The Number Operator for Generalized Quons

Miroslav Dorešić

Department of Theoretical Physics,
Rudjer Bošković Institute, P.O.B. 1016,
41001 Zagreb, CROATIA

Abstract

We construct the number operator for particles obeying infinite statistics, defined by a generalized q-deformation of the Heisenberg algebra, and prove the positivity of the norm of linearly independent state vectors.

PACS numbers: 03.65.-w, 05.30.-d

1e-mail address: doresic@thphys.irb.hr
The approach to particle statistics based on deformations of the bilinear Bose and Fermi commutation relations has attracted considerable interest during the last few years [1, 2, 3, 4]. The particles obeying this type of statistics are called "quons". The quon algebra (or the $q$–mutator) is given by

$$a_i a_j^\dagger - qa_j^\dagger a_i = \delta_{ij}, \quad (\forall i, j), \quad (1)$$

and interpolates between Bose and Fermi algebras as the deformation parameter $q$ goes from 1 to $-1$ on the real axis. When supplemented by the vacuum condition

$$a_i |0> = 0, \quad (\forall i), \quad (2)$$

the quon algebra determines a (Fock-like) representation in a linear vector space. For $q \in [-1,1]$, the squared norms of all vectors made by the limits of the polynomials of the creation operators $a_k^\dagger$ are strictly positive. No commutation relation can be imposed on $a_ia_j$ or $a_i^\dagger a_j^\dagger$. Furthermore, no such rule is needed to calculate the vacuum matrix elements of the polynomials in the $a$'s and $a^\dagger$'s. All such matrix elements can be calculated by moving the annihilation operators to the right using (1), until they, according to (2), annihilate the vacuum [5].

The aim of this paper is to construct the number operator for a generalized $q$–deformation of the Heisenberg algebra which is characterized by the following relations

$$a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij}, \quad (\forall i, j), \quad (3)$$

$$a_i |0> = 0 \quad (\forall i), \quad (4)$$

$$q_{ji}^* = q_{ij}, \quad (5)$$

with the deformations parameters $q_{ij}$ being, in general, complex numbers. The statistics based on the commutation relations (3) generalize classical Bose and Fermi
statistics, which correspond to \( q_{ij} = 1 \) (\( \forall i, j \)) and \( q_{ij} = -1 \) (\( \forall i, j \)), respectively. The relation (5) follows from the consistency requirement of the relation (3).

For each \( k \), the \( k^{th} \) number operator \( N_k \) satisfies the relations

\[
N_k^\dagger = N_k, \quad N_k |0> = 0,
\]
\[
[N_k, a_l^\dagger] = \delta_{kl} a_l^\dagger \quad (\forall l), \tag{6}
\]

which is equivalent to

\[
N_k (a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0>) = s (a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0>), \tag{7}
\]

where \( s \) is the index number \( i_j \), such that \( i_j = k \). The most general expression for the number operator \( N_k \) is of the form

\[
N_k = a_k^\dagger a_k + \sum_{n=2}^{\infty} \sum_{(i_{n-1})} \sum_{\pi(k,i_{n-1})} \sum_{\sigma(k,i_{n-1})} c_{\pi(k,i_{n-1}),\sigma(k,i_{n-1})} \cdot a_{\pi(i_1)}^\dagger a_{\pi(i_2)}^\dagger \cdots a_{\pi(i_{n-1})}^\dagger a_{\sigma(i_1)} a_{\sigma(i_2)} \cdots a_{\sigma(i_{n-1})}, \tag{8}
\]

where \( i_{n-1} \equiv (i_1, \ldots, i_{n-1}) \) is an arbitrary choice of \( n-1 \) indices, including their repetitions, whereas \( \pi, \sigma \) are permutations of \( n \) indices. Making use of the relation (6), one finds that the condition satisfied by the coefficients \( c_{i_1,\ldots,i_n;j_1,\ldots,j_n} \) can be written in the form of the following matrix equation

\[
[(c_{i_1,\ldots,i_n;j_1,\ldots,j_n})] \times M_n(q) = Q_n(q). \tag{9}
\]

The matrix \( M_n \) is defined by

\[
(M_n)_{i_1,\ldots,i_n;j_1,\ldots,j_n} = <0 | a_{i_n} \cdots a_{i_1} a_{j_1}^\dagger \cdots a_{j_n}^\dagger |0>. \tag{10}
\]

