GLOBAL STABILITY OF BOLTZMANN EQUATION WITH LARGE EXTERNAL POTENTIAL FOR A CLASS OF LARGE OSCILLATION DATA

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Abstract. In this paper, we investigate the stability of Boltzmann equation with large external potential in $\mathbb{T}^3$. For a class of initial data with large oscillations in $L^\infty_{x,v}$ around the local Maxwellian, we prove the existence of a global solution to the Boltzmann equation provided the initial perturbation is suitably small in $L^2$-norm. The large time behavior of the Boltzmann solution with exponential decay rate is also obtained. This seems to be the first result on the perturbation theory of large-amplitude non-constant equilibriums for large-amplitude initial data.

1. Introduction

In this paper, we consider the Boltzmann equation with external potential

$$F_t + v \cdot \nabla_x F - \nabla \Phi(x) \cdot \nabla_v F = Q(F, F),$$

supplemented with initial data

$$F(0, x, v) = F_0(x, v).$$

The unknown $F = F(t, x, v) \geq 0$ represents for the density distribution function of gas particles with position $x \in \mathbb{T}^3$ and particle velocity $v \in \mathbb{R}^3$ at time $t > 0$. The collision term $Q(F, F)$ is an integral with respect to velocity variable only, and it takes the non-symmetric bilinear form

$$Q(F_1, F_2) = \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \omega) F_1(u') F_2(v') d\omega d\mu - \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \omega) F_1(u) F_2(v) d\omega d\mu$$

$$\triangleq Q_+(F_1, F_2) - Q_-(F_1, F_2).$$

Here the post-collision velocity pair $(v', u')$ and the pre-collision velocity pair $(v, u)$ satisfy the relation

$$u' + v' = u + v \quad \text{and} \quad |u'|^2 + |v'|^2 = |u|^2 + |v|^2.$$

The collision kernel $B(v - u, \omega)$ depends only on the relative velocity $|v - u|$ and $\cos \theta := (v - u) \cdot \omega/|v - u|$, and it is assumed throughout the paper to take the form

$$B(v - u, \omega) = |v - u|^\gamma b(\theta),$$

with $b(\theta)$ satisfying the angular cutoff assumption

$$0 \leq b(\theta) \leq C|\cos \theta|.$$
Without loss of generality, we assume that $\Phi(x) \geq 0$, otherwise one can replace $\Phi$ by $M + \Phi$ while the equation (1.1) remains the same.

For given external potential, it is direct to check that the local Maxwellian

$$
\mu_E(x, v) = \exp \left\{ -\frac{|v|^2}{2} - \Phi(x) \right\} = \mu(v)e^{-\Phi(x)},
$$

is a steady solution to the Boltzmann equation (1.1). We define the perturbation function

$$
f(t, x, v) := \frac{F(t, x, v) - \mu_E(x, v)}{\sqrt{\mu_E(x, v)}},
$$

then the Boltzmann equation is rewritten as

$$
f_t + v \cdot \nabla_x f - \nabla \Phi \cdot \nabla_v f + e^{-\Phi}Lf = e^{-\frac{\Phi}{2}}\Gamma(f, f),
$$

where $L$ is the standard linearized operator given by

$$
Lf = -\frac{1}{\sqrt{\mu}} \{Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)\} = \nu(v)f - Kf,
$$

with $K = K_2 - K_1$ defined by

$$
K_1f(v) = \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \omega) \sqrt{\mu(u)f(u)} \, d\omega \, du,
$$

$$
K_2f(v) = \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \omega) \sqrt{\mu(u)f(u')f(u')} \, d\omega \, du
data(1.5)
$$

and the collision frequency

$$
\nu(v) = \int_{\mathbb{R}^3} \int_{S^2} |v - u|^\gamma \mu(u)b(\theta) \, d\omega \, du \sim (1 + |v|)^\gamma, \quad 0 \leq \gamma \leq 1.
$$

The nonlinear term $\Gamma(f, f)$ is given by

$$
\Gamma(f, f) := \frac{1}{\sqrt{\mu}} Q_+(\sqrt{\mu}f, \sqrt{\mu}f) - \frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu}f, \sqrt{\mu}f)
$$

$$\Gamma_+(f, f) - \Gamma_-(f, f).
$$

Let $F$ be a solution of the Boltzmann equation (1.1). Since an external potential $\Phi$ is included, generally we only have the conservation of mass and energy:

$$
\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \{F(t, x, v) - \mu_E(x, v)\} \, dv \, dx = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \{F_0(x, v) - \mu_E(x, v)\} \, dv \, dx = M_0,
$$

$$
\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left[ \frac{|v|^2}{2} + \Phi(x) \right] \{F(t, x, v) - \mu_E(x, v)\} \, dv \, dx
$$

$$\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left[ \frac{|v|^2}{2} + \Phi(x) \right] \{F_0(x, v) - \mu_E(x, v)\} \, dv \, dx = E_0,
$$

as well as the entropy inequality:

$$
\mathcal{H}(F(t)) - \mathcal{H}(\mu_E) \leq \mathcal{H}(F_0) - \mathcal{H}(\mu_E),
$$

where $\mathcal{H}(F) \triangleq \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} F \ln F \, dv \, dx.$
As pointed out in [21], generally, one does not have the conservation of momentum. In fact, it is easy to derive from the equation that

\[
\frac{d}{dt} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v F(t,x,v) dv \, dx + \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \nabla \Phi(x) F(t,x,v) dv \, dx = 0, \tag{1.10}
\]

which implies that the momentum is not conserved generally. However, if the external potential \( \Phi \) is independent of some \( x_i \), we obtain the conservation of momentum for those \( v_i \); that is, if we denote the orthogonal complement space of the vector \( \nabla \Phi(x) \) by \( \nabla \Phi(x) \perp \), we define \( \Lambda \) the degenerate subspace of \( \nabla \Phi \) by

\[
\Lambda \triangleq \cap_{x \in \mathbb{T}^3} \nabla \Phi(x) \perp.
\]

Upon reorienting the coordinates, we assume that \( \Lambda \) is spanned by \( \{e_1, \ldots, e_{n_0}\} \) as long as \( n_0 = \dim \Lambda > 0 \). In other words, \( \frac{\partial \Phi}{\partial x_1} = \cdots = \frac{\partial \Phi}{\partial x_{n_0}} \equiv 0 \), which together with (1.10) shows the conservation of momentum for degenerate \( \{v_1, \ldots, v_{n_0}\} \)

\[
\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v_i \{F(t,x,v) - \mu_E(x,v)\} dv \, dx = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} v_i \{F_0(x,v) - \mu_E(x,v)\} dv \, dx = \mathbf{J}_0, \tag{1.11}
\]

for \( i = 1, \cdots, n_0 \). We point out that the momentum conservation (1.11) is important for us to obtain the decay estimate.

The Boltzmann equation is a fundamental model in the collisional kinetic theory. There have been extensive results on the studies of well-posedness, especially on the global existence theory and large time asymptotic behavior. Among these works, we only mention some known results with external potential. For the classical Boltzmann equation, we refer the interested readers to [3, 4, 16, 17, 18, 19, 20, 23, 25, 26] and the references therein. For the Boltzmann equation with external potential, the local well-posedness was studied in [2, 5]; the global well-posedness was established in [6, 9, 10, 22, 24, 27, 29] around a local Maxwellian with small perturbation and some smallness assumption for the external potential \( \Phi \) by the energy method. Guo [13] investigated the global well-posedness of Vlasov-Poisson-Boltzmann solution with a small self-contained external potential, see also [12, 13] and the references therein. For the large amplitude external potential case, it is hard to use the energy method to study the global existence of solution to the Boltzmann equation since the derivatives of Boltzmann solution may grow in time. Recently, Guo [17, 18] developed a new \( L^2 \cap L^\infty \) theory to study the global stability of Boltzmann solutions, which can avoid the derivatives estimates. By using the \( L^2 \cap L^\infty \) method, Kim [21] proved the global well-posedness of Boltzmann equation and large time behavior with large external potential under small perturbation around the local Maxwellian. It is noted that the characteristic lines are heavily bended in [21], then to use the \( L^2 \cap L^\infty \) method, one has to analyze the possible singular point for the change of variable

\[
u \rightarrow X(\tau; s, X(s; t, x, v), u).
\]

This difficulty was overcome in [21] by using Asano’s almost transversality result in classical dynamic system [1]. More precisely, Kim proved

\[
\det \left\{ \frac{\partial X(\tau; s, X(s; t, x, v), u)}{\partial u} \right\} \neq 0, \tag{1.12}
\]

for almost every \( (\tau, s, u) \in (0, s) \times (0, t) \times \mathbb{R}^3 \) for all \( X(\tau; s, X(s; t, x, v), u) \in \mathbb{T}^3 \). We also apply the key estimate (1.12) in our present paper.

We remark that in those works in the perturbation framework, the initial data are required to have small amplitude around the Maxwellian due to the difficulty coming from the nonlinear term. Recently, Duan-Huang-Wang-Yang [7] developed a new \( L^\infty \cap L^1 \) method to obtain the global well-posedness of Boltzmann equation in \( \mathbb{T}^3 \) or \( \mathbb{R}^3 \) for a class of initial data with large amplitude oscillations. Our aim in the paper is to extend the result of [21] to a class of initial
data with large amplitude oscillations in $L_{x,v}^\infty$. Due to the appearance of large external potential, even though it is still in periodic box $\mathbb{T}^3$, one is hard to prove the existence of a global solution to the Boltzmann equation by using the idea of [7]. In fact, one of the main ideas in [7] is to bound the $L_{x,v}^\infty$ norm of the solution $f(t,x,v)$ after the local existence time $t_1(\approx \frac{1}{T - A_0})$ which depends only on the initial $L^\infty$ bound $A_0$, i.e.,

$$\int_{\mathbb{R}^3} |f(t,x,v)| \, dv \leq \int_{\mathbb{R}^3} |f_0(x - vt,v)| \, dv + \cdots \leq \frac{1}{t_1^2} \|f_0\|_{L^1_{x,v}} + \cdots, \quad \forall \ t \geq t_1, \ x \in \mathbb{T}^3, \quad (1.13)$$

where $\|f_0\|_{L^1_{x,v}}$ is demanded to be sufficiently small. Generally, it is difficult to carry the same procedure in [7] since the external force could significantly bend the characteristic lines, then one cannot use the change of variable as in (1.13) at all time $t \geq t_1 > 0$. Recently, Duan-Huang-Wang-Zhang [8] studied the global existence of solution to the Boltzmann equation in bounded domain for a class of initial data with large amplitude oscillations in $L_{x,v}^\infty$, where the similar problem also occurs due to the complex reflections of particles at physical boundary, and they overcame this problem basing on a new Gronwall-type argument. We point out that the perturbation of [8] is made around a non-constant equilibrium (neither global Maxwellian nor local Maxwellian) which is induced by a given boundary temperature with small oscillations around constant temperature, and it is a big open problem to show the existence and dynamical stability of non-constant equilibriums for large-oscillating boundary temperature. In the current paper, the situation is quite different, as the non-constant equilibrium state induced by the stationary potential of large-amplitude naturally exists, and the perturbation approach developed in [8, 11] is still applicable. Motivated by [11, 8], we shall use the Gronwall-type argument to investigate the existence of global solution to the Boltzmann equation with large external potential for a class of initial data with large oscillations.

