Analytical solutions of the geodesic equation in the space-time of a black hole surrounded by perfect fluid in Rastall theory

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Abstract: In this paper, we investigate the geodesic motion of massive and massless test particles in the vicinity of a black hole space-time surrounded by perfect fluid (quintessence, dust, radiation, cosmological constant and phantom) in Rastall theory. We obtain analytical solutions of the equations of motion for geodesics in vicinity of space-time of this black hole. For all cases of perfect fluid, we consider some different values of Rastall coupling constant $k$, for which the equations of motion have integer powers of $\tilde{r}$ and can be solved analytically. These analytical solutions are presented in the form of elliptic and also hyperelliptic functions. Also, by using analytical solutions, effective potential and $L-E^2$ diagrams, we plot some examples of possible orbits. Moreover, different orbits are classified using angular momentum, conserved energy, electrical charge and Rastall parameters. Furthermore, we show that when Rastall parameter becomes zero ($N = 0$), our results are consistent with the analysis of a Reissner–Nordström black hole, however; when both Rastall parameter and electric charge vanish ($N = Q = 0$), the results are the same as the analysis of a Schwarzschild black hole. In addition, the application of astrophysics of these results has also been investigated.

Keywords: Black hole; Geodesic motion; Analytical solutions; Effective potential; Elliptic functions

1. Introduction

Einstein’s theory of general relativity is a geometric gravitational theory which describes all solar system observations, dynamic cosmos and gravitation as a geometrical curvature [1]. The existence of black hole, which is predicted by the equations of general relativity, is one of the important issues in physics. Black holes and the metrics that explain the space-time around them are very important fields of study and research, because of having a gravitational influence on motion of light ray and test particles, information about the last step of the star life and discussion of the dark matter [2]. Researchers have expressed that the Universe contains dark matter [3, 4] and dark energy [5–7] which are two important problems of the standard present cosmological model and can explain the accelerating expansion of the cosmos [5, 6]. Dark matter is a scalar field (25% of energy content in the Universe) composed of weak interacting massive particles, whereas dark energy (70% of energy content in the Universe) [8] known as an exotic fluid, and a type of dynamical quantum vacuum energy or a kind of self-repulsive mysterious force with negative pressure [8–10]. Observational evidence such as cosmic microwave background radiation [11], the large-scale structure of the Universe [12–15] and luminosity distance of supernova type Ia [5, 6, 16–18] will be known as accelerating expansion phase reasons. Dark energy was proposed to interpret the accelerating rate [1] by a very small positive cosmological constant with a state parameter $\omega = -1$ [9, 10, 19–26]. Recent observations enable the existence of cosmological model including dark energy with an equation of state $\omega < -1$. Quintessence as a candidate for dark energy with state parameters in the range of $-1 < \omega_q < -\frac{1}{3}$ [21, 22, 27] and phantom field with $\omega = -\frac{4}{3}$ are two exotic matters which try to explain the nature of dark energy [9, 27]. A black hole might be surrounded by regular matter like radiation with $\omega = \frac{1}{3}$ and dust with state parameter $\omega = 0$ or exotic matter like cosmological constant, quintessence and phantom fields or combination of them [28].

One of the important consequences obtained from Einstein’s fields equations is a null divergence of the energy–momentum tensor in the form $T^\nu_{\;\nu} = 0$ [28]. Due to the

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violation of the usual classical conservation laws and verification of the condition $T^{\mu}_{\nu} \neq 0$ by particle creation in cosmology [29–36], a new formulation of the energy–momentum tensor has been suggested by quantities related to the curvature of the space-time [37]. In 1972, Rastall [38] proposed a modified theory of general relativity with a new formulation of connection between energy–momentum tensor, $T^{\mu}_{\nu}$, and the derivative of Ricci scalar, i.e. $T^{\mu}_{\nu} \propto R^{\nu}$ [28], which get back to the Einstein’s basic assumptions in the empty Universe [39] and represent the Mach principle [40]. Rastall assumed in curved space-time, the usual conservation laws used in general relativity are collapsed [41]. In other words, for a non-minimal way that the matter and geometry fields are joined together, $T^{\mu}_{\nu} = \lambda R^{\nu}$ where $\lambda$ called the Rastall free parameter which describes the deviation from the Einstein’s theory of general relativity and defined from observations [38, 42].

For the Universe expansion, Rastall theory can regenerate some loop quantum cosmological features [42–47] and also has agreement with the cosmic accelerating expansion [48]. Moreover, the effects of quantum fields in curved space-time in a covariant approach can be studied in this theory [28, 37, 49, 50]. Also, this theory can be consistent with the observational data of the Universe age, helium nucleosynthesis and Hubble parameter [51–53]. However, the Rastall theory anyway has attracted great attention and interests in different areas within the context [54–61], generalized in the literature and comparing with standard general relativity [62, 63]. Due to the correlation of Rastall’s theory with the high curvature environments and also, for having more information about the nature of a non-minimal coupling between matter fields and geometry in the Rastall hypothesis, studying the physical quantities and properties of black holes in Rastall gravity in more details and comparing results with general relativity, can be useful [28]. Therefore, in this paper, we are going to study the geodesic motion in the space-time of a black hole in Rastall theory.

Since only light and particles are detectable, studying their orbits in space-time near a black hole is an important tool for investigating physical properties and the features of solutions of Einstein’s field equations and also for tests of general relativity [64]. Also, the Hawking radiation and quantum tunnelling of massive and massless test particles from black holes and differences and resemblances between them have been investigated physically [65–67]. Moreover, study of geodesic equation of motion is especially useful for analysing the properties of space-time and predicts some observational events such as perihelion shift, light deflection and gravitational time-delay [68]. In 1916, Schwarzschild discovered the first exact solution to Einstein’s equations in the case of spherically symmetric black hole in four-dimensional space-time [69]. All analytical solutions of the geodesic equation in a Schwarzschild (AdS) space-time and gravitational field have been presented by Hagihara in 1931 [70].

Many different space-times in theory of general relativity and also in modified theory, such as four-dimensional Schwarzschild–de Sitter [1, 71], higher-dimensional Schwarzschild, Schwarzschild–(anti-)de Sitter, Reissner–Nordström, Reissner–Nordström–(anti-)de Sitter [71–73], Kerr [74], Kerr–de Sitter [75] and black holes in $f(R)$ gravity [76], static and rotating dilaton black hole [68], $(2 + 1)$-dimensional charged BTZ [2], static cylindrically and spherically symmetric conformal gravity [77, 78], the higher-dimensional Myers–Perry space-time [79–81], geodesics in the space-time of a rotating charged black hole [82], black string [83] and three-dimensional rotating black holes [84], have been studied and their geodesic equation solved analytically.

