Cascades and Dissipative Anomalies in Nearly Collisionless Plasma Turbulence

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We develop first-principles theory of kinetic plasma turbulence governed by the Vlasov-Maxwell-Landau equations in the limit of vanishing collision rates. Following an exact renormalization-group approach pioneered by Onsager, we demonstrate the existence of a “collisionless range” of scales (lengths and velocities) in 1-particle phase space where the ideal Vlasov-Maxwell equations are satisfied in a “coarse-grained sense”. Entropy conservation may nevertheless be violated in that range by a “dissipative anomaly” due to nonlinear entropy cascade. We derive “4/5th-law” type expressions for the entropy flux, which allow us to characterize the singularities (structure-function scaling exponents) required for its non-vanishing. Conservation laws of mass, momentum and energy are not afflicted with anomalous transfers in the collisionless limit. In a subsequent limit of small gyroradii, however, anomalous contributions to inertial-range energy balance may appear due both to cascade of bulk energy and to turbulent redistribution of internal energy in phase space. In that same limit the “generalized Ohm’s law” derived from the particle momentum balances reduces to an “ideal Ohm’s law”, but only in a coarse-grained sense that does not imply magnetic flux-freezing and that permits magnetic reconnection at all inertial-range scales. We compare our results with prior theory based on the gyrokinetic (high gyro-frequency) limit, with numerical simulations, and with spacecraft measurements of the solar wind and terrestrial magnetosphere.

I. INTRODUCTION

In turbulent plasmas at very high temperatures and low densities the collisions of constituent particles are so infrequent that fluid models assuming small mean-free path lengths are invalid and the plasma must be described by kinetic equations for the particle distribution functions [1]. A prime example of societal importance is magnetic-confinement fusion [2], where turbulent transport essentially limits performance but where mean free path lengths at typical operating conditions are ~10 km, much larger than the size of the device. The solar wind is one of the best-studied examples of a turbulent plasma in Nature, with a wealth of in situ spacecraft measurements showing turbulent-like spectra down to lengths of order a kilometer, but the mean-free-path for electron collisions in the near-Earth solar wind is ~1 AU [3]. The terrestrial magnetosphere is likewise a nearly collisionless plasma with turbulence occurring either typically (magnetosheath) or sporadically (magnetopause) [4]. The Magnetospheric Multiscale mission [5, 6] is currently measuring proton and electron velocity distribution functions in this environment at high phase-space resolution and cadence. Exploration of this velocity-space has been described as the “next frontier” of kinetic heliophysics [7]. More generally, turbulent, nearly collisionless plasma environments are ubiquitous in astrophysics. The interstellar medium exhibits an approximately Kolmogorov spectrum of electron density over ~13 orders of magnitude, the so-called “Big Power Law in the Sky” [8-10] but almost a third of this range lies below the ion mean-free path length ~10^7 km.

These diverse physics challenges call for fundamental theory of kinetic plasma turbulence. Recently, a first-principles paradigm has emerged for fluid turbulence, based upon a mathematical analysis pioneered by Onsager for incompressible fluids [11,13]. In this “ideal turbulence” theory, the dissipative anomaly—or non-vanishing dissipation of kinetic energy in the inviscid limit—is explained as a consequence of nonlinear energy cascade for “coarse-grained” or “distributional” solutions of the incompressible Euler equations. Onsager’s analysis can be understood as an exact, non-perturbative application of the principle of renormalization-group (RG) invariance [14-16], and it predicts the fluid Holder singularities necessary for turbulent energy cascade. This “ideal turbulence” theory for incompressible fluids has recently been supported by rigorous mathematical developments following from the Nash-Kuiper theorem and Gromov’s h-principle [17-18]. The physical domain of the Onsager theory has also been extended recently to compressible fluids, both non-relativistic [19,20] and relativistic [21], with cascade of thermodynamic entropy and anomalous entropy production as central concepts.

The purpose of this paper is to develop a similar exact theory for kinetic turbulence of nearly collisionless plasmas. It was suggested already some time ago by Krommes & Hu [22-23] that collisional production of kinetic entropy should remain non-zero in plasmas with vanishing collision rates. Empirical evidence for such “anomalous entropy production” has since been obtained by numerical simulations of gyrokinetic turbulence, both forced and decaying (see section VII B for a review). In the gyrokinetic formulation, Schekochihin et al. [24,25] made an explicit analogy with the inertial-range of incompressible turbulence and proposed a gyrokinetic “entropy cascade” through a range of scales in phase-space where collisions can be neglected. We here derive this picture for the full Vlasov-Maxwell-Landau equations of a
weakly-coupled, multi-species plasma, by extending Onsager’s exact, non-perturbative RG analysis to phase-space. The “collisionless range” of scales shall be shown to be governed by “coarse-grained” or “distributional” solutions of the Vlasov-Maxwell kinetic equations, with an entropy-production anomaly due to a nonlinear entropy cascade. We derive expressions for the entropy flux through phase-space scales that are analogous to the 4/5th-law of Kolmogorov for energy flux in incompressible turbulence and we exploit them to deduce the singularities of particle distributions and electromagnetic fields that are required in order to sustain the cascade of entropy.

Such a careful, systematic mathematical framework for kinetic plasma turbulence is valuable not only for its predictive power and conceptual clarity, but also is necessary to avoid inconsistencies and apparent contradictions that arise from naive, informal discussions. There is an analogy with the theory of collisional transport in plasmas which, prior to the systematic derivation by Braginskii, led frequently to “paradoxes which have been the source of various errors and ambiguities” (p.213). An even closer analogy is the situation in elementary particle physics prior to the discovery of the axial anomaly in quantum gauge theories. Naively, both the vector and axial-vector currents are conserved in massless spinor electrodynamics, but the simultaneous assumption of both conservation laws leads to the Veltman-Sutherland “paradox” and the “forbidden” soft pion decay $\pi^0 \rightarrow \gamma\gamma$. As is well-known, this paradox is resolved by the chiral anomaly, which modifies the naive conservation of axial current and which accounts for the experimentally observed neutral-pion decay in a pseudovector coupling calculation. The origin of the axial anomaly lies in the ultraviolet divergences that appear when quantum fields, which exist only as distributions or generalized functions, are naively multiplied pointwise. Careful regularization of these divergences, e.g. by gauge-invariant “point-splitting” of the spinor fields, yields the anomaly in axial charge conservation. The turbulent dissipative anomaly in the naive conservation of kinetic energy arises in a very similar manner, as stressed by Polyakov and already understood long ago by Onsager.

The need for sophistication in treating kinetic plasma turbulence is quite clear from the fact that the “collisionless range” of scales is governed by the Vlasov-Maxwell equations, in a certain sense, but entropy is nevertheless not conserved, as it would be for smooth solutions of the Vlasov-Maxwell equations in the standard sense. Similar cautionary remarks apply not just to entropy conservation in a turbulent plasma, but also to other quantities which are naively conserved. For example, it is true that the ideal Ohm’s law is valid (in a certain sense) in the inertial range of the solar wind, at scales much larger than the ion gyroradius, but this law does not hold in a manner that implies conservation of magnetic flux at those scales, as is frequently asserted. This fact has important implications for the problem of magnetic reconnection in a turbulent plasma. We shall treat this problem here in the framework of the Vlasov-Maxwell-Landau kinetic theory, by considering the momentum conservation of the various charged particle species and the “generalized Ohm’s law” derived from them. We shall also discuss the energy balances of the particle species and of the electromagnetic fields, in order to investigate the possibility of energy cascades in kinetic plasma turbulence. Energy and momentum in totality (particles + fields) are conserved in the Vlasov-Maxwell-Landau model, so that no dissipative anomaly of total energy or momentum is possible. There can, however, be anomalous transfers between different components of energy (electromagnetic, kinetic energy of bulk velocities, kinetic energy of fluctuation velocities) and also in phase-space. We investigate this possibility in the limit of vanishing collision rates and also in subsidiary limits, such as vanishingly small gyroradii.

A notable aspect of the analysis presented here is that it involves almost no discussion of the rich array of linear waves supported by a plasma (shear-Alfvén waves, slow/fast magnetosonic waves, ion acoustic waves, kinetic Alfvén waves, whistler waves, etc.). This contrasts with the vast majority of works, where plasma turbulence is regarded by default as an array of interacting linear waves. The dominance of this wave point of view is due in part to its great empirical success, with the imprint of linear waves, such as their dispersion relations and eigenmodes, often clearly observed even in strongly interacting turbulent plasmas. On the other hand, it is also true that the mathematics of linear plasma waves is very familiar and well-developed, whereas exact nonlinear theory of kinetic plasma turbulence is less straightforward and far fewer works are devoted to it. In their recent discussion of the wave-turbulence dichotomy, Coburn et al. have remarked that:

“For most of the space age our view of solar wind fluctuations (magnetic, velocity, density, etc.) has been based on the theory of plasma waves. Attempts to incorporate turbulence concepts into this thinking have often been treated as little more than an afterthought that is either a secondary dynamic or a concept in direct conflict with the wave interpretation.” — [39], p.1.

It is the main purpose of the present paper to satisfy this need and to provide exact, first-principles theory of the nonlinear cascades in kinetic turbulence of nearly collisionless plasmas. We shall remain mostly silent on the linear wave aspects, but this involves no rejection of their importance. A complete theory of kinetic plasma turbulence will certainly require a full synthesis of the linear wave and nonlinear cascade points of view.
II. VLASOV-MAXWELL-LANDAU EQUATIONS

The theory of kinetic plasma turbulence in the present paper will be developed within the framework of the Vlasov-Maxwell-Landau equations for a weakly-coupled plasma, with a large Debye number or plasma parameter, \( \Lambda = n \lambda_D^3 \gg 1 \) (where \( \Lambda_D \) is the Debye length). In order to provide background and to set notations, we briefly describe this system and its basic properties, the dimensionless number groups which characterize its solutions, and important prior work on the collisionless limit.

A. Basic Equations

The Vlasov-Maxwell-Landau equations describe the evolution of the distribution functions \( f_s(x, v, t) \) in one-particle phase-space of \( S \) species of particles with charges \( q_s \) and masses \( m_s \), \( s = 1, \ldots, S \), and of the smoothed electromagnetic fields \( E(x, t), B(x, t) \), conditionally averaged over microscopic molecular states with given particle distributions \( f_s, s = 1, \ldots, S \); see e.g. \([42]\). These equations in the non-relativistic case have the form of a Boltzmann-type kinetic equation for each species

\[
\partial_t f_s + v \cdot \nabla_x f_s + q_s E_s \cdot \nabla_p f_s = C_s(f)
\]  

or

\[
\partial_t f_s + \nabla_x \cdot (v f_s) + \nabla_p \cdot (q_s E_s f_s) = C_s(f),
\]

for \( s = 1, \ldots, S \) and the conditionally-averaged Maxwell equations

\[
\begin{aligned}
\nabla_x \cdot E & = 4\pi \sum_s q_s n_s \\
\nabla_x \times B & = \frac{1}{c} \partial_t E = \frac{4\pi}{c} j \\
\n\nabla_x \cdot E & = \frac{1}{c} \partial_t B = 0, \quad \nabla_x \cdot B = 0.
\end{aligned}
\]

with electric field in the rest frame of the particle population with velocity \( v \) given by:

\[
E_s = E + \frac{1}{c} v \times B,
\]

with particle number density:

\[
n_s(x, t) = \int d^3 v f_s(x, v, t),
\]

mass density \( \rho_s = m_s n_s \), and momentum density:

\[
\rho_s(x, t) u_s(x, t) = \int d^3 v m_s v f_s(x, v, t)
\]

for \( s = 1, \ldots, S \) and with total electric current density:

\[
J(x, t) = \sum_s q_s n_s(x, t) u_s(x, t).
\]

The equations \((\text{II.1})\) and \((\text{II.2})\) are equivalent because the vector-field \((v, q_s E_s)\) is Hamiltonian and has zero phase-space divergence \( \nabla_x v + \nabla_p (q_s E_s) = 0 \). Note that we avoid additional factors of \( m_s \) in the equations by introducing the momentum variable \( p = m_s v \) for each species. To complete the description, we need to specify the collision operator for species \( s \), given by

\[
C_s(f) = \sum_{s'} C_{ss'}(f_s, f_{s'})
\]

summed over collisions with species \( s' \). Here we shall consider the Landau collision operator:

\[
C_{ss'}(f_s, f_{s'}) = 2\pi q_s^2 q_{s'}^2 \ln \Lambda \times \nabla_p \cdot \left[ \int d^3 v \frac{\Pi_{v-v'}}{|v-v'|} (\nabla_p - \nabla_{p'}) (f_s f_{s'}) \right],
\]

where \( f_s = f_s(x, v, t), f_{s'} = f_{s'}(x, v', t), \) where \( \Pi_{v-v'} = I - \Pi_{w} = I - w w' / |w|^2 \) is the projection orthogonal to \( w \), and where the plasma parameter \( \Lambda \) arises as a cut-off in the collision integral for impact factors greater than the Debye length (and, in principle, depends upon \( s, s' \) pairs). Although this is a standard kinetic model for a plasma, it has never been rigorously derived from a microscopic description and global existence of (strong) solutions is an open problem. Physically, alternative collision-integrals such as that of Balescu-Lenard might give improved accuracy when large-velocity bumps or tails develop in the distribution functions. However, so long as these improved collision integrals satisfy an \( H \)-theorem and have similar differential form as the Landau operator (albeit with higher-order nonlinearity), then the analysis of the present paper will carry over.

B. Conservation Laws and \( H \)-Theorem

Essential properties of the kinetic equations are the local conservation laws for the various quantities preserved by collisions. The mass for each particle species is conserved when collisions do not transform one species to another, so that \( \int d^3 v C_{ss'} = 0 \), and the \( m_s \)-moment of \((\text{II.1})\) in integration over velocity \( v \) then gives

\[
\partial_t \rho_s + \nabla_x \cdot (\rho_s u_s) = 0.
\]

Momentum balance for species \( s \) is obtained from the first moment of \((\text{II.1})\) with \( m_s v \), or:

\[
\partial_t (\rho_s u_s) + \nabla_x \cdot (\rho_s u_s u_s + P_s) = q_s n_s E_{ss} + R_s
\]

where the pressure tensor is

\[
P_s = \int d^3 v m_s (v - u_s) (v - u_s) f_s,
\]

the electric field in the bulk rest-frame of species \( s \) is

\[
E_{ss} = E + \frac{1}{c} u_s \times B.
\]


and the drag force on species $s$ is
\begin{equation}
\mathbf{R}_s = \sum_s \int d^3v \, m_s v \, \mathbf{C}_{ss'}.
\end{equation}

When $\sum_s \mathbf{R}_s = 0$, the total momentum of the particles and fields is conserved. Finally, taking the moment of (II.11) with $(1/2)m_s|v|^2$ gives kinetic energy balances for each species $s$:
\begin{equation}
\partial_t E_s + \nabla \cdot (E_s \mathbf{u}_s + \mathbf{P}_s \cdot \mathbf{u}_s + \mathbf{q}_s) = j_s \cdot \mathbf{E} + \mathbf{R}_s \cdot \mathbf{u}_s + Q_s,
\end{equation}
with $j_s = q_s n_s \mathbf{u}_s$ the partial electric current of species $s$, with kinetic energy density
\begin{equation}
E_s = \int d^3v \, \frac{1}{2} m_s |v|^2 f_s,
\end{equation}
with heat flux,
\begin{equation}
\mathbf{q}_s = \int d^3v \, \frac{1}{2} m_s |v - \mathbf{u}_s|^2 (\mathbf{v} - \mathbf{u}_s) f_s
\end{equation}
and with collisional heat exchange with other species
\begin{equation}
Q_s = \sum_{s'} \int d^3v \, \frac{1}{2} m_s |v - \mathbf{u}_s|^2 \mathbf{C}_{ss'}.
\end{equation}

Total energy of the particles and fields is conserved when collisions are elastic and $\sum_s (\mathbf{R}_s \cdot \mathbf{u}_s + Q_s) = \sum_{ss'} \int d^3v \, (1/2)m_s |v|^2 \mathbf{C}_{ss'} = 0$. The Landau operator $\mathbf{Q}$, as well known, has all of these properties.

One can further subdivide the energy density $E_s$ of species $s$ into a bulk kinetic energy density $(1/2)\rho_s |\mathbf{u}_s|^2$ and an “internal” or fluctuation energy density
\begin{equation}
\epsilon_s := \frac{1}{2} \text{Tr} \mathbf{P}_s = \int d^3v \, \frac{1}{2} m_s |v - \mathbf{u}_s|^2 f_s.
\end{equation}

Note that $\epsilon_s = (3/2)\rho_s$ if the pressure tensor is decomposed into a scalar pressure $p_s$ and a traceless, anisotropic pressure tensor $\mathbf{P}_s$, as $\mathbf{P}_s = p_s \mathbf{I} + \mathbf{P}_s$. It is easy using (II.9), (II.11) to derive the balance equation for bulk kinetic energy of species $s$:
\begin{equation}
\partial_t (\frac{1}{2} \rho_s |\mathbf{u}_s|^2) + \nabla \cdot \left( \frac{1}{2} \rho_s |\mathbf{u}_s|^2 \mathbf{u}_s + \mathbf{P}_s \cdot \mathbf{u}_s \right) = \mathbf{P}_s \cdot \nabla \mathbf{u}_s + j_s \cdot \mathbf{E} + \mathbf{R}_s \cdot \mathbf{u}_s,
\end{equation}
and then by subtracting (II.19) from (II.15) to obtain the balance equation for internal/fluctuation energy:
\begin{equation}
\partial_t \epsilon_s + \nabla \cdot (\epsilon_s \mathbf{u}_s + \mathbf{q}_s) = -\mathbf{P}_s \cdot \nabla \mathbf{u}_s + Q_s
\end{equation}

As well-known, this quantity simply counts the number of microstates of particle species $s$ compatible with the given macroscopic distribution $f_s$. Using (II.1), the density $\mathcal{J}(f_s)$ is easily shown to satisfy the phase-space balance equation:
\begin{equation}
\partial_t \mathcal{J}(f_s) + \nabla_x \cdot (\mathcal{J}(f_s) \mathbf{v}) + \nabla_p \cdot (q_s \mathbf{E} \cdot \mathcal{J}(f_s)) = -\left( \ln f_s + 1 \right) C_s(f).
\end{equation}

When integrated over $\mathbf{v}$ and summed over $s$, this gives the balance of total particle entropy density in space
\begin{equation}
s_{\text{tot}}(f) = \sum_s \int d^3v \, \mathcal{J}(f_s)
\end{equation}
of the form
\begin{equation}
\partial_t s_{\text{tot}} + \nabla \cdot \mathbf{J}_s = \sigma,
\end{equation}
with spatial entropy current density
\begin{equation}
\mathbf{J}_s = \sum_s \int d^3v \, \mathbf{v} \mathcal{J}(f_s)
\end{equation}
and local entropy production rate
\begin{equation}
\sigma(\mathbf{x}, t) := -\sum_s \int d^3v \ln f_s C_s
= \sum_{ss'} \frac{\Gamma_{ss'}}{2} \int d^3v \int d^3v' \frac{|\mathbf{I} - \mathbf{v} \cdot \mathbf{v}' - (\mathbf{ \nabla } \mathbf{v} - \mathbf{ \nabla } \mathbf{v}' ) (f_s f_{s'})|^2}{f_s f_{s'} |\mathbf{v} - \mathbf{v}'|}
\geq 0.
\end{equation}

Here we have introduced the shorthand notation $\Gamma_{ss'} = q^2_s q^2_{s'} \ln \Lambda$. The non-negativity of the entropy production in (II.27) is the statement of the $H$-theorem for the Landau collision operator.

C. Dimensionless Quantities

We now consider the Vlasov-Maxwell-Landau equations in a dimensionless form. For each species $s = 1, ..., S$, we take as characteristic length the largest scale of variation $L_s$ of the distribution function of species $s$. The characteristic velocity for species $s$ will be taken to be its thermal velocity $v_{th,s}$ and the characteristic time to be $\tau_s = L_s / v_{th,s}$. The characteristic magnitude of $f_s$ will be taken as $\langle n_s \rangle / v_{th,s}^3$, where $\langle n_s \rangle$ is the mean density of species $s$. We thus introduce dimensionless variables:
\begin{equation}
\hat{x} = x / L_s, \quad \hat{t} = t / \tau_s, \quad \hat{v} = v / v_{th,s}, \quad \hat{f}_s = v^3_{th,s} f_s / \langle n_s \rangle
\end{equation}
for each separate species $s = 1, ..., S$. In order to non-dimensionalize electromagnetic variables we introduce an effective density $n_0$ and length-scale $L_0$ so that typical field magnitudes are $E_0 \sim B_0 \sim c n_0 L_0$ and we then take
\begin{equation}
\hat{x} = x / L_0, \quad \hat{t} = ct / L_0, \quad \hat{E} = E / E_0, \quad \hat{B} = B / B_0
\end{equation}
Using $q_s = Z_s e$, the inhomogeneous Maxwell equations in these rescaled variables become

$$\nabla_s \hat{\mathbf{E}} = 4\pi \sum_s \frac{\langle n_s \rangle}{n_0} Z_s \hat{n}_s$$

$$\nabla_s \times \mathbf{B} - \partial_t \hat{\mathbf{E}} = 4\pi \sum_s \frac{\langle n_s \rangle}{n_0} \frac{v_{th,s}}{c} Z_s \hat{n}_s \hat{u}_s$$  \hspace{1cm} (II.30)

while the homogeneous Maxwell equations are unchanged in form. Note that the length-scale $L_0$ drops out of the rescaled equations (II.30) and one of the factors $n_0$, $L_0$ can be chosen as desired, e.g., to be a typical magnitude of $(n_s)$ or of $L_s$, if these are of similar orders of magnitude for all $s = 1, ..., S$. With the rescaled variables in (II.28) and the rescaled field-strengths in (II.29), one then obtains the dimensionless kinetic equation for species $s$ as

$$\partial_t \hat{f}_s + \hat{\mathbf{v}} \cdot \nabla_s \hat{f}_s + (Z_s/\beta_{0s}) \hat{\mathbf{E}}_s \cdot \nabla_p \hat{f}_s = \sum_{s'} \Gamma_{ss'} \hat{C}(\hat{f}_s, \hat{f}_{s'})$$  \hspace{1cm} (II.31)

where

$$\hat{\mathbf{E}}_s = \hat{\mathbf{E}} + \frac{v_{th,s}}{c} \hat{\mathbf{v}} \times \mathbf{B},$$

$$\beta_{0s} = \frac{m_s v_{th,s}^2}{e B_0 L_s} = \frac{m_s v_{th,s}^2 \langle n_s \rangle}{B_0^2} \left( \frac{n_0}{\langle n_s \rangle} \right) \left( \frac{L_0}{L_s} \right)$$  \hspace{1cm} (II.32)

$$\Gamma_{ss'} = \frac{2\pi q_s^2 q_{s'}^2 \langle n_s \rangle \ln \Lambda}{\mu_{ss'}^2 (v_{red,s'}^2)}$$  \hspace{1cm} (II.33)

Note that the standard beta parameter for species $s$ is $\beta_s = m_s v_{th,s}^2 \langle n_s \rangle / (B_0^2/4\pi)$ and is nearly the same as the quantity $\beta_{0s}$ defined in (II.33). The meaning of the constants $\Gamma_{ss'}$ is elucidated by recalling that the Spitzer-Harm collision rate $[53, 54]$ for particle pair $s, s'$ is:

$$\nu_{ss'} = \frac{2\pi q_s^2 q_{s'}^2 \langle n_s \rangle \ln \Lambda}{\mu_{ss'}^2 (v_{red,s'}^2)^3}$$  \hspace{1cm} (II.34)

up to a prefactor of order unity, where $\mu_{ss'}$ is the reduced mass for pairs $s, s'$ given by $1/\mu_{ss'} = (1/m_s) + (1/m_{s'})$ and where $v_{red}$ is the typical relative velocity of particles of species $s, s'$, or max{$v_{th,s}, v_{th,s'}$} on order of magnitude. Thus, the quantity $\Gamma_{ss'}$ defined in (II.34) is essentially equal to $\nu_{ss'} \tau_s$, or the ratio of the characteristic time $\tau_s$ of species $s$ and and the mean-free-time for its collisions with species $s'$. We follow [24, 25] in referring to $D_{0s} = 1/\Gamma_{ss'}$ as the Dorland number for the pair $s, s'$ [23]. In terms of the mean-free-path $\ell_{ss'} = v_{th,s}/\nu_{ss'}$ for collisions of species $s$ with $s'$ we can also write the Dorland number as $\ell_{ss'}/L_s$. We thus see that $D_{0s} = 1$ is a measure of the collisionality of the plasma, with the plasma being nearly collisionless when $1/\Gamma = D_0 := \min_{s', s''} D_{0s'} \gg 1$. Hereafter we consider this weakly collisional regime with all other dimensionless parameters $(\beta_{0s}, (n_s)/n_0, \text{etc.})$ assumed to have magnitudes of order unity. We remove hats $(\cdot)$ on all variables for simplicity of notations.

**D. Collisionless Limit and Dissipative Anomaly**

The collisional terms in the kinetic equation (II.1) formally disappear in the limit $\Gamma := \max_{s', s''} \Gamma_{s,s'} \to 0$ and its solutions may be expected to converge, in a certain sense, to solutions of the collisionless Vlasov-Maxwell equations. Naively, the entropy production (II.27) also vanishes in this limit because the prefactors $\Gamma_{s,s'} \to 0$. However, this need not be the case if the velocity-gradients of the distribution functions that appear in the collision integral diverge in the same limit. The simplest mechanism for producing large velocity-gradients is the free-streaming or ballistic advection of spatial structure, which underlies linear Landau damping [56] and which has long been known to produce “velocity-space filamentation” in collisionless Vlasov simulations (e.g., see the review [57] with many earlier references). In the papers of Krommes & Hu [22, 23] and Krommes [24, 25] it was pointed out that entropy production rates in a long-time statistical steady-state of a plasma obtained by taking the limit $t \to \infty$ first are determined entirely by the forcing and thus must remain constant in the subsequent limit $\Gamma \to 0$. The papers [22, 23] argued that the required fine structure in velocity space could be produced by ballistic streaming and drew an explicit analogy with non-vanishing viscous dissipation of kinetic energy for fluid turbulence in the high-Reynolds number limit, or what is called the “dissipative anomaly” [10]. In following work of Schekochihin et al. [24, 25] within the gyrokinetic approach to plasma turbulence, it was pointed out that the analogue of a high Reynolds-number “inertial-range” can exist at sub-ion scales in position and velocity space for high Dorland-number plasma turbulence, with ion entropy cascading through that range by a nonlinear perpendicular phase-mixing mechanism [58]. Employing phenomenological arguments, the authors of [24, 25] argued that small-scales in velocity-space are produced more efficiently by nonlinear entropy cascade than by the simpler ballistic phase-mixing mechanism.

In the present paper we shall further develop the connection between high Reynolds-number turbulence and nearly collisionless (high Dorland-number) plasma kinetics, but without making the more restrictive assumptions necessary for validity of a gyrokinetic description (i.e., without assuming evolution time-scales for any species $s$ long compared with its gyrofrequency). We shall show that existence of a turbulent cascade of entropy emerges as a natural consequence of the conjecture of [24, 25] that collisional entropy production persists in the collisionless limit. We formalize the latter conjecture as the precise hypothesis that the entropy production (II.27) converges in the collisionless limit

$$\lim_{\Gamma \to 0} \sigma(x, t) = \sigma^s(x, t)$$  \hspace{1cm} (II.35)

as a measure in $\mathbf{x}$-space for each $t$. This formulation is motivated by the analogy with energy dissipation in incompressible fluid turbulence [16] and also by the case
of compressible fluids where, for shock solutions, the entropy production converges in exactly this fashion in the infinite Reynolds-number limit \(21\). There is, however, a strengthened version of the hypothesis which is also natural and which involves the collisional entropy-production density in the 2-particle phase-space, or
\[
\zeta(x, v, v', t) := \sum_{ss'} \frac{\Gamma_{ss'}(2(\nabla_{v'} - \nabla_{p'})(f_s f_{s'}))^2}{f_s f_{s'}|\nabla_v|},
\]
so that \(\sigma(x, t) = \int d^3v \int d^3v' \zeta(x, v, v', t).\) This density involves only a single position variable \(x\), since a pair of particles must pass through the same space point (to within a Debye radius) in order to experience an unscreened Coulomb collision. As obvious from the definition \(22\), this phase-space density involves only velocity-gradients of the particle distributions and not space-gradients. It may therefore be expected to remain a continuous function of \(x\) in the limit \(\Gamma \to 0\), if the particle distributions likewise remain continuous in \(x\) and \(v\) (e.g., as gyrokinetic theory suggests; see section \(\text{VII}A\)). In that case, it is reasonable to make the stronger hypothesis that the 2-particle phase-space density of entropy production converges
\[
\lim_{\Gamma \to 0} \zeta(x, v, v', t) = \zeta(x, v, v', t)
\]
as a finite measure in \((v, v')\)-space for every \((x, t)\). Of course, this assumption implies that in \(23\), but now even pointwise in \(x\) rather than simply as a measure. The validity of both these hypotheses can be explored in numerical simulations of the Vlasov-Maxwell-Landau system, similarly as in \(52\). In the present paper we explore their theoretical consequences. As we shall see, the Onsager “ideal turbulence theory” \(10\) carries over under these assumptions to plasma kinetics and predicts properties of the collisionless limit of Vlasov-Maxwell-Landau (VML) solutions with anomalous entropy production. This analysis leads to the concept of “weak” or “coarse-grained” solutions of the Vlasov-Maxwell (VM) equations with irreversible entropy production by non-linear entropy cascade in phase-space.

### III. PHASE-SPACE COARSE-GRAINING

The most obvious requirement for non-vanishing of the entropy production as in \(24\) or \(25\) is divergence of velocity-gradients of the particle distribution functions in the limit \(\Gamma \to 0\), or an “ultraviolet divergence” at small-scales in velocity space. One should furthermore expect that space-gradients of the particle distribution functions will diverge as well in the collisionless limit. Note that the characteristic curves of the VM equation are the Hamiltonian particle motions in an electromagnetic field and, for non-trivial fields, these will generally lead to large space-gradients as well as large velocity-gradients. Such divergences make it impossible to interpret the VML equations naively in this limit and pursuit of a dynamical description which can remain valid requires a suitable regularization. We shall here follow closely the discussion for hydrodynamic turbulence in \(10\) and make use of a similar “coarse-graining” or “block-spin” regularization in the 1-particle phase space.

