Higher regularity and asymptotic behavior of 2D magnetic Prandtl model in the Prandtl-Hartmann regime

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Abstract

In this paper, we investigate the higher regularity and asymptotic behavior for the 2-D magnetic Prandtl model in the Prandtl-Hartmann regime. Due to the degeneracy of horizontal velocity near boundary, the higher regularity of solution is a tricky problem. By constructing suitable approximated system and establishing closed energy estimate for a good quantity (called “quotient” in \[R\]), our first result is to solve this higher regularity problem. Furthermore, we show the global well-posedness and global-in-\(x\) asymptotic behavior when the initial data are small perturbation of the classical Hartmann layer in Sobolev space. By using the energy method to establish closed estimate for the quotient, we overcome the difficulty arising from the degeneracy of horizontal velocity near boundary. Due to the damping effect, we also point out that this global solution will converge to the equilibrium state (called Hartmann layer) with exponential decay rate.

Keywords: Magnetic Prandtl equation, higher regularity, asymptotic behavior

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1 Introduction

In this paper, we are concerned with the higher regularity and asymptotic behavior for the mixed Prandtl-Hartmann boundary layer equations that were derived in [10] from the classical incompressible magnetohydrodynamic (MHD) system in two-dimensional domain with flat boundary. Specifically, we are concerned with the following Prandtl-Hartmann equation in $[0, L] \times \mathbb{R}_+$:

$$
\begin{align*}
&u \partial_x u + v \partial_y u - \partial_y u = -\partial_x p_E(x, 0) + \partial_y b, \\
&\partial_y u + \partial_y b = 0, \\
&\partial_x u + \partial_y v = 0,
\end{align*}
$$

(1.1)

where $(u, v)$ denotes the velocity field, and $b$ is the corresponding tangential magnetic component. The quantity $p_E(x, 0)$ above is considered prescribed, and $\partial_x p_E(x, 0)$ evidently acts as a forcing term to the Prandtl equations.

In this paper, we are concerned with the homogeneous Prandtl equations, that is

$$
\partial_x p_E(x, 0) = 0.
$$

The Prandtl-Hartmann Eq. (1.1) are though of as evolution equation, with $x$ being a time-like variable, and $y$ being space-like. Thus, the Eq. (1.1) are supplemented with boundary conditions at $y = 0, y = +\infty$, and initial data at $x = 0$. The quantity $L$ appearing in (1.1) is though of as the time over which we are considering the evolution.

For the Eq. (1.1), we will consider this system with the initial data

$$
u(x, y)|_{x=0} = u_0(y),
$$

(1.2)

and the Dirichlet boundary conditions

$$
u|_{y=0} = v|_{y=0} = b|_{y=0} = 0.
$$

(1.3)

The far field is taken as a uniform constant state, and hence, we also suppose

$$
\lim_{y \to +\infty} u(x, y) = 1, \quad \lim_{y \to +\infty} b(x, y) = 1, \quad \text{uniformly with respect to } x.
$$

(1.4)

Using the boundary conditions (1.3), (1.4) and the second equation in (1.1), one can obtain the relationship

$$
\partial_y b = -(u - 1).
$$

(1.5)

Substituting this equation (1.5) into the original Eq. (1.1), we have

$$
\begin{align*}
&u \partial_x u + v \partial_y u - \partial_y u - u - 1 = 0, \\
&\partial_x u + \partial_y v = 0,
\end{align*}
$$

(1.6)

with the boundary conditions

$$
u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y \to +\infty} u(x, y) = 1.
$$

(1.7)

Since the vertical velocity $v(x, y) = -\int_0^y \partial_x u(x, y')dy'$ creates a loss of $x-$derivative, the classical method employed by Oleinik is to pass to the following change of coordinates, known as the well-known von-Mises transform:

$$
(x, \psi) = (x, \int_0^y u(x, y')dy').
$$

(1.8)

In terms of new variables, the original Eq. (1.6) translate into a quasilinear, degenerate diffusion equation as follows

$$
(u^2)_x - u(u^2)_{\psi\psi} = 2(1 - u).
$$

(1.9)

The corresponding boundary and initial conditions are

$$
u|_{\psi=0} = 0, \quad \lim_{\psi \to +\infty} u(x, \psi) = 1, \quad u(0, \psi) = u_0(\psi).
$$

(1.10)

In this paper, the words “local” and “global” refer to the $x-$direction. Similar to the classical Prandtl equation, higher regularity is a tricky problem that arises from the degeneracy of velocity near the boundary. Thus, our first target is to establish the local well-posedness with higher regularity for the equations (1.1)-(1.4).
Theorem 1.1 (Higher Regularity Result). Assume that \( u_0(y) \in C^\infty(\mathbb{R}^+) \) satisfies the following conditions:

\[
 u_0(y) > 0, \ y > 0, \ u_0(0) = 0, \ u_0(y) \to 1, \ y \to \infty,
\]

and

\[
 u_0'(0) > 0, \ u_0''(0) + 1 = 0, \ u_0'''(0) - u_0'(0) = 0.
\]

Assume the initial data \( |u_0(y) - 1| \) and \( \partial_y^mu_0(y) \) decay exponentially for all \( m \geq 1 \). Assume also the generic compatibility conditions at the corner \( (0,0) \) up to order \( 2k - 1 \). Then there exists a positive \( 0 < L \ll 1 \) depending on the initial data such that on \( 0 \leq x \leq L \), the solution \((u,v,b)\) of Prandtl-Hartmann Eqs. \[
\begin{align*}
(1.1) - (1.3)
\end{align*}
\]
obeys the following estimates for \( 0 < 2\alpha + \beta \leq 2k \) and for \( \gamma \leq k - 1 \) that

\[
\|\partial_x^\alpha \partial_y^\beta u(y)^4\|_{L_x^\infty L_y^\infty} + \|\partial_x^\alpha \partial_y^\beta v(y)^{\gamma - 1}\|_{L_x^\infty L_y^\infty} + \|\partial_x^\alpha \partial_y^\beta b(y)^4\|_{L_x^\infty L_y^\infty} \leq C_0,
\]

where the weight constant \( l \geq 1 \) and \( C_0 \) depends on \( k, l \) and \( u_0 \).

Next, our second target is to study asymptotic behavior of solution to it tends to the equilibrium state. More precisely, we study the global stability of the Hartmann layer \((\bar{u}, \bar{v}) = (1 - e^{-y}, 0)\). It is easy to check that \((\bar{u}, \bar{v})\) is a special solution of the system \( (1.10) - (1.13) \). Since the higher regularity result in Theorem 1.1 requires the compatibility condition on the initial data, thus we only investigate the asymptotic behavior result of solution in low regularity initial data. Specifically, we have the asymptotic behavior result as follows:

Theorem 1.2 (Asymptotic Behavior Result). Assume that \( u_0(y) > 0 \) for \( y > 0 \); \( u_0(0) = 0, \ u_0'(0) > 0, \ u_0(y) \to 1 \) as \( y \to \infty \); \( u_0(y), \ u_0'(y), \ u_0''(y) \) are bounded for \( 0 \leq y < \infty \) and satisfy the H"older condition. Moreover, assume that for small \( y \) the following compatibility condition is satisfied at the point \( (0,0) \):

\[
-u_0''(0) + u_0(0) - 1 = O(y^2).
\]

For a given real number \( \gamma_0 > 1 \), suppose that the initial data satisfy \( f(0) \leq \gamma_0 \). Furthermore, there exists a positive constant \( \sigma_0 \) depending on \( \gamma_0 \) such that

\[
\|\phi_0 u_0^{\frac{3}{2}}\|_{L_x^3} + \|\phi_0 v_0\|_{L_x^3} + \|\frac{L\phi_0}{u_0^{\frac{3}{2}}}\|_{L_x^3} + \|(L\phi_0) v\|_{L_x^3} \leq \sigma_0,
\]

where \( \phi(x,\psi) := u^2(x,y(x,\psi)) - \bar{u}^2(\psi(y)), \ f(x) := \sup_{x' \in I} \left\{ \|\frac{\phi}{u}\|_{L_x^3}, \|\frac{\phi}{u}\|_{L_x^3} \right\} \), the operator \( L\phi := -u\phi v - 2\frac{\phi}{u(u+\Phi)} \) and \((\bar{u}, \bar{v}) := (1 - e^{-y}, 0)\) be the Hartmann layer. Then there exists a global strong solution \((u,v)\) to the Prandtl-Hartmann system \( (1.10) \). Moreover, the solution satisfies the decay estimate:

\[
\|u(x,y) - \bar{u}(y)\|_{H_x^3} + \|u_y(x,y) - \bar{u}_y(y)\|_{L_x^\infty} \leq C e^{-x},
\]

where \( C \) is a positive constant independent of \( x \).

Remark 1.1. Since we apply the von-Mises transformation to overcome the loss of \( x \) derivative, the difference \( \phi(x,\psi) \) satisfies the new system

\[
\phi_x + L\phi = 0
\]

in the new variable \((x,\psi)\). Then, we establish the closed estimate for the system \( (1.16) \) under the small condition \( (1.14) \). This condition can be expressed in terms of the initial data \( u_0(y) \); however, we only give condition \( (1.14) \) for the sake of simplicity.

Remark 1.2. Denote \( \tilde{b}_y := (1 - \bar{u}) \), then we apply the relation \( (1.4) \) and decay estimate \( (1.15) \) to obtain

\[
\|b_y(x,y) - \tilde{b}_y(y)\|_{H_x^3} + \|b_{yy}(x,y) - \tilde{b}_{yy}(y)\|_{L_x^\infty} \leq C e^{-x},
\]

where \( C \) is a positive constant independent of \( x \).
We now review some related works involving the classical Prandtl and MHD Prandtl type equations to the problem studied in this paper.

(I) Some results for nonstationary boundary layer equations. The vanishing viscosity limit of the incompressible Navier-Stokes equations that, in a domain with Dirichlet boundary condition, is an important problem in both physics and mathematics. As the viscosity coefficient tends to zero, the solution undergoes a sharp transition from a solution of the Euler system to the zero non-slip boundary condition on boundary of the Navier-Stokes system. This sharp transition will lead to the formation of the boundary layer. Indeed, Prandtl [43] derived the Prandtl equations for boundary layer from the incompressible Navier-Stokes equations with non-slip boundary condition. Now, let us introduce the related matter of well-posedness results for the Prandtl equation. If the tangential velocity field in the normal direction to the boundary satisfies the monotonicity condition, Oleinik [41, 42] applied the Crocco transform to establish the global-in-time regular solutions on \([0, L] \times \mathbb{R}_+\) for small \(L\), and local-in-time solutions on \([0, L] \times \mathbb{R}_+\) for arbitrary large by finite \(L\). Under the monotonicity condition and a favorable pressure gradient of the Euler flow, the global-in-time weak solutions were obtained in [49] for arbitrarily \(L\). It should be noted that all of the above results are achieved by using Crocco transformations. By taking care of the cancelation in the convection term to overcome the loss of derivative in the tangential direction of velocity, the researchers in [11, 39] independently used the simply energy method to establish well-posedness theory for the two-dimensional Prandtl equations in the framework of Sobolev space. For more results in this direction, the interested readers can refer to the well-posedness results in the analytic or Gevrey setting without monotonicity \([8, 8, 22, 29, 31, 58, 44, 45]\), ill-posedness results in the Sobolev setting without monotonicity \([7, 16]\), generic invalidity of boundary layer expansions in the Sobolev spaces \([11, 10, 19]\) and references therein.

(II) Some results for stationary boundary layer equations. For the classical stationary Prandtl equation, the local-in-time well-posedness result was obtained by Oleinik [40], who used the von-Mises transformation and maximum principle. Due to the degenerate property of horizontal velocity near the boundary, it is hard to obtain the higher regularity for the stationary Prandtl equation. This problem was settled by Guo and Iyer [18] since they found the good unknown quantity to establish the closed energy estimate. For the incompressible steady Navier-Stokes equations, Guo and Nguyen [20] justified the boundary layer expansion for the flow with a non-slip boundary condition on a moving plate. This result has been extended to the case of a rotating disk and to the case of non-shear Euler flows \([23, 24]\). Recently, Guo and Iyer [17] studied the boundary layer expansion for the small viscous flows with the classical no slip boundary conditions or on the static plate. This work was extend to the case of global theory in the \(x\)-variable for a large class of boundary layer with sharp decay rates. For more results about the boundary layer expansion, the reader should consult \([8, 22, 29, 31, 58, 44, 45]\). In terms of the asymptotic behavior of the solution, Serrin [46] used the maximum principle techniques to single out that similarity solution as those which asymptotically develop downstream, whatever may be the state of motion at the initial position at \(x = 0\). In the case of localized data near the Blasius solution, Iyer [28] applied the energy method instead of maximum principle method for the good known quantity (called “quotient”) to establish specific convergence rate.

(III) Some results for Prandtl type equation influencing magnetic field. Under the influence of electro-magnetic field, the system of magnetohydrodynamics (denoted by MHD) is a fundamental system to describe the movement of electrically conducting fluid, for example plasmas and liquid metals (cf. [2]). By asymptotic analysis of the incompressible MHD system, Gérard-Varet and Prestipino [3] established a systematic derivation of boundary layer models in MHD. The most important point is that they performed some stability analysis for the boundary layer system, and emphasized the stabilizing effect of the magnetic field. At the same time, Liu, Xie and Yang [35] studied the local-in-time well-posedness for the Prandtl type equation in MHD that is built if both the hydrodynamic Reynolds numbers and magnetic Reynolds numbers tend to infinity at the same rate. Since they found that magnetic field has good effect on stability, the well-posedness result holds true under the condition that the initial tangential magnetic field is not zero instead of the monotonicity condition on the tangential velocity field requiring in classical Prandtl equation [41]. This well-posedness result is generalized to the case of inhomogeneous incompressible flow that is Hyperbolic-parabolic coupled equations in [6]. Furthermore, the Prandtl ansatz boundary layer expansion for the unsteady MHD system was justified [30] when no-slip boundary and perfect conducting boundary conditions are imposed on velocity field and magnetic field respectively. Recently there are many mathematical results on two dimensional MHD boundary layer system, for the almost global existence [32], ill-posedness results [33, 34], lifespan of solution with analytic perturbation of general shear flow [48], boundary layer expansions for steady MHD over...
a moving plate [4], and local-in-time well-posedness for compressible MHD boundary layer [21]. For the 2D MHD boundary layer equations in the mixed Prandtl and Hartmann regime derived by formal multiscale expansion in [10], Xie and Yang [47] obtained the global existence of solutions with analytic regularity. If the initial data is near the Hartmann layer, the solution of Prandtl-Hartmann system is globally well-posedness in Sobolev space and will converge to the Hartmann layer with exponent decay rate in [3]. However, there is no result investigating the degeneracy of horizontal velocity near the boundary $y = 0$.

Thus, motivated by [18], our first target is to establish the higher regularity result of solution for the Prandtl-Hartmann system (1.6). The advantage of our method here is to constructed new approximated system such that higher regularity result can be obtained without using the property of lower regular solution. Motivated by the recent work [28], our second target is to use the energy method to build the asymptotic behavior of global solution for the Prandtl-Hartmann system (1.6). Due to the damping effect, we point out that this global solution will converge to the equilibrium state(called Hartmann profile) with exponent decay rate.

Notations: Through this paper, all constant $C$ may be different in different lines. Subscript(s) of a constant illustrates the dependence of the constant, for example, $C_s$ is a constant depending on $s$ only. We write $o_L(1)$ to refer to a constant that is bounded by some unspecified, perhaps small, power of $L$: that is, $a = o_L(1)$ if $|a| \leq CL^θ$ for some $θ > 0$. We also write the $C^\infty$ cutoff function:

$$\chi(y) := \begin{cases} 1, & y \in [0, 1), \\ 0, & y \in (2, \infty), \end{cases}$$

and $\chi'(y) \leq 0$ for all $y > 0$. $A \lesssim B$ ($A \gtrsim B$) represents that there exists a positive constant $C$ such that $A \leq CB$ ($A \geq CB$). And $A \sim B$ stands for $A \lesssim B$ and $A \gtrsim B$. Finally, $P_i(\cdot, \cdot)$ stands for a polynomial function independent of $\epsilon$, and the index $i$ denote it changing from line to line.

2 Difficulties and outline of our approach

The main goal of this section is to explain main difficulties of proving Theorems 1.1 and 1.2 as well as our strategies for overcoming them. In order to establish the well-posedness in some higher regularity Sobolev space and asymptotic behavior for the Prandtl type Eq. (1.6), the main difficulty comes from the vertical velocity $v = -\partial_y^{-1}\partial_x u$ creating a loss of $x-$derivative. Thus, we will introduce the method to overcome this difficulty.

First of all, we state the main idea to establish higher regularity of Prandtl type Eq. (1.6) in Theorem 1.1. In order to overcome the loss of $x-$derivative coming from the vertical velocity, the classical method is to apply the so-called von-Mises transformation that transforms the original Eq. (1.6) into a single quasi-linear parabolic equation. Due to the degenerate property of horizontal velocity on the boundary, it is not easy to establish higher regularity for this quasi-linear parabolic equation. In this respect, Guo and Iyer [18] settled this tricky subject, arising from the classical steady Prandtl equation, by establishing the energy estimate for the good unknown quantity $q := \frac{\theta}{\epsilon}$ via the linear derivative Prandtl equation. Here the quantity $\theta$ is the solution of classical steady Prandtl equation obtained by Oleinik [42]. In other words, this method depends on the property of lower regular solution of original steady Prandtl equation. Since there is no result about the well-posedness of Eq. (1.6), we hope to find a method to establish the well-posedness in higher regularity space without using the property of lower regular solution. Thus, in this paper, we consider the following approximated system

$$\begin{cases} u'^{\epsilon}\partial_x u'^\epsilon + v'^\epsilon\partial_y u'^\epsilon - \partial_y u' + u' - 1 - \epsilon = 0, \\ \partial_x u'^\epsilon + \partial_y v'^\epsilon = 0, \end{cases}$$

with the boundary condition

$$u'^\epsilon|_{y=0} = \epsilon, \quad \lim_{y \to +\infty} u'^\epsilon = 1 + \epsilon, \quad v'^\epsilon|_{y=0} = 0,$$

and the initial data

$$u'^\epsilon|_{x=0} = u_0'(y) := u_0(y) + \epsilon,$$

for any parameter $\epsilon > 0$. Since the initial and boundary conditions of horizontal velocity are positive, we can follow the idea as Guo and Nguyen(see Lemma 2.2 in [20]) to obtain the local well-posedness in higher regularity space
for any fixed $\epsilon > 0$. However, it is worth nothing that the life time interval $[0, L^*]$ depends on the parameter $\epsilon$. As we hope that the solution $(u^\epsilon, v^\epsilon)$ of approximate system (2.1) with positive boundary condition will converge to the solution $(u, v)$ of original Prandtl type Eq. (1.6) as $\epsilon$ tends to zero. To achieve the target, we need to obtain the uniform a priori estimates of solution $(u^\epsilon, v^\epsilon)$ on existence time independent of $\epsilon$. Thanks to the recent paper [18], the so called quotient $\partial_y \left( \frac{v^\epsilon}{u^\epsilon} \right)$ is a good unknown for us to avoid the loss of derivative and the degeneracy of quasi-linear parabolic equation to establish closed a priori estimate. And hence, we need to control the quantity $\frac{v^\epsilon}{u^\epsilon}$ by the good unknown $\partial_y \left( \frac{v^\epsilon}{u^\epsilon} \right)$ by using the Hardy inequality. Thus, our method here is to establish the energy estimate including energy part and dissipative one both with suitable weighted $(y)^j$. Then, the local well-posedness of the Eq. (1.6) in higher regular space follows directly as the parameter $\epsilon$ tends to zero.

Next, we state the main approach to obtain the asymptotic behavior of solution for the Prandtl Eq. (1.6) in Theorem 1.2. For classical Prandtl equation, in the case of localized data near the Blasius solution, Iyer [28] applied the energy method instead of maximum principle method for the good known quantity(called “quotient”) to establish specific convergence rate for the quantity $(x, \psi) := u^2(x, \psi) - \bar{u}^2(\psi)$. However, he did not build any convergence rate for the quantity $u(x, y) - \bar{u}(y)$. Since there is no result about the asymptotic behavior of Eq. (1.6), we hope to find a method to establish the convergence rate in physical variable. By using the method as in [28], we first show the global existence of the solution for 2-D Prandtl-Hartmann system via standard continuity argument. It should be noted that the term $\frac{\partial \phi}{\partial (u + v)}$ in the equation (1.16) plays an important role as a damping term. Thus, we can get that the difference $\phi$ decays exponentially in $x$ variable, that is

$$\|\phi\|^2_{H^1_x} + \|\phi\|^2_{L^2_t} + \|\phi_x\|^2_{L^2_{x,y}} + \|\phi_{xx}\|^2_{H^1_{x,y}} + \|\phi_{y}\|^2_{L^2_{x,y}} + \|\phi_{yy}\|^2_{L^2_{x,y}} \leq Ce^{-x}. \quad (2.4)$$

More details of the proof for this decay estimate can be found in Proposition 4.2 in Section 4.

The asymptotic behavior (2.4) occurs at equal values of the stream function $\psi$, but rather than at equal values of the physical variable $y$. In order to obtain the decay estimate in $x$ variable of the difference of $u$ and $\bar{u}$ as functions of $(x, y)$, as required for (1.15), we need further and more arguments. For any given value of the stream function $\psi$, we denote $y_1$ and $y_2$ represent the corresponding physical variable for the flows $u$ and $\bar{u}$, respectively. Motivated by [40], we decompose the normal gradient of the tangential velocity as the following two parts:

$$|u_y(x, y_1) - \bar{u}_y(y_1)| \leq |u_y(x, y_1) - \bar{u}_y(y_2)| + |\bar{u}_y(y_2) - \bar{u}_y(y_1)|.$$  

The first part can be transformed into von-Mises variable and then controlled by the exponential decay estimate (2.4). After some basic calculations, the second part about the solution $\bar{u}$ of Hartmann boundary layer can also be controlled by the exponential decay estimate (2.4). Therefore, we conclude that $(u_y - \bar{u}_y)$ decays exponentially in $x$ variable in $L^\infty_y$–norm. It then follows from the similar way above to show that $(u - \bar{u})$ also decay exponentially in $x$ variable in $H^2_y$–norm. The main ideas and strategy of establishing the decay estimates in $x$ variable of the difference of $u$ and $\bar{u}$ as functions of $(x, y)$ will be given in Section 4.3.

3 Local existence and uniqueness of Prandtl-Hartmann system

In this section, we will establish the local existence and uniqueness of solution with higher regularity for the Prandtl-Hartmann system (1.1)-(1.4) in Theorem 1.1. In section 3.1 we will establish some uniform estimates with respect to the parameter $\epsilon$ for the approximated system (2.1). The local existence and uniqueness of original Prandtl-Hartmann system are be investigated in sections 3.2 and 3.3 respectively.

3.1 Uniform estimate for approximated system

In this subsection, we will establish uniform estimate with respect to the parameter $\epsilon$ for the approximated system (2.1) such that there exists a positive time $L$ independent of $\epsilon$. Motivated by the recent work [18], we used the divergence-free condition (2.1) to rewrite the approximated system (2.1) as follows

$$-(u^\epsilon)^2 \partial_y \left( \frac{v^\epsilon}{u^\epsilon} \right) - \partial_y u^\epsilon + u^\epsilon - 1 - \epsilon = 0.$$
Apply the $\partial_x$–operator to the above equation and use the divergence-free condition \((2.1)\), we have
\[ -\partial_x \left\{ (u^e)^2 \partial_y \left( \frac{v^e}{u^e} \right) \right\} + \partial_y^2 v^e \partial_y v^e = 0. \]

Let us denote the notation (called “quotient” in \([18]\))
\[ q^e := \frac{v^e}{u^e}, \tag{3.1} \]

then it holds
\[ -\partial_x \left\{ (u^e)^2 \partial_y q^e \right\} + \partial_y^3 v^e \partial_y v^e = 0. \tag{3.2} \]

Due to the conditions of initial data $u_0(y)$ in Theorem \([11]\) there exists a small $0 < \delta_0 \ll 1$ such that
\[ \frac{1}{2} y \leq u_0(y) \leq \frac{3}{2} y, \quad \forall y \in [0, \delta_0] \]

and
\[ u_0(y) \geq \frac{1}{2} \delta_0, \quad \forall y \in [\delta_0, +\infty). \]

Then, the initial data $u^e_0(y)$ (see the definition in \((2.3)\)) will satisfy
\[ \frac{1}{2} (y + \epsilon) \leq u^e_0(y) \leq \frac{3}{2} (y + \epsilon), \quad \forall y \in [0, \delta_0] \tag{3.3} \]

and
\[ u^e_0(y) \geq \frac{1}{2} (\delta_0 + \epsilon), \quad \forall y \in [\delta_0, +\infty). \tag{3.4} \]

Let us define the norms:
\[ E_t^k(x) := \sum_{j=0}^k \| u^e \partial_x^j \partial_y q^e(y)^l \|_{L^\infty_x L^2_y} + \sum_{j=0}^k \| \sqrt{u^e} \partial_x^j \partial_y^2 q^e(y)^l \|_{L^2_x L^2_y}, \tag{3.5} \]
\[ X_t^k(x) := E_t^k(x) + \sum_{j=0}^{k-1} \sum_{l=1}^3 \| \partial_x^j \partial_y q^e(y)^l \|_{L^2_x L^2_y} + \| \partial_y^3 v^e \chi \left( \frac{y}{\delta} \right) \|_{L^2_x L^2_y}, \tag{3.6} \]

and
\[ B_t^k(x) := \sum_{j=1}^2 \| \partial_y^j u^e(x, y) \partial_x (y)^l \|_{L^\infty_x([0, x]) L^2_y} + \| \partial_y^3 u^e(x, y) \chi \left( \frac{y}{\delta} \right) \|_{L^\infty_x([0, x]) L^2_y} \]
\[ + \sum_{j=1}^3 \| \partial_x^{(k-2)} \partial_y^j q^e(x, y) \|_{L^\infty_x([0, x]) L^2_y} + \| \partial_x^{(k-1)} \partial_y q^e(x, y) \|_{L^\infty_x([0, x]) L^2_y} \]
\[ + \| \partial_x^{(k-1)} \partial_y^2 q^e(x, y) \chi \left( \frac{y}{\delta} \right) \|_{L^\infty_x([0, x]) L^2_y}. \tag{3.7} \]

Thus, the parameter $\delta$ appearing in cutoff function $\chi \left( \frac{y}{\delta} \right)$ (see \((1.17)\)) will be chosen to satisfy $\delta \leq \delta_0$ such that the assumption conditions \((3.3)\) and \((3.4)\) hold on.

**Proposition 3.1 (Uniform estimate).** Let $k \geq 2$ be an integer, $l \geq 1$ be a real number and $\epsilon \in (0, 1]$, and $(u^e, v^e)$ be sufficiently smooth solution, defined on $[0, L^e]$, to the non-degenerated approximated system \((2.1)-(2.3)\). The initial data $u^e_0(y)$ is defined by \((2.3)\) and satisfies the conditions \((3.3)\) and \((3.4)\). Then, there exists a time $L_a = L_a(k, l, \delta_0, u_0) > 0$ independent of $\epsilon$ such that the following a priori estimate holds true for all $x \in [0, L_a]$:
\[ X_t^k(x)^2 + B_t^k(x)^2 \leq 2C_{k, l, \delta_0}(1 + C(u_0)), \tag{3.8} \]
and
\[ \frac{1}{4} y \leq u^e(x, y) \leq 2(y + \epsilon), \quad (x, y) \in [0, L_a] \times [0, \delta_0]; \quad u^e(x, y) \geq \frac{1}{4} \delta_0, \quad (x, y) \in [0, L_a] \times [\delta_0, \infty). \tag{3.9} \]

Furthermore, it also holds
\[ \sum_{0 \leq 2\alpha + \beta \leq 2k + 1} \| \partial_x^\alpha \partial_y^\beta q^e(y)^l \|_{L^2_x L^2_y} \leq C_{k, l, \delta_0}(1 + C(u_0)), \tag{3.10} \]
where $C(u_0)$ is a constant depends on the initial data $u_0$. 
Throughout this subsection, for some positive constants $k_*$ and $k^* (> k_*)$, we assume that the following a priori assumptions:

$$ k_*(y + \epsilon) \leq u^i(x, y) \leq k^*(y + \epsilon), \quad (x, y) \in [0, L^*] \times [0, \delta_0], $$

and

$$ u^i(x, y) \geq k_*(\delta_0 + \epsilon), \quad (x, y) \in [0, L^*] \times [\delta_0, +\infty) $$

hold on. Here the parameter $\delta_0$ is defined before.

