Multi-resonance cyclotron-undulator electron acceleration

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Abstract. In this work, we propose a method of particle acceleration by the electromagnetic wave pulse propagating in the waveguide. The method is based on a “superposition” the electron cyclotron oscillations in the uniform longitudinal magnetic field and their bounce-oscillations in the periodical transverse magnetic field (undulator). Correspondingly, two mechanisms of the electron accelerations by the wave are provided simultaneously, namely, the well-known cyclotron acceleration and the undulator acceleration in the regime of the inversed free-electron laser. The multi-resonance character of this system leads to a chaotic behaviour of the electron motion in the field of the wave. In the process of this motion, electrons can get a very high energy (as compared to the case of the purely cyclotron acceleration by the same wave pulse.

1. Introduction

The effect of the cyclotron autoresonance acceleration has been discovered for a long time and is well studied [1-5]. This effect is realized, when electrons move along helical trajectories in the longitudinal magnetic field and captured by the external electromagnetic wave pulse. If the autoresonance condition (the wave phase velocity is equal to the speed of light \(c\) and the magnetic field is equal to the resonance value) takes place, this type of acceleration is especially effective, and the extracted energy determined only the amplitude of accelerating wave pulse and its duration. Moreover, there are several methods of the mismatch compensation, if the electron-wave interaction doesn’t occur in a vacuum (the wave phase velocity differs from \(c\)) [4,5].

In this work, we propose to combine the well-known cyclotron acceleration and the undulator acceleration in the regime of the inversed free-electron laser. It is known that an undulator with axial magnetic field may have unique properties in the case of the relative proximity of cyclotron and undulatory resonances [6-10]. In this work, we show that including of an undulator in the well-known system of the cyclotron acceleration significantly complicates the acceleration process, and that provides the appearance of additional energy levels, which are caused by the simultaneous presence of periodical undulator magnetic field and uniform longitudinal magnetic field. The multi-resonance character of this system leads to a chaotic behavior of the electron motion in the field of the wave. Moreover, we have found that this system contains not two (cyclotron+undulator) but several resonant states. Some electrons can move from one resonance level to another and thus accelerate to much higher (compared to the purely cyclotron acceleration) energy levels. In this paper we give the brief description of the such type of acceleration and estimations for “new” energy levels in several approximations.
2. Cyclotron-undulator acceleration

We consider the following system: the electromagnetic wave pulse propagates in the waveguide; the transverse structure of radiation accords to TE mode. The pulse hits electrons initially at the rest, and captured electrons are accelerated.

In the case of the purely cyclotron acceleration, the waveguide is immersed in the uniform axial magnetic field $B_0$ (see fig. 1, a). In this case, the electrons oscillate in the axial magnetic field. If the cyclotron rotation phase is equal to the wave phase, the electrons accelerate effectively. This effect of the cyclotron acceleration is well known. Obviously, the cyclotron acceleration is most effective, when the phase of the electron cyclotron rotation is equal to the wave phase during the whole electron-wave interaction process. In this case, the electron energy depends only on the wave amplitude and the accelerating pulse duration. Such situation is realized, when the autoresonance condition is provided (the phase velocity of the wave is equal to $c$, $V_{ph} = c$, and the magnetic field is equal to the resonance value). In a waveguide the phase wave velocity is not equal to $c$, and a mismatch between electron and wave phases arises in the acceleration process. The problem of the phases mismatch arising could be solved by the magnetic field adjusting [5]. The maximal energy level of the acceleration is determined by the cyclotron resonance [4,5] ($\gamma_r = \Omega_c/\omega$ – the resonance value of the relativistic gamma-factor, here $\omega$ is the accelerating wave frequency, $\Omega_c = eB_0/(mc)$ is the electron cyclotron frequency; the relativistic gamma-factor is $\gamma = 1/\sqrt{1-\beta^2}$, $\beta = V_e/c$ is the normalized electron velocity). All captured electrons are accelerated equally in this case.

Figure 1. (a) – cyclotron acceleration of electrons by the electromagnetic wave in a waveguide immersed in the uniform axial magnetic field. (b) – cyclotron-undulator acceleration in a waveguide immersed in the uniform axial magnetic field and in the undulator.

