Entanglement in Dilute Flux Line Liquids

A. M. Ettouhami

Department of Physics, University of Colorado, Boulder, CO 80309

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We construct, within the framework of classical statistical mechanics, a mean field theory of dilute flux line liquids which goes beyond linear hydrodynamics. Within our approach, we find that interactions between vortices produce a massive term in the Hamiltonian of the internal modes of the flux lines which confines their transverse fluctuations. This suggests that the flux line liquid, at least in the low density limit considered in this paper, might very well be in a weakly entangled state, where the average width \( \langle u^2 \rangle \) of flux lines can be much larger than the average distance between the lines but does not diverge with the sample thickness \( L \). Consequences on the physics of flux line liquids are briefly discussed.

I. INTRODUCTION

The discovery, by Bednorz and Müller\[1\] in 1986, of the new family of cuprate-based high temperature superconductors (HTSC), has had a profound impact on research in the field of superconductivity. In particular, due to their anisotropy, short coherence lengths and high critical temperatures, it quickly became obvious that thermal fluctuations will play a key role in the physics of interacting flux lines in these materials. Early experiments\[2\] suggested, and it was later unambiguously confirmed\[3\], that a phase transition takes place where the three dimensional Abrikosov\[4\] flux line lattice (FLL) lattice melts into a flux line liquid, and the location of the melting temperature \( T_m \) as a function of the applied magnetic field \( H \) in the \((H,T)\) phase diagram of the flux line system became the subject of intense experimental and theoretical investigations (see Blatter et al.\[5\] for a recent review).

One of the early approaches aimed at describing the melting transition and the physics of melted flux line liquids was the theory of Nelson and Seung\[6\], which is based on the observation that the partition function of a system of interacting flux lines in 2 + 1 dimensions can be mapped onto the imaginary time partition function of a system of interacting quantum mechanical bosons in two dimensions. This formal mapping was used to obtain a complete description of flux line liquids by using well known techniques from boson physics. The physical picture of the flux line liquid which emerges from this description suggests that there is considerable wandering of flux lines as they traverse the sample and that the flux line liquid is heavily entangled over much of the liquid phase. Such a picture has been observed in numerical simulations, and it is now generally believed that the mean square relative displacement \( \langle |r_z(z) - r_z(0)|^2 \rangle \) of the transverse position \( r_z(z) \) of a given flux line in a flux line liquid has the same form as the corresponding quantity for an isolated flux line, namely:

\[
\langle |r_z(z) - r_z(0)|^2 \rangle = 2D|z|, \tag{1.1}
\]

the only effect of the interactions between lines in a dense liquid being to reduce the “diffusion” constant \( D \) from its bare value \( D_0 \) for a single flux line, which is given by\[7\]:

\[
D_0 = \frac{k_BT}{\bar{\varepsilon}_1}
\]

where \( k_B \) is Boltzmann’s constant, \( T \) is the temperature and \( \bar{\varepsilon}_1 \) is the tilt modulus of a single flux line. Similarly, the mean square projected area occupied by the lines is found, within the boson picture, to be of order\[8\]:

\[
\langle u^2 \rangle \approx \frac{2\pi k_BT}{\bar{\varepsilon}_1} L \tag{1.2}
\]

where \( u(z) = r(z) - \langle r(z) \rangle \) and \( L \) is the sample thickness in the direction of the flux lines (which we take parallel to the \( z \) axis). This last result is to be contrasted with the mean square projected area of a given flux line in the crystalline phase, which is given by\[9\]:

\[
\langle u^2 \rangle = \frac{k_BT a}{2\sqrt{\pi} \varepsilon_0} \tag{1.3}
\]

where \( a \) is the average spacing between flux lines in the vortex lattice, \( \varepsilon = \lambda/\lambda_c \) is the ratio of the London penetration depths in the \((ab)\) plane and along the direction of the \( c \) axis respectively (we remind the reader that most HTSC have uniaxial symmetry, and we denote by \( c \) the principal axis of symmetry of the crystal), and \( \varepsilon_0 = (\phi_0/4\pi\lambda)^2 \), with \( \phi_0 = hc/2e \) the flux quantum.\[10\]

Comparing the results \( (1.2) \) and \( (1.3) \), we see that, while the mean square projected area of flux lines \( \langle u^2 \rangle \) in the crystalline phase is finite, in the liquid phase it diverges with the sample thickness \( L \).

Despite a good deal of both theoretical\[11\] and experimental work\[12,13\] and rather extensive numerical studies\[14,15\], several questions regarding the physics of flux line liquids remain open. One of the most important questions which has been the subject of intense debate during the past few years\[16,17\], is related...
to whether a disentangled flux line liquid can survive above melting. Although early numerical simulations have provided support for the proposal by Feigel’man et al.\(^7\) that superconducting coherence could survive in the liquid phase, more extensive simulations by several authors\(^8\-10\) found that longitudinal coherence vanishes simultaneously with melting, and that the vortex liquid is characterized by very short entanglement lengths in the direction of the flux lines. However, analyses of experiments on high quality untwinned single crystals of YBa\(_2\)Cu\(_3\)O\(_{7-\delta}\) (YBCO) by Righi et al.\(^2\) and by Moore\(^4\) have suggested that longitudinal correlations in the liquid phase may be of surprisingly larger micron scale.

At the fundamental level, it is very important to understand, from a purely conceptual point of view, under what circumstances a disentangled state can emerge from a theoretical description, and what kind of theories are best suited to yield such a disentangled state. We first note that, for a disentangled state to appear, it is necessary that flux line conformation variables \(\mathbf{r}_i(z)\) have correlations which do not obey equations (1.1)-(1.2). In particular, the average projected area of a given flux line \((u^2)\) has to be independent of the sample thickness \(L\). In the crystalline phase, transverse fluctuations of a given flux line are confined thanks to the repulsive action of the other, neighboring lines in the FLL. In a similar way, physical intuition suggests that the interactions between flux lines should strongly reduce and might even suppress line wandering if the repulsion energy between flux lines is high enough compared to \(k_B T\). In such a disentangled state with large repulsion forces between the lines, a given flux line will experience the repulsive potential of its surrounding neighbors in much the same way as in a lattice. As a result, the internal fluctuations of that line will be confined in about the same way they are confined in regular Abrikosov FLL. In particular, there should be a regime where the mean square internal fluctuations \((u^2)\) does not differ much, at least locally, from the corresponding quantity in a FLL at the same density. For us to be able to describe a putative disentangled phase of a flux line liquid, that is, a phase where fluctuations of flux lines are confined, we need to concentrate on the individual conformation variables of the flux lines, and try to evaluate the effect of interactions on these variables in the most accurate way possible.

In fact, most theoretical studies to date have concentrated on hydrodynamic quantities, such as, e.g. the density, and have used approximations pertinent to linear hydrodynamics in order to study the physics of flux line liquids. Hydrodynamic observables being coarse-grained quantities, such approaches are intrinsically unable to give an accurate picture of the correlations of flux line conformation variables, such as \(u_1(z)\). Even if we go beyond linear hydrodynamics, such as in ref.\(^1\), because of the use of coarse-grained variables, we expect not to be able to describe the confinement of individual flux line fluctuations that would take place in a hypothetical, disentangled flux line liquid. To the most, hydrodynamic approaches can only yield a renormalization of the tilt modulus of flux lines in a line liquid, leaving the qualitative (analytic) form of the correlations (1.1)-(1.2) unchanged.

In view of the above remarks, it seems to us that it is highly desirable to construct a theory which keeps track in a better way of conformation variables of the flux lines, and which would therefore be able to yield more accurate information about the correlations of these variables. Such a theory would undoubtedly help us gain a better understanding of the properties of flux line liquids.

In this paper, which is largely motivated by the conclusions of references\(^4\), we would like to lay out the path for the construction of such a theory. We show, in particular, that a classical theory of flux liquids with a better handling of the internal modes of flux lines can be formulated which is simple enough to allow for ease of mathematical treatment, and which is transparent enough to allow for approximations to be made with a good level of confidence. Within this approach, and under certain assumptions, we find that a confinement of the internal fluctuations of the lines can indeed take place, at least in the limit of a dilute flux line liquid. More specifically, we find that, due to the interactions between flux lines, the internal modes may actually become massive, leading to a finite mean square width \((u^2)\) of flux lines, and to a qualitative change in the behaviour of the correlation function (1.1).

This paper is organized as follows. In section II, we consider a liquid of interacting flux lines in a weakly anisotropic HTSC. Performing a perturbative expansion of the interaction energy in terms of the internal fluctuations of the flux lines, we show that the interactions with neighboring lines induce a mass term in the Hamiltonian of the internal modes. As a result, we find that the internal fluctuations of the flux lines are strongly suppressed compared with the free flux line result (1.2). We self-consistently determine the range of validity of our perturbative expansion, which is found to be quite large for moderately anisotropic superconductors. We also calculate the structure factor of the flux line liquid and our result is compared to previous derivations. Section III contains a discussion of our findings, which we think can be reconciled with the results of numerical simulations, along with our conclusions.

