Simpler Proofs For Approximate Factor Models of Large Dimensions

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Abstract

Estimates of the approximate factor model are increasingly used in empirical work. Their theoretical properties, studied some twenty years ago, also laid the ground work for analysis on large dimensional panel data models with cross-section dependence. This paper presents simplified proofs for the estimates by using alternative rotation matrices, exploiting properties of low rank matrices, as well as the singular value decomposition of the data in addition to its covariance structure. These simplifications facilitate interpretation of results and provide a more friendly introduction to researchers new to the field. New results are provided to allow linear restrictions to be imposed on factor models.

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1 Introduction

An active area of research in the last twenty years is analysis of panel data with cross-section dependence, where the panel has dimension $T \times N$, and where $T$ (the time) and $N$ (the cross-section) dimensions are both large. Classical factor models studied by Anderson and Rubin (1956) and Lawley and Maxwell (1974) among others are designed to capture cross-section dependence when either $T$ or $N$ is fixed, and that errors are iid across time and units. The approximate factor model formulated in Chamberlain and Rothschild (1983) relaxes many of these assumptions, so what remains is to be able to take the theory to the data. Connor and Korajczyk (1993) suggest to estimate the factors by the method of asymptotic principal components (APC). Consistency proofs were subsequently given in Stock and Watson (2002a), Bai and Ng (2002) under the assumption that $N, T \to \infty$ with $\sqrt{N}/T \to \infty$. Bai and Ng (2006) provide the conditions under which the factor estimates can be treated in subsequent regressions as though they were observed. Novel uses of the factor estimates such as diffusion index forecasting pioneered in Stock and Watson (2002b)) and factor-augmented autoregressions such as considered in Bernanke et al. (2005), along with the natural role that common factors play in many theoretical models in economics and finance have contributed to the popularity of large dimensional factor analysis.

Arguably, the three fundamental results in this literature are i) the consistency proof of the estimated factor space at rate $\min(N, T)$, ii) consistent estimation of the number of factors, and (iii) $\sqrt{N}, \sqrt{T}$, and $\min(\sqrt{N}, \sqrt{T})$ asymptotic normality of the estimated factors, the loadings, and the common component, respectively. The point of departure in these results, given Bai and Ng (2002) and Bai (2003), is an analysis of the factor estimates relative to a specific rotation of the true factors first considered in Stock and Watson (1998) that is defined from the covariance structure of the data. This leads to a decomposition of the estimation error into four terms and carefully deriving the limit for each of them. Though a large body of research is built on these theoretical results, the arguments are lengthy and often not particularly intuitive.

In this paper, we show that the key results can be obtained using simpler arguments and under higher level assumptions. It turns out that inspection of the norm of the $T \times T$ population covariance of the errors is already sufficient to establish that the factor space can be consistently estimated at rate $\min(N, T)$ from which consistent estimation of the number of factors can be easily established. Exploiting the eigen-decomposition of the data and not only its covariance leads to different representation of the factor estimates that also simplify the analysis. Most important is the recognition that the rotation matrix is not unique.
We present four asymptotically equivalent rotation matrices that simplify the proofs for asymptotic normality. It will be shown that the asymptotic variance of the factor estimates can be represented in many ways. This little known fact makes it possible to conduct inference using an estimate of the variance that the researcher finds most computationally convenient. The simplified arguments, presented in consistent notation, should help students and researchers new to the field better understand the role that large $N$ and $T$ play in estimation of approximate factor models.

Economic analysis sometimes impose specific restrictions on the model. Because we can only estimate the factor space up to a rotation matrix, the problem is a bit more tricky. We provide results for estimation of factor models with linear restrictions. These results should be of interest as factor estimation finds more ways into economic applications.

2 Model Setup and Assumptions

We use $i = 1, \ldots, N$ to index cross-section units and $t = 1, \ldots, T$ to index time series observations. Let $X_i = (X_{i1}, \ldots, X_{iT})'$ be a $T \times 1$ vector of random variables and $X = (X_1, X_2, \ldots, X_N)$ be a $T \times N$ matrix. In practice, $X_i$ is transformed to be stationary, demeaned, and often standardized. The normalized data $Z = \frac{X}{\sqrt{NT}}$ has singular value decomposition (SVD)

$$Z = \frac{X}{\sqrt{NT}} = U_{NT} D_{NT} V_{NT}' = U_{NT,r} D_{NT,r} V_{NT,r}' + U_{NT,N-r} D_{NT,N-r} V_{NT,N-r}'. $$

In the above, $D_{NT,r}$ is a diagonal matrix of $r$ singular values $d_{NT,1}, \ldots, d_{NT,r}$ arranged in descending order, $U_{NT,r}, V_{NT,r}$ are the corresponding left and right singular vectors respectively. Note that while the $r$ large singular values of $X$ diverge and the remaining $N-r$ ones are bounded, the $r$ largest singular values of $Z$ are bounded and the remaining ones tend to zero because the singular values of $Z$ are those of $X$ divided by $\sqrt{NT}$. The Eckart and Young (1936) theorem posits that the best rank $k$ approximation of $Z$ is $U_{NT,k} D_{NT,k} V_{NT,k}'$. The nonzero eigenvalues of $Z'Z$ are the same as those $ZZ'$, which when multiplied by $NT$, equal the nonzero eigenvalues of $X'X$ and $XX'$.

We are interested in the low rank component of $X$ viewed from the perspective of a factor model. The static factor representation of the data is

$$X = FA' + e. \quad (1)$$

The common component $C = FA'$ has reduced rank $r$ because $F$ and $A$ both have rank $r$. Let $e_i' = (e_{i1}, e_{i2}, \ldots, e_{iT})$ and $e_i' = (e_{1t}, e_{2t}, \ldots, e_{Nt})$. The factor representation for data of
each unit $i$ is

$$X_i = F\Lambda_i + e_i.$$ 

The $N \times N$ covariance matrix of $X$ takes the form

$$\Sigma_X = \Lambda \Sigma_F \Lambda^T + \Sigma_e = \Sigma_C + \Sigma_e.$$

A strict factor model obtains when $\Sigma_e$ is a diagonal matrix, which holds when the errors are cross-sectionally and serially uncorrelated. The classical factor model studied in Anderson and Rubin (1956) uses the stronger assumption that $e_{it}$ is iid and normally distributed. For economic analysis, this error structure is overly restrictive. We work with the approximate factor model formulated in Chamberlain and Rothschild (1983), which allows the idiosyncratic errors to be weakly correlated in both the cross-section and time series dimensions. In such a case, $\Sigma_e$ need not be a diagonal matrix.

The defining characteristic of an approximate factor model is that the $r$ population eigenvalues of $\Sigma_C$ diverge with $N$ while all eigenvalues of $\Sigma_e$ are bounded. Since $r$ can be consistently estimated, we will assume that $r$ is known. To simplify notation, the subscripts indicating that $F$ is $T \times r$ and $\Lambda$ is $N \times r$ will be suppresed when the context is clear. Estimation of $F$ and $\Lambda$ in an approximate factor model with $r$ factors proceeds by minimizing the sum of squared residuals:

$$\min_{F,\Lambda} \text{SSR}(F,\Lambda; r) = \min_{F,\Lambda} \frac{1}{NT} \|X - F\Lambda'\|^2_F$$

$$= \min_{F,\Lambda} \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T (x_{it} - \Lambda_i'F_t)^2.$$ 

As $F$ and $\Lambda$ are not separately identified, we impose the normalization restrictions

$$\frac{F'F}{T} = I_r, \quad \frac{\Lambda'\Lambda}{N} \text{ is diagonal.} \tag{2}$$

Even with these restrictions, the problem is not convex and is difficult to solve. But we can iteratively solve two bi-convex problems: (i) conditional on $F$, minimizing the objective function with respect to $\Lambda$ suggests that time series regressions of $X_i$ on $F$ will give estimates of $\Lambda_i$ for each $i = 1, \ldots N$; (ii) conditional on $\Lambda$, doing $T$ cross-section regressions of $X_t$ on $\Lambda$ will given estimates of $F_t$ for each $t$. That is, we iteratively compute

$$\tilde{F} = X\tilde{\Lambda}(\tilde{\Lambda}'\tilde{\Lambda})^{-1}, \tag{3a}$$

$$\tilde{\Lambda}' = (\tilde{F}'\tilde{F})^{-1}\tilde{F}'X = \frac{1}{T}\tilde{F}'X. \tag{3b}$$

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The solution upon convergence is the (static) asymptotic principal components (APC):

\[
(\hat{F}, \hat{\Lambda}) = (\sqrt{T}U_{NT,r}, \sqrt{N}V_{NT,r}D_{NT,r}).
\] (3c)

Evidently, the solution involves eigenvectors because the algorithm is an implementation of ‘orthogonal subspace iteration’ algorithm for computing eigenvectors, Golub and Loan (2012, Algorithm 8.2). A related method is the ‘alternating least squares’ developed in De Leeuw (2004) and refined in Unkel and Trendafilov (2010) that treats \( e \) as unknowns to be recovered. Provided that a low rank structure exists, the error bounds for these algorithms can be shown without probabilistic assumptions about \( F, \Lambda, \) and \( e \). We will need these assumptions to obtain distribution theory, and will treat \( e \) as residuals rather than choice variables.

