A PICARD-LINDELÖF THEOREM FOR SMOOTH PDE

PAOLO GIORDANO AND LORENZO LUPERI BAGLINI

Abstract. We prove that Picard-Lindelöf iterations for an arbitrary smooth normal Cauchy problem for PDE converge if we assume a suitable Weissinger-like sufficient condition. This condition includes both a large class of non-analytic PDE or initial conditions, and more classical real analytic functions. The proof is based on a Banach fixed point theorem for contractions with loss of derivatives. From the latter, we also prove an inverse function theorem for locally Lipschitz maps with loss of derivatives in arbitrary graded Fréchet spaces.

1. Introduction

Starting from the work of H. Lewy [21], it is clear that a general Picard-Lindelöf theorem (PLT) for Cauchy problems of the form:

\[
\begin{align*}
\partial_t^d y(t,x) &= F\left[t, x, (\partial_x^\alpha \partial_t^\gamma y)_{|\alpha| \leq L}, \gamma \leq P\right], \\
\partial_t^j y(t_0,x) &= y_0^j(x) \quad j = 0, \ldots, d - 1,
\end{align*}
\]

(1.1)

is not possible (see also e.g. [5] and references therein for the more general problem of solvability of partial differential operators). In (1.1), we consider \( y, \ y_0^j, \ F \) as arbitrary (\( \mathbb{R}^m \)-valued) smooth functions, \( (t, x) \in T \times S \subseteq \mathbb{R} \times \mathbb{R}^s \), \( \alpha \in \mathbb{N}^s \), \( \gamma \in \mathbb{N} \), \( p, L \in \mathbb{N} \), \( d \in \mathbb{N}_{>0} \), and we assume that \( p < d \). In this paper, we show the convergence of Picard-Lindelöf iterations of the general problem (1.1) under a suitable sufficient condition depending both on the initial conditions \( y_0^j \) and the function \( F \). We also prove that this condition includes non-trivial cases where \( F \) could be non-analytic, and a large class of smooth non-analytic initial conditions \( y_0^j \).

According to [7, 8], one of the main problems in trying to solve (1.1) using Picard-Lindelöf iterations is that the corresponding fixed point integral operator \( P \) has \( L \in \mathbb{N} \) loss of derivatives, i.e. satisfies \( \| P^{n+1}(y_0) - P^n(y_0) \|_k \leq \alpha_k n \| P(y_0) - y_0 \|_{k+n+L} \) for all \( k, n \in \mathbb{N} \) (here we are using the notion of “loss of derivatives” as in [25, 7, 8], and not as e.g. in [26, 18, 19]; see Def. 2 below for a formal definition). For this reason, in Sec. 2, we first generalize the Banach fixed point theorem (BFPT) to contractions with loss of derivatives, and we will see that the aforementioned sufficient condition corresponds to a Weissinger-like assumption, [35]. In Sec. 3, we hence apply this BFPT to prove an inverse function theorem in arbitrary graded...
Fréchet spaces (not necessarily of tame type or with smoothing operators, like in Nash-Moser theorem, see [25, 15]) and for locally Lipschitz maps with loss of derivatives (non necessarily differentiable maps, like in Ekeland inverse function theorem, see [7]). In Sec. 4, this BFPT with loss of derivatives is used to prove a PLT for normal PDE. In Sec. 5, we apply this PLT to a family of PDE including both a non-analytic $F$ or non-analytic initial conditions. Finally, in Sec. 6, we present a preliminary study of the notion of contraction with loss of derivatives.

In the following, we say that the Cauchy problem (1.1) is in normal form to specify that the highest derivative in $t$ (called normal variable) can be isolated on the left hand side of the PDE (some authors call this problem in Kowalevskian form).

If $y : X \rightarrow \mathbb{R}^m$, then $y^k : X \rightarrow \mathbb{R}$ is the $h = 1, \ldots, m$ component of $y$, and in $\mathbb{N} = \{0, 1, 2, \ldots\}$ we always include zero. Therefore, the notations $(\partial^\gamma_t \partial^j_i y)(t, x)$ used in (1.1) include cases where some $\alpha_j = 0$, $j = 1, \ldots, s$, or $\gamma = 0$. Finally, $C^k(X, \mathbb{R}^m)$ denotes the set of all the $C^k$ functions $f : X \rightarrow \mathbb{R}^m$, whereas $C^k(X) := C^k(X, \mathbb{R})$.

2. A Banach fixed point theorem with loss of derivatives

The idea to extend the classical Banach fixed point theorem to sequentially complete subsets $X$ of Hausdorff locally convex linear spaces $(E, \| - \|_{\alpha \in A})$ dates back to [3]. Here, a contraction is a map $P : X \rightarrow X$ satisfying

$$\forall \alpha \in A \exists k_\alpha \in [0, 1) \forall x, y \in X : |P(x) - P(y)|_\alpha \leq k_\alpha |x - y|_\alpha.$$ 

The notion of contraction has also been extended to uniform spaces ([29, 30]) and to condensing maps on Hausdorff locally convex linear spaces via the notion of measure of non-compactness (see e.g. [2] and references therein). See also [1] for a recent survey, and [10, 33, 6, 34] for updated references framed in locally convex linear spaces.

In the present section, we want to prove a Banach fixed point theorem for contractions with loss of derivatives in graded Fréchet spaces. In this paper, by a graded Fréchet space $(\mathcal{F}, (\| - \|_k)_{k \in \mathbb{N}})$ we mean a Hausdorff, complete topological vector space whose topology is defined by an increasing sequence of seminorms: $\| - \|_k \leq \| - \|_{k+1}$ for all $k \in \mathbb{N}$. We denote by $B^k_r(x) := \{y \in X \mid \|x - y\|_k < r\}$ the ball of radius $r \in \mathbb{R}_{>0}$ defined by the $k$-norm.

A first trivial and well known result we will use is the following:

\textbf{Lemma 1.} Let $(\mathcal{F}, \tau)$ be a topological space, $P : X \rightarrow \mathcal{F}$ be a continuous function, and assume that there is $y_0 \in X$ such that $\exists \lim_{n \rightarrow +\infty} P^n(y_0) \in X$. Then $\lim_{n \rightarrow +\infty} P^n(y_0)$ is a fixed point of $P$. In particular, this applies if $\mathcal{F}$ is a Fréchet space and $(P^n(y_0))_{n \in \mathbb{N}}$ is a Cauchy sequence of points of $X$, where $X \subseteq \mathcal{F}$ is a Cauchy complete subspace.

\textbf{Proof.} The usual proof works: $\overline{y} := \lim_{n \rightarrow +\infty} P^n(y_0) \in X$ exists by assumption, and since $P : X \rightarrow \mathcal{F}$ is continuous we have

$$P(\overline{y}) = P\left(\lim_{n \rightarrow +\infty} P^n(y_0)\right) = \lim_{n \rightarrow +\infty} P^{n+1}(y_0) = \overline{y}. \quad \square$$

In particular, this general Lem. 1 applies to contractions with loss of derivatives in Fréchet spaces: A key idea in defining this notion is that it has to depend on the starting point $y_0$ of the iterations:
Definition 2. Let \((\mathcal{F}, (\| - \|_k)_{k \in \mathbb{N}})\) be a Fréchet space, \(X\) be a closed subset of \(\mathcal{F}\), \(y_0 \in X\) and \(L \in \mathbb{N}\). We say that \(P\) is a contraction with \(L\) loss of derivatives starting from \(y_0\) (and we simply write \(P \in C(X, L, y_0)\)) if the following conditions are fulfilled:

(i) \(P : X \rightarrow \mathcal{F}\) is continuous;

(ii) \(P^n(y_0) \in X\) for all \(n \in \mathbb{N}\);

(iii) For all \(k, n \in \mathbb{N}\) there exist \(\alpha_{kn} \in \mathbb{R}_{>0}\) such that
\[
\|P^{n+1}(y_0) - P^n(y_0)\|_k \leq \alpha_{kn}\|P(y_0) - y_0\|_{k+nL},
\] (2.1)
and

(iv) For all \(k \in \mathbb{N}\), the following Weissinger condition holds:
\[
\sum_{n=0}^{+\infty} \alpha_{kn}\|P(y_0) - y_0\|_{k+nL} < +\infty.
\] (W)

Note that if we actually have only one norm \(\| - \|_k = \| - \|_0\) and \(L = 0\) (ODE case), then (W) reduces to the classical Weissinger condition, [35].

We first note that condition (2.1) trivially holds for \(n = 0\) by taking \(\alpha_{k,0} = 1\). On the other hand, thinking at (W), we are clearly interested only at the asymptotic behavior of \(\alpha_{kn}\) as \(n \rightarrow +\infty\). Secondly, Def. 2 is weaker than the usual definition of contraction because of the following first three remarks:

(i) We have a loss \(L \geq 0\) of derivatives. If \(L = 0\), Def. 2 actually tells us that, for all \(k \in \mathbb{N}\), there exists \(N_k \in \mathbb{N}\) such that \(\sum_{n=N_k}^{+\infty} \alpha_{kn} < 1\), namely \(P\) is an ordinary contraction with respect to \(\| - \|_k\) when restricted to \(\{P^n(x_0) \mid n \geq N_k\}\).

(ii) We will see only in Sec. 5 that condition Def. 2.(ii) is not a simple generalization of the usual stronger \(P : X \rightarrow X\), but is essential for the choice of the radii in the PL Thm. 12.

(iii) Both conditions (2.1) and (W) depend on the initial point \(y_0 \in X\). In contrast to the classical BFPT, this underscores, in an abstract setting, that for PDE the property to have a contraction with loss of derivatives depends on the initial condition \(y_0 \in X\). Moreover, in this paper we are solely interested in existence results for fixed points of contractions with loss of derivatives; uniqueness results would require conditions closer to the classical BFPT (see e.g. Lemma 3).

(iv) Since we want to take \(n \rightarrow +\infty\), a condition such as Def. 2.(iii) intuitively implies that we have to consider all the derivatives controlled by \(\| - \|_k\) for all \(k \in \mathbb{N}\). It is for this reason that in the present work we deal only with smooth solutions of (1.1).

More classically, contraction property (2.1) is implied by one of the following stronger conditions:

Lemma 3. Let \(X\), \(y_0\) and \(L\) be as in Def. 2. Then, the following sufficient conditions hold:

(i) If \(P : X \rightarrow \mathcal{F}\) satisfies the property
\[
\forall k, n \in \mathbb{N} \exists \alpha_{kn} \in \mathbb{R}_{>0} \forall u, v \in X : \|P^n(u) - P^n(v)\|_k \leq \alpha_{kn}\|u - v\|_{k+nL},
\] (2.2)
then condition Def. 2.(iii) holds for all \(y_0 \in X\) with the same contraction constants \(\alpha_{kn}\).
(ii) If (2.2) holds only for \( n = 1 \) with \( \alpha_k := \alpha_{k1} \), then condition Def. 2.(iii) holds for all \( y_0 \in X \) with contraction constants \( \tilde{\alpha}_{kn} := \prod_{j=0}^{n-1} \alpha_{k+jL} \).

Moreover, if \( L = 0 \) and \( y_1, y_2 \) are fixed points of \( P \), then \( \|y_1 - y_2\|_k = 0 \) for all \( k \in \mathbb{N} \). In particular, if at least one of \( \| - \|_k \) is a norm, this entails that \( y_1 = y_2 \).

Proof. (i): In fact, (2.2) yields
\[
\|P^{n+1}(y_0) - P^n(y_0)\|_k = \|P^n(P(y_0)) - P^n(y_0)\|_k \leq \alpha_k \|P^n(y_0)\|_{k+L}.
\]

Taking \( n = 1 \) in (2.2), we have \( \|P(u) - P(v)\|_k \leq \alpha_{k,1}\|u - v\|_{k+L} \) and hence \( P(B_{\alpha_{k,1}}(u) \cap X) \subset B_{\alpha_k}(P(u)) \cap X \), so that \( P \) is continuous.

(ii): If (2.2) holds only for \( n = 1 \), then we can prove the claim by induction on \( n \). For \( n = 0 \) the conclusion is trivial since \( \tilde{\alpha}_{k0} = 1 \). For the inductive step, we have
\[
\|P^{n+2}(y_0) - P^{n+1}(y_0)\|_k = \|P(P^{n+1}(y_0)) - P(P^n(y_0))\|_k \leq \alpha_k \|P^{n+1}(y_0) - P^n(y_0)\|_{k+L} \leq \alpha_k \prod_{j=0}^{n-1} \alpha_{k+jL}\|P(y_0) - y_0\|_{k+L+nL} = \left( \prod_{j=0}^{n} \alpha_{k+jL} \right) \|P(y_0) - y_0\|_{k+(n+1)L}.
\]

Continuity of \( P \) can be proved as before.

Finally, if \( P(y_1) = y_1 \), then \( \|P^n(y_1) - P^n(y_2)\|_k = \|y_1 - y_2\|_k \leq \alpha_{kn}\|y_1 - y_2\|_k \leq \|y_1 - y_2\|_k \) because the convergence \( \sum_{n=0}^{+\infty} \alpha_{kn} < +\infty \) implies \( \alpha_{kn} \leq 1 \) for some \( n \in \mathbb{N} \). Thereby, \( \|y_1 - y_2\|_k = 0 \), which entails \( y_1 = y_2 \) if \( \| - \|_k \) is a norm.