II. MEAN-FIELD THEORY OF (PUTATIVE) DISENTANGLED FLUX LINE LIQUIDS

In this section, we shall be considering the statistical mechanics of an assembly of flux lines in a sample of thickness \(L\) of a uniaxial HTSC, with both the principal axis of anisotropy \(c\) and the external magnetic field aligned with the \(z\) axis. Such a system can be described by the Hamiltonian\(^4\).
\[ H = \sum_{i=1}^{N} \int_{0}^{L} dz \left[ \frac{1}{2} \hat{\varepsilon}_1 \left( \frac{dr_i}{dz} \right)^2 + \frac{1}{2} \sum_{i \neq j} \int_{0}^{L} dz \ V(r_i(z) - r_j(z)) \right] \]

where \( \hat{\varepsilon}_1 \approx \varepsilon^2 \varepsilon_{\perp} \ln(a/\xi) \) denotes the tilt modulus per unit length of the flux lines (here \( \xi \) and \( a \) are the coherence length and the average distance between flux lines in the \( (ab) \) plane respectively), and where \( V(r) = 2\varepsilon_{\perp} K_0(r/\lambda) \) is the interaction potential between flux line elements at equal height, with \( K_0 \) a modified Bessel function. (Henceforth, we shall neglect the logarithmic factor \( \ln(a/\xi) \), of order unity, in the definition of the tilt modulus, writing \( \hat{\varepsilon}_1 \approx \varepsilon^2 \varepsilon_{\perp} \)). The Hamiltonian above may be viewed as a simplified version of the most general Hamiltonian of arbitrarily curved and tilted flux lines in the London regime of anisotropic superconductors (with nonlocal interactions in the \( z \) direction). The tilt modulus per unit length \( \hat{\varepsilon}_1 \) on the other hand can be viewed as the effective tilt modulus for long-wavelength \( q_L \) distortions which accounts for the most relevant fluctuations of the flux lines near melting, with transverse wavelength \( q_L \approx q_{BZ} \left( q_{BZ} \right) \) being the transverse wavenumber at the Brillouin zone boundary, as can be seen by replacing \( q_{L} \approx q_{BZ} = (4\pi B/\phi_0)^{1/2} \), taking the limit \( q_{L} \to 0 \) in the general (nonlocal) expression of the tilt modulus per unit volume \( c_{44}(q) = B^2 / 4\pi(1 + (\lambda/\xi)^2 q^2_L + \lambda^2 q^2_z) \), and multiplying by the elementary area per vortex \( (\phi_0/B) \). Below, it will prove useful to write the following Fourier decomposition of \( r_i(z) \) into Rouse modes:

\[ r_i(z) = \sum_{n=-\infty}^{\infty} r_i(q_n) e^{i q_n z} \quad (2.2) \]

where \( q_n = 2\pi n/L \), and where the coefficients \( r_i(q_n) \) are given by:

\[ r_i(q_n) = \frac{1}{L} \int_{0}^{L} dz \ r_i(z) e^{-i q_n z} \quad (2.3) \]

as can be verified by using the orthogonality relation

\[ \int_{0}^{L} dz \ e^{i q_n z} (e^{i q_m z})^* = L\delta_{n,m} \quad (2.4) \]

It will also be convenient to write \( r_i(z) \) as the sum

\[ r_i(z) = r_{0i}(z) + u_i(z) \quad (2.5) \]

where

\[ r_{0i}(z) = r_i(q_n = 0) = \frac{1}{L} \int_{0}^{L} dz \ r_i(z) \]

is the center of mass (CM) position, while \( u_i(z) \) is the displacement of the flux line at height \( z \) with respect to the center of mass position. In terms of the Fourier modes \( \{r_i(q_n)\} \), the displacement vector \( u_i(z) \) can be written in the form (here \( c.c. \) denotes complex conjugation):

\[ u_i(z) = \sum_{n=1}^{\infty} \{r_i(q_n) e^{i q_n z} + c.c.\} \quad (2.6) \]

Using the above decomposition of the flux line positions \( \{r_i(z)\} \) into CM and internal modes, equation \( (2.3) \), the interaction term between flux line elements at \( r_i(z) \) and \( r_j(z) \) can be written as

\[ V(r_i(z) - r_j(z)) = V(r_{0i} - r_{0j} + u_i(z) - u_j(z)) \quad (2.7) \]

Since the interaction potential \( V(r) \) is a smooth function which varies very slowly on length scales much smaller than \( \lambda \), we see that if

\[ |u_i(z) - u_j(z)| \ll \lambda \quad (2.8) \]

then we can write, to a very good approximation:

\[ V(r_i(z) - r_j(z)) \approx V(r_{0i} - r_{0j}) + [u_i(z) - u_j(z)] \cdot \nabla V(r_{0i} - r_{0j}) + \frac{1}{2} [ (u_i(z) - u_j(z)) \cdot \nabla ]^2 V(r_{0i} - r_{0j}) \quad (2.9) \]

If we integrate this last equation over \( z \), taking into account the fact that \( \int_{0}^{L} dz \ u_i(z) = 0 \), we obtain

\[ \int_{0}^{L} dz \ V(r_i(z) - r_j(z)) \approx L V(r_{0i} - r_{0j}) + \frac{1}{2} \int_{0}^{L} dz \ [u_{i\alpha}(z) u_{i\beta}(z) + u_{j\alpha}(z) u_{j\beta}(z) + u_{i\alpha}(z) u_{j\beta}(z) + u_{j\alpha}(z) u_{i\beta}(z)] \partial_{\alpha} \partial_{\beta} V(r_{0i} - r_{0j}) \]

In the Boson language, this last equation is reminiscent of the expansion of the action of a quantum particle around a classical path. The Hamiltonian \( (2.1) \) now can be written in the form:

\[ H = H_0 + H_1 \quad (2.10) \]

where

\[ H_0 = \frac{1}{2} \sum_{i \neq j} LV(r_{0i} - r_{0j}) \quad (2.11) \]

is the Hamiltonian of a system of perfectly straight, interacting lines of length \( L \), and

\[ H_1 = \sum_{i=1}^{N} \int_{0}^{L} dz \left[ \frac{1}{2} \hat{\varepsilon}_1 \left( \frac{du_i}{dz} \right)^2 + \frac{1}{2} \mu^{(i)}_{\alpha\beta} u_{i\alpha}(z) u_{i\beta}(z) \right] + \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \int_{0}^{L} dz \mu^{(ij)}_{\alpha\beta} u_{i\alpha}(z) u_{j\beta}(z) \quad (2.12) \]
is the Hamiltonian of the internal modes of the flux lines. In equation (2.12), we defined the following quantities:

\[ \mu_{\alpha\beta}^{(i)} = \sum_{j(\neq i)=1}^{N} \partial_{\alpha} \partial_{\beta} V(r_{0i} - r_{0j}) \]  
(2.13)

\[ \mu_{\alpha\beta}^{(ij)} = -\partial_{\alpha} \partial_{\beta} V(r_{0i} - r_{0j}) \]  
(2.14)

As can be seen from equation (2.13), the interactions between flux lines have generated a mass term \( \frac{1}{2} \mu_{\alpha\beta}^{(ij)} u_{\alpha}(z) u_{\beta}(z) \) for the internal modes \( \{ u_i(z) \} \) as well as additional, two-body terms, \( \frac{1}{2} \mu_{\alpha\beta}^{(ij)} u_{\alpha}(z) u_{\beta}(z) \), which couple the internal modes of different vortices \( (i \neq j) \). As they stand, the coefficients \( \mu_{\alpha\beta}^{(ij)} \) and \( \mu_{\alpha\beta}^{(ij)} \) of equations (2.13) and (2.14) depend on the specific configuration of all the center of mass coordinates \( \{ r_{0i} \} \) considered. In order to obtain the value of these coefficients relevant to a liquid of flux lines, we need to average equations (2.13) and (2.14) over all possible configurations of the CM coordinates compatible with a liquid structure. In order to carry out such an average, let us write the partition function of the flux line liquid (\( \beta = 1/k_B T \) is the inverse temperature) in the form

\[ Z = T_0 T_u (e^{-\beta H}) \]  
(2.15)

where we denote by \( T_0 \) and \( T_u \) the trace over the center of mass and internal modes, respectively. More explicitly:

\[ T_0 = \int dr_{01} \cdots dr_{0N} \]  
(2.16)

\[ T_u = \prod_{n \geq 1} dr_1(q_n) \cdots \prod_{n \geq 1} dr_N(q_n) \]  
(2.17)

where \( dr_1(q_n) \) stands for \( dr_{1,rc}(q_n) dr_{1,im}(q_n) \); \( r_{1,rc}(q_n) \) and \( r_{1,im}(q_n) \) being the real and imaginary parts of \( r_1(q_n) \), respectively. Noting that \( H = H_0 + H_1 \) in (2.13), and taking the trace over the CM mode only, we obtain

\[ T_0 (e^{-\beta H}) = Z_0 (e^{-\beta H_1})_0 \]  
(2.18)

where \( Z_0 = T_0 (e^{-\beta H_0}) \), and where the average \( \langle \cdots \rangle_0 \) is taken with respect to the probability distribution \( \exp(-\beta H_0)/Z_0 \), i.e. for an arbitrary function \( f(\{ r_{0i} \}) \) of the CM coordinates \( \{ r_{01}, \cdots, r_{0N} \} \),

\[ \langle f(\{ r_{0i} \}) \rangle_0 = \frac{1}{Z_0} \int dr_1 \cdots dr_N f(\{ r_{0i} \}) e^{-\beta H_0} \]  
(2.19)