In this paper, we study the geodesic equations of motion of light ray and test particles in the space-time of a black hole surrounded by five perfect fluids such as quintessence, dust (energy matter), radiation, cosmological constant and phantom fields in Rastall theory. We show our analytical solution here in the form of elliptic and also hyperelliptic functions. In Sect. 2, we give a brief review of a black hole surrounded by perfect fluids in Rastall field equations. In Sect. 3, we investigate the analytical solution of the equations of motion for timelike and null geodesic equations with some possible values of Rastall coupling constant for five surrounding fields in five subsections. In Sect. 4, we use the analysis provided in the previous sections for geodesic equations and their analytical solutions, to plot $L$–$E^2$ diagrams, effective potential, and also to analyse the possible orbit types, their classification, astrophysical application, and plot some examples of possible orbits. We represent conclusion in Sect. 5.

2. Field equations in Rastall theory of gravity

The field equations for a space-time with Ricci scalar $R$, an energy momentum source of $T^{\mu}_{\nu}$, and for a space-time metric $g_{\mu\nu}$, in the context of gravitational Rastall theory, can be written as

$$G_{\mu\nu} + k \lambda g_{\mu\nu} R = k T^{\mu}_{\nu}. \quad (2.1)$$

So, the spherical symmetric space-time metric is

$$ds^2 = -f_s(r)dt^2 + \frac{dr^2}{f_s(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.2)$$

with the general metric function $f_s(r)$ in the framework of Rastall theory [28].

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The geodesic differential equation is in general of the form
\[ f_s(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_s}{r^{8m+1}}. \]  
\[ (2.3) \]

Here, \( f(R) \) is dependent on the \( k \) and \( \lambda \) (Rastall parameters), radial coordinate \( r \), mass \( M \), the electric charge of the black hole \( Q \), equation of state parameter \( \omega_s \), and surrounding field structure parameter \( N_s \) [28]. Equations (2.2) and (2.3), for \( k = 8\pi G_N \) and \( \lambda = 0 \), are converted into
\[ ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_s}{r^{8m+1}}\right)dr^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_s}{r^{8m+1}}} + r^2d\Omega^2, \]  
\[ (2.4) \]

which represent the Reissner–Nordström black hole surrounded by a surrounding field in general relativity [27]. By comparing metrics (2.2)–(2.3) with (2.4), some interesting features with introducing an “effective equation of state” have been studied in detail in [28]. The subscript “s” indicates the general surrounding field.

3. Geodesics

The geodesic differential equation is in general of the form
\[ \frac{d^2x^\alpha}{dz^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{dz} \frac{dx^\gamma}{dz} = 0, \]  
\[ (3.1) \]

where \( \Gamma^\alpha_{\beta\gamma} \) are the Christoffel symbols. By using the normalization condition \( g_{\alpha\beta} \frac{dx^\alpha}{dz} \frac{dx^\beta}{dz} = \epsilon \) (where for massive particles \( \epsilon = 1 \) and for light \( \epsilon = 0 \)) and two constants of motion, energy \( E \) and the angular momentum \( L \), as
\[ E = g_{tr} \frac{dr}{dz} = f_s(r) \frac{dr}{dz}, \quad L = g_{\phi r} \frac{d\phi}{dz} = r^2 \frac{d\phi}{dz}, \]  
\[ (3.2) \]

the equations of motion in an equatorial plane, \( \theta = \frac{\pi}{2} \), can be written as
\[ \left( \frac{dr}{dz} \right)^2 = E^2 - f_s(r) \left( \epsilon + \frac{L^2}{r^2} \right), \]  
\[ (3.3) \]
\[ \left( \frac{d\phi}{dz} \right)^2 = \frac{L^2}{E^2} \left( E^2 - f_s(r) \left( \epsilon + \frac{L^2}{r^2} \right) \right) =: R(r), \]  
\[ (3.4) \]
\[ \left( \frac{dr}{dz} \right)^2 = \frac{f_s^2(r)}{E^2} \left( E^2 - f_s(r) \left( \epsilon + \frac{L^2}{r^2} \right) \right). \]  
\[ (3.5) \]

The effective potential \( V_{\text{eff}} \) can be obtained from Eq. (3.3) as
\[ V_{\text{eff}} = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_s}{r^{8m+1}}\right) \left( \epsilon + \frac{L^2}{r^2} \right). \]  
\[ (3.6) \]

We rewrite the equations with new dimensionless parameters
\[ \tilde{r} = \frac{r}{M}, \quad \tilde{L} = \frac{M^2}{L^2}, \quad \tilde{Q} = \frac{Q}{M}, \]  
\[ (3.7) \]
and then
\[ ds^2 = -f_s(\tilde{r})d\tilde{r}^2 + \frac{d\tilde{r}^2}{f_s(\tilde{r})} + \tilde{r}^2d\Omega^2, \]  
\[ (3.8) \]
\[ f_s(\tilde{r}) = 1 - \frac{2\tilde{r}}{\tilde{r}^2} - \frac{\tilde{N}_s}{\tilde{r}^{8m+1}}. \]  
\[ (3.9) \]

Also, Eq. (3.4) with the generic metric of Rastall theory [Eq. (2.3)] takes the following form
\[ \left( \frac{d\tilde{r}}{d\phi} \right)^2 = \tilde{r}^4 \tilde{L} \left( E^2 - \left(1 - \frac{2\tilde{r}}{\tilde{r}^2} - \frac{\tilde{N}_s}{\tilde{r}^{8m+1}}\right) \left( \epsilon + \frac{1}{\tilde{L}^2} \right) \right) = R(\tilde{r}). \]  
\[ (3.10) \]

To solve this equation and investigate its results, we study analytical solutions of geodesic equations of a black hole surrounded by quintessence, dust, cosmological constant, radiation and phantom fields.

3.1. The black hole surrounded by the quintessence field

In this section, we obtain the equations of motion for possible values of \( k\lambda \) for the quintessence surrounding field. By putting \( \omega_s = \omega_q = -\frac{2}{3} \) [27], Eqs. (3.8) and (3.9) are converted into the following equations
\[ ds^2 = -f_q(\tilde{r})d\tilde{r}^2 + \frac{d\tilde{r}^2}{f_q(\tilde{r})} + \tilde{r}^2d\Omega^2, \]  
\[ (3.11) \]
\[ f_q(\tilde{r}) = 1 - \frac{2\tilde{r}}{\tilde{r}^2} - \tilde{N}_q \frac{1}{\tilde{r}^{8m+1}}. \]  
\[ (3.12) \]

The equation of effective state parameter \( \omega_{\text{eff}} \) can be obtained by comparing Eqs. (3.11) and (3.12) with the original Kieslemy metric [Eq. (2.4)] [28]
\[ \omega_{\text{eff}} = \frac{1}{3} \left( -1 + \frac{2k\lambda}{1 - k\tilde{r}} \right). \]  
\[ (3.13) \]