In Thm. 11 and in the proof of Thm. 12, we will see that for the normal smooth Cauchy problem (1.1), the corresponding fixed point integral operator \( P \) always satisfies the stronger condition (2.2). Therefore, in all these cases the real dependence on \( y_0 \) actually lies in conditions (W) and (ii).

Even in the simple case of the transport equation \( \partial_t y = c \cdot \partial_x y \), where \( c, y \in C^\infty([0, a] \times S), S \subset \mathbb{R} \), we can recognize the appearance of a loss of derivative \( L = 1 \) due to the occurrence of the term \( \partial_t y \) on the right hand side of the PDE. In fact, set \( P(u)(t, x) := y_0(x) + \int_0^t c(s, x) \cdot \partial_x u(s, x) \, ds \) for a fixed \( y_0 \in C^\infty(S) \) and for any \( u \in C^\infty([0, a] \times S) \). Considering on \( C^\infty([0, a] \times S) \) the family of norms
\[
\|u\|_k := \max_{|\alpha| + |\beta| \leq k} \max_{(t, x) \in [0, a] \times S} \left| \frac{\partial^\alpha \partial_x^\beta u(t, x)}{t} \right|,
\]
we would like to argue in the following way (where, for simplicity, we consider only the case \( n = 1 \) in property (2.2)):
\[
\|P(u) - P(v)\|_k = \left\| \int_0^t c \cdot \partial_x (u - v) \, ds \right\|_k \leq a \cdot \|c\|_k \cdot \||\partial_x (u - v)\|_k \leq a \cdot \|c\|_k \cdot \||u - v\|_{k+1}.
\]
anyway, respect the same basic ideas (see Def. 7), and where the estimates (2.3) hold.

Def. 2 has been tuned to allow the proof of the following result, whose proof is surprisingly simple:

**Theorem 4** (BFPT with loss of derivatives). In the assumptions of Def. 2, if 
$P \in C(X,L,y_0)$, then $(P^n(y_0))_{n \in \mathbb{N}}$ is a Cauchy sequence, and hence

$$\overline{y} := \lim_{n \to +\infty} P^n(y_0) \in X$$

is a fixed point of $P$. Moreover, for all $k, n \in \mathbb{N}$ we have that

$$\| \overline{y} - P^n(y_0) \|_k \leq \sum_{j=n}^{+\infty} \alpha_{kj} \| P(y_0) - y_0 \|_{k+jL}.$$ 

**Proof.** If we prove that $(P^n(y_0))_{n \in \mathbb{N}}$ is a Cauchy sequence, the claim follows from Lem. 1. Let $m, n, k \in \mathbb{N}$ with $m > n$. Then

$$\| P^m(y_0) - P^n(y_0) \|_k \leq \| P^m(y_0) - P^{m-1}(y_0) \|_k + \cdots + \| P^{n+1}(y_0) - P^n(y_0) \|_k \leq \sum_{j=n}^{m-1} \alpha_{kj} \| P(y_0) - y_0 \|_{k+(m-1)L} + \cdots + \alpha_{kn} \| P(y_0) - y_0 \|_{k+nL}.$$ 

We conclude using (W) of Def. 2. The final claim holds by taking $m \to +\infty$ in (2.4) as \( \overline{y} = \lim_{m \to +\infty} P^m(y_0). \) □

Clearly, the chain of inequalities in (2.4) could be stopped in several different ways. For example, as \( \| P(y_0) - y_0 \|_{k+jL} \leq \| P(y_0) - y_0 \|_{k+(m-1)L} \cdot \sum_{j=n}^{+\infty} \alpha_{kj} \), we can continue arriving at a final term of the form \( \| P(y_0) - y_0 \|_{k+(m-1)L} \cdot \sum_{j=n}^{+\infty} \alpha_{kj} \). Actually, this would lead us to consider a limit of the form \( \lim_{m \to +\infty} p_m \cdot q_n \), which never exists if \( p_m \to +\infty \) and \( q_n = a^n \), because we can take \( n \to +\infty \) depending on \( p_n \). On the contrary, in condition (W) the summation index \( n \) links the two factors in the series; looking at Lem. 3 and next Thm. 11, we can also say that condition (W) links the right hand side \( F \) and the initial conditions \( y_0 \) of the Cauchy problem (1.1). On the other hand, it is clear that the proof of previous Thm. 4 is quite standard, and this underscores that the key step lies in Def. 2 of contraction with loss of derivatives \( L \) starting from \( y_0 \).

## 3. Solutions of equations

In this Section, we want to use the Banach fixed point Thm. 4 with loss of derivatives to solve equations of the form \( f(x) = y \) in arbitrary graded Fréchet spaces. We can also inscribe this problem as the proof of local surjection in inverse function theorems.

In the following, given a sequence \( R = (r_k)_{k \in \mathbb{N}}, r_k \in \mathbb{R}_{>0} \cup \{+\infty\} \), and a point \( x_0 \) in a graded Fréchet space \( \mathcal{F} \), we set

$$\mathcal{B}_R(x_0) := \{ x \in \mathcal{F} \mid \forall k \in \mathbb{N} : \| x_0 - x \|_k \leq r_k \}.$$ 

Note that \( \mathcal{B}_R(x_0) \) is closed in \( \mathcal{F} \) as it is a countable intersection of closed sets. Moreover, \( \mathcal{B}_R(x_0) \) trivially generalizes the space usually used in the proof of the
PLT for ODE, where we only have \( r_k = r_0 < +\infty \). The first trivial consequence of Thm. 4 reformulates \( f(x) = y \) as a fixed point of the map \( P(x) := x - f(x) + y \):

**Corollary 5.** Let \( (\mathcal{F}, (\| - \|)_k)_{k \in \mathbb{N}} \) be a graded Fréchet space. Let \( X \) be a closed subset of \( \mathcal{F} \), \( f : X \to \mathcal{F} \) be a continuous map, \( y_0 \in \mathcal{F} \) and \( L \in \mathbb{N} \). Set \( P(x) := x - f(x) + y_0 \) and assume that for all \( k, n \in \mathbb{N} \), we have

\[
P^n(y_0) \in X,
\]

\[
\exists \alpha_{kn} \in \mathbb{R}_{>0} : \left\| P^{n+1}(y_0) - P^n(y_0) \right\|_k \leq \alpha_{kn} \left\| P(y_0) - y_0 \right\|_{k+nL},
\]

\[
\sum_{n=0}^{+\infty} \alpha_{kn} \cdot \left\| P(y_0) - y_0 \right\|_{k+nL} < +\infty,
\]

then, there exists \( x_0 \in X \) such that \( f(x_0) = y_0 \).

In spite of its triviality, we will see in Sec. 5 that this result allows us to solve PDE with the same scope of the next PL Thm. 12 (which, on the other hand, already includes in its proof the verification of property (3.3)). Moreover, let us now note that in Cor. 5 we do not require differentiability of \( f \) let alone the existence of some inverse of its differential \( df(x) \).

On the other hand, simply by generalizing the derivation of the inverse function theorem from the classical BFPT in Banach spaces (see e.g. [16, 4, 20]), we obtain the following theorem:

**Theorem 6.** Let \( (\mathcal{X}, (\| - \|)_k)_{k \in \mathbb{N}} \), \( (\mathcal{Y}, (\| - \|)_k)_{k \in \mathbb{N}} \) be graded Fréchet spaces, \( x_0 \in \mathcal{X}, R = (r_k)_{k \in \mathbb{N}}, r_k \in \mathbb{R}_{>0} \cup \{ +\infty \} \) and set \( \bar{r}_{k+L_D} := \frac{r_k}{\delta_{k+L_D}} \), \( \bar{R} := (\bar{r}_{k+L_D})_{k \in \mathbb{N}} \).

Let \( D : \mathcal{Y} \to \mathcal{X} \) (dextrum) and \( S : \mathcal{X} \to \mathcal{Y} \) (sinistrum) be linear maps, \( f : \overline{B}_R(x_0) \to \mathcal{Y} \) and set \( P_y(x) := x - D[f(x) - y] \) for all \( y \in \overline{B}_R(f(x_0)) \) and all \( x \in \overline{B}_R(x_0) \). Assume that for all \( k \in \mathbb{N} \) we have:

(i) \( \| D(f(x)) - D(f(\bar{x})) \| - (x - \bar{x}) \| \leq \alpha_k \cdot \| x - \bar{x} \|_{k+L} \) for some \( L \in \mathbb{N} \), \( \alpha_k > 0 \) and all \( x, \bar{x} \in \overline{B}_R(x_0) \);

(ii) \( D \) is the right inverse of \( S \), i.e. \( S \circ D = 1_Y \);

(iii) \( \| S(x) - S(\bar{x}) \|_k \leq \sigma_k \cdot \| x - \bar{x} \|_{k+L_S} \) for some \( L_S \in \mathbb{N} \), some \( \sigma_k > 0 \) and all \( x, \bar{x} \in \mathcal{X} \);

(iv) \( \| D(y) - D(\bar{y}) \| \leq \delta_{k+L_D} \cdot \| y - \bar{y} \|_{k+L_D} \) for some \( L_D \in \mathbb{N} \), some \( \delta_k > 0 \) and all \( y, \bar{y} \in \mathcal{Y} \);

(v) \( \| P_y^{-1}(x_0) - P_y(x_0) \|_k \leq \alpha_{kn} \cdot \| P_y(x_0) - x_0 \|_{k+nL} \) for all \( n \in \mathbb{N} \), and some \( \alpha_{kn} > 0 \);

(vi) \( P_y(x_0) \in \overline{B}_R(x_0) \) for all \( n \in \mathbb{N} \) and all \( y \in \overline{B}_R(f(x_0)) \);

(vii) \( \sum_{n=0}^{+\infty} \alpha_{kn} \cdot \| P_y(x_0) - x_0 \|_{k+nL} < +\infty \).

Then, the following properties hold:

(viii) \( \| f(x) - f(\bar{x}) \|_k \leq \sigma_k \cdot (\alpha_k \cdot \| x - \bar{x} \|_{k+L_S} + \| x - \bar{x} \|_{k+L_S}) \) for all \( k \in \mathbb{N}, x, \bar{x} \in \overline{B}_R(x_0) \), i.e. \( f \) is locally Lipschitz with loss of derivatives;

(ix) \( \| f(x) - f(\bar{x}) \|_{k+L_D} \geq \frac{1}{\delta_{k+L_D}} \cdot \| \bar{x} - \bar{x}_0 \| - \alpha_k \cdot \| x - \bar{x} \|_{k+L_D} \) for all \( k \in \mathbb{N}, x, \bar{x} \in \overline{B}_R(x_0) \);

(x) \( \forall y \in \overline{B}_R(f(x_0)) \exists x \in \overline{B}_R(x_0) : f(x) = y \).

(xi) If \( g : \overline{B}_R(f(x_0)) \to \overline{B}_R(x_0) \) is any right-inverse of \( f \), i.e. \( f \circ g = 1_{\overline{B}_R(f(x_0))} \), and \( (\mathcal{X}, (\| - \|)_k)_{k \in \mathbb{N}} = (\mathcal{X}, \| - \|) \) is a Banach space, then \( \| g(y) - g(\bar{y}) \| \leq \frac{\delta_{k+L_D}}{1 - \alpha_k} \cdot \| y - \bar{y} \|_{k+L_D} \) for all \( k \in \mathbb{N}, y, \bar{y} \in \overline{B}_R(f(x_0)) \).
In particular: 1) if $r_{k+L} \leq r_k$ for all $k \in \mathbb{N}$ then $P_y : \bar{B}_R(x_0) \to \bar{B}_R(x_0)$ for all $y \in \bar{B}_R(f(x_0))$, and hence assumption (vi) always holds; 2) if we set $\alpha_{kn} := \prod_{j=0}^{n-1} \alpha_{k+jL}$, then (v) always holds.

Note that we do not require that the spaces $\mathcal{X}$, $\mathcal{Y}$ are tame or admit smoothing operators like in the Nash-Moser theorem, see [15]; we also do not require that $f$ is differentiable as in Ekeland inverse function theorem [7]. Moreover, its proof is a generalization of [16], so that it includes the inverse function theorem for Lipschitz maps in Banach spaces if $L = L_D = L_S = 0$ and $r_k = r_0$ for all $k \in \mathbb{N}$.

Proof. (viii): From (ii) and (iii), we can write

$$|f(x) - f(\bar{x})|_k = |S(D(f(x)) - S(D(f(\bar{x})))_k | \leq \sigma_k \|D(f(x)) - D(f(\bar{x}))\|_{k+L_S}$$

$$= \sigma_k \|D(f(x)) - D(f(\bar{x})) - (x - \bar{x}) + (x - \bar{x})\|_{k+L_S} \leq \sigma_k (\alpha_k \|x - \bar{x}\|_{k+L_S} + \|x - \bar{x}\|_{k+L_S}),$$

where we used (v).

(ix): Once again from (v), we get

$$\|x - \bar{x}\|_k \leq \|D(f(x)) - D(f(\bar{x})) - (x - \bar{x})\|_k + \|D(f(x)) - D(f(\bar{x}))\|_k$$

$$\leq \alpha_k \|x - \bar{x}\|_{k+L_D} + \delta_{k+L_D} \|f(x) - f(\bar{x})\|_{k+L_D}$$

because of (iv).

