THEORETICAL STATUS OF THE MUON $g - 2 \alpha$

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We review the present status of the theoretical evaluation of the anomalous magnetic moment of the the muon within the Standard Model. We mainly focus on the hadronic contributions in the muon $g - 2$ due to vacuum polarization effects, light-by-light scattering and higher order electroweak corrections. We discuss some recent calculations together with their uncertainties and limitations and point out possible improvements in the future. In view of the inconsistent values for the hadronic vacuum polarization based on $e^+e^-$ and $\tau$ data, no conclusion can be drawn yet, whether the apparent discrepancy between the current experimental and theoretical values for the muon $g - 2$ points to physics beyond the Standard Model.

1 Introduction

For a particle with spin $1/2$, the relation between its magnetic moment and its spin reads $\vec{\mu} = g(e/2m)\vec{s}$. The Dirac equation predicts for the gyromagnetic factor $g = 2$, but radiative corrections to the lepton-photon-lepton vertex in quantum field theory can shift the value slightly. The anomalous magnetic moment is then defined as $a \equiv (g - 2)/2$. There has been a fruitful interplay between experiment and theory over many decades. The results for the anomalous magnetic moments of leptons have provided important insights into the structure of the fundamental interactions (Dirac equation, QED, Standard Model, ...).

As we will briefly discuss below, the electron anomalous magnetic moment $a_e$ provides a stringent test of QED and leads to the most precise determination of the fine structure constant $\alpha$. A weighted average of various measurements for $a_{e^+}$ and $a_{e^-}$, which is dominated by the latest results of Ref. [4], leads to

$$a_e^{\text{exp}} = 11 \ 596 \ 521 \ 88.3(4.2) \times 10^{-12} \ [3.7 \ 	ext{ppb}].$$

(1)

The anomalous magnetic moment of the muon $a_\mu$, on the other hand, allows to test the Standard Model as a whole, since all sectors contribute. The current experimental world average, dominated by the recent measurements of the $g - 2$ collaboration at Brookhaven, reads

$$a_\mu^{\text{exp}} = 11 \ 659 \ 203(8) \times 10^{-10} \ [0.7 \ \text{ppm}].$$

(2)

The final goal is to reach an experimental precision of $4 \times 10^{-10}$. In principle, $a_\mu$ is very sensitive to new physics beyond the Standard Model. Since $a_l \sim (m_l/M_{NP})^2$, therefore $a_\mu$ is about $(m_\mu/m_e)^2 \sim 4 \times 10^4$ times more sensitive to the scale of new physics, $M_{NP}$, than $a_e$, which makes up for the factor of 200 due to less precision in the

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measurement. Unfortunately, the hadronic contributions lead to the largest source of error in the Standard Model prediction for $a_\mu$, about $8 \times 10^{-10}$, and they are very difficult to control. This hinders at present all efforts to extract a clear sign of new physics from $a_\mu$. The hadronic contributions will be the main topic of this article. For more details on the subject of $g - 2$, we refer to the recent reviews\textsuperscript{1,2,3}, whereas the developments before 1990 can be traced from Ref.\textsuperscript{4}. We will not discuss at all here potential new physics contribution to $a_\mu$, like supersymmetry, but refer instead to the article\textsuperscript{5} and references therein.

2 Electron anomalous magnetic moment and $\alpha$

The result for $a_e$ with up to four loops in QED can be written as follows

\[
a_e = \frac{1}{2} \left( \frac{\alpha}{\pi} \right) - 0.32847844400 \ldots \left( \frac{\alpha}{\pi} \right)^2 + 1.1812340 \ldots \left( \frac{\alpha}{\pi} \right)^3 - 1.7502(384) \left( \frac{\alpha}{\pi} \right)^4 + 1.70(3) \times 10^{-12}.
\]

(A) Above we have also included small, mass-dependent corrections due to internal vacuum polarization diagrams with $\mu$ and $\tau$ loops at order $(\alpha/\pi)^2$ and higher order vacuum polarization and light-by-light scattering contributions from the muon and the tau at order $(\alpha/\pi)^3$. In general, these contributions decouple in QED as $(m_e/m_\mu,\tau)^2$. The results are known analytically up to three loops. Explicit expressions for most of the QED contributions can be found in Ref.\textsuperscript{1,2}, together with the original references. The error induced by the experimental uncertainty in $m_e/m_\mu,\tau$ is smaller than the digits given in the second and third term. As input values we used $m_e = 0.510998902(21)$ MeV, $m_\mu = 105.658357(5)$ MeV and $m_\tau = 1776.99^{+0.29}_{-0.26}$ MeV from the PDG\textsuperscript{9}, but the independent determination $m_\mu/m_e = 206.768277(24)$ from Ref.\textsuperscript{10}. The four loop result is only known numerically. Very recently an error in some parts of the contribution has been found, see Ref.\textsuperscript{11}, which changed the coefficient of the term $(\alpha/\pi)^4$ by $-0.24$. A numerical evaluation of all terms is under way\textsuperscript{11} to reduce the error quoted in Eq. (3). Certainly, an independent check of this coefficient by some other group would be highly welcome.