Analysis of the APC estimates in a setting of large \( N \) and large \( T \) must overcome two new challenges not present in the classical factor analysis of Anderson and Rubin (1956). The first pertains to the fact that the errors are now allowed to be cross-sectionally correlated. The second pertains to the fact that covariance matrix of \( X \) or \( X' \) are of dimensions \( T \times T \) and \( N \times N \) respectively, which are of infinite dimensions when \( N \) and \( T \) are large. The asymptotic properties of the factor estimates were first studied in Stock and Watson (2002a); Bai and Ng (2002); Bai (2003). Though the theory is well developed, the derivations are quite involved.

In what follows, we will establish the properties of \( \hat{F} \) and \( \hat{\Lambda} \) using simpler proofs and under weaker assumptions than previously used. Throughout, we let

\[
\delta_{NT} = \min(\sqrt{N}, \sqrt{T}).
\]

Unless otherwise stated, \( \|A\| \) is understood to be the squared Frobenius norm of a \( m \times n \) matrix \( A \). That is, \( \|A\|^2 = \|A\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |A_{ij}|^2 = \text{Tr}(AA^\prime) \). The factor model can also be represented as

\[
X_{it} = \Lambda_i^tF_t + e_{it}.
\]

A strict factor model assumes that \( E[e_{jt}e_{js}] = 0 \) for \( s \neq t \). An approximate factor model relaxes this requirement.

**Assumption A1:** Let \( F^0 \) and \( \Lambda^0 \) be the true values of \( F \) and \( \Lambda \). Let \( M < \infty \), not depending on \( N \) and \( T \).

i. Mean independence: \( E(e_{it}|\Lambda^0_i, F^0_t) = 0 \).

ii. Weak (cross-sectional and serial) correlation in the errors.
(a) $E\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{it}e_{is} - E(e_{it}e_{is})]\right]^2 \leq M,$
(b) For all $i$, $\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}|E(e_{it}e_{is})| \leq M$,
(c) For all $t$, $\frac{1}{N\sqrt{T}}\|e_t^ie\| = O_p(\delta_{NT}^{-1})$ and for all $i$, $\frac{1}{N\sqrt{T}}\|e_t^ie\| = O_p(\delta_{NT}^{-1})$.

**Assumption A2:** (i) $\lim_{T\to\infty}\frac{\sum_{i=1}^{N}\lambda_i\lambda_i^*}{T} = \Sigma_F > 0$; (ii) $\lim_{N\to\infty}\frac{\sum_{i=1}^{N}\lambda_i\lambda_i^*}{N} = \Sigma_A > 0$; (iii) the eigenvalues of $\Sigma_A\Sigma_F$ are distinct.

**Assumption A3:** (i) For each $t$, $E\|N^{-1/2}\sum_{i=1}^{N}\lambda_i^0 e_{it}\|^2 \leq M$ and $\frac{1}{\sqrt{NT}}e_t^ie^F = O_p(\delta_{NT}^{-2})$; (ii) for each $i$, $E\|T^{-1/2}\sum_{t=1}^{T}\lambda_i^0 e_{it}\|^2 \leq M$ and $\frac{1}{K}e_t^i\lambda^0 = O_p(\delta_{NT}^{-2})$.

Assumption A2 implies that $\|F^0\|^2/T = O_p(1)$ and $\|\Lambda^0\|^2/N = O_p(1)$, and that all $r$ eigenvalues of $\Lambda^0\Lambda^0$ diverge at the same rate of $N$. The conditions ensure a strong factor structure which is needed for identification. Under Assumption A3, the following holds:

\[
\frac{1}{NT}\sum_{i=1}^{N} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} e_t^0 e_{it} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} e_t^0 e_{it} \right)^\prime = O_p(1/T) \tag{4}
\]

\[
\frac{1}{NT} \sum_{i=1}^{N} \left( \frac{1}{\sqrt{N}} \sum_{t=1}^{T} \lambda_i^0 e_{it} \right) \left( \frac{1}{\sqrt{N}} \sum_{t=1}^{T} \lambda_i^0 e_{it} \right)^\prime = O_p(1/N) \tag{5}
\]

**Lemma 1** Under Assumption A,

\[
\|ee^\prime\|_F^2 = O_p\left(\frac{1}{T}\right) + O_p\left(\frac{1}{N}\right) = O_p(\delta_{NT}^{-2}).
\]

Lemma 1 establishes that the normalized sum of squared covariances of the errors is of stochastic order that depends on the size of the panel in both dimensions. The proof comes from observing that $ee^\prime$ is a $T \times T$ matrix with $\sum_{j=1}^{N} e_{jt} e_{js}$ as its $(t, s)$ entry. Thus

\[
\|\sqrt{NT}ee^\prime\|_F^2 = \frac{1}{N^2T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} \left( \sum_{j=1}^{N} e_{jt} e_{js} \right)^2
\]

\[
= \frac{1}{T} \left[ \frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{N} \sum_{j=1}^{N} e_{jt}^2 \right)^2 \right] + \frac{1}{N} \left[ \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s \neq t} \left( \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e_{jt} e_{js} \right)^2 \right].
\]

The first term is $O_p(1/T)$. The second term is $O_p(1/N)$ in the special case that $e_{jt}$ are serially uncorrelated. In general, the second term is $O_p(1/N) + O_p(1/T)$, which can be proved by adding and subtracting $E(e_{jt} e_{js})$ and use Assumption A1(ii)(b). Hence under Assumption A, the idiosyncratic errors can only have limited time and cross-section correlations.
3 Consistency Results

From $\frac{1}{NT}XX' = U_{NT}D_{NT}^2U'_{NT}$, we have $\frac{1}{NT}XX'\tilde{F} = \tilde{D}D_{NT,r}^2$. Plugging in $X = F^0\Lambda^0 + e$ and expanding terms give

$$\frac{F^0(\Lambda^0\Lambda^0)}{N} \frac{F^0'\tilde{F}}{T} + \frac{F^0\Lambda^0'e'\tilde{F}}{NT} + \frac{e\Lambda^0F^0'\tilde{F}}{NT} + \frac{ee'\tilde{F}}{NT} = \tilde{F}D_{NT,r}^2.$$  \hspace{1cm} (6)

Various results will be obtained from this useful identity. Define the rotation matrix

$$H_{NT,0} = \left(\frac{\Lambda^0\Lambda^0}{N}\right)\left(\frac{F^0'\tilde{F}}{T}\right)D_{NT,r}^{-2}.$$  

Note that this is the transpose of the one defined in Bai and Ng (2002).

3.1 Consistent Estimation of the Factor Space

We want to establish that $\tilde{F}_t$ is close to $F_t$ and $\tilde{\Lambda}_i$ is close to $\Lambda_i$ in some well-defined sense.

