BACH-FLAT $h$-ALMOST GRADIENT RICCI SOLITONS

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ABSTRACT. On an $n$-dimensional complete manifold $M$, consider an $h$-almost gradient Ricci soliton, which is a generalization of a gradient Ricci soliton. We prove that if the manifold is Bach-flat and $dh/du > 0$, then the manifold $M$ is either Einstein or rigid. In particular, such a manifold has harmonic Weyl curvature. Moreover, if the dimension of $M$ is four, the metric $g$ is locally conformally flat.

1. Introduction

The notion of an $h$-almost Ricci soliton was introduced by Gomes, Wang, and Xia [6]. Such a soliton is a generalization of an almost Ricci soliton presented in [2] and [8]. An $h$-almost Ricci soliton is a complete Riemannian manifold $(M^n, g)$ with a vector field $X$ on $M$, a soliton function $\lambda : M \to \mathbb{R}$ and a signal function $h : M \to \mathbb{R}^+$ satisfying the equation

$$r_g + \frac{h}{2} \mathcal{L}_X g = \lambda g,$$

where $r_g$ is the Ricci curvature of $g$. A function is called signal if it has only one sign; in other words, it is either positive or negative on $M$. Let $(M, g, X, h, \lambda)$ denote an $h$-almost Ricci soliton. In particular, $(M, g, \nabla u, h, \lambda)$ for some smooth function $u : M \to \mathbb{R}$ is called an $h$-almost gradient Ricci soliton with potential function $u$. In this case, we have

$$(1.1)\quad r_g + h D_g du = \lambda g.$$

Here, $D_g du$ denotes the Hessian of $u$. Note that if we take $u = e^{-\frac{m}{u}}$ and $h = -\frac{m}{u}$, then (1.1) becomes

$$\text{Ric}_f^m = r_g + D_g df - \frac{1}{m} df \otimes df = \lambda g.$$

In other words, the $(\lambda, n + m)$-Einstein equation is a special case of (1.1). Here, $\text{Ric}_f^m$ is called the $m$-Bakry-Emery tensor. For further details of $h$-almost Ricci solitons, we refer to [6].

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In this paper we consider Bach-flat $h$-almost gradient Ricci solitons. The Bach tensor was introduced by R. Bach and this notion plays an important role in conformal relativity. On any $n$-dimensional Riemannian manifold $(M, g)$, $n \geq 4$, the Bach tensor is defined by

$$B = \frac{1}{n-3} \delta^p \delta^q W + \frac{1}{n-2} \hat{W}z,$$

where $W$ is the Weyl tensor, $z$ is the traceless Ricci tensor, and $\hat{W}z$ is defined by

$$\hat{W}z(X, Y) = \sum_{i=1}^{n} z(W(X, E_i)Y, E_i)$$

for some orthonormal basis $\{E_i\}_{i=1}^{n}$. It is easy to see that if $(M, g)$ is either locally conformally flat or Einstein, then it is Bach-flat: $B = 0$. When $n = 4$, it is well known that Bach-flat metrics on a compact manifold $M$ are critical points of the following functional

$$g \mapsto \int_{M} |W|^2 dv_g.$$

It is clear that when $h = 1$ and $\lambda$ is a positive constant, an $h$-almost gradient Ricci soliton reduces to a gradient shrinking Ricci soliton. Cao and Chen proved that a complete Bach-flat gradient shrinking Ricci soliton is either Einstein or rigid [4]. On the other hand, Qing and Yuan classified Bach-flat static spaces [9].

Our main result is as follows, which can be considered as a generalization of [4].

**Theorem 1.1.** Let $(M, g, \nabla u, h, \lambda)$ be an $n$-dimensional Bach-flat $h$-almost gradient Ricci soliton with potential function $u$. Assume that each level set of $u$ is compact and $h$ is a function of $u$ only. Then, $(M, g, \nabla u, h, \lambda)$ is either

1. Einstein with constant functions $u$ and $h$, or
2. locally isometric to a warped product with $(n-1)$-dimensional Einstein fibers if $\frac{dh}{du} > 0$ on $M$.

For example, when $m > 0$, $h = -\frac{m}{u} < 0$ satisfies the condition of Theorem 1.1 since

$$\frac{dh}{du} = \frac{m}{u^2} > 0.$$

This recovers the result of [5]. It will be interesting if one can weaken the condition of Theorem 1.1.

