A Geometric View of the Sieve of Eratosthenes

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Abstract

We study the geometry of the Sieve of Eratosthenes. We introduce some concepts as Focals and Extremes. We find a symmetry in the distribution of the Focals (all the information about the primes is contained into a small set of numbers). We find that there is a geometric order in the Sieve and we give a formula for the greatest remainder that returns the same quotient.
Notation

\( \text{rem} \left( \frac{p}{q} \right) \): the remainder of \( \frac{p}{q} \);

\( \lfloor \frac{p}{q} \rfloor \): the floor function, the greatest integer value of \( \frac{p}{q} \);
The Sieve

**Definition 1.** (a number on the plane)

Let be an application from the set of the first \( p^2 \) positive integers (p is a prime number) to the plane:

\[
f_p(n) = \begin{cases} 
(\text{rem}\left(\frac{n}{p}\right), -\left\lfloor \frac{n}{p} \right\rfloor), & \text{if } n \neq kp \\
(p, -\left\lfloor \frac{n}{p} \right\rfloor + 1), & \text{if } n = kp 
\end{cases}
\]  

(0.1)

We will often make no distinction between the image of \( n \) and \( n \).

**Definition 2.** (focal line)

A focal line of order \((k, \text{rem}\left(\frac{p}{a}\right))\), \(1 < a < p\), \(1 \leq k \leq p\), is a line of the form:

\[y = \text{rem}^{-1}\left(\frac{p}{a}\right)(x - ka)\]  

(0.2)

We also define a unique vertical line \( x = p \). By definition, this is the zeroth-order focal line.

**Theorem 3.** (intersection of an integer number with a focal line)

If a focal line of order \((k, \text{rem}\left(\frac{p}{a}\right))\) intersects an integer \( n \), \( p < n < p^2 \), then \( n \) is a multiple of \( a \). The reciprocal is also true.

**Proof.** The trivial case when \( n \) is a multiple of \( p \) it stays on the zeroth-order focal line (see Definition 1 and 2). Let now prove the theorem for nontrivial cases:

i. \( \Rightarrow \)

Let there be a number \( n \) intersected by a focal line. It follows from (0.2) that

\[-\left\lfloor \frac{n}{p} \right\rfloor = \text{rem}^{-1}\left(\frac{p}{a}\right)\left(\text{rem}\left(\frac{n}{p}\right) - ka\right)\]

\[-\left\lfloor \frac{n}{p} \right\rfloor \text{rem}\left(\frac{p}{a}\right) = \text{rem}\left(\frac{n}{p}\right) - ka\]

\[-\left\lfloor \frac{n}{p} \right\rfloor (p - a \left\lfloor \frac{p}{a} \right\rfloor) = n - p \left\lfloor \frac{n}{p} \right\rfloor - ka\]

\[n = a\left(\left\lfloor \frac{n}{p} \right\rfloor \left\lfloor \frac{p}{a} \right\rfloor + k\right)\]  

(0.3)

ii. \( \Leftarrow \)

Let there be an integer multiple of \( a \), \( ma \), expressed by its image:

\[
\left(\text{rem}\left(\frac{ma}{p}\right), -\left\lfloor \frac{ma}{p} \right\rfloor\right)
\]

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Let prove that it satisfies equation (0.2):

\[- \left\lfloor \frac{ma}{p} \right\rfloor = \frac{1}{p} \text{rem} \left( \frac{ma}{p} \right) - \frac{ma}{p} = \frac{1}{p} \text{rem} \left( \frac{p}{a} \right) \left( \frac{1}{p} \text{rem} \left( \frac{ma}{p} \right) - \frac{ma}{p} \right) =
\]

\[
= \frac{1}{\text{rem} \left( \frac{p}{a} \right)} \left( \text{rem} \left( \frac{ma}{p} \right) - ma - a \left\lfloor \frac{p}{a} \right\rfloor \text{rem} \left( \frac{ma}{p} \right) + \frac{ma^2}{p} \left\lfloor \frac{p}{a} \right\rfloor \right) =
\]

\[
= \frac{\text{rem} \left( \frac{ma}{p} \right)}{\text{rem} \left( \frac{p}{a} \right)} - \frac{a}{\text{rem} \left( \frac{p}{a} \right)} \left( m + \frac{p}{a} \left\lfloor \frac{ma}{p} \right\rfloor \right) \left( \text{rem} \left( \frac{ma}{p} \right) - ma \right) =
\]

\[
= \frac{\text{rem} \left( \frac{ma}{p} \right)}{\text{rem} \left( \frac{p}{a} \right)} - \frac{a}{\text{rem} \left( \frac{p}{a} \right)} \left( m - \left\lfloor \frac{p}{a} \right\rfloor \left\lfloor \frac{ma}{p} \right\rfloor \right) = \text{rem}^{-1} \left( \frac{p}{a} \right) \left( \text{rem} \left( \frac{ma}{p} \right) - ka \right).
\]

