RATE OF CONVERGENCE IN TROTTER’S APPROXIMATION THEOREM AND ITS APPLICATIONS

RYUYA NAMBA

Abstract. The celebrated Trotter approximation theorem provides a sufficient condition for the convergence of a sequence of operator semigroups in terms of the corresponding sequence of infinitesimal generators. There exist a few results on the rate of convergence in Trotter’s theorem under some constraints. In the present paper, a new rate of convergence in Trotter’s theorem in full generality is given. Moreover, we see that this rate of convergence works well to obtain quantitative estimates for some limit theorems in probability theory.

1. Introduction and main results

There has been a number of interests in approximation theory for semigroups of linear operators on Banach spaces among several branches of mathematics such as functional analysis, partial differential equations, probability theory and so on. Trotter provided a remarkably useful sufficient condition for the convergence of a sequence of operator semigroups in terms of the corresponding sequence of infinitesimal generators in [15]. Afterwards, several extensions of Trotter’s approximation theorem have been discussed by noting some relations among operator semigroups, resolvents and generators. We refer to e.g., [12, 8] for related early works and [14, 7, 4] for good textbooks with extensive references therein.

We now recall a general statement of Trotter’s approximation theorem according to [12]. In the following, we denote by \( \| \mathfrak{A} \| \) the usual operator norm of a bounded linear operator \( \mathfrak{A} \) defined on some Banach space. Let \(( B_n, \| \cdot \|_{B_n} ), n \in \mathbb{N} \), and \(( E, \| \cdot \|_E) \) be Banach spaces. Let \( P_n : E \to B_n, n \in \mathbb{N}, \) be a bounded linear operator with \( \| P_n \| \leq 1 \).

Definition 1.1. We say that the sequence of pairs \( \{(B_n, P_n)\}_{n=1}^{\infty} \) approximates the Banach space \( E \) if \( \| P_n f \|_{B_n} \to \| f \|_E \) as \( n \to \infty \) for every \( f \in E \).

The definition above means that each \( P_n, n \in \mathbb{N}, \) is regarded as an isomorphism between \( B_n \) and \( E \) when passing to the limit in some sense. Therefore, \( P_n \) is occasionally called an...
approximating operator of $E$. Let $f_n \in B_n$ and $f \in E$. We also say that $f = \lim_{n \to \infty} f_n$ if
\[ \|f_n - P_n f\|_{B_n} \to 0 \]
as $n \to \infty$. We then define the limit $\mathfrak{A}$ of a sequence of linear operators $\mathfrak{A}_n$ with the domain $\text{Dom}(\mathfrak{A}_n)$ and the range $\text{Ran}(\mathfrak{A}_n)$ in $B_n$ by putting
\[ \mathfrak{A} f := \lim_{n \to \infty} \mathfrak{A}_n P_n f \]
for all $f \in E$, for which this limit exists. We put
\[ \text{Dom}(\mathfrak{A}) = \{ f \in E \mid \text{there exists } \lim_{n \to \infty} \mathfrak{A}_n P_n f \} \]
Then we have the following.

**Proposition 1.2** (cf. [12, Theorem 2.13]). Let $T_n$, $n \in \mathbb{N}$, be a bounded linear operator on $B_n$ with $\|T_n\| \leq 1$. Let $\{\ell(n)\}_{n=1}^{\infty}$ be a sequence of positive numbers and $\mathfrak{A}_n := (T_n - I)/\ell(n)$ for $n \in \mathbb{N}$. Suppose that $\ell(n) \to 0$ as $n \to \infty$ and that $\mathfrak{A}$ is defined by the closure of the limit $\lim_{n \to \infty} \mathfrak{A}_n$. If the domain $\text{Dom}(\mathfrak{A})$ is dense in $E$ and the range $\text{Ran}(\lambda - \mathfrak{A})$ is dense in $E$ for some $\lambda > 0$, then there exists a $C_0$-semigroup $(T_t)_{t \geq 0}$ on $E$ such that
\[ \lim_{n \to \infty} \|T^{[t/\ell(n)]}_n P_n f - P_n T_t f\|_{B_n} = 0, \quad t \geq 0, \; f \in E. \]

Note that the contractivity $\|T_n\| \leq 1$, $n \in \mathbb{N}$, is imposed for a convenience. Indeed, Proposition 1.2 can be stated under slightly weaker assumptions. See also (1.1) in Theorem 1.3 below.

We should emphasize that Trotter’s approximation theorem itself did not provide any quantitative estimates for the convergences of semigroups. To obtain such estimates should be one of the main problems of interest in a number of parts of approximation theory. So some authors have tried to consider this problem. As far as we know, Mangino and Rasa gave the first result on the rate of convergence in Trotter’s approximation theorem in [13]. Moreover, Campiti and Tacelli also established a refinement of Trotter’s approximation theorem in [1, Theorem 1.1] under a special condition $B_n \equiv E$ for $n \in \mathbb{N}$. On the other hand, the assumption for the linear operator $T_n$, $n \in \mathbb{N}$, imposed in [1] was not sufficient in general. Therefore, they wrote an additional paper [2], where the result has already improved properly and an application to Bernstein operators has been given. See also [3] for a related result on the rate of convergence in Trotter’s theorem. However, we note that the cases where each approximating Banach space $B_n$ differs for every $n \in \mathbb{N}$ are still left, though they should have a number of applications of this rate of convergence to very extensive areas of mathematics.
Inspired by these circumstances, we obtain the following rate of convergence, which corresponds to a refinement of Proposition 1.2 and is also regarded as a certain extension of [10, Theorem 1.1] to considerable cases.

