The Higson-Roe sequence for étale groupoids
I. Dual algebras and compatibility with the BC map

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Abstract

We introduce the dual Roe algebras for proper étale groupoid actions and deduce the expected Higson-Roe short exact sequence. When the action is co-compact, we show that the Roe $C^*$-ideal of locally compact operators is Morita equivalent to the reduced $C^*$-algebra of our groupoid, and we further identify the boundary map of the associated periodic six-term exact sequence with the Baum-Connes map, via a Paschke-Higson map for groupoids. For proper actions on continuous families of manifolds of bounded geometry, we associate with any $G$-equivariant Dirac-type family, a coarse index class which generalizes the Paterson index class and also the Moore-Schochet Connes’ index class for laminations.

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1 Introduction

This paper is a first of a series of articles where we systematically investigate the expected universal Higson-Roe analytic surgery exact sequence for Hausdorff étale groupoids. This first paper is dedicated to the introduction of the dual Roe algebras for proper groupoid actions, and to the identification of the boundary maps appearing in the associated periodic $K$-theory exact sequence, yielding to the notion of coarse $G$-index for Paterson’s continuous $G$-families of bounded geometry manifolds.

The $K$-theory index map can nowadays be more efficiently defined using the language of étale groupoids with their actions on spaces, mostly manifolds [Co94]. For a countable discrete group $\Gamma$ for instance, this correspondence goes back to the work of Mischenko and Kasparov and yields for any $\Gamma$-cover $\tilde{M} \to M$ over a closed manifold $M$, to a map from the $K$-homology group of $M$ to the $K$-theory of the $C^*$-algebra of the group $\Gamma$:

$$\text{Ind}_\Gamma : K_*(M) \to K_*(C^*_\Gamma).$$

Any such $\Gamma$-cover is uniquely determined (up to isomorphism) by a (homotopy class of a) continuous map from $M$ to the classifying space $BG$ with its universal $\Gamma$-cover $EG \to BG$, and one can assemble
all these maps into a universal map \([BC:00]\):

\[
\mu_\Gamma : RK^*_G(EG) \rightarrow K_*(C^*_\Gamma),
\]

where \(RK^*_G(EG) \simeq RK_*(BG)\) stands for the topological \(K\)-homology group of \(BG\), defined using the inductive system of cocompacts closed subspaces, while \(K_*(C^*_\Gamma)\) is the usual topological \(K\)-theory of the reduced \(C^*\)-algebra of \(\Gamma\). This assembly map is known, when \(\Gamma\) is torsion free, to be an isomorphism for a large class of groups, and the so-called Baum-Connes conjecture states that this assembly map should be an isomorphism for all torsion-free countable discrete groups. When the group has torsion, then one has to replace the universal space \(EG\) by the universal space for proper \(\Gamma\)-actions, usually denoted \(\mathcal{E}\Gamma\), and the Baum-Connes conjecture again states that the similar assembly map \(\mu_\Gamma : RK^*_G(\mathcal{E}\Gamma) \rightarrow K_*(C^*_\Gamma)\) is an isomorphism. For more details on this conjecture, see for instance [BCH:94, BC:00].

In their seminal work on “Mapping Surgery to Analysis” [HR:05, HRI:05, HRII:05], N. Higson and J. Roe proved that the assembly map fits in a six-term periodic exact sequence as a boundary map, this allowed them to obtain their universal periodic exact sequence for any such \(\Gamma\). In analogy with the deep fundamental exact sequence of surgery [Wa:70], N. Higson and J. Roe called their sequence the analytic surgery exact sequence and they in particular introduced an analytic structure group which plays the role of the structure group and an obstruction group which vanishes whenever the group \(\Gamma\) satisfies the Baum-Connes conjecture. These results rely in particular on the Voiculescu theorem for ample representations [Vo:78] and also on the Paschke-Higson duality theorem which extends Poincaré duality to the non-smooth setting [P:81, Hig:95]. By using signature operators, N. Higson and J. Roe went further and constructed an explicit “commutative diagram” relating the fundamental sequence of surgery with their analytic exact sequence, so bringing surgery to analysis. In the same lines, P. Piazza and T. Schick proved later on a similar commutative diagram relating now the Stolz exact sequence with the Higson-Roe sequence, and deduced some important results on the moduli space of metrics of positive scalar curvature [PS:13]. Important applications to rigidity conjectures of reduced eta invariants have then been efficiently explored with the help of these new tools, see for instance [HR:10, PS:13, BR:15].

On the other hand, when \(\Gamma\) acts on a locally compact Hausdorff space \(Z\), the assembly map corresponding to this topological dynamical system can be defined using a natural extension of \(\mu_\Gamma\) where one adds the \(C^*\)-algebra \(C_0(Z)\) as coefficients. This natural generalization of the Baum-Connes map can actually be interpreted as the universal index map for an \(\acute{e}tale\) groupoid, namely the transformation groupoid \(Z \rtimes \Gamma\). More generally, there is a well defined assembly map for any locally compact (\(\acute{e}tale\)) groupoid \(G\) which uses a locally compact model for the classifying space \(E\Gamma\) of proper groupoid actions [Tu:99, LeGall:99]. If the unit space is denoted \(X = G^{(0)}\), then the Baum-Connes assembly map for \(G\) is a map:

\[
\mu_G : RK^*_G(E\Gamma, X) \rightarrow K_*(C^*_\Gamma),
\]

where the LHS, say the group \(RK^*_G(E\Gamma, X)\), is again defined using an inductive system for the classifying space \(E\Gamma\) of proper \(G\)-actions, now a limit of bivariant \(KK\)-groups \(KK_*(Y, X)\) over cocompact \(G\)-subspaces \(Y\) of \(E\Gamma\). The RHS is again the \(K\)-theory of the reduced \(C^*\)-algebra of the groupoid \(G\). With no surprise, the construction of the assembly map relies again on the index theory for groupoid actions, see [Co:94]. Notice that the language of groupoid actions allows to encompass the case of discrete countable groups as well as that of foliations and even laminations.

Following the Higson-Roe program, the next step is to introduce the six-term exact sequence for proper actions of \(\acute{e}tale\) groupoids which would incorporate the previous Baum-Connes assembly map for \(G\) as a boundary map. It is our goal here to extend the Higson-Roe constructions and introduce the dual Roe algebras for proper groupoid actions on families of metric spaces \(\rho : Y \rightarrow G^{(0)}\), so as to encompass new geometric situations. Given such a \(G\)-proper space \((Y, \rho)\) such that the anchor map
Theorem 1.2. The compatibility theorem can then be stated as follows:

\[ 0 \to C^*_G(Y, \mathcal{E}) \to D^*_G(Y, \mathcal{E}) \to Q^*_G(Y, \mathcal{E}) \to 0. \]

Our candidate six-term exact sequence is then deduced by applying the topological $K$-functor and by using Bott periodicity:

\[ K_*(C^*_G(Y, \mathcal{E})) \xrightarrow{\partial_*} K_*(D^*_G(Y, \mathcal{E})) \xrightarrow{\cong} K_*(Q^*_G(Y, \mathcal{E})) \]

As an important application, we consider the Paterson category of families of smooth bounded geometry manifolds with $G$-actions and we deduce that the coarse $G$-index of $G$-invariant families $\mathcal{D}$ of fully elliptic operators acting on the sections of a $C^{0,0}$-bundle $E$ is well defined:

\[ \text{Ind}_{G}(\mathcal{D}) \in K_*(C^*_G(Y, \mathcal{E}_Y,E)). \]

The next important result proved in the present paper is the compatibility of the boundary map $\partial_*$ with the Baum-Connes map for the groupoid $G$. To this end, one has to choose specific Hilbert modules which are naturally associated with $Y$. More precisely, given a full $G$-equivariant $\rho$-system $(\mu_x)_{x \in X}$ on $Y$, and denoting by $\mathcal{E}_Y^\prime$ the Hilbert $G$-module corresponding to the continuous field of Hilbert spaces over $X$ given by $\left( L^2(Y_x, \mu_x) \otimes \ell^2(G^*) \right)_{x \in X}$, we obtain the following theorem.

**Theorem 1.1.** Assume that $Y$ is $G$-compact, then the dual Roe algebra $C^*_G(Y, \mathcal{E}_Y^\prime)$ is Morita equivalent to the reduced $C^*$-algebra of the groupoid $G$. In particular, we have an isomorphism

\[ \mathcal{M}_* : K_*(C^*_G(Y, \mathcal{E}_Y^\prime)) \xrightarrow{\cong} K_*(C^*_G). \]

When the action of $G$ on $Y$ is for instance free, then this theorem holds already with the Hilbert $G$-module $\mathcal{E}_Y$ associated with the continuous field of Hilbert spaces $\left( L^2(Y_x, \mu_x) \right)_{x \in X}$. Using the isomorphism, we see that our class $\text{Ind}_{G}(\mathcal{D})$ extends to the non-cocompact bounded geometry case, the Paterson $G$-index and hence in the case of bounded geometry laminations the Moore-Schochet Connes’ index class, which was previously only defined when the ambient space is compact. In the presence of a fiberwise $G$-invariant metric of uniformly positive scalar curvature and under the usual spin assumption, we also define a secondary class living in the structure group $K_*(D^*_G(Y, \mathcal{E}_Y,E))$, extending results of Roe-Higson and Piazza-Schick [HRIII:05, HR:10, PS:13].

On the other hand, the Paschke-Higson morphism can be defined for any Hilbert $G$-module $\mathcal{E}$ as above with the non-degenerate $G$-equivariant representation of $C^*_0(Y)$, and is now valued in a $G$-equivariant $KK$-group [LeGal:99]:

\[ \mathcal{P}_* : K_*(Q^*_G(Y, \mathcal{E})) \xrightarrow{\cong} KK^*_G(Y, X), \]

The compatibility theorem can then be stated as follows:

**Theorem 1.2.** For $* = 0$ and $* = 1$, the following diagram commutes:

\[ \begin{array}{ccc}
K_*(Q^*_G(Y, \mathcal{E}_Y^\prime)) & \xrightarrow{\partial_*} & K_{*+1}(C^*_G(Y, \mathcal{E}_Y^\prime)) \\
\| & \downarrow & \| \\
KK^*_Y(Y, X) & \xrightarrow{\mu_{*+1}} & K_{*+1}(C^*_\text{red}G)
\end{array} \]
When $G = X$ is a space groupoid such that $X$ is compact and metrizable, our results are closely related with the classical results of Pimsner-Popa-Voiculescu [PPV:79]. The case when $G$ is a discrete countable group has also been studied in the recent article [GWY:16] as well as in [Z:17], we thank the referee for pointing out Zenobi’s paper to us. For free and proper discrete group actions, Zenobi showed, using the Paschke isomorphism, that his structure group, roughly speaking the homotopy fiber of the assembly map, is isomorphic to the Higson-Roe structure group. In fact, Zenobi went further and introduced in his thesis a structure group even for Lie groupoids while our definition in the present paper is the exact extension of the Higson-Roe approach to the category of locally compact groupoids. It is therefore an interesting task to compare the Zenobi structure group with ours in the case of Lie groupoids. There always exists an obvious group morphism from our structure group to the Zenobi one, and if the groupoid is torsion-free and satisfies Paschke-Higson duality, then one easily shows that this morphism is an isomorphism, but the general case remains to be investigated. Finally, in order to prove the universal Higson-Roe exact sequence and the corresponding universal commutative diagram, further properties of the Paschke-Higson map are needed. In order to keep this paper in a reasonable size, these properties together with some geometric corollaries, will be investigated in the next articles of this series.

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Preliminaries and notations. The groupoid $G$ will be a locally compact Hausdorff étale groupoid. We shall denote by $X := G^{(0)}$ the space of units of the groupoid $G$ and by $G^{(1)}$ the space of arrows of $G$. Then $X$ is identified with a closed (and open) subspace of $G^{(1)}$ and we shall sometimes also denote by $G$ the space $G^{(1)}$ of arrows of the groupoid $G$. The source and range maps are denoted $s$ and $r$ respectively and are then étale maps from $G^{(1)}$ to $G^{(0)}$. Given subsets $A$ and $B$ of $X$, we denote by $G_A$ and $G_B$ the subspaces of $G$ defined as $s^{-1}A$ and $r^{-1}B$ respectively. The intersection $G_A \cap G_B$ is denoted $G_{A,B}$ and if $A = \{x\}$ then we denote $G_A$ as simply $G_x$ and similarly for the obvious notations $G^x$ and $G^x_{x'}$.

Given a locally compact Hausdorff space $Z$, we denote by $C_0(Z)$ the $C^*$-algebra of continuous complex valued functions vanishing at infinity. As usual, $C_b(Z)$ denotes the $C^*$-algebra of bounded continuous functions on $Z$. For simplicity, and in order to avoid some annoying technicalities, we shall assume that the Tietze theorem applies for all our spaces and all commutative $C^*$-algebras will have countable approximate units. If $A$ is a given $C^*$-algebra and $E$ is a Hilbert $A$-module,
then we denote by $\mathcal{L}_A(\mathcal{E})$ the $C^*$-algebra of adjointable operators on $\mathcal{E}$, while $\mathcal{K}_A(\mathcal{E})$ is the ideal of $A$-compact operators, see [Ka:80, La:95] for more details on the properties of adjointable operators.

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2 Dual algebras for étale groupoids

We shall freely use the results given in Appendix A, this is an overview of some needed constructions on $G$-algebras, Hilbert $G$-modules and $G$-representations.

2.1 $G$-spaces

The first class of $G$-algebras that we shall use in this paper is given by the commutative ones.

Definition 2.1 (G-space). A (Hausdorff) topological space $Y$ is a (right) $G$-space if we are given:

1. a continuous map $\rho : Y \to X$, called the anchor map;
2. a continuous map $\lambda : Y \rtimes_r G \to Y$ such that:
   - $\rho(\lambda(y, \gamma)) = s(\gamma)$ and $\lambda(y, x) = y$, $x \in X$ is identified with its image in $G^{(1)}$;
   - $\lambda(\lambda(y, \gamma), \gamma') = \lambda(y, \gamma \gamma')$, if $(\gamma, \gamma') \in G^{(2)}$ with $\rho(y) = r(\gamma)$,
where $Y \rtimes_r G := \{(y, \gamma) \in Y \times G / \rho(y) = r(\gamma)\}$.

We shall write $\lambda(y, \gamma)$ as simply $y_\gamma$ and refer to the $G$-space $(Y, \rho, \lambda)$ simply as $(Y, \rho)$ or sometimes as just $Y$. Associated with any such $G$-space, there is an equivalence relation $\sim$ defined by

$$y \sim y' \iff \exists \gamma \in G^{\rho(y)}, y_\gamma = y'_1.$$

The equivalence classes are also called $G$-orbits and the quotient space is then endowed with its quotient topology.

Remark 2.2. We may assume in the previous definition that $\rho$ is surjective since only the $G$-saturated subspace $\rho(Y)$ of $X$ and the subgroupoid $G^{\rho(Y)}$ would be involved. Moreover, in most of the interesting examples for us, the anchor map is open and surjective. We shall therefore restrict ourselves to the case of an open surjective anchor map so as to avoid u.s.c. fields which are not continuous in the sense of Dixmier.

The following proposition is then standard and the proof is a straightforward verification which is omitted.

Proposition 2.3. Let $Y$ be a $G$-space as above. Then $C_0(Y)$ is a $G$-algebra, more precisely:

1. We have $C_0(G)$-algebra identifications
   $$s^* C_0(Y) \cong C_0(Y \rtimes_s G) \text{ and } r^* C_0(Y) \cong C_0(Y \rtimes_r G).$$
2. The $G$-algebra structure $\alpha_Y : s^* C_0(Y) \to r^* C_0(Y)$ is defined by
   $$\alpha_Y(\varphi)(y, g) = \varphi(yg, g), \text{ for } \varphi \in C_c(Y \rtimes_s G).$$
Remark 2.4. The proof of Proposition 2.3 uses the fact that the restriction maps $C_0(Y \times G) \to C_0(Y \times_s G)$ and $C_0(Y \times G) \to C_0(Y \times_s G)$ induce the announced isomorphisms.

A locally compact Hausdorff $G$-space $Y$ is called a proper $G$-space (we also say that the $G$-action is proper) when the map

$$Y \times_r G \longrightarrow Y \times Y$$

given by $(y, \gamma) \mapsto (y\gamma, y)$, is proper. The $G$-space $Y$ is a free $G$-space (we also say that the $G$-action is free) if the above map is injective, i.e. for any $y \in Y$, the isotropy group

$$G(y) := \{ \gamma \in G^{\rho(y)}, y\gamma = y \},$$

is reduced to $\{ \rho(y) \}$. The space of orbits for a proper $G$-action on $Y$ is then Hausdorff. A proper locally compact Hausdorff $G$-space $Y$ is $G$-compact if the quotient space of orbits is compact.

Definition 2.5 (Cutoff function). A cutoff function for a $G$-space $(Z, \rho)$ is a continuous map $c : Z \to [0, 1]$ such that:

$$\sum_{g \in G^{\rho(z)}} c(zg) = 1, \quad \forall z \in Z.$$

and such that for any compact subspace $K$ of $Z$ the space $(K \cdot G) \cap \text{Supp}(c)$ is compact.

We recall that proper $G$-spaces always have cutoff functions [Tu:99], in the $G$-compact case, these cutoff functions are then compactly supported.

2.2 Metric $G$-spaces and Roe algebras

All the $G$-spaces that will be considered are associated with $G$-equivariant continuous fields of commutative $C^*$-algebras over $X$ in the sense of Dixmier. This imposes for such $G$-space $(Y, \rho)$ that the anchor map $\rho$ is open. As explained above, we shall assume that it is also surjective.