We now perform a cumulant expansion in equation (2.18), with the following result to leading order in \( H_1 \),

\[ T_0 (e^{-\beta H}) \simeq Z_0 e^{-\beta (H_1)_0} \]  
(2.20)

and hence we see that the total partition function \( Z \) can be written in the form

\[ Z \simeq Z_0 Z_1 \]  
(2.21)

where \( Z_1 = T_u (e^{-\beta (H_1)_0}) \) is the effective partition function of the internal modes. The quantity

\[ H_{eff} = \langle H_1 \rangle_0 \]  
may be thought of as the effective Hamiltonian of the internal modes, averaged over all possible CM liquid configurations. In order to find this effective Hamiltonian, all we need to do is to calculate the averages \( \langle \mu_{\alpha\beta}^{(ij)} \rangle_0 \) and \( \langle \mu_{\alpha\beta}^{(ij)} \rangle_0 \). Applying the definition of the average (2.19) to the “mass” coefficients \( \mu_{\alpha\beta}^{(ij)} \), we obtain

\[ \langle \mu_{\alpha\beta}^{(ij)} \rangle_0 = \frac{1}{Z_0} \int \prod_{k=1}^N dr_{0k} \sum_{j(\neq i)=1}^N \partial_{\alpha} \partial_{\beta} V(r_{0i} - r_{0j}) e^{-\beta H_0} \]  
\[ = \frac{1}{Z_0} \int \prod_{j(\neq i)=1}^N dr_{0j} \partial_{\alpha} \partial_{\beta} V(r_{0i} - r_{0j}) \times \]  
\[ \times \int \prod_{k \neq i,j} dr_{0k} e^{-\beta H_0} \]  
(2.22)

The coordinates \( \{ r_{0i} \} \) under the integral sign being just dummy variables, we see that the rhs of equation (2.22) consists of a sum of \( (N-1) \) identical terms which can be rewritten in the form

\[ \langle \mu_{\alpha\beta}^{(ij)} \rangle_0 = \frac{(N-1)}{Z_0} \int dr_{01} dr_{02} \partial_{\alpha} \partial_{\beta} V(r_{01} - r_{02}) \times \]  
\[ \times \int \prod_{k \neq 0} dr_{0k} e^{-\beta H_0} \]  
(2.23)

We recognize on the second line of equation (2.23) the two-particle density \( \rho^{(2)}(r_1, r_2) \) of liquid state theory.

\[ \rho^{(2)}(r_0, r_0) = \frac{(N-1)}{Z_0} \int \prod_{k \neq 0} dr_{0k} e^{-\beta H_0} \]  
(2.24)

This last quantity is related to the pair distribution function of the CM mode \( g_0(r_{01}, r_{02}) \) by

\[ \rho^{(2)}(r_0, r_0) = \rho^2 g_0(r_0, r_0) \]  
(2.25)

where \( \rho = B/\phi_0 \) is the average density of flux lines in the system. At equilibrium, we expect the flux line liquid to be translationally invariant, which implies that \( g_0(r_{01}, r_{02}) \) will depend only on the difference \( (r_{01} - r_{02}) \), i.e. \( g_0(r_{01}, r_{02}) = g_0(r_{01} - r_{02}) \). Under these conditions, we obtain from equation (2.23)

\[ \langle \mu_{\alpha\beta}^{(ij)} \rangle_0 = \frac{\rho^2}{N} \int dr_{01} dr_{02} g_0(r_{01} - r_{02}) \partial_{\alpha} \partial_{\beta} V(r_{01} - r_{02}) \]  
\[ = \rho \int dr g_0(r) \partial_{\alpha} \partial_{\beta} V(r) \]  
(2.25)
Introducing the pair correlation function \( h_0(r) = g_0(r) - 1 \), the rhs of the last equation becomes

\[
\langle \mu^{(i)}_{\alpha\beta} \rangle_0 = \rho \int dr \ h_0(r) \partial_\alpha \partial_\beta V(r) \tag{2.26}
\]

where we used the fact that \( \int dr \ \partial_\alpha \partial_\beta V(r) = 0 \). If we further use the fact that both \( h_0 \) and \( V \) are rotationally invariant functions, \( h_0(r) = h_0(\rho(r)) \) and \( V(r) = V(\rho(r) \), we obtain (here \( d_\perp = 2 \) is the number of transverse dimensions)

\[
\langle \mu^{(i)}_{\alpha\beta} \rangle_0 = \mu \delta_{\alpha\beta} \tag{2.27}
\]

\[
\mu = \frac{\rho}{d_\perp} \int dr \ h(r) \nabla^2 \perp V(r) \tag{2.28}
\]

where \( \nabla^2 \perp = \partial^2_\alpha + \partial^2_\beta \) is the Laplacian in the transverse directions. Equation (2.28) is the main result of this paper. It gives the effective mass of the internal modes of flux lines, averaged over all possible configurations of the centers of mass of the vortices in the flux line liquid.

From equation (2.28) above, we now want to derive an explicit expression for the averaged mass \( \mu \). In order to be able to do so, we need to choose a suitable analytic form for the pair correlation function \( h_0(r) \). It is well known from liquid state theory \( \text{[2,3]} \) that the pair distribution function \( g(r) \) of an ordinary classical liquid has peaks of decreasing amplitude at \( r \approx a, 2a, 3a, \ldots \); \( a = 1/\sqrt{\rho} \) being the average spacing between particle in the liquid phase; and that it has the limiting behaviors \( g(r) \ll 1 \) for \( r < a \), and \( g(r) \to 1 \) when \( r \to \infty \). A common (mean field) approximation for the pair distribution function used in the literature consists in neglecting the fine structure of \( g(r) \) for \( r > a \), and approximating

\[
g(r) \approx \theta(r - a) \tag{2.29}
\]

where \( \theta \) is the Heaviside unit step function \( (\theta(x) = 1 \text{ if } x > 0 \text{ and zero otherwise}) \). This approximation, which is obviously relevant to liquids at not very high densities, retains the important feature of \( g(r) \) that it is very small for \( r < a \), which expresses the fact that particles in a liquid of density \( \rho \) have a small probability of being within a distance \( a = 1/\sqrt{\rho} \) from each other. Here, we expect such a behaviour of the pair distribution function \( g_0(r) \) in a dilute flux liquid to be valid if the repulsive interactions between flux lines are strong enough. To quantify how strong the interactions must be, we introduce the following typical energy scale

\[
E_c(T) = \varepsilon_0 \ell_c(T) \tag{2.30}
\]

where \( \ell_c(T) = \max(s, \xi_c(T)) \), \( s \) being the distance between superconducting planes in our layered material. \( E_c \) represents the typical energy barrier for the crossing of two flux lines nearly parallel to \( c \), and if \( E_c(T) \gg k_B T \) we expect interactions between lines to strongly reduce cutting and crossing of lines, and as a consequence, to lead to the kind of confinement of the internal modes we find above. In this regime, equation (2.29) should be a good approximation to the pair distribution function \( g_0(r) \) in a dilute flux liquid. Although the condition \( E_c > k_B T \) may prove to be too stringent for actual flux lines in a HTSC, we here consider this limit as an extreme case for which a disentangled state might be realized.

For the purpose of mathematical tractability, we shall here use the approximation (2.29) in the form (figure 1)

\[
g_0(r) = 1 - \exp(-\alpha r^2/a^2) \tag{2.31}
\]

which gives for the pair correlation function

\[
h_0(r) = -\exp(-\alpha r^2/a^2) \tag{2.32}
\]

The numerical coefficient \( \alpha \) inside the exponential is to be chosen so that \( |h_0(r = a)| \ll 1 \). Using this form of the pair correlation function into equation (2.28), we obtain:

\[
\mu = \frac{\pi}{d_\perp} (\rho a^2) \int \frac{d^2 k}{(2\pi)^2} k^2 V(k) e^{-k^2 a^2/2\alpha} \tag{2.33}
\]

Inserting the Fourier transform of the interaction potential

\[
V(k) = \frac{4\pi \varepsilon_0}{k^2 + \lambda^2} \tag{2.34}
\]

we obtain, after a few manipulations (Appendix A)

\[
\mu \approx \frac{2\pi}{d_\perp} \rho \varepsilon_0 \tag{2.35}
\]

where we assumed that \( a < \lambda \). It is interesting to note that the final result is, to a very good approximation, independent of the particular choice of \( \alpha \) in equation (2.32), as long as this last condition \( (a < \lambda) \) is satisfied. It is also interesting to note that a nontrivial structure in \( g_0(r) \) is necessary to obtain a finite value for the “mass” \( \mu \) : inserting in equation (2.28) the trivial (hydrodynamic) approximation

\[
g_0(r) = 1, \quad 0 < r < \infty
\]

would lead to a vanishing result for \( \mu \) and to the superfluid kind of behaviour described by equations (1.1) - (1.2).
We now turn our attention to the off-diagonal terms \( \mu_{\alpha\beta}^{(ij)} \) which couple the internal modes of vortices \( i \) and \( j \). We have:

\[
(\mu_{\alpha\beta}^{(ij)}) = \frac{1}{Z_0} \int \prod_{k=1}^{N} dr_{0k} \left[ -\partial_{\alpha} \partial_{\beta} V(r_{0i} - r_{0j}) \right] e^{-\beta H_0} = -\frac{1}{Z_0} \int dr_{01} dr_{02} \partial_{\alpha} \partial_{\beta} V(r_{01} - r_{02}) \times \int dr_{03} \ldots dr_{0N} e^{-\beta H_0}
\]

Comparing the \( rhs \) of the last equation with the \( rhs \) of equation (2.23), we see that

\[
(\mu_{\alpha\beta}^{(ij)}) = -\frac{1}{(N-1)} (\mu_{\alpha\beta}^{(i)})
\]

(2.36)

which, in view of the fact that the mass \( \mu = (\mu^{(i)}) \) is finite (i.e., independent of \( N \)), implies that the off diagonal terms vanish in the thermodynamic limit \( N \to \infty \).