**Theorem 1.3.** Let $B_n, n \in \mathbb{N}$, be a Banach space endowed with $\| \cdot \|_{B_n}$ and $P_n : E \to B_n, n \in \mathbb{N}$, a bounded linear operator with $\|P_n\| \leq 1$. Suppose that $\{(B_n, P_n)\}_{n=1}^{\infty}$ approximates a Banach space $E$. Let $T_n, n \in \mathbb{N}$, be a bounded linear operator on $B_n$ satisfying

$$\|T_n^k\| \leq M e^{\omega k/n}, \quad n, k \in \mathbb{N},$$

for some $M \geq 1$ and $\omega \geq 0$ independent of $k$ and $n$. Suppose that $D$ is a dense subspace of $E$ and $\mathcal{A} : (D \subset) \text{Dom}(\mathcal{A}) \to E$ is a linear operator. If $\text{Ran}(\lambda - \mathcal{A})$ is dense in $E$ for some $\lambda > \omega$, then the closure of $(\mathcal{A}, D)$ generates a $C_0$-semigroup $(\mathcal{T}_t)_{t \geq 0}$ on $E$ satisfying $\|T_t\| \leq M e^{\omega t}$ for $t \geq 0$. Moreover, suppose that

$$\|n(T_n - I)P_nf\|_{B_n} \leq \varphi_n(f), \quad f \in D,$$  \quad (1.2)

and the following Voronovskaja-type formula holds:

$$\|n(T_n - I)P_nf - P_n\mathcal{A}f\|_{B_n} \leq \psi_n(f), \quad f \in D,$$  \quad (1.3)

where $\varphi_n : D \to [0, \infty)$ and $\psi_n : D \to [0, \infty)$ are semi-norms on $D$ with $\lim_{n \to \infty} \psi_n(f) = 0$ for $f \in D$. Then, for every $t \geq 0$ and for every increasing $\{k(n)\}_{n=1}^{\infty}$ of non-negative integers, we have

$$\|T_n^{k(n)}P_nf - P_n\mathcal{T}_tf\|_{B_n} \leq M \exp(2\omega e^{\omega/n}k(n)/n) \left( \frac{\omega k(n)}{n} + \sqrt{\frac{k(n)}{n}} \right) \varphi_n(f) + M \exp(\omega t e^{\omega/n}) \left| \frac{k(n)}{n} - t \right| \varphi_n(f) + M \exp(\omega t e^{\omega/n}) \int_0^t \exp(-\omega s e^{\omega/n}) \psi_n(\mathcal{T}_s f) \, ds \quad (1.4)$$

for all $f \in D_0 := \{g \in D \mid \mathcal{T}_t g \in D, \ t \geq 0\}$, where we put $t_n := \max\{t, k(n)/n\}$.

As is seen, the estimates (1.2) and (1.3) play important roles when we obtain (1.4). The condition (1.2) corresponds to an estimate of the operator norm of the infinitesimal generator of a discrete semigroup itself. On the other hand, the condition (1.3) indicates the estimate of the norm of the difference between the discrete infinitesimal generator and the limiting one, which should converge to zero as $n \to \infty$ by virtue of Proposition 1.2.

The most typical choice of the sequence $\{k(n)\}_{n=1}^{\infty}$ is that $k(n) := [nt]$ for $n \in \mathbb{N}$ and $t \geq 0$. Since it holds that $k(n) = [nt] \leq nt, t_n = t$ and $|[nt]/n - t| \leq 1/n$, Inequality (1.4)
\[ \| T_n^{[nt]} P_n f - P_n T_t f \|_{B_n} \leq M \exp(2\omega t e^{\omega/n}) \left( \frac{\omega t}{n} + \sqrt{\frac{t}{n}} \right) \varphi_n(f) + M \frac{\exp(\omega t e^{\omega/n}) \varphi_n(f)}{n} + M \exp(\omega t e^{\omega/n}) \int_0^t \exp(-\omega s e^{\omega/n}) \psi_n(T_s f) \, ds \]

for \( f \in D_0 \) and \( t \geq 0 \). Moreover, if we can especially take \( M = 1 \) and \( \omega = 0 \), then the estimate above can be also written as the following:

\[ \| T_n^{[nt]} P_n f - P_n T_t f \|_{B_n} \leq \sqrt{\frac{t}{n}} \varphi_n(f) + \frac{1}{n} \varphi_n(f) + \int_0^t \psi_n(T_s f) \, ds \quad (1.5) \]

for \( f \in D_0 \) and \( t \geq 0 \).

In Section 2, we give the proof of Theorem 1.3. Since the Banach space \( B_n \) on which the operator \( T_n \) is defined may vary as \( n \) does, one may think of the proof as more complicated than that given in [1]. However, we can see that such generality does not essentially affect the proof itself, which implies that we can obtain many kinds of rates of convergences under various settings. Moreover, we also give a rate of convergence under an additional assumption on the limiting semigroup \( (T_t)_{t \geq 0} \) as an immediate consequence of the main result. Section 3 is devoted to applications of Theorem 1.3 to the rates of convergences for central limit theorems (CLTs, in short) in probability theory. The speed rate of the CLT is called the Berry–Esseen type bound and it corresponds to a certain refinement of the CLT. So far, a lot of ways to establish this kind of bound are known. See e.g., [5, Chapter XVI] for the proof of the Berry–Esseen type bound based on the convergence of characteristic functions. On the other hand, there is an alternative representation of the CLT in terms of the convergence of semigroups, whose proof is given by employing Trotter’s approximation theorem. We give the Berry–Esseen type bound for the semigroup CLT by using Theorem 1.3. As a further problem, we also consider a CLT for magnetic transition operators on crystal lattices discussed in [9]. We give a quantitative estimate of the CLT by applying Theorem 1.3 as well.

2. Proof of Theorem 1.3

The proof of Theorem 1.3 is given in this section. We basically follow the argument given in [1, Theorem 1.1]. Here, we should pay an attention to the proof since the Banach space \( B_n \) may vary for each \( n \in \mathbb{N} \) in our setting. However, as we can see below, somewhat surprisingly, such general settings do not essentially affect the proof. Moreover, it is pointed out in [2] that some commutativeness conditions like \( T_n T_m = T_m T_n, \, m, n \in \mathbb{N} \), play an important role in the proof. On the other hand, in our setting, such a condition does not make sense since the domains of \( T_n \) and \( T_m \) may be distinct. A key to our proof
is a condition that not only $f \in D$ but also $T_tf \in D$, $t \geq 0$, holds, which is introduced as an alternative condition in [2].

Proof of Theorem 1.3. The existence of the $C_0$-semigroup $(T_t)_{t \geq 0}$ generated by the closure of $(\mathfrak{A}, D)$ has been shown in [12, Theorem 2.13]. Therefore, we concentrate on the proof of (1.4). We split the proof into four steps.

