Fracton Matter

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We review a burgeoning field of “fractons” - a class of models where quasi-particles are strictly immobile or display restricted mobility that can be understood through generalized multipolar symmetries and associated conservation laws. Focusing on just a corner of this fast-growing subject, we will demonstrate how one class of such theories - symmetric tensor and coupled-vector gauge theories surprisingly emerge from familiar elasticity of a two-dimensional quantum crystal. The disclination and dislocation crystal defects respectively map onto charges and dipoles of the fracton gauge theory. This fracton-elasticity duality leads to predictions of fractonic phases and quantum phase transitions to their descendants, that are duals of the commensurate crystal, supersolid, smectic, hexatic liquid crystals, as well as amorphous solids, quasi-crystals and elastic membranes. We show how these dual gauge theories provide a field theoretic description of quantum melting transitions through a generalized Higgs mechanism. We demonstrate how they can be equivalently constructed as gauged models with global multipole symmetries. We expect extensions of such gauge-elasticity dualities to generalized elasticity theories provide a route to discovery of new fractonic models and their potential experimental realizations.

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I. INTRODUCTION AND MOTIVATION

Characterization and classification of phases of matter and phase transitions between them is a central pursuit of condensed matter physics. The simplest and most ubiquitous organization of matter is according to Landau’s symmetry-breaking paradigm. Such phases - crystals, magnets, superfluids, panoply of liquid crystal phases and many others - are distinguished by patterns of spontaneous symmetry breaking, characterized by a local order parameter and a short-range entangled, nearly product many-body wavefunction.
Stimulated by an ever-growing class of unusual quantum materials, that appear to lie outside of this Landau’s symmetry-breaking and Fermi liquids paradigms, much effort has been directed at exploring models, that exhibit quantum phases with fractionalized anyonic quasiparticles, robust spectral degeneracy sensitive only to the topology of space, and other unusual properties common to the so-called topological quantum liquids. Such exotic phenomenology is captured by conventional gauge theories, where fractionalized quasiparticles appear at the ends of effective field lines, free to move by growing the corresponding tensionless string, much like charges in Maxwell’s electrodynamics.

Motivated by a continued interest in topological quantum matter, quantum glasses, and by a search of fault-tolerant quantum memory, more recently, a new class of theoretical models has been discovered. These feature system-size-dependent ground state degeneracy, gapped quasiparticles with restricted mobility, and many other highly unusual properties. The first, and most famous, example is the strictly immobile excitation, dubbed “fracton”. Fractons and other subdimensional particles – lineons and planeons, restricted to move in one- and two-dimensions – were originally discovered in fully gapped models of commuting projector (stabilizer codes) lattice spin Hamiltonians, reviewed a few years ago in Refs. 18–21.

Such fractonic models present a challenge to our understanding of the relationship between phases of matter and quantum field theories (QFT) for a number of reasons. For instance, until recently it was a piece of conventional wisdom that any gapped phase of matter has a low-energy topological QFT (TQFT) description. However, gapped fracton phases have a robust ground state degeneracy, where on a spatial $d$-torus the dimension of the ground state degeneracy grows exponentially with system size. This is incompatible with a TQFT description. Absence of a continuum field theory description for the Haah model can also be seen from the bifurcating nature of the renormalization group flow. There is, however, a description in terms of TQFT supplemented with a defect network. Moreover, the number of superselection sectors (i.e., distinct types of particle-like excitations) also diverges. This is suggestive of an infinite number of fields required in the continuum, with a nontrivial dependence on lattice scale, i.e., no obvious continuum field theory limit.

More recently, it was realized that (even if incomplete) such exotic excitations have a natural theoretical description in the language of higher-rank symmetric tensor gauge theories and complementary coupled-vector gauge theories, which exhibit restricted mobility due to an unusual set of higher (e.g., dipole) moment charge conservation. In contrast to above models, that are based on discrete symmetries, this class of $U(N)$ fractonic tensor gauge-theories exhibits gapless degrees of freedom. These are related to discrete models through a condensation of higher charge matter. Rapid recent progress in the field has established connections with numerous other areas of physics, such as localization, gravity, holography, quantum Hall systems, and deconfined quantum criticality, among many other theoretical developments.

![FIG. 1. (b) Immobility of fractonic charges enforced by dipole conservation illustrated in (a), with subdimensional mobility of dipole in (c).](image1)

![FIG. 2. The fracton-elasticity dictionary: topological defects, phonons and strains of a two-dimensional quantum crystal (right) are in one-to-one relation to charges, gauge fields and fields of the scalar-charge rank-2 tensor-gauge theory (left).](image2)
fying properties of fractonic quasi-particles. The central component of the review appears in Sec. III, where we present dualities between various quantum elastic systems such crystals, supersolids and liquid crystals and their fractonic gauge theories. In Sec. IV we utilize these gauge theoretic descriptions to discuss phase transitions between these quantum phases of matter, and most interestingly a gauge-theoretic formulation of quantum melting. Variety of field-theoretic constructions that give rise to gapless and gapped fracton phases are summarized and reviewed in Sec. V. Synthesis, open questions and future directions are relegated to the concluding Sec. VI.

II. GENERAL PERSPECTIVE ON RESTRICTED MOBILITY

As we will see throughout the Colloquium, fracton excitations emerge in a wide variety of very different physical systems. Consequently, it is useful to decouple the phenomenon of restricted mobility of excitations from a specific model or a physical origin that enforces it. Thus, in this section we present a general, realization-independent formulation of excitations with restricted mobility, taking a symmetry-based approach. In all systems we consider it will be assumed that the total number (or charge) of fractons is conserved. The conservation is the fracton-disclination (gauge-elasticity) duality summarized in Fig.2.

\[ Q_{i_1i_2\ldots i_n} = \int d^dx \, x_{i_1}x_{i_2}\ldots x_{i_n} \rho(x) . \]  

In order to enforce such conservation law we demand the current to be of a special form

\[ J^i = \partial_{i_1} \partial_{i_2} \ldots \partial_{i_n} J^{i_1i_2\ldots i_n} , \]  

where \( J^{i_1i_2\ldots i_n} \) is a symmetric tensor of rank \( n+1 \). It describes the transport of \( n \)-th multipole moment \( Q_{i_1i_2\ldots i_n} \) in the direction \( \hat{x}_i \). Combining Eqs.1–4 we find that all multipole moments up to the \( n \)-th moment are conserved (assuming that all fields decay to 0 at infinity)

\[ \partial_0 Q_{i_1i_2\ldots i_k} = 0, \quad k \leq n . \]  

Conservation law (4) implies restricted mobility because generic motion of charges will change the multipole moment of the system. The symmetry leading to these conservation laws is an extension of spatial symmetries, dubbed multipole algebra, and has non-trivial commutation relations with generators of rotation and translations. Tensor and coupled vector gauge theories discussed later emerge upon gauging this symmetry.

![Superfluid-Superconductor Duality](image)

**FIG. 3.** The boson-vortex duality in a 2+1d XY model (also known as superfluid-superconductor duality), whose tensor generalization is the fracton-disclination (gauge-elasticity) duality summarized in Fig.2.

\[ \begin{align*}
\text{Superfluid} & : vortices, \quad \nabla \times j = \rho \\
\text{Maxwell Gauge Theory (with matter)} & : \text{particles, Gauss law: } \nabla \cdot E = \rho \\
\text{Goldstone mode } \phi & : H = \frac{1}{2} \int d^2x \, [\nabla \phi]^2 + n^2 \\
\text{Photon } A & : H = \frac{1}{2} \int d^2x \, [E]^2 + (\nabla \times A)^2 \\
[n, \phi] & = i \\
[A_i, E_j] & = i \delta_{ij}
\end{align*} \]

To make this generic formulation more concrete we consider a theory with a conserved dipole moment, \( Q_i \), and examine its two fracton excitations of equal and opposite charge located at \( x_1 \) and \( x_2 \). Let \( T_i(a_i) \) be an operator that translates \( i \)-th fracton by vector \( a_i \). Then dipole conservation implies

\[ \langle x_1, x_2 \vert T_1(a_1) T_2(a_2) \vert x_1, x_2 \rangle \propto \delta^{(d)}(a_1 - a_2) . \]
The same conclusion can also be reached directly from the constraint on the current \( J^i = \partial_i J^{i\dagger} \). For simplicity focussing on 2d, we consider a wide strip, extended in \( \hat{x}_2 \) direction, and assume that the system is in a homogeneous steady state. The total charge current flowing through a cross section of a strip in \( \hat{x}_1 \) direction is

\[
J^i_{tot} = \int_{-\infty}^{\infty} dx_2 (\partial_1 J^{i\dagger} + \partial_2 J^{2i}) = \partial_1 \int_{-\infty}^{\infty} dx_2 J^{1i} = 0,
\]

where in the last step we used homogeneity. Indeed, if the charges can only move around in the form of bound dipoles, the total current through any cross section in a homogeneous state will always be 0.

Complementarily, we can define a microscopic dipole current \( J^{ij} = x^i J^j \), with the dipole density \( \rho^j \) satisfying

\[
\partial_0 \rho^j + \partial_j J^{ij} = J^i.
\]

This demonstrates that motion of monopoles \( (J^i = 0) \) generates dipoles, thereby violating their continuity equation, and equivalently dipole conservation demands a vanishing of monopole current, \( J^i = 0 \), i.e., immobility of fractonic monopoles \( \text{(162)} \).

The conservation laws discussed above do not cover all variety of mobility-constrained systems. There are two important generalizations one has to consider. First, Eq.\((6)\) assumes that the tensor current transforms in a representation of \( SO(d) \). Since most systems supporting fractons are initially formulated on a lattice, there is no \textit{a priori} reason for \( SO(d) \) symmetry to be relevant. Indeed, the current may transform in representations of a point group symmetry. This is crucial to fit lattice models with discrete symmetries, like the X-cube, Chamon and Haah codes into this framework. Second, the charge density \( \rho \) itself can be assumed to be a tensor transforming in some representation of either \( SO(d) \) or its point subgroup. In this case the relevant multipole moments are the elementary excitations and cannot be divided further into smaller moments or point charges. Finally, when the charge conservation is broken down to \( \mathbb{Z}_p \), the multipole conservation still imposes modulo-\( p \) constraints on the system. However, a general theory of such systems is not yet developed.

## III. FRACTONS AS CRYSTALLINE DEFECTS

In this Section we describe an intriguing connection of fractonic dynamics to the restricted mobility of positional (dislocations) and orientational (disclinations) topological defects in a familiar quantum two-dimensional crystal \( \text{(163)} \). Thus, a quantum crystal provides the only physical realization of a fractonic system known to date. Building on this we will further show how the corresponding coupled \( \text{U}(1) \) vector (and the equivalent symmetric tensor-) gauge theory provides a theory of quantum melting of a crystal and discuss intermediate liquid crystal phases. This Section will introduce the main ideas from the theory of particles with restricted mobility as well as tensor and multipole gauge theories.

### A. Particle-vortex duality

As a warm up for fractonic duality of crystals and liquid crystals we briefly review the standard 2+1d particle-vortex, or equivalently \( \text{(162)} \) XY-to-Abelian-Higgs model duality \( \text{(163)} \). The low energy effective Lagrangian describing 2+1d bosons is given by \( \text{(163)} \)

\[
\mathcal{L} = \frac{K}{2} (\partial_\mu \phi)^2,
\]

where \( \phi \) is the superfluid phase, and we used the units in which the speed of sound is 1. In the presence of vortices, the boson current \( \partial_\mu \phi \) can be decomposed into a smooth and singular (vortex) parts, \( \partial_\mu \phi = \partial_\mu \phi_s + v_\mu \).

The circulation of the latter,

\[
\epsilon^{\mu\nu\rho} \partial_\nu v_\rho = (\partial \times v)^\mu = j^\mu_v,
\]

gives the vortex 3-current \( j^\mu_v = (\rho^\nu, j^\nu_v) \).

Introducing a Hubbard-Stratonovich field \( J_\mu \), transforms the Lagrangian \( \text{(8)} \) into

\[
\mathcal{L} = \frac{K^{-1}}{2} J_\mu J^\mu + (\partial_\mu \phi_s + v_\mu) J^\mu.
\]

Integrating out the smooth component of \( \phi \) enforces the boson continuity equation

\[
\partial_\mu J^\mu = 0,
\]
solved in terms of a vector potential \( J^\mu = \epsilon^{\mu\nu\rho} \partial_\nu A_\rho \equiv (\partial \times A)^\mu \), with the physical current invariant under a gauge transformation
\[
\delta A_\mu = \partial_\mu \chi .
\] (12)

The Lagrangian [8] then takes a form of a U(1) gauge theory
\[
\hat{\mathcal{L}} = \frac{K^{-1}}{2} (\partial \times A)^2 + j^\nu_\mu A_\mu ,
\]
\[
= \frac{K^{-1}}{2} (E^2 - B^2) + j^\nu_\mu A_\mu ,
\] (13)

where the electric field \( E_\mu = \partial_\mu A_\mu \) and the magnetic field \( B = \epsilon^{\mu\nu\rho} \partial_\nu A_\rho \) respectively describe the spatial components of bosonic current and number densities, and under the duality the vortex 3-current \( j^\nu_\mu \) enters Lagrangian as the dual matter.

