Solving bio-heat transfer multi-layer equation using Green’s Functions method

Eduardo Peixoto de Oliveira
School of Mechanical Engineering, Federal University of Uberlândia, Uberlândia, MG, Brazil, 38400-902
E-mail: eduardopeixoto@ufu.br

Gilmar Guimarães
School of Mechanical Engineering, Federal University of Uberlândia, Uberlândia, MG, Brazil, 38400-902
E-mail: gguima@ufu.br

Abstract. An analytical method using Green’s Functions for obtaining solutions in bio-heat transfer problems, modeled by Pennes’ Equation, is presented. Mathematical background on how treating Pennes’ equation and its $\mu^2T$ term is shown, and two contributions to the classical numbering system in heat conduction are proposed: inclusion of terms to specify the presence of the fin term, $\mu^2T$, and identify the biological heat transfer problem. The presentation of the solution is made for a general multi-layer domain, deriving and showing general approaches and Green’s Functions for such n number of layers. Numerical examples are presented to simplify human skin as a two-layer domain: dermis and epidermis, accounting metabolism as a heat source, and blood perfusion only at the dermis. Time-independent summations in the series-solution are written in closed forms, leading to better convergence along the boundaries. Details on obtaining the two-layer solution and its eigenvalues are presented for boundary conditions of prescribed temperature inside the body and convection at the surface, such as its intrinsic verification.

1. Introduction
In the past decades, many authors aim to describe analytically heat conduction problems. The handbook and guide for this kind of work was primarily presented and discussed by Carslaw and Jaeger [1]. Many others authors wrote books that are inserted in this context such as Özisik [2] and Cole et. al. [3]. Beck [4] proposed a numerical system to identify the boundary conditions and other influences in the heat conduction problem. One type of problem that leads to difficulties in solution are multi-layer problems. In 1980, Huang & Chang [5] presented different solutions for infinite and semi-infinite solids multi-layer bodies for permanent and transient regimes with and without periodic behaviors. Obtaining their solutions using Green’s Functions. In 2002, Haji-Sheikh & Beck [6] presented multi-dimensional and multi-layer problems solutions also using Green’s Functions exploring different types of boundary conditions.

Other class of problems that intrigues the researches in the past century is how to describe the heat transfer in biological tissues, and investigate the thermal behavior of biological bodies and
tissues, specially human. Aiming to describe this heat transfer problems, Pennes [7] proposed from an energy balance in a resting human forearm, Eq. (1), known as the Bioheat Equation.

\[ k \nabla^2 T + \omega_b \rho_b c_b (T_a - T) + Q_m = \rho c \frac{\partial T}{\partial t} \]  

(1)

In Eq. (1), \( \rho \), \( c \) and \( k \) are, respectively, the specific mass, the specific heat and the heat conductivity of the tissue, \( \omega_b \) is the blood perfusion rate and, \( \rho_b \) and \( c_b \) are the blood’s specific mass and specific heat; \( T_a \) is the arterial temperature and \( T \) is the temperature distribution. As emphasized by Wissler [8], one of the Pennes goals were the evaluate the applicability of blood flow theory in human forearm in terms of the local heat generation and blood perfusion rates. In this sense, many papers were produced investigating different applications for the Pennes equation for different type of conditions and many techniques such as separation of variables and Laplace’s transforms. Shih et al. [9] proposed a solution to Eq. (1) in a semi-infinite body under a sinusoidal heat flux at the surface, using Laplace transform, evaluating the effects of such type of heat flux as boundary condition in temperature behavior over time and in predicting blood perfusion as a response the phase shift between the surface heat flux and surface temperature, concluding that this kind of prediction is unsuitable for this problem. Talaee & Kabiri [10] presented an analytical solution the the Pennes’ bio-heat equation for a non-homogeneous layer of a biological tissue considering only the blood perfusion term under a moving heat source at the surface, they used a separation of variables technique supported by a variable change, founding a series solution for temperature as function of time and position.

Others, such as Kaplan et al. [11] produced experimental analysis for characterizing both mechanical and thermal properties in biological domains. Kaplan et al. [11] presented experimental data for characterization of silk proteins and its structure. This kind of analysis can also be adapted to investigate human-tissue properties and other biological samples.

To analyze analytically heat conduction in multi-layer bodies, Yang & Liu [12] proposed a closed-form series solution for hollow cylinders, this approach is here used in someway for a biological cartesian domain. In their work, Yang & Liu [12] used a transfer function method, analyzing both steady and transient solution, the last under a Green’s Function approach. In this paper, the Green’s Function method is used to find the solution for one-dimensional multi-layer biological domain both space and time depending.

To illustrate other analytical technique, Ionescu et al. [13] proposed a review paper discussing the hole of fractional calculus in biological heat transfer phenomena. In their work, Ionescu et al. [13] used as model lung tissue, modeling the physical problem as non-integer differ-integral solutions of the diffusion equation. They modeled the lung tissue under a power-law for stress relaxation and constant-phase impedance, building analogies between mechanical and thermal-diffusion systems.

