NEW GENERALIZED APOSTOL-FROBENIUS-EULER POLYNOMIALS AND THEIR MATRIX APPROACH

MARÍA JOSÉ ORTEGA, WILLIAM RAMÍREZ, AND ALEJANDRO URIELES

Abstract. In this paper, we introduce a new extension of the generalized Apostol-Frobenius-Euler polynomials $H_{[m−1,α]}(x; c, a; λ; u)$. We give some algebraic and differential properties, as well as, relationships between this polynomials class with other polynomials and numbers. We also, introduce the generalized Apostol-Frobenius-Euler polynomials matrix $U_{[m−1,α]}(x; c, a; λ; u)$ and the new generalized Apostol-Frobenius-Euler matrix $U_{[m−1,α]}(c, a; λ; u)$, we deduce a product formula for $U_{[m−1,α]}(x; c, a; λ; u)$ and provide some factorizations of the Apostol-Frobenius-Euler polynomial matrix $U_{[m−1,α]}(x; c, a; λ; u)$, which involving the generalized Pascal matrix.

1. Introduction

It is well-known that generalized Frobenius-Euler polynomial $H^{(α)}_n(x; u)$ of order $α$ is defined by means of the following generating function

\[
\left(\frac{1−u}{e^z−u}\right)^α e^{xz} = \sum_{n=0}^{∞} H^{(α)}_n(x; u) \frac{z^n}{n!},
\]

where $u \in \mathbb{C}$ and $α \in \mathbb{Z}$. Observe that $H_n^{(1)}(x; u) = H_n(x; u)$ denotes the classical Frobenius-Euler polynomials and $H_n^{(α)}(0; u) = H_n^{(α)}(u)$ denotes the Frobenius-Euler numbers of order $α$. $H_n(x; −1) = E_n(x)$ denotes the Euler polynomials (see [2,7]).

For parameters $λ, u \in \mathbb{C}$ and $a, b, c \in \mathbb{R}^+$, the Apostol type Frobenius-Euler polynomials $H_n(x; λ; u)$ and the generalized Apostol-type Frobenius-Euler polynomials are
defined by means of the following generating functions (see \[8\]):

\[
\begin{align*}
(1 - u) e^{xz} &= \sum_{n=0}^{\infty} H_n(x; \lambda; u) \frac{z^n}{n!}, \\
\left( a^z - u \right)^{\alpha} c^{xz} &= \sum_{n=0}^{\infty} H_n^{(\alpha)}(x; a, b, c; \lambda; u) \frac{z^n}{n!}.
\end{align*}
\]

If we set \( x = 0 \) and \( \alpha = 1 \) in (1.3), we get

\[
\frac{a^z - u}{\lambda b^z - u} = \sum_{n=0}^{\infty} H_n(a, b, c; \lambda; u) \frac{z^n}{n!},
\]

\( H_n(a, b, c; u; \lambda) \) denotes the generalized Apostol-type Frobenius-Euler numbers (see \[8\]).

In the present paper, we introduce a new class of Frobenius-Euler polynomials considering the work of \[8\], we give relationships between this polynomials with other polynomials and numbers, as well as the generalized Apostol-Frobenius-Euler polynomials matrix.

The paper is organized as follows. Section 2 contains the definitions of Apostol-type Frobenius-Euler and generalized Apostol-Frobenius-Euler polynomials and some auxiliary results. In Section 3, we define the generalized Apostol-type Frobenius-Euler polynomials and prove some algebraic and differential properties of them, as well as their relation with the Stirling numbers of second kind. Finally, in Section 4 we introduce the generalized Apostol-type Frobenius-Euler polynomial matrix, derive a product formula for it and give some factorizations for such a matrix, which involve summation matrices and the generalized Pascal matrix of first kind in base \( c \), respectively.

2. Previous Definitions and Notations

Throughout this paper, we use the following standard notions: \( \mathbb{N} = \{1, 2, \ldots\} \), \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \), \( \mathbb{Z} \) denotes the set of integers, \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{C} \) denotes the set of complex numbers. Furthermore, \( (\lambda)_0 = 1 \) and

\[
(\lambda)_k = \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + k - 1),
\]

where \( k \in \mathbb{N} \), \( \lambda \in \mathbb{C} \). For the complex logarithm, we consider the principal branch. All matrices are in \( M_{n+1}(\mathbb{K}) \), the set of all \( (n+1) \times (n+1) \) matrices over the field \( \mathbb{K} \), with \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). Also, for \( i, j \) any nonnegative integers we adopt the following convention

\[
\binom{i}{j} = 0, \quad \text{whenever} \quad j > i.
\]

Now, let us give some properties of the generalized Apostol-type Frobenius-Euler polynomials and generalized Apostol-type Frobenius-Euler polynomials with parameters \( \lambda, a, c \), order \( \alpha \) (see \[4, 8, 11\]).
Proposition 2.1. For a \( m \in \mathbb{N} \), let \( \{H^{(α)}(x; u)\}_{n≥0} \) and \( \{H_n(x; λ; u)\}_{n≥0} \) be the sequences of generalized Apostol-type Frobenius-Euler polynomials, generalized Frobenius-Euler polynomials respectively. Then the following statements hold.