For later use, we define the weight function

$$w_\beta(x,v) = \left( \frac{|v|^2}{2} + \Phi(x) + 1 \right)^{\frac{\beta}{2}} \approx \left( \frac{|v|^2}{2} + 1 \right)^{\frac{\beta}{2}}.$$

Our main result is

**Theorem 1.1.** Let $0 \leq \gamma \leq 1$, $\beta \geq 4$, and $\Phi$ be a periodic $C^3(\mathbb{T}^3)$ function with (1.3). Assume $F_0(x,v) = \mu_E(x,v) + \sqrt{\mu_E(x,v)}f_0(x,v) \geq 0$, the conservations of mass (1.7), energy (1.8) and momentum for degenerate $\{v_1, \cdots, v_{n_0}\}$ (1.11) are valid for initial data $F_0$ with

$$(M_0, J_0, E_0) = (0, 0, 0) \in \mathbb{R} \times \mathbb{R}^{n_0} \times \mathbb{R}.$$

For any given $A_0 \geq 1$, there exists $\kappa_0 > 0$, depending only on $\beta, M$ and $A_0$, such that if

$$\|w_\beta f_0\|_{L^\infty} \leq A_0 \quad \text{and} \quad \|f_0\|_{L^2} \leq \kappa_0,$$

the Boltzmann equation (1.1) admits a unique global solution $F(t,x,v) = \mu_E(x,v) + \sqrt{\mu_E(x,v)}f(t,x,v) \geq 0$ satisfying

$$\|w_\beta f(t)\|_{L^\infty} \leq \tilde{C}_0 A_0^2 \exp \left\{ \frac{\tilde{C}_0}{\tilde{\nu}_0} A_0^2 \right\} e^{-\tilde{\lambda} t},$$

for all $t \geq 0$, where $\tilde{C}_0, \tilde{\nu}_0$ and $\tilde{\lambda}$ are some positive constants. In addition, the conservations of mass (1.7), energy (1.8), and momentum (1.11) as well as the additional entropy inequality (1.9) hold.

The paper is organized as follows. In section 2, we present some results which will be used later. The proof of main theorem 1.1 is given in section 3. We prove the local existence of solution to Boltzmann equation with large external potential in the appendix.
Notations: Throughout the paper, $C$ denotes a positive constant depending only on $\beta, \gamma, M$ which may vary from line to line. $C_{a,b}$ denotes a positive constant if it further depends on $a, b$. For convenience, we denote $\| \cdot \|_{L^p(\mathbb{T}^3 \times \mathbb{R}^3)}$ by $\| \cdot \|_{L^p}$ for the rest of the paper.

2. Preliminaries

2.1. Backward characteristics. Given $(t, x, v) \in (0, +\infty) \times \mathbb{T}^3 \times \mathbb{R}^3$, let $[X(s), V(s)]$ be the backward bi-characteristics for the Boltzmann equation (1.1), which is determined by the following ODEs

$$\begin{cases}
\frac{dX}{ds} = V(s), \\
\frac{dV}{ds} = -\nabla \Phi(X(s)), \\
X(t) = x, V(t) = v.
\end{cases}$$  (2.1)

It is noted that there exists a Hamiltonian to the system (2.1) given by

$$H(x, v) = \frac{|v|^2}{2} + \Phi(x).$$

One can directly check out that the Hamiltonian $H(x, v)$ is preserved along the characteristics, i.e,

$$\frac{|V(s)|^2}{2} + \Phi(X(s)) = \frac{|v|^2}{2} + \Phi(x),$$

which, together with (1.3), yields that

$$|v|^2 - 2M \leq |V(s)|^2 = |v|^2 + 2\Phi(x) - 2\Phi(X(s)) \leq |v|^2 + 2M.$$  (2.2)

With the above characteristic line, one can express the mild solution of (1.4) as

$$f(t, x, v) = e^{-\int_0^t g(\tau) d\tau} f_0(X(0), V(0)) + \int_0^t e^{-\int_s^t g(\tau) d\tau - \Phi(X(s))} Kf(s, X(s), V(s)) ds$$

$$+ \int_0^t e^{-\int_s^t g(\tau) d\tau - \frac{\Phi(X(s))}{2}} \Gamma(f, f)(s, X(s), V(s)) ds,$$  (2.3)

where $g$ is defined as

$$g(\tau) = e^{-\Phi(X(\tau))} \nu(V(\tau)).$$  (2.4)

By using the almost transversality theorem of [1], C. Kim [21] proved the following useful lemma for Boltzmann equation with steady large external potential.

Lemma 2.1 (Kim [21]). Assume $\Phi$ is a periodic $C^3$-function on $\mathbb{T}^3$. Fix $\varepsilon > 0$, $t_0 > 0$ and $N > 0$. There are disjoint open interval partitions of the time interval $[0, t_0] : \mathcal{D}_{i1}^1 \subset [0, t_0]$ for $i^1 \in \{1, 2, \ldots, M_1\}$ and disjoint open box partitions of $[-N, N]^3 : \mathcal{D}_{i2}^2 \subset [-N, N]^3$ for $I^2 = (i_2^1, i_2^2, i_2^3) \in \{1, 2, \ldots, M_2\}^3$ and disjoint open box partitions of $[-N, N]^3 : \mathcal{D}_{i3}^3 \subset [-N, N]^3$ for $I^3 = (i_3^1, i_3^2, i_3^3) \in \{1, 2, \ldots, M_3\}^3$. For each $i^1, I^2, I^3$ we have $t_{j,i^1,i^2,i^3} \in \mathcal{D}_{i1}^1$ for $j = 1, 2, 3$ so that

$$\left\{ s \in \mathcal{D}_{i1}^1 : \det \left( \frac{\partial X}{\partial v} \right)(s; t_0, x, v) = 0 \right\} \subset \bigcup_{j=1}^3 \left( t_{j,i^1,i^2,i^3} - \frac{\varepsilon}{4M_1}, t_{j,i^1,i^2,i^3} + \frac{\varepsilon}{4M_1} \right),$$

for all $(x, v) \in \mathcal{D}_{i2}^2 \times \mathcal{D}_{i3}^3$. Moreover, there exists a positive constant $\delta_\ast = \delta_\ast(\varepsilon, M_1, M_2, M_3, N, t_0) > 0$ such that

$$\left| \det \left( \frac{\partial X}{\partial v} \right)(s; t_0, x, v) \right| > \delta_\ast, \quad \forall s \notin \bigcup_{j=1}^3 \left( t_{j,i^1,i^2,i^3} - \frac{\varepsilon}{4M_1}, t_{j,i^1,i^2,i^3} + \frac{\varepsilon}{4M_1} \right),$$  (2.5)

if $(s, x, v) \in \mathcal{D}_{i1}^1 \times \mathcal{D}_{i2}^2 \times \mathcal{D}_{i3}^3$ for all $i^1, I^2, I^3$. 
When the initial data is a small perturbation around $\mu_E$ in weighted $L^\infty_{x,v}$, Kim [21] proved the global existence and large time behavior of Boltzmann solution. For later use, we introduce his main result in the following:

**Theorem 2.2** (Kim [21]). Let $\beta > 3$, $\Phi$ be a periodic $C^3$-function on $\mathbb{T}^3$ and $\Phi(x) = \Phi(x_{n_0+1}, \ldots, x_3)$ for some $n_0 \leq 3$. Assume that the conservations of mass (1.7), energy (1.8) and momentum for degenerate $\{v_1, \ldots, v_{n_0}\}$ (1.11) are valid for initial data $F_0 = \mu_E + \sqrt{\mu_E} f_0$ with

$$(M_0, J_0, E_0) = (0, 0, 0) \in \mathbb{R} \times \mathbb{R}^{n_0} \times \mathbb{R},$$

then there exist $\lambda_0 > 0$ and small $\delta > 0$ such that if $\|w_\beta f_0\|_{L^\infty} \leq \delta$, there exists a unique global solution $F(t, x, v) = \mu_E + \sqrt{\mu_E} f(t, x, v) \geq 0$ for the Boltzmann equation (1.1) with

$$\sup_{0 \leq t < +\infty} \{e^{\lambda_0 t}\|w_\beta f(t)\|_{L^\infty}\} \leq C_0\|w_\beta f_0\|_{L^\infty},$$

where $C_0$ is some positive constant depending only on $M$.

2.2. **Useful inequalities.** Recall $K = K_2 - K_1$ in (1.5) and (1.6). Let $k(\cdot, \cdot)$ be the kernel of $K$, i.e.,

$$K f(v) = \int_{\mathbb{R}^3} k(v, u) f(u) \, du,$$

with the symmetric property $k(v, u) = k(u, v).

**Lemma 2.3** ([14, 17]). For $0 \leq \gamma \leq 1$, one has

$$|k(v, u)| \leq C \{\|v - u\| + \|v - u\|^{-1}\} e^{-\frac{|v-u|^2}{8} e^{-\frac{|v-u|^2}{8\gamma}}}. \tag{2.6}$$

By the same calculations as in [17], it is straightforward to check that for $\alpha \geq 0$ and $\theta \in [0, \frac{1}{8})$, the following inequality holds for some positive constant $C_\alpha$

$$\int_{\mathbb{R}^3} \left| k(v, u) \cdot \frac{(1 + |v|^{\alpha} e^{\theta|v|})^2}{(1 + |u|^{\alpha} e^{\theta|u|})^2} \right| \, du \leq C_\alpha (1 + |v|)^{-1}. \tag{2.7}$$

**Lemma 2.4** (Duan-Wang [11]). For $0 \leq \gamma \leq 1$, there is a generic constant $C_\alpha > 0$ such that

$$|(1 + |v|^{\alpha}) \Gamma_+(f, f)(v)| \leq \frac{C_\alpha\|(1 + |\cdot|^{\alpha}) f\|_{L^\infty}}{1 + |v|} \left( \frac{\int_{\mathbb{R}^3} (1 + |u|)^4 |f(u)|^2 \, du}{\int_{\mathbb{R}^3} (1 + |u|)^4 |f(u)|^2 \, du} \right)^{\frac{1}{2}}, \tag{2.8}$$

for all $v \in \mathbb{R}^3$. In particular, for $\alpha \geq 4$, one has

$$|(1 + |v|^{\alpha}) \Gamma_+(f, f)(v)| \leq \frac{C_\alpha\|(1 + |\cdot|^{\alpha}) f\|_{L^\infty}^2}{1 + |v|}, \tag{2.9}$$

for all $v \in \mathbb{R}^3$.

**Lemma 2.5** ([11, 28]). Assume $0 \leq \gamma \leq 1$, then it holds that

$$\|\Gamma(f, f)\|_{L^2_v} \leq C \|f\|_{L^2_v} \|\nu f\|_{L^2_v}, \tag{2.10}$$

where the positive constant $C$ depends only on $\gamma$. 
3. Global Stability

In this section, we consider the global stability of Boltzmann equation (1.1). By using the local existence of unique solution to (1.1) with arbitrarily large initial data established in Theorem 4.1, it suffices to obtain uniform estimates on solutions.

Recall that the initial data satisfies \( \|w_{\beta}f_0\|_{L^\infty} \leq A_0 \) for a given positive constant \( A_0 \geq 1 \) which could be large. Let \( f(t, x, v) \) be the solution to the Boltzmann equation (1.4) with initial data \( F_0 = \mu_E + \sqrt{\mu_E}f_0 \geq 0 \) over the time interval \( [0, T) \) for \( T \in (0, +\infty) \). Throughout this section, we make the a priori assumption:

\[
\sup_{0 \leq t < T} \|h(t)\|_{L^\infty} \leq A_1, \tag{3.1}
\]

with \( h(t, x, v) := w_{\beta}(x, v)f(t, x, v) \), where \( A_1 \geq 1 \) is a large positive constant depending only on \( A_0 \), and will be determined later in section 3.4.

It follows from (1.4) that

\[
\frac{\partial h}{\partial t} + v \cdot \nabla_x h - \nabla\Phi \cdot \nabla_v h + e^{-\Phi} \nu(v)h = e^{-\Phi}K_{\beta}h + e^{-\Phi}2\Gamma_{\beta}(h, h), \tag{3.2}
\]

where the weighted operator \( K_{\beta}, \Gamma_{\beta} \) are defined by

\[
K_{\beta}h = w_{\beta}K\left(\frac{h}{w_{\beta}}\right) \quad \text{and} \quad \Gamma_{\beta}(h, h) = w_{\beta}\Gamma\left(\frac{h}{w_{\beta}}, \frac{h}{w_{\beta}}\right).
\]

Integrating along the backward trajectory defined in (2.1), one obtains the mild formula for \( h(t, x, v) \):

\[
h(t, x, v) = e^{-\int_{t}^{0}g(\tau)d\tau}h_0(X(0), V(0)) + \int_{0}^{t} e^{-\int_{s}^{t}g(\tau)d\tau} - \Phi(X(s))K_{\beta}h(s, X(s), V(s))ds,
\]

\[
+ \int_{0}^{t} e^{-\int_{s}^{t}g(\tau)d\tau} - \Phi(X(s))2\Gamma_{\beta}(h, h)(s, X(s), V(s))ds,
\]

where \( g(\tau) \) is defined in (2.4).

