Twistor–like Formulation of the Supermembrane in D=11 *

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Abstract. We propose a new formulation of the D=11 supermembrane theory that involves commuting spinors (twistor–like variables) and exhibits a manifest $n$–extended world volume supersymmetry ($1 \leq n \leq 8$). This supersymmetry replaces $n$ components of the usual $\kappa$–symmetry. We show that this formulation is classically equivalent to the standard one.

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1 - Introduction

Recently there has been some interest in an alternative approach \cite{1}-\cite{12} to superparticles and G.S. superstrings in the special dimensions \( D = 3, 4, 6, 10 \). This approach involves commuting spinors, i.e. twistor–like variables* and exhibits both manifest target space supersymmetry and \( n \)-extended world line/world sheet supersymmetry \((1 \leq n \leq D-2)\). The latter replaces \( n \) components (and therefore provides a geometrical meaning) of the \( \kappa \)-symmetry in the standard formulation. It has been shown by Berkovits \cite{10} that the \( n = 2, D = 10 \) twistor–string model \cite{4} gives rise to a consistent quantization of the D=10, G.S. heterotic string with vanishing conformal anomaly.

At least at the classical level, the maximally extended models with \( n = D-2 \) are of special interest since in this case the whole \( \kappa \)-symmetry is replaced by world line/world sheet supersymmetry. The correspondence between the critical dimensions \( D = 3, 4, 6, 10 \) and the division algebras of real, complex, quaternionic and octonionic numbers respectively has been pointed out several times \cite{15},\cite{16}. However the non associativity of the octonions has represented an obstacle to get maximally extended models in \( D = 10 \). Nevertheless a way to overcome this \( D = 10 \) obstruction has been found recently for superparticles \cite{7} and heterotic strings \cite{8},\cite{9}. This construction makes use of eight twistors suitably constrained. Its geometrical meaning is clear from refs. \cite{6},\cite{7}: the eight twistors parametrize the sphere \( S^8 \), considered as the coset manifold \( SO(9,1)/SO(8) \otimes S^{\uparrow}(1,1) \times K \) where \( K \) represents the eight conformal boosts.

Unfortunately the \( D = 10 \) twistor–like heterotic string model is incomplete. Indeed a consistent world sheet supersymmetric treatment of the heterotic fermions is still lacking for \( n > 2 \).

Clearly it is worthwhile to extend the twistor approach to other models where \( \kappa \)-symmetry is present, as non heterotic superstrings and supermembranes \cite{17}. A formulation of type II superstrings with \( n = 1 \) world sheet supersymmetry has been described in ref. \cite{11}. Supermembranes have been considered in a different but related approach in ref. \cite{12}.

The extension of the approach to supermembranes is interesting for at least two reasons. Firstly, for supermembranes the critical dimension is \( D = 11 \) so that a supermembrane twistor–like model in \( D = 11 \) would provide an example of the

* The usefulness of twistors in string theory has been pointed out by several authors \cite{13}-\cite{16}.
use of commuting spinors (twistors) in a dimension \(D = 11\) where the cyclic \(\Gamma\)–matrix identity (eq. (1.1) below) does not hold (it is replaced by eq. (1.2) below). Secondly, from such a model one can get, by dimensional reduction, the twistor–like action for type II A superstrings in \(D = 10\).

In this paper we give a twistor–like formulation of \(D = 11\) supermembranes. It involves commuting spinors and shows \(n\)–extended world volume supersymmetry with \(1 \leq n \leq 8\).

In sect. 2 we review the main ingredients involved in the twistor–like approach for the heterotic strings. In sect. 3 we discuss the constraints and describe the classical twistor–like action for supermembranes in \(D = 11\). In sect. 4 we show that this action give rise to the same field equations of the standard one.

In a forthcoming paper we shall derive from this new supermembrane formulation the twistor–like action for the \(D = 10\), type II A superstrings by performing, as in ref. [18], a simultaneous reduction of one world volume and one target space dimension.

As for our notations, vector and spinor indices are denoted by Latin and Greek letters respectively and Capital letters stand for both kind of indices. Moreover we shall follow the convention of ref. [9] to write indices of the target space \(\mathcal{M}(D|N)\) (of the world manifold \(\mathcal{M}(d|n)\)) as underlined (non underlined) letters. Letters from the beginning of the alphabet are kept for the tangent spaces. \(d\) and \(D\) (\(n\) and \(N\)) are the bosonic (fermionic) dimensions of \(\mathcal{M}(d|n)\) and \(\mathcal{M}(D|N)\) respectively. We shall consider only those dimensions \(D\) where Majorana spinors exist. In these dimensions \(\Gamma^a_{\alpha\beta} = (C\gamma^a)_{\alpha\beta}\) and \(\Gamma^{\alpha\beta a} = (\gamma^a C^{-1})_{\alpha\beta}\) are symmetric (\(\gamma^a\) are Dirac matrices and \(C\) is the charge conjugation matrix in \(D\) dimensions). The corresponding matrices in \(d = 3\) are denoted by \(\sigma^a_{\alpha\beta}\) and \(\sigma^{\alpha\beta a}\). In particular \(\sigma_0 = (1 0)\); \(\sigma_1 = (0 1)\), \(\sigma_2 = (1 0)\) and \(\sigma_+ = \frac{1}{2}(\sigma_0 + \sigma_2) = (1 0)\); \(\sigma_- = \frac{1}{2}(\sigma_0 - \sigma_2) = (0 1)\). Indices between round brackets (square brackets) are symmetrized (antisymmetrized). However the antisymmetric product of \(p\) \(\Gamma\)'s (i.e. \(p\gamma^a\) times \(C\) or \(C^{-1}\)) is denoted by \(\Gamma^{\alpha_1\ldots\alpha_p}_{\alpha_p}\). The same for \(\sigma_{ab}\). In \(D = 3, 4, 6, 10\) one has the fundamental cyclic identity

\[
\Gamma_{a(\alpha\beta}\Gamma^{a}_{\gamma\delta}) = 0
\]  

