On Cardy states in the (2,2,2,2) Gepner model

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Abstract

We study Cardy states in the (2,2,2,2) Gepner model from both the algebraic and geometric vantage points. We present the full list of primaries of this model together with their characters. The effects of fixed point resolution are analyzed. Annulus partition functions between various Cardy states are calculated. Using the equivalent description in terms of the $T^4/Z_4$ orbifold, the corresponding geometric realization is partially found.

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1 Introduction and summary

The latest developments in String Theory demonstrated the importance of understanding properties of D-branes in curved backgrounds. Despite widespread effort our knowledge of D-branes properties is still limited to the simplest backgrounds, like tori or toroidal orbifolds, group manifolds etc. It turned out that in most applications of D-branes to string theory detailed understanding of the D-branes on more complicated backgrounds, first of all Calabi-Yau manifolds, is necessary. Unfortunately for general Calabi-Yau manifolds not much is known on special Lagrangian submanifolds or holomorphic cycles wrapped by A- and B-type branes respectively.

However it is known that at certain points in their moduli spaces, Calabi-Yau compactifications can be described by rational conformal field theories, known as Gepner models [1]. By now a sophisticated technique for constructing boundary states on rational conformal field theories exists [2–11]. Starting from the pioneering papers [12,13], a number of papers [14–28] were devoted to boundary states in Gepner models. Much of the work was devoted to figure out geometrical properties of branes given by specific boundary states. These papers also brought to light that in the presence of minimal models with even levels some of the boundary states defined in [13] should be modified or, better, resolved in order to dispose of fixed point ambiguities [23], [24]. General formulae for boundary states in conformal field theories with simple current modular invariants have been given in [29,30].

As usual before tackling complicated cases some simple models were analyzed. In [14] the \((1,1,1) \sim (1,4)\) and \((2,2)\) models, which are equivalent to \(T^2\) compactifications at the \(SU(3)_1\) and \(SO(4)_1\) enhancement points respectively, were discussed. In [28] B-branes in the \((2,2,2,2)\) model, which admits orbifold description as \(T^4/Z_4\) [31], [32], were discussed.

In this paper we study in depth the Cardy states corresponding to \(D0\) branes in the \((2,2,2,2)\) model.

The paper is organized as follows.

In section 2 we review necessary background material on the simple current extensions.

In section 3 we review Gepner models via simple current extension formalism.

In section 4 we write down all the necessary information on the \((2,2,2,2)\) model: orbit representatives, characters, conformal weights. Using the resolved
characters we compute the torus partition function and show that it coincides with the one computed in the appendix C as an orbifold partition function at the $SU(2)^4$ point. Using the general formulae of section 2 we also derive the annulus partition functions between different Cardy states, paying special attention to the peculiarities caused by the presence of the fixed points.

In section 5 we study $D0$ branes on the orbifold $T^4/Z_4$. We compute all the annulus partition functions between $D0$ branes located at points in $T^4/Z_4$ orbifold that are fully or partially fixed under the orbifold group action. Using previously derived formulae for the annulus partition functions between Cardy states of the $(2,2,2,2)$ model we establish a partial dictionary between Cardy states and $D0$ branes. This is the main result of this paper.

The necessary formulae on theta functions are reviewed in appendices A and B.

2 Simple current extension: brief review

Let us briefly remind the meaning of the simple current extension by simple currents of integral conformal weight [33–37], [24]. A primary $J$ is called a simple current if, fused with any other primary $\lambda$, it yields just a single field $J\lambda$. Simple current extension means the combination of two operations:

- Projection. We keep only fields which obey $Q_J(\lambda) = 0$ where

$$Q_J(\lambda) = \Delta_\lambda + \Delta_J - \Delta_{J\lambda} \pmod{Z} \quad (1)$$

- Extension. We extend the chiral algebra by including the simple current $J$. This means that we organize the fields surviving the projection into orbits derived as a result of fusion with the simple current $J$.

Before writing the torus partition function we should discuss the important issue of fixed point resolution. If all the primaries form orbits of the same length, equal to the order $|G|$ of the full group $G$ generated by the simple currents, or in other words have the same number of images under the repeated fusion with the simple current, the characters could be labelled by the primaries chosen, one from each orbit, called orbit representatives, and have the form:

$$\tilde{\chi}_\lambda = \sum_{J \in G} \chi_{J\lambda} \quad (2)$$
The unitary matrix representing modular transformations on the extended theory is:

\[ \tilde{S}_{\hat{a},\hat{b}} = |G|S_{ab} \tag{3} \]

where with hatted variables we denoted the orbit representatives. However it may happen that some of the primaries have a non-trivial stabilizer \( S_\lambda \), i.e. be fixed under the action of currents of a subgroup \( S_\lambda \subset G \). In this case the freely acting group is the factor \( G_a = G/S_a \) and the orbit length is given by

\[ |G_a| = \frac{|G|}{|S_a|} \tag{4} \]

and therefore varies from orbit to orbit. The simple formula (3) for the modular transformation matrix does not work anymore. It turns out that in order to construct a unitary matrix representation of the modular transformation in this case one needs to resolve the primaries with non-trivial stabilizer, i.e. one should consider together with the orbit \( \hat{a} \) additional \(|S_a|\) orbits\(^1\). Labelling the additional orbits by \( i \) we find the characters:

\[ \tilde{\chi}_{\lambda,i} = m_{\lambda,i} \sum_{J \in G/S_\lambda} \chi_{J\lambda} = \frac{m_{\lambda,i}}{|S_\lambda|} \sum_{J \in G} \chi_{J\lambda} \tag{5} \]

where \( m_{i,a} \) are usually equal to 1, but we keep them explicitly so as to keep track of the different resolved orbits.

The diagonal modular invariant torus partition function of the extended theory reads

\[ Z_{\text{ext}} = \sum_{\lambda,i} \left| \tilde{\chi}_{\lambda,i} \right|^2 = \sum_{\text{orbits } Q(\lambda)=0} |S_\lambda| \cdot \sum_{J \in G/S_\lambda} \left| \chi_{J\lambda} \right|^2 \tag{6} \]

where we used that

\[ |S_a| = \sum_i (m_{a,i})^2 \tag{7} \]

The unitary matrix representation of the modular transformation \( S \) on the characters (5), was constructed in [33], [35]. The following ansatz was suggested

\[ \tilde{S}_{(a,i),(b,j)} = m_{a,i}m_{b,j} \frac{|G_a||G_b|}{|G|}S_{a,b} + \Gamma_{(a,i),(b,j)} \tag{8} \]

\(^1\)Actually each primary should be resolved by the order of the subgroup \( U_a \) of the stabilizer, called untwisted stabilizer [35], on which a certain alternating \( U(1) \)-valued bihomomorphism, or discrete torsion, on the stabilizer \( S_a \) vanishes. It is well-known that discrete torsions are classified by the second \( U(1) \)-valued cohomology group \( H^2(S_a, U(1)) \) [38], and since in Gepner models with diagonal (or charge conjugation) torus partition function – the situation of our interest below – the stabilizers are all isomorphic to the \( \mathbb{Z}_2 \) group, for which \( H^2(\mathbb{Z}_2, U(1)) = 0 \), one finds that the untwisted stabilizer coincides with stabilizer.
where $\Gamma_{(a,i),(b,j)}$ satisfies the equation
\[
\sum_j \Gamma_{(a,i),(b,j)} m_{b,j} = 0 \quad (9)
\]
and it is therefore different from zero only between fixed points. It was found in [33] that unitarity requires $\Gamma_{(a,i),(b,j)}$ to satisfy the condition:
\[
\sum_{\text{orbits } Q(b)=0} \Gamma_{(a,i),(b,j)} \Gamma^*_{(c,k),(b,j)} = \delta_{ac} \left( \delta_{ik} - \frac{m_{a,i} m_{a,k}}{|S_a|} \right). \quad (10)
\]

The derivation of equation (10) is reviewed in appendix D.

Using the matrix (8) one can compute the fusion rule coefficients using Verlinde formula and the annulus partition functions for the Cardy states. After some algebra, reviewed in appendix D, we arrive at the expression:

\[
A_{(a,i),(d,e)} = \sum_{\text{orbits } Q(c)=0} \sum_J \frac{m_{a,i} m_{d,e} N_{J,a,c}^d}{|S_a||S_d|} \sum_{K \in G_c} \chi_{Kc} 
+ \sum_{\text{orbits } Q(c)=0} \sum_{\text{orbits } Q(b)=0,j} \frac{\Gamma_{(a,i),(b,j)} \Gamma^*_{(c,k),(b,j)} S_{c,b} \Gamma^*_{(b,j),(d,e)}}{S_{0,b} \sum_{K \in G_c} \chi_{Kc}} \quad (11)
\]

Given that the resolving matrix $\Gamma_{(a,i),(b,j)}$ are different from zero only between fixed points we observe that formula (11) simplifies if one of the states is not fixed. When $a$ is not fixed and $d$ fixed (11) simplifies to

\[
A_{(a),(d,e)} = \sum_{\text{orbits } Q(c)=0} \sum_J \frac{m_{d,e} N_{J,a,c}^d}{|S_d|} \sum_{K \in G_c} \chi_{Kc} \quad (12)
\]

When neither $a$ nor $d$ are fixed (11) further simplifies to

\[
A_{ad} = \sum_{\text{orbits } Q(c)=0} \sum_J N_{J,a,c}^d \sum_{K \in G_c} \chi_{Kc} \quad (13)
\]

For later application to Gepner models let us discuss the matrix $\Gamma_{(a,i),(b,j)}$ and the second term in (11) in the case when all the fixed points have a stabilizer isomorphic to $\mathbb{Z}_2$. In this case equations (9) and (10) can be satisfied by taking $\Gamma_{(a,i),(b,j)}$ in the form:
\[
\Gamma_{(a,\psi),(b,\psi')} = \frac{|G_a||G_b|}{|G|} S_{ab} \psi \psi' \delta_{aj} \delta_{bf} \quad (14)
\]
where \( \psi \) is the resolving index which takes two values \( \pm \), and \( \hat{S}_{ab} \) is a unitary matrix. Plugging (14) in (11) for the the second term one can write:

\[
\frac{1}{|S_a| |S_d|} \sum_{\text{Orbits } Q(c)=0} \sum_b \sum_{J \in G} \hat{S}_{Ja,b} S_{c,b} \hat{S}_{b,d} \left( \sum_{K \in G_c} \chi_{Kc} \right) \delta_{af} \delta_{bf} \delta_{df} \tag{15}
\]

We also show in appendix D that formulae (13) and (11) are actually equivalent to the formulae for the A-type annulus partition functions derived in [13] and [23].

