Monogenic functions in 5-dimensional spacetime used as first principle: gravitational dynamics, electromagnetism and quantum mechanics

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Monogenic functions are functions of null vector derivative and are here analysed in the geometric algebra of 5-dimensional spacetime, $G_{4,1}$, in order to derive several laws of fundamental physics. The paper introduces the working algebra and the definition of monogenic functions, showing that these generate two 4-dimensional spaces, one with Euclidean signature and the other one with Minkowski signature. The equivalence conditions between the two spaces are studied and relativistic dynamics, not entirely coincident with Einstein’s general theory of relativity, is demonstrated. The monogenic condition is then shown to produce Maxwell’s equations and electrodynamics both classical and quantized.

PACS numbers: 02.40.Yy; 03.65.Pm

I. INTRODUCTION

Our goal is to show how the important equations of physics, such as relativity equations and equations of quantum mechanics, can be put under the umbrella of a common mathematical approach\[^2\]. We use geometric algebra as the framework but introduce monogenic functions with their null derivatives in order to advance the concept. Furthermore, we clarify some previous work in this direction and identify the steps to take in order to complete this ambitious project.

Since A. Einstein formulated dynamics in 4-dimensional spacetime, this space is recognized by the vast majority of physicists as being the best for formulating the laws of physics. However, mathematical considerations lead to several alternative 4-D spaces. For example, the 4-dimensional space called 4-D optics (4DO) is equivalent to the 4-D spacetime of the general theory of relativity (GTR) when the metric is static, and therefore the geodesics of one space can be mapped one-to-one with those of the other. Then one can choose to work in the space that is more suitable.

In the case of a central mass, we can examine how the Schwarzschild metric in GTR can be transposed to 4DO. The usual form of the metric is

$$dr^2 = \left(1 - \frac{2m}{\chi}\right) dt^2 - \left(1 - \frac{2m}{\chi}\right)^{-1} d\chi^2 - \chi^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right);$$

(1)

where $m$ is the spherical mass and $\chi$ is the radial coordinate, not the distance to the centre of the mass. This form is non-isotropic but a change of coordinates can be made that returns the expression to isotropic form (see D’Inverno\[^3\] section 14.7):

$$r = \left(\chi - m + \sqrt{\chi^2 - 2m\chi}\right)/2;$$

(2)

and the new form of the metric is

$$dr^2 = \left(1 - \frac{m}{2r}\right)^2 \left(1 + \frac{m}{2r}\right)^4 \left[dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right].$$

(3)

From this equation we immediately define two coefficients, which are called refractive index coefficients,

$$n_4 = \frac{1 + \frac{m}{2r}}{1 - \frac{m}{2r}}, \quad n_r = \frac{1 + \frac{m}{2r}}{1 - \frac{m}{2r}}.$$

(4)

We devote the first part of this paper to deriving them from a geometric algebra approach in a special 5D space with null geodesics, thereby establishing that there is a 4DO Euclidean metric space equivalent to the Schwarzschild metric space. We build upon previous work by ourselves and by other authors about null geodesics, regarding the condition that all material particles must follow null geodesics of 5D space:

The implication of this for particles is clear: they should travel on null 5D geodesics. This idea has recently been taken up in the literature, and has a considerable future. It means that what we perceive as massive particles in 4D are akin to photons in 5D.\[^4\]

Accordingly, particles moving on null paths in 5D ($ds^2 = 0$) will appear as massive particles moving on timelike paths in 4D ($ds^2 > 0$) . . . \[^5\]

We actually improve on these null displacement ideas by introducing the more fundamental monogenic condition, deriving the former from the latter and establishing a common first principle.
II. SOME GEOMETRIC ALGEBRA

Geometric algebra is not usually taught in university courses and its presence in the literature is scarce; good reference works are [1, 2, 3]. We will concentrate on the algebra of 5-dimensional spacetime because this will be our main working space; this algebra incorporates as subalgebras those of the usual 3-dimensional Euclidean space, Euclidean 4-space and Minkowski spacetime. We begin with the simpler 5D flat space and progress to a 5D spacetime of general curvature (see Appendix C for more details.)

The geometric algebra $G_{4,1}$ of the hyperbolic 5-dimensional space we consider is generated by the coordinate frame of orthonormal basis vectors $\sigma_\alpha$ such that

$$
\begin{align*}
(s_0)^2 &= -1, \\
(s_i)^2 &= 1, \\
\sigma_\alpha \cdot \sigma_\beta &= 0, \quad \alpha \neq \beta.
\end{align*}
$$

(5)

Note that the English characters $i, j, k$ range from 1 to 4 while the Greek characters $\alpha, \beta, \gamma$ range from 0 to 4. See the Appendix A for the complete notation convention used.

Any two basis vectors can be multiplied, producing the new entity called a bivector. This bivector is the geometric product or, quite simply, the product, and it is distributive. Similarly to the product of two basis vectors, the product of three different basis vectors produces a trivector and so forth up to the fivevector, because five is the dimension of space.

We will simplify the notation for basis vector products using multiple indices, i.e. $\sigma_\alpha \sigma_\beta \equiv \sigma_{\alpha\beta}$. The algebra is 32-dimensional and is spanned by the basis

- 1 scalar, 1,
- 5 vectors, $\sigma_\alpha$,
- 10 bivectors (area), $\sigma_{\alpha\beta}$,
- 10 trivectors (volume), $\sigma_{\alpha\beta\gamma}$,
- 5 tetravectors (4-volume), $i \sigma_\alpha$,
- 1 pseudoscalar (5-volume), $i \equiv s_0 s_{1234}$.

Several elements of this basis square to unity:

$$
(s_i)^2 = (s_0 n_\alpha)^2 = (s_0 n_\alpha)^2 = (i \sigma_i)^2 = 1. 
$$

(6)

The remaining basis elements square to $-1$:

$$
(s_0)^2 = (s_{ij})^2 = (s_{ijk})^2 = (i \sigma_i)^2 = i^2 = -1.
$$

(7)

Note that the pseudoscalar $i$ commutes with all the other basis elements while being a square root of $-1$; this makes it a very special element which can play the role of the scalar imaginary in complex algebra.

In 5-dimensional spacetime of general curvature, spanned by 5 coordinate frame vectors $g_\alpha$, the indices follow the conventions set forth in Appendix A. We will also assume this spacetime to be a metric space whose metric tensor is given by

$$
g_{\alpha\beta} = g^{\alpha\beta};
$$

(8)

the double index is used with $g$ to denote the inner product of frame vectors and not their geometric product. The space signature is $(-+++)$, which amounts to saying that $g_{00} < 0$ and $g_{ii} > 0$. A reciprocal frame is defined by the condition

$$
g^{\alpha\beta} = \delta^{\alpha\beta}.
$$

(9)

Defining $g^{\alpha\beta}$ as the inverse of $g_{\alpha\beta}$, the matrix product of the two must be the identity matrix; using Einstein’s summation convention this is

$$
g^{\alpha\gamma} g_{\beta\gamma} = \delta^{\alpha\beta}.
$$

(10)

Using the definition $\sigma_\alpha$ we have

$$
(g^{\alpha\gamma} g_{\beta\gamma}) = \delta^{\alpha\beta};
$$

(11)

comparing with Eq. 4 we determine $g^{\alpha}$ with

$$
g^{\alpha} = g^{\alpha\gamma} g_{\gamma}.
$$

(12)

If the coordinate frame vectors can be expressed as a linear combination of the orthonormed ones, we have

$$
g_\alpha = n^{\beta} \sigma_\beta,
$$

(13)

where $n^{\beta} \sigma_\beta$ is called the refractive index tensor or simply the refractive index; its 25 elements can vary from point to point as a function of the coordinates. When the refractive index is the identity, we have $g_{\alpha} = \sigma_\alpha$ for the main or direct frame and $g^{\beta} = -\sigma_\alpha$, $g^0 = \sigma_0$ for the reciprocal frame, so that Eq. (1) is verified. In this work we will not consider spaces of general curvature but only those satisfying condition 13.