We immediately note that (viii) and (iv) imply the continuity of $f$ and $D$ resp. and hence also of the map $P_y(x) := x - D[f(x) - y]$, $P_y : \bar{B}_R(x_0) \to \mathcal{X}$. We have $P_y(x) = x$ if and only if $D[f(x) - y] = 0$ and hence if and only if $f(x) = y$ from (ii). Assumption (v) corresponds exactly to the request that this map contracts with loss of derivatives $L$. Weisssinger condition is assumption (vii). This proves that $P_y$ is a contraction with loss of derivatives $L$, hence (x) holds.

In particular, if $r_{k+L} \leq r_k$ we can that prove that $P_y : \bar{B}_R(x_0) \to \bar{B}_R(x_0)$. In fact, if $y \in \bar{B}_S(f(x_0))$ and $x \in \bar{B}_R(x_0)$, then

$$\|P_y(x) - x_0\|_k = \|x - x_0 - D[f(x) - f(x_0)] - D[f(x_0) - y]\|_k$$

$$\leq \|x - x_0 - D[f(x) - f(x_0)]\|_k + \|D(f(x_0)) - D(y)\|_k$$

$$\leq \alpha_k \|x - x_0\|_{k+L} + \delta_{k+L_D} \|f(x_0) - y\|_{k+L_D} \leq$$

$$\leq \alpha_k \cdot r_{k+L} + \delta_{k+L_D} \cdot r_{k+L_D} \cdot \frac{r_{k+L} (1 - \alpha_k)}{\delta_{k+L_D}} = r_{k+L} \leq r_k,$$

where we used $\|f(x_0) - y\|_{k+L_D} \leq \bar{r}_{k+L_D} = \frac{r_{k+L} (1 - \alpha_k)}{\delta_{k+L_D}}$. Finally, (i) yields:

$$\|P_y(x) - P_y(\bar{x})\|_k = \|x - D[f(x) - y] - \bar{x} + D[f(\bar{x}) - y]\|_k =$$

$$\|D[f(x) - f(\bar{x})] - (x - \bar{x})\|_k \leq \alpha_k \|x - \bar{x}\|_{k+L}.$$

From Lem. 3.(ii) and assumption (vi), we hence obtain that $P_y$ satisfies Def. 2.(iii) with contraction constants $\alpha_{kn} := \prod_{j=0}^{n-1} \alpha_{k+jL}$.

(xii): This follows directly from (ix) for $x := g(y)$ and $\bar{x} := g(\bar{y})$ considering that $\| - \|_k = \| - \|_{k+L} = \| - \|$. \hfill \□

Even if the previous statement allows us to take $r_k = +\infty$, it is now Weisssinger condition (vii) that forces to take $y$ near $f(x_0)$: the factor $\|P_y(x_0) - x_0\|_{k+nL} = \|D[f(x_0) - y]\|_{k+nL}$ is small if $y$ is near $f(x_0)$; note also that (vii) is implied by
the stronger condition $\sum_{n=0}^{+\infty}a_k r_{k+nL} < +\infty$ because of (vi). On the other hand, assumption (vii) is in principle compatible with growing term $\|P_y(x_0) - x_0\|_{k+nL}$ as $k+nL \to +\infty$, even if the Lipschitz factors $a_k$ must keep the series convergent.

4. A Picard-Lindelöf theorem for PDE

In the following, considering the Cauchy problem (1.1), we always set and assume

\[
\hat{L} := \text{Card}\{(\alpha, \gamma) \in \mathbb{N}^s \times \mathbb{N}_{\leq p} \mid |\alpha| \leq L\}
\]

\[
a, b \in \mathbb{R}_{>0}, \quad [t_0 - a, t_0 + b] \times S =: T \times S \subseteq \mathbb{R}^{1+s}
\]

\[
F \in \mathcal{C}^\infty(T \times S \times \mathbb{R}^{m\hat{L}}, \mathbb{R}^m)
\]

\[
y_{0j} \in \mathcal{C}^\infty(S, \mathbb{R}^m) \quad \forall j = 0, \ldots, d - 1,
\]

where $p \leq d - 1$ denotes the maximum order of derivatives $\partial_y^{\alpha} y(t, x) \in \mathbb{R}^m$ appearing on the right hand side of (1.1).

4.1. Supremum norms of integral functions. We have already mentioned that the first inequality in (2.3) is generally wrong. Let us construct a counterexample for the space of one variable functions $\mathcal{C}^\infty([0, a])$ with the norms $\|u\|_k := \max_{t \in [0, a]} \|u(t)\|$. Take e.g. $a = \frac{1}{2}$ and consider the straight line $y = 1$. Then

\[
\left\| \int_0^1 y \right\|_1^{(-)} = \max_{t \in [0, \frac{1}{2}]} \left\| \int_0^t 1 \right\|_1 = 1
\]

and

\[
\|y\|_1 = \max_{t \in [0, \frac{1}{2}]} \left\{ |1|, \max_{t \in [0, \frac{1}{2}]} |0| \right\} = 1,
\]

therefore

\[
\left\| \int_0^1 y \right\|_1^{(-)} = 1 > a \cdot \|y\|_1 = \frac{1}{2}.
\]

It is not hard to prove that, actually, $\|\int_0^1 y\|_k > a \cdot \|y\|_k$ for all $k \geq 1$.

This remark allows us to understand, once again, why in the classical proof of the smooth PLT we consider only the space $\mathcal{C}^0([0, a])$ of continuous functions with the supremum norm $\| - \|_0$: in fact, even if we aim to get a smooth solution $y$ (so that we would have to control all its derivatives), the normal form of the equation recursively yields the smoothness of $y$ starting from a continuous solution of the corresponding integral problem.

Similarly, we can argue for normal PDE: considering the corresponding integral problem

\[
y(t, x) = i_0(t, x) + \int_{t_0}^t ds_d \cdot d \int_{t_0}^{s_2} F \left[ s_1, x, (\partial_x^\alpha \partial_t^\gamma y)_{|\alpha| \leq L} \right] ds_1,
\]

\[
i_0(t, x) := \sum_{j=0}^{d-1} \frac{y_{0j}(x)}{j!} (t - t_0)^j.
\]

we only need that the function $y$ is of class $\mathcal{C}^p$ in $t$ and smooth in $x$: smoothness in $t$ recursively follows from (4.1), and we only have to control all its derivatives in $x$. This motivates the introduction of a space with this kind of functions.
4.2. Spaces of separately regular functions. As we mentioned above, instead of considering functions which are jointly regular in both variables \((t, x)\), we need to consider separate degree of regularity in each variable.

**Definition 7.**

(i) If \(X \subseteq \mathbb{R}^n\) is an arbitrary subset and \(q \in \mathbb{N} \cup \{\infty\}\), we say that \(f \in C^q(X, \mathbb{R}^m)\) if for each \(x \in X\) there exists an open neighborhood \(U \subseteq \mathbb{R}^n\) and a function \(F \in C^q(U, \mathbb{R}^m)\) such that \(F|_{U \cap X} = f|_{U \cap X}\).

(ii) Let \(T \times S \subseteq \mathbb{R}^{1+s}\). Set
\[
N_p^{1+s} := \left\{ \beta \in \mathbb{N}^{1+s} \mid \beta_1 \leq p \right\},
\]
and denote by \(C_p^0C_x^\infty(T \times S, \mathbb{R}^m)\) the set of continuous functions \(y \in C^0(T \times S, \mathbb{R}^m)\) such that
\[
\forall \beta \in N_p^{1+s} : \exists \partial^\beta y \in C^0(T \times S, \mathbb{R}^m).
\]

The functions in \(C_p^0C_x^\infty(T \times S, \mathbb{R}^m)\) are called *separately \(C^0C_x^\infty\) regular*. This space is endowed with the countable family of norms \(||-||_k, k \in \mathbb{N}\), defined by
\[
||y||_k := \max_{1 \leq k \leq m} \max_{|\beta| \leq k} \max_{(t, x) \in T \times S} |\partial^\beta y(t, x)|.
\]

In problem \((1.1)\), we could also consider a reduction to first order: setting \(y^1 := y\), \(y^{j+1} := \partial_t y^j, j = 1, \ldots, p\), problem \((1.1)\) is equivalent to
\[
\begin{align*}
\partial_t Y(t, x) &= \bar{F} \left[ t, x, (\partial_x Y^{\gamma+1})_{|\alpha| \leq L, \gamma \leq p} \right], \\
Y(t_0, x) &= Y_0(x),
\end{align*}
\]
where, as usual, we mean \(\partial_x Y^{\gamma+1} = \partial_x^\beta Y^{\gamma+1}(t, x)\), and
\[
\begin{align*}
Y(t, x) &:= (y^1(t, x), \ldots, y^{p+1}(t, x)) \\
Y_0(x) &:= (y_0^1(x), \ldots, y_0^{d-1}(x)) \\
\bar{F}^d \left[ t, x, (u^{\alpha, \gamma})_{|\alpha| \leq L, \gamma \leq p} \right] &:= F \left[ t, x, (u^{\alpha, \gamma})_{|\alpha| \leq L, \gamma \leq p} \right] \\
\bar{F}^j \left[ t, x, (u^{\alpha, \gamma})_{|\alpha| \leq L, \gamma \leq p} \right] &:= y^{j+1}
\end{align*}
\]
for \(j = 1, \ldots, p\). In the corresponding integral problem \((4.1)\), we could assume \(d = 1\) and hence we only need that the function \(y\) is of class \(C^0C_x^\infty\). On the one hand, this would simplify our next statements. However, we would obtain a PLT with assumptions that are clear only for \(d = 1\), and to prove from this a corresponding result for \(d > 1\) is not so easy. For this reason, we prefer to directly proceed with the generic problem \((4.1)\) without implementing a reduction to first order.

**Lemma 8.** *In the notations of Def. 7, \((C_p^0C_x^\infty(T \times S, \mathbb{R}^m), \|\|_k)_{k \in \mathbb{N}}\) is a graded Fréchet space.*

**Proof.** The only non trivial property to check is that the topology induced by the family of norms \(||-||_k\)_{k \in \mathbb{N}}\) is Cauchy complete. Let \((y_n)_{n \in \mathbb{N}}\) be a Cauchy sequence...
in \((C^0_p \mathcal{C}_x^\infty(T \times S, \mathbb{R}^m))\), so that for \(k = 0\), the sequence \((y_n)_{n \in \mathbb{N}}\) converges uniformly. Let \(y : T \times S \to \mathbb{R}^m\) be the continuous function defined by

\[
y(t, x) := \lim_{n \to +\infty} y_n(t, x) \quad \forall (t, x) \in T \times S.
\]

For all \(\beta \in \mathbb{N}_1^{1+s}\), we have \(\|\partial^{\beta} y_0 - \partial^{\beta} y_n\|_0 \leq \|y_0 - y_n\|_h\) and hence \((\partial^{\beta} y_n)_{n \in \mathbb{N}}\) is a uniformly convergent Cauchy sequence in \(C^0(T \times S, \mathbb{R}^m)\) that converges to \(\partial^{\beta} y \in C^0(T \times S, \mathbb{R}^m)\). This shows that \(y \in C^0_p \mathcal{C}_x^\infty(T \times S, \mathbb{R}^m)\). It remains to prove that \(y_n \to y\) with respect to the norms defined in (4.4). For \(k = 0\), we simply recall that the limit in (4.6) is actually a uniform limit. For \(k > 0\), we note that for all \(\beta \in \mathbb{N}_1^{1+s}\) with \(|\beta| \leq k\), the sequence \((\partial^{\beta} y_n)_{n \in \mathbb{N}}\) converges uniformly in \(T \times S\) to \(\partial^{\beta} y\).

