On the combinatorial and rank properties of certain subsemigroups of full contractions of a finite chain

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Abstract

Let \( [n] = \{1, 2, \ldots, n\} \) be a finite chain and let \( \mathcal{C}T_n \) be the semigroup of full contractions on \([n]\). Denote \( \mathcal{O}CT_n \) and \( \mathcal{OCT}_n \) to be the subsemigroups of order preserving reversing and the subsemigroup of order preserving contractions, respectively. It was shown in [17] that the collection of all regular elements (denoted by, \( \text{Reg}(\mathcal{O}CT_n) \) and \( \text{Reg}(\mathcal{OCT}_n) \), respectively) and the collection of all idempotent elements (denoted by \( \text{E}(\mathcal{O}CT_n) \) and \( \text{E}(\mathcal{OCT}_n) \), respectively) of the subsemigroups \( \mathcal{O}CT_n \) and \( \mathcal{OCT}_n \), respectively are subsemigroups. In this paper, we study some combinatorial and rank properties of these subsemigroups.

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1 Introduction

Denote \( [n] = \{1, 2, \ldots, n\} \) to be a finite chain and let \( T_n \) denote the semigroup of full transformations on \([n]\). A transformation \( \alpha \in T_n \) is said to be order preserving (resp., order reversing) if (for all \( x, y \in [n] \)) \( x \leq y \) implies \( x\alpha \leq y\alpha \) (resp., \( x\alpha \geq y\alpha \)); order decreasing if (for all \( x \in [n] \)) \( x\alpha \leq x \); an isometry (i.e., distance preserving) if (for all \( x, y \in [n] \)) \( |x\alpha - y\alpha| = |x - y| \); a contraction if (for all \( x, y \in [n] \)) \( |x\alpha - y\alpha| \leq |x - y| \). Let \( \mathcal{C}T_n = \{\alpha \in T_n : (\text{for all } x, y \in [n]) \ |x\alpha - y\alpha| \leq |x - y| \} \) be the semigroup of full contractions on \([n]\), as such \( \mathcal{C}T_n \) is a subsemigroup of \( T_n \). Certain algebraic and combinatorial properties of this semigroup and some of its subsemigroups have been studied, for example see [1, 5, 6, 15, 16, 17, 19, 20, 21].

Let

\[ \mathcal{OCT}_n = \{\alpha \in \mathcal{C}T_n : (\text{for all } x, y \in [n]) x \leq y \implies x\alpha \leq y\alpha \}, \]

and

\[ \mathcal{ORCT}_n = \mathcal{OCT}_n \cup \{\alpha \in \mathcal{C}T_n : (\text{for all } x, y \in [n]) x \leq y \implies x\alpha \geq y\alpha \} \]
be the subsemigroups of order preserving full contractions and of order preserving or reversing full contractions on $[n]$, respectively. These subsemigroups are both known to be non-regular left abundant semigroups \(^{17}\) and their Green’s relations have been characterized in \(^{2}\). The ranks of $\text{ORCT}_n$ and $\text{OCT}_n$ were computed in \(^{15}\) while the ranks of the two sided ideals of $\text{ORCT}_n$ and $\text{OCT}_n$ were computed by Leyla \(^{5}\).

In 2021, Umar and Zubairu \(^{17}\) showed that the collection of all regular elements (denoted by $\text{Reg}(\text{ORCT}_n)$) of $\text{ORCT}_n$ and also the collection of idempotent elements (denoted by $\text{E}(\text{ORCT}_n)$) of $\text{ORCT}_n$ are both subsemigroups of $\text{ORCT}_n$. The two subsemigroups are both regular, in fact $\text{Reg}(\text{ORCT}_n)$ has been shown to be an $L$– unipotent semigroup (i.e., each $L$–class contains a unique idempotent). In fact, it was also shown in \(^{17}\) that the collection of all regular elements (denoted by $\text{Reg}\text{OCT}_n$) in $\text{OCT}_n$ is a subsemigroup. However, combinatorial as well as rank properties of these semigroups are yet to be discussed, in this paper we discuss these properties, as such this paper is a natural sequel to Umar and Zubairu \(^{17}\). For basic concepts in semigroup theory, we refer the reader to \(^{10}\), \(^{12}\), \(^{14}\).

