Transient analysis of the subordinated chain of a state dependent pure birth process

Andrea Monsellato

September 2, 2011

Abstract
Consider a pure birth process with intensities $\lambda_k = \frac{1}{1+k}$, with $k = 0, 1, 2, ..., $ we show that the subordinated chain is assimilable to a Bernoullian scheme with dependent successes probabilities, also we show a direct link with a degenerate Polya urn replacement scheme [3]. We compute explicitly transition probabilities, by generating function method, of the subordinated chain and give some interesting bounds using the centering sequence approach proposed by MacDiarmid [2].

1 Introduction
A pure birth process with intensities $\lambda_0, \lambda_1, ... \geq 0$ is a process $\{X(t) : t \geq 0\}$ taking value in $S = \{0, 1, 2, ...\}$ such that:

- $X(0) \geq 0$; if $s < t$ then $X(s) \leq X(t)$,
- $P(X(t + h) = k + m|X(t) = k) = $ \begin{cases} \lambda_m h + o(h), & \text{if } m = 1 \\ o(h), & \text{if } m > 1 \\ 1 - \lambda_m h + o(h), & \text{if } m = 0 \end{cases}$ if $s < t$ then, conditional on the value of $X(s)$, the increment $X(t) - X(s)$ is independent of all arrivals prior to $s$.

If $\lambda_n = \lambda$ for all $n$ a birth process with intensity $\lambda_0, \lambda_1, ...$ is called Poisson process.

Now consider a pure birth process with intensities $\lambda_k = \frac{\lambda}{1+k}$ with $k \geq 0$ where $\lambda > 0$.
Let $p(k, t) = P(X(t) = k)$ where $\{X(t) : t \geq 0\}$ then

\[
\begin{cases}
    p'(0, t) = \lambda p(0, t) \\
    p'(k, t) = \frac{\lambda}{k} p(k - 1, t) - \frac{1}{1+k} p(k, t), & \text{if } k \neq 0
\end{cases}
\]

(1)

In the following, without loss of generality, we choose $\lambda = 1$.

It is known that this problem has a unique positive solution and because $\sum_{i=0}^{+\infty} \frac{1}{\lambda_i} = +\infty$ the solution is a proper probability distribution see [1].
Proposition 1.1. Let
\[
\begin{align*}
    p(0, t) &= e^{-t}, \\
    p(k, t) &= \frac{1}{k!} \sum_{j=1}^{k+1} (-1)^{k+1-j} j^k (k+1)^{-j} e^{-\frac{j}{k+1}}, \quad \text{if } k \geq 1 
\end{align*}
\]
then (2) is the solution of (1).

Proof

Obviously \( p(0, t) = e^{-t} \). Now taking Laplace transform
\[
\hat{p}(k, \theta) := \int_0^{+\infty} e^{-\theta t} p(k, t) dt
\]
from (1) we have
\[
\begin{align*}
    \hat{p}(0, \theta) &= 1 \\
    \hat{p}(k, \theta) &= \frac{1}{k+1} \hat{p}(k-1, \theta), \quad k \in [1, n] 
\end{align*}
\]
so that
\[
\hat{p}(k, \theta) = \frac{1}{k!} \left[ A_1 \frac{\theta + 1}{\theta + 1} + A_2 \frac{\theta + \frac{1}{2}}{\theta + \frac{1}{2}} + \cdots + A_{k+1} \frac{\theta + \frac{k+1}{k+1}}{\theta + \frac{k+1}{k+1}} \right]
\]
where \( A_i, i = 1, \ldots, k + 1 \), satisfy the following equations
\[
A_1 \prod_{i=1; i \neq j}^{k+1} \left( \theta + \frac{1}{i} \right) + A_2 \prod_{i=1; i \neq 2}^{k+1} \left( \theta + \frac{1}{i} \right) + A_{k+1} \prod_{i=1; i \neq k+1}^{k+1} \left( \theta + \frac{1}{i} \right) = 1
\]
Choosing \( \theta = -\frac{j}{k+1}, j = 1, \ldots, k + 1, \) we obtain:
\[
\begin{align*}
    A_l \prod_{i=1; i \neq j}^{k+1} \left( -\frac{j}{k+1} + \frac{1}{i} \right) &= 0, \quad \forall j \neq l \\
    A_j &= \frac{1}{\prod_{i=1; i \neq j}^{k+1} \left( -\frac{j}{k+1} + \frac{1}{i} \right)}
\end{align*}
\]
Anti-transforming then for \( k \geq 1 \)
\[
p(k, t) = \frac{1}{k!} \sum_{j=1}^{k+1} A_j e^{-\frac{j}{k+1}} = \frac{1}{k!} \sum_{j=1}^{k+1} j^k e^{-\frac{j}{k+1}} \prod_{i=1; i \neq j}^{k+1} \frac{i}{j-i}
\]
From (4) we know that
\[
A_j = j^k \prod_{i=1; i \neq j}^{k+1} \frac{i}{j-i} = j^k (-1)^{k+1-j} \prod_{i=1; i \neq j}^{k+1} \frac{i}{j-i}
\]
Let