Multiplying $D_{NT,r}^{-2}$ to both sides of (6) and using the definition of $H_{NT,0}$, we have

$$\tilde{F} - F^0H_{NT,0} = \left(\frac{F^0\Lambda^0'e'\tilde{F}}{NT} + \frac{e\Lambda^0F^0'\tilde{F}}{NT} + \frac{ee'\tilde{F}}{NT}\right)D_{NT,r}^{-2}.$$  \hspace{1cm} (7)

Taking the norm on both sides. we have

$$\frac{1}{T}||\tilde{F} - F^0H_{NT,0}||^2 \leq \left\{2\left(\frac{||F^0||^2}{T^2}\right)\left(\frac{1}{T}\|\frac{1}{N}\Lambda^0'e'\|)^2 + \frac{\|\tilde{F}\|}{NT}\right\} \|D_{NT,r}^{-2}\|^2,$$

**Proposition 1** Under Assumption A, the following holds in squared Frobenius norm

(i). \hspace{1cm} $\frac{1}{T}||\tilde{F} - F^0H_{NT,0}||^2 = \frac{1}{T}\sum_{t=1}^{T} ||\tilde{F}_t - H'_{NT,0}F^0_t||^2 = O_p(\delta_{NT}^{-2})$

(ii). \hspace{1cm} $\frac{1}{N}\|\tilde{\Lambda} - \Lambda^0(H'_{NT,0})^{-1}\|^2 = \frac{1}{N}\sum_{i=1}^{N} ||\tilde{\Lambda}_i - H'_{NT,0}^{-1}\Lambda^0_i||^2 = O_p(\delta_{NT}^{-2})$

(iii). \hspace{1cm} $\frac{1}{NT}\|\tilde{C} - C^0\|^2 = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} ||\tilde{C}_{it} - C^0_{it}||^2 = O_p(\delta_{NT}^{-2})$.

Part (i) of Proposition 1 says that the average squared deviation between $\tilde{F}$ and the space spanned by the true factors will vanish at rate $\min(N, T)$, which is the smaller of the sample size in the two dimensions. This result corresponds to Theorem 1 of Bai and Ng (2002), but
the argument is now simpler. It uses the fact that \( \| F^0 \|^2 / T = O_p(1) \) by Assumption A2, \( \| \tilde{F} \|^2 / T = r \) by normalization, \( \| \tilde{D} \| = O_p(1) \), \( \| \Lambda' e \|^2 = O_p(\frac{1}{N}) \) from equation (5) and \( \| \frac{1}{NT} ee' \|^2 = O_p(\frac{1}{T^2}) + O_p(\frac{1}{N}) \) by Lemma 1. Part (ii) follows by symmetry. Part (iii) does not depend on \( H_{NT,0} \) and is a consequence of (i) and (ii).

Part (i) is weaker than uniform convergence of \( \tilde{F}_t \) to \( F^0_t \). However, this result is sufficient to validate many uses of \( \tilde{F}_t \), the most important being consistent estimation of the number of factors, and being able to treat \( \tilde{F} \) as \( F^0 \) in factor augmented regressions.

3.2 The Limit of \( \tilde{F}'F^0/T \)

An important quantity in determining the properties of the factor estimates is \( \tilde{F}'F^0/T \).

**Proposition 2** Let the \( r \times r \) matrix \( \Sigma \) denote \( \Sigma = \Sigma_\Lambda^{-1/2} \Sigma_F \Sigma_\Lambda^{-1/2} \) and its spectral decomposition \( \Sigma = \Upsilon D_r^2 \Upsilon' \) with \( \Upsilon' \Upsilon = I_r \). Under Assumption A, then \( \lim_{N,T \to \infty} D_{NT,r}^2 = D_r^2 \) and

\[
\tilde{F}'F^0/T \overset{p}{\to} Q = D_r \Upsilon \Sigma_\Lambda^{-1/2}.
\]

**Proof.** The proof of \( \lim_{N,T \to \infty} D_{NT,r}^2 = D_r^2 \) is given in Stock and Watson (1998). We focus on the limit of \( \tilde{F}'F^0/T \). Multiply \( \frac{1}{T} F^0 \) on both sides of (6), we have

\[
\left( \frac{F^0}{T} \right)' \left( \frac{\Lambda^0}{N} \right) \left( \frac{F^0}{T} \right) + \left( \frac{F^0}{T} \right)' \left( \frac{\Lambda^0 e'}{NT} \right) \left( \frac{F^0}{T} \right) + \left( \frac{F^0}{T} \right)' \left( \frac{\Lambda e}{NT} \right) \left( \frac{F^0}{T} \right) + \left( \frac{F^0}{T} \right)' \left( \frac{e e'}{NT^2} \right) = \frac{F^0}{T} - D_{NT,r}^2.
\]

The second and third terms on the left hand side are negligible since the \( r \times r \) matrix

\[
\frac{F^0 e \Lambda^0/N}{NT} = \frac{1}{NT} \sum_t \sum_i F_i \Lambda_i e_{it} = O_p(\delta_{NT}^{-2}).
\]

The fourth term is also negligible because

\[
\frac{F^0 ee' \tilde{F}}{NT^2} = \frac{F^0 e e' F^0}{NT^2} H_{NT,0} + \frac{F^0 ee' (\tilde{F} - F^0 H)}{NT^2}
\]

and each term is negligible. This implies that

\[
\left( \frac{F^0}{T} \right)' \left( \frac{\Lambda^0}{N} \right) \left( \frac{F^0}{T} \right) + o_p(1) = \frac{F^0}{T} - D_{NT,r}^2.
\]

If we left multiply \( \left( \frac{\Lambda^0}{N} \right) \left( \frac{F^0}{T} \right) \) on each side and define

\[
\Sigma_{NT} = \left( \frac{\Lambda^0}{N} \right)^{1/2} \left( \frac{F^0}{T} \right) \left( \frac{\Lambda^0}{N} \right)^{1/2},
\]

\[
\Upsilon_{NT} = \left( \frac{\Lambda^0}{N} \right)^{1/2} \left( \frac{F^0}{T} \right),
\]

\[1 \text{Proposition 2 corresponds to Proposition 1 of Bai (2003) which is stated in terms of } V \text{ instead of } D_r^2.\]
Lemma 2

Under Assumption A, \( \sum_{NT} \mathbf{Y}_{NT} + o_p(1) = \mathbf{Y}_{NT} \mathbf{D}_{NT,r}^2. \)

Now \( \mathbf{Y}_{NT} \) can be interpreted as the (non-normalized) eigenvectors of matrix \( \sum_{NT} \). These eigenvectors do not have unit length even asymptotically because \( \mathbf{Y}_{NT}' \mathbf{Y}_{NT} \xrightarrow{p} \mathbf{I}_r \). We can define normalized eigenvectors \( \mathbf{Y}_{NT} \) as \( \mathbf{Y}_{NT} = \mathbf{Y}_{NT} \mathbf{D}_{NT,r}^{-1} \) so that \( \mathbf{Y}_{NT}' \mathbf{Y}_{NT} \xrightarrow{p} \mathbf{I}_r \).

Since \( \mathbf{A}' \mathbf{A}/N \xrightarrow{p} \mathbf{A} \) and \( \mathbf{F}' \mathbf{F}/T \xrightarrow{p} \mathbf{F} \), \( \sum_{NT} \) converges to \( \sum = \sum_{A}^{1/2} \sum_{F} \sum_{A}^{1/2} \). From \( \sum_{NT} \mathbf{Y}_{NT} + o_p(1) = \mathbf{Y}_{NT} \mathbf{D}_{NT,r}^2 \), taking the limit yields \( \sum \mathbf{Y} = \mathbf{Y} \mathbf{D}_r^2 \), where \( \mathbf{Y} \) is the limit of \( \mathbf{Y}_{NT} \) (note that since the eigenvalues of \( \sum \) are distinct, \( \mathbf{Y} \) is unique up to a column sign change, depending the column sign of \( \mathbf{F}' \)). So \( \mathbf{D}_r^2 \) is the diagonal matrix consisting of the eigenvalues of \( \sum \), and \( \mathbf{Y} \) is the matrix of eigenvectors with \( \mathbf{Y}' \mathbf{Y} = \mathbf{I}_r \). We have

\[
\frac{\mathbf{F}' \mathbf{F}}{T} = \left( \frac{\mathbf{A}' \mathbf{A}}{N} \right)^{-1/2} \mathbf{Y}_{NT} \mathbf{D}_{NT,r} \xrightarrow{p} \sum_{A}^{-1/2} \mathbf{Y} \mathbf{D}_r \equiv \mathbf{Q}'.
\]

Note that \( \mathbf{Q} \) is not, in general, an identity matrix. Proposition 2 implies two useful results for what is to follow:

\[
\begin{align*}
Q'D_r^{-2} &= \sum_{A}^{-1} Q^{-1} \quad (8a) \\
\sum_{F}^{-1} Q' &= Q^{-1}. \quad (8b)
\end{align*}
\]

The first identity follows from the definition of \( \mathbf{Q} \) that \( Q'D_r^{-2}Q = \sum_{A}^{-1/2} \mathbf{Y} D_r D_r^{-2} D_r \mathbf{Y} \sum_{A}^{-1/2} = \sum_{A}^{-1} \). The second identity uses \( Q \sum_{F}^{-1} Q' = D_r \mathbf{Y} (\sum_{A}^{-1/2} \sum_{F}^{-1} \sum_{A}^{-1/2}) \mathbf{Y} D_r = D_r \mathbf{Y} \sum_{A}^{-1} \mathbf{Y} D_r \) which simplifies to \( D_r D_r^{-2} D_r = \mathbf{I}_r \). The two identities can equivalently be stated as \( Q'D_r^{-2}Q = \sum_{A}^{-1} \) and \( Q \sum_{F}^{-1} Q' = \mathbf{I}_r \), respectively.