Taking into account eqs. (3), (4), (5) and (10), and using the method of mathematical induction, one arrives at the closed-form expression

\[
(M_n)_{\pi(1,\ldots,n);\sigma(1,\ldots,n)} = \prod_{r,s=1}^{n} P^{(r,s)}_{\pi(r)\sigma(s)}, \quad (\pi(r) \neq \sigma(s)), \tag{11}
\]
where
\[ P(r, s) = \theta[-(r - s)((\sigma^{-1} \cdot \pi)(r) - (\sigma^{-1} \cdot \pi)(s))], \tag{12} \]
and \( \theta(x) \) designates the function defined by
\[
\theta(x) = \begin{cases} 
1 & \text{if } x > 0, \\
0 & \text{if } x \leq 0.
\end{cases} \tag{13}
\]
The determinant of the matrix \( M_n \) is
\[
\det M_n = \prod_{k=1}^{n-1} \left[ \prod_{\{i_1, \ldots, i_{k+1}\}} \left( 1 - \prod_{\{i_\alpha i_\beta\}} |q_{i_\alpha i_\beta}|^2 \right)^{(k-1)!/(n-k)!} \right], \tag{14}
\]
where the set \( \{i_1, \ldots, i_{k+1}\} \) denotes a choice of \( k + 1 \) different indices out of \( n \) such indices, whereas \( \{i_\alpha i_\beta\} \) is any of its subsets. Unlike in the case of the matrix \( M_n(q) \), explicitly given by eqs. (11) and (12), the closed-form expression for the matrix \( Q_n(q) \) cannot be written down. In fact, this matrix is obtained as a result of the action of the lower-order terms \( a_{i_1}^+ \cdots a_{i_s}^+ a_{j_1} \cdots a_{j_s}, s < n \), entering expression (8) for the number operator on the eigenvector \( a_{k_1}^+ \cdots a_{k_n}^+ |0 > \). Thus, to completely determine the matrix \( Q_n(q) \), knowledge of all the lower-order coefficients \( c_{i_1,\ldots,i_s;j_1,\ldots,j_s} \), \( s = 2, 3, \ldots, n - 1 \) is required.

As a special case of the general formula (14), we give the expression for the determinant of the matrix \( M_4 \), corresponding to \( n = 4 \) and (\( k_1, k_2, k_3, k_4 \), \( k \equiv (k, l, m, p) \)):
\[
det M_4 = (1 - |q_{kl}|^2)^6 (1 - |q_{km}|^2)^6 (1 - |q_{kp}|^2)^6 (1 - |q_{lm}|^2)^6 \\
\times (1 - |q_{lp}|^2)^6 (1 - |q_{mp}|^2)^6 \\
\times (1 - |q_{kl}|^2 |q_{km}|^2 |q_{kp}|^2 |q_{lm}|^2)^2 (1 - |q_{kl}|^2 |q_{kp}|^2 |q_{lp}|^2)^2 \\
\times (1 - |q_{km}|^2 |q_{kp}|^2 |q_{lp}|^2 |q_{mp}|^2)^2 \\
\times (1 - |q_{kl}|^2 |q_{km}|^2 |q_{kp}|^2 |q_{lm}|^2 |q_{lp}|^2 |q_{mp}|^2). \tag{15}
\]
It is evident from eq. (14), and especially from the special case (13), that if the deformation parameters are such that \( |q_{ij}| < 1 \ (\forall i,j) \), the matrix \( M_n(q) \) is regular.
and positively definite. Consequently, the coefficients $c_{i_1, \ldots, i_n; j_1, \ldots, j_n}$, appearing in expression (8) for the number operator $N_k$, exist and the norm of the state vector $a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger \mid 0 \rangle$ is positive. That being the case, one is now allowed to rewrite eq. (8) in the form

$$C_n(q) = Q_n(q) \times [M_n(q)]^{-1},$$

(16)

where, for notational simplicity,

$$C_n(q) = [(c_{i_1, \ldots, i_n; j_1, \ldots, j_n})].$$

(17)

Equation (16) represents the main result of this paper.

As an example, in the following we illustrate the calculation of the coefficients $c_{i_1, i_2, i_3; j_1, j_2, j_3}$, which amounts to finding the matrix coefficients $C_3(q)$. According to (16), this matrix is given by

$$C_3(q) = Q_3(q) \times [M_3(q)]^{-1}.$$  

(18)

There are four different cases to be considered.

