Numerical Results for Ground States of Spin Glasses on Bethe Lattices

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The average ground state energy and entropy for ±J spin glasses on Bethe lattices of connectivities \( k+1 = 3 \ldots 26 \) at \( T = 0 \) are approximated numerically. To obtain sufficient accuracy for large system sizes (up to \( n = 2048 \)), the Extremal Optimization heuristic is employed which provides high-quality results not only for the ground state energies per spin \( e_{k+1} \) but also for their entropies \( s_{k+1} \). The results show considerable quantitative differences between lattices of even and odd connectivities. The results for the ground state energies compare very well with recent one-step replica symmetry breaking calculations. These energies can be scaled for all even connectivities \( k+1 \) to within a fraction of a percent onto a simple functional form, \( e_{k+1} = E_{SK}\sqrt{k+1} - (2E_{SK} + \sqrt{2})/\sqrt{k+1} \), where \( E_{SK} = -0.7633 \) is the ground state energy for the broken replica symmetry in the Sherrington-Kirkpatrick model. But this form is in conflict with perturbative calculations at large \( k+1 \) which do not distinguish between even and odd connectivities. We find non-zero entropies \( s_{k+1} \) at small connectivities. While \( s_{k+1} \) seems to vanish asymptotically with \( 1/(k+1) \) for even connectivities, it is indistinguishable from zero already for odd \( k+1 \geq 9 \).

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I. INTRODUCTION

In this paper we study the ground state \( (T = 0) \) properties of ±J spin glasses on \( k+1 \)-Bethe lattices \( \mathbb{B} \). The Bethe lattices in this case are \( r \)-regular graphs \( \mathbb{B} \) with \( r = k+1 \). These are randomly connected graphs consisting of \( n \) vertices in which each vertex has a fixed connectivity of \( k+1 \). This constraint contrasts with “random graphs” \( \mathbb{G} \) in which pairs of vertices are randomly connected, leading to a Poissonian distribution of connectivities around a mean of \( c \); these graphs will be studied numerically elsewhere \( \mathbb{G} \). We explore the large- \( n \) regime of low-connectivity graphs, \( k+1 = 3 \ldots 26 \), which are of great theoretical interest as finite-connected, mean-field models for low-dimensional lattice spin glasses \( \mathbb{B} \). A great number of studies have focused on various aspects of this conceptually simple model to hone the complex mathematical techniques required to treat disordered systems \( \mathbb{G} \) or optimization problems \( \mathbb{P} \). In this paper we will try to provide a independent numerical check on the validity and accuracy of those techniques.

Our results, in turn, reflect on the flexibility of the extremal optimization (EO) heuristic \( \mathbb{E} \) in finding approximate but high-quality solutions for ground states of spin glasses on an arbitrary graphical structure in a reasonable computational time. These are often NP-hard optimization problems which are believed to require a computational effort that rises faster than any power of \( n \) to obtain provably exact solutions \( \mathbb{E} \). Thus, exact methods as of yet are not able to provide results for large- \( n \) problems, at least not with significant statistics \( \mathbb{E} \), except for some special cases \( \mathbb{E} \). Furthermore, there is only a small number of capable approximate algorithms available for the study of \( T = 0 \) properties of spin glasses \( \mathbb{E} \), mostly restricted to \( d \)-dimensional lattice models, and EO provides a distinct alternative which will increase the confidence in the numerical results available. In previous papers, we have demonstrated the capabilities of EO in determining near-optimal solutions by reproducing existing results for 3d and 4d spin glasses and obtaining new results for the coloring problem \( \mathbb{E} \) and the graph partitioning problem \( \mathbb{E} \). The results in this paper show that EO is not only capable of approximating ground states well but also of sweeping the entire configuration space efficiently to determine the degeneracy of ground states \( \mathbb{E} \). Unlike other methods, EO never “freezes” into local minima and proves to be limited only by the ability to store new ground states.

We find that our results for the ground state energies are consistent with the theoretical results of the one-step replica symmetry-broken (1RSB) solution of ±J spin glasses on \( k+1 \)-connected Bethe lattices. Our numerical result for \( k+1 = 3 \) below clearly excludes the replica symmetric (RS) solution and are consistent with the 1RSB results \( \mathbb{E} \). Beyond that our results suggest subtle differences between even and odd values of the connectivity \( k+1 \), with no obvious way to continue smoothly between them \( \mathbb{E} \). These oscillations may doom perturbative calculation for \( k+1 \rightarrow \infty \). While the expectation has been raised that the entropy per spin of the ground states in this model should be vanishing for any \( k+1 \), we find that the entropy is finite and decaying like \( 1/(k+1) \) for large, even \( k+1 \). For odd \( k+1 \) it is non-zero only for small values and may be vanishing already beyond some finite, odd connectivity.