Definition 2.6. A locally compact (right) $G$-space $(Y, \rho)$ with the anchor map $\rho : Y \to X$ is a $G$-family of proper metric spaces if we are given a continuous scalar valued function $d_Y$ on the closed subset $\{(y, y') \in Y^2, \rho(y) = \rho(y')\}$ of $Y^2$ such that

1. For any fiber $Y_x := \rho^{-1}(x)$ of $\rho$, the restriction $d_x$ of $d_Y$ to $Y_x \times Y_x$ is a distance which defines the induced topology of $Y_x$.

2. (properness) Any closed bounded subset $Z$ of $Y$ such that $\rho(Z)$ is compact in $X$, is itself compact in $Z$. Boundedness means that $\sup_{x \in X} \text{diam}_{d_{\rho(x)}}(\rho^{-1}(x) \cap Z) < +\infty$.

3. (invariance) For any $g \in G$ and any $(y, y') \in Y^2_{r(g)}$, we have $d_{s(g)}(gy, gy') = d_{r(g)}(y, y')$.

In his proof of the Novikov conjecture for (hyper)bolic groupoids [Tu:99], Jean-Louis Tu introduced in the general framework of topological spaces the notion of a continuous family of metric spaces together with an isometric action of a (étale) groupoid $G$ by imposing the first and third axioms above. In our case, we restrict ourselves to the locally compact case and we moreover impose the properness axiom which is the usual condition needed to define the coarse Roe algebras. So the terminology “$G$-family” includes already that $G$ acts isometrically. Since $G$ acts properly on $Y$, an easy argument shows indeed that any metric structure $d_Y$ satisfying the first and second axioms gives rise a metric structure $d'_Y$ which also satisfies the last axiom.

If $Z_1, Z_2$ are two subspaces of $Y$, then the distance $d_Y(Z_1, Z_2)$ will be the fiberwise distance given when $\rho(Z_1) \cap \rho(Z_2) \neq \emptyset$ by

$$d_Y(Z_1, Z_2) = \inf\{d_Y(z_1, z_2) | (z_1, z_2) \in Z_1 \times Z_2, \rho(z_1) = \rho(z_2)\}.$$
So, if $\rho(Z_1) \cap \rho(Z_2) = \emptyset$, then the convention is $d_Y(Z_1, Z_2) = +\infty$.

Recall that the continuous anchor map $\rho$ is open and surjective and that $(Y, d_Y)$ is a $G$-family of proper metric spaces. Let $E$ be a Hilbert $G$-module which corresponds to the continuous field of Hilbert spaces $(H_x)_{x \in X}$ over $X$ in the sense of Dixmier. So, any element $g \in G$ yields a unitary isomorphism $V_g : H_{s(g)} \rightarrow H_{r(g)}$ with the relation over $G^{(2)}$ recalled in Appendix. Consider now a non-degenerate $G$-equivariant $C_0(X)$-representation $\pi : C_0(Y) \rightarrow \mathcal{L}_{C_0(X)}(E)$.

**Definition 2.7** (Finite propagation). An operator $T \in \mathcal{L}_{C_0(X)}(E)$ has finite propagation with respect to $d_Y$ if there exists a constant $R > 0$ such that

$$
\pi(f)T\pi(g) = 0 \text{ whenever } d_Y(\text{supp}(f), \text{supp}(g)) > R.
$$

for all $f, g \in C_0(Y)$. The least such constant $R > 0$ is the propagation of $T$.

**Remark 2.8.** Since $\pi$ is non-degenerate, it can be extended to a $*$-homomorphism $\tilde{\pi} : C_0(Y) \rightarrow \mathcal{L}_{C_0(X)}(E)$. In the previous definition, we may then equivalently use functions $f, g$ from $C_0(Y)$ and the extended representation $\tilde{\pi}$.

Recall the $C^*$-algebra $[\mathcal{L}_{C_0(X)}(E)]^G$ of $G$-invariant adjointable operators on $E$ defined in the appendix. Recall also that $\mathcal{K}_{C_0(X)}(E)$ is the ideal of $C_0(X)$-compact operators in the Hilbert module $E$.

**Definition 2.9.**

- The equivariant Roe-algebra $D^*_G(Y, E)$ is the norm closure (in $\mathcal{L}_{C_0(X)}(E)$) of the space
  $$
  \{T \in \mathcal{L}_{C_0(X)}(E) \mid T \text{ has finite propagation and } [T, \pi(f)] \in \mathcal{K}_{C_0(X)}(E), \forall f \in C_0(Y)\}.
  $$

- The subspace $C^*_G(Y, E)$ is defined as
  $$
  C^*_G(Y, E) := \{T \in D^*_G(Y, E), T\pi(f) \in \mathcal{K}_{C_0(X)}(E), \forall f \in C_0(Y)\}.
  $$

**Remark 2.10.** Notice that by definition, finite propagation means uniform finite propagation with respect to the $X$-variable.

For any operator $T = (T_x)_{x \in X}$ of $D^*_G(Y, E)$, the operators $T_x$ all belong to the $C^*$-algebras $D^*(Y_x, E_x)$ and satisfy an obvious equivariance property. If we consider the $C^*$-algebras $D^*_X(Y, E)$ and $C^*_X(Y, E)$ defined using the pointwise groupoid structure of the space $X$, which are $G$-algebras, then it is tempting to rather consider the $C^*$-algebra $D^*_X(Y, E)^G$ and $C^*_X(Y, E)^G$ composed of the $G$-invariant elements of $D^*_X(Y, E)$ and $C^*_X(Y, E)$ respectively. However, and even for discrete countable groups, our definition better suits with the Baum-Connes map as we shall see, see also [HRII:05, PS:13].

**Lemma 2.11.** For any Hilbert $G$-module $E$ with a non-degenerate $G$-equivariant $C_0(X)$-representation of the $G$-algebra $C_0(Y)$, $D^*_G(Y, E)$ is a (unital) $C^*$-algebra and $C^*_G(Y, E)$ is closed two-sided involutive ideal in $D^*_G(Y, E)$. Hence, denoting by $Q^*_G(Y, E)$ the quotient $C^*$-algebra, we have the following short exact sequence of $C^*$-algebras:

$$
0 \rightarrow C^*_G(Y, E) \rightarrow D^*_G(Y, E) \rightarrow Q^*_G(Y, E) \rightarrow 0.
$$

**Proof.** If we admit that $D^*_G(Y, E)$ is a unital $C^*$-algebra, then it is clear that $C^*_G(Y, E)$ is a closed two-sided involutive ideal in $D^*_G(Y, E)$, as a consequence of the fact that $\mathcal{K}_{C_0(X)}(E)$ is a closed two-sided involutive ideal in $\mathcal{L}_{C_0(X)}(E)$. Hence, we only need to check that $D^*_G(Y, E)$ is a (unital) $C^*$-algebra. Moreover, by standard arguments, only the proof that composition (and adjoint) of finite propagation operators are finite propagation operators. For elements $S, T \in D^*_G(Y, E)$ having finite propagations $R_S$ and $R_T$ respectively, the propagation of the adjoint $T^*$ coincides with that of $R_T$. 

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and the operator $ST$ also has finite propagation, in fact $\leq R_S + R_T$. The proof is standard. Let for instance $f, g \in C_c(Y)$ be such that $d_V(supp(f), supp(g)) > R_S + R_T$ and set $d_V(supp(f), supp(g)) - (R_S + R_T) = 2\epsilon > 0$. Define the subsets $U_f = \{y \in Y, d_V(y, supp(f)) \leq R_T + \epsilon\}$ and $U_g = \{y \in Y, d_V(y, supp(g)) \leq R_S + \epsilon\}$. From the properness of $d = d_V$, we know that the subsets $U_f$ and $U_g$ are compact (disjoint) subspaces of $Y$. Let then $\varphi$ be a continuous compactly supported function on $Y$ such that

$$\varphi|_{U_f} = 1 \text{ and } \varphi|_{U_g} = 0.$$ 

So, $d_V(supp(f), supp(1 - \varphi)) > R_T + \epsilon > R_T$ and $d_V(supp(g), supp(\varphi)) > R_S + \epsilon > R_S$. Hence, we may write

$$\pi(f)T\pi(g) = \pi(f)T\pi(\varphi)S\pi(g) + \pi(f)T\pi(1 - \varphi)S\pi(g) = 0.$$ 

Therefore, $R_T S \leq R_T + R_S$ as announced. 

\[\square\]

**Example 2.12.** Take $G = X \rtimes \Gamma$ to be an action groupoid, where $\Gamma$ is a countable discrete group which acts by homeomorphisms on a compact space $X$. Consider a space $Y$ of the form $Y = X \times Z$, where $Z$ is a locally compact $\Gamma$-proper space, and take $E$ to be $C(X) \otimes H$ for a fixed unitary $\Gamma$-representation Hilbert space $H$ in which we also have a non-degenerate representation of $C_0(Z)$. Then the $C^*$-algebra $L_{C(X)}(C(X) \otimes H)$ coincides with the $C^*$-algebra $C(X, \mathcal{L}(H)_{s-\text{str}})$ of $\ast$-strongly continuous fields of operators. The Roe algebra $D_{E}(Y, E)$ is then the closure of the space of (uniform) finite propagation $\Gamma$-equivariant elements of $C(X, \mathcal{L}(H)_{s-\text{str}})$.

### 2.3 Relation with the groupoid $C^*$-algebra

Denote again by $\rho : Y \to X$ the open surjective continuous anchor map for the proper $G$-space $Y$. Associated with the proper $G$-space $Y$, there is the locally compact proper $G$-proper space, and take $E$ to be $C(X) \otimes H$ for a fixed unitary $\Gamma$-representation Hilbert space $H$ in which we also have a non-degenerate representation of $C_0(Z)$.

A choice of such system allows to construct a continuous field of Hilbert spaces over $X$, or equivalently a $C_0(X)$-Hilbert module $\mathcal{E}_Y$, as usual. The Hilbert $C_0(X)$-module $\mathcal{E}_Y$ corresponds to the continuous field of Hilbert spaces $(L^2(Y_x, \mu_x), \rho)_x \in X$. This is more precisely defined by completing the pre-Hilbert $C_c(X)$-module $C_c(Y)$ of compactly supported continuous functions on $Y$ [GTX04].

Recall that

$$(\xi f)(y) = \xi(y)f(\rho(y)), \quad f \in C_c(X), \xi \in C_c(Y);$$

and the $C_c(X)$-valued inner product is given by:

$$<\eta, \xi>(x) = \int_{Y_x} \eta(y), \xi(y) > d\mu_x(y), \quad \eta, \xi \in C_c(Y).$$

Notice that the map $y \mapsto \langle \eta(y), \xi(y) \rangle$ then belongs to $C_c(Y)$ and hence by the continuity property of the Haar system, we deduce that $<\eta, \xi>$ belongs to $C_c(X)$. So we get in this way that the completion $\mathcal{E}_Y$ of $C_c(Y)$ with respect to the above pre-Hilbert $C_c(X)$-module structure, is a Hilbert $C_0(X)$-module. Notice that, under our assumptions, none of the Hilbert spaces $L^2(Y_x, \mu_x)$ is trivial and hence by a classical argument, the Hilbert module $\mathcal{E}_Y$ is a full Hilbert module.

Using the properness of the $G$-action on $Y$, it is easy to ensure in addition that the $\rho$-system $\{(\mu_x)_{x \in X}\}$ be $G$-equivariant (see [W15, Proposition 2.5]), i.e.
We shall also call such $\rho$-system an equivariant Haar system. Then the following statement is clear.

**Proposition 2.13.** The module $\mathcal{E}_Y$ is a Hilbert $G$-module and the representation $\pi_Y : C_0(Y) \to \mathcal{L}_{C_0(X)}(\mathcal{E}_Y)$ given by multiplication operators is a $G$-equivariant non-degenerate representation.

**Proof.** That $\pi_Y$ is non-degenerate is clear. Recall the spaces

$$
Y \rtimes_s G := \{(y, g) \in Y \times G, \rho(g) = s(g)\} \quad \text{and} \quad Y \rtimes_r G := \{(y, g) \in Y \times G, \rho(g) = r(g)\}.
$$

There is an isomorphism of Hilbert $C_0(G)$-modules between $s^*\mathcal{E}_Y$ and the completion of $C_c(Y \rtimes_s G)$ with respect to the expected structures. The similar statement holds for $r^*\mathcal{E}_Y$ and the completion of $C_c(Y \rtimes_r G)$. Thus, admitting these identifications, we can define the unitary $V : s^*\mathcal{E}_Y \to r^*\mathcal{E}_Y$, which will automatically be a $C_0(G)$-linear map by setting for any continuous compactly supported function $\eta$ on $Y \rtimes_s G$:

$$(V\eta)(y, g) := \eta(yg, g)
$$

The pre-Hilbert $C_c(G)$-module structure on $C_c(Y \rtimes_s G)$ (and similarly on $C_c(Y \rtimes_r G)$) is given for $\xi, \xi_1, \xi_2 \in C_c(Y \rtimes_s G)$ and $f \in C_c(G)$ by

$$(\xi f)(y, g) := \xi(y, g)f(y) \quad \text{and} \quad (\xi_1, \xi_2) := \int_{Y \rtimes_s G} \overline{\xi_1(y, g)}\xi_2(y, g) \, d\mu_s(y, g).$$

A straightforward verification then shows that the natural map

$$C_c(Y) \otimes C_c(G) \to C_c(Y \times G),$$

induces the above identifications of $s^*\mathcal{E}_Y$ and $r^*\mathcal{E}_Y$. We check using the $G$-invariance of the measures $(\mu_\alpha)_{\alpha \in X}$ that $(V^*\xi)(y, g) = \xi(yg^{-1}, g)$ and hence that $V$ extends to a unitary operator. Again by direct inspection, we obtain $d^*_{ij}s^*\mathcal{E}_Y$ and $d^*_{ij}r^*\mathcal{E}_Y$ by completing respectively with respect to the appropriate $C_0(G^{(2)})$-valued inner product the space of continuous compactly supported functions on respectively the spaces

$$Y \rtimes_s, \pi_{ij} G^{(2)} = \{(y, g_1, g_2) \in Y \times G^{(2)}, s\pi_{ij}(g_1, g_2) = \rho(y)\}
$$

and

$$Y \rtimes_r, \pi_{ij} G^{(2)} = \{(y, g_1, g_2) \in Y \times G^{(2)}, r\pi_{ij}(g_1, g_2) = \rho(y)\}.
$$

Hence we can write

$$V_{01}\eta(y, g_1, g_2) = \eta(ng_1, g_2), V_{12}\eta(y, g_1, g_2) = \eta(ng_2, g_1, g_2), V_{02}\eta(y, g_1, g_2) = \eta(ng_1, g_2, g_1, g_2),$$

which shows that $V$ satisfies the allowed relation for $\mathcal{E}_Y$ to be a Hilbert $G$-module.

Recall on the other hand that the structure of $G$-algebra of $C_0(Y)$ is given by the similar $C_0(G)$-isomorphism $\alpha_Y : \varphi \mapsto [(y, g) \mapsto \varphi(yg, g)]$. Now for any $\varphi \in C_c(Y \rtimes_s G)$ and any $\xi \in C_c(Y \rtimes_r G)$:

$$(V \circ (s^*\pi_Y)(\varphi) \circ V^*)(\xi)(y, g) = \varphi(ng_1, g_2)\xi(y, g) = (\alpha_Y\varphi)(y, g)\xi(y, g) = (r^*\pi_Y)(\alpha_Y(\varphi))(\xi)(y, g),$$

which shows that $\pi_Y$ is a $G$-equivariant representation. \qed
So we get using the specific Hilbert $G$-module $\mathcal{E}_Y$ the dual $C^*$-algebra $D_G^*(Y,\mathcal{E}_Y)$ together with its closed two-sided involutive ideal $C_G^*(Y,\mathcal{E}_Y)$ and the quotient $C^*$-algebra $Q_G^*(Y,\mathcal{E}_Y)$. On the other hand, using the proper (and free) $G$-space $Y' := Y \times_r G$, we also obtain the Hilbert $G$-module $\mathcal{E}_{Y'}$ which corresponds to the field of Hilbert spaces over $Y$ whose fiber at $x \in X$ is $L^2(Y_x,\mu_x) \otimes \ell^2(G^x)$, with the obvious extended representation $\pi_Y \otimes \text{id}$ of $C_0(Y)$ in $\mathcal{E}_{Y'}$. Hence, we also get the corresponding Roe $C^*$-algebras $D_G^*(Y,\mathcal{E}_{Y'})$, $C_G^*(Y,\mathcal{E}_{Y'})$ and $Q_G^*(Y,\mathcal{E}_{Y'})$. Notice indeed that the anchor map for the $G$-space $Y'$ is $\rho' = \rho \circ p_1$ with $p_1 : Y \times_r G \to Y$ being the first projection. On the other hand, recall as well the groupoid structure on $Y'$ which is the crossed product structure, so with $r(y,g) = y$ and $s(y,g) = yg$ and with unit space $Y$. This structure is pulled back from $G$ and we have commutative diagrams where $\pi_2$ is the second projection ($\pi_2(y,g) = g$):

$$
\begin{array}{ccc}
Y \times_r G & \xrightarrow{s,r} & Y \\
\downarrow \pi_2 & & \downarrow \rho \\
G & \xrightarrow{s,r} & X 
\end{array}
$$

We can now state (Compare [Roe:02]):

**Theorem 2.14.** Assume that $Y$ is a proper $G$-space which is $G$-compact. Then the $C^*$-algebras $C^*_c(G(G)$ and $C^*_G(Y,\mathcal{E}_{Y'})$ are Morita equivalent.

