PURE SHEAVES AND KLEINIAN SINGULARITIES

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ABSTRACT. Grothendieck proved that any locally free sheaf on a projective line over a field (uniquely) decomposes into a direct sum of line bundles. Ishii and Uehara construct an analogue of Grothendieck’s theorem for pure sheaves on the fundamental cycle of the Kleinian singularity $A_n$. We first study the analogue for the other Kleinian singularities. We also study the classification of rigid pure sheaves on the reduced scheme of the fundamental cycles. The classification is related to the classification of spherical objects in a certain Calabi-Yau 2-dimensional category.

1. INTRODUCTION

Let $E$ be a locally free sheaf on a projective line $\mathbb{P}^1$ over a field $k$. As was proven by Grothendieck [Gr57], the sheaf $E$ decomposes into a direct sum of line bundles on $\mathbb{P}^1$ and the decomposition is unique up to isomorphisms. Hence we have a complete classification not only of locally free sheaves but also of indecomposable sheaves on $\mathbb{P}^1$.

It may be natural to study an analogue of Grothendieck’s theorem for $\mathbb{P}^n$, but it seems difficult. In fact, if $n > 1$ then there exists an indecomposable locally free sheaf on $\mathbb{P}^n$ whose rank is greater than 1. A simple example of such a sheaf is the tangent sheaf on $\mathbb{P}^n$. Moreover the classification of indecomposable locally free sheaves is more difficult in the case of lower rank (cf. [Har79]).

Though higher dimensional analogue of Grothendieck’s theorem is difficult, Ishii and Uehara prove a beautiful analogue for the fundamental cycle $Z_{A_n}$ of the Kleinian singularity $A_n$. To recall their result, a non-zero sheaf $F$ on a scheme $Y$ is said to be pure if the support of any non-trivial subsheaf of $F$ has the same dimension of $Y$. We note that if $Y$ is smooth and 1-dimensional then a pure sheaf on $Y$ is equivalent to a locally free sheaf on $Y$. Thus a pure sheaf is a natural generalization of locally free sheaves for reducible schemes such as $Z_{A_n}$. Ishii and Uehara prove the following:

**Theorem 1.1** ([IU05, Lemma 6.1]). Let $E$ be a pure sheaf on $Z_{A_n}$. Then $E$ decomposes into a direct sum of invertible sheaves on connected subtrees of $Z_{A_n}$. Moreover, the decomposition is unique up to isomorphisms.

We first study an analogue of Theorem 1.1.

**Theorem 1.2** (=Corollary 3.6). Let $Z$ be the fundamental cycle of a Kleinian singularity except for $A_n$. Then

$$\max\{\text{rank}_Z E \mid E \text{ is an indecomposable pure sheaf on } Z\} = \infty.$$  

We remark that the usual rank of sheaves is not appropriate since our scheme is reducible. Thus we introduce more suitable “rank” of sheaves in our setting (see Definition 3.1). By using it the first half of Theorem 1.1 can be restated that the maximum of the...
rank of indecomposable pure sheaves is 1. It may be natural to expect that the maximum of the rank of indecomposable pure sheaves are bounded for the other Kleinian singularities. Our theorem gives a counter-example of the expectation.

The second aim of this note is to study the classification of \( \mathcal{O}_X \)-rigid pure sheaves on \( Z \). The classification is related to the classification of spherical objects in a certain category (for the definition of spherical objects, see also [Huy06] or [ST01]).

To explain the relation, let \( X \) be the minimal resolution of a Kleinian singularity. It is well-known that the fundamental cycle \( Z \) of the singularity is the schematic fiber of the singularity by the resolution. Since \( Z \) is a subscheme of \( X \), we have a natural embedding \( \iota : Z \to X \). We denote by \( D_Z^b(X) \) the bounded derived category of coherent sheaves on \( X \) supported on \( Z \). A coherent sheaf \( E \) on \( Z \) is said to be \( \mathcal{O}_X \)-rigid if the push forward \( \iota_* E \) by \( \iota \) is rigid, that is, \( \operatorname{Ext}^1_X(\iota_* E, \iota_* E) = 0 \).

Ishii and Uehara show that each cohomology (with respect to the standard \( t \)-structure) of spherical objects in \( D_Z^b(X) \) is the push forward \( \iota_* E \) of a pure sheaf \( E \) on \( Z \) which is \( \mathcal{O}_X \)-rigid. If the singularity is \( A_n \), then the classification of \( \mathcal{O}_X \)-rigid sheaves is a direct consequence of Theorem 1.1. By using the classification, Ishii and Uehara classify spherical objects in \( D_Z^b(X) \) for the Kleinian singularity \( A_n \) (the details are in [IU05, Proposition 1.6]). One might hope to classify spherical objects for the other Kleinian singularities following Ishii and Uehara’s approach, by the first classifying indecomposable pure \( \mathcal{O}_X \)-rigid sheaves. Theorem 1.2 is an evidence that this likely to be a rather difficult problem and we do not solve it in this paper. However we do prove the following result, which leaves hope that such a classification might be achieved in the future.

**Theorem 1.3.** Let \( E \) be an indecomposable pure sheaf on the reduced scheme \( Z_r \) of the fundamental cycle of a Kleinian singularity except for \( A_n \). If \( E \) is \( \mathcal{O}_X \)-rigid, then \( \operatorname{rank}_{Z_r} E \leq 3 \) and the inequality is best possible.

The proof of Theorem 1.3 will be postponed till the end of Section 5. The essential part is in the proof of Propositions 4.6 and 4.10.

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2. Notations and Conventions

Throughout this note, our field \( k \) is algebraically closed and Kleinian singularities are given by \( \text{Spec} k[[x, y, z]]/f(x, y, z) \) where \( f(x, y, z) \) is one of the following:

- \( A_n \) for \( n \geq 1 \)
- \( D_n \) for \( n \geq 4 \)
- \( E_6 \)
- \( E_7 \)
- \( E_8 \)

Let \( Z \) be the fundamental cycle of the singularity \( D_n \). The \( i \)-th irreducible component \( C_i \) of \( Z \) is denoted as in Figure 1. Then it is well-known that \( Z \) is \( C_1 + C_2 + \sum_{i=3}^{n-1} 2C_i + C_n \). Similarly the \( j \)-th irreducible component \( C_j \) of the fundamental cycle of the singularities \( E_6 \), \( E_7 \) or \( E_8 \) is denoted as in Figure 2.

**Remark 2.1.** We note that the chain \( \sum_{i=2}^5 C_i \) in Figure 2 gives the reduced scheme of the fundamental cycle of the singularity \( D_4 \). We use this identification in the proof of Proposition 5.1.
Let \( D \) be a \( k \)-linear triangulated category. We denote by \( \text{hom}^p(E,F) \) the dimension of the vector space \( \text{Hom}^p_D(E,F) = \text{Hom}_D(E,F[p]) \) for \( E \) and \( F \in D \). The category \( D \) is said to be of finite type if the sum \( \sum_{p \in \mathbb{Z}} \text{hom}^p(E,F) \) is finite for any \( E,F \in D \). If \( D \) is of finite type then the Euler characteristic
\[
\chi(E,F) = \sum_{p \in \mathbb{Z}} (-1)^p \text{hom}^p(E,F)
\]
is well-defined. If the Serre functor of \( D \) is isomorphic to the double shift \([2]\), then \( D \) is said to be \( 2 \)-dimensional Calabi-Yau (for simplicity CY2). If \( D \) is CY2, then we have \( \text{hom}^p(E,F) = \text{hom}^{2-p}(F,E) \). One of the best example of CY2 categories is \( D_Z(X) \).

