An Exact Algorithm for the Optimal Integer Coefficient Matrix in Integer-Forcing Linear MIMO Receivers

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Abstract—The integer-forcing (IF) linear multiple-input and multiple-output (MIMO) receiver is a recently proposed suboptimal receiver which nearly reaches the performance of the optimal maximum likelihood receiver for the entire signal-to-noise ratio (SNR) range and achieves the optimal diversity multiplexing tradeoff for the standard MIMO channel with no coding across transmit antennas in the high SNR regime [8]. The optimal integer coefficient matrix $A^\star \in \mathbb{Z}^{N_t \times N_t}$ for IF maximizes the total achievable rate, where $N_t$ is the column dimension of the channel matrix. To obtain $A^\star$, a successive minima problem (SMP) on an $N_t$-dimensional lattice that is suspected to be NP-hard needs to be solved. In this paper, an efficient exact algorithm for the SMP is proposed. For efficiency, our algorithm first uses the LLL reduction to reduce the SMP. Then, different from existing SMP algorithms which form the transformed $A^\star$ column by column in $N_t$ iterations, it first initializes with a suboptimal matrix which is the $N_t \times N_t$ identity matrix with certain column permutations that guarantee this suboptimal matrix is a good initial solution of the reduced SMP. The suboptimal matrix is then updated, by utilizing the integer vectors obtained by employing an improved Schnorr-Euchner search algorithm to search the candidate integer vectors within a certain hyper-ellipsoid, via a novel and efficient algorithm until the transformed $A^\star$ is obtained in only one iteration. Finally, the algorithm returns the matrix obtained by left multiplying the solution of the reduced SMP with the unimodular matrix that is generated by the LLL reduction. We then rigorously prove the optimality of the proposed algorithm by showing that it exactly solves the SMP. Furthermore, we develop a theoretical complexity analysis to show that the complexity of the new algorithm in big-O notation is an order of magnitude smaller, with respect to $N_t$, than that of the existing most efficient algorithm. Finally, simulation results are presented to illustrate the optimality and efficiency of our novel algorithm.

Index Terms—Integer-forcing linear receiver, optimal integer coefficient matrix, sphere decoding, successive minima problem, achievable rate.

I. INTRODUCTION

Due to the exponential growth of mobile traffic and subscribers globally, current and future wireless systems require ongoing improvements in capacity, quality and coverage. Multiple-input multiple-output (MIMO) technology uses multiple antenna arrays both in transmitters and receivers and thus exploits the space dimension to improve wireless capacity and reliability. However, to actually realize these gains, optimal or near optimal receiver designs are critical. The maximum likelihood (ML) receiver is optimal and achieves the highest data rate and smallest error probability. However, its complexity is exponential in the number of antennas. Consequently, minimum mean square error (MMSE), successive interference cancellation (SIC) and zero-forcing (ZF) receivers have been developed, which achieve low-complexity, albeit with a performance loss. Although lattice reductions (such as LLL reduction [1], SRQD [2] and V-BLAST [3]) usually improve their performance (see, e.g., [4]–[6]), performance losses can be significant, especially in the low signal-to-noise ratio (SNR) regime. One reason is that these conventional linear receivers attempt to roughly invert the channel, which results in the amplification of the additive noise.

To overcome these issues, a new linear receiver structure, called integer-forcing (IF), has been proposed by Zhan et al. [7] [8]. It exploits the fact that any integer linear combination of lattice points is still a lattice point. Based on this insight, it decodes integer combinations of transmit data, which are digitally solved for the original data. Since the IF linear receiver can equalize the channel to any full-rank integer matrix $A$, it can be optimized over the choice of $A$ to maximize the achievable rate. It has been shown that the IF linear receiver nearly reaches the performance of the optimal joint ML receiver for the entire SNR range and that it achieves the optimal diversity multiplexing tradeoff for the standard MIMO channel with no coding across transmit antennas in the high SNR regime [8].

Let $A^\star$ be the optimal $A$ which maximizes the achievable rate [8], then searching for $A^\star$ is equivalent to solving a Shortest Independent Vectors Problem (SIVP) [8]; given a lattice which is the space formed by all the integer combinations of some linearly independent vectors, find a maximal set of approximately shortest linearly independent lattice vectors. The SIVP problem is suspected to be NP-hard. Thus, for large scale problems with MIMO applications, this search
must be done for each coherence time of the channel. If the MIMO dimension is large, the implementation complexity of the search can be overwhelming. Thus, it is essential to develop efficient algorithms.

Wei et al. [9] developed an algorithm for \( A^* \) by solving a successive minima problem (SMP): given a lattice, find a maximal set of shortest linearly independent lattice vectors, whose solution also solves the SIVP with the same lattice. This algorithm first creates a set \( \Omega \) of candidate vectors for \( A^* \) by using the Fincke-Pohst algorithm [10] search algorithm to search all the integer vectors within a hyper-ellipsoid, and then finds \( A^* \in \mathbb{Z}^{N_t \times N_t} \) by choosing \( N_t \) shortest independent vectors from the set \( \Omega \), where \( N_t \) is the column dimension of the channel matrix. Since this algorithm does not preprocess with a lattice reduction, the choice of initial radius, which is used to create \( \Omega \), is sub-optimal, thus it is slow. Recently, another exact algorithm for \( A^* \) by solving a SMP was proposed in [11]. For efficiency, this algorithm first uses the LLL reduction [1] to preprocess the SMP. Then similar to [12], it finds the transformed optimal integer coefficient matrix column by column. Finally, it obtains \( A^* \) by left multiplying the transformed optimal integer coefficient matrix with the unimodular matrix generated by the LLL reduction. Although this algorithm is generally more efficient than the algorithm developed in [9], a highly efficient faster algorithm is still desirable.

There are also some suboptimal algorithms for \( A^* \) which are of lower complexity than the optimal ones at the expense of performance loss, such as lattice reduction based algorithms [14] and the slowest descent method [15]. Some variants of IF are also proposed, see, e.g., [16]-[18].

This paper focuses on developing an efficient algorithm for \( A^* \) for IF receiver by exactly solving a SMP. Specifically, the contributions are summarized as follows:

- An efficient exact algorithm for the SMP is proposed. For efficiency, the algorithm first uses the LLL reduction to reduce the SMP. Then, different from the algorithms in [12] and [11] which form the transformed \( A^* \in \mathbb{Z}^{N_t \times N_t} \) column by column, our algorithm initializes with a suboptimal matrix which is the \( N_t \times N_t \) identity matrix with certain column permutations that ensure the suboptimal matrix is a good initial solution of the reduced SMP. The suboptimal matrix is then updated by a novel algorithm which uses an improved Schnorr-Euchner search algorithm [19] to search for candidates of the columns of \( A^* \) and uses a novel and efficient algorithm to update the suboptimal matrix until the transformed \( A^* \) is obtained in only one iteration. Finally, the algorithm returns the matrix obtained by left multiplying the solution of the reduced SMP with the unimodular matrix that is generated with the LLL reduction. See Section III.

- We rigorously show the optimality of the new algorithm by showing that it exactly solves the SMP. See Theorem 4.

- We theoretically show that the complexity of the new algorithm in big-O notation is an order of magnitude smaller, with respect to \( N_t \), than that of the most efficient existing algorithm which was proposed in [11]. See Section V.

- Simulation tests show the optimality of the proposed algorithm and indicate that the new exact algorithm is much more efficient than existing optimal algorithms. See Section VI.

