Exactly solvable models of growing interfaces and lattice gases: the Arcetri models, ageing and logarithmic sub-ageing

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Abstract. Motivated by an analogy with the spherical model of a ferromagnet, the three Arcetri models are defined. They present new universality classes, either for the growth of interfaces, or else for lattice gases. They are distinct from the common Edwards–Wilkinson and Kardar–Parisi–Zhang universality classes. Their non-equilibrium evolution can be studied by the exact computation of their two-time correlators and responses. In both interpretations, the first model has a critical point in any dimension and shows simple ageing at and below criticality. The exact universal exponents are found. The second and third model are solved at zero temperature, in one dimension, where both show logarithmic sub-ageing, of which several distinct types are identified. Physically, the second model describes a lattice gas and the third model describes interface...
Exactly solvable models of growing interfaces and lattice gases: the Arcetri models, ageing and logarithmic sub-ageing growth. A clear physical picture on the subsequent time and length scales of the sub-ageing process emerges.

**Keywords:** correlation functions, critical exponents and amplitudes, exact results, kinetic growth processes

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1. Introduction

The physics of the growth of interfaces is a paradigmatic example of the emergence of non-equilibrium cooperative phenomena, with widespread applications in domains as different as the deposition of atoms on a surface, solidification, flame propagation, population dynamics, crack propagation, chemical reaction fronts or the growth of cell colonies [3, 17, 32, 51, 52, 76, 80, 81]. Several universal growth and roughness exponents characterise the morphology of the growing interface and the time-dependent properties are quite analogous to phenomena encountered in the physical ageing in glassy and non-glassy systems [20, 37, 77]. Several universality classes of interface growth have been identified, and the best-known of these are characterised in terms of stochastic equations for the height profile

\[ \frac{\partial h}{\partial t} = \nu \nabla^2 h + \eta; \ \text{Edwards–Wilkinson (EW) [23]} \]

\[ \frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\mu}{2}(\nabla h)^2 + \eta; \ \text{Kardar–Parisi–Zhang (KPZ) [46]}, \]

where \( \nabla \) is the spatial gradient, \( \eta \) is a centred Gaussian white noise, with covariance

\[ \langle \eta(t, r)\eta(t', r') \rangle = 2\nu T \delta(t-t') \delta(r-r'), \]

and \( \nu, \mu, T \) are material-dependent constants.

While the exact solution of the Edwards–Wilkinson (EW) equation is straightforward, extracting the long-distance and/or long-time properties of interfaces in the Kardar–Parisi–Zhang (KPZ) class is considerably more difficult, and several aspects of the problem still remain unresolved. Remarkable progress has been achieved in recent years concerning the exact solution of the KPZ equation in \( d = 1 \) dimension. In particular, several spatial correlators have been found exactly and a deep relationship of the probability distribution \( P(h) \) of the fluctuation \( h - \bar{h} \) with the extremal value statistics of the largest eigenvalue of random matrices has been derived (see [12, 13, 33, 34, 43, 48, 68]). Remarkably, it was possible to confirm these mathematical results experimentally in several physically distinct systems [2, 35, 36, 40, 41, 72–75, 82]. Still, this impressive progress seems to rely on specific properties of the 1D case. Therefore, one might consider whether further classes of exactly solvable models of interface growth could be defined, distinct from both the EW as well as the KPZ universality class, and what physical insights the study of such models might provide.

Indeed, a new class of models can be defined, with the help of some inspiration from the definition of the well-studied spherical model of a ferromagnet [7, 56]. Therein, the traditional Ising spins \( \sigma_i = \pm 1 \), attached to the sites \( i \) of a lattice with \( N \) sites, are replaced by continuous spins \( S_i \in \mathbb{R} \) and subject to the spherical constraint \( \sum_i S_i^2 = N \).

A conventional nearest-neighbour interaction leads to an exactly solvable model, which undergoes a non-trivial phase transition in \( 2 < d < 4 \) dimensions [7, 44]. The relaxational properties can likewise be analysed exactly, see e.g. [14, 18, 19, 22, 26, 27, 29, 30, 64, 66]. In order to identify an analogy with growing interfaces, we restrict ourselves here to \( d = 1 \) dimensions for simplicity. Consider a lattice representation of the KPZ class, where the height differences between two nearest neighbours obey the so-called rsos constraint \( h_{i+1}(t) - h_i(t) = \pm 1 \). It is well established that in the continuum limit this model is described by the KPZ equation [3, 32, 76] (see [8] for a rigorous derivation). The dynamic deposition rule is sketched in figure 1, which makes it clear that

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in this kind of lattice model, the slopes $u_{i+1/2}(t) := h_{i+1}(t) - h_i(t)$ should be considered as the analogues of the Ising spins $\sigma_i = \pm 1$ in ferromagnets. For the slopes, in the continuum limit, from the KPZ equation follows the (noisy) Burgers equation [10] (for a discrete analogue, see [4]):

$$\partial_t u = \nu \nabla^2 u + \mu u \nabla u + \nabla \eta.$$  \hspace{1cm} (1.3)

A spherical model variant of the KPZ universality class now stipulates that we should relax the $\mathrm{rsos}$ constraints $u_i = \pm 1$ to a ‘spherical constraint’ $\sum_i \langle u_i^2 \rangle = \mathcal{N}$ [39]'7. However, for growing interfaces, several equivalent descriptions can give rise to several new models, which may or may not be in the same universality class. Heuristically, the following possibilities may occur.

1. One may start from the Burgers equation and replace its nonlinearity as follows:

$$\partial_t u = \nu \nabla^2 u + \mu u \nabla u + \nabla \eta \mapsto \partial_t u = \nu \nabla^2 u + \mathfrak{z}(t) u + \nabla \eta,$$  \hspace{1cm} (1.4)

with a Lagrange multiplier $\mathfrak{z}(t) \sim \langle \nabla u \rangle$, which might be seen as some kind of ‘averaged curvature’ of the interface. Its value is determined by the mean spherical constraint $\sum \langle u^2 \rangle = \mathcal{N}$. This is the ‘first Arcetri model’, defined [39]'9 and analysed in [39]'10. In any dimension $d > 0$, there is a ‘critical temperature’ $T_c(d) > 0$ such that long-range correlations build up for $T \leq T_c(d)$. At the critical point $T = T_c(d)$, the interface is rough for $d < 2$ and smooth for $d > 2$. For $T < T_c(d)$, the interface is always rough. The model is also related to the gaps in the spectra of random matrices [28] and to the spherical spin glass [19].

2. An alternative way to treat the Burgers equations might proceed as follows:

$$\partial_t u = \nu \nabla^2 u + \mu u \nabla u + \nabla \eta \mapsto \partial_t u = \nu \nabla^2 u + \mathfrak{z}(t) \nabla u + \nabla \eta,$$  \hspace{1cm} (1.5)

where the Lagrange multiplier $\mathfrak{z}(t) \sim \langle u \rangle$ might now be viewed as some kind of ‘averaged slope’. Its value is again determined by the constraint $\sum \langle u^2 \rangle = \mathcal{N}$. This would define a ‘second Arcetri model’.

3. Finally, we might have started directly from the KPZ equation:

$$\partial_t h = \nu \nabla^2 h + \frac{1}{2} \mu (\nabla h)^2 + \eta \mapsto \partial_t h = \nu \nabla^2 h + \mathfrak{z}(t) \nabla h + \eta,$$  \hspace{1cm} (1.6)

where $\mathfrak{z}(t) \sim \langle \nabla h \rangle$ might again be interpreted as an averaged slope and will be found from a constraint $\sum \langle (\nabla h)^2 \rangle = \mathcal{N}$. This would be a ‘third Arcetri model’.

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7 An old observation by Oono and Puri [62] gives additional motivation: treating the Allen–Cahn equation of phase-ordering, after a quench to $T < T_c$, along the lines of the celebrated Ohta–Jasnow–Kawasaki approximation, but for a finite thickness of the domain boundaries, leads to a kinetic equation in the universality class of the spherical model.

8 In this section, the average $\langle \cdot \rangle$ is understood to be taken over both the ‘thermal’ and the ‘initial’ noise.

9 The name comes from the location of the Galileo Galilei Institute of Physics, where this model was conceived.

10 It can be shown that $\mathfrak{z}(t) \sim t^{-1}$ for sufficiently long times, whenever $T \leq T_c(d)$.
However, such a simplistic procedure would lead to undesirable properties of the height and slope profiles in the stationary state, as well as internal inconsistencies. We shall therefore reconsider this correspondence carefully in section 2, where the precise definitions of the second and third Arcetri models will be given.

In one spatial dimension, the slope profile has an interesting relationship with the dynamics of interacting particles. To see this, write the slope as

\[ u(t, r) = 1 - 2\varrho(t, r) \]

where \( \varrho(t, r) \) denotes the particle density at time \( t \in \mathbb{R}^+ \) and position \( r \in \mathbb{R} \). In the KPZ universality class, when the rsos-constraint \( u(t, r) = \pm 1 \) holds on the lattice, denote by • an occupied site with \( \varrho = 1 \) and by ◦ an empty site with \( \varrho = 0 \). Then the only admissible reaction between neighbouring sites is the directed jump • ◦ \( \rightarrow \) ◦ •. The stochastic process described by these interacting particles is a totally asymmetric exclusion process (TASEP) (see e.g. [21, 31, 55, 57]), which is integrable via the Bethe ansatz. Here, we are interested in the situation where the exact rsos constraint is relaxed to the mean spherical constraint \( \langle \sum_r u(t, r)^2 \rangle = N \). In terms of the noise-averaged particle density, this becomes

\[ \sum_r \langle \varrho(t, r) \rangle = \sum_r \langle \varrho(t, r)^2 \rangle, \]

where the sums run over all sites of the lattice. Hence, on any site, neither \( \langle \varrho(t, r) \rangle \) nor the difference \( \langle \varrho(t, r) \rangle - 1 \) can become very large, since the spherical constraint prohibits the condensation of almost all particles onto a very small number of sites. In particular, if one takes a spatially translation-invariant initial condition, then spatial

Figure 1. Illustration of the growth of an interface obeying the RSOS condition. In (a) the interface is shown before the adsorption of a particle and in (b) the same interface is shown after the adsorption process. Below interfaces (a) and (b), the slopes \( u_j = h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} \), defined on the dual lattice \( j \in \mathbb{Z} + \frac{1}{2} \), are indicated. The adsorption process is described by a move \((-+) \rightarrow (+-)\) in terms of the slopes, where the participating slopes are indicated in red before the adsorption (a) and in green afterwards (b).
translation-invariance is kept for all times. Because of constraint (1.7), the average (position-independent) particle density
\[
\rho(t) := \frac{1}{N} \sum_r \langle g(t, r) \rangle \geq 0 \tag{1.8}
\]
is always non-negative. We point out that while the non-averaged density variable \(g(t, r) \in \mathbb{R}\) has no immediate physical interpretation, constraint (1.7) guarantees that the measurable disorder-averaged observables take physically reasonable values.

The long-time non-equilibrium relaxation behaviour is analysed as follows. In models of interface growth, one usually starts from a flat, horizontal interface with uncorrelated heights \([3, 32, 36, 38, 39, 47, 48, 61, 67]\). One then studies the average height \(\langle h(t, r) \rangle \sim t^\beta\), the interface width \(w^2(t) = \langle (h(t, r) - \langle h(t, r) \rangle)^2 \rangle \sim t^{2\beta}\), and the two-time height autocorrelator and auto-response of the height with respect to a change in the height
\[
C(t, s) := \langle (h(t, r) - \langle h(t, r) \rangle) (h(s, r) - \langle h(s, r) \rangle) \rangle = s^{-b} f_C \left( \frac{t}{s} \right) \tag{1.9}
\]
\[
R(t, s) := \left. \frac{\delta \langle h(t, r) \rangle}{\delta j(s, r)} \right|_{j=0} = s^{-a} f_R \left( \frac{t}{s} \right). \tag{1.10}
\]
The scaling forms \([24]\) used here are those of simple ageing and apply in the long-time limit \(t, s \to \infty\) with \(y = t/s\), being kept fixed. The scaling functions are expected to have the asymptotic behaviour
\[
f_C(y)^{y^b_0} \sim y^{-\lambda_C/z}, \quad f_R(y)^{y^b_1} \sim y^{-\lambda_R/z}, \tag{1.11}
\]
where \(z\) is the dynamical exponent. From these relations the exponents \(\beta, a, b, \lambda_C, \lambda_R\) are defined. In table 1, some values of these exponents are collected\(^{11}\). Starting from the Langevin equation (1.4) of the first Arcetri model, formulated in terms of the the slopes \(u\) and using \(u = \nabla h\), an analogous Langevin equation for the heights \(h\) is found, if the spherical constraint is now written as \(\sum ((\nabla h)^2) = N\). In what follows, we shall call this the Arcetri 1H model. Its relaxational behaviour undergoes (simple) ageing for both \(T = T_c\) and for \(T < T_c\), in agreement with the expected scaling forms (1.9) and (1.10). In appendix A, we briefly outline how to find the exponents. The logarithmic sub-scaling exponents \([49]\) in \(w(t)\) of the third Arcetri model are discussed in section 4.

The main focus of this work will be on defining (see section 2 for the precise definitions) and analysing the second and the third Arcetri models. At temperature \(T = 0\), we shall see that the simple ageing behaviour of equations (1.9)–(1.11) does not apply. Rather, we shall find a logarithmic sub-ageing behaviour\(^{12}\), in the scaling limit where both times \(t, s \to \infty\), but such that the scaling variable \(y\) of two-time scaling

\(11\) The 2D KPZ universality class is realised by the octahedron model [61]. For height correlators and responses, the results of the random sequential (RS) update and of the two-sublattice stochastic dynamics (SCA) update are consistent, confirming the expected universality \((\lambda_C, RS = 1.98(5), \lambda_C, SCA = 2.01(2)) [48]\). Comparison with the recent result \(z = 1.613(2) [63]\) gives an a posteriori indication of the presently achieved numerical precision.

\(12\) Sub-ageing behaviour is defined by the scaling variable \(y - 1 := \frac{\langle g(t, r) \rangle}{\langle g(t, r) \rangle^2}\), where \(0 < \mu < 1\) is the sub-ageing exponent and \(\mu = 1\) gives back simple ageing [20, 78]. See [37, table 1.2] for a list of experimentally measured values of \(\mu\). A basic rigorous inequality excludes the case \(\mu > 1\) (‘super-ageing’) [53].