### 3.3 Equivalent Rotation Matrices

As seen above, \( \mathbf{F}' \) is based on \( \mathbf{U}_r \), the left singular vectors of \( \mathbf{X} \) and thus all linear transformations of \( \mathbf{U}_r \) are also solutions. The following Lemma will be useful in establishing that \( \mathbf{H}_{NT,0} \) has asymptotically equivalent representations.

**Lemma 2** Under Assumption A, \( \frac{\mathbf{F}' \mathbf{e}' \mathbf{F}}{NT^2} = O_p(\delta_{NT}^{-2}). \)

**Proof:** From (4), \( \frac{\mathbf{F}' \mathbf{e}' \mathbf{F}}{NT^2} = O_p(1/T) \). Now adding and subtracting terms,

\[
\frac{\mathbf{F}' \mathbf{e}' \mathbf{F}}{NT^2} = \frac{(\mathbf{F} - \mathbf{F}' \mathbf{H})' \mathbf{e} \mathbf{e}' (\mathbf{F} - \mathbf{F}' \mathbf{H})}{NT^2} + \frac{\mathbf{H} \mathbf{F}' \mathbf{e} \mathbf{e}' (\mathbf{F} - \mathbf{F}' \mathbf{H})}{NT^2} + \frac{(\mathbf{F} - \mathbf{F}' \mathbf{H})' \mathbf{e} \mathbf{e}' \mathbf{F}' \mathbf{H}}{NT^2}
= a + b + c + d.
\]
\[ \|a\| \leq \frac{\|\tilde{F} - F^0 H\|^2}{T} \frac{\|e e'\|}{NT} = O_p(\delta_{NT}^{-2})O_p(\delta_{NT}^{-1}) \]
\[ \|b\| \leq \frac{\|\tilde{F} - F^0 H\|}{\sqrt{T}} \frac{\|e e'\| \|F^0\|}{NT} \frac{\|H\|}{\sqrt{T}} = O_p(\delta_{NT}^{-1})O_p(\delta_{NT}^{-1})O_p(1) = O_p(\delta_{NT}^{-2}) \]
\[ \|b\| \equiv \|c\| \]
\[ \|d\| \leq \|H\|^2 \frac{\|F^0 e e' F^0\|}{NT^2} = O_p(\delta_{NT}^{-2}). \]

We are now in a position to consider asymptotically equivalent rotation matrices:

**Lemma 3** Let \( H_{NT,0} = \left( \frac{\lambda'^0 \lambda^0}{N} \right) \left( \frac{F^0 e}{T} \right) D_{NT,r}^{-2} \) and define

\[
H_{NT,1} = (\lambda'^0 \lambda^0) \tilde{\Lambda}^0 (\lambda'^0 \lambda^0)^{-1}, \quad H_{NT,1} = (\tilde{\Lambda}^0 \lambda^0) (\lambda'^0 \lambda^0)^{-1},
\]
\[
H_{NT,2} = (F^0 F^0)^{-1} (F^0 \tilde{F}), \quad H_{NT,2} = (F^0 \tilde{F})^{-1} (F^0 F^0)
\]
\[
H_{NT,3} = (\tilde{F}^0 F^0)^{-1} (\tilde{F}^0 \tilde{F}) = (F^0 F^0 / T)^{-1} \quad H_{NT,3} = (\tilde{F}^0 F^0 / T) = (\tilde{F}^0 \tilde{F})^{-1} (\tilde{F}^0 F^0)
\]
\[
H_{NT,4} = (\lambda'^0 \tilde{\Lambda}) (\lambda'^0 \tilde{\Lambda})^{-1} = (\lambda'^0 \tilde{\Lambda} / N) D_{NT,r}^2, \quad H_{NT,4} = D_{NT,r}^2 (\lambda'^0 \tilde{\Lambda} / N)^{-1}.
\]

Under Assumption A, the following holds for \( \ell = 1, 2, 3, 4 \)

i. \( H_{NT,\ell} = H_{NT,0} + O_p(\delta_{NT}^{-2}) \);

ii. \( H_{NT,\ell} \xrightarrow{p} Q^{-1} \).

**Proof:** Part (ii) follows from Proposition 2 that \( \tilde{F}^0 F^0 / T \xrightarrow{p} Q \). It remains to show that all alternative rotation matrices are asymptotically equivalent.

We begin with \( \ell = 1, 3 \). Recall that \( D_{NT,r}^2 \) is the matrix of eigenvalues of \( \frac{XX'}{NT} \) associated with the eigenvectors \( \tilde{F} \). Using the normalization \( \tilde{F}' \tilde{F} = T I_r \), we have \( \tilde{F}' \left( \frac{XX'}{NT} \right) \tilde{F} = T D_{NT,r}^2 \). Substituting \( X = F^0 \lambda'^0 + e \) into the above, we have

\[
D_{NT,r}^2 = \left( \frac{F^0 e}{T} \right) \left( \frac{\lambda'^0 \lambda^0}{N} \right) \left( \frac{F^0 e}{T} \right) + \frac{1}{T} \left( \frac{F^0 e e' \tilde{F}}{NT} \right) + O_p(\delta_{NT}^{-2}) \tag{9}
\]

where the last \( O_p(\delta_{NT}^{-2}) \) term represents the cross product term, which is dominated. The second on the right hand side is \( O_p(\delta_{NT}^{-2}) \) by Lemma 2. Substituting \( \left( \frac{F^0 e}{T} \right)^{-1} (\lambda'^0 \lambda^0)^{-1} \left( \tilde{F}^0 F^0 \right)^{-1} + O_p(\delta_{NT}^{-2}) \) for \( D_{NT,r}^2 \) into \( H_{0,NT} \) gives

\[
H_{NT,0} = \left( \frac{\tilde{F} F^0}{T} \right)^{-1} + O_p(\delta_{NT}^{-2}).
\]

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Next, left and right multiplying $X = F^0\Lambda^0 + e$ by $\tilde{F}'$ and $\Lambda^0$ respectively, dividing by $NT$, and using $\tilde{\Lambda} = \tilde{F}'X/T$, we obtain

$$\frac{\tilde{\Lambda}'\Lambda^0}{N} = \left(\frac{\tilde{F}'F^0}{T}\right)\left(\frac{\Lambda^0\Lambda^0}{N}\right) + O_p(\delta_{NT}^{-2}).$$

Substituting $\left(\frac{\tilde{\Lambda}'\Lambda^0}{N}\right)^{-1} = \left(\frac{\Lambda^0\Lambda^0}{N}\right)^{-1} \left(\frac{\tilde{F}'F^0}{T}\right)^{-1} + O_p(\delta_{NT}^{-2})$ into $H_{NT,1} = (\frac{\Lambda^0\Lambda^0}{N})(\tilde{\Lambda}'\Lambda^0)^{-1}$, we obtain

$$H_{NT,1} = \left(\frac{\tilde{F}'F^0}{T}\right)^{-1} + O_p(\delta_{NT}^{-2}).$$

Thus $H_{NT,0}$ and $H_{NT,1}$ have the same asymptotic expression.