(a) Special values: for \( n \in \mathbb{N}_0 \),

\[
H^{(0)}_n(x; u) = x^n.
\]

(b) Summation formulas:

\[
H^{(α)}_n(x; u; a, b, c; λ) = \sum_{k=0}^{n} \binom{n}{k} H^{(α)}_k(x; u; a, b, c; λ)(x \ln c)^{n-k},
\]

\[
H^{(α+β)}_n(x + y; u; a, b, c; λ) = \sum_{k=0}^{n} \binom{n}{k} H^{(α)}_k(x; u; a, b, c; λ)H^{(β)}_{n-k}(y; u; a, b, c; λ),
\]

\[
((x + y) \ln c)^n = H^{(α)}_{n-k}(y; u; a, b, c; λ)H^{(-α)}_{k}(x; u; a, b, c; λ),
\]

\[
H^{(-α)}_n(x; \alpha^2; a^2, b^2, c^2; \lambda^2) = \sum_{k=0}^{n} \binom{n}{k} H^{(-α)}_k(x; u; a, b, c; λ)H^{(-α)}_{n-k}(x; -u; a, b, c; λ).
\]

Definition 2.1. ([5, p. 207]). For \( n \in \mathbb{N}_0 \) and \( x \in \mathbb{C} \), the Stirling numbers of second kind \( S(n, k) \) are defined by means of the following expansion

\[
x^n = \sum_{k=0}^{n} \binom{n}{k} k! S(n, k).
\]

The Jacobi polynomials of the degree \( n \) y orde \( (α, β) \), with \( α, β > -1 \), the \( n \)-th Jacobi polynomial \( P_n^{(α, β)}(x) \) may be defined through Rodrigues’ formula

\[
P_n^{(α, β)}(x) = (1 - x)^{-α}(1 + x)^{-β}\frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \left\{ (1 - x)^{n+α} (1 + x)^{n+β} \right\}
\]

and the values in the end points of the interval \([-1, 1]\) is given by

\[
P_n^{(α, β)}(1) = \binom{n + α}{n}, \quad P_n^{(α, β)}(-1) = (-1)^n \binom{n + β}{n}.
\]

The relationship between the \( n \)-th monomial \( x^n \) and the \( n \)-th Jacobi polynomial \( P_n^{(α, β)}(x) \) may be written as

\[
x^n = n! \sum_{k=0}^{n} \binom{n + α}{n - k} (-1)^k \frac{(1 + α + β + 2k)}{(1 + α + β + k)^{n+1}} P_k^{(α, β)}(1 - 2x).
\]

Proposition 2.2. For \( λ \in \mathbb{C} \) and \( m \in \mathbb{N} \), let \( \{B_n^{[m-1]}(x)\}_{n≥0} \), \( \{G_n(x)\}_{n≥0} \) and \( \{E_n(x; λ)\}_{n≥0} \) be the sequences of generalized Bernoulli polynomials of level \( m \), Genocchi polynomials and Apostol-Euler polynomials, respectively, we have the relationships:

(a) [12, Equation (4)]

\[
x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(k + m)!} B_{n-k}^{[m-1]}(x);
\]
(b) [9, Remark 7]
\[ x^n = \frac{1}{2(n+1)} \left[ \sum_{k=0}^{n+1} \binom{n+1}{k} G_k(x) + G_{n+1}(x) \right] ; \]

(c) [10, Equation (32)]
\[ x^n = \frac{1}{2} \left[ \lambda \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_k(x; \lambda) + \mathcal{E}_n(x; \lambda) \right]. \]

**Definition 2.2.** Let \( x \) be any nonzero real number. For \( c \in \mathbb{R}^+ \), the generalized Pascal matrix of first kind in base \( c \) \( P_c[x] \) is an \((n+1) \times (n+1)\) matrix whose entries are given by (see [13,14])
\[
p_{i,j,c}(x) := \begin{cases} 
(i \ln c)^{i-j}, & i \geq j, \\
0, & \text{otherwise}
\end{cases}
\]

When \( c = e \), the matrix \( P_e[x] \) coincides with the generalized Pascal matrix of first kind \( P[x] \). Furthermore, if we adopt the convention \( 0^0 = 1 \), then \( P_c[0] = I_{n+1} \), with \( I_{n+1} = \text{diag}(1,1,\ldots,1) \).

An immediate consequence of the remarks above is the following proposition.

**Proposition 2.3** (Addition Theorem of the argument). For \( x, y \in \mathbb{R} \) is fulfilled
\[ P_c[x + y] = P_c[x]P_c[y]. \]

**Proposition 2.4.** For \( c \in \mathbb{R}^+ \), let \( P_c[x] \) be the generalized Pascal matrix of first kind in base \( c \) and order \( n+1 \). Then the following statements hold.

(a) \( P_c[x] \) is an invertible matrix and its inverse is given by
\[ P_c^{-1}[x] := (P_c[x])^{-1} = P_c[-x]. \]

(e) The matrix \( P_c[x] \) can be factorized as follows
\[ P_c[x] = G_{n,c}[x]G_{n-1,c}[x] \cdots G_{1,c}[x], \]
where \( G_{k,c}[x] \) is the \((n+1) \times (n+1)\) summation matrix given by
\[
G_{k,c}[x] = \begin{cases} 
I_{n-k} & k = 1, \ldots, n-1, \\
0 & k = n,
\end{cases}
\]

being \( S_{k,c}[x] \) the \((k+1) \times (k+1)\) matrix whose entries \( S_{k,c}(x; i, j) \) are given by
\[
S_{k,c}(x; i, j, c) := \begin{cases} 
(x \ln c)^{i-j}, & i \geq j, \\
0, & j > i,
\end{cases} \quad 0 \leq i, j \leq k.
\]
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3. Generalized Apostol-Frobenius-Euler Polynomials

\[ \mathcal{H}_{n}^{[m-1,\alpha]}(x; c, a; \lambda; u) \]