3. INDECOMPOSABLE PURE SHEAVES

**Definition 3.1.** Let \( Z' \) a 1-dimensional closed subscheme of the fundamental cycle \( Z \) of a Klein singularity and \( i': Z' \to X \) be the embedding to the minimal resolution \( X \) of the singularity. We define the rank of a sheaf \( E \) on \( Z' \) as follows:
\[
\text{rank}_{Z'}E := \min\{a \in \mathbb{Z}_{\geq 0} \mid c_1(i'_*E) \leq a \cdot Z'\},
\]
where \( c_1 \) is the first Chern class.

**Remark 3.2.** We would like to define a rank so that the structure sheaf of any \( Z' \) has rank 1. One of a naive generalization of the usual rank is the following: The rank of a sheaf \( E \) on \( Z' \) is the maximum rank of \( E \) on each irreducible components of \( Z' \). Such a generalization does not satisfy our requirement if \( Z' \) is the fundamental cycle \( Z \) of a Kleinian singularity except for \( A_n \).

By using Definition 3.1 the first half of Theorem 1.1 can be restated as follows
\[
\max\{\text{rank}_{Z_n}E \mid E \text{ is an indecomposable pure sheaf on } Z_n\} = 1.
\]
Contrary to the singularity \( A_n \), we prove the following for the singularity \( D_4 \):

**Theorem 3.3.** Let \( E \) be a pure sheaf on the reduced scheme \( Z_r \) of the fundamental cycle \( Z \) of the singularity \( D_4 \). Then there exists an indecomposable pure sheaf \( E \) on \( Z_r \) of rank \( r \) for any \( r \in \mathbb{N} \).

Before the proof we denote by \( \mathcal{O}_{C_1+C_2+C_3}(a_1,a_2,a_3) \) an invertible sheaf on the chain \( C_1 + C_2 + C_3 \) such that the degree on each irreducible component \( C_i \) is \( a_i \).

**Remark 3.4.** A key ingredient of the proof of Theorem 3.3 is a choice of particular sheaves \( \mathcal{L}_n \), where \( \mathcal{L}_n = \mathcal{O}_{C_1+C_2+C_3}(n,-n,0) \) for \( n \in \mathbb{Z} \). Any pair \( (\mathcal{L}_n, \mathcal{L}_m) \) (for \( n \neq m \)) has the following property: For any morphism \( f: \mathcal{L}_n \to \mathcal{L}_m \), the induces morphism \( f_*: \text{Ext}^1_{Z_r}(\mathcal{O}_{C_1}, \mathcal{L}_n) \to \text{Ext}^1_{Z_r}(\mathcal{O}_{C_1}, \mathcal{L}_m) \) is zero (the details are in Lemma 3.5). If the singularity is \( A_n \), such a pair does not exist.
Proof. Take an invertible sheaf $L_n = \mathcal{O}_{C_1 + C_2 + C_3}(n, -n, 0)$ on $C_1 + C_2 + C_3$ for an integer $n \in \mathbb{Z}$. It is easy to see $\text{Ext}^1_{\mathbb{Z}_r}(\mathcal{O}_{C_i}, L_n) \cong \text{Ext}^1_{\mathbb{Z}_r}(\mathcal{O}_{C_i}, \mathcal{O}_{C_j})$. We wish to describe $\text{Ext}^1_{\mathbb{Z}_r}(\mathcal{O}_{C_1}, \mathcal{O}_{C_3})$ in an explicit way. By the locally free resolution of $\mathcal{O}_{C_3}$ as $\mathcal{O}_{\mathbb{Z}_r}$-module, we see $\text{Hom}^4_{\mathbb{Z}_r}(\mathcal{O}_{C_1}, \mathcal{O}_{C_3}) = 0$ and $\text{Ext}^4_{\mathbb{Z}_r}(\mathcal{O}_{C_1}, \mathcal{O}_{C_3}) \cong k(x)$ where $x \in C_3 \cap C_4$. Thus we have $\text{Ext}^1_{\mathbb{Z}_r}(\mathcal{O}_{C_1}, \mathcal{O}_{C_3}) \cong H^0(k(x))$ by the local-to-global spectral sequence.

Let $I$ be an arbitrary finite subset of $\mathbb{Z}$. The vector space $\bigoplus_{n \in I} \text{Ext}^1_{\mathbb{Z}_r}(\mathcal{O}_{C_1}, L_n)$ is denoted by $V_I$. Any extension class $[\mathcal{E}] \in V_I$ can be identified with a column vector with respect to a natural basis of $V_I$. Take the universal extension $[U_I]$, that is, $[U_I]$ is a vector whose components are all 1. We wish to prove that $U_I$ is indecomposable.

Suppose to the contrary that $U_I$ decomposes into $\mathcal{F} \oplus \mathcal{G}$. We can assume $\text{Supp } \mathcal{G} \supset C_4$ without loss of generality. Then we have $\text{Hom}_{\mathbb{Z}_r}(\mathcal{F}, \mathcal{O}_{C_3}) = 0$ since $\text{Supp } \mathcal{F} \subset C_1 + C_2 + C_3$. Hence the natural morphism $f : \mathcal{F} \oplus \mathcal{G} \to \mathcal{O}_{C_3}$ splits into $0 \oplus f$ where 0 is the zero morphism from $\mathcal{F}$ and $\tilde{f} : \mathcal{G} \to \mathcal{O}_{C_3}$.

Let $K$ be the kernel of the morphism $\tilde{f}$. Then we have $\mathcal{F} \oplus K \cong \bigoplus_{n \in I} L_n,$ and see that $\mathcal{F} \oplus K$ is a pure $\mathcal{O}_{C_1 + C_2 + C_3}$-module. Thus, by Theorem 1.1 we see $\mathcal{F} \cong \bigoplus_{n \in I} L_n$ and $K \cong \bigoplus_{n_j \in I'} L_{n_j}$ where $I' \sqcup I'' = I$. In particular we have the following diagram of distinguished triangles:

$$
\begin{array}{ccc}
\mathcal{F} \oplus \mathcal{G} & \xrightarrow{f} & \mathcal{O}_{C_3} \\
\downarrow & & \downarrow \\
\mathcal{F} & \xrightarrow{u} & \bigoplus_{n \in I} L_n[1] \\
\end{array}
$$

Hence the composite $\pi[1] \circ u$ is zero in the derived category $D(\mathbb{Z}_r)$ on $\mathbb{Z}_r$. Thus the corresponding coefficients $[U_I] \in \text{Ext}^1_{\mathbb{Z}_r}(\mathcal{O}_{C_3}, \mathcal{F}) \oplus \text{Ext}^1_{\mathbb{Z}_r}(\mathcal{O}_{C_3}, \mathcal{G})$ to $\mathcal{F}$ should be 0. Moreover we see that the natural representation of the automorphism group $\text{Aut}(\bigoplus L_n)$ on $V_I$ is contained in diagonal matrices by Lemma 3.5 (below). This contradicts the definition of $U_I$. \[\Box\]

Lemma 3.5. We denote by $\mathcal{L}_i$ a pure sheaf $\mathcal{O}_{C_1 + C_2 + C_3}(i, -i, 0)$ for an integer $i$. For any finite subset $I \subset \mathbb{Z}$, the vector space $\bigoplus_{i \in I} \text{Ext}^1_{\mathbb{Z}_r}(\mathcal{O}_{C_i}, \mathcal{L}_i)$ is denoted by $V_I$. Then the image of a natural representation

$$
\rho : \text{End}_{\mathbb{Z}_r}(\bigoplus_{i \in I} \mathcal{L}_i) \to \text{End}_k(V_I)
$$

is contained in diagonal matrices with respect to a natural basis of $V_I$.