The rest of the paper is organized as follows. In Section II, we introduce the optimal integer coefficient matrix design problem for IF receiver design. We propose our new optimal algorithm for the optimal matrix in Section III, show the optimality of the algorithm in Section IV, and analyze its complexity in Section V. We provide a comparative performance evaluation of the proposed and existing algorithms in Section VI. Finally, conclusions are given in Section VII.

Notation. Let \( \mathbb{R}^{m \times n} \) and \( \mathbb{Z}^{m \times n} \) respectively stand for the spaces of the \( m \times n \) real and integer matrices. Let \( \mathbb{R}^n \) and \( \mathbb{Z}^n \) denote the spaces of the \( n \)-dimensional real and integer column vectors, respectively. Matrices and column vectors are respectively denoted by uppercase and lowercase letters. For a matrix \( A \), let \( a_{ij} \) denote its element at row \( i \) and column \( j \), \( A_{i,j} \) be its \( i \)-th column and \( A_{1,i} \) be the submatrix of \( A \) formed by its first \( i \) columns. For a vector \( x \), let \( x_i \) be its \( i \)-th element, and \( x_{ij} \) be the subvector of \( x \) formed by entries \( i, i+1, \ldots, j \), and let \( \lfloor x \rfloor \) denote the nearest integer vector of \( x \), i.e., each entry of \( x \) is rounded to its nearest integer (if there is a tie, the one with smaller magnitude is chosen).

II. PROBLEM STATEMENT

In this paper, we consider a slow fading channel model where the channel remains unchanged over the entire block length. Since a complex MIMO system can be readily transformed to an equivalent real system, without loss of generality, we consider the real-valued channel model only.

![Fig. 1. The block diagram of an IF MIMO system](image_url)

In the IF MIMO system (see Figure 1), the \( m \)-th transmitter antenna is equipped with a lattice encoder \( \varepsilon_m \), which maps the length-\( k \) message \( w_m \) into a length-\( n \) lattice codeword \( x_m \in \mathbb{R}^n \), i.e.,

\[
\varepsilon_m : \mathbb{F}_p^k \rightarrow \mathbb{R}^n, \quad w_m \rightarrow x_m,
\]

where the entries of \( w_m \) are independent and uniformly distributed over a prime-size finite field \( \mathbb{F}_p = \{0, 1, \cdots, p-1\} \), i.e.,

\[
w_m \in \mathbb{F}_p^k, \quad m = 1, 2, \cdots, N_t.
\]

\(^1\)This part will be presented at the 2017 IEEE International Conference on Communications (ICC) conference [20].
All transmit antennas employ the same lattice code. Each codeword satisfies the power constraint:

\[ \frac{1}{n} \| x_m \|^2 \leq P. \] (1)

Let \( X = [x_1, \ldots, x_n] \in \mathbb{R}^{N \times n} \), then the received signal \( Y \in \mathbb{R}^{N_r \times n} \) is given by

\[ Y = HX + Z \]

where \( H \in \mathbb{R}^{N_r \times N} \) is the channel matrix, and \( Z \in \mathbb{R}^{N_r \times n} \) is the noise matrix. All the elements of both \( H \) and \( Z \) are independent and identically follow the standard Gaussian distribution \( \mathcal{N}(0, 1) \).

The receiver aims to find a coefficient matrix \( A \in \mathbb{Z}^{N \times N_t} \) and a filter matrix \( B \in \mathbb{R}^{N \times N_r} \). With the matrix \( B \), the received message \( Y \) will be projected to the more effective received vector for further decoding. The \( m \)-th filter outputs

\[ y_{\text{eff},m} = b^T_m Y = a^T_m X + (b^T_m H - a^T_m) X + b^T_m Z, \]

where

\[ z_{\text{eff},m} = (b^T_m H - a^T_m) X + b^T_m Z \]

is the effective noise, \( a^T_m \) and \( b^T_m \) respectively denote the \( m \)-th rows of \( A \) and \( B \).

The receiver recovers the original \( N_t \) messages \( W = [w_1, \ldots, w_{N_t}]^T \) by decoding \( Y = [y_{\text{eff},1}, \ldots, y_{\text{eff},N_t}]^T \) in parallel based on the algebraic structure of lattice codes, i.e., the integer combination of lattice codewords is still a codeword. In this way, we first recover the linear equation \( u_m = [a^T_m W] \mod p \) from the \( y_{\text{eff},m} \) at one decoder, i.e.,

\[ \pi_m : \mathbb{R}^n \rightarrow \mathbb{F}_p^k, \quad y_{\text{eff},m} \rightarrow \hat{u}_m. \]

The original messages can be recovered free of error as long as all the lattice equations are correctly detected, i.e.,

\[ [\hat{w}_1, \ldots, \hat{w}_{N_t}]^T = A_p^{-1} [\hat{u}_1, \ldots, \hat{u}_{N_t}]^T, \]

where \( A_p = [A] \mod p \) is full-rank over \( \mathbb{Z}_p \).

Hence, the design of IF receiver is the construction of a full rank IF matrix \( A \in \mathbb{Z}^{N \times N_t} \) such that the achievable rate is maximized. For more details, see [8] and [21].

By [8], at the \( m \)-th decoder \( \pi_m \), the achievable rate is,

\[ R_m = \frac{1}{2} \log \left( \frac{P}{P \| H^T b_m - a_m \|_2^2 + \| b_m \|_2^2} \right), \]

where \( \log^+(x) \triangleq \max \{ \log(x), 0 \} \). Moreover,

\[ b^T_m = a^T_m H^T (H H^T + I/P)^{-1} \]

and the achievable rate is

\[ R_m = \frac{1}{2} \log^+ \left( \frac{1}{\| a^T_m G a_m \|} \right), \]

where

\[ G = I - H^T (H H^T + I/P)^{-1} H. \] (2)

Furthermore, the total achievable rate is

\[ R_{\text{total}} = N_t \min \{ R_1, R_2, \ldots, R_{N_t} \}. \] (3)

In this paper, we want to find \( A^* \) to maximize the total achievable rate. Equivalently, we need to solve the following optimization problem,

\[ A^* = \arg \min_{A \in \mathbb{Z}^{N \times N_t}} \max_{1 \leq m \leq N_t} \| a^T_m G a_m \|, \] (4)

where \( G \) is defined in (2).

III. A NOVEL OPTIMAL ALGORITHM FOR (4)

In this section, we propose a novel algorithm for (4). Our algorithm first uses the LLL reduction [1] to reduce (4). Then different from [11] which finds the transformed \( A^* \) column by column in \( N_t \) iterations, it initializes with a suboptimal matrix and then updates it, during the process of an improved Schnorr-Euchner enumeration [19], until the transformed \( A^* \) is obtained. Finally, it returns \( A^* \).

We first briefly review some necessary background of lattice and introduce the LLL reduction to reduce (4) into a reduced SMP in Section III-A. Then, we develop an efficient algorithm to update the suboptimal solution in Section III-B. Furthermore, to better explain our novel algorithm for the SMP, we introduce the Schnorr-Euchner enumeration algorithm in Section III-C. Finally, the novel algorithm for (4) is developed in Section III-D.

A. Preliminaries of lattices

This subsection briefly review some necessary background of the lattice material.

