On the stability of swelling porous elastic soils with a viscoelastic damping

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Abstract. The present work studies a swelling porous-elastic system with viscoelastic damping. We establish a general and optimal decay estimate which generalizes some recent results in the literature. Our result is established without imposing the usual equal-wave-speed condition associated with similar problems in literature.

Keywords: Optimal decay, General decay, Swelling porous problem; Memory term, General decay.

AMS Subject Classifications: 93D20; 35B40

1. Introduction

The basic field equations describing the linear theory of swelling porous elastic soils, see Ieșan [?] and Quintanilla [?] are given by

\[ \rho_\varphi \varphi_{tt} = F_1 x - K_1 + L_1, \]
\[ \rho_\psi \psi_{tt} = F_2 x + K_2 + L_2, \]  
(1.1)

where the constituents \( \varphi = \varphi(x,t) \) and \( \psi = \psi(x,t) \) are respectively, the displacement of the fluid and the elastic solid material. The physical parameters \( \rho_\varphi \) and \( \rho_\psi \) are the densities of each constituent. The functions \( (F_1, K_1, L_1) \) and \( (F_2, K_2, L_2) \) are the partial tensions, internal body forces, and external forces acting on the displacement and the elastic solid respectively. In addition, the constitutive equations of partial tensions are given by

\[ \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix} \begin{pmatrix} \varphi_x \\ \psi_x \end{pmatrix}, \]  
(1.2)

where \( b_1, b_3 \) are positive constants and \( b_2 \neq 0 \) is a real number. The matrix \( \Lambda \) is positive definite (that is \( b_1 b_3 > b_2^2 \)). Quintanilla [?] studied \( \varphi = \varphi(x,t) \) and \( \xi = \xi(x,t) \) are respectively, the displacement and the elastic solid material. The physical parameters \( \rho_\varphi \) and \( \rho_\psi \) are the densities of each constituent. The functions \( (F_1, K_1, L_1) \) and \( (F_2, K_2, L_2) \) are the partial tensions, internal body forces, and external forces acting on the displacement and the elastic solid respectively. In addition, the constitutive equations of partial tensions are given by

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\[ K_1 = K_2 = \xi(\varphi_t - \psi_t), \quad L_1 = b_3 \varphi_{xxt}, \quad L_2 = 0, \]  
where \( \xi \) is a positive coefficient and established an exponential stability result. Wang and Guo [?] investigated \( \varphi = \varphi(x,t) \) and \( \gamma(x) \) is an internal viscous damping function with positive mean. The authors in [?] used the spectral method to prove an exponential stability result. For more related results, we refere the reader to [?] and the references cited there in.

The present work aims at studying \( \varphi = \varphi(x,t) \) with null internal body forces, where the external force is acting only on the elastic solid as a viscoelastic force, that is:

\[ K_1 = K_2 = 0, \quad L_1 = 0, \quad L_2 = -\int_0^t g(t - s)\psi_{xx}(s)ds, \]  
(1.3)

where \( g \) is a given kernel to be specified later (also known as the relaxation function). Substituting \( \varphi = \varphi(x,t) \) into \( \varphi = \varphi(x,t) \), we arrive at

\[ \begin{cases} 
\rho_\varphi \varphi_{tt} - b_1 \varphi_{xx} - b_2 \psi_{xx} = 0, & \text{in } (0,1) \times (0,\infty), \\
\rho_\psi \psi_{tt} - b_3 \psi_{xx} - b_2 \varphi_{xx} + \int_0^t g(t - s)\psi_{xx}(x,s)ds = 0, & \text{in } (0,1) \times (0,\infty). 
\end{cases} \]  
(1.4)
We supplement (2) with the following boundary conditions
\[ \psi(0, t) = \psi(1, t) = \varphi(0, t) = \varphi(1, t) = 0, \quad t \geq 0 \] (1.5)
and initia data
\[ \varphi(x, 0) = \varphi_0(x), \quad \varphi_1(x, 0) = \varphi_1(x), \quad \psi(x, 0) = \psi_0(x), \quad \psi_1(x, 0) = \psi_1(x), \quad x \in [0, 1]. \] (1.6)

The novelty of this paper is to improve the work established by Apalara [2, 3], where he considered (2) and proved a general decay result when \( g \) satisfied \( g'(t) \leq -\xi(t)g(t) \). This present work considers a very general condition with minimal assumption on \( g \) and prove an optimal decay estimate from which the result in [2, 3] is a particular case. For more related results or background of porous elastic swelling soil theory, we refer the reader to [2, 3] and the references cited there.

The rest of this work is organized as follows: In Section 3, we present preliminary materials which will be helpful in obtaining our results. In Section 4, we establish some useful lemmas. In Section 5, we study the decay rate of the energy functional associated to problem (2.5)-(2.6).

2. Assumptions and functional setting

From now on, the variables \( C \) denote a positive constants that may change within lines or from line to line. We denote by \( ||.||_2 \) the usual norm in \( L^2(0, 1) \). We consider the following assumptions on \( g \)

\( (A_1) \) The kernel \( g : [0, +\infty) \to (0, +\infty) \) is a \( C^1 \) decreasing function satisfying

\[ g_0 := \int_0^\infty g(s)ds < b \]

where \( b = b_3 - \frac{b_2^2}{b_1} > 0 \).

