LOCAL WELL-POSEDNESS FOR THE PERIODIC MKDV IN $H^{1/4^+}$

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ABSTRACT. We study the mKdV equation with periodic boundary conditions. We establish low regularity well-posedness in $H^{1/4^+}(T)$. The proof involves a non-linear, solution dependent gauge transformation, similar to the one considered in [5].

1. INTRODUCTION

The main result of this paper is invalidated, due to the failure of the estimate (16) below. Consider the real-valued modified Korteweg-de Vries equation with periodic boundary condition

\[
\begin{align*}
\frac{\partial u}{\partial t} + u_{xxx} + u^2 \partial_x u &= 0, \\
u(0) &= f \in H^s(T).
\end{align*}
\]

Note that if $f$ is real-valued, then

\[
f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{-2\pi i k x}, \ \ \ \ \hat{f}(k) = \hat{f}(-k).
\]

Even though there were quite a few results dealing with the well-posedness of this model with standard energy methods, it was Bourgain, who has initiated in [1], the study of the well-posedness of such models at low regularity. The main new technical idea was the introduction of adapted to the evolution function spaces (coined $X^{s,b}$ spaces), which are more sensitive than the standard energy spaces for the problems at consideration. We should mention that in the case of the problem on $\mathbb{R}^1$, better results are achieved by using the local smoothing estimates associated with the Airy equation, as shown in [3].

The problem for obtaining local well-posedness in spaces with less and less Sobolev regularity has received lots of attention by many authors in the last twenty years. Since Bourgain has showed his basic trilinear estimate (which coupled with his method gives the local well-posedness in $H^{1/2}(T)$), it was shown by Kenig-Ponce-Vega, [4] that this estimate actually fails in $H^s(T), s < 1/2$. In fact, not only this estimate fails, but the solution map was shown to be not uniformly continuous when $f \in H^s(T), s < 1/2$, [2].

However, this does not necessarily mean that the local well-posedness fails. Takaoka-Tsutsumi, [6] have considered the problem in $H^s, s > 3/8$ and they have shown the local well-posedness, by using an iteration argument in $X^{s,b}$ type spaces, which depends on
the initial data. This results were further extended in the work of Nakanishi-Takaoka-Tsutsumi, [5], where the authors have been able to push the l.w.p. results to $H^{1/3+}(T)$. Note that the authors have been able to provide existence results in $H^{1/4+}$, under some additional restrictions on the growth of the Fourier coefficients of the data. The main goal of this paper is to consider data in $H^{4+}(T)$ and to show local well-posedness.

We start with some standard reductions. For nice solutions $u$ of (1), we have conservation of $L^2$ norm. By changing the spatial variable $x$ to $x + ct$ where $c = \frac{1}{2\pi}\|u_0\|_{L^2}^2$, we have

$$\begin{align*}
\partial_t u + \partial_x^2 u + (u^2 - \frac{1}{2\pi} \int_T u^2(t, x) dx)\partial_x u &= 0 \\
u(0) &= f.
\end{align*}$$

This is the equation that we consider from now on. On the Fourier side, the equation is

$$\partial_t \hat{u}(t, k) - ik^3 \hat{u}(t, k) = -i\frac{k}{3} \sum_{k_1 + k_2 + k_3 = k, \ k_j, k \neq 0} \hat{u}(k_1) \hat{u}(k_2) \hat{u}(k_3) + ik|\hat{u}(k)|^2 \hat{u}(k).$$

The first term is called non-resonant, while the other term is referred to as resonant. The non-resonant trilinear term $\mathcal{N}\mathcal{R}$ is introduced to be

$$\mathcal{N}\mathcal{R}(v_1, v_2, v_3)(k) := -i\frac{k}{3} \sum_{k_1 + k_2 + k_3 = k, \ k_j, k \neq 0} \hat{v}_1(k_1) \hat{v}_2(k_2) \hat{v}_3(k_3)$$

where

$$\mathcal{N}\mathcal{R}(h) := \mathcal{N}\mathcal{R}(h, h, h).$$

1.1. **Change of variables.** We start with a general discussion about the change of variables that is required. Basically, one needs to hide the resonant term $ik|\hat{u}(k)|^2 \hat{u}(k)$. To that end, introduce the change of variables,

$$\hat{u}(t, k) := \hat{v}(t, k) + \hat{f}(k) e^{i(tk^3 + \int_0^t |\hat{u}(s, k)|^2 ds)}.$$

Denote for convenience $P(t, k) := tk^3 + k\int_0^t |\hat{u}(s, k)|^2 ds$. This would transform the equation into a new one for $v$, in the form

$$\begin{align*}
\partial_t \hat{v}(k) - ik^3 + k|\hat{f}(k)|^2 \hat{v}(k) &= +ik|\hat{v}(t, k)|^2 \hat{v}(t, k) + 2ik\Re(\hat{f}(k) e^{iP(t, k)} \hat{v}(t, k)) \hat{v}(k) + \\
\mathcal{N}\mathcal{R}(\otimes_{j=1}^3 \hat{f}(k_j) e^{iP(t, k_j)} + \hat{v}(k_j))
\end{align*}$$

$$v(0, k) = 0$$

The disadvantage of this equation for $v$ is that the old variable $u$ is still present inside at the phase function $P$. Nevertheless, for uniqueness purposes, it is good to consider exactly (3).

For existence results however, we seek to introduce a new variable $z$, so that the phase variable (denoted $Q$ below) is dependent only upon the new variable $z$ and which does not contain a reference to the old one $u$. We need the following

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1For more details about this derivation, the reader may consult [5], p. 1639.
Lemma 1. Let \( f \in H^{s_0}(T) \), \( s_0 > 0 \). Let \( \{ \tilde{z}(t, k) \} \) be given continuous functions, defined on an interval \([0, T]\). Assuming that there exists \( C \), so that
\[
\sup_{0 < t < T} \sup_{k} < k >^{1-s_0} |\tilde{z}(t, k)| \leq C.
\]
then for the infinite system of (non-linear) ODE’s
\[
Q'(z; t, k) = k^3 + k|\hat{f}(k)|e^{ijQ(z; t, k)} + \tilde{z}(t, k), \quad Q(z; 0, k) = 0, \quad k \in \mathbb{Z}
\]
there exists a time interval \([0, T_0]\), \( T_0 \geq \min(T; \frac{1}{100\|f\|_{H^{s_0}}}) \), so that it has unique solution \( \{Q(z; k, t)\}_{k \in \mathbb{Z}} : [0, T_0] \to \mathbb{R}^1 \). In particular, the condition (4) is satisfied if \( z = \sum_k \tilde{z}(t, k)e^{ikx} \in L_t^\infty H^{1-s_0} \).

Remark: For the most part, we will suppress the dependence of \( Q \) on \( z \) in our notations.

Proof. The existence argument is easy and it can be justified, based on the theory of non-linear ODE with Lipschitz right hand sides. The non-trivial part of the statement is the common interval of existence, which is independent of \( k \).

To that end, rewrite the system of ODE’s as equivalent system of integral equations
\[
Q(t, k) = t(k^3 + k|\hat{f}(k)|^2) + k \int_{0}^{t} (2\Re(\hat{f}(k)e^{iQ(s, k)}\tilde{z}(s, k)) + |\tilde{z}(s, k)|^2) ds
\]
In order to check that the fixed point argument produces a solution in an interval \([0, T_0]\), we need to check the contractivity of \( Q \to \Sigma(Q) := k \int_{0}^{t} (2\Re(\hat{f}(k)e^{iQ(s, k)}\tilde{z}(s, k)) ds \).

Indeed,
\[
\sup_{0 < t < T_0} |\Sigma(Q_1) - \Sigma(Q_2)(t)| \leq 10T_0|k||\hat{f}(k)|| \sup_{0 < s < T_0} |Q_1(s) - Q_2(s)| \sup_{0 < s < T_0} |\tilde{z}(s, k)| < \leq 10\|f\|_{H^{s_0}}C_0T_0 \sup_{0 < s < T_0} \|Q_1(s) - Q_2(s)\|,
\]
since
\[
|k||\hat{f}(k)|| \sup_{0 < \tau < T_0} |\tilde{z}(\tau, k)| \leq C\|f\|_{H^{s_0}} \sup_{k, \tau} < k >^{1-s_0} |\tilde{z}(\tau, k)| \leq C.
\]
It follows that \( \Sigma \) is a contraction, whenever \( T_0 < 1/(20C_0\|f\|_{H^{s_0}}) \), \( T_0 < T \) and the lemma is proved.

