Some properties of multi-qubit systems under effect of the Lorentz transformation
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Abstract

Effect of Lorentz transformation on some properties of multi-qubit systems is investigated. It is shown that, properties like, the fidelity and entanglement decay as the Wigner’s angles increase, but can be improved, if all the transformed particles are transformed with the same Wigner’s angles. The upper bounds of the average capacity of the GHZ state increases while it decreases and more robust with the W-state as the Wigner’s angle of the observer decreases. Under Lorentz transformation, the tripartite states transform into another equivalent states and hence no change on the efficiency of these states to perform quantum information tasks.

keywords: Lorentz transformation, Multi-qubits, Entanglement, Fidelity, Capacity

1 introduction

It is well known that entanglement represents the corner stone of most applications of quantum information [1, 2, 3]. There are several studies devoted to investigate the possibility of generating entanglement between different types of particles [4, 5, 6]. Quantifying the degree of entanglement which may be generated between these particles assimilates another area for many researchers, where several measurements of entanglement have been found. These measures depend on the type of the generated entangled state, pure or mixed, in two dimensions or higher. The most common measures for two qubit systems are, the concurrence [7], entanglement of formation [8, 9] and negativity [10, 11] etc, while for the tripartite states, tangle represents an acceptable measure [12].

Recently, it has been shown that entanglement is treated from relativistic point of view. For example, the relativistic entanglement of two massive particles is investigated by N.Friis et.al [13]. Saldan and Vedral [14] have discussed the spin quantum correlations of relativistic particles. The behavior of the spin fidelity of the three qubit Greenberge -Horne -Zeingler and W-state under Lorentz transformation is studied by Esfahani and M. Aghaee [15]. The change of entanglement under the effect of Lorentz transformation of a two spin-one particle system is studied by Ruiz and N. Achar [16].

In this contribution, we investigate the effect of Lorentz transformation on two classes of multi-qubit systems: GHZ and W-states. The behavior of the fidelities and the channel capacities are investigated for different values of Wigner angles. Due to the effect of Lorentz transformation the entanglement of these multi-qubit systems decays. Therefor, we investigate the robustness of these states under the action of Lorentz transformation.
The paper is organized as follows: In Sec. 2, the suggested model and its evolution under the effect of the Lorentz transformation is discussed. Sec. 3, is devoted to investigate the immutability of the multi-qubit states by discussing the behavior of the fidelities and the average capacities of these states. The amount of entanglement of the transformed states is quantified by using the three-tangle \[12\] in Sec. 4. Finally, our results are summarized in Sec. 5.

2 The Suggested Model

We assume that, a source supplies a three users with a three massive particles. It is assumed that, the spin part is given by GHZ or W-states, while the momentum part is a superposition between the three qubits. Therefore, the total system can be described by the following state \[17\]

\[
|\psi_{\text{system}}\rangle = |\psi_{\text{mom}}\rangle |\psi_{\text{spin}}\rangle,
\]

where, \(|\psi_{\text{mom}}\rangle\) represents the state vector of the momentum given by

\[
|\psi_{\text{mom}}\rangle = \sin \alpha |p_1^-, p_2^-, p_3^-\rangle + \cos \alpha |p_1^+, p_2^+, p_3^+\rangle,
\]

while the spin part is given by one of the following states

\[
|\psi_{\text{spin}}\rangle = \begin{cases} 
|\psi_g\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \\
|\psi_g'\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle), \\
|\psi_w\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle), \\
|\psi_w'\rangle = \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle),
\end{cases}
\]

