ON A DISTRIBUTED CONTROL PROBLEM FOR A COUPLED CHEMOTAXIS-FLUID MODEL

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Abstract. In this paper we analyze an optimal distributed control problem where the state equations are given by a stationary chemotaxis model coupled with the Navier-Stokes equations. We consider that the movement and the interaction of cells are occurring in a smooth bounded domain of $\mathbb{R}^n$, $n = 2, 3$, subject to homogeneous boundary conditions. We control the system through a distributed force and a coefficient of chemotactic sensitivity, leading the chemical concentration, the cell density, and the velocity field towards a given target concentration, density and velocity, respectively. In addition to the existence of optimal solution, we derive some optimality conditions.

1. Introduction. Chemotaxis is understood as the biological process in which the presence of living organisms activates the production of a certain chemical substance. In this process, the movement of organisms in response to a chemical stimulus can be given towards a higher or lower concentration of the chemical substance (positive or negative chemotaxis, respectively). The typical example for chemotaxis is the amoebae Dictyostelium, which is a specie of soil-living amoeba belonging to the phylum Mycetozoa. When they are running out of nutrients, they produce a chemical, cyclic Adenosine Monophosphate, attracting other amoebae in order to perform some kind of transition to a multicellular organism [11]. The most classical model in the framework of chemotactical movements is the Patlak-Keller-Segel

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system [18, 19], which is given by
\[
\begin{aligned}
\begin{cases}
\partial_t c = D_c \Delta c + \alpha \rho - \beta c & \text{in } \Omega, \quad t > 0, \\
\partial_t \rho = D_\rho \Delta \rho - \nabla \cdot (\chi \rho \nabla c) & \text{in } \Omega, \quad t > 0, \\
\frac{\partial c}{\partial n} = \frac{\partial \rho}{\partial n} = 0 & \text{on } \partial \Omega, \quad t > 0,
\end{cases}
\end{aligned}
\]
(1)
where \( c \) is the chemical concentration and \( \rho \) denotes the cell density. The function \( \alpha \rho - \beta c \) models the production-consumption rate of the chemical; \( \alpha \) is a positive constant denoting the rate of attractant production, and \( \beta > 0 \) is a parameter which measures the self-degradation of the chemical. The chemical and the cell fluxes are given respectively by \( J_c = -D_c \nabla c \) and \( J_\rho = \chi \rho \nabla c - D_\rho \nabla \rho \), where \( D_c > 0 \), \( D_\rho > 0 \) and \( \chi > 0 \) are constants. Therefore, the cells perform a biased random walk in the direction of the chemical gradient, and the chemical diffuses (it is produced by the cells and it degrades) [11]. The term \( \chi \rho \nabla c \) models the transport of cells towards the higher concentrations of chemical signal if \( \chi > 0 \), and towards the lower concentrations of chemical signal if \( \chi < 0 \); \( \chi \rho \) is the so called sensitivity function. Model (1) has been modified in last decades with the aim of improving its consistency with biological reality and understand the implications of chemotaxis in different processes, as for instance, pathological and ecological processes, aggregation patterns, stability of nonconstant stationary states, blow up phenomena, and so on [3, 14, 16, 20, 28, 31, 32].

Stationary Patlak-Keller-Segel models related to (1) have been widely analyzed (see, for instance, [9, 17, 22, 26, 27, 30] and references therein). In this case, since \( \int_\Omega \rho(x, t) \, dx = \int_\Omega \rho_0(x) \, dx \) for all \( t > 0 \), by virtue of (1)\(_2\) and (1)\(_3\), the stationary problem associated to (1) reads
\[
\begin{aligned}
\begin{cases}
D_c \Delta c = \beta c - \alpha \rho & \text{in } \Omega, \\
D_\rho \Delta \rho = \nabla \cdot (\chi \rho \nabla c) & \text{in } \Omega,
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
\begin{cases}
c > 0, \rho > 0 & \text{in } \Omega, \\
\frac{\partial c}{\partial n} = \frac{\partial \rho}{\partial n} = 0 & \text{on } \partial \Omega, \\
\int_\Omega \rho(x) \, dx = m_0 > 0.
\end{cases}
\end{aligned}
\]
where \( m_0 \) is a given constant. On the other hand, in nature, cells frequently live in a viscous incompressible fluid and the chemical substances are transported by the fluid. In this context, it will be interesting to analyze the situation in which the chemical and cell fluxes are given by \( J_c = -D_c \nabla c + c \mathbf{u} \) and \( J_\rho = \chi \rho \nabla c - D_\rho \nabla \rho + \rho \mathbf{u} \), respectively. Thus, we are led to the following nonlinear stationary model
\[
\begin{aligned}
\begin{cases}
-D_c \Delta c + \mathbf{u} \cdot \nabla c + \beta c = \alpha \rho & \text{in } \Omega, \\
-D_\rho \Delta \rho + \mathbf{u} \cdot \nabla \rho + \nabla \cdot (\chi \rho \nabla c) = 0 & \text{in } \Omega, \\
-\nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = -\rho \mathbf{f} & \text{in } \Omega, \\
\nabla \cdot \mathbf{u} = 0 & \text{in } \Omega,
\end{cases}
\end{aligned}
\]
(2)
System (2) is completed with homogeneous Neumann conditions for the concentration and the density, and no-slip boundary condition for the velocity, that is,
\[
\begin{aligned}
\begin{cases}
\mathbf{u} = 0 & \text{on } \partial \Omega, \\
\frac{\partial \rho}{\partial n} = 0 & \text{on } \partial \Omega, \\
\int_\Omega \rho \, dx = m_0, \\
\frac{\partial c}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]
(3)
Here the unknowns are the chemical concentration $c$, the cell density $\rho$, the velocity $u$ and the pressure $\pi$ of the fluid; the equations for $(u, \pi)$ are described by the Navier-Stokes equations. The coupling of chemotaxis and the fluid is realized through the terms $u \cdot \nabla c$, $u \cdot \nabla \rho$ and $-\rho f$, representing the transport of chemical substances, the transport of cells, and the forcing term exerted on the fluid by cells, respectively. The constant $\nu > 0$ determines the viscosity of the fluid, $D_c$ and $D_\rho$ are positive constants representing the chemical diffusion and cell diffusion coefficients, respectively; $\chi$ measures the chemotactic sensitivity, and $m_0$ is a given positive constant. As it was said at the beginning, $\alpha$ is a constant denoting the rate of attractant production, and $\beta > 0$ is a parameter which measures the self-degradation of the chemical. We consider that the movement and interaction of cells are occurring in a bounded domain $\Omega$ of $\mathbb{R}^n$, $n = 2, 3$, with boundary $\partial \Omega$ smooth enough. Non-stationary chemotaxis models, including the coupling with the fluid velocity, have been recently addressed (see, for instance, [23, 34, 36] and references therein).

In this paper we deal with the mathematical formulation and analysis of a distributed optimal control problem of a chemotaxis process under the effect of a viscous and incompressible fluid. We consider the minimization of a general cost functional subject to constraints, where the state equations are given by the stationary model (2)-(3). We control the system through a distributed force and a coefficient of chemotaxis sensitivity leading the chemical concentration, the cell density and the velocity field towards a given target concentration, density and velocity, respectively. Additionally, the objective functional contains two penalty terms which are given by the norms of the controls in their respective spaces (see Theorems 2.2, 2.4 and 2.5 below). The exact mathematical formulation will be given in Section 2. Regularity in $L^p$-spaces for elliptic problems with homogeneous Neumann boundary conditions are used in order to obtain a solution for the minimization problem (10). The control of velocity, the proliferation of organisms and the concentration of chemicals in diverse environments have significant applications in science and biological processes. In fact, in several applications, the respective biological setting requires to control the proliferation and death of cells, for example, bacterial pattern formation [33, 35] or endothelial cell movement and growth in response to a chemical substance known as tumor angiogenesis factor (TAF), which have a significant role in the process of cancer cell invasion of neighboring tissue [5, 6, 25].