**Step 1.** Consider the bounded linear operator $\mathfrak{A}_n = n(T_n - I)$ on $B_n$ for $n \in \mathbb{N}$, which generates a $C_0$-semigroup $(S_t^{(n)})_{t \geq 0}$ on $B_n$ given by

$$S_t^{(n)} = e^{\mathfrak{A}_n} = e^{nt}e^{nt} = e^{nt}\left(\sum_{k=0}^{\infty} \frac{(nt)^k}{k!}T_t^k\right).$$

Note that (1.1) implies

$$\|S_t^{(n)}\| \leq e^{-nt}\left(\sum_{k=0}^{\infty} \frac{(nt)^k}{k!}\|T_t^k\|\right) \leq M \exp\left(nt(e^{\omega/n} - 1)\right), \quad n \in \mathbb{N}, t \geq 0. \quad (2.1)$$

Recall that $D_0 = \{g \in D \mid T_tg \in D, t \geq 0\}$. Let $\{k(n)\}_{n=1}^{\infty}$ be an increasing sequence of positive integers and $f \in D_0$. We then have

$$\|T_t^{(k(n))}P_nf - P_nT_tf\|_{B_n} \leq \|T_t^{(k(n))}P_nf - S_{k(n)/n}^{(n)}P_nf\|_{B_n} + \|S_{k(n)/n}^{(n)}P_nf - S_t^{(n)}P_nf\|_{B_n} + \|S_t^{(n)}P_nf - P_nT_tf\|_{B_n} =: I_1(n) + I_2(n) + I_3(n). \quad (2.2)$$

We now try to estimate each term on the right-hand side of (2.2).

**Step 2.** We here give an estimation of $I_1(n)$. By applying [14, Lemma III.5.1] and an elementary inequality $e^x - 1 \leq xe^x$ for $x \geq 0$, we obtain

$$\|T_t^{(k(n))}P_nf - S_{k(n)/n}^{(n)}P_nf\|_{B_n} = \|e^{k(n)(T_t-I)}P_nf - T_t^{(k(n))}P_nf\|_{B_n} \leq M \exp\left(\omega(k(n) - 1)/n\right) \exp\left(k(n)(e^{\omega/n} - 1)\right) \times \sqrt{k(n)^2(e^{\omega/n} - 1)^2 + k(n)e^{\omega/n}}\|T_t-I\|P_nf\|_{B_n} \leq \frac{M}{n} \exp\left(\omega(k(n) - 1)/n\right) \exp\left(k(n)(e^{\omega/n} - 1)\right) \times \left(k(n)(e^{\omega/n} - 1) + \sqrt{k(n)e^{\omega/n}}\right)\varphi_n(f) \leq M \exp\left(\omega(k(n) - 1)/n\right) \exp\left(e^{\omega/n}k(n)/n\right)\left(\frac{\omega k(n)}{n}e^{\omega/n} + \frac{\sqrt{k(n)}}{n}e^{\omega/2n}\right)\varphi_n(f) = M \exp\left(\omega(e^{\omega/n} + 1)k(n)/n\right)e^{-\omega/n}\left(\frac{\omega k(n)}{n}e^{\omega/n} + \frac{\sqrt{k(n)}}{n}e^{\omega/2n}\right)\varphi_n(f)$$
\[ \leq M \exp \left( 2\omega \frac{e^{\omega/n}k(n)}{n} \frac{k(n)}{n} + \frac{\sqrt{k(n)}}{n} \right) \varphi_n(f). \] (2.3)

Moreover, the estimation of \( I_2(n) \) in (2.2) is given by

\[
\| S_{k(n)/n}^{(n)} P_n f - S_t^{(n)} P_n f \|_{B_n}
= \left\| \int_t^{k(n)/n} S_s^{(n)} (n(T_n - I)) P_n f ds \right\|_{B_n}
\leq M \exp \left( nt_n (e^{\omega/n} - 1) \right) \frac{k(n)}{n} - t \varphi_n(f)
\leq M \exp \left( \omega t_n e^{\omega/n} \right) \frac{k(n)}{n} - t \varphi_n(f), \] (2.4)

where we recall that \( t_n := \max\{t, k(n)/n\} \).

**Step 3.** Since both operators \( S_{t-s}^{(n)} : B_n \to B_n \) and \( P_n T_t : E \to B_n \) are bounded, we have

\[
S_t^{(n)} P_n f - P_n T_t f = - \int_0^t \frac{d}{ds} (S_{t-s}^{(n)} P_n T) f ds
= \int_0^t (S_{t-s}^{(n)} \mathcal{A}_n P_n - S_{t-s}^{(n)} P_n \mathcal{A}) f ds
= \int_0^t S_{t-s}^{(n)} (\mathcal{A}_n P_n - P_n \mathcal{A}) T_s f ds
\]

for \( t \geq 0 \), where we used the fact that \( T_s \) and its generator \( \mathcal{A} \) commute. Therefore, it follows from (2.1), (1.3) and \( T_s f \in D \) that

\[
\| S_t^{(n)} P_n f - P_n T_t f \|_{B_n}
= \left\| \int_0^t S_{t-s}^{(n)} (\mathcal{A}_n P_n - P_n \mathcal{A}) T_s f ds \right\|_{B_n}
\leq M \int_0^t \exp \left( n(t-s)(e^{\omega/n} - 1) \right) \| (\mathcal{A}_n P_n - P_n \mathcal{A}) T_s f \|_{B_n} ds
\leq M \int_0^t \exp \left( \omega (t-s) e^{\omega/n} \right) \psi_n(T_s f) ds
\leq M \exp \left( \omega e^{\omega/n} \right) \int_0^t \exp \left( - \omega s e^{\omega/n} \right) \psi_n(T_s f) ds \] (2.5)

for \( t \geq 0 \).
Step 4. We combine (2.2) with (2.3), (2.4) and (2.5). Then we obtain
\[
\|T_n^{k(n)} P_n f - P_n T_t f\|_{B_n}
\]
\[
\leq M \exp \left( 2\omega e^{\omega/n} k(n)/n \right) \left( \frac{\omega k(n)}{n} + \frac{\sqrt{k(n)}}{n} \right) \varphi_n(f)
\]
\[
+ M \exp \left( \omega_t e^{\omega/n} \right) \left| \frac{k(n)}{n} - t \right| \varphi_n(f)
\]
\[
+ M \exp \left( \omega_t e^{\omega/n} \right) \int_0^t \exp \left( -\omega s e^{\omega/n} \right) \psi_n(T_s f) \, ds
\]
for all \( f \in D_0 \) and \( t \geq 0 \), which is the very desired estimate (1.4). □

Before closing this section, we give a corollary of Theorem 1.3, under an additional assumption that the limiting semigroup \((T_t)_{t \geq 0}\) preserves \(D\) and the seminorm \(\psi_n\) and the limiting semigroup are commutative in a sense.

