On the estimates of the Dunkl Kernel

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Abstract
In this paper, we are interested in the estimates of the Dunkl Kernel on some special sets, following the work of M.F.E. de Jeu and M. Rösler in [3].

1 Introduction

Dunkl (1990-1991) originally defined a family of differential-difference operators associated to a finite reflection group on a Euclidean space, that eventually became associated with his name. The eigenfunctions of the Dunkl operators, that known as Dunkl kernel, were first considered by Dunkl [6] and have been later intensively studied and investigated by several authors [2, 5, 3, 8]. One of the principal problem that arises in the study of the Dunkl’s kernels is the asymptotic behaviors of these functions, which were known only for the reflection group $\mathbb{Z}^2_n$, and conjectured to have an extensions to all reflection groups. In this work we take up this problem, we obtain sharp estimates when we restricted us to cones Lie in the interior of the Weyl chamber.

Let us begin with a few definitions and results as preliminary material. General references are [1, 6, 3, 8, 9, 11].

Let $G \subset O(\mathbb{R}^n)$ be a finite reflection group associated to a reduced root system $R$ and $k : R \to [0, +\infty)$ be a $G$–invariant function (called multiplicity function). Let $R^+$ be a positive root subsystem. The Dunkl operators $D^k_\xi$ on $\mathbb{R}^n$ are the following $k$–deformations of directional derivatives $\partial_\xi$ by difference operators:

$$D^k_\xi f(x) = \partial_\xi f(x) + \sum_{\nu \in R^+} k(\nu) \langle \nu, \xi \rangle \frac{f(x) - f(\sigma_\nu x)}{\langle \nu, x \rangle}, \quad (1.1)$$

where here $\sigma_\nu$ is the reflection with respect to the hyperplane orthogonal to $\nu$ and $\langle . , . \rangle$ is the usual Euclidean inner product with $| . |$ its induced norm. The operators

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\( \partial_\xi \) and \( D_k^\xi \) are intertwined by a Laplace–type operator

\[
V_k f(x) = \int_{\mathbb{R}^n} f(y) \, d\nu_x(y)
\]

associated to a family of compactly supported probability measures \( \{ \nu_x \mid x \in \mathbb{R}^n \} \).

Specifically, \( \nu_x \) is supported in the convex hull \( \text{co}(G.x) \).

For every \( y \in \mathbb{C}^n \), the simultaneous eigenfunction problem

\[
D_k^\xi f = \langle y, \xi \rangle f, \quad \forall \xi \in \mathbb{R}^n,
\]

has a unique solution \( f(x) = E_k(x, y) \) such that \( E_k(0, y) = 1 \), called the Dunkl kernel and is given by

\[
E_k(x, y) = V_k(e^{\langle \cdot, y \rangle})(x) = \int_{\mathbb{R}^n} e^{\langle z, y \rangle} \, d\nu_x(z) \quad \forall x \in \mathbb{R}^n. \tag{1.2}
\]

When \( k = 0 \) the Dunkl kernel \( E_k(x, y) \) reduces to the exponential \( e^{\langle x, y \rangle} \). Furthermore this kernel has a holomorphic extension to \( \mathbb{C}^d \times \mathbb{C}^d \) and the following estimate hold:

(i) \( E_k(x, y) = E_k(y, x) \),

(ii) \( E_k(\lambda x, y) = E_k(x, \lambda y) \), for \( \lambda \in \mathbb{C} \)

(iii) \( E_k(g.x, g.y) = E_k(x, y) \), for \( g \in G \).

(iv) If we design by \( x^+ \) the intersection point of any orbit \( G.x \) with the closure of the weyl chamber \( \overline{C} \), then for \( z \in \mathbb{C} \)

\[
|E_k(zx, y)| \leq e^{Re(z)(x^+, y^+)}. \tag{1.3}
\]

In dimension \( d = 1 \), these functions can be expressed in terms of hypergeometric function \(_1F_1\), specially

\[
E_k(x, y) = E_k(xy) = e^{\gamma y} \, _1F_1(2k, 2k + 1, -2xy)
\]

and from the behavior of \(_1F_1\) (see, e.g. [1]) one can deduce the estimates

\[
|E_k(xy)| \leq c \frac{e^{\gamma |xy|}}{|xy|^k}; \quad |E_k(ixy)| \leq c |xy|^{-k}, \quad x, y \in \mathbb{R} \setminus \{0\}. \tag{1.4}
\]

