REDUCIBILITY OF 1-D QUANTUM HARMONIC OSCILLATOR
WITH DECAYING CONDITIONS ON THE DERIVATIVE OF
PERTURBATION POTENTIALS

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ABSTRACT. We prove the reducibility of 1-D quantum harmonic oscillators in \( \mathbb{R} \) perturbed by a quasi-periodic in time potential \( V(x, \omega t) \) under the following conditions, namely there is a \( C > 0 \) such that

\[
|V(x, \theta)| \leq C, \quad |x \partial_2 V(x, \theta)| \leq C, \quad \forall (x, \theta) \in \mathbb{R} \times T^n_\sigma.
\]

The corresponding perturbation matrix \((P_i^j(\theta))\) is proved to satisfy \((1 + |i - j|)|P_i^j(\theta)| \leq C\) and \(\sqrt{|j|} |P_{i+1}^j(\theta) - P_i^j(\theta)| \leq C\) for any \( \theta \in T^n_\sigma \) and \( i, j \geq 1 \). A new reducibility theorem is set up under this kind of decay in the perturbation matrix element \( P_i^j(\theta) \) as well as the discrete difference matrix element \( P_{i+1}^j(\theta) - P_i^j(\theta) \). For the proof the novelty is that we use the decay in the discrete difference matrix element to control the measure estimates for the thrown parameter sets.

1. INTRODUCTION

1.1. The Statement of Main Result. In this paper we consider the linear Schrödinger equation

\[
i \partial_t \psi = (-\partial_x^2 + x^2)\psi + \epsilon V(x, \omega t)\psi, \quad \psi = \psi(x, t), \quad x \in \mathbb{R},
\]

where \( \epsilon \geq 0 \) is a small parameter and frequency vector \( \omega \) of the forced oscillations is regarded as a parameter in \( \mathcal{D}_\omega := [0, 2\pi)^n \). We assume that the potential \( V : \mathbb{R} \times T^n \ni (x, \theta) \mapsto V(x, \theta) \in \mathbb{R} \) is \( C^1 \) smooth and analytic in \( \theta \in T^n \), where \( T^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n \) denotes the n dimensional torus. For \( \sigma > 0 \), the function \( V(x, \theta) \) analytically in \( \theta \) extends to the strip \( T^n_\sigma = \{ a + bi \in \mathbb{C}^n / 2\pi \mathbb{Z}^n : |b| < \sigma \} \) and is bounded on \( \mathbb{R} \times T^n_\sigma \) as well as \( x \partial_x V \), namely there is a \( C > 0 \) such that

\[
|V(x, \theta)| \leq C, \quad |x \partial_x V(x, \theta)| \leq C, \quad \forall (x, \theta) \in \mathbb{R} \times T^n_\sigma.
\]

To state our main results we first introduce some notations. Let \( H = -\partial_x^2 + x^2 \) be the Hermite operator. It is well-known that \( H \) has a simple pure point spectrum \( \nu_i = 2i - 1 \) so that \( Hh_i(x) = \nu_i h_i(x) \), where \( h_i(x) \) is the so-called Hermite function and satisfies \( \|h_i\|_{L^2} = 1 \) for \( i \geq 1 \). Clearly, \( \{h_i\}_{i \geq 1} \) forms an orthonormal basis of \( L^2(\mathbb{R}) \), called the Hermite basis. For \( s \geq 0 \), denote by \( \mathcal{H}^s = \mathcal{D}(H^{1/2}) = \{ f \in L^2(\mathbb{R}) : H^{1/2} f \in L^2(\mathbb{R}) \} \) the domain of \( H^{1/2} \) endowed by the graph norm. For negative \( s \), the space \( \mathcal{H}^s \) is the dual of \( \mathcal{H}^{-s} \). For Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), we will denote by \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) the space of bounded linear operators from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) and write \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1) \) as \( \mathcal{B}(\mathcal{H}_1) \) for simplicity.

Let \( s \in \mathbb{R} \), we define the complex weighted \( \ell^2 \)-space

\[
\ell^2_s := \{ \xi = (\xi_i \in \mathbb{C}, i \in \mathbb{N}_0) : \|\xi\|_s < \infty \} \text{ with } \|\xi\|_s^2 = \sum_{i \in \mathbb{N}_0} i^s |\xi_i|^2.
\]

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Notice for later use that to a function $f \in \mathcal{H}^s$ we formulate $f(x) = \sum_{i \geq 1} \xi_i h_i(x)$, where the sequence $\xi = (\xi_i)_{i \geq 1} \in \ell_2^s$ is the so-called Hermite coefficients. We will identify $\mathcal{H}^s$ with $\ell_2^s$ by endowing both spaces with the norm $\|f\|_{\mathcal{H}^s} = \|\xi\|_s = (\sum_{i \geq 1} i^s |\xi_i|^2)^{\frac{1}{2}}$. Then our main results are the following.

**Theorem 1.1.** Assume that a $C^1$ smooth function $V(x, \theta)$ analytically in $\theta$ extends to $\mathbb{T}_\sigma^n$ for $\sigma > 0$ and fulfills (1.2). There exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ there is a subset $\mathcal{D}_\epsilon \subset \mathcal{D}_0 = [0, 2\pi)^n$ of asymptotically full measure such that for all $\omega \in \mathcal{D}_\epsilon$, the linear Schrödinger equation (1.1) reduces to a linear equation with constant coefficients (w.r.t. time variable) in $L^2(\mathbb{R})$.

More precisely, let $p \in [0, 3]$ then for any $\omega \in \mathcal{D}_\epsilon$ there exists a linear isomorphism $\Psi_{\omega,\epsilon}(\theta) \in \mathcal{B}(\mathcal{H}^p)$, unitary on $L^2$, which analytically depends on $\theta \in \mathbb{T}_\sigma^{n/2}$ such that $t \mapsto \psi(\cdot, t) \in \mathcal{H}^p$ satisfies (1.1) if and only if $t \mapsto \zeta(\cdot, t) = \Psi_{\omega,\epsilon}^{-1}(\omega t)\psi(\cdot, t) \in \mathcal{H}^p$ satisfies the autonomous equation $i\partial_t \zeta = H^\infty \zeta$ with $H^\infty = \text{diag}(\lambda_i^\infty)_{i \geq 1}$. Furthermore, there is a $C > 0$ such that

$$\text{Meas}(\mathcal{D}_0 \setminus \mathcal{D}_\epsilon) \leq C \epsilon^{\frac{n}{2}},$$

$$|\lambda_i^\infty - \nu_i| \leq C \epsilon, \quad \forall i \geq 1,$$

$$\|\Psi_{\omega,\epsilon}^{-1}(\theta) - \text{Id}\|_{\mathcal{B}(\mathcal{H}^p)} \leq C \epsilon^{\frac{n}{2}}, \quad \forall \theta \in \mathbb{T}_\sigma^{n/2}.$$

**Remark 1.2.** We give for example $V(x, \theta) := g(\theta) F\left(\langle x \rangle^{-\mu}\right)$ to fulfill (1.2), where $\mu > 0$, $\langle x \rangle := \sqrt{1 + x^2}$, $g(\theta)$ is a real analytic and bounded function on $\mathbb{T}_\sigma^n$ with $\sigma > 0$. The real function $F(\cdot)$ is $C^1$ smooth on $\mathbb{R}$ which satisfies $|F(\cdot)|, |F'(\cdot)| \leq C$ on $\mathbb{R}$ with some $C > 0$.

Consequently, we have the following corollaries concerning the Sobolev norm estimations on the solution of (1.1) and the spectra of the corresponding Floquet operator defined by $K_F := -i \sum_{j=1}^n \Omega_j \partial_{\theta_j} - \partial_x^2 + x^2 + \epsilon V(x, \theta)$.

**Corollary 1.3.** Assume that a $C^1$ smooth function $V(x, \theta)$ analytically in $\theta$ extends to $\mathbb{T}_\sigma^n$ for $\sigma > 0$ and fulfills (1.2). There exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ and $\omega \in \mathcal{D}_\epsilon$, the linear Schrödinger equation (1.1) has a unique solution $\psi(t) \in C^0([0, \infty), \mathcal{H}^p)$ subject to the initial datum $\psi(0) \in \mathcal{H}^p$ with $p \in [0, 3)$. Moreover, $\psi$ is almost-periodic in time and satisfies for a $C > 0$

$$(1 - C\epsilon)\|\psi(0)\|_{\mathcal{H}^p} \leq \|\psi(t)\|_{\mathcal{H}^p} \leq (1 + C\epsilon)\|\psi(0)\|_{\mathcal{H}^p}.$$

**Corollary 1.4.** Assume that a $C^1$ smooth function $V(x, \theta)$ analytically in $\theta$ extends to $\mathbb{T}_\sigma^n$ for $\sigma > 0$ and fulfills (1.2). There exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ and $\omega \in \mathcal{D}_\epsilon$, the spectrum of the Floquet operator $K_F$ is pure point.

1.2. Discussions. The boundedness of Sobolev norms via reducibility, as well as the pure point nature of Floquet operator, was firstly considered for 1-D quantum harmonic oscillator (‘QHO’ for short) with time periodic smooth perturbations [13, 16, 19, 27]. As for time quasi-periodic perturbations, we can refer to [26, 39, 40], in which the perturbation potentials are all supposed to be decaying to zero when $x$ goes to $\infty$.

Eliasson [17] asked the following open question: “Is the 1-D quantum harmonic oscillators (1.1) reducible for the bounded perturbation i.e. $|V(x, \theta)| \leq C$, where $V(x, \theta)$ is smooth on $x$ and analytic on $\theta \in \mathbb{T}^n$? ” This question follows naturally from the results in [4] and [32] and keeps still unsolved till now except some special cases(see [28]).

Theorem 1.1 gives an answer for Eliasson’s question if we assume further $|xV_x(x, \theta)| \leq C$.

For 1-D QHO with unbounded perturbations, see Bambusi’s recent papers [2, 3] based
on the pseudo-differential calculus. In fact Bambusi’s results in 1-D quantum harmonic oscillators [3] can be translated into matrix language. In [12] a discrete difference operator $\Delta$ on a function $K : N_0 \times N_0 \to \mathbb{R}$ is defined as $(\Delta K)^n_m := \Delta K_m^n = K_{m+1}^n - K_m^n$, where $N_0 := \{1, 2, \ldots \}$. Write $\Delta^M$ to signify applying the difference operator $M$ times. Chodosh [12] called that a function $K : N_0 \times N_0 \to \mathbb{R}$ is a symbol matrix of order $r$ if for all $M, N \in N_0$, there is $C_{M,N} > 0$ such that

\[
|(\Delta^M K)^n_m| \leq C_{M,N}(1 + m + n)^{r-M}(1 + |m - n|)^{-N}.
\]

Chodosh [12] proved that a operator $A$ with the symbol $a \in S'(N_3)$, $f = 1$ if and only if the “matrix of A” $K(A) : N_0 \times N_0 \to \mathbb{R}$ defined by $(m, n) \to \langle Ah_m, h_n \rangle_{L^2(\mathbb{R})}$ is an order of $r/2$ symbol matrix. Bambusi’s results [3] together with [12] implied that the discrete difference matrix should play a very important role in the reducibility. In the following we will give a heuristic proof how the discrete difference operator “$\Delta$” and the corresponding decay in the difference matrix element work in the proof of Theorem 1.1.