By considering two values of \( \omega_{\text{eff}} \leq -\frac{1}{3} \) and \( w_{\text{eff}} \geq \frac{1}{3} \) in Eq. (2.4) [28], the range values of \( k\lambda \) in Eq. (3.13) are discernible as \( -\frac{1}{2} \leq k\lambda < 1 \) and \( k\lambda \leq -\frac{1}{3} \cup k\lambda > 1 \), respectively. Of course, in this paper for all surrounding fields cases, we consider the possible values of \( k\lambda \), so that \( f_s(\tilde{r}) \) in Eq. (3.9) has included integer powers of \( \tilde{r} \) and also Eq. (3.10) can be solved analytically. For other values of \( k\lambda \), Eq. (3.10) has some terms with fractional powers of \( \tilde{r} \), which in our ability cannot be solved analytically but may be solved numerically same as applied technique in...
Ref. [85]. Considering the weak energy condition, for $\omega_{\text{eff}} \geq -\frac{1}{3}$ and $k\lambda = -2$ [28], then

$$d\mathbf{s}^2 = -f_q(\tilde{r})d\tilde{r}^2 + \frac{d\tilde{\phi}^2}{f_q(\tilde{r})} + \tilde{r}^2 d\Omega^2, \quad (3.14)$$

$$f_q(\tilde{r}) = 1 - \frac{2 + \tilde{N}_d}{\tilde{r}} + \frac{\tilde{Q}^2}{\tilde{r}^2}, \quad (3.15)$$

and

$$V_{\text{eff}} = \left(1 - \frac{2 + \tilde{N}_d}{\tilde{r}} + \frac{\tilde{Q}^2}{\tilde{r}^2}\right) \left(\epsilon + \frac{1}{\tilde{L}^2}\right). \quad (3.16)$$

So, Eq. (3.10) can be written as

$$\left(\frac{d\tilde{r}}{d\tilde{\phi}}\right)^2 = (E^2 - \epsilon)\tilde{L}^4 + (2 + \tilde{N}_d)e\tilde{L}^3 - (1 + \tilde{Q}^2 e\tilde{L})\tilde{r}^2 + (2 + \tilde{N}_d)\tilde{r} - \tilde{Q}^2 = R_q(\tilde{r}). \quad (3.17)$$

3.1.1. Analytical solution of geodesic equations

In this section, we present the analytical solution of the geodesic Eq. (3.17). For both light ray ($\epsilon = 0$) (Null Geodesics) and the massive particle ($\epsilon = 1$) (Timelike geodesics), Eq. (3.17) is a polynomial of degree four in the form $\left(\frac{d\tilde{r}}{d\tilde{\phi}}\right)^2 = \sum_{i=0}^{4} a_i \tilde{r}^i$, which, by substitution $\tilde{r} = \frac{1}{u} + \tilde{r}_R$, where $\tilde{r}_R$ is a zero of $R$, is converted into a polynomial $R_3$ of degree three

$$\left(\frac{du}{d\tilde{\phi}}\right)^2 = R_3(u) = \sum_{j=1}^{3} b_j u^j, \quad u(\tilde{\phi}_0) = u_0, \quad (3.18)$$

where

$$b_j = \frac{1}{(4-j)!} \frac{d^{(4-j)}R}{d\tilde{r}^{4-j}} (\tilde{r}_R), \quad (3.19)$$

in which $b_j$ ($j = 1, 2, 3$) is an arbitrary constant of the relevant metric. Next, substitution $u = \frac{1}{b_3} (4y - b_2)$ transforms $R_3(u)$ to elliptical type differential equation as [82]

$$\left(\frac{dy}{d\tilde{\phi}}\right)^2 = 4y^3 - g_2 y - g_3 = p_3(y). \quad (3.20)$$

Equation (3.20) is known as the Weierstrass form, and

$$g_2 = \frac{1}{16} \left(\frac{4}{3} b_2^2 - 4b_1 b_3\right),$$

$$g_3 = \frac{1}{16} \frac{1}{3} b_1 b_2 b_3 - \frac{2}{27} b_2^3 b_3^2, \quad (3.21)$$

are the Weierstrass invariants. So, the answer of Eq. (3.20), using the Weierstrass function, is

$$y(\tilde{\phi}) = y(\tilde{\phi} - \tilde{\phi}_{\text{in}}; g_2, g_3), \quad (3.22)$$

in which $\tilde{\phi}_{\text{in}} = \phi_0 + \int_0^\infty \frac{dv}{\sqrt{v^2 - g_2v^2 - g_3}}$, with $\phi_0 = \frac{1}{4} \left(\frac{b_2}{r_0} + \frac{b_2}{2}\right)$ depends only on the initial value $\phi_0$ and $\tilde{r}_0$. Eventually, the solution of polynomials of degree four is [1]

$$\tilde{r}(\tilde{\phi}) = \frac{b_3}{4} \sqrt{\epsilon(\phi - \phi_{\text{in}}; g_2, g_3) - b_2} + \tilde{r}_R. \quad (3.23)$$

This analytic solution is obtained for null geodesic in quintessence surrounding field in Rastall theory and is reliable in all regions of these space-times. The explanation and properties presented in this section are applied to solve all geodesic equations of elliptic type in this paper. We use these analytical solutions to plot some examples of possible orbits of test particles and light ray, but before that we need to plot $\tilde{L} - E^2$ diagram for each case. Solving $R_q(\tilde{r}) = 0$ and $\frac{dR_q(\tilde{r})}{d\tilde{r}} = 0$, for massive particle ($\epsilon = 1$) with $k\lambda = -2$, we get

$$\tilde{L} = -\frac{4\tilde{Q}^2 - 3(2 + \tilde{N}_d)\tilde{r} + 2\tilde{r}^2)^2}{(2\tilde{Q}^2 - \tilde{N}_d\tilde{r} + 2\tilde{r}^2)^2}, \quad (3.24)$$

$$E^2 = \frac{2(\tilde{Q}^2 - 3(2 + \tilde{N}_d)\tilde{r} + 2\tilde{r}^2)^2}{(4\tilde{Q}^2 - 3(2 + \tilde{N}_d)\tilde{r} + 2\tilde{r}^2)^2}, \quad (3.25)$$

and for massless particles ($\epsilon = 0$)

$$\tilde{L} = \frac{\tilde{Q}^2 + \tilde{r}^2 - (2 + \tilde{N}_d)\tilde{r}}{E^2\tilde{r}^4}. \quad (3.26)$$

Plots of $\tilde{L} - E^2$ diagrams [Eqs. (3.24), (3.25)] are shown in Fig. 1. Moreover, a summary of possible orbits type with numbers of zero points in each region for $k\lambda = -2$ is shown in Table 1. Also, effective potential diagrams [Eq. (3.16)] are shown in Fig. 2.

