Random walk versus random line

Joël De Coninck(1), François Dunlop(2), Thierry Huillet(2)

Abstract: We consider random walks $X_n$ in $\mathbb{Z}_+$, obeying a detailed balance condition, with a weak drift towards the origin when $X_n \searrow \infty$. We reconsider the equivalence in law between a random walk bridge and a 1+1 dimensional Solid-On-Solid bridge with a corresponding Hamiltonian. Phase diagrams are discussed in terms of recurrence versus wetting. A drift $-\delta X_n^{-1} + \mathcal{O}(X_n^{-2})$ of the random walk yields a Solid-On-Solid potential with an attractive well at the origin and a repulsive tail $\frac{\delta(2+\delta)}{8}X_n^{-2} + \mathcal{O}(X_n^{-3})$ at infinity, showing complete wetting for $\delta \leq 1$ and critical partial wetting for $\delta > 1$.

KEYWORDS: Random walk, recurrence, SOS model, pinning, wetting

AMS subject classification: 60J10, 82B41

1. Introduction

We consider a random walk on $\mathbb{Z}$ or $\mathbb{Z}_+$ as defined by transition probabilities $\mathbb{P}(X_{n+1}|X_n)$, so that the probability distribution of a random walk bridge of length $N$ is

$$
\mathbb{P}(X_1, \ldots, X_{N-1}|X_0 = X_N = 0) = \prod_{n=0}^{N-1} \mathbb{P}(X_{n+1}|X_n) \mathbb{P}(X_N = 0|X_0 = 0) \quad (1.1)
$$

We consider a random line making a bridge of length $N$, in the form of a Solid-On-Solid model, as defined by a probability distribution of the form

$$
\mathbb{P}(X_1, \ldots, X_{N-1}|X_0 = X_N = 0) = Z_N^{-1} \prod_{n=0}^{N-1} e^{-W(X_n, X_{n+1})} \prod_{n=1}^{N} e^{-V(X_n)} \quad (1.2)
$$

with $W(X, Y) = W(Y, X)$ for all $X, Y$, and $Z_N$ the partition function normalising the probability.

We address the question of translating $\mathbb{P}(X_{n+1}|X_n)$ into $W(X, Y)$ and $V(X)$, and conversely, and transferring the information about transience / null recurrence / positive recurrence of the walk to complete wetting / partial wetting of the SOS model, and back. This question is related to the Hamiltonian on random walk trajectories in Ferrari-Martínez [FM].

---

(1) Centre de Recherche en Modélisation Moléculaire, Université de Mons-Hainaut, 20 Place du Parc, 7000 Mons, Belgium. Email: Joel.De.Coninck@crmm.umh.ac.be

(2) Laboratoire de Physique Théorique et Modélisation (CNRS - UMR 8089), Université de Cergy-Pontoise, 95302 Cergy-Pontoise, France. Email: Francois.Dunlop@u-cergy.fr, Thierry.Huillet@u-cergy.fr
We assume that the walk obeys the detailed balance condition with respect to a measure on \( Z \), not necessarily normalisable, which we write as \( \exp(-U(X)) \), so that

\[
P(X_{n+1}|X_n) = e^{U(X_n) - U(X_{n+1})} P(X_n|X_{n+1})
\]

\[
= e^{\frac{1}{2} U(X_n)} \left( P(X_{n+1}|X_n)P(X_n|X_{n+1}) \right)^{\frac{1}{2}} e^{-\frac{1}{2} U(X_{n+1})} \tag{1.3}
\]

\[
≡ e^{\frac{1}{2} U(X_n)} e^{-W(X_n,X_{n+1})} e^{-\frac{1}{2} U(X_{n+1})}
\]

which defines \( W(X,Y) \) from \( P(Y|X) \). The probability of a random walk bridge may now be written as

\[
P(X_1, \ldots, X_{N-1}|X_N=X_0=0) = Z_N^{-1} \prod_{n=0}^{N-1} e^{-W(X_n,X_{n+1})} \tag{1.4}
\]

with \( Z_N = P(X_N=0|X_0=0) \), which is of the form (1.2). The detailed balance condition was used, but the formula implied by (1.3) for the resulting SOS interaction \( W \) does not require the knowledge of the invariant measure \( \exp(-U(X)) \). The interaction \( W(X_n,X_{n+1}) \) typically contains a part of the form \( (V(X_n) + V(X_{n+1}))/2 \), which may be split from \( W \).

