Renormalization-group approach to transverse-momentum dependent parton distribution functions in QCD

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Abstract

We discuss the renormalization of gauge-invariant transverse-momentum dependent (TMD), i.e., unintegrated, parton distribution functions (PDFs) and carry out the calculation of their anomalous dimension at one loop. We show that in the light-cone gauge, TMD PDFs contain UV divergences that may be attributed to the renormalization effect on a cusp-like junction point of the gauge contours at infinity. In order to eliminate the anomalous dimension ensuing from this cusp, we propose to use in the definition of the TMD PDFs, a soft counter term in terms of a path-ordered phase factor along a particular cusped contour extending to transverse light-cone infinity and comprising light-like and transverse segments. We argue that this additional factor is analogous to the “intrinsic” Coulomb phase factor found before in QED.

1 Introduction

Parton distribution functions encode the nonperturbative hadronization dynamics at the amplitude level and are, therefore, of fundamental importance in QCD calculations and phenomenological applications (see [1] for a review). While integrated PDFs can be given an unambiguous gauge-invariant definition in terms of Wilson-line operators (gauge links) [2], the analogous definition for unintegrated, i.e., transverse-momentum dependent, PDFs may depend more critically on the details of the gauge contour. As a result, the renormalization of TMD PDFs is a more demanding task to which the present report is devoted.

Indeed, in order to satisfy factorization, one cannot restore gauge invariance in TMD PDFs by inserting a purely light-like Wilson line joining the quark and antiquark field points directly [3]. The reason is that the gluons emitted from the struck quark along the $x^-$ direction have rapidities that cannot match those of the spectator quarks moving along the $x^+$ direction. Consequently, one is forced to employ a gauge contour that comprises

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segments going off the light cone and joins the quark field points through infinity. Recall in this context that the gauge link resums the contributions due to collinear and transverse gluons between the struck quark and the spectator remnants (see, e.g., [4]).

Quite recently, Belitsky, Ji, and Yuan [5] (see also [6, 7, 8]) have shown that in the light-cone gauge \( A^+ = 0 \), one has to include in the definition of TMD PDFs transverse gauge links at light-cone infinity—as illustrated in Fig. 1. These transverse gauge links cancel when the integration over \( k_\perp \) is performed, so that one recovers the correct integrated PDF. Moreover, it was advocated in [5] that adopting the advanced boundary condition (see below), the transverse field \( A_\perp \) vanishes at \( \xi^- = \infty \) reducing the transverse gauge link to unity. For that particular boundary condition, the light-cone gauge (one-loop) calculation reproduces the Feynman-gauge PDF.

![Figure 1: TMD PDF (shaded oval) in coordinate space. Double lines denote lightlike and transverse gauge links, connecting the quark field points \((0^-, 0_\perp)\) and \((\xi^-, \xi_\perp)\), via a composite contour through light-cone infinity \((\infty^-, \infty_\perp)\).](image)

In these investigations it was tacitly assumed that the lightlike-transverse composite contour going through infinity, illustrated in Fig. 1, is everywhere smooth. However, we have shown in [9,10] by carrying out a one-loop calculation of the gluon radiative corrections to the unpolarized TMD PDF of a quark in a quark in the light-cone gauge that there are UV divergences which are neither related to the quark self energy nor are they caused by the endpoints of the line integral along the gauge contour—as one finds for the direct contour (the “connector” [11,12]). Instead, the origin of these extra UV divergences can be attributed to a cusp obstruction (denoted by the symbol \( \times \) in Fig. 1) in the split gauge contour at transverse light-cone infinity. The concomitant anomalous dimension after renormalization is a local footprint of the cusp and peculiar to the split contour. It turns out to coincide with the leading-order (LO) cusp anomalous dimension [13].

The appearance of this extra anomalous dimension necessitates a modification of the definition of the TMD PDF in order to dispense with it. As pointed out in [9], and further outlined in full detail in [10], this can be achieved by including a path-ordered soft factor, in the sense

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2It remains to be proved that this coincidence persists at the two-loop order and beyond.
of Collins and Hautmann [14], to be evaluated along a specific gauge contour off-the-light cone (see next section). Having described the cornerstones of our approach, let us now have a closer look to its mathematical details.

