The influence of data regularity in the critical exponent for a class of semilinear evolution equations

Marcelo R. Ebert, Cleverson R. da Luz and Maíra F. G. Palma

Abstract. In this paper we find the critical exponent for the global existence (in time) of small data solutions to the Cauchy problem for the semilinear dissipative evolution equations

$$u_{tt} + (-\Delta)^{\delta} u_{tt} + (-\Delta)^{\alpha} u + (-\Delta)^{\theta} u_t = |u_t|^p, \quad t \geq 0, \quad x \in \mathbb{R}^n,$$

with $p > 1$, $2\theta \in [0, \alpha]$ and $\delta \in (\theta, \alpha]$. We show that, under additional regularity $(H^{\alpha + \delta}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)) \times (H^{2\delta}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n))$ for initial data, with $m \in (1, 2]$, the critical exponent is given by $p_c = 1 + \frac{2m\theta}{n}$. The nonexistence of global solutions in the subcritical cases is proved, in the case of integers parameters $\alpha, \delta, \theta$, by using the test function method (under suitable sign assumptions on the initial data).

Mathematics Subject Classification. Primary 35B33, 35B40; Secondary 35L71, 35L90.

Keywords. Semilinear evolution operators, Structural dissipation, Global small data solutions, Critical exponent, Asymptotic behavior of solutions.

1. Introduction

Let us consider the Cauchy problem for the semilinear dissipative evolution equations

$$\begin{cases}
    u_{tt} + (-\Delta)^{\delta} u_{tt} + (-\Delta)^{\alpha} u + (-\Delta)^{\theta} u_t = |u_t|^p, & t \geq 0, \quad x \in \mathbb{R}^n, \\
    (u, u_t)(0, x) = (u_0, u_1)(x),
\end{cases} \tag{1.1}$$

The first author have been partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Grant Number 2017/19497-3. The second author has been partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, Proc. 308868/2015-3 and 314398/2018-0.
with $p > 1$, $2\theta \in [0, \alpha]$ and $\delta \in [0, \alpha]$. Here we denote by $(-\Delta)^{\frac{\delta}{2}} = |D|^b$, with $b \geq 0$, the fractional Laplacian operator defined by its action $|D|^b f = \mathfrak{F}^{-1}(|\xi|^b \mathcal{F} f)$, where $\mathfrak{F}$ is the Fourier transform with respect to the space variable, and $\hat{f} = \mathfrak{F} f$. The case $\alpha = 2$ and $\delta = 0$ in (1.1) is an important model in the literature, it is known as Germain–Lagrange operator, as well as beam operator and plate operator in the case of space dimension $n = 1$ and $n = 2$, respectively.

Models to study the vibrations of thin plates given by the full von Kármán system have been studied by several authors, in particular, see [3,20]. If $\delta = 1$ in (1.1), the term $-\Delta u_{tt}$ is to absorb the effects of rotational inertia on the system at the point $x$ of the plate in a positive time $t$. For the plate equation with exterior damping

$$\begin{cases}
u_{tt} - \Delta u_{tt} + (-\Delta)^{2}u + u_t = f(u, u_t), & t \geq 0, \hspace{0.5cm} x \in \mathbb{R}^n, \\
(u, u_t)(0, x) = (u_0, u_1)(x),
\end{cases}$$

(1.2)

where $f(u, u_t) = |\partial^j_t u|^p, j = 0, 1$, we address the reader to [1,4,18,24] for a detailed investigation of properties like existence, uniqueness, energy estimates for the solution and global existence (in time) of small data solutions. The derived estimates in Sect. 4 for solutions to the associated linear problem to (1.1) could also be applied to generalize the obtained results in [4], namely, problem (1.2) with power nonlinearity $|u|^p$ and, under additional regularity $L^m(\mathbb{R}^n)$ for the initial data, one may expect that the critical exponent for the global existence (in time) of small data solutions is $\bar{p} = 1 + \frac{2m}{n-2m+2\theta}$. But due to the fact that only partial results are obtained in the literature for (1.2) with $f(u, u_t) = |u_t|^p$, in this paper we restrict ourselves to the last power nonlinearities.

It is worth to recall some well known results for dissipative evolution models without the rotational inertia term and with power nonlinearities $|u|^p$. For the classical semilinear damped wave equation

$$u_{tt} - \Delta u + u_t = f(u), \hspace{0.5cm} u(0, x) = u_0(x), \hspace{0.5cm} u_t(0, x) = u_1(x),$$

(1.3)

with $f(u) = |u|^p$, it was proved in [25] the global existence of small data energy solutions in the supercritical range $p > 1 + \frac{2}{n}$, by assuming compactly supported small data from the energy space. Under additional regularity the compact support assumption on the data can be removed. By assuming data in Sobolev spaces with additional regularity $L^1(\mathbb{R}^n)$, a global (in time) existence result was proved in space dimensions $n = 1, 2$ in [14], by using energy methods, and in space dimension $n \leq 5$ in [22], by using $L^r - L^q$ estimates for solutions, with $1 \leq r \leq q \leq \infty$. Nonexistence of weak global (in time) small data solutions is proved in [25] in the subcritical case $1 < p < 1 + \frac{2}{n}$ and in [26] for $p = 1 + \frac{2}{n}$. The exponent $1 + \frac{2}{n}$ is well known as Fujita exponent and it is the critical power for the semilinear parabolic Cauchy problem (see [11]). If one removes the assumption that the initial data are in $L^1(\mathbb{R}^n)$ and we only assume that they are in the energy space, then in [15] the critical exponent is modified to $1 + \frac{4}{n}$ or to $1 + \frac{2m}{n}$ under additional regularity $L^m(\mathbb{R}^n)$, with $m \in (1, 2]$. In [13] the authors proved that, differently from the case $m = 1,$
the critical exponent $p = 1 + \frac{2m}{n}$, with $m \in [m_0, 2]$, for some $m_0 > 1$, belongs to the supercritical case. More recently, under additional regularity $L^1(\mathbb{R}^n)$ for initial data and with $f(u) = |u|^{1+\frac{2}{n}} \mu(|u|)$, in [9] the authors obtained sharp conditions on the modulus of continuity function $\mu$ in order to determine a threshold between global (in time) existence of small data solutions (stability of the zero solution) and blow-up behavior even of small data solutions to problem (1.3).

Now, let us consider the Cauchy problem for the dissipative evolution equation

$$\begin{aligned}
\begin{cases}
 u_{tt} + (-\Delta)^{\alpha} u + (-\Delta)^{\theta} u_t = |\partial_t^j u|^p, & t \geq 0, \ x \in \mathbb{R}^n, \\
(u, u_t)(0, x) = (u_0, u_1)(x),
\end{cases}
\end{aligned}$$

with $j = 0, 1$. The term $(-\Delta)^{\theta} u_t$ represents a damping term. If $\theta > 0$ in (1.4), the damping is said to be structural. The assumption $2\theta \leq \alpha$ means that the damping is effective, according to the classification introduced in [6]. If the damping is effective the multipliers associated to the linearized problem has no oscillations at low frequencies in the phase space. In this case, a diffusion phenomenon appears which make the asymptotic profile of the solution be determined by the solution to an anomalous diffusion problem [16]. In [7], under additional regularity $L^1(\mathbb{R}^n)$ for the initial data, the authors proved that the critical exponent $p_j, j = 0, 1$, for global small data solutions to (1.4) are, respectively, $p_0 \doteq 1 + \frac{2\alpha}{n-2\theta}, n > 2\theta$ and $p_1 \doteq 1 + \frac{2\theta}{n}$.

Having in mind that the asymptotic profile of solutions to the linear part of the equation influences the critical exponent for the problem with power nonlinearity, we consider the linear evolution equation related to (1.1):

$$\begin{aligned}
\begin{cases}
 u_{tt} + (-\Delta)^{\delta} u_{tt} + (-\Delta)^{\alpha} u + (-\Delta)^{\theta} u_t = 0, & t \geq 0, \ x \in \mathbb{R}^n, \\
(u, u_t)(0, x) = (u_0, u_1)(x).
\end{cases}
\end{aligned}$$

The total energy for (1.5) is

$$E(t) = \frac{1}{2} \|u_t(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|(-\Delta)^{\frac{\delta}{2}} u_{tt}(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|(-\Delta)^{\frac{\theta}{2}} u(t, \cdot)\|_{L^2}^2,$n

and it dissipates, i.e.,

$$E'(t) = -\|(-\Delta)^{\frac{\delta}{2}} u_{tt}(t, \cdot)\|_{L^2}^2,$n

so that, a natural space for solutions is $C([0, \infty), H^{\alpha}(\mathbb{R}^n)) \cap C^1([0, \infty), H^{\delta}(\mathbb{R}^n))$, the so-called energy solution space.

If $\delta \leq \theta$, the presence of the structural damping generates a strong smoothing effect on the solution to (1.5), and it guarantees the exponential decay in time of the high-frequencies part of the solution to (1.5). Therefore, the decay rate for (1.5) is only determined by the low-frequencies part of the solution to (1.5), which behaves like the solution to the corresponding anomalous diffusion problem [16]. However, if $\delta > \theta$, the rotational inertia term $(-\Delta)^{\delta} u_{tt}$ creates a structure of regularity-loss type decay in the linear problem (see Theorem 4.1 in Sect. 4) and it is more difficult to apply these linear estimates to
study semilinear problems. This fact can be observed by analysing the structure of the eigenvalues associated with the problem (1.5) in the Fourier space. Hence, to derive estimates for solutions in the region of high frequencies, it is necessary to impose additional regularity on the initial data to obtain the same decay estimates as in the region of low frequencies. Such decay property of the regularity-loss type was also investigated for the dissipative Timoshenko system [12], the plate equation under rotational inertia effects in $\mathbb{R}^n$ [2, 24] and a hyperbolic-elliptic system of a radiating gas model [17].