The subject is then a generalisation of these estimates to any reflection group. Then \([1,4]\) become

\[
|E_k(x, y)| \leq c \frac{e^{\langle x^+, y^+ \rangle}}{w_k(x)w_k(y)}; \quad |E_k(ix, y)| \leq c \frac{e}{\sqrt{w_k(x)w_k(y)}}, \quad x, y \in \mathbb{R}^n \setminus \bigcup_{\alpha \in R^+} H_\alpha. \tag{1.5}
\]

where

\[
w_k(x) = \prod_{\nu \in R^+} \frac{\langle \nu, x \rangle}{| \langle \nu, x \rangle |^{2k(\nu)}}
\]
which is $G$-invariant and homogeneous of degree $2\gamma_k$,

$$\gamma_k = \sum_{\alpha \in \mathbb{R}^+} k(\alpha).$$

In this paper we shall prove that (1.5) hold for every $x, y \in C_\delta, \delta > 0$ where

$$C_\delta = \{ x \in C; \langle x, \alpha \rangle \geq \delta |x| \}$$

with constant $c$ depends only on $\delta$. The motivation of studying the asymptotic behavior at infinity arises from the work of De Jeu and Rösler in [3] where they proved the following result

**Theorem 1.1.** There exists a constant non-zero vector $v = (v_g)_{g \in G}$ such that for all $y \in C$ and $g \in G$

$$\lim_{|x| \to \infty, x \in C_\delta} \sqrt{w_k(x)w_k(y)} e^{-ixgy} E_k(ix, gy) = v_g.$$

2 The main estimates for the Dunkl kernel

In this work we may assume that $\gamma > 0$ and the root system $R$ engender the space $\mathbb{R}^n$. Let $\Delta$ be the basis of $\mathbb{R}^n$ consists of the simple roots of $R$. Recall that every root system has a set of simple root such that for each root may be written as a linear combination of simple roots with coefficients all of the same sign, ( see, e.g. [7]). Consider $(\lambda_i)_{1 \leq i \leq n}$ the dual basis of $\Delta$. Then the fundamental Weyl chamber is given by

$$C = \{ x \in \mathbb{R}^n, x = \sum_{i=1}^{n} x_i \lambda_i, \lambda_i > 0, \forall i = 0, ..., n \}.$$

Let $(v_i)_{1 \leq i \leq n}$ be a family of linearly independent vectors and $\Lambda$ be the convex polytope defined by

$$\Lambda_{v_1, ..., v_n} = \{ x \in \mathbb{R}^n; x = \sum_{i=1}^{n} x_i \nu_i, \ x_i > 0 \}.$$

**Lemma 2.1.** For all $\delta > 0$ there exists a family of linearly independent vectors $(v_i)_{1 \leq i \leq n}$, such that

$$C_\delta \subset \Lambda_{v_1, ..., v_n}$$

**Proof.** Let $\Pi_\delta$ be the set,

$$\Pi_\delta = \{ x \in \mathbb{R}^n; |x| = 1, \langle x, \alpha \rangle \geq \delta |x| \}$$

and put $\lambda = \sum_{i=1}^{n} \lambda_i$. For all integer $p \geq 1$ define the vectors

$$v_{p,i} = \lambda_i + \frac{\lambda}{p}, \ i = 1, ..., n.$$
It is easy to see that the vectors $v_{p,i}$ are linearly independent and

$$v_{p,i} = v_{p+1,i} + \frac{1}{p(2p+1)} \sum_{j=1}^{n} v_{p+1,j}. \quad (2.1)$$

for all $i = 1, \ldots, n$ and $p \geq 1$. Denote $\Lambda^p = \Lambda_{v_{p,1}, \ldots, v_{p,n}}$. It follows from (2.1) that $\Lambda^p \subset \Lambda^{p+1}$.

Next we claim that $\bigcup_p \Lambda^p = C$.

