\end{array}
\]

Here the vertical arrows are the canonical projections. As $\tilde{d} \circ \tilde{d} = 0$ we have $d_r \circ d_r = 0$.

Denote by $F^pH(\tilde{C}, \tilde{d})$ the induced filtration on the homology of $(\tilde{C}, \tilde{d})$, namely

$$F^pH^i(\tilde{C}, \tilde{d}) = \text{Image}(H^i(F^p\tilde{C}, \tilde{d}) \rightarrow H^i(\tilde{C}, \tilde{d}))$$

Note that $F^pH(\tilde{C}, \tilde{d})$ is also a bounded filtration.

The fundamental features of the above construction are the following (see [26]):
(1) The homology of \((E_{r,*}^*, d_r)\) is isomorphic to \(E_{r+1,*}^*\), namely \(H(E_{r,*}^*, d_r) \cong E_{r+1,*}^*\).
(2) \(E_0^{p,q} \cong H^{p+q}(F^p\tilde{C}/F^{p+1}\tilde{C})\).
(3) \(E_\infty^{p,q} \cong F^pH^{p+q}(\tilde{C}, \tilde{d})/F^{p+1}H^{p+q}(\tilde{C}, \tilde{d})\).
(4) For every \(p, q\) there exists \(r_0 = r_0(p, q)\) such that \(E_r^{p,q} = E_{r_0}^{p,q}\) for every \(r \geq r_0\).

If our complex \(\tilde{C}\) consists of vector spaces then summing up \(F^pH^1(\tilde{C}, \tilde{d})/F^{p+1}H^1(\tilde{C}, \tilde{d})\) over all \(p\)'s gives us an isomorphic copy of \(H^1(\tilde{C}, \tilde{d})\), hence by (3), \(\bigoplus_{p+q=1} E_\infty^{p,q} \cong H^1(\tilde{C}, \tilde{d})\).

We now turn to the proof of our theorem, applying the above construction to our complex. A simple computation shows that

\[
Z_0^{p,q} = F^p\tilde{C}^{p,q}, \quad Z_{-1}^{p+1,q-1} = F^{p+1}\tilde{C}^{p+1}, \quad B_0^{p,q} = 0.
\]

Thus in our case \(E_0^{p,q} = C_f^{p+q-pNL} \otimes A^{pNL}\), and \(d_0 : E_0^{p,q} \to E_0^{p,q+1}\) is just \(\partial_0 \otimes 1\). This proves statement 1.

To see 2, write elements \(x \in F^p\tilde{C}^{p+q}\) as finite sums \(x = \sum_{j \geq 0} x_{p+q-(p+j)N} \otimes T^{p+j}\), where \(x_{p+q-(p+j)N} \in C_f^{p+q-(p+j)N}\). A simple computation shows that:

\[
Z_1^{p,q} = \left\{ x = \sum_{j \geq 0} x_{p+q-(p+j)N} \otimes T^{p+j} \mid \partial_0(x_{p+q-pN}) = 0 \right\}
\]

\[
= Z_0(C_f^{p+q-pNL}) \otimes A^{pNL} \bigoplus F^{p+1}\tilde{C}^{p,q},
\]

where \(Z_0(C_f^{p+q-pNL}) = \text{Ker}(\partial_0 : C_f^{p+q-pNL} \to C_f^{p+q+1-pNL})\). Moreover we have:

1. \(Z_0^{p+1,q-1} = F^{p+1}\tilde{C}^{p+q}\),
2. \(B_0^{p,q} = \tilde{d}(F^p\tilde{C}^{p,q-1})\).

It follows that \(Z_0^{p+1,q-1} + B_0^{p,q} = \partial_0(C_f^{p+q-1-pNL}) \otimes A^{pNL} \bigoplus F^{p+1}\tilde{C}^{p,q}\). Thus

\[
E_1^{p,q} = Z_1^{p,q} / (Z_0^{p+1,q-1} + B_0^{p,q}) = H^{p+q-pNL}(L; \mathbb{Z}_2) \otimes A^{pNL}.
\]

To compute \(d_1\), let us describe \(\tilde{d} : Z_1^{p,q} \to Z_1^{p+1,q}\). Write an element \(x \in Z_1^{p,q}\) as

\[
x = x_{p+q-pN} \otimes T^p + x_{p+q-(p+1)N} \otimes T^{p+1} + x',
\]

where \(\partial_0(x_{p+q-pN}) = 0\) and \(x' \in F^{p+2}\tilde{C}^{p+q}\). Then

\[
\tilde{d}x = (\partial_1(x_{p+q-pN}) + \partial_0(x_{p+q-(p+1)N})) \otimes T^{p+1} + \tilde{dx}'.
\]

It follows that \(d_1 : H^{p+q-pNL}(L; \mathbb{Z}_2) \otimes A^{pNL} \to H^{p+1+q-(p+1)NL}(L; \mathbb{Z}_2) \otimes A^{(p+1)NL}\) has the form \([\partial_1] \otimes \tau\), where \([\partial_1]\) is induced from \(\partial_1\). This completes the proof of statement 2.

Statement 3 follows immediately from statement 1 by induction on \(r\). Indeed, note that \(A_{jNL}\) is 1-dimensional for every \(j\). Thus the homomorphism \(d_r : V^{p,q}_r \otimes A^{pNL} \to V^{p+r,q-r+1}_r \otimes A^{(p+r)NL}\) must be of the form \(d_r = \delta_r \otimes \tau^r\) where \(\delta_r\) is a homomorphism \(V^{p,q}_r \to V^{p+r,q-r+1}_r\). As \(d_r \circ d_r = 0\) we also have \(\delta_r \circ \delta_r = 0\). Moreover, the homologies of \(d_r\) and of \(\delta_r\) are related by \(H(V^{p,q}_r \otimes A^{pNL}, d_r) = H(V^{p,q}_r, \delta_r) \otimes A^{pNL} \).
To prove statement 4, note that statement 1 implies that for every \( r, E^r_{p,q} \neq 0 \) only if \( 0 \leq p + q - pN_L \leq \dim L \). Let \( r \geq \nu + 1 \) and \( 0 \leq p + q - pN_L \leq \dim L \). As \( d_r : E^r_{p,q} \to E^{r+1}_{p+r,q-r+1} \) it is enough to show that \((p+r)+(q-r+1)-(p+r)N_L < 0\), i.e. that \( p + q - pN_L + 1 - rN_L < 0 \). But this is immediate because \( rN_L \geq (\nu + 1)N_L > \dim L + 1 \).

The second part of statement 4 follows from the general theory of spectral sequences outlined at the beginning of the proof and the fact that \( H^1(\tilde{C}, \tilde{d}) \cong HF((\mod N_L)) \).

