Pointwise nonlinear stability of nonlocalized modulated periodic reaction-diffusion waves

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Abstract

In this paper, extending previous results of [J1], we obtain pointwise nonlinear stability of periodic traveling reaction-diffusion waves, assuming spectral linearized stability, under nonlocalized perturbations. More precisely, we establish pointwise estimate of nonlocalized modulational perturbation under a small initial perturbation consisting of a nonlocalized modulation plus a localized perturbation decaying algebraically.

1 Introduction

We consider a system of reaction-diffusion equations

\[ u_t = u_{xx} + f(u), \]  

where \((x,t) \in \mathbb{R} \times \mathbb{R}^+\), \(u \in \mathbb{R}^n\) and \(f : \mathbb{R}^n \to \mathbb{R}^n\) sufficiently smooth. We assume that \(u(x,t) = \bar{u}(x - ct)\) is a traveling wave solution of the system (1.1) with a constant speed \(c\) and the profile \(\bar{u}(\cdot)\) satisfies \(\bar{u}(\cdot) = \bar{u}(\cdot + 1)\). In other words, \(\bar{u}(x)\) is a stationary 1-periodic solution of the PDE

\[ \bar{u}_t = \bar{u}_{xx} + c\bar{u}_x + f(\bar{u}). \]  

In [J1], the first author established pointwise Green function bounds on the linearized operator about the underlying solution \(\bar{u}\) and obtained pointwise nonlinear stability of \(\bar{u}\) by estimating the localized modulational perturbation \(v(x,t) = \tilde{u}(x - \psi(x,t), t) - \bar{u}(x)\) \((h_0 = \psi(x,0) = 0)\) under small initial perturbations \(v(x,0) = \tilde{u}(x,0) - \bar{u}(x)\) decaying algebraically for nearby solutions \(\tilde{u}\) of (1.2).

In the present paper, we study the pointwise nonlinear stability of \(\bar{u}\) of (1.2) under small perturbations consisting of a nonlocalized modulation \((h_0(x) = \psi(x,0))\) does not

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decay algebraically, but $\partial_x h_0$ decays algebraically) plus a localized perturbation $v(x,0) = \tilde{u}(x-h_0(x),0) - \bar{u}(x)$ ($v(x,0)$ decays algebraically). Johnson, Noble, Rodrigues and Zumbrun showed $L^p$-nonlinear stability under such nonlocalized modulational perturbations ($h_0 \notin L^1$, but $\partial_x h_0 \in L^1$) for systems of reaction-diffusion equations in [JNRZ1] and of conservation laws in [JNRZ3]. Sandsteds, Scheel, Schneider, and Uecker obtained similar results by rather different methods for systems of reaction-diffusion equations in [SSSU].

Similarly as in [JNRZ1, JNRZ3], here, we determine an appropriate nonlocalized modulation $\psi(x,t)$ by an adaption of the basic nonlinear iteration scheme developed in [JZ]. However, in the absence of cancellation estimates afforded by Hausdorff-Young and Parseval inequalities, we find it necessary to decompose the solution a bit differently than was done in [JNRZ1] in order to estimate sharply the key “modulation” part of the linearized solution operator in response to modulational-type data (see Remark 3.2), and to estimate this modulational part essentially “by hand.” This is the main new difficulty in our analysis beyond those carried out in [J1, JNRZ1].

1.1 Preliminaries

We first linearize the PDE (1.2) about a stationary 1-periodic solution $\bar{u}$ so that we obtain the eigenvalue problem

$$\lambda v = Lv := (\partial_x^2 + c \partial_x + df(\bar{u}))v,$$

operating on $L^2(\mathbb{R})$ with densely defined domains $H^2(\mathbb{R})$. Here, $v$ is considered as a perturbation of $\bar{u}$ defined by $v(x,t) = \bar{u}(x,t) - \bar{u}(x)$ for nearby solutions $\bar{u}$. To characterize the $L^2(\mathbb{R})$-spectrum of $L$ (denoted by $\sigma_{L^2(\mathbb{R})}(L)$), we rewrite (1.3) as the following linear ODE system

$$V_x = \mathbb{A}V, \quad \text{where} \quad V = \begin{pmatrix} v \\ v_x \end{pmatrix} \quad \text{and} \quad \mathbb{A} = \mathbb{A}(x,\lambda) = \begin{pmatrix} 0 & I \\ \lambda I - df(\bar{u}) & -cI \end{pmatrix}.$$  

Since all coefficients of $L$ are 1-periodic, $\mathbb{A}(x+1,\lambda) = \mathbb{A}(x,\lambda)$; so by Floquet theory, the fundamental matrix solution $\Phi(x,\lambda)$ of the linear system (1.4) is

$$\Phi(x,\lambda) = P(x,\lambda)e^{R(\lambda)x},$$

where $R(\lambda) \in \mathbb{C}^{n \times n}$ is a constant matrix and $P(x,\lambda) \in \mathbb{C}^{n \times n}$ is a periodic matrix, $P(x,\lambda) = P(x+1,\lambda)$. In fact, for each eigenvalue $\mu$ (referred to as the Floquet exponent) of $R(\lambda)$, there is a solution to (1.4) of the form $V(x,\lambda) = e^{i\mu x} W(x,\lambda)$, where $W$ is 1-periodic in $x$. Thus, any non-trivial solution $V$ to the system (1.4) does not lie in $L^2(\mathbb{R})$, which means that the $L^2(\mathbb{R})$-spectrum of the linear operator $L$ must be entirely essential. Moreover, $\lambda \in \sigma_{L^2(\mathbb{R})}(L)$ if and only if $R(\lambda)$ is not hyperbolic; thus there is a solution to (1.3) of the form $v = e^{i\xi x} w(x)$ for some neutral eigenvalue $i\xi \in \sigma(R(\lambda))$ and 1-periodic function $w$.

Here, $\xi \in [-\pi, \pi)$ is uniquely defined mod $2\pi$. Plugging $v(x) = e^{i\xi x} w(\xi, x)$ into (1.3), we define the Bloch operators, for each $\xi \in [-\pi, \pi)$,

$$L_\xi := (\partial_x + i\xi)^2 + c(\partial_x + i\xi) + df(\bar{u})$$
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operating on \(L^2_{\text{per}}([0, 1])\) with densely defined domain \(H^2_{\text{per}}([0, 1])\). Indeed, \(L_\xi\) satisfies \(L_\xi w = \lambda(\xi)w\) and \(w\) is 1-periodic in \(x\). Moreover, for each \(\xi \in [-\pi, \pi]\), \(L_\xi\) has a compact resolvent and \(\sigma_{L^2(R)}(L) = \bigcup_{\xi \in [-\pi, \pi]} \sigma_{L^2_{\text{per}}([0, 1])}(L_\xi)\). In other words, for each \(\xi \in [-\pi, \pi]\), \(L_\xi\) operating on \(L^2_{\text{per}}([0, 1])\) has only point spectrum, whereas \(L\) operating on \(L^2(R)\) has only essential spectrum.

We now define the standard diffusive spectral stability conditions (following [S1, S2]). We first notice that 0 is an eigenvalue of \(L_0\) because \(\bar{u}' = 0\) and \(\bar{u}'\) is 1-periodic. Throughout our analysis we assume the following conditions:

(D1) \(\sigma_{L^2(R)}(L) \subset \{ \lambda \in \mathbb{C} : R(\lambda) < 0 \} \cup \{ 0 \}\).

(D2) \(\lambda = 0\) is a simple eigenvalue of \(L_0\).

(D3) There exists a constant \(\theta > 0\) such that \(R\sigma(L_\xi) \leq -\theta |\xi|^2\) for all \(\xi \in [-\pi, \pi]\).

As we mentioned above, 0 is not an isolated eigenvalue of \(L\) but a member of a continuous curve of essential spectrum, so there is no spectral gap between 0 and the rest of the spectrum. This is the reason why we could not use the stability methods used for other types of traveling reaction diffusion waves such as front or pulse. This difficulty was overcome in Swift-Hohenberg equation in [S1, S2] by using the above diffusive spectral stability. Moreover, from the above three conditions, the eigenvalue of \(L_\xi\) bifurcating from 0 at \(\xi = 0\) has the following expression (see [J1, JNRZ1])

(1.6) \(\lambda(\xi) = -ia\xi - b|\xi|^2 + O(|\xi|^3)\) for sufficiently small \(|\xi|\),

where \(a \in \mathbb{R}\) and \(b > 0\).