2
\[ Q_j := \prod_{i=1; i \neq j}^{k+1} \frac{i}{|j-i|} \quad (5) \]

Taking the logarithm of (5) after some algebraic calculations we obtain

\[ Q_j = \frac{(k+1)!}{j!(k+1-j)!} = \binom{k+1}{j} \]

then the thesis follows.

\[ \square \]

2 Subordinated chain

We remember briefly how to construct a generic subordinated chain.

A random point process on the positive half-line is a sequence \( \{T_n\}_{n \geq 0} \) of nonnegative random variable such that, almost surely,

\[
T_0 = 0 \\
0 < T_1 < T_2 < ... \\
\lim_{n \uparrow +\infty} T_n = +\infty
\]

**Definition 2.1.** Let \( \{\hat{X}_n\}_{n \geq 0} \) be a discrete-time HMC with countable state space \( E \) and transition matrix \( K = \{k_{i,j}\}_{i,j \in E} \) and let \( \{T_n\}_{n \geq 1} \) be an HPP on \( \mathbb{R}_+ \) with intensity \( \lambda \) and associated counting process \( N \). Suppose that \( \{\hat{X}_n\}_{n \geq 0} \) and \( \{T_n\}_{n \geq 1} \) are independent. The process \( \{X(t)\}_{t \geq 0} \) with value in \( E \) defined by

\[ X(t) = \hat{X}_{N(t)} \quad (6) \]

is called uniform Markov chain. The Poisson process \( N \) is called the clock, and the chain \( \{\hat{X}_n\}_{n \geq 0} \) is called the subordinated chain.

Let \( \{X_n\}_{n \geq 0} \) a homogeneous discrete Markov chain with countable state space \( N_0 \), considering the transition probabilities

\[
\begin{cases} 
P(X_{n+1} = j | X_n = i) = \frac{1}{i+1}, & j = i + 1 \\
P(X_{n+1} = j | X_n = i) = \frac{i}{i+1}, & j = i 
\end{cases} \quad (7)
\]

then from definition (6) \( \{X_n\}_{n \geq 0} \) is the uniform Markov chain of the birth process \( \{X(t)\}_{t \geq 0} \) with associated differential equation (1).

Let \( p_{n,k} = P(X_n = k) \) from (7) we obtain the following equations:

\[
\begin{cases} 
p_{n+1,k} = \frac{k}{k+1} p_{n,k} + \frac{1}{k+1} p_{n,k-1}, & \forall n \geq 0, \quad 1 \leq k \leq n + 1 \\
p_{n,j} = 0, & i \geq 0, \quad j \geq i + 1 \\
p_{0,0} = 1, \quad p_{n,0} = 0, & n \geq 1
\end{cases} \quad (8)
\]
Proposition 2.2. Let $p_{n,k}$ as in (8) then

$$p_{n,k} = \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} \binom{k+1}{i+1} i^k \left( \frac{i}{i+1} \right)^{n-k}$$

(9)

Proof

See Appendix.

