Asymptotic enumeration of graphs by degree sequence, and the
degree sequence of a random graph

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Abstract
In this paper we relate a fundamental parameter of a random graph, its degree sequence, to
a simple model of nearly independent binomial random variables. This confirms a conjecture
made in 1997. As a result, many interesting functions of the joint distribution of graph degrees,
such as the distribution of the median degree, become amenable to estimation. Our result is
established by proving an asymptotic formula conjectured in 1990 for the number of graphs
with given degree sequence. In particular, this gives an asymptotic formula for the number of
d-regular graphs for all d, as $n \to \infty$.

1 Introduction

We consider the number of graphs with a given degree sequence. In particular, we prove a conjecture
of McKay and Wormald [15] from 1990. It says essentially that a formula known to hold in the
sparse and dense cases as long as the degrees of vertices are somewhat close to each other, also holds
for the remaining cases. The main significance of verifying this conjecture is that it provides a very
simple model for the degree sequence of a random graph. The two most popularly studied models of
random graphs are considered here: $G(n, p)$, in which $n$ vertices have edges included between each
pair of them independently with probability $p$ for each pair, and $G(n, m)$ in which $n$ vertices have $m$
edges included, chosen uniformly at random from the $m$-subsets of the unordered pairs of vertices.
Those classical models of random graphs easily satisfy the conjecture’s restriction on degrees with
high probability, and it follows that the degree sequence of $G(n, m)$ is well approximated by a certain
sequence of independent binomial variables conditioned on summing to $2m$. A similar connection is
provided between the random graph $G(n, p)$ and a slightly twisted sequence of independent binomial
random variables. This makes a very convenient way of proving results about the degree sequence
of $G(n, p)$.

The degree sequence of a random graph has received considerable attention, and indeed was the
first major topic dealt with in Bollobás’ seminal book [5] on the theory of random graphs. This
contained many interesting results, for instance the distribution of the $k^{th}$ largest element $d_k$ of
the sequence was determined quite precisely when $k$ is small. The book [1] by Barbour, Holst and
Janson contained much information on the distribution of the number $D_k$ of vertices of degree $k$.
On the other hand, enumerating graphs with given degree sequence has been the interest of various

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authors over many years. Read [18] found a recursive formula for the number of 3-regular graphs from which he also deduced a simple asymptotic formula. After this, formulae for the number of graphs with given degree sequence $d = (d_1, \ldots, d_n)$ were found by Bender and Canfield [3] and Wormald [19, Theorem 3.3], and for ever-denser ranges of degrees by Bollobás [4], and McKay [13], culminating in papers giving asymptotic formulae for a range of degrees, provided the average degree $d$ is $o(\sqrt{n})$ (by McKay and Wormald [16]) or between $cn\log n$ and $n/2$ for a certain $c$ (by McKay and Wormald [15], also treated more recently by Barvinok and Hartigan [2] for a wider spread of degrees, but similar density). The complementary ranges of $d$ larger than $n/2$ are automatically covered. A case of special interest, which saw no advance since 1990, is the problem of finding the asymptotic number of $d$-regular graphs for $d$ in the range $c\sqrt{n} \leq d = o(n/\log n)$.

In 1990, McKay and Wormald [15] restated the asymptotic formulae from the sparse and dense cases in a common form, which they conjectured to be valid additionally for all densities in between those two cases and hence for all densities except trivial extremely sparse and dense ones. For a precise statement, see the Binomial Approximation Conjecture (Conjecture 1.2) below. It applies in a certain sense to all the typical degree sequences in either model of random graphs defined above. In this paper, we prove the Binomial Approximation Conjecture. In particular, as a very special case, this implies that the number of $d$-regular graphs on $n$ vertices is asymptotically equal to

$$\frac{(n-1)^n}{d} \binom{n}{2m} \frac{\binom{n}{2}}{2m} e^{1/4}$$

for all $1 \leq d \leq n-2$, where $m = dn/2$.

A weakened version of the Binomial Approximation Conjecture was given in 1997 by McKay and Wormald [17] (see Conjecture 1.3) and shown to imply an explicit connection between the degree sequence of a random graph of a given density, and a sequence of independent binomial variables. This work opened up a completely new approach to deriving properties of the degree sequence of a random graph, by considering independent binomials. Many properties that were previously inaccessible, such as the distribution of the median degree, are now within reach for those densities where the conjecture holds.

To present the connection between $D(G)$, where $G = G(n,m)$ or $G(n,p)$, and $B_p(n)$, let us first make some definitions. We assume that a graph on $n$ vertices has vertex set $v_1, \ldots, v_n$ and degree sequence $(d_1, \ldots, d_n)$, so that $d(v_i) = d_i$. If $G$ is any probability space of random graphs, let $D(G)$ be the random vector distributed as the degree sequence of a random graph $G \in G$. Also define $B_p(n)$ to be the random sequence consisting of $n$ independent binomial variables Bin$(n-1,p)$.

Let $A_n$ and $B_n$ be two sequences of probability spaces with the same underlying set for each $n$. Suppose that whenever a sequence of events $H_n$ satisfies $P(H_n) = n^{-O(1)}$ in either model, it is true that $P_{A_n}(H_n) \sim P_{B_n}(H_n)$, where by $f(n) \sim g(n)$ we mean that $f(n)/g(n) \to 1$ as $n \to \infty$. We then say that $A_n$ and $B_n$ are asymptotically quite equivalent (a.q.e.). Throughout this paper we use $\omega$ to be an arbitrary function of $n$ such that $\omega \to \infty$ as $n \to \infty$, perhaps different at each occurrence. As we will see later, our main result, combined with existing results, implies the following. Note for part (i) that $B_p(n) \mid \Sigma = 2m$ is independent of $p$.

**Proposition 1.1.** Let $n, m$ be integers and let $0 < p < 1$. Let $\Sigma$ denote the sum of the components of the random vector $B_p(n)$ in (i) and $B_p(n)$ in (ii).

(i) $D(G(n,m))$ and $B_p(n) \mid \Sigma = 2m$ are a.q.e. provided that $\min\{m, \binom{n}{2} - m\} = \omega \log n$. 

(ii) Let \( \hat{p} \) be randomly chosen according to the normal distribution with mean \( p \) and variance \( p(1 - p)/n(n - 1), \) truncated at 0 and 1. Then \( D(G(n,p)) \) and \( \mathcal{B}_p(n) \) is even are a.g.e. provided that \( p(1 - p) = \omega \log^3 n/n^2. \)

Note that the assumptions on \( p \) and \( m \) merely ensure that the graph, or its complement, has number of edges at least a small power of \( \log n; \) if this fails, the degree sequence is almost trivial and the graph is likely to be uninteresting, either a set of independent edges or its complement.

Proposition 1.1 has significant implications. In particular, it was shown in [17] that a result very similar to Proposition 1.1(ii) can be used, for whatever range of \( p(n) \) it is valid for, to transfer general classes of properties to \( D(G(n,p)) \) from the independent binomial model \( \mathcal{B}_p(n). \) We give details later in this section. It was also observed in [17], using the known asymptotic formulae, that the Binomial Approximation Conjecture holds when \( p = o(1/\sqrt{n}) \) or \( p(1 - p) > n/c \log n. \) However, the rather large gap, where \( p(1 - p) \) is between roughly \( 1/\sqrt{n} \) and \( 1/\log n, \) was still open, as the appropriate enumeration results were lacking. This gap has prevented the new approach being fully utilised. Part (i) of the proposition is also appealing as a direct connection between independent binomials and \( D(G(n,m)) \), however the event \( \Sigma = 2m \) is a fairly “thin” event so some results would require more finesse to be transferred.

In this article we introduce an approach to enumerating graphs by degree sequence that differs significantly from what has previously been applied to this or any similar problems. This new method is versatile enough to be applied to enumeration problems for other discrete structures, as described in Section 8. For this reason, our theoretical results are set in a framework slightly wider than is needed for the enumeration results derived in the present paper.

1.1 Conjectures and results

In 1990 McKay and Wormald [15] unified the existing asymptotic formulae for the sparse and the dense case into one form and conjectured this form to hold also for the gap in the range of degrees, as long as the degree sequences are close to regular. Throughout this paper we use the following notation. Given a sequence \( \mathbf{d} = (d_1, ..., d_n), \) let \( g(\mathbf{d}) \) denote the number of graphs whose degree sequence is \( \mathbf{d}, \) let \( \mu = \mu(\mathbf{d}) = d/(n - 1) \) where \( d = \frac{1}{n} \sum_{i=1}^{n} d_i, \) and let \( \gamma_2 = (n - 1)^{-2} \sum_{i=1}^{n} (d_i - d)^2. \) We refer to the following as the Binomial Approximation Conjecture.

**Conjecture 1.2.** [15, Conjecture 1] For some absolute constant \( \varepsilon > 0, \) if \( \mathbf{d} = \mathbf{d}(n) \) satisfies \( \max_j |d_j - d| = o(n^{\varepsilon} \min\{d, n - d - 1\}^{1/2}), \) \( n \min\{d, n - d - 1\} \to \infty, \) and \( \sum_i d_i \) is even, then

\[
g(\mathbf{d}) \sim \sqrt{2} \exp \left( \frac{1}{4} - \frac{\gamma_2}{4\mu^2(1 - \mu)^2} \right) (\mu^\mu(1 - \mu)^{1 - \mu})^{n(n - 1)/2} \prod_{i=1}^{n} \left( \frac{n - 1}{d_i} \right). \tag{1.1}
\]

**Note.** The permitted domain of \( \mathbf{d} \) is compact for each \( n, \) so one can consider the ‘worst’ sequence \( \mathbf{d}(n) \) for each \( n, \) and arrive at an equivalent way of stating the result: there is a function \( \delta(n) \to 0 \) such that the relative error in “~” is bounded above in absolute value by \( \delta(n) \) for all \( \mathbf{d} \) under consideration. This was the manner of stating Conjecture 1.3 below in [17].