3.2. The black hole surrounded by dust field

When the black hole is surrounded by the dust field, we put $\omega_4 = \omega_5 = 0$ [27] and metric (3.8) can be written as follows:

$$d\mathbf{s}^2 = -f_d(\tilde{r})d\tilde{r}^2 + \frac{d\tilde{\phi}^2}{f_d(\tilde{r})} + \tilde{r}^2 d\Omega^2, \quad (3.26)$$

$$f_d(\tilde{r}) = 1 - \frac{2}{\tilde{r}} + \frac{\tilde{Q}^2}{\tilde{r}^2} - \frac{\tilde{N}_d}{\tilde{r}^{1+\frac{2}{3}}}. \quad (3.27)$$

The equation of effective state parameter $\omega_{\text{eff}}$ can be obtained by comparing this metric with the original Kieslev metric (2.4), as [28]

$$\omega_{\text{eff}} = \frac{1}{3} \left(-1 + \frac{1 - 6k\lambda}{1 - 3k\lambda}\right). \quad (3.28)$$

By considering two values of $\omega_{\text{eff}} \leq -\frac{1}{4}$ and $w_{\text{eff}} \geq -\frac{1}{3}$ in
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Fig. 1 Plots of $L-E^2$ diagram and region of different types of geodesic motion in a quintessence surrounding field with the parameters $k\lambda = -2$, $\tilde{N} = 0.025$ and $\tilde{Q} = \sqrt{0.25}$ corresponding to Table 1 for (a) null and (b) timelike geodesics. The numbers of positive real zeros in these regions are: $I = 1, II = 2, III = 3, IV = 4$

Table 1 Types of orbits of the quintessence surrounding field for $k\lambda = -2$

| Region | pos. zeros | range of $\tilde{r}$ | Orbit   |
|--------|------------|----------------------|---------|
| I      | 1          | [-----•----------]   | TEO     |
| II     | 2          | [-----•--••--]      | MBO, EO |
| III    | 3          | [-----•--•--•--]    | MBO, BO |
| IV     | 4          | [-----•--•--•--]    | MBO, BO |

The lines represent the range of the orbits. The dots show the turning points of the orbits. The positions of the two horizons are marked by a vertical double line. The single vertical line indicates the singularity at $\tilde{r} = 0$

Eq. (2.4) [28], the range values of $k\lambda$ in Eq. (3.28) are discernible as $\frac{1}{6} < k\lambda < \frac{1}{3}$ and $k\lambda < \frac{1}{3} \cup k\lambda > \frac{1}{3}$, respectively. To solve the analytical solution of the equations of motion for surrounding dust field, considering the weak energy condition, for $\omega_{\text{eff}} \leq -\frac{1}{3}$ and $k\lambda = \frac{2}{5}$, metric (3.26) can be written as

$$dx^2 = -f_d(\tilde{r})d\tilde{r}^2 + \frac{d\tilde{r}^2}{f_d(\tilde{r})} + \tilde{r}^2d\Omega^2,$$

$$f_d(\tilde{r}) = 1 - \frac{2}{\tilde{r}} + \frac{\tilde{Q}^2}{\tilde{r}^2} - \tilde{N}d\tilde{r},$$

and

$$V_{\text{eff}} = \left(1 - \frac{2}{\tilde{r}} + \frac{\tilde{Q}^2}{\tilde{r}^2} - \tilde{N}d\tilde{r}\right)\left(\epsilon + \frac{1}{L^2}\right).$$

Then, metric (3.10) gets the form

$$\left(\frac{dr}{d\phi}\right)^2 = \tilde{N}d\tilde{r}^5 + (E^2 - \epsilon)\tilde{r}^4 - (1 + \tilde{Q}^2\epsilon)\tilde{r}^2 + 2\tilde{r} - \tilde{Q}^2 = R_d(\tilde{r}).$$

Null Geodesics For $\epsilon = 0$, Eq. (3.32) is a polynomial of degree four and elliptic type. Therefore, it has an analytical solution same as Sect. 3.1.1

Fig. 2 Plots of the effective potential for the different orbit types of Table 1, for the case of quintessence surrounding field with the parameters $k\lambda = -2$. The horizontal red dashed lines denote the squared energy parameter $E^2$. The vertical black dashed lines show the position of the horizons. The red dot marks denote the zeros of the polynomial $R$, which are the turning points of the orbits. In the cyan area, no motion is possible since $\tilde{R}<0$ (color figure online)
\[ \ddot{r}(\phi) = \frac{b_3}{4\psi(\phi - \phi_m; g_2, g_3)} - \frac{b_3}{2} + \ddot{r}_K. \]  

(3.33)\]

**Timelike Geodesics** For the massive particle \((\epsilon = 1)\), Eq. (3.32) is a polynomial of order five and also of the hyperelliptic type. By substitution \(\ddot{r} = \frac{1}{u}\), this equation of motion can be converted into one of the two forms

\[
\left( \frac{du}{d\phi} \right)^2 = P_5(u),
\]

(3.34)

\[
\left( \frac{d\phi}{d\phi} \right)^2 = P_5(u).
\]

(3.35)

The analytic solution of above equations, which is extensively discussed in [79, 82, 86], is given in the form of derivatives of the Kleinian \(\sigma\) function as

\[
u(\phi) = -\frac{\sigma_1(\varphi_\infty)}{\sigma_2(\varphi_\infty)}|_{\varphi(\varphi_\infty)=0},
\]

(3.36)

with

\[
\varphi_\infty = (\varphi_2, \varphi_3, \varphi_m),
\]

(3.37)

where \(\varphi_m = \varphi_m + \int_{\varphi_m}^{\infty} \frac{du}{P_5(u)}\). The component \(\varphi_2\) is determined by the condition \(\sigma(\varphi_\infty) = 0\). Also, the function \(\sigma_i\) is the \(i\)th derivative of Kleinian \(\sigma\) function and \(\sigma_i\) is

\[
\sigma_i = Ce^{2i\omega \theta[g, \theta]}(2w^{-1}z; \tau),
\]

(3.38)

where \(C\) is a constant, \(\tau\) is the symmetric Riemann matrix, \(\omega\) is the period matrix, \(k = \eta(2)^{-1}\) in which \(\eta\) is the period matrix of the second kind and \(\theta\) is the Riemann function with characteristic \([g, h]\) in which \(2|g, h| = (0, 1)^2 + (1, 1)^2\) [71, 73, 76, 87]. So, the solution of Eq. (3.32) becomes

\[
\ddot{r}(\phi) = -\frac{\sigma_2(\varphi_\infty)}{\sigma_1(\varphi_\infty)}.
\]

(3.39)

This analytic solution is obtained for timelike geodesic in the dust surrounding field in Rastall theory and is reliable in all regions of these space-times. The explanation and properties presented in this section are applied to solve all geodesic equations of hyperelliptic type in this paper.

