ON FACTOR RIGIDITY AND JOINING CLASSIFICATION
FOR INFINITE VOLUME RANK ONE HOMOGENEOUS
SPACES

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ABSTRACT. We classify locally finite joinings with respect to the Burger-Roblin measure and the action of a horospherical subgroup on $\Gamma \backslash G$, where $G = \text{SO}(n,1)^{\circ}$ and $\Gamma$ is a convex cocompact and Zariski dense subgroup of $G$. This extends the classification by Oh and Mohammadi obtained in the case that $G = \text{PSL}_2(\mathbb{R})$ or $\text{PSL}_2(\mathbb{C})$ and $\Gamma$ is geometrically finite and Zariski dense.

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1. INTRODUCTION

In [12], Ratner classified all joinings with respect to horocycle flows on finite volume quotients of $\text{PSL}_2(\mathbb{R})$. In general, the case of infinite volume is not well understood, though there are many works in this direction, for instance [1, 2, 4, 6, 7, 11, 13, 14, 15, 17, 18]. In particular, in [8], Oh and Mohammadi classify joinings on infinite volume geometrically finite quotients of $\text{PSL}_2(\mathbb{R})$ or $\text{PSL}_2(\mathbb{C})$; the purpose of this paper is to extend this result to convex cocompact quotients of $G = \text{SO}(n,1)^{\circ}$, the connected component of the identity in $\text{SO}(n,1)$.

$G$ corresponds to the group of orientation preserving isometries of real hyperbolic space $\mathbb{H}^n$. Throughout the paper, we assume that $\Gamma_1, \Gamma_2$ are

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convex cocompact and Zariski dense discrete subgroups of $G$ with infinite covolume, and define

$$X_i := \Gamma_i \backslash G$$

for $i = 1, 2$ and $X := X_1 \times X_2$.

Let $U$ denote a horospherical subgroup of $G$, and denote by $\Delta(U)$ the diagonal embedding into $G \times G$.

**Definition 1.1.** Let $\mu_i$ be a locally finite $U$-invariant Borel measure on $\Gamma_i \backslash G$ for $i = 1, 2$. A $U$-joining with respect to $(\mu_1, \mu_2)$ is a locally finite $\Delta(U)$-invariant measure $\mu$ on $X$ such that the push-forward measures onto each coordinate are proportional to the corresponding $\mu_i$. If $\mu$ is $\Delta(U)$-ergodic, we call it an ergodic $U$-joining.

The goal of this article is to classify joinings in this infinite volume convex cocompact setting for the pair $(m_{\mathsf{BR}}^1, m_{\mathsf{BR}}^2)$, where $m_{\mathsf{BR}}^i$ is the Burger-Roblin (BR) measure $m_{\mathsf{BR}}^i = m_{\mathsf{BR}}^i(\Gamma_i \backslash G)$ (defined in Section 2.2 - often the subscript will be omitted) on $\Gamma_i \backslash G$, under the action of a horospherical subgroup $U$. In this setting, the BR measure is the natural analogue of the Haar measure used in Ratner’s proof in the case of a lattice. More specifically, the BR measure is the unique locally finite $U$-ergodic measure that is not supported on a closed $U$-orbit, $[3,13,17]$. Note that in this case, the $m_{\mathsf{BR}}^i$‘s are infinite measures $[10]$, so the product measure $m_{\mathsf{BR}}^1 \times m_{\mathsf{BR}}^2$ is not a $U$-joining.

We now restate the definition of a finite cover self-joining as it appears in $[8]$:

**Definition 1.2.** Suppose that there exists $g_0 \in G$ so that $g_0^{-1} \Gamma_1 g_0$ and $\Gamma_2$ are commensurable in $G$. In particular, we have an isomorphism $(g_0^{-1} \Gamma_1 g_0 \cap \Gamma_2) \backslash G \to [(g_0,1_G)] \Delta(G)$ defined by $[g] \mapsto [(g_0 g, g)]$, where $1_G$ denotes the identity in $G$. The pushforward of the BR measure $m_{\mathsf{BR}}^0_{g_0^{-1} \Gamma_1 g_0 \cap \Gamma_2}$ is a $U$-joining, which we call a finite cover self-joining. We also consider any translation of a finite cover self-joining under an element of the form $(u,1_G) \in U \times \{1_G\}$ to be a finite cover self-joining.

Our main result is the following:

**Theorem 1.3.** Suppose that $\Gamma_1, \Gamma_2$ are convex cocompact and Zariski dense discrete subgroups of $G$, and assume that they have infinite covolume. Then every locally finite ergodic $U$-joining on $X = \Gamma_1 \backslash G \times \Gamma_2 \backslash G$ with respect to $(m_{\mathsf{BR}}^1, m_{\mathsf{BR}}^2)$ is a finite cover self-joining.

This article is organized as follows. In Section 2, we define notation that is used throughout the paper, the Patterson-Sullivan (PS), Bowen-Margulis-Sullivan (BMS) and Burger-Roblin (BR) measures, and reference some basic properties of these measures.

In Section 3, we prove general results about the behaviour of PS measure on varieties that will be important in the later proofs. In particular, we prove a kind of absolute continuity between the PS and Lebesgue measures on $U$. 

In Section 4.1, we define the notation and setup for the rest of Section 4. In Section 4.2, we show that the fibers of the projection $\pi_2$ onto the second coordinate must be finite. Finally, in Section 4.3, we prove Theorem 1.3.

