Group analysis and exact solutions for equations of axion electrodynamics

Oksana Kuriksha and A.G. Nikitin$^a$

$^a$Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka Street, Kyiv-4, Ukraine, 01601

Abstract

The group classification of models of axion electrodynamics with arbitrary self interaction of axionic field is carried out. It is shown that the extension of the basic Poincaré invariance is possible only for constant and exponential self interaction terms. With using the three-dimensional subalgebras of the Lie algebra of Poincaré group an extended class of exact solutions for the electromagnetic and axionic fields is obtained. These solutions include arbitrary parameters and arbitrary functions as well, the most general solutions include six arbitrary functions. Among them there are bound and square integrable solutions which propagate faster than light.

---

$^1$E-mail: gaponova@imath.kiev.ua, nikitin@imath.kiev.ua
1 Introduction

Group analysis of partial differential equations is a well developed branch of mathematics including a number of interesting fundamental problems. But maybe the main its value are the powerful tools for construction of exact solutions of complicated nonlinear equations. Sometimes it is the group analysis which gives the only hope to obtain at least some solutions for complicated physical (chemical, biological...) models.

In this preprint we present some results obtained with application of the group analysis to the important physical model called axion electrodynamics. There was a lot of motivations for this research, both physical and mathematical.

Axions are hypothetical particles which belong to main candidates to form the dark matter, see, e.g. [1]. Additional arguments to investigate axionic theories appeared in solid states physics, since the axionic-type interaction terms appears in the theoretical description of crystalline solids called topological insulators [2]. The existence of a pseudoscalar (axion) field can be extracted from the experimental data concerning electric field induced magnetization on Cr$_2$O$_3$ crystals or the magnetic field-induced polarization [3]. In addition, the axion hypothesis makes it possible to resolve the fundamental problem of quantum chromodynamics connected with the CP symmetry violation in interquark interactions [4]-[6]. Thus it is interesting to make group analysis of axionic theories which are requested in the three fundamental branches of physics, i.e., the cosmology, condensed matter physics and quantum chromodynamics.

Let us present two more motivations which are very inspiring for us. Recently a new exactly solvable model for neutral Dirac fermions had been found and other integrable models for such particles had been indicated [8]. But these models involve an external EM field which do not solve Maxwell equations with physically reasonable currents. However, these fields solve equations of axion electrodynamics.

We had classified exactly solvable quantum mechanical systems including shape invariant matrix potentials [9], [10]. Some of these systems also include solutions of field equations of axion electrodynamics.

We have described the finite dimensional indecomposable vector representations of the homogeneous Galilei group and construct Lagrangians which admit the corresponding symmetries [11]-[13]. And exactly the Lagrangian of axion electrodynamics appears to be the relativistic counterpart of some these Galilei invariant Lagrangians [14].

In addition, axion electrodynamics is a nice and complicated mathematical model which needs good group-theoretical grounds. In this preprint we present such grounds and also find an extended class of exact solutions for the related field equations.
2 Equations of axion electrodynamics

Let us consider the following generalized Lagrangian:

\[
L = \frac{1}{2} p_\mu p^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{4} \theta F_{\mu\nu} \tilde{F}^{\mu\nu} - V(\theta). \tag{1}
\]

Here \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \) \( A_\mu \) is the vector-potential of electromagnetic field, \( \tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \) \( \theta \) is the axion field, \( p_\mu = \partial_\mu \theta, \) \( V(\theta) \) is a function of \( \theta \) and \( \kappa \) is a dimensionless constant which is supposed to be nonzero and can be rescaled to the unity.

If \( \theta \equiv 0 \) then formula (1) gives the Lagrangian of Maxwell field. For \( V(\theta) = \frac{1}{2} m^2 \theta^2 \) equation (1) reduces to the standard Lagrangian of axion electrodynamics.

Our goal is to make the group classification of the Euler-Lagrange equations corresponding to the generalized Lagrangian (1) with arbitrary \( V(\theta) \) which have the following form:

\[
\begin{align*}
\nabla \cdot E &= \kappa p \cdot B, \\
\partial_\mu E - \nabla \times B &= \kappa (p_0 B + p \times E), \\
\nabla \cdot B &= 0, \\
\partial_\mu B + \nabla \times E &= 0, \\
\Box \theta &= -\kappa E \cdot B + F.
\end{align*}
\tag{2}
\]

Here \( B \) and \( E \) are vectors of the magnetic and electric fields whose components are expressed via \( F_{\mu\nu} \) as \( E^a = F^a_0, \ B^a = -\frac{1}{2} \varepsilon^{abc} F_{bc}, \) and \( F = -\frac{\partial V}{\partial \theta}, \) \( \Box = \partial^2 - \nabla^2, \) \( \nabla^a = \partial_a = \frac{\partial}{\partial x^a}, \) \( a = 1, 2, 3. \)

Will search for solutions of system (2), (3) with \( \kappa = 1. \) First we notice that equations (2), (3) are invariant with respect to discrete transformations of space reflections \( x_a \rightarrow -x_a, \ E^a \rightarrow -E^a, \ B^a \rightarrow H^a, \ \theta \rightarrow -\theta \) provided \( F \) is an odd function of \( \theta. \) In other words, \( \theta \) transforms as a pseudoscalar.

We will consider also the following system

\[
\begin{align*}
\nabla \cdot E &= \kappa p \cdot E, \\
\partial_\mu E - \nabla \times B &= \kappa (p_0 E + p \times B), \\
\nabla \cdot B &= 0, \\
\partial_\mu B + \nabla \times E &= 0, \\
\Box \theta &= \kappa (B^2 - E^2) + F.
\end{align*}
\tag{4}
\]

which model the generalized axion electrodynamics with a scalar axionic field.

We will present symmetries of equations (2), (3) and (4) with arbitrary \( F(\theta) \) and also their exact solutions, which can be found using these symmetries. We shall present also some results of group classification for more general systems with \( F \) being an arbitrary function of \( \theta \) and \( p_\mu p^\mu. \)
3 Exact solutions: definitions and examples

3.1 Algorithm for finding group solutions

Since the system (2), (3) admits rather extended symmetries, it is possible to find a number of its exact solutions. The algorithm for construction of group solutions of partial differential equations goes back to Sophus Lie. Being applied to system (2), (3) it includes the following steps (compare, e.g., with [15]):

- To find a basis of the maximal Lie algebra \( A_m \) corresponding to continuous local symmetries of the equation.

- To find the optimal system of subalgebras \( SA_\mu \) of algebra \( A_m \). In the case of PDE with four independent variables like system (2), (3) it is reasonable to restrict ourselves to three-dimensional subalgebras. Their basis elements have the unified form \( Q_i = \xi_\mu^i \partial_\mu + \varphi_k^i \partial_{u_k}, \ i = 1, 2, 3 \) where \( u_k \) are dependent variables (in our case we can choose \( u_a = E_a, \ u_{3+a} = B_a, \ u_7 = \theta, \ a = 1, 2, 3 \)).

- Any three-dimensional subalgebra \( SA_\mu \) whose basis elements satisfy the conditions

\[
\text{rank}\{\xi_\mu^i\} = \text{rank}\{\xi_\mu^i, \varphi_k^i\} \tag{5}
\]

and

\[
\text{rank}\{\xi_\mu^i\} = 3 \tag{6}
\]

gives rise to change of variables which reduce system (2), (3) to a system of ordinary differential equation (ODE). The new variables include all invariants of three parameter Lie groups corresponding to the optimal subalgebras \( SA_\mu \).

- Solving if possible the obtained ODE one can generate an exact (particular) solution of the initial PDE.

- Applying to this solution the general symmetry group transformation it is possible to generate a family of exact solutions depending on additional arbitrary (transformation) parameters.

The first step of the algorithm presupposed finding the maximal Lie symmetry of the considered systems. These symmetries are presented in the following subsection.

3.2 Group classification of system (2), (3)

The system of equations (3) includes the arbitrary element \( F(\theta) \) thus its symmetries might be different for different \( F \). The group classification of these equation consists
in complete description of their continuous symmetries together with the specification of all functions \( F \) corresponding to different symmetries.

Applying the classical Lie algorithm (refer, e.g., to [15]), to find symmetries of system (2), (3) w.r.t. continuous groups of transformations \( B \rightarrow B', \ E \rightarrow E', \ \theta \rightarrow \theta' \) \(, x_\mu \rightarrow x'_\mu \) we introduce the infinitesimal operator

\[
Q = \xi^\mu \partial_\mu + \eta^j \partial_{B^j} + \zeta^i \partial_{E^i} + \sigma \partial_\theta, \tag{7}
\]

and its prolongation

\[
Q_{(2)} = Q + \eta^j_i \frac{\partial}{\partial B^j_i} + \zeta^j_i \frac{\partial}{\partial E^j_i} + \sigma_i \partial_\theta_i + \sigma_{ik} \partial_\theta_{ik}, \tag{8}
\]

where \( B^j_i = \partial_i B^j, \ E^j_i = \partial_i E^j, \ \theta_i = \partial_i \theta, \ \theta_{ik} = \partial_i \partial_k \theta \). Functions \( \eta^j_i, \ \zeta^j_i, \ \sigma_i, \ \sigma_{ik} \) can be expressed via \( \xi^\mu, \ \eta^j, \ \zeta^j, \ \sigma \) using the following relations:

\[
\eta^j_i = D_i(\eta^j) - B^k_i D_k(\xi^k), \quad \zeta^j_i = D_i(\zeta^j) - E^k_i D_k(\xi^k),
\]

\[
\sigma_i = D_i(\sigma) - \theta_k D_k(\xi^k), \quad \sigma_{ik} = D_k(\sigma_i) - \theta_d D_d(\xi^j)
\]

where \( D_i = \partial_i + B^j_i \partial_{B^j} + E^j_i \partial_{E^j} + \theta_i \partial_\theta + \theta_{ik} \partial_\theta_k \).

The invariance condition for system (2), (3) can be written in the following form:

\[
Q_{(2)} \mathcal{F} \big|_{\mathcal{F}=0} = 0 \tag{9}
\]

where \( \mathcal{F} \) is the manifold defined by relations (2), (3). Then, equating coefficients for linearly independent functions \( E^j, B^j, \ \theta \) and their derivatives we obtain the following system of determining equations for the coefficients of the infinitesimal operator:

\[
\xi^\mu_{B^\nu} = 0, \quad \xi^\mu_{E^\nu} = 0, \quad \xi^\mu_\theta = 0, \quad \xi^\mu_{x^\nu} = \xi^\nu_{x^\mu}, \quad \xi^\mu_{x^\nu} + \xi^\nu_{x^\mu} = 0, \quad \mu \neq \nu, \tag{10}
\]

\[
\sigma_{E^\nu} = 0, \quad \sigma_{B^\nu} = 0, \quad \sigma_{\theta^\nu} = 0, \tag{11}
\]

\[
\square \sigma + (\sigma_\theta - 2 \xi^0_{x^\nu}) \left( F + kE^a B^a \right) - \kappa(B^a \xi^a + E^a \eta^a) - \sigma F_\theta = 0, \tag{12}
\]

\[
\square \xi^\mu - 2 \sigma_{\theta x^\nu} = 0, \tag{13}
\]

\[
\xi^a_{B^b} + \eta^b_{E^b} = 0, \quad \xi^a_{x^b} + c^b_{E^b} = 0,
\]

\[
\xi^a_{x^b} - \varepsilon_{abc} \eta^c_{E^b} = 0, \quad \xi^a_{x^b} - \varepsilon_{abc} \xi^c_{B^b} = 0,
\]

\[
\partial_\nu \eta^a = 0, \quad \partial_\nu \xi^a + B^a \partial_\nu \sigma = 0,
\]

\[
\eta^a_\theta + \varepsilon_{abc} \xi^c_{x^b} = 0, \quad \xi^a_{x^b} + B^a \sigma_{x^b} - \varepsilon_{abc} (\eta^c_{x^b} + E^b \sigma_{x^b}) = 0,
\]

\[
\eta^a + B^a \sigma_\theta + \xi^a_\theta - B^c \xi^c_{E^b} + \varepsilon_{abc} E^b \xi^c_{x^b} = 0, \tag{14}
\]

\[
\zeta^a - \eta^a_\theta + E^a \sigma_\theta - E^b \xi^a_{E^b} - \varepsilon_{abc} B^b \xi^c_{x^b} = 0,
\]

\[
\eta^b_{E^b} - \eta^b_{B^b} = 0, \quad \eta^b_{E^b} - \zeta^b_{E^b} = 0, \quad \eta^a_\theta - B^a \eta^b_{E^b} = 0, \quad \zeta^a - E^a \eta^b_{E^b} = 0.
\]
Here the subscripts denote the derivatives with respect to the corresponding variables: 
\( \xi_{B}^{\mu} = \frac{\partial c^{\mu}}{\partial B} \), etc, and there are no sums over repeating indices in the last line of (13).

