A new and improved algorithm for online bin packing

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Abstract

We revisit the classic online bin packing problem. In this problem, items of positive sizes no larger than 1 are presented one by one to be packed into subsets called bins of total sizes no larger than 1, such that every item is assigned to a bin before the next item is presented. We use online partitioning of items into classes based on sizes, as in previous work, but we also apply a new method where items of one class can be packed into more than two types of bins, where a bin type is defined according to the number of such items grouped together. Additionally, we allow the smallest class of items to be packed in multiple kinds of bins, and not only into their own bins. We combine this with the approach of packing of sufficiently big items according to their exact sizes. Finally, we simplify the analysis of such algorithms, allowing the analysis to be based on the most standard weight functions. This simplified analysis allows us to study the algorithm which we defined based on all these ideas. This leads us to the design and analysis of the first algorithm of asymptotic competitive ratio strictly below 1.58, specifically, we break this barrier by providing an algorithm AH (Advanced Harmonic) whose asymptotic competitive ratio does not exceed 1.57829.

1 Introduction

Bin packing [5, 6] is the problem of partitioning or packing a set of items of rational sizes in (0, 1] into subsets of items, which are called bins, of total sizes no larger than 1. In the offline variant the list of items is given as a set, and in the online environment items are presented one by one and each item has to be packed into a bin irrevocably before the next item is presented.

For an algorithm $A$, we denote its cost, that is, the number of used bins in its packing on an input $I$ by $A(I)$. The cost of an optimal solution $OPT$, for the same input, is denoted by $OPT(I)$. The asymptotic approximation ratio allows to compare the costs for inputs for which the optimal cost is sufficiently large. The asymptotic approximation ratio of $A$ is defined as follows.

$$R_A = \lim_{N \to \infty} \left( \sup_{I:OPT(I) \geq N} \frac{A(I)}{OPT(I)} \right).$$

In this paper we only consider the asymptotic approximation ratio, which is the common measure for bin packing algorithms. Thus we use the term approximation ratio throughout the paper, with the meaning of asymptotic approximation ratio. Moreover,
the term *competitive ratio* often replaces the term “approximation ratio” in cases where online algorithms are considered. We will use this term for the asymptotic measure. When we discuss the absolute measure \( \sup_I \frac{A(I)}{OPT(I)} \) (the absolute approximation ratio or the absolute competitive ratio), we will mention this explicitly. A standard method for proving an upper bound for the asymptotic approximation ratio or the asymptotic competitive ratio for an algorithm \( A \) is to show the existence of a constant \( C \geq 0 \) independent of the input, such that for any input \( I \), \( A(I) \leq R \cdot OPT(I) + C \) (and then the value of the asymptotic measure is at most \( R \)). Most work on upper bounds on the asymptotic competitive ratio provide in fact an upper bound using this last method, and we will follow this approach as well.

For the offline problem, algorithms with an approximation ratio of \( 1 + \varepsilon \) can be designed \([9, 16]\) for any \( \varepsilon > 0 \). If the first definition is used, a 1-approximation can be designed \([16]\), where the cost of the solution computed by the algorithm is \( OPT(I) + o(OPT(I)) \) (see also recent work on improving the sub-linear function of \( OPT(I) \) by Rothvoss \([21]\) and by Hoberg and Rothvoss \([12]\)).

The classic bin packing problem, which we study here, was presented in the early 1970’s \([24, 13, 14, 15]\). It was introduced as an offline problem, but many of the algorithms initially proposed for it were in fact online. Johnson \([13, 14]\) defined and analyzed the simple algorithm Next Fit (NF), which tries to pack the next item into the last bin that was used for packing, if such a bin exists (in which case such a bin is called “active”) and the item can be packed there, and otherwise it opens a new bin for the item. The competitive ratio of this algorithm is 2 \([13, 14]\). Any Fit (AF) algorithms, as opposed to the behavior of NF which only tests at most one active bin for feasibility of packing a new item there, pack a new item into a nonempty bin unless this is impossible (in which case a new bin is opened). Such algorithms have competitive ratios of at most 2. Next, consider a sub-class of algorithms where one may not select a bin with smallest total size of currently packed items for packing a new item, unless this minimum is not unique or this is the only bin that can accommodate the new item except for an empty bin. The last class of algorithms is called Almost Any Fit (AAF), and they have competitive ratios of 1.7 \([15, 14]\). A well-known algorithm, which is in fact a special case of AAF is Best Fit (BF), which always chooses the fullest bin where the new item can be packed. First Fit (FF) is another important special case of AF (but not of AAF) which selects a minimum index bin for each new item (where it can be packed). The competitive ratio of FF is 1.7 \([15, 7]\).

The pre-sorted versions of these algorithms, called NFD, FFD, BFD, and AFD, were studied as well. In these versions, items are still presented one by one, but they are sorted in a non-increasing order (according to sizes). For example, the approximation ratio of NFD is (approximately) 1.69103 \([2]\) and that of FFD is \( \frac{11}{9} \approx 1.22222 \) \([13]\). For AFD in general, the approximation ratio is at most 1.25 \([13, 14, 15]\). These pre-sorted variants are not online algorithms.

We design and analyze a new algorithm AH (Advanced Harmonic) for online bin packing, and show that its competitive ratio does not exceed 1.57828956. This is the first algorithm whose asymptotic competitive ratio is below 1.58. We use a new type of analysis of algorithms which allows us to split the analysis into cases, while for every case we define only three different values (and even just one value in a large number of cases), and based on those we calculate weights for items. The analysis is split into cases in recent previous work as well, but the analysis of each case is much more difficult. Items are partitioned into classes according to sizes. As in previous work, we sometimes do not pack the maximum number of items of some class into a bin, and leave space for items of another class (possibly arriving later). One new feature of AH is that in previous papers, in the algorithms there were at most two options for every class. For any given class, one option was a bin with the maximum number of items of this class fitting into a bin. For some of the classes there was a second option consisted of a very small number of items from this class (with reserved spaces for items of another class, possibly arriving later). We allow intermediate values as
well with more than two options for some classes and not only two kinds of bins for a given class.

We use simple weight functions for the analysis, rather than the much more complicated tool called weight systems \[22\]. Weight functions are an auxiliary tool used for the analysis of bin packing (and other) algorithms (this technique is also called dual fitting). In this method, a weight is defined for each item (usually, based on its size, and sometimes it is also based on its role in the packing). If there are multiple kinds of outputs, it is possible to define a weight function for each one of them. The total weight of items is then used to compare the numbers of bins in the output of the algorithm and in an optimal solution. The list of weights of one item for different output types, also called scenarios, can be seen as a vector associated with the item. Thus, the weights can be seen as one function from the items to vectors whose dimension is the number of scenarios. Briefly, a weight system is a generalization where the weight function also maps items (or item sizes) to vectors, but in order to compute the weight of some item for a given scenario, another function, called a consolidation function, is used. This last function is a piecewise linear function (mapping real vectors to reals). The slightly simplified approach is to use convex combinations of weights according to subsets of scenarios. It has not been proved that weight systems are a stronger tool than just weights defined for the different scenarios. However, for simple weights every scenario can be analyzed independently from other scenarios. We exploit the simplicity of weight functions to obtain a clean and full analysis, which is easier to implement and verify (compared to the analysis resulting from weight systems). The main advantage is that every case is analyzed in a separate calculation using a standard knapsack solver without considering any other cases at that time. This simplicity allows us to analyze the new features that we introduce. Obviously, as these are cases for one algorithm, they have a common set of parameters, but once the algorithm has been fixed, there is no connection between the various cases.

The significance of our approach is that we combine many existing methods, including that of Babel et al. \[1\] (recently used by Heydrich and van Stee \[11, 10\] for classic bin packing), adding several new features, and applying a simple analysis, which can be verified easily. We define the action of our algorithm AH, we prove a number of invariants and properties of AH in detail, and then we provide the specific parameters and compact representations of the lists of weights. For every possible output type and scenario, there is a small number of values used for the calculation of weights for it. We also provide explicit lists of weights calculated based on the values and the parameters.

To explain the new features of our work, we start with a discussion of the design of harmonic type algorithms. Already in much of the previous work on online algorithms for bin packing, items were partitioned into classes by size. The simplest such classification is based on harmonic numbers, leading to the Harmonic algorithm of Lee and Lee \[17\]. In the harmonic algorithm of index $k$ (for an integer parameter $k \geq 2$), subset $j$ is the intersection of the input and $\left(\frac{1}{j+1}, \frac{1}{j}\right]$ (where $1 \leq j \leq k-1$), and subset $k$ of tiny items is the intersection of the input and $(0, \frac{1}{k})$.

In these algorithms each subset is packed independently from other subsets using NF (so for $j \leq k-1$, any bin for subset $j$, except for possibly the last such bin, has $j$ items, but for subset $k$, every bin except for the last bin for this subset has a total size of items above $\frac{k-1}{k}$), and for $k$ growing to infinity, the resulting competitive ratio is approximately 1.69103 \[17\]. The drawback of those algorithms is that bins of subsets with small values of $j$ can be packed with small sizes of items (for example, a bin of subset 2 may have total size just above $\frac{2}{3}$ and a bin of subset 1 may have just one item of size just above $\frac{1}{2}$).

The first idea which comes to mind is to try to combine items of those two subset into common bins. However, if items of class 2 arrive first, one cannot just pack them one per bin, as this immediately leads to a competitive ratio of 2 (if no items of subset 1 arrive afterwards). Lee and
Lee [17] proposed the following method to overcome this. A fixed fraction of items of subset 2 (up to rounding errors) is packed one per bin and the remaining items are packed in pairs. Thus, there are two kinds of bins for subset 2. The items we refer to here can only be sufficiently small items, so there is a threshold \( \Delta \in (\frac{1}{2}, \frac{3}{4}) \) such that items of sizes in \((\Delta, 1]\) and \((1 - \Delta, \frac{1}{2}]\) are packed as before, while the algorithm tries to combine an item of size in \((\frac{1}{7}, 1 - \Delta]\) with an item of size in \((\frac{1}{7}, \Delta]\). Even if those two items (one item of each one of the two intervals) are relatively small, still their total size is above \(\frac{5}{6} \approx 0.83333\). This last algorithm was called Refined-Harmonic, and its competitive ratio is smaller than 1.636. Ramanan et al. [19] designed two algorithms called Modified Harmonic and Modified Harmonic-2. The first one has a competitive ratio below 1.61562, and it allows to combine items of many subsets with items of sizes above \(\frac{1}{2}\) (and at most \(\Delta\)). The second algorithm does not use only a single value of \(\Delta\), but splits the interval \((\frac{1}{7}, 1]\) further, allowing additional kinds of combinations. Its competitive ratio is approximately 1.612. For most subsets of items (where \(k\) is chosen to be in \([20, 40]\) in all these algorithms), the last two algorithms pack some proportion of the items in groups of smaller sizes, to allow it to be combined with an item of size above \(\frac{1}{7}\). Intuitively, for an illustrative example, assume that \(\Delta = 0.6\), and consider the items of sizes in \((\frac{1}{11}, \frac{1}{10}]\). The items that are not packed into groups of ten items should be packed into groups of four items (the parameters of the algorithms are different from those of this example). For some of the subsets the proportion is zero, and they are still packed using NF. The drawback of such algorithms (as it is exhibited by Ramanan et al. [19]) is that no matter how many thresholds there are, there can be pairs of items that can be combined into bins of optimal solutions while the algorithm does not allow it as it has fixed thresholds. Specifically, such algorithms allow to combine items of different intervals only in the case that the largest items of the two intervals fit together into a bin. This is the case with the next two harmonic type algorithms as well.

The next two papers, that of Richey [20] and that of Seiden [22] deal with a more complicated algorithm where many more subsets can be combined. The general structure is proposed in [20], and a full and corrected algorithm with its analysis is provided in [22]. For illustration, the items packed into smaller groups are called red and those packed into bins with maximum numbers of items of the subset are called blue. The goal is to combine as many bins with blue items with bins having red items as possible. Bins with red items always have small numbers of items, to allow them to be combined with relatively large items of sizes above \(\frac{1}{7}\). The analysis is far from being simple, though it leads to a competitive ratio of at most 1.58889 (Heydrich and van Stee [11, 10] mention that this last value can be decreased very slightly).

The carefully designed subset structure eliminates many worst-case examples, but the drawback mentioned above still remains. Recently, Heydrich and van Stee [11, 10] proposed to use a method introduced by Babel et. al [1], where some items are packed based on their exact size rather than by their subset. The approach of [11, 10] which we adopt is to apply the methods of Babel et. al [1] on the largest items, of sizes in \((\frac{1}{3}, 1]\). This approach means to combine items of sizes above \(\frac{1}{7}\) with items of sizes in \((\frac{1}{7}, \frac{3}{2}]\) based on their exact sizes. Moreover, the approach involves combining pairs of items of subsets of sizes contained in \((\frac{1}{7}, \frac{3}{2}]\) while keeping the smallest items of such a subset to be matched with items of sizes above \(\frac{1}{7}\) (and larger items of such a subset are used to be packed into pairs), as much as possible. Prior to the work of [11, 10], all previous algorithms for classic bin packing that partition items into classes always assumed that an item of a certain subset has the maximum size when its possible packing was examined. This method simplifies the algorithm and its analysis, but it is not always a good strategy as this excludes the option of combining items that can fit together into a bin in many cases. This approach is very different from that of AF algorithms and even from NF. Moreover, an approach similar to that of Babel et. al [1] was used in an online algorithm designed in [3]. Heydrich and van Stee [11, 10] claim a competitive ratio of 1.5815 (see a discussion regarding this in Appendix D).
In algorithm AH, we do not just have red and blue items, but we potentially allow several kinds of bins. For example, for the subset of items of sizes in $\left(\frac{1}{15}, \frac{1}{14}\right]$ we group items into subsets of 14 items or three items or just one item. We also use bins of the smallest items (our value of $k$ is 43) where the total size of items is at most $\frac{17}{60}$, to allow them to be combined (among others) with items of sizes in $\left(\frac{1}{18}, \frac{5}{60}\right)$. These two features are possible due to the simple nature of our analysis, and they are crucial for getting the improved bound. Note that all items of sizes in $\left(0, \frac{1}{23}\right]$ are treated together (by the algorithm and its analysis).

In order to use just a small number of values (one or three) for each scenario, we use the concept of containers. A container is a set of items of one class (in the partition of potential inputs into items of similar sizes, called classes), and it can be complete if its planned number of items has arrived already or incomplete otherwise (but it is treated in the same way in both cases). Containers are of two types, where a container is either positive or negative, and a bin may contain at most one of each of them. The goal is to have as many bins as possible with both a positive and a negative container. Roughly speaking, positive containers have total sizes above $\frac{1}{2}$ and negative containers have total sizes of at most $\frac{1}{2}$. This last statement is imprecise as in most cases we consider volumes and not exact sizes, where volumes are based on the maximum sizes for the corresponding classes. There is one exception which is containers with one item of size above $\frac{1}{3}$, where the exact size is taken into account (both by the algorithm and the analysis), and it is defined to be the volume. A positive container and a negative one fit together if their total volumes does not exceed 1, and does not depend only on the classes. Our positive containers and negative containers have some relation to concepts used in [22].

In our weight based analysis, we assign weights to containers, where the number of different weights is small. Specifically, let the minimum volume of any positive container not packed with a negative container be denoted by $a$. We have two cases. In the simple case where all positive containers packed without negative containers have volumes of at least $\frac{2}{3}$ (i.e., $a \geq \frac{2}{3}$), we define weights as follows. Assign weights of 1 to positive containers packed without negative containers and negative containers packed without positive containers. Since we later base our weights of items on sizes, we assign these weights of 1 to all positive containers of volume at least $a$ and all negative containers of volumes above $1 - a$. We have a variable $w$ ($0 \leq w \leq 1$) such that other positive containers have weights of $w$ and other negative containers have weights of $1 - w$. Those weights are called the required weights of containers (the actual weights can be larger but not smaller). Given the approximate proportions of items of each class packed in every type of container, we compute a weighted average (based on the containers of every item) to define weights of items using the required weights of containers. The case where $a < \frac{2}{3}$ is more interesting as a negative container with one item of size in $\left(\frac{1}{5}, \frac{1}{3}\right]$ and a positive container with one item of size above $\frac{1}{2}$ can be packed into one bin if the total size of the two items does not exceed 1 (i.e., the volumes of their containers are the exact sizes of these two items). Thus, the exact value $a$ is crucial and not only its class, and additionally the class and even the exact value of $1 - a$ play an important role. Here, for other classes we do the same as in the previous case, but for one class we perform a more careful analysis. This is the class containing the value $1 - a$. For this class we define weights of items directly. We let the weight of an item of this class of size at most $1 - a$ be a variable $u$, and otherwise it is a variable $v$, where $v \geq u$ (this class is contained in $\left(\frac{1}{3}, \frac{1}{2}\right]$). For the analysis, we found suitable values for the variables for all scenarios (this was done separately for each scenario), that is, for all possible values of $a$ (the number of scenarios is still finite, as they are based on the dividing points of the algorithm, though not only on the classes). For every scenario where $a < \frac{2}{3}$, there are additional constraints on $u$, $v$, and $w$. As we do not use weights of containers in this case (for the class containing $1 - a$), while the packing of pairs of items of classes contained in $\left(\frac{1}{3}, \frac{1}{2}\right]$ is performed carefully for all such classes. After selection suitable values for those variables, all other
item weights are also computed using the parameters of the algorithm.

It should be noted that there are also improved algorithms based on First Fit. Yao [26] designed an algorithm where certain size based subsets are packed separately, resulting in a competitive ratio of $\frac{3}{2}$. Many years later, an algorithm of absolute competitive ratio $\frac{5}{3}$ was designed [3], which is the best possible with respect to this last measure [27]. The absolute competitive ratios and approximation ratios of other bin packing algorithms were studied as well [23, 7, 8]. The (asymptotic) competitive ratios should be compared to lower bounds on the competitive ratio. The current best such lower bound is 1.5403 [4] (see also [25]).