**Definition 4.** (family of focal lines)

A family of focal lines of order \((k, \left\lfloor \frac{p}{a} \right\rfloor)\), \(y_{\left\lfloor \frac{p}{a} \right\rfloor}^k\), is a set of focal lines that correspond to the same value of the pair \((k, \left\lfloor \frac{p}{a} \right\rfloor)\). \(k\) is the down-order of the family and \(\left\lfloor \frac{p}{a} \right\rfloor\) is the up-order of the family.

**Theorem 5.** Lines in the same family intersect at the same point called focal point of order \((k, \left\lfloor \frac{p}{a} \right\rfloor)\).

\[F_{\left\lfloor \frac{p}{a} \right\rfloor} = \bigcap_{\text{same } k, \left\lfloor \frac{p}{a} \right\rfloor} \left( y_{\left\lfloor \frac{p}{a} \right\rfloor}^k \right) = k \left\lfloor \frac{p}{a} \right\rfloor^{-1} (p, 1) \quad (0.4)\]

**Proof.** Let there be two focal lines corresponding to the same values of \(\left\lfloor \frac{p}{a} \right\rfloor\) and \(k\). We prove that the coordinates of their intersection point only depend on \(\left\lfloor \frac{p}{a} \right\rfloor\) and \(k\).

i. x coordinate:

\[\text{rem}^{-1} \left( \frac{p}{a_1} \right) (x - ka_1) = \text{rem}^{-1} \left( \frac{p}{a_2} \right) (x - ka_2)\]
rem \left( \frac{p}{a_2} \right)(x - ka_1) = rem \left( \frac{p}{a_1} \right)(x - ka_2)

x = \frac{kp}{\left\lfloor \frac{p}{a} \right\rfloor} \quad (0.5)

ii. y coordinate; taking any focal line:

y = \frac{k}{\left\lfloor \frac{p}{a} \right\rfloor} \quad (0.6)

Corollary 6. The focal points stay on the same line called multiplicative axis:

r_F \equiv p^{-1}x \quad (0.7)

It can be easily deduced from (0.5) and (0.6).

Corollary 7. Any focal line has a parallel focal line contained into a family with different down order but the same up order; their corresponding remainders are the same.

It follows from (0.2) and (0.4).

Corollary 8. Let be all the focal points with the same up order \( \{k_i, \left\lfloor \frac{p}{a_1} \right\rfloor = \left\lfloor \frac{p}{a_2} \right\rfloor = ... = \left\lfloor \frac{p}{a} \right\rfloor \} \).

They are distributed as it follows:

\[ F_{k_i}^{\frac{x}{p}} = k_i \left( \frac{p}{x} \cdot \frac{1}{\left\lfloor \frac{p}{a} \right\rfloor} \right) = k_i F_1^{\frac{x}{p}} \quad (0.8) \]

So, once we know the distribution of all the 1st-down-order focals, the others are multiple of these.

Theorem 9. (the Sieve of Eratosthenes)

The complementary of the union of the intersections of first \( p^2 \) numbers with any focal line is the set of the prime numbers greater than \( p \) and less than \( p^2 \).

\[ \bigcup \left( y_k^{\left\lfloor \frac{p}{a} \right\rfloor} \cap f_p(n) \right) = \{q_i | p < q_i < p^2 \text{ and } q_i \text{ is prime}\} \quad (0.9) \]

We don’t prove this theorem since it is well known \[ \blacksquare \].
Figure 0.1: The sieve for $p=101$.

