Locally homogeneous finitely nondegenerate
CR-manifolds

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1. Introduction

In several areas of mathematics, homogeneous spaces are fundamental objects as they often
serve as models for more general objects: Various examples from differential geometry (Rie-
mannian symmetric spaces, principal bundles) and topology (geometric 3-manifolds), to alge-
braic and complex geometry (uniformization theorems, flag manifolds) etc. underline the impor-
tance of spaces, furnished with a structure, compatible under a transitive group action. In this
paper, we investigate homogeneous Cauchy-Riemann manifolds from the local point of view,
more precisely, the germs of CR-manifolds which are locally homogeneous under some finite-
dimensional Lie group.

The most common way of prescribing a CR-manifold is to describe it locally in some $C^\infty$
as the zero set of certain defining functions. The characterization of the geometric properties
of such a manifold, like the signature of the Levi form(s), finite or holomorphic nondegeneracy,
minimality, etc. involves a manipulation of the defining equations, which, in concrete cases, can
be quite hard. A (locally) homogeneous CR-manifold can also be described by a purely algebraic
datum, for instance by a CR-algebra in the sense of [13]. In fact, one can show that there is a
natural equivalence between the category of germs of locally homogeneous CR-manifolds on the
complex geometric side and the category of CR-algebras on the algebraic side, see Section 4 for
further details. In order to characterize the complex-geometric properties of $M$, the knowledge
of the full Lie algebra of local automorphisms of $M$ is not necessary; any Lie group, acting
locally transitively on $M$ will do. The advantage of this point of view is that in general the
manipulation of CR-algebras is easier than the manipulation of the defining equations, provided
that there is a simple “dictionary” which “translates” the algebraic properties of a given CR-
algebra into the complex-geometric properties of the underlying CR-manifold.

In the first part of our paper we pursue this goal and explain how the Levi form of $M$ and
its higher order analogues can be read off the corresponding CR-algebra. This enables us in
Theorem 5.10 to characterize the order of nondegeneracy of a locally homogeneous CR-manifold
$M$, as well as to decide whether or not $M$ is holomorphically degenerate. In Theorem 5.11
the minimality of $M$ is described in terms of the CR-algebra. A basic ingredient in the proofs
is the Main Lemma [2,3] which relates certain canonical tensors and subbundles of $\mathcal{T}M$ and
$\mathcal{T}^4M$ with subspaces of infinitesimal CR-transformations and the corresponding Lie structure.
As a first application we generalize a result of Kaup and Zaitsev stated in [12] (see the paragraph
before 5.12 for the precise statements) for certain irreducible Hermitian symmetric spaces to the
more general case of arbitrary flag manifolds $Z$ with $\mathfrak{h}_Z(Z) = 1$ (Theorem 5.12). Our proof of
this theorem does not use Jordan-theoretical methods.
In the second part of this paper we provide an example of a homogeneous (hence, uniformly) 3-nondegenerate hypersurface $M$ in the 7-dimensional Grassmannian of isotropic 2-planes in $\mathbb{C}^7$: In this example the first order Levi kernel is 3-dimensional and contains the second order kernel which is 1-dimensional. While it is quite easy to produce real-analytic CR-manifolds which are, at some particular point, finitely nondegenerate of an arbitrary high order, our hypersurface seems to be the first known example of a CR-manifold with a uniform order of degeneracy bigger than 2. Note that in \cite{12} orbits of real forms have been studied in a certain subclass of irreducible Hermitian symmetric spaces (of so-called tube type), and the authors prove that all such orbits with a nontrivial CR-structure, (i.e., neither open nor totally real) are 2-nondegenerate. Our example is an orbit in a more general flag manifold and we use methods developed in the first part to determine its kind of nondegeneracy. At this point one might expect to find orbits $M$ in complex flag manifolds $Z$ with uniformly finitely nondegenerate CR-structure of arbitrary high order, provided that the ambient manifold $Z$ is general enough. Surprisingly, at least for hypersurface orbits, this is not the case: In Theorem 6.3 we give a general upper bound for the order of degeneracy that is valid for all finitely nondegenerate hypersurface orbits in arbitrary flag manifolds. For instance, for all classical cases, i.e., where the (connected component of the identity of the) complex group of biholomorphic transformations, $\text{Aut}(Z)$, is a product of classical simple groups, this upper bound is 3. The methods used to determinate the complex-geometric properties of $M$ can be generalized to deal with arbitrary orbits of real forms in arbitrary flag manifolds.

We also like to mention that the methods developed in this paper will be used in the forthcoming article \cite{8}, in which all 5-dimensional 2-nondegenerate germs of locally homogeneous CR-manifolds are classified up to CR-equivalence.

Our paper is organized as follows. In Section 2 we discuss tensors induced by Lie brackets and, following Palais \cite{15}, we recall basic facts on local actions. The main result here is the Main Lemma \cite{2,3}. Section 3 recalls basic geometric notions concerning CR-manifolds, focusing on the condition of being finitely nondegenerate. In Section 4 we recall the definition of the category of CR-algebras (see also \cite{13}) and show that there is an equivalence between this category and the category of germs of locally homogeneous CR-manifolds. The first part of our paper culminates in Section 5, where we provide a “dictionary”, extracting from a given CR-algebra the information necessary to characterize the complex-geometric properties of the underlying CR-geerm. This characterization is used to prove a generalization of the above mentioned result of Kaup and Zaitsev. Finally, in Section 6 we give an example of a homogeneous 3-nondegenerate CR-manifold (as already mentioned above) and indicate a method how arbitrary orbits of real forms in flag manifolds can be handled.

2. Tensors and homogeneous manifolds

**General notation.** Let $X$ be a manifold. Given a vector bundle $E \to X$ over $X$; we write $(X; E)$ for the vector space of smooth sections over $X$. If a further specification is necessary, we write $\mathcal{O}(\cdot)$ or $\mathcal{O}^\infty(\cdot)$ etc. for the real-analytic or holomorphic sections, respectively. By $E_x$ we denote the fibre of $E$ at $x \in X$. As usual, $TX$ stands for the tangent bundle of $X$ and $T_xX$ for the tangent space at $x$. Given a vector field $Z \in \mathfrak{X}(X)$ we write $Z(x)$ for its value at $x$. If not otherwise stated all Lie groups and Lie algebras (except for $\mathfrak{X}(X)$) are assumed to be of finite dimension. In particular, “homogeneous” means
(infinitesimally) homogeneous under a finite dimensional Lie group (algebra). Lie groups are denoted by capital letters $G; H; \vdots$ and the associated Lie algebras by the corresponding fraktur letters $g; h; \vdots$ etc. $G$ stands for the connected component of the identity of a Lie group $G$.

By definition, the Lie bracket in $g$ is given by the Lie bracket of left-invariant vector fields on $G$: By $\text{Ad}$ we denote the adjoint representation of $G$ on $g$ and by $\text{ad}$ its differential, i.e., $\text{ad}_{G}(w) = \{v; w\}$: Given a real vector space $V$, we denote by $V^q := V \otimes \mathbb{R}^q = V \otimes \mathbb{R}^2$ the formal complexification of $V$: If the real vector space $V$ is furnished with an endomorphism $J : V \to V$ satisfying $J^2 = \text{Id}$; we write $V^{\rho; 0}$ for the $(\rho)$-eigenspaces of $J$ in $V^q$.

**Tensors induced by Lie brackets.** Let $E \to X$ be a (smooth) subbundle. It is well-known that the following $\mathbb{R}$–bilinear map

$$\langle X; E \rangle \otimes \langle X; E \rangle \to \langle X; E \rangle \otimes \langle X; E \rangle \to \langle X; E \rangle \twoheadrightarrow \langle X; E \rangle \otimes \langle X; E \rangle \mod \langle X; E \rangle$$

is, in fact, $C^1$–bilinear. Hence, it induces a well-defined fibre-wise bilinear map (tensor)$E_x \to E_x! \to T_xX = E_x$, i.e., $[\cdot; \cdot] \mod E_x$ depends only on the values $x; x$ and not on the choice of the local sections $\cdot$ in $E$.

It turns out that for (locally) homogeneous manifolds $X$ the explicit computation of various tensors naturally attached to $X$; similar to that one given above, can be reduced to a simple algebraic expression. The main application we have in mind is the determination of the Levi form of a (locally) homogeneous CR-manifold $M$ and its “higher-order” analogues, suitable for the characterization of the $k$–nondegeneracy of $M$ in the sense of [5]. In the next paragraphs we fix our notation and briefly recall some basic facts concerning homogeneous manifolds.

**Locally homogeneous manifolds and bundles.** The topics of this subsection are well-known. The reader familiar with the global concepts of a homogeneous space or a homogeneous bundle will have no difficulties to give the local versions of these objects. In the following paragraphs we briefly recall the facts relevant for our purposes. A reference in the local situation is the fundamental paper of Palais [15] (9 is a more up-to-date reference).

All groups occurring in this paper are assumed to be finite-dimensional Lie groups. Let $G$ be such a group. In the global setting, the fundamental objects are $G$–manifolds, i.e., manifolds provided with a (left) $G$–action $: G \times X \to X$: A homogeneous $G$–bundle $E \to X$ over such a manifold $X$ is a vector bundle together with a fibre-wise linear action on $E$ which is a lift of the given $G$–action on $X$ (if $X$ is $G$–homogeneous, i.e., $G$ acts transitively on $X$; we write $G_x$ for the isotropy group at $x \in X$) and $g_x$ for the corresponding isotropy Lie subalgebra. For a homogeneous bundle over a homogeneous manifold the isotropy representation $G_x \to E_x$ determines completely the global structure of the vector bundle $E$ over $X = G = G_x$.

The total space of $E$ is the twisted $G$–product $G \times_x E_x$; Conversely, a representation $X \to \text{GL}(V)$ of a (closed) subgroup of $G$ on some vector space $V$ gives rise to the homogeneous vector bundle $V := G \times H \to V$ over $G = H$.