2 Notation and definitions

Similarly to previous algorithms’ definitions, AH has a sequence of boundary points that are used in its precise definition: $1 = t_0 > t_1 = \frac{1}{2} > t_2 > \cdots > t_b = \frac{1}{3} > \cdots > t_M > t_{M+1} = 0$. That is 1/2 and 1/3 are always boundary points, and there is no boundary point in (1/2, 1).

For every $j$, all items of sizes in the interval $(t_j, t_{j-1}]$ are called items of class $j$. We say that a class of items (and every item of this class) is huge if $j = 1$, it is large if $1 < j \leq b$ (these are all items of sizes above 1/3 and at most 1/2), small if $b < j \leq M$, and tiny if this is the class of items of size at most $t_M$ (i.e., the last class which is the class of tiny items is class $M + 1$, and in general the index of a class corresponds to the index $j$ such that $t_j$ is the infimum size of any item of the class).

Our algorithm will pack items into containers and pack containers into bins. As the algorithm is online, a container will be packed into a bin immediately when it is created, even though it may receive additional items later. In the last case, when we say that an item is packed into a container, this means that the bin containing the container receives that item. Any container will contain items of a single class, and at most two different containers can be combined (packed) into a bin.

We provide additional details on combining two containers into a bin later. Every container of items that are not tiny has a cardinality associated with it, and this is the (maximum) number of items that it is supposed to receive.

Let $\gamma_j = \left\lfloor \frac{t_j - 1}{1 - \frac{1}{j}} \right\rfloor$ for $j \leq M$. For class $j$ that is either large or small (but not huge or tiny, i.e., for values of $j$ such that $2 \leq j \leq M$ holds), and for every $i$ (where $1 \leq i \leq \gamma_j$) there is a nonnegative parameter $\alpha_{ij}$, where $0 \leq \alpha_{ij} \leq 1$. The values $\alpha_{ij}$ will denote the proportions of container numbers of cardinalities $i$ of class $j$ items among the number of container of class $j$ (the term proportion corresponds to the property of the sum of proportions satisfies $\sum_i \alpha_{ij} = 1$ for all $j$). Such containers that will eventually receive $i$ items of class $j$ (unless the input terminate before this becomes possible) will be called type $i$ containers of class $j$. That is, intuitively if we let $x$ denote the number of containers for items of class $j$, we will have approximately $\alpha_{ij} \cdot x$ type $i$ containers each of which having exactly $i$ items of class $j$. For every $j$ such that $2 \leq j \leq M$ and every $i$, we let $A_{i,j} = i \cdot t_{j-1}$. While the values $\alpha_{ij}$ are defined so far only for large and small classes, we see one huge item as a type 1 container. Note that the values of $\alpha_{ij}$ are not proportions of item numbers but of container numbers for class $j$, and the resulting proportions of items can be computed from them (we will prove such bounds accurately later).

For classes of large items the notion of the cardinality of a container is slightly more delicate, and we will have exactly four possible types of containers. The first type is a regular type 2 container (already) containing exactly two items of this class. The second type is a declared type 2 container, where this type consists of containers for which the algorithm already decided to pack two items of this class in the container (so the planned cardinality of the container is 2) but so far only one such item was packed into the container (one of the few next arriving items of this class, if they
exist, will be packed there, in which case the type will be changed into a regular type 2 container). The third is a regular type 1 container, where such a container has one item of the class and cannot ever have (in future steps) an additional item of this class (such a container will be always already combined with a container of another class that is packed into the same bin). The fourth and last type of a container of large items is a temporary type 1 container. A container of this last type currently has one item of the class but sometimes it will get an additional item of this class (and in this case its type will be changed at that time to regular type 2, its type can change to declared type 2 or regular type 1 as well, but in those cases it does not happen as a result of receiving a new item). Given a class of large items, the number of declared type 2 containers will be at most four throughout the execution of the algorithm (as we will prove below) while the numbers of containers of type 1 (of both kinds) and containers of regular type 2 can grow unbounded as the length of the input grows, though we will show certain properties on the relations between their numbers maintained by the algorithm. The set of the union of containers of regular type 2 and declared type 2 are called type 2 containers, and the set of the union of containers of regular type 1 and temporary type 1 are called type 1 containers. The parameters $\alpha_{1j}$ and $\alpha_{2j}$ of a large class $j$ determine the approximate proportions of type 1 containers and type 2 containers, respectively.

For class $M + 1$ (of the tiny items), instead of the definitions above, there is a sequence of $p$ possible upper bounds on the total sizes of items packed into containers of this class: $1 \geq A_{p,M+1} > A_{p-1,M+1} > \cdots > A_{1,M+1} \geq t_M$, and we let the positive parameters $\alpha_{i,M+1} > 0$ for $i = 1, \ldots, p$ denote the proportion of numbers of containers of class $M + 1$ with items of total size in the interval $(A_{i,M+1} - t_M, A_{i,M+1}]$ (this is the planned total size of items for such a container). Such containers will be called type $i$ containers of class $M + 1$.

The volume of a container of type $i$ of class $j$ is defined as follows: If $i = 1$ and $1 \leq j \leq b$ (that is, for items of sizes above 1/3), the volume of the container is the size of its (unique) item, and otherwise ($i = 2$ and $2 \leq j \leq b$ or $i \geq 1$ and $j > b$) it is $A_{i,j}$. That is, the volume is usually simply the largest total size that the container can occupy, but for a container that contains a single large or huge item, the volume is the exact size of the item (there is one exception where the bin already contains one large item and it is planned to contain another item of the same class). In most cases we would like the volume of a container to be known when it is created, which is possible for containers such that their planned contents are known (in the sense that for example type $i$ containers of a non tiny class $j$ are planned to contain $i$ items finally). However, for large items such containers with a single item may be temporary type 1 containers, in which case there is still no planning of contents for them. In this last case, the volume of the container is the size of its unique item. However, the volume of such a container may change in the case the algorithm will decide to pack another item of the same class (no matter if it packs that other item immediately at the time of decision or whether we decide to pack such an item later) into this container and transform it into a type 2 container. The volume of a declared type 2 container of class $j$ is $A_{2,j} = 2 \cdot t_{j-1}$ (the volume is based on its complete contents, no matter whether they are present already or not, as it is the case for classes of small or tiny items).

We say that a container is negative if its volume is at most 1/2 and otherwise it is positive. Obviously, two positive containers cannot be packed into one bin. We will also not pack two or more negative containers into a bin together. Thus, a bin containing two containers will contain one positive container and one negative container, and no bin will contain more than two containers.
3 Algorithm AH

The algorithm AH which we define next will pack items into containers and pack containers into bins according to rules we will define. Recall that the packing of containers into bins will be such that every bin will have at most one positive container and at most one negative container. Obviously, a bin is nonempty if it has at least one container and at most two containers. We say that a nonempty bin is negative if it has a negative container and does not have a positive container, it is positive if it has a positive container and does not have a negative container, and it is neutral if it has both a negative container and a positive container.

It is unknown whether a temporary type 1 container will eventually be positive or negative. Therefore, such a container will not be combined in a bin with another container as long as its type is not changed. Moreover, it is seen as a negative container until it changes its type (so its bin is negative as long as the container is of temporary type 1). Specifically, it remains a negative container if a positive container joins it (and its bin becomes neutral), and in this case it becomes a regular type 1 container (and remains negative), and it becomes a positive container if its type changes to type 2. It can also happen that a temporary type 1 container will remain such till the termination of the input and the action of AH (and its bin remains negative). It is important to note that the difference between regular type 1 containers of a large class and temporary type 1 containers of the same class is that each of the former containers is already packed into a bin with a positive container (of some class), while the latter are not packed with other containers (in fact, the corresponding items are placed into their own bins, one item per bin).

For every class \( j \), we denote by \( n_j \) the number of containers of class \( j \). Let \( n_{ij} \) denote the number of containers of type \( i \) of class \( j \). We also let \( N_j \) denote the number of items of class \( j \) at that moment. We often consider the values \( n_j \) and \( n_{ij} \) just prior to the packing of a new item, when \( N_j \) was already increased but the new item not packed yet so the values \( n_j \) and \( n_{ij} \) are not updated yet.

We say that two containers fit together if their total volume is at most 1. In what follows, when we refer to packing an item \( e \) - or more precisely, packing a container containing \( e \) (which was just created and therefore contains only \( e \)) into existing bins using Best Fit - we refer to packing \( e \) (or the container containing \( e \)) into the bin with a container of largest volume where the existing container and \( e \) (or the container containing \( e \)) fit together. For the original version of Best Fit, actual sizes are taken into account, but here we base this rule on volumes (as for a container with a single large or huge item the volume is equal to the size of the item, if we select one such container among a set of this last kind of containers, our action is equivalent to the standard application of Best Fit).

Next, we define the packing rules of the algorithm when a new item of class \( j \) arrives. The algorithm is defined for each step, based on the class of the new item.

A huge item. Recall that a huge item is immediately packed into a positive container containing only this item. Use Best Fit (applied on volumes, as explained above) to pack the created container into an existing bin, out of existing negative bins, such that the two containers (the new one with the huge item and the negative one of the negative bin) fit together. The only case where the new huge item joins a bin with a large item of some class \( j' \) is the case where the container of class \( j' \) is a temporary type 1 container, and in this case the type of this container of class \( j' \) is changed into regular type 1. If no bin can accommodate the container of the new item according to those packing rules, that is, for every negative bin, the total volume together with the new item is too big (or there is no negative bin at all), then use a new bin for the positive container of the new item (this new bin becomes a positive bin).

An item of a class of small or tiny items. For these classes we define the concept of an
open container. Informally, an open container (of class $j$) can receive at least one additional item of class $j$. As a new container is introduced in order to pack an item, any container (of any type and class) already has at least one item of the corresponding class. If $b < j \leq M$, an open type $i$ container of class $j$ is one where the total number of the items in the container is strictly smaller than $i$. Once such a container receives $i$ items, it is closed. For $j = M + 1$, a type $i$ container of this class will be open starting the time it is created and while the total size of items in it is positive and at most $A_{i,M+1} - t_M$. Once it reaches a total size above $A_{i,M+1} - t_M$, it will be closed. For all cases of packing a small or tiny item, a new container of some class will be used only if there is no open container of the same class, and thus, in particular, there will be at most one open container for each $j$ (and the corresponding value of $i$ will always be one such that $\alpha_{ij} > 0$).

When a new item of class $j$ (such that $j > b$) arrives, if there is an open container of some type $i$ of class $j$, then pack the item there (there can be at most one such container, so there are no ties in this case). Otherwise, open a new container for it (the details of the type are given below). After packing the new item into the container (and packing its container into a bin if it is a new container), close the container if necessary, based on its type and the rules above.

In the case that a new container is used for the item, we define the process of packing the item in more detail. Prior to packing the item, we define the type of the new open container. As the item is not packed yet, $n_j$ is the number of containers of class $j$ excluding the container opened for the new item. Find the minimum value of $i$ such that $\alpha_{ij} > 0$ and so far there are at most $\lfloor \alpha_{ij} \cdot n_j \rfloor$ type $i$ containers of class $j$ (i.e., $n_{ij} \leq \lfloor \alpha_{ij} \cdot n_j \rfloor$, where the values $n_{ij}$ do not include the new container which will be opened). Such an index $i$ exists as otherwise there are more than $n_j$ containers of class $j$. More precisely, since $\sum_i \alpha_{i,j} = 1$, there is always a value of $i$ satisfying that $\alpha_{ij} > 0$ such that so far we opened at most $\lfloor \alpha_{ij} \cdot n_j \rfloor$ type $i$ containers of class $j$. Open a new type $i$ container of class $j$ containing the new item (increasing both $n_j$ and $n_{ij}$). Observe that this opening of a new container defines its volume as well as whether it is a positive container or a negative container.

Next, we decide where to pack this new container. First consider the case where this container is a negative container. Then, if there is a positive bin, such that the new container fits into the bin according to its volume, then use that bin to pack the new container. This last case includes the possibility that the positive container is a type 2 container of a large class (regular or declared). If there are multiple options for choosing a bin, one of them is chosen arbitrarily.

Otherwise (there is no positive bin where the new negative container can be added), the algorithm checks the option of using a bin with a temporary type 1 container of some class of large items. Assume that there is a negative bin $B$ such that the following two conditions are satisfied. The first condition is that the bin $B$ has a temporary type 1 container of class $j'$ such that a positive container of class $j'$ (with two items) will fit together with the new (negative) container. The second condition is that there are at most $\lfloor \alpha_{j',j} \cdot n_{j'} \rfloor - 1$ type 2 containers of class $j'$ (before the packing of the new item is performed). Then, pack the new negative container into $B$, and define the container of class $j'$ packed into $B$ as a declared type 2 container. This last container of class $j'$ will get one of the next items of class $j'$ that will arrive, which will happen before any new container is opened for any new class $j'$ item, see below. If there are multiple options for choosing $B$, one of the classes of large items is chosen arbitrarily (among those that can be used), and a temporary type 1 container of this class with maximum volume is selected, i.e., we use Best Fit in this case. This last packing step is possible as a temporary type 1 container is never packed with another container into a bin (if another container joins it, its type is changed).

Otherwise (if there is no suitable positive bin and no class of large items has a suitable temporary type 1 container that can be used under the required conditions), pack the new negative container into a new bin.
Finally, consider the case where the new container is a positive container. Then, if there is a negative bin whose container is not a temporary type 1 container, such that the new container fits together with it, then use such a bin to pack the new container. Otherwise, if there is a temporary type 1 container with one large item of a class $j'$ where the new container fits, then pack the new positive container into this bin and define the container of class $j'$ in this bin as a regular type 1 container. The class $j'$ can be chosen arbitrarily if there are multiple options, and among the temporary type 1 containers of class $j'$, one of maximum volume (out of those that can be used) is selected, i.e., once again we use Best Fit. Otherwise, pack the new positive container into a new bin.

A large item of a class $j$. If there is a declared type 2 container of class $j$, pack the item there (as a second item) and change it into a regular type 2 container (breaking ties arbitrarily). This packing rule is checked first, and we apply it whenever possible. We continue to the other cases in the situation where there is no such declared type 2 container.

If the number of type 2 containers equals $\lfloor \alpha_2 j \cdot n_j \rfloor$ (that is, we should not increase the number of type 2 containers at this stage), then pack the new item into a new negative container. To pack the container into a bin, do as follows. If there is a positive bin where the new negative container fits, then use Best Fit to pack it as a regular type 1 container of class $j$ (its volume is defined accordingly as the size of the new item) together with a positive container (this positive container is not of large items, as three large items cannot be packed into a bin together). Otherwise the new container is packed into a new bin, in which case it is defined to be a temporary type 1 container.

Otherwise (that is, the number of type 2 containers is strictly smaller than $\lfloor \alpha_2 j \cdot n_j \rfloor$), we will increase the number of regular type 2 containers or the number of declared type 2 containers of this class in the current iteration as follows. If there is a negative bin $B$ where a type 2 container of class $j$ fits, then pack the item into a new declared type 2 container of class $j$ and pack this container into this bin $B$. Otherwise, if there is a temporary type 1 container of class $j$, then we pack the new item using Best Fit (considering only temporary type 1 containers of class $j$, and selecting such a container of largest volume) and change the type of this container into a regular type 2 container. Otherwise (all containers of class $j$ are either regular type 1 or regular type 2, we should increase the number of type 2 containers, and a new container with two items of this class cannot be packed into an existing bin), we open a new declared type 2 container for the new item and open a new bin for this declared type 2 container (and pack it there).

The value $\alpha_2 j$ is strictly positive for every large class $j$ (as packing every item of a certain large class in its own bin will lead to a competitive ratio of 2 for inputs consisting only of such items). However, there may be values of $j$ ($2 \leq j \leq b$) for which $\alpha_{1j} = 0$. In those cases, the algorithm above is still applied. Moreover, in those cases it could happen that there will be a constant number of type 1 containers for class $j$, as we prove below (the general proof is valid in the case $\alpha_{1j} = 0$ too).

Remark 1 Note that the change of types of containers (of large classes) is a unique and delicate feature of AH. While the change of a declared type 2 container into a regular type 2 container when a new item is packed into this container, can be described also by previous approaches, our rules for changing the type of temporary type 1 containers are new and particularly important. We summarize those rules as follows.

1. If a (new) positive container of another class is packed into a bin containing a temporary type 1 container, then we change the type of the temporary type 1 container into a regular type 1 container.

2. If a (new) negative container of another class is packed into a bin containing a temporary
type 1 container, then we change the type of the temporary type 1 container into a declared type 2 container.

3. If a (new) item of the same class of the temporary type 1 container joins the same bin as the temporary type 1 container, then we change the type of the temporary type 1 container into a regular type 2 container.

Furthermore, in all these cases, we pack the new container or large item using Best Fit. That is, we pick the largest temporary type 1 container where the new container or new item fits.

Note that there are, however, previous papers where it was not always decided in advance whether for a class of items for which a bin can contain at most two items of this class, the bin will contain one item or two items. In [1], in the studied problem a bin can never contain more than two items, so there are just two classes of items, larger items of sizes above \( \frac{1}{2} \), and the smaller items, which are all other items. In the algorithm of [1], whenever a new smaller item is to be packed with another smaller item, Best Fit is applied. However, in [1] there is no concept of packing other kinds of items into such bins (as according to their model, those bins already have the maximum number of items). In [11, 10] the difficulty of deciding whether large items should be packed in pairs or alone (in order to be packed with items of other classes) is solved in a slightly different way; there is a provisional decision (so for some large items it is decided that they will be packed in pairs and for others that they will be packed alone). The final decision is set after a sufficient number of items of the class have arrived. If in the meantime some items were combined with items of other classes, for those items the decisions are final. After sufficiently many items arrive without other items being combined with them, a decision is made for all this large set at once.

**Remark 2** Consider a large class \( j \). Consider an iteration \( \ell \) of the algorithm, i.e., the arrival of the \( \ell \)th input item, which is not necessarily of class \( j \). Assume that as a result of packing the \( \ell \)th item a given container of class \( j \) becomes a type 2 container. That is, this last class \( j \) type 2 container either did not exist before the current step, or it was a type 1 container prior to this iteration. Assume also that this container is not packed with a negative container in a bin. Then, the \( \ell \)th item is of class \( j \).

