

A HOMOMORPHISM BETWEEN BOTT-SAMELSON BIMODULES

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Abstract. In the previous paper, we defined a new category which categorifies the Hecke algebra. This is a generalization of the theory of Soergel bimodules. To prove theorems, the existences of certain homomorphisms between Bott-Samelson bimodules are assumed. In this paper, we prove this assumption. We only assume the vanishing of certain two-colored quantum binomial coefficients.

1. Introduction

In recent development of representation theory of algebraic reductive groups, the Hecke category plays central role. Here, the Hecke category means a categorification of the Hecke algebra of Coxeter groups. One can find the importance of the Hecke category in representation theory in Williamson’s survey [Wil18].

There are several incarnations of the Hecke category. They can be roughly divided into two types: geometric ones and combinatorial ones. The geometric Hecke category which appeared in representation theory first is the category of semisimple perverse sheaves on the flag variety. This category is the Hecke category with a field of characteristic zero. Juteau-Mauter-Williamson [JMW14] introduced the notion of parity sheaf. The category of parity sheaves on the flag variety is a geometric incarnation of the Hecke category with any field. When the characteristic of the ground field is zero, parity sheaves are the same as semisimple perverse sheaves.

Soergel [Soe07] introduced a category which is now called the category of Soergel bimodules. Similar to the situation of the geometric ones, if the characteristic of the ground field is zero, this category is the Hecke category. Soergel’s category is equivalent to the category of semisimple perverse sheaves on the flag variety over a field of characteristic zero. This fact is used to prove the Koszul duality of the category $\mathcal{O}$ [BGS96].

Soergel’s category does not behave well over a field of positive characteristic in general. As a generalization of Soergel’s category, the author introduced a new combinatorial category and proved that this category is the Hecke category in more general situation than Soergel’s theory [Abe19]. There is also another combinatorial category defined by Elias-Williamson [EW16] which is defined earlier than [Abe19]. The category is called the diagrammatic Hecke category and it is proved that the category is the Hecke category in general situation. We remark that these categories are equivalent to each others when they behave well [RW18, Abe19, Abe20].

It is proved that these theories works well very general, including most cases over a field of positive characteristic. However, we still need some assumptions. The situation is subtle. In [Abe19], we need one non-trivial assumption which we recall later. One problem is that this assumption is not easy to check. In [Abe19], a sufficient condition for this assumption which we can check easier is given. However the author thought that the assumption holds in more general. The aim of this paper is to prove this assumption under a mild condition. (The situation is also subtle for the diagrammatic Hecke category. See [EW20, 5.1]. We do not discuss about it in this paper.)

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1.1. Soergel bimodules. We recall the category introduced in \cite{Abe19} and the assumption. Let \((W,S)\) be a Coxeter system such that \(\#S < \infty\) and \(\mathbb{K}\) a commutative integral domain. We fix a realization \cite[Definition 3.1]{EW16} \((V,\{\alpha_s\}_{s \in S},\{\alpha_s^\vee\}_{s \in S})\) of \((W,S)\) over \(\mathbb{K}\). Namely \(V\) is a free \(\mathbb{K}\)-module of finite rank with an action of \(W\), \(\alpha_s \in V\), \(\alpha_s^\vee \in \text{Hom}_\mathbb{K}(V,\mathbb{K})\) such that

\begin{enumerate}
  \item \(s(v) = v - \langle \alpha_s^\vee, v \rangle \alpha_s\) for any \(s \in S\) and \(v \in V\).
  \item \(\langle \alpha_s^\vee, \alpha_s \rangle = 2\).
  \item Let \(s,t \in S\) \((s \neq t)\) and \(m_{s,t}\) the order of \(st\). If \(m_{s,t} < \infty\) then the two-colored quantum numbers \([m_{s,t}]_x, [m_{s,t}]_y\) attached to \(\{s,t\}\) are both zero. (See \(3.1\) for the definition of these numbers.)
\end{enumerate}

We also assume the Demazure surjectivity, namely we assume that \(\alpha_s^\vee : \text{Hom}_\mathbb{K}(V,\mathbb{K}) \rightarrow \mathbb{K}\) and \(\alpha_s^\vee : V \rightarrow \mathbb{K}\) are both surjective for any \(s \in S\).

We define the category \(\mathcal{C}\) as follows. Let \(R = S(V)\) be the symmetric algebra and \(Q\) the field of fractions of \(R\). An object of \(\mathcal{C}\) is \((M, (M_Q^x)_{x \in W})\) such that \(M\) is a graded \(R\)-bimodule, \(M_Q^x\) is a \(Q\)-bimodule such that \(mp = x(p)m\) for any \(m \in M_Q^x, p \in Q\) and \(M \otimes_R Q = \bigoplus_{x \in W} M_Q^x\). We also assume that \(M\) is flat as a right \(R\)-module. A morphism \(\varphi : (M, (M_Q^x)_{x \in W}) \rightarrow (N, (N_Q^x)_{x \in W})\) is an \(R\)-bimodule homomorphism \(\varphi : M \rightarrow N\) of degree zero such that \((\varphi \otimes \text{id}_Q)(M_Q^x) \subset N_Q^x\). We often write \(M\) for \((M, (M_Q^x))\).

For \(M, N \in \mathcal{C}\), we define \(M \otimes N \in \mathcal{C}\) as follows. As an \(R\)-bimodule, we have \(M \otimes N = M \otimes_R N\) and \((M \otimes N)_Q = \bigoplus_{x,y \in W} M_Q^x \otimes_Q N_Q^y\).

For each \(s \in S\), we have an object denoted by \(B_s\). As a graded \(R\)-bimodule, \(B_s = R \otimes_R R(1)\) where \((1)\) is the grading shift and \(R^e = \{f \in R \mid s(f) = f\}\). Then \(B_s\) has a unique lift in \(\mathcal{C}\) such that \((B_s)_Q^n = 0\) unless \(x = e, s\). An object of a form

\[B_{s_1} \otimes B_{s_2} \otimes \cdots \otimes B_{s_l}(n)\]

for \(s_1, \ldots, s_l \in S\) and \(n \in \mathbb{Z}\) is called a Bott-Samelson bimodule. Let \(\text{BS}\) denote the category of Bott-Samelson bimodules.

In \cite{Abe19}, we proved that \(\text{BS}\) gives a categorification of the Hecke algebra assuming the following. We refer it as \cite[Assumption 3.2]{Abe19}.

Let \(s, t \in S\), \(s \neq t\) such that \(m_{s,t}\) is finite. Then there exists a morphism

\[\begin{array}{ccc}
B_s 
\otimes 
B_t 
\otimes 
\cdots 
\otimes 
B_{s_l},
\end{array}
\]

\[\begin{array}{ccc}
m_{s,t},
\end{array}
\]

\[\begin{array}{ccc}
\rightarrow,
\end{array}
\]

\[\begin{array}{ccc}
B_t 
\otimes 
B_s 
\otimes 
\cdots 
\otimes 
B_{s_l},
\end{array}
\]

which sends \((1 \otimes 1)(1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1)\) to \((1 \otimes 1)(1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1)\).

We introduce the following assumption.

\textbf{Assumption 1.1.} For any \(s, t \in S\) such that \(m_{s,t} < \infty\), the two-colored quantum binomial coefficients \([m_{s,t}]_x\) and \([m_{s,t}]_y\) are both zero for any \(k = 1, \ldots, m_{s,t} - 1\).

For the definition of two-colored quantum binomial coefficients, see \cite[3.3]{Abe19}. This assumption is related to the existence of Jones-Wenzl projectors. (See Proposition \cite[6.23]{EW20}.) The main theorem of this paper is the following.

\textbf{Theorem 1.2} (Theorem \cite[3.9]{Abe19}). Under Assumption 1.1 \cite[Assumption 3.2]{Abe19} holds.

Note that Assumption 1.1 is a very mild condition. For example, if a realization comes from a root system then it is always satisfied (Proposition \cite[6.7]{Abe19}).

1.2. Diagrammatic category. Let \(\mathcal{D}\) be the diagrammatic Hecke category defined in \cite{EW16}. We assume that the category \(\mathcal{D}\) is "well-defined" \cite[5.1]{EW20}. In \cite{EW16}, under some assumptions \cite[5.3]{EW20}, a functor \(\mathcal{F}\) from \(\mathcal{D}\) to \(\text{BS}\) is constructed. The construction of \(\mathcal{F}\) is deeply related to \cite[Assumption 3.2]{Abe19} as we explain here.
The morphisms in the category \( D \) are defined by generators and relations. So to define \( F \), we have to define the images of generators. Except the generators called \( 2m_{s,t} \)-valent vertices \((s,t) \in S\), the images of generators are given easily. For \( 2m_{s,t} \)-valent vertices, the images should be morphisms in \([Abe19, Assumption 3.2]\). Hence, to prove \([Abe19, Assumption 3.2]\) is almost equivalent to the construction of \( F \). Therefore as a consequence of our main theorem, we can prove the following.

**Theorem 1.3.** Under Assumption \([1.1]\) the category \( D \) is equivalent to \( BS \).

1.3. **Localized calculus.** In the proof, we use localized calculus. Ideas of localized calculus are found in \([EW16, Abe19]\) and more systematic treatment recently appeared in \([EW20]\).

Let \( C_Q \) be the category of \((P^x)_{x \in W}\) where \( P^x \) is a \( Q \)-bimodule such that \( mp = x(p)m \) for \( p \in Q \) and \( m \in P^x \). A morphism \((P^x)_{x \in W} \to (P^x)_{x \in W}\) is \((\varphi_x)_{x \in W}\) where \( \varphi_x: P^x_1 \to P^x_2 \) is a \( Q \)-bimodule homomorphism for any \( x \in W \). Then for \( M \in C \), \((M_Q^x)_{x \in W} \in C_Q \). We denote this object by \( M_Q \). For \( M, N \in C \) and a morphism \( \varphi: M \to N \), we have a morphism \( \varphi_Q: M_Q \to N_Q \). Conversely, assume that \( \varphi_Q: M_Q \to N_Q \) is given and if \( \varphi_Q \) sends \( M \subset M \otimes_R Q = \bigoplus_{x \in W} M^x_Q \) to \( N \), then the restriction of \( \varphi_Q \) to \( M \) gives a morphism \( M \to N \) in \( C \).

Let \( M, N \) be two Bott-Samelson bimodules in \([Abe19, Assumption 3.2]\). A candidate of \( \varphi_Q: M_Q \to N_Q \) is given in \([EW20]\). The hardest part is to prove that \( \varphi_Q \) sends \( M \) to \( N \).

We check that \( \varphi_Q \) gives a desired homomorphism by calculations. One of the things which we need to prove is the following. Let \( s, t \in S \) such that \( m_{s,t} < \infty \). For simplicity, assume that \( V \) is balanced, namely \([m_{s,t} - 1]_x = [m_{s,t} - 1]_y = 1 \). Let \( s_1 \cdots s_{m_{s,t}} \) be a reduced expression of the longest element in the group \( \langle s, t \rangle \) generated by \( \{s, t\} \). Then for any \( g \in \langle s, t \rangle \), we have

\[
\sum_{e = (e_i) \in \{0,1\}^{m_{s,t}}} \prod_{i=1}^{m_{s,t}} s_1^{e_1} \cdots s_{i-1}^{e_{i-1}} \left( \frac{1}{\alpha_{s_i}} \right) = \frac{1}{\prod_{i=1}^{m_{s,t}} s_1 \cdots s_{i-1} (\alpha_{s_i})}.
\]  

(If \( V \) comes from a root system, then this formula can be proved by applying the localization formula to the Bott-Samelson resolution of the flag variety. The author learned this from Syu Kato.)

In Section 2, we calculate the left hand side of \([1.1]\). Moreover, we give an explicit formula of the left hand side for any sequence \((s_1, s_2, \ldots)\) of \( \{s, t\} \). The hardest part of this calculation is to find a correct result. Once we find the correct formulation, the proof is done by induction.

For a general element \( m \in M \), we first give a formula to express \( \varphi_Q(m) \) using the left hand side of \([1.1]\) (with any \( s_1, s_2, \ldots \)). We also have an algorithm to check \( \varphi_Q(m) \in N \) (Lemma 3.8). In Section 3, using this algorithm and an explicit formula obtained in Section 2, we prove the main theorem.

1.4. **On Assumption \([1.1]\).** In \([Abe19]\), a sufficient condition for \([Abe19, Assumption 3.2]\) was given. In \([EW16]\), a sufficient condition for the existence of \( F \) was given. Both conditions are stronger than Assumption \([1.1]\). It was expected that these theorems are proved under the weaker condition \([EW20, Remark 5.6]\) but concrete conditions were not known.

In this paper, we prove these theorems under Assumption \([1.1]\). Moreover, we prove that the theorems are almost equivalent to Assumption \([1.1]\). More precisely, we prove the following. Let \( \varphi_Q: M_Q \to N_Q \) be the morphism in \( C_Q \) introduced above and \( \psi_Q: N_Q \to \)
we define Two-colored quantum numbers. Let \( Z \) be the polynomial ring with two variables over \( \mathbb{Z} \).

