Resourceful Contextual Bandits

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Abstract
We study contextual bandits with ancillary constraints on resources, which are common in real-world applications such as choosing ads or dynamic pricing of items. We design the first algorithm for solving these problems, and prove a regret guarantee with near-optimal statistical properties.

1. Introduction

Contextual bandits is a machine learning framework in which an algorithm makes sequential decisions according to the following protocol: in each round, a context arrives, then the algorithm chooses an action from the fixed and known set of possible actions, and then the reward for this action is revealed; the reward may depend on the context, and can vary over time. Contextual bandits is one of the prominent directions in the literature on online learning with exploration-exploitation tradeoff; many problems in this space are studied under the name multi-armed bandits.

A canonical example of contextual bandit learning is choosing ads for a search engine. Here, the goal is to choose the most profitable ad to display to a given user based on a search query and the available information about this user, and optimize the ad selection over time based on user feedback such as clicks. This description leaves out many important details, one of which is that every ad is associated with a budget which constrains the maximum amount of revenue which that ad can generate. In fact, this issue is so important that in some formulations it is the primary problem (e.g., Devanur and Vazirani, 2004).

The optimal solution with budget constraints fundamentally differs from the optimal solution without constraints. As an example, suppose that one ad has a high expected revenue but a small budget such that it can only be clicked on once. Should this ad be used immediately? From all reasonable perspectives, the answer is “no”. From the user’s or advertiser’s perspective, we prefer that this ad be displayed for the user with the strongest interest rather than for a user who simply has more interest than in other options. From a platform’s viewpoint, it is better to have more ads in the system, since they effectively increases the price paid in a second price auction. And from everyone’s viewpoint, it is simply odd to burn out the budget of an ad as soon as it is available. Instead, a small budget should be parcelled out over time.

To address these issues, we consider a generalization of contextual bandits in which there are one or several resources that are consumed by the algorithm. This formulation has many natu-
rai applications. Dynamic ad allocation follows the ad example described above: here, resources correspond to advertisers’ budgets. In dynamic pricing, a store with a limited supply of items to sell can make customized offers to customers. In dynamic procurement, a contractor with a batch of jobs and a limited budget can experiment with prices offered to the workers, e.g., workers in a crowdsourcing market. The above applications have been studied on its own, but never in models that combine contexts and limited resources.

We obtain the first known algorithm for contextual bandits with resource constraints. As such, we merge two lines of work on multi-armed bandits: contextual bandits and bandits with resource constraints. While significant progress has been achieved in each of the two lines of work (in particular, optimal solutions have been worked out for very general models), the specific approaches break down when applied to our model.

Our model. We define resourceful contextual bandits (in short: RCB), a common generalization of contextual bandits with policy sets (e.g., Dudik et al., 2011) and a general setting for bandits with resource constraints (Badanidiyuru et al., 2013a).

There are several resources that are consumed by the algorithm, with a separate constraint on each. In each round, the algorithm receives a reward and consumes some amount of each resource, in a manner that depends on the context and the chosen action, and may be randomized. We consider a stationary environment: in each round, the context and the mapping from actions to rewards and resource consumption is sampled independently from a fixed joint distribution, called the outcome distribution. Rewards and consumption of various resources can be correlated in an arbitrary way. The algorithm stops as soon as any constraint is violated. Initially the algorithm is given no information about the outcome distribution (except the distribution of context arrivals). In particular, expected rewards and resource consumptions are not known.

An algorithm is given a finite set $\Pi$ of policies: mappings from contexts to actions. We compete against algorithms that must commit to some policy in $\Pi$ before each round. Our benchmark is a hypothetical algorithm that knows the outcome distribution and makes optimal decisions given this knowledge and the restriction to policies in $\Pi$. The benchmark’s expected total reward is denoted $\text{OPT}(\Pi)$. Regret of an algorithm is defined as $\text{OPT}(\Pi)$ minus the algorithm’s expected total reward.

For normalization, per-round rewards and resource consumptions lie in $[0, 1]$.

Discussion of the model. Allowing stochastic resource consumptions and arbitrary correlations between per-round rewards and per-round resource consumptions is essential: this is why our model subsumes diverse applications such as the ones discussed above, and many extensions thereof. Further discussion of the application domains can be found in Appendix A.

Intuitively, the policy set $\Pi$ consists of all policies that can possibly be learned by a given learning method, such as linear estimation or decision trees. Restricting to $\Pi$ allows meaningful performance guarantees even if competing against all possible policies is intractable. The latter is common in real-life applications, as the set of possible contexts can be very large.

Our benchmark can change policies from one round to another without restriction. As we prove, this is essentially equivalent in power to the best fixed distribution over policies. However, the best fixed policy may perform substantially worse.

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1. For example, in dynamic pricing the algorithm receives a reward and consumes a resource only if an item is sold.
2. The expected total reward of the best fixed policy can be half as large as that of the best distribution. This holds for several different domains including dynamic pricing / procurement, even without contexts (Badanidiyuru et al., 2013a). Note that without resource constraints, the two benchmarks are equivalent.
Our stopping condition corresponds to hard constraints: an advertiser cannot exceed his budget, a store cannot sell more items than it has in stock, etc. An alternative stopping condition is to restrict the algorithm to actions that cannot possibly violate any constraint if chosen in the current round, and stop if there is no such action. This alternative is essentially equivalent to the original version.\(^3\) Moreover, we can w.l.o.g. allow our benchmark to use this alternative.

Our contributions: main algorithm. We design an algorithm, called Mixture\_Elimination, and prove the following guarantee on its regret.

**Theorem 1** For all RCB problems with \(K\) actions, \(d\) resources, time horizon \(T\), and for all policy sets \(\Pi\). Algorithm Mixture\_Elimination achieves regret

\[
O \left(1 + \frac{1}{B}\right) \sqrt{dKT \log (dKT|\Pi|)}
\]

where \(B = \min_i B_i\) is the smallest resource constraint.

This regret guarantee is optimal in several regimes. First, we achieve an optimal square-root “scaling” of regret: if all constraints are scaled by the same parameter \(\alpha > 0\), then regret scales as \(\sqrt{\alpha}\). Second, if \(B = T\) (i.e., there are no constraints), we recover the optimal \(\tilde{O}(\sqrt{KT})\) regret. Third, we achieve \(\tilde{O}(\sqrt{KT})\) regret for the important regime when \(\text{OPT}(\Pi)\) and \(B\) are at least a constant fraction of \(T\). In fact, Badanidiyuru et al. (2013a) provide a complimentary \(\Omega(\sqrt{KT})\) lower bound for this regime, which holds in a very strong sense: for any given tuple \((K, B, \text{OPT}(\Pi), T)\).

The \(\sqrt{\log |\Pi|}\) term in Theorem 1 is unavoidable (Dudik et al., 2011). The dependence on the minimum of the constraints (rather than, say, the maximum or some weighted combination thereof) is also unavoidable (Badanidiyuru et al., 2013a). For strongest results, one can rescale per-round rewards and per-round consumption of each resource so that they can be as high as \(1\).\(^4\)

Note that the regret bound in Theorem 1 does not depend on the number of contexts, only on the number of policies in \(\Pi\). In particular, it tolerates infinitely many contexts. On the other hand, if the set \(X\) of contexts is not too large, we can also obtain a regret bound with respect to the best policy among all possible policies. Formally, take \(\Pi = \{\text{all policies}\}\) and observe that \(|\Pi| \leq K^{|X|}\).

Further, Theorem 1 extends to policy sets \(\Pi\) that consist of randomized policies: mappings from contexts to distributions over actions. This may significantly reduce \(|\Pi|\), as a given randomized policy might not be representable as a distribution over a small number of deterministic policies.\(^5\) We assume deterministic policies in the rest of the paper.

Discretization issues. In some applications, such as dynamic pricing and dynamic procurement, the action space is a continuous interval of prices. To immediately apply Theorem 1, the number \(K(\Pi)\) of distinct actions used by policies in \(\Pi\) should be finite and small compared to \(T\). However, we also need to handle problem instances in which \(K(\Pi)\) is prohibitively large or infinite.\(^6\) We use discretization. Essentially, for each \(\epsilon > 0\) we define a discretized policy set \(\Pi_\epsilon\) that approximates \(\Pi\) so that \(K(\Pi_\epsilon) = \frac{1}{\epsilon}\). The \(\epsilon\) controls the tradeoff between the learning rate and the discretization

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3. Each budget constraint changes by at most one, which does not affect our regret bounds in any significant way.
4. E.g., if per-round consumption of each resource is deterministically at most \(\frac{1}{\epsilon}\), then multiplying all per-round consumption by \(10\) would increase the \(B\) by a factor of \(10\), and hence result in a stronger regret bound.
5. We can reduce RCB with randomized policies to RCB with deterministic policies simply by replacing each context \(x\) with a vector \((a(x, n) : \pi \in \Pi)\) such that \(a(x, n) = \pi(x)\), and encoding the randomization in policies through the randomization in the context arrivals. While this blows up the context space, it does not affect our regret bound.
6. The simplest example is dynamic pricing with a single context and the policy set consisting of all prices in \([0, 1]\).
error. We use Theorem 1 to obtain a performance guarantee that depends on \( \epsilon \), we prove a separate result that bounds the discretization error, and then we optimize the choice of \( \epsilon \).