3.1. \( L^2 \) a priori estimate.

Lemma 3.1. Let \( f(t, x, v) \) be the solution to Boltzmann equation (1.4), then there exists a generic positive constant \( \tilde{C}_1 \geq 1 \) such that

\[
\|f(t)\|_{L^2} \leq \tilde{C}_1\|f_0\|_{L^2}e^{\tilde{C}_1A_1t}, \quad \forall t \in [0, T]. \tag{3.3}
\]

Proof. Applying the standard energy estimate to (1.4), we deduce

\[
\frac{d}{dt}\|f(t)\|_{L^2}^2 + 2(e^{-\Phi}f(t), Lf(t)) \leq 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\Phi}f(t, x, v)\Gamma(f, f)(t, x, v)dv dx.
\]

Notice that \( (e^{-\Phi}f(t), Lf(t)) \geq 0 \), then we deduce from Lemma 2.5 and the a priori assumption (3.1) that

\[
\frac{d}{dt}\|f(t)\|_{L^2}^2 \leq C\|\nu f(t)\|_{L^2}\|f(t)\|_{L^2}^2 \leq CA_1\|f(t)\|_{L^2}^2.
\]

Therefore, (3.3) follows from the Gronwall’s inequality, and the proof of Lemma 3.1 is completed. \( \square \)
3.2. $L^\infty_x L^1_v$ estimate. To consider the Boltzmann equation with a class of large amplitude initial data, motivated by [11], it is better to rewrite the Boltzmann equation (3.2) as

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h - \nabla \Phi \cdot \nabla_v h + R(F)h = e^{-\Phi}K_\beta(h) + e^{-\frac{\Phi}{2}}w_\beta \Gamma_+ \left( \sqrt{\mu h}, \sqrt{\nu h} \right), \quad (3.4)$$

where

$$R(F) := \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \omega) F(t, x, u) \, d\omega \, du = \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \omega)[\mu_E(x, u) + \sqrt{\mu_E(x, u)}f(t, x, u)] \, d\omega \, du. \quad (3.5)$$

It follows from the fact $F(t, x, v) \geq 0$ that

$$I(t, s) := \exp \left\{ -\int_s^t R(F(\tau, X(\tau), V(\tau)) \, d\tau \right\} \leq 1.$$

In order to extend the local-in-time solution to a global one with possible large data in $L^\infty_x \times L^1_v$, it is necessary to further obtain the time-decay property of $I(t, s)$, namely,

$$I(t, s) \leq Ce^{-\frac{1}{2}(t-s)}, \quad (3.6)$$

for some generic large positive constant $C$. For solution of large amplitude, it is impossible to obtain the above time-decay property initially since the vacuum could not be excluded. However, if the $L^2$ norm of initial data is small, such time-decay property may still hold in some sense even if the initial data is allowed to have large oscillations.

To prove (3.6), one needs to recover a positive lower bound for $R(F(t))$. Indeed, we aim to obtain a positive lower bound for $R(F(t))$ for time $t$ suitably large. In fact, we have the following lemma.

**Lemma 3.2.** Under the a priori assumption (3.1), there exists a constant $\tilde{C}_2 \geq 1$ such that for any given positive time $T_1 > \tilde{t}$ with

$$\tilde{t} := \frac{2e^M}{\nu_0} \ln(\tilde{C}_2 e^{\frac{M}{2}} A_0), \quad (3.7)$$

there is a small positive constant $\kappa_1 = \kappa_1(A_1, T_1) > 0$, depending only on $A_1$ and $T_1$, such that if $\|f_0\|_{L^2} \leq \kappa_1$, one has

$$R(F(t)) \geq \frac{1}{2}e^{-\Phi} \nu(v), \quad (3.8)$$

for all $t \in [\tilde{t}, \min\{T, T_1\}]$, where $\kappa_1$ decreases in $A_1$ and $T_1$.

**Proof.** To obtain (3.8), it is noted that

$$\int_{\mathbb{R}^3} \int_{S^2} B(v - u, \omega)\sqrt{\mu(u)}f(t, x, u) \, du \leq C_1 \nu(v) \int_{\mathbb{R}^3} e^{-\frac{|u|^2}{8}} |f(t, x, u)| \, du,$$

where $C_1 \geq 1$ is some generic constant. Hence, by using (3.5), it suffices to prove

$$\int_{\mathbb{R}^3} e^{-\frac{|u|^2}{8}} |f(t, x, v)| \, dv \leq \frac{1}{2C_1} e^{-\frac{M}{2}}. \quad (3.9)$$
It follows from (2.3) that
\[
\int_{\mathbb{R}^3} e^{-\frac{|v|^2}{8}} |f(t, x, v)| \, dv \\
\leq \int_{\mathbb{R}^3} e^{-\frac{|v|^2}{8}} e^{-\int_0^t g(\tau) \, d\tau} |f_0(X(0), V(0))| \, dv \\
+ \int_{\mathbb{R}^3} e^{-\frac{|v|^2}{8}} \int_0^t e^{-\int_\tau^t g(\tau) \, d\tau} |Kf(s, X(s), V(s))| \, ds \, dv \\
+ \int_{\mathbb{R}^3} e^{-\frac{|v|^2}{8}} \int_0^t e^{-\int_\tau^t g(\tau) \, d\tau - \Phi(X(s))} |\Gamma(f, f)(s, X(s), V(s))| \, ds \, dv \\
:= I_1 + I_2 + I_3.
\]

Recall \( \nu_0 = \inf_{v \in \mathbb{R}^3} \nu(v) > 0 \) and note that
\[
e^{-\int_0^t g(\tau) \, d\tau} \leq \exp \left\{ -e^{-\nu_0(t-s)} \right\},
\]
it is easy to obtain that
\[
I_1 \leq C \|f_0\|_{L^\infty} \exp \left\{ -e^{-\nu_0 t} \right\}. \tag{3.11}
\]

For \( I_2 \), we split it into three parts.
\[
I_2 \leq \int_0^t \int_{|v| \geq N} e^{-\frac{|v|^2}{8}} \exp \left\{ -e^{-\nu_0(t-s)} \right\} |Kf(s, X(s), V(s))| \, dv \, ds \\
+ \int_0^t \int_{|v| < N} e^{-\frac{|v|^2}{8}} \exp \left\{ -e^{-\nu_0(t-s)} \right\} \int_{|u| \geq 3N} |k(V(s), u)f(s, X(s), u)| \, du \, dv \, ds \\
+ \int_0^t \int_{|v| < N} e^{-\frac{|v|^2}{8}} \exp \left\{ -e^{-\nu_0(t-s)} \right\} \int_{|u| < 3N} |k(V(s), u)f(s, X(s), u)| \, du \, dv \, ds \\
:= I_{21} + I_{22} + I_{23}.
\]

Using (2.7), one obtain that
\[
I_{21} \leq Ce^{-\frac{\nu_0^2}{32}} \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty}. \tag{3.13}
\]

It follows from (2.2) that
\[
|V(s) - u| \geq |u| - |V(s)| \geq N, \quad \text{if} \quad |v| \leq N, |u| \geq 3N, N \gg M,
\]
which yields that
\[
I_{22} \leq Ce^{-\frac{\nu_0^2}{32}} \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty} \int_0^t \int_{|v| < N} e^{-\frac{|v|^2}{8}} \exp \left\{ -e^{-\nu_0(t-s)} \right\} \\
\times \int_{|u| \geq 3N} |k(V(s), u)| e^{-\frac{|V(s) - u|^2}{32}} \, du \, dv \, ds \\
\leq Ce^{-\frac{\nu_0^2}{32}} \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty}, \tag{3.14}
\]
where we have used the following fact
\[
\int_{\mathbb{R}^3} |k(V(s), u)| e^{-\frac{|V(s) - u|^2}{32}} \, du \leq \frac{C}{1 + |V(s)|} \leq \frac{C}{1 + |v|}.
\]
To estimate $I_{23}$, it follows from (2.6) and the Hölder’s inequality that

$$I_{23} \leq C \int_0^t \exp\{-e^{-M} \nu_0(t-s)\} \left\{ \int_{|v|<N} \int_{|u|<3N} e^{-\frac{|u|^2}{8} - \frac{|v(u) - u|^2}{8}} ds \times |f(s, X(T_1 - t + s; T_1, x, v), u)|^2 du dv \right\}^{\frac{1}{2}} ds,$$

(3.15)

where we have used the fact

$$X(s) = X(s; t, x, v) = X(T_1 - t + s; T_1, x, v),$$

(3.16)

since the external potential $\Phi$ is time-independent. We apply Lemma 2.1 for the case $t_0 = T_1$ to deal with the term on the right hand side of (3.15). Assume $x \in \mathcal{D}^2_{I_2}$ for some $I_2 \in \{1, 2, \ldots, M_2\}^3$, then it follows from Lemma 2.1 that

$$I_{23} \leq C \sum_{i_1=1}^{M_1} \sum_{I_3 \in \{1, \ldots, M_3\}^3} \int_0^t \exp\{-e^{-M} \nu_0(t-s)\} 1_{\mathcal{D}_{I_3}^1}(T_1 - t + s) \times \left\{ \int_{v \in \mathcal{D}_{I_3}^3} \int_{|u| \leq 3N} \cdot \cdot \cdot du \right\}^{\frac{1}{2}} ds$$

$$= C \sum_{i_1=1}^{M_1} \sum_{j=1}^{3} \sum_{I_3} \int_0^t 1_{\mathcal{D}_{i_1}^1 \cap (t_j, i_1, t_j, i_3 - \frac{\varepsilon}{4M_1} t_j, i_1, t_j, i_3 + \frac{\varepsilon}{4M_1})} (T_1 - t + s) \times \exp\{-e^{-M} \nu_0(t-s)\} \cdot \left\{ \int_{|v| \leq N} \int_{|u| \leq 3N} 1_{\mathcal{D}_{I_3}^3}(v) \cdot \cdot \cdot du \right\}^{\frac{1}{2}} ds$$

$$+ C \sum_{i_1=1}^{M_1} \sum_{j=1}^{3} \sum_{I_3} \int_0^t 1_{\mathcal{D}_{i_1}^1 \setminus (t_j, i_1, t_j, i_3 - \frac{\varepsilon}{4M_1} t_j, i_1, t_j, i_3 + \frac{\varepsilon}{4M_1})} (T_1 - t + s) \times \exp\{-e^{-M} \nu_0(t-s)\} \cdot \left\{ \int_{|v| \leq N} \int_{|u| \leq 3N} 1_{\mathcal{D}_{I_3}^3}(v) \cdot \cdot \cdot du \right\}^{\frac{1}{2}} ds.$$ (3.17)

A direct calculation shows that the first term on the RHS of (3.17) is bounded by

$$C \sum_{i_1=1}^{M_1} \sum_{j=1}^{3} \frac{\varepsilon}{2M_1} \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty} \leq C \varepsilon \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty},$$

(3.18)

To estimate the second term on the RHS of (3.17), we consider the following change of variables

$$v \rightarrow X(T_1 - t + s; T_1, x, v).$$

(3.19)

It follows from Lemma 2.1 that

$$\frac{\partial X(T_1 - t + s; T_1, x, v)}{\partial v} \geq \delta_\varepsilon(\varepsilon, M_1, M_2, M_3, N, T_1) > 0,$$

(3.20)

for $T_1 - t + s \in \mathcal{D}_{i_1}^1 \setminus (t_j, i_1, t_j, i_3 - \frac{\varepsilon}{4M_1} t_j, i_1, t_j, i_3 + \frac{\varepsilon}{4M_1})$ with $j = 1, 2, 3$ and $i_1 = 1, \ldots, M_1$. Applying the change of variables (3.19) and using (3.20), the second term on the RHS of (3.17) is bounded as

$$C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t \exp\{-e^{-M} \nu_0(t-s)\}\|f(s)\|_{L^2} ds,$$

(3.21)
which, together with (3.18), (3.17), (3.14), (3.13) and (3.12), yields that

\[ I_2 \leq C \left( \varepsilon + e^{-\frac{N^2}{32}} \right) \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty} \]

\[ + C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t \exp\{-e^{-M_1 \nu_0(t-s)}\} \|f(s)\|_{L^2} \, ds. \]  