We shall more precisely identify the $C^*$-algebra $C^*_G(Y,\mathcal{E}_{Y'})$ with the $C^*$-algebra of compact operators of some full Hilbert $C^*_c(G)$-module $L^2_G(Y')$. Let us first give some results which hold for the two Hilbert modules $\mathcal{E}_Y$ and $\mathcal{E}_{Y'}$. For simplicity, we give them for $Y$ and explain later on the needed modifications for $Y'$. The Hilbert module $L^2_G(Y)$ is the Connes-Skandalis Hilbert module obviously extended to the non-smooth case, see [CoSk:84], [Roe:02]. The space $C_c(Y)$ of continuous compactly supported functions on $Y$ can indeed be endowed with the pre-Hilbert module structure over the convolution compactly supported algebra $C_c(G)$ of $G$. The rules are defined by:

- $(\xi f)(y) := \sum_{\gamma \in G_{\rho(y)}} \xi(\gamma^{-1}) f(\gamma)$, for $f \in C_c(G), \xi \in C_c(Y)$.
- $\langle \eta, \xi \rangle (g) := \int_{y \in Y_{\rho(g)}} \overline{\eta(y)} \xi(yg) d\mu_{\rho(g)}(y)$, for $\eta, \xi \in C_c(Y), g \in G$.

Passing to completions, we obtain our Hilbert $C^*_c(G)$-module $L^2_G(Y)$. The similar construction yields the Hilbert $C^*_c(G)$-module $L^2_G(Y')$.

Recall on the other hand the regular representation $\lambda = (\lambda_x)_{x \in X}$ for the groupoid $G$. So $\lambda_x : C^*_c(G) \to \mathcal{L}(\ell^2(G_x))$ is given on the dense subalgebra $C_c(G)$ by:

$$
\lambda_x(f)(\xi)(g) := \sum_{g' \in G_x} f(gg'^{-1}) \xi(g') \quad \text{for } f \in C_c(G), \xi \in C_c(G_x) \text{ and } g \in G_x.
$$

The space of arrows $G(1)$ is itself an interesting example of a proper right $G$-space $Y$ which is moreover a free $G$-space. The anchor map here is the source map $\rho_G = s$ which is surjective and open (since $G$ is étale) so that the action reduces to the composition on $G(2)$. Moreover, the Haar system here corresponds to the counting measures on each $G_x$. We get with this example the (full) Hilbert $C_0(X)$-module $\mathcal{E}_G$ when we simply specify $Y = G(1)$. Then $\mathcal{E}_G$ is associated with the continuous field of Hilbert spaces $(\ell^2(G_x))_{x \in X}$. Notice though that here the fibers of the anchor map are discrete and that the space $G(1)$ is $G$-compact only when $X$ is compact.

**Lemma 2.15.** The regular representation yields an (injective) $\ast$-homomorphism $\lambda : C^*_c(G) \to \mathcal{L}_{C_0(X)}(\mathcal{E}_G)$ which is valued in the space of $G$-invariant operators.

**Proof.** Denote by $V_G$ the unitary of $\mathcal{E}_G$ which defines the $G$-action. Recall that $V_G$ is induced by the map $V_G : C_c(G \times_r G) \to C_c(G(2))$ given by $V_G \varphi(g_1, g_2) = \varphi(g_1, g_2, g_2)$, $(g_1, g_2) \in G(2)$.
Here $G \rtimes_s G = \{(g, g') \in G^2, s(g) = s(g')\}$. But for $k \in C_c(G)$, $\varphi \in C_c(G \rtimes_s G)$ and $(g_1, g_2) \in G^{(2)}$ we can write
\[
(V_G \circ s^* \lambda(k)) (\varphi)(g_1, g_2) = s^* \lambda(k)(\varphi)(g_1g_2, g_2) = \sum_{g \in G \rtimes_s G} k(g_1g_2g^{-1}) \varphi(g, g_2)
\]
On the other hand,
\[
(r^* \lambda(k) \circ V_G)(\varphi)(g_1, g_2) = \sum_{g' \in G \rtimes_s G} k(g_1g'^{-1})(V_G \varphi)(g', g_2) = \sum_{g' \in G \rtimes_s G} k(g_1g'^{-1}) \varphi(g'g_2, g_2).
\]
Setting $g'g_2 = g$ in this last expression gives the equality
\[
r^* \lambda(k) \circ V_G = V_G \circ s^* \lambda(k)
\]

The operator $id \otimes_Y V_G$ denotes the well defined $C_0(X)$-adjointable operator which is a unitary from the Hilbert module $L^2_G(Y) \otimes_{s^* \lambda} s^* \mathcal{E}_G$ to the Hilbert module $L^2_G(Y) \otimes_{r^* \lambda} r^* \mathcal{E}_G$. Now, we have by definition
\[
s^*(L^2_G(Y) \otimes_{s^* \lambda} s^* \mathcal{E}_G) \cong L^2_G(Y) \otimes_{s^* \lambda} s^* \mathcal{E}_G \text{ and } r^*(L^2_G(Y) \otimes_{r^* \lambda} r^* \mathcal{E}_G) \cong L^2_G(Y) \otimes_{r^* \lambda} r^* \mathcal{E}_G.
\]

**Proposition 2.16.** We have an isometric $\ast$-isomorphism of $G$-Hilbert $C_0(X)$-modules:
\[
\Phi : L^2_G(Y) \otimes_{r^* \lambda} r^* \mathcal{E}_G \cong \mathcal{E}_Y.
\]

which is given for $\zeta \in C_c(Y), \xi \in C_c(G)$, and $y \in Y$ by:
\[
\Phi(\zeta \otimes \xi)(y) := \sum_{g \in G \rtimes_s G} \zeta(yg^{-1}) \xi(g).
\]

In the same way, we have the similar isometric $\ast$-isomorphism of $G$-Hilbert $C_0(X)$-modules $\tilde{\Phi} : L^2_G(Y') \otimes_{s^* \lambda} s^* \mathcal{E}_G \cong \mathcal{E}_{Y'}$ given by the similar formula.

**Proof.** We only give the proof for $\mathcal{E}_{Y}$ since the proof for $\mathcal{E}_{Y'}$ is similar. For $f, \xi \in C_c(G)$ and $\zeta \in C_c(Y)$, we have by direct inspection $\Phi(\zeta \otimes \xi) = \Phi(\zeta \otimes \lambda(f) \xi)$. Indeed, the two expressions give at $y \in Y$:
\[
\sum_{g, g' \in G \rtimes_s G} \zeta(yg^{-1}) f(gg'^{-1}) \xi(g'), \quad \text{for } y \in Y.
\]

It is also easy to check that $\Phi$ is an isometry. We complete the proof by pointing out that the space $C_c(Y)$ is contained in the range of $\Phi$. Let $\zeta \in C_c(Y)$ be given and denote by $K$ the image under $\rho$ of the support of $\zeta$, a compact subspace of $X$. With our assumptions on $X$, we can find a continuous compactly supported function $\varphi$ on $X$ which is identically 1 on $K$. Since the unit space $X = G^{(0)}$ is a clopen subspace of $G$, we deduce that $\varphi$ extends trivially to a continuous compactly supported function $\delta$ on $G$. It is then clear that $\Phi(\zeta \otimes \delta) = \zeta$.

It remains to show the $G$-equivariance of $\Phi$. Let $V$ be as in the proof of Proposition 2.13 the unitary which defines the $G$-action on the Hilbert $C_0(X)$-module $\mathcal{E}_Y$, let $\zeta \in C_c(Y)$ and let $k \in C_c(G \rtimes_s G)$ be given. Then we compute for $(y, g) \in Y \rtimes_r G$:
\[
(V \circ s^* \Phi)(\zeta \otimes k)(y, g) = (s^* \Phi)(\zeta \otimes k)(yg, g) = \sum_{g' \in G \rtimes_s G} \zeta(ygg_{1}^{-1}) k(g_1, g).
\]
while in the same way we obtain
\[
(r^* \Phi) \circ (\text{id} \otimes \lambda V_G) (\xi \otimes k)(y, g) = (r^* \Phi)(\xi \otimes V_G k)(y, g) = \sum_{g_2 \in G_{\mu(y)}} \xi(yg_2^{-1}) k(g_2 g). 
\]

Setting in this last expression \(g_2 g = g_1\), the proof is complete. \(\square\)

The representations \(s^* \lambda\) and \(r^* \lambda\) used above are the induced ones on \(s^* \mathcal{E}_G\) and \(r^* \mathcal{E}_G\) respectively by \(\lambda \otimes \text{id}\). Notice that we sometimes denote them simply by \(\lambda\) when no confusion can occur. The isomorphism \(\Phi\) defined above induces the allowed Morita equivalence of Theorem 2.14. More precisely,

**Proposition 2.17.** The map \(\Phi_* : \mathcal{L}_{C^*_red(G)}(L^2_G(Y)) \rightarrow \mathcal{L}_{C_0(X)}(\mathcal{E}_Y)\) given by \(\Phi_*(T) = \Phi \circ (T \otimes \lambda \text{id}) \circ \Phi^{-1}\) induces a \(C^*\)-isomorphism \(K_{C^*_red(G)}(L^2_G(Y)) \cong C^*_0(Y, \mathcal{E}_Y)\).

The same statement holds if we replace \(\mathcal{E}_Y\) by \(\mathcal{E}_{Y'}\). More precisely, we also have the \(C^*\)-isomorphism

\[
\Phi_* : \mathcal{K}_{C^*_red(G)}(L^2_G(Y')) \xrightarrow{\cong} C^*_0(Y, \mathcal{E}_{Y'}). 
\]

**Proof.** Let \(\eta_1, \eta_2 \in C_c(Y) \subset L^2_G(Y)\). Recall the (Hilbert module) rank one operator \(\theta_{\eta_1, \eta_2}\) on \(L^2_G(Y)\) defined by

\[
\theta_{\eta_1, \eta_2}(\xi) := \eta_1 < \eta_2, \xi >.
\]

Notice that \(s^* (\theta_{\eta_1, \eta_2} \otimes \lambda \text{id}) = \theta_{\eta_1, \eta_2} \otimes s^* \lambda \text{id}\). Since \(\Phi\) is \(G\)-equivariant we can write

\[
V \circ s^* \Phi = r^* \Phi \circ (\text{id} \otimes \lambda V_G).
\]

But \((\text{id} \otimes \lambda V_G) \circ (\theta_{\eta_1, \eta_2} \otimes s^* \lambda \text{id}) = (\theta_{\eta_1, \eta_2} \otimes r^* \lambda \text{id}) \circ (\text{id} \otimes \lambda V_G)\) hence

\[
V \circ (s^* \Phi) \circ (\theta_{\eta_1, \eta_2} \otimes s^* \lambda \text{id}) \circ s^* \Phi^{-1} = (r^* \Phi) \circ (\theta_{\eta_1, \eta_2} \otimes r^* \lambda \text{id}) \circ (\text{id} \otimes \lambda V_G) \circ s^* \Phi^{-1} = (r^* \Phi) \circ r^* (\theta_{\eta_1, \eta_2} \otimes \lambda \text{id}) \circ r^* \Phi^{-1} \circ V.
\]

Hence if we denote by \(K_{\eta_1, \eta_2}\) the function on \(Y \times \rho Y\) given by

\[
K_{\eta_1, \eta_2}(y, y') := \sum_{g \in G_{\mu(y)}} \eta_1(yg^{-1}) \eta_2(y'g^{-1}),
\]

then the operator \(\Phi_* \theta_{\eta_1, \eta_2}\) is given by the expression

\[
\Phi_* \theta_{\eta_1, \eta_2}(\xi)(y) = \int_{\rho^{-1}(y)} K_{\eta_1, \eta_2}(y, y') \xi(y') d\mu_{\rho(y)}(y').
\]

It is clear then that \(\Phi_* \theta_{\eta_1, \eta_2}\) also has finite propagation with respect to \(d_Y\). Moreover, for any continuous compactly supported function \(\varphi\) on \(Y\), the operators \(\pi_Y(\varphi) \circ \Phi_*, \theta_{\eta_1, \eta_2}\) and \(\Phi_* \theta_{\eta_1, \eta_2} \circ \pi_Y(\varphi)\) are families of operators on \((L^2(Y_x))_{x \in X}\) which are associated (through the fiberwise integral as above) with the continuous kernels

\[
\varphi K_{\eta_1, \eta_2}(y, y') = \varphi(y) K_{\eta_1, \eta_2}(y, y') \quad \text{and} \quad K^\varphi_{\eta_1, \eta_2}(y, y') = K_{\eta_1, \eta_2}(y, y') \varphi(y').
\]

Notice that the crossed product groupoid \((Y \times \rho Y) \times_s G\) is an étale groupoid so that the counting measures on \(G\) induce \((Y \times \rho Y) \times_s G\) with a continuous Haar system, this shows in turn that the kernels \(\varphi K_{\eta_1, \eta_2}\) and \(K^\varphi_{\eta_1, \eta_2}\) are continuous (and compactly supported) on \(Y \times \rho Y\). Indeed, the properness of the \(G\)-action on \(Y\) shows that the continuous functions on \((Y \times \rho Y) \times_s G\) given by

\[
(y, y'; g) \mapsto \varphi(y) \eta_1(yg^{-1}) \eta_2(y'g^{-1}) \quad \text{and} \quad (y, y'; g) \mapsto \varphi(y) \eta_1(yg^{-1}) \eta_2(y'g^{-1})
\]

are compactly supported. A standard argument then shows that they are both compact operators on the Hilbert module \(\mathcal{E}_Y\). Indeed, any continuous compactly supported kernel as above on \(Y \times \rho Y\)
can be uniformly approximated by linear combinations of kernels from $C_c(Y) \otimes C_c(Y)$ (elementary kernels) that we restrict to $Y \times \rho Y$ and which can even be supposed to be supported within a fixed compact subset of $Y \times \rho Y$. This can be seen for instance using first the Tietze theorem and then the usual approximation property. This allows to prove for instance that the associated operator can be approximated by finite rank operators of the Hilbert module $E_Y$.

Since $\Phi_\ast$ is continuous (actually an isometry) we deduce from the previous discussion that it sends the compact operators of the Hilbert module $L^2_G(Y)$ to operators in $C^*_e(Y, E_Y)$.

To finish the proof, we need to show that the operators $\Phi_\ast \theta_{\eta_1, \eta_2}$ span a dense subspace of $C^*_e(Y, E_Y)$. We use averaging for our proper groupoid $Y \times G$ as follows, see [Pat:07] which extends techniques from [CoMo:82] [Section 1].

Given a compactly supported $P \in \mathcal{L}_{C_c(\mathcal{X})}(E_Y)$ we may consider its well defined average operator $\text{Av}(P) \in \mathcal{L}_{C_c(\mathcal{X})}(E_Y)$ given as

$$\text{Av}(P)_x = \sum_{g \in G^x} V_g P_{s(g)} V_g^{-1}.$$  

The sum is of course finite due to the $G$-properness of the space $Y$ and the compact support of $P$. The resulting operator $\text{Av}(P)$ then has finite propagation. The proof that $\text{Av}(P)$ is an adjointable operator (with adjoint $\text{Av}(P^*)$) and hence belongs to $\mathcal{L}_{C_c(\mathcal{X})}(E_Y)$ is classical, see for instance [Pat:07] [Theorem 4]. Indeed the norm of $\text{Av}(P)$ can be estimated using the norm of $P$ but also its support. Moreover, by construction, the operator $\text{Av}(P)$ is $G$-invariant.

If now $T$ is an element of $C^*_e(Y, E_Y)$ with finite propagation and $c$ is a compactly supported continuous cutoff function (recall that $Y$ is $G$-compact), then the operator $\pi_Y(c)T$ belongs to $\mathcal{K}_{C_c(\mathcal{X})}(E_Y)$ and has compact support contained in some space of the form $A \times \rho A \subset Y \times \rho Y$ with $A$ a compact subspace of $Y$. Moreover, since $T$ is already $G$-invariant, we have the convenient relation:

$$T = \text{Av}(\pi_Y(c)T).$$  

(2.1)

Fix $\epsilon > 0$ and let $(\xi_i, \eta_i)_{i \in I}$ be a finite collection of elements of $E_Y$ (whose supports may be taken inside $A$) such that

$$||\pi_Y(c)T - \sum_{i \in I} \hat{\theta}_{\xi_i, \eta_i}|| < \epsilon,$$

where we have denoted for the sake of clarity by $\hat{\theta}_{\xi_i, \eta_i}$ the rank one operator defined similarly to $\theta_{\eta_1, \eta_2}$ but now acting on the Hilbert module $E_Y$. A density argument allows to further assume that the $\xi_i$'s and $\eta_i$'s live in $C_c(Y)$. Moreover the support of $\sum_{i \in I} \hat{\theta}_{\xi_i, \eta_i}$ can be assumed as close as we please to that of $\pi_Y(c)T$. Therefore, we deduce the existence of a constant $\kappa$ such that

$$||\text{Av} \left( \pi_Y(c)T - \sum_{i \in I} \hat{\theta}_{\xi_i, \eta_i} \right)|| \leq \kappa ||\pi_Y(c)T - \sum_{i \in I} \hat{\theta}_{\xi_i, \eta_i}||.$$

Since $\text{Av}(\pi_Y(c)T) = T$ and $\text{Av} \left( \sum_{i \in I} \hat{\theta}_{\xi_i, \eta_i} \right) = \sum_{i \in I} \Phi_\ast \hat{\theta}_{\xi_i, \eta_i}$, the proof is complete for the Hilbert module $E_Y$.