### 3 Gepner models: generalities

Let us remind the basic facts about Gepner models [1]. The starting point of a Gepner model is the tensor product theory

\[
\mathcal{C}^{s-t}_{k_1,\ldots,k_n} = \mathcal{C}^{s-t} \otimes \mathcal{C}_{k_1} \otimes \cdots \otimes \mathcal{C}_{k_n}, \tag{16}
\]

where \( \mathcal{C}^{s-t} \) is the \( D \) dimensional flat space-time part, and \( \mathcal{C}_k \) is one of the \( N = 2 \) minimal models, whose central charges \( c_k = \frac{3k}{k+2} \) satisfy the relation

\[
\sum_{i=1}^{n} c_{k_i} + \frac{3}{2}(D-2) = 12 \tag{17}
\]

\( N = 2 \) minimal models can be described as cosets \( SU(2)_k \times U(1)_4/U(1)_{2k+1} \). Accordingly the primaries of \( \mathcal{C}_k \) are labelled by three integers \( (l, m, s) \) with ranges \( l \in (0, \ldots, k) \), \( m \in (-k-1, \ldots, k+2) \), \( s \in (-1, 0, 1, 2) \), subject to the selection rule \( l + m + s \in 2\mathbb{Z} \) and the field identification \( (l, m, s) \equiv (k-l, m+k+2, s+2) \). Primaries with even values of \( s \) belong to the NS sector, while primaries with odd values of \( s \) belong to the R sector. The conformal dimension and charge of the primary \( (l, m, s) \) are given by:

\[
h_{l,m,s}^l = \frac{l(l+2) - m^2}{4(k+2)} + \frac{s^2}{8} \pmod{1} \tag{18}
\]

\[
q_{l,m,s}^l = \frac{m}{k+2} - \frac{s}{2} \pmod{2} \tag{19}
\]

The exact dimensions and charges can be read off (18) and (19) using field identifications to bring \( (l, m, s) \) into the standard range

\[
l \in (0, \ldots, k), \quad |m - s| \leq l, \quad s \in (-1, 0, 1, 2) \tag{20}
\]
The characters are given by

\[ \chi_{m,s}^{l(k)}(z,\tau) = \sum_{j=0}^{k-1} c_{m+4j-s}^{l(k)}(\tau) \Theta_{2m+(4j-s)(k+2),2k(k+2)}(z,\tau) \] (21)

where

\[ \Theta_{M,N}(z,\tau) = \theta\left[ \frac{M}{2N} \right] (z,2N\tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n N (n+M/2N)^2} e^{2\pi i z (n+M/2N)} \] (22)

that obviously satisfy the identity

\[ \Theta_{M+2N,N} = \Theta_{M,N} \] (23)

and \( c_{m}^{l(k)} \) are the characters of the parafermionic field theory at level \( k \), satisfying identities:

\[ c_{m}^{l(k)} = c_{-m}^{l(k)} = c_{m+2k}^{l(k)} = c_{k+m}^{l(k)} \] (24)

The fusion coefficients are

\[ \mathcal{N}_{m_1m_2m_3s_1s_2s_3}^{\text{N=2}} l_1l_2l_3 = \]

\[ \mathcal{N}_{l_1l_2}^{\text{SU(2)}} \delta_{m_1-m_2+m_3} \delta_{s_1-s_2-s_3} + \mathcal{N}_{l_1l_2}^{\text{SU(2)}} k-l_3 \delta_{m_1+m_2-(m_3+k+2)} \delta_{s_1-s_2-(s_3+2)} \] (25)

The space-time part can be described in terms of the \( SO(D-2)_1 \) algebra. \( SO(2n)_1 \) algebras have four primaries \( \lambda = (o,v,s,c) \), with conformal dimensions

\[ h_o = 0, \quad h_v = \frac{1}{2}, \quad h_s = h_c = \frac{n}{8} \] (26)

charges

\[ q_o = 0, \quad q_v = 1, \quad q_s = \frac{n}{2}, \quad q_c = \frac{n}{2} - 1 \] (27)

and characters:

\[ \chi_{SO(2n)}^O = \frac{1}{2\eta^n} (\theta_3^n + \theta_4^n) \] (28)

\[ \chi_{SO(2n)}^V = \frac{1}{2\eta^n} (\theta_3^n - \theta_4^n) \]

\[ \chi_{SO(2n)}^s = \frac{1}{2\eta^n} (\theta_2^n + i^{-n}\theta_1^n) \]

\[ \chi_{SO(2n)}^c = \frac{1}{2\eta^n} (\theta_2^n - i^{-n}\theta_1^n) \]

\( O \) and \( V \) primaries belong to the NS sector, while \( S \) and \( C \) belong to the R sector. For future use, let us write down also the fusion rules of the \( SO(2n)_1 \) algebras.
The primaries of the product theory \[^{[16]}\] can be labelled by the following collection of indices

\[(\lambda, \vec{l}, \vec{m}, \vec{s}) = (\lambda, l_1, m_1, s_1, \cdots, l_n, m_n, s_n)\] \[\text{(31)}\]

The Gepner model is the simple current extension of the product \(C_{k_1,\cdots,k_n}^{s-t}\), with the following simple currents:

- supersymmetry current: \(S_{\text{tot}} = (s, (0, 1, 1), \cdots (0, 1, 1))\)
- alignment currents: \(V_i = (v, \cdots (0, 0, 2) \cdots )\), with \((0, 0, 2)\) at the \(i\)th position.

Let us summarize the results of applying the formalism reviewed in the previous section to Gepner models \[^{[34]}, [23], [24]\]. In Gepner models the simple current projection or, in the original Gepner’s language, \(\beta\)-projection with respect to the supersymmetry current \(S_{\text{tot}}\) reads

\[Q_{(\omega, \vec{l}, \vec{m}, \vec{s})} = q_\omega + \sum_{i=1}^{n} q_{l_i, m_i, s_i} = 1 \mod 2\mathbb{Z}\] \[\text{(32)}\]

and is nothing else than the famous GSO projection yielding space-time supersymmetry\[^{[3]}\]. The projection with respect to the alignment current selects only

\[^{2}\text{Actually direct application of the formula }^{[11]}\text{ brings to shift } 1 \text{ with respect to }^{[22]}, \text{ but as explained in }^{[23]} \text{ and }^{[24]} \text{ the shift is absorbed by the superghost part, or alternatively by the bosonic string map.}\]
primaries were all constituent primaries belong to the same sector, either NS either R and guarantees world-sheet supersymmetry.

To analyze the length of the orbits we should consider two cases:

1. all the levels \( k_i \) are odd

   In this case no fixed point occurs, all the primaries have trivial stabilizer, and the length of the \( S_{\text{tot}} \) current is \( K = \text{lcm}\{4, 2k_i + 4\} \). All \( V_i \) currents always act freely and have length 2. But when all the \( k_i \) levels are odd, it turns out that the \( S_{\text{tot}} \) current has an overlap with the \( V_i \) currents, and to cover all orbit it is enough to sum over only \( n - 1 \) of the \( n V_i \) currents. As a result, the orbit length in this case is \( 2^{n-1}K \).

2. one has \( r \neq 0 \) even levels \( k_i \)

   Let us place the even levels in the first \( r \) positions. In this case for a generic primary the orbit length of the supersymmetry current is again \( K = \text{lcm}\{4, 2k_i + 4\} \). But for the primaries with all \( l_i \) at the first \( r \) positions equal \( \frac{k_i}{2} \):

\[
l_i = \frac{k_i}{2} \quad i = 1, \ldots, r
\]

(33)

due to the previously discussed field identification, which for them reads:

\[
\left( \frac{k_1}{2}, m_1, s_1, \ldots, \frac{k_r}{2}, m_r, s_r, l_{r+1}, m_{r+1}, s_{r+1} \ldots, l_n, m_n, s_n \right) \equiv \quad (34)
\]

\[
\left( \frac{k_1}{2}, m_1 + k_1 + 2, s_1 + 2, \ldots, \frac{k_r}{2}, m_r + k_r + 2, s_r + 2, l_{r+1}, m_{r+1}, s_{r+1} \ldots, l_n, m_n, s_n \right)
\]

there is a non-trivial stabilizer:

\[
S_{\vec{m}, \vec{s}}^{\frac{k_1}{2}, \ldots, \frac{k_r}{2}, l_{r+1}, \ldots, l_n} = Z_2.
\]

(35)

We see that the stabilizer depends only on the values of \( l_i \)'s \( i = 1, \ldots, r \) and one can write:

\[
|S_{\vec{l}_{1}, \ldots, \vec{l}_{r}}| = 1 + \delta_{l_1 \frac{k_1}{2}} \cdots \delta_{l_r \frac{k_r}{2}}
\]

(36)

Therefore here we have two kinds of orbits, long orbits with length \( 2^nK \) for generic primary, and short orbits with length \( 2^{n-1}K \) for primaries of type (33). As we explained the short orbits should be resolved and acquire an additional label \( \psi \) taking two values, which we choose to be a sign \( \psi = \pm \).
4 The (2,2,2,2) Gepner model

