Two Particle Quantummechanics in 2+1 Gravity using Non Commuting Coordinates

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Abstract

We find that the momentum conjugate to the relative distance between two gravitating particles in their center of mass frame is a hyperbolic angle. This fact strongly suggests that momentum space should be taken to be a hyperboloid. We investigate the effect of quantization on this curved momentum space. The coordinates are represented by non commuting, Hermitian operators on this hyperboloid. We also find that there is a smallest distance between the two particles of one half times the Planck length.

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1 Introduction

The classical theory of 2+1 gravity was first considered in 1963 by Staruszkiewicz [1]. The subject was revived in 1984 by Deser, Jackiw and ’t Hooft in their cornerstone article [2]. The basic feature that one can find in these articles is that the gravitational field does not carry degrees of freedom. In that sense it is topological theory. Pointparticle solutions can be constructed by a ‘cut and paste’ procedure. For instance for one particle at rest one simply cuts out a wedgelike region from space time and identifies the boundaries. More elaborate configurations are always characterized by elements of the Poincaré group. An elegant procedure to construct multiparticle solutions was put forward by ’t Hooft [3] who glued together flat patches of Minkowski space using Lorentz transformations. Achucarro and Townsend [8] (1986) and Witten [9] (1988) proposed a Chern Simons gauge theory that is equivalent to 2+1 gravity. Many of the subsequent quantum models used this formulation and a great deal of progress was made. These models however concerned mainly with closed universes with non trivial topology (handles) [14] and with the calculation of scattering amplitudes [4, 5, 10, 15]. However no one passed the point of writing down a complete Hilbert space for a multiparticle model. A consistent theory for the creation and annihilation of particles was not considered at all to our knowledge. However this is of course what we would ultimately like to understand. What is the problem we are facing when trying to write down a Hilbert space for a two particle system for instance?

If we solve the Klein Gordon equation on a cone for fixed energy we find a set of fractional Bessel functions times some angular function:

$$\psi_E(r, \theta) = J_{\alpha}(kr) \ e^{i\alpha \theta}$$  \hspace{1cm} (1)

with \(\hbar k = \sqrt{2ME}\) and \(\alpha = 1 - \frac{E}{2\pi}\). However for different energy the \(\alpha\)’s are different which implies that the wave functions cannot be orthonormal. In this paper we will encounter a similar problem.

Another point that was frequently overlooked is the choice of canonically conjugate variables. Because the Hamiltonian is not simply of the Klein Gordon type, the conjugate momentum to for instance the distance of the particle from the origin is not simply \(Mv^a\) (mass times velocity). The first to take into account the correct Hamiltonian and a pair of conjugate phase space variables was ’t Hooft in [16]. He found that the momentum conjugate to the distance of the particle from the origin is actually an angle! Using this he quantized the particle on a spherical momentum space. The Hilbert space is then simply spanned by the spherical harmonics \(Y_{l,m}(\theta, \phi)\).

The idea to use a curved momentum space goes back to 1947. In his paper Snyder proposed de Sitter or anti de Sitter spaces for energy momentum space. He introduced non commuting operators to represent the coordinates \(x, y, z, t\), turning configuration space partly into a lattice. However the model was still
covariant with respect to the full Lorentz group! Recently Demichev considered this issue in the light of quantum groups [11].

In this paper we will follow the same kind of reasoning as in [16]. In section one we calculate the conjugate momentum to the relative distance between two particles and find a hyperbolic angle. This suggests quantization on a hyperboloid. In section two we try to define Hermitian operators that represent the coordinates $x, y$. The fact that momentum space is curved (i.e. a hyperboloid) turns these coordinates into non commuting operators. We also define new momenta which are conjugate to $x, y$ only up to order $\ell_P^2$ where $\ell_P$ is the Planck length. We also calculate the commutator algebra between these operators. In the last section we compute a complete set of basis functions that span the Hilbert space in the case when we do not take the boundary conditions into account properly, but only study the effect of the curvature of momentum space. We find that the particles cannot approach each other closer than $\frac{1}{2}\ell_P$. In the case when we do take into account the correct boundary conditions we find that we are stuck with a similar problem as was the case with the fractional Bessel functions: wave functions with different energy are not orthogonal. Appendix A treats the case of massless particles. In appendix B we derive a Dirac equation.

2 Canonical conjugate coordinates

In 1988 ’t Hooft showed in [4] that we may describe two gravitating particles in their center of mass (c.o.m.) frame by a relative distance vector $\mathbf{r}$ which obeys a peculiar boundary condition. The effect of the particles on the space time is that we should cut out a wedgelike region from space time with a deficit angle $\beta = E$, where $E$ is the energy of the two particles. The boundaries of the wedge should be identified in some way depending on the value of the angular momentum $L$:

$$
\begin{pmatrix}
t' \\
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos E & \sin E \\
0 & -\sin E & \cos E
\end{pmatrix} \begin{pmatrix}
t \\
x \\
y
\end{pmatrix} + \begin{pmatrix}
L \\
0 \\
0
\end{pmatrix}
$$

(2)

When we do quantum mechanics in such a space we need to incorporate this identification as a boundary condition on the wave function. Writing

$$
\psi = \sum_m \psi_m(r)e^{im\vartheta}
$$

(3)

we find:

$$
\psi(r, \vartheta + 2\pi) = e^{imE}\psi(r, \vartheta)
$$

(4)