\( (A_2) \) There exists a \( C^1 \) function \( G : [0, +\infty) \to [0, +\infty) \) which is linear or it is strictly convex \( C^2 \) function on \( (0, \eta], \eta \leq g(t_0) \), for any \( t_0 > 0 \) with \( G(0) = G'(0) = 0 \) and a positive non-increasing differentiable function \( \xi : [0, +\infty) \to (0, +\infty) \), such that

\[ g'(t) \leq -\xi(t)G(g(t)), \quad t \geq 0. \] (2.1)

**Remark 2.1.**

1. From assumption \( (A_1) \) and \( (A_2) \) we see that \( G \) is a strictly increasing convex \( C^2 \)-function on \( (0, r] \), with \( G(0) = G'(0) = 0 \), thus, we can find an extension of \( G \) say

\[ \bar{G} : [0, +\infty) \to (0, +\infty) \]

which is also strictly increasing and strictly convex \( C^2 \)-function. For example, for any \( t > \eta \), we define \( G \) by

\[ \bar{G}(s) = \frac{G''(\eta)}{2}s^2 + (G'(\eta) - G''(\eta)\eta)s + G(\eta) - G'(\eta)\eta + \frac{G''(\eta)}{2}\eta^2. \] (2.2)

2. For any \( t_0 > 0 \), it follows from the fact that \( g \) is continuous, positive and \( g(0) > 0 \), that

\[ \int_0^t g(s)ds \geq \int_0^\infty g(s)ds - c_0 > 0, \quad \forall t \geq t_0. \] (2.3)

3. Using the fact that \( g \) and \( \xi \) are positive, continuous, and nonincreasing, and \( G \) is continuous and positive, there exists \( t_0 > 0 \) such that, for all \( t \in [0, t_0] \),

\[ 0 < g(t_0) \leq g(t) \leq g(0), \quad 0 < \xi(t_0) \leq \xi(t) \leq \xi(0). \]

It follows that

\[ a_1 \leq \xi(t)G(g(t)) \leq a_2, \quad \text{for some } a_1, a_2 > 0. \]

Therefore, \( \forall t \in [0, t_0] \), we get

\[ g'(t) \leq -\xi(t)G(g(t)) \leq -\frac{a_1}{g(0)}g(0) \leq -\frac{a_1}{g(0)}g(t) \] (2.4)

and

\[ \xi(t)g(t) \leq -\frac{g(0)}{a_1}g'(t). \] (2.5)

The well-posedness reads:
Theorem 2.2. Let \((\varphi_0, \varphi_1, \psi_0, \psi_1, \lambda) \in H^1_0(0, 1) \times L^2(0, 1) \times H^1_0(0, 1) \times L^2(0, 1)\) be given and assume conditions (A_1) and (A_2) hold. Then, problem (9) – (10) has a global weak unique solution \((z, u)\) such that

\[
(\varphi, \psi) \in C \left( [0, +\infty), H^1_0(0, 1) \times H^1_0(0, 1) \right) \cap C^1 \left( [0, +\infty), L^2(0, 1) \times L^2(0, 1) \right).
\] (2.6)

Furthermore, if \((\varphi_0, \varphi_1, \psi_0, \psi_1, \lambda) \in H^2(0, 1) \times H^1_0(0, 1) \times H^1_0(0, 1) \times H^2(0, 1) \times H^1_0(0, 1),\)

then the global unique solution of problem (9) – (10) has more regularity in the class

\[
\varphi \in C \left( [0, +\infty), H^2(0, 1) \cap H^1_0(0, 1) \right) \cap C^1 \left( [0, +\infty), H^1_0(0, 1) \right),
\]

\[
\psi \in C \left( [0, +\infty), H^2(0, 1) \cap H^1_0(0, 1) \right) \cap C^1 \left( [0, +\infty), H^1_0(0, 1) \right) \cap C^2 \left( [0, +\infty), L^2(0, 1) \right).
\]

Proof. The result can be establish using the Galerkin approximation method similarly like in [?]

We recall some lemmas which will be useful in establishing the main result.

Lemma 2.3. Let \(w \in L^2_w([0, +\infty), L^2(0, 1))\), there holds

\[
\int_0^1 \left( \int_0^t g(t-s)(w(t)-w(s))ds \right)^2 dx \leq c_p(a-g_0) (g \circ w_x)(t),
\] (2.7)

\[
\int_0^1 \left( \int_0^x w(y, t)dy \right)^2 dx \leq \|w\|_2^2,
\] (2.8)

where

\[
(g \circ w)(t) = \int_0^t g(t-s)\|w(t)-w(s)\|_2^2 ds.
\]

Proof. The result follows easily by Cauchy-Schwarz and Poincaré’s inequalities.

As in [?], for any \(0 < \alpha < 1\), let

\[
h(t) = \alpha g(t) - g'(t) \quad \text{and} \quad A_\alpha = \int_0^{+\infty} \frac{g^2(s)}{\alpha g(s) - g'(s)} ds.
\]

We have the following lemma.

Lemma 2.4. Let \((\varphi, \psi)\) be the solution of problem (9) – (10). Then, for any \(0 < \alpha < 1\), we have

\[
\int_0^1 \left( \int_0^t g(t-s) (\psi(t) - \psi(s)) ds \right)^2 dx \leq A_\alpha (h \circ \psi_x)(t),
\] (2.9)

where

\[
(h \circ \psi_x)(t) = \int_0^t h(t-s)\|\psi_x(t) - \psi_x(s)\|_2^2 ds.
\]

Proof. Using Cauchy-Schwarz and Poincaré’s inequalities, we have

\[
\int_0^1 \left( \int_0^t g(t-s) (\psi(t) - \psi(s)) ds \right)^2 dx
\]

\[
= \int_0^1 \left( \int_0^t g(t-s) \sqrt{h(t-s)} (\psi(t) - \psi(s)) ds \right)^2 dx
\]

\[
\leq \left( \int_0^{+\infty} \frac{g^2(s)}{h(s)} ds \right) \int_0^1 \int_0^t h(t-s) (\psi(t) - \psi(s))^2 ds dx
\]

\[
\leq A_\alpha (h \circ \psi_x)(t).
\] (2.10)

Lemma 2.5. Suppose \(F\) is convex function on a close interval \([a, b]\) and let \(f : \Omega \to [a, b]\) and a function \(j\) such that \(j(x) \geq 0\) and \(\int_\Omega j(x)dx = a > 0\). Then, we have the following Jensen inequality

\[
\frac{1}{a} \int_\Omega F(f(y))j(y)dy \geq F \left( \frac{1}{a} \int_\Omega f(y)j(y)dy \right).
\] (2.11)
3. Essential Lemmas

We state and prove some essential lemmas in this section.