**Lemma 3.2.** Let $(u^i, v^i)$ be the smooth solution, defined on $[0, L^*]$, of the approximated equations (3.10) - (3.12). Under the conditions of (3.11) and (3.12), then we have the following estimates:

\[
\begin{align*}
\|\partial_x^2 \partial_y^q v^i(y)\|_{L^2_{t}L^2_y} &\leq C_{k_*,k^*,l,k,\delta_0}(1 + B_k^i(0)^2 + X_k^i(x)^2); \\
\|\partial_x^{(k)} \partial_y u^i(y)\|_{L^{2}_{t}L^{2}_y} &\leq o_L(1)C_{k_*,l,k,\delta_0}(1 + B_k^i(0)^2 + X_k^i(x)^2); \\
\|\partial_x^{(k)} \partial_y^2 v^i(y)\|_{L^2_{t}L^2_y} &\leq o_L(1)C_{k_*,l,k,\delta_0}(1 + B_k^i(0)^2 + X_k^i(x)^4); \\
\|\partial_x^{(k)} \partial_y^2 v^i(y)\|_{L^2_{t}L^2_y} &\leq o_L(1)C_{k_*,l,k,\delta_0}(1 + B_k^i(0)^2 + X_k^i(x)^4); \\
\|\partial_x^{(k)} \partial_y^2 v^i(y)\|_{L^2_{t}L^2_y} &\leq o_L(1)C_{k_*,l,k,\delta_0}(1 + B_k^i(0)^2 + X_k^i(x)^4); \\
\|\partial_x^{(k)} \partial_y^2 v^i(y)\|_{L^2_{t}L^2_y} &\leq o_L(1)C_{k_*,l,k,\delta_0}(1 + B_k^i(0)^2 + X_k^i(x)^8),
\end{align*}
\]

where $l \geq 1$ and $k \geq 2$.

**Proof.** Step 1: Using the definition of $q^i$ (see (3.1)), we can get

\[
\partial_x^k \partial_y^q v^i = \partial_x^k u^i \partial_y^q + 2 \partial_y u^i \partial_x^{k-1} \partial_y q^i + \partial_x^k \partial_y^2 q^i + \sum_{j=1}^{k} C_j^k \partial_x^j \partial_y^{k-j} \partial_y q^i
\]

\[
+ 2 \sum_{j=1}^{k} C_j^k \partial_x^j \partial_y u^i \partial_x^{k-j} \partial_y q^i + \sum_{j=1}^{k} C_j^k \partial_x^j \partial_y^2 u^i \partial_x^{k-j} q^i
\]

\[
:= I_{11} + I_{12} + I_{13} + I_{14} + I_{15} + I_{16}.
\]

Using estimates (A.4), (A.6) in Appendix A, it is easy to check that

\[
\begin{align*}
&\|I_{11}(y)\|_{L^2_{t}L^2_y} \leq \|\partial_x^2 \partial_y^q v^i(y)\|_{L^2_{t}L^2_y} \|\partial_x^{k} q^i\|_{L^2_{t}L^2_y} \\
&\leq o_L(1)C_{k_*,l}(\|\partial_x^2 \partial_y^q v^i(y)\|_{L^2_{t}L^2_y} + \|\partial_x^2 \partial_y^q v^i(y)\|_{L^2_{t}L^2_y})E_k^i(x); \\
&\|I_{12}(y)\|_{L^2_{t}L^2_y} \leq \|\partial_y u^i\|_{L^2_{t}L^2_y} \|\partial_x^2 \partial_y q^i(y)\|_{L^2_{t}L^2_y} \\
&\leq o_L(1)C_{k_*,l}(\|\partial_y u^i\|_{L^2_{t}L^2_y} + \|\partial_x^2 \partial_y q^i(y)\|_{L^2_{t}L^2_y})E_k^i(x); \\
&\|I_{13}(y)\|_{L^2_{t}L^2_y} \leq \|u^i\|_{L^2_{t}L^2_y} \sqrt{\|\partial_x^2 \partial_y q^i(y)\|_{L^2_{t}L^2_y}} \\
&\leq C(1 + \|\partial_y u^i\|_{L^2_{t}L^2_y})E_k^i(x); \\
&\|I_{15}(y)\|_{L^2_{t}L^2_y} \leq \|\partial_x^{(k-1)} \partial_y u^i\|_{L^2_{t}L^2_y} \|\partial_x^{(k-1)} \partial_y q^i(y)\|_{L^2_{t}L^2_y} \\
&\leq C\|\partial_x^{(k-1)} \partial_y u^i(y)\|_{L^2_{t}L^2_y} \|\partial_x^{(k-1)} \partial_y q^i(y)\|_{L^2_{t}L^2_y} + o_L(1)C_{k_*,l,\delta_0}E_k^i(x); \\
&\|I_{16}(y)\|_{L^2_{t}L^2_y} \leq \|\partial_x^{(k-1)} \partial_y q^i(y)\|_{L^2_{t}L^2_y} \|\partial_x^{(k-1)} \partial_y q^i(y)\|_{L^2_{t}L^2_y} + o_L(1)C_{k_*,l,\delta_0}E_k^i(x).
\end{align*}
\]

Based on the combination of the above estimates, we have for all $l \geq 1$

\[
\|I_{11}(y)\|_{L^2_{t}L^2_y} + \|I_{12}(y)\|_{L^2_{t}L^2_y} + \|I_{13}(y)\|_{L^2_{t}L^2_y} + \|I_{15}(y)\|_{L^2_{t}L^2_y} + \|I_{16}(y)\|_{L^2_{t}L^2_y} \leq C_{k_*,l,\delta_0}(1 + B_k^i(0)^2 + X_k^i(x)^2).
\]

Let us write

\[
I_{14} = \sum_{j=1}^{k} C_j^k \partial_x^j u^i \partial_x^{k-j} \partial_y^2 q^i \chi + \sum_{j=1}^{k} C_j^k \partial_x^j u^i \partial_x^{k-j} \partial_y^2 q^i (1 - \chi) := I_{141} + I_{142}.
\]
First of all, let us deal with the $I_{141}$ term. Due to $k \geq 2$, using Hölder inequality, estimates (A.8) and (A.8), we can obtain
\[
\| I_{141}(y)^{t} \|_{L_{2}^{1}L_{2}^{k}} \leq C_{k}(\delta_{0} + \epsilon)^{\frac{1}{2}} \| \partial_{x}^{(\frac{k-1}{2})}\partial_{x}^{1}v^{t}\|_{L_{2}^{\infty}L_{2}^{k}} \| \partial_{y}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}} \\
+ C_{k}(\delta_{0} + \epsilon)^{\frac{1}{2}} \| \partial_{x}^{(k-1)}\partial_{x}^{1}v^{t}\|_{L_{2}^{\infty}L_{2}^{k}} \| \partial_{y}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}} \\
\leq o_{L}(1)C_{k,k',\delta_{0}}(\| \partial_{x}^{(\frac{k-1}{2})}\partial_{x}^{1}v^{t}\|_{x=0}\|H_{2}^{k} \| + \| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}H_{2}^{k}}) \| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}} \\
+ o_{L}(1)C_{k,k',\delta_{0}}\| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}} \| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}} \\
\leq o_{L}(1)C_{k,k',\delta_{0}}(1 + B_{k}^{k}(0)^{2} + X_{k}^{k}(x)^{2}).
\]  
Here the condition $k \geq 2$ implies directly $[\frac{k-1}{2}] \leq k - 2$. Using the Hölder inequality, we have
\[
\| I_{142}(y)^{t} \|_{L_{2}^{1}L_{2}^{k}} \leq C_{k}\| \partial_{x}^{(k-1)}\partial_{x}^{1}u^{t}(1 - \chi)\|_{L_{2}^{\infty}L_{2}^{k}} \| \partial_{y}^{(k-1)}\partial_{y}^{2}q^{t}(1 - \chi)\langle y \rangle^{t}\|_{L_{2}^{1}L_{2}^{k}}.
\]  
Away from the boundary $y = 0$, we have
\[
\| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}(1 - \chi)\|_{L_{2}^{\infty}L_{2}^{k}} \leq \| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}(1 - \chi)\|_{x=0}\|H_{2}^{k} \| + o_{L}(1)\| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}}
\]  
Substituting the estimate (3.22) into (3.21) and applying the divergence-free condition, we have
\[
\| I_{142}(y)^{t} \|_{L_{2}^{1}L_{2}^{k}} \leq \| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}H_{2}^{k}} \| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}(1 - \chi)\|_{x=0}\|H_{2}^{k} \| + o_{L}(1)C_{k,k',\delta_{0}}\| \sqrt{u^{t}}\partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}}.
\]  
which, together with estimate (3.20), yields directly
\[
\| I_{14}(y)^{t} \|_{L_{2}^{1}L_{2}^{k}} \leq o_{L}(1)C_{k,k',\delta_{0}}(1 + B_{k}^{k}(0)^{2} + X_{k}^{k}(x)^{2}).
\]  
This and the estimate (3.19) imply directly for all $l \geq 1$
\[
\| \partial_{x}^{k}\partial_{y}^{l}v^{t}\|_{L_{2}^{1}L_{2}^{k}} \leq C_{k,k',\delta_{0}}(1 + B_{k}^{k}(0)^{2} + X_{k}^{k}(x)^{2}),
\]  
which yields the estimate (3.18).

Step 2: Obviously, it holds
\[
\partial_{x}^{k}\partial_{y}^{l}v^{t} = \partial_{x}^{k}(\partial_{y}^{l}v^{t}) + \partial_{x}^{k}(u^{t}y^{t})
\]  
\[
= \partial_{y}^{l}\partial_{x}^{k}q^{t} + u^{t}\partial_{x}^{k}\partial_{y}^{l}q^{t} + \sum_{j=1}^{k} C_{j}^{k}\partial_{x}^{j}\partial_{y}^{l-k+j}q^{t} + \sum_{j=1}^{k} C_{j}^{k}\partial_{x}^{j}\partial_{y}^{l-k+j}q^{t} + \sum_{j=1}^{k} C_{j}^{k}\partial_{x}^{j}\partial_{y}^{l-k+j}q^{t}
\]  
\[
:= I_{21} + I_{22} + I_{23}.
\]  
Using Hölder inequality and estimate (A.6), it holds for all $l \geq 1$
\[
\| I_{21}(y)^{t} \|_{L_{2}^{1}L_{2}^{k}} \leq \| \partial_{y}^{l}\partial_{x}^{k}q^{t}\|_{L_{2}^{\infty}L_{2}^{k}} \| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}} + \sqrt{\| \partial_{y}^{l}\partial_{x}^{k}q^{t}\|_{L_{2}^{1}L_{2}^{k}}} \| \partial_{y}^{l}\partial_{x}^{k}q^{t}\|_{L_{2}^{1}L_{2}^{k}}
\]  
Using the divergence-free condition, and estimates (A.6) and (A.7), it holds for $k \geq 2$
\[
\| I_{22}(y)^{t} \|_{L_{2}^{1}L_{2}^{k}} \leq \| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}} \| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}}
\]  
\[
\| I_{23}(y)^{t} \|_{L_{2}^{1}L_{2}^{k}} \leq o_{L}(1)C_{k,k',\delta_{0}}(\| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}}) + o_{L}(1)C_{k,k',\delta_{0}}(\| \partial_{x}^{(k-1)}\partial_{y}^{2}q^{t}\|_{L_{2}^{1}L_{2}^{k}} + C_{k,k',\delta_{0}}(1 + B_{k}^{k}(0)^{2} + X_{k}^{k}(x)^{2})
\]  
\[
\leq o_{L}(1)C_{k,k',\delta_{0}}(1 + B_{k}^{k}(0)^{2} + X_{k}^{k}(x)^{2}).
\]  

2D magnetic Prandtl model in the Prandtl-Hartmann regime
Using the divergence-free condition, and estimates (3.4), we have:

\[
\|I_{23}(y)^l\|_{L^2_y L^2_y} \leq C_k \|\partial_x^{(k-1)} \partial_y \phi^y\|_{L^\infty_y L^\infty_y} \|\partial_x^{(k-1)} \partial_y \phi^y\|_{L^2_y L^2_y} \\
+ C_k \|\partial_x^{(k-1)} \partial_y \phi^y\|_{L^2_y L^2_y} \|\partial_x^{(k-1)} \partial_y \phi^y\|_{L^2_y L^2_y} \\
\leq o_L(1)C_{k,k} \|\partial_x^{(k-1)} \partial_y \phi^y\|_{L^2_y L^2_y} + \|\partial_x^{(k-1)+1} \partial_y \phi^y\|_{L^2_y L^2_y} + o_L(1)C_{k,k} \|\partial_x^{(k-1)+1} \partial_y \phi^y\|_{L^2_y L^2_y} \\
\leq o_L(1)C_{k,k,\delta_0}(1 + B_1^k(0)^2 + X_1^k(x)^2).
\]

Thus, we have for all integer \(k \geq 2\) and \(l \geq 1\)

\[
\|\partial_x^l \partial_y \phi^y\|_{L^2_y L^2_y} \leq o_L(1)C_{k,k,l,\delta_0}(1 + B_1^k(0)^2 + X_1^k(x)^2).
\]

This implies directly the estimate (3.14).

Step 3: Using the equation (3.2), it holds

\[
\|\partial_x^{k-1} \partial_y \phi^y\|_{L^2_y L^2_y} \\
\leq \left(\sum_{j=1}^{k} u' \partial_x^{j-1} \partial_y \phi^y\right) \|\partial_x^{(k-1)j} \partial_y \phi^y\|_{L^2_y L^2_y} \\
+ C_k \sum_{j=1}^{k} \left(\sum_{j_1=1}^{j} \partial_x^{j_1-1} \partial_y \phi^y\right) \|\partial_x^{j_1-1} \partial_y \phi^y\|_{L^2_y L^2_y} \\
:= I_{31} + I_{32} + I_{33} + I_{34}.
\]

It is easy to check that

\[
\|I_{32}\| \leq \left(\sum_{j=1}^{k} u' \partial_x^{j-1} \partial_y \phi^y\right) \|\partial_x^{(k-1)j} \partial_y \phi^y\|_{L^2_y L^2_y} \\
\leq o_L(1)(1 + \|u' - e\|_{L^\infty_y L^\infty_y}) \|\partial_x^{k-1} \partial_y \phi^y\|_{L^2_y L^2_y} \\
\leq o_L(1)(1 + \|\partial_y u_0(y)\|_{L^2_y} + o_L(1)\|\partial_x^{k-1} \partial_y \phi^y\|_{L^2_y L^2_y}) \|\partial_x^{k-1} \partial_y \phi^y\|_{L^2_y L^2_y} \\
\leq o_L(1)(1 + B_1^k(0)^2 + X_1^k(x)^2).
\]

Using divergence-free condition, estimates (3.4), and (3.11), it is easy to check that

\[
\|I_{33}\| \leq C_k \|\partial_x^{(k-2)} \partial_y \phi^y\|_{L^\infty_y L^\infty_y} \|u' \partial_x^{(k-1)} \partial_y \phi^y\|_{L^2_y L^2_y} \\
+ C_k \|\partial_x^{(k-1)} \partial_y \phi^y\|_{L^2_y L^2_y} \|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} \\
\leq o_L(1)C_{k,k} \|\partial_x^{(k-2)} \partial_y \phi^y\|_{L^2_y L^2_y} + \|\partial_x^{(k-1)} \partial_y \phi^y\|_{L^2_y L^2_y} \|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} \\
\leq C_{k,k,\delta_0}(1 + B_1^k(0)^2 + X_1^k(x)^2),
\]

where we have used the estimate in the last inequality

\[
\|\partial_y(u' \partial_y \phi^y\|_{L^\infty_y L^\infty_y} \\
\leq \|\partial_y u' \partial_y \phi^y\|_{L^\infty_y L^\infty_y} + \|u' \partial_y \phi^y\|_{L^\infty_y L^\infty_y}(1 - \chi)\|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} \\
\leq C_{k,k,\delta_0}(1 + B_1^k(0)^2 + X_1^k(x)^2).
\]

Here the above terms can be estimated as follows:

\[
\|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} \\
\leq \|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} \|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} \\
\leq (\|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} + o_L(1)\|\partial_y \phi^y\|_{L^2_y L^2_y} + o_L(1)\|\partial_y \phi^y\|_{L^2_y L^2_y}) \\
\leq C(B_1^k(0)^2 + X_1^k(x)^2),
\]

\[
\|\partial_y u' \partial_y \phi^y\|_{L^\infty_y L^\infty_y} \\
\leq \|\partial_y u' \partial_y \phi^y\|_{L^\infty_y L^\infty_y} \|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} \\
\leq (\|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} + o_L(1)\|\partial_y \phi^y\|_{L^2_y L^2_y} + o_L(1)\|\partial_y \phi^y\|_{L^2_y L^2_y}) \\
\leq C(B_1^k(0)^2 + X_1^k(x)^2),
\]

\[
\|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} \\
\leq \|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} \|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} \\
\leq (\|\partial_y \phi^y\|_{L^\infty_y L^\infty_y} + o_L(1)\|\partial_y \phi^y\|_{L^2_y L^2_y} + o_L(1)\|\partial_y \phi^y\|_{L^2_y L^2_y}) \\
\leq C(B_1^k(0)^2 + X_1^k(x)^2),
\]
and
\[ \| u^* \partial_y^2 q^r(y) \|_{L^\infty - L^2_y} \]
\[ \leq (\| u^* \partial_y^2 q^r(y) \|_{L^\infty - L^2_y} + \| \partial_y u_0(y) \|_{L^2_y} + o_L(1) \| \partial_y^2 q^r(y) \|_{L^2_y} ) \]
\[ \times \left( (\| \partial_y^2 q^r(y) \|_{L^\infty - L^2_y} ) \right) \leq C_{k, \delta_0} (1 + B_k^4(0)^2 + X_k^4(x)^2). \]

Similarly, we can get
\[ |I_{34}| \leq C_k \| \partial_x^{k-2} \partial_y v^r \|_{L^2 - L^2_y} \| \partial_x^{k-1} \partial_y^2 v^r \|_{L^2 - L^2_y} \]
\[ \leq o_L(1) C_{k, k}(\| \partial_x^{k-1} \partial_y^2 v^r \|_{L^2 - L^2_y} + \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y}) \]
\[ \leq o_L(1) C_{k, k, \delta_0}(1 + B_k^4(0)^4 + X_k^4(x)^4), \]
which yields the estimate (3.15).

Step 4: Integrating by part, using Hardy inequality (3.1), Hölder and Cauchy inequalities, it holds true
\[ \int_0^\infty \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y}^2 dy \leq \int_0^\infty \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y}^2 \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y} dy \]
\[ \leq C(1 + \| \partial_y u^r \|_{L^2_y}) \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y} \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y} \]
\[ \leq C(1 + \| \partial_y u^r \|_{L^2_y}) \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y}^2 + \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y}^2. \]

Then, for any integer \( k \geq 2 \) and \( l \geq 1 \), we use the estimates (3.14) and (3.15) to obtain
\[ \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y} \leq o_L(1) C_{k, l, k, \delta_0}(1 + B_k^4(0)^4 + X_k^4(x)^4), \]
which yields the estimate (3.16).

Step 5: Using the equation (3.23), we have for any integer \( k \geq 2 \)
\[ \partial_x^{k-1} \partial_y v^r = \partial_x^{k-1} \partial_y^2 q^r + (u^r)^2 \partial_x^{k-1} \partial_y^2 q^r + 2 \partial_y u^r \partial_x^{k-1} \partial_y q^r + \sum_{j=1}^k C_k^j \partial_x^{k-j} \partial_y u^r \partial_x^{k-j} \partial_y q^r \]
\[ + \sum_{j=1}^k C_k^j \partial_x^{k-j} \partial_y^2 (u^r)^2 \partial_x^{k-j} \partial_y q^r := I_{41} + I_{42} + I_{43} + I_{44} + I_{45}. \]

It is easy to check that
\[ \| I_{42}(y) \|_{L^2 - L^2_y} \leq \| u^r \|_{L^2_y} \| \sqrt{u^r} \partial_x^{k-j} \partial_y q^r \|_{L^2 - L^2_y} \]
\[ \leq C(1 + \| \partial_y u^r \|_{L^2_y}) \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y} \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y} \]
\[ \| I_{43}(y) \|_{L^2 - L^2_y} \leq \| \partial_y u^r \|_{L^2_y} \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y} \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y} \]
\[ \leq o_L(1) \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y} \| \partial_x^{k-1} \partial_y v^r \|_{L^2 - L^2_y}. \]

To deal with the term \( I_{44} \), we can follow the idea as terms \( I_{33} \). Let us write
\[ I_{44} = \sum_{j=1}^k C_k^j \sum_{j=1}^k \sum_{j=1}^{j-1} C_k^j \partial_x^{k-j} \partial_y u^r \partial_x^{k-j} \partial_y q^r \]
\[ + \sum_{j=1}^k \sum_{j=1}^{j-1} C_k^j \partial_x^{k-j} \partial_y u^r \partial_x^{k-j} \partial_y q^r := I_{441} + I_{442} + I_{443}. \]
Using estimate (3.20), it is easy to check that

\[
\|I_{441}(y)^{l}\|_{L^2_x L^2_y} \leq C_k \|\partial_x^{(k-2)} \partial_y v^r\|_{L^\infty_x L^\infty_y} \|u^{r}\|_{L^2_x L^2_y} + C_k \|\partial_x^{(k-1)} \partial^2_y q^v(y)^{l}\|_{L^2_x L^2_y} + C_k \|\partial_x^{(k-1)} \partial^2_y q^u\|_{L^\infty_x L^\infty_y} \\
\leq C_k(\|\partial_x^{(k-2)} \partial_y v^r\|_{L^\infty_x L^\infty_y} + \alpha_l (1) \|\partial_x^{(k-1)} \partial^2_y q^v(y)^{l}\|_{L^2_x L^2_y}) \|u^{r}\|_{L^2_x L^2_y} \leq C_k(1 + B^k_l(0)^4 + X^k_l(x)^4).
\]

Similarly, using estimate (A.4), we can obtain for all \(k \geq 2\)

\[
\|I_{442}(y)^{l}\|_{L^2_x L^2_y} \leq C_k \|\partial_x^{(k-1)} \partial_y v^r\|_{L^\infty_x L^\infty_y} \|\partial_x^{(k-1)} \partial_y q^v(y)^{l}\|_{L^2_x L^2_y} \|u^{r}\|_{L^2_x L^2_y} \\
\leq C_k(\|\partial_x^{(k-1)} \partial_y v^r\|_{L^\infty_x L^\infty_y} + \alpha_l (1) \|\partial_x^{(k-1)} \partial^2_y q^v(y)^{l}\|_{L^2_x L^2_y}) \|u^{r}\|_{L^2_x L^2_y} \leq C_k(1 + B^k_l(0)^4 + X^k_l(x)^4),
\]

and

\[
\|I_{443}(y)^{l}\|_{L^2_x L^2_y} \leq C_k \|\partial_x^{(k-2)} \partial_y v^r\|_{L^\infty_x L^\infty_y} \|\partial_x^{(k-2)} \partial^2_y v^u\|_{L^\infty_x L^\infty_y} \|\partial_x^{(k-1)} \partial_y q^v(y)^{l}\|_{L^2_x L^2_y} \\
\leq C_k(\|\partial_x^{(k-2)} \partial^2_y v^u\|_{L^\infty_x L^\infty_y} + \|\partial_x^{(k-1)} \partial^2_y q^v(y)^{l}\|_{L^2_x L^2_y}) \|u^{r}\|_{L^2_x L^2_y} \leq C_k(1 + B^k_l(0)^4 + X^k_l(x)^4).
\]

Collecting the estimates from terms \(I_{441}\) to \(I_{443}\), we get that

\[
\|I_{44}(y)^{l}\|_{L^2_x L^2_y} \leq C_k, k(1 + B^k_l(0)^4 + X^k_l(x)^4).
\]

Finally, let us deal with the term \(I_{45}\). Indeed, we use the divergence-free condition to write

\[
I_{45} = \sum_{j=1}^{k} 2C_j^g u^r \partial_x^j u^r \partial_x^{j-1} \partial_y q^v + \sum_{j=1}^{k-1} 2C_j^g C_j^d \partial_x^{j-1} \partial_y v^r \partial_x^j u^r \partial_x^{k-j-1} \partial_y v^r \partial_x^{k-j} \partial_y q^v \\
:= I_{451} + I_{452}.
\]

Using divergence-free condition, estimates (3.20) and (A.4), we can get

\[
\|I_{451}(y)^{l}\|_{L^2_x L^2_y} \leq C_k(1 + \|u^r\|_{L^2_x L^2_y}) \|\partial_x^{(k-2)} \partial_y v^r\|_{L^\infty_x L^\infty_y} \sqrt{\|u^r\|_{L^2_x L^2_y}} \|\partial_x^{(k-1)} \partial^2_y q^v(y)^{l}\|_{L^2_x L^2_y} \\
\leq C_k(1 + \|\partial_y v^r\|_{L^2_x L^2_y}) \|\partial_x^{(k-2)} \partial^2_y v^u\|_{L^\infty_x L^\infty_y} \|\partial_x^{(k-1)} \partial^2_y q^v(y)^{l}\|_{L^2_x L^2_y} \\
\leq C_k, k(1 + B^k_l(0)^4 + X^k_l(x)^4),
\]

and

\[
\|I_{452}(y)^{l}\|_{L^2_x L^2_y} \leq C_k \|\partial_x^{(k-2)} \partial_y v^r\|_{L^\infty_x L^\infty_y} \|\partial_x^{(k-1)} \partial_y q^v(y)^{l}\|_{L^2_x L^2_y} \\
\leq C_k, k(1 + B^k_l(0)^4 + X^k_l(x)^4).
\]

Then, we have the estimate

\[
\|I_{45}(y)^{l}\|_{L^2_x L^2_y} \leq C_k, k(1 + B^k_l(0)^4 + X^k_l(x)^4).
\]
The combination of estimates from terms $I_{42}$ to $I_{45}$ and the estimate (3.13), it holds for all $k \geq 2$ and $l \geq 1$
\[ \|\partial_y^k \partial_y^l v^e(y)\|_{L^2 L^\gamma} \leq C_{k,l,k,k,k,\delta}(1 + \mathcal{B}_k^l(0)^4 + X_k^l(x)^4), \]
which yields the estimate (3.14).