Figure 2 illustrates results for the electron cyclotron acceleration by the microwave pulse (the wave length $\lambda = 0.8\text{cm}$) with 1 MW power in a waveguide with radius 1.2cm for two different values of longitudinal magnetic field [(a) is the case of the exact cyclotron resonance $B_0 = 1.25\ T$; (b) is the case when the magnetic field slightly exceeds the resonance value, $B_0 = 1.375\ T$. In the case of the uniform magnetic field, the maximal value of energy of accelerated electrons amounts 3 - 5MeV.

Let’s consider the same acceleration process in the case, when the waveguide is immersed in the axial uniform magnetic field and the transverse periodic undulator magnetic field (see. Fig. 1, b). As simulations show, the addition of undulator in the system provides the emergence of several resonances, caused by the simultaneous presence of longitudinal and undulator magnetic fields (see fig.3). In the part 3, we explain the occurrence of additional resonances.
Figure 2. Electron cyclotron acceleration by the wave with power 1MW, the wave length 0.8 cm, $\beta_{gr} = 0.98$ (the waveguide radius 1.2 cm). – The electron energy as a function of electron axial coordinates in the case of (a) the exact cyclotron resonance and (b) when the magnetic field slightly exceeds the resonance value.

It is important to note, that energy of accelerated electron significantly depends on the electron phase relative to the undulator, moreover electrons with close undulator phases reach different energy levels. In this case, the maximal electron energy exceeds the electron energy acquired due to the cyclotron acceleration by radiation with the same parameters (compare fig. 2 and fig. 3).

Figure 3. Electron cyclotron-undulator acceleration by the radiation with power 1MW, the wave length $\lambda = 0.8$ cm, $\beta_{gr} = 0.98$ (the waveguide radius 1.2 cm), in the system with the axial magnetic field $B_0 = 1.25 T$ and the undulator $d = 8$ cm, $K = 0.4$. Electrons have different initial undulator phases with respect to the wave.

3. Resonance levels of electron energy

Let us describe the emergence of additional resonances (fig. 3) caused by the simultaneous presence of the uniform guiding and the undulator magnetic field. We start from the equation of particles motion in this system

$$\frac{dP}{dt} = -e \frac{c}{c} [V, B_w + B_u + B_0] - eE_w.$$  \hspace{1cm} (1)

Here, $P$ is the electron momentum; $V$ is the electron velocity; $B_w, E_w$ are the magnetic and electric fields of the wave; $B_u$ is the transverse undulator magnetic field, $B_0$ is the uniform axial magnetic field. In the center of the waveguide, the wave vector-potential of the lowest mode TE$_{11}$ corresponds to the plane wave, $A = A_0(\cos(\omega t - ihz) \textbf{x} + \sin(\omega t - ihz) \textbf{y})$, where $\omega$ is the wave frequency, $h = \beta_{gr} k$ is the axial wave number, $\beta_{gr} = V_{gr}/c$ is the normalized wave group velocity, $k = 2\pi/\lambda$ is the wave number.

Equations for the electron moment $p_+ = p_x + ip_y$ has the following form:

$$\frac{dp_+}{d\tau} = \frac{d^2p_+}{d\tau^2} + ik\beta_z \frac{\lambda}{d} \exp \left( i \frac{\lambda}{d} \xi \right) + i \frac{b}{\gamma} p_+.$$  \hspace{1cm} (2)
Here, \( p_{x,y} = \gamma \beta_{x,y}, \) \( \beta_{x,y} = v_{x,y}/c, \) \( \tau = \omega \tau, \) \( \zeta = k_0 z, \) \( a_+ = a_x + i a_y = A_+/(hU_0) = a_0 * \exp(i \omega t - ihz), \) \( d \) is the undulator period, \( K \) is the undulator parameter, \( b = \Omega_c/\omega, \) \( \Omega_c = eB_0/m_e c \) is the non-relativistic electron cyclotron frequency.

The equation of the axial electron motion is rewritten for the normalized momentum \( p_z = \gamma \beta_z, \)

\[
\frac{dp_z}{d\tau} = \text{Re} \left[ \frac{p_z^*}{\gamma} \left( \beta_{gr} \frac{\partial a_+}{\partial \tau} - iK \frac{1}{d} \exp \left( i \frac{\lambda_1}{d} \right) \right) \right].
\]

The electron energy change is determined by the scalar product \( \sim (V, E), \) in the normalized variables:

\[
\frac{dy}{d\tau} = \text{Re} \left( \frac{p_z^*}{\gamma} \frac{\partial a_+}{\partial \tau} \right).
\]

We divide stable states of the system into cyclotron and undulator

\[
p_+ = p_{r,c} + p_{r,u},
\]

then

\[
y = y_c + y_u,
\]