Proof. Let $\mathcal{D}$ be the derived category on $\mathbb{Z}_r$. The vector space $\text{End}_{\mathbb{Z}_r}(\bigoplus_{i \in I} \mathcal{L}_i)$ decomposes into as follows:

$$\text{End}_{\mathbb{Z}_r}(\bigoplus_{i \in I} \mathcal{L}_i) \cong \bigoplus_{i,j \in I} \text{Hom}_{\mathbb{Z}_r}(\mathcal{O}_{C_1 + C_2 + C_3}, \mathcal{L}_{j-i}).$$

By the symmetry for $C_1$ and $C_2$ we can assume $\ell = i - j \geq 0$. \[\Box\]
If $-\ell < 0$ then $H^0(\mathcal{O}_{C_2+C_3}(-\ell,0))$ is zero. Hence the natural inclusion $\mathcal{O}_{C_1}(\ell-1) \to \mathcal{L}_\ell$ induces an isomorphism

\begin{equation}
\text{Hom}(\mathcal{O}_{C_1+C_2+C_3}, \mathcal{L}_\ell) \cong \text{Hom}(\mathcal{O}_{C_1+C_2+C_3}, \mathcal{O}_{C_1}(\ell-1)).
\end{equation}

Thus any morphism $\varphi \in \text{Hom}(\mathcal{O}_{C_1+C_2+C_3}, \mathcal{L}_\ell)$ factors through $\mathcal{O}_{C_1}(\ell-1)$ and hence the induced morphism in $\mathcal{D}$

$$
\varphi_* : \text{Hom}_D^0(\mathcal{O}_{C_1}[-1], \mathcal{O}_{C_1+C_2+C_3}) \to \text{Hom}_D^0(\mathcal{O}_{C_1}[-1], \mathcal{L}_\ell)
$$

factors through $\mathcal{O}_{C_1}(\ell-1)$. Thus the morphism $\varphi_*$ should be zero since the intersection $C_1 \cap C_3$ is empty. Hence the action of $\text{End}_{Z_i}(\oplus_{i \in I} \mathcal{L}_i)$ is contained in diagonal component of $\text{End}_k(V_i)$. \hfill \Box

\textbf{Corollary 3.6.} Let $Z$ be the fundamental cycle of a Kleinian singularity except for $A_n$. Then there is an indecomposable pure sheaf on $Z$ of rank $r$ for any $r \in \mathbb{N}$. In particular the following holds:

$$
\max\{\text{rank}_Z \mathcal{E} \mid \mathcal{E} \text{ is an indecomposable pure sheaf on } Z\} = \infty.
$$

\textbf{Proof.} Let $Z_{4,r}$ be the reduced scheme of the fundamental cycle of the singularity $D_4$. Then $Z_{4,r}$ is a closed subscheme of $Z$.

As in the proof of Theorem 3.3 the universal extension $U_1$ of $\text{Ext}^1_{Z_{4,r}}(\mathcal{O}_{C_4}, \oplus_{i \in I} \mathcal{L}_n)$ is indecomposable pure sheaves on $Z_{4,r}$. The push forward $\iota_* \mathcal{U}$ by the closed embedding $\iota : Z_{4,r} \to Z$ is also a pure sheaf on $Z$. Moreover the push forward $\iota_*$ is a fully faithful functor from $\text{Coh}(Z_{4,r})$ to $\text{Coh}(Z)$ and the full subcategory $\iota_* \text{Coh}(Z_{4,r})$ is closed under direct summands. Thus the assertion holds. \hfill \Box

4. $\mathcal{O}_X$-rigid pure sheaves on $D_n$.

For any closed embedding $f : Z \to X$, the push forward $f_* : \text{Coh}(Z) \to \text{Coh}(X)$ is fully faithful, but the derived push forward $f_* : D(Z) \to D(X)$ is not. To analyze the difference the following lemma is necessary.

\textbf{Lemma 4.1.} Let $f : Z \to X$ be a closed embedding of $Z$ to an algebraic variety $X$. Let $\mathcal{F}$ and $\mathcal{E}$ be sheaves on $Z$. The canonical map $\text{Ext}^1_Z(f_* \mathcal{F}, \mathcal{E}) \to \text{Ext}^1_X(f_* \mathcal{F}, f_* \mathcal{E})$ is injective.

\textbf{Proof.} By the adjunction we have $\text{Ext}^p_Z(f_* \mathcal{F}, f_* \mathcal{E}) \cong \text{Ext}^p_Z(\mathbb{L} f^* f_* \mathcal{F}, \mathcal{E})$ (note that $f$ is an isomorphism). By the canonical morphism $\mathbb{L} f^* f_* \mathcal{F} \to \mathcal{F}$ we have the following distinguished triangle in the derived category $\mathcal{D}$ of coherent sheaves on $Z$:

\[ \mathcal{F} \longrightarrow \mathbb{L} f^* f_* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow [\mathcal{F}][1]. \]

Since $\mathbb{L}^q f^* f_* \mathcal{F} = f^* f_* \mathcal{F} \cong \mathcal{F}$, we see that the $p$-th cohomology (with respect to the standard $t$-structure) of the complex $\mathcal{F}$ vanishes for $p \in \mathbb{Z}_{\geq 0}$. Hence we have $\text{Hom}^p_Z(\mathcal{F}, \mathcal{E}) = 0$ for $q \in \mathbb{Z}_{\leq 0}$.