Next, as discussed in the previous section, we need to plot \(\ddot{L}-E^2\) diagram. So by solving \(R_d(\ddot{r}) = 0\) and \(\frac{dR_d(\ddot{r})}{d\phi} = 0\), for massive particles \((\epsilon = 1)\) with \(k\lambda = \frac{3}{2}\),

\[
\ddot{L} = -\frac{2\ddot{Q}^2 + 2\ddot{r}^2 - 6\ddot{r}}{\ddot{r}^2(\ddot{N}_r^3 + 2\ddot{Q}^2 - 2\ddot{r}^2)},
\]

(3.40)

\[
E^2 = \frac{2(\ddot{N}_r^3 + \ddot{Q}^2 + \ddot{r}^2 - 2\ddot{r})}{(\ddot{N}_r^3 + 4\ddot{Q}^2 + 2\ddot{r}^2 - 6\ddot{r})\ddot{r}^2},
\]

and for massless particles \((\epsilon = 0)\),

\[
\ddot{L} = -\frac{2\ddot{Q}^2 + \ddot{r}^2 - 2\ddot{r}}{E^2\ddot{r}^2}.
\]

(3.41)

Plots of \(\ddot{L}-E^2\) diagram [Eqs. (3.40)–(3.41)] with the region of different types of geodesic motion in the dust surrounding field with the case \(k\lambda = \frac{3}{2}\) are illustrated in Fig. 3. Moreover, a summary of possible orbits type with numbers of zero points in each region is shown in Table 2. Also, effective potential diagrams [Eq. (3.31)] are shown in Fig. 4.

3.3. The black hole surrounded by the radiation field

When the radiation is surrounding field, we put \(w_r = \frac{1}{3}\) [27], so metric (3.8) can be written as

\[
ds^2 = -f_r(r)dr^2 + \frac{dr^2}{f_r(r)} + \frac{r^2}{f_r(r)}d\Omega^2,
\]

(3.42)

\[
f_r(r) = 1 - \frac{2}{r} + \frac{\dot{Q}^2 - N_r}{r^2},
\]

(3.43)

with the effective potential

\[
V_{eff} = \left( 1 - \frac{2}{r} + \frac{\dot{Q}^2 - N_r}{r^2} \right) \left( \epsilon + \frac{1}{L^2} \right).
\]

(3.44)

Therefore, Eq. (3.10) for the black hole surrounded by the Radiation field becomes

\[
\left( \frac{d\phi}{d\phi} \right)^2 = (E^2 - \epsilon)\dot{L}r^3 + 2\epsilon\dot{L}r^3 - ((\dot{Q}^2 - N_r)rL + 1)\dot{r}^2 + 2\dot{r} - (\dot{Q}^2 - N_r) = R_r(\ddot{r}).
\]

(3.45)

**Null and timelike geodesics** Equation (3.45) is a polynomial of degree four and therefore for both massive \((\epsilon = 1)\) and massless \((\epsilon = 0)\) particles has an analytical solution similar to Sect. 3.1.1, as

\[
\ddot{r}(\phi) = \frac{b_3}{4\psi(\phi - \phi_m; g_2, g_3)} - \frac{b_3}{2}.
\]

(3.46)

Then, similar to previous sections, to plot \(\ddot{L}-E^2\) diagram for a black hole surrounded by the radiation field, by solving \(R_r(\ddot{r}) = 0\) and \(\frac{dR_r(\ddot{r})}{d\phi} = 0\), for massive particles \((\epsilon = 1)\)
\[ L = \frac{(\hat{Q}^2 - \hat{N}_r) + \hat{r}^2 - 2\hat{r}}{E^2 \hat{r}^4}. \]  

Plots of \( L - E^2 \) diagrams [Eqs. (3.47)–(3.48)] with the region of different types of geodesic motion, in the radiation surrounding field, are shown in Fig. 5. Also, plots of effective potential [Eq. (3.44)] are shown in Fig. 6. Moreover, a summary of possible orbits type with numbers of zero points in each region for both massive and massless particles is shown in Table 3.

### 3.4. The black hole surrounded by the cosmological constant field

For the cosmological constant surrounding field, we put \( w_c = -1 \) [27], so metric (3.8) can be written as

\[ ds^2 = -f_c(\hat{r})d\hat{r}^2 + f_c(\hat{r})\hat{r}^2 d\Omega^2, \]  

and for massless particles (\( \epsilon = 0 \))

\[ \tilde{L} = \frac{(\hat{Q}^2 - \hat{N}_r) + \hat{r}^2 - 2\hat{r}}{E^2 \hat{r}^4}. \]  

### Table 2

| Region | pos.zeros | range of \( \hat{r} \) | Orbit |
|--------|-----------|-----------------------|-------|
| I      | 1         | \( \infty \) - \( \hat{N}_r \) | TEO   |
| III    | 3         | \( \hat{N}_r \) - \( \hat{L} \) | MBO, EO |
| V      | 5         | \( \hat{L} \) - \( \hat{r} \) | MBO, EO |

The lines represent the range of the orbits. The dots show the turning points of the orbits. The positions of the two horizons are marked by a vertical double line. The single vertical line indicates the singularity at \( \hat{r} = 0 \)

\[ L = \frac{2(\hat{Q}^2 - \hat{N}_r) + \hat{r}^2 - 3\hat{r}}{\hat{r}^2((\hat{Q}^2 - \hat{N}_r) - \hat{r})}, \]

\[ E^2 = \frac{(\hat{Q}^2 - \hat{N}_r) + \hat{r}^2 - 2\hat{r})^2}{(2(\hat{Q}^2 - \hat{N}_r) + \hat{r}^2 - 3\hat{r})\hat{r}^2}, \]

and for massless particles (\( \epsilon = 0 \))

\[ \tilde{L} = \frac{(\hat{Q}^2 - \hat{N}_r) + \hat{r}^2 - 2\hat{r}}{E^2 \hat{r}^4}. \]
\[ f_c(\vec{r}) = 1 - \frac{2}{\vec{r}} + \frac{\vec{Q}^2}{\vec{r}^2} - \vec{N}_c \vec{r}^2, \quad (3.50) \]

with effective potential
\[ V_{\text{eff}} = \left( 1 - \frac{2}{\vec{r}} + \frac{\vec{Q}^2}{\vec{r}^2} - \vec{N}_c \vec{r}^2 \right) \left( \epsilon + \frac{1}{L_c^2} \right). \quad (3.51) \]

So, Eq. (3.10) gets the form
\[ \left( \frac{d\vec{r}}{d\phi} \right)^2 = \vec{N}_c \vec{L} \vec{r}^6 + \left( (E^2 - \epsilon) \vec{L} + \vec{N}_c \right) \vec{r}^4 + 2 \vec{L} \epsilon \vec{r}^3 \]
\[- (1 + \vec{Q}^2 \vec{L}) \vec{r}^2 + 2 \vec{r} - \vec{Q}^2 = R_c(\vec{r}). \quad (3.52) \]

**Timelike Geodesics**

For the massive particle (\( \epsilon = 1 \)), Eq. (3.52) is a polynomial of order six, and by substitutución \( \vec{r} = \frac{1}{\beta} \), this equation of motion can be converted into one of the two forms
\[ \left( \frac{du}{d\phi} \right)^2 = P_3(u), \quad (3.53) \]
\[ \left( \frac{du}{d\phi} \right)^2 = P_3(u). \quad (3.54) \]

So, the analytical solution of (3.52) for massive particles (\( \epsilon = 1 \)) is same as Sec. 3.2,
\[ r(\phi) = \frac{\sigma_2(\phi_\infty)}{\sigma_1(\phi_\infty)}. \quad (3.55) \]