Suppose that transverse cyclotron and undulator momentum have additive contributions in the square of summarize transverse momentum \( p_{l,c}^2 = p_{l,c}^2 + p_{l,u}^2.\) Therefore, \( p_{z,c}^2 = y^2 - 1 - p_{l,c}^2 - p_{l,u}^2, \) here \( p_{l,u} = K/(1 - \Omega_c/\gamma \Omega_u) \), \( \Omega_u = V_2 2\pi/d \) is the bounce frequency. The ratio between the electron axial moment and the energy change in the case of cyclotron acceleration is \( \Delta p_z/\Delta y \sim \beta_{gr}. \)

Therefore,

\[
\frac{1}{2p_z}, \frac{d\left(y^2 - 1 - p_{l,c}^2\right)}{dy} = \beta_{gr},
\]

\[
\frac{dp_z}{dy} = \frac{\beta_{gr}}{1 - \frac{K^2 \alpha}{(p_z - \alpha)^3}}
\]

here \( \alpha = b * d/\lambda. \)

Let us begin from the cyclotron contribution. For the cyclotron phase change

\[
\frac{d\theta}{d\tau} = 1 - \beta_{gr} \beta_p - \frac{b}{\gamma_c},
\]

Figure 4. A characteristic dependence of the axial momentum \( p_z \) on the normalized electron energy \( \gamma \) (green lines). The red dashed line accords to the cyclotron resonance \( \omega = h \star V_2 + \Omega_c/\gamma, \) the blue dashed line depicts the dependence according to the undulator resonance \( \omega = (h + h_u)V_2 - \Omega_c/\gamma, \) the gray dashed line accords to the “combined” resonance \( \omega = (h + h_u)V_2 - \Omega_c/\gamma. \) Intersections of different “resonance” lines with the green line represent resonance energy levels.
The cyclotron resonance dependence of \( p_z \) on \( \gamma \), the condition of the stable state \( d\theta/d\tau = 0 \) [12], following from (9)

\[
p_{z,c} = \frac{\gamma_c - b}{\beta_{gr}},
\]

(10)

This dependence depicted in the figure 4 (red dashed line).

For the undulator part

\[
\frac{d\varphi}{d\tau} = 1 - \beta_{gr}p_z - \frac{\lambda}{d}p_z,
\]

(11)

For the undulator resonance dependence

\[
p_{z,u} = \frac{\gamma_u}{\beta_{gr} + \frac{\lambda}{d}},
\]

(12)

The blue dashed line accords to (12).

Let’s consider several approximations

1. In the case of \( p_z \ll \alpha \), equation (8) is solved as follows

\[
p_z \approx \frac{\beta_{gr}(\gamma - 1)}{1 + K^2/\alpha^2},
\]

(13)

This dependence is depicted in the figure 4 (solid green line).

So, the resonance energy out of (9)

\[
\gamma_{c,r1} \approx \frac{b - \beta_{gr}^2/(1 + K^2/\alpha^2)}{1 - \beta_{gr}^2/(1 + K^2/\alpha^2)},
\]

(14)

This energy value accords to the point of intersection of the line defined the by resonance dependence (10) and the dependence of the axial momentum on the energy (13), in the range of relatively low particle energies (fig. 4). In the case of the cyclotron resonance, \( \gamma_{c,r1} = \gamma_{c,r0} = 1 \).

Coming from (20), the phase change is equal 0, when the electron resonance energy

\[
\gamma_{u,r1} = \frac{\beta_{gr}(\beta_{gr} + \alpha)/(1 + K^2/\alpha^2)}{\beta_{gr}(\beta_{gr} + \alpha)/(1 + K^2/\alpha^2) - 1},
\]

(15)

This energy level exists, if \( \beta_{gr}(\beta_{gr} + \lambda/d) > 1 \).

2. In the case of \( p_z \sim \alpha \) one obtains

\[
p_z \approx \alpha + \sqrt{\frac{K^2\alpha}{2\beta_{gr}(\gamma - 1)}},
\]

(16)

see figure 4 (solid green curve).