Another work that addresses to future problems that will be analyzed in future researches are the analytical solutions of Laplace’s equation presented by Jiang & Zhou [14] for cylindrical layered media. They obtained and presented analytical solutions for this type of media using separation of variables with the Sturm-Liouville theory and Bessel functions coupled, illustrating solutions for different types of boundary conditions, and showed a solution for general boundary conditions, comparing, validating and showing numerical examples for all the solutions proposed.

Nyborg [15] proposed in the end of 80’s a solution for Eq. (1) for a step-function point source obtaining the temperature field from a heat source distribution by superposition and using Green’s Function. This work contributed for a analysis of the boundary conditions and how to interpret the behavior of heat sources.

Numerically, Chan [16] showed how to use the boundary element method analysis to obtain the temperature solution for the bioheat transfer equation for 2-D steady-state problems, comparing them to analytical solutions. This type of work shows the important role of analytical solutions in (bio)engineering problems.
Huang et al. [17] analyzed the bio-heat transfer phenomena in a blood vessel. For a steady state problem in a polar-coordinate system, they proposed a dimensionless solution in terms of the Stanton, Nusselt, Biot and Peclet numbers.

For a one-dimensional biological tissue domain, Deng & Liu [18] obtain a Green Function based solution considering blood perfusion and metabolism effects for a single layer and different types of heat conditions at the surface and inside the body, also using the Green Function method, but solving the equation for a auxiliary variable to simplify the solution. This analysis is useful for a tumor presence investigation, for example. Adapting their approach for a two-layer body, in this paper we show how to interpret different boundary conditions considering both effects, without making variable changes for temperature, instead we proposed a new index to the classic numbering system in heat conduction problems, and the solution for the original variable (usually $T$) of temperature.

As a different type of analysis, Kengne & Lakhssassi [19] investigated, analytically, the heat transport in the peripheral region of a human limb healing after surgery, using a radial coordinate system and separation of variables method, writing the solution in terms of Bessel Function. Fazlali & Ahmadikia [20] presented a non-fourier heat conduction problem, based on the Pennes Equation, analytical solution using Laplace transforms. Rodrigues et al. [21] proposed a study of the the one dimensional multi-layer problem for radial systems and its applications in terms of separation of variables.

In light of the various works in the literature, few of them discussed above, this article presents the mathematical background to develop a Green Function-based solution of multi-layer biological domains and proposes a reformulation to the classic numerical system proposed by Beck [4], to contribute in terms of analytical solutions for bio-engineering issues the arises every day.

2. Mathematical background

Green’s Functions are a mathematical tool that gives an interpretation of the response of a perturbation made at any point in the domain at any time. To obtain a Green’s Function ($\text{GF}$), basically one has to solve an homogeneous problem of the same kind of boundaries conditions established in the original problem. In heat conduction problems, Beck [4] demonstrates how to write the solution for a well-posed problem with any kind of boundary condition and tabulate GF’s for 0-th, 1st, 2nd and 3rd types of boundary conditions combined and based on a numerical system used to identify boundary conditions and other characteristics of the problem.

Given a problem in an $x$ coordinate system along time $t$:

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{k} g(x,t) = \frac{1}{\alpha} \frac{\partial T}{\partial t}, \quad x_1 \leq x \leq x_2, \quad t \geq 0 \quad (2)$$

with initial condition:

$$T(x,0) = F(x) \quad (3)$$

and boundary conditions generically written as:

$$k_i \frac{\partial T}{\partial n} \bigg|_{x=x_i} + h_i T(x_i, t) = f_i(t), \quad t > 0 \quad (4)$$

For 1st (prescribed temperature), 2nd (prescribed flux) and 3rd (convection) types of boundary conditions, $k_i = 0$ and $h_i = 1$, $k_i = k$ and $h_i = 0$, $k_i = k$ and $h_i = h$, respectively. Therefore, the Green Function, $G(x,t|x',\tau)$, is the solution of

$$\frac{\partial^2 G}{\partial x'^2} + \frac{1}{\alpha} \delta(x - x') \delta(t - \tau) = -\frac{1}{\alpha} \frac{\partial G}{\partial \tau} \quad (5)$$

3
given
\[ G(x', -\tau | x, -t) = 0 \] (6)

submitted to homogeneous boundary conditions:
\[ k_i \frac{\partial G}{\partial n'} \bigg|_{x' = x'_i} + h_i G = 0 \] (7)

Once one has the GF, solution shall be written as:
\[ T(x, t) = \int_{x' = x_1}^{x' = x_2} G(x, t | x', 0) F(x') dx' + \alpha \int_{\tau = 0}^{t} \int_{x' = x_1}^{x' = x_2} G(x, t | x', \tau) g(x', \tau) dx' d\tau + \alpha \int_{\tau = 0}^{t} \frac{\partial G}{\partial n_j} \bigg|_{x' = x_j} \] (1st type BC)

\[ \int_{\tau = 0}^{t} \left[ \frac{f_1(\tau)}{k_i} G(x, t | x'_1, \tau) + \frac{f_2(\tau)}{k_i} G(x, t | x'_2, \tau) \right] d\tau \] (2nd and 3rd types BC) (8)

for a XIJ problem, being I and J the type of boundary conditions at \( x = x_1 \) and \( x = x_2 \), respectively.