Now, all the above arguments hold as well for the Hilbert module $E_{Y'}$ with the extended representation $\pi_{Y'} \otimes \rho_{Y'} \text{id}$, but one has to use finite propagation of operators on $E_{Y'}$ according to our definition, say with respect to the representation of $C_0(Y')$. Formula (2.1) then still makes sense for a finite propagation operator $T$ on $E_{Y'}$ although the operator $(\pi_{Y'} \otimes \rho_{Y'} \text{id})(c)T$ is no more compactly supported but is only compactly supported with respect to the $Y$ variable. The rest of the proof is similar.

We have now completed the proof of our theorem. More precisely:

**Proof.** (of Theorem 2.14) Since $Y'$ is a free and proper $G$-space, the Hilbert module $L^2_G(Y')$ is a full Hilbert $C^*_r(G)$-module, see [MRW:87] [Proposition 2.10]. Therefore the assertion of the theorem follows immediately from Proposition 2.17. ∎
Remark 2.18. It is obvious from the above proof that Theorem [2.14] holds with $\mathcal{E}_Y$ instead of $\mathcal{E}_G$, when the action of $G$ on $Y$ is assumed to be free (and proper).

3 Compatibility with the Baum-Connes map

We use the notations of the previous sections, in particular the Hilbert $G$-module $\mathcal{E}_Y$ is associated with the field of Hilbert spaces $L^2(Y, \mu)$. The compatibility theorems proved in the present section hold for $\mathcal{E}_Y$ as well as for $\mathcal{E}_G$ and for simplicity we only give the proofs for the first Hilbert module and leave the easy modifications as an easy verification for the interested reader. The short exact sequence of $C^*$-algebras

$$0 \to C_G^*(Y, \mathcal{E}_Y) \to D_G^*(Y, \mathcal{E}_Y) \to Q_G^*(Y, \mathcal{E}_Y) \to 0,$$

together with Bott periodicity, yields the following periodic six-term exact sequence of $K$-groups

$$\begin{array}{cccccc}
K_0(C_G^*(Y, \mathcal{E}_Y)) & \longrightarrow & K_0(D_G^*(Y, \mathcal{E}_Y)) & \longrightarrow & K_0(Q_G^*(Y, \mathcal{E}_Y)) \\
\partial_1 & & \partial_1 & & \\
K_1(Q_G^*(Y, \mathcal{E}_Y)) & \longleftarrow & K_1(D_G^*(Y, \mathcal{E}_Y)) & \longleftarrow & K_1(C_G^*(Y, \mathcal{E}_Y))
\end{array}
$$

(3.1)

In this section we shall prove the compatibility of the connecting maps $\partial_i$, $i = 0, 1$ with the classical Baum-Connes map, as described for instance in [Tu99]. See [Co94] for a more detailed description of this latter for étale groupoids and its relation with important conjectures in geometry and topology, especially in the study of foliations. Our result, Theorem 3.3 below, is well known for discrete groups [Reed02] and our method is an extension of Roe’s proof to groupoids and Hilbert modules associated with groupoids.

3.1 The Paschke-Higson map

We define here the Paschke-Higson maps,

$$\mathcal{P}_*: K_*(Q_G^*(Y, \mathcal{E}_Y)) \to KK_G^{*+1}(Y, X), \quad * = 0, 1.$$

In the even case, it is easy to see that we can take a class $y \in K_0(Q_G^*(Y, \mathcal{E}_Y))$ which is represented by a self-adjoint operator $P \in D_G^*(Y, \mathcal{E}_Y)$ satisfying the relation

$$P^2 = P \quad \text{modulo } C_G^*(Y, \mathcal{E}_Y).$$

Recall that $C_0(Y)$ and $C_0(X)$ are $G$-algebras, and that $\mathcal{E}_Y$ is a $G$-Hilbert $C_0(X)$-module such that $\pi_Y : C_0(Y) \to \mathcal{L}(\mathcal{E}_Y)$ is a $G$-equivariant representation. Also the operator $2P - 1$ is self-adjoint and satisfies $(2P - 1)^2 = id$ up to $C_G^*(Y, \mathcal{E}_Y)$, moreover it is exactly $G$-invariant (not only up to compacts), hence the triple $(\pi_Y, \mathcal{E}_Y, 2P - 1)$ represents a class in $KK_G^1(Y, X)$.

We thus define the class $\mathcal{P}_0(y) \in KK_G^1(Y, X)$ by setting

$$\mathcal{P}_0(y) := [(\pi_Y, \mathcal{E}_Y, 2P - 1)].$$

Using invariance of $KK$-classes under operator homotopy, the universal property of Grothendieck groups, and that $KK$-classes don’t see the operation of adding degenerate cycles, it is easy to check that $\mathcal{P}_0(y)$ is well defined, that is: it only depends on the class $y$ of $P$ in $K_0(Q_G^*(Y, \mathcal{E}_Y))$.

To define similarly the Paschke-Higson map $\mathcal{P}_1 : K_1(Q_G^*(Y, \mathcal{E}_Y)) \to KK_G^0(Y, X)$ corresponding to the odd case, we let similarly $y \in K_1(Q_G^*(Y, \mathcal{E}_Y))$ be a class which is represented by an operator $u \in D_G^*(Y, \mathcal{E}_Y)$ satisfying

$$uu^* = I \quad \text{and } u^*u = I \quad \text{modulo } C_G^*(Y, \mathcal{E}_Y)$$

and leave the easy modifications as an easy verification for the interested reader. The short exact sequence of $C^*$-algebras

$$0 \to C_G^*(Y, \mathcal{E}_Y) \to D_G^*(Y, \mathcal{E}_Y) \to Q_G^*(Y, \mathcal{E}_Y) \to 0.$$
Then we set
\[ P_1(y) := \left( \pi_Y \oplus \pi_Y, \mathcal{E}_Y \oplus \mathcal{E}_Y, \begin{bmatrix} 0 & u^* \\ u & 0 \end{bmatrix} \right) \]
Again the triple \((\pi_Y \oplus \pi_Y, \mathcal{E}_Y \oplus \mathcal{E}_Y, \begin{bmatrix} 0 & u^* \\ u & 0 \end{bmatrix})\) is obviously a \(\mathbb{Z}_2\)-graded \(G\)-equivariant Kasparov cycle which represents a class in \(KK^*_G(Y, X)\). Moreover, the class \(P_1(y)\) is well defined, i.e. only depends on the class \(y\) of \(u\) in \(K_1(Q_G(Y, \mathcal{E}_Y))\) and not on the representative \(u\).

We now recall the Baum-Connes map associated with the proper \(G\)-compact space \(Y\) \cite{Tu:99}. So, associated with the proper \(G\)-compact space \((Y, \rho)\), there is a (Baum-Connes) index map
\[ \mu^\ast_{BC,Y} : KK^*_G(Y, X) \to K_\ast(C^*_{red}(G)), \]
that we proceed to recall now for the convenience of the reader. See again \cite{Tu:99} and also \cite{Hig:00}.

The map \(\mu^\ast_{BC,Y}\) will be the composite map of two standard constructions that we call respectively “the descent map” and “the KM contraction” in reference to Kasparov-Michschenko, i.e.
\[ \mu^\ast_{BC,Y} : KK^*_G(Y, X) \xrightarrow{\text{descent}} KK^*(C_0(Y) \rtimes_{\text{red}} G, C^*_{red}(G)) \xrightarrow{p_{KM}} KK^*([C_0(Y) \rtimes_{\text{red}} G]) \cong K_\ast(C^*_{red}(G)) \]

The KM contraction is given by reducing to the image of a Michschenko projection \(p_{KM} \in C_0(Y) \rtimes_{\text{red}} G\) and was defined by Kasparov. More precisely such projection defines a class \([p_{KM}]\) in \(K_0(C_0(Y) \rtimes_{\text{red}} G) \cong KK([C_0(Y) \rtimes_{\text{red}} G])\) and the map \(p_{KM}\) is given as the Kasparov product with this class.

On the other hand, the descent map was introduced by Kasparov for groups and extended by Le Gall to groupoids in \cite{LeGal:99}. For a \((\mathbb{Z}_2\text{-graded})\) \(G\)-Hilbert module \(E\), one can define the crossed-product \(G\)-module \(E \rtimes_{\text{red}} G\); it is by definition given by an interior tensor product as follows:
\[ E \rtimes G := E \otimes_{C_0(X), r^*} C^*_{red} G \]
where the action of \(C_0(X)\) on \(C^*_{red} G\) is given via the pull-back map induced by the range map \(r : G \to X\). Note that \(E \rtimes G\) inherits a \(\mathbb{Z}_2\)-grading from \(E\). Any \(G\)-equivariant, degree-preserving representation \(\pi : C_0(Y) \to \mathcal{L}(E, G)\) induces the crossed-product representation
\[ \pi \rtimes \lambda : C_0(Y) \rtimes_{\text{red}} G \to \mathcal{L}_{C^*_{red} G}(E \rtimes G) \]
which is defined as follows. For \(\xi \in C_c(Y)\), \(\phi \in C_c(G)\), \(\eta \in \pi(C_c(Y)) E\), \(\alpha \in C_c(G)\), and \(g \in G\)
\[ \pi \rtimes \lambda(\xi \otimes \phi)(\eta \otimes \alpha)(g) := \sum_{g' \in G^\langle \phi \rangle} \pi(\xi)V^\phi_{g'}(\eta).\phi(g').\alpha(g^{-1}g) \]
where \(V^E \in \mathcal{L}(s^*E, r^*E)\) is the unitary implementing the \(G\)-action on \(E\).

Now suppose that \((\pi, E, F)\) is a triple representing a class in \(KK^*_G(Y, X)\). Then the triple \((\pi \rtimes \lambda, E \rtimes G, F)\) defines a \(KK\)-cycle in \(KK^*(C_0(Y) \rtimes_{\text{red}} G, C^*_{red} G)\) (see e.g. \cite{Tu:99} and \cite{Kas:S}). We end up in this way with the Kasparov descent map
\[ j_G : KK^*_G(Y, X) \to KK^*(C_0(Y) \rtimes_{\text{red}} G, C^*_{red} G) \]
defined by \(j_G([(\pi, E, F)]) := [(\pi \rtimes \lambda, E \rtimes G, F)]\).

Moreover, with \(c\) being the cutoff function defined above for the cocompact proper \(G\)-action on \(Y\), the element \(e \in C_c(Y \rtimes r, G) \subseteq C_0(Y) \rtimes_{\text{red}} G\) given by
\[ e(y, g) := \sqrt{c}(y)\sqrt{c}(yg) \]
is a projection in \(C_0(Y) \rtimes_{\text{red}} G\). Therefore \(e\) defines the Kasparov-Michschenko class \(p_{KM}\) which is viewed as an element of \(KK^*(\mathbb{C}, C_0(Y) \rtimes_{\text{red}} G)\). Kasparov cup-product with this element gives a map:
\[ KK^*(C_0(Y) \rtimes_{\text{red}} G, C^*_{red} G) \xrightarrow{p_{KM} \otimes e} KK^*([\mathbb{C}, C^*_{red} G]) \]
Composition of this map with the descent map $j_G$ is the Baum-Connes map associated with $Y$:

$$\mu_{BC,Y}^*: KK^*_G(Y,X) \to K_*(C^*_\text{red} G).$$

Let us also recall the universal Baum-Connes map for completeness [Tu:99]. If $Y$ is a locally compact proper $G$-space which is not necessarily $G$-compact, then the Baum-Connes map for $Y$ is defined by an inductive limit over $G$-compact closed subspaces. More precisely, for any $G$-compact closed subspace $Y'$ of $Y$, we have the above map $\mu_{BC,Y'}^*$, and if $Y'' \subset Y'$ is an inclusion of $G$-compact closed subspaces of $Y$, then the $G$-equivariant restriction morphism $C^*_0(Y') \subset C^*_0(Y'')$ yields the map $KK^*_G(Y'',X) \to KK^*_G(Y',X)$ which can easily be seen to be compatible with the Baum-Connes maps $\mu_{BC,Y''}^*$ and $\mu_{BC,Y'}^*$. Hence, there is a well defined Baum-Connes map for the $G$-proper locally compact space $Y$ which is well defined on the inductive limit, over all $G$-compact closed subspaces, denoted

$$RK^*_G(Y,X) := \lim_{Y' \subset Y, Y'/G \text{ compact}} KK^*_G(Y',X)$$

In [Tu:99], a locally compact model for the classifying space of proper $G$-actions is constructed and we denote it $EG$.  

**Definition 3.1.** The universal Baum-Connes map for our étale groupoid $G$ is the well defined morphism

$$\mu_{BC}^*: RK^*_G(EG,X) \to K_*(C^*_\text{red} G)$$

**Remark 3.2.** If $F$ is any additional $G$-algebra then we end up using the above construction with the Baum-Connes assembly map with coefficients in $F$:

$$\mu_{BC,F}^*: RK^*_G(EG,F) \to K_*(F \rtimes_{\text{red} G}),$$

so that $\mu_{BC}^* = \mu_{BC,F}(X)$. 

### 3.2 The compatibility theorem

We have proved in the previous section that the $C^*$-algebra $C^*_G(Y,\mathcal{E}_Y)$ is Morita equivalent to the reduced $C^*$-algebra $C^*_\text{red}(G)$ associated with the étale groupoid $G$. This isomorphism result will be needed in the next papers of this series but will not be used here. Recall that if the action of $G$ on $Y$ is free for instance then $\mathcal{E}_Y$ is always full and no need to use the slightly modified Hilbert module $\mathcal{E}_{Y'}$. We shall for simplicity rather give the constructions and proofs for the Hilbert module $\mathcal{E}_Y$ and only point out that all the constructions can be easily modified so as to apply to the Hilbert module $\mathcal{E}_{Y'}$.

Recall that we have constructed an explicit Hilbert $C^*_\text{red}(G)$-module $L^2_G(Y)$ whose compact operators are isomorphic through the map $\Phi_*$ to $C^*_G(Y,\mathcal{E}_Y)$. So, assuming that this module is full, the $K$-theory isomorphism

$$\mathcal{M}_*: K_*(C^*_G(Y,\mathcal{E}_Y)) \to K_*(C^*_\text{red}(G)),$$

induced by this Morita equivalence, can be described using Kasparov’s $KK$-theory as the cup product with the class of the (trivially $\mathbb{Z}_2$-graded) even cycle $(L^2_G(Y), \Phi_*, 0)$.

**Theorem 3.3.** With the previous notations, the following diagram commutes:

$$K_*(Q^*_G(Y,\mathcal{E}_Y)) \xrightarrow{\partial_*} K_{*+1}(C^*_G(Y,\mathcal{E}_Y))$$

$$\downarrow \mathcal{P}_* \quad \downarrow \mathcal{M}_{*+1}$$

$$KK_{*+1}^+G(Y,X) \xrightarrow{\mu_{BC}^*} K_{*+1}(C^*_\text{red} G)$$

where $\partial_*$ is the connecting map in $(\mathbb{Z}_2)$, $\mathcal{P}_*$ is the Pashcke-Higson map and $\mu_{BC}^*$ is the Baum-Connes assembly map recalled in the previous paragraph.
The Paschke-Higson map is known to be an isomorphism for many classes of groupoids, especially for discrete countable groups, and also for groupoids associated with discrete countable group actions on spaces. This latter result is proved in the second paper of this series using an equivariant family version of the Voiculescu theorem. Therefore, Theorem 3.3 relates the Baum-Connes conjecture for the groupoid $G$ with vanishing rigidity results.

### 3.2.1 Proof of Theorem 3.3 in the even case

Our goal is thus to prove the commutativity of the following diagram:

\[
\begin{array}{ccc}
K_0(Q_G^*(Y,\mathcal{E}_Y)) & \xrightarrow{\partial_0} & K_1(C_G^*(Y,\mathcal{E}_Y)) \\
\downarrow{\pi_0} & & \downarrow{\mathcal{M}_1} \\
KK'_G(Y, X) & \xrightarrow{\mu_{GC}} & K_1(C^*_G)
\end{array}
\]  

(3.2)

As already observed, we can start with an operator $P \in D_G^*(Y,\mathcal{E}_Y)$ which satisfies the relations $P = P^*$ and $P^2 - P \in C_G^*(Y,\mathcal{E}_Y)$, and represents a class $[P] \in K_0(Q_G^*(Y,\mathcal{E}_Y))$. No need here to use matrix algebras which would yield to the same argument. Recall the Hilbert module $L^2_G(Y)$ over the reduced $C^*$-algebra $C^*_{red}(G)$ with the representation $\Phi_*$ of $C^*_G(Y,\mathcal{E}_Y)$. The computation of $\mathcal{M}_1 \circ \partial_0$ is given in the following

**Proposition 3.4.** Denote by $\tilde{s}$ the adjointable operator on $L^2_G(Y)$ which corresponds to $2P - I$ through the isomorphism $\Phi_*$ of Proposition 2.17. Then the image of $[P]$ under the composite map $\mathcal{M}_1 \circ \partial_0$ is represented by the odd Kasparov $(\mathbb{C}, C^*_{red}(G))$ cycle

\[(L^2_G(Y), \pi_C, \tilde{s})\]

where $\pi_C$ is the trivial representation by multiplication of scalars.