We are now left with the following averaged Hamiltonian for the internal degrees of freedom of our flux line liquid:

\[
H_{eff} = \sum_{i=1}^{N} \int_0^L dz \left[ \frac{1}{2} \tilde{\varepsilon}_1 \left( \frac{d\mathbf{u}_i}{dz} \right)^2 + \frac{1}{2} \mu \mathbf{u}_i^2(z) \right] + \frac{1}{(N-1)} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \int_0^L dz \left[ \frac{1}{2} \mu \mathbf{u}_i(z) \cdot \mathbf{u}_j(z) \right]
\]

(2.37)

where we used the results (2.27)-(2.36) for the averaged diagonal \( \mu_{\alpha\beta}^{(ij)} \) and off-diagonal \( \mu_{\alpha\beta}^{(ij)} \) terms, in conjunction with equation (2.12). This last expression of \( H_{eff} \) can be rewritten in Fourier space in the form

\[
H_{eff} = \sum_{i=1}^{N} \sum_{n \neq 0} \frac{1}{2} (G^{-1})_{ij}(q_n) \mathbf{u}_i(q_n) \cdot \mathbf{u}_j(-q_n)
\]

(2.38)

with the inverse propagator

\[
(G^{-1})_{ij}(q_n) = \left[ L \tilde{\varepsilon}_1 q_n^2 + \mu + \frac{L\mu}{N-1} \right] \delta_{ij} - \frac{L\mu}{N-1}
\]

From this last equation, the propagator \( G_{ij}(q_n) \) defined through

\[
\langle u_i(q_n) u_j(q_m) \rangle = k_BT G(q_n) \delta_{n,-m}
\]

(2.39)

can be easily obtained using an identity for inverting \( N \times N \) matrices of the form

\[
(A^{-1})_{ij} = a \delta_{ij} + \frac{b}{a(a + bN)}
\]

(2.41)

namely,

\[
G_{ij}(q_n) = \frac{1}{\Gamma(q_n)} \delta_{ij} + \frac{L\mu/(N-1)}{\Gamma(q_n) \left( \Gamma(q_n) - L\mu N/(N-1) \right)}
\]

(2.42)

where we denote by \( \Gamma(q_n) \) the quantity

\[
\Gamma(q_n) = \left[ L \tilde{\varepsilon}_1 q_n^2 + \mu + \frac{L\mu}{N-1} \right]
\]

(2.43)

In the thermodynamic limit \( N \to \infty \), we see from equation (2.42) that \( G_{ij}(q_n) \) reduces to the diagonal form

\[
G_{ij}(q_n) = G(q_n) \delta_{ij},
\]

(2.44)

Hence, the total projected area of a given flux line in the liquid environment is given by

\[
\langle u^2(z) \rangle = d_\perp k_BT \sum_{n \neq 0} G(q_n) = \frac{2d_\perp k_BT}{L \tilde{\varepsilon}_1} \sum_{n=1}^{\infty} \frac{1}{q_n^2 + (\mu/\tilde{\varepsilon}_1)}
\]

(2.45)

Using the result (2.37) in equation (2.45), along with the fact that

\[
\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2} \approx \frac{\pi}{2a} \quad \text{for} \quad a \to \infty,
\]

(2.46)

we obtain (we take \( d_\perp = 2 \))

\[
\langle u^2 \rangle \approx \frac{k_BT}{2\varepsilon_0} \left( \frac{a}{\sqrt{\pi}} \right)
\]

(2.47)

The important thing to note about this result is that, unlike the free flux line, the mean square projected area \( \langle u^2 \rangle \) is not proportional to and in fact does not depend at all on the sample thickness \( L \). More significantly, if we take \( \rho \approx 1/a^2 \), where \( a \) is the average intervortex distance in the liquid phase, the above equation gives us:

\[
\langle u^2 \rangle \approx \frac{k_BT}{2\varepsilon_0} \frac{a}{\sqrt{\pi}}
\]

(2.48)

which is exactly the result one obtains in the crystalline phase, equation (1.3), and is actually what we expect on purely physical grounds for a flux line trapped in the cage potential formed by its surrounding neighbors, assuming that the barriers for cutting and crossing are high enough. That we are able to obtain such a result for \( \langle u^2 \rangle \) gives us confidence in our approach and in the various approximations that were made in the course of our calculation.
Having obtained $\langle u^2 \rangle$, we are now in a position to assess the range of validity of our analysis. Using the fact that $e_0(K/A) = 1.964 \times 10^8/|\lambda(A)|^2$, and the values $\varepsilon = 1/7$, $\lambda(0) = 1400\,\AA$ relevant to YBCO, we find

$$\frac{\langle u^2 \rangle}{a^2} \approx 0.27 \times 10^{-2} \left( \frac{\lambda(0)}{a} \right)$$  \hspace{1cm} (2.49)$$

where we approximated $\sqrt{\mathcal{A}} \approx 1/a$, and where we used the mean field dependence of the London penetration depth on temperature $\lambda(T) = \lambda(0)/\sqrt{1 - T/T_c}$. The width of the flux lines in a dilute liquid ($a \lesssim \lambda$) is hence very small compared to the average inter-vortex distance $a$. We need to go to much higher densities (i.e. to a higher ratio $\lambda/a$) and to temperatures very close to $T_c$ in order to have $\langle u^2 \rangle \approx a^2$. More quantitatively, in order for our perturbative analysis based on the taylor expansion (2.3) to be correct, the condition (2.3) has to be satisfied. This condition can be rewritten in the following, averaged form:

$$\langle (u_i(z) - u_j(z))^2 \rangle < a^2$$ \hspace{1cm} (2.50)$$

But, since $\langle u_i(z) \cdot u_j(z) \rangle = \langle u_i(z) \rangle \cdot \langle u_j(z) \rangle = 0$ for $i \neq j$ in our approach (we remind the reader that the off-diagonal coefficients $\mu_{ij}^{(z)}$ were found to vanish in the thermodynamic limit), the condition above can be rewritten in the form

$$\langle u_i^2 \rangle < a^2/2$$ \hspace{1cm} (2.51)$$

which, by equation (2.49), is sure to be satisfied on a rather large region of the phase diagram $(H, T)$ of the flux line system.

Let us now find the difference correlation function $\langle (u_i(z) - u_i(0))^2 \rangle$, which is given by:

$$\langle (u(z) - u(z'))^2 \rangle = \frac{4d_A k_B T}{L \varepsilon_1} \sum_{n=1}^{\infty} \frac{1 - \cos q_n(z - z')}{q_n^2 + (\mu/\varepsilon_1)}$$

Transforming the sum into an integral, and using the result

$$\int_0^\infty dq \frac{1 - \cos(q z)}{q^2 + a^2} = \frac{\pi}{2a} \left[ 1 - e^{-a |z|} \right] \hspace{1cm} a > 0$$

we obtain, after a few manipulations:

$$\langle (u(z) - u(z'))^2 \rangle \approx \frac{d_A k_B T a}{\sqrt{\tau} \varepsilon_0} (1 - e^{-\frac{\sqrt{\tau}}{2\varepsilon_0} |z - z'|})$$  \hspace{1cm} (2.52)$$

We thus find that the mean square displacement $\langle (u(z) - u(z'))^2 \rangle$ goes to a finite limit as $|z - z'| \to \infty$. This was expected, since in our approach, the internal modes are massive and have bounded fluctuations which do not grow with the sample thickness.