Remark 2.1. We assume that \(T_t(D) \subset D\) for \( t \geq 0 \) and \(\psi_n(T_t f) \leq \|T_t\|\psi_n(f)\) for \( f \in D \) and \( t \geq 0 \). Then we have
\[
\int_0^t \exp \left( -\omega s e^{\omega/n} \right) \psi_n(T_s f) \, ds \leq \int_0^t \exp \left( \omega s (1 - e^{\omega/n}) \right) \psi_n(f) \, ds \leq t \psi_n(f)
\]
for \( f \in D \) and \( t \geq 0 \). Therefore, Formula (1.4) becomes
\[
\|T_n^{k(n)} P_n f - P_n T_t f\|_{B_n}
\]
\[
\leq M \exp \left( 2\omega e^{\omega/n} k(n)/n \right) \left( \frac{\omega k(n)}{n} + \frac{\sqrt{k(n)}}{n} \right) \varphi_n(f)
\]
\[
+ M \exp \left( \omega_t e^{\omega/n} \right) \left| \frac{k(n)}{n} - t \right| \varphi_n(f) + M t \exp \left( \omega te^{\omega/n} \right) \psi_n(f)
\]
(2.6)
for \( f \in D \) and \( t \geq 0 \).

3. Applications of Theorem 1.3

This section is concerned with several applications of the rate of convergence in Trotter’s approximation theorem to obtain some quantitative estimates for limit theorems in probability theory.

3.1. Quantitative estimates of CLTs. It is known that the CLT plays a crucial role in probability theory. Let \( \{\xi_i\}_{i=1}^\infty \) be a sequence of independently and identically distributed (i.i.d., in short) \(\mathbb{Z}^d\)-valued random variables given by
\[
\mathbb{P}(\xi_1 = e_k) = \mathbb{P}(\xi_1 = -e_k) = \frac{1}{2d}, \quad k = 1, 2, \ldots, d,
\]
where \( e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d \) is the unit vector for \( k = 1, 2, \ldots, d \).

The CLT describes the fluctuation of the random variable defined by

\[
X_n := \frac{\xi_1 + \xi_2 + \cdots + \xi_n}{\sqrt{n}}, \quad n \in \mathbb{N},
\]

as \( n \) tends to infinity. More precisely, it asserts the convergence of the distribution of \( X_n \) to the \( d \)-dimensional standard normal distribution \( N(0, I) \) as \( n \to \infty \), where \( I \) denotes the \( d \times d \)-identity matrix. Note that another representation of the CLT is given in terms of the convergence of the discrete semigroups associated with \( X_n \) to the continuous heat semigroup generated by the Laplacian on \( \mathbb{R}^d \). Note that the assertion above is easily extended to the case where the sequence of \( \mathbb{R}^d \)-valued i.i.d. random variables \( \{\xi_i = (\xi_1^i, \xi_2^i, \ldots, \xi_d^i)\}_{i=1}^\infty \) satisfies \( E[\xi_1] = \mu \in \mathbb{R}^d \) and \( \text{Cov}(\xi_i^j, \xi_1^j) = \sigma_{ij} \) so that \( (\sigma_{ij})_{i,j=1}^d \) forms a positive semidefinite symmetric matrix, though we need to replace \( X_n \) by

\[
X_n = \frac{\xi_1 + \xi_2 + \cdots + \xi_n - n\mu}{\sqrt{n}}, \quad n \in \mathbb{N}.
\]

As a refinement of the CLT, the Berry–Esseen type bound is well-known, which gives a rate of convergence of the CLT in the parameter \( n \). We see that the Berry–Esseen type bound is easily obtained by a simple application of Theorem 1.3. Let us put \( B_n \equiv C_\infty(\mathbb{Z}^d) \) for \( n \in \mathbb{N} \) endowed with the sup-norm \( \| \cdot \|_\infty \) and \( E = C_\infty(\mathbb{R}^d) \) with \( \| \cdot \|_{\infty} \). Here, we denote by \( C_\infty(M) \) the space of all functions on a topological space \( M \) vanishing at infinity. We define a bounded linear operator \( P_n : C_\infty(\mathbb{R}^d) \to C_\infty(\mathbb{Z}^d), n \in \mathbb{N}, \) by

\[
P_n f(x) := f(n^{-1/2}x), \quad x \in \mathbb{Z}^d.
\]

Then we easily see that \( \| P_n \| \leq 1, n \in \mathbb{N} \), and the sequence \( \{(C_\infty(\mathbb{Z}^d), P_n)\}_{n=1}^\infty \) approximates the Banach space \( (C_\infty(\mathbb{R}^d), \| \cdot \|_{\infty}) \).

We put \( \mathcal{E} := \{\pm e_1, \pm e_2, \ldots, \pm e_d\} \) and define a linear operator \( T_n, n \in \mathbb{N}, \) on \( C_\infty(\mathbb{Z}^d) \) by

\[
T_n f(x) \equiv \mathcal{L} f(x) := \frac{1}{2d} \sum_{e \in \mathcal{E}} f(x + e), \quad x \in \mathbb{Z}^d.
\]

The operator \( \mathcal{L} \) is called the transition operator associated with \( \{\xi_i\}_{i=1}^\infty \) in the context of probability theory. Note that \( \| \mathcal{L} \| \leq 1 \) holds.