In this work we are interested in the problem (1.1) with the property of regularity-loss and effective damping, i.e., $\delta > \theta$ and $\alpha \geq 2\theta$. Our main goal in this paper is to show that, under additional regularity $(H^{\alpha+\delta}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)) \times (H^{2\delta}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n))$ for initial data, with $m \in (m_0, 2]$ and $m_0 \in [1, 2]$ given by (2.3), the critical exponent for the global existence (in time) of small data solutions to (1.1) is $p_c = 1 + \frac{2m\theta}{n}$. Moreover, we show that for $m \in (m_0, 2]$, $p = p_c$ belongs to the supercritical case, whereas for $m = 1$, $p = 1 + \frac{2\theta}{n}$ is expected to belong to the subcritical case.

The critical exponent for (1.1) in the non-effective case $\alpha < 2\theta$ will be discussed in a forthcoming paper.

**Notation**

Through this paper, we use the following.

**Notation.** Let $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be two functions. We use the notation $f \sim g$ if there exist two constants $C_1, C_2 > 0$ such that $C_1 g(y) \leq f(y) \leq C_2 g(y)$ for all $y \in \Omega$. If the inequality is one-sided, namely, if $f(y) \leq C g(y)$ (resp. $f(y) \geq C g(y)$) for all $y \in \Omega$, then we write $f \lesssim g$ (resp. $f \gtrsim g$).

**Notation.** By $(x)_+$ we denote the positive part of $x \in \mathbb{R}$, i.e. $(x)_+ = \max\{x, 0\}$.

**Notation.** Let $\chi_0, \chi_1$ be $C^\infty(\mathbb{R}^n)$ cut-off nonnegative functions satisfying

$$\chi_0 + \chi_1 = 1, \quad \text{supp} \chi_0 \subset \{|\xi| \leq 1\}, \quad \text{and} \quad \text{supp} \chi_1 \subset \{|\xi| \geq 1/2\}. $$

In particular, it follows that $\chi_0 = 1$ in $\{|\xi| \leq 1/2\}$ and $\chi_1 = 1$ in $\{|\xi| \geq 1\}$. To localize a distribution $g$ at low and high frequencies, we denote $g_{\chi_j} = \mathcal{F}^{-1}(\chi_j \hat{g})$, $j = 0, 1$.

**Notation.** For any $q \in [1, \infty]$, we denote by $L^q(\mathbb{R}^n)$ the usual Lebesgue space over $\mathbb{R}^n$. For any $s \in [0, +\infty)$, we denote by $H^{s,q}$ the Bessel potential space:

$$H^{s,q}(\mathbb{R}^n) = \{ f \in L^q(\mathbb{R}^n) : (1 - |D|)^s f \in L^q(\mathbb{R}^n) \}.$$ 

We recall that $H^{s,q}(\mathbb{R}^n) = W^{s,q}(\mathbb{R}^n)$, the usual Sobolev space, for any $q \in (1, \infty)$ and $s \in \mathbb{N}$. As usual, we denote $H^s(\mathbb{R}^n) \doteq H^{s,2}(\mathbb{R}^n)$ for any $s \geq 0$.

**Notation.** Let $f, g : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ be two regular functions with $\Omega \subset \mathbb{R}^n$. We use the notation $f \ast g$ to indicate the convolution with respect to the space variable of the functions $f$ and $g$, i.e.,

$$(f \ast g)(t, x) = \int_{\Omega} f(t, y) \cdot g(t, x - y) \, dy.$$
2. Main results

Assuming data in a suitable space, one may conclude the local existence of solutions to (1.1) for $p > 1$ (see [1]). The next result explains that for $1 < p < p_c$ this solution can not exist globally in time even if the data are supposed to be very small.

**Theorem 2.1.** Let $\delta, \theta \in \mathbb{N}$, $\alpha \in \mathbb{N} \setminus \{0\}$. Assume that $u_0 \equiv 0$, $u_1$, $(-\Delta)^\delta u_1 \in L^1_{loc}(\mathbb{R}^n)$ and for some $\varepsilon \in (0,1)$ verifies

$$u_1(x) + (-\Delta)^\delta u_1(x) \geq \varepsilon (1 + |x|)^{-\frac{m}{n}} \left(\log (|x|)\right)^{-1}, \quad m \in (1,2]. \quad (2.1)$$

Then there exists no global (in time) weak solution $u \in L^1_{loc}([0,\infty) \times \mathbb{R}^n)$ to (1.1), with $u_t \in L^p_{loc}([0,\infty) \times \mathbb{R}^n)$, for any $p \in \left(1, 1 + \min\{2\theta, \alpha\} \right)$.

In addition, if $u_1$, $(-\Delta)^\delta u_1 \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} (I + (-\Delta)^\delta) u_1(x) \, dx > 0, \quad (2.2)$$

then the conclusion is still true for $m = 1$.

**Remark 2.2.** Hypothesis (2.1) implies that $u_1 + (-\Delta)^\delta u_1 \notin L^{m-\varepsilon}(\mathbb{R}^n)$, for all $\varepsilon > 0$.

Let $n \geq 1$, $2\theta \in (0,\alpha]$ and let us define

$$m_0 \doteq \min \left\{m \in [1,2]; \ n(2-m) \leq 2m\theta \min\{m, \sqrt{2(2-m)}\} \right\}. \quad (2.3)$$

**Remark 2.3.** Conditions (2.3) implies $\frac{2}{m} \leq 1 + \frac{2m\theta}{n}$ and $1 + \frac{2m\theta}{n} \leq \frac{n}{(n-2m\theta)_+}$ for all $m \in [m_0,2]$ with $m_0 < 2$. But since the last inequality should be strict in Theorem 2.4 and we are mainly interested in the case $m > 1$, from now on we are going to assume $m \in (m_0,2]$.

**Example.** If $n = 1,2$ and $\theta = 1$, then $m_0 = 1$ and the admissible interval for $m$ is $(1,2]$. If $n = 3$ and $\theta = \frac{1}{2}$, then $m_0 = \frac{3}{2}$ and the admissible interval for $m$ is $(\frac{3}{2},2]$.

In the next result we show that, under additional $L^m(\mathbb{R}^n)$ regularity for initial data, global small data solutions exist to (1.1) for $2\theta \leq \alpha$ and $p_c \doteq 1 + \frac{2m\theta}{n} < p \leq \frac{n}{2(n-2m\theta)_+}$. In this case, Theorem 2.1 implies a nonexistence result for $1 < p < p_c$ and we conclude that $p = p_c$ is the critical exponent for (1.1).

**Theorem 2.4.** Let $\delta \in (\theta, \alpha)$, $2\theta \in (0,\alpha]$ and $m \in (m_0,2]$, with $m_0$ given be (2.3). If

$$1 + \frac{2m\theta}{n} < p \leq q \leq \frac{n}{2(n-2m\theta)_+},$$


then there exists a sufficiently small \( \varepsilon > 0 \) such that for any data

\[
(u_0, u_1) \in \mathcal{A} \cap \left( H^{\alpha + \delta}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n) \right) \times \left( H^{2\delta}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \right),
\]

with \( \|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon \), there exists a global (in time) energy solution \( u \in C([0, \infty), H^\alpha(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)) \) to (1.1). Also, the solution to (1.1) satisfies the estimates

\[
\|D^\alpha u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{1}{2(n-\frac{\delta}{4})}}(n(\frac{n}{m} - \frac{\delta}{2}) + \alpha - 2\theta) \|(u_0, u_1)\|_{\mathcal{A}}, \quad (2.4)
\]

\[
\|\partial_t^j u(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{1-j-\frac{\theta}{4n}}(\frac{n}{m} - \frac{\delta}{2}) \|(u_0, u_1)\|_{\mathcal{A}}, \quad j = 0, 1, \quad (2.5)
\]

for \( 2 \leq \kappa \leq q \).

In particular, if \( n \leq 2m\theta \) and \( q = +\infty \) we have the following result:

**Corollary 2.5.** Let \( \delta \in (\theta, \alpha] \), \( 2\theta \in (0, \alpha) \), \( m \in (1, 2] \) and assume that \( 1 \leq n \leq 2m\theta \). Let \( p > 1 + \frac{2m\theta}{n} \), then there exists a sufficiently small \( \varepsilon > 0 \) such that for any data

\[
(u_0, u_1) \in \mathcal{A} \cap \left( H^{\alpha + \delta}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n) \right) \times \left( H^{2\delta}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \right),
\]

with \( \|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon \), there exists a global (in time) energy solution \( u \in C([0, \infty), H^\alpha(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \) to (1.1). Also, the solution to (1.1) satisfies the estimates (2.4) and (2.5) for \( \kappa \geq 2 \).

**Remark 2.6.** Let \( u_1, (-\Delta)^\delta u_1 \in L^1_{loc}(\mathbb{R}^n) \) and \( 2\theta \leq \alpha \). If

\[
\int_{\mathbb{R}^n} (I + (-\Delta)^\delta) u_1(x) \, dx > 0,
\]

then Theorem 2.1 implies that one can not have the existence of global solutions to (1.1) with \( u_0 = 0 \) and \( p < 1 + \frac{2\theta}{n} \). If \( u_1 + (-\Delta)^\delta u_1 \notin L^{m-\epsilon}(\mathbb{R}^n) \), for all \( \epsilon > 0 \) such that \( m - \epsilon > 1 \), then Theorem 2.1 implies that one can not have the existence of global solutions to (1.1) with \( u_0 = 0 \) for \( p < 1 + \frac{2\theta(m-\epsilon)}{n} \) for all \( \epsilon > 0 \). This shows that, in general, the assumption \( p > 1 + \frac{2m\theta}{n} \) can not be removed in Theorem 2.4.

**Remark 2.7.** In Theorem 2.4 it appears a loss of regularity with respect to the initial data. This loss of regularity is related to the obtained estimates for solutions to the linear problem at high frequencies. In general one can not avoid this effect, for instance, if the initial data \( u_1 \notin H^{2\delta}(\mathbb{R}^n) \), i.e., \( u_1 + (-\Delta)^\delta u_1 \notin L^2(\mathbb{R}^n) \), then, for \( 2\theta \leq \alpha \), the conclusion of Theorem 2.1 is true for all \( 1 < p < 1 + \frac{4\theta}{n} \) even if \( u_1 \in L^1(\mathbb{R}^n) \).