#### A. Definition of Coarse-Graining

For any time-dependent function \(a(x, v, t)\) on the 1-particle phase-space, we define its coarse-graining \(26\) at position resolution \(\ell\) and velocity resolution \(u\) by
\[
\tau(\mathbf{x}, \mathbf{v}, t) = \int d^3r G_\ell(r) \int d^3w H_u(w) a(\mathbf{x} + r, \mathbf{v} + w, t)
\]
where \(H_u(w) = u^{-3} H(w/u)\) for a kernel \(H\) satisfying the properties:
\[
\begin{align*}
H(w) &\geq 0 \quad \text{(non-negative)} \\
\int d^3w H(w) &= 1 \quad \text{(normalized)} \\
\int d^3w w^2 H(w) &= 0 \quad \text{(centered)} \\
\int d^3w w^2 H(w) &= 1 \quad \text{(unit variance)}.
\end{align*}
\]
We also assume that \(H\) is smooth and rapidly decaying, e.g., \(H\in C^\infty(\mathbb{R}^3)\), and for convenience assume isotropy, or \(H = H(w)\) with \(w = |w|\), so that \(\int d^3w w_i w_j H(w) = (1/3)\delta_{ij}\). In the same manner, \(G_\ell(r) = \ell^{-3} G(r/\ell)\) for a kernel \(G\) satisfying the analogous properties. It is sometimes useful to rewrite the definition \(27\) as
\[
\tau(\mathbf{x}, \mathbf{v}, t) = \left\langle a(\mathbf{x} + r, \mathbf{v} + w, t) \right\rangle_{\ell, u}
\]
where the local average \(\left\langle \cdot \right\rangle_{\ell, u}\) is over displacements \(r, w\) with respect to the distribution \(G_\ell(r) H_u(w)\). In our discussion below we shall also sometimes employ coarse-graining only with respect to position or only with respect to velocity, which we denote by
\[
\begin{align*}
\tau_\ell(\mathbf{x}, \mathbf{v}, t) &= \int d^3r G_\ell(r) a(\mathbf{x} + r, \mathbf{v}, t) \\
&= \left\langle a(\mathbf{x} + r, \mathbf{v}, t) \right\rangle_{\ell} \\
\tau_u(\mathbf{x}, \mathbf{v}, t) &= \int d^3w H_u(w) a(\mathbf{x}, \mathbf{v} + w, t) \\
&= \left\langle a(\mathbf{x}, \mathbf{v} + w, t) \right\rangle_u
\end{align*}
\]
There is consistency between these various notions of coarse-graining if a phase-space function lacks dependence on one variable. For example, if \(b = b(x, t)\) is independent of \(v\), then \(\overline{b} = \overline{b}_v\) and we need not distinguish these two quantities. Likewise, if \(c = c(v, t)\) is independent of \(x\), then \(\overline{c} = \overline{c}_x\).

One more concept that we shall employ extensively in our analysis below is that of coarse-graining cumulants \(\tau(f_1, ..., f_p)\). These are defined as usual \(28\) through the iterative expansion of coarse-grained products into
finite sums of cumulants:

\[ \tau_{1} \cdots \tau_{n} = \sum_{I} \prod_{r=1}^{r_{I}} \tau(a_{i_{r}}, \ldots, a_{i_{p_{r}}}) \]  

(III.5)

where the sum is over all distinct partitions \( I \) of \( \{1, \ldots, n\} \) into \( r_{I} \) disjoint subsets \( \{i_{1}^{(r)}, \ldots, i_{p_{r}}^{(r)}\} \) of \( p_{r} \) members each, \( r = 1, \ldots, r_{I} \), so that \( \sum_{r=1}^{r_{I}} p_{r} = n \) for each partition \( I \). By solving the iterated expansions for cumulants in terms of coarse-grained products one obtains, for example,

\[ \tau(a_{1}, a_{2}) = \langle \delta a_{1} \delta a_{2} \rangle - \langle \delta a_{1} \rangle \langle \delta a_{2} \rangle \]  

(III.7)

and so forth for cumulants of higher order. A relation that is crucial to our analysis is

\[ \tau(a_{1}, a_{2}) = \langle \delta a_{1} \delta a_{2} \rangle - \langle \delta a_{1} \rangle \langle \delta a_{2} \rangle \]  

(III.7)

where \( \delta_{r,w} a(x, v, t) = a(x + r, v + w, t) - a(x, v, t) \) is the increment for a phase-space displacement \( (r, w) \). A similar result holds for the 2nd-order cumulant \( \tau_{r}(b_{1}, b_{2}) \) defined with respect to the average \( \langle \cdot \rangle_{r} \) over \( r \) and with the increment taken to be \( \delta_{r,b} \). The same remark holds for \( \tau_{a}(c_{1}, c_{2}) \), average \( \langle \cdot \rangle_{a} \) over \( w \), and increment \( \delta_{w,c} \). In fact, expressions for higher-order cumulants in terms of increments hold as well, completely analogous to (III.7) for 2nd-order cumulants [15].

The phase-space coarse-graining operation (III.1) clearly regularizes all gradients, so that \( \nabla_{\text{sg}}^{} \) and \( \nabla_{\text{sp}}^{} \) are finite and smooth, even if quantity \( a \) exists only as a distribution on phase-space. Moreover, one can derive expressions for these gradients in terms of increments:

\[ \nabla_{\text{sg}}^{}(x, v, t) = -\frac{1}{\ell} \int d^{3}r \langle \nabla_{v} G \rangle_{(r)} \int d^{3}w H_{a}(w)(\delta_{r,a}(x, v + w, t) \]  

(III.8)

and

\[ \nabla_{\text{sp}}^{}(x, v, t) = -\frac{1}{u} \int d^{3}r G_{(r)} \int d^{3}w \langle \nabla_{v} H \rangle_{a}(w)(\delta_{w,a}(x + r, v, t), \]  

(III.9)

by exploiting \( \int d^{3}r \langle \nabla_{v} G \rangle_{(r)} = \int d^{3}w \langle \nabla_{v} H \rangle_{a}(w) = 0 \). These formulas permit one to estimate the order of magnitude of the coarse-grained gradients. We emphasize that the length scale \( \ell \) and velocity scale \( u \) introduced by our coarse-graining regularization are completely arbitrary. No objective physical fact can depend upon their precise values. The coarse-graining (III.1) is a purely passive operation which corresponds to observing a given phase-space function \( a(x, v, t) \) with some chosen resolutions \( \ell \) in position and \( u \) in velocity. As we see below, the arbitrariness of these regularization scales can be exploited to deduce exact consequences, analogous to RG-invariance in quantum field-theory and statistical physics [14] and analogous to Onsager’s “ideal turbulence” theory for incompressible fluid turbulence [16].

B. Phase-Space Favre Average

In the theory of compressible fluid turbulence, a mass-density weighted average was introduced by Favre [64] within a statistical ensemble approach to compressible fluid turbulence. Density-weighting may be employed also for coarse-graining averages, e.g. [14, 20, 65]. It should be emphasized that the use of density-weighting is not obligatory, but has the advantage that it reduces the number of terms in coarse-grained equations and generally provides each term with an intuitive physical interpretation. Therefore, we employ weighted coarse-graining here as well, but with the novelty that coarse-graining averages are weighted by the phase-space particle distributions rather than by mass-densities. For a field \( a = a(x, v, t) \) we thus define its phase-space Favre average at scales \( \ell, u \) weighted by the particle distribution of species \( s \) as

\[ \bar{a}_{s} := \frac{\ell f_{s}}{\int f_{s}}. \]  

(III.10)

We contrast this with the traditional physical-space Favre average at scale \( \ell \) for a field \( b = b(x, t) \) with no \( v \)-dependence, which is weighted by the mass-density of species \( s \) so that

\[ \bar{b}_{s} := \frac{\rho_{s}}{\int \rho_{s}}. \]  

(III.11)

Even for a purely spatial field \( b = b(x, t) \) with no \( v \)-dependence, these two averages do not agree,

\[ \bar{b}_{s}(x, v, t) \neq \bar{b}_{s}(x, t), \]  

(III.12)

because the correlations between positions and velocities in the distribution function \( f_{s}(x, v, t) \) induce a nontrivial \( v \)-dependence in \( \bar{b}_{s} \). There is, however, an easily derived consistency relation

\[ \int d^{3}v \bar{b}_{s} f_{s} = \bar{b}_{s} \int f_{s} \]  

(III.13)

which holds for any \( b = b(x, t) \).

Just as for unweighted coarse-graining, one may define phase-space Favre cumulants \( \bar{\tau}_{s}(a_{1}, ..., a_{n}) \) through the iterative decompositions

\[ (a_{1} \cdots a_{n})_{s} = \sum_{I} \prod_{r=1}^{r_{I}} \bar{\tau}_{s}(a_{i_{1}}^{(r)}, ..., a_{i_{p_{r}}^{(r)}}) \]  

(III.14)

for \( n = 1, 2, 3, ... \) Likewise, one may define physical-space Favre cumulants \( \bar{\tau}_{s}(b_{1}, ..., b_{n}) \) with respect to the standard Favre average for \( b_{i} = b_{i}(x, t) \), \( i = 1, 2, 3, ... \). Since Favre-averaging is just a convenience, one may always express Favre cumulants in terms of unweighted cumulants, e.g. for \( \bar{\tau}_{s}(a) = \bar{a}_{s} \)

\[ \bar{a}_{s} = \bar{a} + \frac{1}{f_{s}} \bar{\tau}(a, f_{s}), \]  

(III.15)
also
\[ \tilde{\tau}_s(a_1, a_2) = \tau(a_1, a_2) + \frac{1}{f_s} \tau(a_1, a_2, f_s) - \frac{1}{\tilde{f}_s} \tilde{\tau}(a_1, f_s) \tau(a_2, f_s) \] (III.16)
and so forth. Because the unweighted cumulants \( \tau(a_1, ..., a_n) \) can be expressed in terms of increments \( \delta a_i \) \( i = 1, ..., n \) via relations such as (III.17), it follows that the Favre cumulants \( \tilde{\tau}(a_1, ..., a_n) \) can be expressed in terms of increments \( \delta f_s \) and \( \delta a_i, i = 1, ..., n \).

C. Coarse-Grained Distribution

Basic dynamical objects for the coarse-graining regularization are the coarse-grained distributions \( \tilde{f}_s(\mathbf{x}, \mathbf{v}) \) for each particle species \( s = 1, ..., S \). Before we consider their evolution, however, we note some simple properties of the coarse-grained distributions that follow directly from their definition. First, one easily obtains the velocity moments up to quadratic order as
\[ \int d^3 \mathbf{v} \, m_s \tilde{f}_s(\mathbf{x}, \mathbf{v}, t) = \tilde{\rho}_s(\mathbf{x}, t) \] (III.17)
\[ \int d^3 \mathbf{v} \, m_s \mathbf{v} \tilde{f}_s(\mathbf{x}, \mathbf{v}, t) = \tilde{\rho}_s \tilde{u}_s(\mathbf{x}, t) \] (III.18)
\[ \int d^3 \mathbf{v} \, m_s \mathbf{v} \cdot \nabla \tilde{f}_s(\mathbf{x}, \mathbf{v}, t) = \left( \rho_s \mathbf{u}_s \mathbf{u}_s + \mathbf{P}_s + \frac{1}{3} \rho_s u^2 \mathbf{I} \right) (\mathbf{x}, t) \] (III.19)
where to obtain the last two relations we used \( \int d^3 w \, w \, H_s(\mathbf{w}) = 0 \) and \( \int d^3 w \, w \, H_s(\mathbf{w}) = (1/3) u^2 \mathbf{I} \).

Simple consequences of the above three moment conditions are then
\[ \int d^3 \mathbf{v} \, \mathbf{v} \tilde{f}_s / \int d^3 \mathbf{v} \, \tilde{f}_s = \tilde{\mathbf{u}}_s, \] (III.20)
\[ \int d^3 \mathbf{v} \, \nabla \tilde{f}_s / \int d^3 \mathbf{v} \, \tilde{f}_s = \tilde{E}_s + \frac{1}{3} \rho_s u^2, \] (III.21)
and
\[ \int d^3 \mathbf{v} \, m_s (\mathbf{v} - \tilde{\mathbf{u}}_s) (\mathbf{v} - \tilde{\mathbf{u}}_s) \tilde{f}_s = \tilde{\rho}_s \tilde{\mathbf{u}}_s \mathbf{u}_s + \mathbf{P}_s + \frac{1}{3} \rho_s u^2 \mathbf{I}. \] (III.22)

To interpret the last three results, note that \( \tilde{f}_s(\mathbf{x}, \mathbf{v}, t) \) represents an imperfectly measured distribution function for particle species \( s \), observed with resolution \( \ell \) in positions and resolution \( u \) in velocities. The relation (III.20) states that the bulk flow velocity for the measured distribution coincides with the Favre-average of the true bulk velocity. Likewise, the relations (III.21) and (III.22) give the resolved energy density and resolved pressure tensor calculated from the measured distribution. Aside from the extra isotropic term \( (1/3) \tilde{\rho}_s u^2 \mathbf{I} \), the resolved pressure tensor is given by
\[ \mathbf{P}_s = \mathbf{P}_s + \tilde{\rho}_s \tilde{\mathbf{u}}_s \mathbf{u}_s, \] (III.23)
which we call the intrinsic resolved pressure tensor. Note that no calculation involving only the measured distribution function \( \tilde{f}_s(\mathbf{x}, \mathbf{v}, t) \) can yield separately the coarse-grained pressure tensor \( \mathbf{P}_s \) or the subscale stress tensor \( \tilde{\rho}_s \tilde{\mathbf{u}}_s \mathbf{u}_s \) and only the combination is intrinsically defined for the measured distribution. This is similar to the concept of “intrinsic resolved internal energy” that was introduced in [20] for a turbulent compressible fluid, which is likewise the only internal energy that be obtained from coarse-grained observations of the basic fluid variables. In kinetic theory, we may define the intrinsic resolved internal energy by \( \tilde{\tau}_s^* = (1/2) \text{tr} (\mathbf{P}_s) \), or
\[ \tilde{\tau}_s^* = \tilde{\tau}_s + \frac{1}{2} \tilde{\rho}_s \tilde{u}_s \mathbf{u}_s \mathbf{u}_s, \] (III.24)
using the short-hand notation \( \tilde{\tau}(\mathbf{b}; \mathbf{b}') = \sum_{i=1}^3 \tilde{\tau}(b_i, b_i') \). We then see that \( E_s = (1/2) \tilde{\rho}_s |\tilde{\mathbf{u}}_s|^2 + \tilde{\tau}_s^* \). The quantity \( \tilde{\tau}_s^* \) in (III.21) is the only internal/fluctuatational energy that can be obtained from the imperfectly measured distribution function \( \tilde{f}_s(\mathbf{x}, \mathbf{v}, t) \), for which energy in kinetic fluctuations \( \epsilon_s \) and energy in unresolved, turbulent fluctuations of the bulk velocity \( \tilde{\mathbf{u}}_s \) are indistinguishable.

Finally, we note one of the most important properties of the coarse-grained distributions. Because the phase-space entropy density \( \mathcal{S}(f_s) \) is concave in \( f_s \), one has the basic inequality
\[ \mathcal{S}(\tilde{f}_s) \geq \mathcal{S}(f_s). \] (III.25)

Thus, as is well-known (e.g. [64], Chapter XII) the entropy of each species \( s \) can only increase under coarse-graining:
\[ \mathcal{S}(\tilde{f}_s) := \int d^3 \mathbf{v} \, \int d^3 \mathbf{v} \, \mathcal{S}(\tilde{f}_s) \geq \int d^3 x \int d^3 v \, \mathcal{S}(f_s) = S(f_s). \] (III.26)

This result implies that, if increase of total particle entropy \( S_{tot}(f) := \sum_s S(f_s) \) is persistent in the collisionless limit \( \Gamma \rightarrow 0 \), then an observer with only coarse-grained measurements of the phase-space distribution functions at finite resolutions \( \ell, u \) will also observe an increase in \( S_{tot}(\tilde{f}) = \sum_s S(\tilde{f}_s) \). As we show now, however, the entropy production observed at fixed scales \( \ell, u \) is not due to the direct effect of collisions in the limit \( \Gamma \rightarrow 0 \).

IV. COARSE-GRAINED VLASOV-MAXWELL EQUATIONS

The coarse-grained particle distribution functions and coarse-grained electromagnetic fields may have a well-defined dynamics in the collisionless limit, as all of their
gradients necessarily remain finite. The dynamics at fixed resolutions $\ell$, $u$ in fact is governed by a coarse-grained version of the collisionless Vlasov-Maxwell equations, valid for very large (but finite) Dorland number.

**A. Negligibility of Collisions**

The equations for the particle distribution functions coarse-grained at scales $\ell$, $u$ are

$$\partial_t f_s + \nabla_x \cdot (v f_s) + \nabla_B (q_s E_s f_s) = C_s(f),$$  \hspace{1cm} (IV.1)

since the coarse-graining operation commutes with all partial derivatives. The coarse-grained collision operator is given by $C_s = \sum_{s'} C_{ss'}$ with

$$C_{ss'}(x, v, t) = \int d^3r G_2(r) \int d^3v H_s(x + v, v, t) = -\frac{G_{ss'}}{m_s u_s} \int d^3r G_2(r) \int d^3v (\nabla H)_s(v - v').
\int d^3v' \frac{\Pi_{v-v'}}{|v-v'|} (\nabla p - \nabla p') (f_s f_{s'}) (x, t).$$  \hspace{1cm} (IV.2)

Here we have used the specific form of the Landau collision integral \([1,9]\) and integrated by parts once to move the $\nabla_v$ derivative to the kernel $H_s$. In the final expression in \([IV.2]\), $f_s = f_s(x + r, v, t)$, $f_s' = f_s'(x + r, v', t)$.

We now show that $C_{ss'} \to 0$ as $\Gamma \to 0$, by deriving an upper bound. We first factorize the integrand in \([IV.2]\) into a product of two terms to give

$$C_{ss'}(x, v, t) = -\frac{G_{ss'}}{m_s u_s} \int d^3r \int d^3v \int d^3v' \frac{G^{1/2} (r)(\nabla H)_s(v - v')}{|v-v'|} (f_s f_{s'}) (x, t),$$  \hspace{1cm} (IV.3)

and then apply Cauchy-Schwartz inequality to obtain

$$C_{ss'}(x, v, t) \leq \frac{G_{ss'}}{m_s u_s} \int d^3r \int d^3v \int d^3v' G^{1/2} (r)(\nabla H)_s(v - v') (f_s f_{s'}) (x, t).$$  \hspace{1cm} (IV.4)

The integral under the first square-root contains a factor $1/|v-v'|$ in its integrand diverging as $v' \to v$, but this is an integrable singularity in 3D. It is not hard to show under reasonable assumptions on the particle distributions that this integral remains finite as $\Gamma \to 0$ (Appendix \([11]\)). The integral under the second square root is, to within a factor, the spatial coarse-graining of the $s, s'$ term in the local entropy production defined in \([11,27]\). We therefore obtain an upper bound, with $C_{f,u}$ independent of $\Gamma$,

$$|C_{ss'}(x, v, t)| \leq C_{f,u} \sqrt{\Gamma_{ss'}(x, v, t)}$$  \hspace{1cm} (IV.5)

which is vanishing in the limit $\Gamma \to 0$ with $\ell$, $u$ fixed. Since it is the coarse-grained entropy-production which appears in this bound, we only need to assume that $\sigma(x, t) \to \sigma_s(x, t)$ as a measure (eq. \([11,36]\)) and not point-wise in $x$ or in any stronger sense (e.g. \([11,38]\)).

The conclusion of this argument is that for any fixed scales $\ell$, $u$ then for sufficiently large (but finite) Dorland numbers, the fields $f_s$, $s = 1, ..., S$ and $E_s, B_s$ will satisfy, to any desired degree of accuracy, the coarse-grained Vlasov-Maxwell equations:

$$\partial_t f_s + \nabla_x \cdot (v f_s) + \nabla_B (q_s E_s f_s) = 0, \hspace{1cm} s = 1, ..., S.$$

$$\nabla_x E_s = 4\pi \sum_s q_s n_s,$$

$$\nabla_x \times B_s = \frac{4\pi}{c} J_s,$$

$$\nabla_x \times E_s + \frac{1}{c} \partial_t B_s = 0, \hspace{1cm} \nabla_x \times B_s = 0.$$  \hspace{1cm} (IV.6)

The validity of the coarse-grained Maxwell equations is immediate, of course, because of the linearity of the Maxwell equations in $f_s$, $s = 1, ..., S$ and $E_s, B_s$. For any fixed value of the Dorland number $Do \gg 1$, the range of scales $\ell$, $u$ where collisions have no direct effect and where the above “coarse-grained VM equations” are well-satisfied shall be called the “collisionless range” of kinetic turbulence. This concept is completely analogous to the “inertial-range” of hydrodynamic turbulence, where likewise viscosity has no direct effect and “coarse-grained Euler equations” are valid. This is essentially the same analogy suggested in \([24, 25]\) but now derived and interpreted in a precise fashion.

Explicit estimates of the cutoff scales $\ell_c$, $u_c$ where collisions become important can be obtained from our analysis. Since the derivation involves material in later sections of the paper and is somewhat out of logical order, we present the details in \([11]\). Here we just remark briefly that estimate \([IV.5]\) can be improved to:

$$|C_{ss'}(x, v, t)| \leq C'' \sqrt{\nu_{th,s} \tau_{s,t,u}(x, v, t)} f_s(x, v, t) \times \frac{\nu_{th,s'}}{u},$$  \hspace{1cm} (IV.7)

where $\tau_{s,t,u}(x, v, t)$ is a coarse-grained collisional entropy production rate of particle species $s$ per phase-space volume, $\nu_{th,s} = \max\{\nu_{th,s}, \nu_{th,s'}\}$, and $\nu_{th,s'}$ is the Spitzer-Harm collision rate \([11,35]\) for particles of species $s, s'$. By making the stronger hypothesis \([11,38]\) on non-vanishing entropy production, one can infer that $\tau_{s,t,u}(x, v, t)$ remains finite in the limit $Do \to \infty$, so that estimate \([IV.7]\) also implies that collisions can be neglected at fixed $\ell$, $u$ in the limit. Furthermore, from \([IV.7]\) one can infer the following condition to determine cutoff scales $\ell_c, u_c$:

$$\frac{\omega_{s,t,u}^2}{\omega_{s,t,u}} \leq \nu_{th,s'} \left(\frac{\nu_{th,s'}}{u}\right)^2.$$  \hspace{1cm} (IV.8)
where \( \omega_{s,\ell,u}^{edd}(\mathbf{x},\nabla,t) \) is a suitably defined “eddy-turnover rate” and where \( \omega_{s,\ell,u}^{diss}(\mathbf{x},\nabla,t) \) is a coarse-grained collisional “dissipation rate”, at scales \( \ell, u \) in phase-space. See Appendix C. When \( \omega_{s,\ell,u}^{edd} \sim \omega_{s,\ell,u}^{diss} \) the condition (IV.8) essentially coincides with the heuristic criterion proposed in the gyrokinetic literature (see [23], section 2 and [24], eq.(251)) but now derived locally in phase-space and thus consistent with possible intermittency.

Since (IV.8) imposes only a single condition on two parameters \( \ell, u \), an additional relation is required to completely determine \( \ell_c, u_c \). In gyrokinetic turbulence theory this has been taken to be a relation \( u/v_{th,s} \sim \ell/\rho_s \) that connects scaling in position space and velocity space, with \( \rho_s \) the gyroradius for species \( s \). See eq.(17) in [23] and eq.(252) in [24]. From the renormalization-group point of view, however, \( \ell, u \) are two independent regularization scales determined by completely arbitrary choices (just as any curvilinear coordinate system may not be unique). We shall discuss some properties of these known weak solutions further below. We note here only that the weak solutions in the DiPerna-Lions theory are not obtained as collisionless limits of solutions of the VML equations or other Boltzmann-type equations, and that such limits have not to date been mathematically proved (or disproved) to exist [70].

Better mathematical understanding of the collisionless limit would provide important new concepts and tools for the theory of kinetic plasma turbulence. We emphasize, however, that we do not need to assume in this work that limits \( f_s, E, B \rightarrow f_s, E_s, B_s \) must exist for Do \( \rightarrow \infty \). Our principal conclusions are independent of this hypothesis.

### B. Eddy-Drift and Effective Fields

Although the “coarse-grained VM equations” hold to any desired accuracy for fixed \( \ell, u \) when Do \( \gg 1 \), this does not mean that the VM equations in the naive sense hold for the coarse-grained fields \( f_s, s = 1, ..., S \) and \( \overline{E}, \overline{B} \). To explain this point clearly, we shall write the equations (IV.6) in a form as close as possible to the ordinary VM equations. This can be done in a simple way by using the concept of phase-space Favre average introduced in section [11B] to write \( \overline{\nabla f_s} = \hat{\nabla} f_s \) and \( \overline{E_s f_s} = \hat{E}_s f_s \). so that the “coarse-grained Vlasov equation” becomes:

\[
\partial_t \hat{f}_s + \nabla_x \cdot (\hat{v} f_s) + \nabla_p (q_s \hat{E}_s) f_s = 0, \quad s = 1, ..., S
\]

in the sense of distributions. Here we may note that there is rigorous mathematical theory on global existence of weak solutions of the VM equations, the state of the art of which is represented essentially by the work of DiPerna & Lions [69]. Those authors prove that, for any initial data \( f_{0,s}, s = 1, ..., S \) and \( \overline{E_0}, \overline{B_0} \) which satisfy the conditions

\[
\int d^3x \int d^3v |v|^2 f_{0,s} < \infty, \quad \int d^3x \int d^3v f_{0,s}^2 < \infty, \quad s = 1, ..., S
\]

\[
\nabla_x \cdot \overline{E_0} = \sum_s q_s \int d^3v f_{0,s}, \quad \nabla_x \cdot \overline{B_0} = 0
\]

\[
\int d^3x [||\overline{E_0}||^2 + ||\overline{B_0}||^2] < \infty, \quad (IV.11)
\]

then weak/distributional solutions of the VM equations with these initial conditions exist globally in time (but may not be unique). We shall discuss some properties of these known weak solutions further below. We note here only that the weak solutions in the DiPerna-Lions theory are not obtained as collisionless limits of solutions of the VML equations or other Boltzmann-type equations, and that such limits have not to date been mathematically proved (or disproved) to exist [70]. Better mathematical understanding of the collisionless limit would provide important new concepts and tools for the theory of kinetic plasma turbulence. We emphasize, however, that we do not need to assume in this work that limits \( f_s, E, B \rightarrow f_s, E_s, B_s \) must exist for Do \( \rightarrow \infty \). Our principal conclusions are independent of this hypothesis.
\[ = \frac{1}{f_s} \langle w \delta w J_{s,t}(x,v) \rangle_u \]  

(IV.14)

The second expression is obtained by performing first the \( \langle \cdot \rangle_r \)-average over \( r \) and then using the property \( \langle w \rangle_u = 0 \). This expression shall be useful in making estimates of the magnitude of \( \hat{w}_s \). The physical meaning of this “eddy-drift” is that the local mean velocity of the population of particles within distances \( \ell \), \( u \) of the phase point \((x,v)\) does not coincide with \( \nabla v \), and \( \hat{w}_s \) is the average drift velocity of this population relative to \( \nabla v \) itself.

One can likewise derive for the effective fields in (IV.12) the expressions

\[ \hat{E}_{ss} = \hat{E}_s + \frac{1}{c} \nabla \times \hat{B}_s + \frac{1}{c} \langle w \times B \rangle_s \]  

(IV.15)

with

\[ \hat{E}_s(x,v,t) = E(x,t) + \frac{1}{f_s} \tau(E,f_s) \]

\[ = \bar{E}(x,t) + \frac{1}{f_s} \tau(E,\mathcal{I}_{s,u}), \]  

(IV.16)

also

\[ \hat{B}_s(x,v,t) = B(x,t) + \frac{1}{f_s} \tau(B,f_s) \]

\[ = \bar{B}(x,t) + \frac{1}{f_s} \tau(B,\mathcal{I}_{s,u}), \]  

(IV.17)

and

\[ \langle w \times B \rangle_s(x,v,t) \]

\[ = \frac{1}{f_s} \langle w \times B(x + r,t) f_s(x + r, v + w, t) \rangle_{\ell,u} \]

\[ = \frac{1}{f_s} \langle w \times B(x + r,t) \delta w f_s(x + r, v) \rangle_{\ell,u} \]  

(IV.18)

These results are again direct consequences of the definition of Favre coarse-graining. The derivation of (IV.15) is quite similar to that of (IV.13). The first lines in (IV.16), (IV.17) follow by the general relation (III.15) between Favre and unweighted coarse-graining, and the second lines in (IV.16), (IV.17) follow from the \( \nabla \)-dependence of \( E, B \), which allows the \( \langle \cdot \rangle_r \)-average over \( w \) to be performed.

Notice that the Favre-averaged fields \( \hat{E}_s, \hat{B}_s \) become velocity-dependent due to the terms \( \tau(E,\mathcal{I}_{s,u}), \tau(B,\mathcal{I}_{s,u}) \), which account for the fine-scale correlations of particles and fields. This is similar to the velocity-dependence of conditionally-averaged fields in the derivation of the Vlasov-Maxwell system from the BBGKY hierarchy, except that the latter dependence arises from multi-particle statistical correlations and disappear when molecular chaos holds (e.g. [41], section III.1.1). In the “collisionless range” of kinetic turbulence, on the other hand, the correlations arise from turbulent fluctuations in the phase space and they do not vanish under any physically plausible assumptions. As we shall see, these correlations are a major contributor to kinetic turbulent cascades. Similar correlations arise microscopically at the next order in the expansion in the plasma parameter, leading to the collision integral expressed in the form \( C_s(f) = -\epsilon v_p \langle \delta E \delta f_s \rangle \), where the average here is over statistics of the individual ions (e.g. see [41], eq.(26.13)). Thus, the contributions in (IV.12) which arise from the correlation terms \( \tau(E,\mathcal{I}_{s,u}), \tau(B,\mathcal{I}_{s,u}) \) in \( \hat{E}_{ss} \) represent “collisions” of turbulent eddies. It is interesting that in the exact theory presented here at the level of the VML description, these nonlinear wave-particle interaction terms can explicitly drive a cascade in velocity space. In the gyrokinetic approximation there is no corresponding term which can create phase-space fine-scale structure by direct “advection” in velocity space and the necessary fine-structure for persistent entropy dissipation arises instead from the velocity-dependence of ring-averages ([24], p.345).

