Stability of a functional equation deriving from cubic and quartic functions

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Abstract. In this paper, we obtain the general solution and the generalized Ulam-Hyers stability of the cubic and quartic functional equation

\[4(f(3x+y) + f(3x-y)) = -12(f(x+y) + f(x-y))
+ 12(f(2x+y) + f(2x-y)) - 8f(y) - 192f(x) + f(2y) + 30f(2x).\]

1. Introduction

The stability problem of functional equations originated from a question of Ulam [34] in 1940, concerning the stability of group homomorphisms. Let \((G_1,\cdot)\) be a group and let \((G_2,\ast)\) be a metric group with the metric \(d(\ldots)\). Given \(\epsilon > 0\), does there exist a \(\delta > 0\), such that if a mapping \(h : G_1 \rightarrow G_2\) satisfies the inequality \(d(h(x\ast y), h(x) \ast h(y)) < \delta\) for all \(x, y \in G_1\), then there exists a homomorphism \(H : G_1 \rightarrow G_2\) with \(d(h(x), H(x)) < \epsilon\) for all \(x \in G_1\)? In the other words, Under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [9] gave the first affirmative answer to the question of Ulam for Banach spaces. Let \(f : E \rightarrow E'\) be a mapping between Banach spaces such that

\[\|f(x + y) - f(x) - f(y)\| \leq \delta\]

for all \(x, y \in E\), and for some \(\delta > 0\). Then there exists a unique additive mapping \(T : E \rightarrow E'\) such that

\[\|f(x) - T(x)\| \leq \delta\]

for all \(x \in E\). Moreover if \(f(tx)\) is continuous in \(t \in \mathbb{R}\) for each fixed \(x \in E\), then \(T\) is linear. Finally in 1978, Th. M. Rassias [31] proved the following theorem.

Theorem 1.1. Let \(f : E \rightarrow E'\) be a mapping from a normed vector space \(E\) into a Banach space \(E'\) subject to the inequality

\[\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)\]  

(1.1)
for all \( x, y \in E \), where \( \epsilon \) and \( p \) are constants with \( \epsilon > 0 \) and \( p < 1 \). Then there exists a unique additive mapping \( T : E \rightarrow E' \) such that

\[
\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p} \|x\|^p
\]  

(1.2)

for all \( x \in E \). If \( p < 0 \) then inequality (1.1) holds for all \( x, y \neq 0 \), and (1.2) for \( x \neq 0 \). Also, if the function \( t \mapsto f(tx) \) from \( \mathbb{R} \) into \( E' \) is continuous in real \( t \) for each fixed \( x \in E \), then \( T \) is linear.

In 1991, Z. Gajda [5] answered the question for the case \( p > 1 \), which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [1-2], [5-10], [28-30]). On the other hand J. M. Rassias [25-27], generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. According to J. M. Rassias Theorem:

**Theorem 1.2.** If it is assumed that there exist constants \( \Theta \geq 0 \) and \( p_1, p_2 \in \mathbb{R} \) such that \( p = p_1 + p_2 \neq 1 \), and \( f : E \rightarrow E' \) is a map from a norm space \( E \) into a Banach space \( E' \) such that the inequality

\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon \|x\|^{p_1} \|y\|^{p_2}
\]

(1.1p)

for all \( x, y \in E \), then there exists a unique additive mapping \( T : E \rightarrow E' \) such that

\[
\|f(x) - T(x)\| \leq \frac{\Theta}{2 - 2^p} \|x\|^p,
\]

for all \( x \in E \). If in addition for every \( x \in E \), \( f(tx) \) is continuous in real \( t \) for each fixed \( x \), then \( T \) is linear (see [19-26]).

The oldest cubic functional equation, and was introduced by J. M. Rassias (in 2000-2001): [17-18], as follows:

\[
f(x + 2y) + 3f(x) = 3f(x + y) + f(x - y) + 6f(y).
\]

Jun and Kim [11] introduced the following cubic functional equation

\[
f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)
\]

(1.3)

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.3). The function \( f(x) = x^3 \) satisfies the functional equation (1.3), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function \( f \) between real vector spaces \( X \) and \( Y \) is a solution of (1.3) if and only if there exists a unique symmetric multi-additive function \( C : X \times X \times X \rightarrow Y \) such that \( f(x) = C(x, x, x) \) for all \( x \in X \), and \( C \) is symmetric for each fixed one variable and is additive for fixed two variables. The oldest quartic functional equation, and was introduced by J. M. Rassias (in 1999-2000): [16], [27], and then (in 2005) was employed by Won-Gil Park [15] and others, such that:

\[
f(x + 2y) + f(x - 2y) = 4f(x + y) + f(x - y)) + 24f(y) - 6f(x).
\]