In an equivalent Hamiltonian description, illustrated in Fig.3 bosons are described by
\[
\hat{\mathcal{H}} = \frac{1}{2} (\hat{\nabla} \phi)^2 + \frac{1}{2} \hat{\rho}^2 ,
\] (14)

in terms of canonically conjugate \( [\hat{\rho}, \hat{\phi}] = i\delta^{(2)}(x) \), and vortex singularities, \( \nabla \times \nabla \phi = \rho_c \). On the dual side, rotating the boson current \( \nabla \phi \) lines by \( \pi/2 \) transforms them into electric field lines \( E \) and boson density into dual flux density \( B \), with \( [\hat{A}, \hat{E}] = i\delta^{(2)}(x) \). This then transforms the bosonic Hamiltonian into the dual Maxwell theory
\[
\hat{\mathcal{H}} = \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} (\nabla \times \hat{A})^2 - j_\nu A_\nu .
\] (15)

It is more formally obtained from the dual Lagrangian [13] by introducing an independent electric field \( \hat{E} \) as the Hubbard-Stratonovich field to decouple the spatial (electric field energy) part of the Maxwell term. The Integration over the time-component \( A_0 \) then gives the Gauss’s law,
\[
\nabla \cdot E = \rho_c ,
\] (16)

that generates \( \rho \), and is the dual counter-part of the circulation constraint, \( \Psi \).

We thus recover the celebrated Dasgupta-Halperin duality [15, 16] where a bosonic liquid is dual to vortex matter, described by a complex scalar field, \( \Psi \), minimally coupled to a U(1) gauge field \( A_\mu \),
\[
\mathcal{L} = i\Psi^* (\partial_\mu - iA_\mu) \Psi - \frac{1}{2m} (\partial_\mu - iA_\mu) \Psi^2 + V(|\Psi|) + \frac{K^{-1}}{2} (\partial \times A)^2 .
\] (17)

It is also known as the Abelian-Higgs model of a dual superconductor, with \( V \) a generic U(1) symmetric potential. In this duality, a superfluid and Mott insulating phases of bosons thus respectively correspond to a dual-normal (dual-non-superconducting, \( \Psi = 0 \)) state and a dual-superconducting vortex condensate (\( \Psi \neq 0 \) dual-Higgs) phase.

### B. Fracton-discussion duality: commensurate crystal

#### 1. Tensor gauge theory duality

The duality of an elastic medium is an elegant, technically straightforward tensor generalization [25, 26] of the above XY-duality, where the phonons \( u_i \), dislocations and disclinations are respective vector counter-parts of the superfluid phase \( \phi \) and vortices. Since 2d elastic medium is described by phonon Goldstone modes that are spatial vectors, it is not surprising that their dual is captured by a tensor (rather than vector) gauge field \( A_{ij} \).

![FIG. 6. Target-space rotational symmetry of a crystal, encoded in a symmetric strain tensor \( u_{ij} \), appearing in its elastic energy.](image)

To this end [26], we begin with a low-energy elastic description of a 2 + 1 dimensional quantum crystal, captured by a harmonic Lagrangian,
\[
\mathcal{L} = \frac{1}{2} (\partial_\mu u_{ij} \partial^\mu u_{ij}) - \frac{1}{2} C^{ijkl} u_{ij} u_{kl} ,
\] (18)

where \( u_{ij} = \frac{1}{2} (\partial_\mu u_{ij} + \partial_\nu u_{ij}) \) is the linearized symmetric strain tensor (encoding target-space rotational invariance in Fig. 5), rank-4 tensor \( C^{ijkl} \) encodes elastic moduli and underlying crystal symmetry, and we specialized to unit mass density [11].

To include topological defects, we express the displacement in terms of smooth phonon and singular defects components \( u_i = u_i^s + u_i^d \), with latter accounting for disclination density \( \rho \),
\[
\epsilon^{ik} \epsilon^{jl} \partial_\mu \partial_\nu u_{kl} = \rho ,
\] (19)

and implicitly for dislocations as they are dipole bound states of two disclinations [12]. Introducing the Hubbard-Stratonovich momentum, \( \pi^i \) and symmetric stress \( \sigma^{ij} \) fields and integrating over the phonons \( u_i^s \), enforces momentum continuity \( \partial_\mu \pi^i + \partial_\nu \sigma^{ij} = 0 \). To make contact with electromagnetism, it is convenient to introduce dual vector magnetic \( B_i = \epsilon^{ij} \pi_j \) and symmetric tensor electric \( E_i^j = \epsilon^{ikl} \sigma_{kl} \) fields in terms of which the momentum continuity equation takes the form of a generalized Faraday law,
\[
\partial_\mu B^\mu + \epsilon_{ijk} \partial^\nu E_{ij}^k = 0
\] (20)

of the scalar-charge tensor gauge theory. As in a conventional electromagnetism, latter can be solved in terms
of gauge fields, here symmetric tensor, $A_{ij}$ and scalar $A_0$, \[ B_i = \epsilon^{kl} \partial_k A_{li} \,, \quad E_{ij} = -\partial_0 A^{ij} + \partial^\ell \partial_\ell A_0 \,, \tag{21} \]

invariant under the gauge transformations \[ \delta A_{ij} = \partial_i \partial_j \chi \,, \quad \delta A_0 = \partial_0 \chi \,. \tag{22} \]

The elasticity Lagrangian (18) then takes on a Maxwell-like form \[ \mathcal{L} = \frac{1}{2} C_{ijkl} E_{ij} E_{kl} - \frac{1}{2} B^2 + \rho A_0 - J^{ij} A_{ij} \,, \tag{23} \]

where $C^{-1}_{ijkl} = C_{mnpq} \epsilon^i_m \epsilon^j_n \epsilon^k_p \epsilon^l_q$ and $E_{ij} = C^{-1}_{ijkl} F_{kl}$. The dual gauge fields are sourced by crystalline topological defects through the last two terms, (23) with dislocation current given by \[ J^{ij} = \epsilon^{ik} \epsilon^{j\ell} (\partial_\ell \partial_i - \partial_i \partial_\ell) u_\ell \,. \tag{24} \]

Gauge invariance (22) enforces that the dual charge and currents densities $\rho$ and $J^{ij}$ satisfy a continuity equation \[ \partial_\ell \rho + \partial_i \partial_j J^{ij} = 0 \,, \tag{25} \]

with a current indeed constrained in a manner discussed in Section 11 as required on general grounds. Notably, the immobility of fractonic disclination charges $\rho$ is reflected in the absence (vanishing) of fracton current. Gauss’s law generating the gauge transformations (22) can be read out from (21), (23) and appears as a constraint after integration over the scalar potential $A_0$ and takes the form \[ \partial_i \partial_j E_{ij} = \rho \,. \tag{26} \]

It is the counterpart of the topological disclination condition (19), as illustrated in Fig.2.

For an isotropic (e.g., triangular) crystal, the elastic tensor $C_{ijkl}$ reduces to two independent Lamé moduli $\mu$ and $\lambda$ and dual Maxwell-like theory [28] exhibits two propagating gapless degrees of freedom corresponding to transverse and longitudinal phonons. This demonstrates that a quantum crystal is dual to the so-called traceless scalar charge gauge theory [20,21,22], and gives the only known physical realization of fractonic matter.

We note that a conservation of the number of occupied lattice sites, i.e., absence of vacancy and interstitial point defects, characteristic of a “commensurate” crystal implies that $J^{ij}$ is traceless, further constraining the motion of defects [31,32]. Equation (25) then implies that total dual charge, dipole moment and trace of the quadrupole moment are conserved \[ \partial_0 Q = 0 \,, \quad \partial_0 Q_i = 0 \,, \quad \partial_0 \text{tr} (Q_{ij}) = 0 \,. \tag{27} \]

Identification of charges, dipoles and quadrupoles of this fracton phase $F_{U(1)}$ with disclinations, dislocations (with Burgers vector perpendicular to the dipole moment) and vacancies/interstitials [33,73–75] gives an elegant gauge theory formulation of their quantum dynamics in a “commensurate” crystal. Namely, the fracton charges encode the immobility of disclinations, the duality prediction [31] that can also be understood directly in terms of crystal degrees of freedom, as illustrated in Fig. 3. The planeon dipoles, with motion constrained (by the $U(1)$ conservation of the trace of quadrupoles in (27)) transverse to the dipole moment, correspond to the well-known glide-only constraint, that, in the absence of vacancies and interstitials prevents dislocations from climbing perpendicular to their Burgers vector [33,73–75] as illustrated in Figs.4(a) and 12. In Sec. 11 E we will discuss in more detail the relaxation of this $U(1)$ symmetry-enriched constraint, that leads to a qualitatively distinct fractonic state $F$, corresponding to an incommensurate crystal, i.e., a supersolid.

2. Disclination field theory

With crystalline defects appearing as dual matter that sources tensor gauge fields, the formulation extends to a concrete field theoretic representation of fractonic matter, $\rho$ and $J^{ij}$. To this end, we describe disclinations by a complex scalar field $\Phi$, treating them as bosonic excitations. The Lagrangian \[ \mathcal{L} = \mathcal{L}_s [A_{ij}, \Phi] + \mathcal{L}_{\text{Max}} [A_{ij}] \]

is a sum of the dual disclination matter sector, $\mathcal{L}_s$ and the Maxwell-like sector, $\mathcal{L}_{\text{Max}}$, derived in the previous subsection, capturing the phonon degrees of freedom. The dual matter Lagrangian, $\mathcal{L}_s$ takes the form \[ \mathcal{L}_s = i \Phi^\ast (\partial_0 - i A_0) \Phi + g (|D_1(\Phi)|^2 + |D_2(\Phi)|^2) + V(\Phi) \,, \tag{28} \]

where $D_1(\Phi)$ are differential operators, bilinear in $\Phi$, given by \[ D_1(\Phi) = \Pi^i_j (\partial_i \Phi \partial_j \Phi - \Phi \partial_i \partial_j - i A_{ij} \Phi) \,, \quad \Pi^i = (\sigma_1, \sigma_3) \,, \tag{29} \]

where $\sigma_1, \sigma_3$ are the Pauli matrices. In the absence of gauge fields, the Lagrangian (28) exhibits a conservation law [25] due to the presence of an unusual “global” symmetry, \[ \Phi' = e^{i \chi_s(x)} \Phi \,, \quad \chi_s(x) = \alpha + \beta_i x^i + \frac{1}{2} \gamma |x|^2 \,, \tag{30} \]

characterized by phase $\chi_s(x)$, which by the virtue of Noether theorem leads to the conservation of $U(1)$ charge, dipole and trace of the quadrupole moments. In the presence of tensor gauge fields, transforming according to (22), the Lagrangian (28) is also invariant under a general gauge transformation $\Phi \rightarrow e^{i \chi(x)} \Phi$, with arbitrary $\chi(x)$. As in conventional gauge theories, here the disclination density and dislocation current are given as variational derivatives \[ \rho = \frac{\delta S}{\delta A_0} \,, \quad J^{ij} = \frac{\delta S}{\delta A_{ij}} \,. \tag{31} \]

It follows from (29) that the current $J^{ij}$ is indeed traceless as expected [10].
3. Dislocation field theory

The disclination field theory, $\mathcal{L}_s$ is quartic in the fields and thus does not admit a weak-coupling quadratic representation, reflecting its strongly interacting UV degrees of freedom. Indeed, in real crystals disclinations appear in tightly bound lattice-scale dislocation dipoles. To construct their field theory, $\mathcal{L}_d$, we observe that a dislocation with a Burgers vector $b$ (an elementary lattice vector) is created by an operator $\psi_d^+$ labeled by dislocation dipole $d$ (corresponding to $\rho_l$ density of Sec. II and $p$ in the following sections), with the corresponding coherent state field,

$$\psi_d(x) = \Phi \left( x - \frac{d}{2} \right) \Phi^* \left( x + \frac{d}{2} \right),$$

and is bi-local in the disclination fields, with $b = d \times \hat{z}$. Under the global symmetry $G_0$, in the $d \to 0$ limit, the dipole operator transforms as

$$\psi_d(x) \rightarrow e^{i\beta d^i + i\alpha_d x^i} \psi_d(x),$$

where the first factor is a global phase enforcing $U(1)$ symmetry, while the second factor enforces the glide-only constraint.

In constrast to a strongly interacting disclination Lagrangian [28] (that has no noninteracting limit), mobile dislocations admit a weakly interacting Lagrangian. Its form is constrained by the generalized global (subsystem) symmetry [33] as well as the gauge symmetry [22] and is uniquely given by [32, 34]

$$\mathcal{L}_d = \sum_d i\bar{\psi}_d^* \left( \partial_0 - iA_0 \right) \psi_d - \frac{P^{ij}}{2m} \left( \partial_i + id_k A_{ik} \right) \bar{\psi}_d^* \left( \partial_j - id_l A_{jl} \right) \psi_d + V(|\psi_d|^2),$$

where $P^{ij} = \left( \delta^{ij} - \frac{d^i d^j}{d \cdot d} \right)$ is the projector onto axis perpendicular to the dipole moment $d$ (i.e., along the Burgers vector, $b = \hat{z} \times d$), enforcing the glide-only constraint, dictated by [33]. The dipole moment enters as the “charge” of the field $\psi_d$ under the tensor gauge field $A_{ij}$. The Lagrangian [34] is also invariant under discrete transformations that permute primitive lattice vectors, corresponding to the point group symmetry of the crystal.