Using the second lines of each of the formulas (IV.13), (IV.16), (IV.18), we can estimate the magnitudes of all of the contributions to \( \hat{w}_s \) and \( \hat{E}_{ss} \) in (IV.13), (IV.15):

\[ \hat{w}_s(x,v,t) = O(u \delta f_s/f_s), \]  

(IV.19)

\[ \hat{E}_s(x,v,t) = E(x,t) + O(\delta t E \delta f_s/f_s), \]  

(IV.20)

\[ \nabla \times \hat{B}_s(x,v,t) = \nabla \times B(x,t) + O(\delta t B \delta f_s/f_s), \]  

(IV.21)

\[ \langle w \times B \rangle_s(x,v,t) = O(u B \delta f_s/f_s), \]  

(IV.22)

Here we use the short-hand notations

\[ \delta f_s := \sup_{|r|<\ell} |\delta r f_s|, \quad \delta u f_s := \sup_{|\nu|<u} |\delta w f_s| \]  

(IV.23)

and likewise for all other quantities. The estimates (IV.19), (IV.22) are all exact upper bounds, but can also be taken as order-of-magnitude estimates of the terms (IV.13), (IV.16), (IV.18), if one assumes that there are no significant cancellations in the local phase-space averages defining those terms. (As we shall discuss later, this is probably a dubious assumption.) We see explicitly from (IV.19), (IV.22) that the quantities \( \hat{v}_s, \hat{E}_{ss} \) appearing in the “coarse-grained Vlasov equations” are different from \( \nu, \hat{E} + (v/c) \times \hat{B} \) and, thus, \( \mathcal{I}_s, s = 1, ..., S \) and \( E, \hat{B} \) do not satisfy the VM equations in the conventional sense.

From a conceptual point of view, the quantities \( \hat{v}_s, \hat{E}_{ss} \) are scale-dependent “renormalizations” of the “bare” quantities \( v, E \) that appear in the “fine-grained” VML equations (1.1)–(1.3). The particle distribution functions measured in any real experiment will always have some finite resolutions \( \ell, u \) in position- and velocity-space and thus correspond to the coarse-grained distributions \( \mathcal{I}_s(x,v,t) \) and not to the fine-grained distributions \( f_s(x,v,t) \) that exactly satisfy the Vlasov-Landau
equation (I.1). At sufficiently large but finite $D_0$ and with fixed resolutions $\ell$, $u$, these measured distributions $f_s$ will satisfy to any desired degree the renormalized equation (IV.12), which is only equivalent to a Vlasov equation in the “coarse-grained sense” (IV.6). By contrast, any fine-grained distributions $f_{s \times}(x, v, t)$ obtained in the strong limit $D_0 \to \infty$ exactly satisfy the collisionless Vlasov equation (IV.10), but only in a distributional sense. The limits $f_{s \times}$ are singular Vlasov solutions with non-differentiable dependence on position and velocity, which can never be strictly observed in Nature. They are idealized mathematical objects which are approached better and better by the smooth VML solutions as $D_0$ increases and as the fine-grained distributions $f_s$ become more and more nearly singular.

V. ENTROPY CASCADE IN PHASE SPACE

The results in the previous section resolve the “paradox” that the Vlasov-Maxwell equations are valid at fixed scales $\ell$, $u$ as $\Gamma \to 0$, in the sense of eq. (IV.9), and yet entropy $S_{\text{tot}}(\mathcal{J})$ increases at those scales, even without any direct contribution from collisions. As we now show, the entropy production in the coarse-grained description at fixed resolutions $\ell$, $u$ is due to a nonlinear entropy cascade through phase-space scales, in exact analogy to the kinetic-energy cascade in incompressible fluid turbulence.

A. Coarse-Grained Entropy Balance

The first important observation is that the “coarse-grained Vlasov equation” in (IV.6) or (IV.12) satisfies no Liouville theorem, so that $\mathcal{J}_s$ is not conserved along characteristic curves of $\mathbf{v}_s$, $\mathbf{E}_s$. Instead, direct calculation yields along characteristics that

$$\frac{\partial}{\partial t} \mathcal{J}_s + \mathbf{v}_s \cdot \nabla \mathcal{J}_s = q_s \mathbf{E}_s \cdot \nabla \mathcal{J}_s$$

(V.1)

with generally $\mathbf{v}_s \cdot q_s \mathbf{E}_s \neq 0$. Below we give explicit expressions for this phase-space divergence which show clearly that it need not vanish. As a simple consequence of (V.1), one obtains the following phase-space balance equation satisfied by the entropy density of the coarse-grained distribution for species $s$: 

$$\frac{\partial}{\partial t} \sigma_s + \mathbf{v}_s \cdot \nabla \sigma_s = \nabla \mathbf{\cdot} \mathbf{J}_s$$

(V.2)

where

$$\sigma_{\ell,u}^{\text{flux}}(x, v, t) := \nabla \mathbf{\cdot} \mathbf{J}_s$$

(V.3)

The quantity $\sigma_{\ell,u}^{\text{flux}}(x, v, t)$ represents rate of transfer of entropy of species $s$ from unresolved scales $< \ell$, $u$ in the phase-space, where it is created by collisions, up to the resolved scales $> \ell$, $u$, locally for each phase-space point $(x, v)$. It is exactly analogous to the local energy flux $\Pi_e(x, t)$ for incompressible fluid turbulence (III.8), except for a change in sign. Because of the sign-difference, $\sigma_{\ell,u}^{\text{flux}}$ is better regarded as a flux of negentropy, or negative entropy, to small-scales in phase-space, which is there dissipated by collisions. We recall here that the “generalized energy” in gyrokinetics is the electromagnetic field energy minus the entropy of particles (see [24, 25] and the discussion in section VIIA). Negentropy also plays a central role in the “ideal turbulence theory” for compressible fluids [20, 21, 68].

The sign of $\sigma_{\ell,u}^{\text{flux}}(x, v, t)$ will vary from point to point in phase-space and also with scales $\ell$, $u$. However, its integral over velocity and summation over $s$

$$\sigma_{\ell,u}^{\text{flux}}(x) := \sum_s \int d^3 v \sigma_{\ell,u}^{\text{flux}}(x, v, t)$$

(V.4)

must be positive on average. Indeed, velocity integration of (V.2) and summation over $s$ yields

$$\frac{d}{dt} \langle \sigma_{\text{tot}}(f) \rangle = \langle \sigma \rangle \equiv \langle \sigma_s \rangle > 0,$$

(V.5)

with space-density of total resolved entropy

$$\sigma_{\text{tot}} := \sum_s \int d^3 v \mathbf{\cdot} \mathbf{J}_s$$

(V.6)

and with resolved entropy current density

$$\mathbf{J}_s^{\text{res}} := \sum_s \int d^3 v \mathbf{\cdot} \mathbf{J}_s$$

(V.7)

Averaging (II.2) over space, we first choose $D_0$ sufficiently large so that

$$\frac{d}{dt} \langle \sigma_{\text{tot}}(f) \rangle = \langle \sigma \rangle \equiv \langle \sigma_s \rangle > 0,$$

(V.8)

with $\langle \cdot \rangle$ representing the space-average. We then subsequently choose $\ell$, $u$ sufficiently small so that the average of (V.5) over space gives

$$\langle \sigma_{\ell,u}^{\text{flux}} \rangle = \frac{d}{dt} \langle \sigma_{\text{tot}}(f) \rangle = \frac{d}{dt} \langle \sigma_{\text{tot}}(f) \rangle.$$

(V.9)

Comparing the two expressions for $(d/dt)\langle \sigma_{\text{tot}}(f) \rangle$ in (V.8) and (V.9), one concludes that for $D_0 \gg 1$ there is a range of sufficiently small $\ell$, $u$ such that

$$\langle \sigma_{\ell,u}^{\text{flux}} \rangle \equiv \langle \sigma_s \rangle > 0.$$  

(V.10)

Thus, there is a range of nearly constant negentropy flux which, furthermore, is positive, corresponding to a forward cascade of negentropy or an inverse cascade of the standard entropy [72].

We can derive a more general result if we assume that (strong) limits exist $f_s \to f_{s \times}$ as $D_0 \to \infty$. In that case, one has the limiting entropy balance

$$\frac{\partial}{\partial t} \sigma_{\text{tot}}(f_s) + \mathbf{v}_s \cdot \mathbf{J}_s = \sigma_s$$

(V.11)
in the sense of distributions, directly from (12.25). Furthermore, one has in the limit \( \ell, u \to 0 \) that \( s_{\text{tot}}(f_s) \to s_{\text{tot}}(f_s) \) in the sense of distributions for the total entropy defined in (12.24) and likewise as \( \ell, u \to 0 \)

\[
J_{Ss, f_s}^{\text{flux}} = -\sum_s \int d^3 \mathbf{r} \nabla f_s \ln f_s
\]

\[
\to -\sum_s \int d^3 u \nabla f_s \ln f_s = J_{Ss} \quad (V.12)
\]

in the sense of distributions, for the entropy current density defined in (12.29). Because the eq. (V.9) follows for \( \mathbf{J}_{s, f_s}^{\text{flux}} \), one can also conclude that

\[
\lim_{\ell, u \to 0} \sigma_{s, f_s}^{\text{flux}} = \lim_{\ell, u \to 0} \left[ \partial_t s_{\text{tot}}(f_s) + \nabla \cdot J_{Ss, f_s}^{\text{flux}} \right] = \partial_t s_{\text{tot}}(f_s) + \nabla \cdot J_{Ss}
\]

\[
= \sigma_s \quad (V.13)
\]

This is obviously a stronger statement than (V.10), which requires a global space average. The result (V.13) or (V.14) is analogous to the local relation (in the sense of distributions) between kinetic energy flux and viscous energy dissipation derived for incompressible fluid turbulence by Duchon & Robert [73].

The balance for total entropy obtained in (V.11) as \( D\ell \to \infty \), with \( f_{s_s}, s = 1, ..., S \) a set of weak or distributional solutions of the Vlasov-Maxwell equations (14.10) is an example of what is called an “anomalous balance” in quantum field-theory and condensed-matter physics [16, 31, 32]. A positive source term \( \sigma_s > 0 \) implies increasing total entropy for the “weak” solutions, whereas total entropy is conserved for smooth solutions of the Vlasov-Maxwell equations. The non-vanishing entropy-production \( \sigma_s > 0 \) is an example of a “dissipative anomaly”, like that predicted by Onsager [12, 16] for incompressible Euler solutions describing hydrodynamic turbulence as \( Re \to \infty \). As in the fluid case, such anomalies are possible only if the solutions are sufficiently “singular” or “rough”. We next derive the analogue of “4/5th-laws” which express the entropy flux (V.3) in terms of increments of particle distributions and fields and which allow us to establish exact constraints on the degree of singularity/roughness required for the turbulent solutions to sustain a non-vanishing negentropy flux to small scales in phase-space.

### B. 4/5th Laws for Entropy Flux

The formula (V.3) for the entropy flux through scales in phase-space can be further evaluated with the expressions for \( \hat{\mathbf{v}}_s \), \( \hat{\mathbf{E}}_s \) given in (V.13)–(V.18). The net contribution of \( \nabla \) and \( \hat{\mathbf{E}} + (\nabla \times \mathbf{E}) \times \mathbf{B} \) to the divergence in (V.3) is clearly zero, and the non-vanishing contributions arise from the subscale correlation terms. From (V.13)–(V.18), these quantities all have the general form \( \mathbf{A}/f_s \), where \( \mathbf{A} \) is an expression for the sub-scale correlation. Since \( \nabla \cdot (\mathbf{A}/f_s) = \nabla \cdot \mathbf{A} - \mathbf{A} \mathbf{\nabla} \log f_s \), the contributions to the entropy flux \( \mathbf{s}_{\text{flux}} \) consist generally of a total divergence term \( \nabla \cdot \mathbf{A} \) and a second term proportional to \( \nabla f_s \). More precisely,

\[
(\nabla \times \hat{\mathbf{v}}_s) f_s = \nabla \cdot (\hat{\mathbf{w}}_s - \hat{\mathbf{w}}_s) \mathbf{\nabla} f_s \quad (V.15)
\]

and

\[
q_s (\nabla \times \hat{\mathbf{E}}_s) f_s = -\nabla \cdot \mathbf{k}^s + k^s \cdot \nabla \log f_s \quad (V.16)
\]

with

\[
k^s = -q_s \left[ \mathbf{v}_s (\mathbf{E}, \mathbf{J}_s) + \frac{1}{c} \mathbf{\nabla} \times \mathbf{v}_s (\mathbf{B}, \mathbf{J}_s, s) + \frac{1}{c} (\mathbf{w} \times \mathbf{B}) f_s \right] \quad (V.17)
\]

We now make an important observation, that “flux terms” in coarse-grained balance equations are generally defined pointwise in phase space only up to total divergences, which may be considered as contributions to transport in phase-space rather than as transfer between scales. In this spirit, the quantity \( k^s \) defined in (V.17) may be taken to represent a turbulent transport of entropy in momentum-space. Likewise, the quantity

\[
\mathbf{j}^s_{\text{flux}} = -\hat{\mathbf{w}}_s \mathbf{f}_s = -\mathbf{w}_s \mathbf{f}_s \quad (V.18)
\]

may be considered to be turbulent transport of entropy in position-space. Using these definitions, we may now rewrite the coarse-grained entropy balance (V.2) as

\[
\partial_t s_f + \nabla \cdot (\hat{\mathbf{v}}_s f_s ) + j^s_{\text{flux}} + \nabla \cdot (q_s \hat{\mathbf{E}}_s f_s ) + k^s \quad (V.19)
\]

where the source term on the right-hand side

\[
s^s_{\text{flux}} (\mathbf{x}, \mathbf{v}, t) := j^s_{\text{flux}} (\mathbf{x}, \mathbf{v}) = j^s_{\text{flux}} (\mathbf{x}, \mathbf{v}) \mathbf{\nabla} f_s + k^s \cdot \nabla \log f_s \quad (V.20)
\]

is another possible representation of entropy flux across scales \( \mathbf{f}_s \). This expression for entropy flux has an intuitive physical interpretation when expressed in terms of

\[
\lambda[f_s] := \frac{\delta S[f]}{\delta f_s (\mathbf{x}, \mathbf{v})} = -(\log f_s + 1) \quad (V.21)
\]

the potential “entropically conjugate” to \( f_s \). Turbulent entropy production is obviously positive whenever the turbulent transport vectors \( \mathbf{j}^s_{\text{flux}} \), \( k^s \) are anti-aligned with the corresponding gradients \( \nabla \mathbf{x} (\mathbf{f}_s) \), \( \mathbf{\nabla} f_s \). The sign need not be positive everywhere in phase space, of course,
but may often be negative. However, the considerations in the previous section suggest that the sign of $\sigma_{\ell,u}^{\text{flux}}$ all carry over to the corresponding quantity
\[
\sigma_{\ell,u}^{\text{flux}}(\mathbf{x},t) := \sum_s \int d^3v \hat{\sigma}_{\ell,u}^{\text{flux},s}(\mathbf{x},\nabla,\nabla,\mathbf{v},t). \tag{V.22}
\]
This is obvious for the space-average, because the two quantities differ only by a divergence term and thus $\langle \sigma_{\ell,u}^{\text{flux}} \rangle = \langle \sigma_{\ell,u}^{\text{flux},s} \rangle$. Furthermore, the pointwise distributional limits of these two quantities must also coincide, taking first $\rho \to 0$ and then
\[
\lim_{\rho \to 0} \sigma_{\ell,u}^{\text{flux}} = \sigma_s \geq 0, \tag{V.23}
\]
where $\sigma_s$ is the same quantity that appears in (IV.13) as the distributional limit of $\hat{\sigma}_{\ell,u}^{\text{flux},s}$. More generally, distributional limits of $\hat{\sigma}_{\ell,u}^{\text{flux},s}$ and $\hat{\sigma}_{\ell,u}^{\text{flux},s}$ must coincide. This follows again because of the fact that these quantities differ only by terms of the form $\nabla \cdot \mathbf{A}$. The gradient $\nabla$ can always be shifted after smearing in phase space to the test function $\varphi(\mathbf{x},p,t)$, via an integration by parts, whereas estimates (IV.19)-(IV.22) of the correlation terms $\mathbf{A}$ show that each of these vanishes as $\ell$, $u \to 0$ under very mild assumptions, e.g. continuity of the limiting solutions $\mathbf{E}_s$, $\mathbf{B}_s$, $f_{s,s}$, $s = 1,\ldots,S$. \[23\]

The most compelling reason to prefer the modified quantity $\hat{\sigma}_\ell^{\text{flux},s}$ in (V.20) as a measure of "entropy flux" is that the original definition $\sigma_{\ell,u}^{\text{flux},s}$ in (IV.3) suffers large cancellations when integrated over phase space, and the net contribution to the entropy cascade in fact arises from the much smaller quantity $\hat{\sigma}_\ell^{\text{flux},s}$. Indeed, the contributions to $\sigma_{\ell,u}^{\text{flux},s}$ from the $\nabla \cdot \mathbf{A}$ terms are quadratic in increments, like typical turbulent transport terms in space, whereas all of the contributions to $\hat{\sigma}_\ell^{\text{flux},s}$ are cubic in increments, like typical turbulent fluxes, and thus generally smaller in magnitude. Specifically, the entropy flux defined in (V.20) consists of four contributions
\[
\hat{\sigma}_\ell^{\text{flux},s} = -\hat{\mathbf{w}}_\ell \cdot \nabla \hat{\mathbf{T}}_s - (q_s/m_c) \hat{\tau}_\ell(\mathbf{E},\hat{\mathbf{T}}_{s,s}) \cdot \nabla \hat{\mathbf{T}}_s / \hat{\mathbf{T}}_s \\
+ (q_s/m_c) \hat{\tau}_\ell(\mathbf{B},\hat{\mathbf{T}}_{s,s}) \cdot (\nabla \times \nabla \hat{\mathbf{T}}_s) / \hat{\mathbf{T}}_s \\
- (q_s/m_c) (\mathbf{w} \times \mathbf{B})_s \cdot \nabla \hat{\mathbf{T}}_s. \tag{V.24}
\]
These four quantities can all be expressed in terms of phase-space increments of the VML solutions $f_{s,s}$, $s = 1,\ldots,S$ and $\mathbf{E}, \mathbf{B}$ by means of the general relation (III.17) for the correlation terms $\hat{\tau}_\ell(\mathbf{E},\hat{\mathbf{T}}_{s,s})$, $\hat{\tau}_\ell(\mathbf{B},\hat{\mathbf{T}}_{s,s})$, the identities (IV.15), (IV.18) for $\hat{\tau}_\ell(\mathbf{w} \times \mathbf{B})_s$, and the equations (III.8)-(III.9) for the gradients $\nabla \hat{\mathbf{T}}_s$, $\nabla \hat{\mathbf{T}}_s$. These expressions provide exact "4/5-laws" for entropy cascade in kinetic turbulence (see Appendix D for explicit formulas and further discussion), which have previously been obtained only in 2D gyrokinetic turbulence \[76, 77\]. Exploiting them, we can make order-of-magnitude estimates of each of the four terms contributing to the phase-space entropy flux in eq. (V.24):
\[
-\hat{\mathbf{w}}_\ell \cdot \nabla \hat{\mathbf{T}}_s = O \left( \frac{u(\delta_s f_\ell)}{f_\ell} \right), \tag{V.25}
\]
\[
-\frac{q_s}{m_c} \hat{\tau}_\ell(\mathbf{E},\hat{\mathbf{T}}_{s,s}) \cdot \nabla \hat{\mathbf{T}}_s / \hat{\mathbf{T}}_s = O \left( \frac{q_s(\delta_E)(\delta_s f_\ell)}{m_c u f_\ell} \right), \tag{V.26}
\]
\[
\frac{q_s}{m_c} \hat{\tau}_\ell(\mathbf{B},\hat{\mathbf{T}}_{s,s}) \cdot (\nabla \times \nabla \hat{\mathbf{T}}_s) / \hat{\mathbf{T}}_s = O \left( \frac{\Pi q_s(\delta_B)(\delta_s f_\ell)}{c m_c u f_\ell} \right), \tag{V.27}
\]
\[
- \frac{q_s}{m_c} (\mathbf{w} \times \mathbf{B})_s \cdot \nabla \hat{\mathbf{T}}_s = O \left( \frac{q_s B(\delta_s f_\ell)^2}{m_c f_\ell} \right). \tag{V.28}
\]
These estimates all hold as exact upper bounds. One can already see from these estimates the possibility to have a non-vanishing entropy flux as $\ell, u \to 0$, because the diverging factors $1/\ell, 1/u$ in (V.25)-(V.27) that arose from gradients in space and velocity may compensate for the vanishing increment factors. Note that there is an exact cancellation $u/\ell = 1$ in the estimate (V.28), which implies that there can be no such compensation for this particular term, which vanishes whenever the particle distributions $f_{s,s}$, $s = 1,\ldots,S$ remain continuous as $Do \to \infty$. A persistent entropy flux in that limit is therefore expected to arise only from the first three contributions (V.25)-(V.27) to the modified entropy flux $\hat{\sigma}_\ell^{\text{flux},s}$ in (V.20).

Each of the three contributions to entropy flux has a clear physical significance. The two terms (V.25)-(V.27) are entropy transfer due to nonlinear wave-particle interactions, arising from turbulent fluctuations of electric and magnetic fields, respectively. The term (V.28) represents instead entropy transfer due to phase-mixing arising from linear advection. In the theory of Landau damping [\text{50}], linear phase-mixing is well recognized as a mechanism that can transfer entropy to small scales in velocity-space, both in the physics [\text{22, 23, 78, 79}] and mathematics (\text{80}, section 2.7) literatures. To be clear, there is no Onsager-type "entropy dissipation anomaly" in traditional Landau damping with an initially smooth, decaying perturbation of a Vlasov-Maxwell equilibrium, which is an entropy-conserving process. Because the particle distribution remains smooth (but with linearly growing velocity-gradients), the flux of entropy vanishes at sufficiently small scales in velocity-space. In a forced, steady-state, on the other hand, the phase-mixing mechanism can produce an entropy cascade to arbitrarily small scales [\text{22, 23, 78, 79}], but this requires an extremely singular particle distribution. In fact, if we impose the gyrokinetic relation $\ell/p_s \sim u/v_{th,s}$, we see from our eq. (V.25) that the linear-advection contribution to entropy flux is bounded by $(\delta_s f_\ell)^2/f_\ell$ and hence vanishes as $u \to 0$, whenever the distribution function $f_s$ remains continuous or even square-integrable (see footnote [\text{76}]) in the collisionless limit. This general result agrees with the linear kinetic model calculation in [\text{79}], eq.(4.25) showing...
that total “free-energy” diverges in the limit of vanishing collisional damping \[81\]. Our eq. \(V.26\) implies that in nonlinear kinetic turbulence, where particle-distributions are expected to remain even Hölder continuous, the linear advection contribution to entropy flux will generally be sub-dominant compared to the wave-particle interaction contributions \(V.20\)–\(V.27\), although this conclusion obviously can depend upon the arbitrary relation \(IV.9\) which is adopted between scales \(\ell, a\).

As cautioned earlier, the phase-space coarse-graining averages involved in the definitions of the four terms in \(V.24\) may involve substantial cancellations. Furthermore, the four individual terms are all quantities of indefinite sign—although non-negative when summed together and averaged—so that additional cancellations will certainly occur in integrating these over phase space. The bounds \(V.23\)–\(V.28\) on the entropy flux contributions derived above may therefore be considerable overestimates. As we shall see in our discussion of the gyrokinetic predictions in sections \(VIIA\)–\(VIIb\) there are reasons to expect extensive cancellations indeed will occur, which are missed by the above rather crude upper bounds. Despite their giving only upper bounds, the estimates \(V.25\)–\(V.28\) nevertheless suffice to derive non-trivial exact constraints on scaling properties of turbulent solutions in order to be consistent with a non-vanishing entropy flux to small scales in phase-space.

C. Scaling Exponent Constraints

The scaling exponents that we discuss are those which appear in the structure functions of (absolute) increments of phase-space variables \(a(x, v)\), which are defined similarly as for the hydrodynamic case \(\cite{10a},\text{Eq.}(IV.5)\), by

\[
S_p^a(r) := \langle|\delta_r a|^p \rangle, \quad R_p^a(w) := \langle|\delta_w a|^p \rangle. \tag{V.29}
\]

Here the notation \(\langle \cdot \rangle\) stands for a local average over some bounded open region \(O\) in phase-space, that is,

\[
\langle a \rangle := \frac{1}{|O|} \int_O d^3x d^3v \, a(x, v) \tag{V.30}
\]

where \(|O|\) is the phase-volume of the region \(O\). The average depends, of course, on the particular region which is selected. This may be the entire region of phase-space where entropy cascade occurs if it has finite phase-volume \[82\] or any bounded, open subregion. Our results shall give local conditions for entropy cascade to occur within any chosen such domain of phase-space. Note that the structure-functions defined by \(V.29\) are directly related to local \(L^p\)-norms in phase-space:

\[
S_p^a(r) = \|\delta_r a\|_p, \quad R_p^a(w) = \|\delta_w a\|_p. \tag{V.31}
\]

See e.g. \[83\]. Basic properties of such \(L^p\)-norms will be our main analytical tools, in particular the well-known Hölder inequality and also the nesting property of the norms, or \(\|a\|_p \leq \|a\|_{p'}\) for \(p' \geq p\). Note that we may consider the above structure-functions as well for variables \(b\) that are functions of \(x\) only, or variables \(c\) that are functions of \(v\) only. If the region considered has product form \(O = O_x \times O_v\) for bounded open subsets \(O_x\) and \(O_v\) of position-space and velocity-space, respectively, then the local phase-space structure functions reduce to the corresponding (local) structure-functions in position-space or velocity-space.

We seek conditions that must hold in order for there to be constant entropy flux as in \(V.10\), that is, for space-average \(\langle \sigma_{s, a}^{\text{flux}}(x, v) \rangle = \langle \sigma_s \rangle\) in a range of scales \(\ell, u\), which extends down to \(\ell, u = 0\) for \(Do \to \infty\). In light of \(V.4\), this can occur only if for some region \(O\) and some \(s\)

\[
\lim_{\ell, u \to 0} \langle \sigma_{s, a}^{\text{flux}}(x, v) \rangle \neq 0 \tag{V.32}
\]

As we show now, this condition imposes constraints on the structure-function scaling exponents \(\zeta^E_p, \zeta^B_p, \zeta^f_p, \zeta^r_p, s = 1, \ldots, S\) of the solution variables \(a = E, B, f_s, s = 1, \ldots, S\). For any such variable \(a\), we can define the exponents by assuming that scaling laws hold of the form

\[
S_p^a(r) \sim C_p a_{rms}^p \left(\frac{|r|}{L_a}\right)^{\zeta_p^a}, \quad R_p^a(w) \sim D_p a_{rms}^p \left(\frac{|w|}{V_a}\right)^{\zeta_p^a} \tag{V.33}
\]

for \(|r| \sim \ell, |w| \sim u\) in the range of \(\ell, u\) where non-vanishing flux condition \(V.32\) holds. Equivalently, and somewhat more conveniently, we may discuss exponents \(\sigma^E_p, \sigma^B_p, \sigma^f_p, \sigma^r_p, s = 1, \ldots, S\) defined by the scaling laws

\[
\|\delta_r a\|_p \sim C_p a_{rms}^{1/p} \left(\frac{|r|}{L_a}\right)^{\zeta_p^a}, \quad \|\delta_w a\|_p \sim D_p a_{rms}^{1/p} \left(\frac{|w|}{V_a}\right)^{\zeta_p^a} \tag{V.34}
\]

with \(\sigma^E_p = \zeta^E_p/p\) and \(\sigma^r_p = \zeta^r_p/p\). Although it is natural to assume that scaling laws such as \(V.33\) or \(V.34\) hold, this assumption is not necessary. If the infinite-\(Do\) limit variable \(a\) exists and its \(p\)-th order moments \(\langle |a|^p \rangle\) are finite, then we can instead take

\[
\sigma_p^a = \liminf_{|r| \to 0} \frac{\log \|\delta_r a\|_p}{\log |r|}, \quad \rho_p^a = \liminf_{|w| \to 0} \frac{\log \|\delta_w a\|_p}{\log |w|} \tag{V.35}
\]

where the limit-infimun is guaranteed to exist. The exponents defined by \(V.35\) coincide with those given by the scaling laws \(V.33\) or \(V.34\) whenever the latter hold. Otherwise, \(\sigma_p^a\) and \(\rho_p^a\) give the (fractional) smoothness in position and velocity, respectively, of the phase-space variable \(a\) in \(L_p\)-mean sense, or the maximal “Besov exponents”. See \[84\]–\[88\].

We now show that non-smoothness or “roughness” of the solutions \(E, B, f_s, s = 1, \ldots, S\) is required in order to permit a non-vanishing flux as in \(V.32\). For this, it is enough to obtain bounds on the norms

\[
\|\delta_r a_{\ell, u}^{\text{flux}, s}\|_p \leq \|\delta_r a_{\ell, u}^{\text{flux}, s}\|_{p/3}, \quad p \geq 3, \tag{V.36}
\]

that vanish if the solutions are too smooth. By the triangle-inequality we need bounds on the \(L_{p/3}\)-norms of
the three contributions to entropy flux in \((V.16)-(V.19)\) (noting that the fourth contribution \((V.20)\) to flux will always vanish as \(\ell, u \to 0\) when \(p\)-th-moments of \(B\) and \(f_s\) are finite). Simple applications of the nesting property and the Hölder inequality give

\[
\left\| \hat{\omega} \cdot \nabla \mathbf{f}_s \right\|_{p/3} = O \left( \frac{u^{3/2} (\delta_{f_s} + \|\delta_{f_s}\|_p)}{\ell \min\{f_s\}} \right), \quad (V.37)
\]

\[
\left\| \frac{q_s}{m_s c} \nabla \left( \mathbf{E} \cdot \mathbf{f}_s \right) \right\|_{p/3} = O \left( \frac{m_s u \min\{f_s\}}{\max\{\tau\} q_s \|\delta_{f_s} B\|_p \|\delta_{f_s} f_s\|_p} \right), \quad (V.38)
\]

\[
\left\| \frac{q_s}{m_s c} \nabla \left( \mathbf{B} \cdot \mathbf{f}_s \right) \right\|_{p/3} = O \left( \frac{c m_s u \min\{f_s\}}{m_s u \min\{f_s\}} \right), \quad (V.39)
\]

Here we defined

\[
\|\delta_{f_s} f_s\|_p := \sup_{\|\mathbf{r}\| < \ell} \|\delta_{f_s} f_s\|_p, \quad \|\delta_{f_s} f_s\|_p := \sup_{\|\mathbf{w}\| < u} \|\delta_{f_s} f_s\|_p. \quad (V.40)
\]

We have also assumed strict positivity of the distribution, or \(\min\{f_s\} = \min_{s, x, v} f_s(x, v, t) > 0\), which means that there are no “perfect holes” in the distribution function of species \(s\) where \(f_s = 0\). This does not, of course, rule out conventional phase-space holes where the density \(f_s\) becomes much smaller than the density in surrounding regions but remains non-zero \([67]\).