As one example of this general approach, we work out a paradigmatic special case: contextual dynamic pricing with a single product. We achieve regret \( O \left( (BT)^{1/3} \log(BT|\Pi|) \right) \). This is optimal (up to log factors) for the important regime \( B \geq \Omega(T) \).

**Computational issues.** This paper is focused on proving the existence of solutions to this problem, and the mathematical properties of such a solution. The algorithm is specified as a mathematically well-defined mapping from histories to actions; we do not provide a computationally efficient implementation. Such “information-theoretical” results are common for the first solutions to new, broad problem formulations (e.g. Kleinberg et al., 2008; Kleinberg and Slivkins, 2010; Dudik et al., 2011). In particular, for RCB without resource constraints there exists an algorithm with \( \tilde{O}(\sqrt{KT}) \) regret (Auer et al., 2002; Dudik et al., 2011), but for all known computationally efficient algorithms regret scales with \( T \) as \( T^{2/3} \) (Langford and Zhang, 2007).

**Our contributions: partial lower bound.** We derive a partial lower bound: we prove that RCB is essentially hopeless for the regime \( O(\Pi) \leq B \leq \sqrt{KT}/2 \). The condition \( O(\Pi) \leq B \) is satisfied, for example, in dynamic pricing with limited supply.

**Theorem 2**  No algorithm for RCB can achieve regret \( o(O(\Pi)) \) for all problem instances such that \( O(\Pi) \leq B \leq \sqrt{KT}/2 \) (using the notation from Theorem 1).

The above lower bound is specific to the general (“contextual”) case of RCB. In fact, it points to a stark difference between RCB and the context-free version: in the latter, \( o(\Pi) \) regret is achievable as long as (for example) \( B \geq \log T \) (Badanidiyuru et al., 2013a).

While Theorem 2 is concerned with the regime of small \( B \), note that in the “opposite” regime of very large \( B \), namely \( B \gg \sqrt{KT} \), the regret achieved in Theorem 1 is quite low: it can be expressed as \( \tilde{O}(\sqrt{KT} + \epsilon \cdot O(\Pi)) \), where \( B = \frac{1}{\epsilon} \sqrt{KT} \).

**Further discussion: main challenges.** The central issue in bandit problems is the tradeoff between exploration: acquiring new information, and exploitation: making seemingly optimal decisions based on this information. In this paper, we resolve the explore-exploit tradeoff in the presence of contexts and resource constraints. Each of the three components (explore-exploit tradeoff, contexts, and resource constraints) presents its own challenges, and we need to deal with all these challenges simultaneously. Below we describe these individual challenges one by one.

A well-known naive solution for explore-exploit tradeoff, which we call pre-determined exploration, decides in advance to allocate some rounds to exploration, and the remaining rounds to exploitation. The decisions in the exploration rounds do not depend on the observations, whereas the observations from the exploitation rounds do not impact future decisions. While this approach is simple and broadly applicable, it is typically inferior to more advanced solutions based on adaptive exploration – adapting the exploration schedule to the observations, so that many or all rounds serve both exploration and exploitation.\(^7\) Thus, the general challenge in most explore-exploit settings is to design an appropriate adaptive exploration algorithm.

Resource constraints are difficult to handle for the following three reasons. First, an algorithm’s ability to exploit is constrained by resource consumption for the purpose of exploration; the latter

\(^7\) For example, the difference in regret between pre-determined and adaptive exploration is \( \tilde{O}(\sqrt{KT}) \) vs. \( O(K \log T) \) for stochastic \( K \)-armed bandits, and \( \tilde{O}(T^{3/4}) \) vs. \( \tilde{O}(B^{2/3}) \) for dynamic pricing with limited supply.
is stochastic and therefore difficult to predict in advance. Second, the expected per-round reward is no longer the right objective to optimize, as the action with the highest expected per-round reward could consume too much resources. Instead, one needs to take into account the expected reward over the entire time horizon. Third, the best fixed policy is no longer the right benchmark; instead, the algorithm should search over distributions over policies, which is a much larger search space.

In contextual bandit problems, an algorithm effectively chooses a policy \( \pi \in \Pi \) in each round. Naively, this can be reduced to a non-contextual bandit problem in which “actions” correspond to policies. However, the action space in the reduced problem has size \(|\Pi|\); accordingly, regret scales as \( \sqrt{\|\Pi\|} \) in the worst case. The challenge in contextual bandits is to reduce this dependence. In particular, note that we replace \( \sqrt{\|\Pi\|} \) with \( \log |\Pi| \), an exponential improvement.

Related work. Multi-armed bandits have been studied since Thompson (1933) in Operations Research, Economics, and several branches of Computer Science, see (Gittins et al., 2011; Bubeck and Cesa-Bianchi, 2012) for background. This paper unifies two active lines of work on bandits: contextual bandits and bandits with resource constraints.

Contextual Bandits (Auer, 2002; Langford and Zhang, 2007) add contextual side information which can be used in prediction. This is a necessary complexity for virtually all applications of bandits since it is far more common to have relevant contextual side information than no such information. Several versions have been studied in the literature, see (Bubeck and Cesa-Bianchi, 2012; Dudik et al., 2011; Slivkins, 2011) for a discussion. For contextual bandits with policy sets, there exist two broad families of solutions, based on multiplicative weight algorithms (Auer et al., 2002; McMahan and Streeter, 2009; Beygelzimer et al., 2011) or confidence intervals (Dudik et al., 2011; Agarwal et al., 2012). We rework the confidence interval approach, incorporating and extending the ideas from the work on resource-constrained bandits (Badanidiyuru et al., 2013a).

Prior work on resource-constrained bandits includes dynamic pricing with limited supply (Babaioff et al., 2012; Besbes and Zeevi, 2009, 2012), dynamic procurement on a budget (Badanidiyuru et al., 2012; Singla and Krause, 2013; Slivkins and Vaughan, 2013), dynamic ad allocation with advertisers’ budgets (Slivkins, 2013), and bandits with a single deterministic resource (Guha and Munagala, 2007; Gupta et al., 2011; Tran-Thanh et al., 2010, 2012). Badanidiyuru et al. (2013a) define and optimally solve a common generalization of all these settings: the context-free version of RCB. An extensive discussion of these and other applications, including applications to repeated auctions and network routing, can be found in (Badanidiyuru et al., 2013a).

Our setting can be seen as a contextual bandit version of stochastic packing (e.g. Devanur and Hayes, 2009; Devanur et al., 2011). The difference is in the feedback structure: in stochastic packing, full information about each round is revealed before that round.

Organization of the paper. We define the main algorithm, prove its correctness, and describe the key steps of regret analysis (Sections 3-5). Then we prove the lower bound (Section 7) and discuss discretization issues (Section 6). The appendices contain a discussion of the application domains (Appendix A), the missing proofs for the regret analysis (Appendix B).

2. Problem formulation and preliminaries

We consider an online setting where in each round an algorithm observes a context \( x \) from a possibly infinite known set of possible contexts \( X \) and chooses an action \( a \) from a finite known set \( A \). The world then specifies a reward \( r \in [0, 1] \) and the resource consumption. There are \( d \) resources that can
be consumed, and the resource consumption is specified by numbers \( c_i \in [0, 1] \) for each resource \( i \). Thus, the world specifies the vector \( (r; c_1, \ldots, c_d) \), which we call the outcome vector; this vector can depend on the the chosen action \( a \) and the round. There is a known hard constraint \( B_i \in \mathbb{R}_+ \) on the consumption of each resource \( i \); we call it a budget for resource \( i \). The algorithm stops at the earliest time \( \tau \) when any budget constraint is violated; its total reward is the sum of the rewards in all rounds strictly preceding \( \tau \). The goal of the algorithm is to maximize the expected total reward.

We are only interested in regret at a specific time \( T \) (time horizon) which is known to the algorithm. Formally, we model time as a specific resource with budget \( T \) and a deterministic consumption of 1 for every action. W.l.o.g., \( B_i \leq T \) for every resource \( i \).