1.2 Bloch transform

We now recall the Bloch transform, as described for example in [J1, JNRZ1, JNRZ2, JNRZ3]. By the inverse Fourier transform, we have for any \(g \in L^2(R)\),

\(g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} \hat{g}(\xi, x) d\xi,\)

where \(\hat{g}(\xi, x) = \sum_{j \in \mathbb{Z}} e^{i2\pi jx} \hat{g}(\xi + 2\pi j)\) (referred to as the Bloch transform) and \(\hat{g}(\cdot)\) denotes the Fourier transform of \(g\) with respect to \(x\). By the definition of \(L_\xi\) in (1.5), \(L(e^{i\xi x} f) = e^{i\xi x}(L_\xi f)\) for periodic functions \(f\) and \(\hat{g}(\xi, x)\) is 1-periodic in \(x\); so we have the Bloch solution formula for the linear operator \(L\) in (1.3)

(1.7) \(S(t)g(x) := e^{Lt}g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} e^{L_{\xi t} \hat{g}}(\xi, x) d\xi,\)

for any \(g \in L^2(R)\).
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1.3 Main result

With these preparations, we state the main theorem of this paper. Here, and throughout
the paper, \( h_\infty \) and \( h_-\infty \) denote the end states of the initial modulation \( h_0(x) \) as \( x \to \infty \)
and \( x \to -\infty \), respectively, and \( h_{\pm \infty} \) denotes a piecewise constant function defined by

\[
(1.8) \quad h_{\pm \infty}(x) := \begin{cases} 
  h_-\infty, & x \leq 0; \\
  h_+\infty, & x > 0.
\end{cases}
\]

**Theorem 1.1.** Suppose that a stationary 1-periodic solution \( \tilde{u}(x) \) of (1.2) satisfies spectral
stability conditions (D1) \( \sim \) (D3). For \( r \geq \frac{3}{2} \) and sufficiently small \( E_0 > 0 \), we assume that
the initial data \( \tilde{u}_0(x) \) and \( h_0(x) \) satisfy

\[
(1.9) \quad |\tilde{u}_0(x - h_0(x)) - \tilde{u}(x)| \leq E_0(1 + |x|)^{-r}, \quad \text{and} \quad |h_+\infty| = |h_-\infty| \leq E_0 \quad \text{and} \quad v_0 := \tilde{u}_0(x - h_0(x)) - \tilde{u}(x) \in H^2(\mathbb{R}).
\]

Then for all initial data \( \tilde{u}_0 \) satisfying (1.9), the corresponding solution \( \tilde{u}(x,t) \) to (1.2)
satisfies

\[
(1.10) \quad |\tilde{u}(x - \psi(x,t),t) - \tilde{u}(x)| \leq CE_0 \left[ (1 + |x - at| + \sqrt{t})^{-r} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{M' t^p}} \right]
\]

and

\[
(1.11) \quad |\partial_x\psi(x,t)|, |\partial_t\psi(x,t)| \leq CE_0 \left[ (1 + |x - at| + \sqrt{t})^{-r} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{M' t^p}} \right],
\]

for an appropriate modulation \( \psi(x,t) \in W^{2,\infty} \) with \( \psi(x,0) = h_0(x) \) (which is determined in
Section 4). Here, \( M' > M > 0 \) is a sufficiently large number (\( M \) denotes the constant in
Theorem 2.1) and the constant \( a \) is from (1.6).

**Remark 1.2.** This extends the result of [J1] to nonlocalized modulations. If \( h_0(x) \) decays
algebraically, we obtain the same result as [J1]. Here, compared with the localized case
[J1], one can see that nonlocalized modulated perturbations decay at a slower, heat kernel
rate. Moreover, the initial data conditions (1.9) satisfy the conditions in [JNRZ1], so that
integrating the bound in (1.10) with respect to \( x \) gives the same \( L^p \)-bound as in [JNRZ1]
which is \( (1 + t)^{-\frac{1}{2} (1 - \frac{1}{p})} \) for all \( 2 \leq p < \infty \).

**Remark 1.3.** Here, without loss of generality, one can assume \( h_-\infty = -h_+\infty \). If not,
\( h_-\infty + c = -(h_+\infty + c) \) with \( c = -\frac{h_+\infty + h_-\infty}{2} \). This assumption is for handling nonlocalized
functions in the Bloch transform framework.

**Remark 1.4.** For the algebraically decaying data, \((1 + |x|)^{-r} \) with \( r > 1 \) is enough for
linear stability. However, we need \( r \geq \frac{3}{2} \) for nonlinear stability when we use the nonlinear
iteration scheme in Section 4.
1.4 Nonlinear perturbation equations and outline of the analysis

We first look at the nonlinear equation of modulated perturbations of $\tilde{u}$ to obtain the strategy of this paper. As mentioned in the previous section, we define the modulated perturbations

$$v(x,t) = \tilde{u}(x - \psi(x,t), t) - \bar{u}(x)$$

for nearby solutions $\tilde{u}(x,t)$ to (1.2) and unknown function $\psi(x,t) : \mathbb{R}^2 \to \mathbb{R}$ with $\psi(x,0) = h_0(x)$ to be determined in Section 4. This is exactly what we want to estimate in the main theorem; so, we first state the nonlinear perturbation equation about $v$ which is already established in [JZ, JNRZ1].

**Lemma 1.5** (Nonlinear perturbation equations, [JZ, JNRZ1]). For $v$ defined in (1.12) and the linear operator $L$ in (1.3), we have

$$(\partial_t - L)v = -(\partial_t - L)\tilde{u}(x)\psi + Q + R_x + (\partial_x^2 + \partial_t)Z + T,$$

where

$$Q := f(v(x,t) + \bar{u}(x)) - f(\bar{u}(x)) - df(\bar{u}(x))v = O(|v|^2),$$

$$R := -v\psi_t - v\psi_{xx} + (\bar{u}_x + v_x)\frac{\psi_x^2}{1 - \psi_x},$$

$$Z := v\psi_x = O(|v||\psi_x|),$$

and

$$T := -(f(v + \bar{u}) - f(\tilde{u})) \psi_x = O(|v||\psi_x|).$$

We now briefly give the plan of this paper. Setting $v_0(x) = \tilde{u}_0(x - h_0(x)) - \bar{u}(x)$ and $N(x,t) = (Q + R_x + (\partial_x^2 + \partial_t)Z + T)(x,t)$ in (1.13), by Duhamel’s principle, we have

$$v(x,t) = -\tilde{u}(x)\psi(x,t) + e^{Lt}(v_0 + \tilde{u}'h_0) + \int_0^t e^{L(t-s)}N(s)ds.$$  \hspace{1cm} (1.18)

For localized data $v_0$, in order to estimate $e^{Lt}v_0$, we use the pointwise Green function bounds obtained in [J1]; so we first recall one of the main theorems of [J1] in Section 2. Since we defined $\psi(x,t)$ with $h_0(x) = \psi(x,0) = 0$ in [J1], the main new ingredient in this paper compared to [J1] is the pointwise linear behavior under modulational data $\tilde{u}'h_0$. In other works, the main difficulty here is to estimate $e^{Lt}(\tilde{u}'h_0)$ in terms of the localized data $|\partial_x h_0|$, $|\partial_x^2 h_0|$ or $|h_0 - h_{\pm\infty}|$ in Section 3. After we estimate the linear level, we define an appropriate $\psi(x,t)$ with $\psi(x,0) = h_0(x)$; so we finally obtain pointwise bounds of $v$ by the nonlinear iteration scheme in Section 4 and Section 5.
1.5 Discussion and open problems

Compared with [JNRZ1] (Lp-stability estimates for nonlocalized modulations), the assumptions $|\tilde{u}_0(x - h_0(x)) - \tilde{u}(x)|, \sum_{k=1,2} |q^k h_0(x)| \leq E_0 (1 + |x|)^{-r}$ in (1.9) are very natural for pointwise estimates. However, the assumptions on $h - h_{\pm \infty}$ might appear unfamiliar. The reason for these is that it is still an open problem how to establish pointwise estimates on the linearized solution operator directly from the Bloch representation

\[
e^{Lt} g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} e^{L\xi} \tilde{g}(\xi, x) d\xi,
\]

for $|x| >> Ct$ (sufficiently large $C > 0$) even for the localized data $g$. These additional assumptions allow us to obtain estimates by a different route. This is also the reason why we obtained pointwise Green function bounds $G(x, t; y)$ on the linear operator $L$ in [J1, J2] (the cases of localized modulations), without use of the Bloch representation for $|x - y| >> Ct$. Thus, if we could find out how to handle the Bloch solution operator for $|x| >> Ct$, this would be a nice improvement both in Theorem 1.1, and in the analysis of the previous work [J1, J2].