2.1. Fin-term \( \mu^2 T \)

If a term proportional to the temperature such as \( \mu^2 T \) is added to eq. (2):
\[ k_i \frac{\partial T}{\partial n} \bigg|_{x = x_i} + h_i T(x_i, t) - \mu^2 T = f_i(t) \,, \, t > 0 \] (9)

the problem type will be addressed as XIJf, for \( f \) indicating the presence of this kind of term that usually appears when solving extended surfaces conduction problems.

The procedure to obtain the Green Function is the same shown above, except that the GF obtained has to be multiplied by \( \exp[-\alpha \mu^2 (t - \tau)] \).

2.2. Multi-layer solution

For a domain that can be split in \( n \) layers with different thermal properties it is also possible to obtain a general solution in terms of Greens’ Functions, based on Chang [5]. To illustrate that, consider a 2-layer domain in one-dimensional coordinate, \( y \), e.g., \( -y_1 < y < 0 \) e \( 0 < y < y_2 \) that may have a heat source in each of it: \( g_i(y, t) \). In the \( i \)-th layer, \( T_i(y, t) \) behaves like eq. (10).
\[ \frac{\partial^2 T_i}{\partial y^2} + \frac{g_i(y, t)}{k_i} - \mu^2 T_i = \frac{1}{\alpha_i} \frac{\partial T_i}{\partial t} \] (10)

with initial condition
\[ T_i(y, 0) = F_i(y) \] (11)

and generic boundary conditions:
\[ k_1 \frac{\partial T_1}{\partial y} - h_1 T_1 = f_1(t) \,, \, y = -y_1 \] (12)
\[ k_2 \frac{\partial T_2}{\partial y} + h_2 T_2 = f_2(t) \,, \, y = y_2 \] (13)
To show how to build the GF in multi-layer problems, it will be considered that between the layers there is no thermal resistance, i.e., perfect thermal contact.

\[ k_1 \frac{\partial T_1}{\partial y} = k_2 \frac{\partial T_2}{\partial y} , \quad y = 0 \]  

(14)

and

\[ T_1 = T_2 , \quad y = 0 \]  

(15)

In classical conduction numerical system, this problem would be indicated as YICJ, but the authors consider that is important to highlight the type of contact, including a 0 index after ‘C’, and the presence of the fin term adding ‘f’ at the end. Therefore, this kind of problem will be indicated as YIC0Jf.

The GF associated to this type of problem, \( G_{ij} \), are the solutions of a homogeneous boundary-value problem submitted to the same boundary conditions and indicated in equations (16) to (21), where \( \delta_{ij} \) is the Kronecker’s delta function (\( \delta_{ij} = 1 \) for \( i = j \) and \( \delta_{ij} = 0 \) for \( i \neq j \)), \( i \) and \( j \) indicate the layer that is being taken account.

\[ \frac{\partial^2 G_{ij}}{\partial y^2} + \delta(y - y')\delta(t - \tau)\delta_{ij} - \frac{\mu_i^2}{\alpha_i} G_{ij} = \frac{1}{\alpha_i} \frac{\partial G_{ij}}{\partial t} \]  

(16)

\[ G_{ij} = 0 , \quad t = 0 \]  

(17)

\[ k_1 \frac{\partial G_{1j}}{\partial y} - h_1 G_{1j} = 0 , \quad y = -y_1 \]  

(18)

\[ k_2 \frac{\partial G_{2j}}{\partial y} + h_2 G_{2j} = 0 , \quad y = y_2 \]  

(19)

\[ k_1 \frac{\partial G_{1j}}{\partial y} = k_2 \frac{\partial G_{2j}}{\partial y} , \quad y = 0 \]  

(20)

and

\[ G_{1j} = G_{2j} , \quad y = 0 \]  

(21)

The general form of the GF associated to multi-layer heat conduction problems is:

\[ G_{ij}(y,t|y',\tau) = \sum_{n=1}^{\infty} \rho_j c_p j Y_i(y) Y_j(y') \exp \left[ -\lambda_n^2 (t - \tau) \right] \frac{\rho_i c_p i Y_i^2}{N_{y,n}} \]  

(22)

where \( Y_i \) are the eigenfunctions of the homogeneous equation associated to eq. (10) in the form:

\[ Y_i = A_i \cos(\beta_i y) + B_i \sin(\beta_i y) \]  