Recall that \( \mathcal{B}_p(n) \) yields a random sequence consisting of \( n \) independent binomial variables \( \text{Bin}(n-1,p). \) Let \( \mathcal{B}_m = \mathcal{B}_m(n) \) denote \( \mathcal{B}_p(n) \) conditioned on the sum of the sequence being \( 2m \) and note that for a given sequence \( \mathbf{d} \) with \( \sum_{i=1}^{n} d_i = 2m \) we have that

\[
\mathbb{P}_{\mathcal{B}_m}(\mathbf{d}) = \binom{n(n-1)}{2m}^{-1} \prod_{i=1}^{n} \binom{n-1}{d_i}.
\]
which is, we recall, independent of $p$. Multiplying by $|\mathcal{G}(n,m)|$ and using Stirling’s approximation shows that the formula

$$
P_{D(\mathcal{G}(n,m))}(d) \sim P_{B_m}(d) \exp \left( \frac{1}{4} - \frac{\gamma^2}{4\mu^2(1-\mu)^2} \right)$$

(1.2)

is equivalent to the asymptotic formula in (1.1), as long as $m$ and $n(n-1) - 2m$ both tend to infinity.

Proposition 1.1 can be shown to follow from a conjecture made in 1997, which we present below, that is weaker than Conjecture 1.2. To state it, we define the model $\mathcal{E}_p$ to be $B_p(n)$, conditioned on even sum, and then construct $\mathcal{E}_p'$ from $\mathcal{E}_p$ by weighting each $d$ with a weight depending only on $2m = \sum d_i$, such that the value $m$ has distribution $\text{Bin}(n(n-1)/2, p)$.

**Definition** A probability $p = p(n)$ is acceptable if $p(1-p)n^2 = \omega \log n$ and there is a set-valued function $R_p(n)$ of integer sequences of length $n$ with even sum, such that both of the following hold.

(i) For $d \in R_p(n)$

$$
P_{D(\mathcal{G}(n,p))}(d) \sim P_{\mathcal{E}_p'}(d) \exp \left( \frac{1}{4} - \frac{\gamma^2}{4\mu^2(1-\mu)^2} \right).$$

(1.3)

(ii) In each of the models $\mathcal{E}_p$ and $D(\mathcal{G}(n,p))$, we have $P(R_p(n)) = 1 - n^{-\omega}$.

The conjecture in [17] is as follows.

**Conjecture 1.3** (McKay and Wormald [17]). If $p(1-p)n^2 = \omega \log n$ then $p(n)$ is acceptable.

The fact that the exponential factor in (1.3) is sharply concentrated near 1 was used to prove [17, Theorem 2.6(b)] (see Section 3 of that paper), which we do not state here. The first part of [17, Theorem 2.6(b)] gives bounds on the differences between the expected values of random variables in $D(\mathcal{G}(n,p))$ and the model $B_p(n)$ $|\Sigma$ is even of Proposition 1.1(ii) for any acceptable $p$. The second part of that theorem similarly bounds the difference between expectations in $D(\mathcal{G}(n,m))$ and $B_m(n)$, for any $m$ such that $2m/n(n-1)$ is acceptable. In particular, Conjecture 1.3 implies Proposition 1.1 via [17, Theorem 2.6(b)] with $X_n$ defined as the indicator of the event $H_n$ that appears in the definition of a.e. We omit further details of this, since for applications of our main result, one can look at all the consequences of Conjecture 1.3 in [17], of which Proposition 1.1 is just an example. In particular, further results in [17, Section 4] show that for many properties in $\mathcal{G}(n,p)$, the conditioning on parity in Proposition 1.1(ii) has negligible effect, so one can consider purely independent binomials. Sometimes the integration required to deal with the distribution of $\bar{p}$ also has negligible effect, and in any case, [17, Theorems 3.7 and 3.8] give some general results on carrying out the integration. To our knowledge, there are so far no detailed applications of the consequence for $\mathcal{G}(n,m)$ given in Proposition 1.1(i).

It was shown in [17, Theorem 2.5] that $p$ is acceptable if either $\omega \log n/n^2 \leq p(1-p) = o(n^{-1/2})$ or $p(1-p) \geq c/\log n$ for some $c > 2/3$, using the known enumeration results in the respective range. The essence of the proof of [17, Theorem 2.5] shows that if (1.1) of Conjecture 1.2 holds for all $d$ near some $d_0$, then by known concentration results, $p = d_0/(n-1)$ is acceptable. The same considerations prove that Conjecture 1.3 follows from Conjecture 1.2.

In this paper, we prove Conjecture 1.2 and hence Conjecture 1.3 and Proposition 1.1 in the remaining gap range. Our main theorem is the following.

**Theorem 1.4.** Let $\mu_0 > 0$ be a sufficiently small constant, and let $1/2 \leq \alpha < 3/5$. Let $n$ and $m$ be integers and set $d = 2m/n$ and assume that $\mu = d/(n-1)$ satisfies $\mu \leq \mu_0$ and, for all fixed
\( K > 0, (\log n)^K/n = O(\mu) \). Let \( \mathcal{D} \) be the set of sequences \( \mathbf{d} \) of length \( n \) satisfying \( \sum_i d_i = 2m \) and \( |d_i - d| \leq d^3 \) for all \( i \in [n] \). Then uniformly for all \( \mathbf{d} \in \mathcal{D} \) we have

\[
P_{\mathcal{D}(\mathcal{G}(n,m))}(\mathbf{d}) = P_{\mathcal{B}_m}(\mathbf{d}) \exp \left( \frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1-\mu)^2} \left( 1 + O \left( \frac{\log^2 n}{\sqrt{n}} + d^{5\alpha - 3} \right) \right) \right).
\]

**Corollary 1.5.** Conjectures 1.2 and 1.3 are both true.

In Section 6 we provide a hand-checkable proof of Theorem 1.4 under the restriction that \( \max(|d_i - d|) = O(\sqrt{d\log n}) \) and \( \sum_i (d_i - d)^2 \leq 2dn \). (See Theorem 6.3.) As a corollary of this and known results, we obtain Conjecture 1.3. Extending the expansions of functions involved to more terms using computer algebra lets us obtain a proof of Conjecture 1.2.

Our method is markedly different from those previously used for this problem, the most successful of which used either the “configuration model” with “switchings”, or the estimation of integrals representing coefficients of appropriate generating functions. Instead, we derive a set of equations relating the numbers of graphs with almost the same degree sequence. The equations are derived by examining the operation of moving one end of an edge from one vertex to another, and the operation of deleting an edge. These, together with one more equation derived from the degree constraints, are analysed from the perspective of fixed points of contraction mappings.

Note that if we build up a random graph by inserting \( m \) edges using the method of choosing each endpoint of each edge independently at random, the resulting multigraph has exactly the multinomial degree distribution, that is, \( \mathcal{B}_m(n) \) (see for example Cain and Wormald [7]). The only difficulty is that the result can contain loops and multiple edges. For Proposition 1.1, one would need to show that conditioning on the absence of loops and multiple edges does not significantly affect the degree distribution. This is difficult since the probability of no loop or multiple edges occurring is extremely small even for moderate densities. We believe that our approach to this enumeration problem gives an intuitive explanation of why the distribution of degrees in a random graph is so close to the binomial model, since it shows directly that the distribution of graph counts behaves “locally” in the appropriate way. Such an explanation is absent from the main methods used previously, in [16] and [15].

As a by-product of our proof, we obtain asymptotic formuale for the edge probabilities in a random graph with a given degree sequence. For a sequence \( \mathbf{d} \) of length \( n \), let \( \mathcal{G}(\mathbf{d}) \) be a graph chosen uniformly at random from all graphs that have degree sequence \( \mathbf{d} \), and let \( P_{ab}(\mathbf{d}) \) be the probability that the edge \( v_a v_b \) is present in \( \mathcal{G}(\mathbf{d}) \).

**Theorem 1.6.** Let \( \mathcal{D} \) be as in Theorem 1.4 and let \( a, b \in [n], a \neq b \). Then for all \( \mathbf{d} \in \mathcal{D} \)

\[
P_{ab}(\mathbf{d}) = \frac{d_a d_b}{d(n-1)} \left( 1 - \frac{(d_a - d)(d_b - d)}{d(n-1 - d)} \right) + O \left( \frac{\sqrt{d\log n}}{n^2} + \frac{(\sqrt{d\log n})^3}{n^3} \right).
\]

We also give a more accurate but more complicated formula for \( P_{ab}(\mathbf{d}) \) in Lemma 7.1(ii), and an approximation for sparser degree sequences in (4.4).

In Section 2 we provide the notation we use in this paper and some preliminary results. In Section 3 we derive certain recursive formulae that are the core of our method. In Section 4 we reprove enumeration results for the sparse case to illustrate how the recursive formulae from Section 3 are to be used to obtain explicit formulae. We also take this opportunity to provide a general template on how our proofs in the later parts are structured. In Section 5 the recursive formulae are turned into operators. Section 6 contains the proof of Theorem 1.4 under stricter assumptions on the degree spreads, and gives the proof of Conjecture 1.3, all using easily hand-checkable calculations. In Section 7 we provide the full proof for Theorem 1.4. We finish the paper with some concluding remarks in Section 8, and the Appendix gives details of some routine calculations.
1.2 Further comments

Studying the degree sequence of a random graph is not the only significant potential use of enumeration formulæ. Many properties of random regular graphs have been shown using them, in particular, results on subgraphs of random regular graphs or random graphs with given degrees. (See McKay [14] for examples.) Additionally, Kim and Vu [10, Equation (3.1)] heavily used the asymptotic formula for the number of $d$-regular graphs, in the known range, to give strong relations between random graphs and random regular graphs.

We note that Isaev and McKay [8] have given a major further development of the methods in [15] and [2] to obtain further results useful in the case of very dense graphs. They have also (unpublished) announced progress in pushing this method towards sparser cases. Also after our project was under way, Burstein and Rubin [6] presented an idea of comparing numbers of graphs with given degree sequences that is somewhat related to the techniques in the present paper, but differs in several significant ways. Their approach would give a formula that is valid up to maximum degree $n^{1-\delta}$ for any fixed $\delta > 0$, using a finite amount of computation. However, their general result is not explicit enough to enable the derivation of results as simple as the formula (1.1). In particular, we are not aware of any claims of extending the range of validity of the Binomial Approximation Conjecture, apart from the present paper.