Figure 0.2: Detail on the sieve for $p=101$. Uncrossed integers are the primes between $p$ and $p^2$. 
The distribution of the focals

As we have seen in Corollary 8, once we know the distribution of the focals of 1st-down-order, we know the distribution of all focals, by multiplication. So, we will only study the distribution of the 1st-order focals. The 1st-down-order focals are points of coordinates \( \left( \left\lfloor \frac{p}{a} \right\rfloor, \frac{1}{\left\lfloor \frac{p}{a} \right\rfloor} \right) \). So, given a \( p \), we only have to study the distribution of \( \left\lfloor \frac{p}{a} \right\rfloor \).

**Definition 10.** (extreme)

Let there be a prime number \( p \) and \( 1 < a < p \). A point \( \left( a, \left\lfloor \frac{p}{a} \right\rfloor \right) \) is said to be an extreme of order \( \left( a, \left\lfloor \frac{p}{a} \right\rfloor \right) \), if \( \left\lfloor \frac{p}{a} + 1 \right\rfloor < \left\lfloor \frac{p}{a} \right\rfloor \).

**Theorem 11.** \( E_{a, \left\lfloor \frac{p}{a} \right\rfloor} \) and \( E_{a, \left\lfloor \frac{p}{a} \right\rfloor}^{a} \) are symmetrical with respect to the principal bisector.

Now, since all the information about any \( \left( a, \left\lfloor \frac{p}{a} \right\rfloor \right) \) is contained in its extremes, this theorem says that we only have to study the distribution of half of the extremes, let say those up the principal bisector. The exception is the case when \( a > \left\lfloor \frac{p}{a} \right\rfloor \), but in this case \( \left\lfloor \frac{p}{a} \right\rfloor = 1 \).

**Theorem 12.** Any extreme situated on or up the principal bisector must satisfy that \( a \leq \left\lfloor \sqrt{p} \right\rfloor \).

**Proof.** Let be an extreme, \( \left( a, \left\lfloor \frac{p}{a} \right\rfloor \right) \), it is on or up the principal bisector if \( \left\lfloor \frac{p}{a} \right\rfloor \geq a \). So \( a \leq \left\lfloor \sqrt{p} \right\rfloor \). \( \square \)

**Remark 13.** It is interesting to remark that all \( \left( a, \left\lfloor \frac{p}{a} \right\rfloor \right) \) (not only the extremes, but all) are the integer points contained between the curves \( y = \frac{p}{a} \) and \( y = \frac{p}{a} - 1 \), for \( 1 < x < p \) (see figure 3).

**Conjecture 14.** It it impossible to find any exact simplification for \( \left\lfloor \frac{p}{a} \right\rfloor \) when \( a \leq \left\lfloor \sqrt{p} \right\rfloor \).
Figure 0.3: The distribution of $\lfloor \frac{p}{a} \rfloor$ for $p=11$. $\lfloor \frac{11}{11} \rfloor$ are the blue and green points while the extremes are only the green points. We can see that they are the integer points contained by the zone limited at the top by $\frac{p}{a}$ and down by $\frac{p}{a} - 1$. We can also see that all of them are contained by the 'reflective' line between the curves (in red those horizontal lines that contain integers). We can also see that the extremes are symmetrics with respect the principal bisector.
The distribution of the focal lines

Now, let suppose that the distribution of the focal points is known. The focal lines that pass through $F_1 \left\lfloor \frac{p}{a} \right\rfloor$ have slope $\text{rem}^{-1} \left( \frac{p}{a} \right)$. So, given a quotient, we only have to worry about the remainders that return the same quotient.

**Proposition 15. (maximum remainder corresponding to the same quotient)**

Given a $p$ and $\left\lfloor \frac{p}{a} \right\rfloor$ the maximum remainder that corresponds to $\left\lfloor \frac{p}{a} \right\rfloor$ is:

$$\max \left[ \text{rem} \left( \frac{p}{a} \right) \right] = p - \left\lfloor \frac{p}{a} \right\rfloor \left( \left\lfloor \frac{p}{a} \right\rfloor + 1 \right). \quad (0.10)$$

**Proof.** The maximum remainder is:

$$\max \left[ \text{rem} \left( \frac{p}{a} \right) \right] = \max \left[ p - \left\lfloor \frac{p}{a} \right\rfloor a \right] = p - \min \left[ \left\lfloor \frac{p}{a} \right\rfloor a \right], \quad (0.11)$$

and as $\left\lfloor \frac{p}{a} \right\rfloor$ is a constant:

$$\max \left[ \text{rem} \left( \frac{p}{a} \right) \right] = p - \left\lfloor \frac{p}{a} \right\rfloor \min \left[ a \right]. \quad (0.12)$$