In past years, significant progress has been made in mathematical analysis and numerics of optimal control problems for viscous flows described by the Navier-Stokes equations and related models (see for instance, [1, 13, 15, 21, 24] and references therein). However, as far as we know, this kind of optimal control problems related to chemotaxis-fluid model have not been studied previously. In spite of, we remark that in [7] and [8] the authors study some results related to the controllability for the nonstationary Keller-Segel system (similar to model (1)) and the nonstationary chemotaxis-fluid model with consumption of chemoeattractant substance associated to a system of type (2)-(3), based on Carleman estimates for the solutions of the adjoint systems. But clearly, the aims, the methods used and the analysis developed there differ markedly from those used here.

We will prove the solvability of the optimal control problem and state first-order optimality conditions by using the Lagrange multipliers method; we derive some optimality conditions satisfied by the optimal controls. In order to obtain
these conditions, we will use a penalty method. This technique has been used in [1, 15, 21] in order to derive optimality conditions for optimal control problems associated to the stationary state of the Navier-Stokes and Boussinesq equations, when the relation control-state is multivalued. Through this method, we introduce a family of approximate control problems which approximates the initial control problem; then, we analyze their optimality conditions and, finally, we pass to the limit in the parameter of approximation in order to derive the desired optimality conditions.

The remaining of this paper is arranged as follows. Having established the optimal distributive control problem, in Subsection 2.1 we analyze the existence of an optimal solution. In Subsection 2.2 we obtain an optimality system to the control problem (10). We prove the existence of Lagrange multipliers through a penalty method.

2. Optimal distributive control problem. The aim of this work is the study of an optimal distributive control problem for (2)-(3). First of all, we recall some functional spaces which will be used throughout this paper. We will consider the usual Sobolev spaces \( H^m(\Omega) \) and Lebesgue spaces \( L^p(\Omega) \), \( 1 \leq p \leq \infty \), with the usual notations for norms \( \| \cdot \|_{H^m} \) and \( \| \cdot \|_p \), respectively. If \( H \) is a Hilbert space, we denote its inner product by \( \langle \cdot, \cdot \rangle_H \); in particular, the \( L^2(\Omega) \)-norm and the \( L^2(\Omega) \)-inner product, will be represented by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \), respectively. Corresponding Sobolev spaces of vector valued functions will be denoted by \( H^1(\Omega) \), \( L^2(\Omega) \), and so on. We also use the solenoidal Banach space \( V = \{ v \in H^1(\Omega): \nabla \cdot v = 0 \} \), with the norm \( \| u \|_V = \| \nabla u \| \). If \( Z \) is a general Banach space, its topological dual will be denoted by \( Z' \). In the paper, the letter \( K \) will denote different positive constants (independent of the coefficients of the model: \( D_c, D_p, \nu, \alpha, \beta, \gamma \)) which may change from line to line or even within the same line.

Now we introduce the notion of weak solution and establish the distributive control problem related to the chemotaxis-fluid system.

**Definition 2.1.** Let \( f \in L^{3/2}(\Omega) \). A triple \( (c, \rho, u) \in W^{1,3}(\Omega) \times H^1(\Omega) \times V \), with \( c > 0 \) and \( \rho > 0 \), is said a weak solution of (2)-(3) if

\[
\begin{align*}
(u \cdot \nabla c, \xi_1) + D_c(\nabla c, \nabla \xi_1) + \beta(c, \xi_1) &= \alpha(\rho, \xi_1), \quad \forall \xi_1 \in H^1(\Omega), \\
(u \cdot \nabla \rho, \xi_2) + D_\rho(\nabla \rho, \nabla \xi_2) - \chi(\rho \nabla c, \nabla \xi_2) &= 0, \quad \forall \xi_2 \in H^1(\Omega), \\
(u \cdot \nabla u, \psi) + \nu(\nabla u, \nabla \psi) &= -(\rho f, \psi), \quad \forall \psi \in V, \\
\int_\Omega \rho \ dx &= m_0.
\end{align*}
\]

We denote by \( H^1_+(\Omega) \) and \( H^2_+(\Omega) \) the sets \( H^1_+(\Omega) = \{ \rho \in H^1(\Omega): \rho \geq 0 \} \) and \( H^2_+(\Omega) = \{ z \in H^2(\Omega): z \geq 0 \} \). Assume that \( \gamma_i, i = 1, \ldots, 5 \), are constants such that \( \gamma_2, \gamma_4, \gamma_5 > 0 \), and \( \gamma_1, \gamma_3 \geq 0 \). Then, we consider the following objective functional \( J: W^{1,3}(\Omega) \times H^1_+(\Omega) \times V \times L^2(\Omega) \times \mathbb{R} \to \mathbb{R} \) defined by:

\[
J(c, \rho, u, f, \chi) = \frac{\gamma_1}{2} \| c - c_d \|^2 + \frac{\gamma_2}{6} \| \rho - \rho_d \|^6 + \frac{\gamma_3}{2} \| u - u_d \|^2 + \frac{\gamma_4}{2} \| f \|^2 + \frac{\gamma_5}{2} | \chi |^2,
\]

where the constants \( \gamma_k \) measure the cost of the control. Here

\[
c_d \in L^2(\Omega) \quad \rho_d \in L^6(\Omega) \quad \text{and} \quad u_d \in L^2(\Omega)
\]

are given functions which correspond to the desired states.
We study the following constrained minimization problem related to problem (2)-(3):

\[
\begin{cases}
\text{Find } (c, \rho, u, f, \chi) \in W^{1,3}(\Omega) \times H^1_+(\Omega) \times V \times L^2(\Omega) \times \mathbb{R} \text{ such that} \\
\text{the functional} J(c, \rho, u, f, \chi) \text{ reaches its minimum over the solutions of the equations (4)-(7).}
\end{cases}
\tag{10}
\]

The set of admissible solutions of problem (10) is defined by

\[
\mathcal{S}_{ad} = \{ s \equiv (c, \rho, u, f, \chi) \in W^{1,3}(\Omega) \times H^1_+(\Omega) \times V \times L^2(\Omega) \times \mathbb{R} \text{ such that} \\
J(s) < \infty \text{ and (4)-(7) are satisfied} \}. 
\]

**Remark 1.** The functional $J$ in (8) includes several cases of objective functionals depending on the value of the coefficients $\gamma_i$, $i = 1, \ldots, 5$. In particular, we can consider $\gamma_1 = \gamma_3 = 0$, and obtain an optimal density controlled by $\chi$ and $f$. In this case, given the optimal $\rho$, implicitly, by uniqueness in the equation for $c$, we also obtain a controlled concentration. On the other hand, following our analysis, it is possible to consider the case in which the only control is $\chi$ (in this case, $J$ does not depend on $f$, i.e. $\gamma_4 = 0$), or the case in which the only control is $f$ (in this case, $J$ does not depend on $\chi$, i.e. $\gamma_5 = 0$). The distributive control on $\chi$ can be interesting from a physical point of view because it allows to control the chemotactic sensitivity in order to force the movement of cells towards the increasing or decreasing chemical gradient. Notice that $\chi$ could provide minima with different meanings: a positive $\chi$ provides a model where the cells perform a biased random walk towards the increasing chemical gradient, while a negative $\chi$ provides a model where the cells perform a biased random walk towards the decreasing chemical gradient.

**Remark 2.** The norms $\frac{2^4}{3} \|c - c_d\|^2$, $\frac{2^4}{3} \|u - u_d\|^2$ and $\frac{2^4}{3} \|f\|^2$ in the functional $J$ in (8) can be taken in another suitable $L^p$-spaces. However, in our analysis, the norm $\frac{2^4}{3} \|\rho - \rho_d\|^6$ must be considered in the $L^q$-space in order to obtain a uniform $L^2$ estimate for $\nabla \rho^m$, where $\rho^m$ is a minimizing density sequence whose limit reaches the minimum density (see estimate (12) below).

### 2.1. Existence of optimal solutions

In this section we will prove the existence of an optimal solution for problem (10).

