A Möbius invariant discretization and decomposition of the Möbius energy

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Abstract
The Möbius energy, defined by O’Hara, is one of the knot energies, and named after the Möbius invariant property which was shown by Freedman-He-Wang. The energy can be decomposed into three parts, each of which is Möbius invariant, proved by Ishizeki-Nagasawa. Several discrete versions of Möbius energy, that is, corresponding energies for polygons, are known, and it showed that they converge to the continuum version as the number of vertices to infinity. However already-known discrete energies lost the property of Möbius invariance, nor the Möbius invariant decomposition. Here a new discretization of the Möbius energy is proposed. It has the Möbius invariant property, and can be decomposed into the Möbius invariant components which converge to the original components of decomposition in the continuum limit. Though the decomposed energies are Möbius invariant, their densities are not. As a by-product, it is shown that the decomposed energies have alternative representation with the Möbius invariant densities.

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1 Introduction
Let $f$ be a closed curve in $\mathbb{R}^n$, parametrized by the arc-length, with the total length $L$. The Möbius energy

$$\mathcal{E}(f) = \iint_{(\mathbb{R}/L)^2} \left( \frac{1}{\|f(s_1) - f(s_2)\|_{\mathbb{R}^n}} - \frac{1}{\mathcal{D}(f(s_1), f(s_2))^2} \right) ds_1 ds_2$$

was proposed by O’Hara [4] to determine the canonical configuration of knots, that is, closed curves embedded in $\mathbb{R}^3$. The energy is also defined for curves

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In the above formula, \(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n}\) and \(\mathcal{D}(f(s_1), f(s_2))\) are the extrinsic and intrinsic distance between two points \(f(s_1)\) and \(f(s_2)\) on the curve respectively. The energy is invariant under the Möbius transformation, and this fact played an important role to show the existence of minimizers in each prime knot class \([2]\). We consider a discrete energy, that is, the energy defined for polygons. There are at least two discrete versions of the energy, one was defined by Kim-Kusner \([5]\), and another was by Simon \([9]\). These energies converge to the Möbius energy under the continuum limit \((7, 8)\), however they lost the Möbius invariance (for the definition of Möbius invariant of discrete energy, see below, and it is not hard work to see that Kim-Kusner's energy and Simon's energy lost the Möbius invariant properties). Recently the authors proposed a new discrete energy and showed its convergence \([1]\). This has the Möbius invariant property.

In this paper, another discrete Möbius energy is proposed, which keeps the Möbius invariant property and also has the Möbius invariant decomposition. The Möbius energy can be decomposed into

\[
\mathcal{E}(f) = \mathcal{E}_1(f) + \mathcal{E}_2(f) + 4,
\]

\[
\mathcal{E}_1(f) = \frac{1}{2} \int_{(\mathbb{R}/\mathbb{Z})^2} \frac{\|\Delta v\|_R^2}{\|\Delta f\|_{\mathbb{R}^n}} ds_1 ds_2,
\]

\[
\mathcal{E}_2(f) = \int_{(\mathbb{R}/\mathbb{Z})^2} \frac{2}{\|\Delta f\|_{\mathbb{R}^n}} \left\langle \tau(s_1) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}}, \tau(s_2) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} \right\rangle_{\Lambda^2\mathbb{R}^n} ds_1 ds_2,
\]

where \(\tau\) is the unit tangent vector, and the notation \(\Delta\) means

\[
\Delta v = v(s_1) - v(s_2)
\]

for a function on \(\mathbb{R}/\mathbb{Z}\). This was shown by the second and third authors \([3, 4]\).

Each decomposed part is also Möbius invariant. The newly proposed discrete energy in this paper can be decomposed into Möbius invariant components, and each component converges to the corresponding part of the original decomposition.

Let \(\mathcal{F}\) be a discrete energy for polygons in \(\mathbb{R}^n\), and let \(T: \mathbb{R}^n \to \mathbb{R}^n\) be a Möbius transformation. Since the image \(T(P)\) of a polygon \(P\) is not a polygon, we cannot define \(\mathcal{F}(T(P))\). Hence we define the Möbius invariance of discrete energies as follows.

**Definition 1.1** Let \(f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n\) be a closed curve. The parameter may not be necessarily arc-length one. For \(0 \leq \theta_1 < \theta_2 < \cdots < \theta_m < 1\), \(P_m(f, \{\theta_i\}_{i=1}^m)\) is an \(m\)-polygon with the \(i\)-th vertex \(f(\theta_i)\), simply denoted by \(P_m(f)\) hereafter. We say that a discrete energy \(\mathcal{F}\) is Möbius invariant if

\[
\mathcal{F}(P_m(f)) = \mathcal{F}(P_m(Tf))
\]

holds for and all \(f\), all \(\{\theta_i\}\), and all Möbius transformations \(T\) which maps \(\text{Im}(f)\) into \(\mathbb{R}^n\).
The main result of this paper is as follows.

**Theorem 1.1** There exist energies $E^m$, $E^m_1$, and $E^m_2$ for m-polygons such that

- They are Möbius invariant in the sense of Definition 1.1.
- Assume that $f \in W^{2,\infty}$ satisfies the bi-Lipschitz estimate. Let $P_m(f)$ be an m-polygon with the i-th vertex $f(\theta_i)$ satisfying the equi-lateral condition. Then $E^m(P_m(f))$, $E^m_1(P_m(f))$, and $E^m_2(P_m(f))$ converges to $E(f)$, $E_1(f)$, and $E_2(f)$ as $m \to \infty$ expectively.

Before defining discrete energies we study the Möbius invariant properties of the conformal angle in next section. This is necessary for defining Möbius invariant discrete energies. As a by-product, we present an alternative proof of the Möbius invariant property of $E_1$ and $E_2$. We explain how to define the discrete energies and show their Möbius invariance in section 3. In the final section we shall prove their convergence as $m \to \infty$.

## 2 The Möbius invariant property

**Definition 2.1** Let $C_{ij}$ be the circle through $s_j$ which is tangent to $\text{Im } f$ at $f(s_i)$. The angle between $C_{ij}$ and $C_{ji}$ at $f(s_i)$ or $f(s_j)$ is called the conformal angle, and we denote it $\varphi = \varphi_{f}(s_i, s_j)$.

By elementary calculation we find the relation

$$\cos \varphi(s_1, s_2) = \frac{2(\Delta f \cdot \tau(s_1))(\Delta f \cdot \tau(s_2))}{\| \Delta f \|^2_{\mathbb{R}^n}} - \tau(s_1) \cdot \tau(s_2)$$

$$= -\| f(s_1) - f(s_2) \|^2_{\mathbb{R}^n} (\mathcal{M}_1(f) + \mathcal{M}_2(f)) + 1,$$

where $\mathcal{M}_i(f)$ is the energy density of $E_i(f)$. From this, we easily obtain an expression of the Möbius energy by using the conformal angle, which is known as the cosine formula.

**Proposition 2.1 (the cosine formula)** It holds that

$$\int_{(\mathbb{R}/\mathbb{Z})^2} 1 - \cos \varphi(s_1, s_2) \| f(\theta_1) - f(\theta_2) \|^2_{\mathbb{R}^n} ds_1 ds_2 = E(f) - 4.$$

The Möbius invariance of $\varphi$ follows from that of cross ratio.

**Lemma 2.1** It holds that

$$\cos \varphi(\theta_1, \theta_2) = \frac{\| \Delta f \|^2_{\mathbb{R}^n}} {2\| f(\theta_1) \|_{\mathbb{R}^n} \| f(\theta_2) \|_{\mathbb{R}^n}} \frac{\partial}{\partial \theta_1 \partial \theta_2} \frac{\| \Delta f \|^2_{\mathbb{R}^n}} {\| f(\theta_1) \|_{\mathbb{R}^n} \| f(\theta_2) \|_{\mathbb{R}^n}}.$$

In particular, $\cos \varphi(\theta_1, \theta_2)$ is invariant under Möbius transformations.
Proof. The direct calculation derives
\[
\frac{\partial}{\partial \theta_1 \partial \theta_2} \log \| \Delta f \|_{\mathbb{R}^n}^2 = 2 \frac{\| \hat{f}(\theta_1) \|_{\mathbb{R}^n} \| \hat{f}(\theta_2) \|_{\mathbb{R}^n} \cos \varphi(\theta_1, \theta_2)}{\| \Delta f \|_{\mathbb{R}^n}^2}.
\]
Since
\[
\frac{\partial}{\partial \theta_1 \partial \theta_2} \log \| \hat{f}(\theta_1) \|_{\mathbb{R}^n} \| \hat{f}(\theta_2) \|_{\mathbb{R}^n} = \frac{\partial}{\partial \theta_1 \partial \theta_2} \left( \log \| \hat{f}(\theta_1) \|_{\mathbb{R}^n} + \log \| \hat{f}(\theta_2) \|_{\mathbb{R}^n} \right) = 0,
\]
we have
\[
\frac{\partial}{\partial \theta_1 \partial \theta_2} \log \| \Delta f \|_{\mathbb{R}^n}^2 = \frac{\partial}{\partial \theta_1 \partial \theta_2} \log \frac{\| \Delta f \|_{\mathbb{R}^n}^2}{\| \hat{f}(\theta_1) \|_{\mathbb{R}^n} \| \hat{f}(\theta_2) \|_{\mathbb{R}^n}}.
\]
Therefore we get the first half of assertion. Since a Möbius transformation keeps the cross ration of distinct four points in \( \mathbb{R}^d \), it holds that
\[
\| f(\theta_1 + \delta \theta_1) - f(\theta_1) \|_{\mathbb{R}^n} \| f(\theta_2 + \delta \theta_2) - f(\theta_2) \|_{\mathbb{R}^n}
= \| f(\theta_1 + \delta \theta_1) - f(\theta_2 + \delta \theta_2) \|_{\mathbb{R}^n} \| f(\theta_1) - f(\theta_2) \|_{\mathbb{R}^n}
\]
is a Möbius invariance for sufficiently small \( |\delta \theta_1| \) and \( |\delta \theta_2| \) provided \( f(\theta_1) \neq f(\theta_2) \). Dividing by \( \delta \theta_1 \delta \theta_2 \) and passing \( \delta \theta_1 \to 0, \delta \theta_2 \to 0 \), we find the quantity
\[
\frac{\| f(\theta_1) \|_{\mathbb{R}^n} \| f(\theta_2) \|_{\mathbb{R}^n}}{\| \Delta f \|_{\mathbb{R}^n}^2}
\]
is also a Möbius invariance. Consequently, the second assertion has been shown. \( \square \)

Put
\[
g = g(\theta_1, \theta_2) = \frac{\| \hat{f}(\theta_1) \|_{\mathbb{R}^n} \| \hat{f}(\theta_2) \|_{\mathbb{R}^n}}{\| \Delta f \|_{\mathbb{R}^n}^2}.
\]
Note that we can define \( \frac{\partial^2 g}{\partial \theta_1 \partial \theta_2} \) in the classical sense for \( f \in C^2(\mathbb{R}/\mathbb{Z}) \), in the weak sense for \( f \in W^{2,1}_{\text{loc}}(\mathbb{R}/\mathbb{Z} \setminus \{\theta_1 = \theta_2\}) \). The density of the Möbius energy can be written by the Möbius invariance \( g \) and its derivatives:
\[
\mathcal{E}(f) - 4 = \int_{(\mathbb{R}/\mathbb{Z})^2} \left\{ \frac{1}{g} - \frac{1}{2} \frac{\partial}{\partial \theta_1} \left( \frac{1}{g} \frac{\partial g}{\partial \theta_2} \right) \right\} \, d\theta_1 \, d\theta_2.
\]
This gives an alternative proof of the Möbius invariance of decomposed energies. It is not difficult to see
\[
\frac{1}{g} - \frac{1}{2} \frac{\partial}{\partial \theta_1} \left( \frac{1}{g} \frac{\partial g}{\partial \theta_2} \right) = \frac{1}{g} \left( 1 + \frac{1}{2} \frac{\partial^2 g}{\partial \theta_1 \partial \theta_2} \right) = \frac{1}{2g^2} \det \begin{pmatrix}
\partial_1 \partial_2 & \partial_1 g \\
\partial_2 g & 2g
\end{pmatrix}
\]
We define the energies \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) by
\[
\mathcal{E}_1(f) = \int_{(\mathbb{R}/\mathbb{Z})^2} \frac{1}{g} \left( 1 + \frac{1}{2} \frac{\partial^2 g}{\partial \theta_1 \partial \theta_2} \right) \, d\theta_1 \, d\theta_2,
\]
\[
\mathcal{E}_2(f) = -\int_{(\mathbb{R}/\mathbb{Z})^2} \frac{1}{2g^2} \det \begin{pmatrix}
\partial_1 \partial_2 & \partial_1 g \\
\partial_2 g & 2g
\end{pmatrix} \, d\theta_1 \, d\theta_2.
\]
These are Möbius invariant energies, since so is \( g \).
Theorem 2.1 We have $\mathcal{E}_1(f) = \hat{\mathcal{E}}_1(f)$, and $\mathcal{E}_2(f) = \hat{\mathcal{E}}_2(f)$.

Proof. Since $\mathcal{E}_1 + \mathcal{E}_2 = \hat{\mathcal{E}}_1 + \hat{\mathcal{E}}_2 = \mathcal{E} - 4$, it is enough to show the relation for $\mathcal{E}_1(f)$. Put

$$g_n = \|\Delta f\|_{R^n}, \quad g_d = \|f(\theta_1)\|_{R^n} \|f(\theta_2)\|_{R^n}.$$  

Then we have $g = g_n / g_d$, and

$$\mathcal{E}_1(f) = \frac{1}{2} \iint_{(R/\mathbb{Z})^2} \left\| \tau(s_1) - \tau(s_2) \right\|^2_{R^n} ds_1 ds_2$$

$$= \iint_{(R/\mathbb{Z})^2} \frac{1 - \tau(s_1) \cdot \tau(s_2)}{\|\Delta f\|_{R^n}} ds_1 ds_2$$

$$= \iint_{(R/\mathbb{Z})^2} \frac{1}{g} \left( 1 - \frac{f(\theta_1) \cdot f(\theta_2)}{\|f(\theta_1)\|_{R^n} \|f(\theta_2)\|_{R^n}} \right) d\theta_1 d\theta_2$$

$$= \iint_{(R/\mathbb{Z})^2} \frac{1}{g_n} \left( 1 + \frac{1}{2g_d} \frac{\partial^2 g_n}{\partial \theta_1 \partial \theta_2} \right) d\theta_1 d\theta_2.$$

On the other hand, we have

$$\hat{\mathcal{E}}_1(f) = \iint_{(R/\mathbb{Z})^2} \frac{g_d}{g_n} \left( 1 + \frac{1}{2g_d} \frac{\partial^2 g_n}{\partial \theta_1 \partial \theta_2} \right) d\theta_1 d\theta_2.$$

Therefore the difference between $\mathcal{E}_1(f)$ and $\hat{\mathcal{E}}_1(f)$ is

$$\mathcal{E}_1(f) - \hat{\mathcal{E}}_1(f)$$

$$= \frac{1}{2} \iint_{(R/\mathbb{Z})^2} \frac{g_d}{g_n} \left( \frac{\partial^2 g_n}{\partial \theta_1 \partial \theta_2} - \frac{\partial^2 g_n}{g_d \partial \theta_1 \partial \theta_2} \right) d\theta_1 d\theta_2$$

$$= \frac{1}{2} \iint_{(R/\mathbb{Z})^2} \left\{ \left( \frac{\partial}{\partial \theta_1} \log g_d \right) \left( \frac{\partial}{\partial \theta_2} \log g_n \right) + \left( \frac{\partial}{\partial \theta_1} \log g_n \right) \left( \frac{\partial}{\partial \theta_2} \log g_d \right) \right.\right.$$  

$$\left. - 2 \left( \frac{\partial}{\partial \theta_1} \log g_d \right) \frac{\partial}{\partial \theta_2} \log g_n \right\} d\theta_1 d\theta_2.$$

We can show

$$\iint_{(R/\mathbb{Z})^2} \left\{ \left( \frac{\partial}{\partial \theta_1} \log g_d \right) \left( \frac{\partial}{\partial \theta_2} \log g_n \right) + \left( \frac{\partial}{\partial \theta_1} \log g_n \right) \left( \frac{\partial}{\partial \theta_2} \log g_d \right) \right\} d\theta_1 d\theta_2 = 0$$

in a manner similar to the proof of \cite{3} Theorem 3.2. Since $\log g_d$ does not have singularity anywhere,

$$\iint_{(R/\mathbb{Z})^2} \left( \frac{\partial}{\partial \theta_1} \log g_d \right) \left( \frac{\partial}{\partial \theta_2} \log g_d \right) d\theta_1 d\theta_2 = 0.$$
We can perform the integration by parts to get
\[
\int \int \frac{1}{g} \frac{\partial^2 g}{\partial \theta_1 \partial \theta_2} \, d\theta_1 \, d\theta_2 = \int \int \left( \frac{\partial}{\partial \theta_1} \log g \right) \left( \frac{\partial}{\partial \theta_2} \log g \right) \, d\theta_1 \, d\theta_2 = 0.
\]

□

Though Theorem 2.1 is not our main purpose of this paper, it seems to be interesting. The decomposed energies \( E_i \) are Möbius invariant, but their densities are not. Theorem 2.1 says that \( E_i \) can be rewritten as the energies with Möbius invariant densities. The domain of \( \hat{E}_i \) is narrower that that of \( E_i \), hence we can regard \( E_i \) as a relaxation of \( \hat{E}_i \).