(1.1)

In \(D = 11\) the cyclic identity is

\[
\Gamma_{ab(\alpha\beta}\Gamma^{ab}_{\gamma\delta}) = 0
\]  

(1.2)
Before discussing supermembranes, it is convenient to review the two main ingredients on which is based the twistor–like formulation of the G.S. heterotic strings. These models describe the embedding of the superworld \( \mathcal{M}(2|n) \) into the target superspace \( \mathcal{M}(D|2(D-2)) \), where \( D = 3, 4, 6, 10 \) and \( 1 \leq n \leq D-2 \).

\( \mathcal{M} \) is parametrized locally by \( \zeta^I \equiv (\xi^{(\pm)}, \eta^{(q)}) \) where \( \xi^{(\pm)} \) are the coordinates of the world sheet and \( \eta^{(q)} \) are real Grassman parameters \( (q = 1, \ldots, n) \). The superzweinbeins \( e^A \equiv (e^+, e^q) \) define a preferred frame in the cotangent space of \( \mathcal{M} \), with structure group \( SO(1,1) \otimes SO(n) \). \( SO(1,1) \) is the Lorentz group in \( d = 2 \) and \( SO(n) \) acts on \( e^q \). The indices \( \pm \) and \( q \) are raised and lowered with the light–cone metric and the euclidean metric respectively.

\[ d^2 = e^+ d^+ + d^- d^- + e^q d^q \]

\[ \Delta = e^+ \Delta_+ + e^- \Delta_- + e^q \Delta_q \]

The torsion \( T^A = \Delta e^A \) has the following structure

\[ T^+ = 0; \quad T^- = e^q \wedge e_q; \quad T^q = e^+ \wedge e^- T^q_+ \]

which is compatible with the relevant Bianchi identities.

The target superspace \( \mathcal{M} \) is parametrized locally by the string supercoordinates \( Z^M(\zeta) \) which are world sheet superfields. The tangent space geometry of \( \mathcal{M} \) is described by the supervielbeins \( E^A(Z) \), the Lorentz valued superconnection \( \Omega_A^B(Z) \), the Lie–G valued superconnection \( A(Z) \) and the two–superform \( B(Z) \) with their curvatures \( T^A, R_A^B, F \) and \( H = dB \). The intrinsic components of these curvatures are restricted by the SUGRA–SYM constraints

\[ T^A_{\alpha \beta} = 2\Gamma^A_{\alpha \beta}; \quad F_{\alpha \beta} = 0; \quad H_{\alpha \beta \gamma} = 0 \] \hfill (2.2a)

\[ T^\alpha_{\beta \gamma} = 0 = T^\alpha_{b \beta} \]

\hfill (2.2b)

\[ H_{\alpha \beta \gamma} - \phi(Z)\Gamma_{\alpha \beta \gamma} = 0; \quad H_{\alpha \beta \gamma} = -\frac{1}{2}(\Gamma_{\alpha \beta})_{\gamma}^\delta \Delta^\beta_\delta \phi \] \hfill (2.2c)

Eq. (2.2b) are conventional constraints and eqs. (2.2c) follow from eqs. (2.2a), (2.2b) using the Bianchi identities. \( \phi(Z) \) is the dilaton background superfield.

Moreover \( E_A^\Delta \equiv (E^A_+, E^A_q) \) are the intrinsic components of the pull–back of \( E^A \) on \( \mathcal{M} \) and \( E_A^\Delta_{\vert_{\eta^{(p)}=0}} \equiv (E^A_+, \lambda^A_q) \). Notice that \( \lambda^A_q \) are commuting spinors (twistors). Twistors enter in these formulations through the relation
which, due to the cyclic identity (1.1), implies the Virasoro constraint $\mathcal{E}_-^{\underline{a}} \mathcal{E}^{\underline{a}}_{-} = 0$.

The first key ingredient is to implement eq. (2.3) as a world sheet superfield constraint in order to preserve the world sheet supersymmetry. This can be done by requiring that the components of the pull–back of the vector supervielbeins $E^{\underline{a}}$ along the world sheet tangent space spinor directions vanish, i.e. by imposing the following twistor constraint:

$$E^{\underline{a}}_{\eta} = 0 \quad (2.4)$$

In fact the condition $\Delta (\rho E^{\underline{a}}_{\eta})|_{\eta=0} = 0$, that follows from eq. (2.4), reproduces eq. (2.3). The twistor constraint (2.4) is implemented by the action term

$$I^{(c)} = \int_{\mathcal{M}} P^{\underline{a}}_{\underline{\eta}} E^{\underline{a}}_{\underline{\eta}} \quad (2.5)$$

where the lagrangian multipliers $P^{\underline{a}}_{\underline{\eta}}$ are anticommuting world sheet superfields. For superparticles (not coupled to a super Maxwell background) this is the whole story and (the analog of) $I^{(c)}$ represents the full superparticle action. However for the heterotic strings $I^{(c)}$ must be supplemented with two further terms, $I^{(B)}$ that involves the two superform $B = \frac{1}{2} E^{A} \wedge E^{B} B_{BA}$ and $I^{(h)}$ that describes the heterotic fermions. The problem is to write $I^{(B)}$ and $I^{(h)}$ without breaking the world sheet supersymmetry.