Clearly $\bigcup_p \Lambda^p \subset C$. If $x \in C$, $x = \sum_{i=1}^{n} x_i \lambda_i$ with $x_i > 0$ then there exists $p \geq 1$ such that

$$x_i - \sum_{i=1}^{n} \frac{x_i}{p} > 0$$

and so

$$x = \sum_{i=1}^{n} \left( x_i - \sum_{i=1}^{n} \frac{x_i}{p} \right) v_{p,i} \in \Lambda^p.$$

Now since $\Pi_\delta \subset \bigcup_p \Lambda^p$

and $\Pi_\delta$ is compact then there exists $p_0$ such that $\Pi_\delta \subset \Lambda^{p_0}$ and by convexity $C_\delta \subset \Lambda^{p_0}$. This conclude the proof of Lemma 2.1.

**Lemma 2.2.** For all $x, y \in C$ and $g \in G$ the function $t \to t^{\gamma} e^{-t(x,y)} E_k(tx, gy)$, $t \geq 0$ is bounded.

**Proof.** The lemma follows using the Phragmén-Lindelöf Theorem (see, e.g. [11, section 5.61]) by considering the functions of a complex variable

$$g(z) = z^{\gamma} e^{-z(x,y)} E_k(zx, gy), \quad z \in H = \{ z \in \mathbb{C}, \ Re(z) \geq 0 \}.$$

Indeed, by (1.3) we have

$$|g(z)| \leq |z|^{\gamma} = O(e^{z^\delta}), \quad |z| \to \infty, \quad \forall \, \delta > 0$$

and from [3 Corollary 1], the function $t \to t^{\gamma} e^{-z(x,y)} E_k(zx, gy)$ is bounded.

**Lemma 2.3.** Given a polytope convex $\Lambda_{v_1, \ldots, v_n}$ there exists a constant $c > 0$ such that for all $g \in G$ and $x, y \in \Lambda_{v_1, \ldots, v_n}$, $x = \sum_{i=1}^{n} x_i v_i$, $y = \sum_{i=1}^{n} y_i v_i$.

$$0 \leq E_k(x, g, y) \leq c \frac{e^{(x, y)}}{\prod_{i,j=1}^{n} (x_i y_i)^{\gamma/n}} \quad (2.2)$$
**Proof.** By using Holder’s inequality in the integral formula (1.2) it follows that

\[
E_k(x, y) \leq \prod_{i=1}^{n} E_k(nx_i v_i, y) \left(\frac{1}{n} \right)^{1/n} \leq \prod_{i=1}^{n} \prod_{j=1}^{n} E_k(n^2 x_i y_j v_i, g v_j) \left(\frac{1}{n^2} \right)^{1/n^2} \leq \prod_{i=1}^{n} \prod_{j=1}^{n} \left(\left(\frac{n^2(x_i y_j)}{\gamma_k} e^{-n^2 x_i y_j(v_i g v_j)} E_k(n^2 x_i y_j v_i, g v_j)\right) \left(\frac{1}{n^2} \right)^{1/n^2} \right) e^{(x, y)} \prod_{i,j=1}^{n} \frac{e^{(x, y)}}{\prod_{i,j=1}^{n} |x_i y_j|^{\gamma_k/n^2}}.
\]

Then conclude Lemma 2.3 from Lemma 2.2.

Next for \(x = \sum_{i=1}^{n} x_i v_i\) we put

\[x^* = \sum_{i=1}^{n} |x_i| v_i.\]

Proceeding as in the above lemma one can state

**Lemma 2.4.** If \(G\) contains \(-1_{\mathbb{R}^n}\) then the Dunkl kernel satisfies the following estimate

\[0 \leq E_k(x, y) \leq c \frac{e^{(x^*, y^*)}}{\prod_{i,j=1}^{n} |x_i y_j|^\gamma_{k/n}} \]

For all \(x, y \in \mathbb{R}^n\) if \(G\) contains \(-1_{\mathbb{R}^n}\) then the Dunkl kernel satisfies the following estimate

**Theorem 2.5.** Given a polytope convex \(\Lambda_{\xi_1, \ldots, v_n}\) there exists a constant \(c > 0\), depending only on the choice of the vectors \(v_i\) such that

\[E_k(x, g y) \leq C \frac{e^{(x, y)}}{\sqrt{w_k(x) w_k(y)}}, \quad (2.3)\]

for all \(x, y \in \Lambda_{\xi_1, \ldots, v_n}\) and \(g \in G\).