Conversely, given a SOS probability distribution of the form (1.2), where we let \( W \) absorb \( V \) like in (1.4), we look for a set of random walk probability transitions of the form

\[
P(X_{n+1}|X_n) = \frac{e^{-W(X_n,X_{n+1}) - \frac{1}{2} U(X_{n+1}) + \frac{1}{2} U(X_n)}}{Z(X_n)} \tag{1.5}
\]

These would lead to

\[
P(X_1, \ldots, X_{N-1}|X_0=X_N=0) = \prod_{n=0}^{N-1} \frac{e^{-W(X_n,X_{n+1})}}{Z(X_n)} \tag{1.6}
\]

which agrees with (1.4) only if \( Z = \text{const.} \), which requires \( \exp(-\frac{1}{2} U) \) to be an eigenvector of the symmetric kernel \( \exp(-W(X,Y)) \):

\[
\sum_X e^{-\frac{1}{2} U(X)} e^{-W(X,Y)} = \rho e^{-\frac{1}{2} U(Y)} \tag{1.7}
\]

The Perron-Frobenius theorem [S] indicates that (1.7) should have a solution \((\rho, U)\). In any case, (1.7) is equivalent to \( \exp(-U) \) being a left-eigenvector of the (non-symmetric) kernel (1.5) with \( Z = \text{const.} \):

\[
\sum_X e^{-U(X)} e^{-W(X,Y)} - \frac{1}{2} U(Y) + \frac{1}{2} U(X) = \rho e^{-U(Y)} \tag{1.8}
\]

Therefore (1.5) with \( Z = \text{const.} \) and \( U \) obeying (1.7) or (1.8) is an answer to formulating an SOS random line with probability (1.2), written as (1.4), in terms of a random walk.
However, it does require the knowledge of the measure \( \exp(-U(X)) \), with respect to which the walk will obey the detailed balance condition. This is related to the transfer matrix solution of the 1+1 dimensional SOS models of wetting derived in the early eighties [AD, Bu, C, CW, LH, VL] and further elaborated with path space limit theorems in the late nineties [Bo, DGZ, IY, V] and references therein. Expressing an SOS bridge in terms of a random walk, asymptotically as \( N \to \infty \), was used also in the proof of the Wulff shape for SOS models (Theorem 1 in [DDR]).

In the following sections we consider examples, translating from random walk to SOS model, when

\[
\mathbb{P}(X_{n+1} < X_n|X_n) - \mathbb{P}(X_{n+1} > X_n|X_n) \sim \frac{\delta}{X_n} \quad \text{as} \quad X_n \to \infty \quad (1.9)
\]

and discuss recurrence versus wetting. Interest into such random walks goes back to Lamperti [L1, L2]. Detailed properties of the random walk are available [DDH, H] in special instances of (1.9), yielding the corresponding properties in the corresponding SOS models. Some of these examples admit constructions for bridges not using the detailed balance formula.

2. Bridge with \( X_{n+1} - X_n = \pm 1 \): from random walk to random line

Let

\[
\varphi : \{\frac{1}{2}, \frac{3}{2}, \ldots \} \to \mathbb{R}
\]

Consider a random walk \( X_n \) with state space \( \mathbb{Z}_+ = \{0, 1, 2, \ldots \} \), starting at \( X_0 = 0 \), with transition probabilities

\[
\mathbb{P}(X_{n+1}|X_n) = \frac{e-(X_{n+1}-X_n)\varphi(X_{n+1}+X_n)}{e^{-\varphi(X_n+\frac{1}{2})} + e^{\varphi(X_n-\frac{1}{2})}} \quad \text{when} \quad X_n \geq 1 \quad \text{and} \quad X_{n+1} = X_n \pm 1 \quad (2.1)
\]

and reflection at the origin: \( X_{n+1} = 1 \) whenever \( X_n = 0 \). Any random walk with transition probabilities \( p_x = \mathbb{P}(X_{n+1} = x + 1|X_n = x) \) and \( q_x = 1 - p_x = \mathbb{P}(X_{n+1} = x - 1|X_n = x) \) may be written in the form (2.1): take \( \varphi(\frac{1}{2}) \) arbitrarily, and then solve recursively

\[
\varphi(x + \frac{1}{2}) = -\varphi(x - \frac{1}{2}) + \ln \frac{q_x}{p_x} \quad , \quad x \geq 1
\]

