# A new kind of locked circuit: the Quasi-Periodic Locked Loop (Q-PLL)

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A new kind of locked circuit: the Quasi-Periodic Locked Loop (Q-PLL)

Diego Luis González, Lorenzo Grassi, and Alberto Maurizi

Abstract—A new nonlinear circuit with frequency locking capability in the case of a generic quasi-periodic input, is presented. Due to this capability the circuit is called a Quasi-Periodic Locked Loop (Q-PLL). The locked frequency is parametrically selected from among those prescribed by the theory of resonances in dynamical systems. In particular, the locked frequency forms a three-frequency resonance with the frequencies of the quasi-periodic input. The circuit is able to lock also in case of deterministic perturbation (harmonics of the input frequencies) and stochastic perturbation (wide-band noise). The circuit is closely related to the pitch perception of complex sound in humans and, as such, can be considered a bio-inspired technology. From the point of view of applications, it may be considered as an extension of the Phase Locked Loop (PLL) with the additional ability of locking simultaneously to more than one frequency. Due to the dynamical and structural robustness of the locked states, the Q-PLL represents a tangible advance for the development of specific applications, for example, in medicine (hearing aids, and cochlear implants), in robotics (artificial senses), and in industrial and consumer electronics (improvement of speech intelligibility, pitch-based processing, etc.).

Index Terms—Quasi-periodic signals, Frequency locking, Pitch perception

I. INTRODUCTION

PHASE-Locked Loop (PLL) circuits are widely used in electronics. PLL circuits can be used as frequency multipliers or dividers, tracking generators, or clock recovery circuits. Their main technological applications derive from a capacity to recover very weak periodic signals, which would otherwise be lost in background noise, and the detection and/or synthesis of frequency and phase-controlled signals. Notwithstanding their ubiquitous presence in modern electronics, PLLs have a basic limitation in their locking capability: they can lock only to one periodic signal at a time. PLLs fail to lock when the input is perturbed by additional deterministic frequencies of competing amplitudes. However, theoretically, it is possible to have a system locked simultaneously to two or more periodic signals which are usually mutually incommensurate.

In fact, the PLL in its locked regime should be considered as a synchronized oscillator [1]–[4]. Usually the working regime is with 1/1 synchronisation, in which forcing and response frequencies are the same. However, it is not unusual, depending on the kind of PLL and its operating regime, to have locked states corresponding to other synchronised conditions such as, for example, 1/n, where the loop is locked to a sub-harmonic of the forcing signal. The locked responses are indeed resonances of a system with two frequencies (two-frequency resonances).

Dynamical systems theory provides a framework for systems with n-frequency resonances. It has been demonstrated [5] that locking in the case of 3-frequency resonances is possible, i.e., the response synchronises with two independent forcing frequencies.

Moreover, it has also been shown that three-frequency resonances are related to a relevant aspect of neural dynamics, namely the pitch perception of complex sounds [6], [7]. This aspect is relevant for applications because it opens up the possibility of technological developments in various fields including medicine, artificial senses, and robotics.

Our main objective here is to develop, on the basis of a biologically oriented paradigm, a loop circuit able to lock simultaneously to two independent frequencies. Such a circuit would extend the capabilities of the well-known phase-locked loop (PLL) and overcome the structural stability limitation of previously studied dynamical systems (coupled nonlinear oscillators, coupled phase-locked loops and coupled circle maps [5], [8], [9]). This new non-linear circuit is termed a Quasi-Periodic Locked Loop (Q-PLL, patent pending [10], [11]) due to its main property, that is, its ability to stably lock on a quasi-periodic input given by two independent frequencies. The circuit is interesting for applications envisaging real time responses of the auditory system, and also shows promise for other electronic applications representing a significant step forward with respect to standard PLL circuits.

First, the properties of three-frequency resonances from a dynamical systems point of view will be recalled and their implications for pitch perception outlined. The structural stability problems of known implementations will be discussed. Then, the circuit architecture will be presented and the strategy of the implementation for obtaining robust three-frequency resonant responses will be explained. After this, the response of the Q-PLL circuit in parameter space will be studied and its deterministic (signal perturbations) and structural (parameter perturbations) stability properties analysed. Finally, several implementations and possible future developments and applications will be discussed.