Exactly because we have \(\beta_1 \leq p < d\) in (4.3), we can now have the desired estimate in considering the norm of an integral function:

**Lemma 9.** Let \(f \in C^0_p \mathcal{C}_x^\infty(T \times S, \mathbb{R}^m)\) and, for every \(k \in \mathbb{N}\), let \(M_k \in C^0(T \times S)\) be such that

\[
|\partial^{\nu} f^h(t, x)| \leq M_k(t, x)
\]

for all \((t, x) \in T \times S, h = 1, \ldots, m\), and all \(\nu \in \mathbb{N}^s\) such that \(|\nu| \leq k\). Set

\[
\tilde{M}_{kj}(t, x) := \left| \int_{t_0}^t ds_d \cdot \int_{t_0}^{s_2} M_k(s_1, x) \, ds_1 \right| \quad \forall (t, x) \in T \times S \forall j = 1, \ldots, d.
\]

Then, with respect to the norms in the space \(C^0_p \mathcal{C}_x^\infty\) defined as in (4.4), we have

\[
(i) \quad \left\| \int_{t_0}^t \int_{t_0}^{s_2} f(s_1, -) \, ds_1 \right\|_k \leq \max_{x \in S} \tilde{M}_{kj}(t_0 + \max(a, b), x).
\]

In particular, if \(M_k = \|f\|_k\):

\[
(ii) \quad \left\| \int_{t_0}^t \int_{t_0}^{s_2} f(s_1, -) \, ds_1 \right\|_k \leq \max(a, b) \cdot \|f\|_k.
\]

**Proof.** (i): Clearly, the notation \(\int_{t_0}^t \int_{t_0}^{s_2} f(s_1, -) \, ds_1\) denotes the function

\[
(t, x) \in T \times S \mapsto \int_{t_0}^t \int_{t_0}^{s_2} f(s_1, x) \, ds_1 \in \mathbb{R}^m \in C^0_p \mathcal{C}_x^\infty(T \times S, \mathbb{R}^m)
\]

(actually, this is a \(C^0_p \mathcal{C}_x^\infty\)-function, but in the statement we are considering the norms \(\| \cdot \|_k\) of the space \(C^0_p \mathcal{C}_x^\infty\)). For some \(\beta \in \mathbb{N}_1^{1+s}\) with \(|\beta| \leq k\), and some \(h = 1, \ldots, m\), we have

\[
\left\| \int_{t_0}^t \int_{t_0}^{s_2} f(s_1, -) \, ds_1 \right\|_k = \max_{(t, x) \in T \times S} \left| \partial^{\beta} \int_{t_0}^t \int_{t_0}^{s_2} f(s_1, x) \, ds_1 \right|.
\]

But \(\beta_1 \leq p < d\), and hence, setting \(\nu := (\beta_2, \ldots, \beta_s)\), the operator \(\partial^{\beta} = \partial^{\beta_1}_x \partial^{\beta_2}_x \cdots \partial^{\beta_s}_x\) deletes \(\beta_1\) integrals in (4.8); set \(\tilde{j} := d - \beta_1 > 0\). Differentiation under the integral
sign yields
\[ \left\| \int_{t_0}^{s_2} ds_d \cdot \int_{t_0}^{s_2} f(s_1, -) ds_1 \right\|_k = \max_{(t,x) \in T \times S} \left| \int_{t_0}^{t} ds_j \cdot \int_{t_0}^{s_2} \partial_x^r h(s_1, x) ds_1 \right| \]
\[ \leq \max_{(t,x) \in T \times S} \text{sgn}(t - t_0) \int_{t_0}^{t} ds_j \cdot \int_{t_0}^{s_2} |\partial_x^r f^h(s_1, x)| ds_1 \]
\[ \leq \max_{(t,x) \in T \times S} \text{sgn}(t - t_0) \int_{t_0}^{t} ds_j \cdot \int_{t_0}^{s_2} M_k(s_1, x) ds_1 \]
\[ = \max_{(t,x) \in T \times S} \tilde{M}_{kj}(t, x) \]
\[ \leq \max_{x \in S} \tilde{M}_{kj}(t_0 + \max(a, b), x). \]

Note that if \( t > t_0 \), then \( t_0 < s_j < t \); if \( t < t_0 \), then \( t < s_j < t_0 \), and in both cases \( \text{sgn}(t - t_0) = \text{sgn}(s_j - t_0) \). Similarly, we can proceed for the other integration variables \( s_q \).

(ii): Condition (4.7) holds if \( M_k = \|f\|_k \), and we have
\[ \tilde{M}_{kj}(t, x) = \frac{\|f\|_k (t - t_0)^j}{j!}. \]

Thereby, \( \max_{x \in S} \tilde{M}_{kj}(t_0 + \max(a, b), x) = \max(a, b) \|f\|_k. \)

To solve problem (1.1) or, equivalently, the integral problem (4.1), let us introduce the following simplified notation
\[ G(t, x, y) := F \left[ t, x, (\partial_x^\gamma \partial_t^p y)|_{\gamma \leq \ell} (t, x) \right] \in \mathbb{R}^m, \quad (4.9) \]
for all \((t, x) \in T \times S\) and all \( y \in C^p_{x,t} \infty(T \times S, \mathbb{R}^m)\). Explicitly note that the smooth function \( G(t, x, y) \) is given by composition of \( F \left[ t, x, (z^{\alpha, \gamma}) |_{\gamma \leq \ell} \right] \) with the derivatives \( (\partial_x^\gamma \partial_t^p y)(t, x) = z^{\alpha, \gamma} \in \mathbb{R}^m \) that actually appear in (1.1). On the contrary, when we use the variables \( G(t, x, z) \), we mean that \( z = (z^{\alpha, \gamma}) |_{\gamma \leq \ell} \in \mathbb{R}^{m \cdot \ell} \).

We now introduce the following definition of Lipschitz map:

**Definition 10.** Let \( B \subseteq C^p_{x,t} \infty(T \times S, \mathbb{R}^m) \). We say that a map \( G : T \times S \times B \to \mathbb{R}^m \) is Lipschitz on \( B \) with loss of derivatives (LOD) \( L \) and Lipschitz factors \( (\Lambda_k)_{k \in \mathbb{N}} \) if

(i) \( \forall y \in B : G(-, -, y) \in C^p_{x,t} \infty(T \times S, \mathbb{R}^m) \);

(ii) \( \Lambda_k \in C^0(T \times S) \) for all \( k \in \mathbb{N} \);

(iii) If \( k \in \mathbb{N}, \nu \in \mathbb{N}^s, |\nu| \leq k, h = 1, \ldots, m, u, v \in B, (t, x) \in T \times S \), then
\[ |\partial_x^r G^h(t, x, u) - \partial_x^r G^h(t, x, v)| \leq \Lambda_k(t, x) \cdot \max_{l=1, \ldots, m} \max_{|\alpha| \leq k + L, \gamma \leq \ell} |\partial_x^\alpha \partial_t^\gamma (u^l - v^l)(t, x)|. \]

(4.10)
We simply say that $G$ is Lipschitz on $B$ with LOD $L$ if the previous conditions (ii) and (iii) hold for some $(\Lambda_k)_{k\in\mathbb{N}}$.

In the next theorem, we prove that if $G$ is defined by (4.9) and all the radii $r_k < +\infty$, then $G$ is always Lipschitz with respect to constant factors $(\Lambda_k)_{k\in\mathbb{N}}$ in the space $B_R(i_0) \subseteq C^\infty_r(T \times S, \mathbb{R}^m)$ defined in (3.1) and with loss of derivatives $L$ given, as in (1.1), by the maximum order of derivatives in $x$ that appears in our PDE. The space $B_R(i_0)$ is suitable for the proof of the PLT if we are also able to prove that for these finite radii the Picard iterates $P^n(i_0) \in B_R(i_0)$. On the other hand, in Sec. 5 we will show examples of PDE with constant Lipschitz factors $\Lambda_k$ but where we are free to also take $r_k \leq +\infty$. In other words, the following result is only a sufficient condition.

**Theorem 11.** Let $r_k \in \mathbb{R}_{>0}$ for all $k \in \mathbb{N}$. Set $R := (r_k)_{k\in\mathbb{N}}$, $i_0$ as in (4.2) and $B_R(i_0)$ as in (3.1), i.e:

$$B_R(i_0) := \{ u \in C^\infty_r(T \times S, \mathbb{R}^m) \mid \|u - i_0\|_k \leq r_k \forall k \in \mathbb{N} \}.$$  

(4.11) Then the function $G$ defined in (4.9) is Lipschitz in $B_R(i_0)$ with loss of derivatives $L$ and constant Lipschitz factors.

**Proof.** We only have to prove condition (iii) of Def. 10, so that we consider $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^s$, $|\alpha| \leq k$, $h = 1, \ldots , m$, $u, v \in B_R(i_0)$, $(t, x) \in T \times S$. Note that

$$\partial^\alpha_x G^h(t, x, u) = \partial^\alpha_x \left\{ F^h \left[ t, x, (\partial^\mu_x \partial^\nu_t u)_{|\alpha| \leq L} \right] \right\}. \quad (4.12)$$

We first prove the case $|\nu| = 0$. Since $u \in B_R(i_0)$, we have $\|u - i_0\|_{L+\nu} \leq r_{L+\nu}$ and hence $\partial^\alpha_x \partial^\nu_t u(t, x) \in B_{R\nu} \subseteq C_{\nu, \gamma} =: C_0 \subseteq \mathbb{R}^m$ for all $|\alpha| \leq L$ and $\gamma \leq p$ because $r_{L+\nu} < +\infty$. Similarly, $\partial^\alpha_x \partial^\gamma_t v(t, x) \in C_0$. Thereby, using (4.12), we have

$$\left| \partial^\alpha_x G^h(t, x, u) - \partial^\alpha_x G^h(t, x, v) \right| \leq \|F\|_1 \cdot \max_{l=1,\ldots,m} \max_{|\alpha| \leq L} \left| \partial^\mu_x \partial^\gamma_t (u^l - v^l)(t, x) \right| \leq \|F\|_1 \cdot \max_{l=1,\ldots,m} \max_{|\alpha| \leq L} \left| \partial^\mu_x \partial^\gamma_t (u^l - v^l)(t, x) \right|,$$

where the norm $\|F\|_1$ is taken on $T \times S \times C^m_0 \subseteq \mathbb{R}^D$, $D := \dim(\text{dom}(F)) = 1 + s + m\bar{L}$. We firstly set $\Lambda_k(\nu) := \|F\|_1$, and now consider the case $|\nu| > 0$.

From Faà di Bruno’s formula

$$\partial^\nu_x G^h(t, x, u) = \sum_{1 \leq |\mu| \leq |\nu|} \partial^\mu F^h \left[ t, x, (\partial^\nu_x \partial^\gamma_t u)_{|\alpha| \leq L} \right] \cdot B_{\mu\nu} \left[ (\partial^\mu_x \partial^\nu_t u(t, x))_{\gamma} \right], \quad (4.13)$$

where $B_{\mu\beta}(\gamma)$ are Bell’s like polynomials such that $|\mu| \leq |\nu| + |\alpha| \leq k + L$ for all $\mu$. For simplicity, set

$$\partial^\nu F^h(t, x, u) := \partial^\nu F^h \left[ t, x, (\partial^\mu_x \partial^\nu_t u)_{|\alpha| \leq L} \right] \quad (4.14)$$

$$Q_{\mu\nu}(t, x, u) := B_{\mu\nu} \left[ (\partial^\mu_x \partial^\nu_t u(t, x))_{\gamma} \right], \quad (4.15)$$
so that we can estimate
\[
|\partial^\nu_x G^h(t, x, u) - \partial^\nu_x G^h(t, x, v)| = \\
= \left| \sum_\eta \partial^\eta F^h(t, x, u) \cdot Q_{\eta \nu}(t, x, u) - \sum_\eta \partial^\eta F^h(t, x, v) \cdot Q_{\eta \nu}(t, x, v) \right| \\
\leq \left| \sum_\eta \partial^\eta F^h(t, x, u) \cdot Q_{\eta \nu}(t, x, u) - \sum_\eta \partial^\eta F^h(t, x, u) \cdot Q_{\eta \nu}(t, x, v) \right| + \\
+ \left| \sum_\eta \partial^\eta F^h(t, x, v) \cdot Q_{\eta \nu}(t, x, v) - \sum_\eta \partial^\eta F^h(t, x, v) \cdot Q_{\eta \nu}(t, x, v) \right|.
\]

For some $L_{\eta \nu} > 0$ depending on $Q_{\eta \nu}$, the first summand yields
\[
|\partial^\eta F^h(t, x, u)| \cdot |Q_{\eta \nu}(t, x, u) - Q_{\eta \nu}(t, x, v)| \leq \leq \|F\|_k \cdot L_{\eta \nu} \cdot \max_{l=1, \ldots, m} \max_{|\alpha| \leq k+L} |\partial^\nu_x \partial^\gamma_x (u^l - v^l)(t, x)|.
\]
The second summand gives
\[
|\partial^\eta F^h(t, x, u) - \partial^\eta F^h(t, x, v)| \cdot |Q_{\eta \nu}(t, x, v)| \leq \leq \|F\|_{k+1} \cdot \max_{l=1, \ldots, m} \max_{|\alpha| \leq k+L} |\partial^\nu_x \partial^\gamma_x (u^l - v^l)(t, x)| \cdot N_k,
\]
where $N_k := \max_{|\nu| \leq |\nu| \leq k} \max_{(t, x, v) \in \mathcal{T} \times \mathcal{S} \times C^k_{\eta \nu}} |Q_{\eta \nu}(t, x, v)|$ and, as we did above, $v \in \tilde{B}_R(i_0)$ yields some $C^k_{\eta \nu} \subseteq \mathbb{R}^n$ such that $\partial^\nu_x \partial^\gamma_x v(t, x) \in C^k_{\eta \nu}$ for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k + L$. We finally obtain
\[
|\partial^\nu_x G^h(t, x, u) - \partial^\nu_x G^h(t, x, v)| \leq \sum_{1 \leq |\eta| \leq |
u|} \left( \|F\|_k \cdot \max_{|\eta| \leq |\nu| \leq k} L_{\eta \nu} + \|F\|_{k+1} \cdot N_k \right) \cdot \max_{l=1, \ldots, m} \max_{|\alpha| \leq k+L} |\partial^\nu_x \partial^\gamma_x (u^l - v^l)(t, x)|.
\]
Setting
\[
\Lambda_k(\nu) := \sum_{1 \leq |\eta| \leq |
u|} \left( \|F\|_k \cdot \max_{|\eta| \leq |\nu| \leq k} L_{\eta \nu} + \|F\|_{k+1} \cdot N_k \right),
\]
\[
\Lambda_k := \max_{|\nu| \leq k} \Lambda_k(\nu),
\]
we get the conclusion.

4.3. The Picard-Lindelöf theorem for smooth normal PDE. A natural method to solve PDE is to transform it into an infinite-dimensional ODE and then apply a PLT, see e.g. [27]. On the other hand, our approach can be considered simpler because we do not transform partial derivatives into ordinary ones in infinite dimensional spaces.