For technical convenience, we assume there exists a null action: an action with 0 reward and 0 consumption except for time. That is, an algorithm can spend one round without doing anything.\(^8\)

**Stochastic assumptions.** We assume that there exists an unknown distribution \( D(x, r, c_i) \), called the outcome distribution, from which each round’s observations are created independently and identically, where the vectors are indexed by individual actions. In particular, context \( x \) is drawn from the marginal distribution \( D_x(\cdot) \), and the observed reward and resource consumptions for each action \( a \) are drawn from the conditional distribution \( D(r_a, c_a|x) \). For clarity, we assume that the marginal distribution over contexts \( D(x) \) is known.

**Policy sets and the benchmark.** An algorithm is given a finite set \( \Pi \) of policies – mappings from contexts to actions. Our benchmark is a hypothetical algorithm that knows the outcome distribution \( D \), and makes optimal decisions given this knowledge. The benchmark is restricted to policies in \( \Pi \): before each round, it must commit to some policy \( \pi \in \Pi \), and then choose action \( \pi(x) \) upon arrival of any given context \( x \). The expected total reward of the benchmark is denoted \( \text{OPT}(\Pi) \). Regret of an algorithm is \( \text{OPT}(\Pi) \) minus the algorithm’s expected total reward.

**Uniform budgets.** We say that the budgets are uniform if \( B_i = B \) for each resource \( i \). Any problem instance can be reduced to one with uniform budgets by dividing all consumption values for every resource \( i \) by \( B_i / B \), where \( B = \min_i B_i \). (That is tantamount to changing the units in which we measure consumption of resource \( i \).) We assume uniform budgets \( B \) from here on.

**Notation.** Let \( r(\pi) = E_{(x,r) \sim D}[r_{\pi(x)}] \) and \( c_i(\pi) = E_{(x,c_i) \sim D}[c_{\pi(x)}] \) be the expected per-round reward and the expected per-round consumption of resource \( i \) for policy \( \pi \). Similarly, define \( r(P) = E_{\pi \sim P}[r(\pi)] \) and \( c_i(P) = E_{\pi \sim P}[c_{\pi(\pi)}] \) as the natural extension to a distribution \( P \) over policies.

The tuple \( \mu = ( (r(\pi); c_1(\pi), \ldots, c_d(\pi)) : \pi \in \Pi ) \) is called the expected-outcomes tuple.

For a distribution \( P \) over policies, let \( P(\pi) \) is the probability that \( P \) places over policy \( \pi \). By a slight abuse of notation, let \( P(a|x) = \sum_{\pi(x) = a} P(\pi) \) be the probability that \( P \) places on action \( a \) given context \( x \). Thus, each context \( x \) induces a distribution \( P(\cdot|x) \) over actions.

### 2.1. Linear approximation and the benchmark

We set up a linear relaxation that will be crucial throughout the paper. As a by-product, we (effectively) reduce our benchmark to the best fixed distribution over policies.

A given distribution \( P \) over policies defines an algorithm \( \text{Alg}_P \): in each round a policy \( \pi \) is sampled independently from \( P \), and the action \( a = \pi(x) \) is chosen. The value of \( P \) is the total reward of this algorithm, in expectation over the outcome distribution.

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\(^8\) The null action is needed to ensure the existence of an “LP-perfect distribution” (defined in Section 2.1).
As the value of \( P \) is difficult to characterize exactly, we approximate it (generalizing the approach from \( \text{Babaioff et al., 2012; Badanidiyuru et al., 2013a} \)) for the non-contextual version. We use a linear approximation where all rewards and consumptions are deterministic and the time is continuous. Let \( r(P, \mu) \) and \( c_i(P, \mu) \) be the expected per-round reward and the expected per-round consumption of resource \( i \) for policy \( \pi \sim P \), given expected-outcomes tuple \( \mu \). Then the linear approximation corresponds to the solution of a simple linear program:

\[
\begin{align*}
\text{Maximise} & \quad t r(P, \mu) \\
\text{subject to} & \quad t c_i(P, \mu) \leq B \\
& \quad t \geq 0.
\end{align*}
\]

(1)

The solution to this LP, which we call the LP-value of \( P \), is

\[
\text{LP}(P, \mu) = r(P, \mu) \min_i B/c_i(P, \mu).
\]

(2)

Denote \( \text{OPT}_{\text{LP}} = \sup_P \text{LP}(P, \mu) \), where the supremum is over all distributions \( P \) over \( \Pi \).

**Lemma 3** \( \text{OPT}_{\text{LP}} \geq \text{OPT}(\Pi) \).

Therefore, it suffices to compete against the best fixed distribution over \( \Pi \) (as approximated by \( \text{OPT}_{\text{LP}} \)), even though our benchmark \( \text{OPT}(\Pi) \) allows unrestricted changes over time.

A distribution \( P \) over \( \Pi \) that attains the supremum value \( \text{OPT}_{\text{LP}} \) is called LP-optimal. Such \( P \) is called LP-perfect if furthermore \( |\text{support}(P)| \leq d + 1 \) and \( c_i(P, \mu) \leq B/T \) for each resource \( i \).

We find it useful to consider LP-perfect distributions throughout the paper.

**Lemma 4** An LP-perfect distribution exists for any instance of RCB.

Lemma 3 and Lemma 4 hold for the non-contextual version of RCB (Badanidiyuru et al., 2013a). The general case can be reduced to the non-contextual version via a standard reduction where actions in the new problem correspond to policies in \( \Pi \) in the original problem.

3. The algorithm: Mixture Elimination

The algorithm’s goal is to converge on a LP-perfect distribution over policies. The general design principle is to explore as much as possible while avoiding obviously suboptimal decisions.

**Overview of the algorithm.** In each round \( t \), the following happens.

1. **Compute estimates.** We compute high-confidence estimates for the per-round reward \( r(\pi) \) and per-round consumption \( c_i(\pi) \), for each policy \( \pi \in \Pi \) and each resource \( i \). The collection \( \mathcal{I} \) of all expected-outcomes tuple that are consistent with these high-confidence estimates is called the confidence region.

2. **Avoid obviously suboptimal decisions.** We prune away all distributions \( P \) over policies in \( \Pi \) that are not LP-perfect with high confidence. More precisely, we prune all \( P \) that are not LP-perfect for any expected-outcomes tuple in the confidence region \( \mathcal{I} \); the remaining distributions are called potentially LP-perfect. Let \( \mathcal{F} \) be the convex hull of the set of all potentially LP-perfect distributions.
Algorithm 1 MixtureElimination

1: **Parameters:** #actions $K$, time horizon $T$, budget $B$, benchmark set $\Pi$, context distribution $\mathcal{D}_x$.
2: **Data structure:** “confidence region” $\mathcal{I}$ ← {all feasible expected-outcomes tuples}.
3: **For** each round $t = 1 \ldots T$ do
4:  \[ \Delta = \{ \text{distributions } P \text{ over II: } P \text{ is LP-perfect for some } \mu' \in \mathcal{I} \} . \]
5:  Let $\mathcal{F}$ be the convex hull of $\Delta$.
6:  Let $\alpha_\pi = \max_{P \in \mathcal{F}} P(\pi), \forall \pi \in \Pi$.
7:  Choose a “balanced” distribution $P \in \mathcal{F}$; any $P$ such that $\forall \pi \in \Pi$
8:  \[ \mathbb{E}_{x \sim \mathcal{D}_x} \left[ \frac{1}{(1 - q_0)} P(\pi(x)|x) + \frac{q_0}{K} \right] \leq \frac{2K}{\alpha_\pi}, \quad \text{where } q_0 = \min \left( \frac{1}{2}, \sqrt{\frac{K}{T} \log(K T \Pi)} \right) \quad \text{(3)} \]
9:  **Observe** context $x$; **choose** action $a$ to ”play”:
10:  with probability $q_0$, draw $a$ u.a.r. in $A$; else, draw $\pi \sim P$ and let $a = \pi(x)$.
11:  **Observe** outcome vector $(r, c_1, \ldots, c_d)$.
12:  **Halt** if one of the resources is exhausted.
13:  Eliminate expected-outcomes tuples from $\mathcal{I}$ according to equations (4-5).

3. **Explore as much as possible.** We choose a distribution $P \in \mathcal{F}$ which is balanced, in the sense that no action is starved; see Equation (3) for the precise definition. Note that balanced distributions are typically not LP-perfect.

4. **Select an action.** We choose policy $\pi \in \Pi$ independently from $P$. Given context $x$, the action $a$ is chosen as $a = \pi(x)$. The algorithm adds some random noise: with probability $q_0$, the action $a$ is instead chosen uniformly at random, for some parameter $q_0$.

   The algorithm halts as soon as the time horizon is met, or one of the resources is exhausted.

   The pseudocode can be found in Algorithm 1.