Furthermore, we have included in Eq. (3) the small hadronic correction $1.67(3) \times 10^{-12}$ (after correcting the sign in the light-by-light scattering contribution, see Ref.\textsuperscript{2}) and the electroweak contribution $0.03 \times 10^{-12}$. Since the QED part dominates over the hadronic and electroweak contribution, one can invert the relation (3) in order to get $\alpha$, by comparing $a_e$ with the experimental value from Eq. (1). In this way one obtains

\[
\alpha^{-1}(a_e) = 137.035 \ 998 \ 75(50)(13) = 137.035 \ 998 \ 75(52) \ [3.8 \text{ ppb}].
\]

(4)

The error found in Ref.\textsuperscript{11} has a very big effect on $a_e$, shifting it by about $-7.0 \times 10^{-12}$, which changes $\alpha^{-1}$ by $-8.3 \times 10^{-7}$. This corresponds to 6.1 ppb, i.e. 1.6 standard deviations. The errors given in Eq. (4) come from the experimental uncertainty in $a_e$ and from the error in the fourth order coefficient in Eq. (3), respectively.

3 Muon anomalous magnetic moment

The Standard Model contributions are usually split into three parts: $a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{had}}$. We will now discuss in turn the three types of contributions.

3.1 QED contribution

A general feature here is that electron loops are enhanced due to logarithms $\ln(m_\mu/m_e) \sim 5.3$. These are short-distance logarithms from vacuum polarization and infrared logarithms, for instance in the light-by-light scattering contribution. The latter effect completely dominates\textsuperscript{12} the
contribution at order \((\alpha/\pi)^3\): \(a_\mu^{(3)}(\text{Ibyl}) = [(2/3)\pi^2 \ln(m_\mu/m_e) + \ldots] (\alpha/\pi)^3 = 20.947 \ldots (\alpha/\pi)^3\). Loops with \(\tau\)-leptons are again suppressed. The result in QED up to 5-loops reads
\[
a_\mu^{\text{QED}} = 0.5 \times \left(\frac{\alpha}{\pi}\right) + 0.765 \times 857 \times 399(45) \times \left(\frac{\alpha}{\pi}\right)^2 + 24.050 \times 509 \times 5(23) \times \left(\frac{\alpha}{\pi}\right)^3 + 125.08(41) \times \left(\frac{\alpha}{\pi}\right)^4 + 930(170) \times \left(\frac{\alpha}{\pi}\right)^5
\]
\[
= 11658 \times 470.35(05)(12)/(11) \times 10^{-10} = 11658 \times 470.35(28) \times 10^{-10}.
\] (5)

The errors given in the second and third term on the right-hand side are due to the uncertainty in the experimental values of \(m_\mu/m_{e,\tau}\), the one in the fourth term from the numerical integration and the one in the last term from a renormalization group estimate of that coefficient. The error found in Ref. 11 has not such a big effect here. The coefficient of the term \((\alpha/\pi)^4\) changed by about \(-1.0\). This reduces \(a_\mu\) by \(-0.29 \times 10^{-10}\). On the other hand, in the first term one gets a shift of \(+0.07 \times 10^{-10}\) due to the change in \(\alpha\). The errors in the last line come from \(\alpha\), cf. Eq. 4, from the fourth and from the fifth order coefficient, respectively. To be on the safe side, we have added them linearly to obtain the final result, see also Ref. 11.

3.2 Hadronic contributions

a) Hadronic vacuum polarization

We will sketch here only the main issues and give the numerical results of several recent evaluations.\[12\] More details can be found in these references and in Ref. 15.

The hadronic corrections induce the largest uncertainties in \(a_\mu\), since they are theoretically not very well under control. The problem is that quarks are bound by strong gluonic interactions into hadrons at low energies, relevant for the muon \(g - 2\). In particular for the light quarks \(u, d, s\) one cannot use perturbative QCD. The coupling \(\alpha_s(Q^2)\) is large for \(Q^2 \sim 1\ \text{GeV}^2\) and grows for \(Q^2 \to 0\). For instance in the hadronic vacuum polarization contribution \(a_\mu^{\text{had. v.p.}}\), depicted in Fig. 1, one cannot simply equate the hadronic “blob” with a quark loop as it is possible for a lepton loop. In the present case there is, however, a way out by using the optical theorem (unitarity) to relate the imaginary part of the diagram to the measurable scattering cross section \(\sigma(e^+ e^- \to \gamma^* \to \text{hadrons})\). From a dispersion relation one then obtains the spectral representation\[16\]
\[
a_\mu^{\text{had. v.p.}} = \frac{\alpha}{\pi} \epsilon^2 \int_0^\infty ds \frac{1}{s} \text{Im}\Pi(s) \int_0^1 dx \frac{x^2(1-x)}{x^2 + \frac{m_\mu^2}{1-x}},
\] (6)
\[
\frac{1}{\pi} \text{Im}\Pi(s) = \frac{s}{16\pi^3\alpha^2} \sigma(e^+ e^- \to \gamma^* \to \text{hadrons}),
\] (7)
\[
(q_\mu q_\nu - q^2\eta_{\mu\nu})\Pi(q^2) = i \int d^4x e^{iqx} \langle \Omega | T \{j_\mu(x)j_\nu^\ast(0)\} | \Omega \rangle.
\] (8)