We now try to get the tightest bound on the entropy flux by minimizing the sum of the bound \((V.37)\) on the advective phase-mixing contribution and the bound on the total field-particle interaction contribution

\[
\left\| \frac{q_s}{m_s c} \left[ \nabla \left( \mathbf{E} \cdot \mathbf{f}_s \right) \right] + \frac{1}{c} \nabla \times \tau \left( \mathbf{B} \cdot \mathbf{f}_s \right) \right\|_{p/3} = O \left( \frac{m_s u \min\{f_s\}}{m_s u \min\{f_s\}} \right), \quad (V.41)
\]

obtained by combining estimates \((V.38), (V.39)\) and by noting that \(\max\{\tau\} \leq c\). As we have emphasized throughout this work, there is complete freedom in choosing the two scales \(\ell, u\), as long as they are sufficiently small. They represent an arbitrary choice of resolution of the turbulent cascade process. Hence, we can exploit this arbitrariness and choose \(u\) to be the value which minimizes the sum of the bounds \((V.37)\) and \((V.41)\), with \(\ell\) fixed. Elementary calculus gives

\[
u = [\ell \max\{\|\delta_{f_s} E\|_p, \|\delta_{f_s} B\|_p\}]^{1/2} = O \left( \frac{\rho_f^{1/2}}{\delta f_s^{1/2}} \right), \quad (V.42)
\]

which also coincides with the choice of \(u\) for which the two bounds \((V.37)\) and \((V.41)\) are “balanced” or have comparable magnitudes. In \((V.42)\) we have introduced the exponent \(\sigma_F^p = \min\{\sigma_F^p, \sigma_B^p\}\) which gives the minimal \(p\)-th-order smoothness of the electromagnetic field. Putting together all of the previous estimates, then for the choice of \(u\) determined by \((V.42)\) we have

\[
\left\| \mathcal{S}_{\ell, u}^{\text{flux}, s} \right\|_{p/3} = O \left( \frac{u^{3/2} (\delta_{f_s} + \|\delta_{f_s}\|_p)}{\ell \min\{f_s\}} \right) = O \left( \rho_f \sigma_F^p \sigma_F^p + \rho_B \sigma_B^p \right). \quad (V.43)
\]

Clearly the upper bound \((V.43)\) for \(p \geq 3\) will vanish as \(\ell, u \to 0\) if

\[
\frac{1}{2} (\sigma_F^p - 1) + \frac{1}{2} \rho_f \sigma_B^p (\sigma_F^p + 1) > 0. \quad (V.44)
\]

We thus arrive at the exponent inequality

\[
\sigma_F^p + 2\rho_f \sigma_B^p (\sigma_F^p + 1) \leq 1, \quad p \geq 3, \quad (V.45)
\]

as a necessary condition for non-vanishing entropy cascade to small scales in phase-space.

If we assume, for simplicity, that \(\sigma_F^p = \sigma_B^p = \rho_f \sigma_B^p = \sigma_p\) for all fields, with some single \(\sigma_p\) (“uni-scaling”), then the above inequality \((V.45)\) requires that \(\sigma_p \leq 1\) or \(\sigma_B^p \leq \sigma_{cr} = \frac{1}{3} \sum_{i} \sigma_i \geq 0.2361\) as the condition for non-vanishing entropy cascade. This result must not be interpreted as a prediction that the “mean-field” value \(\sigma_{cr} \approx 0.2361\) will be the scaling that physically occurs. Our result \((V.45)\) should be compared with the inequality for velocity scaling exponents \(\zeta_{x} \leq \frac{3}{p} \sum_{i} \sigma_i \leq \frac{1}{3}\) when \(p \geq 3\), which was first derived by Constantin et al. \([88]\) (see also \([16, 89]\)) as a necessary condition for kinetic energy cascade in incompressible fluid turbulence. Empirical results from experiments and simulations in that case indicate that \(\sigma_{cr} \approx \frac{1}{3}\) (just slightly smaller) but that \(\sigma_{cr} \approx \frac{1}{3}\) for \(p \gg 3\) is considerably smaller than the Kolmogorov value \(1/3\). This is due to the effect of “intermittency” in which the energy cascade rate becomes strongly fluctuating in space and time \([16, 27]\). For very large \(p\) values the scaling of velocity structure functions is determined by more singular structures with \(\sigma_{cr} \approx \frac{1}{3}\) much less than \(1/3\). However, these singular structures are also more sporadic and thus contribute relatively little to energy cascade. There are presumably similar phase-space intermittency effects in the entropy cascade of kinetic plasma turbulence, e.g., associated to sheets of strong electric current density \([90]\). Thus, our exponent inequality \((V.45)\) is probably far from equality for \(p \gg 3\).

In gyrokinetic turbulence, we expect that even for \(p \approx 3\) the physically observed exponents \(\sigma_F^p, \sigma_B^p, \sigma_f^p, \rho_f^p, s = 1, \ldots, S\) will satisfy the bound \((V.45)\) as an inequality, with a sizeable gap, rather than as an equality. As we shall see in section VII.A, the gyrokinetic predictions for scaling exponents in various entropy cascade ranges satisfy our bound \((V.45)\) easily, with a considerable gap. This should be expected, because our estimates...
take into account no physical effects of plasma wave oscillations or fast particle gyrations which could lead to strong depletion of nonlinearity. For example, in weak wave turbulence, rapid wave oscillations are known to cancel completely all nonlinear wave interactions except those with resonant wave frequencies $\omega_j$. In general, effects of wave oscillations or particle gyrations will lead to large cancellations in the exact expression for entropy flux, so that the upper bounds will be large overestimates. Because of the depletion of nonlinearity, more singular structures must develop to support the entropy cascade and the physically occurring exponents will not yield an equality in our condition \[(V.24)\]. For the same reason, our equation \[(V.24)\] cannot be regarded as a physical relation between position and velocity scales $\ell$, $u$ in a gyrokinetic entropy cascade range $\sigma_{cr}$. As we discuss in section VIIA, further analytical progress on gyrokinetic turbulence will require the control of delicate cancellations in \[(V.24)\], our exact “$4/5$th-law” expressions for entropy flux.

In summary, our analysis shows that the solutions $E$, $B$, $f_s$, $s = 1, \ldots, S$ of the VML equations cannot remain smooth if there is persistent entropy production in the limit $D_o \to \infty$. In fact, the solutions cannot have even a fractional smoothness which remains too high, or else entropy cascade is not possible. It is important to emphasize that the singularities that are required by our analysis need not develop in finite time from smooth Vlasov-Maxwell solutions with regular initial data. This is obvious for the collisionless limit of long-time steady-states as first considered by Krommes & Hu \cite{krommes2014, hu2014}, which corresponds to the limit first $t \to \infty$ and then $D_o \to \infty$. In this limit, phase-space mixing by ballistic streaming or other mechanisms has an infinite time to create fine structure down to collisional scales, and only subsequently are the collisional scales taken to zero. In freely-decaying turbulence without external forcing, singularities may be input as initial data, e.g. the solar wind originating in the superheated corona might have pre-existing turbulent fluctuations at all scales down to the Debye length. If smooth solutions of the collisionless Vlasov-Maxwell equations can indeed blow up in finite-time, then this would provide an additional source of singularities. It is still unknown whether initially smooth solutions of the (semi-relativistic) system \[(II.1)-(II.3)\] at vanishing collisionality will remain smooth, although it is known that any singularity formation requires particles moving with velocities near light-speed (see \cite{krommes2014}, Proposition 9).

More directly relevant for kinetic turbulence are theorems on the regularity of weak solutions of the Vlasov-Maxwell equations. The current best results seem to be those of \cite{pernaleons2019} for the DiPerna-Lions weak solutions of the (relativistic) Vlasov-Maxwell system, under an assumption that the particle energy densities $E_{\nu}(x, t)$ are square-integrable functions. By an application of averaging lemmas \cite{bcns} and “non-resonant smoothing” for particles with velocities bounded away from light-speed \cite{bcns}, the latter paper proves that electromagnetic fields have regularity exponent $\sigma_k^E > 6/(14 + \sqrt{142}) \approx 0.2315$. This value is remarkably close numerically to the critical value $\sigma_{cr} = \sqrt{5} - 2 \approx 0.2361$ for non-vanishing entropy cascade, which we have shown to require $\sigma_p \leq \sigma_{cr}$ for $p \geq 3$, under the additional assumption that all solution fields scale with the same exponent. Of course, there is no reason that such “uni-scaling” must hold and, even if it does, intermittency of the cascade could allow $\sigma_k^E > \sigma_{cr}$. However, the above numerical coincidence does show that monofractal (non-intermittent), uni-scaling solutions of the Vlasov-Maxwell equations with non-vanishing entropy production can exist in a narrow range only (if at all). Further conditional regularity results along the lines of \cite{bcns, bcns1} would be very valuable, for example, assuming some regularity exponents $\sigma_k^E$, $\rho_k^F$ of particle distributions and deriving corresponding minimal regularity exponents $\sigma_p^F$ of the electromagnetic fields. Such results would cast considerable light on the range of scaling exponents allowed for the dissipative weak solutions of Vlasov-Maxwell equations hypothesized in this work.

VI. BALANCES OF CONSERVED QUANTITIES IN THE COLLISIONLESS LIMIT

In this section we discuss the collisionless limit dynamics of quantities conserved for the total system (particles + fields) governed by the VML equations \[(IV.1)-(IV.3)\], namely, the mass of each particle species, the total momentum, and the total energy. Since these quantities are absolutely conserved for any degree of collisionality, the weak solutions of the VM equations \[(IV.10)\] obtained in the limit $D_o \to \infty$ cannot develop any anomalies in the balances of these quantities of the same sort as the entropy-production anomaly \[(V.3)\]. On the other hand, there are collisional conversions of one form of these conserved quantities into other forms and these conversion terms may, in principle, remain non-zero and “anomalous” as $D_o \to \infty$. Such a situation occurs in the infinite Reynolds-number limit of compressible fluids, for example, where total energy (kinetic + internal) is conserved but energy cascade leads to anomalous conversion of kinetic energy into internal energy \cite{andreussi2019, andreussi2019b}. We show here that such anomalous conversion does not occur in kinetic turbulence of nearly collisionless plasmas and that all collisional conversion terms vanish in the limit $D_o \to \infty$, under reasonable assumptions. We establish this both from the fine-grained point of view and in the coarse-grained description with finite resolutions $\ell$, $u$ in position- and velocity-space.

The results of the present section confirm naive expectations on the collisionless limit, while taking into account non-differentiability of limiting solutions. Results that are less expected can emerge, however, when one considers subsequent limits such as $\rho_i/L_i \ll 1$ (well-satisfied in the solar wind) and $\rho_e/\rho_i \ll 1$ (marginally satisfied in the solar wind), where $\rho_i$ and $\rho_e$ are ion and electron gyroradii, respectively. In these secondary
limits, anomalies by energy cascade through scales or anomalous conversion between different forms of energy may appear which are described by the scale-resolved energy balance in phase-space that we derive below. Likewise, the coarse-grained balance of electron momentum that we derive is the generalized Ohm’s law valid in a turbulent plasma at a given length-scale, which can lead to anomalous breakdown of magnetic flux-conservation and of the “frozen-in” property of field-lines [15, 24].

The limits $p_i/L_i \ll 1$ and $p_i/p_r \ll 1$ mentioned above have been discussed for a turbulent plasma generally within a gyrokinetic description, which becomes valid for gyrofrequencies much larger than rates of change of resolved scales [24, 25]. In these gyrokinetic analyses, energy and entropy balances are intertwined, whereas in the full kinetic description by VML equations their balance equations are completely separate in general. Nevertheless, our coarse-graining in phase-space provides a regularization of short-distance divergences that can appear in these subsidiary limits and it thus provides a suitable non-perturbative tool for analysis of gyrokinetic turbulence. We shall discuss gyrokinetics briefly in the following section, after we derive the collisionless limit of the basic conservation laws here.

A. Mass Balances

Since we have assumed that collisions do not transform one particle species into another, there is no contribution from the collision integral to fine-grained mass balances (II.10). Assuming that strong limits of VML solutions exist as $D_0 \to \infty$, the distributional mass balance equations $\partial_t \rho_s + \nabla \cdot \left( \rho_s \mathbf{u}_s \right) = 0$ hold as a direct limit of (II.10). This result may be obtained by integrating over $\mathbf{v}$ the weak Vlasov equation (IV.10) for the limiting particle distribution $f_{ss}$.

The coarse-grained mass balance at length-scale $\ell$ for each particle species $s$,

$$\partial_t \rho_s + \nabla \cdot \left( \rho_s \mathbf{u}_s \right) = 0,$$  \hspace{1cm} (VI.1)

can be easily derived, either by coarse-graining the fine-grained balance (II.10) or by integrating the coarse-grained Vlasov equation (IV.12) over $\mathbf{v}$ and using $\int d^3 \mathbf{v} \tilde{\mathbf{v}}_s \tilde{f}_s = \int d^3 \mathbf{v} \nabla f_s = \rho_s \mathbf{u}_s$. In terms of spatial Favre averages, this can be written as:

$$\partial_t \rho_s \mathbf{u}_s + \nabla \cdot \left( \rho_s \mathbf{u}_s \mathbf{u}_s \right) = 0$$  \hspace{1cm} (VI.2)

This is the same equation which holds for coarse-grained mass densities in compressible fluid theories [19, 20].

B. Momentum Balances

We now derive the momentum balances that hold in the collisionless limit $D_0 \to \infty$. The total momentum density $\sum_s \rho_s \mathbf{u}_s + (1/4\pi c) \mathbf{E} \times \mathbf{B}$ of particles and fields satisfies a local conservation law for any degree of collisionality, so that it is not possible to have a “dissipative anomaly” of total momentum. However, it is possible, in principle, that collisional momentum transfers between different particle species might remain non-vanishing due to the divergence of velocity-gradients in the limit. We show that this does not happen under mild conditions.

1. Fine-Grained Momentum Balances

The drag force on species $s$ from collisions with species $s'$ can be estimated for the Landau collision integral (II.9) by using integration by parts and the Cauchy-Schwartz inequality, in a similar fashion as for the estimation of $\mathbf{C}_{ss'}$ in eqs. (IV.2), (IV.3):

$$\mathbf{R}_{ss'} := \int d^3 \mathbf{v} \; m_s \mathbf{v} \mathbf{C}_{ss'}$$

$$= -\Gamma_{ss'} \int d^3 \mathbf{v} \int d^3 \mathbf{v}' \left( \frac{\mathbf{v} - \mathbf{v}'}{|\mathbf{v} - \mathbf{v}'|} \right) \left( \mathbf{P}_p - \mathbf{P}_{p'} \right) / f_{ss} f_{s's'}$$

$$= -\Gamma_{ss'} \int d^3 \mathbf{v} \int d^3 \mathbf{v}' \left( f_{ss} f_{s's'} \right)^{1/2}$$

$$\times \frac{\mathbf{P}_{\mathbf{v} - \mathbf{v}'} \left( \mathbf{P}_p - \mathbf{P}_{p'} \right) (f_{s} f_{s'})}{|\mathbf{v} - \mathbf{v}'|^{1/2}}$$  \hspace{1cm} (VI.3)

so that

$$|\mathbf{R}_{ss'}(\mathbf{x},t)| \leq \sqrt{\Gamma_{ss'} \times}$$

$$\sqrt{\int d^3 \mathbf{v} \int d^3 \mathbf{v}' \frac{f_{s} f_{s'}}{|\mathbf{v} - \mathbf{v}'| \times}}$$

$$\Gamma_{ss'} \int d^3 \mathbf{v} \int d^3 \mathbf{v}' \left( \frac{\mathbf{v} - \mathbf{v}'}{|\mathbf{v} - \mathbf{v}'|} \right) \left( \mathbf{P}_p - \mathbf{P}_{p'} \right)^2 / f_{s} f_{s'}$$

$$\leq C \sqrt{\Gamma_{ss'} \sigma(\mathbf{x},t)}$$  \hspace{1cm} (VI.4)

As shown in Appendix [32] the integral under the first square-root factor remains finite as $D_0 \to \infty$ under very mild assumptions on the particle distribution functions. The integral under the second square-root is $\sigma(\mathbf{x},t)$ as defined in [II.27] and, invoking the hypothesis [II.38] on the entropy production in 2-particle phase-space, this quantity remains finite pointwise in $(\mathbf{x},t)$ as $D_0 \to \infty$. Thus, the collisional drag force $\mathbf{R}_{ss'}$ vanishes $\propto \sqrt{\Gamma_{ss'}}$ for all $s, s'$ in the collisionless limit. Assuming that a suitable strong limit exists $f_{ss}, \mathbf{E}, \mathbf{B} \to f_{ss}, \mathbf{E}_s, \mathbf{B}_s$ as $D_0 \to \infty$, which thus satisfies the Vlasov-Maxwell equations (IV.10), then the fine-grained momentum balance for species $s$ in that limit solution becomes

$$\partial_t (\rho_s \mathbf{u}_s) + \nabla \times (\rho_s \mathbf{u}_s \mathbf{u}_s) + \mathbf{P}_s = q_s \mathbf{u}_s (\mathbf{E}_s + \mathbf{B}_s)$$  \hspace{1cm} (VI.5)

This is just the result that would be naively expected in the collisionless limit, with all interspecies momentum transfer due to collisionless wave-particle interactions.
2. Coarse-Grained Momentum Balances

A phase-space momentum balance at fixed resolutions \( \ell \), \( u \) can be obtained by multiplying the coarse-grained kinetic equation (VI.6) with \( \vec{v} \) to obtain

\[
\partial_t (m_s \vec{v}_s f_s^\ell) + \nabla \cdot (m_s \vec{v}_s \vec{v}_s f_s^\ell) + \nabla \cdot (m_s q_s \vec{E}_s \vec{v}_s f_s^\ell) = q_s \vec{E}_s \vec{v}_s \vec{f}_s^\ell + m_s \vec{v} \vec{C}_s^\ell(f).
\]

In the limit as \( \Do \to \infty \) recall from (VI.3) that \( \vec{C}_s^\ell(f) \to 0 \) pointwise in phase-space, so that one may neglect the final term in the nearly collisionless limit for fixed \( \ell \), \( u \). By integrating (VI.6) over velocities, it follows that

\[
\partial_t (\rho_s \vec{u}_s) + \nabla \cdot (\rho_s \vec{u}_s \vec{u}_s + \vec{P}_s) = q_s (n_s \vec{E}_s)_{s},
\]

for any fixed \( \ell \), \( u \) and sufficiently large \( \Do \). Here we have used the fact that the coarse-grained drag force \( \vec{R}_s = \int d^3 \vec{v} \vec{v} \vec{C}_s^\ell(f) \to 0 \) in the limit as \( \Do \to \infty \), assuming some uniform integrability in velocity of \( \nabla \vec{C}_s^\ell(f) \). In the idealized limit \( \Do \to \infty \) at fixed \( \ell \) one therefore obtains

\[
\partial_t (\rho_s \vec{u}_s) + \nabla \cdot (\rho_s \vec{u}_s \vec{u}_s + \vec{P}_s) = q_s n_s (\vec{E}_s)_{s},
\]

a result consistent with (VI.5) and which could also be obtained by coarse-graining that equation after first taking the collisionless limit. The previous two equations can both be rewritten in terms of spatial Favre averages, with (VI.7), for example, expressed equivalently as

\[
\partial_t (\overline{\rho}_s \overline{\vec{u}}_s) + \nabla \cdot (\overline{\rho}_s \overline{\vec{u}}_s \overline{\vec{u}}_s + \overline{\vec{P}}_s) = q_s \overline{n}_s (\overline{\vec{E}}_s)_{s},
\]

using the definitions (HI.11) and (HI.23). These equations for \( s = 1, \ldots, S \) fully specify the coarse-grained momentum balances of the particles in the collisionless limit.

On the other hand, the momentum balance for the electromagnetic fields resolved to a spatial scale \( \ell \) follows from the coarse-grained Vlasov-Maxwell equations (IV.6):

\[
\partial_t \left( \frac{1}{4 \pi c} \vec{E} \times \vec{B} \right) + \nabla \cdot \left[ \frac{1}{4 \pi} \left( \vec{E} \vec{E} - \frac{1}{2} \vec{B}^2 \mathbf{I} \right) + \frac{1}{4 \pi} \left( \vec{E} \vec{E} - \frac{1}{2} \vec{E}^2 \mathbf{I} \right) \right] = - \left( \tau \mathbf{F} + \frac{1}{c} \mathbf{j} \times \vec{B} \right),
\]

where the Lorentz reaction force on the righthand side acts as a source/sink of electromagnetic field momentum. It contains the coarse-grained charge and electric current densities, which are obtained from

\[
\tau = \sum_s q_s \tau_s, \quad \mathbf{J} = \sum_s q_s \tau_s \overline{\vec{u}}_s.
\]

An opposing Lorentz force is obtained by summing the righthand sides of (VI.9) over \( s = 1, \ldots, S \), so that the coarse-grained balance of total momentum from (VI.9), (VI.10) becomes

\[
\partial_t \left( \sum_s \overline{\rho}_s \overline{\vec{u}}_s + \frac{1}{4 \pi c} \vec{E} \times \vec{B} \right)
\]

C. Energy Balances

We finally derive the energy balances that hold in the collisionless limit \( \Do \to \infty \). Since total energy density \( \sum_s E_s + \frac{1}{\mathbf{j}} (|\vec{E}|^2 + |\vec{B}|^2) \) of particles and fields is locally conserved by solutions of the VML system (II.1)-(II.3) for any degree of collisionality, there can be no anomaly in the conservation of total energy as \( \Do \to \infty \). Just as for momentum conservation, however, there are collisional conversions of energy from one type to another which might remain non-zero in the collisionless limit. We show here that such anomalous energy conversion does not occur in the limit \( \Do \to \infty \), even if large velocity-gradients develop in the particle distribution functions. We show this both in the fine-grained description and for the coarse-grained equations at fixed position and velocity resolutions \( \ell \), \( u \) in the collisionless limit. Our energy balance equations will describe the transfers of energy simultaneously in phase-space and across scales \( \ell \), \( u \) in phase-space. We thus recover and generalize previous work of Howes et al. [3, 99] on fine-grained kinetic energy balance in phase space and of Yang et al. [51, 100] on coarse-grained kinetic energy balance of bulk plasma flows in physical space and in length-scale \( \ell \).

1. Fine-Grained Energy Balances

A phase-space density of kinetic energy for particle-species \( s \) was defined in [3, 99] as \( w_s(x, v, t) = \)
(1/2)m_s^2v^2f_s(x, v, t). The evolution of this density is easily obtained from the Vlasov-Landau kinetic equation (VI.11) to be
\[ \partial_t w_s + \nabla_x \cdot (vw_s) + \nabla_v \cdot (q_s(E_s)w_s) = q_s v \cdot E_s + (1/2)m_s\|v\|^2C(f). \] (VI.13)

The second term on the right arising from collision integral \( C_s = \sum_s' C_{ss'} \) can be rewritten using the identity
\[ \frac{1}{2}m_s\|v\|^2C_{ss'} = \nabla_v \cdot \left\{ \frac{1}{2} \Gamma_{ss'}\|v\|^2 \int d^3v' \frac{\Pi_v' \cdot (\nabla_p - \nabla_{p'}) (f_s f_{s'})}{|v - v'|} \right\} + \mathcal{R}_{ss'}(x, v, t), \] (VI.14)

with the divergence term representing a flux of kinetic energy in velocity space produced by collisions and with the second term representing the (signed) conversion of kinetic energy of species \( s' \) by collisions at phase-point \((x, v)\) into kinetic energy of species \( s'' \), given by
\[ \mathcal{R}_{ss'}(x, v, t) := -\Gamma_{ss'} \int d^3v' \frac{\Pi_v' \cdot (\nabla_p - \nabla_{p'}) (f_s f_{s'})}{|v - v'|} \left( v + v' \right) \left( f_s f_{s'} \right) \] (VI.15)

The expression in the second line is obtained by writing \( v = \frac{1}{2}(v + v') + \frac{1}{2}(v - v') \) and using \( w' \cdot \Pi_{w'} = 0 \). A simple estimate of this conversion term may be obtained by grouping the integrand into factors as
\[ \mathcal{R}_{ss'} = -\frac{1}{2} \Gamma_{ss'} \int d^3v' \left( v + v' \right) \left( f_s f_{s'} \right)^{1/2} \frac{\Pi_v' \cdot (\nabla_p - \nabla_{p'}) (f_s f_{s'})}{|v - v'|} \] (VI.16)

and applying the Cauchy-Schwartz inequality to obtain
\[ \int d^3v |\mathcal{R}_{ss'}(x, v, t)| \leq \Gamma_{ss'} \times \left[ \int d^3v \left| \frac{\Pi_v' \cdot (\nabla_p - \nabla_{p'}) (f_s f_{s'})}{|v - v'|} \right|^2 d^3v' \right]^{1/2} \times \left[ \int d^3v \left| \frac{\Pi_v' \cdot (\nabla_p - \nabla_{p'}) (f_s f_{s'})}{|v - v'|} \right|^2 d^3v' \right]^{1/2} \leq C \sqrt{\Gamma_{ss'} \sigma(x, t)}, \] (VI.17)

where the integral under the first square root is shown in Appendix (VI.13) to be finite under mild assumptions. It follows that \( \mathcal{R}_{ss'} \to 0 \) in the sense of distributions as \( Do \to \infty \). Note that the divergence term in (VI.14) can also be shown to vanish in the sense of distributions, by using an argument very similar to that for the term \( \nabla_x \cdot \) in (VI.12). We therefore conclude that in the limit \( Do \to \infty \) the phase-space energy density satisfies
\[ \partial_t w_s + \nabla_x \cdot (vw_s) + \nabla_v \cdot (q_s(E_s)w_s) = q_s v \cdot E_s f_s, \] (VI.18)

This is formally identical with the equation for \( w_{ss} \) argued to hold in the collisionless limit by [3], eq. (5) or [20], eq. (2.6), but rewritten in a form that is meaningful and valid (in the distributional sense) even when, as expected, the particle distribution \( f_{ss} \) becomes non-differentiable in position and velocity.

Since the physical-space energy density of particle species \( s \) is given by \( E_s = \int d^3v w_s \), we obtain from (VI.13) by integrating over velocities and by using definitions (VI.14), (VI.17) that
\[ \partial_t E_s + \nabla_x \cdot (E_s \frac{d}{dt} u_s + P_s \cdot u_s + q_s) = j_{ss} E_s, \] (VI.19)

This same equation can be obtained from the \( Do \to \infty \) limit of equation (VI.15) for \( E_s \), noting that its collisional contribution
\[ Q_{ss'} + R_{ss'} \cdot u_s = \int d^3v \frac{1}{2} m_s |v|^2 C_{ss'} \] (VI.20)

vanishes as \( Do \to \infty \) by an estimate identical to (VI.17). Similarly, since \( R_{s} \cdot u_s \to 0 \) as \( Do \to \infty \), one obtains from (VI.20) the limiting equation for the bulk kinetic energy:
\[ \partial_t \left( \frac{1}{2} \rho_{ss} |u_s|^2 \right) + \nabla_x \cdot \left( \frac{1}{2} \rho_{ss} |u_s|^2 u_s + P_{ss} u_s \right) = P_{ss} \cdot \nabla u_s - j_{ss} E_s. \] (VI.21)

From the vanishing of (VI.20) we infer also that \( Q_s \to 0 \) as \( Do \to \infty \) and thus obtain from (VI.21) the limiting balance equation for the internal/fluuctuational energy:
\[ \partial_t \epsilon_{ss} + \nabla_x \cdot (\epsilon_{ss} u_s + q_s) = -P_{ss} \cdot \nabla u_s. \] (VI.22)

The results (VI.19), (VI.21), (VI.22) coincide, formally, with the results naively expected in the collisionless regime but are derived without assuming space-differentiability of solutions.

Notice that the pressure-strain term on the righthand sides of (VI.21), (VI.22) must be carefully defined as a distributional limit \( P_{ss} \cdot \nabla u_s = D \lim_{Do \to \infty} P_{ss} \cdot \nabla u_s \). For the similar situation with compressible fluid turbulence, see (20). If the limiting fields \( P_{ss} \) and \( \nabla_x u_s \) exist as ordinary functions, then this distributional product will coincide with the ordinary pointwise product of functions. If \( u_s \) is not classically differentiable, however, then this notion of product differs from the naive one. The degree of smoothness of \( u_s \) is a priori not entirely obvious. The inequality (V.45) on scaling exponents of \( E_s, B_s, f_s \) shows that these fields cannot be space-differentiable if there is a non-vanishing entropy production anomaly for species \( s \). The velocity field \( u_s \), on the other hand, is obtained from 0th and 1st velocity-moments of \( f_{ss} \) by the formulas (VI.3), (VI.6) and such moments are generally smoother than the particle distribution function appearing in the integrand (e.g., see section 3 of [59]). It is thus possible that \( \nabla_x f_{ss} \) exists only as a distributionalized function, while \( \nabla_x u_s \) exists as an ordinary function [60]. Further detailed investigation, both analytical and empirical, is required to settle this issue.
2. Coarse-Grained Energy Balances

We now consider the energy balances for solutions of the coarse-grained VM equations (IV.6) that are obtained in the nearly collisionless limit.