**Remark 2.8.** The condition \( q \leq \frac{nm}{n-2m\theta} \) in Theorem 2.4 implies \( n \left( \frac{1}{m} - \frac{1}{q} \right) \leq 2\theta \). Hence, with the assumed regularity for initial data in Theorem 2.4, Theorem 4.1 (iii) implies that solutions to the linear problem (1.5) satisfies (2.5) (see Remark 4.3). Similarly, the condition \( n \leq 2m\theta \) in Corollary 2.5 is in order that the \( L^\infty \) norm of the partial derivative in time of solutions to the linear problem (1.5) has the decay given by \( (1 + t)^{-\frac{n}{4n-\theta}} \) (see Theorem 4.1).

**Example.** Let us assume that initial data has additional regularity \( L^m(\mathbb{R}^n) \). Then:
The critical exponent for the beam equation with strong damping and rotational inertia effects, i.e., $n = 1$, $\delta = 1$, $\alpha = 2$ and $\theta = 1$ is $p_c = 1 + 2m$.

The critical exponent for plate equation with strong damping and rotational inertia effects, i.e., $n = 2$, $\delta = 1$, $\alpha = 2$ and $\theta = 1$ is $p_c = 1 + m$.

In both cases (1, 2] is the admissible interval for $m$.

In the previous results, one may feel the influence of additional $L^m(\mathbb{R}^n)$ regularity in the critical exponent, with $m \in (1, 2]$. In the case $\theta = 0$ we no longer have this effect, so in the next result we assume only data in the $L^2(\mathbb{R}^n)$ basis:

**Theorem 2.9.** Let $\theta = 0$, $0 < \delta \leq \alpha$ and $n < 4\delta$. Then, for all $p > 1$ there exists a sufficiently small $\varepsilon > 0$ such that for any data

$$(u_0, u_1) \in \mathcal{A} \doteq H^{\alpha + \delta}(\mathbb{R}^n) \times H^{2\delta}(\mathbb{R}^n), \quad \|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon,$$

there exists a global (in time) energy solution $u \in C([0, \infty), H^{\alpha}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n))$ to (1.1), with $q \geq 2p$. Also, the solution to (1.1) satisfies the estimates

$$\|\partial_t^j u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-j}\|(u_0, u_1)\|_{\mathcal{A}}, \quad j = 0, 1, \quad (2.6)$$

$$\|D^\alpha u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{\alpha}{2}}\|(u_0, u_1)\|_{\mathcal{A}}, \quad (2.7)$$

$$\|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-\min\{\frac{n}{2m}(\frac{3}{2} - \frac{\alpha}{q}), \frac{\alpha + \delta}{2\delta} - \frac{n}{2\delta}(\frac{3}{2} - \frac{1}{q})\}}\|(u_0, u_1)\|_{\mathcal{A}}, \quad (2.8)$$

$$\|\partial_t u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{\frac{n}{2\delta}(\frac{3}{2} - \frac{1}{q})^{-1}}\|(u_0, u_1)\|_{\mathcal{A}}, \quad (2.9)$$

for all $q \geq 2p$.

**Remark 2.10.** As in previous theorems, estimates (2.6) and (2.7) coincide with the obtained estimates for solutions to the corresponding linear problem (1.5) at low frequencies. However, with the required regularity, estimate (2.8) and (2.9) may coincide with the obtained estimate for solutions to (1.5) at high frequencies.

In the next result we show that, for initial data with additional regularity $L^m(\mathbb{R}^n)$, $p = 1 + \frac{2m \theta}{n}$ belongs to the supercritical case. For the classical damped wave equation, this phenomenon has been investigated in [13].

**Theorem 2.11.** Let $\delta \in (\theta, \alpha]$, $2\theta \in (0, \alpha]$ and $m \in (m_0, 2]$, with $m_0$ given be (2.3). Let $p = 1 + \frac{2m \theta}{n} \leq \frac{q}{2} < \frac{n m}{2(n - 2m \theta)}$. Then there exists a sufficiently small $\varepsilon > 0$ such that for any data

$$(u_0, u_1) \in \mathcal{A} \doteq (H^{\alpha + \delta}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)) \times (H^{2\delta}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)),$$

with $\|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon$, there exists a global (in time) energy solution $u \in C([0, \infty), H^{\alpha}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n))$ to (1.1). Also, the solution to (1.1) satisfies the estimates

$$\|D^\alpha u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{1}{2n - \theta j}(\frac{3}{2} - \frac{\alpha}{q}) + \alpha - 2\theta} \log(e + t)\|(u_0, u_1)\|_{\mathcal{A}},$$

$$\|\partial_t^j u(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{-j - \frac{n}{2\delta}(\frac{3}{2} - \frac{1}{q})}\|(u_0, u_1)\|_{\mathcal{A}}, \quad j = 0, 1,$$

for all $2 \leq \kappa \leq q$. If $\alpha = 2\theta$ the loss of decay can be avoid, namely

$$\|D^\alpha u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2\delta}(\frac{3}{2} - \frac{1}{q})}\|(u_0, u_1)\|_{\mathcal{A}}.$$
3. Non-existence via test function method

For the proof of the next result one may follow as in [7] (see also [8]), but in order to explain the influence of the rotational inertia term we sketch the proof.

Proof. (Theorem 2.1) We fix a nonnegative, non-increasing, test function \( \varphi \in \mathcal{C}_c^\infty([0, \infty)) \) with \( \varphi = 1 \) in \([0, 1/2]\) and \( \text{supp} \varphi \subset [0, 1] \), and a nonnegative, radial, test function \( \psi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \), such that \( \psi = 1 \) in the ball \( B_{1/2} \), and \( \text{supp} \psi \subset B_1 \). We also assume \( \psi(x) \leq \psi(y) \) when \(|x| \geq |y|\). Here \( B_r \) denotes the ball of radius \( r \), centered at the origin. We may assume (see, for instance, [8, 21]) that

\[ \varphi - \frac{\psi}{r} |\varphi'|^{p'}, \quad \psi - \frac{\psi}{r} (|\Delta^\delta \psi|^{p'} + |\Delta^\theta \psi|^{p'} + |\Delta^\alpha \psi|^{p'}) \text{ are bounded}, \]  

where \( p' = p/(p - 1) \). We remark that the assumption that \( \delta, \theta \) and \( \alpha \) are integers plays a fundamental role here. Then, for \( R \geq 1 \), we define:

\[ \varphi_R(t) = \varphi(R^{-\kappa} t), \quad \psi_R(x) = \psi(R^{-1} x), \]

for some \( \kappa > 0 \) which we will fix later.

Let \( \Phi_R \in \mathcal{C}_c^\infty([0, \infty)) \) be the test function defined by

\[ \Phi_R(t) = \int_t^\infty \varphi(s) \, ds. \]

(Indeed, we notice that \( \text{supp} \Phi_R \subset [0, R^\kappa] \), since \( \text{supp} \varphi_R \subset [0, R^\kappa] \)). In particular, \( \Phi'_R = -\varphi_R \).

Let us assume that \( u \in L^1_{\text{loc}}([0, T] \times \mathbb{R}^n) \), with \( u_t \in L^p_{\text{loc}}([0, T] \times \mathbb{R}^n) \) is a (local or global) weak solution to (1.1). Let \( R > 0 \), and also assume that \( R \leq T^\kappa \), if \( u \) is a local solution in \([0, T] \times \mathbb{R}^n\). Integrating by parts, and recalling that \( u_0 = 0 \) and \( \varphi_R(0) = 1 \), we obtain

\[ I_R = \int_0^\infty \int_{\mathbb{R}^n} u_t (\psi_R + \varphi_R(-\Delta)^\theta \psi_R + \Phi_R(-\Delta)^\alpha \psi_R - \varphi'_R(-\Delta)^\delta \psi_R) \, dx \, dt \]

\[ - \int_{\mathbb{R}^n} \psi_R(x)(I + (-\Delta)^\delta)u_1(x) \, dx, \]

where:

\[ I_R = \int_0^\infty \int_{\mathbb{R}^n} |u_t|^p \varphi_R \psi_R \, dx \, dt. \]

We may now apply Young inequality to estimate:

\[ \int_0^\infty \int_{\mathbb{R}^n} |u_t|(|\varphi_R| \psi_R + |\varphi_R||(-\Delta)^\theta \psi_R| + \Phi_R|(-\Delta)^\alpha \psi_R| + |\varphi'_R(-\Delta)^\delta \psi_R|) \, dx \, dt \]

\[ \leq \frac{1}{p'} I_R + \frac{1}{p} \int_0^\infty \int_{\mathbb{R}^n} (\varphi_R \psi_R)^{-\frac{p'}{p'}} \left(|\varphi'_R \psi_R| + |\varphi_R(-\Delta)^\theta \psi_R| + |\Phi_R(-\Delta)^\alpha \psi_R| + |\varphi'_R(-\Delta)^\delta \psi_R|\right)^{p'} \, dx \, dt. \]

Due to

\[ \varphi'_R(t) = R^{-\kappa} \varphi'(R^{-\kappa} t), \quad (-\Delta)^k \psi_R(x) = R^{-2k} ((-\Delta)^k \psi)(R^{-1} x), \]

for integers \( k \).
recalling (3.1), we may estimate
\[ \int_0^\infty \int_{\mathbb{R}^n} (\varphi_R \psi_R)^{-\frac{p'}{p}} |\varphi_R' \psi_R|' \, dx \, dt \leq C \, R^{-\kappa \delta + n + \kappa}, \]
\[ \int_0^\infty \int_{\mathbb{R}^n} (\varphi_R \psi_R)^{-\frac{p'}{p}} |\varphi_R(-\Delta)\psi_R|' \, dx \, dt \leq C \, R^{-2\theta \delta + n + \kappa}, \]
\[ \int_0^\infty \int_{\mathbb{R}^n} (\varphi_R \psi_R)^{-\frac{p'}{p}} |\varphi_R(-\Delta)\delta \psi_R|' \, dx \, dt \leq C \, R^{-(\kappa + 2\delta) \delta + n + \kappa}. \]