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Table 1. Non-equilibrium exponents, as defined in the text, for several universality classes of growing-interface models. The Arcetri 1h class at $T = T_c$ for $d > 2$ is identical to the EW class [11, 23, 42, 48, 67]. For the Arcetri 3 class, there are three distinct logarithmic sub-ageing scaling regimes, which are characterised by the value of the logarithmic sub-ageing parameter $\vartheta$, as indicated. For $\vartheta < 1$, the autoresponse function does not display scaling behaviour, as indicated by DNS (does not scale). For empty entries, no estimate is known. The initial state is flat on average, with uncorrelated heights.

| Model      | $d$  | $a$   | $b$   | $\lambda_C$ | $\lambda_R$ | $z$ | $\beta$ | Reference         |
|------------|------|-------|-------|-------------|-------------|-----|---------|------------------|
| KPZ        | 1    | $-1/3$| $-2/3$| 1           | 1           | $3/2$| $1/3$   | [38, 45, 46, 50]  |
| KPZ        | 2    | $-0.483$| 1.91(6)| 1.61(2) | 0.241(1)   |     |         |                  |
| KPZ        | 2    | 0.30(1) $-0.483(3)$| 1.97(3) | 0.241(15) |     |      | [61]          |
| KPZ        | 2    | 0.24(1) $-0.483(3)$| 1.97(3) | 0.241(15) |     |      | [47]          |
| KPZ        | 2    | 0.24(1) $-0.4828(4)$| 1.98(5) | 0.241(14) |     |      | [48]          |

| Arcetri 1h $T = T_c$ <2 | $d$ | $a$ | $b$ | $\lambda_C$ | $\lambda_R$ | $z$ | $\beta$ | Reference         |
|-------------------------|-----|-----|-----|-------------|-------------|-----|---------|------------------|
|                         | $\frac{d}{2} - 1$ | $\frac{d}{2} - 1$ | $\frac{d}{2} + 1$ | $\frac{d}{2} + 1$ | $2$ | $\frac{1}{4}(2 - d)$ | [39] |
| Arcetri 1h $T = T_c$ >2 | $d$ | $d$ | $d$ | 2           | 0           |     |         |                  |
| Arcetri 1h $T < T_c$    | $d$ | $\frac{d}{2} - 1$ | $-1$ | $\frac{d}{2} - 1$ | $2$ | $\frac{1}{2}$ | [39] |
| Arcetri 3 $T = 0$       | 1   | $-\frac{1}{2}$ | 0   | 0           | 2           |     | $\frac{1}{2}$ |                  |
| Arcetri 3 $T = 0$       | 1   | DNS         | 0   | DNS         | 2           |     | $\frac{1}{2}$ |                  |
| Arcetri 3 $T = 0$       | 1   | DNS         | 0   | $\infty$   | 2           |     | $\frac{1}{2}$ |                  |

$$y - 1 := \frac{t - s}{s} \ln^\vartheta \kappa_2 s$$

(1.12)

is being kept fixed ($\kappa_2$ is a model-dependent constant). It turns out that several types of logarithmic sub-ageing exist for the Arcetri models, which are characterised by different values of the logarithmic sub-ageing exponent $\vartheta > 0$\(^{13}\). With the scaling variable (1.12), the asymptotic scaling forms (1.11) often remain applicable and the corresponding exponent values are quoted in tables 1 and 2. Logarithmic sub-ageing arises from the presence of several time-dependent length scales, which differ by factors that are logarithmic in time, a phenomenon also referred to as multiscaling [14]. If the autocorrelator scaling function $f_C(y)$ decays with $y$ faster than a power law (exponentially or stretched exponentially), the value $\lambda_C = \infty$ is quoted. See section 5 for a fuller discussion.

Analogously, if one considers a system of interacting particles, one usually assumes an initial state of uncorrelated particles (uncorrelated flat slopes $\langle u(t, r) \rangle = 0$ in the present terminology) with an average particle density $\varrho = \langle \varrho(t, r) \rangle = \frac{1}{2}$, equivalent to a vanishing initial slope [21, 48]. One considers the two-time slope (connected) auto-correlator $C(t, s)$, which is related to the density–density autocorrelator, and the linear auto-responses $R(t, s), \mathcal{R}(t, s)$ of the slope with respect to a change $k = \nabla j$ in the slope or a change $j$ in the height, respectively.

\(^{13}\) For $\vartheta = 0$, one is back to simple ageing.
\[ C(t, s) := \langle u(t, r)u(s, r) \rangle = 4 \left( \langle \varrho(t, r) - \frac{1}{2} \rangle \langle \varrho(s, r) - \frac{1}{2} \rangle \right) = s^{-b} f_C \left( \frac{t}{s} \right) \]  

\[ R(t, s) := \left. \frac{\delta \langle u(t, r) \rangle}{\delta k(s, r)} \right|_{k=0} = s^{-1-a} f_R \left( \frac{t}{s} \right) \]  

\[ R(t, s) := \left. \frac{\delta \langle u(t, r) \rangle}{\delta j(s, r)} \right|_{j=0} = s^{-1-a} f_R \left( \frac{t}{s} \right), \]  

along with the expected behaviour of simple ageing in the scaling limit. Equation (1.11) applies again and, analogously, one anticipates \( f_R(y) \sim y^{-\lambda_R/2} \) for \( y \gg 1 \). Considering numerical simulations of the 2D octahedron model, however, it appears that for the slope correlations the two update schemes RS and SCA lead to different values of the autocorrelation exponent—and this for model realisation in both the KPZ and the EW universality classes [48]. The first Arcetri model with initially uncorrelated slopes will be called the \textit{Arcetri 1u model}. It is informative to compare the corresponding exponent values with those of the EW universality class. Some values of these exponents are listed in table 2 (see appendix A for the outline of the calculations in the Arcetri 1u model). We also include results from the spherical model with a conserved order parameter (‘model B’) at \( T = 0 \) [9, 14]. It also becomes apparent how much less is known about the responses of the slope variables compared to the height variables.

This work is organised as follows. In section 2, the second and third Arcetri models are carefully defined. Since the first Arcetri model has already been studied [39], we merely outline its treatment in appendix A and quote the results in tables 1 and 2, where the two possible interpretations are taken into account. Section 3 explains the solution of the second and third models. The explicit spherical constraints and a closed form for correlators and responses are derived. In section 4, the asymptotic analysis at temperature \( T = 0 \) and the emergence of the different types of logarithmic sub-ageing in the second and third models are presented. We conclude in section 5 with a detailed presentation of the kinetic phase diagram and the various scales on which different aspects of logarithmic sub-ageing occur. Technical calculations are treated in several appendices. Appendix A contains a short summary of the first model, both for an interface and for a lattice gas. Appendices B and C derive the various distinct sub-ageing scaling forms of correlators and responses, respectively. Several mathematical identities are derived in appendices D and E, and some basics of discrete cosine and sine transformations are collected in appendix F.

2. The second and third Arcetri models

2.1. Preliminaries

Why are equations (1.5) and (1.6) physically unsatisfactory? In order to understand this, and in consequence the necessity for a better definition of the models, consider for a moment the behaviour of the stationary profiles, as they would follow from equations (1.4)–(1.6). Let \( z_\infty \) denote the stationary value of the Lagrange multiplier. Then the noise-averaged slope profile of the first Arcetri model (1.4) is oscillatory
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Table 2. Non-equilibrium exponents for several universality classes of lattice-gas models, where ‘octa’ stands for ‘octahedron model’, RS stands for random sequential update and SCA stands for two-sub-lattice stochastic dynamics. The model realisations in the KPZ and EW universality classes are indicated. For the Arcetri 2 and conserved spherical classes, there are three logarithmic sub-ageing scaling regimes, which are characterised by the value of the logarithmic sub-ageing parameter \( \vartheta \), as indicated. For \( \vartheta < 1 \), the autoresponse functions do not display scaling behaviour, as indicated by DNS. For empty entries, there is no estimate. The initial state has an average particle density \( \varrho = \frac{1}{2} \) and uncorrelated particles.

| Model     | \( d \) | \( a \) | \( a_R \) | \( b \) | \( \lambda_C \) | \( \lambda_R \) | \( \lambda_\vartheta \) | \( z \) | Reference |
|-----------|---------|---------|-----------|---------|---------------|---------------|----------------|------|-----------|
| TASEP     | 1       | 2/3     | 3         |         | 3/2           |               |                 |      | [21]      |
| Octa RS   | KPZ     | 2       | 0.76(2)   | 3.8(2)  | 1.611(3)      |               |                 |      | [48]      |
| Octa SCA  | KPZ     | 2       | 1.25(2)   |         | 1.611(3)      |               |                 |      | [48]      |
| Octa SCA  | EW      | 2       | 1.1(2)    | \( \approx 4 \) | 2              |               |                 |      | [48]      |
| Octa RS   | EW      | 2       | 1.1(2)    | 1.4(4)  | 2              |               |                 |      | [48]      |
| Arcetri 1u| \( T = T_c \) | \( d \) | \( d/2 + 1 \) | \( d/2 \) | \( d + 2 \) | \( d \) | \( d + 2 \) | 2    |           |
| Arcetri 1u| \( T < T_c \) | \( d \) | \( d/2 - 1 \) | \( d/2 \) | \( d/2 \) | \( d/2 + 2 \) | 2    |           |
| Arcetri 2  | \( T = 0 \) | 1       | -1/2      | 0       | 0             | 0             | 1               | 1    | \( \vartheta > 1 \) |
| Arcetri 2  | \( T = 0 \) | 1       | DNS       | DNS     | 0             | DNS           | DNS             | 2    | \( \frac{1}{2} < \vartheta < 1 \) |
| Arcetri 2  | \( T = 0 \) | 1       | DNS       | DNS     | 0             | \( \infty \)  | DNS             | DNS | \( \vartheta = \frac{1}{2} \) |
| spherical  | \( T = 0 \) | \( d \) | \( (d - 2)/4 \) | 0       | 0             | \( d + 2 \)   | 4               | \( \vartheta > 1 \) |
| spherical  | \( T = 0 \) | \( d \) | DNS       | 0       | 0             | DNS           | 4               | \( \frac{1}{2} < \vartheta < 1 \) |
| spherical  | \( T = 0 \) | \( d \) | DNS       | \( \infty \) | 0             | DNS           | 4               | \( \vartheta = \frac{1}{2} \) |

\( u_{\text{stat}}(r) \sim \cos \left( \frac{r}{\lambda} + \phi_0 \right) \), with the finite wavelength \( \lambda = \sqrt{\nu/4_\infty} \), as one would have expected. On the other hand, equation (1.5) would produce a spatially strongly variable stationary slope profile \( u_{\text{stat}}(r) \sim \exp \left( -r/r_0 \right) \), with a finite length scale \( r_0 = \nu/3_\infty \). Finally, equation (1.6) gives an analogous result for the stationary height profile. This is in apparent contradiction with the expectation of essentially flat profiles for both the height and the slope.

2.2. Definition of the second Arcetri model

How can one formulate a physically sensible spherical model variant of the Burgers equation? Begin with a decomposition of the slope profile \( u(t,r) \), with \( r \in \mathbb{R} \), into its even and odd parts:

\[
u(t,r) = a(t,r) + b(t,r),
\]

where

\[
a(t,r) := \frac{1}{2} \left( u(t,r) + u(t,-r) \right) = a(t,-r) \text{ even}
\]

\[
b(t,r) := \frac{1}{2} \left( u(t,r) - u(t,-r) \right) = -b(t,-r) \text{ odd}.
\]

For definiteness, we shall formulate the defining equations of motion of the second Arcetri model on a periodic chain of \( N \) sites. They read

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\[ \partial_t a_n(t) = \nu (a_{n+1}(t) + a_{n-1}(t) - 2a_n(t)) + \frac{1}{2} \bar{z}(t) (b_{n+1}(t) - b_{n-1}(t)) + \frac{1}{2} (\eta^+_{n+1}(t) - \eta^-_{n-1}(t)) \]

\[ \partial_t b_n(t) = \nu (b_{n+1}(t) + b_{n-1}(t) - 2b_n(t)) - \frac{1}{2} \bar{z}(t) (a_{n+1}(t) - a_{n-1}(t)) - \frac{1}{2} (\eta^+_{n+1}(t) - \eta^-_{n-1}(t)), \]

where \( \eta^\pm_n(t) := \frac{1}{2} (\eta_n(t) \pm \eta_{n-N}(t)) \) is the parity-symmetrised and anti-symmetrised white noise \( \eta_n(t) \), with the moments

\[ \langle \eta_n(t) \rangle = 0, \quad \langle \eta_n(t) \eta_m(t') \rangle = 2\nu T \delta_{n,m} \delta(t - t'). \]

Hence one has the (anti-)symmetrisated noise correlators

\[ \langle \eta^\pm_n(t) \rangle = 0, \quad \langle \eta^\pm_n(t) \eta^\pm_m(t') \rangle = 2\nu T \delta(t - t') [\delta_{n,m} \pm \delta_{n,N-m}], \quad \langle \eta^\pm_n(t) \eta^\mp_m(t') \rangle = 0 \]  

(2.5)

(clearly, the indices \( n, m \) are to be taken modulo \( N \)). The second Arcetri model will be considered as a variant of the Burgers equation and its associated TASEP. Therefore, a natural choice of initial conditions is to admit initially uncorrelated slopes, distributed according to a Gaussian, and with the moments

\[ \langle a_n(0) \rangle = \langle b_n(0) \rangle = 0, \quad \langle a_n(0) b_m(0) \rangle = \frac{1}{2} (\delta_{n,m} + \delta_{n,N-m}). \]

(2.6)

The Lagrange multiplier \(\bar{z}(t)\) is determined from the mean spherical constraint on the slopes

\[ \left\langle \left\langle \sum_{n=1}^{N} (a_n(t) + b_n(t))^2 \right\rangle \right\rangle = N \]  

(2.7)

which is averaged over both sources of noise present in the model, as indicated by the brackets \(\langle\rangle\) for the average over \(\eta\) and \(\langle\langle\rangle\rangle\) for the average over the initial conditions. We stress that the even and odd parts are treated in a slightly different way. In this way, two essential properties of the Burgers equation, namely (i) the conservation law and (ii) the non-invariance under the parity transformation \( x \mapsto -x \) are kept. The initial conditions (2.6) are natural if one wishes to interpret the slope \( u(t,r) = 1 - 2\varrho(t,r) \) in terms of the density \( \varrho(t,r) \) of a model of interacting particles, with the average density \( \varrho = \langle \langle \varrho(t,r) \rangle \rangle = \frac{1}{2} \).

Equations (2.3), (2.6) and (2.7), together with the noise correlator (2.5), define the second Arcetri model.

Formally, one might also arrive at these equations by introducing a complex velocity \( u = a + ib \) into the modification (1.5) of the Burgers equation, with a complex Lagrange multiplier \( \bar{z}(t) = \bar{z}_1(t) - i\bar{z}_2(t) \) and a complex noise \( \eta(t,r) = i(\eta^+(t,r) - i\eta^-(t,r)) \). Separating into real and imaginary parts, this would give

\[ \partial_t a = \nu \partial_r^2 a + \bar{z}_1 \partial_r a + \bar{z}_2 \partial_r b + \partial_r \eta^- \]

\[ \partial_t b = \nu \partial_r^2 b + \bar{z}_1 \partial_r b - \bar{z}_2 \partial_r a - \partial_r \eta^+. \]
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Only if one chooses \( J_1 = 0 \) does one obtain an oscillatory equation \( v^2 p'' = -i(0) \) for the derivative \( p := \lim_{t \to \infty} \partial_t \langle a \rangle \) of the noise-averaged stationary slope, and similarly for \( q := \lim_{t \to \infty} \partial_t \langle b \rangle \). The effect of this formally ‘imaginary’ Lagrange multiplier is included in the equations of motion (2.3).

Conservation laws become explicit by rewriting the complex equations of motion (1.5)

\[
\partial_t (a + ib) = \partial^2_r (a + ib) + \bar{J}(t) \partial_r (a + ib) + \partial_r (\eta^+ - i\eta^-) \\
= \partial_r \left[ (\partial_r - i\bar{J}(t))(a + ib) - i(\eta^+ + i\eta^-) \right]
\]

in the form of a continuity equation. Using \( u = a + ib \) and its formal complex conjugate \( u^* = a - ib \), along with \( \zeta := \eta^+ + i\eta^- \) and \( \zeta^* := \eta^+ - i\eta^- \), we have the pair of equations

\[
\partial_t u = \partial_r [(\partial_r - i\bar{J}(t))u - i\zeta] \quad , \quad \partial_t u^* = \partial_r [(\partial_r + i\bar{J}(t))u^* + i\zeta^*] \quad (2.8)
\]

and identify the densities \( j = (\partial_r - i\bar{J}(t))u - i\zeta \) and \( j^* = (\partial_r + i\bar{J}(t))u^* + i\zeta^* \) of the conserved currents, such that the ‘conserved charges’ \( U = \int_R dr \ u(t, r) \) and \( U^* = \int_R dr \ u^*(t, r) \) are time-independent, namely \( \partial_t U = \partial_t U^* = 0 \).