For the following discussion we will introduce some notations. Let $\alpha \in \mathbb{R}$, denote by $\mathcal{M}_\alpha$ the set of infinite matrices $A : N_0 \times N_0 \to \mathbb{C}$ that satisfy

\[
|A|_{\alpha} := \sup_{i,j \in N_0} (1 + |i - j|)\alpha |A_{ij}| < \infty.
\]

For $\alpha, \beta \in \mathbb{R}$ we define $\mathcal{M}_{\alpha,\beta}$, the subset of $\mathcal{M}_\alpha$ as the following: an infinite matrix $A$ belongs to $\mathcal{M}_{\alpha,\beta}$ if

\[
|A|_{\alpha,\beta} := \sup_{i,j \in N_0} (1 + |i - j|)\alpha |A_{ij}| + \sup_{i,j \in N_0} (ij)^\beta |\Delta A_{ij}| < \infty.
\]

Following Bambusi and Graffi [4], we will prove the reducibility of the equation

\[
\dot{u}(t) = (A_0 + P_0(\omega t))u(t), \quad u \in \ell^2_0,
\]

where $\omega \in D_0 := [0, 2\pi]^n$ and $A_0 = \text{diag}(\nu_j)_{j \geq 1}$ and $\nu_j = j$ for simplicity, and the map $\mathbb{T}^n \ni \theta \mapsto P_0(\theta) \in \mathcal{M}_{\alpha,\beta}$ is analytic on $\mathbb{T}^n$, with $0 < \beta < \alpha$ and $(P_0^j)_i(\theta) = (P_0^i)_j(\theta)$ for $\theta \in \mathbb{T}^n$. Let us describe the iteration step. We begin from the equation

\[
\dot{u}(t) = (A + P(\omega t))u(t), \quad u \in \ell^2_0, \quad P(\theta) \in \mathcal{M}_{\alpha,\beta}.
\]

Now as [4] if we can find a coordinate transformation $u = e^Bv$ to conjugate the above system into $i\dot{v} = (A^* + P^*)v$, where $A = \text{diag}(\lambda_j(\omega))_{j \geq 1}$ and $A^*$ is still diagonal and $P^*$ still in $\mathcal{M}_{\alpha,\beta}$, which is higher order than $P$, one finishes one step of iteration. As [4], one should seek the solutions of the homological equation

\[
[A, B] - i\dot{B} = \tilde{A} - P + R.
\]

From the standard Fourier analysis the center of the reducibility problem is to estimate the measure of the following set

\[
\bigcup_{|k| \leq K, i > j \geq 1} \{\omega \in D_0 : |\langle k, \omega \rangle + \lambda_i(\omega) - \lambda_j(\omega)| < \kappa (1 + |i - j|)\}.
\]  

(1.3)

We recall some existent methods to estimate the above set. In [26], since the perturbation potential $|V(x, \theta)| \leq \frac{c}{(x_\theta)^{\delta_1}}$ with $\delta_1 > 0$, it follows that $\lambda_i = \nu_i + \epsilon \delta_2$ with some $\delta_2 > 0$. The measure estimate for (1.3) is standard in this case, which is close to 1-D wave equations(see [11]). In [40], the estimate is similar since $\lambda_i = \nu_i + \epsilon \delta_3 (\ln i)^{-\delta_3}$ with some positive $\delta_3$ depending on $n$ if we choose small divisor conditions suitably.
Another way to estimate the above set (1.3) is called quasi-Töplitz method. For the 1-D derivative wave equation Berti, Biasco and Procesi [9]-[36]) proved that
\[
\lambda_j = j + a_+(\xi) + \frac{m}{2j} + O(\gamma^{2/3}/j) \text{ for } j \geq O(\gamma^{-1/3}).
\]
When \(i > j \geq O(|k|^\gamma \gamma^{-1/3})\), they could obtain
\[
|\langle k, \omega(\xi) \rangle + \lambda_i - \lambda_j| = |\langle k, \omega(\xi) \rangle + i - j + \frac{m(i - j)}{2ij} + O(\gamma^{2/3}/j)|
\]
and the following estimate is close to 1-D wave equation since they could impose first order Melnikov conditions \(|\langle k, \omega(\xi) \rangle + h| \geq 2\gamma^{2/3}/|k|^\gamma\) for any \(h \in \mathbb{Z}\).

In the concerned problem we will use a different method to estimate the set (1.3). The trick is to use the discrete difference matrix and its element decay from the norm estimates. In fact from the iteration step (see (4.5)) one can prove
\[
|\Delta(A - A_0)^{j+1}_j| = |(A - A_0)^{j+1}_j - (A - A_0)^j_j| \leq Cj^{2\beta},
\]
which is proved to be important for measure estimate as the following. Note
\[
|k \cdot \omega + \lambda_i - \lambda_j| \geq |k \cdot \omega + \nu_i - \nu_j| - |(A - A_0)^j_j - (A - A_0)^i_i|
\]
\[
\geq |k \cdot \omega + i - j| - \sum_{l=j}^{i-1} |\Delta(A - A_0)^i_l| \geq |k \cdot \omega + i - j| - \frac{C|i - j|}{j^{2\beta}}.
\]
The following estimate is similar as above.

In the end let us review the previous works on the reducibility and the behaviors of solutions in Sobolev spaces. Reducibility for PDEs in high dimension was initiated by Eliasson-Kuksin [18]. We can refer to [25], [29] and [30] for higher-dimensional QHO with bounded potential. The reducibility result for n-D QHO with polynomial perturbations was first set up in [5]. Montalto [35] obtained the first reducibility result for linear wave equations with unbounded perturbations on \(\mathbb{T}^d\), which can be applied to the linearized Kirchhoff equation in higher dimension. For transportation equations with unbounded perturbations, see Bambusì, Langella and Montalto [7]-[21]). Recently, there are also some research papers [22, 23] with a linear Schrödinger equation on Zoll manifold with unbounded potential. By implementing the above techniques the KAM-type results of quasi-linear PDEs such as incompressible Euler flows in 3-D and forced Kirchhoff equation on \(\mathbb{T}^d\) have been established in [1] and [14] respectively.

The reducibility results usually imply the boundedness of Sobolev norms. We refer to the papers [5, 15, 20, 24, 31, 33, 38] for the growth rate of the solutions of QHO including the upper-lower bound. There are also many literatures, e.g. [6, 8, 10, 34], which are closely relative to the upper growth bound of the solution in Sobolev space.

The rest paper will be organized as follows. In Section 2, a new reducibility theorem is presented. In Section 3, we check all the hypotheses of the reducibility theorem are satisfied which follows the main theorem. In the following section we prove the reducibility theorem. Finally, the appendix contains some technical Lemmas.

2. Reducibility theorem

In this section we will state an abstract reducibility theorem for a quasi-periodic in time system of the form \(i \dot{u} = (A + \epsilon P(\omega t))u\) with \(A = \text{diag}(\nu_i)_{i \geq 1}\).
2.1. Setting. We first introduce some spaces and work out various properties.

Infinite matrices. In the introduction we have introduced the Banach space $\mathcal{M}_\alpha$ and its subspace $\mathcal{M}_{\alpha,\beta}$. As [26] we denote by $\mathcal{M}_\alpha$ the set of infinite matrices $A : \mathbb{N}_0 \times \mathbb{N}_0 \mapsto \mathbb{C}$ that satisfy $|A|_{\alpha} := \sup_{i,j \in \mathbb{N}_0} (1 + |i - j|)^{\alpha+1}|A_{ij}| < \infty$. Similarly, we denote by $\mathcal{M}_{\alpha+\beta}$ the subspace of $\mathcal{M}_{\alpha+}$: an infinite matrix $A$ belongs to $\mathcal{M}_{\alpha+}$ if

$$|A|_{\alpha+} := \sup_{i,j \in \mathbb{N}_0} (1 + |i - j|)^{\alpha+1}|A_{ij}| + \sup_{i,j \in \mathbb{N}_0} (1 + |i - j|)(ij)^{2}\Delta A_{ij} < \infty.$$ 

We remark that $\mathcal{M}_{\alpha+} \subset \mathcal{M}_\alpha$ and $\mathcal{M}_{\alpha+} \subset \mathcal{M}_{\alpha,\beta}$ for any $\alpha, \beta \in \mathbb{R}$.

The following structural lemma is proved in Appendix A.

**Lemma 2.1.** Let $\alpha \geq \beta > 0$. There exists a constant $C \equiv (\alpha, \beta) > 0$ such that

(i) Let $A, B \in \mathcal{M}_{\alpha+}$. Then $AB$ belongs to $\mathcal{M}_{\alpha+}$ and $\|AB\|_{\alpha+} \leq C|A|_{\alpha+}\|B\|_{\alpha+}$.

(ii). Let $A \in \mathcal{M}_{\alpha,\beta}$ and $B \in \mathcal{M}_{\alpha+}$. Then $AB$ and $BA$ belong to $\mathcal{M}_{\alpha,\beta}$ and

$$\|AB\|_{\alpha,\beta}, \|BA\|_{\alpha,\beta} \leq C|A|\|B\|_{\alpha+}.$$

(iii). Let $A \in \mathcal{M}_{\alpha+}$. Then we have $\|A\|_{(\ell^2)} \leq C|A|_{\alpha+}$ for any $s \in (-2\alpha - 1, 2\alpha + 1)$.

(iv). Let $A \in \mathcal{M}_\alpha$. Then we have

$$\|A\|_{(\ell^2)} \leq C|A|_{\alpha}, \quad \text{if} \quad \alpha \in (0, \frac{1}{2}],$$

$$\|A\|_{(\ell^{s,2})} \leq C|A|_{\alpha}, \quad \forall \ s < 2\alpha - 2, \quad \text{if} \quad \alpha \in (\frac{1}{2}, 1],$$

$$\|A\|_{(\ell^2)} \leq C|A|_{\alpha}, \quad \text{if} \quad \alpha \in (1, \infty).$$

Parameter. In this paper $\omega$ will play the role of a parameter belonging to $\mathcal{D}_0 := [0, 2\pi)^n$. All the constructed maps will depend on $\omega$ with $C^1$ regularity. When a map is only defined on a Cantor subset of $\mathcal{D}_0$ the regularity has to be understood in the Whitney sense.

Let $\sigma > 0$ and $\mathcal{D} \subset \mathcal{D}_0$. We denote by $\mathcal{M}_{\alpha,\beta}(\mathcal{D}, \sigma)$ the set of $C^1$ mappings $D \times \mathbb{T}_\sigma \ni (\omega, \theta) \mapsto P(\omega, \theta) \in \mathcal{M}_{\alpha,\beta}$ which is real analytic in $\theta \in \mathbb{T}_\sigma$ and is equipped with the norm $\|P\|_{\alpha,\beta,D,\sigma} := \sup_{|\omega| < \sigma, \omega \in \mathcal{D}, l=0,1} |\partial_2^l P(\omega, \theta)|_{\alpha,\beta}$. The subset of $\mathcal{M}_{\alpha,\beta}(\mathcal{D}, \sigma)$ formed by the mappings $B$ such that $B(\omega, \theta) \in \mathcal{M}_{\alpha+}$ is denoted by $\mathcal{M}_{\alpha+}(\mathcal{D}, \sigma)$ and equipped with the norm $\|B\|_{\alpha+}(\mathcal{D}, \sigma) := \sup_{|\omega| < \sigma, \omega \in \mathcal{D}, l=0,1} |\partial_2^l B(\omega, \theta)|_{\alpha+}$. The space of mappings $A \in \mathcal{M}_{\alpha,\beta}(\mathcal{D}, \sigma)$ that are independent of $\theta$ will be denoted by $\mathcal{M}_{\alpha,\beta}(\mathcal{D})$ and equipped with the norm $\|A\|_{\alpha,\beta} := \sup_{|\omega| \leq 1} |\partial_\omega^l A(\omega)|_{\alpha,\beta}$.