**Lemma 3.1.** Let \((\varphi, \psi)\) be the solution given by Theorem 2. Then, the energy functional \(E\), defined by

\[
E(t) = \frac{1}{2} \int_0^1 \left[ \rho_\varphi \varphi_t^2 + b_1 \varphi_x^2 + \rho_\psi \psi_t^2 + \left( b_3 - \int_0^1 g(s)ds \right) \psi_x^2 + 2b_2 \varphi_x \psi_x \right] dx + \frac{1}{2} (g \circ \psi_x)(t),
\]

satisfies

\[
E'(t) = \frac{1}{2} (g' \circ \psi_x)(t) - \frac{1}{2} g(t) \int_0^1 \psi_x^2 dx \leq \frac{1}{2} (g' \circ \psi_x)(t) \leq 0, \quad \forall \ t \geq 0.
\]

**Lemma 3.2.** Let \((\varphi, \psi)\) be the solution given by Theorem 2. Then, the functional

\[
F_1(t) := \rho_\psi \int_0^1 \psi_t \psi dx - \frac{b_2}{b_1} \rho_\varphi \int_0^1 \varphi_t \psi dx
\]

satisfies, for any positive \(\sigma_1\),

\[
F_1'(t) \leq -\frac{b_0}{2} \int_0^1 \psi_x^2 dx + \sigma_1 \int_0^1 \varphi_t^2 dx + \left( \rho_\psi + \frac{b_2^2 \rho_\varphi^2}{4 \sigma_1 b_1^2} \right) \int_0^1 \psi_x^2 dx + \frac{A_n}{2b_0} (h \circ \psi_x)(t),
\]

where \(b_0 > 0\) is a constant.

**Proof.** Direct differentiation and using integration by parts yields

\[
F_1'(t) = - \left( b_3 - \frac{b_2^2}{b_1} \right) \int_0^1 \psi_x^2 dx + \rho_\psi \int_0^1 \psi_t^2 dx - \frac{b_2 \rho_\varphi}{b_1} \int_0^1 \varphi_t \psi_t dx + \int_0^1 g(t-s) \psi_x(s) ds dx
\]

Using Cauchy-Schwarz and Young’s inequalities, we get

\[
J_1 \leq \sigma_1 \int_0^1 \varphi_t^2 dx + \frac{b_2^2 \rho_\varphi^2}{4 \sigma_1 b_1^2} \int_0^1 \psi_x^2 dx.
\]

Recalling Lemma 3.1–3.2, then Cauchy-Schwarz and Young’s inequalities give

\[
J_2 = \int_0^1 g(s)ds \int_0^1 \psi_x^2 dx - \int_0^1 \psi_x \int_0^1 g(t-s) (\psi_x(t) - \psi_x(s)) ds dx
\]

\[
\leq \left( \int_0^t \left( g(s) ds + \varepsilon_0 \right) \right) \int_0^1 \psi_x^2 dx + \frac{1}{4 \varepsilon_0} \int_0^1 \left( \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds \right)^2 dx
\]

\[
\leq \left( \int_0^t \left( g(s) ds + \varepsilon_0 \right) \right) \int_0^1 \psi_x^2 dx + \frac{A_n}{4 \varepsilon_0} (h \circ \psi_x)(t).
\]

By substituting the estimates in (3.2) and (3.3) into (3.4), we arrive at

\[
F_1'(t) \leq - \left( b_3 - \frac{b_2^2}{b_1} - \int_0^t g(s)ds - \varepsilon_0 \right) \int_0^1 \psi_x^2 dx + \sigma_1 \int_0^1 \varphi_t^2 dx
\]

\[
+ \left( \rho_\psi + \frac{b_2^2 \rho_\varphi^2}{4 \sigma_1 b_1^2} \right) \int_0^1 \psi_x^2 dx + \frac{A_n}{4 \varepsilon_0} (h \circ \psi_x)(t).
\]

From condition \((A_1)\), we have

\[
\int_0^t g(s)ds < g_0 < b = b_3 - \frac{b_2^2}{b_1},
\]

Thus, it follows that \(b_3 - \frac{b_2^2}{b_1} - \int_0^t g(s)ds = b_0 > 0\). Therefore, by choosing \(\varepsilon_0 = \frac{b_0}{2}\), we obtain (3.5). \(\square\)

**Lemma 3.3.** Let \((\varphi, \psi)\) be the solution given by Theorem 2. Then, for any \(t_0 > 0\), the functional

\[
F_2(t) := -\rho_\psi \int_0^1 \psi_t \int_0^t g(t-s) (\psi(t) - \psi(s)) ds dx
\]
satisfies, for any positive \(\sigma_2, \sigma_3\), the estimate

\[
F_2^\prime(t) \leq -\frac{\rho_0 c_0}{2} \int_0^1 \psi_1^2 dx + \sigma_2 \int_0^1 \psi_2^2 dx + \sigma_3 \int_0^1 \psi_3^2 dx + \left( \frac{\rho_0}{c_0} + A_{\alpha} \left( 1 + \frac{b^2}{\sigma_2} + \frac{b_2^2}{4\sigma_3} + \frac{\rho_0 c_0^2}{c_0} \right) \right) (h \circ \varphi_x) (t).
\]

(3.9)

where \(c_0 = \int_0^t g(s) ds\).

Proof. Differentiation of \(F_2\), using (3.8), and then integrating by parts, we get

\[
F_2^\prime(t) = -\rho_0 \int_0^t g(s) ds \int_0^1 \psi_1^2 dx + b_3 \int_0^1 \psi_2 \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s)) ds dx
\]

\[
- \int_0^1 \int_0^t g(t-s) \varphi_x(s) ds \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s)) ds dx \bigg|_{s=0}^{s=t}
\]

\[
- \rho_0 \int_0^1 \psi_t \int_0^t g'(t-s)(\varphi(t) - \varphi(s)) ds dx + b_2 \int_0^1 \varphi_x \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s)) ds dx.
\]

(3.10)

Using Lemma 3.3, it follows by Young’s, and Cauchy-Schwarz, and Poincaré’s inequalities that for any \(\sigma_2, \sigma_3, \varepsilon_1 > 0\),

\[
J_3 \leq \frac{\sigma_2}{2} \int_0^1 \psi_1^2 dx + \frac{b_3^2}{2\sigma_2} \int_0^1 \left( \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s)) ds \right)^2 dx
\]

\[
\leq \frac{\sigma_2}{2} \int_0^1 \psi_1^2 dx + \frac{b_3^2 A_{\alpha}}{2\sigma_2} (h \circ \varphi_x) (t),
\]