### 4 Analysis

#### 4.1 Properties of the packing of positive and negative containers

In the analysis, we see a pair of a negative container and a positive container, packed together in a bin, as matched to each other, and each one of them is seen as matched (while every container packed into a bin without another container is unmatched). Our next goal is to prove the properties of this matching. Let \( a' = 1 - s_{\text{min}}/2 \) where \( s_{\text{min}} \) is the smallest item size in the examined input, and let \( a \) be the smallest volume of a positive container that is unmatched, if it exists. If no unmatched positive container exists, let \( a = a' \). If \( a > a' \), decrease the value of \( a \) to be \( a' \). A simple property of the algorithm is that it tries to match a positive container and a negative container whenever possible.

**Lemma 3** Consider some time during the execution of the algorithm, just after an item has been packed. If there exists at least one positive bin and at least one negative bin, let \( \theta_{\text{neg}} \) denote the smallest volume of any container of a negative bin and let \( \theta_{\text{pos}} \) denote the smallest volume of any container of a positive bin. Then, \( \theta_{\text{neg}} + \theta_{\text{pos}} > 1 \).
Proof. For a negative container of a large class, its volume is above $\frac{1}{3}$, and for a positive container of a large class, its volume is above $\frac{2}{3}$. Thus, if both containers are of large classes, we are done. It is left to consider several cases, based on whether one of the containers is of a large class (this can be the negative container or the positive container, or none of them), and on which of the two containers was created first (in the case of a positive container of a large class, we also need to consider the time when this container changes its type to type 2).

If none of the two containers is of a large class, by the rules of packing a new container of a class that is tiny, small, or huge, a new positive or negative container is packed into an empty bin only if the total volume of the new container and the container in any relevant bin (a relevant bin is a positive bin if the new container is negative, and it is a negative bin if the new container is negative) is above 1.

If the negative container is of a large class, then since its bin is negative until the current time, it is of temporary type 1 until the current time. If the positive container was created after this negative container, then as it was not combined with the negative container of the large item, their total volume is above 1. Otherwise, when the temporary type 1 container was created, it was not possible to combine it with a positive container, and therefore the total volume is above 1 in this case as well.

If the positive container is of a large class, it is of type 2, and we consider three cases. If the container of the large class becomes of type 2 before the time when the negative container is created, then it is already a positive container when the negative container is created, and therefore by the packing rules their total volume is above 1. Assume now that the negative container was created before the time when the type 2 container was defined as type 2 (either by changing its type from a temporary type 1 container to type 2, or by the creation of a new declared type 2 container). A temporary type 1 container becomes of type 2 without being combined with a negative container only in the case where no negative container that can be combined with a type 2 container of this large class exists. A new declared type 2 container is packed into a new bin only if there is no negative container in a negative bin such that they could be packed together. Thus, in all three cases the total volume of the two containers is above 1.

Lemma 4 Every type 1 container of a large class $j$, where its unique item has size no larger than $1 - a$ is combined with a positive container in the output, and in particular, all such containers are of regular type 1 in the output.

Proof. If $a = a'$, the claim is trivial. Otherwise, by the definition of $a$, there is a positive container that is not packed with a negative container, whose volume is $a$. Thus, at termination, by Lemma 3 there are no negative containers of volumes at most $1 - a$ that are not combined with positive containers. For classes of large items, negative containers that are not combined with another container into the same bin are only temporary type 1 containers (recall that a declared type 2 container is a positive one). Thus, there may be regular type 1 containers for large classes of all possible volumes (that are combined with positive containers), and there can be temporary type 1 containers of volumes strictly above $1 - a$ (but not smaller) that are not combined with positive containers.

Then, by Lemmas 3 and 4, every negative container of volume at most $1 - a$ is matched, and every positive container of volume strictly smaller than $a$ is matched (by the definition of $a$).

4.2 The set of scenarios

We define a finite set of scenarios according to the value of $a$ so that in particular the index of the scenario will reveal the class that contains the value $1 - a$ and so that the index of the scenario
will determine for each container containing at least two items (of the class of the container) the
relation between the volume of this container and the values of \(a\) and \(1 - a\).

To do that we define a set of values \(V\) as follows. \(V = \{A_{i,j}, 1 - A_{i,j} : j = 2, 3, \ldots, M + 1, \alpha_{ij} > 0\} \cup \{t_1, t_2, \ldots, t_M, t_{M+1}\}\) and \(V' = \{x \in V : x \leq 1/2\}\) (in particular, \(1/2 \in V'\)). Note that the set \(V'\) contains (among other) all boundary points \(t_j\) (for all \(j \geq 1\)), even for values of \(j\) for which \(\alpha_{ij} = 0\). The name of a scenario is an interval \((x, y]\) between consecutive values in \(V'\). For a given value of \(a\), we find the values \(x, y\) such that \(1 - a \in (x, y]\), and \(y > x\) are two values in \(V'\) such that \((x, y] \cap V' = \emptyset\). The analysis is based on the value of \((x, y]\), where the motivation is that given this value, it is known exactly which positive bins we may have (excluding containers with one item which is large or huge, for which the volumes are not limited to those in \(V\)). We define the index \(k\) of the threshold class to be the value of \(k\) such that \((x, y] \subseteq (t_k, t_{k-1}]\). As \(V\) and \(V'\) contain values that are not in \(\{t_1, t_2, \ldots, t_M\}\), it is possible that \(x > t_k\) or \(y < t_{k-1}\) or both.

The analysis process in the remainder of this paper is performed for every possible value of \((x, y]\), and the overall asymptotic competitive ratio is the maximum value of \(R\) resulting in the next procedure for a given value of \((x, y]\). That is, we analyze the algorithm with respect to all possible scenarios (where there is a large number of scenarios, but they can be analyzed independently given the parameters of the algorithm), and as it is not known in advance which scenario will occur, a worst-case assumption is applied.

### 4.3 Defining the weight function via a linear program

The first step for analyzing each scenario is to obtain a good weight function for the scenario, in the sense that the analysis will be as tight as possible and can be done using a computer assisted proof within a small running time. The weight function defines size based weights for values in \((0, 1]\). The goal is to define weights such that the cost of the algorithm is roughly the total weight of all input items (formally, the total weight should be at least the number of bins minus a constant \(c\) independent of the input), and if the target competitive ratio is \(R\), the cost of an optimal solution is at least the total weight divided by \(R\) (this can be proved by showing that no bin can contain items of total weight above \(R\)). Then, for an input \(I\), letting \(w(I)\) denote its total weight, (and as defined above, letting \(OPT(I)\) the optimal cost for \(I\), and \(A(I)\) the number of bins used by \(A\)), we will have \(A(I) \leq w(I) + c, OPT(I) \geq \frac{w(I)}{R}\), which shows that \(A(I) \leq R \cdot OPT(I) + c\). This last argument is the standard argument for weight functions based analysis. In \([22]\) generalizations of weight functions were used, but we just use the approach of \([13, 14, 15, 17, 19]\).

In order to define a suitable function, we will solve a linear program defined below (this linear program has only four variables \(w, u, v\) and \(R\), and in some cases it actually has only two variables \(w\) and \(R\)). More precisely, we will provide a feasible solution for this linear program that is very close to the optimal one (but we only use its feasibility and do not prove that it is almost optimal). The weights of specific sizes will be based on the values \(w, u, v\) (or just on \(w\), if the others are undefined), and on some of the parameters of the algorithm (the \(\alpha_{ij}\) values for the given class). The variable \(w\) will be required to satisfy \(0 \leq w \leq 1\). For \(u\) and \(w\), we also require \(0 \leq u, v \leq 1\), for the scenarios where these variables are defined.

#### 4.3.1 The (minimum) required weight of a container

We say that a class is basic if it is a small or tiny class, and also if it is a large class that is not the threshold class. Thus, if the threshold class is a large class it is not basic, and otherwise it is basic. We define a quantity for each container. This quantity will be called the required weight of the container, and its goal is to introduce a uniform value such that weights of items are defined
based on these values, in order to satisfy all requirements. This quantity is defined for any basic class and for the class of huge items. If the threshold class \( k \) is a large class, we keep this quantity undefined for that class. In all other cases it is defined as follows.

For a positive container of volume at least \( a \), the required weight of the container is 1. Note that this means that the weight of a huge item of size at least \( a \) will be 1, and for every other positive container of volume at least \( 1 - x \), we will ensure a weight of 1 for the container. Recall that for any positive container that is not a container of a huge item (for which the volume is the exact size of the item), its volume is an element of \( V \). For a positive container of volume in the interval \((1/2, a)\), the required weight of the container is denoted as \( w \). This will be a decision variable of the forthcoming linear program. Thus, for a positive container of a class of items that are not huge, the required weight of the container is 1 if its volume is at most \( 1 - y \) and the required weight of the container is 1 if its volume is at least \( 1 - x \) (and there are no containers of volumes in \((1 - y, 1 - x)\), except for possibly containers of huge items). The last definition does not depend on the exact value of \( a \) but it depends on the scenario index. However, for a container of one huge item, its required weight depends on the exact value of \( a \) (and the size of the huge item). The reasoning is that a positive container of volume at least \( a \) may be packed into a positive bin, while other positive containers are packed in neutral bins.

Next, we consider negative containers. The intuitive definition of the required weight of a negative container is that it is 1 if we cannot guarantee that it is matched to a positive container, and otherwise it is \( 1 - w \).

Formally, we partition the definition of required weight of a negative container to the following cases.

Assume that \( a \geq 2/3 \). Here, the threshold class is of small or tiny items (it is a basic class). For a negative container of volume in the interval \((1 - a, 1/2]\), the required weight is 1. Since \( 1 - a \leq 1/3 \), this also means that the required weight of a negative container of volume larger than \( 1/3 \) is 1. The required weight of the negative containers of volume in the interval \([0, 1 - a]\) is \( 1 - w \). All containers of volume at most \( 1 - a \) are of small or tiny items, and their volumes are in \( V \). Thus, no negative container has volume in the open interval \((x, y)\) (but there might be negative containers with volume \( x \) or volume \( y \)), and thus we can refine the statement of the required weight of a negative container in this case as follows. For a negative container of volume at least \( y \), the required weight is 1, while for a negative container of volume at most \( x \), the required weight is \( 1 - w \).

Assume that \( a < 2/3 \). Here, the threshold class \( k \) is of large items (and it is not basic). Recall that the required weight of a container of a large class is defined for all large classes except for \( k \).

For a negative container of volume in the interval \((1 - a, 1/2]\), the required weight is 1. The required weight of the negative containers of volume in the interval \([0, 1 - a]\) is \( 1 - w \). This last rule can be stated in terms of \( x \) and \( y \) for a negative container of a class that is not the threshold class. For such negative container, we define the required weight of the container to be \( 1 - w \) if its volume is at most \( x \), and otherwise its required weight is 1. The threshold class (for this case where it is not basic) is discussed later in more detail.

### 4.3.2 The (amortized) weight of an item.

In order to define weights of items, for each class separately, we will define the weight of an item of this class in an amortized way that ensures that the sum of the weights of items of this class will be approximately the sum of the required weights of (all) its containers (excluding a constant number of such containers per class). The weight of an item in the threshold class (if it is not basic) will
be defined without using the required weight of the containers of this class.

For a huge item, since its container is always a positive container, its weight is 1 if its size is at least \( a \) and \( w \) if it is smaller than \( a \).

Consider next classes of items that are not huge. For a basic class \( j \), we let \( r_{x,y}(i, j) \) be the required weight of a type \( i \) container of class \( j \) (recall that we deal with a specific scenario defined by \((x, y)\)). Note that \( r_{x,y}(i, j) \in \{w, 1-w, 1\} \), and we have already defined this value for every possible values of \( x, y, i, j \) such that \( j \neq k \) or \( j = k > b \). For such an item of a non tiny class \( j \), let the weight of the item be

\[
\omega_j = \frac{\sum_i \alpha_{ij} \cdot r_{x,y}(i, j)}{\sum_i i \cdot \alpha_{ij}}.
\]

This term is the ratio between the average required weight of a container of this class and the average number of items in a container of this class. As \( r_{x,y}(i, j) \leq 1 \) for all \( x, y, i, j \) and \( i \geq 1 \), we get \( \omega_j \leq 1 \) for all basic classes \( j \leq M \). Similarly, for \( j = M + 1 \), we define the weight of an item of this class to be its size multiplied by

\[
\rho = \frac{\sum_i \alpha_{i,M+1} \cdot r_{x,y}(i, M + 1)}{\sum_i (A_{i,M+1} - t_M) \cdot \alpha_{i,M+1}}.
\]

The parameters will always be chosen such that \( \rho \leq 2 \), as otherwise the value of the competitive ratio will exceed 2.

In order to define the weight of items of the threshold class \( k \) for scenarios where it is not basic (this means that \( 1/3 < 1 - a < 1/2 \)), we introduce the last two decision variables \( u \) and \( v \) (these are decision variables of the linear program below, and together with \( w \) and the competitive ratio \(\text{value} \ R \) for this scenario this will conclude the introduction of the decision variables). For such an item, we let its weight be \( u \) if its size is at most \( 1-a \) and otherwise its weight is \( v \). We will impose the constraint

\[
u \leq v.
\]

Intuitively, we can see the output as if for every type 1 container of this class we have a collection of \( \frac{\omega_{2k}}{\alpha_{1k}} \) (fractions of) containers of type 2 associated with it (as this is the ratio between the fractions of containers of the two types out of the total number of containers of class \( k \)), such that one of the following conditions hold:

The first option is that the type 1 container is a regular type 1 container and it is matched to some positive container. For this case we assume that its weight is \( u \) and each item in the associated type 2 containers has weight \( u \), however the additional weight of \( w \) of the positive container (the one that is matched to the type 1 container, where we do not know its class) helps us to obtain a sufficient total weight, which is the total number of bins. Our actual claim will be simpler and this discussion is provided just to motivate the constraints (a full set of properties and proofs is given later).

That is, we will have the constraint

\[
u \cdot (\alpha_{1k} + 2\alpha_{2k}) + w \cdot \alpha_{1k} \geq \alpha_{1k} + \alpha_{2k}.
\]

As \( \alpha_{1k} + \alpha_{2k} = 1 \), this inequality is equivalent to

\[
u \cdot (1 + \alpha_{2k}) + w \cdot \alpha_{1k} \geq 1
\]

Otherwise, that is, the type 1 container is not matched to a positive container (and thus it is a temporary type 1 container). In this case, the item of this type 1 container is of size larger than \( 1-a \), and moreover (as we formally justify below) every container of type 2 in its associated
containers of type 2 satisfies either that it has (at least) one item of size larger than $1 - a$ or it is matched to a negative container. In our constraints we consider (only) the two extreme cases where all associated containers of type 2 are of a common case (as the constraint for every intermediate case is a convex combination of the two extreme constraints that we explain now). For the first case where all associated containers of type 2 (as well as the type 1 container) have at least one item of size larger than $1 - a$, we have the constraint

$$u \cdot \alpha_{2k} + v \cdot (\alpha_{1k} + \alpha_{2k}) \geq \alpha_{1k} + \alpha_{2k}. \quad (4)$$

This inequality is equivalent to

$$u \cdot \alpha_{2k} + v \geq 1. \quad (5)$$

For the other case where every type 2 container is matched to a negative container and the type 1 container has an item of size larger than $1 - a$, we have the constraint

$$v \cdot \alpha_{1k} + 2 \cdot u \cdot \alpha_{2k} + (1 - w) \cdot \alpha_{2k} \geq \alpha_{1k} + \alpha_{2k}. \quad (6)$$

The last inequality is equivalent to

$$v \cdot \alpha_{1k} + 2 \cdot u \cdot \alpha_{2k} + (1 - w) \cdot \alpha_{2k} \geq 1. \quad (7)$$

We impose all the constraints (1), (2), (4), and (6) to ensure that we allocate sufficient weight in all cases. While we presented these constraints using a pictorial fractional allocation of type 2 containers to the type 1 containers, our proof is not based on such arguments. We will prove that these constraints are sufficient to guarantee that the resulting weight function satisfies that the cost of the algorithm is at most the total weight of the items plus a constant (which is independent of the input) for each scenario.

### 4.4 The linear program

In addition to these four constraints (1), (2), (4), and (6) (ensuring that this is indeed a valid weight function, as we show below), we have the knapsack constraints expressing the following properties.

For every subset of items that can fit into one bin, the total weight of the items is at most $R$. Next, we elaborate further on these knapsack constraints. We intend to ensure that the total weight of items in any bin in an optimal solution is at most $R$. To do this, we consider every possible subset of items that may fit into a bin, and for each non tiny item we replace its size by the infimum size of an item of the same weight. The resulting set of non tiny items has total size strictly smaller than 1. We may consider only sets with total size strictly smaller than 1 and not sets with total size of exactly 1 as the infimum size of an item of the same weight is never attained (except for one special case, see below), and thus for every nonempty set of such items we can strictly decrease the total size of the items in the set and obtain another set of the same weight and smaller size. There is one case where the minimum is attained, which is a huge item of size $a$. For decreasing the total size of items strictly below 1 it is sufficient for the bin to have at least one other non tiny item, and if the huge item of size $a$ is the only non tiny item of a bin, its size is already below 1. We will consider sets of non tiny items, and to find an upper bound on the total weight that could result from this set, we add to the multiset of items sand of (strictly) positive total size consisting of an arbitrary set of tiny items of total size that equals 1 minus the total size of the multiset of items we consider.