II. THREE-FREQUENCY RESONANCES AND PITCH PERCEPTION

Synchronization was discovered by Christiaan Huygens in 1665 [12]. He found that the pendulums of two clocks, fixed
on the same mounting, after some time swung synchronously. Moreover, Huygens found that the phases of clock oscillations showed an asymptotic tendency towards a certain fixed value of the phase difference. Because of this tendency, synchronization is also known as phase locking.

Synchronization is a well-known universal phenomenon occurring widely in nature, from heartbeats to the movement of celestial bodies [13], [14]. The most common case, well described by the Huygens clocks, is when the forcing and response frequencies are equal. This is called 1/1 synchronization and corresponds, for example, to the case of orbital and rotational synchronized frequencies characterizing the movement of our moon and responsible for the fact that we see always the same face.

Synchronization can appear in dynamical systems with more than one external periodic forcing. Consider the general case of a nonlinear system forced by \( n \) independent external forces.

For \( n = 1 \), a periodic response corresponds to a synchronized, or locked, state. As said above, this resonance can differ from simple 1/1 synchronization. In fact, when the quotient between the proper frequency \( f_0 \) of the unperturbed system and the frequency \( f_1 \) of the external forcing, approach a rational number, i.e., \( f_0/f_1 \approx p/q \) with \( p, q \in \mathbb{N} \), a solution characterized by a sub-harmonic of the forcing frequency, i.e., \( f_1/q \) can arise. When the response locks exactly to the \( q \)-subharmonic, then \( f_0/f_1 = p/q \) and thus \( pf_0 - qf_1 = 0 \). For \( p = q = 1 \), it reduces to 1/1 synchronization which implies \( f_0 = f_1 \). This last relationship is a special case of the more general case represented by

\[
\sum_{k=0}^{N-1} m_k f_k = 0 \tag{1}
\]

where \( m_k \) are integers. A system which satisfies Equation (1) non-trivially, i.e., \( m_k \neq 0 \) for all values of \( k \), is said to be \( N \)-resonant (the frequencies form an \( N \)-resonance).

Analog phase-locked loop circuits represent an electronic realization of a 2-resonant system showing responses locked to the forcing frequency, i.e., the case with \( N = 2 \) and \( |m_k| = 1 \) in Equation 1.

The next case in increasing dimensionality (\( N = 3 \)) corresponds to quasi-periodic forcing, i.e., a system forced with two independent external frequencies \( f_1 \) and \( f_2 \). Quasiperiodically forced systems exhibit a rich variety of dynamical behaviour [15]. These range from periodic behaviour, to 2- and 3-frequency quasi-periodic responses, to strange non-chaotic, and chaotic attractors. The Ruelle-Takens-Newhouse theorem establishes that quasi-periodic attractors with \( 3 \) incommensurate frequencies are on the border of chaos. These quasi-periodic attractors imply that Equation (1) with \( N = 3 \) has only the trivial solution \( m_k = 0 \) for all \( k \) values. The theorem implies that in the immediate neighborhood of these responses, chaotic attractors must be present. As a consequence, quasi-periodic attractors with three-frequencies would be unstable in parameter space.

However, when Equation (1) has non-trivial solutions for \( N = 3 \), the three frequencies satisfy a resonant condition, i.e., are not independent and, therefore, the Ruelle-Takens-Newhouse theorem does not apply. Thus, the system is expected to have resonant responses with stability regions of non-zero size in parameter space. This has already been shown to hold in several nonlinear quasi-periodically forced systems [8].

Although in the study of nonlinear dynamical systems the characterization of deterministic chaos has attracted a great deal of attention, more regular dynamics are found to be important for modelling many real processes. In particular, the study of 3-frequency resonances was found to be particularly important for modelling the auditory perception of complex sounds [6], [7].