From now on we will specialize to the case of the (2, 2, 2, 2) Gepner model, that corresponds to a compactification down to six dimensions. The flat part is described by an $SO(4)_1$ algebra. In order to write down the characters of the model, first of all we note that using the fusion rules (30) one can check that the subgroup generated by the currents $S^2_{\text{tot}}$ and $V_i V_j$ has trivial action on the space-time part. The length of the $S^2_{\text{tot}}$ current is $\frac{K}{2} = 4$. Using (32) we find it convenient to choose the primaries in the form
\[
\{v, (l_1, m_1, s_1), \ldots, (l_n, m_n, s_n)\}
\]
with prescribed space-time part $v$, and neutral internal part, i.e.
\[
\sum_{i=1}^{4} q^{l_i}_{m_i, s_i} = 0 \pmod{2Z} \tag{37}
\]
Now one can express the Gepner extension characters $\chi^{G\vec{l}}_{(\vec{m}, \vec{s})}$ in the form
\[
\chi^{G\vec{l}}_{(\vec{m}, \vec{s})} = \frac{1}{|S^F_{\vec{m}, \vec{s}}|} (\mathcal{X}_v - \mathcal{X}_c + \mathcal{X}_o - \mathcal{X}_s) \tag{38}
\]
where
\[
\mathcal{X}_v = \chi^{SO(4)}_v \frac{A(m_1, s_1, m_2, s_2, m_3, s_3, m_4, s_4)}{\eta^4} \tag{39}
\]
\[
\mathcal{X}_c = \chi^{SO(4)}_c \frac{A(m_1 + 1, s_1 + 1, m_2 + 1, s_2 + 1, m_3 + 1, s_3 + 1, m_4 + 1, s_4 + 1)}{\eta^4} \tag{40}
\]
\[
\mathcal{X}_o = \chi^{SO(4)}_o \frac{A(m_1 + 1, s_1 + 2, m_2, s_2, m_3, s_3, m_4, s_4)}{\eta^4} \tag{41}
\]
\[
\mathcal{X}_s = \chi^{SO(4)}_s \frac{A(m_1 + 1, s_1 + 3, m_2 + 1, s_2 + 1, m_3 + 1, s_3 + 1, m_4 + 1, s_4 + 1)}{\eta^4} \tag{42}
\]
with
\[
A(m_i, s_i) = \sum_{\nu_0 = 0}^{3} \sum_{\nu_1 = 0}^{2} \sum_{\nu_2 = 0}^{2} \sum_{\nu_3 = 0}^{2} \chi^{(m_1 + 2\nu_0, s_1 + \nu_1 + \nu_2 + \nu_3 + 2\nu_3)}_{m_1 + 2\nu_0, s_1 + \nu_1 + \nu_2 + \nu_3 + 2\nu_3} (z_1) \cdot \chi^{m_2 + 2\nu_0, s_2 + \nu_1 + 2\nu_3)}_{m_2 + 2\nu_0, s_2 + \nu_1 + 2\nu_3} (z_2) \cdot \chi^{m_3 + 2\nu_0, s_3 + \nu_2 + 2\nu_3)}_{m_3 + 2\nu_0, s_3 + \nu_2 + 2\nu_3} (z_3) \cdot \chi^{m_4 + 2\nu_0, s_4 + \nu_3 + 2\nu_3)}_{m_4 + 2\nu_0, s_4 + \nu_3 + 2\nu_3} (z_4) \tag{43}
\]
and, as explained above,
\[
|S^F_{\vec{m}, \vec{s}}| = 1 + \delta_{l_1} \delta_{l_2} \delta_{l_3} \delta_{l_4} \delta_{l_{12}} \tag{44}
\]
Using (21), (24) and (97) for the characters of the $k = 2$ minimal model one obtains the following simple expression

$$\chi^{(2)}_{m,s}(z) = c^{(2)}_{m-s}(\tau) \Theta_{4q}(z)$$

where $q = m_4 - s_2$, and $c^{(2)}_{m}$ are related to the Ising characters:

$$c^{0(2)}_{0} = \frac{1}{2\eta} \left( \sqrt{\frac{\theta_3}{\eta}} + \sqrt{\frac{\theta_4}{\eta}} \right)$$

$$c^{0(2)}_{2} = c^{0(2)}_{2} = \frac{1}{2\eta} \left( \sqrt{\frac{\theta_3}{\eta}} - \sqrt{\frac{\theta_4}{\eta}} \right)$$

$$c^{1(2)}_{1} = \frac{1}{\eta} \sqrt{\frac{\theta_2}{2\eta}}$$

Now let us compute $A(m_i, s_i)$. Repeatedly using theta functions product formulae from appendix A, we have

$$A(m_i, s_i) = \Theta_{q_{tot}, 1} \left( \frac{z_{tot}}{8}, \tau \right) B(m_i, s_i)$$

where

$$z_{tot} = z_1 + z_2 + z_3 + z_4$$

$$q_{tot} = q_1 + q_2 + q_3 + q_4 = \text{even}$$

and

$$B(m_i, s_i) = \sum_{\nu_1, \nu_2, \nu_3 = 0, 2} c^{l_1(2)}_{m_1 - (s_1 + \nu_1 + \nu_2 + \nu_3)} c^{l_2(2)}_{m_2 - (s_2 + \nu_1)} c^{l_3(2)}_{m_3 - (s_3 + \nu_2)} c^{l_4(2)}_{m_4 - (s_4 + \nu_3)} \cdot \sum_{a=0, 2} \Theta_{(q_1 - q_2 + q_3 - q_4) - \nu_2 + a, 2}(y_1, 2\tau) \cdot \Theta_{(q_1 + q_2 - q_3 - q_4) - \nu_1 + a, 2}(y_3, 2\tau)$$

where

$$y_1 = \frac{z_1 - z_2 + z_3 - z_4}{4}, \quad y_2 = \frac{z_1 - z_2 - z_3 + z_4}{4}, \quad y_3 = \frac{z_1 + z_2 - z_3 - z_4}{4}$$
Note that $B(m, s) = B(m + 1, s + 1)$. Using \((46), (28), (100), (102)\), this allows us to write for (38):

$$
\chi_{G\vec{l}}(\vec{m}, \vec{s}) = \frac{1}{\eta^6 |S_{\vec{m}, \vec{s}}^l|} \left[ \left( \frac{\theta_2^2(2\tau)}{8} B(m, s) - \theta_2(2\tau) \theta_3(2\tau) B(m, s + 1) \right) + \left( \frac{\theta_3^2(2\tau)}{8} B(m + 1, s + 1) - \theta_2(2\tau) \theta_3(2\tau) B(m + 1, s + 2) \right) \right]
$$

We see that whenever

$$z_1 + z_2 + z_3 + z_4 = 0$$

the Gepner extension characters are supersymmetric. This plays a role in the study of coisotropic/magnetized D-branes and in the computations of threshold connections [39], [40].

From now on we put all $z_i = 0$. For this case the character \((51)\) can be equivalently written as

$$
\chi_{G\vec{l}}^{(\vec{m}, \vec{s})} = \frac{\mathcal{J}}{2|S_{\vec{m}, \vec{s}}^l|\eta^6} \left( \frac{B(m, s)}{\theta_3(0, 2\tau)} + \frac{B(m + 1, s + 2)}{\theta_2(0, 2\tau)} \right)
$$

where $\mathcal{J} = \frac{1}{2}(\theta_3^4(0, \tau) - \theta_2^4(0, \tau) - \theta_1^2(0, \tau))$ is zero thanks to Jacobi \textit{aequatio identica satis abstrusa}. Using \((49)\) and taking into account that

$$
\Theta_{\nu, 2}(z, \tau) = \eta \nu_{SO(2)}^{SO(2)}(\frac{\vec{z}}{2}, \tau)
$$

as well as \((43), (44), (45)\), we are now in a position to compute the characters for the various orbits.

To this end, we are going to present all the primaries of the model, or in other words to list all the orbit representatives. Surely one can pick up orbit representatives in many different ways. To be sure that we have not taken two primaries, belonging to the same orbit, one can resort to some kind of “gauge fixing”. The gauge fixing chosen here, is the following.

1. We take the space-time part to be always $\nu$, as mentioned above.
2. we take $s_2 = s_3 = s_4 = 0$
3. we limit $m_1$ to the values 0 and 1.
4. to avoid taking primaries equivalent due to field identification, we always limit the values of the $l_i$ to be 0 or 1.
The final picture is the following.

In this model we can divide primaries in 4 big groups.

The first group has \( l_1 = l_2 = l_3 = l_4 = 0 \), \( s_1 = s_2 = s_3 = s_4 = 0 \) and contains 16 primaries. We can divide them into three groups: \( K_1 \), \( K_2 \) and \( K_3 \). All primaries in the same group have the same conformal weights and characters. The results are presented in the tables below. It is understood that all the entries should be multiplied by \( \frac{\eta}{\tau} \).

The table for \( K_1 \) is as follows:

| \( K_1 \) | \( h_{K_1} = \frac{1}{2} \) |
|---|---|
| \( K_1 = (v)(0,0,0)(0,0,0)(0,0,0)(0,0,0) \) | \( \chi_{K_1}^G = \frac{\theta_1^2(0,\tau)+\theta_1^4(0,\tau)}{16} + 3\frac{\theta_2(0,\tau)\theta_4(0,\tau)}{8} \) |

The table for \( K_2 \) is as follows:

| \( K_2 \) | \( h_{K_2} = 1 \) |
|---|---|
| \( K_{2a} = (v)(0,0,0)(0,0,0)(0,-2,0)(0,2,0) \) | \( \chi_{K_2}^G = \frac{\theta_1^4(0,\tau)-\theta_4^4(0,\tau)}{16} \) |
| \( K_{2b} = (v)(0,0,0)(0,-2,0)(0,0,0)(0,2,0) \) |
| \( K_{2c} = (v)(0,0,0)(0,2,0)(0,-2,0)(0,0,0) \) |
| \( K_{2d} = (v)(0,0,0)(0,0,0)(0,2,0)(0,-2,0) \) |
| \( K_{2e} = (v)(0,0,0)(0,-2,0)(0,2,0)(0,0,0) \) |
| \( K_{2f} = (v)(0,0,0)(0,2,0)(0,0,0)(0,-2,0) \) |
| \( K_{2g} = (v)(0,0,0)(0,4,0)(0,2,0)(0,2,0) \) |
| \( K_{2h} = (v)(0,0,0)(0,2,0)(0,4,0)(0,2,0) \) |
| \( K_{2k} = (v)(0,0,0)(0,2,0)(0,2,0)(0,4,0) \) |
| \( K_{2l} = (v)(0,0,0)(0,4,0)(0,-2,0)(0,-2,0) \) |
| \( K_{2m} = (v)(0,0,0)(0,-2,0)(0,4,0)(0,-2,0) \) |
| \( K_{2n} = (v)(0,0,0)(0,-2,0)(0,-2,0)(0,4,0) \) |