2 One-loop radiative corrections to gauge-invariant TMD PDFs

Taking into account the findings of [5], the strictly gauge-invariant operator definition of the TMD distribution of a quark with momentum \(k_\mu = (k^+, k^-, k_\perp)\) in a quark with momentum \(p_\mu = (p^+, p^-, 0_\perp)\), with non-lightlike Wilson lines to light-cone infinity included, reads

\[
\begin{align*}
\frac{f_{q/q}(x, k_\perp)}{2} = & \int \frac{d\xi_\perp d^2\xi_\perp}{2\pi(2\pi)^2} \exp \left( -ik^+\xi^- + ik_\perp \cdot \xi_\perp \right) \langle q(p) | \bar{\psi}(\xi^-) | q(p) \rangle \langle q(p) | \bar{\psi}(\xi^-) | q(p) \rangle \\
& \times \left[ \gamma^+ \langle q(p) | \bar{\psi}(0^-) q(p) \rangle \right]_{\xi^+ = 0}.
\end{align*}
\]

Here the gauge links, in the lightlike and the transverse direction, respectively, are defined by the following path-ordered exponentials

\[
[\infty^-; z_\perp; z^-; z_\perp] \equiv P \exp \left[ ig \int_0^\infty d\tau n^- A^a_\tau t^a (n^- + n^- \tau) \right]
\]

\[
[\infty^-; \infty_\perp; \infty^-; \xi_\perp] \equiv P \exp \left[ ig \int_0^\infty d\tau l \cdot A^a_\tau t^a (\xi_\perp + l \tau) \right],
\]

where the two-dimensional vector \(l\) is arbitrary with no influence on the (local) anomalous dimensions we are interested in.

\[\text{(1)}\]

Employing the light-cone gauge \(A^+ = (A \cdot n^-) = 0\), \((n^-)^2 = 0\), we calculated in [9, 10] gluon radiative corrections to \(f_{q/q}(x, k_\perp)\) at the one-loop level and identified its UV divergences (see Fig. 2). We found that those contributions stemming from the interactions with the gluon field of the transverse gauge link cancel all terms that bear a dependence

Figure 2: One-loop radiative corrections (curly lines) contributing UV-divergences to \(f_{q/q}(x, k_\perp)\) in a general covariant gauge. Double lines denote lightlike and transverse gauge links. Diagrams (b) and (c) are absent in the light-cone gauge, while the Hermitian conjugate (“mirror”) diagrams (not shown) are abbreviated by \((h.c.)\).
on the pole prescription applied to regularize the light-cone singularities of the gluon propagator. In the intermediate steps of the calculation the light-cone singularities of the gluon propagator

\[ D^{\text{LC}}_{\mu\nu}(q) = \frac{-i}{q^2 - \lambda^2 + iq^+} \left( g_{\mu\nu} - \frac{q_\mu n_\nu + q_\nu n_\mu}{[q^+]} \right) \]  

(3)

are taken into account by means of the term \(1/[q^+]\) subject to boundary conditions on the gauge potential. In the present work we apply the following regularization prescriptions to the pole at \(q^+\) [5]:

\[ \left. \frac{1}{[q^+]} \right|_{\text{Ret/Adv}} = \frac{1}{q^+ \pm i\eta}, \quad \left. \frac{1}{[q^+]} \right|_{\text{PV}} = \frac{1}{2} \left[ \frac{1}{q^+ + i\eta} + \frac{1}{q^+ - i\eta} \right], \]  

(4)

where \(\eta\) is a mass-scale parameter kept small but finite. The total UV-divergent contribution is obtained by including also the Hermitian conjugate contributions of diagrams (a) and (d) in Fig. 2. Then, we obtain

\[ \Sigma^{(a+d)}_{\text{UV}}(p, \mu, \alpha_s; \epsilon) = -\frac{\alpha_s}{\pi} C_F \frac{1}{\epsilon} \left[ \frac{1}{4} - \frac{\gamma^+ \hat{p}^+}{2p^+} \left( 1 + \ln \frac{\eta}{p^+} - \frac{i\pi}{2} - i\pi C_\infty + i\pi C_\infty \right) \right] \]

\[ = -\frac{\alpha_s}{\pi} C_F \frac{1}{\epsilon} \left[ 1 - \frac{\gamma^+ \hat{p}^+}{2p^+} \left( 1 + \ln \frac{\eta}{p^+} - \frac{i\pi}{2} \right) \right], \]  