Let \( G \) in (2) have the following Cholesky factorization whose algorithms can be found in many references (see, e.g., [22]):

\[ G = R^T R, \] (5)

where \( R \in \mathbb{R}^{N \times N_t} \) is an upper triangular matrix. Then, (4) can be transformed to:

\[ A^* = \arg \min_{A \in \mathbb{Z}^{N \times N_t}} \max_{1 \leq m \leq N_t} \| R a_m \|_2. \] (6)

To solve (6), it is equivalent to find an invertible matrix \( A^* = [a_1^*, \ldots, a_{N_t}^*] \in \mathbb{Z}^{N \times N_t} \) such that \( \max_{1 \leq m \leq N_t} \| R a_m \|_2 \) is as small as possible. In lattice theory, this value is called the \( N_t \)-th successive minimum of lattice \( \mathcal{L}(R) = \{ Ra | a \in \mathbb{Z}^{N_t} \} \).

More generally, the \( k \)-th \((1 \leq k \leq N_t)\) successive minimum \( \lambda_k \) of \( \mathcal{L}(R) \) is the smallest \( r \) such that the closed \( N_t \)-dimensional ball \( \mathbb{B}(0, r) \) of radius \( r \) centered at the origin contains \( k \) linearly independent lattice vectors. Mathematically, we have

\[ \lambda_k = \min \{ r | \dim (\mathcal{R} \cap \mathbb{B}(0, r)) \geq k \}. \]

Thus, solving (6) is equivalent to solving a SIVP which is defined as:

**Definition 1.** SIVP: finding an invertible matrix \( A^* = [a_1^*, \ldots, a_{N_t}^*] \in \mathbb{Z}^{N \times N_t} \) such that

\[ \max_{1 \leq m \leq N_t} \| R a_m^* \|_2 \leq \lambda_{N_t}. \]
As pointed out in [11], instead of solving the SIVP, it is advantageous to solve a SMP whose solution also solves the corresponding SIVP. The SMP is defined as:

Definition 2. SMP: finding an invertible matrix $A^* = [a_1^*,\ldots,a_{N_t}^*] \in \mathbb{Z}^{N_t \times N_t}$ such that

$$\|Ra^*_i\|_2 = \lambda_i, \quad i = 1, 2, \ldots, N_t.$$  

In addition to integer-forcing, solving a SMP is needed in many other applications including physical-layer network coding [12], the expanded compute-and-forward framework [23] and compute-compress-and-forward [24].

In this paper, we will focus on developing an efficient algorithm to solve the SMP to get $A^*$. For efficiency, we use the LLL reduction [1] to preprocess the SMP. Concretely, after the Cholesky factorization of $G$, the LLL reduction reduces $R$ in (5) to $\tilde{R}$ through

$$\tilde{Q}^T R Z = \tilde{R},$$

where $\tilde{Q} \in \mathbb{R}^{N_t \times N_t}$ is orthogonal, $Z \in \mathbb{Z}^{N_t \times N_t}$ is unimodular (i.e., $Z$ also satisfies $|\det(Z)| = 1$), and $\tilde{R} \in \mathbb{R}^{N_t \times N_t}$ is upper triangular which satisfies

$$|\tilde{r}_{ik}| \leq \frac{1}{2}|r_{ik}|, \quad i = 1, 2, \ldots, k - 1,$n
$$\delta \tilde{r}_{k-1,k-1}^2 \leq \tilde{r}_{k-1,k}^2 + \tilde{r}_{kk}^2, \quad k = 2, 3, \ldots, N_t,$$

where $\delta$ is a constant satisfying $1/4 < \delta \leq 1$. The matrix $\tilde{R}$ is said to be LLL reduced.

The LLL reduction algorithm can be found in [1] and its properties have been studied in [4]-[6].

After using the LLL reduction (7), the SMP can be transformed to the following reduced SMP (RSMP):

Definition 3. RSMP: finding an invertible integer matrix $C^* = [c_1^*,\ldots,c_{N_t}^*] \in \mathbb{Z}^{N_t \times N_t}$ such that

$$\|\tilde{R}c_i^*\|_2 = \lambda_i, \quad 1 \leq i \leq N_t.$$  

Clearly $A^*$ and $C^*$ satisfy $A^* = ZC^*$.

B. Preliminaries of the novel algorithm

Since our new algorithm for the RSMP starts with a suboptimal matrix $C$ and updates it during the process of a sphere decoding, an efficient algorithm to update $C$ will be provided in this subsection.

The following theorem shows that for each given invertible matrix $C$ and nonzero vector $c$, there always exists at least one index $j$ such that the matrix obtained by replacing $c_j$ with $c$ is still invertible.

Theorem 1. Let $C \in \mathbb{R}^{n \times n}$ be an arbitrary invertible matrix and $c \in \mathbb{R}^n$ be an arbitrary nonzero vector such that $\tilde{C}_{[i,i+1]}^{[j]}$ is full column rank for some $i$ with $0 \leq i \leq n-1$, where

$$\tilde{C} = [c_1 \ldots c_i \ c \ c_{i+1} \ldots c_n].$$

Then there exists at least one $j$ with $i+2 \leq j \leq n+1$ such that $\tilde{C}_{[\setminus j]}$ is also invertible, where $\tilde{C}_{[\setminus j]}$ is the matrix obtained by removing $c_j$ from $\tilde{C}$.

Proof. Since $C$ is invertible, there exist $f_i \in \mathbb{R}, 1 \leq i \leq n$, such that

$$c = f_1c_1 + \ldots + f_nc_n.$$  

Since $\tilde{C}_{[i,i+1]}$ is full column rank, there exists at least one $j$ with $i+2 \leq j \leq n+1$, such that $f_{j-1} \neq 0$. Otherwise, by the aforementioned equation, $c$ is a linear combination of $c_1,\ldots,c_i$, which leads to that $\tilde{C}_{[i,i+1]}$ is not full column rank, contradicting the assumption. Then, we have

$$\begin{pmatrix} c_1 & c_2 & \ldots & c_{j-2} & c & c_j & \ldots & c_n \end{pmatrix} = \begin{pmatrix} 1 & f_1 & \ldots & f_{j-2} & f_{j-1} & f_j & 1 \end{pmatrix} \begin{pmatrix} c \ c_1 \ \ldots \ c_{n} \end{pmatrix}$$

which implies that

$$\begin{pmatrix} c_1 & c_2 & \ldots & c_{j-2} & c & c_j & \ldots & c_n \end{pmatrix}$$

is invertible. Equivalently, $\tilde{C}_{[\setminus j]}$ is invertible. \hfill \Box

Note that if $i = 0$, $\tilde{C}_{[1,i+1]}$ reduces to $c$ which is full column rank, and Theorem 1 reduces to:

Corollary 1. Let $C \in \mathbb{R}^{n \times n}$ be an arbitrary invertible matrix and $c \in \mathbb{R}^n$ be an arbitrary nonzero vector, then there exists at least one $j$ with $1 \leq j \leq n$ such that the matrix obtained by removing $c_j$ from $[c \ c_1 \ \ldots \ c_n]$ is also invertible.

As will be seen in the next subsection, to efficiently solve the RSMP, we need to develop a fast algorithm for the following problem: for any given nonsingular matrix $C \in \mathbb{R}^{n \times n}$ and nonzero vector $c \in \mathbb{R}^n$ that satisfy

$$\|c_1\|_2 \leq \|c_2\|_2 \leq \ldots \leq \|c_n\|_2 \text{ and } \|c\|_2 < \|c_n\|_2,$$

we need to get an invertible matrix $\tilde{C} \in \mathbb{R}^{n \times n}$, whose columns are chosen from $c_1, c_2, \ldots, c_n$ and $c$, such that $\|\tilde{c}_i\|_2$ are as small as possible for all $1 \leq i \leq n$ and

$$\|\tilde{c}_1\|_2 \leq \|\tilde{c}_2\|_2 \leq \ldots \leq \|\tilde{c}_n\|_2.$$  

We first show the problem is well-defined.