**Case 1:** $n=3 \ (k_1, k_2, k_3) \equiv (k, k, k)$.

This case is trivial, since $Q_3(q)$ and $M_3(q)$ are represented by the numbers (1—square matrices)

$$(Q_3(q))_{kkk,kkk} = (1 - q_{kk})^2(1 + q_{kk}),$$

(19)

$$(M_3(q))_{kkk,kkk} = (1 + q_{kk})(1 + q_{kk} + q_{kk}^2),$$

(20)

so that, in view of (18),

$$c_{kkk,kkk} = \frac{(1 - q_{kk})^2}{1 + q_{kk} + q_{kk}^2}.$$  

(21)

**Case 2:** $n=3 \ (k_1, k_2, k_3) \equiv (k, k, l)$.

In this case, $Q_3(q)$ and $M_3(q)$ are 3—square matrices, the rows and columns of which
are indexed in the order \((k,k,l), (k,l,k)\) and \((l,k,k)\). The matrix \(M_3(q)\) is given by

\[
M_3(q) = \begin{bmatrix}
1 + q_{kk} & q_{kl}(1 + q_{kk}) & q_{kl}^2(1 + q_{kk}) \\
q_{lk}(1 + q_{kk}) & 1 + q_{kk} | q_{kl}|^2 & q_{kl}(1 + q_{kk}) \\
q_{lk}^2(1 + q_{kk}) & q_{lk}(1 + q_{kk}) & (1 + q_{kk})
\end{bmatrix},
\] (22)

with the determinant

\[
\det M_3 = (1 + q_{kk})^2 \left(1 - |q_{kl}|^2\right)^2 \left(1 - q_{kk} | q_{kl}|^2\right).
\] (23)

On the basis of eqs. (22) and (23), the inverse matrix of \(M_3(q)\) is found to be

\[
[M_3(q)]^{-1} = \frac{1}{(1 + q_{kk})(1 - q_{kk} | q_{kl}|^2)}
\times \begin{bmatrix}
1 & -q_{kl}(1 + q_{kk}) & q_{kk} q_{kl}^2 \\
-q_{lk}(1 + q_{kk}) & (1 + q_{kk}) (1 + | q_{kl}|^2) & -q_{kl}(1 + q_{kk}) \\
q_{kk} q_{lk}^2 & -q_{lk}(1 + q_{kk}) & 1
\end{bmatrix}.
\] (24)

The matrix \(Q_3\) can be obtained using the following lower-order coefficients:

\[
c_{kk,kk} = \frac{1 - q_{kk}}{1 + q_{kk}},
\]

\[
c_{kl,kl} = -\frac{q_{kl}}{1 - | q_{kl}|^2}, \quad c_{lk,kl} = \frac{1}{1 - | q_{kl}|^2},
\]

\[
c_{kl,lk} = \frac{| q_{kl}|^2}{1 - | q_{kl}|^2}, \quad c_{lk,kl} = -\frac{q_{lk}}{1 - | q_{kl}|^2}.
\] (25)

The result is

\[
Q_3(q) = \begin{bmatrix}
0 & -q_{kl} & -q_{kl}^2(1 - q_{kk}) \\
0 & 1 + q_{kk} | q_{kl}|^2 & 0 \\
0 & -q_{kk} q_{lk} & -1 + q_{kk}
\end{bmatrix}.
\] (26)

The coefficient matrix \(C_3(q)\) is now obtained by substituting eqs. (24) and (26) into eq.(18). To see what the structure of these coefficients is like, we only give the
elements of the first row of this matrix:

\[
\begin{align*}
  c_{kk\ell,k\ell} &= 2q_{kk}q_{k\ell}^2 \cdot \Delta, \\
  c_{k\ell\ell,kk} &= -q_{k\ell}(1 + q_{kk})(1 + q_{kk} | q_{k\ell}|^2) \cdot \Delta, \\
  c_{k\ell\ell,k\ell} &= |q_{k\ell}|^2(1 + q_{kk} - q_{kk} | q_{k\ell}|^2 + q_{kk}^2 | q_{k\ell}|^2) \cdot \Delta,
\end{align*}
\]

with

\[
\Delta = \frac{1}{(1 + q_{kk})(1 - q_{kk} | q_{k\ell}|^2)}.
\]

**Case 3:** \(n=3\) \((k_1, k_2, k_3) \equiv (k, k, l)\).

In this case, the results are of the same structure as those in the preceding case, so we do not give them here.