We can now state our main local existence result for smooth normal systems of PDE.
Theorem 12. Let \( \Lambda_k \in C^0(T \times S) \), \( r_k \in \mathbb{R}_{>0} \cup \{+\infty\} \) for all \( k \in \mathbb{N} \), and assume that \( \dot{S} \) is dense in \( S \). Define \( R := (r_k)_{k \in \mathbb{N}}, \dot{B}_R(i_0) \) as in (4.11), and \( P : \dot{B}_R(i_0) \to C^0_p C_x^\infty(T \times S, \mathbb{R}^m) \) by

\[
P(y)(t,x) := i_0(t,x) + \int_{t_0}^t \, ds \cdot \frac{d}{ds} \int_{t_0}^{s_2} G(s_1, x, y) \, ds_1.
\]

Assume that \( G \) is Lipschitz on \( \dot{B}_R(i_0) \) with loss of derivatives \( L \) and Lipschitz factors \( (\Lambda_k)_{k \in \mathbb{N}} \), and for all \( (t,x) \in T \times S \), all \( n \in \mathbb{N} \) and all \( j = 1, \ldots, d \), set

\[
\Lambda^j_{k,0} := 1,
\]

\[
\Lambda^j_{k,n+1}(t,x) := \left( \int_{t_0}^s ds \cdot \int_{t_0}^{s_2} \Lambda_k(s_1, x) \cdot \max_{0 < l \leq d} \Lambda^l_{k+L,n}(s_1, x) \, ds_1 \right), \tag{4.16}
\]

\[
\tilde{\Lambda}_{k,n} := \max_{x \in S} \left( \Lambda^j_{k,n}(t_0 + \max(a,b), x) \right).
\]

Finally, assume that the following conditions are fulfilled for all \( k \in \mathbb{N} \):

(i) \( P^n(i_0) \in \dot{B}_R(i_0) \) for all \( n \in \mathbb{N} \);

(ii) \( \sum_{n=0}^{+\infty} \tilde{\Lambda}_{k,n} \cdot \|P(i_0) - i_0\|_{k+nL} < +\infty \).

Then, there exists a smooth solution \( y \in \dot{B}_R(i_0) \cap C^\infty(T \times S, \mathbb{R}^m) \) of the problem

\[
\left\{ \begin{array}{l}
\partial_t^j y(t,x) = F\left[ t,x,(\partial_x^p \partial_t^q y)|_{|\nu| \leq L} \right], \\
\partial_t^j y(t_0, x) = y_0(t) \quad j = 0, \ldots, d - 1,
\end{array} \right. \tag{4.17}
\]

given by \( y = \lim_{n \to +\infty} P^n(i_0) \) in \( C^0_p C_x^\infty(T \times S, \mathbb{R}^m), (\|\cdot\|_k)_{k \in \mathbb{N}} \), which satisfies

\( \forall k, m \in \mathbb{N} : \|y - P^m(i_0)\|_k \leq \sum_{n=m}^{+\infty} \tilde{\Lambda}_{k,n} \cdot \|P(i_0) - i_0\|_{k+nL} \).

In particular, if \( M_k \in C^0(T \times S) \), we set

\[
M_{kj}(t,x) := \left| \int_{t_0}^t ds \cdot \int_{t_0}^{s_2} M_k(s_1, x) \, ds_1 \right|,
\]

and we also assume

(iii) \( |\partial_x^p \partial_t^q \Gamma(t,x,u)| \leq M_k(t,x) \) for all \( u \in \dot{B}_R(i_0), (t,x) \in T \times S \) and all \( \nu \in \mathbb{N}^s \) such that \( |\nu| \leq k \);

(iv) \( \max_{x \in S} \hat{M}_{k,j}(t_0 + \max(a,b), x) \leq r_k \);

Then \( P : \dot{B}_R(i_0) \to \dot{B}_R(i_0) \) and hence (i) always holds.

Proof. We prove that \( P \) actually satisfies the stronger contraction property (2.2) with contraction constants \( \Lambda_{kn} \). We firstly show, by induction on \( n \in \mathbb{N} \), that for each \( k \in \mathbb{N} \), \( u, v \in \dot{B}_R(i_0), (t,x) \in T \times S, h = 1, \ldots, m, \) and \( \beta \in \mathbb{N}^{l+1}_p \) with \( |\beta| \leq k \), we have

\[
\left| \partial^\beta [P^n(u)^h - P^n(v)^h](t,x) \right| \leq \|u - v\|_{k+nL} \cdot \max_{0 < j \leq d} \Lambda^j_{kn}(t,x). \tag{4.18}
\]
For \( n = 0 \), (4.18) reduces to \( |\partial^\beta(u^h - v^h)(t, x)| \leq \|u - v\|_k \cdot \max_{0 \leq j \leq d} \Lambda_{k,j}^j(t, x) \) which holds because \(|\beta| \leq k\) and \( \Lambda_{k,0}^j = 1 \). To prove the inductive step, we consider

\[
|\partial^\beta [P^{n+1}(u)^h - P^{n+1}(v)^h](t, x)| \leq |\partial^\beta \left\{ \int_{t_0}^t ds_d \cdot \int_{t_0}^{s_2} G^h(s_1, x, P^n(u)) \, ds_1 \right. \\
- \left. \int_{t_0}^t ds_d \cdot \int_{t_0}^{s_2} G^h(s_1, x, P^n(v)) \, ds_1 \right\}| =: (1^*) (4.19)
\]

Since \( \beta \in \mathbb{N}^{1+s}_p \), we can write \( \partial^\beta = \partial^\gamma \partial^\delta \), where \( \nu := (\beta_2, \ldots, \beta_s) \) and \( \beta_1 \leq p < d \).

The operator \( \partial^\nu \) deletes \( \beta_1 \) integrals in (4.19); set \( j := d - \beta_1 > 0 \), and take \( \partial^\nu \) inside the integrals to get

\[
(1^*) \leq \text{sgn}(t - t_0)^j \int_{t_0}^t ds_j \cdot \int_{t_0}^{s_2} \big| \partial^\nu G^h(s_1, x, P^n(u)) - \partial^\nu G^h(s_1, x, P^n(v)) \big| \, ds_1 =: (2^*).
\]

Since \( G \) is Lipschitz on \( \bar{B}_R(i_0) \) with factors \((\Lambda_k)_{k \in \mathbb{N}}\), we get

\[
(2^*) \leq \text{sgn}(t - t_0)^j \int_{t_0}^t ds_j \cdot \int_{t_0}^{s_2} \Lambda_k(s_1, x) \cdot \max_{i=1, \ldots, m} \max_{|\gamma| \leq p} \big| \partial^\gamma \partial^\nu [P^n(u)^l - P^n(v)^l](s_1, x) \big| \, ds_1 =: (3^*).
\]

Using inductive hypothesis (4.18) (with \( k + L \) instead of \( k \) and \( (\gamma, \alpha) \) instead of \( \beta \))

\[
(3^*) \leq \text{sgn}(t - t_0)^j \int_{t_0}^t ds_j \cdot \int_{t_0}^{s_2} \Lambda_k(s_1, x) \cdot \|u - v\|_{k+L+nL} \cdot \\
\max_{0 < l \leq d} \Lambda_{k+L,n}^l(s_1, x) \, ds_1 \]

\[
= \|u - v\|_{k+(n+1)L} \cdot \left| \int_{t_0}^t ds_j \cdot \int_{t_0}^{s_2} \Lambda_k(s_1, x) \right| \cdot \\
\max_{0 < l \leq d} \Lambda_{k+L,n}^l(s_1, x) \, ds_1 \]

\[
\leq \|u - v\|_{k+(n+1)L} \cdot \Lambda_{k,n+1}^j(t, x) \cdot \max_{0 < j \leq d} \Lambda_{k,n+1}^j(t, x),
\]

which proves our claim.

Finally, we prove (2.2): for some \( \beta \in \mathbb{N}^{1+s}_p \), \(|\beta| \leq k\), some \( h = 1, \ldots, m \) and some \((t, x) \in T \times S\), from (4.18) we have

\[
\|P^n(u) - P^n(v)\|_k = |\partial^\beta [P^n(u)^h - P^n(v)^h](t, x)| \leq \\
\leq \|u - v\|_{k+nL} \cdot \max_{0 < j \leq d} \Lambda_{k,n}^j(t, x) \leq \\
\leq \|u - v\|_{k+nL} \cdot \Lambda_{k,n}.
\]

This shows the claim on \( P \) with contraction constants \( \Lambda_{k,n} \). The conclusion with \( y \in \bar{B}_R(i_0) \) hence follows from Weissinger condition (ii) and Thm. 4. It only remains
to prove that actually \( y \) is smooth. Since \( y \) is a fixed point of \( P \), we have
\[
y(t, x) = i_0(t, x) + \int_0^t \int_{t_0}^{s_2} G(s_1, x, y) \, ds_1 \, ds_2.
\]

But \( y \in C_t^\infty (T \times S, \mathbb{R}^m) \) and hence \( (\partial_x^\alpha \partial_t^\beta y)|_{\alpha \leq L} \in C^0(T \times S, \mathbb{R}^m) \). By induction (4.20) proves that \( y \) is smooth at interior points of \( T \times S \) and hence also at boundary points by continuity of derivatives on \( T \times S \).

In particular, if we assume both (iii) and (iv), we can prove that \( P : \hat{B}_R(i_0) \to \hat{B}_R(i_0) \) using Lem. 9. In fact, for any \( u \in \hat{B}_R(i_0) \) and \( k \in \mathbb{N} \), from (iii) and (iv) we have
\[
\|P(u) - i_0\|_k = \left\| \int_{t_0}^{t_1} ds_d . \int_{t_0}^{s_2} G(s_1, -, u) \, ds_1 \right\|_k \\
\leq \max_{x \in S} \hat{M}_k(t_0 + \max(a, b), x) \leq M_k.
\]
\( \square \)

Note that, on the contrary with respect to the more classical conditions (iii) and (iv) (inherited from the classical PLT for ODE) depending both on the choice of upper bounds \( M_k \) and radii \( r_k \), assumption (i) depends only on the radii \( r_k \). In Sec. 5, we will see that requirement (i) leads us to the correct choice of these radii \( r_k \) (one more time, depending on the initial conditions \( r_k = r_k(i_0) \)).

If the radii \( r_k \ll +\infty \), we can also consider as bounds \( M_k \) of (iii) and (iv) the minimal constant functions. This is considered in the following result, which allows us to understand that to have a local solution using the latter part of the PLT, we have to avoid that \( \frac{\hat{M}_k}{M_k} \to 0 \):

**Corollary 13.** Let \( \Lambda_k \in C^0(T \times S) \), \( r_k \in \mathbb{R}_{>0} \) for all \( k \in \mathbb{N} \). Assume that \( \hat{S} \) is dense in \( S \) and \( G \) is Lipschitz on \( \hat{B}_R(i_0) \) with loss of derivatives \( L \) and Lipschitz factors \( (\Lambda_k k)_{k \in \mathbb{N}} \), define \( \Lambda_{kn} \) as in (4.16) and
\[
C_k := T \times S \times \bigcup_{|\nu| \leq k + L} \hat{B}_{r_k + \gamma, L + p} (\partial_x^\nu \partial_t^\gamma i_0(T \times S)) \\
M_k := \max_{(t, x, z) \in C_k} \max_{|\nu| \leq k} |\partial_x^\nu G(t, x, z)|
\]
Finally, assume that the following conditions are fulfilled:
(i) \( \max(a, b) \leq \inf_{k \in \mathbb{N}} \frac{\hat{M}_k}{M_k} \);
(ii) \( \sum_{n=0}^{+\infty} \Lambda_{kn} \cdot \|P(i_0) - i_0\|_{k+nL} < +\infty \) for all \( k \in \mathbb{N} \).
Then, there exists a smooth solution \( y \in \hat{B}_R(i_0) \cap C^\infty(T \times S, \mathbb{R}^m) \) of problem (4.17).

**Proof.** Note explicitly that \( C_k \in \mathbb{R}^{1+s+m} \) because \( r_k + L + p < +\infty \). As we proved in Thm. 11, for all \( u \in \hat{B}_R(i_0) \), all \( |\nu| \leq k + L \) and all \( \gamma \leq p \), we have
\[
|\partial_x^\nu \partial_t^\gamma u(t, x) - \partial_x^\nu \partial_t^\gamma i_0(t, x)| \leq \|u - i_0\|_{|\nu| + \gamma} \leq \|u - i_0\|_{k+L+p} \leq r_k + L + p,
\]
and hence \( \partial_x^\nu \partial_t^\gamma u(t, x) \in C_k \). Thereby, condition Thm. 12.(iii) holds for the chosen constant \( M_k \) (see also Rem. 14 just below). Therefore, \( \hat{M}_{kj}(t, x) = \frac{(t-x)^j}{j!} M_k \) and
max_{x \in S} \bar{M}_k(t_0 + \max(a, b), x) = \max(a, b) \cdot M_k \leq r_k \text{ for all } k \in \mathbb{N} \text{ by (i). We can finally apply Thm. 12.} \hfill \square

Remark 14. To avoid misunderstandings, we explicitly note that the simplified notation \( \partial_x^p G(t, x, z) \) denotes the function obtained by the following process:

(i) Consider and arbitrary \( u \in \bar{B}_R(i_0) \), and the derivative \( \partial_x^p(G(t, x, u))(t, x) \) given by (4.13);

(ii) In the formula (4.13) obtained after the computation of this derivative, substitute the variables \( z_{\alpha \gamma} := (\partial_x^{\mu} \partial_x^{\nu} u)_{|\alpha \gamma| \leq L} \) and \( z_{\mu \gamma} := (\partial_x^{\mu} \partial_x^{\nu} u(t, x))_{|\mu \gamma|} \) to obtain \( \partial_x^p G(t, x, z) \), where the variable \( z \) represents all the \( z_{\alpha \gamma} \) and \( z_{\mu \gamma} \).