### 3 A Möbius invariant discritization

For the discretization we use notation \( \Delta^j_i f \) and \( \Delta_i f \) to mean
\[
\Delta^j_i f = f(\theta_j) - f(\theta_i), \quad \Delta_i f = \Delta^{i+1}_i f = f(\theta_{i+1}) - f(\theta_i).
\]

We discritize the Möbius invariance \( g \) as
\[
g_{ij} = \frac{\| \Delta^j_i f \|_{\mathbb{R}^n} \| \Delta^{i+1}_i f \|_{\mathbb{R}^n}}{\| \Delta_i f \|_{\mathbb{R}^n} \| \Delta_j f \|_{\mathbb{R}^n}}.
\]

Since it is the cross ratio of for points \( f(\theta_i), f(\theta_{i+1}), f(\theta_j), \) and \( f(\theta_{j+1}) \), it is also a Möbius invariance.

Using \( g_{ij} \) and difference operation instead of \( g \) and differential operation, we consider discrete energies
\[
E^m_1(f) = \sum_{i \neq j} \frac{1}{g_{ij}} \left( 1 + \frac{1}{2} \Delta_i \Delta_j g_{ij} \right),
\]
\[
E^m_2(f) = -\sum_{i \neq j} \frac{1}{2g_{ij}} \det \begin{pmatrix} \Delta_i \Delta_j g_{ij} & \Delta_j g_{ij} \\ \Delta_j g_{ij} & 2g_{ij} \end{pmatrix},
\]
\[
E^m(f) = E^m_1(f) + E^m_2(f) + 4.
\]

Since \( E^m_1 \) and \( E^m_2 \) are a discrete version of \( E_1 \) and \( E_2 \), they are expected to convergent as \( m \to \infty \) under suitable assumption. Indeed \( E^m_1 \) converges to \( E_1 \), but \( E^m_2 \), and therefore \( E^m \) do not. A regular \( m \)-polygon converges to a right circle, but
\[
E^m(\text{a regular } m\text{-polygon}) \to \infty \neq 4 = E(\text{a right circle})
\]
as \( m \to \infty \). To recover this situation, we modify the definition as follows.
Definition 3.1 We define $E^m_1$, $E^m_2$, and $E^m$ by
\[
\begin{align*}
E^m_1(f) &= \sum_{i \neq j} \frac{1}{g_{ij}} \left( 1 + \frac{1}{2} \Delta_i \Delta_j g_{ij} \right), \\
E^m_2(f) &= -\sum_{i \neq j} \frac{1}{2g_{ij}} \left\{ \det \left( \begin{array}{cc}
\Delta_i \Delta_j g_{ij} & \Delta_i g_{ij} \\
\Delta_j g_{ij} & 2g_{ij}
\end{array} \right) + 1 \right\}, \\
E^m(f) &= E^m_1(f) + E^m_2(f) + 4.
\end{align*}
\]

Proposition 3.1 The energies $E^m_1(f)$, $E^m_2(f)$ and $E^m(f)$ are Möbius invariant.

Proof. Their Möbius invariance follows from that of $g_{ij}$. \hfill \Box

4 Convergence of discrete energies under the equi-lateral condition

In this section we would like to show the convergence of $E^m_1$ and $E^m - 4$. As a corollary we get the convergence of $E^m_2$ also.

4.1 Preparations

We put for a positive sequence $u = \{u_i\}$
\[
\begin{align*}
A_i(u) &= \frac{u_i + u_{i+1}}{2} \quad \text{(the arithmetic mean)}, \\
G_i(u) &= \sqrt{u_i u_{i+1}} \quad \text{(the geometric mean)}, \\
H_i(u) &= 2 \left( \frac{1}{u_i} + \frac{1}{u_{i+1}} \right)^{-1} \quad \text{(the harmonic mean)}.
\end{align*}
\]

We use similar notations for a positive sequence with two subscripts:
\[
\begin{align*}
A_{ij}(u) &= \frac{u_{ij} + u_{(i+1)j}}{2}, \quad G_{ij}(u) = \sqrt{u_{ij} u_{(i+1)j}}, \quad H_{ij}(u) = 2 \left( \frac{1}{u_{ij}} + \frac{1}{u_{(i+1)j}} \right)^{-1}, \\
A_{ij}(u) &= \frac{u_{ij} + u_{i(j+1)}}{2}, \quad G_{ij}(u) = \sqrt{u_{ij} u_{i(j+1)}}, \quad H_{ij}(u) = 2 \left( \frac{1}{u_{ij}} + \frac{1}{u_{i(j+1)}} \right)^{-1},
\end{align*}
\]

and
\[
\begin{align*}
A_{ij}(u) &= \frac{u_{ij} + u_{(i+1)j} + u_{i(j+1)} + u_{(i+1)(j+1)}}{4}, \\
G_{ij}(u) &= \sqrt{u_{ij} u_{(i+1)j} u_{i(j+1)} u_{(i+1)(j+1)}}, \\
H_{ij}(u) &= 4 \left( \frac{1}{u_{ij}} + \frac{1}{u_{(i+1)j}} + \frac{1}{u_{i(j+1)}} + \frac{1}{u_{(i+1)(j+1)}} \right)^{-1}.
\end{align*}
\]
In most cases we use, for example, $A_i(u_{ij})$ or $A_i u_{ij}$ instead of $A_i(\{u_{ij}\})$. Furthermore $A_iG_j(u_{ij}) = A_i(G_j(u_{ij}))$, and so on. It is easy to see the next identities.

**Fact 4.1** There hold

$$\Delta_i(u_i v_i) = (\Delta_i u_i)A_i(v) + A_i(u)(\Delta_i v_i),$$

$$\Delta_i \frac{u_i}{v_i} = \frac{\Delta_i u_i}{H_i(v)} + A_i(u) \left( \Delta_i \frac{1}{v_i} \right),$$

$$A_i(\{u_{ij}\}) = \frac{\Delta_i \Delta_j u_{ij}}{H_{ij} v} + \Delta_j(A_i u) \Delta_i \left( \frac{1}{H_i v} \right)$$

$$+ \Delta_i(A_i u) \Delta_j \left( \frac{1}{H_i v} \right) + (A_i u) \Delta_i \frac{1}{v_{ij}}.$$

### 4.2 Decomposition of densities of discrete energies

Put

$$g_{ij,d} = \|\Delta_i f\|_R \|\Delta_j f\|_R,$$  

$$g_{ij,n} = \|\Delta_i f\|_R \|\Delta_j^{+1} f\|_R,$$  

$$g_{ij,n} = \|\Delta_i f\|_R^2.$$

Let $\mathcal{M}_{1,ij}$ and $\mathcal{M}_{ij}$ be the densities of $\mathcal{E}^m$ and $\mathcal{E}^m - 4$ respectively:

$$\mathcal{E}^m_1(f) = \sum_{i \neq j} \mathcal{M}_{1,ij}(f),$$  

$$\mathcal{E}^m(f) - 4 = \sum_{i \neq j} \mathcal{M}_{ij}(f),$$

Then

$$\mathcal{M}_{1,ij}(f) = \frac{g_{ij,d}}{g_{ij,n}} \left( 1 + \frac{1}{2} \Delta_i \Delta_j \frac{g_{ij,n}}{g_{ij,d}} \right),$$

$$\mathcal{M}_{ij}(f) = \frac{g_{ij,d}}{g_{ij,n}} \left[ 1 - \frac{1}{2} \Delta_i \Delta_j \frac{g_{ij,n}}{g_{ij,d}} + \frac{g_{ij,d}}{2g_{ij,n}} \left\{ \left( \Delta_i \frac{g_{ij,n}}{g_{ij,d}} \right) \left( \Delta_j \frac{g_{ij,n}}{g_{ij,d}} \right) - 1 \right\} \right].$$

We decompose them into

$$\mathcal{M}_{1,ij}(f) = \mathcal{P}_{1,ij}(f) + \mathcal{R}_{1,ij}(f),$$  

$$\mathcal{M}_{ij}(f) = \mathcal{P}_{ij}(f) + \mathcal{R}_{ij}(f),$$

where

$$\mathcal{P}_{1,ij}(f) = \frac{1}{2} \frac{g_{ij,d}}{g_{ij,n}} \left\{ \left( 1 - \frac{\Delta_i f}{\|\Delta_i f\|_R} \cdot \frac{\Delta_j f}{\|\Delta_j f\|_R} \right) + \left( 1 - \frac{\Delta_i^{+1} f}{\|\Delta_i^{+1} f\|_R} \cdot \frac{\Delta_j^{+1} f}{\|\Delta_j^{+1} f\|_R} \right) \right\},$$

$$\mathcal{P}_{ij}(f) = \frac{g_{ij,d}}{g_{ij,n}} \left[ 1 - \frac{1}{2g_{ij,n}} \left\{ \Delta_i \Delta_j \frac{g_{ij,n}}{g_{ij,d}} + \frac{4 \left( \Delta_i f \cdot \Delta_j f \right) \left( \Delta_i^{+1} f \cdot \Delta_j^{+1} f \right)}{g_{ij,n}} \right\} \right].$$

Then we will show

$$\sum_{i \neq j} \mathcal{P}_{1,ij}(f) \to \mathcal{E}_1(f),$$  

$$\sum_{i \neq j} \mathcal{P}_{1,ij}(f) \to 0,$$

4
Similarly we have (4.2) \[
\sum_{i \neq j} \mathcal{R}_{ij}(f) \to \mathcal{E}(f) - 4, \quad \sum_{i \neq j} \mathcal{R}_{ij}(f) \to 0
\]
as \( m \to \infty \).

First of all, we clarify how \( \mathcal{M}_{1,ij} \) comes from \( \mathcal{M}_{1,ij} \). Using Fact 4.1, we decompose \( \mathcal{M}_{1,ij}^m(f) \) into four parts as

\[
\mathcal{M}_{1,ij}^m(f) = I + II + III + IV,
\]

\[
I = \frac{g_{ij,d}}{g_{ij,n}} \left\{ 1 + \frac{\Delta_i \Delta_j g_{ij,n}}{2(H_i \| \Delta_i f \|_{\mathbb{R}^n})(H_j \| \Delta_j f \|_{\mathbb{R}^n})} \right\},
\]

\[
II = \frac{g_{ij,d}}{g_{ij,n}} \frac{\Delta_i (\Delta_j g_{ij,n})}{2H_j \| \Delta_j f \|_{\mathbb{R}^n}} \Delta_i \left( \frac{1}{H_i \| \Delta_i f \|_{\mathbb{R}^n}} \right),
\]

\[
III = \frac{g_{ij,d}}{g_{ij,n}} \frac{\Delta_i (A_j g_{ij,n})}{2H_i \| \Delta_i f \|_{\mathbb{R}^n}} \Delta_j \left( \frac{1}{H_j \| \Delta_j f \|_{\mathbb{R}^n}} \right),
\]

\[
IV = \frac{g_{ij,d}}{g_{ij,n}} \frac{A_{ij,n}}{2} \left\{ \Delta_i \left( \frac{1}{\| \Delta_i f \|_{\mathbb{R}^n}} \right) \right\} \left\{ \Delta_j \left( \frac{1}{\| \Delta_j f \|_{\mathbb{R}^n}} \right) \right\}.
\]

Next we decompose I into two parts. Using Fact 4.1 we have

\[
\Delta_j g_{ij,n} = \left( \Delta_j \| \Delta_i f \|_{\mathbb{R}^n} \right) \left( A_j \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right) + \left( A_j \| \Delta_i f \|_{\mathbb{R}^n} \right) \left( \Delta_j \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right),
\]

\[
\Delta_i \Delta_j g_{ij,n} = \left( \Delta_i \Delta_j \| \Delta_i f \|_{\mathbb{R}^n} \right) \left( A_{ij} \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right) + \left( \Delta_i \Delta_j \| \Delta_i f \|_{\mathbb{R}^n} \right) \left( A_{ij} \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right) + \left( \Delta_i A_{i,j} \| \Delta_i f \|_{\mathbb{R}^n} \right) \left( \Delta_j \Delta_i \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right)
\]

It holds that

\[
\Delta_i \| \Delta_i^j f \|_{\mathbb{R}^n}^2 = -2 A_i \Delta_i^j f \cdot \Delta_i f.
\]

On the other hand,

\[
\Delta_i \| \Delta_i^j f \|_{\mathbb{R}^n}^2 = 2 A_i \| \Delta_i^j f \|_{\mathbb{R}^n} \Delta_i \| \Delta_i^j f \|_{\mathbb{R}^n}.
\]

Therefore we obtain

\[
\Delta_i \| \Delta_i^j f \|_{\mathbb{R}^n} = -\frac{A_i \Delta_i^j f \cdot \Delta_i f}{A_i \| \Delta_i^j f \|_{\mathbb{R}^n}}.
\]

Similarly we have

\[
\Delta_i \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} = -\frac{A_i \Delta_i^{j+1} f \cdot \Delta_i^{j+1} f}{A_i \| \Delta_i^{j+1} f \|_{\mathbb{R}^n}},
\]

\[9\]
\[
\Delta_j \| \Delta_i^j f \|_{R^n} = \frac{A_j \Delta_i^j f \cdot \Delta_j f}{A_j \| \Delta_i^j f \|_{R^n}}, \quad \Delta_j \| \Delta_i^{j+1} f \|_{R^n} = \frac{A_j \Delta_i^{j+1} f \cdot \Delta_{j+1} f}{A_j \| \Delta_i^{j+1} f \|_{R^n}}.
\]