This leads to a revised quantization for the angular momentum:

$$
m = \frac{2\pi n}{2\pi - E}, \quad n \in \mathbb{Z}
$$

(5)
He then proceeds to calculate the scattering amplitude using the Hamiltonian:

\[ H = \sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \]  

(6)

Although the amplitude does not depend on the choice of this Hamiltonian it is not the correct one and only valid in the low energy regime. In this paper we want to investigate the two particle Hilbert space using a proper Hamiltonian. If we define time at infinity, Carlip [15] has shown that the Hamiltonian is given by the total deficit angle at infinity\(^2\). It is expressed in the following way:

\[ e^{iH J_0} = e^{i\mathcal{P}_a J_a} e^{i\mathcal{P}_b J_b} \]

(7)

The \(J_a\) are the generators of the group SO(2,1) and

\[ p_i^a = (M_i \cosh \xi_i, M_i \sinh \xi_i \cos \varphi_i, M_i \sinh \xi_i \sin \varphi_i) \]  

(8)

where \(\xi\) is the rapidity of the particle. This can be rewritten as:

\[
\begin{align*}
\cos \left( \frac{H}{2} \right) &= \cos \mu_1 \cos \mu_2 - \sin \mu_1 \sin \mu_2 \left( \frac{p_1^a p_2^a}{M_1 M_2} \right) \\
\sin \left( \frac{H}{2} \right) \frac{P^a}{|P|} &= \cos \mu_1 \sin \mu_2 \frac{p_1^a}{M_1} + \cos \mu_2 \sin \mu_1 \frac{p_2^a}{M_2} + \sin \mu_1 \sin \mu_2 \varepsilon^{abc} p_{1b} p_{2c} \frac{1}{M_1 M_2}
\end{align*}
\]

(9)

(10)

where we introduced \(\mu_i \equiv \frac{M_i}{2}\). To simplify things a bit we will now assume that both particles have the same mass: \(\mu_1 = \mu_2\). In appendix A we will treat the massless case, i.e. \(\mu_1 = \mu_2 = 0\). Furthermore, in the c.o.m. frame we have that \(E_1 = E_2\), but as we will show shortly, not \(\vec{p}_1 = -\vec{p}_2\). Let us parametrize:

\[
\begin{align*}
p_1 &= (E, p, 0) \\
p_2 &= (E, p \cos \varphi, p \sin \varphi)
\end{align*}
\]

(11)

(12)

with \(E = \sqrt{p^2 + M^2}\). In the c.o.m. frame we also have: \(P^i = 0\). Using this fact we can calculate the the angle \(\varphi\):

\[
\cos \varphi = \frac{E^2 \sin^2 \mu - M^2 \cos^2 \mu}{E^2 \sin^2 \mu + M^2 \cos^2 \mu}
\]

(13)

This is clearly an effect of the gravitational field, not present in the case of two non gravitating particles. Equation (8) can be written as:

\[
\cos \left( \frac{H}{2} \right) = \cos^2 \mu - \sin^2 \mu (E^2 - p^2 \cos \varphi)
\]

(14)

\(^2\)Waelbroeck has shown however, using his lattice model for 2+1 gravity, that different choices for time may lead to different Hamiltonians and different quantizations. For one choice time is quantized, for another it is not!
Using (13) we find for this:

\[ \cos\left(\frac{H}{2}\right) = \frac{\cos M - \sin^2 \mu \sinh^2 \xi}{1 + \sin^2 \mu \sinh^2 \xi} \]  

(15)

One can check that:

\[ \xi \to 0 \Rightarrow H \to 2M \]  

(16)

\[ \xi \to \infty \Rightarrow H \to 2\pi \]  

(17)

as is to be expected. Now that we know the Hamiltonian as a function of \( p \) (or \( \xi \)) we can calculate the canonically conjugate momentum \( q \) to the relative distance \( r \). The other coordinate we will use is an angle \( \vartheta \) and the conjugate angular momentum \( L \). Because the particles are in free motion we know what the velocity is:

\[ \frac{d}{dt} r = \sqrt{(\vec{v}_1^2 - \vec{v}_2^2)^2} \]  

(18)

with

\[ \vec{v}_1 = (\tanh \xi, 0) \]  

(19)

\[ \vec{v}_2 = (\tanh \xi \cos \varphi, \tanh \xi \sin \varphi) \]  

(20)

We find:

\[ \frac{d}{dt} r = \frac{2 \cos \mu \tanh \xi}{\sqrt{1 + \sin^2 \mu \sinh^2 \xi}} \]  

(21)

Next we solve:

\[ \frac{d}{dt} r = \frac{dH}{dq} \Rightarrow q = \int d\xi \frac{-2}{r \sin\left(\frac{H}{2}\right)} \frac{d\cos\left(\frac{H}{2}\right)}{d\xi} \]  

(22)

\[ = \int d\xi \frac{2 \sin \mu \cosh \xi}{\sqrt{1 + \sin^2 \mu \sinh^2 \xi}} = 2 \cosh^{-1}[\sin \mu \sinh \xi] \]  

(23)

In other words:

\[ \sinh\left(\frac{q}{2}\right) = \sin \mu \sinh \xi \]  

(24)

So we find that the momentum \( q \) conjugate to the relative distance is a hyperbolic angle. Substituting this in the formula for the energy (15) we see:

\[ \cos\left(\frac{H}{4}\right) = \frac{\cos \mu}{\cosh\left(\frac{q}{2}\right)} \]  

(25)