In fact they proved that a function \( f \) between real vector spaces \( X \) and \( Y \) is a solution of (1.4) if and only if there exists a unique symmetric multi-additive function \( Q : X \times X \times X \times X \rightarrow Y \) such that \( f(x) = Q(x, x, x, x) \) for all \( x \) (see also [3,4], [12-15], [33]). It is easy to show that the function \( f(x) = x^4 \) satisfies the functional equation (1.4), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.
We deal with the following functional equation deriving from quartic and cubic functions:

\[ 4(f(3x + y) + f(3x - y)) = -12(f(x + y) + f(x - y)) + 12(f(2x + y) + f(2x - y)) \\
- 8f(y) - 192f(x) + f(2y) + 30f(2x). \quad (1.5) \]

It is easy to see that the function \( f(x) = ax^4 + bx^3 \) is a solution of the functional equation (1.5). In the present paper we investigate the general solution and the generalized Hyers-Ulam-Rassias stability of the functional equation (1.5).

2. General solution

In this section we establish the general solution of functional equation (1.5).

**Theorem 2.1.** Let \( X,Y \) be vector spaces, and let \( f : X \to Y \) be a function. Then \( f \) satisfies (1.5) if and only if there exists a unique symmetric multi-additive function \( Q : X \times X \times X \times X \to Y \) and a unique function \( C : X \times X \times X \to Y \) such that \( C \) is symmetric for each fixed one variable and is additive for fixed two variables, and that \( f(x) = Q(x,x,x,x) + C(x,x,x) \) for all \( x \in X \).

**Proof.** Suppose there exists a symmetric multi-additive function \( Q : X \times X \times X \times X \to Y \) and a function \( C : X \times X \times X \to Y \) such that \( C \) is symmetric for each fixed one variable and is additive for fixed two variables, and that \( f(x) = Q(x,x,x,x) + C(x,x,x) \) for all \( x \in X \). Then it is easy to see that \( f \) satisfies (1.5). For the converse let \( f \) satisfy (1.5). We decompose \( f \) into the even part and odd part by setting

\[ f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)), \]

for all \( x \in X \). By (1.5), we have

\[ 4f_e(3x + y) + 4f_e(3x - y) = \frac{1}{2}[4f(3x + y) + 4f(-3x - y) + 4f(3x - y) + 4f(-3x + y)] \]
\[ = \frac{1}{2}[4f(3x + y) + 4f(3x - y)] + \frac{1}{2}[4f((-3x) + (-y)) + 4f((-3x) - (-y))] \]
\[ = \frac{1}{2}[12f(2x + y) + 12f(2x - y) - 12f(x + y) - 12f(x - y)] \\
- 8f(y) - 192f(x) + f(2y) + 30f(2x)] \\
+ \frac{1}{2}[12f(-2x - y) + 12f((-2x) + y) - 12f(-x - y) - 12f(-x + y)] \\
- 8f(-y) - 192f(-x) + f(-2y) + 30f(-2x)] \\
= 12[\frac{1}{2}(f(2x + y) + f((-2x + y))) + 12[\frac{1}{2}(f(2x - y) + f((-2x - y)))] \\
- 12[\frac{1}{2}(f(x + y) + f((-x + y))) - 12[\frac{1}{2}(f(x - y) + f((-x - y)))] \\
- 8[\frac{1}{2}(f(y) + f(-y))] - 192[\frac{1}{2}(f(x) + f(-x))] \\
+ \frac{1}{2}[f(2y) + f(-2y)] + 30[\frac{1}{2}(f(2x) + f(-2x))] \\
= 12[f_e(2x + y) + f_e(2x - y)) - 12(f_e(x + y) + f_e(x - y)) \\
- 8f_e(y) - 192f_e(x) + f_e(2y) + 30f_e(2x)) \]
for all \(x, y \in X\). This means that \(f_e\) satisfies (1.5), or

\[
4(f_e(3x + y) + f_e(3x - y)) = -12(f_e(x + y) + f_e(x - y)) + 12(f_e(2x + y) + f_e(2x - y))
- 8f_e(y) - 192f_e(x) + f_e(2y) + 30f_e(2x).
\]  
(1.5(e))

Now putting \(x = y = 0\) in (1.5(e)), we get \(f_e(0) = 0\). Setting \(x = 0\) in (1.5(e)), by evenness of \(f_e\) we obtain

\[
f_e(2y) = 16f_e(y)
\]
(2.1)

for all \(y \in X\). Hence (1.5(e)) can be written as

\[
f_e(3x + y) + f_e(3x - y) + 3(f_e(x + y)
+ f_e(x - y)) = 3(f_e(2x + y) + f_e(2x - y)) + 72f_e(x) + 2f_e(y)
\]
(2.2)

for all \(x, y \in X\). With the substitution \(y := 2y\) in (2.2), we have

\[
f_e(3x + 2y) + f_e(3x - 2y) + 3f_e(x + 2y)
+ 3f_e(x - 2y) = 48f_e(x + y) + 48f_e(x - y) + 72f_e(x) + 32f_e(y).
\]
(2.3)