2. Preliminaries and notation

For convenience, we remind the reader of the following notation that appeared in the introduction:

- $G = \text{SO}(n, 1)^0$ is the connected component of the identity in $\text{SO}(n, 1)$, and $1_G$ denotes the identity element in $G$.
- $\Gamma_1, \Gamma_2$ are convex cocompact and Zariski dense discrete subgroups of $G$ with infinite co-volume.
- $X_i := \Gamma_i \backslash G$ and $X := X_1 \times X_2$.
- For $H \subset G$, $\Delta(H)$ denotes the diagonal embedding of $H$ into $G \times G$.

Define

$$A = \{ a_s : s \in \mathbb{R} \} \text{ where } a_s = \begin{pmatrix} e^s & I_{n-1} \\ I_{n-1} & e^{-s} \end{pmatrix},$$

where $I_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix.

We denote by $U$ the expanding horospherical subgroup, that is,

$$U = \{ g \in G : a_s g a_{-s} \to 1_G \text{ as } s \to \infty \}.$$

Similarly, we denote by $U^-$ the contracting horospherical subgroup,

$$U^- = \{ g \in G : a_s g a_{-s} \to 1_G \text{ as } s \to -\infty \}.$$

Both groups are isomorphic to $\mathbb{R}^{n-1}$, and we use the following parametrizations:

$$U = \{ u_t : t \in \mathbb{R}^{n-1} \} \text{ where } u_t = \begin{pmatrix} 1 \\ t^T \\ \frac{1}{2} |t|^2 \\ t \\ 1 \end{pmatrix} \text{ and }$$

$$U^- = \{ v_t : t \in \mathbb{R}^{n-1} \} \text{ where } v_t = \begin{pmatrix} 1 \\ t \\ \frac{1}{2} |t|^2 \\ I \\ t^T \end{pmatrix}.$$

We also define

$$M = \left\{ \begin{pmatrix} 1 \\ m \\ 1 \end{pmatrix} : m \in \text{SO}(n-1) \right\}.$$

We will often abuse notation by writing $m \in M$ to refer to the matrix $\begin{pmatrix} 1 \\ m \\ 1 \end{pmatrix}$. Observe that these parametrizations satisfy

$$a_{-s} u_t a_s = u_{te^s} \text{ and } m^{-1} u_t m = u_{tm}.$$
For $T > 0$, we denote balls in $G$ by

$$B(T) = \{g \in G : \|g - I\| \leq T\}$$

where $\|\cdot\|$ is the max norm on $G$, and in $U$ by

$$B_U(T) = \{u_t \in U : |t| \leq T\},$$

where $|\cdot|$ is the max norm on $\mathbb{R}^{n-1}$.

We write $t \in B_U(T)$ as shorthand for $u_t \in B_U(T)$.

2.1. PS measure. We use many definitions and notations as in [8], Section 2, but provide a paraphrased version here for the convenience of the reader. See [8], Section 2, for more details about these constructions.

Let $\partial(\mathbb{H}^n)$ denote the geometric boundary of $\mathbb{H}^n$. For any discrete subgroup $\Gamma$ of $G$, we can define the limit set of $\Gamma$, $\Lambda(\Gamma)$, as the accumulation points of any orbit in $\partial(\mathbb{H}^n)$, that is, $\Lambda(\Gamma) = \overline{\{v - \Gamma v : v \in \mathbb{H}^n\}}$. This is independent of $v$ because $\Gamma$ acts by isometries on $\mathbb{H}^n$. We denote by $\Lambda_r(\Gamma)$ the set of radial limit points of $\Gamma$. $\xi \in \Lambda(\Gamma)$ is a radial limit point if some (hence every) geodesic ray towards $\xi$ has accumulation points in some compact subset of $\Gamma \backslash G$. In the case of $\Gamma$ convex cocompact, $\Lambda(\Gamma) = \Lambda_r(\Gamma)$.

As in [8], we fix a reference point $o \in \mathbb{H}^n$ and a reference vector $w_0 \in T_o(\mathbb{H}^n)$, the unit tangent space of $\mathbb{H}^n$ at $o$. Consider the maximal compact subgroup $K := \text{Stab}_G(o)$. Then $\mathbb{H}^n$ can be viewed as $G/K$. Similarly, $M = \text{SO}(n, 1)$ is $\text{Stab}_G(w_0)$, and $T^1(\mathbb{H}^n)$ can be identified with $G/M$.

With these identifications and the parametrizations in Section 2, $A$ implements the geodesic flow on $T^1(\mathbb{H}^n)$. That is, if $\{g^t : t \in \mathbb{R}\}$ is the geodesic flow on $T^1(\mathbb{H}^n)$, then $g^t(w_0) = [a_sM]$, where $[\cdot]$ denotes the coset in $G/M$.

For $w \in T^1(\mathbb{H}^n)$, $w^\pm \in \partial(\mathbb{H}^n)$ denotes the forward or backward endpoints of the geodesic $w$ determines, i.e. $w^\pm = \lim_{t \to \pm\infty} g^t(w)$. For $g \in G$, we define

$$g^\pm := gw_0^\pm.$$

For $x = [g] \in \Gamma \backslash G$, we write $x^\pm \in \Lambda(\Gamma)$ if $g^\pm \in \Lambda(\Gamma)$ for some representative of the coset. This is well-defined by definition of $\Lambda(\Gamma)$.

Let $x, y \in \mathbb{H}^n$ and $\xi \in \partial(\mathbb{H}^n)$. The Busemann function is the function

$$\beta_\xi(x, y) = \lim_{t \to \infty} d(\xi_t, x) - d(\xi_t, y),$$

where $d$ is the hyperbolic metric and $\xi_t$ is a geodesic ray in $\mathbb{H}^n$ towards $\xi$.