The first use we will make of the reciprocal frame is for the definition of two derivative operators. In flat space we define the vector derivative

$$
\nabla = \sigma^\alpha \partial_\alpha.
$$

(14)

It will be convenient, sometimes, to use vector derivatives in subspaces of 5D space; these will be denoted by an upper index before the $\nabla$ and the particular index used determines the subspace to which the derivative applies; For instance $m \nabla = \sigma^m \partial_m = \sigma^1 \partial_1 + \sigma^2 \partial_2 + \sigma^3 \partial_3$. In 5-dimensional space it will be useful to split the vector derivative into its time and 4-dimensional parts

$$
\nabla = -\sigma_0 \partial_0 + \partial^i \partial_i = -\sigma_0 \partial_0 + \nabla.
$$

(15)

The second derivative operator is the covariant derivative, sometimes called the Dirac operator, and it is defined in the reciprocal frame $g^{\alpha}$

$$
D = g^{\alpha} \partial_\alpha.
$$

(16)
Taking into account the definition of the reciprocal frame \textsuperscript{4}, we see that the covariant derivative is also a vector. In cases such as those we consider in this work, where there is a refractive index, it will be possible to define both derivatives in the same space.

We define also second order differential operators, designated Laplacian and covariant Laplacian respectively, resulting from the inner product of one derivative operator by itself. The square of a vector is always a scalar and the vector derivative is no exception, so the Laplacian is a scalar operator, which consequently acts separately in each component of a multivector. For 4 + 1 flat space it is

\[ \nabla^2 = -\frac{\partial^2}{\partial t^2} + \nabla^2. \]  

(17)

One sees immediately that a 4-dimensional wave equation is obtained by zeroing the Laplacian of some function

\[ \nabla^2 \psi = \left( \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \psi = 0. \]  

(18)

This procedure was used in Ref. \textsuperscript{1} for the derivation of special relativity and will be extended here to general curved spaces.

III. THE MONOGENIC CONDITION

There is a class of functions of great importance, called monogenic functions \textsuperscript{2}, characterized by having null vector derivative; a function \( \psi \) is monogenic in flat space if and only if

\[ \nabla \psi = 0. \]  

(19)

A monogenic function is not usually a scalar and has by necessity null Laplacian, as can be seen by dotting Eq. \textsuperscript{19} with \( \nabla \) on the left. We are then led to Eq. \textsuperscript{18}, which can also be written as

\[ \sum_i \partial_i \psi = \partial_{00} \psi. \]  

(20)

This relation can be recognized as a wave equation in the 4-dimensional space spanned by the \( \sigma_i \) which will accept plane wave type solutions of the general form

\[ \psi = \psi_0 e^{i(p_\alpha x^\alpha + \delta)}, \]  

(21)

where \( \psi_0 \) is an amplitude whose characteristics we shall not discuss for now, \( \delta \) is a phase angle and \( p_\alpha \) are constants such that

\[ \sum_i (p_i)^2 - (p_0)^2 = 0. \]  

(22)

When working in curved spaces the monogenic condition is naturally modified, replacing the vector derivative \( \nabla \) with the covariant derivative \( D \). A generalized monogenic function is then a function that verifies the equation

\[ D\psi = 0. \]  

(23)

Similarly to what happens in flat space, the covariant Laplacian is a scalar and a monogenic function must verify the second order differential equation

\[ D^2 \psi = 0. \]  

(24)

It is possible to write a general expression for the covariant Laplacian in terms of the metric tensor components (see \textsuperscript{2} Section 2.11) but we will consider only situations where that complete general expression is not needed.

When Eq. \textsuperscript{23} is multiplied on the left by \( D \), we are applying second derivatives to the function, but we are simultaneously applying first order derivatives to the reciprocal frame vectors present in the definition of \( D \) itself. We can simplify the calculations if the variations of the frame vectors are taken to be much smaller than those of function \( \psi \) so that frame vector derivatives can be neglected. With this approximation, the covariant Laplacian becomes \( D^2 = g^{\alpha\beta} \partial_{\alpha\beta} \) and Eq. \textsuperscript{24} can be written

\[ g^{\alpha\beta} \partial_{\alpha\beta} \psi = 0. \]  

(25)

This equation can have a solution of the type given by Eq. \textsuperscript{21} if again the derivatives of \( p_\alpha \) are neglected. This approximation is usually of the same order as the former one and should not be seen as a second restriction. Inserting Eq. \textsuperscript{21} one sees that it is a solution if

\[ g^{\alpha\beta} p_\alpha p_\beta = 0. \]  

(26)

This equation means that the square of vector \( p = g^{\alpha}p_\alpha \) is zero, that is, \( p \) is a vector of zero length and is called a null vector or nilpotent. Vector \( p \) is the momentum vector and should not be confused with 4-dimensional conjugate momentum vectors defined below.

IV. EQUIVALENCE BETWEEN 4DO AND GTR SPACES

By setting the argument of \( \psi \) constant in Eq. \textsuperscript{21} and differentiating we can get the differential equation

\[ p_\alpha dx^\alpha = 0. \]  

(27)

The lhs can equivalently be written as the inner product of the two vectors \( p \cdot dx = 0 \), where \( dx = g_{\beta}dx^\beta \) is a general 5D elementary displacement. In 5D hyperbolic space the inner product of two vectors can be null when the vectors are perpendicular but also when the two vectors are null. Since we have established that \( p \) is a null vector, Eq. \textsuperscript{27} can be satisfied either by \( dx \) normal to \( p \) or by \( (dx)^2 = 0 \). In the former case the condition describes a 3-volume called wavefront and in the latter case it describes the wave motion. Notice that the wavefronts
are not surfaces but volumes, because we are working with 4-dimensional waves.

The condition describing 4D wave motion can be expanded as

$$g_{\alpha\beta}dx^\alpha dx^\beta = 0. \quad (28)$$

This condition effectively reduces the spatial dimension to four but the resulting space is non-metric because all displacements have zero length. We will remove this difficulty by considering two special cases. First let us assume that vector $g_0$ is normal to the other frame vectors so that all $g_{ij}$ factors are zeroed; condition (28) becomes

$$g_{00}(dx^0)^2 + g_{ij}dx^i dx^j = 0. \quad (29)$$

All the terms in this equation are scalars and we are allowed to rewrite it with $(dx^0)^2$ in the lhs

$$(dx^0)^2 = - \frac{g_{ij}}{g_{00}} dx^i dx^j. \quad (30)$$

We could have arrived at the same result by defining a 4-dimensional displacement vector

$$dx^0 \nu = \frac{-1}{\sqrt{g_{00}}} g_i dx^i; \quad (31)$$

and then squaring it to evaluate its length; $\nu$ is a unit vector called velocity because its definition is similar to the usual definition of 3-dimensional velocity; its components are

$$v_i = \frac{dx^i}{dx^0}. \quad (32)$$

Being unitary, the velocity can be obtained by a rotation of the $\sigma_4$ frame vector

$$v = \tilde{R} \sigma_4 R. \quad (33)$$

The rotation angle is a measure of the 3-dimensional velocity component. A null angle corresponds to $v$ directed along $\sigma_4$ and null 3D component, while a $\pi/2$ angle corresponds to the maximum possible 3D component. The idea that physical velocity can be seen as the 3D component of a unitary 4D vector has been explored in several papers but see [10].