Now consider the case of $\ell = 2, 4$. From $H_{NT,1} = H_{NT,0} + O_p(\delta_{NT}^{-2})$, we have

$$\left(\frac{\Lambda^0\Lambda^0}{N}\right)\left(\frac{\tilde{\Lambda}'\Lambda^0}{N}\right)^{-1} = \left(\frac{\tilde{F}'F^0}{T}\right)^{-1} + O_p(\delta_{NT}^{-2}).$$

Taking transpose and inverse, and substituting into the original definition of $H_{NT,0}$ yield

$$H_{NT,0} = \left(\frac{\Lambda^0\tilde{\Lambda}}{N}\right)D_{NT,r}^{-2} + O_p(\delta_{NT}^{-2}).$$

This proves part (iv). Now multiply $X = F^0\Lambda^0 + e$ by $F^0'$ on the left and $\Lambda^0$ on the right and divide by $NT$, we obtain

$$\frac{F^0'X\tilde{\Lambda}}{NT} = \frac{F^0'F^0\Lambda^0\tilde{\Lambda}}{T} + \frac{F^0'e\tilde{\Lambda}}{NT}. $$

Now $X\tilde{\Lambda} = X\tilde{\Lambda}(\tilde{\Lambda}'\tilde{\Lambda})^{-1}(\tilde{\Lambda}'\tilde{\Lambda}) = \tilde{F}(\tilde{\Lambda}'\tilde{\Lambda}) = \tilde{F}D_{NT,r}^2N$. Thus $(\frac{F^0'F^0}{T})D_{NT,r}^2 = (\frac{F^0'F^0}{T})(\frac{\Lambda^0\tilde{\Lambda}}{N}) + O_p(\delta_{NT}^{-2})$, or equivalently,

$$\left(\frac{F^0'F^0}{T}\right)^{-1} \left(\frac{F^0'\tilde{F}}{T}\right) = \left(\frac{\Lambda^0\tilde{\Lambda}}{N}\right)D_{NT,r}^{-2} + O_p(\delta_{NT}^{-2}).$$

But the left hand side is equal to $H_{NT,0} + O_p(\delta_{NT}^{-2})$.

These alternative rotation matrices, first used in Bai and Ng (2019), help understand what is meant by consistent estimation of the factor space. For example, since $H_{NT,2}$ is obtained by regressing $F_0$ on $\tilde{F}$, $H_{NT,1}F_0^0$ is asymptotically the fit from projecting $\tilde{F}_t$ on the space spanned by $F_0$. Similarly, $H_{NT,1}$ is obtained by regressing $\Lambda_0$ on $\tilde{\Lambda}$. Hence $H_{NT,1}^{-1}\Lambda^0_t$ is asymptotically the fit from projecting $\tilde{\Lambda}_t$ on the space spanned by $\Lambda^0_t$.  

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4 Distribution Theory

Consider again $X_i = F^0 \Lambda_0^0 + e_i$. As we do not observe $F^0$ or $\Lambda^0$, we need an inferential theory for $\tilde{F}_t$, $\tilde{\Lambda}_i$, and $\tilde{C}_{it} = \tilde{F}_t \tilde{\Lambda}_i$. The following assumption will be used to derive the limiting distributions.

Assumption B. As $N, T \to \infty$, the following holds for each $i$ and $t$:

$$
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Lambda_0^0 e_{it} \xrightarrow{d} N(0, \Gamma_t)
$$

$$
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} F^0 e_{it} \xrightarrow{d} N(0, \Phi_i).
$$

Theorems 1 and 2 of Bai (2003) establish the limiting distribution of $\tilde{F}_t$ and $\tilde{\Lambda}_i$ based on the rotation matrix $H_{NT,0}$ as follows:

$$
\sqrt{N}(\tilde{F}_t - H_{NT,0}^r F^0_t) \xrightarrow{d} N(D_{r-2}^r Q_t Q' D_{r-2}^r)
$$

$$
\sqrt{T}(\tilde{\Lambda}_i - H_{NT,0}^{-1} \Lambda^0_i) \xrightarrow{d} N(0, Q'^{-1} \Phi_i Q^{-1}).
$$

(10a) (10b)

We will use alternative rotation matrices to obtain the limiting distributions. To proceed, we need the following, shown in the Appendix.

Lemma 4 Suppose that Assumption A holds. We have, for $\ell = 0, 1, 2, 3, 4$,

$$
i \quad \frac{1}{T} F^0 \tilde{F}'(\tilde{F} - F^0 H_{NT,\ell}) = O_p(\delta^2_{NT})
$$

$$
ii \quad \frac{1}{N} \Lambda^0 (\tilde{\Lambda} - \Lambda_0 H_{NT,\ell}^{-1}) = O_p(\delta^2_{NT}).
$$

$$
iii \quad \frac{1}{T} (\tilde{F} - F^0 H_{NT,\ell})' e_i = O_p(\delta_{NT}) \text{ for each } i,
$$

$$
iv \quad \frac{1}{N} e_i' (\tilde{\Lambda} - \Lambda_0 H_{NT,\ell}^{-1}) = O_p(\delta_{NT}) \text{ for each } t.
$$

To obtain the limiting distribution of $\tilde{\Lambda}_i$, we multiply $\frac{1}{T} \tilde{F}'$ to both sides of $X = F^0 \Lambda^0 + e$ to obtain

$$
\frac{1}{T} \tilde{F}' X = (\tilde{F}' F^0 / T) \Lambda^0 + \tilde{F}' e / T
$$

$$
\tilde{\Lambda}' = H_{NT,3}^{-1} \Lambda^0 + \tilde{F}' e / T
$$

$$
= H_{NT,3}^{-1} \Lambda^0 + H_{3,NT}^r F^0 e / T + (\tilde{F} - F^0 H_{NT,3})' e / T.
$$

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This implies $\tilde{\Lambda}_i - H_{NT,3}^{-1} \Lambda_i^0 = H'_{NT,3} \sum_{t=1}^T F_t^0 e_{it} + O_p(\delta_{NT}^{-2})$. For the distribution of $\tilde{F}_t$, we multiply $\tilde{\Lambda}(\tilde{\Lambda}')^{-1}$ to both sides of $X = F^0 \Lambda^0 + e$:

$$X \tilde{\Lambda}(\tilde{\Lambda}')^{-1} = F^0 \Lambda^0 \tilde{\Lambda}(\tilde{\Lambda}')^{-1} + e \tilde{\Lambda}(\tilde{\Lambda}')^{-1}$$

$$\tilde{F} = F^0 H_{NT,4} + e \tilde{\Lambda}(\tilde{\Lambda}')^{-1}$$

$$= F^0 H_{NT,4} + e \Lambda^0 H'_{NT,4}(\tilde{\Lambda})^{-1} + e (\tilde{\Lambda} - \Lambda^0 H'_{NT,4}(\tilde{\Lambda})^{-1})$$

This implies $\tilde{F}_t - H'_{NT,4} F_t^0 = (\tilde{\Lambda}^0 / N)^{-1} H_{NT,4}^{-1} \sum_{i=1}^N \Lambda_i^0 e_{it} + O_p(\delta_{NT}^{-2})$. Putting the results together,

$$\sqrt{T}(\tilde{\Lambda}_i - H_{3,NT}^{-1} \Lambda_i^0) = H'_{NT,3} \frac{1}{\sqrt{T}} \sum_{t=1}^T F_t^0 e_{it} + \sqrt{T} O_p(\delta_{NT}^{-2}) \quad (11a)$$

$$\sqrt{N}(\tilde{F}_t - H'_{NT,4} F_t^0) = \left( \frac{\tilde{\Lambda}^0 / N}{N} \right)^{-1} H_{NT,4}^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^N \Lambda_i^0 e_{it} + \sqrt{N} O_p(\delta_{NT}^{-2}). \quad (11b)$$

Assumption B then implies that $(\tilde{F}, \tilde{\Lambda})$ are asymptotically normal with asymptotic variances given in (10a) and (10b). But from $H'_{NT,3} = H_{NT,2}^{-1} (F^0 F^0 / T)^{-1}$ and using (11a), it also holds that

$$\sqrt{T}(\tilde{\Lambda}_i - H_{3,NT}^{-1} \Lambda_i^0) = H_{NT,2}^{-1} \left( \frac{F^0 F^0}{T} \right)^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^T F_t^0 e_{it} + \sqrt{T} O_p(\delta_{NT}^{-2}).$$

Now since $(\tilde{\Lambda}' / N)^{-1} H_{NT,4}^{-1} = (\Lambda^0 / N)^{-1} = H'_{1} (\Lambda^0 / N)^{-1} + O_p(\delta_{NT}^{-2})$, we also have

$$\sqrt{N}(\tilde{F}_t - H'_{NT,4} F_t^0) = H'_{NT,1} \left( \frac{\Lambda^0 / N}{N} \right)^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^N \Lambda_i^0 e_{it} + \sqrt{N} O_p(\delta_{NT}^{-2}).$$