2 Preliminaries and Notation.

For the reader’s convenience, we solidify some notation here. Our graphs are simple, that is, they have no loops or multiple edges. We write $a \sim b$ to mean that $a/b \to 1$, $f = O(g)$ if $|f| \leq Cg$ for some constant $C$, and $f = o(g)$ if $f/g \to 0$. We use $\omega$ to mean a function going to infinity, possibly different in all instances. Also $\binom{n}{2}$ denotes the set of 2-subsets of the set $[n] = \{1, \ldots, n\}$. We often consider a vector $d = (d_1, \ldots, d_n)$, and use $\Delta$ or $\Delta(d)$ to denote max, $d_i$, in line with the notation for maximum degree of a graph. Parity is an important issue, so we say a vector $d$ is even if $\sum_{i=1}^n d_i$ is even, and odd otherwise. Finally, in this paper multiplication by juxtaposition has precedence over “/”, so for example $j/\mu n^2 = j/\mu n^2$.

We first state a simple result by which we leverage an enumeration result from comparisons of related numbers.

**Lemma 2.1.** Let $S$ and $S'$ be probability spaces with the same underlying set $\Omega$. Let $G$ be a graph with vertex set $\mathcal{W} \subseteq \Omega$ such that $P_S(v), P_{S'}(v) > 0$ for all $v \in \mathcal{W}$. Suppose that $\varepsilon_0, \delta > 0$ such that $\min\{P_S(\mathcal{W}), P_{S'}(\mathcal{W})\} > 1 - \varepsilon_0 > 1/2$, and such that for every edge $uv$ of $G$,

$$\frac{P_{S'}(u)}{P_{S'}(v)} = e^{O(\varepsilon_0)} \frac{P_S(u)}{P_S(v)}$$

where the constant implicit in $O(\cdot)$ is absolute. Let $r$ be an upper bound on the diameter of $G$ and assume $r < \infty$. Then for each $v \in \mathcal{W}$ we have

$$P_{S'}(v) = e^{O(r \delta + \varepsilon_0)} P_S(v),$$

with again a bound uniform for all $v$.

**Proof.** For any $u, v \in \mathcal{W}$ we may take a telescoping product of ratios along a path joining $u$ to $v$ of length at most $r$. This gives $P_{S'}(u)/P_{S'}(v) = e^{O(r \delta)} P_S(u)/P_S(v)$. Summing over all $u \in \mathcal{W}$, gives

$$\frac{P_{S'}(\mathcal{W})}{P_{S'}(v)} = e^{O(r \delta)} \frac{P_S(\mathcal{W})}{P_S(v)},$$

and the claim follows using the lower bound $1 - \varepsilon_0$ on $P_S(\mathcal{W})$ and $P_{S'}(\mathcal{W})$. □
Using the lemma calls for evaluating the ratios of probabilities in an “ideal” probability space, the one by which we are approximating the “true” space. This leads to computing ratios in the conjectured formulae. One we need several times is the following. Let $H(d)$ denote the conjectured formula in the right hand side of (1.2), apart from the error term, that is

$$H(d) = P_{E_m}(d) \exp \left( \frac{1}{4} \frac{\gamma_2^2}{4\lambda^2(1-\lambda)^2} \right).$$

Noting that $\mu = d/(n-1)$ where $d = M_1/n$ (a function of the degree sequence) has the same value for both cases $d - e_a$ and $d - e_b$, and recalling $\gamma_2 = \gamma_2(d) = (n-1)^{-2}\sum_i (d_i - d)^2$, we have (for all odd $d$ with $\Delta = \max d_i \leq n/2$ and $\mu > 0$)

$$\frac{H(d - e_a)}{H(d - e_b)} = \frac{d_a(n - d_b)}{d_b(n - d_a)} \exp \left( \frac{(d_a - d_b)(\gamma_2 + O(\Delta/n^2))}{(n-1)^2\mu^2(1-\mu)^2} \right) = \frac{d_a(n - d_b)}{d_b(n - d_a)} \exp \left( \frac{(d_a - d_b)\gamma_2}{d^2(1-\mu)^2} + O(\Delta^2/(dn)^2) \right). \quad (2.1)$$

Next we turn to some issues involving existence of graphs with a given degree sequence. For counting purposes we will be considering graphs with vertex set $V = [n]$. Let $A = A(n) \subseteq \binom{[n]}{2}$ be a set which we call allowable pairs. Note that as usual we regard the edge joining vertices $u$ and $v$ as the unordered pair $\{u, v\}$, and denote this edge by $uv$ following standard graph theoretic notation. A sequence $d := (d_1, \ldots, d_n)$ is called $A$-realisable if there is a graph $G$ on vertex set $V$ such that vertex $a \in V$ has degree $d_a$ and all edges of $G$ are allowable pairs. In this case, we say $G$ realises $d$ over $A$. In standard terminology, if $d$ is $\binom{V}{2}$-realisable, it is graphical. Let $G_A(d)$ be the set of all graphs that realise $d$ over $A$. In this paper, we are particularly interested in the case that $A = A^{\binom{V}{2}} = \binom{V}{2}$. Then $G_A(d)$ is the set of all graphs $G$ on vertex set $[n]$ that have degree sequence $d$. By using different definitions of $A$, it is also possible to model other enumeration problems. Throughout this paper, all graphs are finite and simple (i.e. have no loops or multiple edges).

Let $E \subseteq A$, i.e. a subset of the allowable edges. We write $N_E(d)$ and $N_E^d(d)$ for the number of graphs $G \in G_A(d)$ that contain, or do not contain, the edge set $E$, respectively. We abbreviate $N_E(d)$ to $N_{ab}(d)$ if $E = \{ab\}$ (i.e. contains the single edge $ab$), and put $N(d) = |G_A(d)|$. (When $N$ and similar notation is used, the set $A$ should be clear by context.)

We pause for a notational comment. In this paper, a subscript $ab$ is always interpreted as an ordered pair $(a, b)$ rather than an edge (and similar for triples). This is irrelevant for $N_{ab}(d) = N_{ba}(d)$ since the two ordered pairs signify the same edge, but the distinction is important with other notation.

Let

$$P_E(d) = \frac{N_E(d)}{N(d)},$$

which is the probability that the edges in $E$ are present in a graph $G$ that is drawn uniformly at random from $G_A(d)$. Of particular interest are the probability of a single edge $av$ and a path $avb$, for which we simplify the notation to

$$P_{av}(d) = P_{\{av\}}(d), \quad P_{avb}(d) = P_{\{av, bv\}}(d).$$

We use $e_a$ to denote the elementary unit vector with 1 in its $a^{th}$ coordinate. We will use the following trick several times to switch between degree sequences of differing total degree.
Lemma 2.2. Let \( av \in \mathcal{A} \) and let \( \mathbf{d} \) be a sequence of length \( N \). Then

\[
N_{av}(\mathbf{d}) = N(\mathbf{d} - e_a - e_v) - N_{av}(\mathbf{d} - e_a - e_v) = \begin{cases} N(\mathbf{d} - e_a - e_v)(1 - P_{av}(\mathbf{d} - e_a - e_v)) & \text{if } N(\mathbf{d} - e_a - e_v) \neq 0 \\ 0 & \text{otherwise.} \end{cases}
\]

Proof. Removing an edge \( av \) from a graph in \( \mathcal{G}_\mathcal{A}(\mathbf{d}) \) shows that the number of graphs with that edge is the same as the number of graphs in \( \mathcal{G}_\mathcal{A}(\mathbf{d} - e_a - e_v) \) and no edge between \( a \) and \( v \). (The general form does not apply when \( N(\mathbf{d} - e_a - e_v) = 0 \) only because \( P_{av}(\mathbf{d} - e_a - e_v) \) is then technically undefined.)

For vertices \( a, b \in V \), if \( \mathbf{d} \) is a sequence such that \( \mathbf{d} - e_b \) is \( \mathcal{A} \)-realisable, we define

\[
R_{ab}(\mathbf{d}) = \frac{N(\mathbf{d} - e_a)}{N(\mathbf{d} - e_b)}.
\]

Furthermore, for any sequence \( \mathbf{d} \) we write \( M_1 = M_1(\mathbf{d}) = \sum_{a \in V} d_a \). If \( \mathbf{d} \) is \( \mathcal{A} \)-realisable, this is the total degree, or twice the number of edges, of a graph \( G \) in \( \mathcal{G}_\mathcal{A}(\mathbf{d}) \). We also write \( \mu = \frac{1}{2}M_1/|\mathcal{A}| \), which is the edge density in the case of graphical degree sequences. Finally, for a vertex \( a \in V \), we set \( \mathcal{A}(a) = \{ v \in V : av \in \mathcal{A} \} \), the “projection” of \( \mathcal{A} \) onto the edges incident with vertex \( a \), and, with \( \mathbf{d} \) understood, we use \( \mathcal{A}^*(a) \) for the set of \( v \in \mathcal{A}(a) \) such that \( N_{av}(\mathbf{d}) > 0 \).

We can bound the probability of an edge in a simple way using the following switching argument.

Lemma 2.3. Let \( \mathbf{d} \) be a graphical sequence of length \( n \) with \( \sum d_i = dn \). Then for any \( a \) and \( v \) in \([n]\) we have

\[
P_{av}(\mathbf{d}) \leq \frac{\Delta^2}{dn(1 - \Delta(\Delta + 2)/dn)}.
\]

Proof. For each graph \( G \) with degree sequence \( \mathbf{d} \) and an edge joining \( a \) and \( v \), we can perform a switching (of the type used previously in graphical enumeration) by removing both \( av \) and another randomly chosen edge \( a'v' \), and inserting the edges \( aa' \) and \( vv' \), provided that no loops or multiple edges are so formed. The number of such switchings that can be applied to \( G \) with the vertices of each edge ordered, is at least

\[
dn - 2(\Delta + 1)\Delta
\]

since there are \( dn \) ways to choose \( a' \) and \( v' \) as the ordered ends of any edge, whereas the number of such choices that are ineligible is at most the number of choices with \( a' \) being \( a \) or a neighbour of \( a \) (which automatically rules out \( a' = v \)), or similarly for \( v' \). On the other hand, for each graph \( G' \) in which \( av \) is \textit{not} an edge, the number of ways that it is created by performing such a switching backwards is at most \( \Delta^2 \). Counting the set of all possible switchings over all such graphs \( G \) and \( G' \) two different ways shows that the ratio of the number of graphs with \( av \) to the number without \( av \) is at most

\[
\beta := \frac{\Delta^2}{dn - 2\Delta(\Delta + 1)}.
\]

Hence \( P_{av}(\mathbf{d}) \leq \beta/(1 + \beta) \), and the lemma follows in both cases.