We now continue with the precise definition of the transformation. In the new variable \( z : [0, T] \to C \), let \( Q = Q_z \) as in Lemma 1. That is, let \( Q \) be the solution of (5). Clearly, \( z \) needs to be in \( H^{1-s_0} \), which will be established a-posteriori. Set
\[
\hat{u}(t, k) := \tilde{z}(t, k) + \hat{f}(k)e^{iQ(t, k)}.
\]
Note \( \hat{z}(0, k) = 0 \), since \( \hat{u}(0, k) = \hat{f}(k) \), \( Q(0, k) = 0 \). In terms of \( z \), the equation equivalent to the original equation (2)
\[
\partial_t \hat{z}(k) + i(k^3 + k|\hat{f}(k)|^2)\hat{z}(k) = ik|\hat{z}(t, k)|^2\hat{z}(t, k) +
\]
\[
+2ik\Re(\hat{f}(k)e^{iQ(t, k)}\tilde{z}(t, k))\hat{z}(k) +
\]
\[
+\mathcal{N}\mathcal{R}(\otimes_{j=1}^{3} \hat{f}(k_j)e^{iQ(t, k_j)} + \hat{z}(k_j))
\]
\( z(0, k) = 0 \)
We are now ready to give the definition of local existence that we will be working with.
For the remainder of this paper we will tacitly assume that torus uitous Bourgain spaces, initially defined for the pure KdV evolution for functions on the tion spaces, in which the solutions will live. Naturally, these will be versions of the ubiq-
equation, with initial data

\[
\hat{z}(t, k) = \int_0^t e^{i(t-s)((k^3 + k|\hat{f}(k)|^2)[ik|\hat{z}(s, k)|^2\hat{z}(s, k) + 2ik\text{Re}(\hat{f}(k)e^{iQ(s,k)z(t,k)})\hat{z}(k)]}ds + 
+ \int_0^t e^{i(t-s)((k^3 + k|\hat{f}(k)|^2)[N \text{Re}(\otimes_{j=1}^3 \hat{f}(k_j)e^{iQ(t,k_j)} + \hat{z}(k_j))]ds.
\]

1.2. Function spaces. Since we study a local well-posedness question, we introduce function spaces, in which the solutions will live. Naturally, these will be versions of the ubiquitous Bourgain spaces, initially defined for the pure KdV evolution for functions on the torus \( z : \mathbb{R}^1 \times T \rightarrow C, z(t, x) = \sum_k z_k(t)e^{ikx} \)

\[
\|z\|^2_{X^{s,b}} = \sum_k \int_{\mathbb{R}^1} < \tau - k^3 > 2b < k > 2s |\hat{z}(\tau, k)|^2d\tau.
\]

In addition, we introduce the modified Bourgain space \( Y^{s,b} \) as follows

\[
\|z\|^2_{Y^{s,b}} = \sum_k \int_{\mathbb{R}^1} < \tau - k^3 - k|\hat{f}(k)|^2 > 2b < k > 2s |\hat{z}(\tau, k)|^2d\tau.
\]

It will also be convenient to use the local version of these spaces, namely for any \( T > 0 \), define (for any \( \Lambda = X^{s,b}, Y^{s,b} \))

\[
\|v\|_{\Lambda_T} = \inf\{\|u\|_{\Lambda}, u \in \Lambda, u = v \text{ on } (-T, T)\}
\]

For the remainder of this paper we will tacitly assume that \( T < 1 \).

1.3. Main result. The following is the main result of this work.

Theorem 1. Let \( s_0 > \frac{1}{2} \) and \( 0 < \delta << s_0 - \frac{1}{2} \), \( f \in H^{s_0}(\mathbb{T}) \). Then, there exists a solution in the sense of Definition\[7\] In addition, we have the following smoothing effects:

\[
\sum_k \left[ \hat{u}(t, k) - \hat{f}(k)e^{i(k^3 + k|\hat{f}(k)|^2)(s)ds} \right] e^{ikx} \in L^\infty_t H^{3s_0-},
\]

\[
\sum_k |k||\hat{u}(t, k)|^2 - |\hat{f}(k)|^2 | < \infty.
\]

Assuming that \( u \in L^2(\mathbb{T}) \) obeys

\[
\sup_k |k||\hat{u}(t, k)|^2 - |\hat{f}(k)|^2 | < \infty,
\]

the equation\[5\] has an unique solution \( v \), which is in \( Y^{s_0,b} \cap L^\infty H^{3s_0-} \).

The uniqueness holds in the following sense - let \( v_1, v_2 \) be the two solutions of \(5\), corresponding to \( u_1, u_2 \in L^\infty T H^{s_0}(\mathbb{T}) \) and satisfying\[9\], with \( u_j(0) = f \), then there exists \( T > 0 \), so that \( v_1|_{[0,T]} = v_2|_{[0,T]} \).
Remark: We can upgrade (8) to
\begin{align}
\sum_k |k|^{\min(4s_0,1+s_0)} ||\hat{u}(t,k)||^2 - ||\hat{f}(k)||^2 < \infty.
\end{align}

One should compare the smoothing condition (10) to the smoothing condition (9), which was proved in [5], under the assumption \( s_0 > 1/3 \).

Let us outline the plan for the paper. In Section 2, we give some preliminary estimates, including an adaptation of the trilinear Bourgain estimate for the non-resonant terms. In Section 3, we give the main estimates in this work, which quantify the smoothing of the non-resonant terms as well as the contribution of the resonant terms. In Section 4, we put together the estimates from Section 3, to justify an iteration argument, which provides the existence of the solution \( z \) of (7) (and hence of \( u \)). Then, we show that the equation (3) has unique solution, for fixed \( u \). This is however not enough for uniqueness, but shows that the correspondence \( u \to v \) is well and uniquely defined. Finally, for uniqueness, we show that if two solutions \( u_1, u_2 \), with common initial data \( f \) produce \( v_1, v_2 \), then \( v_1 = v_2 \) in some eventually smaller time interval and hence \( u_1 = u_2 \).

2. Preliminary Estimates

We have the following linear estimate.

Lemma 2. Let \( z \) solves the following equation
\[ \partial_t z_k(t) - i(k^3 + k|\hat{f}(k)|^2)z_k(t) = F_k(t). \]
in the sense that
\[ z_k(t) = e^{it(k^3 + k|\hat{f}(k)|^2)}z_k(0) + \int_0^t e^{i(t-s)(k^3 + k|\hat{f}(k)|^2)}F_k(s)ds. \]
Then for every \( \delta > 0 \),
\[ \|z\|_{Y^{s,b}} \leq C_\delta T^\delta (\|z(0,x)\|_{H^s(T)} + \|F\|_{Y^{s,b-1+\delta}}). \]

We now state a straightforward extension of a well-known estimate by Bourgain, which will be crucial for our approach in the sequel. More precisely, it was proved that
\begin{align}
\|\mathcal{N}\mathcal{R}(u_1,u_2,u_3)\|_{X^{s,-1/2}} \leq C\|u_1\|_{X^{s,1/2}}\|u_2\|_{X^{s,1/2}}\|u_3\|_{X^{s,1/2}}
\end{align}
whenever \( s > 1/4 \). Similar estimate, with \( X^{s,b} \) replaced by \( Y^{s,b} \), was established by [5], see Lemma 2.2, p. 3017. We need a variant of (12), namely

Lemma 3. Let \( s > 1/4, b > 1/2 \) and \( 0 < \delta << s - 1/4 \). Then, there exists a constant \( C = C_\delta \), so that
\begin{align}
\|\mathcal{N}\mathcal{R}(u_1,u_2,u_3)\|_{Y^{s,b-1+\delta}} \leq C_{b,\delta,s}\|u_1\|_{Y^{s,b}}\|u_2\|_{Y^{s,b}}\|u_3\|_{Y^{s,b}}.
\end{align}

Proof. In the proof of (11), the crux of the matter is the resonant identity
\begin{align}
(\tau_1 + \tau_2 + \tau_3) - (k_1 + k_2 + k_3)^3 = \sum_{j=1}^3 (\tau_j - k_j^3) - 3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1).
\end{align}

\[ 2 \text{~ although not explicitly stated, see the remarks (b) after Proposition 8.37} \]
which guarantees that
\[ \max(\tau - k^3, \tau_1 - k^3, \tau_2 - k^3, \tau_3 - k^3) \geq |(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)|. \]

The corresponding ingredient needed for the proof of (12), is
\[ \max(\tau - k^3 - k|\hat{f}(k)|^2, \tau_j - k_j^3 - k_j|\hat{f}(k_j)|^2, j = 1, 2, 3) \geq |(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)|. \]
This is however satisfied by an identity similar to (13), since for \( k_1, k_2, k_3 : (k_1 + k_2)(k_2 + k_3)(k_3 + k_1) \neq 0, \)
\[ |(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)| \geq k_{\max} >> O(k_{\max}^1 - 2) = |k_j||\hat{f}(k_j)|^2 \]
Thus, (12) is established.