Finally, statement 5 follows from the following symmetries of the spectral sequence which are easy to check:

1. \( \tilde{d} \circ \tau = \tau \circ \tilde{d} \).
2. \( Z^{p+1,q+N_L-1}_r = \tau(Z^p,q), B^{p+1,q+N_L-1}_r = \tau(B^p,q) \) for every \( p, q \).
3. \( E^{p+1,q+N_L-1}_\infty = \tau(E^\infty,q) \) for every \( p, q \).

\[ \square \]

6. Proof of the main results

**Proof of Theorem A** Since \( H_1(L; \mathbb{Z}) \) is 2-torsion it is easy to see that \( L \subset \mathbb{C}P^n \) must be monotone with \( N_L = k(n + 1) \) for some \( k \in \mathbb{N} \).

Consider the polarization \( \mathcal{P} = (M = \mathbb{C}P^n, \sigma, J; \Sigma = \mathbb{C}P^n) \) where \( \sigma \) is the standard Kähler form on \( \mathbb{C}P^n \), \( J \) is the standard complex structure and \( \Sigma \subset \mathbb{C}P^n \) is a linear hyperplane. By Example 3.6.1 \( \mathcal{P} \) is a subcritical polarization. Put \( V = M \setminus \Sigma \) and consider the Lagrangian circle bundle \( \Gamma_L \subset V \) as constructed in Section 4.1. By Proposition 4.1.A \( \Gamma_L \) is monotone and \( N_{\Gamma_L} = N_L = k(n + 1) \). By Theorem 4.2.A we have \( HF(\Gamma_L, \Gamma_L) = 0 \).

We now claim that \( N_{\Gamma_L} = n + 1 \) (namely \( k = 1 \)). Indeed, if \( k \geq 2 \) then, due to dimension reasons, \( d = \partial_0 \), hence \( HF(\Gamma_L, \Gamma_L) = \bigoplus_{i=0}^{n+1} H^i(\Gamma_L; \mathbb{Z}_2) \neq 0 \). Contradiction. This proves that \( N_{\Gamma_L} = n + 1 \), hence \( d = \partial_0 + \partial_1 \).

Let \( \{ E^r_{*,*}, d_r \} \) be the spectral sequence of Section 5.2. By Theorem 5.2.A the sequence collapses at stage \( r = 2 \), hence \( E_2^{*,*} = \ldots = E_{\infty}^{*,*} = 0 \). In particular, the following sequence is exact: \( \ldots \to d_{1} \to E_1^{0,q} \to d_{1} E_1^{1,q} \to d_{1} E_1^{2,q} \to d_{1} E_1^{3,q} \to \ldots \). Substituting \( E_1^{p,q} = H^{p+q-pN_{\Gamma_L}}(\Gamma_L; \mathbb{Z}_2) \otimes A^{pN_{\Gamma_L}}, d_1 = [\partial_1] \otimes \tau \) we obtain:

\[
H^1(\Gamma_L; \mathbb{Z}_2) \cong H^{n+1}(\Gamma_L; \mathbb{Z}_2) = \mathbb{Z}_2,
\]
\[
H^n(\Gamma_L; \mathbb{Z}_2) \cong H^0(\Gamma_L; \mathbb{Z}_2) = \mathbb{Z}_2,
\]
\[
H^i(\Gamma_L; \mathbb{Z}_2) = 0 \quad \text{for every } 1 < i < n.
\]

In order to recover the cohomology of \( L \) itself we use the Gysin sequence of the circle bundle \( \Gamma_L \to L \). Note that the second Stiefel-Whitney class of the vector bundle corresponding to \( \Gamma_L \to L \) is just \( \alpha = a|_L \in H^2(L; \mathbb{Z}_2) \), where \( a \in H^2(\mathbb{C}P^n; \mathbb{Z}_2) \) is the generator.
Next note that \( H_1(L; \mathbb{Z}) \) cannot be 0 since if it were than \( N_L = 2(n + 1) \). Thus \( H_1(L; \mathbb{Z}) \) is a non-trivial 2-torsion group hence \( H^1(L; \mathbb{Z}_2) \neq 0 \).

Substituting this into the \( \mathbb{Z}_2 \)-coefficients Gysin sequence we obtain that \( H^i(L; \mathbb{Z}_2) \xrightarrow{\cup \alpha} H^{i+2}(L; \mathbb{Z}_2) \) is an isomorphism for every \( 0 \leq i \leq n - 2 \) and that \( H^1(L; \mathbb{Z}_2) \cong H^1(\Gamma_L; \mathbb{Z}_2) \).

But \( H^1(\Gamma_L; \mathbb{Z}_2) = \mathbb{Z}_2 \) hence we conclude that \( H^i(L; \mathbb{Z}_2) = \mathbb{Z}_2 \) for every \( 0 \leq i \leq n \), which, additively, is precisely the \( \mathbb{Z}_2 \)-cohomology of \( \mathbb{R} P^n \).

Finally, suppose that \( n \) is even, say \( n = 2m \). Denote by \( \beta \in H^1(L; \mathbb{Z}_2) \) the generator. We have seen that \( \beta \cup \alpha^{m-1} \neq 0 \in H^{n-1}(L; \mathbb{Z}_2) \). By Poincaré duality, \( \beta \cup \beta \cup \alpha^{m-1} \neq 0 \in H^n(L; \mathbb{Z}_2) \), hence \( \beta \cup \beta \neq 0 \). Therefore \( \beta \cup \beta = \alpha \), and it follows that \( \beta \) generates the cohomology ring of \( L \), exactly as for \( \mathbb{R} P^n \).

\textbf{Proof of Theorem B} Consider \( \mathbb{CP}^n \times X \subset \mathbb{CP}^{n+1} \times X \), and let \( \Gamma_L \subset (\mathbb{CP}^{n+1} \setminus \mathbb{CP}^n) \times X \) be the Lagrangian circle bundle corresponding to \( L \subset \mathbb{CP}^n \times X \). By our assumptions on \( L \) and on \( X \), \( L \) is monotone and \( N_L = 2(n + 1) \). By Proposition 4.1.A \( \Gamma_L \) is monotone too and \( N_{\Gamma_L} = 2(n + 1) \). (See the second remark after Proposition 4.1.A for the case \( n = 1 \).)

The rest of the proof is very similar to that of Theorem A. From Theorem 4.2.A we obtain \( HF(\Gamma_L, \Gamma_L) = 0 \). Then a similar computation via the spectral sequence gives:

\[
H^1(\Gamma_L; \mathbb{Z}_2) \cong H^{2n+2}(\Gamma_L; \mathbb{Z}_2) = \mathbb{Z}_2,
\]

\[
H^{2n+1}(\Gamma_L; \mathbb{Z}_2) \cong H^0(\Gamma_L; \mathbb{Z}_2) = \mathbb{Z}_2,
\]

\[
H^i(\Gamma_L; \mathbb{Z}_2) = 0 \quad \text{for every } 1 < i < 2n+1.
\]

The proof now continues in the same way, using the Gysin sequence, only that now \( H^1(L; \mathbb{Z}_2) = 0 \). This implies that \( H^i(L; \mathbb{Z}_2) = 0 \) for every \( 0 < i < 2n+1 \), which is exactly the \( \mathbb{Z}_2 \)-cohomology of \( S^{2n+1} \).

