Presheaves on VI, nil-closed unstable algebras and their centres
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Abstract

A nil-closed, noetherian, unstable algebra $K$ over the Steenrod Algebra is determined, up to isomorphism, by the functor $\text{Hom}_{K,f.g.}(K, H^*(\_))$, which is a presheaf on the category $\mathcal{VI}$ of finite dimensional vector spaces and injections, by the theory of Henn-Lannes-Schwartz. In this article, we use this theory to study the centre, in the sense of Heard, of a nil-closed noetherian unstable algebra.

For $F$ a presheaf on $\mathcal{VI}$, we construct a groupoid $G_F$ which encodes $F$. Then, taking $F := \text{Hom}_{K,f.g.}(K, H^*(\_))$, we show how the centre of $K$ is determined by the associated groupoid.

We also give a generalisation of the second theorem of Adams-Wilkerson, defining sub-algebras $H^*(W)^G$ for appropriate groupoids $G$.

There is a $H^*(C)$-comodule structure on $K$ that is associated with the centre. For $K$ integral, we explain how the algebra of primitive elements of this $H^*(C)$-comodule structure is also determined by the groupoid associated with $\text{Hom}_{K,f.g.}(K, H^*(\_))$. Along the way, we prove that this algebra of primitive elements is also noetherian.

1 Introduction

1.1 The two theorems of Adams-Wilkerson

For $p$ a prime number, $W$ a finite dimensional $\mathbb{F}_p$-vector space, and for $BW$ the classifying space of $W$, $W \mapsto H^*(W) := H^*(BW; \mathbb{F}_p)$ defines a functor from $\mathcal{VI}$, the category of finite dimensional vector spaces, to $K$, the category of unstable algebra over the Steenrod algebra over $\mathbb{F}_p$. Then, for $G$ a sub-group of $\text{Gl}(W)$, $G$ acts on $H^*(W)$ and we can define $H^*(W)^G \in K$, the unstable algebra of invariant elements of $H^*(W)$ under the action of $G$. For $K$ an integral, noetherian, unstable algebra of transcendence degree $\dim(W)$, the first theorem of Adams-Wilkerson states that there always exists an injection $\phi$ from $K$ to $H^*(W)$ and that this injection induces a structure of finitely generated $K$-module on $H^*(W)$. Then, for $\text{Gal}(\phi)$ the sub-group of $\text{Gl}(W)$ whose elements are the automorphisms $\alpha$ such that $\alpha^*\phi = \phi$ with $\alpha^*\phi$ the composition of $\phi$ with the isomorphism induced by $\alpha$ on $H^*(W)$, the image of $\phi$ is a sub-algebra of $H^*(W)^{\text{Gal}(\phi)}$. $H^*(W)^{\text{Gal}(\phi)}$ is a first approximation of $K$, but usually $\phi$ does not define an isomorphism between $K$ and $H^*(W)^{\text{Gal}(\phi)}$.

The category of unstable modules over the Steenrod algebra admits a localizing subcategory $\mathcal{Nil}$ (cf [Sch94]), whose objects are called nilpotent unstable modules. We call an unstable module nil-closed if its localization away from $\mathcal{Nil}$ is an isomorphism. The algebras $H^*(W)^G$ are nil-closed, hence $\phi$ cannot be an isomorphism when $K$ is not nil-closed. In this case, and when $p = 2$ the
second theorem of Adams-Wilkerson gives a condition for \( \phi \) to be an isomorphism, when \( K \) is nil-closed. Namely, if \( K \) is

1. noetherian,
2. integral,
3. nil-closed,
4. integrally closed in its field of fractions,

the morphism \( \phi \) of the first theorem of Adams-Wilkerson is an isomorphism between \( K \) and \( H^\ast(W)^{\text{Gal}(\rho)} \). (There is a similar statement when \( p \) is odd.)

The first objective of this article is the following, for \( \mathcal{G} \) a groupoid whose objects are the sub-spaces of \( W \) and whose morphisms satisfy a restriction property, we define an object \( H^\ast(W)^{\mathcal{G}} \) that generalises the algebra of invariants \( H^\ast(W)^{\mathcal{G}} \). We will show that the \( H^\ast(W)^{\mathcal{G}} \) give a complete list of the nil-closed, noetherian, unstable sub-algebras of \( H^\ast(W) \).

### 1.2 The centre of an unstable algebra

In [DW92b], Dwyer and Wilkerson introduced the notion of a central element of an unstable algebra, this notion allowed them to exhibit the only exotic finite loop space at prime 2 in [DW92a]. In the case where \( K \) is noetherian and connected, the set of central elements of \( K \) coincides with the set of pairs \( (V, \psi) \) such that

1. \( \psi \in \text{Hom}_K(K, H^\ast(V)) \),
2. \( K \) admits a structure \( \kappa \) of \( H^\ast(V) \)-comodule in \( K \), such that the following diagram commutes:

\[
\begin{array}{ccc}
K & \xrightarrow{\kappa} & K \otimes H^\ast(V) \\
\downarrow{\phi} & & \downarrow{\epsilon_K \otimes \text{id}} \\
H^\ast(V) & & 
\end{array}
\]

where \( \epsilon_K \) denotes the augmentation of \( K \) (which is uniquely defined because of the connectedness of \( K \)).

In [Hea21], Heard showed that for \( K \) noetherian, \( K \) admits a unique (up to isomorphism) central element \( (C, \gamma) \) such that \( \gamma \) induces a structure of finitely generated \( K \)-module on \( H^\ast(C) \) and \( \dim(C) \) is maximal among such central elements. Heard called this central element the centre of \( K \). The centre of an unstable algebra have been shown to be an important invariant. In [Kuh07] and [Kuh13], Kuhn used it to approximate the depth of \( K \) as well as invariants \( d_0(K) \) and \( d_1(K) \) introduced by Henn, Lannes and Schwartz in [HLS95], in the case where \( K \) is the cohomology of a group. Heard generalised those results for \( K \) noetherian in [Hea20] and [Hea21].

For \( K \) noetherian, since the centre of \( K \) is associated with a \( H^\ast(C) \)-comodule structure on \( K \), it gives rise to a second invariant: the sub-algebra of primitive elements of \( K \) under this \( H^\ast(C) \)-comodule structure. The second objective of this article is to explain how the centre of \( H^\ast(W)^{\mathcal{G}} \) and its sub-algebra of primitive elements are determined by \( \mathcal{G} \). We will then be able to classify nil-closed, noetherian sub-algebras of \( H^\ast(W) \) with a given centre and algebra of primitive elements.
1.3 The category $\mathcal{S}et^{(V\mathcal{I})^{op}}$

We consider $\mathcal{K}/\mathcal{Nil}$ the localization of $\mathcal{K}$ with respect to morphisms whose kernels and cokernels are nilpotent unstable modules. In [HLS93], Henn, Lannes and Schwartz proved that the functor which sends an unstable algebra $K$ to $\text{Hom}_K(K, H^*(\_))$, the functor which maps $V \in \mathcal{V}^f$ to $\text{Hom}_K(K, H^*(V))$, induces an equivalence of category between $\mathcal{K}/\mathcal{Nil}$ and a certain sub-category of $\mathcal{S}et^{(V^f)^{op}}$ the category of contravariant functors from $\mathcal{V}^f$ to $\mathcal{S}et$. Furthermore, they used abundantly the fact that, when $K$ is noetherian, the functor $\text{Hom}_K(K, H^*(\_))$ is fully determined by the $\text{Hom}_K(K, H^*(V))$ for $V$ running through $\mathcal{V}^f$, where $\text{Hom}_K(K, H^*(V))$ is the subset of $\text{Hom}_K(K, H^*(V))$ whose objects are the morphisms from $K$ to $H^*(V)$ which turn $H^*(V)$ into a finitely generated $K$-module.

When $K$ is noetherian, $\text{Hom}_K(K, H^*(\_))$ defines a contravariant functor from $\mathcal{V}^f$ to $\mathcal{S}et$, where $\mathcal{V}^f$ is the wide sub-category of $\mathcal{V}^I$ whose morphisms are the injective morphisms. The category $\mathcal{S}et^{(V^f)^{op}}$ is a lot easier to understand than the category $\mathcal{S}et^{(V^I)^{op}}$, and that is going to be our main tool.

The two first sections of this article consist of recollections about the equivalences of categories constructed in [HLS93] and the definition of central elements of an unstable algebra. In the third section, we will introduce the category $\mathcal{S}et^{(V\mathcal{I})^{op}}$ and its connections with noetherian unstable algebras. Furthermore, we will define a notion of central elements for functors in $\mathcal{S}et^{(V\mathcal{I})^{op}}$. For $F = \text{Hom}_K(K, H^*(\_))$, with $K$ a noetherian nil-closed unstable algebra, the central elements of $F$ will coincide with the central elements $(V, \phi)$ of $K$, such that $\phi$ turns $H^*(V)$ into a finitely generated $K$-module.

**Theorem 4.51** For $K$ a noetherian, nil-closed unstable algebra, $F = \text{Hom}_K(K, H^*(\_))$ and $\phi \in F(V)$, $(V, \phi)$ is central for $K$ if and only if it is central for $F$.

1.4 The groupoid $\mathcal{G}_F$

An injection $\phi$ from a noetherian unstable algebra $K$ to some $\prod_{i \in I} H^*(W_i)$, where $\prod$ denotes the product in the category of connected unstable algebras and such that $\phi$ turns each $H^*(W_i)$ into a finitely generated $K$-module, induces a surjection from $\coprod_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(\_, W_i)$ to $\text{Hom}_{K/\mathcal{E}}(K, H^*(\_))$. For $I$ a set, we consider $(W_i)_{i \in I} \in \mathcal{S}et^{(V\mathcal{I})^{op}}$, the category whose objects are pairs $(F, q_F)$ with $F \in \mathcal{S}et^{(V\mathcal{I})^{op}}$ and $q_F$ a natural surjection from $\coprod_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(\_, W_i)$ to $F$.

In the fourth section, we define an application which sends an object $(F, q_F) \in (W_i)_{i \in I} \mathcal{S}et^{(V\mathcal{I})^{op}}$ to a groupoid $\mathcal{G}_{(F, q_F)}$ whose set of objects is the disjoint union of the sub-spaces of the $W_i$. This groupoid satisfies a property called the restriction property. The first main result of this article is that the isomorphism classes of objects in $(W_i)_{i \in I} \mathcal{S}et^{(V\mathcal{I})^{op}}$ are in one-to-one correspondence with such groupoids. This is stated in the following theorem, where $\sim_\mathcal{G}$ is an equivalence relation on $\coprod_{i \in I} \text{Hom}_{\mathcal{V}\mathcal{I}}(\_, W_i)$ characterised by the groupoid $\mathcal{G}$.

**Theorem 5.17** 1. For $\mathcal{G}$ a groupoid whose objects are the sub-spaces of the $W_i$ and whose
morphisms are isomorphisms of vector spaces such that \( \mathcal{G} \) has the restriction property,
\[
\mathcal{G}(\bigcup_{i \in I} \operatorname{Hom}_V(-, W_i)/_{\sim_{\mathcal{G}, q}}) = \mathcal{G},
\]
for \( q \) the canonical surjection from \( \bigcup_{i \in I} \operatorname{Hom}_V(-, W_i) \) to \( \bigcup_{i \in I} \operatorname{Hom}_V(-, W_i)/_{\sim_{\mathcal{G}}} \).

2. Conversely, let \( F \in \bigcup_{(W_i) \in I} \mathcal{S}_{\operatorname{Set}(\mathcal{V}^\mathcal{G})}^r \). Then, \( F \) is isomorphic to \( \bigcup_{i \in I} \operatorname{Hom}_V(-, W_i)/_{\sim_{\mathcal{G}, F}} \).

We also show, in Theorem \ref{thm:central_elements}, how the central elements of \( \bigcup_{i \in I} \operatorname{Hom}_V(-, W_i)/_{\sim_{\mathcal{G}}} \) are determined by the groupoid \( \mathcal{G} \).

Now, for \( K \) a noetherian, nil-closed, unstable sub algebra of \( H^*(W) \) with transcendence degree \( \dim(W) \), the inclusion of \( K \) in \( H^*(W) \) turns \( \operatorname{Hom}_{K_{\operatorname{op}}}(K, H^*(\_)) \) into an object of \( \mathcal{U}_{\operatorname{Set}(\mathcal{V}^\mathcal{G})} \). This implies the following:

**Theorem \ref{thm:noetherian-nil-closed}** For all \( W \in \mathcal{V}^f \), there is a one-to-one correspondence between the set of nil-closed and noetherian sub-algebras of \( H^*(W) \) whose transcendence degree is \( \dim(W) \) and the set of groupoids with the restriction property, whose objects are the sub-vector spaces of \( W \).

Then, for \( \mathcal{G} \) a groupoid with the restriction property whose objects are sub-spaces of \( W \), we will denote by \( H^*(W)^{\mathcal{G}} \) the noetherian, nil-closed sub-algebra of \( H^*(W) \) such that the groupoid associated with the surjection \( \operatorname{Hom}_V(-, W) \to \operatorname{Hom}_{K_{\operatorname{op}}}(H^*(W)^{\mathcal{G}}, H^*(\_)) \) is \( \mathcal{G} \).

By Theorems \ref{thm:central_elements} and \ref{thm:noetherian-nil-closed}, the central elements of \( H^*(W)^{\mathcal{G}} \) are determined by \( \mathcal{G} \). In the fifth section we will prove the following, where \( P(K, x) \) denotes the algebra of primitive elements of \( K \) for the \( H^*(V) \)-comodule structure induced by a central element \( (V, x) \).

**Theorem \ref{thm:central_elements}** Let \( K \) be a noetherian unstable sub algebra of \( H^*(W) \) of finite transcendence degree \( \dim(W) \) such that \( (V, \delta^* \phi) \) is central, for \( \phi \) the inclusion of \( K \) in \( H^*(W) \) and \( \delta \) some morphism from \( H^*(W) \) to \( H^*(V) \). Then, \( P(K, \delta^* \phi) \) is nil-closed and noetherian.

In this context, we will identify naturally \( P(K, \delta^* \phi) \) with a sub algebra of \( H^*(W/\operatorname{Im}(\delta)) \) of transcendence degree \( \dim(W/\operatorname{Im}(\delta)) \). By Theorem \ref{thm:noetherian-nil-closed}, \( P(K, \delta^* \phi) \) identifies with some \( H^*(W/\operatorname{Im}(\delta))^{\mathcal{G}} \). In the case where \( K = H^*(W)^{\mathcal{G}} \), we explain in Theorem \ref{thm:central_elements} how to determine \( \mathcal{G}' \) from \( \mathcal{G} \).

We conclude this article by giving examples on how to use those constructions to determine nil-closed, noetherian, integral, unstable algebras whose transcendence degree is fixed, with a \( H^*(V) \)-comodule structure whose primitive elements are isomorphic to some \( H^*(U)^{\mathcal{G}} \) with \( U \) in \( \mathcal{V}^f \) and \( \mathcal{G}' \) a groupoid with the restriction property and whose objects are the sub spaces of \( U \). For example:

**Theorem \ref{thm:central_elements}** Let \( K \) be a noetherian, nil-closed, integral, unstable algebra of transcendence degree \( d \). We assume that the centre of \( K \) is of dimension \( d - 1 \). Then, there exists \( G \), a sub-group of \( \operatorname{Gl}(W) \), such that \( K \) is isomorphic to the algebra of invariant elements \( H^*(W)^{\mathcal{G}} \) with \( \dim(W) = d \). Furthermore, \( G \) satisfies that the set of element \( x \in W \) such that \( g(x) = x \) for all \( g \in G \) is a sub-vector space of \( W \) of dimension \( d - 1 \).
This Theorem implies that for $K$ noetherian, nil-closed, integral of transcendence degree $d$ whose centre is of dimension $d - 1$ and for $\phi$ the injection of Adams-Wilkerson, $\phi$ is an isomorphism from $K$ to $H^* (W)^{\text{Gal}(\phi)}$.

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2 Unstable algebras over the Steenrod algebra and the functor $f$

In this section, we recall some known facts about Lannes’ $T$ functor as well as results from [HLS93] about the localization of the categories of unstable modules and algebras modulo away from nilpotent objects. Recollections about unstable algebras, unstable modules and nilpotent objects can be found in [Sch94]. In the following, $A$ denotes the Steenrod algebra over $\mathbb{F}_p$ with $p$ a prime number, $U$ and $K$ denote the category of unstable modules and unstable algebras over $A$ and $Nil$ denotes the class of nilpotent objects in $U$.

2.1 The $T$ functor

Let us recall the definition of Lannes’ $T$ functor.

Theorem 2.1. [Lan87, Proposition 2.1] For $V$, a finite dimensional vector space the functor $- \otimes H^* (V)$ has a left adjoint $T_V$.

Proposition 2.2. [Sch94, Proposition 3.8.4]

1. Let $K$ be an unstable algebra, then $T_V (K)$ is given in a natural way a structure of unstable algebra.

2. Then, $T_V$ defines a functor from $K$ to $K$ which is left adjoint to the tensor product with $H^* (V)$ in $K$.

Example 2.3. [Sch94, 3.9.1] For $V$ and $W$ two finite dimensional $\mathbb{F}_p$-vector spaces, there is an isomorphism of unstable algebras, natural in both $V$ and $W$, $T_V (H^* (W)) \cong H^* (W) \otimes \mathbb{F}_p^{	ext{Hom}(V,W)}$. By the adjunction property, we get that $(\mathbb{F}_p^{	ext{Hom}(V,W)})^\wedge \cong \text{Hom}_{U} (H^* (W), H^* (V))$. In other words, $\mathbb{F}_p [\text{Hom}(V,W)] \cong \text{Hom}_{U} (H^* (W), H^* (V))$ which is a theorem first proved by Adams, Gunawardena and Miller.

2.2 $Nil$-localisation of unstable modules

The class of nilpotent modules is a Serre class in $U$, we recall the existence of an equivalence of categories between $U/Nil$ (defined as in [Gab62]) and a category of functors. The proofs can be found in [HLS93].

Theorem 2.4. [HLS93, Part I.4] There is an adjunction of functors:

$$r_1 : U \longrightarrow U/Nil : s_1,$$
such that, for a morphism of unstable modules, $r_1(\phi)$ is an isomorphism if and only if ker(\phi) and coker(\phi) are objects in \text{Nil}.

Then, $U/\text{Nil}$ satisfies the following universal property: for $A$ an abelian category and $F : U \to A$ an exact functor such that for all $M \in \text{Nil}$, $F(M) = 0$, there exists a unique $G : U/\text{Nil} \to A$ such that $F = G \circ r_1$.

**Definition 2.5.** For an unstable module, $l_1(M) := s_1 \circ r_1(M)$ is the nil-localisation of $M$, and we say that $M$ is nil-closed if the unit of the adjunction $M \to l_1(M)$ is an isomorphism.

For $F$ the category of functors from the category $\mathcal{V}$ of finite dimensional $F_p$-vector spaces to the category $\mathcal{V}$ of $F_p$-vector spaces, we consider $f : U \to F$ the functor which assigns to $M$ in $U$, $f(M) : V \mapsto T_V(M)^0$.

The class $\text{Nil}$ satisfies that $f(M) = 0$ if and only if $M \in \text{Nil}$. Then, $f$ induces a functor $f'$ from $U/\text{Nil}$ to $F$ such that $f = f' \circ r_1$.

We will denote by $F_\omega$ the essential image of $f$ in $F$.

**Theorem 2.6.** [Sch94, Theorem 5.2.6] The functor $f'$ induces an equivalence of categories between $U/\text{Nil}$ and the category $F_\omega$.

In [HLS93], the authors exhibit a right adjoint to $f'$, $m'$ such that the restriction of $m'$ to $F_\omega$ is an inverse of the equivalence of categories induced by $f'$.

**Definition 2.7.** Let $m : F \to U$ be the composition of $m'$ with $s_1$.

**Definition 2.8.** For $F \in F$ and $V \in \mathcal{V}$, let $\Delta_V F$ denote the object of $F$ such that $\Delta_V F(W) = F(V \oplus W)$ and $\Delta_V F(\alpha) = F(id_V \oplus \alpha)$ for all $W \in \mathcal{V}$ and for all morphism $\alpha$ in $\mathcal{V}$. We recall the following results from [Kuh94].

**Proposition 2.9.** For $M \in U$ and $V \in \mathcal{V}$, $f(T_V(M)) = \Delta_V(f(M))$.

For $F$ an object of $F$, $T_V(m(F)) \cong m(\Delta_V(F))$.

**Corollary 2.10.** For $M \in U$ nil-closed, $T_V(M)$ is also nil-closed.

### 2.3 $\text{Nil}$-localisation of unstable algebras

Since $K$ is not abelian, one cannot define a localized category of $K$ in the sense of [Gab62]. In [HLS93], Henn, Lannes and Schwartz constructed a localized category $K/\text{Nil}$ with respect to the morphisms whose kernels and cokernels are in $\text{Nil}$, in the sense of [KS05]. Then, the functor $f$ restricted to $K$ factorises through a functor from $K/\text{Nil}$ to $F$. The authors of [HLS93] identified the essential image of $f$ restricted to $K$ and they deduced an equivalence of category between $K/\text{Nil}$ and a category of contravariant functors from $\mathcal{V}$ to the category of profinite sets.

**Definition 2.11.** A $p$-boolean algebra, is an algebra $B$ over $F_p$, such that, for all $x \in B$, $x^p = x$. 

Since $T_V(K)$ is an unstable algebra, $T_V(K)^0$ is a $p$-boolean algebra. We can then use standard results on $p$-boolean algebras to study $f(K)$.

For $\mathcal{B}$ the category of $p$-boolean algebras and $B$ a $p$-boolean algebra, we consider $\text{Hom}_\mathcal{B}(B, \mathbb{F}_p)$ the set of morphisms of $\mathbb{F}_p$-algebras from $B$ to $\mathbb{F}_p$. Then $B$ is the direct limit of its finite dimensional subalgebras $B_\alpha$. Therefore, $\text{Hom}_\mathcal{B}(B, \mathbb{F}_p)$ is the inverse limit of the $\text{Hom}_\mathcal{B}(B_\alpha, \mathbb{F}_p)$ which are finite. $\text{Hom}_\mathcal{B}(B, \mathbb{F}_p)$ inherits a structure of profinite set.

**Proposition 2.12.** [HLS93] For $\mathcal{P}\text{fin}$ the category of profinite sets, the functor $\text{spec} : \mathcal{B}^{\text{op}} \to \mathcal{P}\text{fin}$, where $\text{spec}(B) := \text{Hom}_\mathcal{B}(B, \mathbb{F}_p)$, is an equivalence of categories whose inverse is the functor which sends $S$ to the algebra of continuous maps from $S$ to $\mathbb{F}_p^\mathbb{F}_p$.

In particular, for $K \in K$, by adjunction $\text{Hom}_K(T_V(K)^0, \mathbb{F}_p) \cong \text{Hom}_K(K, H^*(V))$ comes from the fact that $K$ is the direct limit of the unstable sub-algebra of $K$ which are finitely generated as $A$-algebras. Then, $T_V(K)^0$ is isomorphic as a $p$-boolean algebra to $F_{\text{Hom}_K(K, H^*(V))}$.

**Definition 2.13.** 1. Let $\mathcal{P}\text{fin}^{(V^f)^{\text{op}}}$ be the category of functors from $(V^f)^{\text{op}}$ to $\mathcal{P}\text{fin}$,

2. let $\mathcal{L}$ be Lannes’ linearization functor from $(\mathcal{P}\text{fin}^{(V^f)^{\text{op}}})^{\text{op}}$ to $\mathcal{F}$ defined by $\mathcal{L}(F)(V) := \mathbb{F}_p^{F(V)}$,

3. let $g : K \to (\mathcal{P}\text{fin}^{(V^f)^{\text{op}}})^{\text{op}}$ be the functor which sends $K$ to the functor $g(K) : V \mapsto \text{Hom}_K(K, H^*(V))$.

We have a commutative diagram of functors:

$$
\begin{array}{ccc}
K & \xrightarrow{g} & (\mathcal{P}\text{fin}^{(V^f)^{\text{op}}})^{\text{op}} \\
\downarrow & & \downarrow \mathcal{L} \\
\mathcal{U} & \xrightarrow{f} & \mathcal{F},
\end{array}
$$

where the functor from $K$ to $\mathcal{U}$ is the forgetful functor. We denote by $\mathcal{P}\text{fin}_\omega^{(V^f)^{\text{op}}}$ the full subcategory of $\mathcal{P}\text{fin}^{(V^f)^{\text{op}}}$, whose objects are those whose image under $\mathcal{L}$ are in $\mathcal{F}_\omega$.

The functor $g$ has a unique factorisation of the following form:

$$
K \to K/\mathcal{N}\text{il} \to \mathcal{P}\text{fin}_\omega^{(V^f)^{\text{op}}} \to \mathcal{P}\text{fin}^{(V^f)^{\text{op}}}.
$$

**Theorem 2.14.** [HLS93, Theorem 1.5 of Part II] The functor from $K/\mathcal{N}\text{il}$ to $\mathcal{P}\text{fin}_\omega^{(V^f)^{\text{op}}}$ induced by $g$ is an equivalence of categories.

The following lemma will be of importance in the following.

**Lemma 2.15.** The functor $g$ turns injections into surjections and finite inverse limits into direct limits.

**Proof.** Since $f$ is exact, $f$ sends injections into injections and commutes with finite inverse limits. The result is then a consequence of the isomorphism $f(K) \cong \mathbb{F}_p^{f(K)}$. 

3 Connected components of $T_V(K)$ and central elements of an unstable algebra

In this section, we recall the definition of central elements of an unstable algebra first introduced by Dwyer and Wilkerson in [DW92b].

3.1 Connected components of $T_V(K)$

For $K$ an unstable algebra, we recall the definition of the connected components of $T_V(K)$ which is exposed in [Hea20] and [Hea21]. Such a decomposition exists for any graded algebra over a $p$-boolean algebra.

Lemma 3.1. For $K$ an unstable algebra which is finitely generated as an algebra over $A$, $\text{Hom}_K(K, H^*(V))$ is finite.

Definition 3.2. For $V \in V_f$ and $\phi \in \text{Hom}_K(K, H^*(V))$, let $T_{(V,\phi)}(K) := T_V(K) \otimes_{T_V(K)_0} \mathbb{F}_p(\phi)$, where the structure of $T_V(K)_0$-module over $\mathbb{F}_p(\phi)$ is induced by the morphism from $T_V(K)_0$ to $\mathbb{F}_p$ adjoint to $\phi$.

Proposition 3.3. [Hea20, Equation (2.6)] For $K$ an unstable algebra finitely generated as an algebra over $A$ and $V \in V_f$, we have the following natural isomorphism of unstable algebra

$$T_V(K) \cong \prod_{\phi \in \text{Hom}_K(K, H^*(V))} T_{(V,\phi)}(K).$$

Lemma 3.4. For $K$ a nil-closed unstable algebra, $V \in V_f$ and $\phi \in \text{Hom}_K(K, H^*(V))$, $T_{(V,\phi)}(K)$ is nil-closed.