Step 6: Using the equation (3.23), then we have
\[ \|\partial_y^4 v^e\|_{L^2 L^\gamma} \leq \|\partial_y^2 v^e\|_{L^2 L^\gamma} + 2\|\partial_y^2 u^e \partial_y^2 v^e\|_{L^2 L^\gamma} + \|\partial_y^2 u^e \partial_y q^e\|_{L^2 L^\gamma} + 2\|u^e \partial_y u^e \partial_y^2 q^e\|_{L^2 L^\gamma} + \|u^e \partial_y u^e \partial_y q^e\|_{L^2 L^\gamma} + \|u^e \partial_y u^e \partial_y q^e\|_{L^2 L^\gamma} + \|u^e \partial_y u^e \partial_y q^e\|_{L^2 L^\gamma} + \|u^e \partial_y u^e \partial_y q^e\|_{L^2 L^\gamma} + \|u^e \partial_y u^e \partial_y q^e\|_{L^2 L^\gamma} \]
\[ := I_{51} + I_{52} + I_{53} + I_{54} + I_{55} + I_{56}. \]

Using the a priori assumption (3.11), it is easy to check that
\[ I_{52} \leq C(\|\partial_y^2 u^e\|_{L^2 L^\gamma} + o_L(1)\|\partial_y^2 v^e\|_{L^2 L^\gamma} + \|\partial_y q^e\|_{L^2 L^\gamma} + o_L(1)\|\partial_y q^e\|_{L^2 L^\gamma} + \|\partial_y q^e\|_{L^2 L^\gamma} + o_L(1)\|\partial_y q^e\|_{L^2 L^\gamma}), \]
\[ I_{53} \leq k^*(\delta + \varepsilon)\|\partial_y^2 v^e\|_{L^2 L^\gamma} + o_L(1)\|\partial_y^2 v^e\|_{L^2 L^\gamma} + \|\partial_y^2 q^e\|_{L^2 L^\gamma} + o_L(1)\|\partial_y^2 q^e\|_{L^2 L^\gamma}, \]
\[ I_{54} \leq \sqrt{L}\|\partial_y^2 v^e\|_{L^2 L^\gamma} + o_L(1)\|\partial_y^2 v^e\|_{L^2 L^\gamma} + \|\partial_y^2 q^e\|_{L^2 L^\gamma}, \]
\[ I_{55} \leq \sqrt{L}(\|\partial_y^2 u^e\|_{L^2 L^\gamma} + o_L(1)\|\partial_y^2 v^e\|_{L^2 L^\gamma} + \|\partial_y^2 q^e\|_{L^2 L^\gamma}) + \|\partial_y^2 q^e\|_{L^2 L^\gamma}, \]
\[ I_{56} \leq (k^*)^2(\delta + \varepsilon)^2\|\partial_y^2 v^e\|_{L^2 L^\gamma}. \]

Combining estimates from $I_{52}$ to $I_{56}$ and using the estimates (A.4) and (3.16), we can choose $\varepsilon = \sqrt{L}$ to obtain
\[ \|\partial_y^k v^e\|_{L^2 L^\gamma} \leq o_L(1)\mathcal{C}_{k,l,k,\delta}(1 + \mathcal{B}_k^l(0)^8 + X_k^l(x)^8), \]
which yields the estimate (3.18). Therefore, we complete the proof of this lemma. 

Next, we will establish the energy estimate for the approximated equation (2.1).

**Lemma 3.3.** For any smooth solution $(u^e, v^e)$ of equations (2.1)-(2.3), it holds true for all $x \in [0, L^\gamma]$
\[ \mathcal{E}_k^l(x)^2 \leq \mathcal{E}_k^l(0)^2 + C_{k,k^*,k,l,\delta}(1 + \mathcal{B}_k^l(0)^8 + X_k^l(x)^8), \]
where $k \geq 2$ and $l \geq 1$.

**Proof.** Taking $\partial_y^k$ differential operator to the equation (3.22), we have
\[ \partial_y^{k+1} (u^e)^2 \partial_y q^e + \partial_y^k \partial_y v^e = 0. \]

Multiplying the above equation by $\partial_y^k \partial_y q^e(y)^2$ and integrating over $[0, x] \times [0, \infty)$, we have
\[ \begin{align*}
&\frac{1}{2} \int_0^x \left( (u^e)^2 \partial_y^k \partial_y q^e(y)^2 \right)(x, y) dy - \int_0^x \int_0^{\infty} \partial_y^k \partial_y v^e \cdot \partial_y^k \partial_y q^e(y)^2 d\tau dy \\
&+ \int_0^x \int_0^{\infty} u^e \partial_y^k \partial_y q^e(y)^2 d\tau dy \\
&= \frac{1}{2} \int_0^x \left( (u^e)^2 \partial_y^k \partial_y q^e(y)^2 \right)(0, y)^2 dy + II_1 + II_2 + II_3 + II_4,
\end{align*} \]
where the terms $II_i (i = 1, 2, 3, 4)$ are defined by
\[ II_1 := \int_0^x \int_0^{\infty} u^e \partial_y q^e(y)^2 d\tau dy, \]
\[ II_2 := - \sum_{j=1}^{k+1} C_j \int_0^x \int_0^{\infty} \partial_y^j (u^e)^2 \partial_y^k \partial_y q^e(y)^2 d\tau dy, \]
\[ II_3 := - \sum_{j=1}^{k} C_j \int_0^x \int_0^{\infty} \partial_y^j u^e \partial_y^k \partial_y q^e(y)^2 d\tau dy, \]
\[ II_4 := - \int_0^x \int_0^{\infty} \partial_y^k (\partial_y u^e)^2 \partial_y q^e(y)^2 d\tau dy. \]
Next, we deal with the dissipative term. For any $\epsilon > 0$, we use the boundary condition (3.22) to obtain
\[
\partial^k_x \partial_y q^r |_{y=0} = \partial^k_x \left\{ \frac{\partial_y v^r u^r - v^r \partial_y u^r}{(u^r)^2} \right\} |_{y=0} = 0.
\] (3.30)

Then, we integrate by part to get
\[
- \int_0^x \int_0^\infty \partial^k_x \partial_y v^r \cdot \partial^k_x \partial_y q^r(y)^{2l} d\tau dy \\
= \int_0^x \partial^k_x \partial^2_y v^r \cdot \partial^k_x \partial_y q^r(y)^{2l} \bigg|_{y=0} d\tau + \int_0^x \int_0^\infty \partial^k_x \partial^2_y v^r \cdot \partial_y \{ \partial^k_x \partial_y q^r(y)^{2l} \} d\tau dy \\
= \int_0^x \int_0^\infty \partial^k_x \partial^2_y u^r q^r + 2\partial_y u^r \partial_y q^r + u^r \partial^2_y q^r \cdot \partial_y \{ \partial^k_x \partial_y q^r(y)^{2l} \} d\tau dy.
\] (3.31)

Then, substituting the equality (3.31) into (3.23), we have
\[
\sup_{0 \leq x \leq L} \int_0^\infty (u^r)^2 |\partial^k_x \partial_y q^r|^2 (y)^{2l} dy + 2 \int_0^L \int_0^\infty u^r |\partial^k_x \partial_y q^r| (y)^{2l} dx dy + 2 \int_0^L \int_0^\infty u^r |\partial^k_x \partial^2_y q^r| (y)^{2l} dx dy
\]
\[
\leq \int_0^\infty ((u^r)^2 |\partial^k_x \partial_y q^r|^2 (0, y)^{2l} dy + 2 \sum_{i=1}^{i=8} |I_i|,
\] (3.32)

where the terms $I_i (i = 5, 6, 7, 8)$ are defined by
\[
I_5 := - \int_0^L \int_0^\infty \partial^k_x (\partial^2_y u^r q^r) \cdot \partial^k_x \partial^2_y q^r(y)^{2l} dx dy,
\]
\[
I_6 := -2 \int_0^L \int_0^\infty \partial^k_x (\partial_y u^r \partial_y q^r) \cdot \partial^k_x \partial^3_y q^r(y)^{2l} dx dy,
\]
\[
I_7 := - \sum_{j=1}^{k} C_j^k \int_0^L \int_0^\infty \partial^j_x u^r \partial^k_x \partial^l_x q^r \cdot \partial^k_x \partial^2_y q^r(y)^{2l} dx dy,
\]
\[
I_8 := -2 \int_0^L \int_0^\infty \partial^k_x \partial^2_y u^r q^r + 2\partial_y u^r \partial_y q^r + u^r \partial^2_y q^r \cdot \partial_y \{ \partial^k_x \partial_y q^r(y)^{2l} \} d\tau dy.
\]

Now, we claim that the terms $I_i (i = 1, ..., 8)$ can be estimated as follows:
\[
|I_1| \leq o_L(1) C_{k, l, k_0} (1 + B^k(0)^8 + X^k(x)^8),
\] (3.33)
\[
|I_2| \leq o_L(1) C_{k, l, k_0} (1 + B^k(0)^4 + X^k(x)^4),
\] (3.34)
\[
|I_3| \leq o_L(1) C_{k, k_0} (1 + B^k(0)^4 + X^k(x)^4),
\] (3.35)
\[
|I_4| \leq o_L(1) C_{k, l, k_0} (1 + B^k(0)^4 + X^k(x)^4),
\] (3.36)
\[
|I_5| \leq o_L(1) C_{k, k_0} (1 + B^k(0)^8 + X^k(x)^8),
\] (3.37)
\[
|I_6| \leq o_L(1) C_{k, k_0} (1 + B^k(0)^8 + X^k(x)^8),
\] (3.38)
\[
|I_7| \leq o_L(1) C_{k, k_0} (1 + B^k(0)^8 + X^k(x)^8),
\] (3.39)
\[
|I_8| \leq o_L(1) C_{k, l, k_0} (1 + B^k(0)^4 + X^k(x)^4).
\] (3.40)

Then, substituting the estimates (3.33) - (3.40) into (3.32), it is easy to obtain the estimate (3.28). □

In order to complete the proof of Lemma 3.3, it remains to show the claim estimates (3.33) - (3.40) as follows.

**Proof of estimate (3.33).** It is easy to deduce that
\[
|I_1| \leq \| \frac{\partial_y v^r}{u^r} \|_{L^2 L^\infty_y} \| u^r \partial^2_x \partial_y q^r(y)^{l} \|_{L^2 L^2_y}^2,
\]
which, together with the estimates (3.31), (3.32), (3.34) and (3.35), yields directly
\[
|I_1| \leq C_{k_0} \| \partial_y v^r \|_{L^2 H^2_y} \| u^r \partial^2_x \partial_y q^r(y)^{l} \|_{L^2 L^2_y}^2 \leq o_L(1) C_{k, l, k_0} (1 + B^k(0)^8 + X^k(x)^8),
\]
which is the inequality (3.33). □
Proof of estimate (3.31). Let us write

\[
II_2 = - \sum_{j=1}^{k+1} \sum_{j_1=0}^{j-1} C_{j_1}^{j+1} C_j \int_0^L \int_0^\infty \partial_j^1 u \partial_j^{1-j} \partial_y q^y \cdot \partial_x \partial_\varphi q^\varphi \langle y \rangle^2 \, dx \, dy
\]

\[
- \sum_{|j-j_1|=1}^{k+1} \sum_{j_1=0}^{j-1} C_{j_1}^{j+1} C_j \int_0^L \int_0^\infty \partial_j^1 u \partial_j^{1-j} \partial_y q^y \cdot \partial_x \partial_\varphi q^\varphi \langle y \rangle^2 \, dx \, dy
\]

\[
:=II_{21} + II_{22}.
\]

Applying the estimates (A.3), (3.14), (3.15) and (3.16), we have

\[
|II_{21}| \leq \|\partial_j^{(k+1)} u^x\|_{L^\infty L^2} \|\partial_j^k \partial_y q^y \|_{L^\infty L^2} + \|\partial_j^{(k+1)} u^c\|_{L^\infty L^2} \|\partial_j^k \partial_y q^\varphi \|_{L^\infty L^2}
\]

\[
\leq C_k \partial_j^{(k+1)} \|\partial_y \|_{L^\infty H^2} + \|\partial_j^{(k-1)} \|_{L^\infty H^2} \|\partial_j^k \|_{L^\infty L^2} \|\partial_y q^\varphi \|_{L^\infty L^2}
\]

\[
\leq o_L(1) C_k \partial_j \|\partial_y \|_{L^\infty L^2} (1 + B^k(0)^{16} + X^k(x)^{16})
\]

Using Sobolev inequality, divergence-free condition, estimates (A.3), (A.4) and (3.13), we have

\[
|II_{22}| \leq \|\partial_j^{(k+1)} u^x\|_{L^\infty L^2} \|\partial_j^k \partial_y q^y \|_{L^\infty L^2} + \|\partial_j^{(k+1)} u^c\|_{L^\infty L^2} \|\partial_j^k \partial_y q^\varphi \|_{L^\infty L^2}
\]

\[
\leq C_k \partial_j^{(k+1)} \|\partial_y \|_{L^\infty L^2} + \|\partial_j^{(k-1)} \|_{L^\infty L^2} \|\partial_j^k \|_{L^\infty L^2} \|\partial_y q^\varphi \|_{L^\infty L^2}
\]

\[
\leq o_L(1) C_k \partial_j \|\partial_y \|_{L^\infty L^2} (1 + B^k(0)^{16} + X^k(x)^{16})
\]

Therefore, we can obtain the following estimate

\[
|II_2| \leq o_L(1) C_{k, k, k, \delta_0} (1 + B^k(0)^{16} + X^k(x)^{16}),
\]

which is inequality (3.34). \(\square\)

Proof of estimate (3.35). Apply the divergence-free condition, we have

\[
II_3 = k \sum_{j=1}^{k} C_j \int_0^\infty \int_0^\infty \partial_x^j \partial_y q^x \cdot \partial_x \partial_\varphi q^\varphi \langle y \rangle^2 \, dx \, dy.
\]

Due to \(k \geq 2\), using the Sobolev inequality, estimates (A.4), (A.3), (3.13) and (3.14), we have

\[
|II_3| \leq \sum_{j=1}^{k} C_j \int_0^\infty \int_0^\infty \partial_x^j \partial_y q^x \cdot \partial_x \partial_\varphi q^\varphi \langle y \rangle^2 \, dx \, dy
\]

\[
\leq C_k \|\partial_x^{(k-1)} \partial_y q^x \|_{L^\infty L^2} \|\partial_x^{(k-1)} \partial_y q^\varphi \|_{L^\infty L^2} \|\partial_x \partial_\varphi q^\varphi \|_{L^\infty L^2}
\]

\[
+ C_k \|\partial_x^{(k-1)} \partial_y q^x \|_{L^\infty L^2} \|\partial_x^{(k-1)} \partial_y q^\varphi \|_{L^\infty L^2} \|\partial_x \partial_\varphi q^\varphi \|_{L^\infty L^2}
\]

\[
\leq o_L(1) C_{k, k} \|\partial_x^{(k-1)} \partial_y q^x \|_{L^\infty L^2} \|\partial_x \partial_\varphi q^\varphi \|_{L^\infty L^2}
\]

\[
\leq o_L(1) C_{k, k, \delta_0} \|\partial_x^{(k-1)} \partial_y q^x \|_{L^\infty L^2} \|\partial_x \partial_\varphi q^\varphi \|_{L^\infty L^2}
\]

This yields inequality (3.35) directly. \(\square\)
Proof of estimate (3.36). Using the divergence-free condition, we can write

\[
II_4 = - \int_0^L \int_0^\infty \partial_y u^r \partial_y q^r \cdot \partial_y q^r \ (y)^2 \ d\tau dy \\
+ \sum_{j=1}^k C_j \int_0^\infty \int_0^\infty \partial_x^{-1} \partial_y v^r \partial_x^{-j} q^r \cdot \partial_x q^r \ (y)^2 \ d\tau dy
\]

\[:=II_{41} + II_{42}.
\]

Using Hölder inequality, divergence-free condition, estimates (A.4) and (A.6), it holds true

\[
|II_{41}| \leq |\partial_y u^r (y)^j|_{L^\infty L^2} \| \partial_x q^r \|_{L^2 L^\infty} \| \partial_x \partial_y q^r (y)^j \|_{L^2 L_{(\infty)}}
\leq o_L(1) C_{k,j} |(\partial_y u_0^L (y)^j|_{L^2} + \| \partial_y v^r (y)^j \|_{L^2 L_{(\infty)}}) \mathcal{C}^k_f(x)^2.
\]

(3.41)

Due to \(k \geq 2\), using the Hölder inequality, estimates (A.4), (A.6), (A.7), (3.13) and (3.14), we have

\[
|II_{42}| \leq C_k \| \partial_y^{(k-1)} q^r (y)^j \|_{L^2 L_{(\infty)}} \| \partial_x q^r \|_{L^2 L^\infty} \| \partial_x \partial_y q^r (y)^j \|_{L^2 L_{(\infty)}}
+ C_k \| \partial_x^{(k-1)} q^r (y)^j \|_{L^2 L_{(\infty)}} \| \partial_x q^r \|_{L^2 L^\infty} \| \partial_x \partial_y q^r (y)^j \|_{L^2 L_{(\infty)}}
\leq o_L(1) C_{k,l} |(\partial_y u_{(0)} (y)^j|_{L^2} + \| \partial_y v^r (y)^j \|_{L^2 L_{(\infty)}}) \mathcal{C}^k_f(x)^2
\]

(3.42)

Then, the combination of estimates (3.41) and (3.42) yields the inequality (3.36).

Proof of estimate (3.37). Using the divergence-free condition, it holds true

\[
II_5 = - \int_0^L \int_0^\infty \partial_y u^r \partial_y q^r \cdot \partial_x \partial_y q^r (y)^2 \ dy dx \\
+ \sum_{j=1}^k C_j \int_0^\infty \int_0^\infty \partial_x^{-1} \partial_y v^r \partial_x^{-j} q^r \cdot \partial_x \partial_y q^r (y)^2 \ dy dx
\]

\[:=II_{51} + II_{52}.
\]

First of all, let us deal with the term \(II_{51}\). We can write

\[
II_{51} = - \int_0^L \int_0^\infty \partial_y u^r \partial_y q^r \cdot \partial_x \partial_y q^r (\chi (y)^2) \ dy dx - \int_0^L \int_0^\infty \partial_y u^r \partial_x q^r \cdot \partial_x \partial_y q^r (1 - \chi (y)^2) \ dy dx
\]

\[:=II_{511} + II_{512}.
\]

Using the Hölder inequality, it holds true

\[
|II_{511}| \leq C \| \frac{y}{\sqrt{u}} \partial_y u^r \chi \|_{L^\infty L^\infty} \| \partial_y q^r \|_{L^2 L^\infty} \| \sqrt{u} \partial_x \partial_y q^r \|_{L^2 L^\infty}.
\]

(3.43)

Using the Hardy inequality and estimate (A.4), it is easy to check that

\[
\| \frac{y}{\sqrt{u}} \partial_y q^r \|_{L^2 L_{(\infty)}} \leq \| \partial_x q^r \|_{L^2 L_{(\infty)}} \leq o_L(1) C_k \| |u^r \partial_y q^r (y)^j \|_{L^2 L_{(\infty)}} + \| \sqrt{u} \partial_x \partial_y q^r \|_{L^2 L_{(\infty)}}.
\]

(3.44)

On the other hand, using divergence-free condition, relations (5.11) and (5.12), it holds true

\[
\| \frac{y}{\sqrt{u}} \partial_y u^r \chi \|_{L^\infty L^\infty} \leq C_k \| \sqrt{y} \partial_y q^r \chi \|_{L^\infty L^\infty}
\leq C_k \| \partial_y u_0^L \chi \|_{L^2} + \| \partial_y q^r \chi \|_{L^2 L_{(\infty)}} + o_L(1) C_k \| \partial_y v^r \chi \|_{L^2 L_{(\infty)}} + \| \partial_y \partial_y q^r \|_{L^2 L_{(\infty)}}.
\]

(3.45)
Then, we can get that
\[
|II_{511}| \leq o_L(1)C_k(1 + B^k_t(0)^4 + X^k_t(x)^4).
\]

Using Hölder inequality and estimate (A.4), it holds true
\[
|II_{512}| \leq C_{k^*} \| \partial_x^2 u^i(y)^i \|_{L_x^2 L_y^2} \| \partial_y^3 q^i(y)^i \|_{L_x^2 L_y^\infty} \| \sqrt{u^i} \partial_x^2 \partial_y^3 q^i(y)^i \|_{L_x^2 L_y^2}
\]
\[
\leq o_L(1)C_{k^*} \| \partial_x^2 u^i(y)^i \|_{L_x^2 L_y^\infty} + o_L(1) \| \partial_y^3 q^i(y)^i \|_{L_x^2 L_y^\infty} \| \sqrt{u^i} \partial_x^2 \partial_y^3 q^i(y)^i \|_{L_x^2 L_y^2}
\]
\[
\leq o_L(1)C_{k^*} (1 + B^k_t(0)^4 + X^k_t(x)^4),
\]
which, together with estimate (3.46), yields directly
\[
|II_{511}| \leq o_L(1)C_{k^*} (1 + B^k_t(0)^4 + X^k_t(x)^4).
\]

Integrating by part and using boundary condition (3.30), we have
\[
II_{52} = - \sum_{j=1}^{k} C_j \int_0^L \int_0^\infty \partial_x^{j-1} \partial_y^3 v^e \partial_x^{j-1} q^e \cdot \partial_x \partial_y q^e(y)^i \ |y|^2 dy dx
\]
\[
- \sum_{j=1}^{k} C_j \int_0^L \int_0^\infty \partial_x^{j-1} \partial_y^3 v^e \partial_x^{j-1} \partial_y q^e \cdot \partial_x \partial_y q^e(y)^i \ |y|^2 dy dx
\]
\[
- 2l \sum_{j=1}^{k} C_j \int_0^L \int_0^\infty \partial_x^{j-1} \partial_y^3 v^e \partial_x^{j-1} q^e \cdot \partial_x \partial_y q^e(y)^i \ |y|^2 dy dx
\]
\[
:= II_{521} + II_{522} + II_{523}.
\]

Using the Hölder inequality, estimates (A.4), (A.7) and (3.17), we have
\[
|II_{521}| \leq C_k \| \partial_x^{(k-1)} \partial_y^3 v^e(y)^i \|_{L_x^2 L_y^2} \| \partial_x^{(k-1)} q^e(y)^i \|_{L_x^2 L_y^\infty} \| \sqrt{u^i} \partial_x^k \partial_y q^e(y)^i \|_{L_x^2 L_y^2}
\]
\[
\leq o_L(1)C_{k^*} \| \partial_x^{(k-1)} \partial_y^3 v^e(y)^i \|_{L_x^2 L_y^\infty} \| \partial_x^{(k-1)} q^e(y)^i \|_{L_x^2 L_y^\infty} + o_L(1) \| \sqrt{u^i} \partial_x^k \partial_y q^e(y)^i \|_{L_x^2 L_y^\infty} \| \partial_x \partial_y q^e(y)^i \|_{L_x^2 L_y^\infty}
\]
\[
\leq o_L(1)C_{k^*} (1 + B^k_t(0)^8 + X^k_t(x)^8).
\]

Similarly, using (A.4), (A.6), (A.7) and (3.15), we also have
\[
|II_{522}| \leq C_k \| \partial_x^{(k-1)} \partial_y^3 v^e(y)^i \|_{L_x^2 L_y^2} \| \partial_x^{(k-1)} \partial_y q^e(y)^i \|_{L_x^2 L_y^\infty} \| \sqrt{u^i} \partial_x^k \partial_y q^e(y)^i \|_{L_x^2 L_y^2}
\]
\[
\leq C_k \| \partial_x^{(k-1)} \partial_y^3 v^e(y)^i \|_{L_x^2 L_y^\infty} \| \partial_x^{(k-1)} \partial_y q^e(y)^i \|_{L_x^2 L_y^\infty} \| \sqrt{u^i} \partial_x^k \partial_y q^e(y)^i \|_{L_x^2 L_y^\infty}
\]
\[
\leq o_L(1)C_{k^*} (1 + B^k_t(0)^8 + X^k_t(x)^8).
\]

Finally, we use the estimates (A.4) and (A.7) to obtain
\[
|II_{523}| \leq C_k \| \partial_x^{(\frac{j-1}{2})} \partial_y^3 v^e(y)^i \|_{L_x^2 L_y^2} \| \partial_x^{(k-1)} q^e(y)^i \|_{L_x^2 L_y^\infty} \| \sqrt{u^i} \partial_x^k \partial_y q^e(y)^i \|_{L_x^2 L_y^2}
\]
\[
+ C_k \| \partial_x^{(k-1)} \partial_y^3 v^e(y)^i \|_{L_x^2 L_y^\infty} \| \partial_x^{(\frac{k-1}{2})} q^e(y)^i \|_{L_x^2 L_y^\infty} \| \sqrt{u^i} \partial_x^k \partial_y q^e(y)^i \|_{L_x^2 L_y^2}
\]
\[
\leq o_L(1)C_{k^*} \| \partial_x^{(k-1)} \partial_y^3 v^e(y)^i \|_{L_x^2 L_y^\infty} \| \partial_x^{(k-1)} q^e(y)^i \|_{L_x^2 L_y^\infty} \| \sqrt{u^i} \partial_x^k \partial_y q^e(y)^i \|_{L_x^2 L_y^\infty} \| \partial_x \partial_y q^e(y)^i \|_{L_x^2 L_y^\infty}
\]
\[
\leq o_L(1)C_{k^*} (1 + B^k_t(0)^8 + X^k_t(x)^8).
\]

Based on the above estimates from terms II_{521} to II_{523}, we can obtain the estimate
\[
|II_{52}| \leq o_L(1)C_{k^*} (1 + B^k_t(0)^8 + X^k_t(x)^8),
\]
which, together with estimate (3.30), yields the following estimate directly
\[
|II_5| \leq o_L(1)C_{k^*} (1 + B^k_t(0)^8 + X^k_t(x)^8).
\]

Therefore, we complete the proof of estimate (3.30). \qed
Proof of estimate (3.38). Obviously, we have

\[ II_6 = -2 \int_0^L \int_0^\infty \partial_y u' \partial_y \partial_y q' \cdot \partial_y^2 \partial_y q' \langle y \rangle^2 \mathsf{d}x \mathsf{d}y, \]

\[ + 2 \sum_{j=1}^k C_j \int_0^L \int_0^\infty \partial_y u' \partial_y^j \partial_y q' \cdot \partial_y^2 \partial_y q' \langle y \rangle^2 \mathsf{d}x \mathsf{d}y \]

\[ := II_61 + II_62. \]

Let us write

\[ II_61 = -2 \int_0^L \int_0^\infty \partial_y u' \partial_y \partial_y q' \cdot \partial_y^k \partial_y q' \langle y \rangle^2 \chi \mathsf{d}x \mathsf{d}y \]

\[ - 2 \int_0^L \int_0^\infty \partial_y u' \partial_y \partial_y q' \cdot \partial_y^k \partial_y q' \langle y \rangle^2 (1 - \chi) \mathsf{d}x \mathsf{d}y \]

\[ := II_611 + II_612. \]

Integrating by part and using boundary condition (3.31), we have

\[ II_611 = \int_0^L \int_0^\infty |\partial_y^k \partial_y q'|^2 \partial_y \{ \partial_y u' \langle y \rangle^2 \chi \} \mathsf{d}x \mathsf{d}y \]

\[ = \int_0^L \int_0^\infty |\partial_y^k \partial_y q'|^2 \{ \partial_y^2 u' \langle y \rangle^2 \chi + 2l \partial_y u' \langle y \rangle^{2l-1} \chi + \frac{1}{\delta} \partial_y u' \langle y \rangle^2 \chi \} \mathsf{d}x \mathsf{d}y. \]

Then, choosing \( \delta = \delta_0 \) and using the estimate (A.4), we get

\[ |II_611| \leq C_{k, \delta_0}(\| \partial_y^2 u' \chi \|_{L^\infty L^1} + \| \partial_y^2 u' \|_{L^2 L^2} \| \partial_y^k \partial_y q' \langle y \rangle \|_{L^2 L^2})^2 \]

\[ \leq o_L(1) C_{k, \delta_0}(\| \partial_y^2 u' \chi \|_{L^\infty L^1} + \| \partial_y^2 u' \|_{L^2 L^2} + \| \partial_y^3 v' \|_{L^2 L^2} + \| \partial_y^4 v' \|_{L^2 L^2}) \mathcal{E}^k(x)^2. \]

Using the Hölder inequality, lower bound assumption (3.12) and divergence-free condition, we have

\[ |II_612| \leq o_L(1) C_{k, \delta_0}(\| \partial_y^2 u' \langle y \rangle \|_{L^2 L^2} \| \partial_y^2 \partial_y q' \langle y \rangle \|_{L^2 L^2}) \mathcal{E}^k(x)^2 \]

\[ \leq o_L(1) C_{k, \delta_0}(\| \partial_y^2 u' \langle y \rangle \|_{L^2 L^2} + o_L(1) \| \partial_y^3 v' \|_{L^2 L^2}) \mathcal{E}^k(x)^2. \]

Then, we have the estimate

\[ |II_61| \leq o_L(1) C_{k, \delta_0}(1 + B^2 \mathfrak{f}(0)^4 + X^k(x)^4). \]