In models with $n=1$ this can be done easily [3]:

$$I^{(B)} + I^{(h)} = \int d^2 \xi \ d\eta (\text{sdet } e) \left[ E^{A}_{+} E^{B}_{+} B_{BA} + i \bar{\psi} D_1 \psi \right]$$

where $\psi$ is a set of $N_h$ world–sheet Weyl–Majorana spinors (heterotic fermions) and $D_\psi = (\Delta - A) \psi$ (in $D = 10$, $N_h = 32$).

However for $n > 1$ $I^{(B)}$ and $I^{(h)}$ cannot be written so simply as superspace integrals.

The second key ingredient that allows to write the world sheet supersymmetric action $I^{(B)}$ is an interesting property [9] carried by $B(Z)$ if the SUGRA–SYM and twistor constraint holds. Indeed let us consider the two superform:

$$\tilde{B} = B - \frac{1}{2n} e^+ \wedge e^- H_{+q}^q \quad (2.6)$$

where
\[ H_{+qp} = E_+^A E_+^B E_+^C H_{CBA} = E_+^a E_+^b E_+^c \phi(Z) \Gamma_{a\beta\gamma} \]

Then, taking into account eqs. (2.1)-(2.4) together with the cyclic identity (1.1), it is easily to verify that the pull-back of \( d\tilde{B} \) on the super world sheet vanishes:

\[ d\tilde{B}|_\mathcal{M} = 0 \quad (2.7) \]

A similar property has been met \[^{[19]}\], some years ago, in the framework of SYM theories. This property, called there Weyl triviality, was essential to derive the consistent chiral anomaly (the non trivial BRS cocycle with ghost number one) in SYM theories using the method of the descend equation. In a similar way the property expressed by eqs. (2.6), (2.7) allows to get \( I^{(B)} \) (the non trivial BRS cocycle with ghost number zero) and to verify that it is invariant under world sheet local supersymmetry. This can be done in two different but equivalent ways:

i) Let us call \( \mathcal{M}_0 \) the slide of \( \mathcal{M} \) at \( \eta^{(q)} = 0 = d\eta^{(q)} \). Then \[^{[8]}\]

\[ I^{(B)} = \int_{\mathcal{M}_0} \tilde{B} = \int_{\mathcal{M}_0} e^+ \wedge e^- \left[ -\frac{1}{2n} \mathcal{E}_+^a (\lambda^q \Gamma_{a} \lambda_q) \cdot \phi + \mathcal{E}_-^A \mathcal{E}_+^B B_{BA} \right] \quad (2.8) \]

Under the infinitesimal world sheet supereparametrization

\[ \zeta^I \partial_I \rightarrow \zeta^I \partial_I + \epsilon^A(\xi) D_A \]

one has

\[ \delta_\epsilon \tilde{B}|_\mathcal{M} = (i_\epsilon d\tilde{B} + di_\epsilon \tilde{B})|_\mathcal{M} = di_\epsilon \tilde{B}|_\mathcal{M} \]

where eq. (2.7) has been taken into account. Then

\[ \delta_\epsilon I^{(B)} = 0 \quad (2.9) \]

Therefore \( I^{(B)} \) is invariant under local supersymmetry even if it is not written as a full superspace integral. One should notice that, due to eq. (2.3), eq. (2.8) coincides with the G.S. action (without heterotic fermions).

ii) It follows from eq. (2.7) that locally

\[ \tilde{B}|_\mathcal{M} = dQ|_\mathcal{M} \quad (2.10) \]
By imposing eq. (2.10) as a full superspace constraint \cite{9}\cite{20}, one has

\[ I'(B) = \int_{\mathcal{M}} P^{IJ} [\tilde{B}_{J1} - (dQ)_{J1}] \] (2.11)

As shown in \cite{9}, \( I'(B) \) is invariant under a set of abelian local transformations involving the superfields \( P^{IJ} \). The gauge fixing of these transformations allows to reduces \( I'(B) \) to \( I(B) \).

Further abelian transformations involve the lagrangian multipliers \( P^q_a \) in eq. (2.5) and as a consequence of them the action \( I^{(c)} + I^{(B)} \) gives rise to the usual field equations of the heterotic strings (without heterotic fermions). As for the heterotic action \( I^{(h)} \), a consistent formulation exists \cite{4} for \( n = 2 \). Unfortunately, up to now, a consistent supersymmetric version of \( I^{(h)} \) for \( n > 2 \) is still lacking.