The pitch of a complex sound is a psychoacoustic quantity, i.e., a subjective placement of the sound along an inner template that goes from bass to high tones, as in a musical scale. If a correspondence between pitch and the frequency spectrum of the stimulus is sought, for the majority of musical

\[
\Delta \omega = 2\Delta \omega/(p+q)
\]

Fig. 1. Schematic representation of the relationship between tones and pitch for different stimuli. Musical sounds are complex tones consisting of a lowest frequency, called fundamental (\( \omega_0 \) here) and integer multiples of it, called overtones or harmonics. Panel a) reports the complex tone time series and power spectrum of the first 10 harmonics of a violin whose pitch coincides with the fundamental. In panel b) the fundamental and the first few higher components of the spectrum are removed. The pitch does not change with respect to a): it remains that of the fundamental (missing fundamental). Panel c) displays the effect of uniformly shifting the harmonics of panel b) by a quantity \( \Delta \omega \). Although the difference combination tones remain unchanged, the pitch shifts by a quantity \( \Delta P \). Panel d) reports the relationship between the frequency shift and the perceived pitch according to [6] superimposed on data from the pitch shift experiment in [16].
sounds, *i.e.*, those composed of a fundamental and a series of harmonics, the task is relatively easy: the pitch can be well described by the fundamental component of the sound (see Figure 1a). However, in some cases, the pitch is not trivially connected to the lowest frequency of the stimulus. In fact, even when a few of the lowest components of a musical sound are removed, the perceived pitch still corresponds to that of the (removed) fundamental (see Figure 1b). This phenomenon is called *missing fundamental* or residue. In order to explain this behaviour, a static nonlinear theory was formulated by Helmholtz [17]: the difference tone (one of the possible combination tones), produced within the ear, plays the role of the missing fundamental: the difference between two adjacent partials equals the fundamental component. As an example, in Figure 1b, \( f_5 - f_4 \equiv 5f_0 - 4f_0 = f_0 \). Nevertheless, *pitch shift* experiments, consisting of the observation that the pitch shifts when all the frequency components are shifted by the same amount (see Figure 1c where it is shown that the pitch is also shifted by the amount \( \Delta f \)) [16], demonstrated the fallacy of this view (Figure 1c). This fact was interpreted historically as a failure of the nonlinear approach [18]–[20]. However, more recently, a nonlinear dynamical theory [6], [7] (as opposed to the static view of Helmholtz) was successful in explaining the key feature of pitch shift (Figure 1d). This new theory is based on the concept of dynamical resonances. The main results of the theory and its application to pitch perception modelling are briefly outlined here below (see [6], [7] for a more detailed description).

When a generic dynamical system characterized by an intrinsic frequency \( f_0 \) is forced by two external independent frequencies \( f_1 \) and \( f_2 \), a web of three-frequency resonances is generated in its parameter space. The general procedure for finding the possible resonances starts with the continued fraction development of the quotient of the external frequencies, \( r = f_1 / f_2 \) (see, *e.g.*, [21]):

\[
r = [a_0; a_1, ..., a_n, ...].
\]

The development is finite if the quotient is rational and infinite otherwise. Successive truncations up to a given finite order of the development give the so-called approximants of the ratio,

\[
r_n = p_n / q_n = [a_0; a_1, ..., a_n].
\]

For any approximant a main three-frequency resonance exists which is given by:

\[
f_{0n} = f_1 + f_2 / p_n + q_n
\]

The hierarchy of the frequencies \( f_{0n} \), characterizing the web of three-frequency resonances is described by a generalized Farey sum operation. The usual Farey sum operation on rational numbers, \((p/q) \oplus (r/s) = (p + r)/(q + s)\), describes the hierarchy of the web of synchronized responses in periodically forced oscillators (two-frequency resonances). Such hierarchical organization is known as the Devil’s staircase.

Analogously, for three-frequency resonances, there is a hierarchical organization given by the generalized Farey sum operation [8] defined as follows: for a given convergent
some kind of dynamic control must be implemented in order to enhance the existence intervals of the main resonances. The following Section describes an electronic circuit, based on a sampling locked loop control displaying very large stability intervals for the main three-frequency resonances, and which can lock not only on periodic but also on quasi-periodic inputs.

III. CIRCUIT DESCRIPTION

The Q-PLL circuit robustly implements three-frequency resonator dynamics. The main idea underlying the circuit design is to use a sub-Nyquist approach taking into account that the desired three-frequency resonances are below the Nyquist frequency of the forcing. Within this framework, a feedback system is developed which locks stably to one of the possible three-frequency resonances.