Finally, let us deal with term \( II_62 \). Due to \( k \geq 2 \), integrating by part and using boundary condition (3.30), we have

\[ II_62 = -2 \sum_{j=1}^k C_j \int_0^L \int_0^\infty \partial_y^{j-1} \partial_y^3 v' \partial_y^{k-j} \partial_y q' \cdot \partial_y^k \partial_y q' \langle y \rangle^2 \mathsf{d}x \mathsf{d}y \]

\[ - 2 \sum_{j=1}^k C_j \int_0^L \int_0^\infty \partial_y^{j-1} \partial_y^3 v' \partial_y^{k-j} \partial_y q' \cdot \partial_y^k \partial_y q' \langle y \rangle^2 \mathsf{d}x \mathsf{d}y \]

\[ - 4l \sum_{j=1}^k C_j \int_0^L \int_0^\infty \partial_y^{j-1} \partial_y^3 v' \partial_y^{k-j} \partial_y q' \cdot \partial_y^k \partial_y q' \langle y \rangle^{2l-1} \mathsf{d}x \mathsf{d}y \]

\[ := II_621 + II_622 + II_623. \]

Using the Sobolev inequality, estimates (A.4), (A.8), (3.15) and (3.17), it is easy to deduce

\[ |II_621| \leq C_k \| \partial_x^{(k-1)} \partial_y^3 v' \langle y \rangle \|_{L^\infty L^2} \| \partial_x^{(k-1)} \partial_y^3 q' \langle y \rangle \|_{L^2 L^2} \| \partial_y^k \partial_y q' \langle y \rangle \|_{L^2 L^2} \]

\[ + C_k \| \partial_x^{(k-1)} \partial_y^3 v' \|_{L^2 L^2} \| \partial_x^{(k-1)} \partial_y^3 q' \langle y \rangle \|_{L^2 L^2} \| \partial_y^k \partial_y q' \langle y \rangle \|_{L^2 L^2} \]

\[ \leq o_L(1) C_{k, \delta_0}(\| \partial_x^{(k-1)} \partial_y^3 v' \|_{L^\infty L^1} \| \partial_x^{(k-1)} \partial_y^3 q' \langle y \rangle \|_{L^2 L^2}) \]

\[ + o_L(1) C_{k, \delta_0}(\| \partial_x^{(k-1)} \partial_y^3 v' \|_{L^2 L^2} + \| \partial_x^{(k-1)} \partial_y^3 v' \|_{L^2 L^2}) \| \partial_x^{(k-1)} \partial_y^3 q' \langle y \rangle \|_{L^2 L^2} \mathcal{E}^k(x) \]

\[ \times (\| \partial_x^{(k-1)} \partial_y^3 q' \langle y \rangle \|_{L^2 L^2} + \| \partial_x^{(k-1)} \partial_y^3 q' \langle y \rangle \|_{L^2 L^2}) \]

\[ \leq o_L(1) C_{k, \delta_0}(1 + B^2 \mathfrak{f}(0)^8 + X^k(x)^8). \]
Similarly, for the case of $k \geq 2$, it is easy to check that

$$
|II_{623}| \leq C_k \| \partial_x \left( \frac{1}{2^{2(k-1)}} \partial_y^2 q' \right) \|_{L^\infty \rightarrow L^\infty} \| \partial_x^{(k-1)} \partial_y q' \langle y \rangle \|_{L^2 \rightarrow L^2} \| \partial_x^k \partial_y q' \langle y \rangle \|_{L^2} L^2 \nonumber
$$

$$
+ C_k \| \partial_x^{(k-1)} \partial_y^2 v' \|_{L^2 \rightarrow L^2} \| \partial_y^2 q' \langle y \rangle \|_{L^2 \rightarrow L^2} \| \partial_x^k \partial_y q' \langle y \rangle \|_{L^2} L^2
$$

$$
\leq o_L(1) C_{k, k, L} \left( \| \partial_x \left( \frac{1}{2^{2(k-1)}} \partial_y^2 q' \langle y \rangle \right) \|_{L^2 \rightarrow L^2} \right)^2 L^2 \nonumber
$$

$$
+ o_L(1) C_{k, k, L} \left( \| \partial_x \left( \frac{1}{2^{2(k-1)}} \partial_y^2 q' \langle y \rangle \right) \|_{L^2 \rightarrow L^2} \right)^2 L^2 \nonumber
$$

$$
\leq o_L(1) C_{k, k, L, L} \left( 1 + B_1^k \right)^4 + X_1^k(x^4) \nonumber
$$

and

$$
|II_{622}| \leq C_k \| \partial_x \left( \frac{1}{2^{2(k-1)}} \partial_y^2 q' \right) \|_{L^\infty \rightarrow L^\infty} \| \partial_x^{(k-1)} \partial_y q' \langle y \rangle \|_{L^2 \rightarrow L^2} \| \partial_x^k \partial_y q' \langle y \rangle \|_{L^2} L^2 \nonumber
$$

$$
+ C_k \| \partial_x^{(k-1)} \partial_y^2 v' \|_{L^2 \rightarrow L^2} \| \partial_y^2 q' \langle y \rangle \|_{L^2 \rightarrow L^2} \| \partial_x^k \partial_y q' \langle y \rangle \|_{L^2} L^2
$$

$$
\leq o_L(1) C_{k, k, L} \left( \| \partial_x \left( \frac{1}{2^{2(k-1)}} \partial_y^2 q' \langle y \rangle \right) \|_{L^2 \rightarrow L^2} \right)^2 L^2 \nonumber
$$

$$
+ o_L(1) C_{k, k, L} \left( \| \partial_x \left( \frac{1}{2^{2(k-1)}} \partial_y^2 q' \langle y \rangle \right) \|_{L^2 \rightarrow L^2} \right)^2 L^2 \nonumber
$$

$$
\leq o_L(1) C_{k, k, L, L} \left( 1 + B_1^k \right)^4 + X_1^k(x^4) \nonumber
$$

where we have used the following inequality in the last inequality

$$
\| \partial_x^{(k-1)} \partial_y^2 q' \langle y \rangle \|_{L^2} \leq C_{k, k, L} \left( \| \partial_x \left( \frac{1}{2^{2(k-1)}} \partial_y^2 q' \langle y \rangle \right) \right) \left( 1 - \chi \right) \|_{L^2} + \| \partial_x \left( \frac{1}{2^{2(k-1)}} \partial_y^2 q' \langle y \rangle \right) \|_{L^2} \nonumber
$$

Thus, we can obtain the following estimate for all $k \geq 2$

$$
|II_{62} \leq o_L(1) C_{k, k, L, L} \left( 1 + B_1^k \right)^4 + X_1^k(x^4) \nonumber
$$

(3.49)

Thus, the combination of estimates (3.38) and (3.49) yields directly

$$
|II_6 \leq o_L(1) C_{k, k, L, L} \left( 1 + B_1^k \right)^4 + X_1^k(x^4) \nonumber
$$

which is the inequality (3.38).

Proof of estimate (3.39). Let us write

$$
II_7 = - \sum_{j=1}^k C_j^k \int_0^L \int_0^\infty \partial_x^j u' \partial_x^{k-j} \partial_y^2 q' \cdot \partial_y^2 \partial_y q' \langle y \rangle \chi \left( \frac{y}{\theta_0} \right) \langle y \rangle^2 dy dx
$$

$$
- \sum_{j=1}^k C_j^k \int_0^L \int_0^\infty \partial_x^j u' \partial_x^{k-j} \partial_y^2 q' \cdot \partial_y^2 \partial_y q' \langle y \rangle \chi \left( \frac{y}{\theta_0} \right) \langle y \rangle^2 dy dx
$$

$$
:= II_{71} + II_{72} \nonumber
$$

First of all, let deal with the term $II_{71}$. Using Hölder inequality, estimates (3.38) and (3.39), we can obtain

$$
|II_{71}| \leq \sum_{j=1}^k C_j^k \int_0^L \int_0^\infty \partial_x^j u' \partial_x^{k-j} \partial_y^2 q' \cdot \partial_y^2 \partial_y q' \langle y \rangle \chi \left( \frac{y}{\theta_0} \right) \langle y \rangle^2 dy dx \nonumber
$$

$$
\leq C_k \left( \frac{1}{2^{2(k-1)}} \partial_y^2 q' \langle y \rangle \right) \left( 1 - \chi \right) \|_{L^2} + \| \partial_x \left( \frac{1}{2^{2(k-1)}} \partial_y^2 q' \langle y \rangle \right) \|_{L^2} \nonumber
$$

$$
= o_L(1) C_{k, k, L, L} \left( 1 + B_1^k \right)^4 + X_1^k(x^4) \nonumber
$$

(3.50)
Using the Hölder inequality, we have

\[ |II_7| = \sum_{j=1}^{k} C_j^k \int_0^L \int_0^\infty \partial_j^k u^r \partial_x^{k-j} \partial_y^2 q^r \cdot \partial_x^k \partial_y^2 q^r (y)^{2l}(1 - \chi) dy dx \]
\[ \leq C_k \| \partial_x^{(k-1)} \partial_y u^r (1 - \chi) \|_{L^2_\infty L^2_\infty} \| \partial_y^2 q^r (1 - \chi) \|_{L^2_\infty L^2_\infty} \| \partial_x^k \partial_y^2 q^r (1 - \chi) \|_{L^2_\infty L^2_\infty}. \]

Away from the boundary \( y = 0 \), we have

\[ \| \partial_x^{(k-1)} \partial_y^2 q^r (1 - \chi) \|_{L^2_\infty L^2_\infty} \]
\[ \leq \| \partial_x^{(k-1)} \partial_y^2 q^r (1 - \chi) \|_{L^2_\infty L^2_\infty} + o_L(1) \| \partial_x^{(k)} \partial_y q^r (1 - \chi) \|_{L^2_\infty L^2_\infty} \]
\[ \leq \| \partial_x^{(k-1)} \partial_y^2 q^r (1 - \chi) \|_{L^2_\infty L^2_\infty} + o_L(1) C_k \| \sqrt{u^r} \partial_x^{(k)} \partial_y^2 q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty}. \]

Then, we have

\[ |II_7| \leq \| \partial_x^{(k-1)} \partial_y u^r \|_{L^2_\infty H^1_2} \| \sqrt{\partial_x^2 u^r} \partial_x^k \partial_y^2 q^r \|_{L^2_\infty L^2_\infty} \]
\[ \times (\| \partial_x^{(k-1)} \partial_y^2 q^r (1 - \chi) \|_{L^2_\infty L^2_\infty} + o_L(1) C_k \| \sqrt{u^r} \partial_x^{(k)} \partial_y^2 q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty}) \]
\[ \leq o_L(1) C_k, k^*, k, \delta(1 + B^k_0(0)^8 + X^k_1(x)^8), \]

which, together with the estimate \( 3.50 \), yields directly

\[ |II_7| \leq o_L(1) C_k, k^*, k, \delta(1 + B^k_0(0)^8 + X^k_1(x)^8), \]

which is the inequality \( 3.39 \).

**Proof of estimate \( 3.30 \).** Recall

\[ II_8 := -2L \int_0^L \int_0^\infty \partial_x^k (\partial_y^2 u^r \cdot \partial_x^k \partial_y q^r (y)^{2l-1}) dy dx - 4L \int_0^L \int_0^\infty \partial_x^k (\partial_y u^r \cdot \partial_x^k \partial_y q^r (y)^{2l-1}) dy dx \]
\[ := II_81 + II_82 + II_83. \]

Using the divergence-free condition, estimates \( A.21 \), \( A.23 \), \( A.6 \) and \( A.7 \), we have

\[ |II_81| \leq C_{k,l} \| \partial_x^k u^r \|_{L^2_\infty L^2_\infty} \| \partial_x^2 q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} \| \partial_x^k \partial_y q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} \]
\[ + C_{k,l} \| \partial_x^{(k-1)} q^r \|_{L^2_\infty L^2_\infty} \| \partial_x^{(k-1)} \partial_y^2 q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} \| \partial_x^k \partial_y q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} \]
\[ \leq o_L(1) C_{k,l,k} \| \partial_x^k u^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} + o_L(1) \| \sqrt{\partial_x^2 u^r} (y)^{1_l} \|_{L^2_\infty L^2_\infty} \]
\[ + C_{k,l} \| \partial_x^{(k-1)} \partial_y^2 q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} + o_L(1) \| \sqrt{\partial_x^2 u^r} (y)^{1_l} \|_{L^2_\infty L^2_\infty} \]
\[ \leq o_L(1) C_{k,l,k,l,\delta}(1 + B^k_0(0)^4 + X^k_1(x)^4). \]

Similarly, it is easy to get that

\[ |II_82| \leq C_{k,l} \| \partial_y u^r \|_{L^2_\infty L^2_\infty} \| \partial_x^k \partial_y q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} \| \partial_x^k \partial_y q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} \]
\[ + C_{k,l} \| \partial_x^k \partial_y^2 q^r \|_{L^2_\infty L^2_\infty} \| \partial_x^{(k-1)} \partial_y q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} \| \partial_x^k \partial_y q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} \]
\[ \leq o_L(1) C_{k,k,l} \| \partial_y u^r \|_{L^2_\infty L^2_\infty} + o_L(1) \| \sqrt{\partial_y^2 u^r} \|_{L^2_\infty H^1_2} \]
\[ + C_{k,l} \| \partial_x^{(k-1)} \partial_y^2 q^r \|_{L^2_\infty H^1_2} \| \partial_x^{(k-1)} \partial_y q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} + \| \partial_x^k \partial_y q^r (y)^{1_l} \|_{L^2_\infty L^2_\infty} \]
\[ \leq o_L(1) C_{k,k,l,\delta}(1 + B^k_0(0)^4 + X^k_1(x)^4). \]

Finally, let us deal with the term \( II_83 \). Indeed, Let us write

\[ II_83 = -2L \int_0^L \int_0^\infty u^r \partial_x^k \partial_y^2 q^r \cdot \partial_x^k \partial_y q^r (y)^{2l-1} dy dx \]
\[ - 2L \sum_{j=1}^{k} C_j^k \int_0^L \int_0^\infty \partial_j^k u^r \partial_x^{k-j} \partial_y^2 q^r \cdot \partial_x^k \partial_y q^r (y)^{2l-1} dy dx \]
\[ := II_831 + II_832. \]
Using the Hölder inequality, we get
\[ |I_{831}| \leq C_1 \| u^\epsilon \|_{L^2} \| y^\epsilon \|_{L^2} \| \nabla u^\epsilon \|_{L^2} \| \nabla y^\epsilon \|_{L^2} \].

Using the equivalent relation (3.11) and lower bound (3.12), it holds for all \( \delta \leq \delta_0 \)
\[ \| \nabla u^\epsilon \|_{L^2} \| \nabla y^\epsilon \|_{L^2} \leq \| \nabla u^\epsilon \|_{L^2} \| \nabla y^\epsilon \|_{L^2} + \| \nabla u^\epsilon \|_{L^2} \| \nabla y^\epsilon \|_{L^2} \| \nabla \chi \|_{L^2} \leq \frac{\delta}{\delta_0} \| \nabla u^\epsilon \|_{L^2} \| \nabla y^\epsilon \|_{L^2} + \| \nabla u^\epsilon \|_{L^2} \| \nabla y^\epsilon \|_{L^2} \| \nabla \chi \|_{L^2} \],

where we have chosen \( \delta = \delta_0 \) in the last inequality. Thus, it holds
\[ |I_{831}| \leq o_L(1) C_{k+1} X_k(x)^2. \]

Using Sobolev inequality, estimates (3.11), (3.13), (3.14) and (3.15), we get
\[ |I_{832}| \leq C_{k+1} \| \nabla u^\epsilon \|_{L^2} \| \nabla y^\epsilon \|_{L^2} \| \nabla u^\epsilon \|_{L^2} \| \nabla y^\epsilon \|_{L^2} \| \nabla \chi \|_{L^2} \leq o_L(1) C_{k+1} \| \nabla u^\epsilon \|_{L^2} \| \nabla y^\epsilon \|_{L^2} \| \nabla \chi \|_{L^2} \leq o_L(1) C_{k+1} X_k(x)^2. \]

Thus, we can obtain the following estimate
\[ |I_{83}| \leq o_L(1) C_{k+1} X_k(x)^2. \]

This, together with the estimates (3.11) and (3.12), yields directly
\[ |I_{88}| \leq o_L(1) C_k X_k(x)^2, \]

which is the inequality (3.40).

Now let us give the proof for the Proposition 3.1

**Proof of Proposition 3.1**
Based on the estimates obtained so far, we can complete the proof of Proposition 3.1 in this subsection. First of all, we give the proof for the estimate. For three parameters \( R, k_\star \), and \( k' \), which will be defined later, we have
\[ L'_k := \sup \{ L \in [0, 1] \mid X_k(x)^2 + B_k(x)^2 \leq R, \ u^\epsilon(x, y) \geq k_\star(\delta_0 + \epsilon), \ (x, y) \in [0, L'] \times [\delta_0, +\infty) \}; \]
\[ k_\star(\delta_0 + \epsilon) \leq u^\epsilon(x, y) \leq k'(\delta_0 + \epsilon), \ (x, y) \in [0, L'] \times [0, \delta_0] \}, \]

where the constant \( \delta_0 \) is defined by the property of initial data \( u_0(y) \). Recall the estimates (3.14), (3.15), (3.16) and (3.18) in Lemma 3.2, we have
\[ \sum_{j=1}^{3} \| \nabla u^\epsilon \|_{L^2} \leq o_L(1) C_{k+1} (1 + B_k(0)^{16} + X_k(16)), \]

and estimate (3.26) in Lemma 3.3
\[ X_k(16) \leq X_k(0)^2 + o_L(1) C_{k_\star,k',l,l_0} (1 + B_k(0)^{16} + X_k(16)), \]
then we can obtain the following estimate
\[ X_t^k(x)^2 \leq \varepsilon_t^k(0)^2 + o_L(1)C_{k_*k^*,k,l,d_0}(1 + B_t^k(0)^{16} + X_t^k(x)^{16}). \] (3.53)
On the other hand, the quantity \( B_t^k(x) \) can be controlled as follows
\[ B_t^k(x) \leq B_t^k(0) + o_L(1)C_{k_*k}X_t^k(x). \] (3.54)
Then, the combination of estimates (3.53) and (3.54) yields directly
\[ X_t^k(x)^2 + B_t^k(x)^2 \leq \varepsilon_t^k(0)^2 + C_{k_*k^*,k,l,d_0}(1 + B_t^k(0)^{16}) + o_L(1)C_{k_*k^*,k,l,d_0}X_t^k(x)^{16}. \] (3.55)
It is easy to check that there exists a constant only depends on the initial data \( u_0 \) such that
\[ \varepsilon_t^k(0) + B_t^k(0) \leq C(u_0), \] (3.56)
which can be found in (6.11) in Appendix C. Then, the combination of (3.55) and (3.56) yields directly
\[ X_t^k(x)^2 + B_t^k(x)^2 \leq C_{k_*k^*,k,l,d_0}(1 + C(u_0)) + o_L(1)C_{k_*k^*,k,l,d_0}X_t^k(x)^{16}. \]
Thus, we conclude for all \( L \leq L^* \) that
\[ X_t^k(L)^2 + B_t^k(L)^2 \leq C_{k_*k^*,k,l,d_0}(1 + C(u_0)) + o_L(1)C_{k_*k^*,k,l,d_0}R^{16}. \]
Let us choose constants \( k_* = \frac{1}{8}, \) \( k^* = 4 \) and \( R = 4C_{k_*k^*,k,l,d_0}(1 + C(u_0)) \), then we have
\[ X_t^k(L_1)^2 + B_t^k(L_1)^2 \leq C_{k,l,d_0}(1 + C(u_0)) + o_L(1)C_{k,l,d_0}(4(1 + C(u_0)))^{16}. \] (3.57)
Here we denote \( C_{k,l,d_0} := C_{k,*k^*,k,l,d_0}|_{(k,k^*)=(\frac{1}{8},4)} \). Choose the time \( L_1 \) small enough such that
\[ o_L(1)C_{k,l,d_0}(4(1 + C(u_0)))^{16} \leq C_{k,l,d_0}(1 + C(u_0)), \]
and hence, we deduce from (3.57) that
\[ X_t^k(L_1)^2 + B_t^k(L_1)^2 \leq 2C_{k,l,d_0}(1 + C(u_0)) = \frac{R}{2}. \] (3.58)
It is easy to check that
\[ u^\varepsilon(x,y) = u^\varepsilon(0,y) + \int_0^x \partial_y u^\varepsilon(s,y)ds, \] (3.59)
and
\[ \partial_x u^\varepsilon(x,y) = \partial_x u^\varepsilon(x,0) + \int_0^y \partial_y \partial_x u^\varepsilon(x,\tau)d\tau. \]
Then, using the boundary condition (2.22), we have for all \( y \in [0, \delta_0] \)
\[ |\partial_x u^\varepsilon(x,y)| \leq y\|\partial_y \partial_x u^\varepsilon(x,y)\|_{L^\infty([0,y])} \leq y\|\partial_y^2 \partial_x u^\varepsilon(x,y)\|_{L^2([0,\infty))}, \]
and hence, it is easy to check that
\[ |\int_0^x \partial_x u^\varepsilon(x,y)dy| \leq \sqrt{x}y\|\partial_y^2 u^\varepsilon(y)\|_{L^2_L^2}. \] (3.60)
Then, the combination of representation (3.59) and estimate (3.60) yields directly
\[ u^\varepsilon(x,y) \geq u^\varepsilon(0,y) - \int_0^L \partial_x u^\varepsilon(s,y)ds \geq \frac{1}{2}(y + \epsilon) - \sqrt{L}y\|\partial_y^2 u^\varepsilon(y)\|_{L^2_L^2}. \]
Choose \( L_2 := \min\{\frac{1}{32C_{k,l,d_0}(1 + C(u_0))}, L_1\} \), then we have
\[ u^\varepsilon(x,y) \geq \frac{1}{2}(y + \epsilon) - \frac{1}{4}y \geq \frac{1}{4}(y + \epsilon) = 2k_*(y + \epsilon), \] for all \( (x,y) \in [0, L_2] \times [0, \delta_0] \).
Similarly, it is easy to check that
\[
    u'(x, y) \leq u'(0, y) + \int_0^L \partial_x u'(s, y) ds \leq \frac{3}{2}(y + \epsilon) + \sqrt{L} y \| \partial^3_y u^\epsilon(y) \|_{L_2^2 L_2^2}.
\]
Choose \( L_3 = \min\{L_2, \frac{1}{6C_{k,l,\delta_0}(1+|C(u_0)|)}\} \), then we have
\[
    u'(x, y) \leq \frac{3}{2}(y + \epsilon) + \frac{1}{2} y = 2(y + \epsilon) = \frac{1}{2} k'(y + \epsilon), \quad \text{for all } (x, y) \in [0, L_3] \times [0, \delta_0].
\]
Finally, using the Sobolev inequality, Hölder inequality and divergence-free condition, we have
\[
    \left| \int_0^L \partial_x u'(x, y) dx \right| \leq \int_0^L \| \partial_y \partial_x u^\epsilon(y) \|_{L_2^2} dx \leq \sqrt{L} \| \partial^2_y u^\epsilon(y) \|_{L_2^2 L_2^2},
\]
which, together with (3.59), yields directly
\[
    u'(x, y) \geq u'(0, y) - \left| \int_0^L \partial_x u'(s, y) ds \right| \geq (\delta_0 + \epsilon) - \sqrt{L} \| \partial^2_y u^\epsilon(y) \|_{L_2^2 L_2^2}.
\]
Choose \( L_4 := \min\{L_3, \frac{\delta_0^2}{32C_{k,l,\delta_0}(1+|C(u_0)|)}\} \), then we have
\[
    u'(x, y) \geq \frac{1}{2} (\delta_0 + \epsilon) - \frac{1}{2} \delta_0 \geq \frac{1}{4} (\delta_0 + \epsilon) = 2k_\ast(\delta_0 + \epsilon), \quad \text{for all } (x, y) \in [0, L_4] \times [\delta_0, +\infty).
\]
Obviously, we conclude that there exists \( L_4 > 0 \) depending on \( k, l, \delta_0 \) and the initial data(hence independent of parameter \( \epsilon \)) such that for all \( L \geq \min\{L_4, L'\} \), the estimates (3.8) and (3.9) in Proposition 3.1 hold true. Of course, it holds that \( L_4 \leq L' \). Indeed otherwise, our criterion about the continuation of the solution would contradict the definition of \( L_4^* \). Then, taking \( L = L_4 \), we obtain the estimates (3.8) and (3.9). Furthermore, due to the equation (3.62), we can apply the estimates (3.58) and (3.19) (see Proposition A.3 in Appendix A) to get
\[
    \sum_{0 \leq 2\alpha + 2\beta \leq 2k+1} \| \partial_y \partial_x^\alpha \partial_y \partial_x^\beta u^\epsilon(y) \|_{L_2^2 L_2^2} \leq C_{k,l,\delta_0}(1+|C(u_0)|),
\]
where \( C(u_0) \) is a constant depends on the initial data \( u_0 \). Therefore, the proof of Proposition 3.1 is completed.  