**Some details.** After each round $t$, we estimate the per-round consumption $c_i(\pi)$ and the per-round reward $r(\pi)$, for each policy $\pi \in \Pi$ and each resource $i$, using the following unbiased estimators:

\[ \hat{c}_t,i(\pi) = \frac{1}{t-1} \sum_{s=1}^{t-1} \hat{c}_{s,i}(\pi) \quad \text{and} \quad \hat{r}_t(\pi) = \frac{1}{t-1} \sum_{s=1}^{t-1} \hat{r}_s(\pi). \]

The corresponding time-averages up to round $t$ are denoted

\[ \hat{c}_{t,i}(\pi) = \frac{1}{t-1} \sum_{s=1}^{t-1} \hat{c}_{s,i}(\pi) \quad \text{and} \quad \hat{r}_t(\pi) = \frac{1}{t-1} \sum_{s=1}^{t-1} \hat{r}_s(\pi). \]

We show that with high probability these time-averages are close to their respective expectations. To express the confidence term in a more lucid way, we use the following shorthand, called confidence radius: $\text{rad}_t(\nu) = \sqrt{C_{\text{rad}} \nu / t}$, where $C_{\text{rad}} = \Theta(\log(dK T | \Pi))$ is a parameter which we will fix later. We show that w.h.p. the following holds:

\[ |r(\pi) - \hat{r}_t(\pi)| \leq \text{rad}_t \left( \frac{K}{\alpha_\pi} \right), \quad \text{(4)} \]
\[ |c_i(\pi) - \hat{c}_{t,i}(\pi)| \leq \text{rad}_t \left( \frac{K}{\alpha_\pi} \right) \quad \text{for all } i. \quad \text{(5)} \]

(Here $\alpha_\pi = \max_{P \in \mathcal{F}} P(\pi)$, as in Algorithm 1.)
4. Correctness of the algorithm

We need to prove that in each round $t$, some $P \in \mathcal{F}$ satisfies (3), and Equations (4-5) hold for all policies $\pi \in \Pi$ with high probability.

**Notation.** Let $\alpha_{\pi,t}$ denote the $\alpha_{\pi}$, $\pi \in \Pi$ computed in round $t$, and let $P_t$ be the distribution over $\Pi$ chosen in this round. Let $q_0$ be the noise probability. The “noisy version” of $P_t$ is defined as

$$P'_t(a|x) = (1 - q_0) P_t(a|x) + q_0 / K \quad (\forall x \in X, a \in A).$$

Then action $a$ in round $t$ is drawn from distribution $P'_t(\cdot|x_t)$.

**Lemma 5** In each round $t$, some $P \in \mathcal{F}$ satisfies (3).

**Proof** We extend the minimax argument from Dudik et al. (2011). Let $\mathcal{F}_1$ be the set of all distributions over $\Pi$. Our proof works for any $q_0 \in [0, \frac{1}{2}]$ and any compact and convex set $\mathcal{F} \subset \mathcal{F}_1$.

Equation (3) holds for a given $P \in \mathcal{F}$ if and only if for every distribution $Z \in \mathcal{F}_1$ we have that

$$f(P, Z) \triangleq \mathbb{E}_{x \sim D_x} \mathbb{E}_{\pi \sim Z} \left[ \frac{\alpha_{\pi}}{P'(\pi(x)|x)} \right] \leq 2K,$$

where $P'$ is the noisy version of $P$. It suffices to show that

$$\min_{P \in \mathcal{F}} \max_{Z \in \mathcal{F}_1} f(P, Z) \leq 2K. \quad (6)$$

We use a min-max argument: noting that $f$ is a convex function of $P$ and a concave function of $Z$, by the Simons min-max theorem we have that

$$\min_{P \in \mathcal{F}} \max_{Z \in \mathcal{F}_1} f(P, Z) = \max_{Z \in \mathcal{F}_1} \min_{P \in \mathcal{F}} f(P, Z). \quad (7)$$

For each policy $\pi \in \Pi$, let $\beta_{\pi} \in \arg\max_{\beta \in \mathcal{F}} \beta(\pi)$ be a distribution which maximizes the probability of selecting $\pi$. Such distribution exists because $\beta \mapsto \beta(\pi)$ is a continuous function on a convex set $\mathcal{F}$. Recall that $\alpha_{\pi} = \beta_{\pi}(\pi)$.

Given any $Z \in \mathcal{F}_1$, define distribution $P_Z \in \mathcal{F}_1$ by $P_Z(\pi) = \sum_{\phi \in \Pi} Z(\phi) \beta_{\phi}(\pi)$. Note that $P_Z$ is a convex combination of distributions in $\mathcal{F}$. Since $\mathcal{F}$ is convex, it follows that $P_Z \in \mathcal{F}$. Also, note that $P_Z(a|x) \geq \sum_{\pi \in \Pi; \pi(x) = a} Z(\pi) \alpha_{\pi}$. Letting $P'_Z$ be the noisy version of $P_Z$, we have:

$$\min_{P \in \mathcal{F}} f(P, Z) \leq f(P_Z, Z) = \mathbb{E}_{x \sim D_x} \left[ \sum_{\pi} Z(\pi) \alpha_{\pi} \frac{P'_Z(\pi(x)|x)}{P'(\pi(x)|x)} \right]$$

$$= \mathbb{E}_{x \sim D_x} \left[ \sum_{a \in A} \sum_{\pi \in \Pi; \pi(x) = a} Z(\pi) \alpha_{\pi} \frac{P'(a|x)}{P_Z(a|x)} \right]$$

$$= \mathbb{E}_{x \sim D_x} \left[ \sum_{a \in X} \alpha_{\pi} \frac{1 - q_0}{1 - q_0} + q_0 / K \right]$$

Thus, by Equation (7) we obtain Equation (6). \qed

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We use amin-max argument: noting that $f$ is a convex function of $P$ and a concave function of $Z$, by the Simons min-max theorem we have that

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$$\min_{P \in \mathcal{F}} f(P, Z) \leq f(P_Z, Z) = \mathbb{E}_{x \sim D_x} \left[ \sum_{\pi} Z(\pi) \alpha_{\pi} \frac{P'_Z(\pi(x)|x)}{P'(\pi(x)|x)} \right]$$

$$= \mathbb{E}_{x \sim D_x} \left[ \sum_{a \in A} \sum_{\pi \in \Pi; \pi(x) = a} Z(\pi) \alpha_{\pi} \frac{P'(a|x)}{P_Z(a|x)} \right]$$

$$= \mathbb{E}_{x \sim D_x} \left[ \sum_{a \in X} \alpha_{\pi} \frac{1 - q_0}{1 - q_0} + q_0 / K \right]$$

Thus, by Equation (7) we obtain Equation (6). \qed
To analyze Equations (4-5), we will use Bernstein’s inequality for martingales (Freedman, 1975), via the following formulation from Bubeck and Slivkins (2012):

**Lemma 6** Let \( \mathcal{G}_1 \subseteq \ldots \subseteq \mathcal{G}_n \) be a filtration, and \( X_1, \ldots, X_n \) be real random variables such that \( X_t \) is \( \mathcal{G}_t \)-measurable, \( \mathbb{E}(X_t | \mathcal{G}_{t-1}) = 0 \) and \( |X_t| \leq b \) for some \( b > 0 \). Let \( V_n = \sum_{t=1}^{n} \mathbb{E}(X_t^2 | \mathcal{G}_{t-1}) \). Then with probability at least \( 1 - \frac{1}{n} \) it holds that

\[
\sum_{t=1}^{n} X_t \leq \sqrt{4V_n \log(n\delta^{-1}) + 5b^2 \log^2(n\delta^{-1})}.
\]

**Lemma 7** With probability at least \( 1 - \frac{1}{n} \), Equations (4-5) hold for all rounds \( t \) and policies \( \pi \in \Pi \).

**Proof** Let us prove Equation (4). (The proof of (5) is similar.) Fix round \( t \) and policy \( \pi \in \Pi \). We bound the conditional variance of the estimators \( \tilde{r}_t(\pi) \). Specifically, let \( \mathcal{G}_t \) be the \( \sigma \)-algebra induced by all events up to (but not including) round \( t \). Then

\[
\mathbb{E} \left[ \tilde{r}_t(\pi)^2 \mid \mathcal{G}_t \right] = \mathbb{E} \left[ \mathbb{E} \left[ \frac{r_t^2 1_{(\pi(x) = a)}}{P_t'(a|x)^2} \mid x \sim D_t, \omega \sim P_t \right] \right] \leq \mathbb{E} \left[ \mathbb{E} \left[ \frac{1}{P_t'(\pi(x)|x)} \mid x \sim D_t \right] \right] \leq \frac{2K}{\alpha_{\pi,t}}.
\]

The last inequality holds by the algorithm’s choice of distribution \( P_t \). Since the confidence region \( \mathcal{I} \) in our algorithm is non-increasing over time, it follows that \( \alpha_{\pi,t} \) is non-increasing in \( t \), too. We conclude that \( \text{Var} \left[ \tilde{r}_s(\pi) \mid \mathcal{G}_t \right] \leq 2K/\alpha_{\pi,t} \) for each round \( s \leq t \). Therefore, noting that \( \tilde{r}_t(\pi) \leq 1/P_t'(\pi(x_t)|x_t) \leq K/q_0 \), we obtain Equation (4) by applying Lemma 6 with \( X_t = \tilde{r}_t(\pi) - r(\pi) \). \( \Box \)

### 5. Regret analysis: proof of Theorem 1

We provide the key steps of the proof; the remaining pieces can be found in Appendix B.