We now state a lemma, which allows us to place the terms like \( \sum_k \hat{f}(k)e^{iQ(t,k)}e^{ikx} \) in the space \( Y_{s_0, t}^{\frac{1}{2}} \).

**Lemma 4.** Let \( b \leq 1, \ z \in H^{1-s_0}(T) \) and let \( \{Q(k, t)\}_k \) be the family guaranteed to exist on \([0, T_0]\) by Lemma 7. Then
\[ \| \sum_k \hat{f}(k)e^{iQ(t,k)}e^{ikx} \|_{Y_{s_0, t}^{\frac{1}{2}}} \leq C\sqrt{T_0}(1 + \|z\|_{H^{1-s_0}}\|f\|_{H^{s_0}(T)})\|f\|_{H^{s_0}(T)}. \]

**Proof.** From the integral equation (6), we have
\[ \hat{f}(k)e^{iQ(t,k)} = \hat{f}(k)e^{it(k^3 + k|\hat{f}(k)|^2)}g(t, k), \]
where
\[ g(t, k) = \exp(i(k \int_0^t (2\Re(\hat{f}(k)e^{iQ(s,k)}z(s,k)) + |\hat{z}(s,k)|^2)ds) \]
Note \( |g(t, k)| = 1 \). Denote for conciseness \( \phi_k = k^3 + k|\hat{f}(k)|^2 \), so that
\[ \hat{f}(k)e^{iQ(t,k)} = e^{it\phi_k} \hat{f}(k)g(t, k) = \exp(it\phi_k)\hat{h}(t, k). \]
Taking Fourier transform in \( t \), we have
\[ \hat{f}(k)e^{iQ(-,k)}(\tau) = \hat{h}(\tau - \phi_k, k). \]
Thus,
\[ \| \sum_k \hat{f}(k)e^{iQ(t,k)}e^{ikx} \|^2_{Y_{s_0, t}^{\frac{1}{2}}} = \sum_k < k >^{2s_0} \int < \tau - \phi_k >^{2b} |\hat{f}(k)e^{iQ(-,k)}(\tau)|^2d\tau = \]
\[ = \sum_k < k >^{2s_0} \int < \tau - \phi_k >^{2b} |\hat{h}(\tau - \phi_k, k)|^2d\tau = \]
\[ = \|h\|^2_{H^{\frac{1}{2}}(0, T_0)H_2^1}. \]
We have
\[ (14) \|h\|^2_{H^{\frac{1}{2}}(0, T_0)H_2^1} \leq \|h\|^2_{H^1(0, T_0)H_{s_0}^0} \leq \sum_k < k >^{2s_0} \hat{f}(k)^2(\int_0^{T_0}(1 + |g'(t, k)|^2)^2dt). \]
It is therefore, enough to show \( \sup_k |g'(t, k)| \leq C \). But,
\[ |g'(t, k)| \leq |k||\hat{z}(t, k)||\hat{f}(k)| + |\hat{z}(t, k)|) \leq \]
\[ \leq |k| < k >^{s_0 - 1} \|z\|_{H^{1-s_0}} < k >^{-s_0} \|f\|_{H^{s_0}} \leq C\|z\|_{H^{1-s_0}}\|f\|_{H^{s_0}}, \]
whence we obtain the desired estimate.

3. Estimates for the Nonlinear Terms

Let $\frac{1}{2} < b$ be fixed, and define the solution space $\mathcal{X} = Y^{s_0, b} \cap L_t^\infty H_x^{s_1}$, where $\frac{1}{4} < s_0 < \frac{1}{2}$ and $\frac{1}{2} < 1 - s_0 < s_1 < \min(1, 3s_0)$. That is

$$\| \cdot \|_\mathcal{X} := \| \cdot \|_{Y^{s_0, b}} + \| \cdot \|_{L_t^\infty H_x^{s_1}}.$$ 

Note that the assumption $s_0 > 1/4$ is used in a crucial way to ensure that such $s_1$ exists. On the other hand $\mathcal{X} \hookrightarrow L_t^\infty H^{1-s_0}$, which is used in Lemma 1 to justify the existence of the generalized phase function $Q_z$.

We state several lemmas. Lemma 5 allows us to estimate the contribution of all non-resonant terms, i.e. all terms appearing out of the trilinear term $N R$. The second lemma, Lemma 6 estimates the contribution of the non-resonant terms.

3.1. Estimates of the non-resonant contributions.

Lemma 5. Let $\frac{1}{4} < s_0 < \frac{1}{2}$. Take $\delta : 0 < \delta < s_0 - 1/4$, $b = \frac{1}{2} + 2\delta$ and $\frac{1}{2} < 1 - s_0 < s_1 < \min(1, 3s_0)$. For the solution to

$$\begin{align*}
\partial_t \bar{U}(k) - i(k^3 + k|\hat{f}(k)|^2)\hat{U}(k) &= N\mathcal{R}(u_1, u_2, u_3)(k), \\
U(0, k) &= 0
\end{align*}$$

(15)

$$\|U\|_{Y^{s_0, b} \cap L^{\infty}_t H^{s_1}} \leq C T^\delta \|u_1\|_{Y^{s_0, b}} \|u_2\|_{Y^{s_0, b}} \|u_3\|_{Y^{s_0, b}}$$

(16)

Proof. The first estimate (15) is nothing but a combination of Lemma 2 and Lemma 3. We have

$$\|U\|_{Y^{s_0, b}} \leq C_T \|N\mathcal{R}(u_1, u_2, u_3)\|_{Y^{s_0, b} \cap L^{\infty}_t H^{s_1}} \leq C_T \|u_1\|_{Y^{s_0, b}} \|u_2\|_{Y^{s_0, b}} \|u_3\|_{Y^{s_0, b}}.$$ 

We now take on the estimates in $L^\infty H^{s_1}$. We will show (16) by reducing to the case when $u_1, u_2, u_3$ are free solutions in the corresponding evolutions. This is done through the well-known method of averaging (valid for general dispersion relations), which we now describe. Let $\mu(k)$ be a real-valued symbol, so that

$$X_{\mu}^{s,b} = \{ f : T \times \mathbb{R} \to \mathcal{C} : \|u\|_{X_{\mu}^{s,b}}^2 := \sum_k \int \tau - \mu(k) > 2b \|\hat{u}(\tau, k)|^2 d\tau < \infty \}$$

Write

$$u(t, x) = \int e^{i\lambda t} u_\lambda(t, x) d\lambda,$$

where $\hat{u}_\lambda(\tau, k) = \delta(\tau - \mu(k))\hat{u}(\tau + \lambda, k)$. Clearly, $\hat{u}_\lambda(t, k) = e^{i\lambda t} \hat{u}(\lambda + \mu(k), k)$, that is $u_\lambda(t, x)$ is a free solution of the equation

$$(\partial_t - i\mu(-i\partial_x))u_\lambda(t, x) = 0, u_\lambda(0, x) = \sum_k \hat{u}(\lambda + \mu(k), k) e^{ikx}.$$
Suppose that we can prove estimates for (16), where \( u_j = \sum_k e^{it\mu(k)} \hat{f}_j(k) e^{ikx} \), \( j = 1, 2, 3 \) are free solutions, for \( \mu(k) = k^3 + k|\hat{f}(k)|^2 \).