2.2. Hypothesis on the spectrum. Now we formulate our hypothesis on $\nu_i, \ i \in \mathbb{N}_0$.

**Hypothesis H1** (asymptotics). Assume that there exist some absolute positive constants $c_0, c_1$ and $\delta$ such that for all $i, j \in \mathbb{N}_0$

$$|\nu_i - \nu_j| \geq c_0|i - j| \quad \text{and} \quad |\nu_{i+1} - \nu_i + \nu_j - \nu_{j+1}| \leq c_1|i - j|^{\delta}.$$

**Hypothesis H2** (second Melnikov condition in measure). There exist some absolute positive constants $\tau_1, \tau_2$ and $c_2$ such that the following holds: for each $\gamma > 0$ and $K \geq 1$ there exists a closed subset $\mathcal{D} = \mathcal{D}(\gamma, K) \subset \mathcal{D}_\gamma$ satisfying $\text{Meas}(\mathcal{D}_\gamma \setminus \mathcal{D}) \leq c_2 \gamma^{\tau_1} K^{\tau_2}$ such that for all $\omega \in \mathcal{D}$, all $k \in \mathbb{Z}$ with $0 < |k| \leq K$ and all $i, j \in \mathbb{N}_0$ we have

$$|k \cdot \omega + \nu_i - \nu_j| \geq \gamma(1 + |i - j|).$$
2.3. The reducibility theorem. Consider the non-autonomous system in $\mathcal{H}$:

$$i\dot{u} = (A + \epsilon P(\omega_1 t, \omega_2 t, \ldots, \omega_n t))u,$$

where $A = \text{diag}(\nu_i)_{i \geq 1}$ and $P$ is defined for $i, j \in \mathbb{N}_0$ by

$$P_j^i(\omega t) = \int_{\mathbb{R}} V(x, \omega t)h_i(x)h_j(x) \, dx,$$

and we define

$$\mathcal{H} = \begin{cases} \ell_1^2, & \text{for } 0 < \alpha \leq 1/2, \\ \ell_0^2, & \text{for } \alpha > 1/2. \end{cases}$$

**Theorem 2.2.** Assume that $(\nu_i)_{i \geq 1}$ satisfies Hypotheses H1, H2 and that the perturbation matrix $P$ is hermitian and $P \in \mathcal{M}_{\alpha, \beta}(D_0, \sigma)$ for some $0 < \beta \leq \min\{\alpha, \delta\}$ and $\sigma > 0$. There exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ there is a subset $\mathcal{D}_\epsilon \subset D_0 = [0, 2\pi]^n$ of asymptotically full measure such that for all $\omega \in \mathcal{D}_\epsilon$, the system (2.1) reduces in $\mathcal{H}$ to an autonomous system

$$i\dot{v} = A^\infty(\omega)v, \quad A^\infty(\omega) = \text{diag}(\lambda_i^\infty(\omega))_{i \geq 1}$$

by a coordinate transformation $u = \Phi(\omega, \epsilon t)v$, where $\Phi(\omega, \epsilon) = \Phi(\omega, \epsilon_1)\Phi(\omega, \epsilon_2)$ is unitary on $\ell_0^2$ and real analytic in $\theta \in \mathbb{T}_n/2$ and $\lambda_i^\infty(\omega) \in \mathbb{R}$ is $C^1$ smooth in $\omega$ and close to $\nu_i$ for all $i \in \mathbb{N}_0$.

More precisely, let $p \in [0, 2\alpha + 1]$ then there is a $C > 0$ such that

$$\text{Meas}(D_0 \setminus \mathcal{D}_\epsilon) \leq Ce^{\delta p} \text{ with } \nu_i = \frac{\beta}{\beta + 1} \max\{\tau_1, 1\}, \quad |\lambda_i^\infty - \nu_i| \leq Ce, \quad \forall \, i \geq 1,$$

$$\|\Phi(\omega, \epsilon) - \text{Id}\|_{\mathcal{B}(\ell_p)} \leq Ce^{\delta p}, \quad \forall \, \theta \in \mathbb{T}_n/2.$$

3. Application to Quantum Harmonic Oscillator on $\mathbb{R}$

In this section we will prove Theorem 1.1 as a direct corollary of the reducibility theorem. Before the checks on all the hypotheses, we expand the Schrödinger equation (1.1) on the Hermite basis $(h_i)_{i \geq 1}$ and then it is equivalent to the system (2.1) with $\nu_i = 2i - 1$ for $i \geq 1$.

3.1. Verification of the spectrum. For 1-D quantum harmonic oscillator we have $\nu_i = 2i - 1$ for $i \geq 1$. Then H1 is verified with $c_0 = c_1 = \delta = 1(\delta$ can be any positive number). Besides, we invoke Lemma B.2 to confirm H2 for $\tau_1 = 1$ and $\tau_2 = n + 1$. Then we only remain to verify that $P$ defined by (2.2) belongs to $\mathcal{M}_{\alpha, \beta}(D_0, \sigma)$ for some $\alpha, \beta$ with $0 < \beta \leq \min\{\alpha, \delta\}$, which follows from the subsequent lemma.

Before the proof we introduce two important operators and some facts. From Lemma 3.1 in [12], one obtains for any $i \geq 1,$

$$\partial_x h_i(x) = -\sqrt{\frac{i}{2}} h_{i+1}(x) + \sqrt{\frac{i-1}{2}} h_{i-1}(x),$$

$$x h_i(x) = \sqrt{\frac{i}{2}} h_{i+1}(x) + \sqrt{\frac{i-1}{2}} h_{i-1}(x).$$

Denote $T = \partial_x + x$ and $T^\dagger = -\partial_x + x$. From (3.1), (3.2) and a straightforward computation, we have $T^* = T^\dagger, TT^\dagger = H + \text{Id}$ and

$$Th_i(x) = \sqrt{2(i-1)} h_{i-1}(x), \quad T^\dagger h_i(x) = \sqrt{2i} h_{i+1}(x), \quad TT^\dagger h_i(x) = 2i h_i(x).$$
Lemma 3.1. Assume that a $C^1$ smooth function $V(x, \theta)$ analytically in $\theta$ extends to $\mathbb{T}_\sigma^n$ for $\sigma > 0$ and fulfills (1.2), then the perturbation matrix $P(\theta)$ defined by (2.2) is real analytic from $\mathbb{T}_\sigma^n$ to $\mathcal{M}_{1, \frac{1}{2}}$.

**Proof.** To simplify notation we will denote by $\langle \cdot, \cdot \rangle$ the standard scalar product in $L^2(\mathbb{R})$ and by superscript $"u"$ the partial derivative (w.r.t.$x$). Clearly, $|P_i^j(\theta)| = |\langle Vh_i, h_j \rangle| \leq C$ by (1.2). We hereafter let $i \neq j$. Since $Hh_i = (2i - 1)h_i, i \geq 1$, from (2.2) we have

$$
(2i - 2j)P_i^j(\theta) = \langle Vh_i, h_j \rangle - \langle Vh_j, h_i \rangle = -\langle V\partial_x h_i, h_j \rangle + \langle Vh_i, \partial_x h_j' \rangle
$$

$$
= \langle V'h_i', h_j \rangle + \langle V'h_j', h_j' \rangle - \langle V'h_j, h_j' \rangle = \langle V'h_i', h_j' \rangle - \langle V'h_j', h_i \rangle.
$$

In addition, $(x)^{-1}\partial_x$ is a pseudo-differential operator of order 0, which implies the boundedness from $L^2$ into itself (see [2, 3, 12, 37] for details). Also, by (1.2) we obtain

$$
|\langle V'h_i', h_j \rangle| \leq \sup_{\|f\|_{L^2}=1} |\langle V'(x)(x)^{-1}\partial_x f, g \rangle| \leq \|V'(x)(x)\|_{B(L^2)} \|\langle x \rangle^{-1}\partial_x\|_{B(L^2)} \leq C,
$$

where $\langle x \rangle = \sqrt{1 + x^2}$. Similarly, $|\langle V'h_j', h_i \rangle| \leq C$ by (1.2). It follows that there is a $C > 0$ such that for all $i, j \geq 1$

$$
(1 + |i - j|)P_i^j(\theta) \leq C, \quad \forall \theta \in \mathbb{T}_\sigma^n.
$$

We now commence the estimates on $\Delta P_i^j(\theta)$. From the definition we have

$$
\Delta P_i^j(\theta) = \langle Vh_{i+1}, h_{j+1} \rangle - \langle Vh_i, h_j \rangle = \frac{1}{2\sqrt{i}}(VT'h_i, T'h_j) - \langle Vh_i, h_j \rangle
$$

$$
= \frac{1}{2\sqrt{i}}\langle TV'T'h_i, h_j \rangle - \langle Vh_{i+1}, h_j \rangle = \frac{1}{2\sqrt{i}}\langle VTT'h_i, h_j \rangle - \langle Vh_i, h_j \rangle + \frac{1}{2\sqrt{i}}\langle V'T'h_i, h_j \rangle
$$

$$
= \left(\frac{-1}{\sqrt{j}}\right)\langle Vh_i, h_j \rangle + \frac{1}{2\sqrt{i}}\langle V'T'h_i, h_j \rangle = \frac{i - j}{\sqrt{i} + \sqrt{j}}\langle Vh_i, h_j \rangle + \frac{1}{2\sqrt{i}}\langle V'T'h_i, h_j \rangle.
$$

We likewise have $|\langle V'T'h_i, h_j \rangle| \leq C$ by (1.2). Collecting the last estimate and (3.3) leads to that for all $i, j \geq 1$

$$
\sqrt{i}j|\Delta P_i^j(\theta)| \leq (1 + |i - j|)|\langle Vh_i, h_j \rangle| + |\langle V'T'h_i, h_j \rangle| \leq C, \quad \forall \theta \in \mathbb{T}_\sigma^n.
$$

Combining (3.3) with (3.4), it follows that $P(\theta) \in \mathcal{M}_{1, \frac{1}{2}}$. The rest proof is clear. \(\square\)

3.2. **Proof of Theorem 1.1.** Expanded on Hermite basis $(h_i)_{i \geq 1}$ the Schrödinger equation (1.1) reads as the non-autonomous system (2.1) with $\nu_i = 2i - 1$. By subsection 3.1 if set $\alpha = \delta = 1$ and $\beta = \frac{1}{2}$, then all the hypotheses of Theorem 2.2 are satisfied. Since $\alpha = 1$, the system (2.1) reduces in $L_0^2$ to an autonomous system (2.3). More precisely, in the new coordinates given by Theorem 2.2 $u = \Phi_{\omega, \epsilon}(\omega t)v$, the original system (2.1) becomes an autonomous system $i\dot{v} = A^\omega v$ with $A^\omega = \text{diag}(\lambda^\omega)_{i \geq 1}$. We solve the Cauchy problem (1.1) subject to the initial datum $\psi(x, 0) = \sum_{i \geq 1} u_i(0)h_i(x) \in L^2$, which reads

$$
\psi(x, t) = \sum_{i \geq 1} u_i(t)h_i(x) \in L^2 \text{ with } u(t) = \Phi_{\omega, \epsilon}(\omega t)e^{-itA^\omega}\Phi_{\omega, \epsilon}^{-1}(0)u(0).
$$

Now define the coordinate transformation $\Psi_{\omega, \epsilon}(\theta) \in B(L^2)$ by

$$
\Psi_{\omega, \epsilon}(\theta)\left(\sum_{i \geq 1} v_ih_i(x)\right) = \sum_{i \geq 1} (\Phi_{\omega, \epsilon}(\theta)v)_i h_i(x),
$$

where
and let \( p \in [0, 3) \). If the initial data \( \zeta(0) = \sum_{j \geq 1} v_j(0)h_j(x) \in \mathcal{H}^p \), then one has \( \zeta(\cdot, t) = \sum_{j \geq 1} v_j(t)h_j(x) \in C^0(\mathbb{R}, \mathcal{H}^p) \) satisfies \( i\partial_t \zeta = H^\infty \zeta \) with \( H^\infty = \text{diag}(\lambda^\infty)_{i \geq 1} \), where \( v_j(t) = e^{-i\lambda^j_t}v_j(0) \) for all \( j \geq 1 \). For \( \omega \in \mathcal{D}_\epsilon \), if denote \( \psi(\cdot, t) = \Psi_{\omega,t}(\omega t) \zeta(\cdot, t) \) then it is clear that \( \psi(\cdot, t) \in C^0(\mathbb{R}, \mathcal{H}^p) \) and satisfies (1.1). The other side is similar. Corollary 1.3 and Corollary 1.4 are similar as [26].