Any set of non tiny items (of total size strictly below 1) belongs to one of the following cases:
1. Assume that there is a huge item of size at least \( a \) in the set. Such a set of items contains (beside the huge item) only items smaller than \( 1 - a \), of total size below \( 1 - a \). Thus, the remaining items have total size below \( y \) (and in the calculation of total weight we will obviously take into account the huge item, whose weight is 1). If class \( k \) is large, the set could possibly contain an item of size in \((t_k, t_{k-1}]\), but such an item has size smaller than \( 1 - a \leq y \), so its size is in \((t_k, 1 - a]\). We define a set of sizes with weights, where the set is called \( \Delta \), such that for every set of non tiny items of sizes in \((t_M, 1 - a]\), there is a multiset of items of \( \Delta \) of the same weight, such that its total size is not larger. The set \( \Delta \) consists of all sizes \( t_j \) for \( k \leq j \leq M \). The weight of \( t_j \) for \( k + 1 \leq j \leq M \) is defined to be the weight of an item of size in \((t_j, t_{j-1}]\). If \( k \) is a basic class, the weight of \( t_k \) is defined to be the weight of an item of size in \((t_k, t_{k-1}]\), and otherwise (\( k \) is a large class), the weight of \( t_k \) is defined to be \( u \). Given a set of items of sizes in \((t_M, 1 - a]\) of total size below \( y \), replacing any item whose size is in \((t_j, t_{j-1}]\) with an item of size \( t_j \) results in a total size that is not larger than the original total size, and it has the same weight. Given a set of items of sizes in \((t_M, 1 - a]\) of total size \( z < y \) and a set of tiny items of total size \( y - z \), we obtain a set of items that has a weight that is no larger than the weight of the corresponding multiset of items of \( \Delta \), whose total size is \( z' \leq z \) plus tiny items of total size \( y - z' \geq y - z \geq 1 - a - z \). Thus, we can consider multisets of \( \Delta \) instead of arbitrary sets of non tiny items, and an upper bound on the weights of such multisets together with tiny items, such that their total size is exactly \( y \) (plus the weight 1 of the huge item) is an upper bound on the total weight of any packed bin with a huge item of size at least \( a \).

2. Assume that there is no huge item of size at least \( a \) in a considered set of items. In this case, the total sizes of sets of non tiny items should be below 1, and no item has size of \( a \) or more. The set \( \Delta \) consists of all values \( t_j \) for \( j = 1, 2, \ldots, M \), and the value \( x \) is included as well if \( k \) is a large class (unless \( x = t_k \)). For any other class \( j \neq 1, k \) and also for \( j = k \) if \( k \) is not a large class, the weight of an item of size \( t_j \) of \( \Delta \) is equal to the weight of an item of size in \((t_j, t_{j-1}]\). Assume that \( k \) is a large class. In this case another two elements of \( \Delta \) is an item of size \( t_k \) and weight \( u \), and an item of size \( x \) and weight \( v \). If \( t_k = x \), there is only one additional element whose size is \( t_k = x \) and whose weight is \( v \) (since \( u \leq v \)). The choice of the weight of \( x \) is based on the property that an item of size in \((t_k, t_{k-1}]\) either has weight of \( v \) or of \( u \leq v \). The weight of an item of size \( t_1 = \frac{1}{2} \) in \( \Delta \) is \( w \) (as every item of size above \( \frac{1}{2} \) has at least this weight).

In this case, we consider all multisets of items of sizes \( t_j \) for \( 1 \leq j \leq M \) and \( x \), where a set of tiny items whose size is the complement to 1 is added for the purpose of weight calculation of a multiset. Once again the resulting value for \( \Delta \) is an upper bound on the value we would like to compute for the original items (as we may have increased the weight of some items from \( u \) to \( v \), and the total size of tiny items may have increased).

This provides us with two knapsack problems for each scenario, one resulting from case 1 and the other from case 2. In the first one, the target total size is \( y \) (such that the total size of items of \( \Delta \) is strictly below \( y \)), and the upper bound on \( R \) is 1 plus the upper bound on the weight of the multiset of items of \( \Delta \) plus tiny items complementing the total size to \( y \). In the second one, the target total size is 1 (such that the total size of items of \( \Delta \) is strictly below 1), and the upper bound on \( R \) is the upper bound on the weight of the multiset of items of \( \Delta \) plus tiny items complementing the total size to 1. Under a worst-case assumption, we calculate the maximum of the two values for each scenario (and then the maximum of the upper bounds for all scenarios).
5 Some properties of the algorithm

We state and prove a number of properties, where some of these properties were mentioned above. The goal of this section is to prove formally that the weight function we define (for the given scenario \((x, y)\)) is indeed a valid weight function.

5.1 Bounding the number of containers of each type

We will bound the number of containers of each type in terms of the number of items of the class for small and large items, and in terms of total size for tiny items.

Our first goal is to bound the values of \(n_{2j}\) and \(n_{1j}\) for a large class \(j\) in terms of the number of items of class \(j\). To do that, we use the fact that large items can be packed in type 1 or type 2 containers, and thus for such a class \(j\), there are only two values of \(\alpha\), namely, \(\alpha_{1j}\) and \(\alpha_{2j}\) (whose sum is 1). All values are analyzed just after an item has been packed or just before any item arrived. All properties proved in this section for a large class \(j\) hold even if \(\alpha_{1j} = 0\).

Lemma 5 For any large class \(j\), at any time just after an item was packed, \(n_{2j} \leq \lfloor \alpha_{2j} \cdot n_j \rfloor\), and \(n_{1j} \leq \lfloor \alpha_{1j} \cdot n_j \rfloor + 2\).

Proof. We use induction. Initially, \(n_j = n_{1j} = n_{2j} = 0\) and the two properties hold.

Consider the kind of all possible modifications in the set of containers of class \(j\). In the proof we use \(n_j\), \(n_{1j}\), and \(n_{2j}\) for the values before a modification and \(n'_{1j}\), \(n'_{2j}\), and \(n''_{2j}\) for the values after the modification. Thus, we will show \(n'_{2j} \leq \lfloor \alpha_{2j} \cdot n'_{j} \rfloor\), and \(n'_{1j} \leq \lfloor \alpha_{1j} \cdot n'_{j} \rfloor + 2\) (assuming that \(n_{2j} \leq \lfloor \alpha_{2j} \cdot n_j \rfloor\), and \(n_{1j} \leq \lfloor \alpha_{1j} \cdot n_j \rfloor + 2\) hold before the modifications). The first kind of modifications is the result of the arrival of a new item of class \(j\). In this case, the following actions of the algorithm are possible. The first one is that the new item is added to a declared type 2 container to become a regular type 2 container. In this case neither the number of type 2 containers of class \(j\) nor the total number of containers for this class change (so \(n'_{j} = n_j\), \(n'_{1j} = n_{1j}\), and \(n'_{2j} = n_{2j}\)). The next option is where \(n_{2j} = \lfloor \alpha_{2j} \cdot n_j \rfloor\) and we will not place the new item into a type 2 container. In this case a new container of type 1 is formed, and as \(n_{2j} = n_{1j} + 1\) and \(n''_{2j} = n_{2j}\), we find \(n''_{2j} = n_{2j} \leq \lfloor \alpha_{2j} \cdot n_j \rfloor \leq \lfloor \alpha_{2j} \cdot n'_{j} \rfloor\). Moreover, \(n'_{1j} = n_j - n''_{2j} = n_j - \lfloor \alpha_{2j} \cdot (n'_{j} - 1) \rfloor \leq n'_{j}(\alpha_{1j} + \alpha_{2j}) - \alpha_{2j} \cdot (n'_{j} - 1) + 1 = \lfloor \alpha_{1j} \cdot n_j + 2 - \alpha_{1j} = \lfloor \alpha_{1j} \cdot n_j + 2 \leq \lfloor \alpha_{1j} \cdot n_j \rfloor + 3\) (and therefore \(n_{1j} \leq \lfloor \alpha_{1j} \cdot n_j \rfloor + 2\), as both the first and the last expressions in the sequence of inequalities are integral).

Finally, if \(n_{2j} \leq \lfloor \alpha_{2j} \cdot n_j \rfloor - 1\), there are two options where we might put the item into a type 2 container. In the first option, one new (declared) type 2 container is created, and we have \(n''_{2j} = n_{2j} + 1, n'_{1j} = n_{1j},\) and \(n'_{j} = n_{j} + 1\). In this case, we have \(n'_{2j} = n_{2j} + 1 \leq \lfloor \alpha_{2j} \cdot n_j \rfloor \leq \lfloor \alpha_{2j} \cdot n'_{j} \rfloor\), and \(n'_{1j} = n_{1j} \leq \lfloor \alpha_{1j} \cdot n_j \rfloor + 2 \leq \lfloor \alpha_{1j} \cdot n'_{j} \rfloor + 2\). In the second option, a (temporary) type 1 container is transformed into a (regular) type 2 container, and in this case, \(n''_{2j} = n_{2j} + 1, n'_{1j} = n_{1j} - 1,\) and \(n'_{j} = n_{j}\). In this case, we have \(n'_{2j} = n_{2j} + 1 \leq \lfloor \alpha_{2j} \cdot n_j \rfloor + 1 = \lfloor \alpha_{2j} \cdot n'_{j} \rfloor\), and \(n'_{1j} = n_{1j} - 1 \leq \lfloor \alpha_{1j} \cdot n_j \rfloor + 2 = \lfloor \alpha_{1j} \cdot n'_{j} \rfloor + 2\).

A possible second kind of modifications is a result of the arrival of a small or tiny item. In this case, the change can be that a temporary type 1 container of class \(j\) becomes a declared type 2 container, or that a temporary type 1 container of class \(j\) becomes a regular type 1 container. In the latter case there is no change in the numbers of containers of types 1 and 2. In the former case, \(n'_{j} = n_{j}\), and the change is performed only if \(n_{2j} \leq \lfloor \alpha_{2j} \cdot n_j \rfloor - 1\) (see Remark \[\square\] for the summary of the relevant steps of the algorithm). In this case \(n''_{2j} = n_{2j} + 1, n'_{1j} = n_{1j} - 1,\) and such a situation was already considered. ■
Corollary 6 For any large class $j$, at any time just after packing an item, $n_{2j} \geq \alpha_{2j} \cdot n_j - 2$. Additionally, $n_{1j} \geq \alpha_{1j} \cdot n_j$.

Proof. By Lemma 5, $n_{2j} = n_j - n_{1j} \geq n_j - \lfloor (\alpha_{1j} \cdot n_j) + 2 \rfloor \geq n_j - \alpha_{1j} \cdot n_j - 2 = \alpha_{2j} n_j - 2$, and $n_{1j} = n_j - n_{2j} \geq n_j - \lfloor \alpha_{2j} \cdot n_j \rfloor \geq \alpha_{1j} n_j$.

Recall that $N_j$ denotes the number of items of class $j$ (that arrived so far). We next bound the number of containers of a large class $j$ in terms of $N_j$.

Lemma 7 During the action of the algorithm, for any large class $j$, it holds that $n_j \leq \frac{N_j}{1+\alpha_2} + 2 = \frac{N_j}{2 - \alpha_{1j}} + 2$ and $n_j \geq \frac{N_j + \beta}{1+\alpha_2} = \frac{N_j + \beta}{2 - \alpha_{1j}}$, where $\beta$ is the number of declared type 2 containers at this time.

Proof. Initially $n_j = N_j = \beta = 0$ and the two inequalities hold. A new container of class $j$ may be created when a new item of this class arrives, that is, when $N_j$ increases (but a new item of class $j$ does not always cause the creation of a new container, as in some cases it is packed into an existing container for this class). No existing containers can be destroyed, and therefore we only consider an arrival of an item of class $j$. Assume that item $i$ of class $j$ has just arrived and packed. Let $n_j$, $n_{2j}$, $n_{1j}$, $N_j$, and $\beta$ be the values of these variables prior to the arrival of $i$ and let $n_j'$, $n_{2j}'$, $n_{1j}'$, $N_j'$, and $\beta'$ be their values after the arrival and packing of $i$, and thus, $N_j' = N_j + 1$. Furthermore, if a new container is created, then $n_j' = n_j + 1$ and otherwise $n_j' = n_j$.

Consider the first inequality. If $n_j' = n_j$, we are done. Since the option of adding the new item into a declared type 2 container is tested first, the creation of a new container means that there were no such declared type 2 containers prior to the arrival of item $i$. Thus, just before $i$ is presented to the algorithm, every type 2 container has two items. We have $N_j = n_{1j} + 2 \cdot n_{2j} = 2 n_j - n_{1j}$. By Lemma 5, $n_{1j} \leq \alpha_{1j} \cdot n_j + 2$, and therefore $N_j \geq (2 - \alpha_{1j}) \cdot n_j - 2 = (1 + \alpha_{2j}) \cdot n_j - 2$, proving $N_j' - 1 \geq (1 + \alpha_{2j}) \cdot (n_j' - 1) - 2$, or alternatively, $N_j' \geq (1 + \alpha_{2j}) \cdot (n_j' - 1) - 1 \geq (1 + \alpha_{2j}) \cdot (n_j - 2)$, as required.

Consider the second inequality. We prove this inequality directly (i.e., without induction). We have $N_j = n_{1j} + 2 \cdot n_{2j} - \beta$, due to the numbers of items in the different types of containers. By Lemma 5, $n_{2j} \leq \alpha_{2j} n_j$, and we have $N_j = n_j + n_{2j} - \beta \leq n_j (1 + \alpha_{2j}) - \beta$. The inequality results from rearranging.

Corollary 8 During the action of the algorithm, for any large class $j$, it holds that $n_{2j} \leq \alpha_{2j} \cdot \left(\frac{N_j}{1+\alpha_2}\right) + 2$, $n_{1j} \leq \alpha_{1j} \cdot \left(\frac{N_j}{1+\alpha_2}\right) + 4$, $n_{2j} \geq \alpha_{2j} \cdot \left(\frac{N_j}{1+\alpha_2}\right) - 2$, and $n_{1j} \geq \alpha_{1j} \cdot \left(\frac{N_j}{1+\alpha_2}\right)$.

Proof. The inequalities follow from Lemma 5, Corollary 6 and Lemma 7 using $\beta \geq 0$.

Lemma 9 For every large class $j$, there are at most four declared type 2 containers at each time.

Proof. Assume by contradiction that at a given time there are at least five declared type 2 containers of class $j$. Then, using the numbers of large items of class $j$ in all four types of containers for this class, $N_j \leq 2 \cdot n_{2j} + n_{1j} - 5 = n_j + n_{2j} - 5$. Using Lemma 7 and Corollary 8, we have $N_j \leq n_j + n_{2j} - 5 \leq (\frac{N_j}{1+\alpha_2})(1 + \alpha_{2j}) - 1 < N_j$, a contradiction.

Next, we consider the case where $j$ is a small or a tiny class.

Lemma 10 For any class $j$ of small items, there is at most one value $i$ (such that $\alpha_{ij} > 0$), for which there is a container of class $j$ and type $i$ with less than $i$ items, and for which there is at most one such container (and for any $i' \neq i$, every container of class $j$ and type $i'$ has exactly $i'$ items). For the class of tiny items, there is at most one value of $i$ for which there is a container of class $j$ and type $i$ with total size at most $A_{i,M+1} - t_M$, and there is at most one such container (and for any $i' \neq i$, every container of class $j$ and type $i'$ has a total size larger than $A_{i',M+1} - t_M$).
Proof. The lemma holds because for \( j \geq b \) there is at most one open container of class \( j \) and (if \( j \leq M \)) such a container is of some type \( i \) for which \( \alpha_{ij} > 0 \) (as the algorithm does not open a new container of class \( j \) until the previous open container of this class is closed). 

5.2 Analysis of the total weight of bins of the algorithm

Our analysis is partitioned into two cases. We first analyze the total weight of items that are packed in bins containing containers of the threshold class (assuming it is not a basic class, otherwise there is no special analysis for this class). We consider all other bins afterwards.

5.2.1 Bins containing containers of the threshold class

We start with the analysis of bins containing containers of the threshold class, for the case where the threshold class is a large class.

Consider the threshold class \( k \) and assume that it is a large class. We say that a temporary type 1 container of this class is smaller if its volume is at most \( 1-a \), and that it is bigger if its volume is above \( 1-a \). We note that smaller temporary type 1 containers do not exist at termination, but we will analyze arbitrary times during the execution of the algorithm.

For this class, we use the following notation for the analysis. Let \( \nu(i) \) denote the number of containers of class \( k \) after \( i \) items have arrived (and have been packed, this time is called time \( i \)). Out of those containers, let \( \nu_1(i) \) and \( \nu_2(i) \) denote the numbers of containers of types 1 and 2, respectively (so that the assignment of item \( i \) is based on \( n_{ik} = \nu(i-1) \) for \( \ell = 1,2 \), and \( \nu(i) = \nu_1(i) + \nu_2(i) \)). Furthermore, let \( \nu_1^s(i), \nu_1^a(i), \) and \( \nu_1^{tb}(i) \), denote the numbers of regular type 1 containers of class \( k \), smaller temporary type 1 containers of class \( k \), and bigger temporary type 1 containers of class \( k \), respectively, after item \( i \) has been packed (so \( \nu_1(i) = \nu_1^s(i) + \nu_1^a(i) + \nu_1^{tb}(i) \)). Let \( \tau \) be the minimum index of an item such that for any \( i > \tau \), \( \nu_1^{tb}(i) > 0 \) (note that the bigger temporary type 1 container created at time \( \tau + 1 \) may change its type later on, we only guarantee that there will always be a bigger temporary type 1 container at all times after \( \tau \)). Letting \( \mu \) denote the total number of items in the input, and we use \( \tau = \mu \) if at termination there are no bigger temporary type 1 containers of class \( k \) (i.e., if \( \nu_1^{tb}(\mu) = 0 \)). Since (as argued above) at termination there are no smaller temporary type 1 containers, \( \nu_1^{tb}(\mu) = 0 \) means that all type 1 containers of the output are regular type 1 containers (this special case can be analyzed more easily, but it will be included in the general analysis). The case \( \tau = 0 \) is possible, and in this case there is always a bigger temporary type 1 container of class \( k \).

Consider the case \( \tau < \mu \), that is, there is at least one additional input item after item \( \tau \). By the definition of \( \tau \), item \( \tau + 1 \) is of class \( k \), its size is above \( 1-a \), and a temporary type 1 container is created for it. Assume that there exists at least one smaller temporary type 1 container of class \( k \) after item \( \tau \) has been packed. All the smaller temporary type 1 containers existing after item \( \tau \) is packed will exist also after item \( \tau + 1 \) has been packed. In the next lemma we show that all these containers will become regular type 1 containers.