In the following we first introduce the Bethe lattices we used in the numerical calculations. In Sec. \( \mathbb{E} \) we briefly describe the EO algorithm which is amply discussed elsewhere \( \mathbb{E} \). In Sec. \( \mathbb{E} \) we present a few simulations to reproduce known results to gauge our procedure. In
we present our numerical results, followed by an extensive discussion in Sec. V. Some conclusions are presented in Sec. VI.

II. SPIN GLASSES ON BETHE LATTICES

Disordered spin systems on random graphs have been investigated as mean-field models for low-dimensional spin glasses or optimization problems, since variables are long-range connected yet have a small number of neighbors. Particularly simple are Bethe lattices of connectivities $k+1$, also called fixed-valence or $r$-regular random graphs. These are graphs consisting of $n$ vertices where each vertex possesses a fixed number $k+1$ of bonds with randomly selected other vertices. In comparison to the otherwise more familiar random graphs studied by Erdős and Rényi, Bethe lattices at a given $n$ and $k$ avoid fluctuations in the connectivities of vertices and in the total number of bonds.

There are slight variations in the generation of Bethe lattices. For instance, to add a bond one could choose at random two vertices of connectivities $< k+1$ to link until all vertices are $k+1$-connected. Instead, we have used the method described in Ref. 1 to generate these graphs. Here, all the terminals on the vertices form a list of $n(k+1)$ independent variables. For each added bond two available terminals are chosen at random to be linked and removed from the list. Furthermore, for algorithmic convenience, we reject graphs with possess self loops, bonds that connect two terminals of the same vertex. Multiple bonds between any pair of vertices are allowed, otherwise it is too hard to generate feasible graphs for, say, $n=32$ and $k+1=20$. Since $k+1$ remains finite for $n \to \infty$, the energy and entropy per spin would only be effected to $O(1/n)$ by the differences between these choices.

Once a graphical instance is generated, we assign randomly chosen but fixed couplings $J_{i,j} \in \{-1,+1\}$ to existing bonds between neighboring vertices $i$ and $j$. Each vertex $i$ is occupied by a spin variable $x_i \in \{-1,+1\}$. The energy of the system is defined as the difference in number between violated bonds and satisfied bonds,

$$H = - \sum_{\{bonds\}} J_{i,j} x_i x_j,$$

and in this paper we will focus on the energy per spin,

$$e_{k+1}(n) = \frac{H}{n},$$

as a function of $k+1$ in the limit of $n \to \infty$. Each instance can have a large degeneracy $\Omega$ in the configurations exhibiting its ground state energy, and we also sample the average entropy,

$$s_{k+1}(n) = \frac{1}{n} \ln \Omega,$$

for these instances.

III. $\tau$-EO ALGORITHM FOR BETHE LATTICES

The extremal optimization algorithm, $\tau$-EO, which we employ in this paper, has been discussed previously in Ref. 21, and in Refs. 22, 32 with regard to the setting of its one free parameter, $\tau$. Here, we merely describe the implementation of $\tau$-EO without further justification.

To obtain the numerical results in Secs. V we used the following implementation of $\tau$-EO: For a given spin configuration on a graph, assign to each spin $x_i$ a “fitness”

$$\lambda_i = -\#\text{violated bonds} = -0, -1, -2, \ldots, -(k+1),$$

so that

$$e = -\frac{1}{2n} \sum_i \lambda_i$$

is satisfied. Each spin falls into one of only $k+2$ possible states. Say, currently there are $n_{k+1}$ spins with the worst fitness, $\lambda = -(k+1)$, $n_k$ with $\lambda = -k$, and so on up to $n_0$ spins with the best fitness $\lambda = 0$. Now draw a “rank” $l$ according to the distribution

$$P(l) = \frac{\tau - 1}{1 - n^{1-\tau}} l^{-\tau} \quad (1 \leq l \leq n).$$

Then, determine $0 \leq j \leq (k+1)$ such that $\sum_{i=j+1}^{k+1} n_i < l \leq \sum_{i=j}^{k+1} n_i$. Finally, select any one of the $n_j$ spins in state $j$ and reverse its orientation unconditionally. As a result, it, and its neighboring spins change their fitness. After all the affected $\lambda$s and $n$s are reevaluated, a new spin is chosen for an update.