**Definition 3.1.** For \( m \in \mathbb{N}, \alpha, \lambda, u \in \mathbb{C} \) and \( c, a \in \mathbb{R}^+ \), the generalized Apostol-type Frobenius-Euler polynomials in the variable \( x \), parameters \( c, a, \lambda \), order \( \alpha \) and level \( m \), are defined through the following generating function

\[
\left( \frac{m-1}{h!} \frac{z \ln a}{\lambda e^z - u} \right)^\alpha e^{xz} = \sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\alpha]}(x; c; a; \lambda; u) \frac{z^n}{n!},
\]

where \( |z| < \left| \frac{\ln(u^m)}{\ln(c)} - \frac{\ln(\lambda)}{\ln(c)} \right| \).

For \( x = 0 \) we obtain, the generalized Apostol-Frobenius-Euler numbers of parameters \( \lambda \in \mathbb{C}, a, c \in \mathbb{R}^+ \), order \( \alpha \in \mathbb{C} \) and level \( m \in \mathbb{N} \)

\[ \mathcal{H}_{n}^{[m-1,\alpha]}(c, a; \lambda; u) := \mathcal{H}_{n}^{[m-1,\alpha]}(0; c, a; \lambda; u). \]

According to the Definition 3.1, with \( e = \exp(1) \), we have (1.1) and (1.2)

\[
\mathcal{H}_{n}^{[0,\alpha]}(x; e, 1; 1; u) = H_n^{(\alpha)}(x; \lambda; u),
\]

\[
\mathcal{H}_{n}^{[0,1]}(x; e, 1; 1; u) = H_n^{(1)}(x; \lambda; u).
\]

**Example 3.1.** For any \( \lambda \in \mathbb{C}, m = 2, c = 2, a = 3, \alpha = \frac{1}{2} \) and \( u = 2 \) the first the generalized Apostol-type Frobenius-Euler polynomials in the variable \( x \), parameters \( c, a, \lambda \), order \( \alpha \) and level \( m \) are:

\[
\mathcal{H}_{0}^{[1,\frac{1}{2}]}(x; 2, 3; \lambda; 2) = \sqrt{\frac{3}{\lambda - 4}},
\]

\[
\mathcal{H}_{1}^{[1,\frac{1}{2}]}(x; 2, 3; \lambda; 2) = \sqrt{-\frac{3}{\lambda - 4}} x \left[ \frac{1}{2} \left( \frac{\ln 3}{\lambda - 4} + \frac{3\lambda \ln 2}{(\lambda - 4)^2} \right) + x \ln 4 \right],
\]

\[
\mathcal{H}_{2}^{[1,\frac{1}{2}]}(x; 2, 3; \lambda; 2) = \frac{1}{2} x^2 \left[ \left( -\frac{3}{4} \sqrt{-\frac{3}{\lambda - 4}} \left( \frac{\ln 3}{\lambda - 4} + \frac{3\lambda \ln 2}{(\lambda - 4)^2} \right) \right)^2 + \frac{1}{2} \sqrt{-\frac{3}{\lambda - 4}} \left( \frac{\ln 3}{\lambda - 4} + \frac{3\lambda \ln 2}{(\lambda - 4)^2} \right) + x \ln 2 \sqrt{-\frac{3}{\lambda - 4}} \left( \frac{\ln 3}{\lambda - 4} + \frac{3\lambda \ln 2}{(\lambda - 4)^2} \right) + x^2 \ln 4 \sqrt{-\frac{3}{\lambda - 4}} \right].
\]

**Example 3.2.** For any \( \lambda \in \mathbb{C}, m = 4, c = 2, a = 3, \alpha = 1 \) and \( u = 2 \) the first the generalized Apostol-type Frobenius-Euler polynomials in the variable \( x \), parameters \( c, a, \lambda \), order \( \alpha \) and level \( m \) are:

\[
\mathcal{H}_{0}^{[3,1]}(x; 2, 3; \lambda; 2) = \frac{-15}{\lambda - 16}.
\]
\[
\mathcal{H}_1^{[3,1]}(x; 2, 3; \lambda; 2) = x \left[ \frac{\ln 3}{\lambda - 16} + \frac{\lambda 15 \ln 2}{(\lambda - 16)^2} - \frac{x 15 \ln 2}{\lambda - 16} \right],
\]
\[
\mathcal{H}_2^{[3,1]}(x; 2, 3; \lambda; 2) = \frac{1}{2} x^2 \left[ \frac{\ln 9}{\lambda - 16} - \lambda \frac{2 \ln 3 \ln 2}{(\lambda - 16)^2} + \frac{2 \ln 3 \ln 2}{\lambda - 16} - \lambda^2 \frac{30 \ln 4}{(\lambda - 16)^3} + x \frac{30 \lambda \ln 4}{(\lambda - 16)^2} + \lambda \frac{15 \ln 4}{(\lambda - 16)^2} - x^2 \frac{2 \text{ln} 15}{\lambda - 16} \right].
\]