Define

$$\xi^F_{it} = \left( \frac{F^0 F^0}{T} \right)^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^T F_t^0 e_{it} \xrightarrow{d} N(0, \Sigma_F \Phi, \Sigma_F),$$

$$\xi^\Lambda_{it} = \left( \frac{\Lambda^0 / N}{N} \right)^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^N \Lambda_i^0 e_{it} \xrightarrow{d} (0, \Sigma_\Lambda^{-1} \Gamma, \Sigma_\Lambda^{-1}).$$

A compact way to summarize the estimation error is

$$\sqrt{N}(\tilde{F}_t - H'_{NT,4} F_t^0) = H'_{NT,1} \xi^\Lambda_{it} + o_p(1) \quad (12a)$$

$$\sqrt{T}(\tilde{\Lambda}_i - H_{3,NT}^{-1} \Lambda_i^0) = H_{NT,2}^{-1} \xi^F_{it} + o_p(1). \quad (12b)$$

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Proposition 3 Under Assumptions A and B and the normalization that \( F'F/T = I \), and \( \Lambda'\Lambda/N \) is diagonal, we have, as \( N, T \to \infty \),
\[
\sqrt{N}(\tilde{F}_t - H_{NT}F_t^0) \xrightarrow{d} N(0, Q^{-1}\Sigma_\Lambda^{-1}\Gamma_t\Sigma_\Lambda^{-1}Q^{-1})
\]
\[
\sqrt{T}(\tilde{\Lambda}_t - H_{NT,3}\Lambda_t^0) \xrightarrow{d} N(0, Q\Sigma_F^{-1}\Phi_t\Sigma_F^{-1}Q').
\]

Although the limiting covariance matrices are different from those given in (10a) and (10b), they are mathematically identical because of the different ways to represent \( Q \), as shown in (8a) and (8b). Regardless of the choice of the rotation matrix, the factor estimates are all asymptotically normal. However, as long as \( \tilde{F} \) are used as regressors, there is only one way to construct the confidence intervals in augmented regressions as all rotation matrices are asymptotically the same.

It would seem convenient to assume that \( H_{NT} \) is an identity matrix in making inference. But from Proposition 2, any of the \( H_{NT} \) considered is \( I \), only if the true \( (F^0, \Lambda^0) \) satisfy \( 1/T F^0' F^0 \) and \( \Lambda^0' \Lambda^0 \) is a diagonal matrix, which are strong identification assumptions. As pointed out in Bai and Ng (2013), these assumptions will affect not just where we center the limiting distribution of the factor estimates, but also their asymptotic variances.\(^2\) Hence these restrictions are not innocuous.

While there are many ways to represent the sampling error of \( \tilde{F}_t \) and \( \tilde{\Lambda}_t \), the properties of \( \tilde{C}_{it} \) are invariant to the choice of \( H_{NT,t} \), so we can simply write \( H_{NT} \). By definition, \( C_{it}^0 = \Lambda_i^0 F_t^0 \) and \( \tilde{C}_{it} = \tilde{\Lambda}_i \tilde{F}_t \). Thus
\[
\tilde{C}_{it} - C_{it}^0 = \Lambda_i^0 H_{NT}^{-1}(\tilde{F}_t - H_{NT} F_t^0)' + (\tilde{\Lambda}_i - H_{NT}^{-1} \Lambda_i^0)' \tilde{F}_t
\]
\[
= \Lambda_i^0 H_{NT}^{-1}(\tilde{F}_t - H_{NT} F_t^0)' + F_{it}^0 H_{NT}(\tilde{\Lambda}_i - H_{NT}^{-1} \Lambda_i^0) + O_p(\delta_{NT}^{-2}).
\]

Using the results for \( \tilde{F}_t \) and \( \tilde{\Lambda}_t \),
\[
(\tilde{C}_{it} - C_{it}^0) = \frac{1}{\sqrt{N}} \Lambda_i^0 H_{NT}^{-1} H_{NT}' \left( \frac{\Lambda_i^0 \Lambda_i^0}{N} \right)^{-1} \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \Lambda_i^0 e_{it}
\]
\[
+ \frac{1}{\sqrt{T}} F_{it}^0 H_{NT} H_{NT}^{-1} \left( \frac{F_{it}^0 F_{it}^0}{T} \right)^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_{it}^0 e_{it} + O_p(\delta_{NT}^{-2})
\]
\[
= \frac{1}{\sqrt{N}} \Lambda_i^0 e_{it} + \frac{1}{\sqrt{T}} F_{it}^0 e_{it} + o_p(1).
\]

Now \( F_{it}^0 e_{it} \xrightarrow{d} N(0, W_{it}^F) \) and \( \Lambda_i^0 e_{it} \xrightarrow{d} N(0, W_{it}^\Lambda) \), where \( W_{it}^F = F_{it}^0 \Sigma_F^{-1} \Phi_t \Sigma_F^{-1} F_{it}^0 \), and \( W_{it}^\Lambda = \Lambda_i^0 \Sigma_\Lambda^{-1} \Gamma_t \Sigma_\Lambda^{-1} \Lambda_i^0 \). This leads to a the distribution theory for the estimated common components.

\(^2\)It is possible to relax some of these diagonality restrictions so long as they are replaced by a sufficient number of linear restrictions as in Bai and Wang (2014).
Proposition 4 Under Assumptions A and B and the normalization that \( F'F / T = I \), and \( \Lambda'\Lambda / N \) is diagonal, we have, as \( N, T \to \infty \),

\[
\sqrt{\frac{1}{N} \tilde{W}_{NT, it}^A + \frac{1}{T} \tilde{W}_{NT, it}^F} \rightarrow \mathcal{N}(0, 1)
\]

where \( \tilde{W}_{NT, it}^A \) and \( \tilde{W}_{NT, it}^F \) are consistent estimates of \( W_{it}^A \) and \( W_{it}^F \), respectively.

Proposition 4 characterizes the sampling uncertainty of \( \tilde{C}_{it} \) for each \( i = 1, \ldots, N \) and \( t = 1, \ldots, T \). This error is also asymptotically normal but the convergence rate is unusual: it is the smaller of the sample size in the two dimensions, being \( \delta_{N,T} = \min(\sqrt{N}, \sqrt{T}) \). The sampling distribution allows confidence intervals to be constructed for each or a collection of \( \tilde{C}_{it} \). Such an analysis is possible because of Assumptions A and B.

The results thus far are derived for the APC estimates where the principal components taken to be \( U_r \), where we recall that these are the left eigenvectors of \( Z = \frac{X}{\sqrt{NT}} \). But some textbooks such as Hastie et al. (2001) define principal components as \( U_rD_r \). Though the two definitions will yield principal components that are perfectly correlated, they are based on different normalizations. As normalizing \( F \) to be unit length can be restrictive for some purposes, Bai and Ng (2019) define the principal components estimator (PC) as

\[
\tilde{F} = \sqrt{T}U_{NT,r}D_{NT,r}^{1/2}, \quad \tilde{\Lambda} = \sqrt{N}V_{NT,r}D_{NT,r}^{1/2}.
\]

The PC estimates are related to APC estimates:

\[
\tilde{F} = \tilde{F}D_{NT,r}^{1/2}, \quad \tilde{\Lambda} = \tilde{\Lambda}D_{NT,r}^{-1/2}.
\]

The limiting distribution of the PC estimates follow immediately from those for \( (\tilde{F}, \tilde{\Lambda}) \). Why consider the PC estimates? Because \( \tilde{F}'\tilde{F} = \frac{\tilde{\Lambda}'\tilde{\Lambda}}{N} = D_{NT,r} \), so the factor estimates are no longer unit length. This opens the possibility for constrained estimation. For example, nuclear-norm regularization yields

\[
(\tilde{F}_z, \tilde{\Lambda}_z) = \arg\min_{F, \Lambda} \frac{1}{2} \left( \|Z - F'\Lambda\|^2_F + \gamma \|F\|^2_F + \gamma \|\Lambda\|^2_F \right).
\]

This set up is of interest because it is a convexified formulation of the minimum-rank problem which has a long standing history in factor analysis and has received renewed interest in the machine learning literature in recent years. See ten Berge and Kiers (1991), Saunderson et
al. (2012) and Bertsimas et al. (2017) among others. The solution entails truncating small singular values. Define the singular value thresholding operator (SVT) as

\[ D_{NT,r}^\gamma = \left[ D_{NT,r} - \gamma I_r \right]_+ \equiv \max(D_{NT,r} - \gamma I_r, 0). \] (15)

The robust principal components estimator (RPC) is defined as:

\[ \bar{F} = \sqrt{T} U_{NT,r}(D_{NT,r}^\gamma)^{1/2} \] (16a)
\[ \bar{\Lambda} = \sqrt{N} V_{NT,r}(D_{NT,r}^\gamma)^{1/2}. \] (16b)

Since \((\bar{F}, \bar{\Lambda}) = (\tilde{F}(D_{NT,r}^\gamma)^{1/2}, \tilde{\Lambda} \Delta_{NT}) \) where \(\Delta_{NT}^2 = D_{NT,r}^\gamma D_{NT,r}^{-1}\), this penalized objective function can be used to obtain a robust estimate of the number of factors.