Lemma 11 Consider a smaller temporary type 1 container of class \( k \) existing at time \( \tau \). This container will become a regular type 1 container before termination (and in particular it will not become a type 2 container).

Proof. As there are no smaller temporary type 1 containers at termination, this container changes its type some time during the arrival of items \( \tau + 2, \ldots, \mu \). We will show that it does not become a
type 2 container. In all cases where a temporary type 1 container becomes a type 2 container (no matter whether it becomes a regular type 2 container or a declared type 2 container), the largest available temporary type 1 container of the class is selected (see Remark 1). Here, any temporary type 1 container of class \(k\) can be used, in the sense that all type 2 containers of class \(k\) have the same volume, so the chosen temporary type 1 container of class \(k\) is always the largest one. Thus, if a smaller temporary type 1 container is chosen, this means that there is no bigger temporary type 1 container, contradicting the choice of \(\tau\).

Let \(N_k(i)\) denote the number of class \(k\) items out of the first \(i\) arriving items (the class \(k\) items existing at time \(i\)).

We say that a type 2 container of a large class \(k\) is convenient if it is combined in a bin with a negative container or if it has at least one item of size above \(1 - a\).

In what follows, we will say that a type 2 container is created at a certain time if this container was just defined at this time and it was defined as a declared type 2 container immediately (of class \(k\)) or if it was a temporary type 1 container and its type was just changed to type 2 (regular or declared). That is, whenever the number of type 2 containers of class \(k\) increases, the container responsible for this change is considered to be created. Note that type 2 containers remain type 2 containers (of the same class) till termination. For a type 1 container it is created simply when an item is packed into it.

**Lemma 12** Every type 2 container of class \(k\) created at time \(\tau + 1\) or later is convenient.

**Proof.** The only case where a type 2 container of class \(k\) is created and it is not combined with a negative container in a bin immediately is the case where an item of class \(k\) just arrived (see Remark 2). Starting time \(\tau + 1\) there is always a temporary type 1 container (of class \(k\)), so a declared type 2 container of class \(k\) cannot be created unless it is combined with a negative container in a bin immediately. Thus, it remains to consider the case where the new item of class \(k\) is added to a temporary type 1 container to create a regular type 2 container (of class \(k\)). Since after this is done there is still a bigger temporary type 1 container of class \(k\) (by the choice of \(\tau\)) and such a container of maximum volume was selected to become a regular type 2 container, the created type 2 container also has an item of size above \(1 - a\) (its first item is such).

**Lemma 13** The total number of temporary type 1 containers of class \(k\) at termination of the algorithm is below

\[
\alpha_{1k} \frac{N_k(\mu) - N_k(\tau)}{1 + \alpha_{2k}} + 5.
\]

The total number of convenient type 2 containers of class \(k\) is larger than

\[
\alpha_{2k} \frac{N_k(\mu) - N_k(\tau)}{1 + \alpha_{2k}} - 5.
\]

**Proof.** Consider a temporary type 1 container of class \(k\) that is present at termination. As all temporary type 1 containers of class \(k\) existing at termination are bigger, it was created no earlier than time \(\tau + 1\) (i.e., not prior to the packing of the \(\tau + 1\)-th item). Indeed, after it is created, there will be such bigger temporary type 1 container present at all times (as this container is present at termination), and moreover, there is at least one bigger temporary type 1 container of class \(k\) at all times after the creation of such a container when packing the \(\tau + 1\)-th item (but that specific container created at time \(\tau + 1\) does not necessarily remain a temporary type 1 container until termination).
We will use Corollary 8 for times $\tau$ and $\mu$. After $N_k(\tau)$ items of class $k$ have arrived, there are at most $\alpha_{2k} \cdot \left( \frac{N_k(\tau)}{1+\alpha_{2k}} \right) + 2$ containers of type 2 and at least $\alpha_{1k} \cdot \left( \frac{N_k(\tau)}{1+\alpha_{2k}} \right)$ containers of type 1 of class $k$. At termination, using the same corollary, there are at least $\alpha_{2k} \cdot \left( \frac{N_k(\mu)}{1+\alpha_{2k}} \right) - 2$ containers of type 2 and at most $\alpha_{1k} \cdot \left( \frac{N_k(\mu)}{1+\alpha_{2k}} \right) + 4$ containers of type 1 of class $k$.

Assume that there are at least $\alpha_{1k} \cdot \frac{N_k(\mu)-N_k(\tau)}{1+\alpha_{2k}} + 5$ temporary type 1 containers of class $k$ at termination. Every type 1 container of class $k$ existing at time $\tau$ (all of them are either regular type 1 containers or smaller temporary type 1 containers at time $\tau$) will still be a type 1 container at termination (regular type 1 containers remain such, and by Lemma 11 temporary type 1 containers become regular type 1 containers). Thus, the total number of type 1 containers of class $k$ at termination is at least their number after packing the $\tau$-th item plus the number of temporary type 1 containers of this class (at termination), as those were created at time $\tau+1$ or later. In total, we get at least

$$\alpha_{1k} \cdot \frac{N_k(\mu)-N_k(\tau)}{1+\alpha_{2k}} + 5 + \alpha_{1k} \cdot \frac{N_k(\tau)}{1+\alpha_{2k}} = \alpha_{1k} \cdot \frac{N_k(\mu)}{1+\alpha_{2k}} + 5$$

containers of type 1 of class $k$ at termination, a contradiction.

Next, assume that at most $\alpha_{2k} \cdot \frac{N_k(\mu)-N_k(\tau)}{1+\alpha_{2k}} - 5$ type 2 containers of class $k$ are created starting time $\tau+1$ (by Lemma 12 they are all convenient). All type 2 containers remain such, so the total number of type 2 containers of class $k$ is at least the number of these containers at time $\tau$ plus the number of such containers created starting time $\tau+1$. This number is at most

$$\alpha_{2k} \cdot \frac{N_k(\tau)}{1+\alpha_{2k}} + 2 + \alpha_{2k} \cdot \frac{N_k(\mu)-N_k(\tau)}{1+\alpha_{2k}} - 5 = \alpha_{2k} \cdot \frac{N_k(\mu)}{1+\alpha_{2k}} - 3,$$

a contradiction. ■

Recall that our parameters are selected such that $\alpha_{2j} > 0$ for every large class $j$.

**Corollary 14** Let $C$ be the number of convenient type 2 containers of class $k$ at termination. Then, the number of temporary type 1 containers of class $k$ at termination is at most $\frac{\alpha_{1k}}{\alpha_{2k}}(C+5) + 5$.

Next, we consider the weight function we defined using the values of $u$, $v$, $w$ that have real values in $[0, 1]$ and satisfy the constraints (1), (2), (4), and (5). We consider the total weight of the items of class $k$ together with the required weight of the containers (of other classes) that are packed together (i.e., in common bins) with the containers of class $k$. We let $\phi_k$ denote the total weight of the items of class $k$ together with the required weight of the containers (of other classes) that are packed together with the containers of class $k$. Recall that $\nu(i)$ is the number of bins containing containers of class $k$ after $i$ items are packed.

**Lemma 15** If $\alpha_{1k} = 0$, then $\phi_k \geq \nu(\mu) - 3$.

**Proof.** Since $\alpha_{1k} = 0$, by Lemma 5, every container of class $k$ is a type 2 container, except for at most two containers (and thus there are at most six containers of this class with exactly one item). By Lemma 3 the number of items in these $\nu(\mu)$ containers of class $k$ is at least $2\nu(\mu) - 6$, and each such item has weight of at least $u$ (using constraint (1)). Thus, $\phi_k \geq 2u \cdot (\nu(\mu) - 3) \geq \nu(\mu) - 3$ where the last inequality holds by constraint (2) which is equivalent for this case to the constraint $2u \geq 1$. ■

Thus, we next assume that $\alpha_{1k} > 0$. Thus, in the next lemma we assume that $\alpha_{1k}, \alpha_{2k} > 0$. Let $\lambda = 5 \cdot \frac{\alpha_{1k}}{\alpha_{2k}} + 5$.

**Lemma 16** Assume that $\alpha_{1k} > 0$ holds. Then $\phi_k \geq \nu(\mu) - 2\lambda - 8 - \frac{\lambda}{\alpha_{1k}}$.  

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Proof. The number of containers of class $k$ at termination is $\nu(\mu)$, and this is also the number of bins containing such containers. To ease the description of the proof below, we let the weight of a container of a class not equal to $k$ but packed with an item of class $k$ in a bin to be its required weight. Thus, the total weight of a bin $B$ is the total weight of items of class $k$ packed into $B$ together with the required weight of a container packed into $B$ of a class not equal to $k$ (if there is such a container).

For a declared type 2 container of class $k$, the total weight of the container is at least $u$. For a regular type 2 container of class $k$, there are three cases. If this container is convenient in the sense that it has an item of size above $1 - a$, then the weight of the bin is at least $u + v$. If it is convenient in the sense that this bin also contains a negative container, then the weight of the bin is at least $2u + (1 - w)$. In any case, the weight of a bin containing a regular type 2 container is no smaller than $2u$ (using constraint (1) and $0 \leq w \leq 1$).

Let $C_1$ and $C_2$ denote the numbers of the two kinds of convenient type 2 containers, respectively (where $C_1$ is the number of convenient containers with an item of size above $1 - a$ and $C_2$ is the number of all other convenient containers). Then, as there are at most four declared type 2 containers, the total weight of bins containing type 2 containers of class $k$ is at least

$$2u \cdot (\nu_2(\mu) - 4 - C_1 - C_2) + 4u + C_1 \cdot (u + v) + C_2 \cdot (2u + 1 - w). \tag{8}$$

Here, the first expression in the sum (i.e., $2u \cdot (\nu_2(\mu) - 4 - C_1 - C_2) + 4u = 2u \cdot (\nu_2(\mu) - 2 - C_1 - C_2)$) is a lower bound on the total weight of bins containing containers of class $k$ that are not convenient.

For a temporary type 1 container of class $k$, as all bins containing such a containers have (at termination) exactly one item, and its size is above $1 - a$, the weight of such a bin is $v$. Recall that a type 1 container that is matched to a positive container is always a regular type 1 container. Thus, for a regular type 1 container, it is combined with a positive container in its bin, and therefore the total weight of such a bin is at least $u + w$ (once again using constraint (1) and $w \leq 1$). Letting $C_3$ denote the final number of temporary type 1 containers of class $k$, the total weight of all bins containing a type 1 container of class $k$ is at least

$$(w + u) \cdot (\nu_1(\mu) - C_3) + vC_3 \tag{9}$$

because $\nu_1(\mu) - C_3$ is the number of regular type 1 containers and $C_3$ is the number of temporary type 1 containers.

Thus, we have

$$\phi_k \geq 2u \cdot (\nu_2(\mu) - 2 - C_1 - C_2) + C_1 \cdot (u + v) + C_2 \cdot (2u + 1 - w) + (w + u) \cdot (\nu_1(\mu) - C_3) + v \cdot C_3. \tag{10}$$

If $C_3 - \lambda < 0$, we use simpler properties as follows. The total weight of bins with type 2 containers is at least $2u \cdot (\nu_2(\mu) - 4) + 4u$. In total, we get a lower estimation on the total weight of bins containing items of class $k$ of $\phi_k \geq 2u \cdot (\nu_2(\mu) - 2) + (w + u) \cdot (\nu_1(\mu) - C_3)$, similarly to (8) and (9). As $(w + u)C_3 \leq 2C_3 < 2\lambda$ because of our assumption and by $4u \leq 4$, we have using Corollary 6 that

$$\phi_k \geq 2u \cdot \nu_2(\mu) + (w + u) \cdot \nu_1(\mu) - 2\lambda - 4 \geq 2u \cdot (\alpha_{2k} \nu(\mu) - 2) + (w + u) \cdot (\alpha_{1k} \nu(\mu)) - 2\lambda - 4$$

$$\geq \nu(\mu)(u(1 + \alpha_{2k}) + w\alpha_{1k}) - 2\lambda - 8 \geq \nu(\mu) - 2 \lambda - 8,$$

where in the second inequality we used $\nu_1(\mu) \geq \alpha_{1k} \nu(\mu)$ and $\nu_2(\mu) \geq \alpha_{2k} \nu(\mu) - 2$ (which holds by Lemma 8), and the last inequality follows by constraint (3), and the lemma follows.

Thus, in the remaining part of the proof, we assume that $C_3 - \lambda \geq 0$. 

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By Corollary 14, \( C_3 - \lambda \leq \frac{\alpha_{2k}}{\alpha_{1k}}(C_1 + C_2) \). Let \( C'_1 = \frac{\alpha_{2k}}{\alpha_{1k}} \cdot (C_3 - \lambda) - C_2 \leq C_1 \) and \( C'_2 = C_2 \). If \( C'_1 < 0 \), instead of these values we let \( C'_1 = 0 \) and \( C'_2 = \frac{\alpha_{2k}}{\alpha_{1k}}(C_3 - \lambda) \leq C_2 \) (and \( C'_2 \geq 0 \), as the case where this last value is negative was considered earlier in the case \( C_3 - \lambda < 0 \).)

Based on (5), the total weight of bins with type 2 containers is at least \( 2u \cdot (\nu_2(\mu) - 2 - C_1 - C_2) + C_1(u + v) + C_2(2u + 1 - w) \geq 2u \cdot (\nu_2(\mu) - 2 - C_1 - C_2) + C'_1(u + v) + C'_2(u + v) + (C_1 - C'_1)2u + C'_2(2u + 1 - w) + (C_2 - C'_2) \cdot (2u) = 2u \cdot (\nu_2(\mu) - 2 - C'_1 - C'_2) + C'_1(u + v) + C'_2(2u + 1 - w). \)

In total, we get similarly to (10) that \( \phi_k \geq 2u \cdot (\nu_2(\mu) - 2 - C'_1 - C'_2) + C'_1(u + v) + C'_2(2u + 1 - w) + (w + u) \cdot (\nu_1(\mu) - C_3) + vC_3. \)

Since in both possible definitions of \( C'_1 \) and \( C'_2 \),

\[
C_3 = \frac{\alpha_{1k}}{\alpha_{2k}} \cdot (C'_1 + C'_2) + \lambda.
\]

We finally get

\[
\phi_k \geq 2u \cdot (\nu_2(\mu) - 2 - C'_1 - C'_2) + C'_1(u + v) + C'_2(2u + 1 - w) + (w + u) \cdot (\nu_1(\mu) - C_3) + vC_3
\]

\[
\geq 2u \cdot (\alpha_{2k}\nu(\mu) - C'_1 - C'_2) + (w + u) \cdot (\alpha_{1k}\nu(\mu) - C_3)
\]

\[
+C'_1(u + v) + C'_2(2u + 1 - w) + vC_3 - 8u
\]

\[
= 2u \cdot (\alpha_{2k}\nu(\mu) - \frac{\alpha_{2k}}{\alpha_{1k}}(C_3 - \lambda)) + (w + u) \cdot (\alpha_{1k}\nu(\mu) - C_3)
\]

\[
+C'_1(u + v) + C'_2(2u + 1 - w) + vC_3 - 8u
\]

\[
= (\nu(\mu) - \frac{C_3}{\alpha_{1k}}) \cdot (u + \alpha_{2k}) + \alpha_{1k}w
\]

\[
+C'_1(u + v) + C'_2(2u + 1 - w) + vC_3 - 8u + 2u \cdot \frac{\alpha_{2k}}{\alpha_{1k}}\lambda
\]

\[
\geq \nu(\mu) - \frac{C_3}{\alpha_{1k}} + C'_1(u + v) + C'_2(2u + 1 - w) + vC_3 - 8u + 2u \cdot \frac{\alpha_{2k}}{\alpha_{1k}}\lambda
\]

\[
\geq \nu(\mu) - \frac{C_3}{\alpha_{1k}} + C'_1(u + v) + \frac{C'_2}{\alpha_{2k}} - \frac{\alpha_{1k}}{\alpha_{2k}} \cdot vC'_2 + vC_3 - 8u + 2u \cdot \frac{\alpha_{2k}}{\alpha_{1k}}\lambda
\]

\[
= \nu(\mu) - \frac{C_3}{\alpha_{1k}} + C'_1(u + v) + \frac{C'_2}{\alpha_{2k}} - \frac{\alpha_{1k}}{\alpha_{2k}} \cdot vC'_2
\]

\[
+v \cdot \frac{\alpha_{1k}}{\alpha_{2k}}(C'_1 + C'_2) - 8u + 2u \cdot \frac{\alpha_{2k}}{\alpha_{1k}}\lambda + v\lambda
\]

\[
= \nu(\mu) - \frac{C_3}{\alpha_{1k}} + C'_1(u + v + \nu \frac{\alpha_{1k}}{\alpha_{2k}}) + \frac{C'_2}{\alpha_{2k}} - 8u + 2u \cdot \frac{\alpha_{2k}}{\alpha_{1k}}\lambda + v\lambda
\]

\[
\geq \nu(\mu) - \frac{C_3}{\alpha_{1k}} + C'_1 + C'_2 - 8u + 2u \cdot \frac{\alpha_{2k}}{\alpha_{1k}}\lambda + v\lambda
\]

\[
= \nu(\mu) - 8u + 2u \cdot \frac{\alpha_{2k}}{\alpha_{1k}}\lambda + v\lambda - \frac{\lambda}{\alpha_{1k}},
\]

where (12) follows by \( \nu_1(\mu) \geq \alpha_{1k}\nu(\mu) \) and \( \nu_2(\mu) \geq \alpha_{2k}\nu(\mu) - 2 \) (by Lemma 8), (13) follows by (11), (14) holds by simple algebraic transformation and by substituting \( \alpha_{1k} + \alpha_{2k} = 1 \), (15) holds by constraint (2), (16) follows by constraint (7) and by applying \( \alpha_{1k} + \alpha_{2k} = 1 \) again, (17) follows by (11), (18) holds by simple algebraic transformations, (19) holds because \( u + v + v \frac{\alpha_{2k}}{\alpha_{2k}} = \frac{1}{\alpha_{2k}}(\alpha_{2k}u + v) \geq \frac{1}{\alpha_{2k}} \) where the last inequality holds using constraint (5), and (20) holds by (11). The claim follows using \( u, v \geq 0 \).
5.2.2 The total weight of items of basic classes and the class of huge items

**Lemma 17** For any bin (excluding bins containing at least one item of class \(k\), if \(k\) is not a basic class), the total required weight of the containers that are packed in this bin is at least 1.