In accordance with equations (10) functions \( \xi^{\mu} \) are Killing vectors in the space of independent variables:

\[
\xi^{\mu} = 2x^{\mu} f^{\nu} x_{\nu} - f^{\mu} x_{\nu} x^{\nu} + e^{\mu} x_{\nu} + dx^{\mu} + e^{\mu}
\]  
(15)

where, \( f^{\mu}, e^{\mu} \) and \( e_{\mu\nu} = -e_{\nu\mu} \) are arbitrary constants.

In accordance with (11) \( \sigma = \varphi_{1} \theta + \varphi_{2} \), where \( \varphi_{1} \) and \( \varphi_{2} \) are functions of \( x_{\mu} \).

Substituting this expression into (12) we obtain:

\[
\varphi_{1} \theta F_{\theta} + \varphi_{2} F_{\theta} + 2 \left( \xi_{x_{0}} \right) - 2 \kappa \left( \xi_{x_{0}} \right) E_{a} B_{a} + \kappa \left( B_{a} \xi^{a} + E_{a} \eta^{a} \right) - \theta \square \varphi_{1} - \square \varphi_{2} - 2 \rho^{\mu} \partial_{\mu} \varphi_{1} = 0.
\]  
(16)

If the terms \( \theta F_{\theta}, F_{\theta}, F \) and 1 are linearly independent, then it follows from (16) that

\[
\sigma = \xi_{x_{0}}^{0} = 0, \quad B^{a} \eta^{a} + E_{a} \eta^{a} = 0.
\]  
(17)

Substituting (17) and (15) into (13) we obtain that \( f^{\nu} = 0 \), hence (15) reduces to the form

\[
\xi^{\mu} = e^{\mu} x_{\nu} + e^{\mu}.
\]  
(18)

It follows from (14), (17) and (18) that

\[
\eta^{a} = c^{ab} B^{b} + \epsilon^{abc} c^{0b} E^{c}, \quad \xi^{a} = c^{ab} E^{b} - \epsilon^{abc} c^{0b} B^{c}.
\]  
(19)

In accordance with (18) and (19) the infinitesimal operator (7) is reduced to a linear combination of the following basis elements:

\[
P_{0} = \partial_{0}, \quad P_{a} = \partial_{a},
\]

\[
J_{ab} = x_{a} \partial_{b} - x_{b} \partial_{a} + B^{a} \partial_{B^{b}} - B^{b} \partial_{B^{a}} + E^{a} \partial_{E^{b}} - E^{b} \partial_{E^{a}},
\]

\[
J_{0a} = x_{0} \partial_{a} - x_{a} \partial_{0} + \epsilon_{abc} \left( E^{b} \partial_{E^{c}} - B^{b} \partial_{B^{c}} \right)
\]  
(20)

where \( \epsilon_{abc} \) is the unit antisymmetric tensor, \( a, b, c = 1, 2, 3 \).

For some particular functions \( F \), namely, for \( F = 0, F = c \) and \( F = b \exp(a \theta) \) the symmetry of system (2), (3) is more extended. The corresponding Lie algebra includes the following additional basis elements:

\[
P_{4} = \partial_{\theta}, \quad D = x_{0} \partial_{0} + x_{i} \partial_{i} - B^{i} \partial_{B^{i}} - E^{i} \partial_{E^{i}} \text{ if } F(\theta) = 0,
\]

\[
P_{4} = \partial_{\theta} \text{ if } F(\theta) = c,
\]

\[
X = aD - 2P_{4} \text{ if } F(\theta) = be^{a \theta}.
\]  
(21)

(22)

(23)
Using the standard algorithm of group classification (see, e.g., \[15\]) we can find symmetries of a more general system \(2\), \(3\) with arbitrary element \(F\) being a function of both \(\theta\) and its derivatives \(p_\mu\). Restricting ourselves to the case of Poincaré-invariant systems we find that \(F\) can be an arbitrary function of \(\theta\) and \(p_\mu p^\mu\). Moreover, all cases when this symmetry can be extended are presented by the following formulae:

\[
F = \kappa p_\mu p^\mu, \\
F = f(p_\mu p^\mu), \\
F = e^{ab} f (p_\mu p^\mu e^{-ab})
\]

where \(f(.)\) is an arbitrary function on the argument given in the brackets and \(\kappa\) is an arbitrary constant. Symmetry algebras of system \(2\), \(3\) where \(F\) is invariant w.r.t. Poincaré group for arbitrary \(\mu\) can be found for the first time in paper \[17\]. We use a more advanced classification of subalgebras of algebra \(p(1,3)\) defined up to the group of internal automorphism has been found in \[18\]. In accordance with \[18\] there exist 30 non-equivalent three-dimensional subalgebras \(A_1, A_2, \cdots A_{30}\) of algebra \(p(1,3)\) which we present in the following formulae by specifying their basis elements:

\[
\begin{align*}
A_1 : & \langle P_0, P_1, P_2 \rangle; & A_2 : & \langle P_1, P_2, P_3 \rangle; & A_3 : & \langle P_0 - P_3, P_1, P_2 \rangle; \\
A_4 : & \langle J_{03}, P_1, P_2 \rangle; & A_5 : & \langle J_{03}, P_0 - P_3, P_1 \rangle; & A_6 : & \langle J_{03} + \alpha P_0, P_0, P_3 \rangle; \\
A_7 : & \langle J_{03} + \alpha P_2, P_0 - P_3, P_1 \rangle; & A_8 : & \langle J_{12}, P_0, P_3 \rangle; \\
A_9 : & \langle J_{12} + \alpha P_0, P_1, P_2 \rangle; & A_{10} : & \langle J_{12} + \alpha P_3, P_1, P_2 \rangle; \\
A_{11} : & \langle J_{12} - P_0 + P_3, P_1, P_2 \rangle; & A_{12} : & \langle G_1, P_0 - P_3, P_2 \rangle; \\
A_{13} : & \langle G_1, P_0 - P_3, P_1 + \alpha P_2 \rangle; & A_{14} : & \langle G_1 + P_2, P_0 - P_3, P_1 \rangle; \\
A_{15} : & \langle G_1 - P_0, P_0 - P_3, P_2 \rangle; & A_{16} : & \langle G_1 + P_0, P_1 + \alpha P_2, P_0 - P_3 \rangle; \\
A_{17} : & \langle J_{03} + \alpha J_{12}, P_0, P_3 \rangle; & A_{18} : & \langle \alpha J_{03} + J_{12}, P_1, P_2 \rangle; \\
A_{19} : & \langle J_{12}, J_{03}, P_0 - P_3 \rangle; & A_{20} : & \langle G_1, G_2, P_0 - P_3 \rangle; \\
A_{21} : & \langle G_1 + P_2, G_2 + \alpha P_1 + \beta P_2, P_0 - P_3 \rangle; & A_{22} : & \langle G_1, G_2 + P_1 + \beta P_2, P_0 - P_3 \rangle; \\
A_{23} : & \langle G_1, G_2 + P_2, P_0 - P_3 \rangle; & A_{24} : & \langle G_1, J_{03}, P_2 \rangle; \\
A_{25} : & \langle J_{03} + \alpha P_1 + \beta P_2, G_1, P_0 - P_3 \rangle; & A_{26} : & \langle J_{12} - P_0 + P_3, G_1, G_2 \rangle; \\
A_{27} : & \langle J_{03} + \alpha J_{12}, G_1, G_2 \rangle; & A_{28} : & \langle G_1, G_2, J_{12} \rangle; \\
A_{29} : & \langle J_{01}, J_{02}, J_{12} \rangle; & A_{30} : & \langle J_{12}, J_{23}, J_{31} \rangle.
\end{align*}
\]

3.3 Optimal subalgebras

Thus, to generate exact solutions of system \(2\), \(3\) we can exploit its invariance w.r.t. the Poincaré group whose generators are presented in equation \(20\). The subalgebras of algebra \(p(1,3)\) defined up to the group of internal automorphism has been found for the first time in paper \[17\]. We use a more advanced classification of these subalgebras proposed in \[18\]. In accordance with \[18\] there exist 30 non-equivalent three-dimensional subalgebras \(A_1, A_2, \cdots A_{30}\) of algebra \(p(1,3)\) which we present in the following formulae by specifying their basis elements:
Here \( P_\mu \) and \( J_{\mu\nu} \) are generators given by relations (20), \( G_1 = J_{01} - J_{13} \), \( G_2 = J_{02} - J_{23} \), \( \alpha \) and \( \beta \) are arbitrary parameters.

Using subalgebras (27) we can deduce exact solutions for system (2), (3). Notice that to make an effective reduction using the Lie algorithm, we can use only such subalgebras whose basis elements satisfy conditions (5). This condition is satisfied by basis element of algebras \( \text{A}_1 - \text{A}_{27} \) but is not satisfied by \( \text{A}_{28}, \text{A}_{29}, \text{A}_{30} \) and \( \text{A}_6 \) with \( \alpha = 0 \). Nevertheless, the latter symmetries also can be used to generate exact solutions in frames of the weak transversality approach discussed in [19]. We will use also a certain generalization of this approach.

In the following sections we present the complete list of reductions and find exact solutions for system (2), (3) which can be obtained using reduction w.r.t. the subgroups of Poincaré group. We will find also some solutions whose existence is caused by symmetry of this system with respect to the extended Poincaré group.

### 3.4 Plane wave solutions

Let us find solutions of system (2), (3) which are invariant w.r.t. subalgebras \( \text{A}_1, \text{A}_2 \) and \( \text{A}_3 \). Basis elements of all these subalgebras can be represented in the following unified form

\[
A : \quad \langle P_1, P_2, kP_0 + \varepsilon P_3 \rangle
\]

where \( \varepsilon \) and \( k \) are parameters. Indeed, setting in (28) \( \varepsilon = -k \) we come to algebra \( A_3 \), for \( \varepsilon^2 < k^2 \) or \( k^2 < \varepsilon^2 \) algebra (28) is equivalent to \( A_1 \) or \( A_2 \) correspondingly.

Starting from this point we mark the components of vectors \( B \) and \( E \) by subindices, i.e., \( B = (B_1, B_2, B_3) \) and \( E = (E_1, E_2, E_3) \).

To find the related invariant solutions we need invariants of the group whose generators are given in (28). These invariants include the dependent variables \( E_a, B_a, \theta \) \((a = 1, 2, 3)\) and one independent variable \( \omega = \varepsilon x_0 - k x_3 \). Let us search for solutions of (2), (3) which are functions of \( \omega \). Then equations (2) are reduced to the following system:

\[
\begin{align*}
\dot{B}_3 &= 0, \quad \dot{E}_3 = \theta B_3, \quad k \dot{E}_2 = -\varepsilon \dot{B}_1, \quad k \dot{E}_1 = \varepsilon \dot{B}_2, \\
\varepsilon \dot{E}_1 - k \dot{B}_2 &= \theta (k E_2 + \varepsilon B_1), \quad k \dot{B}_1 + \varepsilon \dot{E}_2 = \theta (\varepsilon B_2 - k E_1)
\end{align*}
\]

(29)

where \( \dot{B}_3 = \frac{\partial B_3}{\partial \omega} \).

The system (29) is easily integrated. If \( \varepsilon^2 = k^2 \neq 0 \) then

\[
E_1 = \frac{\varepsilon}{k} B_2 = F_1, \quad E_2 = -\frac{\varepsilon}{k} B_1 = F_2, \quad E_3 = c \theta + b, \quad B_3 = c
\]

(30)

where \( F_1 \) and \( F_2 \) are arbitrary functions of \( \omega \) while \( c \) and \( b \) are arbitrary real numbers. The corresponding equation (3) is reduced to the form \( e^2 \theta = F(\theta) - b e \), i.e., \( \theta \) is proportional to \( F(\theta) - b e \) if \( e \neq 0 \). If both \( e \) and \( F \) equal to zero then \( \theta \) is an arbitrary function of \( \omega \).
For $\varepsilon^2 \neq k^2$ solutions of (29) have the following form:

\begin{equation}
B_1 = k c_1 \theta - k b_1 + \varepsilon c_2, \quad B_2 = k c_2 \theta - k b_2 - \varepsilon c_1, \quad B_3 = c_3, \\
E_1 = \varepsilon c_2 \theta - \varepsilon b_2 - k c_1, \quad E_2 = -\varepsilon c_1 \theta + \varepsilon b_1 - k c_2, \quad E_3 = c_3 \theta - b_3 (\varepsilon^2 - k^2) \tag{31}
\end{equation}

where $b_a$ and $c_a$ $(a = 1, 2, 3)$ are arbitrary constants. The corresponding equation (3) takes the form

\begin{equation}
\ddot{\theta} = -b \dot{\theta} + c + \tilde{F} \tag{32}
\end{equation}

where $b = \left( c_1^2 + c_2^2 + \frac{c_3^2}{\varepsilon^2 - k^2} \right)$, $\tilde{F} = \frac{F}{(\varepsilon^2 - k^2)}$ and $c = c_1 b_1 + c_2 b_2 + c_3 b_3$.