**Example 3.3.** For any \( \lambda \in \mathbb{C} \), \( m = 2 \), \( c = 3 \), \( a = e \), \( \alpha = \frac{1}{3} \), and \( u = 5 \) the first the generalized Apostol-type Frobenius-Euler polynomials in the variable \( x \), parameters \( c, a, \lambda, \) order \( \alpha \) and level \( m \) are:

\[
\mathcal{H}_0^{[1,\frac{1}{3}]}(x; 3, e; \lambda; 5) = \sqrt[3]{\frac{-24}{\lambda - 25}},
\]
\[
\mathcal{H}_1^{[1,\frac{1}{3}]}(x; 3, e; \lambda; 5) = x \left[ \frac{1}{3} \sqrt{\frac{\lambda - 25}{-24}} \left( \frac{\omega}{\lambda - 25} + \frac{24 \ln 3}{(\lambda - 25)^2} \right) \right] + x \ln 3 \sqrt[3]{\frac{-24}{\lambda - 25}},
\]
\[
\mathcal{H}_2^{[1,\frac{1}{3}]}(x; 3, e; \lambda; 5) = \frac{1}{2} x^2 \left[ \left( \frac{\lambda - 25}{-24} \right)^{\frac{2}{3}} \left( \frac{\omega}{\lambda - 25} + \frac{24 \ln 3}{(\lambda - 25)^2} \right) \right] + \frac{2}{3} x \sqrt{\frac{\lambda - 25}{-24}} \ln 3 \left( \frac{\omega}{\lambda - 25} + \frac{24 \ln 3}{(\lambda - 25)^2} \right) + \frac{1}{3} \sqrt[3]{\frac{\lambda - 25}{-24}} \left( -2 \ln 3 \frac{\omega}{(\lambda - 25)} - \lambda^2 \frac{-48 \ln 9}{(\lambda - 25)^3} + \lambda \frac{24 \ln 9}{(\lambda - 25)^2} \right) + x^2 \ln 9 \sqrt[3]{\frac{-24}{\lambda - 25}},
\]

where \( \omega = \ln \left( \frac{3060513329434037}{1125899906842624} \right) \).

**Theorem 3.1.** For \( m \in \mathbb{N} \), let \( \{\mathcal{H}_n^{[m-1,\alpha]}(x; c; a; \lambda; u)\}_{n \geq 0} \) be the sequence of generalized Apostol-type Frobenius-Euler polynomials, with parameters \( \lambda, u \in \mathbb{C} \) and \( a, c \in \mathbb{R}^+ \), order \( \alpha \in \mathbb{C} \) and level \( m \). Then the following statements hold.

(a) For every \( \alpha = 0 \) and \( n \in \mathbb{N}_0 \)

\( \mathcal{H}_n^{[m-1,0]}(x; c; a; \lambda; u) = (x \ln c)^n. \)

(b) For \( \alpha, \lambda \in \mathbb{C} \) and \( n, k \in \mathbb{N}_0 \), we have the relationship

\( \mathcal{H}_n^{[m-1,\alpha]}(x; c; a; \lambda; u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k}^{[m-1,\alpha]}(c; a; \lambda; u)(x \ln c)^k \).
Making an adequate modification

(c) Differential relations. For \( m \in \mathbb{N} \) and \( n, j \in \mathbb{N}_0 \) with \( 0 \leq j \leq n \), we have

\[
[\mathcal{H}_n^{[m-1,\alpha]}(x; c; a; \lambda; u)]^{(j)} = \frac{n!}{(n-j)!} (\ln c)^j \mathcal{H}_n^{[m-1,\alpha]}(x; c; a; \lambda; u).
\]

(d) Integral formulas. For \( m \in \mathbb{N} \), is fulfilled

\[
\int_{x_0}^{x_1} \mathcal{H}_n^{[m-1,\alpha]}(x; c; a; \lambda; u) \, dx = \frac{\ln c}{n+1} \left[ \mathcal{H}_n^{[m-1,\alpha]}(x_1; c; a; \lambda; u) - \mathcal{H}_n^{[m-1,\alpha]}(x_0; c; a; \lambda; u) \right].
\]

(e) Addition theorem of the argument.

\[
\mathcal{H}_n^{[m-1,\alpha+\beta]}(x+y; c; a; \lambda; u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k^{[m-1,\alpha]}(x; c; a; \lambda; u) \mathcal{H}_n^{[m-1,\beta]}(y; c; a; \lambda; u),
\]

\[
\mathcal{H}_n^{[m-1,\alpha]}(x+y; c; a; \lambda; u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_n^{[m-1,\alpha]}(y; c; a; \lambda; u)(x \ln c)^k,
\]

\[
((x+y) \log c)^n = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_n^{[m-1,\alpha]}(y; c; a; \lambda; u) \mathcal{H}_n^{[m-1,-\alpha]}(x; c; a; \lambda; u).
\]

Proof. (3.2) From Definition 3.1, we have

\[
\sum_{n=0}^{\infty} \mathcal{H}_n^{[m-1,\alpha+\beta]}(x+y; c; a; \lambda; u) \frac{t^n}{n!} = \left[ \sum_{h=0}^{m-1} \frac{(\ln a)^h}{h!} - u^m \right]^{(\alpha+\beta)} \lambda^{cz} - u^m \cdot c^{(x+y)z}
\]

\[
= \left[ \sum_{h=0}^{m-1} \frac{(\ln a)^h}{h!} - u^m \right]^{\alpha} \lambda^{cz} - u^m \cdot c^{yz} \left( \sum_{h=0}^{m-1} \frac{(\ln a)^h}{h!} - u^m \right)^{\beta} = \sum_{n=0}^{\infty} \mathcal{H}_n^{[m-1,\alpha]}(x; c; a; \lambda; u) z^n \frac{n!}{n!} \sum_{n=0}^{\infty} \mathcal{H}_n^{[m-1,\beta]}(y; c; a; \lambda; u) z^n \frac{n!}{n!}.
\]

Proof. (3.4) Making an adequate modification \( \beta = -\alpha \) and apply (3.2)

\[
\sum_{n=0}^{\infty} \mathcal{H}_n^{[m-1,\alpha+\beta]}(x+y; c; a; \lambda; u) \frac{z^n}{n!},
\]
Theorem 3.2. Therefore, (3.4) holds.