The state vector \(|\psi_{g,g'}\rangle\) and \(|\psi_{w,w'}\rangle\) are known by Greenberge-Horn-Zeilinger (GHZ) and W-states respectively. The computational basis ”0 and ”1” represent spins polarized up and down along the z-axis. The action of an arbitrary Lorentz transformation \(\Lambda\) on the initial state \(|\psi_{\text{system}}\rangle\) is given by \[13\] \[17\]

\[
\Lambda |\psi_{\text{system}}\rangle = \sum_{i} D(W(\Lambda, p_i)) |\psi_{\text{mom}}\rangle |\psi_{\text{spin}}\rangle
\]

where, \(D(W(\Lambda, p_i))\) represents the Wigner rotation operator is defined by

\[
D(W(\Lambda, p_i)) = \cos \frac{\Omega_i}{2} + i \vec{\sigma} \cdot \vec{n} \sin \frac{\Omega_i}{2}
\]

where \(\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)\), and \(\Omega_i, i = 1, 2, 3\) are the Wigner angles. The operator \(W(\lambda, p) = L^{-1}(\Lambda p) \Lambda L(p)\) is the Winger’s little group element, \(L(p)\) is the standard boost that transform a particle of mass \(m\) from the rest to four momenta \(\vec{p}\) \[13\] \[15\]. If the momenta are chosen
such that $\vec{p}_1 = \vec{p}_2 = \vec{p}_3 = p\hat{e}_z$, i.e., the momenta are polarized in the $z$-axis, then in the computational basis "0" and "1", the unitary operator takes the following form,

$$D(W(A, p_i)) = C_i |0\rangle \langle 0| + |1\rangle \langle 1| + S_i (|1\rangle \langle 0| - |0\rangle \langle 1|)$$

(6)

where $C_i = \cos \frac{\Omega_i}{2}, S_i = \sin \frac{\Omega_i}{2}, i = 1, 2, 3$. Using the initial state $|\psi_g\rangle$, the Lorentz transformation (6) and tracing out the momentum degree of freedom, one gets the evolution of the spin part.

With the source supplies the user with a GHZ state, of the type $|\psi_g\rangle$ as defined in (6). It has been shown that this class of entangled states turns into separable states if one of its particle is traced out [20]. Under the effect of the Lorentz transformation (6) the initial state $|\psi_g\rangle$ is transformed into,

$$|\psi_{gf}\rangle = A_1 |000\rangle + A_2 |001\rangle + A_3 |010\rangle + A_4 |011\rangle + A_5 |100\rangle + A_6 |101\rangle + A_7 |110\rangle + A_8 |111\rangle,$$

(7)

where

$$A_1 = \frac{1}{\sqrt{2}}(C_1 C_2 C_3 - S_1 S_2 S_3), \quad A_2 = \frac{1}{\sqrt{2}}(C_1 C_2 S_3 + S_1 S_2 C_3),$$

$$A_3 = \frac{1}{\sqrt{2}}(C_1 S_2 C_3 + S_1 C_2 S_3), \quad A_4 = \frac{1}{\sqrt{2}}(C_1 S_2 S_3 - S_1 C_2 C_3),$$

$$A_5 = \frac{1}{\sqrt{2}}(S_1 C_2 S_3 + C_1 S_2 C_3), \quad A_6 = \frac{1}{\sqrt{2}}(S_1 C_2 S_3 - C_1 S_2 C_3),$$

$$A_7 = \frac{1}{\sqrt{2}}(S_1 S_2 C_3 - C_1 C_2 S_3), \quad A_8 = \frac{1}{\sqrt{2}}(C_1 C_2 C_3 + S_1 S_2 S_3).$$

(8)

On the other hand, if we assume that the source supplies the three users with W-state, then under the effect of Lorentz transformation, the initial $|\psi_w\rangle$ state turns into the state $|\psi_{wf}\rangle$,

$$|\psi_{wf}\rangle = B_1 |000\rangle + B_2 |001\rangle + B_3 |010\rangle + B_4 |011\rangle + B_5 |100\rangle + B_6 |101\rangle + B_7 |110\rangle + B_8 |111\rangle$$

