Elastic Solution of Stress Boundary Problem for Orthotropic Materials

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Abstract. The elastic solution of common stress boundary problem is discussed for the plate with orthotropic materials. Essential equations are introduced on the basis of linear elastic mechanics and complex variable method. The complex variable functions including the material parameters are analyzed for solving the partial derivation equation and to meet the needs of the stress boundary conditions. By constructing new stress functions, the mechanic analysis of the orthotropic plate is carried out, and also the stress boundary problem is resolved. The general expressions of the stress fields are determined finally.

1. Introduction
Nowadays the anisotropic materials may really be more and more important in application to many engineering structures. And the orthotropic plates have been for the base of composite materials in common use [1, 2]. The valid method to solve the stress field problems in anisotropic materials must be to use the complex analytic function theory. The complex variable theory may provide a very powerful tool for the solution of many boundary value problems in the elastic body and the results have been reported in isotropic materials [3, 4]. Furthermore, the complex variable technique has also been expanded to use for composite materials. The feasible method to solve the stress field problems in anisotropic materials have to use the analytic function theory. Many investigations have been introduced in some detail [5~7]. The plane stress state of composite sheets is common and to be very importance for the engineering application. It is the key point to solve stress field problems in the orthotropic materials. But the general solutions of the elastic mechanics for composite materials have not given completely or perfectly. Therefore, it is necessary to make up new and general solutions for the composite plates. Particularly, the study of ordinary boundary problem as shown in Figure 1 must be very typical example for the mechanics of composite materials.

Figure 1. Scheme of the plate and coordinates
2. Basic Equations on Elastic Body

Let us now consider the solution of the two-dimensional elastic body as indicated in above figure. The normal pressures are uniformly distributed along the upper half edge. The boundary conditions for the stress components can be represented by:

\[
\begin{align*}
\sigma_y &= -q, \quad \tau_{xy} = 0 \quad (\theta = 0, \; \text{or} \; x > 0, \; y = 0) \\
\sigma_y &= \tau_{xy} = 0 \quad (\theta = \pi, \; \text{or} \; x < 0, \; y = 0)
\end{align*}
\]

(1)

2.1. Partial Derivation Equations

Suppose the principal elastic directions of the plate coincide with the rectangular coordinate directions shown in Fig.1, and let the principal directions (1, 2) parallel to the coordinate axes (x, y) respectively. Now consider the linear elastic strain-stress relations for the orthotropic materials, the constitutive equations in plane stress state are given as follows:

\[
\epsilon_x = S_{11} \sigma_x + S_{12} \sigma_y, \quad \epsilon_y = S_{12} \sigma_x + S_{22} \sigma_y, \quad \gamma_{xy} = S_{66} \tau_{xy}
\]

(2)

Where, \( S_{11} = \frac{1}{E_1} \), \( S_{22} = \frac{1}{E_2} \), \( S_{12} = -\frac{\mu_{12}}{E_1} = -\frac{\mu_{21}}{E_2} \), \( S_{66} = \frac{1}{G_{12}} \) are representative of elasticity coefficients. And we know that compatibility condition of strains must be satisfied by:

\[
\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}
\]

(3)

In the case of ignoring body forces, the differential equations of three-stress equilibrium for the plane problems are in the following form:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0
\]

(4)

The usual method of solving the differential equations may be by introducing a real function \( F(x, y) \), called the stress function. It is easily checked that the equilibrium equations can be satisfied for the stress components by using the following expressions:

\[
\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}
\]

(5)

For the plane stress problems, the stress function \( F \) must be selected reasonably in terms of the given boundary conditions. In order to solve two-dimensional elastic problems of orthotropic plate by using the stress function, we can substitute the expressions (5) into equations (2) and again to (3). Then, the governing equation of the strain compatibility condition can be transformed into:

\[
\frac{\partial^4 F}{\partial y^4} + 2B \frac{\partial^4 F}{\partial x^2 \partial y^2} + C \frac{\partial^3 F}{\partial x^4} = 0
\]

(6)

Where, \( B = \frac{S_{66} + 2S_{12}}{2S_{11}} = \frac{E_1}{2G_{12}} - \mu_{12} \), \( C = \frac{S_{22}}{S_{11}} = \frac{E_1}{E_2} \).