For example, for the PDE \( \partial_t^2 y = a(t) \cdot \partial_x^2 \partial_y^2 y \), we calculate the derivatives as \( \partial_x^p G(t, x, u) = a(t) \cdot \partial_x^{p+L} \partial_x^1 u(t, x) \), and here substituting \( z \) for \( \partial_x^{p+L} \partial_x^1 u(t, x) \) we get \( \partial_x^p G(t, x, z) = a(t) \cdot z \). For the PDE \( \partial_t^2 y = (\partial_x^2 \partial_y^2 y)^2 \), we have e.g. \( \partial_z G(t, x, u) = 2\partial_x^2 \partial_x^1 u(t, x) \partial_x^{L+1} \partial_x^1 u(t, x) \), and \( \partial_z G(t, x, z_0, z_1) = 2z_0z_1 \).

On the contrary with respect to the Cauchy-Kowalevski theorem, in the PL Thm. 12, it would appear that we do not need to assume \( d \geq L \). However, this clearly cannot hold in general, and in Sec. 5 we show that such type of assumption is implicitly contained in the convergence request of Weissinger condition Thm. 12.(ii). A first partial confirmation going in this direction, can be glimpsed by computing the iteration \( P^n(i_0)(t, x) \) in case of simple linear PDE, and then taking \( n \to +\infty \):

Example 15. Assuming all the needed hypotheses to apply the PL Thm. 12 for each one of the following cases (where \( a \in \mathbb{R} \neq 0 \)), we have:

(i) If \( \partial_y y = a \cdot \partial_x^2 y \), then \( y(t, x) = \sum_{n=0}^{+\infty} \frac{\partial_y y_{n0}(x)}{n!} a^n \cdot (t - t_0)^n \);

(ii) If \( \partial_t^2 y = a \cdot \partial_x^2 y \), then \( y(t, x) = \sum_{n=0}^{+\infty} \frac{\partial_y y_{n0}(x)}{n!} a^n (t - t_0)^n + \sum_{n=0}^{+\infty} \frac{\partial_y y_{n1}(x)}{(n+1)!} a^n (t - t_0)^{n+1} \);

(iii) If \( \partial_y y = a \cdot \partial_x y \), then \( y(t, x) = \sum_{n=0}^{+\infty} \frac{\partial_y y_{n0}(x)}{n!} a^n \cdot (t - t_0)^n \);

(iv) If \( \partial_t^2 y = a \cdot \partial_y \partial_x y \), then \( y(t, x) = \sum_{n=0}^{+\infty} \frac{\partial_y y_{n0}(x)}{n!} a^n (t - t_0)^n + \sum_{n=0}^{+\infty} \frac{\partial_y y_{n1}(x)}{(n+1)!} a^n (t - t_0)^{n+1} \);

(v) If \( \partial_t^2 y = a \cdot \partial_x^2 y \), then \( y(t, x) = \sum_{n=0}^{+\infty} \frac{\partial_y y_{n0}(x)}{n!} a^n \cdot (t - t_0)^n + \sum_{n=0}^{+\infty} \frac{\partial_y y_{n1}(x)}{(2n)!} a^n (t - t_0)^{2n+1} \).

The ideas of the proof of Thm. 12 are a simple generalization of the classical proof for ODE, only adapted to contractions with LOD and a countable family of norms. Indeed, for \( L = 0 \) and \( \| - \|_k = \| - \|_0 \) the proof reduces to the classical proof for ODE and assumptions (iii), (iv), (ii) reduce to the usual ones for the PLT for ODE with Weissinger condition, see e.g. [31]. On the other hand, the compact set \( S \subseteq \mathbb{R}^d \) (with \( S \) dense in \( S \) is completely arbitrary: we can hence say that our deduction proves that, with respect to the PLT, PDE can be simply treated as ODE depending on a parameter \( x \in S \).
If the Lipschitz factors $\Lambda_k \in \mathbb{R}$ and the upper bounds $M_k \in \mathbb{R}$ are constant (the proof of Thm. 11 and Cor. 13 show that this is not a loss of generality), then

$$M_{kj}(t, x) = M_k \frac{|t - t_0|^j}{j!},$$

$$\Lambda_{kn}(t, x) = \frac{|t - t_0|^{nd_j}}{(nd + j)!} \prod_{j=0}^{n-1} \Lambda_{k+jL},$$

$$\bar{\Lambda}_{kn} = \max(a, b)^{nd} \frac{n-1}{(nd)!} \prod_{j=0}^{n-1} \Lambda_{k+jL}.$$  

Thereby, Weissinger condition Thm. 12.(ii) becomes

$$\sum_{n=0}^{+\infty} \frac{\max(a, b)^{nd}}{(nd)!} \|P(i_0) - i_0\|_{k+nL} \prod_{j=0}^{n-1} \Lambda_{k+jL} < +\infty \quad \forall k \in \mathbb{N}. \quad (4.21)$$

For ODE, we have $L = 0$ and $\| - \|_k = \| - \|_0$, and (4.21) reduces to

$$\|P(i_0) - i_0\|_0 \sum_{n=0}^{+\infty} \frac{\Lambda_0^{nd}}{(nd)!} < +\infty,$$

which always holds.

Remark 16.

(i) We believe it is worth mentioning that in a non-Archimedean setting such as that of generalized smooth functions theory and Robinson-Colombeau ring $\mathbb{R}$, see e.g. [12, 13, 22, 14], we can repeat the proof of the PLT with exactly the same formal steps (but with the ring $\mathbb{R}$ instead of the field $\mathbb{R}$). In addition, we can take $S = [-\iota, \iota]^s \supseteq \mathbb{R}^s$, where $\iota$ is an infinite number (this kind of sets behave as compact sets for generalized smooth functions, see [11]). In this way, we get a global solution in $x \in S$ (but clearly, we need initial conditions on $S$, or equivalently boundary conditions that hold for all $x \in \mathbb{R}$). Moreover, in the setting of the non-Archimedean ring $\mathbb{R}$, the generalized number (equivalence class in the quotient ring $\mathbb{R}$) $d\rho := [\rho\varepsilon] \in \mathbb{R}$ is an infinitesimal number since $\rho\varepsilon \to 0^+$ as $\varepsilon \to 0^+$ (and hence $d\rho^{-Q} = [\rho^{-Q}]$ is an infinite number for all $Q \in \mathbb{N}_{>0}$). Since in every Cauchy complete non-Archimedean ring a series converges if and only if the general term tends to zero (see e.g. [17]), condition (4.21) is equivalent to $\lim_{n \to +\infty} \frac{\max(a, b)^{nd}}{(nd)!} \|P(i_0) - i_0\|_{k+nL} \prod_{j=0}^{n-1} \Lambda_{k+jL} = 0$. Assuming that for some $Q \in \mathbb{N}_{>0}$ and some $q \in \mathbb{N}_{>0}$, we have

$$\frac{\max(a, b)^{nd}}{(nd)!} \|P(i_0) - i_0\|_{k+nL} \prod_{j=0}^{n-1} \Lambda_{k+jL} \leq d\rho^{-Q}, \quad (4.22)$$

then $\frac{\max(a, b)^{nd}}{(nd)!} \|P(i_0) - i_0\|_{k+nL} \prod_{j=0}^{n-1} \Lambda_{k+jL} \leq d\rho^{-Q} \frac{\max(a, b)^{nd}}{(nd)!} \to 0$ as $n \to +\infty$. Thereby, since for ordinary smooth normal Cauchy problem (even Lewy-Mizohata examples) always has a solution in an infinitesimal interval (see [14] for greater details).
Lewy-Mizohata examples imply that Weissinger condition in these cases does not hold. Moreover, since the non-existence of a solution does not depend on the initial condition $i_0$, [21, 24], and taking $a = -1, b = 1$, we necessarily must have

$$\exists k \in \mathbb{N} : \sum_{n=0}^{+\infty} \frac{1}{n!} \prod_{j=0}^{n-1} \Lambda_{k+j} = +\infty$$

(4.23)

for all Lipschitz factors $(\Lambda_j)_{j \in \mathbb{N}}$ (that always exist because of Thm. 11) (note that $d = 1 = L, m = 2$ for both counter-examples). Condition (4.23) strongly recall the non-analytic nature of $F$ in these cases.

5. Examples

The main aim of this section is to show at least one example of normal PDE (1.1) where either the right hand side $F$ or one of the initial conditions $y_{0j}$ are not analytic functions.

The class of examples we are going to consider is

$$\partial_d^d t y(t, x) = p(t) \cdot \partial^\mu_x \partial_\gamma t y(t, x) + q(t, x),$$

(5.1)

where $y(t, x) \in \mathbb{R}^m, \mu \in \mathbb{N}^s, |\mu| = L > 0, 0 \leq \gamma < d, p \in C^\infty(T, \mathbb{R}^{m \times m})$ is an arbitrary matrix valued smooth function, and $q$ is a smooth function with uniformly bounded derivatives in $x$:

$$\exists Q \in \mathbb{R}_{>0} \forall \nu \in \mathbb{N}^s \forall (t, x) \in T \times S : |\partial^\nu_x q(t, x)| \leq Q. \quad (5.2)$$

Clearly, wave, heat and Laplace equations are particular cases of (5.1). Explicitly note that also Mizohata’s counterexample [24] $\partial_t y^1 = t \partial_x y^2 + q^1(t, x), \partial_t y^2 = -t \partial_x y^1 + q^2(t, x)$ is exactly of the form (5.1), but $q = (q^1, q^2)$ is not analytic, so it does not satisfy condition (5.2). On the other hand, in Lewy’s counterexample the coefficient $p = p(t, x_1, x_2)$ also depends on $x$ and the term $q$ is again not analytic.

We want to directly apply the first part of the PL Thm. 12, i.e. to check properties (i) and (ii). We start focusing on the latter, and arriving at estimates that are independent from the radii $r_k$.

Estimate of Lipschitz constants $\Lambda_k$. For arbitrary radii $R = (r_k)_{k \in \mathbb{N}}$ and all $u, v \in \bar{B}_R(i_0)$ and all $\nu \in \mathbb{N}^s$, with $|\nu| \leq k$, we have:

$$|\partial^\mu_x G(t, x, u) - \partial^\mu_x G(t, x, v)| \leq \|p\|_0 \cdot \max_{|\alpha| \leq k+L} |\partial^\alpha_x u(t, x) - \partial^\alpha_x v(t, x)|.$$

We can hence consider $\Lambda_k := \|p\|_0$, so that setting $\tilde{T} := \max(a, b)$, we get

$$\Lambda_{kn} = \frac{\tilde{T}^{nd}}{(nd)!} \prod_{j=0}^{n-1} \Lambda_{k+jL} = \frac{\tilde{T}^{nd}}{(nd)!} \|p\|_0^n. \quad (5.3)$$
Estimate of the terms $\|P(i_0) - i_0\|_{k+nL}$. We have

$$P(i_0)(t, x) - i_0(t, x) = \int_{t_0}^t ds_d \cdot \int_{t_0}^{s_2} [p(s_1) \cdot \partial_x^{\beta_s} i_0(s_1, x) + q(s_1, x)] \, ds_1$$

$$\partial_x^{\beta_s} i_0(s_1, x) = \sum_{j=\gamma}^{d-1} \frac{\partial_x^{\beta_s} y_{0j}(x)}{j!} \frac{(j-1) \cdot \ldots \cdot (j-\gamma+1) \cdot (t-t_0)^{j-\gamma}}{(j-\gamma)!}$$

$$= \sum_{j=\gamma}^{d-1} \frac{\partial_x^{\beta_s} y_{0j}(x)}{(j-\gamma)!} (t-t_0)^{j-\gamma}. $$

For $\beta \in \mathbb{N}^n$, $|\beta| \leq k + nL$, $\beta_x := (\beta_2, \ldots, \beta_n)$, we thus have

$$\partial^{\beta} [P(i_0)(t, x) - i_0(t, x)] = \sum_{j=\gamma}^{d-1} \frac{\partial_x^{\beta_s} y_{0j}(x)}{(j-\gamma)!} \int_{t_0}^t ds_d \frac{d}{d \cdot s_1} \int_{t_0}^{s_2} p(s_1) \cdot (s_1 - t_0)^{j-\gamma} \, ds_1 + \int_{t_0}^t ds_d \frac{d}{d \cdot s_1} \int_{t_0}^{s_2} \partial_x^{\beta_s} q(s_1, x) \, ds_1.$$

Using assumption (5.2):

$$|\partial^{\beta} [P(i_0)(t, x) - i_0(t, x)]| \leq \|p\|_0 \sum_{j=\gamma}^{d-1} \frac{\|\partial_x^{\beta_s} y_{0j}(x)\|}{(j-\gamma)!} \frac{T^{j-\gamma+d-\beta_s}}{(j-\gamma + d - \beta_s)!}(j-\gamma)! + Q \frac{T^{d-\beta_s}}{(d-\beta_s)!}. $$

