Modulation instability analysis of a nonautonomous $(3 + 1)$-dimensional coupled nonlinear Schrödinger equation

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Abstract We have investigated the modulation instability (MI) analysis of a nonautonomous $(3 + 1)$-dimensional coupled nonlinear Schrödinger (NLS) equation with time-dependent dispersion and phase modulation coefficients. By employing standard linear stability analysis, we have obtained an explicit expression for the MI gain as a function of dispersion, phase modulation, perturbation wave numbers and an initial incidence power. A nonautonomous coupled NLS equation is found to be modulationally unstable for the same sign of dispersion and phase modulation coefficients. This equation is modulationally stable for zero dispersion and or phase modulation. For nonzero dispersion, the equation is found to be modulationally stable/unstable on distinct bandwidth of wave numbers. The trigonometric, exponential, algebraic functions of time and constant have been chosen as test functions for dispersion and phase modulation to study the effect on the MI analysis. The effect of focusing and defocusing medium on the MI analysis has also been investigated. The MI bandwidth in the focusing medium is found to be larger than defocusing medium. It is found that the MI of the equation can be managed by proper choice of the dispersion and phase modulation parameters.

Keywords Nonautonomous $(3 + 1)$-dimensional coupled nonlinear Schrödinger equation · Modulation instability analysis · Focusing medium · Defocusing medium

1 Introduction

Modulation instability is one of the critical processes that initiate the formation of periodic structures in nonlinear dispersive media. In this mechanism, a background carrier wave is modulated due to periodic seeding perturbations that initially grow but may also decay. New frequencies are generated during this process, and intensive energy exchange occurs between the participating waves. The MI process can produce an unlimited number of new frequency components, resulting in a periodic train of localized waves or pulses [1]. This effect has widespread manifestations in natural phenomena and applications in technology. The MI has also been studied in optical fibers [2–4], hydrodynamics [5], plasmas [6] and biology [7]. In nonlinear optics, MI has been the subject of extensive study because of its inherent connection with short pulse train generation in nonlinear optical media, as well as optical parametric amplification. The underlying mechanism involved in the initial stage of supercontinuum generation is closely connected to noise-driven MI. It has been revealed that MI is responsible for the emergence of an optical rogue wave in the supercontinuum generation process.

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There has been an increasing interest in the study of the variable-coefficient nonlinear evolution equations in recent years. Most of the real nonlinear wave equations possess variable coefficients that are more realistic in physical situations than their constant-coefficient counterparts. This is because the constant-coefficient models can only describe the propagation of wave groups in perfect systems. The year 1973 witnessed the first theoretical observation on optical solitons by Hasegawa and Tappert [8,9], and its experimental verification came in 1980 [10]. The mathematical description of the above observation is given by the NLS equation:

$$iu_t + \frac{1}{2}u_{xx} + \sigma |u|^2u = 0,$$  \hspace{1cm} (1)

where $u = u(x,t)$ represents a slowly varying electric field and $x$, $t$ stand for the spatial and temporal variables, respectively. Here, $u_t$, $u_{xx}$ and $|u|^2u$ are the time evolution, group velocity dispersion and self-phase modulation terms. $\sigma$ is the self-phase modulation coefficient. Eq. (1) physically illustrates the pulse propagation in the polarization-preserving nonlinear optical fiber. Moreover, Eq. (1) is essential for studying the evolution of water waves and other nonlinear waves. A single nonlinear equation cannot appropriately describe the behavior of solitons in optical fiber. In the present work, therefore, we have chosen a nonautonomous $(3 + 1)$-dimensional coupled NLS equation [11]

$$iu_t + A(t)(u_{xx} + u_{yy} + u_{zz}) + B(t)(|u|^2 + |v|^2)u = 0,$$
$$iv_t + A(t)(v_{xx} + v_{yy} + v_{zz}) + B(t)(|u|^2 + |v|^2)v = 0,$$  \hspace{1cm} (2)

where $u = u(x, y, z, t)$ and $v = v(x, y, z, t)$ represent the complex amplitude of circularly polarized waves, $A(t)$ and $B(t)$ are dispersion and phase management coefficients, respectively. The terms self-phase modulation and cross-phase modulation are represented by $|u|^2u$, $|v|^2v$ and $|u|^2v$, $|v|^2u$, respectively. The nonlinearities $|u|^2u$ and $|v|^2v$ and dispersions $u_{xx}$, $u_{yy}$, $u_{zz}$, $v_{xx}$, $v_{yy}$ and $v_{zz}$ that are present in Eq. (2) play a vital role in affecting the amplitude and phase modulation of a continuous wave. System (2) physically represents the transmission of coupled wave packets and “optical solitons” in nonlinear optical fibers and also describes transverse effects in nonlinear optical systems [12]. The closed-form soliton solutions of a $(3 + 1)$-dimensional coupled NLS equation were constructed by Huang et al. [13] and Lan [14] for constant coefficients and Yu et al. [15] for time-dependent variable coefficients by Hirota method. Liu et al. [16] studied the soliton solutions of a $(3 + 1)$-dimensional generalized Kadomtsev–Petviashvili equation. Recently, Kumar and Patel [17] obtained dispersion and phase-managed optical soliton solutions of Eq. (2) by complex amplitude ansatz and He’s semi-inverse methods.