From (2.1) we get

\[
\mathbb{P}(X_1, \ldots, X_{N-1}, X_N = 0|X_0 = 0) = \prod_{n=0}^{N-1} \mathbb{P}(X_{n+1}|X_n)
\]

\[
= \prod_{n=1}^{N} e^{\varphi(\frac{1}{2})} \prod_{n=1}^{N} \frac{1}{e^{-\varphi(X_{n+1}+\frac{1}{2})} + e^{\varphi(X_{n-1}+\frac{1}{2})}} \prod_{n=0}^{N-1} 1_{|X_{n+1}-X_n|=1}
\]

\[
= 2^{-N} \prod_{n=1}^{N} 2 e^{\varphi(\frac{1}{2})} \prod_{n=1}^{N} \frac{2}{e^{-\varphi(X_{n+1}+\frac{1}{2})} + e^{\varphi(X_{n-1}+\frac{1}{2})}} \prod_{n=0}^{N-1} 1_{|X_{n+1}-X_n|=1}
\]

\[
= 2^{-N} \prod_{n=1}^{N} e^{-V(X_n)} \prod_{n=0}^{N-1} 1_{|X_{n+1}-X_n|=1}
\]
with

\[ V(X) = -\left(\ln 2 + \varphi\left(\frac{1}{2}\right)\right) 1_{X=0} + \ln \frac{e^{-\varphi(X+\frac{1}{2})} + e^{\varphi(X-\frac{1}{2})}}{2} 1_{X \geq 1} \] (2.4)

The key point in the computation (2.3), instead of using the detailed balance condition, was the pairing of edge factors, one factor corresponding to going up the edge and the other factor going down the edge, leading to the cancellation of factors from the numerator in (2.1). This exact cancellation is restricted to bridges, and requires the coupling \( \varphi \) in (2.1) to be associated with the un-oriented edge \( \{X_n, X_{n+1}\} \) or to the midpoint \( (X_n + X_{n+1})/2 \).

Example (see Fig 1):

\[ \varphi(x) = \frac{\delta}{2x} \quad \Rightarrow \quad V(X) = -(\ln 2 + \delta) 1_{X=0} + \ln \frac{e^{-\frac{\delta}{2X+1}} + e^{\frac{\delta}{2X-1}}}{2} 1_{X \geq 1} \] (2.5)

\[ \sim \frac{\delta(2+\delta)}{8X^2} \quad \text{as} \quad X \to \infty \]

Such a potential for \( \delta > 0 \), having short range attraction at the wall and long range repulsion far from the wall, is reminiscent of van der Waals liquids with a positive Hamaker constant [dG, p846]. The 1+1 dimensional SOS model may be considered a crude effective interface model where some dimensions and degrees of freedom have been integrated out in a mean field approximation.

![Graph](image-url)

Fig. 1: \( V(X) \) as (2.4) with \( \delta = 1.2, 0.5, -0.2, -1.2 \).
Example:

$$\varphi(x) = \frac{\delta}{2x} + \frac{\gamma}{x^2} + O\left(\frac{1}{x^3}\right) \quad \text{as} \quad x \to \infty \quad (2.6)$$

Such random walks should have a phase diagram (transience / null recurrence / positive recurrence) independent of $\gamma$, and also independent of the behaviour of $\varphi$ for small $x$. Hence the corresponding SOS models should have a phase diagram (complete / partial wetting) independent of $\gamma$: partial wetting if and only if $\delta > 1$. However, unlike the square well model in the partial wetting regime (cf. next section), the height distribution will not decay exponentially, but as a power law with an exponent depending upon $\delta$ [DDH], hence the term “critical partial wetting”.