Equation (31) projects the original 5-dimensional space into a space with 4 dimensions, with Euclidean signature, where an elementary displacement is given by the variation of coordinate $x^0$. In the particular case where $g_0 = \sigma_0$ the displacement vector simplifies to $dx^0 \nu = g_i dx^i$ and we can see clearly that the signature is Euclidean because the four $g_i$ have positive norm. Although it has not been mentioned, we have assumed that none of the frame vectors is a function of coordinate $x^0$.

Returning to Eq. (28) we can now impose the condition that $g_4$ is normal to the other frame vectors in order to isolate $(dx^4)^2$ instead of $(dx^0)^2$, as we did before;

$$(dx^4)^2 = - \frac{g_{ij}}{g_{44}} dx^\mu dx^\nu. \quad (34)$$

We have now projected onto 4-dimensional space with signature $(+ -- -)$, known as Minkowski signature. In order to check this consider again the special case with $g_0 = \sigma_0$ and the equation becomes

$$(dx^4)^2 = \frac{1}{g_{44}} (dx^0)^2 - \frac{g_{mn}}{g_{44}} dx^m dx^n; \quad (35)$$

the diagonal elements $g_i$ are necessarily positive, which allows a verification of Minkowski signature. Contrary to what happened in the previous case, we cannot now obtain $(dx^4)^2$ by squaring a vector but we can do it by consideration of the bivector

$$dx^4 \nu = g_{44}^{\frac{1}{2}} g_{ij} dx^i.$$ 

All the products $g_{ij} dx^i$ are bivectors because we imposed $g_4$ to be normal to the other frame vectors. When $(dx^4)^2$ is evaluated by an inner product we notice that $g_0 g_4^4$ has positive square while the three $g_m g^4$ have negative square, ensuring that a Minkowski signature is obtained. Naturally we have to impose the condition that none of the frame vectors depends on $x^4$. Bivector $\nu$ is such that $\nu^2 = \nu \nu = 1$ and it can be obtained by a Lorentz transformation of bivector $\sigma_{04}$.

$$\nu = \tilde{T} \sigma_{04} T, \quad (37)$$

where $T$ is of the form $T = \exp(B)$ and $B$ is a bivector whose plane is normal to $\sigma_4$. Note that $T$ is a pure rotation when the bivector plane is normal to both $\sigma_0$ and $\sigma_4$.

In special relativity it is usual to work in a space spanned by an orthonormed frame of vectors $\gamma_\mu$ such that $(\gamma_0)^2 = 1$ and $(\gamma_m)^2 = -1$, producing the desired Minkowski signature [10]. The geometric algebra of this space is isomorphic to the even sub-algebra of $G_{3,1}$ and so the area element $dx^2 \nu dx^4$ can be reformulated as a vector called relativistic 4-velocity.

Equations (30) and (34) define two alternative 4-dimensional spaces, those of 4-dimensional optics (4DO), with metric tensor $-g_{ij}/g_{00}$ and general theory of relativity (GTR) with metric tensor $-g_{\mu\nu}/g_{44}$, respectively; in the former $x^0$ is an affine parameter while in the latter it is $x^4$ that takes such role. In fact Eq. (34) only covers the spacelike part of GTR space, because $(dx^4)^2$ is necessarily non-negative. Naturally there is the limitation that the frame vectors are independent of both $x^0$ and $x^4$, equivalent to imposing a static metric, and also that $g_{04} = g_{44} = 0$. Provided the metric is static, the geodesics of 4DO can be mapped one-to-one with space-like geodesics of GTR and we can choose to work on the space that best suits us for free fall dynamics. For a physical interpretation of geometric relations it will frequently be convenient to assign new designations to the 5D coordinates that acquire the role of affine parameter in the null subspace. We will then make the assignments $x^0 \equiv t$ and $x^4 \equiv \tau$. Total derivatives with respect to these coordinates will also receive a special notation: $dt/dt = f$.
and \( df/d\tau = \dot{f} \). Special units conventions used in this paper are detailed in appendix \[.]

Unless otherwise specified, we will assume that the frame vector associated with coordinate \( x^0 \) is unitary and normal to all the others, that is \( g_0 = \sigma_0 \) and \( g_{0i} = 0 \). Recalling from Eq. \[\text{[30]}\], these conditions allow the definition of 4DO space with metric tensor \( g_{ij} \). Although we could try a more general approach, we would lose the possibility of interpreting time as a line element and this, as we shall see, provides very interesting and novel interpretations of physics equations. In many cases it is also true that \( g_4 \) is normal to the other frame vectors and we have seen that in those cases we can make metric conversions between GTR and 4DO; it will be interesting, however, to examine one or two situations with non-normal \( g_4 \) and so we leave this possibility open.

For the moment we will concentrate on isotropic space, characterized by orthogonal refractive index vectors \( g_i \) whose norm can change with coordinates but is the same for all vectors. Normally we relax this condition by accepting that the three \( g_m \) must have equal norm but \( g_4 \) can be different. The reason for this relaxed isotropy is found in the parallel we make with physics by assigning dimensions 1 to 3 to physical space. Isotropy in a physical sense need only be concerned with these dimensions and ignores what happens with dimension 4. We could try a more general approach, we would lose interesting, however, to examine one or two situations with non-normal \( g_4 \) within the relaxed isotropy concept but we will not do so for the moment.

Equation \[\text{[30]}\] can now be written in terms of the isotropic refractive indices as

\[
\frac{dt^2}{(n_r)^2} \sum_m (dx^m)^2 + (n_4 d\tau)^2. \tag{38}
\]

Spherically symmetric static metrics play a special role; this means that the refractive index can be expressed as functions of \( r \) if we adopt spherical coordinates. The previous equation then becomes

\[
dt^2 = (n_r)^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] + (n_4 d\tau)^2. \tag{39}
\]

Since we have \( g_4 \) normal to the other vectors we can apply metric conversion and write the equivalent quadratic form for GTR

\[
d\sigma^2 = \left( \frac{dt}{n_4} \right)^2 - \left( \frac{n_r}{n_4} \right)^2 \star \left[ d\tau^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \tag{40}
\]

As we stated in the introduction, the usual form of Schwarzschild’s metric is given by Eq. \[\text{[1]}\] but a more interesting, isotropic form is the one in Eq. \[\text{[3]}\]. The latter can be compared to Eq. \[\text{[4]}\] allowing the derivation of the refractive indices in Eqs. \[\text{[4]}\]. These refractive indices provide a 4DO Euclidean space equivalent to Schwarzschild metric, allowing 4DO to be used as an alternative to GTR. Recalling that we derived trajectories from solutions \[\text{[24]}\] of a 4-dimensional wave equation \[\text{[25]}\], it becomes clear that orbits can also be seen as 4-dimensional guided waves by what could be described as a 4-dimensional optical fibre. Modes are to be expected in these waveguides and we shall say something about them later on.