We finish the section with some simple sufficient conditions for a sequence to be graphical. We need this since our degree switching arguments require that \( N(\mathbf{d}') > 0 \) for several sequences \( \mathbf{d}' \) that are related to a root sequence \( \mathbf{d} \).
Lemma 2.4. For $\varepsilon > 0$, the following holds for $n$ sufficiently large. Let $d_i \geq 0$ be integers for all $1 \leq i \leq n$, with $\sum_{1 \leq i \leq n} d_i$ even. Then there exists a graph with degrees $d_1, \ldots, d_n$ provided that either of the following hold.

(a) There exists $0 < \mu < 1 - \varepsilon$ with $\max |\mu - d_i/n| < \varepsilon \mu$.

(b) We have $1 \leq d_i \leq 2\sqrt{n} - 2$ for $1 \leq i \leq n$.

Proof. Koren [11, Section 1] observed that the classical Erdős-Gallai conditions for the existence of a graph with degrees $d_1, \ldots, d_n$ are equivalent to the following: $\sum d_i$ is even, and whenever $S \cap T = \emptyset \neq S \cup T \subseteq [n]$ we have

$$\sum_{i \in S} d_i - \sum_{j \in T} d_j \leq s(n - 1 - t)$$

where $s = |S|$ and $t = |T|$.

For (a), it suffices to have $s(n - 1 - t) - s\mu n (1 + \varepsilon) + t \mu n (1 - \varepsilon) \geq 0$. We only need to check the extreme values of $t$, i.e. $t = 0$ and $t = n - s$. In both cases, the function is easily seen to be non-negative for all appropriate $s$ and $n$ sufficiently large, noting that $\mu \leq 1 - \varepsilon$.

For (b), we only need $s\Delta - t \leq s(n - 1 - t)$ (recall that $1 \leq d_i \leq \Delta$). Again taking the extreme values of $t$, only the larger one gives a restriction, being $s(\Delta + 2 - s) \leq n$, which is satisfied because $\Delta \leq 2\sqrt{n} - 2$.  

3 Recursive relations

In this section, we derive certain recursive formulae for the probability and ratio functions $P_{av}(d)$ and $R_{ab}(d)$. These identities will serve as a motivation for operators that we define in the next section. With a view to further applications of this work elsewhere, we consider an arbitrary set $\mathcal{A}$ of allowable pairs.

Our first result expresses the probability $P_{av}$ and the ratio $R_{ab}$ in terms of each other and the path probabilities $P_{ivj}$. The latter in turn are expressed in terms of $P_{av}$ in the next result.

Proposition 3.1. Let $d$ be a sequence of length $n$ and let $\mathcal{A} \subseteq \binom{[n]}{2}$.

(a) Let $a, v \in V$. If $N_{av}(d) > 0$ then

$$P_{av}(d) = d_v \left( \sum_{b \in \mathcal{A}^* (v)} R_{ba} (d - e_v) \frac{1 - P_{av} (d - e_b - e_v)}{1 - P_{av} (d - e_a - e_v)} \right)^{-1}. $$

(b) Let $a, b \in V$. If $d - e_b$ is $\mathcal{A}$-realisable then

$$R_{ab}(d) = \frac{d_a}{d_b} \frac{1 - B(a, b, d - e_b)}{1 - B(b, a, d - e_a)}, \quad (3.1)$$

where

$$B(i, j, d') = \frac{1}{d_i} \left( \sum_{v \in \mathcal{A}(i) \setminus \mathcal{A}(j)} P_{iv} (d') + \sum_{v \in \mathcal{A}(i) \cap \mathcal{A}(j)} P_{ivj} (d') \right). \quad (3.2)$$

provided that $B(b, a, d - e_a) \neq 1$.  

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Remarks

1. In (a), the summation over all $b$ in $A^*(v)$, instead of $A(v)$, is merely for the technicality that $P_{bv}(d - e_b - e_v)$ is otherwise undefined.

2. The condition $N_{av} > 0$ in part (a) does not reduce the practical usefulness of the lemma since the degree sequences where this condition fails for allowable edges $av$ with $d_a, d_v > 0$ are pathological enough that our method fails on those for other reasons.

3. For the range of the summations in $B(i, j, d')$, when $A = \binom{[n]}{2}$ (as in the applications in this paper) we have $A(i) \setminus A(j) = \{j\}$ and $A(i) \cap A(j) = [n] \setminus \{i, j\}$.

Proof. To prove part (a) of the lemma, let $d$ be an $A$-realisable sequence. Then every graph $G \in G_A(d)$ contributes exactly $d_v$ to $\sum_{b \in A^*(v)} N_{bv}(d)$. Hence, since $N_{av}(d) > 0$ we may write

$$d_v = \sum_{b \in A^*(v)} \frac{N_{bv}(d)}{N(d)} = P_{av}(d) \sum_{b \in A^*(v)} \frac{N_{bv}(d)}{N_{av}(d)} \frac{N(d - e_b - e_v)(1 - P_{bv}(d - e_b - e_v))}{N(d - e_a - e_v)(1 - P_{av}(d - e_a - e_v))}.$$

by Observation 2.2, noting that $N(d - e_b - e_v) \geq N_{bv}(d) > 0$ by definition of $A^*(v)$. Part (a) follows since by definition

$$\frac{N(d - e_b - e_v)}{N(d - e_a - e_v)} = R_{ba}(d - e_v),$$

and also noting that the summation is non-zero because $a \in A^*(v)$.

To prove part (b) of the lemma, assume that $d - e_b$ is $A$-realisable. Let $J_1$ be the set of graphs in $G_A(d - e_b)$ with a distinguished edge incident to vertex $a$, and $J_2$ the set of graphs in $G_A(d - e_a)$ with a distinguished edge incident to $b$. Then $|J_1| = d_a N(d - e_b)$ and $|J_2| = d_b N(d - e_a)$. Applying a degree switching to $G \in J_1$ consists of deleting the distinguished edge $av$ and adding a new distinguished edge $bv$, to produce a graph $G' \in J_2$. The degree switching cannot be performed, i.e. is not valid, if $bv \notin A$, which includes the case that $v = b$, or $bv$ is an edge of $G$. Now let $G \in G_A(d - e_b)$ be picked uniformly at random and let $v$ be a random neighbour of $a$ in $G$. Let $E$ be the event $E_A \cup E_D$ where $E_A$ is the event that $bv \notin A$ and $E_D$ is the event that $bv$ is an edge of $G$, and define $B(a, b, d - e_b) = P(E)$ (which we show further below to satisfy (3.2)). Then the number of valid degree switchings is

$$d_a N(d - e_b)(1 - B(a, b, d - e_b)).$$

We may count the same switchings from the other direction, i.e. starting with an element of $J_2$, using the same argument, and in this case $N(d - e_a) = 0$ is permissible. Equating the two counts gives

$$\frac{N(d - e_a)}{N(d - e_b)} = \frac{d_a}{d_b} \frac{1 - B(a, b, d - e_b)}{1 - B(b, a, d - e_a)},$$

where the denominator is non-zero by the hypotheses of (b), noting that $d_b > 0$ because $d - e_b$ is $A$-realisable. This gives (3.1).

The event $E_D$ can only happen if the vertex $v$ is a neighbour both of $a$ and of $v$, and hence

$$P(E_D) = \frac{1}{d_a} \sum_{v \in A(a) \cap A(b)} P_{avb}(d - e_b).$$
On the other hand, the vertex \( v \) is always a neighbour of \( a \) in a graph \( G \in \mathcal{G}_A(d) \), and thus
\[
P(E_A) = P(\{ v \notin A(b) \}) = P(\{ v \in A(a) \setminus A(b) \}) = \frac{1}{d_a} \sum_{v \in A(a) \setminus A(b)} P_{av}(d - e_v).
\]
Noting that \( E_A \cap E_D = \emptyset \), so \( P(E) = P(E_A) + P(E_D) \), we obtain the stated formula for \( B(a, b, d - e_b) \). The case of \( B(b, a, d - e_a) \) is the same but in reverse. \( \Box \)

We now address the evaluation of \( P_{avb}(d) \). As usual, the empty product is 1.

**Lemma 3.2.** Let \( n \) be an integer, \( A \subseteq \binom{[n]}{2} \), and let \( a, v, b \in [n] \) such that \( a \neq b \) and \( av, bv \in A \). Let \( k_0 \geq 1 \) and assume that \( 0 < N_{av}(d - ke_a - ke_v) < N(d - ke_a - ke_v) \) for \( 0 \leq k \leq k_0 \). Then
\[
P_{avb}(d) \geq \frac{P_{av}(d)}{1 - P_{av}(d - e_a - e_v)} \cdot \sum^{k_0}
\]
where
\[
\sum^{k_0} = \sum_{k=1}^{k_0} (-1)^{k-1}P_{bv}(d - k(e_a + e_v)) \prod_{j=1}^{k-1} \frac{P_{av}(d - j(e_a + e_v))}{1 - P_{av}(d - (j + 1)(e_a + e_v))},
\]
with \( \triangleright \) denoting ‘\( = \)’ when \( N_{(av, bv)}(d - k_0(e_a + e_v)) = 0 \), and otherwise ‘\( \leq \)’ for \( k_0 \) odd and ‘\( \geq \)’ for \( k_0 \) even.