If we set \( q = 2\eta \) and \( H_P = \frac{1}{2}H \) this is precisely the mass shell condition for one particle in the polygon approach to 2+1 gravity \[3, 12\]. We can therefore

\[ \text{In the polygon approach the conjugate variables are } L \text{ and } 2\eta. \text{ } L \text{ is the length of an edge of the wedge; } \eta \text{ is the rapidity with which this edge moves perpendicular to } L. \]
conclude that the configuration space for the relative coordinate \( r \) is a ‘spinning cone’. The value of the ‘spin’ is the angular momentum of the two particles, the excised angle equals the total energy (see (2)) and the mass shell condition is given by (25). We stress that the configuration space for this ‘effective c.o.m. particle’ is not space time surrounding the two physical particles. This is a ‘double cone’. The quantization of this effective particle in the following can therefore also be interpreted as an alternative program for the quantization on a sphere as was done by ’t Hooft [7]. It might seem confusing that he found that the momentum conjugate to the distance of a particle as seen from a fixed origin is actually an angle instead of a hyperbolic angle. The relative distance between two particles (or the edge length \( L \) in the case of one particle) is however a different coordinate and as such has a different conjugate momentum. Another peculiar aspect of (25) is that \( H \) is an angle and ranges from \( 2M \) to \( 2\pi \). This fact was used by ’t Hooft in [7] to show that time is discrete. We will also adopt this point of view although it is not essential in this paper. Note finally that for small momentum and mass we can expand the ‘mass shell’ relation (25):

\[
H = 2\sqrt{q^2 + M^2} \quad q, M << 1
\]  

which is of course the Klein Gordon equation, only valid in the low energy regime.

3 Curved momentum space and non commuting coordinates

In the previous section we have seen that a convenient set of canonically conjugate coordinates is given by:

\[
q \leftrightarrow r
\]  

\[
\vartheta \leftrightarrow L
\]  

Looking at the mass shell equation we see that \( \frac{\eta}{2}(\equiv \eta) \) only appears as the argument of a sinehyperbolic. It is therefore wise to choose a basis in the Hilbert space upon which \( \sinh \eta \) has a natural action. In much the same spirit as ’t Hooft does in [10] we will introduce a curved momentum space on which we will perform the quantization and study its consequences. The surface we choose as our momentum space is a hyperboloid embedded in \( \mathbb{R}^3 \) as follows:

\[
Q_1 = \sinh \eta \cos \vartheta
\]  

\[
Q_2 = \sinh \eta \sin \vartheta
\]  

\[
Q_3 = \cosh \eta
\]  

\[
Q_3^2 - Q_1^2 - Q_2^2 = 1
\]
Because we like to keep translational invariance in the timelike direction we take
as our energy momentum space $\mathcal{M}_{H,p} = H^2_1 \times S^1$. $H^2_1$ denotes a hyperboloid
with two negative signs and one positive sign in (32). $S^1$ is a circle because the
energy is an angle. To compare, ’t Hooft considered quantization on $S^2 \times S^1$
and $S^3$. Actually quantization on $H^2_2 \sim SL(2, R)$ is also possible [13]. Because
we will do harmonic analysis on the hyperboloid it is useful to remark
that $H^2_2 \sim SO(2,1)/SO(2)$. Fortunately there is a lot of literature on harmonic analysis
on groups and cosets of groups at our disposal. Remember that the group space
is momentum space and not configuration space as is usually considered.

If we define, in addition to the $Q_i$ above, the following $Q_0$:

$$Q_0 = \cos \mu \tan \tilde{H}$$

where $\tilde{H} \equiv \frac{H}{4}$, we find that the mass shell condition becomes the Klein Gordon
equation:

$$Q_0^2 - Q_1^2 - Q_2^2 = \sin^2 \mu$$

which is of course invariant with respect to a Lorentz transformation of $(Q_1, Q_2, Q_0)$

In appendix B we will construct a linear Dirac equation. The fact that this is
indeed a Lorentz vector can be seen from the following:

$$Q_1 = \sinh \eta \cos \vartheta = \sin \mu \sinh \xi \cos \vartheta$$
$$Q_2 = \sinh \eta \sin \vartheta = \sin \mu \sinh \xi \sin \vartheta$$
$$Q_0 = \sqrt{\sin^2 \mu + \sinh^2 \eta} = \sin \mu \cosh \xi$$

A boost is given by $\xi \rightarrow \xi + \varepsilon \xi_0$ and a rotation by $\vartheta \rightarrow \vartheta + \varepsilon \vartheta_0$. While the
boost transformations mix energy and momentum space the rotations leave the
momentum space hyperboloid invariant. They are symmetry operations on the
surface and generated by:

$$L = i\hbar (Q_2 \frac{\partial}{\partial Q_1} - Q_1 \frac{\partial}{\partial Q_2})$$

Although we did not quantize anything yet, we already added the $\hbar$ in the def-
ition of the operator to make it suitable for the quantum theory. Usually we
have two additional translational invariances in momentum space and one in the
direction of the energy, generated by $(\frac{\partial}{\partial p_0}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2})$ These generators will then be
identified with the coordinates $t, x, y$. This is however no longer true due to the
fact that momentum space is a hyperboloid. The operators $i\hbar \frac{\partial}{\partial Q_i}$ would simply
move us out of the energy momentum surface defined by (32). Because we chose
energy momentum space as a direct product of energy and momentum space, the
definition of a time operator (for an off shell particle) is simple:

$$T = i\hbar \frac{\partial}{\partial H}$$

\[\text{This is a Lorentz transformation of the effective particle which is not a Lorentz transfor-
mation of ordinary space time}\]
There are however two more operators that leave momentum space invariant besides $L$. We will call them $X,Y$ for reasons that become clear later:

\begin{align*}
X &= i\hbar(Q_3 \frac{\partial}{\partial Q_1} + Q_1 \frac{\partial}{\partial Q_3}) \quad (40) \\
Y &= i\hbar(Q_3 \frac{\partial}{\partial Q_2} + Q_2 \frac{\partial}{\partial Q_3}) \quad (41)
\end{align*}