(9)

where

$$B_1 = \frac{1}{\sqrt{3}}(S_1 C_2 S_3 + S_1 S_2 C_3 + C_1 S_2 S_3), \quad B_2 = \frac{1}{\sqrt{3}}(S_1 S_2 S_3 - S_1 C_2 C_3 - S_1 C_2 C_3),$$

$$B_3 = \frac{1}{\sqrt{3}}(S_1 C_2 S_3 - S_1 C_2 C_3 - S_1 C_2 S_3), \quad B_4 = \frac{1}{\sqrt{3}}(C_1 C_2 C_3 - S_1 S_2 C_3 - S_1 C_2 S_3),$$

$$B_5 = \frac{1}{\sqrt{3}}(S_1 S_2 C_3 - C_1 S_2 C_3 - C_1 C_2 S_3), \quad B_6 = \frac{1}{\sqrt{3}}(C_1 C_2 C_3 - S_1 S_2 C_3 - C_1 S_2 C_3),$$

$$B_7 = \frac{1}{\sqrt{3}}(C_1 C_2 C_3 - C_1 S_2 C_3 - C_1 S_2 C_3), \quad B_8 = \frac{1}{\sqrt{3}}(S_1 C_2 C_3 + C_1 S_2 C_3 + C_1 C_2 S_3).$$

(10)

On the other hand, if we consider that the spin part is given by $|\psi_{gf}\rangle$ or $|\psi_{wf}\rangle$, one gets a similar expression for the final states of $|\psi_{gf}\rangle$ and $|\psi_{wf}\rangle$ which are given by Eq.(7) and Eq.(9) respectively.
In the following sections, we investigate the effect of Lorentz transformation on some properties related to the above model within the context of quantum information and computation. Specifically, we examine the three quantities (i) fidelity, which measures the closeness of the initial and final states, (ii) the channel capacity, which measures how much information can be sent by using these final state, and the (iii) entanglement, which quantify the degree of correlation between the subsystems of these states.

### 3 Robustness of the transformed states

#### 3.1 Fidelity

Now, it is important to shed the light on the robustness of the these multi-qubit states against the Lorentz transformation. One of the important properties of the GHZ state is investigating the behavior of its fidelity. The closeness of the initial GHZ state $\rho_g = |\psi_g\rangle\langle\psi_g|$ to the final state $\rho_g f = |\psi_g f\rangle\langle\psi_g f|$ is defined by

$$F_{g}^{\pm} = \frac{1}{\sqrt{2}}(|A_1|^2 \pm A_1 A_8^* \pm A_8 A_1^* + |A_8|^2),$$

where $F_{g}^{+}$ and $F_{g}^{-}$ represent the fidelity of $|\psi_g f\rangle$ with respect to $|\psi_g\rangle|\psi_g'^{\prime}\rangle$ respectively.

The behavior of the fidelity $F_{g}^{\pm}$, is described in Fig. 1a, where it is assumed that the Wigner angles $\Omega_1 = \Omega_2 = \Omega'$. It is clear that, at $\Omega' = \Omega_3 = 0$ the fidelity $F_{g}^{+} = 1$ (maximum). However as the Lorentz transformation is acted on, the fidelity decreases and completely vanishes at $\Omega' = 0$ and $\Omega_3 = \pi$. The vanishing of the fidelity is also seen at $\Omega' = \pi$, $\Omega_3 = 0$ and at $\Omega' = \Omega_3 = \pi$. On the other hand, the fidelity $F_{g}^{\pm}$ of the GHZ state is maximum i.e., $F_{g}^{+} = 1$ when $\Omega' = 2\pi$ and $\Omega_3 = 0$ and at $\Omega' = \Omega_3 = 2\pi$.

If we consider the GHZ state given by $|\psi_{g}^{-}\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$, then the fidelity $F_{g}^{-} = tr\{|\rho_g \rho_g^{-}\}$, The behavior of this fidelity is displayed in Fig.(1b). It is clear that, $F_{g}^{-} = 0$ is at $\Omega_1 = \Omega_3 = 0$ or $2\pi$. The maximum value of $F_{g}^{-}$ is reached at $\Omega' = \Omega_3 = \pi$.

Fig. 1c, displays the behavior of the fidelity of the GHZ state when exposed to Lorentz transformation, where we assume that $\Omega_1 = \Omega_2 = \Omega' \in [0,2\pi]$. It is clear that at $\Omega' = 0$, the fidelity $F_{g}^{-}$ is maximum. As $\Omega'$ increases the fidelity decreases to reach its minimum value at $\Omega' = \pi$. However as $\Omega'$ increases the fidelity increases gradually to reach its maximum value at $\Omega' = 2\pi$.

