Fragility of the Laughlin state in an anharmonically-trapped Bose-Einstein condensate

A. Roussou1, J. Smyrnakis2, M. Magiropoulos2, N. K. Efremidis1, W. von Klitzing3, and G. M. Kavoulakis2
1Department of Applied Mathematics, University of Crete, GR-71004, Heraklion, Greece
2Technological Education Institute of Crete, P.O. Box 1939, GR-71004, Heraklion, Greece
3Institute of Electronic Structure and Laser (IESL), Foundation for Research and Technology (FORTH), N. Plastira 100, Vassilika Vouton, 70013, Heraklion, Crete, Greece

When a Bose-Einstein condensate rotates in a purely harmonic potential with an angular frequency which is close to the trap frequency, its many-body state becomes highly correlated, with the most well-known being the bosonic Laughlin state. To take into account that in a real experiment no trapping potential is ever exactly harmonic, we introduce an additional weak, quartic potential and demonstrate that the Laughlin state is highly sensitive to this extra potential. Our results imply that achieving these states experimentally is essentially impossible, at least for a macroscopic atom number.

PACS numbers: 03.75.Lm, 05.30.Jp, 67.85.Hj, 67.85.Jk

An interesting problem in the field of cold atomic gases, which has been investigated both theoretically, see, e.g., Refs. [1–14], as well as experimentally [15, 16], is the realization of highly-correlated states of rotating Bose-Einstein condensates trapped in harmonic potentials. Theses states show up in the limit where the angular frequency of the rotation of the trap $\Omega$ approaches the trap frequency $\omega$, and the centrifugal potential exactly cancels the trapping potential. In this limit there exists even an analytic expression for the many-body state, which is the bosonic version of the Laughlin state [17] that appears in the quantum Hall effect [18, 19].

The theoretical model that leads to these correlated states assumes an exactly harmonic trapping potential. However, in a real experiment the trapping potential always exhibits anharmonic corrections. Thus, a question that arises naturally is whether these states persist, even for “weak” deviations from a purely harmonic potential. References [20, 21] have considered such deviations. In particular, motivated by the experiment of Ref. [16], the theoretical study of Ref. [20] has considered the effect of an additional, repulsive Gaussian potential, which acts at the trap center. Contrary to the model considered in the present study, the potential of Ref. [20] is still harmonic for large distances from the center of the trap. In addition, the more mathematically-rigorous studies of Refs. [21] considered a quadratic-plus-quartic potential, and values of the angular momentum which go way beyond the one where the Laughlin state appears and other correlated states show up. In these studies the authors used trial states in order to derive conditions on the parameters of their model for its ground state to be asymptotically strongly correlated. Numerous theoretical studies have examined the rotational response of a Bose-Einstein condensate that is confined in a quadratic-plus-quartic potential for low rotational frequencies of the trap, where the system is well described by the mean-field approximation. They have shown that there are three phases, namely a vortex lattice, giant-vortex states, and a “mixed” phase, i.e., a vortex lattice with giant vortices in the middle of the trap, see, e.g., Refs. [22, 31]. These phases appear depending on the value of $\Omega$ and the interaction strength.

In the present study we assume that in addition to a harmonic potential, there is a weak quartic potential and focus on its effect on the (bosonic) Laughlin state. We start by evaluating this many-body state in a purely harmonic potential, which is our reference state. We then identify the effect of the anharmonic part of the potential on the many-body state. We stress that while we have focused on the Laughlin state, our results are more general, at least for the states with an angular momentum higher than, but of the same order as the Laughlin state.

In what follows we first introduce our model Hamiltonian, which includes the usual harmonic, and a weak quartic trapping potential, while the interatomic interactions are modelled as the usual contact potential. Since the (bosonic) Laughlin state is highly correlated, we necessarily use the method of diagonalization of the many-body Hamiltonian, considering small atom numbers. We focus on the limit of rapid rotation and investigate the effect of the quartic potential on the energy, the single-particle density distribution, the density matrix, and the pair-correlation function of the evaluated (lowest-energy) many-body state. Finally, we calculate the overlap of this many-body state with the Laughlin state and the giant-vortex state, and find that there is a competition between them.

The novelty of our results is thus basically twofold. First of all, our study provides a very clear picture of the behaviour of the system in the limit of rapid rotation, in the presence of an anharmonic potential and the transition from a correlated state to a mean-field state. Equally important is the conclusion that the correlated states, which in purely harmonic potentials are theoretically expected for rapidly-rotating Bose-Einstein condensates, are extremely fragile and as a result their experimental realization is very difficult.