We want to argue that these operators are the most natural ones to represent the coordinates $x$ and $y$. Strictly speaking we should define these operators differently. From (27) we have:

\begin{align*}
q_1 &= q \cos \vartheta \quad x = i\hbar \frac{\partial}{\partial q_1} \quad (42) \\
q_2 &= q \sin \vartheta \quad y = i\hbar \frac{\partial}{\partial q_2} \quad (43)
\end{align*}

If we work this out a bit further we have:

\begin{align*}
x &= i\hbar(\cos \vartheta \frac{\partial}{\partial q} - \sin \vartheta \frac{\partial}{\partial \vartheta}) \quad (44) \\
y &= i\hbar(\sin \vartheta \frac{\partial}{\partial q} + \cos \vartheta \frac{\partial}{\partial \vartheta}) \quad (45)
\end{align*}

With respect to the Lorentz invariant measure on the hyperboloid:

\[ d\mu = \sinh \eta \, d\eta \, d\vartheta \quad q = 2\eta \quad (46) \]

these operators are not Hermitian (due to the free hyperbolic angle $q$ in these formulas). It makes more sense to break away from the usual quantization and define $x$ and $y$ as the generators of the symmetry transformations on the hyperboloid \[10,11\] which are Hermitian with respect to this measure. Moreover we think that we have a certain freedom to do this as long as in the low energy limit things converge to the well known quantization schemes. What does actually happen in the low energy limit? Clearly we want that $H, \eta \ll 1$. This implies that we are close to the bottom of the hyperboloid where everything seems flat (see figure (11)). Approximately we have:

\begin{align*}
Q_0 &\simeq \tilde{H} \quad (47) \\
Q_1 &\simeq \eta \cos \vartheta = \eta_1 \quad (48) \\
Q_2 &\simeq \eta \sin \vartheta = \eta_2 \quad (49) \\
Q_3 &\simeq 1 \quad (50)
\end{align*}

\[5\]In this paper we take $8\pi G = 1$ unless it is stated differently.
The mass shell becomes the K.G. equation as we have seen in section 1 (26). Moreover we have:

\[ X \simeq i\hbar \frac{\partial}{\partial \eta_1} \]  \hspace{1cm} (51)  
\[ Y \simeq i\hbar \frac{\partial}{\partial \eta_2} \]  \hspace{1cm} (52)  

So in the limit we have indeed the usual quantization rules.

Next we will show that the operators \( X \) and \( Y \) do transform properly under rotations. For that purpose we will introduce the Planck length \( \ell_P = G \hbar \) (in units where \( c = 1 \)). It implies that the operators \( X, Y \) become proportional to \( \ell_P \) (i.e. \( \hbar \to \ell_P \) in (40,41)). We find:

\[
[L, X] = i\hbar Y \]  \hspace{1cm} (53)  
\[
[L, Y] = -i\hbar X \]  \hspace{1cm} (54)  
\[
(L, P_1) = i\hbar Q_3 = i\sqrt{\hbar^2 + (P_1 \ell_P)^2 + (P_2 \ell_P)^2} \simeq i\hbar(1 + 0(\ell_P^2)) \]  \hspace{1cm} (55)

The commutator between the coordinates \( X \) and \( Y \) is quite peculiar and explains the term ‘non commuting coordinates’ in the title:

\[
[X, Y] = -i\frac{\ell_P^2}{\hbar} L \]  \hspace{1cm} (56)

In the large distance limit (low energy limit) \( \ell_P \ll 1 \) so that the term proportional to \( \ell_P^2 \) is negligible. We find thus that the commutator vanishes there. The non commutativity of the coordinates may be regarded as a result of the peculiar definition of the coordinates that we made. However on very small distances it is not clear if a distance has a precise meaning at all. It is more important that we have proper variables to parametrize our phase space. The interpretation in terms of geometry has probably only meaning for large distances as compared with \( \ell_P \). For this same reason it does not matter what is the precise form of the momenta \( P_1, P_2 \) as long as we have the right low energy limit. Whatever we choose we will never find precisely the same commutators as in the ‘classical’ case. The simplest choice is:

\[
P_i = \frac{\hbar}{\ell_P} Q_i \hspace{1cm} i = 1, 2 \]  \hspace{1cm} (57)

We then find:

\[
[P_i, P_j] = 0 \]  \hspace{1cm} (58)  
\[
[L, P_1] = i\hbar P_2 \]  \hspace{1cm} (59)  
\[
[L, P_2] = -i\hbar P_1 \]  \hspace{1cm} (60)  
\[
[X, P_1] = i\hbar Q_3 = i\sqrt{\hbar^2 + (P_1 \ell_P)^2 + (P_2 \ell_P)^2} \simeq i\hbar(1 + 0(\ell_P^2)) \]  \hspace{1cm} (61)
[X, P_2] = 0 \quad (62)

[Y, P_1] = 0 \quad (63)

[Y, P_2] = i\hbar Q_3 = i\sqrt{\hbar^2 + (P_1 \ell P)^2 + (P_2 \ell P)^2} \simeq i\hbar(1 + \mathcal{O}(\ell^2)) \quad (64)