It follows from
\[
\Delta_j \| \Delta_i^j f \|_{R^n} = 2A_j \Delta_i^j f \cdot \Delta_j f
\]
that
\[
\Delta_i \Delta_j \| \Delta_i^j f \|_{R^n}^2 = 2A_j \Delta_i \Delta_i^j f \cdot \Delta_j f = -2\Delta_i \cdot \Delta_j f.
\]

On the other hand,
\[
\Delta_j \| \Delta_i^j f \|_{R^n}^2 = 2A_j \| \Delta_i^j f \|_{R^n} \Delta_j \| \Delta_i^j f \|_{R^n}
\]
derives
\[
\Delta_i \Delta_j \| \Delta_i^j f \|_{R^n}^2 = 2 \left( \Delta_i A_j \| \Delta_i^j f \|_{R^n} \right) \left( \Delta_j A_i \| \Delta_i^j f \|_{R^n} \right)
\]
\[
+ \left( A_{ij} \| \Delta_i^j f \|_{R^n} \right) \left( \Delta_i A_i \| \Delta_i^j f \|_{R^n} \right).
\]

Hence
\[
\Delta_i \Delta_j \| \Delta_i^j f \|_{R^n} = -\frac{\Delta_i \cdot \Delta_j f + \left( \Delta_i A_j \| \Delta_i^j f \|_{R^n} \right) \left( \Delta_j A_i \| \Delta_i^j f \|_{R^n} \right)}{A_{ij} \| \Delta_i^j f \|_{R^n}}.
\]

We obtain
\[
\Delta_i \Delta_j \| \Delta_i^{j+1} f \|_{R^n} = -\frac{\Delta_{j+1} f \cdot \Delta_j f + \left( \Delta_i A_j \| \Delta_i^{j+1} f \|_{R^n} \right) \left( \Delta_j A_i \| \Delta_i^{j+1} f \|_{R^n} \right)}{A_{ij} \| \Delta_i^{j+1} f \|_{R^n}}
\]
similarly. Consequently,
\[
\Delta_i \Delta_j \delta_{ij,n} = -\frac{A_{ij} \| \Delta_i^{j+1} f \|_{R^n} \Delta_i \cdot \Delta_j f}{\Delta_{ij} \| \Delta_i^j f \|_{R^n}} - \frac{A_{ij} \| \Delta_i^j f \|_{R^n} \Delta_i \cdot \Delta_j f}{\Delta_{ij} \| \Delta_i^{j+1} f \|_{R^n}} + \frac{A_{ij} \| \Delta_i^j f \|_{R^n} \Delta_i \cdot \Delta_j f}{\Delta_{ij} \| \Delta_i^{j+1} f \|_{R^n}}
\]
\[
- \left( A_{ij} \| \Delta_i^{j+1} f \|_{R^n} \right) \left( \Delta_i A_j \| \Delta_i^{j+1} f \|_{R^n} \right) \left( \Delta_j A_i \| \Delta_i^{j+1} f \|_{R^n} \right)
\]
\[
+ \left( \Delta_i A_i \| \Delta_i^{j+1} f \|_{R^n} \right) \left( \Delta_j A_j \| \Delta_i^{j+1} f \|_{R^n} \right) \left( \Delta_j A_i \| \Delta_i^{j+1} f \|_{R^n} \right).
\]
Using this, we decomposed $I$ as

$$I = I_1 + I_2,$$

where

$$I_1 = \frac{g_{j,d}}{g_{j,n}} \left[ 1 - \frac{1}{2(H_i \|\Delta_i f\|_{\mathbb{R}^n}) (H_j \|\Delta_j f\|_{\mathbb{R}^n})} \right] \times \left\{ \frac{A_{ij} \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}}{A_{ij} \|\Delta_i^j f\|_{\mathbb{R}^n}} (\Delta_i f \cdot \Delta_j f) + \frac{A_{ij} \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}}{A_{ij} \|\Delta_i^j f\|_{\mathbb{R}^n}} (\Delta_{i+1} f \cdot \Delta_j f) \right\},$$

and

$$I_2 = \frac{1}{2} \frac{g_{j,d}}{g_{j,n}} \left( H_i \|\Delta_i f\|_{\mathbb{R}^n} \right) \left( H_j \|\Delta_j f\|_{\mathbb{R}^n} \right) \times \left\{ \frac{A_{ij} \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}}{A_{ij} \|\Delta_i^j f\|_{\mathbb{R}^n}} (\Delta_i A_i \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}) - \frac{A_{ij} \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}}{A_{ij} \|\Delta_i^j f\|_{\mathbb{R}^n}} (\Delta_i A_i \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}) + \frac{A_{ij} \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}}{A_{ij} \|\Delta_i^j f\|_{\mathbb{R}^n}} (\Delta_i A_i \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}) \right\}.$$

Furthermore, $I_1$ is decomposed into three parts:

$$I_1 = I_{11} + I_{12} + I_{13},$$

where

$$I_{11} = \frac{1}{2} \frac{g_{j,d}}{g_{j,n}} \left\{ \left( 1 - \frac{\Delta_i f}{\|\Delta_i f\|_{\mathbb{R}^n}} \cdot \frac{\Delta_j f}{\|\Delta_j f\|_{\mathbb{R}^n}} \right) + \left( 1 - \frac{\Delta_{i+1} f}{\|\Delta_{i+1} f\|_{\mathbb{R}^n}} \cdot \frac{\Delta_{i+1} f}{\|\Delta_{i+1} f\|_{\mathbb{R}^n}} \right) \right\},$$

$$I_{12} = \frac{1}{2} \frac{g_{j,d}}{g_{j,n}} \left\{ \left( 1 - \frac{A_{ij} \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}}{A_{ij} \|\Delta_i^j f\|_{\mathbb{R}^n}} \right) \frac{\Delta_i f}{\|\Delta_i f\|_{\mathbb{R}^n}} \cdot \frac{\Delta_j f}{\|\Delta_j f\|_{\mathbb{R}^n}} + \left( 1 - \frac{A_{ij} \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}}{A_{ij} \|\Delta_i^j f\|_{\mathbb{R}^n}} \right) \frac{\Delta_{i+1} f}{\|\Delta_{i+1} f\|_{\mathbb{R}^n}} \cdot \frac{\Delta_{i+1} f}{\|\Delta_{i+1} f\|_{\mathbb{R}^n}} \right\},$$

and

$$I_{13} = \frac{1}{2} \frac{g_{j,d}}{g_{j,n}} \left\{ \frac{1}{\|\Delta_i f\|_{\mathbb{R}^n}, \Delta_j f\|_{\mathbb{R}^n}} - \frac{1}{(H_i \|\Delta_i f\|_{\mathbb{R}^n})(H_j \|\Delta_j f\|_{\mathbb{R}^n)})} \times \left\{ \frac{A_{ij} \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}}{A_{ij} \|\Delta_i^j f\|_{\mathbb{R}^n}} (\Delta_i f \cdot \Delta_j f) + \frac{A_{ij} \|\Delta_{i+1}^j f\|_{\mathbb{R}^n}}{A_{ij} \|\Delta_i^j f\|_{\mathbb{R}^n}} (\Delta_{i+1} f \cdot \Delta_j f) \right\}. $$

Now put

$$R_{i,j}^m(f) = I_{11}, \quad R_{i,j}^m(f) = I_{12} + I_{13} + I_2 + II + III + IV.$$

Since

$$R_{i,j}^m(f) = \frac{1}{4} \frac{g_{j,d}}{g_{j,n}} \left\{ \|\Delta_i f\|_{\mathbb{R}^n} - \|\Delta_j f\|_{\mathbb{R}^n} \right\}^2 + \|\Delta_{i+1} f\|_{\mathbb{R}^n} - \|\Delta_{i+1} f\|_{\mathbb{R}^n} \right\}^2 \geq 0,$$

holds.
we have
\[ \mathcal{P}_{1,i,j}^m(f) \to \frac{1}{2} \frac{\| \Delta^i_j \tau \|^2_{R^n}}{\| f(\theta_i) \|_{R^n} \| f(\theta_j) \|_{R^n}} \, d\theta_i d\theta_j \]
as \( m \to \infty \) formally, which is the energy density of \( \mathcal{E}_1(f) \) at \((\theta_i, \theta_j)\). Similarly \( \mathcal{P}_{1,i,j}^m(f) \to 0 \) as \( m \to \infty \) formally. Hence we expect (4.1).

Similarly \( \mathcal{M}_{ij}^m \) is decomposed as
\[ \mathcal{M}_{ij}^m(f) = J_1 + J_2 + J_3 + J_4 + J_5, \]
where
\[
J_1 = \frac{g_{ij,d}}{g_{ij,n}} \left[ 1 - \frac{1}{2g_{ij,d}} \left( \Delta_i \Delta_j g_{ij,n} - \frac{(\Delta_i g_{ij,n}) (\Delta_j g_{ij,n})}{g_{ij,n}} \right) - \frac{g_{ij,d}}{2g_{ij,n}} \right],
\]
\[
J_2 = \frac{g_{ij,d}}{2g_{ij,n}} \left\{ 1 - \frac{1}{H_i \| \Delta_j f \|_{R^n}} - \frac{1}{H_j \| \Delta_i f \|_{R^n}} \right\} \left\{ \Delta_i \Delta_j g_{ij,n} - \frac{(\Delta_i g_{ij,n}) (\Delta_j g_{ij,n})}{g_{ij,n}} \right\},
\]
\[
J_3 = - \frac{g_{ij,d}}{2g_{ij,n} H_i \| \Delta_j f \|_{R^n}} \left\{ \Delta_i A_i g_{ij,n} \left( \Delta_i H_i \| \Delta_j f \|_{R^n} \right) - A_i g_{ij,n} \left( \Delta_i g_{ij,n} \right) \left( \Delta_i \| \Delta_j f \|_{R^n} \right) \right\},
\]
\[
J_4 = - \frac{g_{ij,d}}{2g_{ij,n} H_j \| \Delta_i f \|_{R^n}} \left\{ \Delta_j A_j g_{ij,n} \left( \Delta_j H_j \| \Delta_i f \|_{R^n} \right) - A_j g_{ij,n} \left( \Delta_j g_{ij,n} \right) \left( \Delta_j \| \Delta_i f \|_{R^n} \right) \right\},
\]
\[
J_5 = - \frac{g_{ij,d}}{2g_{ij,n}} \left( \Delta_i A_i g_{ij,n} - \frac{(A_i g_{ij,n}) (A_i g_{ij,n})}{g_{ij,n}} \right) \left( \Delta_i \| \Delta_j f \|_{R^n} \right) \left( \Delta_j \| \Delta_i f \|_{R^n} \right).
\]

Furthermore, we decomposed \( J_1 \) as
\[
J_1 = J_{11} + J_{12} + J_{13} + J_{14},
\]
\[
J_{11} = \frac{g_{ij,d}}{g_{ij,n}} \left[ 1 - \frac{1}{2g_{ij,d}} \left( \Delta_i \Delta_j g_{ij,n} + \frac{4 \left( \Delta^i_j f \cdot \Delta_i f \right) \left( \Delta^i_j f \cdot \Delta_j f \right)}{g_{ij,n}} \right) \right],
\]
\[
J_{12} = g_{ij,d} \left( \frac{1}{g_{ij,n}} - \frac{1}{g_{ij,n}} \right),
\]
\[
J_{13} = - \frac{1}{2g_{ij,n}} \Delta_i \Delta_j g_{ij,n} + \frac{1}{2g_{ij,n}} \Delta_i \Delta_j g_{ij,n},
\]
\[
J_{14} = \frac{1}{2} \left\{ \frac{(\Delta_i g_{ij,n}) (\Delta_j g_{ij,n})}{g_{ij,n}^2} + \frac{4 \left( \Delta^i_j f \cdot \Delta_i f \right) \left( \Delta^i_j f \cdot \Delta_j f \right)}{g_{ij,n}^2} - \left( \frac{g_{ij,d}}{g_{ij,n}} \right)^2 \right\}.
\]

We put
\[ \mathcal{P}_{ij}^m(f) = J_{11}, \quad \mathcal{M}_{ij}^m(f) = J_{12} + J_{13} + J_{14} + J_2 + J_3 + J_4 + J_5. \]
Since \( J_{11} \) can be rewritten as

\[
J_{11} = \frac{1}{2} g_{ij,d} \bar{g}_{ij,n} \left\| \frac{\Delta_i^j \mathbf{f}}{\|\Delta_i^j \mathbf{f}\|_{\mathbb{R}^n}} \right\|^2 \wedge \left( \frac{\Delta_i \mathbf{f}}{\|\Delta_i \mathbf{f}\|_{\mathbb{R}^n}} + \frac{\Delta_j \mathbf{f}}{\|\Delta_j \mathbf{f}\|_{\mathbb{R}^n}} \right) \right\|^2_{\wedge^2 \mathbb{R}^n}
\]

\[
+ \left\{ \frac{\Delta_i^j \mathbf{f}}{\|\Delta_i^j \mathbf{f}\|_{\mathbb{R}^n}} \left( \frac{\Delta_i \mathbf{f}}{\|\Delta_i \mathbf{f}\|_{\mathbb{R}^n}} - \frac{\Delta_j \mathbf{f}}{\|\Delta_j \mathbf{f}\|_{\mathbb{R}^n}} \right) \right\}^2
\]

it formally holds that

\[
\mathcal{E}_m(\mathbf{f}) \to \frac{1}{2} \left\{ \left\| \frac{\Delta_i^j \mathbf{f}}{\|\Delta_i^j \mathbf{f}\|_{\mathbb{R}^n}} \wedge (\tau(\theta_i) + \tau(\theta_j)) \right\|^2_{\wedge^2 \mathbb{R}^n} + \left( \frac{\Delta_i^j \mathbf{f}}{\|\Delta_i^j \mathbf{f}\|_{\mathbb{R}^n}} \cdot \Delta_i^j \tau \right) \right\}^2
\]

\[
\times \frac{\|\dot{f}(\theta_i)\|_{\mathbb{R}^n} \|\dot{f}(\theta_j)\|_{\mathbb{R}^n}}{\|\Delta_i^j \mathbf{f}\|_{\mathbb{R}^n}} d\theta_i d\theta_j
\]

as \( m \to \infty \), which is the energy density of \( \mathcal{E}(\cdot) - 4 \) at \((\theta_i, \theta_j)\). On the other hand, we can expect \( \mathcal{E}_m(\mathbf{f}) \to 0 \) as \( m \to \infty \).

### 4.3 Estimates

We would like to show the convergence of our discrete energies. In this subsection, we will derive several estimates for proving the convergence. It is known that \( \mathbf{f} \) is bi-Lipschitz if \( \mathcal{E}(\mathbf{f}) < \infty \). Therefore it is natural to assume the bi-Lipschitz continuity of \( \mathbf{f} \). Then there exist positive constant \( L_1 \) and \( L_2 \) independent of \( i, j \), and \( m \) such that

\[
L_1 \frac{d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j)}{\Delta_i \theta} \leq \|\Delta_i^j \mathbf{f}\|_{\mathbb{R}^n} \leq L_2 \frac{d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j)}{\Delta_i \theta}.
\]

Here \( d_{\mathbb{R}/\mathbb{Z}}(\cdot, \cdot) \) is the distance of \( \mathbb{R}/\mathbb{Z} \).