□

2.1 Urn interpretation via dependent Bernoullian scheme

Consider an urn containing only one white ball and an arbitrary number of red balls. If we draw the white ball then we add a red ball, while if we draw a red ball we do not do anything. In both cases we reinsert the drawn ball and proceed to the next drawing.

This urn is a special type of Polya urn, see [3], corresponding to the following replacement scheme:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This Polya urn is assimilable to a sequence of dependent Bernoulli random variables, in fact if we consider that the number of red balls in the urn corresponds to the number of previous successes then the probability of next success depends only on this number.

Proposition 2.3. Let $Z_n = \sum_{i=0}^{n} Y_i$, $n \geq 0$, where

$$Y_n|Y_0, \ldots, Y_{n-1} \sim Ber\left(1 + \sum_{j=0}^{n-1} Y_j\right); \quad Y_0 = 0$$

(10)

then $Z_n \overset{d}{=} X_n$, where the probability distribution of $X_n$ is given by [7].

Proof

It is sufficient to observe that the distribution of $Z_n$ satisfies [3].

□

Remark

The limit distribution of $Z_n$ is normal, see [3] theorem 1.5.

Now we recover the mean of the subordinated chain, i.e. the mean number of red balls in the urn after $n$ steps.

Let $\{\tau_i\}_{i \geq 1}$ be a increasing sequence of positive r.v. such that $Y_n = 1$ if and only if $n = \tau_i$ for some $i$.

We have that:

$$P(\tau_i = n) = P(Y_{\tau_i-1+1} = 0, ..., Y_{\tau_i-1} = 0, Y_n = 1)$$

where $Y_m \sim Ber\left(\frac{1}{i}\right)$, with $m \in [\tau_i-1 + 1, n]$.

Then we have that $\tau_i \sim Geom\left(\frac{1}{i}\right)$. 
Knowing that \( P(X_n = k) = P(\sum_{i=1}^{n} Y_i = k) \iff P(\sum_{i=1}^{k} \tau_i \leq n), \) if we consider the r.v. \( T = \sum_{i=1}^{k} \tau_i, \) i.e. the number of trials necessary to obtain \( k \) successes, we have

\[
E(T) = E \left( \sum_{i=1}^{k} \tau_i \right) = \sum_{i=1}^{k} E(\tau_i)
\]

Solving the equation \( E(T) = n, \) with respect to the variable \( k, \) i.e. \( \frac{k(k+1)}{2} = n, \) we find

\[
E(X_n) = \frac{-1 + \sqrt{1 + 8n}}{2}
\]

We want to give a result of weak concentration for the embedded process \( X_n. \) For this purpose we give an upper bound for the variance.

Consider (10), for the second moment of \( X_n \) holds

\[
E(\sum_{i=1}^{k} \tau_i) = E\left( E(\sum_{i=1}^{k} \tau_i | X_n) \right) = E\left( E\left( \frac{X_n + 1}{1 + X_n} \right) \right) = 1 + E(\sum_{i=1}^{k} \tau_i)
\]

so that

\[
E(X_{n+1}^2) \leq 2 + E(X_n^2)
\]

Being \( E(X_n^2) \leq 1 \) we obtain \( E(X_{n+1}^2) \leq 2n \) then

\[
Var(X_n) \leq 2n - \left( \frac{-1 + \sqrt{1 + 8n}}{2} \right)^2 = -\frac{1}{2} + \sqrt{1 + 8n}
\]

Proposition 2.4. Let \( \{X_n\}_{n \geq 0} \) the process with probability distribution function as in (5), then for \( n \to +\infty \)

\[
P(|X_n - E(X_n)| \geq \epsilon E(X_n)) \leq \frac{Var(X_n)}{\epsilon^2 E^2(X_n)} \sim \frac{\sqrt{2}}{\epsilon^2 \sqrt{n}} \to 0
\]

Proof

It is an obvious consequence of (11), (12) and Chebychev’s inequality.