Therefore, taking for simplicity $T \leq 1$:

$$\|P(i_0) - i_0\|_{k+nL} \leq \|p\|_0 \sum_{j=\gamma}^{d-1} \frac{\|y_{0j}\|_{k+(n+1)L}}{(j-\gamma + d)!} + Q. \quad (5.4)$$

**Weissinger condition.** Based on (5.3) and (5.4), we can estimate Weissinger condition as

$$\sum_{n=0}^{+\infty} \Lambda_{kn} \cdot \|P(i_0) - i_0\|_{k+nL} \leq \sum_{n=0}^{+\infty} \frac{T^{nd}}{(nd)!} \|p\|_0^{n+1} \sum_{j=\gamma}^{d-1} \frac{\|y_{0j}\|_{k+(n+1)L}}{(j-\gamma + d)!} + \sum_{n=0}^{+\infty} \frac{T^{nd}}{(nd)!} \|p\|_0^n Q =: S_1 + S_2.$$

Since the latter series $S_2$ is convergent, we focus on the first one:

$$S_1 = \|p\|_0 \sum_{n=0}^{+\infty} \frac{(T^d \|p\|_0)^n}{(nd)!} \sum_{j=\gamma}^{d-1} \frac{\|y_{0j}\|_{k+(n+1)L}}{(j-\gamma + d)!} \frac{1}{(j-\gamma + d)!}. \quad (5.5)$$

In this series, the only potentially problematic terms are the fractions $\frac{\|y_{0j}\|_{k+(n+1)L}}{(nd)!}$ as $n \to +\infty$ for $j = \gamma, \ldots, d - 1$, because the remaining part surely yields a convergent series for $T$ sufficiently small. This also yields that all the possible initial conditions $y_{0j}$ for $j = 0, \ldots, \gamma - 1$ can be freely chosen (see also Example 15.(iv)).
Estimate of radii $r_k$. If $f = f(t)$ is a function of $t$, for simplicity we first set

$$I_d[f(t)] := \int_{t_0}^t ds_d \cdot \int_{s_0}^{s_1} \int_{s_1}^{s_2} f(s_1) \, ds_1.$$  

For $j \geq \gamma$ and $h \in \mathbb{N}$, we set

$$\mu_{j, \gamma, 0}(t) := p(t) \cdot (t - t_0)^{j - \gamma}$$

$$\mu_{j, \gamma, h+1}(t) := I_d[p(t) \cdot \partial^h_t \mu_{j, \gamma, h}(t)]$$

$$\eta_0(t, x) := q(t, x)$$

$$\eta_{h+1}(t, x) := I_d[p(t) \cdot \partial^h_x \partial^h_t \eta_h(t, x)].$$

By induction on $n \in \mathbb{N}$, we can then prove that

$$P^n(i_0)(t, x) = \sum_{j=0}^{d-1} \frac{y_0_j(x)}{j!} (t-t_0)^j + \sum_{h=1}^{\infty} \sum_{j=\gamma}^{\infty} \frac{\partial^h_y y_0_j(x)}{(j - \gamma)!} \mu_{j, \gamma, h}(t) + \sum_{h=0}^{\infty} \eta_h(t, x). \quad (5.6)$$

Thereby, we get in this way a possible definition of the radii $r_k$ as

$$\|P^n(i_0) - i_0\|_k \leq \sum_{h=1}^{\infty} \sum_{j=\gamma}^{\infty} \frac{\|y_0_j\|_{k+hL}}{(j - \gamma)!} \|\mu_{j, \gamma, h}\|_k + \sum_{h=0}^{\infty} \|\eta_h\|_k =: r_k \quad \forall k \in \mathbb{N}.$$  

Using assumption (5.2), the latter series converges because

$$\|\eta_h\|_k \leq \|p\|_0^h \cdot \frac{\hat{T}^{(h+1)(d-\gamma)}}{((h+1)(d-\gamma))!}.$$  

In the former series, for $h \geq 1$ we have instead

$$\|\mu_{j, \gamma, h}\|_k \leq \|p\|_0^h \cdot \frac{\hat{T}^{h(d-\gamma)+j}}{((d-\gamma))!}.$$  

so that

$$\sum_{h=1}^{\infty} \sum_{j=\gamma}^{\infty} \frac{\|y_0_j\|_{k+hL}}{(j - \gamma)!} \|\mu_{j, \gamma, h}\|_k \leq \sum_{h=1}^{\infty} \left( \frac{\|p\|_0 \hat{T}^{d-\gamma}}{(d-\gamma)!} \right)^h \cdot \sum_{j=\gamma}^{\infty} \frac{\|y_0_j\|_{k+hL}}{(j - \gamma)!} \hat{T}^j.$$  

If this series converges (and this mainly depends on the growing of $\|y_0_j\|_{k+hL}$), we can hence have $r_k < +\infty$, otherwise we simply take $r_k = +\infty$.

A case of exponentially growing initial conditions. If all the functions $y_0_j, j = \gamma, \ldots, d-1$, satisfy for some $C_j \in \mathbb{R}_{>0}$

$$\|y_0_j\|_{k+(n+1)L} \leq C_j^{k+(n+1)L} \quad \forall j = \gamma, \ldots, d-1 \forall k \in \mathbb{N}, \quad (5.7)$$

then the series (5.5) converges and we have

Theorem 17. If the initial conditions $y_0_j$ satisfy (5.7), whereas $y_0_j$ for $j = 0, \ldots, \gamma - 1$ are arbitrary smooth functions, then there exists a smooth solution of (5.1) in $B_R(i_0)$ for $\hat{T}$ sufficiently small and all $x \in S$. In this case, we do not have constraints on $d, L$. 

Theorem 18. If \( d \geq L \), then there exists a smooth solution of (5.1) in \( B_R(i_0) \) with analytic initial conditions \( y_0 \) if \( j = \gamma, \ldots, d - 1 \) and arbitrary smooth \( y_0 \) if \( j = 0, \ldots, \gamma - 1 \), for \( T \) sufficiently small and all \( x \in S \).

Note explicitly that already this theorem yields more general results with respect to the classical Cauchy-Kowalevski theorem because both the matrix coefficient \( p(t) \) in (5.1) and the initial conditions \( y_0 \) for \( j = 0, \ldots, \gamma - 1 \) can be arbitrary smooth functions.

A case of non-analytic initial conditions. Now, let us assume that our initial conditions which are not analytic satisfy

\[
\|y_0\|_{k+(n+1)L} \sim (nL)^{\sigma_j/nL}, \quad \sigma_j > 0 \quad \forall j = \gamma, \ldots, d - 1 \forall k \in \mathbb{N}.
\]

(5.8)

Note that \( \lim_{n \to +\infty} \frac{C_{n+1}^{nL}}{n} = 0 \), so that each function \( y_0 \) satisfying (5.8) cannot be an analytic function. We have

\[
\|y_0\|_{k+(n+1)L} \sim \frac{C_{n+1}^{nL}}{(nd)!} \sim \frac{C_{n+1}^{nL}}{(nd)!} \frac{1}{(nd)^{(d-\sigma_j)L}}.
\]

We therefore have the following

Theorem 19. If the initial conditions \( y_0 \), \( j = 0, \ldots, \gamma - 1 \), are arbitrary smooth functions, whereas \( y_0 \) for \( j = \gamma, \ldots, d - 1 \) are analytic or they satisfy (5.8), and if in the latter case we have \( d > \sigma_jL \), then there exists a smooth solution of (5.1) in \( B_R(i_0) \) for \( T \) sufficiently small and all \( x \in S \).

Taking \( n \to +\infty \) in (5.6), we also obtain the following generalization of Example 15:

Corollary 20. In the assumptions of each one of Thm. 17, 18, 19, the solution \( y \) of Picard-Lindelöf iterations is given by the formula:

\[
y(t, x) = \sum_{j=0}^{\gamma-1} \frac{y_0(x)}{j!}(t-t_0)^j + \sum_{h=0}^{\infty} \sum_{j=\gamma}^{d-1} \frac{\partial_x^h y_0(x)}{(j-\gamma)!} \mu_j \mu_h, t(t) + \sum_{h=0}^{\infty} \eta_h(t, x).
\]

(5.9)

In particular, if the functions \( p \) and \( q \) are constant, then

\[
y(t, x) = \sum_{j=0}^{\gamma-1} \frac{y_0(x)}{j!}(t-t_0)^j + \sum_{h=0}^{\infty} \sum_{j=\gamma}^{d-1} \frac{\partial_x^h y_0(x)}{(j-\gamma)!} \frac{p^h (t-t_0)^h (d-\gamma) + j}{[(d-\gamma)!]^h} + q \sum_{h=0}^{\infty} \frac{p^h (t-t_0)^{h+1} (d-\gamma)}{[(h+1)(d-\gamma)]^h}.
\]

Let \( y(t, x; \varepsilon) \) be the solution defined by (5.9) corresponding to initial conditions \( y_0(x; \varepsilon) \), where \( \varepsilon \in (-1, 1) \). If we can exchange \( \lim_{x \to 0} \) and \( \sum_{h=0}^{\infty} \), e.g. if the
sequence of derivatives \( (\partial_x^{h_j}y_{0j}(x;\varepsilon))_{h_j \in \mathbb{N}} \) pointwise converges in a dominated way as \( h \to +\infty \), i.e. for all \( h \in \mathbb{N}, x \in S, j = 0, \ldots, d - 1 \), and \( \varepsilon \in (-1,1) \) we have

\[
\exists \lim_{\varepsilon \to 0} \partial_x^{h_j}y_{0j}(x;\varepsilon) = \partial_x^{h_j}y_{0j}(x;0)
\]

\[
|\partial_x^{h_j}y_{0j}(x;\varepsilon)| \leq g_h(x;\varepsilon)
\]

\[
\sum_{h=0}^{+\infty} g_h(x;\varepsilon) < +\infty,
\]

then \( \lim_{\varepsilon \to 0} y(t,x;\varepsilon) = y(t,x;0) \).

Note that our estimates above of the Weissinger condition and the radii \( r_k \), allow us also to state that conditions \((3.4)\) and \((3.2)\) of Cor. 5 hold. Moreover, the proof of PL Thm. 12 shows that also \((3.3)\) holds. Therefore, to solve \((5.1)\) in the space \( X = B_R(i_0) \) we can also apply Cor. 5, as we stated above in Sec. 3.

We close this section by noting that for the PDE

\[
\partial_t^\mu y(t,x) = y(t,x) \cdot \partial_x^\mu y(t,x)
\]

(5.10)

with \(|\mu| = L\), we can use ideas similar to those of Thm. 11 to show that setting

\[
\bar{C}_{k+L} := \bigcup_{|\alpha| \leq k+L} \partial_x^\alpha i_0(T \times S) \subset \mathbb{R}^m
\]

\[
\|i_0(T \times S)\|_{k+L} := d(\bar{C}_{k+L},0)
\]

\[
\Lambda_k := 2^k (r_{k+L} + \|i_0(T \times S)\|_{k+L})
\]

then (5.10) has \((\Lambda_k)_{k \in \mathbb{N}}\) as Lipschitz constants with \( L \) loss of derivatives. Thereby, \( \bar{\Lambda}_n = \frac{\lambda_n}{(nd!)} 2^n \prod_{j=0}^{n-1} (r_{k+j+L} + \|i_0(T \times S)\|_{k+j+L}) \). However, in the case \( L = 1 \) and \( y_{0j} \) satisfying (5.8) with \( \sigma_j = 1 \), we get \( \bar{\Lambda}_n = \frac{\lambda_n}{(nd!)} 2^n H(n - 1) \), where \( H(n - 1) = \prod_{j=0}^{n-1} j^3 \) is the hyperfactorial function. Since \( H(n) = O(n^{n^2/2}) \), Weissinger condition never holds. This clearly left open the possibility of better estimates of different Lipschitz factors.

6. SOME REMARKS ABOUT THE LOSS OF DERIVATIVES CONDITION

As we discussed in the previous sections, Def. 2 of contraction with LOD is at the core of our version of the BFPT, i.e. Thm. 4. We start this section with a discussion of this notion of contraction.

**Definition 21.** We call minimal loss for \( P \) from \( y_0 \) the quantity

\[
L_P (y_0) := \min \{ L \in \mathbb{N} \mid P \in C(X,L,y_0) \}.
\]

**Lemma 22.** Let \((\mathcal{F}, (\| - \|_k)_{k \in \mathbb{N}})\) be a Fréchet space, \( X \) be a closed subset of \( \mathcal{F} \), \( y_0 \in X \) and \( P : X \to X \) be a continuous map. Assume that \( \| - \|_0 \) (hence, \( \| - \|_k \) for every \( k \in \mathbb{N} \)) is a norm. If there exists \( N \in \mathbb{N}_{>0} \) such that \( P^N(y_0) \) is a fixed point of \( P \), then \( P \in C(X,0,y_0) \), i.e. \( P \) is a contraction with 0 loss of derivatives starting from \( y_0 \).