This EO implementation updates spins with a $(\tau$-dependent) bias against poorly adapted spins on behalf of Eq. (6). This process is “extremal” in the sense that it focuses on atypical variables, and it forms the basis of the EO method. The only adjustable parameter in this algorithm is the power-law exponent $\tau$. For $\tau = 0$, randomly selected spins get forced to update, resulting in a random walk through the configuration space which would yield poor results. For $\tau \to \infty$, only spins in the worst state get updated which quickly traps the update process to a small region of the configuration space which may be far from a near-optimal solution. The arguments given in Ref. 23 and a few experiments indicate that $\tau = 1.3$ is a good choice to find ground states efficiently on Bethe lattices.

The algorithm never converges or “freezes” into a particular state but perpetually explores new near-optimal configurations. It is, of course, easy to simply store the lowest energy state found so far in a given run of $\tau$-EO and terminate when desired. Previous experience with optimizing spin glasses with EO 21, and a few experiments, suggest a typical number of updates of $O(n^3)$ for an EO-run to obtain saturation in the values found for ground states, at least up to the system sizes $n \approx 10^8$ obtainable here. Instead of pushing to attain larger values
of $n$, we opt here for obtaining better statistics by sampling more instances at smaller values of $n$ while spending even more time on each instance than may seem to be required, in an attempt to ensure accuracy. In particular, our implementation restarts for each instance at least $r_{\text{max}} = 4$ times with new random initial spin-assignments, executing $\approx 0.1 n^3$ updates per run. If a new, lower-than-previous energy state is encountered in run $r$, we adjust $r_{\text{max}} = 2 + 2r$ for that instance so that EO runs at least twice as many restarts as were necessary to find the lowest state in the first place. Especially for small $n$, $r_{\text{max}}$ hardly ever exceeds 4; for larger $n$ a few graphs require up to 25 restarts before termination.

Since EO perpetually explores new configurations it is well suited to explore also the degeneracy of low-energy states. In this case we not only store the first configuration found with the lowest energy for that instance. Instead, we consider each configuration with the lowest energy, retaining new ones and rejecting all others. This procedure is somewhat inefficient and at best allows system sizes up to $n = 256$ beyond which the degeneracy exceeds memory constraints. But it provides a fast way to also determine the $T = 0$ entropy of the ground states with moderate accuracy. In these runs, we used a similar approach to the above, except for setting $r_{\text{max}} = 8 + 2r$ where $r$ is the latest run in which another new configuration of the lowest energy was located. Here, for some highly degenerate instances at larger $n$, $r_{\text{max}}$ could reach up into the 100’s, further limiting attainable system sizes.

### IV. NUMERICAL TEST

To evaluate the proposed $\tau$-EO algorithm, we have run a series of tests. First, we can defer to some already published results [21–23]. In Ref. [21] we have calculate approximations to the ground state energy for $\pm J$ spin glasses on a hypercubic lattice for $d = 3$ and $d = 4$ for systems up to $n = 12^3 = 1728$ which for each $n$ reproduced previous results obtained with sophisticated genetic algorithms [27–28] (although there we used a fixed $r_{\text{max}}$).

To evaluate the ability of the algorithm to determine the degeneracy of low-energy states found, we have reproduced within statistical error the results of Ref. [30] up to $n = 63$ beyond which EO ran out of time and memory to sample states completely. (Ref. [30] used a more efficient way to estimate the entropy from sampling only a small number of states.) And it took EO only a fraction of a second to find all 60 ground states of a $4^3$ instance that had been exactly enumerated in Ref. [23].

To gauge $\tau$-EO’s performance for larger $n$, we have run our implementation also on two 3d lattice instances, $\text{toruspm}3$-8-50 and $\text{toruspm}3$-15-50, with $n = 8^3 = 512$ and $n = 15^3 = 3375$, considered in the 7th DIMACS challenge for semi-definite problems [17]. Bounds [18] on the ground-state cost established for the larger instance are $H_{\text{lower}} = -6138.02$ (from semi-definite programming) and $H_{\text{upper}} = -5831$ (from branch-and-cut). EO found $H_{\text{EO}} = -6049$ (or $H/n = -1.7923$), a significant improvement on the upper bound and already lower than $\lim_{n \to \infty} H/n \approx 1.786$... found in Refs. [21–27, 28]. Furthermore, we collected $10^5$ such states, which roughly segregate into $3$ clusters with a mutual Hamming distance of at least 100 distinct spins; at best a small sample of the $\approx 10^7$3 ground states expected [18]! For the smaller instance the bounds given are -922 and -912, resp., while EO finds -916 (or $H/n = -1.7891$) and was terminated after finding 10$^5$ such states. While this run (including sampling degenerate states) took only a few minutes of CPU (at 800MHz), the results for the larger instance required about 16 hours.