Another, very interesting, open problem is to determine not only pointwise derivative decay of the modulation $\psi$, but also its pointwise behavior to lowest order, similarly as done in the Lp context in [JNRZ2]. Indeed, this might be a route also to the elimination of hypotheses on decay of $h - h_{\pm \infty}$, since subtracting off this principal behavior would leave only localized terms more amenable to pointwise estimates. However, as noted earlier, our definition of the phase, being adapted to the pointwise analysis, is somewhat different from that in [JNRZ1, JNRZ2], and so we cannot immediately apply the earlier analysis to obtain such a result.

2 Linear estimates for the localized data $v_0$

In this section, we recall the pointwise Green function bounds of the linear operator $L$ from [J1].

**Theorem 2.1** (Pointwise Green function bounds, [J1]). The Green function $G(x, t; y)$ for the evolution equations $(\partial_t - L)v = 0$ for linear operator (1.3) satisfies the estimates:

\[
G(x, t; y) = \tilde{u}'(x) E(x, t; y) + \tilde{G}(x, t; y),
\]

where

\[
E(x, t; y) = \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-y-\alpha t|^2}{4bt}} \tilde{q}(y, 0) \chi(t),
\]

\[
|\tilde{G}(x, t; y)| \lesssim \left( (1 + t)^{-1} + t^{-\frac{1}{2}} e^{-\eta t} \right) e^{-\frac{|x-y-\alpha t|^2}{4Mt}}
\]

and

\[
|\tilde{G}_y(x, t; y)| \lesssim t^{-1} e^{-\frac{|x-y-\alpha t|^2}{4Mt}},
\]
uniformly on $t \geq 0$, for some sufficiently large constant $M > 0$ and $\eta > 0$. Here $\hat{q}$ is the periodic left eigenfunction of $L_0$ at $\lambda = 0$ and $\chi(t)$ is a smooth cutoff function such that $\chi(t) = 0$ for $0 \leq t \leq \frac{1}{2}$ and $\chi(t) = 1$ for $t \geq 1$.

**Remark 2.2.** In the main theorem in [J1], the Green function $G(x, t; y)$ has no cutoff function $\chi(t)$. However, there is no difference between Theorem 2.1 and the original theorem in [J1] because $(1 - \chi(t))\tilde{u}'(x)\frac{1}{\sqrt{4\pi t}}e^{\frac{|y-x-at|^2}{4bt}}\tilde{q}(y, 0)$ is included in $\tilde{G}(x, t; y)$.

In (1.18), for the localized data $v_0$, we estimate $e^{Lt}v_0$ as

$$(e^{Lt}v_0)(x) = \tilde{u}'(x)\int_{-\infty}^{\infty} E(x, t; y)v_0(y)dy + \int_{-\infty}^{\infty} \tilde{G}(x, t; y)v_0(y)dy.$$  

Here, we assume algebraic decay of the initial localized data $v_0$; so we need to look at the linear behavior of $L$ under the algebraically decaying data which was completed in [J1, HZ]. We re-prove it here in the following lemma because it is used throughout this paper.

**Lemma 2.3 ([J1], [HZ]).** Let $r > 1$. Then for any $x \in \mathbb{R}$ and $t \geq 0$,

$$(2.1) \int_{-\infty}^{\infty} t^{-\frac{1}{2}}e^{-\frac{|x-y-at|^2}{6t}}(1 + |y|)^{-r}dy \lesssim (1 + |x - at| + \sqrt{t})^{-r} + (1 + t)^{-\frac{1}{2}}e^{-\frac{|x-at|^2}{4bt}}.$$

**Proof.** If $x - at = 0$, then it is trivial because

$$\int_{-\infty}^{\infty} t^{-\frac{1}{2}}e^{-\frac{|x-y-at|^2}{6t}}(1 + |y|)^{-r}dy \lesssim (1 + t)^{-\frac{1}{2}}(1 + t)^{-\frac{1}{2}} = (1 + t)^{-1}e^{-\frac{|x-at|^2}{4bt}}.$$

We assume $x - at \neq 0$. Since $|x - y - at| \geq ||x - at| - |y||$,

$$\int_{-\infty}^{\infty} t^{-\frac{1}{2}}e^{-\frac{|x-y-at|^2}{6t}}(1 + |y|)^{-r}dy \leq \int_{-\infty}^{\infty} t^{-\frac{1}{2}}e^{-\frac{||x-at|-|y||^2}{6t}}(1 + |y|)^{-r}dy$$

$$= 2\int_{0}^{\infty} t^{-\frac{1}{2}}e^{-\frac{||x-at|-|y||^2}{6t}}(1 + |y|)^{-r}dy$$

$$\approx \int_{0}^{\frac{|x-at|}{2}} \cdots + \int_{\frac{|x-at|}{2}}^{2|x-at|} \cdots + \int_{2|x-at|}^{\infty} \cdots.$$

Noting first that the left-hand side of (2.1) and $\int_{0}^{\infty} (1 + y)^{-r}dy$ with $r > 1$ are bounded,

$$\int_{0}^{\frac{|x-at|}{2}} \cdots + \int_{\frac{|x-at|}{2}}^{2|x-at|} \cdots \lesssim (1 + t)^{-\frac{1}{2}}e^{-\frac{|x-at|^2}{4bt}}\int_{0}^{\infty} (1 + |y|)^{-r}dy \lesssim (1 + t)^{-\frac{1}{2}}e^{-\frac{|x-at|^2}{4bt}}.$$

We now estimate $\int_{\frac{|x-at|}{2}}^{2|x-at|} \cdots$ in two cases. If $|x - at| \leq \sqrt{t}$, then

$$e^{-\frac{|x-at|^2}{4bt}} = e^{-\frac{1}{4t}(\frac{|x-at|}{\sqrt{t}})^2} \geq e^{-\frac{1}{8t}} > 0;$$
so
\[ \int_{\frac{|x-at|}{2}}^{2|x-at|} (\cdots) \lesssim (1 + t)^{-\frac{1}{2}} \lesssim (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{4bt}}. \]

If \(|x-at| > \sqrt{t}|\), then
\[ \int_{\frac{|x-at|}{2}}^{2|x-at|} (\cdots) \lesssim (1 + |x-at|)^{-r} \lesssim (1 + 2|x-at|)^{-r} \lesssim (1 + |x-at| + \sqrt{t})^{-r}. \]

3 Linear estimates for nonlocalized modulational data \( \hat{u}'h_0 \)

For the modulational data \( \hat{u}'h_0 \), we recall (1.7) and decompose the solution operator \( S(t) \) into
\[ S(t) = S_s^p(t) + \hat{S}_s(t), \quad S_s^p(t) = \hat{u}' s_s^p(t) \]
with
\[ s_s^p(t)(\hat{u}'h_0) \coloneqq \int_{-\infty}^{\infty} e^{i\xi x} e^{(-iaξ-bξ^2)t}\hat{h}_0(ξ)dξ \]
and
\[ \hat{S}_s(t)(\hat{u}'h_0) = (S(t) - S_s^p(t))(\hat{u}'h_0). \]

We re-express \( s_s^p(\hat{u}'h_0) \) as \( s_s^p(\hat{u}'h_0) = IFT\left(e^{(-iaξ-bξ^2)t}h_0\right) = \int_{-\infty}^{\infty} (4\pi bt)^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{4bt}} h_0(y) dy. \]

Here, \( IFT \) denotes the inverse Fourier transform and \( * \) denotes the convolution. Similarly, we have
\[ |\partial_x s_s^p(t)(\hat{u}'h_0)| = \left| \int_{-\infty}^{\infty} e^{i\xi x} e^{(-iaξ-bξ^2)t}(iξ\hat{h}_0)(ξ)dξ \right| \]

\[ \leq \int_{-\infty}^{\infty} (4\pi bt)^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{4bt}} |\partial_y h_0(y)| dy, \]

(3.1)

\[ |\partial_x^2 s_s^p(t)(\hat{u}'h_0)| = \left| \int_{-\infty}^{\infty} e^{i\xi x} e^{(-iaξ-bξ^2)t}(iξ)^2\hat{h}_0(ξ)dξ \right| \]

\[ \leq \int_{-\infty}^{\infty} (4\pi bt)^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{4bt}} |\partial^2_y h_0(y)| dy, \]

(3.2)

and
\[ |\partial_t s_s^p(t)(\hat{u}'h_0)| = \left| \int_{-\infty}^{\infty} e^{i\xi x} e^{(-iaξ-bξ^2)t}(-iaξ-bξ^2)\hat{h}_0(ξ)dξ \right| \]

\[ \lesssim \int_{-\infty}^{\infty} (4\pi bt)^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{4bt}} \left( |\partial_y h_0(y)| + |\partial^2_y h_0(y)| \right) dy. \]

(3.3)
We estimate \((\tilde{S}_s(t)(\tilde{u}'h_0))(x)\) separately in two cases \(|x| \gg Ct\) and \(|x| \ll Ct\) for sufficiently large \(C > 0\). Recalling (1.8), we begin by estimating \((\tilde{S}_s(t)(\tilde{u}'h_0))(x)\) for \(|x| \gg Ct\) by using the fact that \(\tilde{u}'h_{-\infty}\) and \(\tilde{u}'h_{+\infty}\) are stationary solutions of \(S(t)\).