In the case of (2) in Theorem 1.1, a warped product metric has vanishing Cotton tensor (see (2.4) below) since its fiber is Einstein. Thus, as a consequence of Theorem 1.1 we have the following.

**Corollary 1.2.** Let $(M, g, \nabla u, h, \lambda)$ be an $n$-dimensional Bach-flat $h$-almost gradient Ricci soliton with potential function $u$. Assume that each level set
of \( u \) is compact and \( h \) is a function of \( u \) only. If \( \frac{dh}{du} > 0 \) on \( M \), then \( (M, g) \) has harmonic Weyl curvature.

In particular, when \( n = 4 \), the Einstein fibers in Theorem 1.1 have constant curvature. A computation shows that such a metric is locally conformally flat, which proves the following theorem.

**Theorem 1.3.** Let \( (M, g, \nabla u, h, \lambda) \) be a 4-dimensional Bach-flat \( h \)-almost gradient Ricci soliton with potential function \( u \). Assume that each level set of \( u \) is compact and \( h \) is a function of \( u \) only with \( \frac{dh}{du} > 0 \). Then, \( (M, g) \) is locally conformally flat.

We remark that, as in [5], Theorems 1.1, Corollary 1.2, and Theorem 1.3 can be extended to the case in which \( M \) has a non-empty boundary.

2. Preliminaries

In this section, we derive several useful identities containing various curvatures and the Cotton tensor.

We start with basic definitions of differential operators acting on tensors. Let us denote by \( C^\infty(S^2 M) \) the space of sections of symmetric 2-tensors on a Riemannian manifold \( M \). Let \( D \) be the Levi-Civita connection of \( (M, g) \). Then the differential operator \( d^D \) from \( C^\infty(S^2 M) \) into \( C^\infty(\Lambda^2 M \otimes T^* M) \) is defined as

\[
d^D \omega(X, Y, Z) = (D_X \omega)(Y, Z) - (D_Y \omega)(X, Z)\]

for \( \omega \in C^\infty(S^2 M) \) and vectors \( X, Y, \) and \( Z \). Let us denote by \( \delta^D \) the formal adjoint operator of \( d^D \).

For a function \( f \in C^\infty(M) \) and \( \omega \in C^\infty(S^2 M) \), \( df \wedge \omega \) is defined as

\[
(df \wedge \omega)(X, Y, Z) = df(X) \omega(Y, Z) - df(Y) \omega(X, Z).
\]

Here, \( df \) denotes the usual total differential of \( f \). We also denote by \( \delta \) the negative divergence operator so that \( \Delta f = -\delta df \).

Taking the trace of (1.1) gives

\[
s_g + h \Delta u = n \lambda.
\]

Thus,

\[
ds_g + \Delta u dh + h d\Delta u = n d\lambda.
\]

By taking the divergence of (1.1), we have

\[
-\frac{1}{2} ds_g - D_g du(\nabla h, \cdot) - h r_g(\nabla u, \cdot) - h d\Delta u = -d\lambda.
\]

By adding the previous two equations, we have

\[
(2.1) \quad \frac{1}{2} ds_g - D_g du(\nabla h, \cdot) - h r_g(\nabla u, \cdot) + \Delta u dh = (n - 1) d\lambda.
\]

By adding the previous two equations, we have

\[
\delta(hr_g(\nabla u, \cdot)) = -r_g(\nabla u, \nabla h) - \frac{h}{2} (\nabla s_g, \nabla u) + |r_g|^2 - \lambda s_g.
\]
Therefore, we have the following equality.

**Proposition 2.1.** On $M$ we have

$$(n - 1)\Delta \lambda = \frac{1}{2} \Delta s_g + |r_g|^2 - \lambda s_g - \frac{h}{2} \langle \nabla s_g, \nabla u \rangle$$

$$+ \left( \Delta u - \frac{\lambda}{h} \right) \Delta h + \frac{1}{h} (r_g, D_g dh) - 2 r_g (\nabla u, \nabla h).$$

On the other hand, by applying $dD$ to (1.1), we have

$$\tag{2.3} dD r_g - \frac{1}{h} dh \wedge r_g + h \tilde{i}_{\nabla u} R = d\lambda \wedge g - \frac{\lambda}{h} dh \wedge g.$$  