Due to \( \Phi_R(t) \leq \Phi_R(0) \leq R^\kappa \), and being \( \Phi_R' \varphi_R \) bounded one gets:
\[ \int_0^\infty \int_{\mathbb{R}^n} (\varphi_R \psi_R)^{-\frac{p'}{p}} |\varphi_R(-\Delta)\delta \psi_R|' \, dx \, dt \leq C \, R^{-2\theta \delta + \kappa \delta + n + \kappa}. \]

We may now fix \( \kappa = \min\{2\theta, \alpha\} \), so that, summarizing, we proved that
\[ \frac{1}{p'} I_R \leq C \, R^{-\kappa \delta + n + \kappa} - \int_{\mathbb{R}^n} \psi_R(x)(I + (-\Delta)\delta)u_1(x) \, dx. \]

Recalling the assumption (2.1), there exists \( c > 0 \), independent of \( R \), such that
\[ \int_{\mathbb{R}^n} \psi_R(x)(I + (-\Delta)\delta)u_1(x) \, dx \geq \varepsilon \int_{\mathbb{R}^n} (1 + |x|)^{-\frac{n}{m}} (\log \langle x \rangle)^{-1} \psi_R(x) \, dx \geq c \varepsilon \, R^n - \frac{n}{m} (\log \langle R \rangle)^{-1}. \]

As a consequence:
\[ \frac{1}{p'} I_R \leq C \, R^{-\kappa \delta + n + \kappa} - c \varepsilon \, R^n - \frac{n}{m} (\log \langle R \rangle)^{-1} \]
\[ = R^n \left( C \, R^{-(p' - 1)\kappa} - c \varepsilon \, R^{-\frac{n}{m}} (\log \langle R \rangle)^{-1} \right). \]

Assume, by contradiction, that the solution \( u \) is global. In the subcritical case \( p < 1 + \frac{\min\{2\theta, \alpha\}}{m} m \), it follows that \( (p' - 1)\kappa > \frac{n}{m} \) and \( I_R < 0 \), for any sufficiently large \( R \), and this contradicts the fact that \( I_R \geq 0 \). Similarly, under the assumption (2.2), it follows that \( I_R < p'CR^{-\kappa \delta + n + \kappa} \) and for \( p < 1 + \frac{\min\{2\theta, \alpha\}}{n} \) it produces again a contradiction. Therefore, \( u \) cannot be a global (in time) solution and this concludes the proof. \( \square \)

4. Estimates for solutions to the linear problem

We consider the inhomogeneous linear problem
\[
\begin{cases}
   u_{tt} + (-\Delta)\delta u_{tt} + (-\Delta)\alpha u + (-\Delta)\beta u_t = f(t, x), & t \geq 0, \ x \in \mathbb{R}^n, \\
   (u, u_t)(0, x) = (u_0, u_1)(x).
\end{cases}
\]

(4.1)

We introduce the Fourier multipliers
\[ K_0(t, \xi) = \frac{\lambda_+ e^{t\lambda_+} - \lambda_- e^{t\lambda_-}}{\lambda_+ - \lambda_-}, \quad \text{and} \quad K_1(t, \xi) = \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-}, \]

(4.2)
with
\[ \lambda_{\pm} = \frac{|\xi|^{2\theta}}{2(1 + |\xi|^{2\theta})} \left( -1 \pm \sqrt{1 - 4|\xi|^{2(\theta - 2\delta)}(1 + |\xi|^{2\delta})} \right). \]

The solution to (4.1) may be written as
\[ u(t, x) = K_0(t) * u_0 + K_1(t) * u_1 + \int_0^t E_1(t - s, x) * f(s, x) \, ds, \]
where \( K_0(t, x) = \mathcal{F}^{-1}[\hat{K}_0(t, \cdot)](x) \), \( K_1(t, x) = \mathcal{F}^{-1}[\hat{K}_1(t, \cdot)](x) \) and
\[ E_1(t, x) = (I + (-\Delta)^{\delta})^{-1} K_1(t, x). \]

Some estimates in the following result was already discussed in [19] for \( \eta = 1 \), but in order to deal with the semilinear problem we had to derive estimates for a large range of parameters.

**Theorem 4.1.** Let \( \alpha > 0, 2\theta \leq \alpha, \delta \in [0, \alpha], \eta \in [1, 2], q \in [2, +\infty], \gamma \in \mathbb{N}^n \) and \( j = 0, 1 \). Then the kernels defined by (4.2) satisfy:

(i) \[ ||\partial_x^j \partial_t^j K_0(t, \cdot) * \psi||_{L^q} \lesssim (1 + t)^{-\frac{1}{n(\frac{1}{q} - \frac{1}{2})} + |\gamma| - j} ||\psi||_{L^n} + g(t)||\psi||_{H^{r_j}}. \]

(ii) If \( n \left( \frac{1}{\eta} - \frac{1}{q} \right) + |\gamma| - 2\theta \geq 0 \) then
\[ ||\partial_x^j \partial_t^j K_1(t, \cdot) * \psi||_{L^q} \lesssim (1 + t)^{-\frac{1}{n(\frac{1}{q} - \frac{1}{2})} + |\gamma| - 2\theta - \frac{1}{2}} ||\psi||_{L^n} + g(t)||\psi||_{H^{(r_j - 2\delta)_+}}. \]

A special exception is given in the case \( j = 0, \eta = 1, q \geq 2 \) and \( n \left( \frac{1}{\eta} - \frac{1}{q} \right) + |\gamma| - 2\theta = 0 \), namely,
\[ ||\partial_x^j K_1(t, \cdot) * \psi||_{L^q} \lesssim \log(e + t)||\psi||_{L^1} + g(t)||\psi||_{H^{r_0, 0}}, \]

and
\[ ||\partial_x^j E_1(t, \cdot) * \psi||_{L^q} \lesssim \log(e + t)||\psi||_{L^1} + g(t)||\psi||_{H^{(r_{0 - 2\delta})_+}}. \]

(iii) If \( n \left( \frac{1}{\eta} - \frac{1}{q} \right) + |\gamma| - 2\theta < 0 \), then
\[ ||\partial_x^j \partial_t^j K_1(t, \cdot) * \psi||_{L^q} \lesssim (1 + t)^{1 - j - \frac{1}{2\theta} n(\frac{1}{\eta} - \frac{1}{q}) + |\gamma|} ||\psi||_{L^n} + g(t)||\psi||_{H^{r_j}}, \]

and
\[ ||\partial_x^j \partial_t^j E_1(t, \cdot) * \psi||_{L^q} \lesssim (1 + t)^{1 - j - \frac{1}{2\theta} n(\frac{1}{\eta} - \frac{1}{q}) + |\gamma|} ||\psi||_{L^n} + g(t)||\psi||_{H^{(r_{j - 2\delta})_+}}. \]

Here
\[ g(t) = \begin{cases} t^{-\frac{n}{n(\theta - \delta)}(\frac{1}{2} - \frac{1}{\theta})} e^{-ct} & \text{if } \delta < \theta \\ \int (1 + t)^{-\frac{n}{n(\theta - \delta)}(\frac{1}{2} - \frac{1}{\theta})} (1 + t)^{-\frac{1}{2\theta}} & \text{if } \theta < \delta \text{ with } 0 < \beta < \frac{\delta - \theta}{n} \frac{2q}{(q - 2)_+}, \end{cases} \]
\[ s_0 = \begin{cases} |\gamma| & \text{if } \delta \leq \theta \\ |\gamma| + \frac{\delta - \theta}{\beta} & \text{if } \theta < \delta, \end{cases} \quad r_1 = \begin{cases} |\gamma| & \text{if } \delta \leq \theta \\ |\gamma| + \frac{\delta - \theta}{\beta} & \text{if } \theta < \delta. \end{cases} \]

\( r_0 = s_0 + \delta - \alpha \) and \( s_1 = r_1 + \alpha - \delta \).
Remark 4.2. In the case $\delta < \theta$, under additional regularity of the initial data one may avoid the singular behaviour at $t = 0$ for the $L^q$ norm of solutions, with $q \neq 2$, coming from the estimates at high frequencies (see Proposition 4.2 in [7]). For $\delta = \theta$ we no longer have the smoothing effect, so we have an exponential decay in the high frequencies estimates only under additional regularity for the initial data.