We define a subspace \( D \) by

\[
D := C_\infty(\mathbb{R}^d) = \bigcap_{k=1}^\infty \left\{ f \in C_\infty(\mathbb{R}^d) : \lim_{|x| \to \infty} \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}(x) = 0, \quad i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, d\} \right\},
\]
the set of $C^\infty$-functions on $\mathbb{R}^d$ all of whose partial derivatives of an arbitrary order vanish at infinity. We easily see that $D$ is a dense subspace of $C^\infty(\mathbb{R}^d)$. Moreover, it is known that $\text{Ran}(\lambda - \Delta)$ is dense in $C^\infty(\mathbb{R}^d)$ for some $\lambda > 0$ and the closure of $(\mathfrak{A} = \Delta, C^\infty(\mathbb{R}^d))$ generates a heat semigroup $(T_t = e^{t\Delta})_{t \geq 0}$ (see e.g., [6, Proposition 4.4.4]). Here $\Delta = \sum_{i=1}^d (\partial^2 / \partial x_i^2)$ stands for the (negative) Laplacian on $\mathbb{R}^d$. Under these settings, the CLT can be also written as follows:

$$\lim_{n \to \infty} \|L[n]P_nf - P_ne^{t\Delta}f\|_\infty = 0, \quad f \in C^\infty(\mathbb{R}^d), \quad t \geq 0. \quad (3.1)$$

We can show that

$$\|n(L - I)P_nf\|_\infty \leq \varphi_n(f) := \|\Delta f\|_\infty + \frac{d}{6\sqrt{n}} \max_{i=1,2,\ldots,d} \left\| \frac{\partial^3 f}{\partial x_i^3} \right\|_\infty$$

and

$$\|n(L - I)P_nf - P_n\Delta f\|_\infty \leq \psi_n(f) := \frac{d}{6\sqrt{n}} \max_{i=1,2,\ldots,d} \left\| \frac{\partial^3 f}{\partial x_i^3} \right\|_\infty$$

for $f \in C^\infty(\mathbb{R}^d)$. Indeed, by applying the Taylor formula to the function $f$ at $x/\sqrt{n}$, we have

$$n(L - I)P_nf(x)$$

$$= \frac{n}{2d} \sum_{e \in \mathcal{E}} f\left( \frac{x + e}{\sqrt{n}} \right) - n f\left( \frac{x}{\sqrt{n}} \right)$$

$$= \frac{1}{2d} \sum_{e \in \mathcal{E}} \left\{ \sqrt{n} \sum_{i=1}^d \frac{\partial f}{\partial x_i}\left( \frac{x}{\sqrt{n}} \right) e^i + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}\left( \frac{x}{\sqrt{n}} \right) e^i e^j \right. \right.$$

$$+ \frac{1}{6\sqrt{n}} \sum_{i,j,k=1}^d \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(\theta) e^i e^j e^k \left. \right\}$$

$$= \frac{1}{2d} \sum_{e \in \mathcal{E}} \left\{ \sqrt{n} \sum_{i=1}^d \frac{\partial f}{\partial x_i}\left( \frac{x}{\sqrt{n}} \right) e^i + \frac{1}{2} \sum_{i=1}^d \frac{\partial^2 f}{\partial x_i^2}\left( \frac{x}{\sqrt{n}} \right) (e^i)^2 + \frac{1}{6\sqrt{n}} \sum_{i=1}^d \frac{\partial^3 f}{\partial x_i^3}(\theta)e^i e^j e^k \right\}$$

for any $f \in C^\infty(\mathbb{R}^d)$ and some $\theta = \theta(e) \in \mathbb{R}^d$, where $e^i, \ i = 1, 2, \ldots, d$, denotes the $i$th component of $e$. By virtue of

$$\sum_{e \in \mathcal{E}} e^i = 0, \quad \sum_{e \in \mathcal{E}} (e^i)^2 = 2, \quad i = 1, 2, \ldots, d,$$

we have

$$n(L - I)P_nf(x) = P_n\Delta f(x) + \frac{1}{12d\sqrt{n}} \sum_{e \in \mathcal{E}} \sum_{i=1}^d \frac{\partial^3 f}{\partial x_i^3}(\theta)(e^i)^3.$$
Hence, we conclude
\[
\| n(\mathcal{L} - I)P_n f \|_\infty \leq \| \Delta f \|_\infty + \frac{d}{6\sqrt{n}} \max_{i=1,2,\ldots,d} \left\| \frac{\partial^3 f}{\partial x_i^3} \right\|_\infty,
\]
\[
\| n(\mathcal{L} - I)P_n f - P_n \Delta f \|_\infty \leq \frac{d}{6\sqrt{n}} \max_{i=1,2,\ldots,d} \left\| \frac{\partial^3 f}{\partial x_i^3} \right\|_\infty
\]
for all \( f \in C^\infty_0(\mathbb{R}^d) \), which are the desired estimates (3.2) and (3.3).

Since \( T_t(D) \subset D \) and \( \psi_n(T_t f) \leq \psi_n(f) \) hold for \( t \geq 0 \) and \( f \in D \) by definition, Theorem 1.3 (in particular, Equation (2.6) in Remark 2.1) allows us to establish the following refinement of (3.1).

Theorem 3.1. Suppose that \( f \in C^\infty_0(\mathbb{R}^d) \) and \( t \geq 0 \). Then, there exists a positive constant \( C = C(t, f, d) > 0 \) such that
\[
\| \mathcal{L}^{[nt]} P_n f - P_n e^{t\Delta} f \|_\infty \leq \left( \sqrt{\frac{t}{n}} + \frac{1}{n} \right) \left( \| \Delta f \|_\infty + \frac{d}{6\sqrt{n}} \max_{i=1,2,\ldots,d} \left\| \frac{\partial^3 f}{\partial x_i^3} \right\|_\infty \right) + \frac{td}{6\sqrt{n}} \max_{i=1,2,\ldots,d} \left\| \frac{\partial^3 f}{\partial x_i^3} \right\|_\infty \leq \frac{C}{\sqrt{n}}
\]
for all \( n \in \mathbb{N} \).

This theorem implies that the rate of the convergence in the usual CLT is of order \( n^{-1/2} \), which is a fundamental result in numerical calculations of some discrete approximation schemes of diffusion processes such as Brownian motions with values in \( \mathbb{R}^d \). We should emphasize that a lot of studies to establish error bounds similar to Theorem 3.1 are known. However, the Berry–Esseen type bound for the CLT in terms of semigroups has not appeared in existing literatures. We refer to e.g., [3, Chapter XVI] for the proof of the Berry–Esseen type bound based on the convergence of characteristic functions.