**Proposition 3.1.** Suppose \(|(h_0 - h_{\pm\infty})(x)| \leq E_0(1 + |x|)^{-r}\) and \(|h_{+\infty}| = |h_{-\infty}| \leq E_0\) for \(r > 1\) and sufficiently small \(E_0 > 0\). Then if \(|x| \gg Ct\) for sufficiently large \(C > 0\),

\[
|\tilde{S}_s(t)(\tilde{u}'h_0))(x)| \lesssim E_0 \left[(1 + |x - at| + \sqrt{t})^{-r} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{M't}}\right],
\]

for a sufficiently large number \(M' > M > 0\) \((M\) denotes the constant in Theorem 2.1).

**Proof.** Let us first consider \(x \leq 0\). Since \(\tilde{u}'h_{-\infty}\) is a stationary solution of \(S(t)\), \(S(t)(\tilde{u}'h_{-\infty}) = e^{-\frac{|x-y|}{Mt}}(\tilde{u'}h_{-\infty})\); and so \(\tilde{S}_s(t)(\tilde{u}'h_0) = S(t)(\tilde{u}'(h_0 - h_{-\infty})) - S^p(t)(\tilde{u}'(h_0 - h_{-\infty}))\).

From the pointwise bounds on the Green function of \(S(t)\) in Theorem 2.1,

\[
|S(t)(\tilde{u}'(h_0 - h_{-\infty})))| + |S^p(t)(\tilde{u}'(h_0 - h_{-\infty})))| \\
\lesssim \int_0^0 t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{2Mt}} |(h_0 - h_{-\infty})(y)| dy + \int_0^\infty t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{2Mt}} |(h_0 - h_{-\infty})(y)| dy \\
\lesssim \int_0^0 t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{2Mt}} |(h_0 - h_{-\infty})(y)| dy + E_0 \int_0^\infty t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{2Mt}} dy.
\]

The first term is done from the assumption \(|(h_0 - h_{\pm\infty})(x)| \leq E_0(1 + |x|)^{-r}\), \(r > 1\) and Lemma 2.1. Since \(x \ll -Ct\) and \(y \geq 0\), \(|x - y| \gg Ct\); and so

\[
(3.4) \quad \frac{|x - y - at|}{2t} \leq \frac{|x - y|}{t} \leq \frac{2|x - y - at|}{t},
\]

for sufficiently large \(C > 0\). Thus,

\[
\int_0^\infty t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{2Mt}} dy \leq \int_0^\infty t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{2Mt}} dy \leq e^{-\eta t} e^{-\frac{|x-y|^2}{2Mt}} \int_0^\infty t^{-\frac{1}{2}} e^{-\frac{M't}{2}} dy \leq e^{-\eta t} e^{-\frac{|x-y|^2}{2M't}},
\]

for some positive constant \(M' > 16M > 0\). Here, the last inequality is from the fact \(|x| \gg Ct\) and (3.4) again. Similarly, we argue the case \(x > 0\) with a stationary solution \(\tilde{u}'h_{+\infty}\) instead of \(\tilde{u}'h_{-\infty}\).

For the case \(|x| \ll Ct\), we first denote the right and left eigenfuctions of \(L_\xi\) corresponding to \(\lambda(\xi)\) by \(q(\xi, x)\) and \(\tilde{q}(\xi, x)\), respectively, for sufficiently small \(|\xi|\). In particular, \(q(0, x) = \tilde{q}(\xi)\) since \(L_0\tilde{u}' = 0\). Moreover, for sufficiently small \(|\xi|\), let \(\Pi(\xi)(\cdot) = q(\xi)(\tilde{q}(\xi), \cdot) L^2([0, 1])\) which is the eigenprojection onto the right-eigenspace, \(\text{span}\{q(\xi)\}\). In order to estimate \(\tilde{S}_s(t)\) for \(|x| \gg Ct\), we decompose again \(\tilde{S}_s(t)\) into \(\tilde{S}_1(t)\) and \(\tilde{S}_2(t)\) with

\[
\tilde{S}_1(t)(u'h_0)) := \int_{-\pi}^\pi e^{ix(x)} e^{i\lambda(\xi)} t \alpha(\xi) \tilde{q}(\xi, \cdot) d\xi - \int_{-\infty}^\infty e^{i\xi} e^{-i\alpha - \beta(\xi) t} \tilde{h}_0(\xi) d\xi,
\]

and

\[
\tilde{S}_2(t)(u'h_0)) := \int_{-\pi}^\pi e^{ix(x)} e^{i\lambda(\xi)} t \alpha(\xi) \tilde{q}(\xi, \cdot) d\xi + \int_{-\infty}^\infty e^{i\xi} e^{-i\alpha - \beta(\xi) t} \tilde{h}_0(\xi) d\xi.
\]
Remark 3.2. Comparing our decomposition of $S_j$ worst term at $\alpha$ omitting the cutoff function $\bar{\Pi}$ stationary solutions

\[
\tilde{S}_2(t)(\tilde{u}'h_0) := \int_{-\pi}^{\pi} e^{i\xi x}(1 - \alpha(\xi))e^{L\xi t}(\tilde{u}'h_0(\xi, x))d\xi
\]

(3.5)
\[
+ \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)\bar{\Pi}(\xi)e^{L\xi t}(\tilde{u}'h_0(\xi, x))d\xi
\]
\[
+ \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}(q(\xi, x) - q(0, x))\langle \bar{q}(\xi, \cdot), \tilde{u}'h_0(\xi, \cdot) \rangle_{L^2[0,1]}d\xi;
\]

where $\alpha(\xi)$ is a smooth cutoff function such that $\alpha(\xi) = 1$ for sufficiently small $|\xi|$.

**Remark 3.2.** Comparing our decomposition of $S(t) = \tilde{u}'s^p_{\ast}(t) + \tilde{S}_p(t) = \tilde{u}'s^p_{\ast}(t) + \tilde{S}_1(t) + \tilde{S}_2(t)$ with the decomposition of $S(t) = \tilde{u}'s^p_{\ast}(t) + S^p(t)$ in [JNRZ1], we see that $\tilde{u}'s^p_{\ast}(t) + \tilde{S}_1(t) = \tilde{u}'s^p(t)$ and $\tilde{S}_2(t) = S^p(t)$. Here, we set $s^p_{\ast}$ as the principal, Gaussian, part of the worst term at $j = 0$ in

\[
(s^p(t)(\tilde{u}'h_0))(x) = \int_{-\pi}^{\pi} e^{i\xi x}e^{\lambda(\xi)t}\alpha(\xi)\langle \bar{\phi}(\xi, \cdot), \tilde{u}'h_0(\xi, \cdot) \rangle_{L^2[0,1]}d\xi
\]
\[
= \sum_{j \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{i\xi x}e^{\lambda(\xi)t}\alpha(\xi)\langle \bar{\phi}(\xi, \cdot)\tilde{u}'(\cdot), e^{i2\pi j} \rangle_{L^2[0,1]}\tilde{h}_0(\xi + 2\pi j)d\xi.
\]

omitting the cutoff function $\alpha(\xi)$. This is to estimate $S(t)$ for $|x| \gg Ct$ by using the stationary solutions $\tilde{u}'h_+ \text{ and } \tilde{u}'h_-$ in Proposition 3.1. Actually, our decomposition might work also in the analysis of [JNRZ1].

**Proposition 3.3.** Suppose $|\partial_x h_0(x)| \leq E_0(1 + |x|)^{-r}$ for $r > 1$ and sufficiently small $E_0 > 0$. Then if $|x| < Ct$ for sufficiently large $C > 0$,

\[
|\tilde{S}_1(t)(\tilde{u}'h_0)| \lesssim \int_{-\infty}^{\infty} \left[ (1 + |x - y - at| + \sqrt{t})^{-2} + t^{-\frac{1}{2}}e^{\frac{|x - y - at|^2}{Mt}} \right] |\partial_y h_0(y)|dy
\]
\[
\lesssim E_0 \left[ (1 + |x - at| + \sqrt{t})^{-r} + (1 + t)^{-\frac{1}{2}}e^{\frac{|x - at|^2}{M't}} \right],
\]

for sufficiently large $M' > M > 0$. 