(3.11)

\[
J_4 = -\int_0^t g(s) ds \int_0^1 \psi_x \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s)) ds dx + \int_0^1 \left( \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s)) ds \right)^2 dx
\]

\[
\leq \frac{\sigma_2}{2} \int_0^1 \psi_2^2 dx + \frac{1}{2\sigma_2} \left( \int_0^t g(s) ds \right)^2 \int_0^1 \left( \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s)) ds \right)^2 dx + A_{\alpha} (h \circ \varphi_x) (t)
\]

\[
\leq \frac{\sigma_2}{2} \int_0^1 \psi_2^2 dx + \left( 1 + \frac{b_2^2}{2\sigma_2} \right) A_{\alpha} (h \circ \varphi_x) (t),
\]

(3.12)

\[
-J_5 = -\rho_0 \varepsilon_1 \int_0^1 \psi_t \int_0^t g(t-s)(\varphi(t) - \varphi(s)) ds dx + \rho_0 \int_0^1 \psi_t \int_0^t h(t-s)(\varphi(t) - \varphi(s)) ds dx
\]

\[
\leq \rho_0 \varepsilon_1 \int_0^1 \psi_1^2 dx + \frac{\rho_0 c_0^2}{2\varepsilon_1} \int_0^1 \left( \int_0^t g(t-s)(\varphi(t) - \varphi(s)) ds \right)^2 dx
\]

\[
+ \frac{\rho_0}{2\varepsilon_1} \int_0^1 \left( \int_0^t h(t-s)(\varphi(t) - \varphi(s)) ds \right)^2 dx
\]

\[
\leq \rho_0 \varepsilon_1 \int_0^1 \psi_1^2 dx + \frac{\rho_0 c_0^2}{2\varepsilon_1} A_{\alpha} (h \circ \varphi_x) (t) + \frac{\rho_0}{2\varepsilon_1} (h \circ \varphi_x) (t)
\]

\[
= \rho_0 \varepsilon_1 \int_0^1 \psi_1^2 dx + \frac{\rho_0 c_0^2}{2\varepsilon_1} A_{\alpha} + \frac{\rho_0}{2\varepsilon_1} (h \circ \varphi_x) (t),
\]

and

\[
J_6 \leq \sigma_3 \int_0^1 \psi_3^2 dx + \frac{b_2^2}{4\sigma_3} \int_0^1 \left( \int_0^t g(t-s)(\varphi(t) - \varphi(s)) ds \right)^2 dx
\]

\[
\leq \sigma_3 \int_0^1 \psi_3^2 dx + \frac{b_2^2}{4\sigma_3} A_{\alpha} (h \circ \varphi_x) (t).
\]

(3.13)
Substituting (??)–(??) into (??), we end up with
\[
F_2'(t) \leq -\rho_\varphi \left( \int_0^t g(s) ds - \varepsilon_1 \right) \int_0^1 \psi_\varphi^2 dx + \frac{b_2^2 + b_3^2 + \rho_\psi \left( b_1 - b_3 \right)^2}{2 \rho_\varphi} \int_0^1 \psi_\varphi^2 dx + \frac{2 A_\alpha}{\rho_\psi} (h \circ \psi_x)(t).
\]
(3.15)
Recalling (??), it follows that for any \( t \geq t_0 > 0 \)
\[
c_0 = \int_0^t g(s) ds \leq \int_0^t g(s) ds.
\]
(3.16)
Thus, we select \( \varepsilon_1 = \frac{c_0}{2} \) to get (??).

\[ \Box \]

**Lemma 3.4.** Let \((\varphi, \psi)\) be the solution given by Theorem 3.3. Then, the functional
\[
F_3(t) := -b_2 \int_0^1 (\psi \varphi_t - \psi_t \varphi) dx
\]
satisfies
\[
F_3'(t) \leq -\frac{b_2^2}{2 \rho_\varphi} \int_0^1 \varphi_\varphi^2 dx + \frac{b_2^2 + b_3^2 + \rho_\psi \left( b_1 - b_3 \right)^2}{2 \rho_\varphi} \int_0^1 \psi_\varphi^2 dx + \frac{2 A_\alpha}{\rho_\psi} (h \circ \psi_x)(t).
\]
(3.17)