All the above notions can be appropriately “localized”. A local action of $G$ on a manifold $X$ is a map $U \to X$ such that $U \times G M$ is an open neighbourhood of $x \in M$, the identity $e \times x = x$ holds for all $x \in X$ as well as $h (g \times) = (h \times g) \times$ when both sides are defined. Without loss of generality we may assume that $G$ is simply connected, which we do for all what
A local action induces a Lie algebra homomorphism \( \varphi : g \to \mathfrak{X}(\mathbb{T}X) \); see (2.2). A given Lie algebra homomorphism \( \varphi : g \to \mathfrak{X}(\mathbb{T}X) \) is called an infinitesimal action of \( g \) on \( X \) and \( X \) a \( g \)-space. As shown in [13], an infinitesimal action induces a local action of \( G \) on \( X \) (say, \( G \) is the simply connected Lie group with Lie algebra \( g \)); consequently, local and infinitesimal actions are equivalent objects. It is known that not globalizable local actions exist, see [9], p.105 for further details. All above notions can also be applied to germs of manifolds. To fix the notation, we write \( \mathfrak{X}(\mathbb{T}X) \) for a germ at the base point \( x \) and \( X \) for a representative of the germ. Further, \( \mathfrak{X}(\mathbb{T}X) \) stands for the germ of a \( g \)-space (where the homomorphism \( \varphi : g \to \mathfrak{X}(\mathbb{T}X) \) describes the infinitesimal action).

By a morphism between the \( g \)-spaces \( X \) and \( Y \) we mean a pair \( ( ; ; ) \); consisting of a map \( : X \to X^0 \) in the given category (smooth, real-analytic, holomorphic) and a Lie algebra homomorphism \( : g \to g^0 \) such that \(( (v)_x = 0( (v))_x) \) for all \( v \in g \). Every \( g \)-equivariant map (i.e., \( = \mathfrak{X} \)) is an example of a morphism between two \( g \)-spaces. A morphism between the germs \( \mathfrak{X}(\mathbb{T}X); \mathfrak{X}(\mathbb{T}Y) \) of two locally homogeneous spaces \( X \) and \( Y \) is then an equivalence class \( [ ; ; ] \) induced by a base point preserving equivariant morphism \( ( ; ; ) : X \to Y \).

We call an infinitesimal (or local) action of \( g \) (resp. \( G \)) on \( X \) effective if the map \( \varphi \) is injective. A global action \( G \cdot X \) is effective in this sense if and only if the subgroup, formed by all elements \( g \in G \) which act as the identity on \( X \); is discrete. Clearly, dividing \( g \) (or \( G \)) by the ineffectivity ideal \( \mathfrak{i} = \ker \varphi \) (resp. by the connected component of \( \mathfrak{X}(\mathbb{T}X) \)), every non-effective action can be modified into an effective action with the same orbits (resp. Nagano-leaves). A local, or equivalently, infinitesimal action on \( X \) is called transitive if the evaluation map \( : g \to \mathbb{T}X ; v \mapsto (v)_x \) is surjective for all \( x \in X \). Then we say that \( X \) is locally homogeneous or \( g \)-homogeneous. We call a germ \( \mathfrak{X}(\mathbb{T}X) \) homogeneous if there exists a locally homogeneous representative \( X \).

It is known that for every pair \( h \) of finite-dimensional Lie algebras, there is a germ \( \mathfrak{X}(\mathbb{T}X) \) with a transitive infinitesimal action \( \varphi : g \to \mathfrak{X}(\mathbb{T}X) \) such that \( h = \ker \varphi \). We call \( g = g_\mathfrak{X} \) the infinitesimal model for \( \mathfrak{X}(\mathbb{T}X) \): We say that the action or the infinitesimal model is effective if the action of \( g \) on some representative \( X \) has this property. In the case when \( g \) is infinite dimensional, we do not know (even if \( \dim g = g_\mathfrak{X} < 1 \)) whether it is always possible to construct in a meaningful way a germ of a (finite dimensional) manifold with a local transitive action of some group “associated” with \( g \).

Finally, a vector bundle \( E \) over a \( g \)-homogeneous manifold is called locally homogeneous if the local action of \( G \) lifts to a local action on \( E \) in such a way that the corresponding local transformations are fibre-wise linear. A germ of a locally homogeneous bundle (we use the notation \( \mathfrak{X}(\mathbb{T}X;\mathbb{T}E) \) for it) is determined by the linear representation \( \xi : g_\mathfrak{X} \cdot \mathfrak{gl}(\mathfrak{E}_\mathfrak{X}) \) of the isotropy Lie algebra \( g_\mathfrak{X} \) on the fibre \( \mathfrak{E}_\mathfrak{X} \). On the other hand, any representation \( \xi : g_\mathfrak{X} \cdot \mathfrak{gl}(\mathfrak{V}) \) gives rise to a (germ of a) locally homogeneous vector bundle \( \mathfrak{V} \) over the germ \( \mathfrak{X}(\mathbb{T}X) \) of a \( g \)-homogeneous manifold with \( \mathfrak{V}_\mathfrak{X} = \mathfrak{V} \).

Recall that each (local) \( G \)-action on \( X \) induces the so-called fundamental vector fields on \( X \): The following map

\[
(2.2) \quad \varphi : g \to \mathfrak{X}(\mathbb{T}X); \quad v \mapsto \varphi ; \quad \text{given by} \quad \frac{d}{dt} \bigg|_{t=0} f(\exp(tv)) \; y \; ;
\]
where the \( f \)'s run through smooth functions defined in a neighborhood of \( y \); is a Lie algebra homomorphism. For each \( v \in \mathfrak{g} \) the vector field \( \mathcal{v} \in \mathfrak{g} \) is called fundamental. Unfortunately, the fundamental vector fields and (locally) homogeneous vector bundles on a \( \mathfrak{g} \)-space seem to be unrelated. For instance, the fundamental vector fields are not invariant under the local group action. Consequently, given a homogeneous \( G \)-subbundle \( TX = \mathfrak{g} \) and a fundamental vector field \( \mathcal{v} \) such that \( \mathcal{v} \in \mathfrak{g} \) for some \( \mathfrak{x} \); the values \( \mathcal{v} \) may not belong to \( \mathfrak{g} \) for \( y \) close to \( x \): Since in general the fundamental vector fields do not generate a homogeneous subbundle \( \mathfrak{g} \); they cannot be used for the calculation of the Lie brackets in situations similar to \( \mathfrak{g} \).

2.1 Nevertheless the following lemma is valid, which is the main result of this section:

**Main Lemma 2.3** Let \( X \) be a locally homogeneous \( G \)-manifold, \( x \in X \) a base point and \( \mathfrak{g} = \mathfrak{g}_x \) the corresponding infinitesimal model. Let \( E^1;E^2;\mathfrak{d} \) \( \mathfrak{g} \) be the corresponding \( \text{ad} (\mathfrak{g}_x) \)-stable linear subspaces such that \( E^1_x = \mathfrak{g}^1 = \mathfrak{g}_x \) and \( D_x = \mathfrak{d} = \mathfrak{g}_x \): Assume that the bracket map

\[
[\;]; (X;E^1) \times (X;E^2) \rightarrow (X;TX) = (X;\mathfrak{d})
\]

is \( C^1 \times \mathfrak{g} \)-bilinear, i.e., it defines a tensor \( b : E^1 \rightarrow E^2 \times TX \). For arbitrarily given tangent vectors \( 1, 2 \in E^1_x \); \( 2 \in E^2_x \); choose representatives \( u^1 \in E^1 \) and \( u^2 \in E^2 \). Then, identifying \( T_xX = \mathfrak{g}_x \) with \( \mathfrak{g} = \mathfrak{d} \), we have

\[
b_x (1, 2) = [u^1; u^2]_g \mod \mathfrak{d}:
\]

Here, the bracket is taken in the Lie algebra \( \mathfrak{g} \); and the right-hand side does not depend on the choice of the representatives \( u^1 \);

**Proof.** For simplicity, we carry out the proof for globally homogeneous \( X \); i.e., \( X = G = \mathfrak{g}_x \); where \( \mathfrak{g}_x \) stands for the isotropy subgroup at the base point \( x \): It relies on the construction of particular local vector fields \( 1, 2 \) around \( x \) and works equally well in the locally homogeneous case. By construction, the tensor \( b \) is \( G \)-invariant. Hence, it suffices to compute it at one point only. Denote by \( :G \rightarrow G = \mathfrak{g}_x \) the projection map and by \( :T \mathfrak{g} \rightarrow T (G = \mathfrak{g}_x) \) its differential. In particular, \( b \) yields a surjection \( \mathfrak{g} \rightarrow T_xX \): Select once and for all a linear subspace \( \mathfrak{g}_2 \) complementary to \( \mathfrak{g}_x \): Let \( v^1 \in E^1_x \) and \( v^2 \in E^2_x \) be arbitrarily given. Since \( b \) is alternating, we may assume without loss of generality that \( v^1, v^2 \) are linearly independent. Select \( w^1, \ldots, w^n \in \mathfrak{g}_2 \) such that \( (w^n) = v^1 \); extend \( w^1, w^2 \times \mathfrak{g}_2 \) to a basis \( w^1, \ldots, w^n \) of \( \mathfrak{g}_2 \) and let \( w^1, \ldots, w^n \) be a basis of \( \mathfrak{g}_x \):

By assumption, the bracket \( [1, 2] \mod D_x \) does not depend on the choice of the vector fields \( 1, 2 \in E^1 \times \mathfrak{g} \) which, for \( j \in 1, 2 \); \( 2g \); extend \( v^1 \) in some neighborhood \( U \) of \( x \): The key point here is the construction of appropriate local extensions \( 1, 2 \) of \( v^1 \) and \( v^2 \): To accomplish this we first construct certain -projectable vector fields \( \mathcal{v} \) on an open set in \( G \) and then define \( \mathcal{v}^j \in \mathfrak{g} \):

**CONSTRUCTION OF THE VECTOR FIELD FOR A GIVEN \( \mathcal{v} \in \mathfrak{g} \):** Select a convex open neighborhood \( \mathfrak{W} \rightarrow \mathfrak{W} \rightarrow \mathfrak{g} \) of \( 0 \) such that

the map \( \exp : \mathfrak{W} ! Y \rightarrow \exp (\mathfrak{W}) \) is a diffeomorphism onto the locally closed submanifold \( Y \rightarrow \mathfrak{G}, \) and

the restriction \( :Y \rightarrow X \) is a diffeomorphism onto a neighborhood \( V \rightarrow \mathfrak{G} = \mathfrak{G}_x \) of \( x \); i.e., \( Y \) is the (image of a) local section in the principal bundle \( :G \rightarrow \mathfrak{G} = \mathfrak{G}_x \):