\textbf{Proof of Theorem C} Since \( H_1(L; \mathbb{Z}_2) = 0 \), it is easy to see that \( L \subset \mathbb{CP}^n \times \mathbb{CP}^n \) is monotone with \( N_L = 2(n + 1) \). Consider \( \mathbb{CP}^n \times \mathbb{CP}^n \subset \mathbb{CP}^{n+1} \times \mathbb{CP}^n \) and let \( \Gamma_L \subset (\mathbb{CP}^{n+1} \setminus \mathbb{CP}^n) \times \mathbb{CP}^n \) be the Lagrangian circle bundle over \( L \). By Proposition 4.1.A \( \Gamma_L \) is monotone too and \( N_{\Gamma_L} = 2(n + 1) \) (see the second remark after Proposition 4.1.A for the case \( n = 1 \)). By Theorem 4.2.A we have \( HF(\Gamma_L, \Gamma_L) = 0 \). As in the proof of Theorem A the spectral sequence \( \{ E_{r,s}^* \} \) collapses at stage \( r = 2 \), hence \( E_{2,*}^* = \ldots = E_{\infty,*}^* = 0 \), and we obtain the following exact sequences for every \( q \in \mathbb{Z} \):

\[
\ldots \xrightarrow{[\partial]} H^{q-1+N_{\Gamma_L}}(\Gamma_L; \mathbb{Z}_2) \xrightarrow{[\partial]} H^q(\Gamma_L; \mathbb{Z}_2) \xrightarrow{[\partial]} H^{q+1-N_{\Gamma_L}}(\Gamma_L; \mathbb{Z}_2) \xrightarrow{[\partial]} \ldots
\]

As \( \dim \Gamma_L = 2n+1 \) and \( N_{\Gamma_L} = 2n+2 \) we get \( H^i(\Gamma_L; \mathbb{Z}_2) = 0 \) for every \( 0 < i < 2n+1 \) and \( H^0(\Gamma_L; \mathbb{Z}_2) = H^{2n+1}(\Gamma_L; \mathbb{Z}_2) = \mathbb{Z}_2 \). Substituting this into the Gysin sequence of the circle bundle \( \Gamma_L \to L \) we obtain that \( H^i(L; \mathbb{Z}_2) \xrightarrow{\cup \alpha} H^{i+2}(L; \mathbb{Z}_2) \) is an isomorphism for every
0 \leq i \leq 2n - 2$, where $\alpha \in H^2(L; \mathbb{Z}_2)$ is the second Stiefel-Whitney class of the vector bundle corresponding to $\Gamma_L \to L$. It follows that $H^i(L; \mathbb{Z}_2) = \mathbb{Z}_2$ for every $i$ even and $H^i(L; \mathbb{Z}_2) = 0$ for every $i$ odd. Moreover, $\alpha \in H^2(L; \mathbb{Z}_2)$ clearly generates the algebra $H^*(L; \mathbb{Z}_2)$, exactly as for $H^*(\mathbb{C}P^n; \mathbb{Z}_2)$.

Finally, it is easy to see that $\alpha = a|_L$ where $a \in H^2(\mathbb{C}P^n \times \mathbb{C}P^n; \mathbb{Z}_2)$ is the generator of $H^2$ of any of the factors of $\mathbb{C}P^n \times \mathbb{C}P^n$. (As $L$ is Lagrangian, it does not matter which factor we take.)

Before we go on to the proofs of the rest of the theorems we shall need the following proposition.

**Proposition 6.A.** Let $(V^{2k}, \omega)$ be a tame symplectic manifold of dimension $k \geq 2$ and $K^k \subset (V^{2k}, \omega)$ a monotone Lagrangian submanifold with $N_K \geq 2$. Suppose that $HF(K, K) = 0$. Then:

1. If $K$ is a $\mathbb{Z}_2$-homology sphere, namely $H^*(K; \mathbb{Z}_2) \cong H^*(S^k; \mathbb{Z}_2)$, then $N_K \mid k + 1$.
2. If $H^i(K; \mathbb{Z}_2) \neq 0$, $H^i(K; \mathbb{Z}_2) = 0$ for every $i \neq 0, 1, k - 1, k$, and $k \geq 3$, $N_K \geq 3$, then $N_K \mid k$.

**Remark.** The cohomological condition in statement 2 is satisfied whenever $K$ is a circle bundle over a sphere of dimension $\geq 3$.

**Proof of Proposition 6.A** We start by proving the second statement. Consider the spectral sequence $\{E^*_r, d_r\}$ of Section 5.2. Note that since $\dim K = k$, $E^*_{r, k+r-1} = 0$ for every $r \geq 1$. Therefore at the $r$’th step the differential $d_r$ behave as follows:

\[
0 \to E^*_{r, k} \to E^*_{r, k+r+1}.
\]

Since $HF(K, K) = 0$, by Theorem 5.2.A (statement 5) we have $E^*_{\infty} = 0$. On the other hand $E^0_{1, k} = H^k(K; \mathbb{Z}_2) \neq 0$. Denote by $r_0$ the minimal $r \geq 1$ for which $d_r : E^0_{r, k} \to E^0_{r, k+r+1}$ is not 0. It follows from Theorem 5.2.A and (3) that $E^0_{r_0, k} \cong H^k(K; \mathbb{Z}_2)$ and the homomorphism

\[
d_{r_0} = \delta_{r_0} \otimes \tau_{r_0} : H^k(K; \mathbb{Z}_2) \otimes A^0 \to V^{r_0, k-r_0+1} \otimes A^{r_0 N_K}
\]

is not 0. As $H^i(K; \mathbb{Z}_2) = 0$ for every $i \neq 0, 1, k - 1, k$, we have $V^{r_0, k-r_0+1} \neq 0$ only if $r_0 - r_0 N_K + k - r_0 + 1 \in \{0, 1, k - 1, k\}$. As $N_K \geq 3$ it follows that:

1. Either $k + 1 - r_0 N_K = 1$, hence $N_K \mid k$;
2. Or $k + 1 - r_0 N_K = 0$, hence $N_K \mid k + 1$.

Since $H^1(K; \mathbb{Z}_2) \neq 0$, $E^0_{1, k-1} = H^{k-1}(K; \mathbb{Z}_2) \otimes A^0$ is also not 0 by Poincaré duality. Applying the same arguments as above this time to

\[
0 \to E^0_{r, k-1} \to E^0_{r, k-r}
\]
and denoting by $r_1$ the minimal $r \geq 1$ for which $d_r : E^{0,k-1}_r \to E^{r,k-r}_r$ is not 0 we obtain:

(1') Either $k - r_1 N_K = 0$, hence $N_K | k$;
(2') Or $k - r_1 N_K = 1$, hence $N_K | k - 1$.