For $\Gamma < G$ discrete, a $\Gamma$-invariant conformal density of dimension $\delta > 0$ is a family $\{\mu_x : x \in \mathbb{H}^n\}$ of pairwise mutually absolutely continuous finite measures on $\partial(\mathbb{H}^n)$ satisfying

$$\gamma_\ast \mu_x = \mu_{\gamma x} \text{ and } \frac{d\mu_x}{d\mu_y}(\xi) = e^{-\delta \beta_\xi(x, y)}$$

for all $x, y \in \mathbb{H}^n$ and $\xi \in \partial(\mathbb{H}^n)$, where $\gamma_\ast \nu_x(E) = \nu_x(\gamma^{-1}(E))$ for all Borel measurable subsets $E \subseteq \partial(\mathbb{H}^n)$.

Let $\delta_\Gamma$ denote the critical exponent of $\Gamma$. Up to scalar multiplication, there exists a unique $\Gamma$-invariant conformal density of dimension $\delta_\Gamma$, denoted by $\{\nu_x : x \in \mathbb{H}^n\}$ and called the Patterson-Sullivan density.
For each $g \in G$, this density allows us to define the Patterson-Sullivan (PS) measure on a horocycle $gU$ by

$$d\mu_{gU}^{\text{PS}}(gu_t) = e^{\delta \beta_{(gu_t)+}(o,gu_t(o))} d\nu_o(gu_t)^+. $$

In general, there is some subtlety in defining the PS measure on $xU \subseteq \Gamma \backslash G$; see [8, Section 2.3] for more discussion of this. We note that $\mu_{gU}^{\text{PS}}$ can be viewed as a measure on $U \cong \mathbb{R}^{n-1}$ via $d\mu_{gU}(t) = d\mu_{gU}^{\text{PS}}(gu_t)$.

The Lebesgue density is $\{m_x : x \in \mathbb{H}^n\}$, where $m_x$ is the unique probability measure on $\partial(\mathbb{H}^n)$ that is invariant under $\text{Stab}_G(x)$. The Lebesgue density is a $G$-invariant conformal density of dimension $n - 1$. We similarly define the Lebesgue measure on $gU$:

$$d\mu_{gU}^{\text{Leb}}(gu_t) = e^{(n-1)\beta_{(gu_t)+}(o,gu_t(o))} dm_o(gu_t)^+. $$

This is independent of the orbit and is in fact a scalar multiple of the Lebesgue measure on $U \cong \mathbb{R}^{n-1}$, denoted by $dt$.

Note that for every Borel measurable subset $E \subseteq U$, every $g \in G$, and every $s \in \mathbb{R}$, the properties of conformal densities imply that

$$\mu_{gU}^{\text{PS}}(E) = e^{\delta \beta_{U}(o,s)\rho_{g^{-1}a_s^{-1}}(a_sEa_s)}. $$

In particular,

$$\mu_{gU}^{\text{PS}}(B_U(e^s)) = e^{\delta \beta_{U}(o,s)} \mu_{g\delta_{a_s}}^{\text{PS}}(B_U(1)). $$

We record the following properties of PS measure:

**Lemma 2.1.** [5, Lemma 4.1, Cor. 4.2]

1. For every $g \in G$, every proper subvariety of $U$ is a null set for $\mu_{gU}^{\text{PS}}$.
2. $g \mapsto \mu_{gU}^{\text{PS}}$ is a continuous map from $\{g \in G : g^+ \in \Lambda(\Gamma)\}$ to $\mathcal{M}(U)$, where $\mathcal{M}(U)$ is the space of all regular Borel measures on $U$ with the topology $\mu_n \to \mu \iff \mu_n(f) \to \mu(f)$ for all $f \in C_c(U)$.

**Corollary 2.2.** [8, Cor. 2.2] For any compact set $\Omega \subseteq G$ and any $T > 0$,

$$0 < \inf_{g \in \Omega, g^+ \in \Lambda(\Gamma)} \mu_{gU}^{\text{PS}}(B_U(T)) \leq \sup_{g \in \Omega, g^+ \in \Lambda(\Gamma)} \mu_{gU}^{\text{PS}}(B_U(T)) < \infty. $$

### 2.2. BMS and BR measures

This section is again largely a condensed version of [8, Section 2.4].

The map $w \mapsto (w^+, w^-, \beta_{w^+}(o, \pi(w)), \beta_{w^-}(o, \pi(w)))$, where $\pi(w) \in \mathbb{H}^n$ is the base point of $w$, is a homeomorphism between $T^1(\mathbb{H}^n)$ and $(\partial(\mathbb{H}^n) \times \partial(\mathbb{H}^n)) \setminus \{(\xi, \xi) : \xi \in \partial(\mathbb{H}^n)\} \times \mathbb{R}$. This identification allows us to define the BMS and BR measures on $T^1(\mathbb{H}^n) \cong \Gamma \backslash G$, denoted by $m^{\text{BMS}}$ and $m^{\text{BR}}$,

$$dm^{\text{BMS}}(w) = e^{\delta \beta_{w^+}(o, \pi(w)) + \delta \beta_{w^-}(o, \pi(w))} d\nu_o(w^+) d\nu_o(w^-) ds $$

$$d\tilde{m}^{\text{BR}}(w) = e^{(n-1)\beta_{w^+}(o, \pi(w)) + \delta \beta_{w^-}(o, \pi(w))} dm_o(w^+) d\nu_o(w^-) ds. $$

By lifting to $G$, these will induce locally finite Borel measures on $\Gamma \backslash G$, denoted by $m^{\text{BMS}}$ and $m^{\text{BR}}$; see [8, Section 2.4] for more details. $m^{\text{BMS}}$ is a
finite measure \([10]\) (which we will assume to be normalized to a probability measure) and \(m^{BR}\) is infinite if \(\Gamma\) is not a lattice, \([10]\).