**Proof.** The assumptions imply that \( N_{av}(d) > 0 \), and hence by definition,
\[
P_{avb}(d) = \frac{N_{av, bv}(d)}{N(d)} = \frac{N_{av, bv}(d)}{N_{av}(d)} \cdot \frac{N_{av}(d)}{N(d)}.
\]
(3.3)

Immediately (c.f. the proof of Lemma 2.2), and then using iteration for the second line,
\[
N_{(av, bv)}(d) = N_{be}(d - e_a - e_v) - N_{(av, bv)}(d - e_a - e_v)
\]
\[
= (-1)^{k_0}N_{(av, bv)}(d - k_0e_a - k_0e_v) + \sum_{k=1}^{k_0} (-1)^{k-1}N_{bv}(d - ke_a - ke_v).
\]

Since \( N(d - ke_a - ke_v) > 0 \) for \( 0 \leq k \leq k_0 \) by assumption, we have \( P_{bv}(d - ke_a - ke_v) \) well defined and so
\[
N_{(av, bv)}(d) = (-1)^{k_0} \xi + \sum_{k=1}^{k_0} (-1)^{k-1}N(d - ke_a - ke_v) \cdot P_{bv}(d - ke_a - ke_v)
\]
where \( \xi = N_{(av, bv)}(d - k_0e_a - k_0e_v) \geq 0 \). Applying Lemma 2.2 and (3.3) now gives
\[
P_{avb}(d) \triangleright P_{av}(d) \sum_{k=1}^{k_0} (-1)^{k-1}N(d - ke_a - ke_v) \cdot \frac{P_{bv}(d - ke_a - ke_v)}{1 - P_{av}(d - e_a - e_v)},
\]
(with “\( \triangleright \)” defined as in the lemma statement). The lemma follows once we have shown that
\[
\frac{N(d - ke_a - ke_v)}{N(d - e_a - e_v)} = \prod_{j=1}^{k-1} \frac{P_{av}(d - j(e_a + e_v))}{1 - P_{av}(d - (j + 1)(e_a + e_v))}.
\]
(3.4)

For \( d' = d - j(e_a + e_v) \), we have
\[
P_{av}(d') = \frac{N_{av}(d')}{N(d')} = \frac{N(d' - e_a - e_v)}{N(d')} \cdot (1 - P_{av}(d' - e_a - e_v)),
\]
and we have \( 1 - P_{av}(d' - e_a - e_v) > 0 \) by the assumption that \( N_{av}(d - ke_a - ke_v) < N(d - ke_a - ke_v) \). Hence, the product in (3.4) telescopes to become the left hand side. \( \Box \)
4 The general method and the sparse case for graphs

In this section we present a simple application of the recursive relations found in Section 3. This is a completely new derivation of a known formula for the number of sparse graphs with a given degree sequence. We first give a template of the method, since it is used again in the other proofs in this paper and can be used elsewhere.

Template of the method

Step 1. Obtain an estimate of the ratio between the numbers of graphs of related degree sequences, using Proposition 3.1. This step is the crux of the whole argument.

Step 2. By making suitable definitions, we cause this ratio to appear as the expression $P_{S'}(u)/P_{S'}(v)$ in an application of Lemma 2.1. Thus, the probability space $S'$ is the set of degree sequences, with probabilities determined by the random graph under consideration, and the graph $G$ in the lemma has a suitable vertex set $\mathcal{M}$ of such sequences. Each edge of $G$ is in general a pair of degree sequences $\mathbf{d} - \mathbf{e}_a$ and $\mathbf{d} - \mathbf{e}_b$ of the form occurring in the definition of $R_{ab}(\mathbf{d})$. Having defined $G$, we may call any two such degree sequences adjacent.

Step 3. Another probability space $S$ is defined on $\mathcal{M}$, by taking a probability space $\mathcal{B}$ directly from a joint binomial distribution, together with a function $\tilde{H}(\mathbf{d})$ that varies quite slowly, and defining probabilities in $S$ by the equation $P_S(\mathbf{d}) = P_{\mathcal{B}}(\mathbf{d})\tilde{H}(\mathbf{d})/E_{\mathcal{B}}\tilde{H}$.

Step 4. Using sharp concentration results, show that $P(\mathcal{M}) \approx 1$ in both of the probability spaces $S$ and $S'$ (where, by $\approx$, we mean approximately equal to, with some specific error bound in each case). As part of this, we show that $E_{\mathcal{B}}\tilde{H} \approx 1$. At this point, we may specify $\varepsilon_0$ for the application of Lemma 2.1.

Step 5. Apply Lemma 2.1 and the conclusions of the previous steps to deduce $P_{S'}(\mathbf{d}) \approx P_S(\mathbf{d}) \approx P_{\mathcal{B}}(\mathbf{d})\tilde{H}(\mathbf{d})$. Upon estimating the errors in the approximations, which includes bounding the diameter of the graph $G$, we obtain an estimate for the probability $P_{S'}(\mathbf{d})$ of the random graph having degree sequence $\mathbf{d}$ in terms of a known quantity.

Recall that given a sequence $\mathbf{d}$ we write $\Delta(\mathbf{d}) = \max_i d_i$ and $M_1(\mathbf{d}) = \sum_i d_i$. Note that the condition $\Delta(\mathbf{d})^6 + n^{\varepsilon} = o(nd^2)$ in the following result implies in particular that $\mu \sqrt{n} \to \infty$. This restriction is imposed just for simplicity; the technique can still apply in the (less interesting) extremely sparse case. Also recall the probability spaces of random sequences $\mathcal{G}(n,m)$ and $\mathcal{B}_m(n)$ from Section 1.

**Theorem 4.1.** Let $\varepsilon > 0$, let $n$ and $m$ be integers and set $d = 2m/n$. Let $\mathcal{D}$ be any set of sequences of length $n$ with $\sum_i d_i = 2m$ for all $\mathbf{d} \in \mathcal{D}$ such that $\Delta(\mathbf{d})^6 + n^{\varepsilon} = o(nd^2)$ uniformly for all $\mathbf{d} \in \mathcal{D}$. Then uniformly for $\mathbf{d}^* \in \mathcal{D}$ we have

$$P_{\mathcal{G}(n,m)}(\mathbf{d}^*) = P_{\mathcal{B}_m}(\mathbf{d}^*)\exp\left(\frac{1}{4} - \frac{\gamma^2}{4\mu^2(1-\mu)^2}\right) \left(1 + O\left(\frac{\Delta(\mathbf{d})^6 + n^{\varepsilon}}{nd^2} + n^{\varepsilon-1/2}\right)\right).$$

**Proof.** We can clearly assume that $\mathcal{D}$ is nonempty and we fix $\mathbf{d}^* \in \mathcal{D}$. We first estimate ratios of probabilities for adjacent sequences that are typical in the binomial model $\mathcal{B}_m$, and sequences close to $\mathbf{d}^* \in \mathcal{D}$ (as per Step 1 in the template above). We make the following definitions. Let

$$\Delta_1 = 2\Delta(\mathbf{d}^*) + n^{\varepsilon/6}$$

and define $\mathcal{D}^+$ to be the set of all sequences $\mathbf{d} \in \mathbb{Z}_+^n$ with $\Delta(\mathbf{d}) \leq \Delta_1$ and $M_1(\mathbf{d}) = 2m$. For an integer $r \geq 0$, denote by $Q_r^{(0)}$ (or $Q_r^{(1)}$) the set of all even (or odd, respectively) sequences in $\mathbb{Z}_+^n$ that have $L^1$ distance at most $r$ from some sequence in $\mathcal{D}^+$. (Recall that we defined the parity of $\mathbf{d}$ to be the parity of $\sum d_i$.) We will estimate the ratio of the probabilities of adjacent degree sequences in the random graph model using the following. Define $R_{ab}$ as in Section 2 with $A = \binom{1}{2}$. 

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Claim 4.2. Uniformly for all sequences \( d \in Q^1 \) and for all \( a, b \in [n] \)
\[
R_{ab}(d) = \frac{d_a}{d_b} \left( 1 + \frac{(d_a - d_b)(M_1 + M_2)}{M_1^2} \right) \left( 1 + O \left( \frac{\Delta_1^6}{d^3 n^2} \right) \right),
\]
where \( M_1 = M_1(d) \) and \( M_2 = M_2(d) = \sum_{v \in [n]} d_v (d_v - 1) \).

Proof. First, let \( d \in Q^+ \), let \( n_1 \) be the number of non-zero coordinates in \( d \). By summing vertex degrees we find \( d_n = M_1(d) \leq n_1 \Delta(d) \). By assumption we therefore have
\[
\Delta(d) \leq \Delta_1 \ll d^{1/3} n^{1/6} = \frac{(dn)^{1/3}}{n^{1/6}} \leq \frac{(\Delta(d)n_1)^{1/3}}{n^{1/6}},
\]
which readily implies that \( \Delta(d) = O(n_1^{1/2} / n^{1/4}) \). Lemma 2.4(b) applied to the sequence formed by the non-zero coordinates of \( d \) now implies that \( N(d) > 0 \) for \( n \) sufficiently large. We can deduce the same conclusion for all \( d \in Q^0_1 \), since \( \Delta(d) \) and \( n_1(d) \) can only change by bounded factors when moving from such \( d \) to the closest member of \( Q^+ \). Similarly, \( M_1(d) = \sum d_v = d_n + O(1) \) for all \( d \in Q^0_1 \). It is now clear by Lemma 2.3 and (4.2) that
\[
P_{av}(d) = O(\Delta^2 / dn) = O(1) \quad \text{for all } d \in Q^0_1 \text{ and all } a \neq v.
\]

Next consider any distinct \( a, v, b \in [n] \) and \( d \in Q^0_1 \), with \( d_a > 0 \) and \( d_v > 0 \). Then \( d - e_a - e_v \in Q^0_1 \) and hence \( N(d - e_a - e_v) > 0 \) from above, and also \( P_{av}(d - e_a - e_v) = O(\Delta^2 / dn) \) using (4.3). Thus, for \( n \) sufficiently large, \( N_{av}(d) > 0 \) by Lemma 2.2, and we have \( N_{av}(d) < N(d) \) since this establishes the hypotheses for \( N_{av}(d) \) and \( N(d) \) in Lemma 3.2 and in Proposition 3.1 whenever they are needed below.