Proof. By corollary 2.10 $T_V(K)$ is nil-closed. By the isomorphism of proposition 3.3, $T_{(V,\phi)}(K)$ is the kernel of the morphism from $T_V(K)$ to $\bigoplus_{\phi \neq \psi} T_V(K)$ which sends $x$ to the direct sum of the components of $x$ in each $T_{(V,\psi)}(K)$ with $\phi \neq \psi$. Since $\bigoplus_{\phi \neq \psi} T_V(K)$ is nil-closed and $l_1$ is left exact, $l_1(T_{(V,\phi)}(K))$ is the kernel of the same morphism, thus, it is isomorphic to $T_{(V,\phi)}(K)$. \qed

3.2 Central elements of an unstable algebra

The notion of a central element of an unstable algebra $K$ is defined by Dwyer and Wilkerson in [DW92b] and they used it in [DW92a] to exhibit the only exotic finite loop space at the prime 2. The centre of $K$ has been studied in details in [Hea20] and [Hea21].

The aim of this subsection is to recall some known facts about central elements of an unstable algebra.

Notation 3.5. Let $M$ be an unstable module, $V$ a finite dimensional vector space, $K$ an unstable algebra and $\phi \in \text{Hom}_K(K, H^*(V))$. Denote by:

- $\eta_{M,V} : M \to T_V(M) \otimes H^*(V)$ the unit of the adjunction between $T_V$ and $- \otimes H^*(V)$:
Remark 3.6. The morphism \( \rho \) is said to be central if

\[
\rho_{M, V} : M \xrightarrow{\eta_{M, V}} T_V(M) \otimes H^*(V) \xrightarrow{id \otimes \epsilon_V} T_V(M),
\]

where \( \epsilon_V \) denote the augmentation of \( H^*(V) \);

• \( \rho_{K, (V, \phi)} \) the composition of \( \rho_{K, V} \) with the projection onto \( T_{(V, \phi)}(K) \).

Remark 3.7. The morphism \( \rho_{M, V} \) identifies with \( T_{V_0}(M) : M \cong T_0(M) \rightarrow T_V(M) \), the morphism induced by naturality of \( T_V(M) \) with respect to \( V \) by \( i_0^V \), the injection from 0 to \( V \).

Definition 3.9. Let \( K \) be an unstable algebra and \( \phi \in \text{Hom}_K(K, H^*(V)) \). Then, the pair \( (V, \phi) \) is said to be central if \( \rho_{K, (V, \phi)} : K \rightarrow T_{(V, \phi)}(K) \) is an isomorphism.

Let \( C(K) \) be the set of central elements of \( K \).

The classical example and first motivation for studying the centre of an unstable algebra is the example of \( H^*(G) \), the cohomology of a group \( G \). The details can be found in [Hen01].

Example 3.8. For \( G \) a discrete group or a compact Lie group, \( \text{Hom}_K(H^*(G), H^*(V)) \cong \mathbb{F}_p \left[ \text{Rep}(V, G) \right] \), where \( \text{Rep}(V, G) \) denote the conjugacy classes of morphisms from \( V \) to \( G \).

Let \( \rho \) represent a conjugacy class in \( \text{Rep}(V, G) \). We consider the morphism \( V \times C_G(\rho) \rightarrow G \), where \( C_G(\rho) \) denote the centraliser in \( G \) of the image of \( \rho \), which sends \((v, g)\) to \( \rho(v) \cdot g \). It induces a morphism from \( H^*(G) \rightarrow H^*(V) \otimes H^*(C_G(\rho)) \). By adjunction, it gives us a morphism \( T_V(H^*(G)) \rightarrow H^*(C_G(\rho)) \) which depends only on the conjugacy class of \( \rho \). This morphism induce an isomorphism between \( T_{(V, \rho)}(H^*(G)) \) and \( H^*(C_G(\rho)) \), and

\[
\rho_{H^*(G), (V, \rho)} : H^*(G) \rightarrow T_{(V, \rho)}(H^*(G)) \cong H^*(C_G(\rho))
\]

is the morphism induced by the injection \( C_G(\rho) \rightarrow G \).

Hence, \( (V, \rho) \) is central if and only if the injection \( C_G(\rho) \rightarrow G \) induces an isomorphism in cohomology.

Definition 3.10. Let \( K \) be an unstable algebra, \( K \) is connected if \( K \) has an augmentation \( \epsilon_K : K \rightarrow \mathbb{F}_p \) which induces an isomorphism \( K^0 \cong \mathbb{F}_p \).

Remark 3.11. The connected components \( T_{(V, \phi)}(K) \) are connected, hence, if \( K \) is not connected, \( C(K) = \emptyset \).

Example 3.12. The functor \( T_0 \) is the identity, hence, if \( K \) is connected, \( T_{(0, \epsilon_K)}(K) \cong K \), for \( \epsilon_K : K \rightarrow \mathbb{F}_p \) the unique morphism of unstable algebra from \( K \) to \( \mathbb{F}_p \). Hence \( (0, \epsilon_K) \) is central.

Notation 3.13. Let \( \epsilon_{K, V} \), be the composition of \( \epsilon_K \) with the injection from \( \mathbb{F}_p \) to \( H^*(V) \).

For \( K \) a connected unstable algebra, \( I(K) \) denotes the augmentation ideal of \( K \). Then the module of indecomposable elements of \( K \) is defined by \( Q(K) := I(K)/I(K)^2 \). An unstable module \( M \) is said to be locally finite if, for all \( x \in M \), \( Ax \) is finite.

In [DW90], Dwyer and Wilkerson exhibit how, when \( Q(K) \) is locally finite, central elements of \( K \) are related to \( H^*(V) \)-comodule structures on \( K \).
Proposition 3.13. [DW90, Proof of Theorem 3.2]

Let $K$ be a connected unstable algebra such that $Q(K)$ is locally finite as an unstable module, then $(V, \epsilon_{K,V})$ is central for all $V \in \mathcal{V}$. In particular, if $K$ is a connected, noetherian, unstable algebra, then $(V, \epsilon_{K,V})$ is central for all vector space $V$.

We recall the following results of [DW92b].

Proposition 3.14. [DW92b, Proposition 3.4]

Let $K$ be a connected unstable algebra such that $Q(K)$ is locally finite. Then, for $\phi \in \text{Hom}_K(K, H^*(V))$, $(V, \phi)$ is central if and only if there exists a morphism from $K$ to $K \otimes H^*(V)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
K & \xrightarrow{\phi} & K \otimes H^*(V) \\
\downarrow{id} & & \downarrow{id \otimes \epsilon_{H^*(V)}} \\
K \otimes H^*(V) & & H^*(V).
\end{array}
$$

Corollary 3.15. [DW92b] Let $K$ be a connected unstable algebra such that $Q(K)$ is locally finite. For $\phi \in \text{Hom}_K(K, H^*(V))$, $(V, \phi)$ is central if and only if $K$ has a structure of $H^*(V)$-comodule $\kappa$ in $K$, such that the following diagram commutes:

$$
\begin{array}{ccc}
K & \xrightarrow{\kappa} & K \otimes H^*(V) \\
\downarrow{\phi} & & \downarrow{\epsilon_K \otimes id} \\
K \otimes H^*(V) & & H^*(V).
\end{array}
$$

In particular, this implies:

Proposition 3.16. Let $K$ be an unstable algebra such that $Q(K)$ is locally finite, then for $\phi \in \mathcal{C}(K)$ and $\alpha : V \rightarrow E$ a morphism in $\mathcal{V}$, $(V, \alpha^* \circ \phi) \in \mathcal{C}(K)$.

Example 3.17. For $W \in \mathcal{V}$, the addition in $W$, $\nabla_W$, induces on $H^*(W)$ a coalgebra structure in $K$. Then, for every morphism of unstable modules $\phi$ from $H^*(W)$ to $H^*(V)$, one can take the composition of $\nabla_W^*$ with $id_{H^*(W)} \otimes f$ to define a $H^*(V)$-comodule structure on $H^*(W)$ satisfying the hypothesis of corollary 3.15. Therefore $(V, \phi)$ is central.

4 The category $\text{Set}^{(\mathcal{V})^{op}}$

In [Rec84], Rector used the fact that for a noetherian unstable algebra $K$, the functor which maps $V$ to $\text{Hom}_K(K, H^*(V))$ is fully determined by the functor which maps $V$ to $\text{Hom}_{K^{fg}}(K, H^*(V))$, where $\text{Hom}_{K^{fg}}(K, H^*(V))$ is the set of morphism from $K$ to $H^*(V)$ which makes $H^*(V)$ a finitely generated $K$-module. The functor $V \mapsto \text{Hom}_{K^{fg}}(K, H^*(V))$ is defined on the category $\mathcal{V}$, whose objects
are finite dimensional vector spaces on \( \mathbb{F}_p \) and whose morphisms are injective morphisms. In this section, we start by recalling the notion of a regular element of a functor in \( \mathcal{P}_{\text{fin}}(V^f)^{\text{op}} \), introduced in [HLS93]. This allows us to understand the passage from \( \text{Hom}_K(K, H^*(\_)) \) to \( \text{Hom}_{K_{f.g.}}(K, H^*(\_)) \) as a construction on functors in \( \mathcal{P}_{\text{fin}}(V^f)^{\text{op}} \).

Then, we define a shift functor for contravariant functors on the category \( \mathcal{VI} \). This will allow us to define a notion of centrality for objects of \( \text{Set}(\mathcal{VI})^{\text{op}} \), which coincides for noetherian unstable algebras, with the notion of centrality "away from \( \mathcal{N}il \)" for pairs \( (V, \phi) \) such that \( \phi \in \text{Hom}_{K_{f.g.}}(K, H^*(V)) \).

### 4.1 The functor \( \text{Hom}_{K_{f.g.}}(K, H^*(\_)) \)

**Definition 4.1.** Let \( \mathcal{VI} \) be the wide sub-category of \( \mathcal{V}^f \) with only injective morphisms.

In the following, we will navigate between different presheaf categories from \( \mathcal{V}^f \) or \( \mathcal{VI} \) to variants of \( \text{Set} \). Let us define all those categories.

**Definition 4.2.**
1. Let \( \text{Set}(\mathcal{V}^f)^{\text{op}} \) and \( \text{Set}(\mathcal{VI})^{\text{op}} \) be the categories of contravariant functors from \( \mathcal{V}^f \) and \( \mathcal{VI} \) to the category of sets.
2. Let \( \mathcal{F}_{\text{fin}}(\mathcal{V}^f)^{\text{op}} \) and \( \mathcal{F}_{\text{fin}}(\mathcal{VI})^{\text{op}} \) be the categories of contravariant functors from \( \mathcal{V}^f \) and \( \mathcal{VI} \) to the category of finite sets.
3. Let \( \mathcal{P}_{\text{fin}}(\mathcal{V}^f)^{\text{op}} \) and \( \mathcal{P}_{\text{fin}}(\mathcal{VI})^{\text{op}} \) be the categories of contravariant functors from \( \mathcal{V}^f \) and \( \mathcal{VI} \) to the category of profinite sets.

**Definition 4.3.** For \( K \in \mathcal{K} \), we define \( \text{Hom}_{K_{f.g.}}(K, H^*(\_)) \) in \( \text{Set}(\mathcal{VI})^{\text{op}} \) which maps \( V \in \mathcal{VI} \) to the set of morphisms \( \phi \) from \( K \) to \( H^*(V) \) such that \( \phi \) makes \( H^*(V) \) a finitely generated \( K \)-module.

**Remark 4.4.** If \( K \) is finitely generated as an algebra over \( \mathcal{A} \), then, for all finite dimensional vector spaces \( V \), \( \text{Hom}_K(K, H^*(V)) \) is finite. Thus \( \text{Hom}_K(K, H^*(\_)) \) and \( \text{Hom}_{K_{f.g.}}(K, H^*(\_)) \) are respectively in \( \mathcal{F}_{\text{fin}}(\mathcal{V}^f)^{\text{op}} \) and \( \mathcal{F}_{\text{fin}}(\mathcal{VI})^{\text{op}} \).

In particular, this is the case for \( K \) noetherian.

### 4.2 Regular elements of an object in \( \text{Set}(\mathcal{V}^f)^{\text{op}} \)

In this sub-section we recall the notion of a regular element of an object of \( \text{Set}(\mathcal{V}^f)^{\text{op}} \). The interest of this notion is that, for \( F \) in \( \text{Set}(\mathcal{V}^f)^{\text{op}} \) satisfying a noetherianity condition introduced in [HLS93], the regular elements of \( F \) define an object in the much simpler category \( \text{Set}(\mathcal{VI})^{\text{op}} \). Moreover, if \( F \cong \text{Hom}_K(K, H^*(\_)) \) for \( K \) a noetherian unstable algebra, the regular elements of \( F \) are given by \( \text{Hom}_{K_{f.g.}}(H, H^*(\_)) \) and we can show that \( F \) is fully determined by its regular elements. This will allow us to make the connection between the study of the \( \mathcal{N}il \)-localisation of noetherian unstable algebras, and the study of functors in \( \mathcal{F}_{\text{fin}}(\mathcal{V}^f)^{\text{op}} \).

**Proposition 4.5.** [HLS93 Proposition-Definition 5.1] Let \( G \in \text{Set}(\mathcal{V}^f)^{\text{op}} \), \( V \in \mathcal{V}^f \) and \( s \in G(V) \). Then, there exists a unique sub-vector space \( U \) of \( V \), denoted by \( \ker(s) \), such that:

1. For all \( t \in G(W) \) and all morphism \( \alpha : V \to W \) such that \( s = G(\alpha)(t) \), \( \ker(\alpha) \subset U \).
2. There exists \( W_0 \) in \( (\mathcal{V})^{\text{op}} \), \( t_0 \in G(W_0) \) and \( \alpha_0 : V \to W_0 \) such that \( s = G(\alpha_0)(t) \) and \( \ker(\alpha_0) = U \).

3. There exists \( t_0 \in G(V/U) \) such that \( s = G(\pi)(t_0) \), where \( \pi \) is the projection of \( V \) onto \( V/U \).

**Definition 4.6.** Let \( G \in \mathbf{Set}^{(\mathcal{V})^{\text{op}}} \), \( V \in \mathcal{V} \) and \( s \in G(V) \). We say that \( s \) is regular if \( \ker(s) = 0 \).

Let \( \text{reg}(G)(V) := \{ x \in G(V) : \ker(x) = 0 \} \).

We recall the definition of a noetherian functor from [HLS93].

**Definition 4.7.** Let \( F \) be in \( \mathcal{P}\text{fin}^{(\mathcal{V})^{\text{op}}} \), we say that \( F \) is noetherian if it satisfies the following:

1. \( F \in \mathcal{F}\text{in}^{(\mathcal{V})^{\text{op}}} \),
2. there exists an integer \( d \) such that \( \text{reg}(F)(V) = \emptyset \) for \( \dim(V) > d \),
3. for all \( V \in \mathcal{V} \) and \( s \in F(V) \) and for all morphisms \( \alpha \) which takes values in \( V \), \( \ker(F(\alpha)s) = \alpha^{-1}(\ker(s)) \).

**Proposition 4.8.** [HLS93, Theorem 7.1]

1. If \( K \in \mathcal{K} \) is noetherian, \( \text{Hom}_K(K, H^*(\_)) \) is noetherian.
2. If \( F \in \mathbf{Set}^{(\mathcal{V})^{\text{op}}} \) is noetherian, then \( m \circ L(F) \in \mathcal{K} \) is noetherian.

**Lemma 4.9.** Let \( F \) be a noetherian functor, then, \( \text{reg}(F) \) is an object in \( \mathcal{F}\text{in}^{(\mathcal{V})^{\text{op}}} \).

**Proof.** We only have to prove that if \( x \in \text{reg}(F)(V) \), and if \( \alpha \) from \( E \) to \( V \) is an injection, \( \alpha^* x \) is regular. But, since \( F \) is supposed noetherian, \( \ker(\alpha^* x) = \alpha^{-1}(\ker(x)) \) which is equal to \( \{0\} \) since \( x \) is regular and \( \alpha \) is an injection.

**Remark 4.10.** \( \text{reg} \) is not a functor, since the image under a natural transformation of a regular element is not necessarily regular. For example, fix \( V \) and \( U \) where \( U \neq 0 \) is a sub-vector space of \( V \), and consider the natural transformation from \( \text{Hom}_{\mathbb{F}_2}(\_, V) \) to \( \text{Hom}_{\mathbb{F}_2}(\_, V/U) \) induced by the projection from \( V \) to \( V/U \). Then, \( \text{id}_V \) is regular in \( \text{Hom}_{\mathbb{F}_2}(V, V) \) but \( \ker(\pi \circ \text{id}_V) = U \) in \( \text{Hom}_{\mathbb{F}_2}(V, V/U) \).

**Proposition 4.11.** [DW92b, Proposition 4.8] For \( K \) a noetherian unstable algebra, we have a natural isomorphism \( \text{reg} \circ \text{Hom}_K(K, H^*(\_)) \cong \text{Hom}_{\mathbb{F}_2}(K, H^*(\_)) \).

Let us denote by \( \mathcal{O} \) the forgetful functor from \( \mathbf{Set}^{(\mathcal{V})^{\text{op}}} \) to \( \mathbf{Set}^{(\mathcal{V})^{\text{op}}} \) induced by restriction to \( (\mathcal{V})^{\text{op}} \). We define the left Kan extension of \( \mathcal{O} \) along the identity.

**Proposition 4.12.** The functor \( \mathcal{O} \) has a left adjoint, which maps a functor \( F \in \mathbf{Set}^{(\mathcal{V})^{\text{op}}} \) to \( \tilde{F} \), which is defined by \( \tilde{F}(V) := \bigsqcup_{U \in \mathcal{S}(V)} F(V/U) \), where \( \mathcal{S}(V) \) is the set of sub-vector spaces of \( V \).

**Proof.** Let us first prove that \( \tilde{F} \) defines an object in \( \mathbf{Set}^{(\mathcal{V})^{\text{op}}} \). To a morphism \( \alpha : V \to W \) we associate \( \tilde{F}(\alpha) \) in the following way: for \( x \in F(W/U) \) we consider \( \pi \circ \alpha \) where \( \pi \) is the projection from \( W \) to \( W/U \), we factorise \( \pi \circ \alpha \) as \( \tilde{\alpha} \circ \psi \) with \( \psi \) the projection from \( V \) to \( V/\ker(\pi \circ \alpha) \), then \( \tilde{\alpha} \) is injective and we define \( \tilde{F}(\alpha)(x) = F(\tilde{\alpha})(x) \in F(V/\ker(\pi \circ \alpha)) \).
Let $\phi$ be a natural transformation from $\tilde{F}$ to $A$, where $A$ is an object in $\text{Set}^{{V^I}^\text{op}}$. For all $V \in V^I$, we have a morphism in $\text{Set}$,

$$\phi_V : \bigsqcup_{U \in S(V)} F(V/U) \to A(V).$$

Then,

$$\phi_V|_{F(V)}$$

induces a morphism $\iota(\phi)_V$ from $F(V)$ to $\mathcal{O}(A)(V)$. $\iota(\phi)$ is a natural transformation in $\text{Set}^{{V^I}^\text{op}}$. Conversely, for $\phi$ a natural transformation from $F$ to $\mathcal{O}(A)$, we define

$$\gamma(\phi)_V : \tilde{F}(V) \to A(V)$$

in the following way: for $x \in F(V/U)$ we define $\gamma(\phi)_V(x) := A(\pi)(\phi_{V/U}(x))$, for $\pi$ the projection from $V$ to $V/U$. Then, $\gamma(\phi)$ is a natural transformation in $\text{Set}^{{V^I}^\text{op}}$, and $\gamma$ and $\iota$ are mutually inverse.

**Remark 4.13.** For $F \in \text{Set}^{{V^I}^\text{op}}$, $\tilde{F}$ always satisfies the third condition in the definition of a noetherian functor. Moreover, $\tilde{F}$ is in $\mathcal{F}\text{in}^{{V^I}^\text{op}}$ if and only if $F \in \mathcal{F}\text{in}^{{V^I}^\text{op}}$ and there is an integer $d$ such that $F(V) = \emptyset$ for $\dim(V) \geq d$, hence $\tilde{F}$ is noetherian if and only if $F \in \mathcal{F}\text{in}^{{V^I}^\text{op}}$ and there exists $d \in \mathbb{N}$ such that $F(V)$ is empty for $\dim(V)$ greater than $d$.

**Proposition 4.14.** For $F$ a noetherian functor, $\widehat{\text{reg}(F)} \cong F$.

**Proof.** Let $F$ be a noetherian functor. For $V$ a finite dimensional vector space, we define the following morphism from $\widehat{\text{reg}(F)}(V)$ to $F(V)$:

$$\widehat{\text{reg}(F)}(V) \to F(V) \quad \text{where} \quad x \in \text{reg}(F)(V/U) \mapsto F(\pi)(x),$$

where $\pi$ is the projection from $V$ to $V/U$. Then, Proposition 4.5 implies that this morphism is an isomorphism, and the fact that $F$ is noetherian implies that it is natural in $V$.

**Corollary 4.15.** For $K$ a noetherian unstable algebra, there is a natural bijection with respect to $V$,

$$\text{Hom}_K(K, H^*(V)) \cong \bigsqcup_{U \in S(V)} \text{Hom}_{K,f.g.}(K, H^*(V/U)).$$

### 4.3 Central elements of a noetherian algebra

In the last subsection, we explained why, for $K$ a noetherian unstable algebra, $\text{Hom}_K(K, H^*(\_))$ is fully determined by its regular elements, given by $\text{Hom}_{K,f.g.}(K, H^*(\_))$. In this section, we prove that for $K$ noetherian, one can deduce $\mathbf{C}(K)$ from the central elements $(V, \phi)$ of $K$ such that $\phi$ is regular.

We recall the definition of the centre of a noetherian unstable algebra from [Hea21] which is in some sense the maximal regular central element of $K$.

We state the following results for noetherian algebras, since it is in this case that we will use those, but all of them are true if we only suppose that $\mathcal{Q}(K)$ is locally finite.
**Proposition 4.16.** Let $K$ be a noetherian unstable algebra and let $\phi$ be in $\text{Hom}_K(K, H^*(E))$, then if $(E, \phi)$ is central, let $\phi_0$ be the only element in $\text{Hom}_K(K, H^*(E/\ker(\phi)))$ such that $\phi = \pi^*\phi_0$. Then, $\phi_0$ is regular and $(E/\ker(\phi), \phi_0)$ is central.

Conversely, if $(E/U, \phi)$ is central with $\phi$ regular, $(E, \pi^*\circ \phi)$ is central. Thus, $(E, \phi)$ is central if and only if $(E/\ker(\phi), \phi_0)$ is central.

**Proof.** We consider $s$ a section of $\pi$, the projection from $E$ to $E/\ker(\phi)$, then $\phi_0 = s^*\circ \phi$. Then, by Proposition 3.16 if $(E, \phi)$ is central, so is $(E/\ker(\phi), \phi_0)$. The converse is a direct consequence of Proposition 3.16.

**Lemma 4.17.** [DW92b, Lemma 4.6] Let $K$ be a noetherian unstable algebra, let $f$ be in $\text{Hom}_K(K, H^*(E))$ and $(C, g) \in C(K)$, then there exists a unique pair $(E \oplus C, f \boxplus g)$ such that $f \boxplus g$ composed respectively with the projections on $H^*(E)$ and $H^*(C)$ gives $f$ and $g$.

\[
\begin{array}{c}
K \\
\downarrow \text{f} \boxplus \text{g} \\
H^*(C) \oplus H^*(E) \\
\downarrow \text{id} \oplus \epsilon_{H^*(E)} \\
H^*(E).
\end{array}
\]

**Remark 4.18.** In Lemma 4.17 even if $f$ and $g$ are regular, $f \boxplus g$ is not regular in general.

We recall now the definition of the centre of a noetherian algebra.

**Definition 4.19.** [Hea21, Definition 3.9] Let $K$ be a noetherian unstable algebra, let $f$ be in $\text{Hom}_K(K, H^*(E))$ and $(C, g) \in C(K)$, we define $(E \circ C, \sigma(f, g))$ by $E \circ C = (E \oplus C)/\ker(f \boxplus g)$, and $\sigma(f, g) : K \rightarrow E \circ C$ such that the composition of $\sigma(f, g)$ by $H^*(E \oplus C)/\ker(f \boxplus g)$ is equal to $f \boxplus g$, for $\pi_{(E \oplus C)/\ker(f \boxplus g)}$ the projection on $(E \oplus C)/\ker(f \boxplus g)$.

**Proposition 4.20.** [Hea21, Corollary 3.11] Let $K$ be a noetherian unstable algebra and $(C, g)$ and $(E, f)$ be two central elements of $K$. Then $(E \oplus C, f \boxplus g)$ and $(E \circ C, \sigma(f, g))$ are central.

**Theorem 4.21.** [Hea21, Theorem 3.13] Let $K$ be a connected, noetherian, unstable algebra. Then, up to isomorphism, there is a unique couple $(C, \gamma) \in C(K)$ which is regular and satisfy the following: for all central element $(E, f)$ with $f$ regular, there exists an injection $\iota$ from $E$ to $C$, such that $f = \iota^*\gamma$. We will call $(C, \gamma)$ the centre of $K$.

The idea of the proof is to consider $(C, \gamma)$ a regular central element of maximal dimension. Then, show that for $(E, f)$ regular and central, either $(E, f) = (E, \iota^*\gamma)$ for $\iota$ an injection from $E$ to $C$, or the dimension of $E \circ C$ is greater than that of $C$ which contradicts the assumption.

**Corollary 4.22.** For $E \in V^f$ and $f \in \text{Hom}_K(K, H^*(V))$, $(E, f)$ is central if and only if there is a morphism $\alpha$ from $E$ to $C$, with $(C, \gamma)$ the centre of $K$, such that $f = \alpha^*\gamma$.

**Proof.** It is a direct consequence of Proposition 3.16 and Theorem 4.21.
4.4 The shift functor

We want to be able to discuss centrality away from $\mathcal{N}il$, and define accordingly a notion of central elements for objects in $\text{Set}^{(V)}$.

The aim of this subsection is to define, for $F \in \text{Set}^{(V)}$, $V \in \mathcal{V}$ and $\phi \in F(V)$ a functor $\sigma_{(V,\phi)}F$, in such a way that for $F = \text{Hom}_{\mathcal{K}f.g.}(K, H^*(\_))$, $\sigma_{(V,\phi)}F \cong \text{Hom}_{\mathcal{K}f.g.}(T_{(V,\phi)}(K), H^*(\_))$.

We start by defining functors $\Sigma_V$ from $\text{Set}^{(V)^{op}}$ to $\text{Set}^{(V)^{op}}$ such that, for all unstable algebras $K$, $\Sigma_V \text{Hom}_{\mathcal{K}}(K, H^*(\_)) \cong \text{Hom}_{\mathcal{K}}(T_V(K), H^*(\_))$, and such that $\Sigma_V \text{Hom}_{\mathcal{K}}(K, H^*(\_))$ naturally decomposes in $\Sigma_{(V,\phi)} \text{Hom}_{\mathcal{K}}(K, H^*(\_)) \cong \text{Hom}_{\mathcal{K}}(T_{(V,\phi)}(K), H^*(\_))$, with $\phi$ running through $\text{Hom}_{\mathcal{K}}(K, H^*(V))$.

