Informational correlation between two parties of a quantum system. Short spin-1/2 chains with $XY$ Hamiltonian

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Abstract

We introduce the informational correlation $E^{AB}$ between two interacting quantum subsystems $A$ and $B$ of a quantum system as the number of arbitrary parameters $\varphi_i$ of a unitary transformation $U^A$ (locally performed on the subsystem $A$) which may be detected in the subsystem $B$ by the local measurements. This quantity indicates whether the state of the subsystem $B$ may be effected by means of the unitary transformation applied to the subsystem $A$. Emphasize that $E^{AB} \neq E^{BA}$ in general. The informational correlations in systems with tensor product initial states are studied in more details. In particular, it is shown that the informational correlation may be changed by the local unitary transformations of the subsystem $B$. However, there is some non-reducible part of $E^{AB}(t)$ which may not be decreased by any unitary transformation of the subsystem $B$ at a fixed time instant $t$. Two examples of the informational correlations between two parties of the four node spin-1/2 chain are studied.

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I. INTRODUCTION

Quantum correlations are considered to be responsible for advantages of quantum devices in comparison with their classical analogues. To quantify these advantages, a special measures of quantum correlations have been introduced. The entanglement [1–5] and discord [6–10] are known as two basic measures. However, the role of such correlations is not completely clarified. Of course, they have to be available in the system. But, up to now, it is not clear whether the quantum correlations must be large in order to reveal all advantages of quantum information devices. Moreover, there are many verification of the hypothesis that the quantum correlations measured either by entanglement or discord shouldn’t be large. For instance, there are quantum states without entanglement revealing a quantum nonlocality [11–13]. In addition, speed-up of certain calculations may be organized with negligible entanglement [14–18]. These interesting results together with an observation that almost all quantum states posses the non-vanishing quantum discord [19] might lead us to the conclusion (which perhaps causes doubts) that almost all quantum systems may be effectively used in the quantum information devices.

Thus, the above examples suggest us to conclude that even the quantum devices based on the states with minor (but non-zero) entanglement and/or discord may exhibit advantages in comparison with their classical counterparts. Such behavior may be explained by the observation that the spread of information about the state of a given subsystem throughout the whole quantum system does not require the large values of either entanglement or discord [20]. In other words, if we change a state of a given subsystem $A$ at the initial time instant $t = t_0$ (for instance, applying the unitary transformation), then, generally speaking, any other subsystem of the whole quantum system will know about the new state of $A$ at (almost) any instant $t > t_0$. In turn, namely this information provides the overall mutual relations among all parties of a quantum system. Moreover, it is valuable that the spread (or distribution) of information throughout the whole system does not require the fine adjustment of the parameters of a quantum system, unlike the large discord and/or entanglement, when the minor deviation of the system’s parameters from the required values may crucially decrease the values of these measures. It is worthwhile to say that the spread of information is observed even in the system with separable initial state when there is neither discord no entanglement between subsystems initially [20]. Of course the discord...
and entanglement may appear in the course of evolution, but their values are not crucial for the information distribution. The measure of correlations introduced in this paper is based on the outlined above information distribution.

Before proceed to the the subject of our paper, let us notice that the system-environment states with vanishing discord give impact to study of the evolution of a system as a completely positive map \([21]\) from the initial state of this system to its evolving state \([22, 24]\). However, although originally the completely positive maps where found for the states with initially vanishing discord \([22, 23]\), it was shown later that the vanishing discord is not necessary for this \([24]\). Thus, here the situation is opposite to the quantum nonlocality and speed-up.

Similar to the above refs. \([22–24]\), we consider the evolution of the state of a given subsystem \(B\) of some quantum system as a map of the initial state of another subsystem \(A\) and show that this map is responsible for the information distribution between two chosen subsystems. Then, correlations (which absent initially) appear as result of such evolution. Our basic study is referred to the systems having the tensor product initial states (and zero discords), when the above map becomes completely positive one. However, our algorithm may be extended to systems with general initial states (this problem is briefly discussed in Appendix C, Sec.\([\text{V}C]\).

In this paper, we do not separate quantum and classical effects. Instead, we study the possibility to handle the state of some subsystem \(B\) at some instant \(t\) by means of the local unitary transformation, \(U^A\), of another subsystem \(A\) at instant \(t_0 < t\). We refer to the measure quantifying this effect as the informational correlation \(E^{AB}\) between two parties \(A\) and \(B\) of a quantum system. In this regard, one has to mention Ref. \([25]\) where the quantumness of operations has been studied without the direct relation to the entanglement and discord.

Unlike the usual definition of the information through the entropy \([26]\), we define the measure of informational correlation as the number of parameters \(\varphi_i\) of the local unitary transformation \(U^A(\varphi^A)\), \(\varphi^A = \{\varphi_1, \varphi_2, \ldots\}\) that may be detected in the subsystem \(B\) by means of the local measurements. Vise-verse, the influence of the subsystem \(B\) on the subsystem \(A\) may be characterized by the informational correlation \(E^{BA}\) which equals the number of parameters \(\varphi_i\) in the local transformation \(U^B(\varphi^B)\), \(\varphi^B = \{\varphi_1, \varphi_2, \ldots\}\), of the subsystem \(B\) that may be detected in the subsystem \(A\) by means of the local measurements. We assume that namely these measures \(E^{AB}\) and \(E^{BA}\) characterize the strength of those
quantum correlations which are responsible for the information exchange between parties. In turn, namely the mutual exchange of the parameters may be used in the realization of elementary logical gates. Emphasize that, in general, \( \dim A \neq \dim B \) and \( E^{AB} \neq E^{BA} \), i.e. this measure is not symmetrical, similar to the discord, which depends on the particular subsystem chosen for the local classical measurements. For the tensor product initial state \( \rho(0) = \rho^A(0) \otimes \rho^C(0) \otimes \rho^B(0) \) (where \( \rho^A \) and \( \rho^B \) are respectively the density matrices of the subsystems \( A \) and \( B \), while \( \rho^C \) is the density matrix of the rest of the quantum system), it will be shown that \( E^{AB} = E^{BA} = 0 \) only if both initial density matrices of subsystems \( A \) and \( B \) are proportional to the unit matrix. Note, that the unitary invariant discord \(^{27}\) possesses the same property.

Below, we study the informational correlation \( E^{AB} \) only. For the tensor product initial state \( \rho(0) = \rho^A(0) \otimes \rho^{CB}(0) \) (where \( \rho^{CB} \) is the density matrix of the subsystem \( C \cup B \)), it will be shown, in particular, that the informational correlation is invariant with respect to the initial local unitary transformations of the subsystem \( A \) (i.e. \( E^{AB} \) depends only on the eigenvalues of the initial density matrix \( \rho^A(0) \)) and might be changed by the initial local unitary transformations of the subsystem \( B \). In addition, we reveal such part of the informational correlation which may not be decreased by the local unitary transformations of the subsystem \( B \) at a given time instant \( t \) (the so-called non-reducible informational correlation).

The paper is organized as follows. In Sec[III], we introduce the definition of the informational correlation and discuss the non-reducible informational correlation. Examples of the informational correlations in the four node spin-1/2 homogeneous chain governed by the XY Hamiltonian with the nearest neighbor interactions are considered in Sec[III]. In Sec[IV], we collect the basic properties of the informational correlation. Some additional information and calculations are given in the Appendix, Sec[V].
II. DEFINITION OF INFORMATIONAL CORRELATION

For a given quantum system $S$ we introduce the following notations:

- $U^S$ is the unitary transformation performed on the system $S$, (1)
- $N^S = \dim S$ is the dimensionality of the system, (2)
- $M^S = N^S(N^S - 1)/2$ is the number of off-diagonal elements in the $N^S \times N^S$ matrix, (3)
- $D^S = (N^S)^2 - 1$ is the total number of arbitrary parameters parameterizing the group $SU(N^S)$, (4)
- $\varphi^S = \{\varphi_1, \ldots, \varphi_{D^S}\}$ is the set of parameters parameterizing the group $SU(N^S)$, (5)

where $\bar{G}^S$ is the closed region in the space of the parameters $\varphi_i$, $i = 1, \ldots, D^S$. As usual, $G^S$ denotes the appropriate open region. Hereafter we assume that the whole quantum system is splitted into three subsystems $A$, $B$ and $C$, where $A$ and $B$ are the subsystems we are interested in, while $C$ is the rest of our quantum system. Thus, the total system is $A \cup C \cup B$. In particular, the subsystem $C$ may be absent. Let the state of the whole quantum system be described by the density matrix $\rho$. In turn, as usual, the states of the subsystems $A$, $B$, $C$ and $C \cup B$ are represented by the reduced density matrices $\rho^A$, $\rho^B$, $\rho^C$ and $\rho^{CB}$ respectively:

$$\rho^A = \text{Tr}_{BC} \rho, \quad \rho^B = \text{Tr}_{AC} \rho, \quad \rho^C = \text{Tr}_{AB} \rho, \quad \rho^{CB} = \text{Tr}_{A} \rho. \quad (6)$$

Suppose that we want to effect on the state of the subsystem $B$ by means of the unitary transformation $U^A(\varphi^A)$ of the subsystem $A$ at the initial instant $t = 0$. Let us determine how many parameters of the arbitrary transformation $U^A(\varphi^A) \in SU(N^A)$ may be detected in the subsystem $B$ at this instant. For this purpose we, first of all, fix the state of the system at $t = 0$ by the initial density matrix $\rho(0)$. The local transformation $U^A(\varphi^A) \in SU(N^A)$ transforms the initial density matrix $\rho(0)$ of the whole system $A \cup C \cup B$ into the density matrix $\rho(\varphi^A, 0)$ as follows:

$$\rho(\varphi^A, 0) = (U^A(\varphi^A) \otimes I_C \otimes I_B)\rho(0)((U^A(\varphi^A))^+ \otimes I_C \otimes I_B). \quad (7)$$

Now the initial density matrix $\rho^A(\varphi^A, 0)$ at $t = 0$ reads

$$\rho^A(\varphi^A, 0) = \text{Tr}_{BC} \rho(\varphi^A, 0) = U^A(\varphi^A)\rho^A(0)(U^A(\varphi^A))^+, \quad (8)$$
while the initial density matrix $\rho^B(\varphi^A, 0)$ remains the same,

$$\rho^B(\varphi^A, 0) = \text{Tr}_{AC} \rho(\varphi^A, 0) = \rho^B(0),$$  

(9)

which means that no parameters $\varphi_i$ may be detected in $\rho^B$ at $t = 0$.

Eqs. (8) and (9) are valid for any subsystems $A$ and $B$ no matter whether there is quantum interaction between them. Next, if the subsystems $A$ and $B$ do not interact, then no information about the state of the subsystem $A$ propagates into the subsystem $B$, so that no parameters $\varphi_i$ of the applied transformation $U^A(\varphi^A)$ may be detected in the subsystem $B$. In other words, the performed transformation will not effect on the state of the subsystem $B$.

However, information about the state of the subsystem $A$ propagates into the subsystem $B$ if there is quantum interaction between these subsystems. Owing to this interaction, some of the parameters of the unitary transformation $U^A(\varphi^A)$ may be transfered into the subsystem $B$. This interaction is represented by the unitary $t$-dependent transformation applied to the whole system and leads to the evolution of the density matrix:

$$\rho(\varphi^A, t) = V(t)\rho(\varphi^A, 0)V^+(t).$$  

(10)

In particular, if the evolution of our quantum system is governed by the stationary Hamiltonian $\mathcal{H}$, then

$$V(t) = \exp \left( -i\mathcal{H}t \right)$$

(11)

in accordance with the Liouville equation. In this case, the state of the subsystem $B$ is represented by the following reduced evolution density matrix:

$$\rho^B(\varphi^A, t) = \text{Tr}_{AC} \left( V(t) \left( U^A(\varphi^A) \otimes I_C \otimes I_B \right) \rho(0) \left( (U^A(\varphi^A))^+ \otimes I_C \otimes I_B \right) V^+(t) \right) \neq \rho^B(0).$$  

(12)

Hereafter we consider the initial density matrix $\rho(0)$ in the form of the tensor product of two initial density matrices:

$$\rho(0) = \rho^A(0) \otimes \rho^{CB}(0),$$

(13)

where $\rho^A(0) = \{\rho^A_{k_A,n_A}(0)\}$ and $\rho^{CB}(0) = \{\rho^{CB}_{k_B,n_B;k_C,n_C}(0)\}$ are the initial density matrices of the subsystem $A$ and of the joined subsystems $C$ and $B$, respectively. In these notations, the
indices with subscripts \( A \), \( B \) and \( C \) are related with the subsystems \( A \), \( B \) and \( C \) respectively.

Then we may write eq. (12) in terms of the matrix elements as follows:

\[
\rho^{B}_{ij}(\phi^{A}, t) = \sum_{i,A,k_A,n_A=1}^{N^A} \sum_{i,C,k_C,n_C=1}^{N^C} \sum_{k_B,n_B=1}^{N^B} V_{i_Ai_Ci_B;k_Ak_Ck_B}(t) \rho^{A}_{k_A;n_A}(\phi^{A}, 0) \times \rho^{CB}_{k_B;n_Cn_B}(0) V^{+}_{n_A n_C n_B;i_Ai_Cj_B}(t). \tag{14}
\]

For the subsequent analysis, we collect the \( t \)- and \( \phi^{A} \)-dependences in the rhs of eq. (14) into two different matrices and represent this equation in the following compact form

\[
\rho^{B}_{ij}(\phi^{A}, t) = \sum_{k,n=1}^{N^A} T_{ij;kn}(t) \rho^{A}_{k;n}(\phi^{A}, 0), \quad i, j = 1, \ldots, N^B, \tag{15}
\]

where independent on \( \phi^{A} \) elements \( T_{ij;nm} \) are defined by the formulas:

\[
T_{ij;nm}(t) = \sum_{i_A=1}^{N^A} \sum_{i_C,k_C,n_C=1}^{N^C} \sum_{k_B,n_B=1}^{N^B} V_{i_Ai_Ci_B;k_Ak_Ck_B}(t) \rho^{CB}_{k_B;n_Bn_C}(0) V^{+}_{n_A n_C n_B;i_Ai_Cj_B}(t), \tag{16}
\]

The important feature of eq. (15) is that the \( \phi^{A} \)-dependence appears only through the initial density matrix of the subsystem \( A \), \( \rho^{A}(\phi^{A}, 0) \), which becomes possible because of the tensor product initial state (13). Such separation of \( t \)- and \( \phi^{A} \)-dependences is impossible for other initial states, see Appendix C, Sec. V C.