**Proof.** If the bin contains both a positive container and a negative container, we are done, as their required weights are at least \(w\) and \(1 - w\), respectively (using \(0 \leq w \leq 1\)). A positive container that was not combined with a negative one has volume of at least \(a\), and a negative container that was not combined with a positive one has volume above \(1 - a\). Such containers have required weights of 1. ■

Next, we show that for a basic class and the class of huge items, we have that the total required weight of the containers of class \(j\) is at most the total weight of the items of class \(j\) plus a constant.

**Lemma 18** If \(j = 1\), then the total required weight of the containers of class \(j\) equals the total weight of the items of class \(j\).

**Proof.** The lemma follows by our definition of a weight of a huge item (it is \(w\) if its size is smaller than \(a\) and 1 otherwise). ■

For a basic class \(2 \leq j \leq M + 1\), let \(\zeta_j\) be the number of strictly positive values of \(\alpha_{ij}\). For \(j = M + 1\), we let \(\gamma_{M+1} = \zeta_{M+1} = p\). For \(j \leq M\), we have \(\zeta_j \leq \gamma_j\), and for our parameters we actually have \(\zeta_j \leq 3\) for all \(j\), and \(\zeta_j = 2\) for most values of \(j\) (but we sometimes have \(\zeta_j = 3\) and this is an important new feature of our algorithm). Let \(R_j\) be the total required weight of all containers of class \(j\), and let \(W_j\) be the total weight of items of class \(j\). By definition, letting \(n_{ij}\) denote the number of containers of class \(j\) and type \(i\) (for class \(M + 1\) it is denoted by \(n_{i,M+1}\)), \(R_j = \sum_{i=1}^{\gamma_j} r_{x,y}(i, j) \cdot n_{ij}\) and for \(j \leq M\), \(W_j = \omega_j \cdot N_j\).

**Lemma 19** For any small or tiny class \(j\), \(R_j \leq n_j \cdot \sum_{i=1}^{\gamma_j} \alpha_{ij} \cdot r_{x,y}(i, j) + \zeta_j\). For any basic large class \(j\), \(R_j \leq n_j \cdot \sum_{i=1}^{\gamma_j} \alpha_{ij} \cdot r_{x,y}(i, j) + 2\).

**Proof.** For any small or tiny class \(j\), a container of type \(i\) is opened only in the case where there are at most \([\alpha_{ij} \cdot \hat{n}_j]\) such containers, where \(\hat{n}_j\) is the number of containers of class \(j\) before the new container is opened. Moreover, it is never opened if \(\alpha_{ij} = 0\). Thus, the last container of class \(j\) and type \(i\) was opened when there were at most \([\alpha_{ij} \cdot (n_j - 1)]\) such containers and finally there are at most \([\alpha_{ij} \cdot (n_j - 1)] + 1 < \alpha_{ij} n_j + 1\) such containers. By \(r_{x,y}(i, j) \leq 1\) we have

\[
R_j = \sum_{i=1}^{\gamma_j} n_{ij} \cdot r_{x,y}(i, j) \leq \sum_{i: \alpha_{ij} > 0} (\alpha_{ij} n_j + 1) \cdot r_{x,y}(i, j) \leq n_j \cdot \sum_{i=1}^{\gamma_j} \alpha_{ij} \cdot r_{x,y}(i, j) + \zeta_j.
\]

For any basic large class \(j\), \(n_{ij} \leq 1 + n_{j'}\) and \(n_{2j} \leq 2\), by Lemma 18. Thus, \(R_j = \sum_{i=1}^{\gamma_j} n_{ij} \cdot r_{x,y}(i, j') \leq \sum_{i=1}^{\gamma_j} \alpha_{ij} n_{j'} \cdot r_{x,y}(i, j') + 2r_{x,y}(1, j') \leq n_j \cdot \sum_{i=1}^{\gamma_j} \alpha_{ij} \cdot r_{x,y}(i, j') + 2\) (by \(r_{x,y}(i, j') \leq 1\)).

**Lemma 20** For the tiny class \(M + 1\), \(W_{M+1} \geq n_{M+1} \cdot \sum_{i=1}^{p} \alpha_{i,M+1} \cdot r_{x,y}(i, M + 1) - 2p\).

**Proof.** For class \(M + 1\), we have \(W_{M+1} \geq \rho \cdot (\sum_{i=1}^{p} (A_{i,M+1} - t_M) \cdot n_{i,M+1} - 1)\), as a class \(M + 1\) type \(i\) container has items of total size at least \(A_{i,M+1} - t_M\), except for at most one container of class \(M + 1\) and some type (and \(A_{i,M+1} - t_M \leq 1\) for all \(i\)). Let \(n_{i,M+1} = \alpha_{i,M+1} n_{M+1} + \delta_{i,M+1}\). For every \(i\) we have \(\delta_{i,M+1} \leq 1\). As

\[
n_{M+1} = \sum_{i=1}^{p} n_{i,M+1} = \sum_{i=1}^{p} (\alpha_{i,M+1} n_{M+1} + \delta_{i,M+1})
\]
\[
= n_{M+1} \sum_{i=1}^{p} \alpha_{i,M+1} + \sum_{i=1}^{p} \delta_{i,M+1} = n_{M+1} + \sum_{i=1}^{p} \delta_{i,M+1},
\]
we get \( \sum_{i=1}^{p} \delta_{i,M+1} = 0 \), which implies \( \sum_{i=1}^{p} \min\{0, \delta_{i,M+1}\} + \sum_{i=1}^{p} \max\{0, \delta_{i,M+1}\} = 0 \). If for all \( i \) we have \( \delta_{i,M+1} \geq 0 \), then \( \delta_{i,M+1} = 0 \) for all \( i \). Otherwise, if there is a value \( i \) such that \( \delta_{i,M+1} > 0 \), there is also at least one negative value. Thus, \( -\sum_{i=1}^{p} \min\{0, \delta_{i,M+1}\} = \sum_{i=1}^{p} \max\{0, \delta_{i,M+1}\} \leq p - 1 \), and
\[
\sum_{i=1}^{p} (A_{i,M+1} - t_M) \cdot n_{i,M+1} = \sum_{i=1}^{p} (A_{i,M+1} - t_M) \cdot (\alpha_{i,M+1}n_{M+1} + \delta_{i,M+1})
\]
\[
= n_{M+1} \sum_{i=1}^{p} (A_{i,M+1} - t_M) \cdot \alpha_{i,M+1} + \sum_{i=1}^{p} (A_{i,M+1} - t_M) \cdot \delta_{i,M+1}
\]
\[
\geq n_{M+1} \sum_{i=1}^{p} (A_{i,M+1} - t_M) \cdot \alpha_{i,M+1} + \sum_{i=1}^{p} \min\{0, \delta_{i,M+1}\}.
\]
Therefore, \( W_{M+1} \geq \rho(\sum_{i=1}^{p} (A_{i,M+1} - t_M) \cdot n_{i,M+1} - 1) \geq n_{M+1} \sum_{i=1}^{p} \alpha_{i,M+1} \cdot r_{x,y}(i, M+1) - 2p \), as \( -\sum_{i=1}^{p} \min\{0, \delta_{i,M+1}\} \leq p - 1 \) and \( \rho \leq 2 \).

**Lemma 21** For any basic large class \( j \), \( W_j \geq n_j \cdot \sum_{i=1}^{\gamma_j} \alpha_{ij} \cdot r_{x,y}(i,j) - 4 \).

**Proof.** For a (basic) large class \( j \), we have \( \sum_i i \cdot \alpha_{ij} = \alpha_{1j} + 2\alpha_{2j} = 1 + \alpha_{2j} \). Using Corollary [S] we find \( n_j = n_{1j} + n_{2j} \leq \alpha_{1j} \cdot \left( \frac{N_j}{1 + \delta_{2j}} + 4 \right) + \alpha_{2j} \cdot \left( \frac{N_j}{1 + \delta_{2j}} + 2 \right) \leq \frac{N_j}{\gamma_j}, \) as a class \( j \) is at least
\[
\omega_j \cdot N_j = \sum_{i=1}^{\gamma_j} \alpha_{ij} \cdot r_{x,y}(i,j) \cdot N_j \geq (n_j - 4) \sum_{i=1}^{\gamma_j} \alpha_{ij} \cdot r_{x,y}(i,j)
\]
\[
\geq n_j \cdot \sum_{i=1}^{\gamma_j} r_{x,y}(i,j) \alpha_{ij} - 4,
\]
\[
\text{as } \sum_i r_{x,y}(i,j) \alpha_{ij} \leq 1, \text{ by } r_{x,y}(i,j) \leq 1 \text{ for all } x,y,i.
\]

**Lemma 22** For any small class \( j \), \( W_j \geq n_j \cdot \sum_{i=1}^{\gamma_j} \alpha_{ij} \cdot r_{x,y}(i,j) - \gamma_j \cdot \zeta_j \).

**Proof.** For a small class \( j \), we have \( W_j \geq \omega_j (\sum_{i=1}^{\gamma_j} i \cdot n_{ij} - (\gamma_j - 1)) \), as a class \( j \) type \( i \) container has \( i \) items, except for at most one container of class \( j \) and some type, which has at least one item instead \( i \leq \gamma_j \) items of class \( j \). Let \( n_{ij} = \alpha_{ij} n_{ij} + \delta_{ij} \), for some (positive or negative or zero) value \( \delta_{ij} \). For \( i \) such that \( \alpha_{ij} = 0 \) we have \( \delta_{ij} = 0 \). For every \( i \) such that \( \alpha_{ij} > 0 \), we have \( \delta_{ij} \leq 1 \). As
\[
n_j = \sum_{i=1}^{\gamma_j} n_{ij} = \sum_{i=1}^{\gamma_j} (\alpha_{ij} n_{ij} + \delta_{ij}) = n_j \sum_{i=1}^{\gamma_j} \alpha_{ij} + \sum_{i=1}^{\gamma_j} \delta_{ij} = n_j + \sum_{i=1}^{\gamma_j} \delta_{ij},
\]
we get \( \sum_{i=1}^{\gamma_j} \delta_{ij} = 0 \), which implies \( \sum_{i=1}^{\gamma_j} \min\{0, \delta_{ij}\} + \sum_{i=1}^{\gamma_j} \max\{0, \delta_{ij}\} = 0 \). If for all \( i \) we have \( \delta_{ij} \geq 0 \), then \( \delta_{ij} = 0 \) for all \( i \). Otherwise, if there is a value \( i \) such that \( \delta_{ij} > 0 \), there is at least one negative value as well. Thus, \( -\sum_{i=1}^{\gamma_j} \min\{0, \delta_{ij}\} = \sum_{i=1}^{\gamma_j} \max\{0, \delta_{ij}\} \leq \zeta_j - 1 \), and
\[
\sum_{i=1}^{\gamma_j} i \cdot n_{ij} = \sum_{i=1}^{\gamma_j} i \cdot (\alpha_{ij} n_{ij} + \delta_{ij}) = n_j \sum_{i=1}^{\gamma_j} i \cdot \alpha_{ij} + \sum_{i=1}^{\gamma_j} i \cdot \delta_{ij} \geq n_j \sum_{i=1}^{\gamma_j} i \cdot \alpha_{ij} + \sum_{i=1}^{\gamma_j} i \cdot \min\{0, \delta_{ij}\}.
\]
Therefore, \( W_j \geq \omega_j (\sum_{i=1}^{\gamma_j} i \cdot n_{ij} - (\gamma_j - 1)) \geq n_j \sum_{i=1}^{\gamma_j} \alpha_{ij} \cdot r_{x,y}(i,j) - \gamma_j \cdot \zeta_j \), as \( \omega_j \leq 1 \), and
\[
\sum_{i=1}^{\gamma_j} i \cdot \min\{0, \delta_{ij}\} \geq -\gamma_j (\zeta_j - 1).
\]

**Corollary 23** For any class \( j \), we have \( R_j \leq W_j + \zeta_j \), where \( \zeta_j \) is a constant independent of the input such that for \( j = 1 \), \( \xi_1 = 0 \), for \( j = M+1 \), \( \xi_{M+1} = 3p \), for any small class \( j \), \( \xi_j \leq (\gamma_j + 1)\zeta_j \), and for any basic large class \( j \), \( \xi_j \leq 6 \).
5.2.3 The relation between $W$ and the cost of the algorithm

We have proved the next theorem, which follows from Lemmas 15, 16 and from Corollary 23. The theorem shows that our weight function is valid, and it remains to find an upper bound on the supremum total weight of any bin (of the optimal solution). We showed that there is a constant $\Psi$ that is independent of the input (and depends on our set of parameters) such that the following holds.

**Theorem 24** Assume that for input $I$, the output of the algorithm belongs to the scenario of index $(x, y)$. If $y > \frac{1}{3}$ and $u, v, w$ satisfy $0 \leq u, v, w \leq 1$ and the constraints (1), (2), (4), (6), then assigning weights to the items according to our definition in section 4.3 satisfies that the final number of bins of the algorithm (applied on input $I$) is at most $W + \Psi$, where $\Psi$ is a constant independent of the input. If $y \leq \frac{1}{3}$ and $w$ satisfies $0 \leq w \leq 1$, then assigning weights to the items according to our definition in section 4.3 satisfies that the final number of bins of the algorithm (applied on input $I$) is at most $W + \Psi$, where $\Psi$ is a constant independent of the input.

5.3 Analysis of weights of bins of optimal solutions

We provide the remaining part of the proof, where given our sets of weights (which are based on our set of parameters), we find upper bounds on total weights of bins.

In Appendix A we provide a table with all boundary points and all strictly positive values of $\alpha_{ij}$ (all other values of $\alpha_{ij}$ are equal to zero). It can be seen that $\zeta_j \leq 3$ for all $j$. For classes 173, 174, 176, 184, 190, 191 indeed $\zeta_j = 3$, and for classes 2 and 171 we have $\zeta_j = 1$. For all other classes $\zeta_j = 2$ (for large classes the case $\zeta_j = 3$ is impossible as $\gamma_j = 2$). For class $M + 1$ (the class of tiny items), we have $\zeta_{M+1} = 2$, which is an interesting feature of AH. There are containers where the total size of tiny items is at most $\frac{17}{60} \approx 0.283333$ (and at least $\frac{17}{60} - \frac{1}{43} \approx 0.26$, except for at most one container of tiny items of type 1). There is a relatively big number of classes of large items whose sizes are in $(\frac{1}{3}, 0.35]$. The reason for this is that the volume of type 1 containers of these classes is defined by the exact size of an item, while type 2 containers are defined by $2 \cdot t_j - 1$, though we still would like the volume to be close to the total size and to $2 \cdot t_j$ (the difference $t_j - 1 - t_j$ is small, much smaller than the size of a tiny item, and this limits the possible bin kinds of optimal solutions).

To solve the knapsack problems, which are standard knapsack problems, we use a branch and bound type approach similar to that of Ramanan et al. [19] and Seiden [22]. Note that we could use existing solvers for knapsack, and we actually did so (in addition to the branch and bound algorithm described here) in order to verify the results (using rounded values). However, as we were interested in precise results, we represented all our parameters as big fractions with integer numerators and denominators, with a common denominator of $q$ for an appropriate value of $q$. Then, after representing every size in the form $\frac{p}{q}$, we allow the total size of a multiset of items of $\Delta$ to be at most $\frac{q-1}{q}$ in the case where their total size should be below 1 (and an analogous condition is given in the case that the total size should be below $1 - a$). The branch and bound approach is standard as well, where the branching rule is according to the size of the next item, and the bounding rule is according to density, which is the ratio between weight and size. That is, items are sorted by non increasing density. Then, the algorithm iteratively generates the possible packing patterns (multisets of items of $\Delta$) using this sorted order. When a new item is added to the actual pattern, an upper bound is calculated estimating the largest possible weight of the patterns containing the given items. The bound is based on upper bounding the weight of the remaining space in the knapsack by assigning the weight of the current item to it. This has the following meaning. If the maximum density of further items that can still be packed into the bin is such
that no matter what additional items the bin will contain, its total weight is no larger than the
maximum weight of any bin calculated so far, there is no need to compute an exact maximum (or
even a supremum) of the possible weight of a bin containing the items already inserted into the
bin, but it is sufficient to use the resulting upper bound. If the estimated weight is lower than the
current maximal weight, no further items are added to this pattern. After all possible patterns are
generated and checked, the algorithm finds the one with the largest weight. The pseudo-code of
the algorithm is given as Algorithm 1.

Algorithm 1 Branch and Bound Knapsack Solver
Input: sizes[N], weights[N] sorted by weights[i]/sizes[i] in non increasing order
Output: worstbound, worstpat
Require: actpat = ∅, worstbound = 0
Procedure KnapsackSolver(i)
If i = N + 1 Then
    totalweight ← The total weight of a bin containing the items of actpat
    and filled with tiny items
    If totalweight > worstbound Then
        worstbound ← totalweight
        worstpat ← actpat
    End If
Else
    es ← The empty space in a bin containing the items of actpat
    actpw ← The total weight of the items of actpat
    li ← The index of the last item of actpat
    If actpat <> ∅ Then
        If actpw + weights[li]/sizes[li] · es < worstbound Then
            Return
        End If
    End If
    counter ← The maximal number of the items of sizes[i]
    that can be packed into a bin containing the items of actpat
    For j = counter to 1 step − 1
        Add sizes[i] to actpat
        KnapsackSolver(i + 1)
    End For
    Remove all items of sizes[i] from actpat
    KnapsackSolver(i + 1)
End If
End Procedure

Using this branch and bound procedure, for each scenario we calculate the weight function
 corresponding to the values of u, v, and w (for scenarios where the threshold class is a large class)
or the value of w (for the other scenarios). In Appendix B we report the values of u, v, and w that
we use, and the resulting upper bound on the competitive ratio of the algorithm. In this way, we
prove that the competitive ratio of AH is at most 1.57828956.
A  All parameters of the algorithm

We provide all required data for defining the algorithm and its analysis according to our method of analysis. Recall that we use exact values of parameters and exact calculations. In many cases we write an approximate value in the table in order to provide intuition, but these values were not used in our calculations.