The role of a closed-form solution or a numerical approximation may be replaced by MI analysis. The MI analysis has substantial significance in the study of numerical techniques employed in solving the solution of a nonautonomous $(3 + 1)$-dimensional coupled NLS equation and other similar equations. The procedure begins with forming an equilibrium state and ends with capturing the modulation instability gain spectrum. During the process, reasonable differential equations are derived by introducing suitable perturbations in the equilibrium state. The nonlinear dispersion relation thus obtained is solved to arrive at the final step of the MI gain for a nonautonomous $(3 + 1)$-dimensional coupled NLS equation.

Classical studies on MI had been traced back to light waves in dielectric material [3] and nonlinear hydromagnetic $n$ waves of right-hand polarization in a cold plasma [18]. In [19] a detailed study on different physical models had been conducted. Many researchers have used the MI analysis for various models such as nonautonomous Lenells–Fokas model [20], a variable-coefficient NLS equation with fourth-order effects [21], a variable-coefficient fourth-order NLS system [22], an integrable coupled NLS system [23], 2D quantum ultracold atoms [24], a deformed Fokas–Lenells equation [25], coupled derivative NLS equation [26], coupled NLS equation [27], linearly coupled complex cubic quintic Ginzburg Landau equations [28,29], nonlinear coupled NLS equations [30], an inhomogeneous NLS equation including a pseudo-stimulated-Raman-scattering term [31], perturbed NLS-Hirota equation [32], Mel’nikov system [33], cubic-quintic nonlinear Helmholtz equation [34], coupled Zakharov–Kuznetsov [35], coupled generalized NLS equations [36] and many other equations from [37]. In 2017, Mustafa et al. [38] investigated the MI analysis of the $(1 + 1)$-dimensional coupled NLS equation with constant coefficients. In 2020, Sulaiman and Bulut [39] analyzed the MI analysis of the $(1 + 1)$-dimensional...
coupled NLS equation with constant coefficients. In any of the articles cited above or elsewhere, we have not found to the best of our knowledge the discussion about the MI analysis of a nonautonomous (3 + 1)-dimensional coupled NLS equation.

The main objective of the present work is to investigate the MI analysis of a nonautonomous (3 + 1)-dimensional coupled NLS equation by using the standard linear stability analysis. The effect of dispersion and phase modulation coefficients on the MI analysis has been studied for different choices of test functions. The dependence of the MI gain on the dispersion and phase management coefficients, perturbation wave numbers and initial incidence power has also been investigated.

2 Modulation instability analysis

Modulation instability refers to the exponential growth of certain modulation sidebands of the nonlinear plane waves propagating in a dispersive medium due to the interplay between nonlinearity and dispersion. In the MI process, the perturbation of the continuous wave generates the short pulses due to the breakup of the wave into a periodic pulse train. The MI of a plane wave in a nonautonomous (3 + 1)-dimensional coupled NLS equation is investigated by studying the stability of its amplitude in the presence of a sufficiently small perturbation so that one can linearize the equation of the envelope and the carrier wave. The study of the above-mentioned exciting aspects in the nonautonomous (3 + 1)-dimensional coupled NLS equation is done by introducing a small perturbation in the amplitude and the phase both and then investigated the solution of Eq. (2).

For modulation instability analysis, we consider an equilibrium state solution of the nonautonomous (3 + 1)-dimensional coupled NLS equation (2) as:

\[ u = U_0 \times e^{i\phi_1(t)}, \quad v = V_0 \times e^{i\phi_2(t)}, \]

where \( U_0 \) and \( V_0 \) are the real constant-amplitudes (initial incidence power). Using equilibrium state solution (3) into Eq. (2), we obtain the time-dependent nonlinear phase shift \( \phi_1(t) \) and \( \phi_2(t) \) as

\[ \phi_1(t) = \phi_2(t) = \left( U_0^2 + V_0^2 \right)^{1/2} \int_0^t B(t')dt'. \]