**Definition 2.1** (two-colored quantum numbers, \cite[Definition 3.6]{EW16}). For \( n \in \mathbb{Z}_{\geq 0} \), we define \([n]_X, [n]_Y \in Z[X, Y] \) by

\[
[0]_X = [0]_Y = 0, \quad [1]_X = [1]_Y = 1, \\
[n + 1]_X = X[n]_Y - [n - 1]_X, \\
[n + 1]_Y = Y[n]_Y - [n - 1]_Y.
\]

Note that \([2]_X = X \) and \([2]_Y = Y \). Define \( \sigma : \{X, Y\} \to \{X, Y\} \) by \( \sigma(X) = Y \) and \( \sigma(Y) = X \). Then for \( Z \in \{X, Y\} \), we have

\[ [n + 1]_Z = [2]_Z[n]_{\sigma(Z)} - [n - 1]_Z. \]

We prove some properties of these polynomials which we will use later. Some of them are known well or immediately follow from known results. We give proofs for the sake of completeness.

**Lemma 2.2.** Let \( n \in \mathbb{Z}_{\geq 0} \).

1. If \( n \) is odd, then \([n]_X = [n]_Y \).
2. If \( n \) is even, \([n]_X/X, [n]_Y/Y \in Z[X, Y] \) and \([n]_X/X = [n]_Y/Y \).
3. We have \([n]_Z = [n]_{\sigma(Z)} \) for \( Z \in \{X, Y\} \).
4. We have \([n]_X, [n]_Y \neq 0 \) if \( n > 0 \).

**Proof.** The first two statements follow from the definition using induction. For the third, if \( n \) is odd then it follows from (1). If \( n \) is even then it is obvious. We also have \([n]_X(2, 2) = [n]_Y(2, 2) = n \) which follows easily by induction. Hence \([n]_X, [n]_Y \neq 0 \). \( \square \)

An obvious consequence of (1) which will be used several times in this paper is the following. For \( k_1, \ldots, k_r, l_1, \ldots, l_s \in \mathbb{Z}_{\geq 0} \) such that \( #(2\mathbb{Z} \cap \{k_1, \ldots, k_r\}) = #(2\mathbb{Z} \cap \{l_1, \ldots, l_s\}) \), then \((k_1)[X] \cdots [k_r][X]/([l_1][X] \cdots [l_s][X]) = ([k_1][Y] \cdots [k_r][Y])/([l_1][Y] \cdots [l_s][Y]).

**Lemma 2.3.** Let \( m, n \in \mathbb{Z}_{\geq 0} \) and \( Z \in \{X, Y\} \). Then we have

\[ [m + n + 1]_{\sigma(Z)} = [m + 1]_Z[n + 1]_{\sigma(Z)} - [m]_{\sigma(Z)}[n]_Z. \]
Proof. We prove by induction on $n$. The cases of $n = 0$ and $n = 1$ follow from the definitions. Assume the lemma holds for $n - 1, n - 2$. Then

$$[m + n + 1]_{\sigma^n(Z)} = [2]_{\sigma^n(Z)}[m + n]_{\sigma^{n+1}(Z)} - [m + n - 1]_{\sigma^n(Z)}$$

$$= [2]_{\sigma^n(Z)}([m + 1]_Z[n]_{\sigma(Z)} - [m]_{\sigma(Z)}[n - 1]_Z)$$

$$- ([m + 1]_Z[n - 1]_{\sigma(Z)} - [m]_{\sigma(Z)}[n - 2]_Z)$$

$$= [m + 1]_Z([2]_{\sigma^n(Z)}[n]_{\sigma(Z)} - [n - 1]_{\sigma(Z)})$$

$$- [m]_{\sigma(Z)}([2]_{\sigma^n(Z)}[n - 1]_Z - [n - 2]_Z).$$

By Lemma 2.2 (4), we have $[n]_{\sigma(Z)} = [n]_{\sigma^{n+1}(Z)}$ and $[n - 1]_{\sigma(Z)} = [n - 1]_{\sigma^n(Z)}$. Hence $[2]_{\sigma^n(Z)}[n]_{\sigma(Z)} - [n - 1]_Z = [2]_{\sigma^n(Z)}[n]_{\sigma^{n+1}(Z)} - [n - 1]_{\sigma^n(Z)} = [n + 1]_{\sigma^n(Z)} = [n + 1]_{\sigma(Z)}$. In the last we used Lemma 2.2 (4) again. Similarly we have $[2]_{\sigma^n(Z)}[n - 1]_Z - [n - 2]_Z = [2]_{\sigma^n(Z)}[n - 1]_{\sigma^{n+1}(Z)} - [n - 2]_{\sigma^n(Z)} = [n]_{\sigma^n(Z)} = [n]_Z.$

Lemma 2.4. Let $m, n \in \mathbb{Z}_{\geq 0}$ and $Z \in \{X, Y\}$. Then we have

$$[m]_{\sigma^n(Z)} = [m + n]_Z[n + 1]_{\sigma(Z)} - [m + n + 1]_{\sigma(Z)}[n]_Z.$$ 

Proof. By the previous lemma, we have $[m + n + 1]_{\sigma(Z)}[n]_Z = [m + 2n]_{\sigma^n(Z)} + [m + n]_Z[n - 1]_{\sigma(Z)}$. By swapping $m$ and $n$ with $n$, we have $[m + n]_Z[n + 1]_{\sigma(Z)} = [m + 2n]_{\sigma^{n+1}(Z)} + [m + n - 1]_{\sigma(Z)}[n]_Z$. By Lemma 2.2 (4), we have $[m + n]_Z[n + 1]_{\sigma(Z)} = [m + 2n]_{\sigma^n(Z)}$. Hence $[m + n]_Z[n + 1]_{\sigma(Z)} = [m + n + 1]_{\sigma(Z)}[n]_Z$. Therefore, by induction on $n$, $[m + n]_Z[n + 1]_{\sigma(Z)} = [m + n + 1]_{\sigma(Z)}[n]_Z = [m]_{\sigma^n(Z)}[1]_{\sigma^{n+1}(Z)} - [m + 1]_{\sigma^{n+1}(Z)}[0]_{\sigma^n(Z)} = [m]_{\sigma^n(Z)}$.

Lemma 2.5. For $m, n \in \mathbb{Z}_{\geq 0}$ and $Z \in \{X, Y\}$, we have

$$[m + n + 1]_{\sigma^n(Z)}[m + n]_Z - [m + 1]_Z[m]_{\sigma^n(Z)} = [n]_Z[2m + n + 1]_{\sigma^{n+1}(Z)}$$

$$= [n]_{\sigma^n(Z)}[2m + n + 1]_{\sigma^{n+1}(Z)}.$$

$$[m + n + 1]_{\sigma^{n+1}(Z)}[m + n + 1]_Z - [m]_{\sigma^{n+1}(Z)}[n]_Z = [n + 1]_Z[2m + n + 1]_{\sigma^n(Z)}$$

Proof. Applying Lemma 2.3 to $[m + n + 1]_{\sigma^n(Z)}$ (resp. Lemma 2.4 to $[m]_{\sigma^n(Z)}$), we have

$$[m + n + 1]_{\sigma^n(Z)}[m + n]_Z - [m + 1]_Z[m]_{\sigma^n(Z)} = ([m + 1]_Z[n + 1]_{\sigma(Z)} - [m]_{\sigma(Z)}[n]_Z)[m + n]_Z$$

$$- [m + 1]_Z([m + n]_Z[n + 1]_{\sigma(Z)} - [m + n + 1]_{\sigma(Z)}[n]_Z)$$

$$= -[m]_{\sigma(Z)}[n]_Z[m + n]_Z + [m + 1]_Z[m + n + 1]_{\sigma(Z)}[n]_Z$$

$$= [n]_Z([m + n + 1]_{\sigma(Z)}[m + n + 1]_Z - [m + n]_Z[m]_{\sigma(Z)}].$$

The first formula of the lemma follows from Lemma 2.3. The second follows from the first and Lemma 2.2 (4). The third formula follows from a similar calculation.

For $m, n \in \mathbb{Z}_{\geq 0}$ such that $n \leq m$ and $Z \in \{X, Y\}$, define the two-colored quantum binomial coefficient $\begin{bmatrix} m \\ n \end{bmatrix}_Z$ [EW20, Definition 6.7] by

$$\begin{bmatrix} m \\ n \end{bmatrix}_Z = \frac{[m]_Z[m - 1]_Z \cdots [m - n + 1]_Z}{[n]_Z[n - 1]_Z \cdots [1]_Z}.$$ 

By the lemma (2) below and induction, we have $\begin{bmatrix} m \\ n \end{bmatrix}_Z \in \mathbb{Z}[X, Y]$.

Lemma 2.6. Let $m, n \in \mathbb{Z}$ such that $1 \leq n \leq m$ and $Z \in \{X, Y\}$.

$$\begin{bmatrix} m \\ n \end{bmatrix}_Z = \begin{bmatrix} m + 1 \\ n \end{bmatrix}_{\sigma^n(Z)}[n + 1]_Z - \begin{bmatrix} m \\ n - 1 \end{bmatrix}_Z[2]_{\sigma^{n+1}(Z)}.$$
Therefore follows from Lemma 2.2. □

Lemma 2.7. 

If 

\[ m + 1 \]
\[ n \]
\[ Z \]
\[ \sigma^n(Z) \]
\[ [m + 1]_{\sigma^n(Z)} \]
\[ [n + 1]_Z \]
\[ [m + 2]_{\sigma^n+1(Z)} \]
\[ [m - n]_{\sigma^n(Z)} \]
\[ [m - n]_{\sigma^n+1(Z)} \]

Proof. (1) By Lemma 2.4, we have 

\[ [m - n + 1]_Z = [m + 1]_{\sigma^n(Z)} [n + 1]_{\sigma^n(Z)} - [m + 2]_{\sigma^n+1(Z)} [n]_{\sigma^n(Z)} \]

By Lemma 2.2, we have 

\[ [n]_{\sigma^n(Z)} = [n]_Z \]
\[ [n + 1]_{\sigma^n+1(Z)} = [n + 1]_Z \]

Therefore it is sufficient to prove 

\[ [m + 1]_{\sigma^n(Z)} [m]_Z \cdots [m - n + 2]_Z \]
\[ [n]_Z \cdots [1]_Z \]
\[ [m + 1]_{\sigma^n(Z)} [m]_{\sigma^n(Z)} \cdots [m - n + 2]_{\sigma^n(Z)} \]
\[ [n]_{\sigma^n(Z)} \cdots [1]_{\sigma^n(Z)} \]

If \( n \) is even then we have nothing to prove. If \( n \) is odd, then 

\[ \#(2\mathbb{Z} \cap \{m, \ldots, m-n+2\}) = \#(2\mathbb{Z} \cap \{n, \ldots, 1\}) \]

Hence it follows from Lemma 2.2.

(2) By Lemma 2.3, we have 

\[ [m+1]_Z = [m-n+1]_{\sigma^n(Z)} [n+1]_{\sigma^n(Z)} - [m-n]_{\sigma^n(Z)} [n]_{\sigma^n(Z)} \]

By Lemma 2.2, we have 

\[ [n + 1]_{\sigma^n+1(Z)} = [n + 1]_Z \]
\[ [n]_{\sigma^n(Z)} = [n]_Z \]

Hence it is sufficient to prove 

\[ [m]_Z \cdots [m - n + 2]_Z [m - n + 1]_{\sigma^n(Z)} \]
\[ [n]_Z \cdots [1]_Z \]
\[ [m]_{\sigma^n(Z)} \]

If \( n \) is even we have nothing to prove. If \( n \) is odd, then we have 

\[ \#(2\mathbb{Z} \cap \{m, m-1, \ldots, m-n+2\}) = \#(2\mathbb{Z} \cap \{n, \ldots, 1\}) \]

Hence we get (2) by Lemma 2.2. □

**Lemma 2.7.** We have 

\[ \frac{[2m + n + 1]_{\sigma^n+1(Z)}}{[m]_{\sigma^n(Z)}} \frac{[2m + n + 1]}{[m - 1]_{\sigma^n(Z)}} = \frac{[2m + n + 1]}{[m]_{\sigma^n(Z)}} \frac{[m + n + 1]}{[m - 1]_{\sigma^n(Z)}} \]

Proof. Replacing \( Z \) with \( \sigma^n(Z) \), the lemma is equivalent to 

\[ \frac{[2m + n + 1]_{\sigma^n+1(Z)} [2m + n + 1] \cdots [m + 2]_Z}{[m]_Z [m - 1]_Z \cdots [1]_Z} = \frac{[2m + n + 1]_{\sigma(Z)} \cdots [m + 2]_{\sigma(Z)}}{[m]_{\sigma(Z)} [m - 1]_{\sigma(Z)} \cdots [1]_{\sigma(Z)}} \]

If \( m+n+1 \) is even, then we have 

\[ \#(2\mathbb{Z} \cap \{2m+n+1, \ldots, m+n+2\}) = \#(2\mathbb{Z} \cap \{m, \ldots, 1\}) \]

Hence the lemma follows from Lemma 2.2. If \( m+n+1 \) is odd, then \( \sigma^{m+n+1}(Z) = \sigma(Z) \) and 

\[ \#(2\mathbb{Z} \cap \{m, \ldots, m+n+2\}) = \#(2\mathbb{Z} \cap \{m, \ldots, 1\}) \]

Hence again the lemma follows from Lemma 2.2. □
2.2. A formula. Let \((W, S)\) be the universal Coxeter system of rank two, namely the group \(W\) is generated by the set of two elements \(S = \{s, t\}\) and defined by relations \(s^2 = t^2 = 1\). The length function is denoted by \(\ell\) and the Bruhat order is denoted by \(\leq\). Let \(V = \mathbb{Z}[X, Y]_\alpha s \oplus \mathbb{Z}[X, Y]_\alpha t\) be the free \(\mathbb{Z}[X, Y]\)-module of rank two with a basis \(\{\alpha s, \alpha t\}\). We define an action of \(W\) on \(V\) by

\[
\sigma(s) = -s, \quad \sigma(t) = t + Xs, \quad \tau(s) = s + Yt, \quad \tau(t) = -t.
\]

Let \(\Phi = \{w(\alpha s), w(\alpha t) \mid w \in W\}\) be the set of roots and the set of positive roots \(\Phi^+\) is defined by \(\Phi^+ = \{w(\alpha s) \mid ws > w\} \cup \{w(\alpha t) \mid wt > w\}\). For each \(\alpha \in \Phi\), we have the reflection \(s_\alpha \in W\). This is defined as \(s_{\alpha s} = s, s_{\alpha t} = t, s_{w(\alpha)} = ws_\alpha w^{-1}\) for \(\alpha \in \{\alpha s, \alpha t\}\) and \(w \in W\).