### 2.3. Definition of the third Arcetri model

Analogously, for the third Arcetri model we start from the height profile \( h(t, r) \), decomposed into even and odd parts

\[
h(t, r) = a(t, r) + b(t, r) \quad (2.9)
\]

and write down the defining equations of motion (on a discrete chain of \( N \) sites)

\[
\partial_t a_n(t) = \nu \left( a_{n+1}(t) + a_{n-1}(t) - 2a_n(t) \right) + \frac{1}{2} \bar{J}(t) (b_{n+1}(t) - b_{n-1}(t)) + \eta^+_n(t) \\
\partial_t b_n(t) = \nu \left( b_{n+1}(t) + b_{n-1}(t) - 2b_n(t) \right) - \frac{1}{2} \bar{J}(t) (a_{n+1}(t) - a_{n-1}(t)) + \eta^-_n(t) \quad (2.10)
\]

with the symmetrised noise (2.5). In this physical context, it appears natural to use initially uncorrelated Gaussian slopes

\[
\langle a_n(0) \rangle = H_0, \quad \langle b_n(0) \rangle = 0, \quad \langle a_n(0)b_m(0) \rangle = 0 \\
\langle a_n(0)a_m(0) \rangle_c = \frac{1}{2} H_1 (\delta_{n,m} + \delta_{n,N-m}), \quad \langle b_n(0)b_m(0) \rangle = \frac{1}{2} H_1 (\delta_{n,m} - \delta_{n,N-m}) \quad (2.11)
\]

The Lagrange multiplier \( \bar{J}(t) \) is found from the mean spherical constraint on the slopes

\[
\left\langle \left\langle \sum_{n=1}^{N} (\nabla a_n(t) + \nabla b_n(t))^2 \right\rangle \right\rangle = N, \quad (2.12)
\]

where \( \nabla f_n = \frac{1}{2} (f_{n+1} - f_{n-1}) \) is the symmetrised spatial difference. The initial conditions (2.11) are natural for an interpretation of \( h(t,r) \) as the height of a growing and fluctuating interface, which is flat on average.

Equations (2.10)–(2.12), together with the noise correlator (2.5), define the third Arcetri model.

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Formally, one might obtain this from the modified KPZ equation (1.6) by introducing a complex height \( h = a + ib \), a complex Lagrange multiplier \( \hat{\gamma}(t) = \hat{\gamma}_1(t) - i\hat{\gamma}_2(t) \) and a complex noise \( \eta(t, r) = \eta^+(t, r) + i\eta^-(t, r) \). As before, only if one chooses \( \hat{\gamma}_1 = 0 \) does the derivative \( p := \lim_{t \to \infty} \partial_t \langle a \rangle \) of the stationary height obey an oscillatory equation \( \nu^2 p'' = -\hat{\gamma}_2^2 p \). Because of the ‘non-conserved’ noise, there are no obvious conservation laws for \( T \neq 0 \).

All definitions were only made explicit in \( d = 1 \) spatial dimensions. Eventual extensions to \( d > 1 \) are left for future work.

3. Solution

We begin our discussion with the second Arcetri model. The treatment of the third Arcetri model being fairly analogous, we shall simply quote the relevant results in section 3.4.

3.1. Second model: general form

The first step to the solution to equation (2.3) proceeds via the Fourier transform, but we must take into account the specific parity of the \( a_n \) and \( b_n \). Therefore, we use the representation in terms of discrete cosine and sine transforms,

\[
a_n(t) = \frac{1}{N} \sum_{k=0}^{N-1} \cos \left( \frac{2\pi}{N} kn \right) \hat{a}(t, k) , \quad b_n(t) = \frac{1}{N} \sum_{k=0}^{N-1} \sin \left( \frac{2\pi}{N} kn \right) \hat{b}(t, k).
\]

(See appendix F for details.) Using equations (F.3)–(F.6), the equations of motion turn into

\[
\partial_t \hat{a}(t, k) = -2\nu \left( 1 - \cos \left( \frac{2\pi}{N} k \right) \right) \hat{a}(t, k) + \hat{\gamma}(t) \sin \left( \frac{2\pi}{N} k \right) \hat{b}(t, k) + \sin \left( \frac{2\pi}{N} k \right) \hat{\eta}^-(t, k)
\]
\[
\partial_t \hat{b}(t, k) = -2\nu \left( 1 - \cos \left( \frac{2\pi}{N} k \right) \right) \hat{b}(t, k) + \hat{\gamma}(t) \sin \left( \frac{2\pi}{N} k \right) \hat{a}(t, k) + \sin \left( \frac{2\pi}{N} k \right) \hat{\eta}^+(t, k).
\]

(3.2)

Although we shall use the same notation for both cosine and sine transforms, the parity must be taken into account for the inverse transformation. We shall use the shorthand

\[
\omega(k) := 1 - \cos \left( \frac{2\pi}{N} k \right) , \quad \lambda(k) := \sin \left( \frac{2\pi}{N} k \right) , \quad Z(t) := \int_0^t d\tau \hat{\gamma}(\tau).
\]

(3.3)

Later, when taking the continuum limit, it will be enough to simply replace \( \omega(k) \to 1 - \cos k \) and \( \lambda(k) \to \sin k \), and to consider \( k \in (-\pi, \pi) \) instead of \( k = 0, 1, \ldots, N - 1 \) on the chain.

The above equations (3.2) are decoupled by going over to the combinations

\[
\hat{f}_\pm(t, k) := \hat{a}(t, k) \pm \hat{b}(t, k),
\]

which obey the equations

\[
\partial_t \hat{f}_\pm(t, k) = (2\nu \omega(k) \pm \lambda(k) \hat{\gamma}(t)) \hat{f}_\pm(t, k) + \lambda(k) \left( \hat{\eta}^-(t, k) \pm \hat{\eta}^+(t, k) \right)
\]

(3.4)
with the solutions
\[ \hat{f}_{\pm}(t, k) = \hat{f}_{\pm,00}(k) \exp \left[ -2\nu \omega(k) t \pm \lambda(k) Z(t) \right] \]
\[ + \int_0^t d\tau \lambda(k) \left( \hat{\eta}^-(\tau, k) \pm \hat{\eta}^+(\tau, k) \right) \exp \left[ -2\nu \omega(k) (t - \tau) \pm \lambda(k) (Z(t) - Z(\tau)) \right] \]
and where the functions \( \hat{f}_{\pm,00} \) are to be found from the initial conditions. Going back to the parity eigenstates, using the fact that \( \hat{a} = \frac{1}{2} \left( \hat{f}_+ + \hat{f}_- \right) \) and \( \hat{b} = \frac{1}{2} \left( \hat{f}_+ - \hat{f}_- \right) \), we have explicitly
\[ \hat{a}(t, k) = \frac{1}{2} \left[ \hat{f}_{+,00}(k) e^{-2\nu \omega(k) t + \lambda(k) Z(t)} + \hat{f}_{-,00}(k) e^{-2\nu \omega(k) t - \lambda(k) Z(t)} \right] \]
\[ + \frac{1}{2} \int_0^t d\tau \lambda(k) \left[ (\hat{\eta}^-(\tau, k) + \hat{\eta}^+(\tau, k)) e^{-2\nu \omega(k) (t - \tau) + \lambda(k) (Z(t) - Z(\tau))} \right. \]
\[ + \left. (\hat{\eta}^-(\tau, k) - \hat{\eta}^+(\tau, k)) e^{-2\nu \omega(k) (t - \tau) - \lambda(k) (Z(t) - Z(\tau))} \right] \]
\[ \hat{b}(t, k) = \frac{1}{2} \left[ \hat{f}_{+,00}(k) e^{-2\nu \omega(k) t + \lambda(k) Z(t)} - \hat{f}_{-,00}(k) e^{-2\nu \omega(k) t - \lambda(k) Z(t)} \right] \]
\[ + \frac{1}{2} \int_0^t d\tau \lambda(k) \left[ (\hat{\eta}^-(\tau, k) + \hat{\eta}^+(\tau, k)) e^{-2\nu \omega(k) (t - \tau) + \lambda(k) (Z(t) - Z(\tau))} \right. \]
\[ - \left. (\hat{\eta}^-(\tau, k) - \hat{\eta}^+(\tau, k)) e^{-2\nu \omega(k) (t - \tau) - \lambda(k) (Z(t) - Z(\tau))} \right] \]
and a cosine or sine transformation, respectively, will bring back \( a_n(t) \) and \( b_n(t) \). For the chosen initial conditions, we simply have \( \langle a(t, k) \rangle = \langle b(t, k) \rangle = 0 \), which in turn implies \( \langle a_n(t) \rangle = \langle b_n(t) \rangle = 0 \), that is, the interface is always flat on average.

### 3.2. Second model: spherical constraint

The next step in the solution of the model consists of casting the spherical constraint into an equation for \( Z(t) \). To do so, the constraint (2.7) is rewritten in Fourier space
\[ \left\langle \left( \sum_{n=1}^N \left( a_n(t) + b_n(t) \right)^2 \right) \right\rangle = \frac{1}{N} \left\langle \left( \sum_{k=0}^{N-1} \left[ \hat{a}(t, k)\hat{a}(t, k) + \hat{b}(t, k)\hat{b}(t, k) \right] \right) \right\rangle = N. \] (3.8)
The initial conditions must be such that the spherical constraint is respected at \( t = 0 \), hence
\[ \frac{1}{N} \left\langle \sum_{k=0}^{N-1} \left[ \hat{a}(0, k)\hat{a}(0, k) + \hat{b}(0, k)\hat{b}(0, k) \right] \right\rangle = \frac{1}{2N} \left\langle \sum_{k=0}^{N-1} f_{+,00}^2(k) + f_{-,00}^2(k) \right\rangle = N, \] (3.9)
where the solutions (3.6) and (3.7) were used. From the initial conditions (2.6) of initially uncorrelated slopes, we have
\[ \left\langle \hat{f}_{+,00}(k) \right\rangle = 0, \quad \left\langle \hat{f}_{+,00}(k)^2 \right\rangle = N, \quad \left\langle \hat{f}_{+,00}(k)\hat{f}_{-,00}(k') \right\rangle = 0. \] (3.10)
The non-vanishing noise correlators read in Fourier space.
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\[
\langle \tilde{\eta}^\gamma(t, k) \tilde{\eta}^\gamma(t', k') \rangle = N \nu T \delta(t - t') [\delta_{k+k', 0} \pm \delta_{k-k', 0}],
\]

such that the constraint can be re-expressed as follows, for this kind of initial condition:

\[
1 = \frac{1}{2N} \sum_{k=0}^{N-1} \left\{ \left( e^{2\lambda(k)Z(t)} + e^{-2\lambda(k)Z(t)} \right) e^{-4\nu \omega(k)t} + 2\nu T \lambda^2(k) \int_0^t d\tau \ c e^{-4\nu \omega(k)(t-\tau)} \left[ e^{2\lambda(k)(Z(\tau)-Z(t))} + e^{-2\lambda(k)(Z(t)-Z(\tau))} \right] \right\}. \tag{3.12}
\]

The asymptotic analysis of this equation is greatly simplified in the continuum limit, when it takes the form

\[
\int_0^\pi dk \ c \left[ \cosh(2\lambda(k)Z(t)) \ e^{-4\nu \omega(k)t} + 2\nu T \lambda^2(k) \int_0^t d\tau \ c \ e^{-4\nu \omega(k)(t-\tau)} \right] = \pi, \tag{3.13}
\]

where the auxiliary functions (3.3) now stand for their continuum versions \( \omega(k) = 1 - \cos k \) and \( \lambda(k) = \sin k \).

In what follows, we shall require the following identities, with \( a \in \mathbb{N} \):

\[
\begin{align*}
\mathcal{J}_{2a}(A, Z) &:= \frac{1}{\pi} \int_0^\pi dk \ c e^{A \cos k} \cosh(Z \sin k) \ (\sin k)^{2a} = \frac{\partial^{2a} \mathcal{J}_0(A, Z)}{\partial Z^{2a}} = \frac{\partial^{2a} I_0(\sqrt{A^2 + Z^2})}{\partial Z^{2a}}, \\
\mathcal{J}_{2a+1}(A, Z) &:= \frac{1}{\pi} \int_0^\pi dk \ c e^{A \cos k} \sinh(Z \sin k) \ (\sin k)^{2a+1} = \frac{\partial^{2a+1} I_0(\sqrt{A^2 + Z^2})}{\partial Z^{2a+1}}, \tag{3.14}
\end{align*}
\]

which are proven in appendix D and where \( I_0 \) is a modified Bessel function [1]. The constraint (3.13) can be written more compactly as follows:

\[
e^{-4\nu t} \mathcal{J}_0(4\nu t, 2Z(t)) + 2\nu T \int_0^t d\tau \ e^{-4\nu (t-\tau)} \mathcal{J}_2(4\nu (t-\tau), 2Z(t) - 2Z(\tau)) = 1. \tag{3.15}
\]

In contrast to the first Arcetri model [39], or the well-known kinetic spherical models [19, 29, 66], this equation does not take the form of an easily solved Volterra integral equation.

### 3.3. Second model: observables

The observables we are interested in are the two-time correlation and response functions and they shall be defined carefully.

For the correlation function, as the order parameter is the local slope \( u_n(t) \), one might expect that \( u_n(t)u_m(s) \) should describe the two-time temporal–spatial correlator \( C_{n,m}(t, s) \). However, a physically sensible definition of correlators must obey two symmetry conditions. (1) For equal times, the purely spatial correlator \( C_{n,m}(t, t) = C_{m,n}(t, t) \) is symmetric. (2) The two-time autocorrelator \( C_{n,n}(t, s) = C_{n,n}(s, t) \) is symmetric. Therefore, we recall the decomposition \( u_n(t) = a_n(t) + b_n(t) \) into an even and an odd part and adopt the definition\(^{14}\):

\[\tag{3.16}
^14 If we were to consider a complex-valued solution \( u_n(t) = a_n(t) + ib_n(t) \) of the Burgers equation (see section 2), the definition (3.16) would correspond to \( \langle u_n(t)u_n(s) \rangle \).
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\( C_{n,m}(t, s) := \langle \langle a_n(t) a_m(s) \rangle \rangle + \langle \langle b_n(t) b_m(s) \rangle \rangle \).  \hfill (3.16)

Now, using (3.6) and (3.7) together with the cosine and sine transforms, we find that

\[
C_{n,m}(t, s) = \frac{1}{N} \sum_{k=0}^{N-1} \cos \left( \frac{2\pi}{N} k(n - m) \right) \left[ e^{-2\nu \omega(k)(t+s)} \cosh \left( \lambda(k)(Z(t) + Z(s)) \right) + 2\nu T \int_{0}^{\min(t,s)} d\tau \, \lambda^2(k) \, e^{-2\nu \omega(k)(t+s-2\tau)} \cosh \left( \lambda(k)(Z(t) + Z(s) - 2Z(\tau)) \right) \right],
\]

which in the continuum limit \( N \to \infty \) becomes, where \( n,m \) are still considered as integers,

\[
C_{n,m}(t, s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(k(n - m)) \left[ e^{-2\nu (1-\cos(k))(t+s)} \cosh \left( \sin(k)(Z(t) + Z(s)) \right) + 2\nu T \int_{0}^{\min(t,s)} d\tau \, \sin^2(k) \, e^{-2\nu (1-\cos(k))(t+s-2\tau)} \cosh \left( \sin(k)(Z(t) + Z(s) - 2Z(\tau)) \right) \right].
\]  \hfill (3.17)

The required symmetries mentioned above are now obvious. Furthermore, spatial translation-invariance is now manifest and we can write \( C_{n,m}(t, s) = C_{n-m}(t, s) \). A more explicit form is obtained by using the identity, valid for \( n \in \mathbb{N} \) and \( A, Z \in \mathbb{C} \),

\[
\mathcal{C}_n(A, Z) := \frac{1}{\pi} \int_{0}^{\infty} dk \, e^{A \cos k} \cosh(Z \sin k) \cos(nk) = I_n \left( \sqrt{A^2 + Z^2} \right) \cos \left( n \arctan \left( \frac{Z}{A} \right) \right),
\]

which is proven in appendix E. This gives, where, for notational simplicity we let \( t > s \),

\[
C_n(t, s) = e^{-2\nu(t+s)} \mathcal{C}_n(2\nu(t+s), Z(t) + Z(s)) + 2\nu T \int_{0}^{s} d\tau \, e^{-2\nu(t+s-2\tau)} \frac{\partial^2}{\partial Z(t) \partial Z(s)} \mathcal{C}_n(2\nu(t+s-2\tau), Z(t) + Z(s) - 2Z(\tau)).
\]  \hfill (3.19)

In this work, we shall concentrate on the case \( T = 0 \). Then, with (3.19), the two-time slope–slope correlator reads explicitly, in terms of the integrated Lagrange multiplier \( Z(t) \),

\[
C_n(t, s) = e^{-2\nu(t+s)} \mathcal{C}_n(2\nu(t+s), Z(t) + Z(s)) = e^{-2\nu(t+s)} I_n \left( \sqrt{(2\nu(t+s))^2 + (Z(t) + Z(s))^2} \right) \cos \left( n \arctan \left( \frac{Z(t) + Z(s)}{2\nu(t+s)} \right) \right),
\]

and we shall extract its long-time scaling behaviour in the next section, after having found \( Z(t) \) from (3.15). Remarkably, in the long-time limit \( t, s \to \infty \), we shall see that the time–space behaviour simplifies in the sense that the leading term of the correlator

\[
C_n(t, s) \simeq C(t, s) \exp \left( -\frac{n^2}{4\nu(t+s)} \right) \cos \left( n \arctan \left( \frac{Z(t) + Z(s)}{2\nu(t+s)} \right) \right)
\]  \hfill (3.22)

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factorises into the autocorrelator \( C(t,s) = C_0(t,s) \) and a second factor, which alone determines the spatial behaviour.