In Figure 3 a possible realization of the circuit in terms of a Simulink® model is shown. It also represents well the general functional scheme of the system. The quasi-periodic input signal \( F(t) \) is sent to a triggered sampler (Sample and Hold, S/H). The sampling frequency of the S/H is determined by the output frequency of a Voltage Controlled Oscillator (VCO). The sampled signal \( F'(t) \) is then multiplied \((\times)\) by the non-sampled one and, subsequently sent to a time integrator \((1/s)\). Then, the integrated signal \( x'(t) \) is sent to the VCO to determine the VCO frequency as \( f_{\text{VCO}} = G x'(t) + f_0 \), where \( G \) and \( f_0 \) are the sensitivity and quiescent frequency of the VCO, respectively. The output of the VCO, once transformed into a square wave, is sent to the trigger port of the S/H. The response of the system is the output signal of the VCO \( x(t) \) with instantaneous frequency \( f(t) \).

\[
\dot{x}(t) = A_0 \cos (2\pi [f_Q + G \int_0^t F'(\tau) d\tau] t + \varphi_0)
\]  

where \( \varphi_0 \) is the initial phase of the VCO, \( F(t) \) is the input (forcing) and \( F'(t) \), the sampled signal, is formally expressed by \( F'(t) = F(t_i) \) for \( t_i \leq t < t_{i+1} \) with \( t_i = \{t \mid d_i H(x(t)) > 0\} \) where \( H \) is the Heaviside function and \( d_i \) is the distributional derivative. The sampled signal function \( F'(t) \) makes this equation, analytically speaking, rather intractable. Features of the model will be studied using numerical simulations.

Figure 4 reports, as an example, the power spectrum of the Q-PLL output (continuous line) along with that of the input signal (dashed line) for an input \( f_1 = 200 \text{ Hz} \) and \( f_2 = \phi f_1 \) where \( \phi = (1 + \sqrt{5})/2 \) is the golden ratio. This irrational number ensures that the frequency ratio is as far as possible from a rational ratio (quasi-periodic forcing).

In the general case, the sampling frequency \( f_s \equiv f_{\text{VCO}} \) including its initial value \( f_Q \), is below the Nyquist frequency of the input.

It is well known from sampling theory (see, e.g., [23]), that sampling a sinusoidal signal of frequency \( f_1 \) with a sampling frequency \( f_s < 2f_1 \) produces a spectrum with aliasing components. The sequence \( f_{ak} \) of aliases (on the positive frequency axis) is based on the quantity \( \Delta_1 \equiv |f_1 - k f_s| \) \((k \text{ being the nearest integer to } f_1/f_s)\):

\[
f_{ak} = \Delta_1 \{\Delta_1 + k f_s; k \in \mathbb{N}\}.
\]  

As a direct consequence, for \( k = \hat{k}_1 \), it results that either \( \Delta_1 + \hat{k}_1 f_s = f_1 \) or \( -\Delta_1 + \hat{k}_1 f_s = f_1 \), in other words, the resulting signal contains the input frequency.
module) is replaced by a controlled frequency in Figure 5, where the triggering signal (input to the S/H as shown below.

The analysis of these shaded areas enclose the intervals defined by the first three approximants of the frequency input ratio. The transform of the product signal produces a signal \( F' \) with a non-null zero-frequency component \( F(0) \) of the power spectrum. This Direct Current (DC) term is the sum of the contribution resulting from the zero-frequency components, \( F_1(0) \) and \( F_2(0) \), of the two input signals.

Furthermore, along with the DC component, the two lowest alias frequencies \( \Delta_1 \) and \( \Delta_2 \) constitute the low frequency part of the spectrum which survives the subsequent low-pass filtering produced by the integrator. The analysis of these quantities allows a deeper insight into the locking mechanism as shown below.

Consider the open submodel, i.e., without feedback, shown in Figure 5, where the triggering signal (input to the S/H module) is replaced by a controlled frequency \( f_s \). The Fourier transform of the product signal \( F_x \) is computed and the zero-frequency component \( F(0) \) is considered. Calculation of this quantity is performed for different values of the sampling signal frequency \( f_s \) over a range covering the convergence frequency interval, while its phase \( \varphi_s \) is varied over the full range \([0 : 2\pi]\). Results are reported in Figures 6–8.