Further, in order to determine the number of parameters transferred from \( A \) to \( B \), it is convenient to represent eq. (15) in a different form. The matter is that eq. (15), as well as the density matrices \( \rho^{A} \) and \( \rho^{B} \), is complex while the parameters \( \phi_i \) are real. Therefore we split eq. (15) into the real and imaginary parts and write the result in terms of the real and imaginary parts of the density matrices \( \rho^{A} \) and \( \rho^{B} \). For this purpose we introduce the following notations:

\[
T^1_{ij;nm} = T_{ij;nm}, \quad T^2_{ij;nm} = T_{ij;nm}, \quad T^3_{ij;mn} = T_{ij;mn} - T_{ij;nm}, \quad i, j = 1, \ldots, N^B, \quad n, m = 1, \ldots, N^A, \quad m > n. \tag{17}
\]
and write eq.(15) as the following three subsystems:

\[
\Re \rho_{ij}^B(\varphi^A, t) = \sum_{n,m=1}^{N_A} \left( \Re T_{ij;nm}^1 \Re \rho_{nm}^A(\varphi^A, 0) - \Im T_{ij;nm}^2 \Im \rho_{nm}^A(\varphi^A, 0) \right) + \sum_{n=1}^{N^A} \Re (T_{ij;nn}^1 \Re \rho_{nn}^A(\varphi^A, 0), \ i, j = 1, \ldots, N^B, \ j > i
\]

\[
\Im \rho_{ij}^B(\varphi^A, t) = \sum_{n,m=1}^{N_A} \left( \Im T_{ij;nm}^1 \Re \rho_{nm}^A(\varphi^A, 0) + \Re T_{ij;nm}^2 \Im \rho_{nm}^A(\varphi^A, 0) \right) + \sum_{n=1}^{N^A} \Im (T_{ij;nn}^1 \Re \rho_{nn}^A(\varphi^A, 0), \ i, j = 1, \ldots, N^B, \ j > i,
\]

\[
\rho_{ii}^B(\varphi^A, t) = \sum_{n,m=1}^{N_A} \left( \Re T_{ii;nm}^1 \Re \rho_{nm}^A(\varphi^A, 0) - \Im T_{ii;nm}^2 \Im \rho_{nm}^A(\varphi^A, 0) \right) + \sum_{n=1}^{N^A} T_{ii;nn} \Re \rho_{nn}^A(\varphi^A, 0), \ i = 1, \ldots, N^B - 1,
\]

where subsystem (18) is the real offdiagonal part of eq.(15), subsystem (19) is the imaginary offdiagonal part of the same equation, and subsystem (20) is the diagonal (real) part of eq.(15). We also take into account the relation Tr\(\rho^B = 1\), which leaves \(N^B - 1\) independent equations in subsystem (20). Therewith elements \(T^k_{ij;nm}\), \(k = 1, 2\), possess the following symmetry with respect to the indices \(i\) and \(j\):

\[
T^1_{ij;nm} = (T^1_{ji;nm})^*, \quad T^2_{ij;nm} = -(T^2_{ji;nm})^*, \quad T_{ij;nn} = T^*_{ji;nn},
\]

where star means the complex conjugate. Now, for any \(N \times N\) density matrix \(\rho\), we construct three vectors \(X(\rho), Y(\rho)\) and \(Z(\rho)\) with elements \(X_\alpha(\rho), Y_\alpha(\rho), \alpha = 1, \ldots, M, M \equiv N(N-1)/2\), and \(Z_i(\rho), i = 1, \ldots, N-1\) defined as

\[
X_{\sum_{l=1}^{N-1}(N-l)+j-i}(\rho) = \Re \rho_{ij}, \ i, j = 1, \ldots, N, \ j > i,
\]

\[
Y_{\sum_{l=1}^{N-1}(N-l)+j-i}(\rho) = \Im \rho_{ij}, \ i, j = 1, \ldots, N, \ j > i,
\]

\[
Z_i(\rho) = \rho_{ii}, \ i = 1, \ldots, N-1.
\]
Then eqs. (18–20) get the following forms:

\[
X_\alpha(\rho^B(\varphi^A, t)) = \sum_{\beta=1}^{M^A} \left( T_{\alpha\beta}^{11}(t) X_\beta(\rho^A(\varphi^A, 0)) + T_{\alpha\beta}^{12}(t) Y_\beta(\rho^A(\varphi^A, 0)) \right) + \sum_{n=1}^{N^A-1} \left( T_{\alpha\beta}^{13}(t) - T_{\alpha\beta}^{13}(t) \right) Z_n(\rho^A(\varphi^A, 0)) + T_{\alpha\beta}^{13}(t), \quad \alpha = 1, \ldots M^B,
\]

\[
Y_\alpha(\rho^B(\varphi^A, t)) = \sum_{\beta=1}^{M^A} \left( T_{\alpha\beta}^{21}(t) X_\beta(\rho^A(\varphi^A, 0)) + T_{\alpha\beta}^{22}(t) Y_\beta(\rho^A(\varphi^A, 0)) \right) + \sum_{n=1}^{N^A-1} \left( T_{\alpha\beta}^{23}(t) - T_{\alpha\beta}^{23}(t) \right) Z_n(\rho^A(\varphi^A, 0)) + T_{\alpha\beta}^{23}(t), \quad \alpha = 1, \ldots M^B,
\]

\[
Z_i(\rho^B(\varphi^A, t)) = \sum_{\beta=1}^{M^A} \left( T_{\alpha\beta}^{31}(t) X_\beta(\rho^A(\varphi^A, 0)) + T_{\alpha\beta}^{32}(t) Y_\beta(\rho^A(\varphi^A, 0)) \right) + \sum_{n=1}^{N^A-1} \left( T_{\alpha\beta}^{33}(t) - T_{\alpha\beta}^{33}(t) \right) Z_n(\rho^A(\varphi^A, 0)) + T_{\alpha\beta}^{33}(t), \quad i = 1, \ldots, N^B - 1,
\]

where we introduce the matrices \( T^{ij} \) with the following elements:

\[
T_{ij}^{11} = \sum_{l=1}^{(N^B - 1) + j - i} \sum_{n=1}^{(N^A - 1) + m - n} \text{Re} T_{ij;nm}^{1}, \quad T_{ij}^{12} = -\text{Im} T_{ij;nm}^{2}, \quad T_{ij}^{13} = \text{Re} T_{ij;nm}^{3}, \quad T_{ij}^{21} = \text{Im} T_{ij;nm}^{1}, \quad T_{ij}^{22} = \text{Re} T_{ij;nm}^{2}, \quad T_{ij}^{23} = -\text{Im} T_{ij;nm}^{3}, \quad T_{ij}^{31} = \text{Re} T_{ij;nm}^{1}, \quad T_{ij}^{32} = \text{Im} T_{ij;nm}^{2}, \quad T_{ij}^{33} = \text{Re} T_{ij;nm}^{3}.
\]

Thus \( T^{11}, T^{12}, T^{21}, T^{22} \) are \( M^B \times M^A \) matrices, \( T^{13} \) and \( T^{23} \) are \( M^B \times N^A \) matrices, \( T^{31} \) and \( T^{32} \) are \( (N^B - 1) \times M^A \) matrices and \( T^{33} \) is \( (N^B - 1) \times N^A \) matrix. Next, we construct the column-vectors \( \hat{X}(\rho^A) \) and \( \hat{X}(\rho^B) \) of \( D^A = (N^A)^2 - 1 \) and \( D^B = (N^B)^2 - 1 \) elements, respectively:

\[
\hat{X}(\rho^A(\varphi^A, 0)) = \begin{pmatrix} X(\rho^A(\varphi^A, 0)) \\ Y(\rho^A(\varphi^A, 0)) \\ Z(\rho^A(\varphi^A, 0)) \end{pmatrix}, \quad \hat{X}(\rho^B(\varphi^A, t)) = \begin{pmatrix} X(\rho^B(\varphi^A, t)) \\ Y(\rho^B(\varphi^A, t)) \\ Z(\rho^B(\varphi^A, t)) \end{pmatrix}, \quad \hat{X}(\rho^B(\varphi^A, t)) = \begin{pmatrix} X(\rho^B(\varphi^A, t)) \\ Y(\rho^B(\varphi^A, t)) \\ Z(\rho^B(\varphi^A, t)) \end{pmatrix},
\]

and the column vector of \( D^B \) elements \( \hat{T}^0(t) \):

\[
\hat{T}^0(t) = \begin{pmatrix} T^{10}(t) \\ T^{20}(t) \\ T^{30}(t) \end{pmatrix}.
\]
where the elements of the vectors $T^{k0}(t)$, $k = 1, 2, 3$, are defined as $T_{\alpha}^{i0}(t) = T_{\alpha,N}^{33}$, $i = 1, 2$, $\alpha = 1, \ldots, M^B$, and $T_{i}^{30}(t) = T_{i,N}^{33}$, $i = 1, \ldots, N^B - 1$. We also introduce the $D^B \times D^A$ block matrix $\hat{T}(t)$:

$$
\hat{T}(t) = \begin{pmatrix}
T^{11}(t) & T^{12}(t) & T^{13}(t) \\
T^{21}(t) & T^{22}(t) & T^{23}(t) \\
T^{31}(t) & T^{32}(t) & T^{33}(t)
\end{pmatrix},
$$

where the elements of the matrices $\hat{T}^{k3}(t)$ are defined as $\hat{T}_{\alpha n}^{k3} = T_{\alpha n}^{k3} - T_{\alpha N^A}^{k3}$, $k = 1, 2$, $\hat{T}_{in}^{33} = T_{in}^{33} - T_{i N^A}^{33}$, $\alpha = 1, \ldots, M^B$, $i = 1, \ldots, N^B - 1$, $n = 1, \ldots, N^A - 1$. Now system (23) may be represented as the following single vector equation:

$$
\dot{X}(\rho^B(\varphi^A, t)) = \hat{T}(t)\dot{X}(\rho^A(\varphi^A, 0)) + \hat{T}^0(t).
$$

Thus, if $\hat{T}$ is the square matrix ($D^B = D^A$) and ran $\hat{T} = D^A$ (i.e. the rank equals the length of the vector $\dot{X}(\rho^A(\varphi^A, 0))$), then the matrix $\rho^A(\varphi^A, 0)$ can be uniquely reconstructed from the matrix $\rho^B(\varphi^A, t)$ [20]. In other words, all the parameters encoded into the density matrix $\rho^A(\varphi^A, 0)$ can be transferred into the subsystem $B$ (the complete information transfer).

However, it may be shown that the maximal number $\tilde{D}^A$ of arbitrary parameters encoded into $\rho^A(\varphi^A, 0)$ is less than the length of the column $\dot{X}$. In fact, we may represent an arbitrary matrix $\rho^A(0)$ in the form

$$
\rho^A(0) = W\Lambda^A W^+, \ W \in SU(N^A),
$$

where $\Lambda^A = \text{diag}\{\lambda_1^A, \ldots, \lambda_{N^A}^A\}$ is the diagonal matrix of the eigenvalues and $W$ is the matrix of eigenvectors of $\rho^A(0)$. Then we may write

$$
\rho^A(\varphi^A, 0) = U^A(\varphi^A) W \Lambda^A W^+ (U^A(\varphi^A))^+ = \tilde{U}^A(\tilde{\varphi}^A) \Lambda^A (\tilde{U}^A(\tilde{\varphi}^A))^+ \equiv \rho^A(\tilde{\varphi}^A, 0),
$$

where $\tilde{U}^A(\tilde{\varphi}^A) = U^A(\varphi^A) W \in SU(N^A)$, and $\tilde{\varphi}^A = \tilde{\varphi}^A(\varphi^A) = \{\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{\tilde{D}^A}\}$ is the set of redefined (in terms of $\varphi_i$) parameters of the group $SU(N^A)$. Now we may calculate the number of arbitrary parameters $\tilde{\varphi}_i$ encoded into the matrix $\rho^A(\tilde{\varphi}^A, 0)$ as follows. The maximal possible number of the independent real parameters in the $N^A \times N^A$ dimensional matrix $\rho^A$ is $(N^A)^2 - 1$. But $N^A - 1$ of them are related with the eigenvalues of the density matrix (the $N^A - 1$ diagonal elements of the matrix $\Lambda^A$ in eq. (30) where we take into account
the relation \( \text{Tr} \rho^A(0) = \sum_{i=1}^{N^A} \lambda_i^A \equiv 1 \). These elements are fixed since the density matrix \( \rho^A(0) \) is given initially. Consequently, we stay with

\[
\tilde{D}^A = (N^A)^2 - 1 - (N^A - 1) = N^A(N^A - 1)
\]

(31)

arbitrary real parameters in the density matrix \( \rho^A(\varphi^A, 0) \). These parameters are related with the same number of the parameters \( \tilde{\varphi}_i \) in the transformation \( \tilde{U}(\tilde{\varphi}^A) \). Thus, only \( \tilde{D}^A \) arbitrary parameters of the group \( SU(N^A) \) may be encoded into the density matrix \( \rho^A(\tilde{\varphi}^A) \).

Hereafter we consider only the diagonal matrix \( \rho^A(0) \) without loss of generality and do not write the tilde over \( \varphi^A \).

The determined above parameter \( \tilde{D}^A \) indicates the maximal possible number of parameters \( \varphi_i \) encoded into the matrix \( \rho^A(\varphi^A, 0) \). However, this maximum is not always achievable. In fact, if all \( \lambda_i \) are different, then the number of parameters encoded into the subsystem \( A \) is, really, \( \tilde{D}^A \), i.e. \( E^{AA} = \tilde{D}^A \). Now we assume that there is one \( K \)-fold eigenvalue, \( K \leq N^A \). Then \( \Lambda^A \) is invariant with respect to the proper group \( SU(K) \) which possesses \( d = K(K - 1) \) parameters. These \( d \) parameters may not be encoded into \( \rho^A \). Consequently, the number of encoded parameters \( E^{AA} \) becomes less then \( \tilde{D}^A \):

\[
E^{AA} = \tilde{D}^A - K(K - 1).
\]

(32)

Formula (32) may be readily extended to the case of \( Q \) roots with multiplicities \( K_i > 1 \), \( i = 1, \ldots, Q \):

\[
E^{AA} = \tilde{D}^A - \sum_{i=1}^{Q} K_i(K_i - 1), \quad \sum_{i=1}^{Q} K_i \leq N^A.
\]

(33)

This formula for \( E^{AA} \) will be used in Sec.III.

We see that the measure of the informational correlation is defined by the eigenvalues of the density matrix \( \rho^A(0) \) rather then by its elements themselves, which is similar to the unitary invariant discord [27]. It is interesting to note that the value \( \tilde{D}^A \) (31) is less then the length of the vector \( \tilde{X}(\rho^A(\varphi^A, 0)) \) which is \( D^A \) (4). This means that not all elements of the density matrix \( \rho^A(\varphi^A, 0) \) must be transfered into the matrix \( \rho^B(\varphi^A, t) \) in order to detect all \( \tilde{D}^A \) parameters \( \varphi_i \) in the subsystem \( B \). Therefore, the complete information transfer [20], in principle, is not required in order to transfer the maximal possible number of arbitrary parameters \( \varphi_i \) from the subsystem \( A \) into the subsystem \( B \).
a. Zero informational correlations, \( E^{AB} = E^{BA} = 0 \), in system with tensor product initial state \((13)\). Thus, we measure the informational correlation by the number \( E^{AB} \) of arbitrary parameters \( \varphi_i \) of the unitary transformation \( U^A(\varphi^A) \subset SU(N_A) \) which may be deduced from the analysis of the matrix \( \rho^B(\varphi^A, t) \) describing the state of the subsystem \( B \). Notice that, if all \( \lambda_i \) are the same, i.e. \( \Lambda^A \) is proportional to the unit matrix, then \( \rho^A(\varphi^A, 0) \) is also proportional to the unit matrix and consequently no parameters \( \varphi_i \) may be encoded into \( \rho^A(\varphi^A, 0) \). Therefore, no parameters of the unitary transformation \( U^A \) may be transferred to the subsystem \( B \), i.e. \( E^{AB} = 0 \). However, the parameters might be transferred in the opposite direction (from \( B \) to \( A \)). In fact, let us assume that

\[
\rho^{CB}(0) = \rho^C(0) \otimes \rho^B(0)
\]

for simplicity. If not all eigenvalues of the initial density matrix \( \rho^B(0) \) are the same (i.e. the matrix \( \rho^B(0) \) is not proportional to the identity matrix), then at least some of the parameters of the unitary transformation performed on the subsystem \( B \) may be encoded into the density matrix \( \rho^B(\varphi^B, 0) \) and then transferred to the subsystem \( A \) (although this still depends on the \( t \)-evolution operator), i.e. \( E^{BA} \neq 0 \). Thus, no parameters may be transferred in both directions if only both \( \rho^A(0) \) and \( \rho^B(0) \) are proportional to the identity matrices. Emphasize that this conclusion holds only for the tensor product initial state \( \rho(0) = \rho^A(0) \otimes \rho^C(0) \otimes \rho^B(0) \). Note that the unitary invariant discord [27] is zero in such systems as well. The zero informational correlations in systems with more general initial states are not considered in this paper.