4 Two particle wavefunctions

Now that we have established what our momentum space looks like and what our ‘quantum coordinates’ are we will investigate the Hilbert space. Because we have non commuting coordinates in configuration space it proves easier to work in momentum space. What are the most natural Hermitian operators to consider on the hyperboloid? The analogon with the sphere might help here. There one takes the Laplacian $\hat{L}^2$ on the sphere together with the angular momentum $L_z$. This is precisely the situation that ’t Hooft considered in [7]. These operators form a complete set of operators on the sphere. We will do the same thing here and consider the Laplace (-Beltrami) operator, $\Delta$, on the hyperboloid together with the angular momentum $L_z$:

$$\Delta = -\hbar^2 \left( \frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} \sinh \eta \frac{\partial}{\partial \eta} + \frac{1}{\sinh^2 \eta} \frac{\partial^2}{\partial \vartheta^2} \right)$$

$$= (\lambda^2 + \frac{1}{4})\hbar^2 \quad \lambda \in [0, \infty) \quad (65)$$

The invariant measure on the hyperboloid is

$$d\mu = \sinh \eta \, d\eta \, d\vartheta \quad (67)$$

With respect to this measure we can write:

$$\lambda^2 + \frac{1}{4} = <\psi|\Delta\psi> = <O\psi|O\psi> = \|O\psi\|^2 \quad (68)$$

where:

$$O = i \frac{\partial}{\partial \eta} + \frac{1}{\sinh \eta} \frac{\partial}{\partial \vartheta} \quad (69)$$

So we can write $\Delta = O^\dagger O$ which proves that $\Delta$ is Hermitian and positive definite. Let us introduce the angular functions:

$$\phi_m(\vartheta) = \frac{1}{\sqrt{2\pi}} e^{im\vartheta} \quad (70)$$

They are of course the eigenfunctions of the Hermitian operator for the angular momentum:

$$L = -i\hbar \frac{\partial}{\partial \vartheta} \Rightarrow L\phi_m(\vartheta) = m\hbar \phi_m(\vartheta) \quad (71)$$
This is the same operator as (38) written in polar coordinates. The possible (discrete) values of \( m \) depend on the boundary condition on \( \phi \). First we will consider the (wrong) boundary condition: \( \psi(\vartheta + 2\pi) = \psi(\vartheta) \) which gives integer valued values for \( m \), i.e. we neglect the cut out, wedgelike region in order to study the influence of the curved momentum space separately. Later we will consider the right boundary condition (4) leading to fractional angular momentum.

We can separate variables in the usual way:

\[
\mathcal{M}_m^\lambda(\eta, \vartheta) = F_m^\lambda(\eta) \phi_m(\vartheta) \quad (72)
\]

So that (63) becomes:

\[
-h^2 \left( \frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} \sinh \eta \frac{\partial}{\partial \eta} - \frac{m^2}{\sinh^2 \eta} \right) F_m^\lambda(\eta) = \left( \lambda^2 + \frac{1}{4} \right) h^2 F_m^\lambda(\eta) \quad \lambda \in [0, \infty) \quad (73)
\]

The eigenvalues can be found by the following substitution:

\[
\tanh^2 \eta = z \quad (74)
\]

\[
\sinh^2 \eta = \frac{z}{1 - z}
\]

\[
\cosh^2 \eta = \frac{1}{1 - z}
\]

and

\[
F_m^\lambda(z) = z^{\frac{\lambda}{2}} \left( 1 - z \right)^{\frac{1}{2} \left( \frac{1}{2} - i\lambda \right)} V_m^\lambda(z) \quad (75)
\]

The Laplace equation for \( F_m^\lambda(z) \) is then mapped to a hypergeometric equation for \( V_m^\lambda(z) \) from which we can read off the solution (see [18]):

\[
F_m^\lambda(\eta) = \left| \frac{\Gamma\left[\frac{1}{2}(\vert m \vert + i\lambda + \frac{1}{2})\right]\Gamma\left[\frac{1}{2}(\vert m \vert + i\lambda + \frac{3}{2})\right]}{\sqrt{2\pi} \Gamma(i\lambda) \Gamma(\vert m \vert + 1)} \right| \quad (76)
\]

\[
\tanh \eta \vert m \vert \cosh \eta \left( \frac{1}{2} - i\lambda \right) \quad F_m^\lambda(\vert m \vert - i\lambda + \frac{1}{2}; \frac{1}{2}(\vert m \vert - i\lambda + \frac{3}{2}), \vert m \vert + 1; \tanh^2 \eta)
\]

Altogether we have a complete orthonormal basis for our Hilbert space: \( \{ \mathcal{M}_m^\lambda(\eta, \vartheta) = F_m^\lambda(\eta) \phi_m(\vartheta) \} \):

\[
\int_0^{2\pi} \int_0^\infty \sinh \eta \, d\eta \, d\vartheta \, \mathcal{M}_m^\lambda(\eta, \vartheta) \overline{\mathcal{M}_{m'}^\lambda(\eta, \vartheta)} = \delta(\lambda - \lambda') \delta_{m, m'} \quad (77)
\]

\[
\sum_{m = -\infty}^\infty \int_0^{2\pi} \frac{d\lambda}{2\pi} \mathcal{M}_m^\lambda(\eta, \vartheta) \overline{\mathcal{M}_m^\lambda(\eta', \vartheta')} = \delta(\cosh \eta - \cosh \eta') \delta(\vartheta - \vartheta') \quad (78)
\]