From (2.1) and Proposition 2.2 we deduce some algebraic relations connecting the polynomials $\mathcal{H}^{[m-1,\alpha]}_n(x;c,a;\lambda;u)$ with other families of polynomials.

**Theorem 3.2.** For $m \in \mathbb{N}$, the generalized Apostol-type Frobenius-Euler polynomials of level $m$ $\mathcal{H}^{[m-1,\alpha]}_n(x;c,a;\lambda;u)$, are related with the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, by means of the identity.

\begin{equation}
\mathcal{H}^{[m-1,\alpha]}_n(x+y,c,a;\lambda;u) = \sum_{k=0}^{n-j} \binom{n}{j,k} \mathcal{H}^{[m-1,\alpha]}_j(y,c,a;\lambda;\mu) P_k^{(\alpha,\beta)}(1-2x).
\end{equation}

**Proof.** By substituting (2.1) into the right-hand side of (3.3) and using appropriate binomial coefficient identities (see, for instance [1,5,6]), we see that

\begin{align*}
\mathcal{H}^{[m-1,\alpha]}_n(x+y,c,a;\lambda;u) &= \sum_{j=0}^{n} \binom{n}{j} \mathcal{H}^{[m-1,\alpha]}_j(y,c,a;\lambda;u)(n-j)!(\ln c)^{n-j} \sum_{k=0}^{n-j} (-1)^k \binom{n}{n-j+k} (1 + \alpha + \beta + 2k) \mathcal{H}^{[m-1,\alpha]}_{n-j}(y,c,a;\lambda;\mu) P_k^{(\alpha,\beta)}(1-2x) \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \binom{n}{j,k} \mathcal{H}^{[m-1,\alpha]}_j(y,c,a;\lambda;u)(n-j)!(\ln c)^{n-j} (-1)^k \binom{n-j+k}{n-j-k} (1 + \alpha + \beta + 2k) \mathcal{H}^{[m-1,\alpha]}_{n-j}(y,c,a;\lambda;\mu) P_k^{(\alpha,\beta)}(1-2x).
\end{align*}
Therefore, (3.5) holds. □

**Theorem 3.3.** For \( m \in \mathbb{N} \), the generalized Apostol-type Frobenius-Euler polynomials of level \( m \) \( \mathcal{H}_{n}^{[m-1,a]}(x; c, a; \lambda; u) \), are related with the generalized Bernoulli polynomials of level \( m \) \( B_{n}^{[m-1]}(x) \), by means of the following identity

\[
\mathcal{H}_{n}^{[m-1,a]}(x + y; c, a; \lambda; u) = \sum_{k=0}^{n} \binom{n}{k} k!(\ln c)^{j} \binom{j}{k} \mathcal{H}_{n-k}^{[m-1,a]}(y; c, a; \lambda; u)(\ln c)^{j-k} G_{k+1}(x).
\]

**Proof.** By substituting (2.2) into the right-hand side of (3.3), it suffices to follow the proof given in Theorem 3.2, making the corresponding modifications. □

**Theorem 3.4.** For \( m \in \mathbb{N} \), the generalized Apostol-type Frobenius-Euler polynomials of level \( m \) \( \mathcal{H}_{n}^{[m-1,a]}(x; c, a; \lambda; u) \), are related with the Genocchi polynomials \( G_{n}(x) \), by means of

\[
\mathcal{H}_{n}^{[m-1,a]}(x; c, a; \lambda; u)
\]

(3.6)

\[
= \sum_{j=0}^{n} \binom{n}{j} \mathcal{H}_{j}^{[m-1,a]}(y; c, a; \lambda; u) \frac{(\ln c)^{n-j}}{2(n-j+1)} \left[ \sum_{k=0}^{n-j} \binom{n-j+1}{k+1} G_{k+1}(x) + G_{n-j+1}(x) \right]
\]

**Proof.** By substituting (2.3) into the right-hand side of (3.3), we see that

\[
\mathcal{H}_{n}^{[m-1,a]}(x; c, a; \lambda; u)
\]

(3.6)

\[
= \sum_{j=0}^{n} \binom{n}{j} \mathcal{H}_{j}^{[m-1,a]}(y; c, a; \lambda; u) \frac{(\ln c)^{n-j}}{2(n-j+1)} \left[ \sum_{k=0}^{n-j} \binom{n-j+1}{k+1} G_{k+1}(x) + G_{n-j+1}(x) \right]
\]

Then, using appropriate combinatorial identities and summations (see, for instance [1, 5, 6]), we obtain

\[
\mathcal{H}_{n}^{[m-1,a]}(x; c, a; \lambda; u)
\]
by means of the following identity