**Theorem 2.2.** Assume (9). Then, there exists a solution to the minimization problem (10), that is, there exists at least a $\hat{s} \equiv (\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}) \in \mathcal{S}_{ad}$ such that

\[
J(\hat{s}) = \min_{s \in \mathcal{S}_{ad}} J(s). 
\]

**Proof.** Evidently, the pair $(c, \rho) = (E_1, E_2)$, where $E_1, E_2$ are the constants

\[
E_1 = \frac{\alpha}{\beta} E_2, \quad E_2 = \frac{m_0}{|\Omega|}, 
\]

is a solution of (4), (5) and (7), for any $u \in V$ for (6), is well-known. Therefore, the set of admissible solutions $\mathcal{S}_{ad}$ is not empty. Denote by $s_m = (c_m, \rho_m, u_m, f_m, \chi_m) \in \mathcal{S}_{ad}, m \in \mathbb{N}$, a minimizing sequence for which $\lim_{m \to +\infty} J(s_m) = \inf_{s \in \mathcal{S}_{ad}} J(s)$. Then, taking into account that $\gamma_2, \gamma_4, \gamma_5 > 0$, we get that there exist some constants $K_1, K_2$ and $K_3$, independent of $m$, such that $\|f_m\| \leq K_1$, $|\chi_m| \leq K_2$ and $\|\rho_m - \rho_d\|_6 \leq K_3$. Then, $\|\rho_m\|_6 - \|\rho_d\|_6 \leq K_3$, and we conclude $\|\rho_m\|_6 \leq K_3 + \|\rho_d\|_6 = K$. Observe that if $(c, \rho, u) \in$
implies that \( \nu \| u \|_V \leq K \| \rho \|_6 \| f \| \), 
\( D_\rho \| \nabla \rho \| \leq |\chi| \| \rho \|_6 \| \nabla c \|_3 \),
where the constant \( K \) is independent of \( c, \rho, u, f \) and \( \chi \). Therefore, from (11), we can conclude that there exists a constant \( K_4 \), independent of \( m \), such that \( \| u_m \|_V \leq K_4 \). Now we will obtain some strong estimates for \((c, \rho)\) in the class \( H^2(\Omega) \times H^2(\Omega) \).

Analogously, since \( \nabla c \in W^{1,3}(\Omega) \times V \) is a weak solution of (4)-(7), we can easily obtain the following \textit{a priori} estimates
\[
\nu \| u \|_V \leq K \| \rho \|_6 \| f \|,
\]
\[
D_\rho \| \nabla \rho \| \leq |\chi| \| \rho \|_6 \| \nabla c \|_3,
\]
where we obtain that \( c \), solution of the problem
\[
\begin{aligned}
-D_c \Delta c + \beta c &= -u \cdot \nabla c + \alpha \rho \quad \text{in } \Omega,
\frac{\partial c}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
belongs to the space \( W^{\frac{3}{2}}(\Omega) \). Regularity results for elliptic problems in \( L^p \)-spaces can be found, for instance, in the Appendix 10.2 of [10]; see also [2]. From (4), we have
\[
\beta \| c \| \leq \alpha \| \rho \|,
\]
\[
\| c \|_{H^1} \leq \frac{\alpha}{\tau} \| \rho \|, \quad \tau = \min \{ D_c, \beta \}.
\]
Therefore, using (13), (14) and the Sobolev embedding \( L^2(\Omega) \hookrightarrow L^\frac{3}{2}(\Omega) \), we can obtain the following estimate
\[
\| \Delta c \|_{\frac{3}{2}} \leq \frac{1}{D_c} \| -u \cdot \nabla c - \beta c + \alpha \rho \|_{\frac{3}{2}}
\]
\[
\leq \frac{K}{D_c} \left( \| u \|_6 \| c \|_{H^1} + \beta \| c \| + \alpha \| \rho \| \right)
\]
\[
\leq \frac{K}{D_c} \left( \frac{1}{\tau} \| u \|_6 + 1 \right) \alpha \| \rho \|,
\]
where \( K \) is a constant independent of \( \rho \) and \( u \). Moreover, from the embedding \( H^1(\Omega) \hookrightarrow W^{1,\frac{3}{2}}(\Omega) \), (14) and (15), we have
\[
\| c \|_{W^{2,\frac{3}{2}}} \leq K(\| \Delta c \|_{\frac{3}{2}} + \| c \|_{W^{1,\frac{3}{2}}}) \leq K(\| \Delta c \|_{\frac{3}{2}} + \| c \|_{H^1})
\]
\[
\leq \frac{K}{\tau} \left( \frac{1}{\tau} \| u \|_6 + 1 \right) \alpha \| \rho \|.
\]
Analogously, since \( \nabla c \in W^{1,\frac{3}{2}}(\Omega) \hookrightarrow L^3(\Omega) \), we have \(-u \cdot \nabla c + \alpha \rho \in L^3(\Omega) \), which implies that \( c \in H^2(\Omega) \). Since \( \| \nabla c \|_3 \leq K \| c \|_{W^{2,\frac{3}{2}}} \), using (13) and (16), we obtain
\[
\| \Delta c \| \leq \frac{K}{D_c} \left( \frac{1}{\tau^2} \| u \|_6^2 + \frac{1}{\tau} \| u \|_6 + 1 \right) \alpha \| \rho \|,
\]
and therefore
\[
\| c \|_{H^2} \leq \frac{K \alpha}{\tau} \sum_{i=0}^2 \frac{\| u \|_i^i}{\tau^i} \| \rho \|,
\]
where \( \tau = \min \{ D_c, \beta \} \) and the constant \( K \) is independent of \( c, \rho, u, f \) and \( \chi \).

On the other hand, in order to obtain \( \rho \in H^2(\Omega) \), we need more regularity for \( c \). For this reason, first we prove that \( c \in W^{2,3}(\Omega) \). Since \( \nabla c \in W^{1,3}(\Omega) \hookrightarrow L^3(\Omega) \), we have that \(-u \cdot \nabla c + \alpha \rho \in L^3(\Omega) \), and therefore, we obtain that \( c \in W^{2,3}(\Omega) \).
Moreover, using the Hölder inequality, the Sobolev embeddings $H^1(\Omega) \hookrightarrow L^3(\Omega)$, $H^1(\Omega) \hookrightarrow L^6(\Omega)$, $H^2(\Omega) \hookrightarrow W^{3,3}(\Omega)$ and (17), we obtain

$$\|c\|_{W^{2,3}} \leq \frac{K\alpha}{\tau} \left(\frac{\beta}{\tau} + \sum_{i=0}^{3} \frac{\|u\|_6^i}{\tau^i}\right) \|\rho\|_{3}. \quad (18)$$

Taking into account that $c \in W^{2,3}(\Omega)$, we will prove that $\rho \in H^2(\Omega)$. Now we rewrite Equation (2)$_2$ in the form $-D_\rho \Delta \rho + D_\rho \rho = D_\rho \rho - \mathbf{u} \cdot \nabla \rho - \nabla \cdot (\chi \rho \nabla c)$. Thus, using the Hölder inequality and Sobolev embeddings, we can see that $D_\rho \rho - \mathbf{u} \cdot \nabla \rho - \nabla \cdot (\chi \rho \nabla c) \in L^2(\Omega)$, and from classical elliptic regularity, we obtain that $\rho \in W^{2,2}(\Omega)$. Moreover, using (17) we can obtain

$$\|\rho\|_{W^{2,2}} \leq \frac{K}{D_\rho} \|\nabla \rho\|_3 \left(\|u\|_6 + |\nabla c|_6\right) + \frac{K\alpha|\chi|}{D_\rho} \left(\frac{\sum_{k=0}^{2} \|u\|_6^k}{\tau^k}\right) \|\rho\|_2 + K\|\rho\|. \quad (19)$$

Analogously, as $\nabla \rho \in W^{1,2}(\Omega) \hookrightarrow L^3(\Omega)$, we can prove that $D_\rho \rho - \mathbf{u} \cdot \nabla \rho - \nabla \cdot (\chi \rho \nabla c) \in L^2(\Omega)$, which implies $\rho \in H^2(\Omega)$. Furthermore, using (18), we can get

$$\|\rho\|_{H^2} \leq \frac{K}{D_\rho} \|\nabla \rho\|_3 \left(\|u\|_6 + |\nabla c|_6\right)$$

$$+ \frac{K\alpha|\chi|}{D_\rho} \left(\frac{\sum_{k=0}^{2} \|u\|_6^k}{\tau^k}\right) \|\rho\|_2 + K\|\rho\|. \quad (20)$$