Proof. For small frequencies $|\xi| \leq 1$ we have that
\[
\lambda_+ \approx -|\xi|^{2(\alpha - \theta)}, \quad \lambda_- \approx -|\xi|^{2\theta}, \quad \lambda_+ - \lambda_- \approx |\xi|^{2\theta},
\]
and if $t|\xi|^{2\theta} \leq 1$ we have
\[
|\hat{K}_1(t, \xi)| \lesssim |\xi|^{-2\theta} e^{t\lambda_-} \left(e^{t(\lambda_+ - \lambda_-)} - 1\right) \lesssim te^{t\lambda_-},
\]
whereas for $t|\xi|^{2\theta} \geq 1$
\[
|\hat{K}_1(t, \xi)| \lesssim |\xi|^{-2\theta} e^{t\lambda_+}.
\]
For any $\eta \in [1, 2]$, we define $\eta' = \eta/(\eta - 1)$, its Hölder conjugate, and $r \in [2, \infty]$ by
\[
\frac{1}{r'} = \frac{1}{q'} - \frac{1}{\eta'} = \frac{1}{\eta} - \frac{1}{q}, \quad (4.3)
\]
where $q' = q/(q - 1)$. Now, Hausdorff–Young inequality comes into play
\[
\|(-\triangle)^{\frac{\alpha}{2}} \hat{\chi}_0 \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^q} \lesssim \|\xi|^{k} \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^{q'}(|\xi| \leq 1)}
\]
and we estimate $L^{q'}$ in the regions $t|\xi|^{2\theta} \leq 1$ and $t^{-\frac{1}{\eta'}} \leq |\xi| \leq 1$:
\[
\|\xi|^{k} \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^{q'}(|\xi|^{2\theta} \leq 1)} \lesssim t\|\xi|^{k} e^{t\lambda_-} \|_{L^{r}(t|\xi|^{2\theta} \leq 1)} \|\hat{\psi}\|_{L^{q'}}
\]\n\[
\lesssim (1 + t)^{-\frac{1}{\eta'}(\frac{n}{2} + k)}\|\hat{\psi}\|_{L^{q'}},
\]
whereas
\[
\|\xi|^{k} \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^{q'}(|\xi|^{2\theta} \leq 1)} \lesssim \|\xi|^{k - 2\theta} e^{t\lambda_+} \|_{L^{r}(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)} \|\hat{\psi}\|_{L^{q'}}
\]\n\[
\lesssim (1 + t)^{-\frac{1}{\eta'}(\frac{n}{2} + k)}\|\hat{\psi}\|_{L^{q'}},
\]
for $n \left(\frac{1}{\eta} - \frac{1}{q}\right) + k - 2\theta > 0$,
\[
\|\xi|^{k} \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^{q'}(|\xi|^{2\theta} \leq 1)} \lesssim \|\xi|^{k - 2\theta} e^{t\lambda_+} \|_{L^{r}(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)} \|\hat{\psi}\|_{L^{q'}}
\]\n\[
\lesssim (1 + t)^{-\frac{1}{\eta'}(\frac{n}{2} + k)}\|\hat{\psi}\|_{L^{q'}},
\]
for $n \left(\frac{1}{\eta} - \frac{1}{q}\right) + k - 2\theta < 0$, and using (4.3)
\[
\|\xi|^{k} \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^{q'}(|\xi|^{2\theta} \leq 1)} \lesssim \|\xi|^{k - 2\theta} e^{t\lambda_+} \|_{L^{r}(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)} \|\hat{\psi}\|_{L^{q'}}
\]\n\[
\lesssim \log(e + t)\|\hat{\psi}\|_{L^{q'}},
\]
for $n \left(\frac{1}{\eta} - \frac{1}{q}\right) + k - 2\theta = 0$. However, in some cases the last inequality may be improved. Indeed, the case $k - 2\theta = 0$ is immediately, i.e.,
\[
\|(-\triangle)^{\frac{\alpha}{2}} \hat{\chi}_0 \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^{2}} \lesssim \|\hat{\psi}\|_{L^{2}}.
\]
So, let us suppose that \( k - 2\theta < 0 \). If \( 2 \leq q < \infty \) and \( 1 < \eta < \frac{n}{2\theta - k} \), by using the Riesz potential mapping properties \( I_{2\theta - k} f = \mathfrak{F}^{-1}(|\xi|^{-2(\theta - k)}\hat{f}) \) we get [23]

\[
\|(-\Delta)^{\frac{\theta}{2}} \mathfrak{F}^{-1}(\chi_0 \hat{K}_1(t, \cdot))\|_{L^q} = \|I_{2\theta - k}(-\Delta)^{\frac{\theta}{2}} \mathfrak{F}^{-1}(\chi_0 |\xi|^{2\theta - k} \hat{K}_1(t, \cdot))\|_{L^q}
\]

\[
\lesssim \|\mathfrak{F}^{-1}(\chi_0 |\xi|^{2\theta} \hat{K}_1(t, \cdot))\|_{L^q}, \quad \frac{1}{\eta} - \frac{1}{q} = \frac{2\theta - k}{n}.
\]

For \( \beta \in \mathbb{N}^n \) we may estimate

\[
|\partial^\beta_x \chi_0 |\xi|^{2\theta} \hat{K}_1(t, \xi)| \lesssim |\xi|^{-|\beta|},
\]

hence, the Mikhlin–Hörmander multiplier theorem implies

\[
\|\mathfrak{F}^{-1}(\chi_0 |\xi|^{2\theta} \hat{K}_1(t, \cdot))\|_{L^q} \lesssim \|\psi\|_{L^q},
\]

and

\[
\|(-\Delta)^{\frac{\theta}{2}} \mathfrak{F}^{-1}(\chi_0 \hat{K}_1(t, \cdot))\|_{L^\beta} \lesssim \|\psi\|_{L^q}.
\]

If \( q = \infty \) and \( \eta > 1 \), by taking \( s > 0 \) such that \( 2\theta - k - 1 < s < \frac{n}{\eta} \) and applying Lemma 3.1 of [5] we have

\[
\|(-\Delta)^{\frac{\theta}{2}} \mathfrak{F}^{-1}(\chi_0 \hat{K}_1(t, \cdot))\|_{L^\infty} = \|\mathfrak{F}^{-1}(\chi_0 |\xi|^{k+s} \hat{K}_1(t, \cdot))|\xi|^{-s} \hat{\psi}\|_{L^\infty}
\]

\[
\lesssim (1 + t)^{-\frac{1}{2(\alpha - \eta)}(\frac{\theta}{\eta} + k + s - 2\theta)} \| I_s \hat{\psi}\|_{L^\beta} \lesssim \|\psi\|_{L^q}, \quad \frac{1}{\eta} - \frac{1}{q} = \frac{s}{n},
\]

thanks to \( \frac{n}{\eta} + k - 2\theta = 0 \). Furthermore, we have

\[
\| |\xi|^k \hat{K}_0(t, \cdot) \hat{\psi}\|_{L^{\alpha'}(|\xi| \leq 1)} \lesssim \| |\xi|^k e^{t\lambda_+} \|_{L^{\alpha'}(|\xi| \leq 1)} \|\hat{\psi}\|_{L^{\alpha'}}
\]

\[
\lesssim (1 + t)^{-\frac{1}{2(\alpha - \eta)}(\frac{\theta}{\eta} + k)} \|\psi\|_{L^q}.
\]

For time derivatives of the kernels the desired estimates follows thanks to

\[
|\partial_t \hat{K}_0(t, \xi)| = |\lambda_+ - \lambda_-| \left| e^{t\lambda_+} - e^{t\lambda_-} \right| \lesssim |\xi|^{2(\alpha - \theta)} e^{t\lambda_+}
\]

\[
|\partial_t \hat{K}_1(t, \xi)| = \left| \frac{\lambda_- e^{t\lambda_+} - \lambda_+ e^{t\lambda_-}}{\lambda_+ - \lambda_-} \right| \lesssim e^{t\lambda_-} + |\xi|^{2(\alpha - 2\theta)} e^{t\lambda_+}.
\]

In the low frequency region the estimates obtained for \( \hat{E}_1 \) and \( \hat{K}_1 \) are the same, since \( \hat{E}_1 \approx \hat{K}_1 \) for all \( t \geq 0 \).

At high frequencies the roots of the full symbol are complex-valued, and

\[
Re \lambda_\pm \approx -|\xi|^{2(\theta - \delta)}, \quad |\lambda_\pm| \approx |\xi|^{\alpha - \delta}, \quad |\lambda_+ - \lambda_-| \approx |\xi|^{\alpha - \delta}, \quad |\xi| \to \infty.
\]

Using the equivalences (4.4), we have

\[
|\partial_t^j \hat{K}_t(t, \xi)| \lesssim |\xi|^{(j - t)(\alpha - \delta)} e^{tRe \lambda_\pm}.
\]

Hence, using the smoothing effect for \( \theta > \delta \), we have an exponential decay for \( \|\|\xi|^{\gamma} |\partial_t^j \hat{u}(t, \cdot)|\|_{L^q(|\xi| \geq 1)} \), but without additional regularity for the initial data, in the short time estimates, it produces a singularity at \( t = 0 \) if \( q \neq 2 \) (see Proposition 4.3 in [7]).
On the other hand, if \( \delta > \theta \), for \( \beta > 0 \) we may estimate
\[
e^{-t|\xi|^{2(\theta-\delta)}} = t^{-\frac{1}{\beta}} (t |\xi|^{2(\theta-\delta)})^{\frac{1}{\beta}} e^{-t|\xi|^{2(\theta-\delta)}} |\xi|^{\frac{\delta-\theta}{\beta}}
\lesssim t^{-\frac{1}{\beta}} |\xi|^{\frac{\delta-\theta}{\beta}}, \quad t > 0
\]
whereas it is bounded for \( t \in [0,1] \). So, under additional regularity \( \frac{\delta-\theta}{\beta} \) on initial data, we have a polynomial decay \((1+t)^{-\frac{1}{\beta}}\) for the \( L^2 \) norm
\[
|| |\xi|^k \partial_t^j \hat{K}_\ell(t,\cdot)\hat{\psi}||_{L^2(|\xi|\geq 1)} \lesssim || |\xi|^{k+(j-\ell)(\alpha-\delta)} e^{-t \Re \lambda t} \hat{\psi}||_{L^2(|\xi|\geq 1)}
\lesssim || |\xi|^{k+(j-\ell)(\alpha-\delta) + \frac{\delta-\theta}{\beta}} (1+t)^{-\frac{1}{\beta}} \hat{\psi}||_{L^2(|\xi|\geq 1)}
\lesssim (1+t)^{-\frac{1}{\beta}} \hat{\psi}||_{H^s},
\]
with \( s = s_j \) if \( \ell = 0 \) and \( s = r_j \) if \( \ell = 1 \). For \( q \geq 2 \) and \( q' = q/(q-1) \), we may use again Hausdorff–Young inequality
\[
\|(-\Delta)^{\frac{1}{2}} \partial_t^j \hat{K}_\ell(t,\cdot)\hat{\psi}\|_{L^q} = \| |\xi|^k \partial_t^j \hat{K}_\ell(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\geq 1)} \]
\( \ell, j = 0, 1 \). Since \( q' \in [1, 2) \) and \( n < [-k + s - (j - \ell) (\alpha - \delta)] \frac{2q'}{2-q'} \)
\[
\| |\xi|^k \partial_t^j \hat{K}_\ell(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\geq 1)} \lesssim || |\xi|^{k+(j-\ell)(\alpha-\delta)} e^{t \Re \lambda t} \hat{\psi}||_{L^{q'}(|\xi|\geq 1)}
\lesssim || |\xi|^{k+s+(j-\ell)(\alpha-\delta)} || \hat{\psi}||_{L^{2}(|\xi|\geq 1)}
\lesssim (1+t)^{\frac{1}{\beta-q'}} \left( n \frac{2-q'}{2-q'} \right) || \hat{\psi}||_{H^{s}}, \quad n = 1, 2 \]
Now calling \( s = \frac{\delta-\theta}{\beta} + k + (j-\ell)(\alpha-\delta) \) (with \( \beta > 0 \)), if \( n < \frac{2q'(\delta-\theta)}{(2-q')(\alpha-\delta)} \) and \( q' \in [1, 2) \), it follows that
\[
|| |\xi|^k \partial_t^j \hat{K}_\ell(t,\cdot)\hat{\psi}||_{L^{q'}(|\xi|\geq 1)} \lesssim (1+t)^{\frac{n(2-q')}{2-q'} \frac{1}{\beta-q'}} \hat{\psi}||_{H^{s}} \lesssim g(t) || \hat{\psi}||_{H^{s}}, \quad \ell = 0, 1,
\]
with \( s = s_j \) if \( \ell = 0 \) and \( s = r_j \) if \( \ell = 1 \).