Comparing cases (1),(2) with (1'),(2') above we conclude that the only possibility is $N_K | k$.

The proof of statement 1 is similar (and actually simpler).

**Proof of Theorem D** Put $m = \dim_\mathbb{C} X$. We shall first assume that $n + m \geq 3$, i.e. that $\dim L \geq 3$.

Embed $\mathbb{C}P^n \times X \subset \mathbb{C}P^{n+1} \times X$ and let $\Gamma_L \subset (\mathbb{C}P^{n+1} \setminus \mathbb{C}P^n) \times X$ be the Lagrangian circle bundle over $L$. As in the proof of Theorem D, $L$ and $\Gamma_L$ are both monotone with $N_L = N_{\Gamma_L} = 2n + 2$. (Note that $L$ being a sphere of dimension $n + m \geq 3$ is simply connected.) By Theorem 4.2.A, $HF(\Gamma_L, \Gamma_L) = 0$.

As $\Gamma_L$ is a circle bundle over a sphere of dimension $\geq 3$ we have $H^i(\Gamma_L; \mathbb{Z}_2) \neq 0$ and $H^i(\Gamma_L; \mathbb{Z}_2) = 0$ for every $i \neq 0,1,n+m,n+m+1$. Note also that $N_{\Gamma_L} \geq 4$.

By Proposition 6.A, $2n + 2 | n + m + 1$ or equivalently $m \equiv n + 1(\text{mod} 2n + 2)$.

It remains to deal with the case $n + m = 2$, namely $L$ being a Lagrangian sphere in $\mathbb{C}P^1 \times X$ where $\dim_\mathbb{C} X = 1$. We claim that this is impossible under the assumptions of the Theorem. Indeed, $\pi_2(X) = 0$ hence the homotopy class $[L] \in \pi_2(\mathbb{C}P^1 \times X) \cong \pi_2(\mathbb{C}P^1)$ comes entirely from $\pi_2(\mathbb{C}P^1)$. But $\omega([L]) = 0$ hence $[L] = 0$ which is impossible since a Lagrangian 2-sphere must have self-intersection $-2$.

**Proof of Theorem E** Theorem E is a special case of Theorem 7.A which will be proved in Section 7 below.

**Proof of Theorem F** The proof is very similar to that of Theorem D only that now $N_L = 2 \gcd(n + 1, N_M)$.

**Proof of Theorem G** Consider the polarization $\mathcal{P} = (M = \mathbb{C}P^{n+1}, \sigma, J; \Sigma = Q^n)$, where $\Sigma = Q^n$ is the quadric. By Example 3.6.2 the Lagrangian trace is exactly the Lagrangian sphere $\Lambda_Q \subset \Sigma$.

Let $L \subset \Sigma$ be a Lagrangian submanifold with $H_1(L; \mathbb{Z})$ either 0 or a non-trivial 2-torsion group, and assume that $L \cap \Lambda_Q = \emptyset$. The minimal Chern number of $\Sigma$ is $n$, hence $N_L = kn$ for some $k \in \mathbb{N}$. Let $\Gamma_L \subset M \setminus \Sigma$ be the Lagrangian circle bundle over $L$. By Proposition 4.1.A $N_{\Gamma_L} = N_L = kn$. We first claim that $k = 1$. Indeed, by Theorem 4.2.B $HF(\Gamma_L, \Gamma_L) = 0$. Hence, if $k \geq 2$ then since $n \geq 3$ all the differentials $d_r : E^{*,*}_r \to E^{*,*+r,r-1}_r$ of the spectral sequence must vanish for every $r \geq 1$ which is impossible since $E^{*,*}_\infty = 0$ and $E^{*,*}_1$ is not 0. This proves $N_{\Gamma_L} = N_L = n$. Note that this implies that $H_1(L; \mathbb{Z}) \neq 0$, for otherwise $N_L$ would be 2n.
By dimension reasons we have $d = \partial_0 + \partial_1$, hence $E_2^{*,*} = E_\infty^{*,*} = 0$. Following the differentials in the spectral sequence we obtain from Theorem 5.2.1 the following exact sequences for every $q \in \mathbb{Z}$:

$$H^{q-1+n}(\Gamma_L; \mathbb{Z}_2) \xrightarrow{[\partial_1]} H^q(\Gamma_L; \mathbb{Z}_2) \xrightarrow{[\partial_1]} H^{q+1-n}(\Gamma_L; \mathbb{Z}_2).$$

Assume first that $n > 3$. From (4) we obtain:

$$H^2(\Gamma_L; \mathbb{Z}_2) \cong H^{n+1}(\Gamma_L; \mathbb{Z}_2) = \mathbb{Z}_2, \quad H^{n-1}(\Gamma_L; \mathbb{Z}_2) \cong H^0(\Gamma_L; \mathbb{Z}_2) = \mathbb{Z}_2,$$

and $H^i(\Gamma_L; \mathbb{Z}_2) = 0$ for every $2 < i < n - 1$.

Consider now the Gysin sequence of the circle bundle $\Gamma_L \to L$. The first Chern class of the normal bundle $N_{\Sigma/M}$ is $c_1^{N_{\Sigma/M}} = 2a|\Sigma$ where $a \in H^2(\mathbb{C}P^{n+1}; \mathbb{Z})$ is the positive generator. Therefore the second Stiefel-Whitney class of the vector bundle associated to $\Gamma_L \to L$ is 0. From the Gysin sequence we get $H^i(L; \mathbb{Z}_2) = 0$ for every $2 < i < n - 2$, as well as the following exact sequence:

$$0 \to H^2(L; \mathbb{Z}_2) \to \mathbb{Z}_2 \to H^1(L; \mathbb{Z}_2) \to 0.$$

Since $H_1(L; \mathbb{Z}) \neq 0$ is 2-torsion we conclude that $H^1(L; \mathbb{Z}_2) \neq 0$. (Actually this follows also from the fact that $N_L = n$ and $N_Q = n$.) It now easily follows that $H^1(L; \mathbb{Z}_2) = \mathbb{Z}_2$ and that $H^2(L; \mathbb{Z}_2) = 0$. This proves that $H^*(L; \mathbb{Z}_2) \cong A_Q^*$.

Assume now that $n = 3$. From (4) we obtain $H^2(\Gamma_L; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Using the Gysin sequence and similar arguments to the preceding ones we obtain $H^i(L; \mathbb{Z}_2) = \mathbb{Z}_2$ for every $0 \leq i \leq 3$, hence $H^*(L; \mathbb{Z}_2) \cong A_Q^*$.