**Null geodesics**

For the test particle (\( \epsilon = 0 \)), Eq. (3.52) is a polynomial of degree four, which has analytical solution like Sect. 3.1.1
\[ \vec{r}(\phi) = \frac{b_3}{4\lambda(\phi - \phi_\infty; g_2, g_3) - b_1} + \vec{r}_R. \quad (3.56) \]

Next, solving \( R_c(\vec{r}) = 0 \) and \( \frac{dR_c(\vec{r})}{d\vec{r}} = 0 \) gives us \( E^2 \) and \( \vec{L} \) diagram. Thus, for massive particles (\( \epsilon = 1 \)), in the black hole surrounded by the cosmological constant surrounding field, we have
\[ \vec{L} = -\frac{2\vec{Q}^2 + \vec{r}^2 - 3\vec{r}}{\vec{r}^2(\vec{N}_c \vec{r}^3 + \vec{Q}^2 - \vec{r})}, \quad E^2 = \frac{(-\vec{N}_c \vec{r}^4 + \vec{Q}^2 + \vec{r}^2 - 2\vec{r})^2}{(2\vec{Q}^2 + \vec{r}^2 - 3\vec{r})\vec{r}^2}, \quad (3.57) \]

and for massless particles (\( \epsilon = 0 \)), we have
\[ \vec{L} = -\vec{N}_c \vec{r}^4 + \vec{Q}^2 + \vec{r}^2 - 2\vec{r}. \quad (3.58) \]

Plots of \( \vec{L} - E^2 \) diagram [Eqs. (3.57), (3.58)] for the black hole surrounded by the cosmological constant background are shown in Fig. 7. Also, plots of effective potential Eq. (3.51) are shown in Fig. 8.

3.5. The black hole surrounded by the phantom field

For the phantom surrounding field, we put \( \omega_c = -\frac{3}{2} \) [27], so metric (3.8) can be obtained as
\[ ds^2 = -f_p(\vec{r})d\vec{r}^2 + \frac{dr^2}{f_p(\vec{r})} + \vec{r}^2 d\Omega^2, \quad (3.59) \]
\[ f_p(\vec{r}) = 1 - \frac{2}{\vec{r}} + \frac{\vec{Q}^2}{\vec{r}^2} - \frac{\vec{N}_p}{\vec{r}^{1 + \kappa \lambda}}. \quad (3.60) \]

The equation of effective state parameter \( \omega_{\text{eff}} \) can be obtained by comparing Eq. (3.59) with the Kieslev metric (2.4) [28],
\[ \omega_{\text{eff}} = \frac{1}{3} \left( -1 - \frac{3 - 2k \lambda}{1 + k \lambda} \right). \quad (3.61) \]

Now, by considering two values of \( \omega_{\text{eff}} \leq -\frac{1}{2} \) and \( \omega_{\text{eff}} \geq -\frac{1}{2} \) in Eq. (2.4) [28], the range values of \( k \lambda \) in Eq. (3.61) are discernible as \(-1 \leq k \lambda < \frac{1}{2} \) and \( k \lambda \leq -1 \cup k \lambda \geq \frac{3}{2} \), respectively.

---

**Fig. 5** Plots of \( L - E^2 \) diagram and regions of different types of geodesic motion for a black hole surrounded by radiation field with the parameters \( N = 0.12 \) and \( \vec{Q} = \sqrt{0.25} \) corresponding to Table 3 for (a) null (\( \epsilon = 0 \)) and (b) timelike geodesics (\( \epsilon = 1 \)). The numbers of positive real zeros in these regions are:

- Region I: \( N = 1 \), \( II = 2 \), \( III = 3 \), \( IV = 4 \).
Considering the weak energy condition, for 
\[ \frac{L}{C^2} = 0.25, \quad \epsilon = 1, \quad \tilde{N} = 0.12, \quad \tilde{Q} = \sqrt{0.25}, \quad E = \sqrt{0.25} \]
and 
\[ \tilde{k} = \frac{2}{3}, \]
we have
\[ ds^2 = \frac{1}{f_p^2(r)} dr^2 + \frac{1}{f_p^2(r)} r^2 d\Omega^2, \quad (3.62) \]

Equation (3.65) is exactly similar to case \( k\tilde{\kappa} = \frac{2}{5} \) in the dust surrounding field (Eq. (3.32)), except with difference between metric coefficient.

**Null Geodesics** The analytical solution for \( \epsilon = 0 \) is same as Sect. 3.1.1,

\[ f_p(r) = 1 - \frac{2}{r} + \frac{\tilde{Q}^2}{r^2} - \tilde{N}_p r, \quad (3.63) \]
with effective potential
\[ V_{\text{eff}} = \left( 1 - \frac{2}{r} + \frac{\tilde{Q}^2}{r^2} - \tilde{N}_p r \right) \left( \epsilon + \frac{1}{L^2} \right). \quad (3.64) \]

So, Eq. (3.10) gets the form
\[ \left( \frac{d\hat{r}}{d\varphi} \right)^2 = \tilde{N}_p \epsilon \hat{L}\hat{r}^3 + (E^2 - \epsilon) \hat{L}\hat{r}^3 + (2\hat{L}\epsilon + \tilde{N}_p) \hat{r}^3 \]
\[ - (1 + \tilde{Q}^2 \epsilon \hat{L}) \hat{r}^2 + 2\hat{r} - \tilde{Q}^2 = R_p(\hat{r}). \quad (3.65) \]

**Table 3** Types of orbits of the radiation surrounding field

| Region | pos.zeros | range of \( \tilde{r} \) | Orbit |
|--------|-----------|-----------------------|-------|
| I      | 1         | [0, \infty)           | TEO   |
| II     | 2         | [0, \infty)           | MBO   |
| III    | 3         | [0, \infty)           | MBO, EO |
| IV     | 4         | [0, \infty)           | MBO, BO |

The lines represent the range of the orbits. The dots show the turning points of the orbits. The positions of the two horizons are marked by a vertical double line. The single vertical line indicates the singularity at \( \tilde{r} = 0 \)

- Considering the weak energy condition, for \( \omega_{\text{eff}} \leq \frac{1}{3} \) and \( k\tilde{\kappa} = \frac{2}{3} \),
we have
\[ ds^2 = -f_p(\tilde{r}) d\tilde{r}^2 + \frac{d\tilde{r}^2}{f_p(\tilde{r})} + \tilde{r}^2 d\Omega^2, \quad (3.62) \]
Fig. 8 Plots of the effective potential for the different orbit types of Table 4, for the case of cosmological surrounding field. The horizontal red dashed lines denote the squared energy parameter $E^2$. The vertical black dashed lines show the position of the horizons. The red dot marks denote the zeros of the polynomial $R$, which are the turning points of the orbits. In the cyan area, no motion is possible since $R<0$ (color figure online).

\[ \frac{\ddot{r}}{\dot{r}} = \frac{b_3}{4\sqrt{(\phi - \phi_{in}, g_2, g_3) - \frac{b_1}{r^2}} + \ddot{r}R. \]  

(3.66)

Timelike Geodesics For $\epsilon = 1$, Eq. (3.65) is of order five and has the analytical solution same as Sec. 3.2

\[ \ddot{r}(\phi) = \frac{\sigma_2(\phi_\infty)}{\sigma_1(\phi_\infty)}. \]  

(3.67)

- The next possible value considering the weak energy condition in the range $\omega_{eff} \geq -\frac{1}{2}$ is $k\lambda = 4$.