\[ \text{Proof.} \]
Differentiating \( F_3 \), using (??) and then integration by parts give
\[
F_3'(t) = -\frac{b_2^2}{2 \rho_\varphi} \int_0^1 \varphi_\varphi^2 dx + \frac{b_2^2 + b_3^2 + \rho_\psi \left( b_1 - b_3 \right)^2}{2 \rho_\varphi} \int_0^1 \psi_\varphi^2 dx + b_2 \left( \frac{b_1}{\rho_\varphi} - \frac{b_3}{\rho_\psi} \right) \int_0^1 \varphi_\varphi \psi_x dx
\]
\[ + \frac{b_2}{\rho_\varphi} \int_0^1 \varphi_\varphi \int_0^t g(t-s) \psi_x(s) ds dx. \]
(3.18)
Keeping in mind Lemma ??–??, Young’s and Cauchy-Schwarz inequalities yield for any \( \varepsilon_2 > 0 \),
\[
J_7 \leq \frac{b_2^2 \varepsilon_2^2}{2} \int_0^1 \varphi_\varphi^2 dx + \frac{1}{2 \varepsilon_2} \left( \frac{b_1}{\rho_\varphi} - \frac{b_3}{\rho_\psi} \right)^2 \int_0^1 \psi_\varphi^2 dx,
\]
\[
J_8 = \frac{b_2}{\rho_\varphi} \int_0^1 \varphi_\varphi \int_0^1 g(t-s) (\psi_x(s) - \psi_x(t)) ds dx + b_2 \left( \frac{b_1}{\rho_\varphi} - \frac{b_3}{\rho_\psi} \right) \int_0^1 \varphi_\varphi \psi_x dx
\]
\[ \leq \frac{b_2^2 \varepsilon_2^2}{2} \int_0^1 \varphi_\varphi^2 dx + \frac{1}{2 \varepsilon_2^2 \rho_\varphi^2} \left( \int_0^1 g(t-s) (\psi_x(s) - \psi_x(t)) ds \right)^2 dx + \frac{1}{2 \varepsilon_2^2 \rho_\varphi^2} \left( \int_0^1 g(s) ds \right)^2 \int_0^1 \psi_\varphi^2 dx
\]
\[ \leq \frac{b_2^2 \varepsilon_2^2}{2} \int_0^1 \varphi_\varphi^2 dx + \frac{A_\alpha}{\varepsilon_2^2 \rho_\varphi} (h \circ \psi_x)(t) + \frac{b_2^2}{\varepsilon_2^2 \rho_\varphi} \int_0^1 \psi_\varphi^2 dx.
\]
Substituting \( J_7 \) and \( J_8 \) into (??), we end up with
\[
F_3'(t) \leq -b_2 \left( \frac{1}{\rho_\varphi} - \varepsilon_2 \right) \int_0^1 \varphi_\varphi^2 dx + \frac{b_2^2 + b_3^2 + \left( b_1 - b_3 \right)^2}{2 \varepsilon_2} \left( \frac{b_1}{\rho_\varphi} - \frac{b_3}{\rho_\psi} \right) \int_0^1 \psi_\varphi^2 dx + \frac{A_\alpha}{\varepsilon_2 \rho_\varphi} (h \circ \psi_x)(t)
\]
Finally, by selecting \( \varepsilon_2 = \frac{1}{2 \rho_\varphi} \), we obtain (??).
\[ \Box \]

**Lemma 3.5.** Let \((\varphi, \psi)\) be the solution given by Theorem 3.3. Then, the functional
\[
F_4(t) := -\rho_\varphi \int_0^1 \varphi_\varphi \varphi dx
\]
satisfies
\[
F_4'(t) \leq -\rho_\varphi \int_0^1 \varphi_\varphi^2 dx + 2 b_1 \int_0^1 \varphi_\varphi^2 dx + \frac{b_2^2}{4 b_1} \int_0^1 \psi_\varphi^2 dx.
\]
(3.19)
Lemma 3.6. Assume conditions (A_1) and (A_2) hold. Then, the functional

$$F_0(t) = \int_0^1 \int_0^t f(t-s) \psi_x^2(s) ds dx,$$

where

$$f(t) = \int_t^{+\infty} g(s) ds$$

satisfies, along the solution of Problem (??), the estimate

$$F'_0(t) \leq 3g_0 \int_0^1 \psi_x^2(t) dx - \frac{1}{2} (g \circ \psi_x)(t). \quad (3.20)$$

Proof. Differentiating $F_0$ and observing that $f'(t) = -g(t)$, we infer

$$F'_0(t) = \int_0^1 \int_0^t f'(t-s) \psi_x^2(s) ds dx + f(0) \int_0^1 \psi_x^2 dx$$

$$= - \int_0^1 \int_0^t g(t-s) (\psi_x(s) - \psi_x(t))^2 ds dx$$

$$- 2 \int_0^1 \psi_x(t) g(t-s) (\psi_x(s) - \psi_x(t)) dx + f(0) \int_0^1 \psi_x^2(t) dx$$

But,

$$-2 \int_0^1 \psi_x(t) g(t-s) (\psi_x(s) - \psi_x(t)) dx \leq 2g_0 \int_0^1 \psi_x^2(t) dx + \frac{1}{2} (g \circ \psi_x)(t).$$

Since $f$ is decreasing, so $f(t) \leq f(0) = g_0$. Thus, we get

$$F'_0(t) \leq 2g_0 \int_0^1 \psi_x^2(t) dx - \frac{1}{2} (g \circ \psi_x)(t).$$

Lemma 3.7. Let $(\varphi, \psi)$ be the solution given by Theorem ???. Then, for suitable positive parameters $N, N_j, j = 1, 2, 3, 4$ to be chosen later, the functional

$$F(t) = NE(t) + N_1 F_1(t) + N_2 F_2(t) + N_3 F_3(t) + N_4 F_4(t)$$

satisfies

$$c_1 E(t) \leq F(t) \leq c_2 E(t), \quad \forall \ t \geq 0 \quad (3.31)$$

and

$$F'(t) \leq -\beta \int_0^1 \left( \varphi_t^2 + \varphi_x^2 + \psi_t^2 + \psi_x^2 \right) dx + \frac{1}{2} (g \circ \psi_x)(t), \quad \forall \ t \geq t_0, \quad (3.32)$$

where $c_1, c_2$ and $\beta$ are some positive constants.

Proof. With simple routine estimation using Young's, Cauchy-Schwarz, and Poincare's inequalities, we obtain (??) easily. Using (??) (??), (??), (??), (??) and recalling that $h = \alpha g - g'$, we obtain

$$F'(t) \leq - \left[ \rho_\varphi N_4 - \sigma_1 N_1 \right] \int_0^1 \varphi_t^2 dx - \left[ \frac{\rho_\psi c_0 N_2}{2} - N_1 \left( \rho_\psi + \frac{b_4^2 \rho_\psi^2}{4a_1 b_1} \right) \right] \int_0^1 \psi_t^2 dx$$

$$- \left[ \frac{b_2 N_3}{2 \rho_\varphi} - \sigma_2 N_2 - 2b_1 N_4 \right] \int_0^1 \varphi_x^2 dx + \frac{\alpha N_2}{2} (g \circ \psi_x)(t)$$

$$- \left[ \frac{b_0 N_1}{2} - \sigma_2 N_2 - N_3 \left( \frac{b_2^2}{\rho_\varphi} + \frac{b_2^2}{\rho_\psi} + \rho_\psi \left( \frac{b_1}{\rho_\varphi} - \frac{b_2}{\rho_\psi} \right)^2 \right) \right] - \frac{b_2 N_3}{2 b_1} \int_0^1 \psi_x^2 dx$$