**Proof.** Recall that $P \in D_G^*(Y,\mathcal{E}_Y)$, and descends to a self-adjoint idempotent in the quotient $C^*$-algebra $Q^*_G(Y,\mathcal{E}_Y)$. In order to deduce the expected representative of the Kasparov class $\partial[P]$ in $KK_1(\mathbb{C}, C^*_G(Y,\mathcal{E}_Y))$, we simply apply Proposition 17.5.5 in [Bl:86], as suggested to us by the referee. Indeed, let us denote by $\iota : D^*_G(Y,\mathcal{E}_Y) \hookrightarrow M^*(C^*_G(Y,\mathcal{E}_Y))$ the inclusion in the multiplier $C^*$-algebra $M^*(C^*_G(Y,\mathcal{E}_Y))$ (with its strict topology). See again [Bl:86]. Then $\iota$ induces the inclusion, still denoted $\iota$, of $Q^*_G(Y,\mathcal{E}_Y)$ in the quotient algebra $Q^*(C^*_G(Y,\mathcal{E}_Y))$ given by $M^*(C^*_G(Y,\mathcal{E}_Y))/C^*_G(Y,\mathcal{E}_Y)$. We thus have the following commutative diagram of $C^*$-algebra exact sequences:

\[
\begin{array}{ccc}
0 & \rightarrow & C^*_G(Y,\mathcal{E}_Y) \\
\downarrow{=} & & \downarrow{=} \\
0 & \rightarrow & M^*(C^*_G(Y,\mathcal{E}_Y)) \rightarrow Q^*(C^*_G(Y,\mathcal{E}_Y)) \rightarrow 0
\end{array}
\]  

(3.3)

The boundary map for the first sequence can then be deduced from the boundary map for the second sequence, indeed, one has

\[
\partial_0 = \partial \circ \iota : K_0(Q^*_G(Y,\mathcal{E}_Y)) \xrightarrow{\iota} K_0(Q^*(C^*_G(Y,\mathcal{E}_Y))) \xrightarrow{\partial} K_1(C^*_G(Y,\mathcal{E}_Y)),
\]

where $\partial$ is the isomorphism described in [Bl:86]. From this discussion we deduce that when viewed through the isomorphism $K_1(C^*_G(Y,\mathcal{E}_Y)) \simeq KK_1(\mathbb{C}, C^*_G(Y,\mathcal{E}_Y))$, the class $\partial_0[P]$ is represented by the cycle

\[
(C^*_G(Y,\mathcal{E}_Y), F) \text{ where } F = 2P - 1.
\]

Notice that $F^2 - I = 4(P^2 - P) \in C^*_G(Y,\mathcal{E}_Y)$. Now the transport of $F$ through the identification $C^*_G(Y,\mathcal{E}_Y) \otimes C^*_G(Y,\mathcal{E}_Y) \rightarrow L^2_G(Y)$ is the operator $\tilde{s}_0$ defined, on the dense submodule generated by $\Phi^{-1}_*(T)(\zeta)$ with $T \in C^*_G(Y,\mathcal{E}_Y)$ and $\zeta \in L^2_G(Y)$, by

\[
\tilde{s}_0(\Phi^{-1}_*(T)(\zeta)) = \Phi^{-1}_*(F \circ T)(\zeta).
\]
In order to show that $\mathcal{F}_0 = \mathcal{F}$, it suffices to use the isomorphism $L_G^2(Y) \otimes_\lambda \mathcal{E}_G \simeq \mathcal{E}_Y$ of Proposition 2.16 and to compute the resulting operator arising from $\mathcal{F}_0 \otimes_\lambda \text{id}$ on $\mathcal{E}_Y$. Let then $S \in K_{C^*_\text{red}(G)}(L_G^2(Y))$ be such that $\Phi_* S = \Phi \circ (S \otimes \lambda \text{id}) \circ \Phi^{-1} = T$ and let $\xi \in \mathcal{E}_G$ be given. Then we obtain

$$(\mathcal{F}_0 \otimes_\lambda \text{id})(S(\xi) \otimes \lambda \xi) = (\mathcal{F}_0 \circ S)(\xi) \otimes \lambda \xi$$

while

$$(F \circ \Phi)(S\xi \otimes \lambda \xi) = [\Phi \circ (\mathcal{F}_0 \otimes_\lambda \text{id})](S\xi \otimes \lambda \xi).$$

Hence the proof of Proposition 3.4 is complete.

\[\square\]

**Remark 3.5.** We have used in the previous proof Proposition 17.5.5 in [Bl:86] to identify $\theta[P]$. Since only the idea of the proof of that proposition is given in [Bl:86], we point out that $\theta[P]$ is by definition represented in $K_1(C^*_G(Y, \mathcal{E}_Y))$ by the multiplier unitary $e^{2\pi i P}$, which obviously differs from $I$ by an element of $C^*_G(Y, \mathcal{E}_Y)$ and also belongs to $D^*_G(Y, \mathcal{E}_Y)$. Hence the identification of the corresponding Kasparov cycle in $KK_1(C, C^*_G(Y, \mathcal{E}_Y))$ is standard.

We now proceed to compute $(\mu_{BC} \circ P_0)[P]$. Let again $Y \rightrightarrows G$ be the groupoid induced by the $G$-action on $Y$, i.e. pulled back using $r$ and defined in Section 2.2. We construct the Hilbert $C^*_\text{red}(G)$-module $\mathcal{E}_Y \rtimes G$ as usual, see for instance [Tu:99]. The inner product and module structure of this module are described explicitly by

1. For $\varphi \in C_c(G)$ and $\xi \in C_c(Y \rtimes r G)$:

   $$(\xi \varphi)(y, g) := \sum_{g' \in G_{\mu(y)}} \xi(y, g'^{-1}) \varphi(g' g)$$

2. For $\eta, \xi \in C_c(Y \rtimes r G)$ and $g \in G$:

   $$<\eta, \xi > (g) := \sum_{g' \in G_{\mu(y)}} \int_{y \in Y_{r(g')}} \eta_2(y, g') \eta_1(y, g') d\mu_{r(g')}(y).$$

The Hilbert $C^*_\text{red}(G)$-module $\mathcal{E}_Y \rtimes G$ carries a representation $\pi_{Y \rtimes G}$ of $C_0(Y \times_\text{red} G \cong C^*_\text{red}(Y \rtimes r G)$ given, for $\phi \in C_c(Y \rtimes r G) \subseteq C^*_\text{red}(Y \rtimes r G)$ and $\xi \in C_c(Y \rtimes r G) \subseteq \mathcal{E}_Y \rtimes G$, by:

$$\pi_{Y \rtimes G}(\phi)(\xi)(y, g) := \sum_{g' \in G_{\mu(y)}} \phi(y, g') \xi(yg', g'^{-1}g) = \sum_{g_1 \in G_{\mu(y)}} \phi(y, yg_1^{-1}) \xi(yg_1^{-1}, g_1).$$

Another interpretation of $\mathcal{E}_Y \rtimes G$ is by considering the composition Hilbert module over $C^*_\text{red}(G)$ given by $\mathcal{E}_Y \otimes_{C_0(X), r} C^*_\text{red}(G)$ where we view $C^*_\text{red}(G)$ as a Hilbert $C^*_\text{red}(G)$-module where $C_0(X)$ represents through multiplication with pull-backs by $r^*$. More precisely, the map $\Psi : C_c(Y) \otimes_{C_0(X), r} C_c(G) \to C_c(Y \rtimes r G)$ given by:

$$\Psi(\xi \otimes f)(y, g) := \xi(y)f(g), \quad \text{for } \xi \in C_c(Y), f \in C_c(G),$$

(3.4)

can be easily seen to be an isometric isomorphism of Hilbert modules.

Recall the cut-off function $c \in C_c([0, 1])$ from Definition 2.3.

**Proposition 3.6.** There is an isometry of Hilbert $C^*_\text{red}(G)$-modules $I : L_G^2(Y) \to \mathcal{E}_Y \rtimes G$ given by:

$$I(\xi)(y, g) := \sqrt{c}(y)\xi(yg), \quad \text{for } \xi \in C_c(Y).$$

The range of $I$ coincides with the Hilbert submodule which is the range of $\pi_{Y \rtimes G}(e)$ with $e$ the self-adjoint idempotent in $C_c(Y \rtimes r G)$ given by $e(y, g) := \sqrt{c}(y)\sqrt{c}(yg)$. 

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This is more precisely and for a fixed \( h \) identifying the operator \( F \) allows to conclude. Recall from Proposition 2.17 that the map \( \Phi \) on the Hilbert module \( E_Y \) is closed and since the proof only uses the fact that \( F \) is \( G \)-invariant, it is easy to see that the operator \( \Phi \) coincides with the operator \( F \) on the the common dense domain \( C_c(Y) \). In order to complete the proof of the commutativity of Diagram (3.2.2), we prove the following

**Lemma 3.7.** We have \( I^* \circ \tilde{F} \circ I - \tilde{F} \in K_{C^*_\text{red}(G)} \left( L^2_G(Y) \right) \)

**Proof.** We may assume in this proof that \( F \) has finite propagation. Indeed since \( K_{C^*_\text{red}(G)} \left( L^2_G(Y) \right) \) is closed and since the proof only uses the fact that \( F \in D_{C^*_\text{red}(Y)}(E_Y) \), an easy density argument then allows to conclude. Recall from Proposition 2.17 that the map \( \Phi \) is an isometric isomorphism from \( K_{C^*_\text{red}(G)} \left( L^2_G(Y) \right) \) to the \( C^* \)-algebra \( C^*_G(Y, E_Y) \). So, we shall prove that the operator

\[
\Phi_*(I^* \circ \tilde{F} \circ I) - F = \Phi \circ (I^* \circ \tilde{F} \circ I) \otimes \lambda \text{id}_{E_Y}) \circ \Phi^{-1} - F \in C_0^*(Y, E_Y).
\]

But if we fix \( \xi \in C_c(Y) \subset L^2_G(Y) \) and \( \varphi \in C_c(G) \subset E_Y \) and we set \( \eta := \Phi(\xi \otimes \varphi) \) then we may write

\[
\Phi_*(I^* \circ \tilde{F} \circ I)(\eta)(y) = \sum_{g \in G_{\rho(y)}} (I^* \circ \tilde{F} \circ I)(\xi)(yg^{-1}) \varphi(g)
\]

\[
= \sum_{g \in G_{\rho(y)}} \varphi(g) \sum_{k \in G_{\rho(g)}} \sqrt{c}(yg^{-1}k^{-1}) \tilde{F}(I\xi)(yg^{-1}k^{-1}, k).
\]

But identifying the operator \( F \) with the corresponding field \( (F_x)_{x \in X} \), we can write

\[
\tilde{F}(I\xi)(yh^{-1}, h) = F(z \mapsto (I\xi)(z, h))(yh^{-1})
\]

\[
= F(\sqrt{c}(h\xi))(yh^{-1})
\]

\[
= [F, \pi_Y(\sqrt{c})(h\xi)(yh^{-1})] + (\pi_Y(\sqrt{c}) \circ F)(h\xi)(yh^{-1})
\]

\[
= [F, \pi_Y(\sqrt{c})(h\xi)(yh^{-1})] + \sqrt{c}(yh^{-1})F(\xi)(y).
\]

This is more precisely and for a fixed \( h \in G_{\rho(y)} \) given by

\[
[F, \pi_Y(\sqrt{c})]_{r(h)}(h\xi)(y) + \sqrt{c}(yh^{-1})F_{\rho(y)}(\xi_{\rho(y)})(y),
\]

where only the restriction of \( \xi \) to \( Y_{\rho(y)} \) is involved. We deduce from this computation

\[
(I^* \circ \tilde{F} \circ I)(\xi)(y) = \sum_{h \in G_{\rho(y)}} \sqrt{c}(yh^{-1}) ([F, \pi_Y(\sqrt{c})](h\xi)(yh^{-1}) + \sqrt{c}(yh^{-1})F(\xi)(y))
\]

\[
= F_{\rho(y)}(\xi)(y) + \sum_{h \in G_{\rho(y)}} \sqrt{c}(yh^{-1})[F, \pi_Y(\sqrt{c})]_{r(h)}(h\xi)(y)
\]

\[= F_{\rho(y)}(\xi)(y) + \sum_{h \in G_{\rho(y)}} \sqrt{c}(yh^{-1})[F, \pi_Y(\sqrt{c})]_{r(h)}(h\xi)(y).
\]

\[= F_{\rho(y)}(\xi)(y) + \sum_{h \in G_{\rho(y)}} \sqrt{c}(yh^{-1})[F, \pi_Y(\sqrt{c})]_{r(h)}(h\xi)(y).
\]
Therefore,

\[ \Phi_{\ast}[I^* \circ \bar{F} \circ I](\eta)(y) = \sum_{g \in G_{\rho(y)}} \varphi(g) \sum_{k \in G_{r(y)}} \sqrt{c}(yk^{-1}k^{-1}) \]

\[ \left( [F, \pi_Y(\sqrt{c})](k\xi)(yk^{-1}) + \sqrt{c}(yk^{-1}k^{-1})F(\xi)(yk^{-1}) \right) \]

Hence we get

\[ \Phi_{\ast}[I^* \circ \bar{F} \circ I](\eta)(y) = \Phi(\mathfrak{H} \otimes \varphi)(y) + \sum_{g \in G_{\rho(y)}} \varphi(g) \sum_{k \in G_{r(y)}} \sqrt{c}(yk^{-1}k^{-1})[F, \pi_Y(\sqrt{c})]_{r(y)}(k\xi)(yk^{-1}). \]

Now, \( \Phi(\mathfrak{H} \otimes \varphi) \) is nothing but \( F(\eta) \) by definition of \( \mathfrak{H} \). On the other hand, we define out of our operator \( F \) the operator \( T \) on \( C_c(Y) \subset \mathcal{E}_Y \) by setting for our \( \xi \in C_c(Y) \):

\[ T(\xi)(y) := \sum_{k \in G_{\rho(y)}} \sqrt{c}(yk^{-1})[F, \pi_Y(\sqrt{c})]_{r(y)}(k\xi)(yk^{-1}) = \sum_{k \in G_{\rho(y)}} \left\{ k^{-1}(\pi_Y(\sqrt{c}) \circ [F, \pi_Y(\sqrt{c})]) \right\}_{r(y)}(\xi)(y). \]

Then we check that \( T \) is well defined since \( \sqrt{c} \) is compactly supported, and it is obviously \( C_c(X) \)-linear. Also it extends to a \( C_0(X) \)-adjointable operator on \( \mathcal{E}_Y \) by direct inspection, in fact it is self-adjoint since \( F \) and \( \pi_Y(\sqrt{c}) \) are self-adjoint. Moreover, since the commutator \( [F, \pi_Y(\sqrt{c})] \) is compact on the Hilbert module \( \mathcal{E}_Y \), so is \( T \). Indeed, if \( \eta_1, \eta_2 \) are elements of \( C_c(Y) \subset \mathcal{E}_Y \) then

\[ \sum_{k \in G_{\rho(y)}} \sqrt{c}(yk^{-1})\theta_{\eta_1, \eta_2}(k\xi)(yk^{-1}) = \left( \sum_{k \in G_{\rho(y)}} \pi_{\rho(y)}(k^{-1}\sqrt{c}) \circ (\theta_{k^{-1}\eta_1, k^{-1}\eta_2})_{r(y)} \right)(\xi)(y). \]

Now notice that for any compactly supported function \( \zeta \) on \( Y \), the set \( K_{\zeta, c} \) defined by

\[ K_{\zeta, c} := \{ g \in G \mid \text{there exists } y \in \text{supp}(c) \text{ such that } yg \in \text{supp}(\zeta) \}, \]

is compact in \( G \), since the \( G \)-action on \( Y \) is proper. So, we see that the above sum is finite and hence we get a compact operator. One then deduces that \( T \) is compact and also by easy verification that this operator \( T \) has finite propagation if \( F \) does and is in general an element of \( C^*(Y, \mathcal{E}_Y) \). We thus get for any \( x \in X \)

\[ [\Phi_{\ast}[I^* \circ \bar{F} \circ I](\eta) - F(\eta)]_x = \sum_{g \in G_x} T(g^{-1}\xi)\varphi(g) = T_x(\Phi(\xi \otimes \lambda \varphi)) = T_x(\eta). \]

\[ \square \]

**Corollary 3.8.** The KK-cycles \((I(L^2_G(Y)), \lambda_G, \pi_Y \otimes G(e))\) and \((L^2_G(Y), \lambda_G, I)\) are equivalent.