**Case 4:** \(n=3\) \((k_1, k_2, k_3) \equiv (k, l, m)\).

In this case, \(Q_3(q)\) and \(M_3(q)\) are 6–square matrices, the rows and columns of which are indexed by the elements of the permutation group in the order \((k,l,m)\), \((l,k,m)\), \((k,m,l)\), \((l,m,k)\), \((m,k,l)\) and \((m,l,k)\). Again, the matrix \(Q_3(q)\) can be obtained with the help of the lower-order coefficients (25), and is given by

\[
Q_3(q) = \begin{bmatrix}
q_{k\ell}q_{km}q_{lm} & 0 & 0 & 0 & 0 \\
-q_{km}q_{lm} & 0 & -q_{km} & 0 & 0 \\
0 & 0 & q_{k\ell}q_{km}q_{ml} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-q_{k\ell} & 0 & -q_{k\ell}q_{ml} & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}.
\]
On the basis of eq. (11), the matrix $M_3(q)$ is found to be

$$
M_3(q) = \begin{bmatrix}
1 & q_{kl} & q_{lm} & q_{kl}q_{km} & q_{km}q_{lm} & q_{kl}q_{km}q_{lm} \\
q_{lk} & 1 & q_{lk}q_{lm} & q_{km} & q_{lm}q_{lk}q_{km} & q_{lm}q_{km}q_{lk} \\
q_{ml} & q_{kl}q_{ml} & 1 & q_{lk}q_{km}q_{lm} & q_{km} & q_{km}q_{lk}q_{lm} \\
q_{lk}q_{mk} & q_{mk} & q_{lk}q_{lm}q_{mk} & 1 & q_{lm}q_{lk} & q_{lm} \\
q_{mk}q_{ml} & q_{ml}q_{mk}q_{kl} & q_{mk} & q_{lm}q_{kl} & 1 & q_{kl} \\
q_{mk}q_{ml}q_{lk} & q_{ml}q_{mk}q_{kl} & q_{mk}q_{lk} & q_{ml} & q_{lk} & 1
\end{bmatrix},
$$

(30)

with the determinant

$$
\det M_3(q) = (1 - |q_{kl}|^2)^2 (1 - |q_{km}|^2)^2 (1 - |q_{lm}|^2)^2 \times (1 - |q_{kl}|^2 |q_{km}|^2 |q_{lm}|^2).
$$

(31)

Inserting the matrices $Q_3(q)$, given by eq. (29), and the matrix $M_3^{-1}(q)$, obtainable from eqs. (30) and (31), into eq. (18), one finds the coefficient matrix $C_3(q)$. Here, as in the Case 2, we only exhibit the elements comprising its first row. They are

$$
c_{klm,klm} = (M_3^{-1})_{klm,klm},
$$

$$
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$$

$$
c_{klm,klm} = (M_3^{-1})_{klm,klm},
$$

(32)

where

$$
(M_3^{-1})_{klm,klm} = q_{kl}q_{km}q_{lm}(1 - |q_{km}|^2)(1 - |q_{kl}|^2 |q_{lm}|^2),
$$

$$
(M_3^{-1})_{klm,klm} = -q_{kl}q_{km}q_{lm}(1 - |q_{km}|^2)(1 - |q_{kl}|^2 |q_{lm}|^2),
$$

$$
(M_3^{-1})_{klm,klm} = -q_{kl}q_{km}q_{lm}(1 - |q_{km}|^2)(1 - |q_{kl}|^2 |q_{lm}|^2),
$$

(33)
\[(M_3^{-1})_{imk,mlk} = -q_{im}(1 - |q_{kl}|^2)(1 - |q_{km}|^2),\]
\[(M_3^{-1})_{mkl,mlk} = -q_{kl}(1 - |q_{km}|^2)(1 - |q_{lm}|^2),\]
\[(M_3^{-1})_{mlk,mlk} = (1 - |q_{km}|^2)(1 - |q_{kl}|^2),\]

are the relevant matrix elements of \(M_3^{-1}(q)\).

Next we derive a relation that will make it possible to rewrite expression (8), for the number operator \(N_k\), in a more compact and elegant form.

Making use of eqs. (11) and (25), one finds that, for the case \(n = 2\) and \((k_1, k_2) \equiv (k, l)\), the following relation holds:
\[
\sum_{\pi(k,l)} \sum_{\sigma(k,l)} c_{\pi(k,l)\sigma(k,l)} a_{\pi(k)}^\dagger a_{\pi(l)}^\dagger a_{\sigma(k)} a_{\sigma(l)} = (M_2^{-1})_{kl,kl} \tilde{a}_{kl}^\dagger \tilde{a}_{kl}. \tag{34}
\]