$$- \left[ \frac{N_2}{2} - A \left( \frac{N_1}{2b_0} + N_2 \left( 1 + \frac{b_2^2}{\sigma_2} + \frac{b_2^2}{4\sigma_3} + \frac{\rho_\psi \alpha_3}{\sigma_0} \right) + \frac{2N_3}{\rho_\psi} - \frac{\rho_\psi N_2}{c_0} \right) \right] (h \circ \psi_x)(t).$$
Taking \[ N_4 = 1, \quad \sigma_1 = \frac{\rho_{\phi}}{2N_1}, \quad \sigma_2 = \frac{b_3}{2N_2}, \quad \sigma_3 = \frac{b_1}{N_2}, \quad N_3 = \frac{8b_1\rho_{\phi}}{b_2}, \]
we obtain
\[
F'(t) \leq -\frac{\rho_{\phi}}{2t} \varphi^2 \, dx - \left[ b_0N_1 - \frac{8b_1\rho_{\phi}}{b_2} \left( \frac{b^2}{\rho_{\phi}} + \frac{b^2}{\rho_{\phi}} + \frac{b_1}{\rho_{\phi}} - \frac{b_3}{\rho_{\phi}} \right)^2 \right] \int_0^1 \varphi^2 \, dx - b_3 \int_0^1 \varphi^2 \, dx + \frac{\alpha N_1}{2} (g \circ \psi_z) (t) - \frac{\rho_{\phi}N_2}{c_0} (h \circ \psi_z) (t).
\]

Now we state the main stability result of this paper. First, we select \( N_1 \) big enough such that
\[
\frac{b_0N_1}{2} - \frac{8b_1\rho_{\phi}}{b_2} \left( \frac{b^2}{\rho_{\phi}} + \frac{b^2}{\rho_{\phi}} + \frac{b_1}{\rho_{\phi}} - \frac{b_3}{\rho_{\phi}} \right)^2 - b_3 > 0,
\]
then we choose \( N_2 \) so large that
\[
\frac{\rho_{\phi}c_0N_2}{2} - N_1 \left( \rho_{\phi} + \frac{b_3^2\rho_{\phi}N_1}{2b_1^2} \right) > 0.
\]

From \( h = \alpha g - g' \), it follows that \( \frac{\alpha g^2(s)}{h(s)} = \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s) \); thus application of dominated convergence theorem gives
\[
\alpha A_\alpha = \int_0^{+\infty} \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} \, ds \to 0 \quad \text{as} \quad \alpha \to 0. \tag{3.23}
\]
Therefore, we can find \( 0 < \alpha_0 < 1 \) so that if \( \alpha < \alpha_0 < 1 \), then
\[
\alpha A_\alpha < \frac{1}{4 \left( \frac{N_1}{2b_0} + N_2 \left( 1 + \frac{b_2^2N_2}{4b_2} + \frac{b_3^2N_2}{4b_1} + \frac{\rho_{\phi}\alpha^2}{c_0} \right) + \frac{16b_1}{b_2} \right)}.
\]
Finally, we choose \( N \) so large and take \( \alpha = \frac{1}{2N} \) such that (3.23) remains true and
\[
N_2 - A_\alpha \left( \frac{N_1}{2b_0} + N_2 \left( 1 + \frac{b_2^2N_2}{4b_2} + \frac{b_3^2N_2}{4b_1} + \frac{\rho_{\phi}\alpha^2}{c_0} \right) + \frac{16b_1}{b_2} \right) - \frac{\rho_{\phi}N_2}{c_0} > 0.
\]

\[
\square
\]

4. MAIN DECAY RESULT

Now, we state the main result of this paper.

**Theorem 4.1.** Under the assumptions \((A_1)\) and \((A_2)\), there exist positive constants \( m_1 \) and \( m_2 \) such that the solution energy functional \((??)\) satisfies
\[
E(t) \leq m_2 G_1^{-1} \left( m_1 \int_0^t \xi(s) \, ds \right), \quad \text{where} \quad G_1(t) = \int_t^r \frac{1}{s G'(s)} \, ds
\]
and \( G_1 \) is a strictly convex function that is non-increasing on \((0, r]\), where \( r = g(t_0) > 0 \) with \( \lim_{t \to 0} G_1(t) = +\infty \).

**Proof.** Using \((??)\) and \((??)\), if follows that for any \( t \geq t_0 \),
\[
\int_0^{t_0} g(s) \int_0^1 (\psi_x(t) - \psi_x(t - s))^2 \, dx \, ds \leq -\frac{g(0)}{a_1} \int_0^{t_0} g'(s) \int_0^1 (\psi_x(t) - \psi_x(t - s))^2 \, dx \, ds \leq -c E'(t).
\]
Thus, estimates \((??)\) and \((??)\) give
\[
F'(t) \leq -\beta E(t) + \frac{1}{2} \int_0^{t_0} g(s) \int_0^1 (\psi_x(t) - \psi_x(t - s))^2 \, dx \, ds + \frac{1}{2} \int_0^1 g(s) \int_0^1 (\psi_x(t) - \psi_x(t - s))^2 \, dx \, ds
\]
\[
\square
This yields

$$R_1'(t) \leq -\beta(t) + \frac{1}{2} \int_{t_0}^t g(s) \int_0^1 (\psi_x(t) - \psi_x(t-s))^2 \, dx \, ds,$$

where $R_1 = F + cE$ and $R_1 \sim E$ by virtue of (4.2). The proof completed by considering two cases:

**Case I: $G$ is linear.** Multiplying (4.2) by $\xi(t)$, recalling (4.3) and (A2), we have

$$\xi(t)R_1'(t) \leq -\beta(t)\xi(t)E(t) + \frac{1}{2} \int_{t_0}^t \xi(t)g(s) \int_0^1 (\psi_x(t) - \psi_x(t-s))^2 \, dx \, ds$$

$$\leq -\beta(t)\xi(t)E(t) + \frac{1}{2} \int_{t_0}^t \xi(t)g(s) \int_0^1 (\psi_x(t) - \psi_x(t-s))^2 \, dx \, ds$$

$$\leq -\beta(t)\xi(t)E(t) - \frac{1}{2} \int_{t_0}^t g'(s) \int_0^1 (\psi_x(t) - \psi_x(t-s))^2 \, dx \, ds$$

$$\leq -\beta(t)\xi(t)E(t) - cE'(t), \forall \, t \geq t_0.$$

(4.4)