We will provide later an almost explicit solution of (16), a trilinear form
\[
\mathcal{M}(f_1, f_2, f_3)(t, x) = \sum_k \mathcal{M}(f_1, f_2, f_3)(t, k) e^{ikx}.
\]
That is, we will construct
\[
\left| (\partial_t - i(k^3 + k|\hat{f}(k)|^2)) \mathcal{M}(f_1, f_2, f_3)(t, k) = \mathcal{N}R(\otimes_{j=1}^3 e^{it\mu(k_j)} \hat{f}_j(k_j)), \right| k = k_1 + k_2 + k_3
\]
Assume for the moment the validity of
\[
\mathcal{M}(f_1, f_2, f_3)(t, x) = \sum_k \mathcal{M}(f_1, f_2, f_3)(t, k) e^{ikx}.
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\]
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\]
We show that (16) follows. Indeed, employing the representation (17) for each of \( u_j, j = 1, 2, 3 \), we have that the solution \( U \) of (16) will take the form
\[
U(t, x) = \int e^{it\lambda_1 + \lambda_2 + \lambda_3} \mathcal{M}(u_{\lambda_1}(0), u_{\lambda_2}(0), u_{\lambda_3}(0)) d\lambda_1 d\lambda_2 d\lambda_3.
\]
Taking \( L^\infty_t H^{s_1}_x \) norms and applying (18) yields the bound
\[
\|U\|_{L^\infty_t H^{s_1}_x} \leq \int \|\mathcal{M}(u_{\lambda_1}(0), u_{\lambda_2}(0), u_{\lambda_3}(0))\|_{L^\infty_t H^{s_1}_x} d\lambda_1 d\lambda_2 d\lambda_3 \leq C \int \|u_{\lambda_1}\|_{H^{s_0}} d\lambda_1 \int \|u_{\lambda_2}\|_{H^{s_0}} d\lambda_2 \int \|u_{\lambda_3}\|_{H^{s_0}} d\lambda_3.
\]
But
\[
\int \|u_{\lambda}\|_{H^{s_0}} d\lambda \leq \left( \int < \lambda >^{1+2\delta} \|u_{\lambda}\|^2_{H^{s_0}} d\lambda \right)^{1/2} \left( \int < \lambda >^{-1-2\delta} d\lambda \right)^{1/2} \leq C_\delta \sum_k < k >^{2s_0} \int < \lambda >^{1+2\delta} |\hat{u}(\lambda + \mu(k), k)|^2 d\lambda \right)^{1/2} = C_\delta \|u\|_{X^{s_0}+_{T, \mu}}.
\]
Since \( \|u\|_{X^{s_0}+_{T, \mu}} \leq C_\delta T^\delta \|u\|_{X^{s_0}+_{T, \mu}}, \) we have reduced matters to the construction of the trilinear form \( \mathcal{M} \) and the proof of (18).

3.1.1. Proof of (18). Introduce a notation for the free solutions
\[
R[g](t, x) := \sum_k e^{it(k^3 + k|\hat{f}(k)|^2)} \hat{g}(k) e^{ikx}.
\]
Note the algebraic identity
\[
\tau - k^3 - k|\hat{f}(k)|^2 = \sum_{j=1}^3 (\tau_j - k_j^3 - k_j|\hat{f}(k_j)|^2) - 3(k_1 + k_2 + k_3)(k_3 + k_1)
\]
\[
+ \left( \sum_{j=1}^3 k_j|\hat{f}(k_j)|^2 \right) - k|\hat{f}(k)|^2
\]
for \( \tau = \tau_1 + \tau_2 + \tau_3, k = k_1 + k_2 + k_3 \). Denote 
\[ k_{\text{max}} := \max(|k_1|, |k_2|, |k_3|) \quad \text{and} \quad k_{\text{min}} := \min(|k_1|, |k_2|, |k_3|), \]
\[ E(k_1, k_2, k_3) = k_1|\hat{f}(k_1)|^2 + k_2|\hat{f}(k_2)|^2 + k_3|\hat{f}(k_1)|^2 - k|\hat{f}(k)|^2. \]
Notice that if \( f \in H^{s_0}(T) \),
\[ |(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)| \gtrsim k_{\text{max}}, \quad |E(k_1, k_2, k_3)| \leq C\|f\|^2_{H^{s_0}(T)} \leq k_{\text{max}} \]
Thus, there exists \( K_0 = K_0(\|f\|_{H^{s_0}(T)}) \) so that for all \( k_{\text{max}} > K_0 \), we have that
\[ | - 3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1) + E(k_1, k_2, k_3) | \gtrsim k_{\text{max}} > 1. \]
This allows us to define the function 
\[ \hat{h}(t, x) = \sum_k \hat{h}(t, k) e^{ikx} \]
\[ \hat{h}(t, k) = -\frac{i}{3} \sum_{k = k_1 + k_2 + k_3 \neq 0, k_{\text{max}} > K_0 \atop (k_1 + k_2)(k_2 + k_3)(k_3 + k_1) \neq 0} \frac{(k_1 + k_2 + k_3)R[f_1](k_1)R[f_2](k_2)R[f_3](k_3)}{-3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1) + E(k_1, k_2, k_3)}, \]
since the denominator is guaranteed to stay away from zero.
From the algebraic identity displayed above, we see that \( h \) satisfies
\[ (\partial_t - i(k^3 + k|\hat{f}(k)|^2))\hat{h}(t, k) = \mathcal{N}\mathcal{R}^{>K_0}(R[f_1], R[f_2], R[f_3])(k), \]
and
\[ \hat{h}(0, k) = -\frac{i}{3} \sum_{k = k_1 + k_2 + k_3 \neq 0, k_{\text{max}} > K_0 \atop (k_1 + k_2)(k_2 + k_3)(k_3 + k_1) \neq 0} \frac{(k_1 + k_2 + k_3)\hat{f}_1(k_1)\hat{f}_2(k_2)\hat{f}_3(k_3)}{-3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1) + E(k_1, k_2, k_3)}, \]
where we have used the notation
\[ \mathcal{N}\mathcal{R}^{\leq K_0}(v_1, v_2, v_3) := -\frac{i}{3} \sum_{k_1 + k_2 + k_3 = k, \ k_j, k \neq 0 \atop (k_1 + k_2)(k_2 + k_3)(k_3 + k_1) \neq 0} \hat{v}_1(k_1)\hat{v}_2(k_2)\hat{v}_3(k_3), \]
\[ \mathcal{N}\mathcal{R}^{> K_0}(v_1, v_2, v_3) := \mathcal{N}\mathcal{R}(v_1, v_2, v_3)(k) - \mathcal{N}\mathcal{R}^{\leq K_0}(v_1, v_2, v_3) \]
Note that \( h \) is a trilinear form acting on \( f_1, f_2, f_3 \). The construction of the \( h \) provides the major step toward the construction of the \( \mathcal{M} \), for which we need to establish the estimate (18). In fact, we can quickly describe the remaining pieces of \( \mathcal{M} \). Let \( h_1 = \sum_k \hat{h}_1(t, k) e^{ikx} = h_1(f_1, f_2, f_3) \) satisfies
\[ |(\partial_t - i(k^3 + k|\hat{f}(k)|^2))\hat{h}_1(t, k) = \mathcal{N}\mathcal{R}^{\leq K_0}(R[f_1], R[f_2], R[f_3])(t, k), \]
\[ h_1(0, k) = 0 \]
That is
\[ \hat{h}_1(t, k) = \int_0^t e^{i(t-s)(k^3 + k|f(k)|^2)}\mathcal{N}\mathcal{R}^{\leq K_0}(R[f_1], R[f_2], R[f_3])(s, k) ds \]
Finally, let \( h_2 = h_2(f_1, f_2, f_3) \) solves
\[
\begin{align*}
\left( \partial_t - i(k^3 + k|\hat{f}(k)|^2) \right) h_2(t, k) &= 0, \\
h_1(0, k) &= -h(0, k).
\end{align*}
\]
That is
\[
h_2(t, k) = -e^{it(k^3 + k|\hat{f}(k)|^2)} \hat{h}(0, k).
\]
Clearly,
\[
\mathcal{M}(f_1, f_2, f_3) = h(f_1, f_2, f_3) + h_1(f_1, f_2, f_3) + h_2(f_1, f_2, f_3).
\]
We claim that the required estimate (18) follows from
\[
\text{Thus, by Lemma 2,}
\]
\[
\| \hat{h}(f_1, f_2, f_3) \|_{L^\infty H^{s_1}} \leq C \prod_{j=1}^3 \| f_j \|_{H^{s_0}}.
\]
Indeed, assuming (19), we have in particular
\[
\| h(0, \cdot) \|_{H^{s_1}} \leq \| h(f_1, f_2, f_3)(t, \cdot) \|_{L^\infty H^{s_1}} \leq C \prod_{j=1}^3 \| f_j \|_{H^{s_0}}.
\]
Thus, by Lemma 2
\[
\| h_2(t, \cdot) \|_{L^\infty H^{s_1}} \leq \| h_2(t, \cdot) \|_{Y^{s_1, b}} \leq C \| h(0, \cdot) \|_{H^{s_1}} \leq C \prod_{j=1}^3 \| f_j \|_{H^{s_0}}.
\]
Regarding \( h_1 \), we have by energy estimates
\[
\| h_1(t, \cdot) \|_{L^\infty H^{s_1}} \leq C \| \mathcal{R} \|_{L^1 H^{s_1}}
\]
But, by Hölder's and Sobolev embedding
\[
\| \mathcal{R} \|_{L^1 H^{s_1}} \leq C T \left( \sum_{|k| \leq 3 K_0} < k >^{2s_1} \left( \sum_{k = k_1 + k_2 + k_3} \| \hat{f}_1(k_1) \| \| \hat{f}_2(k_2) \| \| \hat{f}_3(k_3) \| \right)^2 \right)^{1/2}
\]
\[
\leq C K_0^{s_1} T \| (\hat{f}_1)_{K_0} \| \| (\hat{f}_2)_{K_0} \| \| (\hat{f}_3)_{K_0} \| \leq C K_0^{s_1} \prod_{j=1}^3 \| (\hat{f}_j)_{K_0} \| \leq
\]
\[
\leq C K_0^{s_1} \prod_{j=1}^3 \| (\hat{f}_j)_{K_0} \| \leq C T K_0^{s_1} \prod_{j=1}^3 \| \hat{f}_j \| \leq C T K_0^{s_1} \prod_{j=1}^3 \| f_j \|,
\]
where we have used the notations \( \hat{g}(x) := \sum_{k} |\hat{g}(k)| e^{ikx} \) and \( g_{\leq K_0} := \sum_{|k| \leq K_0} \hat{g}(k) e^{ikx} \).
The estimates for \( h_1, h_2 \), in addition to (19) implies (18). Thus, it remains to establish (19).