Remarks. (1) Using the multiplicative structure of the spectral sequence (see the second remark after Theorem 5.2.1), it seems that the following should hold for $n = 3$: if $a \in H^1(L; \mathbb{Z}_2)$ is the generator then $a \cup a = 0$.

(2) The same proof as above with small changes actually shows that if $L \subset Q^n$, $n \geq 3$, is a monotone Lagrangian submanifold with $N_L = n$ then either $H^*(L; \mathbb{Z}_2) \cong A_Q^*$ or $L \cap \Lambda_Q \neq \emptyset$.

\begin{theorem} \[ \text{Proof of Theorem H} \]
Throughout the proof we set $\Sigma = \Sigma_d$. Consider the polarization $\mathcal{P} = (M = \mathbb{C}P^{n+1}, \sigma, J; \Sigma)$. By Example 3.6.4 the Lagrangian trace $\Lambda_{\mathcal{P}} \subset \Sigma$ consists of at most $d^{n+1}$ immersed Lagrangian spheres.

Let $L \subset \Sigma$ be a monotone Lagrangian submanifold and suppose that $L \cap \Lambda_{\mathcal{P}} = \emptyset$. Let $\Gamma_L \subset \mathbb{C}P^{n+1}\setminus \Sigma$ be the Lagrangian circle bundle over $L$. By Theorem 12.1 $HF(\Gamma_L; \Gamma_L) = 0$. \end{theorem}
To prove statement 1 assume first that $2d \leq n + 1$ and let $L \subset \Sigma$ be a Lagrangian submanifold with $H_1(L; \mathbb{Z}) = 0$. A simple computation shows that $N_L = 2(n + 2 - d)$ hence $N_L \geq n + 3$. As $\dim \Gamma_L = n + 1$ it follows that the Floer differential is $d = \partial_0$, hence $HF(\Gamma_L, \Gamma_L) \cong H^*(\Gamma_L; \mathbb{Z}_2) \neq 0$ which is a contradiction. This proves that $L \cap \Lambda_P \neq \emptyset$.

The second part of statement 1 (i.e. $d \geq \frac{3}{2}(n + 1)$) will be treated towards the end of the proof.

To prove statements 2 and 3, assume that $2d \leq n + 1$ and that $H_1(L; \mathbb{Z})$ is 2-torsion. A simple computation shows that $N_L = k(n + 2 - d)$ for some $k \in \mathbb{N}$. We first claim that $k = 1$. Indeed, if $k \geq 2$ then $N_L \geq n + 3$ and we would obtain $HF(\Gamma_L, \Gamma_L) \cong H^*(\Gamma_L; \mathbb{Z}_2)$ contradicting the vanishing of Floer homology. Thus $N_L = n + 2 - d$. As $2d \leq n + 1$ we have $d = \partial_0 + \partial_1$ and so $E_2^{*,*} = E_\infty^{*,*} = 0$. Following the differentials of the spectral sequence we get:

$$\begin{align*}
H^{d+1}(\Gamma_L; \mathbb{Z}_2) = \ldots = H^{n-d}(\Gamma_L; \mathbb{Z}_2) &= 0, \\
H^i(\Gamma_L; \mathbb{Z}_2) &\cong H^{n-d+1-i}(\Gamma_L; \mathbb{Z}_2), \quad \text{for every } 0 \leq i \leq d.
\end{align*}$$

Consider now the Gysin sequence of $\Gamma_L \to L$. Note that the second Stiefel-Whitney class $w$ of the vector bundle corresponding to $\Gamma_L \to L$ is $da|_L \in H^2(L; \mathbb{Z}_2)$ where $a \in H^2(\mathbb{C}P^{n+1}; \mathbb{Z}_2)$ is the generator. When $d = e$ we have $w = 0$, hence the Gysin sequence gives the exact sequence:

$$0 \to H^i(L; \mathbb{Z}_2) \to H^i(\Gamma_L; \mathbb{Z}_2) \to H^{i-1}(L; \mathbb{Z}_2) \to 0 \quad \text{for every } i.$$

Combining with (5) we obtain $H^j(L; \mathbb{Z}_2) = 0$ for every $d \leq j \leq n - d$. Moreover, we have $\beta_i(L) + \beta_{i-1}(L) = \beta_i(\Gamma_L)$. By (5) and Poincaré duality we have $\beta_{d-i}(\Gamma_L) = \beta_i(\Gamma_L)$ for every $0 \leq i \leq d$, hence

$$\beta_i(L) + \beta_{i-1}(L) = \beta_{d-i}(L) + \beta_{d-i-1}(L) \quad \text{for every } 0 \leq i \leq d.$$ 

Putting $i = 0$ we obtain $\beta_0(L) = \beta_{d-1}(L)$ (because $\beta_d(L) = 0$). Next, $\beta_1(L) + \beta_0(L) = \beta_{d-1}(L) + \beta_{d-2}(L)$, hence $\beta_1(L) = \beta_{d-2}(L)$. Continuing by induction we obtain $\beta_i(L) = \beta_{d-i-1}(L)$ for every $0 \leq i \leq d - 1$.

The statement for $d = odd$ follows at once from (5) and the fact that the Stiefel-Whitney class $w = da|_L$ equals $a|_L$ since $d$ is odd.

Statement 4 follows easily from statement 2 of Proposition 6.A applied to $\Gamma_L$.

To prove the second part of statement 1 and statement 5 we shall use an extension of Floer homology for so called strongly negative Lagrangian submanifolds, due to Lazzarini [25]. Let $K \subset (V, \omega)$ be a Lagrangian in a tame symplectic manifold. $K$ is called strongly negative if there exists $\lambda < 0$ such that $\mu_K = \lambda |\omega|$ on $\pi_2(V, K)$ and in addition the following conditions are satisfied:
(1) $c_1^V(A) \leq 2 - \dim \mathbb{C} V$ for every $A \in \pi_2(V)$ with $\omega(A) > 0$.

(2) $\mu_K(A) \leq 2 - \dim \mathbb{C} V$ for every $A \in \pi_2(V, K)$ with $\omega(A) > 0$.

Under these assumptions the Floer homology $HF(K, K')$ is well defined for every Lagrangian $K'$ which is Hamiltonianly isotopic to $K$ and moreover

$$HF(K, K) \cong HF(K, K') \cong H^*(K; \mathbb{Z}_2).$$

The reason for this, roughly speaking, is that under the above negativity assumptions there are no $J$-holomorphic spheres or discs with boundary on $K$ for generic almost complex structure $J$. Hence, the Floer differential for $CF(K, K')$ is the Morse-homology differential $d = \partial_0$. The non-existence of spheres and discs is due to negative dimension of the moduli spaces of $J$-holomorphic spheres and discs. For the dimension formulae to hold one has to work with regular almost complex structures $J$. Regularity may be achieved by generic perturbations of $J$ as long as the $J$-holomorphic discs/spheres in question are somewhere injective (see \[28\]). Thus an essential ingredient in applying the dimension argument is a procedure which enables to extract a somewhere injective disc from a given pseudo-holomorphic disc. Such procedures have been developed by Lazzarini \[24\] and by Kwon and Oh \[23\].