A. Calculation of the parameters \( E^{AA} \) and \( E^{AB} \)

As mentioned above, we take the diagonal initial density matrix \( \rho^A(0) \),

\[
\rho^A(0) \equiv \Lambda^A, \quad \lambda_1^A \geq \lambda_2^A \geq \cdots \geq \lambda_{NA}^A,
\]

without loss of generality. Obviously, \( E^{AB} \) may not exceed \( E^{AA} \), the number of parameters \( \varphi_i \) encoded into the density matrix \( \rho^A(\varphi^A, 0) \), \( E^{AA} \leq \bar{D} \). In turn, \( E^{AA} \) is uniquely defined by the multiplicity of the eigenvalues of the density matrix \( \rho^A(0) \). Let us calculate the informational correlation \( E^{AB} \) following its definition as the number of arbitrary parameters transferred from the subsystem \( A \) to the subsystem \( B \). This number equals to the number
of parameters $\varphi_i$ which might be found from vector eq.(28) with known lhs (the matrix $\rho^B$ in the lhs must be determined by the local measurements). This equation is a matrix transcendental equation, whose global solution may not be given analytically. However, we may define the number of different detectable parameters in the close neighborhood of any fixed point $\varphi^A \in G^A$. This is the number of functionally independent elements in the vector $\hat{X}(\rho^B(\varphi^A, t))$, which, in turn, equals to the rank of the Jacobian matrix,

$$J(\rho^B(\varphi^A, t)) = \frac{\partial(\hat{X}_1(\rho^B(\varphi^A, t)), \ldots, \hat{X}_{(NB)^2-1}(\rho^B(\varphi^A, t)))}{\partial(\varphi_1, \ldots, \varphi_{DA})},$$  

(36)

calculated in the above fixed point $\varphi^A \in G^A$. Therefore, we determine the informational correlation as

$$E^{AB}(\varphi^A, t) = \text{ran } J(\rho^B(\varphi^A, t)).$$  

(37)

Similarly, the introduced above $E^{AA}$ may be determined as the rank of the Jacobian matrix $J(\rho^A(\varphi^A, t))$,

$$J(\rho^A(\varphi^A, t)) = \frac{\partial(\hat{X}_1(\rho^A(\varphi^A, t)), \ldots, \hat{X}_{(NA)^2-1}(\rho^A(\varphi^A, t)))}{\partial(\varphi_1, \ldots, \varphi_{DA})},$$  

(38)

as follows:

$$E^{AA}(\varphi^A) = \text{ran } J(\rho^A(\varphi^A, 0)).$$  

(39)

Moreover, we may readily write the relations between two Jacobian matrices $J(\rho^B(\varphi^A, t))$ and $J(\rho^A(\varphi^A, 0))$ differentiating eq.(28) with respect to the parameters $\varphi_i$:

$$J(\rho^B(\varphi^A, t)) = \hat{T}(t)J(\rho^A(\varphi^A, 0)).$$  

(40)

From eq.(40) in virtue of eq.(39) it follows that

$$E^{AB}(\varphi^A, t) \leq \min \left(\text{ran } \hat{T}(t), E^{AA}(\varphi^A)\right), \quad \varphi^A \in G^A.$$

(41)

All in all, it is demonstrated that the informational correlation $E^{AB}(\varphi^A, t)$ depends on two factors.

1. The number of arbitrary parameters $\varphi_i$ which might be encoded into the density matrix $\rho^A(\varphi^A, 0)$ (quantity $E^{AA}$).
2. The number of arbitrary parameters which can be transferred from the subsystem \( A \) to the subsystem \( B \). If the information is completely transferred, then \( E^{AB} = E^{AA} \). Otherwise \( E^{AB} \leq E^{AA} \).

Note, that \( E^{AA} \) defined by eq.(39) does not really depend on \( \varphi^A \) (if only \( \varphi^A \in G^A \)) owing to the mutually unique map \( \rho^A(\varphi^A,0) \leftrightarrow \varphi^A \). This unique map means that, for a given set of eigenvalues of \( \rho^A(0) \), we have the appropriate number of the independent elements of the vector \( \varphi^A \), which uniquely parametrize the matrix \( \rho^A(\varphi^A,0) \) (8). This number equals to the number of functionally independent elements in the vector \( X(\rho^A(\varphi^A,0)) \), which equals to the rank of the Jacobian matrix \( J(\rho^A(\varphi^A,0)) \) and must be the same for all \( \varphi^A \) at least inside of \( G^A \), where \( J(\rho^A(\varphi^A,0)) \) is well defined.

Regarding the informational correlation \( E^{AB}(\varphi^A,t) \) given by eq.(37), it really depends on \( \varphi^A \) in the case of general initial state \( \rho(0) \), as shown in Appendix C, Sec.VC. However, regarding initial state (13), we may readily conclude that \( E^{AB} \) does not depend on \( \varphi^A \), \( \varphi^A \in G^A \). The reason is that initial state (13) results in the separation of \( t \)- and \( \varphi^A \)-dependence in eq.(40) relating \( J(\rho^B(\varphi^A,t)) \) with \( J(\rho^A(\varphi^A,0)) \). It is important that \( \varphi^A \)-dependence is concentrated in \( J(\rho^A(\varphi^A,0)) \). Therefore, multiplying \( J(\rho^A(\varphi^A,0)) \) by \( \hat{T}(t) \) we only recombine rows of the matrix \( J(\rho^A(\varphi^A,0)) \). Consequently, the rank of the resulting matrix \( J(\rho^B(\varphi^A,t)) \) does not depend on \( \varphi^A \), but it depends on \( t \). For this reason, we will not write \( \varphi^A \) in the arguments of \( E^{AB} \) and \( E^{AA} \) (except for the Appendix C, Sec.VC), i.e.

\[
E^{AA}(\varphi^A) \equiv E^{AA}, \quad E^{AB}(\varphi^A,t) \equiv E^{AB}(t).
\] (42)

Two simple examples of informational correlations in the 4-node spin-1/2 chain will be considered in Sec.III.

b. Local initial unitary transformation of subsystem \( C \cup B \) and \( E^{AB} \). While the initial local transformations of the subsystem \( A \) do not effect the informational correlation (they only lead to the redefinition of the independent parameters \( \varphi_i \)) the initial local unitary transformation of the subsystem \( C \cup B \) may change the informational correlation \( E^{AB} \). In this paper we study only the diagonal initial state of the subsystem \( C \cup B \), \( \rho^{CB}(0) = \text{diag}(\lambda_1^{CB}, \ldots, \lambda_N^{CB}) \), and demonstrate the effect of the unitary transformation \( U^{CB} \) considering the particular examples in Sec.III.A.3 and in the end of Sec.III.B.2.

c. Normalization of informational correlation. As mentioned above, the informational correlation \( E^{AB}(t) \) in systems with tensor product initial state (13) may not exceed the
maximal possible number $\tilde{D}^A$ of the parameters $\varphi_i$ encoded into the density matrix $\rho^A(\varphi^A, 0)$. To indicate the discrepancy between $E^{AB}(t)$ and $\tilde{D}^A$, we introduce the normalized measure $E_{\text{norm}}^{AB}$ as the ratio

$$E_{\text{norm}}^{AB} = \frac{E^{AB}}{\tilde{D}^A}. \quad (43)$$

Thus, the maximal value of $E_{\text{norm}}^{AB}$ is unit at least in the quantum systems with the tensor product initial state $|13\rangle$, for which inequality (41) holds. This is, generally speaking, not valid in the case of an arbitrary initial state, see Appendix C, Sec. V C. Similarly, we normalize $E^{AA}$:

$$E_{\text{norm}}^{AA} = \frac{E^{AA}}{\tilde{D}^A}. \quad (44)$$

**B. Non-reducible informational correlation**

It is noted above that the informational correlation is sensitive to the initial unitary transformation of the subsystem $B$. Moreover, it is simple to show that, unlike the entanglement and discord, the informational correlation $E^{AB}(t)$ determined at some instant $t$ may be decreased by the local unitary transformation of the subsystem $B$ at the same instant $t$. In fact, let us represent the density matrix $\rho^B(\varphi, t)$ in the form

$$\rho^B(\varphi^A, t) = U^B(\varphi^A, t)\Lambda^B(\varphi^A, t)(U^B(\varphi^A, t))^+, \quad (45)$$

$$\Lambda^B = \text{diag}(\lambda_1, \ldots, \lambda_{N^B}), \quad (46)$$

where $\Lambda^B$ is the diagonal matrix of the eigenvalues and $U^B$ is composed of the eigenvectors of the matrix $\rho^B(\varphi^A, t)$. Note that we do not write the superscript $B$ in the notation $\lambda_i$ to defer these eigenvalues from the eigenvalues of the initial density matrix $\rho^B(0)$, which will be considered in Sec. III. Representation (45) suggests us to split the whole set of transferred parameters into two subsets. The first subset collects those parameters which may be detected from the analysis of the matrix $U^B$ at instant $t$ (the subset $\varphi^U$), while the second one collects those parameters which may be detected from the analysis of the matrix $\Lambda^B$ at the same time instant (the subset $\varphi^A$). Therewith some of the parameters might appear in both subsets, other parameters might not appear in these subsets at all, so that $E^{AB}$ equals the number of different parameters in two subsets $\varphi^U$ and $\varphi^A$. Moreover, eq. (45) shows that
one can eliminate parameters $\varphi^U$ from the reduced density matrix $\rho^B(\varphi^A, t)$ at instant $t$ performing the local unitary transformation $(U^B(\varphi^A, t))^+$, which transforms the matrix $\rho^B$ to the diagonal form,

$$
\rho^B(\varphi^A, t) \to \Lambda^B(\varphi^A, t),
$$

thus reducing the set of transferred parameters to $\varphi^A$. This means that the informational correlation $E^{AB}$ is also reduced to $E^{AB;\text{min}}(\varphi^A, t)$ which equals the number of parameters in $\varphi^A$. Of course, by definition, this part of the informational correlation may not be decreased by any local unitary transformation of the subsystem $B$ at instant $t$. We refer to $E^{AB;\text{min}}(\varphi^A, t)$ as the non-reducible informational correlation. Obviously, the following upper estimation is valid:

$$
E^{AB;\text{min}}(\varphi^A, t) \leq N^B - 1,
$$

because there are only $N^B - 1$ independent eigenvalues owing to the relation $\text{Tr}\rho^B = 1$. Similarly to $E^{AB}$ and $E^{AA}$ (see eqs.(37) and (39) respectively), the non-reducible informational correlation $E^{AB;\text{min}}(\varphi^A, t)$ may be calculated as the rank of the Jacobian matrix $J^B_\Lambda(\varphi^A, t)$,

$$
J^B_\Lambda(\varphi^A, t) = \frac{\partial (\lambda_1(\varphi^A, t), \ldots, \lambda_{N^B-1}(\varphi^A, t))}{\partial (\varphi_1, \ldots, \varphi_{\tilde{D}^A})},
$$

i.e. by the formula

$$
E^{AB;\text{min}}(\varphi^A, t) = \text{ran} J^B_\Lambda(\varphi^A, t),
$$

where $\lambda_i(\varphi^A, t), i = 1, \ldots, N^B - 1$ are the independent eigenvalues. This correlation may be also normalized:

$$
E^{AB;\text{min}}_{\text{norm}}(\varphi^A, t) = \frac{E^{AB;\text{min}}(\varphi^A, t)}{D^A}.
$$

Let us give more applicable form to eq.(50) in terms of the principal minors of the matrix $\rho^B(\varphi^A, t)$. We start with the characteristic equation for the matrix $\rho^B(\varphi^A, t)$:

$$
\prod_{i=1}^{N^B}(\lambda - \lambda_i) = 0,
$$

and

$$
\lambda^{N^B} - \lambda^{N^B-1} + \sum_{i=0}^{N^B-2} a_i(\varphi^A, t)\lambda^i = 0,
$$

where $a_i(\varphi^A, t), i = 0, \ldots, N^B - 2$ are the coefficients of the characteristic polynomial.
where

\[ a_0(\varphi^A, t) = (-1)^{N^B} \det \rho^B(\varphi^A, t), \quad a_i(\varphi^A, t) = (-1)^{N^B-i} S_{N^B-i}(\varphi^A, t), \quad i = 1, \ldots, N^B - 2 \]

and \( S_j \) means the sum of all the \( i \)th-order principal minors of the matrix \( \rho^B(\varphi^A, t) \). In eq. (53), we take into account that \( \text{Tr} \rho^B = 1 \). Differentiating eq. (53) with respect to the parameter \( \varphi_k \) and solving the resulting equation for \( \frac{\partial \lambda}{\partial \varphi_k} \) we obtain:

\[
\frac{\partial \lambda}{\partial \varphi_k} = -\sum_{i=0}^{N^B-2} \frac{\partial a_i(\varphi^A, t)}{\partial \varphi_k} \lambda^i \left( N^B \lambda^{N^B-1} - (N^B - 1)\lambda^{N^B-2} + \sum_{i=1}^{N^B-2} i a_i(\varphi^A, t) \lambda^{i-1} \right) .
\]

Therefore, for the matrix \( J^B_\Lambda(\varphi^A, t) \), one has

\[
J^B_\Lambda(\varphi^A, t) = \frac{1}{J_0(\varphi^A, t)} \begin{pmatrix}
\sum_{i=0}^{N^B-2} \frac{\partial a_i(\varphi^A, t)}{\partial \varphi_1} \lambda_1^i(\varphi^A, t) & \cdots & \sum_{i=0}^{N^B-2} \frac{\partial a_i(\varphi^A, t)}{\partial \varphi_D^A} \lambda_1^i(\varphi^A, t) \\
\cdots & \cdots & \cdots \\
\sum_{i=0}^{N^B-2} \frac{\partial a_i(\varphi^A, t)}{\partial \varphi_1} \lambda_{N^B-1}^i(\varphi^A, t) & \cdots & \sum_{i=0}^{N^B-2} \frac{\partial a_i(\varphi^A, t)}{\partial \varphi_D^A} \lambda_{N^B-1}^i(\varphi^A, t)
\end{pmatrix} = \hat{\Lambda}^B(\varphi^A, t)H(\varphi^A, t),
\]

where

\[
J_0(\varphi^A, t) = (-1)^{N^B-1} \prod_{j=1}^{N^B-1} \left( N^B \lambda_j^{N^B-1}(\varphi^A, t) - (N^B - 1)\lambda_j^{N^B-2}(\varphi^A, t) + \sum_{i=1}^{N^B-2} i a_i(\varphi^A, t) \lambda_j^{i-1}(\varphi^A, t) \right) \neq 0,
\]

while \( \hat{\Lambda}^B \) and \( H \) are the \( (N^B - 1) \times (N^B - 1) \) and \( (N^B - 1) \times D^A \) matrices respectively:

\[
\hat{\Lambda}^B(\varphi^A, t) = \begin{pmatrix}
1 & \lambda_1(\varphi^A, t) & \cdots & \lambda_1^{N^B-2}(\varphi^A, t) \\
\cdots & \cdots & \cdots & \cdots \\
1 & \lambda_{N^B-1}(\varphi^A, t) & \cdots & \lambda_{N^B-1}^{N^B-2}(\varphi^A, t)
\end{pmatrix},
\]

\[
H(\varphi^A, t) = \begin{pmatrix}
\frac{\partial a_0(\varphi^A, t)}{\partial \varphi_1} & \cdots & \frac{\partial a_0(\varphi^A, t)}{\partial \varphi_D^A} \\
\cdots & \cdots & \cdots \\
\frac{\partial a_{N^B-2}(\varphi^A, t)}{\partial \varphi_1} & \cdots & \frac{\partial a_{N^B-2}(\varphi^A, t)}{\partial \varphi_D^A}
\end{pmatrix}.
\]
If all \( \lambda_i, i = 1, \ldots, N^B - 1 \), are different and nonzero, then \( \det \hat{\Lambda}^B \neq 0 \) and

\[
E^{AB; \text{min}}(\varphi^A, t) = \text{ran} J^B_\Lambda(\varphi^A, t) = \text{ran} H(\varphi^A, t). \tag{60}
\]

It is not difficult to prove that eq. (60) holds even for the case of multiple and/or zero eigenvalues \( \lambda_i, i = 1, \ldots, N^B - 1 \), which is shown in the Appendix B, Sec V.B.