Furthermore we assume the equilateral condition, i.e.,

\[
\|\Delta_i \mathbf{f}\|_{\mathbb{R}^n} = \frac{L_m}{m} \quad (i = 1, \cdots, m).
\]

Here \( L_m \) is the total length of \( m \)-polygon with the \( i \)-th vertex \( \mathbf{f}(\theta_i) \).

\[
L_m \leq L
\]

is obvious. We may assume that \( m \) is sufficiently large so that

\[
d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_{i+1}) = \Delta_i \theta.
\]

It follows from (4.4) and (4.5) that

\[
L_1 \Delta_i \theta \leq \|\Delta_i \mathbf{f}\|_{\mathbb{R}^n} = \frac{L_m}{m} \leq L_2 \Delta_i \theta;
\]

as \( m \to \infty \), which is the energy density of \( \mathcal{E}(\cdot) - 4 \) at \((\theta_i, \theta_j)\). On the other hand, we can expect \( \mathcal{E}_m(\mathbf{f}) \to 0 \) as \( m \to \infty \).
and hence we have
\[
\frac{L_m}{L_2m} \leq \Delta_i \theta \leq \frac{L_m}{L_1m} \leq \frac{L}{L_1m}.
\]

In particular,
\[
\Delta_i \theta \to 0 \quad \text{as} \quad m \to \infty
\]
holds. we can estimate \(\Delta_i \theta\) from below by \(\frac{L}{m}\). To show this, it it sufficient to see \(L_m \to L\) as \(m \to \infty\).

Let \(I, J \subset \mathbb{R}/\mathbb{Z}\) be measurable sets, not necessarily intervals, and we put
\[
[u]_{H^{\frac{1}{2}}(I \times J)} = \left( \iint_{I \times J} \frac{\|\Delta_i^j u\|_{\mathbb{R}^n}^2}{|\Delta_i^j \theta|^2} d\theta_id\theta_j \right)^{\frac{1}{2}}
\]
for \(H^{\frac{1}{2}}(I \times J)\). When we consider an \(\mathbb{R}^n\)-valued function, we define
\[
[u]_{H^{\frac{1}{2}}(I \times J)} = \left( \iint_{I \times J} \frac{\|\Delta_i^j u\|_{\mathbb{R}^n}^2}{|\Delta_i^j \theta|^2} d\theta_id\theta_j \right)^{\frac{1}{2}}.
\]

Put \(I_i = [\theta_i, \theta_i+1], I_j = [\theta_j, \theta_j+1]\).

**Lemma 4.1** It holds for \(f \in H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z})\) that
\[
0 \leq L - L_m \leq \frac{2L}{\sqrt{6}m} \|[f]_{H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z})}\|.
\]

In particular
\[
(4.6) \quad \frac{L}{2} \leq L_m
\]
holds for sufficiently large \(m\).

**Proof.** Let \(\theta_* \in [\theta_i, \theta_i+1]\). Then we have
\[
\int_{\theta_i}^{\theta_i+1} \|f(\theta)\|_{\mathbb{R}^n} d\theta - \left\| \int_{\theta_i}^{\theta_i+1} \dot{f}(\theta) d\theta \right\|_{\mathbb{R}^n}
\leq \int_{\theta_i}^{\theta_i+1} \|\dot{f}(\theta) - \dot{f}(\theta_*\})\|_{\mathbb{R}^n} d\theta + \left\| \int_{\theta_i}^{\theta_i+1} \left( \dot{f}(\theta) - \dot{f}(\theta_*\}) \right) d\theta \right\|_{\mathbb{R}^n}
\leq 2 \int_{\theta_i}^{\theta_i+1} \|f(\theta) - \dot{f}(\theta_*\})\|_{\mathbb{R}^n} d\theta.
\]
We integrate this with respect to \( \theta^* \) on \( I_i \), and divide the result by \( \Delta_i \theta \) to get
\[
\int_{\theta_i}^{\theta_{i+1}} \left\| \int_{\theta_i}^{\theta_{i+1}} \right\| \left\| f(\theta) \right\| d\theta - \left\| \int_{\theta_i}^{\theta_{i+1}} \right\| \left\| f(\theta) \right\| d\theta
\]
\[
\leq \frac{2}{\Delta_i \theta} \int_{\theta_i}^{\theta_{i+1}} \left\| f(\theta) - \int_{\theta_i}^{\theta_{i+1}} \frac{f(\theta) - f(\theta^*)}{|\theta - \theta^*|^2} d\theta d\theta^* \right\|^{\frac{1}{2}} \int_{\theta_i}^{\theta_{i+1}} \left\| f(\theta) - \int_{\theta_i}^{\theta_{i+1}} \frac{f(\theta) - f(\theta^*)}{|\theta - \theta^*|^2} d\theta d\theta^* \right\|^{\frac{1}{2}}
\]
\[
\leq \frac{2\Delta_i \theta}{\sqrt{6}} \| f \|_{H^2((I_i \times I_i)}.
\]
Summing with respect to \( i \), we obtain
\[
0 \leq L - L_m \leq \sum_{i=1}^{m} \left( \int_{\theta_i}^{\theta_{i+1}} \left\| f(\theta) \right\| d\theta - \left\| \int_{\theta_i}^{\theta_{i+1}} \right\| \left\| f(\theta) \right\| d\theta \right) \leq \frac{2L}{\sqrt{6} m} \| f \|_{H^2((\mathbb{R}/\mathbb{Z})}.
\]

By this lemma, we may assume
\[
(4.7) \quad \frac{L}{2L_1 m} \leq \Delta_i \theta \leq \frac{L}{L_2 m}
\]
for sufficiently large \( m \).

**Lemma 4.2** Under (4.5),
\[
\| \Delta_i f - \Delta_{i+1} f \|_{\mathbb{R}^n} \leq 2\Delta_i^{i+2} \theta \| f \|_{H^2((\theta_i, \theta_{i+2})^2)}
\]
holds for \( f \in H^2((\mathbb{R}/\mathbb{Z})\).**

**Proof.** Using (4.5), we have
\[
\| \Delta_i f - \Delta_{i+1} f \|_{\mathbb{R}^n}
\]
\[
= \frac{L_m}{m} \left\| \Delta_i f \|_{\mathbb{R}^n} - \Delta_{i+1} f \|_{\mathbb{R}^n} \right\|_{\mathbb{R}^n}
\]
\[
\leq \frac{L_m}{m} \left\{ \left\| \Delta_i f \|_{\mathbb{R}^n} - \tau(\theta^*) \right\|_{\mathbb{R}^n} + \left\| \tau(\theta^*) - \Delta_{i+1} f \|_{\mathbb{R}^n} \right\|_{\mathbb{R}^n} \right\}
\]
Here we choose \( \theta^* \) from the interval \( [\theta_i, \theta_{i+2}] \). We integrate this with respect to \( \theta^* \), and divide the result by \( \Delta_i^{i+2} \theta \) to get
\[
\| \Delta_i f - \Delta_{i+1} f \|_{\mathbb{R}^n}
\]
\[
\leq \frac{L_m}{m} \frac{1}{\Delta_i^{i+2} \theta} \int_{\theta_i}^{\theta_{i+2}} \left\{ \left\| \Delta_i f \|_{\mathbb{R}^n} - \tau(\theta^*) \right\|_{\mathbb{R}^n} + \left\| \tau(\theta^*) - \Delta_{i+1} f \|_{\mathbb{R}^n} \right\|_{\mathbb{R}^n} \right\} d\theta^*.
\]
It is clear that
\[
\frac{\Delta_i f}{\|\Delta_i f\|_{\mathbb{R}^n}} - \tau(\theta_*)
= \frac{1}{\|\Delta_i f\|_{\mathbb{R}^n}} \int_{\theta_i}^{\theta_{i+1}} \left( \dot{f}(\theta) - \dot{f}(\theta_*) \right) d\theta + \frac{\Delta_i \theta \dot{f}(\theta_*)}{\|\Delta_i f\|_{\mathbb{R}^n} \|f(\theta_*)\|_{\mathbb{R}^n}} \left( \|f(\theta_*)\|_{\mathbb{R}^n} - \|\Delta_i f\|_{\mathbb{R}^n} \right).
\]

We use (4.5) again, and get
\[
\left\| \frac{\Delta_i \theta \dot{f}(\theta_*)}{\|\Delta_i f\|_{\mathbb{R}^n} \|f(\theta_*)\|_{\mathbb{R}^n}} \left( \|f(\theta_*)\|_{\mathbb{R}^n} - \|\Delta_i f\|_{\mathbb{R}^n} \right) \right\|_{\mathbb{R}^n} \leq \frac{m}{L_m} \int_{\theta_i}^{\theta_{i+1}} \left\| \dot{f}(\theta) - \dot{f}(\theta_*) \right\|_{\mathbb{R}^n} d\theta.
\]

On the other hand, it holds that
\[
\left\| \frac{\Delta_i \theta \dot{f}(\theta_*)}{\|\Delta_i f\|_{\mathbb{R}^n} \|f(\theta_*)\|_{\mathbb{R}^n}} \left( \|f(\theta_*)\|_{\mathbb{R}^n} - \|\Delta_i f\|_{\mathbb{R}^n} \right) \right\|_{\mathbb{R}^n} \leq \frac{m}{L_m} \left\| \Delta_i \theta \dot{f}(\theta_*) - \Delta_i \theta \dot{f}(\theta_*) \right\|_{\mathbb{R}^n}
\]

Consequently we obtain
\[
\frac{L_m}{m} \Delta_i^{i+2} \theta \int_{\theta_i}^{\theta_{i+2}} \left( \left\| \frac{\Delta_i f}{\|\Delta_i f\|_{\mathbb{R}^n}} - \tau(\theta_*) \right\|_{\mathbb{R}^n} + \left\| \tau(\theta_*) - \frac{\Delta_i+1 f}{\|\Delta_i+1 f\|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^n} \right) d\theta_i
\leq 2 \Delta_i^{i+2} \theta \left\{ \int_{\theta_i}^{\theta_{i+1}} \int_{\theta_i}^{\theta_{i+2}} \left\| \dot{f}(\theta) - \dot{f}(\theta_*) \right\|_{\mathbb{R}^n} d\theta d\theta_\ast + \int_{\theta_i}^{\theta_{i+1}} \int_{\theta_i}^{\theta_{i+2}} \left\| \ddot{f}(\theta) - \ddot{f}(\theta_*) \right\|_{\mathbb{R}^n} d\theta d\theta_\ast \right\}
\leq 2 \int_{\theta_i}^{\theta_{i+2}} \int_{\theta_i}^{\theta_{i+2}} \left\| \dot{f}(\theta) - \dot{f}(\theta_*) \right\|_{\mathbb{R}^n} d\theta d\theta_\ast
\leq 2 \left( \int_{\theta_i}^{\theta_{i+2}} \int_{\theta_i}^{\theta_{i+2}} \left\| \dot{f}(\theta) - \dot{f}(\theta_*) \right\|_{\mathbb{R}^n} d\theta d\theta_\ast \right)^{\frac{1}{2}}
\leq 2 \Delta_i^{i+2} \theta [\dot{f}]_{H^{\frac{1}{2}}([\theta_i, \theta_{i+2}])}.
\]

\[
\square
\]

**Definition 4.1** For \( u \in H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z}) \) and positive constants \( \varepsilon_1, \varepsilon_2 \), we define \( K(u, \varepsilon_1, \varepsilon_2) \) by
\[
K(u, \varepsilon_1, \varepsilon_2) = \sup \left\{ |u|_{H^{\frac{1}{2}}(I \times J)} \left| I \subset \mathbb{R}/\mathbb{Z}, J \subset \mathbb{R}/\mathbb{Z}, |I| \leq \varepsilon_1, |J| \leq \varepsilon_2 \right. \right\}.
\]

When \( \varepsilon_1 = \varepsilon_2 \), we will write the quantity simply by \( K(u, \varepsilon_1) \).
Corollary 4.1 Let \( f \in H^2( \mathbb{R}/\mathbb{Z} ) \) satisfy (4.4) and (4.5). Then we have

\[
\| \Delta_i f - \Delta_j f \|_{\mathbb{R}^n} \leq \frac{CL|i-j|}{m} K \left( \dot{f}, \frac{2L}{L_2 m} \right),
\]

\[
\left\| \frac{\Delta_i f}{\| \Delta_i f \|_{\mathbb{R}^n}} - \frac{\Delta_j f}{\| \Delta_j f \|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^n} \leq C|i-j| K \left( \dot{f}, \frac{2L}{L_2 m} \right).
\]

Proof. These estimates are shown as

\[
\| \Delta_i f - \Delta_j f \|_{\mathbb{R}^n} \leq \sum_{k=i}^{j-1} \| \Delta_k f - \Delta_k f_{k+1} \|_{\mathbb{R}^n}
\]

\[
\leq \sum_{k=i}^{j-1} L_{k+2} \theta^2 [f]_{H^2(\mathbb{R}^2 \times \{ \theta_k \})^{2}}
\]

\[
\leq \frac{CL|i-j|}{m} K \left( \dot{f}, \frac{2L}{L_2 m} \right),
\]

\[
\left\| \frac{\Delta_i f}{\| \Delta_i f \|_{\mathbb{R}^n}} - \frac{\Delta_j f}{\| \Delta_j f \|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^n} = \frac{m}{L_m} \| \Delta_i f - \Delta_j f \|_{\mathbb{R}^n}
\]

\[
\leq C|i-j| K \left( \dot{f}, \frac{2L}{L_2 m} \right).
\]

It follows from (4.4) and (4.5) that

\[
L_1 \Delta_k \theta \leq \| \Delta_k f \|_{\mathbb{R}^n} = \frac{L_m}{m} \leq \| \Delta_k \theta \|_{\mathbb{R}^n} \leq \| \Delta_k f \|_{\mathbb{R}^n}.
\]

When \( d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j) = \theta_j - \theta_i \), we sum the above estimate with respect to \( k \) from \( i \) to \( j-1 \). The result is

\[
L_1 d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j) \leq \frac{j-i}{m} L_m \leq L_2 d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j).
\]

From (4.4) we have

\[
\frac{L_m L_1 j - i}{m} \leq L_1 d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j) \leq \| \Delta^j f \|_{\mathbb{R}^n} \leq L_2 d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j) \leq \frac{L_m L_2 j - i}{L_1} m.
\]

When \( d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j) = \theta_{i+m} - \theta_j \), we have similarly

\[
L_1 d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j) \leq \frac{m+i-j}{m} L_m \leq L_2 d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j),
\]

and

\[
\frac{L_m L_1 m + i - j}{m} \leq L_1 d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j) \leq \| \Delta^j f \|_{\mathbb{R}^n} \leq L_2 d_{\mathbb{R}/\mathbb{Z}}(\theta_i, \theta_j) \leq \frac{L_m L_2 m + i - j}{L_1} m.
\]
Hence we find
\[
\frac{LL_1 \min \{\lvert i - j \rvert, m - \lvert i - j \rvert \}}{2L_2} \leq \| \Delta_i^j f \| R^n \leq \frac{LL_2 \max \{\lvert i - j \rvert, m - \lvert i - j \rvert \}}{L_1}
\]
for sufficiently large \( m \). We consider the summation with respect to \( i \) and \( j \)
\[
\sum_{i \neq j} (\cdots) = \sum_{i=1}^{m} \left( \sum_{j=1}^{i-1} + \sum_{j=i+1}^{i+\lceil \frac{m}{2} \rceil} \right) (\cdots)
\]
when \( m \) is odd; and
\[
\sum_{i \neq j} (\cdots) = \sum_{i=1}^{m} \left( \sum_{j=i-\lceil \frac{m}{2} \rceil+1}^{i-1} + \sum_{j=i+1}^{i+\lceil \frac{m}{2} \rceil} \right) (\cdots)
\]
when \( m \) is even. Thus we may assume \( \lvert i - j \rvert \leq \lceil \frac{m}{2} \rceil \).