V. FERMAT’S PRINCIPLE IN 4 DIMENSIONS

Fermat’s principle applies to optics and states that the path followed by a light ray is the one that makes the travel time an extremum; usually it is the path that minimizes the time but in some cases a ray can follow a path of maximum or stationary time. These solutions are usually unstable, so one takes the view that light must follow the quickest path. In Eq. \[\text{[30]}\] we have defined a time interval associated with a 4-dimensional elementary displacement, which allows us to determine, by integration, a travel time associated with displacements of any size along a given 4-dimensional path. We can then extend Fermat’s principle to 4D and impose an extremum requirement in order to select a privileged path between any two 4D points. Taking the square root to Eq. \[\text{[30]}\]

\[
dt = \sqrt{-\frac{g_{ij}}{g_{00}}} \, dx^i dx^j. \tag{41}
\]

Integrating between two points \( P_1 \) and \( P_2 \)

\[
t = \int_{P_1}^{P_2} \sqrt{-\frac{g_{ij}}{g_{00}}} \, dx^i dx^j = \int_{P_1}^{P_2} \sqrt{-\frac{g_{ij}}{g_{00}}} \, \dot{x}^i \dot{x}^j dt. \tag{42}
\]

In order to evaluate the previous integral one must know the particular path linking the points by defining functions \( x^i(t) \), allowing the replacement \( dx^i = \dot{x}^i dt \). At this stage it is useful to define a Lagrangian

\[
L = \frac{g_{ij}}{2g_{00}} \, \dot{x}^i \dot{x}^j. \tag{43}
\]

The time integral can then be written

\[
t = \int_{P_1}^{P_2} \sqrt{2L} \, dt. \tag{44}
\]

Time has to remain stationary against any small change of path; therefore we envisage a slightly distorted path defined by functions \( x^i(t) + \varepsilon \chi^i(t) \), where \( \varepsilon \) is arbitrarily small and \( \chi^i(t) \) are functions that specify distortion. Since the distortion must not affect the end points, the distortion functions must vanish at those points. The time integral will now be a function of \( \varepsilon \) and we require that

\[
\frac{dt(\varepsilon)}{d\varepsilon} \bigg|_{\varepsilon=0} = 0. \tag{45}
\]
Now, the Lagrangian (43) is a function of $x^i$, through $g_{\alpha\beta}$ and also an explicit function of $\dot{x}^i$. Allowing for a path change, through $\varepsilon$ makes $t$ in Eq. (44) a function of $\varepsilon$

$$t(\varepsilon) = \int_{P_1}^{P_2} \sqrt{2L(x^i + \varepsilon \dot{x}^i + \varepsilon \chi^i)} \, dt. \quad (46)$$

This can now be derived with respect to $\varepsilon$

$$\frac{dt(\varepsilon)}{d\varepsilon} \bigg|_{\varepsilon=0} = \left[ \int_{P_1}^{P_2} \frac{1}{\sqrt{2L}} \left( \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i + \frac{\partial L}{\partial x^i} \chi^i \right) \right]_{\varepsilon=0}. \quad (47)$$

Note that the first term on the rhs can be written

$$\int_{P_1}^{P_2} \frac{1}{\sqrt{2L}} \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i \, dt = \int_{P_1}^{P_2} \left( \frac{\partial \sqrt{2L}}{\partial \chi^i} \right) \dot{x}^i \, dt. \quad (48)$$

This can be integrated by parts

$$\int_{P_1}^{P_2} \frac{\partial \sqrt{2L}}{\partial \dot{x}^i} \dot{x}^i \, dt = \left[ \frac{\partial \sqrt{2L}}{\partial \chi^i} \dot{x}^i \right]_{P_1}^{P_2} - \int_{P_1}^{P_2} \frac{d}{dt} \left( \frac{\partial \sqrt{2L}}{\partial \dot{x}^i} \right) \dot{x}^i \, dt. \quad (49)$$

The first term on the second member is zero because $\chi^i$ vanishes for the end points; replacing in Eq. (47)

$$\frac{dt(\varepsilon)}{d\varepsilon} \bigg|_{\varepsilon=0} = \frac{1}{\sqrt{2}} \int_{P_1}^{P_2} \left[ \frac{d}{dt} \left( \frac{1}{\sqrt{L}} \frac{\partial L}{\partial \dot{x}^i} \right) + \frac{1}{\sqrt{L}} \frac{\partial L}{\partial x^i} \right] \dot{x}^i \, dt. \quad (50)$$

The rhs must be zero for arbitrary distortion functions $\chi^i$, so we conclude that the following set of four simultaneous equations must be verified

$$\frac{d}{dt} \left( \frac{1}{\sqrt{L}} \frac{\partial L}{\partial \dot{x}^i} \right) = \frac{1}{\sqrt{L}} \frac{\partial L}{\partial x^i}; \quad (51)$$

these are called the Euler-Lagrange equations.

Consideration of Eqs. (41) and (44) allows us to conclude that the Lagrangian defined by (43) can also be written as $L = v^2/2$ and must always equal $1/2$. From the Lagrangian one defines immediately the conjugate momenta

$$v_i = \frac{\partial L}{\partial \dot{x}^i} = -g_{ij} \dot{x}^j. \quad (52)$$

Notice the use of the lower index ($v_i$) to represent momenta while velocity components have an upper index ($v^i$). The conjugate momenta are the components of the conjugate momentum vector

$$\nu = \frac{g^{ij} v_i}{\sqrt{-g_{00}}} \quad (53)$$

and from Eq. (10)

$$\sqrt{-g_{00}} v_i = g^{ij} g_{ij} \dot{x}^j = g_{ij} \dot{x}^j. \quad (54)$$

The conjugate momentum and velocity are the same but their components are referred to the reciprocal and refractive index frames, respectively. Notice also that by virtue of Eq. (22) it is also

$$v_i = \frac{p_i}{p_0}. \quad (55)$$

The Euler-Lagrange equations (51) can now be given a simpler form

$$\dot{\nu}_i = \partial_i L. \quad (56)$$

This set of four equations defines trajectories of minimum time in 4DO space as long as the frame vectors $g_{\alpha}$ are known everywhere, independently of the fact that they may or may not be referred to the orthonormed frame via a refractive index. By definition these trajectories are the geodesics of 4DO space, spanned by frame vectors $g_{ij}/\sqrt{-g_{00}}$, with metric tensor $-g_{ij}/g_{00}$.

Following an exactly similar procedure we can find trajectories which extremize proper time, defined by taking the positive square root of Eq. (44). The Lagrangian is now defined by

$$L = -\frac{1}{2} g_{\mu\nu} \ddot{x}^\mu \ddot{x}^\nu. \quad (57)$$

Consequently the conjugate momenta are

$$\nu_\mu = \frac{\partial L}{\partial \ddot{x}^\mu} = -\frac{g_{\mu\nu}}{g_{00}} \ddot{x}^\nu. \quad (58)$$

From Eq. (52) we have $\nu_\mu = p_\mu/p_0$; the associated Euler-Lagrange equations are

$$\ddot{\nu}_\mu = \partial_\mu L. \quad (59)$$

"These are, by definition, spacelike geodesics of GTR with metric tensor $-g_{\mu\nu}/g_{00}$ and we have thus defined a method for one-to-one geodesic mapping between 4DO and spacelike GTR. Recalling the conditions for this mapping to be valid, all the frame vectors must be independent of both $t$ and $\tau$ and $g_{00}$ and $g_{04}$ must be normal to the other 3 frame vectors. In tensor terms, all the $g_{\alpha\beta}$ must be independent from $t$ and $\tau$ and $g_{0i} = g_{04} = 0.$"

VI. THE SOURCES OF REFRACTIVE INDEX

The set of 4 equations (56) defines the geodesics of 4DO space; particularly in cases where there is a refractive index, it defines trajectories of minimum time but does not tell us anything about what produces the refractive index in the first place. Similarly the set of equations (59) defines the geodesics of GTR space without telling us what shapes space. In order to analyse this question
we must return to the general case of a refractive frame $g_{\alpha}$ without other impositions besides the existence of a refractive index.