4. Coupled vector gauge theory duality

A complementary, more convenient and transparent formulation of crystal’s dual gauge theory is that in terms of coupled $U(1)$ vector gauge fields, introduced and detailed in Ref. [34]. The idea is in fact quite simple, and is based on the observation that elasticity, formulated in terms of the (unsymmetrized) strain $\partial_i u_k$ has a form of two flavored $u_x$ and $u_y$ XY-models. Of course independent XY-models would dualize to conventional non-fractonic $U(1)$ vector gauge theories. Thus, it is the non-trivial “flavor-space” phonon coupling (symmetrization of $\partial_i u_k$ in the conventional formulation) that is responsible for the appearance of fractons [22, 34].

To get to an equivalent flavored vector gauge theory description, we reformulate the conventional elastic theory [18] in terms of “minimally”-coupled quantum XY-models, introducing the orientational bond-angle field, $\theta$ and its canonically conjugate angular momentum density $L$. The Lagrangian density (for simplicity taking elastic constants tensor $C_{ijkl}$ to be characterized by a single elastic constant $C$) is given by

$$\mathcal{L} = \pi_k \partial_0 u_k + L \partial_0 \theta - \frac{1}{2} \pi_k^2 - \frac{1}{2} C (\partial_1 u_k - \theta \epsilon_{ik})^2 - \frac{1}{2} L^2 - \frac{1}{2} K (\nabla \theta)^2.$$  

Coulping of the unsymmetrized strain $\partial_i u_k$ to $\theta \epsilon_{ik}$, “Higgses” its anti-symmetric part below a scale set by $C$, reducing $\mathcal{L}$ to standard crystal elasticity in terms of the symmetrized strain $u_{ik}$ [15], which is the starting point of Ref. [31]. This reformulates 2d elasticity in terms of two translational XY models for two phonons, $u_k$, coupled by the orientational XY model for the orientational bond field, $\theta$. To dualize $\mathcal{L}$, we decouple the elastic and orientational energies in [35] via Hubbard-Stratonovich vector fields, stress $\sigma_k$ (with flavor index $k$ inherited from $\nabla u_k$) and torque $\tau$. We then introduce disclinations $\nabla \times \nabla \theta^\sigma = \frac{2\pi s}{n} \delta^2(r) \equiv \rho(r)$ with charge $2\pi s/n$ (s = ±Z and n = 6 for hexagonal crystal) and their dipoles, dislocations, $\nabla \times \nabla u^\sigma_k = b_k \delta^2(r) \equiv b_k(r)$ with Burgers charge $b_k$ and integrate over the single-valued elastic components of $\theta$ and $u_k$. This enforces the conservation of linear and angular momenta, $\partial_0 \pi - \nabla \cdot \sigma = 0$ and $\partial_0 \tau - \nabla \cdot \tau = \epsilon_{ij} \epsilon_{kl} \equiv \sigma_{ij}$.

Expressing this linear momentum constraint in terms of dual magnetic and electric fields, $\pi_k = \epsilon_{kj} B_j$, $\sigma_k = -\epsilon_{ij} \epsilon_{kl} E_{jk}$, gives $k$-flavored Faraday equations, $\partial_0 B_k + \nabla \times E_k = 0$, solved by $k$-flavored vector $A_k$ and scalar $A_{0k}$ gauge potentials, $B_k = \nabla \times A_k$, $E_k = -\partial_0 A_k - \nabla A_{0k}$. We emphasize that, in contrast to the symmetric tensor approach [32, 33] here, the $k = (x, y)$-flavored vector gauge field $A_k$ has components $A_{ik}$ that form an unsymmetrized tensor field.

The conservation of angular momentum can now be solved with another set of vector $a$ and scalar $a_0$ gauge fields,

$$L = \nabla \times a + A_a, \quad \tau_k = \epsilon_{kj} (\partial_0 a_j + \partial_j a_0 - A_{0j}),$$

leading to the dual Lagrangian density,

$$\mathcal{L}_{\text{cr}} = \frac{1}{2} C^{-1} (\partial_0 A_k + \nabla A_{0k})^2 - \frac{1}{2} (\nabla \times A_k)^2 + \frac{1}{2} K^{-1} (\partial_0 \theta_k + \partial_0 a_0 - A_{0k})^2 - \frac{1}{2} (\nabla \times a + A_a)^2 + A_k \cdot J_k - A_{0k} p_k + a \cdot j - a_0 \rho,$$
where the dipole charge $p_k$ is given by the dislocation density $p_k = (\mathbf{z} \times \mathbf{b})_k$, the fracton charge $\rho$ is the disclination density, and the corresponding currents are given by $J_k = \epsilon_{ijk} \mathbf{z} \times (\partial_\theta \nabla u_l - \nabla \partial_\theta u_l)$ and $\mathbf{j} = \mathbf{z} \times (\partial_\theta \mathbf{b} - \nabla \partial_\theta \mathbf{b})$.

The corresponding Hamiltonian density

$$
\mathcal{H}_{cr} = \frac{1}{2} C |\mathbf{E}_k|^2 + \frac{1}{2} (\nabla \times \mathbf{A}_k)^2 + \frac{1}{2} K |\mathbf{e}|^2 \\
+ \frac{1}{2} (\nabla \times (\mathbf{a} + \mathbf{A}_k))^2 - \mathbf{A}_k \cdot \mathbf{J}_k - \mathbf{a} \cdot \mathbf{j},
$$

(38)

involves three $U(1)$ vector gauge fields with electric fields $\mathbf{E}_k$ (flavors $k = x,y$) and $\mathbf{e}$, and corresponding canonically conjugate vector potentials $\mathbf{A}_k$ and $\mathbf{a}$, with former gauging the latter through a 2-form $A_a = \epsilon_{ik} A_{ik}$ minimal coupling, $(da - A)^2$. This translational and orientational gauge fields coupling encodes a semi-direct product of spatial translations and rotations, constrained by the generalized gauge invariance.\(^\text{131}\)

The Hamiltonian is supplemented by Gauss’s laws,

$$
\nabla \cdot \mathbf{E}_k = p_k - e_k, \quad \nabla \cdot \mathbf{e} = \rho.
$$

(39)

Crucially, the components of the electric field $e_k$ (that would be generated by the motion of charges $\rho$) appear as an additional dipole charge in the dipole gauge’s law for $\mathbf{E}_k$,\(^\text{39}\), and its conservation thus encodes the fractonic immobility of disclination charges, $\rho$. Equivalently, we note that the continuity equation for dipole densities $p_k$,

$$
\partial_\theta p_k + \nabla \cdot \mathbf{J}_k = \mathbf{j}, \quad \partial_\theta \rho + \nabla \cdot \mathbf{j} = 0,
$$

(40)

is violated by the charge current $\mathbf{j}$, which thus must vanish in the absence of dipole charges, i.e., $p_k = 0 \rightarrow j_k = 0$, leading to immobility of fractonic charges, enforced by gauge-invariance.

Although above elasticity-gauge duality only works in 2+1d dimensions, the generalization of the gauge dual to $d$ dimensions is straightforward (though does not correspond to any physical elasticity) and consists of $d + 1$ U(1) gauge fields obeying the same Gauss’s laws but with $k = 1, \ldots, d$. The main difference in the Hamiltonian is that $(\nabla \times (\mathbf{a} + \mathbf{A}_d))^2$ is replaced by a sum of the $d(d-1)/2$ terms of the 2-form minimal coupling $(\partial_i a_j - \partial_j a_i + A_{ij} - A_{ji})^2 = (da - A)^2$. At low energies this coupled $U(1)$ vector gauge theory reduces to the $d$-dimensional scalar-charge tensor gauge theory.\(^\text{131}\)

C. Fracton-disclination duality: smectic

A smectic state of matter, characterized by a uniaxial spontaneous breaking of rotational and translational symmetries is ubiquitous in classical liquid crystals of highly anisotropic molecules (e.g., classical 5CB).\(^\text{43}\) Its quantum realizations range from “striped” states of a two-dimensional electron gas at half-filled high Landau levels\(^\text{83,101}\) and “striped” spin and charge states of weakly doped correlated quantum magnets\(^\text{82,91}\) to the putative Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) paired superfluids\(^\text{14,55}\) in imbalanced degenerate atomic gases\(^\text{46,57}\), ferromagnetic transition in one-dimensional spin-orbit-coupled metals\(^\text{39}\), spin-orbit coupled Bose condensates\(^\text{67,100}\), as well as helical states of bosons or spins on a frustrated lattice.\(^\text{101}\) We thus next review a smectic dual gauge theory representation and the associated quantum melting transitions from surrounding crystal and nematic phases.

A simplest description of a 2+1d quantum smectic,\(^\text{66,67}\) is in terms of a single phonon Goldstone mode, with a Lagrangian density,

$$
\mathcal{L}_{sm} = \frac{1}{2} \nabla u \cdot \nabla u - \frac{\kappa}{2} (\partial_\theta u)^2 - \frac{K}{2} (\partial_\theta^2 u)^2,
$$

(41)

a close cousin of the quantum Lifshitz model.\(^\text{102,112}\)

This more familiar low-energy universal description naturally emerges from a more “microscopic” (low-energy equivalent) formulation in terms of a phonon (layer displacement) $u = u(r) \hat{y}$ and the unit-normal (layer orientation) $\mathbf{n}(r) = -\hat{x} \sin \theta + \hat{y} \cos \theta \equiv \hat{y} + \delta \hat{n}$ field operators, and the corresponding canonically conjugate linear and angular momentum fields, $\pi(r)$ and $L(r)$, with the Hamiltonian density,

$$
\mathcal{H}_{sm} = \frac{1}{2} \pi^2 + \frac{1}{2} L^2 + \frac{1}{2} \kappa (\nabla u + \delta \mathbf{n})^2 + \frac{1}{2} K (\nabla \mathbf{n})^2,
$$

(42)

where $\kappa, K$ are elastic constants.

Working in a phase-space path-integral formulation, the corresponding Lagrangian density is,

$$
\mathcal{L}_{sm} = \pi \partial_\theta u + L \partial_\theta \theta - \frac{\kappa}{2} \pi^2 - \frac{1}{2} L^2 + \frac{1}{2} \kappa^{-1} \sigma^2 + \frac{1}{2} K^{-1} \tau^2 \\
- \mathbf{\sigma} \cdot (\nabla u - \hat{x} \theta) - \mathbf{\tau} \cdot \nabla \theta.
$$

(43)

In above, we neglected $\theta$ nonlinearities, took the $x$-axis to be along the smectic layers, and, as with a crystal duality of previous subsection, for later convenience introduced Hubbard-Stratonovich fields $\mathbf{\sigma}$ and $\mathbf{\tau}$, corresponding to the local stress and torque, respectively. Integrating over the auxiliary fields $\pi, L, \mathbf{\sigma}, \mathbf{\tau}$ easily recovers the phonon-only Lagrangian in Eq. (41).

The form (43) allows us to separate Goldstone modes into smooth and singular (defects) components and to functionally integrate over the smooth, single-valued parts of the phonon $u$ and orientation $\theta$ fields. With this, we obtain coupled linear and angular momenta conservation constraints,

$$
\partial_\theta \pi - \nabla \cdot \mathbf{\sigma} = 0, \quad \partial_\theta L - \nabla \cdot \mathbf{\tau} = \hat{x} \cdot \mathbf{\sigma},
$$

(44)

Solving these in terms of gauge fields,

$$
\pi = \hat{z} \cdot (\nabla \times \mathbf{A}), \quad \mathbf{\sigma} = \hat{z} \times (\partial_\theta \mathbf{A} + \nabla A_0),
$$

(45)

$$
L = \hat{z} \cdot (\nabla \times \hat{x} \times \mathbf{A}), \quad \mathbf{\tau} = \hat{z} \times (\partial_\theta \mathbf{A} + \nabla a_0 - \hat{x} A_0).
$$

(46)

This allows us to express smectic’s Lagrangian density in terms of these Goldstone-mode encoding gauge fields, and to
obtain the Maxwell part of the smectic dual Lagrangian,
\[
\mathcal{L}^\text{sm}_M = \frac{1}{2K} (\partial_\mu A + \nabla A_\mu)^2 - \frac{1}{2} (\nabla \times A)^2 
+ \frac{1}{2K} (\partial_\mu a + \nabla a_\mu - A_\mu \hat{x})^2 - \frac{1}{2} (\nabla \times a - \hat{x} \times A)^2,
\]
Similarly to crystal's gauge-dual of previous subsection, the smectic's gauge dual displays a nontrivial "minimal" coupling between the translational and orientational gauge fields, and exhibits a generalized gauge invariance under transformations,
\[
(A_\mu, A) \rightarrow A'_\mu = (A_\mu - \partial_\mu \phi, A + \nabla \phi),
\]
\[
(a_\mu, a) \rightarrow a'_\mu = (a_\mu - \partial_\mu \chi, a + \nabla \chi - \hat{x} \phi).
\]
The six gauge field degrees of freedom \(A_\mu, a_\mu\) reduce to two physical Goldstone modes after gauge fixing \(\phi, \chi\) and implementing two Gauss's law constraints \([51]\).