(3.22)

For \( I_3 \), we split it into several parts

\[ I_3 \leq \int_0^t \int_{|v| \geq N} e^{-\int_0^t g(\tau) \, d\tau - \frac{\Psi(X(s))}{2}} e^{-\frac{|v|^2}{4}} |\Gamma(f, f)(s, X(s), V(s))| \, dv \, ds \]

\[ + \int_0^t \int_{|v| < N} e^{-\int_0^t g(\tau) \, d\tau - \frac{\Psi(X(s))}{2}} e^{-\frac{|v|^2}{8}} \int_{|u| \geq 3N} (\Lambda_+ + \Lambda_-)(s, t, x, v, u) \, du \, dv \, ds \]

\[ + \int_0^t \int_{|v| < N} \exp\{-e^{-M_1 \nu_0(t-s)}\} e^{-\frac{|v|^2}{8}} \int_{|u| < 3N} \Lambda_- (s, t, x, v, u) \, du \, dv \, ds \]

\[ + \int_0^t \int_{|v| < N} \exp\{-e^{-M_1 \nu_0(t-s)}\} e^{-\frac{|v|^2}{8}} \int_{|u| < 3N} \Lambda_+ (s, t, x, v, u) \, du \, dv \, ds \]

\[ := I_{31} + I_{32} + I_{33} + I_{34}, \]

where

\[ \Lambda_+ (s, t, x, v, u) = \int_{\mathbb{S}^2} B(V(s) - u, \omega) e^{-\frac{|v|^2}{4}} |f(s, X(s), u') f(s, X(s), v')| \, d\omega, \]

\[ \Lambda_- (s, t, x, v, u) = \int_{\mathbb{S}^2} B(V(s) - u, \omega) e^{-\frac{|v|^2}{4}} |f(s, X(s), u) f(s, X(s), v)| \, d\omega, \]

with \( u' = u - [(u - V(s)) \cdot \omega] \omega \) and \( v' = V(s) + [(u - V(s)) \cdot \omega] \omega \).

Noting the fact \( \nu(V(s)) \equiv \nu(v) \), we have that

\[ |\Gamma(f, f)(s, X(s), V(s))| \leq C\nu(V(s)) \|f(s)\|^2_{L^\infty} \leq C\nu(v) \|f(s)\|^2_{L^\infty}, \]  

(3.24)

and

\[ \int_0^t e^{-\int_0^t g(\tau) \, d\tau - \frac{\Psi(X(s))}{2}} \nu(V(s)) \, ds \leq C. \]  

(3.25)

Now by using (3.24) and (3.25), it holds that

\[ I_{31} \leq C \sup_{0 \leq s \leq t} \|f(s)\|^2_{L^\infty} \int_{|v| \geq N} e^{-\frac{|v|^2}{4}} \, dv \int_0^t e^{-\int_0^t g(\tau) \, d\tau - \frac{\Psi(X(s))}{2}} \nu(V(s)) \, ds \]

\[ \leq Ce^{-\frac{N^2}{12}} \sup_{0 \leq s \leq t} \|f(s)\|^2_{L^\infty}, \]  

(3.26)

and

\[ I_{32} \leq C \sup_{0 \leq s \leq t} \|f(s)\|^2_{L^\infty} \int_0^t \int_{|v| < N} e^{-\int_0^t g(\tau) \, d\tau - \frac{\Psi(X(s))}{2}} e^{-\frac{|v|^2}{8}} \int_{|u| \geq 3N} |V(s) - u|^7 e^{-\frac{|v|^2}{4}} \, du \, dv \, ds \]

\[ \leq Ce^{-\frac{N^2}{9}} \sup_{0 \leq s \leq t} \|f(s)\|^2_{L^\infty} \int_{|v| < N} e^{-\frac{|v|^2}{4}} \int_0^t e^{-\int_0^t g(\tau) \, d\tau - \frac{\Psi(X(s))}{2}} \nu(V(s)) \, ds \, dv \]

\[ \leq Ce^{-\frac{N^2}{9}} \sup_{0 \leq s \leq t} \|f(s)\|^2_{L^\infty}. \]  

(3.27)
For $I_{33}$, it follows from the Holder’s inequality and similar arguments as in (3.15)-(3.21) that
\[
I_{33} \leq C \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty} \int_0^t \exp\{-e^{-M}\nu_0(t-s)\}
\times \left( \int_{|u|<N} \int_{|u|<3N} e^{-\frac{|u|^2}{\nu_0}} |f(s, X(T_1 - t + s; T_1, x, v), u)|^2 \, du \, dv \right)^{\frac{1}{2}}
\leq C \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty} \left\{ \varepsilon \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty}
+ C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t \exp\{-e^{-M}\nu_0(t-s)\} \|f(s)\|_{L^2} \, ds \right\}
\leq C\varepsilon \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty}^2
+ C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t \exp\{-e^{-M}\nu_0(t-s)\} \|f(s)\|_{L^2}^2 \, ds.
\]

For $I_{34}$, using the standard change of variables, one has
\[
e^{-\frac{|u|^2}{\nu_0}} \int_{|u|<3N} A_+(s, t, x, v, u) \, du \leq C \|f(s)\|_{L^\infty} \left( \int_{|\eta| \leq 4N} |f(s, X(s, \eta))|^2 \, d\eta \right)^{\frac{1}{2}},
\]
which, together with similar arguments as in (3.15)-(3.21), yields that
\[
I_{34} \leq C \left( \varepsilon + e^{-\frac{\nu_0^2}{M^2}} \right) \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty}^2
+ C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t \exp\{-e^{-M}\nu_0(t-s)\} \|f(s)\|_{L^2}^2 \, ds.
\]

Plug (3.26), (3.27), (3.28) and (3.29) into (3.23) to deduce
\[
I_3 \leq C \left( \varepsilon + e^{-\frac{\nu_0^2}{M^2}} \right) \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty}^2
+ C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t \exp\{-e^{-M}\nu_0(t-s)\} \|f(s)\|_{L^2}^2 \, ds.
\]

Substituting (3.11), (3.22) and (3.30) into (3.10), we obtain
\[
\int_{\mathbb{R}^3} e^{-\frac{|u|^2}{\nu_0}} |f(t, x, v)| \, du
\leq C_2 \|f_0\|_{L^\infty} \exp\{-e^{-M}\nu_0 t\} + C_2 \left( \varepsilon + e^{-\frac{\nu_0^2}{M^2}} \right) \left\{ \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty} + \sup_{0 \leq s \leq t} \|f(s)\|_{L^\infty}^2 \right\}
\]
\[
+ C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t \exp\{-e^{-M}\nu_0(t-s)\} \left[ \|f(s)\|_{L^2} + \|f(s)\|_{L^2}^2 \right] \, ds,
\]
where $C_2 \geq 1$ is a generic positive constant. Take
\[
\tilde{t} := \frac{2e^M}{\nu_0} \ln(4C_1C_2e^{\frac{M}{2}}A_0),
\]
then the first term of (3.31) is bounded by
\[
C_2 \|f_0\|_{L^\infty} \exp\{-e^{-M}\nu_0 t\} \leq \frac{1}{4C_1} e^{-\frac{M}{2}}, \quad \forall \, t \geq \tilde{t}.
\]
On the other hand, by using \((3.1)\) and \((3.3)\), the remaining term of \((3.31)\) is bounded as
\[
C_2 \left( \varepsilon + e^{-N^2/2} + C(\varepsilon, M_1, M_2, M_3, N, T_1) e^{2\tilde{C}_1 A_1 T_1} \right) \left\{ \|f_0\|_{L^2} + \|f_0\|_{L^2}^2 \right\},
\]
for \(t \in [0, T_1]\). We first take \(\varepsilon > 0\) small enough, then \(N \geq 1\) suitably large, and finally take initial data \(f_0\) such that \(\|f_0\|_{L^2} \leq \kappa_1\) with \(\kappa_1 = \kappa_1(A_1, T_1) > 0\) sufficiently small, so that \((3.33)\) is bounded by
\[
C_2 \left( \varepsilon + e^{-N^2/2} \right) \left\{ A_1 + A_1^2 \right\} + C(\varepsilon, M_1, M_2, M_3, N, T_1) e^{2\tilde{C}_1 A_1 T_1} \left\{ \kappa_1 + \kappa_1^2 \right\} \leq \frac{1}{4C_1} e^{-\frac{M}{2}}. \tag{3.34}
\]
Notice that \(\kappa_1(A_1, T_1)\) can be decreasing in \(A_1\) and \(T_1\). Combining \((3.31)\), \((3.32)\), \((3.33)\) and \((3.34)\), one obtains that
\[
\int_{\mathbb{R}^3} e^{-\frac{|t|^2}{8}} |f(t, x, v)| \, dv \leq \frac{1}{2C_1} e^{-\frac{M}{2}},
\]
which yields \((3.9)\). Set \(\tilde{C}_2 := 4C_1 C_2\), then the proof of Lemma \(3.2\) is completed. \(\square\)

### 3.3. \(L^\infty\)-estimate

In this subsection, we focus on the \(L^\infty\) \textit{a priori} estimate for \(h\). Using \((3.4)\), we rewrite the mild form of Boltzmann equation as
\[
h(t, x, v) = e^{-\int_0^t \tilde{g}(\tau) \, d\tau} h_0(X(0), V(0)) + \int_0^t \exp \left\{ -\int_s^t \tilde{g}(\tau) \, d\tau - \Phi(X(s)) \right\} K h(s, X(s), V(s)) \, ds \tag{3.35}
\]
\[
+ \int_0^t \exp \left\{ -\int_s^t \tilde{g}(\tau) \, d\tau - \frac{\Phi(X(s))}{2} \right\} w_{\beta} \Gamma \left( \frac{h}{w_{\beta}}, \frac{h}{w_{\beta}} \right)(s, X(s), V(s)) \, ds,
\]
where
\[
\tilde{g}(\tau) = R(F)(\tau, X(\tau), V(\tau)).
\]
To treat the nonlinear \(L^\infty\) estimate for the solution \(f(t, x, v)\) of Boltzmann equation \((1.4)\) with \(L^\infty\) large amplitude initial data, one needs to use the time-decay property, i.e., \(\tilde{g}(\tau)\) has positive lower bound. Even though \(\tilde{g}(0)\) may not have positive lower bound for all time, by using Lemma \(3.2\) one obtains that
\[
\exp \left\{ -\int_s^t \tilde{g}(\tau) \, d\tau \right\} \leq \begin{cases} 1, & 0 \leq s \leq t \leq \tilde{t}, \\ \exp \left\{ -\frac{1}{2} e^{-\tilde{M} \tilde{\nu}_0 (t - \tilde{\tau})} \right\}, & 0 \leq s \leq \tilde{\tau} \leq t \leq T_1, \\ \exp \left\{ -\frac{1}{2} e^{-M \tilde{\nu}_0 (t - s)} \right\}, & 0 \leq \tilde{\tau} \leq s \leq t \leq T_1, \\ \leq \exp \left\{ -\frac{1}{2} e^{-M \tilde{\nu}_0 (t - s)} \right\} \exp \left\{ \frac{1}{2} e^{-M \nu_0 \tilde{t}} \right\}, & 0 \leq s \leq t \leq T_1, \end{cases}
\]
where \(\tilde{t}\) is defined in \((3.7)\) and we denote \(\tilde{\nu}_0 := e^{-M \nu_0}\) in the last inequality for simplicity of presentation.
Lemma 3.3. Assume \( \|f_0\|_{L^2} \leq \kappa_1 = \kappa_1(A_1,T_1) \), then there exists a generic positive constant \( \bar{C}_3 \geq 1 \) such that
\[
\|h(t)\|_{L^\infty} \leq \bar{C}_3 A_0^2 \left[ 1 + \int_0^t \|h(s)\|_{L^\infty} \, ds \right] \exp \left\{ -\frac{1}{4} \delta_0 t \right\} \\
+ A_0 \left( \frac{\bar{C}_3}{\sqrt{N}} + C_N \cdot \varepsilon \right) \sup_{0 \leq s \leq t} \left\{ \|h(s)\|_{L^\infty} + \|h(s)\|_{L^2}^2 \right\} \\
+ A_0 C(\varepsilon, M_1, M_2, M_3, N, T_1) \sup_{0 \leq s \leq t} \left\{ \|f(s)\|_{L^2} + \|f(s)\|_{L^2}^2 \right\} ,
\]
for all \( t \in [0, \min\{T,T_1\}] \), where \( \varepsilon > 0 \) is some small parameter to be chosen later, and \( N > 0 \) is some large number determined later.