From (A2), $\xi$ is non-increasing, so we get

$$(\xi R_1 + cE)'(t) \leq -\beta(t)\xi(t)E(t), \forall \, t \geq t_0$$

(4.5)

and

$$\xi R_1 + cE \sim E \text{ since } R_1 \sim E.$$  

(4.6)

It follows that

$$R_2'(t) \leq -\beta(t)\xi(t)E(t) - m\xi(t)R_2(t), \forall \, t \geq t_0.$$  

(4.7)

where $R_2 = \xi R_1 + cE$ and $m$ is some positive constant. Integrating (4.3) over (0, $t$) and recalling (4.3), we obtain

$$E(t) \leq m_2 \xi(t) \int_{t_0}^t ds = m_2 G_1^{-1}(m_1 \int_{t_0}^t \xi(s) ds).$$

**Case II: $G$ is nonlinear.** We define the functional $L(t) = F(t) + F_0(t).$ Then, it follows from Lemma 2 and estimate (4.9), that for some $\lambda > 0,$

$$L'(t) \leq -\lambda E(t), \forall \, t \geq t_0,$$

(4.8)

which gives

$$\lambda \int_{t_0}^t E(s) ds \leq L(t_0) - L(t) \leq L(t_0).$$

It follows that

$$\int_0^{+\infty} E(s) ds < \infty.$$  

(4.9)

Using (4.9), we can choose $0 < \mu < 1$ such that

$$q(t) := \mu \int_{t_0}^t \int_0^1 (\psi_x(t) - \psi_x(t-s))^2 \, dx \, ds,$$

satisfies

$$q(t) \leq 1, \forall \, t \geq t_0.$$  

(4.10)

From here onward, we assume $q(t) > 0$ for all $t \geq t_0$; otherwise estimate (4.10) yields an exponential decay result. Now, we define the function $u(t)$ by

$$u(t) = -\int_{t_0}^t g'(s) \int_0^1 (\psi_x(t) - \psi_x(t-s))^2 \, dx \, ds.$$

It follows from (4.10) that $u(t) \leq -cE'(t), \forall \, t \geq t_0.$ Thanks to remark 2, condition (A2), (4.9), and Jensen’s inequality, we obtain

$$u(t) = -\int_{t_0}^t q(t)(-g'(s)) \int_0^1 (\psi_x(t) - \psi_x(t-s))^2 \, dx \, ds$$

$$u(t) = -\frac{1}{\mu q(t)} \int_{t_0}^t q(t)(-g'(s)) \int_0^1 (\psi_x(t) - \psi_x(t-s))^2 \, dx \, ds$$

Therefore, $u(t) \leq 0, \forall \, t \geq t_0.$
Thus estimate (4.1) takes the form
\[
R_1'(t) \leq -\beta E(t) + c\tilde{G}^{-1}\left(\frac{\mu u(t)}{\xi(t)}\right), \quad \forall \ t \geq t_0.
\]

Let \(\eta < \eta \leq g(t_0),\) to be chosen later, and define \(R_2(t)\) by
\[
R_2(t) := \tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right)R_1(t) + E(t) \sim E(t) \text{ since } R_1 \sim E.
\]

By using (4.10) and recalling that
\[
E'(t) \leq 0, \quad \tilde{G}'(t) > 0, \quad \tilde{G}''(t) > 0,
\]
we obtain
\[
R_2'(t) = \eta_0 E'(t)\frac{E(0)}{E(0)} \tilde{G}''\left(\frac{\eta_0 E(t)}{E(0)}\right)R_1(t) + \tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right)R_1'(t) + E'(t)
\]
\[
\leq -\beta E(t)\tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right) + c\tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right)\tilde{G}^{-1}\left(\frac{\mu u(t)}{\xi(t)}\right) + E'(t), \quad \forall \ t \geq t_0.
\] (4.13)

To estimate the term \(J,\) we consider the convex conjugate \(\tilde{G}'\) of \(\tilde{G}\) in the sense of Young, see [7], defined by
\[
\tilde{G}^*(s) = \tau(\tilde{G}')^{-1}(s) - \tilde{G}[(\tilde{G}')^*(s)]
\] (4.14)
and satisfies the generalized Young inequality
\[
AB \leq \tilde{G}^*(A) + \tilde{G}(B).
\] (4.15)

We set \(A = \tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right)\) and \(B = \tilde{G}^{-1}\left(\frac{\mu u(t)}{\xi(t)}\right),\) then using (4.10) and (4.10)-(4.10), we obtain
\[
R_2'(t) \leq -\beta E(t)\tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right) + c\tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right) + c\mu u(t) + E'(t)
\]
\[
\leq -\beta E(t)\tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right) + c\eta_0 E(t)\frac{E(0)}{E(0)}\tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right) + c\mu u(t) + E'(t).
\] (4.16)

Next, multiplying (4.10) by \(\xi(t),\) recalling \(\eta_0 E(t)\frac{E(0)}{E(0)} < \eta,\) then
\[
\tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right) = \tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right),
\]
and we arrive at
\[
\xi(t)R_2'(t) \leq -\beta \xi(t)E(t)\tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right) + c\eta_0 E(t)\frac{E(0)}{E(0)}\xi(t)\tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right) + c\mu u(t) + \xi(t)E'(t)
\]
\[
\leq -\beta \xi(t)E(t)\tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right) + c\eta_0 E(t)\frac{E(0)}{E(0)}\xi(t)\tilde{G}'\left(\frac{\eta_0 E(t)}{E(0)}\right) - cE'(t).
\] (4.17)