To include dislocations and disclinations we allow for the nonsingle-valued component of \(u\) and \(\theta\), respectively defined by
\[
p = \hat{z} \cdot \nabla \times \nabla u, \quad J = \hat{z} \times (\nabla \partial_\mu u - \partial_\mu \nabla u),
\]
\[
\rho = \hat{z} \cdot \nabla \times \nabla \theta, \quad j = \hat{z} \times (\nabla \partial_\mu \theta - \partial_\mu \nabla \theta).
\]
This together with \(\mathcal{L}^\text{sm}_M(A_\mu, a_\mu)\) gives the dual Lagrangian density for the quantum smectic,
\[
\mathcal{L} = \mathcal{L}^\text{sm}_M(A_\mu, a_\mu) + A \cdot J - A_\mu p + a \cdot j - a_\mu \rho,
\]
corresponding to the Hamiltonian density
\[
\mathcal{H}^\text{sm} = \frac{1}{2} \kappa E^2 + \frac{1}{2} (\nabla \times A)^2 + \frac{1}{2} K \epsilon^2
+ \frac{1}{2} (\nabla \times a - \hat{x} \times A)^2 - A \cdot J - a \cdot j,
\]
supplemented by the generalized Gauss's law constraints,
\[
\nabla \cdot E = p - e \cdot \hat{x}, \quad \nabla \cdot e = \rho.
\]
\(\rho\) and \(J\) are \(x\)-dipole charge and current densities, representing \(y\)-dislocations, while \(\rho\) and \(j\) are fractonic charge and current densities, corresponding to disclinations. The generalized gauge invariance of \([50]\) imposes coupled continuity equations for the densities
\[
\partial_\mu p + \nabla \cdot J = -j \cdot \hat{x}, \quad \partial_\mu \rho + \nabla \cdot j = 0,
\]
where dipole conservation is violated by a nonzero charge current, \(j \cdot \hat{x}\) along smectic layers.

These equations thus transparently encode the restricted mobility of the disclination charges \(\rho\), illustrated in Fig. 10 with a relaxation rate \(I_k = DK^2 + \gamma k^4\), resulting in slow subdiffusive decay \(\rho(t) \sim t^{-3/4}\), and mobility and diffusion coefficient vanishing along the \(x\)-directed smectic layers, i.e., \(j \cdot \hat{x} = 0\) in the absence of dislocation dipoles\([29,113,117]\).

### D. Quantum melting

Starting with the gauge-dual of the quantum crystal (derived in the earlier subsections \([III B 1]\)), either in tensor, \([34]\) or in its equivalent coupled-vector gauge theory form, \([37]\), and minimally coupling it to dynamic dislocation-dipole matter, \(\psi_d\), gives a quantum Lagrangian density,
\[
\mathcal{L}_\text{ct} = \sum_{k=1,2} \frac{J_k}{2} |(\partial_\mu - iA_\mu^a)\psi_k|^2 - V(\psi_k)
+ \mathcal{L}_M^s(A_1, A_2, a_1, a_2),
\]
where \(\psi_{k=1,2}\) correspond to \(p_{1,2}\) dipoles (two elementary dislocations), \(A_{k,1,2,\mu}\) gauge fields capture the \(k\)-th phonons and bond orientational Goldstone modes (with unit dipole-charges \(p_k\) absorbed in the definitions of \(A_{k,\mu}\)), and \(V(\psi_1, \psi_2)\) a \(U(1)\) symmetric Landau potential, with a dual Maxwell Lagrangian \([37]\). This field theory\([105]\) thus gives a complete description of the quantum melting transitions out of the fractonic crystal state. The nature of such quantum transition is then dictated by the form of the dipole interaction potential, \(V(\psi_1, \psi_2)\).

#### 1. Crystal-to-hexatic Higgs transition

In a 2d hexagonal (square) crystal, the \(C_6\) (\(C_4\)) invariance enforces the symmetry between \(\psi_1\) and \(\psi_2\) dislocation dipoles, i.e., their interaction potential \(V(\psi_1, \psi_2)\) is \(1 \leftrightarrow 2\) symmetric. Thus, as illustrated in Fig. 13 driven by quantum fluctuations, both types of dipoles unbind and Bose-condense at a single Higgs transition, that is the quantum crystal-to-hexatic (tetratic) melting, the counterpart of the famous 2d classical thermal melting transition, predicted by Kosterlitz-Thouless, Halperin-Nelson, and Young\([106,115,119,157]\). The Higgs transition thus gaps out both translational gauge fields, \(A_{1,\mu}\) and \(A_{2,\mu}\), which can therefore be safely integrated out, or to lowest order effectivley set to zero. This reduces the coupled gauge theory to a conventional Maxwell form for the remaining rotational gauge field \(a_\mu\), with
\[
\mathcal{L}_\text{hex}^s(a_\mu) \approx \mathcal{C}_M^{ct}(A_{1,2,\mu} \approx 0, a_\mu) = \frac{1}{2} K^{-1} \epsilon^2 - \frac{1}{2} (\nabla \times a)^2.
\]
As expected it is the dual to the quantum XY-model of the orientationally-ordered quantum hexatic (tetratic) liquid\([158]\), \(\mathcal{L}_\text{hex} = \frac{1}{2} (\partial_\mu \theta)^2 - \frac{1}{2} K (\nabla \theta)^2\). As with the conventional \(U(1)\) Higgs (normal-superconductor) transition, mean-field approximation breaks down for \(\delta+1 \leq 4\), and may be driven first-order by translational gauge-fields \(A_{k,\mu}\) fluctuations\([121,122]\). Analysis of the non-mean-field criticality of this quantum crystal-superhexatic (supertetrical) melting transition remains an open problem.
FIG. 7. Illustration of quantum melting of a 2D crystal into a smectic, followed by smectic-to-nematic melting, respectively driven by a condensation of $b_x (\psi_y$ dipoles) and of $b_y (\psi_x$ dipoles) dislocations.

2. Crystal-to-smectic Higgs transition

An alternative to the above direct, continuous crystal-to-hexatic (or to-tetratic) quantum melting scenario is a two-stage transition that takes place in a uniaxial crystal where $C_6$ or $C_4$ symmetry is broken down to $C_2$, as illustrated in Fig. 7. The reduced uniaxial symmetry is necessarily encoded in the Landau potential $V(\psi_1, \psi_2)$, specifically, controlled by the quadratic term $g_k |\psi_k|^2$, with $g_2 < g_1$ leading to a condensation of $\psi_2$ dislocation dipoles, with $\psi_1$ remaining gapped.

Alternatively, this breaking of $C_6$ (or $C_4$) symmetry down to $C_2$ may happen spontaneously, with $g_1 = g_2$, instead controlled by the sign of the $v$ coupling in the dislocation interaction, $v|\psi_x|^2|\psi_y|^2$. For $v > 0$, only one of the two dipole species condenses, say $\psi_2 \neq 0$ with $\psi_1 = 0$. This Higgs transition thus only gaps out $A_{2,\mu}$, which can therefore be safely integrated out. To lowest order this corresponds to $A_{2,\mu} \approx 0$, reducing crystal’s Maxwell Lagrangian to that of the quantum smectic [41], with

$$L_{M}^{sm}(A_{1,\mu}, a_{\mu}) \approx L_{M}^{s}(A_{1,\mu}, A_{2,\mu} \approx 0, a_{\mu}). \quad (55)$$

FIG. 8. Quantum crystal-smectic duality relations and the associated quantum melting transition.

3. Quantum smectic-to-nematic Higgs transition

As illustrated in Fig. 7, the above crystal-to-smectic transition is then naturally followed by quantum melting into a nematic superfluid by condensation of $\psi_1$-dipoles (aligned with the smectic layers), i.e., a proliferation of $b_1$ dislocations with Burgers vectors transverse to smectic layers. The resulting $\psi_1 \neq 0$ Higgs phase gaps out the remaining smectic translational gauge field $A_{\mu} (= A_{\mu}^x)$, which can therefore be safely integrated out. This reduces the model to a conventional Maxwell form for the rotational gauge field $a_{\mu}$, with

$$L_{M}^{nem}(a_{\mu}) \approx L_{M}^{s}(A_{\mu} \approx 0, a_{\mu}) = \frac{1}{2} K^{-1} e^2 - \frac{1}{2} (\nabla \times a)^2, \quad (56)$$

that is a dual to the quantum XY-model of the nematic, $L_{nem} = \frac{1}{2}(\partial_\theta \theta)^2 - \frac{1}{2} K (\nabla \theta)^2$. Fluctuation corrections lead to an anisotropic stiffness and subdominant higher order gradients. As with the conventional U(1) Higgs (normal-superconductor) transition, mean-field approximation breaks down for $d+1 \leq 4$, and may be driven first-order by translational gauge-field, $A_{\mu}$, fluctuations. Analysis of the non-mean-field criticality of the quantum smectic-nematic transition also remains an open problem.

E. Supersolid, superhexatic, supersmectic: vacancies and interstitials

Vacancies and interstitials. As discussed in Sec. III B 1, so far we have neglected vacancies and interstitials, (see

While this phase transition faithfully captures the crystal-smectic melting at the mean-field level, as with the crystal-hexatic Higgs transition, its true critical properties are expected to be nontrivial and remain to be analyzed. The two ways of obtaining the smectic gauge dual – by dualizing a smectic Lagrangian and through the above Higgs melting transition of the crystal’s gauge dual – are summarized in Fig. 8.
Fig. 10, which physically corresponds to a restriction to their Mott-insulating, commensurate-crystal state. This clearly misses the additional atomic sector of the system, encoded by a Bose-Hubbard (quantum XY) model or its gauge-dual Abelian Higgs model from Sec. III A. The need for this missing vacancies/interstitials (atomic) sector is clear as quantum melting a crystal of bosons, at zero temperature and in the absence of substrate or disorder generically leads to a gapless superfluid.

![Image](321x450 to 558x519)

**FIG. 10.** Mobile vacancy (and interstitial) defects of a crystal, necessary to faithfully capture the associated supersolid and superfluid phases.

Thus, as discussed in detail in Refs. 33, 73, and 74 and illustrated in the phase diagram of Fig. 13 at zero temperature two qualitatively distinct - commensurate and incommensurate quantum crystals are possible, respectively distinguished as Mott-insulating and superfluid states of this atomic (vacancies/interstitials) sector. In the former, the U(1) symmetry-enriching constraint imposes a glide-only motion of dislocation (illustrated in Figs. 4 and 12), that is broken in the latter, where dipole dislocation motion is unconstrained.

Under duality, these two types of quantum crystals then map onto two distinct fractonic phases, F$_{U(1)}$ and F, respectively, with and without quadrupole-imposed restriction on the dipole glide-only and unrestricted motion, as illustrated in Fig. 12 b,c.

This dipole constrained motion condition is concisely encoded in the Ampere’s law of the corresponding tensor gauge theory,

$$\partial_\mu E^{ij} + \frac{1}{2} (\epsilon^{ijk} B^j + \epsilon^{ijk} B^k) = -J^{ij}$$

whose trace can be expressed in terms of vacancies-interstitials density $n_d = E^d_i + n_0 \partial_i u_0$ and the corresponding current, $J^d_i = \pi^i$,

$$\partial_\mu n_d + \partial_i \pi^i = -J^d_i$$

where we used $E^d_i = n_d - n_0 \partial_i u_0 \approx n_d$, first derived in Ref. 123. With this vacancies-interstitials continuity equation (sourced by $J^d_i$, corresponding to the longitudinal (along-dipole) motion of dipoles), Eq. 58 encodes that climb of dislocations creates vacancy/interstitial defects. Conversely, Mott-insulating (commensurate-crystal) state of the latter, restricts the motion of dislocations to glide-only lineon type, dualizing to a symmetry-enriched fracton state F$_{U(1)}$. It can then undergo a quantum phase transition to a distinct fracton state F, when vacancies and interstitials Bose-condense, thereby breaking the atom-number U(1) symmetry. The corresponding phase diagram is illustrated in Fig. 13.

In contrast to a crystal, that allows F$_{U(1)}$ ground state, we observe that hexatic and smectic states are dislocation condensates (corresponding to condensation of both or just one of the x– and y– dislocation dipoles (created by $b^\pm_b$)). Thus, vacancies and interstitials (created by $a^\dagger$, illustrated in Fig. 11), consisting of pairs of oppositely-charged dislocations (disclination charge quadrupoles), are necessarily driven to Bose-condense by the allowed coupling $a_b^\dagger b^\pm_b$. Thus, a hexatic and a smectic are necessarily incommensurate “super-hexatic” and “super-smectic”, respectively, and their dual gauge theory is implicitly understood to be coupled (via axion-like E – B, $B = E$ couplings) to a conventional U(1) gauge theory with fields $E,B$ – a dual to the liquid of vacancies and interstitials.

![Image](323x645 to 556x740)

**FIG. 11.** A disclination quadrupole, constructed as a bound state of two equal and opposite dislocations with Burgers vectors $b$ and $-b$, carries a unit of vacancy (atom) number - a local defect that can be seen as a deficiency of an atom in the middle of the configuration, as illustrated in Fig. 10. This construction demonstrates the allowed $b^\dagger_b b^\pm_b$ operator, which encodes that condensation of dipoles is necessarily accompanied by condensation of vacancies and interstitials and thus leading to superfluidity of the fluid phases.

