Modelling of dynamics of mechanical systems with regard for constraint stabilization

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Abstract. The main purpose of dynamical processes modelling is to formulate the motion equations of the system with regard for active forces and constraints restricted its movement. Desirable properties of system's motion, which are provided by the influence of additional forces and by the variation of inertial system's properties, can be specified by the constraint equations. Nikolay Zhukovskiy studied two main problems on constructing motion equations: defining the force function, that determines a set of motion trajectories, and analysing its stability. The representation of constraint equations as partial integrals of motion equations allows to provide an asymptotic stability of the corresponding integral manifold and to solve the problem of constraint stabilization at numerical solution of dynamics equations.

1. Introduction
The basis of the mathematical model of mechanical system’s dynamics is the system of motion equations. If the set of constraint equations allows to uniquely represent the kinematic state of a system using generalized coordinates and velocities, then the system’s behavior is determined with some particular level of accuracy that depends on applicable methods. Lagrange multipliers can be applied for accounting the constraint influence in the case of impossibility of generalized coordinates and velocities introduction. At the same time, constraint reactions can be considered as control forces providing the realization of constraint equations.

The problem of defining additional control forces, which allow the motion of a mechanical system to have appropriate properties, is related to the inverse dynamical problems [1]. So, based on the properties of planetary motion sir Isaak Newton [2] established the form of the gravity force and later it was discovered [3] that the motion of a material point on a conic section is the effect of a central force, depending on the point's position [4], [5]. The problem of determining a force function, corresponding to a holonomic system with given integrals, was considered by Gavriil Suslov [6]. Also, Nikolay Zhukovskiy established the method of determining a force function, based on a given set of trajectories of a material point on a curved surface, and gave a solution to the problem of motion strength of a representation point [8], using indicators of the kinetic energy system's force function. General Lyapunov's theory of motion stability [9] allowed to formulate the stability criteria of a bunch of trajectories [10] and to develop some new methods of constructing the systems of differential equations, having a given stable integral manifold [11].

Determining the Lagrange multipliers constraint equations are usually considered as the first integrals of motion equations so that the initial data corresponds to them. Deviations of initial data and an application of some approximation methods cause the disruption of constraint equations. The problem
of a constraint stabilization was firstly mentioned by Joachim Baumgarte in his paper [12]. An asymptotic stability of corresponding to the constraint equations integral manifold of a system of dynamics equations is a necessary condition for solving this problem. The solution of this problem can be obtained by introducing additional forces or by changing the inertial properties of a system [13], [14]. The problem of a constraint stabilization leads to the problem of constructing the system of differential equations admitting constraint equations as partial integrals and defining asymptotic stable invariant set [16] or integral manifold [17] of this system. It is quite possible to construct a system of motion equations with required accuracy [18] of constraints deviations using a general approach of solving inverse dynamical problems. Some relevant dynamical analogies allow us to apply methods and equations of classical mechanics to solve problems of modeling and dynamical control of systems of different nature.

2. Problem statement

Let the state of a mechanical system be determined by generalized coordinates \( q = (q_1, \ldots, q_n) \), velocities \( v = (v_1, \ldots, v_n) \), \( dq_i / dt = v_i \), \( i = 1, \ldots, n \), Lagrange function \( L = L(q, v, t) \) and non-potential generalized forces, interacting with a mechanical system, \( Q = (Q_1, \ldots, Q_s) \), \( Q = Q(q, v, t) \). Let’s consider that the system has both holonomic and nonholonomic constraints \( f(q, t) = 0 \), \( f(q, t) = 0 \), \( i = 1, \ldots, n \). So the problem is to determine dynamical equations of this mechanical system that provide the constraint stabilization during the numerical integration.

3. Dynamical equations of an extended system

Let’s introduce new variables: \( \bar{q} = (q^{n+1}, \ldots, q^{n+m}) \), \( \bar{q} = (q^{n+m+1}, \ldots, q^{n+s}) \), \( \bar{v} = (v^{n+1}, \ldots, v^{n+s}) \), \( \bar{v} = (v^{n+1}, \ldots, v^{n+s}) \), such as

\[
\begin{align*}
\bar{q} - f(q, t) &= 0, \\
\bar{v} - g(q, v, t) &= 0,
\end{align*}
\]

The system of equations that defines the virtual displacements can be constructed based on the equation (4) and it takes form

\[
G \delta q = \delta \bar{v}, \quad G = \frac{\partial g}{\partial q}.
\]

