AN INVERSE EIGENVALUE PROBLEM FOR AN ARBITRARY MULTIPLY CONNECTED BOUNDED REGION: AN EXTENSION TO HIGHER DIMENSIONS

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ABSTRACT. The basic problem in this paper is that of determining the geometry of an arbitrary multiply connected bounded region in $\mathbb{R}^3$ together with the mixed boundary conditions, from the complete knowledge of the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ for the negative Laplacian, using the asymptotic expansion of the spectral function $\Theta(t) = \sum_{j=1}^{\infty} \exp(-t \lambda_j)$ as $t \to 0$.

KEY WORDS AND PHRASES. Inverse problem, Laplace’s operator, eigenvalue problem and spectral function.

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1. INTRODUCTION.

The underlying problem is to deduce the precise shape of a membrane from the complete knowledge of the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ for the negative Laplacian $-\Delta_3 = -\sum_{i=1}^{3} \left( \frac{\partial}{\partial x_i} \right)^2$ in the $(x_1, x_2, x_3)$ - space.

Let $\Omega \subseteq \mathbb{R}^3$ be a simply connected bounded domain with a smooth bounding surface $S$. Consider the Dirichlet/Neumann problem

$$\begin{align*}
(\Delta_3 + \lambda)u &= 0 \text{ in } \Omega, \\
u &= 0 \text{ or } \frac{\partial u}{\partial n} &= 0 \text{ on } S,
\end{align*}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to $S$. Denote its eigenvalues, counted according to multiplicity, by

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \to \infty \text{ as } j \to \infty. \quad (1.3)$$

The problem of determining the geometry of $\Omega$ has been discussed by Pleijel [4], McKean and Singer [3], Waechter [5], Gottlieb [1], Hsu [2] and Zayed [6-8, 11], using the asymptotic expansion of the spectral function

$$\Theta(t) = \sum_{j=1}^{\infty} \exp(-t \lambda_j) \text{ as } t \to 0. \quad (1.4)$$

It has been shown that, in the case of Dirichlet boundary conditions (D.b.c)

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{|S|}{16\pi t^{1/2}} + \frac{1}{12\pi^{3/2}} \int_{\partial H} dS + o(1) \text{ as } t \to 0, \quad (1.5)$$

while, in the case of Neumann boundary conditions (N.b.c.),

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{|S|}{16\pi t^{1/2}} + \frac{1}{12\pi^{3/2}} \int_{\partial H} dS + o(1) \text{ as } t \to 0, \quad (1.6)$$

In these formulae, $V$ and $|S|$ are respectively the volume and the surface area of $\Omega$, while
$H = \frac{1}{2} \left( \frac{1}{R_1^2} + \frac{1}{R_2^2} \right)$ is the mean curvature of $S$, where $R_1$ and $R_2$ are the principal radii of curvature.

Furthermore, the constant term $a_0$ in (1.5) and (1.6) has the following forms:

\[
a_0 = \begin{cases} 
\frac{1}{2\pi} \int_S \left( \frac{1}{R_1^2} - \frac{1}{R_2^2} \right)^2 dS, & \text{in the case of D.b.c. (see [5])}, \\
\frac{7}{2\pi} \int_S \left( \frac{1}{R_1^2} - \frac{1}{R_2^2} \right)^2 dS, & \text{in the case of N.b.c. (see [2])}.
\end{cases}
\]  

(1.7)

In terms of the mean curvature $H$ and Gaussian curvature $N = \frac{1}{R_1 R_2}$, (1.7) may be rewritten in the forms:

\[
a_0 = \begin{cases} 
\frac{1}{12\pi} \int_S (H^2 - N) dS, & \text{in the case of D.b.c.}, \\
\frac{7}{12\pi} \int_S (H^2 - N) dS, & \text{in the case of N.b.c.}.
\end{cases}
\]  

(1.8)

The object of this paper is to discuss the following more general inverse problem: Let $\Omega$ be an arbitrary multiply connected bounded region in $\mathbb{R}^3$ which is surrounded internally by simply connected bounded domains $\Omega_i$, with smooth bounding surfaces $S_i$, $i = 1, 2, ..., m - 1$, and externally by a simply connected bounded domain $\Omega_m$ with a smooth bounding surface $S_m$. Suppose that the eigenvalues (1.3) are given for the eigenvalue equation

\[
(\Delta_3 + \lambda)u = 0 \text{ in } \Omega,
\]  