**Corollary 4.2** There exist constants \( C \) depending on \( L_1 \) and \( L_2 \) such that
\[
\left| A_{ij} \left( \| \Delta_{i+1}^j f \| R^n - \| \Delta_i^j f \| R^n \right) \right| \leq A_{ij} \| \Delta_{i+1}^j f - \Delta_i^j f \| R^n \leq CL\frac{|i - j|}{m} K \left( \frac{\hat{f}}{2L}, \frac{2L}{L_2m} \right).
\]

**Proof.** From Corollary 4.1 we have
\[
\left| \| \Delta_{i+1}^j f \| R^n - \| \Delta_i^j f \| R^n \right| \leq \| \Delta_{i+1}^j f - \Delta_i^j f \| R^n = \| \Delta_j f - \Delta_i f \| R^n \leq \frac{CL\frac{|i - j|}{m}}{K \left( \frac{\hat{f}}{2L}, \frac{2L}{L_2m} \right)}.
\]
Similarly we obtain
\[
\left| \| \Delta_{i+1}^{j+1} f \| R^n - \| \Delta_i^{j+1} f \| R^n \right| \leq \frac{CL\frac{|i - j - 1|}{m}}{K \left( \frac{\hat{f}}{2L}, \frac{2L}{L_2m} \right)},
\]
\[
\left| \| \Delta_{i+2}^{j+1} f \| R^n - \| \Delta_i^{j+1} f \| R^n \right| \leq \frac{CL\frac{|i - j + 1|}{m}}{K \left( \frac{\hat{f}}{2L}, \frac{2L}{L_2m} \right)},
\]
\[
\left| \| \Delta_{i+2}^{j+2} f \| R^n - \| \Delta_i^{j+1} f \| R^n \right| \leq \frac{CL\frac{|i - j|}{m}}{K \left( \frac{\hat{f}}{2L}, \frac{2L}{L_2m} \right)}.
\]
The assertion follows by calculating the arithmetic mean of these. \( \square \)
Lemma 4.3 If \( m \) is sufficiently large, and if \( |i - j| \leq \left\lfloor \frac{m}{2} \right\rfloor \), then

\[
\begin{align*}
A_i \| \Delta_i^0 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{|i - j| - \frac{1}{2}}{m}, \\
A_i \| \Delta_i^1 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{|i - j| - \frac{1}{2}}{m}, \\
A_{ij} \| \Delta_i^0 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{|i - j| - \frac{1}{2}}{m}, \\
A_{ij} \| \Delta_i^1 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{2L_2} \frac{|i - j|}{m}.
\end{align*}
\]

Proof. When \( i < j < i + \left\lfloor \frac{m}{2} \right\rfloor - 1 \), we have

\[
\begin{align*}
0 < j - i < \left\lfloor \frac{m}{2} \right\rfloor - 1, & \quad 0 < j + 1 - i \leq \left\lfloor \frac{m}{2} \right\rfloor - 1, \\
0 \leq j - i - 1 < \left\lfloor \frac{m}{2} \right\rfloor - 1, & \quad 0 < (j + 1) - (i + 1) < \left\lfloor \frac{m}{2} \right\rfloor - 1 \\
0 < (j + 2) - i \leq \left\lfloor \frac{m}{2} \right\rfloor, & \quad 0 < (j + 2) - (i + 1) \leq \left\lfloor \frac{m}{2} \right\rfloor - 1.
\end{align*}
\]

Hence

\[
\begin{align*}
\| \Delta_i^0 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{j - i}{m}, & \| \Delta_i^1 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{j + 1 - i}{m}, \\
\| \Delta_{i+1}^0 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{j - i - 1}{m}, & \| \Delta_{i+1}^1 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{j - i}{m}, \\
\| \Delta_i^{j+2} f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{j + 2 - i}{m}, & \| \Delta_{i+1}^{j+2} f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{j + 1 - i}{m},
\end{align*}
\]

hold. Calculating the arithmetic mean, we obtain

\[
\begin{align*}
A_i \| \Delta_i^0 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{j - i - \frac{1}{2}}{m}, \\
A_i \| \Delta_i^1 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{j - i + \frac{1}{2}}{m}, \\
A_{ij} \| \Delta_i^0 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{j - i}{m}, \\
A_{ij} \| \Delta_i^1 f \|_{\mathbb{R}^n} &\geq \frac{L_m L_1}{L_2} \frac{j + 1 - i}{m}.
\end{align*}
\]

In case of \( j = i + \left\lfloor \frac{m}{2} \right\rfloor - 1 \), it follows from

\[
\begin{align*}
j - i &= \left\lfloor \frac{m}{2} \right\rfloor - 1, & \quad j + 1 - i &= \left\lfloor \frac{m}{2} \right\rfloor, \\
j - i - 1 &= \left\lfloor \frac{m}{2} \right\rfloor - 2, & \quad (j + 1) - (i + 1) &= \left\lfloor \frac{m}{2} \right\rfloor - 1, \\
(j + 2) - i &= \left\lfloor \frac{m}{2} \right\rfloor + 1, & \quad (j + 2) - (i + 1) &= \left\lfloor \frac{m}{2} \right\rfloor.
\end{align*}
\]
\[
\begin{align*}
\|\Delta_j^i f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j - i}{L_2 m}, & \|\Delta_{j+1}^i f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j + 1 - i}{L_2 m}, \\
\|\Delta_{j+1}^i f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j - i - 1}{L_2 m}, & \|\Delta_{j+1}^{i+1} f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j - i}{L_2 m}, \\
\|\Delta_{j+1}^{i+2} f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 m + i - j - 2}{L_2 m} \geq \frac{L_m L_1 \left\lceil \frac{m}{2} \right\rceil - 1}{L_2 m} = \frac{L_m L_1 j - i}{L_2 m} \\
\|\Delta_{j+1}^{i+2} f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j + 1 - i}{L_2 m}.
\end{align*}
\]

Therefore we have

\[
\begin{align*}
A_i \|\Delta_j^i f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j - i - \frac{1}{2}}{L_2 m}, \\
A_i \|\Delta_{j+1}^i f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j - i + \frac{1}{2}}{L_2 m}, \\
A_{ij} \|\Delta_j^i f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j - i}{L_2 m}, \\
A_{ij} \|\Delta_{j+1}^i f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j - i + \frac{1}{2}}{L_2 m}.
\end{align*}
\]

When \( j = i + \left\lceil \frac{m}{2} \right\rceil \), we have

\[
\begin{align*}
j - i & = \left\lceil \frac{m}{2} \right\rceil, & j + 1 - i & = \left\lceil \frac{m}{2} \right\rceil + 1, \\
j - i - 1 & = \left\lceil \frac{m}{2} \right\rceil - 1, & (j + 1) - (i + 1) & = \left\lceil \frac{m}{2} \right\rceil \\
(j + 2) - i & = \left\lceil \frac{m}{2} \right\rceil + 2, & (j + 2) - (i + 1) & = \left\lceil \frac{m}{2} \right\rceil + 1,
\end{align*}
\]

which yield

\[
\begin{align*}
\|\Delta_j^i f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j - i}{L_2 m}, \\
\|\Delta_{j+1}^i f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 m + i - j - 1}{L_2 m} \geq \frac{L_m L_1 \left\lceil \frac{m}{2} \right\rceil - 1}{L_2 m} = \frac{L_m L_1 j - i - 1}{L_2 m}, \\
\|\Delta_{j+1}^{i+1} f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j - i}{L_2 m}, & \|\Delta_{j+1}^{i+1} f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 j - i}{L_2 m} \\
\|\Delta_{j+1}^{i+2} f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 m + i - j - 2}{L_2 m} \geq \frac{L_m L_1 \left\lceil \frac{m}{2} \right\rceil - 2}{L_2 m} = \frac{L_m L_1 j - i - 2}{L_2 m}, \\
\|\Delta_{j+1}^{i+2} f\|_{\mathbb{R}^n} & \geq \frac{L_m L_1 m + i + 1 - j - 2}{L_2 m} \geq \frac{L_m L_1 \left\lceil \frac{m}{2} \right\rceil - 1}{L_2 m} = \frac{L_m L_1 j - i - 1}{L_2 m},
\end{align*}
\]
and hence

\[ A_i \| \Delta_i^1 f \|_{\mathbb{R}^n} \geq \frac{L_m L_1 (j - i)}{L_2 m}, \]

\[ A_i \| \Delta_i^{i+1} f \|_{\mathbb{R}^n} \geq \frac{L_m L_1 (j - i - \frac{1}{2})}{L_2 m}, \]

\[ A_{ij} \| \Delta_i^1 f \|_{\mathbb{R}^n} \geq \frac{L_m L_1 (j - i - \frac{1}{3})}{L_2 m}, \]

\[ A_{ij} \| \Delta_i^{i+1} f \|_{\mathbb{R}^n} \geq \frac{L_m L_1 (j - i - \frac{1}{3})}{L_2 m}. \]

Since \( \left| \frac{m}{2} \right| - 1 \geq \frac{1}{2} \left| \frac{m}{2} \right| \) for \( m \geq 4 \), we have

\[ A_{ij} \| \Delta_i^{i+1} f \|_{\mathbb{R}^n} \geq \frac{L_m L_1 (j - i - 1)}{L_2 m} = \frac{L_m L_1 \left( \frac{m}{2} \right) - 1}{2L_2 m} = \frac{L_m L_1 j - i}{2L_2 m}. \]

The assertion can be proved for \( i > j \) in the same way. \( \square \)

### 4.4 The proof of convergence

**Lemma 4.4** Assume that \( f \in W^{2, \infty}(\mathbb{R}/\mathbb{Z}) \) with (4.4), and (4.5). Then it holds that

\[ \lim_{m \to \infty} \sum_{i,j \neq j} \mathcal{P}_{1,ij}^m(f) = \mathcal{E}_1(f). \]

**Proof.** Recall that

\[ \mathcal{P}_{1,ij}(f) = \frac{1}{4} g_{ij,n} \left\{ \frac{\Delta_i f}{\| \Delta_i f \|_{\mathbb{R}^n}} - \frac{\Delta_j f}{\| \Delta_j f \|_{\mathbb{R}^n}} \right\}^2 + \left\{ \frac{\Delta_i^{i+1} f}{\| \Delta_i^{i+1} f \|_{\mathbb{R}^n}} - \frac{\Delta_j^{i+1} f}{\| \Delta_j^{i+1} f \|_{\mathbb{R}^n}} \right\}^2 \}

Let \( \chi_{ij} \) be the characteristic function of the set \([\theta_i, \theta_{i+1}) \times [\theta_j, \theta_{j+1})\). Then

\[ \mathcal{E}_1^m(f) = \int_{(\mathbb{R}/\mathbb{Z})^2} \sum_{i,j} \frac{1}{4 g_{ij,n}} \left\{ \frac{\Delta_i f}{\| \Delta_i f \|_{\mathbb{R}^n}} - \frac{\Delta_j f}{\| \Delta_j f \|_{\mathbb{R}^n}} \right\}^2 + \left\{ \frac{\Delta_i^{i+1} f}{\| \Delta_i^{i+1} f \|_{\mathbb{R}^n}} - \frac{\Delta_j^{i+1} f}{\| \Delta_j^{i+1} f \|_{\mathbb{R}^n}} \right\}^2 \}

\times \left| \frac{\Delta_i f}{\| \Delta_i f \|_{\mathbb{R}^n}} \right| \left| \frac{\Delta_j f}{\| \Delta_j f \|_{\mathbb{R}^n}} \right| \chi_{ij}(\theta_1, \theta_2) d\theta_1 d\theta_2.

It holds for a.e. \((\theta_1, \theta_2)\) that

\[ \sum_{i,j} \frac{1}{4 g_{ij,n}} \left\{ \frac{\Delta_i f}{\| \Delta_i f \|_{\mathbb{R}^n}} - \frac{\Delta_j f}{\| \Delta_j f \|_{\mathbb{R}^n}} \right\}^2 + \left\{ \frac{\Delta_i^{i+1} f}{\| \Delta_i^{i+1} f \|_{\mathbb{R}^n}} - \frac{\Delta_j^{i+1} f}{\| \Delta_j^{i+1} f \|_{\mathbb{R}^n}} \right\}^2 \}

\times \left| \frac{\Delta_i f}{\| \Delta_i f \|_{\mathbb{R}^n}} \right| \left| \frac{\Delta_j f}{\| \Delta_j f \|_{\mathbb{R}^n}} \right| \chi_{ij}(\theta_1, \theta_2) \]

\[ + \frac{1}{2} \| f(\theta_1) - f(\theta_2) \|_{\mathbb{R}^n} \| \hat{f}(\theta_1) \|_{\mathbb{R}^n} \| \hat{f}(\theta_2) \|_{\mathbb{R}^n} \]
as $m \to \infty$. Since $\dot{f}$ is Lipschitz, we have

$$K \left( \dot{f}, \frac{2L}{L_2 m} \right) \leq \frac{C}{m}$$

for some positive constant independent of $m$. It follows from Corollary 4.1 that

$$\left\| \Delta_i f - \Delta_j f \right\|_{\mathbb{R}^n} \leq \frac{C|i-j|}{m}.$$ A similar estimate holds for

$$\left\| \Delta_i f - \Delta_j f \right\|_{\mathbb{R}^n} \leq \frac{C|i-j|}{m}.$$ Combining this with

$$\left\| \Delta_i f \right\|_{\mathbb{R}^n} \geq \frac{C|i-j|}{m}$$

for $|i-j| \leq \left[ \frac{m}{2} \right]$ and (4.4) we obtain

$$\sum_{i,j} \frac{1}{4g_{i,j,n}} \left\{ \left\| \frac{\Delta_i f}{\left\| \Delta_i f \right\|_{\mathbb{R}^n}} - \frac{\Delta_j f}{\left\| \Delta_j f \right\|_{\mathbb{R}^n}} \right\|^2 + \left\| \frac{\Delta_i f}{\left\| \Delta_i f \right\|_{\mathbb{R}^n}} - \frac{\Delta_j f}{\left\| \Delta_j f \right\|_{\mathbb{R}^n}} \right\|^2 \right\}$$

$$\times \frac{\left\| \Delta_i f \right\|_{\mathbb{R}^n}}{|\Delta_i \theta|} \frac{\left\| \Delta_j f \right\|_{\mathbb{R}^n}}{|\Delta_j \theta|} \chi_{ij}(\vartheta_1, \vartheta_2)
$$

$$\leq C \sum_{i,j} \frac{1}{m^2 |i-j|^2} \left( \frac{|i-j|}{m} \right)^2 L_2^2 \chi_{ij}(\vartheta_1, \vartheta_2) \leq C.$$ Consequently the assertion is derived from Lebesgue’s convergence theorem.