This result shows that the nonlinear phase shifts \( \phi_1(t) \) and \( \phi_2(t) \) of the equilibrium state solution are equal and found to be function of the phase modulation coefficient \( B(t) \) only. Now, we introduce small perturbations in Eq. (3) to perform the stability analysis of the steady-state solution, giving rise to the following

\[
egin{align*}
  u &= [U_0 + \epsilon U_1(x, y, z, t)] e^{i[(U_0^2 + V_0^2)^{1/2} B(t')]dt'}, \\
  v &= [V_0 + \epsilon V_1(x, y, z, t)] e^{i[(U_0^2 + V_0^2)^{1/2} B(t')]dt'}.
\end{align*}
\]

In Eq. (4), \( U_1 \) and \( V_1 \) represent the complex amplitude of the perturbation, \( |U_1(x, y, z, t)| \ll U_0 \) and \( |V_1(x, y, z, t)| \ll V_0 \). Substituting Eq. (4) into Eq. (2) and linearizing the equation with respect to \( U_1, U_1^* \) and \( V_1, V_1^* \), where the asterisk stands for the corresponding complex conjugate, we obtain the following evolution equations for the perturbations as

\[
\begin{align*}
  iU_1 + A(t) [U_{1x} + U_{1yy} + U_{1zz}] &+ B(t) \left[ U_0 (U_1 + U_1^*) + V_0 (V_1 + V_1^*) \right] U_0 = 0, \\
  iV_1 + A(t) [V_{1x} + V_{1yy} + V_{1zz}] &+ B(t) \left[ U_0 (U_1 + U_1^*) + V_0 (V_1 + V_1^*) \right] V_0 = 0.
\end{align*}
\]

Eq. (5) is a coupled linear partial differential equation. Therefore, we assume the solution of Eq. (5) in the form

\[
U_1(x, y, z, t) = r_0(t) e^{i[Kx+Ly+Mz-\int_0^t A(t')dt']} \\
+ r_1(t) e^{-i[Kx+Ly+Mz-\int_0^t A(t')dt']}
\]

\[
V_1(x, y, z, t) = s_0(t) e^{i[Kx+Ly+Mz-\int_0^t A(t')dt']} \\
+ s_1(t) e^{-i[Kx+Ly+Mz-\int_0^t A(t')dt']},
\]

where \( r_0(t), r_1(t), s_0(t), s_1(t) \) and \( A(t) \) are real valued functions of \( t \). The term \( Kx+Ly+Mz-\int_0^t A(t')dt' \) represents the phase of modulation, \( K, L \) and \( M \) are the perturbation wave numbers in \( x, y \) and \( z \) directions, and \( A(t) \) is the time-dependent perturbation wave frequency of the modulation waves. Substituting Eq. (6) into Eq. (5) and separating the coefficients of \( e^{i[Kx+Ly+Mz-\int_0^t A(t')dt']} \) and \( e^{-i[Kx+Ly+Mz-\int_0^t A(t')dt']} \), we get the following time-dependent linear homogeneous equations in \( r_0(t), r_1(t), s_0(t), \) and \( s_1(t) \) as
\[ \alpha_0 r_0(t) + B(t) U_0^2 r_1(t) + B(t) U_0 V_0 s_0(t) + B(t) U_0 V_0 s_1(t) + i r_0, = 0, \]
\[ B(t) U_0^2 r_0(t) + \beta_0 r_1(t) + B(t) U_0 V_0 s_0(t) + B(t) U_0 V_0 s_1(t) + i r_1, = 0, \]
\[ B(t) U_0 V_0 r_0(t) + B(t) U_0 V_0 r_1(t) + \gamma_0 s_0(t) + B(t) V_0^2 s_1(t) + i s_0, = 0, \]
\[ B(t) U_0 V_0 r_0(t) + B(t) U_0 V_0 r_1(t) + B(t) V_0^2 s_0(t) + \delta_0 s_1(t) + i s_1, = 0. \] (7)

where
\[ \alpha_0 = \gamma_0 = \left[ B(t) U_0^2 + A(t) - A(t)(K^2 + L^2 + M^2) \right] \]
and
\[ \beta_0 = \delta_0 = \left[ B(t) V_0^2 + A(t) - A(t)(K^2 + L^2 + M^2) \right]. \]

Equating the real and imaginary parts in Eq. (7), we get \( r_0 = r_1, = s_0 = s_1, = 0. \) Further, the obtained system of equations can be represented in matrix form as
\[
\begin{pmatrix}
\alpha_0 & g_1 & g_3 & g_3 \\
g_1 & \beta_0 & g_3 & g_3 \\
g_3 & g_3 & \gamma_0 & g_2 \\
g_3 & g_3 & g_2 & \delta_0
\end{pmatrix}
\begin{pmatrix}
r_0 \\
r_1 \\
s_0 \\
s_1
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}. \] (8)

where \( g_1 = B(t) U_0^2, g_2 = B(t) V_0^2 \) and \( g_3 = B(t) U_0 V_0. \) In order to obtain a nontrivial solution, the determinant of the coefficient matrix may be allowed to vanish, which gives the following time-dependent dispersion relations
\[ A(t) = \pm A(t) (K^2 + L^2 + M^2); \]
\[ A(t) = \pm \sqrt{\frac{A(t)(K^2 + L^2 + M^2) \times \left[ A(t)(K^2 + L^2 + M^2) - 2B(t)(U_0^2 + V_0^2) \right]}{[2B(t)(U_0^2 + V_0^2) - A(t)(K^2 + L^2 + M^2)]}}. \] (9)