In order to define the linear response of the order-parameter, here identified with the local slope, a choice must be made for the external perturbation. Here, we consider the effect of a small perturbation \( j_0(t) \) of the slope on the slope itself. In generalising the equations of motion, the external perturbation must be decomposed into the even and odd parts \( j^+_n(t) \) and \( j^-_n(t) \), respectively,

\[
\partial_t a_n(t) = \nu (a_{n+1}(t) + a_{n-1}(t) - 2a_n(t)) + \frac{3(t)}{2} (b_{n+1}(t) - b_{n-1}(t))
\]

\[
+ \frac{1}{2} (\eta_{n+1}(t) - \eta_{n-1}(t)) + j^+_n(t) \tag{3.23}
\]

\[
\partial_t b_n(t) = \nu (b_{n+1}(t) + b_{n-1}(t) - 2b_n(t)) - \frac{3(t)}{2} (a_{n+1}(t) - a_{n-1}(t))
\]

\[
- \frac{1}{2} (\eta_{n+1}(t) - \eta_{n-1}(t)) + j^-_n(t), \tag{3.24}
\]

where \( j^+_n(t) \) and \( j^-_n(t) \), respectively, are the conjugate fields associated with the even and odd parts of the order parameter \( a_n(t) \) and \( b_n(t) \). The solution of these equations follows the same lines that led to (3.6) and (3.7), with the replacements \( \lambda(k)\hat{\eta}^+ \rightarrow \lambda(k)\hat{\eta}^+ + \hat{f}^\pm \).

The response function is defined as

\[
R_{n,m}(t,s) := \left\langle \frac{\delta a_n(t)}{\delta j^+_m(s)} \right|_{j=0} + \left\langle \frac{\delta b_n(t)}{\delta j^-_m(s)} \right|_{j=0} \right\rangle \tag{3.25}
\]

and clearly, only the average over the initial condition (2.6) needs to be taken, with the thermal average becoming trivial. This also implies that the temperature \( T \) does not enter explicitly into the response function. Inserting the explicit solution, we readily find, also writing the causality condition \( t > s \) through the Heaviside function \( \Theta \),

\[
R_{n,m}(t,s) = \frac{\Theta(t-s)}{N} \int_0^t d\tau \sum_{k,\ell,m} e^{2i\nu\omega(k)(t-\tau)} \cosh \left( \lambda(k)(Z(t)-Z(\tau)) \right)
\]

\[
\times \left[ \cos \left( \frac{2\pi}{N} k n \right) \cos \left( \frac{2\pi}{N} k \ell \right) + \sin \left( \frac{2\pi}{N} k n \right) \sin \left( \frac{2\pi}{N} k \ell \right) \right] \delta(\tau-s) \delta_{\ell,m}
\]

\[
= \frac{\Theta(t-s)}{N} \sum_{k=0}^{N-1} e^{2i\nu\omega(k)(t-s)} \cosh \left( \lambda(k)(Z(t)-Z(s)) \right) \cos \left( \frac{2\pi}{N} k(n-m) \right), \tag{3.26}
\]

which is spatially translation-invariant, as it should be, hence \( R_{n,m}(t,s) = R_{n-m}(t,s) \). In the \( N \to \infty \) limit, using (3.19), this simplifies into

\[
R_n(t,s) = \frac{\Theta(t-s)}{\pi} \int_0^\pi dk \, e^{-2\nu(1-\cos k)(t-s)} \cosh \left( (Z(t)-Z(s)) \sin k \right) \cos kn
\]

\[
= \Theta(t-s) \, e^{-2\nu(t-s)} I_n \left( \sqrt{4\nu^2(t-s)^2 + (Z(t)-Z(s))^2} \right) \cos \left( n \arctan \left( \frac{Z(t)-Z(s)}{2\nu(t-s)} \right) \right). \tag{3.27}
\]

In the next section, using (3.15), the asymptotic long-time scaling behaviour will be analysed. Again, we find that for large times, the leading term simplifies.
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\[ R_n(t, s) \simeq R(t, s) \exp \left( -\frac{n^2}{4\nu(t-s)} \right) \cos \left( n \arctan \left( \frac{Z(t) - Z(s)}{2\nu(t-s)} \right) \right) \]  

(3.28)

into a product of the autoresponse \( R(t,s) = R_0(t,s) \) and a second factor, which determines the spatial dependence alone.

Alternatively, one might also consider how the local slope will respond to a small change in the height variable. In this case, it is enough to replace \( \eta \mapsto \eta + j \) in the equations of motion (2.3). The formal definition of the response function is still given by (3.25), although the physical meaning of the external fields \( j_n^\pm(t) \) has changed. The explicit calculation is analogous to the previous ones and we just quote the result:

\[ R_n(t, s) = \frac{\Theta(t-s)}{N} \sum_{k=0}^{N-1} \lambda(k) e^{-2\nu\omega(k)(t-s)} \sinh \left( \lambda(k) (Z(t) - Z(s)) \right) \cos \left( \frac{2\pi kn}{N} \right) \]

\[ = \Theta(t-s) e^{-2\nu(t-s)} \]

\[ \times \frac{\partial}{\partial Z(t)} \left[ I_n \left( \sqrt{4\nu^2(t-s)^2 + (Z(t) - Z(s))^2} \right) \cos \left( n \arctan \left( \frac{Z(t) - Z(s)}{2\nu(t-s)} \right) \right) \right], \]

(3.29)

where we have used (3.19) again. The asymptotic behaviour follows from the one of \( Z(t) \).

### 3.4. Third model

In order to define the observables of the third Arcetri model, rewrite first the (anti-)symmetrised equations of motion in Fourier space

\[ \partial_t \hat{a}(t, k) = -2\nu\omega(k)\hat{a}(t, k) + \mathcal{Z}(t)\lambda(k)\hat{b}(t, k) + \hat{\eta}^+(t, k) \]

\[ \partial_t \hat{b}(t, k) = -2\nu\omega(k)\hat{b}(t, k) + \mathcal{Z}(t)\lambda(k)\hat{a}(t, k) + \hat{\eta}^-(t, k), \]

(3.30)

where we have used the abbreviations (3.3). The solution of these equations reads

\[ \hat{a}(t, k) \]

\[ \hat{b}(t, k) \]

\[ = \frac{1}{2} \left[ \hat{f}_{+,00}(k) e^{-2\nu\omega(k)t+\lambda(k)Z(t)} + \hat{f}_{-,00}(k) e^{-2\nu\omega(k)t-\lambda(k)Z(t)} \right] \]

\[ + \frac{1}{2} \int_0^t d\tau \left[ (\hat{\eta}^+(\tau, k) + \hat{\eta}^- (\tau, k)) e^{-2\nu\omega(k)(t-\tau)+\lambda(k)(Z(t)-Z(\tau))} \right] \]

\[ \pm \left( \hat{\eta}^+(\tau, k) - \hat{\eta}^-(\tau, k) \right) e^{-2\nu\omega(k)(t-\tau)-\lambda(k)(Z(t)-Z(\tau))} \right], \]

(3.31)

where the upper signs correspond to \( \hat{a} \) and the lower signs correspond to \( \hat{b} \). The spherical constraint is now given by equation (2.12) and takes the form

\[ \frac{1}{N} \left\langle \sum_{k=0}^{N-1} \lambda(k)^2 \left[ \hat{a}(t, k)\hat{a}(t, k) + \hat{b}(t, k)\hat{b}(t, k) \right] \right\rangle = N. \]

(3.32)

This can be evaluated along the same lines as before. We merely quote the end result, in the continuum limit.
e^{-4\nu t} \mathcal{J}_2(4\nu t, 2Z(t)) + 2\nu T \int_0^t d\tau \ e^{-4\nu(t-\tau)} \mathcal{J}_2(4\nu(t-\tau), 2Z(t) - 2Z(\tau)) = 1. \quad (3.33)

If \( T = 0 \), the conservation laws (2.8) obtained for the second model also hold for the third. This implies a constant height profile \( \langle h(t, r) \rangle = H_0 \), where for simplicity we used the initial conditions (2.11). From now on, we set \( H_0 = 0 \) in (2.11), without restriction to the generality.

The time–space correlator is defined analogously to the way it was in the second model, by equation (3.16), but now using the decomposition \( h_n(t) = a_n(t) + b_n(t) \) of the height in an even and an odd part. Although we reuse the formal definition, equation (3.16), the physical interpretation is now a different one and gives a height–height correlator. The main computational difference with respect to the second model is the absence of the factor \( \lambda(k) \) before the thermal noise \( \hat{\eta}^\pm \). For the initial conditions (2.11), spatial translation-invariance holds for all times \( t, s > 0 \), so that we can write \( C_{n-m}(t, s) = C_{n,m}(t, s) \). We finally have, using (3.15),

\[
C_n(t, s) = e^{-2\nu(t+s)} \mathcal{C}_n(2\nu(t+s), Z(t) + Z(s)) + 2\nu T \int_0^{\min(t,s)} d\tau \ e^{-2\nu(t+s-2\tau)} \mathcal{C}_n(2\nu(t+s-2\tau), Z(t) + Z(s) - 2Z(\tau)).
\]

In particular, for \( T = 0 \), we recover the same abstract expression, equation (3.21), as for the slope–slope correlator \( C_n^{(ff)}(t, s) \), but with the difference that \( Z(t) \) now has to be found from the spherical constraint (3.33).

For the calculation of the linear response of the height with respect to a small change in the height, one replaces \( \hat{\eta}^\pm \mapsto \hat{\eta}^\pm + \tilde{\eta}^\pm \) in the equation of motion (3.31). We can simply reuse the definition (3.25) and formally recover the abstract form

\[
R_n(t, s) = \Theta(t - s) \ e^{-2\nu(t-s)} \times I_n \left( \sqrt{4\nu^2(t-s)^2 + (Z(t) - Z(s))^2} \right) \cos \left( n \arctan \left( \frac{Z(t) - Z(s)}{2\nu(t-s)} \right) \right) \quad (3.35)
\]

identical to (3.27). In particular, in the long-time limit and for \( T = 0 \), both the height correlator and the height response factorise, as in (3.22) and (3.28), respectively. Again, \( Z(t) \) is found from (3.33).

4. Long-time behaviour

4.1. Spherical constraint at \( T = 0 \)

First, we have to determine the long-time behaviour of the Lagrange multiplier \( Z(t) \) from the spherical constraints, equations (3.15) and (3.33). From now on, we shall restrict ourselves to the case \( T = 0 \).

For the second model, equation (3.15) reduces to

\[
e^{-4\nu t} I_0 \left( 4\nu \sqrt{1 + \left( \frac{Z(t)}{2\nu t} \right)^2} \right) = 1. \quad (4.1)
\]
In appendix B, we shall show that for large times \( Z(t) \simeq \sqrt{\nu t \ln(8\pi \nu t)} \).

For the third model, we must solve equation (3.33). In appendix B it is shown that this is equivalent to

\[
\begin{align*}
&\left[ I_1 \left( \sqrt{(4\nu t)^2 + (2Z(t))^2} \right) (4\nu t)^2 \\
&+ 2Z(t)^2 \sqrt{(4\nu t)^2 + (2Z(t))^2} \left( I_0 \left( \sqrt{(4\nu t)^2 + (2Z(t))^2} \right) + I_2 \left( \sqrt{(4\nu t)^2 + (2Z(t))^2} \right) \right) \right] \\
&= e^{4\nu t} \left[ (4\nu t)^2 + (2Z(t))^2 \right]^{-3/2}.
\end{align*}
\] (4.2)

To leading order, this would give the solution \( Z(t) \simeq \sqrt{3\nu t \ln((32\pi e)^{1/3} \nu t)} \). Then, this could be combined with the second model as follows: for \( t \) large enough, we have

\[
Z(t) = \sqrt{\kappa_1 \nu t \ln(\kappa_2 t)}, \quad \begin{cases} 
\kappa_1 = 1, & \text{second model} \\
\kappa_1 = 3, & \text{third model}
\end{cases}
\] \( \kappa_2 = 8\pi \nu \) second model \( \kappa_2 = (32\pi e)^{1/3} \nu \) third model (4.3)

However, as is also shown in appendix B, it is advisable to include the next-to-leading terms as well. If that is done, we find, for the third model and \( t \) large enough

\[
Z(t) \simeq 2tW( (32\pi e)^{1/2} t^{3/2} ) - t \simeq \sqrt{3t \ln(\kappa_2 t)} \left[ 1 - \frac{2 \ln(\kappa_2 t)}{3 \ln(\kappa_2 t)} - \frac{1}{3} \ln(\kappa_2 t) \right] (4.4)
\]

where \( W(x) \) denotes Lambert’s \( W \) function \([16, 54]\). Results (4.3) and (4.4) for the third model are the basis for the entire asymptotic analysis. Formally, for truly enormous times \( t \gg 1 \), the distinction between the second and the third model merely comes from the values of the two constants \( \kappa_{1,2} \), as given in (4.3), and both models can be analysed together. However, the further logarithmic corrections in (4.4) will lead to some significant differences between the second and the third model, even for \( T = 0 \), as we shall now see. In the main text, we merely quote our results and refer to appendices B and C for the calculations.

### 4.2. Zero-temperature correlators

We now turn to the correlators. For a vanishing temperature \( T = 0 \), we have already shown in (3.21) and (3.22) that the time–space correlator \( C_n(t,s) \) factorises into the autocorrelator \( C(t,s) \) and a space-dependent part. The autocorrelator takes the form

\[
C(t,s) = e^{-2\nu(t+s)} I_0 \left( 2\nu(t+s) \sqrt{1 + \left( \frac{Z(t) + Z(s)}{2\nu(t+s)} \right)^2} \right), (4.5)
\]

where \( Z(t) \) is given by (4.3). This autocorrelator does not obey the scaling of simple ageing, where \( t, s \to \infty \) and \( y = t/s > 1 \) is kept fixed. Rather we must consider a different scaling behaviour, where again \( t, s \to \infty \), such that a certain scaling variable \( y \) is kept fixed. We find two possibilities.

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1. The time difference is given by
\[ \tau = t - s = (y - 1) \frac{s}{\ln^{1/2}(\kappa_2 s)}. \] (4.6)

and we have the scaling form
\[ C(t, s) = C^{(0)}(\ln s)^{(\kappa_1 - 1 - \delta_{s,1,3})/2} \exp \left( - \frac{\kappa_1}{32} (y - 1)^2 \right) \] (4.7)

with the constant \( C^{(0)} = \sqrt{\kappa_2^{\pi}}/(8\pi \nu) \) for the second model and \( C^{(0)} = \sqrt{\kappa_2^{\pi}} \ e^{1/2}/(12\pi \nu) \) for the third model. The scaling (4.6) and (4.7) was seen before in the phase-separation kinetics of the spherical model, at temperature \( T = 0 \), with a conserved order parameter (model B dynamics) [9].