(23)

being \( A_i \) and \( B_i \) constants that depend on the boundary conditions of the problem, \( \beta_i \equiv \beta_{i,n} \) is the \( n \)-th auxiliary eigenvalue that is related to the \( n \)-th eigenvalue of the problem, \( \lambda_n \), as shown in eq. (24).

\[ \beta_{i,n}^2 = \frac{\lambda_n^2}{\alpha_i} - \mu_i^2 \]  

(24)

and \( N_{y,n} \) is the norm of the problem defined as

\[ N_{y,n} = \int_{-y_1}^{y_2} \rho_j c_p j Y_i^2 dy \]  

(25)

The YIC0Jf problem general solution for the \( i \)-th layer can be demonstrated and written as

\[ T_i(y,t) = \int_{y'=-y_1}^{y_2} F_j(y') G_{ij}(y,t|y',0) dy' + \int_{\tau=0}^{t} \int_{y'=-y_1}^{y_2} \alpha_j k_j g_j(y',\tau) G_{ij} dy' d\tau + \int_{\tau=0}^{t} \int_{y'=-y_1}^{y_2} \alpha_j \left( \frac{\partial^2 T_j}{\partial y'^2} - T_j \frac{\partial^2 G_{ij}}{\partial y'^2} \right) dy' d\tau \]  

(26)
2.3. Y12C03fBIO problem

To make explicit that the problem runs in biological domains, it is inserted in the numerical system ‘BIO’ after the identification of boundaries condition and terms of equation indexes. Considering a small piece of human skin as a 2-layer domain (dermis and epidermis) with different thermal properties $k_i, c_{pi}$ and $\rho_i$ subjected to convection and inner temperature $T_C$, called body temperature, with average values of blood perfusion ($\omega_b$) and metabolism ($Q_m$) in dermis (first layer, $-y_1 < y < 0$) and perfect contact between dermis and epidermis as shown is Figure 1, Pennes’ Equation can be written for each layer as in equations (27) and (29).

Figure 1. Schematic model of the Y12C03fBIO problem. Dermis ($-y_1 < y < 0$) and epidermis ($0 < y < y_2$).

\[
\frac{\partial^2 T_1}{\partial y^2} + \frac{g_1(y, t)}{k_1} - \mu_1^2 T_1 = \frac{1}{\alpha_1} \frac{\partial T_1}{\partial t}, \quad -y_1 < y < 0
\]

(27)

defining

\[
g_1(y, t) \equiv Q_m + \omega_b \rho_b c_b T_a \quad \text{and} \quad \mu_1^2 \equiv \frac{\omega_b \rho_b c_b}{k_1}
\]

(28)

and

\[
\frac{\partial^2 T_2}{\partial y^2} = \frac{1}{\alpha_2} \frac{\partial T_2}{\partial t}, \quad 0 < y < y_2
\]

(29)

initial condition

\[
T_1(y, 0) = T_2(y, 0) = T_C
\]

(30)

and boundary conditions

\[
T_1 = T_c, \quad y = -y_1
\]

(31)

\[
\begin{aligned}
T_1 &= T_2, \\
k_1 \frac{\partial T_1}{\partial y} &= k_2 \frac{\partial T_2}{\partial y}
\end{aligned}
\]

(32)

and

\[
k_2 \frac{\partial T_2}{\partial y} + h T_2 = h T_\infty, \quad y = y_2
\]

(33)
The solutions, in terms of eq. (26), are:

\[
T_1(y, t) = T_c \int_{-y_t}^{0} G_{11}(y, t|y', 0)dy' + T_c \int_{0}^{y_t} G_{12}(y, t|y', 0)dy' + \\
+ \frac{\alpha_1}{k_1} (Q_m + \omega_b \rho_b c_b T_a) \int_{-y_t}^{t} \int_{0}^{0} G_{11}(y, t|y', \tau)dy'd\tau + \\
+ \alpha_1 T_c \int_{\tau=0}^{t} \left. \frac{\partial G_{11}}{\partial y'} \right|_{y'=-y_t} d\tau + \frac{\alpha_2 hT_{\infty}}{k_2} \int_{\tau=0}^{t} G_{12}(y, t|y_2, \tau)d\tau
\]

\[\text{(34)}\]

\[
T_2(y, t) = T_c \int_{-y_t}^{0} G_{21}(y, t|y', 0)dy' + T_c \int_{0}^{y_t} G_{22}(y, t|y', 0)dy' + \\
+ \frac{\alpha_1}{k_1} (Q_m + \omega_b \rho_b c_b T_a) \int_{-y_t}^{t} \int_{0}^{0} G_{21}(y, t|y', \tau)dy'd\tau + \\
+ \alpha_1 T_c \int_{\tau=0}^{t} \left. \frac{\partial G_{21}}{\partial y'} \right|_{y'=-y_t} d\tau + \frac{\alpha_2 hT_{\infty}}{k_2} \int_{\tau=0}^{t} G_{22}(y, t|y', \tau)d\tau
\]