We now show $\sum_{i \neq j} \mathcal{R}_{1,j}^m(f) \to 0$. Under (4.5) $I_{13} = II = III = IV = 0$. Therefore

$$\mathcal{R}_{1,j}^m(f) = I_{12} + I_2$$

**Lemma 4.5** Suppose that $f \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z})$, $\tau \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z})$, (4.4), and (4.5). Furthermore we assume

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-1} K(f, \varepsilon)^2 = 0.$$ Then we have

$$\lim_{m \to \infty} \sum I_{12} = 0.$$
Proof. $I_{12}$ can be written as

\[
I_{12} = \frac{1}{4} g_{ij,\cdot} \left( \frac{A_{i+j} \left( \| \Delta_i f \|_{\mathbb{R}^n} - \| \Delta_i^{+1} f \|_{\mathbb{R}^n} \right) }{A_{i+j} \| \Delta_i f \|_{\mathbb{R}^n} A_{i+j} \| \Delta_i^{+1} f \|_{\mathbb{R}^n}} \right)^2
\times \left( \frac{\Delta_i f}{\| \Delta_i f \|_{\mathbb{R}^n}} \cdot \frac{\Delta_j f}{\| \Delta_j f \|_{\mathbb{R}^n}} + \frac{\Delta_i^{+1} f}{\| \Delta_i^{+1} f \|_{\mathbb{R}^n}} \cdot \frac{\Delta_j^{+1} f}{\| \Delta_j^{+1} f \|_{\mathbb{R}^n}} \right)
+ \frac{1}{4} \left( \frac{1}{A_{i+j} \| \Delta_i f \|_{\mathbb{R}^n} A_{i+j} \| \Delta_i^{+1} f \|_{\mathbb{R}^n}} \right)^2
\times \left( \frac{\Delta_i^{+1} f}{\| \Delta_i^{+1} f \|_{\mathbb{R}^n}} - \frac{\Delta_{i+j}^{+1} f}{\| \Delta_{i+j}^{+1} f \|_{\mathbb{R}^n}} \right). \]

Now we use the formula

\[
a \cdot b - c \cdot d = - \frac{1}{2} \left( (a - c) \cdot (b - d) \right) \cdot (a - b + c - d)
\]

for four unit vectors $a, b, c, d$ to get

\[
I_{12} = \frac{1}{4} g_{ij,\cdot} \left( \frac{A_{i+j} \left( \| \Delta_i f \|_{\mathbb{R}^n} - \| \Delta_i^{+1} f \|_{\mathbb{R}^n} \right) }{A_{i+j} \| \Delta_i f \|_{\mathbb{R}^n} A_{i+j} \| \Delta_i^{+1} f \|_{\mathbb{R}^n}} \right)^2
\times \left( \frac{\Delta_i f}{\| \Delta_i f \|_{\mathbb{R}^n}} \cdot \frac{\Delta_j f}{\| \Delta_j f \|_{\mathbb{R}^n}} + \frac{\Delta_i^{+1} f}{\| \Delta_i^{+1} f \|_{\mathbb{R}^n}} \cdot \frac{\Delta_j^{+1} f}{\| \Delta_j^{+1} f \|_{\mathbb{R}^n}} \right)
- \frac{1}{2} \left( \frac{1}{A_{i+j} \| \Delta_i f \|_{\mathbb{R}^n} A_{i+j} \| \Delta_i^{+1} f \|_{\mathbb{R}^n}} \right)^2
\times \left( \frac{\Delta_i^{+1} f}{\| \Delta_i^{+1} f \|_{\mathbb{R}^n}} - \frac{\Delta_{i+j}^{+1} f}{\| \Delta_{i+j}^{+1} f \|_{\mathbb{R}^n}} \right). \]

As said before we may assume $1 \leq |i - j| \leq \left\lfloor \frac{m}{T} \right\rfloor$. From (4.4) we have

\[
g_{ij,\cdot} = \left( \frac{L_m}{m} \right)^2 \leq \left( \frac{L}{m} \right)^2.
\]

For sufficient large $m$

\[
g_{ij,\cdot} \geq \left( \frac{LL_1 |i - j|}{2L_2 \cdot m} \right)^2.
\]

It follows from Corollary 4.2 that

\[
\left\{ A_{i+j} \left( \| \Delta_i^{+1} f \|_{\mathbb{R}^n} - \| \Delta_i f \|_{\mathbb{R}^n} \right) \right\}^2 \leq \left\{ \frac{CL |i - j|}{m} K \left( \frac{2L}{L_2 \cdot 2m} \right) \right\}^2.
\]
Lemma 4.3 gives us the estimate

\[ A_{ij} \| \Delta_i^j f \|_{\mathbb{R}^n} A_{ij} \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} \geq \left( \frac{LL_1 |i-j| - \frac{1}{4}}{2L_2 m} \right)^2 \]

for large \( m \). We find from these estimates that

\[
\frac{1}{4} g_{ij,d} \left( \frac{1}{A_{ij} \| \Delta_i^j f \|_{\mathbb{R}^n} A_{ij} \| \Delta_{i+1}^j f \|_{\mathbb{R}^n}} \right) \frac{1}{A_{ij} \| \Delta_i^j f \|_{\mathbb{R}^n} A_{ij} \| \Delta_{i+1}^j f \|_{\mathbb{R}^n}} \left( \| \Delta_i^j f \|_{\mathbb{R}^n} - \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} \right) \cdot \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} \right) \cdot \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} \right)
\]

\[
\leq \left( \frac{CL|i-j|}{m LL_1 |i-j|} \right) K \left( \frac{2L}{L_{2m}} \right) K \left( \frac{2L}{L_{2m}} \right) \left( \frac{2L_{2m}}{LL_1 (|i-j| - \frac{1}{4})} \right)^2
\]

On the other hand 4.1 implies

\[
\left\| \frac{\Delta_i f}{\| \Delta_i^j f \|_{\mathbb{R}^n}} - \frac{\Delta_{i+1} f}{\| \Delta_{i+1}^j f \|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^n} \leq CK \left( \frac{2L}{L_{2m}} \right).
\]

Therefore

\[
\frac{1}{4} g_{ij,d} \left( \frac{1}{A_{ij} \| \Delta_i^j f \|_{\mathbb{R}^n} A_{ij} \| \Delta_{i+1}^j f \|_{\mathbb{R}^n}} \right) \frac{1}{A_{ij} \| \Delta_i^j f \|_{\mathbb{R}^n} A_{ij} \| \Delta_{i+1}^j f \|_{\mathbb{R}^n}} \left( \| \Delta_i^j f \|_{\mathbb{R}^n} - \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} \right) \cdot \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} \right) \cdot \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} \right)
\]

\[
\leq \left( \frac{CL|i-j|}{m LL_1 |i-j|} \right) K \left( \frac{2L}{L_{2m}} \right) K \left( \frac{2L}{L_{2m}} \right) \left( \frac{2L_{2m}}{LL_1 (|i-j| - \frac{1}{4})} \right)^2
\]

As a result we obtain

\[
\sum |I_{12}| \leq \sum_{i=1}^{m} \sum_{k=1}^{[\frac{m}{4}]} \frac{C}{(k-\frac{1}{4})^2} \left( \frac{2L}{L_{2m}} \right)^2 \to 0 \ (m \to \infty).
\]

\[ \square \]
Lemma 4.6

To estimate \( \sum_{i \neq j} I_2 \), we decomposed \( I_2 \) into

\[
I_2 = I_{21} + I_{22},
\]

\[
I_{21} = -\frac{1}{2} A_{ij} \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_i f \|_{\mathbb{R}^n} \right)
\]

\[
\times \left\{ \left( \Delta_i A_j \| \Delta_i^j f \|_{\mathbb{R}^n} \right) \left( \Delta_j A_i \| \Delta_j^i f \|_{\mathbb{R}^n} \right) - \frac{\left( \Delta_i A_j \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} \right) \left( \Delta_j A_i \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} \right)}{g_{ij,n}} \right\},
\]

\[
I_{22} = -\frac{1}{2} \left\{ \Delta_i A_j \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_i^j f \|_{\mathbb{R}^n} \right) \right\} \left\{ \Delta_j A_i \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_i^j f \|_{\mathbb{R}^n} \right) \right\} \left( \frac{A_{ij} \| \Delta_i^j f \|_{\mathbb{R}^n}}{g_{ij,n}} \right).
\]

Here we use (4.3). To estimate each part, we need the following lemma.

**Lemma 4.6** Assume that \( m \) is sufficient large. When \( |i - j| \leq \left\lceil \frac{m}{2} \right\rceil \),

\[
\left| \Delta_j A_i \left( \| \Delta_i^j f \|_{\mathbb{R}^n} - \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} \right) \right| \leq \frac{CL}{m} K \left( \hat{f}, \frac{2L}{2m} \right)
\]

holds.

**Proof.** We have

\[
\Delta_j A_i \left( \| \Delta_i^j f \|_{\mathbb{R}^n} - \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} \right) = A_i \left[ A_j \left( \| \Delta_i^j f - \Delta_{i+1}^j f \|_{\mathbb{R}^n} \right) \cdot \Delta_j f + \frac{A_j \Delta_{i+1}^j f \cdot (\Delta_j f - \Delta_{j+1} f)}{A_j \| \Delta_i^j f \|_{\mathbb{R}^n}} \right]
\]

We want to estimate each term.

It holds that

\[
\Delta_i A_j \| \Delta_i^j f \|_{\mathbb{R}^n} = A_j \left( \frac{A_i \Delta_i^j f \cdot \Delta_i f}{A_i \| \Delta_i^j f \|_{\mathbb{R}^n}} \right) = A_j \left( \frac{A_i \Delta_i^j f}{A_i \| \Delta_i^j f \|_{\mathbb{R}^n}} \right) \cdot \Delta_i f.
\]

Since

\[
\left\| \frac{A_i \Delta_i^j f}{A_i \| \Delta_i^j f \|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^n} \leq \frac{\| \Delta_i^j f + \Delta_{i+1}^j f \|_{\mathbb{R}^n}}{\| \Delta_i^j f \|_{\mathbb{R}^n} + \| \Delta_{i+1}^j f \|_{\mathbb{R}^n}} \leq 1,
\]

we get

\[
\left| \Delta_i A_j \| \Delta_i^j f \|_{\mathbb{R}^n} \right| \leq \| \Delta_i f \|_{\mathbb{R}^n}.
\]

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Corollary 4.1 yields
\[ \| \Delta_{i+1} f - \Delta_i f \|_{\mathbb{R}^n} \leq \frac{CL|i-j|}{m} K \left( \dot{f}, \frac{2L}{L_2m} \right), \]
and hence
\[ \left\| A_j \left( \Delta_{i+1} f - \Delta_i f \right) \right\|_{\mathbb{R}^n} \leq \frac{CL|i-j|}{m} + \frac{1}{2} K \left( \dot{f}, \frac{2L}{L_2m} \right) \leq \frac{CL|i-j|}{m} K \left( \dot{f}, \frac{2L}{L_2m} \right). \]

Here we use \( \frac{1}{2} < |i-j| \), which is obvious from \( |i-j| \geq 1 \). Consequently we obtain
\[
\left\| A_i \left[ \frac{A_j \left( \Delta_i f - \Delta_{i+1} f \right)}{A_j \| \Delta_i f \|_{\mathbb{R}^n}} \cdot \Delta_j f \right] \right\|_{\mathbb{R}^n}
\leq A_i \left( \frac{CL|i-j|}{m} K \left( \dot{f}, \frac{2L}{L_2m} \right) \frac{1}{A_j \| \Delta_i f \|_{\mathbb{R}^n}} \right) \| \Delta_j f \|_{\mathbb{R}^n}
\leq \frac{CL^2}{m^2} K \left( \dot{f}, \frac{2L}{L_2m} \right) A_i \left( \frac{|i-j|}{A_j \| \Delta_i f \|_{\mathbb{R}^n}} \right).
\]

Lemma 4.3 shows
\[ A_i \left( \frac{|i-j|}{A_j \| \Delta_i f \|_{\mathbb{R}^n}} \right) \leq \frac{L_2 m |(i-j) + \frac{1}{2} |}{L L_1 (|i-j|-\frac{1}{2})} \leq \frac{Cm}{L}. \]

Therefore we have
\[
\left\| A_i \left[ \frac{A_j \left( \Delta_i f - \Delta_{i+1} f \right)}{A_j \| \Delta_i f \|_{\mathbb{R}^n}} \cdot \Delta_j f \right] \right\|_{\mathbb{R}^n} \leq \frac{CL m}{L} K \left( \dot{f}, \frac{2L}{L_2m} \right).
\]

Since
\[ \| \Delta_{i+1} f \|_{\mathbb{R}^n} \leq \frac{L|i-j|}{m}, \quad \| \Delta_{i+2} f \|_{\mathbb{R}^n} \leq \frac{L(|i-j|+1)}{m} \leq \frac{2L|i-j|}{m} \]
for \( |i-j| \leq \left[ \frac{m}{2} \right] \), we have
\[ A_j \| \Delta_{i+1} f \|_{\mathbb{R}^n} \leq \frac{CL|i-j|}{m}.
\]

By Lemma 4.2 we get
\[ \| \Delta_j f - \Delta_{j+1} f \|_{\mathbb{R}^n} \leq \frac{CL m}{L} K \left( \dot{f}, \frac{2L}{L_2m} \right). \]
Consequently it holds that
\[
\left\| A_i \left( \frac{A_j \Delta_{j+1}^i f \cdot (\Delta_i f - \Delta_{j+1}^i f)}{A_j \| \Delta_i^i f \|_{\mathbb{R}^n}} \right) \right\|_{\mathbb{R}^n} \\
\leq A_i \left( \frac{CL \| i - j \| CL}{m} K \left( \frac{\dot{f}}{2Lm}, \frac{2L}{L_{2m}} \right) \frac{1}{A_j \| \Delta_i^i f \|_{\mathbb{R}^n}} \right) \\
\leq \frac{CL}{m} K \left( \frac{\dot{f}}{2Lm} \right).
\]

The estimate
\[
\left\| A_i \left[ \left\{ \frac{A_j \left( \| \Delta_{j+1}^i f \|_{\mathbb{R}^n} - \| \Delta_i^i f \|_{\mathbb{R}^n} \right) \right\} A_j \Delta_{j+1}^i f \cdot \Delta_{j+1}^i f \right] \left( A_j \| \Delta_i^i f \|_{\mathbb{R}^n} \right) \left( A_j \| \Delta_{j+1}^i f \|_{\mathbb{R}^n} \right) \right\|_{\mathbb{R}^n} \\
\leq A_i \left( \left\{ \frac{CL \| i - j \|}{m} K \left( \frac{\dot{f}}{2Lm} \right) \right\} \frac{\| A_j \Delta_{j+1}^i f \|_{\mathbb{R}^n} \| \Delta_{j+1}^i f \|_{\mathbb{R}^n}}{A_j \| \Delta_i^i f \|_{\mathbb{R}^n} \left( A_j \| \Delta_{j+1}^i f \|_{\mathbb{R}^n} \right)} \right) \\
\leq \frac{CL^3}{m^3} K \left( \frac{\dot{f}}{2Lm} \right) A_i \left( \frac{\| i - j \|^2}{A_j \| \Delta_i^i f \|_{\mathbb{R}^n} \left( A_j \| \Delta_{j+1}^i f \|_{\mathbb{R}^n} \right)} \right)
\]
can be shown similarly. Hence we have
\[
A_i \left( \frac{\| i - j \|^2}{A_j \| \Delta_i^i f \|_{\mathbb{R}^n} \left( A_j \| \Delta_{j+1}^i f \|_{\mathbb{R}^n} \right)} \right) \\
\leq \left\{ \left[ \frac{m \| i - j \|}{L \left( \| i - j \| - \frac{1}{2} \right)} \right]^2 + \left\{ \frac{m \| i - j \|}{L \left( \| i - j \| - \frac{1}{2} \right)} \right\}^2 \right\} \leq C \left( \frac{m}{L} \right)^2.
\]