It now follows that \( P_{av}(d) = O(\Delta^2 / d^2 n^2) \) for \( d \in Q^0_1 \), if \( d_a \) or \( d_v \) is 0 then this is immediate, and otherwise it follows from Lemma 3.2 with \( k_0 = 1 \) in view of (4.3). Next, definition (3.2) yields \( B(a, b, d) = O(\Delta^4 / d^2 n^2) \) for all distinct \( a, b \in [n] \) and all \( d \in Q^0_1 \) with \( d_a > 0 \). (In the current setting \( A(i) \cap A(j) \) = \{j\} when \( i \neq j \), and \( d \leq \Delta_1 \).) Thus (3.1) gives \( R_{ab}(d) = d_a / d_b (1 + O(\Delta^4 / d^2 n^2)) \) for all \( d \in Q^1_1 \) and all distinct \( a, b \) such that \( d_a, d_b > 0 \). If now \( d \in Q^0_1 \) and \( d_v > 0 \), we have \( N_{be}(d) > 0 \) as noted above. Therefore \( \sum_{b \in A^v} d_b = M_1(d) - d_v \) for such \( d \). Thus, Proposition 3.1 gives
\[
P_{av}(d) = d_a d_v / M_1 (1 + O(\Delta^4 / d^2 n^2))
\]
for all \( d \in Q^0_1 \) and \( a \neq v \). (If \( d_a \) and \( d_v \) are both nonzero, the proposition applies as mentioned above, and if either is 0, the claim holds trivially.) Using a similar argument, Lemma 3.2 with \( k_0 = \min\{2, d_a, d_v\} \) gives
\[
P_{ab}(d) = d_a [d_v]^{2d_b} / M_1 (1 + O(\Delta^4 / d^2 n^2))
\]
for all \( d \in Q^0_2 \) and all distinct \( a, v, b \in [n] \). Applying these results to the definition (3.2) of \( B \) for \( d \in Q^0_2 \) and distinct \( a, b \in [n] \), and recalling that \( d \leq \Delta(d) + 2 \), now gives the sharper estimate
\[
B(a, b, d) = \left( \frac{d_b}{M_1} + \frac{d_b M_2}{M_1^2} \right) \left( 1 + O \left( \frac{\Delta^4}{d^2 n^2} \right) \right),
\]
which we note is \( O(\Delta^2 / dn) \) as \( M_2 \leq M_1 \Delta \), where \( \Delta, M_1 \) and \( M_2 \) are with respect to \( d \). Thus, for all \( d \in Q^1_1 \) and all \( a, b \), and noting that \( M_2 \) changes by a negligible additive term \( O(\Delta_1) \) under bounded perturbations of the elements of the sequence \( d \),
\[
R_{ab}(d) = \frac{d_a}{d_b} \frac{(1 - (d_b - 1)/M_1 - (d_b - 1) M_2 / M_1^2)}{(1 - (d_a - 1)/M_1 - (d_a - 1) M_2 / M_1^2)} \left( 1 + O \left( \frac{\Delta^6}{d^3 n^2} \right) \right),
\]
which implies the claim. \( \blacksquare \)
We next make the definitions of probability spaces necessary to apply Lemma 2.1 (see Steps 2 and 3 in the template). Let \( \Omega \) be the underlying set of \( \mathcal{B}_m(n) \), \( \mathcal{W} = \mathcal{D}^+ \) and \( \mathcal{S}' = \mathcal{D}(\mathcal{G}(n, m)) \). Let \( H(d) = P_{\mathcal{B}_m}(d) \tilde{H}(d) \), where
\[
\tilde{H}(d) = \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1 - \mu)^2}\right),
\]
and define the probability function in \( \mathcal{S} \) by
\[
P_S(d) = H(d) / \sum_{d' \in \Omega} H(d') = \frac{H(d)}{E_{\mathcal{B}_m} \tilde{H}}.
\] (4.6)

Let \( G \) be the graph with vertex set \( \mathcal{W} \) and with two vertices (sequences) adjacent if they are of the form \( d - e_a \), \( d - e_b \) for some \( a, b \in [n] \) and odd \( d \).

We need to estimate the probability of \( \mathcal{W} \) in the two probability spaces (see Step 4 in the template). In \( \mathcal{G}(n, m) \) each vertex degree is distributed hypergeometrically with expected value \( d = 2m/n \). Also note that, letting \( \Delta^* = \Delta(d^*) \), we have by definition \( \Delta_1 \geq \Delta^* + n^{\epsilon/12} \sqrt{\Delta^*} \geq d + n^{\epsilon/12} \sqrt{\Delta^*} \). Thus, for \( d \in \mathcal{D} \),
\[
P_{\mathcal{S}'}(d_i > \Delta_1) \leq P_{\mathcal{S}'}(d_i > d + n^{\epsilon/12} \sqrt{\Delta^*}) = o(n^{-\omega})
\]
by [9, Theorems 2.10 and 2.1] for example (and noting \( \Delta^* \to \infty \)). The union bound, applied to each \( i \), now gives \( P_{\mathcal{S}'}(\mathcal{W}) = 1 - o(n^{-\omega}) \).

For similar reasons \( P_{\mathcal{B}_m}(\mathcal{W}) = 1 - o(n^{-\omega}) \). To deal with the exponential factor \( \tilde{H}(d) \), we claim that if \( d \) is chosen according to \( \mathcal{B}_m(n) \) then \( \gamma_2(d) = \mu(1 - \mu)(1 + O(\xi)) \) with probability \( 1 - o(n^{-\omega}) \), where \( \xi = O(n^{\epsilon - 1/2}) \). Indeed, this follows from the forthcoming Lemma 6.2(ii) in which we may take \( \alpha = \log^2 n/\sqrt{n} \) and note that \( \log^3 n = o(dn) \) is implied by \( m/\sqrt{n} \to \infty \). Thus, for such \( d \in \mathcal{B}_m(n) \), the exponential factor \( \tilde{H}(d) \) is \( 1 + O(\xi) \) with probability \( 1 - o(n^{-\omega}) \), and it is always at most \( e^{1/4} \). We deduce that \( E_{\mathcal{B}_m} \tilde{H} = 1 + O(\xi) \) and additionally, \( P_{\mathcal{S}}(\mathcal{W}) = 1 - o(n^{-\omega}) \). Thus, we may set \( \epsilon_0 = O(1/n) \) in Lemma 2.1 (with apologies to the function \( n^{-\omega} \), ending its life in this proof dominated by \( 1/n \)).

To apply Lemma 2.1 (see Step 5 in the template above), the final condition we need to show is that the ratios of probabilities satisfy
\[
\frac{P_{\mathcal{S}'}(d - e_a)}{P_{\mathcal{S}'}(d - e_b)} = e^{O(\delta)} \frac{P_{\mathcal{S}}(d - e_a)}{P_{\mathcal{S}}(d - e_b)}
\] (4.7)
whenever \( d - e_a \) and \( d - e_b \) are elements of \( \mathcal{D}^+ \), for a particular \( \delta = \delta(\Delta_1) \) independent of \( d \) and specified below, where the constant implicit in \( O() \) is independent of \( d \) and \( d^* \).

To evaluate the right hand side of (4.7) we observe that
\[
\frac{P_{\mathcal{S}}(d - e_a)}{P_{\mathcal{S}}(d - e_b)} = \frac{H(d - e_a)}{H(d - e_b)} = \frac{d_a(n - d_b)}{d_b(n - d_a)} \exp\left( \frac{(d_a - d_b)\gamma_2}{d^2(1 - \mu)^2} + O\left( \frac{\Delta^2}{(dn)^2} \right) \right),
\] (4.8)
for all \( d \in Q^1 \) by (2.1), where \( \bar{d} = M_1(d)/n \), and \( \gamma_2, \mu \) and \( \Delta \) are defined with respect to \( d \). Note that \( \bar{d} = d(1 + O(1/nd)) \), \( \mu = d(1 + O(1/nd))/\bar{d} - 1 \) and \( d \leq \Delta + 1 \), since \( d \in Q^1 \). Thus, we also have \( \gamma_2 = (M_2 + M_1 - dM_1)/(n - 1)^2 = O(d\Delta/n) \). Hence, the argument of the exponential factor in (4.8) is
\[
\frac{(d_a - d_b)\gamma_2}{d^2} + O(\Delta^2/n^2) = \frac{(d_a - d_b)(M_2 + M_1 - dM_1)}{M_1^2} + O(\Delta^2/n^2).
\]
Combining this with (4.8) and Claim 4.2 it follows that for all \( d \in Q_1^+ \)
\[
R_{ab}(d) = \frac{H(d - e_a)}{H(d - e_b)} \left( 1 + O\left( \frac{\Delta^6}{d^6n^2} \right) \right),
\]
where we use that \( (n-d_b)/(n-d_a) = \exp\left( (d_a-d_b)/n + O(\Delta^2/n^2) \right) \), \( M_2 = O(\Delta M_1) \) and \( \Delta^4/M_1^2 \leq \Delta^6/d^6n^2 \), \( \Delta \leq \Delta_1 \), and the most significant error term derives from Claim 4.2.

Equation (4.9) now implies that
\[
P_{S'}(d - e_a) \over P_{S'}(d - e_b) = R_{ab}(d) = e^{O(\delta)} P_S(d - e_a) \over P_S(d - e_b)
\]
whenever \( d - e_a \) and \( d - e_b \) are elements of \( \mathcal{M} = \mathcal{D}^+ \), where we may take \( \delta = \Delta^6/d^6n^2 \).

It is clear that the diameter of \( G \) is at most \( r \) := \( m = nd/2 \). Lemma 2.1 then implies that \( P_{S'}(d) = e^{O(r\delta + \varepsilon_0)} P_S(d) \) for \( d \in \mathcal{D}^+ \). To proceed from here, since we found that \( E_{\mathcal{G}_n}[\hat{H}] = 1 + O(\xi) \), equation (4.6) implies \( P_S(d) = H(d)(1 + O(\xi)) \) for \( d \in \mathcal{D}^+ \). Hence,
\[
P_{S'}(d^*) = e^{O(r\delta + \varepsilon_0 + \xi)} H(d^*).
\]
Note that \( \xi = O(n^{\varepsilon - 1/2}) \), and \( r\delta + \varepsilon_0 = O(\Delta^6/d^2n) = O((\Delta(d^*)^6 + n^\varepsilon)/d^2n) \). The theorem follows since \( S' = \mathcal{D}(\mathcal{G}(n,m)) \).

The result in Conjecture 1.2 or (1.2) follows from this in the sparse case, with different error terms, as long as \( d \) is appreciably above \( 1/\sqrt{n} \). For smaller \( d \), the analysis could be adjusted to obtain results, however the random graph is quite uninteresting here, typically having most vertices far short of \( o(n) \).

We note that the above result applies in the case of \( d \)-regular graphs only for \( d = o(n^{1/4}) \), far short of \( o(\sqrt{n}) \) as reached in [16]. It is also quite straightforward to reach past \( \sqrt{n} \) using our method, by carrying the calculations a little further, iterating several more the recursive equations that are only used twice in the proof above. In fact, this is how we first obtained the formulae for \( P \) and \( R \) in later sections. Having derived those “limiting” formulae, our proofs can completely avoid considering the iterated versions of the formulae, as shown in the next section.