For the case \(n = 3\) and \((k_1, k_2, k_3) \equiv (k, l, m)\), an analogous relation can be established on the basis of the eqs. (33) i (34). It reads
\[
\sum_{\pi(k,l,m)} \sum_{\sigma(k,l,m)} c_{\pi(k,l,m)\sigma(k,l,m)} a_{\pi(k)}^\dagger a_{\pi(l)}^\dagger a_{\pi(m)}^\dagger a_{\sigma(k)} a_{\sigma(l)} a_{\sigma(m)} = \sum_{\pi(l,m)} \sum_{\sigma(l,m)} (M_3^{-1})_{k,\sigma(l,m);k,\pi(l,m)} a_{\kappa,\pi(l,m)}^\dagger a_{\kappa,\sigma(l,m)}, \tag{35}
\]
with the notation
\[
\tilde{a}_{k,l} = a_k a_l - q_{kl} a_l a_k, \tag{36}
\]
\[
\tilde{a}_{k,l,m} = \tilde{a}_{k,l} a_m - q_{mk} q_{ml} a_m \tilde{a}_{k,l}. \tag{37}
\]

Generalizing eqs. (34) and (35) for arbitrary \(n\), one arrives at the following relation:
\[
\sum_{\pi(k,i_{n-1})} \sum_{\sigma(k,i_{n-1})} c_{\pi(k,i_{n-1})\sigma(k,i_{n-1})} a_{\pi(k)}^\dagger a_{\pi(i_1)}^\dagger \cdots a_{\pi(i_{n-1})}^\dagger a_{\sigma(i_1)} \cdots a_{\sigma(i_{n-1})} = \sum_{\pi(i_{n-1})} \sum_{\sigma(i_{n-1})} (M_n^{-1})_{k,\sigma(i_{n-1});k,\pi(i_{n-1})} a_{\kappa,\pi(i_{n-1})}^\dagger a_{\kappa,\sigma(i_{n-1})}. \tag{38}
\]
where
\[
\tilde{a}_{k,j_1,\ldots,j_{n-1}} = \tilde{a}_{k,j_1,\ldots,j_{n-2}}a_{j_{n-1}} - q_{j_{n-1}k}q_{j_{n-1}j_1} \cdots q_{j_{n-1}j_{n-2}}a_{j_{n-1}}\tilde{a}_{k,j_1,\ldots,j_{n-2}}. 
\] (39)

Apart from the elegance of eqs. (34), (35) and (38), the expression of the number operator makes it easier to relate our results to the results of Greenberg [2], corresponding to the \( q_{ij} = 0 \) (\( \forall i,j \)) statistics.

All the results obtained above correspond to the case when the deformation parameters satisfy the condition
\[
| q_{ij} | < 1 \ (\forall i,j). 
\] (40)

Before concluding, we briefly discuss the cases where we have the following conditions, instead of (40):
\[
| q_{ij} | > 1 \ (\forall i,j) \quad (41)
\] or
\[
| q_{ij} | = 1, \text{ i.e. } q_{ij} = e^{i\Phi_{ij}} \ (\forall i,j). 
\] (42)

If the case (41) is realized, the number operator exists. This is clearly seen from eqs. (14) and (16). However, as it is evident from eq.(15), the norm of the state vector \( a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger | 0 > \) is not positive, making this case physically unacceptable.

However, if \( q_{ij} \)'s are such that the condition (42) is satisfied, even though the matrix \( M_n(q) \) is singular, the existence of the number operator is not excluded. In this case, the existence of the operator \( \bar{N}_k \) depends on whether or not it is possible to impose such a \( q \)-mutator on the operator pairs \( a_i, a_j \ (\forall i,j) \) that all terms higher than \( a_k^\dagger a_k \) cancel. This is precisely what happens for \( q_{ij} = \pm 1 \), corresponding to Bose and Fermi statistics, respectively. Further investigation regarding this point is unquestionably of interest and is the subject of another study. For the statistics
characterized by eqs. (12), it should be pointed out that anyons (particles existing in 2+1 dimensions) can alternatively be represented as a $q-$deformation of an underlying bosonic algebra. This can be viewed as an extension of Greenberg’s approach with $q$ being a complex number $|q| = 1$ [7].

To conclude, in this paper we have studied the generalized $q-$deformation of the Heisenberg algebra defined by eqs. (3), (4) and (5). For the case when deformation parameters are such that $|q_{ij}| < 1 (\forall i,j)$ we have proved the existence and presented a method to explicitly construct the number operator for particles obeying the corresponding statistics. We have also proved the positivity of the norm of linearly independent state vectors.
Acknowledgment

I express my gratitude to S.Pallua and S. Meljanac for introducing me to the subject of q–deformed algebras. I wish to thank V. Bardek for stimulating discussions and comments, and B. Nižić for careful reading of the manuscript. I would also like to thank the other members of the Theoretical Physics Department of the Rudjer Bošković Institute for their encouragement and support during the course of this work.

This work was supported by the Croatian Ministry of Science and Technology under contract No. 1-03-199.

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