3.2. Local existence of Prandtl-Hartmann system

In this subsection, we will give the proof of local existence of original system (1.13). For any \( \epsilon > 0 \) and some large \( N > 0 \), we introduce an artificial truncation at \( y = N \) and then send \( N \to \infty \). For any \( \epsilon \in (0, 1] \) and large \( N \), let us consider the problem on \( (x, y) \in (0, L) \times (0, N) \),
\[
    \begin{cases}
        u^\epsilon,N \partial_x u^\epsilon,N + u^\epsilon,N \partial_y u^\epsilon,N - \partial_y u^\epsilon,N + u^\epsilon,N - u(N) = 0, \\
        \partial_x u^\epsilon,N + \partial_y v^\epsilon,N = 0,
    \end{cases}
\]
with the boundary conditions
\[
    u^\epsilon,N(x, y)_{|y=0} = \epsilon, \quad u^\epsilon,N(x, y)_{|y=N} = u(N) + \epsilon, \quad v^\epsilon,N(x, y)_{|y=0} = v^\epsilon,N(x, y)_{|y=N} = 0, \quad x \in (0, L),
\]
and the initial data
\[
    u^\epsilon,N(x, y)_{|x=0} := u_0(y) + \epsilon, \quad y \in [0, N].
\]
Applying the classical von-Mises transformation (i.e., \( \eta := \int_0^y u^\epsilon,N(x, y')dy' \)), the system (3.61) can be translated into a quasi-linear parabolic equation that will not cause the loss of \( x \)-derivative. Then, it is easy to obtain the local-in-time well-posedness with higher order Sobolev regularity for this translated system. Due to the non-degeneracy boundary of horizontal velocity in (3.62), the variable \( \eta \) after von-Mises transformation is equivalent to the vertical variable \( y \) in the original Eulerian coordinates. Then, the system (3.61) - (3.63) be obtained local well-posedness with higher Sobolev regularity on the life span \([0, L']\). For more details on the results of this local
existence, the reader can refer to the Lemma 2.2 in [20]. Then, applying the a priori estimates given in Proposition 3.1 to the solution \((u^\epsilon, v^\epsilon, N)\), we obtain a uniform life span time \(L_\epsilon > 0\) and a constant \(C_1\) (independent of \(\epsilon\) and \(N\)) such that it holds true

\[
\sum_{0 \leq 2\alpha + \beta \leq 2k+1} \|\partial^\alpha_x \partial^\beta_y u^\epsilon\|_{L^2_t L^2_x[0,N]} \leq C_1, \tag{3.64}
\]

and

\[
\frac{1}{4} y \leq u^\epsilon(x, y) \leq 2(y + \epsilon), \quad (x, y) \in [0, L_\epsilon] \times [0, \delta_0],
\]

\[
u^\epsilon(x, y) \geq \frac{1}{4} \delta_0, \quad (x, y) \in [0, L_\epsilon] \times [\delta_0, N]. \tag{3.65}
\]

Based on the uniform estimates for \((u^\epsilon, v^\epsilon, N)\), one can pass the limit \(\epsilon \to 0^+\) and \(N \to +\infty\) to get a solution \((u, v)\) satisfying (1.8) by using a strong compactness arguments. Indeed, it follows from estimate that \(\partial_y v^\epsilon\) is bounded uniformly in \(L^\infty([0, L_\epsilon]; H^{2k+1}_x([0, N]))\), where \(v^\epsilon\) is bounded uniformly in \(L^2([0, L_\epsilon]; H^{2k-2}_x([0, N]))\). Then, it follows from a strong compactness argument, after taking a subsequence \(\epsilon_k \to 0^+\),

\[
\partial_y v^{\epsilon_k} \to u^N \quad \text{in} \quad L^2([0, L_\epsilon]; H^{2k+1}_x([0, N])),
\]

and

\[
\partial_y v^{\epsilon_k} \to u^N \quad \text{in} \quad C([0, L_\epsilon]; H^{2k-2}_x([0, N])),
\]

where \(u^N \in L^2([0, L_\epsilon]; H^{2k+1}_x([0, N])) \cap L^\infty([0, L_\epsilon]; H^{2k-2}_x([0, N]))\). Let us define \(v^N(x, y) := \int_0^x w^N(x', y)dx'\) and \(u^N(x, y) := u_0(y) - \int_0^x w^N(x', y)dx'\), it holds true

\[
\partial_x u^N(x, y) + \partial_y v^N(x, y) = 0.
\]

Furthermore, it is easy to deduce that

\[
u^{\epsilon_k}(x, y) - u^N(x, y) = -\int_0^x \partial_y v^{\epsilon_k}(x', y)dx' + u_0(y) + \epsilon_k - u_0(y) + \int_0^x w(x', y)dx'
\]

\[
= -\int_0^x (\partial_y v^{\epsilon_k}(x', y) - w^N(x', y))dx' + \epsilon_k,
\]

and hence, we have

\[
\|u^{\epsilon_k} - u^N\|_{L^\infty_t H^{2k-2}_x([0, N])} \leq \int_0^x (\partial_y v^{\epsilon_k}(x', y) - w^N(x', y))dx'\|_{L^\infty_t H^{2k-2}_x([0, N])} + \epsilon_k \sqrt{N}
\]

\[
\leq L_\epsilon \|\partial_y v^{\epsilon_k} - w^N\|_{L^\infty_t H^{2k-2}_x([0, N])} + \epsilon_k \sqrt{N}.
\]

Therefore, it holds true

\[
u^{\epsilon_k} \to u^N \quad \text{in} \quad L^\infty([0, L_\epsilon]; H^{2k-2}_x([0, N])).
\]

Therefore, the solution \((u^N(x, y), v^N(x, y))\) satisfies

\[
\left\{
\begin{array}{ll}
\partial_x u^N + v^N \partial_y u^N - \partial_y u^N + u^N - u_0(N) = 0, & (x, y) \in [0, L_\epsilon] \times (0, N); \\
\partial_x u^N + \partial_y v^N = 0, & (x, y) \in [0, L_\epsilon] \times (0, N),
\end{array}
\right.
\]

with the boundary condition

\[
u^N(x, y)|_{y=0} = 0, \quad u^N(x, y)|_{y=N} = u_0(N), \quad v^N(x, y)|_{y=0} = 0, \quad x \in (0, L_\epsilon),
\]

and the initial data

\[
u^N(x, y)|_{x=0} = u_0(y), \quad y \in [0, N].
\]

Furthermore, the solution \((u^N, v^N)\) also satisfies the estimates

\[
\sum_{0 \leq 2\alpha + \beta \leq 2k+1} \|\partial^\alpha_x \partial^\beta_y u^N\|_{L^2_t L^2_x[0,N]} \leq C_{k, l, \delta_0} (1 + C(u_0)), \tag{3.66}
\]

and

\[
\frac{1}{4} y \leq u^N(x, y) \leq 2y, \quad (x, y) \in [0, L_\epsilon] \times [0, \delta_0]; \quad u^N(x, y) \geq \frac{1}{4} \delta_0, \quad (x, y) \in [0, L_\epsilon] \times [\delta_0, N]. \tag{3.67}
\]

Since the estimate (3.66) is bounded independent of \(N\), and similarly taking \(N \to +\infty\), it is easy to obtain the solution \((u, v)\) to the original problem Eqs. (1.6)-(1.7). At the same time, the estimates (3.66) and (3.67) hold on over the domain \([0, +\infty)\). Therefore, we complete the proof of local existence and estimate 1.13 in Theorem 1.4.
3.3. Uniqueness of Prandtl-Hartmann system

In this subsection, we will show the uniqueness of solution to the original system (1.6). Let \((u_1, v_1)\) and \((u_2, v_2)\) be two solutions in the existence time \([0, L_a]\) constructed in the previous subsection, with respect to the same initial data. Let us set
\[
(\tilde{u}, \tilde{v}, \tilde{q}) := (u_2 - u_1, v_2 - v_1, \frac{v_2 - v_1}{u_2})
\]
then they satisfy the following evolution
\[
(u_2)^2 \partial_y \tilde{q} + \partial_y^2 \tilde{u} - \tilde{u} = \tilde{u} \partial_x u_1 + v_1 \partial_y \tilde{u}.
\]
Taking \(\partial_z\) operator to the above equation and using the divergence-free condition, we have
\[
\partial_x \{(u_2)^2 \partial_y \tilde{q} \} - \partial_y^2 \tilde{v} + \partial_y \tilde{v} = \partial_x (\tilde{u} \partial_x u_1 + v_1 \partial_y \tilde{u}),
\]
with the zero boundary condition and initial data
\[
(\tilde{u}, \tilde{v})|_{y=0} = 0, \quad \lim_{y \to +\infty} \tilde{u} = 0, \quad \tilde{u}|_{x=0} = 0.
\]

Next, we will establish the following estimate.

**Proposition 3.4.** Let \((u_1, v_1)\) and \((u_2, v_2)\) be two solutions of the equation (1.6) with the same initial data (1.2), and satisfying the estimate (3.68), respectively. Then, the quantity \((\tilde{u}, \tilde{v}, \tilde{q})\) given by satisfies
\[
\|(u_2 \partial_y \tilde{q})(x)\|_L^2 + \int_0^x \|\sqrt{u_2 \partial_y \tilde{q}}(\tau)\|_L^2 d\tau + \int_0^x \|\sqrt{u_2 |v_2|} \partial_y \tilde{q}\|_L^2 d\tau \leq C \int_0^x \|(u_2 \partial_y \tilde{q})(\tau)\|_L^2 d\tau,
\]
for any \(x \in (0, L_a]\).

**Proof of Uniqueness.** Applying the Gronwall’s lemma to the estimate (3.70), we obtain \((u_2 \partial_y \tilde{q})(x, y) \equiv 0\). Due to the fact \(u_2 > 0\) in the fluid domain, then we have
\[
v_2 - v_1 = u_2 w
\]
for some function \(w := w(x)\). Using the assumption \(u_2 > 0\) in the fluid domain and boundary condition \(\tilde{v}|_{y=0} = 0\), we know that \(w \equiv 0\) and hence \(v_2 \equiv v_1\). Using the divergence-free condition, it holds
\[
u_2(x, y) - u_1(x, y) = \int_0^x \partial_y \{u_2(\tau, y) - u_1(\tau, y)\} d\tau = - \int_0^x \partial_y \{v_2(\tau, y) - v_1(\tau, y)\} d\tau = 0.
\]
Then, we have \(u_2(x, y) \equiv u_1(x, y)\). This proves the uniqueness of solution in Theorem 1.1. \(\square\)

In the rest of this subsection, we will give the proof of Proposition 3.4 as follows.

**Proof of Proposition 3.4** Multiplying the equation (3.68) by \(\partial_y \tilde{q}\) integrating over \([0, x] \times [0, +\infty)\) and integrating by part, we have
\[
\frac{1}{2} \int_0^\infty \int_0^\infty (u_2^2 |\partial_y \tilde{q}|^2)(x, y) dy \int_0^\infty \int_0^\infty (u_2 |\partial_y \tilde{q}|^2) dy d\tau \int_0^x \int_0^\infty u_2 |\partial_y \tilde{q}|^2 dy d\tau + \int_0^x \int_0^\infty 2\partial_y u_2 |\partial_y \tilde{q}|^2|_{y=0} d\tau,
\]
\[
= - \int_0^\infty \int_0^\infty \partial_x u_2 |\partial_y \tilde{q}|^2 dy d\tau - \int_0^\infty \int_0^\infty (2\partial_y u_2 \partial_y \tilde{q} + \partial_y^2 u_2 \tilde{q}) \cdot \dot{\partial}_y \tilde{q} dy d\tau
\]
\[
- \int_0^\infty \int_0^\infty \partial_y u_2 \dot{\tilde{q}} \dot{\tilde{q}} dy d\tau - \int_0^\infty \int_0^\infty (\partial_x v_1 \partial_y \tilde{u} - v_1 \partial_y^2 \tilde{v} + \partial_y \tilde{u} \partial_y v_1 - \tilde{u} \partial_x \partial_y v_1) \partial_y \tilde{q} dy d\tau,
\]
\[
:= III_1 + III_2 + III_3 + III_4 + III_5.
\]
Similar to the estimates in Lemma 3.3, we can obtain

\[ ||III_1|| \leq (1 + \|v_2\|_{L^2_y}) \left( \|\nabla_x \phi \|_{L^2_y} + \|\nabla_x^2 \phi \|_{L^2_y} \right) \int_0^\infty \int_0^\infty |\nabla_y \tilde{q}|^2 dy dt,
\]

\[ ||III_2|| \leq \left( \int_x^\infty \nabla_y u_2 \nabla_y \tilde{q}^2 \big|_{y=0} dx + C(\|\nabla_x^2 \phi \|_{L^2_y} + \|\nabla_x \phi \|_{L^2_y}^2) \right) \int_0^\infty \int_0^\infty |\nabla_y \tilde{q}|^2 dy dt,
\]

\[ ||III_3|| \leq \left( \|\nabla_x^2 u_2 \|_{L^2_y} + \|\nabla_x \phi \|_{L^2_y}^2 \right) \int_0^\infty \int_0^\infty |\nabla_y \tilde{q}|^2 dy dt,
\]

\[ ||III_4|| \leq \left( \|\nabla_x^2 u_2 \|_{L^2_y} + \|\nabla_x \phi \|_{L^2_y} \right) \int_0^\infty \int_0^\infty |\nabla_y \tilde{q}|^2 dy dt.
\]

It is easy to check that

\[ ||III_5|| \leq \left( \|\nabla_x^2 u_2 \|_{L^2_y} + \|\nabla_x \phi \|_{L^2_y}^2 \right) \int_0^\infty \int_0^\infty |\nabla_y \tilde{q}|^2 dy dt.
\]

Recall the definition \( \tilde{q} := \frac{\partial u_2}{\partial u_2} = \frac{\hat{\phi}}{\partial_2} \), then it holds true

\[ \|\nabla_x^2 \tilde{q}\|_{L^2_y} \leq \left( \|\nabla_x^2 u_2 \|_{L^2_y} + \|\nabla_x \phi \|_{L^2_y}^2 \right) \int_0^\infty \int_0^\infty |\nabla_y \tilde{q}|^2 dy dt.
\]

Thus, collecting the estimates from III1 to III5, we can get

\[ |III_1 + III_2 + III_3 + III_4 + III_5| \leq \int_0^\infty \nabla_y u_2 \nabla_y \tilde{q}^2 \big|_{y=0} dy + \frac{1}{4} \int_0^\infty \int_0^\infty |\nabla_y \tilde{q}|^2 dy dt + C \|\nabla_y \tilde{q}\|_{L^2_y}^2. \tag{3.72}
\]

Similar to the estimate (3.72), it is easy to obtain

\[ \|\nabla_y \tilde{q}\|_{L^2_y} \leq 4 \|u_2 \|_{L^2_y} \|\nabla_y \tilde{q}\|_{L^2_y} + \frac{1}{4} \int_0^\infty \int_0^\infty |\nabla_y \tilde{q}|^2 dy dt,
\]

which, together with estimate (3.72), yields directly

\[ |III_1 + III_2 + III_3 + III_4 + III_5| \leq \int_0^\infty \nabla_y u_2 \nabla_y \tilde{q}^2 \big|_{y=0} dy + \frac{1}{2} \int_0^\infty \int_0^\infty |\nabla_y \tilde{q}|^2 dy dt + 2 \|u_2 \|_{L^2_y} \|\nabla_y \tilde{q}\|_{L^2_y} + C \|\nabla_y \tilde{q}\|_{L^2_y}^2. \tag{3.73}
\]

Substituting the estimate (3.73) into the inequality (3.71), then we obtain the claim estimate (3.70). Therefore, we complete the proof of Proposition 3.4.

\[ \square \]

4 Asymptotic behavior of Prandtl-Hartmann system

In this section, we aim to investigate the asymptotic behaviour of velocity profiles in the 2-D magnetic Prandtl boundary layer. More specifically, we would like to establish the decay estimate in \( x \) variable, thus \( C \) denotes a generic positive constant independent of \( x \) which may vary in following different estimates in this section.

Then an important question is whether the system (1.6) is globally well-posed in Sobolev space when the initial data are near the Hartmann layer. More precisely, we study the global stability of the Hartmann layer \((\tilde{u}, \tilde{v}) = (1 - e^{-y}, 0)\). Thus, we need to define an appropriate notion of the difference between two pairs \((u, v)\) and \((\tilde{u}, \tilde{v})\). The notion we used in our case does not coincide with pointwise perturbations \(u(x, y) - \tilde{u}(y)\) and \(v(x, y)\), but rather the modulated perturbation \(\phi(x, \psi) := u^2(x, \psi) - \tilde{u}^2(\psi)\). Then it follows from (1.9) and (1.10) that the difference \(\phi\) satisfies

\[
\begin{align*}
\phi_x - u\phi_{uv} + 2\phi_{\psi} &= 0, & (x, \psi) &\in \mathbb{R}_+ \times \mathbb{R}_+, \\
\phi|_{x=0} &= \phi_{uv}(\psi) := u_0^2(\psi) - \tilde{u}_0^2(\psi), & \psi &\in \mathbb{R}_+, \\
\phi|_{x=\infty} &= 0, & \phi|_{\psi=0} &= 0, & x &\in \mathbb{R}_+.
\end{align*}
\]
We rewrite (1.1), as
\[ \phi_x + L\phi = 0, \]
where the operator \( L\phi := -u\phi_{\psi\psi} + 2\frac{\phi}{u(u + \psi)} \). The global existence and the asymptotic behaviour of the solution \((u, v)(x, y)\) to (1.6) can be easily translated into that of the perturbed solution \(\phi(x, \psi)\). And it is universally known that the global existence of solutions for system (1.1) will be obtained by combining the local existence result with some a priori estimates and then employing the standard continuity argument. Thus, we first review the classical local existence result, which can be obtained by a similar way to Oleinik’s result in [42]. And we omit details of proof here.

**Proposition 4.1.** Assume that \(u_0(y) > 0\) for \(y > 0\); \(u_0(0) = 0\), \(u'_0(0) > 0\), \(u_0(y) \to 1\) as \(y \to \infty\); \(u_0(y), u'_0(y)\), \(u''_0(y)\) are bounded for \(0 \leq y < \infty\) and satisfy the Hölder condition. Moreover, assume that for small \(y\) the following compatibility condition is satisfied at the point \((0, \bar{\psi})\):
\[ -u''_0(y) + u_0(y) - 1 = O(y^2). \]
Then, for some \(X > 0\) there exists a solution \((u, v)\) of problem (1.6) and (1.7) in \(D := \{0 < x < X, 0 < y < \infty\}\), which has the following properties: \((u, v)\) is bounded and continuous in \(\bar{D}\), \(u > 0\) for \(y > 0\); \(u_y > m > 0\) for \(0 < y \leq y_0\), where \(m\) and \(y_0\) are constants; \(u_y\) and \(u_{yy}\) are bounded and continuous in \(D\); \(u_x\), \(v\) and \(v_y\) are bounded and continuous in any finite portion of \(D\).

It is easy to check that the local existence above is also valid for system (1.1) in new variables \((x, \psi)\). First, we set \(I := [0, x]\), and denote the norms
\[
\begin{align*}
\| u \|_{L^\infty} &:= \sup_{x \in I} \left\{ \| u \|_{L^\infty}, \| \bar{u} \|_{L^\infty} \right\}, \\
\mathcal{E}(x) &:= \sum_{k=0,1} \sup_{\psi \in I} \left\{ \| \partial_x^k \phi \|_{L^2_x}, \| \partial_x^k \phi \|_{L^2_x} \right\}, \\
\mathcal{D}(x) &:= \mathcal{E}(x) + \sum_{k=0,1} \left\{ \| \partial_x^k \phi \|_{L^2_x} \right\} + \sum_{k=0,1} \left\{ \| \partial_x^k \phi \|_{L^2_x} \right\}.
\end{align*}
\]

Based on the local existence result, we can obtain the global existence and asymptotic stability results as follow.

**Proposition 4.2.** (Global well-posedness and asymptotic stability in von-Mises variable). Under the conditions of Proposition 4.1, assume that initial data satisfies \(f(0) \leq \gamma_0\) with any positive constant \(\gamma_0 > 1\). Furthermore, there exists a small positive constant \(\sigma_0\) such that
\[ \| \phi_0 \|_{L^2_x} + \| \frac{\phi_0}{u_0^{3/2}} \|_{L^2_x} + \| \frac{L\phi_0}{u_0^{3/2}} \|_{L^2_x} \leq \sigma_0, \]
then the system (1.1) has a global-in-\(x\) solution \(\phi\) satisfying for any \(0 < x < \infty\),
\[ f(x) \leq 2\gamma_0, \quad \mathcal{D}(x) \leq C_1\mathcal{E}(0), \]
and the decay estimate
\[ \mathcal{E}(x) \leq C_2 e^{-x}. \]

Here \(C_1\) and \(C_2\) are positive constants independent of \(x\).

**Remark 4.1.** The initial condition \(f(0) \leq \gamma_0\) implies that \(u\) and \(\bar{u}\) tend to zero in the same order near the boundary \(\{\psi = 0\}\). And in general \(u \neq \bar{u}\), thus it is reasonable to assume that \(\gamma_0 > 1\).

**Remark 4.2.** Indeed, we require the initial data satisfy the compatibility conditions: \((\phi_x - L\phi)|_{x=0} = (\phi_{x\psi} - L(\psi_{\psi}))|_{x=0} = 0\) in Proposition 4.1. At the same time, we apply the fact \(u \lesssim 1\) to check that
\[ \mathcal{E}(0) \leq \| \phi_0 \|_{L^2_x} + \| \frac{\phi_0}{u_0^{3/2}} \|_{L^2_x} + \| \frac{L\phi_0}{u_0^{3/2}} \|_{L^2_x}. \]
This is reason why we require the small initial data (1.2) instead of \(\mathcal{E}(0) \leq \sigma_0\).
4.1. A priori estimates

The global existence of solution will be obtained by combing the local existence result with a priori estimates. Since the local existence has been built in Proposition 4.1, our main task is to establish the closed a priori estimate. Therefore, for any small constant $\sigma > 0$ and positive constant $\gamma > 1$, we assume a priori estimates

$$f(x) \leq 2\gamma, \quad E(x) \leq \sigma.$$  (4.4)

Thus, we will establish some estimates for the system (4.1) under the a priori estimates (4.4). Let us denote $\omega := \phi \frac{\partial u}{\partial x} + \bar{u} = u - \bar{u}.$

Next, we analyze the properties of the solution $\bar{u}$ for classical Hartmann boundary layer. We claim that there exists a positive constant $\eta_0$ such that

$$\bar{u} \sim \sqrt{\psi} \quad \text{for} \quad 0 < \psi \leq \eta_0 \quad \text{and} \quad \bar{u} \gtrsim 1 \quad \text{for} \quad \psi \geq \eta_0,$$  (4.5)

where we recall again that $A \lesssim B$ ($A \gtrsim B$) represents that there exists a positive constant $C$ such that $A \leq CB$ ($A \geq CB$), and $A \sim B$ stands for $A \lesssim B$ and $A \gtrsim B$ (see the end of Section 1). The parameter $\eta_0$ appearing in (4.5) will be chosen in the following proof, which is related to the Taylor expansion of $\bar{u}(y)$.

Indeed, we suppress the dependence of the solution $\bar{u}$ on $x$ in the following calculations for simplicity. Using Taylor expansion, $\bar{u}|_{y=0} = 0$ and $\bar{u}|_{y=0} = 1 > 0$, we find

$$\bar{u}(y) = y + o(y).$$

Then, there exists a positive constant $\eta_1$ such that

$$\frac{1}{2}y \leq \bar{u}(y) \leq \frac{3}{2}y, \quad \text{for} \quad 0 < y \leq \eta_1.$$  (4.6)

Since $\psi = \int_0^y \bar{u}(y')dy'$, then a direct calculation yields for $0 < y \leq \eta_1$,

$$\psi \leq \frac{3}{2} \int_0^y y'dy' = \frac{3}{4}y^2, \quad \psi \geq \frac{1}{2} \int_0^y y'dy' = \frac{1}{4}y^2.$$  (4.7)

The estimates (4.4), together with another equivalent relation (4.5), yields that there exists a positive constant $\eta_0 = \frac{1}{4}\eta_1^2$ such that

$$\bar{u}(\psi) \sim \sqrt{\psi}, \quad \text{for} \quad 0 < \psi \leq \eta_0.$$  (4.8)

The fact $\psi \geq \eta_0$ implies that there exists a positive constant $\eta_2$ such that $y \geq \eta_2$. And it is easy to check that

$$\bar{u}(y) = 1 - e^{-y} \geq 1 - e^{-\eta_2} \gtrsim 1, \quad \text{for} \quad y \geq \eta_2,$$

which immediately implies that

$$\bar{u}(\psi) \gtrsim 1, \quad \text{for} \quad \psi \geq \eta_0.$$  (4.9)

This completes the proof of the claim (4.5).

First, our aim is to show that $f(x) \leq \gamma$. And the proofs of following estimates in Lemmas 4.3-4.4 are interconnected because of their nonlinear nature, thus it will be convenient for us to give the following notation:

$$\alpha(x) := \sup_{x' \in I} \frac{\|\omega\|_{L^\infty}}{u}.$$  (4.10)

**Lemma 4.3.** Under the assumption (4.4), we have

$$\alpha(x) \leq \tilde{C} f^{1/2}(x) E(x),$$  (4.11)

where $\tilde{C}$ is a positive constant independent of $x$. 28
Proof. It follows from equation (4.1), that
\[
\| u \frac{\phi}{u^2} \psi \|_{L^2_v} \leq C \left( \| \sqrt{u} \phi_x \|_{L^2_v} + \| \frac{\phi}{u^2} \|_{L^2_v} \right) \leq C f(x) \left( \| \phi \|_{L^2_v} + \| \sqrt{u} \psi \|_{L^2_v} \right),
\]
and
\[
\| \sqrt{u} \phi \psi \|_{L^2_v} \leq C \left( \| \phi \|_{L^2_v} + \| \sqrt{u} \psi \|_{L^2_v} \right) \leq C f(x) \left( \| \phi \|_{L^2_v} + \| \sqrt{u} \psi \|_{L^2_v} \right). (4.10)
\]
It is easy to check from the definition of $\mathcal{E}(x)$ that the second term on the right handside of (4.10) can be controlled by $\mathcal{E}(x)$. For the first term, consider the case where $\psi \geq \eta_0$,
\[
\| \frac{\phi(x, \psi)}{u^2} (1 - \chi(\frac{\psi}{\eta_0})) \|_{L^2_v} \leq C \| \phi \|_{L^2_v},
\]
where the definition of the cutoff function $\chi$ can be found in the end of Section 1. And for the other case where $0 \leq \psi \leq \eta_0$, using equivalent relation $\bar{u} \sim \sqrt{\psi}$ and Hardy inequality, we get
\[
\| \frac{\phi(x, \psi)}{u^2} \chi(\frac{\psi}{\eta_0}) \|_{L^2_v} \leq C f^2(x) \left( \| \phi \|_{L^2_v} + \| \sqrt{u} \phi \|_{L^2_v} \right)
\]
where the positive constant $C$ may depend on $\eta_0$. Combining two estimates above, and using the fact that $f(x) \geq 1$, we find that
\[
\| \frac{\phi}{u^2} \|_{L^2_v} \leq C f^2(x) \left( \| \phi \|_{L^2_v} + \| \sqrt{u} \phi \|_{L^2_v} \right). (4.11)
\]
Inserting (4.11) into (4.9), we then obtain that
\[
\| u \frac{\phi}{u^2} \psi \|_{L^2_v} \leq C f^3(x) \left( \| \phi \|_{L^2_v} + \| \sqrt{u} \phi \|_{L^2_v} + \| \sqrt{u} \omega \|_{L^2_v} \right), (4.12)
\]
Now returning to estimate (4.10), it is easy to find the second term on the right handside of (4.10) can be controlled by $\mathcal{E}(x)$. To obtain the estimate of the first term, we calculate the far-field contribution immediately by using the fact that $u \geq 1$ in the support of $1 - \chi$:
\[
\| \frac{\phi(x, \psi)}{u^2} (1 - \chi(\frac{\psi}{\eta_0})) \|_{L^2_v} \leq \| \phi \|_{L^2_v}.
\]
And then, we can localize based on the location of $\psi$ and use Hardy inequality to find
\[
\| \frac{\phi(x, \psi)}{u^2} \chi(\frac{\psi}{\eta_0}) \|_{L^2_v} \leq C f^3(x) \left( \| \phi \|_{L^2_v} + \| \sqrt{u} \phi \|_{L^2_v} \right)
\]
where, we have used the estimate (4.12) and the fact that $u \geq 1$ in the last inequality. Hence, we obtain that
\[
\| \frac{\phi}{u^2} \|_{L^2_v} \leq C f^3(x) \left( \| \phi \|_{L^2_v} + \| \sqrt{u} \phi \|_{L^2_v} + \| \phi_x \|_{L^2_v} \right). (4.13)
\]
Upon inserting into (4.10) yields that
\[
\| \sqrt{u} \phi \psi \|_{L^2_v} \leq C f^8(x) \left( \| \phi \|_{L^2_v} + \| \sqrt{u} \phi \|_{L^2_v} + \| \phi_x \|_{L^2_v} \right). (4.14)
\]
And it then follows from (4.12) that
\[
\|\frac{\phi_\psi}{\sqrt{u}}\|_{L^2_\phi} \leq CF^\frac{\psi}{u}(x)\|\frac{\phi_\psi(x,\psi)}{\psi\frac{\psi}{\eta_0}}\|_{L^2_\phi} + C\|\frac{\phi_\psi(x,\psi)}{\sqrt{u(x,\psi)}}(1 - \chi(\frac{\psi}{\eta_0})\|_{L^2_\phi}.
\]
\[
\leq CF^\frac{\psi}{u}(x)(\|\phi_\psi(x,\psi)\|_{L^2_\phi} + \|\phi_\psi(x,\psi)\|_{L^2_\phi}) + C\|\phi_\psi\|_{L^2_\phi}.
\]
(4.15)

From Sobolev inequality, we deduce from the estimates (4.14)-(4.15) that
\[
|\phi_\psi|^2 \leq CF^\frac{\psi}{u}(1 - \chi(\frac{\psi}{\eta_0}))\|\phi_\psi\|_{L^2_\psi} + \|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi}.
\]
from which, we obtain
\[
\|\phi_\psi\|_{L^2_\psi} \leq CF^\frac{\psi}{u}(x)(\|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi}).
\]
(4.16)

We then deduce from (4.16) that
\[
\frac{\phi(x,\psi)}{u^\frac{\psi}{\eta_0}}\chi(\frac{\psi}{\eta_0}) \leq CF^\frac{\psi}{u}(x)\|\phi_\psi\|_{L^2_\phi} \leq CF^\frac{\psi}{u}(x)(\|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi}).
\]
(4.17)

The far-field term can be estimated easily as follows:
\[
\|\frac{\phi(x,\psi)}{u^\frac{\psi}{\eta_0}}(1 - \chi(\frac{\psi}{\eta_0}))\|_{L^2_\psi} \leq \|\phi_\psi\|_{L^2_\psi} \leq CF^\frac{\psi}{u}(x)(\|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi}),
\]
(4.18)

which, together with (4.17), yields that
\[
\|\frac{\phi}{u^2}\|_{L^2_\psi} \leq CF^\frac{\psi}{u}(x)(\|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi}).
\]
(4.19)

Recalling the definition of $\omega$, we deduce from (4.19) that
\[
\|\frac{\omega}{u}\|_{L^2_\psi} = \|\frac{\omega}{u^2}\|_{L^2_\phi} \leq \|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi} \leq CF^\frac{\psi}{u}(x)(\|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi} + \|\phi_\psi\|_{L^2_\phi}),
\]
which, together with the definition of $\alpha(x)$, gives (4.8) immediately.