Finally, we note that we have considered the algorithm for making Bethe lattices previously in Refs. [31, 35]. In Ref. [21] we have studied the graph bipartitioning problem and found that the ground state energy was well above previous RS calculations from Ref. [1], but only minutely below numerical calculations obtained using simulated annealing [19]. In Ref. [35] we have considered some variations in the generation of Bethe lattices and found that they effect the results only in next-to-leading order.

### V. NUMERICAL RESULTS FOR BETHE LATTICES

We have simulated Bethe lattices with the algorithm described in Sec. II for $k + 1$ between 3 and 26, and graph sizes $n = 2^l$ for $l = 5, 6, \ldots, 10$ to obtain results for ground state energies, and for $n \in [16 \ldots 256]$ to determine their entropy. In the following, we present the results for ground-state energies and entropies from those simulations. The results are discussed in detail in Sec. IV.C.

#### A. Ground State Energies

To reach relative statistical errors of our averages roughly uniform with $n$ we generated initially a number of $10^5/\sqrt{n}$ instances for each $n$ and $k + 1$. Fortunately, deviations appear to narrow much faster than $1/\sqrt{n}$, and thus we added more instances at smaller $n$ with small extra computational cost to obtain narrow error bars there as well. In Tab. I we list the values of average energies according to Eq. (2), $\langle e_{k+1}(n) \rangle$, for each $k + 1$ and $n$, the number of instances used, and the average number of update steps required. Tab. I lists a few properties of the computations. The results for the number of updates has been also averaged over all connectivities $k + 1$, although lower-connected graphs require typically fewer updates. Note that this is the minimal number of updates needed to obtain the listed results, the actual number of updates taken up by each run of EO to ensure convergence was at least twice of that but could be much larger, according to the specification of the algorithm in Sec. IV.
We find that for the whole range of connectivities $k + 1$ studied here, the scaling corrections appear to be consistent with $\nu = 2/3$ within a few percent, except for two outliers at $k + 1 = 10$ and 25. Thus, we have plotted for each $k+1$ the values of $e_{k+1}(n)$ as a function of $1/n^{2/3}$ in Figs. 1 and 2. Although the extrapolation appears to be linear on that scale for each $k+1$, we have fitted the data with the more general form of Eq. (7). (Fits were weighted according to $n$ and to the inverse of the standard deviation for each point.) These fits are also shown as dashed lines in each of the Figs. 1 and 2.

The extrapolation results for the ground state energies appear to be quite stable under variation of the scaling form, for instance, when fitting with fixed $\nu = 2/3$ instead of Eq. (7). We estimate that each has a relative error of about $0.3\%$. Exceptions to this estimate we have to grant for the cases of $k+1 = 10$ and 25, in which case we also observe significant differences to the $\nu = 2/3$ corrections to scaling.

We can compare our results with existing theoretical predictions at the RS and the 1RSB level at least for the case of $k+1 = 3$. For this case, a recently published calculation yielded $e_3 = -1.2777$ at the RS, and $e_3 = -1.2717$ at the 1RSB level (further replica corrections are expected to be small). These values are also indicated in Fig. 1. Clearly, our result for $k+1 = 3$ is consistent with the 1RSB results, but certainly inconsistent with the RS result. Further 1RSB results for other values $k+1$ are currently being calculated [39]. We will discuss a more detailed analysis of the extrapolated values of $e_{k+1}$ at $n \to \infty$ in Sec. V C.