Let $S$ be a semigroup and $U$ be a subset of $S$, then $|U|$ is said to be the rank of $S$ (denoted as $\text{Rank}(S)$) if

$$|U| = \min\{|A| : A \subseteq S \text{ and } \langle A \rangle = S\}.$$ 

The notation $\langle U \rangle = S$ means that $U$ generate the semigroup $S$. The rank of several semigroups of transformation were investigated, see for example, \(^{3}, 4, 7, 8, 9, 11, 13\). However, there are several subsemigroups of full contractions which their ranks are yet to be known. In fact the order and the rank of the semigroup $\text{CT}_n$ is still under investigation.

Let us briefly discuss the presentation of the paper. In section 1, we give a brief introduction and notations for proper understanding of the content of the remaining sections. In section 2, we discuss combinatorial properties for the semigroups $\text{Reg}(\text{ORCT}_n)$ and $\text{E}(\text{ORCT}_n)$, in particular we give their orders. In section 3, we proved that the rank of the semigroups $\text{Reg}(\text{ORCT}_n)$ and $\text{E}(\text{ORCT}_n)$ are 4 and 3, respectively, through the minimal generating set for their Rees quotient semigroups.

## 2 Combinatorial Properties of $\text{Reg}(\text{ORCT}_n)$ and $\text{E}(\text{ORCT}_n)$

In this section, we want to investigate some combinatorial properties of the semigroups, $\text{Reg}(\text{ORCT}_n)$ and $\text{E}(\text{OCT}_n)$. In particular, we want to compute their Cardinalities. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_p \end{pmatrix} \in \text{T}_n \ (1 \leq p \leq n),$$

then the rank of $\alpha$ is defined and denoted by rank $(\alpha) = |\text{Im } \alpha| = p$, so also, $x_i^{-1} = A_i$ $(1 \leq i \leq p)$ are equivalence classes under the relation $\text{ker } \alpha = \{(x, y) \in [n] \times [n] : x\alpha = y\alpha\}$. Further, we denote the partition $(A_1, \ldots, A_p)$ by $\text{Ker } \alpha$ and also, $\text{fix}(\alpha) = \{x \in [n] : x\alpha = x\}$, a subset $T_\alpha$ of $[n]$ is said to be a transversal of the partition $\text{Ker } \alpha$ if $|T_\alpha| = p$, and $|A_i \cap T_\alpha| = 1$ $(1 \leq i \leq p)$. A transversal $T_\alpha$ is said to be convex if for all $x, y \in T_\alpha$ with $x \leq y$ and if $x \leq z \leq y$ $(z \in [n])$, then $z \in T_\alpha$.

Before we proceed, let’s describe some Green’s relations on the semigroups $\text{Reg}(\text{ORCT}_n)$ and $\text{E}(\text{ORCT}_n)$. It is worth noting that the two semigroups, $\text{Reg}(\text{ORCT}_n)$ and $\text{E}(\text{ORCT}_n)$ are both regular subsemigroups of the Full Transformation semigroup $\text{T}_n$, therefore by \(^{14}\), Prop. 2.4.2 they automatically inherit the Green’s $\mathcal{L}$ and $\mathcal{R}$ relations of the semigroup $\text{T}_n$, but not necessary $\mathcal{D}$ relation, as such we have the following lemma.

**Lemma 1.** Let $\alpha, \beta \in S \in \{\text{Reg}(\text{ORCT}_n), \ E(\text{ORCT}_n)\}$, then

1. $\alpha \mathcal{R} \beta$ if and only if $\text{Im } \alpha = \text{Im } \beta$;
2. $\alpha \mathcal{L} \beta$ if and only if $\text{ker } \alpha = \text{ker } \beta$. 


2.1 The Semigroup $\text{Reg}(\text{ORCT}_n)$

Before we begin discussing on the semigroup $\text{Reg}(\text{ORCT}_n)$, let us first of all consider the semigroup $\text{Reg}(\text{OCT}_n)$ consisting of only order-preserving elements. Let $\alpha$ be in $\text{Reg}(\text{OCT}_n)$, from [17, Lem. 12], $\alpha$ is of the form

$$\alpha = \left( \begin{array}{cccccc}
{1, \ldots, a+1} & a+2 & \ldots & a+p-1 & \{a+p, \ldots, n\} \\
1 & x+1 & \ldots & x+p-1 & \{x+p\}
\end{array} \right).$$