Proof. Recalling $\tilde{h}_0(\xi, x) = \sum_{j \in \mathbb{Z}} e^{i2\pi jx} \tilde{h}_0(\xi + 2\pi j)$, $\tilde{S}_1(t)$ is separated into four parts

$$
\tilde{S}_1(t)(\tilde{u}'h_0) = \int_{-\pi}^{\pi} e^{i\xi x} e^{\lambda(\xi)t} \alpha(\xi) \langle \tilde{\phi}(\xi, \cdot), \tilde{u}'h_0(\xi, \cdot) \rangle_{L^2[0,1]} d\xi - \int_{-\infty}^{\infty} e^{i\xi x} e^{(-ia\xi-b\xi^2)t} \tilde{h}_0(\xi) d\xi
$$

$$
= \sum_{j \in \mathbb{Z}\{0\}} \int_{-\pi}^{\pi} e^{i\xi x} e^{\lambda(\xi)t} \alpha(\xi) \langle \tilde{\phi}(\xi, \cdot), \tilde{u}'h_0(\xi, \cdot) \rangle_{L^2[0,1]} \tilde{h}_0(\xi + 2\pi j) d\xi
$$

$$
= I + II + III + IV.
$$

**Estimate I.** We first notice that by (1.6),

$$
e^{\lambda(\xi)t} = e^{-ia\xi-b\xi^2} O(|\xi|^3)t = e^{-ia\xi-b\xi^2} \left(1 + O(|\xi|^3t)\right).
$$

We separate again $I$ into two parts.

$$
I = \sum_{j \in \mathbb{Z}\{0\}} \int_{-\pi}^{\pi} e^{i\xi x} e^{\lambda(\xi)t} \alpha(\xi) \langle \tilde{\phi}(\xi, \cdot), \tilde{u}'h_0(\xi, \cdot) \rangle_{L^2[0,1]} \tilde{h}_0(\xi + 2\pi j) d\xi
$$

$$
= \sum_{j \in \mathbb{Z}\{0\}} \int_{-\pi}^{\pi} e^{i\xi x} e^{\lambda(\xi)t} \frac{1}{\xi + 2\pi j} \alpha(\xi) \tilde{\phi}(\xi) \tilde{u}'_j e^{i2\pi jx} \partial_x h_0(x)(\xi) d\xi
$$

$$
= \sum_{j \in \mathbb{Z}\{0\}} \int_{-\pi}^{\pi} e^{i\xi x} e^{(-ia\xi-b\xi^2)t} \frac{1}{\xi + 2\pi j} \alpha(\xi) \tilde{\phi}(\xi) \tilde{u}'_j e^{i2\pi jx} \partial_x h_0(x)(\xi) d\xi
$$

$$
+ \sum_{j \in \mathbb{Z}\{0\}} \int_{-\pi}^{\pi} e^{i\xi x} e^{(-ia\xi-b\xi^2)t} O(|\xi|^3t) \frac{1}{\xi + 2\pi j} \alpha(\xi) \tilde{\phi}(\xi) \tilde{u}'_j e^{i2\pi jx} \partial_x h_0(x)(\xi) d\xi
$$

$$
= A + B.
$$

Here, $\tilde{\phi}(\xi) \tilde{u}'_j$ denotes the complex conjugate of $j$-th Fourier coefficient of the 1-periodic function $\tilde{\phi}(\xi) \tilde{u}'_j$. In order to estimate $A$, we first set

$$
\beta_j := \frac{1}{\xi + 2\pi j} \alpha(\xi) \tilde{\phi}(\xi) \tilde{u}'_j
$$
which is bounded and in Schwartz class. Then we estimate \(A\) as
\[
|A| \leq \sum_{j \in \mathbb{Z}} |\text{IFT}(e^{(-i\alpha \xi - \beta k^2)t}) * \beta_j + e^{-i2\pi j \cdot \partial_x h_0}|
\]
\[
\leq |\text{IFT}(e^{(-i\alpha \xi - \beta k^2)t})| * \sum_{j \in \mathbb{Z}} |\beta_j| * |e^{-i2\pi j \cdot \partial_x h_0}|
\]
\[
\approx \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi b}} e^{-\frac{|x - y - at|^2}{4bt}} \sum_{j \in \mathbb{Z}} |\beta_j(y)| dy * |\partial_x h_0(x)|;
\]
so it is enough to estimate \(\sum_{j \in \mathbb{Z}} |\beta_j(x)|.\) Noting that
\[
|x^2 \beta_j(x)| = |\partial^2_x \beta_j(x)| \leq \sum_{k=0}^{2} \frac{1}{1+|j|} |\partial^k_x \phi(\xi) u'_j|
\]
and recalling \(\beta_j(\xi)\) has a smooth cut-off function \(\alpha(\xi),\) we have
\[
|x^2 \beta_j(x)| = \left| \int_{-\infty}^{\infty} e^{i\xi x} x^2 \beta_j(x)(\xi) d\xi \right| \leq \left| \int_{|\xi| \leq \epsilon} \partial^2_x \beta_j(\xi) |d\xi| \right| \leq \sup_{|\xi| \leq \epsilon} \left( \sum_{k=0}^{2} \frac{1}{1+|j|} |\partial^k_x \phi(\xi) u'_j| \right).
\]
Since \(\partial^k_x \phi(\xi, \cdot) u'(\cdot)\) and \(\partial^k_x \phi(\xi, \cdot) u''(\cdot)\) are also periodic, \(\partial_x \phi(\xi) u'_j\) and \(\partial^2_x \phi(\xi) u'_j\) are Fourier coefficients of \(\partial_x \phi(\xi) u'\) and \(\partial^2_x \phi(\xi) u',\) respectively. For each \(k = 0, 1, 2,\) by using the Cauchy-Schwarz estimate,
\[
\sum_{j \in \mathbb{Z}} \frac{1}{1+|j|} |\partial^k_x \phi(\xi) u'_j| \leq \sqrt{\sum_{j \in \mathbb{Z}} (1+|j|)^{-2} \sum_{j \in \mathbb{Z}} |\partial^k_x \phi(\xi) u'_j|^2}
\]
\[
\leq C \| \partial^k_x \phi(\xi) u' \|_{L^2([0,1])};
\]
and so
\[
\sum_{j \in \mathbb{Z}} |\beta_j(x)| \leq (1+|x|)^{-2} \sup_{|\xi| \leq \epsilon} \sum_{j \in \mathbb{Z}} \left( \sum_{k=0}^{2} \frac{1}{1+|j|} |\partial^k_x \phi(\xi) u'_j| \right)
\]
\[
\leq (1+|x|)^{-2} \sup_{|\xi| \leq \epsilon} \sum_{k=0}^{2} \| \partial^k_x \phi(\xi) u' \|_{L^2([0,1])}
\]
\[
\leq C(1+|x|)^{-2}.
\]
Thus, by Lemma 2.3,
\[
|A| \lesssim \left[ (1+|x - at| + \sqrt{t})^{-2} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x - at|^2}{16bt}} \right] \cdot |\partial_x h_0(x)|
\]
\[
= \int_{-\infty}^{\infty} \left[ (1+|x - y - at| + \sqrt{t})^{-2} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x - y - at|^2}{16bt}} \right] |\partial_x h_0(y)| dy.
\]
If \(|\partial_y h_0(y)| \leq (1 + |y|)^{-r}\) with \(r > 1\), then
\[
|A| \leq E_0 \left[ (1 + |x - at| + \sqrt{t})^{-r} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{Mt}} \right]
\]
for sufficiently large \(M > 0\). Here, we compute \(\int_{-\infty}^{\infty} (1 + |x - y - at| + \sqrt{t})^{-2}(1 + |y|)^{-r} dy\) in two cases \(|x - at| \leq \sqrt{t}\) and \(|x - at| > \sqrt{t}\) similarly as in Lemma 2.3.