These relations reveal the dependency of the time-dependent oscillations \( e^{iA(t)} \) on the spatial oscillations \( e^{iKx}, e^{iLy}, \) and \( e^{iMz} \). The stability of the perturbed steady-state solution depends on the real or complex value of \( A(t) \). For \( B(t) = 0 \), the second dispersion relation reduces into the first. The \( A(t) = \pm A(t) (K^2 + L^2 + M^2) \) is always real, and therefore, the corresponding modulation instability does not occur. Similarly, for \( A(t) = 0 \) we get \( A(t) = 0 \), so the system is again modulationally stable. The steady-state solution becomes unstable whenever \( A(t) \) has an imaginary part as in this case, perturbation grows exponentially along the fiber length. This phenomenon is called modulation instability [4,33,34] as it leads to modulation of the steady-state solution. From Eq. (9), we can observe that the steady-state solution of a nonautonomous \((3+1)\)-dimensional coupled NLS equation will be modulationally stable or unstable whenever \( \left( A(t) \frac{A(t)(K^2 + L^2 + M^2) \times \left[ A(t)(K^2 + L^2 + M^2) - 2B(t)(U_0^2 + V_0^2) \right]}{[2B(t)(U_0^2 + V_0^2) - A(t)(K^2 + L^2 + M^2)]} \right) \geq 0 \) or \( < 0 \), respectively. It is found that \( A(t) \) is imaginary when either \( A(t) \) and \( B(t) \) are both positive or both negative. The MI gain \( G \) of a nonautonomous \((3+1)\)-dimensional coupled NLS equation can be written as
\[ G = 2 \text{Im}(A(t)) = 2 \sqrt{\frac{A(t)(K^2 + L^2 + M^2) \times \left[ A(t)(K^2 + L^2 + M^2) - 2B(t)(U_0^2 + V_0^2) \right]}{[2B(t)(U_0^2 + V_0^2) - A(t)(K^2 + L^2 + M^2)]}}. \] (10)

Clearly, the MI gain \( G \) depends on the dispersion coefficient \( A(t) \), phase modulation coefficient \( B(t) \), perturbation wave numbers \( K, L, M \) and incidence power \( U_0 \) and \( V_0 \) in the equilibrium state solution. For \( A(t) = 0 \), the gain \( G = 0 \) for the wave number \( K = K_s = \pm \sqrt{\frac{2B(t)(U_0^2 + V_0^2)}{A(t)}} \) and the coupled NLS equation is modulationally stable for bandwidth \( |K| \geq |K_s| \) and unstable for instability bandwidth \( |K| < |K_s| \). The width of the instability band depends on the dispersion and phase managed coefficients, incidence power \( (U_0, V_0) \) and perturbation wave numbers \( L \) and \( M \). The MI gain attains a minimum value
\[ G_{\text{min}} = \sqrt{\frac{A(t)(L^2 + M^2) \times \left[ 2B(t)(U_0^2 + V_0^2) - A(t)(L^2 + M^2) \right]}{[2B(t)(U_0^2 + V_0^2) - A(t)(L^2 + M^2)]}}; \]

at wave number \( K = 0 \) and maximum value
\[ G_{\text{max}} = 2B(t) \sqrt{2(U_0^2 + V_0^2)}, \]

at \( K = \pm \sqrt{\frac{B(t)(U_0^2 + V_0^2)}{A(t)}} - (L^2 + M^2) \). It is found that \( G_{\text{min}} \) and \( K \) can be controlled by the dispersion and phase modulation both but \( G_{\text{max}} \) can only be controlled by phase managed coefficient. Further, \( G_{\text{max}} \) is inde-
pendent of the wave numbers $L$ and $M$. The gain spectra $G$ are plotted against wave number $K$ in Figs. 1, 2, 4, 5, 7, 10 and 12; and against wave numbers $K$ and $L$ in Figs. 3, 6, 9 and 11 for different choices of dispersion $A(t)$, phase modulation $B(t)$, fixed values of wave numbers $L$ and/or $M$ and incidence power $(U_0, V_0)$. The effects of focusing and defocusing are plotted in Figs. 8, 9, 10, 11, 12. The MI analysis of the steady-state solution of a nonautonomous $(3 + 1)$-dimensional coupled NLS equation is discussed in the next section.