For W-state the behavior of the fidelity is described in Fig.2, where we assume that, the particles are transformed with the same Wigner angles for GHZ state. It is clear that, at $\Omega' = \Omega_3 = 0$ i.e., before switching on the Lorentz transformation, the fidelity $F_{w} = 1$ (maximum). However as $\Omega'$ or $\Omega_3$ increase the fidelity decreases smoothly to vanish completely for the first time at $\Omega' \simeq \frac{3\pi}{8}$. The fidelity, $F_{w}$ re-increases again to reach its upper bound at $\Omega' = \frac{3\pi}{8}$. On the other hand, as $\Omega_3$ increases the fidelity $F_{w}$ vanishes once for $\Omega_3 \in [0,2\pi]$. It is clear
Figure 1: The fidelity of the GHZ states where $\Omega_1 = \Omega_2 = \Omega_3 = \Omega$. The solid and dot curves for $F_g^+$ and $F_g^-$, respectively.

that the fidelity of W-state reaches its maximum values either at $\Omega' = \Omega_3 = 0, 2\pi$ or $\Omega' = 0$ and $\Omega_3 = 2\pi$ or $\Omega' = \pi$ and $\Omega_3 = 0$.

It is well know that the W-state has another form defined by $|\psi'_w\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle)$. In Fig.(2b), we plot the fidelity $F_{w'}$ of the state (8) with respect to $|\psi'_w\rangle$. The general behavior of the fidelity $F_{w'}$ shows that as $F_w = 1$ (as shown in Fig.(2a)), the fidelity $F_{w'} = 0$ (minimum). This behavior is depicted for all values of $\Omega'$ and $\Omega_3$.

Fig.2c, summarizes clearly these results which depicted in Figs. (2a&2b). It is clear that at $\Omega' = 0$ the fidelity $F_w = 1$(maximum), $F_{w'} = 0$ (minimum). However, as $\Omega'$ increases the fidelity $F_w$ completely vanishes twice at $\Omega' = \frac{3\pi}{8}$ and $\frac{2\pi}{3}$, while $F_{w'}$ reaches its maximum bounds ($F_{w'} = 1$). This behavior completely changes i.e., at the values of $\Omega'$ which maximize $F_{w}$, the fidelity $F_{w'}$ is minimized.

From Figs.(1) and (2), one concludes that, although the Lorentz transformation causes a decay of the fidelity, it generates an equivalent state which has a maximum fidelity when the initial one has a minimum fidelity. Therefore, the Lorentz transformation has no effect on the efficiency of these classes of the tripartite states within the context of quantum information.
3.2 The average capacity

In this subsection, we investigate the ability of using the tripartite state under Lorentz transformation to send information. For this aim we quantify the average capacity of the final transformed states. For the state $\rho_{abc}$, we have three possible channels between each two users: $\rho_{ab}, \rho_{ac}$ and $\rho_{bc}$, where $a, b$ and $c$ represent the three users. This means that each two users share a two-qubit state. The channel capacity of a two qubit state is defined as,

$$C_p^{(k)}(\rho_{ij}) = \log_i D + S(\rho_i^{(k)}) - S(\rho_{ij}^{(k)}),$$

where $\rho_i = tr_j{\rho_{ij}}$, $D = 2$ is the dimension of $\rho_i$, $i = a, b, c, ij = ab, ac, bc$ and $S(.)$ is the von Numann entropy. The superscript $k$ stands for the GHZ or W- state. For tripartite state the average capacity between the three users can be considered as a measure of the average capacity of the state $\rho_{abc}$ is defined as,

$$C_p^{(k)}(\rho_{abc}) = \frac{1}{3} \left( C_p^{(k)}(\rho_{ab}) + C_p^{(k)}(\rho_{bc}) + C_p^{(k)}(\rho_{ac}) \right).$$