Proof. It follows from (3.35) and (3.36) that
\[
|h(t,x,v)| \leq C A_0^2 e^{-\frac{1}{2} \delta_0 t} + C A_0 \int_0^t e^{-\frac{1}{2} \delta_0 (t-s)} \int_{\mathbb{R}^3} |k_\beta(V(s),u)h(s,X(s),u)| \, du \, ds \\
+ C A_0 \int_0^t e^{-\frac{1}{2} \delta_0 (t-s)} \left| w_\beta \Gamma_+ \left( \frac{h}{w_\beta}, \frac{h}{w_\beta} \right) (s,X(s),V(s)) \right| \, ds
\]
(3.37)
where \( k_\beta(v,u) \) is bounded as
\[
|k_\beta(v,u)| \leq C |k(v,u)| \frac{(1 + |v|^2)^{\frac{3}{2}}}{(1 + |u|^2)^{\frac{1}{2}}}
\]

Now we estimate the terms on the RHS of (3.37). For \( J_1 \), similar as in [18], we use (3.37) again to get that
\[
J_1 \leq C A_0^2 \int_0^t e^{-\frac{1}{2} \delta_0 (t-s)} \int_{\mathbb{R}^3} |k_\beta(V(s),u)| e^{-\frac{1}{2} \delta_0 s} \, du \, ds \\
+ C A_0 \int_0^t \int_0^s e^{-\frac{1}{2} \delta_0 (t-r)} \int_{\mathbb{R}^3} \left| k_\beta(V(s),u)k_\beta(\bar{V}(\tau),\xi)h(\tau,\bar{X}(s),\xi) \right| \, d\xi \, d\tau \, ds \\
+ C A_0 \int_0^t \int_0^s e^{-\frac{1}{2} \delta_0 (t-r)} \int_{\mathbb{R}^3} \left| k_\beta(V(s),u) \right| \, \left| w_\beta \Gamma_+ \left( \frac{h}{w_\beta}, \frac{h}{w_\beta} \right) (\tau,\bar{X}(\tau),\bar{V}(\tau)) \right| \, d\tau \, ds \\
:= C A_0 \left( A_0 e^{-\frac{1}{2} \delta_0 t} + J_{11} + J_{12} \right),
\]
(3.38)
where we have denoted \( \bar{X}(\tau), \bar{V}(\tau) := [X(\tau; s, X(s), u), V(\tau; s, X(s), u)] \).

For \( J_{11} \), we divide it into the following several cases.
Case 1. For \( |v| \geq N \gg 1 \), it follows from (2.22) and (2.4) that
\[
J_{11} \leq \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} \int_0^t \int_0^s e^{-\frac{1}{2} \delta_0 (t-r)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left| k_\beta(V(s),u)k_\beta(\bar{V}(\tau),\xi) \right| \, d\xi \, d\tau \, ds \\
\leq \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} \cdot \sup_{0 \leq s \leq t} \frac{C}{1 + |V(s)|} \leq \frac{C N}{1 + |v|} \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty}
\]
(3.39)
\[
\leq \frac{C}{N} \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty}.
\]
Case 2. For either $|v| < N, |u| \geq 3N$ or $|u| < 3N, |\xi| \geq 5N$, it follows from (2.2) that we have either $|V(s) - u| \geq N, \forall s \in [0, t]$ or $|\tilde{V}(\tau) - \xi| \geq N, \forall \tau \in [0, s]$. Therefore, either of the following is valid

$$
|k_\beta(V(s), u)| \leq C e^{-\frac{N^2}{32} V(s) - \frac{|u|^2}{32}}, \quad \forall s \in [0, t],
$$

$$
|k_\beta(\tilde{V}(\tau), \xi)| \leq C e^{-\frac{N^2}{32} \tilde{V}(\tau) - \frac{|\xi|^2}{32}}, \quad \forall \tau \in [0, s].
$$

(3.40)

Hence, by using (2.6), a direct calculation shows that

$$
\int_{\mathbb{R}^3} k_\beta(V(s), u) e^{-\frac{|V(s)|^2}{32}} \left| \int_{\mathbb{R}^3} k_\beta(\tilde{V}(\tau), \xi) e^{-\frac{|\tilde{V}(\tau)|^2}{32}} d\xi \right| du \leq C (1 + |V(s)|)^{-1} \leq C (1 + |v|)^{-1}, \quad \forall s \in [0, t],
$$

$$
\int_{\mathbb{R}^3} |k_\beta(\tilde{V}(\tau), \xi)| e^{-\frac{|\tilde{V}(\tau)|^2}{32}} d\xi \leq C (1 + |\tilde{V}(\tau)|)^{-1} \leq C (1 + |u|)^{-1}, \quad \forall \tau \in [0, s].
$$

(3.41)

Now it follows from (3.40) and (3.41) that

$$
\int_0^t \int_0^s e^{-\frac{1}{2} \phi_0(t - \tau)} \left\{ \int_{|u| \geq 3N} \int_{|\xi| < 3N} \int_{|\xi| \geq 5N} \right\} \times \left| k_\beta(V(s), u) k_\beta(\tilde{V}(\tau), \xi) h(\tau, X(s), \xi) \right| d\xi d\tau ds 
\leq C e^{-\frac{N^2}{32} \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty}}.
$$

(3.42)

Case 3. $|v| < N, |u| < 3N, |\xi| < 5N$. This is the last remaining case. Recalling that $\Phi$ is independent of time, similar as in (3.16), it holds that

$$
\tilde{X}(\tau) = X(\tau; s, X(s), u) = X(T_1 - s + \tau; T_1, X(s), u).
$$

(3.43)

Using (2.6), it is easy to get that

$$
\int_{\mathbb{R}^3} |k_\beta(v, u)|^2 du \leq C \int_{\mathbb{R}^3} |k(v, u)|^2 \frac{(1 + |v|^2)^\beta}{(1 + |u|^2)^\beta} du \leq C (1 + |v|)^{-1},
$$

(3.44)

which, together with Hölder’s inequality and (3.43) yields that

$$
\int_0^t \int_0^s e^{-\frac{1}{2} \phi_0(t - \tau)} \int_{|u| < 3N} \int_{|\xi| < 5N} \left| k_\beta(V(s), u) k_\beta(\tilde{V}(\tau), \xi) h(\tau, X(s), \xi) \right| d\xi du d\tau ds
\leq \int_0^t \int_0^s e^{-\frac{1}{2} \phi_0(t - \tau)} \left\{ \int_{|u| < 3N} \int_{|\xi| < 5N} \left| k_\beta(V(s), u) k_\beta(\tilde{V}(\tau), \xi) \right|^2 d\xi du \right\} \frac{1}{2}
\times \left\{ \int_{|u| < 3N} \int_{|\xi| < 5N} \left| h(\tau, \tilde{X}(\tau), \xi) \right|^2 d\xi du \right\} \frac{1}{2} d\tau ds
\leq C \int_0^t \int_0^s e^{-\frac{1}{2} \phi_0(t - \tau)} \left\{ \int_{|u| < 3N} \int_{|\xi| < 5N} \left| h(\tau, X(T_1 - s + \tau; T_1, X(s), u), \xi) \right|^2 d\xi du \right\} \frac{1}{2} d\tau ds.
$$

(3.45)
Using Lemma 2.1, we split the term on the RHS of (3.45) as

\[
\int_0^t \int_0^s e^{-\frac{1}{2}\nu_0(t-\tau)} \left\{ \int_{|u|<3N} \int_{|\xi|<5N} |h(\tau, X(T_1-s+\tau; T_1, X(s), u), \xi)|^2 \, d\xi \, du \right\}^{\frac{1}{2}} \, d\tau \, ds
\]

\[
\leq \sum_{i_1=1}^{M_1} \sum_{i_2 \in \{1, \ldots, M_2\}} \sum_{i_3} \int_0^t 1_{\{X(s) \in \mathcal{G}_{i_2}^2\}}(s) \, ds \int_0^s e^{-\frac{1}{2}\nu_0(t-\tau)} 1_{\mathcal{G}_{i_1}^1}(T_1-s+\tau) \, d\tau
\]

\[
\times \left\{ \int_{|u|<3N} \int_{|\xi|<5N} |h(\tau, X(T_1-s+\tau; T_1, X(s), u), \xi)|^2 1_{\mathcal{G}_{i_3}^3}(u) \, d\xi \, du \right\}^{\frac{1}{2}}
\]

\[
= \sum_{j=1}^3 \sum_{i_1} \sum_{i_2} \sum_{i_3} \int_0^t 1_{\{X(s) \in \mathcal{G}_{i_2}^2\}}(s) \, ds \int_0^s e^{-\frac{1}{2}\nu_0(t-\tau)} 1_{\mathcal{G}_{i_1}^1}(T_1-s+\tau)
\]

\[
\times \left\{ \int_{|u|<3N} \int_{|\xi|<5N} |h(\tau, X(T_1-s+\tau; T_1, X(s), u), \xi)|^2 1_{\mathcal{G}_{i_3}^3}(u) \, d\xi \, du \right\}^{\frac{1}{2}},
\]

Noting

\[
\sum_{i_2 \in \{1, \ldots, M_2\}} \int_0^t 1_{\{X(s) \in \mathcal{G}_{i_2}^2\}}(s) \, ds = \int_0^t ds,
\]

then the first term on the RHS of (3.46) is bounded by

\[
C_N \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} \sum_{j=1}^3 \sum_{i_1} \sum_{i_2} \int_0^t 1_{\{X(s) \in \mathcal{G}_{i_2}^2\}}(s) \, ds \int_0^s e^{-\frac{1}{2}\nu_0(t-\tau)} 1_{\mathcal{G}_{i_1}^1}(T_1-s+\tau)
\]

\[
\times \left\{ \int_{|u|<3N} \int_{|\xi|<5N} |h(\tau, X(T_1-s+\tau; T_1, X(s), u), \xi)|^2 1_{\mathcal{G}_{i_3}^3}(u) \, d\xi \, du \right\}^{\frac{1}{2}}.
\]

On the other hand, it follows from (2.5) that

\[
\left| \frac{\partial X(T_1-s+\tau; T_1, X(s), u)}{\partial v} \right| \geq \delta_*(\varepsilon, M_1, M_2, M_3, N, T_1) > 0,
\]

where \(\delta_*(\varepsilon, M_1, M_2, M_3, N, T_1)\) is a positive constant depending on \(\varepsilon, M_1, M_2, M_3, N, T_1\).
for \(X(s) \in \mathcal{G}_{12}^2\), \(u \in \mathcal{G}_{13}^3\) and \(T_1 - s + \tau \notin (t_{j,1}, t_{2,1} - \frac{\varepsilon}{2M\tau}, t_{j,1}, t_{2,1} + \frac{\varepsilon}{2M\tau})\), which immediately yields that the second term on the RHS of (3.46) is bounded by

\[
\frac{C_N}{\delta_\varepsilon} \sum_{j=1}^{3} \sum_{i_1} \sum_{i_2} \sum_{j_3} \int_0^t \int_0^s e^{-\frac{1}{2}\rho_0(t-\tau)} 1 \{X(s) \in \mathcal{G}_{12}^2\}(s) ds \int_0^s e^{-\frac{1}{2}\rho_0(t-\tau)} 1 \{X(s) \in \mathcal{G}_{12}^2\}(s) ds \quad \text{(3.48)}
\]

where

\[
\begin{align*}
\tau &\leq C\bigg(\sum_{j=1}^{3} \sum_{i_1} \sum_{i_2} \sum_{j_3} \int_0^t \int_0^s e^{-\frac{1}{2}\rho_0(t-\tau)} 1 \{X(s) \in \mathcal{G}_{12}^2\}(s) ds \int_0^s e^{-\frac{1}{2}\rho_0(t-\tau)} 1 \{X(s) \in \mathcal{G}_{12}^2\}(s) ds \bigg)^{\frac{1}{2}} d\tau ds \\
&\leq C_N \cdot \varepsilon \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} + C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t e^{-\frac{1}{2}\rho_0(t-s)} \|f(s)\|_{L^2} ds. \quad \text{(3.49)}
\end{align*}
\]

Combining (3.39), (3.42), (3.45) and (3.49), one obtains that

\[
J_{11} \leq C \left(\frac{1}{N} + C_N \cdot \varepsilon\right) \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} + C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t e^{-\frac{1}{2}\rho_0(t-s)} \|f(s)\|_{L^2} ds. \quad \text{(3.50)}
\]

For \(J_{12}\), it follows from Lemma 2.3 that

\[
J_{12} \leq C \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} \int_0^t \int_0^s e^{-\frac{1}{2}\rho_0(t-\tau)} \int_{\mathbb{R}^3} |k_\beta(V(s), u)| d\tau ds \quad \text{(3.51)}
\]

We divide the estimation of (3.51) into the following several cases.