(1.9)

together with one of the following mixed boundary conditions:

\[
\frac{\partial u}{\partial n_i} = 0 \text{ on } S_i, \quad i = 1, ..., k, \quad u = 0 \text{ on } S_{i+k}, \quad i = k + 1, ..., m,
\]  

(1.10)

or

\[
u = 0 \text{ on } S_i, \quad i = 1, ..., k, \quad \frac{\partial u}{\partial n_i} = 0 \text{ on } S_{i+k}, \quad i = k + 1, ..., m,
\]  

(1.11)

where $\frac{\partial}{\partial n_i}$ denote differentiations along the inward pointing normals to $S_i$, $i = 1, ..., m$. Determine the geometry of $\Omega$ from the asymptotic form of the spectral function $\Theta(t)$ for small positive $t$.

Note that problem (1.9)-(1.11) has been investigated recently by Zayed [11] in the special case when $\Omega$ is an arbitrary doubly connected region (i.e., $m = 2$).

2. STATEMENT OF RESULTS.

Suppose that the bounding surfaces $S_i$ ($i = 1, ..., m$) of the region $\Omega$ are given locally by infinitely differentiable functions $x^n = \rho^n(\sigma_i), n = 1, 2, 3$, of the parameters $\sigma_i = \text{constants}$, are lines of curvature, the first and second fundamental forms of $S_i$ ($i = 1, ..., m$) can be written respectively in the following forms:

\[
\Pi_1(\sigma_i, \Delta \sigma_i) = a_{1i}(\sigma_i)(\Delta \sigma_i)^2 + b_{1i}(\sigma_i)(\Delta \sigma_i)^2,
\]  

and

\[
\Pi_2(\sigma_i, \Delta \sigma_i) = b_{2i}(\sigma_i)(\Delta \sigma_i)^2 + b_{2i}(\sigma_i)(\Delta \sigma_i)^2.
\]  

In terms of the coefficients $a_{1i}, b_{1i}, a_{2i}, b_{2i}$ the principal radii of curvatures for $S_i$ ($i = 1, ..., m$) are given by:

\[
R_{1i} = a_{1i}/b_{1i}, \quad \text{and} \quad R_{2i} = a_{2i}/b_{2i}.
\]

Consequently, the mean curvatures $H_i$ and Gaussian curvatures $N_i$ of the bounding surfaces $S_i$ ($i = 1, ..., m$) are defined by:

\[
H_i = \frac{1}{2} \left( \frac{1}{R_{1i}} + \frac{1}{R_{2i}} \right) \quad \text{and} \quad N_i = \frac{1}{R_{1i} R_{2i}}.
\]

Let $|S_i|$, ($i = 1, ..., m$) be the surface areas of the bounding surfaces $S_i$ ($i = 1, ..., m$) respectively. Then, the results of problem (1.9)-(1.11) can be summarized in the following cases:

CASE 1. (N.b.c. on $S_i$, $i = 1, ..., k$ and D.b.c. on $S_i$, $i = k + 1, ..., m$)

\[
\Theta(t) = -\frac{V}{(4\pi)^{3/2}} + \frac{1}{16\pi} \left\{ \sum_{i=1}^k \frac{|S_i|}{|S_i|} - \sum_{i=k+1}^m \frac{|S_i|}{|S_i|} \right\} + \frac{1}{12\pi^{3/2}} \sum_{i=1}^m \int_{S_i} H_i dS_i
\]

\[+\frac{7}{12\pi} \left\{ \sum_{i=1}^k \int_{S_i} (H_i^2 - N_i) dS_i + \sum_{i=k+1}^m \int_{S_i} (H_i^2 - N_i) dS_i \right\} \]
CASE 2. (D.b.c. on $S_i, i = 1, ..., k$ and N.b.c. on $S_j, i = k + 1, ..., m$)

In this case, the asymptotic expansion of $\Theta(t)$ as $t \to 0$ follows directly from (2.1) with the interchanges $S_i, i = 1, ..., k \to S_j, i = k + 1, ..., m$.

With reference to formulae (1.5), (1.6) and to the articles [1], [2], [7], [11], the asymptotic expansion (2.1) may be interpreted as follows:

(i) $\Omega$ is an arbitrary multiply connected bounded region in $R^3$ and we have the mixed boundary conditions (1.10) or (1.11) as indicated in the specifications of the two respective cases.