The table for \( K_3 \) is as follows:

| \( K_3 \) | \( h_{K_3} = \frac{3}{2} \) |
|---|---|
| \( K_{3a} = (v)(0,0,0)(0,0,0)(0,4,0)(0,4,0) \) | \( \chi_{K_3}^G = \frac{\theta_1^2(0,\tau)+\theta_1^4(0,\tau)}{16} - \frac{\theta_2(0,\tau)\theta_4(0,\tau)}{8} \) |
| \( K_{3b} = (v)(0,0,0)(0,4,0)(0,0,0)(0,4,0) \) |
| \( K_{3c} = (v)(0,0,0)(0,4,0)(0,4,0)(0,0,0) \) |

(55)

(56)

(57)
The second group has \( l_1 = l_2 = l_3 = l_4 = 0, s_1 = 2, s_2 = s_3 = s_4 = 0 \) and also contains 16 primaries, which again can be divided into 3 subgroups, in such a way that all primaries inside each group have the same characters.

| \( L_1 \) | \( h_{L_1} = 1 \) |
|---|---|
| \( L_{1a} = (v)(0, 0, 2)(0, 4, 0)(0, 4, 0)(0, 4, 0) \) | \( \chi_{L_1}^G = \frac{\theta_1^1(0, r) - \theta_2^1(0, r)}{16} \) |
| \( L_{1b} = (v)(0, 0, 2)(0, 0, 0)(0, 0, 0)(0, 4, 0) \) |
| \( L_{1c} = (v)(0, 0, 2)(0, 0, 0)(0, 4, 0)(0, 0, 0) \) |
| \( L_{1d} = (v)(0, 0, 2)(0, 4, 0)(0, 0, 0)(0, 0, 0) \) |

(58)

| \( L_2 \) | \( h_{L_2} = \frac{1}{2} \) |
|---|---|
| \( L_{2a} = (v)(0, 0, 2)(0, 4, 0)(0, 2, 0)(0, 0, 0) \) | \( \chi_{L_2}^G = \frac{\theta_1^1(0, r) + \theta_2^1(0, r)}{16} + \frac{\theta_3^1(0, r) \theta_4^1(0, r)}{8} \) |
| \( L_{2b} = (v)(0, 0, 2)(0, 0, 0)(0, 2, 0)(0, 4, 0) \) |
| \( L_{2c} = (v)(0, 0, 2)(0, 0, 0)(0, 4, 0)(0, 2, 0) \) |
| \( L_{2d} = (v)(0, 0, 2)(0, 4, 0)(0, 0, 0)(0, 2, 0) \) |
| \( L_{2e} = (v)(0, 0, 2)(0, 2, 0)(0, 0, 0)(0, 4, 0) \) |
| \( L_{2f} = (v)(0, 0, 2)(0, 2, 0)(0, 2, 0)(0, 0, 0) \) |

(59)

| \( L_3 \) | \( h_{L_3} = \frac{3}{2} \) |
|---|---|
| \( L_{3a} = (v)(0, 0, 2)(0, 0, 0)(0, 2, 0)(0, 2, 0) \) | \( \chi_{L_3}^G = \frac{\theta_1^1(0, r) + \theta_2^1(0, r)}{16} - \frac{\theta_3^1(0, r) \theta_4^1(0, r)}{8} \) |
| \( L_{3b} = (v)(0, 0, 2)(0, 2, 0)(0, 0, 0)(0, 2, 0) \) |
| \( L_{3c} = (v)(0, 0, 2)(0, 2, 0)(0, 2, 0)(0, 0, 0) \) |
| \( L_{3d} = (v)(0, 0, 2)(0, 0, 0)(0, 0, 0)(0, 2, 0) \) |
| \( L_{3e} = (v)(0, 0, 2)(0, 0, 0)(0, 0, 0)(0, 2, 0) \) |
| \( L_{3f} = (v)(0, 0, 2)(0, 0, 0)(0, 0, 0)(0, 2, 0) \) |

(60)

The third group containing 48 primaries with any two of \( l_i \) equal to 1, and other two of them to 0. This group consists of 6 subgroups:

\[
l_1 = l_2 = 1 \quad l_3 = l_4 = 0
\]

(61)
\[
\begin{align*}
l_1 &= l_3 = 1 \quad l_2 = l_4 = 0 \\
l_1 &= l_4 = 1 \quad l_2 = l_3 = 0 \\
l_2 &= l_3 = 1 \quad l_1 = l_4 = 0 \\
l_2 &= l_4 = 1 \quad l_1 = l_3 = 0 \\
l_3 &= l_4 = 1 \quad l_1 = l_2 = 0
\end{align*}
\]

Each such a subgroup consists of 8 primaries and can be derived from, let’s say, the first of them by permutations, so we will write down only one of them, the one with \(l_1 = l_2 = 1\) and \(l_3 = l_4 = 0\). We schematically denote the primaries in this group as \(\Phi_{1a}^{1,1,\cdots}\), indicating explicitly in the superscript which \(l_i\) are equal to 1.

| \(\Phi_1\) | \(h_{\Phi_1} = \frac{3}{4}\) |
|---|---|
| \(\Phi_{1a}^{1,1,\cdots} = (v)(1,1,0)(1,3,0)(0,2,0)(0,0,0)\) |  |
| \(\Phi_{1b}^{1,1,\cdots} = (v)(1,1,0)(1,−1,0)(0,0,0)(0,0,0)\) | \(\chi_{\Phi_1}^G = \frac{\theta_2^2(0,\tau)(\theta_4^2(0,\tau) + \theta_4^2(0,\tau))}{8}\) |

| \(\Phi_2\) | \(h_{\Phi_2} = \frac{5}{4}\) |
|---|---|
| \(\Phi_{2a}^{1,1,\cdots} = (v)(1,1,0)(1,3,0)(0,0,0)(0,4,0)\) |  |
| \(\Phi_{2b}^{1,1,\cdots} = (v)(1,1,0)(1,−1,0)(0,−2,0)(0,2,0)\) | \(\chi_{\Phi_2}^G = \frac{\theta_2^2(0,\tau)(\theta_4^2(0,\tau) − \theta_4^2(0,\tau))}{8}\) |

| \(\Phi_3\) | \(h_{\Phi_3} = 1\) |
|---|---|
| \(\Phi_{3a}^{1,1,\cdots} = (v)(1,1,0)(1,−3,0)(0,2,0)(0,0,0)\) |  |
| \(\Phi_{3b}^{1,1,\cdots} = (v)(1,1,0)(1,−3,0)(0,0,0)(0,2,0)\) | \(\chi_{\Phi_3}^G = \frac{\theta_4^2(0,\tau) − \theta_4^2(0,\tau)}{8}\) |

| \(\Phi_4\) | \(h_{\Phi_4} = \frac{1}{2}\) |
|---|---|
| \(\Phi_{4a}^{1,1,\cdots} = (v)(1,1,0)(1,1,0)(0,−2,0)(0,0,0)\) |  |
| \(\Phi_{4b}^{1,1,\cdots} = (v)(1,1,0)(1,1,0)(0,0,0)(0,−2,0)\) | \(\chi_{\Phi_4}^G = \frac{\theta_4^2(0,\tau) + \theta_4^2(0,\tau)}{8}\) |
Finally we have a small group containing only 4 elements with \( l_1 = l_2 = l_3 = l_4 = 1, s_1 = s_2 = s_3 = s_4 = 0 \). All primaries in this group, as we explained in section 3, have a short orbit and should be resolved. After resolution we end up with 8 primaries. The \( \pm \) in the notations refers to the resolution process.

\[
\begin{array}{|c|c|}
\hline
R_1 & h_{R_1} = 1 \\
\hline
R_{1a \pm} = (v)(1,1,0)(1,-1,0)(1,1,0)(1,-1,0) & \\
R_{1b \pm} = (v)(1,1,0)(1,-1,0)(1,-1,0)(1,1,0) & \\
R_{1c \pm} = (v)(1,1,0)(1,1,0)(1,-1,0)(1,-1,0) & \\
\hline
\end{array}
\]  
(66)

\[
\begin{array}{|c|c|}
\hline
R_2 & h_{R_2} = \frac{1}{2} \\
\hline
R_{2 \pm} = (v)(1,1,0)(1,1,0)(1,-3,0)(1,1,0) & \\
\hline
\end{array}
\]  
(67)

We see that before fixed points resolution we had 84 orbits: 31 orbits with conformal dimension 1, 12 orbits with conformal dimension \( \frac{3}{4} \), 12 orbits with conformal dimension \( \frac{5}{4} \), 20 orbits with conformal dimension \( \frac{1}{2} \), 9 orbits with conformal dimension \( \frac{3}{4} \). After the fixed points resolution we have 88 primaries: 34 orbits with conformal dimension 1, 12 orbits with conformal dimension \( \frac{3}{4} \), 12 orbits with conformal dimension \( \frac{5}{4} \), 21 orbits with conformal dimension \( \frac{1}{2} \), 9 orbits with conformal dimension \( \frac{3}{4} \) [41].

Collecting all the above results, we can write down the torus amplitude:

\[
\begin{align*}
Z &= \left| \frac{\mathcal{J}}{\eta^{12}} \right|^2 \left[ \left| \frac{\theta_3^4(0,\tau) - \theta_4^4(0,\tau)}{16} \right|^2 + 9 \left| \frac{\theta_3^4(0,\tau) + \theta_4^4(0,\tau)}{16} - \frac{\theta_2^2(0,\tau)\theta_4^2(0,\tau)}{8} \right|^2 \\
&\quad + 6 \left| \frac{\theta_3^4(0,\tau) + \theta_4^4(0,\tau)}{16} \right|^2 + 18 \left| \frac{\theta_3^4(0,\tau) - \theta_4^4(0,\tau)}{8} \right|^2 \\
&\quad + 14 \left| \frac{\theta_2^2(0,\tau)(\theta_3^2(0,\tau) - \theta_4^2(0,\tau))}{8} \right|^2 \right] + \\
&= \left| \frac{\mathcal{J}}{\eta^{12}} \right|^2 \left[ \frac{1}{16} \left( |\theta_3(0,\tau)|^4 + |\theta_4(0,\tau)|^4 + |\theta_2(0,\tau)|^4 \right) \right]^2
\end{align*}
\]  
(68)

15
The partition function (68), as first noted in [31], coincides with the partition function of the $T^4/Z_4$ orbifold at the $SU(2)^4$ point, which we review in appendix C.

Now we elaborate on the expression (11) for the annulus partition function for the $(2,2,2,2)$ Gepner model.