The last quantity we shall be interested in is the density-density correlation function

$$S(r - r'; z - z') = \langle \hat{\rho}(r, z) \hat{\rho}(r', z') \rangle$$  \hspace{1cm} (2.53)$$

where $\hat{\rho}(r, z)$ is the local density operator at height $z$

$$\hat{\rho}(r, z) = \sum_{i=1}^{N} \delta(r - r_i(z))$$  \hspace{1cm} (2.54)$$

According to the theory of reference [3], the partial Fourier transform $S(q, z)$ should behave as

$$S(q, z) \approx S(q, 0) e^{-\varepsilon(q)|z|}$$ \hspace{1cm} (2.55)$$

where $\varepsilon(q)$ is the Bogoliubov spectrum of the corresponding bosons. The above form of $S(q, z)$ suggests that the densities at two different heights $z$ and $z'$ become more decorrelated as $|z - z'|$ grows. Here, due to the fact that the internal modes are massive, we expect the density to remain correlated on the whole longitudinal length of the sample. Indeed, in Appendix B we show that the structure factor $S(q, z)$ within our model is given by:

$$S(q, z) = \rho^2 g_0(q) e^{-\varepsilon(q)^2 \frac{k_B T a}{\sqrt{\tau} \varepsilon_0}} + \rho \exp \left( -\frac{q^2 k_B T a}{2\sqrt{\tau} \varepsilon_0} (1 - e^{-\frac{\sqrt{\tau}}{2\varepsilon_0} |z|}) \right)$$ \hspace{1cm} (2.56)$$

which shows that the structure factor consists of two pieces: a CM piece, proportional to $\rho^2$, which corresponds to a liquid of more or less straight flux lines and includes, through $g_0(q)$, nontrivial correlations between the CM positions of flux lines, and a second piece linear in the density $\rho$, which describes the internal fluctuations of the individual lines and which incorporates the effect of interactions between vortices through the confining mass $\mu$.

Like the structure factor of the boson analogy, our $S(q, z)$ does decrease as a function of $|z|$, with the important difference, however, that $S(q, z)$ here goes to a finite limit (much like in a crystal) as $|z|$ grows very large, in contrast to the result of ref. [3] where $S(q, z) \to 0$ as $|z| \to \infty$, indicating that flux lines remain correlated over rather long length scales in a disentangled flux line liquid.

The most important implication of the theory developed in this section is related to the issue of entanglement of flux lines. Assuming that the typical energy scale for the interactions between flux lines is much higher than $k_B T$ and neglecting thermally nucleated vortex loops, we have derived the statistical mechanics of a dilute line liquid using a simple Taylor expansion of the Hamiltonian in terms of the internal modes of the flux lines. We have calculated the mean square projected area of a given flux line $\langle u^2 \rangle$ and we have found that it is of the same order of magnitude as what is usually obtained in a flux line lattice. The physical picture of a flux line in a liquid environment that emerges from our analysis, under the above assumptions, is that of a roughly straight object whose internal modes are contained in a tube of finite radius $\langle u^2 \rangle \lesssim a$, no matter how thick the sample might be. At this stage, we can already assert that we have achieved the goal stated in the Introduction, and that we now have
at our disposal a detailed and yet simple theory which is able to describe a completely disentangled flux line liquid. As we mentioned earlier, the assumption $E_c \gg k_B T$ is, however, too strong for real flux liquids on the melting line $H_m(T)$ (as can be easily checked using material parameters relevant to most HTSC), and our theory needs to be adjusted to take this fact into account. In the following section, we discuss and compare our findings with previous work, and we try to construct a consistent picture of what we expect the physics of flux line liquids to be based on our approach.

### III. DISCUSSION AND CONCLUSIONS

We shall start this section by briefly reviewing and discussing the results of numerical simulations of flux line liquids. A number of these simulations have used the mapping of the Ginzburg-Landau free energy of a superconductor onto the Hamiltonian of the uniformly frustrated three dimensional (3D) XY model:

$$H_{XY} = -\sum_{i,j} J_{i,j} \cos(\theta_i - \theta_j)$$

where, in a discrete lattice, $\theta_i$ is the phase of the complex Ginzburg-Landau order parameter $\psi(r,z) = |\psi(r,z)| e^{i\theta(r,z)}$ at site $i$, $A_{ij} = (2\pi/\phi_0) \int_i^r A \cdot dl$ is the line integral of the magnetic vector potential along a path linking site $i$ to site $j$, and the sum is over nearest-neighbor sites. The Hamiltonian (3.1) results from making the London approximation $\psi_0 = |\psi_0(x,z)| = \text{Cst.}$ in the discretized Ginzburg-Landau free energy of an anisotropic superconductor in the regime $\lambda/\xi \rightarrow \infty$, so that the magnetic induction $B = \hat{B} \hat{z}$ inside the superconductor can be taken as a constant. For a layered superconductor with the principal axis $c$ along the $\hat{z}$ axis, the couplings $J_{i,j}$ are given by $J_{i,j} = J_{\perp}$ when sites $i$ and $j$ are located on the same superconducting plane, with $J_{\perp} = \phi_0^2 s/(16\pi^2 \lambda^2)$, and $J_{i,j} = J_z = \varepsilon \phi_0^2 \xi^2/(16\pi^2 \lambda^2)$ when sites $i$ and $j$ are located at $\pm \hat{z}$ from each other, where $s$ is the spacing between superconducting layers.

Within the model (3.1), evidence for an entangled state is deduced from various dynamical quantities, such as the helicity modulus $\Upsilon (\mathbf{q})$ which gives the linear response of the supercurrent $\mathbf{j}$ to a transverse perturbation in the vector potential $A^{ext}$ of the externally applied magnetic field:

$$j_{\mu}(q_{\nu}) = -\Upsilon_{\mu,\nu}(q_{\nu}) \delta A^{ext}_{\mu}(q_{\nu}), \quad \mu \neq \nu$$

The helicity modulus measures the response of the system to an imposed phase twist $\hat{z}$, and is therefore related to the correlation function (here $\mathbf{R} = (r,z)$)

$$\Psi(\mathbf{R}) = \left\langle \psi(\mathbf{R}) \psi^* (\mathbf{0}) \exp \left(-\frac{2\pi i}{\phi_0} \int_0^R \mathbf{A} \cdot d\mathbf{l} \right) \right\rangle$$

which measures the coherence of the phase degrees of freedom in the system. Also measured in numerical simulations of model (3.1) is the structure factor $S(\mathbf{q}) = \langle \hat{\rho}(\mathbf{r},z) \hat{\rho}(\mathbf{0},0) \rangle$, where the density operator $\hat{\rho}(\mathbf{r},z) = N^{-1} \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i(z))$ is given in terms of the phase variables $\hat{\theta}(\mathbf{r},z)$ by

$$\hat{\rho}(\mathbf{x},z) = \frac{1}{2\pi} \left[ \nabla \times \nabla \theta(\mathbf{x},z) \right] \cdot \hat{z}$$

Information about entanglement and longitudinal correlations is then deduced from the behaviour of the structure factor $S(\mathbf{0},z) = \langle \hat{\rho}(\mathbf{r},z) \hat{\rho}(\mathbf{0},0) \rangle$ at zero transverse separations.

Another model which has been used in numerical studies of flux liquids is the so-called lattice London model, which is defined by the Hamiltonian (3.2)

$$H = 2\pi^2 J_{\perp} \sum_{i,j} \sum_{\mu=x,y,z} G_{\mu}(\mathbf{R}_i - \mathbf{R}_j) n_{\mu}(\mathbf{R}_i) n_{\mu}(\mathbf{R}_j) \tag{3.4}$$

where $n_{\mu}(\mathbf{R}_i)$ is the integer vorticity through plaquette $\mu$ at site $\mathbf{R}_i = (x,z)$ of a cubic mesh of points, and $J_{\perp}$ is the same coupling as the one defined above in the context of the uniformly frustrated 3D XY model. The lattice London interactions $G_{\mu}$ have Fourier transforms:

$$G_{\mu}(\mathbf{q}) = \frac{\varepsilon^2}{Q_x^2 + Q_y^2 + \varepsilon^2 (Q^2 + \lambda^2/\lambda^2)}$$

$$G_{\mu}(\mathbf{q}) = \frac{Q_x^2 + Q_y^2 + \varepsilon^2 (Q^2 + \lambda^2/\lambda^2)}{Q_z^4 + \varepsilon^2 (Q^2 + \lambda^2/\lambda^2)} \tag{3.5}$$

where $Q_{\mu} = 2 \sin(q_{s}/2)$ and $Q^2 = \sum_{\mu} Q_{\mu}^2$. Unlike the 3D XY model, which has Coulomb like interactions between vortex elements, which and is only valid in the regime $\lambda \gg a$ of dense systems, the lattice London model takes screening into account and can therefore be used for dense as well as for dilute flux liquids.

The last kind of simulation which has been performed is the Monte Carlo simulation of Nordborg and Blatter, who use the boson mapping to simulate a system of interacting flux lines. In this approach, the authors adapt an algorithm which has been used in the past to study superfluidity in quantum-mechanical Bose systems, involving large scale permutations of vortex trajectories subject to the “periodic” boundary conditions $\mathbf{r}_i(0) = \mathbf{r}_i(L)$, i.e. every line ends either on itself or on another line. Below we comment on this Monte Carlo method and the associated algorithm, but, before doing so, we want to consider the two other simulation methods, models (3.2) and (3.4).