Considering the momentum vector

$$ p = p_\alpha g^\alpha = p_\alpha n_\beta \sigma^\beta, \quad (60) $$

with $n_\alpha \gamma n_\beta = \delta_\alpha^\beta$, we will now take its time derivative. Using Eq. (54)

$$ \dot{p} = \dot{x} \cdot (Dp) = \dot{x} \cdot G. \quad (61) $$

By a suitable choice of coordinates we can always have $g^0 = \sigma^0$. We can then invoke the fact that for an elementary particle in flat space the momentum vector components can be associated with the concepts of energy, 3D momentum and rest mass as $p = E \sigma^0 + p + m \sigma^4$ (see [1, 11] and Sec. VIII). If this consequence is extended to curved space and to mass distributions, we write $p = E \sigma^0 + p + m \sigma^4$, where now $E$ is energy density, $p = p_m \sigma_m$ is 3D momentum density and $m$ is mass density. The previous equation then becomes

$$ \dot{E} \sigma^0 + \dot{p} + m \sigma^4 = \dot{x} \cdot G. \quad (62) $$

When the Laplacian is applied to the momentum vector the result is still necessarily a vector

$$ D^2 p = S. \quad (63) $$

Vector $S$ is called the sources vector and can be expanded into 25 terms as

$$ S = (D^2 n_\alpha) \sigma_\beta p^\beta = S_\alpha^\beta \sigma_\beta p^\alpha; \quad (64) $$

where $p^\alpha = \sigma^\alpha \beta p_\beta$. Tensor $S_\alpha^\beta$ contains the coefficients of the sources vector and we call it the sources tensor. The sources tensor influences the shape of geodesics as we shall see in one particularly important situation. One important consequence that we don’t pursue here is that by zeroing the sources vector one obtains the wave equation $D^2 p = 0$, which accepts gravitational wave solutions.

If $\sigma^0$ is normal to the other frame vectors we can write $p = E(\sigma^0 + v)$ in the reciprocal frame, with $v$ a unit vector or $p = E(-\sigma^0 + v)$ in the direct frame. Equation (64) can then be given the form

$$ \dot{E}(\sigma^0 + v) + E \dot{v} = \sigma_0 + v \cdot G. \quad (65) $$

Since $G$ can have scalar and bivector components, the scalar part must be responsible for the energy change, while the bivector part rotates the velocity $v$. The bivector part of $G$ is generated by $D \wedge p$, which allows a simplification of the previous equation to

$$ \dot{v} = v \cdot (D \wedge v), \quad (66) $$

if the frame vectors are independent of $t$. This equation is exactly equivalent to the set of Euler-Lagrange equations [56] but it was derived in a way which tells us when to expect geodesic movement or free fall.

We will now investigate spherically symmetric solutions in isotropic conditions defined by Eq. (39); this means that the refractive index can be expressed as functions of $r$. The vector derivative in spherical coordinates is of course

$$ D = \frac{1}{n_\tau} \left( \sigma_\tau \partial_\tau + \frac{1}{r} \sigma_\theta \partial_\theta + \frac{1}{r \sin \theta} \sigma_\phi \partial_\phi \right) - \frac{1}{n_\gamma} \sigma_\gamma \partial_\gamma. \quad (67) $$

The Laplacian is the inner product of $D$ with itself but the frame vectors’ derivatives must be considered; all the derivatives with respect to $r$ are zero and the others are

$$ \partial_\theta \sigma_\theta = \sigma_\theta, \quad \partial_\phi \sigma_\phi = \sin \theta \sigma_\phi, $$

$$ \partial_\theta \sigma_\phi = -\sigma_\phi, \quad \partial_\phi \sigma_\theta = \cos \theta \sigma_\theta, $$

$$ \partial_\theta \sigma_\phi = 0, \quad \partial_\phi \sigma_\phi = -\sin \theta \sigma_\phi - \cos \theta \sigma_\theta. \quad (68) $$

After evaluation the curved Laplacian becomes

$$ D^2 = \frac{1}{(n_\gamma)^2} \left( \partial_\gamma + \frac{2 n_\gamma}{r} \partial_\gamma - \frac{n_\gamma^2}{n_\tau} \partial_\tau + \frac{1}{r^2} \partial_\theta + \frac{\cot \theta}{r^2} \partial_\theta + \frac{\sin ^2 \theta}{r^2} \partial_\phi \phi \right) - \frac{1}{(n_\gamma)^2} \partial_\tau. \quad (69) $$

The search for solutions of Eq. (63) must necessarily start with vanishing second member, a zero sources situation, which one would implicitly assign to vacuum; this is a wrong assumption as we will show. Zeroing the second member implies that the Laplacian of both $n_\tau$ and $n_\gamma$ must be zero; considering that they are functions of $r$ we get the following equation for $n_\tau$

$$ n_\tau + \frac{2 n_\gamma^2}{r} - \frac{(n_\gamma^2)^2}{n_\tau} = 0, \quad (70) $$

with general solution $n_\tau = b \exp(a/r)$. It is legitimate to make $b = 1$ because the refractive index must be unity at infinity. Using this solution in Eq. (63) the Laplacian becomes

$$ D^2 = e^{-a/r} \left( \partial_\tau + \frac{a}{r^2} \partial_\tau + \frac{1}{r^2} \partial_\theta + \frac{\cot \theta}{r^2} \partial_\theta + \frac{\sin ^2 \theta}{r^2} \partial_\phi \phi \right) - \frac{1}{(n_\gamma)^2} \partial_\tau; \quad (71) $$

which produces the solution $n_\gamma = n_\tau$. So space must be truly isotropic and not relaxed isotropic as we had allowed. The solution we have found for the refractive index components in isotropic space can correctly model Newton dynamics, which led the author to adhere to it for some time [10]. However if inserted into Eq. (42) this solution produces a GTR metric which is verifiably in disagreement with observations; consequently it has purely geometric significance.

The inadequacy of the isotropic solution found above for relativistic predictions deserves some thought, so that
we can search for solutions guided by the results that are expected to have physical significance. In the physical world we are never in a situation of zero sources because the shape of space or the existence of a refractive index must always be tested with a test particle. A test particle is an abstraction corresponding to a point mass considered so small as to have no influence on the shape of space; in reality a point particle is a black hole in GTR, although this fact is always overlooked. A test particle must be seen as source of refractive index itself and its influence on the shape of space should not be neglected in any circumstances. If this is the case the solutions for vanishing sources vector may have only geometric meaning, with no connection to physical reality.