Theorem 3.5. For \( m \in \mathbb{N} \), the generalized Apostol-type Frobenius-Euler polynomials of level \( m \) \( \mathcal{H}_n^{[m-1,\alpha]}(x; c, a; \lambda; u) \), are related with the Apostol-Euler polynomials \( \mathcal{E}_n(x; \lambda) \), by means of the following identity

\[
\mathcal{H}_n^{[m-1,\alpha]}(x + y; c, a; \lambda; u) = \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} \mathcal{H}_j^{[m-1,\alpha]}(y; c, a; \lambda; u)(\ln c)^{n-j} \left( \frac{1}{2} \right) \mathcal{E}_j(x; \lambda) + \mathcal{E}_{n-j}(x; \lambda).
\]

Proof. By substituting (2.4) into the right-hand side of (3.3), we can see that

\[
\mathcal{H}_n^{[m-1,\alpha]}(x + y; c, a; \lambda; u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k^{[m-1,\alpha]}(y; c, a; \lambda; u)(\ln c)^{n-k} \left( \frac{1}{2} \right) \sum_{j=0}^{n-k} \binom{n-k}{j} \mathcal{E}_j(x; \lambda) + \mathcal{E}_{n-k}(x; \lambda).
\]

The first sum in (3.8) becomes

\[
\sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k^{[m-1,\alpha]}(y; c, a; \lambda; u)(\ln c)^{n-k} \left( \frac{1}{2} \right) \sum_{j=0}^{n-k} \binom{n-k}{j} \mathcal{E}_j(x; \lambda) = \sum_{j=0}^{n} \binom{n}{j} \left( \frac{1}{2} \right) \mathcal{E}_j(x; \lambda) \sum_{k=0}^{n-j} \binom{n-j}{k} \mathcal{H}_k^{[m-1,\alpha]}(y; c, a; \lambda; u)
\]

\[
= \sum_{j=0}^{n} \left( \frac{1}{2} \right) \mathcal{E}_j(x; \lambda) \sum_{k=0}^{n-j} \binom{n-j}{k} \mathcal{H}_k^{[m-1,\alpha]}(y; c, a; \lambda; u)(\ln c)^{n-k}
\]

\[
= \sum_{j=0}^{n} \left( \frac{1}{2} \right) \mathcal{E}_j(x; \lambda) \mathcal{H}_j^{[m-1,\alpha]}(y + 1; c, a; \lambda; u).
\]

For the second sum in (3.8), we obtain

\[
\sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k^{[m-1,\alpha]}(y; c, a; \lambda; u)(\ln c)^{n-k} \left( \frac{1}{2} \right) \mathcal{E}_{n-k}(x; \lambda) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k^{[m-1,\alpha]}(y; c, a; \lambda; u)(\ln c)^k \mathcal{E}_k(x; \lambda).
\]
Combining (3.9) and (3.10) we get
\[
\mathcal{H}_n^{[m-1,\alpha]}(x + y; c, a; \lambda; u) = \left(\frac{\lambda}{2}\right)^n \sum_{j=0}^{n} \binom{n}{j} \mathcal{E}_j(x; \lambda) \mathcal{H}_n^{[m-1,\alpha]}(y + 1; c, a; \lambda; u)
\]
\[
+ \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} \mathcal{H}_n^{[m-1,\alpha]}(y; c, a; \lambda; u)(\ln c)^j \mathcal{E}_j(x; \lambda)
\]
\[
= \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} \left[ \lambda \mathcal{H}_n^{[m-1,\alpha]}(y + 1; c, a; \lambda; u) + (\ln c)^j \mathcal{H}_n^{[m-1,\alpha]}(y; c, a; \lambda; u) \right] \mathcal{E}_{n-j}(x; \lambda).
\]

Therefore, (3.7) holds.

\[
\text{Proposition 3.1. For } m \in \mathbb{N}, \alpha, \lambda, u \in \mathbb{C}, a, c \in \mathbb{R}^+ \text{ and } n \in \mathbb{N}_0, \text{ we have}
\]
\[
\mathcal{H}_n^{[m-1,\alpha]}(x + y; c, a; \lambda; u) = \sum_{k=0}^{n} k! \binom{x}{k} \sum_{j=0}^{n-k} \binom{n}{j} \mathcal{H}_j^{[m-1,\alpha]}(y; c, a; \lambda; u)(\ln c)^{n-j} S(n - j, k)
\]
\[
= \sum_{k=0}^{n} k! \binom{x}{k} \sum_{j=k}^{n} \binom{n}{j} \mathcal{H}_{n-j}^{[m-1,\alpha]}(y; c, a; \lambda; u)(\ln c)^{j} S(j, k).
\]

4. THE GENERALIZED APOSTOL-FROBENIUS-EULER POLYNOMIALS MATRIX

\[
\text{Definition 4.1. The generalized } (n+1) \times (n+1) \text{ Apostol-Frobenius-Euler polynomials matrix } \mathcal{U}^{[m-1,\alpha]}(x; c, a; \lambda; u) \text{ with } m \in \mathbb{N}, \alpha, \lambda, u \in \mathbb{C} \text{ and } a, c \text{ positive real numbers is defined by}
\]
\[
\mathcal{U}^{[m-1,\alpha]}(x; c, a; \lambda; u) = \left\{ \begin{array}{ll}
\binom{i}{j} \mathcal{H}_i^{[m-1,\alpha]}(x; c, a; \lambda; u), & i \geq j, \\
0, & \text{otherwise.}
\end{array} \right.
\]

While, the matrices
\[
\mathcal{U}^{[m-1]}(x; c, a; \lambda; u) := \mathcal{U}^{[m-1,\alpha]}(x; c, a; \lambda; u),
\]
\[
\mathcal{U}^{[m-1]}(c, a; \lambda; u) := \mathcal{U}^{[m-1]}(0; c, a; \lambda; u)
\]
are called the Apostol-Frobenius-Euler polynomial matrix and the Apostol-Frobenius-Euler matrix, respectively.

