CORRIGENDUM

Étale cohomology, purity and formality with torsion coefficients

(J. Topol. 15 (2022), no. 4, 2270–2297)

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Abstract
Proposition 6.9 in (J. Topol. 15 (2022), no. 4, 2270–2297) is incorrect without a connectivity assumption. In this note, we provide a counter-example, give a correct proof of the modified proposition and explain the other changes that need to be made to [1].

MSC 2020
55Pxx (primary), 55S30 (secondary)

Proposition 6.9 in [1] must be replaced by the following proposition.

Proposition. Every “simply connected” dg-algebra in $gr^{(h)} DGA_{k}^{\alpha\text{-pure}}$ is $N$-formal, with $N = \lfloor \frac{(h-1)}{\alpha} \rfloor$.

Proof. Our first task is to construct a convenient model for $A$. We claim that there exists an $N$-quasi-isomorphism $f : M \to A$ in $\mathrm{Ch}^{>0}(gr^{(h)} k)^{\alpha\text{-pure}}$ such that:

1. $M$ is generated by elements of degree $\leq N$;
2. the differential on $M^\alpha_n$ is trivial for all $\alpha n \leq h - 1$;
3. let $\alpha n < p \leq h - 1$. If $M^\alpha_p \neq 0$ then $p \leq \alpha(2n - 2)$.

We use the standard theory of free models for dg-algebras over an arbitrary field (see [2]) in order to construct such an $M$. We will define, inductively over the degree $n \geq 0$, a sequence of dg-algebras $M\langle n \rangle$ with $gr^{(h)}$-weight decompositions $M\langle n \rangle^k = \bigoplus M\langle n \rangle_p^k$ and morphisms $f_n : M\langle n \rangle \to A$ compatible with weight decompositions and satisfying the following conditions.

(a) The dg-algebra $M\langle n \rangle$ is a free extension of $M\langle n-1 \rangle$ by generators of degree $n$ and weights $\alpha n \leq p \leq \alpha(2n - 2)$.

(b) The map $H^i(f_n)$ is an isomorphism for all $i \leq n$ and a monomorphism for $i = n + 1$.

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Then the morphism
\[ f : \bigcup_n f_n : \bigcup_{n \leq N} M(n) \to A \]
will be our model for \( A \).

Let \( M(1) = k \) concentrated in weight 0 and degree 0 and define \( f_1 : M(1) \to A \) to be the unit map. Conditions \((a_1)\) and \((b_1)\) are trivially satisfied. Assume inductively that we have defined \( f_{n-1} : M(n-1) \to A \) satisfying \((a_{n-1})\) and \((b_{n-1})\). For each \( 0 \leq p < h \), let \( V_p := H^n(C(f_{n-1}))_p \) and consider it as a bigraded vector space of degree \( n \) and pure weight \( p \). Define a differential \( d : V_p \to M(n-1)_p^{n+1} \) and a map \( f_n : V_p \to A_p^n \) by taking a section of the projection
\[ H^n(C(f_{n-1}))_p \to Z^n(C(f_{n-1}))_p \subset M(n-1)_p^{n+1} \oplus A_p^n. \]

These define a differential on
\[ M(n) := M(n-1) \cup T(V) \]
and a map \( f_n : M(n) \to A \) compatible with the weight decompositions. By classical arguments, as \( A \) is simply connected and \( H^n(A) = \bigoplus H^n(A)_p \), condition \((b_n)\) is satisfied. Let us prove \((a_n)\).

Note that elements in \( V_p \) arise either from the cokernel of \( H^n(f)_p \), which by \( \alpha \)-purity, is non-trivial only for \( p = \alpha n \), or from elements in the kernel of \( H^{n+1}(f)_p \). As \( M^1 = 0 \), elements in this kernel are represented by sums of products \( x_1 \cdots x_k \in M(n-1)_p^{n+1} \) with \( k \geq 2 \). Let \( n_i := |x_i| \). By induction hypothesis, we have \( w(x_i) \equiv p_i \mod h \) with \( \alpha n_i \leq p_i \leq \alpha(2n_i - 2) \). We get
\[ \alpha(n+1) = \alpha \left( \sum_{i=1}^k n_i \right) \leq \sum_{i=1}^k p_i \leq \alpha \sum_{i=1}^k (2n_i - 2) \leq 2\alpha(n+1) - 2\alpha k \leq \alpha(2n - 2). \]

Therefore, the generators of \( V_p \) have degree \( n \) and weights \( w \equiv p \mod h \) with \( \alpha n \leq p \leq (2n - 2)\alpha \) and condition \((a_n)\) is satisfied. This ends the inductive step.

We now prove that the differential on \( M^{n+1}_{\alpha n} \) is trivial for all \( \alpha n \leq h - 1 \). In fact, we will show that \( M^{n+1}_{\alpha n} = 0 \) for all \( \alpha n \leq h - 1 \). Assume that \( x \in M^{n+1}_{\alpha n} \) with \( \lambda \in \mathbb{Z}_{>0} \) and \( \alpha(n+1) \leq p \leq 2n\alpha \). This gives the condition \( \lambda h \leq \alpha n \), which contradicts the condition that \( \alpha n \leq h - 1 \). Therefore, \( x = 0 \).