We have that

\[
\text{Supp} m^{BMS} = \{ x \in \Gamma \backslash G : x^\pm \in \Lambda(\Gamma) \} \quad \text{and} \quad \text{Supp} m^{BR} = \{ x \in \Gamma \backslash G : x^- \in \Lambda(\Gamma) \}.
\]
Convex cocompactness is equivalent to \(\text{Supp} m^{BMS}\) being compact.

3. VARIETIES AND ABSOLUTE CONTINUITY

**Lemma 3.1.** Let \(V \subseteq U\) be a proper subvariety and let \(K \subseteq X_1\) be compact.
For all \(\eta > 0\), there exists \(\varepsilon' > 0\) such that for all \(y \in K\), \(\mu_y^\text{PS}(N_{\varepsilon'}(V)) < \eta\).

**Proof.** Suppose not. Then there exists \(\eta > 0\) such that for all \(m \in N\), we can find \(y_m \in K\) satisfying \(\mu_{y_m}^\text{PS}(N_{1/m}(V)) \geq \eta\).

Since \(K\) is compact, by dropping to a subsequence if necessary, we may assume that there exists some \(y_\infty \in K\) such that \(y_m \to y_\infty\). Since \(V\) is a variety, \(\mu_y^\text{PS}(V) = 0\) by Lemma 2.1, and so there exists \(\varepsilon'\) such that \(\mu_{y_\infty}^\text{PS}(N_{\varepsilon'}(V)) < \eta/2\).

Let \(0 \leq f \leq 1\) be continuous such that \(1_{N_{\varepsilon'}/2}(V) \leq f\) and \(\text{Supp}(f) \subseteq N_{\varepsilon'}(V)\). By continuity of the map \(g \mapsto \mu_g^\text{PS}\) (Lemma 2.1), we have that \(\mu_{y_\infty}^\text{PS}(f) < \eta/2\).

Then for all \(m\) such that \(1/m < \varepsilon'/2\), \(\eta \leq \mu_{y_m}^\text{PS}(N_{1/m}(V)) \leq \mu_{y_m}^\text{PS}(N_{\varepsilon'/2}(V)) \leq \mu_{y_\infty}^\text{PS}(f) < \eta/2\), a contradiction.

For \(V \subseteq U\) and \(r > 0\), define \(N_r(V) := VB_U(r) = \{ u_su_t : s \in V, t \in B_U(r) \}\).

**Lemma 3.2.** Let \(V_0 \subseteq U\) be a proper subvariety, and let \(V = V_0 \cap B_U(1)\).
For every \(f \in C_c(X_1)\) and for every \(\eta > 0\), there exists \(\varepsilon' > 0\) and \(T_0 = T_0(f, \varepsilon') > 0\) such that for all \(T > T_0\) and all \(y \in \text{Supp} m^{BMS}\),
\[
e^{(n-1-\delta_1)s} \int_{a \in \Lambda(\Gamma)} f(yu_t) dt = e^{(n-1-\delta_1)s} \int_{N_{\varepsilon'}(V)} f(ya^-s) u_a dt \ll_f \eta,
\]
where \(s = \log T\). \((\ll_f \eta\) means \(\leq k\eta\) for some constant \(k\) that depends only on \(f\).)

To prove Lemma 3.2, we need a fact from \([8]\), which requires the following definition. Let \(P = AMU^-\) and let \(P_r\) denote the ball of radius \(r\) in \(P\). For \(\varepsilon_0, \varepsilon_1 > 0\), we say that \(zP_{\varepsilon_1}B_U(\varepsilon_0)\) is an admissible box if it is the injective image of \(P_{\varepsilon_1}B_U(\varepsilon_0)\) in \(\Gamma \backslash G\) under the map \(g \mapsto zg\), and \(\mu_{z^p}^\text{PS}(zP_{\varepsilon_1}B_U(\varepsilon_0)) \neq 0\) for all \(p \in \varepsilon_{\varepsilon_1}\).

**Fact 3.3.** \([8]\) Claim A in Theorem 4.6] Let \(\xi \in C_c(X_1)\) and assume that it is supported within the admissible box \(zP_{\varepsilon_1}B_U(\varepsilon_0)\) with \(0 < \varepsilon_1 \leq \varepsilon_0\) (by a partition of unity argument, there is no loss of generality). Suppose that \(x^- \in \Lambda_r(\Gamma_1)\), and let \(\Omega\) be a compact set such that there exists \(t_n \to \infty\) with \(xa^-t_n \in \Omega\) for all \(n\). Assume that \(x_0 := xa^-s_0 \in \Omega\).
For $\rho > 0$ and each $y \in x_0 U$, suppose that $f_y \in C(yB_U(\rho))$ is such that $0 \leq f_y \leq 1$ and $f = 1$ on $yB_U(\rho/8)$. Then there exists $c > 1$ such that for all $y \in x_0 U$,

$$e^{(n-1-\delta_1)s_0} \int_U \xi(yu_4) f_y(yu_4) dt \ll_{\phi} \mu^\text{PS}_{y, \psi}(f_y, e^{-s_0 \varepsilon_1})$$

where $f_{y, \psi} = \sup_{u' \in B_U(\eta)} f_y(yu')$. Moreover, by [9] Lemma 6.2, $c$ depends only on the injectivity radii of $\text{Supp} \xi$ and $\text{Supp} f_y$. In the convex cocompact case, there is a uniform lower bound on the injectivity radii, so there is one $c$ that works universally.

**Proof of Lemma 3.2** Let $f \in C_c(X_1)$. As discussed in the remark above, we may assume that $\text{Supp}(f)$ is contained in some admissible box $zP_{x_1} U_{x_0}$.