\[\text{(35)}\]

the eigenfunctions used to build the GF are \(Y_1(y) = \cos(\eta y) + B_1 \sin(\eta y)\) and \(Y_2 = \cos(\gamma y) + B_2 \sin(\gamma y)\), where \(B_1 = \cot(\eta y_1)\), \(B_2 = (k_1/k_2)(\eta/\gamma) \cot(\eta y_1)\), \(\eta = \sqrt{\lambda_n^2/\alpha_1 - \mu_1^2}\) and \(\gamma = \lambda_n/\sqrt{\alpha_2}\). And the calculated norm, defined in eq. (25),

\[
N_{y,n} = \rho_1 c_p \left\{ \frac{B_1}{2\eta} \cos(2\eta y_1) - 1 \right\} + \frac{(1 - B_1^2)}{4\eta} \sin(2\eta y_1) + \frac{(1 + B_1^2)}{2} \eta_1 + \\
+ \rho_2 c_p \left\{ \frac{B_2}{2\gamma} [1 - \cos(2\gamma y_2)] + \frac{(1 - B_2^2)}{4\gamma} \sin(2\gamma y_2) + \frac{(1 + B_2^2)}{2} \eta_2 \right\}
\]

\[\text{(36)}\]

And the general solution in each layer can be written as

\[
T_1(y, t) = \rho_1 c_p T_c \sum_{n=1}^{\infty} \frac{\cos(\eta y) + B_1 \sin(\eta y)}{N_{y,n}} \cdot \frac{\sin(\eta y_1) + B_1 \cos(\eta y_1) - B_1}{\eta} \cdot \exp(-\lambda_n^2 t) + \\
+ \rho_2 c_p T_c \sum_{n=1}^{\infty} \frac{\cos(\eta y) + B_1 \sin(\eta y)}{N_{y,n}} \cdot \frac{\sin(\gamma y_2) - B_2 \cos(\gamma y_2) + B_2}{\gamma} \cdot \exp(-\lambda_n^2 t) + \\
+ (Q_m + \omega_b \rho_b c_b T_a) \sum_{n=1}^{\infty} \frac{\cos(\eta y) + B_1 \sin(\eta y)}{N_{y,n}} \cdot \frac{\sin(\eta y_1) + B_1 \cos(\eta y_1) - B_1}{\eta} \cdot \frac{1 - \exp(-\lambda_n^2 t)}{\lambda_n^2} + \\
+ k_1 T_c \sum_{n=1}^{\infty} \frac{\cos(\gamma y_2) + B_1 \cos(\gamma y_2)}{N_{y,n}} \cdot \frac{[\eta \sin(\eta y_1) + B_1 \eta \cos(\eta y_1)]}{\lambda_n^2} \cdot \frac{1 - \exp(-\lambda_n^2 t)}{\lambda_n^2} + \\
+ hT_{\infty} \sum_{n=1}^{\infty} \frac{\cos(\eta y) + B_1 \sin(\eta y)}{N_{y,n}} \cdot \frac{\cos(\gamma y_2) + B_2 \sin(\gamma y_2)}{\gamma} \cdot \frac{1 - \exp(-\lambda_n^2 t)}{\lambda_n^2}
\]

\[\text{(37)}\]
and

\[ T_2(y,t) = \rho_1 c_p T_c \sum_{n=1}^{\infty} \frac{\cos(\gamma y) + B_2 \sin(\gamma y)}{N_{y,\eta}} \cdot \left[ \frac{\sin(\eta y_1) + B_1 \cos(\eta y_1) - B_1}{\eta} \right] \cdot \exp(-\lambda_n^2 t) + \]

\[ + \rho_2 c_p T_c \sum_{n=1}^{\infty} \frac{\cos(\gamma y) + B_2 \sin(\gamma y)}{N_{y,\gamma}} \cdot \left[ \frac{\sin(\gamma y_2) - B_2 \cos(\gamma y_2) + B_2}{\gamma} \right] \cdot \exp(-\lambda_n^2 t) + \]

\[ + (Q_n + \omega \rho_0 c_b T_a) \sum_{n=1}^{\infty} \frac{\cos(\gamma y) + B_2 \sin(\gamma y)}{N_{y,\eta}} \cdot \left[ \frac{\sin(\eta y_1) + B_1 \cos(\eta y_1) - B_1}{\eta} \right] \cdot \frac{1 - \exp(-\lambda_n^2 t)}{\lambda_n^2} + \]

\[ + k_1 T_c \sum_{n=1}^{\infty} \frac{\cos(\gamma y) + B_2 \sin(\gamma y)}{N_{y,\gamma}} \cdot \left[ \frac{\sin(\gamma y_2) + B_2 \cos(\gamma y_2) + B_2}{\gamma} \right] \cdot \frac{1 - \exp(-\lambda_n^2 t)}{\lambda_n^2} + \]

\[ + hT_\infty \sum_{n=1}^{\infty} \frac{\cos(\gamma y) + B_2 \sin(\gamma y)}{N_{y,\eta}} \cdot \left[ \frac{\cos(\gamma y_2) + B_2 \sin(\gamma y_2)}{\gamma} \right] \cdot \frac{1 - \exp(-\lambda_n^2 t)}{\lambda_n^2} \]