**Lemma 4.4.** Under the assumption (4.4), we have
\[
f(x) \leq \gamma, \quad \text{for some } \gamma > 0.
\]
(4.20)

provided that $\sigma \leq \min \left\{ \frac{1}{2\gamma}, \frac{\gamma - 1}{2\gamma}, \frac{\gamma - 1}{\gamma - 1 + C} \right\}$.

**Proof.** The estimate (4.8) and the assumption (4.4) lead us to obtain
\[
\alpha(x) \leq C\|\phi_\psi\|_{L^2_\phi}(x)E(x) \leq 2\frac{\psi}{u^2}(x)C\sigma \leq \frac{1}{4},
\]
(4.21)

provided that $\sigma \leq \frac{1}{2\gamma - 1 + C}$. On the one hand, it is easy to check that
\[
\|\frac{\omega}{u}\|_{L^2} = \|\frac{\omega}{u}\|_{L^2} + \|\omega\|_{L^2} \leq \|\omega\|_{L^2} + \|\omega\|_{L^2} \leq 1 + \alpha(x)\|\frac{\omega}{u}\|_{L^2},
\]
which implies that
\[
\|\frac{\omega}{u}\|_{L^2} \leq \frac{1}{1 - \alpha(x)}.
\]
(4.22)
On the other hand, we find
\[ \| \frac{\bar{u}}{u} \|_{L^\infty_v} \leq \| \frac{\bar{u}}{u + \omega} \|_{L^\infty_v} \leq \| \frac{\bar{u}}{u - |\omega|} \|_{L^\infty_v} \leq \| \frac{1}{1 - |\omega|/u} \|_{L^\infty_v} \leq \frac{1 - \alpha(x)}{1 - 2\alpha(x)}. \] (4.23)

Then, recalling the definition of \( f(x) \) and combining the estimates (4.21)-(4.23), we have
\[ f(x) \leq \frac{1 - \alpha(x)}{1 - 2\alpha(x)} \leq \frac{1}{1 - 2\frac{19}{31} \frac{17}{2} \frac{19}{2} C}. \]

This immediately implies that \( f(x) \leq \gamma \), provided that
\[ \sigma \leq \gamma - \frac{1}{2 \frac{19}{31} \frac{17}{2} \frac{19}{2} C}. \]

Thus, we have completed the proof of this lemma.

From Lemmas 4.3-4.4, we may deduce that

**Corollary 4.5. Under the assumption (4.4), we have**
\[ \alpha(x) \leq CE(x), \]
where \( C \) is a positive constant independent of \( x \).

From now on, with (4.20), (4.24) and our bootstrap assumption (4.4) at hand, we will use the fact that \( f(x) \lesssim 1 \) and \( \alpha(x) \lesssim 1 \) in forthcoming estimates. Next, we will devote ourselves to showing standard energy estimate, quotient estimate and other estimate in \( L^2 \) level, that is Lemmas 4.6-4.8. The first one is \( L^2 \) standard energy estimate.

**Lemma 4.6. Under the assumption (4.4), we have**
\[ \sup_{x \in I} \| \phi \|^2_{L^2_v} + \int_I (\| \sqrt{u} f \phi \|^2_{L^2_v} + \| \frac{\phi}{u} \|^2_{L^2_v}) \, dx \leq \| \phi_0 \|^2_{L^2_v}. \] (4.25)

Moreover, we have the following exponential decay estimate:
\[ \sup_{x' \in I} e^{x'} \| \phi \|^2_{L^2_v} + \int_I e^{x'} \| \phi \|^2_{L^2_v} \, dx' \leq \| \phi_0 \|^2_{L^2_v}. \] (4.26)

**Proof.** Multiplying (4.11) by \( \phi \) and integrating with respect to \( \psi \), then we compute that
\[ \frac{1}{2} \frac{d}{dx} \| \phi \|^2_{L^2_v} + (\| \sqrt{u} f \phi \|^2_{L^2_v} + \| \frac{\phi}{u} \|^2_{L^2_v}) - \frac{1}{2} \int_0^\infty \bar{u} \phi \psi |\phi|^2 \, d\psi \leq \int_0^\infty |\omega f \phi \psi| \, d\psi. \] (4.27)

The last term on the left handside of (4.27) is positive, since
\[ \bar{u} \phi \psi = \bar{u} \psi_y - \bar{u} |\bar{u} \phi|^2 \leq 0. \]

Now returning to estimate the term on the right handside of (4.27) as follows:
\[ \int_0^\infty |\omega f \phi \psi| \, d\psi \leq C \| u \omega \|_{L^\infty_v} \| \frac{\phi}{u^2} \|_{L^2_v} \| \sqrt{u} \phi \psi \|_{L^2_v}. \] (4.28)

We recall the definition of \( \omega \) and Sobolev inequality, to find
\[ \| u \omega \|_{L^\infty_v} \leq C \| \phi \|_{L^\infty_v} \leq C \| \frac{\phi}{\sqrt{u}} \|_{L^2_v} \| \sqrt{u} \phi \psi \|_{L^2_v}. \] (4.29)

Combining the estimates (4.11) and (4.20), we find that
\[ \| \frac{\phi}{u^2} \|_{L^2_v} \leq C (\| \phi \|_{L^2_v} + \| \sqrt{u} \phi \|_{L^2_v}). \] (4.30)
Next, we note from (4.14) and (4.20) that

\[
\|\sqrt{u}\phi\psi\|_{L^2_v} \leq C\left(\|\phi\|_{L^2_v} + \|\sqrt{u}\phi\psi\|_{L^2_v} + \|\frac{\phi_x}{\sqrt{u}}\|_{L^2_v}\right). \tag{4.31}
\]

Inserting the estimates (4.28)-(4.31) into (4.27) yields the estimate

\[
\frac{1}{2} \frac{d}{dx} \|\phi\|_{L^2_v}^2 + (\|\sqrt{u}\phi\psi\|_{L^2_v}^2 + \|\frac{\phi}{\sqrt{u(u + \bar{u})}}\|_{L^2_v}^2) \leq C\left(\|\phi\|_{L^2_v} + \|\sqrt{u}\phi\psi\|_{L^2_v} + \|\frac{\phi_x}{\sqrt{u}}\|_{L^2_v}\right)(\|\sqrt{u}\phi\psi\|_{L^2_v}^2 + \|\frac{\phi}{u}\|_{L^2_v}^2)
\]

\[
\leq CE(x)\left(\|\sqrt{u}\phi\psi\|_{L^2_v}^2 + \|\frac{\phi}{u}\|_{L^2_v}^2\right).
\]

It follows the definition of \( f(x) \) and (4.20) that \( u \sim \bar{u} \). Since \( E(x) \leq \sigma \), by choosing \( \sigma \) small enough, then we have

\[
\frac{d}{dx} \|\phi\|_{L^2_v}^2 + (\|\sqrt{u}\phi\psi\|_{L^2_v}^2 + \|\frac{\phi}{u}\|_{L^2_v}^2) \leq 0,
\]

which, integrating over \( I \), yields (4.26) immediately. Since \( u \lesssim 1 \), the estimate above implies that

\[
\frac{d}{dx} \|\phi\|_{L^2_v}^2 + \|\phi\|_{L^2_v}^2 \leq 0.
\]

Therefore, using Gronwall inequality, we can obtain the decay estimate (4.26).

By using the method in [28], we then derive the following quotient estimate.

\textbf{Lemma 4.7.} Under the assumption (4.4), we have

\[
\sup_{x \in I} \left\| \frac{\phi}{\sqrt{u}}\right\|_{L^2_v}^2 + \int_I \left(\|\phi\psi\|_{L^2_v}^2 + \|\frac{\phi}{u}\|_{L^2_v}^2\right) dx' \leq \left\| \frac{\phi_0}{\sqrt{u_0}}\right\|_{L^2_v}^2. \tag{4.32}
\]

Moreover, we have the following exponential decay estimate:

\[
\sup_{x' \in I} e^{x'} \left\| \frac{\phi}{\sqrt{u}}\right\|_{L^2_v}^2 + \int_I e^{x'} \left(\|\phi\psi\|_{L^2_v}^2 + \|\frac{\phi}{u}\|_{L^2_v}^2\right) dx' \leq \left\| \frac{\phi_0}{\sqrt{u_0}}\right\|_{L^2_v}^2. \tag{4.33}
\]

\textbf{Proof.} We multiply (4.11) by \( \frac{\phi}{u} \) and integrate with respect to \( \psi \), to discover

\[
\frac{1}{2} \frac{d}{dx} \|\phi\|_{L^2_v}^2 + (\|\sqrt{u}\phi\psi\|_{L^2_v}^2 + \|\frac{\phi}{\sqrt{u(u + \bar{u})}}\|_{L^2_v}^2) \leq C \int_0^\infty \left[\frac{\phi^2 \omega_x}{u^2}\right] d\psi \leq C\|u\omega_x\|_{L^2_v} \|\frac{\phi}{u}\|_{L^2_v}^2. \tag{4.34}
\]

Now we write

\[
\omega_x = \left(\frac{\phi}{u + \bar{u}}\right)_x = \frac{\phi_x}{u + \bar{u}} - \frac{\phi_x}{(u + \bar{u})^2},
\]

which gives the estimate

\[
\|u\omega_x\|_{L^2_v} \leq C\|\phi_x\|_{L^2_v} + C\|\frac{\phi}{u^2}\|_{L^2_v} \|u\omega_x\|_{L^2_v}. \tag{4.35}
\]

Using Sobolev inequality, we get

\[
\|\phi_x\|_{L^2_v} \leq C\|\phi_x\|_{L^2_v}^2 + C\|\phi_x\|_{L^2_v}^2 \|\phi_x\|_{L^2_v}^2. \tag{4.36}
\]

\textbf{Invoking (4.19) and (4.20), we obtain}

\[
\|\frac{\phi}{u^2}\|_{L^2_v} \leq C\left(\|\phi\|_{L^2_v}^2 + \|\phi\|_{L^2_v}^2 + \|\phi_x\|_{L^2_v}^2\right) \leq CE(x). \tag{4.37}
\]

We substitute the estimates (4.37)-(4.38) into (4.36), to discover

\[
\|u\omega_x\|_{L^2_v} \leq C\|\phi_x\|_{L^2_v}^\frac{1}{2} \|\phi_{x\psi}\|_{L^2_v} + CE(x) \|u\omega_x\|_{L^2_v} \leq C\|\phi_x\|_{L^2_v}^\frac{1}{2} \|\phi_{x\psi}\|_{L^2_v}^\frac{1}{2} + C\sigma \|u\omega_x\|_{L^2_v}.
\]

We choose \( \sigma \) small enough to find

\[
\|u\omega_x\|_{L^2_v} \leq C\|\phi_x\|_{L^2_v}^\frac{1}{2} \|\phi_{x\psi}\|_{L^2_v}^\frac{1}{2} \leq CE(x). \tag{4.39}
\]
Utilizing (4.39) in (4.34) and using the fact that $u \sim \bar{u}$, we then obtain
\[
\frac{d}{dx} \left( \frac{\phi}{\sqrt{u}} \right)^2 L^2_x + \left( \|\phi\|^2_{L^2_x} + \|\phi_{x}\|^2_{L^2_x} \right) \leq C \mathcal{E}(x) \frac{\phi}{u^2} L^2_x.
\]
Since $\mathcal{E}(x) \leq \sigma$, by choosing $\sigma$ small enough, we can deduce that
\[
\frac{d}{dx} \left( \frac{\phi}{\sqrt{u}} \right)^2 L^2_x + \left( \|\phi\|^2_{L^2_x} + \|\phi_{x}\|^2_{L^2_x} \right) \leq 0,
\]
which implies that (4.32). According to the fact that $u \lesssim 1$, we conclude
\[
\frac{d}{dx} \left( \frac{\phi}{\sqrt{u}} \right)^2 L^2_x + \|\phi_{x}\|^2_{L^2_x} \leq 0.
\]
Then using Gronwall inequality, we obtain the decay estimate (4.33) immediately.

Finally, we also need the following energy estimate.

**Lemma 4.8.** Under the assumption (4.4), we have
\[
\sup_{x \in I} \left( \|\phi\|^2_{L^2_x} + \|\phi_{x}\|^2_{L^2_x} \right) + \int_I \left( \frac{\phi}{\sqrt{u}} \right)^2 L^2_x dx \leq \mathcal{E}^2(0) + C \mathcal{E}(x) D^2(x),
\]
where $C$ is a positive constant independent of $x$.

**Proof.** Multiplying (4.1) by $\frac{\phi}{u}$, using equivalent relation $u \sim \bar{u}$, and integrating by parts, we deduce
\[
\frac{1}{2} \frac{d}{dx} \left( \|\phi\|^2_{L^2_x} + \|\phi_{x}\|^2_{L^2_x} \right) \leq C \int_0^\infty \frac{d^2}{u^2} \frac{d\psi}{2} \leq C \frac{\omega_x}{u^2} L^2_x \frac{\phi}{u^2} \|\phi\|_{L^2_x}.
\]
Recalling the definition of $\mathcal{E}(x)$, then combining (4.35) with (4.38), we get
\[
\|\omega\|_{L^2_x} \leq \left| \frac{\phi_{x}}{u} \right|_{L^2_x} + \|\phi_{x}\|_{L^2_x} \|\omega\|_{L^2_x} \leq \|\phi_{x}\|_{L^2_x} + C \mathcal{E}(x) \|\omega\|_{L^2_x}.
\]
Recalling assumption (4.4) that $\mathcal{E}(x) \leq \sigma$, choosing $\sigma$ small enough, and using Sobolev inequality, we then have
\[
\|\omega\|^2_{L^2_x} \leq C \left( \|\phi_{x}\|^2_{L^2_x} \right) \leq C \left( \|\phi_{x}(x, \psi)\|^2_{L^2_x} \right) \left( 1 - \chi(\frac{\psi}{\eta_0}) \right) \|\phi_{x}\|^2_{L^2_x} + \|\phi_{x}(x, \psi)\|^2_{L^2_x} \chi(\frac{\psi}{\eta_0}) \|\phi_{x}\|^2_{L^2_x} \leq C \left( \|\phi_{x}\|^2_{L^2_x} + \|\phi_{x}\|^2_{L^2_x} + \|\sqrt{u} \phi_{x}\|^2_{L^2_x} \right),
\]
where we have used that
\[
\left| \frac{\phi_{x}(x, \psi)}{u(x, \psi)} \chi(\frac{\psi}{\eta_0}) \right| \leq C \frac{\psi}{\eta_0} \int_0^\infty |\phi_{x}(x, \psi)| d\psi' \leq C \frac{\psi}{\eta_0} \int_0^\psi |\phi_{x}(x, \psi)| \chi(\frac{\psi}{\eta_0}) d\psi' \leq C \frac{\psi}{\eta_0} \|\phi_{x}\|^2_{L^2_x} \int_0^\psi \psi' d\psi' \leq C \|\phi_{x}\|^2_{L^2_x} \|\phi_{x}\|^2_{L^2_x} \leq C \|\phi_{x}\|^2_{L^2_x} \|\phi_{x}\|^2_{L^2_x} \leq C \|\phi_{x}\|^2_{L^2_x} \|\phi_{x}\|^2_{L^2_x}.
\]
Next differentiate (4.11) with respect to $x$, then
\[
\phi_{xx} - u \phi_{x}\psi + 2 \frac{\phi_{x}}{u(x + \bar{u})} - \omega_x \phi_{x}\psi - 2 \frac{\phi_{x}}{u(x + \bar{u})} = 0.
\]
Combining this equation, (4.13), (4.20), (4.31) with (4.42), for any small \( \epsilon \) (\( \epsilon \) will be determined later), it holds
\[
\|u^2 \phi_x \psi\|_{L^2_v} \leq C \left( \|\sqrt{u} \phi\|_{L^2_{x,v}} + \|\phi_x\|_{L^2_{x,v}} + \|u \phi\|_{L^2_{x,v}} \right)
\]
\[
\leq C \left( \|\sqrt{u} \phi\|_{L^2_{x,v}} + \|\phi_x\|_{L^2_{x,v}} + \|u \phi\|_{L^2_{x,v}} \right),
\]
where we have used the fact that \( u \leq 1 \) and
\[
\|\phi\|_{L^2_{x,v}} + \|\sqrt{u} \phi\|_{L^2_{x,v}} + \|\phi_x\|_{L^2_{x,v}} \leq C\mathcal{E}(x) \leq C\sigma.
\]
We then choose \( \epsilon \) small enough, to find
\[
\|u^2 \phi_x \psi\|_{L^2_v} \leq C \left( \|\phi\|_{L^2_{x,v}} + \|\sqrt{u} \phi\|_{L^2_{x,v}} + \|\phi_x\|_{L^2_{x,v}} + \|u \phi\|_{L^2_{x,v}} \right). \tag{4.44}
\]
Recalling (4.39) and (4.38), we deduce
\[
\|\omega_x \|_{L^2_{x,v}} \leq \|\frac{\phi_x}{u^2}\|_{L^2_{x,v}} + \|\frac{\phi_{xx}}{u^2}\|_{L^2_{x,v}} \leq \|\frac{\phi_x}{u^2}\|_{L^2_{x,v}} + C\mathcal{E}(x) \|\omega_x\|_{L^2_{x,v}}. \tag{4.45}
\]
It then follows from a similar way as (4.43) that
\[
\|\frac{\phi_x}{u^2}\|_{L^2_{x,v}} \leq \|\frac{\phi_x(x, \psi)}{u^2}(1 - \chi(\frac{\psi}{\eta_0})\|_{L^2_{x,v}} + \|\frac{\phi_x(x, \psi)}{u^2}\chi(\frac{\psi}{\eta_0})\|_{L^2_{x,v}} \leq C \left( \|\phi_x\|_{L^2_{x,v}} + \|\phi_x(x, \psi)\psi\|_{L^2_{x,v}} \right), \tag{4.46}
\]
where we have used the estimate (4.44) in the last inequality. Therefore, combining (4.46) with (4.45), and choosing \( \sigma \) small enough, we discover
\[
\|\omega_x\|_{L^2_{x,v}} \leq C \left( \|\phi\|_{L^2_{x,v}} + \|\sqrt{u} \phi\|_{L^2_{x,v}} + \|\phi_x\|_{L^2_{x,v}} + \|\sqrt{u} \phi_x\|_{L^2_{x,v}} \right). \tag{4.47}
\]
Using Sobolev inequality, we have
\[
\|\phi\|_{L^\infty_{x,v}} \leq \|\frac{\phi_x}{L^2_{x,v}}\|_{L^2_{x,v}} \tag{4.48}
\]
We then conclude from (4.39), (4.40), (4.47) and (4.48) that
\[
\int_0^\infty \frac{\|\phi_x\|_{L^2_{x,v}}}{u^4} d\psi \leq C \left( \|\phi\|_{L^2_{x,v}} + \|\sqrt{u} \phi\|_{L^2_{x,v}} + \|\phi_x\|_{L^2_{x,v}} + \|\sqrt{u} \phi_x\|_{L^2_{x,v}} \right),
\]
At the \( H^1 \) level, comparing equation (4.11) versus equation (4.38), the nonlinear terms changes and thus requires a new treatment, see Lemmas (4.14-4.16) below. Firstly, we derive the following standard energy estimate in \( H^1 \) level.
Lemma 4.9. Under the assumption \((4.4)\), we have
\[
\sup_{x' \in I} \|\phi_{x'}\|^2_{L^2_0} + \int_I (\|\sqrt{u_0} \phi_{x'}\|^2_{L^2_0} + \|\frac{\partial \phi_{x'}}{u_0}\|^2_{L^2_0}) \, dx' \leq \mathcal{E}^2(0) + \mathcal{C}(x)D^2(x),
\]
(4.49)
where \(C\) is a positive constant independent of \(x\).

Proof. Multiplying \((4.43)\) by \(\phi_x\) and integrating with respect to \(\psi\), we deduce that
\[
\frac{1}{2} \frac{d}{dx} \|\phi_x\|^2_{L^2_0} + \|\sqrt{u_0} \phi_x\|^2_{L^2_0} + \|\frac{\phi_x}{\sqrt{u_0}}\|^2_{L^2_0} - \frac{1}{2} \int_0^\infty \bar{u}_\psi \phi_x |d\psi|
\leq \int_0^\infty \left( |\omega_x \phi_x \phi_x| + \frac{|\phi_x \omega_x|}{u_0^2} + |\omega_\psi \phi_x \phi_x| \right) d\psi.
\]
(4.50)
Since \(\bar{u}_\psi \leq 0\), we have
\[
\frac{1}{2} \int_0^\infty \bar{u}_\psi \phi_x |d\psi| \leq 0.
\]
We now proceed to estimate three terms on the right handside of \((4.50)\). According to \((4.31)\) and \((4.39)\), we find
\[
\int_0^\infty \left| \omega_x \phi_x \phi_x \right| d\psi \leq C \|\sqrt{u_0} \phi_x\|^2_{L^2_0} \|\frac{\phi_x}{u_0^2}\|^2_{L^2_0} \|\omega_x\|_{L^\infty_0}
\leq C (\|\phi\|_{L^2_0} + \|\sqrt{u_0} \phi_x\|_{L^2_0} + \|\frac{\phi_x}{u_0}\|_{L^2_0} ) \|\phi_x\|^2_{L^2_0} \|\phi_x\|^\frac{1}{2}_{L^2_0} \|\phi_x\|^\frac{1}{2}_{L^2_0}.
\]
(4.51)
Next, we deduce from \((4.13)\), \((4.39)\) and \((4.46)\) that
\[
\int_0^\infty \frac{|\phi_x \omega_x|}{u_0^3} d\psi \leq C \|\frac{\phi_x}{u_0^2}\|^2_{L^2_0} \|\frac{\phi_x}{u_0}\|^2_{L^2_0} \|\omega_x\|_{L^\infty_0}
\leq C (\|\phi\|_{L^2_0} + \|\sqrt{u_0} \phi_x\|_{L^2_0} + \|\phi_x\|_{L^2_0} ) (\|\phi\|_{L^2_0} + \|\sqrt{u_0} \phi_x\|_{L^2_0} + \|\frac{\phi_x}{u_0}\|_{L^2_0} + \|\phi_{\psi x}\|_{L^2_0})
\]
(4.52)
In order to estimate the last term on the right handside of \((4.50)\), we first control the following term by using a similar way as \((4.17)\) and \((4.18)\)
\[
\|\frac{\phi_x}{u_0}\|_{L^\infty_0} \leq C (\|\phi\|_{L^2_0} + \|\sqrt{u_0} \phi_x\|_{L^2_0}).
\]
(4.53)
It then follows from a similar way as \((4.30)\) that
\[
\|\frac{\phi_x}{u_0}\|^2_{L^2_0} \leq C (\|\phi\|_{L^2_0} + \|\sqrt{u_0} \phi_x\|_{L^2_0}).
\]
(4.54)
For the enhanced localized \(\frac{\phi_x}{u_0}\) estimate, we deduce from \((4.31)\) and \((4.52)\) that
\[
\|\frac{\phi_x}{\sqrt{u_0}}\|^2_{L^2_0} \leq C \|\phi_x\|^2_{L^2_0} + \|\phi_x\|^2_{L^2_0} + \|\frac{\phi_x}{u_0}\|^2_{L^2_0}.
\]
(4.55)
Substituting \((4.48)\) and \((4.55)\) into \((4.53)\), we have
\[
\|\frac{\phi_x}{u_0}\|_{L^\infty_0} \leq C (\|\phi\|_{L^2_0} + \|\phi_x\|_{L^2_0} + \|\frac{\phi_x}{u_0}\|^2_{L^2_0}).
\]
(4.56)
We then estimate \( \| \sqrt{u} \omega \phi \|_{L^\infty_x} \). Owing to

\[
2u \omega \phi = \phi' - 2 \frac{\tilde{u}}{u} \omega,
\]

we then conclude from (4.50), (4.51), (4.52) and (4.53) that

\[
\| \sqrt{u} \omega \phi \|_{L^\infty_x} \leq C \left( \| \frac{\phi'}{u} \|_{L^\infty_x} + \| \tilde{u} \omega \|_{L^\infty_x} \right)
\]

\[
\leq C \left( \left(1 - \chi \left( \frac{\psi}{\eta_0} \right) \right) \right) + \| \phi \|_{L^\infty_x} + \| \frac{\phi}{u^2} \|_{L^\infty_x}
\]

\[
\leq C \left( \left(1 - \chi \left( \frac{\psi}{\eta_0} \right) \right) \right) + \| \phi \|_{L^2_x} + \| \frac{\phi}{u^2} \|_{L^\infty_x}
\]

\[
\leq C \left( \| \phi \|_{L^2_x} + \| \phi \|_{L^2_x} + \| \phi \|_{L^\infty_x} \right).
\]

(4.57)

Invoking Sobolev inequality and (4.57), we obtain

\[
\int_0^\infty |\omega \phi \phi_x | dx \leq C \| \sqrt{u} \omega \phi \|_{L^\infty_x} \| \frac{\phi_x}{u} \|_{L^2_x} \| \sqrt{u} \phi \phi_x \|_{L^2_x}
\]

\[
\leq C \left( \| \phi \|_{L^2_x} + \| \phi \|_{L^2_x} + \| \phi \|_{L^\infty_x} \right) \| \frac{\phi_x}{u} \|_{L^2_x} \| \sqrt{u} \phi \phi_x \|_{L^2_x}.
\]

This, combining (4.50) and (4.52), leads to the estimate (4.49) by using equivalent relation \( u \sim \tilde{u} \) and recalling the definition of \( E(x) \) and \( D(x) \). Thus, we have completed the proof of this lemma.

\[
\square
\]

Next, we can also obtain the following quotient estimate in \( H^1 \) level.

**Lemma 4.10.** Under the assumption (4.3), we have

\[
\sup_{x' \in \Gamma} \| \frac{\phi_v}{u} \|_{L^2_{x'}} + \int_{\Gamma} \left( \| \phi_{vx} \|^2_{L^2_{x'}} + \| \frac{\phi_v}{u^2} \|^2_{L^2_{x'}} \right) dx' \leq E^2(0) + C E(x) D^2(x),
\]

where \( C \) is a positive constant independent of \( x \).