By taking $\mathbb{R} \text{Hom}_D(-, \mathcal{E})$ to the above sequence we have the following exact sequence:

$$
\text{Hom}_D^0(\mathcal{F}, \mathcal{E}) \longrightarrow \text{Hom}_D^1(\mathcal{F}, \mathcal{E}) \longrightarrow \text{Hom}_D(\mathbb{L} f^* f_* \mathcal{F}, \mathcal{E}) \longrightarrow \text{Hom}_D^1(\mathcal{F}, \mathcal{E}).
$$

Note that the canonical morphism $\text{Ext}^1_Z(\mathcal{F}, \mathcal{E}) \to \text{Ext}^1_X(f_* \mathcal{F}, f_* \mathcal{E})$ is given by $\kappa$. Since $\text{Hom}_D^p(\mathcal{F}, \mathcal{E}) = 0$ the morphism $\kappa$ is injective. \hfill \Box

\textbf{Corollary 4.2.} Let $Z$ be a chain of rational curves in $X$ and let $\{C_i\}_{i=1}^n$ be a set of irreducible components of $Z$. Then for $i \neq j$, we have

$$
\text{Ext}^1_Z(\mathcal{O}_{C_i}(d_i), \mathcal{O}_{C_j}(d_j)) \cong \text{Ext}^1_X(f_* \mathcal{O}_{C_i}(d_i), f_* \mathcal{O}_{C_j}(d_j)).
$$
Proof. A locally free resolution of \( f_\ast \mathcal{O}_{C_i}(d_i) \) is given by

\[
0 \longrightarrow \mathcal{O}_X(D - C_i) \longrightarrow \mathcal{O}_X(D) \longrightarrow 0,
\]
where \( D \) is a divisor on \( X \) such that \( D.C_i = d_i \). Thus \( \mathbb{L}f_\ast \mathcal{O}_{C_i}(d_i) \) is given by the following:

\[
0 \longrightarrow \mathcal{O}_Z(D - C_i) \overset{\delta}{\longrightarrow} \mathcal{O}_Z(D) \longrightarrow 0.
\]

In particular \( \mathbb{L}^{-1}f_\ast \mathcal{O}_{C_i}(d_i) \) is the kernel of \( \delta \) which is isomorphic to \( \mathcal{O}_{C_i}(D - Z) \). Moreover \( \mathfrak{F} \) is isomorphic to \( \mathcal{O}_{C_i}(D - Z)[1] \). Since \( C_i \neq C_j \) we have

\[
\text{Hom}^1_Z(\mathfrak{F}, \mathcal{O}_{C_j}(d_j)) = \text{Hom}^0_Z(\mathcal{O}_{C_i}(D - Z), \mathcal{O}_{C_j}(d_j)) = 0.
\]

This is the desired conclusion. \( \square \)

We first introduce a relation on a collection of sheaves and secondly prove that the relation defines an order:

**Definition 4.3.** Let \( \{N_i\}_{i \in I} \) and \( \{L_j\}_{j \in J} \) be finite collections of isomorphism classes of sheaves on a scheme \( Y \). Suppose that \( \mathcal{F} \) and \( \mathcal{L} \) satisfy the following:

(a) Endomorphism rings \( \text{End}_Y(L_j) \) and \( \text{End}_Y(N_i) \) are generated by the identity for each \( i \) and \( j \).

(b) Each pair \( (L_j, N_i) \) satisfies \( \dim \text{Ext}^1_Y(L_j, N_i) = 1 \).

1. If there is a morphism \( f : L_{j_2} \to L_{j_1} \) such that \( f^* : \text{Ext}^1_Y(L_{j_1}, N_i) \to \text{Ext}^1_Y(L_{j_2}, N_i) \) is nonzero for all \( i \in I \) then we define a relation \( L_{j_1} \leq L_{j_2} \) on \( \{L_j\}_{j \in J} \).

2. Dually if there is a morphism \( g : N_{i_1} \to N_{i_2} \) such that the induced morphism \( g_* : \text{Ext}^1_Y(L_j, N_{i_1}) \to \text{Ext}^1_Y(L_j, N_{i_2}) \) is nonzero for all \( j \in J \) then we define a relation \( N_{i_1} \leq N_{i_2} \) on \( \{N_i\}_{i \in I} \).

**Proposition 4.4.** The relations on \( \{N_i\}_{i \in I} \) and \( \{L_j\}_{j \in J} \) in Definition 4.3 respectively define orders. In particular both are posets.

Proof. Since the proof is similar, it is enough to show the claim for \( \{L_j\}_{j \in J} \). The reflexivity is obvious since the identity gives the identity on \( \text{Ext}^1_Y(L_j, N_i) \).

Suppose \( L_{j_1} \leq L_{j_2} \) and \( L_{j_2} \leq L_{j_3} \). Then there exist morphisms \( f_1 : L_{j_2} \to L_{j_1} \) and \( f_2 : L_{j_1} \to L_{j_2} \). By the condition (b) in Definition 4.3 both \( f_1^\ast \) and \( f_2^\ast \) are isomorphisms. In particular the compositions \( (f_2 \circ f_1)^\ast \) and \( (f_2 \circ f_1)^\ast \) are nonzero morphisms. Thus two morphisms \( f_1 \circ f_2 \in \text{End}(L_{j_1}) \) and \( f_2 \circ f_1 \in \text{End}(L_{j_2}) \) are not zero. By the condition (a) in Definition 4.3 we see that \( f_1 \circ f_2 \) and \( f_2 \circ f_1 \) are identities up to scalar and hence \( L_{j_1} \cong L_{j_2} \).

For the transitivity let us suppose \( L_{j_1} \leq L_{j_2} \) and \( L_{j_2} \leq L_{j_3} \). Similarly as above the composition \( f_1 \circ f_2 \) of two morphisms \( f_1 : L_{j_2} \to L_{j_1} \) and \( f_2 : L_{j_3} \to L_{j_2} \) induces a non-zero morphism \( (f_1 \circ f_2)^\ast : \text{Ext}^1_Y(L_{j_3}, N_i) \to \text{Ext}^1_Y(L_{j_1}, N_i) \). Thus we obtain \( L_{j_1} \leq L_{j_3} \). \( \square \)

**Remark 4.5.** We are interested in the classification of \( \mathcal{O}_X \)-rigid pure sheaves and study the classification in Lemma 4.4 and Proposition 5.1. Any pure sheaf \( \mathcal{E} \) on \( Z \) of a Kleinian singularity is given by a successive extension of pure sheaves on subtrees (see the filtration (4.1) below). The poset structure defined in Proposition 4.4 is convenient to analyze the successive extension.

We are ready to prove our main proposition in this section.

**Proposition 4.6.** Let \( X \) be the minimal resolution of the singularity \( D_n \) and \( Z_r \) be the reduced scheme of the fundamental cycle \( Z \) of \( D_n \). Suppose that \( \mathcal{E} \) is an indecomposable pure sheaf on \( Z_r \). If \( \mathcal{E} \) is \( \mathcal{O}_X \)-rigid then we have \( \text{rank}_{Z_r}(\mathcal{E}) \leq 3. \)
Proof. We have $Z_r = \sum_{i=1}^{n} C_i$ by the definition. Take a pure sheaf $\mathcal{E}$ on $Z_r$ which is not necessarily indecomposable but $O_X$-rigid. We shall show that the rank of each direct summand of $\mathcal{E}$ is at most 3.

Let $\mathcal{F}$ be the kernel of the restriction $\mathcal{E} \to \mathcal{E} \otimes O_{C_4+\cdots+C_n}$. By taking saturation if necessary, we can assume that the sheaf $\mathcal{E}$ fits into the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

where $\mathcal{F}$ is a pure sheaf on $C_1 + C_2 + C_3$ and $\mathcal{G}$ is a pure sheaf on $C_4 + \cdots + C_n$. Both $\mathcal{F}$ and $\mathcal{G}$ are respectively direct sums of invertible sheaves of subtrees by Theorem 4.1 since both trees $C_1 + C_2 + C_3$ and $C_4 + \cdots + C_n$ is the fundamental cycle of respectively $A_3$ and $A_{n-3}$:

$$\mathcal{F} = \bigoplus_{i \in I} \mathcal{N}_i \text{ and } \mathcal{G} = \bigoplus_{j \in J} \mathcal{L}_j.$$ 

Without loss of generality we can assume the following

- The support of each $\mathcal{N}_i$ contains $C_3$.
- The support of each $\mathcal{L}_j$ contains $C_4$ and is connected.