2. The time difference is given by, with \( \vartheta > \frac{1}{2} \),
\[ \tau = t - s = \frac{s}{\ln^{1/2}(\kappa_2 s)} \sqrt{W \left( (y - 1)^2 \ln^{1-2\vartheta} (\kappa_2 s) \right)}, \] (4.8)

where \( W(x) \) is, again, Lambert’s \( W \) function. For \( s \to \infty \), this gives \( \tau \sim (y - 1)s \ln^{-\vartheta}(\kappa_2 s) \). The autocorrelator becomes
\[ C(t, s) = C^{(0)}(\ln(\kappa_2 s))^{(\kappa_1 - 1 - \delta_{s,1,3})/2}. \] (4.9)

As we shall see, we must further distinguish the cases \( \frac{1}{2} < \vartheta < 1 \) and \( \vartheta > 1 \) here.

A scaling behaviour according to (4.6) and (4.8) corresponds to logarithmic sub-ageing, since the time difference \( \tau = t - s \) grows more slowly than \( s \) by a logarithmic factor. Although the forms (4.6) and (4.8) are distinct from simple ageing as described in section 1, we shall cast the autocorrelator into a scaling form \( C(t, s) = s^{-b} \ln^{-b} s f_C(y) \) and check for an asymptotic form \( f_C(y) \sim y^{-\lambda_C/z} \ln^{\lambda_C/z} s \) as \( y \to \infty \). If these forms apply, the exponents \( b, \lambda_C \) are quoted in tables 1 and 2. By analogy with equilibrium critical phenomena, we also introduce the logarithmic sub-scaling exponents \( \hat{b}, \hat{\lambda}_C \) [49]. We find that \( \hat{b} = \hat{\lambda}_C = 0 \) for the second Arcetri model and \( \hat{b} = -\frac{1}{2}, \hat{\lambda}_C = 0 \) for the third Arcetri model.

Concerning the equal-time correlator \( C(t, t) = (\ln(\kappa_2 t))^{(\kappa_1 - 1 - \delta_{s,1,3})/2} \) in both scaling regimes, there is a difference in interpretation between the second and third models. In the second model, \( C(t, t) = 1 \) because of constraint (4.1), which is consistent with a probabilistic interpretation, either in terms of the slopes or else in terms of particles and holes. Indeed, one has \( \kappa_1 = 1 \) in this case. For the third model, \( C(t, t) = w^2(t) = (\ln(\kappa_2 t))^{1/2} \) is simply the square of the interface width \( w(t) \sim t^\beta \ln^2 t \). Hence for \( d = 1 \), where \( \kappa_1 = 3 \), the interface is logarithmically rough, with growth exponents \( \beta = 0 \) and \( \hat{\beta} = \frac{1}{2} \).

15 The notation is chosen such that one returns to simple ageing if the logarithmic factors are dropped.
16 Simple inequalities exclude the opposite case of super-ageing, where \( t - s \) would grow faster than \( s \) [53].
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The time–space-dependent slope–slope correlator $C_n(t,s)$ is given by (3.22). This yields the following long-time behaviour:

$$C_n(t, s) \simeq C(t, s) \exp\left(-\frac{1}{2} \left(\frac{n}{\sqrt{2\nu(t + s)}}\right)^2\right) \cos\left(\frac{n}{\sqrt{2\nu(t + s)}}\sqrt{\frac{\kappa_1}{2}(\ln \kappa_2 t + \ln \kappa_2 s)}\right).$$

In particular, this gives the equal-time correlator

$$C_n(t) := C_n(t, t) = e^{-4\nu t} I_n\left(4\nu t \sqrt{1 + \left(\frac{Z(t)}{2\nu t}\right)^2}\right) \cos\left(n \arctan\frac{Z(t)}{2\nu t}\right)$$

$$\Rightarrow C_0(t, t) \exp\left(-\left(\frac{n}{L_1(t)}\right)^2\right) \cos\left(\frac{n}{L_2(t)}\right) \quad (4.10)$$

with the equal-time autocorrelator $C_0(t) := C_0(t, t)$ and the two distinct length scales

$$L_1(t) := \sqrt{8\nu t} \quad , \quad L_2(t) := \sqrt{\frac{4\nu}{\kappa_1 \ln(\kappa_3 t) F(t)}} \quad (4.11)$$

where $F(t)$ is defined in appendix B and gives double-logarithmic corrections to scaling for the third model. The presence of two logarithmically different length scales indicates a breaking of dynamical scaling. This becomes even more explicit when considering the structure factor

$$S(t, k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dn \ e^{-i kn} C_n(t)$$

$$= C_0(t) L_1(t) \frac{1}{\sqrt{2}} \exp\left(-\frac{1}{4} L_1^2(t)\right) \exp\left(-L_1^2(t) k^2\right) \cosh\left(\frac{L_1(t) L_1(t) k}{2L_2(t)}\right)$$

$$\Rightarrow C_0(t) L_1(t) \frac{1}{\sqrt{2}} \left(\frac{\kappa_2 t}{\kappa_1}\right)^{\kappa_1/2} e^{-L_1(t)(k/2)^2} \cosh\left(\frac{L_1(t) L_1(t) k}{2L_2(t)}\right) \left\{\begin{array}{ll}
1 & \text{; second model} \\
\frac{3}{2} e^{\ln \kappa_2 t} & \text{; third model}
\end{array}\right. \quad (4.12)$$

where the last logarithmic factor comes from the auxiliary function $F(t)$. Working out the long-time behaviour of the two lengths, we find, for the second and third model, respectively

$$S^{(II)}(t, k) = S^{(0)}(t, k) e^{-2\nu t k^2} \cosh\left(\sqrt{4\nu t \ln 8\pi \nu t} \ k\right) \quad (4.14a)$$

$$S^{(III)}(t, k) = S^{(0)}(t, k) e^{-2\nu t k^2} \left(\frac{\ln^{3/2} \kappa_2 t}{t}\right) \cosh\left(k \sqrt{4\nu t \left(3\ln \kappa_2 t - 2\ln(\ln \kappa_2 t) - \left(1 + 2\ln \frac{3}{2}\right)\right)}\right) \quad (4.14b)$$

with known constants $S^{(0)}$ and $\kappa_2$ as defined above.

The explicit expressions (4.14) permit a clear understanding of the distinct length scales involved. Since for both models the structure factor contains two factors with different $k$-dependence, one expects a peak at some time-dependent lieu $k_m(t)$. Figure 2 shows that this is indeed the case, notably in the inset of figure 2(b), which illustrates

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how for increasing times the peak becomes sharper and is progressively shifted towards $k \to 0$. Equation (4.13) implies that $k_m(t) \simeq L_2^{-1}(t) \sim \left(\frac{\ln t}{t}\right)^{1/2}$. If one attempts to scale the structure factors with respect to $L_2(t)$, as is done in figures 2(a) and (b), one might believe at first sight that a scaling behaviour would result (at least for the second model). However, the presence of the diffusive length $L_1(t) \sim t^{1/2}$ means that it is impossible to achieve a collapse and dynamical scaling does not hold for all wave numbers $k \geq 0$. This kind of behaviour is completely analogous to that of phase separation in the $T = 0$ kinetic spherical model with a conserved order parameter (model B dynamics) [5, 14]. These two length scales also describe the time–space correlator: while $L_2$ is seen in the spatial modulation, the scale $L_1$ describes the overall spatial decay.

The scaling of the autocorrelator introduces at least one more scale, $L_{\text{corr}}(t) \sim (t/\ln^\vartheta t)^{1/2}$, with $\vartheta \geq 1/2$. Several distinct regimes must be distinguished, as we shall see in section 5.

### 4.3. Zero-temperature response

From (3.27) and (3.28), we have for large times the factorisation into the autoresponse

$$R(t, s) = e^{-2\nu(t-s)} I_0 \left( 2\nu(t-s) \sqrt{1 + \left( \frac{Z(t) - Z(s)}{2\nu(t-s)} \right)^2} \right)$$

(4.15)

and the time–space response

$$R_n(t, s) = e^{-2\nu(t-s)} I_n \left( 2\nu(t-s) \sqrt{1 + \left( \frac{Z(t) - Z(s)}{2\nu(t-s)} \right)^2} \cos \left( n \arctan \frac{Z(t) - Z(s)}{2\nu(t-s)} \right) \right)$$

$$\simeq R(t, s) \exp \left[ -\frac{1}{2} \left( \frac{n}{\sqrt{2\nu(t-s)}} \right)^2 \right].$$

(4.16)

The scaling is obtained as follows and corresponds to (4.8). We have for the time difference

$$\tau = t - s = \frac{s}{\ln(\kappa_2 s)} W \left( (y - 1) \ln^{1-\vartheta}(\kappa_2 s) \right),$$

(4.17)

where $\vartheta > 1$, in terms of Lambert’s $W$ function. This gives $\tau \sim (y - 1) s \ln^{-\vartheta}(\kappa_2 s)$ for large times. The autoresponse (3.27) takes the scaling form

$$R(t, s) = (4\pi \nu s)^{-1/2} \ln^{\vartheta/2}(\kappa_2 s) \left( y - 1 \right)^{-1/2},$$

(4.18)

which is so close to the one found in systems undergoing simple ageing (as in the first Arcetri model, see appendix A), up to a logarithmic prefactor, that we read off the exponents $a = -\frac{1}{2}$ and $\lambda_R/z = \frac{1}{2}$ (see table 1). There is no correspondence in the autoresponse to the scaling (4.6) of the autocorrelator.

\[\text{https://doi.org/10.1088/1742-5468/aa9a53}\]
We also observe that the response $R_n(t, s)$ defined in (3.29) has the same factorisation into the autoresponse $R(t, s)$ and a spatial part as in (3.28). The autoresponse is readily found and reads, in the scaling limit with $\vartheta > 1$,

$$R(t, s) = \left(8\pi^{1/2}\nu s\right)^{-1} \ln^{(1+\vartheta)/2}(\kappa_2 s) (y - 1)^{-1/2},$$

and we read off $a_R = 0$ and $\lambda_R = 1$.

In the time–space responses, we merely find a single further length scale $L_{\text{diff}}(t) \sim t^{1/2}$, which describes the overall decay, but no spatial modulation of the response.

5. Discussion and perspectives

Triggered by an analogy with the spherical model of a ferromagnet [7], we have used the Burgers and Kardar–Parisi–Zhang equations to define two new models, which we have called the second Arcetri model and the third Arcetri model (see sections 2.2 and 2.3). Because of the natural initial conditions (2.6) and (2.11), respectively, the second model is interpreted as a lattice gas model, whereas the third model appears to describe a growing interface. At vanishing ‘temperature’ $T = 0$, we have found the exact two-time correlators and responses. Unexpectedly, the scaling behaviour of these turned out not to be given by simple ageing, but rather by a subtle modification of this, described by logarithmic sub-ageing and characterised by the presence of several logarithmically distinct time-dependent length scales.

As we shall now see, this is a very fortunate circumstance, since the separation of several length scales, which coalesce for simple ageing, allows for a much clearer understanding of how the ageing process takes place. Figure 3 summarises this as a kinetic phase diagram.

Equations in physics should be read as instructions for how to carry out experiments. In the case of ageing, the central equation describes the scaling of the difference between observation time $t$ and waiting time $s$.

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\[ \tau = t - s = \frac{s}{\ln \vartheta} f(s, y) \underset{s \to \infty}{\sim} \frac{s}{\ln \vartheta} (y - 1) \]  

(5.1)
in terms of a certain function \( f = f(s, y) \) that we determined explicitly for the second and third Arcetri models in section 4. In combination with the scaling form of the autocorrelator \( C = C(t, s) \) or the autoresponse \( R = R(t, s) \), the meaning of equation (5.1) is:

‘For a large waiting time \( s \), and a fixed scaling variable \( y > 1 \), compute the observation time scale \( t = s + \tau(s, y) \) in order to observe the corresponding scaling behaviour of the observable in question’.

For systems with logarithmic sub-ageing, as realised in the second and third Arcetri models at \( T = 0 \), this leads to the following insights.

1. Phase I is characterised by an exponent \( \vartheta > 1 \) in (5.1)\(^{19} \). Both the correlators and the responses scale, and the asymptotic forms of the scaling variables are compatible. The scaling functions are given by equations (4.9) and (4.18), respectively, and the corresponding exponents are listed in tables 1 and 2. These relatively small time differences also imply a corresponding length scale \( L_{\text{corr}}(t) \sim \sqrt{t/\ln \vartheta} \).

On these time and space scales, the autocorrelation is perfect and the system is spatially homogeneous. The most rapid events occur in the slow decay of the response to an external, localised perturbation.

2. The end of phase I is seen when one goes to larger scales by choosing \( \vartheta = 1 \). At this length scale, which seen from phase I would correspond to a \( y \to \infty \) limit, the responses have decayed. The new feature is the onset of the spatial modulation of the spatial correlators (4.11), which occurs at the scale \( L_{\text{max}} \sim \sqrt{t/\ln t} \), and is signalled by a strong peak in the structure function at wave number \( k_{\text{max}} \) (see figure 3). Since the scale \( L_{\text{diff}}(t) \gg L_{\text{max}}(t) \), these modulations occur with constant amplitude. And because the scaling form (4.8) of the autocorrelator still holds, autocorrelations do not dissipate away.

3. Phase II is characterised by the intermediate range \( \frac{1}{2} < \vartheta < 1 \) between the onset of spatial modulations and a new scaling form of the autocorrelator, which in this scaling limit is still perfect. In contrast to phase I, at this time scale the responses have decayed away and no longer scale.

4. The end of phase II is seen at large length scales \( L_{\text{corr}}(t) \), which correspond to \( \vartheta = \frac{1}{2} \). At these scales, the system consists of many small spatial units, each of them still fully ordered. Now the autocorrelator decays according to the scaling form (4.6), but since it is still the case that \( L_{\text{diff}} \gg L_{\text{corr}}(t) \), each spatial unit remains fully ordered. The responses to external perturbations are vanishingly small on these scales.

5. On scales so large that they correspond to \( 0 < \vartheta < \frac{1}{2} \), there is no more scaling form and temporal correlations are lost. The eventual decay of the amplitude of

\(^{19}\) The precise scaling prescriptions are given by (4.8) and (4.17), see also figure C1.
spatial correlations at distances corresponding to $\vartheta = 0$ has no effect on a system that at these scales is already disordered.

A completely analogous behaviour can be found in the kinetic spherical model with a conserved order parameter (model B dynamics) at temperature $T = 0$, which describes spinodal decomposition, in $d > 2$ dimensions. It has a dynamical exponent $z = 4$, and hence is distinct from the Arcetri models. However, the breaking of the scale-invariance of the equal-time spin–spin correlator through two logarithmically distinct length scales is analogous to \[(4.11)\] \[14\]. Furthermore, the scaling of the magnetic autocorrelator $C(t, s)$ and of the magnetic autoresponse $R(t, s)$ \[9\] is completely analogous to the Arcetri models. The scaling variable is, again, defined by $\tau = t - s = (y - 1)s \ln \vartheta s$. If $\vartheta > 1$ for sufficiently large $s$, we find that the autocorrelator $C(t, s) = 1$ and the autoresponse $R(t, s) \sim s^{-(d+2)/4} \left( \ln \vartheta (d+2)/4 \right)^{-1} (y - 1)^{-1}$. If $\vartheta = 1$, one still has $C(t, s) \sim \exp \left( \frac{-d}{64} (y - 1)^2 \ln (1-2\vartheta) \right)^{s^{-\infty}}$ and the autoresponse does not scale. Finally, for $\vartheta = \frac{1}{2}$, one has $C(t, s) \sim \exp \left( -\frac{d}{64} (y - 1)^2 \right)$ \[9\]. See also table 2.

Logarithmic sub-ageing might also occur in the ageing of glassy materials. It is common to fit experimental data on relaxation phenomena in glasses through sub-ageing, with a sub-ageing exponent $\mu \lesssim 1$ (see \[71, 78\] and references therein), which in practice may become very difficult to distinguish from logarithmic sub-ageing \[9\].

The presence of two logarithmically different length scales is called multiscaling \[14\]. In O($n$)-symmetric magnets with a conserved order parameter, quenched to temperature $T = 0$, multiscaling only occurs exactly in the spherical limit $n = \infty$ but does not arise for $n$ finite \[5, 6, 58\]. It appears to be plausible that a similar effect may also arise for the second and third Arcetri models at $T = 0$, in view of the conservation law \[(2.8)\]. It has been argued that multiscaling is present in systems such as diffusion-limited aggregation (DLA) \[15\], but apparently no consensus has been reached on whether multiscaling in DLA is genuine \[60\] or rather an effective finite-size effect \[59, 70\].