3.2. Quantitative estimates of CLTs for the magnetic transition operator. In this subsection, we give another application of Theorem 1.3 to find out the rate of convergence of CLTs for magnetic transition operators on crystal lattices. Before fixing the setting, we briefly review the magnetic Schrödinger operator on \( \mathbb{R}^d \). Let \( B \) be a closed 2-form on \( \mathbb{R}^d \), which is called a magnetic field on \( \mathbb{R}^d \). Let \( A \) be a vector potential for \( B \), that is, \( dA = B \), where \( d \) is the exterior derivative. We put \( \nabla_A := d - \sqrt{-1} A \). Then the magnetic Schrödinger operator is given by \( \nabla_A^* \nabla_A \). We see that the magnetic field \( B \) is periodic with respect to \( \mathbb{Z}^d \) if and only if \( \sigma^* A - A = df_\sigma, \sigma \in \mathbb{Z}^d \) for some \( f_\sigma \in C^\infty(\mathbb{R}^d) \). Moreover, if it holds that \( B = \sum_{1 \leq i < j \leq d} b_{ij} dx_i \wedge dx_j \) with some \( b_{ij} \in \mathbb{R} \), we then take a linear vector potential \( A = \sum_{i,j=1}^d a_{ij} x_j dx_i \), where \( b_{ij} = a_{ji} - a_{ij} \) for \( i, j = 1, 2, \ldots, d \).

A crystal lattice is defined to be a covering graph \( X = (V, E) \) of a finite graph \( X_0 = (V_0, E_0) \) whose covering transformation group is isomorphic to \( \mathbb{Z}^d \). Here, \( V \) (resp. \( V_0 \)) is the set of all vertices and \( E \) (resp. \( E_0 \)) is the set of all oriented edges of \( X \) (resp. \( X_0 \)). For
an edge $e \in E$, we denote by $o(e), t(e), \tau$ the origin, the terminus and the inverse edge of $e$, respectively. We put $E_x := \{ e \in E \mid o(e) = x \}$ for $x \in V$. Intuitively, a crystal lattice is an infinite graph with a fundamental pattern consisting of finite number of edges and vertices, which appears periodically.

Let us consider a discrete analogue of the semigroup generated by the Schrödinger operator with periodic magnetic field. Let $p : E \to (0, 1]$ be a $\mathbb{Z}^d$-invariant transition probability on $X$, that is, $\sum_{e \in E_x} p(e) = 1$ for $x \in V$ and $p(\gamma e) = p(e)$ for $\gamma \in \mathbb{Z}^d$ and $e \in E$. Here, $\gamma e$ means the parallel translation of $e$ along $\gamma \in \mathbb{Z}^d$. Note that $p$ is also induced on the finite quotient graph $X_0 = \mathbb{Z}^d \setminus X$ through the covering map $\pi : X \to X_0$. Then the Perron–Frobenius theorem implies the unique existence of the normalized invariant measure $m$ on $V_0$. Namely, $m$ is a positive function on $V_0$ satisfying

$$\sum_{e \in (E_0)_x} p(\tau)m(t(e)) = m(x), \quad x \in V_0, \quad \text{and} \quad \sum_{x \in V_0} m(x) = 1.$$ 

In the present paper, we assume the detailed balanced condition

$$p(e)m(o(e)) = p(\tau)m(t(e)), \quad e \in E_0.$$ 

Then the random walk induced by $p$ is said to be $(m)$-symmetric. We define the magnetic transition operator $H_\omega : C_\infty(X) \to C_\infty(X)$ by

$$H_\omega f(x) := \sum_{e \in E_x} p(e) \exp(\sqrt{-1}\omega(e))f(t(e)), \quad x \in V,$$

where $\omega : E \to \mathbb{R}$ is a 1-cochain on $X$ satisfying $\omega(\tau) = -\omega(e)$ for $e \in E$. We set the following technical but natural conditions for 1-cochains $\omega : E \to \mathbb{R}$.

**(A1)**: $\omega$ is weakly $\mathbb{Z}^d$-invariant, that is, the cohomology class $[\omega] \in H^1(X, \mathbb{R})$ is $\mathbb{Z}^d$-invariant, where $H^1(X, \mathbb{R})$ is the first cohomology group of $X$.

**(A2)**: For every $\sigma \in \mathbb{Z}^d$, it holds that

$$\sum_{e \in E_x} p(e)(\omega(\sigma^{-1}e) - \omega(e)) = 0, \quad x \in V.$$ 

**(A3)**: It holds that $\sigma_1(\sigma_2 \omega - \omega) = \sigma_2 \omega - \omega$ for $\sigma_1, \sigma_2 \in \mathbb{Z}^d$.

Both (A1) and (A3) essentially mean the invariance of a 1-cochain $\omega$ under the $\mathbb{Z}^d$-action. On the other hand, a 1-cochain satisfying (A2) is said to be harmonic, which corresponds to a discrete analogue of a harmonic form on Riemannian manifolds. In fact,
for \( b \in \mathbb{R} \), the classical Harper operator on \( \mathbb{Z}^2 \) defined by

\[
(H_b f)(m, n) := \frac{1}{4} \left\{ \exp \left( \frac{1}{2} \sqrt{-1} bn \right) f(m + 1, n) + \exp \left( - \frac{1}{2} \sqrt{-1} bm \right) f(m, n + 1) \right. \\
+ \exp \left( - \frac{1}{2} \sqrt{-1} bm \right) f(m, n + 1) + \exp \left( \frac{1}{2} \sqrt{-1} bn \right) f(m - 1, n) \}
\]

for \((m, n) \in \mathbb{Z}^2 \) satisfies (A1), (A2) and (A3). Hence, the operator \( H_\omega \) with these conditions is also called the generalized Harper operator on \( X \).

A piecewise linear map \( \Phi : V \to \mathbb{R}^d \) is called a periodic realization of a crystal lattice \( X \) if it satisfies \( \Phi(\sigma x) = \Phi(x) + \sigma \) for \( x \in V \) and \( \sigma \in \mathbb{Z}^d \). By noting geometric features of crystal lattices, Kotani obtained the following CLT of semigroup type for magnetic transition operators.