**Proof.** Choose a vectors \(\xi_1, \ldots, \xi_n \in C\), linearly independent such that

\[\Lambda_{\xi_1, \ldots, v_n} \subset \Lambda_{\xi_1, \ldots, \xi_n}.\]

Consider \(\xi_1, \ldots, \xi_n\) as a basis of \(\mathbb{R}^n\) and let the sets

\[H_{p, i} = \{ x \in \Lambda_{\xi_1, \ldots, \xi_n}, \ x = \sum_{i=1}^{n} x_i \xi_i, \ x_1/p \leq x_i \leq px_1 \}, \quad p \in \mathbb{N}^*.\]

Clearly \(H_{p, i} \uparrow \Lambda_{\xi_1, \ldots, \xi_n}\) as \(p \to +\infty\). Thus one can find \(p_i\) such that

\[\Lambda_{\xi_1, \ldots, v_n} \subset H_{p_i, i}. \quad (2.4)\]
For $x = \sum_{i=1}^{n} x_i \xi_i \in \Lambda_{\varepsilon_1, \ldots, \varepsilon_n}$ and $\alpha \in R^+$, with the use of (2.4) we have
\[
\langle x, \alpha \rangle = \sum_{i=1}^{n} x_i \langle \xi_i, \alpha \rangle \leq \max_{i} \langle \xi_i, \alpha \rangle \sum_{i=1}^{n} x_i \leq c \left( \prod_{i=1}^{n} x_i \right)^{1/n}
\]
Now from Lemma 2.3
\[
E_k(x, g, y) \leq c \frac{e^{(x, y)}}{\prod_{i,j=1}^{n} (x_i y_i)^{\gamma_k/n}} 
\leq c \frac{e^{(x, y)}}{\prod_{\alpha \in R^+} (\prod_{i=1}^{n} x_i)^{\kappa_\alpha/n} (\prod_{i=1}^{n} y_i)^{\kappa_\alpha/n}} 
\leq c \frac{e^{(x, y)}}{\sqrt{w_k(x)w_k(y)}}
\]
which is the desired estimate.

We came now to the second part of this work, that is the behavior of the kernel $E_k(ix, y)$. The main result is the following

Theorem 2.6. For $g \in G$ there exists a constant non-zero vector $v_g$ such that for each $\delta > 0$
\[
\lim_{|x| \times |y| \to +\infty; \ x, y \in C_\delta} \sqrt{w_k(x)w_k(y)} e^{-i(x, gy)} E_k(ix, gy) = v_g. \tag{2.5}
\]
Proof. We proceed as in the proof of [3] Theorem 1. Keeping the notations of [3] we consider the function
\[
F_g(x, y) = \sqrt{w_k(x)w_k(y)} e^{-i(x, gy)} E_k(ix, gy); \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n
\]
and $F = (F_g)_{g \in G}$. According to [3] Lemma 1, if $\xi = (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^n$, then
\[
\partial_\xi F_g(x, y) = \sum_{\alpha \in R^+} k(\alpha) \left( \frac{\langle \alpha, \xi_1 \rangle}{\langle \alpha, x \rangle} + \frac{\langle \alpha, g \xi_2 \rangle}{\langle \alpha, y \rangle} \right) e^{-i(\alpha, x)\langle \alpha, gy \rangle} F_{\sigma_\alpha g}(x, y).
\]
Let $\delta > 0$ and $\kappa = (\kappa_1, \kappa_2)$ be a curve of $\mathbb{R}^n \times \mathbb{R}^n$ such that $\kappa_1, \kappa_2 : (0, +\infty) \to C_\delta$ are two admissible curves in the sense given in [3]. Define $F^\kappa(t) = (F_g)_{g \in G}$ where $F^\kappa_g(t) = F_g(\kappa_1(t), \kappa_2(t))$. We have
\[
(F^\kappa_g)'(t) = \sum_{\alpha \in R^+} k(\alpha) \left( \frac{\langle \alpha, \kappa_1'(t) \rangle}{\langle \alpha, \kappa_1(t) \rangle} + \frac{\langle \alpha, g \kappa_2'(t) \rangle}{\langle \alpha, g \kappa_2(t) \rangle} \right) e^{-i(\alpha, \kappa_1(t))\langle \alpha, g \kappa_2(t) \rangle} F_{\sigma_\alpha g}^\kappa(t)
\]
and $F^\kappa$ satisfies the differential equation
\[
(F^\kappa)'(t) = A^\kappa(t) F^\kappa(t)
\]
where the matrix $A^\kappa(t)$ is given by $A(t) = \sum_{\alpha \in R^+} k(\alpha) B^\kappa_{\alpha}(t)$ and
\[
B_{g,h}(t) = \left\{ \begin{array}{ll}
\langle \alpha, \kappa_1'(t) \rangle + \langle \alpha, g \kappa_2'(t) \rangle & \text{if } h = \sigma_\alpha g, \\
0 & \text{otherwise.}
\end{array} \right.
\]
We will try to apply the proposition 1 of [3]. For arbitrary $t > 0$