![Image](327x82)

**FIG. 12.** (a) Fracton motion is forbidden as it requires emission of a conserved dipole. (b) In the F$_{U(1)}$ charge enriched fractonic phase (commensurate crystal) longitudinal dipole motion (dislocation climb) is forbidden as it requires emission of a linear quadrupole carrying conserved vacancy/interstitial number, corresponding to local compression of the crystal, i.e., a vacancy defect. c) Transverse dipole motion (dislocation glide) is allowed as it creates a square quadrupole, corresponding to a local shear.

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$$\partial_\mu n_d + \partial_i \pi^i = -J^d_i$$

where we used $E^d_i = n_d - n_0 \partial_i u_0 \approx n_d$, first derived in Ref. 123. With this vacancies-interstitials continuity equation (sourced by $J^d_i$, corresponding to the longitudinal (along-dipole) motion of dipoles), Eq. 58 encodes that climb of dislocations creates vacancy/interstitial defects. Conversely, Mott-insulating (commensurate-crystal) state of the latter, restricts the motion of dislocations to glide-only lineon type, dualizing to a symmetry-enriched fracton state F$_{U(1)}$. It can then undergo a quantum phase transition to a distinct fracton state F, when vacancies and interstitials Bose-condense, thereby breaking the atom-number U(1) symmetry. The corresponding phase diagram is illustrated in Fig. 13.

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FIG. 13. A schematic phase diagram illustrating phases derived from the supersolid (a $U(1)$-symmetry broken fracton phase, $F$). Upon condensation of vortex defects, bosons can transition to a commensurate crystal (a $U(1)$-symmetric fracton phase, $F_{U(1)}$), super-hexatic, or superfluid phase. Note that $U(1)$-symmetric liquid and hexatic phases are forbidden at zero temperature for reasons summarized in Fig. 11.

F. External perturbations

With an eye to experimental probes, we briefly discuss the role of external perturbations. Crystal’s analog of a chemical potential is the imposed velocity — a “momentum chemical potential”, which imposes a nonzero density of finite-momentum boson density, $n_G$, i.e., a nonzero momentum on the crystal. On the dual side this corresponds to an external dual-magnetic field. In the dual-non-superconducting state of vanishing disclination and dislocation density, the response is linear, as expected by crystal’s Galilean invariance. In contrast, the dual-superconductor expels the imposed flux, either completely in the dual-Meissner or “mixed” Abrikosov-like state, respectively corresponding to a viscous response of a fluid or as a lattice of dislocations, carrying a nonzero momentum.

A crystal can also be subjected to a compressive or shear stress, which on the dual side is an imposition of a tensor electric field. A dislocation will generically be set in motion by the associated Peach-Koehler force, $E_{ij} p_j$ encoded as an imposed dual electrostatic field on a charged dipole particle, $p_j$. In the absence of dipoles this probes the response of external tensor field across the dual “dielectric”. For stress above a critical value, a dual dielectric breakdown will take place, corresponding to a proliferation of dipole dislocations. The response of a smectic is more complex—it is dual Meissner-like along the same, however the glide constraint is modified due to the possibility of vortex creation in the superfluid, akin to earlier discussion of vacancies and interstitials in an incommensurate atomic crystal.

A substrate also plays a qualitatively interesting role, as it breaks the underlying rotational and translational symmetries, thereby breaking angular and linear momentum conservations. Repeating the duality analysis for a translationally incommensurate substrate, we find that, it reduces the orientational $U(1)$ sector ($\mathbf{e}, \mathbf{a}$) to a discrete $Z_n$ gauge theory (for $n$-fold orientational commensurability), coupled to a noncompact $U(1)$ translational-sector gauge theory $(\mathbf{E}_k, \mathbf{A}_k)$. For $n = 1$ a la Polyakov confinement in 2+1d, the orientational degree are eliminated and the translational sector reduces to two conventional decoupled $U(1)$ gauge theories, for $k = x, y$. These are compact in the presence of a translationally $p$-fold commensurate substrate and will also be reduced to discrete $Z_p$ gauge theories, and fully confined, i.e., pinned for $p = 1$.

G. Vortex crystal

We now consider constrained dynamics of vortices. The first system of interest is a vortex crystal. We assume that it was formed by nucleating a large number of vortices in a superfluid. The elasticity of a vortex crystal is described in terms of both superfluid phase degree of freedom and vortex-lattice phonons. The low energy Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{2} \Gamma \bar{n} \epsilon^{ij} u_i \partial_0 u_j - \frac{1}{2} C_{ijkl} u_i u_k \partial_0 + \Gamma \epsilon^i u^i + \frac{1}{g^2} (\epsilon^2 - b^2) ,$$

(59)

where $\Gamma$ is the vorticity, $\bar{n}$ is superfluid density, and dual electromagnetic fields, $\epsilon_i$ and $b$ capture the Goldstone modes of the superfluid (see Section III A). The first term (single time derivative Berry’s phase-like term) in (59) is unique to the vortex crystals and explicitly breaks parity. It originates from the Magnus force (associated with a nonzero boson density — seen by vortices as an effective magnetic field) experienced by vortices, encoding non-commutativity of $u_x$ and $u_y$.

The duality transformation follows the steps similar to the previous sections and the final dual Lagrangian is

$$\tilde{\mathcal{L}} = \frac{1}{2 \Gamma n} \epsilon^{ij} \left( B_i - \Gamma \epsilon^k a_k \right) \partial_0 \left( B_j - \Gamma \epsilon^j a_l \right)$$

$$+ \frac{1}{2} \bar{C}_{ijkl} \left( \epsilon^{ij} + \Gamma \delta^{ij} a_0 \right) \left( \epsilon^{kl} + \Gamma \delta^{kl} a_0 \right) .$$

(60)

The tensor and vector gauge sectors are coupled through a non-trivial minimal-like coupling akin to the coupled vector gauge theories of a crystal, Sec. III B 4 and smectic, Sec. III C duals 33,66,67. The gauge transformations act as follows

$$\delta A_{ij} = \partial_0 \partial_j \alpha + \Gamma \delta_{ij}, \quad \delta A_0 = - \partial_0 \alpha, \quad \delta a_{\mu} = \partial_\mu \beta .$$

(61)

The first term in (60) has no analogue in ordinary electromagnetism. The dipole conservation law (25) remains the same, however the glide constraint is modified due to the possibility of vortex creation in the superfluid, akin to earlier discussion of vacancies and interstitials in an incommensurate atomic crystal, [59].

$$\partial_\mu j^\mu = j^\mu ,$$

(62)

where $j^\mu$ is the superfluid vortex current. The glide constraint (62) states that the dislocations can climb at the
expense of creating vortices (i.e., violating vortex continuity equation $\partial_\mu j^\mu = 0$)\textsuperscript{124}.

Detailed analysis of vortex lattice melting, similar to the discussions of Sect. III D, was presented in\textsuperscript{25}.

## II. Vortex fluid

Remarkably, a classical system of interacting vortices (or, equivalently electric charges in a strong magnetic field) conserves dipole moment on its own, and is thus fractonic. We demonstrate this on the example of a classical system of $N$ vortices.

On increasing vortex density (e.g., by rotation), we expect a vortex crystal to melt into a vortex fluid. Neglecting dissipation, at zero temperature a vortex system be approximated as Hamiltonian for any number of vortices and is described by the following Lagrangian

\[
L = 2\pi \sum_{\alpha} \gamma_{\alpha} x_1^\alpha \partial_\alpha x_2^\alpha - 2\pi \sum_{\alpha<\beta} \gamma_{\alpha} \gamma_{\beta} \ln |x^\alpha - x^\beta|.
\] (63)

We note that this Lagrangian neglects effects of vortex drag and the normal component of a superfluid.

Due to translational and rotation invariance, the total linear and angular momenta are conserved. However, due to non-commutative nature of vortex coordinates in the linear and rotational symmetries. This follows directly from the definition\textsuperscript{125}

\[
[\nabla_i, \nabla_j]v^a = T^b_{ij} \partial_b v^a + R^a_{b;ij} v^b.
\] (65)

Equation (65) states that transporting a vector $v^a$ around a small loop leads to an infinitesimal rotation by $R^a_{b;ij}$ (Franck angle) and translation by $T^a_{ij}$ (Burgers vector).

The relation between dislocations and disclinations dipoles is built into the structure of RC geometry and is phrased as a relation between the Levi-Civita curvature and torsion

\[
2R = \partial_i (\epsilon^{a} \epsilon_{\alpha} T^\alpha_{i}),
\] (66)

where $\epsilon^a_\alpha$ is the frame field, $T^b = T^b_{ij} \epsilon^{ij}$ and $R$ is the Ricci scalar curvature constructed from the curvature two-form $R^a_{b;ij}\textsuperscript{126}$. A similar relation plays the foundational role in the teleparallel formulation of gravity, in which the spacetime geometry is described using torsion.\textsuperscript{127} RC geometry further supplies us with a geometric formulation of \textsuperscript{13}, \textsuperscript{40}, \textsuperscript{52} by the virtue of the Bianchi identity. We illustrate the relationship in two spatial dimensions. It becomes physically transparent when we define the dislocation and disclination currents in terms of torsion and curvature according to

\[
J^\mu = \epsilon^{\mu\nu\rho} T_{a;\nu\rho}, \quad \Theta^\mu = \epsilon^{\mu\nu\rho} R_{\nu\rho}.
\] (67)

Then the Bianchi identity takes form\textsuperscript{7}

\[
\nabla_\mu J^\mu = \epsilon_{\alpha \beta \rho} \epsilon^b \Theta^\rho.
\] (68)

Finally, we should discuss the origins of the glide constraint in the RC language. The glide constraint requires extra information regarding the conservation of total number of lattice sites. The latter can be formulated in an elegant geometric way as follows. First we introduce a current of lattice sites as follow

\[
J^\mu = \epsilon^{\mu\nu\rho} \epsilon_{ab} \epsilon^a_\rho \epsilon^b_{\nu}.
\] (69)

Then the conservation of the number of lattice sites takes the form of continuity equation

\[
\partial_\mu J^\mu = 0.
\] (70)

The conserved quantity is the total volume, which translates to the total number of lattice sites

\[
V = \int d^2 x J^0 = \int d^2 x \det(\epsilon^a_\alpha).
\] (71)

The glide constraint becomes more transparent after writing out (70) as

\[
\partial_0 \det(\epsilon^a_\alpha) + 2 \epsilon^a_0 \epsilon_{ab} J^b = 0,
\] (72)
where $J_{0}^{a} = \sum_{I} b^{a} \delta(x - x_{I})$ is the dislocation density defined by $[67]$ and temporal frame $e_{0}^{a}$ plays the role of velocity field. Thus local volume $\det(e_{0}^{ij})$ changes when the dislocations are carried in the direction perpendicular to the Burgers vector. Eq. (70) has to be postulated in addition to the RC structure.

**J. Diverse realizations of tensor gauge theories**

Since original identification of fractons with crystals and liquid crystals, fractons have naturally appeared in a number of other elastic systems that support geometric defects. Again, fractons emerge after an appropriate duality transformation. Here we review a few of these interesting connections.

1. **Fragile amorphous solids**

A symmetric tensor gauge theory and its associated fractonic order also recently found application in amorphous fragile solids and granular media. These are highly nonequilibrium and heterogeneous solid states, that can sustain external shear $[72, 129]$. The effective long wavelength elasticity $[130]$ emerges from local force and torque balance constraints of mechanical equilibrium on every grain when force chain of contacting grains percolate. In a continuum these can be encoded through a condition of mechanical equilibrium on the local symmetric stress tensor $\sigma_{ij}(r)$ and external force $f_{j}(r)$, satisfying,

$$
\partial_{i} \sigma_{ij}(r) = f_{j}(r).
$$

(73)

As with crystalline solids this static equilibrium condition $[73]$ can be naturally interpreted as a generalized Gauss’s law,

$$
\partial_{i} E_{ij}(r) = \rho_{i}(r).
$$

(74)

for a vector-charge U(1) rank-2 tensor gauge theory $[28]$ with a symmetric electric field tensor, $E_{ij}$ and the vector charge density $\rho_{i}$ describing external force $f_{i}$ $[130]$. This formulation then automatically encodes the net force and torque balance through vector charge and dipole moment neutrality.

The amorphous solid elasticity is then postulated to be governed by the pseudo-electrostatics, with energy density $H = \frac{1}{2} C_{ijkl} E_{ij} E_{kl}$ and curl-free condition on the electric field coming from the electrostatic limit of the Faraday law. The latter implies the existence of an electrostatic potential that plays the role of an effective phonon-like field that, unlike crystals arises in the absence of spontaneous breaking of translational symmetry. The rank-4 elastic tensor $C_{ijkl}$ is to be determined experimentally and is generically heterogeneous and anisotropic. The formulation then allows an efficient computation of the stress-stress correlations associated with a distribution, geometry and topology of the force-chain network via $\langle E_{ij} E_{kl} \rangle$ correlator. The latter gives 4-fold pitch-point singularities characteristic of the tensor gauge theory $[33, 152]$.

In contrast to a tensor gauge theory of crystalline solids that, as we reviewed here can be derived explicitly through duality $[81]$, this gauge theory formulation of amorphous solids is a conjecture that needs the support of numerics and experiments. Indeed the measured averaged stress-stress correlations are well fit by the electric field correlator of the vector-charge tensor gauge theory $[130]$. Fitting this to numerics and experiments $[131, 132]$ allows one to extract the average pseudo-dielectric tensor $C_{ijkl}$ that fully characterizes the emergent static elasticity of the amorphous solid. The resulting tensor gauge theory can then be used to further explore solid’s phenomenology, such as response to perturbations, melting, and dynamics.