Coming back to the proof of the second part of statement 1, assume that $d \geq \frac{3}{2}(n+1)$, and $n \geq 3$. Note that since $H_1(L; \mathbb{Z}) = 0$ we have $\mu_L = 2(n+2-d)[\omega_\Sigma]$. As $n+2-d < 0$, for every $A \in \pi_2(\Sigma, L)$ with $\omega_\Sigma(A) > 0$ we have $\mu_L(A) \leq 2(n+2-d)$. By Proposition 4.1.A, for every $A \in \pi_2(\mathbb{C}P^{n+1}\setminus \Sigma, \Gamma_L)$ with $\omega(A) > 0$ we have $\mu_{\Gamma_L}(A) \leq 2(n+2-d)$. But since $d \geq \frac{3}{2}(n+1)$ we have

$$2(n+2-d) \leq 2 - (n+1) = 2 - \dim \mathbb{C}(\mathbb{C}P^{n+1}\setminus \Sigma).$$

Thus $\Gamma_L \subset \mathbb{C}P^{n+1}\setminus \Sigma$ is strongly negative. It follows that $HF(\Gamma_L, \Gamma_L) \cong H^*(\Gamma_L; \mathbb{Z}_2)$ which contradicts the vanishing of Floer homology. This proves that $L \cap \Lambda_d \neq \emptyset$.

We turn to the proof of statement 5. Since $H_1(L; \mathbb{Z})$ is $t$-torsion and $n+2-d < 0$ we have $\mu_L(A) \leq \frac{2(n+2-d)}{t}$ for every $A \in \pi_2(\Sigma, L)$ with $\omega_\Sigma(A) > 0$. By our assumption that $d \geq \frac{t(n-1)}{2} + n+2$ and Proposition 4.1.A, for every $A \in \pi_2(\mathbb{C}P^{n+1}\setminus \Sigma, \Gamma_L)$ with $\omega(A) > 0$ we have $\mu_{\Gamma_L}(A) \leq \frac{2(n+2-d)}{t} \leq 1-n = 2 - \dim \mathbb{C}(\mathbb{C}P^{n+1}\setminus \Sigma)$, hence $\Gamma_L$ is strongly negative. The rest of the proof is very similar to the preceding arguments.

**Proof of Theorem 4.** Consider the polarization $\mathcal{P} = (M = \mathbb{C}P^n \times \mathbb{C}P^n, \omega = \sigma \oplus \sigma, J; \Sigma)$. By Example 3.6.3 the Lagrangian trace $\Lambda_\mathcal{P}$ is exactly the Lagrangian sphere $\Lambda_\Sigma$.

Let $L \subset \Sigma$ be a Lagrangian submanifold with $H_1(L; \mathbb{Z}) = 0$ and suppose that $L \cap \Lambda_\Sigma = \emptyset$. A simple computation shows that $L$ is monotone with $N_L = 2n$. Consider the Lagrangian circle bundle $\Gamma_L \rightarrow L$. By Proposition 4.1.A, $\Gamma_L \subset (\mathbb{C}P^n \times \mathbb{C}P^n) \setminus \Sigma$ is monotone too and $N_{\Gamma_L} = N_L = 2n$. By Theorem 4.2.13, $HF(\Gamma_L, \Gamma_L) = 0$. 
In a similar manner to the proofs of Theorems A and B we recover the cohomology of $\Gamma_L$ via the spectral sequence, obtaining:

$$H^0(\Gamma_L; \mathbb{Z}_2) \cong H^1(\Gamma_L; \mathbb{Z}_2) \cong H^{2n-1}(\Gamma_L; \mathbb{Z}_2) \cong H^{2n}(\Gamma_L; \mathbb{Z}_2) \cong \mathbb{Z}_2,$$

$$H^i(\Gamma_L; \mathbb{Z}_2) = 0 \text{ for every } 1 < i < 2n - 1.$$

Next note that the restriction of the complex line bundle $N_{\Sigma/CP^n \times CP^n}$ to $L$ is trivial, since its first Chern class is just $[\omega]_L$ and $H^2(L; \mathbb{Z})$ is torsion-free because $H_1(L; \mathbb{Z}) = 0$. Thus the circle bundle $\Gamma_L \to L$ is trivial, hence $H^*(\Gamma_L) = H^*(L) \otimes H^*(S^1)$, from which it follows that $H^*(L; \mathbb{Z}_2) \cong H^*(S^{2n-2}; \mathbb{Z}_2).$ \hfill $\Box$

7. Generalizations

Below we present miscellaneous generalizations of some of the theorems from Section 6. Since the proofs are rather analogous to those of Section 6 we shall only outline the proofs omitting repeated details.

The following theorem generalizes Theorem E.

**Theorem 7.A.** Let $X$ be a symplectic manifold that has a covering which is symplectomorphic to a domain in a subcritical Stein manifold. Let $M$ be a closed spherically monotone symplectic manifold. Assume $\dim M, \dim X > 0$. If $M \times X$ has a Lagrangian sphere then $2N_M | \dim_{\mathbb{C}} M + \dim_{\mathbb{C}} X + 1$.

Examples of subcritical Stein manifolds, other than $\mathbb{C}^n$, can be found in [5, 9]. (Note however, that after completion all subcritical Stein manifolds are split [12].)

**Proof of Theorem 7.A** Let $Y \to X$ be a covering of $X$ by a symplectic manifold $Y$ which is (symplectomorphic to) a domain in a subcritical Stein manifold $V$. Let $L \subset M \times X$ be a Lagrangian sphere. By assumption $\dim L \geq 2$, hence $L$ is simply connected. Consider the lift $\tilde{L} \subset M \times Y \subset M \times V$ of $L$. Clearly $\tilde{L}$ is also an embedded Lagrangian sphere, and $N_{\tilde{L}} = 2N_M$. As $V$ is subcritical we have $HF(\tilde{L}, \tilde{L}) = 0$. By Proposition 6.A $2N_M | \dim_{\mathbb{C}} M + \dim_{\mathbb{C}} X + 1$. \hfill $\Box$

A symplectic manifold $(\Sigma, \omega_\Sigma)$ is called monotone if there exists $\lambda > 0$ such that $[\omega_\Sigma] = \lambda c_1^\Sigma \in H^2(\Sigma; \mathbb{R})$, where $c_1^\Sigma$ is the first Chern class of the tangent bundle of $\Sigma$. We denote by $N_H^{\Sigma} \in \mathbb{N}$ the positive generator of the subgroup $\{c_1^\Sigma(A) \mid A \in H_2(\Sigma; \mathbb{Z})\}$.