A vector of virtual displacements \( \delta q \) can be defined due to the system (5) considering the vector \( \delta \bar{v} \) to be arbitrary. If the columns of the matrix \( G \) are linear independent, then the solution of the system (5) \( \delta q = (\delta q)^T \) consists of a general solution \( (\delta q)^T = \delta \bar{v}[GC]\) of the homogeneous equation and a partial solution \( (\delta q)^T = G^* \delta \bar{v} \) of the nonhomogeneous equation

\[
\delta q = \delta \bar{v}[GC] + G^* \delta \bar{v}.
\]

Here \( \delta \bar{v} \) is an arbitrary scalar value, \( [GC] \) is a cross product of the vectors \( g^\sigma = (g_1^\sigma, \ldots, g_s^\sigma) \), \( \sigma = 1, \ldots, s \), composing the columns of the matrix \( G \) and arbitrary vectors \( c^\gamma = (c_1^\gamma, \ldots, c_n^\gamma) \), \( \gamma = s + 1, \ldots, n - 1 \), \( G^* = G^T (GG^T)^{-1} \).
Let's consider that \( L = L(q, v, t) \) is a Lagrange function, \( Q = (Q_1, \ldots, Q_n) \), \( Q_i = Q_i(q, v, t) \) are non-potential generalized forces, interacting with a mechanical system, \( R = (R_1, \ldots, R_n) \), \( R_i \) are components of the vector of constraint reaction. Let's define the functions \( \tilde{L}(q, v, \tilde{q}, \tilde{v}, t) \) and \( \tilde{D} = \tilde{D}(q, v, \tilde{q}, \tilde{v}, t) \) that satisfy the following conditions: \( \dot{\tilde{L}}(q, v, 0, 0, t) = L(q, v, t) \), \( \tilde{D}(q, v, 0, 0, t) = 0 \) and \( \tilde{D}(q, v, \tilde{q}, \tilde{v}, t) \geq D(q, \tilde{v}) > 0 \) if \( \tilde{q}, \tilde{v} \) do not go to zero simultaneously and \( D(0, 0) = 0 \). If \( R = (R_1, \ldots, R_n) \), \( R_i \) are components of the vector of constraint reaction, then D’Alembert’s principle for the extended system with the Lagrange function \( \tilde{L} \) takes form

\[
E(q, v) - Q - R = 0, \tag{7}
\]

\[
E(\tilde{q}, \tilde{v}) + \frac{\partial \tilde{D}}{\partial \tilde{v}} = 0, \tag{8}
\]

\[
E(q, v) = d \frac{\partial \tilde{L}}{dt} - \frac{\partial \tilde{L}}{\partial q} \tilde{q} = (q, \tilde{q}).
\]

We can define the sum of elementary works by scalar multiplying equations (7) and (8) by \( \delta q \) and \( \delta \tilde{v} \) correspondingly

\[
\left( E(q, v) - Q - R \right)^T \delta q + \left( E(\tilde{q}, \tilde{v}) + \frac{\partial \tilde{D}}{\partial \tilde{v}} \right)^T \delta \tilde{v} = 0, \tag{9}
\]

then we can rewrite it with regard for (6):

\[
\left( E(q, v) - Q - R \right)^T [GC] \delta l + \left( E(q, v) - Q - R \right)^T G^+ + \left( E(\tilde{q}, \tilde{v}) + \frac{\partial \tilde{D}}{\partial \tilde{v}} \right)^T \delta \tilde{v} = 0. \tag{10}
\]

The equality (10) can be accomplished only if the following conditions are satisfied

\[
\left( E(q, v) - Q - R \right)^T [GC] = 0, \tag{11}
\]

\[
\left( E(q, v) - Q - R \right)^T G^+ + \left( E(\tilde{q}, \tilde{v}) + \frac{\partial \tilde{D}}{\partial \tilde{v}} \right)^T = 0. \tag{12}
\]