To gain an idea of how the locking mechanism acts, it is important to observe that a necessary condition for the output of the Q-PLL circuit in Figure 3 to lock to a fixed value, is that the input to the VCO be constant. Therefore, the input signal to the integrator must vanish. A necessary condition for this is that the DC component of \( F_x \) vanishes. Figure 6 displays the DC component, \( F(0) \), over a wide range of \( f_s \) that cover the first three approximants of the frequency input ratio. The shaded areas enclose the intervals defined by \([f1/p_k, f2/q_k]\), for \((p_k, q_k) = (1, 2), (2, 3), (3, 5)\), with \( p_k/q_k \) being the \( k \)-th convergent of \( f_1/f_2 \), and \( k = 1, 2, 3 \). The vertical lines represent the variations in \( F'(0) \), caused by sampling signal phase change, which the system produces at exactly \( f_s = f_{0k} \) (and only for those values). To closely analyse this behaviour, Figure 7 displays a zoom of Figure 6 around the convergence region of the second approximant \((p_k, q_k) = (2, 3)\). Consider the values of the two components of \( F(0) \), namely \( F_1 \) and \( F_2 \) : they are in some relationship with \( f_s \). Outside the shaded region in Figure 7, for \( f_s < f_1/p \) they are both positive while they are both negative for \( f_s > f_2/q \). Inside the shaded region,
their sign is opposite and, for a given \( f_s = \hat{f} \), they equal each other in magnitude making \( F(0) \) vanish. However, Figure 7 shows that \( \hat{f} \) does not correspond to the main resonance and, therefore, the locking mechanism must rely on some additional features of the system.

In fact, \( F(0) \) is, in general, a function of both the frequency \( f_s \) and the phase \( \varphi_s \) of the sampling signal, and Figure 7 should be viewed as a projection along the \( \varphi_s \) axis. However, it can be observed that over the range represented in Figure 7, except for \( f_s = f_0 \), \( F(0) \) does not depend on \( \varphi_s \). Nevertheless, a strong dependence appears for \( f_s = f_0 \) as shown by the thick vertical line in Figure 7. It is known from Section II that in this case \( \Delta_1 = \Delta_2 \) and the two alias frequency sequences coincide. The plot inserted in Figure 7 represents the section of \( F(0) \) along the \( \varphi_s \) axis at fixed \( f_s = f_0 \) and displays explicitly this dependence. It is important to observe that the variations of \( F(0) \) due to \( \varphi_s \) are such that \( F(0) \) crosses the zero value, thus providing a mechanism of convergence towards the main resonance frequency \( f_0 \).

It is remarkable that this very special phase dependence happens exactly at \( f_s = f_0 \), which underlines the strong connection between three-frequency resonances and the properties of the signal \( F_\varphi \) and, in turn, with the sub-Nyquist-sampled signal \( F' \). Another remarkable feature is that the periodicity of \( F(0) \) as a function of \( \varphi_s \) is exactly \( p + q \). The insert in Figure 7 corresponds to the convergent \((p_k,q_k) = (2,3)\) and the function presents 5 cycles. As another example of this feature, Figure 8 shows the case corresponding to the convergent \((p_k,q_k) = (3,5)\) for which the periodicity of the phase response results to be 8.

Consider now the closed circuit (Figure 3). At the beginning of the locking cycle, if \( f_Q < \min(f_k,f_0) \), the resulting DC is positive and forces the system to increase the sampling frequency \( f_s \) (in the opposite case, if \( f_Q > \max(f_k,f_0) \) \( f_s \) decreases). When \( f_s \) crosses \( f_0 \), a phase-locking mechanism is activated and the system adjusts the phase of the sampling signal to find one of the zeroes of the DC component (Figures 7 and 8). This corresponds to a relative phase which cancels the two low frequency components \((\Delta_1 = \Delta_2)\) of the aliases, ensuring a zero input to the integrator. When far from the resonance, the two low frequency components \((\Delta_1 \neq \Delta_2)\) provide a perturbation which leads the system to cross \( f_s = f_0 \) and then to switch to the phase-locking mechanism described above. The parameter acting as an amplifier of the perturbation is the VCO sensitivity \( G \) which, if too small does not allow the system to move far enough from the DC = 0 solution and cross the resonance value \( f_0 \). On the other hand, if \( G \) is too large the circuit becomes unstable.