**Proposition 3.2** (cf. [9] Theorem 4]). Let \( \Phi_0 : X \to \mathbb{R}^d \) be a periodic realization of \( X \) satisfying

\[
\sum_{e \in E_x} p(e) \left\{ \Phi_0(t(e)) - \Phi_0(o(e)) \right\} = 0, \quad x \in V.
\]

Suppose that \( \omega \) satisfies (A1), (A2) and (A3). Then, there exists a flat Riemannian metric \( g \) on \( \mathbb{R}^d \), a linear vector potential \( A = \sum_{i,j=1}^d a_{ij}x_j dx_i \) on \( (\mathbb{R}^d, g) \) and a harmonic 1-form \( \omega_0 \) on \( X_0 \) such that

\[
\omega(e) = -\langle A \Phi_0(o(e)), v_e \rangle_g - \frac{1}{2} \langle A v_e, v_e \rangle_g + \pi^* \omega_0(e), \quad e \in E,
\]

where \( v_e := \Phi_0(t(e)) - \Phi_0(o(e)) \) for \( e \in E \) and \( A = (a_{ij})_{i,j=1}^d \). Moreover, we have

\[
\lim_{n \to \infty} \| H^{[nt]}_{\frac{\pi \omega}{}^\pi} P_n f - P_n e^{t \nabla_A \nabla_A} f \|_\infty = 0
\]

for every \( f \in C_\infty(\mathbb{R}^d) \) and \( t \geq 0 \), where \( P_n : C_\infty(\mathbb{R}^d) \to C_\infty(X) \) is an approximation operator given by

\[
P_n f(x) := f \left( \frac{1}{\sqrt{n}} \Phi_0(x) \right), \quad x \in V, \quad n \in \mathbb{N}.
\]

We note that, if \( \omega = 0 \), then the operator \( \nabla_A \nabla_A \) becomes the usual (negative) Laplacian \( \Delta \) on \( (\mathbb{R}^d, g) \). The flat metric \( g \) on \( \mathbb{R}^d \) above is called the Albanese metric. See e.g., [11] for its geometric meaning as well as its explicit construction.

By applying Theorem 1.3, we show the following quantitative estimate of Proposition 3.2.

**Theorem 3.3.** For \( f \in C_\infty(\mathbb{R}^d) \) and \( t \geq 0 \), there exists a positive constant \( C = C(t, f, \Phi_0) > 0 \) such that

\[
\| H^{[nt]}_{\frac{\pi \omega}{}^\pi} P_n f - P_n e^{t \nabla_A \nabla_A} f \|_\infty \leq \frac{C}{\sqrt{n}}, \quad n \in \mathbb{N}.
\]
Before giving the proof of Theorem 3.3, we show the following lemma.

**Lemma 3.4.** Let $\Phi_0 : V \to \mathbb{R}^d$ be a periodic realization satisfying (3.4). Then there exists a positive constant $C = C(f, \Phi_0) > 0$ such that

$$\|n(H_{\pi n} - I)P_n f\|_\infty \leq \varphi_n(f) = \|(\nabla A^* \nabla A)f\|_\infty + \frac{C}{\sqrt{n}}, \quad f \in C_\infty^\infty(\mathbb{R}^d),$$

(3.6)

and

$$\|n(H_{\pi n} - I)P_n f - P_n(\nabla A^* \nabla A)f\|_\infty \leq \psi_n(f) = \frac{C}{\sqrt{n}}, \quad f \in C_\infty^\infty(\mathbb{R}^d).$$

(3.7)

**Proof.** By applying the Taylor formula to $\exp(\sqrt{-1} / n \omega(e)/n)$ and by noting (3.5), we have

$$\exp\left(\sqrt{-1} / n \omega(e)\right) = 1 - \sqrt{-1} / n \left\langle A\left(1 / n \Phi_0(o(e))\right), v_e \right\rangle_g$$

$$- \frac{1}{2n} \sqrt{-1} \langle A v_e, v_e \rangle_g + 2\sqrt{-1} \pi^* \omega_0(e)$$

$$+ \left\langle A\left(1 / n \Phi_0(o(e))\right), v_e \right\rangle_g^2 + J_n(\Phi_0, e),$$

where $J_n(\Phi_0, e)$ satisfies that $|J_n(\Phi_0, e)| \leq Cn^{-3/2}$ for some $C = C(\Phi_0) > 0$ independent of $e \in E$. Denote by $x_i$ the $i$th coefficient of $x \in \mathbb{R}^d$ with respect to the Albanese metric. Then, another use of the Taylor formula gives

$$n(H_{\pi n} - I)P_n f = -\sqrt{n} \sum_{e \in E} p(e) \left\{ \sqrt{-1} \left\langle A\left(1 / n \Phi_0(x)\right), v_e \right\rangle_g f\left(1 / n \Phi_0(x)\right) \right.$$

$$+ \frac{d}{dx_i} \left(1 / n \Phi_0(x)\right) (v_e)_i \right\}$$

$$+ \frac{1}{2} \sum_{e \in E} p(e) \left\{ \frac{\partial f}{\partial x_i} \left(1 / n \Phi_0(x)\right) (v_e)_i (v_e)_j \right.$$}

$$- 2\sqrt{-1} \left\langle A\left(1 / n \Phi_0(x)\right), v_e \right\rangle_g \sum_{i=1}^d \frac{\partial f}{\partial x_i} \left(1 / n \Phi_0(x)\right) (v_e)_i$$

$$- \frac{1}{2} \left(\sqrt{-1} \langle A v_e, v_e \rangle_g + 2\sqrt{-1} \pi^* \omega_0(e) \right.$$

$$+ \left\langle A\left(1 / n \Phi_0(x)\right), v_e \right\rangle_g^2 f\left(1 / n \Phi_0(x)\right) \right\} + \tilde{J}_n(\Phi_0, x),$$

(3.8)
where \( \tilde{J}_n(\Phi_0, f, x) \) satisfies \( \| \tilde{J}_n(\Phi_0, f, \cdot) \|_\infty \leq Cn^{-1/2} \) for some \( C > 0 \). We easily see that the first term of the right-hand side of (3.8) is zero since
\[
\sum_{e \in E_x} p(e)v_e = 0 \quad \text{and} \quad \sum_{e \in E_x} p(e)\omega(e) = 0, \quad x \in V.
\]
As for the second term of the right-hand side of (3.8), we can show that it is equal to
\[
-\frac{d}{\sqrt{n}} \frac{\partial^2 f}{\partial x^2_i}\left( \frac{1}{\sqrt{n}} \Phi_0(x) \right) + 2\sqrt{-1} \sum_{i,j=1}^d a_{ij}x_j \frac{\partial f}{\partial x_i}\left( \frac{1}{\sqrt{n}} \Phi_0(x) \right)
+ \left( \sqrt{-1} \sum_{i=1}^d a_{ii} + \sum_{j=1}^d \left( \sum_{i=1}^d a_{ij}x_j \right)^2 \right) f\left( \frac{1}{\sqrt{n}} \Phi_0(x) \right) + \tilde{J}_n'(\Phi_0, f, x)
= P_n(\nabla^*_A \nabla_A)f(x) + \tilde{J}_n'(\Phi_0, f, x)
\]
by following the same discussion as [9 pp. 473 and 474], where \( \tilde{J}_n'(\Phi_0, f, \cdot) \) satisfies \( \| \tilde{J}_n'(\Phi_0, f, \cdot) \|_\infty \leq Cn^{-1/2} \) for some \( C > 0 \). We note that the ergodic theorem for the transition operator acting on \( \ell^2(X_0) = \{ f : V_0 \to \mathbb{C} \} \) plays a crucial role. This completes the proof.