Write \( (\mathfrak{g}^1; u) \) for elements in \( TG = \mathfrak{g} \rightarrow \mathfrak{g} \) with respect to the trivialization by left-invariant
vector fields. For an arbitrary given $v \in T_xX$ let $w \in \mathfrak{w}$ be the unique element with $(w) = v$: For such $w$ define along $Y$ simply by requiring $g = (g^g w)$ for all $g \in Y$ and then extend to a vector field on $\mathfrak{g} = Y \mathfrak{g}_x$ by

$$g h = (g h; \text{Ad}_h : (w)) \quad g \in Y; h \in \mathfrak{g}_x.$$ 

Note that is invariant under the action of $\mathfrak{g}_x$ from the right; hence, it is -projectable, and we have $g h = L^g h \mathfrak{w}$, where $g \in Y; h \in \mathfrak{g}_x$; In particular, for the tangent vectors $v^1; v^2$ as above we write $^1; ^2$ for the above constructed vector fields on $^1; ^2$. From the above follows that the vector fields

$$^j \mathfrak{V} \quad j = 1; 2$$

on $\mathfrak{V} \times X$ are local sections in the $G$-bundles $E \times \mathfrak{V}$ with $^1 = v^1; ^2 = v^2$: (In general, the 's are neither left- nor right-invariant.) The -projectable vector fields satisfy

$$[^1; ^2] = [^1; ^2].$$

We claim that $[^1; ^2]$ has a simple expression in terms of the Lie brackets in $\mathfrak{g}$ (by definition with respect to the left-invariant vector fields). Since $w^1; j = 1; n$; form a basis of $\mathfrak{g}$; the vector fields $^j$ can be written as linear combinations of left-invariant vector fields, i.e.,

$$^1 = P \sum_{j=1}^{n} a_j w^1, \quad ^2 = P \sum_{j=1}^{n} b_j w^2$$

with $a_j; b_j \in C^\infty (\mathfrak{g})$. By construction, all these functions are constant on $Y$ and we have in particular $a_k = 0$ for $k \in 1$; and $b_k = 0$ for $k \in 2$. The following identity is valid at an arbitrary point $y \in Y$:

$$[^1; ^2] = \sum_{j=1}^{n} a_j w^1, \quad \sum_{j=1}^{n} b_j w^2 = \sum_{j=1}^{n} \left[ a_j (y) b_k (y) [w^1, w^2] + \right. \sum_{j=1}^{n} \left. a_j (y) (w^1) b_k (y) (w^2) a_j (y) \right]$$

with $a_j; b_j \in C^\infty (\mathfrak{g})$. By construction, all these functions are constant on $Y$ and we have in particular $a_k = 0$ for $k \in 1$; and $b_k = 0$ for $k \in 2$. The following identity is valid at an arbitrary point $y \in Y$:

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Since $\exp \mathfrak{w} \in Y$ are the local integral curves at $\mathfrak{g}$ for $w^1$ and $w^2$; it follows $w^1 b_k (e) = w^2 a_k (e) = 0$ for all $k$; and the above formula, evaluated at $e$, implies $[^1; ^2] = [w^1, w^2]$. This identity together with (2.6) concludes our proof.

Corollary 2.7 (of the Proof of the Main Lemma)

(i) Assume that $\mathfrak{g} = TX$ is a (locally) $G$-homogeneous vector subbundle over a (locally) homogeneous space $X$ and $J : E \to E$ is a (locally) $G$-equivariant bundle endomorphism. Let $v \in T_xX$ be arbitrary and $v^0 = J_x v$. Select $w : \mathfrak{w} \to \mathfrak{w}$ with $(w) = v$; $(w^0) = v^0$ and define $; \in C^\infty$. Then for the corresponding vector fields $(; = v; ; = \mathfrak{w}$) the relation holds at all points of $X$.

(ii) The statement of the Main Lemma remains true if $TX$ is replaced by its formal complexification $X = TX \mathfrak{C} = \mathfrak{g} \mathfrak{C}$ and $E^1; E^2; E^3$ are $G$-homogeneous subbundles, corresponding to the linear subspaces $\mathfrak{e}^1; \mathfrak{e}^2; \mathfrak{e}^3$ of $\mathfrak{g}$. Further, the Main Lemma remains true if the tensor $\mathfrak{b}$ is defined by a linear combination of brackets (even if every single bracket, which occurs in such an expression, does not yield a well-defined tensor).
In the next section we apply the formula stated in the Main Lemma to locally homogeneous CR-manifolds for the computation of their Levi forms and certain higher order analogues. This will enable us to give a simple characterization of the (non)degeneracy type for locally homogeneous CR-manifolds.

3. CR-manifolds and nondegeneracy conditions

In this section we briefly recall some basic facts concerning CR-manifolds and certain geometric properties of them. In particular, we closely examine the condition of being finitely nondegenerate, which plays a major role in the next sections. As a general reference for CR-manifolds, see [4] and [7].

Definition 3.1 An abstract CR-manifold is a smooth manifold $M$ together with a subbundle $H \subset TM$ (we call it the complex subbundle) and a vector bundle endomorphism $J : H \to H$ with $J^2 = \mathbb{I}$ (the so-called partial almost complex structure) such that for all $X \in \mathfrak{X}(M)$ it follows $[X; J] = [J; X]$.

If, in addition, the Nijenhuis tensor $N(J)(X;X) = [J; J]X$ vanishes, we call $(M; H, J)$ formally integrable.

In this paper we almost exclusively investigate manifolds which are locally homogeneous under some Lie group. Every smooth manifold furnished with a smooth locally transitive action of a finite dimensional Lie group automatically carries a real-analytic structure, compatible with the group action. We assume from now on (if the contrary is not explicitly stated) that all manifolds, actions and subbundles are real-analytic and the CR-manifolds are formally integrable. However, the sections in such subbundles may be only smooth.

Two "extreme" classes of CR-manifolds are the following: Complex manifolds $\mathbb{Z}$ are precisely those formally integrable CR-manifolds with maximal possible complex subbundle: $H = T\mathbb{Z}$: Here, $J : T\mathbb{Z} \to T\mathbb{Z}$ is the complex structure, induced by the multiplication with $i = \sqrt{-1}$ in local coordinate charts. On the other hand, every real manifold, furnished with the trivial CR-structure $H = 0$ is CR and called totally real as a CR-manifold.

From the local point of view complex manifolds as well as real manifolds with $H = 0$ are not very interesting. Hence, apart from few exceptions, the CR-manifolds considered in this paper do not belong to any of the above two classes. A wide class of CR-manifolds consists of real submanifolds $M$ of complex manifolds $(\mathbb{Z}; J)$ such that $H_x := T_xM \setminus JT_xM$ and $\dim H_x$ is a constant function of $x$. Such a CR-manifold is formally integrable (since $(\mathbb{Z}; J)$ has this property). On the other hand, due to the well-known embedding theorem of Andreotti-Fredricks (23), every formally integrable real-analytic CR-manifold admits a generic CR-embedding into a complex manifold $\mathbb{Z}$: Hence, without loss of generality we assume in the following that all CR-manifolds under consideration are (locally) closed submanifolds $M \subset \mathbb{Z}$ and fulfill the above conditions together with $T\mathbb{Z} \neq H = TM + JT \mathbb{M}$ (genericity).

Infinitesimal CR-transformations. Let $\mathfrak{M} = (\mathfrak{M}; H, J)$ be a real-analytic CR-manifold. There is a particular Lie subalgebra of $(\mathfrak{M}; TM)$, related to the CR-structure: Call a vector field $\mathfrak{X} \subset (\mathfrak{M}; TM)$ an infinitesimal CR-transformation if the corresponding local 1-parameter subgroup $e^t \mathfrak{X}$ acts by local CR-transformations of $\mathfrak{M}$. Write $\mathfrak{M}(\mathfrak{O})$ for
the germ at \( \partial Z \) of \( M \); Define \( \text{holol} M \) \( \,^1(\mathfrak{g} ; TM) \) (resp. \( \text{holol} M ; \partial) \); if dealing with germs) as the subspace consisting of (germs of) infinitesimals CR-transformations of \( M \) (or \( \mathfrak{g} M ; \partial) \), respectively; the elements in \( \text{holol} M ; \partial \) not necessarily vanish at \( \partial \). The spaces \( \text{holol} M \) and \( \text{holol} M ; \partial \) are Lie algebras, with Lie structure induced by the usual Lie brackets of vector fields. In the above definition we do not require that the infinitesimal CR-transformation on an embedded CR-manifold, \( M \neq Z \); are restriction of holomorphic vector fields on \( Z \); However, due to Proposition 12.4.22 in \([4]\), this follows automatically. Finally, by a holomorphic vector field on a complex manifold \( Z \) we mean a holomorphic section in the real tangent bundle \( T Z \). Given a manifold \( M \) with some structure \( C \); we write \( \text{Aut}_C \mathfrak{g} M \); or simply \( \text{Aut} M \) for the group of all automorphisms of \( M \) preserving this structure and \( \text{Aut} M \) for the corresponding Lie algebra.

The notion of \( k \)-nondegeneracy. A basic invariant of a CR-manifold is its vector-valued Levi form \( L M \), or equivalently with respect to the encoded information, the canonical alternating 2-form \( ! M : H \times H ! TM =_H \); This 2-form is simply the tensor induced by Lie brackets (as in \([2]\)). The (classical) Levi form\(^2 \) \( L M \); which is a \( J \)-invariant sesquilinear tensor

\[
L M : H \times H ! TM =_H \mathfrak{g} ;
\]

and \( ! M \) are related: \( L M (u;v) = ! M (u;v) + i! M (J u;v) \); A complexified version of the Levi form is the tensor \( \mathfrak{g} M : H^{01} ; H^{10} ! TM =_H \mathfrak{g} \) induced by Lie brackets of local sections in \( H^{01} \) and \( H^{10} ; \)

Set

\[
F^{01}_0 = H^{01} ; F^{01}_1 = f \quad 2 H^{01} : L^1 (\, , H^{10}) = 0 \mathfrak{g} ;
\]

A CR-manifold is called Levi-nondegenerate or 1–nondegenerate at \( \times 2 M \) if the fibre of \( F^{01}_1 \) at \( \times \) is zero.

The notion of \( k \)-nondegeneracy of \( M \) at a point \( \times \) has been originally defined in \([5]\) (see also Sec. 11.1 in \([4]\)) for arbitrary CR-manifolds. In general, the order \( k \) of nondegeneracy at \( \times 2 M \) varies from point to point and can be arbitrarily high. For the class of CR-manifolds of “uniform degeneracy” (i.e., the dimensions of all fibre-wise defined subspaces \( (F^{01}_0) / x \times T^m M \), as constructed below, do not depend on \( \times 2 M \) and form well-defined subbundles of \( T^m M \)) which includes all locally homogeneous CR-manifolds, \( k \)-nondegeneracy can be expressed as the nondegeneracy of certain tensors \( L^{k+1} \); The latter tensors can be considered as a generalization of the Levi form \( L^1 \); This has already been explained in the Appendix of \([12]\). For convenience, we recall this construction in a form suitable for our purposes.