So, we must now prove that, given a constant $\left\lfloor \frac{p}{a} \right\rfloor$, the minimum $a$ is:

$$\min \left[ \frac{p}{a} \right] = \left( \left\lfloor \frac{p}{a} \right\rfloor + 1 \right) + 1. \quad (0.13)$$

This is that:

$$\left\lfloor \frac{p}{\left\lfloor \frac{p}{a} \right\rfloor + 1} \right\rfloor = \left\lfloor \frac{p}{a} \right\rfloor, \quad (0.14)$$

while:

$$\left\lfloor \frac{p}{\left\lfloor \frac{p}{a} \right\rfloor + 1} \right\rfloor > \left\lfloor \frac{p}{a} \right\rfloor. \quad (0.15)$$

Let first prove (0.14):

$$\left\lfloor \frac{p}{\left\lfloor \frac{p}{a} \right\rfloor + 1} \right\rfloor - \left\lfloor \frac{p}{a} \right\rfloor = \left\lfloor \frac{p}{\left\lfloor \frac{p}{a} \right\rfloor + 1} \right\rfloor - \left\lfloor \frac{p}{a} \right\rfloor = \left\lfloor \frac{p}{\left\lfloor \frac{p}{a} \right\rfloor + 1} \right\rfloor - \left\lfloor \frac{p}{a} \right\rfloor = \left\lfloor \frac{p}{\left\lfloor \frac{p}{a} \right\rfloor + 1} \right\rfloor - \frac{p}{\left\lfloor \frac{p}{a} \right\rfloor + 1}$$

which is zero since the denominator is greater than the numerator and both positive.
\[
\left\lfloor \frac{p+\left\lfloor \frac{N}{p} \right\rfloor}{\left\lfloor \frac{N}{p} \right\rfloor+1} \right\rfloor - \frac{p+\left\lfloor \frac{N}{p} \right\rfloor}{\left\lfloor \frac{N}{p} \right\rfloor+1} = \left\lfloor \frac{p+\left\lfloor \frac{N}{p} \right\rfloor}{\left\lfloor \frac{N}{p} \right\rfloor+1} \right\rfloor \frac{p}{\left\lfloor \frac{N}{p} \right\rfloor+1} - p = \left\lfloor \frac{p+N}{N} \right\rfloor (N-p = \left\lfloor \frac{p}{\left\lfloor \frac{N}{p} \right\rfloor+1} \right\rfloor N - p + N = N - \text{rem} \left( \frac{p}{N} \right) > 0.
\]

Let now prove (0.15):
\[
\left\lfloor \frac{p}{\left\lfloor \frac{N}{p} \right\rfloor+1} \right\rfloor - \frac{p}{\left\lfloor \frac{N}{p} \right\rfloor+1} = \left\lfloor \frac{p}{\left\lfloor \frac{N}{p} \right\rfloor+1} \right\rfloor - \frac{p}{\left\lfloor \frac{N}{p} \right\rfloor+1} = \left\lfloor \frac{p-N}{\left\lfloor \frac{N}{p} \right\rfloor+1} \right\rfloor ,
\]
and since the difference between the numerator and the denominator is:
\[
p - \left\lfloor \frac{p}{\left\lfloor \frac{N}{p} \right\rfloor+1} \right\rfloor - \left\lfloor \frac{p}{\left\lfloor \frac{N}{p} \right\rfloor+1} \right\rfloor = p - \left\lfloor \frac{p}{\left\lfloor \frac{N}{p} \right\rfloor+1} \right\rfloor \left\lfloor \frac{p}{\left\lfloor \frac{N}{p} \right\rfloor+1} \right\rfloor = \text{rem} \left( \frac{p}{\left\lfloor \frac{N}{p} \right\rfloor+1} \right),
\]
which is greater than zero for any a between 2 and p-1. □

**Remark 16.** The other remainders can be obtained by resting an integer multiple of \( \left\lfloor \frac{p}{\alpha} \right\rfloor \).
Figure 0.4: The distribution of \( \left( \left\lfloor \frac{p}{q} \right\rfloor, \text{rem} \left( \frac{p}{q} \right) \right) \) for \( p=101 \).
References

[1] Weisstein, Eric W. "Sieve of Eratosthenes." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/SieveofEratosthenes.html