Emphasize that, unlike \( E^{AB} \) and \( E^{AA} \), the non-reducible correlation \( E^{AB; \text{min}}(\varphi^A, t) \) depends on \( \varphi^A \) even for the product initial state \( |\psi\rangle \). This means that there might be such points \( \varphi^A_1 \) and \( \varphi^A_2 \) in the space of \( \varphi^A \) that \( E^{AB; \text{min}}(\varphi^A_1, t) \neq E^{AB; \text{min}}(\varphi^A_2, t) \). However, there might exist such \( g \subset G^A \) and appropriate time intervals \( T_{1i} < t < T_{2i}, i = 1, 2, \ldots \) that \( E^{AB; \text{min}} \) is independent on \( \varphi^A \in g \) inside of the above time intervals. In this case, it might be reasonable to write \( E^{AB; \text{min}}(g, t) \) instead of \( E^{AB; \text{min}}(\varphi^A, t) \), see Secs III A 4 and III B 3.

The non-reducible correlation may be also normalized:

\[
E^{AB; \text{min}}(\varphi^A, t) = \frac{E^{AB; \text{min}}(\varphi^A, t)}{D}. \tag{61}
\]

A particular example of single parameter in the subset \( \varphi^A \) is considered analytically in Sec III A 4. In general, the calculation of \( E^{AB; \text{min}} \) is a numerically solvable problem, which is partially discussed in the Appendix C, Sec V.C and in two examples of Sec III B 3.

C. Removable informational correlation

The non-reducible informational correlation is the analogy of classical correlations in the calculation of discord \cite{6–8}. However, we do not identify the non-reducible informational correlation with the classical part of the informational correlation, because \( E^{AB; \text{min}} \) might be related with some quantum effects as well. Now, having \( E^{AB} \) and \( E^{AB; \text{min}} \), we define the removable informational correlation as the increment

\[
\Delta E^{AB}(\varphi^A, t) = E^{AB}(t) - E^{AB; \text{min}}(\varphi^A, t). \tag{62}
\]

This correlation may be normalized as

\[
\Delta E^{AB}_{\text{norm}}(\varphi^A, t) = \frac{\Delta E^{AB}(\varphi^A, t)}{D^A} = E^{AB}_{\text{norm}}(t) - E^{AB; \text{min}}_{\text{norm}}(\varphi^A, t). \tag{63}
\]

Since the removable informational correlation is defined by those parameters \( \varphi_i \) which may be detected only from the matrix \( U^B(\varphi^A, t) \) at instant \( t \), it may be considered as an analogy.
of the quantum correlations in the calculation of discord [6–8] (in other words, $\Delta E^{AB}$ is the analog of discord). However, we do not state that this measure characterizes the pure quantum effects.

III. FOUR-NODE HOMOGENEOUS SPIN-1/2 CHAIN

We consider the spin-1/2 system of four nodes whose evolution is governed by the following $XY$ Hamiltonian

$$\mathcal{H} = -\sum_{i=1}^{3} \frac{d}{2}(I_i^+ I_{i+1}^- + I_i^- I_{i+1}^+),$$

with the nearest neighbor interaction. Here $d$ is the coupling constant between the nearest neighbors, $I_i^\pm = I_{x,i} \pm iI_{y,i}$ and $I_{\alpha,i}$, $\alpha = x, y, z$, are the projection operators of the total spin angular momentum. We put $d = 1$ without loss of generality. Hamiltonian (64) must be used in the evolution operator $V(t) \in SU(16)$ defined by eq. (11).

A. One-node subsystems $A$ and $B$

Let the subsystems $A$ and $B$ be represented by the first and the last nodes respectively, while the subsystem $C$ consist of two middle nodes. Thus $N^A = N^B = 2$ and $N^C = 4$, so that $SU(2)$ is the group of unitary transformations of the subsystem $A$.

We consider the initial density matrix having the structure (13) with the tensor product matrix $\rho^{CB}(0)$ given by eq. (34), so that the initial density matrix $\rho(0)$ reads

$$\rho(0) = \rho^A(0) \otimes \rho^C(0) \otimes \rho^B(0).$$

In accordance with Sec II A we take the diagonal initial density matrix $\rho^A(0)$,

$$\rho^A(0) = \text{diag}(\lambda^A, 1 - \lambda^A),$$

and choose the diagonal matrices $\rho^B(0)$ and $\rho^C(0)$ as well:

$$\rho^B(0) = \text{diag}(\lambda^B, 1 - \lambda^B), \quad \rho^C(0) = \text{diag}(\lambda_1^C, \lambda_2^C, \lambda_3^C, 1 - \lambda_1^C - \lambda_2^C - \lambda_3^C).$$
1. Number of parameters encoded into the subsystem $A$, $E^{AA}$

The general form of $SU(2)$ transformation reads \[ U^A(\varphi^A) = \begin{pmatrix} \cos \varphi_1 & -e^{-i\varphi_2} \sin \varphi_1 \\ e^{i\varphi_2} \sin \varphi_1 & \cos \varphi_1 \end{pmatrix} e^{i\sigma_3 \varphi_3}, \quad \sigma_3 = \text{diag}(1, -1), \] \[ G^A : \ 0 < \varphi_1, \varphi_3 < \frac{\pi}{2}, \ 0 < \varphi_2 < 2\pi, \] where we do not consider the boundary values of the parameters $\varphi_i$, $i = 1, 2, 3$. Since $\rho^A(0)$ is diagonal, the parameter $\varphi_3$ does not appear in $\rho^A(\varphi^A, 0)$. Therefore only two parameters can be encoded into the density matrix $\rho^A(\varphi^A, 0)$, i.e. $D = 2$. The same result may be obtained directly using the formula (39) with the Jacobian matrix calculated by the formula (38), where the vector $X(\rho^A(\varphi^A, 0))$, defined by eqs. (22, 25), reads

\[ \dot{X}(\rho^A(\varphi^A, 0)) = \{ \text{Re} \rho_{12}^A(\varphi, 0), \text{Im} \rho_{12}^A(\varphi^A, 0), \rho_{11}^A(\varphi^A, 0) \}^T = \begin{pmatrix} (2\lambda^A - 1) \sin \varphi_1 \cos \varphi_1 \cos \varphi_2, \\ -(2\lambda^A - 1) \sin \varphi_1 \cos \varphi_1 \sin \varphi_2, \\ \frac{1}{2} \{1 + (2\lambda^A - 1) \cos(2\varphi_1)\} \end{pmatrix}^T. \]

Here the superscript $T$ means the matrix transposition. Thus, the parameter $\varphi_1$ appears in three entries of the column $\dot{X}$, while $\varphi_2$ appear only in two entries. This observation suggests us to consider the parameter $\varphi_1$ as a more reliable parameter for establishing the informational correlations. In fact, it will be shown in Sec. III A 4 that namely this parameter is responsible for the non-reducible informational correlation between our subsystems $A$ and $B$ (i.e. between the first and the last nodes of our 4-node spin chain). All in all we have

\[ E^{AA} = \begin{cases} 2, & \lambda^A \neq \frac{1}{2}, \\ 0, & \lambda^A = \frac{1}{2}. \end{cases} \] \[ (71) \]

2. Relation between the rank of $\dot{T}$ and the informational correlation $E^{AB}$

Now we turn to the whole quantum system $A \cup C \cup B$ and consider the evolution of the density matrix $\rho(\varphi^A, t)$ of this system,

\[ \rho(\varphi^A, t) = V(t) \rho(\varphi^A, 0) V^+(t), \]

\[ (72) \]
where
\[ \rho(\varphi^A, 0) = \left( \tilde{U}^A(\varphi^A) \times I_4 \times I_2 \right) \rho(0) \left( (\tilde{U}^A(\varphi^A))^+ \times I_4 \times I_2 \right), \quad (73) \]
the evolution operator \( V(t) \in SU(16) \) is given by eq. (11), and \( \rho(0) \) is given by eq. (65). To calculate \( E^{AB} \) we refer to eqs. (36 – 40) and obtain all matrices used in these equations. The vector \( X(\rho_B(\varphi^A, t)) \) associated with the local density matrix \( \rho_B(\varphi^A, t) = \text{Tr}_{AC} \rho(\varphi^A, t) \) is defined by eqs. (22, 25) as follows:
\[ X(\rho_B(\varphi^A, t)) = \begin{pmatrix} \text{Re}\rho_{12}(\varphi^A, t), \text{Im}\rho_{12}(\varphi^A, t), \rho_{11}(\varphi^A, t) \end{pmatrix}^T. \quad (74) \]
The matrices \( \hat{T} \) and \( \hat{T}^0 \) read:
\[ \hat{T}(t) = \begin{pmatrix} 0 & a_1(t) & 0 \\ -a_1(t) & 0 & 0 \\ 0 & 0 & a_2(t) \end{pmatrix}, \quad \hat{T}^0(t) = \begin{pmatrix} 0 \\ b(t) \end{pmatrix}, \quad (75) \]
where
\[ a_1(t) = \frac{(2\lambda^B - 1)(2\lambda^C_3 + 2\lambda^C_2 - 1)}{5 + \sqrt{5}} r(t), \quad (76) \]
\[ a_2(t) = \frac{r(t)^2}{10(3 + \sqrt{5})}, \]
\[ b(t) = \lambda^B \left( \frac{3 + 2 \cos \frac{\sqrt{5} t}{2}}{5} - a_2(t) \right) + \frac{2 \sin^2 \frac{\sqrt{5} t}{4}}{5} \left( 2\lambda^C_1 + \lambda^C_2 + \lambda^C_3 + (\lambda^C_3 - \lambda^C_2) \cos \frac{t}{2} \right) \]
with
\[ r(t) = 2 \sin \left( \frac{1 + \sqrt{5})t}{4} + (3 + \sqrt{5}) \sin \frac{(1 - \sqrt{5})t}{4}. \quad (77) \]
Formulas (76) for the entries of the matrix \( \hat{T} \) suggest us to consider only such time instants that satisfy the condition
\[ r(t) \neq 0, \quad (78) \]
because otherwise the rank of \( \hat{T} \) is zero. The first positive root of the expression in the lhs of condition (78) is \( t = 9.070 \). Thus, a suitable time interval for the parameter detection in the subsystem \( B \) might be the following one: \( 0 < t < 9.070 \). Under condition (78), eqs. (75) and (76) yield
\[ \text{ran} \hat{T}(t) = \begin{cases} 3, & \lambda^B \neq \frac{1}{2} \text{ and } 2\lambda^C_3 + 2\lambda^C_2 - 1 \neq 0 \\ 1, & \lambda^B = \frac{1}{2} \text{ or } 2\lambda^C_3 + 2\lambda^C_2 - 1 = 0 \end{cases}. \quad (79) \]
If \( \text{ran} \hat{T} = 3 \) (i.e. \( \det \hat{T} \neq 0 \)), then we have the complete information transfer from the first to the last node (i.e. from A to B) and, consequently, all parameters encoded into \( \rho^A(\varphi^A, 0) \) will be transferred to the subsystem B, so that \( E^{AB}(t) = E^{AA} \).

If \( \text{ran} \hat{T} = 1 \), (i.e. either \( \lambda^B = \frac{1}{2} \) or \( 2\lambda^C_3 + 2\lambda^C_2 = 1 \)) then there is only one nonzero element in the matrix \( J(\rho^B(\varphi^A, t)) \) (the first order minor):

\[
M_1 = -\frac{(2\lambda^A_1 - 1)r^2(t)\sin(2\varphi_1)}{10(3 + \sqrt{5})},
\]

which is nonzero for \( \varphi_1 \in G^A \). Thus eq.(37) yields \( E^{AB}(t) = 1 \) if only \( \lambda^A \neq \frac{1}{2} \). Otherwise, if \( \lambda^A = \frac{1}{2} \), then the rank of the matrix \( J(\rho^B(\varphi^A, t)) \) equals zero, and eq.(37) yields \( E^{AB}(t) = 0 \).

All in all, we may write the following formula for the informational correlation \( E^{AB}(t) \):

\[
E^{AB}(t) = \begin{cases} 
2, & \lambda^A \neq \frac{1}{2}, \lambda^B \neq \frac{1}{2}, 2\lambda^C_3 + 2\lambda^C_2 - 1 \neq 0 \\
1, & \lambda^A \neq \frac{1}{2} \quad \text{and} \quad \left( \lambda^B = \frac{1}{2}, \quad \text{or} \quad 2\lambda^C_3 + 2\lambda^C_2 - 1 = 0 \right) \\
0, & \lambda^A = \frac{1}{2}
\end{cases}
\]  

(81)

In addition, for the normalized correlation \( E^{AB}_{\text{norm}}(t) \) introduced by eq.(43) we have \( (\tilde{D} = 2) \):

\[
E^{AB}_{\text{norm}}(t) = \begin{cases} 
1, & \lambda^A \neq \frac{1}{2}, \lambda^B \neq \frac{1}{2}, 2\lambda^C_3 + 2\lambda^C_2 - 1 \neq 0 \\
\frac{1}{2}, & \lambda^A \neq \frac{1}{2} \quad \text{and} \quad \left( \lambda^B = \frac{1}{2}, \quad \text{or} \quad 2\lambda^C_3 + 2\lambda^C_2 - 1 = 0 \right) \\
0, & \lambda^A = \frac{1}{2}
\end{cases}
\]  

(82)

Eq.(81) allows us to conclude that the informational correlation \( E^{AB} \) is very sensitive to the multiplicity of the eigenvalues of the matrices \( \rho^A(0) \) and \( \rho^B(0) \).