### 3 Results and discussion

The stability of the steady-state solution of a nonautonomous $(3 + 1)$-dimensional coupled NLS equation has been investigated by the method of MI analysis. The MI gain $G$ is calculated and plotted against time $t$, and the perturbation wave numbers $K$, $L$ for a fixed value of $M$, $U_0$ and $V_0$. From Eq. (10), we have found that the equilibrium state $(U_0, V_0)$, normalized wave numbers $K$, $L$, $M$, dispersion $A(t)$ and phase modulation $B(t)$ have a significant effect on the MI gain $G$. In the absence of dispersion $(A(t) = 0)$ and/or phase modulation $(B(t) = 0)$, the system is found to be modulationally stable. It has also been found that the MI gain exists only when either $A(t) > 0$, $B(t) > 0$ or $A(t) < 0$, $B(t) < 0$. This observation is similar to the results of [4] for constant coefficients. The effects of the distinct time-dependent dispersion and phase modulation coefficients on the MI gain have been discussed under the following subsections:

#### 3.1 Impact of trigonometric dispersion and phase modulation on MI

Figures 1a–2a give the variation of the MI gain for trigonometric function $A(t) = \sin t$, $B(t) = \cos t$, whereas Fig. 2b–d gives for $A(t) = \cos t$, $B(t) = \sin t$. The surface plots (see Fig. 1a–b) of the MI gain give the clear pattern of the MI of a coupled NLS equation and its variation with time $t$ and perturbation wave number $K$. At a particular time $t = \pi/100$ (see Fig. 2a), the MI gain witnessed dual-band, which depicts that it is independent of the sign of the $K$. There exist two local maxima in the MI gain and an instability band for wave number $K$ with a fixed value of the initial incidence power $(U_0, V_0)$. With an increase in the initial incidence power $(U_0, V_0)$, the local maxima and the width of the instability band both increase. The contour plot of the MI is shown in Fig. 1c. Figure 2b–d gives the variation in the MI gain with the change in values of perturbation wave numbers $L$ and $M$. With an increase in the value of incidence power $(U_0, V_0)$, there develops a single MI band for $L = 0.2$ and $M = 0.5$ (see Fig. 2b). For $U_0 = 1.0$ and $V_0 = 2.0$, there exist
local MI gain maxima at $K = 0$. But for $U_0 = 1.5$ and $V_0 = 2.5$, the MI gain achieves the maxima in the interval $|K| < 0.2$. With a decrease in the value of $L$ and $M$, the single MI band (see Fig. 2b) split into two fully developed sidebands (see Fig. 2d). The partially developed sidebands are shown in Fig. 2c. The width of the instability band for wave number $K$ increases with the decrease in the perturbation wave number $L$ and $M$. For $L = 0.2$ and $M = 0$, the MI gain $G$ attains local minima at $K = 0$ (see Fig. 2c). For $L = 0$ and $M = 0$, the local minima in the MI gain $G$ hits the zero gain at $K = 0$ (see Fig. 2d). Figure 3 shows that the MI gain and instability bandwidth increases with the increase in initial incidence power but decreases with time. The contour plots (see Fig. 3b and d) also confirm it.

3.2 Impact of trigonometric and exponential dispersion and phase modulation on MI

The surface plots of the MI gain (see Fig. 4a–b) for $A(t) = \sin t$ and $B(t) = -\exp(t)$ for $\pi < t < 2\pi$ show a similar pattern as in Fig. 1a–b, but the gain increases exponentially with time $t$. For $t = 3\pi/2$, Fig. 5a depicts the two-dimensional variation of the MI gain. For $A(t) = -\exp(t)$ and $B(t) = \sin t$, the two-dimensional plot of the MI gain is given in Fig. 5b–d. The two local maxima in the MI gain and MI bandwidth increase with phase-managed coefficient $B(t) = -\exp(t)$ (see Fig. 5a), keeping a similar pattern of the MI gain spectrum for $B(t) = \sin t$ (see Fig. 2a). The contour plot of the MI gain is shown in Fig. 4c. Figure 5b gives the zero MI gain for all the values of $K$, opposite to the case of a single band of Fig. 2b for corresponding values of $L$ and $M$ and initial incidence power $(U_0, V_0)$. It is found that for initial incidence power $(U_0, V_0) > (2.6, 2.9)$, the single MI gain band originates. Therefore, we may conclude that the dispersion coefficient $A(t)$ and incidence power $(U_0, V_0)$ largely affect the MI gain of the coupled NLS equation. Figure 5c–d shows a similar pattern of the MI gain as in Fig. 2c–d with higher values of local maxima and MI bandwidth. Figures 3 and 6 show that MI gain and instability bandwidth both are symmetric with respect to wave numbers.