**Remark 4.3.** We assume the hypotheses in Theorem 2.4 that the solutions to (4.1), with \( f \equiv 0 \), satisfy (2.4)–(2.5). Indeed, the assumed maximum regularity \( H^{r} \) for the second data, with \( r = \max\{r_0, r_1\} \), is in order that
\[
||| D^{\alpha} K(t,\cdot) * \psi |||_{L^2} \lesssim (1+t)^{-\frac{1}{\alpha}} (n(\frac{1}{m} - \frac{1}{q}) + \alpha - 2\theta) || \psi ||_{H^{r_0} \cap L^m}
\]
for \( n \left( \frac{1}{m} - \frac{1}{q} \right) \leq 2\theta \), i.e.,
\[
r_0 = \delta + \delta - \theta \beta = \delta + \frac{\delta - \theta}{\alpha - \theta} \left( n \left( \frac{1}{m} - \frac{1}{2} \right) + \alpha - 2\theta \right)
\leq \delta + \frac{\delta - \theta}{\alpha - \theta} \left( n \left( \frac{1}{m} - \frac{1}{q} \right) + \alpha - 2\theta \right)
\leq \delta + \frac{\delta - \theta}{\alpha - \theta} \alpha \leq 2\delta,
\]
and
\[
|| \partial_t K(t,\cdot) * \psi ||_{L^2} \lesssim (1+t)^{-\frac{1}{\alpha}} (\frac{1}{m} - \frac{1}{q}) || \psi ||_{H^{r_1} \cap L^m},
\]
i.e.,
\[ r_1 = \frac{\delta - \theta}{\beta} = (\delta - \theta) \left[ \frac{n}{\delta - \theta} \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{n}{\theta} \left( \frac{1}{m} - \frac{1}{q} \right) \right] \]
\[ \leq n \left( \frac{1}{m} - \frac{1}{q} \right) + n \frac{(\delta - \theta) (\frac{1}{m} - \frac{1}{q})}{\theta} \leq 2\delta. \]

This implies the required regularity \( H^s \) for the first data, with \( s = \max\{s_0, s_1\} \), since \( s_j = r_j + \alpha - \delta, j = 0, 1 \).

**Remark 4.4.** We assume the hypotheses in Theorem 2.9 that the solutions to (4.1), with \( f \equiv 0 \), satisfy estimates (2.8) and (2.9). Indeed, if \((u_0, u_1) \in H^{s_1}(\mathbb{R}^n) \times H^{r_1}(\mathbb{R}^n)\), with \( s_1 = \alpha + \delta \) and \( r_1 = 2\delta \), we have
\[ \|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-\frac{n}{2\delta} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{1}{2\gamma} \left( \frac{1}{2} - \frac{1}{q} \right)} \left( \|u_0\|_{L^2} + \|u_1\|_{L^2} \right) \]
\[ + (1 + t)^{-\frac{n}{2\delta} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{1}{2\gamma} \left( \frac{1}{2} - \frac{1}{q} \right)} \left( \|u_0\|_{H^{s_1}} + \|u_1\|_{H^{r_1}} \right) \]
because \( r_1 = \frac{\delta}{\beta} \), i.e., \( \beta = \frac{1}{2} \) implies \( n \frac{1}{2\delta} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{1}{2\gamma} \left( \frac{1}{2} - \frac{1}{q} \right) = -\frac{n}{2\alpha} \left( \frac{1}{2} - \frac{1}{q} \right) - 1 \).

On the other hand, if \((u_0, u_1) \in H^{s_0}(\mathbb{R}^n) \times H^{r_0}(\mathbb{R}^n)\), with \( s_0 = \alpha + \delta \) and \( r_0 = \beta \), we have
\[ \|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-\frac{n}{2\delta} \left( \frac{1}{2} - \frac{1}{q} \right)} \left( \|u_0\|_{L^2} + \|u_1\|_{L^2} \right) \]
\[ + (1 + t)^{-\frac{n}{2\delta} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{1}{2\gamma} \left( \frac{1}{2} - \frac{1}{q} \right)} \left( \|u_0\|_{H^{s_0}} + \|u_1\|_{H^{r_0}} \right), \]
where \( \beta = \frac{\delta}{\alpha + \delta} \). We note that, depending on the parameters \( \alpha, \delta \) and in the space dimension \( n \), the last estimate may be determined at low frequency or at high frequency.

**5. Proof of the global existence results**

By Duhamel’s principle, a function \( u \in Z \), where \( Z \) is a suitable space, is a solution to (1.1) if, and only if, it satisfies the equality
\[ u(t, x) = u^{\text{lin}}(t, x) + \int_0^t E_1(t - s, x) * f(u_t(s, x)) \, ds, \quad \text{in } Z, \quad (5.1) \]
with \( f(u_t(s, x)) = |u_t(s, x)|^p \) and
\[ u^{\text{lin}}(t, x) = K_0(t, x) * u_0(x) + K_1(t, x) * u_1(x), \]
is the solution to the linear Cauchy problem (1.5). The proof of our global existence results is based on the following scheme. We define an appropriate data function space
\[ A = (H^{\alpha + \delta}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)) \times (H^{2\delta}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)) , \quad (5.2) \]
and an evolution space for solutions
\[ Z(T) \ni C([0, T], H^{\alpha}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)) \cap C^1([0, T], L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)), \quad (5.3) \]
equipped with a norm relate to the estimates of solutions to the linear problem (1.5) such that

$$\|u^{\text{lin}}\|_Z \leq C \|(u_0, u_1)\|_A.$$  \hspace{1cm} (5.4)

We define the operator $F$ such that, for any $u \in Z$,

$$Fu(t, x) = \int_0^t E_1(t - s, x) \ast f(u_t(s, x)) \, ds,$$

then we prove the estimates

$$\|Fu\|_Z \leq C \|u\|^p_Z,$$  \hspace{1cm} (5.5)

$$\|Fu - Fv\|_Z \leq C \|u - v\|_Z (\|u\|^{p-1}_Z + \|v\|^{p-1}_Z).$$  \hspace{1cm} (5.6)

By standard arguments, since $u^{\text{lin}}$ satisfies (5.4) and $p > 1$, from (5.5) it follows that $u^{\text{lin}} + F$ maps balls of $Z$ into balls of $Z$, for small data in $A$, and that estimates (5.5)–(5.6) lead to the existence of a unique solution to (5.1), that is, $u = u^{\text{lin}} + Fu$, satisfying (5.4). We simultaneously gain a locally in time for large data and globally in time for small data existence result [10].

The information that $u \in Z$ plays a fundamental role to estimate $f(u_t(s, \cdot))$ in suitable norms. We will employ the following well-known result (for instance, see [7]).

**Lemma 5.1.** Let $\kappa \leq 1$. Then it holds

$$\int_0^t (1 + t - s)^{-\kappa} (1 + s)^{-\mu} \, ds \lesssim \begin{cases} (1 + t)^{-\kappa} & \text{if } \mu > 1 \\ (1 + t)^{-\kappa} \log(e + t) & \text{if } \mu = 1. \end{cases}$$

**Proof.** (Theorem 2.4) We have to prove (5.4), (5.5) and (5.6), with $A$ as in (5.2) and $Z(T)$ as in (5.3) for all $q \geq 2p$, equipped with the norm

$$\|u\|_{Z(T)} = \sup_{t \in [0, T]} \left\{ (1 + t)^{-1 + \frac{n}{2p} \left(\frac{1}{m} - \frac{1}{2}\right)} \|u(t, \cdot)\|_{L^2} + (1 + t)^{-1 + \frac{n}{2p} \left(\frac{1}{m} - \frac{1}{2}\right)} \|u_t(t, \cdot)\|_{L^q} + (1 + t)^{\frac{1}{2p} \left(\frac{1}{m} - \frac{1}{2}\right)} \|u_t(t, \cdot)\|_{L^q} \right\}.$$

Thanks to Theorem 4.1 and Remark 4.3, $u^{\text{lin}} \in Z(T)$ and it satisfies (5.4).

Let us prove (5.5). We omit the proof of (5.6), since it is analogous to the proof of (5.5).