(ii) For the first five terms, $\Omega$ is an arbitrary multiply connected bounded region in $R^3$ of volume $V$.

In Case 1, the bounding surfaces $S_i, i = 1, ..., k$ are of surface areas $\sum_{i=1}^k |S_i|$, mean curvatures $H_i$ and Gaussian curvature $N_i$ together with Neumann boundary conditions, while the bounding surfaces $S_{i+}, i = k + 1, ..., m$ are of surface areas $\sum_{i=k+1}^m |S_i|$, mean curvatures $H_i$ and Gaussian curvature $N_i$ together with Dirichlet boundary conditions.

We close this section with the following remarks:

REMARK 2.1. On setting $\frac{\partial}{\partial t} = 0$ in (2.1) with the usual definition that $\frac{\partial}{\partial t} = 0$, we obtain the result of D.b.c. on $S_i, i = 1, ..., m$.

REMARK 2.2. On setting $k = m$ in (2.1) with the usual definition that $\frac{\partial}{\partial t} = 0$, we obtain the result of N.b.c. on $S_i, i = 1, ..., m$.

3. FORMULATION OF THE MATHEMATICAL PROBLEM.

In analogy with the two-dimensional problem (see [9, 10]), it is easy to show that $\Theta(t)$ associated with problem (1.9)-(1.11) is given by:

$$\Theta(t) = \int_{\Omega} \int G(\xi, \xi'; t) \, d\xi .$$

where $G(\xi, \xi'; t)$ is Green’s function for the heat equation

$$\left( \Delta_3 + \frac{\partial}{\partial t} \right) = 0,$$

subject to the mixed boundary conditions (1.10) or (1.11) and the initial condition

$$\lim_{t \to 0} G(\xi, \xi; t) = \delta(\xi - \xi'),$$

where $\delta(\xi - \xi')$ is the Dirac delta function located at the source point $\xi$.

Let us write

$$G(\xi, \xi'; t) = G_0(\xi, \xi'; t) + x(\xi, \xi; t),$$

where

$$G_0(\xi, \xi'; t) = (4\pi t)^{-3/2} \exp\left\{ - \frac{|\xi - \xi'|^2}{4t} \right\}.$$

is the “fundamental solution” of the heat equation (3.2) while $x(\xi, \xi'; t)$ is the “regular solution” chosen so that $G(\xi, \xi'; t)$ satisfies the mixed boundary conditions (1.10) or (1.11).

On setting $\xi = \xi_2 = \xi$, we find that

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + K(t).$$

where

$$K(t) = \int_{\Omega} \int x(\xi, \xi'; t) \, d\xi .$$

In what follows, we shall use Laplace transforms with respect to $t$, and use $s^2$ as the Laplace transform parameter; thus we define

$$\tilde{G}(\xi, \xi; s^2) = \int_0^\infty e^{-s^2 t} G(\xi, \xi; t) \, dt .$$
An application of the Laplace transform to the heat equation (3.2) shows that \( \tilde{G}(\xi, \xi^*; s^2) \) satisfies the membrane equation
\[
(\Delta - s^2)\tilde{G}(\xi, \xi^*; s^2) = -\delta(\xi_1 - \xi_2) \quad \text{in } \Omega.
\]
(3.9)
together with the mixed boundary conditions (1.10) or (1.11).

The asymptotic expansion of \( K(t) \) as \( t \to 0 \), may then be deduced directly from the asymptotic expansion of \( \tilde{K}(s^2) \) as \( s \to \infty \), where
\[
\tilde{K}(s^2) = \int \int \int \chi(\xi, \xi^*; s^2) d\xi.
\]
(3.10)