Let

$$K_p = \{ \alpha \in \text{Reg}(\text{OCT}_n) : |\text{Im} \alpha| = p \} \quad (1 \leq p \leq n),$$

and suppose that $\alpha \in K_p$, by [18, Lem. 12] $\text{Ker} \alpha = \{1, \ldots, a+1, a+2, \ldots, a+p-1, \{a+p, \ldots, n\}\}$ have an admissible traversal (A transversal $T_\alpha$ is said to be admissible if and only if the map $A_i \mapsto t_i$ ($t_i \in T, \ i \in \{1, 2, \ldots, p\}$) is a contraction, see [2]) $T_\alpha = \{a+i : 1 \leq i \leq p\}$ such that the mapping $a+i \mapsto x+i$ is an isometry. Therefore, translating the set $\{x+i : i \leq 1 \leq p\}$ with an integer say $k$ to $\{x+i+k : 1 \leq i \leq p\}$ will also serve as image set to $\text{Ker} \alpha$ as long as $x+1-k \leq 1$ and $x+p+k \geq n$.

For example, if we define $\alpha$ as:

$$\alpha = \left( \begin{array}{cccccc}
{1, \ldots, a+1} & a+2 & \ldots & a+p-1 & \{a+p, \ldots, n\} \\
1 & 2 & 3 & \ldots & p-1 & \{p\}
\end{array} \right),$$

then we will have $n-p$ other mappings in $K_p$ that will have the same domain as $\alpha$.

In similar manner, suppose we fix the image set $\{x+i : 1 \leq i \leq p\}$ and consider $\text{Ker} \alpha$, then we can refine the partition $\{\{1, \ldots, a+1\}, \{a+2, \ldots, a+p-1\}, \{a+p, \ldots, n\}\}$ by $i$-shifting to say $\{\{1, \ldots, a+i\}, \{a+i+1, \ldots, a+p-i\}, \{a+p-i+1, \ldots, n\}\}$ for some integer $1 \leq i \leq p$ which also have an admissible convex traversal.

For the purpose of illustrations, if for some integer $j$, $\{\{1, \ldots, a+1\}, \{a+2, \ldots, a+p-1\}, \{a+p, \ldots, n\}\} = \{\{1, 2, \ldots, j\}, \{j+1, \ldots, n\}\}$, then the translation $\{\{1, 2, \ldots, j-1\}, \{j\}, \{j+1, \ldots, n-1, n\}\}$ will also serve as domain to the image set of $\alpha$. Thus, for $p \neq 1$ we will have $n-p+1$ different mappings with the same domain set in $K_p$.

To see what we have been explaining, consider the table below: For $n \geq 4, 2 \leq p \leq n$ and $j = n-p+1$, the set $K_p$ can be presented as follows:

$$
\begin{array}{cccccccc}
\{1, \ldots, j\} & j+1 & \ldots & n-1 & n \\
1 & 2 & \ldots & p-1 & p \\
\{1, \ldots, j\} & j+1 & \ldots & n-1 & n \\
2 & 3 & \ldots & p & p+1 \\
\vdots \\
\{1, \ldots, j\} & j+1 & \ldots & n-1 & n \\
j-1 & j & \ldots & n-2 & n-1 \\
\{1, \ldots, j\} & j+1 & \ldots & n-1 & n \\
j & j+1 & \ldots & n-1 & n
\end{array}
\begin{array}{cccccccc}
\{1, 2\} & 3 & \ldots & p-1 & \ldots & n \\
1 & 2 & \ldots & p & \ldots & p+1 \\
\{1, 2\} & 3 & \ldots & p-1 & \ldots & n \\
2 & 3 & \ldots & p & \ldots & p+1 \\
\vdots \\
\{1, 2\} & 3 & \ldots & p-1 & \ldots & n \\
\vdots \\
\{1, 2\} & 3 & \ldots & p-1 & \ldots & n \\
j-1 & j & \ldots & n-2 & n-1 \\
\{1, 2\} & 3 & \ldots & p-1 & \ldots & n \\
j & j+1 & \ldots & n-1 & n
\end{array}
$$

(6)

From the table above, we can see that for $p = 1$, $|K_p| = n - p + 1 = n$, while for $2 \leq p \leq n$, $|K_p| = (n-p+1)^2$.

The next theorem gives us the cardinality of the semigroup $\text{Reg}(\text{OCT}_n)$.