The following formula can be proved by induction.

**Lemma 2.8.** We have

\[
(st)^k \alpha s = [2k + 1]X\alpha s + [2k]Y\alpha t, \quad (ts)^k \alpha t = [2k]X\alpha s + [2k + 1]Y\alpha t.
\]

**Lemma 2.9.** Let \(\gamma \in \Phi^+\) and \(g = s_\gamma\).

1. If \(sg > g\), then

\[
\gamma = \left[\frac{\ell(g) - 1}{2}\right]_X \alpha s + \left[\frac{\ell(g) + 1}{2}\right]_Y \alpha t.
\]

2. If \(tg > g\), then

\[
\gamma = \left[\frac{\ell(g) + 1}{2}\right]_X \alpha s + \left[\frac{\ell(g) - 1}{2}\right]_Y \alpha t.
\]

**Proof.** We have \(\gamma = (ts)^k(\alpha t)\) or \(t(st)^k(\alpha s)\) or \((st)^k(\alpha s)\) or \(s(ts)^k(\alpha t)\). If \(\gamma = (ts)^k(\alpha t)\), then \(sg > g\) and \(\ell(g) = 4k + 1\). The lemma follows from the previous lemma. If \(\gamma = t(st)^k(\alpha s)\), then \(sg > g\) and \(\ell(g) = 4k + 3\). We have

\[
\gamma = t([2k + 1]X\alpha s + [2k]Y\alpha t) = [2k + 1]X(\alpha s + Y\alpha t) - [2k]Y\alpha t
\]

\[
= [2k + 1]X\alpha s + ([2k + 1]X - [2k]Y)\alpha t = [2k + 1]X\alpha s + [2k + 2]Y\alpha t
\]

and the lemma follows. The proof of the other cases are similar. \(\square\)

We define some elements which will be needed for our main formula. We use the following notation for sequences in \(S\). A sequence in \(S\) will be written with the underline like \(w = (s_1, \ldots, s_t)\). We write \(w = s_1 \cdots s_t\). For \(w \in S\), put \((w, u) = (s_1, \ldots, s_t, u)\). For \(e = (e_1, \ldots, e_l) \in \{0, 1\}^l\), we put \(we = s_1^{e_1} \cdots s_t^{e_t}\). We set \(\ell(w) = l\).

For \(g, w \in W\), we put

\[
X_g^w = \{\alpha \in \Phi^+ \mid s_\alpha g \leq w\}.
\]

Let \(w = (s_1, \ldots, s_t) \in S^t\) be a sequence of elements in \(S\) and \(g \in W\). For a real number \(r\), let \([r]\) be the integral part of \(r\). We define \(k^w_g \in \mathbb{Z}[X, Y]_{\alpha s}\) as follows. If \(s_i = s_{i+1}\) for some \(i\) or \(g \not\leq w\), then \(k^w_g = 0\). If \(\ell(w) = 0\), then \(k^w_g = 1\) and \(k^w_g = 0\) if \(g \neq 1\). Otherwise we put

\[
k^w_g = \begin{cases} 
\left[\frac{\ell(w) - 1}{\ell(w) - \ell(s_1)}\right] s^{\ell(w) - 1}(Z) & \text{if } s_1 g > g, \\
\left[\frac{\ell(w) - 1}{\ell(w) - \ell(s_1)}\right] s^{\ell(w) - 1}(Z) & \text{if } s_1 g < g,
\end{cases}
\]

where \(Z = X\) if \(s_1 = s\) and \(Z = Y\) if \(s_1 = t\).
Let \( R \) be the symmetric algebra of \( V \) and \( R^0 = \Phi^{-1}R \) the ring of fractions. We define an element \( a^w(g) \) of \( R^0 \) by

\[
a^w(g) = \sum_{w-y=g} \frac{1}{\alpha_{s_1}} s_1 \left( \frac{1}{\alpha_{s_2}} s_2 \left( \cdots \frac{1}{\alpha_{s_{l-1}}} s_{l-1} \left( 1 \frac{1}{\alpha_{s_l}} \cdots \right) \right) \right) = \sum_{w-y=g} \prod_{i=1}^{l} (s_{e_1}^{e_i} \cdots s_{e_{l-1}}^{e_{l-1}}) \left( \frac{1}{\alpha_{s_i}} \right).
\]

**Lemma 2.10.** If \( s_i = s_{i-1} \) for some \( i \), namely if \( w \) is not a reduced expression, then \( a^w(g) = 0 \).

**Proof.** Set \( A = \{ e \in \{0,1\}^l \mid w^e = g \} \). Define \( f : A \to A \) by \( f(e) = (e'_1, \ldots, e'_l) \) where \( e'_i = 1 - e_i \) for \( i < j \) and \( e'_j = e_i \) for \( j = i \). Set \( b_{e,j} = (s_{e_1}^{e_i} \cdots s_{e_{j-1}}^{e_{j-1}}) \left( \frac{1}{\alpha_j} \right) \). If \( j < i \) then obviously we have \( b_{e,j} = b_{f(e),j} \). If \( j > i \) then since \( s_{i-1}^{e_{i-1}} s_{i-1}^{e_i} = s_{i-1}^{e_{i-1}} s_{i}^{e_i} \), we have \( b_{e,j} = b_{f(e),j} \). If \( j = i \) then

\[
b_{f(e),i} = (s_{e_1}^{e_i} \cdots s_{e_{i-1}}^{e_{i-1}}) s_{i-1} \left( \frac{1}{\alpha_i} \right) = -b_{f(e),i}
\]

since \( s_{i-1}^{e_{i-1}} = s_{i-1}^{e_{i-1}} s_{i-1} \) and \( s_{i-1} = s_i \). Therefore \( b_e = \prod_{i=1}^{l} b_{e,i} \) satisfies \( b_f(e) = -b_e \). Let \( B \) be a set of complete representatives of \( A/(f) \). Then \( a^w(g) = \sum_{e \in A} b_e = \sum_{e \in B} (b_e + b_{f(e)}) = 0. \)

The aim of this section is to prove the following theorem.

**Theorem 2.11.** For \( w \in S_l \), we have

\[
a^w(g) = \frac{k^w}{\prod_{\alpha \in X^w} \alpha}.
\]

From the above lemma, we may assume \( s_{i-1} \neq s_i \) for any \( i \). By definitions, we also may assume \( g \leq w \), otherwise both sides are zero.

2.3. **Proof of Theorem 2.11.** In this subsection we prove Theorem 2.11.

We split the sum in the definition of \( a^w(g) \) to \( e_l = 0 \) part and \( e_l = 1 \) part. If \( e_l = 0 \), then \( s_{e_1}^{e_1} \cdots s_{e_{l-1}}^{e_{l-1}} = g \). Hence \( (s_{e_1}^{e_1} \cdots s_{e_{l-1}}^{e_{l-1}}) \left( \frac{1}{\alpha_l} \right) = g \left( \frac{1}{\alpha_l} \right) \). Therefore

\[
\prod_{i=1}^{l} (s_{e_1}^{e_i} \cdots s_{e_{i-1}}^{e_{i-1}}) \left( \frac{1}{\alpha_i} \right) = g \left( \frac{1}{\alpha_l} \right) \prod_{i=1}^{l} (s_{e_1}^{e_i} \cdots s_{e_{i-1}}^{e_{i-1}}) \left( \frac{1}{\alpha_i} \right)
\]

Similarly if \( e_l = 1 \) then \( (s_{e_1}^{e_1} \cdots s_{e_{l-1}}^{e_{l-1}}) \left( \frac{1}{\alpha_l} \right) = gs_l \left( \frac{1}{\alpha_l} \right) = -g \left( \frac{1}{\alpha_l} \right) \). Therefore we have

\[
a^w(g) = \frac{1}{g(\alpha_l)} \left( \sum_{s_{e_1}^{e_1} \cdots s_{e_{l-1}}^{e_{l-1}} = g} \prod_{i=1}^{l-1} (s_{e_1}^{e_i} \cdots s_{e_{i-1}}^{e_{i-1}}) \left( \frac{1}{\alpha_i} \right) - \sum_{s_{e_1}^{e_1} \cdots s_{e_{l-1}}^{e_{l-1}} = gs_l} \prod_{i=1}^{l-1} (s_{e_1}^{e_i} \cdots s_{e_{i-1}}^{e_{i-1}}) \left( \frac{1}{\alpha_i} \right) \right)
\]

We change the notation slightly and we get the following lemma.

**Lemma 2.12.** Let \( w \in S_l \) and \( u \in S \). Then we have

\[
a^w(w^u)(g) = \frac{1}{g(\alpha_u)} (a^w(g) - a^w(gu)).
\]

To prove the theorem we need the following lemmas.

**Lemma 2.13.** Let \( w, g \in W \) and \( u \in S \) such that \( wu > w \), \( sw < w \), \( g, gu \leq w \).

1. There exists unique \( \beta \in X^w_g \) such that \( s_\beta \in \{wg^{-1}, swg^{-1}\} \).
2. There exists unique \( \gamma \in X^w_g \) such that \( s_\gamma \in \{wg^{-1}, swg^{-1}\} \).
(3) We have \( X_g^w \setminus \{ \beta \} = X_{g^w}^w \setminus \{ \gamma \} \) and \( X_{g^w}^w = X_g^w \cup \{ \gamma \} \).

Proof. Since our Coxeter system has rank two, for \( x \in W \), there exists \( \alpha \in \Phi^+ \) such that \( s_\alpha = x \) if and only if \( \ell(x) \) is odd. One of elements in \( w g^{-1}, sw g^{-1} \) has the odd length. Hence there exists \( \beta \in \Phi^+ \) such that \( s_\beta \in \{ w g^{-1}, sw g^{-1} \} \). If \( s_\beta = w g^{-1} \), then \( s_\beta g = w \leq w \). If \( s_\beta = sw g^{-1} \), then \( s_\beta g = sw \leq w \). Hence \( \beta \in X_g^w \) and we get (1). The proof of (2) is similar.

We prove (3). Let \( \delta \in X_g^w \). Then \( s_\delta g \leq w \). Since our Coxeter system is of rank two, if \( \ell(s_\delta g u) \leq \ell(w) - 1 \), we have \( s_\delta g u \leq w \). Hence \( \delta \in X_g^w \). Therefore if \( \ell(s_\delta g) \leq \ell(w) - 2 \), then \( \ell(s_\delta g u) \leq \ell(s_\delta g) + 1 \), we have \( \delta \in X_g^w \).

Let \( u' \) be the element in \( S \) which is not \( u \). Then we have \( sw < w, wu' < w \).

- If \( \ell(s_\delta g) = \ell(w) - 1 \), then \( s_\delta g = sw \) or \( wu' \). If \( s_\delta g = sw \), then \( s_\delta = sw g^{-1} \), hence \( \delta = \beta \). If \( s_\delta g = wu' \) and \( w \neq u' \), the reduced expression of \( wu' \) ends with \( u \).

In any case, if \( \delta \in X_g^w \), then \( \delta = \beta \) or \( \delta \in X_{g^w}^w \). Hence \( X_g^w \setminus \{ \beta \} \subset X_{g^w}^w \). If \( \delta = \gamma \), the element \( s_\delta g \) is \( wu \) or \( swu \). Since \( wu > w \), we have \( s_\delta g \leq w \) only when \( s_\delta g = swu = w \).

Lemma 2.14. Let \( w, g \in W, u \in S \) such that \( wu > w, sw < w, g \leq w \) and \( gu \not\leq w \). Then \( X_{g^w}^w = X_g^w \cup \{ g(\alpha_u) \} \).

Proof. By Property Z in [Deo77], for any \( x \in W \), \( x \leq w \) implies \( xu \leq wu \). Applying this to \( x = s_\gamma g \) for \( \gamma \in X_g^w \), we have \( X_g^w \subset X_{g^w}^w \). Since \( g \leq w \), we have \( s_\gamma(\alpha_u) g = gu \leq wu \). Therefore \( g(\alpha_u) \in X_{g^w}^w \). Hence \( X_{g^w}^w \cup \{ g(\alpha_u) \} \subset X_{g^w}^w \).

In any case, we have \( s_\delta = g u g^{-1} \) and \( \delta = g(\alpha_u) \).

Lemma 2.15. Let \( w = (s_1, \ldots, s_l) \in S^t \) such that \( s_{i-1} \neq s_i \) for any \( i \) and \( g \in W \). Set \( u = s_l \).

1. \( a_{\Xi}(g) = a_{\Xi}(gu) \).
2. \( k^w_g = k^{w}_{g^w} \).
3. \( X_g^w = X_{g^w}^w \).