**Proof.** Let \( N \) be the smallest number such that \( P^N(y_0) \) is a fixed point of \( P \).

If \( N = 1 \), \( P(y_0) = y_0 \) hence \( \| P^{n+1}(y_0) - P^n(y_0) \|_k = 0 \) for every \( k, n \in \mathbb{N} \), so our claim follows just by setting each \( \alpha_{kn} := 0 \).
If $N > 1$, $\|P(y_0) - y_0\|_0 \neq 0$ since $\|\cdot\|_0$ is a norm, therefore $\|P(y_0) - y_0\|_k \neq 0$ for every $k \in \mathbb{N}$, as the norms $\|\cdot\|_k$ are increasing. For every $k$, $n \in \mathbb{N}$ set

$$\alpha_{kn} := \begin{cases} \frac{\|P^{n+1}(y_0) - P^n(y_0)\|_k}{\|P^n(y_0) - y_0\|_{k+nL}}, & \text{if } n < N; \\ \frac{1}{n^2 \|P^n(y_0) - y_0\|_{k+nL}}, & \text{otherwise.} \end{cases} \tag{6.1}$$

With this choice, Def. 2.(iii)) and Def. 2.(iv)) are easily verified because $\alpha_{kn} \|P(y_0) - y_0\|_{k+nL} = \frac{1}{n^2} \geq \|P^{n+1}(y_0) - P^n(y_0)\|_k = \|y_0 - y_0\|_k$ if $n \geq N$, hence $P \in C(X,0,y_0)$. $\Box$

The following is a rather surprising fact that holds, e.g., in $C^1_0 C^\infty(T \times S, \mathbb{R}^m)$.

**Theorem 23.** Let $(F, (\|\cdot\|_k)_{k \in \mathbb{N}})$ be a Fréchet space, let $X$ be a closed subset of $F$, let $y_0 \in X$ and let $P : X \rightarrow X$. Assume that $\|\cdot\|_0$ (hence, $\|\cdot\|_k$ for every $k \in \mathbb{N}$) is a norm. The following properties are equivalent:

(i) There exists $L \in \mathbb{N}$ such that $P \in C(X,L,y_0)$;

(ii) $P$ is continuous and for all $k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} \|P^{n+1}(y_0) - P^n(y_0)\|_k < +\infty; \quad \text{ (W')}$$

(iii) $P \in C(X,0,y_0)$.

*Proof.* (i) $\Rightarrow$ (ii): If there exist $k$, $N \in \mathbb{N}$ such that $\|P^{N+1}(y_0) - P^N(y_0)\|_k = 0$, then $P^N(y_0)$ is a fixed point of $P$ as $\|\cdot\|_k$ is a norm. But then, for every $m \geq N$ and for every $k \in \mathbb{N}$ we have $\|P^{m+1}(y_0) - P^m(y_0)\|_k = 0$, and hence $\sum_{n=0}^{\infty} \|P^{n+1}(y_0) - P^n(y_0)\|_k = \sum_{n=0}^{N-1} \|P^{n+1}(y_0) - P^n(y_0)\|_k < +\infty$. Otherwise, $\|P(y_0) - y_0\|_{k+nL} \neq 0$ for every $k$, $N \in \mathbb{N}$, hence by Def. 2.(iii) we get that $\alpha_{kn} \geq \frac{\|P^{n+1}(y_0) - P^n(y_0)\|_k}{\|P^n(y_0) - y_0\|_{k+nL}} \in \mathbb{R}_{>0}$ which, substituted in Def. 2.(iv), gives (W').

(ii) $\Rightarrow$ (iii): If there exist $k$, $N \in \mathbb{N}$ such that $\|P^{N+1}(y_0) - P^N(y_0)\|_k = 0$, then $P^N(y_0)$ is a fixed point of $P$ as $\|\cdot\|_k$ is a norm, and we conclude by Lemma 22. Otherwise, in particular $\|P(y_0) - y_0\|_k \neq 0$ for every $k \in \mathbb{N}$. For every $k$, $N \in \mathbb{N}$, we set $\alpha_{kn} := \frac{\|P^{n+1}(y_0) - P^n(y_0)\|_k}{\|P^n(y_0) - y_0\|_{k+nL}}$. Then Def. 2.(iii) holds trivially, and Def. 2.(iv) holds as, by construction

$$\sum_{n=0}^{\infty} \alpha_{kn} \|P(y_0) - y_0\|_k = \sum_{n=0}^{\infty} \|P^{n+1}(y_0) - P^n(y_0)\|_k < +\infty$$

by assumption.

(iii) $\Rightarrow$ (i): This is trivial. $\Box$

In particular, this result shows that, if $\|\cdot\|_k$ are norms, $L_P(y_0) = 0$ whenever $P \in C(X,L,y_0)$ for some $L$. Note that, in general, this does not entail the uniqueness of the fixed point of $P$, since such uniqueness would require a much stronger condition on $P$ than Def. 2.(iii) or condition (W'), see e.g. Lem. 3.

We also note that condition (W') implies that $(P^n(y_0))_{n \in \mathbb{N}}$ is a Cauchy sequence as we did in (2.4), and this, together with the continuity of $P$, yields that $\overline{7} := \lim_{n \rightarrow +\infty} P^n(y_0)$ is a fixed point of $P$ by Lem. 1.

On the other hand, the previous Thm. 23 does not imply that we can take $L = 0$ in the PLT Thm. 12, because the assumption that the right hand side $G$ of the PDE is Lipschitz on $\hat{B}_R(y_0)$ with loss of derivatives $L = 0$ in general is not satisfied. In
other words: The natural loss of derivatives $L > 0$ corresponds to the maximum order of derivatives in $x$ appearing in the PDE (1.1), and the natural Lipschitz constants $\alpha_{kn}$ are derived in the proof of Thm. 12, e.g. using the Lipschitz factors $(\Lambda_k)_{k \in \mathbb{N}}$ for the right hand side of the PDE derived from Thm. 11. Using these natural constants, Weissinger condition (ii) is easier to estimate than condition (W') or the use of (6.1).

7. Conclusions

Starting from the classical Kowalevski counter-example for the heat equation or Hadamard’s results on the Cauchy problem for the Laplace equation, one can think that a PDE links in a given relation $\partial_t y$ and $\partial_x y$ and hence it necessarily forces the solution, in general, in a space of functions whose derivatives growth in a restricted way, these constraints being related to the PDE itself. This implies that the initial conditions cannot be freely chosen but must be taken into another constrained space. We could say that we do not have to find a suitable space of generalized solutions for our PDE, but conditions stating when it has a solution or not; only at the philosophical level, this is similar to the point of view of nonlinear differential Galois theory, see e.g. [23], or the formal theory of differential equations, see e.g. [28].

The PLT we proved in this paper goes exactly in this direction, by showing that the existence of a solution (by Picard-Lindelöf iterations) depends on the initial conditions we start with: Def. 2 of contraction with loss of derivatives, the closure with respect to iterations (ii), Weissinger condition (W), the definition of the radii (5.6), all go in this direction. Examples considered in Sec. 5 show a first link between the syntax of the PDE (in the term (nd)!}) and the order of growth of the derivatives of the initial conditions $\|y_0\|_{k+nL}$. On the other hand, exactly as up to fourth order algebraic equations are solvable in radicals, if the order $d = 1$ the method of characteristics allows one to solve a large class of PDE for any initial condition.

It is now natural to ask for a generalization to more singular normal (nonlinear) PDE, e.g. where the right hand side $F$ or some of the initial conditions $y_0$ are some kind of generalized functions. In order to get this generalization by following the ideas of the present work, we would need a space of generalized functions which is closed with respect to composition and with a complete topology generated by norms; this space must clearly be non-trivial, e.g. containing all Sobolev-Schwartz distributions. In our opinion, this target can be fully accomplished in a beautiful and simple setting by considering the Grothendieck topos of non-Archimedean generalized smooth function, see e.g. [13, 22, 14]. We plan to realize this goal in future works.

References

[1] Agarwal, R.P., Frigon, M., O’Regan, D., A survey of recent fixed point theory in Fréchet spaces, Nonlinear analysis and applications: to V. Lakshmikanth on his 80th birthday, Vol. 1, 2, Kluwer Acad. Publishers (2003), 75-88.
[2] Banas, J., Goebel, K., Measure of noncompactness in Banach spaces, M. Dekker, 1980.
[3] Caín, G.L. Jr., Nashed, M.Z., Fixed points and stability for a sum of two operators in locally convex spaces, Pacific J. Math. 39 (1971), 581–592.
[4] Clarke, F.H., On the inverse function theorem, Pacific Journal of Mathematics, Vol. 64, No. 1, 1976.
[5] Dencker, N., The resolution of the Nirenberg-Treves conjecture. Ann. of Math. (2) 163 (2006), 405-444.

[6] Dudek, S., Fixed point theorems in Fréchet algebras and Fréchet spaces and applications to nonlinear integral equations, Applicable Analysis and Discrete Mathematics, Vol. 11, No. 2 (2017), 340-357.

[7] Ekeland, I., An inverse function theorem in Fréchet spaces, Ann. I. H. Poincaré – AN 28 (2011) 91–105.

[8] Ekeland, I., Sére, É., A surjection theorem for maps with singular perturbation and loss of derivatives. Journal of the European Mathematical Society, Vol. 23, No. 10, 3323–3349.

[9] Frigon, M., Fixed point results for generalized contractions in gauge spaces and applications, Proc. American Math. Soc., Vol. 128, No. 10 (2000), 2957–2965.

[10] Giordano, P., Kunzinger, M., A convenient notion of compact sets for generalized functions. Proceedings of the Edinburgh Mathematical Society, Volume 61, Issue 1, February 2018, pp. 57-92.

[11] Giordano, P., Kunzinger, M., Vernaeve, H., Strongly internal sets and generalized smooth functions. Journal of Mathematical Analysis and Applications, volume 422, issue 1, 2015, pp. 56–71.

[12] Giordano, P., Kunzinger, M., Vernaeve, H., A Grothendieck topos of generalized functions I: Basic theory. See: arXiv 2101.04492.

[13] Giordano, P., Luperi Baglini, L., A Grothendieck topos of generalized functions III: Normal PDE, preprint, See: https://www.mat.univie.ac.at/~giordap7/ToposIII.pdf.

[14] Hamilton, R., The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. (1) 7 (1982) 65–222.

[15] Howard, R., The inverse function theorem for Lipschitz maps. Lecture Notes, 1997. See https://people.math.sc.edu/howard/Notes/inverse.pdf

[16] Kobliitz, N., p-adic Numbers, p-adic Analysis, and Zeta-Functions, Graduate Texts in Mathematics (Book 58), Springer; 2nd edition, 1996.

[17] Kohn, J.J., Hypoellipticity and loss of derivatives, Annals of Math. 162(2005), 943–986.

[18] Kohn, J.J., (2013) Loss of Derivatives. In: Farkas H., Gunning R., Knopp M., Taylor B. (eds) From Fourier Analysis and Number Theory to Radon Transforms and Geometry. Developments in Mathematics, vol 28. Springer, New York, NY.

[19] Leach, E.B., A Note on Inverse Function Theorems, Proceedings of the American Mathematical Society, Vol. 12, No. 5 (Oct., 1961), pp. 694-697.

[20] Lewy, H., An Example of a Smooth Linear Partial Differential Equation Without Solution, Annals of Mathematics, Second Series, Vol. 66, No. 1 (Jul., 1957), pp. 155-158.

[21] Luperi Baglini, L., Giordano, P., A Grothendieck topos of generalized functions II: ODE, preprint. See: https://www.mat.univie.ac.at/~giordap7/ToposII.pdf.

[22] Malgrange, B., On nonlinear differential Galois theory. Dedicated to the memory of Jacques-Louis Lions. Chinese Ann. Math. Ser. B 23, no. 2, 219–226 (2002).

[23] Mizohata, S., Solutions milles et solutions non analytiques, J. Math. Kyoto Univ. 1-2 (1962), pp. 271-302.

[24] Moser, J., A new technique for the construction of solutions of nonlinear differential equations, Proc. Nat. Acad. Sci. USA 47 (1961), 1824-1831.

[25] Parenti, C., Parmeggiani, A., On the hypoellipticity with a big loss of derivatives, Kyushu J. Math. 59 (2005), 155–230.

[26] Seifert, C., Trostorf, S., Waurick, M., Evolutionary Equations. Picard’s Theorem for Partial Differential Equations, and Applications. Birkhäuser 2022.

[27] Seiler, W.M., Involutive. The Formal Theory of Differential Equations and its Applications in Computer Algebra. Springer-Verlag Berlin Heidelberg 2010.

[28] Teschl, G., Ordinary Differential Equations and Dynamical Systems, AMS, 2013.

[29] Treves, F., Topological Vector Spaces, Distributions and Kernels, Pure and Applied Mathematics, Vol. 25, Academic Press 2016.
[33] Wang, F., Zhou, H., Fixed point theorems and the Krein-Šmulian property in locally convex spaces, Fixed Point Theory and Applications (2015) 2015:154.
[34] Wang, F., Zhou, H., Weng, S., Existence of weak solutions for an infinite system of second order differential equations. Adv. Oper. Theory 4 (2019), No. 2, 514-528.
[35] Weissinger, J., Zur Theorie und Anwendung des Iterationsverfahrens, Math. Nachr., 8 (1952), pp. 193-212.

Faculty of Mathematics, University of Vienna, Austria, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

Faculty of Mathematics, Università di Milano, Italy, Via Cesare Saldini 50, 20133 Milano, Italy

Email address: paolo.giordano@univie.ac.at, lorenzo.luperi@unimi.it