Therefore, from (17), (12), (19) and (20), we conclude that there exist constants $K_5$ and $K_6$, independent of $m$, such that $\|c_m\|_{H^2} \leq K_5$ and $\|\rho_m\|_{H^2} \leq K_6$. From the above, the sequence $s_m = (c_m, \rho_m, \mathbf{u}_m, \mathbf{f}_m, \chi_m)$ is uniformly bounded in $H^2(\Omega) \times V \times L^2(\Omega) \times \mathbb{R}$. Thus, taking into account that $\rho_m \geq 0$ for all $m \in \mathbb{N}$, we obtain an element $\tilde{s} \equiv (\tilde{c}, \tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\mathbf{f}}, \tilde{\chi}) \in H^2(\Omega) \times H^2(\Omega) \times V \times L^2(\Omega) \times \mathbb{R}$ such that, for some subsequence of $(s_m)_{m \in \mathbb{N}}$, still denoted by $(s_m)_{m \in \mathbb{N}}$, we have

$c_m \rightharpoonup \tilde{c}$ in $H^2(\Omega)$ and strongly in $L^s(\Omega)$, $s \in [1, \infty)$,

$\rho_m \rightharpoonup \tilde{\rho}$ in $H^2(\Omega)$ and strongly in $L^l(\Omega)$, $l \in [1, \infty)$,

$\mathbf{u}_m \rightharpoonup \tilde{\mathbf{u}}$ in $V$ and strongly in $L^p(\Omega)$, $p \in [1, 6)$,

$\mathbf{f}_m \rightharpoonup \tilde{\mathbf{f}}$ in $L^2(\Omega)$,

$\chi_m \rightharpoonup \tilde{\chi}$ in $\mathbb{R}$.

A standard procedure allows to pass to the limit in (4)-(7), as $m$ goes to $\infty$, and thus we obtain that $\tilde{s}$ satisfies (4)-(7). Consequently we have that $\tilde{s} \equiv (\tilde{c}, \tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\mathbf{f}}, \tilde{\chi}) \in S_{ad}$, and recalling that the functional $\mathcal{J}$ is weakly lower semicontinuous on $S_{ad}$, we have that

$$\mathcal{J}(\tilde{s}) = \min_{s \in S_{ad}} \mathcal{J}(s).$$

In next lemma, we shall show that $\tilde{c} > 0$ and $\tilde{\rho} > 0$. This conclusion is consistent with the physical meaning of control problem (10) and model (2)-(3).

**Lemma 2.3.** Let $(\tilde{c}, \tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\mathbf{f}}, \tilde{\chi}) \in H^2(\Omega) \times H^2(\Omega) \times V \times L^2(\Omega) \times \mathbb{R}$ an optimal solution of problem (10). Then $\tilde{c} > 0$ and $\tilde{\rho} > 0$.

**Proof.** In order to prove the positivity of $\tilde{c}$ and $\tilde{\rho}$, we argue in three steps.

**First step. Smoothness of $\tilde{c}$ and $\tilde{\rho}$.** We can obtain more regularity for the
velocity \( \hat{u} \) using the classical \( L^p \)-regularity for the Stokes problem \([4]\). In fact, taking into account that \( \hat{\rho} \in H^2(\Omega) \rightarrow L^\infty(\Omega) \), we have that \( \hat{u} \cdot \nabla \hat{u} + \hat{\rho} \hat{f} \in L^\frac{5}{3}(\Omega) \), and from the regularity of the Stokes problem, we conclude that \( \hat{u} \in W^{2,\frac{5}{3}}(\Omega) \). Analogously, since \( \nabla \hat{u} \in W^{1,2}(\Omega) \rightarrow L^3(\Omega) \), and \( H^2(\Omega) \rightarrow L^\infty(\Omega) \) we have \( -\hat{u} \cdot \nabla \hat{u} + \hat{\rho} \hat{f} \in L^2(\Omega) \), and thus, we get \( \hat{u} \in H^2(\Omega) \rightarrow C(\bar{\Omega}) \). We know that \( \hat{c} \in W^{2,3}(\Omega) \). This implies in particular that \( -\hat{u} \cdot \nabla \hat{c} + \alpha \hat{\rho} \in L^4(\Omega) \), and consequently, by elliptic regularity, we get \( \hat{c} \in W^{2,4}(\Omega) \). Using the last regularity we get \( -\hat{u} \cdot \nabla \hat{c} + \alpha \hat{\rho} \in C(\bar{\Omega}) \) and the elliptic regularity again, implies that \( \hat{c} \in C^2(\bar{\Omega}) \). In the same spirit, using that \( \hat{u} \in C(\bar{\Omega}) \) and \( \hat{c} \in C^2(\bar{\Omega}) \), we conclude that \( \hat{\rho} \in C^2(\bar{\Omega}) \).

**Second step. Positivity of \( \hat{c} \).** Note that the minimum \( \hat{\rho} \) obtained in Theorem 2.2 satisfies \( \hat{\rho} \geq 0 \) in \( \Omega \). Then, we first claim that \( \hat{c} \geq 0 \) on \( \bar{\Omega} \). Indeed, if not, since \( \hat{c} \in C(\bar{\Omega}) \), there exists a \( \mathbf{x}_0 \in \bar{\Omega} \) such that

\[
\hat{c}(\mathbf{x}_0) = \min_{\mathbf{x} \in \Omega} \hat{c}(\mathbf{x}) < 0.
\]

Consider the operator

\[
L[c] = D_x \Delta c - \hat{u} \cdot \nabla c, \quad h = -\beta < 0.
\]

Then, from \((2)_1\), we have that

\[
(L + h)[-\hat{c}] = \alpha \hat{\rho} \geq 0.
\]

If \( \mathbf{x}_0 \in \Omega \), then by Theorem 6 in \([29]\) (p. 64) (see also, \([12]\)), \( \hat{c} \equiv \hat{c}(\mathbf{x}_0) < 0 \). Hence, from \((2)_1\), we deduce that \( \hat{\rho} \) must be a constant whose value is \( \hat{\rho} = \frac{\beta}{\alpha} \hat{c}(\mathbf{x}_0) \), and therefore, \( \int_\Omega \hat{\rho} d\mathbf{x} = \frac{\beta}{\alpha} \hat{c}(\mathbf{x}_0) |\Omega| < 0 \), which contradicts the condition \( \int_\Omega \hat{\rho} d\mathbf{x} > 0 \) in \((3)_2\). On the other hand, if \( \mathbf{x}_0 \in \partial\Omega \), then Theorem 8 in \([29]\) (p. 67) (see also, \([12]\)), implies that \( \hat{c} \equiv \hat{c}(\mathbf{x}_0) < 0 \) because of \( \frac{\partial \hat{c}}{\partial \mathbf{n}}(\mathbf{x}_0) = 0 \), which contradicts again the condition \( \int_\Omega \hat{\rho} d\mathbf{x} > 0 \). This proves that \( \hat{c} \geq 0 \) on \( \bar{\Omega} \). Now, it only remains to prove that \( \hat{c} > 0 \). Assume first that \( \hat{c}(\mathbf{y}) = 0 \) for some \( \mathbf{y} \in \Omega \). From Theorem 6 in \([29]\), we deduce that \( \hat{c} \equiv 0 \). But, in this case, from \((2)_1\), we deduce that \( \hat{\rho} \equiv 0 \), which is not possible due the integral in \((3)_2\). Then, assume that \( \hat{c}(\mathbf{y}) = 0 \) for some \( \mathbf{y} \in \partial\Omega \). In this case, from Theorem 8 in \([29]\), we get \( \hat{c} \equiv 0 \) because of \( \frac{\partial \hat{c}}{\partial \mathbf{n}}(\mathbf{y}) = 0 \), which leads again to a contradiction because \( \hat{\rho} \) cannot vanish. Therefore, \( \hat{c} > 0 \).