Usually, the relation is expressed as an integral involving the ratio \(R(s) = \sigma(e^+ e^- \to \gamma^* \to \text{hadrons})/\sigma(e^+ e^- \to \gamma^* \to \mu^+ \mu^-)\) multiplied with a known, positive kernel function peaked at low-energy.

Figure 1: Hadronic vacuum polarization contribution to \(a_\mu\).
Information on the spectral function $\text{Im} \Pi(s)$ in Eq. (6) can also be obtained from hadronic $\tau$ decays, like $\tau^{-} \rightarrow \nu_{\tau} \pi^{-} \pi^{0}$. One has, however, to apply corrections due to isospin violations, since $m_{u} \neq m_{d}$ and because of electromagnetic radiative corrections, see Refs. 17, 12, 15.

The most recent estimates are collected in Table 1. One observes that the evaluations based on $e^{+}e^{-}$ and $\tau$ data in Ref. 12 are inconsistent with each other at the 2.5 $\sigma$ level. Note that at least part of this discrepancy could be due to an error in the cross section measured by the CMD-2 collaboration 19. This measurement with its small uncertainty dominates at present the low-energy region around the $\rho$-peak and therefore the final result for $a_{\mu}^{\text{had. v.p.}}$. After correcting the error, the value will then shift towards the one based on $\tau$ data. Still, there remain quite large discrepancies between the spectral functions derived from $e^{+}e^{-}$ and $\tau$ data in some energy regions above the $\rho$ which are not yet understood and which cannot be explained by the known sources of isospin violation. Presumably, only future measurements of $\sigma(e^{+}e^{-} \rightarrow \text{hadrons})$, either using the radiative return method 20 at KLOE (Daphne) or BABAR or with a new scan at VEPP-2000, will be able to resolve the puzzle. In any case, it will be necessary to better understand the implementation of radiative corrections to the hadronic final state 21.

Averaging the results that use $e^{+}e^{-}$ data only, we obtain

$$a_{\mu}^{\text{had. v.p.}}(e^{+}e^{-}) = 683.8(7.5) \times 10^{-10}. \tag{9}$$

Recently, a first evaluation of $a_{\mu}^{\text{had. v.p.}}$ on the lattice appeared, although still with very large uncertainties 22

$$a_{\mu}^{\text{had. v.p.}} \big|_{u,d,s} = 460(78) \times 10^{-10}. \tag{10}$$

Note that the error is only statistical. Large systematical errors from the quenching approximation, unphysically large quark masses and finite volume effects are not accounted for. It will probably take a very long time to even come down to a 10% error.

Finally, there are higher order vacuum polarization effects, if additional photonic corrections or fermion loops (leptons and hadrons) are added to the diagram in Fig. 1. They have been evaluated in Ref. 23 with the result

$$a_{\mu}^{\text{h.o.-h.v.p.}} = -10.0(0.6) \times 10^{-10}. \tag{11}$$

b) Hadronic light-by-light scattering

The present picture of hadronic light-by-light scattering, as reviewed recently in Ref. 21, is shown in Fig. 2 and the corresponding contributions to $a_{\mu}$ are listed in Table 2 taking into account the corrections made in the two full evaluations 25, 26, after we had discovered the sign error in the pion-pole contribution 27, 28.

There are three classes of contributions to the hadronic four-point function [Fig. 2(a)], which can be understood from an effective field theory (EFT) analysis of hadronic light-by-light scattering 23, 28: (1) a charged pion loop [Fig. 2(b)], where the coupling to photons is dressed by some form factor ($\rho$-meson exchange, e.g. via vector meson dominance (VMD)), (2) the pseudoscalar pole diagrams [Fig. 2(c)] together with the exchange of heavier resonances ($f_{0}, a_{1}, \ldots$)

| Authors          | Contribution to $a_{\mu} \times 10^{10}$ |
|------------------|------------------------------------------|
| Davier et al. 12 | $(e^{+}e^{-} + \tau)$ 709.0 ± 5.9        |
| Davier et al. 12 | $(e^{+}e^{-})$ 684.7 ± 7.0              |
| Hagiwara et al. 13 | $(e^{+}e^{-})$ 683.1 ± 6.2              |
| Jegerlehner 14   | $(e^{+}e^{-})$ 683.62 ± 8.61             |

Table 1: Recent evaluations of $a_{\mu}^{\text{had. v.p.}}$.
and, finally, (3) the irreducible part of the four-point function which was modeled in Refs. 25, 26 by a constituent quark loop dressed again with VMD form factors [Fig. 2(d)]. The latter can be viewed as a local contribution $\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$ to $a_1$. The two groups used similar, but not identical models which explains the slightly different results for the dressed charged pion and the dressed constituent quark loop, although their sum seems to cancel to a large extent and the final result is essentially given by the pseudoscalar exchange diagrams. We take the difference of the results as indication of the error due to the model dependence.