By (9), one can see that there exists $i$ with $0 \leq i \leq n-1$ such that $\|c_i\|_2 \leq \|c\|_2 < \|c_{i+1}\|_2$ (note that if $i = 0$, it means $\|c\|_2 < \|c_1\|_2$). Then, one can see that

$$\|\tilde{c}_1\|_2 \leq \|\tilde{c}_2\|_2 \leq \ldots \leq \|\tilde{c}_{n+1}\|_2,$$

where $\tilde{C}$ is defined in (8). Hence, the question is equivalent to finding the largest $j$ with $1 \leq j \leq n+1$ such that $\tilde{C}_{[\setminus j]}$ is invertible. Specifically, after finding $j$, set $\tilde{C} = \tilde{C}_{[\setminus j]}$; then $\|\tilde{c}_j\|_2$ are as small as possible for all $1 \leq i \leq n$. Moreover, by (11), (10) holds.

If $\tilde{C}_{[1,i+1]}$ is not full column rank, then $j = i + 1$, i.e., $c$ should be removed from $C$, and the resulting matrix is $C$ which is invertible by assumption. This is because, no matter which $c_j$ is removed for $i + 2 \leq j \leq n + 1$, the resulting
matrix contains \( \tilde{C}_{[1,i+1]} \) as a submatrix, and hence it is not invertible. On the other hand, if \( \tilde{c}_j \) is removed for \( 1 \leq j \leq i \), then it is not the column with the largest index being removed.

If \( \tilde{C}_{[1,i+1]} \) is full column rank, then by Theorem 1, there exists at least one \( j \) with \( i + 2 \leq j \leq n + 1 \) such that \( \tilde{C}_{[\backslash j]} \) is invertible.

By the above analysis, \( j \) exists no matter whether \( \tilde{C}_{[1,i+1]} \) is full column rank or not. Thus, the aforementioned problem is well-defined.

By the above analysis, a natural method to find the desire \( j \) is to check whether \( \tilde{C}_{[\backslash j]} \) is invertible for \( j = n+1, n, \ldots, i+1 \) until finding an invertible matrix. Clearly, this approach works, but the main drawback of this method is that its worst complexity is \( O(n^4) \) which is too high. Concretely, in the worst case, i.e., in the case that only \( \tilde{C}_{[1,i+1]} \) is invertible, then \( n \) matrices need to be checked. Since the complexity of checking whether an \( n \times n \) matrix is invertible or not is \( O(n^3) \) flops, the whole complexity is \( O(n^4) \) flops.

In the following, we introduce a method which can find \( j \) in \( O(n^3) \) flops. By the above analysis, \( C = \tilde{C}_{[\backslash (i+1)]} \) if \( \tilde{C}_{[1,i+1]} \) is not full column rank. Thus, in the sequel, we only consider the case that \( \tilde{C}_{[1,i+1]} \) is full column rank. We begin with introducing the following theorem which shows that if the matrix formed by a given nonzero vector \( c \) and the first \( j \) columns of a given nonsingular matrix \( C \) is not full column rank, then \( \tilde{C}_{[\backslash j]} \) is invertible.

**Theorem 2.** Let \( C \in \mathbb{R}^{n \times n} \) be an arbitrary invertible matrix and \( c \in \mathbb{R}^n \) be an arbitrary nonzero vector such that \( \tilde{C}_{[1,i+1]} \) (see (8)) is full column rank for some \( i \) with \( 0 \leq i \leq n-1 \). If there exists some \( j \) with \( i + 2 \leq j \leq n \) such that \( \tilde{C}_{[\backslash j]} \) is not full column rank, then \( \tilde{C}_{[\backslash j]} \) is invertible.

**Proof.** We prove it by contradiction. We assume \( \tilde{C}_{[\backslash j]} \) is not invertible, and show that \( C \) is not invertible either, which contradicts the assumption.

Since \( \tilde{C}_{[\backslash j]} \) is not invertible, \( \tilde{c}_{j+1} \) (i.e., \( c_j \)) is a linear combination of vectors

\[
\{ \tilde{c}_k | 1 \leq k \leq n + 1, k \neq j, j + 1 \}
\]

which is actually

\[
\{ c | \tilde{c}_k | 1 \leq k \leq n, k \neq j - 1, j, \}
\]

Similarly, as \( \tilde{C}_{[\backslash j]} \) is not full column rank, \( c \) is a linear combination of vectors

\[
\{ \tilde{c}_k | 1 \leq k \leq j - 1 \}.
\]

Thus, \( c_j \) is a linear combination of vectors

\[
\{ c_k | 1 \leq k \leq n, k \neq j \},
\]

and this implies that \( C \) is not invertible, which contradicts the assumption that \( C \) is invertible. Thus, the assumption is not correct. In other words, \( \tilde{C}_{[\backslash j]} \) is invertible.

The following theorem shows how to find the desire \( j \).

**Theorem 3.** Let \( C \in \mathbb{R}^{n \times n} \) be an arbitrary invertible matrix and \( c \in \mathbb{R}^n \) be an arbitrary nonzero vector such that \( \tilde{C}_{[1,i+1]} \) (see (8)) is full column rank for some \( i \) with \( 0 \leq i \leq n-1 \). Suppose that \( j \) is the smallest integer with \( i + 2 \leq j \leq n \) such that \( \tilde{C}_{[1,j]} \) is not full column rank, then \( j \) is the largest integer with \( j \leq n \) such that \( \tilde{C}_{[\backslash j]} \) is invertible.

**Proof.** By Theorem 2, \( \tilde{C}_{[\backslash j]} \) is invertible. In the following, we show that \( \tilde{C}_{[\backslash k]} \) is not invertible for any \( k \) with \( j < k \leq n + 1 \) by contradiction.

Suppose that there exists \( k \) with \( j < k \leq n + 1 \) such that \( \tilde{C}_{[\backslash k]} \) is invertible. Then \( \tilde{C}_{[1,j]} \) is full column rank which contradicts the assumption. Thus, \( j \) is the largest integer with \( j \leq n \) such that \( \tilde{C}_{[\backslash j]} \) is invertible.

**Remark 1.** In Theorem 3, we assumed \( j \leq n \) which is because if \( \tilde{C}_{[1,j]} \) is full column rank for all \( j \) with \( i + 2 \leq j \leq n \), then \( \tilde{C}_{[\backslash (n+1)]} \) is invertible, leading to \( j = n + 1 \).

By Theorem 3, one of the methods to find \( j \) is to check whether \( \tilde{C}_{[1,j]} \) is full column rank for \( j = i + 1, \ldots, n \) until finding a \( j \) such that it is not full column rank. If \( \tilde{C}_{[1,j]} \) is full column rank for all \( i + 1 \leq j \leq n \), then set \( j = n + 1 \). But again, the worst complexity is \( O(n^4) \) flops. Concretely, when \( i = 1 \) and \( j = n \), then \( \tilde{C}_{[1,j]} \) should be checked for all \( 2 \leq j \leq n \), and hence the complexity is \( O(n^4) \) flops.

In the following, we develop an algorithm to find \( j \) with the complexity of \( O(n^3) \) flops. To clearly explain the algorithm, we introduce the definition of the row echelon form of matrices which can be found in many linear algebra books (see, e.g., [25, p.39]).