Now, we are ready to prove the proposition. It suffices to prove that the model \( M \) that we have just constructed is \( N \)-formal. As a bigraded vector space, \( M \) may be decomposed into \( M = A \oplus D \oplus B \) where \( D \) denotes the diagonal \( p = \alpha n \) truncated up to degree \( \alpha n < h \), and \( A \) and \( B \) are the direct sum of all vector spaces above and below the diagonal, respectively, by letting
\[ I_d := \{(n, p); p = \alpha n \leq h - 1\} \text{ and } I_a := \left\{ (n, p); n < \frac{h-1}{\alpha}, p > \alpha n \right\} \]
we may write
\[ D = \bigoplus_{(n, p) \in I_d} M^n_p, A = \bigoplus_{(n, p) \in I_a} M^n_p \text{ and } B = \bigoplus_{(n, p) \in I_d \cup I_a} M^n_p, \]
By degree-weight reasons, \( B \) is a dg-algebra ideal of \( M \). By construction of \( M \), the differential of \( D \) is trivial and we have \( H^n(B) = 0 \) for all \( n \leq (h-1)/\alpha \). Therefore, the morphism of dg-algebras \( \pi : M \to M' := M/B \) induces an isomorphism in cohomology \( H^n(\pi) \) for all \( n \leq (h-1)/\alpha \). Consider the projection morphism \( M' \to H^*(M') \) given by \( A \mapsto 0 \) and \( D \mapsto D/\text{Im}(d) \). To see that it is a quasi-isomorphism of dg-algebras, it suffices to see that \( A \times M' \subseteq A \). Let \( x \in A \) and \( y \in M' \). By construction of \( M \) we have

\[
\begin{align*}
w(x) &\equiv p_x \pmod{h} \quad \text{with} \quad \alpha|x| < p_x \leq (2|x| - 2)\alpha, \\
w(y) &\equiv p_y \pmod{h} \quad \text{with} \quad \alpha|y| \leq p_y \leq (2|y| - 2)\alpha.
\end{align*}
\]

Assume that \( 0 \neq x \cdot y \in D \) and let \( n \equiv |x| + |y| \). Then there is \( \lambda \in \mathbb{Z} \) such that \( p_x + p_y = \alpha n + \lambda h \). As \( \alpha n < p_x + p_y \) we have \( \lambda \geq 0 \). This gives \( \alpha n + h \leq \alpha n + \lambda h \leq 2\alpha n - 4\alpha \) and hence \( h \leq \alpha(n - 4) \) which is a contradiction, as \( \alpha n \leq h - 1 \).

This mistake affects subsequent results in the paper. The following changes should be made.

1. In part (iii) of Theorem 7.2, the statement only holds if the algebra is simply connected.
2. In part (iii) of Theorem 7.12, the statement only holds if the algebra is simply connected.
3. In Theorem 7.14, the hypothesis that \( d \geq 2 \) should be added.
4. In Corollary 7.15, the hypothesis that \( d \geq 2 \) should be added.
5. Example 7.16 is incorrect, as the algebra of cochains on \( F_m(\mathbb{C}) \) is never simply connected if \( m \geq 2 \). However, the statement about Massey products is correct.

**Example.** As a counter-example to [1, Proposition 6.9] consider the following commutative dg-algebra:

\[
B = \Lambda(x, y, z_0, z_1, z_2), \quad dx = dy = 0, dz_0 = xz_2, dz_1 = xz_0, dz_2 = xy.
\]

Let \( I \) be the dg-ideal of \( B \) generated by \( yz_2, yz_0 \) and \( xz_0 z_1 \) and let \( A = B/I \).

A straightforward computation shows that the non-zero cohomology groups of \( A \) are as follows

\[
\begin{align*}
H^0(A) &= \mathbb{k}[1] \\
H^1(A) &= \mathbb{k}[x] \oplus \mathbb{k}[y] \\
H^2(A) &= \mathbb{k}[xz_1] \oplus \mathbb{k}[yz_1] \oplus \mathbb{k}[z_0 z_2] \\
H^3(A) &= \mathbb{k}[xyz_1] \oplus \mathbb{k}[z_2 z_0 z_1]
\end{align*}
\]

(where we denote by \([a]\) the cohomology class of a cocycle \( a \)).

If we assign weight 0 to \( z_0 \), weight 1 to \( x, y, z_1 \) and weight 2 to \( z_2 \), the algebra \( B \) becomes an object of \( gr^{(3)} \text{DGA}_k \). As the ideal \( I \) is generated by homogeneous elements, the algebra \( A \) is also in \( gr^{(3)} \text{DGA}_k \), and by inspecting its cohomology we see that it is in \( gr^{(3)} \text{DGA}_k^{1\text{-pure}} \).

On the other hand, we claim that \( A \) is not 2-formal. This can be seen by observing that there is a non-trivial 5-tuple Massey product of elements in \( H^1(A) \), given by

\[\langle [x], [x], [x], [x], [y] \rangle = \{[xz_1]\} \]
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