$$\int_{t}^{+\infty} B_{g,\sigma,g}(s) ds = \int_{t}^{+\infty} \left( \frac{\langle \alpha, \kappa_{1}(t) \rangle + \langle \alpha, g\kappa_{2}(t) \rangle}{\langle \alpha, \kappa_{1}(t) \rangle} \right) e^{-i(\alpha,\kappa_{1}(t))\langle \alpha, g\kappa_{2}(t) \rangle}$$

$$= \int_{[\langle \alpha, \kappa_{1}(t) \rangle \langle \alpha, g\kappa_{2}(t) \rangle]}^{+\infty} \frac{e^{-i \text{sign}(g^{-1}\alpha) u}}{u} du$$

where for $\beta \in R$

$$\text{sign}(\beta) = \begin{cases} 1, & \text{if } \beta \in R^{+} \\ -1, & \text{if } \beta \in R^{-}. \end{cases}$$

This integral exists. Next we are led to examine the integrability of

$$I_{a,\beta,g}(t) = \left( \frac{\langle \alpha, \kappa_{1}(t) \rangle + \langle \alpha, g\kappa_{2}(t) \rangle}{\langle \alpha, \kappa_{1}(t) \rangle} \right) e^{-i(\alpha,\kappa_{1}(t))\langle \alpha, g\kappa_{2}(t) \rangle} \int_{[\langle \beta, \kappa_{1}(t) \rangle \langle \beta, \sigma_{a}g\kappa_{2}(t) \rangle]}^{+\infty} \frac{e^{-i \text{sign}(g^{-1}\alpha) u}}{u} du$$

with $g \in G$ and $\alpha, \beta \in R^{+}$. Observe that

$$\left| \int_{[\langle \beta, \kappa_{1}(t) \rangle \langle \beta, \sigma_{a}g\kappa_{2}(t) \rangle]}^{+\infty} \frac{e^{-i \text{sign}(g^{-1}\alpha) u}}{u} du \right| \leq \frac{2}{|\langle \beta, \kappa_{1}(t) \rangle \langle \beta, \sigma_{a}g\kappa_{2}(t) \rangle|} \leq \frac{c}{|\langle \alpha, \kappa_{1}(t) \rangle \langle \alpha, g\kappa_{2}(t) \rangle|}$$

since $|\langle \beta, \sigma_{a}g\kappa_{2}(t) \rangle| = |\langle g^{-1}\sigma_{a}\beta, \kappa_{2}(t) \rangle| \geq \delta |\kappa_{2}(t)| \geq \delta |\langle \alpha, g\kappa_{2}(t) \rangle| \sqrt{2}$ and similarly $|\langle \beta, \kappa_{1}(t) \rangle| = \delta |\langle \alpha, \kappa_{1}(t) \rangle| \sqrt{2}$. Then

$$|I_{a,\beta,g}(t)| \leq c \text{sign}(g^{-1}\alpha) \left( \frac{\langle \alpha, \kappa_{1}(t) \rangle \langle \alpha, g\kappa_{2}(t) \rangle + \langle \alpha, g\kappa_{2}(t) \rangle \langle \alpha, \kappa_{1}(t) \rangle}{(\langle \alpha, \kappa_{1}(t) \rangle \langle \alpha, g\kappa_{2}(t) \rangle)^{2}} \right)$$

and for $t_{0} > 0$

$$\int_{t_{0}}^{+\infty} |I_{a,\beta,g}(t)| dt \leq \frac{c}{(\langle \alpha, \kappa_{1}(t_{0}) \rangle \langle \alpha, g\kappa_{2}(t_{0}) \rangle)} .$$