At this point, it is worth mentioning that the particular form of the free solutions \( R[f_j] \) as entries in \( \hat{h} \) will not be important anymore, other than the fact that they belong to the
space $H^{s_0}(T)$. Thus, upon introducing the new trilinear form

$$H(v_1, v_2, v_3) := \sum_{k = k_1 + k_2 + k_3 \neq 0, k_{\text{max}} > k_0 \atop (k_1 + k_2)(k_2 + k_3)(k_3 + k_1) \neq 0} \frac{(k_1 + k_2 + k_3)\hat{v}_1(k_1)\hat{v}_2(k_2)\hat{v}_3(k_3)}{-3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1) + E(k_1, k_2, k_3)},$$

we will show the more general estimate

$$\|H(v_1, v_2, v_3)\|_{L^\infty H_x^{s_1}} \leq C \|v_1\|_{H^{s_0}(T)} \|v_2\|_{H^{s_0}(T)} \|v_3\|_{H^{s_0}(T)},$$

which of course implies (19) with $v_j = R[f_j]$, since $\|v_j\|_{H^{s_0}} = \|f_j\|_{H^{s_0}}$.

Recall $|(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)| \gtrsim k_{\text{max}} \gg |E(k_1, k_2, k_3)|$. Thus, we have the following inequalities

$$\frac{|k_1 + k_2 + k_3|}{-3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1) + E(k_1, k_2, k_3)} \leq \frac{|k_1 + k_2 + k_3|}{(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)} \leq \frac{C}{|k_1 + k_2||k_2 + k_3| + |k_2 + k_3||k_3 + k_1| + |k_1 + k_2||k_3 + k_1|}.$$

We need to consider two cases - $k_{\text{min}} \sim k_{\text{max}}$ and the case $k_{\text{min}} << k_{\text{max}}$.

**Case I:** $k_{\text{min}} \sim k_{\text{max}}$ or $|k_1| \sim |k_2| \sim |k_3|$. In this case $|k| \lesssim |k_j|, j = 1, 2, 3$. We only consider the term $\frac{1}{(k_1 + k_2)(k_2 + k_3)} = \frac{1}{(k_1 + k_2)(k - k_1)}$, the others being symmetric.

By Cauchy-Schwartz, we have

$$|H(v_1, v_2, v_3)(k)|^2 \leq \left( \sum_{k_1, k_2: |k_1| \sim |k_2| \gtrsim |k|} |\hat{v}_1(k_1)|^2|\hat{v}_2(k_2)|^2 \right) \times \left( \sum_{k_1, k_2: |k_1| \sim |k_2| \sim |k - k_1 - k_2| \gtrsim |k|} \frac{\hat{v}_3(k - k_1 - k_2)^2}{|k_1 + k_2|^2|k - k_1|^2} \right)$$

It is now easy to estimate

$$\|H(v_1, v_2, v_3)\|_{L^\infty H_x^{s_1}}^2 \leq C \sum_k < k >^{2s_1} \left( \sum_{k_1: |k_1| \gtrsim |k|} |\hat{v}_1(k_1)|^2 \right) \left( \sum_{k_2: |k_2| \gtrsim |k|} |\hat{v}_2(k_2)|^2 \right) \times \left( \sum_{k_1, k_2: |k_1 - k_2| \gtrsim |k|} \frac{|\hat{v}_3(k - k_1 - k_2)|^2}{|k_1 + k_2|^2|k - k_1|^2} \right) \leq C \left( \sum_{k_1 < k_1 >^{2s_1/3} |\hat{v}_1(k_1)|^2} \left( \sum_{k_2 < k_2 >^{2s_1/3} |\hat{v}_2(k_2)|^2 \right) \times \left( \sum_{\mu, k_1, k_2: (k_1 + k_2)(\mu + k_2) \neq 0} \frac{|\hat{v}_3(\mu)|^2}{|k_1 + k_2|^2|\mu + k_2|^2} \right) \leq C \|v_1\|_{H^{s_0}}^2 \|v_2\|_{H^{s_0}}^2 \|v_3\|_{H^{s_0}}^2.$
provided $s_1 < 3s_0$, since
\[ \sum_{k_1, k_2: (k_1 + k_2) (\mu + k_2) \neq 0} \frac{1}{|k_1 + k_2|^2 |\mu + k_2|^2} < \infty. \]

**Case II:** $k_{\text{min}} < k_{\text{max}}$. In this case, we have that for all $i \neq j \neq l \neq i$, $|(k_i + k_j) (k_j + k_l)| \gtrsim k_{\text{max}}$. Thus
\[ |H(v_1, v_2, v_3)(k)| \leq C < k >^{-1} \sum_{k_1, k_2} |\hat{v}_1(k_1)||\hat{v}_2(k_2)||\hat{v}_3(k - k_1 - k_2)|. \]

Since $1 > s_1 > 1/2$. We have by Sobolev embedding\(^4\)
\[ \|H(v_1, v_2, v_3)\|_{H^{s_1}} \leq C\|\partial_x \hat{v}_1\|_{L^2} \leq C\|\hat{v}_1\|_{L^3} \leq C\|v_1\|_{H^{s_0}}, \]
where $\frac{1}{q} - \frac{1}{2} = 1 - s_1$, so that $q \in (1, 2)$. Under the restriction $s_1 < \min(3s_0, 1)$, it follows by Sobolev embedding
\[ \|\hat{v}_j\|_{L^{3q}} \leq C\|\hat{v}_j\|_{H^{s_1/3}} \leq C\|v_j\|_{H^{s_0}}. \]

so that $\frac{1}{2} - \frac{1}{3q} = \frac{s_1}{3} < s_0$. This finishes the proof of the estimate (20) and hence the proof of Lemma 5. \(\square\)