We first observe that a direct comparison between our results and simulations of the uniformly frustrated 3D XY model (3.1) is made difficult by the fact that, while this model is relevant to the case $\lambda \gg a$, we here consider the opposite limit $a \lesssim \lambda$ of a dilute flux line liquid. In addition, and as we already mentioned, most of the simulations above were concerned with the measurement
of the helicity modulus, which is essentially a measure of phase correlations in the system. It should be noted, however, that phase correlations can be very different from density correlations. To illustrate this statement, we refer the reader to ref. 2, where it was found within a simple elastic approach that the order parameter for phase correlations $\langle \hat{\rho}_K(r,z) \rangle$ decays exponentially in the direction of the superconducting planes in a three dimensional flux line lattice 8, although the density order parameter $\langle \hat{\rho}_K(r,z) \rangle$ is known to be finite in such a lattice. This example shows clearly that there is no “one to one” correspondence between phase and density correlations, and that the vanishing of the helicity modulus in a flux line liquid for example does not necessarily mean exponential decay of density correlations along the $\hat{z}$ axis or correlations between flux line positions of the type of equations (3.1)-(3.2). This is even more so if we allow for the formation of vortex loops. Thermally excited vortex loop excitations were found to destroy phase coherence and to lead to a vanishing of the helicity modulus even in an otherwise perfectly ordered flux line lattice.

Finally, it should be realized that the simulations of models (3.1) and (3.4) are highly simplified representations of considerably more complex physics. For example, simulations are usually carried out at a given density of flux quanta $f = (B^2\xi^2/\phi_0) = (B/2\pi H_{c2})$ (we remind the reader that the upper critical field for $H|c$ is given by $H_{c2} = \phi_0/2\pi\xi^2$, with values of $f$ ranging from 1/12 to 1/30 most frequently used. Values of $f$ in this range correspond to values of the induction $B$ in the range $\frac{1}{2}H_{c2} - \frac{1}{3}H_{c2}$. Apart from the fact that the use of models (3.1) and (3.4), which are based on the assumption of a constant amplitude of the order parameter, becomes questionable in this range of fields, we also note that at such high fields vortices in adjacent superconducting planes in the liquid phase might very well be already decoupled (note also that the temperature dependence of the coherence length $\xi$ in $f = (B^2\xi^2/\phi_0)$ and of the London penetration depth $\lambda$ in $J_1$ and $J_2$ is generally left out as $T$ is varied). Moreover, in most of these simulations, which are carried out on a cubic mesh, the mesh constant in the $\hat{z}$ direction is associated with the distance $s$ between superconducting planes in a layered material, and no mention is made whatsoever of the coherence length $\xi_c$ along the $\hat{c}$ axis. In particular, the question regarding whether the same results, e.g. for density correlations in the liquid phase, would follow in the regimes $\xi_c(T) < s$ and $\xi_c(T) > s$, has largely remained untouched. The fact that most of these simulations consider implicitly the quasi-two-dimensional case $\xi_c(T) < s$ does not allow us to draw any conclusions on the interesting regime $\xi_c(T) \gg s$ where the average tilt angle of flux lines $\langle |d\mathbf{r}/dz|^2 \rangle \sim (k_B T/\xi^2\xi_c(T))$ can be very small and where we expect superconducting coherence, whether it be for the phase of the superconducting order parameter or for the density, to survive on much longer length scales.

Several remarks are due regarding the Monte Carlo simulation of Nordborg and Blatter 6, who use the boson mapping 1 to simulate a system of interacting flux lines. First, the use of the “periodic” boundary conditions $\mathbf{r}_i(L) = \mathbf{r}_i(0)$ imposes additional constraints on the system which are not present in a real flux line liquid. Actually, in order to “capture the effects of Bose statistics”, the flux lines in this simulation were made to switch their endpoints by hand, and this might very well introduce an artificial entanglement in the system. More specifically, the switching of endpoints is achieved by cutting out sufficiently long segments of a number of lines and trying different ways of connecting the loose ends while satisfying the periodic boundary conditions. While this procedure and the corresponding algorithm are appropriate for the imaginary time paths of quantum mechanical bosons, we find it more realistic that the flux line system should be allowed to equilibrate with free (as opposed to periodic) boundary conditions and, more importantly, without explicit switching of endpoints. Within the context of quantum mechanical bosons, superfluidity is brought about in numerical simulations by precisely these kind of manipulations. In the context of vortices, we expect these manipulations to lead to an overestimation of the effect of entanglement in flux liquids, and the “superfluid” behaviour found in reference 6 might therefore be a direct consequence of the specific bosonic algorithm used in this simulation.

In view of all the above remarks, and the results of section II, it seems to us that there is still room, at least in the dilute limit considered in this paper and neglecting vortex loops, for a new phase of the flux line liquid which has not been considered in the past, and which we might describe as weakly entangled. By “weakly entangled” flux liquid, we mean a liquid phase in which $\langle u^2 \rangle$ can be very large, but does not actually diverge with the sample thickness $L$. For this to happen, a small value of the confining mass $\mu$, much smaller than the one found in equation (2.35), is needed. In the following paragraph, we try to understand how such small values of $\mu$ can emerge from our model, and how a weakly entangled phase can be reconciled with what is observed in numerical simulations.

The key quantity in the derivation of the confining mass of section II is the pair distribution function $g_0(r)$ of the CM mode. In equation (2.34), we used an approximation for $g_0(r)$ which is relevant to a situation where the average distance between the centers of mass of the flux lines is $a = 1/\sqrt{\rho}$, and where the CM positions are strongly anti-correlated for $r \leq a$, i.e. if the CM of the $i$th flux line is at location $\mathbf{r}_i$, then the centers of mass of neighboring vortices have a very small probability to be within a distance $a$ from $\mathbf{r}_i$. While this is a perfectly legitimate way of thinking for actual flux line elements and for the regime $E_c(T) \gg k_B T$ considered in section II, since the CM is only a mathematical construct on one hand, and since the condition $E_c(T) \gg k_B T$ is usually not satisfied near melting on another (making situa-
where the CM positions of different flux lines are very close to each other (rather frequent), the CM distribution function does not have to be very small \((g_0(r) \ll 1)\) for \(r \leq a\). However, because of the repulsion between flux lines, we cannot completely neglect correlations between the center of mass positions and simply take \(g_0(r) = 1\) for \(0 < r < \infty\), in which case we would recover a fully entangled line liquid with the correlations \((1.1)-(1.2)\). Since there is obviously no easy way to derive an analytic expression for the pair distribution function \(g_0(r)\), we here are bound to speculate about its possible shape using our physical intuition and infer the resulting physics. One possible shape for \(g_0(r)\) which allows for close encounters between the centers of mass of different flux lines is the one shown in figure 2, and corresponds to the approximation

\[
g_0(r) \simeq 1 - \eta \exp\left(-\alpha r^2/a^2\right) \tag{3.6}
\]

where \(\eta\) is a numerical constant such that \(0 < \eta < 1\) (in figure 2, we use \(\eta = 0.1\)). Such a form of \(g_0(r)\) would yield a confining mass which is \(\eta\) times smaller than the one found previously, equation \((2.33)\), and would therefore lead to values of the average projected area \(\langle u^2 \rangle\)

\[
\langle u^2 \rangle \simeq \frac{k_BTa}{2\eta \sqrt{\pi \varepsilon_0}} \tag{3.7}
\]

which are much larger than the square of the average distance \(a\) between flux lines. On the other hand, equation \((3.6)\) would lead to a relative displacement

\[
\langle u(z) - u(z')^2 \rangle \simeq \frac{d_{\perp} k_BT a}{\eta \sqrt{\pi \varepsilon_0}} \left(1 - e^{-\frac{\sqrt{\varepsilon_0} |z-z'|}{\varepsilon}}\right) \tag{3.8}
\]

For \(\eta \ll 1\) there will be a substantial range of separations

\[
|z-z'| < \ell_z = \frac{\varepsilon a}{\eta \sqrt{\pi}} \tag{3.9}
\]

where the rhs of equation \((3.8)\) can very well be approximated by

\[
\langle u(z) - u(z')^2 \rangle \simeq \frac{d_{\perp} k_BT}{\varepsilon^2 \varepsilon_0} |z-z'| \tag{3.10}
\]

(none that factors of \(\eta\) have cancelled each other in this last equation) and for which density correlations \(\langle \hat{\rho}(q,z) \hat{\rho}(-q,0) \rangle\) will decrease exponentially as a function of \(|z|\), leading to the structure factor

\[
S(q,z) \simeq \rho e^{-q^2 \frac{d_{\perp} k_BT}{\varepsilon^2 \varepsilon_0} |z|} \tag{3.11}
\]

Note, in particular that for moderately anisotropic superconductors, and if \(\eta \ll 1\), the length \(\ell_z\) can be quite large, maybe even larger than the relatively small thicknesses used in numerical simulations, giving the semblance that the entangled state is characterized by the correlations \((1.1)-(1.2)\) with an entanglement correlation length

\[
\xi_v \simeq \frac{\varepsilon^2 \varepsilon_0 a^2}{d_{\perp} k_BT} \tag{3.12}
\]

On the other hand, for values of \(\eta\) close to unity \((\eta \lesssim 1)\), we recover a flux line liquid where correlations extend over much longer distances, which corresponds to the situation analyzed by Righi et al.\(\textsuperscript{4}\). We thus see that our model can describe, depending on what \(g_0(r)\) in reality is, very different situations corresponding to nearly disentangled \((\eta \lesssim 1)\), weakly entangled \((\eta \ll 1)\) or fully entangled \((\eta = 0; g_0(r) = 1)\) flux line liquids.