We can use this basis to define a Fourier transform from momentum space to configuration space and back:

\[
\hat{\psi}(\lambda, m) = \int_0^\infty \int_0^{2\pi} \sinh \eta \, d\eta \, d\vartheta \, \psi(\eta, \vartheta) \mathcal{M}_m^\lambda(\eta, \vartheta) \quad (79)
\]

\[
\psi(\eta, \vartheta) = \sum_{m = -\infty}^\infty \int_0^{2\pi} \frac{d\lambda}{2\pi} \hat{\psi}(\lambda, m) \mathcal{M}_m^\lambda(\eta, \vartheta) \quad (80)
\]
The coordinates that parametrize configuration space are the angular momentum $m$ and the spectral parameter $\lambda$. But what does this $\lambda$ mean physically? From section two we have:

$$\frac{\Delta}{\hbar^2} = -\frac{L_z^2}{\hbar^2} + \frac{X^2}{\ell_P^2} + \frac{Y^2}{\ell_P^2} = \lambda^2 + \frac{1}{4}$$  \hfill (81)

Defining the operator for the distance between the particles: $R^2 = X^2 + Y^2$ we find:

$$R^2 = (\frac{\Delta}{\hbar^2} + \frac{L_z^2}{\hbar^2})\ell_P^2 = (m^2 + \lambda^2 + \frac{1}{4})\ell_P^2 \geq 0$$  \hfill (82)

Notice that for the classical limit we have

$$R \sim \lambda \ell_P \quad \lambda^2 >> (m^2 + \frac{1}{4})$$  \hfill (83)

In the classical limit (low energy limit) we therefore find that $\lambda$ represents the distance between the particles. Notice furthermore that there is a smallest distance in our theory:

$$R \geq R_{\text{min}} = \frac{1}{2} \ell_P$$  \hfill (84)

This implies that the particles can never get closer to one another than $\frac{1}{2} \ell_P$. This number thus serves as a natural cutoff in our theory.

Let us return to the point where we had to choose a boundary condition for the function $\phi_m(\vartheta)$. The correct boundary condition is of course:

$$\psi(\vartheta + 2\pi) = e^{imE} \psi(\vartheta)$$  \hfill (85)

giving rise to the quantization condition:

$$m = \frac{2\pi n}{2\pi - E} \equiv \frac{n}{\alpha} \quad n \in \mathbb{Z}$$  \hfill (86)

This gives for $\phi_m(\vartheta)$:

$$\phi_m(\vartheta) = e^{\frac{im\vartheta}{\alpha}}$$  \hfill (87)

Let us consider an effective particle on the mass shell and with a definite direction $\vartheta$. The solution in momentum space is:

$$\psi_E(\eta, \vartheta) = \delta(\cosh \eta - \cosh \eta_0)\delta(\vartheta - \vartheta_0)$$  \hfill (88)

$$\cosh \eta_0 = \frac{\cos \mu}{\cos(\frac{\mu}{4})}$$  \hfill (89)

What does this solution look like in our configuration space? Because we consider fixed energy the value of $\alpha$ is fixed. In order to find a complete set of
eigenfunctions on this energy surface we solve the Laplace Beltrami equation on hyperboloid with a wedgelike region cut out (see figure (1)).

So we have to solve again (73) with $m = \frac{n}{\alpha}$. This is done by the same substitutions (74,75) and the solution is again (76) but with $m$ replaced by $\frac{n}{\alpha}$. This is again a complete set of eigenfunctions and we will use them to convert solution (88) to the configuration space:

$$
\psi_E(\lambda, n) = \int_1^{\infty} \int_0^{2\pi} d\cosh \eta \, d\vartheta \, \delta(\cosh \eta - \cosh \eta_0) \, \delta(\vartheta - \vartheta_0) \, F_{\frac{n}{\alpha}}^\lambda(\eta) e^{i\frac{\lambda}{\alpha} \vartheta_0} (92)
$$

$$
= F_{\frac{n}{\alpha}}^\lambda(\eta_0) e^{i\frac{\lambda}{\alpha} \vartheta_0} (93)
$$

Notice that these functions are the analogon of the fractional Besselfunctions

$^6$What we mean with an effective particle is explained in the text below equation 24

$^7$It is easily shown that equation (73) is of the Schrodinger (or Sturm Liouville) type by the following substitution: $\phi = \sqrt{\sinh \eta} \, F$. The equation then becomes:

$$
- \frac{\partial^2}{\partial \eta^2} \phi + \left( \frac{1}{4} \cosh^2 \eta + m^2 - \frac{1}{2} \right) \phi = (\lambda^2 + \frac{1}{4}) \phi (90)
$$

This equation has a complete set of eigenfunctions given by the hypergeometric solutions (76) with $m = \frac{n}{\alpha}$. In the limit where $\eta \gg 1$ we have, by using the transformation formula's for the hypergeometric function, the following asymptotic behaviour:

$$
F_{\frac{n}{\alpha}}^\lambda(\eta) \simeq e^{-\frac{1}{2} \eta} \left\{ A(\lambda, \frac{n}{\alpha}) e^{i\lambda \eta} + B(\lambda, \frac{n}{\alpha}) e^{-i\lambda \eta} \right\} (91)
$$