(5)

where \(C_F = (N_c^2 - 1)/(2N_c) = 4/3\) and the parameter \(C_\infty\) encodes the adopted pole prescription (cf. Eq. (4)). This expression can be further simplified using

\[ \frac{\gamma^+ \hat{p}^+}{2p^+} = \gamma^+ \]

and recalling that the mirror counterparts of the evaluated diagrams yield complex-conjugated contributions. As a result, the imaginary terms in Eq. (5) mutually cancel and one is left with

\[ \Sigma^{(a+d)}_{\text{UV}}(\alpha_s, \epsilon) = 2\frac{\alpha_s}{\pi} C_F \left[ \frac{1}{\epsilon} \left( \frac{3}{4} + \ln \frac{\eta}{p^+} \right) - \gamma_E + \ln 4\pi \right] \]  

(6)

The key contribution here is the term \(\sim \ln \frac{\eta}{p^+}\) which gives rise to the one-loop anomalous dimension in the light-cone (LC) gauge \((\gamma = \frac{\mu}{2Z} \frac{\partial Z}{\partial \alpha_s})\) :

\[ \gamma^\text{LC}_\text{1-loop} = \frac{\alpha_s}{\pi} C_F \left( \frac{3}{4} + \ln \frac{\eta}{p^+} \right) = \gamma^\text{smooth} - \delta\gamma. \]

(7)

Here \(\gamma^\text{smooth}\) is the anomalous dimension one would obtain in a covariant gauge, or, equivalently, the anomalous dimension associated with a direct smooth contour between the quark fields (i.e., with the connector correction). The term \(\delta\gamma\) is the anomalous-dimensions defect entailed by the cusp, we have to compensate in order to recover the same expression as in a covariant gauge according to the factorization proof. Consistent with this finding, one has to modify the multiplication rule for gauge links (or, equivalently, the way of decomposing gauge contours) [10]:

\[ \gamma_c = \gamma_1^\infty \cup \gamma_2^\infty + \gamma_{\text{cusp}} \quad \leftrightarrow \quad [2, 1|C] = [2, \infty|C_1^\infty]^{[\infty, 1|C_2^\infty]} e^{i\Phi_{\text{cusp}}}. \]

(8)
Figure 3: Renormalization effect on the junction point due to gluon corrections (illustrated by a shaded oval with gluon lines attached to it) for (a) two smoothly joined gauge contours $C_1$ and $C_2$ at point 3 and (b) the same for two contours joined by a cusp (indicated by the symbol $\otimes$) at infinite transverse distance (marked by the earth symbol) off the light cone. All contours shown are assumed to be arbitrary non-lightlike paths in Minkowski space.

The graphics at right of Fig. 3 helps the eye catch the key features of the situation involving two non-lightlike contours $C_1$ and $C_2$. For comparison, the smooth decomposition of a purely lightlike contour is shown in the left panel. In that case the junction point 3 creates no anomalous dimension and the standard multiplication rule for gauge links applies.

In the above expression, $\Phi_{\text{cusp}}$ contains a phase entanglement ensuing from the renormalization effect on the cusp-like junction point at infinity. One may associate this phase with final (or initial) state interactions, as proposed by Ji and Yuan in [7], and also by Belitsky, Ji, and Yuan in [5]. However, these authors (and also others) did not recognize that the junction point in the split contour (the latter stretching to light-cone infinity) is no more a simple point, but a cusp obstruction that entails an anomalous dimension $\sim \ln p^+$. More precisely, we have

$$\gamma_{\text{cusp}}(\alpha_s, \chi) = \frac{\alpha_s}{\pi} C_F (\chi \coth \chi - 1),$$

$$\frac{d}{d \ln p^+} \delta\gamma = \lim_{\chi \to \infty} \frac{d}{d\chi} \gamma_{\text{cusp}}(\alpha_s, \chi) = \frac{\alpha_s}{\pi} C_F,$$