Then, we define a shift functor $\sigma_V$ in the category $\text{Set}^{(V)^{op}}$ and we identify the desired functors $\sigma_{(V,\phi)}F$ as the connected components of $\sigma_V F$. It is worth pointing out that since the decomposition of $\text{Hom}_{\mathcal{K}f.g.}(T_V(K), H^*(\_))$ is induced by elements $(V, \phi)$ with $\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$ (and not only elements in $\text{Hom}_{\mathcal{K}f.g.}(K, H^*(V))$), $\sigma_V \text{Hom}_{\mathcal{K}f.g.}(K, H^*(\_))$ is only a sub functor of $\text{Hom}_{\mathcal{K}f.g.}(T_V(K), H^*(\_))$.

**Definition 4.23.** For $V \in \mathcal{V}$, define the functor $\Sigma_V$ from $\text{Set}^{(V)^{op}}$ to $\text{Set}^{(V)^{op}}$ by

$$\Sigma_V F(W) = F(V \oplus W),$$

and $\Sigma_V F(\alpha) = F(\text{id}_V \oplus \alpha)$, for $F \in \text{Set}^{(V)^{op}}$, $W \in \mathcal{V}$ and $\alpha$ a morphism in $\mathcal{V}$.

**Lemma 4.24.** For $K \in \mathcal{K}$ and $V \in \mathcal{V}$, there is an isomorphism natural both in $K$ and $V$, $\text{Hom}_{\mathcal{K}}(T_V(K), H^*(\_)) \cong \Sigma_V \text{Hom}_{\mathcal{K}}(K, H^*(\_))$.

**Proof.** This is a direct consequence of the definition of $T_V$ as the adjoint of the tensor product by $H^*(V)$ and of the natural isomorphism $H^*(V) \otimes H^*(W) \cong H^*(V \oplus W)$. \hfill $\square$

**Definition 4.25.** For $F \in \text{Set}^{(V)^{op}}$, $V \in \mathcal{V}$ and $\phi \in F(V)$, we consider $\Sigma_{(V,\phi)} F(W)$ to be the fibre over $\{\phi\}$ of the morphism $\Sigma_V F(W) \to \Sigma_V F(0) \cong F(V)$ induced by the injection from 0 to $W$.

**Lemma 4.26.** Let $K$ be an unstable algebra, $V \in \mathcal{V}$ and $\phi \in \text{Hom}_{\mathcal{K}}(K, H^*(V))$. There is a natural isomorphism $\Sigma_{(V,\phi)} \text{Hom}_{\mathcal{K}}(K, H^*(W)) \cong \text{Hom}_{\mathcal{K}}(T_{(V,\phi)}(K), H^*(W))$.

**Proof.** We have a commutative diagram

$$
\begin{array}{ccc}
\Sigma_V \text{Hom}_{\mathcal{K}}(K, H^*(W)) & \cong & \text{Hom}_{\mathcal{K}}(T_V(K), H^*(W)) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathcal{K}}(K, H^*(V)) & \cong & \text{Hom}_{\mathcal{K}}(T_{(V,\phi)}(K), H^*(W)),
\end{array}
$$

where the vertical maps are induced by the injection from 0 to $W$, and the horizontal ones are given by the natural isomorphism of Lemma 4.24. By definition, $\Sigma_{(V,\phi)} \text{Hom}_{\mathcal{K}}(K, H^*(W))$ is the fibre over $\phi$ of the left map, and by construction $\text{Hom}_{\mathcal{K}}(T_{(V,\phi)}(K), H^*(W))$ is the fibre over the adjoint of $\phi$ of the right one. This concludes the proof. \hfill $\square$

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Proposition 4.27. If $F$ is noetherian, for $x$ an element in $F(V)$, $\Sigma VF$ and $\Sigma_{(V,x)}F$ are also noetherian.

Proof. If $F$ is finite, $\Sigma VF$ and $\Sigma_{(V,x)}F$ are obviously finite, we only have to prove the second condition. Let $a$ be an element in $\sigma VF(W)$, then $a \in F(V \oplus W)$. There is an ambiguity in considering $\ker(a)$, since $a$ can be viewed either as an element in $F$ or $\Sigma VF$. Let $\ker^1(a)$ denote the kernel of $a$ considered as an element in $F$, and $\ker^2(a)$ its kernel in $\Sigma VF$. We notice that $\ker^1(a)$ is a sub-vector space of $V \oplus W$, whereas $\ker^2(a)$ is a sub-space of $W$. Then, $\ker^2(a) = \ker^1(a) \cap W$. Indeed, for $t$ regular such that $a = \pi^*t$ for $\pi$ the projection from $V \oplus W$ to $(V \oplus W)/\ker^1(a)$, $\pi$ factorises as $\pi_2 \circ \pi_1$, for $\pi_1$ the projection from $V \oplus W$ to $V \oplus (W/(\ker^1(a) \cap W))$ and $\pi_2$ the projection from $V \oplus (W/(\ker^1(a) \cap W))$ to $(V \oplus W)/\ker^1(a)$, then $a = \Sigma VF(\pi')(\pi^*_2t)$, for $\pi'$ the projection from $W$ to $W/(\ker^1(a) \cap W)$. Therefore, $\ker^1(a) \cap W \subset \ker^2(a)$.

Conversely, if $\pi$ is now the projection from $W$ to $W/\ker^2(a)$, let $t \in F(V \oplus (W/\ker^2(a)))$ such that $a = \Sigma VF(\pi)(t)$. Then, $a = (\id_V \oplus \pi)^*t$, hence $\ker^2(a) \subset \ker^1(a)$. Since, we also have $\ker^2(a) \subset W$, $\ker^2(a) = \ker^1(a) \cap W$. Then, for $\alpha$ a morphism from a finite dimensional vector space $U$ to $W$, $\ker^2(\Sigma VF(\alpha)(a)) = \ker^1((\id_V \oplus \alpha)^*a) \cap W$ which is equal to $(\id_V \oplus \alpha)^{-1}(\ker^1(a)) \cap W$, since $F$ is noetherian. Hence, it is equal to $\alpha^{-1}(\ker^1(a) \cap W) = \alpha^{-1}(\ker^2(a))$. This proves that $\Sigma VF$ is noetherian. The proof for $\Sigma_{(V,x)}F$ is similar. \qed

Corollary 4.28. If $K$ is noetherian and nil-closed, $T_V(K)$ and $T_{(V,\phi)}K$ are noetherian, for $V \in V^f$ and $\phi \in \text{Hom}_K(K, H^*(V))$.

Proof. By Proposition 4.27, $g(T_V(K))$ and $g(T_{(V,\phi)}(K))$ are noetherian, then by Proposition 4.8, $l_1(T_V(K)) \cong (m \circ L \circ g)(T_V(K))$ and $l_1(T_{(V,\phi)}(K))$ are noetherian. Therefore, since by Corollary 2.10 and Lemma 3.4, $T_V(K)$ and $T_{(V,\phi)}(K)$ are nil-closed, they are noetherian. \qed

We want to identify a shift functor $\sigma_V$ from $\text{Set}^{(V^f)^{op}}$ to $\text{Set}^{(V^f)^{op}}$, in so that $\sigma_V \text{Hom}_{K, \mathbb{Z}}(K, H^*(\_))$ captures the behaviour of $T_V(K)$ away from nilpotent objects for $K$ noetherian, and such that for $F$ in $\text{Set}^{(V^f)^{op}}$, $\sigma_V F$ comes with a decomposition in $\sigma_{(V,\phi)}F$ with $\phi$ running through $F(V)$.

In order to define this shift functor, we have to discuss pushouts in $V^f$ and $V^I$. It is worth noticing that the pull-back in $V^f$ of a diagram whose morphisms are in $V^I$ is also a pull-back in $V^I$.

Remark 4.29. Pushouts usually don’t exist in $V^I$, for example, if one consider the following diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & V \\
\downarrow & & \downarrow \\
V & \longrightarrow & V
\end{array}
$$

the pushout “should be” $V \oplus V$, but since non injective morphisms are not in $V^I$ the commutative square

$$
\begin{array}{ccc}
0 & \longrightarrow & V \\
\downarrow & & \downarrow \\
V & \longrightarrow & V
\end{array}
$$

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does not give rise to a morphism from $V \oplus V$ to $V$ satisfying the universal property of the pushout. So, in the following, we will consider pushouts in $\mathcal{V}^I$ of diagrams in $\mathcal{V}I$.

The following simple lemma deals with this.

**Lemma 4.30.** Consider a pushout square in $\mathcal{V}^I$:

\[
\begin{array}{c}
M \xrightarrow{\omega} W \\
\downarrow \nu \\
V \xrightarrow{\mu} P
\end{array}
\]

Suppose that $\nu$ and $\omega$ are injections. Then, for all pullback squares

\[
\begin{array}{c}
M \xrightarrow{\omega} W \\
\downarrow \nu \\
V \xrightarrow{\mu} N
\end{array}
\]

such that the morphisms from $V$ and $W$ to $N$ are injections, the morphism from $P$ to $N$ induced by the universal property of the pushout in $\mathcal{V}^I$, is a morphism in $\mathcal{V}I$.

Let us also recall the pasting law for pullbacks:

**Lemma 4.31.** ([AHS90, Proposition 11.10] We consider a diagram of the following shape in a category $\mathcal{C}$:

\[
\begin{array}{cc}
A & B \\
\downarrow & \downarrow \\
D & E
\end{array} \xrightarrow{C} \begin{array}{cc}
B & C \\
\downarrow & \downarrow \\
E & F
\end{array}
\]

If the right square is a pullback square, then the outer square is a pullback square if and only if the left square is.

**Definition 4.32.** For $V$ and $W$ some fixed objects of $\mathcal{V}^I$, let $B(V,W)$ be the set of triples $(M, \nu, \omega)$, where $M \in \mathcal{V}^I$ and $\nu$ and $\omega$ are morphisms from $M$ to $V$ and $W$ in $\mathcal{V}I$. Let also $\equiv$ be the relation on $B(V,W)$ defined by $(M, \nu, \omega) \equiv (M', \nu', \omega')$ if there exists an isomorphism $\mu$ from $M'$ to $M$ such that $\nu' = \nu \circ \mu$ and $\omega' = \omega \circ \mu$.

The following lemma is obvious.

**Lemma 4.33.** $\equiv$ is an equivalence relation on $B(V,W)$.

**Definition 4.34.** For $V$ and $W$ two objects of $\mathcal{V}I$, let $B(V,W)$ be the set of equivalence classes for $\equiv$ in $B(V,W)$. For $[M, \nu, \omega] \in B(V,W)$, let $V \oplus_{\nu, \omega} W$ denote the pushout of the following diagram in $\mathcal{V}^I$:

\[
\begin{array}{c}
M \xrightarrow{\omega} W \\
\downarrow \nu \\
V
\end{array}
\]

let also denote by $\iota_V^{V \oplus_{\nu, \omega} W}$ and $\iota_W^{V \oplus_{\nu, \omega} W}$ the induced injections from $V$ and $W$ to $V \oplus_{\nu, \omega} W$. When there is no ambiguity, we will denote them by $\iota_V$ and $\iota_W$. 

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Remark 4.35. $V \oplus_{\nu, \omega} W$ does not depend on the choice of $(M, \nu, \omega) \in [M, \nu, \omega]$.

Proposition 4.36. $B(\omega, \nu)$ is a bifunctor on $\mathcal{V}I^{op}$.

Proof. Let $\alpha : V' \to V$ and $\beta : W' \to W$ be morphisms in $\mathcal{V}I$, for $[M, \nu, \omega] \in B(V, W)$ we define $B(\alpha, \beta)([M, \nu, \omega]) \in B(V', W')$ in the following way. We consider the following diagram, where $M_V, M_W$ and $M'$ are defined by pullback.

Then, $\nu'$ and $\omega'$ are given by the compositions $M' \to M_V \to V'$ and $M' \to M_W \to W'$.

Remark 4.37. By the pasting law of pullbacks (lemma 4.31, that we used two times) we can show that the outer square of the following diagram is a pullback square:

Definition 4.38. Let $\alpha : V' \to V$ and $\beta : W' \to W$ be injections in $\mathcal{V}I$, for $[M, \nu, \omega] \in B(V, W)$ and for $[M', \nu', \omega'] := B(\alpha, \beta)([M, \nu, \omega]) \in B(V', W')$ we define

the injective (cf lemma 4.30) morphism induced by the universal property of the pushout in $\mathcal{V}I$ of the following pullback square:

Definition 4.39. For $F \in \text{Set}^{(\mathcal{V}I)^{op}}$ and $V$ and $W$ two objects in $\mathcal{V}I$, we define

Even though the roles of $V$ and $W$ are symmetric in the definition of $\sigma_V F(W)$, we use an asymmetric notation to reflect that of $T_V(K)$. 18
Proposition 4.40. For \( F \in \mathcal{S}et(VI)^{op} \), \( \sigma F \) is a bifunctor on \( VI^{op} \).

Proof. For \( \alpha : V' \to V \) and \( \beta : W' \to W \) two morphisms in \( VI \) we have to define \( \sigma_\alpha(\beta) \) from \( \sigma_V F(W) \) to \( \sigma_V F(W') \). For \( x \in F(V \oplus_{\nu,\omega} W) \) let \( \sigma_\alpha F(\beta)(x) = \sigma(\alpha \oplus_{\nu,\omega} \beta))(x) \in F(V' \oplus_{\nu',\omega'} W') \). Then, the fact that \( \sigma F \) is a bifunctor comes from the fact that \( \text{id}_V \oplus_{\nu,\omega} \text{id}_W = \text{id}_{V \oplus_{\nu,\omega} W} \) and that for \( \alpha' : V'' \to V' \) and \( \beta' : W'' \to W' \), \( \sigma(\alpha' \oplus_{\nu',\omega'} \beta') = (\alpha \circ \alpha') \oplus_{\nu,\omega} (\beta \circ \beta') \).

Corollary 4.41. For \( F \in \mathcal{S}et(VI)^{op} \) and \( V \) and \( W \) some fixed objects in \( VI \), \( \sigma_V F \) and \( \sigma F(W) \) are objects in \( \mathcal{S}et(VI)^{op} \).

Let us now define the analogue of \( T_{(V,\phi)} \) in the category \( \mathcal{S}et(VI)^{op} \).

Lemma 4.42. For \( F \in \mathcal{S}et(VI)^{op} \) and \( V \) and \( W \) in \( VI \), we have natural isomorphisms \( \sigma_0 F(W) \cong F(W) \) and \( \sigma_V F(0) \cong F(V) \).

Definition 4.43. For \( F \in \mathcal{S}et(VI)^{op} \) and \( V \) and \( W \) two objects of \( VI \), the morphism from 0 to \( W \) induces a morphism \( \sigma_V F(W) \to \sigma_V F(0) \cong F(V) \). For \( x \in F(V) \), let \( \sigma_{(V,x)} F(W) \) be the inverse image of \( \{ x \} \) under this morphism.

Lemma 4.44. For \( F \in \mathcal{S}et(VI)^{op} \), \( V \) and \( W \) objects in \( VI \) and for \( x \in F(V) \), \( \sigma_{(V,x)} F \) is a subfunctor of \( \sigma_V F \).

Proof. We only have to prove that for \( \alpha : U \to W \) a morphism in \( VI \) and for \( y \in \sigma_{(V,x)} F(W) \), \( \sigma_V F(\alpha)(y) \in \sigma_{(V,x)} F(U) \). This follow directly from the fact that the morphism 0 \to W factorizes through 0 \to U \to W.

As we stated in the beginning of this sub-section,

\[
\sigma_V \text{Hom}_{K,I,G}(K,H^*(\_)) \not\cong \text{Hom}_{K,I,G}(T_V(K),H^*(\_)).
\]

Nonetheless, as we will see in corollary 4.46 for \( \phi \in \text{Hom}_{K,I,G}(K,H^*(V)) \), we have that

\[
\sigma_{(V,\phi)} \text{Hom}_{K,I,G}(K,H^*(\_)) \cong \text{Hom}_{K,I,G}(T_{(V,\phi)}(K),H^*(\_)).
\]

Proposition 4.45. For \( F \in \mathcal{S}et(VI)^{op} \), \( V \in VI \) and \( x \in F(V) \), \( \sigma_{(V,x)} F \cong \Sigma_{(V,x)} \tilde{F} \).

Proof. Let \( W \) be a finite dimensional vector space. Let us first notice that both \( \sigma_{(V,x)} F(W) \) and \( \Sigma_{(V,x)} \tilde{F}(W) \) can be seen as sub-sets of \( \tilde{F}(V \oplus W) = \bigsqcup_{H \in \mathcal{S}(V \oplus W)} F((V \oplus W)/H) \). It is obvious for \( \Sigma_{(V,x)} \tilde{F}(W) \). For \( \sigma_{(V,x)} F(W) \subset \sigma_{(V,x)} F(W) \), for every \( V \oplus_{\nu,\omega} (W/U) \) the universal property of the product induces a projection \( \pi_{\nu,\omega} \) from \( V \oplus W \) to \( V \oplus_{\nu,\omega} (W/U) \). It induces an isomorphism from \( (V \oplus W)/\ker(\pi_{\nu,\omega}) \) to \( V \oplus_{\nu,\omega} (W/U) \). We identify \( F(V \oplus_{\nu,\omega} (W/U)) \) with \( F((V \oplus W)/\ker(\pi_{\nu,\omega})) \subset \tilde{F}(V \oplus W) \) through this isomorphism. Then we only have to prove that those sub-sets are equals.

\( \Sigma_{(V,x)} \tilde{F}(W) \) identifies with the set of elements \( \gamma \) in \( \tilde{F}(V \oplus W) \) such that \( \tilde{F}(\nu,\omega)(\gamma) = x \in F(V) \). Since \( \tilde{F} \) is noetherian and \( x \) is regular, \( \ker(\gamma) \cap V = \{0\} \). Then, there exists \( U \) a subspace of \( W \) and a class \([M,\nu,\omega] \in \mathcal{B}(V,W/U) \) such that the canonical projection from \( V \oplus W \)
to $V \oplus_{\nu, \omega} (W/U)$ induces an isomorphism $(V \oplus W)/\ker(\gamma) \cong V \oplus_{\nu, \omega} (W/U)$. Then, up to this isomorphism $\gamma \in F(V \oplus_{\nu, \omega} (W/U))$ and $(i_{V}^{\oplus_{\nu, \omega}(W/U)})^{*}\gamma = x$, so $\gamma \in \sigma_{(V,x)}F(W/U) \subset \sigma(V,x)F(W)$.

Conversely, for $\gamma \in \sigma_{(V,x)}F(W/U) \subset \sigma_{(V,x)}F(W)$, there is $[M, \nu, \omega] \in \mathcal{B}(V, W/U)$ such that $\gamma \in F(V \oplus_{\nu, \omega} (W/U))$ and $(i_{V}^{\oplus_{\nu, \omega}(W/U)})^{*}\gamma = x$, then for $H$ the kernel of the projection from $V \oplus W$ to $V \oplus_{\nu, \omega} (W/U)$, we have the following isomorphism induced by the first isomorphism theorem $(V \oplus W)/H \cong V \oplus_{\nu, \omega} (W/U)$. Up to this isomorphism, $\gamma \in F((V \oplus W)/H) \subset \tilde{F}(V \oplus W)$. Then, by construction and up to the isomorphism from $(V \oplus W)/H$ to $V \oplus_{\nu, \omega} (W/U)$, $\tilde{F}(i_{V}^{\oplus W})(\gamma) = (i_{V}^{\oplus_{\nu, \omega}(W/U)})^{*}\gamma = x$. Then, $\gamma \in \Sigma_{(V,x)}\tilde{F}(W)$.

\[ \square \]

**Corollary 4.46.** For $K$ a noetherian algebra, $V \in \mathcal{V}^I$ and $\phi \in \text{Hom}_{K,f.g.}(K, H^*(V))$,

\[ \sigma_{(V,\phi)}\text{Hom}_{K,f.g.}(K, H^*(\_)) \cong \text{Hom}_{K,f.g.}(T_{(V,\phi)}(K), H^*(\_)). \]

**Proof.** This is a direct consequence of Lemma 4.26 of Proposition 4.45 and of the fact that, if $K$ is noetherian, $T_{(V,\phi)}(K)$ is also noetherian.

**Remark 4.47.** Since the morphism $\sigma_{V}F(W) \rightarrow F(V)$ is a surjection, $\sigma_{V}F(W)$ is the disjoint union of the fibres over singletons in $F(V)$, we then have an isomorphism which is natural in both $K$ and $W$, $\sigma_{V}\text{Hom}_{K,f.g.}(K, H^*(W)) \cong \bigsqcup_{\phi \in \text{Hom}_{K,f.g.}(K, H^*(V))} \text{Hom}_{K,f.g.}(T_{(V,\phi)}(K), H^*(W))$.

### 4.5 Central elements of an object of $\text{Set}^{(\mathcal{V}^I)^{op}}$

In Definition 3.7, we defined the notion of a central elements of an unstable algebra $K$. In the following, we define a notion of centrality "away from $\mathcal{N}il$".

**Definition 4.48.** For $K \in \mathcal{K}$, $V \in \mathcal{V}^I$ and $\phi \in \text{Hom}_{K}(K, H^*(V))$, we will say that $(V, \phi)$ is central away from $\mathcal{N}il$, if $g(\rho_{K,(V,\phi)})$ (or equivalently $f(\rho_{K,(V,\phi)})$) is an isomorphism.

**Remark 4.49.** Since, by lemma 3.4 for $K$ nil-closed, $T_{(V,\phi)}(K)$ is also nil-closed, if $K$ is nil-closed $(V, \phi)$ is central away from $\mathcal{N}il$ if and only if it is central.

The centrality away from $\mathcal{N}il$ can be characterised by properties of the functor $\text{Hom}_{K,f.g.}(K, H^*(\_)) \in \text{Set}^{(\mathcal{V}^I)^{op}}$. For $F \in \text{Set}^{(\mathcal{V}^I)^{op}}$, $V \in \mathcal{V}^I$ and $\phi \in F(V)$, we want to define $\rho_{F,(V,\phi)} : \sigma_{(V,\phi)}F \rightarrow F$ in such a way that, when $F \cong \text{Hom}_{K,f.g.}(K, H^*(\_))$ where $K$ is a noetherian unstable algebra, $\rho_{F,(V,\phi)}$ is an isomorphism if and only if $g(\rho_{K,(V,\phi)})$ is.

**Definition 4.50.** For $F \in \text{Set}^{(\mathcal{V}^I)^{op}}$, $V$ and $W$ two objects of $\mathcal{V}^I$, let $\rho_{F,V} : \sigma_{V}F \rightarrow F$ be the natural transformation induced by the morphism $0 \rightarrow V$ and the natural isomorphism $\sigma_{0}F \cong F$.

For $x \in F(V)$, we also define

\[ \rho_{F,(V,x)} : \sigma_{(V,x)}F \rightarrow F \]

as the restriction of $\rho_{F,V}$ to $\sigma_{(V,x)}F$.

If $\rho_{F,(V,x)}$ is an isomorphism we will say that $(V, x)$ is a central element of $F$. We will denote by $C(F)$ the set of central elements of $F$.
**Theorem 4.51.** For $K$ a noetherian unstable algebra and $\phi \in \text{Hom}_{K, \delta}(K, H^*(V))$, $(V, \phi)$ is central away from $\mathcal{N}il$ for $K$ if and only if it is central for $\text{Hom}_{K, \delta}(K, H^*(\_))$.

**Proof.** By construction,

$$g(\rho_{K,(V,\phi)}) : \text{Hom}_K(T(V,\phi)(K), H^*(\_)) \cong \Sigma_{(V,\phi)} g(K) \to g(K),$$

is the morphism which sends $\delta \in \Sigma_{(V,\phi)} \text{Hom}_K(K, H^*(W))$ to $(\iota_W^{V\oplus W})^* \delta \in \text{Hom}_K(K, H^*(W))$. Then, for $F_K = \text{Hom}_{K,K}(K, H^*(\_))$ and for $\phi$ regular, by construction the isomorphisms $\sigma_{(V,\phi)} F_K(W) \cong \Sigma_{(V,\phi)} \text{Hom}_K(K, H^*(W))$ and $\widehat{F}_K(W) \cong \text{Hom}_K(K, H^*(W))$ (Propositions 4.14 and 4.11), fit into the following commutative diagram, which is natural with respect to $K$:

$$\Sigma_{(V,\phi)} \text{Hom}_K(K, H^*(\_)) \xrightarrow{g(\rho_{K,(V,\phi)})} \text{Hom}_K(K, H^*(\_)) \xrightarrow{\approx} \Sigma_{(V,\phi)} \widehat{F}_K.$$ 

This concludes the proof. \qed

**Definition 4.52.** For $F \in \text{Set}^{(V\mathcal{I})^{op}}$, we will say that $F$ is connected if $F(0)$ is reduced to one element.

**Remark 4.53.** For $F \in \text{Set}^{(V\mathcal{I})^{op}}$, $V \in \mathcal{V}\mathcal{I}$ and $x \in F(V)$, $\sigma_{(V,x)} F(0) = \{x\}$, therefore it is connected. For $F$ not connected, $C(F) = \emptyset$.

We give an alternative criterion for the centrality of $(V,x)$. This alternative criterion will often prove to be easier to check, when we have to prove the centrality of a given element.

**Lemma 4.54.** Let $F \in \text{Set}^{(V\mathcal{I})^{op}}$, $V \in \mathcal{V}\mathcal{I}$ and $x \in F(V)$. Then, $(V,x)$ is central if for all $W \in \mathcal{V}\mathcal{I}$ and $y \in F(W)$ there is a unique class $[M,\nu,\omega] \in B(V,W)$ and a unique $b \in F(V \oplus_{\nu,\omega} W)$ such that $i^*_V b = x$ and $i^*_W y$.

**Proof.** By construction, $\rho_{F,V}$ is the morphism which maps $b \in F(V \oplus_{\nu,\omega} W)$ to $i^*_W b$, then $b \in \sigma_{F,(V,x)} F(W)$ if and only if $i^*_V b = x$. So, $y$ has a unique element in its inverse image under $\rho_{F,(V,x)}$ if there is a unique $(M,\nu,\omega) \in B(V,W)$ and a single $b \in F(V \oplus_{\nu,\omega} W)$ such that $i^*_V b = x$ and $i^*_W b = y$. \qed

In the rest of this subsection, we want to prove an analogue of Theorem [4.21] for objects in $\text{Set}^{(V\mathcal{I})^{op}}$. Namely, we want to prove that, for $F \in \text{Set}^{(V\mathcal{I})^{op}}$ such that $F(V) = \emptyset$ when $\dim(V)$ is greater than some integer $d$, there is, up to isomorphism, a unique maximal central element $(C,c)$ in the following sense, for all central element $(V,x)$ there is an injection $\iota$ from $V$ to $C$ such that $x = \iota^* c$.