3 - Twistor–like supermembrane: the action

Twistor–like supermembrane models describe the embedding of the superworld volume \( \mathcal{M}(3|2n)(1 \leq n \leq 8) \) into the target superspace \( \mathcal{M}(11|32) \). \( \mathcal{M}(3|2n) \) is parametrized locally by the supercoordinates \( \zeta^M \equiv (\xi^m, \eta^{q\mu}) \) where \( m = 0, 1, 2; \ q = 1,...,n; \ \mu = 1, 2 \) and \( \eta^{q\mu} \) are real Grassman parameters. \( e^A = d\zeta^M e^A_M \equiv (e^a, e^{q\alpha}) \) define a local frame in the cotangent space of \( \mathcal{M} \) and \( T^A = \Delta e^A \) is the torsion. \( \Delta \) is the covariant differential with respect to the structure group \( SO(2,1) \otimes SO(n) \) and the rheonomic parametrization of \( T^A \) is

\[ T^a = e^{q\alpha} \wedge e^a_{\alpha\beta}; \quad T^q = e^a \wedge e^{p\beta} T^a_{q\beta} + e^a \wedge e^b T^a_{q\beta} \] (3.1)

\( \mathcal{M}(11|32) \) is parametrized locally by the supercoordinates \( Z^M \equiv (X^m, \theta^\underline{m}) \) \( (m = 0,..,10; \mu = 1,..,32) \) which are world volume superfields. We are interested on supermembranes in \( D = 11 \) supergravity background. \( D = 11 \) supergravity is described by the supervielbeins \( E^A = dZ^M E^A_M(Z) \), the Lorentz superconnection \( \Omega^B_{\underline{c}A} = E^C_{CA} \underline{B}(Z) \) and the 3–superform \( B = \frac{1}{3!} E^A \wedge E^B \wedge E^C B_{CBA}(Z) \) with their curvatures i.e. the torsion \( T^A_{\underline{a}} \), the Lorentz curvature \( R^B_{\underline{a}A} \) and the B–curvature \( H = dB \). The intrinsic components of these curvatures are restricted by the SUGRA constraints:

\[ T^a_{\beta\gamma} = 2\Gamma^a_{\beta\gamma}; \quad H_{\alpha\beta\gamma\delta} = 0 = H_{a\beta\gamma\delta} \] (3.2)

\[ H_{ab\beta\gamma} = -\frac{1}{3} (\Gamma_{ab})_{\beta\gamma} \] (3.2b)
that imply the field equations for the $D = 11$, SUGRA.

For the pull–back of $E_A^d$ we write

$$\left. E_A^d \right|_{\mathcal{M}} = e^A E_A^d$$

and

$$\left( E_a^d, E_q^a \right)_{\eta^{p\mu} = 0} = \left( \xi^d_a, \lambda^a_q \right).$$

$\eta_{ab}$ and $\eta_{ab}$ are the flat Minkowski metrics in three and eleven dimensions respectively.

Finally let us recall that the standard supermembrane action is $^{[17]}$

$$I_{SM} = \int_{\mathcal{M}_0} d^3 \xi (-detG)^{1/2} + \int_{\mathcal{M}_0} E_A^A \wedge E_B^B \wedge E_C^{CBA} \quad (3.3)$$

where $\mathcal{M}_0$ is the slide of $\mathcal{M}$ at $\eta^{q\mu} = 0 = d\eta^{q\mu}$ and $G_{mn}$ is the world volume metric induced by $E^a_m$:

$$G_{mn} = E^a_m \eta_{ab} E^b_n \quad (3.4)$$

Like in twistor models for superparticles and heterotic strings, we now impose the twistor constraint

$$E_q^{a\alpha} = 0 \quad (3.5)$$

By taking the derivative $\Delta_{p\beta}$ of eq. (3.5) one gets

$$(E_q^{a\alpha} \Gamma^a_{p\beta} E_{p\beta}) = \delta_{qp} \sigma^a_{\alpha\beta} E_a^a \quad (3.6)$$

Eqs. (3.5), (3.6) give for $\eta^{q\mu} = 0$

$$\lambda_{q\alpha}^a = 0 \quad (3.7a)$$

$$(\lambda_{q\alpha} \Gamma^a_{p\beta} \lambda_{p\beta}) = \delta_{qp} \sigma^a_{\alpha\beta} \xi_a^a \quad (3.7b)$$

Before discussing the twistor–like supermembrane action, let us describe some useful identities that follow essentially from the cyclic identity (1.2) together with the eq. (3.6). Our first identity is
\[
\frac{1}{2} \sigma_c^{\alpha\beta} (E_{q\alpha} \Gamma_{ab} E_{p\beta}) E_a^a E_b^b = \delta_{qp} \epsilon_{abc} M 
\]  
(3.8)

where

\[
M = -\frac{1}{6} \epsilon^{abc} F_{c(qp)}^{ab} E_a^a E_b^b 
\]  
(3.9)

\[
F_{c(qp)}^{ab} = \frac{1}{2} \sigma_c^{\alpha\beta} (E_{q\alpha} \Gamma_{ab} E_{p\beta}) 
\]  
(3.10)

and

\[
F_{c}^{ab} = \frac{1}{n} F_{c(qp)}^{ab} \delta^{qp}
\]