Replacing \(y\) by \(x + 2y\) in (2.2), we obtain

\[
16f_e(2x + y) + 16f_e(x - y) + 48f_e(x + y)
+ 48f_e(y) = 3f_e(3x + 2y) + 3f_e(x - 2y) + 2f_e(x + 2y) + 72f_e(x).
\]
(2.4)

Substituting \(-y\) for \(y\) in (2.4) gives

\[
16f_e(2x - y) + 16f_e(x + y) + 48f_e(x - y)
+ 48f_e(y) = 3f_e(3x - 2y) + 3f_e(x + 2y) + 2f_e(x - 2y) + 72f_e(x).
\]
(2.5)

By utilizing equations (2.3), (2.4) and (2.5), we obtain

\[
4f_e(2x + y) + 4f_e(2x - y) + f_e(x + 2y)
+ f_e(x - 2y) = 20f_e(x + y) + 20f_e(x - y) + 90f_e(x).
\]
(2.6)

Interchanging \(x\) and \(y\) in (2.3), we get

\[
f_e(2x + 3y) + f_e(2x - 3y) + 3f_e(2x + y)
+ 3f_e(2x - y) = 48f_e(x + y) + 48f_e(x - y) + 32f_e(x) + 72f_e(y).
\]
(2.7)

If we add (2.3) to (2.7), we have

\[
f_e(2x + 3y) + f_e(3x + 2y) + f_e(2x - 3y) + f_e(3x - 2y)
+ 3f_e(2x + y) + 3f_e(x + 2y) + 3f_e(2x - y)
+ 3f_e(x - 2y) = 96f_e(x + y) + 96f_e(x - y) + 104f_e(x) + 104f_e(y).
\]
(2.8)

And by utilizing equations (2.4), (2.5) and (2.8), we arrive at
\[
3f_e(x + y) + 3f_e(x - y) = -25f_e(x + y) - 25f_e(x - y)
\]
\[
-4f_e(x - 2y) - 4f_e(x + 2y) + 224f_a(x + y)
+ 224f_e(x - y) + 456f_e(x) + 216f_e(y).
\]

Let us interchange \(x\) and \(y\) in (2.9). Then we see that
\[
3f_e(3x + 2y) + 3f_e(3x - 2y) = -25f_e(x + 2y) - 25f_e(x - 2y)
\]
\[
-4f_e(2x - y) - 4f_e(2x + y) + 224f_a(x + y)
+ 224f_e(x - y) + 456f_e(y) + 216f_e(x).
\]

Comparing (2.10) with (2.3), we get
\[
4f_e(2x - y) + 4f_e(2x + y) = -16f_e(x + 2y)
- 16f_e(x - 2y) + 80f_e(x + y) + 80f_e(x - y) + 360f_e(y).
\]

If we compare (2.11) and (2.6), we conclude that
\[
f_e(x + 2y) + f_e(x - 2y) + 6f_e(x) = 4f_e(x + y) + 4f_e(x - y) + 24f_e(y).
\]

This means that \(f_e\) is quartic function. Thus there exists a unique symmetric multi-additive function \(Q : X \times X \times X \times X \rightarrow Y\) such that \(f_e(x) = Q(x,x,x,x)\) for all \(x \in X\). On the other hand, we can show that \(f_o\) satisfies (1.5), or
\[
4(f_o(3x + y) + f_o(3x - y)) = -12(f_o(x + y) + f_o(x - y)) + 12(f_o(2x + y) + f_o(2x - y))
- 8f_o(y) - 192f_o(x) + f_o(2y) + 30f_o(2x).
\]

Now setting \(x = y = 0\) in (1.5(o)) gives \(f_o(0) = 0\). Putting \(x = 0\) in (1.5(o)), then by oddness of \(f_o\), we have
\[
f_o(2y) = 8f_o(y).
\]

Hence (1.5(o)) can be written as
\[
f_o(3x + y) + f_o(3x - y) + 3f_o(x + y)
+ 3f_o(x - y) = 3f_o(2x + y) + 3f_o(2x - y) + 12f_o(x)
\]

for all \(x, y \in X\). Replacing \(x\) by \(x + y\), and \(y\) by \(x - y\) in (2.13) we have
\[
8f_o(2x + y) + 8f_o(x + 2y) + 24f_o(x)
+ 24f_o(y) = 3f_o(3x + y) + 3f_o(x + 3y) + 12f_o(x + y)
\]

and interchanging \(x\) and \(y\) in (2.13) yields
\[
f_o(x + 3y) - f_o(x - 3y) + 3f_o(x + y)
- 3f_o(x - y) = 3f_o(x + 2y) - 3f_o(x - 2y) + 12f_o(y).
\]

Which on substitution of \(-y\) for \(y\) in (2.13) gives
\[ f_o(3x - y) + f_o(3x + y) + 3f_o(x - y) \\
+ 3f_o(x + y) = 3f_o(2x - y) + 3f_o(2x + y) + 12f_o(x). \]  
(2.16)