### 3.2. Estimate of the resonant contributions.

**Lemma 6.** Let $\frac{1}{2} > s_0 > \frac{1}{2}, \delta, \delta, s_0 - 1/4, b = 1/2 + \delta, 1 - s_0 < s_1 < \min(1, 3s_0)$. Assume that $F_1, F_2; G_1, G_2 \in L_T^\infty L^{s_1}(T)$, whereas $v_1, v_2 \in Y^{s_0,b}$. For the solution of
\[ \partial_t \hat{V}(k) - i(k^3 + k|f(k)|^2) \hat{V}(k) = c_1 k \hat{v}_1(k) \hat{F}_1(k) \hat{F}_2(k) + c_2 k \hat{v}_2(k) \hat{G}_1(k) \hat{G}_2(k), \]
we have the estimates, with $C = C(c_1, c_2)$

(21) $\|V\|_{Y_{T_0}^{s_0,b}} \leq C(\|v_1\|_{Y^{s_0,b}} \|F_1\|_{L^\infty L^{s_1}} \|F_2\|_{L^\infty L^{s_1}} + \|v_2\|_{Y^{s_0,b}} \|G_1\|_{L^\infty L^{s_1}} \|G_2\|_{L^\infty L^{s_1}})$

(22) $\|V\|_{L^\infty L^{s_1}} \leq C(\|v_1\|_{Y^{s_0,b}} \|F_1\|_{L^\infty L^{s_1}} \|F_2\|_{L^\infty L^{s_1}} + \|v_2\|_{Y^{s_0,b}} \|G_1\|_{L^\infty L^{s_1}} \|G_2\|_{L^\infty L^{s_1}})$

### 3.3. Proof of Lemma 6

The proof of Lemma 6 is fairly easy. Denote the right hand side of the equation by $RHS$. By Lemma 2,
\[ \|V\|_{Y_{T_0}^{s_0,b}} \leq C_T^\delta \|RHS\|_{Y_{T_0}^{s_0,b} - 1 + \delta} \leq CT^\delta \|RHS\|_{L_{T_0}^3 L_{t}^{s_0}} \leq CT^{\delta + 1/2} \sup_t \|RHS\|_{H^{s_0}}. \]

By energy estimates
\[ \|V\|_{L_{T}^\infty L^{s_1}} \leq C \|RHS\|_{L_{t}^{1} L_{x}^{s_1}} \leq CT \sup_t \|RHS\|_{H^{s_1}}. \]

Thus, recalling that $T < 1$, $\|V\|_{Y_{T_0}^{s_0,b}} + \|V\|_{L_{T}^\infty L^{s_1}} \leq C T \sup_t \|RHS\|_{H^{s_1}}$, so it suffices to estimate this quantity. We also estimate only say the first quantity of $RHS$, since they

\(^4\)recall that we use the notation $\hat{v}(x) = \sum_k |\hat{v}(k)| e^{ikx}$. 
are symmetric from the point of view of the required estimates. We have
\[ \|RHS\|_{H_t^2}^2 \leq C \sum_k < k >^{2(1+s_1)} |\hat{v}_1(k)|^2 |\hat{F}_1(k)|^2 |\hat{F}_2(k)|^2 \]
\[ \leq C (\sup_k < k >^{s_0} |\hat{v}_1(k)|)^2 (\sup_k < k >^{1-s_0} |\hat{F}_2(k)|)^2 \sum_k < k >^{2s_1} |\hat{F}_1(k)|^2 \]
\[ \leq C \|v_1\|_{H_t^{s_0}}^2 \|F_1\|_{H_t^{1-s_0}}^2 \|F_1\|_{H_t^1}^2 \]
The estimate follows since \( 1 - s_0 < s_1 \).

4. PROOF OF THEOREM

4.1. Existence of the solution. We start with the existence of the solution \( z \) in the sense of Definition[1]. We produce it by an iteration argument as follows[5]. Start with \( z_0 = 0 \) and \( Q_0(t) = t(k^3 + k|\hat{f}(k)|^2) \) as prescribed in Lemma[1]. Define iteratively, \( z_{m+1}, m = 0, \ldots \)
by producing the next iterate from the previous one, namely
\[ \hat{z}_{m+1}(t, k) = \int_0^t e^{i(t-s)(k^3+k|\hat{f}(k)|^2)} [ik|\hat{z}_m(s, k)|^2 \hat{z}_m(s, k) + \]
\[ + \int_0^t e^{i(t-s)(k^3+k|\hat{f}(k)|^2)} 2i k \Re(\hat{f}(k)e^{iQ_m(s,k)}\hat{z}_m(t, k)) \hat{z}_m(k) ds + \]
\[ + \int_0^t e^{i(t-s)(k^3+k|\hat{f}(k)|^2)} [N^j R(\otimes_{j=1}^3 \hat{f}(k_j) e^{iQ_m(t,k_j)} + \hat{z}_m(k_j))] ds. \]
By the definition,
\[ \hat{z}_1(t, k) = \int_0^t e^{i(t-s)(k^3+k|\hat{f}(k)|^2)} [N^j R(\otimes_{j=1}^3 \hat{f}(k_j) e^{i(t-k^3+k|\hat{f}(k)|^2)] ds. \]
According to the estimates in Lemma[5] we have that
\[ \|z_1\|_X \leq C \|f\|_{H_t^{s_0}}^3. \]
Denote \( K := \|z_1\|_X < C \|f\|_{H_t^{s_0}}^3. \) We will show that with the right choice of \( T \) (to be made precise below), we will have that \( \|z_T\|_X \leq 2K. \)
We need to estimate \( \|z_{m+1} - z_m\|_X. \) The right hand side of the equation for \( z_{m+1} \) has a multilinear structure, which allows us (by adding and subtracting appropriate terms) to use the estimates of Lemma[5] and Lemma[6]. Denote for conciseness
\[ F_m(t, x) := \sum_k \hat{f}(k)e^{iQ_m(t,k)}e^{ikx}. \] We have
\[ \|z_{m+1} - z_m\|_X \lesssim T^\delta \|z_m - z_{m-1}\|_X (\|z_m\|_X + \|z_{m-1}\|_X)^2 + \]
\[ + T^\delta \|z_m - z_{m-1}\|_X (\|z_m\|_X + \|z_{m-1}\|_X + \|F_m\|_{Y^{\alpha,0}} + \|F_{m-1}\|_{Y^{\alpha,0}})^2 \]
\[ + T^\delta \|F_m - F_{m-1}\|_{Y^{\alpha,0}} (\|z_m\|_X + \|z_{m-1}\|_X + \|F_m\|_{Y^{\alpha,0}} + \|F_{m-1}\|_{Y^{\alpha,0}})^2. \]
Further, similar to Lemma[4] (more specifically (14)), we estimate,
\begin{equation}
(23) \quad \|F_m - F_{m-1}\|_{Y^{\alpha,0}} \lesssim \left( \sum_k < k >^{2s_0} |\hat{f}(k)|^2 \int_0^T (1 + |g'(t, k)|^2) dt \right)^{1/2}
\end{equation}
\[ ^5 \text{recall that } Q = Q(z) \text{ is constructed for a given } z \text{ in Lemma[1]} \]
where $g(t, k) = e^{iQ_m(t, k)} - e^{iQ_{m-1}(t, k)}$. But
\[
|g'(t, k)| \leq C[|Q'_m(t, k)|Q_m(t, k) - Q_{m-1}(t, k)| + |Q'_m(t, k) - Q'_{m-1}(t, k)|].
\]

From (5), we have
\[
|Q'_m(t, k) - Q'_{m-1}(t, k)| \leq C \sup_{0 \leq \tau < t} |k| \hat{f}(k) \times \left(|Q_m(t, k) - Q_{m-1}(t, k)| + |z_m(t, k) - z_{m-1}(t, k)| \right)
\]

Employing the estimates of Lemma 4, namely the bound
\[
\sup_{0 \leq \tau < T} |k| \hat{f}(k) |\hat{z}(t, k)| \leq C \|f\|_{L^0} \|z\|_{X},
\]
we conclude
\[
|Q'_m(t, k) - Q'_{m-1}(t, k)| \leq C \|f\|_{L^0} \left(\|z_m\|_{X} + \|z_{m-1}\|_{X}\right) |Q_m(t, k) - Q_{m-1}(t, k)| +
\]
\[
\quad + C \|f\|_{L^0} \left(\|z_m\|_{X} + \|z_{m-1}\|_{X}\right) \|z_m - z_{m-1}\|_{X}.
\]