Define recursively the subbundles

\[
(3.2) \quad F^{01}_0 = f \quad 2 F^{01}_1 : L^k (\, , H^{10}) = 0 \mathfrak{g} ;
\]

and the following maps, induced by Lie brackets:

\[
(3.3) \quad L^{k+1} : F^{01}_0 \times H^{10} ! F^{01}_1 \times H^{10} \quad F^{01}_0 \times H^{10} \quad H \times H^{01} \quad H^{01} \times H^{10} ;
\]

The fact that all \( L^{k+1} \)'s are well-defined tensors follows from the formula \( \mathfrak{d} (\, , \, ) = (\mathfrak{d} \, , \, \, ) \), where \( \mathfrak{d} \) and \( \mathfrak{d} \) are local sections in \( F^{01}_0 \) and \( H^{10} \), respectively, and the \( \mathfrak{d} \)’s run

\(^2\text{We took the definition from \([11]\). It differs from the Levi form considered by some other authors, see for instance \([4]\), by the factor } i=2\)
through all 1-forms \( \Theta^1 \) which vanish on \( H^{0;1}_{\mathbb{R}} \). By construction, for each CR-manifold of uniform degeneracy there is the following filtration of \( H^{0;1} \) by complex subbundles: 
\[
H^{0;1} = F^{0;1}_{(0)} \rightarrow F^{0;1}_{(1)} \rightarrow F^{0;1}_{(2)} \rightarrow \cdots
\]
: The property of being \( k \)-nondegenerate is characterized in the following

**Proposition 3.4** Let \( F^{0;1}_{(j)} \), \( j = 0;1;2;\cdots \) be the subbundles as defined in 3.2. A CR-manifold \( M \) of uniform degeneracy is \( k \)-nondegenerate if and only if \( F^{0;1}_{(k-1)} \not\cong F^{0;1}_{(k)} = 0 \):

For locally homogeneous CR-manifolds the subbundles and tensors, as defined in 3.2 and 3.3, respectively, can be characterized in Lie algebraic terms. In particular, the geometric notion of \( k \)-nondegeneracy can be completely described in terms of a filtration of certain subalgebras, as will be shown in Section 5.

### 4. Homogeneous CR-germs and CR-algebras

In this section we show that each germ \( M ; o \) of a locally homogeneous real-analytic CR-manifold (homogeneous CR-germ, for short) can be described by an algebraic datum, for instance by a CR-algebra. Vice versa, every CR-algebra gives rise to a homogeneous CR-germ and all these assignments are functorial. We start by recalling the definition of the category of CR-algebras, essentially following [13].

**The category of CR-algebras.** To fix notation, let \( g \) stand for a real Lie algebra, let \( g^f = g \otimes \mathbb{C} \) be its complexification and \( f \) the complexification of a real homomorphism \( g \rightarrow \mathbb{C}^2 \). As before, we write \( l \) for the complexification \( g^f \) and \( 1 \) for the unique complex conjugation \( \mathbb{C}^2 \). Fixing the real form \( g \), \( l \):

A pair, consisting of a finite-dimensional real Lie algebra \( g \) and a complex subalgebra \( q \) of \( l = g^f \), is called a CR-algebra. In contrast to [13], here we require the finite dimensionality of \( g \). A morphism \( (g; q) \rightarrow (g'; q') \) is a Lie algebra homomorphism \( g \rightarrow g' \) with \( f : g \rightarrow g' \). We refer to the category in which the objects are CR-algebras and the morphisms are as just described as the category of CR-algebras, or, for short, \( A_{CR} \).

On the geometric side there is the category of **homogeneous CR-germs.** The objects in this category are homogeneous CR-germs \( (M; o) \) and the morphisms \[ \] are as defined in the subsection “locally homogeneous manifolds and bundles” of section 2. Note that automatically is a CR-map. We refer to this category as to the category of homogeneous CR-germs (and write \( CR_{ho} \), for short).

Discarding for a moment local actions, there is also the category \( CR_o \), consisting of germs of real-analytic CR-manifolds as objects and real-analytic (germs of) base point preserving CR-maps \( \Phi M ; o \) as morphisms. We have then the obvious forgetful functor \( CR_{ho} \rightarrow CR_o \). Note, however, that the notion of an isomorphism is different in these two categories: Two homogeneous CR-germs \( (M; o) \) and \( (M'; o') \) may be non-isomorphic in \( CR_{ho} \), though the underlying CR-germs are CR-equivalent, i.e., isomorphic in \( CR_o \). To distinguish these two notions of an isomorphism, we refer to \( (M; o) \) and \( (M'; o') \) as isomorphic if there is an isomorphism between them in \( CR_{ho} \) and as CR-equivalent if \( (M; o) \) and \( (M'; o') \) are isomorphic in \( CR_o \). This fine point plays a role in [8], where 5-dimensional 2-nondegenerate
homogeneous CR-germs are classified up to CR-equivalence, and this classification is reduced to the classification of $\mathfrak{g}$-homogeneous CR-germs with $\dim \mathfrak{g}$ as small as possible.

**Functors.** There is a functor $\mathcal{G}$ from the category of CR-algebras to the category of homogeneous CR-germs (this has also been remarked in [13]). Given a CR-algebra $(\mathfrak{g};\mathfrak{q})$ set $\mathcal{G}! : (\mathfrak{g};\mathfrak{q}) \to \mathfrak{g}$. Let $(\mathfrak{g};\mathfrak{q})$ be the germ of a complex homogeneous manifold with the infinitesimal model $\mathfrak{q}$ and $\mathfrak{q}$ a locally homogeneous representative. The CR-germ $(\mathfrak{g};\mathfrak{q})$ is then determined as the germ at $\mathfrak{o}$ of the real-analytic Nagano leaf $\mathfrak{M}$ through $\mathfrak{o}$ in $\mathfrak{q}$ with respect to $\mathfrak{g}$. Hence, the restriction of $\mathfrak{M}$ to $\mathfrak{M}$ is a real-analytic CR-map and yields a morphism between the homogeneous CR-germs $(\mathfrak{g};\mathfrak{q})$ and $(\mathfrak{g};\mathfrak{q})$:

There exists also a functor $\mathcal{K}$ in the opposite direction. Let a $\mathfrak{g}$-homogeneous CR-germ $(\mathfrak{g};\mathfrak{q})$ be given. Due to [2], there exists a complex manifold $\mathfrak{M}$ such that a representative $\mathfrak{M}$ is generically CR-embedded in $\mathfrak{M}$. Hence, possibly after shrinking $\mathfrak{M}$, we may assume that for each $\mathfrak{v} \in \mathfrak{g}$ the vector field $\mathfrak{v}$ is the restriction of a holomorphic vector field on $\mathfrak{M}$. Therefore, we can consider $\mathfrak{g}$ as a Lie algebra homomorphism $\mathfrak{g} : (\mathfrak{g};\mathfrak{q}) \to (\mathfrak{g};\mathfrak{q})$. Since the Lie algebra $(\mathfrak{g};\mathfrak{q})$ is complex, extends to a complex homomorphism $\mathfrak{f} : (\mathfrak{g};\mathfrak{q}) \to (\mathfrak{g};\mathfrak{q})$. Define the complex isotropy subalgebra $\mathfrak{a} = \mathfrak{f} \mathfrak{w} \in \mathfrak{g}$: The pair $((\mathfrak{g};\mathfrak{q}) = (\mathfrak{g};\mathfrak{q}))$ is a CR-algebra and we call it the CR-algebra associated with $(\mathfrak{g};\mathfrak{q})$. Define $\mathfrak{q}_{\mathfrak{a}} = \mathfrak{q} \setminus \mathfrak{a}$; Observe that $\mathfrak{g} = \mathfrak{q}_{\mathfrak{a}}$ is the infinitesimal model for $(\mathfrak{g};\mathfrak{q})$ and $\mathfrak{q}_{\mathfrak{a}}$ the infinitesimal model for $(\mathfrak{g};\mathfrak{q})$. A word of caution: Even if $\mathfrak{g} : (\mathfrak{g};\mathfrak{q})$ is injective, i.e., the original $\mathfrak{g}$-action is effective, the complexification $\mathfrak{q}$ may not be injective, i.e., the sum $\mathfrak{q} + \mathfrak{J}$ may not be direct.

It follows that an equivariant morphism $(\mathfrak{g};\mathfrak{q}) : (\mathfrak{g};\mathfrak{q})$ induces a morphism of the associated CR-algebras: The only point which has to be checked is that the complexification of $(\mathfrak{g};\mathfrak{q}) : (\mathfrak{g};\mathfrak{q})$ maps $\mathfrak{q}$ to $\mathfrak{q}_{\mathfrak{a}}$. Again by the extension results from [2], a representative $\mathfrak{J} : \mathfrak{M} \to \mathfrak{M}$ extends to a holomorphic map $\mathfrak{b} : \mathfrak{J} : \mathfrak{M} \to \mathfrak{M}$: By the identity principle, $\mathfrak{b}$ is equivariant with respect to $\mathfrak{a}$ and $\mathfrak{b}$. Since $\mathfrak{b}$ preserves the base points, the inclusion $\mathfrak{q}$ follows from $\mathfrak{b}$ to $\mathfrak{q}_{\mathfrak{a}}$: Summarizing, we have

**Proposition 4.1** The above defined covariant functors $\mathcal{A}_{CR} : \mathcal{G}$ and $\mathcal{R}_{ho} : \mathcal{G}$ are mutually quasi-inverse and yield an equivalence of the two categories.

**Remark.** There exist (locally) homogeneous manifolds with non-integrable CR-structures. A germ of such a more general CR-manifold can also be described by purely algebraic data, for instance by a quadruple $(\mathfrak{g}_0;\mathfrak{g};\mathfrak{H};\mathfrak{J})$ consisting of the Lie algebras $\mathfrak{g}_0$; $\mathfrak{g}$; an ad $(\mathfrak{g}_0)$-stable subspace $\mathfrak{H}$ of $\mathfrak{g}$ and an endomorphism $\mathfrak{J} : \mathfrak{H} \rightarrow \mathfrak{g}_0$, such that $\mathfrak{J}$ is ad $(\mathfrak{g}_0)$-equivariant.
\[ J^2 = \mathbb{I} \] and \( \{ P; J \} 2 H \) holds for all \( v; w \in H \) and some linear lift \( \mathcal{P} : H \to H \) of \( J \) with \( \mathcal{P}(g_o) = g_o \). However, such quadruples \( (g_o; g; H; J) \) seem to be less convenient to deal with than CR-algebras.