Proof. We may assume \( g < gu \) by replacing \( g \) with \( gu \) if necessary. We also may assume that \( s_1 = s \) by swapping \( s \) with \( t \) if necessary. (1) follows from Lemma 2.12 and \( a_{\Xi}(w)(g) = 0 \).

For (2), first we assume \( sg > g \) and \( g \neq 1 \). Then the reduced expression of \( g \) has a form \( g = t \cdots u' \) where \( u' \in S \) is the element which is not \( u \), namely the reduced expression starts with \( t \) and ends with \( u' \). Since \( w = (s_1, \ldots, u) \) and \( s_{i-1} \neq s_i \) for any \( i \), we have
Hence it is sufficient to prove that
\[
\ell(g) = \ell(w) \pmod{2}.
\]
Hence the lemma follows from the definition of \(k_g^w\). The proof in the case of \(sg < g\), \(g \neq 1\) is similar.

Assume \(g = 1\). If \(u = s\), then \(s_1 = s_t = s\), hence \(\ell(w)\) is odd. If \(u = t\), then \(s_1 = s\) and \(s_t = t\). Hence \(\ell(w)\) is even. In both cases, we can confirm \(k_g^w = k_g^{wu}\) by the definition.

Since \(wu < w\), by Property Z in [Deo77] we have \(s_\gamma g \leq w\) if and only if \(s_\gamma gu \leq w\). (3) follows.

**Proof of Theorem 2.11.** We prove the theorem by induction on \(\ell(w)\). If \(\ell(w) = 0\), then this is trivial. Let \(u \in S\) and we prove that the theorem is true for \((w, u)\) assuming that the theorem is true for \(w\). If \((w, u)\) is not a reduced expression, then both sides of the theorem are zero. Hence we may assume \((w, u)\) is a reduced expression. By the previous lemma, we also may assume \(u > g\). If \(g \not\leq w\), then by Property Z [Deo77], \(g \not\leq wu\). Hence both sides are zero.

Take \(s_1, \ldots, s_t \in S\) such that \(w = (s_1, \ldots, s_t)\). If \(g \leq w\) and \(gu \not\leq w\), then \(a(w)(gu) = 0\).

By Lemma 2.12 inductive hypothesis and Lemma 2.14
\[
a'(w)(u)g = \frac{a(w)(g)}{g(\alpha_u)}(a(w)(g) - a(w)(gu)) = \frac{1}{g(\alpha_u)} \left( \frac{k_g^w}{\prod_{\gamma \in X_g^w} g(\alpha_u)} - \frac{k_g^{wu}}{\prod_{\gamma \in X_g^{wu}} g(\alpha_u)} \right).
\]

As in the proof of Lemma 2.14 we have \(g = w\) or \(g = s_1w\) (the latter does not happen when \(l = 0\)). Hence \(k_g^w = k_g^{wu} = 1\) from the definitions. Therefore the theorem holds in this case.

We assume \(g, gu \leq w\). Then \(\ell(w) > 0\). We may assume \(s_1 = s\) by swapping \((s, X)\) with \((t, Y)\) if necessary. By Lemma 2.12 and inductive hypothesis, we have
\[
a'(w)(u)(g) = \frac{1}{g(\alpha_u)}(a(w)(g) - a(w)(gu)) = \frac{1}{g(\alpha_u)} \left( \frac{k_g^w}{\prod_{\gamma \in X_g^w} g(\alpha_u)} - \frac{k_g^{wu}}{\prod_{\gamma \in X_g^{wu}} g(\alpha_u)} \right).
\]

Take \(\beta, \gamma \in \Phi^+\) as in Lemma 2.13. Then by Lemma 2.13 the right-hand side is
\[
\frac{1}{\prod_{\delta \in X_g^w \setminus \{\beta\}} g(\alpha_u)}(k_g^w \gamma - k_g^{wu} \delta) = \frac{1}{\prod_{\delta \in X_g^{wu}} g(\alpha_u)}(k_g^w \gamma - k_g^{wu} \delta).
\]

Hence it is sufficient to prove that \(k_g^w \gamma - k_g^{wu} \delta = k_g^{(w,u)} \gamma(\alpha_u)\). Since \(gu > g\), the reduced expression of \(g\) ends with the simple reflection which is not \(u\). Hence the reduced expression of \(gug^{-1}\) can be obtained by concatenating the reduced expressions of \(g, u\) and \(g^{-1}\). Therefore we have \(\ell(gug^{-1}) = \ell(g) + \ell(u) + \ell(g^{-1}) = 2\ell(g) + 1\). Moreover, if \(sg > g\), then we have \(sgug^{-1} > gug^{-1}\).

First we assume \(sg > g\) and \(g \neq 1\). Then \(ssg(u) = sgug^{-1} > gug^{-1}\). Hence
\[
g(u) = [\ell(g)]_X \alpha_s + [\ell(g) + 1]_Y \alpha_t
\]
by Lemma 2.9. Since \(gu > g\) and \(wu > w\), the reduced expressions of \(g\) and \(w\) ends with the same simple reflection. Namely if \(u' \in S\) is the element which is not \(u\), then the reduced expression of \(w\) is \(w = s \cdots u'\) and the reduced expression of \(g\) is \(g = t \cdots u'\) since we assumed \(sg > g\). Since \(g \leq w\), the last \(\ell(g)\)-letters of the reduced expression of \(w\) is the reduced expression of \(g\). Hence \(\ell(wg^{-1}) + \ell(g) = \ell(w)\) and the reduced expression of \(wg^{-1}\) starts with \(s\) and ends with \(t\). Therefore \(twg^{-1} > wg^{-1}\) and \(s_\beta = wg^{-1}\). Hence by Lemma 2.9 we have
\[
\beta = \left[ \frac{\ell(wg^{-1}) + 1}{2} \right]_X \alpha_s + \left[ \frac{\ell(wg^{-1}) - 1}{2} \right]_Y \alpha_t
\]

\[
= \left[ \frac{\ell(w) - \ell(g) + 1}{2} \right]_X \alpha_s + \left[ \frac{\ell(w) - \ell(g) - 1}{2} \right]_Y \alpha_t.
\]
A calculation of $\gamma$ is similar. We have $\ell(w_{ug}^{-1}) = \ell(g) + \ell(u) + \ell(w^{-1})$ and the reduced expression of $w_{ug}^{-1}$ starts with $s$ and ends with $t$. Therefore $\ell(s_{ug}^{-1}) = \ell(w_{ug}^{-1}) - 1$, $s_{\gamma} = swg^{-1}$ and $s(sw_{ug}^{-1}) > swg^{-1}$. Hence by Lemma 2.9, we have

$$
\gamma = \left[ \frac{\ell(w) + \ell(g) - 1}{2} \right]_X \alpha_s + \left[ \frac{\ell(w) + \ell(g) + 1}{2} \right]_Y \alpha_t.
$$

Put $m = (\ell(w) - \ell(g) - 1)/2$ and $n = \ell(g)$. Then we have

- $g(\alpha_u) = [n]_X \alpha_s + [n + 1]_Y \alpha_t$,
- $\beta = [m + 1]_X \alpha_s + [m]_Y \alpha_t$,
- $\gamma = [m + n]_X \alpha_s + [m + n + 1]_Y \alpha_t$.

Therefore we have

$$
k^w_{\gamma} - k^w_{\beta} = (k^w_{\gamma}[m + n]_X - k^w_{\beta}[m + 1]_X)\alpha_s + (k^w_{\gamma}[m + n + 1]_Y - k^w_{\beta}[m]_Y)\alpha_t.
$$

By the definition, we have

$$
k^w_{\gamma} = \left[ \frac{2m + n}{m} \right]_{\sigma^{2m+n}(X)} = \left[ \frac{m + n + 1}{m} \right]_{\sigma^{2m+n}(X)} \left[ \frac{2m + n}{m} \right]_{\sigma^{2m+n}(X)} - 1 \left[ \frac{m + n + 1}{m} \right]_{\sigma^{2m+n}(X)}
$$

Hence,

$$
k^w_{\gamma} - k^w_{\beta} = \frac{k^w_{\gamma}}{[m]_{\sigma^n(X)}} \left( ([m + n + 1]_{\sigma^n(X)}[m + n]_X - [m + 1]_X[m]_{\sigma^n(X)})\alpha_s + ([m + n + 1]_{\sigma^n(X)}[m + n + 1]_Y - [m]_Y[m]_{\sigma^n(X)})\alpha_t \right).
$$

By Lemma 2.5 this is equal to

$$
\frac{k^w_{\gamma}}{[m]_{\sigma^n(X)}} ([m]_X[2m + n + 1]_{\sigma^{m+1}(X)}\alpha_s + [n + 1]_Y[2m + n + 1]_{\sigma^{m+1}(X)}\alpha_t)
$$

$$
= \frac{k^w_{\gamma}}{[m]_{\sigma^n(X)}} [2m + n + 1]_{\sigma^{m+1}(X)} g(\alpha_u).
$$

Hence it is sufficient to prove

$$
k^w_{\gamma} \left[ \frac{2m + n + 1}{m} \right]_{\sigma^{m+1}(X)} = k^{(w,u)}_{\gamma}.
$$

This follows immediately from Lemma 2.7.

The case of $tg > g$ is similar. By Lemma 2.9, we have

$$
g(\alpha_u) = [\ell(g) + 1]_X \alpha_s + [\ell(g)]_Y \alpha_t.
$$

The reduced expressions of $w$ and $g$ end the same reflection, hence $\ell(wg^{-1}) = \ell(w) - \ell(g)$. The reduced expression of $g$ starts with $s$. Hence the reduced expression of $wg^{-1}$ starts with $s$, ends with $t$. Hence $s_{\beta} = swg^{-1}$, $s(sw_{ug}^{-1}) > swg^{-1}$ and $\ell(s_{\beta}) = \ell(w) - \ell(g) - 1$. Hence by Lemma 2.9 we have

$$
\beta = \left[ \frac{\ell(w) - \ell(g)}{2} - 1 \right]_X \alpha_s + \left[ \frac{\ell(w) - \ell(g)}{2} \right]_Y \alpha_t.
$$
We have $\ell(wug^{-1}) = \ell(w) + \ell(g) + 1$ and the reduced expression starts with $s$ and ends with $s$. Hence $s_\gamma = wug^{-1}$, $ts_\gamma > s_\gamma$, and $\ell(s_\gamma) = \ell(g) + \ell(w) + 1$. Therefore by Lemma 2.9 we have

$$
\gamma = \left[\frac{\ell(w) + \ell(g)}{2} + 1\right] X_{s} + \left[\frac{\ell(w) + \ell(g)}{2}\right] Y_{\alpha_t}.
$$

Put $m = (\ell(w) - \ell(g))/2 - 1$ and $n = \ell(g) + 1$. Then

$$
g(\alpha_u) = \left[n\right]_x \alpha_s + \left[n-1\right]_y \alpha_t,
\beta = \left[m\right]_x \alpha_s + \left[m+1\right]_y \alpha_t,
\gamma = \left[m + n + 1\right]_x \alpha_s + \left[m + n\right]_y \alpha_t.
$$

We have

$$
k_g^w = \frac{[m + n]_{\sigma^m(X)} k_g^w}{[m + n]_{\sigma^m(X)}}.
$$

Therefore, by Lemma 2.5 we have

$$
k_g^w \gamma - k_g^w \beta
= \frac{k_g^w}{[m + 1]_{\sigma^m(X)}} \left(\left[m + n\right]_{\sigma^m(X)} [m + n + 1]_x - \left[m\right]_x [m + 1]_{\sigma^m(X)} \alpha_s
+ \left[m + n\right]_y \left[m + n\right]_{\sigma^m(X)} - \left[m + 1\right]_y [m + 1]_{\sigma^m(X)} \alpha_t\right).
$$

$$
= \frac{k_g^w}{[m + 1]_{\sigma^m(X)}} \left([m]_x [2m + n + 1]_{\sigma^m(X)} + [n - 1]_y [2m + n + 1]_{\sigma^m(X)}\right)
$$

$$
= \frac{k_g^w}{[m + 1]_{\sigma^m(X)}} [2m + n + 1]_{\sigma^m(X)} g(\alpha_u).
$$

Therefore it is sufficient to prove

$$
\frac{[2m + n + 1]_{\sigma^m(X)} k_g^w}{[m + 1]_{\sigma^m(X)}} = k_{g^u}
$$

which is again an immediate consequence of Lemma 2.7.

We assume $g = 1$ and $u = t$. Then one can check that formulas for $g(\alpha_u), \beta, \gamma$ in the case of $sg > g, g \neq 1$ hold. Hence the theorem follow from the calculations in this case.

If $g = 1$ and $u = s$, then one can use the calculations in the case of $tg > g, g \neq 1$. 

\section{A homomorphism between Bott-Samelson bimodules}

\subsection{Finite Coxeter group of rank two and a realization.}

We add the tilde to the notation in the previous section, namely $(\tilde{W}, \tilde{S})$ is the universal Coxeter system of rank 2, $\tilde{V}$ is the free $\mathbb{Z}[\tilde{X}, \tilde{Y}]$-module with the action of $\tilde{W}$, $[n]_\tilde{X}, [n]_\tilde{Y} \in \mathbb{Z}[\tilde{X}, \tilde{Y}]$ is the two-colored quantum numbers, etc.