The next table contains the values \(\alpha_{ij}\) for all \(j\) such that \(2 \leq j \leq 5\) and \(j \geq 166\). The values \(\alpha_{ij}\) are only given for \(i\) such that \(\alpha_{ij} \neq 0\).

For \(6 \leq j \leq 165\), \(\alpha_{ij} = \frac{22145926}{78181827} \approx 0.2832618\), \(\alpha_{2j} = \frac{56035901}{78181827} \approx 0.71673816\). For these values of \(j\), \(t_j = 0.35 - \frac{j-5}{9600}\) and \(t_{j-1} = 0.35 - \frac{j-6}{9600}\). For class 166, the right endpoint is \(0.35 - \frac{166}{9600} = \frac{1}{3}\).

There are many boundary points between \(\frac{1}{3}\) and 0.35 as AH packs such items carefully, and we would like very similar pairs of such items to be packed together in one bin of the algorithm.

| Class index: \(j\) | Left endpoint \(t_j\) | Right endpoint \(t_{j-1}\) | \(i\) | \(\alpha_{ij}\) or \(\alpha_{i,j}\) |
|-------------------|---------------------|---------------------------|-----|---------------------|
| 1                 | \(\frac{1}{2} = 0.5\) | \(\frac{1}{2} = 0.5\)     | 1   | 1                   |
| 2                 | \(\frac{3}{120} \approx 0.2857\) | \(\frac{1}{2} = 0.5\)     | 2   | 31755722 \(\approx 0.2115698824262585\) |
| 3                 | \(\frac{43}{120} \approx 0.3583\) | \(\frac{3}{5} \approx 0.4285\) | 1   | 118339862 \(\approx 0.7884300117573741\) |
| 4                 | \(\frac{5}{100} \approx 0.3554\) | \(\frac{43}{120} \approx 0.3583\) | 1   | 33032666 \(\approx 0.2212104811917158\) |
| 5                 | \(\frac{7}{20} = 0.35\) | \(\frac{5}{100} \approx 0.3554\) | 1   | 59023851 \(\approx 0.2361913999431558\) |
| 166               | \(\frac{721}{960} \approx 0.2826\) | \(\frac{1}{3} \approx 0.3333\) | 1   | 1430926 \(\approx 0.002403008384307076\) |
| 167               | \(\frac{1}{4} = 0.25\) | \(\frac{721}{960} \approx 0.2826\) | 1   | 14309260 \(\approx 0.9975969916156929\) |
| 168               | \(\frac{97}{480} \approx 0.2020\) | \(\frac{1}{4} = 0.25\)     | 1   | 4109057 \(\approx 0.247418183318368\) |
| 169               | \(\frac{1}{5} = 0.2\) | \(\frac{97}{480} \approx 0.2020\) | 1   | 2265824 \(\approx 0.269412481855349\) |
| 170               | \(\frac{12}{88} \approx 0.1704\) | \(\frac{1}{5} = 0.2\)     | 5   | 1                   |
| 171               | \(\frac{1}{6} \approx 0.1667\) | \(\frac{15}{88} \approx 0.1704\) | 1   | 7697265 \(\approx 0.0677149208923630\) |
| 172               | \(\frac{3}{20} = 0.15\) | \(\frac{1}{6} \approx 0.1667\) | 2   | 21797889 \(\approx 0.129824430649552\) |
| 173               | \(\frac{12}{88} \approx 0.1445\) | \(\frac{3}{20} = 0.15\)   | 2   | 15897552 \(\approx 0.328808487423192\) |
| 174               | \(\frac{1}{7} \approx 0.1429\) | \(\frac{12}{88} \approx 0.1445\) | 2   | 5682611 \(\approx 0.379519847343640\) |
| Fraction | Approximation | Error | Fraction | Approximation | Error |
|----------|---------------|-------|----------|---------------|-------|
| $\frac{11}{83}$ | ≈ 0.13253 | | $\frac{1}{7}$ | ≈ 0.14286 | |
| $\frac{1}{8}$ | = 0.125 | | $\frac{11}{83}$ | ≈ 0.13253 | |
| $\frac{1}{9}$ | ≈ 0.11111 | | $\frac{1}{8}$ | = 0.125 | |
| $\frac{1}{10}$ | = 0.1 | | $\frac{1}{9}$ | ≈ 0.11111 | |
| $\frac{1}{11}$ | ≈ 0.09091 | | $\frac{1}{10}$ | = 0.1 | |
| $\frac{1}{12}$ | ≈ 0.08333 | | $\frac{1}{11}$ | ≈ 0.09091 | |
| $\frac{1}{13}$ | ≈ 0.07692 | | $\frac{1}{12}$ | ≈ 0.08333 | |
| $\frac{1}{14}$ | ≈ 0.07143 | | $\frac{1}{13}$ | ≈ 0.07692 | |
| $\frac{1}{15}$ | ≈ 0.06667 | | $\frac{1}{14}$ | ≈ 0.07143 | |
| $\frac{1}{16}$ | = 0.0625 | | $\frac{1}{15}$ | ≈ 0.06667 | |
| $\frac{1}{17}$ | ≈ 0.05882 | | $\frac{1}{16}$ | = 0.0625 | |
| $\frac{1}{18}$ | ≈ 0.05556 | | $\frac{1}{17}$ | ≈ 0.05882 | |
| $\frac{1}{19}$ | ≈ 0.05263 | | $\frac{1}{18}$ | ≈ 0.05556 | |
| $\frac{1}{20}$ | = 0.05 | | $\frac{1}{19}$ | ≈ 0.05263 | |
| $\frac{1}{21}$ | ≈ 0.04762 | | $\frac{1}{20}$ | = 0.05 | |
| $\frac{1}{22}$ | ≈ 0.04545 | | $\frac{1}{21}$ | ≈ 0.04762 | |
| $\frac{1}{23}$ | ≈ 0.04348 | | $\frac{1}{22}$ | ≈ 0.04545 | |
| 191 | $\frac{1}{22} \approx 0.04167$ | $\frac{1}{22} \approx 0.04348$ | 22 | $\frac{181510035}{254170714} \approx 0.7141263866308705$ |
| 192 | $\frac{1}{22} \approx 0.04167$ | $\frac{1}{22} \approx 0.04348$ | 23 | $\frac{17968099}{63082909} \approx 0.287456496474916$ |
| 193 | $\frac{1}{22} = 0.04$ | $\frac{1}{22} \approx 0.04167$ | 6 | $\frac{14568370}{49692809} \approx 0.7172543503255084$ |
| 194 | $\frac{1}{22} \approx 0.03846$ | $\frac{1}{22} = 0.04$ | 7 | $\frac{6808688}{2028377} \approx 0.3132270802710437$ |
| 195 | $\frac{1}{22} \approx 0.03704$ | $\frac{1}{22} \approx 0.03846$ | 24 | $\frac{15606949}{60028771} \approx 0.6867729197289564$ |
| 196 | $\frac{1}{22} \approx 0.03704$ | $\frac{1}{22} \approx 0.03846$ | 25 | $\frac{16935853}{23559574} \approx 0.3211485046376475$ |
| 197 | $\frac{1}{22} \approx 0.03448$ | $\frac{1}{22} \approx 0.03571$ | 7 | $\frac{5483252}{17198095} \approx 0.32791319249554893$ |
| 198 | $\frac{1}{22} \approx 0.03333$ | $\frac{1}{22} \approx 0.03448$ | 26 | $\frac{53174541}{17372911} \approx 0.720786807504451$ |
| 199 | $\frac{1}{22} \approx 0.03333$ | $\frac{1}{22} \approx 0.03448$ | 27 | $\frac{20611149}{67582056} \approx 0.2785782845098278$ |
| 200 | $\frac{1}{22} \approx 0.03333$ | $\frac{1}{22} \approx 0.03448$ | 28 | $\frac{53376547}{16935853} \approx 0.7214217154901722$ |
| 201 | $\frac{1}{22} \approx 0.0303$ | $\frac{1}{22} \approx 0.03125$ | 8 | $\frac{24577815}{71769463} \approx 0.34356925665390536$ |
| 202 | $\frac{1}{22} \approx 0.02941$ | $\frac{1}{22} \approx 0.0303$ | 9 | $\frac{19385660}{63335051} \approx 0.2608320430190504$ |
| 203 | $\frac{1}{22} \approx 0.02857$ | $\frac{1}{22} \approx 0.02941$ | 10 | $\frac{209651}{6725284} \approx 0.25914598281037654$ |
| 204 | $\frac{1}{22} \approx 0.02778$ | $\frac{1}{22} \approx 0.02857$ | 11 | $\frac{743295}{254170714} \approx 0.7408540171896235$ |
| 205 | $\frac{1}{22} \approx 0.02703$ | $\frac{1}{22} \approx 0.02778$ | 12 | $\frac{6808688}{2028377} \approx 0.3132270802710437$ |
| 206 | $\frac{1}{22} \approx 0.02632$ | $\frac{1}{22} \approx 0.02703$ | 13 | $\frac{9905269}{300829683} \approx 0.24332344776739177$ |
| 207 | $\frac{1}{22} \approx 0.02564$ | $\frac{1}{22} \approx 0.02632$ | 14 | $\frac{109022543}{3435692566} \approx 0.756676552326082$ |
| 208 | $\frac{1}{22} \approx 0.02564$ | $\frac{1}{22} \approx 0.02632$ | 15 | $\frac{40741107}{16935853} \approx 0.2485006121862823$ |
| 209 | $\frac{1}{22} \approx 0.02439$ | $\frac{1}{22} \approx 0.02564$ | 16 | $\frac{123006170}{463974729} \approx 0.7514993878137172$ |
| 210 | $\frac{1}{22} \approx 0.02381$ | $\frac{1}{22} \approx 0.02439$ | 17 | $\frac{44138539}{16610862} \approx 0.26471339101029717$ |
B The values $u$, $v$, $w$, and the competitive ratio in all scenarios

First, consider the case where the scenario is such that $k$ is not a large class (it is small or tiny). For all scenarios whose interval $(x, y)$ is contained in $(0, \frac{1}{5})$, we use $w = 0$, and find $R < \frac{820976062891}{920990514430} \approx 1.57820153$. The scenarios contained in $\left( \frac{3}{10}, \frac{3}{5} \right)$ also have common features. Every scenario has the form $\left( \frac{3}{10} + \frac{\ell - 1}{4500}, \frac{3}{10} + \frac{\ell}{4500} \right)$ for $1 \leq \ell \leq 160$, $w = \frac{43913}{521288} \approx 0.8394764$, and

$$R < \frac{10060574276093395247}{6374352691333693440} \approx 1.57828956 .$$

The remaining cases are shown in the following table. The values in the right column (RBB) are the bounds we obtained on the total weight of a bin using the branch and bound procedure.

| Threshold class | Scenario Interval | $w$ | RBB |
|-----------------|-------------------|-----|-----|
| $\frac{1}{15}$ | $\left( 0, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{3}{15}, \frac{1}{5} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{1}{5}, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{3}{15}, \frac{1}{5} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{1}{5}, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{3}{15}, \frac{1}{5} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{1}{5}, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{3}{15}, \frac{1}{5} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{1}{5}, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{3}{15}, \frac{1}{5} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{1}{5}, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{3}{15}, \frac{1}{5} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{1}{5}, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{3}{15}, \frac{1}{5} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{1}{5}, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{3}{15}, \frac{1}{5} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{1}{5}, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{3}{15}, \frac{1}{5} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{1}{5}, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{3}{15}, \frac{1}{5} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{1}{5}, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{3}{15}, \frac{1}{5} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
| $\frac{1}{15}$ | $\left( \frac{1}{5}, \frac{3}{15} \right]$ | $0.00957082229614258$ | $1.577787359487858$ |
The case where $k$ is a large class is given in the next table. Here, for each scenario, we present the values of $u$, $v$, and $w$, and the resulting upper bound on the resulting upper bound on the total weight of a bin as computed by our branch and bound procedure.

| Scenario Interval | $u$ | $v$ | $w$ | RBB |
|-------------------|-----|-----|-----|-----|
| $(\frac{1}{1067}, \frac{1}{1067})$ | 4848971 | 9837910 | 32015240 | $\approx 1.578159755311909$ |
| $1067 \leq 10600 \leq 10600$ | 4848971 | 9837910 | 32015240 | $\approx 1.57807870198628$ |
| $1601 \leq 16000 \leq 16000$ | 4848971 | 9837910 | 32015240 | $\approx 1.577997654973248$ |
| $3203 \leq 32000 \leq 32000$ | 4848971 | 9837910 | 32015240 | $\approx 1.577791654982266$ |
| $641 \leq 6400 \leq 6400$ | 4848971 | 9837910 | 32015240 | $\approx 1.57757313138089$ |
| $1603 \leq 16000 \leq 16000$ | 4848971 | 9837910 | 32015240 | $\approx 1.577349954711995$ |
| $3209 \leq 32000 \leq 32000$ | 4848971 | 9837910 | 32015240 | $\approx 1.57712995797064$ |
| $641 \leq 6400 \leq 6400$ | 4848971 | 9837910 | 32015240 | $\approx 1.57692202734197$ |
| $1603 \leq 16000 \leq 16000$ | 4848971 | 9837910 | 32015240 | $\approx 1.57671861489903$ |
| $3209 \leq 32000 \leq 32000$ | 4848971 | 9837910 | 32015240 | $\approx 1.57650566680742$ |
| $641 \leq 6400 \leq 6400$ | 4848971 | 9837910 | 32015240 | $\approx 1.57634301668013$ |

The values of $u$, $v$, and $w$, and the resulting upper bound on the resulting upper bound on the total weight of a bin as computed by our branch and bound procedure.
\begin{align*}
&\approx 1.5763752430999367 \\
&\approx 1.576294858216417 \\
&\approx 1.576129176498314 \\
&\approx 1.576317305885907 \\
&\approx 1.576050659846777 \\
&\approx 1.575964959021585 \\
&\approx 1.5758882476260425 \\
&\approx 1.575807024632376 \\
&\approx 1.575725910316502 \\
&\approx 1.5757529019598681 \\
&\approx 1.575285629938257 \\
&\approx 1.575204477606314 \\
&\approx 1.575123149518027 \\
&\approx 1.575123387677802 \\
&\approx 1.5750421414374224 \\
&\approx 1.5749609652076608 \\
&\approx 1.5748797699571104 \\
&\approx 1.5747985638451925 \\
&\approx 1.5747176460793634 \\
&\approx 1.5746362463633783 \\
&\approx 1.574550007655421 \\
&\approx 1.5744738854003466 \\
&\approx 1.574392624595173 \\
&\approx 1.5743114728654097 \\
&\approx 1.5742301987283957 \\
&\approx 1.574149033062958 \\
&\approx 1.574067848333426 \\
&\approx 1.573986541623945 \\
&\approx 1.573905343245792 \\
&\approx 1.5738241259453316 \\
&\approx 1.573742905818833 \\
&\approx 1.5736616741842933 \\
&\approx 1.573580432931712 \\
&\approx 1.5734991801710894 \\
\end{align*}
| 3257 | 543 | 1202323 | 4916119 | 3398327 | \(\approx 1.5734179085268438\) |
| 543 | 3259 | 3404215 | 4912237 | 6795959 | \(\approx 1.573315188674645\) |
| 1609 | 9660 | 1191304 | 838668 | 838668 | \(\approx 1.573238552090321\) |
| 3259 | 163 | 9615497 | 9885451 | 15391215 | \(\approx 1.5731525406725584\) |
| 163 | 1087 | 9614133 | 2471607 | 849407 | \(\approx 1.573071334467333\) |
| 1087 | 1631 | 9617029 | 4917032 | 1538809 | \(\approx 1.5729908771514997\) |
| 1631 | 3263 | 4805703 | 9888383 | 15389105 | \(\approx 1.572827420097824\) |
| 3263 | 17 | 4805921 | 9889861 | 6794201 | \(\approx 1.5727461782979537\) |
| 17 | 653 | 4804139 | 9415169 | 15387699 | \(\approx 1.5726647984777584\) |
| 653 | 1633 | 4803657 | 2472829 | 15386995 | \(\approx 1.57258354899548\) |
| 1633 | 1089 | 4803975 | 9892939 | 3396573 | \(\approx 1.572502252538697\) |
| 1089 | 3029 | 4802293 | 9898271 | 3396397 | \(\approx 1.572504368145120\) |
| 3029 | 1200 | 4801611 | 9894249 | 15384885 | \(\approx 1.573239663206862\) |
| 1200 | 3407 | 5590919 | 1493804 | 1077216 | \(\approx 1.572528236679823\) |
| 3407 | 1129 | 4099441 | 618665 | 6793053 | \(\approx 1.571933010550788\) |
| 1129 | 3200 | 4099029 | 9589225 | 15384181 | \(\approx 1.5718769184685635\) |
| 3200 | 109 | 4099775 | 494059 | 15383167 | \(\approx 1.571770268633972\) |
| 109 | 3271 | 4098761 | 4957622 | 15382216 | \(\approx 1.571688852570905\) |
| 3271 | 1637 | 4099035 | 9900117 | 15386663 | \(\approx 1.571607610195833\) |
| 1637 | 3808 | 331 | 5099921 | 1496405 | 15377017 | \(\approx 1.5705724662613\) |
| 3808 | 821 | 5096997 | 4959217 | 6755511 | \(\approx 1.570491178941144\) |
| 821 | 21900 | 5090479 | 4959217 | 6755511 | \(\approx 1.57032789590323\) |
| 21900 | 630 | 4784557 | 9919415 | 15386333 | \(\approx 1.569049586185092\) |
| 630 | 1643 | 5196241 | 1677216 | 15378497 | \(\approx 1.56804781619536\) |
| 1643 | 824 | 5193133 | 9923393 | 15386347 | \(\approx 1.56706343654269\) |
| 824 | 1097 | 5960015 | 2943281 | 15384081 | \(\approx 1.5665110722966\) |
| 1097 | 823 | 5951973 | 9924321 | 846673 | \(\approx 1.566019607729266\) |
Using this table, we completed the proof of the following theorem.