From this we arrive at
\[
\left\| A_i \left[ \left\{ A_{ij} \left( \| \Delta_{j+1}^i f \|_{\mathbb{R}^n} - \| \Delta_i^i f \|_{\mathbb{R}^n} \right) \right\} \left( \Delta_j A_j \| \Delta_{j+1}^i f \|_{\mathbb{R}^n} \right) \left( \Delta_j A_j \| \Delta_{j+1}^i f \|_{\mathbb{R}^n} \right) \right] \right\|_{\mathbb{R}^n} \\
\leq \frac{CL}{m} K \left( \frac{\dot{f}}{2Lm} \right).
\]

\[\square\]

**Lemma 4.7** Suppose that \( f \in H^2(\mathbb{R}/\mathbb{Z}), \ \tau \in H^2(\mathbb{R}/\mathbb{Z}), \ (4.4), \ and \ (4.5). \)
Furthermore we assume
\[
\lim_{\varepsilon \to +0} \varepsilon^{-1} K(\dot{f}, \varepsilon)^2 = 0.
\]

Then we have
\[
\lim_{m \to \infty} \sum I_{21} = \lim_{m \to \infty} \sum I_{22} = 0.
\]
Proof. Terms in braces of $I_{21}$ can be rewritten as

$$
\left( \Delta_i A_j \| \Delta_i^1 f \|_{\mathbb{R}^n} \right) \left( \Delta_i A_i \| \Delta_i^1 f \|_{\mathbb{R}^n} \right) - \left( \Delta_i A_j \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right) \left( \Delta_j A_i \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right)
$$

$$
\begin{aligned}
&= \frac{\left( \Delta_i A_j \| \Delta_i^1 f \|_{\mathbb{R}^n} \right) \left( \Delta_j A_i \left( \| \Delta_i^1 f \|_{\mathbb{R}^n} - \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right) \right)}{A_{ij} \| \Delta_i^1 f \|_{\mathbb{R}^n}} \\
&+ \frac{\left\{ A_{ij} \left( \| \Delta_i^1 f \|_{\mathbb{R}^n} - \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right) \right\} \left( \Delta_j A_i \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right)}{A_{ij} \| \Delta_i^1 f \|_{\mathbb{R}^n}} \\
&+ \frac{\left\{ A_{ij} \left( \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} - \| \Delta_i^1 f \|_{\mathbb{R}^n} \right) \right\} \left( \Delta_j A_i \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right)}{A_{ij} \| \Delta_i^{j+1} f \|_{\mathbb{R}^n}}.
\end{aligned}
$$

It follows from Lammass $4.2, 4.3$ and

$$
\left| \Delta_i A_j \| \Delta_i^1 f \|_{\mathbb{R}^n} \right| \leq \| \Delta_i f \|_{\mathbb{R}^n} = \frac{L}{m}
$$

that

$$
\left| \frac{\left( \Delta_i A_j \| \Delta_i^1 f \|_{\mathbb{R}^n} \right) \left( \Delta_j A_i \left( \| \Delta_i^1 f \|_{\mathbb{R}^n} - \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right) \right)}{A_{ij} \| \Delta_i^1 f \|_{\mathbb{R}^n}} \right| \leq \frac{CL}{m(|i-j| - \frac{1}{4}^2)} K \left( \hat{f}, \frac{2L}{L_{2m}} \right).
$$

Similarly we have

$$
\left| \frac{\left\{ A_{ij} \left( \| \Delta_i^1 f \|_{\mathbb{R}^n} - \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right) \right\} \left( \Delta_j A_i \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right)}{A_{ij} \| \Delta_i^1 f \|_{\mathbb{R}^n}} \right| \leq \frac{CL}{m(|i-j| - \frac{1}{4})^2} K \left( \hat{f}, \frac{2L}{L_{2m}} \right).
$$

Consequently we have

$$
\left| \frac{\left\{ A_{ij} \left( \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} - \| \Delta_i^1 f \|_{\mathbb{R}^n} \right) \right\} \left( \Delta_j A_i \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} \right)}{A_{ij} \| \Delta_i^{j+1} f \|_{\mathbb{R}^n}} \right| \leq \frac{CL}{m(|i-j| - \frac{1}{4})^2} K \left( \hat{f}, \frac{2L}{L_{2m}} \right),
$$

and then

$$
|I_{21}| \leq \frac{1}{2} \left| \frac{A_{ij} \left( \| \Delta_i^{j+1} f \|_{\mathbb{R}^n} - \| \Delta_i^1 f \|_{\mathbb{R}^n} \right)}{g_{ij,n}} \right| \frac{CL}{m(|i-j| - \frac{1}{4})^2} K \left( \hat{f}, \frac{2L}{L_{2m}} \right)
$$

$$
\leq \frac{CL}{m} \left( \hat{f}, \frac{2L}{L_{2m}} \right)^2 \left( \frac{m}{L_{2m}} \right)^2 \frac{1}{(|i-j| - \frac{1}{4})}
$$

$$
\leq \frac{C}{(|i-j| - \frac{1}{4})^3} K \left( \hat{f}, \frac{2L}{L_{2m}} \right)^2.
$$
Thus we know
\[ \sum_{i \neq j} |I_{21}| \leq \sum_{i=1}^{m} \sum_{k=1}^{[\frac{m}{2}]} \frac{C}{(k - \frac{1}{4})^3} K \left( f, \frac{2L}{L_{2m}} \right)^2 \]
\[ \leq C m K \left( f, \frac{2L}{L_{2m}} \right)^2 \rightarrow 0 \quad (m \to \infty). \]
Similarly we have
\[ |I_{22}| \leq \frac{1}{2} \frac{1}{g_{ij,n}} \left\{ \frac{C L}{m} K \left( f, \frac{2L}{L_{2m}} \right) \right\}^2 \leq \frac{C}{(|i-j|-\frac{1}{4})^2} K \left( \tau, \frac{L}{L_{2m}} \right)^2, \]
and
\[ \sum_{i \neq j} |I_{22}| \to 0 \quad (m \to \infty). \]

□

Next we show the convergence of \( \sum_{i \neq j} P_{m}^{ij}(f) \) and \( \sum_{i \neq j} R_{m}^{ij}(f) \). Note that we have \( J_2 = J_3 = J_4 = J_5 = 0 \) under (4.5), and hence
\[ P_{m}^{ij}(f) = J_{11}, \quad R_{m}^{ij}(f) = J_{12} + J_{13} + J_{14} + J_{15}. \]

**Lemma 4.8** Assume that \( f \in W^{2,\infty}(\mathbb{R}/\mathbb{Z}) \) with (4.4), and (4.5). Then it holds that
\[ \lim_{m \to \infty} \sum_{i \neq j} P_{m}^{ij}(f) = \mathcal{E}(f) - 4, \quad \lim_{m \to \infty} \sum_{i \neq j} R_{m}^{ij}(f) = 0. \]

**Proof.** The proof is based on the argument similar to those of Lemma 4.4. It is not difficult to show
\[ \sum_{i,j} P_{m}^{ij}(f) \chi_{ij}(\vartheta_1, \vartheta_2) \]
\[ \to \frac{1}{2} \left[ \left\| \frac{f(\vartheta_1) - f(\vartheta_2)}{\| f(\vartheta_1) - f(\vartheta_2) \|_{\mathbb{R}^n}} \wedge (\tau(\vartheta_1) + \tau(\vartheta_2)) \right\|_{\Lambda^2 \mathbb{R}^n}^2 \right. \]
\[ + \left\{ \frac{f(\vartheta_1) - f(\vartheta_2)}{\| f(\vartheta_1) - f(\vartheta_2) \|_{\mathbb{R}^n}} \cdot (\tau(\vartheta_1) - \tau(\vartheta_2)) \right\}^2 \frac{\| \hat{f}(\vartheta_1) \|_{\mathbb{R}^n} \| \hat{f}(\vartheta_2) \|_{\mathbb{R}^n}}{\| \hat{f}(\vartheta_1) - \hat{f}(\vartheta_2) \|_{\mathbb{R}^n}^2}, \]
\[ \sum_{i,j} R_{m}^{ij}(f) \chi_{ij}(\vartheta_i, \vartheta_j) \to 0 \]
\[ 29 \]
as \( m \to \infty \) for a.e. \((\vartheta_1, \vartheta_2) \in (\mathbb{R}/\mathbb{Z})^2\). The first one is the energy density of \( \mathcal{E}(f) - 4 \). To apply Lebesgue’s theorem we will show uniform boundedness. Let \( i < j \leq \frac{\pi}{\alpha} \). Then

\[
\frac{\Delta_i^4 f}{\|\Delta_i^4 f\|_{\mathbb{R}^n}} - \frac{j - i}{2}\frac{L_m}{\|\Delta_i^4 f\|_{\mathbb{R}^n}} \frac{L_m}{m} \left( \frac{\Delta_i f}{\|\Delta_i f\|_{\mathbb{R}^n}} + \frac{\Delta_j f}{\|\Delta_j f\|_{\mathbb{R}^n}} \right)
\]

\[
= \frac{1}{\|\Delta_i^4 f\|_{\mathbb{R}^n}} \sum_{k=i}^{j-1} \left\{ \Delta_k f - \frac{1}{2} (\Delta_i f + \Delta_j f) \right\}
\]

\[
= \frac{1}{2\|\Delta_i^4 f\|_{\mathbb{R}^n}} \sum_{k=i}^{j-1} \left\{ (\Delta_k f - \Delta_i f) + (\Delta_k f - \Delta_j f) \right\}.
\]

Therefore we have

\[
\left\| \frac{\Delta_i^4 f}{\|\Delta_i^4 f\|_{\mathbb{R}^n}} - \frac{j - i}{2}\frac{L_m}{\|\Delta_i^4 f\|_{\mathbb{R}^n}} \frac{L_m}{m} \left( \frac{\Delta_i f}{\|\Delta_i f\|_{\mathbb{R}^n}} + \frac{\Delta_j f}{\|\Delta_j f\|_{\mathbb{R}^n}} \right) \right\|_{\mathbb{R}^n} \leq \frac{C}{|i - j|} \sum_{k=i}^{j-1} \{|k - i| + |k - j| \} K \left( f, \frac{2L}{L_{2m}} \right) \]

\[
\leq C|i - j| K \left( f, \frac{2L}{L_{2m}} \right).
\]

A similar estimate holds for \( i - \frac{m}{\alpha} \leq j < i \) also. Therefore it holds that

\[
\left\| \frac{\Delta_i^4 f}{\|\Delta_i^4 f\|_{\mathbb{R}^n}} \wedge \left( \frac{\Delta_i f}{\|\Delta_i f\|_{\mathbb{R}^n}} + \frac{\Delta_j f}{\|\Delta_j f\|_{\mathbb{R}^n}} \right) \right\|_{\wedge^2 \mathbb{R}^n}^2 \leq \frac{C}{|i - j|^2} K \left( f, \frac{2L}{L_{2m}} \right)^2.
\]

On the other hand, we have

\[
\left\{ \frac{\Delta_i f}{\|\Delta_i f\|_{\mathbb{R}^n}} \cdot \left( \frac{\Delta_i f}{\|\Delta_i f\|_{\mathbb{R}^n}} - \frac{\Delta_j f}{\|\Delta_j f\|_{\mathbb{R}^n}} \right) \right\}^2 \leq C|i - j|^2 K \left( f, \frac{2L}{L_{2m}} \right)^2.
\]

Hence the estimate

\[
|J_{11}| \leq \frac{C}{|i - j|^2} |i - j|^2 K \left( f, \frac{2L}{L_{2m}} \right)^2 = CK \left( f, \frac{2L}{L_{2m}} \right)^2
\]