□

Considering the Bernoullian scheme representation (2.3) for the subordinated chain, we establish the large deviation bounds using some results due to McDiarmid [2].

To this purpose we recall the definition of centering sequences and some result about them. For more details see [2].

5
Definition 2.5. Given a sequence \( X = (X_1, X_2, \ldots) \) of (integrable) random variables the corresponding difference sequence is \( Y = (Y_1, Y_2, \ldots) \) where \( Y_k = X_k - X_{k-1} \) (and where we always set \( X_0 = 0 \)). Let \( \mu_k(x) = E(Y_k|X_{k-1} = x) \), that is \( \mu_k(X_{k-1}) \) is a version of \( E(Y_k|X_{k-1}) \). We call the sequence \( X \) centering if for each \( k = 2, 3, \ldots \) we may take \( \mu_k(x) \) to be a non-increasing function of \( x \).

Theorem 2.6. Let \( X_1, X_2, \ldots, X_n \) be a centering sequence with corresponding differences \( Y_k = X_k - X_{k-1} \) satisfying \( 0 \leq Y_k \leq 1 \) for each \( k \). Then

\[
P(X_n \geq (1 + \epsilon)E(X_n)) \leq \exp \left( -\frac{1}{3} \epsilon^2 E(X_n) \right) \quad 0 < \epsilon < 1;
\]

\[
P(X_n \leq (1 - \epsilon)E(X_n)) \leq \exp \left( -\frac{1}{3} \epsilon^2 E(X_n) \right) \quad 0 < \epsilon < 1
\]

Proof

See [2]. □

Remark

Considering the Bernoullian scheme as in proposition 2.3, let \( Y_n = X_n - X_{n-1} \) we have \( E(Y_n|X_{n-1}) = \frac{1}{1+X_{n-1}} \), for all \( n \geq 1 \), then \( X_n \) is a centering sequence.

Corollary 2.7. Under the hypothesis of proposition 2.3 we have

\[
P(X_n \geq (1 + \epsilon)E(X_n)) \leq \exp \left( -\frac{1}{3} \epsilon^2 E(X_n) \right) \sim e^{-\frac{1}{2} \epsilon^2 \sqrt{2n}}, \quad 0 < \epsilon < 1 \quad (14)
\]

\[
P(X_n \leq (1 - \epsilon)E(X_n)) \leq \exp \left( -\frac{1}{3} \epsilon^2 E(X_n) \right) \sim e^{-\frac{1}{2} \epsilon^2 \sqrt{2n}}, \quad 0 < \epsilon < 1 \quad (15)
\]

Proof

The thesis is an obvious consequence of (11) and theorem 2.6. □

Proposition 2.8. Let \( X_1, X_2, \ldots, X_n \) be a centering sequence with corresponding differences \( Y_k = X_k - X_{k-1} \) with mean \( \mu_k = Y_k \) and suppose that there are constants \( a_k \) and \( b_k \) such that \( a_k \leq Y_k \leq b_k \) for each \( k \). Then for any \( h > 0 \)

\[
E(\exp(hX_n)) \leq \prod_{k=1}^{n} \left( \frac{b_k - \mu_k}{b_k - a_k} e^{ha_k} + \frac{\mu_k - a_k}{b_k - a_k} e^{hb_k} \right)
\]

(16)

Proof
Corollary 2.9. Under the hypothesis of above proposition, if $a_k = 0$ and $b_k = 1$ then

$$E(exp(hX_n)) \leq (1 - \alpha + \alpha e^h)^n$$

where $\alpha = \frac{E(X_n)}{n}$.