Above basic equations have been appeared in hand books and some references. Notice that the key point may be to find the stress function \( F \) in terms of stress boundary conditions and to be suitable for the governing equation with four order partial derivation.
2.2. Complex Variable Functions

For the plane stress problems solved so far, both rectangular and polar coordinates have proved to be very adequate and advantageous. For the convenience of the solution to the partial differential equation, we introduce the new complex variable numbers \((w, \bar{w})\), namely:

\[
w = x + ihy, \quad \bar{w} = x - ihy
\]

(7)

Where \(h\) is a real arbitrary constant. To give a precise definition, we suppose the constant \(h\) to be positive, \(h > 0\). By the definition, we have that: \(w\bar{w} = x^2 + h^2 y^2\), since \(i^2 = -1\) or \(i = \sqrt{-1}\). If the real constant \(h\) is equal to one, then: \(w = x + iy = z\), that is the habitual complex number. In addition, converting to polar coordinates as in Figure 1, the complex number can be given as:

\[
w = x + ihy = r(\cos \theta + ih \sin \theta)
\]

(8)

And again we introduce two letters \((M, \beta)\) to do the transformation about polar coordinates. The transformation relations may be defined as:

\[
h \sin \theta = M \sin \beta, \quad \cos \theta = M \cos \beta
\]

(9)

Obviously, it is easy to derive the more definitive relations:

\[
h \tan \theta = \tan \beta, \quad \beta = \arctan(h \tan \theta), \quad M = \sqrt{h^2 \sin^2 \theta + \cos^2 \theta}
\]

(10)

Hence it can be seen that the complex numbers become as follows:

\[
w = rM(\cos \beta + i \sin \beta) = rMe^{i\beta}
\]

\[
\bar{w} = rM(\cos \beta - i \sin \beta) = rMe^{-i\beta}
\]

(11)

3. General Solution

On the basis of above definition, the stress function \(F\) can be expressed by the complex variables. The partial derivations of \(F\) with \(x\) or \(y\) can be transformed into the expressions with complex numbers \((w, \bar{w})\):

\[
\frac{\partial F}{\partial x} = \frac{\partial F}{\partial w} \frac{\partial w}{\partial x} + \frac{\partial F}{\partial \bar{w}} \frac{\partial \bar{w}}{\partial x}, \quad \frac{\partial F}{\partial y} = \frac{\partial F}{\partial w} \frac{\partial w}{\partial y} + \frac{\partial F}{\partial \bar{w}} \frac{\partial \bar{w}}{\partial y}
\]

(12)

Still further, we can determine the higher derivatives, namely:

\[
\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial w^2} + 2 \frac{\partial^2 F}{\partial w \partial \bar{w}} + \frac{\partial^2 F}{\partial \bar{w}^2}, \quad \frac{\partial^2 F}{\partial y^2} = -h^2 \left( \frac{\partial^2 F}{\partial w^2} - 2 \frac{\partial^2 F}{\partial w \partial \bar{w}} + \frac{\partial^2 F}{\partial \bar{w}^2} \right)
\]

(13)

\[
\frac{\partial^2 F}{\partial x \partial y} = ih \left( \frac{\partial^2 F}{\partial w^2} - \frac{\partial^2 F}{\partial w \partial \bar{w}} \right)
\]

By substituting above expressions into the governing equation (6), we obtain the partial derivative equation with complex variables in the following way:
\[(h^4 - 2Bh^2 + C)(\frac{\partial^4 F}{\partial w^4} + \frac{\partial^4 F}{\partial w^2}) + 4(C - h^4)(\frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial w^2}) \frac{\partial^2 F}{\partial w^2 \partial w^2} + 2(3h^4 + 2Bh^2 + 3C) \frac{\partial^4 F}{\partial w^2 \partial w^2} = 0\]  

Eq. (14)

Evidently, the coefficients of the first two terms may be equal to zero. So the characteristic value \(h\) and the stress function \(F\) can be determined by different types.

**First case:** Let the coefficients of the first two terms in equation (14) equate to zero. Thus we can obtain the following equations:

\[h^4 - 2Bh^2 + C = 0, \quad C - h^4 = 0, \quad \frac{\partial^4 F}{\partial w^2 \partial w^2} = 0\]

The real constant \(h\) can be determined by: \(h^4 = B^2 = C\). And denote that

\[B = \frac{E_1}{2G_{12}} - \mu_{12}, \quad C = \frac{E_1}{E_2}, \quad h > 0\]

It leads to the result:

\[h = \sqrt[4]{\frac{E_1}{2G_{12}} - \mu_{12}} = \sqrt[4]{\frac{E_1}{E_2}}\]  

Eq. (15)

Next, by using \(\frac{\partial^4 F}{\partial w^2 \partial w^2} = 0\), we can obtain the stress function \(F\) to be of the form:

\[F = D_1w\Phi + D_2w\overline{\Phi} + D_3\Psi + D_4\overline{\Psi}\]  

Eq. (16)

Where, \(\Phi = \Phi(w), \overline{\Phi} = \overline{\Phi}(\overline{w}), \Psi = \Psi(w)\) and \(\overline{\Psi} = \overline{\Psi}(\overline{w})\) are the analytic functions with the complex numbers \(w = x + iy\) and \(\overline{w} = x - iy\). They must have the partial derivatives.