Fix $\eta > 0$. By Lemma 3.1, there exists $\varepsilon' > 0$ such that for all $w \in \text{Supp} \mu_{\text{BMS}}$, we have $\mu_{\text{PS}}^w (N_{2\varepsilon'}(V)) < \eta$. Let $c = c(f) > 1$ be as from Fact 3.3 above. Let $T_0 = T_0(f, \varepsilon') > 0$ be such that $ce^{-s_0 \varepsilon_1} < \varepsilon'/20$, where $s_0 = \log T_0$.

Let $y \in \text{Supp} \mu_{\text{BMS}}$, let $T \geq T_0$, and define $s = \log T$. Let $I_T$ be a maximal set of points in $ya_{-s} N_{\varepsilon'}(V)$ such that the balls $\{zB_U(\varepsilon'/16) : z \in I_T\}$ are disjoint. Thus, $\{zB_U(\varepsilon'/4) : z \in I_T\}$ covers $ya_{-s} N_{\varepsilon'}(V)$. Let $\{f_z : z \in I_T\}$ be a partition of unity subordinate to this cover. Then we have:

$$e^{(n-1-\delta_1)s} \int_{N_{\varepsilon'}(V)} f(ya_{-s} u t a_s) dt \leq e^{(n-1-\delta_1)s} \sum_{z \in I_T} e^{B_U(\varepsilon'/4)} f(z u t a_s) f_z(z u_t) dt$$

by Fact 3.3$\ll_{\phi} \sum_{z \in I_T} \mu_{\text{PS}}^z (f_z, \varepsilon'/20)$

$$\ll_{\phi} \kappa \mu_{\text{PS}}^{ya_{-s}} (N_{2\varepsilon'}(V))$$

$$\ll_{\phi} \eta$$

where $\kappa$ is the multiplicity of the cover given by the Besicovitch covering theorem. $\kappa$ depends only on the dimension $n$. $\square$

4. **Joinings**

4.1. **Notation.** We establish notation and setup used in the rest of the section:

**The measure $\mu$.** $\mu$ is an ergodic $U$-joining on $X = X_1 \times X_2$ for the pair $(m_1^{\text{BR}}, m_2^{\text{BR}})$. Note that we will often omit the subscripts on $m^{\text{BR}}$.

**$\psi$ and $\Psi$.** Fix a non-negative function $\psi \in C_c(X_1)$ with $m^{\text{BR}}(\psi) > 0$. Let $\Psi = \psi \circ \pi_1 \in C(X)$. 

The set \( \Omega_1 \). A compact set \( \Omega_1 \subseteq \text{Supp} m_1^{B_{\text{MS}}} \) with \( m^{B_{\text{MS}}} (\Omega_1) > 0 \) such that \([8, \text{Lemma 4.6}]\) holds for \( \psi \) uniformly across all \( x \in \Omega_1 \). That is, the convergence
\[
\lim_{T \to \infty} \frac{1}{\mu_{x}^{F_{\Omega}}(B_{U}(T))} \int_{B_U(T)} \psi(x u_t) dt = m^{B_{\text{MS}}} (\psi)
\]
holds uniformly for all \( x \in \Omega_1 \). (See \([8, \text{Remark 4.8}]\).)

The set \( Q \). A compact set \( Q \subseteq X \) with \( \mu(Q) \in (0, \infty) \), \( \pi_1(Q) \subseteq \Omega_1 \), and such that for all \( f \in C_c(X) \) and all \( x \in Q \),
\[
\lim_{T \to \infty} \frac{\int_{B_U(T)} f(x \Delta(u_t)) dt}{\int_{B_U(T)} \Psi(x \Delta(u_t)) dt} = \frac{\mu(f)}{\mu(\Psi)}.
\]
Such a set exists by the Hopf ratio ergodic theorem.

\( \varepsilon, \eta_0 \) and the sets \( Q_+, Q_{++} \). Fix \( 0 < \varepsilon < 1 \) satisfying \((1 + 2\varepsilon)^{-2} > 1/2 \) and \( \eta_0 > 0 \) such that \( \mu(Q_{++}) \leq (1 + \varepsilon) \mu(Q) \), where
\[
Q_{++} := Q(B(\eta_0) \times B(\eta_0)) = Q\{(g, g) \in G \times G : ||g - I|| \leq \eta_0\}.
\]

Define
\( Q_+ := Q(B(\eta_0/4) \times B(\eta_0/4)) = Q\{(g, g) \in G \times G : ||g - I|| \leq \eta_0/4\} \).

\( \phi \) and \( \Phi \). Let \( \phi \in C_c(X_1) \) be such that \( 1_{\pi_1(Q_{++})} \leq \phi \leq 1 \), where \( 1_E \) denotes the characteristic function of a set \( E \) in \( X \). Let \( \Phi = \phi \circ \pi_1 \).

The set \( Q_\varepsilon \) and the family \( \mathcal{F} \). Let \( \mathcal{F} = \{1_Q, 1_{Q+}, 1_{Q_{++}}, \Phi\} \). By the Hopf ratio ergodic theorem again together with Egorov’s theorem, there exists a compact set \( Q_\varepsilon \subseteq Q \) with \( \mu(Q_\varepsilon) > (1 - \varepsilon) \mu(Q) \) such that for each \( f \in C_c(X) \cup \mathcal{F} \), the convergence in equation \((1)\) holds uniformly for all \( x \in Q_\varepsilon \). That is, for all \( f \in C_c(X) \cup \mathcal{F} \) and \( \theta > 0 \), there exists \( T_0 = T_0(f, \theta) \) such that if \( T \geq T_0 \), then
\[
\left| \frac{\int_{B_U(T)} f(x \Delta(u_t)) dt}{\int_{B_U(T)} \Psi(x \Delta(u_t)) dt} - \frac{\mu(f)}{\mu(\Psi)} \right| \leq \theta
\]
for all \( x \in Q_\varepsilon \).