Let’s choose the vector \( R \), so that the elementary work of constraint reactions due to the displacements \( (\delta q)^T \) will be equal to zero: \( R^T [GC] = 0 \). This fact denotes the correspondence to the ideal constraints of the initial system \( R = G^T \lambda, \lambda = (\lambda_1, \ldots, \lambda_n) \). From the identity (11) follows the equation, describing the variation of generalized coordinates of the system. Identity (12) can be reduced to the equation of the constraints’ perturbations and, as a result, with regard for kinematic equations, identities (3) - (4) and initial conditions we can obtain the following system of differential-algebraic equations for \( q, v, \tilde{q}, \tilde{v}, \lambda \):

\[
\frac{dq}{dt} = v, \quad \frac{d\tilde{q}}{dt} = \tilde{v}, \quad \frac{d\tilde{L}}{dt} - \frac{\partial \tilde{L}}{\partial \tilde{q}} \tilde{q} = Q + G^T \lambda, \tag{13}
\]

\[
\frac{d\tilde{L}}{dt} = \tilde{v}, \quad \frac{d\tilde{L}}{dt} - \frac{\partial \tilde{L}}{\partial \tilde{q}} \tilde{q} = - \frac{\partial \tilde{D}}{\partial \tilde{v}}, \tag{14}
\]

\[
\tilde{q} = f(q, t), \quad \tilde{q} = (\tilde{q}, \tilde{v}), \quad \tilde{v} = g(q, v, t), \tag{15}
\]

\[
q(t_0) = q_0, \quad \tilde{q}(t_0) = f(q_0, t_0), \quad \tilde{q}(t_0) = \tilde{q}_0, \quad v(t_0) = v_0, \quad \tilde{v}(t_0) = g(q_0, v_0, t_0). \tag{16}
\]

It is necessary to complete the right sides of differential equations (13) - (14) to solve the systems (13) - (16). The values of the forces of constraint reactions are determined by defining the multipliers \( \lambda \) that ensure the equalities (15). If we assume that the values of the deviations from the solution of the system (13), (14) are defined with the help of extra variables \( \tilde{q}, \tilde{v} \), then the solution \( q = 0, \tilde{v} = 0 \) of this
system corresponds to the constraint equations and its stability depends on the choice of the Lagrange function \( \bar{L} \) and dissipative function \( \bar{D} \). Let’s consider that functions \( \bar{L} \) and \( \bar{D} \) take form

\[
2\bar{T} = v^T A(q)v + \bar{v}^T \bar{A}(q,\bar{q})\bar{v}, \quad 2\bar{P} = q^T \bar{H}(q)\bar{q}, \quad 2\bar{D} = \bar{v}^T B(q,\bar{q})\bar{v},
\]

\[
\bar{H}(q) = \begin{pmatrix} \bar{H}(q) & 0 \\ 0 & 0 \end{pmatrix}.
\]

The values of \( \lambda \) are determined from the equations (13) – (15):

\[
\lambda = M^{-1}(q,v,t)h(q,v,t), \quad M = GA^{-1}(q)G^T,
\]

\[
h = \tilde{A}^{-1}(q,\bar{q})\left( \frac{\partial \bar{L}}{\partial \bar{q}} - \frac{d\bar{A}(q,\bar{q})}{dt} \bar{v} - B(q,\bar{q})\bar{v} \right) + GA^{-1}(q)\left( \frac{dA(q)}{dt} v - \frac{\partial \bar{L}}{\partial q} q + \phi \bar{g} v - \bar{g} \phi \right),
\]

\[
q = f(q,t), \quad \bar{v} = g(q,v,t).
\]

Dynamical equations (13) with regard for the values \( \bar{L}, \bar{q}, \bar{v}, \lambda \) as functions of the variables \( q,v,t \) are reduced to the following system of differential equations:

\[
\frac{dq}{dt} = v, \quad \frac{dv}{dt} = A^{-1}(q)\left( \frac{\partial \bar{L}}{\partial \bar{q}} - \frac{dA(q)}{dt} \bar{v} + Q(q,v,t) + G^T(q,v,t)\lambda(q,v,t) \right),
\]

that has the partial integrals:

\[
f(q,t) = 0, \quad g(q,v,t) = 0.
\]

4. Stability of integral manifold

The constraint (1), (2) stabilization requires the asymptotic stability of an integral manifold of the system (18) given by the equalities (19). This stability of an integral manifold can be defined with the help of the following terms.

An integral manifold of the system (18), given by the equalities (19), is stable if for any \( \epsilon > 0 \) there exists such \( \delta > 0 \), so the solution \( q = q(t), \quad v = v(t) \) of the system, corresponding to the initial conditions \( q(t_0) = q_0, \quad \bar{v}(t_0) = v_0 \), \( \|f(q_0,t_0)\| \leq \delta, \quad \|g(q_0,v_0,t_0)\| \leq \delta \), will satisfy the inequalities \( \|f(q(t),t)\| \leq \epsilon, \quad \|g(q(t),v(t),t)\| \leq \epsilon \).