4. CONSTRUCTION OF GREEN'S FUNCTION.

It is well known [7] that the membrane equation (3.9) has the fundamental solution
\[
\tilde{G}_0(\xi, \xi^*; s^2) = \frac{\exp(-sr_{\xi_1, \xi_2})}{4\pi r_{\xi_1, \xi_2}}
\]
where \( r_{\xi_1, \xi_2} = |\xi_1 - \xi_2| \) is the distance between the points \( \xi_1 = (\xi_1^1, \xi_1^2, \xi_1^3) \) and \( \xi_2 = (\xi_2^1, \xi_2^2, \xi_2^3) \) of the domain \( \Omega \). The existence of the solution (4.1) enables us to construct integral equations for \( \tilde{G}(\xi, \xi^*; s^2) \) satisfying the mixed boundary conditions (1.10) or (1.11). Therefore, in Case 1, Green's theorem gives:
\[
\tilde{G}(\xi, \xi^*; s^2) = \frac{\exp(-sr_{\xi_1, \xi_2})}{4\pi r_{\xi_1, \xi_2}} + \frac{1}{2\pi} \sum_{i=1}^{k} \int_{S_i} \tilde{G}(\xi, \xi^*; s^2) \frac{\partial}{\partial n_{\xi^*}} \left[ \frac{\exp(-sr_{\xi_1, \xi_2})}{r_{\xi_1, \xi_2}} \right] dy.
\]
(4.2)

On applying the iteration method (see [7], [9], [11]) to the integral equation (4.2), we obtain the Green's function \( \tilde{G}(\xi, \xi^*; s^2) \) which has the regular part:
\[
\chi(\xi, \xi^*; s^2) = \frac{1}{8\pi^2} \sum_{i=1}^{k} \int_{S_i} \frac{\exp(-sr_{\xi_1, \xi_2})}{r_{\xi_1, \xi_2}} \frac{\partial}{\partial n_{\xi^*}} \left[ \frac{\exp(-sr_{\xi_1, \xi_2})}{r_{\xi_1, \xi_2}} \right] dy.
\]

(4.3)
where
\begin{align}
M_i(y', y) &= \sum_{\nu = 0}^{\infty} K^{(\nu)}_i(y', y), \\
M'_i(y', y) &= \sum_{\nu = 0}^{\infty} K'_i(y', y), \\
L_i(y, y') &= \sum_{\nu = 0}^{\infty} L^{(\nu)}_i(y, y'), \\
L'_i(y, y') &= \sum_{\nu = 0}^{\infty} L'_i(y, y'), \\
K_{i}(y', y) &= \frac{1}{2\pi} \frac{\partial}{\partial n_y} \left[ \frac{\exp(-s r)}{r^{2}} \right], \\
K'_i(y', y) &= \frac{1}{2\pi} \frac{\partial}{\partial n_y} \left[ \frac{\exp(-s r)}{r^{2}} \right], \\
K_{-i}(y', y) &= \frac{1}{2\pi} \frac{\partial^2}{\partial n_y^2} \left[ \frac{\exp(-s r)}{r^{2}} \right], \\
K'_{-i}(y', y) &= \frac{1}{2\pi} \frac{\partial^2}{\partial n_y^2} \left[ \frac{\exp(-s r)}{r^{2}} \right].
\end{align}

In the same way, we can show that in Case 2, the Green's function \( G(\xi_1, \xi_2; z^2) \) has a regular part of the same form (4.3) with the interchanges \( S_i, i = 1, \ldots, k \rightleftharpoons S_{i'}, i = k + 1, \ldots, m \).

On the basis of (4.3) the function \( \chi(\xi_1, \xi_2; z^2) \) will be estimated for \( s \to \infty \). The case when \( \xi_1 \) and \( \xi_2 \) lie in the neighborhood of the bounding surfaces \( S_i, i = 1, \ldots, m \) of \( \Omega \) is particularly interesting. For this case, we need to use the following coordinates.

5. COORDINATES IN THE NEIGHBORHOOD OF \( S_i, i = 1, \ldots, m \).

Let \( h > 0(i = 1, \ldots, m) \) be sufficiently small. Let \( n_i(i = 1, \ldots, m) \) be the minimum distances from a point \( \xi = (x^1, x^2, x^3) \) of the domain \( \Omega \) to its bounding surfaces \( S_i(i = 1, \ldots, m) \) respectively. Let \( n_i(\sigma_i)(i = 1, \ldots, m) \) denote the inward drawn unit normals to \( S_i(i = 1, \ldots, m) \) respectively. We note that the coordinates in the neighborhood of \( S_i(i = k + 1, \ldots, m) \) are in the same form as in Section 5.1 of [11] with the interchanges \( \sigma^1 \rightleftharpoons \sigma^1, \sigma^2 \rightleftharpoons \sigma^2, n_2 \rightleftharpoons n_1, h_2 \rightleftharpoons h_1, I_2 \rightleftharpoons I_1, \mathcal{N}(I_2) \rightleftharpoons \mathcal{N}(I_1) \) and \( \delta_2 \rightleftharpoons \delta_1, (i = k + 1, \ldots, m) \). Thus we have the same formulae (5.1.1)-(5.1.6) of Section 5.1 in [11] with the interchanges \( n_2 \rightleftharpoons n_1, n_2(\sigma_2) \rightleftharpoons n_1(\sigma_1), I_1 \rightleftharpoons I_1, I_2 \rightleftharpoons I_2, H_1 \rightleftharpoons H_1, N_1 \rightleftharpoons N_1, (i = k + 1, \ldots, m) \).