Here, an interior product $\tilde{i}$ of the final factor is defined by

$$\tilde{i}_{\xi} R(X, Y, Z) = R(X, Y, Z, \xi),$$

and we used the identity

$$dD Ddu = \tilde{i}_{\nabla u} R.$$  

Hereafter, we denote $s_g, r_g,$ and $D_g du$ by $s, r,$ and $Ddu,$ respectively. From the curvature decomposition, we can compute that

$$\tilde{i}_{\nabla u} R = \tilde{i}_{\nabla u} W - \frac{1}{n - 2} \tilde{i}_{\nabla u} r \wedge g + \frac{s}{(n - 1)(n - 2)} du \wedge g - \frac{1}{n - 2} du \wedge r,$$

where $\tilde{i}_{\nabla u} r$ denotes the interior product defined by

$$i_{\nabla u} r(X) = r(\nabla u, X).$$

The Cotton tensor $C$ is defined by

$$C = dD r - \frac{1}{2(n - 1)} ds \wedge g.$$  

Then, by (2.1) and (2.3) as well as the fact that

$$s + h \Delta u = n \lambda,$$

we have

$$C + h \tilde{i}_{\nabla u} W = h D + \frac{h}{n - 1} i_{\nabla u} r \wedge g + d\lambda \wedge g - \frac{1}{2(n - 1)} du \wedge g$$

$$+ \frac{1}{h} dh \wedge r - \frac{\lambda}{h} dh \wedge g$$

$$\tag{2.5} = h D + H,$$

where $D$ is defined (as usual) by

$$\tag{2.6} (n - 2) D = du \wedge r + \frac{1}{n - 1} i_{\nabla u} r \wedge g - \frac{s}{n - 1} du \wedge g,$$

and $H$ is defined by

$$H = - \frac{1}{n - 1} i_{\nabla h} Ddu \wedge g + dh \wedge \left( \frac{1}{h} r + \frac{\Delta u}{n - 1} g - \frac{\lambda}{h} g \right)$$

$$= db \wedge r + \frac{1}{n - 1} i_{\nabla b} r \wedge g - \frac{s}{n - 1} db \wedge g.$$
Here, $b = \log |h|$ with $\nabla b = \frac{\nabla h}{h}$. In particular, $g^{ik}H_{ijk} = -g^{ik}H_{ijk} = 0$.

**Proposition 2.2.** Let $(M, g, \nabla u, h, \lambda)$ be an $h$-almost gradient Ricci soliton with potential function $u$. Then

$$C + h\tilde{\nabla}_u W = hD + H.$$ 

In particular, if $h$ is constant or $\frac{dh}{du} = 0$, $H \equiv 0$.

### 3. Bach-flat metrics

In this section, we assume that $g$ is Bach-flat. Note that

$$\delta W = -\frac{n-3}{n-2}C.$$ 

Recall that the Bach tensor is given by

$$B = \frac{1}{n-3} \delta D \delta W + \frac{1}{n-2} W_z = \frac{1}{n-2} \left( -\delta C + \hat{W} z \right).$$ 

Since

$$\delta(h\tilde{\nabla}_u W)(X, Y) = -W(\nabla h, X, Y, \nabla u) + h \delta W(X, Y, \nabla u) + hW(X, E_i, Y, D_{E_i} du)$$

$$= W(X, \nabla h, Y, \nabla u) - \frac{n-3}{n-2} h C(Y, \nabla u, X) - \hat{W} z,$$

by taking the divergence of (2.5) we have

$$-(n-2)B(X, Y) = -W(\nabla h, X, Y, \nabla u) + \frac{n-3}{n-2} h C(Y, \nabla u, X) - i_{\nabla h}D(X, Y) + h \delta D(X, Y) + \delta H(X, Y).$$

Hence,

$$-(n-2)B(\nabla u, \nabla u) = -D(\nabla h, \nabla u, \nabla u) + h \delta D(\nabla u, \nabla u) + \delta H(\nabla u, \nabla u).$$

As a result, from the assumption that $B = 0$ and $h$ is a function of $u$ only,

$$0 = \frac{1}{h} D(\nabla h, \nabla u, \nabla u) = \delta D(\nabla u, \nabla u) + \frac{1}{h} \delta H(\nabla u, \nabla u).$$