4. PROOF OF REDUCIBILITY THEOREM

In this section we will prove Theorem 2.2 by KAM methods with the general strategy in the opening subsection.

4.1. General strategy. Consider the non-autonomous system of the form

\[
iu = (A + P)u, \quad (4.1)
\]

where \( A \) is diagonal. Note that at the beginning of the KAM iteration \( A = A_0 = \text{diag}(\nu_i)_{i \geq 1} \) is independent of \( \omega \), and the perturbation matrix \( P(\theta) \) is of size \( O(\epsilon) \). We search for a coordinate transformation \( u = e^{B}v \) with \( B \) of size \( O(\epsilon) \) to conjugate the system (4.1) into

\[
i\hat{v} = (A^+ + P^+)v,
\]

where \( A^+ \) is still diagonal and \( \epsilon \)-close to \( A \), and the new perturbation \( P^+ \) is of size \( O(\epsilon^2) \) at least formally. More precisely, as a consequence of the construction and Bambusi [3] or [4] we have that

\[
A^+ + P^+ = A + [A, B] - i\hat{B} + P + \int_0^1 e^{-sB}\left([A, B] - i\hat{B} + P\right) + sP, B|e^{sB} \, ds.
\]

So to achieve the goal at least formally we should solve the homological equation

\[
[A, B] - i\hat{B} + P = A^+ - A + R
\]

with \( R \) of size \( O(\epsilon^2) \). Concretely, we hereafter let \( A^+ = A + \tilde{A} \) with \( \tilde{A} = \text{diag}(\overline{P}) = O(\epsilon) \) (The bar denotes Fourier average) and

\[
P^+ = R + \int_0^1 e^{-sB}\left([\tilde{A} + R] + sP, B|e^{sB} \, ds = O(\epsilon^2).
\]

Repeating iteratively the same procedure with \( A^+ \) instead of \( A \), we will construct a coordinate transformation \( u = \Phi_{\omega,v}v \) to conjugate the original non-autonomous system (2.1) into an autonomous system \( i\hat{v} = A^\infty v \) with \( A^\infty \) diagonal and \( \epsilon \)-close to \( A_0 \).

4.2. Homological equation. Following the previous strategy, in this section we will consider a homological equation of the form \( [A, B] - i\hat{B} + P = \text{remainder with } A \text{ diagonal and close to } A_0 = \text{diag}(\nu_i)_{i \geq 1} \) and \( P \in \mathcal{M}_{\alpha, \beta} \) with \( 0 < \beta \leq \min\{\alpha, \delta\} \). Then we will construct a solution \( B \in \mathcal{M}_{\alpha+\beta} \) by the following proposition.

Proposition 4.1. Denote \( \mathcal{D} \subset \mathcal{D}_0 \). Let \( \mathcal{D} \ni \omega \mapsto A(\omega) = \text{diag}(\lambda_i(\omega))_{i \geq 1} \) be a \( C^1 \) mapping that verifies for some \( 0 < c \leq \min \left\{ \frac{1}{4}, \frac{1}{3} \right\} \)

\[
\|A - A_0\|_{\mathcal{M}_{\alpha, \beta}} \leq c.
\]

Let \( P \in \mathcal{M}_{\alpha, \beta}(\mathcal{D}, \sigma) \) with \( 0 < \beta \leq \min\{\alpha, \delta\} \) be hermitian, and \( 0 < \kappa \leq \gamma \leq \min\{\frac{1}{2}, 1\} \) and \( K \geq 1 \). There exists a subset \( \mathcal{D}' = \mathcal{D}'(\kappa, K) \subset \mathcal{D} \) fulfilling

\[
\text{Meas}(\mathcal{D} \setminus \mathcal{D}') \leq C\kappa^{\epsilon_1}K^{\epsilon_2}
\]

(4.3)
and $C^1$ mappings $\tilde{A} : \mathcal{D}' \mapsto \mathcal{M}_{\alpha, \beta}$ diagonal, $R : \mathcal{D}' \times \mathbb{T}^{n}_{\sigma} \mapsto \mathcal{M}_{\alpha, \beta}$ hermitian, $B : \mathcal{D}' \times \mathbb{T}^{n}_{\sigma} \mapsto \mathcal{M}_{\alpha, \beta}$ anti-hermitian, and analytic in $\theta$, such that

$$[A, B] - iB = \tilde{A} - P + R$$

and for any $0 < \sigma' < \sigma$

$$\|\tilde{A}\|_{\alpha, \beta}^{D_{\sigma'}} \leq \|P\|_{\alpha, \beta}^{D_{\sigma}};$$

$$\|R\|_{\alpha, \beta}^{D_{\sigma'}} \leq \frac{Ce^{-\frac{K}{2}(\sigma - \sigma')}}{(\sigma - \sigma')^n} \|P\|_{\alpha, \beta}^{D_{\sigma}};$$

$$\|B\|_{\alpha, \beta}^{D_{\sigma'}} \leq \frac{CK}{\kappa^1(\sigma - \sigma')^n} \|P\|_{\alpha, \beta}^{D_{\sigma}}.$$  

Here the constant $C > 0$ depends on $\alpha, \beta, \delta, c, c_0, c_1, c_2$ and $\iota_1 = \frac{\beta}{\beta + 1} \max\{\iota_1, 1\}, \iota_2 = \max\{\tau_2, n + 1\}$ with all the explicit parameters shown in Hypotheses H1, H2.

**Proof.** Expanding in Fourier series, the homological equation (4.4) reads

$$L\hat{B}(k) = \delta_{k,0} \tilde{A} - \hat{P}(k) + \hat{R}(k),$$

where $\delta_{k,l}$ denotes the Kronecker symbol and $L := L(\omega, k)$ is the linear operator, acting on $\mathcal{M}_0$, defined by

$$LM = k \cdot \omega M + [A(\omega), M].$$

Taking its matrix elements between the eigenvalues of $A$ the equation (4.8) becomes

$$k \cdot \omega \hat{B}_i^j(k) + (\lambda_i - \lambda_j) \hat{B}_i^j(k) = \delta_{k,0} \tilde{A}_i^j - \hat{P}_i^j(k) + \hat{R}_i^j(k).$$

First solve this equation when $|k| + |i - j| = 0$ (i.e. $k = 0$ and $i = j$) by defining

$$\hat{B}_i^j(0) = 0, \quad \hat{R}_i^j(0) = 0$$

and $\tilde{A}_i^j = \hat{P}_i^j(0)$.

Then we set $\tilde{A}_i^j = 0$ for $i \neq j$ in such a way $\tilde{A} \in \mathcal{M}_{\alpha, \beta}$ and fulfills $|\tilde{A}|_{\alpha, \beta} \leq |\hat{P}(0)|_{\alpha, \beta}$. The estimates of the derivatives (w.r.t. $\omega$) are obtained by differentiating the expression of $\tilde{A}$. Taking all the estimations on $\tilde{A}$ leads to (4.5).

Now let us consider the remainder case when $|k| + |i - j| > 0$. We solve equation (4.9) by defining for $i, j \geq 1$

$$\hat{R}_i^j(k) = \begin{cases} 0, & \text{for } |k| \leq K, \\ \hat{P}_i^j(k), & \text{for } |k| > K; \end{cases}$$

$$\hat{B}_i^j(k) = \begin{cases} 0, & \text{for } |k| > K \text{ or } |k| + |i - j| = 0, \\ \frac{\hat{P}_i^j(k)}{k \cdot \omega + \lambda_i - \lambda_j}, & \text{in the other cases}. \end{cases}$$

Before the estimations on such matrices, first note that with this resolution, $R$ and $\tilde{A}$ are hermitian but $B$ is anti-hermitian because $A$ and $P$ are hermitian. From equation (4.10), a canonical Fourier analysis leads to that for any $|3\theta| < \sigma'$

$$|R(\theta)|_{\alpha, \beta} \leq \frac{Ce^{-\frac{K}{2}(\sigma - \sigma')}}{(\sigma - \sigma')^n} \sup_{|3\theta| < \sigma'} |P(\theta)|_{\alpha, \beta}.$$  

In order to estimate $B$, we will use Lemma B.1 stated in Appendix B. Let us face the small divisors $k \cdot \omega + \lambda_i - \lambda_j$, $i, j \in \mathbb{N}_0$. We will distinguish two cases depending on whether $k = 0$ or not.
(a). The case $k = 0$. In this case $i \neq j$ and from (4.2) and Hypothesis H1 we have
\[ |k \cdot \omega + \lambda_i - \lambda_j| \geq |\nu_i - \nu_j| - |\lambda_i - \nu_i| - |\lambda_j - \nu_j| \geq c_0 |i - j| - 2c \geq \kappa (1 + |i - j|). \]
The last estimate and (4.2) allow us to use Lemma B.1 to conclude that for $i \neq j$
\[ (1 + |i - j|)^{a + 1} |\hat{B}_i(0)| + (1 + |i - j|) |\Delta \hat{B}_i(0)| \leq \frac{C}{\kappa^2} |\hat{P}(0)|_{\alpha, \beta}. \]
Together with $\hat{B}_i(0) = 0$ we obtain that $\hat{B}(0) \in \mathcal{M}_{\alpha+, \beta}$ and
\[ |\hat{B}(0)|_{\alpha+, \beta} \leq \frac{C}{\kappa^2} |\hat{P}(0)|_{\alpha, \beta}. \quad (4.13) \]