**Definition 4.** A matrix is in row echelon form if it satisfies the following two conditions:

1. All nonzero rows are above any rows of all zeros (if it exists);
2. The leading coefficient of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

Note that in some textbooks (see, e.g., [26, p.11]), the leading coefficient of each nonzero row is required to be 1 for the definition of row echelon form.

The main reason for introducing the row echelon form of a matrix is whether its submatrices are full column rank or not can be efficiently checked. As a result, the desire \( j \) can also be efficiently found. Specifically, we have the following algorithm for the problem raised in the paragraph under Corollary 1.

By Theorem 3, one can easily see that this algorithm indeed works. Moreover, the complexity of the second step, which dominates the whole algorithm, is \( O(n^3) \) flops via Gaussian elimination [25, p.44], so the total complexity is \( O(n^3) \) flops.

**C. Schnorr-Euchner search algorithm**

As our novel algorithm for the RSMP needs to use the integer vectors obtained by the improved Schnorr-Euchner search algorithm [19] to update the suboptimal solution \( C \), for self-contained, we introduce the Schnorr-Euchner search algorithm [27] for the following shortest vector problem (SVP)

\[
e^* = \min_{e \in \mathbb{Z}^n \setminus \{0\}} \|Re\|_2^2.
\]
Algorithm 1 Efficient algorithm for updating $C$

**Input:** A full column rank matrix $C \in \mathbb{R}^{n \times n}$ and a nonzero vector $c \in \mathbb{R}^n$ that satisfy (9).

**Output:** An invertible matrix $\hat{C} \in \mathbb{R}^{n \times n}$, whose columns are chosen from $c_1, c_2, \ldots, c_n$ and $c$, such that (10) holds and $\|c_i\|_2$ are as small as possible for all $1 \leq i \leq n$.

1. Find $i$ and form $\hat{C}$ (see (8)) such that it satisfies (11);
2. Reduce $\hat{C}$ into its row echelon form by Gaussian elimination, for more details, see, e.g., [25, pp.40-41], and denote it by $\bar{R}$.
3. Check whether $\bar{r}_{jj} = 0$ for $j = i+1, \ldots, n$ until finding a $j$ such $\bar{r}_{jj} = 0$ if it exists, and set $\hat{C} = \hat{C}_{[\bar{j}]}$. Otherwise, i.e., $\bar{r}_{jj} \neq 0$ for all $i+1 \leq j \leq n$, set $\hat{C} = \hat{C}_{[\bar{j}(n+1)]}$.

More details on this algorithm are referred to [28], [29], and its variants can be found in e.g., [30]–[33]. Note that SVP arises from enormous applications including communications (see, e.g., [28], [34], [35]) and cryptography (see, e.g., [36]–[38]).

Suppose that $c^*$ satisfies the following hyper-ellipsoid constraint

$$\|\hat{R}c\|_2^2 < \beta^2,$$

where $\beta$ is a given constant. Let

$$d_{N_i} = 0, d_i = -\frac{1}{\bar{r}_{ii}} \sum_{j=i+1}^{N_i} \bar{r}_{ij}c_j, \quad i = N_i - 1, \ldots, 1. \quad (14)$$

Then (13) can be transformed to

$$\sum_{i=1}^{N_i} \bar{r}_{ii}(c_i - d_i)^2 < \beta^2$$

which is equivalent to

$$\bar{r}_{ii}(c_i - d_i)^2 < \beta^2 - \frac{N_i}{\sum_{j=i+1}^{N_i} \bar{r}_{jj}(c_i - d_j)^2}, \quad (15)$$

for $i = N_i, N_i - 1, \ldots, 1$ which is called as the level index, where $\sum_{j=N_i+1}^{N_i} \bar{r}_{jj} = 0$.

The Schnorr-Euchner search algorithm starts with $\beta = \infty$, and sets $c_i = [d_i]$ ($d_i$ are computed via (14)) for $i = N_i, N_i - 1, \ldots, 1$. Clearly, $c = 0$ is obtained and (15) holds. Since $c^* \neq 0$, $c$ should be updated. To be more specific, $c_1$ is set as the next closest integer to $d_1$. Since $\beta = \infty$, (15) with $i = 1$ holds. Thus, this updated $c$ is stored and $\beta$ is updated to $\beta = \|\hat{R}c\|_2$. Then, the algorithm tries to update the latest found $c$ by finding a new $c$ satisfying (13). Since (15) with $i = 1$ is an equality for the current $c$, $c_1$ only cannot be updated. Thus we try to update $c_2$ by setting it as the next closest integer to $d_2$. If it satisfies (15) with $i = 2$, we try to update $c_1$ by setting $c_1 = [d_1]$ ($d_1$ is computed via (14)) and then check whether (15) with $i = 1$ holds or not, and so on; Otherwise, we try to update $c_3$, and so on. Finally, when we are not able to find a new integer $c$ such that (15) holds with $i = N_i$, the search process stops and outputs the latest $c$, which is actually $c^*$ satisfying (12).

D. A novel algorithm for solving the SMP

In this subsection, we develop a novel and efficient algorithm for (6). We begin with designing the algorithm for the RSMP by incorporating Algorithm 1 into an improved Schnorr-Euchner search algorithm [19].

The proposed algorithm for the RSMP is described as follows: we start with a suboptimal solution $C$ which is the $N_1 \times N_1$ identity matrix with some column permutations such that

$$\|\hat{R}c_1\|_2 \leq \|\hat{R}c_2\|_2 \leq \ldots \leq \|\hat{R}c_{N_1}\|_2,$$

and assume $\beta = \|\hat{R}c_{N_1}\|$. Then we modify the Schnorr-Euchner search algorithm to search the nonzero integer vectors $c$ satisfying (13) to update $C$. Specifically, whenever a zero vector $c$ is obtained, we update $c$ by setting $c_1$ as the next closest integer to $d_1$ to obtain a nonzero integer vector which is also denoted by $c$, and as long as a nonzero integer vector $c$ is obtained (note that $c$ satisfies $\|\hat{R}c\|_2 < \|\hat{R}c_{N_1}\|_2$), we use Algorithm 1 to update $C$ (note that $RC$ and $\hat{R}$ are respectively viewed as $C$ and $c$) to another matrix which is also denoted as $C$. Then, we set $\beta = \|\hat{R}c_{N_1}\|$. Note that $\beta$ decreases only when the last column of $C$ is changed. After this, we use the Schnorr-Euchner search algorithm [19] to update $c$ and then update $C$ with Algorithm 1 (for more details on how to update $c$, see Section III-C). Finally, when $C$ cannot be updated anymore and $\beta$ cannot be decreased anymore (i.e., when we are not able to find a new value for $c_{N_1}$ such that (15) holds with $k = N_1$), the search process stops and outputs $C^*$.

For efficiency, $\|\hat{R}c_i\|_2, 1 \leq i \leq N_i$, can be stored with a vector, say $p$, and $\|\hat{R}c_i\|_2$ can be calculated while using the Schnorr-Euchner enumeration strategy. Then, whenever we need to update $C$, we also update $p$ instead of calculating $\|\hat{R}c_i\|_2$ for all $1 \leq i \leq N_i$.

Clearly, if $C^*$ is a solution to the RSMP, so is $\bar{C}^*$, where $\bar{C}^* = C^*$ except that $\bar{c}_j^* = -c_j^*$ for a $1 \leq j \leq N_i$. Thus, to further speed up the above process, the strategy proposed in [19] can be applied here. Specifically, only the nonzero integer vectors $c$, satisfying $c_{N_1} \geq 0$ and $c_k \geq 0$ if $c_{k+1} \leq 0$, where $1 \leq k \leq N_1 - 1$, are searched to update $C$ in the above process. Note that only the former property of $c$ is exploited in [11], whereas our strategy can prune more vectors while retaining optimality.