Case I. For \(|v| \geq N \gg 1\), it follows from (2.2) and (2.7) that

\[
J_{12} \leq C \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty}^2 \int_0^t \int_0^s e^{-\frac{1}{2}\rho_0(t-\tau)} \int_{\mathbb{R}^3} |k_\beta(V(s), u)| d\tau ds \quad \text{(3.52)}
\]
Case II. For either $|v| < N, |u| \geq 3N$ or $|u| < 3N, |\eta| \geq 5N$, it follows from (2.2) that either $|V(s) - u| \geq N, \forall s \in [0, t]$ or $|\eta| \geq 5N$, which together with (3.40) yields that

$$\begin{align*}
C \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} & \int_0^t \int_0^s e^{-\frac{1}{2} \tilde{\rho}_0(t - \tau)} \int_{|u| \leq 3N} |k_\beta(V(s), u)| \left( \int_{\mathbb{R}^3} \cdots d\eta \right) \frac{1}{2} du \, d\tau \, ds \\
+ C \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} & \int_0^t \int_0^s e^{-\frac{1}{2} \tilde{\rho}_0(t - \tau)} \int_{|u| < 3N} |k_\beta(V(s), u)| \left( \int_{|\eta| \leq 5N} \cdots d\eta \right) \frac{1}{2} du \, d\tau \, ds \\
\leq & \frac{C}{\sqrt{N}} \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty}^2. \quad (3.53)
\end{align*}$$

Case III. $|v| < N, |u| < 3N, |\eta| < 5N$. This is the last remaining case. It follows from (3.43), (3.44) and (3.49) that

$$\begin{align*}
C \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} & \int_0^t \int_0^s e^{-\frac{1}{2} \tilde{\rho}_0(t - \tau)} \int_{|u| < 3N} |k_\beta(V(s), u)| \left( \int_{|\eta| < 5N} \cdots d\eta \right) \frac{1}{2} du \, d\tau \, ds \\
\leq C_N \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} & \int_0^t \int_0^s e^{-\frac{1}{2} \tilde{\rho}_0(t - \tau)} \\
\times & \left\{ \int_{|u| < 3N} \int_{|\eta| < 5N} |h(\tau, X(T_1 - s + \tau - T_1, X(s), u), \eta)|^2 d\eta \, du \right\} \frac{1}{2} d\tau \, ds \quad (3.54) \\
\leq C(\varepsilon, M_1, M_2, M_3, N, T_1) & \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} \int_0^t \int_0^s e^{-\frac{1}{2} \tilde{\rho}_0(t - \tau)} \|f(\tau)\|_{L^2} d\tau \, ds \\
+ C_N \cdot \varepsilon & \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty}^2 \\
\leq C_N \cdot \varepsilon & \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty}^2 + C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t e^{-\frac{1}{4} \tilde{\rho}_0(t - s)} \|f(s)\|_{L^2}^2 ds.
\end{align*}$$

Substituting (3.52), (3.53) and (3.54) into (3.51), one obtains that

$$J_{12} \leq \left( \frac{C}{\sqrt{N}} + C_N \cdot \varepsilon \right) \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty}^2 \\
+ C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t e^{-\frac{1}{4} \tilde{\rho}_0(t - s)} \|f(s)\|_{L^2}^2 ds. \quad (3.55)$$

Substituting (3.50) and (3.55) into (3.38), one gets that

$$J_1 \leq CA_0\|e^{-\frac{1}{2} \tilde{\rho}_0 t} \cdot \frac{1}{\sqrt{N}} + C_N \cdot \varepsilon \sup_{0 \leq s \leq t} \{ \|h(s)\|_{L^\infty} + \|h(s)\|_{L^\infty}^2 \} \\
+ C(\varepsilon, M_1, M_2, M_3, N, T_1) \cdot A_0 \sup_{0 \leq s \leq t} \{ \|f(s)\|_{L^2} + \|f(s)\|_{L^2}^2 \}. \quad (3.56)$$
Next we estimate the nonlinear term \(J_2\). Motivated by (11), we need to make an iteration again in the nonlinear term. In fact, it follows from (2.8) that

\[
J_2 \leq C \int_0^t e^{-\frac{1}{2}\tilde{p}_0(t-s)} \|h(s)\|_{L^\infty} \left\{ \int_{|u| \leq N} (1 + |u|)^{-2\beta+4} |h(s, X(s), u)|^2 \, du \right\}^{\frac{1}{2}} ds \\
\leq \frac{C}{\sqrt{N}} \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} \\
+ C \int_0^t e^{-\frac{1}{2}\tilde{p}_0(t-s)} \|h(s)\|_{L^\infty} \left\{ \int_{|u| \leq N} (1 + |u|)^{-2\beta+4} |h(s, X(s), u)|^2 \, du \right\}^{\frac{1}{2}} ds. \tag{3.57}
\]

Recall \([\tilde{X}(\tau), \tilde{V}(\tau)] := [X(\tau; s, X(s), u), V(\tau; s, X(s), u)],\) we substitute the mild formulation of \(h(s, X(s), u)\) into the second term on the RHS of (3.57) to obtain that

\[
\int_0^t e^{-\frac{1}{2}\tilde{p}_0(t-s)} \|h(s)\|_{L^\infty} \left( \int_{|u| \leq N} (1 + |u|)^{-2\beta+4} |h(s, X(s), u)|^2 \, du \right)^{\frac{1}{2}} ds \\
\leq CA_0^2 \int_0^t e^{-\frac{1}{2}\tilde{p}_0(t-s)} e^{-\frac{1}{2}\tilde{p}_0}\|h(s)\|_{L^\infty} \left\{ \int_{|u| \leq N} (1 + |u|)^{-2\beta+4} \, du \right\}^{\frac{1}{2}} ds \\
+ CA_0 \int_0^t e^{-\frac{1}{2}\tilde{p}_0(t-s)} \|h(s)\|_{L^\infty} \left\{ \int_0^s e^{-\frac{1}{2}\tilde{p}_0(s-\tau)} \, d\tau \\
\times \int_{|u| \leq N} (1 + |u|)^{-2\beta+4} \left( \int_{\mathbb{R}^3} |k_\beta(\tilde{V}(\tau), \xi)h(\tau, \tilde{X}(\tau), \xi)|^2 \, d\xi \right)^{\frac{1}{2}} ds \\
+ CA_0 \int_0^t e^{-\frac{1}{2}\tilde{p}_0(t-s)} \|h(s)\|_{L^\infty} \left\{ \int_0^s e^{-\frac{1}{2}\tilde{p}_0(s-\tau)} \|h(\tau)\|_{L^\infty}^2 \, d\tau \\
\times \int_{|u| \leq N} \int_{\mathbb{R}^3} (1 + |u|)^{-2\beta+4} (1 + |\eta|)^{-2\beta+4} |h(\tau, \tilde{X}(\tau), \eta)|^2 \, d\eta \, du \right\}^{\frac{1}{2}} ds \\
:= J_{21} + J_{22} + J_{23}. \tag{3.58}
\]

A direct calculation shows that

\[
J_{21} \leq CA_0^2 e^{-\frac{1}{2}\tilde{p}_0} \int_0^t \|h(s)\|_{L^\infty} \, ds. \tag{3.59}
\]
For $J_{22}$, it follows from (3.41), (3.43) and (3.44) that
\[
J_{22} \leq CA_0 \int_0^t e^{-\frac{1}{2} \tilde{\nu}_0(t-s)} \|h(s)\|_{L^\infty} \left\{ \int_0^s e^{-\frac{1}{2} \tilde{\nu}_0(s-\tau)} d\tau \right. \\
\times \int_{|u| \leq N} (1 + |u|)^{-2\beta+4} \left( \int_{|\xi| \leq 3N} \left| k_\beta(\tilde{V}(\tau), \xi) h(s, \tilde{X}(\tau), \xi) \right| d\xi \right)^2 du \left\}^\frac{1}{2} ds \\
+ \frac{CA_0}{N} \sup_{0 \leq s \leq t} \|h(s)\|^2_{L^\infty} \\
\leq CA_0 \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} \int_0^t e^{-\frac{1}{2} \tilde{\nu}_0(t-s)} \left\{ \int_0^s e^{-\frac{1}{2} \tilde{\nu}_0(s-\tau)} d\tau \right. \\
\times \int_{|u| \leq N} \int_{|\xi| \leq 3N} (1 + |u|)^{-2\beta+4} |h(s, X(T_1 - s + \tau; T_1, X(s), u, \xi)|^2 d\xi du \left\}^\frac{1}{2} ds \\
+ \frac{CA_0}{N} \sup_{0 \leq s \leq t} \|h(s)\|^2_{L^\infty}. \tag{3.60}
\]
By similar arguments as in (3.46)-(3.49), one has that
\[
\int_0^t e^{-\frac{1}{2} \tilde{\nu}_0(t-s)} \left\{ \int_0^s e^{-\frac{1}{2} \tilde{\nu}_0(s-\tau)} \int_{|u| \leq N} \int_{|\xi| \leq 3N} (1 + |u|)^{-2\beta+4} d\tau \\
\times |h(s, X(T_1 - s + \tau; T_1, X(s), u, \xi)|^2 d\xi du \right\}^\frac{1}{2} ds \\
\leq C_N \cdot \varepsilon \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty} + C(\varepsilon, M_1, M_2, M_3, N, T_1) \left\{ \int_0^t e^{-\frac{1}{2} \tilde{\nu}_0(t-\tau)} \|f(\tau)\|^2_{L^2} d\tau \right\}^\frac{1}{2}, \tag{3.61}
\]
which, together with (3.60), yields that
\[
J_{22} \leq A_0 \left( \frac{C}{N} + C_N \cdot \varepsilon \right) \sup_{0 \leq s \leq t} \|h(s)\|^2_{L^\infty} \\
+ C(\varepsilon, M_1, M_2, M_3, N, T_1) \int_0^t e^{-\frac{1}{2} \tilde{\nu}_0(t-\tau)} \|f(\tau)\|^2_{L^2} d\tau. \tag{3.62}
\]
For $J_{23}$, it follows from (3.61) that
\[
J_{23} \leq CA_0 \sup_{0 \leq s \leq t} \|h(s)\|^2_{L^\infty} \int_0^t e^{-\frac{1}{2} \tilde{\nu}_0(t-s)} \left\{ \int_0^s e^{-\frac{1}{2} \tilde{\nu}_0(s-\tau)} d\tau \int_{|u| \leq N} \int_{|\eta| \leq 3N} \\
\times (1 + |u|)^{-2\beta+4} (1 + |\eta|)^{-2\beta+4} \left| h(\tau, \tilde{X}(\tau), \eta) \right|^2 d\eta du \right\}^\frac{1}{2} ds \\
+ \frac{CA_0}{\sqrt{N}} \sup_{0 \leq s \leq t} \|h(s)\|^3_{L^\infty} \\
\leq A_0 \left( \frac{C}{\sqrt{N}} + C_N \cdot \varepsilon \right) \sup_{0 \leq s \leq t} \|h(s)\|^3_{L^\infty} \\
+ A_0 C(\varepsilon, M_1, M_2, M_3, N, T_1) \left\{ \int_0^t e^{-\frac{1}{2} \tilde{\nu}_0(t-\tau)} \|f(\tau)\|^2_{L^2} d\tau \right\}^\frac{3}{2}. \tag{3.63}
\]
Combining \((3.58), (3.59), (3.62), (3.63)\) with \((3.57)\), one proves that
\[
J_2 \leq CA_0^2 e^{-\frac{1}{2} \varphi_0 t} \int_0^t \|h(s)\|_{L^\infty} \, ds + A_0 \left( \frac{C}{\sqrt{N}} + C_N \cdot \varepsilon \right) \sup_{0 \leq s \leq t} \left\{ \|h(s)\|_{L^2}^2 + \|h(s)\|_{L^\infty}^3 \right\}
\]
\[
+ A_0 C(\varepsilon, M_1, M_2, M_3, N, T_1) \sup_{0 \leq s \leq t} \left\{ \|f(s)\|_{L^2}^2 + \|f(s)\|_{L^2}^3 \right\} ,
\]
which, together with \((3.59)\) and \((3.37)\), yields that
\[
\|h(t)\|_{L^\infty} \leq \tilde{C}_3 A_0^2 \left( 1 + \int_0^t \|h(s)\|_{L^\infty} \, ds \right) e^{-\frac{1}{4} \varphi_0 t} + D,
\]
where \(\tilde{C}_3 \geq 1\) is some generic positive constant. Therefore the proof of Lemma 3.3 is completed. □