Let $\{E_i\}_{i=1}^n$ be a normal geodesic frame. Note that, since

$$hD(E_i, D_{E_i} du, \nabla u) = -D(E_i, E_k, \nabla u) r_{ik} = 0,$$

we have

$$\text{div}(D(\cdot, \nabla u, \nabla u)) = -\delta D(\nabla u, \nabla u) + D(E_i, \nabla u, D_{E_i} du).$$

Furthermore,

$$|D|^2 = \frac{1}{n-2} (du(E_i) r(E_j, E_k) r(E_i, E_k) - du(E_j) r(E_i, E_k)) D_{ijk}$$

$$= \frac{2}{n-2} D(E_i, \nabla u, E_k) r_{ik}$$

$$= \frac{2h}{n-2} D(E_i, \nabla u, D_{E_i} du).$$
Similarly, since
\[ h H(E_i, D_{E_i} du, \nabla u) = -H(E_i, E_k, \nabla u) r_{ik} = 0 \]
and \( h \) is a function of \( u \) only, we have
\[
\text{div} \left( \frac{1}{h} H(\cdot, \nabla u, \nabla u) \right) = -\frac{1}{h} \delta H(\nabla u, \nabla u) + \frac{1}{h} H(E_i, \nabla u, D_{E_i} du).
\]
Moreover,
\[
|H|^2 = -\frac{2}{h} H(E_i, \nabla h, E_k) r_{ik} = -\frac{2}{h} \frac{dh}{du} H(E_i, \nabla u, E_k) r_{ik}
\]
\[ = 2 \frac{dh}{du} H(E_i, \nabla u, D_{E_i} du). \]
Thus,
\[
0 = \int_{t_1 \leq u \leq t_2} \delta D(\nabla u, \nabla u) + \frac{1}{h} \delta H(\nabla u, \nabla u)
\]
\[ = \frac{n-2}{2} \int_{t_1 \leq u \leq t_2} |D|^2 h + \frac{1}{2} \int_{t_1 \leq u \leq t_2} \frac{|H|^2}{h} \frac{dh}{du}.
\]
Since \( h \) is signal, \( h \) is either positive or negative. For each case, we derive \( D = H = 0 \) when \( \frac{dh}{du} > 0 \). Therefore we have the following result.

**Lemma 3.1.** Let \((M, g, \nabla u, h, \lambda)\) be a Bach-flat \( h \)-almost gradient Ricci soliton with potential function \( u \). Assume that each level set of \( u \) is compact and \( h \) is a function of \( u \) only. If \( \frac{dh}{du} > 0 \) on \( M \), then on \( M \) we have
\[
D = H = 0.
\]

Now, since \( D = H = 0 \), by (2.4) and (2.5)
\[
(3.1)
\]
\[ C = -h \tilde{i}_{\nabla u} \mathcal{W}. \]

By taking the divergence of (3.1), we have
\[
\mathcal{W}(X, \nabla h, Y, \nabla u) = \frac{n-3}{n-2} h C(Y, \nabla u, X).
\]

By combining these equations,
\[
\frac{n-3}{n-2} h^2 C(Y, \nabla u, X) = -C(X, \nabla h, Y),
\]
and
\[
\mathcal{W}(X, \nabla h, Y, \nabla u) = -\frac{n-3}{n-2} h^2 \mathcal{W}(X, \nabla u, Y, \nabla u).
\]

Therefore, we have the following.

**Corollary 3.2.** When \( D = H = 0 \), we have
\[
(3.2) \quad \mathcal{W}(\cdot, \nabla u, \cdot, \nabla u) = C(\cdot, \nabla u, \cdot) = 0,
\]
unless
\[
\frac{dh}{du} = -\left( \frac{n-3}{n-2} \right) h^2.
\]
For example, when $h = -\frac{m}{n}$, (3.2) holds if $m \neq 0$ or $-\frac{n-2}{n-3}$. Note that (3.2) also holds if $h$ is constant.

Moreover, we have the following result.

**Lemma 3.3.** Suppose that $\frac{dh}{du} > 0$. Then, for $X$ orthogonal to $\nabla u$,

$$r(X, \nabla u) = 0.$$  

In particular,

$$i\nabla_u r = \alpha du,$$

where $\alpha = r(N, N)$ with $N = \nabla u/|\nabla u|$. 