**Proof.** Multiplying (4.43) by \( \frac{\phi_v}{u} \) and integrating with respect to \( \psi \), we find that

\[
\frac{1}{2} \frac{d}{dx} \left( \frac{\phi_v}{u} \right)^2_{L^2_{x'}} + \frac{\phi_v}{u^2} \| \phi_{vx} \|_{L^2_{x'}} \leq C \int_0^\infty \left( \omega_x \phi_{x} \phi_{vx} \right) \left( \frac{\phi_v}{u^2} \right) dx.'
\]

Utilizing (4.41) into (4.42), we then calculate that

\[
\| \omega_x \|_{L^\infty_x} \leq C \left( \| \phi \|_{L^2_x} + \| \phi \|_{L^2_x} + \| \phi \|_{L^\infty_x} \right). \quad (4.59)
\]

Hence, we get from (4.31) and (4.59) that

\[
\int_0^\infty \frac{\omega_x \phi_{vx} \phi_{vx}}{u} \left( \frac{\phi_v}{u^2} \right) dx' \leq \| \omega_x \|_{L^\infty_x} \left( \frac{\phi_v}{u^2} \right) \left( \frac{\phi_v}{u^2} \right) \left( \sqrt{u} \phi_{vx} \right) \left( \sqrt{u} \phi_{vx} \right)
\]

\[
\leq C \left( \left(1 - \chi \left( \frac{\psi}{\eta_0} \right) \right) \right) + \| \phi \|_{L^2_x} + \| \frac{\phi_v}{u^2} \|_{L^\infty_x} \left( \| \sqrt{u} \phi_{vx} \|_{L^2_x} \right) \left( \frac{\phi_v}{u^2} \right) \left( \sqrt{u} \phi_{vx} \right)
\]

\[
\times \left( \| \phi \|_{L^2_x} + \| \sqrt{u} \phi \|_{L^2_x} + \| \frac{\phi_v}{u^2} \|_{L^\infty_x} \right).
\]

Using (4.59) once again, together with (4.13) and (4.20), we compute

\[
\int_0^\infty \left( \frac{\phi_v \omega_x}{u^4} \right) dx' \leq \| \omega_x \|_{L^\infty_x} \left( \frac{\phi_v}{u^2} \right) \left( \frac{\phi_v}{u^2} \right)
\]

\[
\leq C \left( \left(1 - \chi \left( \frac{\psi}{\eta_0} \right) \right) \right) + \| \phi \|_{L^2_x} + \| \frac{\phi_v}{u^2} \|_{L^\infty_x} \left( \| \sqrt{u} \phi \|_{L^2_x} \right) \left( \frac{\phi_v}{u^2} \right)
\]

\[
\times \left( \| \phi \|_{L^2_x} + \| \sqrt{u} \phi \|_{L^2_x} + \| \sqrt{u} \phi \|_{L^2_x} \right).
\]

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Applying (4.59) once more, we discover that
\[
\int_0^\infty \frac{\phi_x^2 \omega_x}{u^2} d\psi \leq \|\omega_x\|_{L^\infty_v} \|\phi_x\|_{L^2_v} \|\phi_x\|_{L^2_v} \leq C(\|\phi\|_{L_v^2} + \|\phi_v\|_{L_v^2} + \|\phi_x\|_{L_v^2} + \|\sqrt{u} \phi_{xx}\|_{L_v^2}) \frac{\phi_x}{u^2} \|\phi\|_{L_v^2}.
\]
Integrating and utilizing these estimates above, using the equivalent relation \(u \sim \bar{u}\), then recalling the definition of \(E(x)\) and \(D(x)\), we deduce (4.58) directly.

**Lemma 4.11.** Under the assumption (4.3), we have
\[
\sup_{x' \in I} \left(\|\phi_{x'}\|_{L^2_v}^2 + \|\phi_{x'}\|_{L^2_v}^2\right) + \int_I \left(\omega_x \phi_x \phi_{x'}\right) \frac{u}{u \bar{u}} \|\phi\|_{L^2_v} \leq C \sup_{x \in I} \left(\|\phi\|_{L^2_v} + \|\phi_v\|_{L^2_v} + \|\phi_{xx}\|_{L^2_v} + \|\phi_{x'x}\|_{L^2_v}\right)
\]
where \(C\) is a positive constant independent of \(x\).

**Proof.** Multiplying (4.43) by \(\phi_{x'} / u\) and integrating with respect to \(\psi\), we find that
\[
\frac{1}{2} \frac{d}{dx} \left(\|\phi_{x'}\|_{L^2_v}^2 + \|\phi_{x'}\|_{L^2_v}^2 + \|\phi_{x'}\|_{L^2_v}^2\right) + \int_I \left(\frac{\omega_x \phi_x \phi_{x'}\psi}{u} + \frac{\phi_{xx} \omega_{x'}\phi}{u^3} + \frac{\phi_{x}^2 \omega_{x}}{u^3}\right) d\psi \leq C \int_0^\infty \left(\omega_x \phi_x \phi_{x'}\right) \frac{u}{u \bar{u}} \|\phi\|_{L^2_v} \leq \sup_{x \in I} \left(\|\phi\|_{L^2_v} + \|\phi_v\|_{L^2_v} + \|\phi_{xx}\|_{L^2_v} + \|\phi_{x'x}\|_{L^2_v}\right).
\]
Recalling now the equation (4.37), the estimates (4.37) and (4.43) yields
\[
\|u \phi_{x'}\|_{L^\infty_v} \leq C(\|\phi\|_{L^2_v} + \|\phi_x\|_{L^2_v} + \|\phi_{xx}\|_{L^2_v} + \|\phi_{x'x}\|_{L^2_v})
\]
We can calculate from (4.37) and (4.43) that
\[
\int_0^\infty \left(\frac{\omega_x \phi_x \phi_{x'}\psi}{u} + \frac{\phi_{xx} \omega_{x'}\phi}{u^3} + \frac{\phi_{x}^2 \omega_{x}}{u^3}\right) d\psi \leq C \left(\|\phi\|_{L^2_v} + \|\phi_v\|_{L^2_v} + \|\phi_{xx}\|_{L^2_v} + \|\phi_{x'x}\|_{L^2_v}\right) \frac{\phi_x}{u^2} \|\phi\|_{L^2_v}
\]
Then, utilizing (4.38) with (4.47), we deduce
\[
\int_0^\infty \left(\frac{\omega_x \phi_x \phi_{x'}\psi}{u^3} + \frac{\phi_{xx} \omega_{x'}\phi}{u^2} + \frac{\phi_{x}^2 \omega_{x}}{u^2}\right) d\psi \leq C \left(\|\phi\|_{L^2_v} + \|\phi_v\|_{L^2_v} + \|\phi_{xx}\|_{L^2_v} + \|\phi_{x'x}\|_{L^2_v}\right) \frac{\phi_x}{u^2} \|\phi\|_{L^2_v}
\]
Making use of (4.37), (4.46) and (4.47), we discover
\[
\int_0^\infty \left(\frac{\phi_{x}^2 \omega_{x}}{u^4}\right) d\psi \leq C \left(\|\phi\|_{L^2_v} + \|\phi_v\|_{L^2_v} + \|\phi_{xx}\|_{L^2_v} + \|\phi_{x'x}\|_{L^2_v}\right) \frac{\phi_x}{u^2} \|\phi\|_{L^2_v}
\]
Plugging these estimates above into the estimate (4.61), recalling the fact that \(u \sim \bar{u}\) and the definition of \(E(x)\) and \(D(x)\), it is easy to deduce the estimate (4.60).

Under the a priori assumption (4.4), we collect all the estimates in Lemmas 4.4, 4.11 to get that
\[
D(x) \leq E(0) + C^* \mathcal{E} \mathcal{D}(x),
\]
which, together with the a priori assumption (4.4), leads to
\[
D(x) \leq E(0) + C^* \mathcal{E} \mathcal{D}(x).
\]
If we take \( \sigma \leq \min \left \{ \frac{1}{4(\mathcal{C}^*)^2}, \frac{1}{2^{\sigma_1} \gamma^2 \mathcal{C}}, \frac{1}{2^{\sigma_1} \gamma^2 \mathcal{C}} \right \} \), then we have

\[
\mathcal{E}(x) \leq \mathcal{D}(x) \leq 2\mathcal{E}(0).
\]

(4.63)

Taking \( \gamma := 2f(0) \) and \( \sigma := 4\mathcal{E}(0) \), one can apply the a priori estimate

\[
f(x) \leq 4f(0), \quad \mathcal{E}(x) \leq 4\mathcal{E}(0)
\]

(4.64)

to obtain the following estimate

\[
f(x) \leq 2f(0), \quad \mathcal{E}(x) \leq 2\mathcal{E}(0).
\]

(4.65)

Here we require the initial data is small enough to satisfy the condition

\[
\mathcal{E}(0) \leq \sigma_0 = \frac{1}{4} \min \left \{ \frac{1}{4(\mathcal{C}^*)^2}, \frac{1}{2^{\sigma_1} \gamma^2 \mathcal{C}}, \frac{1}{2^{\sigma_1} \gamma^2 \mathcal{C}} \right \}.
\]

Therefore, we have established the closed estimate in this subsection.

4.2. Global existence and exponential decay in von-Mises variable

In this section, our first aim is to establish the global existence of \( \phi \). Next, we will use Lemmas 4.6-4.11 to show that the difference \( u - \bar{u} \), as function of \( (x, \psi) \), decays exponentially in \( x \) variable, which proves the decay estimates (1.3) in Proposition 4.2. This, together the proof of the global existence result, gives the proof of Proposition 4.2.

**Proof of Proposition 4.2** Suppose the assumptions in Proposition 4.2 hold on, it is easy to obtain the local existence of the solution \( \phi \). Next, we use standard continuity argument to show the global well-posedness. From the local existence result in Proposition 4.1 and initial conditions in Proposition 4.2, it holds on

\[
f(x) \leq 4f(0) \quad \text{and} \quad \mathcal{E}(x) \leq 4\mathcal{E}(0), \quad \forall x \in [0, x_1).
\]

Set

\[
x^* := \sup_{x_1 \leq x} \{ x | f(x) \leq 4f(0) \} \quad \text{and} \quad \mathcal{E}(x) \leq 4\mathcal{E}(0), \quad \forall x \in [0, x_1).
\]

We claim that \( x^* = +\infty \). Otherwise, applying the estimate (4.63) and the local existence result in Proposition 4.1, there exists a positive constant \( x^{**} \), such that \( x^{**} > x^* \), it holds that for any \( x^* < x < x^{**} \),

\[
f(x) \leq 4f(0) \quad \text{and} \quad \mathcal{E}(x) \leq 4\mathcal{E}(0), \quad \forall x \in [0, x_1).
\]

This contradicts the definition of \( x^* \). Thus, we complete the proof of the global result in Proposition 4.2. Then we conclude from the estimates of Lemmas 4.6-4.11 that

\[
\frac{d}{dx} \mathcal{E}(x) + \mathcal{D}(x) \leq 0,
\]

where \( \mathcal{D}(x) := \sum_{k=0,1} \left( \| \sqrt{u} \partial_x^k \partial \phi \|_{L^2} + \| \partial_x^k \partial \phi \|_{L^2} + \| \partial_x^k \phi \|_{L^2} + \| \partial_x^{k+1} \phi \|_{L^2} \right) \). Since \( u \lesssim 1 \), by the definition of \( \mathcal{E}(x) \) and \( \mathcal{D}(x) \), we note that there exists a positive constant independent of \( x \) such that \( \mathcal{E}(x) \leq C\mathcal{D}(x) \).

This, together with the above inequality directly, implies that

\[
\frac{d}{dx} \mathcal{E}(x) + \mathcal{E}(x) \leq 0.
\]

We then employ Gronwall inequality, which yields the decay estimate (1.3) immediately. Therefore, we finish the proof of Proposition 4.2. \( \square \)
4.3. Proof of Theorem 1.2

With the help of Proposition 4.2, we are ready to prove Theorem 1.2. By virtue of the strong solution \( \phi \) obtained in Proposition 4.2, we have

\[
u(x, \psi) = \left( \phi(x, \psi) + \bar{u}^2(\psi) \right)^{\frac{1}{2}}.
\]

Then for a given value of \( \psi \), let \( y \) denotes the physical variable for the flows \( u \), that is

\[
y = \int_0^\psi \frac{d\psi'}{u(x, \psi')},
\]

Indeed, this transformation is one-to-one between the regions \((x, \psi) \in [0, +\infty) \times [0, +\infty)\) and \((x, y) \in [0, +\infty) \times [0, +\infty)\). Then for \( 0 \leq x < +\infty \) and \( 0 \leq y < +\infty \), we have

\[
u(x, y) = \nu(x, \psi(x, y)), \quad v(x, y) = \int_0^y \nu_x(x, y') dy'.
\] (4.66)

And the pair \((u, v)\) defines a global strong solution to the magnetic Prandtl model 1.1. Furthermore, we can obtain the strong nonlinear asymptotic stability of classical Hartmann boundary layer in the physical variable, see Theorem 1.2. This is coincide with pointwise perturbations \( u(x, y) - \bar{u}(y) \) and \( v(x, y) \).

More specifically, we have already shown the exponential decay estimate in \( x \) variable of \( u \) and \( \bar{u} \) as functions of \((x, \psi)\) in Section 4.2. In order to obtain that \( u \) and \( \bar{u} \), as functions of \((x, y)\), decay in same rate in \( x \) variable, we need to investigate the relation between \( y \) and \( \psi \). For any given value of \( \psi \), we set

\[
y_1 := \int_0^\psi \frac{1}{u(x, \psi')} d\psi', \quad y_2 := \int_0^\psi \frac{1}{\bar{u}(\psi')} d\psi',
\] (4.67)

where \( y_1 \) and \( y_2 \) represent the physical variables for the flows \( u \) and \( \bar{u} \), respectively. As a function of \((x, \psi)\) or \( y \), it is easy to check that \( y_1(x, \psi) \neq y_2(\psi) \). Clearly, for any given \( y_1 \), we have

\[
|u_y(x, y_1) - \bar{u}_y(y_1)| \leq |u_y(x, y_1) - \bar{u}_y(y_2)| + |\bar{u}_y(y_2) - \bar{u}_y(y_1)|.
\] (4.68)

Recalling the definition (4.67) of \( y_1 \) and \( y_2 \), then using the fact that \( u \leq 1 \), (4.16), (4.20) and the decay estimate (4.3), we deduce that

\[
|u_y(x, y_1) - \bar{u}_y(y_1)| = |uu_y - \bar{u}u_y|(x, \psi) \leq C \|\phi\|_{L^\infty} \leq C \left( \|\phi\|_{L^2} + \|\phi\|_{L^2} + \|\phi\|_{L^2} \right) \leq Ce^{-x}.
\] (4.69)

Next return to estimate the second term on the right handside of (4.68). Without loss of generality, we assume that \( y_1 < y_2 \). Employing Mean value theorem of differential, we conclude that there exists \( y' \in [y_1, y_2] \), such that

\[
|\bar{u}_y(y_2) - \bar{u}_y(y_1)| \leq C |\bar{u}_y(y')(y_1 - y_2)| \leq C |e^{-y'}(y_1 - y_2)|.
\] (4.70)

We recall the definition (4.67) of \( y_1 \) and \( y_2 \), by using the estimates (4.18), (4.20) and the decay estimate (4.3), to deduce that

\[
|e^{-y'}(y_1 - y_2)| \leq \int_0^\psi e^{-y'} |u - \bar{u}| \frac{d\psi'}{u\bar{u}} \leq C \left( \int_0^\psi \frac{|u|^2}{u\bar{u}} d\psi' \right) \frac{1}{2} \left( \int_0^\psi \frac{e^{-2y_1}}{u} d\psi' \right) \frac{1}{2}
\leq C \|\phi\|_{L^2} \|\phi\|_{L^2} \|\phi\|_{L^2} \leq C \|\phi\|_{L^2} \|\phi\|_{L^2} \|\phi\|_{L^2} \leq Ce^{-x}.
\] (4.71)

Combining (4.70) and (4.71), we obtain that

\[
|\bar{u}_y(y_2) - \bar{u}_y(y_1)| \leq Ce^{-x}.
\] (4.72)

It may be a wonder why we can obtain the decay estimate in \( x \) variable of \( \bar{u} \). Although \( \bar{u} \) is a function only depending on the space-like variable \( y \), \( y_1(x, \psi) \) is a function depending on \( x \), see the definition (4.67). Thus, we can get the exponential decay estimate (4.72) in \( x \) variable of \( \bar{u} \). Insert (4.69) and (4.72) into (4.68), we find

\[
\|u_y(x, y) - \bar{u}_y(y)\|_{L^\infty} \leq Ce^{-x}.
\]
It follows from the same way as (4.68) that for any $k = 0, 1, 2$,
\[ \|\partial_y^k(u - \bar{u})(x,y_1)\|_{L^2_{\psi_1}} \leq \|\partial_y^k u(x,y_1) - \partial_y^k \bar{u}(y_2)\|_{L^2_{\psi_1}} + \|\partial_y^k \bar{u}(y_2) - \partial_y^k \bar{u}(y_1)\|_{L^2_{\psi_1}}. \] (4.73)

Invoking (4.70) and (4.71), it is easy to see that the second term on the right hand side of (4.73) can be controlled as follows:
\[ \int_0^y |\partial_y^k \bar{u}(y_2) - \partial_y^k \bar{u}(y_1)|^2 dy_1 \leq C \int_0^y |\partial_y^{k+1} \bar{u}(y^*)(y_1 - y_2)^2 dy_1 \]
\[ \leq C \int_0^y |e^{-y^*}(y_1 - y_2)^2 dy_1 \leq C \int_0^y e^{-y_1} \left| \int_0^\psi e^{-\frac{y_1+y}{u\bar{u}}} d\psi \right|^2 dy_1 \]
\[ \leq C \sup_y \left( \int_0^\psi e^{-\frac{y_1+y}{u\bar{u}}} d\psi \right)^2 \int_0^y e^{-y_1} dy_1 \]
\[ \leq C \int_0^\psi \frac{|\omega|^2}{u^3} d\psi' \int_0^\psi e^{-\frac{y_1+y}{u\bar{u}}} d\psi \leq \|\frac{\omega}{u^2}\|_{L^2_\psi}^{2} \int_0^\psi e^{-\frac{y}{u\bar{u}}} dy \leq C \|\frac{\phi}{u^2}\|_{L^2_\psi}^{2} \leq Ce^{-2x}. \] (4.74)

We now proceed to estimate the first term on the right hand side of (4.73). We then deduce from the decay estimates (4.69), the estimates (4.11) and (4.20) that for $k = 0$,
\[ \int_0^y |u(x,y_1) - \bar{u}(y_2)|^2 dy_1 \leq \int_0^\psi \frac{|u(x,\psi') - \bar{u}(\psi')|^2}{u} d\psi' \leq \|\frac{\omega}{u^2}\|_{L^2_\psi}^{2} + \|\frac{\omega}{u^2}\|_{L^2_\psi}^{2} \leq C \|\frac{\phi}{u^2}\|_{L^2_\psi}^{2} \leq Ce^{-2x}. \]

For $k = 1$, we employ the decay estimate (4.69), the estimates (4.11) and (4.20) once again, and combine with (4.14), to compute
\[ \int_0^y |u_\psi(x,y_1) - \bar{u}_\psi(y_2)|^2 dy_1 \leq \int_0^\psi \frac{|u\psi(x,\psi') - \bar{u}\psi(\psi')|^2}{u} d\psi' \]
\[ \leq C \|\frac{\phi}{u^2}\|_{L^2_\psi}^{2} + C \|\frac{\phi}{u^2}\|_{L^2_\psi}^{2} \leq C \|\frac{\phi}{u^2}\|_{L^2_\psi}^{2} + \|\phi\|_{L^2_\psi}^{2} \leq Ce^{-2x}. \]

For $k = 2$, in view of the equation (4.19), the decay estimate (4.69), the estimates (4.11) and (4.20), we calculate that
\[ \int_0^y |u_\psi y(x,y_1) - \bar{u}_\psi y(y_2)|^2 dy_1 \leq \int_0^\psi \frac{|u_\psi y(x,\psi'y') - \bar{u}_\psi y(\psi'y')|^2}{u(x,\psi'y')} d\psi' \]
\[ \leq C \|\frac{\phi}{u^2}\|_{L^2_\psi}^{2} + \|\frac{\phi}{u^2}\|_{L^2_\psi}^{2} \leq C \|\frac{\phi}{u^2}\|_{L^2_\psi}^{2} + \|\phi\|_{L^2_\psi}^{2} \leq Ce^{-2x}. \]

By collecting these estimates above, we can get that for $k = 0, 1, 2$,
\[ \|\partial_y^k(y - \bar{u})(x,y_1)\|_{L^2_{\psi_1}} \leq Ce^{-x}. \] (4.75)

Substituting (4.74) and (4.75) into (4.73), we then obtain
\[ \|\partial_y^k(u - \bar{u})(x,y_1)\|_{L^2_{\psi_1}} \leq Ce^{-x}, \quad k = 0, 1, 2. \]

Therefore, we complete the proof of Theorem 1.72.

### A Some useful estimates

First, we will state without proof two elementary Hardy type inequalities, refer to Lemma B.1 in [39].

**Lemma A.1.** Let $f : \mathbb{R}^+ \to \mathbb{R}$ be some proper function. Then
(i) If $\lambda > -\frac{1}{2}$ and $\lim_{y \to +\infty} f(y) = 0$, then
\[ \|f(1 + y)^\lambda f\|_{L^2(\mathbb{R}^+)} \leq \frac{2}{2\lambda + 1} \|f(1 + y)^{\lambda+1} \partial_y f\|_{L^2(\mathbb{R}^+)}; \] \[ (A.1) \]
(ii) If $\lambda < -\frac{1}{2}$, then
\[ \|f(1 + y)^\lambda f\|_{L^2(\mathbb{R}^+)} \leq \sqrt{\frac{1}{2\lambda + 1} \left| f|_{y=0} - \frac{1}{2\lambda + 1} \|f(1 + y)^{\lambda+1} \partial_y f\|_{L^2(\mathbb{R}^+)}}. \] \[ (A.2) \]
Second, we prove some estimates as follows.

**Proposition A.2.** Let \((u', v')\) be the smooth solution, defined on \([0, L]\), to the approximated equations (3.12), (3.22) and (3.23). Under the assumption conditions (3.11) and (3.12), then it holds for integers \(s \geq 0\) and \(l \geq 1\) that

\[
\|\frac{\partial^{s+1} u'}{u'}\|_{L^\infty_y L^2_x} \leq C_{k, \delta_0} \|\partial^s_y \partial^y v'\|_{L^2_x H^1_y};
\]

\[
\|\partial^s_y \partial_y q'(y)'\|_{L^2_y L^2_x} \leq C_{k, (1)C_k, (u', \partial^2_y \partial^y q')\|_{L^2_y L^2_x} + \|\sqrt{u'} \partial^s_y \partial_y q'\|_{L^2_x H^1_y};
\]

\[
\|\partial^s_y \partial_y q'(y)'\|_{L^2_y L^2_x} \leq C_{k, (1)C_k, (u', \partial^2_y \partial^y q')\|_{L^2_y L^2_x} + \|\sqrt{u'} \partial^s_y \partial_y q'\|_{L^2_x H^1_y};
\]

\[
\|\partial^s_y \partial^2_y q'(y)'\|_{L^2_y L^2_x} \leq C_{k, \delta_0} \|\sqrt{u'} \partial^s_y \partial^2_y q'\|_{L^2_x H^1_y};
\]

Here we require \(L\) satisfying \(L \leq \delta_0^2\) in estimates (A.3) and (A.6).

**Proof.** Step 1: For any integer \(s \geq 0\), it holds for any cut off function \(\chi\left(\frac{y}{\delta_0}\right)\) (see definition in (1.14))

\[
\|\frac{\partial^{s+1} u'}{u'}\|_{L^\infty_y L^2_x} \leq \|\frac{\partial^s_x \frac{u'}{\delta_0} \chi\left(\frac{y}{\delta_0}\right)\|_{L^2_x L^\infty_y} + \|\frac{\partial^{s+1} u'}{u'}\|_{L^\infty_y L^2_x} \leq \frac{1}{k_s} \|\partial^s_y \chi\left(\frac{y}{\delta_0}\right)\|_{L^\infty_y}.
\]

Using the equivalent relation (3.11), it is easy to check that

\[
\|\frac{\partial^{s+1} u'}{u'}\|_{L^\infty_y L^2_x} \leq \|\frac{\partial^s_x \frac{u'}{\delta_0} \chi\left(\frac{y}{\delta_0}\right)\|_{L^\infty_y} \leq \frac{1}{k_s} \|\partial^s_x \frac{u'}{\delta_0} \chi\left(\frac{y}{\delta_0}\right)\|_{L^\infty_y}.
\]

and

\[
\|\frac{\partial^{s+1} u'}{u'}\|_{L^\infty_y L^2_x} \leq \frac{1}{k_s (\delta_0 + \epsilon)} \|\partial^s_y \|_{L^\infty_y} \leq \frac{C_{k, \delta_0}}{\delta_0} \|\partial^s_y \|_{L^\infty_y}.
\]

Substituting estimates (A.10) and (A.11) into (A.9), and applying the Sobolev embedding inequality, it holds true

\[
\|\frac{\partial^{s+1} u'}{u'}\|_{L^\infty_y L^2_x} \leq C_{k, \delta_0} \|\partial^s_y \|_{L^\infty_y} \leq \frac{C_{k, \delta_0}}{\delta_0} \|\partial^s_y \|_{L^\infty_y}.
\]

Therefore, we complete the proof of estimate (A.3).

Step 2: For any integer \(s \geq 0\) and \(\delta \leq \delta_0\), we can apply the estimate (3.12) to get that

\[
\|\partial^s_y \partial_y q'(y)'\|_{L^2_x L^2_y} \leq \|\partial^s_y \partial_y q'(y)'\|_{L^2_x_L^2_y} + \|\partial^s_y \partial_y q'(y)'\|_{L^2_x L^2_y} + \|\partial^s_y \partial_y q'(y)'\|_{L^2_x L^2_y} + \|\partial^s_y \partial_y q'(y)'\|_{L^2_x L^2_y},
\]

and

\[
\|\partial^s_y \partial_y q'(y)'(1 - \chi\left(\frac{y}{\delta_0}\right))\|_{L^2_x L^2_y} \leq \frac{1}{k_s (\delta_0 + \epsilon)} \|\partial^s_y \partial_y q'(y)'(1 - \chi\left(\frac{y}{\delta_0}\right))\|_{L^2_x L^2_y}.
\]

Integrating by part, we can get that

\[
\|\partial^s_y \partial_y q'(y)'\|_{L^2_x L^2_y} = \int_{0}^{\infty} |\partial^s_y \partial_y q'\|_{L^2_x L^2_y} = \int_{0}^{\infty} \|\partial^s_y \partial_y q'\|_{L^2_x L^2_y},
\]

which, together with the estimate (3.11), yields directly

\[
\|\partial^s_y \partial_y q'(y)'\|_{L^2_x L^2_y} \leq C(\sqrt{\delta}) \|\sqrt{u'} \partial^s_y \partial_y q'\|_{L^2_x L^2_y} + \delta^{-1} \|y^s \partial_y q'\|_{L^2_x L^2_y}.
\]

(A.14)
Substituting estimates (A.14) and (A.15) into (A.12), we have

$$\|\partial_s^2 \partial_y q(y)^l \|_{L_x^2} \leq C_k, (\delta^{-1} \|u^s \partial_x^2 \partial_y q(y)^l \|_{L_x^2} + \sqrt{\delta} \|u^s \partial_x^2 \partial_y^2 q(y)^l \|_{L_x^2}),$$

which, integrating over $[0, L]$, implies that

$$\|\partial_s^2 \partial_y q(y)^l \|_{L_x^2 L_y^2} \leq C_k, (\delta^{-1} L \|u^s \partial_x^2 \partial_y q(y)^l \|_{L_x^2 L_y^2} + \sqrt{\delta} \|u^s \partial_x^2 \partial_y^2 q(y)^l \|_{L_x^2 L_y^2}).$$

Choosing $\delta = \sqrt{L} \leq 2\delta_0$ in the above inequality, then we have

$$\|\partial_s^2 \partial_y q(y)^l \|_{L_x^2 L_y^2} \leq C_k (1) \|u^s \partial_x^2 \partial_y q(y)^l \|_{L_x^2 L_y^2} + \|u^s \partial_x^2 \partial_y^2 q(y)^l \|_{L_x^2 L_y^2}.$$

On the other hand, it is easy to check that

$$\|\partial_s^2 \partial_y q(y)^l \|_{L_x^2 L_y^2} \leq \|\partial_s^2 \partial_y q_0(y)^l \|_{L_x^2} + C_k (1) \|\partial_x^{s+1} \partial_y q(y)^l \|_{L_x^2}.$$

This, together with the estimate (A.15) and $\delta = \delta_0$, yields directly

$$\|\partial_s^2 \partial_y q(y)^l \|_{L_x^2 L_y^2} \leq \|\partial_s^2 \partial_y q_0(y)^l \|_{L_x^2} + C_k (1) \|u^s \partial_x^2 \partial_y q(y)^l \|_{L_x^2 L_y^2} + \|u^s \partial_x^2 \partial_y^2 q(y)^l \|_{L_x^2 L_y^2}.$$

Thus, we complete the proof of claimed estimates (A.4) and (A.5).