3.4. Proof of Theorem 1.1. We take
\[
\tilde{C}_4 = \max \left\{ 2, C_0, \tilde{C}_3 \right\} ,
\]
then we define
\[
A_1 := 4 \tilde{C}_4 A_0^2 \exp \left\{ \frac{4}{\tilde{\nu}_0} \tilde{C}_4 A_0^2 \right\} \quad \text{and} \quad T_1 := \frac{8}{\tilde{\nu}_0} \left( \ln A_1 + |\ln \delta| \right) ,
\]
where \(C_0 > 0\) and \(\delta > 0\) are the constants introduced in Theorem 2.2. We emphasize that the above \(A_1\) in \((3.63)\) depends only on \(A_0\), and \(T_1\) depends only on \(\delta\) and \(A_0\).

Assume that \(\|f_0\|_{L^2} \leq \kappa_1 (A_1, T_1)\) where \(\kappa_1\) is defined in Lemma 3.3. Hence, it follows from \((3.1)\), \((3.61)\) and Lemmas 3.2 and 3.3 that
\[
\|h(t)\|_{L^\infty} \leq \tilde{C}_4 A_0^2 \left( 1 + \int_0^t \|h(s)\|_{L^\infty} \, ds \right) e^{-\frac{1}{4} \varphi_0 t} + D,
\]
where
\[
D := A_0 \left( \frac{\tilde{C}_3}{\sqrt{N}} + C_N \cdot \varepsilon \right) \left\{ A_1 + A_1^3 \right\}
\]
\[
+ A_0 C(\varepsilon, M_1, M_2, M_3, N, T_1) \tilde{C}_3 \left\{ \|f_0\|_{L^2} e^{\tilde{C}_1 A_1 T_1} + \left( \|f_0\|_{L^2} e^{\tilde{C}_1 A_1 T_1} \right)^3 \right\} .
\]
Now we define
\[
H(t) := 1 + \int_0^t \|h(s)\|_{L^\infty} \, ds ,
\]
then, \((3.65)\) is rewritten as
\[
H'(t) \leq \tilde{C}_4 A_0^2 e^{-\frac{1}{4} \varphi_0 t} H(t) + D.
\]
Hence it holds that
\[
\frac{d}{dt} \left( H(t) \exp \left\{ -\frac{4}{\tilde{\nu}_0} \tilde{C}_4 A_0^2 \left( 1 - e^{-\frac{1}{4} \varphi_0 t} \right) \right\} \right) \leq D,
\]
which yields immediately that
\[
H(t) \leq (1 + Dt) \exp \left\{ \frac{4}{\tilde{\nu}_0} \tilde{C}_4 A_0^2 \right\} , \quad \forall t \in [0, T_1].
\]
Substituting (3.67) back into (3.65), one has
\[ \|h(t)\|_{L^\infty} \leq \tilde{C}_4 A_0^2 \exp \left\{ \frac{4}{\tilde{\nu}_0} \tilde{C}_4 A_0^2 \right\} (1 + Dt) e^{-\frac{1}{2} \tilde{\nu}_0 t} + D \]
\[ \leq \frac{1}{4C_4} A_1 \left( 1 + \frac{8}{\tilde{\nu}_0} D \right) e^{-\frac{1}{2} \tilde{\nu}_0 t} + D. \]  
(3.68)

Noting (3.64) and (3.66), we firstly choose \( N > 0 \) large enough, then \( \varepsilon > 0 \) sufficiently small, and finally let \( \|f_0\|_{L^2} \leq \kappa_2 \) with \( \kappa_2 = \kappa_2(\delta, A_0) > 0 \) further sufficiently small, such that
\[ D = \min \left\{ \frac{\tilde{\nu}_0}{32}, \frac{\delta}{8} \right\}, \]
which, together with (3.68) yields that
\[ \|h(t)\|_{L^\infty} \leq \frac{5}{16C_4} A_1 e^{-\frac{1}{2} \tilde{\nu}_0 t} + D \leq \frac{1}{2C_4} A_1, \]  
(3.69)
for all \( t \in [0, T_1] \). Hence, we have closed the \textit{a priori} assumption (3.1) over \( t \in [0, T_1] \) provided that
\[ \|f_0\|_{L^2} \leq \kappa_0 := \min \{ \kappa_1, \kappa_2 \}. \]

Note that \( \kappa_0 > 0 \) depends only on \( \delta \) and \( A_0 \).

Using the uniform estimate (3.69) and the local existence Theorem 4.1, we can extend the Boltzmann solution to the time interval \( t \in [0, T_1] \), see [11] for more details. Next, for the case \( t \geq T_1 \), we note from the first inequality of (3.69) and the definition (3.64) for \( T_1 \) that
\[ \|h(T_1)\|_{L^\infty} \leq \frac{5}{16C_4} A_1 \exp \left\{ \frac{-\tilde{\nu}_0}{8} T_1 \right\} + \frac{\delta}{8} < \frac{5\delta}{16C_4} + \frac{\delta}{8} < \frac{1}{2}. \]

With \( t = T_1 \) as the initial time and applying Theorem 2.2, we can extend the Boltzmann solution \( f(t) \) from \( [0, T_1] \) to \( [0, \infty) \), and thus obtain the unique solution \( f(t) \) globally in time on \( [0, \infty) \) such that \( F(t, x, v) = \mu_E + \sqrt{\mu_E} f(t, x, v) \geq 0 \) and \( \sup_{t \in \mathbb{R}} \|w_\beta f(t)\|_{L^\infty} \leq A_1 \). This proves the global existence and uniqueness of solutions in weighted \( L^\infty \) space.

For the large time behavior of the obtained solution, we note that as an immediate consequence of Theorem 2.2 it holds that
\[ \|h(t)\|_{L^\infty} \leq C_0 \|h(T_1)\|_{L^\infty} e^{-\lambda_0 (t - T_1)} \leq C_0 \delta e^{-\lambda_0 (t - T_1)}, \]  
(3.70)
for all \( t \geq T_1 \). By taking
\[ \tilde{C}_0 := 4\tilde{C}_4^3 \text{ and } \tilde{\lambda}_0 := \min \left\{ \lambda_0, \frac{\tilde{\nu}_0}{8} \right\}, \]
it follows from (3.69), (3.70) and a direct computation that
\[ \|h(t)\|_{L^\infty} \leq \max \left\{ \frac{1}{2}, C_0 \right\} A_1 e^{-\tilde{\lambda}_0 t} \leq \tilde{C}_0 A_1 e^{-\tilde{\lambda}_0 t} \]
\[ = 4\tilde{C}_4^3 A_0^2 \exp \left\{ \frac{4\tilde{C}_4}{\tilde{\nu}_0} A_0^2 \right\} e^{-\tilde{\lambda}_0 t} \]
\[ = \tilde{C}_0 A_0^2 \exp \left\{ \frac{\tilde{C}_0}{\tilde{\nu}_0} A_0^2 \right\} e^{-\tilde{\lambda}_0 t}. \]

Therefore the proof of Theorem 1.1 is completed. \( \Box \)
4. Appendix

The following theorem is devoted to the existence of local solution to Boltzmann equation with large external potential and large $L^\infty$ initial data.

**Theorem 4.1** (Local Existence). Let $0 \leq \gamma \leq 1$. Assume \[1.3\], and

\[ F_0(x,v) = \mu_E(x,v) + \sqrt{\mu_E(x,v)} f_0(x,v) \geq 0, \]

with $\|w_\beta f_0\|_{L^\infty} < +\infty$, then there exists a positive time

\[ T_0 := (8C \left[ 1 + \|w_\beta f_0\|_{L^\infty}\right])^{-1} > 0, \]

such that the Boltzmann equation \[1.1\] admits a unique solution

\[ F(t,x,v) = \mu_E(x,v) + \sqrt{\mu_E(x,v)} f(t,x,v) \geq 0 \]

satisfying

\[ \|w_\beta f(t)\|_{L^\infty} \leq 2\|w_\beta f_0\|_{L^\infty}, \text{ for } 0 \leq t \leq T_0, \]

where $C \geq 1$ is some positive constant depending only on $\beta, \alpha, \gamma, M$. In addition, the conservations of mass \[1.7\], energy \[1.8\], and degenerate momentum \[1.11\] as well as the additional entropy inequality \[1.9\] hold.

**Remark 4.2.** In fact, the above local existence result can be obtained by making a slight modification to the proof of Proposition 2.1 in \[7\]. For completeness, we put its proof in the following.

**Proof of Theorem 4.1.** To prove the local existence for the Boltzmann equation with large external potential and large initial data, we consider the following iteration, for $n = 0, 1, 2, \ldots$,

\[
\begin{align*}
\partial_t F^{n+1} + v \cdot \nabla_x F^{n+1} - \nabla \Phi(x) \cdot \nabla_v F^{n+1} + Q_-(F^n, F^{n+1}) &= Q_+(F^n, F^n), \\
F^{n+1}(0, x, v) &= F_0(x, v) \geq 0,
\end{align*}
\]

(4.1)

where $F_0(t, x, v) = \mu_E(x, v)$. Along the characteristic line \[2.1\], the mild form of \[1.1\] is

\[ F^{n+1}(t, x, v) = e^{-\int_0^t g^n_1(\tau) \, d\tau} F_0(0) + \int_0^t e^{-\int_0^\tau g^n_1(\tau) \, d\tau} Q_+(F_n, F_n)(s, X(s), V(s)) \, ds, \]

where

\[ g^n_1(\tau) = R(F^n)(\tau, X(\tau), V(\tau)), \]

with $R(F)$ defined in \[3.5\]. By the induction argument, it is easy to show that

\[ F^n(t, x, v) \geq 0, \quad \forall n \geq 0. \]

(4.2)

To take the limit $n \to \infty$, one needs to obtain some uniform estimate. In fact, to show the uniform estimate for the approximation sequence $F^n$, we turn to estimate $h^n := w_\beta f^n$ with

\[ f^n := \frac{F^n - \mu_E}{\sqrt{\mu_E}} \quad \text{and} \quad f^n|_{t=0} = f_0 := \frac{F_0 - \mu_E}{\sqrt{\mu_E}}, \quad \forall n \geq 0, \]

then the equation \[1.1\] is rewritten as

\[
\frac{\partial h^{n+1}}{\partial t} + v \cdot \nabla_x h^{n+1} - \nabla \Phi \cdot \nabla_v h^{n+1} + R(F^n) h^{n+1} = e^{-\frac{4}{\gamma} w_\beta \Gamma_+} \left( \frac{\sqrt{\mu} h^n}{w_\beta}, \frac{\sqrt{\mu} h^n}{w_\beta} \right) + e^{-\Phi} w_\beta K \left( \frac{h^n}{w_\beta} \right).
\]
Integrate along the characteristics (2.1) to obtain
\[
\begin{align*}
    h_{n+1}(t, x, v) &= e^{-\int_0^t g_1^*(\tau) \, d\tau} h_0(X(0), V(0)) \\
    &+ \int_0^t e^{-\int_0^s g_1^*(\tau) \, d\tau} \Phi(X(s)) w_\beta(X(s), V(s)) K\left(\frac{h^n}{w_\beta}\right)(s, X(s), V(s)) \, ds \\
    &+ \int_0^t e^{-\int_0^s g_1^*(\tau) \, d\tau} \frac{\Phi(X(s))}{\sqrt{\mu(V(s))}} Q + \left(\frac{\sqrt{\lambda} h^n}{w_\beta}, \frac{\sqrt{\mu} h^n}{w_\beta}\right)(s, X(s), V(s)) \, ds \\
    &:= H_1 + H_2 + H_3.
\end{align*}
\]