3. **Local initial unitary transformations of subsystems C and B.**

First, we shall note that the initial local unitary transformation of the subsystem B having general form (68) with parameters \( \beta_i \) instead of \( \varphi_i \) changes the matrix \( \hat{T}(t) \) given by eq.(75). Namely, factor \( \cos(2\beta_1) \) appears in the expressions for \( a_1 \) given by the first of eqs.(76), expression for \( a_2 \) (the second of eqs.(76)) remains unchanged and two more nonzero entries appear in the third row of \( \hat{T} \). In turn, this changes the conditions in the rhs
of eqs. \((79,81,82)\). For instance, eqs. \((79)\) and \((81)\) now read:

\[
\begin{align*}
\text{ran } \hat{T}(t) &= \begin{cases} 
3, & \lambda^B \neq \frac{1}{2}, \ 2\lambda_3^C + 2\lambda_2^C - 1 \neq 0, \ \beta_1 \neq \frac{\pi}{4} \\
1, & \lambda^B = \frac{1}{2}, \ \text{or } 2\lambda_3^C + 2\lambda_2^C - 1 = 0, \ \text{or } \beta_1 = \frac{\pi}{4} \\
2, & \lambda^A \neq \frac{1}{2}, \lambda^B \neq \frac{1}{2}, \ 2\lambda_3^C + 2\lambda_2^C - 1 \neq 0, \ \beta_1 \neq \frac{\pi}{4} \\
1, & \lambda^A \neq \frac{1}{2}, \ \text{and} \ \left(\lambda^B = \frac{1}{2}, \ \text{or } 2\lambda_3^C + 2\lambda_2^C - 1 = 0, \ \text{or } \beta_1 = \frac{\pi}{4}\right) \\
0, & \lambda^A = \frac{1}{2}
\end{cases},
\end{align*}
\]

\[
E^{AB}(t) = \begin{cases} 
3, & \lambda^B \neq \frac{1}{2}, \ 2\lambda_3^C + 2\lambda_2^C - 1 \neq 0, \ \beta_1 \neq \frac{\pi}{4} \\
1, & \lambda^B = \frac{1}{2}, \ \text{or } 2\lambda_3^C + 2\lambda_2^C - 1 = 0, \ \text{or } \beta_1 = \frac{\pi}{4} \\
2, & \lambda^A \neq \frac{1}{2}, \lambda^B \neq \frac{1}{2}, \ 2\lambda_3^C + 2\lambda_2^C - 1 \neq 0, \ \beta_1 \neq \frac{\pi}{4} \\
1, & \lambda^A \neq \frac{1}{2}, \ \text{and} \ \left(\lambda^B = \frac{1}{2}, \ \text{or } 2\lambda_3^C + 2\lambda_2^C - 1 = 0, \ \text{or } \beta_1 = \frac{\pi}{4}\right) \\
0, & \lambda^A = \frac{1}{2}
\end{cases}.
\]

Next, the initial local transformation of the subsystem \(C\) changes formulas \((79,81,82)\) as well. To demonstrate this effect, we consider a simple example. Let, instead of the diagonal initial state \(\rho^C(0)\) (see eq. \((67)\)), we take the following initial density matrix of the subsystem \(C\):

\[
\tilde{\rho}^C(0) = U^C \rho^C(0)(U^C)^+, \quad U^C = \begin{pmatrix} 
\cos \gamma & \sin \gamma & 0 & 0 \\
-\sin \gamma & \cos \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad 0 < \gamma < 2\pi.
\]

The effect of the local transformation \(U^C\) on the matrix \(\hat{T}\) results in the replacement of the expression \((2\lambda_3^C + 2\lambda_2^C - 1)\) with \((2\lambda_3^C + \lambda_1^C + \lambda_2^C - \lambda_1^C \cos(2\gamma) - 1)\) in eqs. \((79,81,82)\).

Results obtained in this subsection mean, in particular, that applying the local transformation we may handle the informational correlation up to a certain extent. For instance, using the transformation of the subsystem \(B\) with \(\beta_1 = \frac{\pi}{4}\), we may decrease \(E^{AB}\) from 2 to 1. Thus, this transformation may be used as a lock in some gates. Vice-versa, if originally \(\lambda^A \neq \frac{1}{2}\) and the density matrix of the subsystem \(C\) is given by eq. \((67)\) with \(2\lambda_3^C + 2\lambda_2^C - 1 = 0\), then applying the local transformation \(U^C\) with \(\cos(2\gamma) \neq 1\) we increase \(E^{AB}\) from 1 to 2.

4. Non-reducible informational correlation

In this example, the upper estimation \((48)\) yields \(E^{AB;\text{min}} \leq 1\). Remember that the evolution of the density matrix is given by eq. \((72)\) with the initial density matrix \((65)\). To calculate the non-reducible informational correlation \(E^{AB;\text{min}}\) using the formulas of Sec. II B.
we write the density matrix $\rho^B(\varphi^A, t)$ explicitly:

$$
\rho^B(\varphi^A, t) = \begin{pmatrix}
\hat{b}(\varphi^A, t) & \hat{a}(\varphi^A, t)e^{-i\varphi_2} \\
-\hat{a}(\varphi^A, t)e^{i\varphi_2} & 1 - \hat{b}(\varphi^A, t)
\end{pmatrix},
$$

(86)

$$
\hat{b}(\varphi^A, t) = b(t) + \frac{a_2(t)}{2}((2\lambda_1^A - 1)\cos(2\varphi_1) + 1), \quad \hat{a}(\varphi^A, t) = -\frac{ia_1(t)}{2}(2\lambda_1^A - 1)\sin(2\varphi_1),
$$

where $a_1$, $a_2$, and $b$ are given in eq. (76). The characteristic equation for the matrix $\rho^B(\varphi^A, t)$ reads:

$$
\lambda^2 - \lambda + \det(\rho^B(\varphi^A, t)) = 0.
$$

(87)

It is obvious that $\det(\rho^B(\varphi^A, t))$ depends on $\varphi_1$ and does not depend on $\varphi_2$. Consequently, representation (59) of $H$ becomes scalar:

$$
H(\varphi_1, t) = \frac{\partial}{\partial \varphi_1} \det(\rho^B(t)) = a_2(t)(2\lambda_1^A - 1)\sin(2\varphi_1)\left(2b(t) + a_2(t) - 1 + (2\lambda_1^A - 1)(a_2(t) - (2\lambda_1^B - 1)^2(2\lambda_3^C + 2\lambda_2^C - 1)^2)\cos(2\varphi_1)\right).
$$

(88)

Thus we have

$$
E_{AB;\min}(\varphi_1, t) = \text{ran} H(\varphi_1, t) = \begin{cases}
1, & H(\varphi_1, t) \neq 0 \\
0, & H(\varphi_1, t) = 0
\end{cases},
$$

(89)

$$
E_{\text{norm};\min}(\varphi_1, t) = \begin{cases}
\frac{1}{2}, & H(\varphi_1, t) \neq 0 \\
0, & H(\varphi_1, t) = 0
\end{cases}.
$$

(90)

All zeros of the function $H(\varphi_1, t)$ are defined by the following formulas (remember, that $0 < \varphi_1 < \frac{\pi}{2}$ as indicated in eq. (59)):

$$
r = 0,
$$

(90)

$$
\lambda_1^A = \frac{1}{2},
$$

(91)

$$
\sin(2\varphi_1) = 0 \Rightarrow \varphi_1 = 0, \frac{\pi}{2},
$$

(92)

$$
\cos 2\varphi_1 = m(t), \Rightarrow
$$

(93)

$$
\varphi_1 = \frac{1}{2} \arccos m(t),
$$

(94)

$$
m(t) = -\frac{2b(t) + a_2(t) - 1}{(2\lambda_1^A - 1)(a_2(t) - (2\lambda_1^B - 1)^2(2\lambda_3^C + 2\lambda_2^C - 1)^2)}.
$$

(95)

Analyzing eqs. (90-94) we see, first of all, that $H(\varphi_1, t)$ is identical to zero for all $\varphi_1$ at the time instants satisfying condition (90), but these instants are excluded by condition (78).
We obviously shall not use the initial matrix $\rho^A(0)$ with equal eigenvalues, that follows from eq. (91). Next, eq. (92) means that, at any time instant, the boundary values 0 and $\frac{\pi}{2}$ of the parameter $\varphi_1$ may not be uniquely transferred. However, the boundary is not involved into our consideration. Finally, $E^{AB;\min} = 0$ at instants satisfying eq. (93). Thus, if at some instant $t_1$ (provided that $\varphi_1 \in G^A$, $\lambda^A \neq \frac{1}{2}$ and $r(t_1) \neq 0$) we obtain $H(\varphi^A, t_1) = 0$, then this means that the value of the parameter $\varphi_1$ defined by eq. (94) is transferred. To avoid the zero value of $H$, we shall consider such time instants which do not satisfy condition (93) for all $\varphi_1$ from the interval $0 < \varphi_1 < \frac{\pi}{2}$, i.e.

$$|m(t)| > 1.$$  \hspace{1cm} (95)

This may be done for any particular set of fixed eigenvalues of the initial density matrices $\rho^A(0)$, $\rho^C(0)$ and $\rho^B(0)$. For instance, let us define the suitable time interval for the following set of eigenvalues:

$$\lambda^A_1 = \lambda^B_1 = \frac{3}{4}, \quad \lambda^C_1 = \lambda^C_2 = \lambda^C_3 = \frac{1}{4}.$$  \hspace{1cm} (96)

The first positive root of equation $|m(t)| = 1$ is $t = 2.726$. Consequently, the first positive time interval where condition (95) is satisfied (and consequently condition (93) fails) is the following one:

$$0 < t < 2.726.$$  \hspace{1cm} (97)

Interval (97) is suitable for the detection of the parameter $\varphi_1$. In this case $g = G^A$ and we conclude that $E^{AB;\min}(G^A, t) = 1$ if $t$ is inside of the interval (97) for the eigenvalues (96).

We see that the parameter $\varphi_1$ is encoded into the eigenvalue of the matrix $\rho^B(\varphi^A, t)$. It is obvious that $\varphi_1$ is also encoded into the matrix of eigenvectors of $\rho^B(\varphi^A, t)$, because otherwise two matrices with different values of $\varphi_1$ would have the same complete set of independent eigenvectors and consequently they would commute which is not true (this might be checked directly in our example). Therefore, the parameter $\varphi_1$ is most reliable one since it might be detected in the either eigenvalue or eigenvectors of the density matrix $\rho^B$. Meanwhile, the parameter $\varphi_2$ may be transferred only by the eigenvectors of the density matrix $\rho^B(\varphi^A, t)$ and may be removed by the local unitary transformation of the subsystem $B$ at instant $t$. Again, this result is valid provided condition (78) is satisfied. It might be shown that the local transformations performed on the either subsystem $C$ or $B$ may
insert the parameter $\varphi_2$ into the determinant $\det \rho^B(\varphi^A, t) = H(\varphi^A, t)$, which appears as a condition in the rhs of eq. (89). Both parameters $\varphi_1$ and $\varphi_2$ may be used on the equal foot in this case.

**B. Two-node subsystems $A$ and $B$**

Now we consider the informational correlation between two-node subsystems $A$ and $B$ of the four-node homogeneous chain with XY Hamiltonian \(\text{(64)}\). The subsystem $A$ is represented by the first and the second nodes, while the subsystem $B$ is represented by the third and the fourth nodes. Thus $N^A = N^B = 4$, $N^C = 0$ and $SU(4)$ is the group of the local unitary transformations of the subsystem $A$. We consider the initial density matrix in the form \(\text{(13)}\) without the subsystem $C$:

\[
\rho(0) = \rho^A(0) \times \rho^B(0),
\]

where $\rho^A(0)$ and $\rho^B(0)$ are the diagonal matrices:

\[
\rho^A(0) = \text{diag}(\lambda^A_1, \lambda^A_2, \lambda^A_3, \lambda^A_4), \quad \lambda^A_1 \geq \lambda^A_2 \geq \lambda^A_3 \geq \lambda^A_4, \quad \sum_{i=1}^{4} \lambda^A_i = 1, \quad (99)
\]

\[
\rho^B(0) = \text{diag}(\lambda^B_1, \lambda^B_2, \lambda^B_3, \lambda^B_4), \quad \lambda^B_1 \geq \lambda^B_2 \geq \lambda^B_3 \geq \lambda^B_4, \quad \sum_{i=1}^{4} \lambda^B_i = 1.
\]

1. **Number of parameters encoded into the subsystem $A$, $E^{AA}$**

The group $SU(4)$ is the 15-parametric one. However, considering the transformation of the diagonal matrix $\rho^A(0)$ we deal with the 12-parametric representation, $\tilde{D}^A = 12$ (see eq. \(\text{(31)}\) and ref. [29]):

\[
\rho^A(\varphi^A, 0) = U^A(\varphi^A) \Lambda^A(U^A(\varphi^A))^+ = \tilde{U}^A(\varphi^A) \Lambda^A(\tilde{U}^A(\varphi^A))^+,
\]

\[
\tilde{U}^A(\varphi^A) = e^{i\gamma_3 \varphi_1} e^{i\gamma_2 \varphi_2} e^{i\gamma_3 \varphi_3} e^{i\gamma_5 \varphi_4} e^{i\gamma_7 \varphi_5} e^{i\gamma_9 \varphi_6} e^{i\gamma_{10} \varphi_7} e^{i\gamma_2 \varphi_8} e^{i\gamma_3 \varphi_9} e^{i\gamma_5 \varphi_{10}} e^{i\gamma_7 \varphi_{11}} e^{i\gamma_2 \varphi_{12}}. \quad (101)
\]

Here the ranges of the parameters $\varphi_i$ are following [29]

\[
0 \leq \varphi_1, \varphi_3, \varphi_5, \varphi_7, \varphi_9, \varphi_{11} \leq \pi, \quad 0 \leq \varphi_2, \varphi_4, \varphi_6, \varphi_8, \varphi_{10}, \varphi_{12} \leq \frac{\pi}{2}. \quad (102)
\]

The explicit matrix representation of $\gamma_i$ is given in the Appendix A, Sec. [29, 30]. One should note that the expression \(\text{(101)}\) for $\tilde{U}^A$ holds if all eigenvalues $\lambda^A_i$ are different. In this
In addition, for the normalized parameter $E^{AA}$, if some of eigenvalues $\lambda_i^A$ coincide. For instance, if $\lambda_1^A = \lambda_2^A$, then $\gamma_2$ and $\gamma_3$ commute with $\rho^A(0)$, so that eqs. (100,101) reduce to

$$\rho^A(\varphi^A,0) = \tilde{U}_2^A(\varphi^A)\Lambda^A(\tilde{U}_2^A(\varphi^A))^+, \quad \tilde{U}_2^A(\varphi^A) = e^{i\gamma_3\varphi_1}e^{i\gamma_2\varphi_2}e^{i\gamma_1\varphi_3}e^{i\gamma_4\varphi_4}e^{i\gamma_5\varphi_5}e^{i\gamma_6\varphi_6}e^{i\gamma_7\varphi_7}e^{i\gamma_8\varphi_8}e^{i\gamma_9\varphi_9},$$

where $\tilde{U}_2^A$ possesses 10 parameters $\varphi_i$, $i = 1,\ldots,10$, so that only 10 parameters may be encoded into the density matrix $\rho^A(\varphi^A)$, i.e. $E^{AA} = 10$, which agrees with eq. (33).

Next, let $\lambda_1^A = \lambda_2^A$ and $\lambda_3^A = \lambda_4^A$, but $\lambda_1^A \neq \lambda_3^A$. Then eq. (33) yields $E^{AA} = 8$. If $\lambda_1^A = \lambda_2^A = \lambda_3^A$, then $\gamma_2$, $\gamma_3$ and $\gamma_5$ commute with $\rho^A(0)$, so that eq. (103) reduces to

$$\rho^A(\varphi^A,0) = \tilde{U}_3^A(\varphi^A)\Lambda^A(\tilde{U}_3^A(\varphi^A))^+, \quad \tilde{U}_3^A(\varphi^A) = e^{i\gamma_3\varphi_1}e^{i\gamma_2\varphi_2}e^{i\gamma_1\varphi_3}e^{i\gamma_4\varphi_4}e^{i\gamma_5\varphi_5}e^{i\gamma_6\varphi_6},$$

where $\tilde{U}_3^A$ possesses 6 parameters $\varphi_i$, $i = 1,\ldots,6$. Consequently only 6 parameters may be encoded into the density matrix $\rho^A(\varphi^A)$, i.e. $E^{AA} = 6$. This also agrees with eq. (33).