If $F = 0$ or $F = -m^2 \theta$ then (32) is reduced to the linear equation:

\begin{equation}
\ddot{\theta} = -a \dot{\theta} + c \tag{33}
\end{equation}

where $a = c_1^2 + c_2^2 + \frac{c_3^2 + m^2}{\varepsilon^2 - k^2}$. Thus

\begin{align}
\theta &= a_{\mu} \cos \mu \omega + b_{\mu} \sin \mu \omega + \frac{c}{\mu^2}, \quad \text{if} \quad a = \mu^2 > 0, \\
\varphi &= a_{\sigma} e^{\sigma \omega} + b_{\sigma} e^{-\sigma \omega} - \frac{c}{\sigma^2}, \quad \text{if} \quad a = -\nu^2 < 0, \\
\theta &= \frac{1}{2} c \omega^2 + c_1 \omega + c_2 \quad \text{if} \quad a = 0 \tag{34-36}
\end{align}

where $a_{\mu}, b_{\mu}, a_{\sigma}, b_{\sigma}, c_1$ and $c_2$ are arbitrary constants.

One more plane wave solution of equations (2), (3) with $\kappa = 1$ and $F = 0$ can be written as:

\begin{align}
E_2 &= c_k \varepsilon \cos \omega + d_k \varepsilon \sin \omega, \quad E_3 = c_k \varepsilon \sin \omega - d_k \varepsilon \cos \omega, \\
B_2 &= c_k k \sin \omega - d_k k \cos \omega, \quad B_3 = -c_k k \cos \omega - d_k k \sin \omega, \\
E_1 &= \varepsilon, \quad B_1 = 0, \quad \theta = \alpha x_0 + \nu x_1 + c_3, \quad \omega = \varepsilon x_0 + k x_1 \tag{37}
\end{align}

where $\varepsilon, c_k, d_k, \varepsilon, k, \alpha, \nu$ are arbitrary constants restricted by the only relation:

\begin{equation}
\varepsilon^2 - k^2 = \nu \varepsilon - \alpha k. \tag{38}
\end{equation}

If $\varepsilon = k$ then $\alpha = \nu$ and formulae (37) present solutions depending on one light cone variable $x_0 + x_1$. However, for $\varepsilon \neq k$ we have solutions depending on two different plane wave variables, i.e., $\varepsilon x_0 + k x_1$ and $\alpha x_0 + \nu x_1$.

It is interesting to note that for fixed parameters $\alpha$ and $\nu$ solutions (37) for $E_a$ and $B_a$ satisfy the superposition principle, i.e., a sum of solutions with different $\varepsilon, k, c_k$ and $d_k$ is also a solution of equations (2), (3) with $\kappa = 1$ and $F = 0$.

Using symmetries of system (2), (3) it is possible to extend the obtained solutions. Indeed, applying to (30), (31) the rotation transformations

\begin{equation}
E_a \rightarrow E'_a = R_{ab} E_b, \quad B_a \rightarrow B'_a = R_{ab} B_b, \tag{39}
\end{equation}
where \( \{R_{ab}\} \) is an arbitrary orthogonal matrix of dimension \( 3 \times 3 \), and then the Lorentz transformations

\[
E'_a \rightarrow E'_a \cosh \lambda - \varepsilon_{abc} \lambda_b B'_c \frac{\sinh \lambda}{\lambda},
\]

\[
B'_a \cosh \lambda + \varepsilon_{abc} \lambda_b E'_c \frac{\sinh \lambda}{\lambda} + \lambda_a \lambda_b B'_b \frac{1 - \cosh \lambda}{\lambda^2}, \quad \lambda = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}
\]

and transforming \( \omega \rightarrow n_\mu x^\mu \) where \( n_\mu \) are components of the constant vector given by the following relations:

\[
n_0 = \cosh \lambda - \nu \lambda_a R_{a3} \frac{\sinh \lambda}{\lambda},
\]

\[
n_a = \nu R_{a3} - \lambda_a \frac{\sinh \lambda}{\lambda} - \nu \lambda_a \lambda_b R_{b3} \frac{(1 - \cosh \lambda)}{\lambda^2},
\]

we obtain more general solutions of equations (2), (3).

In formulae (39)–(41) summation is imposed over the repeating index \( b, b = 1, 2, 3 \).

3.5 Selected radial and cylindric solutions

Let us present some exact solutions of equations (2), (3) which can be interesting from the physical point of view.

First we consider solutions which include the field of point charge, i.e.

\[
E_a = \frac{q x_a}{r^3}, \quad a = 1, 2, 3
\]

where \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \) and \( q \) is a coupling constant. Notice that up to scaling the dependent variables \( x_a \) we can restrict ourselves to \( q = 1 \). The related vector \( B_a \) is trivial, i.e., \( B_a = 0 \), while for \( \theta \) there are two solutions:

\[
\theta = c_1 x_a r^3 \quad \text{and} \quad \theta = \frac{1}{r} (\varphi_1(x_0 + r) + \varphi_2(x_0 - r))
\]

where \( \varphi_1 \) and \( \varphi_2 \) are arbitrary functions of \( x_0 + r \) and \( x_0 - r \) correspondingly, \( c_a \) are arbitrary constants and summation is imposed over the repeating indices \( a = 1, 2, 3 \). These solutions correspond to trivial nonlinear terms in (2), (3).

Radial solutions which generate nontrivial terms in the r.h.s. of equations (2), (3) with \( F = -m^2 \theta \) can be found in the following form:

\[
B_a = -\frac{q x_a}{r^3}, \quad E_a = -\frac{q \theta x_a}{r^3}, \quad \theta = c_1 \sin(mx_0) e^{-r^2}
\]

where \( c_1 \) and \( q > 0 \) are arbitrary parameters. The components of magnetic field \( B_a \) are singular at \( r = 0 \) while \( E_a \) and \( \theta \) are bounded for \( 0 \leq r \leq \infty \).

Solutions (42)–(44) were obtained with using invariants of algebra \( A_{30} \).
Let us present solutions which depend on two spatial variables but are rather similar to the three dimensional Coulomb field. We denote \(x = \sqrt{x_1^2 + x_2^2}\), then functions

\[
E_1 = -B_2 = \frac{x_1}{x_3}, \quad E_3 = 0, \quad B_1 = E_2 = \frac{x_2}{x_3}, \quad B_3 = b, \quad \theta = \arctan \left( \frac{x_2}{x_1} \right)
\]

where \(b\) is a number, solve equations (2), (3) with \(\kappa = 1\) and \(F = 0\).

A particularity of solutions (45) is that, in spite of their cylindric nature, the related electric field decreases with growing of \(x\) as the field of point charge in the three dimensional space.

Functions (45) solve the standard Maxwell equations with charges and currents also. However, they correspond to the charge and current densities proportional to \(1/x^3\) which looks rather nonphysical. In contrary, these vectors present consistent solutions for equations of axion electrodynamics with zero axion mass.

Solutions (45) can be expressed via invariants of the subgroup of the extended Poincaré group whose Lie algebra is spanned on the basis \(\langle P_0, P_3, J_{12} + P_4 \rangle\), see equations (20), (22) for definitions.

Let us write one more solution of equations (2), (3) with \(F = 0\):

\[
B_1 = \frac{x_1 x_3}{r^2 x}, \quad B_2 = \frac{x_2 x_3}{r^2 x}, \quad B_3 = -\frac{x}{r^2}, \quad \theta = \arctan \left( \frac{x}{x_3} \right),
\]

\[
E_a = \frac{x_a}{r^2}, \quad a = 1, 2, 3
\]

where \(r = \sqrt{x_1^2 + x_2^2 + x_3^2}\), \(x = \sqrt{x_1^2 + x_2^2}\). The electric field (47) is directed like the three dimensional field of point charge but its strength is proportional to \(1/r\) instead of \(1/r^2\).

Let us note that functions (46), (47) solve equations (4) with \(\kappa = 1, F = 0\) also.

Two more stationary exact solutions for these equations can be written as:

\[
E_a = \frac{x_a}{r^2}, \quad a = 1, 2, 3; \quad B_a = 0, \quad \theta = \ln(r)
\]

and

\[
E_a = \frac{x_a}{r}, \quad B_a = b_a, \quad \theta = \ln(r)
\]

where \(b_a\) are constants satisfying the condition \(b_1^2 + b_2^2 + b_3^2 = 1\). Functions (49) solve equations (4) with \(F = 0\) for \(0 < r < \infty\) while formula (45) gives solutions of equation (4) with \(F = p_a p^a\).

The complete list of exact solutions for equations (2), (3) obtained using symmetries w.r.t. the 3-dimensional subalgebras of the Poincaré algebra is presented in the following section.
4 Complete list of invariant solutions

In this section we present all exact solutions for equations (2), (3) which can be obtained using symmetries w.r.t. the 3-dimensional subalgebras of the Poincaré algebra. Basis elements of these subalgebras are given by relations (27).

We shall consider equations (2), (3) with the most popular form of function $F$, i.e., $F = -m^2 \theta$, which is the standard choice in axion electrodynamics. In addition, up to scaling the dependent variables, we can restrict ourselves to the case $\kappa = 1$.

Under these conventions the system (2), (3) can be rewritten in the following form:

\begin{align}
\nabla \cdot \mathbf{E} &= p \cdot \mathbf{B}, \\
\partial_0 \mathbf{E} - \nabla \times \mathbf{B} &= p_0 \mathbf{B} + p \times \mathbf{E}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\partial_0 \mathbf{B} + \nabla \times \mathbf{E} &= p_0 \mathbf{B} + p \times \mathbf{E}, \\
\Box \theta &= -\mathbf{E} \cdot \mathbf{B} - m^2 \theta \\
\end{align}

In the following we present exact solutions just for equations (50), (51) for both nonzero and zero $m$.

Solutions corresponding to algebras $A_1 - A_3$ have been discussed in previous subsection. Here we apply the remaining subalgebras from the list (27), grouping them into classes which correspond to similar reduced equations.

4.1 Reductions to algebraic equations

Let us consider subalgebras $A_{11}, A_{20}, A_{26}$ and show that using their invariants the system (50), (51) can be reduced to algebraic equations.

Algebra $A_{11}$: \\ (< J_{12} - P_0 + P_3, P_1, P_2 >

Invariants $I$ of the corresponding Lie group are functions of the dependent and independent variables involved into system (50), (51), which satisfy the following conditions

\begin{align}
P_1 I = P_2 I = 0, \quad (J_{12} - P_0 + P_3) I = 0. \quad (52)
\end{align}

The system (52) is non-degenerated thus there are eight invariants which we choose in the following form:

\begin{align}
I_1 &= E_1 \sin \zeta - E_2 \cos \zeta, \quad I_2 = E_2 \sin \zeta + E_1 \cos \zeta, \\
I_3 &= B_1 \sin \zeta - B_2 \cos \zeta, \quad I_4 = B_2 \sin \zeta + B_1 \cos \zeta, \\
I_5 &= E_3, \quad I_6 = B_3, \quad I_7 = \theta, \quad I_8 = \omega = x_0 + x_3 \\
\end{align}

where $\zeta = \frac{1}{2}(x_3 - x_0)$ and $I_a, \quad a = 1, 2, ..., 7$ are arbitrary functions of $\omega$. Solving (52) for $E_a, B_a$ and $\theta$ and using (50) we obtain

\begin{align}
E_1 &= B_2 = c_1 \sin \zeta + c_2 \cos \zeta, \quad E_2 = -B_1 = c_2 \sin \zeta - c_1 \cos \zeta, \\
\end{align}
\[ E_3 = c_3\theta + c_4, \quad B_3 = c_3, \quad \] (55)

where \( c_1, c_2, c_3, c_4 \) are arbitrary real constants and \( \theta \) is a function of \( \omega \) which, in accordance with (54), should satisfy the following linear algebraic relation:

\[ (c_3^2 + m^2)\theta + c_3c_4 = 0. \quad (56) \]

Thus \( \theta = -\frac{c_3c_4}{c_3^2 + m^2} \) if the sum in brackets is nonzero and \( \theta \) is an arbitrary constant provided \( c_3 = m = 0 \).