5. Geometric properties of a germ, given by a CR-algebra

As seen in the previous section, the germ at \( \mathfrak{o} \) of a locally homogeneous CR-manifold \( M \) is completely determined by the corresponding CR-algebra. Consequently, all objects naturally attached to \( M \) and their geometric properties are (at least a priori) completely determined by \( (g; \mathcal{P}) \): In this section we show in an explicit way how the geometric information encoded in a CR-algebra can be extracted. In particular, we give a description of the subbundles \( H \otimes H_1 \otimes H_1^\ast \otimes H \otimes H_1 \otimes H \) of \( T^4M \) in terms of quotients of Lie algebras. The main results of this section are a description of the \( k \)-nondegeneracy and the holomorphic nondegeneracy of a CR-germ \((M; \mathfrak{o})\) as a purely algebraic property of its CR-algebra (Theorem 5.10; see also the following remarks), and Theorem 5.11 in which the minimality of \( M \) is characterized in a similar fashion. As an application we give a simple proof of the following result: each non-extreme \( G \)-orbit in \( Z = \mathbb{C} \otimes \mathbb{Q} \); where \( Z \) is an arbitrary flag manifold with \( \mathfrak{b}_2(\mathbb{Z}) = 1 \); \( L \) a complex subgroup of \( \mathfrak{u}(\mathbb{Z}) \) and \( G \) \( L \) an arbitrary real form, is minimal and holomorphically nondegenerate. This generalizes a theorem of Kaup and Zaitsev, see [12].

Let \( (g; \mathcal{P}) \) be a given CR-algebra. Recall that \( I = g_\mathcal{P}; g_o = g \setminus q \) and \( : ! : \) is the involutive automorphism with \( I = v \otimes 2 I \). \( v = v_\mathcal{P} = g \). Let \((M; \mathfrak{o})\) be the corresponding homogeneous CR-germ which is CR-embedded in \((\mathcal{E}; \mathfrak{o})\); as explained in section 4. Since the vector bundles \( T^M ; H ; H^1 \otimes H^1 \otimes H^1 \otimes H \); etc. are locally homogeneous with respect to the given transitive local actions on \( M \) and \( Z \), they are determined by a single fibre, say at \( \mathfrak{o} \in M \).

As these various fibres are subspaces of the corresponding (complexifications of) tangent spaces \( T^eM = g_\mathcal{P} \setminus g \); \( T^eZ = g_\mathcal{P} \setminus g \); \( T^eM = g_\mathcal{P} \setminus g \); etc., we need to specify the appropriate subspaces of the preceeding quotients of Lie algebras. We proceed with preparatory observations.

The real isotropy Lie algebra \( g \) is a real form of \( q \setminus q \) (this was already observed in [17]). Hence, the complexified tangent space \( \mathbb{T}^4M \setminus q \) is the quotient \( \mathbb{T}^4M \setminus q \):

Define the subspace \( H := (q \setminus q) = (q \setminus g) \setminus g \) of \( g \): Note that \( g_o : H \) and observe that the map \( q \setminus H; : w \setminus w \) is surjective. The quotient \( H = g_o \) coincides with the intersection \( g_\mathcal{P} \setminus i g \). This follows from the equation \( H = f \otimes x \otimes g = h \otimes y \) for some \( y \in g \).

The invariant complex structure \( J : H \to H \) induced by the embedding \( M \to Z \); i.e., the endomorphism \( J_o : H = g_o \to H = g_o \); can be described as follows: Recall that given any \( \mathfrak{u} \in H \) there exists a \( w \in q \) with \( \mathfrak{u} = w + \mathfrak{u} \): Further, since \( 1 = q + i q \); each element in \( 1 \) has the unique decomposition into its real and imaginary parts. Then:

\[
J_o : H = g_o \to H = g_o; \quad (\mathfrak{u} + g_o) = 2y + g_o;
\]

where \( \mathfrak{u} = w + \mathfrak{u} \) mod \( g_o \); \( w \in q \) and \( x + iy \) is the decomposition of \( w \) into its real and imaginary parts.

We summarize the above results, i.e., the identifications of the various fibres at \( \mathfrak{o} \) with the
corresponding quotients of Lie algebras in the following diagram:

\[
\begin{array}{c}
H = q_0 \\ \uparrow \\ q_0 = T_0 M, ! \quad T_0 Z = l q \\
\uparrow \\ \uparrow \\
q_1, ! \quad T_1 Z = f q_1 q \\
\uparrow \\
q_2, ! \quad T_1 Z = q_1 q_2 = l q
\end{array}
\]

(5.2)

**Finite nondegeneracy in terms of CR-algebras.** In the next paragraphs we repeatedly apply the Main Lemma 2.3 to the various tensors associated with a locally homogeneous CR-manifold \( M \) as described in Section 3. We obtain in that way expressions for all \( L_k \)'s in terms of Lie brackets in the Lie algebra \( \mathfrak{g} \). Here, \( 1 = q^f \) comes from the CR-algebra, associated to a given \( g \)-homogeneous CR-germ \( M ; 0 \): Keeping in mind the identifications 5.2, the Main Lemma 2.3 immediately implies

\[
(5.3) \quad !^M : H = q_0, \quad H = q_0, \quad \mathfrak{g} = H; \quad (u; v) \not\equiv (u; v; 0) \mod H.
\]

As already mentioned, the complexification of \( !^M \), restricted to \( H_{0,0} \), \( H_{1,0} \); i.e., the invariant tensor \( L^0 \), is equal to the Levi form up to some factor. Also in this case the Main Lemma together with the identifications 5.2 implies the following formula for \( L^1 \) at \( 0 \):

\[
(5.4) \quad L^1 : q = q^f, q = q, q = q \mod (q^f + q); (u; v) \not\equiv (u; v; 0) \mod (q^f + q).
\]

For short, write \( q^{(0)} = q \); \( q^{(1)} = q \); \( q^{(0)} = q \); \( q^{(1)} = q \); \( q^{(0)} = q \); \( q^{(1)} = q \); where

\[
(5.5) \quad q^{(1)} = f w 2 q : [w; q] + q g q
\]

coincides with the normalizer \( N_q (q^f + q) \) and consequently is a complex subalgebra. Similarly, the recursively defined (3.3) tensors \( L_k \) (which are invariant under the local action) are given by the formulae:

\[
(5.6) \quad L^{k+1} : q^{(k)} = q^{(k)}; q^{(0)} = q^{(0)} + q^{(k)} q^{(0)} q^{(k)} q^{(0)} + q^{(k)} q^{(0)} \mod q^{(k)} q^{(0)} = (u; v; 0) \mod q^{(k)} q^{(0)}
\]

Here and above, the right-hand sides does not depend on the choice of the representatives \( u \) and \( v \): The (left) kernels of \( L_{k+1} \) are the homogeneous subbundles \( F_{k+1} \) (see 3.2); hence, they are determined by the corresponding fibres at \( 0 \): A glance at (5.6) suggests the following definition:

\[
(5.7) \quad q^{(k+1)} = f w 2 q^{(k)} : [w; q^{(0)}] + q^{(k)} + q^{(0)} q
\]

**Observation 5.8** The fibre of \( F_{k+1} \) at \( 0 \) is isomorphic to the quotient \( q^{(k)} q^{(0)} = q^{(k)} q^{(0)} \):
Next, we prove the auxiliary

**Lemma 5.9** Let \((g;\mathfrak{g})\) be a CR-algebra, \(H = (q + \mathfrak{g})\) and let \(q^{(k)}\) be the subspaces of \(q\); defined in \([S.7]\). Then

(i) The real subspace \(F = N_q(H) \setminus H\) is a subalgebra and \((q^{(1)} + \mathfrak{g}^{(1)}) = F\);

(ii) All subspaces occurring in the filtration \(q = q^{(0)} \supseteq q^{(1)} \supseteq q^{(2)} \supseteq \ldots \) are complex subalgebras of \(q\).

**Proof.** \(\text{ad (i): To show that } F\) is a subalgebra, it suffices to show that for \(u;v \in F\); \([u;v]_F\) belongs to \(H\); This follows from \([F;F] \subset [F;H]\); \(H\). For the proof of the second identity note that the inclusion \(q^{(1)} + \mathfrak{g}^{(1)} \ni N_1(q^+ q) \setminus (q^+ q) = \mathfrak{g}^{(1)}\) is obvious. Let now \(u + w N_1(q^+ q) \setminus (q^+ q)\) be an arbitrary element with \(u;v \in F\); If \([u;v]_F \in \mathfrak{g}\); \(i.e.,\) if there were \(a \in \mathfrak{g}\) with \([u; a] \notin \mathfrak{g}\); \(\) then also \([u + w; a] \notin \mathfrak{g}\); \(\) contrary to the construction of \(u + w\); It follows \([u;w]_F \in F^{(1)}\); \(\) ad (ii): Clearly, \(q = q^{(0)}\) and \(q^{(1)}\) are subalgebras. Assume that we have already proven that \(q^{(j)}\) are subalgebras for all \(j < k\); To conclude that \(q^{(k)}\) \(\) is also a subalgebra, note that for \(u;v \in F\); \(q^{(k)}\) we have

\[
\[[u;v]_F + [v;w] = [u;v;w]_F + [v;[u;w]_F + \mathfrak{g}^{(k+1)} + \mathfrak{g}^{(1)} + q^{(k+1)} + q^{(1)} + q
\]

and the proof is complete. \(\square\)

We are now in the position to characterize holomorphic (non)degeneracy in terms of a purely algebraic condition on CR-algebras. As already mentioned, a homogeneous CR-germ \((M;\mathfrak{g})\) is holomorphically nondegenerate if and only it is \(k\)-nondegenerate for some finite \(k\); This follows from Theorem 11.5.1 in \([4]\); applied to the homogeneous case.

**Theorem 5.10** Let \((g;\mathfrak{g})\) be a given CR-algebra and \((M;\mathfrak{g})\) the corresponding homogeneous CR-germ, generically embedded into the germ \(((Z;\mathfrak{g}));\). Let \(q^{(1)}\) be the filtration by subalgebras as in \([S.9]\) ii. Then, for every integer \(k \geq 1\)

(i) \((M;\mathfrak{g})\) is \(k\)-nondegenerate if and only if \(q^{(k+1)} \supseteq q^{(k)}\); \(\) and then \(k = \dim q^{(0)}\);

(ii) \((M;\mathfrak{g})\) is holomorphically degenerate if and only if there exists a complex subalgebra \(r_+ = \mathfrak{g}\) with \(q (r_+ q)\); The latter condition implies the existence of a locally equivariant CR-morphism \(M ! M^{(0)}\); whose fibres are positive-dimensional complex submanifolds of \(Z\); Here, \((M;\mathfrak{g}^{(0)})\) is the CR-germ associated with \((g;\mathfrak{g})\).