An integral manifold of the system (18), given by the equalities (19), is asymptotically stable, if it is stable and the following conditions are satisfied:

\[
\lim_{t \to \infty} \|f(q(t),t)\| = 0, \quad \lim_{t \to \infty} \|g(q(t),v(t),t)\| = 0.
\]

It is obvious that the stability of an integral manifold is determined by the corresponding stability of the trivial solution \( \bar{q}(t) = 0, \quad \bar{v}(t) = 0 \) of the system of the perturbation constraint equations (14). With the regard for values of \( \bar{L}, \bar{D} \) (17) the system (14) can be written in the form:

\[
\frac{d\bar{q}}{dt} = \bar{v}, \quad \frac{d\bar{v}}{dt} = -\bar{\Sigma}(q,\bar{q})\bar{q} - \bar{K}(q,\bar{q})\bar{v} + \tilde{A}^{-1}(q,\bar{q})\left( \bar{v} + \frac{\partial \tilde{A}(q,\bar{q})}{\partial \bar{q}} \bar{v} \right),
\]

\[
\bar{\Sigma}(q,\bar{q}) = \tilde{A}^{-1}(q,\bar{q})\bar{\Sigma}(q), \quad \tilde{K}(q,\bar{q}) = \tilde{A}^{-1}(q,\bar{q})\left( \frac{d\tilde{A}(q,\bar{q})}{dt} + B(q,\bar{q}) \right),
\]

or after expanding the matrices \( \bar{\Sigma}(q,\bar{q}) \) and \( \tilde{K}(q,\bar{q}) \) in a series in powers of \( \bar{q} \):

\[
\bar{\Sigma}(q,\bar{q}) = \bar{\Sigma}(q,0) + \frac{\partial \bar{\Sigma}(q,0)}{\partial \bar{q}} \bar{q} + \bar{\Sigma}^{(2)}(q,0) + \frac{\partial \bar{\Sigma}(q,0)}{\partial \bar{q}} \bar{q} + \bar{\Sigma}^{(3)}(q,0), \quad \tilde{K}(q,\bar{q}) = \tilde{K}(q,0) + \frac{\partial \tilde{K}(q,0)}{\partial \bar{q}} \bar{q} + \tilde{K}^{(2)}(q,0),
\]
it can take the following form:

\[
\frac{dq}{dt} = \dot{q}, \quad \frac{d\dot{q}}{dt} = -S(q)\dot{q} - K(q)\ddot{q} + \dddot{V},
\]

(20)

Let’s introduce the following notation:

\[
y = (\dot{q}, \ddot{q}), \quad W(q) = \begin{pmatrix} 0 & E \\ -S(q) & -K(q) \end{pmatrix}, \quad Y^{(2)} = \begin{pmatrix} 0 \\ \dddot{V} \end{pmatrix},
\]

and rewrite the system (20) in an abbreviated form:

\[
\frac{dy}{dt} = W(q)y + Y^{(2)}.
\]

(21)

If all of the roots of the characteristic polynomial of the matrix \( W(q) \) at all of the possible values of the generalized coordinates \( q_1, \ldots, q_n \) at their domain \( \Omega_q \) have negative real parts then a trivial solution of the equation with primary approximation:

\[
\frac{dy}{dt} = W(q)y
\]

is asymptotic stable. The problem the holonomic constraint stabilization is studied in the paper [12] using the equations of perturbed constraints with the constant matrix

\[
W = \begin{pmatrix} 0 & E \\ -\omega^2 E & -\gamma E \end{pmatrix}.
\]

An algorithm of solving the problem of stabilization with the matrix \( W = W(q,v) \), determined with the help of the matrix \( G \) and its derivative, is established in the paper [19].

In general the method of Lyapunov’s function is applied [11] to define the sufficient conditions of the stability of a trivial solution (21). If the constraints are scleronomic

\[
f(q) = 0, \quad g(q,v) = 0,
\]

then we can take as a Lyapunov’s function a positive definite quadratic form with the constraint matrix with coefficients \( 2V = y^T U y \). If the derivative of this function

\[
\frac{dV}{dt} = y^T UW(q)y + Y^{(3)}
\]

is negative definite then the trivial solution \( y = 0 \) of the equation (21) is asymptotic stable.