Keeping in mind the measure $\sinh \eta \, d\eta$ these functions are normalizable as wave packets. We also see from this that the spectrum of $\Delta$ really starts at $\frac{1}{4}$ because for for smaller values $\lambda$ becomes imaginary and one of the terms in (91) blows up.
What if we consider an effective particle with different energy? For this particle the deficit angle on the hyperboloid is different. This means that we cannot use the same set of eigenfunctions as was used before. Following the same calculation we have for this particle in configuration space:

\[ \psi_{E'}(\lambda, n) = F_{\frac{\lambda}{\sigma'}}(\eta'_0) e^{i\frac{2\pi}{\eta_0}} \]  

And thus the disaster happens that \( \psi_E \) and \( \psi_{E'} \) are not orthogonal. We cannot construct a superposition of wavefunctions with different energy so we cannot construct wavepackets in this theory. This situation is of course no different than in the case where one used flat momentum space and found the same problem for the fractional Besselfunctions. We think however that this problem is not stressed sufficiently in the literature. We are not able to solve this problem now but are convinced that a in full theory of 2+1 quantum gravity this issue must be adressed. What we can do with the present understanding is calculate scattering amplitudes, but ironically this will be the same in the case of curved and flat momentum space. Because we fixed the the value of the momentum \( \eta = \eta_0 \), the distance \( \mathcal{R} \) between the two particles is completely unknown by the Heisenberg uncertainly relations. In other words, because we do not know how to superpose wavefunctions with different energy it is impossible to construct localized wavepackets and thus to conclude anything about the spectrum of the \( \mathcal{R} \) operator. We expect however that the prediction of a smallest distance will survive in the full theory as this is connected with the curvature in momentum space and not with the problems mentioned above.

5 Discussion

In this paper we found that the conjugate momentum to the relative distance between two particles in 2+1 dimensional gravity is a hyperbolic angle. This caused us to consider quantization on a hyperbolic momentum space. The result of this was that the model has a shortest distance built in which serves as a natural cutoff in the theory. The fact that curvature in momentum space can regularize the theory was already suggested by Snyder in 1947 [17]. The beautiful idea behind this is that the model can still be covariant under the full group of Lorenztransformations. The price one has to pay is the introduction of non commuting coordinates. Also translational invariance is lost. In this paper we kept translational invariance in the timelike direction but introduced non commuting coordinates \( X \) and \( Y \). Because the Hamiltonian is an angle we have discrete time. The distance between the particles has a continuous spectrum but starts at \( R_{\text{min}} = \frac{1}{2} \ell_p \).

The boundary conditions on the wavefunctions prevented us from writing down a complete set of orthonormal basisfunctions in the Hilbert space. The
problem is that wavefunctions with different energy obey different boundary conditions and are therefore not orthogonal. In future work we hope to address this issue.

The hope is of course that the introduction of a curved momentum space can act as a covariant cutoff in a more advanced field theory, ultimately resulting in a finite theory for quantum gravity.

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Appendix A: Massless particles

A *massive* particle at position $a$ is described by the following identification rule over a wedge:

$$x' = a + e^{ip^a J_a}(x - a) \quad (95)$$

where:

$$p^a = (M \cosh \xi, M \sinh \xi \cos \vartheta, M \sinh \xi \sin \vartheta) \quad (96)$$

In order to describe a massless particle we take the limit $M \to 0$ and $\xi \to \infty$ in such a way that:

$$p^a \to 2(\sigma, \sigma \cos \vartheta, \sigma \sin \vartheta) \equiv 2\sigma^a \quad (97)$$

The meaning of the parameter $2\sigma$ will become clear in a moment. Clearly we have that $\sigma^a$ is a lightlike vector: $\sigma^a \sigma_a = 0$. Notice also that the particle moves with the speed of light: $v = \tanh \xi = 1$. The energy of this particle is given by the deficit angle of the wedge that we have cut out. From [7] we have the following relations:

$$\tan \frac{H}{2} = \cosh \xi \tan \frac{M}{2} \to \sigma \quad (98)$$

$$\sin \frac{H}{2} = \frac{\tanh \eta}{\tanh \xi} \to \tanh \eta \quad (99)$$

$$\cos \frac{H}{2} = \frac{\cos \frac{M}{2}}{\cosh \eta} \to \frac{1}{\cosh \eta} \quad (100)$$

$$\sinh \eta = \sin \frac{M}{2} \sinh \xi \to \sigma \quad (101)$$

So we must take $\sigma = \sinh \eta$ where $\eta$ represents the rapidity with which the boundary of the wedge moves (see figure (2)). In the low energy limit we have $H = 2\eta$ where $2\eta$ is the momentum canonically conjugate to the edgelength $L$. 

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Figure 2: A ‘gravitating’ particle in 2+1 dimensions deforms space time by cutting out a wedgelike region.

which in the limit for small deficit angle is just the distance from the origin to the particle. This is of course the correct dispersion relation for a massless particle.

If we want to repeat the calculation of section one in the massless case we must take the limit (97) in all formula’s. The angle between the two particles in their c.o.m. frame is then for instance:

$$\cos \varphi = \frac{\sigma^2 - 1}{\sigma^2 + 1}$$  \hspace{1cm} (102)

and the total energy becomes:

$$\cos \frac{H}{4} = \frac{1}{\cosh \eta} = \frac{1}{\cosh \frac{q}{2}}$$  \hspace{1cm} (103)

Notice that in the limit for small energy and momentum we find the correct dispersion $H = 2q$.