Starting with our model, we consider bosonic atoms which are confined in a plane, via a very tight potential in the perpendicular direction, and are also subject to an axially-symmetric trapping potential along their plane of motion, $V(\rho)$, where $\rho$ is the radial coordinate. This trapping potential is assumed to be anharmonic (we set...
where $\lambda$ is the non-mean-field, Laughlin-like state, whilst for even higher values of $\lambda$ there are other correlated states. For $\lambda \geq 20$ the interaction energy vanishes exactly, in agreement with our numerical results.

To see the effect of the quartic potential, we also plot in Fig. 1 the result of the same calculation for $\lambda = 0.05$. Its effect is drastic, and already for such a small value of $\lambda$ one finds a distinctly different spectrum. The most distinctive feature is that it starts to develop a quasi-periodic behaviour, as in a ring potential (in a ring trapping potential the spectrum is periodic on top of a parabola, as Bloch’s theorem implies [35]), developing local minima when $L$ is an integer multiple of $N$, i.e., for $L = 5, 10, 15$, and 20. This result, as well as the ones that follow below, show a transition of the many-body state from the (correlated) Laughlin state to a (mean-field) giant-vortex state as $\lambda$ increases.

Let us now turn to the (axially-symmetric) single-particle density, \( n(\rho) = \langle \Phi^\dagger(\hat{\rho})\Phi(\hat{\rho}) \rangle \), where $\Phi(\hat{\rho})$ is the operator that destroys a particle at $\hat{\rho}$.

Figure 2 shows the single-particle density $n(\rho)$ for $N = 5$ atoms, $L = 20$, $g = 0.1$ and for three values of $\lambda = 0.00$, 0.01, and 0.02. For $\lambda = 0.00$, i.e., for the Laughlin state, we see that indeed the density is roughly constant and close to the expected result $1/(2\pi) \approx 0.16$, while the radius is also close to the expected result $\sqrt{2N} = \sqrt{10} \approx 3.16$, to leading order in $N$ [17].

It is seen clearly that the effect of the anharmonic potential is to create a “hole” in the middle of the cloud. Actually, even for the rather small value of $\lambda = 0.02$, the single-particle density is well approximated by that
of the “giant vortex”, \( N|\psi_4(\rho)|^2 \), as seen also in Fig. 2. The maximum of this is at \( \rho = \sqrt{N - 1} = 2 \), with a value \( N/\sqrt{N - 1}/(\sqrt{2\pi}) \approx 0.32 \).

Further evidence of this transition is also seen from the the eigenvalues of the density matrix, \( \langle a_{m0}^\dagger a_{m0} \rangle \) (which coincide with the occupancies of the single-particle states, since the density matrix is diagonal, due to the axial symmetry of the problem). Figure 3 shows this result for \( N = 5, L = 20, g = 0.1 \), and for the same values of \( \lambda \) considered in Fig. 2, i.e., 0.00, 0.01, and 0.02. As \( \lambda \) increases we observe that the occupancy of the single-particle state \( \psi_{m0} \), with \( m_0 = L/N = 4 \), becomes dominant, which is consistent with the transition to a giant-vortex state.

In both plots the transition from the Laughlin state to the giant vortex takes place when the energy (per atom) due to the quartic part of the potential, \( \lambda N^2 \) is comparable with the interaction energy of the giant vortex. This is \( \sim g n_{2D} \), where the two-dimensional density \( n_{2D} \sim N/R \sim \sqrt{N} \). Here \( R \) is the radius of the (roughly) homogeneous density of the Laughlin state, which is \( \propto \sqrt{N} \). Thus the “threshold” value of \( \lambda \) is \( \sim g/\sqrt{N} \). Figure 4 shows \( g(\rho,\rho') = 2 \) for the lowest-energy eigenstate of the Hamiltonian, for \( N = 4 \) atoms, \( L = 20, g = 0.1 \), and \( \lambda = 0 \) (solid line), 0.001 (dashed line), 0.001 (dotted line), and 0.01 (dotted-dashed line).