The behavior of the average capacity $C_p^{(g)}$ of the GHZ state under Lorentz transformation is described in Fig.(3a) for different values of Wigner’s angles. It is assumed that, the first and second particles are transformed with the same Wigner angle i.e., $\Omega_1 = \Omega_2 = \Omega' \in [0, 2\pi]$, \(\forall\Omega_3\).
while the third particle is transformed with different Wigner’s angle, $\Omega_3$. The initial average capacity $\bar{C}_P^g(0,0,\Omega_3)$ of the GHZ state depends on the setting value of $\Omega_3$. It is clear that, for small values of $\Omega_3$, the initial capacity is large, while it is small for smaller values of $\Omega_3$. On the other hand, as $\Omega'$ increases, the average channel capacity oscillates between its maximum and minimum bounds. The maximum bounds depend on the Wigner’s angle of the third particle, where for larger values of $\Omega_3$ the upper bounds of the average capacity decrease. However, the minimum bounds are reached at $\Omega' = \pi$. If we set $\Omega_3 = 0$, i.e., the third particle is considered as an observer, the average capacity increases to reach its maximum bound, i.e., $\bar{C}_P^g = 2$ for the first time at $\Omega' = \frac{5\pi}{12}$.

For W-state the average channel capacity $\bar{C}_P^w$ is shown in Fig.(3b), where the same values of the Wigner’s angles are used. In this case, the behavior is completely different. In the interval $[0, \pi]$, the average capacity decreases as the Wigner’s angle of the third particle $\Omega_3$ decreases. However the lower bounds of $\bar{C}_P^w$ decrease and consequently the average capacity increases as $\Omega_3$ increases. This shows that as the difference between the Wigner’s angle is small the average channel capacity is large. However, in the interval $[\pi, 2\pi]$, the situation is different i.e., the channel capacity increases for smaller values of $\Omega_3$.

From Fig.(3), one concludes that the average capacity of the GHZ state may increase if we allow for one of these particles to play the role of observer or minimize the value of Wigner’s angle. In this case the GHZ state can be used to send a large amount of information within the context of relativistic quantum information.
4 Entanglement

In this section the effect of Lorentz transformation on the degree of entanglement of GHZ and W-states is investigated. For this task we use a measure called tangle \[18, 19, 20\]. This measure is used to quantify the three way entanglement. For a three qubit state \(|\psi\rangle_{123}\), the three-tangle of this state is given by,

\[
T_{123} = C_{1(23)}^2 - C_{12}^2 - C_{13}^2,
\]

where

\[
C_{1(23)} = 2\sqrt{\det\{\rho_1\}}, \quad \rho_1 = tr_{23}\{|\psi\rangle_{123}\langle\psi|\}
\]

and \(C_{ij}, ij = 12, 13\) is the concurrence of the two qubit states \(\rho_{12} = tr_3\{|\psi\rangle_{123}\langle\psi|\}\) and \(\rho_{13} = tr_2\{|\psi\rangle_{123}\langle\psi|\}\), respectively. The concurrence of a two qubit state \(\rho_{ij}\) is defined as

\[
C(\rho_{ij}) = \max\{0, \sqrt{\lambda_1 - \sqrt{\lambda_2 - \sqrt{\lambda_3 - \sqrt{\lambda_4}}}\}
\]

where \(\lambda_k, k = 1..4\) are the eigenvalues of the matrix \(\rho_{ij}(\sigma_{y}^{(i)} \otimes \sigma_{y}^{(j)})\rho_{ij}^*(\sigma_{y}^{(i)} \otimes \sigma_{y}^{(j)})\) \[7\].