**Theorem 2.** Let $\text{OCT}_n$ be as defined in equation (1), then $|\text{Reg}(\text{OCT}_n)| = \frac{n(n-1)(2n-1)+6n}{6}$.

**Proof.** It is clear that $\text{Reg}(\text{OCT}_n) = K_1 \cup K_2 \cup \ldots \cup K_n$. Since this union is disjoint, we have that

$$|\text{Reg}(\text{OCT}_n)| = \sum_{p=1}^{n} |K_p| = |K_1| + \sum_{p=2}^{n} |K_p| = n + \sum_{p=2}^{n} (n-p+1)^2$$

$$= n + \frac{(n-1)^2 + (n-2)^2 + \ldots + 2^2 + 1^2}{6} = \frac{n(n-1)(2n-1)+6n}{6},$$

as required. 

\[\square\]
Corollary 3. Let $\text{ORCT}_n$ be as defined in equation (2). Then $|\text{Reg}(\text{ORCT}_n)| = \frac{n(n-1)(2n-1)+6n}{3} - n$.

Proof. It follows from Theorem 2 and the fact that $|\text{Reg}(\text{ORCT}_n)| = 2|\text{Reg}(\text{OCT}_n)| - n$. \qed

2.2 The Semigroup $E(\text{ORCT}_n)$

Let $\alpha$ be in $E(\text{ORCT}_n)$, then it follows from [17], Lem. 13 that $\alpha$ is of the form

$$\alpha = \left( \begin{array}{cccccc} 1, \ldots, i & i+1 & i+2 & \ldots & i+j-1 & \{i+j, \ldots, n\} \\ i & i+1 & i+2 & \ldots & i+j-1 & i+j \end{array} \right).$$

(7)

Since $\text{fix}(\alpha) = j+1$, then for each given domain set there will be only one corresponding image set.

Let $E_p = \{ \alpha \in E(\text{ORCT}_n) : |\text{Im} \alpha| = p \} \quad (1 \leq p \leq n).$ (8)

To choose $\alpha \in E_p$ we only need to select the image set of $\alpha$ which is a $p$ consecutive (convex) numbers from the set $[n]$. Thus $|E_p| = n - p - 1$. Consequently, we have the cardinality of the semigroup $E(\text{ORCT}_n)$.

Theorem 4. Let $\text{ORCT}_n$ be as defined in equation (2). Then $|E(\text{ORCT}_n)| = \frac{n(n+1)}{2}$.

Proof. Following the argument of the proof of Theorem 2 we have,

$$|E(\text{ORCT}_n)| = \sum_{p=1}^{n} |E_p| = \sum_{p=1}^{n} (n - p + 1)$$

$$= n + (n-1) + (n-2) + \cdots + 2 + 1$$

$$= \frac{n(n+1)}{2}. \quad \square$$

Remark 5. Notice that idempotents in $\text{ORCT}_n$ are necessarily order preserving, as such $|E(\text{OCT}_n)| = |E(\text{ORCT}_n)| = \frac{n(n+1)}{2}$.

3 Rank Properties

In this section, we discuss some rank properties of the semigroups $\text{Reg}(\text{ORCT}_n)$ and $E(\text{ORCT}_n)$.

3.1 Rank of $\text{Reg}(\text{OCT}_n)$

Just as in section 2 above, let us first consider the semigroup $\text{Reg}(\text{OCT}_n)$, the semigroup consisting of regular elements of order-preserving full contractions. Now, let $K_p$ be defined as in equation (4). We have seen how elements of $K_p$ look like in Table 6 above. Suppose we define:

$$\eta := \left( \begin{array}{cccccc} 1, \ldots, j & j+1 & \ldots & n-1 & n \\ 1 & 2 & \ldots & p-1 & p \end{array} \right),$$

(9)

$$\delta := \left( \begin{array}{cccccc} 1 & 2 & \cdots & p-1 & \{p, \ldots, n\} \\ 1 & 2 & \cdots & p-1 & p \end{array} \right)$$

(10)

and

$$\tau := \left( \begin{array}{cccccc} 1 & 2 & \cdots & p-1 & \{p, \ldots, n\} \\ j & j+1 & \cdots & n-1 & n \end{array} \right)$$