(b). The case $k \neq 0$. Concretely, in this case we only solve the main terms of Fourier series truncated by $K$ and throw out a small set of parameter $\omega$. Using Hypothesis H2, for each $\gamma > 0$ there exists a subset $D_1 := D(2\gamma, K)$ fulfilling $\text{Meas}(D_0 \setminus D_1) \leq C\gamma^\tau_1 K^{\tau_2}$ such that for all $\omega \in D_1$
\[ |k \cdot \omega + \nu_i - \nu_j| \geq 2\gamma (1 + |i - j|), \quad \forall i, j \in \mathbb{N}_0, \forall k \in \mathbb{Z}^n \text{ with } 0 < |k| \leq K. \]
Without loss of generality, let $i \geq j$ in the following discussion. By (4.2) the last equation implies that
\[ |k \cdot \omega + \lambda_i - \lambda_j| \geq |k \cdot \omega + \nu_i - \nu_j| - |(A - A_0)^i - (A - A_0)^j| \]
\[ = |k \cdot \omega + \nu_i - \nu_j| - \sum_{l=j}^{i-1} |\Delta (A - A_0)^i| - |(A - A_0)^j| \]
\[ \geq 2\gamma (1 + |i - j|) - \sum_{l=j}^{i-1} \frac{c}{|l_2^\beta|} \geq 2\gamma (1 + |i - j|) - \frac{c |i - j|}{|j_2^\beta|} \geq \left( 2\gamma - \frac{c}{|j_2^\beta|} \right) (1 + |i - j|) \]
\[ \geq \gamma (1 + |i - j|) \geq \kappa (1 + |i - j|), \quad \text{provided } i \geq j \geq \left( \frac{\kappa}{\gamma} \right)^{\frac{1}{\beta}}. \]
Now let $j \leq \left( \frac{\kappa}{\gamma} \right)^{\frac{1}{\beta}}$. In fact, the inequality $|i - j| \geq CK$ turns out that $|k \cdot \omega + \lambda_i - \lambda_j| \geq \kappa (1 + |i - j|)$. Then we only need to face the small divisors
\[ k \cdot \omega + \lambda_i - \lambda_j, \quad \text{for } 0 \leq i - j \leq CK, \quad j \leq \left( \frac{\kappa}{\gamma} \right)^{\frac{1}{\beta}} \text{ and } k \in \mathbb{Z}^n \text{ with } 0 < |k| \leq K. \]
Clearly, one has $i \leq C\gamma^{-\frac{1}{\beta}}$. Since $|\partial_\omega (k \cdot \omega) \cdot \frac{k}{|k|} | = |k| \geq 1$, then by (4.2) one has
\[ |\partial_\omega (k \cdot \omega + \lambda_i (\omega) - \lambda_j (\omega)) \cdot \frac{k}{|k|} | \geq \frac{|k|}{2}, \]
which allows us to use Lemma B.2 to conclude that for any $i, j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$ with $0 < |k| \leq K$ one has
\[ |k \cdot \omega + \lambda_i - \lambda_j| \geq \kappa (1 + |i - j|), \]
expect a subset $F_{i,j,k} \subset D$ fulfilling $\text{Meas} (F_{i,j,k}) \leq \frac{C\kappa (1 + |i - j|)}{|k|}$. Denoting $D_2$ be the union of $F_{i,j,k}$ for $i \leq C\gamma^{-\frac{1}{\beta}}$, $j \leq C\gamma^{-\frac{1}{\beta}}$ with $|i - j| \leq CK$ and $0 < |k| \leq K$ we have
\[ \text{Meas}(D_2) \leq C\gamma^{-\frac{1}{\beta}} K^{n+1}. \]
Let $D' = D_1 \cup (D \setminus D_2)$ and $\kappa = \gamma^{1+\frac{1}{\beta}}$, $\tau_1 = \frac{\beta}{\beta+1} \max \{ \tau_1, 1 \}$, $\tau_2 = \max \{ \tau_2, n+1 \}$, then
\[ \text{Meas}(D \setminus D') \leq \text{Meas}(D_0 \setminus D_1) + \text{Meas}(D_2) \leq C\gamma^{\tau_1} K^{\tau_2} + C\gamma K^{n+1} \leq C\kappa^{\tau_1} K^{\tau_2}. \]
By construction for all $\omega \in \mathcal{D}'$ we have $|k \cdot \omega + \lambda_i - \lambda_j| \geq \kappa (1 + |i - j|)$, which allows us to use Lemma B.1 to conclude that for all $0 < |k| \leq K$

$$\hat{B}(k) \in \mathcal{M}_{\alpha,\beta}$$

and $|\hat{B}(k)|_{\alpha,\beta} \leq \frac{C}{k^2} |\hat{P}(k)|_{\alpha,\beta}$.

Collecting the last estimate and (4.13) we have for any $|\Im \theta| < \sigma'$

$$|B(\theta)|_{\alpha,\beta} \leq \frac{C}{\kappa^2 (\sigma - \sigma')^n} \sup_{|\Im \theta| < \sigma} |P(\theta)|_{\alpha,\beta}.$$

(4.14)

For the proceeding estimates on derivatives (w.r.t. $\omega$) we differentiate (4.8) to get

$$L \partial_\omega \hat{B}(\omega, k) = \delta_{k,0} \partial_\omega \hat{A}(\omega) - \left( (\partial_\omega L) \hat{B}(\omega, k) + \partial_\omega \hat{P}(\omega, k) \right) + \partial_\omega \hat{R}(\omega, k),$$

which is an equation of the same type as (4.8) for $\partial_\omega \hat{B}(\omega, k)$ and $\partial_\omega \hat{R}(\omega, k)$ with $\hat{P}(k)$ replaced by $Q(\omega, k) := (\partial_\omega L) \hat{B}(\omega, k) + \partial_\omega \hat{P}(\omega, k)$. We can solve this equation by defining

$$\partial_\omega \hat{B}(\omega, k) = \chi_{|k| \leq K} (k) L^{-1}(\omega, k) \left( \delta_{k,0} \partial_\omega \hat{A}(\omega) - Q(\omega, k) \right)$$

$$\partial_\omega \hat{R}(\omega, k) = \chi_{|k| > K} (k) Q(\omega, k) = \chi_{|k| > K} (k) \partial_\omega \hat{P}(\omega, k).$$

By (4.2) we get for all $|k| \leq K$, $|(\partial_\omega L) \hat{B}(\omega, k)|_{\alpha,\beta} \leq CK |\hat{B}(\omega, k)|_{\alpha,\beta}$, which follows

$$|\partial_\omega \hat{B}(\omega, k)|_{\alpha,\beta} \leq \left\| L^{-1}(\omega, k) \right\| \left( |(\partial_\omega L) \hat{B}(\omega, k)|_{\alpha,\beta} + |\partial_\omega \hat{P}(\omega, k)|_{\alpha,\beta} \right) \leq \frac{CK}{\kappa^4} \left( \left| \hat{P}(\omega, k) \right|_{\alpha,\beta} + \left| \partial_\omega \hat{P}(\omega, k) \right|_{\alpha,\beta} \right).$$

A canonical Fourier analysis leads to that for any $|\Im \theta| < \sigma'$

$$|\partial_\omega B(\omega, \theta)|_{\alpha,\beta} \leq \frac{CK}{\kappa^4 (\sigma - \sigma')^n} \sup_{|\Im \theta| < \sigma} |P(\omega, \theta)|_{\alpha,\beta} + \sup_{|\Im \theta| < \sigma} |\partial_\omega P(\omega, \theta)|_{\alpha,\beta},$$

$$|\partial_\omega R(\omega, \theta)|_{\alpha,\beta} \leq \frac{C e^{-\frac{\sigma}{\kappa} (\sigma - \sigma')}}{(\sigma - \sigma')^n} \sup_{|\Im \theta| < \sigma} |\partial_\omega P(\omega, \theta)|_{\alpha,\beta}.$$

Collecting the last two estimates and equations (4.12), (4.14) concludes (4.6), (4.7). □

4.3. The KAM iteration. In this section we will present our iterative KAM procedures following the previous strategy in the opening subsection. Let us begin with the initial non-autonomous system

$$i \dot{u} = (A_0 + P_0(\omega t)) u,$$

where $A_0 = A = \text{diag}(\nu_i)_{i \geq 1}$ fulfilling Hypotheses H1 and H2, $\omega \in \mathcal{D}_0$ and $P_0 = \epsilon P \in \mathcal{M}_{\alpha,\beta}(\mathcal{D}_0, \sigma_0)$ with $0 < \beta \leq \min\{\alpha, \delta\}$ and $\sigma_0 = \sigma$. We will build iteratively coordinate transformation $u = \phi_{m+1} v$ to conjugate the system $i \dot{u} = (A_m + P_m) u$ with $A_m = \text{diag}(A_{m,i})_{i \geq 1}$ and $P_m \in \mathcal{M}_{\alpha,\beta}(\mathcal{D}_m, \sigma_m)$ into $i \dot{v} = (A_{m+1} + P_{m+1}) v$ as follows: assume that the construction has been built up to step $m \geq 0$ then (i), we use Proposition 4.1 to construct $B_{m+1}(\omega, \theta)$ solution of the homological equation

$$[A_m, B_{m+1}] - i \dot{B}_{m+1} = \hat{A}_m - P_m + R_m, \quad (\omega, \theta) \in \mathcal{D}_{m+1} \times \mathbb{T}_{\sigma_m}^{n+1},$$

(4.16)
with $\tilde{A}_m(\omega), R_m(\omega, \theta)$ defined for $(\omega, \theta) \in D_{m+1} \times T^n_{\sigma_{m+1}}$ by

$$\tilde{A}_m(\omega) = \left( \delta_{ij} \left( \tilde{P}_m(\omega, 0) \right)_i^j \right)_{i,j \geq 1}, \quad (4.17)$$

$$R_m(\omega, \theta) = \sum_{|k| > K_{m+1}} \tilde{P}_m(\omega, k) e^{ik\theta}, \quad (4.18)$$

(ii). we define $A_{m+1}, P_{m+1}$ for $(\omega, \theta) \in D_{m+1} \times T^n_{\sigma_{m+1}}$ by

$$A_{m+1} = A_m + \tilde{A}_m, \quad (4.19)$$

$$P_{m+1} = R_m + \int_0^1 e^{-sB_{m+1}}[(1 - s)(A_m + R_m) + sP_m, B_{m+1}]e^{sB_{m+1}} \, ds. \quad (4.20)$$

Notice that the coordinate transformation $\phi_{m+1} : v \mapsto u = e^{B_{m+1}}v$ and by construction if $A_m$ and $P_m$ are hermitian, so are $A_{m+1}, R_m$ and $A_{m+1}$, by resolution of homological equation $B_{m+1}$ is anti-hermitian which implies $P_{m+1}$ is hermitian. Following the general scheme $(4.16)-(4.20)$ we obtain that the coordinate transformation $\Phi_m = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_m : v \mapsto u = e^{B_1} \circ e^{B_2} \circ \cdots \circ e^{B_m}v$ conjugates the original system $(4.15)$ into $i\dot{v} = (A_m + P_m)v$ with $A_m$ diagonal and $P_m \in M_{2\alpha, \beta}(\mathcal{D}_m, \sigma_m)$. At step $m$ the Fourier series are truncated at order $K_m$ and the small divisors are controlled by $\kappa_m$. Now we specify the choice of all the parameters. First let $\sigma_0 = \sigma$ and $\|P_0\|_{\alpha, \beta} = \epsilon_0$. Then choose for $m \geq 1$

$$\epsilon_m = \epsilon^{4/3}_{m-1}, \quad \kappa_m = \epsilon^{1/16}_{m-1}, \quad \sigma_m - \sigma_m = \frac{\sigma_0}{2C_s} m^{-2}, \quad K_m = 2(\sigma_{m-1} - \sigma_m)^{-1} \ln \epsilon^{-1}_{m-1}, \quad (4.21)$$

where $C_s = \sum_{m \geq 1} m^{-2}$.