The behaviour (2.6) implies

$$V(X) = \frac{\delta(2 + \delta)}{8X^2} + O\left(\frac{1}{X^3}\right) \quad \text{as} \quad X \to \infty \quad (2.7)$$

which indeed is independent of $\gamma$.

3. Bridge with $X_{n+1} - X_n = \pm 1$: from random line to random walk

Suppose now that the potential $V(X)$ on $\mathbb{Z}_+$ is given and satisfies $V(X) \to 0$ as $X \to \infty$. We want to find $\varphi : \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right\} \to \mathbb{R}$ such that (2.4) is satisfied up to a constant $\lambda$. Let

$$b_X = e^{V(X)+\lambda}, \quad a_X = e^{-\varphi(X+\frac{1}{2})} \quad (3.1)$$

Then (2.4) with $V(X) + \lambda$ instead of $V(X)$ becomes

$$2b_0 = a_0$$
$$2b_X = a_X + a_{X-1}^{-1}, \quad X \geq 1 \quad (3.2)$$

whose solution is the continued fraction

$$a_0 = 2b_0$$
$$a_1 = 2b_1 - \frac{1}{2b_0}$$
$$\ldots$$
$$a_X = 2b_X - \frac{1}{2b_{X-1} - \frac{1}{2b_{X-2} - \ldots - \frac{1}{2b_0}} \quad (3.3)}$$

acceptable only if $a_X > 0 \forall X$. Consistency may be verified when (2.3)(2.4), converted into (1.4), obeys (1.7), which takes the form

$$\sum_{X=Y \pm 1 \atop X \geq 0} e^{-\frac{1}{2}U(X)-\frac{1}{2}V(X)-\frac{1}{2}V(Y)} = 2\rho e^{-\frac{1}{2}U(Y)}, \quad Y \geq 0 \quad (3.4)$$
so that (3.2)(3.3)(3.4) have the solution \( \lambda = \ln \rho \) and

\[
a_X = e^{-\frac{1}{2}U(X+1)-\frac{1}{2}V(X+1)+\frac{1}{2}U(X)+\frac{1}{2}V(X)} , \quad X \geq 0
\] (3.5)

To conclude this section, we give explicitly the random walks corresponding to the SOS model with a square well or a double step potential at the wall:

- For \( V(X) = v_0 1_{X=0} \), equations (3.1)(3.2) with \( \rho = e^\lambda \) take the form

\[
\begin{align*}
2b_0 &= 2\rho e^{v_0} = a_0 \\
2b_X &= 2\rho = a_X + a_{X-1}^{-1}, \quad X \geq 1
\end{align*}
\] (3.6)

— **First ansatz: \( \rho = 1 \)**

\[
a_X = \frac{(2b_0 - 1)X + 2b_0}{(2b_0 - 1)X + 1} > 0 \quad \forall X \quad \Rightarrow \quad v_0 \geq -\ln 2
\] (3.7)

a transient walk with

\[
\varphi(x) \sim -\frac{1}{x} \quad \text{as} \quad x \to \infty
\] (3.8)

compatible with (2.4), \( \delta = -2 \).

— **Second ansatz: \( a_X = a = \text{const.} \)**

\[
\rho = \frac{a + a^{-1}}{2}, \quad a^{-2} = e^{-v_0} - 1 > 0 \quad \Rightarrow \quad v_0 < 0
\] (3.9)

Both ansatz work when \(-\ln 2 \leq v_0 < 0\), corresponding to transient cases. The wetting transition is at \( v_0 = -\ln 2 \).