(38)

The eigencondition, from the boundary conditions obtained during separation of variables process, for this type of problem is:

\[ f(\lambda) = f(\eta, \gamma) = -k_2^2 \gamma^2 \sin(\eta y_1) \sin(\gamma y_2) + k_1 k_2 \eta \gamma \cos(\eta y_1) \cos(\gamma y_2) + \]

\[ + h k_2 \gamma \sin(\eta y_1) \cos(\gamma y_2) + h k_1 \eta \cos(\eta y_1) \sin(\gamma y_2) \]

(39)

Therefore, the eigenvalues are the roots of eq. (39).

2.4. Closed form solution

To avoid instabilities and convergence issues at the boundaries, it is proposed to overwrite the solutions presented in equations (37) and (38) in terms of closed forms. To do so, the same-type problem was solved for permanent regime, which solutions are the same as those presented when limits are taken for a \( t \rightarrow \infty \), leading to the expressions:

\[ T_1(y,t) = \rho_1 c_p T_c \sum_{n=1}^{\infty} \frac{\cos(\eta y) + B_1 \sin(\eta y)}{N_{y,\eta}} \cdot \left[ \frac{\sin(\eta y_1) + B_1 \cos(\eta y_1) - B_1}{\eta} \right] \cdot \exp(-\lambda_n^2 t) + \]

\[ + \rho_2 c_p T_c \sum_{n=1}^{\infty} \frac{\cos(\eta y) + B_1 \sin(\eta y)}{N_{y,\gamma}} \cdot \left[ \frac{\sin(\gamma y_2) - B_2 \cos(\gamma y_2) + B_2}{\gamma} \right] \cdot \exp(-\lambda_n^2 t) - \]

\[ - (Q_n + \omega \rho_0 c_b T_a) \sum_{n=1}^{\infty} \frac{\cos(\eta y) + B_1 \sin(\eta y)}{N_{y,\eta}} \cdot \left[ \frac{\sin(\eta y_1) + B_1 \cos(\eta y_1) - B_1}{\eta} \right] \cdot \frac{1 - \exp(-\lambda_n^2 t)}{\lambda_n^2} - \]

\[ - k_1 T_c \sum_{n=1}^{\infty} \frac{\cos(\eta y) + B_1 \sin(\eta y)}{N_{y,\gamma}} \cdot \left[ \frac{\sin(\eta y_1) + B_1 \cos(\eta y_1) - B_1}{\eta} \right] \cdot \frac{1 - \exp(-\lambda_n^2 t)}{\lambda_n^2} - \]

\[ - h T_\infty \sum_{n=1}^{\infty} \frac{\cos(\eta y) + B_1 \sin(\eta y)}{N_{y,\eta}} \cdot \left[ \frac{\cos(\gamma y_2) + B_2 \sin(\gamma y_2)}{\gamma} \right] \cdot \frac{1 - \exp(-\lambda_n^2 t)}{\lambda_n^2} + \]

\[ + \left[ k_2 h (T_\infty - T_p) \cosh(\mu y_1) + k_2 h (T_p - T_c) \right] \cdot \sinh(\mu y_1) + \]

\[ + \left[ k_2 h (T_\infty - T_p) \sinh(\mu y_1) + k_1 \mu (k_2 + h y_0) \mu \cosh(\mu y_1) \right] \cdot \sinh(\mu y) + \]

\[ + \left[ k_2 h (T_\infty - T_p) \sinh(\mu y_1) + k_1 \mu (k_2 + h y_0) \mu \cosh(\mu y_1) \right] \cdot \cosh(\mu y) + T_p \]