By the above analysis, the proposed algorithm for the RSMP can be summarized in Algorithm 2, where

$$\text{sgn}(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases} \quad (17)$$

Since if $C^*$ is a solution to the RSMP, then $A^* = ZC^*$ is a solution to the SMP, where $Z$ is the unimodular matrix in (7), the algorithm for $A^*$ for the problem (4) can be described in Algorithm 3.

IV. Optimality of Algorithm 3

In this section, we show that Algorithm 3 is an optimal algorithm for (4) by proving it is an exact algorithm for the SMP.
Algorithm 2 A novel algorithm for the RSMP

Input: A nonsingular upper triangular $R \in \mathbb{R}^{N_t \times N_t}$.
Output: A solution $C^*$ to the RSMP, i.e.,
$$||\bar{R}c_i||_2 = \lambda_i, \quad i = 1, 2, \ldots, N_t,$$
where $\lambda_i$ is the $i$-th successive minimum of lattice $\mathcal{L}(\bar{R})$.

1. Set $k := N_t$, let $C$ be the $N_t \times N_t$ identity matrix with some column permutations such that (16) holds, and let $\beta := \|\bar{R}e_n\|_2$.
2. Set $c_k := [d_k]$ (where $d_k$ is obtained by using (14)) and $s_k := \text{sgn}(d_k - c_k)$ (see (17)).
3. If (15) does not hold, then go to Step 4. Else if $k > 1$, set $k := k - 1$ and go to Step 2. Else, i.e., $k = 1$, go to Step 5.
4. If $k = N_t$, set $C^* := C$ and terminate. Else, set $k := k + 1$ and go to Step 6.
5. If $c \neq \emptyset$, use Algorithm 1 to update $C$, set $\beta := \|\bar{R}e_N\|_2$ and $k := k + 1$.
6. If $k = N_t$ or $c_{k+1:N_t} = \emptyset$, set $c_k := c_k + 1$, and go to Step 3. Otherwise, set $c_k := c_k + s_k$, $s_k := -s_k - \text{sgn}(s_k)$ and go to Step 3.

Algorithm 3 A novel algorithm for (4)

Input: A symmetric positive definite matrix $G \in \mathbb{R}^{N_t \times N_t}$.
Output: A solution $A^*$ to the problem (4).

1. Perform Cholesky factorization to $G$ in (4) to get a nonsingular matrix $R$ (see (5)).
2. Perform LLT reduction to $R$ to get $\bar{R}$ and $Z$ (see (7)).
3. Getting $C^*$ by solving the RSMP with Algorithm 2.
4. Set $A^* := ZC^*$.

We first provide some properties of successive minima which are useful for showing the optimality of Algorithm 3. We begin with introducing the following Lemma.

**Lemma 1.** Suppose that linearly independent vectors $c_1, \ldots, c_i \in \mathcal{L}(\bar{R}) = \{Ra | a \in \mathbb{Z}^{N_t}\}$ satisfy $\|Ra\|_2 = \lambda_i$ for $1 \leq j \leq i$ with $1 \leq i \leq N_t$. Then, they are the first $i$ columns of a solution of the RSMP.

**Proof.** We assume $i < N_t$, otherwise, the lemma holds naturally. To prove the Lemma, it suffices to show there exist $c_{i+1}, \ldots, c_{N_t}$ such that $c_1, \ldots, c_{N_t}$ are linearly independent and $c_j = \lambda_j$ for $i + 1 \leq j \leq N_t$. Let
$$c_{i+1} = \arg \min_{c \in S_i} \|\bar{R}c\|_2,$$
$$\vdots$$
$$c_{N_t} = \arg \min_{c \in S_{N_t}} \|\bar{R}c\|_2,$$
where
$$S_i = \{c \in \mathbb{Z}^{N_t} \mid c, c_1, \ldots, c_{i-1} \text{ are independent}\},$$
$$\vdots$$
$$S_{N_t} = \{c \in \mathbb{Z}^{N_t} \mid c, c_1, \ldots, c_{N_t-1} \text{ are independent}\}.$$

Then, by the proof of [12, Theorem 8], one can see that the above $c_{i+1}, \ldots, c_{N_t}$ satisfy the above requirements. Hence, the Lemma holds.

By Lemma 1, we can get the following useful lemma.

**Lemma 2.** Suppose that the successive minima of lattice $\mathcal{L}(\bar{R}) = \{Ra | a \in \mathbb{Z}^{N_t}\}$ satisfy $\lambda_{i-1} < \lambda_i$ for some $2 \leq i \leq N_t$, and linearly independent vectors $c_1, \ldots, c_i \in \mathcal{L}(\bar{R})$ satisfy $\|\bar{R}c_j\|_2 = \lambda_j$ for $1 \leq j \leq i$. Let $c_1, \ldots, c_{k} \in \mathcal{L}(\bar{R})$ be any linearly independent vectors which either satisfy that $\|\bar{R}c_j\|_2 < \lambda_i$ or $\|\bar{R}c_j\|_2 = \lambda_i$ and $c_j$ is not the $i$-th column of any solution of the RSMP. For all $1 \leq j \leq k$, then $c_1, \ldots, c_k, c_i$ are also linearly independent.

**Proof.** For any $1 \leq j \leq k$, if $\|\bar{R}c_j\|_2 < \lambda_i$, then by the definition of $\lambda_i$ and the assumption that $\lambda_{i-1} < \lambda_i$, one can yield $\bar{c}_j$ is a linear combination of $c_1, \ldots, c_{i-1}$. Otherwise, $\bar{c}_j, c_1, \ldots, c_{i-1}$ are linearly independent, then according to $\lambda_{i-1} < \lambda_i$ and $\|\bar{R}c_j\|_2 < \lambda_i$, one can obtain that $\lambda_i \leq \max\{\|\bar{R}c_1\|_2, \|\bar{R}c_j\|_2\} < \lambda_i$ which is impossible. If $\|\bar{R}c_j\|_2 = \lambda_i$, since it is not the $i$-th column of any solution of the RSMP, by Lemma 1, $\bar{c}_j$ is a linear combination of $c_1, \ldots, c_{i-1}$.

Since $c_1, \ldots, c_k$ are linearly independent, if $c_1, \ldots, c_{k-1}, c_i$ are linearly dependent, then $c_i$ is a linear combination of $c_1, \ldots, c_{k-1}$, contradicting the assumption that $c_1, \ldots, c_i$ are linearly independent. Thus, $c_1, \ldots, c_{k-1}, c_i$ are linearly independent.

We give a remark here. By the definition of $\lambda_i$, one can see that the number of vectors $\bar{c}_j$ satisfying $\|\bar{R}c_j\|_2 < \lambda_i$ is less than $i$.

With Lemma 2, we can prove the following theorem which shows the optimality of Algorithm 2, leading to the optimality of Algorithm 3.

**Theorem 4.** Suppose that $C^* \in \mathbb{Z}^{N_t \times N_t}$ is the invertible matrix returned by Algorithm 2, then
$$\|\bar{R}c_i\|_2 = \lambda_i, \quad i = 1, 2, \ldots, N_t,$$
where $\bar{R}$ is defined in (7) and $\lambda_i$ is the $i$-th successive minimum of lattice $\mathcal{L}(\bar{R})$.