**Pion-pole contribution**

The contribution from the neutral pion intermediate state is given by a two-loop integral that involves the convolution of two pion-photon-photon transition form factors $F_{\pi \gamma^* \gamma^*}$, see Fig. 2(c). We refer to Ref. 27 for all the details. Since no data on the doubly off-shell form factor $F_{\pi \gamma^* \gamma^*}(q_1^2, q_2^2)$ is available, one has to resort to models. In order to proceed with the analytical evaluation of the two-loop integral, we considered a certain class of form factors which includes the ones based on large-$N_C$ QCD that we studied in Ref. 26. For comparison, we have also used a vector meson dominance (VMD) and a constant form factor, derived from the Wess-Zumino-Witten (WZW) term.

![Figure 2: The hadronic light-by-light scattering contribution to the muon $g - 2$.](image)

Table 2: Contributions to $a_1(\times 10^{10})$ according to Fig. 2

| Type | Ref. 25 | Ref. 26 | Ref. 27 | No form factors |
|------|---------|---------|---------|----------------|
| (b)  | -0.5(0.8) | -1.9(1.3) |         | -4.5           |
| (c)  | 8.3(0.6)  | 8.5(1.3)  | 8.3(1.2) | +\infty        |
| $f_0, a_1$ | 0.174$^a$ | -0.4(0.3) |         | ~6             |
| (d)  | 1.0(1.1)  | 2.1(0.3)  |         |                |
| Total | 9.0(1.5)  | 8.3(3.2)  | 8(4)$^b$ |                |

$^a$ Only $a_1$ exchange.

$^b$ Our estimate, using Refs. 25, 26, 27.

Our approach to this problem consists of making an ansatz for the relevant Green’s functions in the framework of large-$N_C$ QCD. In this limit, an infinite set of narrow resonance states contributes in each channel. The pion-photon-photon form factor is then given by a sum over an infinite set of narrow vector resonances, involving arbitrary couplings, although there are constraints at long and short distances. To implement them, we perform a matching of the ansatz with chiral perturbation theory (ChPT) at low energies and with the operator product expansion (OPE) at high momenta in order to reduce the model dependence. In practice, it is sufficient to keep a few resonance states to reproduce the leading behavior in ChPT and the OPE. The normalization is given by the WZW term, $F_{\pi \gamma^* \gamma^*}(0,0) = -N_C/(12\pi^2 F_{\pi})$, whereas the OPE tells us that $\lim_{\lambda \to \infty} F_{\pi \gamma^* \gamma^*}(\lambda^2 q^2, (p - \lambda q)^2) = (2F_{\pi})/(3\lambda^2 q^2) + \mathcal{O}(1/\lambda^3)$. We considered the form factors that are obtained by truncation of the infinite sum to one (lowest meson dominance, LMD), and two (LMD+V), vector resonances per channel, respectively. Some of the parameters in the LMD+V form factor are not fixed by the normalization and the leading term in the OPE.

$^b$In contrast, the calculations in Refs. 25, 26 were based purely on numerical approaches.
We have determined these coefficients phenomenologically. In particular, \( F_{\pi^0\gamma^*\gamma^*}(-Q^2,0) \) with one photon on-shell behaves like \( 1/Q^2 \) for large spacelike momenta, \( Q^2 = -q^2 \). Whereas the LMD form factor does not have such a behavior, it can be reproduced with the LMD+V ansatz. Note that the usual VMD form factor \( F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \sim 1/[(q_1^2 - M_\pi^2)(q_2^2 - M_\pi^2)] \) does not correctly reproduce the OPE.

For the form factors discussed above one can perform all angular integrations in the two-loop integral analytically. The pion-exchange contribution to \( a_\mu \) can then be written as a two-dimensional integral representation, where the integration runs over the moduli of the Euclidean momenta

\[
a^\text{LbyL;had}_{\mu} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \sum_i w_i(Q_1, Q_2) f_i(Q_1, Q_2),
\]

with universal [for the above class of form factors] weight functions \( w_i \) (rational functions, square roots and logarithms). The dependence on the form factors resides in \( f_i \). In this way we could separate the generic features of the pion-pole contribution from the model dependence and thus better control the latter. This is not possible anymore in the final analytical result (as a series expansion) for \( a^\text{LbyL;had}_\mu \) derived in Ref. Note that the analytical result has not the same status here as for instance in QED. One has to keep in mind that there is an intrinsic uncertainty in the form factor of \( 10 - 30\% \), furthermore the VMD form factor used in that reference has the wrong high-energy behavior.