First of all let us show that the left hand side of eq. (3.8) is diagonal in \( q \) and \( p \). Indeed from eq. (3.6) the l.h.s. of eq. (3.8) can be written as

\[
M_{cabc}^{qp} = \frac{1}{8} \sigma_c^{\alpha\beta} \sigma_a^{\gamma\gamma'} \sigma_b^{\delta\delta'} (E_{q\alpha} \Gamma_{ab} E_{p\beta}) (E_{r\gamma} \Gamma_{a} E_{r\gamma'}) (E_{r\delta} \Gamma_{b} E_{r\delta'}) 
\]  
(3.11)

where at least one of the triples \((\beta, \gamma, \gamma')\), \((\beta, \delta, \delta')\), \((\alpha, \gamma, \gamma')\), \((\alpha, \delta, \delta')\) is completely symmetric. Suppose that the triple \((\beta, \gamma, \gamma')\) is symmetric and take \( r = p \). Then from the cyclic identity

\[
(E_{q\alpha} \Gamma_{ab} E_{p\beta}) (E_{r\gamma} \Gamma_{a} E_{r\gamma'}) = - (E_{(r\gamma} \Gamma_{ab} E_{p\beta)} (E_{p\beta}) \Gamma_{a} E_{q\alpha}) 
\]  
(3.12)

and due to eq. (3.6) the r.h.s. of eq. (3.12) vanishes for \( p \neq q \) so that \( M_{cabc}^{qp} = \delta^{qp} M_{abc}^{(q)} \). Now let us take eq. (3.11) for \( q = p = r \). With the notation that \( V_{\{ab\}} \) represents the components of the tensor \( V_{ab} \), symmetric and traceless with respect to \( a \) and \( b \), let us consider \( M_{\{ca\}}^{(q)} \). Since \( \sigma_c^{\alpha\beta} \sigma_a^{\gamma\gamma'} \) is completely symmetric in \( \alpha, \beta, \gamma \) and \( \gamma' \), it follows from the cyclic identity that \( M_{\{ca\}}^{(q)} \) vanishes. This means that the \( SO(2,1) \) irreps “5” and “3” of \( M_{abc}^{(q)} \) vanish so that \( M_{abc}^{(q)} \) is proportional to \( \epsilon_{abc} \). Finally to prove that \( M_{cab}^{(q)} \) is independent from \( q \) it is sufficient to apply the cyclic identity to eq. (3.11), taken for \( q = p \) and \( r \neq q \). This complete the proof of eq. (3.8).

A similar argument allows to derive from the cyclic identity the following interesting relation

\[
\sigma_c^{\alpha\beta} (\Gamma_{ab})_{\alpha\beta} E_{q\beta}^\beta E_{p\beta}^{\beta} E_{\gamma\gamma'}^{\gamma\gamma'} E_{r\gamma}^{\gamma} E_{r\gamma'}^{\gamma'} E_{r\delta}^{\delta} E_{r\delta'}^{\delta'} F_{c}^{ab} \epsilon^{abc} = \sigma_a^{\alpha\beta} (\Gamma_{a})_{\alpha\beta} E_{q\beta}^\beta E_{p\beta}^{\beta} E_{\gamma\gamma'}^{\gamma\gamma'} E_{r\gamma}^{\gamma} E_{r\gamma'}^{\gamma'} E_{r\delta}^{\delta} E_{r\delta'}^{\delta'} F_{c}^{ab} \epsilon^{abc} 
\]  
(3.13)
If we define

\[ R_{qp}^{a} = \frac{1}{2} \varepsilon^{abc} F_{b(qp)}^{ab} E_{bc} \]  

(3.14)

eq. (3.8) can be written as

\[ R_{qp}^{a} E_{ab} = \delta_{qp} \eta_{ab} M \]  

(3.15)

so that we can put

\[ R_{qp}^{a} = \delta_{qp} M E_{b}^{a} (g^{-1})^{ba} + \overline{R}_{qp}^{a} \]  

(3.16)

with

\[ \overline{R}_{qp}^{a} E_{ab} = 0 \]

and \( g_{ab} \) is the tangent space world sheet metric induced by \( E_{a}^{b} \) i.e.

\[ g_{ab} = E_{a}^{b} E_{ab} \]  

(3.17)

We need the technical assumption that this metric (or \( G_{mn} \) in eq. (3.4)) is non-degenerate and indeed that its signature is Minkowskian.

By taking into account eqs. (3.14), (3.16), eq. (3.13) yields

\[
(\sigma_{a})^{\alpha\beta}(1 - \Gamma)_{\alpha\beta}^{\beta} E_{q\beta}^{b} E_{b}(g^{-1})^{ba} (-\text{det} \, g)^{1/2} = \]

\[ (\sigma_{a})^{\alpha\beta}(\Gamma_{b})_{\alpha\beta}^{\beta} E_{q\beta}^{b} E_{b} \{ \delta_{p}^{q} [(-\text{det} \, g)^{1/2} - M E_{b}^{b}(g^{-1})^{ba} - \frac{1}{n} \overline{R}_{q}^{ab}] \} \]  

(3.18)

where

\[
(\overline{\Gamma})_{\alpha}^{\beta} = \frac{\varepsilon^{abc} E_{a}^{a} E_{b}^{b} E_{c}^{c}(\Gamma_{abc})_{\alpha}^{\beta}}{6 (-\text{det} \, g)^{1/2}} \]  

(3.19)

Notice that \( \overline{\Gamma}^{2} = 1 \) so that

\[ Q_{\pm} = \frac{1}{2} (1 \pm \overline{\Gamma}) \]  

(3.20)

are orthogonal projectors. Eq. (3.18) implies

\[ M = (-\text{det} \, g)^{1/2} \]  

(3.21)
\[(1 + \Gamma)\Gamma_b E_{q\beta} \mathcal{R}^{bb}_{qp}(\sigma_b)^{\beta\alpha} = 0 \quad (3.22)\]

Eq. (3.21) is obtained from eq. (3.18) by applying to it $E_{\tilde{q}\alpha}(1 + \Gamma) \frac{\tilde{\alpha}}{\tilde{\alpha}}$ and eq. (3.22) is recovered by acting on eq. (3.18) with the projector $Q_+$ and using eq. (3.21).