5 The Number of Factors

The foregoing results presume that the number of factors \(r\) is unknown which is not usually the case in practice. An informal analysis is to plot the eigenvalues and use the point where the plot changes slope as an estimate of \(r\). This is the ‘scree plot’ first considered in Cattell (1966) and implemented in many software packages. A more formal approach is to balance the cost of adding an additional factor against model complexity. Let \(\text{ssr}(\bar{F}, k)\) be the sum of squared residuals when \(k\) factors are estimated. For given \(r_{\max}\), Bai and Ng (2002) propose to determine \(r\) by

\[ \hat{r} = \min_{k=0, \ldots, r_{\max}} IC(k), \quad \hat{IC}(k) = \log(\text{ssr}(\bar{F}, k)) + k \cdot g(N, T) \]

where \(g(N, T)\) is chosen such that

\[(i). \quad g(N, T) \to 0, \quad (ii). \quad \delta_{NT}^2 g(N, T) \to \infty.\]

The original proof of Lemma 3 in Bai and Ng (2002) is based on \(H_{NT,0}\) matrix and is tedious. But from \(\bar{e} = X - \bar{C}\), it follows from Assumptions A and B that for any fixed \(k \geq r\),

\[ \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \bar{e}_{it}^2 - \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T e_{it}^2 = O_p(\delta_{NT}^2). \]

This implies that

\[ \frac{1}{NT} \left( \text{ssr}(\bar{F}, k) - \text{ssr}(F_0^0H, r) \right) = O_p(\delta_{NT}^{-2}). \]
For \( k < r \), Bai and Ng (2002) shows that, for some \( c > 0 \),
\[
\frac{1}{NT} \left( \text{ssr}(\tilde{F}, k) - \text{ssr}(F^0H, r) \right) \geq c.
\]
These results imply that \( g(N, T) = \frac{\log \delta_{NT}}{\delta_{NT}} \) is appropriate, as are \( \frac{(N+T)\log NT}{NT} \) and \( \frac{(N+T)\log(\frac{NT}{N+T})}{NT} \) since they satisfy the two conditions.

To relate the criterion function above to eigenvalues, recall that by construction, the standardized data have the property that \( \|Z\|_F^2 = d_1^2 + d_2^2 + \cdots + d_{\min\{N, T\}}^2 = 1 \). The PC estimate of a low rank component \( \hat{C}_k \) assumed to be of rank \( k \) satisfies
\[
\|\hat{C}_k\|_F^2 = \|D_{NT,k}\|_F^2 = d_1^2 + d_2^2 + \cdots + d_k^2.
\]
Then \( \text{SSR}_k \) based on PC estimates can be written as
\[
\text{SSR}_k = 1 - \sum_{j=1}^{k} d_j^2 = \|Z - \hat{C}_k\|_F^2,
\]
showing that criteria in the \( IC \) class are also based on eigenvalues. Ahn and Horenstein (2013) consider successive changes in eigenvalues while Onatski (2009) which formalizes the scree plot of Cattell (1966). It is difficult to avoid using eigenvalues to determine \( r \).

Recall that the number of strong factors in an approximate factor model is the number of eigenvalues that increase with \( N \). To take the focus on strong factors one step further, Bai and Ng (2019) use the rank-regularized PC estimates \( \|C_k\| = \|D_{NT,k}\|_F^2 \) in the \( IC \) criterion function. Given \( k \) and \( \gamma > 0 \), the regularized sum of squared residuals is \( \text{SSR}_k(\gamma) = 1 - \sum_{j=1}^{k} (d_j - \gamma)^2 = \|Z - \hat{C}_k\|_F^2 \). This leads to a class of rank-regularized class of criteria
\[
\hat{r} = \min_{k=0, \ldots, r_{\text{max}}} \log \left( 1 - \sum_{j=1}^{k} (d_j - \gamma)^2 \right) + kg(N, T).
\]
Taking the approximation \( \log(1 + x) \approx x \), we see that
\[
\overline{IC}(k) = \hat{IC}(k) + \gamma \sum_{j=1}^{k} \frac{(2d_j - \gamma)}{\text{SSR}_k}.
\]
Since \( d_j \geq d_j - \gamma \geq 0 \), the penalty is heavier in \( \overline{IC}(k) \) than \( \hat{IC}(k) \). The rank constraint adds a data dependent term to each factor to deliver a more conservative estimate of \( r \) that does not require the researcher to make precise the source of the small singular values. They can be due to genuine weak factors, noise corruption, omitted lagged and non-linear interaction of the factors that are of lesser importance.
5.1 Linear Constraints

The minimization problem in (14) has a unique solution under the normalization $F'F = \Lambda'\Lambda = D_r$. However, the unique solution may or may not have economic interpretations. This section considers $m$ linear restrictions on $\Lambda$ of the form

$$R\vec(\Lambda) = \phi$$  \hfill (18)

where $R$ is $m \times Nr$, and $\phi$ is $m \times 1$. Both $R$ and $\phi$ are assumed known a priori. Economic theory may suggest a lower triangular $\Lambda$. By suitable design of $R$, causality restrictions can be expressed as $R\vec(\Lambda) = \phi$ without ordering the data a priori. Cross-equation restrictions such as due to homogeneity of the loadings across individuals or a subgroup of individuals suggested by theory can also be considered. Other restrictions are considered in Stock and Watson (2016). Nos imposing diagonality of $F'F$ and $\Lambda'\Lambda$ for identification (rather than statistical normalizations) actually generate linear constraints on the loadings (18) that can be used as over-identifying restrictions with which we can use to test economic hypothesis. The Appendix provides an example how to implement the restrictions in MATLAB.

The linear restrictions on the loadings we consider here are known a priori. This stands in contrast to sparse principal components (SPC) estimation that either imposes LASSO type penalty on the loadings, or shrinks the individual entries to zero in a data dependent way.\footnote{For SPC, see Jolliffe et al. (2003), Ma (2013), Shen and Huang (2008), and Zou et al. (2006). The SPC is in turn different from the POET estimator of Fan et al. (2013) which constructs the principal components from a matrix that shrinks the small singular values towards zero.}

The constrained factor estimates $(\bar{F}_{\gamma,\tau}, \bar{\Lambda}_{\gamma,\tau})$ are defined as solutions to the penalized problem

$$\left(\bar{F}_{\gamma,\tau}, \bar{\Lambda}_{\gamma,\tau}\right) = \min_{F, \Lambda} \frac{1}{2} \|Z - F\Lambda'\|_F^2 + \frac{\gamma}{2} \left(\|F\|_F^2 + \|\Lambda\|_F^2\right) + \frac{\tau}{2} \|R\vec(\Lambda) - \phi\|_2^2$$  \hfill (19)

where $\gamma$ and $\tau$ are regularization parameters. The linear constraints can be imposed with or without the rank constraints. Imposing cross-equation restrictions will generally require iteration till the constraints are satisfied.