So, metric (3.10) will be as the form

\[ ds^2 = -f_p(\dot{r})d\tau^2 + \frac{d\varphi^2}{f_p(\dot{r})} + \dot{r}^2 d\Omega^2, \]  

(3.68)

\[ f_p(\dot{r}) = 1 - 2 + \frac{\dot{N}_p}{\dot{r}} + \frac{\dot{Q}^2}{\dot{r}^2} \]  

(3.69)

with effective potential

\[ V_{\text{eff}} = \left( 1 - \frac{2 + \dot{N}_p}{\dot{r}} + \frac{\dot{Q}^2}{\dot{r}^2} \right) \left( \epsilon + \frac{1}{E_\infty^2} \right). \]  

(3.70)

Then,

\[ \left( \frac{dR}{d\varphi} \right)^2 = (E^2 - \epsilon)\dot{L}\dot{r}^2 + (\dot{N}_p \epsilon + 2\epsilon)\dot{L}\dot{r}^3 - (1 + Q^2 \epsilon L)^2 \]  

\[ + 2\dot{r} + \dot{N}_p\dot{r} - \dot{Q}^2 = R_p(\dot{r}). \]  

(3.71)

Analytical solution of Eq. (3.71) for both massive and massless geodesic is given by Weierstrass form same as Sect. 3.1.1.

For a phantom surrounding field with $k\lambda = 4$, solving $R_p(\dot{r}) = 0$ and $\frac{dR_p(\dot{r})}{d\dot{r}} = 0$ gives us $L-E^2$ diagram similar to the case $k\lambda = -2$ in quintessence surrounding field. So, for massive particles ($\epsilon = 1$),

\[ \dot{L} = -\frac{4\dot{Q}^2 + 2\dot{r}^2 - 3(2 + \dot{N}_p)\dot{r}}{\dot{r}^2(2Q^2 - 2 + \dot{N}_p)} \]  

(3.72)

\[ E^2 = \frac{2(\dot{Q}^2 + \dot{r}^2 - 2(2 + \dot{N}_p)\dot{r})^2}{(4\dot{Q}^2 + 2\dot{r}^2 - 3(2 + \dot{N}_p)\dot{r})^2}, \]  

and for massless particles ($\epsilon = 0$),

\[ \dot{L} = \frac{(\dot{Q}^2 + \dot{r}^2 - 2(2 + \dot{N}_p))\dot{r}}{E^2\dot{r}^4}. \]  

(3.73)

Therefore, the plots of $L-E^2$ diagram [Eqs. (3.72), (3.73)] with the region of different types of geodesic motion in the phantom surrounding field are shown in Fig. 9. Also plots of effective potential (Eq. (3.70)) are shown in Fig. 10. Moreover, a summary of possible orbits type with numbers of zero points in each region is shown in Table 5.

4. Orbits

In this section, we use the analysis provided in the previous sections for geodesic equations as well as their analytical solutions, effective potentials and $L-E^2$ diagrams, to plot some examples of possible orbit. So, we begin with introducing different types of possible orbits. Suppose $r_-$ be the inner horizon and $r_+$ be the outer event horizon.

1. **Terminating orbit** (TO) with ranges either $\dot{r} \in [0, \infty)$ or $\dot{r} \in [0, r_1)$ with $r_1 \geq r_+$. In other words, $r$ starts in $[0, r_1)$ or comes from $\infty$ and falls into the singularity at $r = 0$ [Fig. 12(i)].

2. **Escape orbit** (EO) with range $\dot{r} \in [r_1, \infty)$ with $r_1 > r_+$. In other words, $r$ starts from $\infty$, then approaches a $r_+$ and goes back to $\infty$ [Fig. 12(c)].
Two-world escape orbit (TEO) with range \( \frac{1}{2} r_1 \), where \( 0 < r_1 < r_- \). In other words, \( r \) starts from \( r_1 \), then approaches \( r_- \) and goes back to \( r_1 \) [Fig. 12(b, d)].

Bound orbit (BO) with range \( \tilde{r} \in \left[ r_1, r_2 \right] \) with
(a) \( r_1, r_2 > r_+ \), or
(b) \( 0 < r_1, r_2 < r_- \).

In other words, \( r \) oscillates between two boundary values \( r_1, r_2 \) [Fig. 12(e, h)] with
(a) \( r_1, r_2 > r_+ \), or
(b) \( 0 < r_1, r_2 < r_- \).

Many-world bound orbit (MBO) with range \( \tilde{r} \in [r_1, r_2] \) where \( 0 < r_1 \leq r_- \) and \( r_2 \geq r_+ \). In other words, \( r \) oscillates between two boundary values \( r_1, r_2 \) with \( 0 < r_1 \leq r_- \) and \( r_2 \geq r_+ \) [Fig. 12(a, f, g)].

For certain parameters \( (E, L, Q, N) \), different types of orbit are dependent on primary location of the test particle or light ray. In the following, we explain the possible orbit types and examples of effective potentials.

1. In region O, there is no real positive zero, so the possible type of orbit is TO [Fig. 12(i)].
2. In region I, there is one real positive zero and the kind of possible orbit is TEO [Fig. 12(b, d)].

3. In region II, there are two real positive zeros and therefore the kind of possible orbit is MBO [Fig. 12(a, f, g)].
4. In region III, there are three real and positive zeros and therefore the kind of two possible orbits is EO [Fig. 12(c)] and MBO [Fig. 12(a), (f) and (g)].
5. In region IV, there are four real and positive zeros, so the kind of two possible orbits is BO [Fig. 12(e), (h)] and MBO [Fig. 12(a) and (f)].
6. In region V, there are five real and positive zeros, so the kind of three possible orbits is EO [Fig. 12(c)], BO [Fig. 12(e), (h)] and MBO [Fig. 12(a), (f) and (g)].

1. Quintessence surrounding field (with \( \kappa \lambda = -2 \)).
2. Dust surrounding field (with \( \kappa \lambda = \frac{3}{5} \)).
3. Radiation surrounding field.
4. Cosmological surrounding field.
5. Phantom surrounding field (with \( \kappa \lambda = 4 \)).