**Third step. Positivity of \( \hat{\rho} \).** Since \( \hat{\rho} \geq 0 \) in \( \Omega \), we claim that \( \hat{\rho} > 0 \) in \( \Omega \). In fact, from \((2)_2\) we have that

\[
(L + h)[-\hat{\rho}] = M \hat{\rho} \geq 0,
\]

where \( L[\hat{\rho}] = D_x \Delta \hat{\rho} - \hat{u} \cdot \nabla \hat{\rho} - \chi \nabla \rho \nabla \hat{c} \), with \( h = -\chi \Delta \hat{c} - M \), for \( M > 0 \) sufficiently large such that \( h < 0 \). Observe that \( \hat{c} \in C^2(\bar{\Omega}) \) implies that \( \Delta \hat{c} \in C^0(\bar{\Omega}) \) and thus \( h \) is well-defined. Then, if we assume that \( \hat{\rho}(\mathbf{y}) = 0 \), and using either Theorem 6 or Theorem 8 in \([29]\), when either \( \mathbf{y} \in \Omega \) or \( \mathbf{y} \in \partial\Omega \) respectively, we can deduce that \( \hat{\rho} \equiv 0 \). But this contradicts integral \((3)_2\).

2.2. **Necessary optimality conditions and an optimality system.** This section is devoted to obtain an optimality system to problem \((10)\). We wish to use the method of Lagrange multipliers to turn the constrained optimization problem \((10)\) into an unconstrained one. In order to prove the existence of Lagrange multipliers, we use a penalty method (see \([1, 15, 21]\)). This method consists in the introduction of a family of penalized problems \((P_\delta)_{\delta \in (0,1)}\), which have at least one optimal solution; we can derive first-order necessary optimality conditions for \((P_\delta)\) and then,
we obtain first-order necessary optimality conditions for the problem (10). First, we introduce the following operators

\[ A : \mathbf{V} \to \mathbf{V}', \]
\[ \langle A(u), \psi \rangle = (\nabla u, \nabla \psi), \quad \forall \psi \in \mathbf{V} \]

\[ B : \mathbf{V} \times \mathbf{V} \to \mathbf{V}', \]
\[ \langle B(u, v), \psi \rangle = (u \cdot \nabla v, \psi), \quad \forall \psi \in \mathbf{V}, \]

\[ F : H^1(\Omega) \times L^2(\Omega) \to \mathbf{V}' \]
\[ \langle F(z, f), \psi \rangle = (zf, \psi), \quad \forall \psi \in \mathbf{V}. \]

Observe that \( A \) is the well-known Stokes operator. In order to simplify the notation, let us denote by \( \mathbb{M} \) the set

\[ \mathbb{M} \equiv H^2(\Omega) \times H^2(\Omega) \times \mathbf{V} \times L^2(\Omega) \times \mathbb{R}. \]

We consider the following family of penalized extremal problems \((P_\delta)\):

\[
\begin{align*}
\mathcal{J}_\delta(c, \rho, u, f, \chi) &= \mathcal{J}(c, \rho, u, f, \chi) \\
&+ \frac{1}{2\delta} \| - D_c \Delta c + u \cdot \nabla c + \beta c - \alpha \rho \|^2 \\
&+ \frac{1}{2\delta} \| - D_\rho \Delta \rho + u \cdot \nabla \rho + \nabla \cdot (\chi \rho \nabla c) \|^2 \\
&+ \frac{1}{2\delta} \| \nu A(u) + B(u, u) + F(\rho, f) \|^2 \\
&+ \frac{1}{2\delta} \left| \int_\Omega \rho \, dx - m_0 \right|^2 + \frac{N}{2} \| c - \hat{c} \|^2_{H^2} \\
&+ \frac{N}{2} \| \rho - \hat{\rho} \|^2_{H^2} + \frac{N}{2} \| u - \hat{u} \|^2_{\mathbf{V}} + \frac{N}{2} \| f - \hat{f} \|^2 \\
&+ \frac{N}{2} \| \chi - \hat{\chi} \|^2; \quad \delta \in (0, 1),
\end{align*}
\]

with

\[ \frac{\partial c}{\partial n} \big|_{\partial \Omega} = 0, \quad \frac{\partial \rho}{\partial n} \big|_{\partial \Omega} = 0, \]

where \( N > 0 \) is a given constant and \( \hat{s} = (\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}) \in \mathbb{M} \) is a solution of problem (10).

The existence of \( (\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}) \) was established in Theorem 2.2. Next proposition establishes the existence of optimal solution for the family of penalized problems (21).

**Proposition 1.** Assume (9). There exists a solution to the minimization problem (21), that is, for any \( \delta \in (0, 1) \), there exists at least a point \( \hat{s}_\delta \equiv (\hat{c}_\delta, \hat{\rho}_\delta, \hat{u}_\delta, \hat{f}_\delta, \hat{\chi}_\delta) \in \mathbb{M} \) such that

\[ \mathcal{J}_\delta(\hat{s}_\delta) = \min_{s \in \mathbb{M}} \mathcal{J}_\delta(s). \]

**Proof.** Let \( \delta \in (0, 1) \) and denote by \( s_m^\delta = (c_m^\delta, \rho_m^\delta, u_m^\delta, f_m^\delta, \chi_m^\delta) \in \mathbb{M}, m \in \mathbb{N}, \) a minimizing sequence for which \( \lim_{m \to \infty} \mathcal{J}_\delta(s_m^\delta) = \inf_{s \in \mathbb{M}} \mathcal{J}_\delta(s) \). Then, taking into account that \( N > 0 \), we get that there exists a constant \( K \), independent of \( m \), such that \( \| c_m^\delta - \hat{c} \|^2_{H^2} \leq K \), which implies that \( \| c_m^\delta \|^2_{H^2} \leq K + \| \hat{c} \|^2_{H^2} = K_1 \). Analogously,
we deduce that there exist some constants $K_2, K_3, K_4$ and $K_5$, independent of $m$, such that \( \|\rho^m\|_{H^2} \leq K_2, \|u^m\|_{V} \leq K_3, \|F^m\| \leq K_4 \) and \( |\chi^m| \leq K_5 \). Thus, the sequence \( s^m_\delta \) is uniformly bounded in \( H^2(\Omega) \times H^2(\Omega) \times V \times L^2(\Omega) \times \mathbb{R} \) and, taking into account that \( \rho^m_\delta \geq 0 \) for all \( m \in \mathbb{N} \), we obtain an element \( \tilde{s}_\delta \equiv (\tilde{c}_\delta, \tilde{\rho}_\delta, \tilde{u}_\delta, \tilde{f}_\delta, \tilde{\chi}_\delta) \in H^2(\Omega) \times H^2_+(\Omega) \times V \times L^2(\Omega) \times \mathbb{R} \) such that, for some subsequence of \( (s^m_\delta)_{m \in \mathbb{N}} \), still denoted by \( (s^m_\delta)_{m \in \mathbb{N}} \), we have

\[
(c^m_\delta, \rho^m_\delta, u^m_\delta, f^m_\delta) \rightarrow (\tilde{c}_\delta, \tilde{\rho}_\delta, \tilde{u}_\delta, \tilde{f}_\delta) \quad \text{weakly in } H^2(\Omega) \times H^2_+(\Omega) \times V \times L^2(\Omega)
\]

and

\[
\chi^m_\delta \rightarrow \tilde{\chi}_\delta \quad \text{in } \mathbb{R}. \tag{22}
\]

Therefore, from (22) and using the fact that \( \mathcal{J}_\delta \) is weakly lower semicontinuous on \( \mathcal{M} \), we deduce that

\[
\mathcal{J}_\delta(\tilde{s}_\delta) = \min_{s \in \mathcal{M}} \mathcal{J}_\delta(s).
\]

\[\square\]

**Theorem 2.4.** For each \( \delta \in (0,1) \), let \( (\hat{c}_\delta, \hat{\rho}_\delta, \hat{u}_\delta, \hat{f}_\delta, \hat{\chi}_\delta) \in \mathcal{M} \) a solution of problem (21) and \( (\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}) \in \mathcal{M} \) a solution of problem (10). Then,

\[
\lim_{\delta \rightarrow 0} \mathcal{J}_\delta(\hat{c}_\delta, \hat{\rho}_\delta, \hat{u}_\delta, \hat{f}_\delta, \hat{\chi}_\delta) = \mathcal{J}(\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}). \tag{23}
\]