What does this massless limit imply for section two? Because we still want to quantize on the hyperboloid, the definition of $Q_1$, $Q_2$ and $Q_3$ does not change. For $Q_0$ we now have:

$$Q_0 = \tan \tilde{H}$$  \hspace{1cm} (104)

This definition indeed gives the massless Klein Gordon equation:

$$Q_0^2 - Q_1^2 - Q_2^2 = 0$$  \hspace{1cm} (105)

What we actually find is that we can safely take the limit $\mu \to 0$ in all the formula’s.

**Appendix B: The Dirac equation**

In this appendix we will derive a Dirac equation that is linear in the $Q'$s. The derivation is valid for one particle in the polygon approach (with variables $L, \eta$)
and for two particles in their center of mass frame. The starting point is the following set of equations:

\[
\begin{align*}
-Q_0^2 + Q_1^2 + Q_2^2 &= -\sin^2 \mu \\
-Q_3^2 + Q_1^2 + Q_2^2 &= -1 \\
Q_3^2 - Q_0^2 &= \cos^2 \mu
\end{align*}
\]

We would like an equation that pushes time forward by one unit of time, i.e. we are looking for an equation of the form:

\[e^{i\hat{H}}\psi = S\psi\]

where \(S\) is a unitary matrix in SU(2). We know from (25) that:

\[
\begin{align*}
Q_3 \cos \hat{H} &= \cos \mu \Rightarrow \quad (B.5) \\
Q_3 \sin \hat{H} &= \sqrt{Q_3^2 - \cos^2 \mu} = Q_0 \\
&= \sqrt{Q_1^2 + Q_2^2 + \sin^2 \mu} \quad (B.6)
\end{align*}
\]

Combining these equations gives:

\[
Q_3 e^{i\hat{H}}\psi = (\cos \mu + i\sqrt{Q_1^2 + Q_2^2 + \sin^2 \mu})\psi \quad (B.8)
\]

Using Dirac’s trick to linearize this equation gives:

\[
Q_3 e^{i\hat{H}}\psi = (\cos \mu + i\beta \sin \mu + i\alpha_1 Q_1 + i\alpha_2 Q_2)\psi \quad (B.9)
\]

Here we can take the Dirac matrices to be the Pauli matrices: \(\beta = \sigma_3\) and \(\alpha_i = \sigma_i\). In matrix notation:

\[
e^{i\hat{H}}\psi = \frac{1}{\cosh \eta} \left( \begin{array}{cc} e^{i\mu} & i \sinh \eta \ e^{-i\alpha} \\ i \sinh \eta \ e^{i\alpha} & e^{-i\mu} \end{array} \right) \psi \quad (B.10)
\]

Clearly the right hand side is an element of SU(2) which assures conservation of probability. It is also clear that this equation contains particles and anti particles as a solution which will become part of our model if we use this equation.

What about covariance? Using (B.6) we can write:

\[
(Q_3 \cos \hat{H} + iQ_0)\psi = \cos \mu \psi + i(\sigma_1 Q_1 + \sigma_2 Q_2 + \sigma_3 \sin \mu)\psi \quad (B.11)
\]

We can rewrite this as:

\[
\sigma_3(Q_3 \cos \hat{H} - \cos \mu)\psi - i(\gamma^a Q_a + \sin \mu)\psi = 0 \quad (B.12)
\]

We take here \(\hat{H} = \frac{\hat{P}}{4}\) in the case of two particles in their c.o.m. frame and \(\hat{H} = \frac{\hat{P}}{2}\) in the case of one particle using polygon variables. Accordingly we rescale time in both cases to \(\hat{t} = 4t\) and \(\hat{t} = 2t\) respectively.
where we have defined: $\gamma^0 = -\sigma_3$ and $\gamma^i = \sigma_3 \sigma_i$. If $\psi$ transforms according to the two dimensional representation of the Lorentzgroup SU(1,1) and $(Q_0, Q_1, Q_2)$ transforms as a Lorentzvector under SO(2,1) we can derive with the usual argument that the second term is covariant. However, the first term does not transform covariantly as we have:

$$Q_3 = \sqrt{\cos^2 \mu + \sin^2 \mu \cosh^2 \xi}$$  \hspace{1cm} (B.13)

and a boost is generated by $\xi \to \xi + \varepsilon \xi_0$. We must conclude that the Dirac equation is only Lorentz invariant on the mass shell: $Q_3 \cos \bar{H} = \cos \mu$.

Another important point is locality of the equation in configuration space. The matrix on the right hand side has by no means a ‘local action’ on the states $F^k_m(\eta) \phi_m(\vartheta)$. So it is not clear if this equation is an improvement over the Klein Gordon equation in that respect.

Still another difficulty is the implementation of the boundary condition. If we fix the energy as in section 3 we have to consider this equation on a hyperboloid with a wedge cut out. The action of $Q_3$ is still well defined but what about $Q_1$ and $Q_2$? They contain $\cos \vartheta$ and $\sin \vartheta$ while the basis is built from the angular functions: $\phi_n(\vartheta) = \frac{1}{\sqrt{2\pi}} e^{i \alpha n \vartheta}$. The action of the operators $Q_{\pm} \equiv Q_1 \pm iQ_2$ on these states is:

$$Q_{\pm} \phi_n(\vartheta) = \sinh \eta \frac{1}{\sqrt{2\pi}} e^{i(\frac{\pm}{2} + 1) \vartheta}$$  \hspace{1cm} (B.14)

These states are however no longer in the basis set, so that $Q_{\pm}$ throws them out of the Hilbert space. Clearly if we want to take the boundary condition into account correctly we need to modify this Dirac equation. For one particle described by polygon variables one can view this Dirac equation as an analogon of the Dirac equation derived by ’t Hooft in [16].

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