Proof

See [2].

Remark

From (11) we have $\alpha \sim \sqrt{\frac{2}{n}}$ recalling (17) then for $n \to +\infty$

$$E(exp(hX_n)) \sim e^{\sqrt{2(e^h-1)}\sqrt{n}}$$

References

[1] Feller, W., An introduction to probability theory and its applications. Vol. I, Third edition, John Wiley & Sons Inc., New York, 1968

[2] McDiarmid, C., Centering sequences with bounded differences, Combin. Probab. Comput., Vol.6, 1997, No.1, 79–86

[3] Janson, S., Limit theorems for triangular urn schemes, Probab. Theory Related Fields, Vol. 134, 2006, No.3, pp. 417-452.
3 Appendix

First we rewrite (8) as

\[ p_{n,k} = p_{n,k-1} + \left( k + \frac{1}{k+1} \right) p_{n,k} - k p_{n+1,k+1}, \quad \forall n \geq 0, \quad 0 \leq k \leq n+1 \]  

(19)

where

\[ p_{n,-1} = 0, \quad \forall n \geq 0 \]

We define

\[ F(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} x^n y^k p_{n,k} = \sum_{n=0}^{\infty} x^n y^k p_{n,k} \]

(20)

substituting into (19):

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} x^n y^k p_{n,k} = \sum_{n=0}^{\infty} x^n y^k (p_{n,k-1} + kp_{n+1,k+1}) + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} x^n y^k \frac{1}{1+k} p_{n,k} \]

(21)

After the computation of the all summands of (21), we recover that:

\[ F(x, y) = yF(x, y) + yF_y(x, y) - \frac{y}{x} F_y(x, y) + \frac{1}{y} \int_0^y F(x,t) dt \]

(22)

Now from (22) multiplying both members by y and computing the derivative respect to y we have

\[ y \frac{1-x}{x} F_{yy}(x, y) + \left( \frac{2-x}{x} - y \right) F_y(x, y) - 2F(x, y) = 0 \]

(23)

**Proposition 3.1.** For \( 0 \leq x < 1, 0 \leq y < 1 \) let

\[ F(x, y) = 1 + \sum_{k=1}^{\infty} \left( k! \prod_{h=0}^{k-1} (1 - \frac{x}{h+1}) \right) \]

(24)

then (24) is a solution of (23) verifying the conditions \( F(0, y) = F(x, 0) = 1 \).

**Proof**

From (23) we note that the variable \( x \) does not appear in the derivatives, therefore we treat it as a parameter.

For convenience we put, for \( 0 < x < 1 \),

\[ a = \frac{1-x}{x}; \quad b = \frac{2-x}{x} \]

(25)
then (23) becomes
\[ F_{yy}(x, y) + \frac{b - y}{ay} F_y(x, y) - \frac{2}{ay} F(x, y) = 0 \] (26)

We follow Frobenius method. The solutions of the indicial equation
\[ \lambda(\lambda - 1) + \frac{b}{a} = 0 \]
are \( \lambda_1 = 0 \) and \( \lambda_2 = 1 - \frac{b}{a} \).
Since \( \lambda_2 < \lambda_1 \) then all the solutions of (26) will be of the type
\[ F(x, y) = C_1(x)F_1(y) + C_2(x)F_2(y) \]
where
\[ F_1(y) = \sum_{k=0}^{+\infty} c_k y^k, \quad c_0 = 1 \]
\[ F_2(y) = y^{1-\frac{b}{a}} \sum_{k=0}^{+\infty} d_k y^k + CF_1(y) \ln(y), \quad d_0 = 1 \]
where the constant \( C \) is equal to zero if \( \lambda_1 - \lambda_2 \) is not an integer, i.e. \( x \neq 1 - \frac{1}{n} \).
After computing the derivatives of \( F_1(y) \) and substituting in (23) we recover that
\[
\left\{
\begin{array}{l}
c_0 = 1 \\
c_k = \frac{k + 1}{\prod_{i=0}^{k-1} (ia + b)}, \quad \forall k \geq 1
\end{array}
\right.
\]
thus
\[ F_1(y) = 1 + \sum_{k=1}^{+\infty} \frac{k + 1}{\prod_{i=0}^{k-1} (ia + b)} y^k \]
From (25) we have
\[ F(x, y) = 1 + \sum_{k=1}^{+\infty} \frac{x^k y^k}{k! \prod_{h=0}^{k} (1 - \frac{hx}{k+1})} \]
is well defined even if \( x = 0 \) and we have \( F(0, y) = F(x, 0) = 1 \).
This conclude the proof.