### IV. Experimental Setup and Results

The circuit features were studied by means of numerical experiments. The implementation of the circuit in terms of Simulink® blocks shown in Figure 3 is used here.

As already shown in the example reported in Figure 4, the system is able to lock to a quasi-periodic input composed of two incommensurate sinusoidal components whose ratio, notably, is the "most irrational" number, i.e., the golden ratio \( \phi \). The ability to lock to different convergents of the \( f_1/f_2 \) ratio over the entire range of the parameter \( f_Q \), is investigated here below.

Before proceeding further, some considerations on the dependence of the system on \( G \) are in order. \( G \) provides the rate of change of the VCO frequency as a function of the integrated signal. Roughly speaking, it must depend on the expected frequency response which, in turn, depends on \( f_Q \). This can be observed through examining the intervals of the DC component around the locking frequencies in Figure 6. As a rule of thumb, the smaller the interval extent, the smaller \( G \) must be in order to avoid jumps outside the given convergence basin and to maximize the existence regions of the main resonances. On this basis, appropriate values for \( G \equiv G(p_k,q_k) \) are selected for each of the \( k \)-th approximant \((p_k,q_k)\).

Thus, by means of a trial-and-error procedure several values of \( G(p_k,q_k) \) were selected: \( G(1,2) = 8000 \text{ Hz/V}, G(2,3) = 2000 \text{ Hz/V}, G(3,5) = 1000 \text{ Hz/V}, G(5,8) = 500 \text{ Hz/V}, G(8,13) = 250 \text{ Hz/V} \). Small variations of \( G \) around the optimal values defined above have no significant effect and, therefore, this parameter is not investigated further.

#### A. Q-PLL features for quasi-periodic input

Figure 9 reports the response frequency \( f_R \) of the Q-PLL as a function of \( f_Q \). It can be observed that \( f_R \) locks exactly to \( f_0 \), the main three-frequency resonance corresponding to the \( k \)-th convergent. The extent of the intervals over which the solutions are stable are very large, much larger than the \([f_1/p_k,f_2/q_k]\) intervals reported as continuous thin lines in Figure 9 (see also the shaded regions in Figure 6). It is worth noting that all other possible solutions corresponding to generalized mediants in the Devil’s staircase-like portrait (see Figure 2) expected to exist between \( f_1/p_k \) and \( f_2/q_k \), are suppressed. This is at variance with what has been found for some systems previously investigated: coupled phase-locked
loops, coupled nonlinear oscillators with piece-wise analytic solution and a quasi-periodically forced circle map [5], [8], [9]. Within the convergence intervals the attractors are stable and, therefore, evolve to asymptotic orbits independently of the initial conditions, mainly the initial phase of the VCO.

The highest reported convergent is \((p_0, q_0) = (8, 13)\). Convergence intervals corresponding to higher convergents are too short to be represented fairly on the same scale. At the frontier of the reported stable solutions there are small regions in which the system is unstable where the responses may depend on the initial conditions. In these regions, however, other stable solutions may also occasionally appear (not reported here) which correspond to generalized mediants of the neighbouring (parent) solutions (see Figure 2).

Experiments to study the stability of the main resonances facing various perturbations of the input signals were also performed. First, the signal of Figure 4 was perturbed by an additive white Gaussian noise with an intensity of about -30 dBm relative to the input (0 dBm). For the present system implementation and reported parameters, the system response remains stable for noise figures below or equal to this level. This can be appreciated as a peak corresponding to the expected resonance at 104.72 Hz in Figure 10. A thorough investigation about the stability to noise perturbation is beyond the scope of this paper but would undoubtedly be interesting for the design of electronics applications.

B. PLL-like behaviour

If the Q-PLL is forced with a signal containing a frequency \(f_1 = 200\) Hz only, selecting \(f_Q\) from a suitable range around \(f_Q = 200\) Hz the system locks at \(f_R = 200\) Hz as would a normal PLL.