5. Numerical solution

If the perturbation constraint equations have asymptotic stable trivial solution then we can limit our choice with the simplest numerical methods of solving the dynamical equations (18). So the application of the finite-difference scheme

\[
x_{k+1} = x_k + \tau X_k, \quad x_k = x(t_k), \quad \tau = t_{k+1} - t_k, \quad x(t_0) = x_0,
\]

\[
x = \begin{pmatrix} q \\ v \end{pmatrix}, \quad X = \begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix}, \quad F(q,v,t) = A^{-1}(q) \left( \frac{\partial L}{\partial q} \frac{dA(q)}{dt} v + Q(q,v,t) + G^T(q,v,t) \lambda(q,v,t) \right)
\]

with regard for (21) leads to the inequality
\[ \| y_{k+1} \| \leq \| E - \tau W_k \| \| y_k \| + \gamma_k^{(2)}, \]

where \( \gamma_k^{(2)} \) is a reminder. From inequality (23) follows the estimation \( \| y_{k+1} \| \leq \varepsilon \) if for all \( k = 0, 1, 2, ..., N \) the following conditions:

\[ \| y_k \| \leq \alpha, \quad \| E - \tau W_k \| \leq \alpha \leq 1, \quad \| \gamma_k^{(2)} \| \leq 2\varepsilon(1 - \alpha) \]

are satisfied.

If solving equations (18) we use the finite difference scheme

\[ x_{k+1} = x_k + \Delta x_k, \quad \Delta x_k = \tau (1 - \sigma) X_k + \tau \sigma \dot{X}_k, \]

\[ \dot{X}_k = X_k (\dot{x}_k, \dot{t}_k + \alpha \tau), \quad \dot{x}_k = x_k + \alpha \tau X_k, \quad \sigma > 0, \quad \alpha > 0, \quad k = 0, 1, 2, ..., N, \]

where \( \alpha, \sigma - \text{const} \) and for all \( k = 0, 1, 2, ..., N \) the following conditions are satisfied

\[ 2\alpha \sigma = 1, \quad \| y_k \| \leq \varepsilon, \quad \| \gamma_k^{(1)} \| \leq 6\varepsilon(1 - \beta), \quad \| x_{k+1} - \tau W_k + \frac{1}{2} \tau^2 \left( W_k + \left( \frac{\partial W}{\partial t} \right)_k \right) \| \leq \beta \leq 1, \]

then we have the estimation \( \| y_{k+1} \| \leq \varepsilon \). The conditions of the constraint stabilization were obtained by the Runge-Kutta method in the paper [21].

6. Example

Our goal is to determine the law of the variation of the force \( F \) providing the stable motion of the rocket on the trajectory \( f(x, y) = 0 \). The rocket is considered as a material point that has coordinates

\[ q = (x, y), \quad v = (v_x, v_y), \quad v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt} \]

and it interacts with the force of gravity \( mg \) directed vertically downwards. The deviation of the point from its trajectory and its derivative are denoted as

\[ \ddot{q} = f(x, y), \quad \ddot{v} = \dot{v}_x + \frac{\partial f}{\partial x} v_x + \frac{\partial f}{\partial y} v_y. \quad (22) \]

Let’s introduce Lagrange and dissipative functions

\[ 2\dot{L} = mv^2 - 2mgv + \ddot{v}^2 - c\dddot{v}^2, \quad 2D = kv^2, \quad (23) \]

\[ v^2 = v_x^2 + v_y^2, \quad c, k, g - \text{const.} \]

From the equalities (22) follows the equation:

\[ \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y = \delta \ddot{v}, \]

that determines virtual displacements of a point depending on the arbitrary values \( \delta s \) and \( \delta \ddot{v} \):

\[ \delta x = -\frac{\partial f}{\partial y} \delta s + \frac{\partial f}{\partial x} \left( \frac{\partial f}{\partial x} \right)^{-1} \delta \ddot{v}, \quad \delta y = \frac{\partial f}{\partial x} \delta s + \frac{\partial f}{\partial y} \left( \frac{\partial f}{\partial x} \right)^{-1} \delta \ddot{v}. \]