The weight functions \( w_i \) in the main contribution are positive and peaked around momenta of the order of 0.5 GeV. There is, however, a tail in one of these functions, which produces for the constant WZW form factor a divergence of the form \( \ln^2 \Lambda \) for some UV-cutoff \( \Lambda \). We will come back to this point below. Other weight functions have positive and negative contributions in the low-energy region, which lead to a strong cancellation in the corresponding integrals.

All form factors lead to very similar results (apart from WZW). Judging from the shape of the weight functions described above, it seems more important to correctly reproduce the slope of the form factor at the origin and the available data at intermediate energies. On the other hand, the asymptotic behavior at large \( Q_i \) seems not very relevant. The results for the LMD+V form factor are rather stable under the variation of the parameters.

With the LMD+V form factor, we then get

\[
a^\text{LbyL;had}_\mu = +5.8(1.0) \times 10^{-10},
\]

where the error includes the variation of the parameters and the intrinsic model dependence. A similar short-distance analysis in the framework of large-\( N_C \) QCD and including quark mass corrections for the form factors for the \( \eta \) and \( \eta' \) was beyond the scope of Ref. We therefore used VMD form factors fitted to the available data for \( F_{\pi^0\gamma^*\gamma^*}(-Q^2,0) \) to obtain our final estimate

\[
a^\text{LbyL;FS}_\mu = a^\text{LbyL;had}_\mu + a^\text{LbyL;\eta}_\text{VMD} + a^\text{LbyL;\eta'}_\text{VMD} = +8.3(1.2) \times 10^{-10}.\]

An error of 15 % for the pseudoscalar pole contribution seems reasonable, since we impose many theoretical constraints from long and short distances on the form factors. Furthermore, we use experimental information whenever available.

**Effective field theory approach to \( a^\text{LbyL;had}_\mu \)**

In Ref. we discussed an EFT approach to hadronic light-by-light scattering based on an effective Lagrangian that describes the physics of the Standard Model well below 1 GeV, see also Ref. It includes photons, light leptons, and the pseudoscalar mesons and obeys chiral symmetry and \( U(1) \) gauge invariance.

The leading contribution to \( a^\text{LbyL;had}_\mu \), of order \( p^6 \), is given by a finite loop of charged pions with point-like electromagnetic vertices, see Fig. (b). Since this contribution involves a loop of hadrons, it is subleading in the large-\( N_C \) expansion.
At order $p^6$ and at leading order in $N_C$, we encounter the divergent pion-pole contribution, diagrams (a) and (b) of Fig. 3 involving two WZW vertices. The diagram (c) is actually finite. The divergences of the triangular subgraphs in the diagrams (a) and (b) are removed by inserting the counterterm $\chi$ from the Lagrangian $\mathcal{L}^{(6)} = (\alpha^2/4\pi^2 F_0) \chi \bar{\psi} \gamma_{\mu} \gamma_{5} \psi \partial^{\mu} \chi + \cdots$, see the one-loop diagrams (d) and (e). Finally, there is an overall divergence of the two-loop diagrams (a) and (b) that is removed by a local counterterm, diagram (f). Since the EFT involves such a local contribution, we will not be able to give a precise numerical prediction for $a_{\mu}^{LbyL;had}$.

Nevertheless, it is interesting to consider the leading and next-to-leading logarithms that are in addition enhanced by a factor $N_C$ and which can be calculated using the renormalization group. The EFT and large-$N_C$ analysis tells us that

$$ a_{\mu}^{LbyL;had} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ f \left( \frac{M_{\pi^\pm}}{m_\mu} - \frac{M_{K^\pm}}{m_\mu} \right) + N_C \left( \frac{m_\mu^2}{16\pi^2 F_\pi^2} \frac{N_C}{3} \right) \left[ \ln^2 \frac{\mu_0}{m_\mu} + c_1 \ln \frac{\mu_0}{m_\mu} + c_0 \right] + \mathcal{O} \left( \frac{m_\mu^4}{\mu_0^4} N_C \times \log's \right) \right\}, \tag{15} $$

where $f(M_{\pi^\pm}/m_\mu, M_{K^\pm}/m_\mu) = -0.038$ represents the charged pion and kaon-loop that is formally of order one in the chiral and $N_C$ counting and $m_0$ denotes some hadronic scale, e.g. $M_\rho$. The coefficient $\mathcal{C} = (N_C^2 m_0^2)/(48\pi^2 F_\pi^2) = 0.025$ of the log-square term in the second line is universal and of order $N_C$, since $F_\pi = \mathcal{O}(\sqrt{N_C})$. The value given corresponds to $N_C = 3$.