Now we claim the following:

**Lemma 4.7.** Let $\mathcal{M}$ be the collection of direct summands $\{\mathcal{N}_i\}_{i \in I}$ of $\mathcal{F}$ and $\mathcal{L}$ the collection of direct summands $\{\mathcal{L}_j\}_{j \in J}$ of $\mathcal{G}$.

1. Both $\mathcal{M}$ and $\mathcal{L}$ are posets with respect to the relation in Definition 4.3.
2. There exist at most 3 minimal elements in any subposet of $\mathcal{M}$ and the poset $\mathcal{L}$ is totally ordered.

Before the proof of Lemma 4.7, we finish the proof. Note that $\mathcal{E}$ defines a class $[\mathcal{E}]$ in $\bigoplus_{i \in I, j \in J} \text{Ext}^1_{Z_r}(\mathcal{L}_j, \mathcal{N}_i)$ denoted by $V_{IJ}$. Put $m = \# I$ and $n = \# J$. Clearly $V_{IJ}$ can be identified with the set of $m \times n$ matrices and we can write $[\mathcal{E}]$ by a matrix

$$[\mathcal{E}] = \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{m1} & e_{m2} & \cdots & e_{mn} \end{pmatrix}.$$ 

If $\mathcal{N}_{i_1} \leq \mathcal{N}_{i_2}$ and $e_{i_1, j} \neq 0$ then we can assume $e_{i_2, j} = 0$ by row fundamental transformations induced by a morphism $\mathcal{N}_{i_1} \to \mathcal{N}_{i_2}$. Since $\mathcal{M}$ has at most 3 minimal elements by Lemma 4.7, there exists at most 3 components $e_{ij}$ such that $e_{ij} = 1$ in each row.

Similarly if $\mathcal{L}_{j_1} \geq \mathcal{L}_{j_2}$ and $e_{i, j_1} \neq 0$ then we can assume $e_{i, j_2} = 0$ by column fundamental transformations. Since $\mathcal{L}$ is totally ordered by Lemma 4.7, there exists at most 3 components $e_{i'j'}$ such that $e_{i'j'} = 1$ in each column. This means that the rank of each direct summand of $\mathcal{E}$ is at most 3 since $Z_r$ is reduced. □

**Proof of Lemma 4.7.** Let $\iota : Z_r \to X$ be the embedding to the minimal resolution of the singularity. Since $\text{Hom}_{Z_r}(\mathcal{F}, \mathcal{G})$ is zero the push forwards $\iota_* \mathcal{F}$ and $\iota_* \mathcal{G}$ are rigid by [BB13, Lemma 2.5]. Each direct summand of $\mathcal{G}$ is an invertible sheaf on a connected subtree of $C_4 + \cdots + C_n$. Then the order introduced in [IU05, Section 6.1] gives the order in Definition 4.3. In particular $\mathcal{L}$ is totally ordered.

To determine $\mathcal{M}$, similarly as before, take a filtration of $\mathcal{F}$

$$(4.1) \quad 0 \subset \mathcal{F}_1 \subset \mathcal{F}_3 \subset \mathcal{F}_2 = \mathcal{F}$$

such that
• \( \mathcal{F}_1 \) and \( \mathcal{F}_3 \) are pure on respectively \( C_1 \) and \( C_1 + C_3 \),
• \( \mathcal{F}_3/\mathcal{F}_1 \) is pure on \( C_3 \) and
• \( \mathcal{F}_2/\mathcal{F}_3 \) is a pure sheaf on \( C_2 \).

Similarly, \( \mathcal{F}_3/\mathcal{F}_1 \) and \( \mathcal{F}_2/\mathcal{F}_3 \) are rigid by Lemma \cite[11]{4} and \cite[Lemma 2.5]{13}. Since \( \mathcal{F}_1 \) is rigid and pure, there exists an integer \( a_1 \) such that \( \mathcal{F}_1 = O_{C_1}(a_1) \oplus O_{C_1}(a_1 + 1) \). A similar statement applies to \( \mathcal{F}_3/\mathcal{F}_1 \) and \( \mathcal{F}_2/\mathcal{F}_3 \). So there are three integers \( \{a_1, a_2, a_3\} \) such that every summand of \( \mathcal{F} \) is one of the 18 possibilities listed in Table 1.

| \( O_{C_1}(a_3) \) | \( O_{C_1+C_2+C_3}(a_1+1,a_2,a_3+1) \) | \( O_{C_1+C_2+C_3}(a_1+2,a_2,a_3+1) \) | \( O_{C_1+C_2+C_3}(a_1+2,a_2+1,a_3+1) \) |
| --- | --- | --- | --- |
| \( O_{C_1+C_2+C_3}(a_1+1,a_2,a_3+1) \) | | | |
| \( O_{C_1+C_2+C_3}(a_1+2,a_2,a_3+1) \) | | | |
| \( O_{C_1+C_2+C_3}(a_1+2,a_2+1,a_3+1) \) | | | |
| \( O_{C_1+C_2}(a_1,a_2+1,a_3+2) \) | | | |
| \( O_{C_1+C_2}(a_1+2,a_2,a_3+2) \) | | | |
| \( O_{C_1+C_2}(a_1+2,a_2+1,a_3+2) \) | | | |
| \( O_{C_1+C_2}(a_1+3) \) | | | |
| \( O_{C_2+C_3}(a_2,a_3+2) \) | | | |
| \( O_{C_2+C_3}(a_2+1,a_3+2) \) | | | |

Table 1. We denote by \( N_{ij} \) the \( i \)-th column and \( j \)-the row component in the table. For instance, \( N_{31} = O_{C_3}(a_3) \).

We prove the first assertion. Recall that \( \text{Ext}^1_{\mathcal{O}_x}(\mathcal{L}_j, \mathcal{N}_i) \) is isomorphic to \( H^0(\mathcal{O}_x) \) where \( x \in C_3 \cap C_4 \). Since the support of each \( \mathcal{N}_i \) contains \( C_3 \), both \( \mathfrak{M} \) and \( \mathfrak{L} \) satisfy the conditions (a) and (b) in Definition \cite{3}. If \( \text{Hom}(O_{C_1}(d), O_{C_1}(d')) \) is not zero, where \( d \) and \( d' \in \{a_4, a_4+1\} \), then there exists a morphism \( \psi: O_{C_1}(d) \to O_{C_1}(d') \) which induces a non-zero morphism on \( H^0(\mathcal{O}_x) \). Similarly, if \( \text{Hom}(N_{ij_1}, N_{ij_2}) \) is not zero for \( N_{ij_1} \) and \( N_{ij_2} \) in \( \mathfrak{M} \), then there exists a morphism \( \varphi: N_{ij_1} \to N_{ij_2} \) which induces a non-zero morphism on \( H^0(\mathcal{O}_x) \) since the point \( x \) is not in \( (C_1 \cup C_2) \cap C_3 \). Thus \( \mathfrak{M} \) and \( \mathfrak{L} \) are posets and this gives the proof of the first assertion.