A slightly different situation arises in the quantum dynamics of the quantum spherical model, where the associated Lindblad equation preserves the canonical commutator relations, which might be viewed as an (infinite) set of prescribed conservation laws. Then, at $T = 0$ and for quantum quenches deep into the two-phase coexistence region, a simple scaling behaviour without logarithmic corrections is found for $d = 2$ dimensions, but multiscaling arises for $d \neq 2$ \[79\].

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Our discussion has been restricted to the special case of $T = 0$. The solution of the spherical constraint for $T > 0$ is left as an open problem. Since the conservation laws (2.8) of the second model are maintained for $T \neq 0$, one might anticipate that a sufficiently small change in $T$ should not lead to a drastic modification of the qualitative behaviour of the second model. In the third model, however, for any $T > 0$ the conservation laws (2.8) are broken and as a consequence the behaviour of the model should change notably. Another question that is left open is the extension to any dimension $d$.

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**Appendix A. The first Arcetri model revisited**

We recall and extend the so-called ‘first’ Arcetri model, as originally introduced in [39], in order to clarify the possible interpretations, as either a model of interface growth or interacting particles. For brevity of notation, we restrict ourselves to $d = 1$ dimensions.

A model of growing interfaces is naturally described in terms of the height $h_n(t)$ at the site $n$ of a periodic chain with $N$ sites. The defining equation of motion of the first Arcetri model for the heights (Arcetri 1n) is, along with the spherical constraint on the slopes,

$$\partial_t h_n(t) = \nu (h_{n+1}(t) + h_{n-1}(t) - 2h_n(t)) + \tilde{z}(t)h_n(t) + \eta_n(t)$$

(A.1)

$$\frac{1}{4} \sum_{n=0}^{N-1} \langle\langle (h_{n+1}(t) - h_{n-1}(t))^2 \rangle\rangle = N.$$  

(A.2)

Herein, to each lattice site $n$ a centred Gaussian random variable $\eta_n(t)$ is attached, with variance $\langle\langle \eta_n(t)\eta_m(t') \rangle\rangle = 2\nu T \delta(t - t') \delta_{n,m}$. The Lagrange multiplier $\tilde{z}(t)$ is determined from the spherical constraint (A.2) and $\nu$ and $T$ are constants. A natural initial condition stipulates an initial Gaussian distribution, with spatially translation-invariant moments

$$\langle\langle h_n(0) \rangle\rangle = H_0, \quad \langle\langle h_n(0)h_m(0) \rangle\rangle - \langle\langle h_n(0) \rangle\rangle \langle\langle h_m(0) \rangle\rangle = H_1 \delta_{n,m}.$$  

(A.3)

This describes an initially flat interface of uncorrelated heights and initial mean height $H_0$.

If one rewrites the equation of motion (A.1) in terms of the slopes

$$u_n(t) := \frac{1}{2} (h_{n+1}(t) - h_{n-1}(t)),$$ 

one has

$$\partial_t u_n(t) = \nu (u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)) + \tilde{z}(t)u_n(t) + \frac{1}{2} (\eta_{n+1}(t) - \eta_{n-1}(t))$$

(A.4)

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which together with the initial conditions (A.3), with \( H_0 = 0 \), was the only model studied in [39]. The formal continuum limit of (A.4) is given by (1.4). The first Arcetri model for the particles (or slopes) (ARCETRI 1t) has the defining equation of motion (A.4), the spherical constraint (A.5) and an initial Gaussian distribution, with the moments
\[
\langle \langle u_n(0) \rangle \rangle = 0, \quad \langle \langle u_n(0) u_m(0) \rangle \rangle = U_1 \delta_{n,m}.
\]
This initial condition with zero average slope corresponds to a system of initially uncorrelated particles with average mean density \( \rho = \frac{1}{2} \).

The solution of the equations of motion is standard, see [39]. Let \( g(t) := \exp \left( -2 \int_0^t d\tau \beta(\tau) \right) \), and also define
\[
f(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} d\nu \sin^2 \nu \exp \left( -4\nu (1-\cos \nu)t \right) = \frac{\exp(4\nu t) I_1(4\nu t)}{4\nu t},
\]
\[
F(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} d\nu \exp \left( -4\nu (1-\cos \nu)t \right) = \exp(4\nu t) I_0(4\nu t),
\]
where the \( I_n \) are modified Bessel functions [1]. For \( d \geq 1 \) dimensions, these functions become [39]
\[
f(t) = d \frac{\exp(4\nu t) I_1(4\nu t)}{4\nu t} \left( \exp(4\nu t) I_0(4\nu t) \right)^{d-1}, \quad F(t) = \left( \exp(4\nu t) I_0(4\nu t) \right)^{d}.
\]
The spherical constraints (A.2) and (A.5) reduce to the Volterra integral equations
\[
\begin{cases}
g(t) = H_1 f(t) + 2\nu T \int_0^t d\tau g(\tau) f(t-\tau) & \text{for ARCETRI 1H} \\
g(t) = U_1 F(t) + 2\nu T \int_0^t d\tau g(\tau) f(t-\tau) & \text{for ARCETRI 1U },
\end{cases}
\]
which immediately gives for the Laplace transformation \( \mathcal{G}(p) = \int_0^\infty dt \ e^{-pt} g(t) \)
\[
\mathcal{G}(p) = \begin{cases}
H_1 \mathcal{F}(p)/\left[ 1 - 2\nu T \mathcal{F}(p) \right] & \text{for ARCETRI 1H} \\
U_1 \mathcal{F}(p)/\left[ 1 - 2\nu T \mathcal{F}(p) \right] & \text{for ARCETRI 1U },
\end{cases}
\]
such that the small-\( p \) behaviour of \( \mathcal{G}(p) \) is related by a Tauberian theorem to the long-time asymptotics of \( g(t) \) for \( t \to \infty \) [25]. This gives two distinct physical situations.

(a) If an interpretation in terms of interface growth is sought, one may consider the ARCETRI 1H model, characterised by initially uncorrelated height variables according to (A.3). Then the time-dependent height is
\[
\langle \langle h_n(t) \rangle \rangle = H_0 \ g(t)^{-1/2}.
\]
This average is indeed non-vanishing, since the equation of motion (A.1) is not invariant under the transformation \( h_n(t) \mapsto h_n(t) + \alpha \). The two-time autocorrelator is given by
Exactly solvable models of growing interfaces and lattice gases: the Arcetri models, ageing and logarithmic sub-ageing

\[
C(t, s) := \langle \langle (h_n(t) - \langle h_n(t) \rangle) (h_n(s) - \langle h_n(s) \rangle) \rangle \rangle = H_1 F((t + s)/2) + 2\nu T \int_0^{\min(t,s)} d\tau \frac{g(\tau)}{\sqrt{g(t)g(s)}} F\left(\frac{t + s - \tau}{2}\right),
\]

(A.12)
such that the interface width becomes

\[
w^2(t) = C(t, t) = H_1 F(t) + 2\nu T \int_0^t d\tau \frac{g(\tau)}{g(t)} F(t - \tau).
\]

(A.13)

Finally, the linear autoresponse of the height \( h_n(t) \) with respect to a change \( h_n(s) \rightarrow h_n(s) + j_n(s) \) in the height is independent of the initial distribution and reads

\[
R(t, s) := \left. \frac{\delta \langle h_n(t) \rangle}{\delta j_n(s)} \right|_{j=0} = \Theta(t - s) \sqrt{\frac{g(s)}{g(t)}} F\left(\frac{t - s}{2}\right),
\]

(A.14)

where the Heaviside function \( \Theta(t) \) expresses causality. Equations (A.12) and (A.14) are the analogues of (3.34) and (3.35) in the third model for \( n = 0 \).

(b) If a comparison with the 1D TASEP is sought, one may consider the Arcetri 1V model, characterised by initially uncorrelated slopes and described by (A.6). The slope–slope auto-correlator \( C(t, s) \), related to the connected density-density correlator via (1.13), the linear auto-response \( R(t, s) \) of the slope \( u_n(t) \) with respect to a change \( u_n(s) \rightarrow u_n(s) + j_n(s) \) in the slope, and the linear auto-response \( \mathcal{R}(t, s) \) of the slope \( u_n(t) \) with respect to a change \( j_n(s) \) in the height, respectively, are given by

\[
C(t, s) := \langle \langle u_n(t)u_n(s) \rangle \rangle = \frac{U_1 F((t + s)/2)}{\sqrt{g(t)g(s)}} + 2\nu T \int_0^{\min(t,s)} d\tau \frac{g(\tau)}{\sqrt{g(t)g(s)}} f\left(\frac{t + s - \tau}{2}\right)
\]

\[
R(t, s) := \left. \frac{\delta \langle u_n(t) \rangle}{\delta k_n(s)} \right|_{k=0} = \Theta(t - s) \sqrt{\frac{g(s)}{g(t)}} f\left(\frac{t - s}{2}\right)
\]

\[
\mathcal{R}(t, s) := \left. \frac{\delta \langle u_n(t) \rangle}{\delta j_n(s)} \right|_{j=0} = \Theta(t - s) \sqrt{\frac{g(s)}{g(t)}} f\left(\frac{t - s}{2}\right).
\]

(A.15)

Equation (A.15) gives the analogues of (3.21), (3.27) and (3.29) in the second model.

Following the analysis in [39], the critical exponents are readily found and are listed in tables 1 and 2.

21 In the KPZ universality class, the KPZ ansatz stipulates that \( \langle \langle h_n(t) \rangle \rangle - v_m t \sim t^\beta \) and \( w(t) \sim t^\beta \) scale with the same exponent \( \beta \) [66], and where \( v_m \) is the mean velocity of particle deposition (one may achieve \( v_m = 0 \) by the choice of a co-moving frame of reference, implicit in (1.1)). The KPZ ansatz is satisfied by the Arcetri 1H model at \( T = T_c \), but does not hold for \( T < T_c \). In the third Arcetri model at \( T = 0 \), the KPZ ansatz is broken through logarithmic sub-scaling exponents, see section 4.
Appendix B. Long-time correlator

We derive the long-time behaviour of the correlations in the second and third models at \( T = 0 \).

In what follows, we shall often need the following asymptotic formula [1, 69]:

\[
I_n(x) \simeq \frac{1}{\sqrt{2\pi x}} \exp \left( x - \frac{4n^2-1}{8x} \right) \left( 1 + O(x^{-2}) \right).
\]  

(B.1)

First, we must find \( Z(t) \) for large times from constraints (3.15) or (3.33), respectively. We now prove equation (4.3) in the main text.

For the second model, (3.15) becomes (4.1). Since the Bessel function \( I_0(x) \) increases monotonically with \( x > 0 \), this implies that \( Z(t) \) increases with \( t > 0 \). Then, one can apply (B.1) and one has

\[
e^{4\nu t} = I_0 \left( 4\nu t \sqrt{1 + \frac{Z^2(t)}{4\nu^2 t^2}} \right) \simeq \frac{\exp 4\nu t \sqrt{1 + \frac{Z^2(t)}{4\nu^2 t^2}}}{8\pi \nu t \sqrt{1 + \frac{Z^2(t)}{4\nu^2 t^2}}}^{1/2}
\]

\[
\simeq \exp \left[ 4\nu t \left( 1 + \frac{Z^2(t)}{8\nu^2 t^2} \right) - \frac{1}{2} \ln(8\pi \nu t) - \frac{1}{2} \ln \left( 1 + \frac{Z^2(t)}{8\nu^2 t^2} \right) \right].
\]

Keeping the terms of leading non-vanishing order gives a linear equation for \( Z^2(t) \)

\[
\frac{Z^2(t)}{2\nu t} - \frac{1}{2} \ln(8\pi \nu t) = 0
\]

that is equivalent to the first equation (4.3) in the main text. One might also obtain this result from the integral representation of \( I_0 \) using the saddle-point method.

For the third model, constraint (3.33), using appendix D and standard formulae for the modified Bessel function [1], takes the form

\[
\left[ I_1 \left( \sqrt{(4\nu t)^2 + (2Z(t))^2} \right) (4\nu t)^2 + 2Z(t)^2 \sqrt{(4\nu t)^2 + (2Z(t))^2} \left( I_0 \left( \sqrt{(4\nu t)^2 + (2Z(t))^2} \right) + I_2 \left( \sqrt{(4\nu t)^2 + (2Z(t))^2} \right) \right) \right] = e^{4\nu t} \left[ (4\nu t)^2 + (2Z(t))^2 \right]^{-3/2}.
\]  

(B.2)

As before, we use (B.1) and expand to the lowest required order. We then find

\[
e^{4\nu t} \left[ (4\nu t)^2 + (2Z(t))^2 \right]^{-3/2} = \frac{\exp 4\nu t \sqrt{1 + \frac{Z^2(t)}{4\nu^2 t^2}}}{8\pi \nu t \sqrt{1 + \frac{Z^2(t)}{4\nu^2 t^2}}}^{1/2} \left[ (4\nu t)^2 + (2Z(t))^2 \right] 4\nu t,
\]

which can be further simplified into

\[
\frac{Z(t)^2}{2\nu t} - \frac{1}{2} \ln(2\pi) - \frac{7}{2} \ln(4\nu t) + \ln \left( (4\nu t)^2 \left( 1 + \frac{(2Z(t))^2}{4\nu t} \right) \right) = 0.
\]
The single positive solution of this equation is

\[ Z(t) = \sqrt{2t W\left(\left(32\pi e\right)^{1/2} t^{3/2}\right)} - t, \]  

(B.3)

where \( W(x) \) denotes Lambert’s \( W \) function, defined as the solution of \( W e^W = x \) \([16, 54]\). Throughout, we shall require the following two expansions of Lambert’s function

\[ W(x) \approx \begin{cases} 
    x - x^2 + O(x^3) & \text{for } x \to 0 \\
    \ln x - \ln(\ln x) + O(\ln(\ln x)/\ln(x)) & \text{for } x \to \infty
\end{cases}, \]  

(B.4)

Inserting into (B.3), we finally have the leading asymptotics for \( Z(t) \), including the dominant logarithmic corrections

\[ Z(t) \simeq 3t \ln(\kappa_2 t) \left[ 1 - \frac{2}{3} \ln \left( \frac{3}{2} \ln \kappa_2 t \right) - \frac{1}{3} \frac{1}{\ln \kappa_2 t} \right] =: \sqrt{3t \ln(\kappa_2 t) F(t)}, \]  

(B.5)

and we have derived equations (4.3) and (4.4) in the main text, and especially the values of \( \kappa_1 \) and \( \kappa_2 \) quoted therein. For later use, below and in appendix C, we also defined the function \( F(t) \). This function describes additional modifications of the scaling behaviour of the third model with respect to the second model, where from constraint (4.1) we had seen that \( F(t) = 1 \).

Since the abstract expression (3.21) holds true for both the slope correlator in the second model and the height correlator in the third model, respectively, both can be analysed together. We begin with an analysis of the scaling behaviour of the autocorrelator, which reads

\[ C(t, s) = e^{-2\nu(t+s)} I_0 \left( 2\nu(t + s) \sqrt{1 + \left( \frac{Z(t) + Z(s)}{2\nu(t+s)} \right)^2} \right). \]  

(B.6)

In what follows, we shall use the logarithmic sub-ageing scaling variable

\[ \tau = t - s = \frac{s}{\ln \kappa_2 s} (y - 1) g(s), \]  

(B.7)

which is to be considered in the double limit \( t, s \to \infty \), such that \( y > 1 \) is kept fixed, and a positive constant \( \vartheta > 0 \), in analogy with, but generalising [9]. In certain cases, as we shall particularly see in appendix C when analysing the autoresponse function, the function \( g(s) \) has to be conveniently chosen. For what follows, an important simplification is obtained for the auxiliary function \( F(t) \). Inserting the scaling ansatz and expanding, we find that

\[ F(t) \simeq 1 - \frac{2}{6} \ln \frac{3}{2} - \frac{1}{3} \ln \kappa_2 s - O(\ln^{-2} \kappa_2 s) \simeq F(s) \]  

to this order.