Moreover,

\[
(\hat{c}_\delta, \hat{\rho}_\delta, \hat{u}_\delta, \hat{f}_\delta) \rightarrow (\hat{c}, \hat{\rho}, \hat{u}, \hat{f}) \quad \text{weakly in } H^2(\Omega) \times H^2_+(\Omega) \times V \times L^2(\Omega)
\]

and

\[
\hat{\chi}_\delta \rightarrow \hat{\chi} \quad \text{in } \mathbb{R}. \tag{24}
\]

**Proof.** First, observe that

\[
\mathcal{J}_\delta(\hat{c}_\delta, \hat{\rho}_\delta, \hat{u}_\delta, \hat{f}_\delta, \hat{\chi}_\delta) \leq \mathcal{J}_\delta(\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}) = \mathcal{J}(\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}), \quad \forall \delta \in (0,1). \tag{25}
\]

Thus, from the definition of \( \mathcal{J}_\delta \) in (21), we have

\[
\|\hat{u}_\delta - \hat{u}\|^2_V \leq \frac{2}{N_0} \mathcal{J}(\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}) \leq K,
\]

which implies that \( \|\hat{u}_\delta\|_V - \|\hat{u}\|_V \leq K_1 \), and therefore \( \|\hat{u}_\delta\|_V \leq K_1 \), for all \( \delta \in (0,1) \), where the constant \( K_1 \) is independent of \( \delta \). Similarly, we can verify that there exist constants \( K_2, K_3, K_4 \) and \( K_5 \), independent of \( \delta \), such that \( \|\hat{c}_\delta\|_{H^2} \leq K_2, \|\hat{\rho}_\delta\|_{H^2} \leq K_3, \|\hat{f}_\delta\| \leq K_4 \) and \( |\hat{\chi}_\delta| \leq K_5 \). Then, we conclude that \( \tilde{s}_\delta = (\tilde{c}_\delta, \tilde{\rho}_\delta, \tilde{u}_\delta, \tilde{f}_\delta, \tilde{\chi}_\delta)_{\delta \in (0,1)} \) is uniformly bounded in \( H^2(\Omega) \times H^2_+(\Omega) \times V \times L^2(\Omega) \times \mathbb{R} \). Thus, as \( H^2_+(\Omega) \) and \( V \) are closed subsets of \( H^2(\Omega) \) and \( H^1(\Omega) \) respectively, we obtain the existence of \( (c^*, \rho^*, u^*, f^*, \chi^*) \in H^2(\Omega) \times H^2_+(\Omega) \times V \times L^2(\Omega) \times \mathbb{R} \) such that, for some subsequence of \( (\tilde{s}_\delta)_{\delta \in (0,1)} \), still denoted by \( (\tilde{s}_\delta)_{\delta \in (0,1)} \), it holds

\[
(\tilde{c}_\delta, \tilde{\rho}_\delta, \tilde{u}_\delta, \tilde{f}_\delta) \rightarrow (c^*, \rho^*, u^*, f^*) \quad \text{weakly in } H^2(\Omega) \times H^2_+(\Omega) \times V \times L^2(\Omega)
\]

and

\[
\tilde{\chi}_\delta \rightarrow \chi^* \quad \text{in } \mathbb{R}. \tag{26}
\]

Moreover, from (25) we have that

\[
\| - D_\delta \Delta \hat{c}_\delta + \hat{u}_\delta \cdot \nabla \hat{c}_\delta + \beta \hat{c}_\delta - \alpha \hat{\rho}_\delta \|^2_\mathcal{X} \leq 2\delta \mathcal{J}(\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}),
\]

and therefore, proceeding as in Theorem 2.2, taking the limit as \( \delta \) goes to 0, we get

\[
-D_\delta \Delta c^* + u^* \cdot \nabla c^* + \beta c^* = \alpha \rho^* \quad \text{in } L^2(\Omega).
\]

Similarly, we can verify that

\[
-D_\rho \Delta \rho^* + u^* \cdot \nabla \rho^* + \nabla \cdot (\chi^* \rho^* \nabla c^*) = 0 \quad \text{in } L^2(\Omega),
\]

\[
A(u^*) + B(u^*, u^*) = -F(\rho^*, f^*) \quad \text{in } V',
\]
Let \( \rho^* > 0 \) we deduce that \( c^* > 0 \) and \( \rho^* > 0 \). Consequently we have that \( (c^*, \rho^*, u^*, f^*, \chi^*) \in S_{ad} \). Thus

\[
J(\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}) \leq J(c^*, \rho^*, u^*, f^*, \chi^*). \tag{27}
\]

Finally, since \( \mathcal{J}_\delta \) is weakly lower semicontinuous on \( M \), and using (25) and (27), we have

\[
\mathcal{J}(c^*, \rho^*, u^*, f^*, \chi^*) \leq \mathcal{J}(c^*, \rho^*, u^*, f^*, \chi^*) \leq \liminf_{\delta \to 0} \mathcal{J}_\delta(\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}) \leq \liminf_{\delta \to 0} \mathcal{J}_\delta(\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}) \leq \mathcal{J}(\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}) \leq \mathcal{J}(c^*, \rho^*, u^*, f^*, \chi^*), \tag{28}
\]

and thus we obtain (23). Also, from (28) we conclude that \( \mathcal{J}_\delta(c^*, \rho^*, u^*, f^*, \chi^*) = \mathcal{J}(c^*, \rho^*, u^*, f^*, \chi^*) \) and therefore \( c^* = \hat{c}, \rho^* = \hat{\rho}, u^* = \hat{u}, f^* = \hat{f} \) and \( \chi^* = \hat{\chi} \). Consequently, using (26), we deduce (24). \( \square \)

Now, let us consider the following operators

\[
\begin{align*}
A_1 : L^2(\Omega) &\to (H^2(\Omega))^\prime, \\
G_1 : L^2(\Omega) &\to (H^1(\Omega))^\prime, \\
I_1 : V &\to (H^1(\Omega))^\prime, \\
B_1 : V \times V &\to V, \\
\tilde{B}_1 : L^2(\Omega) \times H^2(\Omega) &\to V,
\end{align*}
\]

defined by

\[
\begin{align*}
\langle A_1(z), \xi \rangle &= (z, \Delta \xi), \quad \forall \xi \in H^2(\Omega), \\
\langle B_1(u, z), \xi \rangle &= (u \cdot \nabla \xi, z), \quad \forall \xi \in H^2(\Omega), \\
\langle G_1(z), \xi \rangle &= (z, \xi), \quad \forall \xi \in H^1(\Omega), \\
\langle I_1(\xi, f), \varphi_2 \rangle &= (\varphi_2 f, \xi), \quad \forall \varphi_2 \in H^1(\Omega), \\
\langle F_1(\chi, \rho, \mu), \xi \rangle &= (\nabla \cdot (\chi \rho \nabla \xi), \mu), \quad \forall \xi \in H^2(\Omega), \\
\langle B_1^x(u, \xi), \varphi_3 \rangle &= (\nabla u^T : \xi, \varphi_3), \quad \forall \varphi_3 \in V, \\
\langle G(\chi, \mu, \varphi_3) \rangle &= (\nabla \cdot (\chi \nabla \varphi_3), \mu), \quad \forall \xi \in H^2(\Omega), \\
\langle \tilde{B}_1(\mu, \rho), \varphi_3 \rangle &= (\varphi_3 \cdot \nabla \rho, \mu), \quad \forall \varphi_3 \in V.
\end{align*}
\]

In next theorem, we derive first-order necessary optimality conditions for the penalized control problem (21), and then by taking the limit as \( \delta \) goes to zero, we obtain first-order necessary optimality conditions for the problem (10).