Let \( \mathcal{I}_t \) and \( \Delta_t \) be, resp., the confidence region \( \mathcal{I} \) and the set \( \Delta \) of potentially LP-perfect distributions computed in round \( t \). Let Conv(\( \Delta_t \)) be the convex hull of \( \Delta_t \).

First we bound the deviations within the confidence region.

**Lemma 8** For any two expected-outcomes tuples \( \mu', \mu'' \in \mathcal{I}_t \) and a distribution \( P \in \text{Conv}(\Delta_t) \):

\[
|c_i(P, \mu') - c_i(P, \mu'')| \leq \text{rad}_t(dK) \quad \text{for each resource } i \tag{8}
\]

\[
|r(P, \mu') - r(P, \mu'')| \leq \text{rad}_t(dK) \tag{9}
\]

**Proof** Let us prove Equation (9). (Equation (8) is proved similarly.) By definition of \( \mathcal{I}_t \):

\[
|r(P, \mu') - r(P, \mu'')| \leq \sum_{\pi \in \Pi} P(\pi) |r(\pi, \mu') - r(\pi, \mu'')| \leq \sum_{\pi \in \Pi} P(\pi) \text{rad}_t(K/\alpha_{\pi,t}).
\]

It remains to prove that the right-hand side is at most \( \text{rad}_t(dK) \). By linearity, it suffices to prove this for \( P \in \Delta_t \). So let us assume \( P \in \Delta_t \) from here on. Recall that \( |\text{support}(P)| \leq d \) since \( P \) is LP-perfect, and \( P(\pi) \leq \alpha_{\pi,t} \) for any policy \( \pi \in \Pi \). Therefore:

\[
\sum_{\pi \in \Pi} P(\pi) \text{rad}_t(K/\alpha_{\pi,t}) \leq \sum_{\pi \in \Pi} \text{rad}_t(KP(\pi)) \leq \text{rad}_t(dK \sum_{\pi \in \Pi} P(\pi)) = \text{rad}_t(dK). \quad \Box
\]
Using Lemma 8 and a long computation (fleshed out in Appendix B), we prove the following.

**Lemma 9** For any two expected-outcomes tuples $\mu', \mu'' \in I_t$ and a distribution $P \in \text{Conv}(\Delta_t)$:

$$LP(P, \mu') - LP(P, \mu'') \leq \left( \frac{1}{T} \right) \cdot \text{rad}_K(dK).$$

Let $\text{REW}_t$ and $C_{t,i}$ be, respectively, the (realized) total reward and average consumption of resource $i$ up to and including round $t$. Recall that $P_t'$ is the noisy version of distribution $P_t$ chosen by the algorithm in round $t$. Given $P_t$, the expected revenue and resource-$i$ consumption in round $t$ is, respectively, $r(P_t', \mu)$ and $c_i(P_t', \mu)$. Denote $\mathfrak{r}_t = \frac{1}{T} \sum_{i=1}^t r(P_t', \mu)$ and $\mathfrak{c}_{t,i} = \frac{1}{T} \sum_{i=1}^t c_i(P_t', \mu)$.

Henceforth we assume a **clean execution** where several high-probability conditions are satisfied. Formally, the algorithm’s execution is clean if in each round $t$ Equations (4-5) are satisfied, and moreover $\min\left(\frac{1}{T} |\text{REW}_t - \mathfrak{r}_t|, |C_{t,i} - \mathfrak{c}_{t,i}|\right) \leq \text{rad}_K(1)$.

In particular, the set $\Delta_t$ of potentially LP-perfect distributions indeed contains a LP-perfect distribution. By Lemma 7 and Azuma-Hoeffding Inequality, clean execution happens with probability at least $1 - \frac{1}{T}$. Thus, it suffices to lower-bound the total reward $\text{REW}_T$ for a clean execution.

The following lemma captures a crucial argument. Denote

$$\Phi_t = \left( 2 + \frac{1}{T} \max_{P \in \mathcal{F}_t, \mu \in I_t} LP(P, \mu) \right) \cdot T \cdot \text{rad}_K(dK)$$

$$\Psi_t = (2 + \frac{1}{T} \text{OPT}_{LP}) \cdot T \cdot \text{rad}_K(dK).$$

**Lemma 10** For any expected-outcomes tuple $\mu^*, \mu^{**} \in I_t$ and distributions $P', P'' \in \text{Conv}(\Delta_t)$:

$$|LP(P', \mu^*) - LP(P'', \mu^{**})| \leq 3\Phi_t. \quad (10)$$

**Proof** Assume $P', P'' \in \Delta_t$. In particular, $P', P''$ are LP-perfect for some expected-outcomes tuples $\mu', \mu'' \in I_t$, resp. Also, some distribution $P^* \in \Delta_t$ is LP-perfect for $\mu^*$. Therefore:

$$LP(P', \mu^*) \geq LP(P', \mu') - \Phi_t \quad \text{(by Lemma 9: } P = P')$$
$$\geq LP(P^*, \mu') - \Phi_t$$
$$\geq LP(P^*, \mu^*) - 2\Phi_t \quad \text{(by Lemma 9: } P = P^*)$$
$$\geq LP(P'', \mu^*) - 2\Phi_t.$$

We proved Equation (10) for $P', P'' \in \Delta_t$. Thus:

$$\min_{P \in \Delta_t} LP(P, \mu^*) - \max_{P \in \Delta_t} LP(P, \mu^*) \leq 2\Phi_t. \quad (11)$$

Next we generalize to $P', P'' \in \text{Conv}(\Delta_t)$.

$$LP(P', \mu^*) \geq \min_{P \in \Delta_t} LP(P, \mu^*) \quad \text{(since } P' \in \text{Conv}(\Delta_t))$$
$$\geq \max_{P \in \Delta_t} LP(P, \mu^*) - 2\Phi_t \quad \text{(by Equation (11))}$$
$$\geq LP(P'', \mu^*) - 2\Phi_t \quad \text{(since } P'' \in \text{Conv}(\Delta_t)).$$

We proved Equation (10) for $\mu^* = \mu^{**}$. We obtain the general case by plugging in Lemma 9. $\square$
Corollary 11  \( \Phi_t \leq 2\Psi_t \), assuming that \( B \geq 6 \cdot T \cdot \text{rad}_t(dK) \).

**Proof**  Follows from Lemma 10 via a simple computation (omitted). \( \square \)

Corollary 12  \( \text{LP}(P_t, \mu) \geq \text{OPT}_{\text{LP}} - 12\Psi_t \), where \( \mu \) is the actual expected-outcomes tuple.

**Proof**  Follows from Lemma 10 and Corollary 11, observing that \( P_t \in \text{Conv}(\Delta_t) \) and \( \text{OPT}_{\text{LP}} = \text{LP}(P^*, \mu) \) for some \( P^* \in \Delta_t \). \( \square \)

In the remainder of the proof (which is fleshed out in Appendix B) we build on the above lemmas and corollaries to prove the following sequence of claims:

\[
\text{REW}_t \geq \frac{t}{T} (\text{OPT}_{\text{LP}} - O(\Psi_t)) \\
C_{t,i} \leq B/T + O(\text{rad}_t(dK)) \\
\text{REW}_T \geq \text{OPT}_{\text{LP}} - O(\Psi_T). \tag{12}
\]

To complete the proof of Theorem 1, we re-write the last equation as \( \text{REW}_T \geq f(\text{OPT}_{\text{LP}}) \) for an appropriate function \( f() \), and observe that \( f(\text{OPT}_{\text{LP}}) \geq f(\text{OPT}) \) because function \( f() \) is increasing.

6. Discretization issues

In order to immediately apply Theorem 1, the set of all actions used by policies in \( \Pi \), denote it \( A(\Pi) \), should be finite and small compared to \( T \). However, in some applications, such as dynamic pricing and dynamic procurement, the set of all possible actions is infinite, so \( A(\Pi) \) may be prohibitively large or infinite, too. One way to handle such problem instances is discretization. Below, we outline the general approach, and work out the paradigmatic special case of contextual dynamic pricing with a single product.