**Lemma 4.55.** For $F \in \text{Set}^{(V\mathcal{I})^{op}}$, $V$ an object in $\mathcal{V}\mathcal{I}$, $x \in F(V)$ and $\alpha : T \to V$ a morphism in $\mathcal{V}\mathcal{I}$, $\alpha^* x$ is always in the image of $\rho_{F,(V,x)}$. Furthermore, the morphism $\iota_V^{T \oplus \iota_{T^{op},\alpha} V}$ is invertible and $((\iota_V^{T \oplus \iota_{T^{op},\alpha} V})^{-1} \alpha^* x$ is an element in the inverse image of $\alpha^* x$. 

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Proof. The morphism $\iota^V_T : V \to T \otimes \text{id}_T, a V$ is an isomorphism. Indeed, the following commutative diagram is a pushout in $\mathcal{V}^f$:

$$
\begin{array}{ccc}
T & \overset{\alpha}{\rightarrow} & V \\
\downarrow{\text{id}_T} & & \downarrow{\text{id}_V} \\
T & \overset{\iota}{\rightarrow} & V.
\end{array}
$$

Then, $((\iota^V_T)^{-1})^* x$ satisfies the two following conditions, $(\iota^V_T)^{*} (\iota(T \otimes \text{id}_T, a V)^{-1})^* x = x$ and $\iota^V_T \otimes \text{id}_T, a V * (\iota(T \otimes \text{id}_T, a V)^{-1})^* x = a^* x$, where the second is a consequence of the following commutative diagram:

$$
\begin{array}{ccc}
T & \overset{\alpha}{\rightarrow} & V \\
\downarrow{\text{id}_T} & & \downarrow{\text{id}_V} \\
T \oplus \text{id}_T, a V & \overset{\iota}{\rightarrow} & V.
\end{array}
$$

Therefore it is in the inverse image of $a^* x$ under $\rho_{\mathcal{F}, (V, x)}$, if $(V, x)$ is central it is the only element in the inverse image.

In the following, for $V, W$ and $T$ finite dimensional vector spaces, we think of spaces of the form $(V \oplus \nu, \omega) \oplus g, T$ as $V + W + T$, where we identified $V, W$ and $T$ with their images in $(V \oplus \nu, \omega) \oplus g, T$, in order to use the associativity $(V + W) + T = V + (W + T)$. The following technical construction aims to identify a canonical isomorphism $\zeta_{\nu, \omega, g, \tau}$ from $(V \oplus \nu, \omega) \oplus g, T$ to the appropriate $V \oplus \nu', g'$ $(W \oplus \omega', \tau')$ $T$.

**Notation 4.56.** For $V, W$ and $T$ in $\mathcal{V}^f$, we denote

$$T_1(V, W; T) := \bigcup_{[M, \nu, \omega] \in B(V \oplus \nu, \omega) W, T} B(V \oplus \nu, \omega) W, T),$$

and

$$T_2(V, W; T) := \bigcup_{[M', \nu', \omega', \tau'] \in B(W \oplus \omega', \tau') T} B(V, W \oplus \omega', \tau') T).$$

**Lemma 4.57.**

1. $T_1(\cdot, \cdot, \cdot)$ and $T_2(\cdot, \cdot, \cdot)$ are trifunctors on $\mathcal{V}^{op}$.

2. There is a natural transformation $\zeta$ from $T_1(\cdot, \cdot, \cdot)$ and $T_2(\cdot, \cdot, \cdot)$,

3. for $[M, g, \tau] \in B(V \oplus v, W, T) \subset T_1(V, W; T)$ and for $(v', g', \omega', \tau')$ such that $\zeta([M, g, \tau]) = [M', v', g'] \in B(V, W \oplus \omega', \tau') T$, there is a natural isomorphism

$$\zeta_{v, \omega, g, \tau} : (V \oplus v, W) \oplus g, T \to V \oplus v', g' (W \oplus \omega', \tau') T,$$

such that the canonical injections from $V$, $W$ and $T$ to $V \oplus v' , g' (W \oplus \omega', \tau') T$ factorizes through $\zeta_{v, \omega, g, \tau}$ and their injections in $(V \oplus v, W) \oplus g, T$.

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Proof. The fact that $T_1(\omega, \tau)$ and $T_2(\omega, \tau)$ are trifunctors is a direct consequence of the fact that $B(\omega, \tau)$ is a bifunctor. Let us construct the natural transformation $\zeta$.

For $[M, g, \tau] \in B(V \oplus_{\nu,\omega} W, T) \subset T_1(V, W; T)$, we consider $\iota_W + \iota_T$ from $W \oplus T$ to $(V \oplus_{\nu,\omega} W) \oplus_{g,\tau} T$. Since $\iota_W$ and $\iota_T$ are injections from $W$ and $T$ to $(V \oplus_{\nu,\omega} W) \oplus_{g,\tau} T$, $\ker(\iota_W + \iota_T) \cap W = \ker(\iota_W + \iota_T) \cap T = \{0\}$. Then, for all $x \in \ker(\iota_W + \iota_T)$, $x$ has a non trivial component both in $W$ and $T$, hence there exists $\omega'(x) \in W\setminus\{0\}$ and $\tau'(x) \in T\setminus\{0\}$ such that $x = \omega'(x) - \tau'(x)$. Then, by the first isomorphism theorem, $\iota_W + \iota_T$ factorises through an injection $\iota_{W \oplus_{\nu',\tau'} T}$ from $W \oplus_{\omega',\tau'} T$ to $(V \oplus_{\nu,\omega} W) \oplus_{g,\tau} T$.

We do the same construction a second time. We consider $\iota_V + \iota_{W \oplus_{\nu',\tau'} T}$ from $V \oplus (W \oplus_{\omega',\tau'} T)$ to $(V \oplus_{\nu,\omega} W) \oplus_{g,\tau} T$. Since the two are injections, there are injections $\nu'$ and $g'$ from $\ker(\iota_V + \iota_{W \oplus_{\nu',\tau'} T})$ to $V$ and $W \oplus_{\omega',\tau'} T$ such that, for all $x \in \ker(\iota_V + \iota_{W \oplus_{\nu',\tau'} T})$, $x = \nu'(x) - g'(x)$. Then, by the first isomorphism theorem, $\iota_V + \iota_{W \oplus_{\nu',\tau'} T}$ factorises through an injection from $(V \oplus_{\nu,\omega} W) \oplus_{g,\tau} T$ to $(V \oplus_{\nu,\omega} W) \oplus_{g,\tau} T$. But, by construction, $\iota_V + \iota_{W \oplus_{\nu',\tau'} T}$ is surjective, therefore this injection is an isomorphism that we denote $\zeta_{\nu,\omega,\nu',\tau'}$. We define $\zeta([M, g, \tau]) := \ker(\iota_V + \iota_{W \oplus_{\nu',\tau'} T}), \nu', g']$.

By construction, $\zeta_{\nu,\omega,\nu',\tau'}$ satisfies the desired factorizations.\hfill $\square$

Lemma 4.58. Let $F$ be an object in $\text{Set}^{\text{VIF}}$, $V$ be an object of $\text{VIF}$ and $x$ be an element of $F(V)$ such that $(V, x)$ is central. For $\alpha : T \to V$ in $\text{VIF}$, $(T, \alpha^* x)$ is central.

Proof. By Lemma 4.54, we have to prove that for every $y \in F(W)$ there is a unique $b \in F(W \oplus_{\omega,\tau} T)$ for some $[M, \omega, \tau] \in B(W, T)$, such that $\iota_V b = y$ and $\iota_W b = \alpha^* x$.

We will prove the existence first. For $a \in F(V \oplus_{\nu,\omega} W)$ satisfying $\iota_V a = x$, we have that $\sigma_{\omega} F(W)(a) = (\alpha \oplus_{\nu,\omega} \text{id}_W)^* a$ is some $F(T \oplus_{\tau,\nu} W)$ and satisfies $\iota_V (\alpha \oplus_{\nu,\omega} \text{id}_W)^* a = \alpha^* \iota_V a = \alpha^* x$, so $\sigma_{\omega} F(W)(a) \in \Sigma(T, \alpha^* x) F(W)$, then if $a$ is the unique element such that $\iota_W a = y$ (which exists since $(V, x)$ is central), then $b := \sigma_{\omega} F(W)(a)$ satisfies $\iota_W b = \alpha^* x$ and $\iota_V b = \iota_V a = y$.

Let us now prove the uniqueness. Let $b$ be in the preimage of $y$ in $\sigma_{F(T, \alpha^* x)} F(W)$, with $b \in F(W \oplus_{\omega,\tau} T)$. Then, since $(V, x)$ is central, there is a unique $[M, \nu, \psi] \in B(W \oplus_{\omega,\tau} T, V)$ and a $c \in F((W \oplus_{\omega,\tau} T) \oplus_{g,\nu} V)$ such that $\iota_V c = x$ and $\iota_W c = b$. We consider the isomorphism $\zeta_{\omega,\psi,\nu,\nu'}$ given by Lemma 4.57 which takes values in some space $W \oplus_{\nu',\omega'} (T \oplus_{\tau',\nu'} V)$. Then, by construction, $\iota_T \zeta_{\omega,\psi,\nu,\nu'} (\psi^* c) \in T \oplus_{\nu',\omega'} V \zeta_{\omega,\psi,\nu,\nu'} c = \alpha^* x$ and $\iota_V \zeta_{\omega,\psi,\nu,\nu'} (\psi^* c) = x$. Therefore, since $(V, x)$ is central, Lemma 4.55 implies that $V \to T \oplus_{\tau',\nu'} V$ is an isomorphism and $(\iota_V^{-1})^* \iota_T \zeta_{\omega,\psi,\nu,\nu'} (\psi^* c) = x$. Then, identifying $W \oplus_{\nu',\omega'} (T \oplus_{\tau',\nu'} V)$ with the appropriate $W \oplus_{\nu',\omega'} V$, we get that $\zeta_{\omega,\psi,\nu,\nu'} c \in F(W \oplus_{\nu',\omega'} V)$ satisfies $\iota_V \zeta_{\omega,\psi,\nu,\nu'} c = y$ and $\iota_W \zeta_{\omega,\psi,\nu,\nu'} c = \omega^* x$. $\zeta_{\omega,\psi,\nu,\nu'} c$ is then the unique element in the inverse image of $y$ by $\rho_{F(V, x)}$ and $b = \sigma_{\omega} F(W)(\zeta_{\omega,\psi,\nu,\nu'} c)$, which proves the uniqueness of $b$.\hfill $\square$
Lemma 4.59. Let $F$ be in $\mathcal{S}(V^2)^{op}$ and $(V, x)$ and $(T, y)$ be central elements of $F$, then there exists, up to isomorphism, a unique pair $(R, z)$, with $R = V \oplus_{\nu, \tau} T$ for some $[M, \nu, \tau] \in B(V, T)$ and with $z \in F(R)$, such that $x = \iota_T^* z$ and $y = \iota_T^* z$. Moreover, $(R, z)$ is central.

Proof. By centrality of $(V, x)$ and by Lemma 4.54 $z$ is necessarily the only element in the inverse image of $y$ under $\rho_{F,(V,x)}$, which prove the existence and the uniqueness of $(R, z)$.

Let us prove that it is central. Let $e$ be in $F(W)$, we take $e'$ in $F(W \oplus_{\omega, \tau} T)$ the only element in the inverse image of $e$ under $\rho_{F,(T,y)}$ and $e'' \in F((W \oplus_{\omega, \tau} T) \oplus_{\omega, \nu} V)$ the only element in the inverse image of $e'$ under $\rho_{F,(V,x)}$. We consider $\zeta_{\omega, \tau, g, \nu}$ as in lemma 4.57, it is an isomorphism from $(W \oplus_{\omega, \tau} T) \oplus_{\omega, \nu} V$ to some $W \oplus_{\omega', \nu'} (T \oplus_{\tau', \nu'} V)$. Then, $\iota_{T \oplus_{\omega', \nu'} V}^* (\zeta_{\omega, \tau, g, \nu}^{-1})^{e''} \in$ the inverse image of $y$ under $\rho_{F,(V,x)}$, since $(V, x)$ is central $\iota_{T \oplus_{\omega', \nu'} V}^* (\zeta_{\omega, \tau, g, \nu}^{-1})^{e''} = x$. Moreover, $\iota_{W}^* (\zeta_{\omega, \tau, g, \nu}^{-1})^{e''} = e$, this imply that $(\zeta_{\omega, \tau, g, \nu}^{-1})^{e''}$ is in the inverse image of $e$ under $\rho_{F,(R,z)}$.

Let us now show the uniqueness of the element in the inverse image of $e$. We take $R$ to be of the form $V \oplus_{\nu, \tau} T$, and we take $a \in F(W \oplus_{\omega,g} (V \oplus_{\nu,\tau} T))$ and $b \in F(W \oplus_{\nu,\tau} (V \oplus_{\nu,\tau} T))$ two elements in the inverse image of $e$ under $\rho_{F,(R,z)}$. We consider the isomorphisms $\zeta_{\omega', \nu', g', \tau'}$ and $\zeta_{\omega', \nu', \gamma', \tau''}$ given by Lemma 4.57 from some vector spaces $(W \oplus_{\omega', \nu'} V') \oplus_{g', \tau'} T$ and $(W \oplus_{\omega', \nu'} V') \oplus_{g', \tau''} T$ to $W \oplus_{\omega,g} (V \oplus_{\nu,\tau} T)$ and $W \oplus_{\omega,g} (V \oplus_{\nu,\tau} T)$. Set $a' = \iota_{W \oplus_{\omega', \nu'} V}^* \zeta_{\omega', \nu', g', \tau'} a \in F(W \oplus_{\omega', \nu'} V)$ and $b' = \iota_{W \oplus_{\omega', \nu'} V}^* \zeta_{\omega', \nu', g', \tau''} b \in F(W \oplus_{\omega', \nu'} V)$. They both satisfy that their image by $\iota_{W}^*$ is $e$ and by $\iota_{V}^*$ is $x$. Then, by the centrality of $(V, x)$ we have that $a' = b'$. Moreover, applying $\iota_{T}^*$ to $\zeta_{\omega', \nu', g', \tau'} a$ and $\zeta_{\omega', \nu', g', \tau''} b$, we obtain $y$. Therefore, $\zeta_{\omega', \nu', g', \tau'} a$ and $\zeta_{\omega', \nu', g', \tau''} b$ are both in the inverse image of $a' = b'$ under $\rho_{F,(T,y)}$. Since $(T, y)$ is also central, $\zeta_{\omega', \nu', g', \tau'} a = \zeta_{\omega', \nu', g', \tau''} b$, hence $a = b$. □

Theorem 4.60. Let $F$ be an object in $\mathcal{S}(V^2)^{op}$ such that $\mathcal{C}(F)$ is not empty and such that $F(V) = \emptyset$ for $\dim(V)$ greater than some integer $d$, then there is a unique central element $(C, c)$ up to isomorphism satisfying that $(V, x)$ is central if and only if there is an injective morphism $\alpha$ from $V$ to $C$ such that $x = \alpha^* c$. We call $(C, c)$ the centre of $F$.

Proof. It is a direct consequence of lemma 4.58 and 4.59. Indeed, for $(C, c)$ a central element with $\dim(C)$ maximal, if $(V, x)$ is a central element such that there is no injection from $V$ to $C$ satisfying $\alpha^* c = x$, then the dimension of $(R, z)$ the unique pair satisfying the assumptions of lemma 4.59 for $(T, y) = (C, c)$ is greater than $\dim(C)$ which is absurd, since $\dim(C)$ is supposed to be maximal among central elements. □

Corollary 4.61. For $K$ a noetherian unstable algebra, if $K$ is nil-closed, the centre of $K$ is equal to the centre of $\hom_{K, f, g}(K, H^*(\Lambda))$.

5 Definition of the groupoid $\mathcal{G}_F$ and application to the computation of the centre of $F$

For $F \in \mathcal{S}(V^2)^{op}$, Yoneda's Lemma implies the existence of surjections from functors of the form $\bigcup_{i \in I} \hom_{V^2}(\lambda, W_i)$ to $F$. This statement is not specific to $\mathcal{S}(V^2)^{op}$, we could make a similar one for $\mathcal{S}(V^2)^{op}$. The interest we have about such surjections in $\mathcal{S}(V^2)^{op}$ is that the simplicity of the category $V\mathbb{Z}$ makes it easy to classify functors in $\mathcal{S}(V^2)^{op}$ with a given surjections from
\[ \bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}T}(\_ , W_i). \] We start this section by explaining how to associate to objects \( F \) in \( \mathcal{S}et^{(\mathcal{V}T)^{op}} \) with a given surjection from some \( \bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}T}(\_ , W_i) \) to \( F \), a groupoid \( \mathcal{G}_F \) with objects the sub-vector spaces of the \( W_i \). We will prove that the isomorphism class of \( F \) in the coslice category with respect to \( \bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}T}(\_ , W_i) \) is determined by \( \mathcal{G}_F \).

In the second sub-section, we will prove that the centre of \( F \) is explicitly determined by the groupoid \( \mathcal{G}_F \) of the first sub-section.

### 5.1 Definition of the groupoid

By Yoneda’s lemma, \( F(W) \) is naturally isomorphic to \( \text{Hom}_{\mathcal{S}et^{(\mathcal{V}T)^{op}}}(\text{Hom}_{\mathcal{V}T}(\_ , W), F) \) under the morphism which sends a natural transformation \( \eta \) to \( \eta(id_V) \). Then, we can always exhibit a surjection from some \( \bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}T}(\_ , W_i) \) to \( F \), where the cardinal of \( I \) may be very big. The most obvious one being defined by \( I = \{ x \in F(\mathbb{F}_p^d) : d \in \mathbb{N} \} \), \( W_x = \mathbb{F}_p^d \) for \( x \in F(\mathbb{F}_p^d) \), and \( q \) is the only natural transformation which sends \( id_{W_x} \) to \( x \) for all \( x \in I \).

So \( F \) is always isomorphic to some functor of the form \( \bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}T}(\_ , W_i) \) with a generating family \( \sim_F \) with \( \sim_F \) an equivalence relation on \( \bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}T}(\_ , W_i) \). In this part, we will show how the equivalence relation \( \sim_F \) associated with a surjection

\[ \bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}T}(\_ , W_i) \twoheadrightarrow F, \]

is encoded by a groupoid whose objects are the sub-spaces of the \( W_i \).

**Definition 5.1.** Let \( F \) be an object in \( \mathcal{S}et^{(\mathcal{V}T)^{op}} \), a generating family of \( F \) is a family \( (W_i)_{i \in I} \) of objects of \( \mathcal{V}T \), together with a surjection \( q : \bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}T}(\_ , W_i) \twoheadrightarrow F. \)

Even though the results of this sub-section do not require any finiteness condition on the cardinal of \( I \), for \( ((W_i)_{i \in I} , q) \) a generating family, the constructions might not be exploitable in the case where \( I \) is not finite.

**Definition 5.2.** An object \( F \) in \( \mathcal{S}et^{(\mathcal{V}T)^{op}} \) is finitely generated if \( F \) admits a generating family \( ((W_i)_{i \in I} , q) \) with \( I \) finite.

Let \( F \) be an object in \( \mathcal{S}et^{(\mathcal{V}T)^{op}} \), and let \( ((W_i)_{i \in I} , q) \) be a generating family for \( F \). We denote by \( F_i \) the image of \( \text{Hom}_{\mathcal{V}T}(\_ , W_i) \) under \( q \). Then, we have \( F = \bigsqcup_{i \in I} F_i \). We also denote by \( \phi^F_i \) (or just \( \phi_i \) when there is no ambiguity) the image of \( id_{W_i} \) under \( q \).

Let us now define the main ingredient of this section.

**Definition 5.3.** For \( F \in \mathcal{S}et^{(\mathcal{V}T)^{op}} \) and \( ((W_i)_{i \in I} , q) \) a generating family of \( F \), let \( \mathcal{G}^{(W_i)_{i \in I} , q}_F \) be the groupoid whose set of objects is the disjoint union of the sets of sub-spaces of the \( W_i \) and whose morphisms from \( U \subset W_i \) to \( U' \subset W_j \) are the isomorphisms \( \alpha \) from \( U \) to \( U' \) such that \( \alpha \circ i^F_{U'} \phi_j = i^F_U \phi_i \).
The groupoid $G_F^{((W_i)_{i \in I}, q)}$ depends heavily on the choice of a generating family $((W_i)_{i \in I}, q)$, so cannot be made functorial on the category $\text{Set}^{(V^\mathsf{op})^p}$. It can nonetheless be made functorial on the category whose objects are provided with a surjection from a fixed $\bigsqcup_{i \in I} \text{Hom}_{V^\mathsf{op}}(\_ , W_i)$.

**Definition 5.4.** Let $((W_i)_{i \in I}, q) \in \text{Set}^{(V^\mathsf{op})^p}$ be the slice category whose objects are pairs $(F, q_F)$ with $F \in \text{Set}^{(V^\mathsf{op})^p}$ and $q_F$ a natural surjection from $\bigsqcup_{i \in I} \text{Hom}_{V^\mathsf{op}}(\_ , W_i)$ to $F$, and whose morphisms from $(F, q_F)$ to $(G, q_G)$ are natural transformations $\eta$ from $F$ to $G$ such that the following diagram commutes:

\[
\begin{array}{ccc}
F & \xrightarrow{q_F} & \bigsqcup_{i \in I} \text{Hom}_{V^\mathsf{op}}(\_ , W_i) \\
\vline & \downarrow \eta & \\
\bigsqcup_{i \in I} \text{Hom}_{V^\mathsf{op}}(\_ , W_i) & \xrightarrow{q_G} & G.
\end{array}
\]

**Proposition 5.5.** $(F, q_F) \mapsto G_{(F,q_F)} := G^{((W_i)_{i \in I}, q_F)}_F$ is a functor from the category $((W_i)_{i \in I}, \text{Set}^{(V^\mathsf{op})^p})$ to the category of groupoids.

**Proof.** For $\eta$ a morphism from $(F, q_F)$ to $(G, q_G)$ two objects in $((W_i)_{i \in I}, \text{Set}^{(V^\mathsf{op})^p})$, we have to define a morphism $G_\eta$ from $G_{(F,q_F)}$ to $G_{(G,q_G)}$. $G_{(F,q_F)}$ has the same objects as $G_{(G,q_G)}$, furthermore for all $i \in I$, $\eta(\phi^F_i) = \phi^G_i$, otherwise the diagram of Definition 5.4 would not commute. Then, for all $\alpha \in G_{(F,q_F)}(U \subset W_i \subset W_j)$, the commutativity of the preceding diagram implies that

\[\alpha^* i^*_U \phi^G_j = \eta(\alpha^* i^*_U \phi^F_j) = i^*_U \phi^G_i,\]

therefore $\alpha \in G_{(G,q_G)}$. Hence, $G_{(F,q_F)}$ is a sub-groupoid of $G_{(G,q_G)}$ and we define $G_\eta$ to be the inclusion of this sub-groupoid. \qed

In the following, we fix a family $((W_i)_{i \in I}$. When there can be no ambiguity, for $(F, q_F) \in ((W_i)_{i \in I}, \text{Set}^{(V^\mathsf{op})^p})$, we will use the notation $G_F$ instead of $G_{(F,q_F)}$. We want to prove that the isomorphism class of $(F, q_F)$ in $((W_i)_{i \in I}, \text{Set}^{(V^\mathsf{op})^p})$ is determined by the groupoid $G_{(F,q_F)}$.

**Lemma 5.6.** Let $(F, q_F)$ be an object in $((W_i)_{i \in I}, \text{Set}^{(V^\mathsf{op})^p})$. Then, $G_F$ satisfies the following property. For $\alpha \in G_F(U \subset W_i \subset U' \subset W_j)$, for $M$ a sub-space of $U$, and for $\alpha_M : M \to \alpha(M)$ the restriction of $\alpha$ to $M$ corestricted to $\alpha(M)$, $\alpha_M \in \mathcal{G}_F(M \subset W_i, \alpha(M) \subset W_j)$.

**Proof.** For $i^*_M U$ the inclusion of $M$ in $U$ (we reserve the notation $i_M$ for the inclusion of $M$ in $W_i$) we have, $\alpha \circ i^*_M U = \alpha_M$. Then, $\alpha^* i^*_M U \phi_j = i^*_M \phi_i$ implies that $\alpha_M^* i^*_M U \phi_j = (i^*_M)^* \alpha^* i^*_M U \phi_j = (i^*_M)^* i^*_M \phi_i = i^*_M \phi_i$. Then, $\alpha_M \in \mathcal{G}_F(M \subset W_i, \alpha(M) \subset W_j)$. \qed

**Example 5.7.** Let $W$ be a finite dimensional vector-space and $G$ be a sub-group of $\text{Gl}(W)$. We consider $F(V) := \text{Hom}_{V^\mathsf{op}}(V, W)/G$ where $G$ acts on $\text{Hom}_{V^\mathsf{op}}(V, W)$ by composition. $F$ is an object in $\text{Set}^{(V^\mathsf{op})^p}$. Then, for $q$ the canonical projection $\text{Hom}_{V^\mathsf{op}}(V, W) \twoheadrightarrow \text{Hom}_{V^\mathsf{op}}(V, W)/G$, $(F, q) \in W\cdot \text{Set}^{(V^\mathsf{op})^p}$. $G_F$ is the groupoid whose objects are the sub-vector spaces of $W$, and whose morphisms from $U$ to $U'$ are the isomorphisms $\alpha$ from $U$ to $U'$ such that $\alpha^* q(i_U) = q(i_U)$. But, $\alpha^* q(i_{U'}) =
Definition 5.8. For \( \mathcal{G} \) a groupoid whose objects are the sub-vector spaces of the \( W_i \) with \( i \in I \), and whose morphisms are isomorphisms of vector spaces, we say that \( \mathcal{G} \) has the restriction property if, for all \( U \subset W_i \), for all \( U' \subset W_j \) and for all \( \alpha \in \mathcal{G}(U \subset W_i, U' \subset W_j) \), \( \alpha_M \) is in \( \mathcal{G}(M \subset W_i, \alpha(M) \subset W_j) \).

Remark 5.9. Thus, Lemma 5.6 asserts that, for \( F \in (W_i)_{i \in I} \mathcal{S}et^{(V^I)^{op}} \), \( \mathcal{G}_F \) has the restriction property.

Remark 5.10. The set of groupoids whose set of objects is the disjoint union of the sets of sub-vector spaces of the \( W_i \) and that satisfies the restriction property, is ordered by inclusion, where \( \mathcal{G} \subset \mathcal{H} \) if, for each pair \( (U \subset W_i, U' \subset W_j) \), \( \mathcal{G}(U \subset W_i, U' \subset W_j) \subset \mathcal{H}(U \subset W_i, U' \subset W_j) \).

Notation 5.11. We denote by \( \text{Groupoid}(W_i)_{i \in I} \) the poset of groupoids whose objects are the sub-spaces of the \( W_i \) and that satisfies the restriction property.

For \( W \) a finite dimensional vector space, let us also denote by \( \text{Group}(W) \) the poset of sub-groups of \( \text{Gl}(W) \).

Definition 5.12. Let \( g \) be the poset preserving map from \( \text{Group}(W) \) to \( \text{Groupoid}(W) \) which sends \( G \) a subgroup of \( \text{Gl}(W) \) to the groupoid \( g(G) \) such that, for \( U \) and \( U' \) subspaces of \( W \) and \( \alpha \) from \( U \) to \( U' \), \( \alpha \in g(G)(U, U') \) if and only if there is \( g \in G \) such that \( \alpha = g \).