(40)
\[ T_2(y,t) = \rho_1 c_p T_c \sum_{n=1}^{\infty} \frac{[\cos(\gamma y) + B_2 \sin(\gamma y)]}{N_{y,n}} \cdot \frac{[\sin(\eta y_1) + B_1 \cos(\eta y_1) - B_1]}{\eta} \cdot \exp(-\lambda^2_n t) + \]
\[ + \rho_2 c_p T_c \sum_{n=1}^{\infty} \frac{[\cos(\gamma y) + B_2 \sin(\gamma y)]}{N_{y,n}} \cdot \frac{[\sin(\eta y_2) - B_2 \cos(\eta y_2) + B_2]}{\gamma} \cdot \exp(-\lambda^2_n t) - \]
\[ -(Q_m + \omega \rho_b c_b T_a) \sum_{n=1}^{\infty} \frac{[\cos(\gamma y) + B_2 \sin(\gamma y)]}{N_{y,n}} \cdot \frac{[\sin(\eta y_1) + B_1 \cos(\eta y_1) - B_1]}{\eta} \cdot \exp(-\lambda^2_n t) - \]
\[ - k_1 T_c \sum_{n=1}^{\infty} \frac{[\cos(\gamma y) + B_2 \sin(\gamma y)] \cdot [\sin(\eta y_1) + B_1 \eta \cos(\eta y_1)]}{N_{y,n}} \cdot \exp(-\lambda^2_n t) - \]
\[ - h T_\infty \sum_{n=1}^{\infty} \frac{[\cos(\gamma y) + B_2 \sin(\gamma y)] \cdot [\cos(\gamma y) + B_2 \sin(\gamma y)]}{N_{y,n}} \cdot \exp(-\lambda^2_n t) + \]
\[ + \frac{k_1 h \mu (T_\infty - T_p) \cosh(\mu y_1) - k_1 h \mu (T_c - T_p)}{k_1 h \sinh(\mu y_1) + k_1 (k_2 + h y_2) \mu \cosh(\mu y_1)} y + \]
\[ + \frac{k_1 (k_2 + h y_2) \mu T_p \cosh(\mu y_1) + k_1 h T_\infty \sinh(\mu y_1) + k_1 (k_2 + h y_2) \mu (T_c - T_p)}{k_1 h \sinh(\mu y_1) + k_1 (k_2 + h y_2) \mu \cosh(\mu y_1)} \]
\[ (41) \]

with \( T_p = (Q_m + \omega \rho_b c_b T_a)/(\omega \rho_b c_b T_a) \).

3. Numerical examples

To show time behavior of the temperature, numerical values have been assigned to each layer as follows (adapted from [18]): thicknesses \( y_1 = 2 \text{ mm} \) and \( y_2 = 0.8 \text{ mm} \), thermal conductivities \( k_1 = 5.2 \cdot 10^{-4} \text{ W/(mm.}°\text{C)} \) and \( k_2 = 2.6 \cdot 10^{-4} \text{ W/(mm.}°\text{C)} \), specific masses: \( \rho_1 = 1.2 \cdot 10^{-3} \text{ g/mm}^3 \), \( \rho_b = 1.06 \cdot 10^{-3} \text{ g/mm}^3 \) and \( \rho_2 = 1.2 \cdot 10^{-3} \text{ g/mm}^3 \); specific heats \( c_p = 3.4 \text{ J/(g.}°\text{C)} \), \( c_b = 4.15 \text{ J/(g.}°\text{C)} \) and \( c_\rho_b = 3.6 \text{ J/(g.}°\text{C)} \); blood perfusion \( \omega = 5 \cdot 10^{-4} \text{ mL/s/mL; metabolism} \)
\( Q_m = 3.38 \cdot 10^{-5} \text{ W/mm}^3 \); both arterial and corporal temperatures set as \( T = T_c = 37°\text{C} \); and outer thermal convection coefficient \( h = 25 \text{ W/(m}^2\text{-K)} \). The eigenvalues were obtained with a second order Newton’s modified method, proposed by Haji-Sheik & Beck [6], from an initial estimation \( \lambda_0 \) for a eigenvalue and the eigencondition (eq. (39)) the functional \( f_0 = f(\lambda_0), f_1 = f'(\lambda_0) \) and \( f_2 = f''(\lambda_0) \) are calculated and the next estimation will be:

\[ \lambda_1 = \lambda_0 - \frac{f_0}{f_1} + \frac{1}{2} f_2 \frac{f_0^2}{f_1^2} \]

usually the numerical computation converges after 4-6 steps.

Figure 2 illustrates de eigenfunction behavior in terms of all the eigenvalues lesser than 500. The first eigenvalues are shown in Figure 3.

Numerical results for the properties above are shown in Figure 4, that illustrates how the temperature behaves along time inside the two-layer domain that simplifies that dermis-epidermis human skin model.

In order to investigate the geometric and heat generation dependence of the solutions presented, in Figure 5 it is show that, once the domain is smaller than the previous one, permanent regime is achieved faster, that also corroborates to the physics of the problem, once the distances are smaller and heat can flow faster.

Figure 6 shows an investigation of a anomalous metabolic heat generation \( (Q_m = 6 \cdot 10^{-4} \text{ W/mm}^3) \), i.e., one order higher than the common values, to illustrate the dermis behavior in presence of exaggerated heat source values that can simulate, e.g., a tumor.

To verify the obtained solutions it is important to perform and intrinsic verification, that is one of the advantages of using the Green Function method. It consists in reducing the solution
Figure 2. Eigenfunction $f(\lambda)$.