If now the quiescent frequency is set, for example, to \(f_Q = 90\) Hz, the system locks to \(f_R = 100\) Hz (Figure 11). This frequency is the first sub-harmonic of the forcing frequency, \(f_R = f_1 / 2\). In general, the system is able to lock to any sub-harmonic \(f_R = f_1 / n\) depending on the value of \(f_Q\). Figure 12 shows the response diagram as a function of \(f_Q\). Solutions are very stable and do not depend on the VCO sensitivity \(G\) (in Figure 12, \(G = 1000\) Hz/V is used). In this situation the Q-PLL behaves, in the limit, as a PLL with frequency-divider capability such as, for example, those in the sampling class of PLLs.

C. Q-PLL as a pitch detector

As the focus of this work is on the ability of the circuit to model the pitch of complex sounds, its ability to lock in the case of the sum of two sine waves with harmonic frequency ratio \(f_1 / f_2 = 2/3\) is analysed. In musical terms, this particular ratio corresponds to a perfect fifth interval. Such a ratio is used as a paradigmatic example being considered one of the most
consonant musical intervals. It should be borne in mind that this interval represents a stimulus with a missing fundamental at $f_1/2 = f_2/3$.

Figure 13 displays the input and response frequencies for the perfect fifth case, $f_1 = 200$ Hz and $f_2 = 300$ Hz with $f_Q = 90$ Hz. The system locks stably to the missing fundamental value $f_R = 100$ Hz.

Next, the more complex case corresponding to a pitch-shift is analysed. Figure 14, reports the result for a non-harmonic signal obtained by applying a pitch shift operator to the harmonic signal used in the perfect fifth example. A small shift $\Delta f = 21$ Hz was selected to keep the ratio $2/3$ as the first approximant of the input frequency ratio. In fact, the continued fractions development of the ratio 221/321 is:

$$[1, 2, 4, 1, 3, 5]$$  

and the corresponding convergents: 1, 3/2, 13/9, ...

Figure 14 shows the power spectrum density of the input and output signals. The system locks exactly (within the numerical precision) to the first mediant of $f_1/2$ and $f_2/3$, i.e., $f_0 = 108.4$ Hz. Another remarkable feature is that, in virtue of the locking mechanism described in Section III, only the residue frequency survives.

Another interesting variation of the input signal, related to the possibility of using the Q-PLL as a model for auditory pitch recognition, is represented by a signal which is a harmonic complex with two-to-several components under pitch shift [7]. This stimulus is the basis of several psychoacoustics experiments (see, e.g., [16], [24]–[27]). Although the argument is very complex and stimuli can be very different, to provide an example: an input signal comprised of the first 7 harmonics of a sine wave of frequency 100 Hz, shifted by the same amount used in previous pitch-shift examples ($\Delta f = 21$ Hz), is fed into the Q-PLL. Figure 15 displays the spectrum of the input and the response signals. The response remains stable to $f_0 = (f_1 + f_2)/(2+3) = 108.4$ Hz under significant variations in the different magnitudes of the input components, provided...
that the amplitudes decay towards high harmonic numbers (typical of musical sounds).

The resulting spectrum is more complex than that reported in Figure 14, displaying not only the locking peak, but also a number of lower power components. This happens because in this case the low frequency components resulting from the sub-Nyquist sampling (see Section III) do not mutually cancel as in the case of a stimulus comprising only two components of the same amplitude.

V. CONCLUSION

A new nonlinear circuit which can maintain a locked state when forced by two incommensurate frequencies has been presented. The circuit was developed in a biologically inspired paradigm within the framework of dynamical systems theory. Locking states are characterized by the generation of a third frequency which forms a so-called three-frequency resonance together with the forcing frequencies. Moreover, it has been demonstrated that these resonances characterize, in turn, the pitch of complex sounds when the fundamental component is absent. Thanks to this property, the circuit can emulate one of the main psycho-acoustical parameters of auditory perception. Thus, it opens the way for the development of real-time auditory processors which can be utilized in a diversity of applications, from medicine (hearing aids, artificial cochlea), to robotics (artificial senses), to industrial and consumer electronics (by including auditory-like processing in audio applications, as a natural extension of the universal PLL). This new circuit, being orbitally and structurally stable, thanks to a control strategy based on sample-and-hold feedback, supersedes former attempts (such as coupled PLLs) and prompts further study regarding optimization of its features and performance in the light of future technological applications.

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