To finish the proof of the second assertion (2), let us denote by \( N_{ij} \) the \( i \)-th column and the \( j \)-th row component of Table 1 and put \( \mathfrak{S} = \{N_{ij}\}_{i \in I, j \in J} \). Clearly \( \mathfrak{M} \) is a subposet of \( \mathfrak{S} \).

We see that each column subposet \( \{N_{ij}\}_{i=1}^6 \) is totally ordered \( \{N_{ij} \leq \cdots \leq N_{6j}\} \) \( (j \in \{1, 2, 3\}) \) and each row subposet is also totally ordered \( \{N_{i1} \leq N_{i2} \leq N_{i3}\} \) \( (i \in \{1, 2, 3\}) \). However the poset \( \mathfrak{S} \) is not totally ordered. For instance the pair \( (N_{31}, N_{22}) \) satisfies \( N_{31} \leq N_{22} \) and \( N_{22} \leq N_{31} \) since \( \text{Hom}_{\mathcal{O}_x}(N_{31}, N_{22}) = \text{Hom}_{\mathcal{O}_x}(N_{22}, N_{31}) = 0 \). Thus, there are at most three minimal elements in any subposet of \( \mathfrak{S} \). In particular \( \mathfrak{M} \) has also at most three minimal elements.

**Remark 4.8.** Let \( Z \) be the fundamental cycle of the singularity \( A_n \). Similarly as in the proof of Lemma \cite[14]{7} a pure sheaf \( \mathcal{E} \) on \( Z \) is obtained from an extension on pure sheaves \( \mathcal{F} \) on \( C_1 + \cdots + C_{n-1} \) and \( \mathcal{G} \) on \( C_n \). The sets of direct summands of \( \mathcal{F} \) and \( \mathcal{G} \) are not only posets but also totally ordered sets (see also \cite[Section 6.1]{5}). This is the essential difference between the singularity \( A_n \) and the other Kleinian singularities.

In the rest of this note we show that the inequality in Proposition \cite[16]{7} is best possible by constructing an \( \mathcal{O}_X \)-rigid sheaf. The following lemma is necessary for the construction.

**Lemma 4.9.** Let \( \mathcal{D} \) be a \( k \)-linear triangulated category. Suppose that \( \mathcal{D} \) is CY2. Let \( \mathcal{F} \) and \( \mathcal{G} \) be in the heart \( \mathcal{A} \) of a \( t \)-structure on \( \mathcal{D} \). Consider an extension class \( [\mathcal{E}] \in \text{Hom}_T(\mathcal{G}, \mathcal{F}) \)

\[
\begin{array}{cccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
\end{array}
\]
such that
\[ \text{hom}^0_D(\mathcal{F}, \mathcal{G}) = \text{hom}_D^1(\mathcal{F}, \mathcal{F}) = \text{Hom}^1_D(\mathcal{G}, \mathcal{G}) = 0. \]

Then the following are equivalent.

(a) \( \mathcal{E} \) is rigid.
(b) the vector space \( \text{Hom}_D^1(\mathcal{G}, \mathcal{F}) \) is generated by \( \epsilon^1(\text{End}_D(\mathcal{G})) \) and \( \epsilon^R(\text{End}_D(\mathcal{F})) \)
where \( \epsilon = [\mathcal{E}] \in \text{Hom}_D^1(\mathcal{G}, \mathcal{F}) \) and \( \epsilon^1 \) (resp. \( \epsilon^R \)) is the left (resp. right) composition :

\[
\epsilon^R : \text{End}_D(\mathcal{F}) \to \text{Hom}_D^1(\mathcal{G}, \mathcal{F}), \quad \epsilon^R(f) = f \circ \epsilon \\
\epsilon^L : \text{End}_D(\mathcal{G}) \to \text{Hom}_D^1(\mathcal{G}, \mathcal{F}), \quad \epsilon^L(g) = \epsilon \circ g 
\]

Proof. Since \( \mathcal{F} \) and \( \mathcal{G} \) are rigid by the assumption, we have

\[ \text{hom}^0_D(\mathcal{F}, \mathcal{F}) = \frac{1}{2} \chi(\mathcal{F}, \mathcal{F}) \quad \text{and} \quad \text{hom}^0_D(\mathcal{G}, \mathcal{G}) = \frac{1}{2} \chi(\mathcal{G}, \mathcal{G}). \]

In particular, the following is obvious since \( \mathcal{E} \) is in the heart \( \mathcal{A} \):

\[ \mathcal{E} \text{ is rigid } \iff \text{hom}^0_D(\mathcal{E}, \mathcal{E}) = \frac{1}{2} \chi(\mathcal{E}, \mathcal{E}). \]

By taking \( \text{Hom}_D(\cdot, \mathcal{G}) \) to the sequence \( \text{(4.2)} \), we have \( \text{Hom}_D(\mathcal{G}, \mathcal{G}) \cong \text{Hom}_D(\mathcal{E}, \mathcal{G}) \) and hence

\[ \text{hom}^0_D(\mathcal{E}, \mathcal{G}) = \frac{1}{2} \chi(\mathcal{G}, \mathcal{G}). \]

Similarly we have the following exact sequence:

\[
\begin{array}{c}
0 \longrightarrow \text{Hom}_D^0(\mathcal{G}, \mathcal{F}) \longrightarrow \text{Hom}_D^0(\mathcal{E}, \mathcal{F}) \longrightarrow \text{Hom}_D^1(\mathcal{F}, \mathcal{F}) \\
\text{Hom}_D^1(\mathcal{G}, \mathcal{F}) \longrightarrow \text{Hom}_D^1(\mathcal{E}, \mathcal{F}) \longrightarrow 0.
\end{array}
\]

We remark that \( \text{hom}^2_D(\mathcal{G}, \mathcal{F}) = \text{hom}^0_D(\mathcal{F}, \mathcal{G}) = 0 \) since \( D \) is CY2. Hence the exact sequence gives the following equation

\[ d_0 - d_1 = \frac{1}{2} \chi(\mathcal{F}, \mathcal{F}) + \chi(\mathcal{G}, \mathcal{F}), \]

where \( d_i = \text{hom}_D^i(\mathcal{E}, \mathcal{F}) \) for \( i \in \{0, 1\} \).