Let us denote the first Cardy state $I$:

$$I = (S_0, L_1, M_1, S_1, \ldots, L_4, M_4, S_4) \quad (69)$$

and the second $J$:

$$J = (\tilde{S}_0, \tilde{L}_1, \tilde{M}_1, \tilde{S}_1, \ldots, \tilde{L}_4, \tilde{M}_4, \tilde{S}_4) \quad (70)$$

Consider first the case when neither the first boundary state nor the second are fixed.

Now using (13) and the fusion coefficients (25) we can easily derive:

$$Z_{IJ} = \sum_{s_0} \sum_{l_i} |S^{l_1 \cdots l_4}| \mathcal{N}_{v(S_0)S_0}^{SO(4)} \prod_{i=1}^4 \mathcal{N}_{L_i L_i}^{SU(2)} \chi_{G \hat{l}_1 \cdots \hat{l}_4}^{M_1 \cdots M_4 \hat{M}_4, s_0, \hat{S}_1 \cdots \hat{S}_4} \quad (71)$$

Actually the sum over $J$ in (13) $\sum_{J \in G} \mathcal{N}_{Ja,c}^d$ is running over the orbit of the primary

$$(s_0, l_1, M_1 - \hat{M}_1, S_1 - \hat{S}_1 \cdots l_4, M_4 - \hat{M}_4, S_4 - \hat{S}_4) \quad (72)$$

while the sum over orbits in (13) runs over the specific representatives, for examples listed in the tables above. It means that generically in this sum only one term will survive, the specific representative of the orbit of the primary (72). If this primary has non-trivial stabilizer, due to field identification the sum over $J$ will produce the representative twice. The fusion $v(S_0)$ in $\mathcal{N}_{v(S_0)S_0}^{SO(4)}$ is due to the bosonic string map [24]. Collecting all pieces we get (71). In practice in order to use formula (71) one needs to compute the primary (72) and then use the action of the simple current to find in the orbit which of the representatives listed in tables above it belongs to, and substitute its character.

Consider next the case when $I$ is not fixed but $J$ is. In this case elaborating
on (12) we obtain:

\[
Z_{IJ} = \frac{1}{|S_I||S_J|} \sum_{s_0} \sum_{l_i} |S|^{S(4)}_{v(s_0)} \prod_{i=1}^{4} \mathcal{N}^{SU(2)}_{i_l} \mathcal{N}^{U(1)}_{S_{i_l}} \chi^{G_{i_1} \ldots i_4}_{M_1 \ldots M_4 \hat{M_1} \ldots \hat{M_4}, s_0, s_1 \ldots s_4, \hat{s}_1 \ldots \hat{s}_4}
\]

(73)

The last case is when both and I and J are fixed points. To elaborate on this case we need the matrices \(S_{c,b}\) and \(\hat{S}_{ab}\) in formula (15).

The matrix \(S_{ab}\) for Gepner models is the product of all the elementary S's and reads:

\[
2^4 S^{SO(4)}_{s_0 s_0'} \prod_{i=1}^{4} S^{SU(2)}_{l_i l_i'} S^{U(1)}_{s_i s_i'} S^{U(1)}_{m_i m_i'}
\]

(74)

The matrix \(\hat{S}_{ab}\) was found in [24]. For the (2,2,2,2) model it has the form

\[
-2^4 S^{SO(4)}_{s_0 s_0'} \prod_{i=1}^{4} S^{U(1)}_{s_i s_i'} S^{U(1)}_{m_i m_i'}
\]

(75)

The numerical factors come from the field identification.

Plugging (74) and (75) in (13) one obtains:

\[
Z_{I\psi, J\psi} = \frac{1}{|S_I||S_J|} \sum_{s_0} \sum_{l_i} |S|^{SO(4)}_{v(s_0)} \left( \prod_{i=1}^{4} \mathcal{N}^{SU(2)}_{i_l} + \psi \psi' \prod_{i=1}^{4} \sin \frac{\pi l_i + 1}{2} \right) \chi^{G_{i_1} \ldots i_4}_{M_1 \ldots M_4 \hat{M_1} \ldots \hat{M_4}, s_0, s_1 \ldots s_4, \hat{s}_1 \ldots \hat{s}_4}
\]

(76)

\[
5 \quad D0\text{-branes on the } T^4/Z_4 \text{ orbifold.}
\]

5.1 Fixed points

Defining complex coordinates \(z_1 = x_1 + ix_2\) and \(z_2 = x_3 + ix_4\) the \(Z_4\) group action can be described as

\[
z_1 \rightarrow e^{\frac{2\pi i k}{4}} z_1 \quad z_2 \rightarrow e^{-\frac{2\pi i k}{4}} z_2
\]

(77)

We can consider it as generated by the \(Z_2\) subgroup acting as \(z_1 \rightarrow -z_1\) and \(z_2 \rightarrow -z_2\) and a \(Z'_2\) subgroup rotating by \(\frac{\pi}{2}\) and \(-\frac{\pi}{2}\) the \((x_1, x_2)\) and \((x_3, x_4)\) planes: \(z_1 \rightarrow iz_1\) and \(z_2 \rightarrow -iz_2\).

The \(Z_2\) group has 16 fixed points \((\pi R e_1, \pi R e_2, \pi R e_3, \pi R e_4)\), where \(e_i = 0, 1\), out of which only 4 are also fixed under \(Z'_2\): \(D0f_1 = (0, 0, 0, 0)\), \(D0f_2 = (\pi R, \pi R, \pi R, \pi R)\), \(D0f_3 = (\pi R, \pi R, 0, 0)\), \(D0f_4 = (0, 0, \pi R, \pi R)\).
To begin with, let us calculate the annulus partition function for open strings having both ends at the same fixed point.

The partition function is given by

$$Z_{D0_fD0_f} = \frac{1}{8} \sum_{k=0}^{3} \text{Tr}(1 + (-)^F) g^k e^{-2\pi L_0} = \frac{1}{4} \frac{J}{\eta^{12}} Z_{\text{windings}} + \frac{1}{8} \sum_{k=1}^{3} 4 \sin^2 \frac{\pi k}{4} Z_{0,k}$$

(78)

where

$$Z_{\text{windings}} = \sum_{n_1,n_2,n_3,n_4} q^{n_1^2+n_2^2+n_3^2+n_4^2} = \theta_3^4(0,2\tau) = \frac{\theta_3^4(0,\tau) + \theta_4^4(0,\tau)}{4} + \frac{\theta_2^2(0,\tau)\theta_4^2(0,\tau)}{2}$$

(79)

and $Z_{0,k}$ can be found in (114), (115), (116) of appendix C. Collecting all the pieces, we obtain:

$$Z_{D0_fD0_f} = \frac{J}{\eta^{12}} \left( \frac{\theta_3^4(0,\tau) + \theta_4^4(0,\tau)}{16} + \frac{3\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{8} \right)$$

(80)

We see that (80) coincides with (55):

$$Z_{D0_fD0_f} = \chi_{K_1}$$

(81)

In order to compute the partition function for strings with ends at different fixed point, we need to recall the partition function for a scalar $X$ compactified at the self-dual radius $R = \frac{1}{\sqrt{2}}$ with Dirichlet boundary conditions placed at $2\pi R \xi_1$ and $2\pi R \xi_2$, so that

$$X = 2\pi R \xi_1 + (2R(\xi_2 - \xi_1) + 2nR) \sigma + \text{oscillators}$$

(82)

The partition function is easily calculated to be

$$Z_{x_1x_2} = \frac{1}{\eta} q^{(\xi_2 - \xi_1)^2} \theta_3(2\tau(\xi_2 - \xi_1),2\tau)$$

(83)

Using (83) we can then compute the annulus partition functions between different fixed points:

$$Z_{D0_1fD0_2f} = \frac{J}{\eta^{12}} \left( \frac{\theta_3^4(0,\tau) + \theta_4^4(0,\tau)}{16} + \frac{3\theta_3^2(0,\tau)\theta_4^2(0,\tau)}{8} \right) = \chi_{L_2}$$

(84)
Now we consider the case when the $D0$ branes lie at a point fixed only under $Z_2$. We have the following list of such branes:

$D0_1 = A_1 + A'_1 : (0, \pi R, 0, 0) + (\pi R, 0, 0, 0)$ (86)
$D0_2 = A_2 + A'_2 : (\pi R, 0, 0, \pi R) + (0, \pi R, \pi R, 0)$
$D0_3 = A_3 + A'_3 : (\pi R, \pi R, 0, \pi R) + (0, \pi R, \pi R, 0)$
$D0_4 = A_4 + A'_4 : (\pi R, 0, \pi R, \pi R) + (0, \pi R, \pi R, 0)$
$D0_5 = A_5 + A'_5 : (0, 0, 0, \pi R) + (0, 0, \pi R, 0)$
$D0_6 = A_6 + A'_6 : (\pi R, 0, \pi R, 0) + (0, \pi R, 0, \pi R)$

The partition functions between branes (86) and fixed point branes are given by equation:

$$Z_{D0,D0_f} = \text{Tr}_{A_0,D0_f} \frac{(1 + (-)^F) (1 + g^2)}{2} e^{-2\pi \tau L_0}$$ (88)

which taking into account (85) simplifies to

$$Z_{D0,D0_f} = \frac{1}{4} e^{-2\pi \tau L_0}$$ (88)

Using (85) we can easily compute all annulus partition functions of this type. The result is presented in the following table:

| Branes | $D0_{1f}$ | $D0_{2f}$ |
|--------|-----------|-----------|
| $D0_{1f}$ | $\frac{\theta_2^2 + \theta_1^2}{16} + \frac{3\theta_2^4 \theta_1^4}{8}$ | $\frac{\theta_2^2 + \theta_1^2}{16} + \frac{\theta_2^4 \theta_1^4}{8}$ |
| $D0_{2f}$ | $\frac{\theta_2^2 + \theta_1^2}{16} + \frac{\theta_2^4 \theta_1^4}{8}$ | $\frac{\theta_2^2 + \theta_1^2}{16} + \frac{3\theta_2^4 \theta_1^4}{8}$ |
| $D0_1$ | $\frac{\theta_2^2 (\theta_2^2 + \theta_1^2)}{8}$ | $\frac{\theta_2^2 (\theta_2^2 + \theta_1^2)}{8}$ |
| $D0_2$ | $\frac{\theta_2^2 - \theta_1^2}{8}$ | $\frac{\theta_2^2 - \theta_1^2}{8}$ |
| $D0_3$ | $\frac{\theta_2^2 (\theta_2^2 - \theta_1^2)}{8}$ | $\frac{\theta_2^2 + \theta_1^2}{8}$ |
| $D0_4$ | $\frac{\theta_2^2 (\theta_2^2 - \theta_1^2)}{8}$ | $\frac{\theta_2^2 (\theta_2^2 - \theta_1^2)}{8}$ |
| $D0_5$ | $\frac{\theta_2^2 (\theta_2^2 + \theta_1^2)}{8}$ | $\frac{\theta_2^2 (\theta_2^2 - \theta_1^2)}{8}$ |
| $D0_6$ | $\frac{\theta_2^2 - \theta_1^2}{8}$ | $\frac{\theta_2^2 - \theta_1^2}{8}$ |
where it is understood that all entries should be multiplied by $\frac{J}{\eta \tau} = \frac{1}{2\eta \tau} (\theta_4^4 - \theta_2^4)$.