The question is then what should go into the second member of Eq. (63) in order to find physically meaningful solutions. If we are testing gravity we must assume some mass density to suffer gravitational influence; this is what is usually designated as non-interacting dust, meaning that some continuous distribution of non-interacting particles follows the geodesics of space. Mass density is expected to be associated with particles which undergoes gravitational influence and is animated with velocity v and inertial mass cannot be defined without some field n4 acting upon it. Complete investigation of the sources tensor elements and their relation to physical quantities is not yet done; it is believed that 16 terms of this tensor have strong links with homologous elements of stress tensor in GTR, while the others are related to electromagnetic field.

VII. ELECTROMAGNETISM IN 5D SPACETIME

Maxwell’s equations can easily be written in the form of Eq. (32) if we don’t impose the condition that g4 should remain normal the other frame vectors; as we have seen in section IV this has the consequence that there will be no GTR equivalent to the equations formulated in 4DO.

We will consider the non-orthonormed reciprocal frame defined by

\[ g^\mu = \sigma^\mu, \quad g^4 = \frac{q}{m} A^\mu \sigma_\mu + \sigma^4; \]

where q and m are charge and mass densities, respectively, and A = A_\mu \sigma^\mu is the electromagnetic vector potential, assumed to be a function of coordinates t and x^m but independent of \( \tau \). The associated direct frame has vectors

\[ g_\mu = \sigma_\mu - \frac{q}{m} A_\mu \sigma_4, \quad g_4 = \sigma_4; \]

and one can easily verify that Eq. (35) is obeyed. The momentum vector in the reciprocal frame is p = E\sigma^0 + p_m \sigma^m + q A_\mu \sigma^\mu + ma^4 and G in the second member of Eq. (31) is \( G = qDA \). We will assume D-A to be zero, as one usually does in electromagnetism; also D can be replaced by \( \imath \nabla \) because the vector potential of the electromagnetic field.

The equivalent GTR space is characterized by the quadratic form

\[ \imath \nabla = e^{-2M/r} dt^2 - e^{2M/r} \sum_m (dx^m)^2. \]

Expanding in series of M/r the coefficients of this metric one would find that the lower order terms are exactly the same as for Schwarzschild and so the predictions of the metrics are indistinguishable for small values of the expansion variable. Montanus [12] arrives at the same solutions with a different reasoning; the same metric is also due to Yilmaz [13, 14, 15].

Equation (73) can be interpreted in physical terms as containing the essence of gravitation. When solved for spherically symmetric solutions, as we have done, the first member provides the definition of a stationary gravitational mass as the factor M appearing in the exponent and the second member defines inertial mass as \( \nabla^2 n_4 \).

Gravitational mass is defined with recourse to some particle which undergoes gravitational influence and is animated with velocity v and inertial mass cannot be defined without some field n4 acting upon it. Complete investigation of the sources tensor elements and their relation to physical quantities is not yet done; it is believed that 16 terms of this tensor have strong links with homologous elements of stress tensor in GTR, while the others are related to electromagnetic field.
This corresponds to a refractive index tensor whose non-zero terms are
\[ n^\alpha_\alpha = 1, \quad n^\mu_\mu = -\frac{q}{m} A_\mu. \] (82)

According to Eq. (82) the sources tensor has all terms null except for the following
\[ S^4_\mu = -\frac{q}{m} D^2 A_\mu; \] (83)
where \( D \) is the covariant derivative given by
\[ D = g^\alpha \partial_\alpha = \sigma^\mu \partial_\mu + (\sigma^4 + \frac{q}{m} A_\mu \sigma^\mu) \partial_4. \] (84)

We can then define the current vector \( J \) verifying
\[ \mu \nabla^2 A = \mu \nabla F = J, \] (85)
where
\[ J = -\frac{m}{q} S^4_\mu \sigma^\mu. \] (86)

Please refer to [1, Chap. 7] or to [8, Part 2] to see how these equations generate classical electromagnetism, particularly how setting the current vector to zero generates electromagnetic waves.

VIII. MONOGENIC FUNCTIONS AND QUANTUM MECHANICS

Dirac equation has been derived from the 5-dimensional monogenic condition in previous works [1, 11]; the motivation for returning to the subject here is the correction of the electro-dynamics equation, which is incorrect in the earlier paper [11] and absent in the later one [1]. Because we are working in geometric algebra, our quantum mechanics equations will inherit that character but the isomorphism between the geometric algebra of 5D spacetime, \( G_{4,1} \), and complex algebra of \( 4 \times 4 \) matrices, \( M(4, C) \), ensures that they can be translated into the more usual Dirac matrix formalism. The equivalence between the two formulations has been amply demonstrated in the two references above.

Recalling the monogenic condition (19), we will now expand it into 3 terms
\[ (\sigma^0 \partial_0 + \sigma^m \partial_m + \sigma^4 \partial_4) \psi = 0. \] (87)

We have already established that this equation accepts solutions in the form of Eq. (21) and we use that to evaluate the derivative with respect to \( x^4 \)
\[ (\sigma^0 \partial_0 + \sigma^m \partial_m - i\sigma^4 p_4) \psi = 0. \] (88)

If the equation is multiplied by \( \sigma^4 \) on the left, the first 4 terms on the first member acquire bivector factors of the form \( \sigma^{4\nu} \), the first of which, \( \sigma^{40} \) squares to unity, while the other 3 square to minus unity. Those bivectors belong to the even sub-algebra of \( G_{4,1} \), which is isomorphic to the algebra of Minkowski spacetime, as we have already stated. It is perfectly legitimate to replace the said bivector factors by Dirac matrices, as was demonstrated in the above cited references. We can then rewrite the monogenic condition as
\[ (\gamma^\mu \partial_\mu + ip_4) \psi = 0, \] (89)
which can be immediately recognized as Dirac’s equation if \( p_4 \) is assigned to the particle’s rest mass. The monogenic function given by Eq. (22) can then be given the usual physical interpretation of a Dirac spinor
\[ \psi = \psi_0 e^{i(\gamma \cdot p \cdot x + m q)}; \] (90)
where \( E \) is energy, \( p \) is 3-dimensional momentum and \( m \) is rest mass.

In order to separate left and right spinor components we use a technique adapted from Ref. [6]. We choose an arbitrary base element which squares to identity, for instance \( \sigma_4 \), with which we form the two idempotents \((1 + \sigma_4)/2 \) and \((1 - \sigma_4)/2 \). The name idempotents means that they reproduce themselves when squared. These idempotents absorb any \( \sigma_4 \) factor; as can be easily checked \((1 + \sigma_4)\sigma_4 = (1 + \sigma_4) \) and \((1 - \sigma_4)\sigma_4 = -(1 - \sigma_4) \). Obviously we can decompose the wavefunction \( \psi \) as
\[ \psi = \psi_{+} \frac{1 + \sigma_4}{2} + \psi_{-} \frac{1 - \sigma_4}{2} = \psi_{+} + \psi_{-}. \] (91)

This apparently trivial decomposition produces some surprising results due to the following relations
\[ e^{i\theta} (1 + \sigma_4) = (\cos \theta + i \sin \theta)(1 + \sigma_4) \]
\[ = (1 \cos \theta + i \sigma_4 \sin \theta)(1 + \sigma_4) \]
\[ = e^{i\sigma_4 \theta} (1 + \sigma_4). \] (92)

and similarly
\[ e^{i\theta} (1 - \sigma_4) = e^{-i\sigma_4 \theta} (1 - \sigma_4). \] (93)

We could have chosen other idempotents, which would produce similar results. The available idempotents generate an \( SU(4) \) group and it has been argued that they may be related to different elementary particles [11].