which makes it apparent that the defect of the anomalous dimension is related to the universal cusp anomalous dimension [13]. To derive this expression, we have used the fact that $p^+ = (p \cdot n^-) \sim \cosh \chi$ defines an angle $\chi$ between the direction of the quark momentum $p_\mu$ and the lightlike vector $n^-$. Then, in the large $\chi$ limit, one has $\ln p^+ \to \chi$. It is worth recalling in this context that the cusp anomalous dimension of Wilson lines controls the Sudakov factor resulting from gluon resummation and is known to the three-loop order [15]. The Sudakov exponent in next-to-leading logarithmic approximation has been calculated in [16] and expressed as an expansion in inverse powers of the first beta-function coefficient.

### 3 How to avert the defect of the anomalous dimension

In this section we will show in more depth how to get rid of the cusp anomalous dimension and refurbish the definition of the TMD PDF. The defect of the anomalous dimension,
ensuing from the cusp-like junction point of the non-lightlike gauge contours, represents a distortion of the gauge-invariant formulation of the TMD PDF in the light-cone gauge. This is best appreciated by inspecting the composite non-smooth contour $C_{\text{cusp}}$, visualized in Fig. 4, and defined by

$$C_{\text{cusp}} : \zeta_\mu = \begin{cases} [p_\mu^+, -\infty < s < 0] \cup [n_\mu^- s', 0 < s' < \infty] \cup [l_\perp \tau, 0 < \tau < \infty] \end{cases} ,$$

with $n^-_\mu$ being the minus light-cone vector. This contour is obviously cusped: at the origin, the four-velocity $p_\mu^+$, which is parallel to the plus light-cone ray, is replaced—non-smoothly—by the four-velocity $n^-_\mu$, which is parallel to the minus light-cone ray. This means that exactly at this point the contour has a cusp, that is characterized by the angle $\chi \sim \ln p^+ = \ln(p \cdot n^-)$, and will generate an anomalous dimension with the opposite sign relative to $\delta \gamma$—cf. Eq. (7). This contour can be used to define a soft counter term in the sense of Collins and Hautmann [14], namely,

$$R \equiv \Phi(p^+, n^-|0)\Phi^\dagger(p^+, n^-|\xi) ,$$

where the eikonal factors are given by

$$\Phi(p^+, n^-|0) = \left\langle 0 \left| \mathcal{P} \exp \left[ ig \int_{C_{\text{cusp}}} d\zeta^\mu A^{a}_\mu(\zeta) \right] \right| 0 \right\rangle ,$$

$$\Phi^\dagger(p^+, n^-|\xi) = \left\langle 0 \left| \mathcal{P} \exp \left[ - ig \int_{C_{\text{cusp}}} d\zeta^\mu t^a A^{a}_\mu(\xi + \zeta) \right] \right| 0 \right\rangle$$

and have to be evaluated along the integration contour $C_{\text{cusp}}$.

Next, we consider the one-loop gluon radiative corrections, contributing to the UV divergences of $R$ and displayed in Fig. 5. Diagrams (a) and (d) give rise to an anomalous dimension that will finally compensate the anomalous-dimensions defect generated by the cusp-like junction point of the contours. On the other hand, by virtue of the light cone gauge $A^+ = 0$, we are employing, diagrams (b) and (c) vanish. The UV parts of diagrams (a) and (d) yield, respectively,

$$\Phi_{\text{UV}}^{(a)}(\eta) = -\frac{\alpha_s}{\pi} C_F \frac{1}{\epsilon} \left( \ln \frac{\eta}{p^+} - \frac{\pi}{2} - i\pi C_\infty \right)$$

and

$$\Phi_{\text{UV}}^{(d)}(\eta) = -\alpha_s C_F i\pi C_\infty \Gamma(\epsilon) \left( -4\pi \frac{\mu^2}{\lambda^2} \right)^\epsilon .$$
Figure 5: Gluon radiative corrections giving rise to UV-divergences contributing to the soft counter term $R$. The designations are as in Fig. 2.