2. **Elastic sheets**

Another interesting connection developed in $[28]$ is the application of fractonic tensor gauge theory to elastic thin sheets and their associated defects like folds and tears. With this the authors presented a tensor gauge theory view of the kirigami mechanics. The fractonic dual theory transparently encodes a number of known properties of such defects in thin sheets. They showed that the observation that folding of a sheet of paper can only be done along a straight line can be interpreted using the language of fractons and restricted mobility of vector-charge tensor gauge theory.

In more detail, in Ref. $[28]$ the sheet elasticity is formulated in terms of its out-of-plane flexural field $h(x, y)$, (questionably) neglecting the in-plane phonon displacements $u(x, y)$. Such model then corresponds to a fluid, rather than an elastic membrane $[133, 134]$. The sheet’s local momentum density $\sim \partial_{\alpha} h$ is identified with the scalar magnetic flux density $B$, sheet’s curvature tensor $\partial_{\alpha} \partial_{\beta} h$ with tensor electric field $E_{\alpha \beta}$, and flexural modes with a quadratically dispersing photon. A “tear” defect is characterized by a nonzero closed line integral of $\partial_{\alpha} h$ around the end of the defect, thereby capturing a (non-quantized) discontinuity $\Delta h$ across a tear. This out-of-plane discontinuity ray has some formal similarity with the in-plane dislocation defect. A “fold” defect - an undeformable line along which there is a sheet’s tangent vector discontinuity across the fold, characterized by a closed line integral of $\partial_{\alpha} \partial_{\beta} h$. It maps onto a fractonic vector charge – endpoint of the fold – of the tensor gauge theory. It is hoped that such formulation can be useful in exploration of quantum dynamics and statistical mechanics of kirigami sheets.

3. **Quasiperiodic systems**

Another interesting example is that of quasicrystals (QC), whose elasticity-fracton duality was investigated
As developed by its pioneers and discussed in detail in Ref., the elasticity of the QCs is characterized by two sets of low-energy modes: phonons, described by the symmetric strain tensor \( u_{ij} \), and phasons, described by a general rank-2 tensor \( w_{ij} \). Consequently, the equations of motion are formulated in terms of two stress tensors, \( T_{ij} \) and \( H_{ij} \). These can be defined as derivatives of the Lagrangian density

\[
T_{ij} = -\frac{\partial L}{\partial u_{ij}}, \quad H_{ij} = -\frac{\partial L}{\partial w_{ij}}, \tag{75}
\]

with, as clear from (75), the stress tensor \( H_{ij} \), non-symmetric. Duality transformation follows the steps reviewed in Sec. \( \text{III B1} \). Under duality each QC stress is described by a dual tensor gauge field, with a traceless scalar charge theory \( A_{ij} \), \( A_0 \) for \( T_{ij} \) and the general rank-2 tensor \( A_{ij}, A_0 \) characterizing \( H_{ij} \). In the dual gauge theory these degrees of freedom are coupled and the Lagrangian takes the Maxwell form, \( i.e. \), it is formulated as a quadratic form in terms of tensor electric and magnetic fields.

Associated with phonons and phasons, QCs exhibit two types of topological defects: those of \( u_{ij} \) \( i.e. \), dislocations and disclinations) and defects of \( w_{ij} \), known as the stacking faults. Disclinations are scalar charges, while stacking faults are vector charges coupled to \( A_{ij}, A_0 \) and \( A_{ij}, A_0 \) tensor gauge fields. Mobility of the dislocations in QCs (and QC SPTs) was carefully studied in Ref.\( \text{142} \), the elasticity of the QCs is characterized by a dual tensor gauge field, with a traceless scalar charge theory \( A_{ij} \), \( A_0 \) described by a dual tensor gauge field, with a traceless scalar charge theory \( A_{ij} \), \( A_0 \) and the general rank-2 tensor \( A_{ij}, A_0 \) characterizing \( H_{ij} \). In the dual gauge theory these degrees of freedom are coupled and the Lagrangian takes the Maxwell form, \( i.e. \), it is formulated as a quadratic form in terms of tensor electric and magnetic fields.

A particular example of a quasiperiodic system is the Moire superlattice generated in twisted bilayer graphene. The phasons in Moire systems correspond to relative displacement of the layers. Singularities of the phason modes in this context were referred to as discompressions in Ref.\( \text{114} \) which were indeed found to be immobile.

### IV. GLOBAL SYMMETRIES AND GAUGE THEORIES

The Lagrangian form is invariant under the multipole algebra — multipole algebra. This approach allows us to generalize tensor gauge theories to multipole gauge theories that are related to both anisotropic liquid crystals and fractal surface codes.

#### A. General symmetric tensor gauge theories

Tensor gauge theories such as arise from gauging a global algebra of conserved charge and multipole moments. Such algebra has to include spatial symmetries such as translations and rotations as well because they do not commute with multipole charges. The simplest case of this algebra includes all multipole charges up to some rank \( r \) and all translations and rotations.

More formally it is described by a set of commutation relations. Let \( T_i \) and \( R_{ij} \) be generators of translations and rotations, and \( Q^{(n)}_{i_1 i_2 \ldots i_n} \) be the charge corresponding to the \( n \)-th multipole moment. The multipole algebra \( \mathfrak{M}_{n,k} \) then takes the form,

\[
[T_i, T_j] = 0, \quad [R_{ij}, T_k] = \delta_k[i]T_j], \quad [R_{ij}, R_{kl}] = \delta_{[i|k}R_{j|l]}], \quad [Q^{(n)}_{i_1 i_2 \ldots i_n}, Q^{(m)}_{i_1 i_2 \ldots i_m}] = 0, \quad \forall n > k
\]

where \( i_r \) indicates that the index \( i_r \) should be omitted and \( n \geq k \) and the commutators in Eq.(76) indicate antisymmetrization \( \delta_{[i|k}T_j] = \delta_k[i]T_j - \delta_j[i]T_k \). Eq.(79) indicates that \( Q^{(k)}_{i_1 i_2 \ldots i_k} \) is the fundamental charge that happens to be a rank-\( k \) tensor.

Given the conserved charges \( Q^{(n)}_{i_1 i_2 \ldots i_n} \) we postulate a local conservation law in a form of continuity equation

\[
\partial_0 \rho_{i_1 \ldots i_k} + \partial_{i_n \ldots i_k} \ldots \partial_{i_{n-k+1}} J_{i_1 \ldots i_k} = 0, \quad n \geq k
\]

where \( J_{i_1 \ldots i_{n-k}} \) is a symmetric tensor current. Eq.\( \text{81} \) implies that a tensor charge \( Q^{(k)}_{i_1 i_2 \ldots i_k} \) is conserved as well as its first \( n - k \) moments. In other words the indivisible unit charge in \( \mathfrak{M}_{n,k} \) is a rank-\( k \) tensor.

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Then, requiring invariance under gauge transformations
\[ \delta A_{i_1\ldots i_n} = \sum_{r=1}^{n} \partial_{i_r} \lambda_{i_1\ldots i_r\ldots i_n}, \]  
and
\[ \delta A_{0,i_2\ldots i_k} = \sum_{r=1}^{n} \partial_0 \lambda_{i_1\ldots i_r\ldots i_n}, \]  
enforces the continuity equation \([81]\).

Precise structure of \([86]-[87]\) depends on \(k\). If the lowest conserved moment is the (scalar) charge \(Q^{(0)}\), i.e. \(k = 0\) the gauge parameter takes general form
\[ \lambda_{i_1\ldots i_{n-1}} = \partial_{i_1} \ldots \partial_{i_{n-1}} \lambda. \]  
If the lowest conserved moment is \(Q^{(k)}_{i_1i_2\ldots i_k}\), the algebra still makes sense assuming that the commutator between \(Q^{(k)}_{i_1i_2\ldots i_k}\) vanishes \([T_{ij}, Q^{(k)}_{i_1i_2\ldots i_k}] = 0\). If that is the case, we say that the theory has a rank-\(k\) tensor charge and the higher moments of this charge are conserved. When \(k = 1\) this is known as vector charge theory.

The density and current can then be found by the usual variational prescription
\[ \rho_{i_1\ldots i_k} = \frac{\delta \mathcal{L}}{\delta A_{0,i_1\ldots i_k}}, \quad J^{i_1\ldots i_n} = \frac{\delta \mathcal{L}}{\delta A_{i_1\ldots i_n}}. \]  
The gauge-invariant electric field \(E_{i_1\ldots i_n}\) is easy to construct. It is given by
\[ E_{i_1\ldots i_n} = \partial_0 A_{i_1\ldots i_n} - \sum_{r=1}^{n} \partial_{i_r} A_{0,i_1\ldots i_r\ldots i_n}. \]  

The Gauss’s law generating \([90]\) takes form
\[ \partial_{i_{k+1}} \ldots \partial_{i_n} E_{i_1\ldots i_n} = \rho_{i_1\ldots i_k}. \]  
Gauge invariant magnetic field can be defined in all of the above cases, however it explicit form depend on the theory and we do not provide its general expression.

In \([79]\) the algebra \([76]-[80]\) was referred to as the maximally symmetric multipole algebra.

### B. Examples of symmetric tensor gauge theories

The first example is the scalar charge theory \([29]\). It requires the conservation of dipole moment only. The multipole algebra takes form
\[ [T_i, T_j] = 0, \quad [R_{ij}, T_k] = \delta_{k[i} T_{j]} - \delta_{k[j} T_{i]}, \]  
\[ [R_{ij}, R_{kl}] = \delta_{[kl} R_{ij]} , \quad [Q^{(1)}_i, Q^{(1)}_j] = 0, \]  
\[ [T_i, Q^{(2)}_j] = \delta_{ij} Q, \quad [T_i, Q] = 0 \]  
\[ [R_{jk}, Q^{(2)}_i] = \delta_{kj} Q^{(1)}_j - \delta_{kj} Q^{(1)}_i. \]  

A field theory invariant under \([92]-[95]\) can readily be written. The only degree of freedom is a real scalar \(\phi\) with the Lagrangian given by
\[ \mathcal{L} = (\partial_0 \phi)^2 - (\partial_i \partial_j \phi)^2, \]  
and is the Lifshitz models with ubiquitous applications. Lagrangian \([96]\) is invariant under a polynomial shift symmetry
\[ \delta \phi = c_0 + c_i x^i, \]  
which by virtue of Noether’s theorem implies the conservation of total charge and total dipole moment. Indeed, there are \(d+1\) independent symmetry parameters leading to \(d+1\) conserved quantities.

To gauge \([97]\) we introduce a tensor gauge fields as follows
\[ \rho = \frac{\delta \mathcal{L}}{\delta A_0} = \partial_0 \phi, \quad J^{ij} = \frac{\delta \mathcal{L}}{\delta A_{ij}} = \partial_i \partial_j \phi \]  
and satisfy the continuity equation
\[ \partial_0 \rho + \partial_i \partial_j J^{ij} = 0, \]  
which implies the dipole conservation.

Our second example is the traceless scalar charge theory. Its symmetry algebra is a sub-algebra of \(\mathfrak{so}_2\) where the conserved quantities are
\[ Q^{(1)}_i, \quad \Delta = \text{tr} \left( Q^{(2)}_{ij} \right). \]  
The only additional (compared to \([92]-[95]\)) non-trivial commutation relation is
\[ [T_i, \Delta] = Q_i. \]  
A Lagrangian invariant under the symmetry algebra \([92]-[102]\) take form
\[ \mathcal{L} = (\partial_0 \phi)^2 - \left( \left[ \partial_i \partial_j - \frac{1}{2} \partial_0^2 \right] \phi \right)^2. \]  
Lagrangian \([103]\) is invariant under a polynomial shift symmetry
\[ \delta \phi = c_0 + c_i x^i + c x_k x^k. \]  
The last term leads to an extra conservation law for the trace of the quadrupole moment. Gauging leads to the traceless scalar charge theory with a traceless tensor potential, while the tensor current satisfies an extra constraint \(\delta_{ij} J^{ij} = 0\).
Finally, we discuss a general scalar charge theory invariant under the multipole algebra $\mathfrak{m}_{a,d}$. The commutation relations are given by (76)-(80) with $k = 0$. The Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\alpha \phi)^2 - \frac{1}{2}(\partial_{i_1} \cdots \partial_{i_n} \phi)^2,$$  

(105)

which is invariant under a general polynomial shift symmetry

$$\delta \phi = c_0 + \sum_{m=1}^n c_{i_1 \cdots i_m} x^{i_1} \cdots x^{i_m}.$$  

(106)

The local conservation law that follows from the Noether’s theorem takes form (81) with $k = 0$.

To gauge (106) we introduce a general rank-$n$ tensor gauge field as follows

$$\mathcal{L} = \frac{1}{2}(\partial_\alpha \phi - A_0)^2 - \frac{1}{2}(\partial_{i_1} \cdots \partial_{i_n} \phi - A_{i_1 \cdots i_n})^2.$$  

(107)

The density and tensor current are then given by (89) with $k = 0$.

### C. General multipole algebra

So far we have assumed that the tensor gauge theories discussed above are invariant under continuous rotations. This does not have to be the case because we expect (at least) some of those theories to emerge from UV lattice models and to thereby inherit the lattice symmetries. We will reduce our discussion to scalar charge theories.