Unfortunately, although the logarithm is sizeable, $\ln(M_\rho/m_\mu) = 1.98$, in $a_{\mu}^{LbyL;\pi^0}$ there occurs a cancellation between the log-square and the log-term. If we fit our result for the VMD form factor for large $M_\rho$ to an expression as given in Eq. (15), we obtain

$$ a_{\mu}^{LbyL;\pi^0} VMD = \left(\frac{\alpha}{\pi}\right)^3 \mathcal{C} \left[ \ln^2 \frac{M_\rho}{m_\mu} + c_1 \ln \frac{M_\rho}{m_\mu} + c_0 \right] \approx \left(\frac{\alpha}{\pi}\right)^3 \mathcal{C} \left[ 3.94 - 3.30 + 1.08 \right] = [12.3 - 10.3 + 3.4] \times 10^{-10} = 5.4 \times 10^{-10}, \tag{16} $$

which is confirmed by the analytical result of Ref. 31 (setting for simplicity $M_{\pi^0} = m_\mu$):

$$ a_{\mu}^{LbyL;\pi^0} VMD = [12 - 8.0 + 1.7] \times 10^{-10} = 5.7 \times 10^{-10}. $$

This cancellation is now also visible in the published version of Ref. 32. In that paper the remaining parts of $c_1$ have been calculated: $c_1 = -2\chi(\mu_0)/3 + 0.237 = -0.93^{+0.67}_{-0.83}$, with our conventions for $\chi$ and $\chi(M_\rho_{\exp}) = 1.75^{+1.25}_{-1.00}$.

Finally, the EFT analysis shows that the modeling of hadronic light-by-light scattering by a constituent quark loop is not consistent with QCD. The latter has a priori nothing to do with the full quark loop in QCD which is dual to the corresponding contribution in terms of hadronic degrees of freedom. Equation (15) tells us that at leading order in $N_C$ any model of QCD has to show the behavior $a_{\mu}^{LbyL;had} \sim (\alpha/\pi)^3 N_C [N_C m_\mu^2/(48\pi^2 F_\pi^2)] \ln^2 \Lambda$, with a universal coefficient $\mathcal{C}$, if we sends the cutoff $\Lambda$ to infinity. From the analytical result for the quark loop, one obtains the behavior $a_{\mu}^{LbyL;CQM} \sim (\alpha/\pi)^3 N_C (m_\mu^2/M_Q^2) + \ldots$, for $M_Q \gg m_\mu$, if we interpret the constituent quark mass $M_Q$ as a hadronic cutoff. Even though one may argue that $N_C/(48\pi^2 F_\pi^2)$ can be replaced by $1/M_Q^2$, the log-square term is not correctly reproduced with
this model. Therefore, the constituent quark model (CQM) cannot serve as a reliable description for the dominant contribution to $a_{\mu}^{LbyL,\text{had}}$, in particular, its sign. Moreover, we note that the pion-pole contribution is infrared finite in the chiral limit; whereas the quark loop shows an infrared divergence $\ln(M_Q/m_\mu)$ for $M_Q \to 0$.

The analysis within the EFT and large-$N_C$ framework, together with the numerical results for all contributions depicted in Fig. 2 and listed in Table 2 leads us to the following (conservative) estimate for the hadronic light-by-light scattering contribution

$$a_{\mu}^{LbyL,\text{had}} = +8(4) \times 10^{-10}. \quad (17)$$

Since the model calculations for the dressed charged pion and the dressed constituent quark loop yield slightly different results we have added the errors linearly.

3.3 Electroweak contribution

The electroweak correction to $a_{\mu}$ lies somehow in between the QED and hadronic contribution. At one loop, the result is reliably calculable

$$a_{\mu}^{\text{EW},(1)} = 19.5 \times 10^{-10}, \quad (18)$$

with a Higgs boson contribution that is very small for $M_H \geq 114.5$ GeV (LEP 2 bound).

Two-loop corrections, see Fig. 4 are potentially large due to factors $\ln(M_Z/m_\mu) \sim 6.8$.

![Figure 4](image)

Figure 4: Two-loop electroweak corrections to $a_{\mu}$ from the first family. The light quark loop is to be understood symbolically, representing again a hadronic "blob" (QCD three-point function).