Another relevant quantity is the pair-correlation function, which is defined as

\[
g^{(2)}(\vec{\rho},\vec{\rho}') = \frac{\langle \Phi^{\dagger}(\vec{\rho})\Phi^{\dagger}(\vec{\rho}')\Phi(\vec{\rho}')\Phi(\vec{\rho}) \rangle}{\langle \Phi^{\dagger}(\vec{\rho})\Phi(\vec{\rho}) \rangle \langle \Phi^{\dagger}(\vec{\rho}')\Phi(\vec{\rho}') \rangle}.
\] (6)

In an uncorrelated, mean-field, state \( g^{(2)}(\vec{\rho},\vec{\rho}') = 2 \), with \( \vec{\rho} \) and \( \vec{\rho}' \) pointing in the same direction, for \( N = 4 \) atoms, \( L = 12, g = 0.001 \) and \( \lambda = 0, 0.0001, 0.001, \) and 0.01. The reference point is chosen to be at \( \rho' = 2 \) (this is roughly where the maximum of the single-particle density distribution is located). That is why for \( \lambda = 0 \) there is a node in \( g^{(2)}(\rho,\rho' = 2) \) at \( \rho = 2 \), as expected from the Laughlin state. As \( \lambda \) increases the node disappears. In addition, for values of \( \rho \) larger than 2, \( g^{(2)}(\rho,\rho' = 2) \) is roughly constant (differing from 3/4 due to the finiteness of \( N \) and the relatively small values of \( \lambda \)), while a local minimum forms at some value of \( \rho \) which is smaller than
\[ \frac{|\langle \Psi | \Psi_L \rangle|}{|\langle \Psi | \Psi_{GV} \rangle|} \]

\[ \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \]

\[ \langle \Psi | \Psi_{GV} \rangle \]

\[ \langle \Psi_L | \Psi \rangle \]

\[ \langle \Psi | \Psi_{GV} \rangle \]

\[ \langle \Psi_L | \Psi \rangle \]

\[ N = 3, \text{ Laughlin} \]

\[ N = 3, \text{ Giant vortex} \]

\[ N = 6, \text{ Laughlin} \]

\[ N = 6, \text{ Giant vortex} \]

\[ \lambda \] has to decrease equally rapidly with increasing \( N \), in order for the Laughlin state to survive. Turning to the overlap of \( |\Psi\rangle \) with the giant vortex, \( |\langle \Psi | \Psi_{GV} \rangle| \) increases rapidly with increasing \( \lambda \), and the slope also increases with increasing \( N \), for small values of \( \lambda \), as seen in Fig. 5.

The effect of the interaction strength on the overlap of \( |\Psi\rangle \) with the Laughlin state, for a fixed value of \( \lambda \) can be seen in Fig. 6. We observe that as the interaction strength increases, the effect of the anharmonic potential is suppressed and thus the overlap with the Laughlin state increases. Still, the question is how the overlap behaves as function of \( N \). In the results shown in Fig. 6 we evaluated the inner product between the Laughlin state (for \( \lambda = 0 \)) and the many-body state for \( \lambda = 0.01 \) as a function of \( g \), for \( N = 3, 4 \) and 5 atoms. The interesting observation here is that the overlap approaches unity as \( g \) increases less rapidly as \( N \) increases.

To conclude, a Bose-Einstein condensate that rotates in a purely harmonic potential undergoes a series of transitions as the rotational frequency of the trap increases. Singly-quantized vortex states enter the cloud, which eventually form a vortex lattice. When the rotational frequency approaches the trap frequency, the system enters a highly-correlated regime, where the number of vortices becomes comparable to the number of atoms.

The question we have posed here is whether these correlated states persist in a harmonic-plus-quartic potential. In such a potential, within the mean-field approximation there are three distinct phases. In the first phase we have a vortex lattice, in the second we have giant-vortex states, while the third is a combination of a lattice with a giant vortex which is located at the trap center.

In the limit of rapid rotation we see that the Laughlin state competes with the giant-vortex state. The transition between them takes place for a “strength” of the quartic part of the confining potential that decreases rapidly as the atom number increases. While we have focused on the bosonic Laughlin state [for \( L = N(N-1) \)], our results are more general and are valid, at least for the states with \( L \gtrsim N(N-1) \).

The transition from the Laughlin state to the giant-vortex state may be attributed to the single-particle density distribution (shown in Fig. 2) of the cloud in the two states, which is rather different in the two states. It is flat and extends up to a radius equal to \( \sqrt{2N} \) in the Laughlin state. It has a Gaussian profile with a width of order unity, and its maximum is located at \( \sqrt{N-1} \) in the giant-vortex state. The Laughlin state thus becomes energetically unfavourable, even in a weakly anharmonic trapping potential [35].