Total Energy: We may define a coarse-grained version of the phase-space kinetic energy density of particle species $s$ as $\mathcal{w}_s(x,\mathbf{v},t) := (1/2)m_s|\mathbf{v}|^2 f_s(x,\mathbf{v},t)$. It follows directly from the coarse-grained Vlasov-Landau equation (IV.4) that this energy density satisfies

$$\partial_t \mathcal{w}_s + \mathbf{v} \cdot \nabla \mathcal{w}_s + \mathbf{v} \cdot \nabla_f \left( q_s \hat{E}_s \mathcal{w}_s \right) = q_s \mathbf{v} \cdot \hat{E}_s f_s + (1/2)m_s |\mathbf{v}|^2 \mathcal{C}_s(f)$$

(IV.23)

The “renormalized” quantities $\hat{v}_s, \hat{E}_s$ are those given in (IV.18)-(IV.16). Because of the vanishing of the coarse-grained collision integral from estimate (IV.5), we see that for fixed $\ell$, $u$ and for sufficiently large (but finite) $Do$ the collisionless equation

$$\partial_t \mathcal{w}_s + \mathbf{v} \cdot \nabla \mathcal{w}_s + \mathbf{v} \cdot \nabla_f \left( q_s \hat{E}_s \mathcal{w}_s \right) = q_s \mathbf{v} \cdot \hat{E}_s f_s$$

(IV.24)

is satisfied to any specified accuracy. In the idealized limit $Do \to \infty$ this becomes

$$\partial_t \left( \frac{1}{2} m_s |\mathbf{v}|^2 \mathcal{C}_s(f) \right) + \mathbf{v} \cdot \nabla_f \left( \frac{1}{2} m_s |\mathbf{v}|^2 \mathcal{C}_s(f) \right) = \mathbf{v} \cdot q_s (\hat{E}_s) f_s$$

(IV.25)

which further reduces to the equation (IV.18) proposed in (99) in the limit as $\ell, u \to 0$. It must be stressed, however, that in dealing with real experimental data at fixed resolutions $\ell, u$, it is the equation (IV.23) which will be satisfied by the measured energy density $\mathcal{w}_s$ and not the equation (IV.18) suggested in (99). The unresolved plasma turbulence at scales below $\ell, u$ may lead to significant renormalization effects in the quantities $\hat{v}_s, \hat{E}_s$ appearing in (IV.24).

The spatial energy distribution of solutions to the coarse-grained Vlasov-Maxwell system (IV.6) is governed, for kinetic energy of particles, by the equation that comes from integrating (IV.24) over $\mathbf{v}$ and using definitions (II.12)-(II.17):

$$\partial_t \mathcal{E}_s + \mathbf{v} \cdot \nabla \left( \mathcal{E}_s \mathbf{v}_s + P_s \cdot \mathbf{u}_s + q_s \right) = \mathcal{J}_s \cdot \mathbf{E}$$

(IV.26)

The same result is also obtained by coarse-graining (II.13) and using $R_s - \mathbf{u}_s + Q_s = \int d^3\mathbf{v} \frac{1}{2} |\mathbf{v}|^2 \mathcal{C}_s(f) \to 0$ as $Do \to \infty$. On the other hand, the evolution of the energy density of the resolved electromagnetic field is obtained from the coarse-grained Maxwell equations by the Poising theorem:

$$\partial_t \left( \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} \right) + \mathbf{v} \cdot \nabla \left( \frac{\mathbf{E} \times \mathbf{B}}{4\pi} \right) = -\mathcal{J}_s \cdot \mathbf{E}$$

(IV.27)

Summing (IV.26) over $s$ and adding (IV.27) gives the balance equation for total energy density of coarse-grained solutions as

$$\partial_t \left( \sum_s \mathcal{E}_s + \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{8\pi} \right) + \mathbf{v} \cdot \nabla \left( \sum_s \mathcal{E}_s \mathbf{u}_s + P_s \cdot \mathbf{u}_s + q_s + \frac{\mathbf{E} \times \mathbf{B}}{4\pi} \right) = \mathcal{J}_s \cdot \mathbf{E}$$

(IV.28)

Just as for the coarse-grained momentum balance (III.12) there is a source-term on the right-hand side of (IV.28) which represents a flux of energy from unresolved scales $< \ell$ to resolved scales $> \ell$. Since total energy (particles + fields) is conserved for the VML system (III.3), this flux of energy must vanish for any collisionless limit of such solutions. Because of the estimate $\mathcal{J}_s(j; \mathbf{E}) \sim (\delta \mathbf{j})(/\delta \mathbf{E})$ from (III.4), the energy flux indeed vanishes as $\ell \to 0$ whenever limits $\mathbf{v}_s, \mathbf{v}_s, \mathbf{E}_s, \mathbf{B}_s$ are spatially continuous or satisfy even weaker regularity conditions [102].

As an aside, we note that current mathematical theory for global solutions of the Vlasov-Maxwell system does not provide weak solutions that conserve energy but instead guarantees only that total energy for solutions is non-increasing in time! Cf. (62), p.740, remark 4. The arguments for energy conservation which we made above may not apply, because the DiPerna-Lions theory guarantees only that $f_s$, $\mathbf{E}$ and $\mathbf{B}$ are square-integrable and that 2nd-moments of $f_s$ with respect to $\mathbf{v}$ exist. Such regularity properties are not enough to allow the equation (IV.28) to be even written down, because they do not guarantee that heat fluxes $\mathbf{q}_s$ (3rd moments) are finite. Even if energy density integrated over all space is considered, which eliminates the undefined $\mathbf{q}_s$ term, the DiPerna-Lions solutions are not guaranteed to satisfy the weak regularity conditions of the type discussed in footnote [99] that imply that $\mathcal{J}_s(j; \mathbf{E}) \to 0$ as $\ell \to 0$. While solutions with decreasing total energy are physically unrealistic as collisionless limits of VML solutions, one cannot rule out that weak Vlasov-Maxwell solutions with decreasing total energy might occur in other physical contexts (e.g. see discussion in section VIII).

Kinetic-Energy of Bulk Velocities: The balance equation (IV.26) describes the dynamics of the total kinetic energy of species $s$ calculated from the particle distribution resolved to scales $\ell, u$. However, one may furthermore divide the energy density $\mathcal{E}_s$ into separate contributions from the resolved bulk velocity $\mathbf{u}_s$, as defined in (III.20) and from the (intrinsic) resolved internal energy $\mathcal{E}_s$, defined in (III.23). In particular, the contributions from the bulk velocity $\mathbf{u}_s$ and from the coarse-grained fields $\mathbf{E}, \mathbf{B}$ are often considered to be the only “turbulent” energy contributions at length-scale $\ell$, because these low-frequency fields are described by “fluid-like” equations and experience a continual, reversible energy exchange due to Alfvénic wave oscillations (e.g. (103), section 2(c)). In this view, $\gamma_s$ represents a quasi-thermal
energy or energy of kinetic fluctuations not directly participating in the “turbulence” at scale \( \ell \). We do not subscribe to this view, but it is nevertheless interesting to consider separately the kinetic energy dynamics of bulk flow and of the fluctuations.

The balance equation for the bulk kinetic energy \((1/2)\rho \bar{u}_s^2\) in the nearly collisionless limit is easily obtained from coarse-grained mass conservation \([VI.2]\) and coarse-grained momentum conservation \([VI.9]\), yielding

\[
\partial_t \left( \frac{1}{2} \rho \bar{u}_s^2 \right) + \nabla \cdot \left( \frac{1}{2} \rho \bar{u}_s^2 \bar{u}_s + \mathbf{P}_s \cdot \mathbf{u}_s \right) = \left( \mathbf{P}_s \cdot \nabla \mathbf{u}_s \right) + q_s \mathbf{E} \cdot \mathbf{E}_s - \mathbf{E}_s \cdot \mathbf{u}_s
\]

\(\text{VI.29}\)

This same equation has been derived earlier in \([100]\) for kinetic plasma turbulence and it is very similar to the analogous equations for resolved kinetic energy in compressible fluid turbulence \([11, 20]\). Obviously, the term \(q_s \mathbf{E} \cdot \mathbf{E}_s \cdot \mathbf{u}_s\) represents resolved wave-particle interactions. Based on the fluid turbulence analogy, the term \(-\mathbf{P}_s \cdot \nabla \mathbf{u}_s\) may be taken to represent energy flux arising from turbulent cascade, while \(-\mathbf{E}_s \cdot \mathbf{u}_s\) represents resolved pressure-work. It should be remembered, however, that only the intrinsic resolved pressure tensor \(\mathbf{P}_s\) is calculable from the distribution function \(f_s\) resolved to scales \(\ell, u\), and it is impossible from such coarse measurements of the particle distributions to compute the separate contributions of \(\mathbf{P}_s \cdot \nabla \mathbf{u}_s\) and \(-\mathbf{E}_s \cdot \mathbf{u}_s\).

The limit in \([VI.20]\) with first \(D_o \to \infty\) and then \(\ell \to 0\) must recover the equation \([VI.21]\) for \((1/2)\rho \bar{u}_s^2\), if the strong limits \(\mathbf{E} \to \mathbf{E}_s, \mathbf{B} \to \mathbf{B}_s, f_s \to f_s\) exist as \(D_o \to 0\). Indeed, since all of the other terms in \([VI.20]\) then converge distributionally to the corresponding terms in \([VI.21]\), one must have

\[
D_o \lim_{\ell \to 0} \mathbf{P}_s \cdot \nabla_\mathbf{x} \mathbf{u}_s \mathbf{u}_s = \mathbf{P}_s \cdot \nabla_\mathbf{x} \mathbf{u}_s \mathbf{u}_s,
\]

\(\text{VI.30}\)

where the product on the righthand side is the same quantity that appears in \([VI.21]\). The result \([VI.30]\), if correct, means that there is no “pressure-work defect” of the type that appears in compressible fluid shocks \([20]\). This result would be expected, in particular, if the gradient \(\nabla_\mathbf{x} \mathbf{u}_s \mathbf{u}_s\) exists as an ordinary function. In that case,

\[
D_o \lim_{\ell \to 0} \rho \bar{u}_s^2 \nabla_\mathbf{x} \bar{u}_s = 0
\]

\(\text{VI.31}\)

as well. This last relation can be interpreted as the statement that there is a vanishing energy flux in the order of limits first \(D_o \to \infty\) and then \(\ell \to 0\). This is a quite reasonable conclusion, since the collisional transfer of energy from species \(s\) to other species, \(R_s \cdot \mathbf{u}_s\), vanishes as \(D_o \to \infty\) according to \([VI.4]\). Thus, there is physically no “sink” for an energy cascade to small scales.

This tentative conclusion, that there is “no energy cascade to small scales in a collisionless plasma”, must be carefully interpreted. The solar wind is a nearly collisionless plasma with Kolmogorov-type spectra observed at scales above the (thermal) ion gyroradius \(\rho_i\), that are generally interpreted as an energy-cascade “inertial range” of, primarily, incompressible shear-Alfvén waves. In fact, there is direct evidence of non-zero energy flux in this range from empirical studies of third-order structure functions \([e.g. 39, 104]\). This cascade is described by the balance equation of the resolved mechanical energy in the bulk velocities of the particles (mostly from protons, or \(H^+\) ions) and electromagnetic fields, obtained by combining the eqs. \([VI.27]\), \([VI.29]\),

\[
\partial_t \left( \sum_s \frac{1}{2} \rho \bar{u}_s^2 + \frac{E^2}{8\pi} + \frac{B^2}{4\pi} \right) + \nabla \cdot \left( \sum_s \frac{1}{2} \rho \bar{u}_s^2 \bar{u}_s + \mathbf{P}_s \cdot \mathbf{u}_s \right) = \sum_s \left( \mathbf{P}_s \cdot \nabla \bar{u}_s + j_s \mathbf{s} \right)
\]

\(\text{VI.32}\)

with \(\mathbf{s}\) an “electromotive force” generated by unresolved turbulent fluctuations of bulk velocity and density for particles of species \(s\):

\[
\mathbf{s} := \frac{1}{c} \mathbf{r}(\mathbf{u}_s \times \mathbf{B}) + \frac{1}{n_s} \mathbf{E} (\mathbf{E}_s \times \mathbf{B}) + \frac{1}{c} \bar{u}_s \times \mathbf{E}_s
\]

\(\text{VI.33}\)

so that \(Q_{\ell, F} := \sum_s j_s \mathbf{s}\) represents a flux of electromagnetic energy to the unresolved scales. Thus for length-scales \(\ell\) in the range \(L_i \gg \ell \gg \rho_i\), one would expect non-vanishing values of the ion kinetic-energy flux \(Q_{\ell, i} := -\mathbf{P}_s \cdot \nabla \bar{u}_s\) and of \(Q_{\ell, F}\). This does not contradict the conclusion \([VI.31]\), which involves the limit \(\ell \to 0\) with \(\rho_i\) fixed or, equivalently, length-scales \(\ell \ll \rho_i\). In order to develop an Onsager-type theoretical description of the energy-cascade “inertial-range” of the solar wind at scales \(\ell \gg \rho_i\), one would need to consider after the limit \(D_o \to \infty\) a subsequent limit \(\rho_i/L_i \to 0\), corresponding to a long energy inertial-range of scales. It is quite plausible that limits exist \(E_s \to E_s, B_s \to B_s, j_s \to j_s\), \(s = i, e\) as \(\rho_i/L_i \to 0\), leading to a kinetic description with a turbulent cascade of ion kinetic nergy:

\[
Q_{s} := D_o \lim_{\ell \to 0} \rho \bar{u}_s^2 \nabla_\mathbf{x} \bar{u}_s \mathbf{u}_s = 0
\]

\(\text{VI.34}\)

More precisely, one expects that this limit lies within the regime of validity \([105]\) of a gyrokinetic description \([24, 25]\). A full treatment of the \(\rho_i/L_i \to 0\) limit is beyond the scope of the current paper, but we shall discuss briefly the relationship of our analysis with gyrokinetic theory in section \(\text{VII A}\). We likewise do not consider in detail the limit \(\rho_i/\rho_i \to 0\) (heavy ion limit) which idealizes the “ion dissipation range” of the solar wind over the interval of length-scales \(\ell\) satisfying \(\rho_i \gg \ell \gg \rho_e\), \(\text{VI.106}\), where a gyrokinetic description is expected to be valid at least for the electrons. See section \(\text{VII A}\) for brief remarks.

Kinetic-Energy of Fluctuations: The balance equation for \(\mathbf{E}_s = \mathbf{E}_s + \frac{1}{2} \rho \bar{u}_s^2 \nabla_\mathbf{x} \bar{u}_s\) can be obtained by subtracting equation \([VI.20]\) for \(\mathbf{E}_s\) and equation \([VI.29]\) for
(1/2)\(\overline{\epsilon_s}|\overline{u}_s|^2\), giving:

\[
\partial_t \overline{\epsilon_s} + \nabla \cdot \left( \epsilon_s \overline{u}_s + \overline{q}_s + \overline{P}_s; \overline{u}_s \right) - \overline{P}_s \cdot \overline{\tau}(\rho_s, \overline{u}_s) \overline{P}_s + \frac{1}{2}\rho_s \overline{\tau}(\overline{u}_s, \overline{u}_s) = - \overline{P}_s \cdot \nabla \overline{u}_s + q_s \overline{\tau}(E_{s*}; \overline{u}_s).
\]

Note that the term \(-\overline{P}_s \cdot \nabla \overline{u}_s\) on the righthand side differs only in sign from the corresponding term on the righthand side of (VI.29), so that this quantity acts to exchange kinetic energy between bulk flow and fluctuations. Even after taking the limit \(Do \to \infty\), (VI.35) is quite distinct from the equation obtained by coarse-graining (VI.22) for the fine-grained limit field \(\overline{\epsilon_s}\), or:

\[
\partial_t \overline{\epsilon_s} + \nabla \cdot (\overline{\epsilon_s} \overline{u}_s + \overline{q}_s) = - \overline{P}_s \cdot \nabla \overline{u}_s. \tag{VI.36}
\]

In particular, note that (VI.35) contains a non-vanishing wave-particle interaction term \(q_s \overline{\tau}(E_{s*}; \overline{u}_s)\) which is entirely absent from (VI.36). These two equations must agree in the limit \(\ell \to 0\), on the other hand, and in that limit the term \(q_s \overline{\tau}(E_{s*}; \overline{u}_s) \to 0\) under plausible regularity assumptions, as in footnote [99].

It is interesting to refine the spatial-balance equation (VI.35) for kinetic energy of fluctuations in order to follow the transfer through phase-space. For that purpose, we define a phase-space density of fluctuation energy at scales \(\ell, u\) by

\[
\overline{\epsilon_s}(\overline{\mathbf{x}}, \overline{\mathbf{v}}, t) := \frac{1}{2} m_s |\overline{v} - \overline{u}_s|^2 f_s(\overline{\mathbf{x}}, \overline{\mathbf{v}}, t) \tag{VI.37}
\]

so that \(\overline{\epsilon_s} = \int d^3\mathbf{v} \overline{\epsilon_s}\). A tedious calculation (see Appendix [A]) yields the following balance equation for \(\overline{\epsilon_s}\):

\[
\partial_t \overline{\epsilon_s} + \nabla \cdot \left( \overline{\epsilon_s} \overline{u}_s + \overline{q}_s \right) - \overline{P}_s \cdot \nabla \overline{u}_s = - m_s \left( \overline{v} \nabla f_s - \nabla \overline{v}_s f_s \right) \cdot \nabla \overline{u}_s
\]

(turbulent redistribution of energy)

\[
+ \overline{P}_s \cdot \nabla \left( \overline{u}_s - \overline{v} \right) f_s \left( \overline{\mathbf{x}}, \overline{\mathbf{v}} \right) \overline{\tau}(\rho_s, \overline{u}_s) \overline{P}_s + \frac{1}{2}\rho_s \overline{\tau}(\overline{u}_s, \overline{u}_s).
\]

(energy redistribution by resolved pressure)

\[
- m_s \left( \overline{\nabla} f_s \cdot \overline{u}_s + \overline{\nabla} f_s \cdot \overline{\mathbf{v}} \right) - \overline{\nabla} f_s \cdot \overline{\mathbf{v}}
\]

(work by mean-velocity gradient)

\[
- m_s \overline{\tau}(\rho_s, \overline{u}_s) \overline{\nabla} f_s \cdot \overline{u}_s
\]

(energy input from turbulent cascade)

\[
+ q_s \left( \overline{\nabla} - \overline{\mathbf{u}}_s \right) \cdot \left( E_{s*} - \overline{E}_{s*} \right) f_s
\]

(energy input & redistribution by EM field)

\[
\tag{VI.38}
\]

The four terms on the righthand side of (VI.40) are taken to be distributional limits of the corresponding first four terms on the righthand side of (VI.38). As one can see, there is a possible anomalous redistribution of energy \(R_{s*}\), which vanishes upon integration over velocities, and a possible anomalous input of energy \(Q_{s*}\) from turbulent cascade. These conclusions must be considered tentative, since they require a rigorous study of the limit \(\rho_i/L_i \to 0\), which we do not attempt here. In the next section we discuss the problem very briefly.

VII. RELATION TO PRIOR WORKS

A. Gyrokinetic Turbulence

All prior work on entropy cascade in plasma turbulence has been developed, essentially, within the framework of gyrokinetics. We therefore must briefly review
gyrokinetic theory and its physical basis, in order to make comparisons with our own work.

1. Concise Review of Gyrokinetic Theory

Nonlinear gyrokinetic equations capable of describing turbulent cascades were first derived in the seminal paper of Frieman & Chen [108] and subsequently extensively investigated theoretically and numerically by the plasma fusion community. Modern approaches to nonlinear gyrokinetics exploit powerful Hamiltonian and geometric methods [109, 110]. The application of gyrokinetics to astrophysical and space plasmas was pioneered in papers of Howes et al. [111], Schekochihin et al. [24, 25], which also first proposed and developed the theory of entropy cascades in plasma turbulence. Our review of gyrokinetic theory and especially the role of entropy in gyrokinetics shall follow closely the discussions in [24, 25, 111]. More general gyrokinetic theories of entropy cascade are possible (e.g. [33]), but the scaling predictions have been less developed in those generalizations and the original theoretical work therefore provides a more adequate basis of comparison with our results.

Although not necessary to achieve a gyrokinetic reduction [109, 110], many treatments, including that of [24, 25, 111] start from a decomposition of fields into “background” and “fluctuation” contributions:

\[ f_s = F_s + \delta f_s, \quad B = B_0 + \delta B, \quad E = \delta E \quad (E_0 = 0) \]  

(VII.1)

with the further assumption of (i) fluctuation amplitudes small relative to backgrounds:

\[ \frac{\delta f_s}{F_s} \sim \frac{\delta B_\perp}{B_0} \sim \frac{\delta B_\parallel}{B_0} \sim \epsilon \frac{\delta E_\perp}{v_{th,s} B_0} \sim \epsilon, \]  

(VII.2)

where \( \epsilon \ll 1 \) is a dimensionless parameter that quantifies this smallness and the subscripts \( \parallel \) and \( \perp \) denote vector components parallel and perpendicular to \( B_0 \), respectively. With \( \frac{u_\perp}{v_{th,s}} \sim \epsilon \) giving the \( E \times B \) drift velocity, the fourth condition in (VII.1) can be restated as \( \frac{u_\perp}{v_{th,s}} \sim \epsilon \). For applications to astrophysical and space plasmas (e.g. the solar wind), the condition (i) is perhaps the most dubious of the various assumptions discussed here. A second assumption, very essential for the validity of gyrokinetics, is (ii) frequency of fluctuations small relative to the gyrofrequency:

\[ \omega/\Omega_s \sim \epsilon. \]  

(VII.3)

This condition is often found to be satisfied over very broad ranges of scales in a turbulent plasma. It imposes no direct restriction on the perpendicular length-scale \( \ell_\perp \) or perpendicular wavenumber \( k_\perp \sim 1/\ell_\perp \) relative to the thermal gyroradius \( \rho_s \), which may be taken to satisfy \( k_\perp \rho_s \sim 1 \). However, if one takes \( \omega \sim v_{th,s} k_\parallel \) in order to admit Landau resonances, then (iii) scale-anisotropy of fluctuations is required:

\[ k_\parallel/k_\perp \sim \epsilon. \]  

(VII.4)

This condition is also often observed to be satisfied over wide ranges of scales. If electric fields are assumed electrostatic, \( \delta E = -\nabla \varphi \), to leading order, then scale-anisotropy implies \( \delta E_\parallel/\delta E_\perp \sim \epsilon \). Whenever the above conditions hold initially, then gyrokinetic theory implies that they are dynamically maintained, with a slow evolution of the background fields on \( \sim 1/\epsilon^3 \Omega_s \) time scales.

Gyrokinetic theory obtains closed evolutionary equations by seeking approximations to solutions of the Vlasov-Maxwell-Landau equations as asymptotic series \( \delta a \sim \sum_{\ell \geq 1} \delta a^{(\ell)} \epsilon^\ell \) for all fluctuation fields \( \delta a \), as \( \epsilon \to 0 \). Following [24, 25, 111], we consider here the simple case where all background distributions are isotropic Maxwellian, \( F_s = n_s (m_s/2\pi T_s)^{3/2} \exp(-m_s \varphi^2/2T_s) \), with temperature \( T_s \) of species \( s \) (in energy units) and where the background magnetic field is uniform, \( B_0 = B_0 \hat{z} \). Then it is found in [24, 25, 111] that

\[ \delta f_s^{(1)} = -\frac{q_s \varphi(X_s,t)}{T_s} f_s(v,t) + h_s(X_s,v,v_\perp,t) \]  

(VII.5)

where the first term in (VII.5) gives the adiabatic, Boltzmann response and the second term \( h_s \) is the ring distribution function which describes for each species \( s \) the distribution of the gyrocenters

\[ X_s = x + v_\perp \times \hat{z}/\Omega_s. \]  

(VII.6)

The time-evolution of the ring distribution functions \( h_s \) is obtained from the gyrokinetic equations:

\[ \frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0} \frac{\partial}{\partial v_\perp} \{ \langle \chi \rangle X_s, h_s \} = q_s F_s \frac{\partial \langle \chi \rangle}{\partial t} \]  

(VII.7)

where the gyrokinetic electromagnetic potential is defined by \( \chi := \varphi - v \cdot A/c \) in terms of the usual scalar \( \varphi \) and vector \( A \) potentials, where

\[ \langle a \rangle X_s = \frac{1}{2\pi} \int_0^{2\pi} d\theta a(X_s - v_\perp(\theta) \times \hat{z}/\Omega_s, v_\parallel, v_\perp(\theta), t) \]  

(VII.8)

is the ring-average over cyclotron motions with velocities \( v_\perp(\theta) = v_\perp |(\sin \theta) \hat{x} + (\cos \theta) \hat{y}| \), the spatial Poisson bracket is defined by \( \{ a,b \} := \hat{z} \cdot (\nabla_X a \times \nabla_X b) \), and \( (\partial h_s/\partial t)_c \) is the collisional contribution from the linearized and gyro-averaged Landau operator. The evolution of the electromagnetic fields \( \varphi, A_\parallel, \delta B_\parallel \) is likewise obtained from the Maxwell equations in a reduced, gyro-averaged form. See [111], eqs.(26)-(28). Together with the kinetic equations for the ring distribution functions \( h_s \), these completely specify the dynamics. One has the freedom in these equations to take \( \int d^3x \varphi = \int d^3x h_s = 0 \), and, in fact, to any order in the expansion in \( \epsilon \), one can impose \( \int d^3x \delta f_s = 0 \). The nonlinear Poisson bracket term arises, of course, from the wave-particle interaction term \( (q_s/m_s) \rho_0 \cdot \nabla \varphi \), in the Vlasov-Landau equation (VII.11). The v-gradient of \( \delta f_s^{(1)} \) contributes an \( X_s \)-gradient of \( h_s \) because of the \( v_\perp \)-dependence of the gyrocenter \( X_s \) in (VII.6). Although there is no direct “advective” term in velocity-space for the gyrokinetic equation, the
Poisson bracket term represents this effect, which creates fine-scale velocity structure.

2. Gyrokinetic H-Theorems

The gyrokinetic H-theorem for entropy has been discussed in \[24, 25, 111\], whose results we briefly summarize. Assuming the smallness of fluctuations, the phase-space entropy density can be Taylor-expanded as

$$\mathcal{S}(F_s) = \mathcal{S}(F_s) - (1 + \ln F_s) \delta f_s - \frac{(\delta f_s)^2}{2F_s}. \quad (VII.9)$$

With \( \int d^3x \delta f_s = 0 \), the entropy of species \( s \) becomes

$$S(F_s) = S(F_s) - \sum_{s'} \int d^3x \int d^3v \ln F_s \, C_{ss'}(f_s, f_{s'}) \geq 0, \quad (VII.10)$$

The second law of can be written as

$$\frac{d}{dt} \sum_s S(F_s) = - \sum_{s'} \int d^3x \int d^3v \ln F_s \, C_{ss'}(f_s, f_{s'}) \geq 0, \quad (VII.11)$$

with the logarithm on the RHS expanded as

$$\ln f_s = \ln F_s + \frac{\delta f_s}{F_s}. \quad (VII.12)$$

For a Maxwellian \( F_s \) with temperature \( T_s \)

$$\ln F_s = -\frac{m_s v^2}{2T_s} + \log(cn_s/T_s^{3/2}), \quad (VII.13)$$

for a constant \( c \), and thus the contribution from \( \ln F_s \) on the RHS of vanishes because of the equations \( \int d^3v \, C_{ss'} = 0 \) and \( \sum_{s'} \int d^3v \, (1/2) m_s |v|^2 C_{ss'} = 0 \). The contribution from \( \delta f_s/F_s \) in then gives the final quadratic-order H-theorem

$$\frac{d}{dt} \sum_s \left[ S(F_s) - \int d^3x \int d^3v \frac{(\delta f_s)^2}{2F_s} \right] = - \sum_s \int d^3x \int d^3v \frac{\delta f_s}{F_s} \left( \frac{\partial \delta f_s}{\partial t} \right) \geq 0 \quad (VII.14)$$

where

$$\left( \frac{\partial \delta f_s}{\partial t} \right) = \sum_{s'} \left[ C_{ss'}(F_s, \delta f_{s'}) + C_{ss'}(\delta f_s, F_{s'}) \right]$$

is the linearized collision integral and the condition \( \int d^3x \delta f_s = 0 \) has been used again to eliminate the contribution from \( C_{ss'}(F_s, F_{s'}) \) on the RHS of

In the work \[24, 25, 111\], this H-theorem has been further reformulated as an equation for the dissipation of a “generalized energy” or “free energy”. Noting that the entropy per volume for the Maxwellian \( F_s \) is

$$S(F_s)/V = n_s \ln(T_s^{3/2}/cn_s) + \frac{3}{2} \frac{n_s}{c}, \quad (VII.16)$$

then \[111\], Appendix B1, shows that to leading order \( dn_s/dt = 0 \) (their Eq.(B3)). Thus, the entropy balance for a single species \( s \) reduces, per volume, to

$$\frac{1}{T_s} \frac{dE_{0s}}{dt} - \frac{d}{dt} \int d^3x \int d^3v \left( \frac{(\delta f_s)^2}{2F_s} \right) = - \int d^3x \int d^3v \frac{\delta f_s}{F_s} \left( \frac{\partial \delta f_s}{\partial t} \right) + \frac{1}{T_s} Q_s, \quad (VII.17)$$

with \( E_{0s} = (3/2)n_sT_s \) the kinetic energy density for the Maxwellian \( F_s \), and \( Q_s \) the collisional heat exchange defined in \[111\]. The above eq.(VII.17) is the “heating equation” derived as eq.(B15) of \[111\], Appendix B2. As already discussed there, this “heating equation” implies that the temperatures \( T_s \) evolve on a time-scale \( \sim 1/e^2\Omega_s \), an order \( O(\epsilon^{-2}) \) longer than the evolution time-scale \( 1/c\Omega_s \) of the ring distribution functions \( h_s \).

Because of the condition \( \int d^3x \delta f_s = 0 \), one has also

$$E_{0s} = \int d^3x \int d^3v \frac{1}{2} m_s |v|^2 f_s, \quad (VII.18)$$

which shows that \( E_{0s} \) is just the volume-average of the particle energy density \( E_s \) defined in \[111\]. Using this equation and the slow time evolution of \( T_s \), eq.(VII.17) is rewritten to leading order as \[24, 25, 111\]

$$\int d^3x \int d^3v \left[ \frac{1}{2} m_s |v|^2 f_s - T_s (\delta f_s)^2 \right] = - \int d^3x \int d^3v \frac{T_s |\delta f_s|^2}{F_s} \left( \frac{\partial \delta f_s}{\partial t} \right) + Q_s, \quad (VII.19)$$

which is equivalent to eq.(B11) in \[111\], Appendix B1, or eq.(9) in \[25\]. Summing over \( s \) gives a valid formulation of the H-theorem for gyrokinetics, but the quantity in the square brackets is sign-indeterminate. Using conservation of total energy with space density \( E_s = \sum_s E_s = \frac{1}{8\pi} (|E|^2 + |B|^2) \), one can instead introduce a free energy or generalized energy with volume-average density

$$W = \int d^3x \int d^3v \sum_s T_s (\delta f_s)^2 + \frac{|E|^2 + |B|^2}{8\pi}, \quad (VII.20)$$

which is non-negative and also dissipated, according to the balance equation

$$\frac{dW}{dt} = \int d^3x \int d^3v T_s \delta f_s \left( \frac{\partial \delta f_s}{\partial t} \right) \leq 0. \quad (VII.21)$$

This coincides with the eq.(B19) derived in \[111\], Appendix B3, or eq.(11) in \[25\] for the case of no external
forcing. Here we have emphasized how (VII.21) arises from the more general Vlasov-Maxwell-Landau model, but it can also be derived directly within the gyrokinetic description for the first-order fluctuations $\delta f_s^{(1)}$ in (VII.5). See eqs.(73),(74) in [24].