Lemma 4.2. Let $0 < \beta \leq \min\{\alpha, \delta\}$ and $\ell_1 = \frac{\beta}{\beta+1} \max\{\tau_1, 1\}$. There is $\epsilon_s \ll 1$ depending on $\sigma, n, \alpha, \beta, \tau_1, \tau_2$ and $A_0$ such that for all $0 \leq \epsilon < \epsilon_s$ the following holds for all $m \geq 1$: there exist $\mathcal{D}_m \subset \mathcal{D}_{m-1}, B_m \in M_{2\alpha, \beta}(\mathcal{D}_m, \sigma_m)$ and $P_m \in M_{2\alpha, \beta}(\mathcal{D}_m, \sigma_m)$ such that

(i). for all $p \in (-2\alpha - 1, 2\alpha + 1)$ the coordinate transformation

$$\phi_m(\omega, \theta) : \ell^2_p \ni v \mapsto e^{B_m}v = u \in \ell^2_p, \quad \forall (\omega, \theta) \in \mathcal{D}_m \times T^n_{\sigma_m}, \quad (4.22)$$

is linear (unitary on $\ell^2_p$) isomorphism conjugating the system at the $(m - 1)^{th}$ KAM step $i\dot{u} = (A_{m-1} + P_{m-1})u$ into the system at the $m^{th}$ step $i\dot{v} = (A_m + P_m)v$;

(ii). we have

$$\text{Meas}(\mathcal{D}_{m-1} \setminus \mathcal{D}_m) \leq \epsilon^{\frac{1}{14}}_{m-1}, \quad (4.23)$$

$$\|\tilde{A}_{m-1}\|_{\alpha, \beta} \leq \epsilon^{\frac{1}{16}}_{m-1}, \quad (4.24)$$

$$\|B_m\|_{\alpha, \beta} \leq \epsilon^{\frac{2}{3}}_{m-1}, \quad (4.25)$$

and for all $p \in (-2\alpha - 1, 2\alpha + 1)$ the coordinate transformation satisfies

$$\|\phi_m(\omega, \theta) - \text{Id}\|_{B(\ell^2_p)} \leq \epsilon^{\frac{2}{3}}_{m-1}, \quad \forall (\omega, \theta) \in \mathcal{D}_m \times T^n_{\sigma_m}. \quad (4.26)$$

Proof. We proceed by induction applying Proposition 4.1. At the first step the initial $A = A_0$ which implies $(4.2)$, then we use Proposition 4.1 to construct $B_1, A_0, R_0$ and $\mathcal{D}_1, \sigma_1$ such that for all $(\omega, \theta) \in \mathcal{D}_1 \times T^n_{\sigma_1}$

$$[A_0, B_1] - i\dot{B}_1 = \tilde{A}_0 - P_0 + R_0.$$
Due to (4.3) we have
\[
\text{Meas}(D_0 \setminus D_1) \leq C \kappa_1^{\gamma_1} K_1^{\gamma_2} \leq C \epsilon_0^{\frac{1}{2}} \left( (\sigma_0 - \sigma_1)^{-1} \ln \epsilon_0^{-1} \right)^{\gamma_2} \leq \epsilon_0^{\gamma_1}.
\]
In view of (4.7) we have
\[
\|B_1\|_{a+\beta}^{D_{1,\sigma_1}} \leq \frac{C K_1}{\kappa_1^{1/2} (\sigma_0 - \sigma_1)^n} \|P_0\|_{a+\beta}^{D_{0,\sigma_0}} \leq \frac{C \ln \epsilon_0^{-1/2}}{\epsilon_0^{1/2}} (\sigma_0 - \sigma_1)^{n+1} \leq C \sigma_0^{-n} (m + 1)^{n+1} \ln \epsilon_0^{-1} \leq \epsilon_0^{2/3}.
\]
Thus, from the definition and Lemma 2.1 we have for all \( p \in (-2\alpha - 1, 2\alpha + 1) \)
\[
\|\phi_1(\omega, \theta) - \text{Id}\|_{B(\ell_2)} = \|e^{B_1} - \text{Id}\|_{B(\ell_2)} \leq C e^{C\|B_1\|_{a+\beta}^{D_{1,\sigma_1}}} \|B_1\|_{a+\beta}^{D_{1,\sigma_1}} \leq \epsilon_0^{2/3}, \quad \forall (\omega, \theta) \in D_1 \times T^{n}_{m+1}.
\]
Collecting equations (4.5) and (4.6) leads to that \( \|\tilde{A}_0\|_{a+\beta}^{D_{1,\sigma_1}} \leq \epsilon_0 \)
\[
\|R_0\|_{a+\beta}^{D_{1,\sigma_1}} \leq \frac{C e^{C\|B_1\|_{a+\beta}^{D_{1,\sigma_1}}} \|P_0\|_{a+\beta}^{D_{0,\sigma_0}}}{\|B_1\|_{a+\beta}^{D_{1,\sigma_1}}} \leq C \sigma_0^{2/3} \epsilon_0^{1/3} \leq \frac{1}{2}\epsilon_0^{4/3}.
\] (4.26)
Besides, by Lemma 2.1 we have for \( s \in [0, 1] \)
\[
\|((1-s)(\tilde{A}_0 + R_0) + sP_0, B_1)\|_{a+\beta}^{D_{1,\sigma_1}} \leq C \|P_0\|_{a+\beta}^{D_{0,\sigma_0}} \cdot \|B_1\|_{a+\beta}^{D_{1,\sigma_1}} \leq C \epsilon_0^{5/3}.
\]
It follows that
\[
\left\| \int_0^1 e^{-s B_1} [(1-s)(\tilde{A}_0 + R_0) + sP_0, B_1] e^{s B_1} \, ds \right\|_{a+\beta}^{D_{1,\sigma_1}} \leq \frac{1}{2} \epsilon_0^{4/3}.
\]
Collecting the last estimate and equation (4.26) concludes \( \|P_1\|_{a+\beta}^{D_{1,\sigma_1}} \leq \epsilon_0^{4/3} = \epsilon_1. \)
Now assume that we have verified Lemma 4.2 up to step \( m > 1 \), then we go from step \( m \) to step \( m + 1 \). Clearly, from the assumption we have
\[
\|A_m - A_0\|_{a+\beta}^{D_{m+1}} = \left\| \sum_{l=0}^{m-1} \tilde{A}_l \right\|_{a+\beta}^{D_{m+1}} \leq \sum_{l=0}^{m-1} \|\tilde{A}_l\|_{a+\beta}^{D_{m+1}} \leq \sum_{l=0}^{m-1} \epsilon_l \leq 2 \epsilon_0,
\]
which allows us to apply Proposition 2.1 to construct \( B_{m+1}, \tilde{A}_m, R_m \) and \( D_{m+1}, \sigma_{m+1} \) such that for all \( (\omega, \theta) \in D_{m+1} \times T^{n}_{m+1} \)
\[
[A_m, B_{m+1}] - i\dot{B}_{m+1} = \tilde{A}_m - P_m + R_m.
\]
Due to (4.3) we have
\[
\text{Meas}(D_m \setminus D_{m+1}) \leq C \kappa_1^{\gamma_1} K_1^{\gamma_2} \leq C \epsilon_0^{4/3} \left( (\sigma_0 - \sigma_{m+1})^{-1} \ln \epsilon_0^{-1} \right)^{\gamma_2} \leq \epsilon_0^{4/3}.
\]
In view of (4.7) we have
\[
\|B_{m+1}\|_{a+\beta}^{D_{m+1,\sigma_{m+1}}} \leq \frac{C K_1}{\kappa_1^{1/2} (\sigma_0 - \sigma_{m+1})^n} \|P_m\|_{a+\beta}^{D_{m,\sigma_m}} \leq \frac{C (\ln \epsilon_0^{-1})^{3/4}}{(\sigma_0 - \sigma_{m+1})^{n+1}} \leq \epsilon_0^{2/3}.
\]
Thus, from the definition and Lemma 2.1 we have for all \( p \in (-2\alpha - 1, 2\alpha + 1) \)
\[
\|\phi_{m+1}(\omega, \theta) - \text{Id}\|_{B(\ell_2)} \leq C e^{C\|B_{m+1}\|_{a+\beta}^{D_{m+1,\sigma_{m+1}}} \|B_{m+1}\|_{a+\beta}^{D_{m+1,\sigma_{m+1}}}} \leq \epsilon_0^{2/3}.
\]
Collecting equations (4.5) and (4.6) leads to that \( \|\tilde{A}_m\|_{a+\beta}^{D_{m+1,\sigma_{m+1}}} \leq \epsilon_m \)
\[
\|R_m\|_{a+\beta}^{D_{m+1,\sigma_{m+1}}} \leq \frac{C e^{K_{m+1}^{1/2}(\sigma_0 - \sigma_{m+1})^{n}} \|P_m\|_{a+\beta}^{D_{m,\sigma_m}}}{\|B_{m+1}\|_{a+\beta}^{D_{m+1,\sigma_{m+1}}}} \leq C \sigma_0^{-n} (m + 1)^{2n} \epsilon_m \leq \frac{1}{2} \epsilon_m^{4/3}.
\] (4.27)
Besides, by Lemma 2.1 we have for $s \in [0, 1]$

$$
\|[(1-s)(\tilde{A}_m + R_m) + sP_m, B_{m+1}]\| \leq C\|P_m\| \leq C\epsilon_{m+1}^{5/3}.
$$

It follows that

$$
\left\| \int_0^1 e^{-sB_{m+1}}[(1-s)(\tilde{A}_m + R_m) + sP_m, B_{m+1}] e^{sB_{m+1}} ds \right\| \leq \frac{1}{2}\epsilon_{m}^{4/3}.
$$

Collecting the last estimate and equation (4.27) leads to $\|P_{m+1}\| \leq \epsilon_{m+1} = \epsilon_{m+1}$.

By induction, we complete the proof.

### 4.4. Proof of Theorem 2.2.

Let $D_\epsilon = \cap_{m \geq 0} P_{m}$. Owing (4.21) it is a Borel set satisfying

$$
\operatorname{Meas}(D_0 \setminus D_\epsilon) \leq \sum_{m \geq 0} \epsilon_{m}^4 \leq 2\epsilon_0^4.
$$

Clearly, $\sigma_\infty = \sigma_0 - \sum_{m \geq 1} (\sigma_{m-1} - \sigma_m) = \frac{\sigma}{2}$. Then we let $(\omega, \theta) \in D_\epsilon \times T_{\sigma/2}^n$ and $p \in (-2\sigma - 1, 2\sigma + 1)$ in the following discussion. Also, $P_m \to 0$ as $m \to \infty$ by (4.24).

Moreover, writing $A^\infty := \text{diag}(\lambda_i^\infty)_{i \geq 1}$ with $\lambda_i^\infty = \lim_{m \to \infty} \lambda_i^{(m)}$ for $i \geq 1$, by (4.22) one has

$$
\|A^\infty - A_0\|_{\alpha, \beta} \leq \sum_{m \geq 0} \epsilon_{m} \leq 2\epsilon_0,
$$

which follows that

$$
|\lambda_i^\infty - \nu_i| \leq 2\epsilon_0, \quad \forall \ i \geq 1.
$$

At last, we only remain to estimate the coordinate transformation $\Phi_{\omega, \epsilon}(\theta)$. First we present two auxiliary lemmas below.