**Theorem 7.B.** Let $\Sigma$ be a closed monotone symplectic manifold (resp. spherically monotone) that participates in some polarization $\mathcal{P}$ (see Section 4.2). Denote $n = \dim_{\mathbb{C}} \Sigma$.

1. Suppose that $\mathcal{P}$ is subcritical. Then:
(a) If $2N^H \Sigma > n + 1$ (resp. $2N^H \Sigma > n + 1$) then there exist no Lagrangian submanifolds $L \subset \Sigma$ with $H_1(L; \mathbb{Z}) = 0$ (resp. $\pi_1(L) = 0$).

(b) If $2N^H \Sigma = n + 1$ (resp. $2N^H \Sigma = n + 1$) then every Lagrangian submanifold $L \subset \Sigma$ with $H_1(L; \mathbb{Z}) = 0$ (resp. $\pi_1(L) = 0$) must satisfy $H^*(L; \mathbb{Z}_2) \cong H^*(S^n; \mathbb{Z}_2)$.

(2) Suppose $\mathcal{P}$ has property $(\mathcal{S})$ (see Section 3.5). Then:

(a) If $2N^H \Sigma > n + 1$ (resp. $2N^H \Sigma > n + 1$) then for every Lagrangian submanifold $L \subset \Sigma$ with $H_1(L; \mathbb{Z}) = 0$ (resp. $\pi_1(L) = 0$) must satisfy $L \cap \Lambda_\mathcal{P} \neq \emptyset$.

(b) If $2N^H \Sigma = n + 1$ (resp. $2N^H \Sigma = n + 1$) then every Lagrangian submanifold $L \subset \Sigma$ with $H_1(L; \mathbb{Z}) = 0$ (resp. $\pi_1(L) = 0$) and with $L \cap \Lambda_\mathcal{P} = \emptyset$ must satisfy $H^*(L; \mathbb{Z}_2) \cong H^*(S^n; \mathbb{Z}_2)$.

We remark that a similar statement to 1(a) above was proved in [9] using a different approach.

**Proof of Theorem 7.B** Note that statement (1) is a special case of statement (2), since for a subcritical polarization $\mathcal{P}$ we have $\Lambda_\mathcal{P} = \emptyset$. We therefore prove statement (2).

Let $L \subset \Sigma$ be a Lagrangian submanifold with $H_1(L; \mathbb{Z}) = 0$ and $L \cap \Lambda_\mathcal{P} = \emptyset$. Note that $L$ is monotone and that $N_L \geq 2N^H \Sigma$ (here we continue to denote by $N_L$ the minimal Maslov number, i.e. the positive generator of the subgroup $\mu_L(\pi_2(\Sigma, L)) \subset \mathbb{Z}$.)

Consider now the Lagrangian circle bundle $\Gamma_L \rightarrow L$ in $M \setminus \Sigma$ (where, $\mathcal{P} = (M, \omega, J; \Sigma)$ is the polarization in which $\Sigma$ participates). By Theorem 4.2.B $HF(\Gamma_L, \Gamma_L) = 0$.

As in the end of the proof of Theorem 11 since $H_1(L; \mathbb{Z}) = 0$ and $L \subset \Sigma$ is Lagrangian, the bundle $\Gamma_L \rightarrow L$ must be trivial. Thus $H^1(\Gamma_L; \mathbb{Z}_2) = \mathbb{Z}_2$. In view of this, the only way the spectral sequence can converge to 0 is if $N_L \leq n + 1$. Thus if $2N^H \Sigma > n + 1$ we arrive at a contradiction. This proves 2(a).

Suppose now that $2N^H \Sigma = n + 1$. Computing using the spectral sequence we obtain:

$$H^0(\Gamma_L; \mathbb{Z}_2) \cong H^1(\Gamma_L; \mathbb{Z}_2) \cong H^n(\Gamma_L; \mathbb{Z}_2) \cong H^{n+1}(\Gamma_L; \mathbb{Z}_2) = \mathbb{Z}_2,$$

$$H^i(\Gamma_L; \mathbb{Z}_2) = 0, \text{ for every } 1 < i < n.$$

It now easily follows that $H^*(L; \mathbb{Z}_2) \cong H^*(S^n; \mathbb{Z}_2)$.

We omit the proof of the statements assuming spherical monotonicity since it is completely analogous to the proof above. \qed

The following theorem generalizes statement 4 of Theorem 11

**Theorem 7.C.** Let $\Sigma$ be a closed spherically monotone symplectic manifold that participates in a polarization $\mathcal{P}$ which has property $(\mathcal{S})$ (see Sections 4.2 and 3.5). Assume that $\dim_\mathbb{C} \Sigma \geq 3$ and $N_2 \Sigma \geq 2$. If $2N_2 \Sigma \uparrow \dim_\mathbb{C} \Sigma + 1$ then every Lagrangian sphere $L \subset \Sigma$ must satisfy $L \cap \Lambda_\mathcal{P} \neq \emptyset$. 
We omit the proof as it is rather similar to that of Theorem \textbf{H}.

The next theorem provides “Euler characteristic” type restrictions on monotone Lagrangians. Let \((V, \omega)\) be a symplectic manifold and \(L \subset (V, \omega)\) a monotone Lagrangian submanifold with minimal Maslov number \(N_L \geq 2\). For every \(j \in \mathbb{Z}\), denote by \(\gamma_j\) the sum of all \(\mathbb{Z}_2\)-Betti numbers of indices that are congruent to \(j\) modulo \(N_L\), namely

\[
\gamma_j = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} H^{kN_L+j}(L; \mathbb{Z}_2).
\]

Next, for every two integers \(s \leq t\) denote:

\[
\chi_{s,t}(L) = \gamma_s - \gamma_{s+1} + \cdots + (-1)^{t-s} \gamma_t.
\]

Given a Morse function \(f : L \to \mathbb{R}\) and \(s \in \mathbb{Z}\), put

\[
\kappa_s(f) = \# \{ p \in \text{Crit}(f) \mid \text{ind}_p(f) \equiv s \pmod{N_L} \}.
\]

Next, for every \(s \leq t \in \mathbb{Z}\) put

\[
\lambda_s(L) = \min_{f \in \text{Morse}} \left\{ \min \{ \kappa_{s-1}(f), \kappa_s(f) \} \right\},
\]

\[
\lambda_{s,t}(L) = \min_{f \in \text{Morse}} \left\{ \min \{ \kappa_{s-1}(f), \kappa_s(f) \} + \min \{ \kappa_t(f), \kappa_{t+1}(f) \} \right\}.
\]

**Theorem 7.D.** Let \((V, \omega)\) be a tame symplectic manifold and \(L \subset (V, \omega)\) be a monotone Lagrangian submanifold with \(N_L \geq 2\). Put \(\nu = \left[ \frac{\dim L+1}{N_L} \right] \). Suppose \(HF(L, L) = 0\). Then for every \(s \leq t\) we have:

1. If \(t - s = \text{even}\):
   a. \(0 \leq \chi_{s,t}(L) \leq \nu \min \{ \gamma_{s-1}, \gamma_s \} + \nu \min \{ \gamma_t, \gamma_{t+1} \} \).
   b. \(\chi_{s,t}(L) \leq \lambda_{s,t}(L) \).