Remark 5.13. We have seen in Example 5.7 that \( \mathcal{G}(\text{Hom}_{V^I}(\cdot, W)/G.q_G) = g(G) \), for \( q_G \) the canonical projection from \( \text{Hom}_{V^I}(\cdot, W) \) to \( \text{Hom}_{V^I}(\cdot, W)/G \).

In the following, we want to show that for \( \mathcal{G} \) a groupoid whose objects are the sub-vector spaces of the \( W_i \) and whose morphisms are isomorphisms of vector spaces, there exists \( F \in (W_i)_{i \in I} \mathcal{S}et^{(V^I)^{op}} \) such that \( \mathcal{G}_F = \mathcal{G} \) if and only if \( \mathcal{G} \) has the restriction property. We will then show that those groupoids are in one-to-one correspondence with isomorphism classes of objects in \( (W_i)_{i \in I} \mathcal{S}et^{(V^I)^{op}} \).

Definition 5.14. For \( \mathcal{G} \in \text{Groupoid}(W_i)_{i \in I} \), let \( \sim_\mathcal{G} \) be the relation on \( \bigsqcup_{i \in I} \text{Hom}_{V^I}(V, W_i) \) defined by \( \zeta \sim_\mathcal{G} \epsilon \) if there is \( \alpha \in \mathcal{G}(\text{Im}(\zeta) \subset W_i, \text{Im}(\epsilon) \subset W_j) \) such that \( \epsilon = \alpha \circ \zeta \), where \( \tilde{\zeta} \) and \( \tilde{\epsilon} \) denote the corestriction of \( \zeta \) and \( \epsilon \) to their images.

Lemma 5.15. For \( \mathcal{G} \in \text{Groupoid}(W_i)_{i \in I} \), \( \sim_\mathcal{G} \) is an equivalence relation.

We will denote by \([\epsilon]_\mathcal{G}\) the equivalence class of \( \epsilon \) for this equivalence relation.

Proof. Since for all \( U \subset W_i \), \( \text{id}_U \in \mathcal{G}(U \subset W_i, U \subset W_i) \), the relation is reflexive. The transitivity comes from the composition of morphisms in \( \mathcal{G} \) and the symmetry from the fact that \( \mathcal{G} \) is a groupoid and hence that every morphism in \( \mathcal{G} \) has an inverse.

Proposition 5.16. For \( \mathcal{G} \in \text{Groupoid}(W_i)_{i \in I} \), \( \bigsqcup_{i \in I} \text{Hom}_{V^I}(\cdot, W_i)/ \sim_\mathcal{G} \) defines an object of \( \mathcal{S}et^{(V^I)^{op}} \).

Moreover, \( \bigsqcup_{i \in I} \text{Hom}_{V^I}(\cdot, W_i)/ \sim_\mathcal{G} \) is connected if and only if, for all \((i, j) \in I^2\), \( \mathcal{G}(0 \subset W_i, 0 \subset W_j) \) contains the only morphism from 0 to 0.
Proof. To prove that $\bigsqcup_{i \in I} \text{Hom}_\mathcal{I}(\omega, W_i)/\sim_\mathcal{G}$ is an object in $\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$, let us prove that if $\rho \sim_\mathcal{G} \zeta$, for $\rho \in \text{Hom}_\mathcal{I}(V,W_i)$ and $\zeta \in \text{Hom}_\mathcal{I}(V,W_j)$ with $V \in \mathcal{V}\mathcal{I}$, and if $\beta : U \to V$ is a morphism in $\mathcal{V}\mathcal{I}$, then $\beta^* \rho \sim_\mathcal{G} \beta^* \zeta$. Let $\alpha$ be in $\mathcal{G}(\text{Im}(\rho) \subset W_i, \text{Im}(\zeta) \subset W_j)$ such that $\tilde{\zeta} = \alpha \circ \tilde{\rho}$. Then since $\mathcal{G}$ has the restriction property, $a_{\rho \circ \beta(U)} : \rho \circ \beta(U) \to \zeta \circ \beta(U)$ is in $\mathcal{G}(\text{Im}(\rho \circ \beta) \subset W_i, \text{Im}(\zeta \circ \beta) \subset W_j)$ and $\tilde{\zeta} \circ \tilde{\beta} = \alpha_{\rho \circ \beta(U)} \circ \tilde{\rho} \circ \tilde{\beta}$ therefore $\beta^* \rho \sim_\mathcal{G} \beta^* \zeta$.

Theorem 5.17. 1. For $\mathcal{G} \in \text{Groupoid}((W_i)_{i \in I})$,

$$\mathcal{G}(\bigsqcup_{i \in I} \text{Hom}_\mathcal{I}(\omega, W_i)/\sim_\mathcal{G}) = \mathcal{G},$$

for $q$ the canonical surjection from $\bigsqcup_{i \in I} \text{Hom}_\mathcal{I}(\omega, W_i)$ to $\bigsqcup_{i \in I} \text{Hom}_\mathcal{I}(\omega, W_i)/\sim_\mathcal{G}$.

2. Conversely, let $F \in (W_i)_{i \in I} \mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$. Then, $F$ is isomorphic to $\bigsqcup_{i \in I} \text{Hom}_\mathcal{I}(\omega, W_i)/\sim_\mathcal{G}_F$.

Proof. Let us prove the first point. Let $\alpha$ be an isomorphism from $U \subset W_i$ to $U' \subset W_j$. $\alpha \in \mathcal{G}(\bigsqcup \text{Hom}_\mathcal{I}(\omega, W_i)/\sim_\mathcal{G}(U \subset W_i, U' \subset W_j)$ if and only if $\alpha^* [\iota_U]_\mathcal{G} = [\iota_U]_\mathcal{G}$. This the case if and only if $\alpha \in \mathcal{G}(U \subset W_i, U' \subset W_j)$.

By construction, for $\alpha$ an isomorphism from $U$ to $U'$ with $U$ and $U'$ sub-spaces of $W_i$ and $W_j$, $\alpha^* [\iota_U]_\mathcal{G} = [\iota_U]_\mathcal{G}$ if and only if $\alpha \in \mathcal{G}(U \subset W_i, U' \subset W_j)$. We only have to prove that $[\iota_U]_\mathcal{G} \mapsto \epsilon^* \phi_i$, for $\epsilon \in \text{Hom}_\mathcal{I}(V, W_i)$, is well defined and defines a bijection from $\bigsqcup_{i \in I} \text{Hom}_\mathcal{I}(V, W_i)/\sim_\mathcal{G}_F$ to $F(V)$ for $V \in \mathcal{V}\mathcal{I}$.

For the second point, if $\epsilon \sim_\mathcal{G}_F \zeta$, there exists $\alpha : \epsilon(V) \to \zeta(V)$ such that $\alpha \in \mathcal{G}_F(\text{Im}(\epsilon) \subset W_i, \text{Im}(\zeta) \subset W_j)$ and $\zeta = \alpha \circ \tilde{\epsilon}$. Then,

$$\zeta^* \phi_j = \tilde{\zeta}^* \iota_{\zeta(V)} \phi_j,$$

$$= \tilde{\epsilon}^* \alpha^* \iota_{\zeta(V)} \phi_j,$$

$$= \iota_{\epsilon(V)} \phi_i,$$

$$= \epsilon^* \phi_i.$$

The map which maps $[\iota_U]_\mathcal{G}_F$ to $\epsilon^* \phi_i$ is then well defined. It is obviously surjective.

Let us prove that it is injective. Let $\epsilon$ and $\zeta$ be morphisms from $V$ to $W_i$ and $W_j$ such that $\epsilon^* \phi_i = \zeta^* \phi_j$. Then, $\tilde{\epsilon}^* \iota_{\epsilon(V)} \phi_i = \tilde{\zeta}^* \iota_{\zeta(V)} \phi_j$. Since $\tilde{\epsilon}$ is an isomorphism from $V$ to the image of $\epsilon$, we have $\iota_{\epsilon(V)} \phi_i = (\tilde{\zeta} \circ \tilde{\epsilon}^{-1})^* \iota_{\zeta(V)} \phi_j$. Therefore, $\tilde{\epsilon} \circ \tilde{\epsilon}^{-1} \in \mathcal{G}_F(\text{Im}(\epsilon) \subset W_i, \text{Im}(\zeta) \subset W_j)$ and $\epsilon \sim_\mathcal{G}_F \zeta$.

The following is a “converse” of example 5.7.

Example 5.18. Let $F$ be an object in $\mathcal{W}\mathcal{S}et^{(\mathcal{V}\mathcal{I})^{op}}$ and consider the group $G = \mathcal{G}_F(W,W)$. We know, since $\mathcal{G}_F$ has the restriction property, that $\mathcal{G}_F \subset \mathcal{G}(G)$ (see Definition 5.12). Suppose that this inclusion is an equality. Then, $\sim_\mathcal{G}$ is the equivalence relation defined by $\rho \sim_\mathcal{G} \zeta$ if and only if there is $g \in G$ such that $g \circ \rho = \zeta$. By Theorem 5.17, $F \cong \text{Hom}_\mathcal{I}(\omega, W)/\mathcal{G}$.
5.2 Computation of $C(F)$ using $\mathcal{G}_F$

In this sub-section, we take $F \in (W_i)_{i \in I} \text{Set}^{(\mathcal{V}^g)^{op}}$. We want to prove that, under some assumptions on the generating family of $F$, the central elements of $F$ are determined by the groupoid $\mathcal{G}_F$.

We start by proving that, if $F$ has a generating family with one element, $\rho_{F,(V,x)}$ is surjective for all $V \in \mathcal{V}^g$ and for all $x \in F(V)$.

**Notation 5.19.** For $F \in \text{Set}^{(\mathcal{V}^g)^{op}}$ and for $(W,q)$ a generating family with one element, we denote by $\phi^F \in F(W)$ the image of id$_W$ under $q$.

On one hand, for $F \in W\text{Set}^{(\mathcal{V}^g)^{op}}$, the functor $F$ is determined up to isomorphism by the groupoid $\mathcal{G}_F$. On the other hand, Lemma 4.54 gives a criterion for the centrality of a pair $(V,x)$ through the sets $F(V \oplus_{\nu,\mu} U)$ with the $V \oplus_{\nu,\mu} U \in \mathcal{B}(V,U)$ which are not sub-vector spaces of $W$. To characterise the centre of $F$ using $\mathcal{G}_F$, we need to reformulate the centrality condition by comparing the $V \oplus_{\nu,\mu} U$ with sub-vector spaces of $W$.

**Remark 5.20.** Consider $\delta : V \to W$ and $\epsilon : U \to W$ two morphisms in $\mathcal{V}^g$. For $\delta + \epsilon$ the morphism from $V \oplus U$ to $W$ which sends $v \to \delta(v) + \epsilon(v)$, since $\delta$ and $\epsilon$ are injective, there exists $\nu$ and $\mu$ from $\ker(\delta + \epsilon)$ to $V$ and $U$ such that for all $v \in \ker(\delta + \epsilon)$, $v = \nu(v) - \mu(v)$. Then, by the universal property of the pushout, $(V \oplus U)/\ker(\delta + \epsilon)$ is isomorphic to $V \oplus_{\nu,\mu} U$. The first isomorphism theorem implies that $\delta + \epsilon$ induces an isomorphism between $V \oplus_{\nu,\mu} U$ and $\delta(V) + \epsilon(U)$ (which is the image of $\delta + \epsilon$).

**Notation 5.21.** For $\delta : V \to W$ and $\epsilon : U \to W$ two morphisms in $\mathcal{V}^g$, let $\nu$ and $\mu$ be as in Remark 5.20. Then, let $\epsilon \uparrow \delta$ denote the isomorphism from $V \oplus_{\nu,\mu} U$ to $\delta(V) + \epsilon(U)$ induced by the first isomorphism theorem. It satisfies $\delta + \epsilon = \epsilon \circ (\nu \uparrow \delta) \circ \epsilon \circ \pi$, for $\pi$ the canonical projection from $V \oplus U$ to $V \oplus_{\nu,\mu} U$ and $\epsilon \circ (\nu \uparrow \delta)$ the inclusion from $\delta(V) + \epsilon(U)$ in $W$.

**Lemma 5.22.** For $F \in W\text{Set}^{(\mathcal{V}^g)^{op}}$, $\delta$ from $V$ to $W$ and $\epsilon$ from $U$ to $W$, $(\epsilon \uparrow \delta)^*\nu_{\delta(V) + \epsilon(U)}\phi^F \in F(V \oplus_{\nu,\mu} U)$ is in the inverse image of $\epsilon^*\phi^F$ under $\rho_{F,(V,x)}$.

**Proof.** By definition of $\nu \uparrow \delta$, we have $\nu_{\delta(V) + \epsilon(U)} \circ (\epsilon \uparrow \delta) \circ \nu_V = \delta$ and $\nu_{\delta(V) + \epsilon(U)} \circ (\epsilon \uparrow \delta) \circ \nu_U = \epsilon$, for $\nu_V$ and $\nu_U$ the inclusions of $U$ and $V$ in $V \oplus_{\nu,\mu} U$. Therefore, $\nu_{\delta(V) + \epsilon(U)} \circ (\epsilon \uparrow \delta)^*\nu_{\delta(V) + \epsilon(U)}\phi^F = \epsilon^*\phi^F$.

This lemma implies two things, firstly that $\rho_{F,(V,x)}$ is always surjective, for $F \in W\text{Set}^{(\mathcal{V}^g)^{op}}$ and $\delta$ from $V$ to $W$. Secondly, that $\epsilon^*\phi^F \in F(V)$ is central if and only if, for all morphisms $\epsilon : U \to W$ in $\mathcal{V}^g$, $(\epsilon \uparrow \delta)^*\nu_{\delta(V) + \epsilon(U)}\phi^F \in F(V \oplus_{\nu,\mu} U)$ is the only element in the inverse image of $\epsilon^*\phi^F$ under $\rho_{F,(V,x)}$.

In the case $F \in (W_i)_{i \in I} \text{Set}^{(\mathcal{V}^g)^{op}}$ with $|I|$ greater than one, we recall that $F_i$ denotes the image of $\text{Hom}_{\mathcal{V}^g}(\_,W_i)$ in $F$. We want to use the surjectivity of each $\rho_{F_i,(V,x)}$ with $i \in I$ and $x \in F_i(V)$.

We start by proving that, under some condition on the surjection $q_F$ from $\bigsqcup_{i \in I} \text{Hom}_{\mathcal{V}^g}(\_,W_i)$ to $F$, if $(V,x)$ is central, then $x \in F_i(V)$ for all $i \in I$.

**Definition 5.23.** Let $F$ be an object in $\text{Set}^{(\mathcal{V}^g)^{op}}$, with generating family $((W_i)_{i \in I}, q)$. We say that $((W_i)_{i \in I}, q)$ is a minimal generating family, if for all $i \neq j$, $F_i \nsubseteq F_j$. 

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Lemma 5.25. Let \( F \) be in \( \mathcal{Sel}(V^Z)^{\oplus} \) and \(((W_i)_{i \in I}, q)\) a minimal generating family of \( F \). Let also \((V, x)\) be a central element of \( F \). Then, \( x \in \bigcap_{i \in I} F_i(V) \) and \((V, x)\) is a central element for each \( F_i \).

Proof. For each \( i \), let \( x_i \in F(W_i \oplus \omega_i, V) \) be in the inverse image of \( \phi_i \) under \( \rho_{F_i(V, x)} \). If \( x_i \in F_j(W_j \oplus \omega_j, V) \), \( \phi_i = \iota_{W_i}^* x_i \) hence \( F_i \) is a sub-functor of \( F_j \). Since we supposed the generating family to be minimal, this implies that \( i = j \). Then, \( x = x_i \) and therefore \( x \in F_i(V) \).

By Lemma 5.22 we get that an element in \( F_i(E) \), for \( E \in V^Z \), has at least one element in its inverse image under \( \rho_{F_i(V, x)} \). If \((V, x)\) is central for \( F_i \), this element is unique and \((V, x)\) is also central for \( F_i \).

Now, let us consider \( x \in \bigcap_{i \in I} F_i(V) \). We want to find a criterion on \( G_F \) to determine whether \((V, x)\) is central. Since, \( x \) is in every \( F_i(V) \) and that each \( F_i \) are generated by one element, Lemma 5.22 implies that every \( \rho_{F_i(V, x)} \) is surjective, and so \( \rho_{F_i(V, x)} \) is surjective. So the only way that \((V, x)\) can fail to be central is if a central element has two elements in its inverse image under \( \rho_{F_i(V, x)} \).

Since \( x \in \bigcap_{i \in I} F_i(V) \), for all \( i \in I \), there is some inclusion \( \delta_i \) from \( V \) to \( W_i \) such that \( x = \delta_i^* \phi_i \) and we have the following lemma.

Lemma 5.26. Let \( F \in \mathcal{Sel}(V^Z)^{\oplus} \) and \((V, x)\) a central element of \( F \), we consider for all \( i \in I \), \( \delta_i \) from \( V \) to \( W_i \) such that \( x = \delta_i^* \phi_i \). Then, for \( \epsilon_k : U \to W_k \) and \( \epsilon_j : U \to W_j \) such that \( \epsilon_k^* \phi_k = \epsilon_j^* \phi_j \), we have \((\epsilon_k \uparrow \delta_k)^* \epsilon_k^* \phi_k = (\epsilon_j \uparrow \delta_j)^* \epsilon_j^* \phi_j \).

Proof. Lemma 5.22 states that \((\epsilon_k \uparrow \delta_k)^* \epsilon_k^* \phi_k = (\epsilon_j \uparrow \delta_j)^* \epsilon_j^* \phi_j \) are in the inverse image of \( \epsilon_k \) and \( \epsilon_j \) respectively under \( \rho_{F_k(V, x)} \) and \( \rho_{F_j(V, x)} \). Hence, each of them are in \( V \) and \( W_i \) are central elements of \( F \). Thus, if \((V, x)\) is central, we necessarily have that \((\epsilon_k \uparrow \delta_k)^* \epsilon_k^* \phi_k = (\epsilon_j \uparrow \delta_j)^* \epsilon_j^* \phi_j \).

This gives us a necessary condition on \( G_F \), for \((V, x)\) to be central. This condition will also be sufficient.

For \( \alpha \in G(U \subset W_i, U' \subset W_j) \), \( \alpha^* \iota_{U'}^* \phi_j = \iota_{U}^* \phi_i \) has two, possibly equal, elements in its inverse image under \( \rho_{F_k(V, x)} \), where \( x = \delta_i^* \phi \) for all \( i \). Namely \((\iota_U \circ \alpha)^* \iota_U^* \phi \) and \((\iota_{U'} \circ \delta_i)^* \iota_{U'}^* \phi \). If \((V, x)\) is central, those two have to be equal, in particular they have to live in the same \( F(V \oplus \omega_{V, U}) \), which gives us a first condition on \( G_F \).

Lemma 5.27. Let \( F \) in \( \mathcal{Sel}(V^Z)^{\oplus} \) and \(((W_i)_{i \in I}, q)\) a minimal generating family of \( F \). Let \((V, x)\) be a central element of \( F \) and for all \( i \in I \), let \( \delta_i : V \to W_i \), such that \( x = \delta_i^* \phi_i \). Then, for \( \alpha \in G(U \subset W_i, U' \subset W_j) \), if \( v \in U \cap V_i, \alpha(v) = \delta_j \circ (\delta_i|V_i)^{-1}(v) \), for \( V_i \) the image of \( V \) under \( \delta_i \).
Proof. We consider
\[(t_U \uparrow \delta_i)^*(t_{U+\lambda V_j}^W \phi_i) \in F_i(V \oplus \nu, \mu)\]
and
\[( (t_U \circ \alpha) \uparrow \delta_j)^*(t_{U' \oplus \lambda V_j}^W \phi_j) \in F_j(V \oplus \nu', \mu'),\]
where \(\mu\) is the injection of \(U \cap V_i\) into \(U\), \(\nu\) is the restriction to \(U \cap V_i\) of \((\delta_i|_{V_i'})^{-1}\), and \(\nu'\) and \(\mu'\) are the restriction to \(U' \cap V_j\) of \((\delta_j|_{V_j'})^{-1}\). They are equal by Lemma 5.26. In particular they need to live in the same set. We get that \(V \oplus \nu, \mu U = V \oplus \nu', \mu' U\), which implies that, for \(v \in U \cap V_i\),
\[\alpha(v) = V_j \cap (\delta_j|_{V_j'})^{-1}(\alpha(v)) = (\delta_j|_{V_j'})^{-1}(v),\]
which means that \(\alpha(v) = \delta_j \circ (\delta_i|_{V_i'})^{-1}(v)\).

If the condition of Lemma 5.27 is satisfied, both \(( (t_U \circ \alpha) \uparrow \delta_j)^*(t_{U+\lambda V_j}^W \phi_j) \) and \((t_U \uparrow \delta_i)^*(t_{U+\lambda V_j}^W \phi_i)\) live in \(F(V \oplus \nu, \mu)\) where \(\mu\) is the inclusion from \(U \cap V_i\) in \(U\) and \(\nu\) is the restriction to \(U \cap V_i\) of \((\delta_i|_{V_i'})^{-1}\). Since by definition \((t_U \uparrow \delta_i)\) and \((t_U \circ \alpha) \uparrow \delta_j\) are isomorphisms from \(V \oplus \nu, \mu U\) respectively to \(V_i + U\) and \(V_j + U'\), \(( (t_U \circ \alpha) \uparrow \delta_j) \circ (t_U \uparrow \delta_i)^{-1}\) is an isomorphism from \(U' + V_j\) to \(U + V_j\) that we can compute.

Lemma 5.28. Let \(\alpha\) be an isomorphism from \(U \subset W_i\) to \(U' \subset W_j\) and let \(\delta_i\) and \(\delta_j\) be two injections from \(V\) respectively to \(W_i\) and \(W_j\) whose images are \(V_i\) and \(V_j\). Then, if for every \(v \in U \cap V_j\), \(\alpha(v) = \delta_j \circ (\delta_i|_{V_i'})^{-1}(v)\), \((t_U \uparrow \delta_i)^{-1}(u) = t_{U+\lambda V_j}^W\) (then \(( (t_U \circ \alpha) \uparrow \delta_j)^* t_{U+\lambda V_j}^W \phi_j = \alpha(u)\), and for \(v \in V_i\), \((t_U \uparrow \delta_i)^{-1}(u) = t_{V \oplus \nu, \mu}^W (t_{U+\lambda V_j}^W (\delta_i|_{V_i'})^{-1}(v))\) and \(( (t_U \circ \alpha) \uparrow \delta_j) (t_{V \oplus \nu, \mu}^W (t_{U+\lambda V_j}^W (\delta_i|_{V_i'})^{-1}(v))) = \delta_j \circ (\delta_i|_{V_i'})^{-1}(v)\).

Proof. The first part of the statement is a direct consequence of the definition of \((t_U \uparrow \delta_i)\) and \((t_U \circ \alpha) \uparrow \delta_j\).

For \(u \in U\), \((t_U \uparrow \delta_i)^{-1}(u) = t_{V \oplus \nu, \mu}^W (t_{U+\lambda V_j}^W (\delta_i|_{V_i'})^{-1}(v))\) (then \(( (t_U \circ \alpha) \uparrow \delta_j)^* t_{U+\lambda V_j}^W \phi_j = \alpha(u)\), and for \(v \in V_i\), \((t_U \uparrow \delta_i)^{-1}(u) = t_{V \oplus \nu, \mu}^W (t_{U+\lambda V_j}^W (\delta_i|_{V_i'})^{-1}(v))\) and \(( (t_U \circ \alpha) \uparrow \delta_j) (t_{V \oplus \nu, \mu}^W (t_{U+\lambda V_j}^W (\delta_i|_{V_i'})^{-1}(v))) = \delta_j \circ (\delta_i|_{V_i'})^{-1}(v)\).

Notation 5.29. Let \(\alpha\) be an isomorphism from \(U \subset W_i\) to \(U' \subset W_j\) and let \(\delta_i\) and \(\delta_j\) be two injections from \(V\) respectively to \(W_i\) and \(W_j\) whose images are \(V_i\) and \(V_j\) and such that for every \(v \in U \cap V_j\), \(\alpha(v) = \delta_j \circ (\delta_i|_{V_i'})^{-1}(v)\). We denote by \(\bar{\alpha}\) the morphism from \(U' + V_j\) to \(U' + V_j\) which sends \(u \in U\) to \(\alpha(u)\) and \(v \in V_i\) to \(\delta_j \circ (\delta_i|_{V_i'})^{-1}(v)\).

If \((V, x)\) is central, we have \(( (t_U \circ \alpha) \uparrow \delta_j) t_{U+\lambda V_j}^W \phi_j = (t_U \uparrow \delta_i) t_{U+\lambda V_j}^W \phi_i\), which implies that \(\bar{\alpha}\) must be in \(G_F(U + V_i \subset W_i; U' + V_j \subset W_j)\). We state the principal theorem of this sub-section.

Theorem 5.30. Let \(F\) be in \(S(I)\) and \((w_i)_{i \in I}, q)\) a minimal generating family of \(F\). Let also \(x \in F(V)\) and \((\delta_i : V \rightarrow W_i)_{i \in I}\) a family of injective morphisms from \(V\) to \(W_i\), such that, for all \(i \in I\), \(x = \delta_i \phi_i\). Then, \((V, x)\) is central if and only if the two following conditions are satisfied:

1. for all \(\alpha \in G_F(U \subset W_i; U' \subset W_j)\), and for all \(v \in U \cap V_i\), \(\alpha(v) = \delta_j \circ (\delta_i|_{V_i'})^{-1}(v)\),

2. for all sub-spaces \(U\) of \(W_i\) and \(U'\) of \(W_j\), and for all isomorphism \(\alpha\) from \(U\) to \(U'\) satisfying \(\alpha(v) = \delta_j \circ (\delta_i|_{V_i'})^{-1}(v)\), for \(v \in U \cap V_i\), \(\alpha \in G_F(U \subset W_i; U' \subset W_j)\) if and only if \(\bar{\alpha} \in G_F(U + V_i \subset W_i; U' + V_j \subset W_j)\).
Proof. Let us show first necessity. We suppose that \((V, x)\) is central, then by Lemma 5.27, the first condition is satisfied. We consider \(g \in G_F(U \subseteq W_i, U' \subseteq W_j)\), then by Lemma 5.26 \((u_U \uparrow \delta_i)^* (i_U^{W_i})^* \phi_i = ((i_U \circ \alpha) \uparrow \delta_i)^* (i_{\alpha(U)+V_j}^{W_j})^* \phi_j\). By Lemma 5.28, this implies that \(\bar{\alpha}^* i_{U+V_j}^* \phi_j = i_{U+V_j}^* \phi_i\). Then, \(\bar{\alpha} \in G_F(U + V_i \subseteq W_i, U' + V_j \subseteq W_j)\). Conversely, if \(\bar{\alpha} \in G_F(U + V_i \subseteq W_i, U' + V_j \subseteq W_j)\), since \(G_F\) has the restriction property, \(\alpha = \bar{\alpha}_U \in G_F(U \subseteq W_i, U' \subseteq W_j)\).