Finally, if $\lambda_1^A = \lambda_2^A = \lambda_3^A = \lambda_4^A = 1/4$, then $\rho^A(\varphi^A,0)$ is proportional to the $4 \times 4$ identity matrix $I_4$,

$$\rho^A(\varphi^A,0) = \frac{1}{4}I_4, \quad (105)$$

and no parameters may be encoded into such $\rho^A(\varphi^A,0)$, i.e. $E^{AA} = 0$.

We collect the above results in the following formula:

$$E^{AA} = \begin{cases} 12, & \lambda_1^A \neq \lambda_2^A \neq \lambda_3^A \neq \lambda_4^A \\ 10, & \lambda_1^A = \lambda_2^A, \lambda_1^A \neq \lambda_3^A \neq \lambda_4^A \\ 8, & \lambda_1^A = \lambda_2^A, \lambda_1^A = \lambda_3^A, \lambda_1^A \neq \lambda_4^A \\ 6, & \lambda_1^A = \lambda_2^A = \lambda_3^A \neq \lambda_4^A \\ 0, & \lambda_1^A = \lambda_2^A = \lambda_3^A = \lambda_4^A = \frac{1}{4} \end{cases}. \quad (106)$$

In addition, for the normalized parameter $E_{\text{norm}}^{AA}$, given by eq. (11), we find

$$E_{\text{norm}}^{AA} = \begin{cases} 1, & \lambda_1^A \neq \lambda_2^A \neq \lambda_3^A \neq \lambda_4^A \\ 5, & \lambda_1^A = \lambda_2^A, \lambda_1^A \neq \lambda_3^A \neq \lambda_4^A \\ 6, & \lambda_1^A = \lambda_2^A, \lambda_1^A = \lambda_3^A, \lambda_1^A \neq \lambda_4^A \\ 3, & \lambda_1^A = \lambda_2^A = \lambda_3^A \neq \lambda_4^A \\ 2, & \lambda_1^A = \lambda_2^A = \lambda_3^A = \lambda_4^A \\ 0, & \lambda_1^A = \lambda_2^A = \lambda_3^A = \lambda_4^A = \frac{1}{4} \end{cases}. \quad (107)$$
Remark that the same expression (106) for $E^{AA}$ may be obtained calculating the matrix $J^A(\varphi^A,0)$ and using eq.(39). Therewith the matrix $\hat{X}(\rho^A(\varphi^A,0))$ may be constructed by its definition (22) and (25), but we do not represent its explicite form here, because it is too complicated.

2. Relation between the rank of $\hat{T}$ and the informational correlation $E^{AB}$

Now we apply the transformation $V(t) \in SU(16)$ defined by eq.(11) with Hamiltonian (64) and find the density matrix $\rho(\varphi^A, t)$ of the system using eq.(72), where

$$\rho(\varphi^A, 0) = (\hat{U}^A(\varphi^A) \times I_4)\rho(0)((\hat{U}^A(\varphi^A))^+ \times I_4),$$

and the initial density matrix $\rho(0)$ is given by eq.(98). Now we may calculate the matrix $\hat{T}$ using eqs.(16,17,24,27). The explicite form of this matrix is very complicated and it is not represent in this paper. Below we study the informational correlation $E^{AB}$ for different ranks of the matrix $\hat{T}$.

a. Complete information transfer, $\text{ran} \hat{T} = 15$ (or $\det \hat{T} \neq 0$). The direct calculations show that the condition for the complete information transfer reads

$$\det \hat{T} = -\frac{(2\lambda_2^B + 2\lambda_3^B - 1)^8}{10^{16}} \left( \cos \frac{\sqrt{5}t}{2} - 5 \cos \frac{t}{2} + 4 \right)^6 \neq 0. \quad (109)$$

In this case all parameters $E^{AA}$ will be transferred into the subsystem $B$, i.e. $E^{AB}(t) = E^{AA}$ and $E^{AB}_{\text{norm}}(t) = E^{AA}_{\text{norm}}$, see eqs.(106,107). Therewith $\text{ran} \hat{T} = 15$. Hereafter we consider such instants $t$ that the second factor in eq.(109) is non-zero, i.e

$$\cos \frac{\sqrt{5}t}{2} - 5 \cos \frac{t}{2} + 4 \neq 0. \quad (110)$$

The first positive root of the expression in the lhs of (110) is $t = 11.909$. Thus, the suitable time interval for the parameter detection might be

$$0 < t < 11.909. \quad (111)$$

b. Partial information transfer, $\text{ran} \hat{T} = 11$. Next, let $\lambda_i^B$ be such that $\det \hat{T} = 0$, i.e.

$$\lambda_3^B = \frac{1}{2} - \lambda_2^B. \quad (112)$$
Then we have to calculate the rank of the matrix $\hat{T}$. It might be readily checked that the 11th order minors of $\hat{T}$ are nonzero and they may be represented by the following formula:

$$M_{11} = \pm \left(\frac{\lambda_B^2(2\lambda_B^2 - 1) - \lambda_A^2(2\lambda_A^2 - 1)}{4 \times 10^{14}}\right)^2 \left(\cos \frac{\sqrt{5}t}{2} - 5 \cos \frac{t}{2} + 4\right)^{10} \times \left(\cos \frac{\sqrt{5}t}{2} + 5 \cos \frac{t}{2} + 4\right)^4.$$  \hspace{1cm} (113)

Thus $\text{ran } \hat{T} = 11$. The informational correlation $E^{AB}(t)$ calculated by the formula (37) depends on $\lambda_i^B$ as follows

$$E^{AB}(t) = \begin{cases} 11, \lambda_1^B \neq \lambda_2^B \neq \lambda_3^B \neq \lambda_4^B \\ 10, \lambda_1^B = \lambda_2^B, \lambda_1^A \neq \lambda_3^A \neq \lambda_4^A \\ 8, \lambda_1^A = \lambda_2^A, \lambda_1^A = \lambda_3^A, \lambda_1^A \neq \lambda_3^A . \\ 6, \lambda_1^A = \lambda_2^A = \lambda_3^A \neq \lambda_4^A \\ 0, \lambda_1^A = \lambda_2^A = \lambda_3^A = \lambda_4^A = \frac{1}{4} \end{cases} \hspace{1cm} (114)$$

In addition, for the normalized informational correlation $E^{AB}_{\text{norm}}(t)$ given by eq.(43), we obtain

$$E^{AB}_{\text{norm}}(t) = \begin{cases} \frac{11}{12}, \lambda_1^A \neq \lambda_2^A \neq \lambda_3^A \neq \lambda_4^A \\ \frac{5}{6}, \lambda_1^A = \lambda_2^A, \lambda_1^A \neq \lambda_3^A \neq \lambda_4^A \\ \frac{1}{3}, \lambda_1^A = \lambda_2^A, \lambda_4^A = \lambda_3^A, \lambda_1^A \neq \lambda_3^A . \\ \frac{1}{2}, \lambda_1^A = \lambda_2^A = \lambda_3^A \neq \lambda_4^A \\ 0, \lambda_1^A = \lambda_2^A = \lambda_3^A = \lambda_4^A = \frac{1}{4} \end{cases} \hspace{1cm} (115)$$

In this case we shall consider such instant $t$ that two last factors in the formula (113) are nonzero. In other words, along with condition (110), the following condition must be satisfied

$$\cos \frac{\sqrt{5}t}{2} + 5 \cos \frac{t}{2} + 4 \neq 0. \hspace{1cm} (116)$$

The first positive root of the expression in the lhs of condition (116) is $t = 5.952$ which restricts interval (111) to $0 < t < 5.952$.

**c. Partial information transfer, ran $\hat{T} = 9$.** Next, we consider such $\lambda_i^B$ that $M_{11} = 0$. This happens in one of two following cases:

$$\lambda_2^B = \lambda_1^B, \text{ or} \hspace{1cm} (117)$$

$$\lambda_2^B = \frac{1}{2} - \lambda_1^B. \hspace{1cm} (118)$$
In the first case, eq. (117), we have two following pairs of equal eigenvalues:

\[ \lambda_4^B = \lambda_3^B = \frac{1}{2} - \lambda_1^B, \quad \lambda_2^B = \lambda_1^B. \]  

(119)

In the second case, eq. (118), we have two other pairs of equal eigenvalues:

\[ \lambda_3^B = \lambda_1^B = \frac{1}{2} - \lambda_2^B, \quad \lambda_4^B = \lambda_2^B. \]  

(120)

In both cases, the 9th order minors \( M_{9i} \) of \( \hat{T} \) are nonzero and all of them are collected in the following formula:

\[
M_{9i} = -\left( \cos \frac{\sqrt{5} t}{2} - 5 \cos \frac{t}{2} + 4 \right)^8 \frac{(4\lambda_i^B - 1)m_i(t)}{10^8}, \quad i = 1, \ldots, 16,
\]

(121)

where \( m_i(t) \) are some explicite functions of \( t \) independent on \( \lambda_i^B \), but they depend on whether eq. (117) or (118) is considered (we do not represent the expressions for these functions). The analysis shows that \( \sum_{i=1}^{16} |m_i(t)| \neq 0 \) for all \( t \) and both cases (117) and (118). Thus ran \( \hat{T} = 9 \), provided condition (110) is satisfied and \( \lambda_1^B \neq \frac{1}{4} \).

The informational correlation \( E^{AB}(t) \) calculated by the formula (37) depends on \( \lambda_i^A \) as follows

\[
E^{AB}(t) = \begin{cases} 
9, & \lambda_1^A \neq \lambda_2^A \neq \lambda_3^A \neq \lambda_4^A \\
9, & \lambda_1^A = \lambda_2^A, \lambda_1^A \neq \lambda_3^A \neq \lambda_4^A \\
8, & \lambda_1^A = \lambda_2^A, \lambda_4^A = \lambda_3^A, \lambda_1^A \neq \lambda_3^A \\
6, & \lambda_4^A = \lambda_3^A \neq \lambda_4^A \\
0, & \lambda_1^A = \lambda_2^A = \lambda_3^A = \lambda_4^A = \frac{1}{4} 
\end{cases}
\]

(122)

In addition, for the normalized informational correlation \( E_{norm}^{AB}(t) \) (see eq. (43)), we obtain

\[
E_{norm}^{AB}(t) = \begin{cases} 
\frac{3}{4}, & \lambda_1^A \neq \lambda_2^A \neq \lambda_3^A \neq \lambda_4^A \\
\frac{3}{4}, & \lambda_1^A = \lambda_3^A, \lambda_1^A \neq \lambda_3^A \neq \lambda_4^A \\
\frac{3}{4}, & \lambda_1^A = \lambda_2^A = \lambda_4^A = \lambda_3^A, \lambda_1^A \neq \lambda_3^A \\
\frac{3}{2}, & \lambda_4^A = \lambda_2^A = \lambda_3^A \neq \lambda_4^A \\
0, & \lambda_1^A = \lambda_2^A = \lambda_3^A = \lambda_4^A = \frac{1}{4} 
\end{cases}
\]

(123)
d. Partial information transfer, \( \text{ran} \hat{T} = 7 \). Finally, let \( \lambda^B_1 = 1/4 \), which means that all \( \lambda^B_i \) are the same and equal 1/4. Then the 7th order minor of \( \hat{T} \) is nonzero:

\[
M_7 = - \left( \cos \frac{\sqrt{5}}{2} t - 5 \cos \frac{t}{2} + 4 \right)^8
\]

Thus \( \text{ran} \hat{T} = 7 \). The informational correlation \( E^{AB}(t) \) calculated by the formula (37) depends on \( \lambda^A_i \) as follows

\[
E^{AB}(t) = \begin{cases} 
7, & \lambda^A_1 \neq \lambda^A_2 \neq \lambda^A_3 \neq \lambda^A_4 \\
7, & \lambda^A_1 = \lambda^A_2, \lambda^A_1 \neq \lambda^A_3 \neq \lambda^A_4 \\
6, & \lambda^A_1 = \lambda^A_2, \lambda^A_1 = \lambda^A_3, \lambda^A_1 \neq \lambda^A_3 \\
5, & \lambda^A_1 = \lambda^A_2 = \lambda^A_3 = \lambda^A_4 \\
0, & \lambda^A_1 = \lambda^A_2 = \lambda^A_3 = \lambda^A_4 = \frac{1}{4}
\end{cases}
\]

In addition, for the normalized informational correlation \( E_{\text{norm}}^{AB}(t) \) given by eq.(43), we obtain

\[
E_{\text{norm}}^{AB}(t) = \begin{cases} 
\frac{7}{12}, & \lambda^A_1 \neq \lambda^A_2 \neq \lambda^A_3 \neq \lambda^A_4 \\
\frac{1}{12}, & \lambda^A_1 = \lambda^A_2, \lambda^A_1 \neq \lambda^A_3 \neq \lambda^A_4 \\
\frac{2}{5}, & \lambda^A_1 = \lambda^A_2, \lambda^A_1 = \lambda^A_3, \lambda^A_1 \neq \lambda^A_3 \\
\frac{1}{12}, & \lambda^A_1 = \lambda^A_2 = \lambda^A_3 = \lambda^A_4 \\
0, & \lambda^A_1 = \lambda^A_2 = \lambda^A_3 = \lambda^A_4 = \frac{1}{4}
\end{cases}
\]

Similar to Sec. III A 2, from the analysis of eqs.(106,114,122,125), we conclude that the informational correlation \( E^{AB} \) is very sensitive to the multiplicity of the eigenvalues of the density matrices \( \rho^A(0) \) and \( \rho^B(0) \).

We emphasize that the represented analysis is valid at the time instants satisfying conditions (110) and, generally speaking, (116).

e. Local unitary transformations of the subsystem \( B \). Similar to the example of Sec. III A 3, the local unitary transformations of the subsystem \( B \) allow one to handle the informational correlation up to a certain extent. Having the initial density matrix \( \rho^B(0) \) with fixed eigenvalues, we may apply the local unitary transformation with the purpose to either increase or decrease the number of parameters transferred from the subsystem \( A \) to the subsystem \( B \). The detailed study of this problem is beyond the scope of this paper.
3. Non-reducible informational correlation

In this example, the upper estimation (13) yields $E^{AB;\min} \leq 3$. We may not carry out all calculations of Sec.II B analytically and therefore we do not study the non-reducible correlation in the full extent. To illustrate some properties of the non-reducible correlations, we consider two particular examples.

a. Example 1: $\text{ran} \hat{T} = 7$, $E^{AB} = 5$ in eq.(125). In this case $E^{AA} = 6$, i.e. there are six parameters in the set $\varphi^A$. To calculate the non-reducible informational correlation by formula (60), we construct the $3 \times 6$ matrix $H$ (59). It may be readily shown that the third order minors of the matrix $H$ are zero. The second order minors are nonzero so that $E^{AB;\min}(\varphi^A, t) = 2$. The analysis of these minors shows that the following pairs of parameters might be transferred from the subsystem $A$ to the subsystem $B$:

$$(\varphi_1, \varphi_2), \ (\varphi_1, \varphi_4), \ (\varphi_1, \varphi_6), \ (\varphi_2, \varphi_3), \ (\varphi_2, \varphi_4), \ (\varphi_2, \varphi_6),$$

$$(\varphi_3, \varphi_4), \ (\varphi_3, \varphi_6), \ (\varphi_4, \varphi_6).$$

(127)

It is remarkable that the parameter $\varphi_5$ does not appear in the above list (127).