Electrodynamics can now be implemented in the same way used in Sec. VII to implement classical electromagnetism. The monogenic condition must now be established with the covariant derivative given by Eq. (24),
\[ \sigma^\mu \partial_\mu \psi + \left( \sigma^4 + \frac{q}{m} A_\mu \sigma^\mu \right) \partial_4 \psi = 0. \] (94)

Multiplying on the left by \( \sigma^4 \) and taking \( \partial_4 \psi = im \psi \)
\[ [\gamma^\mu (\partial_\mu + ip_\mu) + im] \psi = 0. \] (95)

This equation can be compared to what is found in any quantum mechanics textbook.
It is now adequate to say a few words about quantization, which is inherent to 5D monogenic functions. We have already seen that these functions are 4-dimensional waves, that is, they have 3-dimensional wavefronts normal to the direction of propagation. Whenever the refractive index distribution traps one of these waves a 4-dimensional waveguide is produced, which has its own allowed propagating modes. In the particular case of a central potential, be it an atom’s or a galaxy’s nucleus, we expect spherical harmonic modes, which produce the well known electron orbitals in the atom and have unknown manifestations in a galaxy.

**IX. CONCLUSION**

Every physicist dreams of finding a unified formulation for the fundamental laws of physics. It is usually accepted that in order to achieve such objective a new paradigm is needed, meaning that one must surely step back from accepted physics principles and start afresh from new simpler ones. Ideally one should have a small set of principles, valid for all areas of physics, and all the important relations should flow naturally from mathematical reasoning.

In this paper we extend a proposal previously made in that direction, that one should accept 5-dimensional spacetime as the adequate space to formulate the laws of physics and introduce in this space the condition of monogeneity. We had shown in another work that this condition is sufficient to arrive simultaneously at special relativity and the free particle Dirac equation; here we show that by generalizing the monogenic condition to bent spaces one is able to obtain relativistic dynamics not entirely coincident with GTR but also electrodynamics, both classical and quantized.

Maxwell’s equations were also derived from the monogenic condition and were unified to the equations responsible for gravitational dynamics. The procedure is not yet entirely satisfactory, in the sense that an ad hoc proposal had to be made in respect to inertial mass; in future work we hope to find a suitable formulation for the derivation of curvature from first principles.

**APPENDIX A: INDEXING CONVENTIONS**

In this section we establish the indexing conventions used in the paper. We deal with 5-dimensional space but we are also interested in two of its 4-dimensional subspaces and one 3-dimensional subspace; ideally our choice of indices should clearly identify their ranges in order to avoid the need to specify the latter in every equation. The diagram in Fig. 1 shows the index naming convention used in this paper; Einstein’s summation convention will be adopted as well as the compact notation for partial derivatives \( \partial_\alpha = \partial/\partial x^\alpha \).

**APPENDIX B: NON-DIMENSIONAL UNITS**

The interpretation of \( t \) and \( \tau \) as time coordinates implies the use of a scale parameter which is naturally chosen as the vacuum speed of light \( c \). We don’t need to include this constant in our equations because we can always recover time intervals, if needed, introducing the speed of light at a later stage. We can even go a step further and eliminate all units from our equations so that they become pure number equations; in this way we will avoid cumbersome constants whenever coordinates have to appear as arguments of exponentials or trigonometric functions. We note that, at least for the macroscopic world, physical units can all be reduced to four fundamental ones; we can, for instance, choose length, time, mass and electric charge as fundamental, as we could just as well have chosen others. Measurements are then made by comparison with standards; of course we need four standards, one for each fundamental unit. But now note that there are four fundamental constants: Planck constant \( \hbar \), gravitational constant \( G \), speed of light in vacuum \( c \) and proton electric charge \( e \), with which we can build four standards for the fundamental units.

Table 1 lists the standards of this units’ system, frequently called Planck units, which the authors prefer to designate by non-dimensional units. In this system all the fundamental constants, \( \hbar, G, c, e \), become unity, a particle’s Compton frequency, defined by \( \nu = mc^2/\hbar \), becomes equal to the particle’s mass and the frequent term \( GM/(c^2r) \) is simplified to \( M/r \). We can, in fact, take all measures to be non-dimensional, since the standards are defined with recourse to universal constants; this will be our posture. Geometry and physics become relations between pure numbers, vectors, bivectors, etc. and the geometric concept of distance is needed only for graphical representation.

**TABLE I: Standards for non-dimensional units’ system**

| Length | Time | Mass | Charge |
|---|---|---|---|
| \( \sqrt{G\hbar/c^3} \) | \( \sqrt{G\hbar/c^5} \) | \( \hbar c/G \) | \( e \) |

FIG. 1: Indices in the range \( \{0, 4\} \) will be denoted with Greek letters \( \alpha, \beta, \gamma \). Indices in the range \( \{0, 3\} \) will also receive Greek letters but chosen from \( \mu, \nu, \xi \). For indices in the range \( \{1, 4\} \) we will use Latin letters \( i, j, k \) and finally for indices in the range \( \{1, 3\} \) we will use also Latin letters chosen from \( m, n, o \).
APPENDIX C: SOME COMPLEMENTS OF GEOMETRIC ALGEBRA

In this section we expand the concepts given in Sec. II introducing some useful relations and definitions. Starting with the basis elements that square to unity Eq. (6), repeated here,

\[(\sigma_i)^2 = (\sigma_0)^2 = (\sigma_{0ij})^2 = (i\sigma_0)^2 = 1, \quad (C1)\]

it is easy to verify any of the above equations; suppose we want to check that \((\sigma_{0ij})^2 = 1\). Start by expanding the square and remove the compact notation \((\sigma_{0ij})^2 = \sigma_0\sigma_i\sigma_j\sigma_0\sigma_i\sigma_j\), then swap the last \(\sigma_j\) twice to bring it next to its homonymous; each swap changes the sign, so an even number of swaps preserves the sign: \((\sigma_{0ij})^2 = \sigma_0\sigma_i\sigma_j\sigma_0\sigma_i\sigma_j\). From the third equation (5) we know that the squared vector is unity and we get successively \((\sigma_{0ij})^2 = \sigma_0\sigma_i\sigma_j\sigma_0\sigma_i\sigma_j\). Using the first equation (5) we get finally \((\sigma_{0ij})^2 = 1\) as desired.

The remaining basis elements square to \(-1\) as can be verified in a similar manner, Eq. (7):

\[(\sigma_0)^2 = (\sigma_i)^2 = (\sigma_{ijk})^2 = (i\sigma_1)^2 = i^2 = -1. \quad (C2)\]

Note that the pseudoscalar \(i\) commutes with all the other basis elements while being a square root of \(-1\); this makes it a very special element which can play the role of the scalar imaginary in complex algebra.

We can now address the geometric product of any two vectors \(a = a^\alpha\sigma_\alpha\) and \(b = b^\beta\sigma_\beta\) making use of the distributive property

\[ab = \left(-a^0b^0 + \sum_i a^ib^i\right) + \sum_{\alpha\neq\beta} a^\alpha b^\beta \sigma_{\alpha\beta}; \quad (C3)\]

and we notice it can be decomposed into a symmetric part, a scalar called the inner or interior product, and an anti-symmetric part, a bivector called the outer or exterior product.

\[ab = a\cdot b + a\wedge b, \quad ba = a\cdot b - a\wedge b. \quad (C4)\]

Reversing the definition one can write inner and outer products as

\[a\cdot b = \frac{1}{2}(ab + ba), \quad a\wedge b = \frac{1}{2}(ab - ba). \quad (C5)\]

The inner product is the same as the usual "dot product," the only difference being in the negative sign of the \(ab\) term; this is to be expected and is similar to what one finds in special relativity. The outer product represents an oriented area; in Euclidean 3-space it can be linked to the "cross product" by the relation \(\text{cross}(a, b) = -\sigma_{123}a\wedge b\); here we introduced bold characters for 3-dimensional vectors and avoided defining a symbol for the cross product because we will not use it again. We also used the convention that interior and exterior products take precedence over geometric product in an expression.