3.3 Impact of algebraic dispersion and phase modulation on MI

Figure 7a–c gives the variation of the MI gain for algebraic functions $A(t) = t/2$ and $B(t) = t$ at time $t = 0.5$. It shows the existence of the two sidebands in
the MI gain with respect to wave number \( K \). With an increase in initial incidence power \((U_0, V_0)\), the local maxima and the MI bandwidth increase as in earlier cases 3.1 and 3.2 for fixed values of \( L \) and \( M \). For algebraic coefficients, the MI sidebands are already developed for \( L = 0.2 \) and \( M = 0.5 \), whereas in both of the earlier cases the sidebands develop with decrease in the value of \( L \) and \( M \) (see Figs. 2b–d, 5b–d and 7a–c). In algebraic case, the bandwidth remains constant with a decrease in the value of the perturbation wave numbers \( L \) and \( M \). This is opposite to the case of trigonometric and exponential coefficients, where the bandwidth increases. For perturbation wave numbers \( L = 0 \) and \( M = 0 \), the MI gain spectrum hits the zero gain at \( K = 0 \) that is independent of the type of the dispersion and phase modulation (see Figs. 2d, 5d and 7c). The gain spectrum in this case is similar with the corresponding two cases of subsections (3.1) and (3.2).

### 3.4 Effect of focusing and defocusing medium on MI

The focusing effect \((B(t) = +1)\) on the MI gain is shown in Fig. 8a–c at time \( t = \pi/100 \), and the defocusing effect \((B(t) = -1)\) is given in Fig. 10a–c at time \( t = 3\pi/2 \). In focusing as well as defocusing cases of the coupled NLS equation, the two local maxima and the MI bandwidth remain constant with wave numbers \( L \) and \( M \). For \( L = 0.2 \) and \( M = 0.5 \), the MI gain spectrum of a coupled NLS equation hits zero gain at \( K = 0 \) in the focusing case (see Fig. 8a) but not in the defocusing case (see Fig. 10a). When we consider the perturbation in complex amplitude \( U_1 \) and \( V_1 \) only along \( x \)-direction (i.e., \( L = 0, M = 0 \)), then the pattern of the MI gain is identical in the focusing and defocusing cases (see Figs. 8c and 10c). For the corresponding value of incidence power \((U_0, V_0)\), the local maxima in both the focusing and defocusing cases are equal, but the MI bandwidth is quite larger in the focusing case. In focusing medium \((B(t) = +1)\), the MI gain and instability bandwidth both decrease with time (see Fig. 9). In defocusing medium, the gain is not changing significantly, but instability bandwidth decreases for small change in
time (see Fig. 11). Also, in the neighborhood of time $t = \pi$, the MI gain and instability bandwidth are similar in focusing and defocusing medium (see Figs. 9c and 11a).

3.5 Comparative analysis:

We have compared our result of the MI analysis of the nonautonomous $(3+1)$-dimensional coupled NLS equation with [38] and [39]. Figure (12) explains the MI analysis for fixed values of wave numbers $L = 0$, $M = 0$, dispersion coefficient $A(t) = 1$ and phase modulation $B(t) = 1$. The values $L = 0$, $M = 0$ incor-
porate the perturbation in equilibrium state solution only in $x-$direction. It is found that the MI gain spectrum obtained in Fig. (12) is very similar to the spectrum obtained in [39] and better in comparison to the result in [38]. This validates the extension of the MI analysis for the case of a nonautonomous $(3+1)$-dimensional coupled NLS equation in this work.

4 Conclusions

This paper investigates the modulation instability of a nonautonomous $(3+1)$-dimensional coupled NLS equation. We have considered small perturbations in steady-state solution in all the directions $x$, $y$ and $z$ in terms of perturbation wave numbers. It is found that the nonlinear phase shift in steady-state solution depends only on the phase modulation coefficient. Some important observations about the MI of a nonautonomous $(3+1)$-dimensional coupled NLS equation can be summarized as follows:

- The MI gain is found to be a function of time $t$, the initial incidence power and perturbation wave numbers. The gain is symmetric with respect to the wave numbers.
- The system of a nonautonomous coupled NLS equation is modulationally stable in the absence of dispersion and/or phase modulation against weak perturbations. The cross-phase modulation, together with the self-phase modulation, is responsible for novel instability.