Replace \( y \) by \( x + 2y \) in (2.13). Then we have

\[ 8f_o(2x + y) + 8f_o(x - y) + 24f_o(x + y) \\
- 24f_o(y) = 3f_o(3x + 2y) + 3f_o(x - 2y) + 12f_o(x). \]  
(2.17)

From the substitution \( y := -y \) in (2.17) it follows that

\[ 8f_o(2x - y) + 8f_o(x + y) + 24f_o(x - y) \\
+ 24f_o(y) = 3f_o(3x - 2y) + 3f_o(x + 2y) + 12f_o(x). \]  
(2.18)

If we add (2.17) to (2.18), we have

\[ 3f_o(3x - 2y) + 3f_o(3x + 2y) = 8f_o(2x + y) + 8f_o(2x - y) \\
- 3f_o(x + 2y) - 3f_o(x - 2y) + 32f_o(x - y) \\
+ 32f_o(x + y) - 24f_o(x). \]  
(2.19)

Let us interchange \( x \) and \( y \) in (2.19). Then we see that

\[ 3f_o(2x + 3y) - 3f_o(2x - 3y) = 8f_o(x + 2y) - 8f_o(x - 2y) \\
- 3f_o(2x + y) + 3f_o(2x - y) + 32f_o(x + y) \\
- 32f_o(x - y) - 24f_o(y). \]  
(2.20)

With the substitution \( y := x + y \) in (2.13), we have

\[ f_o(4x + y) + f_o(2x - y) + 3f_o(2x + y) \\
- 3f_o(y) = 3f_o(3x + y) + 3f_o(x - y) + 12f_o(x) \]  
(2.21)

and replacing \(-y\) by \( y \) gives

\[ f_o(4x - y) + f_o(2x + y) + 3f_o(2x - y) \\
+ 3f_o(y) = 3f_o(3x - y) + 3f_o(x + y) + 12f_o(x). \]  
(2.22)

If we add (2.21) to (2.22), we have

\[ f_o(4x + y) + f_o(4x - y) = 3f_o(3x + y) + 3f_o(3x - y) \\
- 4f_o(2x - y) - 4f_o(2x + y) + 3f_o(x - y) \\
+ 3f_o(x + y) + 24f_o(x). \]  
(2.23)

By comparing (2.16) with (2.23), we arrive at

\[ f_o(4x + y) + f_o(4x - y) = 5f_o(2x + y) + 5f_o(2x - y) \\
- 6f_o(x + y) - 6f_o(x - y) + 60f_o(x) \]  
(2.24)
and replacing $y$ by $2y$ in (2.13) gives

$$f_o(3x + 2y) + f_o(3x - 2y) = 24f_o(x + y) + 24f_o(x - y)$$
$$- 3f_o(x + 2y) - 3f_o(x - 2y) + 12f_o(x).$$  \hfill (2.25)

By comparing (2.25) with (2.19), we arrive at

$$3f_o(x + 2y) + 3f_o(x - 2y) = 20f_o(x + y) + 20f_o(x - y)$$
$$- 4f_o(2x + y) - 4f_o(2x - y) + 30f_o(x).$$  \hfill (2.26)

Let us interchange $x$ and $y$ in (2.25). Then we see that

$$f_o(2x + 3y) - f_o(2x - 3y) = 24f_o(x + y) - 24f_o(x - y)$$
$$- 3f_o(2x + y) + 3f_o(2x - y) + 12f_o(y).$$  \hfill (2.27)

Thus combining (2.27) with (2.20) yields

$$4f_o(x + 2y) - 4f_o(x - 2y) = 3f_o(2x - y) - 3f_o(2x + y)$$
$$+ 20f_o(x + y) - 20f_o(x - y) + 30f_o(y).$$  \hfill (2.28)

By comparing (2.28) with (2.15), we arrive at

$$4f_o(x + 3y) - 4f_o(x - 3y) = 9f_o(2x - y) - 9f_o(2x + y)$$
$$+ 48f_o(x + y) - 48f_o(x - y) + 138f_o(y).$$  \hfill (2.29)

Which, by putting $y := 2y$ in (2.14), leads to

$$64f_o(x + y) + 8f_o(x + 4y) + 24f_o(x)$$
$$+ 192f_o(y) = 3f_o(3x + 2y) + 3f_o(3x + 6y) + 12f_o(x + 2y).$$  \hfill (2.30)

Replacing $y$ by $-y$ in (2.30) gives

$$64f_o(x - y) + 8f_o(x - 4y) + 24f_o(x)$$
$$- 192f_o(y) = 3f_o(3x - 2y) + 3f_o(3x - 6y) + 12f_o(x - 2y).$$  \hfill (2.31)