For a small undulator parameters \( K \ll 1 \) we obtain

\[
\gamma_{c,r2} \approx b + \frac{\beta_{gr}\alpha}{\lambda},
\]

(17)

\[
\frac{\lambda}{d} \ll 1, \gamma_{u,r2} \approx \alpha,
\]

(18)

\[
\frac{\lambda}{d} \sim 1, \gamma_{u,r2} \approx 2,
\]

(19)

For a big undulator parameters \( K \sim 1 \) we obtain

\[
\gamma_{c,r2} \approx \frac{\beta_{gr}\alpha}{\lambda},
\]

(20)

\[
\frac{\lambda}{d} \ll 1, \gamma_{u,r2} \approx \alpha,
\]

(21)

\[
\frac{\lambda}{d} \sim 1, \gamma_{u,r2} \approx 3,
\]

(22)
In the graphical interpretation, these are the points of intersection the red dashed line defined by \( \gamma_{c,r2} \) or blue dashed line \( \gamma_{u,r2} \) with the solid green line for the axial momentum in the situation described by eq. (16).

3. In the case of the sufficiently high electron energy values \( p_z \gg \alpha \), we obtain

\[
p_z \approx \beta gr \left( \gamma - 1 + \frac{K^2 \alpha}{2} \right),
\]

\[
\gamma_{c,r3} \approx \frac{b + \beta gr^2 K^2 \alpha / 2 - \beta gr^2}{1 - \beta gr^2},
\]

For the undulator resonance

\[
\gamma_{u,r3} = \frac{\beta gr (\beta gr + \alpha) \left( K^2 \alpha / 2 - 1 \right)}{\beta gr (\beta gr + \alpha) - 1}.
\]

This energy level realized, if \( (K^2 \alpha / 2 - 1) / [\beta gr (\beta gr + \lambda / d) - 1] > 0 \).

Simulations show that in the case of high energy levels there is a resonance caused by the existence of the stable state on the phase plane of «combined» phase \( \theta = \omega t - (h + h_u) z + \Omega \gamma t / \gamma \). This state is determined by the condition [12]

\[
\frac{d\theta}{d(\omega t)} = 1 - \left( \frac{\beta gr + \lambda}{\lambda} \right) \beta z + \frac{b}{\gamma} = 0,
\]

The solution of this equation in this case (the resonance dependence of energy on axial momentum) is illustrated by the grey dashed curve in the figure 4.

\[
\gamma_{cur} = \frac{b + \beta gr (\beta gr + \alpha) \left( K^2 \alpha / 2 - 1 \right)}{\beta gr (\beta gr + \alpha) - 1}.
\]

These points of lines intersection in the area of high electron energies (fig. 4).

**Figure 5.** (a) – the \( \gamma \)-factor as a function of axial coordinate (cm), (b) – the cyclotron phase plane, the purple points mark the oscillations around the second cyclotron resonance, orange points mark the phase trajectories around the third cyclotron resonance; (c) – the undulator phase plane; (d) – the “mixed” phase plane, the red points mark the oscillation around the “mixed” resonance.
We considered the electron acceleration (electrons initially at rest) by radiation with the power of 1 MW and the wavelength of 0.8 cm in a waveguide with the radius 1.2 cm ($\beta_{gr} = 0.98$). The axial magnetic field $B_0 = 1.25$ T corresponds to the exact cyclotron resonance, the undulator parameter is $K = 0.4$, and the undulator period is $d = 8$ cm. In order to describe the acceleration process, we consider the acceleration of the electron, which manage to “jump” to all levels (fig. 5, a).

Initially, the electron energy commits several oscillations close to the first cyclotron resonance (coinciding with the null cyclotron resonance, because $b = 1$), after, “jumps” to the first undulator resonance energy level, but there is no anyone full cycle, because it is difficult to identify in the undulator phase plane (fig. 5, c). Then, electron jumps to the second cyclotron (coinciding with the second undulator resonance, as numerical simulations show, this is the optimal case for acceleration), passing the “combined” energy level (26) electron makes one oscillation at the highest third cyclotron level (fig. 5, b). After, the electron gets down to the coinciding second resonances (fig. 5, b, c). Thereafter, the electron ups to the “combined” energy level (fig. 5, d). Black lines in the figure 5 accord to resonance levels estimated analytically (17), (18), (25) and (27). Obviously, analytically estimations in a good agreement with results of numerical simulations.

4. Conclusion
We have considered the scheme of multi-resonance cyclotron-undulator acceleration. This type of acceleration makes it possible to increase the maximal energy of the acceleration by the wave pulse in comparison with the cyclotron acceleration by the same pulse in the uniform magnetic field. However, the energy of accelerated electron significantly depends on the electron phase relative to the undulator, that complicates implementation of this effect.

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