To incorporate the lattice symmetries we observe that every conserved multipole moment is associated to a polynomial $P(x)$. For example, conserved dipole moment is associated to $d$ monomials of degree 1: $x_1, x_2, \ldots, x_d$. Generally a component of the multipole tensor $Q_\alpha^{(n)}$ is obtained by integrating a polynomial of degree $n$, $P_\alpha^{(n)}(x)$ against the charge density

$$Q_\alpha^{(n)} = \int dx P_\alpha^{(n)}(x) \rho(x).$$  

(108)

The general index $I$ may transform in an irreducible representation of a point group. We now turn to a more formal description of the general multipole algebra.

To describe the algebra we introduce the general multipole moment as follows. Let $P_\alpha^{(n)}(x)$ be a homogeneous polynomial of degree $n$. Then multipole moment corresponding to $P_\alpha^{(n)}(x)$ is defined by (108).

Commutation relations between these multipole moments and spatial symmetries form the multipole algebra

$$[T_k, Q_\alpha^{(n)}] = f^{\alpha \beta \gamma} Q_\beta^{(n-1)},$$  

(109)

$$[R_k, Q_\alpha^{(n)}] = g^{\alpha \beta \gamma} Q_\beta^{(n)},$$  

(110)

where $T_k$ is a translation in direction $\hat{r}$, while $R_k$ is rotation about $\hat{r}$, while $f^{\alpha \beta \gamma}$ and $g^{\alpha \beta \gamma}$ are the structure constants. In general, the rotations can include either a subgroup of $SO(d)$ or be discrete. For example, in the gauge theory approach to the Haah code there is an $SO(2)$ rotation symmetry, while $\hat{r} \propto (1, 1, 1)$.

Given the polynomials we define a set of homogeneous differential operators $D_\alpha$ that annihilate all $P_\alpha^{(n)}$ simultaneously

$$D_\alpha P_\alpha^{(n)} = 0 \quad \forall \, I, n.$$  

(111)

Then the local conservation laws take form

$$\partial_\alpha \rho + \sum_{\alpha} D_\alpha^\dagger J^\alpha = 0.$$  

(112)

where $J^\alpha$ are the multipole currents and $D_\alpha^\dagger$ is obtained from $D_\alpha$ via integration by parts.

Gauging procedure follows the same logic as in the previous section. We introduce the gauge fields $A_\alpha$ and $A_0$ conjugate to the multipole current $J^\alpha$ and density $\rho$. These gauge fields are labeled by an abstract index (which also can transform in an irreducible representation of the rotation group), and are neither 1-forms nor symmetric tensors. A general Lagrangian invariant under the multipole algebra is supplemented with

$$\delta \mathcal{L} = \rho A_0 + \mathcal{J}^\alpha A_\alpha.$$  

(113)

The gauge transformation law takes form

$$\delta A_\alpha = D_\alpha \lambda, \quad \delta A_0 = \partial_\alpha \lambda$$  

(114)

and ensures (112).

Using these $D_\alpha$ we can construct the electric field and the Gauss’s law as follows. The electric field is invariant under (114) and is given by

$$E_\alpha = \partial_\alpha A_\alpha - D_\alpha A_0,$$  

(115)

satisfying the Gauss’s law with a generalized divergence,

$$\sum_{\alpha} D_\alpha^\dagger E_\alpha = \rho.$$  

(116)

### D. Relation to the symmetric case

Multipole algebra includes the symmetric case as a special case. Here we illustrate how it works on the example of traceless scalar charge theory. In $d$ spatial dimensions there are $d + 1$ polynomials

$$P_1^{(1)}(x) = x_1, \ldots, P_d^{(1)}(x) = x_d, \quad P_1^{(2)}(x) = x_1 x^1.$$  

(117)

The differential operators are of degree 2. The index $\alpha$ can be represented as a multi-index $\alpha = (i, j)$ with the differential operators given by

$$D_{i,j} = \partial_i \partial_j - \frac{1}{d} \delta_{ij} \partial^2,$$  

(118)

where it is clear that (117) holds, i.e. $D_{i,j}$ annihilates all the polynomials $P_1^{(1)}(x), \ldots, P_d^{(1)}(x), P_1^{(2)}(x).$ This algebra includes all translations and continuous rotations.
E. Gaussian free field with multipole symmetries

Next we discuss an explicit example of a free field theory that is invariant under a general multipole algebra. Consider a real scalar field $\phi$ and a set of homogeneous polynomials $P_I^{(n)}(x)$. We will construct a Lagrangian invariant under the following transformation

$$\phi \rightarrow \phi + P_I^{(n)}(x). \quad (119)$$

To the lowest order in derivatives the most general action takes form

$$\mathcal{L} = \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2} \sum \alpha (D_\alpha \phi)^2 \quad (120)$$

by virtue of (111). The symmetry (119) leads to the conservation of (108).

We can also utilize (120) to obtain the multipole gauge theory structure. Indeed, the symmetry (119) can be gauged by replacing derivatives as follows

$$\mathcal{L} = \frac{1}{2}(\partial_0 \phi - A_0) - \frac{1}{2} \sum \alpha (D_\alpha \phi - A_\alpha)^2, \quad (121)$$

where $\Phi$ is the scalar potential. The action (121) is invariant under the gauge transformation (114). The conserved charges are explicitly given by

$$Q_I^{(n)} = \int \partial_0 \phi P_I^{(n)}. \quad (122)$$

F. Multipole gauge theory of a smectic

Next we discuss a simple example of multipole gauge theory that arises as a dual theory to elasticity of a quantum smectic phase [6][10]. We will show that a 2+1d quantum smectic is dual to a multipole gauge theory. We start with the following Lagrangian density that describes a smectic phase at long distances

$$\mathcal{L} = \frac{1}{2} \partial_0 u^2 - \frac{1}{2} \kappa (\partial_y u)^2 - \frac{1}{2} \kappa (\lambda \partial_x^2 u)^2, \quad (123)$$

where the layers are perpendicular to the $y$ axis and extend along the $x$ axis. We remind the reader that a smectic is a liquid crystal phase that spontaneously breaks rotational symmetry (by a choice of layers’ orientation) and one out of the two translation symmetries. It can be viewed as a periodic array of 1d liquids with a period of order $\lambda$ along $y$ axis. We denote

$$D_1^1 = \partial_w, \quad D_2^1 = \lambda \partial_x^2. \quad (124)$$

In terms of these derivatives the action is

$$\mathcal{L} = \frac{1}{2} \partial_0 u^2 - \frac{1}{2} \kappa (D_1^1 u)^2 - \frac{1}{2} \kappa (D_2^1 u)^2. \quad (125)$$

We introduce auxiliary variables using Hubbard-Stratonovich trick

$$\mathcal{L} = Pu - \frac{P^2}{2} - (D_1^1 u)T_1 - (D_2^1 u)T_2 + \frac{T_1^2}{2} + \frac{T_2^2}{2} \quad (126)$$

Integrating out the phonon $u$ we find a constraint

$$\partial_0 P - D_1 T_1 - D_2 T_2 = 0. \quad (127)$$

This equation is solved by

$$T_1 = \epsilon_{IJ}(\partial_0 A_J - D_J A_0) = \epsilon_{IJ} E_J, \quad (128)$$
$$P = B = \epsilon_{IJ} D_I A_J, \quad (129)$$

where $\epsilon_{IJ}$ is the Levi-Civita symbol and $I, J = 1, 2$. The gauge redundancy of the solution is

$$\delta A_I = D_I \alpha, \quad \delta A_0 = \dot{\alpha}, \quad (130)$$

which is exactly a multipole gauge theory structure. The Gauss’s law that generates (130) is given by

$$D_I^1 E_I = \rho. \quad (131)$$

The defect density $\rho$ is the density of smectic disclinations. The defect matter conserves the dipole moment in $u$ direction, which can be seen directly from (131). Disclination dipole extended in the $u$ direction is a dislocation with the Burgers vector in $w$ direction. The dislocations are completely mobile, whereas the disclinations are 1D particles (also known as lines) that can only move in the $w$ direction. The low energy phonon is described by the multipole gauge theory with the generalized Maxwell action

$$\mathcal{L} = \frac{1}{2} \epsilon(E_1^2 + E_2^2) - \frac{1}{2} B^2. \quad (132)$$

As required $\mathcal{L}$ admits is a single smectic mode with a linear dispersion along the $y$ axis and quadratic dispersion along the $x$ axis.

G. $U(1)$ Haah code in three dimensions

Next we turn to a discussion of the “$U(1)$ Haah code” studied in[12][17]. We begin by postulating the symmetries

$$\delta \phi = c_0 + c_1 P_1^{(1)} + c_2 P_2^{(1)} + c_3 P_2^{(2)} + c_4 P_2^{(2)}, \quad (133)$$

where

$$P_1^{(1)} = x_1 - x_2, \quad P_2^{(1)} = x_1 + x_2 - 2x_3, \quad (134)$$
$$P_1^{(2)} = (x_1 - x_2)(x_1 + x_2 - 2x_3), \quad (135)$$
$$P_2^{(2)} = (2x_1 - x_2 - x_3)(x_2 - x_3). \quad (136)$$

These polynomials have been chosen after examining the elementary fracton configurations in the Haah code [12] and generalizing them to the $U(1)$ conserved charge.
FIG. 14. (a) The elementary charge configurations, corresponding to the invariant derivatives $D_I$, for the effective theory for the $U(1)$ Haah code \[\text{(137)}\] charge configurations. These charge configurations violate conservation of the dipole moment in $(1,1,1)$ direction. (b) A different basis of elementary charge configurations. The first two configurations are precisely the ones studied in \[\text{(70)}\], while the last charge configuration is allowed by symmetries and is linearly independent from others.

These configurations carry dipole moment in the $(1,1,1)$ direction, which leads us to enforce conservation of the dipole moment in the $(111)$ plane. Polynomials of the degree 2 were chosen in a similar manner. We will also enforce the $SO(2)$ symmetry in the $(111)$ plane. This leaves us with three invariant derivatives that take form

$$D_1 = q^i \partial_i, \quad D_2 = q_1^{ij} \partial_i \partial_j, \quad D_3 = q_2^{ij} \partial_i \partial_j,$$

where

$$q^i = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad q_1^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q_2^{ij} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$ 

We then gauge these symmetries, as explained in the previous sections. The Lagrangian describing dynamics of the gauge field is given by

$$L = \sum E_I^2 - B_1^2 - B_2^2,$$

Where the magnetic fields are given by

$$B_I = \epsilon_{IJK} D_J A_K,$$

and the Gauss’s law takes form

$$\sum_\beta D_\beta^\dagger E_\beta = \rho.$$ 

The $U(1)$ Haah model has a hidden infinite symmetry. We need to introduce a bit of notation. First we define a basis $\mu_1^I, \mu_2^I$ in the plane where the dipole moment is conserved. One choice is $\mu_1^I = (1, -1, 0)$ and $\mu_2^I = (1, 1, -2)$. With this basis at hand we introduce new variables $x = \mu_1^I x^I / |\mu_1^I|$ and $y = \mu_2^I x^I / |\mu_2^I|$. Then all invariant derivatives $D_\alpha$ (and, consequently the Lagrangian) are also invariant under an infinite symmetry

$$\delta \phi(z, \bar{z}, x_3) = f(z) + g(\bar{z}), \quad z = x + iy,$$

where $f(z)$ is holomorphic and $g(\bar{z})$ is anti-holomorphic. This is an example of a well-known “sliding” symmetry,\[\text{(141)}\] that appears in physics of smectics,\[\text{(140)}\] and it can be understood as a continuous version of sub-systems symmetries. Finally, the $U(1)$ Haah model exhibits an anisotropic scaling symmetry, which takes form

$$t \to \lambda t, \quad x \to \lambda^\frac{1}{3} x, \quad y \to \lambda^\frac{2}{3} y, \quad x_3 \to \lambda x_3, \quad \phi \to \lambda^{-\frac{1}{3}} \phi.$$ 

(142)
It was shown in Ref. [138] that gauge theory for $U(1)$ Haah code is dual to the smectic-A phase in 3D.

H. Subsystem symmetry

We now turn to an even more exotic class of much larger, extensive symmetries—the so-called subsystem symmetries—that lead to restricted mobility and multipole-gauge theories upon gauging. These symmetries were initially defined on a lattice for various spin models, but, as we illustrate can be extended to the continuum.

While a covariant theory of subsystem symmetries has not yet been developed, we can understand a class of these symmetries as an infinite-dimensional generalization of the multipole algebra discussed in the previous subsection. In our development we assume that the physical model is defined on a flat space and that we are given a set of lines, planes, or hyperplanes that foliate the space.

As a pedagogical example we consider a model of a real scalar field in 2d, $\phi(x_1, x_2)$, with the following transformation as an example of subsystem symmetry,

$$\delta \phi = f_1(x_1) + f_2(x_2),$$  \hspace{1cm} (143)

where $f_1(x_1)$ and $f_2(x_2)$ are arbitrary functions of $x_1$ and $x_2$. The set of lines consists of two families: (i) lines parallel to $x_2$ and (ii) lines parallel to $x_1$. Any 2d lattice provides enough structure to develop a set of subsystem symmetries.