Similarly, the coordinates in the neighborhood of \( S_i(i = 1, \ldots, k) \) are similar to those obtained in Section 5.2 of [11] with the interchanges \( \sigma^1 \rightleftharpoons \sigma^1, \sigma^2 \rightleftharpoons \sigma^2, n_1 \rightleftharpoons n_1, h_1 \rightleftharpoons h_1, I_1 \rightleftharpoons I_1, \mathcal{N}(I_1) \rightleftharpoons \mathcal{N}(I_1) \) and \( \delta_1 \rightleftharpoons \delta_1, (i = 1, \ldots, k) \). Thus, we have the same formulae (5.2.1)-(5.2.5) of Section 5.1 in [11] with the interchanges \( n_2 \rightleftharpoons n_1, n_2(\sigma_2) \rightleftharpoons n_1(\sigma_1), I_1 \rightleftharpoons I_1, I_2 \rightleftharpoons I_{2'}, H_1 \rightleftharpoons H_1, N_1 \rightleftharpoons N_1, (i = k + 1, \ldots, m) \).

6. SOME LOCAL EXPANSIONS.

It now follows that the local expansions of the functions
\[
\frac{\exp(-s r \xi \cdot y)}{r^{2}} \frac{\partial}{\partial n_y} \left[ \frac{\exp(-s r \xi \cdot y)}{r^{2}} \right], \quad i = 1, \ldots, m.
\]
when the distance between \( x \) and \( y \) is small are very similar to those obtained in Section 6 of [11]. Consequently, the local behavior of the kernels

\[
K_i(y', y) \ast K_{-i}(y', y'), \quad (6.2)
\]

and

\[
\ast K_i(y', y) \ast K_{-i}(y', y'), \quad (6.3)
\]

when the distance between \( y \) and \( y' \) is small, follows directly from the local expansions of the functions (6.1).

**DEFINITION 1.** If \( \xi_1 \) and \( \xi_2 \) are points in the half-part \( \xi^3 > 0 \), then we define

\[
\bar{\rho}_{12} = \sqrt{(\xi_1^1 - \xi_2^1)^2 + (\xi_1^2 - \xi_2^2)^2 + (\xi_1^3 + \xi_2^3)^2}.
\]

An \( e^{\lambda}(\xi_1, \xi_2; s) \)-function is defined for points \( \xi_1 \) and \( \xi_2 \) belong to sufficiently small domains \( \Psi (I_i) \) except when \( \xi_1 = \xi_2 \) \( \in I_i (i = 1, \ldots, m) \) and \( \lambda \) is called the degree of this function. For every positive integer \( \Lambda \), it has the local expansion (see [11]):

\[
e^{\lambda}(\xi_1, \xi_2; s) = \Sigma^* f(\xi_1^1, \xi_2^1)(\xi_1^2)^{P_2} \left( \frac{\partial}{\partial \xi_1^1} \right)^{\ell_1_1} \left( \frac{\partial}{\partial \xi_1^2} \right)^{\ell_1_2} \frac{\exp(-\sigma \bar{\rho}_{12})}{\bar{\rho}_{12}} + R^\Lambda(\xi_1, \xi_2; s),
\]

where \( \Sigma^* \) denotes a sum of a finite number of terms in which \( f(\xi_1^1, \xi_2^1) \) are infinitely differentiable functions. In this expansion \( P_1, P_2, \ell_1, \ell_2, \ell_3 \) are integers, where \( P_1 \geq 0, P_2 \geq 0, \ell_1 \geq 0, \ell_2 \geq 0, \ell_3 \geq 0, \lambda = \min (P_1 + P_2 - q), q = \ell_1 + \ell_2 + \ell_3 \) and the minimum is taken over all terms which occur in the summation \( \Sigma^* \). The remainder \( R^\Lambda(\xi_1, \xi_2; s) \) has continuous derivatives of order \( d \leq \Lambda \) satisfying

\[
D^d R^\Lambda(\xi_1, \xi_2; s) = 0 \left[ e^{-\Lambda \exp (-\sigma \bar{\rho}_{12})} \right] \quad \text{as} \quad s \to \infty,
\]

where \( A \) is a positive constant.