However, the second order minors found above depend on $\varphi^A$ and $t$ and, consequently, there might be such $\varphi^A$ and $t$ that these minors equal zero which decreases $E^{AB;\min}$. Thus, as the next step, we must define such region $g \subset G^A$ and such time intervals that at least one of pairs (127) with values from $g$ may be detected in the subsystem $B$ at any instant during these time intervals (only in that case $E^{AB;\min} = 2$).

We do not consider this problem in the full extent. Instead of this, we study the problem of informational correlation between the subsystems $A$ and $B$ by means of a particular pair of parameters from the list (127). For instance, let us consider the pair $(\varphi_2, \varphi_6)$. In other words, we restrict the set $\varphi^A$ to $\hat{\varphi}^A = \{\varphi_2, \varphi_6\}$ and $G^A$ to $\hat{G}^A \in G^A$ defined as follows:

$$\hat{G}^A : 0 < \varphi_2, \varphi_6 < \frac{\pi}{2}.$$ 

(128)

This cut of the whole set of 6 parameters in the unitary transformation of the subsystem $A$ ($E^{AA} = 6$ in our case) is justified by the fact, that not all parameters of the local unitary transformation might be equivalent for the particular quantum information problem. In this example we assume that namely parameters $\varphi_2$ and $\varphi_6$ must be used.

Thus, for the pair $(\varphi_2, \varphi_6)$, we have to determine such time intervals and region $\hat{g}$ in the space of parameters $\hat{\varphi}^A$ that both parameters with values from $\hat{g}$ may be detected at any
instant during the above time intervals. For the sake of simplicity, we put other parameters \( \varphi_i \) to zero. We also take
\[
\lambda_1^A = \lambda_2^A = \lambda_3^A = \frac{5}{16},
\]
so that \( \lambda_4^A = \frac{1}{16} \). In this case only the second and the sixth columns of the matrix \( H(t) \) are nonzero so that we may replace the matrix \( H \) with the following matrix \( \tilde{H} \):
\[
\tilde{H}(\varphi_2, \varphi_6, t) = \begin{pmatrix}
\frac{\partial a_0(\varphi_2, \varphi_6, t)}{\partial \varphi_2} & \frac{\partial a_0(\varphi_2, \varphi_6, t)}{\partial \varphi_6} \\
\frac{\partial a_1(\varphi_2, \varphi_6, t)}{\partial \varphi_2} & \frac{\partial a_1(\varphi_2, \varphi_6, t)}{\partial \varphi_6} \\
\frac{\partial a_2(\varphi_2, \varphi_6, t)}{\partial \varphi_2} & \frac{\partial a_2(\varphi_2, \varphi_6, t)}{\partial \varphi_6}
\end{pmatrix}.
\]

Let us consider the informational correlation over the time interval \( 0 \leq t \leq T = 10 \) and define \( \hat{g}(\varphi_2, \varphi_6) \subset \hat{G}^A \) as follows:
\[
\hat{g} : \varepsilon < \varphi_2, \varphi_6 < \frac{\pi}{2} - \varepsilon.
\]

To find out the time intervals where both parameters \( \varphi_i, i = 2, 6 \), may be detected, we construct the function \( M_{26}^2(t) \) defined as
\[
M_{26}^2(t) = \frac{M_{26}^2(t)}{M_{26}^{max}}, \quad M_2^{26}(t) = \min_{\varphi_2, \varphi_6 \in g} \sum_{i=1}^{3} |M_{2i}^2(\varphi_2, \varphi_6, t)|, \quad M_{26}^{max} = \max_{0 \leq t \leq T} M_2^{26}(t),
\]
where the minimization is over the parameters \( \varphi_2 \) and \( \varphi_6 \) inside of the region \( \hat{g} \), and \( M_{2i}^2(\varphi_2, \varphi_6, t), i = 1, 2, 3 \), are the second order minors of the matrix \( \tilde{H} \):
\[
M_{21}^2 = \begin{vmatrix}
\frac{\partial a_0}{\partial \varphi_2} & \frac{\partial a_0}{\partial \varphi_6} \\
\frac{\partial a_0}{\partial \varphi_2} & \frac{\partial a_0}{\partial \varphi_6}
\end{vmatrix}, \quad M_{22}^2 = \begin{vmatrix}
\frac{\partial a_0}{\partial \varphi_2} & \frac{\partial a_0}{\partial \varphi_6} \\
\frac{\partial a_0}{\partial \varphi_2} & \frac{\partial a_0}{\partial \varphi_6}
\end{vmatrix}, \quad M_{23}^2 = \begin{vmatrix}
\frac{\partial a_1}{\partial \varphi_2} & \frac{\partial a_1}{\partial \varphi_6} \\
\frac{\partial a_1}{\partial \varphi_2} & \frac{\partial a_1}{\partial \varphi_6}
\end{vmatrix},
\]
where we do not write the parameters \( \varphi_i \) in the arguments of the functions for the sake of simplicity. The function \( M_{26}^2(t) \) with \( \varepsilon = \frac{\pi}{160} \) is depicted in Fig[1]. We see that the function \( M_{26}^2(t) \) is positive during the rather long time intervals. However, the time intervals corresponding to the very small values of this function must be disregarded because there might be obstacles to detect parameters \( \varphi_i, i = 2, 6 \) during these intervals (for instance, because of fluctuations). For this reason, we suggest to use time intervals corresponding to \( M_{26}^2 > \frac{1}{2} \). Fig[1] shows that there are two such subintervals inside of the selected interval
FIG. 1: The function $M_{26}(t)$ given by eq.(132) with $M_{26}^{\text{max}} = 1.247 \times 10^{-11}$ and $\varepsilon = \frac{\pi}{160}$, there are two time intervals corresponding to $M_{26}(t) > \frac{1}{2}$: $3.781 < t < 4.827$ and $7.082 < t < 8.678$.

$0 < t < 10$: $3.781 < t < 4.827$ and $7.082 < t < 8.678$. We also may say that the non-reducible informational correlation provided by the parameters $\varphi_2$ and $\varphi_6$ with zero values of other parameters and eigenvalues (129) is $E^{AB;\text{min}}(\hat{g}, t) = 2$ inside of the two above time subintervals (remember that $\lambda_i^B = \frac{1}{4}$, $i = 1, 2, 3, 4$, in this example).

b. **Example 2:** ran $\hat{T} = 9$, $E^{AB} = 6$ in eq.(122). In this case, using formula (59) for $H$, we obtain that the third order minors of the $3 \times 6$ matrix $H$ are nonzero, so that eq.(60) yields $E^{AB;\text{min}} = 3$. The analysis of the third-order minors shows that any triad of the parameters $\varphi_i$, $i = 1, \ldots, 6$ may be transferred from the subsystem $A$ to the subsystem $B$ by the eigenvalues of the density matrix $\rho^B(\varphi^A, t)$ except for the triad of the parameters $\varphi_1, \varphi_3, \varphi_5$, which are introduced in the unitary transformation (104) by the diagonal matrix exponents $e^{i\gamma_i \varphi_i}$, $i = 1, 3, 5$.

Similar to the previous example, we consider the informational correlation established by the three parameters $\varphi_2$, $\varphi_4$ and $\varphi_6$ putting other parameters to zero. Thus the cut set of parameters is $\hat{\varphi}^A = \{\varphi_2, \varphi_4, \varphi_6\}$. Therewith we take $\hat{\varphi}^A \in \hat{G}^A$ where $\hat{G}^A : 0 < \varphi_2, \varphi_4, \varphi_6 <$
\[ \frac{\pi}{2} \]. We also fix eigenvalues as follows:

\[
\begin{align*}
\lambda_1^A &= \lambda_2^A = \lambda_3^A = \frac{4}{15}, & \lambda_4^A &= \frac{1}{5}, \\
\lambda_1^B &= \lambda_2^B = \frac{4}{15}, & \lambda_3^B &= \lambda_4^B &= \frac{7}{30}.
\end{align*}
\]  

(134)

In this case only the second, the fourth and the sixth columns of the matrix \( H(t) \) (59) are nonzero so that we may replace \( H \) by the following matrix \( \tilde{H} \):

\[
\tilde{H}(\varphi_2, \varphi_4, \varphi_6, t) = \begin{pmatrix}
\frac{\partial a_0(\varphi_2, \varphi_4, \varphi_6, t)}{\partial \varphi_2} & \frac{\partial a_0(\varphi_2, \varphi_4, \varphi_6, t)}{\partial \varphi_4} & \frac{\partial a_0(\varphi_2, \varphi_4, \varphi_6, t)}{\partial \varphi_6} \\
\frac{\partial a_1(\varphi_2, \varphi_4, \varphi_6, t)}{\partial \varphi_2} & \frac{\partial a_1(\varphi_2, \varphi_4, \varphi_6, t)}{\partial \varphi_4} & \frac{\partial a_1(\varphi_2, \varphi_4, \varphi_6, t)}{\partial \varphi_6} \\
\frac{\partial a_2(\varphi_2, \varphi_4, \varphi_6, t)}{\partial \varphi_2} & \frac{\partial a_2(\varphi_2, \varphi_4, \varphi_6, t)}{\partial \varphi_4} & \frac{\partial a_2(\varphi_2, \varphi_4, \varphi_6, t)}{\partial \varphi_6}
\end{pmatrix}.
\]  

(135)

Similar to the previous example, we consider the informational correlation during the time interval \( 0 \leq t \leq 10 \) and define \( \hat{g} \subset \hat{G}^A \) as follows:

\[
\hat{g} : \varepsilon < \varphi_2, \varphi_4, \varphi_6 < \frac{\pi}{2} - \varepsilon.
\]  

(136)

To find out the time intervals where all three parameters \( \varphi_i, i = 2, 4, 6, \) may be detected, we construct the function \( M^{246}(t) \) defined by the following formula (similar to eq.(132)):

\[
M^{246}(t) = \frac{M_{246}^{31}(t)}{M_{246}^{max}}, \quad M_{246}^{31}(t) = \min_{\varphi_2, \varphi_4, \varphi_6 \in \hat{g}} |M_{31}^{246}(\varphi_2, \varphi_4, \varphi_6, t)|, \quad M_{246}^{max} = \max_{0 \leq t \leq T} M_{31}^{246}(t),
\]  

(137)

where the minimization is over the parameters \( \varphi_2, \varphi_4 \) and \( \varphi_6 \) inside of the region \( \hat{g} \), and the only nonzero 3rd-order minor of \( \tilde{H} \) is \( \det \tilde{H} \), i.e. \( M_{31}^{246} = \det \tilde{H} \). The function \( M^{246}(t) \) with \( \varepsilon = \frac{\pi}{50} \) is depicted in Fig.2 Similar to the previous example, we select the time intervals with \( M^{246} > \frac{1}{2} \) as the suitable intervals for the parameter detection. We see that there are two such subintervals inside of the interval \( 0 < t < 10 \): \( 3.257 < t < 4.520 \) and \( 7.233 < t < 7.983 \). We may state that the non-reducible informational correlation provided by the parameters \( \varphi_2, \varphi_4 \) and \( \varphi_6 \) with zero values of other parameters and eigenvalues (134) is \( E^{AB;min}(\hat{g}, t) = 3 \) inside of the two above time subintervals.

IV. CONCLUSIONS

We introduce the informational correlation between two subsystems as the possibility to effect on the state of the subsystem \( B \) through the parameters of the unitary transformation
FIG. 2: The function $M^{246}(t)$ given by eq. (137) with $M^{246}_{\max} = 2.292 \times 10^{-25}$, $\varepsilon = \frac{\pi}{50}$; there are two time intervals corresponding to $M^{246}(t) > \frac{1}{2}$: $3.257 < t < 4.520$ and $7.233 < t < 7.983$.

$U^A$ locally performed on the subsystem $A$ and vice-versa. The measure of the informational correlation $E^{AB}$ is the number of parameters of the local unitary transformation $U^A$ which may be detected in the subsystem $B$. We also introduce the normalized measure of the informational correlation $E^{AB}_{\text{norm}}$ showing whether the informational correlation is far from the saturation. The so-called non-reducible informational correlation $E^{AB;\text{min}}(t)$ is of a special interest, because these part of informational correlation is invariant with respect to the local unitary transformations of the subsystem $B$ at the time instant $t$.

All in all, the informational correlation has the following properties (for the tensor product initial density matrix (13)).

1. Unlike the entanglement and discord, the informational correlation represents a dynamical characteristic which is identical to zero at the initial time instant.

2. The informational correlation is invariant with respect to the initial local unitary transformations of the subsystem $A$, similar to the usual entanglement and discord. However, the informational correlation is not invariant with respect to the local unitary transformations of the subsystem $B$ (either initial or $t$-dependent), unlike the
entanglement and discord. Consequently, using the local unitary transformations of the receiver $B$ we may handle (up to a certain extent) the number of the parameters transferred from the subsystem $A$ to the subsystem $B$ and, thus, manipulate the informational correlation $E^{AB}$. The local transformations performed on the subsystem $C$ may also effect $E^{AB}$.

3. $E^{AB}(t) \equiv 0$ only if the initial density matrix $\rho^A(0)$ in formula (13) is proportional to the identity matrix. For the tensor product initial state $\rho^{AB} = \rho^B = \rho^A = 0$ only if both $\rho^A(0)$ and $\rho^B(0)$ are proportional to the identity matrix. The unitary invariant discord possesses the same property [27].

4. The complete information transfer is not required in order to obtain the maximal possible value of $E^{AB}$, because the maximal possible number of arbitrary parameters $\varphi_i$ transferred from $A$ to $B$ is less then $(N^A)^2 - 1$ (the maximal number of different real parameters in the $N^A \times N^A$ density matrix).

5. The informational correlation is sensitive to the multiplicity of the eigenvalues of the matrices $\rho^A(0)$ and $\rho^B(0)$ for the case of the tensor product initial state $\rho^{AB}$.

6. It is interesting that the conditions $E^{AA} < \tilde{D}^A$ and $E^{AB} < E^{AA}$ require the strong relations among the eigenvalues $\lambda^A_i$, $\lambda^B_i$ and $\lambda^C_i$. For the particular examples, these relations have been found in Sec.III, eqs.(71,79,81,106,112,117,118). The minor deviation from these exact relations leads (i) to the encoding of the maximal possible parameters $\tilde{D}^A$ into the subsystem $A$ and (ii) to the spread of the complete information throughout the whole system and consequently to the maximal possible informational correlation $E^{AB} = E^{AA} = \tilde{D}^A$. This phenomenon must be closely related with the fluctuations of the informational correlation and requires more detailed study.

7. There are two subsets of parameters $\varphi_i$ transferred from $A$ to $B$: $\varphi^U$ and $\varphi^A$. The first one may be detected in the matrix of the eigenvectors of the reduced density matrix $\rho^B(\varphi^A, t)$, while the second subset is transfered by the eigenvalues of the same matrix. The subset $\varphi^A$ is most reliable for the purpose of the information transfer, because the number of parameters in this subset may not be decreased by any local unitary transformation performed on the subsystem $B$. Namely this subset is responsible for
the non-reducible informational correlation $E^{AB;\text{min}}$. Note that some of the parameters $\varphi_i$ might be encoded in both subsets $\varphi^U$ and $\varphi^\Lambda$. The informational correlation $E^{AB}$ and the non-reducible informational correlation $E^{AB;\text{min}}$ might be viewed as the analogies of the total and the classical correlations in the definition of the discord. The removable informational correlation $\Delta E^{AB} = E^{AB} - E^{AB;\text{min}}$ is the analog of the discord itself.