By taking \( \mathbb{R} \text{Hom}_D(\mathcal{E}, \cdot) \), we have the following exact sequence:

\[
\begin{array}{c}
0 \longrightarrow \text{Hom}_D^0(\mathcal{E}, \mathcal{F}) \longrightarrow \text{Hom}_D^0(\mathcal{E}, \mathcal{E}) \longrightarrow \text{Hom}_D^0(\mathcal{G}, \mathcal{G}) \longrightarrow \text{Hom}_D^1(\mathcal{E}, \mathcal{F}).
\end{array}
\]

By computation of dimensions and \( \text{(4.4)} \), we see that the surjectivity of \( \delta \) is equivalent to the following

\[ \text{hom}^0_D(\mathcal{E}, \mathcal{E}) = d_0 - d_1 + \frac{1}{2} \chi(\mathcal{G}, \mathcal{G}). \]

By \( \text{(4.5)} \) the equation \( \text{(4.6)} \) is equivalent to the following:

\[
\text{hom}^0_D(\mathcal{E}, \mathcal{E}) = \frac{1}{2} \chi(\mathcal{F}, \mathcal{F}) + \chi(\mathcal{G}, \mathcal{F}) + \frac{1}{2} \chi(\mathcal{G}, \mathcal{G}) \\
= \frac{1}{2} \chi(\mathcal{E}, \mathcal{E}).
\]
Hence $\mathcal{E}$ is rigid if and only if the morphism $\delta$ is surjective by (4.3). Furthermore, the subjectivity of $\delta$ can be understood by the following diagram of exact sequences:

$$
\begin{array}{cccc}
0 & \rightarrow & \text{Hom}^0_D(\mathcal{G}, \mathcal{G}) & \rightarrow & \text{Hom}^0_D(\mathcal{E}, \mathcal{G}) & \rightarrow & 0 \\
& & \downarrow \epsilon^L & & \downarrow \delta & & \\
\text{Hom}^0_D(\mathcal{F}, \mathcal{F}) & \rightarrow & \text{Hom}^0_D(\mathcal{G}, \mathcal{F}) & \rightarrow & \text{Hom}^1_D(\mathcal{E}, \mathcal{F}) & \rightarrow & 0
\end{array}
$$

Since $\pi$ is surjective, $\delta$ is surjective if and only if $\epsilon^L$ is surjective up to the image of $\epsilon^R$. □

**Proposition 4.10.** Let $Z_{4,r}$ be the reduced scheme of the fundamental cycle of $D_4$ and let $Z_r$ be the reduced scheme of the fundamental cycle of the singularity $D_n$.

1. There exists a rank 3 indecomposable pure sheaf on $Z_{4,r}$ which is $O_X$-rigid.
2. The inequality in Proposition 4.8 is best possible. Namely there exists an $O_X$-rigid pure sheaf $\mathcal{E}$ on $Z_r$ with rank $\mathcal{E} = 3$.

**Proof.** We first prove the assertion (1). Let $\iota : Z_{4,r} \rightarrow X$ be the embedding to the minimal resolution of the singularity. Take three pure sheaves $\mathcal{N}_{41} = O_{C_1 + C_3} (a_1 + 1, a_3 + 1), \mathcal{N}_{32} = O_{C_2 + C_3} (a_2, a_3 + 1)$ and $\mathcal{N}_{23} = O_{C_1 + C_2 + C_3} (a_1 + 2, a_2 + 1, a_3 + 1)$ from Table 1 and put $\mathcal{N} = \mathcal{N}_{41} \oplus \mathcal{N}_{32} \oplus \mathcal{N}_{23}$. Consider the universal extension $[\mathcal{U}] \in \text{Ext}^1_{Z_{4,r}} (O_{C_4}, \mathcal{N})$:

$$
\begin{array}{c}
0 \rightarrow \mathcal{N} \rightarrow \mathcal{U} \rightarrow O_{C_4} \rightarrow 0.
\end{array}
$$

We wish to prove that $\mathcal{U}$ is indecomposable and $O_X$-rigid.

The proof of indecomposability is essentially the same as in the proof of Theorem 3.3. It is easily see

(4.7) $\text{Hom}_{Z_{4,r}} (\mathcal{N}_{41}, \mathcal{N}_{23}) = \text{Hom}_{Z_{4,r}} (\mathcal{N}_{32}, \mathcal{N}_{41}) = 0$,

(4.8) $\text{Hom}_{Z_{4,r}} (\mathcal{N}_{41}, \mathcal{N}_{23}) \cong H^0 (O_{C_1 + C_3} (1, -1)) \cong k$ and

(4.9) $\text{Hom}_{Z_{4,r}} (\mathcal{N}_{32}, \mathcal{N}_{23}) \cong H^0 (O_{C_2 + C_3} (1, -1)) \cong k$.

Take non-zero morphisms $\varphi$ and $\varphi'$ respectively in $\text{Hom}_{Z_{4,r}} (\mathcal{N}_{41}, \mathcal{N}_{23})$ and $\text{Hom}_{Z_{4,r}} (\mathcal{N}_{32}, \mathcal{N}_{23})$. Both section $\varphi$ and $\varphi'$ are zero on $C_3$ by (4.3) and (4.9). Moreover, since $\mathcal{N}_{41}$ is left and right orthogonal to $\mathcal{N}_{23}$ by (4.7), the same argument in the proof of Theorem 3.3 shows that $\mathcal{U}$ is indecomposable.

The rigidity of $\iota_* \mathcal{U}$ is a consequence of Lemma 4.9. We first show that $\mathcal{N}$ and $O_{C_4}$ satisfy the assumption in Lemma 4.9. It is enough to show that $\iota_* \mathcal{N}$ is rigid.

Let us denote by $\mathcal{D}$ the derived category $D_Z (X)$. The rigidity of $\iota_* \mathcal{N}$ essentially follows from Riemann-Roch theorem. In fact, by Riemann-Roch theorem, we easily see

(4.10) $\chi (\iota_* \mathcal{N}_{41}, \iota_* \mathcal{N}_{23}) = 0$

(4.11) $\chi (\iota_* \mathcal{N}_{41}, \iota_* \mathcal{N}_{32}) = \chi (\iota_* \mathcal{N}_{32}, \iota_* \mathcal{N}_{23}) = 1$.

By (4.7) and (4.10) we have $\text{Hom}^0_D (\iota_* \mathcal{N}_{41}, \iota_* \mathcal{N}_{32}) = 0$. It is easy to see

(4.12) $\text{Hom}_{Z_{4,r}} (\mathcal{N}_{23}, \mathcal{N}_{41}) \cong H^0 (O_{C_1 + C_3} (-1, 0)) = 0$ and

(4.13) $\text{Hom}_{Z_{4,r}} (\mathcal{N}_{23}, \mathcal{N}_{32}) \cong H^0 (O_{C_2 + C_3} (-1, 0)) = 0$.

Hence we have $\text{Hom}^1_D (\iota_* \mathcal{N}_{23}, \iota_* \mathcal{N}_{41}) = \text{Hom}^1_D (\iota_* \mathcal{N}_{23}, \iota_* \mathcal{N}_{32}) = 0$ by (4.8), (4.9), (4.11), (4.12) and (4.13). Thus we see that $\iota_* \mathcal{N}$ is rigid.
By Corollary 4.12, the push forward $\iota_*\mathcal{U}$ is also a universal extension. Since the projections $p_{ij}: \iota_*\mathcal{N} \to \iota_*\mathcal{N}_{ij}$ to the direct summand of $\iota_*\mathcal{N}$ give a basis $\{[\iota_*\mathcal{U}]^i(p_{ij})\}$ of $\text{Hom}_X^1(\iota_*\mathcal{O}_C, \iota_*\mathcal{N})$, the push forward $\iota_*\mathcal{U}$ is rigid by Lemma 4.13.

For the second assertion (2), let us denote by $j: Z_4 \to Z_r$ the closed embedding. Then the push forward $j_*\mathcal{U}$ is a pure sheaf on $Z_r$ and is $\mathcal{O}_X$ rigid by Lemma 4.11. Thus the second assertion holds.