For \(B\), we set
\[
\tilde{\beta}_j := \frac{1}{\xi + 2\pi j} \alpha_{\tilde{\tau}}(\xi) \tilde{\phi}(\xi) \tilde{u}'_j ;
\]
so
\[
|B| \leq \sum_{j \in \mathbb{Z} \setminus \{0\}} |\text{IFT} \left( e^{(-i\alpha\xi - b\xi^2) t} O(|\xi^3| t) \alpha_{\tilde{\tau}}(\xi) \right) * \tilde{\beta}_j * e^{-i2\pi j \partial_x h_0}| \leq \sum_{j \in \mathbb{Z} \setminus \{0\}} |\tilde{\beta}_j(x)| |\partial_x h_0(x)| \leq \sum_{j \in \mathbb{Z} \setminus \{0\}} |\tilde{\beta}_j(x)| |\partial_x h_0(x)|
\]
Computing similarly as in \(A\), \(\sum_{j \in \mathbb{Z} \setminus \{0\}} |\tilde{\beta}_j(x)| \lesssim (1 + |x|)^{-2}\). For the second integration,
\[
\int_{\xi \leq |\xi| \leq 2\xi} e^{i\xi x} (\ldots) d\xi \leq \int_{\xi \leq |\xi| \leq 2\xi} e^{-b\xi^2 t} O(|\xi^3| t) d\xi \lesssim t^{-\frac{1}{2}} e^{-b\xi^2 t} \leq t^{-\frac{1}{2}} e^{-\tau t} \leq t^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{Mt}}
\]
for some \(\eta > 0\) and sufficiently large \(M > 0\). Here, the last inequality is from the boundedness of \(\frac{|x - at|}{t}\) because \(\frac{|x|}{t}\) is bounded, says \(\frac{|x - at|}{t} < S_1\) for some \(S_1 > 0\). Indeed, for sufficiently large \(M > 0\),
\[
e^{-\frac{|x-at|^2}{Mt}} = e^{-(\frac{|x-at|}{\sqrt{t}})^2 \frac{2\pi}{\sqrt{t}}} \geq e^{-\frac{|x-at|^2}{2\pi t}} \geq e^{-\frac{\eta}{t}}.
\]
We now estimate \(\left| \int_{|\xi| \leq \varepsilon} e^{i\xi x} (\ldots) d\xi \right|\) by using complex contour integrals like [J1]. Since \(\frac{|x|}{t}\) is bounded, we define
\[\tilde{\alpha} := \frac{x - at}{2bt}\]
which is bounded and positive (without loss of generality). Thus, we have
\[
\int_{|\xi| \leq \varepsilon} e^{i\xi x} (\ldots) d\xi = \int_{-\varepsilon}^{\varepsilon} e^{i(\xi_1 + i\tilde{\alpha})(x-at) - b(\xi_1 + i\tilde{\alpha})^2 t} O(|\xi_1 + i\tilde{\alpha}|^3 t) d\xi_1 + \int_{0}^{\varepsilon} e^{i(\varepsilon + i\xi_2)(x-at) - b(\varepsilon + i\xi_2)^2 t} O(|\varepsilon + i\xi_2|^3 t) d\xi_2
\]
which is bounded by \( t^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{Mt}} \) for sufficiently large \( M > 0 \). Thus,

\[
|B| \lesssim \int_{-\infty}^{\infty} \left[ (1 + |x - y - at| + \sqrt{t})^{-2} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{Mt}} \right] |\partial_y h_0(y)| dy
\]

\[
\lesssim E_0 \left[ (1 + |x - at| + \sqrt{t})^{-r} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{Mt}} \right].
\]

**Estimate II, III and IV.** The estimate II, III and IV follows similarly. Since \( \hat{\phi}(\xi, \cdot) - \hat{\phi}(0, \cdot) = O(\|\xi\|) \),

\[
II = \int_{-\pi}^{\pi} e^{i\xi x} e^{t\lambda(\xi)} (1 + i\xi) \hat{h}_0(\xi) d\xi
\]

\[
= \int_{-\pi}^{\pi} e^{i\xi x} e^{t\lambda(\xi)} (1 + i\xi) \hat{h}_0(\xi) d\xi
\]

\[
= \int_{-\pi}^{\pi} \left( \int_{-\pi}^{\pi} e^{i\xi (x-y)} e^{t\lambda(\xi)} (1 + i\xi) \hat{h}_0(y) d\xi \right) \partial_y h_0(y) dy
\]

\[
= \int_{-\pi}^{\pi} \left( \int_{-\pi}^{\pi} e^{i\xi (x-y)} e^{(-ia\xi - b\xi^2)t} (1 + i\xi) \hat{h}_0(y) d\xi \right) \partial_y h_0(y) dy
\]

and

\[
III = \int_{-\pi}^{\pi} e^{i\xi x} e^{(-ia\xi - b\xi^2)t} O(|\xi|^2 t) \alpha(\xi) (i\xi)^{-1} \hat{h}_0(\xi) d\xi
\]

\[
= \int_{-\pi}^{\pi} \left( \int_{-\pi}^{\pi} e^{i\xi x} e^{(-ia\xi - b\xi^2)t} O(|\xi|^2 t) \alpha(\xi) d\xi \right) \partial_y h_0(y) dy.
\]

Computing similarly as in I, by complex contour integrals,

\[
|II + III| \lesssim \int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{Mt}} |\partial_y h_0(y)| dy
\]

\[
\lesssim E_0 \left[ (1 + |x - at| + \sqrt{t})^{-r} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{Mt}} \right].
\]

Notice that

\[
IV = \int_{-\pi}^{\pi} e^{i\xi x} e^{(-ia\xi - b\xi^2)t} (1 - \lambda(\xi))(i\xi)^{-1} \hat{h}_0(\xi) d\xi
\]

\[
= \int_{-\pi}^{\pi} \left( \int_{|\xi|}^{\pi} e^{i\xi (x-y)} e^{(-ia\xi - b\xi^2)t} (1 - \lambda(\xi))(i\xi)^{-1} d\xi \right) \partial_y h_0(y) dy.
\]

By (3.6), we estimate IV; so

\[
|IV| \lesssim E_0 \left[ (1 + |x - at| + \sqrt{t})^{-r} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{Mt}} \right].
\]
We now treat \((\tilde{S}_2(t)(\tilde{u}'h_0))(x)\) in terms of \(\partial_x h_0(x)\) for \(|x| < Ct\) with sufficiently large \(C > 0\).

**Proposition 3.4.** Suppose \(|\partial_x h_0(x)| \leq E_0(1 + |x|)^{-r}\) for sufficiently small \(E_0 > 0\). Then if \(|x| < Ct\) for sufficiently large \(C > 0\),

\[
|\tilde{S}_2(t)(\tilde{u}'h_0)(x)| \lesssim \int_{-\infty}^{\infty} \left(1 + |x - y - at| + \sqrt{t}\right)^{-2} + t^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{Mt}} |\partial_y h_0(y)| dy
\]

(3.8)

for sufficiently large \(M' > M > 0\).

**Proof.** Similarly as in I of Proposition 3.3, re-express (3.5):

\[
(\tilde{S}_2(t)(\tilde{u}'h_0))(x) = \sum_{j \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{ix} (e^{L_{k,t}}(\tilde{u}'e^{2\pi j x}))(x) \frac{1 - \alpha(\xi)}{i(\xi + 2\pi j)} e^{2\pi j x} \partial_x h_0(\xi) d\xi
\]

\[
+ \sum_{j \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{ix} (e^{L_{k,t}}\tilde{\Pi}(\tilde{u}'e^{2\pi j x}))(x) \frac{\alpha(\xi)}{i(\xi + 2\pi j)} e^{2\pi j x} \partial_x h_0(\xi) d\xi
\]

\[
+ \sum_{j \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{ix} e^{\lambda(x)} O(|\xi|) \phi(\xi) \hat{u}'_j \frac{\alpha(\xi)}{i(\xi + 2\pi j)} e^{2\pi j x} \partial_x h_0(\xi) d\xi
\]

= I + II + III.

Setting \(\hat{\beta}_j := \frac{\xi}{\xi + 2\pi j} \alpha^{\frac{1}{2}}(\xi) \phi(\xi) \hat{u}'_j\), we estimate III as before, where we are using the complex contour integrals:

\[
|III| \lesssim \int_{-\infty}^{\infty} e^{ix} e^{-i\alpha\xi - b\xi^2} t (O(1) + O(|\xi|^3)) \alpha^{\frac{1}{2}}(\xi) d\xi |\sum_{j \in \mathbb{Z}} |\hat{\beta}_j(x)| dx |\partial_x h_0(x)|
\]

which is bounded by \(\int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{Mt}} |\partial_y h_0(y)| dy\) as before. The only difference compared to term I in Proposition 3.3 is that the summation contains \(j = 0\), but it is totally okay because we have \(\frac{\xi}{\xi + 2\pi j}\) in \(\hat{\beta}_j\) instead of \(\frac{1}{\xi + 2\pi j}\).