**Proof.** Recall that \( \pi_Y \otimes G(e) = I^* \) and that, by Lemma 3.6, \( I \) is an isometry which identifies \( L^2_G(Y) \) with the range \( I(L^2_G(Y)) \) in \( \mathcal{E}_Y \otimes G \). Therefore, using the unitary isomorphism

\[ I^* : I(L^2_G(Y)) \rightarrow L^2_G(Y), \]

we deduce that the KK-cycle \((I(L^2_G(Y)), \lambda_G, \pi_Y \otimes G(e))F\pi_Y \otimes G(e)\) is conjugate to the KK-cycle \((L^2_G(Y), \lambda_G, I^*F)\). Now, applying Lemma 3.7, we see that \( I^*F \) is a compact perturbation of \( \mathfrak{H} \). This implies the assertion. \( \square \)

Thus we have proved:

**Corollary 3.9.** The image of \([P]\) under \( \mu_{BC}^1 \circ P_0 \) coincides with that under \( \mathcal{M}_1 \circ \partial_0 \).
3.2.2 Proof of Theorem 3.3 in the odd case

The similar result in the odd case can be deduced using the space $Y \times \mathbb{R}$ with the anchor map $\rho \circ p_1$ with $p_1 : Y \times \mathbb{R} \to Y$ being the first projection. Then the groupoid $G$ needs to be replaced by the groupoid $G \times \mathbb{Z}$. This needs though some extra-arguments and since the direct proof is shorter, we chose it here. So, we now prove by a direct computation the analogous result in the odd case, i.e. the commutativity of the following diagram:

$$
\begin{array}{ccc}
K_1(Q^*_G(Y, E_Y)) & \xrightarrow{\partial_1} & K_0(C^*_G(Y, E_Y)) \\
\downarrow p_1 & & \downarrow \mathcal{M}_0 \\
K K^0_G(Y, X) & \xrightarrow{\nu^0_G} & K_0(C^*_G) \\
\end{array}
$$

**Proposition 3.10.** Given an element $u \in D_G^*(Y, E_Y)$ whose projection in $Q^*_G(Y, E_Y)$ is a unitary operator, the image under the map $\mathcal{M}_0 \circ \partial_1$, of the class of $u$ in $K_1(Q^*_G(Y, E_Y))$, is represented in $K K(\mathbb{C}, C^*_\text{red}(G))$, by the Kasparov $KK$-cycle

$$
\left( L^2_G(Y), L^2_G(Y), \left( \begin{array}{cc} 0 & U^* \\ U & 0 \end{array} \right) \right),
$$

where $U$ is the adjointable operator on $L^2_G(Y)$ which corresponds to $u$ under the isomorphism $\Phi_*$ of Proposition 2.7.

**Proof.** We know that $uu^* - 1, u^*u - 1 \in C^*_G(Y, E_Y)$ by assumption. There exists a unitary (actually in the connected component of the identity) $U \in M_2(D_G^*(Y, E_Y))$ such that

$$
\pi(U) = \begin{bmatrix} \pi(u) & 0 \\ 0 & \pi(u^*) \end{bmatrix} \in U_2(Q^*_G(Y, E_Y))
$$

Denote by $P_0$ the projection $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and define the projection $P = UP_0U^*$. Then the class $[P] - [P_0] \in K_0(C^*_G(Y, E_Y))$ is by definition $\partial_1([u])$. The corresponding class in $K K^0(\mathbb{C}, C^*_G(Y, E_Y))$ is thus given by the Kasparov cycle $(\mathcal{E} := \mathcal{E}^+ \oplus \mathcal{E}^-, F)$, where

$$
\mathcal{E}^+ := \text{Im}(P) \subseteq C^*_G(Y, E_Y) \oplus C^*_G(Y, E_Y), \quad \mathcal{E}^- := C^*_G(Y, E_Y) \quad \text{and} \quad F = \begin{bmatrix} 0 & P \circ i_1 \\ \pi_1 \circ P & 0 \end{bmatrix} \in L_{C^*_G(Y, E_Y)}(\mathcal{E}).
$$

The operator $\pi_1 \in L_{C^*_G(Y, E_Y)}(C^*_G(Y, E_Y) \oplus C^*_G(Y, E_Y), C_G^*(Y, E_Y))$ is the projection onto the first factor and the operator $i_1 : C^*_G(Y, E_Y) \to C^*_G(Y, E_Y) \oplus C^*_G(Y, E_Y)$ is the inclusion; we note that $\pi_1^2 = \pi_1$. It is clear from the definition that $F$ is self-adjoint.

Let $U = \begin{bmatrix} u & w \\ u & u^* \end{bmatrix}$, since $\pi(U) = \text{diag}(\pi(u), \pi(u^*))$, we get $v, w \in C^*_G(Y, E_Y)$. Then

$$
F^2 - I = \begin{bmatrix} P \circ i_1 \circ \pi_1 \circ P - P & 0 \\ 0 & \pi_1 \circ P \circ i_1 - I \end{bmatrix}
$$

Since $\pi_1 \circ P \circ i_1 = uu^*$, by the hypothesis on $u$, we get $\pi_1 \circ P \circ i_1 - I \in C^*_G(Y, E_Y)$. On the other hand, the term $P \circ i_1 \circ \pi_1 \circ P - I$ is given by

$$
P \circ i_1 \circ \pi_1 \circ P - P = \begin{bmatrix} uu^*(uu^* - I) & (uu^* - I)u^* \\ vu^*(uu^* - I) & v(u^*u - I)v^* \end{bmatrix}
$$

Since by hypothesis, the elements $uu^* - I$ and $u^*u - I$ belong to $C^*_G(Y, E_Y)$, it is clear that all the entries in the matrix above belong to $C^*_G(Y, E_Y)$. 

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So we have proved that $F^2 - I \in C^*_G(Y,\mathcal{E}_Y)$ and hence is a compact operator on the Hilbert module $\mathcal{E}$. Now consider the operator $U^*: C^*_G(Y,\mathcal{E}_Y) \oplus C^*_G(Y,\mathcal{E}_Y) \to C^*_G(Y,\mathcal{E}_Y) \oplus C^*_G(Y,\mathcal{E}_Y)$, then the map $\Psi$ given by

$$\Psi = \begin{bmatrix} \pi_1 \circ U^* & 0 \\ 0 & I \end{bmatrix}: \mathcal{E}^+ \oplus \mathcal{E}^- \to C^*_G(Y,\mathcal{E}_Y) \oplus C^*_G(Y,\mathcal{E}_Y)$$

is a unitary isomorphism $\Psi: \mathcal{E}^+ \oplus \mathcal{E}^- \to C^*_G(Y,\mathcal{E}_Y) \oplus C^*_G(Y,\mathcal{E}_Y)$. This is an easy consequence of the relation $U^* P = P_0 U^*$ and the verification is omitted. Therefore, the Kasparov cycle $(\mathcal{E}^+ \oplus \mathcal{E}^-, F)$ is unitarily equivalent to $(C^*_G(Y,\mathcal{E}_Y) \oplus C^*_G(Y,\mathcal{E}_Y), \Psi \circ F \circ \Psi^{-1})$. Computing the operator $\hat{F} := \Psi \circ F \circ \Psi^{-1}$, we get that it is given by

$$\hat{F} = \begin{bmatrix} 0 & \pi_1 U^* P \pi_1 \\ \pi_1 PU \pi_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & u^* \\ u & 0 \end{bmatrix}$$

Taking now the Kasparov product with the Morita cycle $M = [L^2_G(Y), 0] \in KK_0(C^*_G(Y,\mathcal{E}_Y), C^*_G)$ that we have already described, with the representation given by the inclusion of $C^*_G(Y,\mathcal{E}_Y)$ in $L^2_{C^*_red(G)}(L^2_G(Y))$ through the Morita isomorphism $C^*_G(Y,\mathcal{E}_Y) \cong \mathcal{K} C^*_red(G)(L^2_G(Y))$, we obtain the following class in $KK_0(C, C^*_red(G))$:

$$L^2_G Y \oplus L^2_G Y, \lambda_C, \begin{bmatrix} 0 & \mathfrak{U}^* \\ \mathfrak{U} & 0 \end{bmatrix},$$

where $\mathfrak{U} = u \otimes C^*_G(Y,\mathcal{E}_Y) \text{id}_{L^2_G(Y)}$. But the representation of $D^*_G(Y,\mathcal{E}_Y)$ in $L^2_G(Y)$ is precisely given by the isomorphism $\Phi^{-1}_*$. □

**Proposition 3.11.** The image of the class of $u$ under the composite map $\mu^0_G \circ \mathcal{P}_1$ is represented by the Kasparov cycle

$$L^2_G Y \oplus L^2_G Y, \lambda_C, \begin{bmatrix} 0 & \mathfrak{U}^* \\ \mathfrak{U} & 0 \end{bmatrix}.$$ 

**Proof.** In order to compute the image of the class of $u$ under the composite map $\mu^0_G \circ \mathcal{P}_1$, we apply the same construction as for the even case. Notice first that the image under the Paschke map $\mathcal{P}_1$ of $u$ is easily described by the even $G$-equivariant Kasparov cycle

$$\left(\mathcal{E}_Y \oplus \mathcal{E}_Y, \pi_Y \oplus \pi_Y, T = \begin{bmatrix} 0 & u^* \\ u & 0 \end{bmatrix}\right)$$

and we need to represent the image of this latter cycle under the Baum-Connes map $\mu^0_G$. The computation is similar to the even case and we get that this image is represented by the Kasparov cycle

$$\left(\pi_Y \otimes_G (e) (\mathcal{E}_Y \otimes_G) \oplus \pi_Y \otimes_G (e) (\mathcal{E}_Y \otimes_G), \pi_Y \otimes_G (e) \circ \mathcal{P}_1 \circ \pi_Y \otimes_G (e) \right),$$

where $\mathcal{P}_1$ is the adjointable operator on $\mathcal{E}_Y \otimes_G$ corresponding to $T \otimes C_0(\mathcal{X}) \rtimes I$ under the isomorphism describe previously, and $e$ is the Michielenko idempotent also described previously. Recall that the isometry $I$ identifies $L^2_G(Y)$ with the orthocomplemented Hilbert submodule $\pi_Y \otimes_G (e) (\mathcal{E}_Y \otimes G)$ of $\mathcal{E}_Y \otimes G$ and it satisfies more precisely that $II^* = \pi_Y \otimes_G (e)$. Therefore, the previous Kasparov cycle is unitarily equivalent to the Kasparov cycle

$$\left(L^2_G(Y) \oplus L^2_G(Y), I^* \mathcal{T} \mathcal{T}^* \right).$$

If now $\Sigma$ is the $C^*_red(G)$-adjointable operator in $L^2_G(Y)$ corresponding to $T$ under the isomorphism $\Phi_*$, then we need to show that

$$I^* \mathcal{T} \mathcal{T}^* - \Sigma \in \mathcal{K} C^*_red(G)(L^2_G(Y)).$$

This is a corollary of Lemma 3.7. □
4 The coarse $G$-index for continuous $G$-families

We explain in this last section how to define the coarse $G$-index of a $G$-equivariant family of Dirac-type operators on a $G$-proper continuous family of bounded geometry smooth riemannian manifolds. Our construction relies on some classical results due to Shubin [Sh:00] and extends the work of Paterson to the coarse category. In particular, the construction given is a generalization of the index class for laminations defined by Moore-Schochet [MoSc:10] to the setting of (uniform) bounded geometry laminations.

4.1 Continuous families of manifolds

The material in this overview subsection is taken from [Pat:07] so we shall be brief.

Definition 4.1 ($C^{\infty,0}$ maps). Let $M, N$ be smooth manifolds and let $X$ be a locally compact Hausdorff space. A function $f : M \times X \to N \times X$ is said to be of class $C^{\infty,0}$ if $f(M \times \{x\}) \subset N \times \{x\}$ for each $x \in X$, and the map $X \ni x \mapsto f(\bullet, x) \in C^\infty(M, N)$ is continuous.

In the previous definition, $C^\infty(M, N)$ is given the usual Fréchet topology of uniform convergence on compact subsets together with derivatives of all orders. Let $Y$ be an second countable locally compact Hausdorff space and $\rho : Y \to X$ be an open surjective map.

Definition 4.2 (Continuous family of smooth manifolds). The triple $(Y, \rho, X)$ is called a continuous family of smooth manifolds if $\exists k \in \mathbb{N}$ and a collection $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ where:

1. $U_\alpha \subset Y$ is open, and $Y = \bigcup_{\alpha \in A} U_\alpha$.
2. for each $\alpha \in A$, $\phi_\alpha : U_\alpha \to \rho(U_\alpha) \times V_\alpha$ is a fiber-preserving homeomorphism with $V_\alpha \subseteq \mathbb{R}^k$ an open subset.
3. for any $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, the maps $\phi_{\alpha\beta} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a $C^{\infty,0}$ function from $\rho(U_\alpha \cap U_\beta) \times (V_\alpha \cap V_\beta)$ to itself.

The pairs $(U_\alpha, \phi_\alpha), \alpha \in A$ will be called local charts. We shall call $Y$ a $C^{\infty,0}$-manifold.

In [Pat:07], the integer $k$ is assumed to be $\geq 1$ but we prefer here to include $k = 0$ which corresponds to $\mathbb{R}^k$ being $\{\ast\}$. The notion of $C^{\infty,0}$-maps between continuous families is defined similarly as well as $C^{\infty,0}$-diffeomorphism for instance. It is also standard to define fibrations of continuous families of manifolds as well as $C^{\infty,0}$-vector bundles or hermitian $C^{\infty,0}$-vector bundles over continuous families of smooth manifolds, see again [Pat:07]. If for instance, $(Y, \rho, X)$ be a continuous family of smooth manifolds, then the space $TY = \bigcup_{x \in X} TY_x$ (as well as all the functorially associated bundles) inherits the structure of $C^{\infty,0}$-vector bundle over $Y$.

Lemma 4.3 ([Pat:07]). Let $U = \{U_\alpha\}_{\alpha \in A}$ be a locally finite open cover of $Y$ consisting of local charts. Then there exists a partition of unity $\{\psi_\alpha\}_{\alpha \in A}$ consisting of $C^{\infty,0}$ functions subordinate to $U$.

Let $G$ be our étale Hausdorff groupoid as before with $G^{(0)} = X$ then $G$ is in fact a continuous family groupoid in the sense of [Pat:07] since we have included the case $k = 0$ in Definition 4.2. Recall the definition of a $G$-space (Definition (2.1)).

Definition 4.4 (Smooth $G$-spaces). Let $Y$ be a $C^{\infty,0}$-manifold which is endowed with a $G$-action, thus making it a $G$-space. The $G$-action is said to be of class $C^{\infty,0}$ if $\rho : Y \to X$ is a continuous family of smooth manifolds, and the structure map $\lambda : Y \rtimes_r G \to Y$ is of class $C^{\infty,0}$.

Notice that the space $Y \rtimes_r G$ carries a natural $C^{\infty,0}$ structure so that the previous definition makes sense. The previous definition makes sense for any continuous family groupoid $G$ [Pat:07], but the general case is not needed in the present paper.
4.2 The coarse $G$-index

We assume from now on that the proper $G$-space $(Y, \rho : Y \to X)$ is a continuous family of smooth riemannian manifolds (also denoted $C^{\infty, 0}$ according to Connes' notation for laminations) such that the action is of class $C^{\infty, 0}$, see [Pat:07] or the short overview given in [4.1]. We are mainly interested in the coarse index for complete laminations and the Paterson formalism will be convenient for us.

We assume moreover that $Y$ has (uniformly over $X$) bounded geometry in the fiber direction, and we also assume that the $G$-action is fiberwise isometric. In particular, we assume that the injectivity radius associated with the induced Riemannian metric on each $Y_x = \rho^{-1}(x)$ is bounded below independently of $x$, so that we have well defined barycentric fiberwise coordinates with $C^{\infty, 0}$-bounded changes over $Y$, and also that the curvature tensor defined on each smooth fiber of $\rho$ is $C^{\infty, 0}$-bounded over $Y$. In particular, the smooth manifolds $Y_x$ are all complete riemannian manifolds and we use the complete riemannian $G$-invariant riemannian metric to see $Y$ as a $G$-family of proper metric spaces. Finally, notice that there exist $C^{\infty, 0}$-bounded partitions of unity which are subordinate to covers by geodesic balls, see [Sh:00] [Pat:07]. All the geometric structure that we shall use in this section are assumed to have uniformly bounded geometry in an obvious sense which extends the classical definitions of [Sh:00] [Pat:07].

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Definition 4.5 (Bounded Propagation Speed). A family $\mathbb{D} = (D_x)_{x \in X}$ of symmetric, first-order elliptic differential operators on $\{ L^2(Y_x, E_x) \}_{x \in X}$ such that the quantity

$$C(\mathbb{D}) := \sup\{ ||\sigma(y, \xi)||_{\mathcal{E}(E_x)} : y \in Y, \xi \in S_y^*Y_{\rho(y)} \}$$

is finite, is said to be of (uniformly) bounded propagation speed.

By elliptic we of course mean here fully elliptic in the sense of Shubin, see [Sh:00]. Using the results of [Sh:00] and [Pat:07], it is a routine argument to show that, under our assumptions, families of Dirac-type operators associated with (uniformly) $C^{\infty, 0}$-bounded structures, do have (uniformly) bounded propagation speed. Observe that a family $\mathbb{D}$ of symmetric, first-order elliptic differential operators induces a family of self-adjoint, closed unbounded operators which we also denote by $\mathbb{D}$, cf. [HR:00], Chapter 10. Suppose that this induced family $\mathbb{D}$ is continuous (see Definition [3.7] and has bounded propagation speed. We denote the induced regular operator on the associated Hilbert module $E_{Y,E}$ by $D$. By the functional calculus of regular operators, we know that for any continuous complex valued bounded function $f$ on $\mathbb{R}$, the operator $f(D)$ is a well defined adjointable operator on the Hilbert module $E_{Y,E}$ associated with the continuous field of Hilbert spaces $\{ L^2(Y_x, E_x) \}_{x \in X}$. In particular, the wave operator $\exp(isD)$ can be defined in this way as an adjointable operator on $E_{Y,E}$.

Lemma 4.6 (Finite propagation property of the wave operator). Let $f, g \in C_0(Y)$ such that $\text{supp}(f) \cap \text{supp}(g) = \emptyset$ and either $f$ or $g$ has compact support. Then there exists $\epsilon > 0$ such that $\pi_Y(f) \exp(isD)\pi_Y(g) = 0$ for $|s| < \epsilon$.