**Lemma 4.3.** Denote $\Phi_m = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_m$ for $m \geq 1$, then we have

$$
\|\Phi_m(\omega, \theta) - \text{Id}\|_{B(\ell_p^2)} \leq \sum_{l=0}^{m-1} 2\epsilon_l^{2/3}.
$$

**Proof.** We proceed by induction applying equation (4.25) in Lemma 4.2. Clearly, (4.25) implies that $\|\Phi_1(\omega, \theta) - \text{Id}\|_{B(\ell_p^2)} \leq \epsilon_0^{2/3} \leq 2\epsilon_0^{2/3}$. Now we assume that Lemma 4.3 has been verified up to step $m > 1$, then we go from step $m$ to $m + 1$. Since

$$
\Phi_{m+1} - \text{Id} = \Phi_m \circ \phi_{m+1} - \text{Id} = \Phi_m \circ (\phi_{m+1} - \text{Id}) + \Phi_m - \text{Id},
$$

then by above assumption and equation (4.25) we have

$$
\|\Phi_{m+1} - \text{Id}\|_{B(\ell_p^2)} \leq \|\Phi_m\|_{B(\ell_p^2)} \cdot \|\phi_{m+1} - \text{Id}\|_{B(\ell_p^2)} + \|\Phi_m - \text{Id}\|_{B(\ell_p^2)} \leq \sum_{l=0}^{m} 2\epsilon_l^{2/3}.
$$

By induction, we complete the proof.

**Lemma 4.4.** The coordinate transformation $(\Phi_m(\omega, \theta))_{m \geq 1}$ is a Cauchy sequence in $B(\ell_p^2)$. Letting $\Phi_{\omega, \epsilon}(\theta) \in B(\ell_p^2)$ be the limit mapping we have

$$
\|\Phi_{\omega, \epsilon}(\theta) - \text{Id}\|_{B(\ell_p^2)} \leq 4\epsilon_0^{2/3},
$$

which follows that $\Phi_{\omega, \epsilon}(\theta)$ is analytic in $\theta \in T_{\sigma/2}^n$. 


Given has reducibility in \( \ell \), we invoke Lemma 4.3 and equation (4.25) we have
\[
\|\Phi_{m+1} - \Phi_m\|_{\mathcal{B}(\ell_2^2)} \leq \|\Phi_m\|_{\mathcal{B}(\ell_2^2)} \cdot \|\phi_{m+1} - \text{Id}\|_{\mathcal{B}(\ell_2^2)} \leq 2\varepsilon_m^{2/3}.
\]

It follows the Cauchy sequence. Then we invoke Lemma 4.3 to complete the proof. \( \square \)

By now we have obtained all the estimates (2.4) of the reducibility theorem, which follows from equations (4.28), (4.29) and (4.30).

In the end we will prove the system (2.1) reduces in \( \ell_1^2 \) or \( \ell_0^2 \) to an autonomous system (2.3). By Lemma 2.1 (iii),(iv) we have \( \mathcal{M}_s \subset \mathcal{B}(\ell_1^2, \ell_{s-1}^2) \) and \( \mathcal{M}_{s+} \subset \mathcal{B}(\ell_2^2), \forall \ s \in [-1, 1], \) provided \( 0 < \alpha \leq 1/2 \). Besides, the coordinate transformation \( u = \Phi_m v \) conjugates the original system \( i\dot{u} = (A + \varepsilon P)u \) into \( i\dot{v} = (A_m + P_m)v \), \( v \in \ell_1^2 \). These follow the identity in \( \mathcal{B}(\ell_1^2, \ell_{s-1}^2) \)
\[
A_m + P_m = \Phi_m^{-1}(A + \varepsilon P)\Phi_m - i\Phi_m^{-1}\partial_t \Phi_m.
\]

By construction \( \Phi_{\omega, \varepsilon} = \lim_{m \to \infty} e^{B_{2^1} \circ e^{B_{2^2}} \circ \cdots \circ e^{B_m}}. \) Letting \( m \to \infty \), one obtains the reducibility identity in \( \mathcal{B}(\ell_1^2, \ell_{s-1}^2) \)
\[
A^\infty(\omega) = \Phi_{\omega, \varepsilon}^{-1}(A + \varepsilon P(\omega t)) \Phi_{\omega, \varepsilon} - i\Phi_{\omega, \varepsilon}^{-1}\partial_t \Phi_{\omega, \varepsilon}, \ \omega \in \mathcal{D}_t. \quad (4.31)
\]

Consequently, if \( v(t) \in C^0(\mathbb{R}, \ell_1^2) \cap C^1(\mathbb{R}, \ell_{s-1}^2) \) satisfies (2.3) and define \( u = \Phi_{\omega, \varepsilon} v \), then by the reducibility identity (4.31) one gets
\[
i\dot{u} = \Phi_{\omega, \varepsilon} i\dot{v} + i(\partial_t \Phi_{\omega, \varepsilon}) v = (\Phi_{\omega, \varepsilon} A^\infty \Phi_{\omega, \varepsilon}^{-1} + i(\partial_t \Phi_{\omega, \varepsilon}) \Phi_{\omega, \varepsilon}^{-1}) v = (A + \varepsilon P(\omega t)) u.
\]

In addition, from \( v(t) \in C^0(\mathbb{R}, \ell_1^2) \cap C^1(\mathbb{R}, \ell_{s-1}^2) \) we can draw \( u(t) \in C^0(\mathbb{R}, \ell_1^2) \cap C^1(\mathbb{R}, \ell_{s-1}^2) \). Conversely, if \( u(t) \in C^0(\mathbb{R}, \ell_1^2) \cap C^1(\mathbb{R}, \ell_{s-1}^2) \) satisfies (2.1) and define \( v = \Phi_{\omega, \varepsilon}^{-1} u \), then one has \( v(t) \in C^0(\mathbb{R}, \ell_1^2) \cap C^1(\mathbb{R}, \ell_{s-1}^2) \) fulfills (2.3) by the identity (4.31). This explains the reducibility in \( \ell_1^2 \) when \( 0 < \alpha \leq 1/2 \).

On the other hand, if \( \alpha > 1/2 \), by Lemma 2.1 (iii),(iv) we have \( \mathcal{M}_s \subset \mathcal{B}(\ell_0^2, \ell_{s-2}^2) \) and \( \mathcal{M}_{s+} \subset \mathcal{B}(\ell_2^2), \forall \ s \in [-2, 2] \). It implies the system (2.1) reduces in \( \ell_0^2 \) in the sense that if \( v(t) \in C^0(\mathbb{R}, \ell_0^2) \cap C^1(\mathbb{R}, \ell_{s-2}^2) \) fulfills (2.3) if and only if \( u = \Phi_{\omega, \varepsilon} v \in C^0(\mathbb{R}, \ell_0^2) \cap C^1(\mathbb{R}, \ell_{s-2}^2) \) fulfills (2.1) because the above identity (4.31) holds in \( \mathcal{B}(\ell_0^2, \ell_{s-2}^2) \). This completes the proof of Theorem 2.2. \( \square \)

**Appendix A. Proof of Lemma 2.1**

Recall that \( \alpha \geq \beta > 0 \), which will be used occasionally.

(i). Since \( A, B \in \mathcal{M}_{\alpha+} \), then by \( \sum_{i \geq 1} \leq \left( \sum_{i+j \geq 1 \cap |i-j| \geq \frac{1}{2}} + \sum_{i+j \geq 1 \cap |i-j| \geq \frac{1}{2}} \right) \) one has
\[
(1 + |i - j|)^{\alpha+1} |(AB)_i^j| \leq C|A|_{\alpha+}|B|_{\alpha+}.
\]

In addition, from the definition we have
\[
\Delta(AB)_i^j = \sum_{l \geq 1} A_{i+l}^l B_{i+l}^j - \sum_{l \geq 1} A_i^l B_{i+l}^j = A_{i+1}^1 B_{i+1}^j + \sum_{l \geq 1} \Delta A_i^l \cdot B_{i+l}^j + \sum_{l \geq 1} A_i^l \cdot \Delta B_{i+l}^j. \quad (A.1)
\]
We estimate it term by term. First see that
\[
|A_{i+1}^i B_{i+1}^j| \leq \frac{|A|_{\alpha+} |B|_{\alpha+}}{(1 + i)^{\alpha+} (1 + j)^{\alpha+}} \leq \frac{|A|_{\alpha+} |B|_{\alpha+}}{ij(ij)^{\alpha+}} \leq \frac{|A|_{\alpha+} |B|_{\alpha+}}{(1 + i - j)(ij)^{\beta}}.
\]
Then turn to the remainder terms. We have that
\[
\left| \sum_{l \geq 1} \Delta A_i^l \cdot B_{i+1}^j \right|, \left| \sum_{l \geq 1} A_i^l \cdot \Delta B_i^j \right| \leq C \frac{|A|_{\alpha+} |B|_{\alpha+}}{(1 + i - j)(ij)^{\beta}},
\]
which follow from
\[
\sum_{l \geq 1} \frac{1}{(1 + |i - l|)^{\alpha+}} \leq C, \quad \sum_{l \geq 1} \frac{1}{(1 + |i - l|)(1 + |j - l|)^{\alpha+}} \leq C
\]
Therefore, we obtain
\[
|AB|_{\alpha+} \leq C |A|_{\alpha+} |B|_{\alpha+}.
\]
(ii). Recall that \( A \in M_{\alpha, \beta}, B \in M_{\alpha, \beta} \). Clearly, we have
\[
(1 + |i - j|)^{\alpha} |(AB)^j_i| \leq C |A|_{\alpha} |B|_{\alpha+}.
\]
We remain to estimate \( \Delta(AB)^j_i \). Recall the equation (A.1) and first observe that
\[
|A_{i+1}^i B_{i+1}^j| \leq \frac{|A|_{\alpha+} |B|_{\alpha+}}{(1 + i)^{\alpha+} (1 + j)^{\alpha+}} \leq \frac{|A|_{\alpha+} |B|_{\alpha+}}{(ij)^{\beta}}.
\]
Also, by the equation (A.2) one has
\[
\left| \sum_{l \geq 1} \Delta A_i^l \cdot B_{i+1}^j \right|, \left| \sum_{l \geq 1} A_i^l \cdot \Delta B_i^j \right| \leq C \frac{|A|_{\alpha+} |B|_{\alpha+}}{(ij)^{\beta}}.
\]
Hence, we have
\[
|AB|_{\alpha+} \leq C |A|_{\alpha, \beta} |B|_{\alpha+}. \quad \text{Clearly, repeating the same procedures we likewise obtain that} \quad |BA|_{\alpha, \beta} \leq C |A|_{\alpha, \beta} |B|_{\alpha+}.
\]
(iii). First let \( s \in (0, 2\alpha + 1) \) and claim that there is a \( C > 0 \) such that
\[
\sum_{j \geq 1} \left( \frac{i}{j} \right)^{\frac{\alpha}{s}} \frac{1}{(1 + |i - j|)^{\alpha+}} \leq C, \quad \sum_{i \geq 1} \left( \frac{i}{j} \right)^{\frac{\alpha}{s}} \frac{1}{(1 + |i - j|)^{\alpha+}} \leq C,
\]
for some \( p, q > 0 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and
\[
\frac{s - \alpha}{s} < \frac{1}{p} \leq \frac{\alpha + 1}{s}.
\]
Taking into account \( A \in M_{\alpha+}, \) we calculate for \( \xi \in \ell^2_p \) that
\[
\|A\xi\|_{s}^{2} \leq \sum_{i \geq 1} i^{s} \left( \sum_{j \geq 1} |A_{i}^j| |\xi_{j}| \right)^{2} \leq |A|_{\alpha+}^{2} \sum_{i \geq 1} \left( \sum_{j \geq 1} \left( \frac{i}{j} \right)^{\frac{\alpha}{s}} \frac{1}{(1 + |i - j|)^{\alpha+}} \right)^{2} |\xi_{j}|^{2}
\leq |A|_{\alpha+}^{2} \sum_{i \geq 1} \left( \sum_{j \geq 1} \left( \frac{i}{j} \right)^{\frac{\alpha}{s}} \frac{1}{(1 + |i - j|)^{\alpha+}} \right)^{2} \sum_{j \geq 1} \left( \frac{j}{i} \right)^{\frac{\alpha}{s}} |\xi_{j}|^{2}
\leq C |A|_{\alpha+}^{2} \sum_{i \geq 1} \sum_{j \geq 1} \left( \frac{i}{j} \right)^{\frac{\alpha}{s}} \frac{j^{\alpha}|\xi_{j}|^{2}}{(1 + |i - j|)^{\alpha+}} = C |A|_{\alpha+}^{2} \sum_{j \geq 1} j^{\alpha}|\xi_{j}|^{2} \sum_{i \geq 1} \left( \frac{j}{i} \right)^{\frac{\alpha}{s}}
\leq C^{2} |A|_{\alpha+}^{2} \sum_{j \geq 1} j^{\alpha}|\xi_{j}|^{2} = C^{2} |A|_{\alpha+}^{2} \|\xi\|^{2}_{s}.
\]
Consequently, collecting all the estimates of above three cases leads to the desired results.