Now we consider I and II which are same estimations. Re-expressing I,

\[
I = \int_{-\pi}^{\pi} e^{ix} (1 - \alpha(\xi))(e^{L_{k,t}}(\tilde{u}'\tilde{h}_0))(\xi, x) d\xi
\]

\[
= \sum_{j \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{ix} (e^{L_{k,t}}(\tilde{u}'e^{2\pi j x}))(x) \frac{1 - \alpha(\xi)}{i(\xi + 2\pi j)} e^{-i2\pi j x} \partial_x h_0(\xi) d\xi
\]

\[
= \sum_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{ix} \left(e^{L_{k,t}}(\tilde{u}'e^{2\pi j x})\right)(x) \frac{(1 - \alpha(\xi))\chi_{[-\pi,\pi]}(\xi)}{i(\xi + 2\pi j)} e^{-i2\pi j x} \partial_x h_0(\xi) d\xi.
\]
Set 
\[ d_j(\xi, x, t) := (e^{L_\xi t}(u^j e^{i2\pi jx}))(x) \]
which is periodic in \( x \) on \([0, 1]\) and setting \( c_{j,k}(\xi, t) \) are Fourier coefficients of \( d_j \), we have 
\[
I = \sum_{j,k \in \mathbb{Z}} e^{i2\pi kx} \int_{-\infty}^{\infty} e^{i\xi x} c_{j,k}(\xi) \left( \frac{1 - \alpha(\xi)}{i(\xi + 2\pi j)} \right) e^{-i2\pi jx} \partial_x h_0(\xi)d\xi
\]
\[
= \sum_{j,k \in \mathbb{Z}} e^{i2\pi kx} \int_{-\infty}^{\infty} e^{i\xi x} c_{j,k}(\xi) \left( \frac{1 - \alpha(\xi)}{i(\xi + 2\pi j)} \right) d\xi \ast e^{-i2\pi jx} \partial_x h_0(x)
\]
and so
\[
|I| \lesssim \sup_{\varepsilon < |\xi| < \pi} \sum_{j \in \mathbb{Z}} |d_j(\xi, x, t) \frac{1 - \alpha(\xi)}{i(\xi + 2\pi j)}| \ast |\partial_x h(x)|;
\]
thus, it is enough to estimate \( \sum_{j \in \mathbb{Z}} |d_j(\xi, x, t) \frac{1 - \alpha(\xi)}{i(\xi + 2\pi j)}| \) independently on \( \xi \). By Cauchy-Schwarz inequality,
\[
\sum_{j \in \mathbb{Z}} |d_j(\xi, x, t) \frac{1 - \alpha(\xi)}{i(\xi + 2\pi j)}| \leq \sqrt{\sum_{j \in \mathbb{Z}} \frac{1}{(1 + |j|)^2} \sum_{j \in \mathbb{Z}} |d_j(\xi, x, t)|^2} \leq C \sqrt{\sum_{j \in \mathbb{Z}} |d_j(\xi, x, t)|^2}.
\]
Noting that \( \text{Re}\sigma(L_\xi) \leq -\eta < 0 \) for any \( |\xi| \geq \varepsilon \), we re-define the sector as \( \Omega \cap \{\text{Re}\lambda \leq -\eta\} \) independently on \( \xi \) and set 
\[ \Gamma = \partial(\Omega \cap \{\text{Re}\lambda \leq -\eta\}). \]
Then we have 
\[
\sum_{j \in \mathbb{Z}} |d_j(\xi, x, t)|^2 = \sum_{j \in \mathbb{Z}} \left| (e^{L_\xi t}(u^j e^{i2\pi jx}))(x) \right|^2
\]
\[
= \sum_{j \in \mathbb{Z}} \left| \int_{\Gamma} e^{\lambda t} (L_\xi - \lambda)^{-1} (u^j(x) e^{i2\pi jx}) d\lambda \right|^2
\]
\[
= \sum_{j \in \mathbb{Z}} \left| \int_{\Gamma} e^{\lambda t} \left( \int_0^1 [G_{\xi,\lambda}(x, z)] u'(z) e^{i2\pi jz} dz \right) d\lambda \right|^2
\]
\[
\leq \left( \sum_{j \in \mathbb{Z}} \left| \int_{\Gamma} e^{\lambda t} \left( \int_0^1 [G_{\xi,\lambda}(x, z)] u'(z) e^{i2\pi jz} dz \right) d\lambda \right| \right)^2,
\]
where the brackets \([\cdot]\) denote the periodic extensions of the given function onto the whole line. Since \( [G_{\xi,\lambda}(x, z)] u'(z) \) is periodic in \( z \) on \([0, 1]\), let’s set 
\[ h_j(\xi, x, \lambda) := \int_0^1 e^{-i2\pi jz} [G_{\xi,\lambda}(x, z)] u'(z) dz \]
which are Fourier coefficients of \([G_{\xi,\lambda}(x,z)]\hat{u}'(z)\). Recall that \(|G_{\xi,\lambda}(x,z)| \leq C|\lambda|^{-\frac{1}{2}}\) and \(|\partial_z G_{\xi,\lambda}(x,z)| \leq C\) in \([J1]\), for \(|\lambda| > R\), \(R\) sufficiently large, and \(|G_{\xi,\lambda}(x,z)|, |\partial_z G_{\xi,\lambda}(x,z)| \leq C\), for \(|\lambda| < R\). Then we have

\[
\sum_{j \in \mathbb{Z}} |h_j^*(\xi, x, \lambda)| \leq C \left( \sum_{j \in \mathbb{Z}} (1 + |j|)^{-2} \sum_{j \in \mathbb{Z}} (1 + |j|^2) |h_j(\xi, x, \lambda)|^2 \right) \leq C \|G_{\xi,\lambda}(x,z)\hat{u}'(z)\|_{H^1\{z;[0,1]\}} \leq C,
\]

where * denote complex conjugate. Thus we have

\[
\sqrt{\sum_{j \in \mathbb{Z}} |d_j(\xi, x, t)|^2} = \int_{\Gamma} e^{Re\lambda t} \sum_{j \in \mathbb{Z}} |h_j(\xi, x, \lambda)| d\lambda
\]
\[
\leq C e^{-\eta t} \int_0^\infty e^{-\eta k} dk
\]
\[
\leq C t^{-1} e^{-\eta t}
\]
\[
\leq C t^{-1} e^{-\frac{\eta t^2}{2M^2}},
\]

for some \(\eta > 0\) and large \(M > 0\). Here, the last inequality is from (3.7) again. \(\square\)

4 Nonlinear iteration scheme

Recalling the nonlinear perturbation equation (1.13) and (1.18), we now define \(\psi(x, t)\) to cancel \(E(x, t; y)\) and \(\hat{u}'s_p(t)\) in \(S(t)v_0\) and \(S(t)\hat{u}'h_0\), respectively,

(4.1) \(\psi(x, t) := s_p^*(t)(\hat{u}'h_0) + \int_{-\infty}^{\infty} E(x, t; y)v_0(y)dy + \int_0^t \int_{-\infty}^{\infty} E(x, t - s; y)N(y, s)dyds.\)

Since there is a cutoff function in \(E\), \(\psi(x, 0) = h_0(x)\) and so we have a new integral representation of \(v(x, t)\):

(4.2) \(v(x, t) = \tilde{S}_s(t)(\hat{u}'h_0) + \int_{-\infty}^{\infty} \tilde{G}(x, t; y)v_0(y)dy + \int_0^t \int_{-\infty}^{\infty} \tilde{G}(x, t - s; y)N(y, s)dyds.\)

Remark 4.1. Similarly as in the localized case, we define \(\psi\) as “bad” terms which have not enough decay rates in the solution operator \(S(t)\) to close a nonlinear iteration. One can actually see \(\psi_x \sim v\). Since \(N\) consists of \(v\) and derivatives of \(\psi\), by (4.1) and (4.2), we prove Theorem 1.1 in the next section.

5 Nonlinear stability

We now prove the main theorem, starting with the following lemma.
Lemma 5.1. For $r \geq \frac{3}{2}$ and sufficiently small $E_0 > 0$, we assume
\[
|\tilde{u}_0(x - h_0(x)) - \tilde{u}(x)| + \sum_{k=1,2} |\partial_x^k h_0(x)| + |h_0(x) - h_{-\infty}| \leq E_0(1 + |x|)^{-r},
\]
\[
|h_{+\infty}| = |h_{-\infty}| \leq E_0 \quad \text{and} \quad v_0 := \tilde{u}_0(x - h_0(x)) - \tilde{u}(x) \in H^2(\mathbb{R}).
\]
For $v$ and $\psi$ defined in Section 4, we define
\[
\zeta(t) := \sup_{0 \leq s \leq t, x \in \mathbb{R}} |(v, \psi, \psi_x, \psi_{xx})(x, s)| \left[ (1 + |x - as| + \sqrt{s})^{-r} + (1 + s)^{-\frac{1}{2}} e^{-\frac{|x-as|^2}{M_s}} \right]^{-1},
\]
for sufficiently large $M > 0$. Then for all $t \geq 0$ for which $\zeta(t)$ is finite, we have
\[
\zeta(t) \leq C(E_0 + \zeta(t)^2)
\]
for some constant $C > 0$.