Proof. By the compatibility of functional calculi [3.10], the adjointable operator $\exp(isD)$ corresponds to the continuous family of bounded operators $\{ \exp(isD_x) \}_{x \in X}$. Then, by [HR:00], Corollary 10.3.3 for finite propagation of wave operators on complete manifolds, we have

$$\pi_x(f|_{Y_x}) \exp(isD_x)\pi_x(g|_{Y_x}) = 0 \text{ for all } s \text{ such that } |s| < 1/c_x$$

where $c_x$ is the propagation speed of $D_x$ and $\pi_x$ is the representation of $C_0(Y_x)$ on $L^2(Y_x, E_x)$ by pointwise multiplication.

Therefore for $|s| < 1/C(\mathbb{D})$, the continuous family of operators $\pi_Y(f|_{Y_x}) \exp(isD_x)\pi_Y(g|_{Y_x})$ is identically zero. Since $\pi_Y(f) \exp(isD)\pi_Y(g)$ corresponds to this family, the proof is concluded. \qed
**Lemma 4.7 (Inverse Fourier representation).** Let $\phi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function such that it has compactly supported Fourier transform. Then, for any $u, v \in C^\infty_c(0)(Y)$ the function $s \mapsto \phi(D)u, v >$ is continuous, and the following relation holds:

$$< \phi(D)u, v > = \int_\mathbb{R} \hat{\phi}(s) < \exp(isD)u, v > ds$$

**Proof.** This is a classical result which can be deduced from the spectral theorem. The reader can also consult the arguments in [HPS:15] Lemma 3.6 which immediately carries over to our situation. □

**Definition 4.8 (Normalizing function).** A function $\chi \in C(\mathbb{R})$ is called normalizing if $\chi$ is an odd function such that $\chi(x) > 0$ for $x > 0$ and $\chi(x) \xrightarrow{x \to \infty} 1$.

**Remark 4.9.** The space of Fourier transforms of smooth compactly supported functions on $\mathbb{R}$, a subspace of the Schwartz space $\mathcal{S}(\mathbb{R})$ is dense in $C_0(\mathbb{R})$. It is easy to see that the inverse Fourier representation of Lemma (4.7) also allows to define the operators $f(D)$ for any $f \in C_0(\mathbb{R})$. Moreover, using the extension of the Fourier transform to distributions, one can also define $\chi(D)$ for any normalizing function $\chi$ by using again the inverse Fourier representation of Lemma 4.7.

**Theorem 4.10 (Roe’s Lemma).** Let $\mathcal{D}$ be a $G$-equivariant family of first-order elliptic differential operators with (uniformly) bounded propagation speed, inducing a continuous field of closed, self-adjoint operators, also denoted $\mathcal{D}$. Denote the regular operator on $\mathcal{E}_{Y,E}$ induced by $\mathcal{D}$ as $\mathcal{D}$. Then for any normalizing function $\chi$, the operator $\chi(D)$ belongs to the Roe algebra $\mathcal{D}_G^*(Y;\mathcal{E}_{Y,E})$. Moreover, for any $\phi \in C_0(\mathbb{R})$, the operator $\phi(D)$ belongs to the ideal $\mathcal{D}_G^*(Y;\mathcal{E}_{Y,E})$. In particular, the class in the quotient $C^*$-algebra $\mathcal{D}_G(Y;\mathcal{E}_{Y,E})$ defined by the operator $\chi(D)$ is independent of the choice of $\chi$.

**Proof.** It is clear that functional calculus preserves $G$-equivariance, so $\chi(D) \in \mathcal{L}_{C_0(\mathcal{X})(\mathcal{E}_{Y,E})^G}$. Let $\chi'$ be a normalizing function such that $\text{supp}(\hat{\chi'})$ is included in $[-\epsilon, \epsilon]$ for $\epsilon > 0$ small enough. Then, $\chi - \chi' \in C_0(\mathbb{R})$ so that corresponding operator $\chi'(D) - \chi'(D)$ is locally compact according to the second item. Therefore it suffices to show that $\chi'(D) \in \mathcal{D}_G^*(Y;\mathcal{E}_{Y,E})$. To show that $[\chi'(D), \pi_Y(f)] \in \mathcal{K}_{C_0(\mathcal{X})(\mathcal{E}_Y)}$ for all $f \in C_0(\mathcal{Y})$, from Lemma 4.11 it suffices to show that $\pi_Y(f)\chi'(D)\pi_Y(g)$ is compact whenever $f,g \in C_0(\mathcal{Y})$ have disjoint supports and at least one of them have compact support. However, to this end we apply the finite propagation property of the wave operator in conjunction with the inverse Fourier representation of $\chi'(D)$ (cf. Lemma 4.9) as

$$\chi'(D) = \int_{-\epsilon}^{\epsilon} \hat{\chi'}(s) \exp(isD)ds$$

Since $\pi_Y(f) \exp(isD)\pi_Y(g) = 0$ for $|s| < \epsilon$ for small enough $\epsilon$, the operator $\pi_Y(f)\chi'(D)\pi_Y(g)$ is in fact zero (thus compact). Lastly, that $\chi(D)$ is a norm-limit of finite propagation operators can be shown following the techniques in [HPS:15], Lemma 3.6.

For the second part of the theorem, it suffices to give the proof for $\phi$ in the subspace $J \subset \mathcal{S}(\mathbb{R})$ of Fourier transforms of smooth compactly supported functions. Now, for any $\psi \in J$, the operator $\psi(D)$ has finite propagation because of the integral representation of Lemma 4.7. By density of $C^\infty_c(0)(Y)$ in $C_0(\mathcal{Y})$, it moreover suffices to show that for any $f \in C^\infty_c(0)(Y)$ and any $\psi \in J$, $\psi(D)\pi_Y(f) \in \mathcal{K}_{C_0(\mathcal{X})(\mathcal{E}_Y)}$. From Proposition 10.5.2 of [HR:00], we know that $\psi(D)\pi_Y(f)_{|_{\mathcal{Y}}}$ is compact for each $x \in \rho(\text{Supp}(f))$ and only the regularity with respect to the $Y$-variable needs to be investigated. Now, a classical argument using convolution reduces the proof to the case where $\hat{\psi}$ is supported within a small enough neighborhood of 0. By the uniform speed propagation, we can thus assume that the finite propagation operator $\psi(D)$ is actually supported within a uniform in $X$ small enough neighborhood of the diagonals of the fibers. Again, assuming that $f$ is also supported within a small enough subset of $Y$, we are reduced to the case of an open trivializing set $U_\alpha$ as in Definition 4.2 with a fiber-preserving homeomorphism

$$\phi_\alpha : U_\alpha \to \rho(U_\alpha) \times V_\alpha \text{ with } V_\alpha \text{ an open set in } \mathbb{R}^k.$$
Now, the proof is reduced to this local situation and we need to show that the family \( (\psi(D_x)\pi_Y(f)\chi_x)_{x \in X} \) is then continuous in \( x \) for the operator norm. But this latter is justified in [Pat:07, Proposition 11] using classical results due to M. Shubin [Sh:00]. We have now shown that the operator \( \psi(D)\pi_Y(f) \) is a compact operator on the Hilbert \( C_0(X) \)-module, i.e. \( \psi(D)\pi_Y(f) \in \mathcal{K}_{C_0(X)}(\mathcal{E}_Y, E) \), and hence that \( \psi(D) \in \mathcal{D}_G(Y; \mathcal{E}_Y, E) \) for any \( \psi \in \mathcal{J} \) and finally also for any \( \psi \in C_0(\mathbb{R}) \) by density.

We end this proof by pointing out that if \( \chi_1 \) and \( \chi_2 \) are two normalizing functions then the difference \( \chi_1 - \chi_2 \) belongs to \( C_0(\mathbb{R}) \) and hence \( \chi_1(D) - \chi_2(D) \in \mathcal{D}_G(Y; \mathcal{E}_Y, E) \). Therefore, the two operators \( \chi_1(D) \) and \( \chi_2(D) \) yield the same class in the quotient Roe algebra \( Q^*_G(Y; \mathcal{E}_Y, E) \).

We recall now the so-called Kasparov’s lemma that was used in the above proof. Let \( \pi : C_0(\mathbb{Y}) \rightarrow \mathcal{L}_{C_0(X)}(\mathcal{E}) \) be a non-generator representation in the Hilbert \( C_0(X) \)-module \( \mathcal{E} \). Recall that % % %
\[ T \in \mathcal{L}_{C_0(X)}(\mathcal{E}) \] is pseudolocal if \( [T, \pi_Y(f)] \in \mathcal{K}_{C_0(X)}(\mathcal{E}) \) for all \( f \in C_0(\mathbb{Y}) \). The following is a straightforward generalization of Kasparov’s Lemma characterizing pseudolocal operators. The proof is a straightforward adaptation of [HR:00] [Lemma 5.4.7] and is omitted.

**Lemma 4.11** (Kasparov’s Lemma). An operator \( T \in \mathcal{L}_{C_0(X)}(\mathcal{E}) \) is pseudolocal if and only if for all \( f, g \in C_0(\mathbb{Y}) \) such that \( \text{supp}(f) \cap \text{supp}(g) = \emptyset \) and at least one of the functions \( f \) and \( g \) have compact support, the operator \( \pi(f)T\pi(g) \in \mathcal{K}_{C_0(X)}(\mathcal{E}) \).

From Theorem 4.10 we conclude that for a normalizing function \( \chi \), the operator \( \chi(D) \in \mathcal{D}_G(Y; \mathcal{E}_Y, E) \), while \( \chi^2(D) - 1 \in \mathcal{D}_G(Y; \mathcal{E}_Y, E) \). Thus, \( \frac{1}{2}(1 + \chi(D)) \) gives a well-defined element in the K-theory group \( K_0(Q^*_G(Y; \mathcal{E}_Y, E)) \). Let \( \partial_0 \) be the boundary map in the K-theory long exact sequence induced by the short exact sequence
\[
0 \rightarrow \mathcal{D}_G(Y; \mathcal{E}_Y, E) \rightarrow \mathcal{D}_G^*(Y; \mathcal{E}_Y, E) \rightarrow Q^*_G(Y; \mathcal{E}_Y, E) \rightarrow 0
\]
We can now give the following definition of the coarse index.

**Definition 4.12** (Coarse G-index). The (odd) coarse G-index \( \text{Ind}_G(D) \) is defined as the class \( \text{Ind}_G(D) \) which is the image of the class \( \frac{1}{2}(1 + \chi(D)) \) under the boundary map
\[
\partial_0 : K_0(Q^*_G(Y; \mathcal{E}_Y, E)) \rightarrow K_1(C_0^*(Y; \mathcal{E}_Y, E)).
\]
So,
\[
\text{Ind}_G(D) := \partial_0 \left( \frac{1}{2}(1 + \chi(D)) \right) \in K_1(C_0^*(Y; \mathcal{E}_Y, E))
\]
Since the boundary map \( \partial_0 \) is an exponential map, it is easy to see that the index class is represented by the unitary associated with \( \chi \) by the formula \( -e^{i\pi \chi(D)} \). In particular, the index class could as well be a priori defined by the Cayley transform \( (i + D)(i - D)^{-1} \). We have treated the odd case which corresponds for families of Dirac-type operators to assuming that the fibers \( \mathbb{Y}_x = \rho^{-1}(x) \) be odd dimensional. The even case is similar and uses the notion of Voiculescu G-isometry. It is worth pointing out that when \( Y \) is G-compact, Morita equivalence allows to deduce a well-defined index class in \( K_*(C^*_r(G)) \), and we recover in this way the index class defined by Paterson in [Pat:07]. Notice that Paterson’s index class EXTENDS the classical construction of Moore-Schochet for laminations of compact spaces [MoSc:10], which in turn extends the Connes-index class for smooth foliations on compact manifolds [Co:79]. We end this paper with the following corollary.

**Corollary 4.13.** Assume that \( D \) is invertible with a gap in its spectrum around 0, then \( D \) has a well defined transgressed coarse G-index class \( [D] \) which lives in \( K_0(D^*_G(Y; \mathcal{E}_Y, E)) \) whose image under the functoriality map \( K_0(D^*_G(Y; \mathcal{E}_Y, E)) \rightarrow K_0(Q^*_G(Y; \mathcal{E}_Y, E)) \) is the class \( \frac{1}{2}(1 + \chi(D)) \).

**Proof.** Indeed, we may then choose \( \chi \) such that \( \chi^2 = 1 \) on the spectrum of \( D \) and then we see that \( \frac{1}{2}(1 + \chi(D)) \) is already a projection in \( D^*_G(Y; \mathcal{E}_Y, E) \) which defines the class \([D]\) and which embodies the vanishing of the index class \( \text{Ind}_G(D) \). □
The previous proposition applies when the $G$-equivariant family of smooth bounded geometry
manifolds is a spin family and admits (uniformly in the $X$-variable) positive scalar curvature. For
foliations this is related with the recent results of [BH:17]. A similar statement happens when we
only assume positive scalar curvature at infinity but using relative Roe algebras. These developments
will be treated in a forthcoming paper in relation with the Gromov-Lawson relative index theorem.

A $G$-representations in $G$-modules

We give the definition and gather the properties of $G$-algebras and representations of $G$-algebras in
$G$-modules, that will be used in the sequel. For most of the material about groupoid actions on
$C^*$-algebras, we refer to [LeGall:99, Tu:99]. Given a $C^*$-algebra $A$, we denote by $MA$ the $C^*$-algebra
of multipliers of $A$ [Ka:80], and the center of $M(A)$ is denoted by $ZM(A)$.

Definition A.1. [LeGall:99, Definition 3.1]

1. A $C_0(X)$-algebra is a couple $(A, \theta)$, where $A$ is a $C^*$-algebra and $\theta: C_0(X) \to ZM(A)$ is a
$\ast$-homomorphism such that $\theta(C_0(X))A = A$.

2. A morphism $\phi: (A, \theta_A) \to (B, \theta_B)$ of $C_0(X)$-algebras is a $\ast$-homomorphism $\phi: A \to B$ such
that the following relation holds:
$$\phi(\theta_A(f)a) = \theta_B(f)(\phi(a)), \quad \text{for } f \in C_0(X).$$

So, a $C_0(X)$-algebra structure on the $C^*$-algebra $A$ endows it with the structure of a $C_0(X)$-
module. Given a $C_0(X)$-algebra $A$, the fibre of $A$ at $x \in X$ is $A_x := A/\theta(C_x)A$, where $C_x$ is the
ideal of functions in $C_0(X)$ vanishing at $x$. This yields an u.s.c. field $(A_x)_{x \in X}$ of $C^*$-algebras.

The $C^*$-algebra $C_0(X)$ is itself a $C_0(X)$-algebra ($\theta$ is just the multiplication operator by the given
function), and more generally for any locally compact Hausdorff space $Y$ with a continuous map
$\rho: Y \to X$, we may endow the $C^*$-algebra $C_0(Y)$ with the $C_0(X)$-algebra structure corresponding
to the $\ast$-homomorphism $\theta = \rho^*$ defined by $\rho^*(f) = f \circ \rho$.

Remark A.2. If $f: Y \to X$ is a continuous map between locally compact Hausdorff spaces, then
the u.s.c. field associated with the $C_0(X)$-algebra $(C_0(Y), f^*)$ corresponds to a continuous field in
the sense of Dixmier if and only if $f$ is open. See for instance [Bl:96, GTX:97].

If we are given a $C_0(X)$-algebra $(A, \theta)$, then we can pull it back to a $C_0(Y)$-algebra $(\rho^*A, \rho^*\theta)$.
Moreover, given a morphism $\phi: (A, \theta_A) \to (B, \theta_B)$ of $C_0(X)$-algebras, we easily get a morphism of
$C_0(Y)$-algebras between the pull-backs
$$\rho^*\phi: (\rho^*A, \rho^*\theta_A) \longrightarrow (\rho^*B, \rho^*\theta_B).$$

See again [LeGall:99] for the details of these standard constructions.

In particular, we may consider the $C^*$-algebras $r^*A$ and $s^*A$ which are endowed with the
structures of $C_0(G)$-algebras. Recall the set of pairs of composable arrows $G^{(2)} = \{(g, g') \in G^2, s(g) =
\rho(g')\}$ for our groupoid $G$. Then we have the three maps $\pi_{01}, \pi_{12}$ and $\pi_{02}$ from $G^{(2)} \to G^{(1)}$
corresponding respectively, to projection onto the first component, projection onto the second component,
and composition of arrows. The following relations hold:
$$r \circ \pi_{01} = r \circ \pi_{02}, \quad s \circ \pi_{12} = s \circ \pi_{02}, \quad \text{and} \quad s \circ \pi_{01} = r \circ \pi_{12}.$$

Definition A.3. A $G$-algebra is a $C_0(X)$-algebra $(A, \theta)$ together with an isomorphism $\alpha: s^*A \to
r^*A$ of $C_0(G)$-algebras such that
$$\pi_{02}^*\alpha = \pi_{01}^*\alpha \circ \pi_{12}^*\alpha,$$
where these maps are seen as $C_0(G^{(2)})$-algebra isomorphisms from $\pi_{02}^*s^*A$ to $\pi_{02}^*r^*A$.
So, we have the following commutative diagram

\[
\begin{array}{ccc}
\pi_{12}^\ast s^\ast A & \xrightarrow{\pi_{12}^\ast} & \pi_{12}^\ast r^\ast A \\
\pi_{02}^\ast & \downarrow & \downarrow \text{id} \\
\pi_{01}^\ast r^\ast A & \leftarrow \pi_{01}^\ast & \pi_{01}^\ast s^\ast A
\end{array}
\]

**Remark A.4.** In term of the corresponding u.s.c. field \((A_x)_{x \in X}\), the above isomorphism \(\alpha : r^\ast A \rightarrow s^\ast A\) of \(C_0(G)\)-algebras satisfies the expected relation \(\alpha_{gg'} = \alpha_g \circ \alpha_{g'}\), for \((g, g') \in G^{(2)}\).