Notice that solution (54) can be generalized to the following form:

\[ E_1 = B_2 = f(\zeta), \quad E_2 = -B_1 = g(\zeta), \quad E_3 = c_3\theta + c_4, \quad H_3 = c_3 \quad (57) \]

where \( f(\zeta), g(\zeta) \) are arbitrary functions and \( \theta \) again is defined by equation (56). However, solution (57) cannot be obtained via symmetry reduction.

In analogous way we obtain solutions corresponding to subalgebras \( A_{20} \) and \( A_{26} \).

Algebra \( A_{20} \): \( \langle G_1, G_2, P_0 - P_3 \rangle \)

\[
B_1 = E_2 - \frac{c_2}{\omega} = \frac{-2c_1x_1x_2 + c_2(x_1^2 - x_2^2) + 2c_3x_1 + 2c_3x_2\theta + 2c_4x_2}{2\omega^3} + \varphi_1, \\
B_2 = -E_1 + \frac{c_1}{\omega} = \frac{c_1(x_1^2 - x_2^2) + 2c_3x_1x_2 + 2c_3x_2 - 2c_3x_1\theta - 2c_4x_1}{2\omega^3} + \varphi_2, \\
B_3 = -c_1x_2 + c_2x_1 + c_3, \quad E_3 = \frac{-c_1x_1 - c_2x_2 + c_3\theta + c_4}{\omega^2}
\]

where \( \varphi_1 \) are functions of \( \omega = x_0 + x_3 \),

\[
\theta = -\frac{(c_1\varphi_1 + c_2\varphi_2)\omega^3 + c_3c_4}{c_3^2 + m^2\omega^4} \quad \text{if} \quad c_3^2 + m^2 > 0; \\
\theta = \varphi_3, \quad c_1\varphi_1 + c_2\varphi_2 = 0 \quad \text{if} \quad c_3^2 + m^2 = 0.
\]

Algebra \( A_{26} \): \( \langle J_{12} - P_0 + P_3, G_1, G_2 \rangle \)

\[
B_1 = \frac{c_1x_1x_2}{\omega^3} \cos \zeta + \frac{c_2x_2}{\omega^3} + \frac{c_1((\dot{\theta} - 2)\omega^2 + 2(x_1^2 - x_2^2))}{4\omega^3} \sin \zeta, \\
B_2 = \frac{c_1x_1x_2}{\omega^3} \sin \zeta + \frac{c_1((\dot{\theta} - 2)\omega^2 - 2(x_1^2 - x_2^2))}{4\omega^3} \cos \zeta, \\
B_3 = \frac{c_1x_1}{\omega^2} \sin \zeta - \frac{c_2x_1}{\omega^3} + \frac{c_1x_2}{\omega^2} \cos \zeta, \quad E_1 = -B_2 - \frac{c_1}{\omega} \cos \zeta, \\
E_2 = B_1 + \frac{c_1}{\omega} \sin \zeta, \quad E_3 = \frac{c_1x_2}{\omega^2} \sin \zeta + \frac{c_2}{\omega^3} + \frac{c_1x_1}{\omega^2} \cos \zeta, \\
\theta = 0, \quad \text{if} \quad m \neq 0, \quad \theta = \varphi(\omega) \quad \text{if} \quad m = 0
\]

where \( \zeta = \frac{x^2}{\omega} + \frac{\phi}{\omega}, \quad x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \) and \( \varphi(\omega) \) is an arbitrary function of \( \omega = x_0 + x_3 \).
4.2 Reductions to linear ODE

The next class includes subalgebras $A_5, A_7, A_{15}, A_{16}$ and $A_{25}$. Using them we shall reduce the system \((50), (51)\) to the only linear ordinary differential equation \((33)\).

Let us start with algebra $A_5$ whose basis elements are \(\langle J_{03}, P_0 - P_3, P_1 \rangle\). The corresponding invariant solutions of equations \((50), (51)\) have the following form:

\[
B_1 = E_2 = (x_0 + x_3)(c_1 \theta + c_2), \quad B_2 = -E_1 = c_1(x_0 + x_3),
\]
\[
B_3 = -c_3 \theta + c_4, \quad E_3 = c_3, \quad c_1 c_2 = 0.
\]

Function $\theta = \varphi(\omega)$ depends on the only variable $\omega = x_2$ and satisfies equation \((33)\) where $a = c_3^2 - m^2$, $c = c_3 c_4$. Its possible explicit forms are given by equations \((34)–(36)\).

Algebra $A_7$ : \(\langle J_{03} + \alpha P_2, P_0 - P_3, P_1 \rangle\)

\[
B_1 = E_2 = \frac{-c_1 \theta + c_2}{x_0 + x_3}, \quad B_2 = -E_1 = \frac{-\alpha c_3 \theta + \alpha c_4 - c_1}{x_0 + x_3},
\]
\[
B_3 = -c_3 \theta + c_4, \quad E_3 = c_3.
\]

Possible functions $\theta = \varphi(\omega)$ again are given by equations \((34)–(36)\) where $a = c_3^2 - m^2$, $c = c_3 c_4$ and $\omega = x_2 - \alpha \ln |x_0 + x_3|$. 

Algebra $A_{15}$ : \(\langle G_1 - P_0, P_0 - P_3, P_2 \rangle\)

\[
B_1 = E_2 = -c_2(x_0 + x_3)\theta - c_1(x_0 + x_3), \quad B_2 = c_2 \theta + c_1,
\]
\[
B_2 = -E_1 = c_3(2\omega - x_1) + c_2(x_0 + x_3), \quad E_3 = c_3(x_0 + x_3) + 2
\]

where $\omega = x_1 + \frac{1}{2}(x_0 + x_3)^2$. The possible $\theta$ are given by equations \((35), (36)\) where $\sigma^2 = m^2 + c_2^2$, $c = c_1 c_2$.

Algebra $A_{16}$ : \(\langle G_1 + P_0, P_1 + \alpha P_2, P_0 - P_3 \rangle\)

\[
B_1 = (x_0 + x_3)(c_3 \theta - c_4) + \frac{1}{2} c_1(x_0 + x_3)^2 + \frac{c_5}{1 + \alpha^2}(\theta - \alpha) + c_2,
\]
\[
B_2 = c_1(\omega - \frac{\alpha}{2}(x_0 + x_3)^2) + c_3(x_0 + x_3) + \frac{c_5}{1 + \alpha^2}(\alpha \theta + 1) + \alpha c_2,
\]
\[
B_3 = -c_3 \theta - c_1(x_0 + x_3) + c_4, \quad E_1 = -B_2 - \alpha c_1,
\]
\[
E_2 = B_1 + c_1, \quad E_3 = -\alpha c_1(x_0 + x_3) + c_3
\]

where $\omega = x_2 - \alpha x_1 - \frac{\alpha}{2}(x_0 + x_3)^2$,

\[
\theta = \frac{1}{\alpha^2 + 1} \left( \frac{c_2^2}{6} \omega^3 + \frac{1}{2} (c_3 c_4 + c_1 c_5) \omega^2 \right) + c_7 \omega + c_8 \quad \text{if} \quad c_3^2 = m^2,
\]
\[
\theta = \varphi + \frac{c_7^2 \omega}{c_3^2 - m^2} \quad \text{if} \quad c_3^2 \neq m^2.
\]
Here \( \varphi \) is the function of \( \omega \) given by equations (34)–(36) where \( \mu^2 = -\sigma^2 = \frac{c^2 - m^2}{\alpha^2 + 1} \), \( c = \frac{c_3 a + c_4 b}{\sigma + 1} \).

Algebra \( A_{25} \): \( \langle J_{03} + \alpha P_1 + \beta P_2, G_1, P_0 - P_3 \rangle \)

\[
B_1 = E_2 = \frac{c_3 + (c_3 \theta + c_2)\zeta}{x_3 + x_0}, \quad B_2 = -E_1 = \frac{\beta c_3 \theta + c_1 + c_3 \zeta}{x_3 + x_0},
\]

\[B_3 = c_3 \theta + c_2, \quad E_3 = -c_3,\]

where \( \zeta = x_1 - \alpha \ln |x_3 + x_0| \) and \( \theta = \varphi(\omega) \) is a function of \( \omega = x_2 - \beta \ln |x_3 + x_0| \) given by equations (34)–(36) where \( c = -c_2 c_3 \) and \( \mu^2 = -\sigma^2 = c_3^2 - m^2 \).

Consider now reductions which can be made with using invariants of subalgebras \( A_4, A_8, A_{19}, A_{24} \) and \( A_{27} \). In this way we will reduce the system (50), (51) to linear ODEs which, however, differ from (33).

Algebra \( A_4 \): \( \langle J_{03}, P_1, P_2 \rangle \).

\[
B_1 = \frac{-c_2 x_3 \theta + c_5 x_3 - c_1 x_0}{\omega^2}, \quad B_2 = \frac{-c_1 x_3 \theta + c_5 x_3 + c_2 x_0}{\omega^2}, \quad B_3 = c_3, \quad E_1 = \frac{-c_1 x_0 \theta + c_5 x_0 + c_2 x_3}{\omega^2}, \quad E_2 = \frac{c_4 x_0 \theta + c_1 x_3 - c_6 x_0}{\omega^2}, \quad E_3 = c_3 \theta + c_4
\]

(58)

where \( c_1, \ldots, c_6 \) are arbitrary constants, \( \theta = \theta(\omega) \) and \( \omega^2 = x_0^2 - x_3^2 \). Substituting (58) into (51) we obtain:

\[
\omega^2 \dot{\theta} + \omega \theta + (\nu^2 + \mu^2 \omega^2) \theta = \delta + \alpha \omega^2
\]

(59)

where \( \nu^2 = c_1^2 + c_2^2, \mu^2 = c_3^2 + m^2, \delta = c_1 c_5 + c_2 c_6, \alpha = c_3 c_4 \) and \( \dot{\theta} = \partial \theta / \partial \omega \).

The general real solution of equation (59) for \( x_0^2 > x_3^2 \) is:

\[
\theta = c_7 (J_{\nu \nu}(\mu \omega) + J_{-\nu \nu}(\mu \omega)) + c_8 (Y_{\nu \nu}(\mu \omega) + Y_{-\nu \nu}(\mu \omega))
\]

\[+ \frac{\delta \pi}{2\nu} \left( \coth \left( \frac{\pi \nu}{2} \right) J_{\nu \nu}(\mu \omega) + i E_{\nu \nu}(\mu \omega) \right) + \frac{\alpha}{\mu^2} L_s(1, i \nu, \mu \omega)
\]

(60)

where \( \omega = \sqrt{x_0^2 - x_3^2} \), \( J_{\nu \nu}(\mu \omega) \) and \( Y_{\nu \nu}(\mu \omega) \) are Bessel functions of the first and second kind, \( L_s(1, i \nu, \mu \omega) \) is the Lommel function \( s, J_{\nu \nu}(\mu \omega) \) and \( E_{\nu \nu}(\mu \omega) \) are Anger and Weber functions.

If \( \mu \nu = 0 \) and \( x_0^2 > x_3^2 \) then solutions of (59) are reduced to the following form:

\[
\theta = c_7 \sin(\nu \ln \omega) + c_8 \cos(\nu \ln \omega) + \frac{\delta}{\nu^2} + \frac{\alpha \omega^2}{\nu^2 + 4} \text{ if } \mu = 0, \nu \neq 0;
\]

\[
\theta = \frac{1}{4} \alpha \omega^2 + \frac{\delta}{2} \ln^2(\omega) + c_7 \ln(\omega) + c_8 \text{ if } \mu = \nu = 0;
\]

\[
\theta = c_7 J_0(\mu \omega) + c_8 Y_0(\mu \omega) + \frac{\alpha}{\mu^2} \text{ if } \nu = \delta = 0, \mu \neq 0.
\]

(61) \quad (62) \quad (63)
We shall not present the cumbersome general solution of equation (59) for \( x_0^2 - x_3^2 < 0 \) but restrict ourselves to the particular case when \( \alpha = \frac{c_2}{m^2} \delta \). Then

\[
\theta = c_7 (I_{1\nu}(\mu \tilde{\omega}) + I_{-1\nu}(\mu \tilde{\omega})) + c_8 (K_{1\nu}(\mu \tilde{\omega}) + K_{-1\nu}(\mu \tilde{\omega})) + \frac{\delta}{\nu^2}
\]

where \( \tilde{\omega} = \sqrt{x_3^2 - x_0^2} \).