- The MI gain exists only for the same sign of dispersion and phase modulation. It can be controlled by choosing desired test function for dispersion and phase modulation coefficients.
- The width of the instability band depends on the dispersion and phase modulation coefficients in a coupled NLS equation, incidence power and perturbation wave numbers in the steady-state solution.
- The value of the MI gain for the exponential coefficient is much higher than the trigonometric coefficient for fixed values of the other parameters. For algebraic and constant dispersion and phase modulation, the MI bandwidth is independent of perturbation wave numbers.
- The local maxima in the MI gain in the focusing and defocusing medium are equal, but the MI bandwidth is found significantly larger for the focusing medium.

The MI analysis of a nonautonomous $(3+1)$-dimensional coupled NLS equation reveals that the modulation instability of the equation can be controlled by the proper choice of the dispersion and phase modulation coefficients, initial incident power and perturbation wave numbers in three dimensions. Although this study is limited in the context of optical fiber, the analysis may find application in other nonlinear dispersive system and in the different branches of physics.

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Declarations

Conflict of interest The authors declared that they have no conflicts of interest in this work.

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References

1. Akhmediev, N., Ankiewicz, A.: Solitons: Nonlinear Pulses and Beams. Chapman & Hall, Cambridge (1997)
2. Hasegawa, A.: Generation of a train of soliton pulses by induced modulational instability in optical fibers. Opt. Lett. 9, 288–290 (1984)
3. Tai, K., Hasegawa, A., Tomita, A.: Observation of modulational instability in optical fibers. Phys. Rev. Lett. 56, 135–138 (1986)
4. Agrawal, G.P.: Modulation instability induced by cross-phase modulation. Phys. Rev. Lett. 59, 880–883 (1987)
5. Benjamin, T.B., Feir, J.E.: The disintegration of wave trains on deep water part 1. Theory. J. Fluid Mech. 27, 417–430 (1967)
6. Zakharov, V.E.: Collapse of Langmuir Waves. Sov. Phys.-Jetp 35, 908–914 (1972)
7. Turing, A.M.: The chemical basis of morphogenesis. Philos. Trans. R. Soc. Lond. B 237, 37–72 (1952)
8. Hasegawa, A., Tappert, F.: Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion. Appl. Phys. Lett. 23, 142–144 (1973)
9. Hasegawa, A., Tappert, F.: Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. II. Normal dispersion. Appl. Phys. Lett. 23, 171–172 (1973)
10. Mollenauer, L.F., Stolen, R.G., Gordon, J.P.: Experimental observation of picosecond pulse narrowing and solitons in optical fibers. Phys. Rev. Lett. 45, 1095–1098 (1980)
11. Mansfield, E.L., Reid, G.J., Clarkson, P.A.: Nonclassical reductions of a 3 + 1-cubic nonlinear Schrödinger system. Comput. Phys. Comm. 115, 460–488 (1998)
12. Abraham, N.B., Firth, W.J.: Overview of transverse effects in nonlinear-optical systems. J. Opt. Soc. Am. B 7, 951–962 (1990)
13. Huang, Z.R., Tian, B., Wang, Y.P., Sun, Y.: Bright soliton solutions and collisions for a (3 + 1)-dimensional coupled nonlinear Schrödinger system in optical-fiber communication. Comput. Math. Appl. 69, 1383–1389 (2015)
14. Lan, Z.-Z.: Dark solitonic interactions for the (3 + 1)-dimensional coupled nonlinear Schrödinger equations in nonlinear optical fibers. Opt. Laser Technol. 113, 462–466 (2019)
15. Yu, W., Liu, W., Triki, H., Zhou, Q., Biswas, A.: Phase shift, oscillation and collision of the anti-dark solitons for the (3 + 1)-dimensional coupled nonlinear Schrödinger equation in an optical fiber communication system. Nonlinear Dyn. 97, 1253–1262 (2019)
16. Liu, S.H., Tian, B., Qu, Q.X., Li, H., Zhao, X.H., Du, X.X., Chen, S.S.: Breather, lump, shock and travelling-wave solutions for a (3 + 1)-dimensional generalized Kadomtsev-Petviashvili equation in fluid mechanics and plasma physics. Int. J. Comput. Math. 1–16, (2020)
17. Kumar, V., Patel, A.: Dispersion and phase managed optical soliton solutions of a nonautonomous (3 + 1)-dimensional coupled nonlinear Schrödinger equation. Optik (2021). https://doi.org/10.1016/j.ijleo.2021.166648
18. Taniuti, T., Washimi, H.: Self-Trapping and Instability of Hydromagnetic Waves Along the Magnetic Field in a Cold Plasma. Phys. Rev. Lett. 21, 209–212 (1968)