- For \( V(X) = v_0 1_{X=0} + v_1 1_{X=1} \), equations (3.1)(3.2) with \( \rho = e^\lambda \) take the form

\[
\begin{align*}
2b_0 &= 2\rho e^{v_0} = a_0 \\
2b_1 &= 2\rho e^{v_1} = a_1 + a_0^{-1} \\
2b_X &= 2\rho = a_X + a_{X-1}^{-1}, \quad X \geq 2
\end{align*}
\] (3.10)

— **First ansatz: \( \rho = 1 \)**

\[
\begin{align*}
a_0 &= 2b_0 \\
a_1 &= 2b_1 - \frac{1}{2b_0} > 0 \\
\frac{(a_1 - 1)X + 1}{(a_1 - 1)X + 2 - a_1} > 0 \quad \forall X \geq 2 \quad \Rightarrow \quad a_1 = 2b_1 - \frac{1}{2b_0} \geq 1
\end{align*}
\] (3.11)

or

\[
4e^{v_1} \geq 2 + e^{-v_0}
\] (3.12)
a transient walk with \( \varphi(x) \sim -\frac{1}{x} \) as \( x \to \infty \). Condition (3.12) coincides with the complete wetting range.

— Second ansatz: \( a_X = a = \text{const.} \forall X \geq 1 \)

\[
\begin{align*}
a_0 &= 2\rho e^{v_0} \\
a &= 2\rho e^{v_1} - \frac{1}{2\rho e^{v_0}} \\
\rho &= \frac{a + a^{-1}}{2}
\end{align*}
\]  

(3.13)

Eliminating \( \rho \) gives

\[
a^4(e^{v_1} - 1) + a^2(2e^{v_1} - e^{-v_0} - 1) + e^{v_1} = 0
\]  

(3.14)

giving a suitable solution for \( v_1 \leq 0 \) and any \( v_0 \) and also for

\[
v_1 \geq 0 , \quad v_0 \leq 0 , \quad v_1 \leq 2 \log \cosh \frac{v_0}{2}
\]  

(3.15)

Whatever \( v_0 \) and \( v_1 \), one or the other or both ansatz provides a solution. There is partial wetting if and only if there is a representation with \( 0 < a < 1 \), equivalent to

\[
4e^{v_1} < 2 + e^{-v_0}
\]  

(3.16)

where only the second ansatz gives a solution, in fact one solution if \( v_1 \leq 0 \) and two solutions if \( v_1 > 0 \).
4. Bridge with $X_{n+1} - X_n \in \{-1, 0, +1\}$, Metropolis algorithm

Let

$$U : \mathbb{Z}_+ \to \mathbb{R}$$

Consider a random walk $X_n$ with state space $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, starting at $X_0 = 0$, with transition probabilities

$$P(X_{n+1}|X_n, X_n \geq 1) = 1_{X_{n+1}=X_n \pm 1} \frac{1}{2} e^{-(U(X_{n+1})-U(X_n))_+}$$

$$+ 1_{X_{n+1}=X_n} \left[ 1 - \frac{1}{2} e^{-(U(X_{n+1})-U(X_n))_+} - \frac{1}{2} e^{-(U(X_n-1)-U(X_n))_+} \right]$$

and reflection at the origin: $X_{n+1} = 1$ whenever $X_n = 0$. Then

$$P(X_1, \ldots, X_{N-1}, X_N = 0|X_0 = 0) = \prod_{n=0}^{N-1} P(X_{n+1}|X_n)$$

$$= \prod_{n=0}^{N-1} \left[ 1 - \frac{1}{2} e^{-(U(X_{n+1})-U(X_n))_+} - \frac{1}{2} e^{-(U(X_n-1)-U(X_n))_+} \right] \cdot \prod_{n=1}^{N-1} \frac{1}{2} e^{-(U(X_{n+1})-U(X_n))_+} \prod_{n=0}^{N-1} \frac{1}{2} e^{-(U(0)-U(1))_+}$$

$$= \prod_{n=0}^{N-1} \left[ 1 - \frac{1}{2} e^{-(U(X_{n+1})-U(X_n))_+} - \frac{1}{2} e^{-(U(X_n-1)-U(X_n))_+} \right]$$

$$\cdot \prod_{n=0}^{N-1} \frac{1}{2} e^{-(U(X_{n+1})-U(X_n))_+} \prod_{n=1}^{N} 2 e^{(U(1)-U(0))_+}$$