The notation without tilde will be used for non-universal version. Let $(W, S)$ be a Coxeter system such that $S = \{s, t\}, s \neq t$. We assume that the order $m_{st}$ of $st$ is finite. Let $\mathbb{K}$ be a commutative integral domain and $(V, \{\alpha_s, \alpha_t\}, \{\alpha^\vee_s, \alpha^\vee_t\})$ a realization [EW16, Definition 3.1], namely $V$ is a free $\mathbb{K}$-module of finite rank with an action of $W$, $\alpha_s, \alpha_t \in V$ and $\alpha^\vee_s, \alpha^\vee_t \in \text{Hom}_\mathbb{K}(V, \mathbb{K})$ such that

\begin{itemize}
  \item \(\langle \alpha^\vee_s, \alpha_s \rangle = \langle \alpha^\vee_t, \alpha_t \rangle = 2\).
  \item \(s(v) = v - \langle \alpha^\vee_s, \alpha_s \rangle v, t(v) = v - \langle \alpha^\vee_t, \alpha_t \rangle v\) for any $v \in V$.
  \item \(\alpha_s, \alpha_t \neq 0\) and \(\alpha^\vee_s, \alpha^\vee_t : V \to \mathbb{K}\) are surjective.
  \item \([m_{st}]_\tilde{X}(-\langle \alpha^\vee_s, \alpha_t \rangle, -\langle \alpha^\vee_t, \alpha_s \rangle) = [m_{st}]_\tilde{Y}(-\langle \alpha^\vee_s, \alpha_t \rangle, -\langle \alpha^\vee_t, \alpha_s \rangle) = 0\).
\end{itemize}
The map \( \tilde{s} \mapsto s, \tilde{t} \mapsto t \) gives a surjective homomorphism \( \tilde{W} \rightarrow W \). Set \( X = -\langle \alpha_i^+, \alpha_i \rangle \), \( Y = -\langle \alpha_i^+, \alpha_i \rangle \). Then \( \hat{\alpha}_s \mapsto \alpha_s, \hat{\alpha}_t \mapsto \alpha_t \) gives a \( \mathbb{Z}[\tilde{X}, \tilde{Y}] \)-module homomorphism \( \tilde{V} \rightarrow V \) which commutes with the actions of \( \tilde{W} \) where we regard \( V \) as a \( \mathbb{Z}[\tilde{X}, \tilde{Y}] \)-module via \( \mathbb{Z}[\tilde{X}, \tilde{Y}] \rightarrow \mathbb{K} \) defined by \( \tilde{X} \mapsto X \) and \( \tilde{Y} \mapsto Y \). The image of \([n]_{\tilde{X}} \) (resp. \([n]_{\tilde{Y}} \)) is denoted by \([n]_X \) (resp. \([n]_Y \)). We also have \([n]_{\tilde{m}} X \) \([n]_{\tilde{m}} Y \) \( \in \mathbb{K} \).

Let \( R \) (resp. \( \tilde{R} \)) be the symmetric algebra of \( V \) (resp. \( \tilde{V} \)). We regard \( R \) as a graded \( \mathbb{K} \)-algebra via \( \text{deg}(V) = 2 \). We put \( \partial_s(p) = (p - u(p)) / \alpha_u \) for \( p \in R \). The maps \( \tilde{V} \rightarrow V \) and \( \mathbb{Z}[\tilde{X}, \tilde{Y}] \rightarrow \mathbb{K} \) induce \( \tilde{R} \rightarrow R \). We defined an element \( \tilde{a}(\tilde{g}) \in \tilde{R}[\tilde{w}(\tilde{a})^{-1} | \tilde{w} \in \tilde{W}, \tilde{u} \in \tilde{S}] \). Let \( Q \) be the field of fractions of \( R \). The image of \( \tilde{a}(\tilde{g}) \) in \( Q \) is denoted by \( a(\tilde{g}) \in Q \).

As some of them are appeared already, objects related to the universal Coxeter system is denoted with the tilde and the corresponding letter without the tilde means the image in the finite Coxeter system. For example, if \( \tilde{w} = (s_1, s_2, \ldots) \) is a sequence of elements in \( \tilde{S} \), then \( w = (s_1, s_2, \ldots) \) is the corresponding sequence in \( S \). As we have already explained, a sequence is denoted with the underline and removing the underline means the product of elements in the sequence. Hence \( \tilde{w} = \tilde{s}_1 \tilde{s}_2 \cdots \in \tilde{W} \) and \( w = s_1 s_2 \cdots \in W \). For each root \( \tilde{\alpha} \in \tilde{\Phi} \), we have \( \tilde{s}_\alpha \in \tilde{W} \) and \( s_\alpha \in W \).

Set \( \tilde{x} = (s_1, \tilde{t}, \ldots) \in S^{m_{s,t}} \) and \( \tilde{y} = (\tilde{t}, \tilde{s}_1, \ldots) \in S^{m_{s,t}} \). The sequences \( x \) and \( y \) are the two reduced expressions of the longest element. In general, for a sequence \( \tilde{w} = (s_1, s_2, \ldots, s_l) \in S^l \), we put

\[
\pi_\tilde{w} = \prod_{i=1}^l s_1 \cdots s_{i-1}(\alpha_s) \in R.
\]

The two elements \( \pi_x \) and \( \pi_y \) are not the same in general. By \( [\text{EW20}, (7.9), (7.10)] \),

\[
\pi_x = \pi_y \text{ if } m_{s,t} \text{ is even and } \pi_y = [m_{s,t} - 1]_X \pi_x \text{ if } m_{s,t} \text{ is odd.}
\]

Put \( \xi = [m_{s,t} - 1]_X \pi_x \) if \( m_{s,t} \) is even and \( \xi = 1 \) if \( m_{s,t} \) is odd. Then we have \( \pi_y = \xi [m_{s,t} - 1]_X \pi_x \). In particular, \( \pi_y \in \mathbb{K}^\times \pi_x' \) \( [\text{EW20}, (6.11), (6.12)] \). The realization is even-balanced if and only if \( \xi = 1 \).

**Lemma 3.1.** We have \([k]_Z [m_{s,t} - 1]_{\sigma^{k-1}(Z)} = [m_{s,t} - k]_Z \).

**Proof.** This follows from \([\text{EW20}, (6.10)] \). \( \square \)

**Lemma 3.2.** Let \( \tilde{g} \in \tilde{W} \) such that \( \tilde{g} \leq \tilde{x} \). Then we have

\[
\Pi_{\delta \in \tilde{X}^+} \delta = \frac{\Pi_{\delta \in \tilde{X}^+} \delta}{\pi_x} = \begin{cases} 
[m_{s,t} - (\ell(g) - 1)]_{\sigma^{i-1}(X)} & (\tilde{s}_i \tilde{g} \geq \tilde{g}), \\
[m_{s,t} - (\ell(g) - 1)]_{\sigma^{i-1}(X)} & (\tilde{s}_i \tilde{g} < \tilde{g}).
\end{cases}
\]

**Proof.** We prove the lemma by backward induction on \( \ell(\tilde{g}) \). If \( \tilde{g} = \tilde{x} \), then by Theorem \( \text{2.11} \) we have \( \alpha(\tilde{x}) = (\Pi_{\delta \in \tilde{X}^+} \delta)^{-1} \). On the other hand, for \( e \in \{0, 1\}^{m_{s,t}} \), we have \( \tilde{x}^e = \tilde{x} \) if and only if \( e = (1, \ldots, 1) \). Hence by the definition of \( \alpha(\tilde{x}) \), we have \( a(\tilde{x}) = 1 / \pi_x \).

Next assume that \( \tilde{g} = \tilde{s}_i \tilde{x} \). Define \( \tilde{s}_i = \tilde{s} \) if \( i \) is odd and \( \tilde{s}_i = \tilde{t} \) if \( i \) is even. Then \( \tilde{x} = (\tilde{s}_1, \ldots, \tilde{s}_{m_{s,t}}) \). For \( e \in \{0, 1\}^{m_{s,t}} \), \( \tilde{x}^e = \tilde{g} \) if and only if \( e = (0, 1, \ldots, 1) \). Hence by the definition, \( \alpha(\tilde{g}) = 1 / \alpha s \Pi_{i=2}^{m_{s,t}} \tilde{s}_i \cdots \tilde{s}_{i-1}(\alpha_s) \). Since \( \tilde{y} = (\tilde{s}_2, \tilde{s}_3, \ldots, \tilde{s}_{m_{s,t} + 1}) \), we have \( \pi_y = \Pi_{i=2}^{m_{s,t} + 1} \tilde{s}_2 \cdots \tilde{s}_{i-1}(\alpha_s) = (1 / \alpha(\tilde{g})) (\tilde{s}_2 \tilde{s}_3 \cdots \tilde{s}_{m_{s,t} + 1}(\alpha_{m_{s,t} + 1})) / \alpha s \). Since \( \tilde{s}_i = \tilde{s} \), \( \tilde{s}_2 \tilde{s}_3 \cdots \tilde{s}_{m_{s,t} + 1}(\alpha_{m_{s,t} + 1}) > \tilde{s}_2 \tilde{s}_3 \cdots \tilde{s}_{m_{s,t} + 1}(\alpha_{m_{s,t} + 1}) \). Hence we have \( \tilde{s}_2 \tilde{s}_3 \cdots \tilde{s}_{m_{s,t} + 1}(\alpha_{m_{s,t} + 1}) = [m_{s,t} - 1]_X \).
$1\lambda \alpha_s + [m_{s,t}]\gamma \alpha_t = [m_{s,t} - 1]\lambda \alpha_s$ by Lemma 2.9. Since $s_1 = s$, we get $\pi_y = [m_{s,t} - 1]\lambda /\alpha \tilde{Z}(\tilde{g})$. By $\pi_y = \xi [m_{s,t} - 1]\lambda \pi_x$, we have $\xi \pi_x \alpha \tilde{Z}(\tilde{g}) = 1$. By Theorem 2.11, the left hand side of the lemma is $(\alpha \tilde{Z}(\tilde{g})\pi_x)^{-1}$. Hence we get the lemma in this case.

Assume that $\tilde{g} \neq \tilde{x}, \tilde{s}\tilde{x}$. Then there exists $\tilde{w} \in \tilde{S}$ such that $\tilde{x} \geq \tilde{y} \tilde{w} > \tilde{g}$. When $\tilde{g} = 1$, we take $\tilde{u} = \tilde{t}$.

• First assume that $\tilde{x} \tilde{u} < \tilde{x}$. By Lemma 2.15, the left hand side is not changed if we replace $\tilde{g}$ with $\tilde{g} \tilde{w}$. We prove that the right hand side is also not changed. Then this gives the lemma by inductive hypothesis.

Assume $\tilde{y} \neq 1$. The reduced expression of $\tilde{x}$ is given as $\tilde{x} = \tilde{s} \cdots \tilde{u}$. Let $\tilde{u} \in \tilde{S}$ be the element which is not $\tilde{u}$. If $\tilde{u} < \tilde{g}$, then the reduced expression of $\tilde{g}$ is $\tilde{g} = \tilde{t} \cdots \tilde{u}$. Hence $\ell(\tilde{g}) \equiv \ell(\tilde{x}) \pmod{2}$. If $\tilde{u} \tilde{u} < \tilde{g}$, then the reduced expression of $\tilde{g}$ is $\tilde{g} = \tilde{s} \cdots \tilde{u}$. Hence $\ell(\tilde{g}) \equiv \ell(\tilde{x}) + 1 \pmod{2}$. Therefore the right hand side is not changed.

Assume $\tilde{x} \tilde{u} > \tilde{x}$. Take $\beta$ and $\gamma$ such that $\tilde{s} \in \{\tilde{x} \tilde{x} \tilde{u}^{-1}, \tilde{s} \tilde{x} \tilde{u}^{-1}\}$ and $\tilde{s} \gamma \in \{\tilde{x} \tilde{u} \tilde{u}^{-1}, \tilde{s} \tilde{x} \tilde{u} \tilde{u}^{-1}\}$. By Lemma 2.13, we have $(\pi_x \tilde{S} \tilde{S})^4 \equiv (\pi_x \tilde{S} \tilde{S})^4 = \beta / \gamma$. We calculate $\gamma / \gamma$. The proof in the case of $\beta / \gamma$.

• If $\tilde{s} \tilde{u} > \tilde{g}$, $\tilde{y} \neq 1$ or $\tilde{g} = 1$, then by the proof of Theorem 2.11, we have $\beta = [(m_{s,t} - \ell(\tilde{g}) + 1)/2] X\alpha_s + [(m_{s,t} - \ell(\tilde{g}) - 1)/2] \gamma \alpha_t$ and $\gamma = [(m_{s,t} + \ell(\tilde{g}) - 1)/2] X\alpha_s + [(m_{s,t} + \ell(\tilde{g}) + 1)/2] \gamma \alpha_t$. Therefore by the previous lemma, we have $\gamma = [(m_{s,t} - 1)] \sigma(m_{s,t} - (\ell(\tilde{g}) - 1)/2) \beta$. We have $[m_{s,t} - 1] X[m_{s,t} - 1] Y = 1$ by [EW20], (6.11),(6.12)]. Hence we have $\beta / \gamma = [m_{s,t} - 1] \sigma(m_{s,t} - (\ell(\tilde{g}) - 1)/2 - 1) X) \beta$. By inductive hypothesis, we get the lemma in this case.