Thus, using methods similar to those obtained in Section 7 of [11], we can show that the functions (6.1) are \( e^{\lambda} \)-functions with degrees \( \lambda = -1, -2 \) respectively. Consequently, the functions (6.2) are \( e^{\lambda} \)-functions with degrees \( \lambda = 0, -1 \) while the functions (6.3) are \( e^{\lambda} \)-functions with degrees \( \lambda = 0, 1 \) respectively.

**DEFINITION 2.** If \( \xi_1 \) and \( \xi_2 \) are points in large domains \( \Omega + S_i \), then we define

\[
\bar{\rho}_{12} = \min_{\xi} \left( r_1^1 + r_2^1 \right) \quad \text{if} \quad \xi \in S_i, i = 1, \ldots, k,
\]

and

\[
\bar{R}_{12} = \min_{\xi} \left( r_1^1 + r_2^1 \right) \quad \text{if} \quad \xi \in S_i, i = k + 1, \ldots, m.
\]

An \( E^\Lambda(\xi_1, \xi_2; s) \)-function is defined and infinitely differentiable with respect to \( \xi_1 \) and \( \xi_2 \) when these points belong to large domains \( \Omega + S_i \) except when \( \xi_1 = \xi_2 \) \( \in S_i, i = 1, \ldots, m \). Thus, the \( E^\Lambda \)-function has a similar local expansion of the \( e^{\lambda} \)-function (see [7], [11]).

With the help of Section 8 in [11], it is easily seen that formula (4.3) is an \( E^{-2}(\xi_1, \xi_2; s) \)-function and consequently

\[
\bar{G}(\xi_1, \xi_2; s^2) = \sum_{i = 1}^{k} 0 \left( \bar{r}_{12}^{2 \exp (-A_s \bar{\rho}_{12})} \right) + \sum_{i = k + 1}^{m} 0 \left( \bar{R}_{12}^{2 \exp (-A_s \bar{R}_{12})} \right), \quad (6.4)
\]

which is valid for \( s \to \infty \), where \( A_s(i = 1, \ldots, m) \) are positive constants. Formula (6.4) shows that \( \bar{G}(\xi_1, \xi_2; s^2) \) is exponentially small for \( s \to \infty \).

With reference to Sections 7 and 9 in [11], if the \( e^{\lambda} \)-expansions of the functions (6.1)-(6.3) are introduced into (4.3) and if we use formulae similar to (7.4) and (7.10) of Section 7 in [11], we obtain the following local behavior of \( \chi(\xi_1, \xi_2; s^2) \) as \( s \to \infty \) which is valid when \( \bar{r}_{12} \) and \( \bar{R}_{12} \) are small:
where, if $\mathcal{E}_1$ and $\mathcal{E}_2$ belong to sufficiently small domains $\mathcal{D}(I_i)$, $i = 1, \ldots, m$, then
\[
\bar{\chi}_{1}(\mathcal{E}_1, \mathcal{E}_2; s^2) = \sum_{i = 1}^{m} \bar{\chi}_i(\mathcal{E}_1, \mathcal{E}_2; s^2),
\] (6.5)

where $\mathcal{D}(I_i)$ are sufficiently small domains. Then we have
\[
\bar{\chi}_{1}(\mathcal{E}_1, \mathcal{E}_2; s^2) = \frac{\exp(-s\tilde{\rho}_{12})}{8\pi\tilde{\rho}_{12}} + O\left(\frac{\exp(-A_s\tilde{\rho}_{12})}{\tilde{\rho}_{12}}\right) \text{ as } s \to \infty.
\] (6.6)