**Proof.** The first part is an immediate consequence of Proposition \([3.4]\) and Observation \([S.8]\). For the proof of the second part of the theorem recall that the holomorphic degeneracy of \((M;\mathfrak{g})\) is equivalent to the fact that \((M;\mathfrak{g})\) is not finitely nondegenerate. Thus, according to (i) and Lemma \([S.9]\) ii, there exists \(n \in \mathbb{N}\) such that \(q^{(n)} = q^{(n+1)} \supseteq q^{(1)}\); This implies \((q^{(n)}; q)\) \(\mathfrak{g}^{(n)}\) \(\); \(q\); Since \(q^{(n)}\) is a subalgebra by Lemma \([S.9]\); \(q^{(n)}\) \(\) is a subalgebra, as well. Define \(r = q^{(n)} + q = (q^{(n)} + q)\) and note that \(r\) \(\) and \(r = q^{(n)} + q\); This proves the existence of \(r\) as claimed. Let \((M;\mathfrak{g}^{(0)}); (Z;\mathfrak{g}^{(0)})\) be the CR-germ, associated with the CR algebra \((g;\mathfrak{g});\) Since \(\mathfrak{g} = \mathfrak{g};\)
the identity map on \( 1 = \mathcal{g}^{\mathfrak{f}} \) induces a morphism \( (g;\mathcal{g}) \rightarrow (g;\mathfrak{z}) \); and by Proposition 4.1, a CR-morphism \( \mathcal{M} ;(\circ) \rightarrow \mathcal{M} ;(\circ) \) which is the restriction of a holomorphic surjective morphism \( \mathcal{B} : (\mathcal{Z};(\circ)) \rightarrow (\mathcal{Z};(\circ)) \): We claim that the germ of the fibre \( H^{k}(\circ) \) coincides with the germ of the fibre \( \mathcal{B} : (\circ) \) : This can be seen by comparing the dimensions: a simple check shows that the injection \( T_{\circ} \mathfrak{q}^{1}(\circ) = \mathcal{g} \setminus \mathfrak{q} \setminus \mathcal{g} ! \mathfrak{q} = T_{\circ} (\mathcal{B} : T_{\circ} (\circ)) \) is also surjective.

Remarks.

In [13], certain purely algebraic nondegeneracy conditions of CR-algebras have been introduced. However, their geometric interpretation, in particular the characterization of holomorphic (non)degeneracy as given in the remark following Prop. 13.3, compare also Theorem 3.2 in [11], contradicts our Theorem 5.10.

A not necessarily homogeneous, holomorphically degenerate CR-manifold \( M \) is, at generic points in sense of [6], locally CR-equivalent to a product of a lower-dimensional CR-manifold and a complex manifold. This is a consequence of Proposition 3.1. in [6].

Minimality in terms of CR-algebras. Recall that a CR-manifold \( \mathcal{M} ;H ;J \) is called minimal at \( (\circ)M \) if for each locally closed submanifold \( Y \subseteq M \) such that \( (\circ)2Y \) and \( H_{\circ}Y \) for \( y \) \( Y \) the identity \( T_{\circ}Y = T_{\circ}M \) holds, i.e., \( \mathcal{M} ;(\circ) = (Y;\circ) \): In the locally homogeneous situation the property of being minimal at one particular point is equivalent to the minimality at all points of \( M \): As before, \( H = (\mathfrak{g} + \mathfrak{q}) \) \( \mathfrak{g} \) and \( H_{\circ} = H_{\circ} \mathfrak{g} \).

Theorem 5.11. Given a CR-algebra \( (g;\mathcal{g}) \); let \( \mathcal{M} ;(\circ) \) be the underlying CR-germ. Then \( M \) is minimal at \( \circ \) if and only if the smallest subalgebra of \( \mathfrak{g} \) which contains \( H; \circ \); is \( \mathfrak{g} \) itself.

Proof. The minimality condition can be reformulated as follows: Define inductively the following ascending chain of subbundles (associated with the locally homogeneous CR-manifold \( M \)):

\[
H^{(0)} = H; \quad H^{(j)} = H^{(j-1)} + \left[ H^{(j-1)} ; H^{(j-1)} \right] \quad \text{for } j > 0;
\]

Here, \( [H^{(j)} ; H^{(j)}] \) stands for the subbundle generated by all brackets \( [ ; ] \); where \( ; \) run through local sections in \( H^{(j)} \): The minimality of \( M \) is equivalent to the condition \( H_{\circ}^{(k)} = \mathcal{T}_{\mathcal{M}} \) : In our situation all subbundles \( H^{(k)} \) are homogeneous; hence, they are completely determined by the fibres at \( (\circ)M \). Let \( H^{(k)} \) \( \mathfrak{g} \) denote the subspaces containing \( \mathfrak{g}_{\circ} \) such that \( H_{\circ}^{(k)} = H^{(k)} ; \mathfrak{g}_{\circ} \) for all \( k \). Observe that the map \( \mathcal{M} ;(H^{(k)}) \rightarrow \mathcal{M} ;(H^{(k)}) \rightarrow \mathcal{M} ;(H^{(k+1)}) = \mathcal{M} ;(H^{(k+1)}) \); given by the Lie brackets is \( C^{1} \) - bilinear. Consequently, we can employ the Main Lemma P.1. The corresponding tensor \( H_{\circ}^{(k)} \rightarrow H^{(k)} \rightarrow H^{(k+1)} \) is simply given by the Lie bracket in \( \mathfrak{g} \); This yields an inductive definition of all \( H^{(k)} \). The subspace \( H^{(k+1)} \) is generated by elements \( u \) \( 2H^{(k)} \) and all Lie brackets \( u ; v \); \( u ; v \) \( 2H^{(k)} \): If the smallest Lie algebra in \( \mathfrak{g} \) which contains \( H^{(0)} \); coincides with \( \mathfrak{g} \) then \( H_{\circ}^{0} H^{(k)} = \mathfrak{g} \); and consequently \( H_{\circ}^{0} H^{(k)} = \mathcal{T}_{\mathcal{M}} ; \circ \); i.e., \( M \) is minimal. The opposite direction, i.e., \( M \) minimal implies \( \mathfrak{g} \) is the smallest subalgebra containing \( H \) is easier to see: The existence of a proper subalgebra of \( \mathfrak{g} \) which contains \( H \); would imply the existence of an integral manifold (Nagano leaf) through \( \circ \); strictly lower-dimensional than \( M \): But this contradicts the minimality of \( M \).
Orbits in flag manifolds. In this subsection let $Z$ stand for a flag manifold, i.e., a projective homogeneous manifold with $b_1(Z) = 0$. Let $L = \text{Aut}_D(Z)$ be a complex subgroup which acts transitively on $Z$; i.e., $Z = L=Q$; since in such a case $L$ is semisimple and the isotropy subgroup $Q$ is parabolic. Select a real form $G$ of $L$; then the $G$-orbits in $Z$ provide a broad class of examples of CR-manifolds. For instance, all bounded symmetric domains $D = Q^{\text{SL}}$ and the pieces of the natural stratification of their boundaries arise as certain orbits of the above type. In [12], global actions of so-called real forms of tube type have been considered in the particular case where $Z$ is a Hermitian compact symmetric space. Recall that a real form $G$ of a complex semisimple Lie group $L$ is called of tube type if $G$ has an open orbit in $Z$; which is biholomorphically equivalent to a bounded symmetric domain of tube type. It has been proven (Theorem 4.7 in [12]) with Jordan algebraic tools that for an $L=Q$ of tube type each $G$–orbit $M$ in an irreducible Hermitian space $Z = L=Q$; which in neither open nor totally real is 2-nondegenerate and minimal. As shown in [17] in each flag manifold $Z$ there is precisely one closed $G$–orbit $Y$; Further, $\dim_R Y = \dim_Q Z$; and the closed orbit is totally real if and only if $\dim_R Y = \dim_Q Z$; as is the case for $G$ of tube type and $Z$ the corresponding Hermitian space.

Natural generalizations of irreducible compact Hermitian symmetric spaces are the flag manifolds $Z = L=Q$ with second Betti number $b_2(Z)$ equal to 1, or equivalently, where $Q$ is a maximal parabolic subgroup. In this situation Theorem 4.7 from [12] can be generalized as follows:

**Theorem 5.12** Let $L$ be a complex simple Lie group, $G = L$ an arbitrary real form and $Q = L$ a parabolic subgroup.

(i) Assume that $Q$ is maximal parabolic. Then every non-open $G$–orbit $M$ in $Z = L=Q$ is holomorphically nondegenerate. All such orbits are also minimal, except for the totally real ones.

(ii) In particular, if $Z = L=Q$ is an irreducible Hermitian space with $L = \text{Aut}_D(Z)$ and $G = L$ an arbitrary real form then every $G$–orbit $M$ which is not open is $k$–nondegenerate with $k \geq 2$: For every such orbit $M$; which in addition is not totally real, $\Phi_M (Z)$ belong to the class $C$ in the sense of 4.4, in [12].

(iii) If $Q$ is not maximal, then there always exist non-open holomorphically degenerate $G$–orbits in $Z$.

**Proof.** Let $\gamma : l \to l$ be the involution given by the real form $G = L$: Let $q_L$ be the isotropy Lie algebra at a point $z \in Z = G=Q$; $M = G$ the orbit with the inherited CR-structure such that neither $q_L + q_L = q$ (i.e., $M$ is not open) nor $q_L + q_L = q$ (i.e., $M$ is not totally real; here we follow the notational convention from [17] and denote the complex isotropy at $z \in Z$ by $q_L$ rather than $\gamma_L$). Since the only Lie algebra, properly containing $q_L$ (and in particular $q_L + q_L$), is $l$ itself, Theorem 5.10 together with Theorem 5.11 imply the first part of the claim. The estimate for the order of nondegeneracy $k$ in the Hermitian case $Z = L=Q$ follows from Theorem 6.3 together with the following well-known technical fact that $c(q) = 1$ ([16], see our notation in the paragraph preceding Theorem 6.3). As the example $F_{2n-1} = \text{Sp}_{2n}(\mathbb{C}) = G = \text{Sp}_{2n}(\mathbb{R}) = G$ shows, complex Lie groups of different dimensions may act transitively on a given flag manifold. If $Q$ is not maximal, there exists a maximal parabolic $Q^0$; containing $Q$; such that $G$ is not transitive on $L=Q^0$: Further, there is the $L$–equivariant holomorphic map $\gamma : L=Q^0 ! L=Q^0 : Z^0$ with positive-dimensional complex fibres. Let $M^0$ be an arbitrary $G$–orbit which is
not open. Then \( M^0 \) consists of finitely many \( G \)-orbits. In particular there exists an orbit \( M \) which is open in \( M^1 \): The fibres of the restriction \( \mathcal{I} : M^0 \rightarrow L^0 \) (which is a CR-map) are then complex manifolds and consequently \( M \) is locally equivalent to a product of a CR-manifold and a positive dimensional complex manifold. This implies that \( M \) is holomorphically degenerate.