Proof. For any $0 \leq s \leq t$, applying Propositions 3.1, 3.3 and 3.4 to the integral representation of $v$ in (4.2),
\[
|\bar{S}_s(s)(\bar{u}^h_0)| \leq C E_0 \left[ (1 + |x - as| + \sqrt{s})^{-r} + (1 + s)^{-\frac{1}{2}} e^{-\frac{|x-as|^2}{M_s}} \right]
\]
for any $x \in \mathbb{R}$; so then for any $0 \leq s \leq t$ and any $x \in \mathbb{R}$,
\[
|\bar{S}_s(s)(\bar{u}^h_0)| \left[ (1 + |x - as| + \sqrt{s})^{-r} + (1 + s)^{-\frac{1}{2}} e^{-\frac{|x-as|^2}{M_s}} \right]^{-1} \leq C E_0.
\]
Reminding the pointwise bounds of $\bar{G}(x,t;y)$ in Theorem 2.1, we have for any $0 \leq s \leq t$ and any $x \in \mathbb{R},$
\[
\left| \int_{-\infty}^{\infty} \bar{G}(x, s; y) v_0(y) dy \right| \leq \int_{-\infty}^{\infty} |\bar{G}(x, s; y)||v_0(y)| dy
\]
\[
\leq \int_{-\infty}^{\infty} \left( (1 + s)^{-r} + s^{-\frac{1}{2}} e^{-\eta} \right) e^{-\frac{|x-y-as|^2}{M_s}} (1 + |y|)^{-r} dy
\]
\[
\leq C E_0 \left[ (1 + |x-as| + \sqrt{s})^{-r} + (1 + s)^{-\frac{1}{2}} e^{-\frac{|x-as|^2}{M_s}} \right].
\]
Recalling (1.14) \sim (1.17) and applying the boundedness of $|v_x|_{L^\infty}$ in the main theorem of [JNRZ1] to $\mathcal{N}$ in (4.2), we have
\[
|(Q, R, S, T)(x, s)| \leq C |(v, \psi, \psi_x, \psi_{xx})(x, s)|^2
\]
\[
\leq C \zeta^2(t) \left[ (1 + |x - as| + \sqrt{s})^{-r} + (1 + s)^{-\frac{1}{2}} e^{-\frac{|x-as|^2}{M_s}} \right]^2.
\]
By using integration by parts in the third term of \( v \),

\[
\left| \int_0^t \int_{-\infty}^{\infty} \tilde{G}(x, t - s; y) N(y, s) dy ds \right|
\]

\[
\leq \int_0^t \int_{-\infty}^{\infty} |\tilde{G}_y(x, t - s; y)||Q, R, S, T)(y, s)| dy ds
\]

\[
\leq C \zeta^2(t) \int_0^t \int_{-\infty}^{\infty} (t - s)^{-1} e^{-\frac{|x - y - a(t - s)|^2}{M(t - s)}} (1 + |y - as| + \sqrt{s})^{-2r} dy ds
\]

\[+ C \zeta^2(t) \int_0^t \int_{-\infty}^{\infty} (t - s)^{-1} e^{-\frac{|x - y - a(t - s)|^2}{M(t - s)}} (1 + s)^{-1} e^{-\frac{|y - as|^2}{Ms}} dy ds
\]

\[= C \zeta^2(t)(I + II).
\]

Noting first that

\[
\int_{-\infty}^{\infty} e^{-\frac{|x - y - a(t - s)|^2}{M(t - s)}} e^{-\frac{|y - as|^2}{Ms}} dy \leq Ct^{-\frac{1}{2}} s^{\frac{1}{2}} (t - s)^{\frac{1}{2}} e^{-\frac{|x - at|^2}{M}}
\]

we easily estimate \( II \) as

\[II \leq e^{-\frac{|x - at|^2}{Ms}} \int_0^t (t - s)^{-\frac{1}{2}} (1 + s)^{-\frac{1}{2}} t^{-\frac{1}{2}} ds \lesssim (1 + t)^{-\frac{1}{2}} e^{-\frac{|x - at|^2}{M}}.
\]

We now estimate \( I \) by separating \( \int_0^t \) into \( \int_0^{t/2} \) and \( \int_{t/2}^t \). By Lemma 2.3, we have

\[
\int_{-\infty}^{\infty} (t - s)^{-\frac{1}{2}} e^{-\frac{|x - (y - as) - at|^2}{M(t - s)}} (1 + |y - as|)^{-r} dy
\]

\[\lesssim (1 + |x - at| + \sqrt{t - s})^{-r} + (1 + t - s)^{-\frac{1}{2}} e^{-\frac{|x - at|^2}{M(t - s)}};
\]

so then

\[
\int_0^{t/2} \lesssim \int_0^{t/2} (1 + s)^{-\frac{1}{2}} (t - s)^{-\frac{1}{2}} [(1 + |x - at| + \sqrt{t - s})^{-r} + (1 + t - s)^{-\frac{1}{2}} e^{-\frac{|x - at|^2}{M(t - s)}}] ds
\]

\[\lesssim \left[ (1 + |x - at| + \sqrt{t})^{-r} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x - at|^2}{M}} \right] \int_0^{t/2} (1 + s)^{-\frac{1}{2}} (t - s)^{-\frac{1}{2}} ds
\]

\[\lesssim (1 + |x - at| + \sqrt{t})^{-r} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x - at|^2}{M}}.
\]
In order to estimate \( \int_{t/2}^{t} \), we re-prove Lemma 2.3 as
\[
\int_{-\infty}^{\infty} (t - s) - \frac{1}{2} e^{-\frac{|y - (y - as)|^2}{Mt^2}} (1 + |y - as| + \sqrt{s})^{-2r} dy
\leq (1 + |x - at| + \sqrt{s})^{-2r} + (1 + t - s) - \frac{1}{2} e^{-\frac{|y - at|^2}{Mt^2}} \int_{0}^{\infty} (1 + y + \sqrt{s})^{-2r} dy
\leq (1 + |x - at| + \sqrt{s})^{-2r} + (1 + t - s) - \frac{1}{2} e^{-\frac{|y - at|^2}{Mt^2}} (1 + \sqrt{s})^{-2r+1}.
\]
Since \( r \geq \frac{3}{2} \), we obtain
\[
\int_{t/2}^{t} (t - s) - \frac{1}{2} \left[ (1 + |x - at| + \sqrt{s})^{-2r} + (1 + t - s) - \frac{1}{2} (1 + \sqrt{s})^{-2r+1} e^{-\frac{|y - at|^2}{Mt^2}} \right] ds
\leq (1 + |x - at| + \sqrt{t})^{-r} + (1 + t - s)^{-r+\frac{1}{2}} e^{-\frac{|y - at|^2}{Mt^2}}
\leq (1 + |x - at| + \sqrt{t})^{-r} + (1 + t - s)^{-\frac{1}{2} e^{-\frac{|y - at|^2}{Mt^2}}}.
\]
Similarly, by (3.1) \sim (3.3) and Theorem 2.1, we estimate \( \psi_t, \psi_x \) and \( \psi_{xx} \) as
\[
|\psi_t, \psi_x, \psi_{xx}(x, s)| \leq C(E_0 + \zeta(t)^2) \left[ (1 + |x - as| + \sqrt{s})^{-r} + (1 + s)^{-\frac{1}{2} e^{-\frac{|x - as|^2}{M^2}}} \right],
\]
for any \( 0 \leq s \leq t \) and \( x \in \mathbb{R} \); thus we complete the proof.

From here, the proof of Theorem 1.1 goes similarly as in the localized case.

**Proof of Theorem 1.1.** Without loss of generality, we assume \( C > \frac{1}{2} \). Then \( \zeta(0) < 2CE_0 \). Since \( \zeta(t) \) is continuous so long as it remains small, by the continuous induction, \( \zeta(t) < 2CE_0 \) for all \( t \geq 0 \) if \( E_0 < \frac{1}{4C^2} \). Indeed, if \( \zeta(t) = 2CE_0 \), then by Lemma 5.1, \( 2CE_0 \leq C(E_0 + 4C^2E_0^3) \); so \( E_0 \geq \frac{1}{4C^2} \) which is a contradiction.

\[ \square \]
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