**Second case:** Suppose the coefficient of the first term in equation (14) to be zero. Hence, the partial derivatives of other terms must be zero. Thus the following equations may be given by:

\[h^4 - 2Bh^2 + C = 0, \quad \frac{\partial^2 F}{\partial w^2 \partial w^2} = 0\]

Therefore, the characteristic value \(h\) can be determined by:

\[h^2 = B \pm \sqrt{B^2 - C} \quad (B = \frac{E_1}{2G_{12}} - \mu_{12}, \quad C = \frac{E_1}{E_2})\]

Once more, we select: \(B^2 > C\) and let \(h_1 > h_2 > 0\). Thus the final solution can be obtained as:

\[h_1 = \sqrt{B + \sqrt{B^2 - C}}, \quad h_2 = \sqrt{B - \sqrt{B^2 - C}}\]  

Eq. (17)

Next, by using \(\frac{\partial^2 F}{\partial w^2 \partial w^2} = 0\), the stress function \(F\) can be determined as:

\[F = D_1\Phi_1 + D_2\overline{\Phi}_1 + D_3\Phi_2 + D_4\overline{\Phi}_2\]  

Eq. (18)

Where, \(\Phi_1 = \Phi_1(w_1), \overline{\Phi}_1 = \overline{\Phi}_1(\overline{w}_1), \Phi_2 = \Phi_2(w_2), \overline{\Phi}_2 = \overline{\Phi}_2(\overline{w}_2)\). And the complex variable numbers are: \(w_1 = x + ih_1y, \overline{w}_1 = x - ih_1y, w_2 = x + ih_2y, \overline{w}_2 = x - ih_2y\). Besides, we can give the complex numbers in the polar coordinates, namely:
\[ w_1 = r \cos \theta + i_h r \sin \theta = r M_1 e^{i \beta_1} \quad \text{and} \quad w_2 = r \cos \theta + i_h r \sin \theta = r M_2 e^{-i \beta_2} \]

\[ \overline{w}_1 = r \cos \theta - i_h r \sin \theta = r M_1 e^{-i \beta_1} \quad \text{and} \quad \overline{w}_2 = r \cos \theta - i_h r \sin \theta = r M_2 e^{i \beta_2} \]

(19)

Where, \( h_1 \sin \theta = M_1 \sin \beta_1 \), \( \cos \theta = M_1 \cos \beta_1 = M_2 \cos \beta_2 \), \( h_2 \sin \theta = M_2 \sin \beta_2 \).

4. Stress Analysis with Complex Function

For the first case, the stress function may be selected according to equation (16). And also it must be suited to meet the boundary condition in Figure 1. After all considering, the real function \( F \) can be determined by the following form:

\[ F = D_1 h \overline{w} (\ln w - \ln \overline{w}) + D_2 w \overline{w} + D_3 (w^2 - \overline{w}^2) + D_4 (w - \overline{w})^2 \]  

(20)

The first order partial derivatives may be easily found as follows:

\[ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial w} = D_1 i (w + \overline{w}) (\ln w - \ln \overline{w}) + D_2 (w + \overline{w}) + (2D_3 - D_1)(w - \overline{w}) \]

\[ \frac{\partial F}{\partial y} = h i \frac{\partial F}{\partial w} - \frac{\partial F}{\partial w} = h D_1 (w - \overline{w}) (\ln w - \ln \overline{w}) - h (D_1 + 2D_3)(w + \overline{w}) + hi(4D_4 - D_2)(w - \overline{w}) \]

And then, the second order partial derivatives are given as:

\[ \frac{\partial^2 F}{\partial x^2} = D_1 i [2(\ln w - \ln \overline{w}) + \overline{w} - \frac{w}{\overline{w}}] + 2D_2 \]

\[ \frac{\partial^2 F}{\partial y^2} = h^2 D_1 i [2(\ln w - \ln \overline{w}) + \frac{w}{\overline{w}} - \frac{\overline{w}}{w}] + 2h^2 D_2 - 8h^2 D_4 \]

\[ \frac{\partial^2 F}{\partial x \partial y} = h D_1_i (w - \overline{w}) \left( \frac{1}{w} - \frac{1}{\overline{w}} \right) - 2h(D_1 + D_3) \]

Substituting the derivations into front stress expressions (5), the stress components can be given by:

\[ \sigma_x = h^2 D_1 i (2 \ln \frac{w}{\overline{w}} + \frac{w}{\overline{w}} - \frac{\overline{w}}{w}) + 2h^2 D_2 - 8h^2 D_4 \]

\[ \sigma_y = D_1 i (2 \ln \frac{w}{\overline{w}} + \frac{w}{\overline{w}} - \frac{\overline{w}}{w}) + 2D_2 \]

\[ \tau_{xy} = h D_1 \left( \frac{(w - \overline{w})^2}{w \overline{w}} \right) + 2h(D_1 + D_3) \]