Using the characters in section 4 one can present table (89) in the form

| Branes | $D0_1 f$ | $D0_2 f$ |
|--------|----------|----------|
| $D0_1 f$ | $\chi_{K_1}$ | $\chi_{L_2}$ |
| $D0_2 f$ | $\chi_{L_2}$ | $\chi_{K_1}$ |
| $D0_1$ | $\chi_{\Phi_1}$ | $\chi_{\Phi_2}$ |
| $D0_2$ | $\chi_{R_1}$ | $\chi_{R_1}$ |
| $D0_3$ | $\chi_{\Phi_2}$ | $\chi_{\Phi_1}$ |
| $D0_4$ | $\chi_{\Phi_1}$ | $\chi_{\Phi_2}$ |
| $D0_5$ | $\chi_{R_1}$ | $\chi_{R_1}$ |

Table (90) already gives a hint for the candidate Cardy states, describing $D0$ branes located at fixed and partially fixed points.

To make things more precise we should compute also the partition functions between the different partially fixed branes (86). They have the form:

$$Z_{D0_i D0_j} = \text{Tr}_{A_i A_j} \left( \frac{1 + (-)^F}{2} \right) \left( 1 + g^2 \right) e^{-2\pi \tau L_0} + \text{Tr}_{A_i A'_j} \left( \frac{1 + (-)^F}{2} \right) \left( 1 + g^2 \right) e^{-2\pi \tau L_0}$$  \hspace{1cm} (91)

which using (115) simplifies to

$$Z_{D0_i D0_j} = \text{Tr}_{A_i A_j} \frac{1 + (-)^F}{4} e^{-2\pi \tau L_0} + \text{Tr}_{A_i A'_j} \frac{1 + (-)^F}{4} e^{-2\pi \tau L_0}$$ \hspace{1cm} (92)

Again using (83) we can present the result in the following table:
where, as before, it is understood that all entries should be multiplied by 
\[ \frac{1}{\eta^2} = \frac{1}{2\eta^2}(\theta_3^4 - \theta_4^4 - \theta_2^4). \]

After some trial and error we can solve these conditions with the following Cardy states:

\[ D_{01f} = |K_1\rangle^{\text{Cardy}} \]  
\[ D_{02f} = |L_{20}\rangle^{\text{Cardy}} \]  
\[ D_{01} = |\Phi_{1a}^{1,1,\cdots}\rangle^{\text{Cardy}} \]  
\[ D_{02} = |R_{10+}\rangle^{\text{Cardy}} \]  
\[ D_{03} = |\Phi_{2b}^{1,1,\cdots}\rangle^{\text{Cardy}} \]  
\[ D_{04} = |\Phi_{2b}^{1,1,\cdots}\rangle^{\text{Cardy}} \]  
\[ D_{05} = |\Phi_{1a}^{1,1,\cdots}\rangle^{\text{Cardy}} \]  
\[ D_{06} = |R_{10-}\rangle^{\text{Cardy}} \]  

Using the formulae (71), (73), (76) we obtain for the annulus partition functions between the states (94) the following table:

| Branes | \( D_{01} \) | \( D_{02} \) | \( D_{03} \) | \( D_{04} \) | \( D_{05} \) |
|--------|--------|--------|--------|--------|--------|
| \( D_{01} \) | \( Z_{11} \) | \( \chi_{\phi_2} + \chi_{\phi_1} \) | 2\( \chi_{R_1} \) | 2\( \chi_{K_3} + \chi_{L_1} \) | 2\( \chi_{R_1} \) |
| \( D_{02} \) | \( \chi_{\phi_2} + \chi_{\phi_1} \) | \( \chi_{K_3} + 3\chi_{K_1} \) | \( \chi_{\phi_2} + \chi_{\phi_1} \) | \( \chi_{\phi_1} + \chi_{\phi_2} \) | \( \chi_{\phi_1} + \chi_{\phi_2} \) |
| \( D_{03} \) | 2\( \chi_{R_1} \) | \( \chi_{\phi_2} + \chi_{\phi_1} \) | \( Z_{11} \) | 2\( \chi_{R_1} \) | 2\( \chi_{K_3} + \chi_{L_1} \) |
| \( D_{04} \) | 2\( \chi_{K_3} + \chi_{L_1} \) | \( \chi_{\phi_1} + \chi_{\phi_2} \) | 2\( \chi_{R_1} \) | \( Z_{11} \) | 2\( \chi_{R_1} \) |
| \( D_{05} \) | 2\( \chi_{R_1} \) | \( \chi_{\phi_1} + \chi_{\phi_2} \) | 2\( \chi_{K_3} + \chi_{L_1} \) | 2\( \chi_{R_1} \) | \( Z_{11} \) |
| \( D_{06} \) | \( \chi_{\phi_2} + \chi_{\phi_1} \) | 4\( \chi_{L_1} \) | \( \chi_{\phi_2} + \chi_{\phi_1} \) | \( \chi_{\phi_1} + \chi_{\phi_2} \) | \( \chi_{\phi_1} + \chi_{\phi_2} \) |
where $Z_{11} = \chi_{K_1} + \chi_{K_3} + 2\chi_{L_1}$, which coincides with table (93).

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A  Theta functions identities

We start by reviewing some useful identities satisfied by Theta functions [42].

\[
\theta \left[ \begin{array}{c} \alpha \\ n_1 \\ 0 \\
\end{array} \right] (x_1, n_1 \tau) \theta \left[ \begin{array}{c} b \\ n_2 \\ 0 \\
\end{array} \right] (x_2, n_2 \tau) = 
\] (95)

\[
\sum_{\mu=0}^{n_1+n_2-1} \theta \left[ \begin{array}{c} n_1 \mu + a + b \\ n_1+n_2 \\ 0 \\
\end{array} \right] (x_1 + x_2, (n_1 + n_2) \tau) \cdot 
\theta \left[ \begin{array}{c} \frac{n_2 \mu + n_2 a - n_1 b}{n_1 n_2 (n_1 + n_2)} \\ 0 \\
\end{array} \right] (n_2 x_1 - n_1 x_2, n_1 n_2 (n_1 + n_2) \tau)
\]

where

\[
\theta \left[ \begin{array}{c} a \\ b \\
\end{array} \right] (x, \tau) = \sum_{n \in \mathbb{Z}} \exp(i \pi (n + a)^2 \tau + 2i \pi (n + a) (x + b)) 
\] (96)

Using the identity

\[
\sum_{\mu=0}^{n-1} \theta \left[ \begin{array}{c} \mu + a \\ n \\ 0 \\
\end{array} \right] (nx, n^2 \tau) = \theta \left[ \begin{array}{c} a \\ 0 \\
\end{array} \right] (x, \tau) 
\] (97)

we can exploit (95) for the case relevant to our analysis i.e. \( n_1 = r_1 n \) and \( n_2 = r_2 n \)

\[
\theta \left[ \begin{array}{c} a \\ r_1 n \\
\end{array} \right] (x_1, r_1 n \tau) \theta \left[ \begin{array}{c} b \\ r_2 n \\
\end{array} \right] (x_2, r_2 n \tau) = 
\] (98)

\[
\sum_{\mu=0}^{r_1+r_2-1} \theta \left[ \begin{array}{c} r_1 \mu \\ r_1+r_2 \\
\end{array} \right] + \frac{a+b}{(r_1+r_2)n} (x_1 + x_2, (r_1 + r_2) n \tau) \cdot 
\theta \left[ \begin{array}{c} \frac{r_2 a - r_1 b}{r_1 r_2 (r_1+r_2)n} \\ 0 \\
\end{array} \right] (r_2 x_1 - r_1 x_2, r_1 r_2 (r_1 + r_2) n \tau)
\]

Let us explicitly write this formula for the most relevant for us case: \( n_1 = n_2 = n \), \( r_1 = r_2 = 1 \)

\[
\theta \left[ \begin{array}{c} a \\ n \\
\end{array} \right] (x_1, n \tau) \theta \left[ \begin{array}{c} b \\ n \\
\end{array} \right] (x_2, n \tau) = 
\] (99)

\[
\sum_{\mu=0}^{1} \theta \left[ \begin{array}{c} \frac{\mu + a + b}{2n} \\ 0 \\
\end{array} \right] (x_1 + x_2, 2n \tau) \theta \left[ \begin{array}{c} \frac{\mu + a - b}{2n} \\ 0 \\
\end{array} \right] (x_1 - x_2, 2n \tau)
\]
B Other relevant identities

Recall the identities:

\[
\begin{align*}
\theta_3^2(\tau) - \theta_1^2(\tau) &= 2\theta_2^2(2\tau) \\
\theta_3^2(\tau) + \theta_1^2(\tau) &= 2\theta_2^2(2\tau) \\
\theta_3(\tau)\theta_4(\tau) &= \theta_1^2(2\tau) \\
\theta_2^2(\tau) &= 2\theta_2(2\tau)\theta_3(2\tau)
\end{align*}
\] (100)