Finally assume that $\tilde{s} \tilde{g} < \tilde{g}$. By the proof of Theorem 2.11, we have $\beta = [m_{s,t} - \ell(\tilde{g}) + 1)/2 - 1] X\alpha_s + [(m_{s,t} - \ell(\tilde{g})) + 1] \gamma \alpha_t$, $\gamma = [(m_{s,t} + \ell(\tilde{g})) - 1) X\alpha_s + [(m_{s,t} + \ell(\tilde{g})) /2 + 1] \gamma \alpha_t$. Hence $\gamma = [m_{s,t} - 1] \sigma(m_{s,t} - (\ell(\tilde{g}) - 1)/2 - 1) X) \beta$. Therefore $\beta = [m_{s,t} - 1] \sigma(m_{s,t} - (\ell(\tilde{g}) - 1)/2 - 1) X) \gamma$ and we get the lemma.

Lemma 3.3. Let $\pi_x \tilde{S} \tilde{S}$. If $0 \leq l \leq m_{s,t}$, then $\pi_x \tilde{S} \tilde{S} \gamma \in R$. If $\tilde{w} = \tilde{x}$, then it follows from Lemma 3.2. By swapping $s$ with $t$, $\pi_x \tilde{S} \tilde{S} \gamma \in R$. Since $\pi_x \tilde{S} \tilde{S}$, we get the lemma for $\tilde{x} = \tilde{y}$. In general, we have $\tilde{w} = \tilde{x} \tilde{u}$ or $\tilde{w} \leq \tilde{y}$. If $\tilde{w} \leq \tilde{x}$ then $X \tilde{S} \tilde{S} \tilde{S}$. Hence $\pi_x / \pi \tilde{S} \tilde{S} \gamma$ is the lemma when $\tilde{w} \leq \tilde{y}$.

3.2. An assumption. To prove the main theorem, we need one more assumption. In this subsection, we discuss on the assumption. We start with the following proposition.

Proposition 3.4. The following are equivalent.

1. $[m_{s,t}]_Z = 0$ for any $1 \leq k \leq m_{s,t} - 1$ and $Z \in \{X, Y\}$.
2. We have $[m_{s,t} - 1]_Z = \prod_{l=1}^k [m_{s,t} - 1]_{\sigma^{-1}(Z)}$ for $0 \leq k \leq m_{s,t} - 1$ and $Z \in \{X, Y\}$.
3. The realization is even-balanced and $[m_{s,t} - 1]_Z = \prod_{l=1}^k [m_{s,t} - 1]_{\sigma^{-1}(Z)}$ for $0 \leq k \leq m_{s,t} - 1$ and $Z \in \{X, Y\}$.

Proof. Assume (1). By Lemma 2.6 and (1), we have $[m_{s,t} - 1]_Z = -[m_{s,t} - 1]_Z$. Hence (2) follows from induction on $k$. 

$$[m_{s,t} + 1]_{\sigma^{k-1}(Z)} = -[m_{s,t} - 1]_{\sigma^{k-1}(Z)}$$
Conversely assume (2) and we prove (1). By Lemma 2.6 we have
\[
\begin{align*}
[m_{s,t}]_Z = & \left[ \begin{array}{c} m_{s,t} - 1 \\ k \end{array} \right]_{\sigma^k(Z)} [k + 1]Z - \left[ \begin{array}{c} m_{s,t} - 1 \\ k - 1 \end{array} \right]_{Z} [m_{s,t} - k - 1]_{\sigma^{k+1}(Z)} \\
= & \prod_{i=1}^{k} [m_{s,t} - 1]_{\sigma^{k-i}(Z)} [k + 1]Z - \prod_{i=1}^{k-1} [m_{s,t} - 1]_{\sigma^{i}(Z)} [m - k - 1]_{\sigma^{k+1}(Z)}.
\end{align*}
\]
By replacing \(i\) with \(k - i\), we have \(\prod_{i=1}^{k} [m_{s,t} - 1]_{\sigma^{k-i}(Z)} = \prod_{i=0}^{k-1} [m_{s,t} - 1]_{\sigma^{i}(Z)} = [m_{s,t} - 1]_{\sigma(Z)} \prod_{i=1}^{k-1} [m_{s,t} - 1]_{\sigma^{1-i}(Z)}\). Therefore it is sufficient to prove \([m_{s,t} - 1]_{\sigma(Z)} [k + 1]Z - [m - k - 1]_{\sigma^{k+1}(Z)} = 0\). By Lemma 2.3 we have \([m_{s,t} - 1]_{\sigma(Z)} [k + 1]Z = [m_{s,t}]Z[k + 2]_{\sigma(Z)} - [m_{s,t} + k + 1]_{\sigma^{k+1}(Z)}\). Since \([m_{s,t}]Z = 0\) and \([m_{s,t} + k + 1]_{\sigma^{k+1}(Z)} = -[m_{s,t} - k - 1]_{\sigma^{k+1}(Z)}\) we get (1).

We assume (2) and we prove (3). By putting \(k = m_{s,t} - 1\), we have \(\prod_{i=1}^{m_{s,t} - 1} [m_{s,t} - 1]_{\sigma^{i-1}(Z)} = 1\). If \(m_{s,t}\) is even, by Lemma 2.2 (1) and \([m_{s,t} - 1]_Z^2 = 1\) we get \([m_{s,t} - 1]Z = 1\). Hence \(V\) is even-balanced and we get (3).

Assume (3) and we prove (2). It is sufficient to prove that \(\prod_{i=1}^{m_{s,t} - 1} [m_{s,t} - 1]_{\sigma^{i-1}(Z)} = 1\). Hence the right hand side is \(\prod_{i=1}^{m_{s,t} - 1} [m_{s,t} - 1]_{\sigma^{i-1}(Z)} = 1\). Here in the last part we replaced \(i\) with \(m_{s,t} - i\) and used \([m_{s,t} - 1]X[m_{s,t} - 1]Y = 1\). By Lemma 2.2 (4), we have \([m_{s,t} - 1]_{\sigma^{m_{s,t}-1}(Z)} = [m_{s,t} - 1]_{\sigma^{1}(Z)}\) and we get (2). \(\square\)

We need the following assumption to prove the main theorem.

**Assumption 3.5.** The equivalent conditions in Proposition 3.4 hold.

We have a sufficient condition of Assumption 3.5

**Proposition 3.6.** If the action of \(W\) on \(\mathbb{K} \alpha_s + \mathbb{K} \alpha_t\) is faithful, then Assumption 3.5 holds.

**Proof.** If \([k]_X = [k]_Y = 0\) for \(1 \leq k \leq m_{s,t} - 1\), then by [16] before Claim 3.2, Claim 3.5. \((st)^k\) is the identity on \(\mathbb{K} \alpha_s + \mathbb{K} \alpha_t\). This is a contradiction. Hence \([k]_X \neq 0\) or \([k]_Y \neq 0\) for any \(1 \leq k \leq m_{s,t} - 1\). For \(1 \leq k \leq m_{s,t} - 1\), we have \([k]_X [m_{s,t}]_{X} = [m_{s,t}]_X [m_{s,t} - 1]_{X} = 0\). Hence if \([k]_X \neq 0\), then \([m_{s,t}]_{X} = 0\). Therefore if \([k]_X, [k]_Y \neq 0\) for any \(1 \leq k \leq m_{s,t} - 1\), we get the proposition. Assume that there exists \(k = 1, \ldots, m_{s,t} - 1\) such that \([k]_X = 0\). Then \([k]_Y \neq 0\). By Lemma 2.2 (1), \(k\) is even and by Lemma 2.2 (2), we have \([k]_Y X = [k]_X Y = 0\). Hence \(X = 0\). Therefore by induction we have \([2n]_X = 0\) and \([2n + 1]_X = (-1)^n\) for any \(n \in \mathbb{Z}_{\geq 0}\). Hence \([2n + 1]_Y = [2n + 1]_X = (-1)^n \neq 0\) for any \(n \in \mathbb{Z}_{\geq 0}\) by Lemma 2.2 (1). We also have \([2n]_Y \neq 0\) if \(1 \leq 2n \leq m_{s,t} - 1\) since \([2n]_X = 0\). Therefore for any \(1 \leq l \leq m_{s,t} - 1\), \([l]_Y \neq 0\). Therefore \([m_{s,t}]_{l} = 0\).

Since \([2n + 1]_X \neq 0\) for any \(n \in \mathbb{Z}_{\geq 0}\), \(m_{s,t}\) is even. Therefore if \(l\) is even, \(\#(2\mathbb{Z} \cap \{m_{s,t}, \ldots, m_{s,t} - l + 1\}) = \#(2\mathbb{Z} \cap \{1, \ldots, l\})\). Hence by Lemma 2.2, we have \([m_{s,t}]_{l} X = [m_{s,t}]_{l} Y\), which is zero as we have proved. On the other hand, if \(l\) is odd, then \([l]_X \neq 0\). Hence \([m_{s,t}]_{l} X = 0\). \(\square\)

Maybe a more useful criterion is the following.

**Proposition 3.7.** If the realization comes from a root datum and \(W\) is the Weyl group, then Assumption 3.5 holds.
We are in one of the following situation.

- \( m_{s,t} = 2 \), \( \langle \alpha_s, \alpha_t^\vee \rangle = \langle \alpha_t, \alpha_s^\vee \rangle = 0 \).
- \( m_{s,t} = 3 \), \( \langle \alpha_s, \alpha_t^\vee \rangle = \langle \alpha_t, \alpha_s^\vee \rangle = -1 \).
- \( m_{s,t} = 4 \), \( \langle \alpha_s, \alpha_t^\vee \rangle = -1, \langle \alpha_t, \alpha_s^\vee \rangle = -2 \).
- \( m_{s,t} = 6 \), \( \langle \alpha_s, \alpha_t^\vee \rangle = -1, \langle \alpha_t, \alpha_s^\vee \rangle = -3 \).

We can check the assumption by direct calculations. \( \square \)

The assumption is related to the existence of Jones-Wenzl projectors. If Assumption 3.5 holds, then \( \left[ m_{i-1} \right]_k \) is invertible by [EW20, (6.11), (6.12)]. If [EW20 Conjecture 6.23] is true, the assumption implies the existence of the Jones-Wenzl projector \( JW_{m_{i,t-1}} \).

3.3. **Soergel bimodules.** For a graded \( R \)-bimodule \( M = \bigoplus_{i \in \mathbb{Z}} M^i \) and \( k \in \mathbb{Z} \), we define the grading shift \( M(k) \) by \( M(k)^i = M^{i+k} \).

We define a category \( \mathcal{C} \) as follows. An object of \( \mathcal{C} \) is \((M,(M_Q^x)_{x \in W})\) where

- \( M \) is a graded \( R \)-bimodule.
- \( M_Q^x \) is a \( Q \)-bimodule such that \( mp = x(p)m \) for \( m \in M_Q^x \) and \( p \in Q \).
- \( M \otimes_R Q = \bigoplus_{x \in W} M_Q^x \).
- There exist only finite \( x \in W \) such that \( M_Q^x \neq 0 \).
- The \( R \)-bimodule \( M \) is flat as a right \( R \)-module.

A morphism \((M,(M_Q^x)) \rightarrow (N,(N_Q^x))\) is an \( R \)-bimodule homomorphism \( \varphi \) of degree zero such that \((\varphi \otimes \text{id}_Q)(M_Q^x) \subset N_Q^x\) for any \( x \in W \). Usually we denote just \( M \) for \((M,(M_Q^x))\).

For \( M, N \in \mathcal{C} \), we define the tensor product \( M \otimes N = (M \otimes_R N,((M \otimes N)_Q^x))\) by \((M \otimes N)_Q^x = \bigoplus_{y,z=x} M_Q^y \otimes_Q M_Q^z\).

Let \( \mathcal{C}_Q \) be the category consisting of objects \((P^x)_{x \in W}\) where \( P^x \) is a \( Q \)-bimodule such that \( mp = x(p)m \) for \( m \in P^x \), \( p \in Q \) and there exists only finite \( x \in W \) such that \( P^x \neq 0 \). A morphism \((P^x_1) \rightarrow (P^x_2)\) in \( \mathcal{C}_Q \) is \((\phi_x)_{x \in W}\) where \( \phi_x : P^x_1 \rightarrow P^x_2 \) is a \( Q \)-bimodule homomorphism. Obviously \( M \mapsto (M_Q^x)_{x \in W} \) is a functor \( \mathcal{C} \rightarrow \mathcal{C}_Q \). We denote this functor by \( M \mapsto M_Q \). Since \( M \rightarrow M \otimes_R Q \) is injective, this functor is faithful.

For \( P_1 = (P_1^x), P_2 = (P_2^x) \in \mathcal{C}_Q \), we define \( P_1 \otimes P_2 = ((P_1 \otimes P_2)_x) \) by \( (P_1 \otimes P_2)_x^y = \bigoplus_{z=x} P_1^z \otimes_Q P_2^z \). We have \((M \otimes N)_Q = M_Q \otimes N_Q \).

For \( x \in W \), we define \( Q_x \in \mathcal{C}_Q \) by

- \((Q_x)^x = Q \) as a left \( Q \)-module and the right action of \( q \in Q \) is given by \( m \cdot q = x(g)m \).
- \((Q_x)^y = 0 \) if \( y \neq x \).

Then any object in \( \mathcal{C}_Q \) is isomorphic to a direct sum of \( Q_x \)'s. We have \( Q_x \otimes Q_y \simeq Q_{xy} \) via \( f \otimes g \mapsto f(x(g)) \).