**Theorem 25** The competitive ratio of AH is at most 1.57828956.

C Several weight functions and examples

The weight functions are calculated easily using our definitions of weights, which are based on the parameters in the previous two sections and on our formulas for weights. Recall that weights of items in all scenarios are based directly on values of the form $a_{ij}$, $u$, $v$, and $w$, and are defined precisely in Section 4.3.2. Here, we provide three examples of such weight functions, while all 412 weight functions can be downloaded from [http://math.haifa.ac.il/lea/WeightFunctionsForBP.pdf](http://math.haifa.ac.il/lea/WeightFunctionsForBP.pdf).

We note that since the large classes in the interval $(1/3, 7/20]$ use common values of $a_{ij}$ (and $a_{ij}$), for all large classes $j$, the number of distinct weights in this interval is at most four. Thus, a table reporting the weight function of a given scenario is sufficiently short.

### C.1 Weight function for the scenario interval $(17/50, 653/1920]$

In this section we discuss the threshold class $(17/50 = 0.34, 653/1920 \approx 0.340104]$.

The large threshold classes that are close to $1/3$ are significant ones in the analysis. In the threshold class discussed here we have $a \approx 0.66$, that is, there can be positive bins whose single huge item has size below $2/3$, and there may be negative bins whose single item has size slightly above $0.34$. If one chooses parameters in such a way that $a_{1k}$ is too big, there may be a big fraction of bins that are relatively empty.

The weights table contains the weights assigned to classes in the case of a given threshold class, based on the $a_{ij}$ values and $u$, $v$, $w$ (if $u$ and $v$ are undefined, obviously weights are not based on them). The set $\Delta$ as it is defined above is based on these weights. Note that in the case where $u$ and $v$ are defined, the threshold class is split if necessary into $(t_k, 1-a)$ and $(1-a, t_k-1]$. While the value of $a$ is unknown, as explained above, we overcome this difficulty by considering bins with an item of size at least $a$ and other items of sizes no larger than $1-a$, and bins without such an item (in which case, the value $1-a$ can be only slightly larger than $x$, but it can also be $y$).

For solutions of the knapsack problem, recall that in the set $\Delta$, the size of every item is of the form $t_\ell$ or $x$, where its weight in $\Delta$ is the weight of an item that is just slightly larger (an item of the following interval in the table).

Consider solutions of the knapsack problems on $\Delta$. We provide several examples in order to allow the reader to see how total weights are computed and to provide some intuition to the knapsack problems.

Recall that tiny items are added to each solution such that the total size becomes 1. The first set of items from $\Delta$ is as follows:

$$\frac{17}{50}, \frac{17}{50}, \frac{1}{4}, \frac{1}{15}, \text{ (and tiny items of total size } \frac{1}{300}) \).$$

The total weight of these items is

$$\frac{4945169}{8388608} + \frac{1080920410736029}{3377699720527872} + \frac{569257302249594179}{8070450532247928832} + \frac{1936246260875168533}{198313198414926848} \approx 1.572827420097824.$$
The second set of items from $\Delta$ contains \( \frac{17}{100} \) an item of size \( a \), and tiny items of total size \( \frac{1}{100} \).

The total weight of these items is \( 1 + \frac{480439}{858608} + \frac{1932646206875616853}{198313194814926848} \cdot \frac{1}{100} \approx 1.572823543. \)

The third set of items from $\Delta$ is as follows:

\[
\begin{align*}
\frac{1}{2} & \quad 17 & \quad \frac{3}{50} & \quad \frac{3}{20} \\
& \quad (\text{and tiny items of total size } \frac{1}{100})
\end{align*}
\]

The total weight of these items is \( \frac{1357699}{10777216} + \frac{4945169}{858608} + \frac{2211215673518099}{1351098882111488} + \frac{1932646206875616853}{198313194814926848} \cdot \frac{1}{100} \approx 1.57282647.
\]

It can be seen that these are all knapsack solutions with total weight very close to the maximum.

| Interval | Weight |
|----------|--------|
| \((0, \frac{1}{33})\) | \(\\rho = \frac{1932646206875616853}{198313194814926848} \approx 0.9763577569423795\) |
| \(\frac{1}{33} \leq \frac{1}{32}\) | \(\approx 0.09763577569423795\) |
| \(\frac{1}{32} \leq \frac{1}{31}\) | \(\approx 0.083577569423795\) |
| \(\frac{1}{31} \leq \frac{1}{30}\) | \(\approx 0.07142857142857143\) |
| \(\frac{1}{30} \leq \frac{1}{29}\) | \(\approx 0.05769230769230769\) |
| \(\frac{1}{29} \leq \frac{1}{28}\) | \(\approx 0.04545454545454545\) |
| \(\frac{1}{28} \leq \frac{1}{27}\) | \(\approx 0.03571428571428571\) |
| \(\frac{1}{27} \leq \frac{1}{26}\) | \(\approx 0.02941176470588235\) |
| \(\frac{1}{26} \leq \frac{1}{25}\) | \(\approx 0.02564102564102564\) |
| \(\frac{1}{25} \leq \frac{1}{24}\) | \(\approx 0.02272727272727273\) |
| \(\frac{1}{24} \leq \frac{1}{23}\) | \(\approx 0.02083333333333333\) |
| \(\frac{1}{23} \leq \frac{1}{22}\) | \(\approx 0.01904761904761905\) |
| \(\frac{1}{22} \leq \frac{1}{21}\) | \(\approx 0.01764705882352941\) |
| \(\frac{1}{21} \leq \frac{1}{20}\) | \(\approx 0.0163125\) |
| \(\frac{1}{20} \leq \frac{1}{19}\) | \(\approx 0.0150625\) |
| \(\frac{1}{19} \leq \frac{1}{18}\) | \(\approx 0.01388888888888889\) |
| \(\frac{1}{18} \leq \frac{1}{17}\) | \(\approx 0.01285714285714286\) |
| \(\frac{1}{17} \leq \frac{1}{16}\) | \(\approx 0.011904761904761905\) |
| \(\frac{1}{16} \leq \frac{1}{15}\) | \(\approx 0.01105263157894737\) |
C.2 Weight function for the scenario interval $(3/7, 1/2]$

In this section we discuss the threshold class $(3/7 \approx 0.42857, 1/2 = 0.5)$.

A set of items from $\Delta$ resulting in a large weight is as follows:

$$\begin{align*}
\frac{1}{2}, \frac{1}{3}, \frac{1}{11}, \frac{1}{37}, \frac{1}{40}, \frac{1}{43} \quad \text{(and tiny items of total size } \frac{997}{2100120} \text{)}.
\end{align*}$$

The total weight of these items is

$$1 + \frac{56935901}{13928614} + \frac{150932657}{140938869} + \frac{66650937}{140928614} + \frac{20848039}{2100120} \approx 1.578279665.$$
Other sets of items give smaller bounds. For example, one could expect that at least one of \( \left\{ \frac{1}{7}, \frac{3}{7}, \frac{11}{14} \right\} \), and \( \left\{ \frac{1}{2}, \frac{1}{3}, \frac{15}{16} \right\} \) should be a subset of \( \Delta \) resulting in a large weight. However, the resulting knapsack solutions are

\[
1 + \frac{8388625}{16777216} + \frac{60903479239}{962072674304} + \frac{1209038869}{1409286144} \cdot \frac{1}{210} \approx 1.567391
\]

and

\[
1 + \frac{56035901}{144217728} + \frac{30171089}{2114486968} + \frac{1209038869}{1409286144} \cdot \frac{1}{105} \approx 1.576955
\]

One can apply arguments of linear programming to observe that as in this case \( u = v \) and \( w = 1 \), there is a single set giving the maximum in this scenario, unlike the previous case we have considered.

| Interval | \( \rho = \frac{1209038869}{1409286144} \approx 0.8579087179331553 \) |
|----------|------------------------------------------------------------------|
| \((0, \frac{1}{16}]\) | \(30206759 \) |
| \((\frac{1}{16}, \frac{1}{8}]\) | \(3020659 \) |
| \((\frac{1}{8}, \frac{1}{4}]\) | \(30206469 \) |
| \((\frac{1}{4}, \frac{1}{2}]\) | \(30206469 \) |
| \((\frac{1}{2}, \frac{3}{4}]\) | \(30206469 \) |
| \((\frac{3}{4}, 1]\) | \(30206469 \) |

\(1 + \frac{8388625}{16777216} + \frac{60903479239}{962072674304} + \frac{1209038869}{1409286144} \cdot \frac{1}{210} \approx 1.567391
\]

\(1 + \frac{56035901}{144217728} + \frac{30171089}{2114486968} + \frac{1209038869}{1409286144} \cdot \frac{1}{105} \approx 1.576955
\)
C.3 Weight function for the scenario interval (2/9, 3/13)

In this section we discuss the threshold class \( (2/9 \approx 0.2222, 3/13 \approx 0.23076923) \).

Consider the two multisets in \( \Delta \): \( \{ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{11} \} \) and \( \{ \frac{1}{4}, \frac{1}{5}, \frac{1}{11}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10} \} \). The resulting knapsack solutions are

\[
\begin{align*}
(\frac{1}{4}, \frac{1}{4}) & \approx 0.059409239888191225 \\
(\frac{1}{4}, \frac{1}{5}) & \approx 0.06330444764275203 \\
(\frac{1}{4}, \frac{1}{11}) & \approx 0.06738886752953896 \\
(\frac{1}{4}, \frac{1}{3}) & \approx 0.0732277308901151 \\
(\frac{1}{12}, \frac{1}{11}) & \approx 0.08197791535745967 \\
(\frac{1}{11}, \frac{1}{10}) & \approx 0.08960108160972595 \\
(\frac{1}{10}, \frac{1}{9}) & \approx 0.09772058410776986 \\
(\frac{7}{8}, \frac{5}{8}) & \approx 0.1083084437996149 \\
(\frac{1}{11}, \frac{1}{8}) & \approx 0.115772432 \\
(\frac{1}{7}, \frac{1}{5}) & \approx 0.1218674864087786 \\
(\frac{1}{8}, \frac{1}{7}) & \approx 0.12898306790365105 \\
(\frac{1}{7}, \frac{1}{6}) & \approx 0.134217728 \\
(\frac{1}{6}, \frac{1}{5}) & \approx 0.13844068348407745 \\
(\frac{5}{9}, \frac{4}{9}) & \approx 0.14050448732450604 \\
(\frac{3}{7}, \frac{2}{5}) & \approx 0.14640329281489053 \\
(\frac{1}{5}, \frac{1}{3}) & \approx 0.19713626932352782 \\
(\frac{3}{10}, \frac{1}{3}) & = 0.2 \\
(\frac{1}{5}, \frac{4}{5}) & \approx 0.2288978286087513 \\
(\frac{1}{4}, \frac{1}{7}) & \approx 0.23101308196783066 \\
(\frac{1}{4}, \frac{1}{6}) & \approx 0.30234069128831226 \\
(\frac{1}{3}, \frac{1}{5}) & \approx 0.3330659039784223 \\
(\frac{1}{3}, \frac{1}{4}) & \approx 0.41749999672174454 \\
(\frac{1}{2}, \frac{1}{5}) & \approx 0.43304505944252014 \\
(\frac{1}{2}, \frac{1}{4}) & \approx 0.4378199391067028 \\
(\frac{1}{2}, \frac{1}{3}) & \approx 0.4408503584563732 \\
(\frac{1}{2}, \frac{1}{2}) & \approx 0.5000010132789612 \\
\end{align*}
\]

\[
\begin{align*}
\left(\frac{1}{10}, \frac{1}{9}\right) & = 0.059409239888191225 \\
\left(\frac{1}{8}, \frac{1}{7}\right) & = 0.1083084437996149 \\
\left(\frac{1}{6}, \frac{1}{5}\right) & = 0.1218674864087786 \\
\left(\frac{1}{4}, \frac{1}{3}\right) & = 0.134217728 \\
\left(\frac{1}{3}, \frac{1}{2}\right) & = 0.2 \\
\left(\frac{1}{2}, \frac{1}{1}\right) & = 0.5
\end{align*}
\]

and

\[
\begin{align*}
3 \cdot \frac{39794075}{100663296} + 10232538759712973 & = \frac{25427139196783563}{180143985094819840} + \frac{1400445382739}{945631002624} \cdot \frac{1}{110} \approx 1.5772432 \\
4 \cdot \frac{39794075}{100663296} + 93992497 & = \frac{536870912}{180143985094819840} + \frac{1400445382739}{945631002624} \cdot \frac{1}{180} \approx 1.576226,
\end{align*}
\]

respectively. Note that these two multisets giving the worst case for this scenario have relatively small items.
## Interval Weight

| Interval | Weight |
|----------|--------|
| $(0, \frac{1}{10})$ | $\rho = \frac{1400445382739}{9560109224} \approx 1.4809639054271178$ |
| $(\frac{1}{10}, \frac{1}{9})$ | $15739215 \approx 0.0295052064073422$ |
| $(\frac{1}{9}, \frac{1}{8})$ | $91298099 \approx 0.0301753152698741$ |
| $(\frac{1}{8}, \frac{1}{7})$ | $1824013513 \approx 0.030886272019283328$ |
| $(\frac{1}{7}, \frac{1}{6})$ | $17696187 \approx 0.03113453302175882$ |
| $(\frac{1}{6}, \frac{1}{5})$ | $83470431 \approx 0.03260253239819878$ |
| $(\frac{1}{5}, \frac{1}{4})$ | $65967767 \approx 0.03301366915976679$ |
| $(\frac{1}{4}, \frac{1}{3})$ | $407682371 \approx 0.03370000163581635$ |
| $(\frac{1}{3}, \frac{1}{2})$ | $21192921290 \approx 0.0354216507400454$ |
| $(\frac{1}{2}, 1)$ | $248268817 \approx 0.03626954570120456$ |
| $(1, 2)$ | $969673 \approx 0.03734060730596895$ |
| $(2, 3)$ | $73333051 \approx 0.038460261861069336$ |
| $(3, 4)$ | $396923301 \approx 0.04445392530291311$ |
| $(4, 5)$ | $117694643 \approx 0.04456034818043311$ |
| $(5, 6)$ | $33882541 \approx 0.043545116252940275$ |
| $(6, 7)$ | $21196653 \approx 0.04504334713731935$ |
| $(7, 8)$ | $184669909 \approx 0.04666696528278331$ |
| $(8, 9)$ | $1377291 \approx 0.0483206178049906$ |
| $(9, 10)$ | $1824013513 \approx 0.0546521704488804$ |
| $(10, 11)$ | $11773433 \approx 0.057385442834911926$ |
| $(11, 12)$ | $117116234117278577 \approx 0.063489157681026$ |
| $(12, 13)$ | $9171943871524336559 \approx 0.06629494314281197$ |
| $(13, 14)$ | $574888 \approx 0.06849623918533325$ |
| $(14, 15)$ | $848959177 \approx 0.07028043079707358$ |
| $(15, 16)$ | $180925206 \approx 0.07929311298272189$ |
| $(16, 17)$ | $90208717 \approx 0.08401341456919909$ |
| $(17, 18)$ | $107341823 \approx 0.08662459030747413$ |
| $(18, 19)$ | $11626557 \approx 0.091137272920$ |
| $(19, 20)$ | $521783501019513497 \approx 0.10101642558806$ |
| $(20, 21)$ | $521783501019513497 \approx 0.10101642558806$ |
| $(21, 22)$ | $9483429 \approx 0.10870377490153679$ |
| $(22, 23)$ | $45761591 \approx 0.11365014066298802$ |
| $(23, 24)$ | $63517591 \approx 0.1147255590467742$ |
| $(24, 25)$ | $2542139561783863 \approx 0.14114897693308961$ |
| $(25, 26)$ | $4084635006536938908 \approx 0.15746027169341759$ |
| $(26, 27)$ | $93992497 \approx 0.1750746860115528$ |
| $(27, 28)$ | $114669903 \approx 0.195331283780535$ |
D A short discussion of the results of [11]

We had some difficulties in verifying the result of [11]. Here, we only address some issues in this last manuscript. Both in the conference proceedings version [10] and in the arxiv version published in September 2016 [11], the authors use a linear program and its dual (see page 27 of [11]). The dual variables $y_1, y_2$ should be non-negative (as the corresponding primal constraints are inequalities). However, instead of proceeding to solving the dual (or finding a feasible solution for it) the authors just fix those variables to values that can be negative in many of the cases. Going back to their primal linear program, these negative values mean that the authors assume (without providing a proof for it) that a pair of patterns (specified in advance) are critical in all scenarios. There is no clear reason as for why this should hold for their setting and we note that in our setting (which is related to their setting, as we also try to pack large items together with huge items as much as possible, and their critical patterns are those where an optimal solution packs such a pair of items together, leaving just a little gap for tiny items according to their definition of tiny). The corresponding patterns of our algorithm and its analysis are in fact not critical in many of the scenarios.

We do not see a simple way to fix this flaw. Setting these dual variables to zeroes instead of the resulting negative values would be incorrect either as the resulting solution possibly becomes infeasible for some scenarios. We believe that due to this, not all required calculations were done, a solution for the dual linear program was not found for all scenarios, and it is possible that the true competitive ratio is higher (maybe even higher than 1.58333, the barrier they claim to break). Moreover, there should be additional linear programs for other cases which are not presented in
the manuscripts (and do not appear in the accessible additional data that the first author provides on her web page), and we cannot tell if any problems of this kind occurs there as well.

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