Let $u \in Z(T)$. Using Theorem 4.1 with $\eta = m$ and Remark 4.3, for $j = 0, 1$ we have

$$\|\partial_t^j E_1(t - s, \cdot) \ast f(u_t(s, \cdot))\|_{L^2} \leq (1 + t - s)^{1-j - \frac{n}{2p} \left(\frac{1}{m} - \frac{1}{2}\right)} \|f(u_t(s, \cdot))\|_{L^2 \cap L^m},$$

$$\|\partial_t^j E_1(t - s, \cdot) \ast f(u_t(s, \cdot))\|_{L^q} \leq (1 + t - s)^{1-j - \frac{n}{2p} \left(\frac{1}{m} - \frac{1}{2}\right)} \|f(u_t(s, \cdot))\|_{L^2 \cap L^m},$$

$$\|D^\alpha E_1(t - s, \cdot) \ast f(u_t)\|_{L^2} \leq (1 + t - s)^{-\frac{n}{2p} \left(\frac{1}{m} - \frac{1}{2}\right) + \alpha - 2\theta} \|f(u_t)\|_{L^2 \cap L^m}.$$  \hspace{1cm} (5.6)

Condition (2.3) implies that $m \left(1 + \frac{2m\theta}{n}\right) \geq 2$, hence, for $\kappa \geq m$ and $1 + \frac{2m\theta}{n} < p \leq \frac{q}{2}$, by interpolation we may estimate

$$\|f(u_t(s, \cdot))\|_{L^\kappa} = \|u_t(s, \cdot)\|_{P_{L^\kappa}} \lesssim (1 + s)^{-\frac{n}{2p} \left(\frac{1}{m} - \frac{1}{2}\right)} \|u\|^p_{Z(T)}.$$
and Lemma 5.1 implies
\[
\|\partial_t^j Fu(t, \cdot)\|_{L^2} \lesssim \|u\|^p_{Z(T)} \int_0^t (1 + t - s)^{1 - j - \frac{n}{2p}(\frac{1}{m} - \frac{1}{2})} (1 + s)^{- \frac{n}{2p}(\frac{1}{m} - \frac{1}{2})} ds
\]
\[
\lesssim (1 + t)^{1 - j - \frac{n}{2p}(\frac{1}{m} - \frac{1}{2})} \|u\|^p_{Z(T)},
\]
\[
\|\partial_t^j Fu(t, \cdot)\|_{L^q} \lesssim \|u\|^p_{Z(T)} \int_0^t (1 + t - s)^{1 - j - \frac{n}{2p}(\frac{1}{m} - \frac{1}{2})} (1 + s)^{- \frac{n}{2p}(\frac{1}{m} - \frac{1}{2})} ds
\]
\[
\lesssim (1 + t)^{1 - j - \frac{n}{2p}(\frac{1}{m} - \frac{1}{2})} \|u\|^p_{Z(T)},
\]
and
\[
\|D^\alpha Fu(t, \cdot)\|_{L^2} \lesssim \|u\|^p_{Z(T)} \int_0^t (1 + t - s)^{- \frac{n}{2p} + \frac{1}{2} \alpha} (1 + s)^{- \frac{n}{2p} \alpha} \|u\|^p_{Z(T)} ds
\]
\[
\lesssim (1 + t)^{- \frac{n}{2p} (\frac{1}{2} + \frac{1}{m})} \|u\|^p_{Z(T)},
\]
thanks again to \( p > 1 + \frac{2m\theta}{n} \) and \( q \leq \frac{nm}{(n - 2m\theta) +}. \]

\[
\square
\]

**Proof.** (Theorem 2.9) We have to prove (5.4), (5.5) and (5.6), with \( A \) as in (5.2) for \( m = 2 \) and \( Z(T) \) as in (5.3) for all \( q \geq 2p \), equipped with the norm
\[
\|u\|_{Z(T)} = \sup_{t \in [0, T)} \left\{ \|u(t, \cdot)\|_{L^2} + (1 + t)^{\frac{1}{2}} \|D^\alpha u(t, \cdot)\|_{L^2} + (1 + t) \|u_t(t, \cdot)\|_{L^q} \right. \]
\[
\left. + (1 + t)^{\min\left\{\frac{n}{2p}(\frac{1}{2} - \frac{1}{q}), \frac{n+1}{2p}, \frac{n-1}{2p}\right\}} \|u(t, \cdot)\|_{L^q} \right\}.
\]

We only discuss the proof of (5.5). Let \( u \in Z(T) \). Using Theorem 4.1 with \( \eta = 2 \), for \( j = 0, 1 \) we have
\[
\|\partial_t^j E_1(t - s, \cdot) \ast f(u_t(s, \cdot))\|_{L^2} \lesssim (1 + t - s)^{- j} \|f(u_t(s, \cdot))\|_{L^2},
\]
\[
\|D^\alpha E_1(t - s, \cdot) \ast f(u_t(s, \cdot))\|_{L^2} \lesssim (1 + t - s)^{- \frac{1}{2}} \|f(u_t(s, \cdot))\|_{L^2},
\]
\[
\|E_1(t - s, \cdot) \ast f(u_t)\|_{L^q} \lesssim (1 + t - s)^{- \min\left\{\frac{n}{2p}(\frac{1}{2} - \frac{1}{q}), \frac{n+1}{2p}, \frac{n-1}{2p}\right\}} \|f(u_t)\|_{L^2}
\]
and
\[
\|\partial_t E_1(t - s, \cdot) \ast f(u_t(s, \cdot))\|_{L^q} \lesssim (1 + t - s)^{- \frac{n}{2p} (\frac{1}{2} - \frac{1}{q}) - 1} \|f(u_t(s, \cdot))\|_{L^2}
\]
\[
+ (1 + t - s)^{\frac{n}{2p} (\frac{1}{2} - \frac{1}{q}) - \frac{1}{2p}} \|f(u_t(s, \cdot))\|_{H^{(\tau_1 - 2\delta)+}}.
\]

Choosing \( \beta = \frac{1}{2} \) in the above estimate we have \( \tau_1 - 2\delta = 0 \) and
\[
\|\partial_t E_1(t - s, \cdot) \ast f(u_t(s, \cdot))\|_{L^q} \lesssim (1 + t - s)^{\frac{n}{2p} (\frac{1}{2} - \frac{1}{q}) - 1} \|f(u_t(s, \cdot))\|_{L^2}.
\]

By interpolation and the definition of \( \|\cdot\|_{Z(T)} \) we have
\[
\|f(u_t(s, \cdot))\|_{L^2} = \|u_t(s, \cdot)\|_{L^{2p}}^p \lesssim (1 + s)^{\frac{n}{2p}(p - 1) - p} \|u\|^p_{Z(T)}.
\]
Lemma 5.1 implies
\[
\| \partial_t^j Fu(t, \cdot) \|_{L^q} \lesssim \| u \|_{Z(T)}^p \int_0^t (1 + t - s)^{-j} (1 + s)^{-\frac{\alpha}{p} + \frac{\delta}{p} - 1} (1 + s)^{\frac{\alpha}{p} - 1 - \frac{\alpha + \delta}{p} - \frac{\alpha}{p} (1 - \frac{1}{q})} - p ds
\]
\[
\lesssim (1 + t)^{-j} \| u \|_{Z(T)}^p,
\]
\[
\| D^{\alpha} Fu(t, \cdot) \|_{L^q} \lesssim \| u \|_{Z(T)}^p \int_0^t (1 + t - s)^{-\frac{\alpha}{p} (1 - \frac{1}{q}) - 1} (1 + s)^{\frac{\alpha}{p} - 1 - \frac{\alpha + \delta}{p} - \frac{\alpha}{p} (1 - \frac{1}{q})} - p ds
\]
\[
\lesssim (1 + t)^{-\frac{\alpha}{p} (1 - \frac{1}{q}) - 1} \| u \|_{Z(T)}^p,
\]
and
\[
\| \partial_t Fu(t, \cdot) \|_{L^q} \lesssim \| u \|_{Z(T)}^p \int_0^t (1 + t - s)^{-\frac{\alpha}{p} (1 - \frac{1}{q}) - 1} (1 + s)^{\frac{\alpha}{p} - 1 - \frac{\alpha + \delta}{p} - \frac{\alpha}{p} (1 - \frac{1}{q})} - p ds
\]
\[
\lesssim (1 + t)^{-\frac{\alpha}{p} (1 - \frac{1}{q}) - 1} \| u \|_{Z(T)}^p,
\]
thanks to \( p > 1 \) and \( n < 4 \delta \).

Proof. (Theorem 2.11) We have to prove (5.4), (5.5) and (5.6), with \( A \) as in (5.2) and \( Z(T) \) as in (5.3) for all \( q \geq 2p \), equipped with the norm
\[
\| u \|_{Z(T)} = \sup_{t \in [0, T]} \left\{ (1 + t)^{-1 + \frac{n}{2p} \left( \frac{1}{q} - \frac{1}{2} \right)} \| u(t, \cdot) \|_{L^2} + (1 + t)^{-1 + \frac{n}{2p} \left( \frac{1}{q} - \frac{1}{2} \right)} \| u_t(t, \cdot) \|_{L^q} + (1 + t)^{-1 + \frac{n}{2p} \left( \frac{1}{q} - \frac{1}{2} \right)} \| u_{tt}(t, \cdot) \|_{L^q} + (1 + t)^{-1 + \frac{n}{2p} \left( \frac{1}{q} - \frac{1}{2} \right)} \| u_{ttt}(t, \cdot) \|_{L^q} + (1 + t)^{-1 + \frac{n}{2p} \left( \frac{1}{q} - \frac{1}{2} \right)} \| h(t) \|_{d^2 u(t, \cdot)} \right\},
\]
(5.7)
where \( h(t) = (\log(e + t))^{-1} \) if \( \alpha > 2 \theta \) and \( h(t) \equiv 1 \) if \( \alpha = 2 \theta \). Thanks to Theorem 4.1 and Remark 4.3, \( u^{\text{lin}} \in Z(T) \) and it satisfies (5.4).