Figure 3. Eigenfunction $f(\lambda)$: first values.

for a well-known solution of simpler cases and guarantees stability and coherence for the series expressions presented. For intrinsic verification two cases were evaluated: (i) reducing the problem to a one-layer domain without heat generation and no blood perfusion - Figures 7, 8 and 9; (ii) no metabolism heat generation for a 2-layer domain - Figures 10, 11 and 12. In both situations, the permanent regime solutions were obtained and compared to the analytical classic
Figure 4. Temperature evolution along time.

Figure 5. Time-dependent temperature for a 1.5 mm dermis.

solutions that are easy-obtained when eliminating biological effects from Pennes Equation (Eq. (1)). Is is showns that the errors are in order of $10^{-12}$ when using closed-form solutions, and even though without them errors of $10^{-5}$ were encountered. These results show that that solutions are in according to the physics and are suitable of improvement of details and can be used as bases for more complex problems.
**Figure 6.** Time-dependent temperature for a 1.5 mm dermis with an anomalous metabolism generation one order high than the common.

**Figure 7.** Time-dependent temperature - 1 layer without metabolism nor blood perfusion (1000 eigenvalues). (a) without closed-form solutions; (b) closed-form solutions.
Figure 8. Permanent-regime solution with Green Functions for 1 layer without metabolism nor blood perfusion (1000 eigenvalues).

Figure 9. Validation of the intrinsic verification for permanent regime without metabolism nor blood perfusion (1000 eigenvalues): (a) without closed-form solutions; (b) closed-form solutions.
Figure 10. Time-dependent temperature for the 2 layer case without metabolism (1000) eigenvalues: (a) without closed-forms; (b) closed-form solutions.

Figure 11. Permanent regime Green Function solution for the 2 layer case without metabolism (1000 eigenvalues).
Figure 12. Validation of the intrinsic verification for permanent regime 2-layer case without metabolism (1000 eigenvalues): (a) without closed-form solutions; (b) closed-form solutions.

4. Conclusions
A Green-function based solution for two-layer biological domain was presented. In terms of closed-forms series solutions, expressions were validated corroborating to the physics validation and mathematical stability of the the solutions. Numerical examples showed the temperature behavior inside human body and can be used as basis to the formulation of three-dimensional problems and as validation of experimental and numerical data.

5. Acknowledgments
The authors would like to acknowledge the Graduate Student Program in Mechanical Engineering at Federal University of Uberlândia, and the sponsor agencies CAPES and CNPq.

6. References
[1] Carslaw HS & Jaeger JC 1959 Conduction of Heat in Solids (Oxford at the Clarendon Press)
[2] Özisik M 1993 Heat Conduction (New York, NY: John Wiley)
[3] Cole K, Beck J, Hajji-SHeikh A and Litkouhi B 2010 Heat Conduction Using Green’s Functions (Series in Computational and Physical Processes in Mechanics and Thermal Sciences) (Boca Raton, FL: CRC Press)
[4] Beck J 1984 J. of Heat and Mass Transf. 27 327-33
[5] Huang S, Chang Y 1980 J. of Heat Transfer 102 742-8
[6] Haji-Sheikh A, Beck J 2002 Intl. J. Heat Transfer 45 1865-77
[7] Pennes HH 1948 JAP 1 5-34
[8] Wissler H 1998 JAP 85 35-41
[9] Shih TC et al. 2007 Medical Eng. & Phys. 29 946-53
[10] Talaee M, Kabiri A 2017 J. of Mech. in Medicine and Biology 17 811-12
[11] Kaplan D, McGill M & Holland G 2016 Macromolecular Rapid Communications 901-14
[12] Yang B, Liu S 2017 Intl. J. of Heat and Mass Transfer 907-17
[13] Ionescu C et al. 2017 Communications in Nonlinear Science and Numerical Simulation
[14] Jiang J, Shou Z 2020 Intl. J. of Mech. Sciences 184
[15] Nyborg WL 1988 Phys in Med and Biology 33 785-792
[16] Chan CL 1992 J of Biomech Engineering 114 358-365
[17] Huang HW, Chan CL 1994 J of Biomech Engineering 116 208-12
[18] Deng Z-S, Liu L 2002 Biomech Engineering 124 638-49
[19] Kengne E, Lakhessassi A 2015 Math Biosciences 269 1-9
[20] Ahmadikia H, Moradi A, Fazlali R et al. 2012 J Mech Sci Technol 26 1937-1947
[21] Rodrigues DB, Pereira PJS, Limão-Vieira PR, Maccarini PF 2013 Int J of Heat and Mass Transfer 62 153-162
[22] De Monte F, Haji-Sheikh A 2017 Int J of Heat and Mass Transfer 113 1291-1305
[23] Hobuny A, Abbas I 2019 Mech Based Design of Structures and Machines 49 430-439
[24] Ahmed N, Shah N A, Ali F, Viery D, Zaman FD 2021 Mathematics 2021 1156
[25] Ma J, Yang X, Sun Y et al. 2021 Sci Rep 21 9958