**Theorem 2.5.** Let \( (\hat{c}, \hat{\rho}, \hat{u}, \hat{f}, \hat{\chi}) \in M \) an optimal solution of problem (10). Then, there exists a non zero Lagrange multiplier \( (\lambda_0, \sigma, \mu, \xi, \zeta) \in [0, \infty) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \) such that

\[
\int_\Omega \rho^* \, dx = m_0.
\]
\( \mathbf{V} \times \mathbb{R} \) such that
\[
-D_c(A_1(\sigma), \varphi_1) + \langle B_1(\hat{u}, \sigma), \varphi_1 \rangle + \beta \langle G_1(\sigma), \varphi_1 \rangle + \langle F_1(\hat{\chi}, \hat{\rho}, \mu), \varphi_1 \rangle \\
+ \lambda_0 \gamma_1 (\hat{c} - c_d, \varphi_1) = 0, \tag{31}
\]
\[
-D_\rho(A_1(\mu), \varphi_2) + \langle B_1(\hat{u}, \mu), \varphi_2 \rangle + \langle G(\hat{\chi}, \mu, \hat{c}), \varphi_2 \rangle - \alpha \langle G_1(\sigma), \varphi_2 \rangle \\
+ \langle I_1(\xi, \hat{f}), \varphi_2 \rangle + \langle I(\zeta), \varphi_2 \rangle + \lambda_0 \gamma_2 (|\hat{\rho} - \rho_d|^5 \text{sgn}(\hat{\rho} - \rho_d), \varphi_2) \geq 0, \tag{32}
\]
\[
\nu(A(\xi), \varphi_3) + \langle B_1(\hat{u}, \xi), \varphi_3 \rangle - \langle B(\hat{u}, \xi), \varphi_3 \rangle + \langle \hat{B}_1(\sigma, \hat{c}), \varphi_3 \rangle \\
+ \langle \hat{B}_1(\mu, \hat{\rho}), \varphi_3 \rangle + \lambda_0 \gamma_3 (\hat{u} - u_d, \varphi_3) = 0, \tag{33}
\]
\[
(\hat{\rho} \xi, \varphi_3) + \lambda_0 \gamma_4 (\hat{f}, \varphi_4) = 0, \tag{34}
\]
\[
\varphi_5 \left[ \lambda_0 \gamma_5 \check{\chi} + \int_\Omega \nabla \cdot (\hat{\rho} \nabla \check{c}) \mu \, dx \right] = 0, \tag{35}
\]
for all \( \varphi_1 \in H^2(\Omega), \varphi_2 \in H^2_+(\Omega), \varphi_3 \in \mathbf{V}, \varphi_4 \in L^2(\Omega) \) and \( \varphi_5 \in \mathbb{R} \).

**Proof.** Let \( \varphi_1 \in H^2(\Omega), \varphi_2 \in H^2_+(\Omega), \varphi_3 \in \mathbf{V}, \varphi_4 \in L^2(\Omega) \) and \( \varphi_5 \in \mathbb{R} \) be arbitrary functions. We introduce the function \( \mathcal{P} \) on \( \mathbb{R} \times [0, \infty) \times \mathbb{R}^3 \) defined by:
\[
\mathcal{P}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = J_3(\hat{c}_3 + \lambda_1 \varphi_1, \hat{\rho}_3 + \lambda_2 \varphi_2, \hat{u}_3 + \lambda_3 \varphi_3, \hat{f}_3 + \lambda_4 \varphi_4, \hat{\chi}_3 + \lambda_5 \varphi_5),
\]
where \( (\hat{c}_3, \hat{\rho}_3, \hat{u}_3, \hat{f}_3, \hat{\chi}_3) \in \mathcal{M} \) is a solution of the problem (21), for all \( \delta \in (0, 1) \).

Clearly, \( \mathcal{P} \) attains a minimum at \( (0, 0, 0, 0, 0) \) on the set \( \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in \mathbb{R} \times [0, \infty) \times \mathbb{R}^3 \} \). Thus, we have
\[
\frac{\partial \mathcal{P}}{\partial \lambda_i}(0, 0, 0, 0, 0) = 0, \quad i = 1, 3, 4, 5. \tag{36}
\]
For simplicity, we consider that \( \|f\|_{\mathbf{V}'} = \|\nabla (A^{-1} f)\| \). Then, from (36), we obtain the optimality system for the penalized control problem (21)
\[
-D_c(A_1(\sigma_3), \varphi_1) + \langle B_1(\hat{u}_3, \sigma_3), \varphi_1 \rangle + \beta \langle G_1(\sigma_3), \varphi_1 \rangle + \langle F_1(\hat{\chi}_3, \hat{\rho}_3, \mu_3), \varphi_1 \rangle \\
+ N(\hat{c}_3 - \hat{c}, \varphi_1) = 0, \quad \forall \varphi_1 \in H^2(\Omega), \tag{37}
\]
\[
-D_\rho(A_1(\mu_3), \varphi_2) + \langle B_1(\hat{u}_3, \mu_3), \varphi_2 \rangle + \langle G(\hat{\chi}_3, \mu_3, \hat{c}_3), \varphi_2 \rangle - \alpha \langle G_1(\sigma_3), \varphi_2 \rangle \\
+ \langle I_1(\xi_3, \hat{f}_3), \varphi_2 \rangle + \langle I(\zeta_3), \varphi_2 \rangle + N(\hat{\rho}_3 - \hat{\rho}, \varphi_2) = 0, \quad \forall \varphi_2 \in H^2_+(\Omega), \tag{38}
\]
\[
\nu(A(\xi_3), \varphi_3) + \langle B_1(\hat{u}_3, \xi_3), \varphi_3 \rangle - \langle B(\hat{u}_3, \xi_3), \varphi_3 \rangle + \langle \hat{B}_1(\sigma_3, \hat{c}_3), \varphi_3 \rangle \\
+ \langle \hat{B}_1(\mu_3, \hat{\rho}_3), \varphi_3 \rangle + N(\hat{u}_3 - \hat{u}, \varphi_3) = 0, \quad \forall \varphi_3 \in \mathbf{V}, \tag{39}
\]
\[
(\hat{\rho}_3 \xi_3, \varphi_4) + \gamma_3 (\hat{u}_3 - u_d, \varphi_3) = 0, \quad \forall \varphi_4 \in \mathbf{V}, \tag{40}
\]
\[
\varphi_5 \left[ \gamma_5 \hat{\chi}_3 + \int_\Omega \nabla \cdot (\hat{\rho}_3 \nabla \hat{c}_3) \mu_3 \, dx + N(\hat{\chi}_3 - \hat{\chi}) \right] = 0, \quad \varphi_5 \in \mathbb{R}, \tag{41}
\]
where

\[ \sigma_\delta = \frac{1}{\delta} (-D_c \Delta \hat{c}_\delta + \hat{u}_\delta \cdot \nabla \hat{c}_\delta + \beta \hat{c}_\delta - \alpha \hat{\rho}_\delta) \in L^2(\Omega), \]

\[ \mu_\delta = \frac{1}{\delta} (-D_\rho \Delta \hat{\rho}_\delta + \hat{u}_\delta \cdot \nabla \hat{\rho}_\delta + \nabla \cdot (\hat{\chi}_\delta \hat{\rho}_\delta \nabla \hat{c}_\delta)) \in L^2(\Omega), \]

\[ \xi_\delta = \frac{1}{\delta} A^{-1} [\nu A(\hat{u}_\delta) + B(\hat{u}_\delta, \hat{u}_\delta) + F(\hat{\rho}_\delta, \hat{\xi}_\delta)] \in \mathbf{V}, \]

\[ \zeta_\delta = \frac{1}{\delta} \left[ \int_\Omega \hat{\rho}_\delta \, dx - m_0 \right] \in \mathbb{R}. \]

In order to take the limit, setting \( K_\delta = \left( \|\sigma_\delta\|^2 + \|\mu_\delta\|^2 + \|\xi_\delta\|_V^2 + |\zeta_\delta|^2 \right)^{\frac{1}{2}} \), we consider two cases:

1. Let \( \lim_{\delta \to 0} K_\delta = +\infty \). Denote \( \hat{\sigma}_\delta = \frac{\sigma_\delta}{K_\delta}, \hat{\mu}_\delta = \frac{\mu_\delta}{K_\delta}, \hat{\xi}_\delta = \frac{\xi_\delta}{K_\delta} \) and \( \hat{\zeta}_\delta = \frac{\zeta_\delta}{K_\delta} \). Then, by the definitions of \( \hat{\sigma}_\delta, \hat{\mu}_\delta, \hat{\xi}_\delta \) and \( \hat{\zeta}_\delta \), we have

\[ \|\hat{\sigma}_\delta\| \leq 1, \quad \|\hat{\mu}_\delta\| \leq 1, \quad \|\hat{\xi}_\delta\|_V \leq 1, \quad |\hat{\zeta}_\delta| \leq 1. \]