**Definition A.5.** Let \((A, \theta_A, \alpha_A)\) and \((B, \theta_B, \alpha_B)\) be \(G\)-algebras. A morphism between these two \(G\)-algebras is a morphism \(\phi\) between the \(C_0(X)\)-algebras \((A, \theta_A)\) to \((B, \theta_B)\) such that

\[
r^\ast \phi \circ \alpha^A = \alpha^B \circ s^\ast \phi.
\]

So the following diagram is assumed to commute:

\[
\begin{array}{ccc}
s^\ast A & \xrightarrow{\alpha^A} & r^\ast A \\
\downarrow s^\ast \phi & & \downarrow r^\ast \phi \\
s^\ast B & \xrightarrow{\alpha^B} & r^\ast B
\end{array}
\]

We now review the notion of a Hilbert \(G\)-module. For the classical material about Hilbert modules over \(C^\ast\)-algebras, we refer the reader for instance to [Kas80] or to the more recent monograph [La95]. Let \(E\) be a Hilbert \(A\)-module where \(A\) is assumed to be a \(G\)-algebra with the isomorphism \(\alpha : s^\ast A \rightarrow r^\ast A\) satisfying the conditions of Definition A.3. Then we may define the fibre of \(E\) at \(x \in X = G^{(0)}\) as being \(E_x := E \otimes_A A_x\). Then \(E_x\) inherits the structure of a Hilbert \(A_x\)-module. We define the Hilbert \(r^\ast A\)-module, denoted \(r^\ast E\) so that its fibre at \(g \in G\) is given by \(E_{r(g)}\). More precisely,

\[
r^\ast E := E \otimes_A r^\ast A,
\]

carries a left module action of \(A\) through multiplication on the first factor. Similarly we can define the Hilbert \(s^\ast A\)-module \(s^\ast E\). In this situation, the notion of adjointable operator between Hilbert modules over isomorphic \(C^\ast\)-algebras is well defined [La95]. In particular, we shall denote by \(\mathcal{L}_\alpha(s^\ast E, r^\ast E)\) the space of adjointable \(\alpha\)-linear operators. More precisely, using the isomorphism \(\alpha\), we endow \(r^\ast E\) with the structure of a Hilbert module over the \(C^\ast\)-algebra \(s^\ast A\) and \(\mathcal{L}_\alpha(s^\ast E, r^\ast E)\) is the space of adjointable operators between the Hilbert \(s^\ast A\)-modules thus obtained. This is the space of linear maps \(T : s^\ast E \rightarrow r^\ast E\) such that \(T(\xi u) = T(\xi)\alpha(u)\) for any \(\xi \in s^\ast E\) and \(u \in s^\ast A\) and which are adjointable in the sense that there exists an \(\alpha^{-1}\)-linear operator \(T^\ast : r^\ast E \rightarrow s^\ast E\) such that \((T(\xi), \eta) = \alpha((\xi, T^\ast(\eta)))\) for any \(\xi \in s^\ast E\) and \(\eta \in r^\ast E\).

**Definition A.6.** Let \((A, \theta, \alpha)\) be a \(G\)-algebra as before. A Hilbert \(A\)-module \(E\) is endowed with the structure of a Hilbert \(G\)-module if we are given a unitary element \(V \in \mathcal{L}_\alpha(s^\ast E, r^\ast E)\) such that (setting \(V_{ij} = \pi_{ij}^\ast V\))

\[
V_{01} \circ V_{12} = V_{02} \text{ as elements of } \mathcal{L}_{C_0(G^{(1)})}(\pi_{02}^\ast s^\ast E, \pi_{02}^\ast r^\ast E).
\]

If we fix a Hilbert \(G\)-module \((E, V)\) over the \(G\)-algebra \((A, \theta, \alpha)\) and let \(\widehat{V} : \mathcal{L}_{C_0(G)}(s^\ast E) \rightarrow \mathcal{L}_{C_0(G)}(r^\ast E)\) be conjugation by \(V\), i.e.

\[
\widehat{V}(T) := V \circ T \circ V^\ast, \quad \text{for } T \in \mathcal{L}_{C_0(G)}(s^\ast E).
\]

Clearly the operator \(\widehat{V}(T)\) is then adjointable on the Hilbert \(r^\ast A\)-module \(r^\ast E\).
Definition A.7. An element \( T \in \mathcal{L}_{\mathcal{C}_0(X)}(E) \) is called \( G \)-equivariant if the following equality holds:
\[
\hat{V}(s^*T) = r^*T
\]
where \( s^*T = T \otimes_{\mathcal{C}_0(X)} s \text{id} \in \mathcal{L}_{\mathcal{C}_0(G)}(s^*E) \) and \( r^*T = T \otimes_{\mathcal{C}_0(X)} r \text{id} \in \mathcal{L}_{\mathcal{C}_0(G)}(r^*E) \).

The space of \( G \)-equivariant elements in \( \mathcal{L}_{\mathcal{C}_0(X)}(E) \) forms a \( C^* \)-subalgebra that we denote by \( \mathcal{L}_{\mathcal{C}_0(X)}(E)^G \).

We can as well use the maps \( \pi_{01}, \pi_{12} \) and \( \pi_{02} \) defined above to pull-back one step further the Hilbert \( \mathcal{C}_0(X) \)-module \( E \) and get Hilbert modules
\[
\pi_{01}(s^*E) = \pi_{12}(r^*E), \quad \pi_{01} r^*E = \pi_{02} r^*E \quad \text{and} \quad \pi_{12} s^*E = \pi_{02} s^*E.
\]
Then the pull-back of the transformation \( \hat{V} \) gives
\[
\mathcal{L}_{\mathcal{C}_0(G(2))}(\pi_{12} s^*E) \xrightarrow{\hat{V}_{12}} \mathcal{L}_{\mathcal{C}_0(G(2))}(\pi_{12} r^*E) = \mathcal{L}_{\mathcal{C}_0(G(2))}(\pi_{01} s^*E) \xrightarrow{\hat{V}_{01}} \mathcal{L}_{\mathcal{C}_0(G(2))}(\pi_{01} r^*E)
\]
and also
\[
\mathcal{L}_{\mathcal{C}_0(G(2))}(\pi_{12} s^*E) = \mathcal{L}_{\mathcal{C}_0(G(2))}(\pi_{02} s^*E) \xrightarrow{\hat{V}_{02}} \mathcal{L}_{\mathcal{C}_0(G(2))}(\pi_{02} r^*E) = \mathcal{L}_{\mathcal{C}_0(G(2))}(\pi_{01} r^*E).
\]
From the properties of the unitary \( V \), we deduce that the map \( \hat{V} \) satisfies the cocycle condition
\[
\hat{V}_{01} \circ \hat{V}_{12} = \hat{V}_{02}.
\]

To end this appendix, we say some words about \( G \)-equivariant representations.

Definition A.8. Let \((A,\theta)\) be a \( \mathcal{C}_0(X) \)-algebra. A \( \mathcal{C}_0(X) \)-representation of \( A \) is given by a Hilbert \( \mathcal{C}_0(X) \)-module \( E \) together with a \( * \)-homomorphism \( \pi : A \to \mathcal{L}_{\mathcal{C}_0(X)}(E) \) such that:
\[
\pi(\theta(f)(a))(e) = \pi(a)(ef) \quad \text{for} \quad f \in \mathcal{C}_0(X), a \in A \quad \text{and} \quad e \in E.
\]

So, equivalently the \( * \)-representation \( \pi \) must satisfy the condition
\[
\pi \circ \theta(f) = R_f \circ \pi : A \to \mathcal{L}_{\mathcal{C}_0(X)}(E), \quad \text{for any} \quad f \in \mathcal{C}_0(X).
\]
Here \( R_f \) is the (adjointable) operator implementing the right module action of \( \mathcal{C}_0(X) \) on \( E \). It is clear by definition that any \( \mathcal{C}_0(X) \)-representation \( \pi \) gives rise to a field of representations in Hilbert spaces \((\pi_x : A_x \to \mathcal{L}(E_x))_{x \in X}\), where \((E_x)_{x \in X}\) is the field of Hilbert spaces over \( X \) which is associated with \( E \).

Definition A.9 (\( G \)-equivariant representation). Let \((A,\theta,\alpha)\) be a \( G \)-algebra and let \((E,V)\) be a Hilbert \( G \)-module. A \( \mathcal{C}_0(X) \)-representation \( \pi : A \to \mathcal{L}_{\mathcal{C}_0(X)}(E) \) is \( G \)-equivariant if the following diagram commutes:
\[
\begin{array}{ccc}
s^*A & \xrightarrow{\alpha} & r^*A \\
s^*\pi & & | \quad r^*\pi \\
\mathcal{L}_{\mathcal{C}_0(G)}(s^*E) & \xrightarrow{\hat{V}} & \mathcal{L}_{\mathcal{C}_0(G)}(r^*E)
\end{array}
\]
In terms of the u.s.c. field of \( C^* \)-algebras associated with \( A \) and the u.s.c. field of Hilbert modules associated with \( E \), the equivariance property can be written as expected
\[
\pi_{r(g)}[a_g(x)] = V_g \circ \pi_{s(g)}(x) \circ V_g^*, \quad \text{for} \quad g \in G \quad \text{and} \quad x \in A_{s(g)}.
\]
B Continuous fields of operators

Let $X$ be a paracompact locally compact Hausdorff space. For each $x \in X$, let $H_x$ be a complex separable Hilbert space.

**Definition B.1** (Continuous field of Hilbert spaces, compare cf. [Dix:77], Definition 10.1.2). A continuous field of Hilbert spaces is a (complex) linear subspace $F \subseteq \prod_{x \in X} H_x$ such that:

1. (totality) the collection $\{u(x) : u \in F\} \subseteq H_x$ is dense in $H_x$,

2. (norm continuity) for every $u \in F$, the map $x \mapsto ||u(x)||$ is a continuous function vanishing at infinity,

3. (closure under uniform local approximability) given $v \in \prod_{x \in X} H_x$, if for each $\epsilon > 0$ and each $x_0 \in X$ there exists an element $u \in F$ and a neighbourhood $U_{x_0}$ of $x_0$ such that $||v(x) - u(x)|| < \epsilon, \forall x \in U_{x_0}$, then $v \in F$.

**Definition B.2** (Continuous field of bounded operators, cf. [Ph:88], Chapter 1). A continuous field of bounded operators on $F$ is a family $t = \{t_x\}_{x \in X} \subseteq \prod_{x \in X} \mathcal{B}(H_x)$ such that:

1. (uniform bound in norm) $||t|| := \sup_{x \in X} ||t_x|| < \infty$

2. (strong-* continuity) for $u \in F$, the elements $tu = \{t_xu_x\}_{x \in X}$ and $t^*u = \{t_x^*u_x\}_{x \in X}$ belong to $F$.

A continuous field of Hilbert spaces $F$ over $X$ gives rise to a full Hilbert $C_0(X)$-module $\mathcal{H}$. We shall denote the collection of such continuous fields of bounded operators by $\mathcal{B}$.

**Remark B.3.** (cf. [Ph:88], Chapter 1) Let $t \in \mathcal{B}$. Then $t$ induces an adjointable operator on $T$ on the Hilbert $C_0(X)$-module $\mathcal{H}$. In turn, any element $T \in \mathcal{L}_{C_0(X)}(\mathcal{H})$ defines a continuous field of bounded operators $t \in \mathcal{B}$.

The following result is standard and will be used later.

**Lemma B.4** (Dense range). Let $\mathcal{S} = (S_x)_{x \in X}$ be a continuous family of bounded operators on $F$, such that $\text{Range}(S_x)$ is dense in $H_x$ for each $x \in X$. Then the adjointable operator $S$ induced by $\mathcal{S}$ has dense range on $\mathcal{H}$.

For a Banach algebra $B$ we denote the spectrum of an element $a \in B$ by $\text{Sp}(a)$. Notice that if $t = \{t_x\}_{x \in X} \subseteq \mathcal{B}$ induces the operator $T \in \mathcal{L}_{C_0(X)}(\mathcal{H})$, then, we have for each $x \in X$, $\text{Sp}(t_x) \subseteq \text{Sp}(T)$.

**Lemma B.5.** Let $T \in \mathcal{L}_{C_0(X)}(\mathcal{H})$ be self-adjoint, inducing an element $t \in \mathcal{B}$ such that each $t_x$ is self-adjoint. Let $f \in C_b(\mathbb{R})$, then the operator $f(T) \in \mathcal{L}_{C_0(X)}(\mathcal{H})$ given by the continuous functional calculus induces a continuous field of bounded operators $s \in \mathcal{B}$ such that $s_x = f(t_x)$.

**Proof.** The statement is obvious for polynomials, and for $f \in C_b(\text{Sp}(T))$ one uses a standard approximation argument.

We now consider fields of closed (unbounded) operators. start with the following proposition/definition.

**Proposition B.6.** [Sk, La:95] Let $T$ be a densely defined, closed, unbounded operator on $\mathcal{H}$, and suppose that $T^*$ is also densely-defined. Then the following conditions are equivalent:

- $I + T^*T$ has dense image.

- $I + T^*T$ is surjective.
• The graph of $T$, denoted $G(T)$, is an orthocomplemented submodule of $H \oplus H$, with $G(T) = \sigma G(T^*)$, where $\sigma : H \oplus H \to H \oplus H$ is the map $(x, y) \mapsto (y, -x)$.

• The projection $p$ onto $G(T)$ is a self-adjoint idempotent in $\mathcal{L}(H \oplus H)$.

An operator satisfying one of the equivalent properties in Proposition B.6 is called a regular operator. To sum up an operator $T$ is regular if it is densely defined as well as its adjoint and if its graph is an orthocomplemented submodule. If $T$ is regular then so is $T^*$ and one has $(T^*)^* = T$. It is also worth pointing out that for such regular operator $T$, the operator $T^*T$ is a self-adjoint regular operator whose spectrum is contained in $\mathbb{R}_+$. Therefore for any continuous bounded function $f : \mathbb{R}_+ \to \mathbb{C}$, there is a well defined adjointable operator $f(T^*T)$ given by the spectral theorem, see again [Sk]. We then set:

$$Q(T) := TW(T)$$

with $W(T) := (I + T^*T)^{-1/2}$ and we define similarly $Q(T^*)$ and $W(T^*)$.

Notice for instance that $Q(T^*) = Q(T)^*$, $W(T^*) = W(T)^*$ and $(I - Q(T^*)Q(T))^{1/2} = W(T)$. See again [Sk]. Recall that we have fixed a continuous field of Hilbert spaces $F \subset \prod_{x \in X} H_x$. Let $(D_x)_{x \in X}$ be a family of closed unbounded operators on the family of Hilbert spaces $(H_x)_{x \in X}$. We denote the dense domains of $D_x$ by $\text{Dom}(D_x)$.

**Definition B.7.** [Tak:75] The field $D := (D_x)_{x \in X}$ of closed operators is called a continuous field if the field $(Q(D_x))_{x \in X}$ determines a continuous field of bounded operators on the continuous field of Hilbert spaces $F$.

**Remark B.8.**

1. One could also ask for the equivalent condition that the field of projections $(p_x)_{x \in X}$ onto the graphs of $(D_x)_{x \in X}$ be a continuous field of bounded operators.

2. When $D$ is a self-adjoint family, then it is continuous if and only if the two fields $((D_x \pm i)^{-1})_{x \in X}$ are strongly continuous.

**Lemma B.9.** A continuous field $D = (D_x)_{x \in X}$ of closed self-adjoint operators is well-defined on $F$ and induces a regular self-adjoint operator on the associated Hilbert module $H$.

**Proof.** The field $(W(D_x) := (I + D_x^2)^{-1/2})_{x \in X}$ is then also continuous and by Lemma B.4, the adjointable operator $R$ induced by this continuous field $(W(D_x))_{x \in X}$ has dense range in $H$, which is thus a dense submodule of $H$. Let $u \in \text{Im}(R)$, then each $u_x \in \text{Im}(W(D_x)) = \text{Dom}(D_x)$, and there exists an element $v \in F$ such that $u_x = W(D_x)v_x$. Therefore, $D_x u_x = Q(D_x)v_x$, which is continuous by hypothesis. Since $(1 - Q(D_x)^2)^{1/2} = W(D_x)$, the regular operator $S$ on the Hilbert module $H$ associated with the continuous field $(Q(D_x))_{x \in X}$ satisfies that $(1 - S^2)^{1/2}$ has dense range in $H$ with the obvious relation $\|S\| \leq 1$. Therefore, applying [Sk] [Théorème 15.10], we deduce that $S$ corresponds uniquely to a regular operator $D$ on $H$ such that $S = Q(D)$.

**Lemma B.10 (Compatibility of functional calculi).** Let $D = (D_x)_{x \in X}$ be a continuous family of closed self-adjoint operators. Let $D$ be the regular operator on $H$ induced by $D$. Then for $f \in C_b(\mathbb{R})$ the adjointable operator $f(D) \in \mathcal{L}_{C_b(X)}(H)$, defined by the functional calculus of regular self-adjoint operators, corresponds to the (continuous) family $(f(D_x))_{x \in X}$ of bounded operators.

**Proof.** This follows from the compatibility of functional calculi of continuous families of bounded operators and adjointable operators (cf. Lemma B.5), since the functional calculus of regular operators is defined via the $Q$-transform.
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