(11)

that is, $\eta$ to be the top left-corner element, $\delta$ be the top right-corner element while $\tau$ be the bottom right corner element in Table 6. And let $R_\eta$ and $L_\delta$ be the respective $R$ and $L$ equivalent classes of $\eta$ and $\delta$. Then
for $\alpha$ in $K_p$ there exist two elements say $\eta'$ and $\delta'$ in $R_\eta$ and $L_\delta$, respectively for which $\alpha$ is $L$ related to $\eta'$ and $R$ related to $\delta'$ and that $\alpha = \eta' \delta'$. For the purpose of illustrations, consider

$$\alpha = \left( \begin{array}{cccc} 1, \ldots, j - 1 & j & j + 1 & \cdots & n - 1, n \end{array} \right),$$

then the elements

$$\left( \begin{array}{cccc} 1, \ldots, j - 1 & j & j + 1 & \cdots & n - 1, n \end{array} \right)$$

and

$$\left( \begin{array}{cccc} 1 & 2 & \cdots & p - 1 & \{p, \ldots, n\} \\
2 & 3 & \cdots & p & p + 1 \end{array} \right)$$

are respectively elements of $R_\eta$ and $L_\delta$ and that

$$\alpha = \left( \begin{array}{cccc} 1, \ldots, j - 1 & j & j + 1 & \cdots & n - 1, n \end{array} \right) \left( \begin{array}{cccc} 1 & 2 & \cdots & p - 1 & \{p, \ldots, n\} \\
2 & 3 & \cdots & p & p + 1 \end{array} \right).$$

Consequently, we have the following lemma.

**Lemma 6.** Let $\eta$ and $\delta$ be as defined in equations (9) and (10), respectively. Then $\langle R_\eta \cup L_\delta \rangle = K_p$.

**Remark 7.** The following are observed from Table 6:

(i) The element $\delta$ belongs to both $R_\eta$ and $L_\delta$;

(ii) $\tau \eta = \delta$;

(iii) For all $\alpha \in R_\eta$, $\alpha \delta = \alpha$ while $\delta \alpha$ has rank less than $p$;

(iv) For all $\alpha \in L_\delta$, $\delta \alpha = \alpha$ while $\alpha \delta$ has rank less than $p$;

(v) For all $\alpha, \beta \in R_\eta \setminus \delta$ (or $L_\delta \setminus \delta$), $\text{rank}(\alpha \beta) < p$.

To investigate the rank of $\text{Reg}(\textbf{OCT}_n)$, let

$$L(n, p) = \{ \alpha \in \text{Reg}(\textbf{OCT}_n) : |\text{Im} \alpha| \leq p \} \quad (1 \leq p \leq n),$$

and let

$$Q_p = L(n, p) \setminus L(n, p - 1).$$

Then $Q_p$ is of the form $K_p \cup \{0\}$, where $K_p$ is the set of all elements of $\text{Reg}(\textbf{OCT}_n)$ whose height is exactly $p$. The product of any two elements in $Q_p$ say $\alpha$ and $\beta$ is of the form:

$$\alpha \ast \beta = \left\{ \begin{array}{ll} \alpha \beta, & \text{if } |h(\alpha \beta)| = p; \\
0, & \text{if } |h(\alpha \beta)| < p \end{array} \right.$$ 

$Q_p$ is called the Rees quotient semigroup on $L(n, p)$. Next, we have the following lemma which follows from Lemma 6 and Remark 7.

**Lemma 8.** $(R_\eta \cup L_\delta) \setminus \delta$ is the minimal generating set for the Rees quotient semigroup $Q_p$.

To find the generating set for $L(n, p)$, we need the following proposition:

**Proposition 9.** For $n \geq 4$, $\langle K_p \rangle \subseteq \langle K_{p+1} \rangle$ for all $1 \leq p \leq n - 2$. 

5
Proof. Let \((A) = K_p\), to proof \(\langle K_p \rangle \subseteq \langle K_{p+1} \rangle\), it suffices to show that \(A \subseteq \langle K_{p+1} \rangle\). From Lemma 8 \(A = (R_\eta \cup L_\delta) \setminus \delta\). Now, let \(\alpha\) be in \(A\):

CASE I: If \(\alpha = \eta\), then \(\alpha\) can be written as \(\alpha = \langle 1, \ldots, j - 1 \rangle \begin{pmatrix} j - 2 & j - 1 & j & \cdots & n - 2 & n - 1 & n \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & \cdots & p & p + 1 \end{pmatrix}\), a product of two elements of \(K_{p+1}\).