From (100) we can derive another couple of useful identities:

\[
\begin{align*}
\theta_3(2\tau)\theta_2^2(\tau) &= \theta_2(2\tau)(\theta_3^2(\tau) + \theta_1^2(\tau)) \\
\theta_2(2\tau)\theta_2^2(\tau) &= \theta_3(2\tau)(\theta_3^2(\tau) - \theta_1^2(\tau))
\end{align*}
\] (101)

\[
\begin{align*}
\Theta_{0,1}(z, \tau) &= \theta_3(z, 2\tau) \\
\Theta_{1,1}(z, \tau) &= \theta_2(z, 2\tau)
\end{align*}
\] (102)

Let us also mention the following formulae.

\[
\begin{align*}
\theta_1 \left( \frac{1}{2}, \tau \right) &= \theta_2(0, \tau) \\
\theta_2 \left( \frac{1}{2}, \tau \right) &= 0 \\
\theta_3 \left( \frac{1}{2}, \tau \right) &= \theta_4(0, \tau) \\
\theta_4 \left( \frac{1}{2}, \tau \right) &= \theta_3(0, \tau)
\end{align*}
\] (103-106)

\[
\begin{align*}
\theta_1 \left( \frac{1}{4}, \tau \right) &= \theta_2 \left( \frac{1}{4}, \tau \right) = \theta_1 \left( \frac{3}{4}, \tau \right) = -\theta_2 \left( \frac{3}{4}, \tau \right) \\
\theta_3 \left( \frac{1}{4}, \tau \right) &= \theta_4 \left( \frac{1}{4}, \tau \right) = \theta_3 \left( \frac{3}{4}, \tau \right) = \theta_4 \left( \frac{3}{4}, \tau \right)
\end{align*}
\] (107-108)

\[
\frac{\theta_3^2(\frac{1}{4}, \tau)}{\theta_2^2(\frac{1}{4}, \tau)} = \frac{\theta_3(0, 2\tau)}{\theta_2(0, 2\tau)} = \frac{\theta_3^2(0, \tau)}{\theta_2^2(0, \tau) - \theta_4^2(0, \tau)}
\] (109)
C Partition function of the $T^4/Z_4$ orbifold

\[ Z = \frac{1}{4} Z_{\text{lattice}} \left| \frac{\mathcal{J}}{\eta^{12}} \right|^2 + \sum_{r,s} n_{r,s} |Z_{r,s}|^2 \]  

(110)

where

\[ Z_{\text{lattice}} = (|\chi_1^{SU(2)}|^2 + |\chi_2^{SU(2)}|^2)^4 = \frac{1}{4} \left( |\theta_3(0, \tau)|^4 + |\theta_4(0, \tau)|^4 + |\theta_2(0, \tau)|^4 \right)^2 \]  

(111)

and

\[ Z_{r,s} = \sum_{\alpha,\beta} c_{\alpha,\beta} \frac{\theta^2 (\begin{array}{c} \alpha \\ \beta \end{array}, 0, \tau) \theta (\begin{array}{c} \alpha + \frac{r}{4} \\ \beta + \frac{s}{4} \end{array}, 0, \tau) \theta (\begin{array}{c} \alpha - \frac{r}{4} \\ \beta - \frac{s}{4} \end{array}, 0, \tau)}{\eta^6} \theta (\begin{array}{c} \frac{1}{2} + \frac{r}{4} \\ \frac{1}{2} + \frac{s}{4} \end{array}, 0, \tau) \theta (\begin{array}{c} \frac{1}{2} - \frac{r}{4} \\ \frac{1}{2} - \frac{s}{4} \end{array}, 0, \tau) \]  

(112)

Consider the Ramond part.

\[ Z^R_{r,s} = \frac{\theta^2 (0, \tau)}{\eta^6} \frac{\theta (\begin{array}{c} \frac{1}{2} + \frac{r}{4} \\ 0 + \frac{s}{4} \end{array}, 0, \tau) \theta (\begin{array}{c} \frac{1}{2} - \frac{r}{4} \\ 0 - \frac{s}{4} \end{array}, 0, \tau)}{\theta (\begin{array}{c} \frac{1}{2} + \frac{r}{4} \\ \frac{1}{2} + \frac{s}{4} \end{array}, 0, \tau) \theta (\begin{array}{c} \frac{1}{2} - \frac{r}{4} \\ \frac{1}{2} - \frac{s}{4} \end{array}, 0, \tau)} \]  

(113)

\[ Z^R_{0,1} = -\theta^4_2(0, \tau) \frac{\theta^2_3(0, \tau) \theta^2_4(0, \tau)}{4\eta^{12}} \]  

(114)

\[ Z^R_{0,2} = 0 \]  

(115)

\[ Z^R_{0,3} = -\theta^4_2(0, \tau) \frac{\theta^2_3(0, \tau) \theta^2_4(0, \tau)}{4\eta^{12}} \]  

(116)

\[ Z^R_{2,0} = \theta^4_2(0, \tau) \frac{\theta^4_3(0, \tau)}{4\eta^{12}} \]  

(117)

\[ Z^R_{2,1} = \theta^4_2(0, \tau) \frac{\theta^2_3(0, \tau) \theta^2_4(0, \tau)}{4\eta^{12}} \]  

(118)

\[ Z^R_{2,2} = \theta^4_2(0, \tau) \frac{\theta^4_3(0, \tau)}{4\eta^{12}} \]  

(119)

\[ Z^R_{2,3} = \theta^4_2(0, \tau) \frac{\theta^2_3(0, \tau) \theta^2_4(0, \tau)}{4\eta^{12}} \]  

(120)

\[ Z^R_{1,0} = \theta^4_2(0, \tau) \frac{\theta^4_3(0, \tau) + \theta^2_3(0, \tau) \theta^2_4(0, \tau)}{4\eta^{12}} \]  

(121)

\[ Z^R_{1,2} = -\theta^4_2(0, \tau) \frac{\theta^4_3(0, \tau) - \theta^2_3(0, \tau) \theta^2_4(0, \tau)}{4\eta^{12}} \]  

(122)
The numbers \( n_{r,s} \) are given by the following formulae: \( n_{0,s} = 4 \sin^4 \frac{\pi s}{4} \), \( n_{r,s} = n_{r,s+r} \), \( n_{r,s} = n_{s,4-r} \).

Plugging all in (110) we get (68).

## D Annulus partition functions for simple current extensions

In this section we mainly follow [35].

Let us start with the partition function:

\[
Z = \sum_{\text{orbits } Q(a) = 0} |S_a| \cdot \sum_{J \in G/S_a} \chi_{Ja}^2
\]

where \( G \) is a group of simple currents and \( S_a \) is the stabilizer of \( a \). Denote by \( G_a = G/S_a \) the factor group acting non-trivially on \( a \). The order \( |G_a| \) of \( G_a \) is \( |G_a| = |G|/|S_a| \). Let us write \( |S_a| \) as a sum of squares:

\[
|S_a| = \sum_i (m_{a,i})^2
\]

where \( i \) labels the different primaries into which \( a \) gets resolved (usually \( m_{a,i} \) has to be independent of \( i \), but to keep track of the different sums we will keep the index \( i \).) Corresponding to this definition we have

\[
\tilde{\chi}_{a,i} = m_{a,i} \sum_{J \in G_a} \chi_{Ja}
\]
so that \( \sum_i |\tilde{\chi}_{a,i}|^2 = |S_a| \cdot |\sum_{J \in G/S_a} \chi_{Ja}|^2 \). The ansatz for the resolved modular \( S \) matrix suggested in [33] is

\[
\tilde{S}_{(a,i),(b,j)} = m_{a,i} m_{b,j} \frac{|G_a||G_b|}{|G|} S_{a,b} + \Gamma_{(a,i),(b,j)} \tag{132}
\]

where \( \Gamma_{(a,i),(b,j)} \) satisfies

\[
\sum_j \Gamma_{(a,i),(b,j)} m_{b,j} = 0. \tag{133}
\]

Now we derive the unitarity condition on \( \tilde{S}_{(a,i),(b,j)} \). Recall that \( S_{a,b} \) satisfies

\[
S_{Ja,b} = e^{2\pi i Q(b)} S_{a,b} \tag{134}
\]

Computing \( \tilde{S}\tilde{S}^\dagger \) we produce four terms

\[
P_{S,S} = \sum_{\text{orbits } Q(b)=0,j} m_{a,i} m_{b,j} m_{c,k} S_{a,b} S_{c,b}^* |G_a||G_b|^2 |G_c|/|G|^2 \tag{135}
\]

\[
P_{S,\Gamma} = \sum_{\text{orbits } Q(b)=0,j} m_{a,i} m_{b,j} \frac{|G_a||G_b|}{|G|} S_{a,b} \Gamma_{(c,k),(b,j)}^* \tag{136}
\]

\[
P_{\Gamma,S} = \sum_{\text{orbits } Q(b)=0,j} \Gamma_{(a,i),(b,j)}^* m_{b,j} m_{c,k} \frac{|G_b||G_c|}{|G|} S_{c,b}^* \tag{137}
\]

\[
P_{\Gamma,\Gamma} = \sum_{\text{orbits } Q(b)=0,j} \Gamma_{(a,i),(b,j)} \Gamma_{(c,k),(b,j)}^* \tag{138}
\]

We see that, due to (133), (136) and (137) are 0. Now we evaluate (133). The sum over \( j \) can be carried out using (130):

\[
P_{S,S} = \sum_{\text{orbits } Q(b)=0} m_{a,i} m_{c,k} S_{a,b} S_{c,b}^* |G_a||G_b|^2 |G_c|/|G| \tag{139}
\]

The sum over \( b \) here runs over representatives of neutral orbits. Using (134) we can extend it to sum over all values of \( b \). Using that \( a, b \) and \( c \) are neutral we get that \( S_{a,b} \) and \( S_{c,b} \) are independent of the specific orbit representative \( S_{a,b} = S_{Ja,Kb} \) and \( S_{c,b} = S_{c,Kb} \). Using this observation we can write

\[
P_{S,S} = \sum_{\text{orbits } Q(b)=0} \sum_{J \in G_a, K \in G_b} m_{a,i} m_{c,k} S_{Ja,Kb} S_{c,Kb}^* |G_c|/|G| \tag{140}
\]