The fragility of the Laughlin state makes its experimental realization virtually impossible for macroscopic atom numbers. For example, for a typical value of \( \lambda = 10^{-3} \), as in the experiments of Refs. [16, 34], \( N \) should be less than roughly \( N = 10 \) in order for the Laughlin state to be achievable. Therefore it is an experimental challenge to realize this state, which could become possible either by reducing the atom number very drastically, or by decreasing the value of \( \lambda \), also very drastically.
[1] N. R. Cooper and N. K. Wilkin, Phys. Rev. B 60, R16279(R) (1999).
[2] N. K. Wilkin and J. M. F. Gunn, Phys. Rev. Lett. 84, 6 (2000).
[3] S. Vielers, T. H. Hansson, S. M. Reimann, Phys. Rev. A 62, 053604 (2000).
[4] N. R. Cooper, N. K. Wilkin, and J. M. F. Gunn, Phys. Rev. Lett. 87, 120405 (2001).
[5] T. L. Ho, Phys. Rev. Lett. 87, 060403 (2001).
[6] J. Sinova, C. B. Hanna, and A. H. MacDonald, Phys. Rev. Lett. 89, 030403 (2002).
[7] N. Regnault and T. Jolicoeur, Phys. Rev. B 69, 235309 (2004).
[8] G. Baym and C. J. Pethick, Phys. Rev. A 69, 043619 (2004).
[9] N. Regnault, C. C. Chang, T. Jolicoeur, and J. K. Jain, J. Phys. B 39, S89 (2006).
[10] N. Barberán, M. Lewenstein, K. Osterloh, and D. Dagnino, Phys. Rev. A 73, 063623 (2006).
[11] L. O. Baksmaty, C. Yannouleas, and Uzi Landman, Phys. Rev. A 75, 023620 (2007).
[12] N. R. Cooper, Advances in Physics 57, 539 (2008).
[13] S. Vielers, J. Phys. 20, 123202 (2008).
[14] M. Roncaglia, M. Rizzi, J. Dalibard, Scientific Reports 1, 43 (2011).
[15] V. Schweikhard, I. Coddington, P. Engels, V. P. Mendoza, and E. A. Cornell, Phys. Rev. Lett. 92, 040404 (2004).
[16] V. Bretin, S. Stock, Y. Seurin, and J. Dalibard, Phys. Rev. Lett. 92, 050403 (2004); S. Stock, B. Battelier, V. Bretin, Z. Hadzibabic, and J. Dalibard, Laser Phys. Lett. 2, 275 (2005).
[17] H. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[18] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980).
[19] H. L. Stormer, D. C. Tsui, and A. C. Gossard, Rev. Mod. Phys. 71, S298 (1999).
[20] A. G. Morris and D. L. Feder, Phys. Rev. Lett. 99, 240401 (2007).
[21] N. Rougerie, J. Yngvason, S. Serfaty, Phys. Rev. A, 87, 023618 (2013); N. Rougerie, S. Serfaty, J. Yngvason, J. Stat. Phys. 154, 2 (2014); E. H. Lieb, N. Rougerie, and J. Yngvason, J. Stat. Phys. 172, 544 (2018).
[22] A. L. Fetter, Phys. Rev. A 64 063608 (2001).
[23] E. Lundh, Phys. Rev. A 65 043604 (2002).
[24] K. Kasamatsu, M. Tsubota, and M. Ueda, Phys. Rev. A 66, 053606 (2002).
[25] U. Fischer and G. Baym, Phys. Rev. Lett. 90, 140402 (2003).
[26] G. M. Kavoulakis and G. Baym, New Journal of Physics 5, 51.1 (2003).
[27] A. D. Jackson, G. M. Kavoulakis, and E. Lundh, Phys. Rev. A 69, 053619 (2004); A. D. Jackson and G. M. Kavoulakis, Phys. Rev. A 70, 023601 (2004).
[28] A. L. Fetter, B. Jackson, and S. Stringari, Phys. Rev. A 71, 013605 (2005).
[29] Jong-kwan Kim and A. L. Fetter, Phys. Rev. A 72, 023619 (2005).
[30] H. Fu and E. Zaremba, Phys. Rev. A 73, 013614 (2006).
[31] M. Correggi, F. Pinski, N. Rougerie, and J. Yngvason, Phys. Rev. A 84, 053614 (2011).
[32] G. M. Kavoulakis, B. Mottelson, and C. J. Pethick, Phys. Rev. A 62, 063605 (2000).
[33] G. F. Bertsch and T. Papenbrock, Phys. Rev. Lett. 83, 5412 (1999).
[34] A. D. Jackson and G. M. Kavoulakis, Phys. Rev. Lett. 85, 2854 (2000).
[35] F. Bloch, Phys. Rev. A 7, 2187 (1973).
[36] In Refs. [21] it has been argued that for $L > N(N−1)$ and under certain conditions, the many-body states remain correlated, with a single-particle density distribution that develops a hole in the middle of the trap. This is energetically favourable and is consistent with the results of the present study.