Thus, we obtain an element \((\hat{\sigma}_\delta, \hat{\mu}_\delta, \hat{\xi}_\delta, \hat{\zeta}_\delta) \in L^2(\Omega) \times L^2(\Omega) \times V \times \mathbb{R}\) such that, for some subsequence of \((\hat{\sigma}_\delta, \hat{\mu}_\delta, \hat{\xi}_\delta, \hat{\zeta}_\delta)_{\delta \in (0,1)}\), still denoted by \((\hat{\sigma}_\delta, \hat{\mu}_\delta, \hat{\xi}_\delta, \hat{\zeta}_\delta)_{\delta \in (0,1)}\), it holds

\[(\hat{\sigma}_\delta, \hat{\mu}_\delta, \hat{\xi}_\delta, \hat{\zeta}_\delta) \to (\hat{\sigma}, \hat{\mu}, \hat{\xi}, \hat{\zeta}) \text{ in } L^2(\Omega) \times L^2(\Omega) \times V \text{ and } \hat{\zeta}_\delta \to \hat{\zeta} \text{ in } \mathbb{R}. \quad (42)\]

Moreover, from \((37)-(41)\) we have that \((\hat{\sigma}_\delta, \hat{\mu}_\delta, \hat{\xi}_\delta, \hat{\zeta}_\delta)\) satisfies the equations

\[ -D_c \langle A_1(\hat{\sigma}_\delta), \varphi_1 \rangle + \langle B_1(\hat{u}_\delta, \hat{\sigma}_\delta), \varphi_1 \rangle + \beta \langle G_1(\hat{\sigma}_\delta), \varphi_1 \rangle \]

\[ + \langle F_1(\hat{\chi}_\delta, \hat{\rho}_\delta, \hat{\mu}_\delta), \varphi_1 \rangle + \frac{N}{K_\delta} \langle \hat{c}_\delta - \hat{c}, \varphi_1 \rangle_{H^2} + \frac{\gamma_3}{K_\delta} \langle \hat{c}_\delta - c_d, \varphi_1 \rangle = 0, \quad (43)\]

\[ -D_\rho \langle A_2(\hat{\mu}_\delta), \varphi_2 \rangle + \langle B_2(\hat{u}_\delta, \hat{\mu}_\delta), \varphi_2 \rangle + \langle F_2(\hat{\chi}_\delta, \hat{\rho}_\delta, \hat{\mu}_\delta), \varphi_2 \rangle \]

\[ - \alpha \langle G_2(\hat{\sigma}_\delta), \varphi_2 \rangle + \langle I_1(\hat{\xi}_\delta, \hat{\xi}_\delta), \varphi_2 \rangle + \langle I(\hat{\zeta}_\delta), \varphi_2 \rangle + \frac{N}{K_\delta} \langle \hat{\rho}_\delta - \hat{\rho}, \varphi_2 \rangle_{H^2} \]

\[ + \frac{\gamma_2}{K_\delta} \langle \hat{\rho} \hat{\rho} \rangle \|\hat{\rho} - \hat{\rho} \|_V^2 \text{sgn}(\hat{\rho} - \hat{\rho}) \|\varphi_2\| \geq 0, \quad (44)\]

\[ \nu \langle A(\hat{\xi}_\delta), \varphi_3 \rangle + \langle B_1^+(\hat{u}_\delta, \hat{\xi}_\delta), \varphi_3 \rangle - \langle B(\hat{u}_\delta, \hat{\xi}_\delta), \varphi_3 \rangle + \langle B_1(\hat{\sigma}_\delta, \hat{c}_\delta), \varphi_3 \rangle \]

\[ + \langle B_1(\hat{\mu}_\delta, \hat{\rho}_\delta), \varphi_3 \rangle + \frac{N}{K_\delta} \langle \hat{u}_\delta - \hat{u}_\delta, \varphi_3 \rangle + \frac{\gamma_3}{K_\delta} \langle \hat{u}_\delta - u_d, \varphi_3 \rangle = 0, \quad (45)\]

\[ \langle \hat{\rho}_\delta \hat{\xi}_\delta, \varphi_4 \rangle + \frac{\gamma_4}{K_\delta} \langle \hat{\xi}_\delta, \varphi_4 \rangle + \frac{N}{K_\delta} \langle \hat{\xi}_\delta - \hat{\xi}, \varphi_4 \rangle = 0, \quad (46)\]

\[ \varphi_5 \left[ \frac{\gamma_5}{K_\delta} \hat{\chi}_\delta + \int_\Omega \nabla \cdot (\hat{\rho}_\delta \nabla \hat{c}_\delta) \hat{\mu}_\delta \, dx + \frac{N}{K_\delta} \langle \hat{\chi}_\delta - \hat{\chi} \rangle \right] = 0, \quad (47)\]

for all \( \varphi_1 \in H^2(\Omega), \varphi_2 \in H^2(\Omega), \varphi_3 \in V, \varphi_4 \in L^2(\Omega) \) and \( \varphi_5 \in \mathbb{R} \). Thus passing to the limit in \((43)-(47)\) as \( \delta \to 0 \), taking into account \((24)\) and \((42)\), we obtain the optimality system \((31)-(35)\) with \( \lambda_0 = 0 \). Observe that the passage to the limit in \((43)-(47)\) as \( \delta \to 0 \) follows from the uniform regularity of
the sequences \((\tilde{\sigma}_\delta, \tilde{\mu}_\delta, \tilde{\xi}_\delta, \tilde{\zeta}_\delta)_{0<\delta<1}\), \((\hat{\sigma}_\delta, \hat{\mu}_\delta, \hat{\xi}_\delta, \hat{\zeta}_\delta)_{0<\delta<1}\), and compactness arguments. Finally, observe that, \(\forall \delta \in (0, 1)\),
\[
\| (\tilde{\sigma}_\delta, \tilde{\mu}_\delta, \tilde{\xi}_\delta, \tilde{\zeta}_\delta) \|_{L^2 \times L^2 \times \mathbb{V} \times \mathbb{R}} = \left( \| \tilde{\sigma}_\delta \|^2 + \| \tilde{\mu}_\delta \|^2 + \| \tilde{\xi}_\delta \|_{\mathbb{V}}^2 + \| \tilde{\zeta}_\delta \|^2 \right)^{\frac{1}{2}} = 1,
\]
and therefore, using (42), we conclude that \((\tilde{\sigma}, \tilde{\mu}, \tilde{\xi}, \tilde{\zeta}) \neq (0, 0, 0, 0)\).

2. Let \(\lim_{\delta \to 0} K_\delta < +\infty\) or, in other words,
\[
\| \sigma_\delta \|^2 + \| \mu_\delta \|^2 + \| \xi_\delta \|^2 + \| \zeta_\delta \|^2 \leq K.
\]
Thus, we obtain an element \((\sigma, \mu, \xi, \zeta) \in L^2(\Omega) \times L^2(\Omega) \times \mathbb{V} \times \mathbb{R}\) such that, for some subsequence of \((\sigma_\delta, \mu_\delta, \xi_\delta, \zeta_\delta)_{\delta \in (0, 1)}\), still denoted by \((\sigma_\delta, \mu_\delta, \xi_\delta, \zeta_\delta)_{\delta \in (0, 1)}\), we have
\[
(\sigma_\delta, \mu_\delta, \xi_\delta) \to (\sigma, \mu, \xi) \text{ in } L^2(\Omega) \times L^2(\Omega) \times \mathbb{V} \text{ and } \zeta_\delta \to \zeta \text{ in } \mathbb{R}.
\]

By (24) and (48), passing to the limit in (37)-(41) as \(\delta \to 0\), we obtain the optimality system (31)-(35) with \(\lambda_0 = 1\).

\[
\square
\]

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