We now prove sufficiency. Suppose that \(F\) and \((V, x)\) satisfy the two conditions. We consider \(\zeta^* \phi_i \in F(V \oplus_{U, U} U)\) in the inverse image of an element of \(\beta^* \phi_j \in F(U)\), where \(j\) is not necessarily equal to \(i\). The first condition implies that, if \(\eta : V \to W_i\) satisfies \(\eta \sim_G \delta_i\), then \(\eta = \delta_i\). But \(\zeta^* \phi_i = \delta^* \phi_i\), hence \(\zeta|_V = \delta_i\). Then, the equality \(\zeta|_V^* \phi_i = \beta^* \phi_j\) implies the existence of a morphism \(\alpha \in G_F(\beta(U), \zeta(U))\) such that \(\zeta|_U = \alpha \circ \beta\), where \(\zeta|_U\) and \(\beta\) are the corestrictions of \(\zeta\) to \(U\) and \(\beta\) to their images. Therefore, \(\zeta = (i_{\zeta(U)+V_j}^* \circ \alpha \circ (\beta \uparrow \delta_j)\), where \((\beta \uparrow \delta_j)^* (i_{\beta(U)+V_j}^* \phi_j)\) is the only element in the inverse image of \(\beta^* \phi_j\) under \(p_{F,V}(x,x)\).

Example 5.31. When \(F \in W_{Set}^{(VW)^p}\), the two conditions become simpler. We consider \(\delta\) from \(V\) to \(W\), such that \((V, \delta^* \phi^F)\) is central.

The first condition becomes that, for \(\alpha \in G(U, U')\) and for \(v \in U \cap \delta(V)\), \(\alpha(v) = v\). Furthermore, in this case, \(\bar{\alpha}\) is the morphism from \(U + \delta(V)\) to \(U' + \delta(V)\) which sends \(u \in U\) to \(\alpha(u)\) and \(v \in \delta(V)\) to \(v\).

We return to Example 5.7

Proposition 5.32. For \(G\) a sub-group of \(Gl(W)\), the centre of \(F = \text{Hom}_{VW}(\cdot, W)/G\) is given by \((C, \iota_C)\) with \(C\) the maximal sub-vector space of \(W\) such that for all \(v \in C\) and \(g \in G\), \(g(v) = v\) and with \(\iota_C\) the inclusion of \(C\) in \(W\).

Proof. As we have seen in Example 5.7 \(G_F = g(G)\) (see Definition 5.12). Let \(\delta\) be an injection from some vector space \(V\) to \(W\) such that \((V, q(\delta))\) is central, for \(q\) the canonical projection from \(\text{Hom}_{VW}(\cdot, W)\) to \(F\). We will denote by \(V'\) the sub-space \(\delta(V)\) of \(W\). The first condition of Theorem 5.30 implies that, for all \(g \in G\), \(g \in G_F(V, W)\), therefore for all \(x \in V'\), \(g(x) = x\). Conversely, if, for all \(g \in G\) and for \(x \in V'\), \(g(x) = x\), since for all sub-spaces \(U\) of \(W\), and for all \(\alpha \in G_F(U, U')\), there exists \(g \in G\) such that \(\alpha = g_U\), \((V, q(\delta))\) satisfies the first condition of the theorem.

In this case, if \((V, q(\delta))\) satisfies the first condition of the theorem, the second condition of Theorem 5.30 is immediately satisfied. Indeed, for \(\alpha \in G_F(U, U')\), \(\bar{\alpha}\) is the morphism from \(U + V'\) to \(U' + V'\) which sends \(u \in U\) to \(\alpha(u)\) and \(v \in V'\) to \(v\). Then, let \(g \in G\) such that \(\alpha = g_U\). Since for all \(v \in V'\), \(g(v) = v\), \(g_{U+V'} = \bar{\alpha}\). Hence, \(\alpha \in G_F(U + V', U' + V')\).

Therefore, \(q(\delta)\) is central if and only if \(\delta\) factorizes through \(\iota_C\). Thus, \((C, \iota_C)\) is the centre of \(F\).

6 The algebras \(H^*(W)^G\)

This section shows how to apply the groupoid \(G_F\) of the last section to classification problems of nil-closed, integral, noetherian, unstable algebras. Before we explain the focus of this section, let us recall the theorem of Adams-Wilkerson.

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Definition 6.1. [[HLS93, Part II.2] For \( K \in \mathcal{K} \), the transcendence degree of \( K \) is \( d \in \mathbb{N} \cup \{ \infty \} \), the supremum of the cardinals of finite sets of homogeneous elements in \( K \) which are algebraically independent.

Remark 6.2. If \( K \) is noetherian, the transcendence degree of \( K \) is finite.

Let us recall the theorem of Adams-Wilkerson.

Theorem 6.3. [[HLS93, Theorem 3] Let \( K \) be an integral, unstable algebra of transcendence degree lesser or equal to \( \dim(W) \), then there exists an injection \( \phi \) from \( K \) to \( H^*(W) \). Furthermore, this injection is regular if and only if the transcendence degree of \( K \) equals \( \dim(W) \).

Therefore, every integral, nil-closed, noetherian, unstable algebra is isomorphic to a nil-closed, noetherian sub unstable algebra of some \( H^*(W) \). In the first sub-section we define \( H^*(W)G \) for \( G \) a groupoid on sub spaces of \( W \) that satisfies the restriction property. Then, \( G \to H^*(W)G \) defines an explicit one-to-one correspondence between the groupoids on sub spaces of \( W \) satisfying the restriction property and the noetherian, nil-closed, unstable sub algebras of \( H^*(W) \) of transcendence degree \( \dim(W) \).

For simplicity, we restrict to the integral case. The constructions generalizes to the non integral case.

Let us now recall the definition of the primitive elements of a comodule.

Definition 6.4. For \( K \in \mathcal{K} \) provided with a \( H^*(V) \)-comodule structure \( \kappa \) in \( \mathcal{K} \), the algebra of primitive elements of \( K \) is the sub-algebra of \( K \) whose elements are those satisfying that \( \kappa(x) = x \otimes 1 \), for 1 the unit of \( H^*(V) \). We will denote by \( P(K, \kappa) \) the algebra of primitive elements of \( K \) for the \( H^*(V) \)-comodule structure \( \kappa \).

Remark 6.5. By Corollary [[3.15] for all \( (V, \phi) \in C(K) \), there is a unique structure \( \kappa_\phi \) of \( H^*(V) \)-comodule on \( K \) such that \( (\epsilon_K \otimes \text{id}_{H^*(V)}) \circ \kappa_\phi = \phi \).

Notation 6.6. We will also denote \( P(K, \kappa_\phi) \) by \( P(K, \phi) \).

The problem that we are interested in is the following. If we fix \( V \) some finite dimensional vector space and \( P \) some unstable algebra, can we classify, under suitable hypothesis, the connected, noetherian, nil-closed unstable algebras \( K \), satisfying that \( K \) admit a \( H^*(V) \)-comodule structure \( \kappa \) in \( \mathcal{K} \), whose algebra of primitive elements is isomorphic to \( P \). Since, every nil-closed, noetherian, integral, unstable algebra of transcendence degree \( \dim(W) \) is isomorphic to some \( H^*(W)G \), we need to be able to identify the primitive elements associated with a regular central element \( (V, \phi) \) of \( H^*(W)G \).

In the second subsection, we consider \( H^*(W)G \) and an inclusion \( \delta \) from some vector space \( V \) to \( W \), such that \( (V, \delta^* \phi) \in C(H^*(W)G) \) for \( \phi \) the inclusion of \( H^*(W)G \) in \( H^*(W) \). Then, we prove that \( P(H^*(W)G, \delta^* \phi) \) is a nil-closed and noetherian sub-algebra of \( \pi^*(H^*(W/\text{Im}(\delta))) \) for \( \pi \) the projection from \( W \) to \( W/\text{Im}(\delta) \). Since \( \pi^* \) is injective, there exists \( H^*(W/\text{Im}(\delta))G' \subset H^*(W/\text{Im}(\delta)) \) such that \( P(H^*(W)G, \delta^* \phi) = \pi^*(H^*(W/\text{Im}(\delta))G') \). We conclude this sub-section by explaining how to determine \( G' \) from \( G \).

Finally, in the last sub-section, we give examples of how to answer the following question: given \( G' \) a groupoid with the restriction property on sub-spaces of \( W/V \), with \( W \) a finite dimensional
vector space and $V$ a sub-space of $W$, what are the groupoids $G$ with the restriction property on sub-spaces of $W$ such that $(V, \iota_V^* \phi) \in C(H^*(W)^G)$, for $\iota_V$ the inclusion of $V$ in $W$, and such that the primitive elements of $H^*(W)^G$ for the $H^*(V)$-comodule structure induced by $\iota_V^* \phi$ are $\pi^*(H^*(W/V)^G)$.  

6.1 Noetherian, nil-closed, unstable sub-algebras of $H^*(W)$

In this sub-section, we give an explicit one-to-one correspondence between the groupoids on sub-spaces of $W$ satisfying the restriction property and the noetherian, nil-closed, unstable sub-algebra of $H^*(W)$ of transcendence degree $\dim(W)$.  

**Theorem 6.7.** For all $W \in \mathcal{V}$, there is a one-to-one correspondence between the set of nil-closed and noetherian sub-algebras of $H^*(W)$ whose transcendence degree is $\dim(W)$ and the set of groupoids with the restriction property, whose objects are the sub-vector spaces of $W$.  

**Proof.** By Theorem 5.17, there is a one-to-one correspondence between isomorphism classes in $W_{\mathbf{Set}}^{(\mathcal{V})^{op}}$ and the set of groupoids satisfying the assumption. Thus, we have to justify that the set of nil-closed and noetherian sub-algebras of $H^*(W)$ of transcendence degree $\dim(W)$ are in one-to-one correspondence with isomorphism classes in $W_{\mathbf{Set}}^{(\mathcal{V})^{op}}$. Let $K$ be a nil-closed, noetherian, sub-algebra of $H^*(W)$ whose transcendence degree is $\dim(W)$. Then, for $\phi_K$ the inclusion of $K$ in $H^*(W)$, since the transcendence degree of $K$ is $\dim(W)$, by the Theorem of Adams-Wilkerson, $\phi_K$ is regular. Since $K$ is noetherian, this implies that we can restrict $\phi_K^* : \text{Hom}_{\mathcal{V}}(\ldots, W) \to \text{Hom}_K(K, H^*(\ldots))$ to regular elements. For $F := \text{Hom}_{\mathcal{V}_R}(K, H^*(\ldots))$ and $q_F : \text{Hom}_{\mathcal{V}}(\ldots, W) \to F$ the restriction of $\phi_K^*$ to regular elements, we have that $\phi_K^* = q_F^*$ and that $(F, q_F)$ is an element in $W_{\mathbf{Set}}^{(\mathcal{V})^{op}}$. This defines a map $h$ from the set of nil-closed and noetherian sub-algebras of $H^*(W)$ whose transcendence degree is $\dim(W)$ to the set of isomorphism classes in $W_{\mathbf{Set}}^{(\mathcal{V})^{op}}$.  

Let us prove that $h$ is injective. We consider, $K$ and $K'$ two nil-closed and noetherian sub-algebras of $H^*(W)$ such that, for $h(K) = (F, q_F)$ and $h(K') = (F', q_{F'})$ are isomorphic in $W_{\mathbf{Set}}^{(\mathcal{V})^{op}}$. Then, there is an isomorphism $\eta$ from $F$ to $F'$ such that $q_{F'} = \eta \circ q_F$. By applying the functor $(\ldots)$, we get the following commutative diagram in $\mathbf{Set}^{(\mathcal{V})^{op}}$:

\[
\begin{array}{ccc}
\text{Hom}_K(K, H^*(\ldots)) & \xrightarrow{\phi_K^*} & \text{Hom}_K(H^*(W), H^*(\ldots)) \\
\downarrow{\text{Hom}(\ldots, W)} & & \downarrow{\eta} \\
\text{Hom}_K(K', H^*(\ldots)) & \xrightarrow{\phi_{K'}^*} & \end{array}
\]
Finally, by applying the functor $m \circ L$, we get the following commutative diagram:

\[
\begin{array}{ccc}
  l_1(K) & \xrightarrow{l_1(\phi_K)} & l_1(H^*(W)) \\
  m \circ L(\tilde{\eta}) & \downarrow & \downarrow \\
  l_1(K') & \xrightarrow{l_1(\phi_{K'})} & l_1(H^*(W))
\end{array}
\]

where, $m \circ L(\tilde{\eta})$ is an isomorphism. Since $K$, $K'$ and $H^*(W)$ are nil-closed, this implies that there is an isomorphism from $K$ to $K'$ such that the following diagram commutes:

\[
\begin{array}{ccc}
  K & \xrightarrow{\phi_K} & H^*(W) \\
  \cong & \downarrow & \downarrow \\
  K' & \xrightarrow{\phi_{K'}} & H^*(W)
\end{array}
\]

so that $K$ and $K'$ are the same sub-algebra of $H^*(W)$, hence $h$ is injective.

We conclude, by exhibiting a right inverse of $h$, for $(F, q_F) \in W\text{Set}^{(\mathcal{V}T)^\text{op}}$, we consider the injection $m \circ L(\tilde{F}) \xrightarrow{m \circ L(q_{\tilde{F}})} H^*(W)$ and we define $j(F, q_F)$ the image of $m \circ L(q_{\tilde{F}})$ in $H^*(W)$, which does not depend on the choice of $(F, q_F)$ in its isomorphism class. By construction, $j(F, q_F)$ is nil-closed, and $g(j(F, q_F)) \cong \tilde{F}$ is noetherian so, by Proposition 4.8, $j(F, q_F)$ is noetherian. $j$ is obviously a right inverse of $h$; since $h$ is injective, it is a bijection.

**Definition 6.8.** For $\mathcal{G}$ a groupoid whose objects are the sub-vector spaces of $W$ and which has the restriction property, for $F := \text{Hom}_{\mathcal{V}T}(\_ , W)/ \sim_{\mathcal{G}}$ and for $q_{F}$ the canonical surjection from $\text{Hom}_{\mathcal{V}T}(\_ , W)$ to $F$, $H^*(W)^{\mathcal{G}}$ is the image of the map $m \circ L(q_{\tilde{F}}) : m \circ L(\tilde{F}) \hookrightarrow H^*(W)$.

**Remark 6.9.** $H^*(W)^{\mathcal{G}}$ is the unique nil-closed, noetherian, sub-algebra of $H^*(W)$, whose transcendence degree is $\dim(W)$ and such that $\mathcal{G}(F, q_F) = \mathcal{G}$, for $F := \text{Hom}_{\mathcal{K}_{\mathcal{L},g}}(H^*(W)^{\mathcal{G}}, H^*(\_))$ and $q_F$ the natural surjection from $\text{Hom}_{\mathcal{V}T}(\_ , W)$ to $\text{Hom}_{\mathcal{K}_{\mathcal{L},g}}(H^*(W)^{\mathcal{G}}, H^*(\_))$ induced by the inclusion from $H^*(W)^{\mathcal{G}}$ to $H^*(W)$.

Furthermore, $\mathcal{G} \mapsto H^*(W)^{\mathcal{G}}$ defines a contravariant functor between $\text{Groupoid}(W)$ (see Notation 5.11) and the poset of nil-closed, noetherian, sub-algebras of $H^*(W)$, whose transcendence degrees are $\dim(W)$, ordered by inclusion.

**Corollary 6.10.** Any nil-closed, integral, noetherian, unstable, algebra whose transcendence degree is equal to $\dim(W)$ is isomorphic to $H^*(W)^{\mathcal{G}}$ for some $\mathcal{G}$.

**Proof.** It is a reformulation of the theorem of Adams-Wilkerson using Theorem 6.7. □
Example 6.11. For $G$ a sub-group of $\text{Gl}(W)$, $H^*(W)^{\mathfrak{g}(G)} = H^*(W)^G$, for $H^*(W)^G$ the algebra of invariant element of $H^*(W)$ under the action of $G$.

Let us identify precisely the sub-algebra $H^*(W)^G$ of $H^*(W)$.

Proposition 6.12. Let $\mathcal{G}$ be a groupoid whose objects are the sub-vector spaces of $W$ with the restriction property. Then,

$$H^*(W)^G = \{ x \in H^*(W) : \alpha^*\iota_{U'}^*(x) = \iota_{U}^*(x) \text{ for all } \alpha \in \mathcal{G}(U, U') \}. $$

Proof. Let $\phi$ be the inclusion of $H^*(W)^G$ in $H^*(W)$ and let $K(\mathcal{G}) = \{ x \in H^*(W) : \alpha^*\iota_{U'}^*(x) = \iota_{U}^*(x) \text{ for all } \alpha \in \mathcal{G}(U, U') \}$. By definition, $\alpha^*\iota_{U'}^*, \phi = \iota_{U}^*, \phi$ for all $\alpha \in \mathcal{G}(U, U')$ and for all sub-spaces $U$ and $U'$ of $W$. Then,

$$H^*(W)^G \subset K(\mathcal{G}).$$

Furthermore, the inclusion from $K(\mathcal{G})$ to $H^*(W)$ induces a surjection from $\text{Hom}_{\mathcal{V}T}(\cdot, W)$ to $g(K(\mathcal{G}))$ which factorises through an isomorphism from $\text{Hom}_{\mathcal{V}T}(\cdot, W) / \sim_\mathcal{G}$ to $g(K(\mathcal{G}))$. The fact that the surjection factorises through $\text{Hom}_{\mathcal{V}T}(\cdot, W) / \sim_\mathcal{G}$ is a direct consequence of the definitions of $K(\mathcal{G})$ and $\sim_\mathcal{G}$, and the fact that this morphism is injective, comes from the fact that the isomorphism from $\text{Hom}_{\mathcal{V}T}(\cdot, W) / \sim_\mathcal{G}$ to $g(H^*(W)^G)$ factorises as the following diagram:

$$\text{Hom}_{\mathcal{V}T}(\cdot, W) / \sim_\mathcal{G} \to g(K(\mathcal{G})) \to g(H^*(W)^G).$$

Then, by applying the functor $m \circ \mathcal{L}$ to the last diagram, we get the bottom line of the following:

$$H^*(W)^G \xrightarrow{\eta_{H^*(W)^G}} K(\mathcal{G}) \xrightarrow{\eta_{K(\mathcal{G})}} H^*(W)$$

where $\eta$ denotes the unit of the adjunction between $f$ and $m$. Then, since $H^*(W)^G$ and $H^*(W)$ are nil-closed, $\eta_{H^*(W)^G}$ and $\eta_{H^*(W)}$ are isomorphisms. Furthermore, $K(\mathcal{G})$ is a sub unstable algebra of $H^*(W)$, hence it does not contains any nilpotent sub module, and $\eta_{K(\mathcal{G})}$ is injective. Then, the commutativity of the diagram implies that $\eta_{K(\mathcal{G})}$ is an isomorphism, and therefore that $H^*(W)^G = K(\mathcal{G})$.  

Definition 6.13. For $g \in \text{Gl}(W)$, and $\mathcal{G} \in \text{Groupoid}(W)$, $g \cdot \mathcal{G}$ is the groupoid in $\text{Groupoid}(W)$ defined by $\beta \in g \cdot \mathcal{G}(R, R')$, for $\beta$ an isomorphisms between subspaces $R$ and $R'$ of $W$, if there exist $\alpha \in \mathcal{G}(U, U')$ for $U = g^{-1}(R)$ and $U' = g^{-1}(R')$, such that the following diagram commutes:

$$\begin{array}{ccc}
U & \xrightarrow{g \mid_U^R} & R \\
\downarrow{\alpha} & & \downarrow{\beta} \\
U' & \xrightarrow{g \mid_{U'}^{R'}} & R'.
\end{array}$$

This defines a poset preserving action of $\text{Gl}(W)$ on $\text{Groupoid}(W)$.
**Remark 6.14.** This action generalises the action by conjugation on $\text{Group}(W)$. Indeed, for $G$ a subgroup of $\text{Gl}(W)$ and $g \in \text{Gl}(W)$, $g \cdot g(G) = g(gGg^{-1})$.

**Proposition 6.15.** For $g \in \text{Gl}(W)$ and $G \in \text{Groupoid}(W)$, $H^*(W)^g_G = (g^{-1})^*(H^*(W)^G)$.

*Proof.* This is a direct consequence of Proposition 6.12. \hfill \Box

**Remark 6.16.** We want to notice that the $(H^*(W)^G)_{G \in \text{Groupoid}(W)}$ does not constitute a minimal list for representing elements of isomorphism classes of nil-closed, integral and noetherian unstable algebras of transcendence degree $\dim(W)$. For $g \in \text{Gl}(W)$ and $G \in \text{Groupoid}(W)$, $g \cdot G$ needs not to be equal to $G$, but, by Proposition 6.15, $H^*(W)^G \cong H^*(W)^{g \cdot G}$.

Conversely, since the inclusion of $H^*(W)^G$ in $H^*(W)$ induces a surjection from $\text{Hom}_K(H^*(W), H^*(W))$ to $\text{Hom}_K(H^*(W)^G, H^*(W))$, and since $g \mapsto g^*$ induces an isomorphism between $H^*(W)^G$ and $\text{Gl}(W)$, we have that if $H^*(W)^G \cong H^*(W)^H$, there exists $g \in \text{Gl}(W)$ such that $H^*(W)^H = (g^{-1})^*(H^*(W)^G)$. By Proposition 6.15, $H = g \cdot G$.

### 6.2 Centrality and primitive elements of $H^*(W)^G$

Throughout this sub-section, we fix $V$ and $W$ two objects in $\mathcal{V}$, as well as an injection $\delta$ from $V$ to $W$.

We consider $K$ a nil-closed, noetherian unstable sub algebra of $H^*(W)$ of transcendence degree $\dim(W)$, such that $(V, \delta^* \phi) \in \mathcal{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. We start by explaining why the $H^*(V)$-comodule structure on $K$ induced by $\delta^* \phi$ is induced from the $H^*(V)$-comodule structure on $H^*(W)$ given by $(\text{id}_W + \delta)^* : H^*(W) \to H^*(W) \otimes H^*(V)$.

Then, for $K = H^*(W)^G$, we explain how to determine the primitive elements of this comodule structure from $G$.

**Proposition 6.17.** Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi) \in \mathcal{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. The $H^*(V)$-comodule structure $\kappa$ on $K$, induced by $\delta^* \phi$ and Corollary 6.15, fits into the following commutative diagram:

$$
\begin{array}{ccc}
K & \xrightarrow{\kappa} & K \otimes H^*(V) \\
\phi \downarrow & & \phi \otimes \text{id}_{H^*(V)} \\
H^*(W) & \xrightarrow{(\text{id}_W + \delta)^*} & H^*(W) \otimes H^*(V).
\end{array}
$$

*Proof.* We consider the following diagram:

$$
\begin{array}{ccc}
K & \xrightarrow{\kappa} & K \otimes H^*(V) \\
\phi \downarrow & & \phi \otimes \text{id}_{H^*(V)} \\
H^*(W) & & H^*(W) \otimes H^*(V).
\end{array}
$$

The existence of a morphism $\psi^*$ from $H^*(W)$ to $H^*(W) \otimes H^*(V)$ which turns it into a commutative diagram is a consequence of the surjectivity of $\phi^*$ from $\text{Hom}_K(H^*(W), H^*(W \oplus V))$ to
Hom\(_K(K, H^*(W \oplus V))\). We only have to justify why we can take \(\psi = \text{id}_W + \delta\). We have that the composition of \((\phi \otimes \text{id}_{H^*(V)}) \circ \kappa\) with \(\epsilon_K \otimes \text{id}_{H^*(V)}\) is equal to \(\delta^* \phi\) and that with \(\text{id}_{H^*(W)} \otimes \epsilon_{H^*(V)}\) is equal to \(\phi\). Hence, since \(\delta^* \phi\) is central, \((\phi \otimes \text{id}_{H^*(V)}) \circ \kappa\) is the unique element in the inverse image of \(\phi\) under \(\rho_{\text{Hom}_K(K, H^*(\_)), (V, \delta^* \phi)}\). But \((\text{id}_W + \delta)^* \phi\) is also in this inverse image of \(\phi\), hence the diagram commutes. \(\square\)

We consider \((\text{id}_W + \delta)^* : H^*(W) \to H^*(W) \otimes H^*(V)\) which is the \(H^*(V)\)-comodule structure on \(H^*(W)\) associated with \((V, \delta) \in \mathcal{C}(K)\).

**Proposition 6.18.** Let \(K\) be a noetherian unstable sub algebra of \(H^*(W)\) of finite transcendence degree \(\dim(W)\) such that \((V, \delta^* \phi) \in \mathcal{C}(K)\), for \(\phi\) the inclusion of \(K\) in \(H^*(W)\). Then, we have a pullback diagram of the following form:

\[
\begin{array}{ccc}
P(K, \delta^* \phi) & \longrightarrow & K \\
\uparrow & & \downarrow \phi \\
H^*(W/\text{Im}(\delta)) & \longrightarrow & H^*(W).
\end{array}
\]

**Proof.** Proposition 6.17 says that the following diagram commutes:

\[
\begin{array}{ccc}
K & \longrightarrow & K \otimes H^*(V) \\
\phi & & \phi \otimes \text{id}_{H^*(V)} \\
H^*(W) & \longrightarrow & H^*(W) \otimes H^*(V).
\end{array}
\]

This means that the \(H^*(V)\)-comodule structure on \(K\) is induced by that on \(H^*(W)\). Hence, the primitive elements of \(K\) are simply the primitive elements of \(H^*(W)\) that are in \(K\). But the comodule structure on \(H^*(W)\) is the morphism \((\text{id}_W + \delta)^*\) whose algebra of primitive elements is the image of \(H^*(W/\text{Im}(\delta))\) under \(\pi^*\), for \(\pi\) the projection from \(W\) to \(W/\text{Im}(\delta)\). \(\square\)

**Corollary 6.19.** Let \(K\) be a noetherian unstable sub algebra of \(H^*(W)\) of finite transcendence degree \(\dim(W)\) such that \((V, \delta^* \phi) \in \mathcal{C}(K)\), for \(\phi\) the inclusion of \(K\) in \(H^*(W)\). Then, the following is a pushout diagram:

\[
\begin{array}{ccc}
\text{Hom}_V(\_, W) & \longrightarrow & \text{Hom}_K(K, H^*(\_)) \\
\downarrow & & \downarrow \\
\text{Hom}_V(\_, W/\text{Im}(\delta)) & \longrightarrow & \text{Hom}_K(P(K, \delta^* \phi), H^*(\_)).
\end{array}
\]

**Proof.** It is a direct consequence of Lemma 2.15 and of Proposition 6.18. \(\square\)

We can thus identify \(\text{Hom}_K(P(K, \delta^* \phi), H^*(\_))\) in this context. In particular, we show that \(P\) is always noetherian.