Algebra \( A_8 : \langle J_{12}, P_0, P_3 \rangle \)

\[
B_1 = \frac{c_2x_2\theta + c_1x_1 - c_6x_2}{\omega^2}, \quad B_2 = \frac{-c_2x_1\theta + c_1x_2 + c_6x_1}{\omega^2}, \quad B_3 = -c_3\theta + c_4,
\]

\[
E_1 = \frac{c_1x_1\theta + c_5x_1 - c_2x_2}{\omega^2}, \quad E_2 = \frac{c_1x_2\theta + c_5x_2 + c_2x_1}{\omega^2}, \quad E_3 = c_3
\]

where \( \omega^2 = x_1^2 + x_2^2 \) and \( \theta \) is a solution of equation (59) with

\[
\nu^2 = c_1^2 - c_2^2, \quad \mu^2 = c_3^2 - m^2, \quad \delta = c_1c_5 + c_2c_6, \quad \alpha = c_3c_4.
\]  

(64)

If \( c_1^2 \geq c_2^2 \) and \( c_3^2 \geq m^2 \) then \( \theta \) is defined by relations (60)–(63) where \( \mu, \nu \) and \( \delta \) are constants given in (64). If \( c_1^2 - c_2^2 = -\lambda^2 < 0 \), \( m^2 < c_3^2 \) and \( \alpha(\alpha\lambda^2 + \delta\mu^2) = 0 \) then

\[
\theta = c_7 J_{\lambda}(\mu \omega) + c_8 Y_{\lambda}(\mu \omega) - \frac{\delta\pi}{2\lambda} \left( \cot \left( \frac{\pi\lambda}{2} \right) J_{\lambda}(\mu \omega) + E_{\lambda}(\mu \omega) \right) + \frac{\alpha}{\mu^2}
\]

where \( J_{\lambda}(\mu \omega) \) and \( Y_{\lambda}(\mu \omega) \) are the Bessel functions of the first and second kind, \( J_{\lambda}(\mu \omega) \) and \( E_{\lambda}(\mu \omega) \) are Anger and Weber functions correspondingly. In addition,

\[
\theta = c_7 I_{\lambda}(\kappa \omega) + c_8 K_{\lambda}(\kappa \omega) + f \quad \text{if} \quad m^2 - c_3^2 = \kappa^2 > 0, \quad c_2^2 \geq c_1^2
\]

(66)

where

\[
f = -\frac{\delta}{\kappa^2} \quad \text{if} \quad \delta = \frac{\alpha}{\lambda^2}, \quad \lambda \neq 0, \quad f = \frac{4\alpha}{m^4x^2} - \frac{\alpha}{m^2} \quad \text{if} \quad \lambda = 2, \quad \delta = 0,
\]

(67)

\( I_{\lambda}(\kappa \omega) \) and \( K_{\lambda}(\kappa \omega) \) are the modified Bessel functions of the first and second kind.

Solutions (66) are valid also for parameters \( \delta \) and \( \lambda \) which do not satisfy conditions presented in (67). The corresponding function \( f \) in (66) can be expressed via the Bessel and hypergeometric functions, but we will not present these cumbersome expressions here.

Algebra \( A_{19} : \langle J_{12}, J_{03}, P_0 - P_3 \rangle \)

\[
B_1 = E_2 = \frac{c_1(x_1 + x_2\theta)}{(x_3 + x_0)(x_1^2 + x_2^2)}, \quad B_2 = -E_1 = \frac{c_1(x_2 - x_1\theta)}{(x_3 + x_0)(x_1^2 + x_2^2)}.
\]
\[ B_3 = -c_3 \theta + c_2, \quad E_3 = c_3, \]

where \( \theta \) is a function of \( \omega = \sqrt{x_1^2 + x_2^2} \) which solves equation (59) with \( \nu = \delta = 0, \quad \mu^2 = c_2^2 - m^2, \quad \alpha = c_2 c_3. \) Its explicit form is given by equations (61) and (66) where \( \delta = 0. \)

Algebra \( A_{24} : \quad \langle G_1, J_{03}, P_2 \rangle \)

\[
\begin{align*}
B_1 &= -x_3 \varphi, \quad B_2 = -\frac{c_2 x_0}{\omega^3} - \frac{c_1}{x_0 + x_3}, \quad B_3 = x_1 \varphi, \\
E_1 &= -\frac{c_2 x_3}{\omega^3} + \frac{c_1}{x_0 + x_3}, \quad E_2 = x_0 \varphi, \quad E_3 = \frac{c_2 x_1}{\omega^3}
\end{align*}
\]

where \( \omega = \sqrt{x_0^2 - x_1^2 - x_2^2}, \quad \varphi = \varphi(\omega). \) Functions \( \varphi \) and \( \theta \) should satisfy the following equations:

\[
\begin{align*}
\omega \ddot{\varphi} + 3 \dot{\varphi} + \left( \frac{c_1}{\omega} + \frac{c_2}{\omega^2} \right) \dot{\theta} &= 0, \\
\ddot{\theta} + \frac{2 \dot{\theta}}{\omega} - \left( \frac{c_1}{\omega} + \frac{c_2}{\omega^2} \right) \varphi + m^2 \theta &= 0.
\end{align*}
\]

If \( c_1 c_2 = 0 \) then this system can be integrated in elementary or special functions:

\[
\begin{align*}
c_1 = 0: \quad \varphi &= -\frac{c_2 \dot{\theta} + c_3}{\omega^3}, \\
\theta &= c_4 \sinh\left( \frac{c_2}{\omega} \right) + c_5 \cosh\left( \frac{c_2}{\omega} \right) \quad \text{if} \quad m = 0, \quad c_2 \neq 0, \\
\theta &= \frac{1}{\omega} (c_4 \sin m \omega + c_5 \cos m \omega) \quad \text{if} \quad m \neq 0, \quad c_2 = 0, \quad \text{and} \\
\theta &= \frac{D}{\omega} \left( c_4 + \int \frac{1}{D^2 \omega} \left( c_5 + c_2 c_3 \int \frac{D dx}{\omega^{5/2}} \right) dx \right) \quad \text{if} \quad m \neq 0, \quad c_2 \neq 0
\end{align*}
\]

where \( D = D(0, m, n, m, f(\omega)) \) is the Heun double confluent function with

\[
m_\pm = m^2 + c_2^2 - \frac{1}{4}, \quad n = 2(m^2 - c_2^2), \quad f(\omega) = \frac{\omega^2 + 1}{\omega^2 - 1}.
\]

Let \( c_2 = 0, \quad c_1 \neq 0, \) then

\[
\begin{align*}
\varphi &= \frac{1}{c_1} \left( \dot{\theta} + \frac{2 \dot{\theta}}{\omega} + m^2 \theta \right), \\
\theta &= c_3 G_1(\omega) + c_4 (G_2 + G_2^* (\omega)) + i c_3 (G_2 - G_2^* (\omega))
\end{align*}
\]

where

\[
\begin{align*}
G_1(\omega) &= F \left( \frac{3 + i c_1}{2}; \frac{3 - i c_1}{2}; \frac{3}{2}; -\frac{m^2 \omega^2}{4} \right), \\
G_2(\omega) &= F \left( 1 + i c_1, 1 - i c_1; 1 + \frac{i c_1}{2}; -\frac{m^2 \omega^2}{4} \right) \omega^{-1 + i c_1},
\end{align*}
\]

16
\[ F(a, b; c; x) \] are hypergeometric functions and the asterisk denotes the complex conjugation.

Algebra \( A_{27} : \langle J_{03} + \alpha J_{12}, G_1, G_2 \rangle \)

\[
B_1 = \frac{\varphi_1}{x_0 + x_3} + \frac{(x_0 + x_3)^2 - x_1^2 + x_3^2}{2(x_0 + x_3)\omega^4} \varphi_3 - \frac{x_1 x_2}{(x_0 + x_3)\omega^4} \varphi_4, \\
B_2 = \frac{\varphi_2}{x_0 + x_3} + \frac{(x_0 + x_3)^2 - x_1^2 + x_3^2}{2(x_0 + x_3)\omega^4} \varphi_4 - \frac{x_1 x_2}{(x_0 + x_3)\omega^4} \varphi_3, \\
E_1 = -B_2 + \frac{\varphi_3(x_0 + x_3)}{\omega^4}, \quad E_2 = B_1 - \frac{\varphi_3(x_0 + x_3)}{\omega^4}, \\
E_3 = \frac{x_2 \varphi_3 - x_1 \varphi_4}{\omega^4}, \quad B_3 = -\frac{x_1 \varphi_3 + x_2 \varphi_4}{\omega^4},
\]

\[ \theta = \frac{1}{\omega} (c_1 J_1(m\omega) + c_2 Y_1(m\omega)) \] if \( m \neq 0, \) \( \omega^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 > 0, \)

\[ \theta = \frac{1}{\omega} (c_1 I_1(m\omega) + c_2 K_1(m\omega)) \] if \( m \neq 0, \) \( \tilde{\omega}^2 = -\omega^2 > 0, \)

\[ \theta = c_1 + \frac{c_2}{\omega^2} \] if \( m = 0, \)

\[ \varphi_1 = c_2 \cos (\alpha \ln(x_0 + x_3)) + c_3 \sin (\alpha \ln(x_0 + x_3)), \]

\[ \varphi_2 = c_2 \sin (\alpha \ln(x_0 + x_3)) - c_3 \cos (\alpha \ln(x_0 + x_3)), \]

\[ \varphi_3 = c_4 \sin \left( \alpha \ln \left( \frac{\omega^2}{x_0 + x_3} \right) \right) + c_5 \cos \left( \alpha \ln \left( \frac{\omega^2}{x_0 + x_3} \right) \right), \]

\[ \varphi_4 = c_4 \cos \left( \alpha \ln \left( \frac{\omega^2}{x_0 + x_3} \right) \right) - c_5 \sin \left( \alpha \ln \left( \frac{\omega^2}{x_0 + x_3} \right) \right), \quad (c_1^2 + c_2^2)(c_3^2 + c_4^2) = 0. \]

### 4.3 Reductions to nonlinear ODE

Using subalgebras \( A_6, A_9, A_{10}, A_{13}, A_{14}, A_{17} \) and \( A_{18} \) we can reduce (50), (51) to systems of ordinary differential equations which however are nonlinear.

Algebra \( A_6 : \langle J_{03} + \alpha P_2, P_0, P_3 \rangle, \quad \alpha \neq 0 \)

\[
B_1 = \varphi_1 \cosh \frac{x_2}{\alpha} - \varphi_2 \sinh \frac{x_2}{\alpha}, \quad B_2 = \alpha \varphi_2 \cosh \frac{x_2}{\alpha} - \alpha \varphi_1 \sinh \frac{x_2}{\alpha}, \\
B_3 = -c_1 \theta + c_2, \\
E_1 = \alpha \varphi_1 \cosh \frac{x_2}{\alpha} - \alpha \varphi_2 \sinh \frac{x_2}{\alpha}, \quad E_2 = \varphi_1 \sinh \frac{x_2}{\alpha} - \varphi_2 \cosh \frac{x_2}{\alpha}, \quad E_3 = c_1
\]

where \( \theta, \varphi_1 \) and \( \varphi_2 \) are functions of \( \omega = x_1 \) which satisfy the following system of nonlinear equations:

\[ \alpha \dot{\theta} \dot{\varphi}_2 = \alpha^2 \dot{\varphi}_2 + \varphi_2, \quad \varphi_1 \dot{\varphi}_2 - \dot{\varphi}_1 \varphi_2 = c_3, \]

and

\[ \ddot{\theta} = (m^2 - c_1^2) \theta + \alpha (\dot{\varphi}_1 \varphi_1 - \dot{\varphi}_2 \varphi_2) + c_1 c_2. \]
We could find only particular solutions of this complicated system, which correspond to some special values of arbitrary constants. First let us present solutions linear in \( \omega \):

\[
\theta = \frac{\omega}{\alpha} - \frac{c_1 c_2}{\nu^2} \pm \frac{\sqrt{1 - c_1^2 c_5}}{\nu}, \quad \varphi_2 = \pm \frac{\nu \omega}{\alpha \sqrt{1 - c_1^2}} + c_5, \quad \varphi_1 = c_4 \varphi_2, \quad c_4^2 < 1 \quad (70)
\]

if \( \nu^2 = m^2 - c_1^2 > 0 \),

\[
\theta = \frac{\omega}{\alpha} + \frac{c_1 c_2}{\mu^2} \pm \frac{\sqrt{c_4^2 - 1} c_5}{\mu}, \quad \varphi_2 = \pm \frac{\mu \omega}{\alpha \sqrt{c_4^2 - 1}} + c_5, \quad c_4^2 > 1 \quad (71)
\]

if \( \mu^2 = c_1^2 - m^2 > 0 \).