5 Function operators and fixed points

In the previous section we used two iterations of the recursive equations from Proposition 3.1, each time applying them to all degree sequences at a certain distance from a root sequence \( d \). This allowed us to determine the ratio \( \mathcal{N}(d - e_a)/\mathcal{N}(d - e_b) \) up to negligible error terms. For denser graphs we would need an unbounded number of iterations to obtain the desired precision of about \( O(1/n \sqrt{d}) \) since the improvement is \( O(\Delta/n) \) each time. Instead of doing this, we define operators based on the recursive identities from Proposition 3.1 and study their behaviour on input functions that are close to the desired functions.

Let \( \mathbb{Z}_+^n \) denote the set of non-negative integer sequences of length \( n \). For a given integer \( n \) and a set \( \mathcal{A} \subseteq \binom{[n]}{2} \) we define \( \tilde{\mathcal{A}} \) to be the set of ordered pairs \((u,v)\) with \( \{u,v\} \in \mathcal{A} \). Ordered pairs are needed here because, although the functions of interest are symmetric in the sense that the probability of an edge \( uv \) is the same as \( vu \), our approximations to the probability do not obey this symmetry.

Given \( p : \tilde{\mathcal{A}} \times \mathbb{Z}_+^n \to \mathbb{R} \), we write \( p_{av}(d) \) for \( p(a,v,d) \), and remind the reader that in this paper, a subscript \( av \) always denotes an ordered pair rather than an edge. We next define an associated
function $p_{ab}(d)$ as follows. For $d \in \mathbb{Z}_+^n$ and $\{a, v\}, \{b, v\} \in A$, and integer $k_0 \geq 1$, we set

$$p_{ab}(d) = (p_{ab}(d))(p) = \frac{p_{av}(d)}{1 - p_{av}(d - e_a - e_v)} \cdot \Sigma^{k_0}(p, d, a, v, b), \quad (5.1)$$

where

$$\Sigma^{k_0}(p, d, a, v, b) = \sum_{k=1}^{k_0} (-1)^{k-1} \sum_{j=1}^{k-1} \frac{p_{av}(d - j(e_a + e_v))}{1 - p_{av}(d - (j+1)(e_a + e_v))}.$$ 

Note that $\Sigma^{k_0}$ in Lemma 3.2 is just $\Sigma^{k_0}(P, d, a, v, b)$. Furthermore, note that we allow $a = b$ in $p_{ab}(d)$ for convenience; its value is irrelevant since $p_{aaa}(d)$ is only called in $R(a, a, d)$ (see (5.4) below) which is then trivially 1. Of course, the functional value of such an operator is undefined where any denominators are zero. The parameter $k_0$ is suppressed from the notation $\Sigma$, and things involving it, for convenience. We define another associated function $bad = bad(p)$ as follows. For $d \in \mathbb{Z}_+^n$ and $a, b \in [n]$, set

$$bad(a, b, d) := \frac{1}{d_a} \left( \sum_{v \in A(a) \setminus A(b)} p_{av}(d) + \sum_{v \in A(a) \cap A(b)} p_{av}(d) \right). \quad (5.2)$$

Given $p : \bar{A} \times \mathbb{Z}_+^n \to [0, 1]$ and $r : [n]^2 \times \mathbb{Z}_+^n \to \mathbb{R}$, we define two operators $P(p, r)$ and $R(p)$ as follows. As with $p$, we write $r_{ab}(d)$ for $r(a, b, d)$. For $d \in \mathbb{Z}_+^n$ and $a, v, b \in [n]$ with $\{a, v\} \in A$ we set

$$P(p, r)_{av}(d) = d_v \left( \sum_{b \in A(v)} r_{ba}(d - e_v) \frac{1 - p_{bv}(d - e_b - e_v)}{1 - p_{av}(d - e_a - e_v)} \right)^{-1} \quad (5.3)$$

$$R(p)_{ab}(d) = \frac{1}{d_b} \cdot \frac{1 - bad(a, b, d - e_b)}{1 - bad(b, a, d - e_b)} \quad (5.4)$$

Proposition 3.1 and Lemma 3.2 say essentially (ignoring the issue of truncating the summations at $k_0$ in (5.1)) that the probability and ratio functions $P$ and $R$ are fixed points of the operators $P$ and $R$. It is very useful for us that $P$ is “contractive”, in a certain sense, in a neighbourhood of this fixed point. Unfortunately, the concept of contraction which we have here uses a slightly different metric before and after applying the operators, stemming from the fact that the value of $R(p)$ at a point $(a, b, d)$ depends on values at a set of $d'$ in a neighbourhood of $d$. This makes it difficult to define a true and useful contraction mapping. Nevertheless, we can exploit the useful features of the situation using the following lemma.

We denote by $Q^0_v(d), Q^0_v(d) \subseteq \mathbb{Z}^n$ the set of even and odd, respectively, vectors of arbitrary integers that have $L^1$-distance at most $r$ from $d$. We use $1 \pm \xi$ to denote a quantity between $1 - \xi$ and $1 + \xi$ inclusively.

Note that from (5.3) we may assume $v \in A(a) \cap A(b)$ when computing $r_{ba}$, and also $R(p)(a, b, \cdot)$. So we restrict to $A(a) \cap A(b) \neq \emptyset$ in the following.

**Lemma 5.1.** Given a constant $C$, there is a constant $\mu_0 > 0$ such that the following holds. Let $n$ be an integer and $A \subseteq [n]^2$ such that $|A(a) \setminus A(b)| < Cd_a$ for all $a$ and $b$ that satisfy $A(a) \cap A(b) \neq \emptyset$. Let $k_0 > 0$ be an integer and $d = (d(n)) \in \mathbb{Z}_+^n$. Let $0 < \xi \leq 1$ and $0 < \mu = \mu(n) < \mu_0$, let $p, p' : \bar{A} \times \mathbb{Z}_+^n \to \mathbb{R}$ be functions such that $p_{av}(d'), p'_{av}(d') \leq \mu$ for all $(a, v) \in \bar{A}$ and $d' \in Q^0_{2k_0+1}(d)$, and let $r, r' : [n]^2 \times \mathbb{Z}_+^n \to \mathbb{R}$. Let $a, v, b \in [n]$ such that $a \neq b$, $A(a) \cap A(b) \neq \emptyset$ and $\{a, v\} \in \bar{A}$.
(a) If $d$ is odd and $p_{cw}(d') = p'_{cw}(d')(1 \pm \xi)$ for all $c, w \in A$ and all $d' \in Q^0_{2k_0+1}(d)$ then
\[ R(p)_{ab}(d) = R(p')_{ab}(d)(1 + O(\mu \xi)). \]

(b) If $d$ is even, $p_{av}(d') = p'_{av}(d')(1 \pm \xi)$ for all $c \in A(v)$ and all $d' \in Q^0_{2}(d)$, and $r_{ea}(d') = r'_{ea}(d')(1 \pm \mu \xi)$ for all $c \in A(v)$ and all $d' \in Q^1_{2}(d)$, then
\[ P(p, r)_{av}(d) = P(p', r')_{av}(d)(1 + O(\mu \xi)). \]

The constants implicit in $O(\cdot)$ are absolute.

Proof. We assume without loss of generality that $\mu_0 < 1/2$, which readily implies that $p_{av}(d'), p'_{av}(d') < 1/2$ for all $d' \in Q^0_{2k_0+1}(d)$ and all $a \neq v \in [n]$. For (a) we use $p_{cw}(d') = p'_{cw}(d')(1 \pm \xi)$ for all $cw \in A$ and all $d' \in Q^0_{2k_0+1}(d)$. Apply this to the right hand side of (5.1) and note that in the summation over $k$, the terms are bounded in size by a geometric series with ratio $O(\mu)$, which can be assumed to be arbitrarily small. This implies, for all $\{b, v\} \in A$ and for all $d' \in Q^0_{2}(d)$, that $p_{av}(d') = p'_{av}(d')(1 + O(\xi))$, and then from (5.2) that $\text{bad}(a, b, d')(p) = \text{bad}(a, b, d')(p')(1 + O(\xi))$ for all such $d'$. Additionally, we observe that $p_{av}(d') = p_{aw}(d')O(\mu)$ and hence
\[ \text{bad}(a, b, d') = \left( |A(a)\setminus A(b)| + \sum_{v \in A(a)\setminus A(b)} p_{av}(d') \right) O(\mu)/d_a = O(\mu) \]

since $\sum_a p_{av}(d) = d_a$ and $|A(a)\setminus A(b)|/d_a \leq C$ by assumption on $\{a, b\}$. Thus $\text{bad}(a, b, d')(p) = \text{bad}(a, b, d')(p') + O(\mu \xi)$ for all $d' \in Q^0_{2}(d)$. Hence, with $\mu$ sufficiently small to ensure $\text{bad}(b, a, d - e_a) < 1/2$ say, part (a) follows from (5.4). The equation for $P(p, r)$ follows immediately from (5.3) since both $p_{be}$ and $p_{av}$ are $O(\mu)$.

Fix $\Omega^{(0)} \subseteq \mathbb{Z}_+^n$ for the following definitions. Let $\Omega^{(s)}$ denote the set of all $d \in \Omega^{(0)}$ for which $Q^0_{s}(d), Q^1_{s}(d) \subseteq \Omega^{(0)}$. We define a set of distance functions, indexed by $s$, on the functions $p : A \times \mathbb{Z}_+^n \rightarrow [0, 1]$ by
\[ \chi^{(s)}(p, p') = \max\{|\log (p_{cw}(d')/p'_{cw}(d'))| : cw \in A, \ d \in \Omega^{(s)}\}. \]

These are extended metrics: they may take the value $\infty$. Clearly $\chi^{(s)}$ is non-increasing in $s$ unless $\Omega^{(s)}$ is empty, in which case we set $\chi^{(s)} = 0$.

Let $\Pi_{\mu}$ denote the set of functions $p : A \times \mathbb{Z}_+^n \rightarrow \mathbb{R}$ such that $0 \leq p_{av}(d') \leq \mu$ for all $(a, v) \in A$ and $d' \in Q^0_{2k_0+1}(d)$. Also, define the compositional operator $C(p) = P(p, R(p))$. Recall that $k_0$ is used in defining $R$.