To summarize, in this paper, using a perturbative expansion of the Hamiltonian of interacting flux lines in a vortex liquid similar to the expansion of the action of a quantum particle around a classical path\(\textsuperscript{2}\), we have constructed a mean field theory of the flux line liquid which, in the author’s opinion, has a better handling of the internal fluctuations of flux lines, and which, through the use of a nontrivial pair distribution function \(g_0(r)\) for the positions of the centers of mass of flux lines, goes beyond Gaussian hydrodynamics, making contact with the standard liquid state theory of classical fluids. Within our approach, we find that a weakly entangled phase might exist, where the average width of flux lines \((w^2)^{1/2}\) can be much larger than the average intervortex distance \(a\), but does not diverge with the sample thickness \(L\). By varying the shape of \(g_0(r)\), we are able to describe situations with very short and rather large entanglement correlation lengths. Since the slightest deviation of the CM pair distribution function from unity at small distances \((r \leq a)\) will necessarily lead to a finite (nonzero) confining mass \(\mu\), a careful measurement of \(g_0(r)\), for example in numerical simulations, would provide an unequivocal way of verifying the predictions of this paper, and to ascertain whether a fully entangled liquid phase with correlations \((1.1)-(1.2)\) or the weakly entangled phase proposed here is the correct ground state of the flux line liquid.

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IV. APPENDIX A : DETAILS OF THE CALCULATION OF THE MASS $\mu$ FOR THE INTERNAL MODES OF VORTICES IN A FLUX LINE LIQUID

In this appendix, we show details of the calculation of the integral on the rhs of equation (2.33) which gives the value of the effective mass $\mu$. Using the expression (2.34) of the interaction potential in Fourier space $V(k)$, and performing the integration over the polar angle, we obtain

$$\mu = \frac{2\pi}{d_\perp \alpha^2} (\rho \varepsilon_0) \int_0^\infty dk \frac{\varepsilon_0 k^2}{k^2 + \lambda^2} e^{-k^2 \alpha^2 / 2 \alpha}$$  \hspace{1cm} (4.1)

Using the change of variables $x = \lambda^2 k^2$, this last expression can be cast in the form

$$\mu = \frac{\pi a^2}{d_\perp \alpha^2} (\rho \varepsilon_0) \int_0^\infty dx \frac{x}{1 + x} e^{-\kappa x}$$  \hspace{1cm} (4.2)

with $\kappa = a^2 / 2 \alpha \lambda^2$. Now, using the result (4.3)\footnote{Where $E_1(x) = \int_0^\infty dt e^{-t / t}$ is the exponential integral, and using the Taylor expansion for small arguments $E_1(x) = -\gamma - \ln x - \sum_{n=1}^\infty \frac{(-1)^n x^n}{n!}$ (here $\gamma = 0.577 \ldots$ is Euler’s constant), we obtain}

$$\int_0^\infty dx \frac{x}{1 + x} e^{-\kappa x} = \frac{1}{\kappa} - e^\kappa E_1(\kappa)$$  \hspace{1cm} (4.3)

where $E_1(x) = \int_0^\infty dt e^{-t / t}$ is the exponential integral, and using the Taylor expansion for small arguments $E_1(x) = -\gamma - \ln x - \sum_{n=1}^\infty \frac{(-1)^n x^n}{n!}$ (here $\gamma = 0.577 \ldots$ is Euler’s constant), we obtain

$$\mu = \frac{\pi a^2}{d_\perp \alpha^2} (\rho \varepsilon_0) \cdot \frac{1}{\kappa} \left(1 + \kappa e^\kappa(\gamma + \ln \kappa) + o(\kappa^2)\right)$$  \hspace{1cm} (4.5)

Using the fact that $\kappa = a^2 / 2 \alpha \lambda^2 < 1$ in the regime $a < \lambda$ (assuming that $\alpha > 1$), we finally obtain

$$\langle \hat{\rho}(q, z) \hat{\rho}(-q', z') \rangle = \frac{\pi a^2}{d_\perp \alpha^2} (\rho \varepsilon_0) \cdot \frac{2\alpha \lambda^2}{a^2} \frac{1}{\kappa} \left(1 + \kappa e^\kappa(\gamma + \ln \kappa) + o(\kappa^2)\right)$$

which is the result quoted in the text, equation (2.35).

V. APPENDIX B : DETAILS OF THE CALCULATION OF THE STRUCTURE FACTOR OF A FLUX LINE LIQUID

In this appendix, we show some details of the calculation of the structure factor

$$S(r, z; r', z') = \langle \hat{\rho}(r, z) \hat{\rho}(r', z') \rangle$$  \hspace{1cm} (5.1)

both for a free flux line liquid of noninteracting flux lines, and for the weakly entangled flux line liquid described in the text. In general, we expect flux liquids at equilibrium to be translationally invariant, so that the structure factor will depend only on the relative coordinates $(r - r')$ and $(z - z')$, i.e. $S(r, z; r', z') = S(r - r', z - z')$. As a consequence, we have for the Fourier transform of the density-density correlation function : $\langle \hat{\rho}(q, z) \hat{\rho}(-q', z') \rangle = (2\pi)^d \delta(q + q') \delta(q_z + q_z') S(q, q_z)$

$$\langle \hat{\rho}(q, z) \hat{\rho}(-q, -q_z) \rangle = \frac{1}{L L_\perp} \langle \hat{\rho}(q, z) \hat{\rho}(-q, -q_z) \rangle$$  \hspace{1cm} (5.3)

where we used equation (5.2) above and the fact that $(2\pi)^d \delta(q = 0) \delta(q_z = 0) \equiv L L_\perp$ in the limit $L, L_\perp \to \infty$. We have :

$$\langle \hat{\rho}(q, z) \hat{\rho}(-q, -q_z) \rangle = \int dr dz \int dr' dz' \langle \hat{\rho}(r, z) \hat{\rho}(r', z') \rangle e^{-i q \cdot (r - r')} e^{-i q_z (z - z')}$$

$$= \sum_{i=1}^N \sum_{j=1}^N \int dr dz \int dr' dz' \langle \hat{\rho}(r - r_i(z)) \hat{\rho}(r' - r_j(z')) \rangle e^{-i q \cdot (r - r')} e^{-i q_z (z - z')}$$

$$= \sum_{i=1}^N \sum_{j \neq i}^N \int dz \int dz' \langle e^{-i q \cdot (r_i(z) - r_j(z'))} \rangle_0 \langle e^{-i q_z \cdot (u_i(z) - u_j(z))} \rangle_1 e^{-i q_z (z - z')}$$

$$+ \sum_{i=1}^N \int dz \int dz' \langle e^{-i q_z \cdot (u_i(z) - u_j(z'))} \rangle_1 e^{-i q_z (z - z')}$$

$$= \sum_{i=1}^N \sum_{j \neq i}^N \int dz \int dz' \langle e^{-i q \cdot (r_i(z) - r_j(z'))} \rangle_0 e^{-i q_z \cdot (u_i(\alpha(z) - u_j(\alpha(z))) [(u_i, \beta(z) - u_j, \beta(z))]_1 e^{-i q_z (z - z')}$$

$$+ \sum_{i=1}^N \int dz \int dz' e^{-i q \cdot (r_i(z) - r_i(z'))} e^{-i q_z \cdot (u_i(\alpha(z) - u_j(\alpha(z))) [(u_i, \beta(z) - u_j, \beta(z))]_1 e^{-i q_z (z - z')}}$$  \hspace{1cm} (5.4)

where $\langle \cdots \rangle_0$ and $\langle \cdots \rangle_1$ denote averages over the CM and internal modes with statistical weights $\exp(-\beta H_0)$ and
we finally obtain : 

$$\langle [u_{i,\alpha}(z) - u_{j,\alpha}(z')] [u_{i,\beta}(z) - u_{j,\beta}(z')] \rangle_1 = \langle u_{i,\alpha}(z) u_{i,\beta}(z) + u_{j,\alpha}(z') u_{j,\beta}(z') \rangle_1$$

$$= \frac{2\delta_{\alpha,\beta}}{d_{\perp}} \langle u^2 \rangle$$ (5.5)

where, in going from the first to the second line, we used the fact that $\langle u_{i,\alpha} u_{i,\beta} \rangle_1 = \delta_{\alpha,\beta} \langle u^2 \rangle$. Equation (5.4) becomes

$$\langle \hat{\rho}(q, q') \hat{\rho}(-q, -q_z) \rangle = \sum_{i=1}^{N} \sum_{j \neq i} \int d\zeta \int d\zeta' \left( e^{-i\mathbf{q} \cdot (\mathbf{r}_0 - \mathbf{r}_0')} \right) e^{-\frac{1}{2}\mathbf{q}^2 \langle u^2 \rangle} \langle q, q' \rangle \rho \rho \theta_0(q) + \sum_{j \neq i} \int d\zeta \int d\zeta' e^{-\frac{1}{2}i\mathbf{q} \cdot (u_{i,\alpha}(z) - u_{i,\alpha}(z')) [u_{i,\beta}(z) - u_{i,\beta}(z')]} e^{-i\mathbf{q} \cdot (z - z')}(5.6)$$