2. If \(t - s = \text{odd}\):
   a. \(-\nu \min \{ \gamma_t, \gamma_{t+1} \} \leq \chi_{s,t}(L) \leq \nu \min \{ \gamma_{s-1}, \gamma_s \} \).
   b. \(-\lambda_t(L) \leq \chi_{s,t}(L) \leq \lambda_s(L) \).

In particular, if \((V, J, \varphi)\) is a Stein manifold which is either subcritical, or has property \((S)\) of Section \(2.4.1\) and \(L \cap \nabla_{\varphi} \text{crit} = \emptyset\), then the above inequalities hold.

To prove Theorem \textbf{7.D.} we shall need the following simple Lemma from linear algebra.

**Lemma 7.E.** Let \((D = \bigoplus_{i \in \mathbb{Z}} D^i, \partial)\) be a complex of vector spaces and \(H(D, \partial) = \bigoplus_{i \in \mathbb{Z}} H^i(D, \partial)\) its cohomology. For every two integers \(s \leq t\) put

\[
\chi_{s,t}(D) = \sum_{i=s}^{t} (-1)^{t-s} \dim D^i, \quad \chi_{s,t}(H(D, \partial)) = \sum_{i=s}^{t} (-1)^{t-s} \dim H^i(D, \partial).
\]

Then \(\chi_{s,t}(D) = \chi_{s,t}(H(D, \partial)) + \dim \partial(D^{s-1}) + (-1)^{t-s} \dim \partial(D^t)\).
In particular, for \( t - s = \text{even} \),
\[
\chi_{s,t}(H(D, \partial)) \leq \chi_{s,t}(D) \leq \chi_{s,t}(H(D, \partial)) + \min\{\dim D^{s-1}, \dim D^s\} + \min\{\dim D^t, \dim D^{t+1}\},
\]
while for \( t - s = \text{odd} \),
\[
\chi_{s,t}(H(D, \partial)) - \min\{\dim D^t, D^{t+1}\} \leq \chi_{s,t}(D) \leq \chi_{s,t}(H(D, \partial)) + \min\{\dim D^{s-1}, D^s\}.
\]

The proof of the lemma is completely straightforward, we therefore omit it.

**Proof of Theorem 7.11** We first prove the second and fourth inequalities. Given a Morse function \( f : L \to \mathbb{R} \), denote by \( C^*_f \) the Morse complex associated to \( f \). For every \( i \in \mathbb{Z} \), put \( C^i = C^i(f \mod N_L) = \bigoplus_{k \in \mathbb{Z}} C^{i+kN_L}_f \). Recall from Section 5.1 that \( C^* \) can be endowed with two differentials: the Morse differential \( \partial_0 \) and the Floer differential \( d = \partial_0 + \cdots + \partial_{\nu} \). Thus we have \( H^*(C, \partial_0) = H^*(\mod N_L)(L; \mathbb{Z}_2) \) and \( H^*(C, d) = HF^{*(\mod N_L)}(L, L) \). By Lemma 7.1 (applied for \( \partial_0 \) and for \( d \)), for every \( s \leq t \) we have:
\[
\chi_{s,t}(C) = \chi_{s,t}(H(C, \partial_0)) + \partial_0(C^{s-1}) + (-1)^{t-s} \dim \partial_0(C^t),
\]
\[
\chi_{s,t}(C) = \chi_{s,t}(H(C, d)) + \dim d(C^{s-1}) + (-1)^{t-s} \dim d(C^t).
\]

By assumption \( H(C, d) = HF(L, L) = 0 \), hence
\[
(6) \quad \chi_{s,t}(L) = \chi_{s,t}(H(C, \partial_0))
\]
\[
= \dim d(C^{s-1}) - \dim \partial_0(C^{s-1}) + (-1)^{t-s}(\dim d(C^t) - \dim \partial_0(C^t)).
\]

Now if \( t - s = \text{even} \) we get
\[
\chi_{s,t}(L) \leq \dim d(C^{s-1}) + \dim d(C^t) \leq \min\{\kappa_{s-1}(f), \kappa_s(f)\} + \min\{\kappa_t(f), \kappa_{t+1}(f)\}.
\]

Taking the minimum over all Morse functions \( f : L \to \mathbb{R} \) we obtain \( \chi_{s,t}(L) \leq \lambda_{s,t}(L) \).

Assume now that \( t - s = \text{odd} \). As \( d = \partial_0 + \cdots + \partial_{\nu} \), a simple dimension computation (using the grading of each \( \partial_k \)) shows that \( \dim d(C^i) \geq \dim \partial_0(C^i) \) for every \( i \in \mathbb{Z} \). Using this with (6) we get
\[
- \min\{\kappa_t(f), \kappa_{t+1}(f)\} \leq - \dim d(C^t) \leq \chi_{s,t}(L) \leq \dim d(C^{s-1}) \leq \min\{\kappa_{s-1}(f), \kappa_s(f)\}.
\]

Since this is true for all Morse functions \( f : L \to \mathbb{R} \) we obtain \( -\lambda_t(L) \leq \chi_{s,t}(L) \leq \lambda_s(L) \).

We now turn to the proof of the first and third inequalities. Let \( \{E_r^{p,q}, d_r\} \) be the spectral sequence of Section 5.2. For every \( r \geq 0 \), \( l \in \mathbb{Z} \), put \( \bar{E}^l_r = \bigoplus_{p+q=l} E_r^{p,q} \). Note that \( \bar{E}_{r+1} = H^*(\bar{E}_r, d_r) \), and \( \bar{E}_1 \cong H^{*(\mod N_L)}(L; \mathbb{Z}_2) \) for every \( l \in \mathbb{Z} \). By Lemma 7.12, applied
ν times we obtain:

\[
\chi_{s,t}(L) = \chi_{s,t}(\bar{E}_1) + \dim d_1(\bar{E}_1^{s-1}) + (-1)^{t-s} \dim d_1(\bar{E}_1^t)
\]

\[
= \ldots = \chi_{s,t}(\bar{E}_{\nu+1}) + \sum_{r=1}^{\nu} (\dim d_r(\bar{E}_r^{s-1}) + (-1)^{t-s} \dim d_r(\bar{E}_r^t)).
\]

Now \(\bar{E}_{\nu+1} = 0\) because \(HF(L, L) = 0\). Note that by Theorem 5.2.A we have that \(\dim d_r(E_l^t) \leq \min\{\gamma_l, \gamma_{l+1}\}\) for every \(l \in \mathbb{Z}\). The desired inequalities now easily follow from (7).

\[\square\]

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