### Table I: Data from the EO simulations for the average ground-state energy per spin $e_{k+1}(n)$, plotted also in Figs. 1 and 2.

| $n$ | $e_3(n)$ | $e_4(n)$ | $e_5(n)$ | $e_6(n)$ | $e_7(n)$ | $e_8(n)$ | $e_9(n)$ | $e_{10}(n)$ | $e_{15}(n)$ | $e_{20}(n)$ | $e_{25}(n)$ |
|-----|----------|----------|----------|----------|----------|----------|----------|-----------|-----------|-----------|-----------|
| 32  | 1.3506(6)| 1.5543(6)| 1.6734(8)| 1.8424(8)| 1.9425(9)| 2.0906(9)| 2.1730(33)| 2.7013(40)| 3.1056(48)| 3.4923(51)|         |
| 64  | 1.2231(3)| 1.3961(4)| 1.5979(4)| 1.7924(5)| 1.8972(5)| 2.0083(6)| 2.1551(6)| 2.2557(23)| 2.8842(28)| 3.2090(33)|         |
| 128 | 1.2426(5)| 1.4245(10)| 1.6269(10)| 1.7885(8)| 1.9549(8)| 2.0782(10)| 2.2204(10)| 2.3234(11)| 2.8774(14)| 3.1861(16)| 3.7215(18)|
| 256 | 1.2542(3)| 1.4417(6)| 1.6434(7)| 1.7885(8)| 1.9549(8)| 2.0782(10)| 2.2204(10)| 2.3234(11)| 2.8774(14)| 3.1861(16)| 3.7215(18)|
| 512 | 1.2608(2)| 1.4534(4)| 1.6548(5)| 1.8020(5)| 1.9685(6)| 2.0934(6)| 2.2379(7)| 2.3488(7)| 2.8993(10)| 3.3435(11)| 3.7505(12)|
| 1024| 1.2644(1)| 1.4603(3)| 1.6612(3)| 1.8110(3)| 1.9762(5)| 2.1035(5)| 2.2470(5)| 2.3605(5)| 2.9092(7)| 3.3551(9)| 3.7612(11)|
| 2048| 1.2673(1)|         |         |         |         |         |         |         |         |         |         |

### Table II: Some properties of the numerical computations.

| $n$ | Instances | $t$ |
|-----|-----------|-----|
| 32  | 19444     | 3.0 $10^2$ |
| 64  | 13750     | 1.5 $10^3$ |
| 128 | 883       | 1.0 $10^4$ |
| 256 | 625       | 1.6 $10^5$ |
| 512 | 441       | 3.1 $10^6$ |
| 1024| 312       | 7.6 $10^7$ |
| 2048| 220       | 1.5 $10^8$ |
FIG. 2: Extrapolation plot for the EO data in Tab. 1 for $k+1 = 4$ to 25, as in Fig. 1. All data seems to extrapolate well linearly in $1/n^{2/3}$. The extrapolated values of $e_{k+1}$ for $n \to \infty$ are also listed in Tab. 1.
While the search for a ground state of an instance is certain to provide a rigorous upper bound to the actual ground state energy, the search for the complete set of ground states for an instance entails the risk of a two competing systematic errors. (1) If EO misses to find the exact ground state, one is likely to vastly over-count the degeneracy, since $\Omega$ is expected to rise exponentially with the energy above the ground state $[3]$. (2) Even if EO finds ground states, it may simply undercount $\Omega$, since such states could be too far separated in configuration space. Therefore, we have implemented EO with the settings described in Sec. II, which emphasize the desire for accuracy over computational efficiency. Accordingly, we were bound to conduct a separate set of simulations from those that determined the energies only. In these simulations we focused on smaller system sizes of $n \leq 256$ for $k+1 = 3, \ldots, 9$ and $10, 14, \ldots, 26$ only. The limit on $n$ for the smaller $k+1$ is mostly dictated by avoiding system sizes at which $\Omega$ typically exceeds $10^6$.

As a test for the accuracy of our implementation, we have run the simulation for $k+1 = 3$ twice on the exactly identical instances, using different initial conditions and $n/5$ more updates in the second run: The results, both for the energies and $\Omega$, were identical for each instance, producing the same set of configurations independent of the starting point of the search. We therefore assume that systematic errors in our data are small and can be neglected.

Since the range of system sizes $n$ is smaller than for the case of the energies, it is more difficult to extrapolate our data for $\langle s_{k+1}(n) \rangle$. Again, it is clear that the corrections are not linear in $1/n$, but instead seems to be scaling close to $1/n^{2/3}$ for all $k+1$, as for the energies above. Considering the limitations on $n$, we assume that the corrections are exactly of that form and extrapolate our data simply with a fit to

$$ s_{k+1}(n) \sim s_{k+1} + \frac{A}{n^{2/3}} \quad (n \to \infty), \quad (8) $$

again, weighting each data point with respect to $n$ and the inverse of its error. While the systematic and statistical uncertainties of our data appears to be small, the uncertainty about the scaling corrections must be considered the most significant limitation on accuracy in our extrapolation. The data and the extrapolation fits according to Eq. (8) are shown in Figs. 3 and 4. The results for $s_{k+1}$ for $n \to \infty$ are listed in Tab. III.