Applying now the proposition of [3], we conclude that

$$\lim_{t \to +\infty} F_{g}^{n}(t)$$

exists and different from zero. It remains to justify that is independent of the choice of the curves $\kappa_{1}$ and $\kappa_{2}$. Also this can be do by the same argument of the proof of Theorem 1 of [3]. Given $\ell_{1}$ and $\ell_{2}$ be an other admissible curves in $C_{\delta}$. One can construct an admissible sequences $(x_{n})_{n}$ and $(y_{n})_{n}$ in $C_{\delta}$ such that $(x_{n+1}, y_{n+1}) \in (\kappa_{1}, \kappa_{2})$ and $(x_{n}, y_{n}) \in (\ell_{1}, \ell_{2})$. Let $r_{1}$ and $r_{2}$ be an interpolating curves respectively of $(x_{n})_{n}$ and $(y_{n})_{n}$. Thus we may have

$$\lim_{t \to -\infty} F_{g}(\kappa_{1}(t), \kappa_{2}(t)) = \lim_{t \to -\infty} F_{g}(\ell_{1}(t), \ell_{2}(t)) = \lim_{t \to -\infty} F_{g}(r_{1}(t), r_{2}(t)).$$

In particular if we choose $\kappa_{1}(t) = tx$ and $\kappa_{2}(t) = ty$ for fixed vectors $x$ and $y$ then from Theorem 1 of [3] there exists a constant non-zero vector $v = (v_{g})_{g \in G} \in \mathbb{C}^{G}$ such that

$$\lim_{t \to -\infty} F_{g}(\kappa_{1}(t), \kappa_{2}(t)) = \lim_{t \to -\infty} F_{g}(t^{2}x, y) = v_{g}.$$
Let us now show that
\[
\lim_{|x| \times |y| \to +\infty; x,y \in C_\delta} \sqrt{w_k(x)w_k(y)} e^{-\langle ix, gy \rangle} E_k(ix, gy) = v_g
\]
Indeed, if it is not, then we can find \( \varepsilon > 0 \) and sequences \((x_n)_n\) and \((y_n)_n\) of \( C_\delta \) such that \(|x_n| \times |y_n| \to +\infty\) and
\[
|F_g(x_n, y_n) - v_g| > \varepsilon
\]
we can assume that \(|x_n| \to +\infty\) and \(|y_n| \to +\infty\), since we have
\[
F_g(x_n, y_n) = F_g(\sqrt{|x_n||y_n|} x'_n, (\sqrt{|x_n||y_n|} y'_n)
\]
where \( x'_n = x_n/|x_n| \) and \( y_n = y'_n/|y_n| \). We may also assume that \((x_n)_n\) and \((y_n)_n\) are admissibles. Thus it yield that
\[
\lim_{t \to \infty} F_g(x_n, y_n) = v_g
\]
which is a contradiction.

**Corollary 2.7.** Let \( \delta > 0 \). There exists a constant \( c > 0 \), such that
\[
|E_k(ix, gy)| \leq \frac{c}{\sqrt{w_k(x)w_k(y)}}
\]
for all \( x, y \in C_\delta \) and \( g \in G \).

**Proof.** From Theorem 2.6 we can find \( M > 0 \) such that for all \( x, y \in C_\delta \), \(|x| \times |y| \geq M\) we have
\[
\sqrt{w_k(x)w_k(y)} |E_k(ix, gy)| \leq 2|v_g|
\]
for all \( g \in G \). When \(|x| \times |y| \leq M\), by use of (1.3), one obtains
\[
\sqrt{w_k(x)w_k(y)} |E_k(ix, gy)| \leq \left( \prod_{\alpha \in R^+} |\alpha| \right)^{2\gamma_k} (|x| \times |y|)^{\gamma_k} \leq M_k^\gamma \left( \prod_{\alpha \in R^+} |\alpha| \right)^{2\gamma_k}
\]
Hence (2.6) follows.

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