Furthermore, as noted in Ref. 34, one cannot separate leptons and quarks anymore, but must treat each generation together because of the cancellation of the triangle anomalies. Therefore earlier estimates were incomplete. A first full two-loop calculation was done in Ref. 36 and recently revisited by two groups 37, 38 to improve on the treatment of the hadronic contributions. Instead of using a simple constituent quark loop, short-distance constraints from the OPE on the relevant QCD three-point functions have been imposed. There is still some disagreement in the details, but the numerical values are very close. Adding the one-loop result from Eq. (18), Ref. 37 obtains

$$a_{\mu}^{\text{EW}} = 15.2(0.1) \times 10^{-10}. \quad (15)$$

The error reflects hadronic uncertainties and the variation of $M_H$. No resummation has been performed in that reference. Ref. 38 gets $a_{\mu}^{\text{EW}} = 15.4(0.1)(0.2) \times 10^{-10}$, where the first error corresponds to the hadronic uncertainty and the second to an allowed Higgs boson mass range of 114 GeV $\leq M_H \leq 250$ GeV, the current top mass uncertainty, and unknown three-loop effects. There are large cancellation in the resummation, therefore the final shift after the resummation is very small. Averaging the two estimates, we obtain

$$a_{\mu}^{\text{EW}} = 15.3(0.2) \times 10^{-10}, \quad (19)$$

which corresponds to quite a large two-loop correction of $a_{\mu}^{\text{EW},(2)} = -4.2(0.2) \times 10^{-10}$. Although not all details have been resolved, the electroweak contribution seems well under control.

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1. This can be shown by studying the low momentum behavior of the weight functions $w_i$ corresponding to the two-loop diagrams 3(a)–(c) and the one-loop diagrams 3(d)+ (e) (given in Ref. 25, for $M_{\nu} \to 0$, see also Ref. 22).

2. The resummation of leading logarithms has been discussed in Ref. 29 and corrected in Ref. 38.

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3.4 Summary

We now collect the results for the different contributions in the Standard Model from Eqs. (5), Table 1, (9), (11), (17), and (19):

\[
a_{\mu}^{SM}(e^+e^-) = (11,659,167.5 \pm 7.5 \pm 4.0 \pm 0.35) \times 10^{-10},
\]

\[
a_{\mu}^{SM}(\tau) = (11,659,192.7 \pm 5.9 \pm 4.0 \pm 0.35) \times 10^{-10},
\]

where we kept the results based on \(e^+e^-\) and \(\tau\) data separately. Comparison with the experimental value from Eq. (2) leads to

\[
a_{\mu}^{exp} - a_{\mu}^{SM}(e^+e^-) = (35.5 \pm 11.7) \times 10^{-10} [3.0 \sigma],
\]

\[
a_{\mu}^{exp} - a_{\mu}^{SM}(\tau) = (10.3 \pm 10.7) \times 10^{-10} [1.0 \sigma].
\]

Is the discrepancy using the \(e^+e^-\) data a sign for new physics beyond the Standard Model? In view of the inconsistencies between the evaluations based on \(e^+e^-\) and \(\tau\) data this conclusion is certainly premature. Furthermore, the error found in the CMD-2 data\(^{18}\) will probably reduce the discrepancy between \(a_{\mu}^{exp}\) and \(a_{\mu}^{SM}(e^+e^-)\) to less than 2 \(\sigma\).

4 Conclusions

We briefly presented the current theoretical value for the anomalous magnetic moment of the electron. An error found recently in the coefficient of the four-loop QED result leads to a 1.6 \(\sigma\) shift in the fine structure constant \(\alpha\) when comparing theoretical and experimental values for \(a_e\).

We then reviewed the prediction for the anomalous magnetic moment of the muon \(a_\mu\) in the Standard Model. In particular, we discussed the uncertainties induced by the hadronic contributions, like vacuum polarization, light-by-light scattering and higher order electroweak corrections. Using \(e^+e^-\) data, an apparent discrepancy between Standard Model value and experimental value exists which could point to new physics. In view of inconsistencies when using \(e^+e^-\) and \(\tau\) data and because there seems to be an error in the latest, most precise \(e^+e^-\) data, such a conclusion cannot be drawn yet. Perhaps some of the hadronic uncertainties are even underestimated. There still remains a lot of work to be done to better control these hadronic contributions, if we want to use the muon \(g-2\) to search for signs of new physics.