**Proof.** By Lemma 3.1, $D = H = 0$. From (2.3), if $X$ is orthogonal to $\nabla u$,

$$d^D r(X, Y, \nabla u) = -\frac{1}{h} dh(Y) r(X, \nabla u) + d\lambda(X) du(Y).$$

Since $C(X, Y, \nabla u) = -h\mathcal{W}(X, Y, \nabla u, \nabla u) = 0$ by (3.1), by (2.4) we have

$$d^D r(X, Y, \nabla u) = \frac{1}{2(n-1)} ds(X) du(Y).$$

Thus, by (2.1)

$$\frac{1}{h} \frac{dh}{du} r(X, \nabla u) = d\lambda(X) - \frac{1}{2(n-1)} ds(X)$$

$$= \frac{1}{(n-1)h} \left( \frac{dh}{du} - h^2 \right) r(X, \nabla u),$$

which implies that

$$\left( (n-2) \frac{dh}{du} + h^2 \right) r(X, \nabla u) = 0.$$ 

This completes the proof of our lemma. 

Note that Lemma 3.3 holds with the assumptions that $D = H = 0$ and

$$\frac{dh}{du} \neq -\frac{1}{n-2} h^2$$

without $\frac{dh}{du} > 0$. For example, in the case of $m$-Bakry-Emery tensor, $h = -\frac{m}{u}$ satisfies (3.4) if $m \neq 2 - n$.

**4. Level sets of $u$**

In this section, we will investigate the structure of regular level sets of the potential function $u$. For a regular value $c$, we denote the level set $u^{-1}(u)$ by $L_c$. On $L_c$, let $\{E_i\}, 1 \leq i \leq n$, be an orthonormal frame with $E_n = N = \nabla u/|\nabla u|$.

Furthermore, throughout the section we assume that $D = H = 0$ with

$$\frac{dh}{du} \neq -\left( \frac{n-3}{n-2} \right) h^2 \quad \text{and} \quad \frac{dh}{du} \neq -\frac{1}{n-2} h^2.$$
Then, by Corollary 3.2 and 3.3 hold. Furthermore, for $X$ orthogonal to $\nabla u$, by the proof of Lemma 3.3,

$$d\lambda(X) = \frac{1}{2(n-1)} ds(X).$$

Thus, $s + 2(1-n)\lambda$ is constant on each level set of $u$. Furthermore,

$$\frac{1}{2} X(|\nabla u|^2) = \langle D_X du, \nabla u \rangle = \frac{1}{h} (\lambda du(X) - r(X, \nabla u)) = 0,$$

which implies that $|\nabla u|^2$ is constant on each level set of $u$. Therefore, we have the following.

**Lemma 4.1.** $|\nabla u|^2$ and $s + 2(1-n)\lambda$ are constant on each regular level set of $u$.

For further investigation, we need the following key lemma.

**Lemma 4.2.**

$$0 = \frac{ns - (n-1)^2\lambda - \alpha}{(n-1)h} r - D\nabla u r - \frac{r \circ r}{h} + \frac{n-3}{2(n-1)} du \otimes ds$$

$$+ \frac{1}{n-1} (ds(u) - \langle \nabla u, \nabla \alpha \rangle) g + s + (1-n)\lambda \frac{h}{(n-1)h} (\alpha - s)g + \frac{1}{n-1} du \otimes d\alpha.$$

**Proof.** To find $\delta D$, by (2.6), we first compute

$$\delta(du \wedge r) = \frac{s - (n-1)\lambda}{h} r - D\nabla u r - \frac{r \circ r}{h} + \frac{1}{2} du \otimes ds.$$

By Lemma 3.3, $i\nabla_u r = \alpha du$. Thus,

$$\delta(i\nabla_u r \wedge g) = -\langle \nabla u, \nabla \alpha \rangle g + \frac{s + (1-n)\lambda}{h} \alpha g + du \otimes d\alpha - \frac{\alpha}{h} r.$$

Similarly,

$$-\delta(s du \wedge g) = ds(u) g - \frac{s^2 + (1-n)s\lambda}{h} g - du \otimes ds + \frac{s}{h} r.$$

Hence, by (2.6) together with (3.3), we have

$$(n-2)\delta D = \frac{ns - (n-1)^2\lambda - \alpha}{(n-1)h} r - D\nabla u r - \frac{r \circ r}{h}$$

$$+ \frac{n-3}{2(n-1)} du \otimes ds + \frac{1}{n-1} du \otimes d\alpha$$

$$+ \frac{1}{n-1} \left( ds(u) - \langle \nabla u, \nabla \alpha \rangle + s + \frac{(1-n)\lambda}{h}(\alpha - s) \right) g.$$

Since $D = \delta D = 0$, the proof follows. \qed

Thus, we have the following.