Now we are ready to come back to the supermembrane action. Under the twistor and SUGRA constraints, the three superform B enjoys the property of Weyl triviality \cite{21}. Indeed one can consider the modified superform

\[\tilde{B} = B + \frac{1}{12n} e^a \wedge e^b \wedge e^c \sigma^a_{\alpha\beta} E_a^{\mu} E_b^{\nu} E_c^{\rho} H_{DCBA} \quad (3.23)\]

and using eqs. (3.1), (3.2), (3.5) (3.6) one can see that the pull-back on $\mathcal{M}$ of $d\tilde{B}$ vanishes

\[d\tilde{B}|_{\mathcal{M}} = 0 \quad (3.24)\]

Then under the infinitesimal superdiffeomorphism $\zeta^I \partial_I \rightarrow \zeta^I \partial_I + \epsilon^A(\xi) D_A$ one has

\[\delta_\epsilon \tilde{B}|_{\mathcal{M}} = di_\epsilon \tilde{B}|_{\mathcal{M}} \]

so that the action

\[I^{(B)} = \int_{\mathcal{M}_0} \tilde{B} \equiv \int_{\mathcal{M}_0} (E_\mathcal{A}^A \wedge E_\mathcal{B}^B \wedge E_\mathcal{C}^C B_{CBA} + \frac{1}{6} e^a \wedge e^b \wedge e^c \epsilon_{cba} M) \quad (3.26)\]

is invariant under local supersymmetry, even if it is not a full superspace integral (recall that $\mathcal{M}_0$ is the slide of $\mathcal{M}$ at $\eta^{a\mu} = 0 = d\eta^{a\mu}$). To get eq. (3.26) the twistor and SUGRA constraints have been taken into account and eq. (3.8) has been used.

In conclusion we propose the following action for the twistor–like supermembrane

\[I = I^{(B)} + I^{(c)} \quad (3.27)\]

where $I^{(c)}$ is given in eq. (3.26) and

\[I^{(c)} = \int_{\mathcal{M}} P_\alpha^{a\mu} E_{\tilde{q}\alpha}^{a} \quad (3.28)\]
implements the twistor constraint (3.5), the superfields $P_{\alpha}^{\alpha}$ being lagrangian multipliers.

The action $I$ is invariant under diffeomorphisms and n–extended local supersymmetry of the world volume. In addition $I$ is also invariant under the local transformations

$$\delta P_{\alpha}^{\alpha} = \Delta_{\beta q} \Lambda^{\{\alpha p, \beta q\}}$$

where the superfields $\Lambda^{\{\alpha p, \beta p\}}$ are symmetric with respect to $(\alpha q)$ and $(\beta p)$ and traceless in $q$ and $p$ (i.e. $(\sigma c)_{\alpha \beta} \delta_{pq} \Lambda^{\{\alpha q, \beta p\}} = 0$). These transformations are similar to those discussed in ref. [7], [9] for superparticles and heterotic strings.

Finally one should notice that, as in the case of the heterotic strings, $I^{(B)}$ can be written as a full superspace integral. Since locally $\tilde{B} = dQ$ one has

$$I^{(B)} = \int_{\mathcal{M}} P^{IJK} \left( \tilde{B}_{IJK} - (dQ)_{IJK} \right)$$

(3.30)

4 - Twistor–like supermembrane: the field equation

From eqs. (3.8), (3.21), the last term in the r.h.s. of eq. (3.26) can be written as

$$\int_{\mathcal{M}_0} e^{a} \wedge e^{b} \wedge e^{c} F_{ab}^{\alpha} e_{\beta}^{a} = \int_{\mathcal{M}_0} d^{3} \xi (-\det e \cdot \det g \cdot \det e)^{1/2} =$$

$$= \int_{\mathcal{M}_0} d^{3} \xi (-\det G)^{1/2}$$

(4.1)

where $G_{mn}$, defined in eq. (3.4), is the world volume metric induced by $E_{m}^{a}$. Indeed, under eq. (3.5),

$$e_{m}^{a} E_{a}^{a} = e_{m}^{A} E_{A}^{a} = E_{m}^{a}$$

Of course the last equality in eq. (4.1) holds modulo terms proportional to $\Lambda_{\alpha}^{\alpha}$. Therefore, by performing a suitable shift of $P_{\alpha}^{\alpha}$, the action $I$ can be rewritten in the form

$$I = \int_{\mathcal{M}_0} d^{3} \xi (-\det G)^{1/2} + \int_{\mathcal{M}_0} e^{A} \wedge e^{B} \wedge e^{C} B_{CBA} + \int_{\mathcal{M}} P_{\alpha}^{\alpha} E_{q}^{\alpha}$$

(4.2)
(We use the same symbol to denote the superfields $P^q_{\alpha}$ in eq. (3.28) and the shifted ones in eq. (4.2)).

The first two integrals in the r.h.s. of eq. (4.2) reproduce the standard supermembrane action $I_{SM}$, eq. (3.3).

A perhaps surprising feature of the twistor–like supermembrane is that the world volume metric $G_{mn}$ induced by the target supervielbeins $E_{a}$ is different from the metric specified by the local frame $e^{\alpha}$, suitable to reveal the hidden world volume supersymmetry of the model.