If we subtract (2.30) from (2.31), we obtain

$$8f_o(x + 4y) - 8f_o(x - 4y) = 3f_o(3x + 2y) - 3f_o(3x - 2y)$$
$$+ 3f_o(x + 6y) - 3f_o(x - 6y) + 12f_o(x + 2y) - 12f_o(x - 2y)$$
$$+ 64f_o(x - y) - 64f_o(x + y) - 384f_o(y).$$  \hfill (2.32)

Setting $x$ instead of $y$ and $y$ instead of $x$ in (2.24), we get

$$f_o(x + 2y) - f_o(x - 2y) = 5f_o(x + y) - 5f_o(x - y)$$
$$+ 6f_o(x - y) - 6f_o(x + y) + 60f_o(y).$$  \hfill (2.33)

Combining (2.32) and (2.33) yields
3f_o(3x + 2y) - 3f_o(3x - 2y) = 28f_o(x + 2y) - 28f_o(x - 2y) + 3f_o(x - 6y) - 3f_o(x + 6y) + 16f_o(x + y) - 16f_o(x - y) + 864f_o(y) 

(2.34)

and subtracting (2.18) from (2.17), we obtain

3f_o(3x + 2y) - 3f_o(3x - 2y) = 3f_o(x + 2y) - 3f_o(x - 2y) + 8f_o(2x + y) - 8f_o(2x - y) + 16f_o(x + y) - 16f_o(x - y) - 48f_o(y). 

(2.35)

By comparing (2.34) with (2.35), we arrive at

3f_o(x + 6y) - 3f_o(x - 6y) = 25f_o(x + 2y) - 25f_o(x - 2y) + 8f_o(2x - y) - 8f_o(2x + y) + 912f_o(y). 

(2.36)

Interchanging y with 2y in (2.29) gives the equation

4f_o(x + 6y) - 4f_o(x - 6y) = 48f_o(x + 2y) - 48f_o(x - 2y) + 72f_o(x - y) - 72f_o(x + y) + 1104f_o(y). 

(2.37)

We obtain from (2.36) and (2.37)

44f_o(x + 2y) - 44f_o(x - 2y) = 32f_o(2x - y) - 32f_o(2x + y) + 216f_o(x + y) - 216f_o(x - y) + 336f_o(y). 

(2.38)

By using (2.28) and (2.38), we lead to

\[ f_o(2x + y) - f_o(2x - y) = 4f_o(x + y) - 4f_o(x - y) - 6f_o(y). \]  

(2.39)

And interchanging x with y in (2.39) gives

\[ f_o(x + 2y) + f_o(x - 2y) = 4f_o(x + y) + 4f_o(x - y) - 6f_o(x). \]  

(2.40)

If we compare (2.40) and (2.26), we conclude that

\[ 8f_o(x + y) + 8f_o(x - y) + 48f_o(x) = 4f_o(2x + y) + 4f_o(2x - y). \]

This means that \( f_o \) is cubic function and that there exits a unique function \( C : X \times X \times X \rightarrow Y \) such that \( f_o(x) = C(x, x, x) \) for all \( x \in X \), and \( C \) is symmetric for each fixed one variable and is additive for fixed two variables. Thus for all \( x \in X \), we have

\[ f(x) = f_o(x) + f_o(x) = C(x, x, x) + Q(x, x, x). \]

This completes the proof of Theorem.

The following Corollary is an alternative result of above Theorem 2.1.

**Corollary 2.2.** Let \( X, Y \) be vector spaces, and let \( f : X \rightarrow Y \) be a function satisfying (1.5). Then the following assertions hold.

a) If \( f \) is even function, then \( f \) is quartic.

b) If \( f \) is odd function, then \( f \) is cubic.
3. Stability

We now investigate the generalized Hyers-Ulam-Rassias stability problem for functional equation (1.5). From now on, let $X$ be a real vector space and let $Y$ be a Banach space. Now before taking up the main subject, given $f: X \rightarrow Y$, we define the difference operator $D_f: X \times X \rightarrow Y$ by

$$D_f(x, y) = 4[f(3x + y) + f(3x - y)] - 12[f(2x + y) + f(2x - y)] + 12[f(x + y) + f(x - y)]$$
$$- f(2y) + 8f(y) - 30f(2x) + 192f(x)$$

for all $x, y \in X$. We consider the following functional inequality:

$$\|D_f(x, y)\| \leq \phi(x, y)$$

for an upper bound $\phi: X \times X \rightarrow [0, \infty)$. 

**Theorem 3.1.** Let $s \in \{1, -1\}$ be fixed. Suppose that an even mapping $f: X \rightarrow Y$ satisfies $f(0) = 0$, and

$$\|D_f(x, y)\| \leq \phi(x, y) \quad (3.1)$$

for all $x, y \in X$. If the upper bound $\phi: X \times X \rightarrow [0, \infty)$ is a mapping such that the series $\sum_{i=0}^{\infty} 2^{4s^i} \phi(0, \frac{x}{2^i})$ converges, and that $\lim_{n \rightarrow \infty} 2^{4s^i} \phi(0, \frac{x}{2^n}) = 0$ for all $x, y \in X$, then the limit $Q(x) = \lim_{n \rightarrow \infty} 2^{4s^i} f(\frac{x}{2^n})$ exists for all $x \in X$, and $Q: X \rightarrow Y$ is a unique quartic function satisfying (1.5), and

$$\|f(x) - Q(x)\| \leq \frac{1}{16} \sum_{i=0}^{\infty} 2^{4s^i+1} \phi(0, \frac{x}{2^{2i+1}}) \quad (3.2)$$

for all $x \in X$.