The algebra of subsystem symmetries is infinite-dimensional. Its action is intermediate between a global symmetry, that acts on full d-dimensional space and gauge redundancy that acts on individual sites. We can also interpret it as an infinite-dimensional generalization of the multipole algebra by representing the functions $f_1, f_2$ as Taylor series in $x_1, x_2$, correspondingly

$$\delta \phi = \sum_{n \geq 0} c_n x_1^n + \sum_{m \geq 0} b_m x_2^m,$$  \hspace{1cm} (144)

where $c_n$ and $b_m$ are arbitrary coefficients. Viewed this way the symmetry implies conservation of arbitrary high multipole moments in one of the axes

$$Q^{(n)}_{11...1}, \quad Q^{(n)}_{22...2}, \quad n \geq 0$$  \hspace{1cm} (145)

The Lagrangian invariant under (143) break rotational symmetry down to a discrete subgroup, $C_4$

$$\mathcal{L} = \frac{1}{2}(\partial_0 \phi^2) - \frac{1}{2}(\partial_1 \partial_2 \phi^2) + \ldots,$$  \hspace{1cm} (146)

and was analyzed in great detail in Ref. [109].

Subsystem symmetries can also be gauged. Indeed gauging (143) requires a single “vector” potential $A_{12}$ that transforms as $\delta A_{12} = \partial_1 \partial_2 \lambda$. There is a single electric field given by $E_{12} = \partial_0 A_{12} - \partial_1 \partial_2 A_0$ and the Gauss’s law takes the form

$$\partial_0 \partial_2 E_{12} = \rho.$$  \hspace{1cm} (147)

Other examples of subsystem symmetries are discussed in Ref. [115].

The conservation law following from (143) is encoded in a continuity equation

$$\partial_0 \rho + \partial_1 \partial_2 J = 0,$$  \hspace{1cm} (148)

where $J$ is the current. It exhibits an infinite number of conserved charges

$$Q_{1x} = \int_{x_1=x} dx_2 \rho(x_1, x_2), \quad Q_{2y} = \int_{x_2=y} dx_1 \rho(x_1, x_2),$$  \hspace{1cm} (149)

which are conserved independently on any line $x$ and $y$, respectively. It is thus clear that this conservation law makes particles that are charged under both $Q_{1x}$ and $Q_{2y}$ completely immobile, while the dipoles can move perpendicular to their dipole moments.

I. Fracton hydrodynamics

Conservation laws (25) and (148) can, in principle, be either microscopic or emergent. In either case, the long-wave, long-time phenomenology is affected if these conservation laws are present. These effects manifest themselves in the transport of charge and momentum, with the simplest manifestation of sub-diffusion.

We illustrate the emergence of sub-diffusion in the simplest case of a conserved dipole moment. As discussed in earlier sections, the corresponding charge continuity equation takes the form (25).

$$\partial_0 \rho + \partial_i \partial_j J^{ij} = 0.$$  \hspace{1cm} (150)

To describe diffusion of charge we relate the dipole current to the charge density. In equilibrium the dipole current must vanish. Consequently, the constitutive relation between $\rho$ and $J^{ij}$ takes form

$$J^{ij} = \chi^{-1} \partial_i \partial_j \rho + \ldots,$$  \hspace{1cm} (151)

where $\chi$ is the susceptibility. The (sub-)diffusion equation then takes form

$$\partial_0 \rho + \chi \partial^4 \rho = 0.$$  \hspace{1cm} (152)

Eq. (152) implies that density perturbation at wavelength $\lambda$ decays at a characteristic time $\tau \sim \chi \lambda^2$ for long wavelengths parametrically far slower than the conventional diffusive time $\sim \lambda^2$. Such slow relaxation time enhancement has been observed in cold atomic gasses in tilted optical lattices. Subdiffusion also emerges in random unitary circuits. Such sub-diffusion straightforwardly generalizes to the case of conservation of the $n$-th multipole moment, where it gives

$$\partial_0 \rho + \chi \partial^2 + 2n \rho = 0,$$  \hspace{1cm} (153)

leading to even slower characteristic time $\tau \sim \chi \lambda^{2+2n}$. 
Another interesting effect appears when subsystem symmetry constrains the diffusion equation. Consider the charge conservation equation \[ \partial_0 \rho + \chi^{-1} \partial_x^2 \partial_y^2 \rho = 0. \] (154)

Thus, the subsystem symmetry breaks the rotational symmetry down to a discrete subgroup of the lattice, with these effects persisting at longest scales. This is in stark contrast with the classic diffusion, that has an emergent rotational symmetry to the lowest order in derivatives.

V. CONCLUSIONS

A. Summary

In this Colloquium we have reviewed theoretically inspired, burgeoning subject of fractonic matter. We began with a model-independent, symmetry- and conservation-based formulation of fractons – excitations with restricted mobility arising in a broad class of exotic models.

The central focus of this review is on the emergence of fractonic order from elasticity-gauge duality of a broad class of quantum elasticity models, that include quantum commensurate and incommensurate supersolid crystals, smectic liquid crystals, hexatic fluids, amorphous solids, quasi-crystals and elastic membranes, all encoding some form of multipolar global symmetries. We also discussed a vortex crystal and a vortex liquid that dualize to a parity-breaking gauge-dual variants. Building on the familiar boson-vortex duality, we explicitly reviewed how such dualities lead to interesting tensor and coupled-vector gauge theories that exhibit fractonic charges, dipoles and higher multipoles as duals of elastic topological defects, with gauge fields encoding gapless phonons.

As we discussed, such elasticity-gauge duality is a powerful tool for discovery of new class of fractonic models. The resulting models can then also be generalized into a broader class, beyond any elastic dual connection. A complementary motivation for the duality studies is that they provide efficient formulation of the quantum elasticity and the topological defects dynamics. One striking example is the prediction of the zero-temperature immobility of disclinations in a 2d crystal (previously unknown despite decades of its studies in the elasticity context), that arose purely through this fractonic gauge theory connection. In the review, we furthermore demonstrated that gauge duals provide the first field-theoretic formulation of quantum melting of a crystal and a smectic through a generalized Higgs mechanism associated with a condensation of dislocations and disclinations.

As we discussed, tensor gauge theories can also be studied in arbitrary dimension and without any relation to elasticity. We have presented a general construction of a large class of such gauge theories based on gauging the multipole algebra — an algebra of spatial symmetries that includes dipole and higher multipole symmetries. The resulting multipole gauge theories include the models obtained from dualities as special cases. We expect that these theories can serve as templates for identifying exotic gapless excitations in spin liquids.

We have also discussed an even more exotic subsystem symmetries and illustrated a formal relation between these symmetries and infinite-dimensional multipole algebras. Finally, we concluded the review with a description of the long-time subdiffusive hydrodynamic of fractonic matter that emerges as a result of multipole conservation laws.

B. Open problems

While the field of fractons and associated tensor/multipole gauge theories has seen a rapid growth in the last ten years, it remains in an early stage of development, with many open theoretical and experimental questions.

1. Mathematical structure

Most fundamentally, the mathematical structure of tensor gauge theories is still not well formulated, currently unclear what replaces the $G$-bundle of the traditional gauge theories. Consequently, the topology of the space of tensor gauge fields is poorly understood. Namely, since the geometric interpretation of tensor gauge fields is unclear, it is not known how to construct topological invariants that generalize the Chern numbers. In fact it is certain that such invariants cannot be purely topological, because any naive generalization of the Chern number will depend on the spatial metric, leading to the metric dependence of the “topological invariant”. One can also generalize the Chern-Simons theory to the higher rank case, but the dependence on the metric appears to be unavoidable. Furthermore, in all cases the inclusion of the metric is in tension with gauge invariance, finding that the magnetic field is only gauge invariant in flat space (or on Einstein manifold in certain cases). It is hoped, however, that coupled vector gauge theory formulation may be more suitable for addressing these questions.

As we have discussed in the Introduction, fractonic gauge theories concisely encode quasi-particle restricted mobility and Gaussian fluctuations. However, these continuum field theories preclude an encoding of their expected exponential in-system-size ground-state degeneracy, and nontrivial quasi-particle quantum statistics — this contrasts strongly with discrete qubit models (e.g., X-cube), where these properties are well defined and have been calculated.
As was recently demonstrated\cite{12,13}, a gauge structure and phenomenology similar to that of fractonic tensor gauge theories also arises in superfluids and the fractional quantum Hall effect. In fact, a similar form and UV/IR mixing characteristic of fractonic gauge theories\cite{104} arise in the non-commutative gauge theories as well as non-commutative matter theories coupled to a gauge field. A development and illumination of these relations remains an open problem.

2. Quantum melting and insights on elasticity

Focussing on specific models and their phenomenology discussed in this review, we have seen that these dual gauge theories are also of interest because they provide a formulation of crystal-to-hexatic, crystal-to-smectic, smectic-to-nematic quantum melting transitions as generalized Higgs transition associated with a condensation of the topological defects. However, these have so far only been analyzed at the mean-field level thus leaving their true criticality to future studies.

3. Generalizations beyond bosonic elasticity

We also note that in all the elastic models reviewed here and studied in the literature, the focus has been on the simplest bosonic realizations. These lead to bosonic statistics of dislocation and disclination defects (i.e., bosonic fracton matter) and superfluidity when crystalline order is lost. It is natural to study extensions of the bosonic duality to that of fermions and anyons and to explore the possibility of nontrivial statistics of topological defects.

4. Experimental void

Of course the field’s most vexing challenge is that of experimental realizations, that so far have been sorely absent. In part this is due to the fact that discrete qubit (e.g., the simplest X-cube) models, are typically formulated in terms of many-spin commuting projectors, that are therefore extremely difficult to implement. A promising alternative direction is that of the $U(1)$ tensor and coupled vector gauge theories that have been the focus of this review. These are related to discrete models through condensation of higher charge matter\cite{15,16}. In fact, the 2+1d crystal-gauge duality strikingly demonstrates a concrete physical realization of fractonic order in a familiar quantum crystal and other elastic media.

In tensor-gauge theories, realization-independent quadrupolar pinch-point singularities have been predicted\cite{130,131,132} (extending neutron scattering predictions for conventional spin-liquids in frustrated magnets\cite{133}) and in fact observed in granular solids\cite{129}. However, so far no smoking gun restricted mobility experiment has been conceived even in these physical instantiations of an in-principle fractonic elastic systems. The immobility of disclinations is not a particularly impressive observation, especially deep inside a crystal phase, where even vacancies and interstitials are energetically immobile and behave classically. Perhaps a study of a He$_4$ crystalline film or 2d lattices of ultra cold bosons close to a low-temperature quantum melting transition are good experimental platforms to explore.

Another obstacle in this direction is that in 2d, a disclination energy is extensively large and thus is not an excitation that can be easily explored. In contrast, the topology of a spherical crystals, such as, for example, the Buckminsterfullerene $C_{60}$ molecule and many other closed structures, as large as $C_{960}$, by Gauss-Bonnet Theorem automatically ensures 12 disclinations in the ground state, whose lattice hopping dynamics can perhaps be studied experimentally. Extension of dual gauge theories to a spherical geometry with a detailed analysis remains an problems, and corresponding experiments is an interesting direction to pursue.

Some further proposals for experimental realization of fractons or tensor gauge fields have appeared in spin liquids\cite{131} and Rydberg atoms\cite{154,155}.

This plethora of exciting open problems bodes well for a bright future of the exciting field of fractons, the current glimpse of which we presented in this Colloquium.

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Osciloscopic fingerprints of gapped quantum spin liquids, both conventional and fractonic, have been studied extensively [131, 132]. Gapped, topologically ordered fracton models can only appear in dimension $d > 2$ [133].

Latter admit high dimensional and lattice generalizations. For $D = 3$, the XY model Lagrangian and its duality can more carefully be treated on a lattice, giving equivalent results. Thus, our streamlined continuum approach is fundamentally justified by lattice regularization [134].

There are two qualitatively distinct classes of models with restricted mobility. As discussed in Sec. IV, one is with highly fragile, fine-tuned global multipolar and subsystem symmetries. The other are gauge-like theories with topological order, that are generally robust over some nonzero range of all local perturbations [135].

More precisely, $J^{13}$ transforms in spin-2 irreducible representation of $SO(2)$.

As discussed in the following subsection, for a complete description it must be supplemented with vacancies/interstitials (atoms) sector, captured by a conventional Abelian-Higgs model with bosonic matter [136].

A classical electrostatic limit of the crystal's gauge dual reduces [137] to a vector sine-Gordon model $H = \frac{1}{2} \sum_{i,j} C_{ijkl} \partial_i \phi \partial_j \phi - g_b \sum_{n=1}^{\infty} \cos(\mathbf{b}_n \cdot \hat{z} \times \nabla \phi) - g_f \cos(s_p \phi)$ that reproduces [138] the two-stage crystal-hexatic-isotropic melting of KTHNY.

Because, a condensation of dislocations necessarily leads to Bose-condensation of bosonic vacancies and interstitials, the resulting hexatic fluid is necessarily a superfluid, i.e., a superhexatic [139].

Here indices $a, b, c, \ldots$ refer to the tangent space, while indices $\mu, \nu, \rho, \ldots$ refer to the spacetime and indices $i, j, k, \ldots$ refer to the space. Unless lattice regularization is introduced [140].

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130. N. E. Myerson-Jain, S. Yan, D. Weld, C. Xu Construction of Fractal Order and Phase Transition with Rydberg Atoms Phys. Rev. Lett. 128, 017601

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