The functions \( \varphi_m \). Suppose that we have a sequence \( g_m \in G - U \) with \( g_m \to 1_G \) and a point \( x = (x_1, x_2) \in Q_\varepsilon \) such that \( (x_1 g_m, x_2) \in Q_\varepsilon \) for all \( m \). For each \( m \geq 0 \), \( \varphi_m(t) := u_t^{-1} g_m u_t \). In particular, \( \varphi_m(t) \) satisfies \( x \Delta(u_t) = x(\varphi_m(t), 1_G) \Delta(u_t) \).

The values \( T_m \). \( T_m := \sup\{T > 0 : \varphi_m(B_U(T)) \subseteq B(1)\} \). Since \( g_m \not\in U = N_G(U) \) (where \( N_G(U) \) denotes the normalizer in \( G \) of \( U \)), \( \varphi_m(t) \) is not constant, and so \( T_m < \infty \). Moreover, \( T_m \to \infty \) because \( g_m \to 1_G \).
The functions $\tilde{\psi}_m$ and $\tilde{\phi}$. On $B_U(1)$, define $\tilde{\psi}_m(t) = \varphi_m(T_m t)$. By definition of $T_m$, each of the entries in the $\tilde{\psi}_m$’s gives rise to a sequence of uniformly bounded polynomials on a compact domain, hence an equicontinuous family. Thus, we may assume that there exists some $\tilde{\phi}$ defined on $B_U(1)$ such that $\tilde{\psi}_m \to \tilde{\phi}$ uniformly on $B_U(1)$. Observe that $\tilde{\phi}$ maps into $\mathcal{N}_G(U) = U$ by construction of the $\varphi_m$’s.

The sets $\tilde{V}$, $N_{\epsilon'}(\tilde{V})$, and $I_{\epsilon'}(T)$. Define $\tilde{V} = \{ t \in B_U(1) : \| \tilde{\phi}(t) - I \| = 0 \}$, and for $\epsilon' > 0$, $N_{\epsilon'}(\tilde{V}) := \tilde{V} B_U(\epsilon') = \{ u_s u_t : s \in \tilde{V}, t \in B_U(\epsilon') \}$. For $T > 0$, let $s = \log T$ and define $I_{\epsilon'}(T) = B_U(T) - a_{\epsilon'} N_{\epsilon'}(\tilde{V}) a_s$.

4.2. Fibers of $\pi_2$ are finite. In this section, we show that it must be the case that almost every fiber of $\pi_2$ is finite, as otherwise $\mu$ would be invariant under a nontrivial connected subgroup of $U \times \{ 1_G \}$, which is impossible by [S, Lemma 7.16]. More precisely, we will establish the following theorem:

**Theorem 4.1** (c.f. [S, Theorem 7.17]). There exists a positive integer $\ell > 0$ and a $m^{\text{BR}}$-null subset $X'_2 \subseteq X_2$ so that $\pi_2^{-1}(x_2)$ has cardinality $\ell$ for every $x_2 \in X'_2$. Moreover, the fiber measures $\mu_{x_2}^{\pi_2}$ are uniform measures for each $x_2 \in X'_2$.

The proof of Theorem 4.1 will follow as in the proof of [S, Theorem 7.17] once we establish the following generalization of [S, Theorem 7.12] to the convex cocompact case:

**Theorem 4.2** (c.f. [S, Theorem 7.12]). Suppose that there exists $x = (x_1, x_2) \in Q_\varepsilon$ and a sequence $g_n \in G - U$ with $g_n \to 1_G$ such that $(x_1 g_m, x_2) \in Q_\varepsilon$ for all $m$. Then $\mu$ is invariant under a nontrivial connected subgroup of $U \times \{ 1_G \}$.

The proof of Theorem 4.2 first requires several lemmas.

**Lemma 4.3.** For every $0 < \eta' < 1$, there exists $\epsilon' > 0$ and $T_0 > 0$ such that for all $T \geq T_0$, for all $F \in \{ \Psi, \Phi \}$, and for all $y \in Q_\varepsilon$, we have that

$$\frac{\int_{I_{\epsilon'}(T_m)} \Psi(y \Delta(u_t)) dt}{\int_{I_{\epsilon'}(T_m)} \Phi(y \Delta(u_t)) dt} \leq c_1 \frac{\eta'}{1 - \eta'},$$

for some constant $c_1 > 0$.

**Proof.** By definition of $\Omega_1$ and $Q_\varepsilon$ in section 4.1 we have in particular that there exists $T_1 > 0$ such that for all $T \geq T_1$ and for all $y \in Q_\varepsilon$, $T \geq T_0$ implies

$$\int_{B_U(T)} \Psi(y \Delta(u_t)) dt = \int_{B_U(T)} \psi(y u_t) dt \geq \frac{1}{2} \mu^{PS}(B_U(T)) m^{\text{BR}}(\psi) \geq \frac{1}{2} T^{\delta'} m^{\text{BR}}(\psi) C > 0,$$

where $C := \inf_{s \in \text{Supp } m^{\text{BMS}}} \mu^{PS}(B_U(1)) > 0$ by Corollary 2.2.
Now, let \( f = \psi \) if \( F = \Psi \), and \( f = \phi \) if \( f = \Phi \). Fix \( 0 < \eta' < 1 \) and let \( \eta = \frac{C(\psi')}{2K} \eta' m(\Phi) \), where \( K \) is the implied constant from Lemma 3.2 for \( \Phi \). Then by Lemma 3.2 there exists \( \varepsilon' > 0 \) and \( T_0 \geq T_1 \) such that for all \( T \geq T_0 \) and all \( y \in \text{Supp} m(\Phi) \), \( e^{\gamma(T)} \frac{C(\psi')}{2K} \eta' m(\Phi) \). In particular, this implies that for all \( T \geq T_0 \) and \( y \in \text{Supp} m(\Phi) \),

\[
\int_{a_{-\varepsilon N(\Phi)_a}} f(yu_t)dt \leq \frac{\eta'}{2} T^{\delta'} C m(\Phi).
\]