If \( c_1^2 = m^2 \) and \( \varphi_1 = \varphi_2 \) then \( \theta \) is given by equation (30) with \( c = -c_1 c_2 \) while \( \varphi_2 \) is a linear combination of Airy functions:

\[
\varphi_2 = c_7 \text{Ai} \left( \lambda (\omega - \nu) \right) + c_8 \text{Bi} \left( \lambda (\omega - \nu) \right) \quad (72)
\]

where \( \lambda = \left( \frac{c_1 c_2}{\alpha} \right)^3 \), \( \nu = \frac{1}{\alpha c_1 c_2} \).

If \( c_1^2 = m^2, c_2 = 0, c_3 \neq 0 \) then we find a particular solution:

\[
\theta = \left( \alpha \mu^2 + \frac{1}{\alpha} \right) x_1 + c_5, \quad \varphi_2 = c_6 \cosh \mu x_1 + c_7 \sinh \mu x_1, \quad \varphi_1 = \frac{1}{\mu} \varphi_2
\]

where \( \mu = \frac{c_1}{\alpha c_2 c_6} \) and \( c_7^2 \neq c_6^2 \).

If \( c_1^2 = m^2 \) and \( \varphi_1 = c_4 \varphi_2, c_4 \neq 1, c_2 = 0 \) then

\[
\theta = \alpha \lambda \int \varphi_2^2 d\omega + \frac{c_5}{\alpha} \omega + c_6 \quad (73)
\]

where \( \lambda = \frac{1}{2} (c_4^2 - 1) \) and \( \varphi_2 \) is an elliptic function which solves the equation

\[
\ddot{\varphi}_2 = \lambda \varphi_2^3 - \kappa \varphi_2 \quad (74)
\]

where \( \kappa = \frac{1 - c_5}{c_4^2} \). In addition to elliptic functions, equation (74) admits particular solutions in elementary ones:

\[
\varphi_2 = \pm \frac{\kappa}{\lambda} \tanh \left( \sqrt{\frac{\kappa}{2}} \omega + c_7 \right) \quad \text{if} \quad c_5 < 1, \quad c_4^2 > 1, \quad (75)
\]

\[
\varphi_2 = \pm \frac{\kappa}{\lambda} \tan \left( \sqrt{\frac{-\kappa}{2}} \omega + c_7 \right) \quad \text{if} \quad c_5 > 1, \quad c_4^2 < 1, \quad (76)
\]

\[
\varphi_2 = \pm \frac{\sqrt{2}}{\sqrt{\lambda} \omega} \quad \text{if} \quad c_5 = 1, \quad c_4 > 1. \quad (77)
\]
If \( c_1^2 = m^2 + \frac{1}{\alpha} \) and \( \varphi_1 = \pm \sqrt{1 + c_1^2} \varphi_2 \), then we can set

\[
\theta = \alpha c_4 \varphi_2 + c_1 c_2 \alpha^2
\]

and \( \varphi_2 \) should satisfy the following equation

\[
\ddot{\varphi}_2 - c_4 \dot{\varphi}_2 \varphi_2 + \frac{1}{\alpha^2} \varphi_2 = 0.
\]

Its solutions can be found in the implicit form:

\[
\omega = c_4 \alpha^2 \int_0^{\varphi_2} \frac{dt}{W\left(c_5 e^{\frac{1}{2} \alpha \nu^2}\right) + 1} + c_6
\]

where \( W \) is the Lambert function, i.e., the analytical at \( y = 0 \) solution of equation \( W(y)e^{W(y)} = y \).

Finally, for \( c_1^2 \neq m^2 \) and \( \varphi_1 = \varphi_2 \) we find the following solutions:

\[
\begin{align*}
\theta & = \frac{1}{2} \left(2 \nu c_5 \sin 2 \nu \omega + c_6 \cosh 2 \nu \omega\right) - \frac{c_1 c_2}{4 \nu^2}, \\
2 \nu & = \sqrt{m^2 - c_1^2}, \\
\varphi_2 & = D \left(0, m_+, n, m_-, \tanh \nu \omega\right) \left(c_7 + c_8 \int \frac{dx_1}{D^2 \left(0, m_+, n, m_-, \tanh \nu \omega\right)}\right)
\end{align*}
\]

(78)

where \( D(0, m_+, n, m_-, \tanh \nu \omega) \) is the Heun double confluent functions with \( m_\pm = \frac{m}{\alpha} \pm \frac{1}{\nu^2 \alpha}, \ n = \frac{c_5}{2 \nu} \). If in (78) \( c_5 = \frac{c_5}{2 \nu} \) and \( \frac{c_5}{2 \nu} = -\frac{\alpha^2}{2} < 0 \) then the corresponding expression for \( \varphi_2 \) is reduced to the following form:

\[
\varphi_2 = c_7 J_{\frac{1}{\nu}} \left(\kappa e^{\nu \omega}\right) + c_8 Y_{\frac{1}{\nu}} \left(\kappa e^{\nu \omega}\right).
\]

(79)

Algebra \( A_9 \): \( \langle J_{23} + \alpha P_0, P_2, P_3 \rangle, \ \alpha \neq 0 \)

\[
\begin{align*}
E_1 & = c_1 \theta + c_2, \\
E_2 & = \varphi_1 \cos \frac{x_0}{\alpha} - \varphi_2 \sin \frac{x_0}{\alpha}, \\
E_3 & = \varphi_1 \sin \frac{x_0}{\alpha} + \varphi_2 \cos \frac{x_0}{\alpha}, \\
B_1 & = c_1, \\
B_2 & = -\alpha \varphi_1 \cos \frac{x_0}{\alpha} + \alpha \varphi_2 \sin \frac{x_0}{\alpha}, \\
B_3 & = -\alpha \varphi_1 \sin \frac{x_0}{\alpha} - \alpha \varphi_2 \cos \frac{x_0}{\alpha}
\end{align*}
\]

where \( \varphi_1, \ \varphi_2 \) and \( \theta \) are functions of \( \omega = x_1 \) which satisfy equations \( \{68\} \) and the following equation:

\[
\ddot{\theta} = (c_1^2 + m^2) \theta - \alpha (\dot{\varphi}_1 \varphi_1 + \dot{\varphi}_2 \varphi_2) + c_1 c_2.
\]

A particular solution of this system is \( \varphi_1 = c_4 \varphi_2 \) and \( \theta, \varphi_2 \) given by equation (71) where \( \mu^2 = m^2 + c_1^2 \).

If \( c_1^2 + m^2 = 0 \) then we obtain solutions given by equations (73), (74), (76) (77) where \( \lambda = -\frac{1}{2}(c_1^2 + 1) \), and the following solutions:

\[
\begin{align*}
\varphi_1 & = c_6 \cos \mu \omega + c_7 \sin \mu \omega, \\
\varphi_2 & = c_6 \sin \mu \omega - c_7 \cos \mu \omega, \\
\theta & = \left(\frac{1}{\alpha} - \alpha \mu^2\right) \omega + c_5
\end{align*}
\]

(80)
where \( \mu = \frac{c_1}{c_4 + c_2} \).

Algebra \( A_{10} : \langle J_{12} + \alpha P_3, P_1, P_2 \rangle, \alpha \neq 0 \)

\[
\begin{align*}
B_1 &= \varphi_1 \cos \frac{x_3}{\alpha} - \varphi_2 \sin \frac{x_3}{\alpha}, \quad B_2 = \varphi_1 \sin \frac{x_3}{\alpha} + \varphi_2 \cos \frac{x_3}{\alpha}, \quad B_3 = c_1, \\
E_1 &= \alpha \dot{\varphi}_1 \cos \frac{x_3}{\alpha} - \alpha \dot{\varphi}_2 \sin \frac{x_3}{\alpha}, \quad E_2 = \alpha \dot{\varphi}_1 \sin \frac{x_3}{\alpha} + \alpha \dot{\varphi}_2 \cos \frac{x_3}{\alpha}, \quad E_3 = c_1 \theta + c_2
\end{align*}
\]

where \( \varphi_1, \varphi_2 \) and \( \theta \) are functions of \( \omega = x_0 \) satisfying (68) and the following equation:

\[
\ddot{\theta} = - (c_1^2 + m^2) \theta - \alpha (\varphi_1 \varphi_1 + \varphi_2 \varphi_2) - c_1 c_2.
\]

If \( c_1^2 + m^2 = 0 \) we again obtain solutions (80) and solutions given by equations (73), (74), (75) where \( \lambda = - \frac{1}{2} (c_1^2 + 1) \).

Algebra \( A_{17} : \langle J_{03} + \alpha J_{12}, P_0, P_3 \rangle, \alpha \neq 0 \)

\[
\begin{align*}
B_1 &= (\alpha \dot{\varphi}_1 x_2 - \varphi_2 x_1)e^{-\frac{c_1}{\alpha} - \omega} + (x_1 \varphi_1 + \alpha \dot{\varphi}_1 x_2)e^{\frac{c_1}{\alpha} - \omega}, \\
B_2 &= -(\alpha \dot{\varphi}_2 x_1 + \varphi_2 x_2)e^{-\frac{c_1}{\alpha} - \omega} - (\alpha \dot{\varphi}_1 x_1 - \varphi_1 x_2)e^{\frac{c_1}{\alpha} - \omega}, \quad B_3 = -c_1 \theta + c_2, \\
E_1 &= (\alpha \dot{\varphi}_2 x_2 - \varphi_1 x_1)e^{-\frac{c_1}{\alpha} - \omega} - (x_1 \alpha \dot{\varphi}_1 - \varphi_1 x_2)e^{\frac{c_1}{\alpha} - \omega}, \\
E_2 &= (\alpha \dot{\varphi}_1 x_2 + \varphi_2 x_1)e^{-\frac{c_1}{\alpha} - \omega} - (\alpha \dot{\varphi}_1 x_1 + \varphi_1 x_2)e^{\frac{c_1}{\alpha} - \omega}, \quad E_3 = c_1,
\end{align*}
\]

where \( \omega = \frac{1}{2} \ln \left( \frac{x_2}{x_1} \right), \quad \zeta = \arctan \frac{x_2}{x_1} \). Functions \( \varphi_1 = \varphi_1(\omega), \varphi_2 = \varphi_2(\omega) \) and \( \theta = \theta(\omega) \) should satisfy (68) and the following equation:

\[
e^{-2\omega} \ddot{\theta} + (m^2 - c_1^2) \theta + 2 \alpha (\varphi_1 \varphi_2 + \varphi_1 \dot{\varphi}_2) + c_1 c_2 = 0. \tag{81}
\]

This rather complicated system has the following particular solutions for \( c_1 = \pm m \):

\[
\theta = \frac{1}{\alpha} \omega + c_4, \quad \varphi_1 = c_5, \quad \varphi_2 = c_6; \tag{82}
\]

\[
\theta = \left( \frac{1}{\alpha} + \alpha k^2 \right) \omega + c_4, \quad \varphi_1 = c_5 e^{\omega}, \quad \varphi_2 = c_6 e^{-\omega} \quad \text{if} \quad 2 c_5 c_6 k + c_3 = 0;
\]

\[
\theta = - \frac{1}{4} c_1 c_2 e^{2\omega} + c_4 \omega + c_5, \quad \varphi_1 = 0, \quad \varphi_2 = c_6 J_\mu (k e^{\omega}) + c_7 Y_\mu (k e^{\omega}),
\]

\[
\mu = \frac{1}{\alpha} \sqrt{c_4 \alpha - 1} \quad \text{if} \quad \frac{c_1 c_2}{2 \alpha} = k^2 > 0,
\]

and

\[
\varphi_1 = \kappa \varphi_2, \quad \theta = \varphi_2 e^{\omega}, \quad \varphi_2 = 2 \mu \tan (\mu e^{\omega}) + c_4 \tag{83}
\]

if \( \kappa = \frac{1}{2\alpha}, \quad c_1 c_2 = 4 \mu^2 > 0, \quad \alpha = \pm 1 \). In (83) we restrict ourselves to the particular value of \( \alpha \) in order to obtain the most compact expressions for exact solutions.
An exact solution of equation (63), (81) for \( m^2 - c_1^2 = 4\lambda^2 > 0 \) and \( c_2 = 0 \) is given by the following equation:

\[
\theta = \frac{e^{4\mu(1+\alpha^2)\omega+2\lambda^2\omega^2}d\omega + c_4}{e^{4\mu(1+\alpha^2)\omega+2\lambda^2\omega^2}d\omega + c_4}, \quad \varphi_2 = \theta e^{\omega}, \quad \varphi_1 = \mu \varphi_2
\]  

where \( \lambda, \mu \) and \( \alpha \) are arbitrary real numbers.