**Corollary 4.3.** $(n-3) s + 2\alpha$ is constant on each regular level set of $u$. 

Proof. Let $X$ be a vector orthogonal to $\nabla u$. By putting $(X, \nabla u)$ in the equation in Lemma 4.2,

\[ D_{\nabla u} r(X, \nabla u) = 0. \]  

(4.1)

Now, by putting $(\nabla u, X)$ in the equation in Lemma 4.2 again, we have

\[ 0 = \frac{n - 3}{2(n - 1)} |\nabla u|^2 ds(X) + \frac{1}{n - 1} |\nabla u|^2 d\alpha(X), \]

since $r(X, \nabla u) = 0$ and

\[ D_{\nabla u} r(\nabla u, X) = D_{\nabla u} r(X, \nabla u). \]

\[ \square \]

Lemma 4.4. $s_g + 2(1-n)\alpha$ is constant on each regular level set of $u$.

Proof. For $X$ orthogonal to $\nabla u$, by (3.2) and (4.1)

\[ 0 = C(X, \nabla u, \nabla u) \]

\[ = D_X r(\nabla u, \nabla u) - \frac{1}{2(n - 1)} ds(X) |\nabla u|^2. \]

Thus,

\[ X(\alpha) = \frac{1}{|\nabla u|^2} X(r(\nabla u, \nabla u)) \]

\[ = \frac{1}{|\nabla u|^2} (D_X r(\nabla u, \nabla u) + 2 r(D_X du, \nabla u)) \]

\[ = \frac{1}{2(n - 1)} ds(X), \]

since

\[ r(D_X du, \nabla u) = \frac{1}{h} (\lambda r(X, \nabla u) - r \circ r(X, \nabla u)) = 0. \]

\[ \square \]

By combining Lemma 4.1, Corollary 4.3, and Lemma 4.4, we have the following.

Theorem 4.5. Let $(M, g, \nabla u, h, \lambda)$ be a Bach-flat $h$-almost gradient Ricci soliton with potential function $u$. Assume that each level set of $u$ is compact and $h$ is a function of $u$ only with $\frac{dh}{du} > 0$. Then $s_g, \alpha, \lambda$ are constant on each regular level set of $u$. In particular, if $h$ is constant, the condition on $\frac{dh}{du}$ is not necessary.

When $D = 0$, the Ricci tensor has the following characterization.

Lemma 4.6. Suppose that $D = 0$. Then the Ricci curvature tensor has at most two eigenvalues.
Proof. Let \{E_i\}, \(1 \leq i \leq n\), be an orthonormal frame with \(E_n = N = \nabla u/|\nabla u|\). Then

\[
II_{ij} = \frac{1}{h|\nabla u|}(\lambda g_{ij} - r_{ij}),
\]
and

\[
m = \text{tr} II = \frac{n-1}{h|\nabla u|}\left(\lambda + \frac{\alpha - s}{n-1}\right).
\]

Thus, \(m\) is constant on each level set of \(u\), and

\[
|II - \frac{m}{n-1} g|^2 = |II|^2 - \frac{m^2}{n-1} = \frac{1}{h^2|\nabla u|^2}\left(|r|^2 - \alpha^2 - \frac{(s - \alpha)^2}{n-1}\right) = \frac{1}{h^2|\nabla u|^2}\left(|r|^2 - \frac{n-1}{n-1}\alpha^2 + \frac{2s\alpha}{n-1} - \frac{s^2}{n-1}\right).
\]

Since \(r \circ r(\nabla u, \nabla u) = \alpha^2|\nabla u|^2\), from the identity

\[
\frac{n-2}{2}|D|^2 = |r|^2|\nabla u|^2 - \frac{n}{n-1}r \circ r(\nabla u, \nabla u) + \frac{2s}{n-1}r(\nabla u, \nabla u) - \frac{s^2}{n-1}|\nabla u|^2,
\]
we have

\[
|D|^2 = \frac{2}{n-2}h^2|\nabla u|^4|II - \frac{m}{n-1} g|^2.
\]
Since \(D = 0\), we have

\[
II_{ij} = \frac{m}{n-1} g_{ij},
\]
which implies that

\[
r_{ij} = \frac{s - \alpha}{n-1} g_{ij}
\]
for \(i = 1, \ldots, n - 1\) by (4.2). This completes the proof of our lemma. 