Combining these UV terms, we find

$$F_{\text{UV}}^{(a+d)}(\eta) = -\frac{\alpha_s}{\pi} C_F \frac{1}{\epsilon} \left( \ln \frac{\eta}{p^+} - i \frac{\pi}{2} - i \pi C_{\infty} + i \pi C_{\infty} \right) = -\frac{\alpha_s}{\pi} C_F \frac{1}{\epsilon} \left( \ln \frac{\eta}{p^+} - i \frac{\pi}{2} \right). \quad (16)$$

Taking into account the Hermitian conjugate (“mirror”) terms, we obtain the total UV-divergent part of the soft factor $R$ in one-loop order:

$$\Phi_{\text{UV}}^{(1-\text{loop})}(\eta) = -\frac{\alpha_s}{\pi} C_F \frac{2}{\epsilon} \ln \frac{\eta}{p^+}. \quad (17)$$

One notices that this expression bears no dependence on the pole prescription, since all $C_{\infty}$-dependent terms have mutually canceled. Indeed, only the cusp-dependent term $\sim \ln \frac{\eta}{p^+}$ survives that will ultimately yield $-\gamma_{\text{cusp}}$.

The above considerations make it apparent that one may use $R$ and redefine the TMD PDF as follows

$$f_{q/q}^{\text{mod}}(x, k_\perp; \mu, \eta) = \frac{1}{2} \int d\xi^- d^2\xi_\perp \frac{\epsilon}{2\pi (2\pi)^2} \exp \left( -i k^+ \xi^- + i k_\perp \cdot \xi_\perp \right) \left< q(p) | \bar{\psi}(\xi^-_-, \xi_\perp) \right| \times \left[ \Phi(p^+, n^-|0^-, 0_\perp) \Phi(p^+, n^-|\xi^-, \xi_\perp) \right]. \quad (18)$$

Before we conclude, let us mention that integrating the above expression over the transverse momenta, we obtain an integrated PDF that coincides with the standard one, containing no artifacts of the cusped contour, and satisfying the DGLAP evolution equation. Moreover, $f_{q/q}^{\text{mod}}(x, k_\perp; \mu, \eta)$ satisfies the simple renormalization-group equation

$$\frac{1}{2} \frac{d}{d\mu} \ln f_{q/q}^{\text{mod}}(x, k_\perp; \mu, \eta) = \frac{3\alpha_s}{4\pi} C_F + O(\alpha_s^2). \quad (19)$$

Note that without the soft counter term, $R$, extra contributions to the anomalous dimension on the right-hand side would appear. In [10] we have outlined the correspondence between the evolution with respect to the scale parameter $\eta$ in our approach and the Collins-Soper evolution equation with respect to the rapidity parameter $\zeta$, establishing the absence of UV singularities entailed by the light-cone gauge.
4 Conclusions

To summarize the results of this report on the renormalization of gauge-invariant TMD PDFs, the following may be said. First, we have elaborately discussed the one-loop calculation of the UV divergences of a typical TMD PDF which contains lightlike and transverse gauge links in order to fully restore gauge invariance. We found that an extra UV divergence appears, not noticed before in the literature, which is unrelated to the quark self energy and the end-point singularities of the contours. Second, we showed that these divergences give rise to an anomalous dimension, which can be regarded as originating from the renormalization effect on a cusp-like junction point of the integration contours in the gauge links at light-cone infinity. At the considered one-loop order, this anomalous dimension coincides with the universal cusp anomalous dimension of Wilson-line operators and is an ingrained property of the split contours. Third, in order to dispense with this anomalous-dimensions defect and recover the well-known results in a covariant gauge (say, in the Feynman gauge) in which $A_\perp$ vanishes at infinity, we have proposed a modified definition of the TMD PDF. This definition includes a Collins-Hautmann soft counter term by means of path-ordered eikonal factors that are evaluated along a specific non-smooth contour off the light cone. This cusped contour suffices to neutralize the cusp artifact encountered in the standard definition of the TMD PDF. Finally, as we outlined in [10], the soft counter term can be given an interpretation akin to the “intrinsic” Coulomb phase found by Jakob and Stefanis [17] in QED. In both cases, a phase entanglement appears, ensuing either from the charged “particle behind the moon” (QED) or from the cusp-like junction point at light-cone infinity (QCD). Recently, Collins [18] has considered possible refinements and modifications in the definition of unintegrated parton densities that deserve further examination. An improved definition of TMD PDFs will have tangible consequences in several areas of QCD.

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