Our first scaling analysis again uses (B.1) and (4.3). We find by expansion, up to the first non-vanishing order, and using (B.8), that
\[ \ln C(t, s) \simeq \frac{(Z(t) + Z(s))^2}{4\nu(t + s)} - \frac{1}{2} \ln(4\pi\nu(t + s)) + O(s^{-1}, t^{-1}) \]

\[ \simeq F(s) \left( \frac{\sqrt{\kappa_1}\nu \ln \kappa_2 t + \sqrt{\kappa_1}\nu s \ln \kappa_2 s}{4\nu(t + s)} \right)^2 - \frac{1}{2} \ln(4\pi\nu(t + s)). \] (B.9)

We use the scaling ansatz (B.7). Inserting into the above, and expanding, we obtain

\[ \ln C(t, s) \simeq -\frac{1}{2} \ln(8\pi\nu) + \frac{\kappa_1}{2} \ln \kappa_2 + \frac{1}{2}(\kappa_1 - 1) \ln s - \frac{\kappa_1}{32}(y - 1)^2 \ln^{1 - 2\vartheta}(\kappa_2 s) \]

\[ - \delta_{\kappa_1,3} \left( \frac{1}{2} \ln \kappa_2 s + \frac{1}{4} \left( 2 \ln \frac{3}{2} - 1 \right) \right) + o(1), \]

where the contributions in the second line only arise for the third model. Multiscaling can only be avoided by choosing \( \vartheta = \frac{1}{2} \) [9]. This gives (4.6) and (4.7) in the main text and is the first type of scaling behaviour of \( C(t, s) \) to be considered.

However, different ways to obtain a scaling behaviour exist. These are found by considering the difference \( \tau = t - s \) between the two times and by making the change of variables

\[ \tau = t - s = \frac{s}{\ln^\vartheta(\kappa_2 s)} f(s, y), \] (B.10)

where the unknown function \( f = f(s, y) \) is assumed to be small compared to \( \ln^\vartheta s \). The function \( f = f(s, y) \) must be found such that \( \tau = \tau(y) \) increases monotonically with \( y \) and the autocorrelator \( C = C(y) \) decreases monotonically. Of course, relation (B.10) is to be understood in the scaling limit \( t, s \to \infty \), with \( y \) being kept fixed. Using (B.9) again and expanding as before, we have

\[ \ln C(t, s) \simeq \frac{\kappa_1}{2} - \frac{1}{2} \delta_{\kappa_1,3} \ln(\ln \kappa_2 s) - \frac{\kappa_1}{32} \ln^{1 - 2\vartheta}(\kappa_2 s) f^2(s, y) - \frac{\delta_{\kappa_1,3}}{4} \left( 2 \ln \frac{3}{2} - 1 \right). \]

In order to obtain a scaling behaviour, we make the following ansatz

\[ \ln C(t, s) = A' \ln(\ln \kappa_2 s) - \ln(y - 1) + \frac{1}{2} \ln f^2(s, y) - \frac{B'}{32} \ln^{1 - 2\vartheta}(\kappa_2 s) f(s, y) - \frac{\delta_{\kappa_1,3}}{4} \left( 2 \ln \frac{3}{2} - 1 \right), \] (B.12)

where the constants \( A', B' \) are to be determined. Consistency between (B.11) and (B.12) gives the condition

\[ f^2 \exp \left[ \frac{\kappa_1 - B'}{16} \ln^{1 - 2\vartheta}(\kappa_2 s) f^2 \right] = (y - 1)^2 \ln^{\kappa_1 - 2A' - 1 - \delta_{\kappa_1,3}}(\kappa_2 s), \] (B.13)

which has the unique solution

\[ f^2 = \frac{16}{\kappa_1 - B'} \ln^{1 - 2\vartheta}(\kappa_2 s) W \left( \frac{\kappa_1 - B'}{16} (y - 1)^2 \ln^{\kappa_1 - 2A' - 2\vartheta - \delta_{\kappa_1,3}}(\kappa_2 s) \right), \] (B.14)

using the Lambert \( W \) function again. Herein, self-consistency requires that \( \kappa_1 - B' > 0 \) and \( \kappa_1 - 2A' - 2\vartheta - \delta_{\kappa_1,3} < 0 \). Then, we find the following asymptotic form for \( s \to \infty \).
\[
\tau = t - s = \frac{s}{\ln(1 + \delta_{3,1} - \kappa_1)/2 + A' + \vartheta/(\kappa_2 s)} (y - 1)
\]

and we see that at least for the autocorrelator we can simply set \( g(s) = 1 \). This explains the chosen ansatz: we have chosen variables such that in the special case where \((1 + \delta_{3,1} - \kappa_1)/2 + A' + \vartheta = 0\), we recover the scaling form \( \tau = t - s = s(y - 1) \) of simple ageing. This discussion will be completed by a comparison with the corresponding results from the autoresponse \( R(t, s) \) (see appendix C).

In order to derive (3.22), we reuse equations (B.1) in (3.21) and also recall that for large times, \( Z(t) \sim t^{1/2} \), up to logarithmic factors. All factors that do not contain \( n \) will be absorbed into the autocorrelator \( C(t, s) \). Therefore, the leading \( n \)-dependent term coming from the Bessel function \( I_n \) in (3.21) is simply \( \exp[-n^2/(4\nu(t + s))] \), whereas those terms that contain \( Z(t) \) or \( Z(s) \) will give rise to finite-time corrections to the leading scaling contribution. Hence (3.22) describes the leading scaling behaviour of the time–space correlator \( C_n(t, s) \).

In order to derive the spatial modulation of the time–space correlator (4.10), we start from (3.22). For large times \( t, s \to \infty \), the modulating factor can be rewritten as follows:

\[
\cos \left( n \frac{Z(t) + Z(s)}{2\nu(t + s)} \right) = \cos \left( \frac{n}{\sqrt{2\nu(t + s)}} \sqrt{\frac{(Z(t) + Z(s))^2}{2\nu(t + s)}} \right).
\]

Herein, since \( Z(t) \sim t^{1/2} \), we used the fact that the argument of the arctan is small enough to keep the lowest order. Now, straightforward expansion of the square root produces the stated form (which is symmetric in \( t \) and \( s \)) and which can be performed up to finite-time corrections.

Finally, the single-time correlator \( C_n(t, t) = \lim_{s \to t} C_n(t, s) \) is read off immediately from (4.10) to produce (4.11).

**Appendix C. Long-time response**

The analysis of the autocorrelator starts from

\[
R(t, s) = e^{-2\nu(t-s)} I_0 \left( \frac{2\nu(t-s)}{\sqrt{1 + \left( \frac{Z(t)-Z(s)}{2\nu(t-s)} \right)^2}} \right).
\]

Expanding the Bessel function via (B.1) and using (4.3), (4.4) and (B.7), we obtain

\[
\ln R(t, s) \simeq \frac{(Z(t)-Z(s))^2}{4\nu(t-s)} - \frac{1}{2} \ln(4\pi\nu(t-s)) + O(s^{-1}, t^{-1})
\]

\[
\simeq F(s) \left( \sqrt{\kappa_1 \nu t \ln \kappa_2 t} - \sqrt{\kappa_1 \nu s \ln \kappa_2 s} \right)^2 - \frac{1}{2} \ln(4\pi\nu(t-s)).
\]

In analogy with our analysis for the autocorrelator in appendix B, we try to find a scaling variable \( y \) such that the time difference \( \tau = \tau(y) \) increases monotonically with \( y \) and the response \( R = R(y) \) decreases monotonically with \( y \). The time difference is written as

https://doi.org/10.1088/1742-5468/aa9a53
\[ \tau = t - s = \frac{s}{\ln(\kappa_2 s)} f(s, y), \]  
(C.3)

where the unknown function \( f = f(s, y) \) plays the role of the scaling variable. We assume that \( f \ln s \ll 1 \). Then we can expand \( R(t, s) \). After several cancellations, we finally arrive for the second model and third models, respectively, at

\[
\ln R(t, s) \simeq \frac{\kappa_1}{16} f(s, y) - \frac{1}{2} \ln f(s, y) - \frac{1}{2} \ln(4\pi \nu s) + \frac{1}{2} \ln(\ln \kappa_2 s) + o(1) \quad (C.4a)
\]

\[
\ln R(t, s) \simeq \frac{\kappa_1 f(s, y)}{16} - \frac{1}{2} \ln f(s, y) - \frac{1}{2} \ln \left( 1 - \frac{\ln \kappa_2 s}{\kappa_1 \ln \kappa_2 s} \right) - \frac{1}{2} \ln(4\pi \nu s) + \frac{1}{2} \ln(\ln \kappa_2 s) + o(1)
\]

\[
\simeq \frac{\kappa_1 f(s, y)}{16} - \frac{1}{2} \ln f(s, y) - \frac{1}{2} \ln(4\pi \nu s) + \frac{1}{2} \ln(\ln \kappa_2 s) + o(1), \quad (C.4b)
\]

where for the third model we redefined the scaling variable \( \tilde{f}(s, y) := f(s, y) \left( 1 - \frac{\ln \kappa_2 s}{\kappa_1 \ln \kappa_2 s} \right)^{s \rightarrow \infty} f(s, y) \). The scaling of the autoresponse of the second model in \((C.4a)\) and of the third model in \((C.4b)\) can be discussed simultaneously by using the scaling variables \( f \) or \( \tilde{f} \), respectively.

Now, we can define a scaling variable \( y > 1 \), for \( s \rightarrow \infty \), through the ansatz (using \( f \) or \( \tilde{f} \), respectively):

\[
\ln R(t, s) \overset{1}{=} -\frac{1}{2} \ln(4\pi \nu s) - \frac{1}{2} \ln(y - 1) + \frac{B}{16} f(s, y) + A \ln(\ln \kappa_2 s), \quad (C.5)
\]

where \( A, B \) are constants. Consistency of \((C.4a)\) and \((C.5)\) leads to

\[
f(s, y) = \frac{8}{B - \kappa_1} W \left( \frac{B - \kappa_1}{8} (y - 1) \ln^{1-2A}(\kappa_2 s) \right), \quad (C.6)
\]

where \( 2A > 1,\ B - \kappa_1 > 0 \), and Lambert’s function \( W(x) \) is used again. For the third model, one simply reads \( \tilde{f} \) instead of \( f \). The response function becomes

\[
\ln R(t, s) \simeq -\frac{1}{2} \ln(4\pi \nu s) + A \ln(\ln \kappa_2 s) - \frac{1}{2} \ln(y - 1) + \frac{B}{16} \ln^{1-2A}(\kappa_2 s) (y - 1)
\]

and we have the final scaling form, with \( A > \frac{1}{2} \)

\[
R(t, s) = (4\pi \nu s)^{-1/2} \ln^{A}(\kappa_2 s) (y - 1)^{-1/2} \quad (C.7)
\]

with the scaling variable

\[
t - s = \frac{s}{\ln \kappa_2 s} \frac{8}{B - \kappa_1} W \left( \frac{B - \kappa_1}{8} (y - 1) \ln^{1-2A}(\kappa_2 s) \right), \quad (C.8)
\]

such that for \( s \rightarrow \infty \), we recover \( t - s \simeq s \ln^{-2A}(\kappa_2 s) (y - 1) \) with \( 2A > 1 \).

To finish the argument, we now compare with the scaling of the autocorrelator, as discussed in appendix B. Because of the condition \( 2A > 1 \), the scaling \((C.8)\) cannot be compatible with \((4.6)\). A contrario, compatibility with \((4.8)\) can be achieved via the condition

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Figure C1. Illustration of the definition of the scaling variables $\tau = \tau(s, y)$ for fixed waiting time $s$. The black curves show the definition (4.17) for the response, for several values of $\vartheta = 2A$, and the blue curves correspond to (4.8) for the correlator. The straight green line is the asymptotic form $\tau = (y - 1)s / \ln s$.

\[ 2A = \frac{1 + \delta_{\kappa_1,3} - \kappa_1}{2} + A' + \vartheta. \]  

(C.9)

It follows that the condition $2A > 1$ implies the bound $\kappa_1 - 2A' - 2\vartheta - \delta_{\kappa_1,3} < 0$ obtained in appendix B. Since we shall only encounter the combination $A' + \vartheta$, we may as well fix one of those two constants. For example, we might insist that the function $f$ in (B.14) should not become singular for $s \to \infty$. This fixes $\kappa_1 - 2A' - 2\vartheta - \delta_{\kappa_1,3} = 0$, hence

\[ \vartheta = 2A , \quad A' = \frac{1}{2}(\kappa_1 - 1 - \delta_{\kappa_1,3}). \]  

(C.10)

This produces the final forms (4.8) and (4.17) in the text, if we choose $B = \kappa_1 + 8$ and $B' = \kappa_1 - 16$. In figure C1 we compare the functions $\tau = \tau(s, y)$, for finite and fixed $s$, for the responses and correlators. This illustrates the fact that unless $s$ becomes enormously large, there are strong nonlinearities in the scaling variable to be taken into account.

Finally, the asymptotic form in (4.16) is derived in completely analogy to the treatment of the correlator in appendix B. The absence of a spatial modulation in (4.16) follows from

\[ \frac{(Z(t) - Z(s))^2}{2\nu(t - s)} = O(\ln^{-1}s). \]  

(C.11)

Appendix D. On some special functions

We compute the functions $\mathcal{J}_a$, with $a \in \mathbb{N}$, defined by

\[ \mathcal{J}_{2a}(A, Z) := \frac{1}{\pi} \int_0^{\pi} dk \ e^{A\cos k} \cosh(Z \sin k) (\sin k)^{2a} = \frac{\partial^{2a}}{\partial Z^{2a}} \mathcal{J}_0(A, Z) \]  

(D.1)

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\[ J_{2a+1}(A, Z) := \frac{1}{\pi} \int_0^\pi dk \ e^{A \cos k} \sinh(Z \sin k) (\sin k)^{2a+1} = \frac{\partial^{2a+1}}{\partial Z^{2a+1}} J_0(A, Z) \] (D.2)

and in principle, it is sufficient to find \( J_0 \) explicitly. This can be done as follows:

\[ J_0(A, Z) = \frac{1}{\pi} \int_0^\pi dk \ e^{A \cos k} \sum_{r=0}^{\infty} \frac{Z^{2r}}{(2r)!} \sin^{2r+1} k \]

\[ = \sum_{r=0}^{\infty} \frac{1}{\pi} \frac{Z^{2r}}{\Gamma(2r+1)} \sqrt{\pi} \Gamma \left( r + \frac{1}{2} \right) \left( \frac{A}{2} \right)^{-r} I_r(A) \]

\[ = \sum_{r=0}^{\infty} \left( \frac{Z^2}{2A} \right)^r \frac{1}{r!} I_r(A) = \sum_{r=0}^{\infty} \left( \frac{A}{2} \right)^r \frac{I_r(A)}{r!} \left[ \left( \frac{Z}{A} \right)^2 \right]^r \]

\[ = I_0 \left( A \sqrt{1 + Z^2/A^2} \right) = I_0 \left( \sqrt{A^2 + Z^2} \right). \] (D.3)

Herein, after expanding the \( \cosh \) in the first line, we used an integral representation \([1, \text{ equation (9.6.18)}]\) of the modified Bessel function \( I_r(A) \) in the second line, simplified in the third line with the help of the duplication formula \([1, \text{ equation (6.1.18)}]\) of the Gamma function, and in the fourth line applied the multiplication formula \([1, \text{ equation (9.6.51)}]\) for the \( I_r \) (see also \((E.7)\) below).

We note the unexpected rotation-symmetry in the \((A, Z)\)-plane of the integral identity

\[ \frac{1}{\pi} \int_0^\pi dk \ e^{A \cos k} \cosh(Z \sin k) = I_0 \left( \sqrt{A^2 + Z^2} \right) \] (D.4)

just derived.