\[ \square \]

**Proposition 3.2.** Let \( F(x, y) \) as in (24) then
\[ F(x, y) = 1 + \sum_{n=1}^{+\infty} \sum_{k=1}^{n} x^n y^k \frac{1}{k!} \sum_{i=1}^{k} A_{i,k} \left( \frac{i}{i+1} \right)^{n-k} \] (27)
where

\[ A_{i,k} = \frac{1}{\prod_{h=1, h \neq i}^{k} (1 - \frac{h + x}{h + 1})} \]  \tag{28}

**Proof**

Let

\[
\begin{align*}
\phi_1 &= \frac{1}{1 - x} \\
\phi_k &= \frac{1}{\prod_{h=0}^{k} (1 - \frac{hx}{h + 1})}, \quad k > 1
\end{align*}
\]

because

\[
\phi_k = \sum_{i=1}^{k} A_{i,k} \frac{1}{1 - \frac{ix}{i + 1}}
\]

then

\[
\sum_{i=1}^{k} A_{i,k} \prod_{h=1, h \neq i}^{k} (1 - \frac{hx}{h + 1}) = 1
\]

Choosing \( x = \frac{i}{i + 1} \) we obtain

\[
A_{i,k} = \frac{1}{\prod_{h=1, h \neq i}^{k} (1 - \frac{h + x}{h + 1})}
\]

From (27)

\[
F(x, y) = 1 + \sum_{k=1}^{+\infty} x^k y^k \frac{1}{k!} \sum_{i=1}^{k} A_{i,k} \frac{1}{1 - \frac{ix}{i + 1}} =
\]

\[
= 1 + \sum_{k=1}^{+\infty} x^k y^k \frac{1}{k!} \sum_{i=1}^{k} A_{i,k} \sum_{r=0}^{+\infty} \left( \frac{i}{i + 1} \right)^r x^r =
\]

\[
= 1 + \sum_{k=1}^{+\infty} x^k y^k \frac{1}{k!} \sum_{i=1}^{k} A_{i,k} \sum_{s=1}^{+\infty} \left( \frac{i}{i + 1} \right)^{s-1} x^{s-1}
\]

Inverting the order of the last summations it follows that

\[
F(x, y) = 1 + \sum_{k=1}^{+\infty} x^k y^k \frac{1}{k!} \sum_{s=1}^{+\infty} x^{s-1} \sum_{i=1}^{k} A_{i,k} \left( \frac{i}{i + 1} \right)^{s-1} =
\]

\[
= 1 + \sum_{k=1}^{+\infty} \sum_{s=1}^{+\infty} x^{k+s-1} y^k \frac{1}{k!} \sum_{i=1}^{k} A_{i,k} \left( \frac{i}{i + 1} \right)^{s-1} =
\]

\[
= 1 + \sum_{k=1}^{+\infty} \sum_{n=k}^{+\infty} x^ny^k \frac{1}{k!} \sum_{i=1}^{k} A_{i,k} \left( \frac{i}{i + 1} \right)^{n-k}
\]
Inverting the order of the first two summations we have

\[ F(x, y) = 1 + \sum_{n=1}^{+\infty} \sum_{k=1}^{n} x^n y^k \frac{1}{k!} \sum_{i=1}^{k} A_{i,k} \left( \frac{i}{i+1} \right)^{n-k} \]

The thesis follows.

\[ \square \]

**Remark**

From (27) taking into account (20) for \( n \geq k \geq 1 \) it follows that

\[ p_{n,k} = \frac{1}{k!} \sum_{i=1}^{k} A_{i,k} \left( \frac{i}{i+1} \right)^{n-k} \quad (29) \]

Now we conclude proving that (29) is the solution of the initial system (8).