CASE II: If \(\alpha \in R_\eta \setminus \eta\), then \(\alpha\) is of the form \(\langle 1, \ldots, j - k - 1 \rangle \begin{pmatrix} j - k & \cdots & n - k, \ldots, n \end{pmatrix} \begin{pmatrix} 1 \ 2 \ \cdots \ p - 1 \ p \ p + 1 \end{pmatrix}\), (\(k = 1, 2, \ldots, j - 2\)). Then \(\alpha\) can be written as:

CASE III: If \(\alpha \in L_\delta \setminus \delta\), then \(\alpha\) is of the form \(\langle 1, \ldots, j - k - 1 \rangle \begin{pmatrix} j - k & \cdots & n - k \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & p - 1 \ p \ p + 1 \end{pmatrix}\), (\(r = 2, 3, \ldots, n - p + 1\)). and it can be written as:

\[
\begin{pmatrix} 1 & 2 & \cdots & p & \{p, p + 1, \ldots, n\} \\ r & r + 1 & \cdots & p + r - 2 & p + r - 1 \end{pmatrix},
\]

due to the proof.

Remark 10. Notice that by the proposition above, the generating set for \(Q_p\) \((1 \leq p \leq n - 1)\) generates the whole \(L(n, p)\).

The next theorem gives us the rank of the subsemigroup \(L(n, p)\) for \(1 \leq p \leq n - 1\).

Theorem 11. Let \(L(n, p)\) be as defined in equation (12). Then for \(n \geq 4\) and \(1 < p \leq n - 1\), the rank of \(L(n, p)\) is \(2(n - p)\).

Proof. It follows from Lemma 8 and Remark 10 above.

Now as a consequence, we readily have the following corollaries.

Corollary 12. Let \(L(n, p)\) be as defined in equation (12). Then the rank of \(L(n, n - 1)\) is 2.

Corollary 13. Let \(\text{OCT}_n\) be as defined in equation (11). Then the rank of \(\text{Reg} (\text{OCT}_n)\) is 3.

Proof. The proof follows from Corollary 12 coupled with the fact that \(\text{Reg} (\text{OCT}_n) = L(n, n - 1) \cup id_{[n]}\), where \(id_{[n]}\) is the identity element on \([n]\).
3.2 Rank of Reg(\(\text{ORCT}_n\))

To discuss the rank of Reg(\(\text{ORCT}_n\)), consider the Table above. Suppose we reverse the order of the image set of elements in that table, then we will have the set of order-reversing elements of Reg(\(\text{ORCT}_n\)). For \(1 \leq p \leq n\), let

\[
J_p = \{\alpha \in \text{Reg}(\text{ORCT}_n) : |\text{Im} \alpha| = p\}
\]

and let

\[
K_p^* = \{\alpha \in J_p : \alpha \text{ is order-reversing}\}.
\]

Observe that \(J_p = K_p \cup K_p^*\). Now define:

\[
\eta^* = \left( \begin{array}{cccc} \{1, \ldots, j\} & j+1 & \cdots & n-1 & n \\ p & p-1 & \cdots & 2 & 1 \end{array} \right),
\]

\[
\delta^* = \left( \begin{array}{cccc} 1 & 2 & \cdots & p-1 & \{p, \ldots, n\} \\ p & p-1 & \cdots & 2 & 1 \end{array} \right)
\]

and

\[
\tau^* = \left( \begin{array}{cccc} 1 & 2 & \cdots & p-1 & \{p, \ldots, n\} \\ n & n-1 & \cdots & j+1 & j \end{array} \right)
\]

i.e., \(\eta^*, \delta^*\) and \(\tau^*\) are respectively \(\eta, \delta\) and \(\tau\) with image order-reversed.

Remark 14. Throughout this section, we will write \(\alpha^*\) to mean a mapping in \(K_p^*\) which has a corresponding mapping \(\alpha\) in \(K_p\) with order-preserving image.

And let \(R_\eta\) and \(L_\delta\) be the respective \(R\) and \(L\) equivalent classes of \(\eta\) and \(\delta\). Then we have the following lemmas which are analogue to Lemma 6.

Lemma 15. Let \(\eta\) and \(\delta^*\) be as defined in equations (13) and (17), respectively. Then \((R_\eta \cup L_\delta^*) = K_p^*\).