The effect of the Lorentz transformation on the degree of entanglement of the GHZ state is displayed in Fig. 4a, where it is assumed that \(\Omega_1 = \Omega_2 = \Omega' \in [0, 2\pi]\) and different values of \(\Omega_3\) are considered. It is clear that, before switching on the Lorentz transformation the tangle \(T_G = 1\). However, as the Wigner angles increase, entanglement reach its lower bound at \(\Omega' = \pi/3\). For larger values of \(\Omega'\), the tangle \(T_G\) increases again to reach its maximum value at \(\Omega' = 4\pi/3\). As \(\Omega'\) increases further the entanglement decreases gradually to reach its lower bound at \(\Omega' \approx 5\pi/3\). Finally the tangle increases to its maximum value i.e., \(T_G = 1\) at \(\Omega' = 2\pi\). For larger values of \(\Omega_3\), the tangle decreases gradually and the minimum bounds decrease (tangle increase) as the Wigner’s angle \(\Omega_3\) increases.

The behavior of the tangle for a system initially prepared in W-state evolves under the effect of Lorentz transformation is displayed in Fig.(4b). Jung et.al \[21\]have shown that the tangle of W-state, \(T_w \approx 0.55\) . The general behavior is similar to that depicted for \(T_G\) as

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The entanglement of(a) GHZ- state (b) W-state . The solid, dot, dash-dot and dash curves for and \(\Omega_3 = 0, \pi/3, \pi/4\) and \(\pi/6\) respectively.}
\end{figure}
shown in Fig. (4a). However, for $\Omega' \in [0, \pi]$, the tangle $T_w$ increases as the Wigner’s angle of the third particle $\Omega_3$ decreases, the lower bounds are larger than that shown for $T_G$. This behavior changes in the interval $[\pi, 2\pi]$, where for larger values of $\Omega_3$, the upper bounds of $T_w$ are larger.

From Fig. 4, one concludes that, as the difference between the Wigner’s angle, $\Delta = \Omega' - \Omega_3$ decreases, the tangle of the GHZ state increases. Due to the structure of W-state, the tangle increases as the difference between the Wigner’s angles increases in the interval $[0, \pi]$ and decreases as the difference $\Delta$ increases in the interval $[\pi, 2\pi]$. The tangle behavior shows that the W-state is more robust than the GHZ state.

5 Conclusion

The effect of the Lorentz transformation on the fidelity, capacity and entanglement of tripartite systems of GHZ and W-states are investigated, where the final state vectors of GHZ and W-states are obtained analytical as functions of Wigner angles. The behavior of these phenomena under the Lorentz transformation, is considered as a measure of robustness of these states to Lorentz transformation. It is shown that, the values of these quantities oscillate between their lower and upper bounds depending on the values of the Wigner's angles.

The behavior of the fidelities of GHZ and W-states shows that, these states turn into an equivalent form, where as soon as the fidelities of the initial state decreases, the fidelity of the equivalent state rebirths. However, when the fidelity of the initial state vanishes completely, the fidelity of the equivalent state becomes maximum. Therefore, Lorentz transformation keeps the entangled properties of the transformed states.

The effect of the Lorentz transformation on the channel capacities shows different behaviors for GHZ and W-states, where the average capacity for GHZ increases as the Wigner angles increases, while it decreases for W-state. For GHZ state, if one particle is considered as an observer, the upper bounds increase as the difference between the Wigners’s angles of this particle and the other two particles increases. However, for W-state, the lower bounds increase (average capacity increases) as the difference between Wigner’s angle increases, i.e., the observer is transformed with a small Wigner’s angle.

The amount of survival entanglement is quantified by means of the tangle as a measure of entanglement between three qubits. Our results show that the tangle decreases gradually as the Wigner’ angles increase. For GHZ state, the lower bounds decreases, i.e., the entanglement increase when the difference between the observer’s Wigner angles and the Wigner angles of the other two particles decreases, while for W-states, the entanglement increases as this difference increases. However, the lower bounds of entanglement for W-state is much larger than that depicted for GHZ state. This shows that the W-state is more robust than GHZ state under the effect of Lorentz transformation.
In conclusion: the entanglement of the tripartite states can be improved or almost kept invariant, if all the particles are transformed with an equal Wigner’s angles. Since for any Wigner’s angles, the fidelity doesn’t vanish, then these states are transformed to an equivalent form at some specific values of Wigner’s angle and consequently can be used to perform quantum information tasks, as teleportation and quantum coding with high efficiency.

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