Since \( \mathcal{H}_n^{[m-1,0]}(x; c, a; \lambda; u) = (x \ln(c))^n \), we have \( \mathcal{U}^{[m-1,0]}(x; c, a; \lambda; u) = P_c[x] \). It is clear that (3.3) yields the following matrix identity:
\[
\mathcal{U}^{[m-1,\alpha]}(x + y; c, a; \lambda; u) = \mathcal{U}^{[m-1,\alpha]}(y; c, a; \lambda; u)P_c[x].
\]

\[
\text{Theorem 4.1. For a fixed } m \in \mathbb{N}, \text{ let } \{\mathcal{H}_n^{[m-1,\alpha]}(x; c, a; \lambda; u)\}_{n \geq 0} \text{ and } \{\mathcal{H}_n^{[m-1,\beta]}(x; c, a; \lambda; u)\}_{n \geq 0} \text{ be the sequences of generalized Apostol-type Frobenius-Euler}
\]
polynomials in the variable $x$, parameters $\lambda, u \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m$. Then satisfies the following product formula:

\[
(4.1) \quad U^{[m-1,\alpha+\beta]}(x + y; c, a; \lambda; u) = U^{[m-1,\alpha]}(x; c, a; \lambda; u) U^{[m-1,\beta]}(y; c, a; \lambda; u)
= U^{[m-1,\beta]}(x; c, a; \lambda; u) U^{[m-1,\alpha]}(y; c, a; \lambda; u)
= U^{[m-1,\beta]}(y; c, a; \lambda; u) U^{[m-1,\alpha]}(x; c, a; \lambda; u).
\]

**Proof.** Let $B^{[m-1,\alpha,\beta]}_{i,j,c}(a; \lambda; u)(x, y)$ be the $(i, j)$-th entry of the matrix product $U^{[m-1,\alpha]}(x; c, a; \lambda; u) U^{[m-1,\beta]}(y; c, a; \lambda; u)$, then by the addition formula (3.2) we have

\[
B^{[m-1,\alpha,\beta]}_{i,j,c}(a; \lambda; u)(x, y) = \sum_{k=0}^{n} \binom{i}{k} \mathcal{H}^{[m-1,\alpha]}(x; c, a; \lambda; u) \binom{j}{k} \mathcal{H}^{[m-1,\beta]}(y; c, a; \lambda; u)
= \sum_{k=0}^{n} \binom{i}{k} \mathcal{H}^{[m-1,\alpha]}(x; c, a; \lambda; u) \binom{j}{k} \mathcal{H}^{[m-1,\beta]}(y; c, a; \lambda; u)
= \binom{i}{j} \mathcal{H}^{[m-1,\alpha]}(x; c, a; \lambda; u) \mathcal{H}^{[m-1,\beta]}(y; c, a; \lambda; u)
= \binom{i}{j} \mathcal{H}^{[m-1,\alpha,\beta]}_{i,j,c}(x + y; c, a; \lambda; u),
\]

which implies the first equality of the theorem. The second and third equalities of can be derived in a similar way. \hfill \square

**Corollary 4.1.** For a fixed $m \in \mathbb{N}$, let $\{\mathcal{H}^{[m-1,\alpha]}(x; c, a; \lambda; u)\}_{n \geq 0}$ and $\{\mathcal{H}^{[m-1,\beta]}(x; c, a; \lambda; u)\}_{n \geq 0}$ be the sequences of generalized Apostol-type Frobenius-Euler polynomials in the variable $x$, parameters $\lambda, u \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m$ and $P_c[x]$ the generalized Pascal matrix of first kind in base $c$. Then

\[
U^{[m-1,\alpha]}(x + y; c, a; \lambda; u) = U^{[m-1,\alpha]}(x; c, a; \lambda; u) P_c[y]
= P_c[x] U^{[m-1,\alpha]}(y; c, a; \lambda; u)
= U^{[m-1,\beta]}(y; c, a; \lambda; u) P_c[x].
\]

In particular,

\[
U^{[m-1]}(x + y; c, a; \lambda; u) = P_c[x] U^{[m-1]}(y; c, a; \lambda; u)
= P_c[y] U^{[m-1]}(x; c, a; \lambda; u).
\]

**Proof.** The substitution $\beta = 0$ into (4.1) yields

\[
U^{[m-1,\alpha]}(x + y; c, a; \lambda; u) = U^{[m-1,\alpha]}(x; c, a; \lambda; u) U^{[m-1,0]}(y; c, a; \lambda; u).
\]

Since $U^{[m-1,0]}(y; c, a; \lambda; u) = P_c[y]$, we obtain

\[
(4.2) \quad U^{[m-1,\alpha]}(x + y; c, a; \lambda; u) = U^{[m-1,\alpha]}(x; c, a; \lambda; u) P_c[y].
\]
A similar argument allows to show that

\[
U^{[m-1,\alpha]}(x + y; c, a; \lambda; u) = P_c[x]U^{[m-1,\alpha]}(y; c, a; \lambda; u) = U^{[m-1,\alpha]}(y; c, a; \lambda; u)P_c[x].
\]

Finally, the substitution \( \alpha = 1 \) into (4.2) and its combination with the previous equations completes the proof. \( \square \)

Using the relation (2.5) and Corollary 4.1 we obtain the following factorization for \( U^{[m-1,\alpha]}(x + y; c, a; \lambda; u) \) in terms of summation matrices.

\[
U^{[m-1,\alpha]}(x + y; c, a; \lambda; u) = U^{[m-1,\alpha]}(x; c, a; \lambda; u)G_{n,c}[y]G_{n-1,c}[y] \cdots G_{1,c}[y].
\]

Under the appropriate choice on the parameters, level and order, it is possible to provide some illustrative examples of the generalized Apostol-Frobenius-Euler polynomials matrices.