Again using (134) we deduce that the sum over \( a \) allows us to extend the sum over \( b \) from neutral orbits to all orbits. The sum over \( a \) projects out the charged
one. Now when we sum over all values of $b$ we can use unitarity of $S$ and finally write

$$P_{S,S} = \frac{\delta_{ac} m_{a,i} m_{a,k}}{|S_a|}. \quad (141)$$

We get that unitarity imposes the following constraint on $\Gamma$

$$\sum_{\text{orbits } Q(b) = 0} \Gamma_{(a,i),(b,j)} \Gamma_{(c,k),(b,j)} = \delta_{ac} \delta_{ik} - \frac{m_{a,i} m_{a,k}}{|S_a|} \quad (142)$$

derived in [33]. Using the same tricks we turn to the computation of the fusion coefficients.

$$N_{(d,e),(a,i),(c,k)}^{(b,j)} = \sum_{(b,j)} \frac{S_{(a,i),(b,j)} S_{(c,k),(b,j)} S^*_{(b,j),(d,e)}}{S_{(0),(b,j)}} \quad (143)$$

We know that the vacuum state has trivial stabilizer. Therefore

$$\tilde{S}_{(0),(b,j)} = m_{b,j} |G_b| S_{0,b} \quad (144)$$

Inserting (132) and (144) in (143) we obtain the following eight terms:

$$P_{SSS} = \sum_{(\text{orbits } Q(b) = 0)} \frac{m_{a,i} m_{b,j}^2 m_{c,k} m_{d,e} |G_a| |G_b|^2 |G_c| |G_d| S_{a,b} S_{c,b} S_{d,e}^*}{|G|^3 S_{0,b}} \quad (145)$$

$$P_{STS} = \sum_{(\text{orbits } Q(b) = 0)} \frac{m_{a,i} m_{b,j} m_{d,e} |G_a| |G_b| |G_d| S_{a,b} \Gamma_{(c,k),(b,j)} S_{d,e}^*}{|G|^2 S_{0,b}} \quad (146)$$

$$P_{TSS} = \sum_{(\text{orbits } Q(b) = 0)} \frac{m_{c,k} m_{b,j} m_{d,e} |G_c| |G_b| |G_d| \Gamma_{(a,i),(b,j)} S_{c,b} S_{d,e}^*}{|G|^2 S_{0,b}} \quad (147)$$

$$P_{TTS} = \sum_{(\text{orbits } Q(b) = 0)} \frac{m_{d,e} |G_d| \Gamma_{(a,i),(b,j)} \Gamma_{(c,k),(b,j)} S_{b,d}^*}{|G| S_{0,b}} \quad (148)$$

$$P_{SST} = \sum_{(\text{orbits } Q(b) = 0)} \frac{m_{a,i} m_{b,j} m_{c,k} |G_a| |G_b| |G_c| S_{a,b} S_{c,b} \Gamma_{(b,j),(d,e)}^*}{|G|^2 S_{0,b}} \quad (149)$$

$$P_{STT} = \sum_{(\text{orbits } Q(b) = 0)} \frac{m_{a,i} |G_a| S_{a,b} \Gamma_{(c,k),(b,j)} \Gamma_{(b,j),(d,e)}^*}{|G| S_{0,b}} \quad (150)$$

$$P_{TST} = \sum_{(\text{orbits } Q(b) = 0)} \frac{m_{c,k} |G_c| \Gamma_{(a,i),(b,j)} \Gamma_{(c,k),(b,j)} S_{b,d}^*}{|G| S_{0,b}} \quad (151)$$

$$P_{TTT} = \sum_{(\text{orbits } Q(b) = 0)} \frac{\Gamma_{(a,i),(b,j)} \Gamma_{(c,k),(b,j)} \Gamma_{(b,j),(d,e)}^*}{m_{b,j} |G_b| S_{0,b}} \quad (152)$$

28
We see that thanks to (133) the three terms containing two $S$ and one $\Gamma$, namely $P_{SSS}, P_{TSS}, P_{SST}$ given by (146), (147) and (149) correspondingly are zero. Further simplifications occur when we consider the annulus amplitude:

$$A_{(a,i),(d,e)} = \sum_{\text{orbits } Q(c)=0,k} \mathcal{N}^{(d,e)}_{(a,i),(c,k)} \chi_{c,k} = \sum_{\text{orbits } Q(c)=0,k} \mathcal{N}^{(d,e)}_{(a,i),(c,k)} m_{c,k} \sum_{J \in G_c} \chi_{Jc}$$  \hspace{1cm} (153)

Now again due to (133) also (148), (150) and (152) have vanishing contribution. We see that for the evaluation of the annulus amplitude it is enough to consider only the term (145) and (151). Using the same tricks as in the check of unitarity we can easily compute (145). First using (130) we perform the sum over $j$ and obtain

$$P_{SSS} = \sum_{\text{orbits } Q(b)=0} m_{a,i} m_{c,k} m_{d,e} |G_a||G_b||G_c| S_{a,b} S_{c,b} S_{b,d}^*$$  \hspace{1cm} (154)

Again using neutrality of $a$, $c$ and $d$ we can extend the sum over $b$ from the representatives of the neutral orbits to the whole orbit, absorbing $|G_b|$, and afterwards using neutrality of $b$ absorbing $G_a$ in the sum of $a$ over orbit:

$$P_{SSS} = \sum_{\text{orbits } Q(b)=0} \sum_{J \in G_a, K \in G_b} m_{a,i} m_{c,k} m_{d,e} |G_c| |G_d| S_{Ja,Kb} S_{c,Kb} S_{b,d}^*$$  \hspace{1cm} (155)

As before the sum over $a$ allows us to extend the sum from the neutral orbits to all orbits. So we get that the sum over $b$ runs over all values. Using Verlinde formula for the modular $S$-matrix we obtain:

$$P_{SSS} = \sum_{J \in G_a} m_{a,i} m_{c,k} m_{d,e} |G_c| |G_d| \mathcal{N}^d_{Ja,c}$$  \hspace{1cm} (156)

Inserting now (151) and (156) in (153) and again using (130) finally we obtain:

$$A_{(a,i),(d,e)} = \sum_{\text{orbits } Q(c)=0} \sum_{J \in G_a} m_{a,i} m_{d,e} |G_d| \mathcal{N}^d_{Ja,c} \sum_{K \in G_c} \chi_{Kc}$$

$$+ \sum_{\text{orbits } Q(c)=0} \sum_{\text{orbits } Q(b)=0,j} \Gamma_{(a,i),(b,j)} S_{c,b} \Gamma_{(b,j),(d,e)}^* S_{0,b} \sum_{K \in G_c} \chi_{Kc}$$  \hspace{1cm} (157)

It is easy to check that $\sum_{J \in G_a} \mathcal{N}^d_{Ja,c}$ is independent on the orbit representative of $c$:

$$\sum_{J \in G_a} \mathcal{N}^d_{Ja,Kc} = \sum_{J \in G_a} \mathcal{N}^d_{Ja,c}$$  \hspace{1cm} (158)
Also it is easy to see that if \( Q(a) = 0 \), and \( Q(d) = 0 \), \( N^d_{a,c} \neq 0 \) only if also \( Q(c) = 0 \). So without changing the result we can sum over all \( c \) in (153) and omit the sum over \( K \):

\[
\sum_{\text{orbits } Q(c) = 0} \sum_{J \in G_a} a^{m_{a,i}}d^{m_{d,e}}N^d_{J a,c} = \sum_{c} \sum_{J \in G_a} a^{m_{a,i}}d^{m_{d,e}}N^d_{J a,c} \chi_c \quad (159)
\]

Let us momentarily consider the case when we have no fixed points. In this case the annulus amplitude is

\[
A_{a,d} = \sum_c \sum_{J \in G} N^d_{J a,c} \chi_c \quad (160)
\]

This result can be easily interpreted. It means that in the simple current orbifold theory the Cardy boundary states are given by

\[
|a\rangle_{\text{orbifold}} = \frac{1}{|G|} \sum_{J \in G} |Ja\rangle \quad (161)
\]

This is the expected result. This was the starting point of the Recknagel-Schomerus boundary states construction in [13].

\( \Gamma_{(a,i),(b,j)} \) also satisfies the condition (134):

\[
\Gamma_{J(a,i),(b,j)} = e^{2\pi i Q(b)} \Gamma_{(a,i),(b,j)} \quad (162)
\]

In (157) sums over \( (b, j) \) and \( c \) run over neutral representatives, but to evaluate this sum in practice, we can use (162) and the same tricks as before to extend this sum over all values of \( b \) and \( c \). So for the second part in (157) we have:

\[
\sum_{\text{orbits } Q(c) = 0} \sum_{J \in G_a} \sum_{b,j} \sum_{c} \frac{\Gamma_{(a,i),(b,j)} S_{c,b} \Gamma^*_{(b,j),(d,e)}}{S_{0,b} |G_b||G_a|} \chi_c \quad (163)
\]

Finally let us take into account that in the case of Gepner models, all the stabilizer equal \( Z_2 \) and the unitarity condition can be satisfied taking \( \Gamma_{(a,\psi),(b,\psi')} \) in the form

\[
\Gamma_{(a,\psi),(b,\psi')} = \frac{|G_a||G_b|}{|G|} \hat{S}_{ab} \psi \psi' \delta_{af} \delta_{bf} \quad (164)
\]

where the resolving index \( \psi \) takes two values \( \pm \) and \( \hat{S}_{ab} \) is some unitary matrix. Putting this in (163) for this part of the annulus amplitude we get:

\[
\frac{1}{|S_d|} \psi \psi'' \sum_c \sum_{b} \sum_{J \in G_a} \frac{\hat{S}_{Ja,b} S_{c,b} \hat{S}^*_{b,d}}{S_{0,b}} \chi_c \delta_{af} \delta_{bf} \delta_{df} \quad (165)
\]
This is the correction required when fixed points are present. Putting all pieces together we have:

$$A_{(a,i),(d,e)} = \sum_{c} \sum_{J \in G} \frac{m_{a,i}m_{d,e}N_{Ja,c}^{d}}{|S_{a}| |S_{d}|} \chi_{c} +$$

$$\frac{1}{|S_{a}| |S_{d}|} \psi' \psi'' \sum_{c} \sum_{b} \sum_{J \in G} \hat{S}_{Ja,b} S_{c,b} \hat{S}_{b,d}^{*} S_{0,b} \chi_{c} \delta_{a_f} \delta_{b_f} \delta_{d_f}$$

This formula was found by Brunner and Schomerus in [23].
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