**Proof.** Let $s = 1$. Putting $x = 0$ in (3.1), we get

$$\|f(2y) - 16f(y)\| \leq \phi(0, y). \quad (3.3)$$

Replacing $y$ by $\frac{x}{2}$ in (3.3), yields

$$\|f(x) - 16f(\frac{x}{2})\| \leq \phi(0, \frac{x}{2}). \quad (3.4)$$

Interchanging $x$ with $\frac{x}{4}$ in (3.4), and multiplying by 16 it follows that

$$\|16f(\frac{x}{4}) - 16^2f(\frac{x}{4^2})\| \leq 16\phi(0, \frac{x}{4}). \quad (3.5)$$

Combining (3.4) and (3.5), we lead to

$$\|16^2f(\frac{x}{4^2}) - f(x)\| \leq \phi(0, \frac{x}{2}) + 16\phi(0, \frac{x}{4}). \quad (3.6)$$

From the inequality (3.4) we use iterative methods and induction on $n$ to prove our next relation:

$$\|16^n f(\frac{x}{2^n}) - f(x)\| \leq \frac{1}{16} \sum_{i=0}^{n-1} 16^{i+1} \phi(0, \frac{x}{2^{2i+1}}). \quad (3.7)$$
We multiply (3.7) by $16^m$ and replace $x$ by $\frac{x}{2^m}$ to obtain that

$$\|16^{m+n} f(\frac{x}{2^{m+n}}) - 16^m f(\frac{x}{2^m})\| \leq \sum_{i=0}^{n-1} 16^{m+i} \phi(0, \frac{x}{2^{m+i+1}}).$$

This shows that $\{16^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in $Y$ by taking the limit $m \to \infty$. Since $Y$ is a Banach space, it follows that the sequence $\{16^n f(\frac{x}{2^n})\}$ converges. We define $Q : X \to Y$ by $Q(x) = \lim_{n \to \infty} 2^n f(\frac{x}{2^n})$ for all $x \in X$. It is clear that $Q(-x) = Q(x)$ for all $x \in X$, and it follows from (3.1) that

$$\|D_Q(x,y)\| = \lim_{n \to \infty} 16^n \|D_f(\frac{x}{2^n}, \frac{y}{2^n})\| \leq \lim_{n \to \infty} 16^n \phi(\frac{x}{2^n}, \frac{y}{2^n}) = 0$$

for all $x, y \in X$. Hence by Corollary 2.2, $Q$ is quartic. It remains to show that $Q$ is unique. Suppose that there exists another quartic function $Q' : X \to Y$ which satisfies (1.5) and (3.2). Since $Q(2^n x) = 16^n Q(x)$, and $Q'(2^n x) = 16^n Q'(x)$ for all $x \in X$, we conclude that

$$\|Q(x) - Q'(x)\| = 16^n \|Q(\frac{x}{2^n}) - Q'(\frac{x}{2^n})\|$$

$$\leq 16^n \|Q(\frac{x}{2^n}) - f(\frac{x}{2^n})\| + 16^n \|Q'(\frac{x}{2^n}) - f(\frac{x}{2^n})\|$$

$$\leq 2 \sum_{i=0}^{\infty} 16^{n+i} \phi(0, \frac{x}{2^{n+i+1}})$$

for all $x \in X$. By letting $n \to \infty$ in this inequality, it follows that $Q(x) = Q'(x)$ for all $x \in X$, which gives the conclusion. For $s = -1$, we obtain

$$\|f(\frac{2^m x}{16^m}) - f(x)\| \leq \frac{1}{16} \sum_{i=0}^{n-2} 2^{3s+1} \phi(0, \frac{2^{s+1} x}{16^{i+1}}),$$

from which one can prove the result by a similar technique. \qed

**Theorem 3.2.** Let $s \in \{1, -1\}$ be fixed. Suppose that an odd mapping $f : X \to Y$ satisfies

$$\|D_f(x, y)\| \leq \phi(x, y)$$

(3.8)

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a mapping such that $\sum_{i=0}^{\infty} 2^{3s+1} \phi(0, \frac{x}{2^n})$ converges, and that $\lim_{n \to \infty} 2^{3s+1} \phi(\frac{x}{2^n}, \frac{y}{2^n}) = 0$ for all $x, y \in X$, then the limit $C(x) = \lim_{n \to \infty} 2^{3s} f(\frac{x}{2^n})$ exists for all $x \in X$, and $C : X \to Y$ is a unique cubic function satisfying (1.5), and

$$\|f(x) - C(x)\| \leq \frac{1}{8} \sum_{i=0}^{\infty} 2^{3s+i+1} \phi(0, \frac{x}{2^{i+1}})$$

(3.9)

for all $x \in X$.