The first order condition with respect to $F$ for a given $\Lambda$ is unaffected by the introduction of the linear constraints on $\Lambda$. Hence, the solution

$$\bar{F}_{\gamma,\tau} = Z\Lambda(\Lambda'\Lambda + \gamma I_r)^{-1}, \quad \forall \tau \geq 0$$  \hfill (20)

can be obtained from a ridge regression of $Z$ of $\Lambda$. To derive the first order condition with respect to $\Lambda$, we rewrite the problem in vectorized form:

$$\|Z - F\Lambda'\|_F^2 = \|\vec(Z') - (F \otimes I_N)\vec(\Lambda)\|_2^2, \quad \|\Lambda\|_F^2 = \|\vec(\Lambda)\|_2^2.$$
The first order condition with respect to \( \text{vec}(\Lambda) \) is
\[
0 = -(F' \otimes I_N)[\text{vec}(Z') - (F \otimes I_N)\text{vec}(\Lambda)] + \gamma \text{vec}(\Lambda) + \tau R' [R \text{vec}(\Lambda) - \phi]
\]
\[
= -\text{vec}(Z'F) - \tau R' \phi + (F'F \otimes I_N) \text{vec}(\Lambda) + \gamma \text{vec}(\Lambda) + \tau R'R \text{vec}(\Lambda).
\]

Solving for \( \text{vec}(\Lambda) \) and denoting the solution by \( \bar{\Lambda}_{\gamma,\tau} \), we obtain
\[
\bar{\Lambda}_{\gamma,\tau} = \left( (F'F \otimes I_N) + \gamma I_{Nr} + \tau R'R \right)^{-1} \left[ \text{vec}(Z'F) + \tau R' \phi \right]
\]
\[
= \left( (F'F + \gamma I_r) \otimes I_N + \tau R'R \right)^{-1} \left[ \text{vec}(Z'F + \tau R' \phi) \right]
\]

where the last line follows from the fact that \((F'F \otimes I_N) + \gamma I_{Nr} = (F'F + \gamma I_r) \otimes I_N\). Equations (20) and (21) completely characterize the solution under rank and linear restrictions. In general, the solution will need to be solved by iterating the two equations until convergence. A reasonable starting value is \((\bar{F}, \bar{\Lambda})\), the solution satisfying the rank constraint and before the linear restrictions are imposed. However, while \(\bar{F}'\bar{F} = \bar{\Lambda}'\bar{\Lambda} = D_{\gamma,0} \) and \(D_{\gamma,0}\) is diagonal, \(\bar{F}_{\gamma,\tau}'\bar{F}_{\gamma,\tau}\) and \(\bar{\Lambda}_{\gamma,\tau}'\bar{\Lambda}_{\gamma,\tau}\) will not, in general, be diagonal when linear restrictions are present.

These constraint will not bind unless \(\tau = \infty\), and we denote by \(\Lambda_{\gamma,\infty}\) the binding solution. Observe that in the absence of linear constraints (i.e. \(\tau = 0\)),
\[
\text{vec}(\bar{\Lambda}_{\gamma,0}) = \left( (F'F + \gamma I_r) \otimes I_N \right)^{-1} \text{vec}(Z'F)
\]
which is a ridge estimator. Furthermore, (20) and (22) are the RPCA estimates when iterated till convergence. An estimator that satisfies both the rank constraint and \(R \text{vec}(\Lambda) = \phi\) can be obtained as follows. For given \(F\), let \(\bar{\Lambda}_{\gamma,\infty}\) be the solution to (19) with \(\tau = \infty\). Also let \(\bar{\Lambda}_{\gamma,0}\) be the solution with \(\tau = 0\). Similar to the usual formula for restricted OLS, the restricted solution is related to the unrestricted one as follows:
\[
\text{vec}(\bar{\Lambda}_{\gamma,\infty}) = \text{vec}(\bar{\Lambda}_{\gamma,0}) - \left[ (F'F + \gamma I_r)^{-1} \otimes I_N \right] R' \left[ R[(F'F + \gamma I_r)^{-1} \otimes I_N] R' \right]^{-1} \left( R \text{vec}(\bar{\Lambda}_{\gamma,0}) - \phi \right)
\]
This implies that a restricted estimate of \(\Lambda\) that satisfies both the rank and linear restrictions can be obtained by imposing the linear restrictions on \(\bar{\Lambda}_{\gamma,0}\), the RPCA solution of \(\Lambda\) that only imposes rank restrictions. It is easy to verify \(\bar{\Lambda}_{\gamma,\infty}\) satisfies restriction (18). Once the restricted estimates are obtained, \(F\) needs to be re-estimated based on (20). The final solution is obtained by iterating (20) and (23). We note again that \(\bar{F}_{\gamma,\infty}'\bar{F}_{\gamma,\infty}\) and \(\bar{\Lambda}_{\gamma,\infty}'\bar{\Lambda}_{\gamma,\infty}\) will not, in general, be diagonal matrices in the presence of linear restrictions.
6 Conclusion

This note has presented simplified proofs for properties of the factor estimates by principal components under the assumption that the factors are strong i.e. $\Lambda'\Lambda/N > 0$ and the population eigenvalues of $\Sigma_X$ increase with $N$. Situations may arise that require a precise documentation of the number of factors, whether they are strong or weak. Onatski (2012) formalizes weak factors as those with loadings satisfying $\Lambda'\Lambda > 0$ as $N$ and $T$ tend to infinity, and so the population eigenvalues of $\Sigma_X$ increase slower than $N$. The model choice of strong versus weak factors depends on the objective and the assumptions that the researcher finds defensible. We have also focused exclusively on estimation of static factors. Dynamic principal components are analyzed in Forni et al. (2000, 2004).
Appendix

Proof of Lemma 4

Proof of (i). Let $H_{NT} = H_{4,NT}$. Then

$$
\tilde{F} - F^0H_{NT} = e\tilde{\Lambda}(\tilde{\Lambda}'\tilde{\Lambda})^{-1} = \frac{e\tilde{\Lambda}}{N} \left( \frac{\tilde{\Lambda}'\tilde{\Lambda}}{N} \right)^{-1} = \frac{e\tilde{\Lambda}}{N} D^{-2}_{NT,r}.
$$

Hence $\frac{1}{T}F^0(\tilde{F} - F^0H_{NT}) = \frac{1}{NT}F^0 e\tilde{\Lambda}D^{-2}_{NT,r}$, where

$$
a = \frac{1}{NT}F^0 e\Lambda^0 H^{-1}_{NT} D^{-2}_{NT,r} = \left( \frac{1}{NT} \sum_{t=1}^T \sum_{i=1}^N F^0_i e_{it} \right) H^{-1}_{NT} D^{-2}_{NT,r} = O_p(\delta^{-2}_{NT})
$$

$$
\|b\| = \frac{1}{NT}F^0 e(\tilde{\Lambda} - \Lambda^0 H^{-1}_{NT}) \leq \frac{1}{\sqrt{N}} \|\tilde{\Lambda} - \Lambda^0 H^{-1}_{NT}\| \frac{1}{\sqrt{NT}} \|F^0 e\|
$$

$$
= O_p(\delta^{-1}_{NT})O_p\left(\frac{1}{\sqrt{T}}\right) = O_p(\delta^{-2}_{NT})
$$

where we use $\frac{1}{\sqrt{NT}}\|F^0 e\| = O_p(T^{-1/2})$ by equation (4). The proof of (ii) follows by symmetry to part (i).

Proof of (iv): Here, we use $H_{NT} = H_{NT,3}$. Then

$$
\tilde{\Lambda}' - H^{-1}_{3,NT}\Lambda^0 = (\tilde{F}'\tilde{F})^{-1} \tilde{F}'X - (\tilde{F}'\tilde{F})^{-1} \tilde{F}'F^0 \Lambda^0 = \tilde{F}'e/T
$$

and $\frac{1}{N}e'_{it}(\tilde{\Lambda} - \Lambda^0 H^{-1}_{NT}) = \frac{1}{NT}e'_{it} \tilde{F} = a + b$, where $a = \frac{1}{NT}e'_{it} F^0 H_{NT}$ and $b = \frac{1}{NT}e'_{it} (\tilde{F} - F^0 H_{NT})$. By Assumption A3, $a = O_p(\delta^{-2}_{NT})$, and

$$
\|b\| \leq \frac{1}{N\sqrt{T}}e'_{it}\|\tilde{F} - F^0 H_{NT}\|/\sqrt{T} = O_p(\delta^{-1}_{NT})O_p(\delta^{-1}_{NT}) = O_p(\delta^{-2}_{NT}),
$$

where $\frac{1}{N\sqrt{T}}e'_{it}\|e\| = O_p(\delta^{-1}_{NT})$ by Assumption A1(ii)(c). Proof of (iii) follows by symmetry.
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