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References

1. A. Czarnecki and W.J. Marciano, Nucl. Phys. B (Proc. Suppl.) 76, 245 (1999).
2. M. Knecht, Lectures delivered at the 41. Internationale Universitätswochen für Theoretische Physik, Schladming, Austria, 22-28 February 2003, Preprint CPT-2003/P.4525.
3. V.W. Hughes and T. Kinoshita, Rev. Mod. Phys. 71, S133 (1999); K. Melnikov, Int. J. Mod. Phys. A 16, 4591 (2001); E. de Rafael, hep-ph/0208251.
4. Quantum Electrodynamics, ed. T. Kinoshita (World Scientific, Singapore, 1990).
5. A. Czarnecki and W. J. Marciano, Phys. Rev. D 64, 013014 (2001).
6. P. J. Mohr and B. N. Taylor, Rev. Mod. Phys. 72, 351 (2000).
7. R.S. Van Dyck, P.B. Schwinberg and H.G. Dehmelt, Phys. Rev. Lett. 59, 26 (1987).
8. F. Gray, these proceedings; G. W. Bennett et al. [Muon g-2 Collaboration], Phys. Rev. Lett. 89, 101804 (2002) [Erratum-ibid. 89, 129903 (2002)].
9. K. Hagiwara et al. [Particle Data Group Collaboration], Phys. Rev. D 66, 010001 (2002).
10. W. Liu et al., Phys. Rev. Lett. 82, 711 (1999).
11. T. Kinoshita and M. Nio, Phys. Rev. Lett. 90, 021803 (2003).
12. M. Davier, S. Eidelman, A. Höcker and Z. Zhang, Eur. Phys. J. C 27, 497 (2003).
13. K. Hagiwara, A. D. Martin, D. Nomura and T. Teubner, Phys. Lett. B 557, 69 (2003).
14. F. Jegerlehner, J. Phys. G 29, 101 (2003).
15. A. Höcker, these proceedings.
16. C. Bouchiat and L. Michel, J. Phys. Radium 22, 121 (1961); L. Durand III, Phys. Rev. 128, 441 (1962) [Erratum-ibid. 129, 2835 (1963)]; S. J. Brodsky and E. de Rafael, ibid. 168, 1620 (1968); M. Gourdin and E. de Rafael, Nucl. Phys. B 10, 667 (1969).
17. V. Cirigliano, G. Ecker and H. Neufeld, Phys. Lett. B 513, 361 (2001); JHEP 0208, 002 (2002).
18. A. Höcker, private communication at this conference.
19. R. R. Akhmetshin et al. [CMD-2 Collaboration], Phys. Lett. B 527, 161 (2002).
20. A. B. Arbuzov, E. A. Kuraev, N. P. Merenkov and L. Trentadue, JHEP 9812, 009 (1998); S. Binner, J. H. Kühn and K. Melnikov, Phys. Lett. B 459, 279 (1999); V. A. Khoze et al., Eur. Phys. J. C 18, 481 (2001); 25, 199 (2002); G. Rodrigo, H. Czyż, J. H. Kühn and M. Szopa, ibid. 24, 71 (2002); H. Czyż, A. Grzelińska, J. H. Kühn and G. Rodrigo, ibid. 27, 563 (2003).
21. A. B. Arbuzov et al., JHEP 9710, 006 (1997); A. Hoefer, J. Gluza and F. Jegerlehner, Eur. Phys. J. C 24, 51 (2002); J. Gluza, A. Hoefer, S. Jadach and F. Jegerlehner, hep-ph/0212386.
22. T. Blum, hep-lat/0212018.
23. B. Krause, Phys. Lett. B 390, 392 (1997); updated in R. Alemany, M. Davier and A. Höcker, Eur. Phys. J. C 2, 123 (1998).
24. A. Nyffeler, hep-ph/0203243; Nucl. Phys. B (Proc. Suppl.) 116, 225 (2003) hep-ph/0210347.
25. M. Hayakawa and T. Kinoshita, Phys. Rev. D 66, 019902 (E) (2002) and hep-ph/0112102.
26. J. Bijnens, E. Pallante, and J. Prades, Nucl. Phys. B626, 410 (2002).
27. M. Knecht and A. Nyffeler, Phys. Rev. D 65, 073034 (2002).
28. M. Knecht, A. Nyffeler, M. Perrottet and E. de Rafael, Phys. Rev. Lett. 88, 071802 (2002).
29. E. de Rafael, Phys. Lett. B 322, 239 (1994).
30. M. Knecht and A. Nyffeler, Eur. Phys. J. C 21, 659 (2001).
31. I. Blokland, A. Czarnecki and K. Melnikov, Phys. Rev. Lett. 88, 071803 (2002).
32. M. Ramsey-Musolf and M.B. Wise, Phys. Rev. Lett. 89, 041601 (2002).
33. R. Jackiw and S. Weinberg, Phys. Rev. D 5, 2396 (1972); I. Bars and M. Yoshimura, ibid. 6, 374 (1972); G. Altarelli, N. Cabibbo and L. Maiani, Phys. Lett. B 40, 415 (1972); W. A. Bardeen, R. Gastmans and B. Lautrup, Nucl. Phys. B 46, 319 (1972); K. Fujikawa, B. W. Lee and A. I. Sandy, Phys. Rev. D 6, 2923 (1972).
34. S. Peris, M. Perrottet and E. de Rafael, Phys. Lett. B 355, 523 (1995).
35. T. V. Kukhto, E. A. Kuraev, Z. K. Silagadze and A. Schiller, Nucl. Phys. B 371, 567 (1992).
36. A. Czarnecki, B. Krause and W. J. Marciano, Phys. Rev. D 52, R2619 (1995); Phys. Rev. Lett. 76, 3267 (1996).
37. M. Knecht, S. Peris, M. Perrottet and E. de Rafael, JHEP 0211, 003 (2002).
38. A. Czarnecki, W. J. Marciano and A. Vainshtein, Phys. Rev. D 67, 073006 (2003).
39. G. Degrassi and G. F. Giudice, Phys. Rev. D 58, 053007 (1998).