By subtracting equation (3) for \( \psi \) from (2), we conclude that for all \( T \geq T_0 \) and for all \( y \in \text{Supp} m(\Phi) \),

\[
\int_{I_\varepsilon(T_m)} \psi(yu_t)dt \geq \frac{1}{2} (1 - \eta') T^{\delta'} C m(\Phi).
\]

Then from equations (3) and (4), we have that for all \( T \geq T_0 \) and \( y \in \text{Supp} m(\Phi) \),

\[
\frac{\int_{a_{-\varepsilon N(\Phi)_a}} f(yu_t)dt}{\int_{I_\varepsilon(T_m)} \psi(yu_t)dt} \leq \frac{\eta'}{2(1 - \eta')} T^{\delta'} C m(\Phi) \leq c_1 \frac{\eta'}{1 - \eta'}
\]

for \( c_1 = \max\{m(\Phi)/m(\Phi), 1\} \), as desired. \( \Box \)

By definition of \( Q_\varepsilon \), we have that for all \( f \in F \) and for all \( \theta > 0 \), there exists \( T_0 = T_0 > 0 \) such that if \( T \geq T_0 \), then for all \( y \in Q_\varepsilon \),

\[
\left| \frac{\int_{B(T)} \frac{f(yu_t)}{\Psi(yu_t)} dt}{\Psi(yu_t) dt} - \frac{\mu(f)}{\mu(\Psi)} \right| \leq \theta.
\]

We can now improve this to integration over sets of the form \( I_\varepsilon(T) \) as follows.

**Corollary 4.4.** For all \( \theta > 0 \), there exists \( \varepsilon' > 0 \) and \( T_0 > 0 \) such for all \( T \geq T_0 \), all \( y \in Q_\varepsilon \), and every \( F \in F = \{1_Q, 1_{Q+}, \Phi\} \), we have that

\[
\left| \frac{\int_{I_\varepsilon(T)} \frac{f(yu_t)}{\Psi(yu_t)} dt}{\Psi(yu_t) dt} - \frac{\mu(F)}{\mu(\Psi)} \right| \leq \theta.
\]

**Proof.** \( \varepsilon' \) and \( T_0 \) come from Lemma 4.3. Let \( T \geq T_0 \) and let \( s = \log T \). Define \( N_s = a_{-s} N(\Phi)_a \cap B_\varepsilon(T) \). The conclusion then follows by writing

\[
\int_{B_\varepsilon(T)} F(yu_t)dt = \frac{\mu(F)}{\mu(\Psi)} \int_{I_\varepsilon(T)} \Psi(yu_t)dt + \frac{\mu(F)}{\mu(\Psi)} \int_{N_s} \Psi(yu_t)dt + \int_{B_\varepsilon(T)} \Theta(T) \Psi(yu_t)dt
\]

where \( \Theta(T) \) is determined from equation (5) and tends to zero uniformly over \( Q_\varepsilon \).
Thus,
\[
\frac{\int_{I_{\ell}(T)} F(y\Delta(u_t))dt}{\int_{I_{\ell}(T)} \Psi(y\Delta(u_t))dt} = \frac{\mu(F)}{\mu(\Psi)} \left(1 + \frac{\int_{N_{\varepsilon}} \Psi(y\Delta(u_t))dt}{\int_{I_{\ell}(T)} \Psi(y\Delta(u_t))dt}\right)
\]
\[+ \Theta(T) \frac{\int_{N_{\varepsilon}} \Psi(y\Delta(u_t))dt}{\int_{I_{\ell}(T)} \Psi(y\Delta(u_t))dt} - \frac{\int_{N_{\varepsilon}} F(y\Delta(u_t))dt}{\int_{I_{\ell}(T)} \Psi(y\Delta(u_t))dt}\]

The conclusion then follows from Lemma 4.3, where for the last term we note that for all \( f \in \mathcal{F}, 0 \leq F \leq \Phi \).

We can now prove Theorem 4.2.

Proof of Theorem 4.2. Recall the notation from section 4.1. Define \( \tau_m = \sup\{\tau > 0 : \varphi_m(B_{\mathcal{U}}(\tau)) \subseteq B(y_0/4)\} \). Note that \( \tau_m \to \infty \) as \( m \to \infty \).

It follows from Corollary 4.4 (by writing out with error terms and dividing) that there exists \( \varepsilon' > 0 \) and \( T_0 > 0 \) such that for every \( T \geq T_0 \), every \( y \in Q_\varepsilon \), and every \( F_1, F_2 \in \mathcal{F} = \{1_Q, 1_{Q_+}, 1_{Q_{++}}\} \),

\[
\left| \frac{\int_{I_{\ell}(T)} F_1(y\Delta(u_t))dt}{\int_{I_{\ell}(T)} F_2(y\Delta(u_t))dt} - \frac{\mu(F_1)}{\mu(F_2)} \right| \leq \varepsilon.
\]

Moreover, this \( T_0 \) can be chosen so that \( \{t \in I_{\varepsilon}(T_0) : x\Delta(u_t) \in Q_{++}\} > 0 \).