Let \( C_x([0, t], \mathbb{R}^d) \) be the set of all continuous functions \( w : [0, t] \to \mathbb{R}^d \) with \( w(0) = x \in \mathbb{R}^d \) and \( \mu \) the usual Wiener measure on \( C_x([0, t], \mathbb{R}^d) \). The Schrödinger semigroup \( (e^{t\nabla^*_A \nabla_A})_{t \geq 0} \) acts on \( C_\infty(\mathbb{R}^d) \) and it is represented as
\[
(e^{t\nabla^*_A \nabla_A}f)(x) = \int_{C_x([0, t], \mathbb{R}^d)} \exp\left( \sqrt{-1} \int_0^t A\left( w(s) \right) \circ dw(s) \right) f(w(t)) \mu(dw)
\]
for every \( f \in C_\infty(\mathbb{R}^d) \) by virtue of the Feynman–Kac formula, where \( \circ dw(s) \) denotes the Stratonovich integral. Moreover, we verify that \( \text{Ran}(\lambda - \nabla^*_A \nabla_A) \) is dense in \( C_\infty(\mathbb{R}^d) \) for some \( \lambda > 0 \) and the closure of \( (\nabla^*_A \nabla_A, C_\infty(\mathbb{R}^d)) \) generates the Schrödinger semigroup \( (e^{t\nabla^*_A \nabla_A})_{t \geq 0} \) (see [9 Section 1]). Since it holds that \( (e^{t\nabla^*_A \nabla_A})(D) \subset D \) and \( \psi_n(e^{t\nabla^*_A \nabla_A}) \leq \psi_n(f) \) for \( t \geq 0 \) and \( f \in D \), Theorem 3.3 is obtained as a consequence of (3.6) and (3.7) in Lemma 3.4.

As far as we know, there seems to be no results establishing the rate of convergence of the (generalized) Harper operators to the magnetic Schrödinger operator. Hence, Theorem 3.3 gives a new contribution to the study of magnetic Schrödinger operators on periodic graphs. Since our main result (Theorem 1.3) is given in full generality, we expect further applications of it in various settings.

Remark 3.5. The periodic realization \( \Phi_0 \) satisfying (3.4) is called the harmonic realization, which was introduced in [10] and was regarded as a discrete analogue of harmonic maps on Riemannian manifolds. It also describes the most natural configurations of a crystal from a geometric perspective. We note that Theorem 3.3 as well as Proposition
3.2 holds even when the given realization $\Phi$ is not always harmonic, since the difference $|\Phi(x) - \Phi_0(x)|$ is uniformly bounded in $x \in V$ due to the periodicities. See also [9] Section 4 for related discussions.

Acknowledgements. The author would like to thank the anonymous referee for providing valuable comments and suggestions. This work is supported by JSPS KAKENHI Grant No. 19K23410.

References

[1] M. Campiti and C. Tacelli: Rate of convergence in Trotter’s approximation theorem, Constr. Approx. 28 (2008), no. 3, 333–341.
[2] M. Campiti and C. Tacelli: Erratum to: Rate of convergence in Trotter’s approximation theorem, Constr. Approx. 31 (2010), no. 3, 459–462.
[3] M. Campiti and C. Tacelli: Trotter’s approximation of semigroups and order of convergence in $C^{2,\alpha}$-spaces, J. Approx. Theory 162 (2010), no. 2, 2303–2316.
[4] K.-J. Engel and R. Nagel: One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics 194, Springer-Verlag, New York, 2000.
[5] W. Feller: An Introduction to Probability Theory and Its Applications, Vol. 2, Second edition, John Wiley and Sons, Inc., New York-London-Sydney, 1971.
[6] G. Kallianpur and P. Sundar: Stochastic Analysis and Diffusion Processes, Oxford Graduate Texts in Mathematics 24, Oxford University Press, Oxford, 2014.
[7] T. Kato: Perturbation Theory for Linear Operators, Second edition, Grundlehren der Mathematischen Wissenschaften, Band 132, Springer-Verlag, Berlin-New York, 1976.
[8] J. Kisyński: A proof of the Trotter–Kato theorem on approximation of semi-groups, Colloq. Math. 18 (1967), 181–184.
[9] M. Kotani: A central limit theorem for magnetic transition operators on a crystal lattice, J. London Math. Soc. (2) 65 (2002), no. 2, 464–482.
[10] M. Kotani and T. Sunada: Standard realizations of crystal lattices via harmonic maps, Trans. Amer. Math. Soc. 353 (2000), no. 1, 1–20.
[11] M. Kotani and T. Sunada: Large deviation and the tangent cone at infinity of a crystal lattice, Math. Z. 254 (2006), no. 4, 837–870.
[12] T. G. Kurtz: Extensions of Trotter’s operator semigroup approximation theorems, J. Funct. Anal. 3 (1969), 354–375.
[13] E. M. Mangino and I. Rasa: A quantitative version of Trotter’s approximation theorem, J. Approx. Theory 146 (2007), no. 2, 149–156.
[14] A. Pazy: Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences 44, Springer-Verlag, New York, 1983.
[15] H. F. Trotter: Approximation of semi-groups of operators, Pac. J. Math. 8 (1958), 887–919.

Department of Mathematics, Faculty of Education, Shizuoka University, 836, Ohya, Suruga-ku, Shizuoka, 422-8529, Japan

Email address: namba.ryuya@shizuoka.ac.jp