$$= 2^{-N} \prod_{n=0}^{N-1} e^{-W(X_n, X_{n+1})} \prod_{n=1}^{N} e^{-V(X_n)}$$

with

$$V(X) = -\left( \ln 2 + (U(1) - U(0))_+ \right) 1_{X=0}$$

$$W(X, X) = -\ln \left[ 2 - e^{-(U(X+1)-U(X))_+} - e^{-(U(X-1)-U(X))_+} \right]$$

$$W(X, X + 1) = W(X + 1, X) = \frac{|U(X + 1) - U(X)|}{2}$$

where $W(X, X)$ is used only with $X \geq 1$. The pairing of edge factors was used, like in Section 2.
Example: \( \delta \geq 0 \) and

\[
U(X) = \delta \ln(X + 1) \quad \Rightarrow \quad V(X) = -(\ln 2 + \delta \ln 2) 1_{X=0}
\]

\[
W(X, X) = -\ln\left(1 - \left(\frac{X + 1}{X + 2}\right)^\delta\right)
\]

\[
W(X + 1, X) = W(X, X + 1) = \frac{\delta}{2} \ln\left(\frac{X + 2}{X + 1}\right)
\]  

(4.4)

Instead of reflection at the origin, let us now choose the full Metropolis algorithm, including at the wall:

\[
P(X_{n+1}|X_n = 0) = \frac{1}{2} e^{-\left(U(1) - U(0)\right)_+} 1_{X_{n+1}=1} + \left(1 - \frac{1}{2} e^{-\left(U(1) - U(0)\right)_+}\right) 1_{X_{n+1}=0}
\]  

(4.5)

Then

\[
P(X_1, \ldots, X_{N-1}, X_N = 0|X_0 = 0) = \prod_{n=0}^{N-1} P(X_{n+1}|X_n)
\]

\[
= \prod_{n=0}^{N-1} \left[1 - \frac{1}{2} e^{-\left(U(X_{n+1}) - U(X_n)\right)_+} - \frac{1}{2} e^{-\left(U(X_{n-1}) - U(X_n)\right)_+}\right] \prod_{n=0}^{N-1} \frac{1}{2} e^{-\frac{|U(X_{n+1}) - U(X_n)|}{2}}
\]

\[= 2^{-N} \prod_{n=0}^{N-1} e^{-W(X_n, X_{n+1})}
\]  

(4.6)

with

\[
W(X, X + 1) = W(X + 1, X) = \frac{|U(X + 1) - U(X)|}{2}
\]

\[
W(X, X) = -\ln\left[2 - e^{-\left(U(X+1) - U(X)\right)_+} - e^{-\left(U(X-1) - U(X)\right)_+}\right]
\]

except:

\[
W(0, 0) = -\ln\left[2 - e^{-\left(U(1) - U(0)\right)_+}\right]
\]  

(4.7)

Example: \( \delta \geq 0 \) and

\[
U(X) = \delta \ln(X + 1) \quad \Rightarrow \quad W(0, 0) = -\ln(2 - 2^{-\delta})
\]  

(4.8)

and the other values same as first Metropolis example.

Remark: The factor \( 1/2 \) in (4.1) could be replaced by any number between 0 and 1/2.
5. Random walk with $X_{n+1} - X_n \in \mathbb{Z}$, Metropolis algorithm

Let $\exp(-W_0(X,Y))$ be a symmetric probability kernel in $\mathbb{Z} \times \mathbb{Z}$,

$$W_0(X,Y) = W_0(Y,X), \quad \sum_{Y \in \mathbb{Z}} e^{-W_0(X,Y)} = 1 \quad (5.1)$$

and

$$U: \mathbb{Z}_+ \rightarrow \{\mathbb{R} \cup \{+\infty\}\}, \quad \text{with:} \quad X < 0 \Rightarrow U(X) = +\infty \quad (5.2)$$

Consider a random walk $X_n$ with state space $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, starting at $X_0 = 0$, with transition probabilities

$$\mathbb{P}(X_{n+1}|X_n) = e^{-W_0(X_{n+1},X_n)-(U(X_{n+1})-U(X_n))_+} \quad \text{if} \quad X_{n+1} \neq X_n$$

$$\mathbb{P}(X_{n+1} = X_n|X_n) = 1 - \sum_{Y \neq X_n} \mathbb{P}(Y|X_n) \quad (5.3)$$