**Proof.** Please see Appendix A.

V. COMPLEXITY ANALYSIS

In this section, we first analyze the complexity of the proposed algorithm, i.e., Algorithm 3, for the SMP, and then show that its complexity in big-O notation is an order of magnitude smaller than that of [11, Alg. 2] with respect to $N_t$.

A. Complexity Analysis of the Proposed Algorithm

In this subsection, we analyze the complexity of Algorithm 3.

We first look at its space complexity. One can easily see that the space complexities of both Algorithms 1 and 2 are $O(N_t^2)$ spaces. All the space complexities of the Cholesky
factorization (see (5)), the LLL reduction and saving $Z$ (see (7)) are $O(N_t^2)$ spaces, thus the space complexity of Algorithm 3 is $O(N_t^2)$ spaces.

In the following, we investigate the time complexity, in terms of flops, of Algorithm 3. Since the complexity of the Cholesky factorization, the expected complexity of the LLL reduction (when $1/4 < \delta < 1$) (see, e.g., [39]) and the complexity of computing $A^* = ZC^*$ are polynomial, and the complexity of Algorithm 2 is exponential, so the complexity of Algorithm 3 is dominated by Algorithm 2.

In the sequel, we study the time complexity of Algorithm 2, which is also the complexity of Algorithm 3 by the above analysis.

From Algorithm 2, one can see that its complexity, denoted by $C(N_t)$, consists of two parts, the complexities of finding and updating integer vector $c$ satisfying (13), and updating $C$ whenever a nonzero integer vector $c$ is obtained, which are respectively denoted by $C_1(N_t)$ and $C_2(N_t)$. Then,

\[ C(N_t) = C_1(N_t) + C_2(N_t). \]

Let $N_t(N_t)$ and $f_k$, $1 \leq k \leq N_t$, respectively be the number of nodes searched by the Schnorr-Euchner enumeration algorithm and the number of elementary operations, i.e., additions, subtractions, multiplications and divisions, that the enumeration performs for each visited node in the $k$-th level. By [40, Sec. IV-B], $f_k = O(k)$, thus,

\[ C_1(N_t) = \sum_{k=1}^{N_t} N_t(N_t)f_k \leq \sum_{k=1}^{N_t} N_1(N_t)O(k) = O(N_t^2N_1(N_t)). \]

Since the number of times that $C$ needs to be updated is $N_1(N_t)$, and by Algorithm 1, each updating costs $O(N_t^2)$ flops, so we obtain

\[ C_2(N_t) = N_1(N_t)O(N_t^2) = O(N_t^3N_1(N_t)). \]

By the aforementioned three equations, we have

\[ C(N_t) = O(N_t^3N_1(N_t)). \tag{18} \]

To compute $C(N_t)$, we need to know $N_1(N_t)$, but unfortunately, exactly computing $N_1(N_t)$ is very difficult if it is not impossible. However, from [28], [40]-[42], the expected value of $N_1(N_t)$, i.e., $E[N_1(N_t)]$, is proportional to

\[ E[\|\hat{R}c_k\|_2] = \sqrt{N_t} \text{ for } 1 \leq k \leq N_t, \text{ thus } E[\|\hat{R}c_n\|_2] \leq \sqrt{N_t}. \]

Hence,

\[ E[N_1(N_t)] = O\left(\frac{\pi^{N_t/2}}{\Gamma(N_t/2 + 1)} N_t^{N_t/2}\right). \tag{19} \]

By Stirling’s approximation and the fact that $\Gamma(n + 1) = n!$ for any positive integers $n$, we obtain

\[ \frac{\pi^{N_t/2}}{\Gamma(N_t/2 + 1)} \approx \frac{1}{\sqrt{N_t} \pi} \left(\frac{2\pi e}{N_t}\right)^{N_t/2}. \]

Hence,

\[ E[N_1(N_t)] \approx O\left(\frac{1}{\sqrt{N_t} (2\pi e)^{N_t/2}}\right). \]

By (18) yields

\[ E[C(N_t)] = O(N_t^{5/2}(2\pi e)^{N_t/2}). \tag{20} \]

B. Comparison of the complexity of the proposed method with that in [11]

In this subsection, we compare the complexity of Algorithm 3 with the one, i.e., Algorithm 2, for the real SMP in [11]. Note that two algorithms, which are respectively for real and complex SMPs, are proposed in [11]. In this paper, we only developed an algorithm for real SMPs since an algorithm for complex SMPs can be similarly designed and we omit its details due to the limitation of spaces.

For better understand [11, Alg. 2], we briefly review it here. It first employs the LLL reduction to reduce the SMP to the RSMP, then it solves the RSMP to get $C^*$, finally it returns $A^* = ZC^*$, where $Z$ is defined in (7). Similar to [12, Alg. 1], $C^*$ is obtained column by column in $N_t$ iterations. To be more concrete, the solution of the SVP (12) forms the first column of $C^*$, and the integer vector which minimizes $\|\hat{R}c\|_2$ over all the integer vectors $c'$ that are independent with the first $k-1$ columns of $C^*$ forms the $k$-th column of $C^*$ for $2 \leq k \leq N_t$ (note that this problem is called a subspace avoiding problem in [43] [44]). These vectors are obtained by a modified Schnorr-Euchner algorithm in [11].

By the above analysis, one can see that its space complexity is also $O(N_t^2)$ spaces. So it has the same space complexity with Algorithm 3.

In the following, we compare their time complexities. By [11, eqs. (15) and (18)], the complexity of [11, Alg. 2] is

\[ O\left(N_t^2 \frac{\pi^{N_t/2}}{\Gamma(N_t/2 + 1)} N_t^{N_t/2}\right). \]

While, by (18) and (19), the complexity of the proposed algorithm is

\[ O\left(N_t^3 \frac{\pi^{N_t/2}}{\Gamma(N_t/2 + 1)} N_t^{N_t/2}\right). \]

Thus, the complexity of the new algorithm is an order of magnitude smaller than that of [11, Alg. 2] with respect to $N_t$.

In fact, the above result can be easily understood. By the above analysis, [11, Alg. 2] solves one SVP and $(N_t - 1)$ subspace avoiding problems. The complexity of solving the subspace avoiding problem to get $c^*_k$ is around $O(k^3)$ times
TABLE I

| Alg. | WC  | DKWZ | New Alg. |
|------|-----|------|----------|
| 2    | 0.0039 | 0.0022 | 0.00073 |
| 4    | 0.0201 | 0.0023 | 0.00078 |
| 6    | 0.2426 | 0.0023 | 0.00080 |
| 8    | 0.1827 | 0.0023 | 0.00080 |
| 10   | 1.810  | 0.0023 | 0.00085 |
| 12   | 13.32  | 0.0023 | 0.00081 |
| 14   | 26.23  | 0.0023 | 0.00089 |
| 16   | 32.69  | 0.0023 | 0.00089 |

of that of solving an $N_t$-dimensional SVP, thus the total complexity is around $O(N_t^4)$ times of that of solving an $N_t$-dimensional SVP. In contrast, the complexity of the Algorithm 3 is around $O(N_t^3)$ times of that of solving an $N_t$-dimensional SVP. Hence, the complexity of the new algorithm is $O(N_t)$ times faster than [11, Alg. 2].