\[
\sum_{i \geq 1} \left( \frac{i}{j} \right)^{-\frac{s}{p}} \frac{1}{(1 + |i - j|)^{\alpha + 1}} \leq C, \quad \sum_{j \geq 1} \left( \frac{i}{j} \right)^{-\frac{s}{q}} \frac{1}{(1 + |i - j|)^{\alpha + 1}} \leq C, \quad \forall \ s \in (0, 2\alpha + 1),
\]

provided (A.4) holds, which follows that \( \|A\|_{\mathcal{B}(\ell^2_{\alpha})} \leq C|A|_{\alpha^+} \) for any \( s \in (0, 2\alpha + 1) \).

(iv). Recall that \( A \in \mathcal{M}_\alpha \). We will distinguish three cases depending on the size of \( \alpha \).

(a). The case \( \alpha \in (0, \frac{1}{2}] \). We compute for \( \xi \in \ell^2_{\alpha} \) that

\[
\|A\xi\|_{\ell^1_{\alpha}}^2 \leq \sum_{i \geq 1} i^{-1} \left( \sum_{j \geq 1} |A_i^j| |\xi_j| \right)^2 \leq |A\|_{\alpha}^2 \sum_{i \geq 1} i^{-1} \left( \sum_{j \geq 1} \frac{1}{j(1 + |i - j|)^\alpha} \right)^2 \leq C|A\|_{\alpha}^2 \sum_{i \geq 1} \sum_{j \geq 1} \frac{i(1 + |i - j|)^\alpha}{|\xi_j|^2} \leq C^2|A\|_{\alpha}^2 \|\xi\|_{\ell^1_{\alpha}}^2.
\]

(b). The case \( \alpha \in (\frac{1}{2}, 1] \). Let \( s < 2\alpha - 2 \) and there exists a \( C > 0 \) such that

\[
\sum_{j \geq 1} \frac{1}{(1 + |i - j|)^{2\alpha}} \leq C, \quad \sum_{i \geq 1} \frac{1}{i(1 + |i - j|)^{2\alpha}} \leq C,
\]

for some \( p, q > 0 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \frac{1}{2\alpha} < \frac{1}{p} < \frac{2\alpha - s - 1}{2\alpha} \). We calculate for any \( \xi \in \ell^2_{\alpha} \) that

\[
\|A\xi\|_{\ell^1_{\alpha}}^2 \leq \sum_{i \geq 1} i^s \left( \sum_{j \geq 1} |A_i^j| |\xi_j| \right)^2 \leq |A\|_{\alpha}^2 \sum_{i \geq 1} i^s \left( \sum_{j \geq 1} \frac{|\xi_j|}{(1 + |i - j|)^{\frac{\alpha + 4}{q}}} \right)^2 \leq C|A\|_{\alpha}^2 \sum_{i \geq 1} \sum_{j \geq 1} i^s \frac{1}{|\xi_j|^2} \frac{1}{(1 + |i - j|)^{\frac{4\alpha}{q}}} \leq C^2|A\|_{\alpha}^2 \|\xi\|_{\ell^1_{\alpha}}^2. \quad \text{by } \frac{2\alpha}{q} - s > 1.
\]

(c). The case \( \alpha \in (1, \infty) \). We compute for \( \xi \in \ell^2_{\alpha} \) that

\[
\|A\xi\|_{\ell^1_{\alpha}}^2 \leq \sum_{i \geq 1} \left( \sum_{j \geq 1} |A_i^j| |\xi_j| \right)^2 \leq |A\|_{\alpha}^2 \sum_{i \geq 1} \left( \sum_{j \geq 1} \frac{|\xi_j|}{(1 + |i - j|)^{\alpha}} \right)^2 \leq C|A\|_{\alpha}^2 \sum_{i \geq 1} \sum_{j \geq 1} \frac{|\xi_j|^2}{(1 + |i - j|)^{\alpha}} \leq C^2|A\|_{\alpha}^2 \|\xi\|_{\ell^1_{\alpha}}^2.
\]

Consequently, collecting all the estimates of above three cases leads to the desired results.
Appendix B. Some technical lemmas

Lemma B.1. Let $Q \in \mathcal{M}_{\alpha,\beta}$ and $(\mu_i)_{i \geq 1}$ is a sequence of real numbers satisfying
\[ |\mu_{i+1} - \mu_i + \nu_i - \nu_{i+1}| \leq \frac{c_{\mu}}{\sqrt[2]{\beta}} \]  \hspace{1cm} (B.1)
for a given $c_{\mu} > 0$, where $0 < \beta \leq \min\{\alpha, \delta\}$, $\delta$ and $\nu_i$ are shown in Hypothesis H1. If $B(k) = (B^j_i(k))_{i,j \geq 1} := (\frac{Q^j_i}{k \cdot \omega + \mu_i - \mu_j})_{i,j \geq 1}$ defined for $k \in \mathbb{Z}^n$ with $|k| \leq K$, $\omega \in \mathbb{R}^n$ and the sequence $(\mu_i)_{i \geq 1}$ satisfies
\[ |k \cdot \omega + \mu_i - \mu_j| \geq \kappa(1 + |i - j|), \forall i, j \geq 1 \text{ and } |k| \leq K, \]  \hspace{1cm} (B.2)
then for $|k| \leq K$, $B(k) \in \mathcal{M}_{\alpha,\beta}$. More clearly, there is a constant $C \equiv 2^{\beta+1}(c_{\mu} + c_1 + 1)$ such that for $i, j \geq 1$ and $|k| \leq K$,
\[ (1 + |i - j|)\alpha^* |B^j_i(k)| + (1 + |i - j|)\beta |\Delta B^j_i(k)| \leq \frac{C|Q|_{\alpha,\beta}}{\kappa^2}. \]

Proof. For simplicity we will omit the variable $k$ if there is no confusion. Since $Q \in \mathcal{M}_{\alpha,\beta}$ with $0 < \beta \leq \min\{\alpha, \delta\}$, then from the definition and (B.2) we have
\[ (1 + |i - j|)\alpha^* |B^j_i| \leq \frac{(1 + |i - j|)^{\alpha^*} |Q^j_i|}{k \cdot \omega + \mu_i - \mu_j} \leq \frac{(1 + |i - j|)^{\alpha^*} |Q^j_i|}{|k \cdot \omega + \mu_i - \mu_j|} \leq \frac{|Q|_\alpha}{\kappa}. \]

Now we commence the estimations on $\Delta B^j_i$. First observe that
\[ \Delta B^j_i = \frac{\Delta Q^j_i}{k \cdot \omega + \mu_i + \mu_j} \cdot \frac{(\mu_i - \mu_j + \mu_{i+1} - \mu_{j+1})Q^j_i}{(k \cdot \omega + \mu_i + \mu_j)(k \cdot \omega + \mu_i - \mu_j)} := \Delta_1 + \Delta_2. \]

Clearly, by (B.2) we have
\[ (1 + |i - j|)\beta |\Delta_1| \leq \frac{(1 + |i - j|)^{\beta} |\Delta Q^j_i|}{k \cdot \omega + \mu_i + \mu_j} \leq \sup_{1 \leq j \leq 1} (ij)^{\beta} |\Delta Q^j_i|. \]

Then turn to $\Delta_2$. We hereafter assume $i \geq j$ without loss of generality and will distinguish two cases depending upon whether $i \leq 2j$ or not.

(a). The case $j \leq i \leq 2j$. We calculate that
\[ \Delta_2 = \frac{(\mu_i - \mu_j + \mu_{i+1} - \mu_{j+1}) + (\nu_j - \nu_{j+1}) + (\nu_{i+1} - \nu_{i+1})Q^j_i}{(k \cdot \omega + \mu_i + \mu_j)(k \cdot \omega + \mu_i - \mu_j)}. \]

Consequently, by equations (B.1),(B.2) and Hypothesis H1 we have
\[ |\Delta_2| \leq \frac{(c_{\mu} + c_{\nu} + c_1)Q^j_i}{\kappa^2(1 + |i - j|)^2} \leq \frac{(c_{\mu} + 2\delta c_{\mu} + c_1)|i - j| |Q^j_i|}{\kappa^2(1 + |i - j|)^2} \leq \frac{2^{\beta+1}(c_{\mu} + c_1)|Q^j_i|}{\kappa^2(1 + |i - j|)^{j^{\beta}}}, \]

which follows that $(1 + |i - j|)\beta |\Delta_2| \leq \frac{2^{\beta+1}(c_{\mu} + c_1)|Q^j_i|}{\kappa^2(1 + |i - j|)^{j^{\beta}}}$ for $j \leq i \leq 2j$.

(b). The case $i > 2j$. In this case $1 + |i - j| = 1 + i - j \geq \frac{i}{2}$. We likewise obtain that
\[ |\Delta_2| \leq \frac{(c_{\mu} + c_{\nu} + c_1)|i - j| |Q^j_i|}{\kappa^2(1 + |i - j|)^2} \leq \frac{2(c_{\mu} + c_1)|i - j| |Q^j_i|}{\kappa^2(1 + |i - j|)^{j^{\beta}}}. \]
which follows that \((1 + |i - j|) (ij)^{\beta} |\Delta_2| \leq \frac{2(c_{\mu} + c_1) |Q|}{\kappa^2} \leq \frac{2^{\beta+1}(c_{\mu} + c_1) |Q|}{\kappa^2} \) for \(i > 2j\).

Collecting all the above estimates concludes that
\[
(1 + |i - j|) (ij)^{\beta} |\Delta_2| \leq \frac{2^{\beta+1}(c_{\mu} + c_1) |Q|}{\kappa^2} \quad \forall \ i, j \geq 1.
\]

As a consequence, we obtain the desired results. □

Lemma B.2. Let \( f : [0, 1] \mapsto \mathbb{R} \) be a \( C^1 \) map satisfying \( |f'(x)| \geq \varsigma > 0 \) for all \( x \in [0, 1] \) then for each \( \kappa > 0 \) we have
\[
\text{Meas} \left( \{ x \in [0, 1] : |f(x)| \leq \kappa \} \right) \leq \frac{2\varsigma}{\kappa}.
\]

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