**Proof.** Let $s = 1$. Set $x = 0$ in (3.8). We obtain

$$\|f(y) - f(2y)\| \leq \phi(0, y).$$

(3.10)

Replacing $y$ by $\frac{x}{2}$ in (3.10) to get

$$\|8f(\frac{x}{2}) - f(x)\| \leq \phi(0, \frac{x}{2}).$$

(3.11)

An induction argument now implies
\| \frac{8^n}{2^{n+1}} f(x) - f(x) \| \leq \frac{1}{8} \sum_{i=0}^{n-1} 8^{i+1} \phi(0, \frac{x}{2^{i+1}}). \quad (3.12)

Multiply (3.12) by $8^m$ and replace $x$ by $\frac{x}{2^m}$, we obtain that
\|
\frac{8^{m+n}}{2^{m+n+1}} f\left( \frac{x}{2^{m+n+1}} \right) - \frac{8^m}{2^{m+1}} f\left( \frac{x}{2^{m+1}} \right) \|
\leq \sum_{i=0}^{n-1} 8^{i+1} \phi(0, \frac{x}{2^{i+1}}).
\quad (3.13)

The right hand side of the inequality (3.13) tends to 0 as $m \to \infty$ because of
\[
\sum_{i=0}^{\infty} 8^i \phi(0, \frac{x}{2^{i+1}}) < \infty
\]
by assumption, and thus the sequence $\{2^{3n} f\left( \frac{x}{2^{n}} \right)\}$ is Cauchy in $\mathcal{Y}$, as desired. Therefore we may define a mapping $C : X \to \mathcal{Y}$ as $C(x) = \lim_{n \to \infty} 2^{3n} f\left( \frac{x}{2^{n}} \right)$. The rest of proof is similar to the proof of Theorem 3.1.

**Theorem 3.3.** Let $s \in \{1, -1\}$ be fixed. Suppose a mapping $f : X \to \mathcal{Y}$ satisfies $f(0) = 0$, and $\|Df(x,y)\| \leq \phi(x,y)$ for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a mapping such that
\[
\sum_{i=0}^{\infty} \left( (|s|+s)2^{4si} \phi(0, \frac{x}{2^{s+1}}) + (|s|-s)2^{4si} \phi(0, \frac{x}{2^{s+1}}) \right) < \infty,
\quad (3.14)
\]
and
\[
\lim_{n \to \infty} \left( (|s|+s)2^{4sn-1} \phi\left( \frac{x}{2^{s+n}}, \frac{y}{2^n} \right) + (|s|-s)2^{4sn} \phi\left( \frac{x}{2^{s+n}}, \frac{y}{2^n} \right) \right) = 0,
\quad (3.15)
\]
for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to \mathcal{Y}$ and a unique cubic function $C : X \to \mathcal{Y}$ satisfying
\[
\| f(x) - Q(x) - C(x) \| \leq \sum_{i=0}^{\infty} \left\{ \frac{2^{4i+1} + 2^{4i+1}}{32} \right\} \left[ \phi(0, \frac{x}{2^{i+1}}) + \phi(0, \frac{-x}{2^{i+1}}) \right] \]
\quad (3.16)

for all $x \in X$.

**Proof.** Let $f_+(x) = \frac{1}{2} f(x) + f(-x)$ for all $x \in X$. Then $f_+(0) = 0$ and $f_+$ is even function satisfying $\|Df_+(x,y)\| \leq \frac{1}{2} [\phi(x,y) + \phi(-x,-y)]$ for all $x, y \in X$. From Theorem 3.1, it follows that there exists a unique quartic function $Q : X \to \mathcal{Y}$ satisfying
\[
\| f_+(x) - Q(x) \| \leq \frac{1}{32} \sum_{i=0}^{\infty} \left\{ 2^{4i+1} + 2^{4i+1} \right\} \left[ \phi(0, \frac{x}{2^{i+1}}) + \phi(0, \frac{-x}{2^{i+1}}) \right]
\quad (3.16)
\]
for all $x \in X$. Let now $f_-(x) = \frac{1}{2} [f(x) - f(-x)]$ for all $x \in X$. Then $f_-$ is odd function satisfying
\[
\| Df_-(x,y) \| \leq \frac{1}{2} [\phi(x,y) + \phi(-x,-y)]
\]
for all $x, y \in X$. Hence in view of Theorem 3.2, it follows that there exists a unique cubic function $C : X \to \mathcal{Y}$ such that
and inequality controlled by the mixed type product-sum function combining (3.16) and (3.17), it follows that introduced by J. M. Rassias (see for example [32]).