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V. APPENDIX

A. A. Explicit form of the matrices $\gamma_i$.

We give the list of matrices $\gamma_i$ representing the basis of the Lie algebra of $SU(4)$:

$$
\gamma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad 
\gamma_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad 
\gamma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
\gamma_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad 
\gamma_5 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad 
\gamma_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
\gamma_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad 
\gamma_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad 
\gamma_9 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},
$$

$$
\gamma_{10} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad 
\gamma_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad 
\gamma_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix},
$$

$$
\gamma_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad 
\gamma_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 \end{bmatrix}, \quad 
\gamma_{15} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.
$$

B. B. Proof of eq.(60) for multiple and zero eigenvalues $\lambda_i$ of the matrix $\rho^B(\varphi^A, t)$

Suppose that there are $(P + 1) < (N^B - 1)$ nonzero different eigenvalues of the matrix $\rho^B(\varphi, t)$. Then, in order to define the non-reducible informational correlation $E^{AB,min}$ we
may replace the characteristic equation (53) with the following polynomial one:

\[
\prod_{i=1}^{P} (\lambda - \lambda_i(\varphi^A, t)) = \lambda^P + \sum_{j=0}^{P-1} \tilde{a}_i(\varphi^A, t) \lambda^j = 0,
\]

(139)

where \(\tilde{a}_i\) are expressed in terms of \(\lambda_i\). In this equation, we take into account only \(P\) different nonzero eigenvalues because of the identity \(\text{Tr} \rho^B = \sum_{i=1}^{P+1} q_i \lambda_i = 1\), where \(q_i\) is the multiplicity of the root \(\lambda_i\). Then the non-reducible informational correlation is defined by the rank of the matrix

\[
J^B_\Lambda(\varphi^A, t) = \frac{\partial(\lambda_1(\varphi^A, t), \ldots, \lambda_P(\varphi^A, t))}{\partial(\varphi_1, \ldots, \varphi_{\tilde{D}^A})},
\]

(140)

so that

\[
E^{AB; \text{min}}(\varphi^A, t) = \text{ran} J^B_\Lambda(\varphi^A, t).
\]

(141)

Differentiating eq. (139) with respect to the parameters \(\varphi_k, k = 1, \ldots, \tilde{D}^A\), and solving the resulting equations for \(\frac{\partial \lambda}{\partial \varphi_k}\) we obtain:

\[
\frac{\partial \lambda}{\partial \varphi_k} = - \frac{\sum_{i=0}^{P-1} \frac{\partial \tilde{a}_i(\varphi^A, t)}{\partial \varphi_k} \lambda^i}{P \lambda^{P-1} + \sum_{i=1}^{P-1} i \tilde{a}_i(\varphi^A, t) \lambda^{i-1}}.
\]

(142)

Therefore, for the matrix \(J^B_\Lambda(\varphi^A, t)\) one has

\[
J^B_\Lambda(\varphi^A, t) = \frac{1}{\tilde{J}_0(\varphi^A, t)} \left( \begin{array}{cccc}
\sum_{i=0}^{P-1} \frac{\partial \tilde{a}_i(\varphi^A, t)}{\partial \varphi_1} \lambda_1^i(\varphi^A, t) & \cdots & \sum_{i=0}^{P-1} \frac{\partial \tilde{a}_i(\varphi^A, t)}{\partial \varphi_{\tilde{D}^A}} \lambda_P^i(\varphi^A, t) \\
\vdots & \ddots & \vdots \\
\sum_{i=0}^{P-1} \frac{\partial \tilde{a}_i(\varphi^A, t)}{\partial \varphi_1} \lambda_P^i(\varphi^A, t) & \cdots & \sum_{i=0}^{P-1} \frac{\partial \tilde{a}_i(\varphi^A, t)}{\partial \varphi_{\tilde{D}^A}} \lambda_P^i(\varphi^A, t)
\end{array} \right) = \tilde{\Lambda}^B(\varphi^A, t) \tilde{H}(\varphi^A, t),
\]

(143)

where

\[
\tilde{J}_0(\varphi^A, t) = (-1)^P \prod_{j=1}^{P} \left( P \lambda_j^{P-1}(\varphi^A, t) + \sum_{i=1}^{P-1} i \tilde{a}_i(\varphi^A, t) \lambda_j^{i-1}(\varphi^A, t) \right) \neq 0,
\]

(144)
while $\tilde{\Lambda}^B$ and $\tilde{H}$ are the $P \times P$ and $P \times \tilde{D}^A$ matrices respectively:

$$\tilde{\Lambda}^B = \begin{pmatrix}
1 & \lambda_1(\varphi^A, t) & \cdots & \lambda_1^{P-1}(\varphi^A, t) \\
\vdots & \ddots & \ddots & \vdots \\
1 & \lambda_P(\varphi^A, t) & \cdots & \lambda_P^{P-1}(\varphi^A, t)
\end{pmatrix},$$  

(145)

$$\tilde{H}(\varphi^A, t) = \begin{pmatrix}
\frac{\partial a_0(\varphi^A, t)}{\partial \varphi_1} & \cdots & \frac{\partial a_0(\varphi^A, t)}{\partial \varphi_{\tilde{D}^A}} \\
\frac{\partial \tilde{a}_{p-1}(\varphi^A, t)}{\partial \varphi_1} & \cdots & \frac{\partial \tilde{a}_{p-1}(\varphi^A, t)}{\partial \varphi_{\tilde{D}^A}}
\end{pmatrix}. $$  

(146)

Then eq.(141) yields

$$E^{AB; \min}(\varphi^A, t) = \text{ran} \Lambda^B(\varphi^A, t) = \text{ran} \tilde{H}(\varphi^A, t).$$  

(147)

Now notice that $P$ coefficients $\tilde{a}_i$ in eq.(139) are defined by $P$ different nonzero eigenvalues $\lambda_i$, $i = 1, \ldots, P$. From another hand, the coefficients $a_i$ in eq.(53) are defined by the same $P$ independent eigenvalues $\lambda_i$, $i = 1, \ldots, P$, and consequently by $P$ coefficients $\tilde{a}_i$, $i = 1, \ldots, P$. The last statement is provided by the relation between sets $\tilde{a}_i$ and $\lambda_i$ following from eq.(139), where

$$\left| \frac{\partial(\tilde{a}_0, \ldots, \tilde{a}_{P-1})}{\partial(\lambda_1, \ldots, \lambda_P)} \right| \neq 0,$$  

(148)

because all $\lambda_i$, $i = 1, \ldots, \lambda_P$ are different by our requirement. Thus, for the matrix $H$ represented by eq.(59) we may write

$$H(\varphi^A, t) = F(\varphi^A, t)\tilde{H}(\varphi^A, t),$$  

(149)

where $F$ is $(N^B - 1) \times P$ matrix,

$$F(\varphi^A, t) = \begin{pmatrix}
\frac{\partial a_0(\varphi^A, t)}{\partial a_0} & \cdots & \frac{\partial a_0(\varphi^A, t)}{\partial \tilde{a}_{p-1}} \\
\vdots & \ddots & \vdots \\
\frac{\partial a_{N^B-2}(\varphi^A, t)}{\partial a_0} & \cdots & \frac{\partial a_{N^B-2}(\varphi^A, t)}{\partial \tilde{a}_{p-1}}
\end{pmatrix}. $$  

(150)

It may be readily shown that the rank of the matrix $F$ takes its maximal possible value, $\text{ran} F = P$, $P \leq (N^B - 1)$. In fact, since $a_i$, $i = 0, 1, \ldots, N^B - 2$, are expressed in terms of $\lambda_i$ (see eqs.(52,53)) and there are only $P$ independent eigenvalues $\lambda_i$, $i = 1, \ldots, P$, then
\[
\text{ran} \frac{\partial(a_0, a_1, \ldots, a_{N^B-2})}{\partial(\lambda_1, \ldots, \lambda_P)} = P. \quad \text{But} \quad \frac{\partial(a_0, a_1, \ldots, a_{N^B-2})}{\partial(\lambda_1, \ldots, \lambda_P)} = F \frac{\partial(\tilde{a}_0, \ldots, \tilde{a}_{P-1})}{\partial(\lambda_1, \ldots, \lambda_P)}. \]
Consequently, in virtue of condition (148), we conclude that \( \text{ran} F = \text{ran} \frac{\partial(a_0, a_1, \ldots, a_{N^B-2})}{\partial(\lambda_1, \ldots, \lambda_P)} = P. \) Thus the rank of the product \( F\tilde{H} \) equals to the rank of \( \tilde{H} \) in eq. (149), which yields
\[
\text{ran} H(\varphi^A, t) = \text{ran} \tilde{H}(\varphi^A, t). \quad (151)
\]
In turn, eq. (151) means that eq. (60) holds for the multiple and/or zero eigenvalues as well.

C. Informational correlation in systems with arbitrary initial state

The results obtained in Secs.II and III are based on the tensor product initial state (13). If the initial state is more general, then eq. (15) is not valid as well as eq. (28). In other words, the \( \varphi^A \)-dependence may not be collected in the density matrix \( \rho^A(\varphi^A, 0) \). In this case we also may introduce informational correlation \( E^{AB} \) by eqs. (36, 37) and the number of parameters encoded into the subsystem \( A, E^{AA} \), by eqs. (38, 39), where the vector \( \hat{X} \) is defined by eq. (25) together with eqs. (22). Again, the number of parameters encoded into \( \rho^A(\varphi^A, 0) \) is defined by the multiplicity of the eigenvalues of \( \rho^A(0) \). However, the representation (23) for \( X(\rho^B(\varphi^A, t)), Y(\rho^B(\varphi^A, t)) \) and \( Z(\rho^B(\varphi^A, t)) \) is not valid any more. Inequality (41) between \( E^{AB} \) and \( E^{AA} \) has no place as well. At first glance, this inequality must be replaced by the more formal one:
\[
E^{AB}(t) \leq E^{AA}. \quad (152)
\]
However, inequality (152) is not evident and might be wrong in general. In fact, applying the local transformation to the subsystem \( A \) we influence on the whole density matrix \( \rho(0) \) yielding the density matrix \( \rho(\varphi^A, 0) \). However, only certain combinations of the elements of \( \rho(\varphi^A, 0) \) appear in \( \rho^A(\varphi^A, 0) \). Thus, some of the parameters \( \varphi_i \) might be missed from the local density matrix \( \rho^A(\varphi^A, 0) \), but might be detected in the whole density matrix \( \rho(\varphi^A, 0) \). This forces us to denote the number of all parameters encoded into the initial density matrix \( \rho(\varphi^A, 0) \) by \( E^A \geq E^{AA} \) and define \( E^A \) by equation (similar to eqs. (37) and (39))
\[
E^A(\varphi^A, t) = \text{ran} J(\rho(\varphi^A, 0)), \quad (153)
\]
where
\[
J(\rho(\varphi^A, 0)) = \frac{\partial(\hat{X}_1(\rho(\varphi^A, 0)), \ldots, \hat{X}_{N^2-1}(\rho(\varphi^A, t)))}{\partial(\varphi_1, \ldots, \varphi_{DA})}. \quad (154)
\]
Thus, there might be such parameters \( \varphi_i \) that are not encoded into the initial reduced density matrix \( \rho^A(\varphi^A, 0) \), but might appear in the reduced density matrix \( \rho^B(\varphi^A, t) \) in the course of evolution. The number of these parameters may not exceed the value \( \delta E^A \),

\[
\delta E^A(\varphi^A, t) = E^A(\varphi^A, t) - E^{AA}.
\]  

(155)

Consequently, instead of (152), the following inequality holds:

\[
E^{AB}(\varphi^A, t) \leq E^A(\varphi^A, t).
\]  

(156)

Emphasize that \( E^{AB} \) depends on \( \varphi^A \) in the case of arbitrary initial state \( \rho(0) \). It is obvious that the normalized informational correlation \( E^{AB}_{\text{norm}} \) defined by formula (43) might be bigger than one in this case.

Now let us calculate \( E^{AB} \) using eqs. (36,37). In general, the rank of the matrix \( J(\rho^B) \) must be calculated numerically. For this purpose, we fix the time interval \( T_1 < t < T_2 \) taken for the parameter detection in the subsystem \( B \) and introduce the set of auxiliary functions

\[
\mathcal{M}_n(t) = \frac{\hat{M}_n(t)}{M_{n;\text{max}}}, \quad \hat{M}_n(t) = \int_{G^A} \sum_i |M_{ni}(\varphi^A, t)| d\Omega(\varphi^A), \quad \hat{M}_{n;\text{max}} = \max_{T_1 < t < T_2} \hat{M}_n(t),
\]  

(157)

where \( M_{ni} \) are the \( n \)th order minors of \( J(\rho^B) \), sum is over all minors, integration is over the whole \( G^A \) and \( \Omega(\varphi^A) \) is some measure. The function \( \mathcal{M}_n(t) \) is positive if only at least some of the \( n \)th order minors \( M_{ni} \) are nonzero over the non-zero volume subregion \( g(t) \) of the region \( G^A \) (note that \( g \) may depend on \( t \)). Then we define \( E^{AB}(g(t), t) \) as the maximal order \( n_0 \) of the positive functions \( \mathcal{M}_n(t) \), \( n = 1, \ldots, n_0 \), i.e.

\[
E^{AB}(g(t), t) = \max_{\mathcal{M}_n(t) > 0} n(t) = n_0(t),
\]  

(158)

so that \( n_0 \) depends on \( t \) in general. For the practical purpose, we might need to replace the positivity condition of \( \mathcal{M}_n(t) \) by the following one:

\[
\mathcal{M}_n(t) \geq \varepsilon,
\]  

(159)

where \( \varepsilon > 0 \) is some parameter predicted by the errors of calculations and/or experiment.

Let us consider the case of stationary region \( \hat{g}, \hat{g} \subset G^A \). The time intervals suitable for the detection of the transferred parameters in the subsystem \( B \) might be defined numerically
by the algorithm similar to that used in examples of Sec. III B 3. First of all we introduce
the auxiliary function $M(t)$ defined as follows:

$$
M(t) = \frac{M_{n_0}(t)}{M_{max}}, \quad M_{n_0}(t) = \min_{\psi^A \in g} \sum_i |M_{n_0}(\psi^A, t)|, \quad M_{max} = \max_{T_1 \leq t \leq T_2} M_{n_0}(t).
$$

(160)

Formally, any time instant corresponding to the positive $M$ is suitable for the parameter
detection. However, if $M$ is too small, then there might be some obstacles for the correct
detection of these parameters (for instance, fluctuations). Thus we take only such time
subintervals inside of the taken interval $T_1 < t < T_2$ that satisfy the following condition:

$$
M(t) > \tilde{\epsilon}.
$$

(161)

Here $\tilde{\epsilon}$ is some positive parameter, predicted by the required accuracy. For instance, $\tilde{\epsilon} = \frac{1}{2}$
in examples of Sec. III B 3.

In a similar way we may study the non-reducible correlations. The formulas (157-161)
hold with replacement $E^{AB} \rightarrow E^{AB;min}$, therewith $M_{ni}$ must be the $n$th order minors of the
matrix $H$ defined by eq. (59).

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