**Lemma 6.20.** For \(S\) a set, and \(\sim_1\) and \(\sim_2\) two equivalence relations on \(S\), we denote by \(\sim\) the smallest equivalence relation on \(S\) (in the sense that \(\{(a, b) \in S \times S : a \sim b\} \subset S \times S\) is the smallest)
such that, for all $a$ and $b$ in $S$ such that $a \sim_1 b$ or $a \sim_2 b$, $a \sim b$. Then, the following is a pushout in $\text{Set}$:

\[
\begin{array}{ccc}
S & \rightarrow & S/ \sim_1 \\
\downarrow & & \downarrow \\
S/ \sim_2 & \rightarrow & S/ \sim.
\end{array}
\]

**Proof.** Let $\Sigma$ denote the pushout of

\[
\begin{array}{ccc}
S & \rightarrow & S/ \sim_1 \\
\downarrow & & \downarrow \\
S/ \sim_2 & \rightarrow & S/ \sim_2.
\end{array}
\]

Then, for $s : S \rightarrow \Sigma$ the composition of the projection from $S$ to $S/ \sim_1$ with the surjective application $S/ \sim_1 \rightarrow \Sigma$, $s$ is surjective. We define $\sim'$ the equivalence relation on $S$ defined by $a \sim' b$ if and only if $s(a) = s(b)$. $\Sigma$ is isomorphic in $\text{Set}$ with $S/ \sim'$ and we will show that $\sim' = \sim$. 

By commutativity of the pushout diagram, for $a$ and $b$ in $S$ such that $a \sim_1 b$ or $a \sim_2 b$, $s(a) = s(b)$. Suppose that $\sim'$ is not the smallest such equivalence relation. Then, there exists $x$ and $y$ with $x \sim' y$ and an equivalence relation $\sim''$, satisfying that for $a$ and $b$ such that $a \sim_1 b$ or $a \sim_2 b$, $a \sim'' b$, and such that $x$ is not equivalent to $y$ for $\sim''$. Then, the following diagram is commutative:

\[
\begin{array}{ccc}
S & \rightarrow & S/ \sim_1 \\
\downarrow & & \downarrow \\
S/ \sim_2 & \rightarrow & S/ \sim'.
\end{array}
\]

and factorise by a morphism $S/ \sim' \rightarrow S/ \sim''$. This is a contradiction, so $\sim' = \sim$. □

**Remark 6.21.** For $\sim_1$ and $\sim_2$ as in Lemma 6.20, and for $S$ finite, the smallest equivalence relation $\sim$ on $S$ such that, for all $a$ and $b$ in $S$ such that $a \sim_1 b$ or $a \sim_2 b$, $a \sim b$, is the equivalence relation defined by $a \sim b$ if there is a finite family $(s_i)_{i \in[1,n]}$ of objects in $S$ such that:

1. $s_1 = a$,
2. $s_n = b$,
3. for all $1 \leq i \leq n$, if $i$ is odd $s_i \sim_1 s_{i+1}$ and if $i$ is even $s_i \sim_2 s_{i+1}$.

We deduce the following proposition.

**Proposition 6.22.** Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi) \in C(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. Then, for $\zeta$ and $\gamma$ in $\text{Hom}_{V^f}(U, V)$, $\gamma^* \phi|_{P(K, \delta^* \phi)} = \zeta^* \phi|_{P(K, \delta^* \phi)} \in \text{Hom}_K(P(K, \delta^* \phi), H^*(U))$ if and only if there exists a family $(\epsilon_i)_{i \in[1,n]} \in \text{Hom}_{V^f}(U, W)^n$ with $n \in \mathbb{N}$ greater than 1, such that:

1. $\gamma = \epsilon_1$, 

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2. $\zeta = \epsilon_n$.

3. for all $1 \leq i \leq n - 1$, $\epsilon_i^* \phi = \epsilon_{i+1}^* \phi$ if $i$ is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if $i$ is even.

Proof. By Corollary \[6.19\], the following is a pushout:

$$
\begin{array}{ccc}
\text{Hom}_{W/}(\cdot, W) & \xrightarrow{k} & \text{Hom}_K(K, H^*(\cdot)) \\
\pi & \downarrow & p \\
\text{Hom}_{W/}(\cdot, W/\text{Im}(\delta)) & \xrightarrow{} & \text{Hom}_K(P(K, \delta^* \phi), H^*(\cdot)),
\end{array}
$$

where $k$ maps $\zeta : U \to \zeta^* \phi : K \to H^*(U)$, $p$ maps $\psi : K \to H^*(U)$ to $\psi|_{P(K, \delta^* \phi)}$ and $\pi_*$ maps $\zeta : U \to W$ to $\pi \circ \zeta : U \to W/\text{Im}(\delta)$. Then, by Lemma \[6.20\], $p \circ k(\zeta) = p \circ k(\gamma)$ if and only if there exists a family $(\epsilon_i)_{1 \leq i \leq [1, n]} \in \text{Hom}_{W/}(U, W)^n$ with $n \in \mathbb{N}$ greater than 1, such that $\gamma = \epsilon_1$, $\zeta = \epsilon_n$, and for all $1 \leq i \leq n - 1$, $k(\epsilon_i) = k(\epsilon_{i+1})$ if $i$ is odd and $\pi_*(\epsilon_i) = \pi_*(\epsilon_{i+1})$ if $i$ is even.

Corollary 6.23. Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi) \in \mathcal{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. Then, for $\zeta \in \text{Hom}_{W/}(U, W)$, $\ker(\zeta^* \phi|_{P(K, \delta^* \phi)}) = \ker(\pi \circ \zeta)$.

Proof. Let $\zeta_0 \in \text{Hom}_{W/}(U/\ker(\zeta^* \phi|_{P(K, \delta^* \phi)}), W)$ such that $\zeta^* \phi|_{P(K, \delta^* \phi)} = \pi^* U \zeta_0^* |_{P(K, \delta^* \phi)}$, with $\pi_U$ the projection from $U$ to $U/\ker(\zeta^* \phi|_{P(K, \delta^* \phi)})$. Let $\epsilon_1 = \zeta_0 \circ \pi_U$, $\epsilon_n = \zeta$ and for all $i$ $\epsilon_i^* \phi = \epsilon_{i+1}^* \phi$ if $i$ is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if $i$ is even. Then, since $\text{Hom}_K(K, H^*(\cdot))$ is noetherian, $\ker(\pi \circ \epsilon_i) = \ker(\pi \circ \epsilon_{i+1})$ for all $1 \leq i \leq n - 1$. Hence, $\ker(\pi \circ \zeta) = \ker(\pi \circ \zeta_0 \circ \pi_U) = \ker(\zeta^* \phi|_{P(K, \delta^* \phi)})$.

Corollary 6.24. Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi) \in \mathcal{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. Then, $\text{Hom}_K(P(K, \delta^* \phi), H^*(\cdot))$ is noetherian.

Proof. The fact that $\text{Hom}_K(P(K, \delta^* \phi), H^*(\cdot))$ is finite is obvious. Let $\zeta^* \phi|_{P(K, \delta^* \phi)}$ in $\text{Hom}_K(P(K, \delta^* \phi), H^*(U))$ and let $\alpha$ be a morphism from a vector space $Y$ to $U$. Then, by Corollary \[6.23\],

$$
\ker(\alpha^* \zeta^* \phi|_{P(K, \delta^* \phi)}) = \ker(\pi \circ \zeta \circ \alpha).
$$

Since $\alpha$ is injective, this is equal to

$$
\alpha^{-1}(\ker(\pi \circ \zeta)) = \alpha^{-1}(\ker(\zeta^* \phi|_{P(K, \delta^* \phi)})).
$$

Theorem 6.25. Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^* \phi) \in \mathcal{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. Then, $P(K, \delta^* \phi)$ is nil-closed and noetherian.

Proof. Since, $P(K, \delta^* \phi)$ is the kernel of $\kappa - 1$ from $K$ to $K \otimes H^*(V)$ which are nil-closed, for $\kappa$ the comodule structure of $K$ associated with $\delta^* \phi$, and since $f$ is exact and $m$ is left-exact, the following is an exact sequence:

$$
0 \to l_1(P(K, \delta^* \phi)) \to l_1(K) \xrightarrow{l_1(\delta^* \phi \otimes 1)} l_1(K \otimes H^*(V)).
$$

Therefore, since $K$ is nil-closed, $P(K, \delta^* \phi)$ is also nil-closed. Then, the noetherianity of $P(K, \delta^* \phi)$ is a consequence of the noetherianity of $\text{Hom}_K(P(K, \delta^* \phi), H^*(\cdot))$ and of Proposition \[4.18\].

\[40\]
Remark 6.26. We have identified $P(K, \delta^*\phi)$ with a sub-algebra of $H^*(W/\text{Im}(\delta))$. Furthermore, we proved that $P(K, \delta^*\phi)$ is nil-closed and noetherian, and (because we took $\delta$ to be an injection) Corollary 6.23 implies that the inclusion from $P(K, \delta^*\phi)$ into $H^*(W/\text{Im}(\delta))$ is regular. Therefore, by Theorem 6.7, $P(K, \delta^*\phi)$ has the form $H^*(W/\text{Im}(\delta))^\mathcal{G}$, for some $\mathcal{G} \in \text{Groupoid}(W/\text{Im}(\delta))$.

This leads to the following question: for $W$ and $V$ in $\mathcal{V}'$, for $\delta$ an inclusion from $V$ to $W$ and for $\mathcal{G}'$ a groupoid with the restriction property and whose objects are the sub-spaces of $(W/\text{Im}(\delta))$, which are the groupoids $\mathcal{G}$ with the restriction property and whose objects are the sub-spaces of $W$, such that

1. $H^*(W)^\mathcal{G}$ is a sub $H^*(V)$-comodule of $H^*(W)$ for the comodule structure induced by $\delta$,
2. the intersection of $H^*(W)^\mathcal{G}$ with $\pi^*(H^*(W/\text{Im}(\delta)))$ is the image under $\pi^*: H^*(W/\text{Im}(\delta)) \to H^*(W)$ of $H^*(W/\text{Im}(\delta))^\mathcal{G}'$.

Remark 6.27. $H^*(W)^\mathcal{G}$ is a sub $H^*(V)$-comodule of $H^*(W)$ for the comodule structure induced by $\delta$ if and only if $\mathcal{G}$ satisfies the two conditions of theorem 5.30.

We recall that, from the beginning of this sub-section, $V$ and $W$ are fixed objects of $\mathcal{V}'$ and $\delta$ a fixed injective morphism from $V$ to $W$.

Theorem 6.28. Let $\mathcal{G}$ be a groupoid with the restriction property and whose objects are the sub-vector spaces of $W$, such that $H^*(W)^\mathcal{G}$ is a sub $H^*(V)$-comodule of $(H^*(W), (\text{id}_W + \delta)^*)$. For $\mathcal{G}'$ the only groupoid whose objects are sub-spaces of $W/\text{Im}(\delta)$ which satisfies that $\pi^*(H^*(W/\text{Im}(\delta))^\mathcal{G})$ is the algebra of primitive elements of $H^*(W)^\mathcal{G}$, the two following conditions are equivalent:

1. $\alpha \in \mathcal{G}'(U, U')$, where $U$ and $U'$ are sub-vector spaces of $W/\text{Im}(\delta)$ and $\alpha$ is an isomorphism from $U$ to $U'$,
2. there exists $N$ and $N'$ sub-spaces of $W$ such that $\pi$ induce isomorphisms from $N$ and $N'$ to $U$ and $U'$, as well as an element $\beta \in \mathcal{G}(N, N')$ such that $\alpha = \pi|_{N'} \circ \beta \circ (\pi|_{N})^{-1}$.

Proof. We consider the pushout diagram of Corollary 6.19

$$
\begin{array}{ccc}
\text{Hom}_{\mathcal{V}'}(\_W) & \xrightarrow{k} & \text{Hom}_K(H^*(W)^\mathcal{G}, H^*(\_)) \\
\pi^* \downarrow & & \downarrow q \\
\text{Hom}_{\mathcal{V}'}(\_, W/\text{Im}(\delta)) & \xrightarrow{p} & \text{Hom}_K(H^*(W/\text{Im}(\delta))^\mathcal{G}', H^*(\_)),
\end{array}
$$

where $\pi^*$ maps $\gamma : U \to W$ to $\pi \circ \gamma$, $k$ maps $\gamma$ to $\gamma^*\phi_\mathcal{G}$ for $\phi_\mathcal{G}$ the inclusion from $H^*(W)^\mathcal{G}$, $q$ maps $\psi : H^*(W)^\mathcal{G} \to H^*(U)$ to $\psi|_{\pi^*(H^*(W/\text{Im}(\delta)))}$ the restriction of $\psi$ to $\pi^*(H^*(W/\text{Im}(\delta)))$ and, finally, $p$ maps $\xi$ from $U$ to $W/\text{Im}(\delta)$ to $\xi^*\phi_\mathcal{G}$, for $\phi_\mathcal{G}$ the inclusion of $H^*(W/\text{Im}(\delta))^\mathcal{G}'$ into $H^*(W/\text{Im}(\delta))$.

We fix a section $s$ from $W/\text{Im}(\delta)$ to $W$. Since $\pi \circ s = \text{id}_{W/\text{Im}(\delta)}$, $\pi^*(s) = \text{id}_{W/\text{Im}(\delta)}$. Then, by commutativity of the pushout diagram, we have $\phi_\mathcal{G}' = q(s^*\phi_\mathcal{G}) = s^*\phi_\mathcal{G}|_{\pi^*(H^*(W/\text{Im}(\delta)))}$.

By construction, there are natural isomorphisms $\text{Hom}_{Kf.g.}(H^*(W)^\mathcal{G}, H^*(\_)) \cong \text{Hom}_{\mathcal{V}'}(\_, W)/\sim_\mathcal{G}$ and $\text{Hom}_{Kf.g.}(H^*(W/\text{Im}(\delta))^\mathcal{G}', H^*(\_)) \cong \text{Hom}_{\mathcal{V}'}(\_, W/\text{Im}(\delta))/\sim_{\mathcal{G}'\mathcal{G}}$. These are the isomorphisms.
that map $\phi_G$ to $[\text{id}_W]_G$ and $\phi_G'$ to $[\text{id}_{W/\text{Im}(\delta)}]_{G'}$ respectively.

Let us first prove $[2] \Rightarrow [1]$. We consider, $\beta \in \mathcal{G}(N, N')$ such that $\pi$ induces isomorphisms $\pi|_{N'}^{U'}$ and $\pi|_{N'}^{U''}$ between $N$ and $U$ and between $N'$ and $U'$. Let $\alpha$ be an isomorphism from $U$ to $U'$ such that $\alpha = \pi|_{N'}^{U''} \circ \beta \circ (\pi|_{N'}^{U'})^{-1}$. Then, $(\pi|_{N'}^{U'})^{-1} \circ \alpha = \beta \circ (\pi|_{N'}^{U'})^{-1}$. Therefore,

$$\alpha^* ((\pi|_{N'}^{U'})^{-1})^* \iota_N^* \phi_G = ((\pi|_{N'}^{U'})^{-1})^* \beta^* \iota_N^* \phi_G = ((\pi|_{N'}^{U'})^{-1})^* \iota_N^* \phi_G.$$ 

We can choose the section $s$ in such a way that $s \circ \iota_U = \iota_{N'} \circ ((\pi|_{N'}^{U'})^{-1})$, then

$$\alpha^* \iota_U^* s^* \phi_G = ((\pi|_{N'}^{U'})^{-1})^* \iota_N^* \phi_G.$$ 

This implies that $\alpha^* \iota_U^* s^* q(\phi_G) = ((\pi|_{N'}^{U'})^{-1})^* \iota_N^* q(\phi_G)$. Furthermore, $\pi \circ s \circ \iota_U = \pi \circ \iota_N \circ (\pi|_{N'}^{U'})^{-1}$.

Hence, we also have, by Proposition 6.22, that $\iota_U^* s^* q(\phi_G) = ((\pi|_{N'}^{U'})^{-1})^* \iota_N^* q(\phi_G)$. Hence,

$$\alpha^* \iota_U^* s^* q(\phi_G) = \iota_U^* s^* q(\phi_G).$$

Since $s^* q(\phi_G) = [\text{id}_{W/\text{Im}(\delta)}]_{G'}$, this implies that $\alpha \in \mathcal{G}'(U, U')$, as required.

Now, let us prove the far more challenging $[1] \Rightarrow [2]$.

We consider $\alpha \in \mathcal{G}'(U, U')$ where $U$ and $U'$ are two sub-spaces of $W/\text{Im}(\delta)$. Then,

$$\alpha^* \iota_U^* [\text{id}_{W/\text{Im}(\delta)}]_{G'} = \iota_U^* [\text{id}_{W/\text{Im}(\delta)}]_{G'},$$

or, equivalently,

$$[\iota_U \circ \alpha]_{G'} = [\iota_U]_{G'}.$$ 

By Proposition 6.22, since $[\text{id}_{W/\text{Im}(\delta)}]_{G'}$ is the image of $s^* q(\phi_G)$ under the isomorphism

$$\text{Hom}_{\mathcal{G}'(\mathcal{G}, \mathcal{G}')} (\text{H}^*(W/\text{Im}(\delta)); \text{H}^*(\_)) \cong \text{Hom}_{\mathcal{G}'(\mathcal{G}, \mathcal{G}')} (\_); \text{H}^*(W/\text{Im}(\delta))/_{\sim};$$

we have that $[\iota_U]_{G'} = [\iota_U]_{G'}$, if and only if there exists a family $(\epsilon_i)_{i \in [1, n]} \in \text{Hom}_{\mathcal{G}'(\mathcal{G}, \mathcal{G}')} (U, W)^n$ with $n \in \mathbb{N}$ greater than 1, such that $s \circ \gamma = \epsilon_1$, $s \circ \zeta = \epsilon_n$, and, for all $1 \leq i \leq n - 1$, $\epsilon_i \phi_G = \epsilon_{i+1} \phi_G$ if $i$ is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if $i$ is even.

So let $(\epsilon_i)_{i \in [1, n]} \in \text{Hom}_{\mathcal{G}'}(U, W)^n$ be such that $\epsilon_1 = s \circ \iota_U$, $\epsilon_n = s \circ \iota_{U'}, \alpha$ and for all $1 \leq i \leq n - 1$, $\epsilon_i \phi_G = \epsilon_{i+1} \phi_G$ if $i$ is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if $i$ is even.

By induction, for all $i \in [1, n]$, $\epsilon_i^* \phi_G$ and $\pi \circ \epsilon_i$ are regular elements respectively of $\text{Hom}_{\mathcal{G}'}(\text{H}^*(W)_{\mathcal{G}'}, \text{H}^*(\_))$ and $\text{Hom}_{\mathcal{G}'}(U, W/\text{Im}(\delta))$. Hence, $\epsilon_i$ and $\pi \circ \epsilon_i$ are injections. For all $i$, let $N_i$ denote the image of $\epsilon_i$ in $W$. We denote also by $\bar{N}_i$ the corestriction of $\epsilon$ to $N_i$. Then, for $i$ odd, $\epsilon_i^* \phi_G = \epsilon_{i+1}^* \phi_G$ implies that there exists $\beta_i$ in $\mathcal{G}(N_i, N_{i+1})$ such that $\beta_{i+1} = \beta_i \circ \epsilon_i$.

We take some moment to explain subtlety in the proof. We would like, for $i$ even, to have $\epsilon_i = \epsilon_{i+1}$. Then, the composition of the $\beta_i$ with $i$ odd would give an isomorphism $\beta$ between $N_1 = s(U)$ and $N_n = s(U')$ such that $\beta \in \mathcal{G}(N_1, N_n)$, since $\mathcal{G}$ is a groupoid and we would have $(s \circ \iota_U)|_{N_1} \circ \alpha = \beta \circ (s \circ \iota_U)|_{N_1}$. Since $(s \circ \iota_U)|_{N_1}$ and $(s \circ \iota_U)|_{N_n}$ are inverse isomorphisms of
\[ \pi|_{N_i}^{U_i} \text{ and } \pi|_{N_i'}^{U_i'} , \] we would have \( \alpha = \pi|_{N_i}^{U_i} \circ \beta \circ (\pi|_{N_i}^{U_i})^{-1}. \] If this were the case, we would have found a \( \beta \) for any \( N \) and \( N' \) such that \( \pi \) induces isomorphisms between \( U \) and \( N \) and between \( U' \) and \( N' \), and we would have done so without using the assumption that \( \delta^* \phi_G \) is central. Unfortunately, this naive approach fails, and \( N \) and \( N' \) must be chosen carefully. The hypothesis on the \( \epsilon_i \) for \( i \) even indicates how to modify our original \( N_1 \) and \( N_n \) to make it work, using the centrality of \( \delta^* \phi_G \).

First notice that, since \( \pi \circ \epsilon_i \) is injective for all \( i \), we always have \( N_i \cap \text{Im}(\delta) = \{0\} \). Then, the assumption that, for \( i \) even, \( \pi \circ \epsilon_i = \pi \circ \epsilon_{i+1} \) implies that there exists \( \rho_i \) from \( U \) to \( W \) whose image is inside \( \text{Im}(\delta) \) and such that \( \epsilon_{i+1} = \epsilon_i + \rho_i \). Now, since \( H^*(W)^\delta \) is a sub \( H^*(V) \)-comodule of \( (H^*(W), (\text{id}_U + \delta)^*) \) we know that \( [\delta]|_G \) is a central element of \( \text{Hom}_G(\_ , W ) / \sim_G \). Then, by Theorem 5.30, for \( i \) odd, we know that the isomorphisms \( \beta_i \) from \( N_i \oplus \text{Im}(\delta) \) to \( N_{i+1} \oplus \text{Im}(\delta) \) defined by \( \tilde{\beta}_i(n) = \beta_i(n) \) for \( n \in N_i \) and \( \tilde{\beta}_i(v) = v \) for \( v \in \text{Im}(\delta) \) satisfy \( \tilde{\beta}_i \in \mathcal{G}(N_i \oplus \text{Im}(\delta), N_{i+1} \oplus \text{Im}(\delta)) \). Moreover, for \( i \) even, \( \pi \circ \epsilon_i = \pi \circ \epsilon_{i+1} \) implies that \( N_i \oplus \text{Im}(\delta) = N_{i+1} \oplus \text{Im}(\delta) \). Then, at each even step \( i \), we can “correct” \( \epsilon_{i-1} \) to get \( \tilde{\beta}_{i-1} \circ \epsilon_{i-1} = \epsilon_{i+1} \) instead of \( \epsilon_i \).

For each \( i \in [1,n] \), we define \( \epsilon'_i \) (the “corrected” \( \epsilon_i \)) by

\[
\epsilon'_i := \iota_{N_i \oplus \text{Im}(\delta)} \circ \left( \epsilon_i \oplus \sum_{\{j \text{ even } : i < j < n\}} \beta_{i \to j}^{-1} \circ \rho_j |_{N_j \oplus \text{Im}(\delta)} \right),
\]

where \( \beta_{i \to j} \) is the composition of all the \( \tilde{\beta}_k \) with \( k \) odd and \( i \leq k < j \). The family \( (\epsilon'_i)_{i \in [1,n]} \) satisfies the following:

1. \( \pi \circ \epsilon'_1 = \iota_U, \pi \circ \epsilon'_n = \iota_{U'}, \alpha \),
2. for all \( i \), if we denote by \( N'_i \) the image of \( \epsilon'_i \), then \( N'_i \oplus \text{Im}(\delta) = N_i \oplus \text{Im}(\delta) \),
3. for \( i \) odd, if we denote by \( \beta'_i \) the restriction of \( \beta_i \) to \( N'_i \) corestricted to \( N'_{i+1} \), \( \epsilon'_{i+1} = \beta'_i \circ \epsilon'_i \), with \( \beta'_i \in \mathcal{G}(N'_i, N'_{i+1}) \), since \( \tilde{\beta}_i \in \mathcal{G}(N_i \oplus \text{Im}(\delta), N_{i+1} \oplus \text{Im}(\delta)) \) and \( \mathcal{G} \) has the restriction property,
4. for \( i \) even, \( \epsilon'_i = \epsilon'_{i+1} \).

Then, let \( N = N'_1, N' = N'_n, \) and \( \beta = (\beta'_0 \circ \ldots \circ \beta'_i \circ \beta'_1) \), where \( k = n - 2 \) if \( n \) is odd, \( k = n - 1 \) otherwise. Then, \( \beta \in \mathcal{G}(N, N') \) and \( \beta \circ \epsilon'_1 = \epsilon'_n \). Finally, \( \pi \circ \epsilon'_1 = \iota_U \) implies that \( \epsilon'_1 = (\pi|_{N'_1})^{-1} \) and \( \pi \circ \epsilon'_n = \iota_{U'} \circ \alpha \) implies that \( \epsilon'_n = (\pi|_{N'_n})^{-1} \circ \alpha \). Hence, \( \alpha = \pi|_{N'} \circ \beta \circ (\pi|_{N'})^{-1} \).

### 6.3 Applications

We end this section by presenting some applications of Theorem 6.28. The first result was already known.

**Proposition 6.29.** Let \( K \) be a noetherian, nil-closed, integral, unstable algebra of transcendence degree \( d \). We assume that the centre of \( K \) is of dimension \( d \). Then, \( K \cong H^*(W)^\delta \) with \( \dim(W) = d \).

**Proof.** By Theorem 6.7 and the theorem of Adams-Wilkerson \( K \cong H^*(W)^\delta \) for some groupoid \( \mathcal{G} \) with the restriction property and whose objects are the sub-spaces of \( W \). Then, since the centre of \( K \) has dimension \( \dim(W) \), up to isomorphism the centre of \( K \) is given by \((W, \phi)\) where \( \phi \) is the inclusion from \( K \) to \( H^*(W) \) induced by the theorem of Adams-Wilkerson. Then, by Theorem 5.30, for all \( \alpha \in \mathcal{G}(U, U') \), where \( U \) and \( U' \) are sub-vector spaces of \( W \), and for all \( x \in U = U \cap W' \), \( \alpha(x) = x \). Hence, \( \mathcal{G} \) is the groupoid in which the only morphisms are identities of sub-spaces of \( W \). Then, \( H^*(W)^\delta = H^*(W) \).

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Let us now consider the case where the centre is of dimension \(d - 1\), for \(d\) the transcendence degree of \(K\).

**Theorem 6.30.** Let \(K\) be a noetherian, nil-closed, integral, unstable algebra of transcendence degree \(d\). We assume that the centre of \(K\) is of dimension \(d - 1\). Then, there exists \(G\) a sub-group of \(\text{Gl}(W)\) such that \(K\) is isomorphic to the algebra of invariant elements \(H^* (W)^G\) with \(\dim(W) = d\). Furthermore, \(G\) satisfies that the set of element \(x \in W\) such that \(g(x) = x\) for all \(g \in G\) is a sub-vector space of \(W\) of dimension \(d - 1\).