Let us prove (5.5). Let \( u \in Z(T) \). To consider the case \( p = p_c = 1 + \frac{2m \theta}{n} \), we have to change the argument done in Theorem 2.4. Condition (2.3) implies that \( m (1 + \frac{2m \theta}{n}) \geq 2 \), hence, for \( \kappa \geq m \) and \( 1 + \frac{2m \theta}{n} = p \leq \frac{q}{2} \), by interpolation and (5.7) we have
\[
\| f(u_t(t, \cdot)) \|_{L^\kappa} = \| u_t(t, \cdot) \|_{L^{\kappa p}} \lesssim (1 + s)^{-\frac{n}{2p} \left( \frac{1}{q} - \frac{1}{2} \right)} \| u \|_{Z(T)}^p,
\]
(5.8)
Applying Theorem 4.1 with $1 < \eta < m$ such that $\eta \left(1 + \frac{2m\theta}{n}\right) > 2$ and $n \left(\frac{1}{\eta} - \frac{1}{q}\right) \leq 2\theta$, we may estimate

$$\|\partial_t^j F u(t, \cdot)\|_{L^2} \lesssim \int_0^t (1 + t - s)^{1 - j - \frac{n}{2^\theta} \left(\frac{1}{\eta} - \frac{1}{q}\right)} \|f(u_t(s, \cdot))\|_{L^2 \cap L^0} ds$$

$$\lesssim \int_0^t (1 + t - s)^{1 - j - \frac{n}{2^\theta} \left(\frac{1}{\eta} - \frac{1}{q}\right)} \|u_t(s, \cdot)\|_{p_c} \|u\|_{L^2 \cap L^{p_c}} ds, \quad j = 0, 1.$$

Using (5.8) we conclude

$$\|\partial_t^j F u(t, \cdot)\|_{L^2} \lesssim \|u\|_{p_c(Z(T))} \int_0^t (1 + t - s)^{1 - j - \frac{n}{2^\theta} \left(\frac{1}{\eta} - \frac{1}{q}\right)} (1 + s)^{- \frac{n}{2^\theta} \left(\frac{1}{m} - \frac{1}{q}\right)} ds.$$

Now we split the integration interval into $[0, t/2]$ and $[t/2, t]$:

$$\int_0^t (1 + t - s)^{1 - j - \frac{n}{2^\theta} \left(\frac{1}{\eta} - \frac{1}{q}\right)} (1 + s)^{- \frac{n}{2^\theta} \left(\frac{1}{m} - \frac{1}{q}\right)} ds$$

$$\lesssim \int_0^{t/2} + \int_{t/2}^t (1 + t - s)^{1 - j - \frac{n}{2^\theta} \left(\frac{1}{\eta} - \frac{1}{q}\right)} (1 + s)^{- \frac{n}{2^\theta} \left(\frac{1}{m} - \frac{1}{q}\right)} ds \lesssim (1 + t)^{1 - j - \frac{n}{2^\theta} \left(\frac{1}{m} - \frac{1}{q}\right)}.$$

On the other hand, we conclude

$$\int_0^t (1 + t - s)^{1 - j - \frac{n}{2^\theta} \left(\frac{1}{\eta} - \frac{1}{q}\right)} (1 + s)^{- \frac{n}{2^\theta} \left(\frac{1}{m} - \frac{1}{q}\right)} ds$$

$$\lesssim \int_0^{t/2} + \int_{t/2}^t (1 + t - s)^{1 - j - \frac{n}{2^\theta} \left(\frac{1}{\eta} - \frac{1}{q}\right)} (1 + s)^{- \frac{n}{2^\theta} \left(\frac{1}{m} - \frac{1}{q}\right)} ds \lesssim (1 + t)^{1 - j - \frac{n}{2^\theta} \left(\frac{1}{m} - \frac{1}{q}\right)}.$$

Hence

$$\|\partial_t^j F u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{1 - j - \frac{n}{2^\theta} \left(\frac{1}{\eta} - \frac{1}{q}\right)} \|u\|_{p_c(Z(T))}, \quad j = 0, 1.$$

Similarly,

$$\|\partial_t^j F u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{1 - j - \frac{n}{2^\theta} \left(\frac{1}{\eta} - \frac{1}{q}\right)} \|u\|_{p_c(Z(T))}, \quad j = 0, 1$$

and

$$\|D^{\alpha} F u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{1 - \frac{1}{2^\theta} \alpha \left(n \left(\frac{1}{m} - \frac{1}{q}\right) + \alpha - 2\theta\right)} \|u\|_{p_c(Z(T))}, \quad \text{if } \alpha = 2\theta.$$

However, compared with the estimates for solutions to the linear problem, in the case $p = p_c$ we have a logarithm loss of decay to the term $\|D^{\alpha} F u(t, \cdot)\|_{L^2}$ if $\alpha > 2\theta$. Applying Theorem 4.1 and Lemma 5.1 with $\mu = 1$ we have

$$\|D^{\alpha} F u(t, \cdot)\|_{L^2} \lesssim \int_0^t (1 + t - s)^{- \frac{1}{2^\theta} \alpha \left(n \left(\frac{1}{m} - \frac{1}{q}\right) + \alpha - 2\theta\right)} \|f(u_t(s, \cdot))\|_{L^2 \cap L^m} ds$$

$$\lesssim \|u\|_{p_c(Z(T))} \int_0^t (1 + t - s)^{- \frac{1}{2^\theta} \alpha \left(n \left(\frac{1}{m} - \frac{1}{q}\right) + \alpha - 2\theta\right)} (1 + s)^{- \frac{n}{2^\theta} (p_c - 1)} ds$$

$$\lesssim (1 + t)^{- \frac{1}{2^\theta} \alpha \left(n \left(\frac{1}{m} - \frac{1}{q}\right) + \alpha - 2\theta\right) \log(e + t)} \|u\|_{p_c(Z(T))}.$$
Acknowledgements

The authors would like to thank the referee for his careful reading of the manuscript and for valuable comments.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Charão, R.C., da Luz, C.R.: Asymptotic properties for a semilinear plate equation in unbounded domains. J. Hyperbolic Differ. Equ. 6, 269–294 (2009)

[2] Charão, R.C., da Luz, C.R., Ikehata, R.: Sharp decay rates for wave equations with a fractional damping via new method in the Fourier space. J. Math. Anal. Appl. 408, 247–255 (2013)

[3] Ciarlet, P.G.: A justification of the von Kármán equations. Arch. Rational Mech. Anal. 73, 349–389 (1980)

[4] D’Abbicco, M.: The critical exponent for the dissipative plate equation with power nonlinearity. Comput. Math. Appl. 74, 1006–1014 (2017)

[5] D’Abbicco, M., Ebert, M.R.: Diffusion phenomena for the wave equation with structural damping in the $L^p - L^q$ framework. J. Differ. Equ. 256, 2307–2336 (2014)

[6] D’Abbicco, M., Ebert, M.R.: A classification of structural dissipations for evolution operators. Math. Methods Appl. Sci. 39, 2558–2582 (2016)

[7] D’Abbicco, M., Ebert, M.R.: A new phenomenon in the critical exponent for structurally damped semi-linear evolution equations. Nonlinear Anal. Theory Methods Appl. 149, 1–40 (2017)

[8] D’Ambrosio, L., Lucente, S.: Nonlinear Liouville theorems for Grushin and Tricomi operators. J. Differ. Equ. 193, 511–541 (2003)

[9] Ebert, M.R., Girardi, G., Reissig, M.: Critical regularity of nonlinearities in semilinear classical damped wave equations. Math. Ann. (2019). https://doi.org/10.1007/s00208-019-01921-5

[10] Ebert, M.R., Reissig, M.: Methods for Partial Differential Equations, Qualitative Properties of Solutions, Phase Space Analysis, Semilinear Models. Birkhäuser/Springer, Cham (2018)

[11] Fujita, H.: On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sect. I(13), 109–124 (1966)

[12] Ide, K., Kawashima, S.: Decay property of regularity-loss type and nonlinear effects for dissipative Timoshenko system. Math. Models Methods Appl. Sci. 18, 1001–1025 (2008)
[13] Ikeda, M., Inui, T., Okamoto, M., Wakasugi, Y.: $L^p - L^q$ estimates for the damped wave equation and the critical exponent for the nonlinear problem with slowly decaying data. Commun. Pure Appl. Anal. 18, 1967–2008 (2019)

[14] Ikehata, R., Miyaoaka, Y., Nakatake, T.: Decay estimates of solutions for dissipative wave equations in $\mathbb{R}^N$ with lower power nonlinearities. J. Math. Soc. Jpn. 56, 365–373 (2004)

[15] Ikehata, R., Ohta, M.: Critical exponents for semilinear dissipative wave equations in $\mathbb{R}^N$. J. Math. Anal. Appl. 269, 87–97 (2002)

[16] Karch, G.: Selfsimilar profiles in large time asymptotics of solutions to damped wave equations. Studia Math. 143, 175–197 (2000)

[17] Kubo, T., Kawashima, S.: Decay property of regularity-loss type and nonlinear effects for some hyperbolic-elliptic system. Kyushu J. Math. 63, 139–159 (2009)

[18] Liu, Y.: Decay of solutions to an inertial model for a semilinear plate equation with memory. J. Math. Anal. Appl. 394, 616–632 (2012)

[19] da Luz, C. R., Palma, M.F.G.: Asymptotic properties for second-order linear evolution problems with fractional laplacian operators (2018). arXiv:1802.01112

[20] Menzala, G.P., Zuazua, E.: Timoshenko’s plate equation as a singular limit of the dynamical von Kármán system. J. Math. Pures Appl. 79, 73–94 (2000)

[21] Mitidieri, E., Pohozaev, S.I.: Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on $\mathbb{R}^n$. J. Evol. Equ. 1, 189–220 (2001)

[22] Narazaki, T.: $L^p - L^q$ estimates for damped wave equations and their applications to semi-linear problem. J. Math. Soc. Jpn. 56, 585–626 (2004)

[23] Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton, NJ (1970)

[24] Sugitani, Y., Kawashima, S.: Decay estimates of solutions to a semi-linear dissipative plate equation. J. Hyperbolic Differ. Equ. 7, 471–501 (2010)

[25] Todorova, G., Yordanov, B.: Critical exponent for a nonlinear wave equation with damping. J. Differ. Equ. 174, 464–489 (2001)

[26] Zhang, Qi S.: A blow-up result for a nonlinear wave equation with damping: the critical case. C. R. Acad. Sci. Paris Sér. I Math. 333, 109–114 (2001)
Cleverson R. da Luz and Maíra F. G. Palma  
Department of Mathematics  
Federal University of Santa Catarina  
Campus Trindade  
Florianópolis SC88040-900  
Brazil  
e-mail: cleverson.luz@ufsc.br

Maíra F. G. Palma  
e-mail: mairagauer@gmail.com

Received: 4 October 2019.  
Accepted: 15 June 2020.