Similarly, one can express \( J_1 \), with the help of \([1, \text{ equations (9.6.18), (6.1.18) and (9.6.51)}]\) as:

\[ J_1(A, Z) = \frac{1}{\pi} \int_0^\pi dk \ e^{A \cos k} \sum_{r=0}^{\infty} \frac{Z^{2r}}{(2r)!} \sin^{2r+1} k \]

\[ = \sum_{r=0}^{\infty} \left( \frac{A}{2} \right)^r \left( \frac{Z}{A} \right)^{2r+1} \frac{I_{r+1}(A)}{r!} \]

\[ = \frac{Z I_1 \left( \sqrt{A^2 + Z^2} \right)}{\sqrt{A^2 + Z^2}} \] (D.5)

and one can also verify that \( J_1(A, Z) = \partial_Z J_0(A, Z) \). Similarly, \( J_2(A, Z) = \partial_Z J_1(A, Z) \).

Appendix E. Proof of an identity

We derive the following identity for all \( n \in \mathbb{N} \) and \( A, Z \in \mathbb{C} \):

\[ C_n(A, Z) := \frac{1}{\pi} \int_0^\pi dk \ e^{A \cos k} \cosh(Z \sin k) \cos(nk) = I_n \left( \sqrt{A^2 + Z^2} \right) \cos \left( n \arctan \left( \frac{Z}{A} \right) \right), \] (E.1)

where \( I_n \) is a modified Bessel function. The special case \( n = 0 \) is the function \( J_0(A, Z) \) derived in appendix D. For the proof, we shall require the following result.

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Lemma. For any integers \( n, \nu \in \mathbb{N} \) and \( z, \xi \in \mathbb{C} \), one has
\[
\frac{1}{\pi} \int_0^\pi d\theta \; \exp(z \cos \theta) \cos(n \theta) \sin^{2\nu} \theta = \frac{(-1)^\nu}{2^{2\nu}} \sum_{k=0}^{2\nu} (-1)^k \binom{2\nu}{k} J_{n+2\nu-2k}(z) \tag{E.2}
\]
and
\[
\frac{1}{\pi} \int_0^\pi d\theta \; \exp(i\xi \cos \theta) \cos(n \theta) \sin^{2\nu} \theta = i^n \frac{1}{2^{2\nu}} \sum_{k=0}^{2\nu} \binom{2\nu}{k} J_{n+2\nu-2k}(\xi), \tag{E.3}
\]
where \( J_n \) is a Bessel function and \( I_n \) is a modified Bessel function \([1]\).

Corollary. Separating the real and imaginary parts in (E.3), for \( n \) even and odd, respectively, gives for \( \xi \in \mathbb{R} \) (\( m, \nu \in \mathbb{N} \)):
\[
\frac{1}{\pi} \int_0^\pi d\theta \; \cos(\xi \cos \theta) \cos(2m \theta) \sin^{2\nu} \theta = \frac{(-1)^m}{2^{2\nu}} \sum_{k=0}^{2\nu} \binom{2\nu}{k} J_{2m+2\nu-2k}(\xi)
\]
\[
\frac{1}{\pi} \int_0^\pi d\theta \; \sin(\xi \cos \theta) \cos((2m+1) \theta) \sin^{2\nu} \theta = \frac{(-1)^m}{2^{2\nu}} \sum_{k=0}^{2\nu} \binom{2\nu}{k} J_{2m+1+2\nu-2k}(\xi)
\]
\[
\int_0^\pi d\theta \; \sin(\xi \cos \theta) \cos((2m+1) \theta) \sin^{2\nu} \theta = \frac{1}{\pi} \int_0^\pi d\theta \; \cos(\xi \cos \theta) \cos((2m+1) \theta) \sin^{2\nu} \theta = 0.
\]

The strategy of proof will be as follows: (E.1) follows from (E.2), which in turn is an immediate consequence of (E.3).

Step 1. To prove (E.3), recall Euler’s formula, \( e^{i \cos \theta} = \cos(\xi \cos \theta) + i \sin(\xi \cos \theta) \), with \( \xi \in \mathbb{C} \). Multiplying with \( \cos n \theta \) and integrating gives (see \([1\), equation (9.1.21)]\), with \( n \in \mathbb{N} \),
\[
i^n J_n(\xi) = \frac{1}{\pi} \int_0^\pi d\theta \; \cos(\xi \cos \theta) \cos(n \theta) + \frac{i}{\pi} \int_0^\pi d\theta \; \sin(\xi \cos \theta) \cos(n \theta). \tag{E.4}
\]
Next, denote the integral on the left-hand side of (E.3) as
\[
C_{n,\nu}(\xi) := \frac{1}{\pi} \int_0^\pi d\theta \; e^{i \xi \cos \theta} \sin^{2\nu} \theta. \]
It is easily verified that one has the differential recurrence relation
\[
C_{n,\nu+1}(\xi) = \partial^2_\xi C_{n,\nu}(\xi) + C_{n,\nu}(\xi). \tag{E.5}
\]
For a fixed \( n \in \mathbb{N} \), the identity (E.3) is the assertion that for all \( \nu \in \mathbb{N} \) one has the identity:
\[
C_{n,\nu}(\xi) = \frac{i^n}{2^{2\nu}} \sum_{k=0}^{2\nu} \binom{2\nu}{k} J_{n+2\nu-2k}(\xi), \tag{E.6}
\]
which we now prove by induction over \( \nu \). For \( \nu = 0 \), the assertion (E.6) is just the relation (E.4). For the induction step \( \nu \mapsto \nu + 1 \), we use\(^{22}\) (E.5) and first apply the Bessel function identities \([1\), equation (9.1.27)]\) and then apply the standard identities of the binomial coefficients several times:

\(^{22}\) In this calculation, we write \( J_n \) instead of fully \( J_n(z) \).

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\[
C_{n,\nu+1}(\zeta) = \left(1 + \frac{\partial^2}{\partial \zeta^2}\right) C_{n,\nu}(\zeta)
\]
\[
= \frac{i^n}{2^{\nu+1}} \sum_{k=0}^{2\nu} \binom{2\nu}{k} J_{n+2\nu-2k} + \frac{1}{4} [J_{n-2+2\nu-2k} - 2J_{n+2\nu-2k} + J_{n+2\nu-2k}]
\]
\[
= \frac{i^n}{2^{\nu+1}} \sum_{k=0}^{2\nu} \binom{2\nu}{k} [J_{n+2\nu-2k} + J_{n-2+2\nu-2k} + J_{n+2\nu-2k} + J_{n+2\nu-2k}]
\]
\[
= \frac{i^n}{2^{\nu+1}} \left\{ \sum_{k=0}^{2\nu} \binom{2\nu}{k} J_{n+2\nu-2k} + \sum_{k=1}^{2\nu} \binom{2\nu}{k-1} J_{n+2\nu-2k} \right\}
\]
\[
= \frac{i^n}{2^{\nu+1}} \left\{ J_{n+2\nu} + \sum_{k=1}^{2\nu} \binom{2\nu+1}{k} J_{n+2\nu-2k} + J_{n+2\nu-2k} \right\}
\]
\[
= \frac{i^n}{2^{\nu+1}} \left\{ J_{n+2\nu} + J_{n-2(\nu+1)} + J_{n+2(\nu+1)} + J_{n-2\nu} \right\}
\]
\[
= \frac{i^n}{2^{\nu+1}} \left\{ J_{n+2\nu} + J_{n-2(\nu+1)} + J_{n+2(\nu+1)} + J_{n-2\nu} \right\}
\]
\[
= \frac{i^n}{2^{\nu+1}} \left\{ J_{n+2\nu} + J_{n-2(\nu+1)} + J_{n+2(\nu+1)} + J_{n-2\nu} \right\}
\]
\[
= \frac{i^n}{2^{\nu+1}} \sum_{k=0}^{2(\nu+1)} \binom{2(\nu+1)}{k} J_{n+2(\nu+1)-2k}(\zeta),
\]

which proves assertion (E.6) for all \(\nu \in \mathbb{N}\) (in the last line we restored the argument \(\zeta\)).

**Step 2.** Starting from (E.3), it is sufficient to set \(\zeta = iz\) and to recall that \(J_n(iz) = i^n I_n(z)\). Equation (E.3) then gives

\[
\frac{1}{\pi} \int_0^\pi d\theta \ e^{-z \cos \theta \cos(n\theta)} \sin^{2\nu} \theta = \frac{(-1)^{n+\nu}}{2^{2\nu}} \sum_{k=0}^{2\nu} \binom{2\nu}{k} (-1)^k I_{n+2\nu-2k}(z).
\]

Going over to \(z \mapsto -z\), along with \(I_n(-z) = (-1)^n I_n(z)\), produces (E.2). This proves the lemma.

**Step 3.** To prove (E.1), we require two more preparations. First, recall the identity [1, equation (9.6.51)] for \(n \in \mathbb{N}\), \(x \in \mathbb{C}\) and \(\lambda \neq 0\)

\[
\sum_{\ell=0}^{\infty} \frac{(\lambda^2 - 1)^{\ell}(x/2)^{\ell}}{\ell!} I_{n+\ell}(x) = \lambda^{n} I_n(\lambda x).
\]

(E.7)
Second, let \( x = \tan \varphi \). Then
\[
\left( \frac{1-ix}{1+ix} \right)^{n/2} + \left( \frac{1+ix}{1-ix} \right)^{n/2} = \exp \left(-2i\varphi \frac{n}{2}\right) + \exp \left(+2i\varphi \frac{n}{2}\right) = 2 \cos n\varphi.
\] (E.8)

Now, denote the left-hand side of (E.1) by \( \mathcal{C}_n = \mathcal{C}_n(A, Z) \). Expanding the cosh in the integral representation of \( \mathcal{C}_n \), we have
\[
\mathcal{C}_n = \frac{1}{\pi} \sum_{m=0}^{\infty} Z^{2m} \int_0^\pi dk \; e^{A \cos k} \sin^{2m} k \cos(kn)
\]
\[
= \sum_{m=0}^{\infty} \sum_{\ell=0}^{2m} \frac{(-1)^{m+\ell}}{(2m-\ell)!\ell!} I_{n+2m-2\ell}(A)
\]
\[
= \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \frac{1}{2} \left[ 1 + (-1)^m \right] \left( \frac{Z}{2} \right)^m \frac{i^{m+\ell}}{(m-\ell)!\ell!} I_{n+m-2\ell}(A),
\]
where in the second line we used (E.2). In the last line, we replaced the even integer \( 2m \) by the integer \( m \in \mathbb{N} \), where the extra factor guarantees that only the even values of \( m \) give a non-vanishing contribution. Now, we can exchange the order of summation and afterwards perform a shift in the summation variable \( m \) to obtain
\[
\mathcal{C}_n = \sum_{\ell=0}^{\infty} \sum_{m=\ell}^{\infty} \frac{1}{2} \left[ 1 + (-1)^m \right] \left( \frac{Z}{2} \right)^m \frac{i^{m+\ell}}{(m-\ell)!\ell!} I_{n+m-2\ell}(A)
\]
\[
= \frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \left[ (-1)^\ell + (-1)^m \right] \left( \frac{Z}{2} \right)^{m+\ell} \frac{1}{m!\ell!} I_{n+\ell+m}(A)
\]
\[
= \frac{1}{2} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( \frac{iZ}{2} \right)^\ell \sum_{m=0}^{\infty} \left[ (-1)^\ell + (-1)^m \right] \frac{1}{m!} \left( \frac{iZ}{A} \right)^m \left( \frac{A}{2} \right)^m I_{y+m}(A)
\]
\[
= \frac{1}{2} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( \frac{iZ}{A} \right)^\ell \left[ (-1)^\ell \left( 1 + i \frac{Z}{A} \right)^{-\ell/2} I_{n-\ell} \left( A \sqrt{1 + i \frac{Z}{A}} \right) + \left( 1 - i \frac{Z}{A} \right)^{-\ell/2} I_{n-\ell} \left( A \sqrt{1 - i \frac{Z}{A}} \right) \right]
\]
\[
= \frac{1}{2} \left( 1 + i \frac{Z}{A} \right)^{-n/2} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( -i \frac{Z}{A} \right)^\ell \left( \frac{1}{2} A \sqrt{1 + i \frac{Z}{A}} \right)^\ell I_{n-\ell} \left( A \sqrt{1 + i \frac{Z}{A}} \right)
\]
\[
+ \frac{1}{2} \left( 1 - i \frac{Z}{A} \right)^{-n/2} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( i \frac{Z}{A} \right)^\ell \left( \frac{1}{2} A \sqrt{1 - i \frac{Z}{A}} \right)^\ell I_{n-\ell} \left( A \sqrt{1 - i \frac{Z}{A}} \right)
\]
\[
= \frac{1}{2} \left[ \frac{1 - iZ/A}{1 + iZ/A} \right]^{n/2} + \frac{1 + iZ/A}{1 - iZ/A} \right]^{n/2} \left( A \sqrt{1 + \frac{Z^2}{A^2}} \right)
\]
\[
= \cos \left( n \arctan \frac{Z}{A} \right) I_n \left( \sqrt{A^2 + Z^2} \right).
\]
Appendix F. Discrete cosine and sine transformations

For the convenience of the reader, we recall some basic properties of discrete cosine and sine transformations. On a periodic chain with $N$ sites, the cosine transformation $\mathcal{C}$ of an even function $a_n(t) = a_{-n}(t)$ is defined, with $k = 0, 1, \ldots, N - 1$, as

$$\hat{a}(t, k) = \mathcal{C}(a_n(t))(k) := \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{N}k n\right) a_n(t), \quad a_n(t) = \frac{1}{N} \sum_{k=0}^{N-1} \cos\left(\frac{2\pi}{N}k n\right) \hat{a}(t, k),$$

and is itself even, namely $\hat{a}(t, k) = \hat{a}(t, -k)$. The sine transformation $\mathcal{S}$ of an odd function $b_n(t) = -b_{-n}(t)$ is defined as

$$\hat{b}(t, k) = \mathcal{S}(b_n(t))(k) := \sum_{n=0}^{N-1} \sin\left(\frac{2\pi}{N}k n\right) b_n(t), \quad b_n(t) = \frac{1}{N} \sum_{k=0}^{N-1} \cos\left(\frac{2\pi}{N}k n\right) \hat{b}(t, k)$$

and is itself odd, namely $\hat{b}(t, k) = -\hat{b}(t, -k)$. Clearly, $\mathcal{C}$ and $\mathcal{S}$ are linear operators. Furthermore, $\mathcal{C}(b_n(t)) = \mathcal{S}(a_n(t)) = 0$ and $\mathcal{C}^2(a_n(t)) = Na_n(t)$ and $\mathcal{S}^2(b_n(t)) = Nb_n(t)$.

In the main text, we shall frequently require the following cosine transformations of the even functions:

$$\mathcal{C} \left( a_{n+1}(t) + a_{n-1}(t) - 2a_n(t) \right)(k) = \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{N}k n\right) (a_{n+1}(t) + a_{n-1}(t) - 2a_n(t))$$

$$= -2 \left[ 1 - \cos\left(\frac{2\pi}{N}k\right) \right] \hat{a}(t, k) \quad \text{(F.3)}$$

$$\mathcal{C} \left( \frac{1}{2} (b_{n+1}(t) - b_{n-1}(t)) \right)(k) = \frac{1}{2} \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{N}k n\right) (b_{n+1}(t) - b_{n-1}(t))$$

$$= \sin\left(\frac{2\pi}{N}k\right) \hat{b}(t, k) \quad \text{(F.4)}$$

and also the sine transformations of the odd functions

$$\mathcal{S} \left( b_{n+1}(t) + b_{n-1}(t) - 2b_n(t) \right)(k) = \sum_{n=0}^{N-1} \sin\left(\frac{2\pi}{N}k n\right) (b_{n+1}(t) + b_{n-1}(t) - 2b_n(t))$$

$$= -2 \left[ 1 - \cos\left(\frac{2\pi}{N}k\right) \right] \hat{b}(t, k) \quad \text{(F.5)}$$

$$\mathcal{S} \left( \frac{1}{2} (a_{n+1}(t) - a_{n-1}(t)) \right)(k) = \frac{1}{2} \sum_{n=0}^{N-1} \sin\left(\frac{2\pi}{N}k n\right) (a_{n+1}(t) - a_{n-1}(t))$$

$$= - \sin\left(\frac{2\pi}{N}k\right) \hat{a}(t, k). \quad \text{(F.6)}$$
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