In order to show that the action $I$ gives rise to the same field equations of $I_{SM}$ we need the following

Lemma. If $V_{\alpha}$ is a target space vector–world volume spinor such that

\[ V_{\alpha} E_{b}^{a} = 0 \]  
\[ Q V_{\alpha} \Gamma^{a} \lambda_{q\alpha} = 0 \]

for some $q$, then $V_{\alpha}$ vanishes.

Of course the lemma holds even if $V_{\alpha}^{(n)}$ carries a free index $(n)$, in particular for a target vector–world volume vector $V^{\alpha\beta}_{a} = \sigma^{\alpha\beta}_{a} V_{a}$ such that

\[ V_{\alpha} E_{b}^{a} = 0 = Q V_{\alpha} \Gamma^{a}_{\beta} \lambda_{q\alpha}. \]

$Q_{\pm}$ are the projectors defined in eq. (3.20).

Let us call $\hat{\Gamma}^{u}(u = 1, \ldots, 8)$ the eight $\Gamma$–matrices that span the eight dimensional subspace orthogonal to $E_{a}^{a}$. They anticommute with $\bar{\Gamma}$. Therefore, by taking into account eq. (4.3a) and with the notation

\[ \lambda_{q\alpha}^{(\pm)} = Q_{\pm} \lambda_{q\alpha} \]

Eq. (4.3b) becomes

\[ V_{\alpha}^{a} \hat{\Gamma}^{u} \lambda_{q\alpha}^{(+)} = 0 \]

or,

\[ L_{\sigma} \frac{\alpha}{\bar{\alpha}} V_{\alpha} = 0 \]

(\sigma = 1, \ldots, 16)
where the $16 \times 16$ matrix $L$ is $L_u \sigma = (\hat{\Gamma}_u \lambda^{(+)})_u$. To prove the Lemma it is sufficient to show that the determinant of $L$ is different from zero. Consider the equation

$$L_u \sigma Y^\alpha_u = 0$$

or more explicitly

$$(\hat{Y}_1 \lambda^{(+)})_u = -(\hat{Y}_2 \lambda^{(+)})_u$$

(4.4)

where $Y^\alpha_u$ are commuting vectors orthogonal to $E^a_u$ and $\hat{Y}_\alpha = Y^\alpha_u + \hat{\Gamma}_u$. Eq. (4.4) yields the following identity

$$(\lambda^{(+)}_1 \hat{Y}_1 \Gamma^a \hat{Y}_1 \lambda^{(+)}) = (\lambda^{(+)}_2 \hat{Y}_2 \Gamma^a \hat{Y}_2 \lambda^{(+)})$$

which can be rewritten as

$$(Y^\alpha_u Y_1^a) E^a_+ = (Y^\alpha_u Y_2^a) E^a_-$$

But $E^a_+$ and $E^a_-$ are linearly independent so that

$$(Y^\alpha_u Y_1^a) = 0 = (Y^\alpha_u Y_2^a)$$

and therefore $Y^\alpha_u = 0$ (in the tangent subspace orthogonal to $E^a_u$ the metric is euclidean). This proves that $\det L \neq 0$. Then $V^\alpha_u$ vanishes independently of its statistic.

Let us consider at first the model with $n = 1$. In this case the action (4.2) reduces to

$$I = \int_{M_0} \{(-(\det G))^{1/2} + B\} + P^a_\alpha \left( \sigma^a_\alpha - \frac{1}{2} \sigma^\alpha_\beta (\lambda^a \Gamma^a \lambda^\beta) \right) +$$

$$+ \frac{1}{2} P^{(0)\alpha} \left( (\sigma^a)_\alpha (\varepsilon^a_\alpha \Gamma^a \lambda^\beta) - 3 (\lambda^a \Gamma^a \lambda^\alpha) \right)$$

(4.6)

where

$$P^a_\alpha = \Delta_\alpha (\sigma^a_\beta P^\beta_\alpha) \bigg|_{\eta^\mu = 0}; \quad P^{(0)\alpha}_\alpha = P^\alpha_\alpha \bigg|_{\eta^\mu = 0}$$

$$Y^A_\lambda = \frac{1}{2} \epsilon^{\beta \gamma} \Delta_\beta E^A_\gamma \bigg|_{\eta^\mu = 0}$$

and $\lambda^a_\alpha, Y^a$ have been eliminated through their field equations. Then the relevant field equations are
\[ \frac{\delta I}{\delta Y_{\alpha}} \equiv P_{\alpha}^{(0)}(\Gamma^{\alpha} \lambda_{\alpha})_{\alpha} = 0 \] (4.7)

\[ e^{I} \frac{\delta I}{\delta e_{\alpha}} \equiv P_{\alpha}^{(0)}(\sigma_{a} \sigma_{b})_{a\alpha} \epsilon^{ab} = 0 \] (4.8)

\[ e^{b} \frac{\delta I}{\delta e_{\alpha}} \equiv P_{\alpha a} E_{b}^{a} + \frac{1}{2} P_{\alpha}^{(0)}(\sigma_{a})_{\alpha}^{\beta}(\lambda_{\beta} \Gamma^{a} \epsilon_{b}) = 0 \] (4.9)