6. A 3-nondegenerate homogeneous CR-manifold

The purpose of this section is to give an explicit example of a homogeneous 3-nondegenerate CR-manifold. Recall that all CR-manifolds which occur in [12] are either holomorphically degenerate or 2-nondegenerate. In the Hermitian symmetric spaces, complementary to those considered in [12] there are also 1-nondegenerate CR-manifolds. Up to our knowledge, there are no known examples of homogeneous 4-nondegenerate CR-manifolds with \( k = 3 \):

Examples, promising in search of homogeneous CR-manifolds with higher nondegeneracy, arise as orbits of real forms in flag manifolds. Note however that the Jordan-algebraic methods used in [12] in the particular case, where \( Z \) is a compact Hermitian symmetric space, cannot be generalized to arbitrary flag manifolds. Nevertheless, this (bigger) class of orbits of real forms with induced CR-structures coming from general flags \( \mathcal{G} = \mathfrak{gl}(7) \mathbb{C} \) is still quite accessible from a computational point of view: This is due to the fact that every complex isotropy Lie algebra \( \mathfrak{q} = \mathfrak{l} \mathbb{C} \) contains a \( \mathfrak{z} \)-stable Cartan subalgebra \( \mathfrak{c} \) (Thm. 2.6). Here, \( \mathfrak{c} \) is the conjugation induced by the real form \( \mathfrak{g} \). Consequently, all subspaces \( \mathfrak{q}^{(j)} \) of \( \mathfrak{c} \) contain this Cartan subalgebra and are direct sums of root spaces. The algebraic manipulation of the corresponding CR-algebra boils down to the combinatorics of root systems. In the next subsection we explain for a particular example all that in greater detail.

The example, described in geometric terms. In the context of flag manifolds, the simplest example of a 3-nondegenerate CR-manifold arises as a (locally closed) hypersurface orbit \( M \) in \( Z \) := \( G \mathbb{C} / \mathcal{G} \). Here, \( V \) is isomorphic to \( \mathbb{C}^7 \); further, \( \mathcal{C} := \mathfrak{g} \mathbb{C} / \mathcal{G} \). A is a symmetric nondegenerate 2-form; it determines the orthogonal group \( L = \mathcal{G} \mathcal{O}(7) \mathbb{C} \) and we write \( \mathcal{G} \mathcal{O}(7) \mathbb{C} = \mathfrak{f} A 2 \mathcal{G} \mathfrak{L}(V) := b(abv;aw) = b(v;w) \) for all \( v;w \). In this Cartan subalgebra and are direct sums of root spaces. The algebraic manipulation of the corresponding CR-algebra boils down to the combinatorics of root systems. In the next subsection we explain for a particular example all that in greater detail.

Finally, define the associated Hermitian 2-form \( \mathcal{H}(v;w) := b(v;w) \): It has signature \( (3;4) \) and \( G = (\mathcal{G}\mathfrak{om}(V;b) \setminus \mathcal{G}\mathfrak{om}(V;h^b)) \):
Let $\mathcal{H} = \mathcal{G}_2^b(V)$ be the set of all planes $E \subseteq \mathcal{G}_2^b(V)$ such that $h^b_{E}L$ is degenerate. This is a (singular) real hypersurface in $\mathcal{Z}$; stable under $G$. The CR-manifold $\mathcal{M}$ is a $G$--orbit in the smooth part of $\mathcal{H}$; open in $\mathcal{H}$. Note that the closed $G$--orbit $Y$ in $\mathcal{G}_2^b(V)$ is totally real and isomorphic to the real Grassmannian $\mathcal{G}_2^b(V^R)$: It is also contained in $\mathcal{H}$.

The geometrically described hypersurfaces $\mathcal{H}$ and $\mathcal{M}$ can also be given in local coordinates as the zero set of a function $c$. Note that $Z$ as a (7-dimensional) flag manifold is covered by Zariski open subsets $U \subset Z$ which are all isomorphic to $\mathcal{C}^7$ and provide coordinate charts on $Z$: We pick one of such charts, $U$; centered in a point of the totally real orbit $Y \subset Z$; and give a defining function for $U \setminus \mathcal{M}$: Write $z = (z_1; z_2; z_3)$ and $w = (w_1; w_2; w_3)$ for (row) vectors in $\mathcal{C}^3$; define the quadratic 2-form $c(z; w) = \frac{1}{2}(z_1w_3 + z_2w_2 + z_3w_1)$; and write $(z, w; u)$ for (row) vectors in $\mathcal{C}^7 = U$: Then the function $\det(z; w; u)$ is polynomial of degree 4 and is given as the determinant of a $2 \times 2$ matrix:

$$
\begin{align*}
\det c(z; w; u) &= c(z; w) + 2c(z; w) + c(z; w) + c(z; w) - z^t u^t, \\
&= c(z; w) + 2c(z; w) + c(z; w) - z^t u^t.
\end{align*}
$$

(6.1)

Note that with respect to our coordinates chart, $\mathcal{R}^7 = U \setminus Y$; i.e., $\mathcal{R}^7 \setminus \mathcal{M} = \emptyset$. However, $(0, 1; 0, 1; 0, 1; 0, 1; 0, 2) \mathcal{M}$.

Instead of a direct examination of this equation (which might be one possibility to check that $\mathcal{M}$ is uniformly 3-nondegenerate), we give a description of the corresponding CR-algebra and use Theorem 5.10 to check that order of nondegeneracy. The method used below can actually be generalized to find the associated CR-algebra of an arbitrary $G$--orbit in an arbitrary flag manifold $\mathcal{L} = \mathcal{Q}$: For simplicity, we restrict our considerations to the particular case of our hypersurface orbit in $\mathcal{G}_2^b(V) = \mathcal{L} = \mathcal{Q}$.

A root theoretical description and further generalization. Our first task is to identify the conjugacy class of the parabolic isotopy subalgebra $q$ of $\mathcal{G}_2^b(V)$) in terms of root subsystems. For the general theory of parabolics we refer to [10]. Recall that every parabolic subalgebra $q$ contains a Borel subalgebra $b$ (a maximal solvable subalgebra of $\mathcal{L}$) and a Cartan subalgebra $c$ (a maximal subalgebra, containing semisimple elements only) such that $c \supset b \supset q$: The Lie algebra $\mathcal{L}$ has a decomposition $\mathcal{L} = c \oplus b \oplus l$; into the root spaces $\mathcal{L} = \mathcal{F} \oplus 2l$; $[c; c] = (c)\mathcal{v}$ for all $c \in 2c$ with respect to the Cartan subalgebra $c$: Here, $c^\perp = (c^\perp c)$ $c$ stands for the set of roots (i.e., non-trivial eigenfunctionals) $c$ $c$ which appear in the root decomposition. It is possible to select a $c^\perp$--stable Cartan subalgebra $c^\perp (c^\perp)$ $c$; In such a case induces a permutation $\mathcal{P}$ of roots. We follow here the general convention and declare $(c^\perp c)$ $c$ to be the negative roots $c$: Let $c^\perp$ denote the corresponding simple roots. The conjugacy classes of parabolic subalgebras of $\mathcal{L}$ are parameterized 1-to-1 by subsets of $c^\perp$. This assignment is given by $c^\perp$. $c^\perp = (c^\perp c \setminus \setminus c)$ (Some authors use the complementary identification $c^\perp = (c^\perp c \setminus \setminus c)$)

In our particular example we have $\mathcal{L} = \text{SO}_7(c)$; and $c^\perp$ is the Dynkin diagram of $c^\perp$. Let $1; 2; 3$ denote the consecutive simple roots, with $3$ short. Then the parabolic isotropy subalgebra, defining $\mathcal{G}_2^b(V)$, corresponds to the subset $c^\perp = f; 1; 3; c$: Let $q = q_c$ be the complex isotropy at $z$ in a given $G$--orbit. In our case, the computation of the various subspace $c_q^{(\perp)}$ and $c_q^{(\perp)}$ and $c_q^{(\perp)}$ can be reduced to the computation of the corresponding subsets $c_q^{(\perp)}$ which, in turn, is pure combinatorial: Select $c^\perp$--stable
Cartan subalgebra and a Borel subalgebra with \( t \bigotimes q_\mathbb{C} \): In our particular example \( M \); the induced action of \( t \bigotimes q \) on the roots \( \Phi \) is depicted in the figure below: For short, the digits stand for the coefficients in the expression of a root with respect to the basis \( 1 \bigotimes 1 \); \( 3 \): For instance, \( 012 \) stands for \( 2 \bigotimes 2 \bigotimes 3 \) and \( 1 \bigotimes 012 = \bigotimes 2 \bigotimes 3 \): The arcs connect all pairs \( i \); \( j \); hence, completely determine \( \Phi \): A glance at that diagram immediately shows that

\[
\begin{align*}
q^{(1)} &= q^{(0)} = t \quad 100 \quad 1 \quad 112 \quad 1 \quad 001 \quad 1 \quad 011 \quad 1 \quad 012 \\
q^{(2)} &= q^{(3)} = 1 \quad 122 \\
q^{(1)} &= q^{(2)} = 1 \quad 010 \quad 1 \quad 111 \\
q &= q^{(0)} = q^{(1)} = 1 \quad 100 \quad 1 \quad 110 \quad \Phi^{01} : \\
\end{align*}
\]

The particular shape of \( q \) could be computed by “brute force” simply by selecting a base point \( y \in M \); describing the corresponding subalgebras \( t = (t) \bigotimes b \); \( q = q_\mathbb{C} \) in terms of \( 7 \times 7 \) matrices and finally computing the induced involution \( (t) \): \( (t) \): A more elegant way, suitable for a generalization to arbitrary orbits in flag manifolds is the following: Start with a point \( y \in M \); on the closed orbit. In our example, for \( t0 \); \( b0 \); \( c0 = q(y) \); the action of \( \Phi \) on \( 0 = (\Phi^{0}) \) is particularly simple: It is the identity (in the general case \( : 0 \); \( \Phi^{0} \) can be read off the Satake diagram of the real form \( q \)). Apply certain partial Cayley transformations to obtain \( \Phi^{0} = (\Phi^{0}) \bigotimes (b0) \bigotimes c(y) = q_\mathbb{C} \bigotimes z \left. \bigotimes \right. \): such that \( \Phi^{0} \) is contained in the orbit under consideration. In our particular case, we perform the partial Cayley transformations with respect to the strongly orthogonal roots \( 1 \bigotimes 1 \bigotimes 2 \bigotimes 3 \) and \( 2 \bigotimes 2 \bigotimes 3 \); i.e., \( c = c_1 \bigotimes c_2 \): The induced involution \( (t) = c(0) \) can be computed by a repeated application of the formula

\[
c(\ ) = c(\ ) \cdot h \bigotimes j \bigotimes i \bigotimes c(\ ) ; \quad 2 \bigotimes 0 ;
\]

where \( \bigotimes \) is the symmetric product on \( 0 \); induced by the Killing form and \( h \bigotimes j \bigotimes i = 2 \bigotimes (i) \bigotimes (j) \bigotimes (\ ) \): This method can be used to handle arbitrary orbits of real forms in arbitrary flag manifolds.