Using the fact that $\int d\zeta \int d\zeta' e^{-i\mathbf{q} \cdot (z - z')} = L^2 \delta_{q_x,0}$, and noticing that

$$\sum_{i=1}^{N} \sum_{j \neq i} (e^{-i\mathbf{q} \cdot \mathbf{r}_0} e^{-i\mathbf{q} \cdot \mathbf{r}_0'}) = (2\pi)^{d_{\perp}} \delta(q + q') \rho^2 \theta_0(q)$$

which gives us here (with $q' = -q$)

$$\sum_{i=1}^{N} \sum_{j \neq i} (e^{-i\mathbf{q} \cdot \mathbf{r}_0} e^{i\mathbf{q} \cdot \mathbf{r}_0}) = (2\pi)^{d_{\perp}} \delta(q = 0) \rho^2 \theta_0(q) = L^2_{\perp} \rho^2 \theta_0(q)$$ (5.8)

we finally obtain :

$$\langle \hat{\rho}(q, q_z) \hat{\rho}(-q, -q_z) \rangle = L^2 \delta_{q_x,0} L^2_{\perp} \rho^2 \theta_0(q) e^{-\frac{1}{2}\mathbf{q}^2 \langle u^2 \rangle} + \sum_{i=1}^{N} \int d\zeta \int d\zeta' e^{-\frac{1}{2}\mathbf{q}^2 \langle u^2 \rangle} \langle [u_{i,\alpha}(z) - u_{i,\alpha}(z')] [u_{i,\beta}(z) - u_{i,\beta}(z')] \rangle_1 e^{-i\mathbf{q} \cdot (z - z')}(5.9)$$

where we used the fact that $\langle [u_{i,\alpha}(z) - u_{i,\alpha}(z')] [u_{i,\beta}(z) - u_{i,\beta}(z')] \rangle_1 = \frac{\delta_{\alpha,\beta}}{d_{\perp}} \langle [u_{i,\alpha}(z) - u_{i,\alpha}(z')]^2 \rangle_1$. From this point on, things will differ depending on whether we consider a liquid of free flux lines or a weakly entangled flux liquid. We shall treat both cases separately, starting with a liquid of free flux lines.

A. Liquid of free flux lines

In a liquid of free flux lines (or in a fully entangled “superfluid” state, for that matter), the average width of flux lines $\langle u^2 \rangle^{\frac{1}{2}}$ diverges with the sample thickness $L$. We can therefore drop the first term on the rhs of equation (5.9), which contains the factor $e^{-\frac{1}{2}\mathbf{q}^2 \langle u^2 \rangle}$. If we in addition use the fact that

$$\langle [u_{i,\alpha}(z) - u_{i}(z')] \rangle_1 = \frac{d_{\perp} k_B T}{\varepsilon_i} \left| z - z' \right|$$ (5.10)

we obtain :

$$\langle \hat{\rho}(q, q_z) \hat{\rho}(-q, -q_z) \rangle = \sum_{i=1}^{N} \int d\zeta \int d\zeta' e^{-\frac{1}{2}\mathbf{q}^2 \frac{2k_B T}{\varepsilon_i} \left| z - z' \right|} e^{-i\mathbf{q} \cdot (z - z')}(5.11)$$

Performing the $Z$ integration, and using the fact that the density $\rho = N/L^2_{\perp}$, we finally obtain
\[
\langle \hat{\rho}(q, z) \hat{\rho}(-q, -z) \rangle = LL^{d_1} \frac{\rho k_B T q^2/\tilde{\epsilon}_1}{q^2 + (k_B T q^2/2\tilde{\epsilon}_1)^2}
\]  

(5.12)

From the structure factor \(S(q, z) = \langle \hat{\rho}(q, z) \hat{\rho}(-q, -z) \rangle / LL^{d_1}\), we can deduce the density-density correlation function \(S(r, z) = \langle \hat{\rho}(r, z) \hat{\rho}(0, 0) \rangle\) in real space. We have:

\[
S(r, z) = \int \frac{d^d q}{(2\pi)^d} \int \frac{dq_z}{2\pi} S(q, z) e^{iqr} e^{iq_z z}
\]

(5.13)

The integrations being quite straightforward for the free flux liquid considered here, we only quote the following intermediate result:

\[
S(q, z) = \rho \exp \left( -\frac{k_B T q^2}{2\tilde{\epsilon}_1} |z| \right)
\]

(5.14)

along with the final result

\[
S(r - r', z - z') = \rho \left( \frac{2\pi\tilde{\epsilon}_1}{k_B T |z - z'|} \right)^{d_1/2} \exp \left( -\frac{\tilde{\epsilon}_1}{2k_B T} (r - r')^2 \right)
\]

(5.15)

which shows that the density-density correlation function \(\langle \hat{\rho}(r, z) \hat{\rho}(r, z') \rangle\) at the same transverse location \(r\) behaves like

\[
S(0, z) = \rho \left( \frac{2\pi\tilde{\epsilon}_1}{k_B T |z - z'|} \right)^{d_1/2}
\]

(5.16)

and thus we see that, even for the fully entangled noninteracting flux line liquid, density-density correlations \(\langle \hat{\rho}(r, z) \hat{\rho}(r, z') \rangle\) at the same transverse location \(r\) decrease only algebraically as a function of the height separation \(|z - z'|\). It is therefore very surprising that \(S(0, z)\) was found to decrease exponentially, like \(\exp(-|z|/\xi_a)\), in the presence of interactions in some numerical simulations of the uniformly frustrated 3D XY model.

### B. Weakly entangled flux liquid

In a weakly entangled flux line liquid, \(\langle u^2 \rangle\) does not diverge with the sample thickness \(L\), and therefore the first term on the rhs of equation (5.9) has to be kept. Using equation (3.8),

\[
\langle \{u(z) - u(z')\}^2 \rangle \simeq \frac{d_1 k_B T a}{\sqrt{\pi \tilde{\epsilon}_0}} \left( 1 - e^{-\frac{\pi u^2}{a^2 |z - z'|}} \right)
\]

we obtain:

\[
\langle \hat{\rho}(q, z) \hat{\rho}(-q, -z) \rangle = L^2 \delta_{q, 0} L^{d_1} \rho^2 g_0(q) e^{-\frac{k_B T a}{\sqrt{\pi \tilde{\epsilon}_0}} q^2} + NL \int dZ \exp \left( -q^2 \frac{k_B T a}{2\sqrt{\pi \tilde{\epsilon}_0}} (1 - e^{-\frac{\pi u^2}{a^2 |z|}}) \right) e^{-iq_z z}
\]

\[
= LL^{d_1} \left\{ \frac{\delta_{q, 0}}{L} \rho^2 g_0(q) e^{-\frac{k_B T a}{\sqrt{\pi \tilde{\epsilon}_0}} q^2} + \rho \int dZ \exp \left( -q^2 \frac{k_B T a}{2\sqrt{\pi \tilde{\epsilon}_0}} (1 - e^{-\frac{\pi u^2}{a^2 |z|}}) \right) e^{-iq_z z} \right\}
\]

from which we see that \(S(q, z) = \frac{1}{LL^{d_1}} \langle \hat{\rho}(q, z) \hat{\rho}(-q, -z) \rangle\) is given by

\[
S(q, z) = L \delta_{q, 0} \rho^2 g_0(q) e^{-\frac{k_B T a}{\sqrt{\pi \tilde{\epsilon}_0}} q^2} + \rho \int dZ \exp \left( -q^2 \frac{k_B T a}{2\sqrt{\pi \tilde{\epsilon}_0}} (1 - e^{-\frac{\pi u^2}{a^2 |Z|}}) \right) e^{-iq_z z}
\]

(5.17)

where we used the fact that \(\langle u^2 \rangle \simeq (d_1 k_B T a/2\eta\sqrt{\pi \tilde{\epsilon}_0})\), equation (3.7). In view of the fact that the integration on the rhs of this last equation cannot be performed exactly, it is more convenient to use the partial Fourier transform:

\[
S(q, z) = \rho^2 g_0(q) e^{-\frac{k_B T a}{\sqrt{\pi \tilde{\epsilon}_0}} q^2} + \rho \int dZ \exp \left( -q^2 \frac{k_B T a}{2\eta\sqrt{\pi \tilde{\epsilon}_0}} (1 - e^{-\frac{\pi u^2}{a^2 |Z|}}) \right)
\]

(5.18)

which, with \(\eta = 1\), is the result (2.56) of section II. On the other hand, for \(\eta \ll 1\), the “Debye-Waller” factor \(\exp\left( -q^2 \frac{k_B T a}{2\eta\sqrt{\pi \tilde{\epsilon}_0}} \right)\) can be very small. Neglecting the first term on the rhs of equation (5.18), we obtain

\[
S(q, z) \simeq \rho \exp \left( -q^2 \frac{k_B T a}{2\eta \sqrt{\pi \tilde{\epsilon}_0}} (1 - e^{-\frac{\pi u^2}{a^2 |z|}}) \right)
\]

which, for \(|z| < \ell_z = \varepsilon a/\eta \sqrt{\pi}\) leads to equation (3.11) of the text.
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