Then, proceeding as in Section 1, we get (1.4) with

$$W(X,Y) = W(Y,X) = W_0(X,Y) + \frac{|U(Y) - U(X)|}{2} \quad \text{if} \quad Y \neq X$$

$$W(X,X) = -\ln \left(1 - \sum_{Y \in \mathbb{Z}} e^{-W_0(X,Y)-(U(Y)-U(X))_+}\right) \quad (5.4)$$

Example:

$$W_0(X,Y) = J|X - Y| + \text{const.}, \quad U(X) = \delta \ln(X + 1) \quad (5.5)$$

again giving partial wetting if and only if $\delta > 1$.

Example: $W_0(X,Y) = \ln 2$ if $|X - Y| = 1$ and $+\infty$ otherwise. This is equivalent to (4.1)(4.5-7).

For the random walk with $X_{n+1} - X_n \in \mathbb{Z}$, edges up and down cannot be paired exactly as in Sections 2 and 4. Approximate pairing would leave a remainder of order $X_n^{-2}$, which one might argue to be “irrelevant”.

Acknowledgments: F. D. and T. H. acknowledge support and kind hospitality from Université de Mons-Hainaut and CRMM.

References

[AD] D.B. Abraham, J. De Coninck: Description of phases in a film-thickening transition, J. Phys. A16, L333–337 (1983).

[Bo] E. Bolthausen: Localization-delocalization phenomena for random interfaces, Proceedings of the ICM, Beijing 2002, vol. 3, pp 25–40.
[Bu] T.W. Burkhardt: Localization–delocalization transition in a solid-on-solid model with a pinning potential, J. Phys. A14, L63–L68 (1981).

[C] J.T. Chalker: The pinning of a domain wall by weakened bonds in two dimensions, J. Phys. A14, 2431–2440 (1981).

[CW] S.T. Chui, J.D. Weeks: Pinning and roughening of one-dimensional models of interfaces and steps, Phys. Rev. B23, 2438–2441 (1981).

[dG] P.G. de Gennes: Wetting: statics and dynamics, Rev. Mod. Phys. 57, 827–863 (1985).

[DDH] J. De Coninck, F. Dunlop, T. Huillet: Random walk weakly attracted to a wall, J. Stat. Phys. 133, 271–280 (2008).

[DDR] J. De Coninck, F. Dunlop, V. Rivasseau : On the Microscopic Validity of the Wulff Construction and of the Generalized Young Equation, Commun. Math. Phys 121, 401–419 (1989).

[DGZ] J-D Deuschel, G. Giacomin, L. Zambotti: Scaling limits of equilibrium wetting models in (1+1)-dimension, Probab. Theory Relat. Fields 132, 471–500 (2005).

[FM] P.A. Ferrari, S. Martínez: Hamiltonians on random walk trajectories, Stochastic Process. Appl. 78, 47–68 (1998).

[H] T. Huillet: Random walk with long-range interaction with a barrier and its dual: Exact results, Preprint hal-00370353.

[IY] Y. Isozaki, N. Yoshida: Weakly pinned random walk on the wall: pathwise descriptions of the phase transition, Stochastic Process. Appl. 96, 261–284 (2001).

[L1] J. Lamperti: Criteria for the recurrence or transience of stochastic process I., J. Math. Anal. Appl. 1, 314–330 (1960).

[L2] J. Lamperti: Criteria for stochastic processes. II. Passage-time moments, J. Math. Anal. Appl. 7, 127–145 (1963).

[LH] J.M.J. van Leeuwen, H.J. Hilhorst: Pinning of rough interface by an external potential, Physica A 107, 319–329 (1981).

[S] E. Seneta: Nonnegative matrices and Markov chains, Springer Series in Statistics, Springer-Verlag, New York, 1981.

[VL] M. Vallade, J. Lajzerowicz: Transition rugueuse pour une singularité linéaire dans un espace à deux ou trois dimensions, J. physique 42, 1505–1514 (1981).

[V] Y. Velenik: Localization and delocalization of random interfaces, Probab. Surv. 3, 112–169 (2006).