As an immediate consequence, on an open set \(\{x \in M \mid \nabla u(x) \neq 0\}\), the Ricci tensor may be written as

\[
r_g = \beta du \otimes du + \left(\frac{s - \alpha}{n-1}\right) g,
\]
where

\[
\beta = \frac{n \alpha - s}{(n-1)|\nabla u|^2}.
\]
Thus, by (1.1) we have

\[
D_g du = \frac{1}{h}\left(\lambda + \frac{\alpha - s}{n-1}\right) g - \frac{\beta}{h} du \otimes du.
\]

Now, we are ready to prove Corollary 1.2 which shows the relationship between Bach-flat metrics and harmonic Weyl metrics.
Proof of Corollary 1.2. Note that, by (3.1) and (3.2)
\[ C(\cdot,\cdot,\nabla u) = C(\cdot,\nabla u,\cdot) = 0. \]
On the other hand, by the Codazzi equation,
\[ \langle R(X,Y)Z,N \rangle = D_Y II(X,Z) - D_X II(Y,Z). \]
Thus, for \( 1 \leq i,j,k \leq n-1 \), by (4.3)
\[ \langle R(E_i,E_j)E_k,N \rangle = E_j(II(E_i,E_k)) - II(D_{E_j}E_i,E_k) - II(E_i,D_{E_j}E_k) - E_i(II(E_j,E_k)) + II(D_{E_i}E_j,E_k) + II(E_j,D_{E_i}E_k) = 0. \]
Therefore, by (2.3)
\[ d^D r(E_i,E_j,E_k) = 0, \]
which implies that
\[ C(E_i,E_j,E_k) = d^D r(E_i,E_j,E_k) - \frac{1}{2(n-1)} ds \wedge g(E_i,E_j,E_k) = 0. \]
Hence, \( C \) is identically zero, and so is \( \delta W \).

The following is a restatement of Theorem 1.1.

Theorem 4.7. Let \((M,g,\nabla u,h,\lambda)\) be a Bach-flat \( h \)-almost gradient Ricci soliton with potential function \( u \). Assume that each level set of \( u \) is compact with \( \frac{dh}{du} > 0 \) on \( M \). Then, either \( g \) is Einstein with constant function \( u \) or the metric can be written as
\[ g = dt^2 + \psi^2(t) \hat{g}_E, \]
where \( \hat{g}_E \) is the Einstein metric on the level set \( E = L_{c_0} \) for some \( c_0 \).

Proof. Assume that \( u \) is not constant. By Lemma 3.1 \( D = H = 0 \). Since \( |\nabla u|^2 \) depends only on \( u \) by Lemma 4.6 as shown in the proof of Theorem 7.9 of [7] with Remark 3.2 of [4], the metric can be locally written as
\[ g = dt^2 + \hat{g}_c. \]
Here, \( \hat{g}_c \) denotes the induced metric on the level set \( L_c = u^{-1}(c) \) for each regular value \( c \). Furthermore, \((L_c,\hat{g}_c)\) is necessarily Einstein; by the Gauss equation
\[ \hat{R}_{ijij} = R_{ijij} + II_{ii} II_{jj} - II_{ijij} = R_{ijij} + \frac{m^2}{(n-1)^2}. \]
Thus,
\[ \hat{r}_{ii} = r_{ii} - R(N,E_i,N,E_i) + \frac{m^2}{n-1}. \]
By (3.2) and (4.4), we have
\[ R(E_i,N,E_i,N) = \frac{1}{n-2} (r_{ii} + \alpha) - \frac{s}{(n-1)(n-2)} = \frac{\alpha}{n-1}. \]
Hence, it follows that
\[ \hat{r}_{ii} = r_{ii} + \frac{m^2 - \alpha}{n - 1} = \frac{1}{n - 1} (s - 2\alpha + m^2) = \hat{\lambda}_0. \]
Since \( s, \alpha, \) and \( m \) are constant along \( L_c, \) this proves that \( (L_c, \hat{g}_c) \) has constant Ricci curvature. As a result, by suitable change of variable, the metric \( g \) can be written as in the statement of Theorem 4.7. □

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