Unfortunately, there is no obvious analogue of this “free energy” for the full VLM model in general. The analogous quantity would seem to be

$$W = \sum_s T_s H(f_s|F_s) + \int \frac{d^3x}{V} \left[ \frac{|E|^2 + |B|^2}{8\pi} \right]$$

where $F_s$ is a global Maxwellian with density $n_{0s}$ and temperature $T_s$ and the relative entropy is

$$H(f_s|F_s) = \int \frac{d^3x}{V} \int d^3v \left( f_s \log(f_s/F_s) - f_s + F_s \right) \geq 0.$$  

Using the conservation of total energy and $dS/F_s/dt = \frac{1}{2}(N_{0s}/T_s)dT_s/dt$, it then follows that

$$\frac{dW}{dt} = \sum_s \frac{dT_s}{dt} \left( S[F_s] - S[f_s] \right) / V - \sum_s T_s \frac{d}{dt} S[f_s] / V.$$  

The first term on the right is positive if, as seems plausible, $dT_s/dt > 0$. The second term on the left also cannot be shown to be negative, because the symmetrization argument using $s \leftrightarrow s'$ and $p \leftrightarrow p'$ with the Landau collision integral giving (VII.27) also takes $T_s \leftrightarrow T'_s$. Only for $T_s = T$ and $dT/dt = 0$ does one obtain

$$\frac{dW}{dt} = \frac{d}{dt} T \int \frac{d^3v}{\sigma} \geq 0$$

holds to leading order because in that case similarly $T_s - T'_s = O(\epsilon^2)$ for $s \neq s'$ and $dT_s/dt = O(\epsilon^3\Omega_s)$ (see [111], footnote 8, p.595).

3. Scaling Exponent Predictions

Gyrokinetics is expected to provide an asymptotic description as $\epsilon \to 0$ of a class of exact solutions of the Vlasov-Maxwell-Landau (VML) equations, including solutions that describe turbulent cascades of energy and entropy. A theory of these cascades may therefore be developed either within the reduced gyrokinetic description or within the more comprehensive VML model. Although energy and entropy are separate quantities with their own distinct balances for VML solutions, these quantities are intertwined into the single invariant $W$ in the works [24, 25] on astrophysical gyrokinetic turbulence. The cascades of $W$ discussed in those works are partially associated to energy cascade in the full VML description, and partially to entropy cascade. However, the flux of $W$ at the smallest collisionless scales which matches onto the anomalous entropy production by collisions (see section 2.5 in [111], section 5 in [25], sections 7.9.3, 7.12 in [24]) must be entirely due to entropy cascade in the VML description, since no energy dissipation anomalies are possible in the $Do \to \infty$ limit.

Anomalous entropy production, both in the gyrokinetic and in the full VML description, requires short-distance divergences of solutions in phase-space, which must be regularized to allow for a dynamical description in the collisionless limit. One may study this limit $Do \to \infty$ either before or after the limit $\epsilon \to 0$. Taking the limit first $\epsilon \to 0$, then $Do \to \infty$ can be achieved with a suitable distributional or “weak” formulation of the gyrokinetic model equations, which we shall not attempt to develop here [112]. Alternatively, one take first the limit $Do \to \infty$ of regularized VML solutions and then take $\epsilon \to 0$ as a subsidiary limit. This second order of limits is required if the collisional phase-space cutoff scales $\ell_c, u_c$ (see section 2.5 in [111], section 5 in [25], sections 7.9.3, 7.12 in [24], and Appendix C) are too small for the gyrokinetic approximation to be valid at those scales. In this order of limits, the coarse-graining regularization of VML solutions employed in the present work applies and all of our rigorous estimates of entropy flux carry over to gyrokinetics. Note that in our estimates should be understood to represent $\ell_1$, when the scale-anisotropy $\ell_\parallel \gg \ell_\perp$ implied by (VII.3) holds in the limit $\epsilon \to 0$. All fields are then smoother along the $B_0$-direction and, for fixed displacement length $r$, increments are likewise dominated by the most singular direction in velocity space.

Based on these remarks, we may directly compare
our exact inequalities \((\text{V.45})\) on the scaling exponents \(\sigma^F_p := \min \{\sigma^E_p, \sigma^B_p, \sigma^f_p\}, \rho^f_p\), and \(\rho^f_p\) of orders \(p \geq 3\), required for entropy cascade, with the scaling predictions for gyrokinetic turbulence in \([24, 23]\). Those papers derive predictions for spectral exponents, or orders \(p = 2\), but their results may be assumed to apply to all orders \(p\) if intermittency effects can be ignored. Since the scaling exponents in question are non-increasing in \(p\), this “mean-field” approximation necessarily overestimates the true exponent values for \(p \geq 3\). Gyrokinetic theory assumes that background fields are smoother than fluctuations, so that \(\sigma^B_p = \sigma^E_p = \sigma^f_p = \rho^f_p\). The first-order gyrokinetic result \((\text{VII.5})\) for \(\delta f_s\) also implies that \(\sigma^f_p := \min \{\sigma^f_p, \sigma^h_p\}\). Another general prediction of gyrokinetics is the relation \(\delta f_s \sim \ell / \rho_s\) that connects scaling in position and velocity space. This relation is a consequence of the non-parallel perpendicular phase-mixing mechanism for entropy cascade in gyrokinetics, in which velocity-space structure arises from position-space structure due to the dependence of ring gyroradii on perpendicular velocity (Figure 1 in \([23]\); Figure 10 in \([24]\)). An immediate consequence is that velocity-space and position-space exponents are equal, or \(\rho^f_p = \sigma^f_p\), in gyrokinetic turbulence.

Specific predictions for scaling exponents in possible entropy cascade ranges of gyrokinetic turbulence have been developed phenomenologically in \([24, 23]\), for the particular case of a Maxwellian, two-species (electron-ion) plasma. The work \([24]\) considered three different situations, which we briefly summarize here:

(a) KAW/ion entropy cascade \((\rho_e \ll \ell \ll \rho_i)\): Section 7.9 of \([24]\) considered an entropy cascade passively driven by a kinetic Alfvén wave (KAW) cascade, assuming \(\rho_i/L_i \lesssim 1, m_e/m_i \ll 1\). Their predictions, expressed in terms of scaling of increments, are:

\[
\sigma^E_p \sim \ell^{-1/3}, \quad \delta f_s \sim \ell^2/3 \tag{VII.28}
\]

\[
\delta f_i \sim \ell^{1/6}, \quad \delta^u f_i \sim u^{1/6} \tag{VII.29}
\]

so that

\[
\sigma^F_p = -\frac{1}{3}, \quad \sigma^f_p = \frac{1}{6}, \quad \rho^f_p = \frac{1}{6}. \tag{VII.30}
\]

Quotation marks “” appear around the electric-field term in \((\text{VII.28})\) because increments no longer suffice to define scaling exponents, in the same manner as in \((\text{VIII.35})\), when the exponents become negative. Instead, one must use some sort of smooth low-pass or band-pass filter, e.g. wavelet coefficients as in \([84, 114, 115]\).

(b) Pure ion entropy cascade \((\rho_e \ll \ell \ll \rho_i)\): Section 7.10 of \([24]\), under the same limit conditions \(\rho_i/L_i \lesssim 1, m_e/m_i \ll 1\) but assuming now no KAW cascade and assuming also \(h_e = 0\), predicted:

\[
\delta f_s \sim \ell^{1/6}, \quad \delta f_i \sim \ell^{1/6}, \quad \delta^u f_i \sim u^{1/6} \tag{VII.31}
\]

\[
\delta f_s \sim \ell^{1/6}, \quad \delta f_i \sim u^{1/6} \tag{VII.32}
\]

so that

\[
\sigma^F_p = \frac{1}{6}, \quad \sigma^f_p = \frac{1}{6}, \quad \rho^f_p = \frac{1}{6}. \tag{VII.33}
\]

In this case magnetic fluctuations are very small over the range considered, so that the entropy cascade is self-driven by the electrostatic fields arising from fluctuations in the ion distribution. Note that the high smoothness of the magnetic field (scaling exponent \(> 2\)) implies that its 1st-order increments scale as \(\delta B_s \sim \ell\). Thus, the scaling exponent as defined in \((\text{VIII.35})\) is \(\sigma^B_p = 1\). To obtain instead \(\sigma^B_p = 13/6\), one must replace the 1st-order increments in \((\text{VIII.35})\) with 3rd-order increments, so that the \(O(\ell), O(\ell^2)\) terms in the Taylor-expansion are cancelled. See \([84, 114]\) for a general discussion.

(c) Electron entropy cascade \((\ell \ll \rho_e)\): Section 7.12 of \([24]\), assuming \(\rho_e \ll \rho_i \lesssim L_i\), considered a pure electron entropy cascade, with contributions of ion distribution \(h_i\) neglected (e.g. because the gyroaveraging makes its contributions subdominant in powers of \(m_e/m_i\))

\[
\delta f_e \sim \ell^{1/6}, \quad \delta^u f_e \sim u^{1/6} \tag{VII.34}
\]

so that

\[
\sigma^F_p = \frac{1}{6}, \quad \sigma^f_p = \frac{1}{6}, \quad \rho^f_p = \frac{1}{6}. \tag{VII.36}
\]

The scaling exponents are identical to those for the pure ion entropy cascade and, indeed, the physics is very similar, with electrostatic fields created by fluctuations in the electron distribution driving cascade of electron entropy.

Comparing these various predictions with our inequalities \((\text{V.45})\), the first observation is that our exact constraints required for an entropy cascade to exist are well-satisfied by the predictions of \([24]\) for all three cases. Secondly, the inequalities are not satisfied as near-equalities, but instead with the predicted exponents yielding a value considerably below the upper bound in \((\text{V.45})\). A somewhat similar situation occurs also in incompressible fluid turbulence, where the corresponding inequality \(\sigma^u_p < 1/3\) is satisfied with values of \(\sigma^u_p\) much smaller than \(1/3\) for \(p \geq 3\). For incompressible turbulence, this is a consequence of space-time intermittency (see e.g. \([89]\)), but, as there, a “mean-field” approximation which neglects effects of intermittency should be approximately valid for exponents of order \(p \approx 3\). We believe that the large gap is due instead to the strong depletion of nonlinearity in gyrokinetics, arising from substantial cancellations in the ring-averages \((\text{VII.8})\), and which is not taken into account in our upper bounds \((\text{VIII.36})-(\text{VIII.39})\) on entropy flux. In order to compensate for the reduced nonlinearity, more singular scaling behavior than what follows from \((\text{V.45})\) is thus required in gyrokinetic turbulence in order to sustain the cascade of entropy.
B. Empirical Studies

We here briefly review the available evidence for kinetic entropy cascades from empirical studies and discuss also some promising situations in space plasmas where they are likely to exist.

Numerical simulations of gyrokinetic turbulence have provided, so far, the best direct evidence for nonlinear entropy cascades in turbulent plasmas. The studies [117, 118] has considered decaying, electrostatic turbulence in a spatially 2D setting, with no variations parallel to \( B_0 \), in order to eliminate damping by the Landau resonance. The spatial domain-size was \( 2\pi \times 2\pi \), with \( p \) the gyroradius. A smooth, unstable initial condition was chosen for \( \delta f \), perturbed by small-amplitude white noise, together with the corresponding electrostatic potential \( \varphi \). This initial configuration was evolved under the gyrokinetic dynamics for three cases with decreasing collisionality (\( Do = 48, 118, 440 \)) and correspondingly increased numerical resolution. The collisional entropy production was found to be only weakly dependent on \( Do \) and spectrally-local, nonlinear fluxes of entropy were observed to small scales in position-space and velocity-space. The scaling behavior found in this study was quite close to that predicted in cases (b), (c) above, with Fourier spectra \( E_h(k_\perp), E_{E_\perp}(k_\perp) \sim k_\perp^{-4/3} \) and identical scaling in the Hankel-transform velocity spectrum of \( h \). A similar study [119] has also considered electrostatic gyrokinetic turbulence, but now in 3D and ion temperature gradient-driven. A statistical steady-state was reached with artificial hyperdiffusion added in position and velocity space. Despite the fact that such dissipation acted effectively at all scales, this study observed scale-local, nonlinear entropy cascade and obtained spectra similar to those in the study [117, 118].

In a different direction, the paper [120] performed a 3D, fully electromagnetic, gyrokinetic simulation of an ion-electron plasma designed to reproduce the turbulent KAW/ion entropy cascade of [24] (case (a) above). The size of the spatial domain was \( L_\perp = 2\pi p_i \) and \( L_\parallel \gg L_\perp \), with a \( 128^3 \) spatial grid able to resolve the electron gyroradius \( \rho_e = p_i/42.8 \). The simulation was driven by an “antenna current” set up to mimic energy input from a critically-balanced cascade of Alfvén waves and collisions were incorporated by a fully conservative, linearized collision operator. The field spectra observed were close to \( E_{E_\perp}(k_\perp) \sim k_\perp^{-1/3} \) and \( E_{B_\perp}(k_\perp), E_{B_\parallel}(k_\perp) \sim k_\perp^{-2.8} \), with the latter somewhat steeper than the \( k_\perp^{-7/3} \) spectrum predicted in [24]. This steepening was plausibly explained by the finiteness of the mass ratio \( m_e/m_i \) and the damping of KAW modes by Landau resonance with electrons, which peaks in the simulation at \( k_\perp p_i \sim 1 \) but is increasing roughly as a power-law over the entire \( k_\perp \)-range. The important point here is that the collisionless input into \( h_\parallel \) by the Landau resonance with ions peaked at \( k_\perp p_i \sim 1 \) but the collisional ion heating peaked at higher wave number \( k_\perp p_i \sim 20 \). This is consistent with the presence of an ion entropy cascade. See also [121, 122].

Entropy cascade should occur not only within numerical simulations but also ubiquitously at small scales in turbulent plasmas of very weak collisionality, with the solar wind and the terrestrial magnetosheath as likely examples. We know of no direct evidence of non-vanishing entropy flux in such environments, although high-resolution measurements of ion distribution functions in the magnetosheath do reveal complex velocity-space structure. Furthermore, in situ observations of magnetic field spectra broadly agree with gyro-simulations exhibiting entropy cascade. As recently reviewed [124], solar wind spectra are well fit as power-laws \( E_B(k_\perp) \sim k_\perp^{-x} \) for \( 1/p_e \lesssim k_\perp \lesssim 1/p_i \) and \( E_B(k_\perp) \sim k_\perp^{-y} \) for \( 1/p_e \lesssim k_\perp \lesssim 1/p_i \), with a distribution of exponents \( x \in [2.5, 3.1] \) peaked at \( x = 2.8 \), and \( y \in [3.5, 5.5] \) peaked at \( y = 4 \). In the terrestrial magnetosheath, paper [124] reports similar scaling but with \( x \in [2.4, 3.5] \) peaked at \( x = 2.9 \), and \( y \in [4.7, 5.5] \) peaked at \( y = 5.2 \). Clearly, the magnetic spectra observed in the range \( 1/p_i \approx k_\perp \lesssim 1/p_e \) for both the solar wind and heliosheath agree reasonably well with the simulation of the KAW/ion entropy cascade in [120]. Another paper [120] reported in the solar wind an electric spectrum \( E_{E_\perp}(k_\perp) \sim k_\perp^{-3.3} \) fitted over the decade \( k_\perp p_i \in [0.43, 4.3] \), roughly consistent with the prediction \( E_{E_\perp}(k_\perp) \sim k_\perp^{-1/3} \) of [24] for the KAW/ion entropy cascade. See as well [124]. At sub-electron scales \( 1/p_e \approx k_\perp \) the magnetic spectra reported for both the solar wind and magnetosheath in these references appear also to be roughly in agreement with the prediction \( E_B(k_\perp) \sim k_\perp^{-10/3} \) of [24] for the electron entropy cascade. Agreement is clearly best for the magnetosheath where, as pointed out in [124, 125], the signal-to-noise ratio of measurements is higher than for the solar wind and where, therefore, the spectral slopes are more reliable.

Although reasonably identified as entropy cascades, these turbulent space plasmas are likely not accurately described by gyrokinetics all the way down to collisional scales. The gyrokinetic approximation is estimated break down in the solar wind at a length-scale between \( p_i \) and \( \rho_e \) [111], but the collisional cutoffs for both ion and electron entropy cascades should lie at much smaller scales. The cutoff scale for the ion entropy cascade is \( \ell_i \sim p_i/Do_i^{-3/5} \) within gyrokinetic theory [24], where ion-scale Dorland number is given by \( Do_i = 1/\nu_i\tau_{pi} \) for ion-ion Coulomb collision rate \( \nu_i \) and eddy-turnover rate \( \tau_{pi} \) at the ion gyroradius. In the solar wind at \( 1 \text{ AU} \) \( \nu_i \approx 3 \times 10^{-7} \text{ Hz} \) and \( \rho_i \approx 100 \text{ km} \). From \( \tau_{pi} \approx \varepsilon^{-1/3} \rho_i^{2/3} \) and using \( \varepsilon \approx 10^4 \text{ m}^2/\text{sec}^2 \) from 3rd-moment measurements [39], one can estimate \( \tau_{pi} \approx 10 \text{ sec} \). Thus, \( Do_i \approx 10^5 \) and the collisional cutoff scale for ion entropy cascade calculated within gyrokinetics is \( \ell_i \sim 10^{-3} \rho_i \) or smaller. Similar estimates apply to the cutoff \( \ell_i \sim \rho_i/Do_i^{-3/5} \) for the electron entropy cascade, with electron-scale Dorland number \( Do_e = 1/\nu_e\tau_{pe} \). Note that the electron-ion collision rate \( \nu_{ei} \) is larger than \( \nu_i \) by a factor of \((m_i/m_e)^{3/2}\).
but the electron-scale turnover rate $\tau_{\rho_e}$ is smaller than $\tau_{\rho_i}$ by a comparable factor. If these various estimates are accurate, entropy cascades in the solar wind and terrestrial magnetosheath must extend down to scales well below those where gyrokinetics is valid.

The description of such kinetic cascades is one of the principal motivations for the theory developed in the present work. Measured magnetic and electric spectra in the solar wind [124, 127] and in the magnetosheath [128, 129] indicate that the turbulence at sub-electron scales in those environments is probably “electrostatic,” with electric fluctuations much larger than magnetic fluctuations. Therefore, the dominant contribution to the entropy flux is presumably the electric-field contribution (V.26) from the wave-particle interaction. Future work will exploit this formalism to elucidate further the physics of this phase-space cascade.

C. Turbulent Magnetic Reconnection

The results on coarse-grained momentum balance in section (VI.14) of this paper also make connection with prior work on turbulent magnetic reconnection and provide it with a deeper theoretical foundation. As is well-known, the momentum balance equations for an electron-ion plasma yield a “generalized Ohm’s law” for the electric field [130–132]. For a turbulent plasma, the coarse-grained momentum balance equations (VI.17) or (VI.19), for the two species $s = i, e$ can be combined, using the formula $j = e(n_iu_i - n_eu_e)$ for the electric current and assuming quasi-neutrality ($n_e = n_i = n$), to give:

$$\vec{E} + \frac{1}{c} \vec{u}_i \times \vec{B} = \frac{1}{n_e} \vec{E} + \frac{m_i}{m_i + m_e} \frac{j \times \vec{B}}{n_e c}$$

$$- \frac{1}{n_e} \nabla \cdot \left[ \frac{m_e \vec{P}_e - m_i \vec{P}_i}{m_i + m_e} \right]$$

$$+ \frac{m_e}{n_e^2 (m_i + m_e)} \left[ \partial_j J + \nabla \cdot (\vec{u}_i u_i - \vec{u}_e u_e - \vec{J}/ne) \right].$$

(VII.37)

Here we have retained the collisional drag forces $\pm \vec{R}$ on the electrons/ions, respectively. Unresolved turbulent eddies can be considered to contribute two new terms to this coarse-grained Ohm’s law. One is the velocity-fluctuation induced electric field defined by

$$\vec{e}_n := \frac{1}{n} (\vec{n}(n, \vec{E}) + \vec{u}_i \times \vec{n}(n, \vec{B}))/c.$$ (VII.40)

Here we have used the general relation $\vec{b} = \vec{B} + \nabla(n, b)/n$ between unweighted and Favre-weighted spatial coarse-graining, analogous to (III.15). This second electric-field contribution from turbulent density fluctuations was pointed out in [15], eq.(6.11). The sum of these two electric fields $\vec{e}_i = \vec{e}_{ui} + \vec{e}_n$ coincides with the “turbulent electromotive force” defined in eq.(VI.33) for $s = i$.

Magnetic reconnection at length-scale $\ell$ in a turbulent plasma is thus governed by the generalized Ohm’s law

$$\vec{E} + \frac{1}{c} \vec{u}_i \times \vec{B} = -\vec{e}_i + \frac{1}{n_e} \vec{R} + \frac{1}{n_e c} \vec{j} \times \vec{B} - \frac{1}{n_e} \nabla \cdot \vec{P}_e$$

$$+ \frac{m_e}{n_e^2} \left[ \partial_j J + \nabla \cdot (\vec{u}_i u_i - \vec{u}_e u_e - \vec{J}/ne) \right],$$

(VII.41)

assuming for simplicity a small mass ratio $m_e/m_i \ll 1$, which recovers eqs.(VI.2),(VI.10) of [15]. In [15], eq.(6.2) the collisional drag force was represented by an Ohmic field $\vec{R}/en = \eta \vec{j}$ with Spitzer resistivity $\eta$ and it was argued from this representation that the drag term is negligible in a weakly collisional plasma such as the solar wind. Strictly speaking, such an argument is only valid for coarse-graining length $\ell \gg \lambda_{mfp,e}$, the mean-free path of the electrons, since it is only at such scales that the drag force is correctly represented by Ohmic resistivity [28]. On the other hand, the estimate (VI.14) in the present work shows more generally that the collisional drag term vanishes as $Do \rightarrow 0$ at any fixed length-scale $\ell$ in the coarse-grained momentum balance equations (VI.17) or (VI.19).

It was further shown in [15] that all of the microscopic non-ideal electric fields terms on the righthand side of the generalized Ohm’s law (VII.37) are negligible in the inertial-range of the solar wind. Assuming the scaling of increments that are observed at length scales $\rho_i \ll \ell \ll L_i$ in the solar wind and that are expected generally for MHD-like turbulence, the analysis showed that the non-ideal terms are all suppressed by powers of $\delta_\ell/\ell$ or $(\delta_i/\ell)^2$ at length scale $\ell$, with $\delta_i$ the ion skin-depth. The non-ideal terms are thus like (infrared) irrelevant variables in the technical RG sense. Here we may note that the plasma dynamics in the “inertial-range” of the solar wind, for $\rho_i \ll \ell \ll L_i$, has been previously argued to be governed by “kinetic RMHD” in the works [24], section 5, and [33], by means of gyrokinetic theory. In particular, the dominant component of incompressible, shear-Alfvén waves in that range was argued to be described by “reduced MHD” (RMHD) and the magnetic field to be governed by the ideal induction equation. Our analysis here and in [15] agrees with the latter conclusion. However, papers [24, 33] both go on to argue that, as a consequence, the magnetic field at inertial-range scales is “frozen in” to the ion flow, e.g. “At $k_i \rho_i \ll 1$, ions (as well as the electrons) are magnetized and the magnetic
field is frozen into the ion flow” [24]. This statement is incorrect. Insofar as the ideal induction equation holds in the inertial-range of the solar wind, it does not imply magnetic flux-freezing at those scales, and insofar as the ideal induction equation implies magnetic flux-freezing, it is not valid in the inertial-range of the solar wind.

As pointed out in [13], an “ideal Ohm’s law” holds in the inertial range of the solar wind only in the sense that the equality

$$\mathbf{E} + \frac{1}{c} \mathbf{u}_i \times \mathbf{B} = 0$$  \hspace{1cm} (VII.42)

is well-satisfied for length scales $\ell \gg \rho_i$. Validity of the ideal Ohm’s law in this “weak” or “coarse-grained” sense, however, does not imply that the magnetic field at those scales is “frozen-in” to the velocity $\mathbf{u}_i$. This becomes obvious if one rewrites the “ideal Ohm’s law” (VII.42) equivalently as

$$\mathbf{E} + \frac{1}{c} \mathbf{u}_i \times \mathbf{B} = -\varepsilon,$$  \hspace{1cm} (VII.43)

which makes apparent that the turbulent electromotive force $\varepsilon$ breaks flux-freezing at those scales. Keeping the contribution $\mathbf{j}_e \varepsilon$ to energy cascade in eq. (VII.42), while discarding $\varepsilon$, spuriously from the Ohm’s law eq. (VII.43) in order to infer “flux-freezing” at scales $\ell \gg \rho_i$ is a fundamental inconsistency. As recognized in the work of Lazarian & Vishniac [134], reconnection must occur for eddies at all scales $\ell$ in a turbulent plasma. In fact, due to the turbulent contributions, magnetic-flux conservation may be anomalous and violated in the limit first max{$\rho_i, \delta_i$}/$L_i \rightarrow 0$, then $\ell/L_i \rightarrow 0$ [97, 135]. Magnetic flux-structures with dimensions much larger than $\rho_i$ or $\delta_i$ which are embedded in a turbulent inertial-range may therefore undergo reconnection at rates which are independent of microscopic physics and determined solely by the inertial-range turbulence. A concrete example of this type has been studied numerically in [36] using a database of incompressible MHD turbulence, where it was shown the electric field $\varepsilon_{ue}$ induced by turbulent velocity fluctuations accounts for the reconnection at inertial-range scales. An empirical study in [13] using spacecraft data suggests that in the solar wind the compressible contribution $\varepsilon_n$ plays a relatively small role and that inertial-range reconnection there is also due primarily to the “ideal” electric field $\varepsilon_{ue}$ induced by velocity fluctuations of unresolved eddies.

Similar remarks hold for reconnection of magnetic structures at sub-ion scales, which is generally treated by rewriting the generalized Ohm’s law to refer to the electron fluid. Turbulent reconnection at sub-ion scales $\ell < \rho_i$ may likewise be treated by by rewriting the coarse-grained Ohm’s law (VII.37) in terms of the electron bulk velocity, yielding

$$\mathbf{E} + \frac{1}{c} \mathbf{u}_e \times \mathbf{B} = \frac{1}{ne^2} \mathbf{R} - \frac{m_e}{m_i + m_e} \frac{\mathbf{j} \times \mathbf{B}}{ne^2}$$

$$- \frac{1}{ne^2} \nabla \cdot \left( \frac{m_e \mathbf{F}_e - m_i \mathbf{F}_i}{m_i + m_e} \right)$$

$$+ \frac{m_e m_i}{ne^2 (m_i + m_e)} \left[ \frac{\partial \mathbf{j} + \nabla \cdot (\mathbf{j} \mathbf{u}_e + \mathbf{u}_e \mathbf{j} + \mathbf{j}/ne)}{ne^2} \right].$$  \hspace{1cm} (VII.44)

For weak collisionality and $m_e/m_i \ll 1$

$$\mathbf{E} + \frac{1}{c} \mathbf{u}_e \times \mathbf{B} = -\varepsilon_e - \frac{m_e}{ne^2} \nabla \cdot \mathbf{F}_e$$

$$+ \frac{m_e m_i}{ne^2} \left[ \frac{\partial \mathbf{j} + \nabla \cdot (\mathbf{j} \mathbf{u}_e + \mathbf{u}_e \mathbf{j} + \mathbf{j}/ne)}{ne^2} \right],$$  \hspace{1cm} (VII.45)

with $\varepsilon_e$ given by eq. (VII.33) for $s = e$. The estimates in [13] show that the contributions from the electron pressure tensor and electron inertia are suppressed by powers of $\delta_e/\ell$ [136]. Therefore, when $\ell \gg \delta_e$, then the Ohm’s law referred to the electron fluid is “ideal” but magnetic fields are nevertheless not frozen-in into the velocity $\mathbf{u}_e$ because of the turbulent contribution $\varepsilon_e = \varepsilon_{ue} + \varepsilon_n$. When $\ell \sim \rho_i \sim \delta_i$ (assuming $\beta_i \sim 1$), then the non-ideal electric fields are suppressed by a factor of only $\sim 1/4\beta$ relative to the turbulent contributions and need not be entirely negligible. When $\ell \sim \delta_e \sim \rho_e$ then the non-ideal contributions will begin to dominate. Price et al. [137, 138] have suggested based upon 3D PIC simulations that the $\mathbf{P}(n, E)/\mathbf{P}$ contribution in $\varepsilon_n$ plays an important (but not dominant) role in dayside magnetopause reconnection observed by MMS, with turbulence self-driven by the reconnection itself. Magnetic reconnection of ion and electron-scale structures is also observed in the terrestrial magnetosheath [139, 140]. There is strong pre-existing turbulence in this environment which should contribute significantly to reconnection of magnetic structures at length-scales $\ell \sim \rho_i \sim \delta_i$.

### VIII. CONCLUSIONS AND OUTLOOK

This paper has systematically explored the hypothesis that entropy production in a weakly coupled, multi-species plasma may remain non-zero in the limit of vanishing collisionality. This hypothesis implies that there will be thermalization of the plasma or a tendency of velocity distribution functions to evolve toward Maxwellian, even as the dimensionless collision rate tends to zero. This tendency is consistent with particle distribution functions for driven systems remaining very far from Maxwellian and with large mean entropy-production in long-time steady states. The earlier conjecture of [24, 27] that such non-vanishing dissipation may occur by a turbulent cascade of entropy through phase-space, based on gyrokinetic theory, has been shown here to be the necessary consequence of an entropy production anomaly. In close analogy with Onsager’s “ideal turbulence” theory for incompressible fluids, we have shown that the dynamics of the plasma at fixed length and velocity scales in the collisionless limit is governed by a “weak” or “coarse-grained” solution of the Vlasov-Maxwell equations. Although smooth solutions of the Vlasov-Maxwell system conserve entropy, the solutions...
suggested by our analysis violate that conservation law by a nonlinear cascade of entropy. We obtain an explicit formula for the entropy flux through phase-space, which we use to predict specific correlations (down-gradient transport) and specific types of singularities/scaling exponents required to sustain a non-vanishing entropy cascade. Our results are consistent with gyrokinetics, but are more general, because they do not require any of the specific conditions assumed for validity of gyrokinetic theory (evolution rates small compared with gyrofrequencies, scale-anisotropy, etc.). Our sole assumption is weak collisionality. Our conclusions are thus widely applicable, holding, for example, at all scales in the solar wind smaller than the Coulomb mean-free-path length and larger than the Debye screening length. The collisionless entropy cascade discussed in this work should occur and be observable at sub-ion and sub-electron scales in the solar wind and the terrestrial magnetosheath.