**Proof.** By Theorem 5.7 and the theorem of Adams-Wilkerson \(K \cong H^*(W)^\mathcal{G}\) for some groupoid \(\mathcal{G}\) with the restriction property and whose objects are the sub-spaces of \(W\). Then, the centre of \(K\) is associated with a \(H^*(V)\)-comodule structure on \(H^*(W)^\mathcal{G}\), with \(\dim(V) = d - 1\). Up to isomorphism, we can suppose that this \(H^*(V)\)-comodule structure is induced by \(\iota_V\) the inclusion of a sub-vector space \(V\) in \(W\) (we can assume that the comodule structure is induced by an injection, because it is associated with the centre of \(K\) which is regular by definition). By Theorem 5.30 for all \(\alpha \in \mathcal{G}(U, U')\) and for all \(x \in U \cap V\), \(\alpha(x) = x\). Furthermore, for all sub-spaces \(U\) and \(U'\) of \(W\), \(\mathcal{G}(U, U')\) is determined from \(\mathcal{G}(V \oplus U, V \oplus U')\) by the restriction property. Since we assumed \(\dim(V) = \dim(W) - 1\), \(\mathcal{G}\) is uniquely determined by \(\mathcal{G}(V, V)\) and \(\mathcal{G}(W, W)\). Then, since for all \(x \in V\) and for all \(\alpha \in \mathcal{G}(V, V)\), \(\alpha(x) = x\), \(\mathcal{G}(V, V)\) is reduced to the identity of \(V\). Therefore, for \(g \in G := \mathcal{G}(W, W), g(x) = x\) for all \(x \in V\), and \(\alpha \in \mathcal{G}(U, U')\) if and only if there is \(g \in G\) such that \(\alpha = g \iota_V\). We get that \(H^*(W)^\mathcal{G} = H^*(W)^G\).

**Corollary 6.31.** Let \(K\) be a noetherian, nil-closed, integral, unstable algebra of transcendence degree \(\dim(W)\), and let \((C, \gamma)\) be its centre. Then if \(K\) is not isomorphic to \(H^*(W)^G\) for some sub-group \(G\) of \(\text{Gl}(W)\), \(\dim(C) \leq d - 2\).

In the next section, we will have examples where \(\dim(C) = d - 2\) and \(K\) is not an algebra of invariant elements under some action on \(H^*(W)\).

7 Two examples

In the following, we focus on the case where \(p = 2\), so that \(H^*(V) \cong \mathbb{F}_2 [x_1, ..., x_n]\), for \((x_1, ..., x_n)\) a basis of \(V^*\).

We take \(V_3 = \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_2 \oplus \mathbb{F}_2 e_3\) and denote by \((x, y, z)\) the dual basis of \((e_1, e_2, e_3)\). The morphism

\[(\text{id}_{V_3} + \iota_{\mathbb{F}_2 e_1})^* : H^*(V_3) \to H^*(V_3) \otimes H^*(\mathbb{F}_2 e_1),\]

which maps \(x\) to \(x \otimes 1 + 1 \otimes x\), \(y\) to \(y \otimes 1\) and \(z\) to \(z \otimes 1\) defines an \(H^*(\mathbb{F}_2 e_1)\)-comodule structure on \(H^*(V_3)\) in \(K\). In each of the following examples we want to find the list of noetherian, nil-closed, unstable sub-algebras of \(H^*(V_3)\) of transcendence degree 3, which are sub \(H^*(\mathbb{F}_2 e_1)\)-comodules of \(H^*(V_3)\) for the comodule structure given by \((\text{id}_{V_3} + \iota_{\mathbb{F}_2 e_1})^*\) and whose algebra of primitive elements is \(\pi^*(H^*(V_3/\mathbb{F}_2 e_1))^\mathcal{G}\) for \(\pi\) the projection from \(V_3\) over \(V_3/\mathbb{F}_2 e_1\), and for \(H^*(V_3/\mathbb{F}_2 e_1)^\mathcal{G}\) a chosen nil-closed, noetherian, sub algebra of \(H^*(V_3/\mathbb{F}_2 e_1)\). By Theorem 6.28 \(\mathcal{G}\) has to satisfy that \(\alpha \in \mathcal{G}(U, U')\) if and only if there is \(\beta \in \mathcal{G}(N, N')\) such that \(\pi|_{N'} \circ \beta = \alpha \circ \pi|_{N}\).

Furthermore, by Theorem 5.30 since \([\iota_{\mathbb{F}_2 e_1}]_G \in \text{Hom}_{k[G]}(H^*(W)^\mathcal{G}, H^*(\mathbb{F}_2 e_1))\) is central, all morphism \(\beta\) in \(\mathcal{G}(N, N')\) is the restriction of a morphism \(\bar{\beta}\) in \(\mathcal{G}(\mathbb{F}_2 e_1 + N, \mathbb{F}_2 e_1 + N')\) defined by
\( \bar{\beta}(e_1) = e_1 \) and \( \bar{\beta}(n) = \beta(n) \) for \( n \in N \). So \( G \) can be deduced from its full sub-groupoid whose objects are \( V_3, \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_2, \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_3, \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 (e_2 + e_3) \) and \( \mathbb{F}_2 e_1 \). This full sub-groupoid can be represented as follows:

\[
\begin{array}{c}
\mathbb{F}_2 e_1 \\
\mathbb{F}_2 e_1 \oplus \mathbb{F}_2 (e_2 + e_3) \\
\mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_3 \\
V_3 \\
G \\
\end{array}
\]

Where \( G \) denotes \( G(V_3, V_3) \) and \( G_{ij} \) denotes \( G(\mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_i, \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_j) \) with \( e_4 := e_2 + e_3 \), and where we omitted \( G_{43}, G_{24}, \) etc., which can be deduced by composition and inversion of the already given sets of morphisms.

We consider first the case where \( G' \) contains only identities.

**Proposition 7.1.** There are exactly 15 nil-closed, noetherian, unstable sub-algebras \( K \) of \( H^*(V_3) \) of transcendence degree 3, which are sub \( \mathbb{F}_2 [x] \)-comodules of \( H^*(V_3) \), for the comodule structure that maps \( x \) to \( x \otimes 1 + 1 \otimes x \), \( y \) to \( y \otimes 1 \) and \( z \) to \( z \otimes 1 \), and such that the algebra of primitive elements of \( K \) is \( \mathbb{F}_2 [y, z] \). These are:

1. \( \mathbb{F}_2 [y, z, x(x + y)(x + z)(x + y + z)] \),
2. \( \mathbb{F}_2 [y, z, x(x + y)] \),
3. \( \mathbb{F}_2 [y, z, x(x + z)] \),
4. \( \mathbb{F}_2 [y, z, x(x + y + z)] \),
5. \( \mathbb{F}_2 [y, z, x(x + y)(x + z)(x + y + z)] + \mathbb{F}_2 [y, z, x(x + y)] y \),
6. \( \mathbb{F}_2 [y, z, x(x + y)(x + z)(x + y + z)] + \mathbb{F}_2 [y, z, x(x + z)] z \),
7. \( \mathbb{F}_2 [y, z, x(x + y)(x + z)(x + y + z)] + \mathbb{F}_2 [y, z, x(x + y + z)] (y + z) \),
8. \( \mathbb{F}_2[x, y, z] \),
9. \( \mathbb{F}_2[x, y, z] (y + z) \oplus \mathbb{F}_2[z, x(x + z)] \),
10. \( \mathbb{F}_2[x, y, z] y \oplus \mathbb{F}_2[z, x(x + z)] \),
11. \( \mathbb{F}_2[x, y, z] z \oplus \mathbb{F}_2[y, x(x + y)] \),
12. \( \mathbb{F}_2[z, x(x + z)] \oplus \mathbb{F}_2[x, y, z] (y + z)y \oplus \mathbb{F}_2[y, x(x + y)] y, \)
13. \( \mathbb{F}_2[y, x(x + y)] \oplus \mathbb{F}_2[x, y, z] (y + z)z \oplus \mathbb{F}_2[z, x(x + z)] z, \)
14. \( \mathbb{F}_2[(y + z), x(x + y + z)] \oplus \mathbb{F}_2[x, y, z] yz \oplus \mathbb{F}_2[z, x(x + z)] z, \)
15. \( \mathbb{F}_2[z, x(x + z)] z \oplus \mathbb{F}_2[y, x(x + y)] y \oplus \mathbb{F}_2[y, x(x + y)] (y + z)y \oplus \mathbb{F}_2[x, y, z] (y + z)yz. \)

**Proof.** In this case, \( \mathbb{F}_2[y, z] = \pi^*(H^*(\mathbb{V}_3/\mathbb{F}_2e_1)) \). Then, the groupoid \( G' \) is the trivial groupoid, which contains only identities. Therefore, by Theorem 6.28, for all \( \beta \in \mathcal{G}(N, N') \), \( \pi \circ \beta = \pi \). For \( \beta \in G_{ij} \) with \( i \) and \( j \) in \( \{2, 3, 4\} \), \( \pi \circ \beta(e_i) = \pi(e_i) \) which is not possible if \( i \neq j \). Hence, \( G_{ij} = \emptyset \) if \( i \neq j \). We only have to determine \( G, G_{22}, G_{33} \) and \( G_{44} \). We have two conditions on \( \beta \in G \) or \( \beta \in G_{ii} \), the first is that \( \beta(e_1) = e_1 \) (by Theorem 5.30 and the centrality of \( [i_{\mathbb{F}_2e_1}]_G \)) and the second that \( \pi \circ \beta = \pi \) (by Theorem 6.28 and the hypothesis on \( G' \)). Therefore, \( \beta \) has a block matrix of the following form:

\[
\begin{pmatrix}
\hat{\beta} & \text{id}_{\mathbb{F}_2e_1} \\
0 & \text{id}_N
\end{pmatrix},
\]

where \( N \in \{\mathbb{F}_2e_2, \mathbb{F}_2e_3, \mathbb{F}_2(e_2 + e_3), \mathbb{F}_2e_2 \oplus \mathbb{F}_2e_3\} \) and \( \hat{\beta} \) is a morphism from \( N \) to \( \mathbb{F}_2e_1 \). Hence, the set of matrices of morphisms in \( G \) in the basis \((e_1, e_2, e_3)\) is a sub group of

\[
H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.
\]

There are five possibilities for \( G \), \( G_1 = \{\text{id}_{\mathbb{V}_3}\}, G_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, G_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, G_4 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \) and \( G_5 \) the full group. The groups \( G_{ii} \) are either trivial or equal to \( H_{ii} \) the group generated by the morphism which sends \( e_1 \) to itself and \( e_i \) to \( e_1 + e_i \). Furthermore for a chosen \( G \), the restriction property (Definition 5.8) requires that some of them are non trivial. Let us summarise the possible values of the \( G_{ii} \) for each value of \( G \).

| \( G \) | \( G_{22} \) | \( G_{33} \) | \( G_{44} \) |
|---|---|---|---|
| \( G_1 \) | \{\text{id}\} or \( H_{22} \) | \{\text{id}\} or \( H_{33} \) | \{\text{id}\} or \( H_{44} \) |
| \( G_2 \) | \( H_{22} \) | \{\text{id}\} or \( H_{33} \) | \( H_{44} \) |
| \( G_3 \) | \{\text{id}\} or \( H_{22} \) | \( H_{33} \) | \( H_{44} \) |
| \( G_4 \) | \( H_{22} \) | \( H_{33} \) | \{\text{id}\} or \( H_{44} \) |
| \( G_5 \) | \( H_{22} \) | \( H_{33} \) | \( H_{44} \) |
We find, 15 possible values for \( (G, G_{22}, G_{33}, G_{44}) \). Each one characterising precisely one \( G \in \text{Groupoid}(V_3) \) such that \( H^*(V_3)^G \) satisfies the required conditions.

We give details for only two computations of \( H^*(V_3)^G \). The other ones are left to the reader.

**First case:** If \( G \) contains all morphisms whose matrix are in \( H \), then by the restriction property \( G_{ii} \) contains all the morphisms \( \beta \) that satisfy the two conditions, for \( i \in \{2, 3, 4\} \). Hence, all morphisms in \( G_{ii} \) are induced by restriction from a morphism in \( G \). Therefore, \( H^*(V_3)^G \) is equal to \( H^*(V_3)^G \), the algebra of invariant elements under the action of \( G \) on \( H^*(V_3) \). Let us start by using that an element of \( H^*(V_3)^G \) has to be invariant under the action \( g_y \) which sends \( x \) to \( x + y \), \( y \) and \( z \) to themselves. Then, every polynomial \( P(x, y, z) \) satisfies that

\[
x^2 P(x, y, z) = x(x + y)P(x, y, z) + yxP(x, y, z).
\]

Hence, we can express every polynomial \( P(x, y, z) \) as \( P_1(x(x + y), y, z) + xP_2(x(x + y), y, z) \). Then, since \( P_1(x(x + y), y, z) \) is invariant under \( g_y \), \( xP_2(x(x + y), y, z) \) also to be invariant, therefore \( P_2 = 0 \). So every element in \( H^*(V_3)^G \) can be expressed as \( P(X, y, z) \) with \( X = x + y \).

But an element of \( H^*(V_3)^G \) is also invariant under the action of \( g_z \) which send \( x \) to \( x + z \) and \( y \) and \( z \) to themselves. \( g_z \) sends \( X \) to \( x + y + z \). We can show as before that \( P(X, y, z) \) can be expressed as \( P_1(X(X + y(y + z)), y, z) + xP_2(X(X + y(y + z)), y, z) \). Since \( P_1(X(X + y(y + z)), y, z) \) is invariant under \( g_z \), \( P_2 = 0 \). We have \( X(X + y(y + z)) = x(x + y)(x + z)(x + y + z) \), so every element in \( H^*(V_3)^G \) is in \( F_2[y, z, x(x + y)(x + z)(x + y + z)] \), but conversely, \( y, z \) and \( x(x + y)(x + z)(x + y + z) \) are invariant under \( G \). Hence, \( H^*(V_3)^G = F_2[y, z, x(x + y)(x + z)(x + y + z)] \).

**Second case:** If \( G = G_2 \), by the restriction property \( G_{22} \) and \( G_{44} \) are the groups of isomorphisms of \( F_2e_1 \oplus F_2e_i \), with \( i = 2 \) or \( 4 \), generated by the morphism whose matrix in the basis \( (e_1, e_i) \) is \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). The only group which is not fully determined by the restriction property is \( G_{33} \) which can either be equal to \( \{id_{F_2e_1} \oplus F_2e_4\} \), or to the subgroup \( B_2 \) of \( \text{Gl}(F_2e_1 \oplus F_2e_3) \) generated by the morphism \( \beta \) whose matrix in the basis \( (e_1, e_3) \) is \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

1. If \( G_{33} = \{id_{F_2e_1} \oplus F_2e_3\} \), then all the sets \( G(U, U') \) are determined by \( G_2 = G(V_3, V_3) \), by the restriction property, hence \( H^*(V_3)^G = H^*(V_3)^{G_2} = F_2[y, z, x(x + y)] \).

2. If \( G_{33} = B_2 \), we get our first example which is not an algebra of invariant elements. By Proposition 6.12,

\[
H^*(V_3)^G = \{ x \in H^*(V_3) \mid \alpha^* \iota_\alpha^*(x) = \iota_\alpha^*(x) \text{ for all } \alpha \in G(U, U') \},
\]

which is equal to

\[
H^*(V_3)^{G_2} \cap \{ x \in H^*(V_3) \mid \beta^* \iota_\beta^* (x) = \iota_\beta^* (x) \}.
\]

Hence, \( H^*(V_3)^G \) contains the element of \( H^*(V_3)^G \) which are in the inverse image of \( H^*(F_2e_1 \oplus F_2e_3)^{G_2} = F_2[x(x + y), z] \) under \( \iota_{F_2e_1 \oplus F_2e_3}^* \) which maps \( x(x + y) \) to \( x^2 \), \( y \) to \( 0 \) and \( z \) to \( z \). We notice that \( \iota_{F_2e_1 \oplus F_2e_3}(F_2[x(x + y), y, z] \cap F_2[x(x + y), z, z] \) and
that ker($\iota_{F_{F_1}bF_{F_2}e_3}^*\otimes F_{F_2}e_2$) = $F_2 [x(x + y), y, z]$. Since $F_2 [x(x + y)(x + z)(x + y + z), y, z]$ is a sub algebra of $F_2 [x(x + y), y, z]$ that surjects onto $F_2 [x^2(x + z)^2, y, z]$ under $\iota_{F_{F_1}bF_{F_2}e_3}^*$, we get that

$$(\iota_{F_{F_1}bF_{F_2}e_3}^*)^{-1}(F_2 [x^2(x + z)^2, y]) = F_2 [y, z, x(x + y)(x + z)(x + y + z)] + F_2 [y, z, x(x + y)] y.$$

Therefore,

$$H^*(V_3)^G = F_2 [y, z, x(x + y)(x + z)(x + y + z)] + F_2 [y, z, x(x + y)] y.$$  

\[ \square \]

**Remark 7.2.** In Proposition 5.8, it is worth noticing that, for example, $F_2 [y, z, x(x + y)]$, $F_2 [y, z, x(x + z)]$, and $F_2 [y, z, x(x + y + z)]$ are conjugates. In particular, they are isomorphic. To get a minimal list of isomorphism classes of nil-closed, noetherian, integral, unstable algebras of transcendence degree 3, with a $F_2 [x]$-comodule structure whose algebra of primitive elements is isomorphic to $F_2 [y, z]$, we should consider the conjugacy classes of the algebras found in Proposition 7.1 (See Remark 6.16).

**Proposition 7.3.** There are exactly 12 nil-closed, noetherian, unstable sub-algebras $K$ of $H^*(V_3)$ of transcendence degree 3, which are sub $F_2 [x]$-comodules of $H^*(V_3)$, for the comodule structure that maps $x$ to $x \otimes 1 + y \otimes 1 + z \otimes 1$, and such that the algebra of primitive elements of $K$ is $F_2 [y, z]$. These are:

1. $F_2 [y, z, x(x + y), x(x + z)(x + y + z)]$,
2. $F_2 [y, z, x(x + y), x(x + z)(x + y + z)] + F_2 [y, z, x(x + y)(x + z)(x + y + z)] (x(x + z) + y^2(y + z))$,
3. $F_2 [y, z, x(x + z)]$,
4. $(F_2 [y, z, x(x + z)] y, z, x(x + z)]) z \oplus F_2 [y, z, x(x + z)]$,
5. $F_2 [z, x, y(y + z)]$,
6. $F_2 [z, x, y(y + z)] y, z, x(x + z)]$,
7. $F_2 [z, x, y(y + z)] y, z, x(x + z)]$,
8. $(F_2 [z, x, y(y + z)] y, z, x(x + z)]) z \oplus F_2 [y, z, x(x + z)](x + y + z)]$,
9. $F_2 [z, x, y(y + z)]$,
10. $(F_2 [z, x, y(y + z)] y, z, x(x + z)]) z \oplus F_2 [y, z, x(x + z)](x + y + z)]$,
11. $F_2 [(x + y), y, z, y(y + z)] y, z, x(x + y) + z)]$,
12. $(F_2 [z, y(y + z)] y, z, x(x + y)]) z \oplus F_2 [y, z, x(x + z)](x + y + z)].$

**Proof.** In this case, $F_2 [z, y(y + z)] = \pi^*(H^*(V_3/F_{F_2}e_3)^B_3)$, for $B_2$ the group generated by the morphism $b_2$ which sends $e_2'$ to itself and $e_3'$ to $e_3' + e_1'$, where $e_2'$ and $e_3'$ are the images of $e_2$ and $e_3$ under the canonical projection. Then, the groupoid $G' = \mathbf{g}(B_2)$ is the following:

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Theorem 6.28 implies that, for $\beta \in G(N, N')$, when $e_2 \in N$, $\pi \circ \beta(e_2) = e'_2$, when $e_3 \in N$, $\pi \circ \beta(e_3) = e'_3$ or $e'_4$ and when $e_4 \in N$, $\pi \circ \beta(e_4) = e'_4$. In this case, $G_{34}$ is not necessarily empty. We have to determine $G_{12}$, $G_{22}$, $G_{33}$, $G_{44}$ and $G_{34}$ ($G_{43}$ being the set of inverses of morphisms in $G_{34}$). By the two conditions on $\beta \in G$, the matrix of $\beta$ is in $H$, for

$$H = \{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \}.$$

Furthermore, since $b_2 \in G'(F_2e'_2 \oplus F_2e'_3, F_2e'_2 \oplus F_2e'_3)$, by Theorem 6.28 there exist $N$ and $N'$ and $\beta \in G(N, N')$ such that $\pi$ induces isomorphisms from $N$ and $N'$ to $F_2e'_2 \oplus F_2e'_3$ and $b_2 \circ \pi = \pi \circ \beta$. Then, by Theorem 5.30, $\beta \in G$. Therefore $G$ contains at least one element among

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

$H$ is isomorphic to the dihedral group $D_4$. It admits ten subgroups, among those the possible values of $G$ are: $G_1 = < \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} >$, $G_2 = < \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} >$, $G_3 = < \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} >$, $G_4 = < \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} >$ and $G_5$ the full group.

For $\beta \in G_{ii}$, we necessarily have $\beta(e_1) = e_1$. By Theorem 6.28, $\beta$ has a block matrix of the following form:

$$\begin{pmatrix} \text{id}_{F_2e_1} & \hat{\beta} \\ 0 & \text{id}_N \end{pmatrix},$$

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where $N \in \{ F_2 e_2, F_2 e_3, F_2 (e_2 + e_3) \}$ and $\hat{\beta}$ is a morphism from $N$ to $F_2 e_1$.

Finally, $G_{34}$ cannot be trivial, otherwise $(b_2)_{g_2 e'_2} \not\in \mathcal{G}(F_2 e'_1, F_2 e'_2)$, and for $\beta \in G_{34}$, $\beta(e_1) = e_1$ and $\beta(e_3) = e_4$ or $e_1 + e_4$. We get that the $G_{ii}$ with $i \in \{2, 3, 4\}$ is a subgroup of $H_{ii} := \{ \text{id}, \beta_{ii} \}$, for $\beta_{ii}$ the morphism which sends $e_1$ to itself and $e_i$ to $e_1 + e_i$, and $G_{34}$ is a non trivial subset of the set $H_{34} := \{ \beta_1, \beta_2 \}$ where $\beta_1$ and $\beta_2$ are the morphisms from $F_2 e_1 \oplus F_2 e_3$ to $F_2 e_1 \oplus F_2 e_4$ whose matrix in the basis $(e_1, e_3)$ and $(e_1, e_4)$ are either \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), or \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

As in Proposition 7.1 (or rather its proof), some values of $G$ imply, by restriction property, the maximality of some of the $G_{ii}$ or the one of $G_{34}$. Furthermore, in this case since the groupoid admits a morphism between $F_2 e_1 \oplus F_2 e_3$ and $F_2 e_1 \oplus F_2 e_4$, for $G$ to be a groupoid we need the compositions of morphisms in $G_{33}$, $G_{34}$, $G_{44}$ and $G_{43}$ to be in $G$. This is equivalent to requiring that $G_{33}, G_{44}$ and $G_{34}$ have the same cardinal.

| $G$ | $G_{22}$ | $G_{33}$ | $G_{44}$ | $G_{34}$ |
|-----|---------|---------|---------|---------|
| $G_1$ | $\{ \text{id} \}$ or $\{ \beta_1 \}$ | $\{ \text{id} \}$ | $\{ \text{id} \}$ | $\{ \beta_1 \}$ |
|    | $H_{22}$ | $H_{33}$ | $H_{44}$ | $H_{34}$ |
| $G_2$ | $\{ \text{id} \}$ or $\{ \beta_2 \}$ | $\{ \text{id} \}$ | $\{ \text{id} \}$ | $\{ \beta_2 \}$ |
|    | $H_{22}$ | $H_{33}$ | $H_{44}$ | $H_{34}$ |
| $G_3$ | $\{ \text{id} \}$ or $H_{22}$ | $H_{33}$ | $H_{44}$ | $H_{34}$ |
|    | $H_{22}$ | $H_{33}$ | $H_{44}$ | $H_{34}$ |
| $G_4$ | $H_{22}$ | $H_{33}$ | $H_{44}$ | $H_{34}$ |
|    | $H_{22}$ | $H_{33}$ | $H_{44}$ | $H_{34}$ |
| $G_5$ | $H_{22}$ | $H_{33}$ | $H_{44}$ | $H_{34}$ |

We find 12 possible values for $(G, G_{22}, G_{33}, G_{44}, G_{34})$. Each one characterising precisely one $G \in \text{Groupoid}(V_3)$ such that $H^*(V_3)^G$ satisfies the required conditions. We leave the computations of the corresponding $H^*(V_3)^G$ to the reader.

\[\square\]

References

[AHS90] Jiri Adamek, Horst Herrlich, and George E. Strecker. Abstract and concrete categories. The joy of cats. John Wiley and Sons, 1990.

[DW90] William G. Dwyer and Clarence W. Wilkerson. Spaces of null homotopic maps. In Miller H.-R., Lemaire J.-M., and Schwartz L., editors, Théorie de l’homotopie, number 191 in Astérisque, pages 97–108. Société mathématique de France, 1990.

[DW92a] W. Dwyer and C. Wilkerson. A new finite loop space at the prime two. Journal of the American Mathematical Society, 6, 09 1992.

[DW92b] W. G. Dwyer and C. W. Wilkerson. A cohomology decomposition theorem. Topology, 31:433–443, 1992.

[Gab62] Pierre Gabriel. Des catégories abéliennes. Bull. Soc. Math. France, 90:(323–448), 1962.

[Hea20] Drew Heard. Depth and detection for noetherian unstable algebras. Trans. Amer. Math. Soc., 373, 2020.
[Hea21] Drew Heard. The topological nilpotence degree of a noetherian unstable algebra. \textit{Selecta Mathematica}, 27, 05 2021.

[Hen01] Hans-Werner Henn. \textit{Cohomology of Groups and Unstable Modules over the Steenrod Algebra}, pages 55–98. Birkhäuser Basel, Basel, 2001.

[HLS93] Hans-Werner Henn, Jean Lannes, and Lionel Schwartz. The categories of unstable modules and unstable algebras over the Steenrod algebra modulo nilpotent objects. \textit{Math. Ann.}, 115:(1053–1106), 1993.

[HLS95] Hans-Werner Henn, Jean Lannes, and Lionel Schwartz. Localizations of unstable $\mathcal{A}$-modules and equivariant mod $p$ cohomology. \textit{American Journal of Mathematics}, 301:(23–68), 1995.

[KS05] Mazaki Kashiwara and Pierre Shapira. \textit{Categories and Sheaves}. Springer, 2005.

[Kuh94] Nicholas J. Kuhn. Generic representations of the finite general linear group and the Steenrod algebra: I. \textit{American Journal of Mathematics}, 116:(327–360), 1994.

[Kuh07] Nicholas J. Kuhn. Primitives and central detection numbers in group cohomology. \textit{Advances in Mathematics}, 216(1):387–442, 2007.

[Kuh13] Nicholas J. Kuhn. Nilpotence in group cohomology. \textit{Proceedings of the Edinburgh Mathematical Society}, 56(1):151–175, 2013.

[Lan87] Jean Lannes. Sur la cohomologie modulo $p$ des $p$-groupes abéliens élémentaires. \textit{Cambridge University Press}, 1987.

[Rec84] D.L. Rector. Noetherian cohomology rings and finite loop spaces with torsion. \textit{Journal of Pure and Applied Algebra}, 32(2):191–217, 1984.

[Sch94] Lionel Schwartz. \textit{Unstable modules over the Steenrod Algebra and Sullivan’s fixed point set conjecture}. Chicago Lectures in Mathematics, 1994.

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