The data clearly shows a different quantitative behavior between odd and even values of $k+1$. This difference for the entropies can be explained in terms of the "free spins": In a highly frustrated system, even near ground states, many spins are stuck in a situation in which they violate many of their constraints, no matter how they are oriented, and changing from one direction to the other may hardly change the energy of the system. In particular, an even-connected spin that happens to violate exactly half of its bonds (with $J = \pm 1$) can flip freely without any change to the energy. Odd-connected spins can only become “free” in a connected pair (that happens to violate exactly half of its external bonds but satisfies their mutual bond) in which both simultaneously flip without changing the energy. The latter situation is naturally far less likely, and thus, purely even-connected graphs exhibit far more potential for degeneracy at the ground state than the corresponding odd-connected graphs. Some preliminary studies for $k+1 = 3$ and 4 show that in ground state configurations the fraction of free spins (zero by design for $k+1 = 3$) converges to a value just around 5% for $k+1 = 4$, while the fraction of free pairs seems to vanish for large $n$ for both, even and odd $k+1$. We have not explored the clustering of these states $\hat{8}$. We will explore the different behaviors for even and odd $k+1$ in the next Section.

### C. Discussion of the Extrapolation Results

In this section, we want to focus on some of the curious properties exhibited by the values of the energies and entropies found by extrapolation in the previous section. We have already noted the difference between the entropies for even and odd values of $k+1$. In fact, there are similar differences, although more subtle, for the energies $e_{k+1}$. These differences become most apparent when we plot the data asymptotically for large $k+1$, where it is known that

$$ \lim_{k+1 \to \infty} \frac{e_{k+1}}{\sqrt{k+1}} = E_{SK}, \quad (9) $$

with $E_{SK} = 0.7633$ being the RSB ground state energy of the Sherrington-Kirkpatrick model $\hat{3} \hat{4}$. In Fig. 3 we have plotted $e_{k+1}/\sqrt{k+1}$ as a function of $1/(k+1)$. On this scale, we notice that the energies split into a set of even and a set of odd values, each located apparently on a straight line. Even though $k+1 \leq 25$ is quite small, each line separately extrapolates very close to the exact value for large $k+1$ indeed: $E_{SK}^{even} \approx -0.763$ and $E_{SK}^{odd} \approx -0.765$. Even more amazing, the value of $e_2 = -1$ [see Eq. (10) below] for the trivial $k+1 = 2$ Bethe lattice is very close to the linear fit for the even EO results. Clearly, a function that would interpolate continuously

| $k+1$ | $s_{k+1}$ | $k+1$ | $s_{k+1}$ |
|-------|------------|-------|------------|
| 3     | 0.0102(10) | 4     | 0.0381(15) |
| 5     | 0.0048(10) | 6     | 0.0291(10) |
| 7     | 0.0020(10) | 8     | 0.0218(10) |
| 9     | 0.0002(15) | 10    | 0.0198(10) |
| 15    | 0.0002(15) | 14    | 0.0126(10) |
| 18    | 0.0095(10) | 22    | 0.0076(10) |
| 26    | 0.0063(15) |       |            |

*TABLE III: Extrapolation results for the entropies per spin for the data plotted in Figs. 3 and 4.*
FIG. 3: Extrapolation plot for the EO data for the entropy $s_{k+1}(n)$ for $k + 1 = 3$ to 15. All data seems to extrapolate well linearly in $1/n^{2/3}$. Note the difference in the results between odd (left) and even (right) $k + 1$. The extrapolated values of $s_{k+1}$ for $n \to \infty$ are listed in Tab. III.
FIG. 4: Extrapolation plot for the EO data for the entropy $s_{k+1}(n)$ for some larger, even $k+1$, similar to Figs. 3.