We have also considered in this paper the balances of the standard collisional invariants: mass, momentum, and energy. Although conserved overall, these quantities can be converted from one form to another by Coulomb collisions of the particles (e.g. momentum may be transferred from one particle species to another). We show that such collisional transfers cannot be anomalous, but instead must vanish in the collisionless limit. Anomalies may appear in subsidiary limits, however, such as gyroradii small compared with turbulence injection scales ($\rho_s/L_s \ll 1$). For example, the electron momentum equation reduces in that limit to an ideal Ohm’s law, but only in a “weak” or “coarse-grained” sense that does not imply the frozen-in property of magnetic flux and that predicts instead reconnection of “magnetic eddies” at all inertial-range scales. Likewise, energy transfers through length-scales and velocity-space may be anomalous in such a small gyroradius limit, including a novel phase-space redistribution effect. The energy balance equations that we derive in this work, resolved simultaneously in phase-space and in scale, generalize and unify previous results in the literature [7, 51, 99, 100]. They provide a basis for the study both of turbulent energy cascade and of nonlinear Landau damping in a turbulent setting.

Because energy is not dissipated by collisions in the Vlasov-Maxwell-Landau theory, it is useful to address briefly the question of the ultimate sink of energy cascaded to small length-scales. The answer to this question is clearly situation-dependent. In some cases, there may be no sink at all, with energy simply accumulating in kinetic velocity fluctuations after cascading to small length-scales. This seems to be the case in the solar wind, where turbulent cascade appears to provide the energy required to offset the “cooling” due to adiabatic processes such as bremsstrahlung. However, the energy transferred to non-Maxwellian tails and supra-thermal particle production. In other cases, e.g. the solar corona, the particle kinetic energy cascaded to small scales may be carried off by electromagnetic radiation. This process is not described within Maxwell-Vlasov-Landau theory, which assumes elastic Coulomb collisions that conserve the total kinetic energy of charged particles. Radiative processes such as bremsstrahlung involve inelastic particle collisions with emission of photons and their treatment requires separate consideration of plasma emissivity [132]. Likewise, thermal radiation which carries off both energy and entropy requires a kinetic model of the photon gas that is coupled with the kinetic equations for the charged particles [143, 144]. A theory of plasma turbulence based upon the Vlasov-Maxwell-Landau equations alone cannot directly answer the question of the ultimate fate of cascaded energy but it should provide the inputs (e.g. particle distribution functions at small scales) necessary to address that question.

The present paper is intended to provide an exact, systematic framework for describing plasma turbulence at collisionless scales and should serve as a useful starting point for further investigations, not only theoretical but also numerical and experimental. Our analysis provides the foundation for numerical modelling of kinetic plasma turbulence by a “large-eddy simulation” (LES) methodology in phase-space [143, 144]. For experimentalists, our results provide a concrete model of “resolution effects”. Our results show that finite-resolution measurements in a turbulent plasma can lead to substantial “renormalizations” of bare quantities that must be taken into account in interpreting observational data. These are all important directions to pursue in future work.

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Appendix A: Derivation of Eq. (VI.38) in the Main Text

We define a phase-space density of fluctuation energy at scales $\ell$, $u$ as in eq. (VI.37) of the main text by

$$\zeta_s(x, u, t) := \int \frac{1}{2} m_s |\nabla \mathbf{u}_s - \tilde{u}_s|^2 f_s(x, u, t)$$

(A.1)

so that $\zeta = \int d^3u \zeta_s$. A phase-space balance equation
for this quantity can be obtained by decomposing it as
\[
\frac{1}{2} m_s |\nabla - \bar{u}_s|^2 T_s = \left( \frac{1}{2} m_s |\nabla|^2 T_s \right) - (m_s \nabla T_s) \cdot \bar{u}_s + \left( \frac{1}{2} m_s |\bar{u}_s|^2 \right) T_s
\]
and finally eq. (A.2), the equations
\[
\dot{\tilde{D}}_{t,s} \left( \frac{1}{2} m_s |\bar{u}_s|^2 \right) + (\bar{u}_s / \pi_s) \cdot \nabla \nabla^2 \tilde{P}_s = q_s \tilde{E}_{ss}
\]
(A.3)
\[
\dot{\tilde{D}}_{t,s} \left( \frac{1}{2} m_s |\bar{u}_s|^2 \right) + (\bar{u}_s / \pi_s) \cdot \nabla \nabla^2 \tilde{P}_s = q_s \tilde{E}_{ss}
\]
(A.4)
following from (A.3), (A.9) with \( \dot{\tilde{D}}_{t,s} := \partial_t + \bar{u}_s \nabla \) and finally eq. (A.12) gives in the nearly collisionless limit
\[
\partial_t \tilde{F}_s + \nabla \cdot \left( \tilde{v}_s \tilde{F}_s + \tilde{P}_s \cdot (\tilde{u}_s - \tilde{v}_s) \tilde{F}_s / \pi_s \right) + \nabla \cdot \left( q_s \tilde{E}_{ss} \tilde{F}_s \right) = \tilde{P}_s : \nabla \tilde{F}_s \left( \tilde{u}_s - \tilde{v}_s \right) / \pi_s
\]
(A.5)
\[
\partial_t \tilde{F}_s + \nabla \cdot \left( \tilde{v}_s \tilde{F}_s + \tilde{P}_s \cdot (\tilde{u}_s - \tilde{v}_s) \tilde{F}_s / \pi_s \right) + \nabla \cdot \left( q_s \tilde{E}_{ss} \tilde{F}_s \right) = \tilde{P}_s : \nabla \tilde{F}_s \left( \tilde{u}_s - \tilde{v}_s \right) / \pi_s
\]
(A.6)
Here the first contribution has a vanishing \( \nabla \)-integral and thus represents a redistribution of fluctuational energy in velocity space, whereas the \( \nabla \)-integral of the second term (and of the sum of the terms) is easily checked to give
\[
\tilde{P}_s : \nabla \tilde{u}_s. \tilde{u}_s \tilde{F}_s. \tilde{u}_s \tilde{F}_s / \pi_s
\]
(A.7)

**Appendix B: Bounds on Phase-Space Integrals**

1. The Integral in Estimate (IV.4)

In the upper bound (IV.3) on the coarse-grained collision integral, there appears the following integral over 2-particle phase-space:
\[
I = \int d^3r d^3v \int d^3v' G_{s}(r) [\nabla H]_s (v - v')^2 f_s f_s' / |v - v'|
\]
We shall show that this integral remains finite as \( \Gamma \to 0 \) under reasonable assumptions. First, we assume that
\[
n_s(x, t) := \int d^3v f_s(x, v, t) < \infty
\]
for all \( s \), uniformly in \( \Gamma \), so that no infinite spatial densities appear in the collisionless limit. Second, we assume that the distributions \( f_s \) are locally square-integrable for all species, so that
\[
\int_B d^3v f_s^2(x, v, t) < \infty
\]
for all bounded open sets of velocities \( B \) and for all \( s \), uniformly in \( \Gamma \). Note that the square-integrability of the distribution functions is generally assumed in theories of gyrokinetic turbulence, so that second-order structure functions and spectra are well-defined [23,24]. Square-integrability is also a natural assumption guaranteeing that the wave-particle term \( E_s f_s \) in the Vlasov-Maxwell equation is pointwise well-defined [66].

Divide the integral \( I \) into two contributions as
\[
I = I_\geq + I_\leq, \text{corresponding to the conditions } |v - v'| \geq 1 \text{ and } |v - v'| \leq 1, \text{ respectively. Then}
\]
\[
I_\geq := \int d^3r d^3v \int d^3v' G_{s}(r) [\nabla H]_s (v - v')^2 f_s f_s' / |v - v'|
\]
\[
\leq \int d^3r d^3v \int d^3v' G_{s}(r) [\nabla H]_s (v - v')^2 f_s f_s' / |v - v'|
\]
\[
\leq \max |(\nabla H)_s |^2 \cdot m_s n_r(x, t)
\]
(B.4)
and is bounded uniformly in \( \Gamma \). On the other hand, applying Cauchy-Schwartz to \( I_\leq \) gives
\[
I_\leq := \int d^3r d^3v \int d^3v' G_{s}(r) [\nabla H]_s (v - v')^2 f_s f_s' / |v - v'|
\]
\[
\leq \int d^3r d^3v \int d^3v' G_{s}(r) [\nabla H]_s (v - v')^2 f_s f_s' / |v - v'|
\]
\[
\times \int d^3r d^3v \int d^3v' G_{s}(r) [\nabla H]_s (v - v')^2 f_s f_s' / |v - v'|
\]
(B.5)
The integral inside the first square-root is finite in 3D and defines a constant depending only upon \( \ell, u \). The integral
inside the second square root has the upper bound
\[
\int d^3r \int d^3v \int d^3v' \, G_\ell(r) |(\nabla H)_s(v - v')|^2 f_s f_{s'}^2 \\
\leq \max |(\nabla H)_u|^2 \int d^3r \, G_\ell(r) \times \left( \int_{B_u(v)} d^3v \, f_s^2 \right) \left( \int_{B_{u+1}(v)} d^3v' \, f_{s'}^2 \right) \quad (B.6)
\]
since the support of |(\nabla H)_u(v - v')|^2 is contained inside the ball \(B_u(v)\) of radius \(u\) around \(v\) in velocity-space, with our assumptions on \(H\). We thus conclude that \(L_c\) is also bounded uniformly in \(\Gamma\).

2. The Integral in Estimate (VI.14)

In the upper bound (VI.14) on the drag force \(R_{s,s'}\) there appears the following integral:
\[
J = \int d^3v \int d^3v' \frac{f_s f_{s'}}{|v - v'|}. \quad (B.7)
\]
We shall show that this integral remains finite as \(\Gamma \to 0\) under reasonable assumptions, which include (B.2) and a strengthening of (B.3), according to which
\[
\int d^3v f_s(x, v, t) \int_{B_1(v)} d^3v' f_{s'}^2(x, v', t) < \infty \quad (B.8)
\]
The proof again proceeds by dividing the integral \(J\) into two contributions \(J_c, J_d\) corresponding to the conditions \(|v - v'| \geq 1\) and \(|v - v'| \leq 1\). Clearly,
\[
J_d := \int d^3v \int d^3v' \frac{f_s f_{s'}}{|v - v'|} \leq n_s(x, t) n_{s'}(x, t) \quad (B.9)
\]
On the other hand,
\[
J_c := \int d^3v \int d^3v' \frac{f_s f_{s'}}{|v - v'|} \\
= \int d^3v f_s \int_{B_1(v)} d^3v' \frac{f_{s'}}{|v - v'|} \leq \sqrt{4\pi} \int d^3v f_s \int_{B_1(v)} d^3v' f_{s'}^2 \quad (B.10)
\]
by applying Cauchy-Schwartz to the inner integral and by using \(\int_{B_1(v)} \frac{d^3v'}{|v - v'|^2} = 4\pi\) in 3D. Now apply Cauchy-Schwartz to the outer integral, giving
\[
J_c < \sqrt{4\pi} \int d^3v f_s \times \int d^3v f_s \int_{B_1(v)} d^3v' f_{s'}^2 < \infty \quad (B.11)
\]
3. The Integral in Estimate (VI.17)

In the upper bound (VI.17) on the conversion term \(R_{s,s'}\) there appears the following integral:
\[
K = \frac{1}{4} \int d^3v \int d^3v' \frac{|v + v'|^2}{|v - v'|} f_s f_{s'} \quad (B.12)
\]
We show that this integral remains finite as \(D\alpha \to \infty\) under the conditions (B.2), (B.8), and with also the further reasonable conditions
\[
K_s(x, t) := E_s(x, t)/m_s = \frac{1}{2} \int d^3v |v|^2 f_s(x, v, t) < \infty \quad (B.13)
\]
and
\[
\int d^3v |v|^2 f_s(x, v, t) \int_{B_1(v)} d^3v' f_{s'}^2(x, v', t) < \infty \quad (B.14)
\]
As with the preceding integrals, we divide the integral \(K\) into two contributions \(K_c, K_d\) corresponding to the conditions \(|v - v'| \geq 1\) and \(|v - v'| \leq 1\) and bound these two integrals separately.

Using \(|v + v'|^2 \leq 2(|v|^2 + |v'|^2)\), we get
\[
K_d := \frac{1}{4} \int d^3v \int d^3v' \frac{|v + v'|^2}{|v - v'|} f_s f_{s'} \leq \frac{1}{2} \int d^3v \int d^3v' (|v|^2 + |v'|^2) f_s f_{s'} = K_s n_s + K_{s'} n_{s'} < \infty. \quad (B.15)
\]
On the other hand, for \(|v - v'| \leq 1\),
\[
|v + v'|^2 = |2v + (v' - v)|^2 \leq (2|v| + 1)^2 \leq 2(4|v|^2 + 1) \quad (B.16)
\]
so that
\[
K_c := \frac{1}{4} \int d^3v \int d^3v' \frac{|v + v'|^2}{|v - v'|} f_s f_{s'} \leq \frac{1}{4} \int d^3v f_s \int_{B_1(v)} d^3v' \frac{|v + v'|^2}{|v - v'|} f_{s'} \leq \frac{1}{2} \int d^3v (1 + 4|v|^2) f_s \int_{B_1(v)} d^3v' f_{s'}^2 \quad (B.17)
\]
In the same manner as for \(J_c\) in (B.10) and (B.11), we apply Cauchy-Schwartz to the inner integral and then apply Cauchy-Schwartz to the outer integral, giving
\[
K_c \leq \sqrt{\pi} \int d^3v (1 + 4|v|^2) f_s \sqrt{\int_{B_1(v)} d^3v' f_{s'}^2} \leq \sqrt{\pi} \int d^3v (1 + 4|v|^2) f_s \leq \sqrt{\pi} \int d^3v (1 + 4|v|^2) f_s \int_{B_1(v)} d^3v' f_{s'}^2 < \infty \quad (B.18)
\]
using (B.2), (B.8), (B.13), (B.14).
Appendix C: Estimating the Collisional-Cutoff or Dissipation Scales

The estimate \( \ell_c \) on the coarse-grained collision integral derived in the main text provides a means to estimate the “cut-off scales” \( \ell_c, w_c \) in phase-space where particle collisions begin to compete with the turbulent renormalization effects due to ideal Vlasov-Maxwell dynamics. We here follow this approach to make more explicit determinations of such collisional-cutoff or dissipation scales. First, however, we shall review the derivation of similar viscous cut-offs in incompressible fluid turbulence, which suggests the approach to be followed also within kinetic turbulence. As we shall see, an improvement of the estimate \( \ell_c \) is required and also an additional phenomenological assumption analogous to Kolmogorov’s “refined similarity hypothesis” in incompressible fluid turbulence.

1. Viscous-Cutoff Scale in Incompressible Fluid Turbulence

In incompressible fluid turbulence, the role of the coarse-grained collision integral is played by the viscous diffusion term \( \nu \nabla^2 \mathbf{u} \) in the coarse-grained Navier-Stokes equation. See [27], eqs.(III.1,2) or [85], Chapter II(D). The Cauchy-Schwartz estimate analogous to \( \ell_c \) for the collision integral is eq.(III.3) in [14] or, in detail,

\[
|\nu \nabla^2 \mathbf{u}(\mathbf{x}, t)| \leq \frac{1}{\ell} \sqrt{\nu \text{ vol}(\text{supp}(G_\ell))} \times \int d^3r |(\nabla G)_\ell(r)|^2 \varepsilon(\mathbf{x} + r, t),
\]

where \( \text{vol}(\text{supp}(G_\ell)) \) is the volume of the compact support of the kernel \( G_\ell \). This volume is \( C^3 \) for some \( \ell \)-independent constant \( C \), so that we may rewrite (C.1) instead as

\[
|\nu \nabla^2 \mathbf{u}(\mathbf{x}, t)| \leq \frac{1}{\ell} \sqrt{\nu} C' \int d^3r \Phi_\ell(r) \varepsilon(\mathbf{x} + r, t),
\]

with \( \Phi := |\nabla G|^2 / \int |\nabla G|^2 \) another \( C^\infty \), compactly-supported, unit-normalized test function, \( \Phi_\ell(r) = (1/\ell^3)\Phi(r/\ell) \), and \( C' = C \int |\nabla G|^2 \) is a new \( \ell \)-independent constant. The integral inside the square root in (C.2) thus represents viscous dissipation (smoothly) averaged over a volume of order \( \sim \ell^3 \) and therefore can be estimated by an appeal to the “Kolmogorov refined similarity hypothesis” as of order

\[
\varepsilon(\mathbf{x}, t) := \int d^3r \Phi_\ell(r) \varepsilon(\mathbf{x} + r, t) \sim (\delta u(\ell))^3 / \ell.
\]

with \( \delta u(\ell) := \sup_{|r| < \ell} |\delta \mathbf{u}(\mathbf{x}, t)| \). This hypothesis is unproved but has enjoyed considerable empirical success; see [27], section 8.6.2. We therefore obtain

\[
|\nu \nabla^2 \mathbf{u}(\mathbf{x}, t)| \leq C'' \sqrt{\frac{\nu (\delta u(\ell))^3}{\ell^3}}
\]

with \( C'' \) a constant of order unity. The bound (C.4) is, in general, a large over-estimate of \( \nu \nabla^2 \mathbf{u} \). A better estimate is provided by

\[
|\nu \nabla^2 \mathbf{u}(\mathbf{x}, t)| \sim \frac{\nu \delta u(\ell)}{\ell^2},
\]

which is established as a rigorous upper bound in [16], footnote [16] or [85], Chapter II(D), but which should also be a good order-of-magnitude estimate.

Interestingly, however, the estimates (C.4) and (C.5) coincide when local Reynolds-number \( Re_\ell := \delta u(\ell) / \nu \sim 1 \), which is also the standard criterion used to identify the local viscous cut-off scale \( \ell_c \) in incompressible fluid turbulence (see e.g. [27], section 8.5.5). This criterion can be rationalized by estimating the “Reynolds-stress” term \( \nabla \cdot \tau(\mathbf{u}, \mathbf{u}) \) that arises in the coarse-grained Navier-Stokes equation as a turbulent renormalization effect of unresolved eddies (see eq.(III.6) in [16]). A rigorous bound can be derived of the form

\[
|\nabla \cdot \tau(\mathbf{u}, \mathbf{u})| \leq C(\delta u(\ell))^2 / \ell,
\]

using cumulant methods (e.g. see [15], Appendix B or the detailed derivation in [85], Chapter II(D)).

The exact upper bound (C.6) should also be a good order-of-magnitude estimate of \( |\nabla \cdot \tau(\mathbf{u}, \mathbf{u})| \), unless there is a substantial depletion of nonlinearity. Equating \( |\nabla \cdot \tau(\mathbf{u}, \mathbf{u})| \sim \nu \nabla^2 \mathbf{u} \) to determine \( \ell \sim \ell_c \) and using (C.6) for \( |\nabla \cdot \tau(\mathbf{u}, \mathbf{u})| \) and either (C.4) or (C.5) for \( |\nu \nabla^2 \mathbf{u}| \), one finds that the condition \( Re_\ell := \delta u(\ell) / \nu \sim 1 \) indeed provides the criterion for appearance of viscous effects locally in the coarse-grained equations.

If there is a substantial depletion of nonlinearity, one may instead proceed by defining an “eddy-turnover rate” \( \omega^{\text{eddy}}_\ell \) and a coarse-grained “dissipation rate” \( \omega^{\text{diss}}_\ell \), at each length-scale \( \ell \), by the equations

\[
\omega^{\text{eddy}}_\ell \delta u(\ell) := |\nabla \cdot \tau(\mathbf{u}, \mathbf{u})|,
\]

\[
\omega^{\text{diss}}_\ell (\delta u(\ell))^2 := \tau_\ell.
\]

Depletion of nonlinearity means that \( \omega^{\text{eddy}}_\ell, \omega^{\text{diss}}_\ell \ll \delta u(\ell)/\ell \). Balancing \( |\nabla \cdot \tau(\mathbf{u}, \mathbf{u})| \) with the sharp estimate (C.5) of \(|\nu \nabla^2 \mathbf{u}| \) then yields

\[
\omega^{\text{eddy}}_\ell \simeq \nu / \ell^2,
\]

i.e. turnover-rate \( \sim \) viscous diffusion rate, as the criterion to determine \( \ell \sim \ell_c \). On the other hand, the bound \( |\nabla \cdot \tau(\mathbf{u}, \mathbf{u})| \) with the looser estimate (C.4) for \(|\nu \nabla^2 \mathbf{u}| \) gives

\[
\omega^{\text{diss}}_\ell (\delta u(\ell))^2 \simeq \nu / \ell^2.
\]

As long as \( \omega^{\text{eddy}}_\ell \sim \omega^{\text{diss}}_\ell \), the two criteria (C.8), (C.9) will select the same \( \ell \sim \ell_c \). Empirical evidence suggests that there is not a strong depletion of nonlinearity in incompressible fluid turbulence, so that \( \omega^{\text{eddy}}_\ell \sim \omega^{\text{diss}}_\ell \sim \delta u(\ell)/\ell \) and the conditions (C.8), (C.9) then coincide with the naive criterion \( Re_\ell := \delta u(\ell) / \nu \simeq 1 \).
2. Improved Estimation of Coarse-Grained Collision Integral

In kinetic theory, on the other hand, there are well-known effects that may lead to depletion of nonlinearity, such as rapid wave oscillations, fast gyromotion of particles, dynamical alignment of vectors, etc. It would be desirable have a sharp estimate of the coarse-grained collision integral analogous to \((C.6)\), in order to obtain a criterion like \((C.8)\), involving only the ideal small-eddy turnover time and consistent with depletion of nonlinearity. Unfortunately, the Landau collision integral has much greater complexity than the viscous diffusion term in hydrodynamic turbulence, so that it is not at all obvious how to derive an estimate of the coarse-grained collision integral similar to \((C.6)\). An exact analogue of the hydrodynamic estimate \((C.4)\) can be derived, however, by a modest improvement of the estimate \((IV.5)\) in the main text and this result can be employed to derive a criterion analogous to \((C.9)\) for collisional cut-off scales \(\ell, u\) in kinetic turbulence.

To obtain the desired improvement of \((IV.5)\), we make a slightly different factorization of the integrand in \((IV.3)\), now moving the \((\nabla H)_a\) into the second factor:

\[
C_{ss'}(\mathbf{x}, \nabla, t) = \Gamma_{ss'} \frac{d^3r}{m_s u} \int d^3r f_s f_{s'} \left| \nabla \cdot \mathbf{v} < C_u \right| \frac{1}{v - v'} G^{1/2}_r \left( \frac{f_s f_{s'}}{\sqrt{v - v'}} \right)^{1/2} \cdot \frac{G^{1/2}_r(\nabla H)_a(v - v') (\nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{v'})(f_s f_{s'})}{(f_s f_{s'})(v - v')} \tag{C.10}
\]

We have assumed here that \(H_s\) has compact support, contained inside a ball of radius \(C u\), so that the \(v\)-integration can be restricted to that ball centered around \(\mathbf{v}\). We then apply the Cauchy-Schwartz inequality to obtain

\[
|C_{ss'}(\mathbf{x}, \nabla, t)| \leq \sqrt{\Gamma_{ss'}} \sqrt{\frac{d^3r}{m_s u} \int d^3r f_s f_{s'} \left| \nabla \cdot \mathbf{v} < C_u \right| \frac{1}{v - v'} G^{1/2}_r \left( \frac{f_s f_{s'}}{\sqrt{v - v'}} \right)^{1/2} \cdot \frac{G^{1/2}_r(\nabla H)_a(v - v') (\nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{v'})(f_s f_{s'})}{(f_s f_{s'})(v - v')} \times \mathbf{r} + \mathbf{v} + \mathbf{w}, \mathbf{v'}, t \tag{C.11}
\]

where \(\zeta_{ss'}(\mathbf{x}, \mathbf{v}, \mathbf{v'}, t)\) is the entropy production rate in the 2-particle phase-space due to collisions of particle species \(s, s'\), which is given by the corresponding term in the sum over \(s, s'\) in Eq. (1.37) in the main text that defines \(\zeta(\mathbf{x}, \mathbf{v}, \mathbf{v'}, t)\). Since the estimates \((B.4)\) - \((B.6)\) in section B.1 of this Supplement in fact depended only upon the compact support property of \(H_s\), they show for the first square-root factor essentially that

\[
\int d^3r \int_{|\mathbf{v} - \mathbf{v'}| < C_u} d^3v \int d^3v' G_r(\mathbf{v}) |f_s f_{s'}| \frac{1}{v - v'}
\]

with \(\nu_{h, ss'} := \max\{\nu_{h, ss}, \nu_{h, s's}\}\) and with \(C'\) a constant depending upon \((\mathbf{x}, \nabla, t)\) and \(f_s, f_{s'}\), but not upon \(\ell, u\). We leave details to the reader and note here only that \(J_s(\mathbf{x}, \nabla)\) and \(\Pi_{ss'}(\mathbf{x}, t)\) represent in fact local r.m.s. values in the averages over \(\mathbf{r}, \mathbf{v}, \mathbf{v'}\), which we for simplicity replaced with the usual coarse-grained values, assuming that they are of similar orders of magnitude. For the second square-root factor in \((C.11)\), we write

\[
|(\nabla H)_a(\mathbf{v} - \mathbf{v'})|^2 = \frac{1}{u^2} \Psi_a(\mathbf{v} - \mathbf{v'}) \times \int |(\nabla H)|^2 \quad \tag{C.13}
\]

with \(\Psi = |(\nabla H)|^2 / \int |(\nabla H)|^2\) another \(C^\infty\), compactly-supported, unit-normalized test function. Putting all of these estimates together, we obtain our final improvement of \((IV.5)\), for some \(\ell, u\)-independent constant \(C''\):

\[
C_{ss'}(\mathbf{x}, \nabla, t) \leq C'' \sqrt{\nu_{ss'} \nu_{s, \ell, u}(\mathbf{x}, \nabla, t) J_s(\mathbf{x}, \nabla, t) \times \nu_{h, ss'}} \frac{\ell}{u} \tag{C.14}
\]

where \(\nu_{ss'} := \Gamma_{ss'} \nu_{ss'}/m_s^2 \nu_{h, ss'}\) is essentially the Spitzer-Harm Coulomb collision rate of particle species \(s\) with particle species \(s'\), and we have defined

\[
\zeta_{s, \ell, u}(\mathbf{x}, \nabla, t) := \sum_{s'} \frac{d^3r}{f_{s'}} \int d^3w \rho_s(\mathbf{w}) \int d^3w' \left( \frac{\delta f_s(\ell, u)}{f_s} \right)^2 := \zeta_{s, \ell, u}(\mathbf{x}, \nabla, t) \tag{C.16}
\]

where

\[
(\Delta \varphi)_{s, \ell, u} := \frac{\partial J_s}{f_s} - \frac{\partial J_s}{f_s} \sim \frac{(\delta f_s(\ell, u))^2}{f_s} \tag{C.17}
\]

is a measure of the kinetic entropy of species \(s\) residing at scales \(\ell, u\) in phase-space, with \(\delta f_s(\ell, u)(\mathbf{x}, \nabla, t) = \sup_{|\mathbf{r}| < \ell, |w| < u} |\delta f_s(\mathbf{x}, \nabla, t)|\). For the estimate on the right side of \((C.17)\), see \[21\], footnote \([132]\). In these terms, the bound \((C.14)\) may be rewritten as

\[
C_{ss'}(\mathbf{x}, \nabla, t) \leq C'' \sqrt{\nu_{ss'} \omega_s \omega_{s'} \ell \delta f_s(\ell, u)} \tag{C.18}
\]

Another way to represent the estimate \((C.14)\) follows from a natural kinetic analogue of the Kolmogorov “refined similarity hypothesis (RHS)”, according to which the coarse-grained entropy production rate \(\zeta_{s, \ell, u}(\mathbf{x}, \nabla, t)\)
should scale in the same manner as the phase-space entropy flux $\dot{q}_s^{\text{flux},l}(\mathbf{x}, \nabla, t)$ given by (V.24) in the main text, so that

$$
\rho_{s,l,u}(\mathbf{x}, \nabla, t) \sim \max \left\{ \frac{u (\delta f_s)(\delta f_s)}{\ell f_s}, \frac{q_s(\delta E)(\delta f_s)}{m_s u f_s}, \frac{m_s u f_s}{c m_s u f_s} \right\}. \tag{C.19}
$$

The three terms on the right side of (C.19) arise from the estimates (V.23)-(V.27) of entropy-flux contributions in the main text and we assume, naturally, that the largest contribution to flux dominates the scaling (neglecting the fourth flux term from (V.28) as always smaller). Assuming this kinetic RHS implies a corresponding estimate of the entropy cascade rate, as an upper bound:

$$
\omega^{\text{diss}}_{s,l,u}(\mathbf{x}, \nabla, t) = \max \left\{ \frac{u}{\ell} \frac{q_s(\delta E)}{m_s u}, \frac{\tau_s(\delta f_s)}{c m_s u} \right\}. \tag{C.20}
$$

The true entropy flux rate (and, assuming the kinetic RHS, the coarse-grained dissipation rate) can be much smaller than this upper limit, if there is substantial depletion of nonlinearity. We therefore prefer to employ the general bound (C.18), without making use of the more specific estimate in (C.20).