Proof. Let \(\alpha^* = \left( \begin{array}{cccc} \{1, \ldots, a+1\} & a+2 & \cdots & a+p-1 & \{a+p, \ldots, n\} \\ x+p & x+p-1 & \cdots & x+2 & x+1 \end{array} \right)\) be in \(K_p^*\), then there exists \(\alpha \in K_p\) such that by Lemma 6 \(\alpha\) can be expressed as the following product:

\[
\left( \begin{array}{cccc} \{1, \ldots, a+1\} & a+2 & \cdots & a+p-1 & \{a+p, \ldots, n\} \\ y+1 & y+2 & \cdots & y+p-1 & y+p \end{array} \right) \left( \begin{array}{cccc} \{1, \ldots, b+1\} & b+2 & \cdots & b+p-1 & \{b+p, \ldots, n\} \\ x+1 & x+2 & \cdots & x+p-1 & x+p \end{array} \right)
\]

a product of elements of \(R_\eta\) and \(L_\delta\), respectively. Therefore, \(\alpha^*\) can be expressed as the following product:

\[
\left( \begin{array}{cccc} \{1, \ldots, a+1\} & a+2 & \cdots & a+p-1 & \{a+p, \ldots, n\} \\ y+1 & y+2 & \cdots & y+p-1 & y+p \end{array} \right) \left( \begin{array}{cccc} \{1, \ldots, b+1\} & b+2 & \cdots & b+p-1 & \{b+p, \ldots, n\} \\ x+1 & x+2 & \cdots & x+p-1 & x+p \end{array} \right)
\]

a product of elements of \(R_\eta\) and \(L_\delta\), respectively. \(\square\)

Lemma 16. Let \(J_p = \{\alpha \in \text{Reg}(\text{ORCT}_n) : |\text{Im} \alpha| = p\}\). Then, \(\langle R_\eta \cup L_\delta^* \rangle = J_p\).

Proof. Since \(J_p = K_p \cup K_p^*\), to proof \(\langle R_\eta \cup L_\delta^* \rangle = J_p\), is suffices by Lemma 19 to show that \(K_p \subseteq \langle K_p^* \rangle\). Now, let

\[
\alpha = \left( \begin{array}{cccc} \{1, \ldots, a+1\} & a+2 & \cdots & a+p-1 & \{a+p, \ldots, n\} \\ b+1 & b+2 & \cdots & b+p-1 & b+p \end{array} \right)
\]

be in \(K_p\), if \(\alpha\) is an idempotent, then there exists \(\alpha^* \in K_p^*\) such that \((\alpha^*)^2 = \alpha\). Suppose \(\alpha\) is not an idempotent, define

\[
\epsilon = \left( \begin{array}{cccc} \{1, \ldots, b+1\} & b+2 & b+3 & \cdots & b+p-1 & \{b+p, \ldots, n\} \\ b+1 & b+2 & b+3 & \cdots & b+p-1 & b+p \end{array} \right)
\]

which is an idempotent in \(K_p\), then \(\alpha\) can be written as \(\alpha = \alpha^* \epsilon^*\). \(\square\)
Before stating the main theorem of this section, let

\[ M(n, p) = \{ \alpha \in \text{Reg}(\text{ORCT}_n) : |\text{Im } \alpha| \leq p \} \quad (1 \leq p \leq n). \] (19)

And let

\[ W_p = M(n, p) \setminus M(n, p - 1) \] (20)

be Rees quotient semigroup on \( M(n, p) \). From Lemma 10 and Remark 7 we have:

**Lemma 17.** \((R_\eta \cup L_\delta^r) \setminus \delta\) is the minimal generating set for the Rees quotient semigroup \( W_p \).

The next proposition is also analogue to Proposition 9 which plays an important role in finding the generating set for the subsemigroup \( M(n, p) \).

**Proposition 18.** For \( n \geq 4 \), \( \langle J_p \rangle \subseteq \langle J_{p+1} \rangle \) for all \( 1 \leq p \leq n - 2 \).

**Proof.** The proof follows the same pattern as the proof of the Proposition 9. We want to show that \((R_\eta \cup L_\delta^r) \subseteq \langle J_{p+1} \rangle \) and by Proposition 9 we only need to show that \( L_\delta^r \subseteq \langle J_{p+1} \rangle \). Now Let \( \alpha \) be in \( L_\delta^r \).