Let \(R_3 = \xi R_2 + cE\) and using the fact that \(R_1 \sim E,\) we get
\[
\kappa_0 R_3(t) \leq E(t) \leq \kappa_1 R_3(t),
\] (4.18)
for some $\kappa_0, \kappa_1 > 0$. Thus, from estimate (??), we obtain
\[ R_3'(t) \leq - (\beta E(0) - c_0 \eta(t)) \frac{E(t)}{E(0)} G'(E(t) \frac{E(t)}{E(0)}), \quad \forall t \geq t_0. \]
Now, we choose $\eta_0 < \eta$ very small so that $\beta E(0) - c_0 \eta > 0$ and we get for some positive $m$,
\[ R_3'(t) \leq - m \eta(t) \frac{E(t)}{E(0)} G'(E(t) \frac{E(t)}{E(0)}), \quad \forall t \geq t_0, \tag{4.19} \]
where $G_2(t) = t G'(\eta t)$. Its easy to see that
\[ G_2'(t) = G'(\eta_0 t) + \eta_0 G''(\eta_0 t). \]
Therefore, using the fact $G$ is strictly convex on $(0, \eta]$, we have $G_2(t) > 0$, $G_2'(t) > 0$ on $(0, \eta]$. Let
\[ R_4(t) = \kappa_0 \frac{R_3(t)}{E(0)}. \]
Then, from (??) and (??), we obtain
\[ R_4(t) \sim E(t) \tag{4.20} \]
and
\[ R_4'(t) = \kappa_0 \frac{R_3'(t)}{E(0)} \leq - m \eta(t) G_2(R_4(t)), \quad \forall t \geq t_0. \tag{4.21} \]
The integration of (??) over $(t_0, t)$ yields
\[ m_1 \int_{t_0}^{t} \frac{\eta(t)}{G_2(R_4(t))} ds \leq \frac{1}{\eta_0} \int_{t_0}^{\eta(t)} \frac{1}{s G'(s)} ds. \tag{4.22} \]
This implies
\[ R_4(t) \leq \frac{1}{\eta_0} \frac{G_1^{-1}(m_1 \int_{t_0}^{t} \eta(t) ds)}{G_1^{-1}(m_1 \int_{t_0}^{t} \eta(t) ds)} \tag{4.23} \]
From (A2), we see that $G_1$ is strictly decreasing on $(0, \eta]$ and
\[ \lim_{t \to 0} G_1(t) = +\infty. \]
Thus, (??) follows from (??) and (??). \qed

**Corollary 4.2.** Under assumptions (A1) and (A2), suppose the function $G$ in assumption (A2) is defined by
\[ G(s) = s^q, \quad p \geq 1. \tag{4.24} \]
Then there exist positive constants $m$ and $\bar{m}$ such that the solution energy (??) satisfies
\[ E(t) \leq \begin{cases} m \exp \left( - \bar{m} \int_{0}^{t} \eta(t) ds \right), & \text{for } p = 1, \\ \frac{m}{(1 + \int_{0}^{t} \eta(t) ds)^{\frac{1}{p}}}, & \text{for } p > 1. \end{cases} \tag{4.25} \]

**5. Concluding Remarks**

The present work improves the result in [??], where the author established a general decay result. The decay results in Theorem ?? is optimal in the sense that it agrees with the decay rate of the memory term $g$, see Remark 2.3 in [??]. This decay result is paramount to the engineers and architects as they might employ it to attenuate the harmful effects of swelling soils. The result in this paper also holds for some other boundary conditions such as
\[ \psi_x(0, t) = \varphi_x(1, t) = \varphi_x(0, t) = \varphi_x(1, t) = 0, \quad \psi(0, t) = \psi_x(1, t) = \varphi(0, t) = \varphi_x(1, t) = 0, \quad \text{and} \quad \psi_x(0, t) = \psi(1, t) = \varphi_x(0, t) = \varphi(1, t) = 0. \]

However, there might be some challenges for the following boundary conditions
\[ \psi_x(0, t) = \psi_x(1, t) = \varphi(0, t) = \varphi(1, t) = 0 \quad \text{and} \quad \psi_x(0, t) = \psi(1, t) = \varphi_x(0, t) = \varphi_x(1, t) = 0, \]
except we impose

\[ \int_0^1 \psi_0 dx = 0 \quad \text{and} \quad \int_0^1 \varphi_0 dx = 0. \]

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**References**

[1] Iqegan, D. *On the theory of mixtures of thermoelastic solids.* J. Thermal stresses, 14(4), 389–408 (1991).

[2] Quintanilla, R. *Exponential stability for one-dimensional problem of swelling porous elastic soils with fluid saturation.* J. Comput. Appl. Math., 145(2), 525–533 (2002).

[3] Wang, J. M. and Guo, B. Z. *On the stability of swelling porous elastic soils with fluid saturation by one internal damping.* IMA J. Appl. Math., 71(4), 565–582 (2006).

[4] Quintanilla, R. *Existence and exponential decay in the linear theory of viscoelastic mixtures.* European J. Mechanics-A/Solids, 24(2), 311–324 (2005).

[5] Quintanilla, R. *Exponential stability of solutions of swelling porous elastic soils.* Meccanica, 39(2), 139-145 (2004).

[6] Quintanilla, R. *On the linear problem of swelling porous elastic soils with incompressible fluid.* Int. J. Eng. Sci., 40(13), 1485–1494 (2002).

[7] Boill, F. and Quintanilla, R. *Anti-plane shear deformations of swelling porous elastic soils.* Int. J. Eng. Sci., 41(8), 801–816 (2003).

[8] Murad, M. A. and Cushman, J. H. *Thermomechanical theories for swelling porous media with microstructure.* Int. J. Eng. Sci., 38(5), 517–564 (2000).

[9] Apalara, T. A. *General stability result of swelling porous elastic soils with a viscoelastic damping,* Z. Angew. Math. Phys. (2020) 71:200.

[10] Eringen, A. C. *A continuum theory of swelling porous elastic soils.* Int. J. Eng. Sci., 32(8), 1337–1349 (1994).

[11] Bedford, A. and Drumheller, D. S. *Theories of immiscible and structured mixtures.* University News, University of Saskatchen, Saskatchewan, Canada, 2003.

[12] A. Eng. Sci., 39(4), 401–411 (2003).