VI. NUMERICAL RESULTS

This section presents simulation results to compare our proposed algorithm (denoted by “New Alg.”), i.e., Algorithm 3, with the two optimal algorithms in [9] and [11] (denoted by “WC” and “DKWZ”, respectively) by using flat Rayleigh fading channel. The average achievable rates and CPU time over 2000 random samples are reported. Specifically, for simplicity, we let $N_r = N_t$, and for any fixed $N_t$ and $P$ which is the power constraint (see (1)), we first generated 2000 $G$’s according to (2) (recall that the entries of $H$ independent and identically follow the standard Gaussian distribution). Then, we respectively used these algorithms to solve (4) for each generated $G$, counted their CPU time and computed their achievable rate according to (3). Finally, we calculated their average CPU time and achievable rates.

All of the simulations were performed on Matlab 2016b on the same desktop computer with Intel(R) Xeon(R) CPU E5-1603 v4 working at 2.80 GHZ.

Figure 2 shows the average achievable rates for the three algorithms with $N_t = 2, 4$. Figure 3 shows the average CPU time for the three algorithms with $N_t = 2$. Since “WC” is time consuming when $N_t = 4$, to clearly see the average CPU time for these algorithms, we display them in Table I.

Figures 4 and 5 respectively show average achievable rates and CPU time for “New Alg.” and “DKWZ” with $P = 1, 10, 20$ dB and $N_t = 6 : 2 : 20$. We did not compare these two algorithms with “WC” because we found it is slower than the “DKWZ”.

Figures 2 and 4 indicate that the average rates for the three algorithms are exactly the same, and this is because all of them are optimal algorithms for $A^*$. From Table I, Figures 3 and 5, one can see that the new algorithm is always most efficient. Although Figure 3 indicates that “DKWZ” is slower than “WC” when $N_t = 2$, we found that the former is always faster than the latter when $N_t = 4 : 2 : 20$ (the case for $N_r = 4$ is showed in Table I).
VII. Conclusion

In this paper, we investigated the design of the optimal integer coefficient matrix $A^* \in \mathbb{Z}^{N_i \times N_t}$, which maximizes the achievable rate, of the IF linear receiver, and developed an efficient exact SMP algorithm to find $A^*$. Similar to the algorithms in [12] and [11], for efficiency, this algorithm first uses the LLL reduction to reduce the SMP. But different from these two algorithms which create the transformed $A^*$ column by column in $N_i$ iterations, it initializes with a suboptimal matrix obtained by performing certain column permutations on the $N_i \times N_t$ identity matrix. The suboptimal matrix is then updated by an algorithm which utilizes an improved Schnorr-Euchner search algorithm [19] to search the candidates of the columns of the transformed $A^*$ and updates the columns of the suboptimal matrix by a novel and efficient algorithm until the transformed $A^*$ is obtained. Finally, the matrix, obtained by left multiplying the solution of the reduced SMP with the unimodular matrix that is generated by the LLL reduction, is returned by the algorithm. We have rigorously proved that the proposed algorithm is optimal for the SMP whose solution is an optimal integer coefficient matrix $A^*$. Theoretical complexity analysis showed that the new algorithm is $O(N_i)$ times faster than the most efficient existing optimal algorithm which was proposed in [11]. Simulation results have been done to confirm the optimality and efficiency of the proposed algorithm.

APPENDIX A

PROOF OF THEOREM 4

Proof. By Algorithm 2, one can see that all the columns of $C^*$, which is a solver of the RSMP and satisfies $c^*_N, j \geq 0$ and $c^*_i, j \geq 0$ if $c^*_i, j+1 = 0$ for each $1 \leq i \leq N_i - 1$ and $1 \leq j \leq N_t$, can be searched by the algorithm. Thus, to show the theorem, it suffices to show that, during the process of Algorithm 2, for each $1 \leq j \leq N_t$, the first $c^*_j$ (note that there may exist several $C^*$’s which are the solutions of the RSMP, and here “the first $c^*_j$” means the first vector that obtained by Algorithm 2 and is the $j$-th column of a solver of the RSMP) will replace a column of $C$ corresponding to the suboptimal solution when $c^*_j$ is obtained by Algorithm 2 (in the following, we assume this $c^*_j$ is not a column of the $N_i \times N_t$ identity matrix, otherwise $c^*_j$ is already a column of the suboptimal solution, and we only need to show the next step), and it will not be replaced by any vector $c \in \mathbb{Z}^{N_i}$.

We first show the conclusion holds for $c^*_j$. Clearly, the matrix $C$ corresponding to the suboptimal solution when the first $c^*_j$ is obtained satisfies $||Rc||_2 < ||Rc^*_j||_2$ for all $1 \leq j \leq N_t$, thus, by Corollary 1 and Algorithm 2, the first $c^*_j$ will replace a column of $C$. Moreover, as there is not any vector $c \in \mathbb{Z}^{N_i}$ satisfying $||Rc||_2 < ||Rc^*_j||_2$, by Algorithm 2, $c^*_j$ will not be replaced by any vector $c \in \mathbb{Z}^{N_i}$.

In the following, we show the conclusion holds for the first $c^*_j$ for any $2 \leq j \leq N_t$ with Lemma 2 by considering two cases: $\lambda_{j-1} < \lambda_j$ and $\lambda_{j-1} = \lambda_j$.

Suppose that $\lambda_{j-1} < \lambda_j$. Let $C$ be the suboptimal solution when the first $c^*_j$ is obtained. Since $C$ is invertible and $c^*_j$ is the first vector that it is the $j$-th column of a solution of the RSMP, by Lemma 2 and Algorithm 2, $c^*_j$ will replace a column of $C$. Moreover, by Lemma 2, all the linearly independent vectors $c \in \mathbb{Z}^{N_i}$ such that $||Rc||_2 < ||Rc^*_j||_2$ are linearly independent with $c^*_j$, thus, by Algorithm 2, $c^*_j$ will not be replaced by any vector $c \in \mathbb{Z}^{N_i}$.

Suppose that $\lambda_{j-1} = \lambda_j$. We consider two cases: all the first $j-1$ successive minima of lattice $L(R)$ are equal, and at least two of the successive minima of lattice $L(R)$ are different.

We first consider the first case. In this case, $\lambda_1 = \ldots = \lambda_j$. By the above analysis, Algorithm 2 can find the first $c^*_j$, use it to replace the the first column of the suboptimal solution corresponding to $c^*_j$, and will not replace it by any other vectors. Since $\lambda_2 = \lambda_1$, there exists at least one $c \in \mathbb{Z}^{N_i}$ such that $||Rc||_2 = ||Rc^*_j||_2$ and $Rc$ and $Rc^*_j$ are linearly independent. Thus, the first such $c$, i.e., $c^*_j$, will replace the second column of the suboptimal solution corresponding to this vector. Similarly, one can show that the first $c^*_j$ will replace a column of $C$ corresponding to the suboptimal solution when it is obtained. Moreover, since there is not any vector $c \in \mathbb{Z}^{N_i}$ satisfying $||Rc||_2 < \lambda_1 = \lambda_j$, by Algorithm 2, $c^*_j$ will not be replaced by any vector $c \in \mathbb{Z}^{N_i}$.

In the following, we consider the second case, and let (if $j \geq 3$) 

$$k = \arg \min_{1 \leq k \leq j-2} \lambda_i < \lambda_{i+1}.$$ 

By the above proof, one can see that Algorithm 2 can find $c^*_j, \ldots, c^*_k$, use them to replace the the first $k$ columns of the suboptimal solution corresponding to them iteratively, and will not replace them with any other vectors. Since $\lambda_k < \lambda_{k+1}$, the conclusion holds clearly for $c^*_k$. Similarly and iteratively, one can show that $c^*_j$ will replace a column of $C$ corresponding to the suboptimal solution when it is obtained, and it will not be replaced by any vector $c \in \mathbb{Z}^{N_i}$.

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