(21)

According to equation (11), the complex numbers are of below relations:

\[ w - \overline{w} = r M (e^{i \beta} - e^{-i \beta}) = 2i r M \sin \beta \quad \text{and} \quad (w - \overline{w})^2 = -4r^2 M^2 \sin^2 \beta \quad \text{and} \quad w \overline{w} = r^2 M^2 \]

\[ \frac{w}{\overline{w}} = e^{2i \beta} \quad \text{and} \quad \frac{w}{w} - \frac{\overline{w}}{\overline{w}} = e^{2i \beta} - e^{-2i \beta} = 2i \sin 2\beta = 4i \sin \beta \cos \beta \quad \text{and} \quad \ln \frac{w}{\overline{w}} = \ln e^{2i \beta} = 2i \beta \]

Hence, the stress components become of the following:
Finally, the stress components can be given by the following form:

\[ \sigma_x = -4h^2D_1(\beta + \sin \beta \cos \beta) + 2h^2D_2 - 8h^2D_4 \]
\[ \sigma_y = -4D_1(\beta - \sin \beta \cos \beta) + 2D_2 \]
\[ \tau_{xy} = -4hD_1 \sin^2 \beta + 2h(D_1 + 2D_4) \]  

On the basis of the definitive equation (9), it shows that: \( \theta = 0 \) then \( \beta = 0 \), and also \( \theta = \pi \) then \( \beta = \pi \). Just now, the stress boundary conditions must be considered. In terms of given conditions in equation (1), the constants in above expressions can be determined as below:

\[ 2D_2 = -q \quad , \quad 2D_3 = -D_1 = \frac{q}{4\pi} \]

Finally, the stress components can be determined by:

\[ \sigma_x = \frac{q}{\pi} h^2 (\beta + \sin \beta \cos \beta) - qh^2 - 8h^2D_4 \]
\[ \sigma_y = \frac{q}{\pi} (\beta - \sin \beta \cos \beta) - q \]
\[ \tau_{xy} = \frac{qh}{\pi} \sin^2 \beta \]  

Distinctly, the arbitrary constant \( D_4 \) remains yet. It is in need of other conditions to fix the constant. This matter may be not considered at present.

For the second case in above mention, to meet the stress boundary condition, the real function \( F \) can be determined by the following form:

\[ F = D_1i w_1^2 \ln w_1 - \ln w_1^2 \ln w_1 + D_2 (w_1^2 + w_1^2) \]
\[ + D_1i w_2^2 \ln w_2 - \ln w_2^2 \ln w_2 + D_4 (w_2^2 + w_2^2) \]  

Similarly, the partial derivatives of the stress function are conducted to find the stresses. Then, by the stress expressions (5), the stress components can be given by:

\[ \sigma_x = -2h_1^2D_1i(\ln w_1 - \ln w_1^2) - 2h_2^2D_2i(\ln w_2 - \ln w_2^2) - 4h_1^2D_2 - 4h_2^2D_4 \]
\[ \sigma_y = 2D_1i(\ln w_1 - \ln w_1^2) + 2D_1i(\ln w_2 - \ln w_2^2) + 4D_2 + 4D_4 \]
\[ \tau_{xy} = 2h_1D_1i(\ln w_1 + \ln w_1^2 + 3) + 2h_2D_1(\ln w_2 + \ln w_2^2 + 3) \]  

The stress boundary conditions as shown in Figure 1 must be considered as well. By the conditions in equation (1), some constants can be determined as below:

\[ h_1D_3 = -h_1D_1 \quad , \quad 4D_2 + 4D_4 = -q \quad , \quad 4D_1 = \frac{h_3q}{\pi(h_1 - h_2)} \]

Finally, the stress components can become of the following form:
\[
\begin{align*}
\sigma_x &= \frac{qh_1h_2}{\pi(h_1-h_2)}(h_1\beta_1 - h_2\beta_2) + qh_2^2 - 4D_2(h_1^2 - h_2^2) \\
\sigma_y &= \frac{qh_1h_2}{\pi(h_1-h_2)}(\frac{\beta_2}{h_2} - \frac{\beta_1}{h_1}) - q \\
\tau_{xy} &= \frac{qh_1h_2}{\pi(h_1-h_2)}\ln\frac{M_1}{M_2}
\end{align*}
\]

(26)

Here and now, \( \beta_k = \arctan(h_k \tan \theta) \) and \( M_k = \sqrt{h_k^2 \sin^2 \theta + \cos^2 \theta} \), \( (k = 1, 2) \). As seen by the equation, the arbitrary constant \( D_2 \) keeps at undetermined value and to be not discussed at present. In brief, the equations (23) and (26) are representative of the elastic solution on common stress fields for the orthotropic materials.

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6. References
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