In particular, we obtain the following corollary:

**Corollary 13**  Consider contextual dynamic pricing with a limited supply \( B \) of a single product. Let \( \Pi \) be the policy set and \( T \) be the time horizon. Then there exists a policy set \( \Pi' \) such that algorithm Mixture Elimination satisfies

\[
\text{REW}(\Pi') \geq \text{OPT}(\Pi) - O \left( (TB)^{1/3} \cdot \log (TB|\Pi|) \right). \tag{13}
\]

Here \( \text{REW}(\Pi') \) is the algorithm’s total expected reward if it is given policy set \( \Pi' \).

Note that if \( B \) is at least a constant fraction of \( T \), we obtain regret \( \tilde{O}(T^{2/3}) \), which matches the lower bound (even) for the non-contextual version (Babaioff et al., 2012). However, the optimal regret for the non-contextual version is \( \tilde{O}(B^{2/3}) \) regret, and we do not match that. With an upper bound of \( \tilde{O}(BT^{1/3}) \) and a lower bound of \( \Omega(B^{2/3}) \), it is not clear what is the optimal regret here.

The general approach is as follows. For each \( \epsilon > 0 \) we define an \( \epsilon \)-discretized policy set \( \Pi_\epsilon \) that approximates \( \Pi \) so that \( |A(\Pi_\epsilon)| \) decreases with \( \epsilon \). The \( \epsilon \) controls the tradeoff between the learning rate and the discretization error. We use Theorem 1 to obtain a performance guarantee that depends on \( \epsilon \). Further, we need a separate result that upper-bounds the discretization error of \( \Pi_\epsilon \) compared to \( \Pi \). Then we can optimize the choice of \( \epsilon \) to provide an algorithm with a specific regret bound.
In what follows we apply this approach to dynamic pricing with \( B \) copies of a single product. Recall that in this setting actions are prices, and the action set is \([0, 1] \cup \{\infty\}\), the set of all possible prices. Here \( p = \infty \) is the special price that corresponds to skipping a round.

First, let us introduce some notation. Let \( \Pi \) be an arbitrary (finite) policy set. Let \( S(p|x) \) be the probability of a sale for price \( p \) and context \( x \). For a randomized policy \( \pi \), define \( S(\pi|x) = \mathbb{E}_{p \sim \pi(x)}[S(p|x)] \). Let \( f_\epsilon(p), p \geq 0 \) be the largest price \( p' \leq p \) such that \( p' \in \epsilon \mathbb{N} \).

Second, for each policy \( \pi \in \Pi \) we define a randomized policy \( \pi_\epsilon \): for each context \( x \),

\[
\pi_\epsilon(x) = \begin{cases} f_\epsilon(\pi(x)) & \text{with probability } \frac{S(\pi|x)}{S(f_\epsilon(\pi(x)) | x)} \\
\infty & \text{with the remaining probability.}
\end{cases}
\]  

The \( \epsilon \)-discretized policy set is defined as \( \Pi_\epsilon = \{ \pi_\epsilon : \pi \in \Pi \} \).

Observe that the \( \pi_\epsilon \) satisfies the following useful properties:

(i) \( A(\pi_\epsilon) \subset \epsilon \mathbb{N} \cap [0, 1] \), so that \( |A(\Pi_\epsilon)| \leq \frac{1}{\epsilon} \).

(ii) \( \pi(x) \geq \pi_\epsilon(x) \geq \pi(x) - \epsilon \) for all contexts \( x \),

(iii) \( S(\pi_\epsilon|x) = S(\pi|x) \) for all contexts \( x \).

Third, let us bound the discretization error.

**Lemma 14** \( \text{OPT}_{\text{LP}}(\Pi) - \text{OPT}_{\text{LP}}(\Pi_\epsilon) \leq \epsilon B, \) for each \( \epsilon > 0 \).

**Proof** Fix \( \epsilon > 0 \) and a distribution \( P \) over policies in \( \Pi \). Define the \( \epsilon \)-discretized distribution \( P_\epsilon \) over \( \Pi_\epsilon \) in a natural way: for each policy \( \pi \in \Pi \), select \( \pi_\epsilon \) with probability \( P(\pi) \). It suffices to prove that \( \text{LP}(P_\epsilon) \geq \text{LP}(P) - \epsilon B \).

Let \( r(P) \) and \( c(P) \) denote the expected per-round reward and the expected per-round consumption for distribution \( P \). Then

\[
c(P_\epsilon) = \mathbb{E}_{x, \pi} [S(\pi_\epsilon|x)] = \mathbb{E}_{x, \pi} [S(\pi|x)] = c(P)
\]

\[
r(P_\epsilon) = \mathbb{E}_{x, \pi} [f_\epsilon(\pi(x)) \cdot S(\pi_\epsilon|x)] 
\geq \mathbb{E}_{x, \pi} [(\pi(x) - \epsilon) \cdot S(\pi_\epsilon|x)] 
\geq \mathbb{E}_{x, \pi} [\pi(x) \cdot S(\pi|x)] - \epsilon \mathbb{E}_{x, \pi} [S(\pi|x)] 
= r(P) - \epsilon c(P_\epsilon).
\]

\[
\text{LP}(P_\epsilon) = \frac{r(P_\epsilon) B}{c(P_\epsilon)} \geq \frac{r(P) B}{c(P)} - \epsilon B. 
\]

Fourth, let us formulate an appropriate version Theorem 1. Recall that we actually prove a somewhat stronger statement that is with respect to \( \text{OPT}_{\text{LP}}(\Pi) \) rather than \( \text{OPT}(\Pi) \). Namely, the algorithm’s expected total reward \( \text{REW}(\Pi) \) satisfies

\[
\text{REW}(\Pi) \geq \text{OPT}_{\text{LP}}(\Pi) - O \left( 1 + \frac{1}{\epsilon} \text{OPT}_{\text{LP}}(\Pi) \right) \sqrt{dKT \log (dKT |\Pi|)}.
\]

For this particular setting we have \( d = 1 \) (single constraint) and \( \text{OPT}_{\text{LP}}(\Pi) \leq B \), which implies

\[
\text{REW}(\Pi) \geq \text{OPT}_{\text{LP}}(\Pi) - O \left( \sqrt{KT \log (KT |\Pi|)} \right).
\]
Finally, plugging in Lemma 14, we obtain regret

$$\text{REW}(\Pi_\epsilon) \geq \text{OPT}_{LP}(\Pi) - \epsilon B - O\left(\sqrt{\frac{T}{\epsilon}} \log \left(\frac{T}{\epsilon} |\Pi_\epsilon|\right)\right), \quad \text{for each } \epsilon > 0. \quad (15)$$

We choose the $\epsilon$ so as to optimize Equation (15).

In particular, we obtain Corollary 13 if we plug in $|\Pi_\epsilon| \leq |\Pi|$ (rather than consider how $|\Pi_\epsilon|$ decreases with $\epsilon$) and pick $\epsilon = (\frac{dT}{B})^{1/3}$.

7. Lower bound: proof of Theorem 2

We will use the following lemma (which follows from simple probability arguments).

**Lemma 15** Consider two collections of $n$ balls $\mathcal{I}_1$ and $\mathcal{I}_2$, each numbered from 1 to $n$. Let $\mathcal{I}_1$ consists of all red balls, while $\mathcal{I}_2$ consist of $n-1$ red balls and 1 green ball (with labels chosen uniformly at random). In this setting, let an algorithm is given access to random samples from one of $\mathcal{I}_i$ with replacement. The algorithm is allowed to first look at the ball’s number and then decide whether to inspect it’s color. Then any algorithm $\mathcal{A}$ which with probability at least 1/2 can distinguish between $\mathcal{I}_1$ and $\mathcal{I}_2$ must inspect color of at least $n/2$ balls in expectation.

Let us define a family of problem instances as follows. Let the set of arms be $\{a_1, a_2, \ldots, a_k\}$. There are $T/B$ different contexts labelled $\{x_1, \ldots, x_{T/B}\}$ and there is a uniform distribution over contexts. There are $T(K-1)/B$ policies. Let them be indexed as $\pi_{i,j}$ for $2 \leq i \leq K$ and $1 \leq j \leq T/B$. Define them as follows: $\pi_{i,j}(x_l) = a_i$ for $l = j$, and $\pi_{i,j}(x_l) = a_1$ for $l \neq j$.

There is just one resource constraint. Pulling arm $a_1$ always costs 0 and arm $a_i, i \neq 1$ always costs 1. Now consider the following problem instances:

- Let $\mathcal{F}_0$ be the instance in which every arm always gives a reward 0. Note that $\text{OPT}(\mathcal{F}_0) = 0$.
- Let $\mathcal{F}_{i,j}$ be the instance in which arm $a_j$ on context $x_j$ gives reward 1, otherwise every arm on every context gives reward 0. Note that in this case the optimal distribution over policies is just to follow $\pi_{i,j}$ and gets reward $\approx B$.