**Example 4.1.** For \( m = 1, c = a = e = \exp(1), \alpha = 1, \lambda = -1 \), The first four polynomials \( \mathcal{H}^{[1-1,1]}(x; e, e; 1; u) \), \( k = 0, 1, 2, 3 \) are

\[
\begin{align*}
\mathcal{H}^{[1-1,1]}_0(x; e, e; 1; u) &= 1, \\
\mathcal{H}^{[1-1,1]}_1(x; e, e; 1; u) &= x - \frac{1}{1 - u}, \\
\mathcal{H}^{[1-1,1]}_2(x; e, e; 1; u) &= x^2 - \frac{2}{1 - u}x + \frac{1 + u}{(1 - u)^2}, \\
\mathcal{H}^{[1-1,1]}_3(x; e, e; 1; u) &= x^3 - \frac{3}{1 - u}x^2 + \frac{3(1 + u)}{(1 - u)^2}x - \frac{u^2 + 4u + 1}{(1 - u)^3}.
\end{align*}
\]

Hence, for \( n = 3 \), we have

\[
U^{[m-1,1]}(x; e, e; 1; u) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
u_{10} & 1 & 0 & 0 \\
u_{20} & u_{21} & 1 & 0 \\
u_{30} & u_{31} & u_{32} & 1
\end{bmatrix},
\]

where

\[
\begin{align*}
u_{10} &= u_{21} = u_{32} = \mathcal{H}^{[1-1,1]}_1(x; e, e; 1; u), \\
u_{20} &= u_{31} = \mathcal{H}^{[1-1,1]}_2(x; e, e; 1; u), \\
u_{30} &= \mathcal{H}^{[1-1,1]}_3(x; e, e; 1; u).
\end{align*}
\]
Example 4.2. For $m = 1$, $c = a = e = \exp(1)$, $\lambda = 1$ and $u = -1$, the first four polynomials $\mathcal{H}_k^{[1,1,a]}(x; e,e; 1; -1)$, $k = 0, 1, 2, 3$, are

\[
\begin{align*}
\mathcal{H}_0^{[1,1,a]}(x; e,e; 1; -1) &= 1, \\
\mathcal{H}_1^{[1,1,a]}(x; e,e; 1; -1) &= x - \frac{\alpha}{2}, \\
\mathcal{H}_2^{[1,1,a]}(x; e,e; 1; -1) &= x^2 - \alpha x + \frac{\alpha(\alpha - 1)}{4}, \\
\mathcal{H}_3^{[1,1,a]}(x; e,e; 1; -1) &= x^3 - \frac{3\alpha}{2} x^2 + \frac{3\alpha(\alpha - 1)}{4} x - \frac{3\alpha^2(\alpha - 1)}{8}.
\end{align*}
\]

Then, for $n = 3$, we have

\[
\mathcal{U}^{[m-1,a]}(x; e,e; 1; -1) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
u_{10} & 1 & 0 & 0 \\
u_{20} & 2u_{21} & 1 & 0 \\
u_{30} & 3u_{31} & 3u_{32} & 1 \\
\end{bmatrix},
\]

where

\[
\begin{align*}
u_{10} &= u_{21} = u_{32} = \mathcal{H}_1^{[1,1,a]}(x; e,e; 1; -1), \\
u_{20} &= u_{31} = \mathcal{H}_2^{[1,1,a]}(x; e,e; 1; -1), \\
u_{30} &= \mathcal{H}_3^{[1,1,a]}(x; e,e; 1; -1).
\end{align*}
\]

Example 4.3. For $\lambda \in \mathbb{C}$, $m = c = 2$, $a = 3$, $\alpha = \frac{1}{2}$, $u = 2$, we have the Example 3.1. Therefore,

\[
\mathcal{U}^{[1,\frac{1}{2}]}(x; 2, 3; \lambda; 2) = \begin{bmatrix}
\mathcal{H}_1^{[1,\frac{1}{2}]}(x; 2, 3; \lambda; 2) & \sqrt{\frac{3}{\lambda-4}} & 0 & 0 \\
\mathcal{H}_2^{[1,\frac{1}{2}]}(x; 2, 3; \lambda; 2) & \sqrt{\frac{3}{\lambda-4}} & 0 & 0 \\
\mathcal{H}_3^{[1,\frac{1}{2}]}(x; 2, 3; \lambda; 2) & 2\mathcal{H}_1^{[1,\frac{1}{2}]}(x; 2, 3; \lambda; 2) & \sqrt{\frac{3}{\lambda-4}} & 0 \\
\end{bmatrix}.
\]

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1Department of Mathematics, Universidad del Atlántico, Km 7 vía Pto. Barranquilla-Colombia

Email address: mortega22@cuc.edu.co

Email address: wramirez4@cuc.edu.co

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2Department of Mathematics, Universidad de la Costa, Barranquilla-Colombia

Email address: alejandbourieles@mail.uniatlantico.edu.co