Case I: \( \alpha \in L_\delta^r \setminus \tau^* \), then \( \alpha \) is of the form

\[
\begin{pmatrix}
1 & 2 & \cdots & p \\
2 & 3 & \cdots & p + 1 \\
\vdots & \vdots & \ddots & \vdots \\
p & p + 1 & \cdots & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
p + r - 1 & p + r - 2 & \cdots & p - 1 \\
p + r & p + r - 1 & \cdots & r + 1
\end{pmatrix}
\]

\( (r = 1, 2, \ldots, n - p) \),

and it can be written as

\[
\alpha = \begin{pmatrix}
1 & 2 & \cdots & p \\
2 & 3 & \cdots & p + 1 \\
\vdots & \vdots & \ddots & \vdots \\
p & p + 1 & \cdots & 2
\end{pmatrix}
\begin{pmatrix}
1 & 2 & \cdots & p \\
2 & 3 & \cdots & p + 1 \\
\vdots & \vdots & \ddots & \vdots \\
p & p + 1 & \cdots & 2
\end{pmatrix}
\]

\( \cdot \begin{pmatrix}
p + r - 1 & p + r - 2 & \cdots & p - 1 \\
p + r & p + r - 1 & \cdots & r + 1
\end{pmatrix}
\]

\( \cdot \begin{pmatrix}
p + r - 1 & p + r - 2 & \cdots & p - 1 \\
p + r & p + r - 1 & \cdots & r + 1
\end{pmatrix}
\]

a product of two elements of \( J_{p+1} \).

Case II: \( \alpha = \tau^* \) then \( \alpha \) can be written as

\[
\alpha = \begin{pmatrix}
1 & 2 & \cdots & p \\
1 & 2 & \cdots & p \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2 & \cdots & p
\end{pmatrix}
\begin{pmatrix}
p + 1 & \cdots & n \\
p + 1 & \cdots & n \\
\vdots & \vdots & \ddots & \vdots \\
p + 1 & \cdots & n
\end{pmatrix}
\]

\[
\begin{pmatrix}
p + r - 1 & p + r - 2 & \cdots & p - 1 \\
p + r & p + r - 1 & \cdots & r + 1
\end{pmatrix}
\]

\( \cdot \begin{pmatrix}
p + r - 1 & p + r - 2 & \cdots & p - 1 \\
p + r & p + r - 1 & \cdots & r + 1
\end{pmatrix}
\]

\( \cdot \begin{pmatrix}
p + r - 1 & p + r - 2 & \cdots & p - 1 \\
p + r & p + r - 1 & \cdots & r + 1
\end{pmatrix}
\]

The first element in the product above is \( \delta \in J_p \), but it was shown in Remark 7 that it can be written as \( \tau \eta \) which were both shown in Proposition 9 that they can be expressed as product of elements of \( J_{p+1} \). Hence the proof.

**Remark 19.** Notice also that, by Proposition 18 above, for \( 2 \leq p \leq n - 1 \) the generating set for \( W_p \) generates the whole \( M(n, p) \).

The next theorem gives us the rank of subsemigroup \( M(n, p) \) for \( 2 \leq p \leq n - 1 \).

**Theorem 20.** Let \( M(n, p) \) be as defined in equation 19. Then for \( n \geq 4 \) and \( 2 < p < n - 1 \), the rank of \( M(n, p) \) is \( 2(n - p) + 1 \).

**Proof.** To prove this, we only need to compute the cardinality of the set \((R_\eta \cup L_\delta^r) \setminus \delta\), which from Table 6 we easily obtain \( (n - p) + (n - p) + 1 = 2(n - p) + 1 \).

As a consequence, we have the following corollaries.

**Corollary 21.** Let \( M(n, p) \) be as defined in equation 19. Then the rank of \( M(n, n - 1) \) is 3.

**Corollary 22.** Let \( \text{ORCT}_n \) be as defined in equation 2. Then the rank of \( \text{Reg}(\text{ORCT}_n) \) is 4.

**Proof.** The only elements of \( \text{Reg}(\text{ORCT}_n) \) that are not in \( M(n, n - 1) \) are the elements \( \{id^*[n], id^*_n\} \). It is clear that \( (id^*)^2 = id \).
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