\[ \frac{\delta I}{\delta \lambda_{\alpha}} \equiv P_{\alpha}^{a}(\sigma_{a})^{\alpha\beta}(\Gamma^{a} \lambda_{\beta})_{\alpha} + \frac{1}{2} P_{\alpha}^{(0)}(\sigma_{a})^{\beta}_{\alpha\beta}(E_{a}^{\beta}(\sigma_{a})_{\beta}^{\alpha} - 3 \delta_{\alpha}^{\alpha} \epsilon_{\beta}^{\beta}) = 0 \] (4.10)

\[ E_{\alpha}^{M} \frac{\delta I}{\delta Z_{\beta}} \equiv L_{\alpha} - \Delta_{a} P_{\alpha}^{a} = 0 \] (4.11)

\[ E_{\alpha}^{M} \frac{\delta I}{\delta Z_{\beta}} \equiv ((1 - \bar{\Gamma})S)_{\alpha} + \frac{1}{2} \Delta_{a}(P_{\alpha}^{(0)}(\sigma_{a})^{\beta}_{\alpha\beta} \Gamma^{a} \lambda_{\beta})_{\alpha} + P_{\alpha}^{a}(\Gamma^{a} \epsilon_{a})_{\alpha} = 0 \] (4.12)

\[ \frac{\delta I}{\delta P_{\alpha}^{a}} \equiv \epsilon_{a}^{\alpha} - \frac{1}{2} \sigma_{a}^{\alpha\beta}(\lambda_{\alpha} \Gamma^{a} \lambda_{\beta}) = 0 \] (4.13)

\[ \frac{\delta I}{\delta P_{\alpha}^{(0)}(a)} \equiv (\sigma^{a})^{\beta}_{\alpha}(\epsilon_{a}^{a} \Gamma^{a} \lambda_{\beta}) - (Y \Gamma^{a} \lambda_{a}) = 0 \] (4.14)

where

\[ L_{\alpha} \equiv \frac{\delta I_{SM}}{\delta Z_{\alpha}} \cdot E_{\alpha}^{M} \] (4.15)

\[ L_{\alpha} \equiv \frac{\delta I_{SM}}{\delta Z_{\alpha}} E_{\alpha}^{M} = ((1 - \bar{\Gamma})S)_{\alpha} \] (4.16)

and

\[ S_{\alpha} = (\epsilon_{a}^{a} \Gamma_{\alpha} \epsilon_{a})_{\alpha} \]

From eq. (4.7) one has

\[ P_{\alpha}^{(0)}(\sigma_{b} \sigma_{a})_{\alpha\beta} \epsilon^{ab} = 0 \]

which, together with eq. (4.8), implies
Due to eqs. (4.7), (4.17) $P^{(0)\alpha}_a$ satisfies the conditions of the Lemma so that $P^{(0)\alpha}_a = 0$. If $P^{(0)\alpha}_a$ vanishes, eqs. (4.9), (4.10) show that $P^a_a$ too fulfills the conditions of the Lemma so that $P^a_a = 0$. Then eqs. (4.11), (4.12) become the standard supermembrane field equations

$$L^a_a = 0 = L^\alpha_\alpha$$

(4.18)

Now let us go back to the general case, $n > 1$.

The $Z^M$ field equations are

$$\frac{1}{n!} (\eta^2)^n L^\alpha_\alpha + (P^{q\alpha}_a \Gamma^a E_{q\alpha})_\alpha = 0$$

(4.19)

$$\frac{1}{n!} (\eta^2)^n L^\alpha_\alpha + \Delta_{q\alpha} P^{q\alpha}_a = 0$$

(4.20)

where

$$\eta^2 = \frac{1}{2} \epsilon_{\alpha\beta} \eta^{q\alpha} \eta^{p\beta}$$

Eq. (4.20) implies

$$P^{q\alpha}_a = \eta^{\beta\gamma} (\eta^2)^{n-1} P^a_a (\sigma^\alpha)_\beta + \Delta_{p\beta} \tilde{\Lambda}^{(q\alpha, p\beta)}_a$$

(4.21)

where $\tilde{\Lambda}^{(q\alpha, p\beta)}_a$ is symmetric with respect to $(q\alpha)$ and $(p\beta)$ and traceless in $q$ and $p$ However the last term in the r.h.s. of eq. (4.21) can be gauged to zero using the local invariance, eq. (3.29), so that only the component $P^a_a$ of $P^{q\alpha}_a$ survives. At this point one can repeat the argument given in the case $n = 1$ to conclude that also $P^a_a$ vanishes and therefore that eqs. (4.18) hold. In conclusion we have shown that the twistor–like supermembrane action, eq. (3.27) with eqs. (3.26), (3.28), is classically equivalent to the standard one. Eq. (3.27) exhibit $n$–extended world volume supersymmetry that replaces $n$ components of the usual $\kappa$–symmetry. Moreover, in our formulation, conformal invariance is manifest, in agreement with a recent result [21] where a new conformal invariant formulation of super $p$–branes has been proposed. This feature is interesting in view of the problem of the quantization and renormalization of supermembrane models. It confirms and, in some sense, explains the conjecture of ref. [22] where, merely on the basis of $\kappa$–symmetry, it has been argued that, at the quantum level, the
standard, \( D = 11 \), supermembrane theory should be renormalizable, despite its lack of conformal invariance.

Another bonus of our formulation is that it provides, by dimensional reduction, a twistor-like formulation for type II A superstrings, as we shall show elsewhere.

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