all the data will have to be very complicated (oscillatory). But could it be that its envelope on the even and the odd integers happens to be simple? Then, in case of the even data [44], we could even write down the exact form of the function for $E_{k+1}$ that would fit the data, since we know it also has to pass $e_2 = -1$ and satisfy Eq. (9):

$$E_{k+1} = \sqrt{k+1} E_{SK} - \frac{2E_{SK} + \sqrt{2}}{\sqrt{k+1}}.$$

To test Eq. (10), we plot the data in Fig. 5 as $e_{k+1}/E_{k+1}$ to study its deviations from the conjecture. While the extrapolated values do not fall exactly within their (estimated) error bars on the proposed form, they are indeed within about 0.1% of it. To judge how close the data is to the proposed functional form in Eq. (10), we utilize a closely related example. The ground-state energy as a function of the (continuous) average connectivity $\langle c \rangle$ is known exactly for the RS case of ordinary random graphs with fluctuating internal connectivities, Eq. (16) in Ref. [42]. If one plots that solution (which involves exponentials and modified Bessel functions) in the same way as $e_{k+1}$ in Fig. 4, one notes that it, too, could be approximated surprisingly well with a straight line, $-\sqrt{2c/\pi + \left(\sqrt{2/\pi - \frac{1}{2}}\right)/\sqrt{c}}$, now crossing the RS ground state energy $-\sqrt{2/\pi}$ for $\langle c \rangle \rightarrow \infty$ and reaching the trivial result of $-1/2$ at the percolation point $\langle c \rangle = 1$. In Fig. 7 we superimpose the relative error of this approximation with respect to the exact RS result with the relative error of our data with respect to the conjecture. It shows that the error of the conjecture is still almost by an order of magnitude smaller than the global bound for the RS example, thus putting a significant bound on any corrections similar in type to Eq. (16) in Ref. [42].

The differences between even and odd connectivities are even more pronounced in case of the entropies, as we have explained in Sec. V. Thus, although our data for the entropy is not nearly as accurate as for the energies, it is still instructive to study it in more detail. In Fig. 8, we plot the extrapolated values of the entropies from Tab. III to explore its decrease for large $k+1$. Despite the large error bars, a significant qualitative difference between even and odd data points is visible: The entropy for even values of $k+1$ decays slowly, apparently linearly with $1/(k+1)$. On the other hand, the entropies for odd $k+1$ drop much more rapidly, and are already indistinguishable from zero (within our errors) for $k+1 = 9$, while it is clearly non-vanishing for $k+1 = 3$ [unless our assumption about the scaling corrections in Eq. (8) are incredibly wrong (see Fig. 3)]. Unfortunately, with only a small, discrete number of data points available that are significantly above zero, it is very hard to decide whether the entropy for odd $k+1$ merely decays exponentially, or
both cases, we significant differences emerged between the data for odd and even values of $k+1$. Based on the numerical results, we showed that the extrapolated energies for all even values of $2 \leq k + 1 \leq \infty$ was well fitted with a simple function, Eq. (10). Furthermore, the data suggests that the entropies are generally non-zero at small $k+1$, but may vanish above a finite $k+1$ for odd values.

VI. CONCLUSION

In this paper we have presented an extensive numerical study of the ground states of spin glasses on Bethe lattices. The available data possessed sufficient accuracy to obtain extrapolated values for ground state energies and entropies at the 0.1% and the 10% level, respectively. In

Of course, there is plenty of reason to doubt that such a simple result as Eq. (10), albeit confined to discrete integer values of $k+1$, could indeed be the solution to a complex RSB problem. In fact, one argument against the conjecture is a discrepancy in its prediction for large $k+1$ at next-to-leading order. Several authors \cite{8, 12} have studied spin glasses on random graphs beyond the RS level perturbatively for $k + 1 = z \to \infty$ to determine the $1/z$ correction $f_1$ to the free energy (at $T > 0$) in $E_{SK} + f_1/z$. It was recently calculated \cite{8}, that the correction for fixed connectivities for $T \to 0$ would be about $f_1 = -0.317$, while Eq. (10) would predict about 0.1124. It should be noted, though, that the $1/z$ expansion implicitly assumes a smooth continuation off the integers which may lead to ambiguities in light of the oscillatory behavior between even and odd integers we found for the ground states in Fig. 5 (similar to the continuation of, say, the function $\cos(2\pi z)/z$ for $z \to \infty$, although this function would not possess a $1/z$ expansion at all).