**Proposition 3.3.** Let \( p_{n,k} \) as in (29) then it solves (8).

**Proof**

We proceed by induction. Let \( k = 1 \) in this case we have

\[ p_{n+1,1} = \frac{1}{2} p_{n,1} \]

from above equality substituting in (29) it follows that

\[ \frac{1}{1!} \sum_{i=1}^{1} A_{i,1} \left( \frac{i}{i+1} \right)^{n+1-1} = \frac{1}{2} \frac{1}{1!} \sum_{i=1}^{1} A_{i,1} \left( \frac{i}{i+1} \right)^{n-1} \]

i.e.

\[ A_{1,1} \left( \frac{1}{2} \right)^n = \frac{1}{2} A_{1,1} \left( \frac{1}{2} \right)^{n-1} \]

then the thesis, remembering that \( A_{1,1} = 1 \).

For \( k = 2, \ldots, n \) and \( n \geq 1 \) substituting in (29) we have

\[ \frac{1}{k!} \sum_{i=1}^{k} A_{i,k} \left( \frac{i}{i+1} \right)^{n+1-k} = \frac{k}{k+1} \frac{1}{k!} \sum_{i=1}^{k} A_{i,k} \left( \frac{i}{i+1} \right)^{n-k} + \frac{1}{k} \frac{1}{(k-1)!} \sum_{i=1}^{k-1} A_{i,k-1} \left( \frac{i}{i+1} \right)^{n-k+1} \]

multiplying both members for \( k! \) and eliminate by algebraic calculation the terms with index \( i = k \) we have

\[ \sum_{i=1}^{k-1} A_{i,k} \left( \frac{i}{i+1} \right)^{n+1-k} = \frac{k}{k+1} \sum_{i=1}^{k-1} A_{i,k} \left( \frac{i}{i+1} \right)^{n-k} + \sum_{i=1}^{k-1} A_{i,k-1} \left( \frac{i}{i+1} \right)^{n-k+1} \]
So that we have to verify that
\[
\sum_{i=1}^{k-1} \left( \frac{i}{i+1} \right)^{n-k} \left[ A_{i,k} \left( \frac{i}{i+1} \right) - \frac{k}{k+1} A_{i,k} - A_{i,k-1} \left( \frac{i}{i+1} \right) \right] = 0
\] (30)

From (30) for \( k = 2 \) we have
\[
\sum_{i=1}^{1} \left( \frac{i}{i+1} \right)^{n-2} \left[ A_{i,2} \left( \frac{i}{i+1} \right) - \frac{2}{2+1} A_{i,2} - A_{i,1} \left( \frac{i}{i+1} \right) \right] = 0
\]

because
\[
A_{1,2} = \frac{1}{\prod_{h=1, h \neq i}^{2} \left( 1 - \frac{h}{n+1} \right)} = -3
\]
and remembering that \( A_{1,1} = 1 \).