Since \(g_1(\tau) = R(F^n)(\tau, X(\tau), V(\tau)) \geq 0\), one has that
\[
|H_1| \leq \|h_0\|_{L^\infty}. \tag{4.4}
\]

For \(H_2\), it follows from (2.7) that
\[
|H_2| \leq C \int_0^t \|h^n(s)\|_{L^\infty} \, ds. \tag{4.5}
\]

For \(H_3\), it follows from (2.9) that
\[
\frac{w_\beta(X(s), V(s))}{\sqrt{\mu(V(s))}} \left| Q + \left(\frac{\sqrt{\lambda} h^n}{w_\beta}, \frac{\sqrt{\mu} h^n}{w_\beta}\right)(s, X(s), V(s)) \right| \leq C \|h^n(s)\|^2_{L^\infty},
\]
which immediately yields that
\[
|H_3| \leq C \int_0^t \|h^n(s)\|^2_{L^\infty} \, ds. \tag{4.6}
\]

Substituting (4.4), (4.5) and (4.6) into (4.3), one obtains that
\[
\|h_{n+1}(t)\|_{L^\infty} \leq \|h_0\|_{L^\infty} + C t \left\{ \sup_{0 \leq s \leq t} \|h^n(s)\|_{L^\infty} + \sup_{0 \leq s \leq t} \|h^n(s)\|^2_{L^\infty} \right\},
\]
where \(C \geq 1\) depends only on \(\beta, \gamma\) and \(M\). Take
\[
T_0 = (4C [1 + \|h_0\|_{L^\infty}])^{-1} < 1,
\]
and by induction arguments, we get the uniform estimate
\[
\|h^n(t)\|_{L^\infty} \leq 2\|h_0\|_{L^\infty}, \quad \forall t \in [0, T_0], \tag{4.7}
\]
which yields that \(\left\{h^n\right\}_{n=1}^\infty \beta_{\sqrt{\lambda}}\) is a sequence in \(L^\infty([0, T_0] \times T^3 \times \mathbb{R}^3)\).

It is noted that the approximation sequence \(h^n\) itself does not converge in \(L^\infty([0, T_0] \times T^3 \times \mathbb{R}^3)\). However, we can prove that \(\left\{h^n\right\}_{n=1}^\infty \beta_{\sqrt{\lambda}}\) is a Cauchy sequence in \(L^\infty([0, T_0] \times T^3 \times \mathbb{R}^3)\). In fact, it
follows from (4.3) that
\[ |(h^{n+1} - h^n)(t, x, v)| \]
\[ \leq |h_0(X(0), V(0))| \int_0^t |(g_2^n - g_2^{n-1})(\tau)| \, d\tau \]
\[ + \int_0^t w_\beta(X(s), V(s)) \left| K \left( \frac{h^n}{w_\beta} \right)(s, X(s), V(s)) \right| \int_s^t |(g_2^n - g_2^{n-1})(\tau)| \, d\tau \, ds \]
\[ + \int_0^t w_\beta(X(s), V(s)) \left| K \left( \frac{h^n - h^{n-1}}{w_\beta} \right)(s, X(s), V(s)) \right| \, ds \]
\[ + \int_0^t w_\beta(X(s), V(s)) \left| \frac{Q_+ \left( \sqrt{\mu h^n}, \sqrt{\mu h^n} \right)(s, X(s), V(s)) \right| \int_s^t |(g_2^n - g_2^{n-1})(\tau)| \, d\tau \, ds \]
\[ + \int_0^t w_\beta(X(s), V(s)) \left| \frac{Q_+ \left( \sqrt{\mu (h^n - h^{n-1})}, \sqrt{\mu h^{n-1}} \right)(s, X(s), V(s)) \right| \, ds \]
\[ := H_4 + H_5 + H_7 + H_8 + H_9. \]

By a direct calculation, we have
\[ |(g_2^n - g_2^{n-1})(\tau)| \leq C \nu(V(\tau)) \left\| \left( \frac{h^n}{\sqrt{w_\beta}} - \frac{h^{n-1}}{\sqrt{w_\beta}} \right)(\tau) \right\|_{L^\infty}. \tag{4.9} \]

It follows from (2.2) that
\[ \int_0^t |\nu(V(\tau))| \, d\tau \leq C t(1 + |v|^2)^{\frac{3}{2}} \leq C t \sqrt{w_\beta(x, v)} \tag{4.10} \]

Using (4.9) and (4.10), a direct calculation shows that
\[ H_4 + H_5 + H_7 \leq C t \sqrt{w_\beta(x, v)} \| h_0 \|_{L^\infty} \cdot \sup_{0 \leq s \leq t} \left\| \left( \frac{h^n}{\sqrt{w_\beta}} - \frac{h^{n-1}}{\sqrt{w_\beta}} \right)(s) \right\|_{L^\infty}. \tag{4.11} \]

Again, (2.2) implies that
\[ w_\beta(X(s), V(s)) \leq C w_\beta(x, v). \tag{4.12} \]

Therefore, it follows from (2.2) that
\[ |H_6| \leq C \int_0^t w_\beta(X(s), V(s)) \left| K \left( \frac{h^n - h^{n-1}}{w_\beta} \right)(s, X(s), V(s)) \right| \, ds \]
\[ \leq C \int_0^t \sqrt{w_\beta(X(s), V(s))} \left\| \left( \frac{h^n}{\sqrt{w_\beta}} - \frac{h^{n-1}}{\sqrt{w_\beta}} \right)(s) \right\|_{L^\infty} \int_{\mathbb{R}^3} |k(V(s), u)| \sqrt{w_\beta(X(s), V(s)) \frac{w_\beta(X(s), u)}{w_\beta(X(s), u)}} \, du \, ds \]
\[ \leq C t \sqrt{w_\beta(x, v)} \sup_{0 \leq s \leq t} \left\| \left( \frac{h^n}{\sqrt{w_\beta}} - \frac{h^{n-1}}{\sqrt{w_\beta}} \right)(s) \right\|_{L^\infty}, \tag{4.13} \]
For $H_8$, we note from (4.12) that
\[
\frac{w_\beta(X(s), V(s))}{\sqrt{\mu(V(s))}} \left| Q_+ \left( \frac{\sqrt{h_n}}{w_\beta}, \sqrt{\mu(h_n - h_{n-1})} \right) (s, X(s), V(s)) \right|
\leq C \sqrt{w_\beta(x_v)} \|h^n(s)\|_{L^\infty} \left\| \left( \frac{h_n}{\sqrt{w_\beta}} - \frac{h_{n-1}}{\sqrt{w_\beta}} \right) (s) \right\|_{L^\infty}
\times \int_{\mathbb{R}^3} \int_{S^2} B(V(s) - u, \omega) \sqrt{\mu(u)} \frac{1}{w_\beta(X(s), u')} \left( \frac{w_\beta(X(s), V(s))}{w_\beta(X(s), v')} \right) d\omega du
\leq C \nu(v) \|h^n(s)\|_{L^\infty} \left\| \left( \frac{h_n}{\sqrt{w_\beta}} - \frac{h_{n-1}}{\sqrt{w_\beta}} \right) (s) \right\|_{L^\infty}
\leq C \sqrt{w_\beta(x_v)} \|h^n(s)\|_{L^\infty} \left\| \left( \frac{h_n}{\sqrt{w_\beta}} - \frac{h_{n-1}}{\sqrt{w_\beta}} \right) (s) \right\|_{L^\infty},
\]
which yields that
\[
H_8 \leq Ct \sqrt{w_\beta(x_v)} \|h_0\|_{L^\infty} \sup_{0 \leq s \leq t} \left\| \left( \frac{h_n}{\sqrt{w_\beta}} - \frac{h_{n-1}}{\sqrt{w_\beta}} \right) (s) \right\|_{L^\infty}.
\tag{4.14}
\]
Similarly, we have
\[
H_9 \leq Ct \sqrt{w_\beta(x_v)} \|h_0\|_{L^\infty} \sup_{0 \leq s \leq t} \left\| \left( \frac{h_n}{\sqrt{w_\beta}} - \frac{h_{n-1}}{\sqrt{w_\beta}} \right) (s) \right\|_{L^\infty}.
\tag{4.15}
\]
Substituting (4.11), (4.13), (4.14), (4.15) into (4.8), one obtains that
\[
\sup_{0 \leq t \leq T_0} \left\| \left( \frac{h_n}{\sqrt{w_\beta}} - \frac{h_{n-1}}{\sqrt{w_\beta}} \right) (t) \right\|_{L^\infty} \leq CT_0 \left( \|h_0\|_{L^\infty} + 1 \right) \sup_{0 \leq t \leq T_0} \left\| \left( \frac{h_n}{\sqrt{w_\beta}} - \frac{h_{n-1}}{\sqrt{w_\beta}} \right) (t) \right\|_{L^\infty}
\leq \frac{1}{2} \sup_{0 \leq t \leq T_0} \left\| \left( \frac{h_n}{\sqrt{w_\beta}} - \frac{h_{n-1}}{\sqrt{w_\beta}} \right) (t) \right\|_{L^\infty}.
\]
This inequality shows that $\sqrt{w_\beta} f^n$ or equivalently $\frac{n}{\sqrt{w_\beta}}$ is a Cauchy sequence in $L^\infty([0, T_0] \times \mathbb{T}^3 \times \mathbb{R}^3)$. Therefore, there exists a limit function $f$ such that
\[
\sup_{0 \leq t \leq T_0} \left\| \left( \sqrt{w_\beta} f^n - \sqrt{w_\beta} f \right) (t) \right\|_{L^\infty} \to 0, \text{ as } n \to \infty,
\]
and the limit function $F := \mu_E(x, v) + \sqrt{\mu_E(x, v)} f(t, x, v)$ is indeed a mild solution to the Boltzmann equation (1.1). Moreover, it follows from (4.2) and (4.7) that
\[
\begin{cases}
F(t, x, v) = \mu_E(x, v) + \sqrt{\mu_E(x, v)} f(t, x, v) \geq 0, \\
\sup_{0 \leq t \leq T_0} \|w_\beta f(t)\|_{L^\infty} \leq 2 \|w_\beta f_0\|_{L^\infty}.
\end{cases}
\]
For the uniqueness. Let $\tilde{F} := \mu_E(x, v) + \sqrt{\mu_E(x, v)} \tilde{f}(t, x, v)$ be another mild solution to the Boltzmann equation (1.1) with initial data (1.2) and satisfy
\[
\sup_{0 \leq t \leq T_0} \|\tilde{h}(t)\|_{L^\infty} < +\infty,
\]
with $\tilde{h} := w_\beta(x, v) \tilde{f}$. By the same argument as in (4.8), we deduce
\[
\left\| \left( \frac{h}{\sqrt{w_\beta}} - \frac{\tilde{h}}{\sqrt{w_\beta}} \right) (t) \right\|_{L^\infty} \leq C \left( \|h_0\|_{L^\infty} + 1 \right) \int_0^t \left\| \left( \frac{h}{\sqrt{w_\beta}} - \frac{\tilde{h}}{\sqrt{w_\beta}} \right) (s) \right\|_{L^\infty} ds,
\]
Hence uniqueness follows immediately from the Gronwall’s inequality.
Finally, multiplying both sides of (4.1) by $1, v, \sqrt{2} + \Phi(x)$ and $F^{n+1}$, integrating by parts and taking the limit $n \to \infty$, we obtain the corresponding result of the conservation of mass, energy and degenerate momentum. □

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