In any case, future calculations like the one in Ref. \cite{1}, but for even $k+1$ will provide a check on both, our extrapolated data and the conjecture.
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Appendix

A. The Case $k+1 = 2$

Clearly, a Bethe lattice in which each vertex has exactly 2 connections can only consists of a collection of disconnected loop graphs. We merely need to determine the number of loops and their size distribution to derive the average ground state energy and entropy. Each loop has a 50% chance of being frustrated, thus, the number of cut bonds is equal to one-half of the number of loops, and the degeneracy is equal to the length of these loops to the power of their number.

To analyze the $k+1 = 2$ case we consider each vertex as a node with two terminals. Adding lines can create two types of objects: strings and loops. We consider, after adding $t$ lines, an individual vertex as a string of length 0, of which there are $l_{0,t}$; in general, we have $l_{i,t}$ strings of length $i$, each possessing two open terminals. In particular, before we added any lines: $l_{0,t=0} = \delta_{i,0}$. A loop of length $i$ is created by addition of a line to both open terminals of a string of length $i-1$. There are $p_{i,t}$ loops of length $i$ after adding $t$ lines which can not evolve further, since they don’t possess any more open terminals. We start with $2n$ open terminals and cover 2 of those with each newly added line. We can identify two constraints:

$$\sum_{i=0}^{\infty} l_{i,t} = n - t, \quad \sum_{i=1}^{\infty} i (l_{i,t} + p_{i,t}) = t.$$  \hspace{1cm} (11)

After adding $t$ lines at random, there are $2(n-t)$ terminals left to accommodate the next line, allowing for $\binom{2(n-t)}{2}$ different choices. Accounting for all possible choices, we obtain

$$l_{0,t+1} = \left[ 1 - \frac{1}{n-t} + \frac{1}{2(n-t)} \right] l_{0,t},$$

$$l_{i,t+1} = \left[ 1 - \frac{2}{n-t} + \frac{1}{2(n-t)} \right] l_{i,t} + \frac{2}{2(n-t)} \sum_{j=0}^{i-1} l_{j,t}l_{i-1-j,t} - l_{\frac{i}{2},t}$$

$$p_{i,t+1} = p_{i,t} + \frac{1}{2(n-t)} l_{i-1,t}.$$  \hspace{1cm} (12)

where $i > 0$. It is easy to show that these equations satisfy the constraints in Eqs. (11).

We can transform these equations by defining $\theta = t/n$, $d\theta = 1/n$, $y(x,\theta) = \frac{1}{n} \sum_{i=0}^{\infty} l_{i,t}x^i$, and $p(x,\theta) = \sum_{i=0}^{\infty} p_{i,t}x^i$. Considering $n$ large and $\theta$ continuous, Eqs. (12) turn into

$$\frac{dy(x,\theta)}{d\theta} = -\frac{2y(x,\theta)}{1-\theta} + \frac{x[y(x,\theta)]^2}{(1-\theta)[1-\theta-1/(2n)]},$$

$$\frac{dp(x,\theta)}{d\theta} = \frac{xy(x,\theta)}{(1-\theta)^2},$$

$$y(0) = 1, \quad p(x,0) = 0.$$  \hspace{1cm} (13)

Luckily, for $n \to \infty$, the equations are easily solved to give

$$y(x,\theta) = \frac{(1-\theta)^2}{1-x\theta}, \quad p(x,\theta) = -\frac{1}{2} \ln(1-x\theta).$$  \hspace{1cm} (14)

Finally, the total number of loops for the (almost) completed graph, $\theta = 1-1/n$, is given by

$$p(1,1-1/n) = \sum_{i=1}^{\infty} p_{i,n-1} \sim \frac{1}{2} \ln(n).$$  \hspace{1cm} (15)

On average, half of these loops will be frustrated, i.e., they will have one of their bonds violated. Since the Hamiltonian in Eq. (3) counts the difference between violated and satisfied bonds, or twice the violated bonds minus the number of all bonds, $n(k+1)/2 = n$, we get

$$e = \frac{H}{n} \sim -1 + \frac{\ln(n)}{2n}.$$  \hspace{1cm} (16)

Similarly, we can calculate the degeneracy $\Omega$ of these ground states, roughly, as the average length of loops,

$$\langle i \rangle = \langle x \rangle = \frac{\ln(n)}{(4n)} \sum_{x=1}^{n} \ln(x),$$

taken to the power of one-half of their number, $\ln(n)/(4n)$, to give

$$s = \frac{1}{4n} \ln(n) \sim \frac{1}{4n} \ln(n)^2.$$  \hspace{1cm} (17)

Clearly, both the number of violated bonds as well as the entropy vanish in the large-$n$ limit.

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