For \( k > 2 \), let us observe that for every \( i = 1, \ldots, k - 1 \) we have
\[
\left( \frac{i}{i+1} - \frac{k}{k+1} \right) \frac{1}{\prod_{h=1, h \neq i}^{k} \left( 1 - \frac{h+i+1}{n+1} \right)} - \left( \frac{i}{i+1} \right) \frac{1}{\prod_{h=1, h \neq i}^{k-1} \left( 1 - \frac{h+i+1}{n+1} \right)} =
\]
\[
= \left( \frac{i}{i+1} - \frac{k}{k+1} \right) \frac{1}{\prod_{h=1, h \neq i}^{k-1} \left( 1 - \frac{h+i+1}{n+1} \right)} \left( 1 - \frac{k}{k+1} \right) - \left( \frac{i}{i+1} \right) \frac{1}{\prod_{h=1, h \neq i}^{k-1} \left( 1 - \frac{h+i+1}{n+1} \right)} =
\]
\[
= \frac{1}{\prod_{h=1, h \neq i}^{k-1} \left( 1 - \frac{h+i+1}{n+1} \right)} \left[ \frac{i}{i+1} - \frac{k}{k+1} - \frac{i}{i+1} \left( 1 - \frac{k}{k+1} \right) \right] =
\]
\[
= \frac{1}{\prod_{h=1, h \neq i}^{k-1} \left( 1 - \frac{h+i+1}{n+1} \right)} \left[ \frac{i}{i+1} - \frac{k}{k+1} - \frac{i}{i+1} + \frac{k}{k+1} \right] = 0
\]
then the thesis follows.

\[\square\]

We proceed to establish a compact form for the \( p_{n,k} \).

**Proposition 3.4.** Under the hypothesis of proposition 3.3 for \( n \geq k \geq 1 \)

\[
p_{n,k} = \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} \left( \frac{k}{i+1} \right) i^{n-k} \left( \frac{i}{i+1} \right)^{n-k}
\] (31)

**Proof**
From (29) we have that

\[
\begin{align*}
A_{i,1} &= 1 \\
A_{i,k} &= \frac{1}{\prod_{h=1, h \neq i}^{k-1} (1 - \frac{h}{i+1})}, \quad \forall k > 1
\end{align*}
\]

so that

\[
\frac{1}{A_{i,k}} = \prod_{h=1, h \neq i}^{k} \left(1 - \frac{h}{i+1}\right) = \prod_{h=1, h \neq i}^{k} \left(\frac{i+1}{i+1-h}\right) = \frac{1}{i^{k-1}} \frac{i+1}{i+1-k}\]

Let \(i = k\): then

\[
\frac{1}{A_{k,k}} = \frac{1}{k^{k-1}} \frac{k+1}{(k+1)!} \prod_{h=1, h \neq k}^{k} (k-h) = \frac{1}{k^{k-1}} \frac{k+1}{(k+1)!} (k-1)! = \frac{1}{k^{k}}
\]

Let \(i < k\): then

\[
\frac{1}{A_{i,k}} = \frac{1}{i^{k-1}} \frac{i+1}{(i+1)!} \prod_{h=1, h \neq i}^{k} (i-h) = \frac{1}{i^{k-1}} \frac{i+1}{(i+1)!} \prod_{h=1}^{i-1} (i-h) \prod_{h=i+1}^{k} (i-h) = \frac{1}{i^{k-1}} \frac{i+1}{(i+1)!} \prod_{h=i+1}^{k} (i-h) = \frac{1}{i^{k-1}} \frac{i+1}{(i+1)!} (-1)^{k-i} \prod_{h=i+1}^{k} (h-i) = \frac{1}{i^{k}} (k+1)! (-1)^{k-i} \prod_{h=i}^{k} (h-i) = \frac{1}{i^{k}} \frac{(i+1)!}{(k+1)!} (-1)^{k-i} \prod_{h=i}^{k} (h-i) = \frac{1}{i^{k}} \frac{(i+1)!}{(k+1)!} \frac{(-1)^{k-i}}{(k-i)!} \prod_{h=i}^{k} (h-i)
\]

so that

\[
A_{i,k} = (-1)^{k-i} \left(\frac{k+1}{i+1}\right)^{i}, \quad \forall k > 1, \quad i = 1, \ldots, k
\]

then the thesis follows.