\[ \| f(x) - C(x) \| \leq \frac{1}{16} \sum_{i=\frac{-s}{2}}^{\infty} \left\{ 2^{3i+1} \phi(0, \frac{x}{2^{s+1}}) + 2^{3i+1} \phi(0, \frac{-x}{2^{s+1}}) \right\} \]  
(3.17)

for all \( x \in X \). On the other hand we have \( f(x) = f_s(x) + f_o(x) \) for all \( x \in X \). Then by combining (3.16) and (3.17), it follows that

\[ \| f(x) - C(x) - Q(x) \| \leq \| f_s(x) - Q(x) \| + \| f_o(x) - C(x) \| \]

\[ \leq \sum_{i=\frac{-s}{2}}^{\infty} \left\{ \left( \frac{2^{3i+1}}{32} + \frac{2^{3i+1}}{16} \right) \phi(0, \frac{x}{2^{s+1}}) + \phi(0, \frac{-x}{2^{s+1}}) \right\} \]

for all \( x \in X \), and the proof of Theorem is complete.

\[ \square \]

We are going to investigate the Hyers-Ulam-Rassias stability problem for functional equation (1.5).

**Corollary 3.4.** Let \( p \in (-\infty, 3) \cup (4, +\infty) \), \( \theta > 0 \). Suppose \( f : X \to Y \) satisfies \( f(0) = 0 \), and inequality

\[ \| D_f(x, y) \| \leq \theta(\|x\|^p + \|y\|^p), \]

for all \( x, y \in X \). Then there exists a unique quartic function \( Q : X \to Y \), and a unique cubic function \( C : X \to Y \) satisfying

\[ \| f(x) - Q(x) - C(x) \| \leq \begin{cases} \theta \|x\|^p(\frac{1}{2^p - 2} + \frac{1}{2^{p-2} - 2}), & p > 4, \\ \theta \|x\|^p(\frac{1}{2^{p-2} - 2} + \frac{1}{2^{p-3} - 2}), & p < 3 \end{cases} \]

for all \( x \in X \).

**Proof.** Let \( s = 1 \) in Theorem 3.3. Then by taking \( \phi(x, y) = \theta(\|x\|^p + \|y\|^p) \) for all \( x, y \in X \), the relations (3.14) and (3.15) hold for \( p > 4 \). Then there exists a unique quartic function \( Q : X \to Y \) and a unique cubic function \( C : X \to Y \) satisfying

\[ \| f(x) - Q(x) - C(x) \| \leq \theta \frac{x^p}{2^p}(\frac{1}{1 - 2^{4-p}} + \frac{1}{1 - 2^{4-p}}) \]

for all \( x \in X \). Let now \( s = -1 \) in Theorem 3.3 and put \( \phi(x, y) = \theta(\|x\|^p + \|y\|^p) \) for all \( x, y \in X \). Then the relations (3.14) and (3.15) hold for \( p < 3 \). Then there exists a unique quartic function \( Q : X \to Y \) and a unique cubic function \( C : X \to Y \) satisfying

\[ \| f(x) - Q(x) - C(x) \| \leq \theta \|x\|^p(\frac{1}{2^p - 2} + \frac{1}{2^{p-2} - 2}) \]

for all \( x \in X \). \[ \square \]

Similarly, we can prove the following Ulam stability problem for functional equation (1.5) controlled by the mixed type product-sum function

\[ (x, y) \mapsto \theta(\|x\|^p \|y\|_Y + \|x\|^p + \|y\|^p) \]

introduced by J. M. Rassias (see for example [32]).
Corollary 3.5. Let $u, v, p$ be real numbers such that $u + v, p \in (-\infty, 3] \cup (4, +\infty)$, and let $\theta > 0$. Suppose $f : X \to Y$ satisfies $f(0) = 0$, and inequality

$$\|Df(x, y)\| \leq \theta (\|x\|_X^u \|y\|_X^v + \|x\|_p^p + \|y\|_p^p),$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$, and a unique cubic function $C : X \to Y$ satisfying

$$\|f(x) - Q(x) - C(x)\| \leq \begin{cases} \theta \|x\|_p^p \left(\frac{1}{2^p - 2} + \frac{1}{2^p - 2}\right), & p > 4, \\ \theta \|x\|_p^p \left(\frac{1}{4^p - 2} + \frac{1}{4^p - 2}\right), & p < 3 \end{cases}$$

for all $x \in X$.

By Corollary 3.4, we solve the following Hyers-Ulam stability problem for functional equation (1.5).

Corollary 3.6. Let $\epsilon$ be a positive real number. Suppose $f : X \to Y$ satisfies $f(0) = 0$, and $\|Df(x, y)\| \leq \epsilon$, for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$, and a unique cubic function $C : X \to Y$ satisfying

$$\|f(x) - Q(x) - C(x)\| \leq \frac{22}{105} \epsilon$$

for all $x \in X$.

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