5. $\mathcal{O}_X$-rigid pure sheaves on $E_{6,7,8}$

**Proposition 5.1.** Let $Z$ be the fundamental cycle of the singularity $E_n$ for $n \in \{6,7,8\}$ and $Z_r$ is the reduced scheme of $Z$. Then the maximal rank of $\mathcal{O}_X$-rigid indecomposable pure sheaves is 3:

$$\max\{\text{rank}_Z \mathcal{E} \mid \mathcal{E} \text{ is an } \mathcal{O}_X \text{-rigid indecomposable pure sheaf} \} = 3.$$

**Proof.** The proof is essentially the same as in Proposition 4.6. Let $\mathcal{F}$ be an $\mathcal{O}_X$-rigid pure sheaf on $Z_r$. By the same argument in the proof of Proposition 4.6, there is a filtration of $\mathcal{F}$

$$0 = F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset F_5 = F$$

such that

- $F_i/F_{i-1}$ is a pure sheaf on $C_i$ for $i \in \{1,2,3,4\}$ and
- $F_5/F_4$ is a pure sheaf on $C_5 + \cdots + C_n$.

The quotient $F_i/F_{i-1}$ is also $\mathcal{O}_X$-rigid since $\mathcal{F}$ is $\mathcal{O}_X$-rigid. Hence for each $i \in \{1,2,3,4\}$ we have

$$F_i/F_{i-1} \cong \mathcal{O}_{C_i}(a_i)^{\oplus n_i} \oplus \mathcal{O}_{C_i}(a_i + 1)^{\oplus n_i}.$$

Moreover we can assume that the support of each $F_i$ contains $C_i$. Hence each direct summand of $F_3$ is one of the Table 2 and each direct summand of $F_4$ supported on $C_4$ is one of Table 3.

Since $F_5$ is pure sheaf on the tree which is isomorphic to the fundamental cycle of $A_{n-4}$, the set of direct summands of $F_5/F_4$ is totally ordered with respect to Definition 4.3. Let $\mathcal{T}$ be the set of sheaves in Table 3. Similarly as in Proposition 4.6 the set $\mathcal{T}$ is a poset. Furthermore each column set $\{N_{ij}\}_{j=1}^{i}$ and each row set $\{N_{ij}\}_{i=1}^{14}$ are totally ordered. Hence any subposet of $\mathcal{T}$ has at most 3 minimal elements. Thus the maximum of the rank of indecomposable pure sheaves on $Z_r$ is at most 3.

Let $\mathcal{U}$ be a pure sheaf constructed in the proof of Proposition 4.10 and $\iota: Z' \to Z_r$ be the closed embedding where $Z' = \sum_{j=2}^{5} C_j$. Then the push forward $\iota_*\mathcal{U}$ gives an indecomposable pure sheaf on $Z_r$ after the change of indexes $(1,2,3,4) \mapsto (2,4,3,5)$.

The sheaf $\iota_*\mathcal{U}$ is $\mathcal{O}_X$-rigid by Lemma 4.11. Thus the opposite inequality holds.

**Proof of Theorem 5.3.** If the singularity is $D_n$ then the inequality holds and best possible by Propositions 4.6 and 4.10. The case of $E_6, E_7$ or $E_8$ follows form Proposition 5.1.

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Autoequivalences of Derived Categories on the Minimal Resolutions of

\[ \text{Table 2.} \]

| \( \mathcal{O}(a_1 - 1, a_2 + 1, a_3) \) | \( \mathcal{O}(a_1 - 1, a_2 + 1, a_3 + 1, a_4 + 1) \) | \( \mathcal{O}(a_1 - 1, a_2 + 1, a_3 + 1, a_4 + 2) \) |
| --- | --- | --- |
| \( \mathcal{O}(a_2 + 1, a_3) \) | \( \mathcal{O}(a_2 + 1, a_3 + 1, a_4) \) | \( \mathcal{O}(a_2 + 1, a_3 + 1, a_4 + 1) \) |
| \( \mathcal{O}(a_1 - 1, a_2 + 2, a_3) \) | \( \mathcal{O}(a_1 - 1, a_2 + 2, a_3 + 1, a_4) \) | \( \mathcal{O}(a_1 - 1, a_2 + 2, a_3 + 1, a_4 + 1) \) |
| \( \mathcal{O}(a_1, a_2 + 2, a_3) \) | \( \mathcal{O}(a_1, a_2 + 2, a_3 + 1, a_4) \) | \( \mathcal{O}(a_1, a_2 + 2, a_3 + 1, a_4 + 1) \) |
| \( \mathcal{O}(a_2 + 2, a_3) \) | \( \mathcal{O}(a_2 + 2, a_3 + 1, a_4) \) | \( \mathcal{O}(a_2 + 2, a_3 + 1, a_4 + 1) \) |
| \( \mathcal{O}(a_3) \) | \( \mathcal{O}(a_3 + 1, a_4) \) | \( \mathcal{O}(a_3 + 1, a_4 + 1) \) |

| \( \mathcal{O}(a_1 - 1, a_2 + 1, a_3 + 1) \) | \( \mathcal{O}(a_1 - 1, a_2 + 1, a_3 + 2, a_4) \) | \( \mathcal{O}(a_1 - 1, a_2 + 1, a_3 + 2, a_4 + 1) \) |
| \( \mathcal{O}(a_1, a_2 + 1, a_3 + 1) \) | \( \mathcal{O}(a_1, a_2 + 1, a_3 + 2, a_4) \) | \( \mathcal{O}(a_1, a_2 + 1, a_3 + 2, a_4 + 1) \) |
| \( \mathcal{O}(a_2 + 1, a_3 + 1) \) | \( \mathcal{O}(a_2 + 1, a_3 + 2, a_4) \) | \( \mathcal{O}(a_2 + 1, a_3 + 2, a_4 + 1) \) |
| \( \mathcal{O}(a_1 - 1, a_2 + 2, a_3 + 1) \) | \( \mathcal{O}(a_1 - 1, a_2 + 2, a_3 + 2, a_4) \) | \( \mathcal{O}(a_1 - 1, a_2 + 2, a_3 + 2, a_4 + 1) \) |
| \( \mathcal{O}(a_1, a_2 + 2, a_3 + 1) \) | \( \mathcal{O}(a_1, a_2 + 2, a_3 + 2, a_4) \) | \( \mathcal{O}(a_1, a_2 + 2, a_3 + 2, a_4 + 1) \) |
| \( \mathcal{O}(a_2 + 2, a_3 + 1) \) | \( \mathcal{O}(a_2 + 2, a_3 + 2, a_4) \) | \( \mathcal{O}(a_2 + 2, a_3 + 2, a_4 + 1) \) |
| \( \mathcal{O}(a_3 + 1) \) | \( \mathcal{O}(a_3 + 2, a_4) \) | \( \mathcal{O}(a_3 + 2, a_4 + 1) \) |

For simplicity we denote by \( \mathcal{O}(b_1, b_2, b_3, b_4) \) an invertible sheaf on \( C_1 + C_2 + C_3 + C_4 \) whose degrees are respectively \( b_i \) on \( C_i \).

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