In all surrounding fields (quintessence, dust, radiation, cosmological constant and phantom fields), when Rastall geometric parameter becomes zero, the results are reduced to a Reissner–Nordström black hole [Fig. 11(a) and Table 6] and when both electric charge and Rastall geometric parameter become zero, the metric and results are same as a Schwarzschild black hole [Fig. 11(b) and Table 7] as our expectation. In addition, the possible types of orbits for a Reissner–Nordström black hole are BO, EO, TEO and MBO, while for a Schwarzschild black hole are TO, EO and BO. However, by comparing between Table 7 and tables of all other cases (Tables 1–6), here, we can see that a terminating orbit (TO) has appeared, in which test particles come from certain point and fall into singularity.
Table 5 Types of orbits of the phantom surrounding field

| Region | pos.zeros | range of $\hat{r}$ | Orbit       |
|--------|-----------|---------------------|-------------|
| I      | 1         | $[\hat{r}_0, \hat{r}_1]$ | TEO         |
| II     | 2         | $[\hat{r}_0, \hat{r}_1]$ | MBO         |
| III    | 3         | $[\hat{r}_0, \hat{r}_1]$ | MBO, EO     |
| IV     | 4         | $[\hat{r}_0, \hat{r}_1]$ | MBO, BO     |

The lines represent the range of the orbits. The dots show the turning points of the orbits. The positions of the two horizons are marked by a vertical double line. The single vertical line indicates the singularity at $\hat{r} = 0$

of a Schwarzschild black hole, while this orbit cannot appear for all other discussed cases included electrical charged.

4.1. Astrophysical applications

In this section, we study the astrophysical application such as the deflection of light for escape orbits [Fig. 12(c)]. Here, to avoid complexity, as an example for all cases in this paper, we consider Eq. 3.14 of Sect. 3.1 and, with the help of Eq. 3.23 of Sect. 3.1.1, investigate the deflection of light.

4.1.1. Deflection of light

To investigate the deflection of light, we use the invariant formula for the cosine of the angle between two coordinate directions $d$ and $\delta$ (the radial direction and the spatial direction of the light ray), as \[88–90\]

$$
\cos(\psi) = \frac{g_{\phi\phi} d^\phi}{(g_{\phi\phi} d^\phi)^2 (g_{\phi\phi} d^\phi)^2},
$$

(4.1)

So, by considering $d$ and $\delta$ as

$$
d = (dr, d\phi) = \left(\frac{dr}{d\phi}, 1\right) d\phi, \quad (d\phi < 0),
$$

(4.2)

$$
\delta = (\delta r, 0) = (1, 0) \delta r,
$$

(4.3)

we have

$$
cos(\psi) = \frac{\left(\frac{dr}{d\phi}\right)^2 + \left(\frac{g_{\phi\phi}}{g_{r\phi}}\right)^2}{\sqrt{\left(\frac{dr}{d\phi}\right)^2 + \left(\frac{g_{\phi\phi}}{g_{r\phi}}\right)^2}},
$$

(4.4)

or more appropriately, this angle can be written as

$$
\tan(\psi) = \sqrt{\frac{g_{\phi\phi}}{g_{r\phi}}},
$$

(4.5)

Thus, according to Eq.(3.14), we have

$$
\tan(\psi) = \sqrt{\frac{\left(1 - \frac{2M + N_2}{r(\phi)} + \frac{Q^2}{r(\phi)^2}\right) r^2}{\left(1 - \frac{2M + N_2}{r_\phi} + \frac{Q^2}{r_\phi^2}\right) r(\phi)^2 - \left(1 - \frac{2M + N_2}{r(\phi)} + \frac{Q^2}{r(\phi)^2}\right) r^2}},
$$

(4.6)

where $r(\phi)$ is the solutions of Eq.(3.17) which are solved in Eq.(3.23) for $\epsilon = 0$,

$$
\tilde{r}(\phi) = \frac{b_3}{4\psi^2(\phi - \phi_0, g_2, g_3)} - \frac{b_3}{3} + \tilde{r}_R.
$$

(4.7)

This result now is valid for all light rays.
4.2. Examples of trajectory of particle motion and light

5. Conclusions

In this paper, the analytical solution of geodesic motion for massless and massive test particles in the vicinity of a black hole space-time surrounded by perfect fluid in the context of Rastall gravity has been presented. The timelike and null geodesics equations of motion for the black hole surrounded by quintessence, dust, radiation, cosmological constant and phantom fields have been studied in detail. In each case, we have obtained the analytical solutions by regarding and constraint on effective state parameter \( \omega_{eff} \) and considering some possible values of Rastall coupling constant \( k \).

For a black hole surrounded by quintessence field with \( \omega_q = -\frac{2}{3} \), the possible case of \( k \lambda \ (k \lambda = -2) \) has been analysed. For the case \( k \lambda = -2 \), the equation of motion has included the term \( N_q \), as Rastall’s correction term. When dust field has been considered as a background fluid with \( \omega_d = 0 \), the possible case of \( k \lambda \ (k \lambda = \frac{2}{3}) \) has been analysed. For the case \( k \lambda = \frac{2}{3} \), the equation of motion has contained Rastall correction term \( N_d \), in which Rastall geometric parameter \( N_d \) can play the role of small- and large-scale physical evidence for the ranges of scalar curvature [76]. When radiation is considered as a background field with \( \omega_r = \frac{1}{3} \), the metric is Reissner–Nordström metric of a black hole, in which the field structure parameter \( N_r \) with the electric charge \( Q \) behaves role of the effective charge of this black hole \( (\sqrt{Q^2 - N_r}) \). When we consider a black hole is surrounded by cosmological constant field with \( \omega_c = -1 \), the metric is Reissner–Nordström metric in general relativity which was achieved previously. Therefore, the term \( N_c \) as Rastall correction term will appear, in which the Rastall geometric parameter \( N_c \) causes the role of accelerating expansion of the Universe. Finally, for a black hole surrounded by the phantom field with \( \omega_p = -\frac{4}{3} \), two cases of \( k \lambda \ (k \lambda = \frac{2}{3}, k \lambda = 4) \) are possible values, which have been analysed. For the case \( (k \lambda = \frac{2}{3}) \), Rastall correction term has appeared in terms of \( N_p \), in which Rastall geometric parameter \( N_p \) again can play the role of small- and large-scale physical evidence for the ranges of scalar curvature [76], but for the case \( k \lambda = 4 \), field structure parameter \( N_p \) behaves mass role, i.e. \( \frac{1}{2} \), same as the quintessence field.
After reviewing the space-time and the corresponding equations of motion, we have classified the complete set of orbit types for massive and massless test particles moving on geodesics for each case. In addition, Table 6 shows that when Rastall geometric parameter vanishes, the metrics reduce to Reissner–Nordström black holes, while Table 7 shows that when the electric charge of a black hole becomes zero, the results decrease to Schwarzschild metric.

The geodesic equations have been solved analytically by Weierstrass elliptic and derivatives of hyperelliptic Kleinian sigma functions. We also have considered all possible types of orbits. Using effective potential techniques and parametric diagrams, the possible types of orbits were derived. For null geodesics, EO, TEO and MBO were possible, while for timelike geodesics, EO, TEO, BO and MBO were possible. So, as it turns out, these two null and timelike geodesics have the same results for EO, TEO and MBO, but are different for the case BO, and this case is not possible for null geodesics. Moreover, some observational phenomena such as the deflection angle of light are the result of these solutions that we have calculated such astrophysical application in Sect. 4.1.

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