Let \( u \in S \) and we put \( R^u = \{ f \in R \mid u(f) = f \} \), \( B_u = R \otimes_R R(1) \). Then there exists a unique decomposition \( B_u \otimes_R Q = (B_u)_Q^u \oplus (B_u)_Q^v \) as in the definition of the category \( \mathcal{C} \). Explicitly, it is given by the following. Take \( \delta_u \in V \) such that \( \langle \alpha_u^\vee, \delta_u \rangle = 1 \). Then

\[
(B_u)_Q^u = (\delta_u \otimes 1 - 1 \otimes u(\delta_u))Q, \\
(B_u)_Q^v = (\delta_u \otimes 1 - 1 \otimes \delta_u)Q.
\]

Therefore \( B_u \in \mathcal{C} \). We have \((B_u)_Q \simeq Q_x \oplus Q_s\) and an isomorphism is given by

\[
f \otimes g \mapsto \left( \begin{array}{c} f(g) \\ u(f(g)) \end{array} \right).
\]

We always use this isomorphism to identify \((B_u)_Q\) with \( Q_x \oplus Q_u \).
Let $M \in C$ and consider $M \otimes B_u$. Then $(M \otimes B_u)_Q \simeq M_Q \otimes Q \oplus M_Q \otimes Q u$. As a left $Q$-module, this is isomorphic to $M_Q \oplus M_Q$. The right action is given by $(m_1, m_2)p = (m_1p, m_2u(p))$ for $p \in Q$.

**Lemma 3.8.** Let $(m_1, m_2) \in M_Q \oplus M_Q$. Then $(m_1, m_2) \in M \otimes B_u$ if and only $m_1 \alpha_u \in M$ and $m_1 - m_2 \in M$.

**Proof.** Let $m \in M$, $p_1, p_2 \in R$. Then the image of $m \otimes (p_1 \otimes p_2) \in M \otimes B_u$ in $(M \otimes B_u)_Q \simeq M_Q \otimes M_Q$ is $(mp_1 p_2 \alpha_u^{-1}, mp_1 u(p_2) \alpha_u^{-1})$. Hence $(mp_1 p_2 \alpha_u^{-1}) \alpha_u = mp_2 p_2 \in M$ and $(mp_1 p_2 \alpha_u^{-1}) - (mp_1 u(p_2) \alpha_u^{-1}) = mp_1 \delta_u(p_2) \in M$.

On the other hand, assume that $m_1 \alpha_u \in M$ and $m_1 - m_2 \in M$. Take $\delta_u \in V$ such that $\langle \alpha_u^\delta, \delta_u \rangle = 1$. Then we have $u(\delta_u) = \delta_u - \alpha_u$. Hence the image of $(m_1 \alpha_u) \otimes (1 \otimes 1) + (m_2 - m_1) \otimes (\delta_u \otimes 1 - 1 \otimes \delta_u)$ is $(m_1, m_1) + ((m_2 - m_1)(\delta_u / \alpha_u), (m_2 - m_1)(\delta_u / \alpha_u)) - ((m_2 - m_1)(\delta_u / \alpha_u), (m_2 - m_1)(u(\delta_u) / \alpha_u)) = (m_1, m_2)$.

In general, for a sequence $w = (s_1, s_2, \ldots, s_l) \in S^l$ of elements in $S$, we put $B_w = B_{s_1} \otimes \cdots \otimes B_{s_l}$. Set $b_w = (1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1) \in B_w$. The main theorem of this paper is the following.

**Theorem 3.9.** Assume Assumption 3.5. There exists a morphism $\varphi : B_{\underline{z}} \to B_{\underline{z}}$ such that $\varphi(b_{\underline{z}}) = b_{\underline{z}}$.

**3.4. Localized calculus.** Since $(B_{\underline{z}})_Q = (B_u)_Q^e \oplus (B_u)_Q^f \simeq Q \oplus Q u$, for $w = (s_1, \ldots, s_l) \in S$, we have

$$
(B_w)_Q \simeq \bigoplus_{e = \langle e \rangle \in \{0, 1\}^l} Q_{s_1} \otimes \cdots \otimes Q_{s_l} \simeq \bigoplus_{e \in \{0, 1\}^l} Q_w^e.
$$

We call the component corresponding to $e$ the $e$-component of $(B_{\underline{z}})_Q$. As an $R$-bimodule, $B_{\underline{z}} = (R \otimes R) \otimes_R (R \otimes R) \otimes_R \cdots \otimes_R (R \otimes_R R) \otimes_R (R \otimes_R R) \otimes_R \cdots \otimes_R R)$. The $e$-component of $p_0 \otimes p_1 \otimes \cdots \otimes p_l \in R \otimes R \otimes R \cdots \otimes R \otimes R$ is

$$
\left( \prod_{i=1}^l s_i^{e_i} \cdots s_i^{e_i} \left( \frac{p_i}{\alpha_{s_i}} \right) \right) s_1^{e_1} \cdots s_l^{e_l} (p_l).
$$

We construct $\varphi : B_{\underline{z}} \to B_{\underline{z}}$ as follows. First we define $\varphi_Q : (B_{\underline{z}})_Q \simeq \bigoplus_{e \in \{0, 1\}^l} Q^e \to \bigoplus_{f \in \{0, 1\}^m} Q^f$ explicitly and we will prove that $\varphi_Q$ satisfies $\varphi_Q(B_{\underline{z}}) \subseteq B_{\underline{z}}$. The definition of $\varphi_Q$ is given in [EW20] 2.6. For $w = (s_1, \ldots, s_l) \in S^l$ and $e = (e_1, \ldots, e_l) \in \{0, 1\}^l$, we put $\zeta_{\underline{z}}(e) = \prod_{i=1}^l s_i^{e_i} \cdots s_i^{e_i} (\alpha_{s_i})$. Then set

$$
G^f_e = \begin{cases} 
\frac{\pi_{\underline{z}}}{\zeta_{\underline{z}}(f)} & (x^e = y^f), \\
0 & (x^e \neq y^f).
\end{cases}
$$

Now we define $\varphi_Q : \bigoplus_{e \in \{0, 1\}^m} Q^e \to \bigoplus_{f \in \{0, 1\}^l} Q^f$ by

$$
\varphi_Q((q_e)) = \left( \sum_{e \in \{0, 1\}^l} G^f_e q_e \right) f = \left( \frac{\pi_{\underline{z}}}{\zeta_{\underline{z}}(f)} \sum_{x^e = y^f} q_e \right) f.
$$

By the same way, we also define $\psi_Q : (B_{\underline{z}})_Q \to (B_{\underline{z}})_Q$. From the definition, we have

$$
\varphi_Q(b_{\underline{z}}) = \left( \frac{\pi_{\underline{z}}}{\zeta_{\underline{z}}(f)} \sum_{e \in \{0, 1\}^l} s_i^{e_1} \cdots s_i^{e_l} \left( \frac{1}{\alpha_{s_i}} \right) \right) f.
$$
Define \( r : W \to \tilde{W} \) as follows. If \( w \in W \) is not the longest element, \( r(w) = \tilde{s}_1 \ldots \tilde{s}_l \) where \( w = s_1 \ldots s_l \) is the reduced expression of \( w \). If \( w \) is the longest element then \( r(w) = \tilde{x} \). Then for \( e \in \{0, 1\}^{m_{s,t}}, x_e^r = g \) if and only if \( \tilde{x}_e^r = r(g) \). Therefore we have

\[
\varphi_Q(b_{\tilde{x}}^r) = \left( \frac{\pi_x}{\zeta_{\tilde{y}}(f)} a^2(r(y^f)) \right).
\]

**Proposition 3.10.** We have \( \varphi_Q(b_{\tilde{x}}^r) = b_{\tilde{x}} \) and \( \psi_Q(b_{\tilde{x}}^r) = b_{\tilde{x}} \) if and only if Assumption 3.5 holds.

**Proof.** Set \( \varepsilon(f) = 1 \) if \( \tilde{s}r(y^f) > r(y^f) \) and \( \varepsilon(f) = 0 \) otherwise. By Theorem 2.11 and Lemma 3.2, the f-component of \( \varphi_Q(b_{\tilde{x}}^r) \) is

\[
\frac{1}{\zeta_{\tilde{y}}(f)} \left[ \frac{m_{s,t} - 1}{2} \right]_{m_{s,t}-\ell(y^f)-\varepsilon(f)} \sigma_{m_{s,t}}(X) \left( \xi(f) \prod_{i=1}^{m_{s,t}} [m_{s,t} - 1]_{\sigma_i^{-1}(X)} \right)^{-1}.
\]

On the other hand, the f-component of \( b_{\tilde{x}}^r \) is \( 1/\zeta_{\tilde{y}}(f) \). Therefore \( \varphi_Q(b_{\tilde{x}}^r) = b_{\tilde{x}}^r \) if and only if

\[
(\varepsilon(f) \prod_{i=1}^{m_{s,t}} [m_{s,t} - 1]_{\sigma_i^{-1}(Z)}) = 1.
\]

for any \( f \in \{0, 1\}^l \) where \( Z = \sigma^{-1}(X) \). Here we used \( [m_{s,t} - 1]_{\sigma_{i}^{-1}(Z)} = [m_{s,t} - 1]_{\sigma_{i}^{-1}(Z)} \) which follows from Lemma 2.2. With \( Z = \sigma_{m_{s,t}}^{-1}(Y) \) we have another equation which is equivalent to \( \psi(b_{\tilde{x}}^r) = b_{\tilde{x}}^r \). Hence \( \varphi(b_{\tilde{x}}^r) = b_{\tilde{x}}^r \) and \( \psi(b_{\tilde{x}}^r) = b_{\tilde{x}}^r \) if and only if

\[
(3.1) \quad \left[ \frac{m_{s,t} - 1}{2} \right]_{m_{s,t}-\ell(y^f)-\varepsilon(f)} \prod_{i=1}^{m_{s,t}} [m_{s,t} - 1]_{\sigma_i^{-1}(Z)} = 1.
\]

We assume that \( \varphi_Q(b_{\tilde{x}}^r) = b_{\tilde{x}}^r \) and \( \psi_Q(b_{\tilde{x}}^r) = b_{\tilde{x}}^r \). Set \( f_k = (1^{m_{s,t}-k-1}, 0^{k+1}) \in \{0, 1\}^{m_{s,t}} \) for \( 0 \leq k \leq m_{s,t} - 1 \). Then \( \tilde{s}r(y^f_k) > r(y^f_k) \) and \( \ell(y^f_k) = m_{s,t} - k - 1 \). Take \( f = f_k \) in (3.1). Then we have \( [m_{s,t} - 1]_{\ell(y^f_k)} = m_{s,t} - k - 1 \). Let \( k = 0 \). Then \( \xi = 1 \). Therefore \( \psi \) is even-balanced. Hence for any \( 0 \leq k \leq m_{s,t} - 1 \), we have \( [m_{s,t} - 1]_{\ell(y^f_k)} = m_{s,t} - k - 1 \). Therefore we have Assumption 3.5. The converse implication is easy to prove.

For \( \tilde{w} = (\tilde{s}_1, \ldots, \tilde{s}_l) \in S_l^l \) and \( c = (c_1, \ldots, c_l) \in \{0, 1\}^l \), we define the sequence \( \tilde{w}(c) \) by removing \( i \)-th entry from \( \tilde{w} \) when \( c_i = 0 \). For \( u \in S \), we put \( D_u^{(0)} = 0 \) and \( D_u^{(1)} = u \).

**Lemma 3.11.** Let \( \tilde{w} = (\tilde{s}_1, \ldots, \tilde{s}_l) \in \tilde{S}_l^l \), \( \tilde{g} \in \tilde{W} \) and \( g \) the image of \( \tilde{g} \) in \( W \). For \( p_1, \ldots, p_l \in R \), we have

\[
\sum_{c \in \{0, 1\}^l} \sum_{c_1} a^{(c)}(\tilde{g})g(D^{(c_l)}(p_l)D^{(c_{l-1})}(\ldots(p_2D^{(c_1)}(p_1))\ldots)).
\]

**Proof.** We prove the lemma by induction on \( l = \ell(w) \). Set \( \tilde{w} = (\tilde{s}_1, \ldots, \tilde{s}_{l-1}) \) and \( p_{l-1}^{(c_l)} = D^{(c_l)}(p_{l-1}D^{(c_{l-2})}(\ldots(p_2D^{(c_1)}(p_1))\ldots)). \) The \( e_l = 0 \) part of the left hand side in the lemma is

\[
\sum_{c \in \{0, 1\}^{l-1}} \sum_{c_1} a^{(c)}(\tilde{g})g(D^{(c_l)}(p_l)D^{(c_{l-1})}(\ldots(p_2D^{(c_1)}(p_1))\ldots)).
\]

by inductive hypothesis and similarly the \( e_l = 1 \) part is

\[
-g^{(c_l)}(\tilde{s}_l p_l) g_{s_l}^{(c_l)} = -g^{(c_l)}(\tilde{s}_l p_l) g_{s_l}^{(c_l)}.
\]
We have
\begin{align*}
g \left( \frac{p_l}{\alpha_s} \right) a^{\varepsilon(c)}(\tilde{g}) g(p^{(c)}_{\tilde{E}}) - g \left( \frac{s_l(p_l)}{\alpha_s} \right) a^{\varepsilon(c)}(\tilde{g} \tilde{s}_l) g s_l(p^{(c)}_{\tilde{E}}) \\
= a^{\varepsilon(c)}(\tilde{g}) g \left( \frac{p_l s_l(p^{(c)}_{\tilde{E}})}{\alpha_s} \right) + a^{\varepsilon(c)}(\tilde{g} \tilde{s}_l) g s_l(p^{(c)}_{\tilde{E}}) \\
= a^{\varepsilon(c)}(\tilde{g}) g(\partial_s(p^{(c)}_{\tilde{E}})) + a^{\varepsilon(c),s_l}(\tilde{g}) g s_l(p^{(c)}_{\tilde{E}}) \quad \text{by Lemma 2.12} \\
= \sum_{d=0}^1 a^{\varepsilon(d,e)}(\tilde{g}) g(D^{(d)}_s(p^{(c)}_{\tilde{E}})).
\end{align*}

We get the lemma. \(\square\)

Therefore we get the following.