Step 3: It is easy to check that

$$\|\partial_s^2 q(y)^l \|_{L_x^2 L_y^2} \leq 2 \int_0^1 |\partial_s^2 q(y)^l \|_{L_x^2} dy \leq 2 \|\partial_s^2 q(y)^l \|_{L_x^2} \|\partial_s \partial_y q(y)^l \|_{L_x^2}.$$

Using the Hardy inequality (A.2), it holds true

$$\|\partial_s^2 q(y)^l \|_{L_x^2 L_y^2} \leq C \|\partial_s^2 \partial_y q(y)^l \|_{L_x^2}.$$ 

Thus, we get that

$$\|\partial_s^2 q(y)^l \|_{L_x^2 L_y^2} \leq C \|\partial_s^2 \partial_y q(y)^l \|_{L_x^2}.$$ 

This together with the estimates (A.4) and (A.5) gives respectively

$$\|\partial_s^2 q(y)^l \|_{L_x^2 L_y^2} \leq \|\partial_s^2 \partial_y q(y)^l \|_{x=0} + C_k (1) \|u^s \partial_x^2 \partial_y q(y)^l \|_{L_x^2 L_y^2} + \|u^s \partial_x^2 \partial_y^2 q(y)^l \|_{L_x^2 L_y^2},$$

and

$$\|\partial_s^2 q(y)^l \|_{L_x^2 L_y^2} \leq C_k (1) \|u^s \partial_x^2 \partial_y q(y)^l \|_{L_x^2 L_y^2} + \|u^s \partial_x^2 \partial_y^2 q(y)^l \|_{L_x^2 L_y^2},$$

for all $L \leq \delta_0$. Therefore, we complete the proof of the claimed estimates (A.6) and (A.7).

Step 4: For any integer $\kappa \geq 0$, it holds

$$\partial_s^2 \partial_y^3 v(y)^l = \partial_s^2 \partial_y^2 q(y)^l \chi\left(\frac{y}{\delta}\right) + \partial_s^2 \partial_y^2 q(y)^l (1 - \chi(\frac{y}{\delta})).$$

Using the assumptions (3.11) and (3.12), is easy to check that

$$\|\partial_s^2 \partial_y^2 q(y)^l (1 - \chi(\frac{y}{\delta})) \|_{L_x^2} \leq \frac{1}{(k_\kappa (\delta + \epsilon))^{\frac{1}{2}}} \|\partial_s^2 \partial_y^2 q(y)^l (1 - \chi(\frac{y}{\delta})) \|_{L_x^2},$$

and

$$\|\partial_s^2 \partial_y^2 q(y)^l \chi(\frac{y}{\delta}) \|_{L_x^2} \leq \|\partial_s^2 \partial_y^2 q(y)^l \chi(\frac{y}{\delta}) \|_{L_x^2} \leq C_k, (k_\kappa) \|\partial_s^2 \partial_y^2 v(y)^l \|_{L_x^2}.$$ 

Therefore, we can get that

$$\|\partial_s^2 \partial_y^2 q(y)^l \|_{L_x^2} \leq C_k, (\delta^{\frac{1}{2}}) \|\partial_s^2 \partial_y^2 q(y)^l (1 - \chi(\frac{y}{\delta})) \|_{L_x^2} + C_k, (\kappa) \|\partial_s^2 \partial_y^2 v(y)^l \|_{L_x^2},$$

which, integrating over $[0, L]$ and choosing $\delta = \delta_0$, yields directly

$$\|\partial_s^2 \partial_y^2 q(y)^l \|_{L_x^2 L_y^2} \leq C_k, (\delta_0) \|\partial_s^2 \partial_y^2 q(y)^l \|_{L_x^2 L_y^2} + C_k, (\kappa) \|\partial_s^2 \partial_y^2 v(y)^l \|_{L_x^2 L_y^2},$$

which implies the claimed estimate (A.8). Therefore, we complete the proof of this lemma.
Let us define
\[ V_1^k(x) := \| \partial_x^{(k)} \partial_y \nu'(y) \|_{L^2 L^2} + \| \partial_x^{(k)} \partial_y^2 \nu'(y) \|_{L^2 L^2} + \| \partial_x^{(k-1)} \partial_y^3 \nu'(y) \|_{L^2 L^2}, \] (A.16)
and
\[ \hat{V}_1^k(0) := \| \partial_y^{(2k-2)} \partial_y^2 \nu'(y) \|_{L^2 L^2} + \sum_{\beta \geq 1, 2\alpha + \beta \leq 2(k-1)} \| \partial_y^\beta \partial_y^2 \nu'(y) \|_{L^2 L^2}. \] (A.17)

**Proposition A.3.** For any smooth solution \((u', \nu')\) of equation (3.2), then the following estimates hold true
\[
\| \partial_x^{(k-1)} \partial_y \nu'(y) \|_{L^2 L^2} \leq C_{k,l}(1 + B_1^k(0)^2 + V_1^k(x)^2),
\] (A.18)
\[
\| \partial_x^{(k-1)} \partial_y^2 \nu'(y) \|_{L^2 L^2} \leq C_{k,l}(1 + B_1^k(0)^2 + \hat{V}_1^k(0)^2 + V_1^k(x)^2),
\] (A.19)
where any integers \(\alpha\) and \(\beta\) satisfying \(0 \leq 2\alpha + \beta \leq 2k\).

**Proof.** Step 1: Applying the differential operator \(\partial_x^{k-1}\) to the equation (3.2), we have
\[
\partial_x^{k-1} \partial_y \nu' = \partial_x^{k-1} \partial_y^2 \nu' + \partial_x^{k}(u' \partial_y^2 \nu') - \partial_x^{k}(\nu' \partial_y^2 u') := J_1 + J_2 + J_3.
\]

Using the divergence-free condition, it is easy to check that
\[
\| J_2(y) \|_{L^2 L^2} \leq \| u' \partial_x^{k} \partial_y^2 \nu'(y) \|_{L^2 L^2} + \sum_{j=1}^{k} C_{j,k} \| \partial_x^{j-1} \partial_y \nu' \partial_x^{k-1} \partial_y^2 \nu'(y) \|_{L^2 L^2}
\]
\[
\leq (1 + \| \partial_y \nu'(y) \|_{L^2} + \| \partial_x^2 \nu'(y) \|_{L^2 L^2}) \| \partial_x^{k} \partial_y^2 \nu'(y) \|_{L^2 L^2}
\]
\[
+ C_k \| \partial_x^{(k-1)} \partial_y \nu'(y) \|_{L^2 L^2} \| \partial_x^{(k-1)} \partial_y^2 \nu'(y) \|_{L^2 L^2}
\]
\[
+ C_k \| \partial_x^{(k-1)} \partial_y^2 \nu'(y) \|_{L^2 L^2} \| \partial_x^{(k-1)} \partial_y^2 \nu'(y) \|_{L^2 L^2} \leq C_k (1 + \| \partial_y \nu'(y) \|_{L^2} + \| \partial_x^{(k-2)} \partial_y^2 \nu'(y) \|_{L^2 L^2}) + C_k \| \partial_x^{(k)} \partial_y^2 \nu'(y) \|_{L^2 L^2}.
\]

and
\[
\| J_3(y) \|_{L^2 L^2} \leq \| \partial_y^2 u'(y) \|_{L^2 L^2} \| \partial_x^{k} \partial_y^2 \nu'(y) \|_{L^2 L^2} + \sum_{j=0}^{k-1} C_{j,k} \| \partial_x^{j} \partial_y^2 \nu'(y) \|_{L^2 L^2}
\]
\[
\leq C_k (1 + \| \partial_y \nu'(y) \|_{L^2} + \| \partial_x^{(k-1)} \partial_y^3 \nu'(y) \|_{L^2 L^2}) + C_k \| \partial_x^{(k-1)} \partial_y^2 \nu'(y) \|_{L^2 L^2}
\]
\[
+ C_k \| \partial_x^{(k-1)} \partial_y^2 \nu'(y) \|_{L^2 L^2} \| \partial_x^{(k-1)} \partial_y^2 \nu'(y) \|_{L^2 L^2} \leq C_k (1 + \| \partial_y \nu'(y) \|_{L^2} + \| \partial_x^{(k-2)} \partial_y^2 \nu'(y) \|_{L^2 L^2}) + C_k \| \partial_x^{(k)} \partial_y^2 \nu'(y) \|_{L^2 L^2}.
\]

Therefore, we can get that for all \(l \geq 1\)
\[
\| \partial_x^{l-1} \partial_y \nu'(y) \|_{L^2 L^2} \leq C_{k,l}(1 + \| \partial_y \nu'(y) \|_{L^2} + \| \partial_x^2 \nu'(y) \|_{L^2 L^2})
\]
\[
+ C_{k,l}(1 + \| \partial_x^{(k-2)} \partial_y^2 \nu'(y) \|_{L^2 L^2} + \| \partial_x^{(k-2)} \partial_y^2 \nu'(y) \|_{L^2 L^2}) + C_{k,l}(1 + \| \partial_x^{(k)} \partial_y^2 \nu'(y) \|_{L^2 L^2}).
\]

This implies the estimate (A.18).

Step 2: We will give the proof of estimate (A.19) by induction. Indeed, the estimate (A.18) implies that (A.19) holds true for the case of \(\beta = 0, 1, 2\). Now we suppose the estimate (A.19) holds true for the case \(\beta \leq 2n\), i.e.,
\[
\| \partial_x^{n} \partial_y^2 \nu'(y) \|_{L^2 L^2} \leq C_{k,l}(1 + B_1^k(0)^2 + V_1^k(x)^2 + \hat{V}_1^k(0)^2),
\] (A.20)
for $2 \leq 2n \leq (k-1)$. This and the relation $2\alpha + 2n \leq 2k$ implies the integer $\alpha \in [0, k-1]$. We need to verify that (A.19) holds on for case of $\beta \leq 2n + 1$ and $\beta \leq 2n + 2$, i.e.

$$\|\partial_x^{\alpha-1}\partial_y^{\beta+2}\psi(y)^I\|_{L^2_y L^2_x} \leq C_{k,l}(1 + B^k_1(0)^2 + V^k_1(x)^2 + \hat{V}^k_1(0)^2),$$

(A.21)

Indeed, the estimate (A.20) implies that (A.21) holds true for the case $\beta \leq 2n$. We just need to prove the estimate (A.21) holds on for the case $\beta = 2n + 1$ and $\beta = 2n + 2$. Applying differential operator $\partial_x^{\alpha-1}\partial_y^{\beta+2}$ to the equation (A.21), then we have

$$\partial_x^{\alpha-1}\partial_y^{2n+3}\psi^y = \partial_x^{\alpha-1}\partial_y^{2n+1}\psi + \partial_x^{\alpha-1}\partial_y^{2n}(u'\partial_y^y\psi) - \partial_x^{\alpha-1}\partial_y^{2n}(v'\partial_y^y u') := K_1 + K_2 + K_3.$$

Using the H"{o}lder inequality and divergence-free condition, then we have

$$\|K_2(y)^I\|_{L^2_y L^2_x} \leq C_k \sum_{j=0}^{2n} \|\partial_y^n u'\partial_y^{2n-j}\partial_y^y \psi(y)^I\|_{L^2_y L^2_x} + C_k \sum_{i=0}^{2n} \|\partial_x^n \partial_y^y \partial_x^{2n-j}\partial_y^y \psi(y)^I\|_{L^2_y L^2_x} \leq C_k \|\partial_y^{2n} u(y)^I\|_{L^2_y L^2_x} \|\partial_x^{2n+1} \psi(y)^I\|_{L^2_y L^2_x} + K_k \|\partial_x^{\alpha-1}\partial_y^{2n+1}\psi(y)^I\|_{L^2_y L^2_x} \|\partial_x^{\alpha-1}\partial_y^{2n+1}\psi(y)^I\|_{L^2_y L^2_x} \leq C_k \|\partial_y^{2n} u(y)^I\|_{L^2_y L^2_x} + \|\partial_x^{\alpha-1}\partial_y^{2n+1} v(y)^I\|_{L^2_y L^2_x} + \|\partial_x^{\alpha-1}\partial_y^{2n+2} (\psi)^I\|_{L^2_y L^2_x}$$

and

$$\|K_3(y)^I\|_{L^2_y L^2_x} \leq C_k \sum_{j=0}^{2n} \|\partial_y^n \partial_y^y \partial_x^{2n-j}\partial_y^y u(y)^I\|_{L^2_y L^2_x} + C_k \sum_{i=0}^{2n} \|\partial_x^n \partial_y^y \partial_x^{2n-j}\partial_y^y u(y)^I\|_{L^2_y L^2_x} \leq C_k \|\partial_y^{2n} u(y)^I\|_{L^2_y L^2_x} \|\partial_x^{2n+1} \psi(y)^I\|_{L^2_y L^2_x} + C_k \|\partial_x^{\alpha-1}\partial_y^{2n+1} \psi(y)^I\|_{L^2_y L^2_x} \|\partial_x^{\alpha-1}\partial_y^{2n+2} \psi(y)^I\|_{L^2_y L^2_x} \leq C_k \|\partial_y^{2n} u(y)^I\|_{L^2_y L^2_x} + \|\partial_x^{\alpha-1}\partial_y^{2n+1} v(y)^I\|_{L^2_y L^2_x} + \|\partial_x^{\alpha-1}\partial_y^{2n+2} \psi(y)^I\|_{L^2_y L^2_x}$$

Then, we can get that

$$\|\partial_x^{\alpha-1}\partial_y^{2n+3} \psi(y)^I\|_{L^2_y L^2_x} \leq C_{k,l}(1 + \|\partial_y^{2n+1} u(y)^I\|_{L^2_y L^2_x} + \|\partial_x^{\alpha-1}\partial_y^{2n+1} v(y)^I\|_{L^2_y L^2_x} + C_{k,l} \|\partial_x^{\alpha-1}\partial_y^{2n+2} v(y)^I\|_{L^2_y L^2_x} \|\partial_x^{\alpha-1}\partial_y^{2n+2} \psi(y)^I\|_{L^2_y L^2_x}.$$

(A.22)

Applying differential operator $\partial_x^{\alpha-1}\partial_y^{2n+1}$ to the equation (A.22), then we have

$$\partial_x^{\alpha-1}\partial_y^{2n+3} \psi = \partial_x^{\alpha-1}\partial_y^{2n+1}(u'\partial_y^y\psi) - \partial_x^{\alpha-1}\partial_y^{2n+1}(v'\partial_y^y u').$$

Then, similar to the estimates of terms $K_2$ and $K_3$, it is easy to check that

$$\|\partial_x^{\alpha-1}\partial_y^{2n+4} \psi(y)^I\|_{L^2_y L^2_x} \leq C_{k,l}(1 + \|\partial_y^{2n+2} u(y)^I\|_{L^2_y L^2_x} + \|\partial_x^{\alpha-1}\partial_y^{2n+2} v(y)^I\|_{L^2_y L^2_x} + \|\partial_x^{\alpha-1}\partial_y^{2n+3} v(y)^I\|_{L^2_y L^2_x} + \|\partial_x^{\alpha-1}\partial_y^{2n+3} \psi(y)^I\|_{L^2_y L^2_x}.$$

(A.23)

Thus, the estimates (A.22) and (A.23) imply estimate (A.21) holds true for $\beta = 2n + 1$ and $\beta = 2n + 2$. Therefore, we complete the proof of lemma.

## B Compatibility of initial data

In this section, we will introduce the definition of compatibility of initial data at the corner $(0,0)$. This can explain the reason that the initial data required in Theorem 1.1. First of all, by evaluating the equation (1.1) at $y = 0$, we obtain the boundary condition:

$$(-\partial_y^y u - 1)|_{y=0} = 0.$$

Thus, we require the initial data $u_0(y)$ satisfies the compatibility condition

$$(-\partial_y^y u_0(y) - 1)|_{y=0} = 0.$$

(B.1)
Then, we use the equation (1.6) to deduce that \( u \frac{\partial u}{\partial y} \big|_{x=0} \in L^2(\mathbb{R}^+) \). Taking \( \partial_y \) operator to the equation (1.6) and applying the divergence-free condition, we have

\[
u \frac{\partial_x}{\partial y} u + v \frac{\partial_y^2}{\partial y^2} u - \partial_y^3 u + \partial_y u = 0.
\]

(B.2)

Evaluating the above equation at \( y = 0 \), then we have

\[ (-\partial_y^3 u + \partial_y u) \big|_{y=0} = 0. \]

Thus, we require the initial data \( u_0(y) \) satisfies the compatibility condition

\[ (-\partial_y^3 u_0 + \partial_y u_0(y)) \big|_{y=0} = 0. \]

(B.3)

Furthermore, we also have that \( \partial_y \left( \frac{\nu}{u} \right) \big|_{x=0} \in L^2(\mathbb{R}^+) \). The above two compatibilities on initial data is devoted to ensuring that the quantities \( u \frac{\partial u}{\partial y} \big|_{x=0} \) and \( \partial_y \left( \frac{\nu}{u} \right) \big|_{x=0} \) is not singularity at the point \( y = 0 \).

Secondly, using the equation (1.6), we have

\[ v(x,y) = u(x,y) \int_0^y -\frac{\partial_y^3 u(x,y') + u(x,y') - 1}{\left[u(x,y')\right]^2} dy'. \]

which implies

\[ v_0(y) := v(0,y) = u_0(y) \int_0^y -\frac{\partial_y^3 u_0(y') + u_0(y') - 1}{\left[u_0(y')\right]^2} dy'. \]

(B.4)

By taking \( \partial_x \) differential operator to the equation (1.6) and using the divergence-free condition, we obtain

\[ \partial_x (u \frac{\partial_x}{\partial y} u + v \frac{\partial_y}{\partial y} u) + \partial_x^3 v - \partial_y v = 0, \]

(B.5)

which, evaluating at \( y = 0 \), yields directly

\[ (\partial_x^3 v - \partial_y v) \big|_{y=0} = 0. \]

The first order compatibility condition that arises from this entails matching now the initial data evaluated at \( y = 0 \) with boundary data evaluated at \( x = 0 \) in the standard manner for initial boundary value problem:

\[ \{ \partial_y^3 v_0(y) - \partial_y v_0(y) \} \big|_{y=0} = 0. \]

(B.6)

On the other hand, using the divergence-free condition, we can rewrite the equation as the form

\[ u^2 \frac{\partial_y}{\partial y} \left( \frac{\nu}{u} \right) = 2u \frac{\partial_y}{\partial y} v \frac{\partial_y}{\partial y} \left( \frac{\nu}{u} \right) + \partial_y^3 v - \partial_y v. \]

It is clear that all quantities on the righthand side of equation (B.5) are vanishing at \( y = 0 \). Then, the first compatibility on initial data implies directly

\[ u \frac{\partial_y}{\partial y} \left( \frac{\nu}{u} \right) \big|_{x=0} \in L^2(\mathbb{R}^+). \]

Taking \( \partial_x \) operator of equation (B.2) and using the divergence-free condition, then we have

\[ \partial_x (u \frac{\partial_x}{\partial y} u + v \frac{\partial_y^2}{\partial y^2} u) + \partial_y^4 v - \partial_y^2 v = 0. \]

which, evaluating at \( y = 0 \), yields directly

\[ (\partial_y^4 v - \partial_y^2 v) \big|_{y=0} = 0. \]

This then gives our second order compatibility condition:

\[ \{ \partial_y^4 v_0(y) - \partial_y^2 v_0(y) \} \big|_{y=0} = 0. \]

(B.7)

Taking \( \partial_y \) operator to the equation, we have

\[ \partial_y \left\{ u^2 \frac{\partial_y}{\partial y} \left( \frac{\nu}{u} \right) \right\} = \partial_y \left\{ 2u \frac{\partial_y}{\partial y} v \frac{\partial_y}{\partial y} \left( \frac{\nu}{u} \right) \right\} + \partial_y^4 v - \partial_y^2 v. \]

(B.8)
It is clear that all quantities on the righthand side of equation (B.8) are vanishing at $y = 0$. Then, the second compatibility on initial data implies directly
\[
\partial_{xy} \left( \frac{v}{u} \right) |_{x=0} \in L^2(\mathbb{R}^+) .
\]
Therefore, we can define the higher order compatibility conditions at the corner $(0, 0)$ in the same manner, and can be stated as follows
\[
\{ \partial_{xy}^m v |_{x=0} - \partial_y (\partial_{xy}^m v |_{x=0}) \} |_{y=0} = 0 , \tag{B.9}
\]
and
\[
\{ \partial_{xy}^m v |_{x=0} - \partial_y (\partial_{xy}^m v |_{x=0}) \} |_{y=0} = 0 , \tag{B.10}
\]
for all $m \geq 0$. It is worth noticing that the higher order $x$-derivative of $v$ at $x = 0$ can be controlled by the lower one. The compatibility conditions (B.9) and (B.10) on the initial data are called the $(2m+1)$th and $(2m+2)$th order compatibility conditions at the corner $(0, 0)$. According to the initial norms $B^k_l(0)$ and $\mathcal{E}^k_l(0)$ (see the definitions in (3.5) and (3.7)), we require the generic compatibility conditions at the corner $(0, 0)$ up to order $2k - 1$ in Theorem 1.1. Finally, we explain the initial data $u_0$ satisfying the compatibility condition. Obviously, we have
\[
\partial_y v = \partial_y u \int_0^y \frac{-\partial_y^2 u + u - 1}{(u)^2} \, d\tau + \frac{-\partial_y^2 u + u - 1}{u} ,
\]
and hence, it holds
\[
\partial_y v_0 |_{y=0} = \left\{ \partial_y u_0 \int_0^y \frac{-\partial_y^2 u_0 + u_0 - 1}{(u_0)^2} \, d\tau + \frac{-\partial_y^2 u_0 + u_0 - 1}{u_0} \right\} |_{y=0} = 0 ,
\]
where we have used the compatibility conditions (B.1) and (B.3). This and the compatibility condition (B.6) implies $\partial_y^3 v_0 |_{y=0} = 0$. Recall that
\[
\partial_y^3 v_0 = \partial_y^2 u_0 \int_0^y \frac{-\partial_y^2 u_0 + u_0 - 1}{u_0^2} \, d\tau + \partial_y^3 u_0 \frac{-\partial_y^2 u_0 + u_0 - 1}{u_0} - \partial_y u_0 \frac{-\partial_y^3 u_0 + \partial_y u_0}{u_0} + \frac{-\partial_y^3 u_0 + \partial_y^3 u_0}{u_0} ,
\]
then we require the initial data of $u_0$ to satisfy
\[
(-\partial_y^4 u_0 + \partial_y^2 u_0) |_{y=0} = (-\partial_y^5 u_0 + \partial_y^3 u_0) |_{y=0} = 0 .
\]
Thus, this ensures $\partial_y^3 v_0 |_{y=0} = 0$ that will satisfy the compatibility condition. Similarly, we require the initial data $u_0$ satisfying
\[
(-\partial_y^5 u_0 + \partial_y^3 u_0) |_{y=0} = (-\partial_y^7 u_0 + \partial_y^5 u_0) |_{y=0} = 0 ,
\]
to ensure the compatibility (B.7). Similarly, we can add conditions to the initial data $u_0$ to satisfy the compatibility conditions (B.9) and (B.10).

C Control of initial data

In this section, we will prove that the initial condition $B^k_l(0) + \mathcal{E}^k_l(0)$ can be controlled be the initial data $u_0$, and hence is independent of $\epsilon$. That is to give the proof for the claim estimate as follows (see (3.56))
\[
\mathcal{E}^k_l(0) + B^k_l(0) \leq C(u_0) , \tag{C.1}
\]
where the constant $C(u_0)$ only depends on the initial data $u_0$.

Proof. First of all, it is easy to check that
\[
|\partial_y \left( \frac{v^\epsilon}{u^\epsilon} \right) |_{x=0} = \left| \frac{-\partial_y^2 u^\epsilon + u^\epsilon - 1 - \epsilon}{(u^\epsilon)^2} \right|_{x=0} = \left| \frac{-\partial_y^2 u_0 + u_0 - 1}{(u_0 + \epsilon)^2} \right| \leq \left| \frac{-\partial_y^2 u_0 + u_0 - 1}{u_0^2} \right| .
\]
This, together with the compatibility conditions (B.1) and (B.3), yields directly

$$\| \partial_y (\frac{v^\epsilon}{u^\epsilon})(y)^l \|_{L^2_y} \leq C(u_0).$$

(C.2)

Secondly, according to the equation (2.1), we have

$$v^\epsilon = u^\epsilon \int_0^y \frac{-\partial_y^2 u^\epsilon + u^\epsilon - 1 - \epsilon}{(u^\epsilon)^2} \, d\tau,$$

which implies directly

$$\partial_y v^\epsilon = \partial_y u^\epsilon \int_0^y \frac{-\partial_y^2 u^\epsilon + u^\epsilon - 1 - \epsilon}{(u^\epsilon)^2} \, d\tau + \frac{-\partial_y^2 u^\epsilon + u^\epsilon - 1 - \epsilon}{u^\epsilon}.$$

Then, it holds

$$|\partial_y v^\epsilon|_{x=0} \leq |\partial_y u_0| \int_0^y \left| \frac{-\partial_y^2 u_0 + u_0 - 1}{(u_0 + \epsilon)^2} \right| \, d\tau + \left| \frac{-\partial_y^2 u_0 + u_0 - 1}{u_0 + \epsilon} \right|,$$

which, together with compatibility condition (B.3), yields directly

$$\| \partial_y v^\epsilon \|_{L^2_y} \leq C(u_0).$$

(C.3)

Similarly, it is easy to deduce that

$$\| \partial_y^2 v^\epsilon \|_{L^2_y} + \| \partial_y^3 v^\epsilon \|_{L^2_y} \leq C(u_0).$$

(C.4)

It is easy to check that

$$\partial_{xy} v^\epsilon = -\partial_y^2 v^\epsilon \int_0^y \frac{-\partial_y^2 u^\epsilon + u^\epsilon - 1 - \epsilon}{(u^\epsilon)^2} \, d\tau + \frac{(\partial_y^3 v^\epsilon - \partial_y v^\epsilon) u^\epsilon + (-\partial_y^2 u^\epsilon + u^\epsilon - 1 - \epsilon) \partial_y v^\epsilon}{(u^\epsilon)^2}$$

$$+ \partial_y u^\epsilon \int_0^y \frac{(\partial_y^3 v^\epsilon - \partial_y v^\epsilon)(u^\epsilon)^2 + 2u^\epsilon(-\partial_y^2 u^\epsilon + u^\epsilon - 1 - \epsilon) \partial_y v^\epsilon}{(u^\epsilon)^2} \, d\tau.$$

This, together with the estimates (C.2), (C.4) and compatibility condition (B.7), yields directly

$$\| \partial_{xy} v^\epsilon \|_{L^2_y} \leq C(u_0).$$

(C.5)

Similarly, it is easy to deduce that

$$\| \partial_x \partial_y v^\epsilon \|_{L^2_y} + \| \partial_x \partial_y^3 v^\epsilon \|_{L^2_y} \leq C(u_0).$$

(C.6)

Finally, using the equation (2.1), it is easy to check that

$$|\partial_y^{\epsilon} \partial_{xy} (\frac{v^\epsilon}{u^\epsilon})|_{x=0} \leq \frac{2 \epsilon u^\epsilon \partial_y (\frac{v^\epsilon}{u^\epsilon}) \partial_y v^\epsilon + \partial_y^3 v^\epsilon - \partial_y v^\epsilon}{u^\epsilon} \bigg|_{x=0} \leq 2 \epsilon \partial_y (\frac{v^\epsilon}{u^\epsilon}) \bigg|_{x=0} \partial_y v^\epsilon \bigg|_{x=0} + \frac{\partial_y^3 v^\epsilon - \partial_y v^\epsilon}{u_0 + \epsilon} \bigg|_{x=0},$$

This, together with the estimates (C.2), (C.4) and compatibility condition (B.7), yields directly

$$\| \partial_x \partial_y (\frac{v^\epsilon}{u^\epsilon}) \|_{L^2_y} \leq C(u_0).$$

Similarly, we can also obtain

$$\| \partial_{xy} (\frac{v^\epsilon}{u^\epsilon}) \|_{L^2} \leq C(u_0).$$

The other terms in $\mathcal{E}_f^\epsilon(0)$ and $\mathcal{B}_f^\epsilon(0)$ can be controlled by the similar way, for the sake of simplicity, we refrain from writing. Therefore, we complete the proof of the claim estimate (C.1).
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