Let \( x \in M \); be a point and \( \text{hol}M \); \((x)\) the Lie algebra of germs of all infinitesimal CR-transformations at \( x \) (see Section 3). By the nondegeneracy of \( M \); we have \( \dim \text{hol}M \); \((x)\) \( < \); \( 1 \); and clearly \( \text{so}(3;4) = q \); \( \text{hol}M \); \((x)\): We do not know, however, if this inclusion is proper. It should be noted that Prop. 3.8 in [12] which uses the existence of nonresonant vector fields does not apply in our situation since \( \mathfrak{g}_2^{10}(\mathfrak{g}^{10}) \) is not a Hermitian space: Due to the following lemma there is no nonresonant vector field on \( \mathfrak{g}_2^{10}(\mathfrak{g}^{10}) \) coming from \( 1 = \text{so}(7;4) \): More precisely.

**Lemma 6.2** Let \( M = G \); \( Z \); \( Z = L = Q \) be an orbit of a real form in an arbitrary flag manifold such that \( 1 \); is simple. Then \( 1 \); contains a nonresonant vector field if and only if \( Z \); is Hermitian and \( 1 = \text{aut}(Z) \): In the Hermitian case there always exists a nonresonant vector field in \( \text{aut}(Z) \); \( \text{hol}(Z;\bar{z}) = \text{hol}(\mathfrak{g}^{10};0) \); with linear part equal to \( \text{Id} \).

**Proof.** It is sufficient to consider the isotropy action of a Cartan subalgebra \( t \); \( q \) on \( T_zZ = q^2 \): This action is diagonalizable and the corresponding eigenfunctions \( 2 \); \( t \); (i.e., roots in \( n \)) determine the eigenvalues of the linear parts of the vector fields, induced by elements in \( t \): We use here the decomposition \( = \bigotimes n \bigotimes [ \bigotimes n \); of the root system \( = (\ell; t) \); induced
by \(q\) such that \((q; t) = r \quad \text{are the roots of the nilpotent resp. reductive part of } q\): If \(q\) is not of Hermitian type (i.e., \(Z = L = Q\) is not a Hermitian symmetric space with \(l = \text{aut}(Z)\)) then there always exist \(\bar{Z} = n^+ \bar{Z}\) with \(\bar{Z}^2 = 0\): This violates the nonresonance condition (as given in [12]). The Hermitian situation is well-known.

The above defined hypersurface \(G\)-orbit \(M\) is a particular example of a finitely nondegenerate CR-manifold. One would expect that there are \(G\)-orbits in flag manifolds which are finitely nondegenerate of arbitrary high order. Surprisingly, at least for hypersurface orbits, this is not true as the following theorem shows. Before we state it, we recall some standard notation: Given a parabolic subalgebra \(q\) \(L\) select \(t\) \(b\) \(q\); (\(t\) a Cartan and \(b = t\) \(l\) a Borel subalgebra), and let \(= f_1; \cdots; q_1; \cdots; n_q\) \((q; t)\) be the corresponding simple roots. If \(q\) is maximal, it is determined by a subset \(r \in f_q = (q; t) \setminus t\) where \(q\) is \(T\) a simple root. Let \(c(q) = \max f_c q(\) ) \(= 2 \cdot \text{c is the } c^\phi \text{th coefficient in the expression } \) \(= q_1 + \cdots + q = q\): For example \(c(q) = 1\) if \(Z = L = Q\) is a Hermitian space with \(\text{Aut}(Z) = L\) (see [15]).

If \(L = L_1\) is a direct product of simple complex Lie groups and \(G = Q\), \(G_2\) a real form of \(L\) such that \(G_2\) is a Hermitian symmetric space in the simple factors \(L\)'s, the corresponding \(G\)-orbit \(M\) in \(Z = L = Q\) is also a direct product \(M = M_1 \otimes M_2\) as a CR-manifold. Consequently, we may restrict our consideration to the case \(L = Q\) where \(L\) is simple:

**Theorem 6.3** Let \(Z = L = Q\) be an arbitrary flag manifold where \(L\) is a simple complex group and \(G\) \(L\) a real form. Let \(M = G z\) be an orbit in \(Z\):

(i) Assume that \(M\) is a real hypersurface in \(Z\): Then \(M\) is holomorphically nondegenerate if and only if \(Q\) is a maximal parabolic subgroup of \(L\):

(ii) Assume that \(d_2(Z) = 1\); i.e., \(q\) is maximal parabolic, and \(M\) is not open in \(Z\): Let \(k(M)\) denote the order of nondegeneracy of \(M\): Then \(k(M)\) \(c(q) + 1\) \(= 7\) (with \(c(q)\) as defined above). In particular, \(k(M)\) \(= 3\) if \(L\) is not an exceptional simple group.

**Proof.** Let \(M = G z\) \(L = Q\) be a hypersurface orbit and \(Q = Q\) the complex isotropy subgroup at \(z\): As explained before, select \(t\) \(b\) \(q\) with \(a\) \(-\)stable Cartan subalgebra \(t\):

The assumption \(\text{codim}_Z M = 1\) implies that \(q^+\) \(q\) is a hyperplane in \(L\); i.e., there exists \(2 \cdot t\) with \(= \text{such that } p^+ = q^+\) \(q = q\): Let \(Q = \{ f_1; \cdots; n_q\} \) be the subset determined by \(q\) and \(= q_{1}; \cdots; n_q\): Write \(\text{supp} (\) \(= = 2 \cdot n_j > 0\): Clearly \(\text{supp} (\) \(\in Q\): Select an element \(= \text{such that } r^2 \in Q\): Then \(r^{\in Q}\) and consequently the parabolic subalgebra \(l\) corresponding to the set \(r\) \(\in Q\) \(\in Q\): If \(q\) \(\in Q\) \(\in Q\): Then \(q\) \(\in Q\) \(\in Q\): Further, the simple root \(q\) determines the following \(Z\)-filtration \(L^\phi \) \(1\): where the homogeneous parts are given by

\[
L^\phi = \begin{cases} 
\mathcal{L} & \text{for } j = 0 \\
\mathcal{L}^\phi_j & \text{for } j > 0 
\end{cases}
\]

\(\mathcal{L}^\phi_j = \mathcal{L}^{-1} \mathcal{L}^\phi_j \text{ for } j = 0 \) (Here, \(q^\phi\) is the reductive part of \(q\) containing \(t\)) and \(J = 0\) for \(j > 0\): We have then \(q = L c q^\phi J\): For short, write...
and note that \( q = q^{\infty} \) \((q \setminus \mathbb{C}) = q^{\infty}\). Let \( \mathcal{Q} = q^{(0)} q^{(1)} \cdots q^{(l)} \) be the filtration as defined in [5.7] and write \( q^{(1)} \cdot q^{(2)} \cdots \) for the corresponding gradation of the \( q^{(1)} \)'s, \( \mathcal{Q}_0 = q^{(1)} \setminus q^{(0)} \). Since \( q^{(1)} \) and all subalgebras \( q^{(1)} \) are \( \text{ad}(q^{(1)}) \)-stable, the condition defining \( q^{(1)} \) (see [5.7]) is equivalent to

\[
q^{(1)} = q^{(1)}_{n+1} \quad \text{with} \quad \mathcal{Q}^{(1)}_{n+1} = q^{(1)}_{n+1} \quad \text{for all} \quad n = 0, \ldots , m.
\]

The last statement of the theorem follows then from the following

**Claim.** For every \( p \geq 0 \) we have \( q^{(p+1)}_p = q^{(p+2)}_p = 0 \).

We prove the claim by induction, using condition (7): For \( p = 0 \) we have \( q^{(1)}_0 = q^{(2)}_0 = 0 \) since the condition \( q^{(1)}_0 \cdot q^{\infty} \) \( q^{\infty} \) does not depend on \( j \); Assume, we have already proved the claim for \( p = 0, 1, \ldots , m \) : Then, for every \( \cdot m + 2 \); we have

\[
q^{(1)}_{m+1} = q^{(1)}_{m} + q^{(m+1)}_{m+1} \quad \text{for all} \quad n = 0, \ldots , m.
\]

and consequently the conditions imposed on \( q^{(1)}_{m+1} q^{(m+2)}_{m+1} \) for each \( m + 2 \) do not depend on \( m \). This proves the claim.

Due to the above claim, at most after \( c(q) + 1 \) steps the filtration \( q^{(1)} \) becomes stationary. The theorem follows now from this observation and Theorem 5.10. The values of \( c(q) \) are bounded by the highest coefficient \( c(q) \) of the highest root of \( \Delta \) A glance at the table of highest roots for the classical and exceptional simple Lie algebras \( \mathfrak{sl} \) yields \( c(q) = c(q) = 2 \) in the classical cases and \( c(q) = c(q) = 3 \); \( c(q) = c(q) = 4 \) and \( c(q) = 6 \) in the exceptional cases.

**Problems.** Let \( M \) stand for a \( G \)-orbit in a flag manifold \( Z = L=Q \); Generalizing the above methods:

(A) Carry out the case where \( CR = \text{codim}_Z M = 2 \) and \( b_2(Z) = 2 \).

(B) Carry out the “group case,” i.e., describe the degeneracy of the \( G \)-orbits \( M \) in \( Z = L=Q \);

where the real form \( G \) carries a complex structure, i.e., \( L = G \) \( G \).

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