follows from (4.3). This implies that we can apply Lebesgue’s convergence theorem to \( \sum_{i \neq j} J_{11} \), and we find that it converges to \( \mathcal{E}(f) - 4 \) as \( m \to \infty \).
Now we estimate each term of $R^m_{ij}(f)$. We begin with
\[
\frac{1}{g_{ij,n}} - \frac{1}{\bar{g}_{ij,n}} = \frac{1}{\|\Delta_i^j f\|_{\mathbb{R}^n}} \left( \frac{1}{\|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n}} - \frac{1}{\|\Delta_i^j f\|_{\mathbb{R}^n}} \right) = \frac{\|\Delta_i^j f\|_{\mathbb{R}^n} - \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n}}{\|\Delta_i^j f\|_{\mathbb{R}^n} \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n}}.
\]
It follows from the proof of Corollary 4.2 that
\[
\left| \frac{1}{g_{ij,n}} - \frac{1}{\bar{g}_{ij,n}} \right| \leq \left( \frac{L}{m} \right)^{-2} C |i - j|^3 K \left( \bar{f}, \frac{2L}{L_2 m} \right) = \left( \frac{L}{m} \right)^{-2} C \frac{2L}{|i - j|^2} K \left( \bar{f}, \frac{2L}{L_2 m} \right).
\]
Hence we can estimate $J_{12}$ as
\[
|J_{12}| \leq \frac{C}{|i - j|^2} K \left( \bar{f}, \frac{2L}{L_2 m} \right).
\]
We decompose $J_{13}$ into
\[
J_{13} = J_{131} + J_{132},
\]
\[
J_{131} = -\frac{1}{2} \left( \frac{1}{g_{ij,n}} - \frac{1}{\bar{g}_{ij,n}} \right) \Delta_i \Delta_j g_{ij,n},
\]
\[
J_{132} = -\frac{1}{\bar{g}_{ij,n}} \Delta_i \Delta_j \left( g_{ij,n} - \bar{g}_{ij,n} \right).
\]
From
\[
\Delta_i \Delta_j g_{ij,n} = -\Delta_i f \cdot \Delta_j f - \Delta_{i+1} f \cdot \Delta_{j+1} f
\]
\[
- A_{ij} \left( \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_j^i f\|_{\mathbb{R}^n} \right) \left( \Delta_i f \cdot \Delta_j f \right)
\]
\[
+ \frac{A_{ij} \left( \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_j^i f\|_{\mathbb{R}^n} \right)}{\|\Delta_j^i f\|_{\mathbb{R}^n}} \left( \Delta_{i+1} f \cdot \Delta_{j+1} f \right)
\]
\[
- \frac{A_{ij} \left( \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_j^i f\|_{\mathbb{R}^n} \right)}{A_{ij} \|\Delta_j^i f\|_{\mathbb{R}^n}} \left( \Delta_i A_j \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n} \right) \left( \Delta_i A_i \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n} \right)
\]
\[
+ \frac{A_{ij} \left( \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_j^i f\|_{\mathbb{R}^n} \right)}{A_{ij} \|\Delta_j^i f\|_{\mathbb{R}^n}} \left( \Delta_i A_j \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n} \right) \left( \Delta_j A_i \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n} \right)
\]
\[
- \left\{ \Delta_i A_j \left( \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_j^i f\|_{\mathbb{R}^n} \right) \right\} \left\{ \Delta_j A_i \left( \|\Delta_{i+1}^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_j^i f\|_{\mathbb{R}^n} \right) \right\}
\]
and \[
|\Delta_i \Delta_j \|\Delta_j^{i} f\|_{\mathbb{R}^n}| \leq \frac{L}{m} \text{ we find}
\]
\[
|\Delta_i \Delta_j g_{ij,n}| \leq C \left( \frac{L}{m} \right)^2.
\]
Hence we can estimate $J_{131}$ as
\[
|J_{131}| \leq \frac{C}{|i - j|^2} K \left( \frac{\bar{f}}{L_2 m} \right).
\]
Since
\[
g_{ij,n} - \bar{g}_{ij,n} = \|\Delta_i^j f\|_{\mathbb{R}^n} \left( \|\Delta_i^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_i^j f\|_{\mathbb{R}^n} \right),
\]
we obtain
\[
\Delta_j (g_{ij,n} - \bar{g}_{ij,n}) = \left( \Delta_j \|\Delta_i^j f\|_{\mathbb{R}^n} \right) A_j \left( \|\Delta_i^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_i^j f\|_{\mathbb{R}^n} \right) + \left( A_j \|\Delta_i^j f\|_{\mathbb{R}^n} \right) \Delta_j \left( \|\Delta_i^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_i^j f\|_{\mathbb{R}^n} \right),
\]
and
\[
\Delta_i \Delta_j (g_{ij,n} - \bar{g}_{ij,n}) = \left( \Delta_i \Delta_j \|\Delta_i^j f\|_{\mathbb{R}^n} \right) A_{ij} \left( \|\Delta_i^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_i^j f\|_{\mathbb{R}^n} \right) + \left( \Delta_i A_i \|\Delta_i^j f\|_{\mathbb{R}^n} \right) \Delta_i A_j \left( \|\Delta_i^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_i^j f\|_{\mathbb{R}^n} \right) + \left( A_i \|\Delta_i^j f\|_{\mathbb{R}^n} \right) \Delta_i A_j \left( \|\Delta_i^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_i^j f\|_{\mathbb{R}^n} \right).
\]
We have already known
\[
\left| A_{ij} \left( \|\Delta_i^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_i^j f\|_{\mathbb{R}^n} \right) \right| \leq L \frac{C}{m |i - j|}
\]
and
\[
\left| A_i \|\Delta_i^j f\|_{\mathbb{R}^n} \right| \leq C \frac{L}{m},
\]
and
\[
\left| A_i \|\Delta_i^j f\|_{\mathbb{R}^n} \right| \leq C \frac{L}{m}.
\]
Furthermore we have
\[
\Delta_i \left( \|\Delta_i^{j+1} f\|_{\mathbb{R}^n} - \|\Delta_i^j f\|_{\mathbb{R}^n} \right) = - \frac{A_i \Delta_i^{j+1} f \cdot \Delta_i^{j+1} f}{A_i \|\Delta_i^{j+1} f\|_{\mathbb{R}^n}} + \frac{A_i \Delta_i^j f \cdot \Delta_i f}{A_i \|\Delta_i^j f\|_{\mathbb{R}^n}}
\]
and
\[
= \frac{A_i \left( \Delta_i^{j+1} f - \Delta_i^j f \right) \cdot \Delta_i^{j+1} f}{A_i \|\Delta_i^{j+1} f\|_{\mathbb{R}^n}} - \frac{A_i \Delta_i^j f \cdot \left( \Delta_i^{j+1} f - \Delta_i^j f \right)}{A_i \|\Delta_i^j f\|_{\mathbb{R}^n}}
\]
and
\[
= \frac{1}{A_i \|\Delta_i^{j+1} f\|_{\mathbb{R}^n}} - \frac{1}{A_i \|\Delta_i^j f\|_{\mathbb{R}^n}} \right) A_i \Delta_i^j f \cdot \Delta_i^{j+1} f.
\]
From estimates

\[
\left| \frac{A_i (\Delta_{i+1}^j f - \Delta_i^j f) \cdot \Delta_{i+1} f}{A_i \| \Delta_{i+1}^j f \|_{\mathbb{R}^n}} \right| \leq C \frac{L}{m} \frac{|i - j| K \left( \bar{f}, \frac{2L}{L_{2m}} \right)}{|i - j|} = C \frac{L}{m} K \left( \bar{f}, \frac{2L}{L_{2m}} \right),
\]

\[
\left| \frac{A_i \Delta_i^2 f \cdot (\Delta_{i+1} f - \Delta_i f)}{A_i \| \Delta_i^j f \|_{\mathbb{R}^n}} \right| \leq C \| \Delta_{i+1} f - \Delta_i f \|_{\mathbb{R}^n} \leq C \frac{L}{m} K \left( \bar{f}, \frac{2L}{L_{2m}} \right),
\]

we obtain

\[
\left| \Delta_i \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_i^j f \|_{\mathbb{R}^n} \right) \right| \leq C \frac{L}{m} K \left( \bar{f}, \frac{2L}{L_{2m}} \right).
\]

Since

\[
\Delta_i \Delta_j \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_i^j f \|_{\mathbb{R}^n} \right)
= - \frac{1}{2} \frac{(\Delta_{i+1} f - \Delta_i f) \cdot (\Delta_{j+1} f + \Delta_j f) + (\Delta_{i+1} f - \Delta_i f) \cdot (\Delta_{j+1} f - \Delta_j f)}{A_i \| \Delta_{i+1}^j f \|_{\mathbb{R}^n}}
+ \left( \frac{1}{A_{ij} \| \Delta_i^j f \|_{\mathbb{R}^n}} - \frac{1}{A_{ij} \| \Delta_{i+1}^j f \|_{\mathbb{R}^n}} \right) \Delta_i f \cdot \Delta_j f
- \frac{1}{2} \left\{ \Delta_i A_j \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_i^j f \|_{\mathbb{R}^n} \right) \right\} \left\{ \Delta_j A_i \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} + \| \Delta_i^j f \|_{\mathbb{R}^n} \right) \right\}
- \frac{1}{2} \left\{ \Delta_i A_j \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} + \| \Delta_i^j f \|_{\mathbb{R}^n} \right) \right\} \left\{ \Delta_j A_i \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_i^j f \|_{\mathbb{R}^n} \right) \right\}
+ \left( \frac{1}{A_{ij} \| \Delta_i^j f \|_{\mathbb{R}^n}} - \frac{1}{A_{ij} \| \Delta_{i+1}^j f \|_{\mathbb{R}^n}} \right) \left( \Delta_i A_j \| \Delta_i^j f \|_{\mathbb{R}^n} \right) \left( \Delta_j A_i \| \Delta_i^j f \|_{\mathbb{R}^n} \right),
\]

it holds that

\[
\left| \Delta_i \Delta_j \left( \| \Delta_{i+1}^j f \|_{\mathbb{R}^n} - \| \Delta_i^j f \|_{\mathbb{R}^n} \right) \right| \leq \frac{L}{m} \frac{C}{|i - j| K \left( \bar{f}, \frac{2L}{L_{2m}} \right)}.
\]
Combining estimates above, we get

$$|J_{132}| \leq \frac{C}{|i-j|^2} K \left( \hat{f}, \frac{2L}{L^2 m} \right).$$

To estimate $J_{14}$, it is decomposed into 4 parts as

$$J_{14} = J_{141} + J_{142},$$

$$J_{141} = \frac{1}{2} \left( g_{ij,n} - g_{j,n} \right) \left( \Delta_i g_{ij,n} \right) \left( \Delta_j g_{ij,n} \right),$$

$$J_{142} = \frac{1}{2} \left( \Delta_i g_{ij,n} \right) \left( \Delta_j g_{ij,n} - g_{ij,n} \right),$$

$$J_{143} = \frac{1}{2} \left( \Delta_i g_{ij,n} \right) \left( \Delta_j g_{ij,n} \right),$$

$$J_{144} = \frac{1}{2} \left( \Delta_i g_{ij,n} \right) \left( \Delta_j g_{ij,n} \right) + 4 \left( \Delta_i f \cdot \Delta_j f \right) \left( \Delta_i f \cdot \Delta_j f \right) - \frac{1}{2} \left( \Delta_i g_{ij,n} \right) \left( \Delta_j g_{ij,n} \right).$$

Since

$$\frac{1}{g_{ij,n}^2} - \frac{1}{g_{j,n}^2} = \frac{1}{\|\Delta f\|^3_{\mathbb{R}^n}} \left( \left\|\Delta f_{i+1} \right\|_{\mathbb{R}^n} - \left\|\Delta f_i \right\|_{\mathbb{R}^n} \right),$$

we have

$$\left| \frac{1}{g_{ij,n}^2} - \frac{1}{g_{j,n}^2} \right| \leq \left( \frac{L}{m} \right)^{-4} C |i-j| K \left( \hat{f}, \frac{2L}{L^2 m} \right) = \left( \frac{L}{m} \right)^{-4} C K \left( \hat{f}, \frac{2L}{L^2 m} \right).$$

The relation

$$\Delta_i g_{ij,n} = -\frac{A_i}{A_i \|\Delta f\|_{\mathbb{R}^n}} A_i \|\Delta f_{i+1}\|_{\mathbb{R}^n} + A_i \|\Delta f\|_{\mathbb{R}^n}$$

implies

$$|\Delta_i g_{ij,n}| \leq C \left( \frac{L}{m} \right)^{2} |i-j|. $$

Hence we obtain the estimate

$$|J_{141}| \leq \frac{C K \left( \hat{f}, \frac{2L}{L^2 m} \right)}{|i-j|^2}.$$
that

\[ |\Delta_i (g_{ij,n} - \bar{g}_{ij,n})| \leq C \left( \frac{L}{m} \right)^2 |i - j| K \left( \frac{\hat{f}}{L_{2m}} \right). \]

Consequently

\[ |J_{142}| \leq C \left( \frac{L}{m} \right)^4 |i - j|^{-4} \left( \frac{L}{m} \right)^2 |i - j| K \left( \frac{\hat{f}}{L_{2m}} \right) \left( \frac{L}{m} \right)^2 |i - j| = \frac{CK \left( \frac{\hat{f}}{L_{2m}} \right)}{|i - j|^2} \]

holds. We have

\[ |J_{143}| \leq C \left( \frac{L}{m} \right)^4 |i - j|^{-4} \left( \frac{L}{m} \right)^2 |i - j| K \left( \frac{\hat{f}}{L_{2m}} \right) \left( \frac{L}{m} \right)^2 |i - j| = \frac{CK \left( \frac{\hat{f}}{L_{2m}} \right)}{|i - j|^2} \]

from

\[ \Delta_i g_{ij,n} = -2A_i \Delta_i^j f \cdot \Delta_i f. \]

Now we use

\[ \frac{1}{4} (\Delta_i \bar{g}_{ij,n}) (\Delta_j \bar{g}_{ij,n}) + \left( \Delta_i^j f \cdot \Delta_i f \right) \left( \Delta_i^j f \cdot \Delta_j f \right) \]

\[ = \frac{1}{4} \left( \frac{L_m}{m} \right)^2 \left\{ \left( \frac{L_m}{m} \right)^2 + 2\Delta_i^j f \cdot (\Delta_j f - \Delta_i f) \right\} \]

to estimate \( J_{144} \):

\[ \sum_{i \neq j} J_{144} = \frac{1}{2} \sum_{i \neq j} \left[ \frac{1}{\bar{g}_{ij,n}^2} \left( \frac{L_m}{m} \right)^2 \left\{ \left( \frac{L_m}{m} \right)^2 + 2\Delta_i^j f \cdot (\Delta_j f - \Delta_i f) \right\} \right] - \frac{1}{\bar{g}_{ij,n}^2}. \]

Let \( i < j \leq i + \frac{m}{2} \). Then we have

\[ \Delta_i^j f \cdot (\Delta_i f - \Delta_i f) \]

\[ = \sum_{k=i}^{j-1} \Delta_k f \cdot \sum_{\ell=i}^{j-1} \Delta_\ell f \]

\[ = \sum_{k=i}^{j-1} \sum_{\ell=i}^{j-1} \left\{ \Delta_k f - \frac{1}{2} (\Delta_\ell f + \Delta_{\ell+1} f) \right\} \cdot \Delta_\ell f \]

\[ = \frac{1}{2} \sum_{k=i}^{j-1} \sum_{\ell=i}^{j-1} \left\{ (\Delta_k f - \Delta_\ell f) + (\Delta_k f - \Delta_{\ell+1} f) \right\} \cdot \Delta_\ell f \]

\[ = \frac{1}{2} \sum_{k=i}^{j-1} \sum_{\ell=i}^{j-1} \left\{ \text{sgn}(k - \ell) \sum_{p=\min\{k,\ell\}}^{\max\{k,\ell\}-1} \Delta_p f + \text{sgn}(k - \ell - 1) \sum_{p=\min\{k,\ell+1\}}^{\max\{k,\ell+1\}-1} \Delta_p f \right\} \cdot \Delta_\ell f. \]

Similar relation holds for \( i - \frac{m}{2} \leq j < i \). Using

\[ \|\Delta_q \Delta_q f\|_{R^n} = \|\Delta_q \Delta_q f - \Delta_{q+1} f - \Delta_q f + \Delta_q f\|_{R^n} \leq \frac{CL}{m} K \left( \frac{\hat{f}}{L_{2m}} \right), \]

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We have
\[ \| \Delta_j f \cdot (\Delta_j - \Delta_i) \|_{R^n} \leq C \left( \frac{L}{m} \right)^2 |i - j|^3 K \left( \frac{2L}{L_{2m}} \right)^2. \]

Consequently we obtain the decay
\[ \sum_{i \neq j} \frac{1}{g_{ij,n}} \left( \frac{L_m}{m} \right)^2 \| \Delta_i f \cdot (\Delta_j - \Delta_i) \|_{R^n} \leq \sum_{i \neq j} C |i - j|^3 \left( \frac{2L}{L_{2m}} \right)^2 \]
\[ \leq C m \log m K \left( \frac{2L}{L_{2m}} \right)^2 \to 0 \quad (m \to \infty). \]

Furthermore we have
\[ \left| \sum_{i \neq j} \left\{ \frac{1}{2g_{ij,n}} \left( \frac{L_m}{m} \right)^4 \frac{1}{2g_{ij}} \right\} \right| = \frac{1}{2} \sum_{i \neq j} \left( \frac{L_m}{m} \right)^4 \left( \frac{1}{g_{ij,n}} - \frac{1}{g_{ij}} \right) \]
\[ \leq C \sum_{i \neq j} K \left( \frac{2L}{L_{2m}} \right)^2 \leq C mK \left( \frac{2L}{L_{2m}} \right). \]

If \( f \in W^{1,1}(\mathbb{R}/\mathbb{Z}) \), then \( mK \left( \frac{2L}{L_{2m}} \right) \) is bounded uniformly on \( m \). Thus Lebesgue’s convergence theorem is applicable to
\[ \sum_{i \neq j} (J_{12} + J_{13} + J_{141} + J_{142} + J_{143} + J_{144}), \]
and we find that it converges to 0 as \( m \to \infty \). \( \square \)

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