Similarly,
\[
|Q'_{m-1}(t, k)| \leq C \sup_{0 \leq \tau < T} |k| |\hat{z}_{m-1}(t, k)| (|\hat{f}(k)| + |\hat{z}_{m-1}(t, k)|) \leq
\]
\[
\leq C \|z_{m-1}\|_{X} \left(\|f\|_{L^0} + \|z_{m-1}\|_{X}\right).
\]

Putting all estimates together yields
\[
|g'(t, k)| \leq C \|f\|_{L^0} \left(\|z_m\|_{X} + \|z_{m-1}\|_{X}\right) |Q_m(t, k) - Q_{m-1}(t, k)| +
\]
\[
\quad + \|f\|_{L^0} \left(\|z_m\|_{X} + \|z_{m-1}\|_{X}\right) \|z_m - z_{m-1}\|_{X}.
\]

Thus, we now need to find a good estimate for $|Q_m(t, k) - Q_{m-1}(t, k)|$. Arguing again from the integral equation (6), we have
\[
|Q_m(t, k) - Q_{m-1}(t, k)| \leq
\]
\[
\leq CT |k| \hat{f}(k) \sup_{0 \leq \tau < t} |Q_m(\tau, k) - Q_{m-1}(\tau, k)| (|\hat{z}_m(\tau, k)| + |\hat{z}_{m-1}(\tau, k)|) +
\]
\[
\quad + CT |k| \sup_{0 \leq \tau < t} (|\hat{z}_m(\tau, k)| + |\hat{z}_{m-1}(\tau, k)|) \|\hat{z}_m(t, k) - \hat{z}_{m-1}(t, k)| \leq
\]
\[
\leq CT \|f\|_{L^0} \left(\|z_m\|_{X} + \|z_{m-1}\|_{X}\right) \sup_{0 \leq \tau < t} |Q_m(\tau, k) - Q_{m-1}(\tau, k)| +
\]
\[
\quad + C \|z_m - z_{m-1}\|_{X} \left(\|z_m\|_{X} + \|z_{m-1}\|_{X}\right).
\]

Now, if $T$ is so small that $CT \|f\|_{L^0} \left(\|z_m\|_{X} + \|z_{m-1}\|_{X}\right) \leq \frac{1}{2}$, we can hide the first term on the right hand side and thus, we obtain the estimate
\[
\sup_{0 \leq t < T} |Q_m(t, k) - Q_{m-1}(t, k)| \leq C \|z_m - z_{m-1}\|_{X} \left(\|z_m\|_{X} + \|z_{m-1}\|_{X}\right).
\]

In all
\[
|g'(t, k)| \leq C \|z_m - z_{m-1}\|_{X} \left(\|f\|_{L^0} \left(\|z_m\|_{X} + \|z_{m-1}\|_{X}\right)\right)^2.
\]

Hence, plugging this back in (23), we obtain
\[
\|F_m - F_{m-1}\|_{Y_{\mu}^0}^b \leq C \|z_m - z_{m-1}\|_{X} \left(\|f\|_{L^0} \left(\|z_m\|_{X} + \|z_{m-1}\|_{X}\right)\right)^2,
\]

under the additional smallness assumption on $T : T \|f\|_{L^0} \left(\|z_m\|_{X} + \|z_{m-1}\|_{X}\right) \ll 1$. 

Going further back to our estimate for \( \|z_{m+1} - z_m\|_X \), and plugging in (24), we have
\[
\|z_{m+1} - z_m\|_X \leq CT^8 \|z_m - z_{m-1}\|_X (1 + \|z_m\|_X + \|z_{m-1}\|_X + \|f\|_{H^{s_0}})^4
\]
where we have also used Lemma 4 to control \( \|F_m\|_{Y^{s_0,b}} \), \( \|F_{m-1}\|_{Y^{s_0,b}} \leq C_T \|f\|_{H^{s_0}} \).

Clearly, one can choose now \( T \), so that \( T \) satisfies the previous assumptions (i.e. \( T \|f\|_{H^{s_0}}(\|z_m\|_X + \|z_{m-1}\|_X) \ll 1 \)) and
\[
T : T^8 (1 + \|z_m\|_X + \|z_{m-1}\|_X + \|f\|_{H^{s_0}})^4 < \frac{1}{2}.
\]
This will ensure that
\[
\|z_{m+1} - z_m\|_X \leq \frac{1}{2} \|z_m - z_{m-1}\|_X,
\]
and thus Cauchyness and the convergence of \( \{z_m\} \), \( z := \lim_m z_m \), where \( z : [0, T] \to C \).

In addition,
\[
\|z\|_X \leq \|z_1\|_X + \sum_{m=2}^\infty \|z_m - z_{m-1}\|_X \leq 2K,
\]
where \( \|z_1\|_X = K \). This completes the existence part of the argument.

### 4.2. Smoothing effects.

The first smoothing effect announced in Theorem 1 follows from \( z \in \mathcal{X} \subseteq L^\infty_t H^{s_1}_x \subset L^\infty H^{3s_0} \).

For (8), we have
\[
|\hat{u}(t, k)|^2 = |\hat{z}(t, k)|^2 + |\hat{f}(k)|^2 + 2\Re(\hat{f}(k) e^{iQ(t,k)} \hat{z}(t, k))
\]
whence, since \( s_1 + s_0 > 1 \)
\[
\sup_t \sum_k |k||\hat{u}(t, k)|^2 - |\hat{f}(k)|^2| \leq 2 \sum_k |k||\hat{z}(t, k)|(|\hat{z}(t, k)| + |\hat{f}(k)|) \leq C \sum_k |k| < 2^{s_0} (|\hat{f}(k)|^2 + |\hat{z}(t, k)|^2)^{1/2} \leq (\sum_k |k| < 2^{s_1} |\hat{z}(t, k)|^2)^{1/2} \leq C \|z\|_X (\|f\|_{H^{s_0}} + \|z\|_{H^{s_0}}) \leq C \|z\|_X (\|f\|_{H^{s_0}} + \|z\|_X).
\]

### 4.3. Uniqueness.

The uniqueness of the solution, in the sense of Definition 1, requires us to analyze (3) in detail. We start with the proof of the well-posedness of (3).

#### 4.3.1. Proof of the well-posedness of (3), for fixed \( u \).

Let us first show that under the condition (9), the equation (3) produces unique local solutions in \( H^{s_1} \), recall \( s_1 < \min(3s_0, 1) \).

The main ingredient that we need here is
\[
\sum_k \hat{f}(k) e^{iP(t,k)} e^{ikx} \in Y^{s_0,b},
\]
which is simply a variant of Lemma 4. Indeed, observe that
\[
P(u; t, k) = it(k^3 + k|\hat{f}(k)|^2) + k \int_0^t (|\hat{u}(s,k)|^2 - |\hat{f}(k)|^2)ds.
\]

Thus, similar to the proof of Lemma 4, we infer the bound
\[
\|\sum_k \hat{f}(k) e^{iP(t,k)} e^{ikx}\|_{Y^{s_0,b}} \leq C \sqrt{T} \|f\|_{H^{s_0}(T)}, \tag{25}
\]
Adding and subtracting terms and using the a-priori bound (9) yields
\[ \sup_{k,t} |(e^{ikf_0'}(|\hat{u}(s,k)|^2 - |\hat{f}(k)|^2)ds)| < C. \]
But by (9)
\[ \sup_{k,t} |(e^{ikf_0'}(|\hat{u}(s,k)|^2 - |\hat{f}(k)|^2)ds)| = \sup_{k,t} |k| \left| \hat{u}(t, k) \right|^2 - |\hat{f}(k)|^2 | < C, \]
and hence the solutions in (3) are in \( H^{s_1} \), in some time interval \([0, T] \), \( T = T(\|f\|_{H^{s_0}}) \). In addition, there is the estimate
\[ \|v\|_{X_T} \leq C_T \|f\|_{H^{s_0}}^3 \]
There is an unique solution \( v \) in this class. Indeed, we have the multilinear structure of the non-linearity, which allows us to use Lemma 5 and Lemma 6 to show that it is a contraction on the space \( X_T \), whence uniqueness follows. This, however does not, by itself imply uniqueness due to its dependence on \( P = P(u) \). Let us explain this point in more detail. So far, we have shown that for a given \( u \), with the property (9), the equation (3) has an unique solution \( v \). For the uniqueness, we need to establish more. Namely that for two different \( u_1, u_2 \) and the corresponding \( v_1, v_2 \), constructed via (3), where \( P(u_j, t, k) \) are involved, we still have \( v_1 = v_2 \) (which then will later easily imply \( u_1 = u_2 \)).