Now consider any algorithm $\mathcal{A}$ and let the expected number of times it pulls arm $a_i$ be $p_i$ on input $\mathcal{F}_0$. Let $i', i'' \neq 1$ be the arm for which this is minimum. Then by simple linearity of expectation we get that $B \geq (K-1)p_{i'}$. It is also simple to see that for the algorithm to get a regret better than $\Omega(\text{OPT})$ it should be able to distinguish between $\mathcal{F}_0$ and $\mathcal{F}_{i'',i'}$ at least with probability 1/2. From lemma 15 this can be done iff $p_{i''} \geq T/(2B)$. Combining the two equations we get $B \geq (K-1)T/(2B)$. Solving for $B$ we get $B \geq \sqrt{KT}/2$.

8. Conclusions and open questions

We define a very general setting for contextual bandits with resource constraints. We design an algorithm for this problem, and derive a regret bound which achieves the optimal root-$T$ scaling in terms of the time horizon $T$, and the optimal $\sqrt{\log |\Pi|}$ scaling in terms of the policy set $\Pi$. Further, we consider discretization issues, and derive a specific corollary for contextual dynamic pricing with a single product; we obtain a regret bound that is optimal in the regime $B \geq \Omega(T)$. Finally, we derive a partial lower bound which establishes a stark difference from the non-contextual version.
The main open question is combining provable regret bounds and a computationally efficient implementation. Note that achieving root-$T$ regret in a computationally efficient way is a major open question for contextual bandits with policy sets, even without the resource constraints.

Several open questions concern our regret bounds. First, it is desirable to achieve the same regret bounds without assuming a known time horizon $T$. However, this might not be feasible due to the fact that in RCB, $T$ is essentially one of the resource constraints. Second, our main regret bound is not tight for an arbitrary tuple $(B,T,OPT)$; likewise, our corollary for dynamic pricing with a single product is not tight for an arbitrary tuple $(B,T)$. In both cases, both upper and lower bounds can potentially be improved. Third, it is important to work out the discretization issues for more general settings of dynamic pricing / procurement. However, these issues are largely unresolved even for the non-contextual version. Finally, if there are no contexts or resource constraints then one can achieve $O(\log T)$ regret with an instance dependent constant; it is not clear whether one can meaningfully extend this result to settings where either contexts or resource constraints are present.

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Appendix A: RCB: applications and special cases

In this section we discuss the application domains of resource-constrained (contextual) bandits in more detail. We focus on the three main application domains: dynamic pricing, dynamic procurement, and dynamic ad allocation. A more extensive discussion of these and other application domains (in the non-contextual version) can be found in Badanidiyuru et al. (2013a,b).

**Dynamic pricing with limited supply.** The algorithm is a monopolistic seller with a limited inventory. In the basic version, there is a limited supply of identical items. In each round, a new customer arrives, the algorithm picks a price, and offers one item for sale at this price. The customer then either buys the item at this price, or rejects the offer and leaves. The “context” represents the available information about the current customer, such as demographics, location, etc. The probability of selling at a given price for a given context (a.k.a. the demand distribution) is fixed over time, but not known to the algorithm. The algorithm optimizes the revenue; it does not derive any utility from the left-over items.

We represent this problem as an instance of RCB as follows. “Actions” are the possible prices, and the “resource constraint” is the number of items. In each round, the outcome vector is a pair (reward, items sold); if the offered price is $p$, the outcome vector is $(p, 1)$ if there is a sale, and $(0, 0)$ otherwise.

Many generalizations of dynamic pricing have been studied in the literature. In particular, RCB subsumes a number of extensions. First, an algorithm can sell multiple items to the same customer, possibly with volume discounts or surcharges. Second, an algorithm can have multiple products for sale, with limited inventory of each. Third, it may be advantageous to offer bundles consisting of different products, possibly with non-additive pricing (mirroring the non-additive valuations of the customers).

**Dynamic procurement on a budget.** The algorithm is a monopolistic buyer with a limited budget. The basic version is as follows. In each round, a new customer arrives, the algorithm picks a price, and offers to buy one item at this price. Then the customer either accepts the offer and sells the item at this price, or rejects the offer and leaves. The “context” is the available information on the
current customer. The probability of buying at a given price for a given context (a.k.a. the “supply distribution”) is fixed over time, but not known to the algorithm. The algorithm maximizes the number of items bought; it has no utility for the left-over money.

An alternative interpretation is that the algorithm is a contractor which hires workers to perform tasks, e.g. in a crowdsourcing market. In each round, a new worker arrives, the algorithm picks a price, and offers the worker to perform one task for this price; the worker then either accepts and performs the task at this price, or rejects and leaves. The relevant “context” for a worker in a crowdsourcing market may include, for example, age, location, language, and task preferences.

Here, “actions” correspond to the possible prices, and the “resource constraint” is the buyer’s budget. In each round, the outcome vector is a pair (items bought, money spent); if the offered price is $p$, then the outcome vector is $(1, p)$ if the offer is accepted, and $(0, 0)$ otherwise.

Dynamic procurement is a rich problem space, both for buying items and for hiring workers (see (Slivkins and Vaughan, 2013) for a discussion of the application to crowdsourcing markets). In particular, RCB subsumes a number of extensions of this basic setting. First, the algorithm may offer several tasks to the same worker, possibly at a discount. Second, there may be multiple types of tasks, each having a different value for the contractor; moreover, there may be additional budget constraints on each task type, or on various subsets of task types. Third, a given worker can be offered a bundle of tasks, consisting of tasks of multiple types, possibly with non-additive pricing. Fourth, there is a way to model the presence of competition (other contractors).

Dynamic ad allocation with budgets. The algorithm is an advertising platform. In the basic version, there is a fixed collection of ads to choose from. In each round, a user arrives, and the algorithm chooses one ad to display to this user. The user either clicks on this ad, or leaves without clicking. The algorithm receives a payment if and only if the ad is clicked; the payment for a given ad is fixed over time and known to the algorithm. The “context” is the available information about the user and the page on which the ad is displayed. The click probability for a given ad and a given context is constant over time, but not known to the algorithm.

Each ad belongs to some advertiser (who is the one paying the algorithm when this ad is clicked). Each advertiser may own multiple ads, and has a budget constraint: a maximal amount of money that can be spent on all his ads. Moreover, an advertiser may specify additional budget constraints on various subsets of the ads. The algorithm maximizes its revenue; it derives no utility from the left-over budgets.

Here, “actions” correspond to ads, and each budget corresponds to a separate resource. In a round when the chosen ad $a$ is clicked, the reward is the corresponding payment $v$, and the resource consumption is $v$ for each budget that involves $a$, and 0 for all other budgets. If the ad is not clicked, the reward and the consumption of each resource is 0.

RCB also subsumes more advanced versions in which multiple non-zero outcomes are possible in each round. For example, the ad platform may record what happens after the click, e.g. the time spent on the page linked from the ad and whether this interaction has resulted in a sale.

Appendix B: Regret analysis: missing proofs

Here we provide the remaining pieces for the regret analysis in Section 5.

B.1. Proof of Lemma 9

We restate the lemma for convenience.
Lemma  For any two expected-outcomes tuples $\mu', \mu'' \in \mathcal{I}_t$ and a distribution $P \in \text{Conv}(\Delta_t)$:

$$LP(P, \mu') - LP(P, \mu'') \leq \left( \frac{1}{B} LP(P, \mu') + 2 \right) \cdot T \cdot \text{rad}_t(dK).$$

Proof  For brevity, we will denote:

$$LP' = LP(P, \mu') \quad \text{and} \quad LP'' = LP(P, \mu'')$$
$$r' = r(P, \mu') \quad \text{and} \quad r'' = r(P, \mu'')$$
$$c_i' = c_i(P, \mu') \quad \text{and} \quad c_i'' = c_i(P, \mu'').$$

By symmetry, it suffices to prove the upper bound for $LP' - LP''$. Henceforth, assume $LP' > LP''$.

We consider two cases, depending on whether $T \leq B/c_i'$ for all resources $i$.

Case 1. Assume Equation (16) holds. Then $LP'' = T r''$. Therefore by Lemma 8

$$LP' - LP'' \leq T r' - T r'' \leq T \text{rad}_t(dK).$$

Case 2. Assume Equation (16) fails. Then $LP'' = B r''/c_i''$ for some resource $i$. We consider two subcases, depending on whether $T \leq B/c_j'$ for all resources $j$.

Subcase 1. Assume Equation (17) holds. Then:

$$LP' = T r'$$
$$LP'' \leq T \cdot \min(r', r'') \leq LP'$$

Equation (19) follows from (18) and $LP' > LP''$.

For $\delta \in [0, c_i'')$, define

$$r(\delta) = r'' + \delta$$
$$c_i(\delta) = c_i'' - \delta$$
$$f(\delta) = B r(\delta)/c_i(\delta).$$

Then $f()$ is monotonically and continuously increasing function, with $f(\delta) \rightarrow \infty$ as $\delta \rightarrow c_i''$. For convenience, define $f(c_i'') = \infty$.