When $\tilde{\rho}_{12} \geq \rho_i > 0$, and $\tilde{R}_{12} \geq \rho_i > 0$, the function $\bar{\chi}_{1}(\mathcal{E}_1, \mathcal{E}_2; s^2)$ is of order $O(s^N)$ as $s \to \infty$, $N > 0$. Thus, since $\lim_{\tilde{\rho}_{12} \to 0} \frac{\tilde{R}_{12}}{\tilde{\rho}_{12}} = \lim_{\tilde{\rho}_{12} \to 0} \frac{\tilde{R}_{12}}{\tilde{\rho}_{12}} = 1$ (see [11]), then the local behavior of the formula (4.3) has the form (6.5), where if $\mathcal{E}_1$ and $\mathcal{E}_2$ belong to large domains $\Omega + \mathcal{S}_i$, $i = 1, \ldots, k$, we get
\[
\bar{\chi}_{1}(\mathcal{E}_1, \mathcal{E}_2; s^2) = \frac{\exp(-s\tilde{\rho}_{12})}{8\pi\tilde{\rho}_{12}} + O\left(\frac{\exp(-A_s\tilde{\rho}_{12})}{\tilde{\rho}_{12}}\right) \text{ as } s \to \infty.
\] (6.7)

while, if $\mathcal{E}_1$ and $\mathcal{E}_2$ belong to large domains $\Omega + \mathcal{S}_i$, $i = k + 1, \ldots, m$, we get:
\[
\bar{\chi}_{1}(\mathcal{E}_1, \mathcal{E}_2; s^2) = \frac{\exp(-s\tilde{\rho}_{12})}{8\pi\tilde{\rho}_{12}} + O\left(\frac{\exp(-A_s\tilde{\rho}_{12})}{\tilde{\rho}_{12}}\right) \text{ as } s \to \infty.
\] (6.8)

7. CONSTRUCTION OF RESULTS.

Since for $\varepsilon^2 > h_i > 0$, the functions $\bar{\chi}_{1}(\mathcal{E}_1, \mathcal{E}_2; s^2)$ are of orders $O(e^{-\varepsilon A_s})$, the integral over $\Omega$ of the function $\bar{\chi}_{1}(\mathcal{E}_1, \mathcal{E}_2; s^2)$ can be approximated in the following way (see (3.10)):

\[
\bar{K}(s^2) = \sum_{i = 1}^{m} \int_{S_i} \bar{\chi}_{1}(\mathcal{E}_1, \mathcal{E}_2; s^2)(1 - 2\varepsilon^2/\rho_{12}) \left(\frac{dH_i}{dS} + (\varepsilon^2)^2 N_i dS_s\right) \int_{S_i} \bar{\chi}_{1}(\mathcal{E}_1, \mathcal{E}_2; s^2)(1 + 2\varepsilon^2/\rho_{12}) \left(\frac{dH_i}{dS} + (\varepsilon^2)^2 N_i dS_s\right).
\] (7.1)

If the $e^s$-expansions of $\bar{\chi}_{1}(\mathcal{E}_1, \mathcal{E}_2; s^2)$ are introduced into (7.1) and with the help of formula (10.2) of Section 10 in [11], we deduce after inverting Laplace transforms, that
\[
K(t) = \frac{a_1}{t} + \frac{a_2}{t^{1/2}} + a_3 + a_4t^{1/2} + O(t) \text{ as } t \to 0,
\] (7.2)

where
\[
a_1 = \frac{1}{16\pi} \left\{ \sum_{i = 1}^{m} \left| S_i \right| - \sum_{i = k + 1}^{m} \left| S_i \right| \right\},
a_2 = \frac{1}{12\pi^{3/2}} \sum_{i = 1}^{m} \int_{S_i} H_i dS_s,
\]

\[
a_3 = \frac{1}{128\pi} \left\{ 7 \sum_{i = 1}^{k} \int_{S_i} (H_i^2 - N_i) dS_s + \sum_{i = k + 1}^{m} \int_{S_i} (H_i^2 - N_i) dS_s \right\},
\]

and
\[
a_4 = \frac{1}{3\pi^{3/2}} \left\{ \frac{13}{1440} \sum_{i = 1}^{k} \int_{S_i} H_i^2 dS_s - \frac{1}{315} \sum_{i = k + 1}^{m} \int_{S_i} H_i^2 dS_s \right\}.
\]

On inserting (7.2) into (3.6) we arrive at our result (2.1).
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