19. Hasewara, A., Matsumoto, M.: Optical Solitons in Fibers. Springer, Berlin (1990)
20. Wang, L., Zhu, Y.-J., Qi, F.-H., Li, M., Guo, R.: Modulational instability, higher-order localized wave structures, and nonlinear wave interactions for a nonautonomous Lenells-Fokas equation in inhomogeneous fibers. Chaos 25, 063111 (2015)
21. Cai, L.Y., Wang, X., Wang, L., Li, M., Liu, Y., Shi, Y.Y.: Nonautonomous multi-peak solitons and modulation instability for a variable-coefficient nonlinear Schrödinger equation with higher-order effects. Nonlinear Dyn. 90, 2221–2230 (2017)
22. Chen, S.-S., Tian, B., Liu, L., Yuan, Y.-Q., Du, X.-X.: Breathers, multi-peak solitons, breather-to-soliton transitions and modulation instability of the variable-coefficient fourth-order nonlinear Schrödinger system for an inhomogeneous optical fiber. Chin. J. Phys. 62, 274–283 (2019)
23. Guo, D., Tian, S.F., Zhang, T.T., Li, J.: Modulation instability analysis and soliton solutions of an integrable coupled nonlinear Schrödinger system. Nonlinear Dyn. 94, 2749–2761 (2018)
24. Dijoufack, Z.I., Fotsa-Ngaffo, F., Tala-Tebue, E., Fendzi-Donfack, E., Kapche-Tagne, F.: Modulational instability in addition to discrete breathers in 2D quantum ultracold atoms loaded in optical lattices. Nonlinear Dyn. 98, 1905–1918 (2019)
25. Wang, X., Wei, J., Wang, L., Zhang, J.: Baseband modulation instability, rogue waves and state transitions in a deformed Fokas–Lenells equation. Nonlinear Dyn. 97, 343–353 (2019)
26. Tao, X., Guoliang, H.: The coupled derivative nonlinear Schrödinger equation: conservation laws, modulation instability and semirational solutions. Nonlinear Dyn. 100, 2823–2837 (2020)
27. Yao, X., Yang, Z.Y., Yang, W.L.: Frequency conversion dynamics of vector modulation instability in normal-dispersion high-birefringence fibers. Nonlinear Dyn. 103, 1035–1041 (2021)
28. Porsezian, K., Murali, R., Malomed, B.: Ganapathy, modulational instability in linearly coupled complex cubic-quintic Ginzburg–Landau equations. Chaos Solitons Fractals 40, 1907–1913 (2009)
29. Alcaraz-Pelegrina, J.M., Rodríguez-García, P.: Modulational instability in two cubic-quintic Ginzburg–Landau equations coupled with a cross phase modulation term. Phys. Lett. A 374, 1591–1599 (2010)
30. Govindarajan, A., Malomed, B.A., Mahalingam, A., Uthayakumar, A.: Modulational Instability in Linearly Coupled Asymmetric Dual-Core Fibers. Appl. Sci. 7, 645 (2017)
31. Calaça, L., Avelar, A.T., Malomed, B.A., Cardoso, W.B.: Influence of pseudo-stimulated-Raman-scattering on the modulational instability in an inhomogeneous nonlinear medium. Eur. Phys. J. Spec. Top. 227, 551–561 (2018)
32. Inc, M., Alyiu, A.I., Yusuf, A., Baleanu, D.: Dispersive optical solitons and modulation instability analysis of Schrödinger–Hirota equation with spatio-temporal dispersion and Kerr law nonlinearity. Superlatt. Microstruct. 113, 319–327 (2018)
33. Kumar, V., Patel, A.: Construction of the soliton solutions and modulation instability analysis for the Mel’nikov system. Chaos Solitons Fractals 140, 110159 (2020)
34. Tamilselvan, K., Kanna, T., Govindarajan, A.: Cubic-quintic nonlinear Helmholtz equation: modulational instability, rogue waves and state transitions in a deformed Fokas–Lenells equation. Nonlinear Dyn. 97, 343–353 (2019)
35. Kumar, V., Patel, A.: Soliton solutions and modulation instability analysis of the coupled Zakharov–Kuznetsov equation. Eur. Phys. J. Plus 134, 170 (2019)
36. Vishnu Priya, N., Senthilvelan, M.: On the characterization of breather and rogue wave solutions and modulation instability of a coupled generalized nonlinear Schrödinger equations. Wave Motion \textbf{54}, 125–133 (2015)
37. Agrawal, G.P.: Nonlinear Fiber Optics, 5th edn. Elsevier, New York (2013)
38. Inc, M., Aliyu, A.I., Yusuf, A., Baleanu, D.: Novel optical solitary waves and modulation instability analysis for the coupled nonlinear Schrödinger equation in monomode step-index optical fibers. Superlatt. Microstruct. \textbf{113}, 745–753 (2018)
39. Sulaiman, T.A., Bulut, H.: Optical solitons and modulation instability analysis of the (1+1)-dimensional coupled nonlinear Schrödinger equation. Commun. Theor. Phys. \textbf{72}, 025003 (2020)

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