Let $\delta_0 = \min(c_i'', \text{rad}_t(dK))$. By Lemma 8, we have $f(\delta_0) \geq Br'/c_i$. Therefore:

$$f(0) = LP'' < LP' \leq Br'/c_i \leq f(\delta_0).$$

Thus, by Equation (19), we can fix $\delta \in [0, \delta_0)$ such that $f(\delta) = T \cdot \min(r', r'')$. 

19
\[ LP'' = B \frac{r''}{c''_i} = B \frac{r(\delta) - \delta}{c_i(\delta) + \delta} \]
\[ \geq B \frac{r(\delta) - \delta}{c_i(\delta)} \left( 1 - \frac{\delta}{c_i(\delta)} \right). \]
\[ f(\delta) - LP'' \leq B \frac{B r(\delta)}{c_i(\delta)} \delta + B \frac{r(\delta)}{c_i(\delta)^2} \delta \]
\[ = \left( 1 + \frac{r(\delta)}{c_i(\delta)} \right) \frac{B \delta}{c_i(\delta)} \]
\[ = \left( 1 + \frac{f(\delta)}{B} \right) \frac{f(\delta) \delta}{r(\delta)} \]
\[ \leq \left( 1 + \frac{T r'}{B} \right) \frac{T r'' \delta}{r(\delta)} \]
\[ \leq \left( 1 + \frac{LP'}{B} \right) T \delta \]
\[ \leq (LP'/B + 1) \cdot T \cdot \text{rad}_i(dK). \]

\[ LP' - f(\delta) = T \cdot \text{rad}_i(dK) \]
\[ LP' - LP'' = (LP' - f(\delta)) + (f(\delta) - LP'') \]
\[ \leq (LP'/B + 2) \cdot T \cdot \text{rad}_i(dK). \]

**Subcase 2.** Assume Equation (17) fails. Then \( LP' = B \frac{r'}{c'_j} \) for some resource \( j \). Note that \( c'_i \leq c'_j \) and \( c''_j \leq c''_i \) by the choice of \( i \) and \( j \).

From these inequalities and Lemma 8 we obtain \( c''_i \leq c'_j + \text{rad}_i(dK) \). Therefore,

\[ B \frac{r''}{c''_i} \geq B \frac{r' - \text{rad}_i(dK)}{c'_j + \text{rad}_i(dK)} \quad \text{(by Lemma 8)} \]
\[ \geq B \frac{r' - \text{rad}_i(dK)}{c'_j} \left( 1 - \frac{\text{rad}_i(dK)}{c'_j} \right). \]

\[ LP' - LP'' = B \frac{r'}{c'_j} - B \frac{r''}{c''_i} \]
\[ \leq \left( \frac{B}{c'_j} + B \frac{r'}{(c'_j)^2} \right) \text{rad}_i(dK) \]
\[ \leq \left( T + \frac{LP'}{B} \right) \text{rad}_i(dK) \]
\[ \leq (LP'/B + 1) \cdot T \cdot \text{rad}_i(dK). \]

\[ □ \]

**B.2. The remainder of the proof after Lemma 10**

We start with Corollary 11, which we restate here for convenience.
Corollary \( \Phi_t \leq 2\Psi_t \), assuming that \( B \geq 6 \cdot T \cdot \text{rad}_t(dK) \).

**Proof** Let \( \gamma = \max_{P \in \mathcal{F}_t, \mu \in \Omega_t} \text{LP}(P, \mu) \). Note that \( \gamma \leq T \). Then from Lemma 10 we obtain:

\[
\gamma - \text{OPT}_{\text{LP}} \leq 3\left(\frac{\gamma}{T} + 2\right) \cdot T \cdot \text{rad}_t(dK) \leq \frac{\gamma}{T} + 6 \cdot T \cdot \text{rad}_t(dK).
\]

(20)

Using (20) andLemma 10 we get the desired bound:

\[
\Phi_t \leq \left(\frac{\gamma}{T} + 2\right) \cdot T \cdot \text{rad}_t(dK) \leq \left(\frac{2 \text{OPT}_{\text{LP}} + 12 \cdot T \cdot \text{rad}_t(dK)}{B} + 2\right) \cdot T \cdot \text{rad}_t(dK) \leq \gamma + 6 \cdot T \cdot \text{rad}_t(dK) = 2\Psi_t.
\]

□

In the remainder of this appendix, we prove the claims in Equation (12) one by one.

**Corollary 16** \( \text{REW}_t \geq \frac{t}{T} (\text{OPT}_{\text{LP}} - o(\Psi_t)) \) for each round \( t \leq \tau \).

**Proof** From Lemma 12 we obtain

\[
T \tau(P'_t, \mu) \geq (1 - q_0) \text{LP}(P_t, \mu) \geq (1 - q_0) (\text{OPT}_{\text{LP}} - 12\Psi_t) \geq \text{OPT}_{\text{LP}} - 13\Psi_t.
\]

Summing up and taking average over rounds, we obtain:

\[
T \bar{\tau}_t \geq \text{OPT}_{\text{LP}} - \frac{13}{\tau} \sum_{s=1}^{t} \Psi_s \geq \text{OPT}_{\text{LP}} - O(\Psi_t).
\]

By definition of clean execution, we obtain:

\[
\text{REW}_t \geq t(\bar{\tau}_t - \text{rad}_t(\bar{\tau}_t)) \geq \frac{t}{T} (\text{OPT}_{\text{LP}} - O(\Psi_t)).
\]

□

**Corollary 17** \( C_{t,i} \leq B/T + O(\text{rad}_t(dK)) \) for each round \( t \leq \tau \).

**Proof** Let \( \mu \) be the (actual) expected-outcomes tuple, and recall that \( P_t \) is LP-optimal for some expected-outcomes tuple \( \mu' \in \Delta_t \). Then, by Lemma 8, it follows that \( c_i(P_t, \mu) \leq c_i(P_t, \mu') + \text{rad}_t(dK) \). Furthermore since \( P_t \) is LP-optimal for \( \mu' \) we have \( c_i(P_t, \mu') \leq \frac{B}{T} \). Therefore:

\[
c_i(P_t, \mu) \leq \frac{B}{T} + \text{rad}_t(dK)
\]

\[
c_i(P'_t, \mu) \leq (1 - q_0) c_i(P_t, \mu) + q_0 \leq \frac{B}{T} + O(\text{rad}_t(dK)).
\]

Now summing up and taking average we obtain \( \overline{c}_{t,i} \leq \frac{B}{T} + O(\text{rad}_t(dK)) \). Using the definition of clean execution, it follows that

\[
C_{t,i} \leq \overline{c}_{t,i} + \text{rad}_t(\overline{c}_{t,i}) \leq \frac{B}{T} + O(\text{rad}_t(dK)).
\]

□
Lemma 18 \( \text{REW}_T \geq \text{OPT}_{\text{LP}} - O(\Psi_T) \).

**Proof** Either \( \tau = T \) or some resource \( i \) gets exhausted, in which case (using Corollary 17)

\[
\tau = \frac{B}{C_{\tau,i}} \geq \frac{B}{\frac{B}{T} + \text{rad}_T(dK)}
\]

\[
\Rightarrow \tau \frac{B}{T} + \tau \text{rad}_T(dK) \geq B
\]

\[
\Rightarrow \tau \frac{B}{T} + T \text{rad}_T(dK) \geq B
\]

\[
\Rightarrow \tau \geq T \left( 1 - \frac{T}{B} \text{rad}_T(dK) \right).
\]

(21)

Using this lower bound and Corollary 16, we obtain the desired bound on the total revenue \( \text{REW}_T \).

\[
\text{REW}_T = \text{REW}_\tau \geq \frac{\tau}{T} \left( \text{OPT}_{\text{LP}} - O(\Psi_T) \right)
\]

\[
\geq \text{OPT}_{\text{LP}} \left( 1 - \frac{T}{B} \text{rad}_T(dK) \right) - \frac{O(\tau \Psi_T)}{T}
\]

\[
\geq \text{OPT}_{\text{LP}} - \Psi_T - \frac{O(\tau \Psi_T)}{T}.
\]

In the above, the first inequality holds by Corollary 16, the second by Equation (21), and the third by definition of \( \Psi_T \).

Finally, we note that \( \tau \Psi_T \) is an increasing function of \( \tau \), and substitute \( \tau \Psi_T \leq T \Psi_T \). \( \square \)

We complete the proof of Theorem 1 as follows. Re-writing Lemma 18 as \( \text{REW}_T \geq f(\text{OPT}_{\text{LP}}) \), for an appropriate function \( f() \), note that \( \text{REW}_T \geq f(\text{OPT}) \) because function \( f() \) is increasing.