When a vector is operated with a multivector the inner product reduces the grade of each element by one unit and the outer product increases the grade by one. We will generalize the definition of inner and outer products below; under this generalized definition the inner product between a vector and a scalar produces a vector. Given a multivector \(a\) we refer to its grade-\(r\) part by writing \(<a>\); the scalar or grade zero part is simply designated as \(<a>\). By operating a vector with itself we obtain a scalar equal to the square of the vector’s length

\[a^2 = aa = a\cdot a + a\wedge a = a\cdot a. \quad (C6)\]

The definitions of inner and outer products can be extended to general multivectors

\[a\cdot b = \sum_{\alpha,\beta} <a>_{\alpha} <b>_{\beta} |_{\alpha - \beta}, \quad (C7)\]

\[a\wedge b = \sum_{\alpha,\beta} <a>_{\alpha} <b>_{\beta} |_{\alpha + \beta}. \quad (C8)\]

Two other useful products are the scalar product, denoted as \(<ab>\) and commutator product, defined by

\[a \times b = ab - ba. \quad (C9)\]

In mixed product expressions we will use the convention that inner and outer products take precedence over geometric products.

We will encounter exponentials with multivector exponents; two particular cases of exponentiation are specially important. If \(u\) is such that \(u^2 = -1\) and \(\theta\) is a scalar

\[e^{u\theta} = 1 + u\theta + \frac{\theta^2}{2!} - u\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots\]

\[= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \{= \cosh \theta\}

\[+u\theta - u\frac{\theta^3}{3!} + \ldots \{= u\sin \theta\} \quad (C10)\]

\[= \cosh \theta + u\sin \theta. \]

Conversely if \(h\) is such that \(h^2 = 1\)

\[e^{h\theta} = 1 + h\theta + \frac{\theta^2}{2!} + h\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots\]

\[= 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots \{= \cosh \theta\}

\[+h\theta + h\frac{\theta^3}{3!} + \ldots \{= h\sinh \theta\} \quad (C11)\]

\[= \cosh \theta + h\sinh \theta. \]

The exponential of bivectors is useful for defining rotations; a rotation of vector \(a\) by angle \(\theta\) on the \(\sigma_{12}\) plane is performed by

\[a' = e^{\sigma_{12}\theta/2}ae^{\sigma_{12}\theta/2} = RaR; \quad (C12)\]
the tilde denotes reversion and reverses the order of all products. As a check we make \( a = \sigma_1 \)

\[
e^{-\sigma_1 \theta/2} \sigma_1 e^{\sigma_1 \theta/2} = \left( \cos \frac{\theta}{2} - \sigma_{12} \sin \frac{\theta}{2} \right) \sigma_1
\]

\[
\times \left( \cos \frac{\theta}{2} + \sigma_{12} \sin \frac{\theta}{2} \right)
= \cos \theta \sigma_1 + \sin \theta \sigma_2.
\]

Similarly, if we had made \( a = \sigma_2 \), the result would have been \(- \sin \theta \sigma_1 + \cos \theta \sigma_2\).

If we use \( B \) to represent a bivector whose plane is normal to \( \sigma_0 \) and define its norm by \( |B| = (BB)^{1/2} \), a general rotation in 4-space is represented by the rotor

\[
R \equiv e^{-B/2} = \cos \left( \frac{|B|}{2} \right) - \frac{B}{|B|} \sin \left( \frac{|B|}{2} \right).
\]

The rotation angle is \( |B| \) and the rotation plane is defined by \( B \). A rotor is defined as a unitary even multivector (a multivector with even grade components only) which squares to unity; we are particularly interested in rotors with bivector components. It is more general to define a rotation by a plane (bivector) then by an axis (vector) because the latter only works in 3D while the former is applicable in any dimension. When the plane of bivector \( B \) contains \( \sigma_0 \), a similar operation does not produce a rotation but produces a boost instead. Take for instance \( B = \sigma_0 \theta/2 \) and define the transformation operator \( T = \exp(B) \); a transformation of the basis vector \( \sigma_0 \) produces

\[
a' = \tilde{T} \sigma_0 T = e^{-\sigma_0 \theta/2} \sigma_0 e^{\sigma_0 \theta/2}
= \left( \cosh \frac{\theta}{2} - \sigma_{01} \sinh \frac{\theta}{2} \right) \sigma_0
\times \left( \cosh \frac{\theta}{2} + \sigma_{01} \sinh \frac{\theta}{2} \right)
= \cos \theta \sigma_0 + \sin \theta \sigma_1.
\]

\textbf{APPENDIX D: TIME DERIVATIVE OF A 4-DIMENSIONAL VECTOR}

If there is a refractive index the wave displacement vector can be written as

\[
dx = g_\alpha dx^\alpha = n^\beta \sigma_\beta dx^\alpha.
\]

Because this vector is nilpotent, by virtue of Eq. \((\ref{eq:20})\), the five coordinates are not independent and we can divide both members by \( dx^0 = dt \) defining the nilpotent vector

\[
\dot{x} = g_0 + g_i \dot{x}^i = n^\alpha \sigma_\alpha + n^\beta \sigma_\beta \dot{x}^i.
\]

Suppose we have a 5D vector \( a = \sigma_\alpha a^\alpha \) and we want to find its time derivative along a path parameterized by \( t \), that is all the \( x^i \) are functions of \( t \). We can write

\[
\dot{a} = \partial_\beta a^\alpha \dot{x}^\beta \sigma_\alpha;
\]

where naturally \( \dot{x}^0 = 1 \). Remembering the definition of covariant derivative \((\ref{eq:10})\) and Eq. \((\ref{eq:12})\) we can modify this equation to

\[
\dot{a} = \dot{x}^\beta g_\beta \cdot g^\alpha \partial_\beta a^\alpha \sigma_\alpha = \dot{x} \cdot (Da).
\]

We have expressed vector \( a \) in terms of the orthonormed frame in order to avoid vector derivatives but the result must be independent of the chosen frame.

This procedure has an obvious dual, which we arrive at by defining

\[
\dot{x} = g_\mu \dot{x}^\mu + g_4.
\]

The proper time derivative of vector \( a \) is then

\[
\dot{a} = \dot{x} \cdot (Da).
\]
[14] H. Yilmaz, *New theory of gravitation*, Phys. Rev. Lett. 27, 1399+, 1971.

[15] M. Ibisin, *The Yilmaz cosmology*, in *1st Crisis in Cosmology Conference, CCC–I*, edited by E. Lerner and J. B. Almeida (American Institute of Physics, Monçao, Portugal, 2005), to be published.

[16] In most cases $g_{00} = -1$, the velocity can be conveniently written $v = g_{i\dot{x}^i}$ and conjugate momenta $v_i = g_{ij}\dot{x}^j$. 