Using D’Alembert-Lagrange principle

\[ \left( \frac{d}{dt} \frac{\partial \dot{L}}{\partial \dot{x}} - \frac{\partial L}{\partial x} \right) \delta x + \left( \frac{d}{dt} \frac{\partial \dot{L}}{\partial \dot{y}} - \frac{\partial L}{\partial y} \right) \delta y + \left( \frac{d}{dt} \frac{\partial \dot{L}}{\partial \ddot{q}} - \frac{\partial L}{\partial \ddot{q}} \right) \delta \ddot{v} = 0 \]

let’s construct the dynamical equations of the rocket

\[ \frac{dy}{dt} = v_y, \quad \frac{d}{dt} (mv_x) = \lambda \frac{\partial f}{\partial x} (mv_y) = -mg + \lambda \frac{\partial f}{\partial y}, \quad (24) \]

and the perturbation constraint equations

\[ \frac{dq}{dx} = \ddot{v}, \quad \frac{d}{dt} (c\dddot{v} - k\ddot{v}) = 0. \quad (25) \]
Let’s introduce $\mu$ as a velocity of a particle separation from the rocket and considering the equality
\[
\frac{d}{dt}(mv) = m\frac{dv}{dt} - \mu \left( \frac{dm}{dt} \right) v
\]
we can rewrite (24) in the following form:
\[
\begin{align*}
\frac{dx}{dt} &= v_x, m\frac{dv_x}{dt} = T_x + \lambda \frac{\partial f}{\partial x}, \\
\frac{dy}{dt} &= v_y, m\frac{dv_y}{dt} = T_y - mg + \lambda \frac{\partial f}{\partial y}, \\
T_x &= \frac{\mu v_x}{\sqrt{v_x^2 + v_y^2}} \frac{dm}{dt}, \\
T_y &= \frac{\mu v_y}{\sqrt{v_x^2 + v_y^2}} \frac{dm}{dt}.
\end{align*}
\] (26)

Right parts of the equations (26) contain the traction force $T = (T_x, T_y)$ directed along the tangent to the motion trajectory and the constraint reaction $R = \lambda \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ directed along the normal. These two forces form the unknown force $F$. From the equations (22), (25), (26) we can determine the Lagrange multipliers
\[
\lambda = -\frac{m}{N^2} \left( \frac{\partial^2 f}{\partial x^2} v_x^2 + 2 \frac{\partial^2 f}{\partial x \partial y} v_x v_y + \frac{\partial^2 f}{\partial y^2} v_y^2 + cf(x, y) + k \left( \frac{\partial f}{\partial x} v_x + \frac{\partial f}{\partial y} v_y \right) \right) + \frac{1}{N^2} \frac{\mu}{\sqrt{v_x^2 + v_y^2}} \left( \frac{\partial f}{\partial x} v_x + \frac{\partial f}{\partial y} v_y \right) \frac{dm}{dt},
\] (27)
\[
N^2 = \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2.
\]

The motion on the trajectory will be asymptotic stable if the roots the characteristic equation of the system (25) $\kappa^2 + k\kappa + \epsilon = 0$ have negative real parts. Numerical solution of the system (26), (27)
\[
x(t_k) = x_k, x_{k+1} = x_k + v_{x_{k+1} k}, y_{k+1} = y_k + v_{y_{k+1} k},
\]
\[
(v_{x_{k+1} k}) = (v_x)_k + \frac{\tau}{m_k} \left( T_x + \lambda \frac{\partial f}{\partial x}_k \right), (v_{y_{k+1} k}) = (v_y)_k + \frac{\tau}{m_k} \left( T_y - mg + \lambda \frac{\partial f}{\partial y}_k \right),
\]
will satisfy the equality $\|y_k\| \leq \epsilon$ for all $k = 0, 1, 2, \ldots, K$ if the conditions $\|y_k\| \leq \epsilon$, $\|E - \tau W\| \leq \alpha \leq 1$, $\|W_k^{(2)}\| \leq 2\epsilon(1 - \alpha)$ are satisfied, where $W$ is a matrix of the coefficients of the system (25), $W_k^{(2)}$ is a reminder of the function’s $\tilde{q} = f(x, y)$ expansion.

7. Conclusions

The methods of solving inverse dynamical problems and the conditions of the stability of a set of trajectories based on the Zhukovskiy’s papers allow us to develop an algorithm of solving dynamical problems of mechanical systems and problems of dynamical processes control in the systems with different nature.

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