Algebra \( A_{18} \) : \( \langle \alpha, j_{03} + j_{12}, P_1, P_2 \rangle, \alpha \neq 0 \)

\[
B_1 = e^{-2\omega} ((\varphi_1 x_0 - \alpha \varphi_2 x_3) \cos \zeta - (\varphi_2 x_0 + \alpha \varphi_1 x_3) \sin \zeta), \\
B_2 = e^{-2\omega} ((x_0 \varphi_1 - \alpha \varphi_2 x_3) \sin \zeta + (x_0 \varphi_2 + \alpha \varphi_1 x_3) \cos \zeta), \quad B_3 = c_1, \\
E_1 = e^{-2\omega} ((-\alpha \varphi_2 x_0 + \varphi_1 x_3) \sin \zeta + (\alpha \varphi_1 x_0 + \varphi_2 x_3) \cos \zeta), \\
E_2 = e^{-2\omega} ((\alpha \varphi_2 x_0 - \varphi_1 x_3) \cos \zeta + (\alpha \varphi_1 x_0 + \varphi_2 x_3) \sin \zeta), \quad E_3 = c_1 \theta - c_2
\]

where \( \omega = \frac{1}{2} \ln(x_0^2 - x_3^2) \), \( \alpha \zeta = \ln(x_0 + x_3) - \ln(x_0 - x_3) \), \( \varphi_1, \varphi_2 \) and \( \theta \) are functions of \( \omega \) which should solve the system including (63) and the following equation:

\[
e^{-2\omega} \dot{\theta} = -(m^2 + c_1^2) \theta + \alpha (\dot{\varphi}_1 \varphi_1 + \dot{\varphi}_2 \varphi_2) + c_1 c_2.
\]

Particular solutions of this system for \( m^2 + c_1^2 = 0 \) are:

\[
\theta = \left( \frac{1}{\alpha} - \alpha \kappa^2 \right) \omega + c_4, \quad \varphi_1 = c_5 \sin(k \omega), \quad \varphi_2 = c_6 \cos(k \omega), \quad \kappa = -\frac{c_3}{c_5 c_6}.
\]

In addition, the solutions (82), (83) and (84) are valid where \( \omega \to \ln(x_0^2 - x_3^2) \).

### 4.4 Reductions to PDE

Finally, let us make reductions of system (50), (51) using the remaining subalgebras, i.e., \( A_6 \) with \( \alpha = 0 \) and \( A_{28} - A_{30} \). Basis elements of these algebras do not satisfy condition (5) and so it is not possible to use the classical symmetry reduction approach. However, to make the reductions we can impose additional conditions on dependent variables which force equations (5) to be satisfied.

This idea is used in the weak transversality approach discussed in [19]. Moreover, in this approach the condition (5) by itself is used to find algebraic conditions for elements of matrices \( \varphi_k \).

We use even more week conditions which we call extra weak transversality. In other words we also look for additional constraints to solutions of equations (50), (51) which force condition (5) to be satisfied. But instead of the direct use of algebraic condition (5) we also take into account their differential consequences. As a result we make all reductions for the system (50), (51) which can be obtained in frames of the weak transversality approach and also some additional reductions.
Let us start with algebra $A_6$ with $\alpha = 0$. The set of the related basis elements $(P_0, P_3, J_{03})$ does not satisfy condition (5). If we consider this condition as an additional algebraic constraint to solutions of equations (50), (51) then components of vectors $E$ and $B$ should satisfy the following relations:

$$E_1 = E_2 = B_1 = B_2 = 0 \quad (85)$$

Substituting (85) into (50), (51) and supposing that $E_3$, $B_3$ and $\theta$ depend only on invariants $x_1$ and $x_2$ we can find the corresponding exact solutions. However, we will obtain more general solutions using the following observation.

To force condition (5) to be satisfied it is possible to apply additional conditions which are weaker than (85). In particular, condition (5) should be satisfied if we impose a constraint on dependent variables which nullifies the term $E_1 \partial B_2 - E_2 \partial B_1 - B_1 \partial E_2 + B_2 \partial E_1$ in $J_{03}$. Up to Lorentz transformations such constraints are exhausted by the following ones:

$$E_1 = 0, \quad E_2 = 0; \quad (86)$$
$$E_1 = 0, \quad B_2 = 0; \quad (87)$$
$$E_1 = B_1, \quad E_2 = B_2. \quad (88)$$

Let relations (86) are fulfilled and $B_2$, $B_3$, $E_1$, $E_3$, $\theta$ depend on the invariant variables $x_1$ and $x_2$. Then the system (50), (51) is solved by the following vectors:

$$E_1 = E_2 = 0, \quad E_3 = c_3, \quad B_1 = \frac{\partial \phi}{\partial x_2}, \quad B_2 = -\frac{\partial \phi}{\partial x_1}, \quad B_3 = -c_3 \theta + c_2 \quad (89)$$

provided function $\theta = \theta(x_1, x_2)$ satisfies the following equation

$$\frac{\partial^2 \theta}{\partial x_1^2} + \frac{\partial^2 \theta}{\partial x_2^2} = (m^2 - c_1^2) \theta + c_1 c_2. \quad (90)$$

and $\phi = \phi(x_1, x_2)$ solves the two-dimension Laplace equation:

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = 0. \quad (91)$$

A particular solution of equation (90) is:

$$\theta = X(x_1) Y(x_2) + \frac{c_1 c_2}{c_1^2 - m^2} \quad \text{if} \quad c_1^2 \neq m^2,$$
$$\theta = X(x_1) Y(x_2) + \frac{c_1 c_2}{2}(x_1^2 + x_2^2) \quad \text{if} \quad c_1^2 = m^2 \quad (92)$$

where

$$X(x_1) = c_3, e^{k_\mu x_1} + c_4, e^{-k_\mu x_1}, \quad Y(x_2) = c_5, \cos(n_\mu x_2) + c_6, \sin(n_\mu x_2). \quad (93)$$
Here \( k_2^2 = m^2 + \mu^2 \), \( n_\mu = c_\mu^2 + \mu^2 \), and \( c_{s,\mu} \), \( s = 3, 4, 5, 6 \) are arbitrary constants. The general solution of equation (90) can be expressed as a sum (integral) of functions (92) over all possible values of \( \mu \) and \( c_{s,\mu} \).

Solutions (89) include an arbitrary harmonic function \( \phi \). Only a very particular case of this solution corresponding to \( \phi = \text{Const} \) can be obtained in frames of the standard weak transversality approach discussed in [10].

Analogously, imposing condition (87) we obtain the following solutions:

\[
E_1 = 0, \quad E_3 = c_1, \quad B_2 = 0, \quad B_3 = -c_1 \theta + c_2 \tag{94}
\]

and

\[
E_2 = c_3 \sum^{1/2} (c_4 \cos \nu x_2 - c_5 \sin \mu x_2) + c_6 \sum^{-1/2} (c_5 \sin \nu x_2 - c_4 \cos \mu x_2), \\
B_1 = c_3 \sum^{1/2} (c_4 \cos \nu x_2 - c_5 \sin \mu x_2) - c_6 \sum^{-1/2} (c_5 \sin \nu x_2 - c_4 \cos \mu x_2), \\
\theta = x_1 (c_4 \cos \nu x_2 + c_5 \sin \mu x_2) + c_7 \sin \mu x_2 + c_8 \cos \mu x_2 + \frac{c_1 c_2}{c_1^2 - m^2}
\]

if \( c_2^2 - m^2 = \mu^2 > 0 \);

\[
E_2 = c_3 \sum^{1/2} (c_4 \cos \nu x_2 - c_5 \sin \mu x_2) + c_6 \sum^{-1/2} (c_5 \sin \nu x_2 - c_4 \cos \mu x_2), \\
B_1 = c_3 \sum^{1/2} (c_4 \cos \nu x_2 - c_5 \sin \mu x_2) - c_6 \sum^{-1/2} (c_5 \sin \nu x_2 - c_4 \cos \mu x_2), \\
\theta = x_1 (c_4 \cos \mu x_2 + c_5 \sin \mu x_2) + c_8 \sin \mu x_2 + c_9 \cos \mu x_2 + \frac{c_6 c_7}{c_6^2 - m^2}
\]

if \( c_6^2 - m^2 = -\nu^2 < 0 \);

\[
E_2 = c_3 \sinh \left( \frac{1}{2} c_4 x_2^2 + c_5 x_2 \right) + c_6 \cosh \left( \frac{1}{2} c_4 x_2^2 + c_5 x_2 \right), \\
B_1 = c_6 \sinh \left( \frac{1}{2} c_4 x_2^2 + c_5 x_2 \right) + c_3 \cosh \left( \frac{1}{2} c_4 x_2^2 + c_5 x_2 \right), \\
\theta = x_1 (c_4 x_2 + c_5) - \frac{1}{2} c_1 c_2 x_2^2 + c_7 x_2 + c_8 \quad \text{if} \quad c_1^2 = m^2.
\]

If conditions (88) are imposed then one obtains the solutions

\[
E_\alpha = B_\alpha = \partial_\alpha \phi, \quad \alpha = 1, 2, \quad E_3 = c_1, \quad B_3 = -c_1 \theta + c_2 \tag{96}
\]

where \( \phi \) is a function satisfying (91), and \( \theta \) is a solution of the following equation:

\[
\frac{\partial^2 \theta}{\partial x_1^2} + \frac{\partial^2 \theta}{\partial x_2^2} = (m^2 - c^2_\mu) \theta + c_1 c_2 + (\partial_1 \phi)^2 + (\partial_2 \phi)^2. \tag{97}
\]

Another solution corresponding to (88) is given by equations (85) for \( B_1, B_2, E_1, E_2 \) and \( E_3 = c_1, \quad B_3 = -c_1 \theta + c_2 \), while \( \theta \) is given by equation (92).

Algebra \( A_{28} \) : \( \langle G_1, G_2, J_{12} \rangle \)

\[
B_1 = E_2 = \frac{1}{(x_0 + x_3)^3} (c_1 x_1 - x_2 (c_1 \theta + c_2)),
\]

23
\[ B_2 = -E_1 = \frac{1}{(x_0 + x_3)^3} \left( c_1 x_2 + x_1 (c_1 \theta + c_2) \right), \]
\[ B_3 = \frac{c_1}{(x_0 + x_3)^2}, \quad E_3 = -\frac{(c_1 \theta + c_2)}{(x_0 + x_3)^3}, \quad \theta = \frac{\varphi}{x_0 + x_1} \]

where \( \varphi \) is a function of two variables \( \omega = \frac{x_0^2 - x_1^2 - x_3^2}{2(x_0 + x_3)} \) and \( \zeta = x_0 + x_3 \), which satisfies the following equation:

\[
\frac{\partial^2 \varphi}{\partial \omega \partial \zeta} = \left( \frac{c_1^2}{\zeta^4} - m^2 \right) \varphi + \frac{c_1 c_2}{\zeta^3}.
\] (98)

Let \( m = c_1 = 0 \) then \( \varphi = \varphi_1(\omega) + \varphi_2(\zeta) \) where \( \varphi_1 \) and \( \varphi_2 \) are arbitrary functions. For \( c_1 = 0, m^2 \neq 0 \) equation (98) admits solutions in separated variables:

\[
\varphi = \sum_{\mu} (a_\mu \sin(\nu_\mu \xi_+ \sin(\mu \xi_-)) + b_\mu \cos(\nu_\mu \xi_+ \cos(\mu \xi_-)) + c_\mu \cos(\nu_\mu \xi_+ \sin(\mu \xi_-)) + d_\mu \sin(\nu_\mu \xi_+ \cos(\mu \xi_-))
\] (99)

where \( \xi_+ = \omega \pm \zeta, \; \nu_\mu^2 = m^2 + \mu^2 \) and \( \mu, S_\mu, a_\mu, b_\mu, c_\mu \) and \( d_\mu \) are arbitrary constants.