3. Estimation of Turbulence-Generated Terms in Coarse-Grained Equations

As emphasized in the main text, the Vlasov-Maxwell equations “in the coarse-grained sense” (IV.12) differ from the naive Vlasov-Maxwell equations, because turbulent renormalization effects from unresolved eddies produce correction terms to the naive equations at each set of scales $\ell$, $u$ in phase-space. The coarse-grained collision integral can be neglected at those scales $\ell$, $u$ where it is much smaller than (the largest of) these turbulence-induced correction terms, and this is the defining characteristic of the “collisionless range” of scales. One can easily see from eqs. (IV.13)-(IV.18) in the main text that the small-eddy contributions to the time-evolution of $f_s$ have the form:

$$
(\partial_t f_s)^{\text{eddy}} = \nabla_{\mathbf{x}} (\dot{\rho}_s f_s) + q_s \nabla_{\mathbf{p}} \cdot (\nabla \cdot f_s) + \frac{1}{c} \nabla \times (\nabla \cdot f_s) + \frac{1}{c} (\nabla \times f_s) \times \mathbf{B}. \tag{C.21}
$$

Simple expressions can be readily obtained for each of the four contributions, which permit their magnitudes to be estimated as follows:

$$
\nabla_{\mathbf{x}} (\dot{\rho}_s f_s) = \frac{1}{\ell} \int d^3r (\nabla \cdot \delta f_s)(\mathbf{r}) \|\mathbf{w} (\nabla \delta w)\| u = O \left( \frac{u (\delta f_s)}{\ell} \right) \tag{C.22}
$$

$$
q_s \nabla_{\mathbf{p}} \nabla \cdot (E, f_s) = \frac{q_s}{m_s} \tau_s (\nabla_{\mathbf{e}} f_s) = O \left( \frac{q_s (\delta f_s)}{m_s u} \right) \tag{C.23}
$$

and

$$
q_s \nabla_{\mathbf{p}} \left( \frac{1}{c} \nabla \times (B, f_s) \right) = \frac{q_s}{m_s c} \epsilon_{ijk} \nabla_i \cdot (B_j, \partial_v \mathbf{T}_{s,u}) = O \left( \frac{q_s (\delta f_s)}{m_s u} \right) \tag{C.24}
$$

with the rightmost terms providing rigorous upper bounds. Note, as usual, that the fourth term is negligible compared with the others (in fact, vanishing exactly when $H$ is radially symmetric) and can be dropped. In these bounds we have introduced the following notation for the maximum double-increment (both in $\mathbf{r}$ and in $\mathbf{w}$):

$$
\delta \delta f_s := \sup_{|\mathbf{r}|<\ell, |\mathbf{w}|<u} |\delta \delta w f_s| \sim \min \{ \delta f_s, \delta \delta f_s \}. \tag{C.26}
$$

The estimate in (C.26) for the double-increment is also seen to be an exact upper bound, using the identities $\delta \delta w f_s(x, v) = \delta e f_s(x, v + w) - \delta e f_s(x, v)$ and likewise $\delta \delta w f_s(x, v) = \delta w f_s(x + r, v) - \delta w f_s(x, v)$, where the first identity is used if $f_s$ is smoother in $\mathbf{x}$ and the second identity if $f_s$ is smoother in $\mathbf{v}$.

An “eddy-turnover rate” $\omega^{\text{eddy}}_{s,l,u}(\mathbf{x}, \nabla, t)$ in phase-space is naturally defined by the equality

$$
\omega^{\text{eddy}}_{s,l,u}(\mathbf{x}, \nabla, t) f_s(\ell, u) := (\partial_t f_s)^{\text{eddy}}(\mathbf{x}, \nabla, t). \tag{C.27}
$$

One can readily see from the estimates (C.22)-(C.25) that an upper bound follows

$$
\omega^{\text{eddy}}_{s,l,u}(\mathbf{x}, \nabla, t) = \left( \frac{\max \left\{ \frac{u}{\ell} \frac{q_s(\delta E)}{m_s u}, \frac{\tau_s(\delta f_s)}{c m_s u} \right\}}{c^C \sup_{|\mathbf{r}|<\ell, |\mathbf{w}|<u} \int d^3w (\nabla \mathbf{H}) f_s \right) \cdot (\mathbf{w} \times (B, \delta w f_s)) \tag{C.28}
$$

of the same form as (C.20) for $\omega^{\text{diss}}_{s,l,u}(\mathbf{x}, \nabla, t)$. When there is large depletion of nonlinearity, however, one can expect that $\omega^{\text{eddy}}_{s,l,u}(\mathbf{x}, \nabla, t)$ is much smaller in magnitude than the bound (C.28). We shall therefore not use the latter bound in our determination of collisional cut-off scales. Even when there is strong nonlinearity depletion, however, it is plausible to expect that $\omega^{\text{eddy}}_{s,l,u}(\mathbf{x}, \nabla, t) \sim \omega^{\text{diss}}_{s,l,u}(\mathbf{x}, \nabla, t)$, with similar magnitudes and identical scaling in $\ell, u$. Despite the physical plausibility of these expectations, it is far from clear how to prove their validity.

4. Criterion for Collisional-Cutoff Scales

The collisionless range of scales for particle species $s$ is characterized by the condition that $|\partial_t f_s^{\text{eddy}}| \gg |\nabla| f_s^{\text{eddy}}|$. 


and, likewise, cut-off scales $\ell_s$, $u_s$ where collisions with particles of species $s'$ become important for species $s$ are specified by

$$
| (\partial \overrightarrow{f}_s)^{eddy}(\overrightarrow{x}, \overrightarrow{v}, t)| \simeq |C_{ss'}(\overrightarrow{x}, \overrightarrow{v}, t)|. \tag{C.29}
$$

Note that (C.29) is a pointwise condition in phase-space and thus the cut-off scales $\ell_s(\overrightarrow{x}, \overrightarrow{v}, t)$, $u_s(\overrightarrow{x}, \overrightarrow{v}, t)$ are local quantities, with fluctuating values reflecting phase-space intermittency of the entropy cascade. Employing the upper bound (C.18) as an estimate of $|C_{ss'}(\overrightarrow{x}, \overrightarrow{v}, t)|$ and recalling the definition (C.27) of $\omega^{eddy}_{s,s',t,u}$ in terms of $(\partial \overrightarrow{f}_s)^{eddy}(\overrightarrow{x}, \overrightarrow{v}, t)$, the condition (C.29) can be approximately rewritten as

$$
\frac{\omega^{eddy}_{s,s',t,u}}{\omega^{diss}_{s,s',t,u}} \simeq \nu_{ss'} \left( \frac{\nu_{th,ss'}}{u} \right)^2. \tag{C.30}
$$

Because (C.18) is only an upper bound on the coarse-grained collision integral, the true values of $\ell_s(\overrightarrow{x}, \overrightarrow{v}, t)$, $u_s(\overrightarrow{x}, \overrightarrow{v}, t)$ defined by (C.29) could be smaller than those specified by (C.30). On the other hand, under the reasonable scaling hypothesis $\omega^{eddy}_{s,s',t,u}(\overrightarrow{x}, \overrightarrow{v}, t) \sim \omega^{diss}_{s,s',t,u}(\overrightarrow{x}, \overrightarrow{v}, t)$, the condition (C.30) reduces to

$$
\omega^{eddy}_{s,s',t,u} \simeq \nu_{ss'} \left( \frac{\nu_{th,ss'}}{u} \right)^2. \tag{C.31}
$$

and thus essentially coincides with the heuristic criterion employed by Schekochihin et al. (2008,2009); see [24], eq.(251) and [25], section 2.

Clearly the conditions (C.29) or (C.30) are satisfied for any fixed $\ell$, $u$ in the formal limit $\nu_{ss'} \to 0$ (or $D_{0ss'} \to \infty$). To determine how large $D_{0ss'}$ must be in order to neglect collisions at specific values of $\ell$, $u$ requires concrete scaling laws for $\omega^{eddy}_{s,s',t,u}$, $\omega^{diss}_{s,s',t,u}$ in terms of $\ell$, $u$, which will depend upon the circumstances (specific plasma parameters) and also, presumably, will fluctuate from point to point in phase space. One important, general point is already clear, however, from the fact that (C.29) or (C.30) provide a single condition to determine two free parameters $\ell$, $u$. There is obviously an underdetermined degree of freedom, which may be taken to be the slope $\beta$ in the $\log(1/\ell) - \log(1/u)$ plane along which $\log(1/\ell), \log(1/u) \to \infty$. In other words, one can impose as an a priori relation, with any choice of $\beta > 0$,

$$
\ell \sim \delta_s(u/\nu_{th,ss'})^\beta, \tag{C.32}
$$

where, for example, $\delta_s$ is the skin-depth for particle species $s$, so that $\ell$, $u$ vanish together. Substituting the relation (C.32) into (C.29) or (C.30) then uniquely determines $\ell_s(\beta)$, $u_s(\beta)$ for that choice of $\beta > 0$. One should expect there to be a non-trivial $\beta$-dependence, since different choices of that free parameter will weight differently the linear advection contribution and the wave-particle interaction contribution to the rates $\omega^{eddy}_{s,s',t,u}$, $\omega^{diss}_{s,s',t,u}$.

### Appendix D: The 4/5th-Law for Entropy Cascade in Kinetic Turbulence

In the main text we derived an explicit expression Eq. (V.24) for entropy flux rate in phase space, or

$$
s^{flux}_{s,s'} = -\tilde{w}_s \cdot \nabla \overrightarrow{f}_s = \frac{q_s}{m_s} \tilde{\tau}(E, \overrightarrow{f}_s) \cdot \nabla \overrightarrow{f}_s + \frac{q_s}{m_s c} \tilde{\tau}(B, \overrightarrow{f}_s) \cdot (\nabla \times \overrightarrow{f}_s) - \frac{q_s}{m_s c} (\nabla \times B) \cdot \nabla \overrightarrow{f}_s. \tag{D.1}
$$

As we remarked there, the four quantities that appear in this entropy flux can all be expressed in terms of phase-space increments of the VML solutions $f_s$, $s = 1, ..., S$ and $E$, $B$ and these formulae provide the kinetic theory analogues of “4/5th laws” for entropy cascade. The concrete connection with turbulent “4/5th-laws” is not needed to derive the scaling exponent constraints Eq. (V.15) in the main text, but we discuss such relations here for their general interest.

We have explained in the main text how to write the entropy flux in terms of increments, by means of the general relation eq. (III.7) for the correlation terms $\tilde{\tau}(E, \overrightarrow{f}_s)$, $\tilde{\tau}(B, \overrightarrow{f}_s)$, the identities eq. (IV.14), eq. (IV.18) for $\tilde{w}_s$, $(\nabla \times B)$, and the equations eqs. (III.8), eq. (III.9) for the gradients $\nabla \overrightarrow{f}_s$, $\nabla \overrightarrow{f}_s$. We shall now write down the explicit expressions in terms of increments, adopting notations that are purposely chosen to make the connection with traditional “4/5th laws” more obvious. Taking $z = (r, w)$ to denote a displacement in six-dimensional phase-space, we use the notation

$$
\langle a_z \rangle := \int d^6 z G_z(r) H_u(w) a_z \tag{D.2}
$$

to indicate the average over $z$ with respect to the kernels $G_z(r) H_u(w)$. Here $a_z$ is any quantity depending upon $z$, possibly through $r$ or $w$ alone. One can then easily check using the aforementioned equations in the main text that

$$
\tilde{w}_s \cdot \nabla \overrightarrow{f}_s = \left\langle \nabla w \cdot \left[ \frac{w (\delta w f_s) (\delta w' f_s)}{f_s} \right] \right\rangle_{z,z'}. \tag{D.3}
$$

$$
\tilde{\tau}(E, \overrightarrow{f}_s) \cdot \nabla \overrightarrow{f}_s = \left\langle \nabla w \cdot \left[ \frac{(\delta r E) (\delta r f_s) (\delta w f_s)}{f_s} \right] \right\rangle_{z,z'}. \tag{D.4}
$$

$$
\tilde{\tau}(B, \overrightarrow{f}_s) \cdot \nabla \overrightarrow{f}_s = \left\langle \nabla w \cdot \left[ \frac{(\delta r B) (\delta r f_s) (\delta w f_s)}{f_s} \right] \right\rangle_{z,z'}. \tag{D.5}
$$
with multiple averages over \( z = (r, w), z' = (r', w') \), etc., indicated by corresponding multiple subscripts. These formulas may be compared with standard expressions for (anisotropic) 4/5th-laws both in incompressible fluid turbulence, such as [27], Eq.(6.8), and in gyrokinetic turbulence, such as [70], Eq.(4.52) or [72], Eq.(6.9). The resemblance is quite clear for the two middle contributions (D.4)-D.5) from nonlinear wave-particle interactions, which are cubic in terms of solutions fields. The other two, (D.3) from linear advection and the last term (D.6), have a similar form but are only quadratic in solution fields. In the main text we in fact sketched the derivation of two different versions of the “kinetic 4/5th law” based upon the above formulas: Eq.(V.10), which is an “ensemble version” (or globally spatially-averaged version) analogous to that of Kolmogorov, and Eq.(V.14), which is a “deterministic, local version” like that of Duchon-Robert [72]. We shall further elaborate on both of these here.

The derivation of the “ensemble version” Eq.(V.10) mostly follows standard arguments for the fluid case, except for the one important difference that there is no “statistical homogeneity” in velocity-space for kinetic turbulence. On the other hand, the total integrals over velocity-space can be presumed to exist, so that one can instead integrate over \( \nabla \) rather than average. Integrating the phase-space entropy balance Eq.(V.19) over velocity then gives a physical-space entropy balance

\[
\frac{\partial}{\partial t} s[J_s] + \nabla \cdot J_{\text{res},s}^{\text{flux},s} = \sigma_{\ell,u}^{\text{flux},s}
\]

with \( \sigma_{\ell,u}^{\text{flux},s} \) the term corresponding to species \( s \) in the sum of Eq.(V.22). Whereas the four terms in Eq.(D.1) all represent entropy production rate per unit phase-space volume and per unit time, the corresponding terms in \( \sigma_{\ell,u}^{\text{flux},s} \) give entropy production rates per unit physical-space volume and per unit time. We may now average over space or, assuming statistical homogeneity, average over an ensemble of solutions and obtain Eq.(V.13) by the arguments in the main text. The resulting “4/5th-law” for kinetic entropy cascade written out in full detail is:

\[
\begin{align*}
\langle \sigma_s \rangle &= \left\langle \nabla_{w^{\prime \prime}} \left[ w \left( \delta_w f_s \left( \delta_w f_s \right) \right) \right] \right\rangle_{\ell,u} \\
&+ \frac{q_s}{m_s c} \left\langle \nabla_{w^{\prime \prime}} \left[ \left( \delta_r E \right) \left( \delta_r f_s \right) \left( \delta_w f_s \right) \right] \right\rangle_{\ell,u} \\
&- \frac{q_s}{m_s} \left\langle \nabla_{w^{\prime \prime}} \left[ \left( \delta_r E \right) \left( \delta_r f_s \right) \left( \delta_w f_s \right) \right] \right\rangle_{\ell,u} \\
&+ \frac{q_s}{m_s c} \left\langle \nabla_{w^{\prime \prime}} \left[ \nabla_x \left( \delta_r B \right) \left( \delta_r f_s \right) \left( \delta_w f_s \right) \right] \right\rangle_{\ell,u}
\end{align*}
\]

valid in the “collisionless range” \( L \gg \ell \gg \ell_c, U \gg u \gg u_c \), where \( (\cdot)_{\ell,u} \) means that all increments \( z, z' \), etc. which appear inside the bracket have been independently averaged with respect to \( G_L H_L \), that \( \nabla \) has been integrated over all of velocity-space, and that \( x \) has been averaged over all of physical-space.

The local, deterministic form of this 4/5th Law in Eq.(V.14) can be likewise derived following the arguments of [73], which are briefly sketched in the main text. The result has exactly the same form as (D.8) except that, on both sides of the equation, averages of \( x \) over all space are replaced with averages of \( (x, t) \) over \( \varphi(x,t) \) for a smooth, compactly supported, normalized function \( \varphi \). The condition on \( u \) for validity of this local relation is unchanged, but the condition on \( \ell \) becomes \( L_\varphi(t) \gg \ell \gg l_c \), where \( L_\varphi(t) \) is the spatial dimension of the support of \( \varphi(\cdot, t) \), which must be held fixed as first \( D_0 \to \infty \) and then \( \ell, u \to 0 \). Although this relation is “space-time local in the sense of distributions”, the spatial average here is over many increment lengths \( \ell \) (which in turn must be much larger than \( \ell_c \)). In particular, the result does not help to justify a “refined similarity hypothesis” of the type \( (C.19) \), which involves on the left-hand side an average of \( \zeta \) over a region of extent \( \ell \) in space and \( u \) in velocity.

A discontented reader might wonder why the kinetic 4/5th-law that we present does not make use of the simple “point-splitting” argument employed in standard derivations for incompressible fluids [27] or for gyrokinetics [76, 77]. The problem is that there is no obvious “point-splitting” of the phase-space entropy density \( \sigma_s[f_s] = -f_s \log f_s \) for which one can show as \( D_0 \to \infty \) that: (i) all terms in the point-split relation remain finite even as \( \nabla_x \)-gradients and \( \nabla_v \)-gradients diverge, and (ii) the contribution of the Landau collision integral can furthermore be neglected. Standard verifications of (i) use essentially the fact that kinetic energy per volume \((1/2)|u|^2\) for incompressible fluids and free-energy per phase-volume \( g^2/2F_0 \) for gyrokinetics (see [70], eq.(4.16)) are quadratic in the solution fields. Careful derivations of the analogue of (ii) for incompressible fluids (e.g. [27], eq.(6.47), or [85], p.5, Ch.II.B) use the simple form \( \nu \Delta_b u \) of viscous diffusion. None of these standard arguments obviously carries over to the entropy density and the Vlasov-Maxwell-Landau kinetic equations, whereas our coarse-graining regularization (III.1) in the main text trivially guarantees (i) and has been shown also to yield (ii). It is worth remarking that the standard point-splitting argument does guarantee (i) for the quadratic quantity \((1/2)|u|^2\), which is an ideal invariant for smooth Vlasov-Maxwell solutions. On the other hand, this quadratic quantity satisfies no \( H \)-theorem for the VML equations and it is also not obvious how to justify (ii) for a point-splitting of this quantity.
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[35] For example, in [24], section 8.1 on inertial-range solar wind turbulence, the authors wrote: “In a plasma such as the solar wind... for k_p < 1, these fluctuations are rigorously described by the RMHD equations. The magnetic flux is frozen into the ion motions...” RMHD indeed governs the shear-Alfvén modes in the inertial-range of the solar wind, under reasonable assumptions, but not in a sense that implies magnetic flux-freezing to the ion flow at those scales. See section [VH].
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Theodore D Drivas, Ethan T Vishniac, and Alexander Lazarian, “Inertial-range reconnection in magnetohydrodynamic turbulence and in the solar wind,” Physical review letters 115, 025001 (2015).

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[42] This may be termed the “semi-relativistic Vlasov-Maxwell-Landau system”, since the full relativistic Maxwell equations are retained for the fields, but the velocities of particles of all species s are approximated by \(v \approx \frac{p}{m_s} \), under the assumption that \( |p| \ll m_s c \) for all momenta in the support of the distribution functions \( f_s(x, p, t) \) for \( s = 1, \ldots, S \).

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[67] We here use the standard terminology of analysis, with “strong convergence” denoting convergence in norms such as the $L^p$ norms and “weak convergence” denoting convergence after smearering with an element of the dual space, e.g. $\|f\|_p$. A remarkable result of the DiPerna-Lions theory [66] is that, even for sequences $f_j^{(n)}$, $E^{(n)}$, $B^{(n)}$ of the Vlasov-Maxwell system that converge only weakly to limits $f_\star$, $E$, $B$, nevertheless the nonlinear wave-particle interaction term $\nabla \cdot (E^{(n)} + (v/c) \times B^{(n)}) f_j^{(n)}$ also converges (distributionally) to $\nabla \cdot (E + (v/c) \times B) f_\star$. The corresponding statement is not true of the advective nonlinear term $\nabla \cdot |u|u|$ for the incompressible Euler equation, nor for most nonlinear PDE’s, but instead depends upon special features of the Vlasov-Maxwell equations. Because of this special “stability” property of VM solutions, it is quite likely that strong limits as $D_\epsilon \to \infty$ of the VML solutions need not be assumed in order to obtain (distributional) solutions of VM. On the other hand, the same remarkable convergence statements need not hold for other nonlinear functions, such as the phase-space entropy densities $J(f_\star)$. Thus, some of our key conclusions, such as the anomalous entropy balance (V.11), may require strong convergence.

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[69] Ronald J DiPerna and Pierre-Louis Lions, “Global weak solutions of Vlasov-Maxwell systems,” Commun. Pure Appl. Math. 42, 729–757 (1989).

[70] Of course, this is not an unusual situation in physics. There are many examples of mathematical objects whose existence is supported by almost overwhelming empirical evidence and theoretical arguments, but which have never been rigorously proved to exist from first principles. These examples include crystalline solid phases at sufficiently low temperatures [68] or the Gaussian renormalization group fixed point believed to describe the critical properties of all equilibrium systems in the universality class of the 3D Ising model [144]. Needless to say, the development of solid state physics did not have to wait for the mathematical proof from many-body quantum theory (still unavailable) that crystals exist!

[71] The phase-space contraction rate, or violation of the Liouville theorem, is well-known in other contexts to be related to the rate of entropy production, e.g. in molecular dynamics theory of non-equilibrium statistical mechanics and transport behavior, cf. [157], section 10.1.

[72] Identities $-\nabla \cdot (\varepsilon_e f_\star) = -\nabla \cdot (E \cdot f_\star) + (E \cdot \nabla E) f_\star$ and $-\nabla \cdot (\varepsilon_w f_\star) = -\nabla \cdot (\varepsilon_w f_\star) + (\varepsilon_w \cdot \nabla E) f_\star$ show that positive entropy production results at a phase-space point when the coarse-grained rate of spreading of the fine-grained distribution is greater there than the rate of spreading of the coarse-grained distribution. It is worth recalling that the particle-field interaction term can be written as $(q/m_e) E \cdot \nabla f = (q/m_e) E \cdot \nabla f - (q/B/m_e) \cdot (\varepsilon_w \cdot \nabla E) f_\star$, with the first term corresponding over a time-increment $dt$ to a translation of the particle distribution in velocity space by $dt(q/m_e)$ and the second term corresponding to a rotation of the distribution in velocity space by rotation vector $-dt(q/B/m_c)$. Of course, the free-streaming term $-(\varepsilon_w \cdot \nabla E) f_\star$ corresponds to translation of the particle distribution in position space by displacement $v dt$ over time $dt$.

[73] Jean Duchon and Raoul Robert, “Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations,” Nonlinearity 13, 249 (2000).

[74] It is worth pointing out that the same expression (V.20) for entropy flux can be obtained without using Favre averaging, by instead treating particle distributions $f_j$ as “advected scalars” for the incompressible flow in phase-space generated by the Hamiltonian equations of a charged particle in an EM field. Standard derivations of 4/5th-type laws for advected scalars in incompressible fluid turbulence using spatial coarse-graining, e.g. eqs.(29)-(33) in [151], recover (V.20).

[75] In fact, a standard “density argument” from real analysis shows that the terms $\nabla \cdot A \to 0$ distributionally as $\epsilon, \eta \to 0$ even under the much weaker assumption that $E_\star$, $B_\star$, $f_{\star,n}$, $s = 1, \ldots, S$ are $L^2$ functions, which is almost the minimal regularity required for the Vlasov-Maxwell equations to make sense.

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[80] Clément Mouhot and Cédric Villani, “On Landau damping,” Acta mathematica 207, 29–201 (2011).

[81] Note, in fact, that the $m_0^{-1/2}$ spectrum of parallel Hermite modes derived in [78, 79] corresponds to exponent $p_L^2 = -1/2$ as defined in our Eq. (V.35). See e.g. [152].

[82] We always assume that the only velocities which occur in the solutions $f_{\star,n}$, $s = 1, \ldots, S$ of the VLM equations have $|v| < c$, the speed of light, or otherwise the semi-relativistic model (II.1)-(II.3) becomes unphysical and must be replaced by a fully relativistic Vlasov-Maxwell model. We also assume that the turbulence occurs in only a bounded region of position-space.

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One can avoid any assumption of the existence of an infinite-Do limit $a_s$ by defining the $p$th-order exponent of $a_{p,s}$ by $\sigma^s_p = \sup\{s : \sup_{Do} \|a_{p,s}\|_{L^p(\mathbb{R}^\infty)} < \infty\}$, which is the largest smoothness exponent $s$ for which $a_{p,s} \in B^s_{\mathbb{R}^\infty}$, uniformly in $Do$. See [155] for an analogous definition in the context of the incompressible Navier-Stokes turbulence.

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For example, if the electromagnetic fields satisfy the finite-energy condition in [IV.11] and if also the particle distributions satisfy $\int d^3 v (1 + |\mathbf{v}|)|f_s(\mathbf{r}, \mathbf{v}, t)|_2 \leq \infty$ for $s = 1, \ldots, S$, where $|\cdot|_2$ is the spatial $L^2$-norm, then the Cauchy-Schwarz inequality $|\langle \mathbf{v}, \mathbf{E} \rangle| \leq 2 \int d^3 r G_t(r) |\delta \mathbf{v}|_2 |\delta \mathbf{E}|_2$ and likewise $|\langle \mathbf{v}, \mathbf{B} \rangle| \leq 2 \int d^3 r G_t(r) |\delta \mathbf{v}|_2 |\delta \mathbf{B}|_2$ imply that these momentum fluxes vanish as $t \to 0$. This follows from square-integrability of $\mathbf{v}$, which is a consequence of the bound $|\langle \mathbf{v}, \mathbf{E} \rangle|_2 + |\langle \mathbf{v}, \mathbf{B} \rangle|_2 \leq \sum_s \int d^3 v (1 + |v|)|f_s(\mathbf{r}, \mathbf{v}, t)|_2$.

The previous considerations apply to any weak solution of the Vlasov-Maxwell equations. Of course, if the solution $f_{s,i}, \mathbf{E}, \mathbf{B}$, arises as a suitable strong limit of a solution of the Vlasov-Maxwell-Landau equations, then $\int \mathbf{v} \cdot \mathbf{E} d\mathbf{v}$ and likewise $\int \mathbf{v} \cdot \mathbf{B} d\mathbf{v}$ are also conserved.

Robert T Glassey and Walter A Strauss, “Singularities and the local Cauchy problem for the Vlasov-Maxwell system,” submitted (2018).
Taking the limit $\ell, u \to 0$ after first taking the collisionless limit $D_0 \to \infty$ simplifies the equation \( \text{(VI.35)} \) to \( \partial_t z_{s+} + \nabla \cdot \left( \nu z_{s+} \mathbf{P}_{S+} (u_{s+} - v_f) f_{s+}/n_{s+} \right) + \nabla \cdot \left( q_p (E_s, z_{s+}) \right) = P_{S+} \nabla \cdot \left( \left( u_{s+} - v \right) f_{s+}/n_{s+} \right) - m_s \left( v - u_{s+} \right) \nabla \cdot \mathbf{u}_{S+} \), which yields eq. \( \text{(VI.22)} \) for $\epsilon_s$ after integration over velocities. This same equation could also be obtained formally from Howes’ equation \( \text{(VI.18)} \) for $w_{s+}$, the collisionless kinetic equation \( \text{(IV.10)} \) for $f_s$, and the momentum balance \( \text{(VI.3)} \) for $u_{s+}$. Note, however, that $f_s$ will not be classically differentiable when there is a non-vanishing entropy cascade for species $s$ and $n_{s+}, u_{s+}$ may not be differentiable as well. Thus, the two terms on the righthand side must be carefully interpreted as distributional limits of the corresponding terms on the righthand side of \( \text{(VI.35)} \) (i.e. the second and third).

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[112] This should be possible, in principle. The weak formulation of the reduced Ampère’s law for $A_j, B_l$ is straightforward. One can define $h_s$ to be a weak solution of the collisionless version of \( \text{(VII.4)} \) if

$$
\int \left[ \frac{\partial}{\partial t} + v_l \frac{\partial}{\partial x_l} + \frac{e}{m} \left\{ (\chi) X_s, \psi \right\} \right] h_s = - \int 2F_{s+} \frac{\partial (\chi) X_s}{\partial t} \, \psi
$$

for all smooth test functions $\psi = \psi(X_s, z, v_l, v_i, t)$, where integration is with respect to the measure $d^2X_s, dz dv_l, dv_i, dt$. This formulation suffices if the electromagnetic scalar and vector potentials $\varphi, A$ are at least once-differentiable in space and time. This is probably a reasonable assumption generally, but seems not to be true for the joint KAW/ion entropy cascade predicted by \( \text{(IV.24)} \) in the limit $m_e/m_i \to 0$, when the scalar potential is expected to become extremely rough. See further discussion below.

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[115] If the electric field has the extreme irregularity predicted in \( \text{(VI.28)} \) as $m_e/m_i \to 0$, then the coarse-graining analysis of the present paper cannot be used to analyze that limit. For example, the phase-space Favre-average electric field $E_s$ that appears in the coarse-grained Vlasov equation \( \text{(V1.15)} \) is no longer well-defined because the fields $E_s$ and $f_s$ are too irregular in the limit $m_e/m_i \to 0$ for their pointwise product $E_s f_s$ to be a priori meaningful. This is largely a technical issue, but it means that the coarse-graining regularization as employed in this paper probably does not suffice to study the KAW/ion entropy cascade in the $m_e/m_i \to 0$ limit.

[116] A recent numerical simulation of a hybrid Vlasov-Maxwell system \( \text{(124)} \) provides the best evidence so far of an ion entropy cascade beyond the gyrokinetic description. This simulation employs a fully kinetic Vlasov-Maxwell equation for ions, but a generalized Ohm’s law for an electron fluid. A full discussion of this simulation would be too lengthy for the current paper, but we note here briefly that the spectra observed in \( \text{(154)} \) over the range $1 \lesssim \rho_i k_{\perp} \lesssim 10$ of perpendicular wavenumbers and $1 \lesssim m_i \lesssim 10$ of Hermite velocity modes are equivalent to $\sigma_{f,1}^2 = -1/6, \sigma_{f,2}^2 = 5/6$, and $\sigma_{f,1}^2 = 1/2$ in our language \( \text{(152)} \). The authors of \( \text{(154)} \) interpreted this range as an ion entropy cascade driven by linear phase-mixing along field-lines, similar to that discussed in \( \text{(78, 79)} \). The observed scaling is consistent with an ion entropy cascade according to our analysis, but we find that all three terms \( \text{(V.37–V.39)} \) of the entropy flux could contribute, with the nonlinear wave-particle interaction terms perhaps even dominant. The range $\rho_i k_{\perp} \gtrsim 10, m_i \gtrsim 10$ observed in the simulation of \( \text{(154)} \) is very unlikely to correspond to an asymptotic ion entropy cascade according to our analysis, unless due to an extreme phase-space intermittency.

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