4.3.2. Estimate on the difference \( v_1 - v_2 \). Taking the difference of \( v_1, v_2 \), we see that it satisfies an equation similar to the one satisfied by \( z_{m+1} - z_m \) that we have considered for the existence part. Using the multilinear structure and the estimates of Lemma 5, Lemma 6 we obtain
\[ \|v_1 - v_2\|_{X_T} \lesssim T^\delta \|v_1 - v_2\|_X^2 (\|v_1\|_X + \|v_2\|_X + \|F_1\|_{Y^{s_0,b}} + \|F_2\|_{Y^{s_0,b}})^2 + T^\delta \|F_1 - F_2\|_{Y^{s_0,b}}^2 (\|v_1\|_X + \|v_2\|_X + \|F_1\|_{Y^{s_0,b}} + \|F_2\|_{Y^{s_0,b}})^2. \]
where again, we have adopted the notation \( P_j(t, k) = P(u_j; t, k) \) and \( F_j := \sum_k \hat{f}(k)e^{ iP_j(t,k) e^{ikx}}. \) In view of our bound (25), we have
\[ \|v_1 - v_2\|_{X_T} \lesssim T^\delta (\|v_1 - v_2\|_X + \|F_1 - F_2\|_{Y^{s_0,b}}^2) (1 + \|f\|_{H^{s_0}}^3)^2. \]
Thus, our main task now is to effectively control \( \|F_1 - F_2\|_{Y^{s_0,b}} \). To that end, represent
\[ F_1 - F_2 = \sum_k \hat{f}(k)(e^{ iP_1(t,k) } - e^{ iP_2(t,k) })e^{ikx} = \sum_k \hat{f}(k)e^{it(3k^2 + |k|\hat{f}(k)^2)}e^{ikx} (e^{ikf_0'}(|\hat{u}_1(s,k)|^2 - |\hat{f}(k)|^2)ds - e^{ikf_0'}(|\hat{u}_2(s,k)|^2 - |\hat{f}(k)|^2)ds). \]
Similar to (23), we can estimate
\[ \|F_1 - F_2\|_{Y^{s_0,b}} \leq C \|f\|_{H^{s_0}} \sup_k |g'(t, k)|, \]
where
\[ g(t, k) = e^{ikf_0'}(|\hat{u}_1(s,k)|^2 - |\hat{f}(k)|^2)ds - e^{ikf_0'}(|\hat{u}_2(s,k)|^2 - |\hat{f}(k)|^2)ds. \]
Adding and subtracting terms and using the a-priori bound (9) yields
\[ |g'(t, k)| \leq C|k| \left| \hat{u}_1(t, k) \right|^2 - |\hat{f}(k)|^2 |k|T \sup_{0 < T < T} \left| \hat{u}_1(1, k) \right|^2 - |\hat{u}_2(1, k)|^2 | + k| \left| \hat{u}_1(t, k) \right|^2 - |\hat{u}_2(t, k)|^2 | \leq \bar{C}(1 + T)|k| \sup_{0 < T \leq T} \left| \hat{u}_1(t, k) \right|^2 - |\hat{u}_2(t, k)|^2 |. \]
Thus, we need control in the form (for say $T \leq 1$)

$$\sup_k |\hat{u}_1(t, k)|^2 - |\hat{u}_2(t, k)|^2 \leq C(\|f\|_{L^2}) \|v_1 - v_2\|_{X_T}.$$  

Let us show that once we assume (27), we can establish the uniqueness. Indeed, plugging (27) in the estimate for $|g'(t, k)|$, we obtain

$$\|F_1 - F_2\|_{X_{\infty_T}} \leq C(\|f\|_{L^2}) \|v_1 - v_2\|_{X_T}.$$  

Going back to (26), we have (for say all $T : T < 1$)

$$\|v_1 - v_2\|_{X_T} \leq C(\|f\|_{L^2}) T^\delta \|v_1 - v_2\|_{X_T} (1 + \|f\|^2_{L^2})^\delta,$$

which imply that for small enough $T = T(\|f\|_{L^2})$, $\|v_1 - v_2\|_{X_T} = 0$.

Thus, again from (27), we obtain that $|u_1(t, k)| = |u_2(t, k)|$, which implies that $P_1(t, k) = P_2(t, k)$. This however means that $u_1 = u_2$, so uniqueness follows.

4.3.3. Proof of (27). Expanding $|\hat{u}_3(t, k)|^2$ and taking the difference yields

$$|\hat{u}_1(t, k)|^2 - |\hat{u}_2(t, k)|^2 = 2\Re(\hat{f}(k)(e^{iP_1(t, k)}\hat{v}_1(t, k) - e^{iP_2(t, k)}\hat{v}_2(t, k))) + |\hat{v}_1(t, k)|^2 - |\hat{v}_2(t, k)|^2.$$

Thus,

$$|\hat{u}_1(t, k)|^2 - |\hat{u}_2(t, k)|^2 \leq C|\hat{f}(k)|(|\hat{v}_1(t, k) - \hat{v}_2(t, k)| + |\hat{v}_1(t, k)||P_1(t, k) - P_2(t, k)|) + |\hat{v}_1(t, k) - \hat{v}_2(t, k)|(|\hat{v}_1(t, k)| + |\hat{v}_2(t, k)|).$$

But

$$|P_1(t, k) - P_2(t, k)| \leq CT |k| \sup_{0 < \tau < t} |\hat{u}_1(t, k)|^2 - |\hat{u}_2(t, k)|^2.$$

Thus, we have

$$|\hat{u}_1(t, k)|^2 - |\hat{u}_2(t, k)|^2 \leq C \sup_{0 < \tau < t} |\hat{u}_1(\tau, k)|^2 - |\hat{u}_2(\tau, k)|^2 |k| |\hat{f}(k)||\hat{v}_1(t, k)| + C|\hat{v}_1(t, k) - \hat{v}_2(t, k)|(|\hat{v}_1(t, k)| + |\hat{v}_2(t, k)| + |\hat{f}(k)|)$$

We can now run a continuity argument in $A(t, k) := \sup_{0 < \tau < t} |\hat{u}_1(\tau, k)|^2 - |\hat{u}_2(\tau, k)|^2$, since (recalling that $s_0 + s_1 > 1$)

$$\sup_k |k| |\hat{f}(k)||\hat{v}_1(t, k)| \leq C \|f\|_{L^2} \|v_1\|_{X_T} \leq C \|f\|_{L^2} \|v_1\|_{X_T}.$$

We have

$$A(t) \leq [CT \|f\|_{L^2} \|v_1\|_{X_T} A(t) + C|\hat{v}_1(t, k) - \hat{v}_2(t, k)|(|\hat{v}_1(t, k)| + |\hat{v}_2(t, k)| + |\hat{f}(k)|).$$

Thus, for $T$ small enough, $T = T(\|f\|_{L^2})$ (recall the bounds on $\|v_1\|_{X_T}$ are in terms of $C\|f\|^2_{L^2}$), we can hide the terms containing $A(t)$ on the right hand side. We obtain

$$|\hat{u}_1(t, k)|^2 - |\hat{u}_2(t, k)|^2 \leq A(t) \leq C|\hat{v}_1(t, k) - \hat{v}_2(t, k)|(|\hat{v}_1(t, k)| + |\hat{v}_2(t, k)| + |\hat{f}(k)|).$$

It follows that (again, since $s_0 + s_1 > 1$)

$$|k| |\hat{u}_1(t, k)|^2 - |\hat{u}_2(t, k)|^2 \leq C |k| |\hat{v}_1(t, k) - \hat{v}_2(t, k)|(|\hat{v}_1(t, k)| + |\hat{v}_2(t, k)| + |\hat{f}(k)|) \leq C \|v_1 - v_2\|_{X_{s_1}} \|v_1\|_{L^2} \|v_2\|_{L^2} + \|f\|_{L^2} \leq C \|v_1 - v_2\|_{X_T} (1 + \|f\|^3_{L^2}).$$
which is (27). Thus, the uniqueness and thus the proof of Theorem 1 is complete.

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