Claim: Let \( h_m = (g_m, 1_G) \). For all \( m \) with \( \tau_m \geq T_0 \) and all \( T_0 \leq T \leq \tau_m \),

\[\{t \in I_{\varepsilon}(T) : x\Delta(u_t), xh_m\Delta(u_t) \in Q\} \neq \emptyset.\]

Proof of claim. Recall that \( \varphi_m(t) = u_t^{-1}g_mu_t \) satisfies

\[xh_m\Delta(u_t) = x\Delta(u_t)(\varphi_m(t), 1).\]

By definition of \( \tau_m \), if \(|t| \leq \tau_m\), then \( \|\varphi_m(t), 1_G\| = 1 \), so

\[
\{t \in I_{\varepsilon}(T) : x\Delta(u_t) \in Q\} \subseteq \{t \in I_{\varepsilon}(T) : xh_m\Delta(u_t) \in Q_+\} \\
\subseteq \{t \in I_{\varepsilon}(T) : x\Delta(u_t) \in Q_{++}\}.
\]

By applying equation (6) to \( F_1 = 1_{Q_{++}} \) and \( F_2 = 1_Q \) with \( y = x \), we have that

\[
\ell(\{t \in I_{\varepsilon}(T) : x\Delta(u_t) \in Q\}) \geq (1 + 2\varepsilon)^{-1}\ell(\{t \in I_{\varepsilon}(T) : x\Delta(u_t) \in Q_{++}\}),
\]

where \( \ell \) is the Lebesgue measure.

And by applying it with \( F_1 = 1_{Q_+} \) and \( F_2 = 1_Q \) with \( y = xh_m \), we have that

\[
\ell(\{t \in I_{\varepsilon}(T) : xh_m\Delta(u_t) \in Q\}) \geq (1 + 2\varepsilon)^{-1}\ell(\{t \in I_{\varepsilon}(T) : xh_m\Delta(u_t) \in Q_+\}) \\
\geq (1 + 2\varepsilon)^{-1}\ell(\{t \in I_{\varepsilon}(T) : x\Delta(u_t) \in Q\}) \\
\geq (1 + 2\varepsilon)^{-2}\ell(\{t \in I_{\varepsilon}(T) : x\Delta(u_t) \in Q_{++}\})
\]

Since \( \{t \in I_{\varepsilon}(T) : x\Delta(u_t) \in Q\} \) and \( \{t \in I_{\varepsilon}(T) : xh_m\Delta(u_t) \in Q\} \) are both subsets of \( \{t \in I_{\varepsilon}(T) : x\Delta(u_t) \in Q_{++}\} \) from the definition of \( \tau_m \), the choice of \( \varepsilon \) implies that both subsets have greater than half the Lebesgue
measure of the larger set (which is positive by choice of \( T_0 \)), and thus their intersection cannot be empty. □

By the claim, for all sufficiently large \( m \), there exists \( t_m \in I_{\epsilon'}(\tau_m) \) such that \( x\Delta(u_{t_m}) \in Q \) and \( xh_{\tau_m}\Delta(u_{t_m}) = x\Delta(u_{t_m})(\varphi_m(t_m), 1_G) \in Q \). By the compactness of \( Q \) and by dropping to a subsequence if necessary, we may assume that there exists \( x_\infty \in Q \) such that \( x_\Delta(u_{t_m}) \to x_\infty \).

Let \( \tilde{t}_m = t_m/\tau_m \in B_{U}(1) \), so that \( \varphi_m(t_m) = \tilde{\varphi}_m(\tilde{t}_m) \). Then again by the compactness of \( B_{U}(1) \) and the uniform convergence of \( \tilde{\varphi}_m \to \tilde{\varphi} \), we may assume that there exists \( t_\infty \in B_{U}(1) \) such that \( \varphi_m(t_m) \to \tilde{\varphi}(t_\infty) \).

By definition of \( I_{\epsilon'}(\tau_m) \), \( t_m \notin N_{\epsilon'}(\tilde{V}) \) for all \( m \), so \( t_\infty \notin \tilde{V} \). In particular, this implies that \( \tilde{\varphi}(t_\infty) \neq I \). Moreover, the image of \( \tilde{\varphi} \) is contained within \( N_{G \times G}(\Delta(U)) \cap (G \times \{1_G\}) = U \times \{1_G\} \), so it follows from [8, Lemma 7.7] that \( \mu \) is quasi-invariant under a nontrivial connected subgroup of \( U \times \{1_G\} \). Strict invariance follows from [8, Lemma 7.3]. □

The proof of Theorem 4.1 now follows as in [8, Theorem 7.17], with Theorem 4.2 replacing references to Theorem 7.12 in that proof.

4.3. Joining classification. Theorem 4.1 tells us that there exists \( \ell > 0 \) such that for \( m^{BR} \)-a.e. \( x_2 \in X_2 \), \( \pi_2^{-1}(x_2) \) has cardinality \( \ell \). This gives rise to a set-valued map \( \Upsilon \) defined by

\[
\Upsilon(x_2) = \pi_2^{-1}(x_2).
\]

The fact that \( \mu \) is \( \Delta(U) \)-invariant implies that \( \Upsilon \) is \( U \)-equivariant. In [5, Prop. 6.7, Lemma 7.1], Flaminio and Spatzier establish a rigidity result in the convex cocompact case for \( U \)-equivariant functions, namely that they are also \( AMU^- \)-equivariant. The argument in [8, Section 7.6] can be adapted to establish the same conclusion for the set-valued function \( \Upsilon \) in this case. Finally, as in [8, Prop. 7.23], \( \Upsilon \) satisfying these conditions will imply that \( \mu \) is a finite cover self-joining, establishing Theorem 1.3. The details of this argument, including a more general factor rigidity statement, will be provided in a later version.

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