For \( c_1 \neq 0 \) we obtain:

\[
\varphi = \frac{c_1 c_2 \zeta}{c_1^2 - m^2 \zeta^4} + \sum_{\mu} R_\mu e^{i\omega - \frac{3m^4 + x_3^2}{\nu c^3}}.
\]

Algebra \( A_{20} : \langle J_{01}, J_{02}, J_{12} \rangle \)

\[
B_1 = x_2 \left( \frac{c_1 \theta + c_2}{\omega^3} \right), \quad B_2 = -\frac{x_1 (c_1 \theta + c_2)}{\omega^3}, \quad B_3 = \frac{c_1 x_0}{\omega^3}, \]
\[
E_1 = \frac{c_1 x_2}{\omega^3}, \quad E_2 = \frac{c_1 x_1}{\omega^3}, \quad E_3 = \frac{x_0 (c_1 \theta + c_2)}{\omega^3}, \quad \theta = \frac{\varphi}{\omega}
\]

where \( \omega^2 = x_0^2 - x_1^2 - x_3^2 \) and \( \varphi \) is a function of \( \omega \) and \( x_3 \) which satisfy the following equation:

\[
\frac{\partial^2 \varphi}{\partial x_3^2} - \frac{\partial^2 \varphi}{\partial \omega^2} = \left( \frac{c_1^2}{\omega^4} + m^2 \right) \varphi + \frac{c_1 c_2}{\omega^3} \quad \text{if} \quad x_0^2 > x_1^2 + x_3^2
\] (100)

where \( \omega = \sqrt{x_0^2 - x_1^2 - x_3^2} \), and

\[
\frac{\partial^2 \varphi}{\partial x_3^2} + \frac{\partial^2 \varphi}{\partial \omega^2} = \left( m^2 - \frac{c_1^2}{\omega^4} \right) \varphi - \frac{c_1 c_2}{\omega^3} \quad \text{if} \quad x_0^2 < x_1^2 + x_3^2
\] (101)

where \( \omega = \sqrt{x_1^2 + x_3^2 - x_0^2} \).

Let \( c_1 = m = 0 \) then the general solution of equation (100) is: \( \varphi = \varphi_1(\omega + x_3) + \varphi_2(\omega - x_3) \) where \( \varphi_1 \) and \( \varphi_2 \) are arbitrary functions. Solutions which correspond to
c₁ = 0, m ≠ 0 can be obtained from (99) by changing ξ₊ → x₃, ξ₋ → ω. If c₁ ≠ 0 and m ≠ 0 then

\[ \varphi = \sum \alpha \left( a_\mu + b_\mu \int \frac{d\omega}{\omega D_\mu^2} \sin(\mu x_3) + (c_\mu + d_\mu \int \frac{d\omega}{\omega D_\mu^2} \cos(\mu x_3)) \right) - c_1c_2 \int \left( \frac{1}{\omega D_0^2} \int \frac{D_0 d\omega}{\omega^{3/2}} \right) d\omega \]  

(102)

where D_µ = D(0, k₋, s, k₊, f(ω)) is the double confluent Heun function with k₊ = m² - μ² + c₁² ± 1/4, s = 2(m² - μ² - c₁²), f(ω) = \( \frac{ω² + 1}{ω² - 1} \), \( \mu, a_\mu, b_\mu, c_\mu \) and \( d_\mu \) are arbitrary constants.

Solutions of equation (101) also can be represented in the form (102) where ω → ˜ω and

\[ k_± = \mu² - m² + c₁² ± 1/4, \ s = 2(\mu² - m² - c₁²), \ f(ω) = \frac{ω² + 1}{ω² - 1}. \]

Algebra \( A_{30} = \{ J_{12}, J_{23}, J_{31} \} \)

\[ B_a = \frac{c_1 x_a}{r^3}, \ E_a = \frac{(c_1 \theta - c_2) x_a}{r^3}, \ \theta = \frac{\varphi}{r}, \]

where \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \) and \( \varphi \) is a function of \( r \) and \( x_0 \) satisfying the following equation:

\[ \frac{\partial^2 \varphi}{\partial r^2} - \frac{\partial^2 \varphi}{\partial x_0^2} = \left( \frac{c_1}{r^2} + m² \right) \varphi - \frac{c_1c_2}{r^3}. \]  

(103)

Solutions of this equation can be represented in the form (102) where ω = r, \( x₃ \rightarrow x₀ \) and \( D_µ = D(0, k₋, s, k₊, f(ω)) \) is the double confluent Heun function with \( k_± = -(m² + μ² + c₁²) ± 1/4, \ s = 2(c₁² - m² - μ²), \ f(ω) = f(r) = \frac{r² + 1}{r² - 1}. \)

A special solution of equation (103) corresponding to \( c_2 = 0 \) and zero constant of variable separation is given in (44).

### 4.5 Solutions with maximal arbitrariness

Solutions considered in the above include arbitrary parameters and in some cases even arbitrary functions. At the end of our analysis a special class of solutions will be presented which depend on six (!) arbitrary functions. This class cover all reductions which can be obtained using subalgebras \( A_{12} - A_{14} \) and \( A_{20} - A_{23} \).

Let us define

\[ B_1 = E_2 = ψ_1(x_1, x_2, ω) - x_1 ϕ_1(ω) - x_2 \left( ϕ_4(ω) - ϕ_5(ω) \right) \theta, \]

\[ B_2 = -E_1 = ψ_2(x_1, x_2, ω) - x₂ϕ_5(ω) + x₁ \left( ϕ_2(ω) - ϕ_1(ω) \right) \theta(ω) \],

\[ B_3 = ϕ_1(ω) + ϕ_5(ω), \ E_3 = ϕ_2(ω) + ϕ_4(ω) \]

(104)
where \( \varphi_1, \ldots, \varphi_5 \) and \( \psi_1, \psi_2 \) are functions of \( \omega = x_0 + x_3 \) and \( x_1, x_2, \omega \) respectively, and

\[
\theta = -\frac{(\varphi_1 + \varphi_5)(\varphi_2 + \varphi_4)}{m^2}, \quad \text{if} \quad F = -m^2\theta, \ m^2 \neq 0, \\
\theta = \varphi_3(\omega), \quad (\varphi_1 + \varphi_5)(\varphi_2 + \varphi_4) = 0 \quad \text{if} \quad F = 0. \quad (105)
\]

Up to restriction present in (105) functions \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) are arbitrary while \( \psi_1 \) and \( \psi_2 \) should satisfy the Caushy–Rieman condition with respect to variables \( x_1 \) and \( x_2 \):

\[
\partial_1 \psi_1 + \partial_2 \psi_2 = 0, \quad \partial_1 \psi_2 - \partial_2 \psi_1 = 0. \quad (106)
\]

It is easily to verify that Ansatz (104), (105) does solves equations (50), (51). Thus any solution of the Caushy-Rieman equations (106) depending on parameter \( \omega \) and three arbitrary functions \( \varphi_1(\omega), \varphi_2(\omega), \varphi_3(\omega) \) satisfying (105) give rise to solution (104) of system (50), (51).

5 Discussion

The main aim of the present paper is to find families of exact solutions of field equations of axion electrodynamics using invariants of three parameter subgroups of the Poincaré group. The complete list of reductions which can be made using these invariants together with obtained solutions are presented in sections 3 and 4. Among them there are solutions including sets of arbitrary parameters and arbitrary functions as well. In addition, it is possible to generate more extended families of exact solutions applying the inhomogeneous Lorentz transformations to the found ones.

For such subalgebras whose basis elements do not satisfy the transversality condition (5) we apply the week and ”extra weak” transversality approach, see section 4.4. As a result we find solutions (89)–(94) which cannot be found applying standard weak transversality conditions discussed in [19].

Making reductions of equations (2), (3) we restrict ourselves to functions \( F \) linear in \( \theta \). However, these reductions do not depend of the choice of \( F \); to obtain reduced equations with \( F \) arbitrary it is sufficient simple to change \( m^2\theta \to -F(\theta) \) or even \( m^2\theta \to -F(\theta, p^\mu p^\mu) \) everywhere.

Except a particular example given by relations (46)-(49) we did not present exact solutions for equations (4). Let us note that reductions of these equations can be made in a very straightforward way. Indeed, making the gauge transformation \( E_a \to e^{\theta}E_a \) and \( B_a \to e^{\theta}B_a \) we can reduce these equations to a system including the Maxwell equation for the electromagnetic field in vacua and the following equation:

\[
\Box \theta = \kappa(B^2 - E^2)e^{-2\theta} + F. \quad (107)
\]
Since reductions of the free Maxwell equations with using three-dimension subalgebras of \( p(1,3) \) have been done in paper [20], to find the related exact solutions for system (4), it is sufficient to solve equation (107) with \( B \) and \( E \) being exact solutions found in [20].

Solutions presented in sections 3 and 4 can have various useful applications. Indeed, the significance of exact solutions, even particular ones, can be rather high. First, they present a certain information about particular properties of the model. Secondly, they can solve an important particular boundary value problem, a famous example of this kind is the Barenblat solution for the diffusion equation [21]. In addition, the particular exact solutions can be used to test the accuracy of various approximate approaches.

The found solutions, especially those which include arbitrary functions or, like (37), satisfy the superposition principle, are good candidates to applications in various initial and boundary value problems of axion electrodynamics. Some of these solutions, e.g., (31), (34) with \( \mu = 1 \) and \( c_1^2 + c_2^2 > 1 \), describe the wave propagation with the group velocity higher than the velocity of light. Moreover, these solutions are smooth and bounded functions which correspond to positive definite and bounded energy density [14].

We believe that the list of exact solutions presented in sections 3 and 4 can find other interesting applications. In particular, solutions, which correspond to algebras \( A_9, A_1, A_{17}, A_{18} \) and \( A_{28} \) generate well visible dynamical contributions to the axion mass. In addition, as it was indicated in [8], the vectors of the electric and magnetic fields described by relations (45) give rise to exactly solvable Dirac equation for a charged particle anomalously interacting with these fields. We plane to present a detailed analysis of the obtained solutions elsewhere.

Acknowledgments

We would like to thank Professors Roman Popovych and Stephen Anco for useful discussions.

References

[1] G.G. Raflet, Phys. Rep. 198, 1 (1990).
[2] X-L. Qi, T. L. Hughes, and S-C. Zhang, Phys. Rev. B 78, 195424 (2008).
[3] F.W. Hehl, Y.N. Obukhov J.-P. Rivera and H. Schmid, Eur. Phys. J. B 71, 321329 (2009).
[4] R. D. Peccei and H. Quinn, Phys. Rev. Lett. 38, 1440 (1977).
[5] S. Weinberg, Phys. Rev. Lett. 40, 223 (1978).
[6] F. Wilczek, Phys. Rev. Lett. **40**, 279 (1978).

[7] F. Wilczek, Phys. Rev. Lett. **58**, 1799 (1987).

[8] E. Ferraro, N. Messina and A.G. Nikitin, Phys. Rev. A **81**, 042108 (2010).

[9] A. G. Nikitin and Y. Karadzhov, J. Phys. A: 44 (2011) 305204.

[10] A. G. Nikitin and Y. Karadzhov, J. Phys. A: 44 (2011) 445202.

[11] M. de Montigny, J. Niederle and A.G. Nikitin, J. Phys. A: Mat. Theor. **39**, 1 (2006).

[12] J. Niederle and A.G. Nikitin, Czech. J. Phys. **56**, 1243 (2006).

[13] J. Niederle and A.G. Nikitin, J. Phys. A: Mat. Theor. **42** 105207 (2009).

[14] A. G. Nikitin and O. Kuriksha. ArXiv 1201.4935

[15] P. Olver, *Application of Lie groups to Differential equations* (Springer-Verlag, N.Y., 1986).

[16] W. I. Fushchich and A. G. Nikitin, *Symmetries of Equations of Quantum Mechanics* (Allerton Press Inc., N.Y. 1994).

[17] I. V. Bel’ko, Izv. Akad. Nauk Bel. SSR **1**, 5 (1971).

[18] J. Patera, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **16** 1597 (1975).

[19] M. A. Grundland, P. Tempesta and P. Winternitz, J.Math.Phys. **44**, 2704 (2003).

[20] H. O. Lahno and V. F. Smalij, Proc. of Institute of Mathematics of NAS of Ukraine **43**, Part 1, 162 (2002).

[21] G. I. Barenblat, *Scaling, Self-Similarity and Intermediate Asymptotics* (Cambridge University Press, Cambridge, U.K., 1996).