**Corollary 3.12.** Take \(s_1, \ldots, s_{m,s} \in S\) such that \(\tilde{g} = (s_1, \ldots, s_{m,s})\). For \(p_1, \ldots, p_{m,s} \in R\), \(\varphi_Q(p_1 \otimes p_2 \otimes \cdots \otimes p_{m,s} \otimes 1)\) is given by
\[
\left( \frac{\pi_{\tilde{E}}}{\zeta_{\tilde{E}}(f)} \right) \sum_{e \in [0,1]^{m,s}} a^{\varepsilon(e)}(r^{(f,s)}) a^{\varepsilon(s)}(D^{(e)}_{s,t} D^{(e,s)}_{s,t-1} (\cdots (p_2 D^{(1)}_{s_1}(p_1)) \cdots)) f.
\]

Hence to prove \(\varphi_Q(B_{\tilde{E}}) \subset B_{\tilde{E}}\), it is sufficient to prove that \((\pi_{\tilde{E}}/\zeta_{\tilde{E}}(f)) a^{\varepsilon(e)}(r^{(f,s)}) a^{\varepsilon(s)}(D^{(e,s)})(p^{(e,s)}(p)) f\) is in \(B_{\tilde{E}}\) for any \(p \in R\). To proceed the induction, we formulate as follows.

**Lemma 3.13.** Assume Assumption 3.5. Let \(p \in R\), \(\tilde{w} \in S^l\) and \(\tilde{w}^l \in S^{l'}\) such that \(l, l' \leq m,s\). We assume that \(l < m,s\) or \((\tilde{w}^l, \tilde{w}) = (\tilde{s}_l, \tilde{g})\). Then we have
\[
\left( \frac{\pi_{\tilde{E}}}{\zeta_{\tilde{E}}(f)} \right) a^{\varepsilon}(r^{(f,s)}(w^{(f,s)}(p)) f \in B_{\tilde{E}}.
\]

**Proof.** We prove the lemma by induction on \(l\). If \(l = 0\), then the lemma means \(\pi_{\tilde{E}} a^{\varepsilon}(1)p \in R\). This is Lemma 3.3.

Take \(\tilde{s}_1, \ldots, \tilde{s}_l \in S\) such that \(\tilde{w} = (\tilde{s}_1, \ldots, \tilde{s}_l)\). Put \(a(g) = a^{\varepsilon}(g)\) and \(\tilde{g} = (\tilde{s}_1, \ldots, \tilde{s}_l-1)\). Then by Lemma 3.8, it is sufficient to prove
\[
\left( \frac{\pi_{\tilde{E}}}{\zeta_{\tilde{E}}(f')} \right) a^{\varepsilon}(r^{(f',s)}(w^{(f',s)}(p)) \cdot \alpha_{s_l}) f' \in [0,1]^{l'-1} \in B_{\tilde{E}}
\]
and
\[
\left( \frac{\pi_{\tilde{E}}}{\zeta_{\tilde{E}}(f')} \right) a^{\varepsilon}(r^{(f',s)}(w^{(f',s)}(p)) - \frac{\pi_{\tilde{E}}}{\zeta_{\tilde{E}}(f')}) a^{\varepsilon}(r^{(f',s+1)}(w^{(f',s+1)}(p)) f' \in [0,1]^{l'-1} \in B_{\tilde{E}}.
\]
We have
\[
\left( \frac{\pi_{\tilde{E}}}{\zeta_{\tilde{E}}(f')} \right) a^{\varepsilon}(r^{(f',s)}(w^{(f',s)}(p)) \cdot \alpha_{s_l} = \frac{\pi_{\tilde{E}}}{\zeta_{\tilde{E}}(f')} a^{\varepsilon}(r^{(f',s)}(w^{(f',s)}(p)) \alpha_{s_l} = \frac{\pi_{\tilde{E}}}{\zeta_{\tilde{E}}(f')}(w^{(f',s)}(p)) f' \cdot \alpha_{s_l} = \frac{\pi_{\tilde{E}}}{\zeta_{\tilde{E}}(f')}(w^{(f',s)}(p)) f' \in [0,1]^{l'-1}
\]
which is in \(B_{\tilde{E}}\) by inductive hypothesis.
Put $g = \psi'$. Then $\omega^{(f',0)} = g$ and $\omega^{(f',1)} = gs_l$. Since $\zeta_\omega((f',0)) = \zeta_\omega((f',1)) = \psi'(\alpha_s)\zeta_\omega(f')$, the $f'$-component of the left hand side of (3.3) is

$$\frac{\pi_\omega}{\zeta_\omega(f')} \frac{1}{g(\alpha_s)} \left( a(r(g))g(p) - a(r(gs_l))gs_l(p) \right)$$

$$= -\frac{\pi_\omega}{\zeta_\omega(f')} \left( a(r(g))g\left(\frac{p-s_l(p)}{\alpha_{s_l}}\right) + \frac{a(r(g)) - a(r(gs_l))}{g(\alpha_{s_l})}gs_l(p) \right)$$

$$= -\frac{\pi_\omega}{\zeta_\omega(f')} \left( a(r(g))g(\partial_{s_l}(p)) + \frac{a(r(g)) - a(r(gs_l))}{g(\alpha_{s_l})}gs_l(p) \right).$$

We prove that

$$\left( \frac{\pi_\omega}{\zeta_\omega(f')} a(r((\psi'))_{\psi'}(\partial_{s_l}(p))) \right)_{f'}, \left( \frac{\pi_\omega}{\zeta_\omega(f')} a(r((\psi')) - a(r((\psi')_{s_l}))_{\psi'}(s_l(p))) \right)_{f'}$$

are in $B_\zeta$. The first one is in $B_\zeta$ by inductive hypothesis.

For the second, we divide into two cases.

- First assume that $l < m_{s,t}$. Then $\ell((\psi')) + \ell(s_l) < m_{s,t}$. Hence $r((\psi')_{s_l}) = \tilde{\psi'}_{s_l}$. Therefore by Lemma 3.4, we have $(a(r((\psi'))) - a(r((\psi')_{s_l})))_{\psi'}(\alpha_{s_l}) = a((\psi',\tilde{\psi'}_{s_l}))(r((\psi')))$. Therefore if $l' < m_{s,t}$ then the second one of (3.4) is in $B_\zeta$ by inductive hypothesis.

  If $l' = m_{s,t}$, we have $\ell((\tilde{\psi'},\tilde{s}_l)) = m_{s,t} + 1$. We also have $\ell((r((\psi')))) \leq \ell(\tilde{\psi'}) = l - 1 \leq m_{s,t} - 2 = \ell((\tilde{\psi'},\tilde{s}_l)) - 3$. Hence $\ell((\tilde{\psi'},\tilde{s}_l)) - \ell((r((\psi')))) \geq 3$. By Theorem 2.11 and Assumption 3.5, we have $a((\psi',\tilde{\psi'}_{s_l}))(r((\psi'))) = 0$. Hence the second one of (3.4) is zero which is in $B_\zeta$.

- Next assume that $l = m_{s,t}$. Then we have $\tilde{\psi'} = \tilde{x}$ and $\tilde{w} = \tilde{y}$. In this case we prove that $a((r((\psi')))) = a((r((\psi')_{s_l})))$.

  If $l' \neq (1, \ldots, 1)$, then the calculation in the case of $l < m_{s,t}$ is still valid. Hence $(a((r((\psi')))) - a((r((\psi')_{s_l}))))_{\psi'}(\alpha_{s_l}) = a((\psi',\tilde{\psi'}_{s_l}))(r((\psi')))$. We have $\ell((\tilde{x},\tilde{s}_l)) = m_{s,t} + 1$ and, since $f' \neq (1, \ldots, 1)$, we have $\ell((r((\psi')))) \leq m_{s,t} - 2$. Therefore $\ell((\tilde{x},\tilde{s}_l)) - \ell((r((\psi')))) \geq 3$. By Theorem 2.11 and Assumption 3.5, we have $a((\psi',\tilde{\psi'}_{s_l}))(r((\psi'))) = 0$.

  We assume that $l' = (1, \ldots, 1)$. By the definition, $r((\psi')_{s_l}) = \tilde{x}$. Hence $a((\tilde{x}) = 1/\pi_\omega$ by Theorem 2.11 and Lemma 3.2. We have $\ell((r((\psi')))) = m_{s,t} + 1 = \ell((\tilde{x}) - 1$. Therefore by Theorem 2.11 and Lemma 3.2, we have $a((\tilde{x}))(r((\psi'))) = 1/\pi_\omega$ as $\xi = 1$.

We finish the proof.

Theorem 3.9 is proved.

3.5. Relation with the diagrammatic Hecke category. Let $(W, S)$ be a general Coxeter system such that $\#S < \infty$ (we allow $\#S \neq 2$) and $(V, \{\alpha_u\}_{u \in S}, \{\alpha_u^\vee\}_{u \in S})$ a realization. We assume that for any $u_1, u_2 \in S$ ($u_1 \neq u_2$) such that the order $m_{u_1,u_2}$ of $u_1u_2$ is finite, we have $\frac{m_{u_1,u_2}}{k} Z = 0$ for any $Z \in \{X, Y\}$ and $1 \leq k \leq m_{u_1,u_2} - 1$. We can define the category $\mathcal{C}, \mathcal{C}_0$ by the same way as in 3.3. Let $BS$ be the full subcategory of $\mathcal{C}$ consisting of objects of a form $B_{s_1} \otimes \cdots \otimes B_{s_l}(n)$. If $u_1, u_2 \in S$, $u_1 \neq u_2$ satisfies $m_{u_1,u_2} \leq \infty$, then we put $B_{u_1,u_2} = B_{u_1} \otimes B_{u_2} \otimes \cdots$ and $B_{u_2,u_1} = B_{u_2} \otimes B_{u_1} \otimes \cdots$. By Theorem 3.9 there exists a homomorphism $\varphi_{u_1,u_2} : B_{u_1,u_2} \rightarrow B_{u_2,u_1}$ which sends $(1 \otimes 1) \otimes (1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1)$. Let $\mathcal{D}$ be the diagrammatic Hecke category defined by Elias-Williamson [EW16]. We also assume that $\mathcal{D}$ is “well-defined”, see [EW20 5.1].
We define a functor $\mathcal{F}: \mathcal{D} \to \mathcal{BS}$ as follows. For an object $(s_1, \ldots, s_l) \in \mathcal{D}$, we define $\mathcal{F}(s_1, \ldots, s_l) = B_{s_1} \otimes \cdots \otimes B_{s_l}$. We define $\mathcal{F}$ on morphisms by

$$
\mathcal{F}(\bullet) = (p \mapsto p \delta_u \otimes 1 - p \otimes u(\delta_u)),
$$

$$
\mathcal{F}(\uparrow) = (p_1 \otimes p_2 \mapsto p_1 p_2),
$$

$$
\mathcal{F}(\downarrow) = (p_1 \otimes p_2 \mapsto p_1 \otimes 1 \otimes p_2),
$$

$$
\mathcal{F}(\ast) = (p_1 \otimes p_2 \otimes p_3 \mapsto p_1 \partial_u(p_2) \otimes p_3),
$$

$$
\mathcal{F}(2m_{u_1, u_2}\text{-valent vertex}) = \varphi_{u_1, u_2}.
$$

for $u, u_1, u_2 \in S$ and $p, p_1, p_2, p_3 \in R$. Here we regard $B_u \otimes B_u = R \otimes_{R^u} R \otimes_{R^u} R (2)$ and $\delta_u \in V$ is an element satisfying $\langle \alpha^\vee_u, \delta_u \rangle = 1$.

**Lemma 3.14.** The functor $\mathcal{F}$ is well-defined.

**Proof.** In [EW20], a functor $\Lambda: \mathcal{D} \to \mathcal{C}_Q$ is defined and it is proved that $\Lambda$ is well-defined. By the construction, we have $\Lambda = (\cdot)_Q \circ \mathcal{F}$. Therefore $(\cdot)_Q \circ \mathcal{F}$ is well-defined and since $(\cdot)_Q: \mathcal{BS} \to \mathcal{C}_Q$ is faithful, $\mathcal{F}$ is also well-defined. □

**Theorem 3.15.** The functor $\mathcal{F}: \mathcal{D} \to \mathcal{BS}$ gives an equivalence of categories.

**Proof.** The proof is the same as that of the corresponding theorem in [Abe19]. It is obviously essentially surjective. In [EW16], for each object $M, N \in \mathcal{D}$, elements in $\text{Hom}_R(M, N)$ called double leaves are defined and proved that it is a basis of $\text{Hom}_D(M, N)$ [EW16 Theorem 6.12]. In [Abe19], the corresponding statement in $\mathcal{BS}$ is proved, namely double leaves in $\text{Hom}_C(\mathcal{F}(M), \mathcal{F}(N))$ are defined and proved that it is a basis. By the definition of $\mathcal{F}$, $\mathcal{F}$ sends double leaves to double leaves. Hence $\mathcal{F}$ gives an isomorphism between morphism spaces. □

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