Corollary 5.2. Given a constant $C$, there is a constant $\mu_0 > 0$ such that the following holds. Let $n, k_0$ and $A$ be as in Lemma 5.1. Let $\xi \leq C$ and $0 < \mu = \mu(n) < \mu_0$. Then for $p, p' \in \Pi_{\mu}$ and $s \geq 0$ with $\chi^{(s)}(p, p') \leq \xi$, we have $\chi^{(s+2k_0+2)}(C(p), C(p')) = O(\mu \xi)$.

Proof. Since $\chi^{(s)}(p, p') \leq \xi$, we have $p_{cw}(d') = p'_{cw}(d')(1 + O(\xi))$ for all $cw \in A$ and all $d \in \Omega^{(s+2k_0+1)}$ and $d' \in Q^0_{2k_0+1}(d)$. By Lemma 5.1(a), $R(p)_{ab}(d) = R(p')_{ab}(d)(1 + O(\mu \xi))$ for all such $d$ and for $a, b$ as in that lemma.

Now let $d \in \Omega^{(s+2k_0+2)}$. Applying Lemma 5.1(b) with $r = R(p)$, $r' = R(p')$ (and $\xi$ replaced by $C'\xi$ for a suitable large constant $C'$) gives
\[ C(p)_{cw}(d) = C(p')_{cw}(d)(1 + O(\mu \xi)) \]

for $(c, w) \in A$ (recalling that $C$ is not necessarily symmetric in $c, w$). Since $\mu \xi \leq \mu_0 C$ is small enough for sufficiently small $\mu_0$, we have $\log(1 + O(\mu \xi)) = O(\mu \xi)$ as required to deduce that $\chi^{(s+2k_0+2)}(C(p), C(p')) = O(\mu \xi)$. 

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6 Proof of the binomial model in the graph case

In this section we prove Conjecture 1.3 for $p$ in the “gap range” which we can describe as $o(n^{-1/2}) < p < c/\log n$. Before doing so we need concentration results for some functions $f(d)$ when $d$ has either independent binomial entries, or is the degree sequence of $G(n,m)$. In [17, Theorem 3.4] it was essentially shown that, when $d$ has independent binomial entries, $\sigma^2 = \sigma^2(d) = \sum_{i=1}^{n} (d_i - d)^2 / n$ is concentrated. We give a more efficient proof of the crucial part of this, using the following result, (McDiarmid) Lemma 6.1 (7.4) and Example (7.3) of McDiarmid [12]. However, since the constants there are not explicit and the framework makes the proof not so easily accessible, we give a proof here.

**Lemma 6.1 (McDiarmid).** Let $c > 0$ and let $f$ be a function defined on the set of subsets of some set $U$ such that $|f(S) - f(T)| \leq c$ whenever $|S| = |T| = m$ and $|S \cap T| = m - 1$. Let $S$ be a randomly chosen $m$-subset of $U$. Then for all $\alpha > 0$ we have

$$P(|f(S) - Ef(S)| \geq \alpha \sqrt{m}) \leq 2 \exp(-2\alpha^2).$$

**Proof.** Consider a process in which the random subset $S$ is generated by inserting $m$ distinct elements one after another, each randomly chosen from the remaining available ones. Let $S_k$ denote the $k$th subset formed in this process, $0 \leq k \leq m$. Consider the Doob martingale process determined by $Y_k = E(f(S_m) \mid S_0, \ldots, S_k) = E(f(S_m) \mid S_k)$. Given $S_{k-1}$, let $X = X_0, \ldots, X_m$ denote the remaining elements added in the process. Let $X_0$ be the random sequence $X$ conditioned on $X_k = x_k \in U$, and $X_1$ the random sequence $X$ conditioned on $X_k = x'_k \in U$. Then $X_0$ can be coupled with $X_1$ by interchanging $x_k$ and $x'_k$ wherever they occur in $X_0$. The values of $f(S_m)$ in the two elements of a couple pair differ by at most $c$ by assumption. Since each possible realisation of $X_k, \ldots, X_m$ has the same probability, it follows that

$$|E(f(S_m) \mid S_{k-1} \wedge X_k = x_k) - E(f(S_m) \mid S_{k-1} \wedge X_k = x'_k)| \leq c,$$

and hence $|Y_{k-1} - Y_k| \leq c$. Azuma's Inequality (see, e.g., [9]), or alternatively [12, Corollary (6.10)], now completes the proof. \[
\]

Recall that by $\omega$ we denote a function that tends to $\infty$ arbitrarily slowly with $n$, and that $B_m(n)$ is a sequence of $n$ i.i.d. random variables each distributed as Bin$(n-1, m)$ conditioned on $\sum d_i = 2m$.

**Lemma 6.2.** Define $d = (d_1, \ldots, d_n)$ as either (a) the degree sequence of a random graph in $G(n,m)$, or (b) a sequence in $B_m(n)$. Let $d = 2m/n$. Then

(i) for $1 \leq i \leq n$ and all $\alpha > 0$ we have

$$P(|d_i - d| \geq \alpha) \leq 2 \exp\left(-\frac{\alpha^2}{2(d + \alpha/3)}\right);$$

(ii) if $\log^3 n = o(\log n^2)$ and $\alpha$ satisfies $(\log n) / \sqrt{n} + (\log^3/2 n) / \sqrt{n} = o(\alpha)$ then we have

$$P(|\sigma^2 - Var d_i| \geq \alpha d + 1/n) = o(n^{-\omega}).$$

Moreover, $Var d_i = d(n-d)/n + O(d/n)$.

**Proof.** We deal with the graph case (a) first. Each vertex degree $d_i$ is distributed hypergeometrically with parameters $\binom{n}{2}$, $m, n-1$, expected value $d$, and hence (i) holds by [9, Theorems 2.10 and 2.1]. For a graph $G$ with degrees $d_1, \ldots, d_n$ define

$$f = f(G) = \sum_{i=1}^{n} \min\{(d_i - d)^2, x\},$$

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where $x > 1$ is specified below. Then increasing or decreasing the value of a single $d_i$ by 1 whilst holding $d$ fixed can only change $f$ by at most $(\sqrt{x})^2 - (\sqrt{x} - 1)^2 < 2\sqrt{x}$. Since $G$ is determined by a random $m$-subset of all possible edges, Lemma 6.1 applies with $c = 8\sqrt{x}$ (as each edge in the symmetric difference of $S$ and $T$ affects two vertex degrees). Replacing $\alpha$ appropriately gives

$$
P(|f(G) - Ef(G)| \geq \alpha d) \leq 2 \exp(-\alpha^2 dn/(32x)) = o(n^{-\omega})$$

provided that $\alpha^2 dn/(x \log n) \to \infty$. On the other hand, let $A$ denote the event $\max_i |d_i - d| \geq \sqrt{x}$. By (i) and the union bound applied over all $n$ values of $i$, we have $P(A) = o(n^{-\omega})$ as long as we choose $x = \omega(d \log n + \log^2 n)$. By the bound on $\alpha$, there exists $x$ satisfying both conditions, and at this point we set $x$ as such. Provided that $A$ does not hold, we have $f(G) = \sum(d_i - d)^2 = n\sigma^2$. We thus conclude

$$
P\left(\frac{\sigma^2 - Ef(G)}{n} \geq \alpha d\right) = o(n^{-\omega}) + O(1)P(f(G) \neq n\sigma^2) = o(n^{-\omega}).$$

Now evidently

$$|Ef(G) - n\sigma^2| = O(n^3)P(f(G) \neq n\sigma^2) = o(n^{-\omega})$$

and thus

$$P\left(\left|\frac{\sigma^2 - Ef(G)}{n}\right| \geq \alpha d + 1/2n\right) = o(n^{-\omega}).$$

Noting that $E(d_i - d)^2 = \text{Var} d_1$, we obtain part (ii) for (a). The estimate for $\text{Var} d_1$ follows from the standard formula for variance of this hypergeometric random variable.

For the binomial random variable case (b), essentially the same argument applies for both (i) and (ii), by regarding $d_i$ as a random variable taking values in $\{0, 1\}$. Conditioning on the sum being $2m$ is equivalent to a uniformly random selection of a $2m$-subset of the $n(n - 1)$ indicator variables.

We shall see that the following establishes Conjecture 1.3 in the gap range with explicit error terms. Recall that $\sigma^2(d) = \frac{1}{n} \sum_{i=1}^n (d - d_i)^2$ where $n$ is the length of the sequence $d$.

**Theorem 6.3.** Let $n$ and $m$ be integers, set $d = 2m/n$, and assume that $\mu = d/(n - 1)$ satisfies $(\log n)^K/n < \mu = o(1/\sqrt{n}^{3/4})$ for all fixed $K > 0$. Let $\mathcal{D}$ be the set of sequences $d$ of length $n$ satisfying the following for some constant $C \geq 2$:

(i) $M_1(d) = 2m,$

(ii) $|d_i - d| \leq C\sqrt{\log n}$ for all $i \in [n],$

(iii) $\sigma^2(d) \leq 2d.$

Then

(a) in each of the models $\mathcal{B}_m(n)$ and $\mathcal{D}(\mathcal{G}(n, m))$ we have $P(\mathcal{D}) = 1 - n^{-h(C)}$, where $h(x) \to \infty$ as $x \to \infty$, and

(b) for $d = d(n) \in \mathcal{D}$ we have

$$P_{D(\mathcal{G}(n, m))}(d) = P_{Sm}(d) \exp \left( \frac{1}{4} \frac{\gamma_2^2}{4\mu^2(1 - \mu)^2} \right) \left( 1 + O \left( \frac{1}{\sqrt{d}} + \frac{d\sqrt{\log n}}{n} + \frac{d^2(\log n)^{3/2}}{n^2} \right) \right),$$

where $\gamma_2 = \gamma_2(d) = \frac{1}{(n - 1)^2} \sum_i (d_i - d)^2$. 

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We note that the constant implicit in $O()$ in (b) of course can, and in fact does, depend on $C$.

Proof. Let $\Omega$ be the underlying set of $\mathcal{B}_m(n)$ and let $d \in \Omega$. We will consider $d$ chosen either according to $\mathcal{D}(G(n, m))$ or $\mathcal{B}_m(n)$. By definition, $M_1(d) = 2m$ for all $d \in \Omega$. Apply Lemma 6.2(i) with $\alpha = C \sqrt{d \log n}$ and the union bound to see that in both $\mathcal{D}(G(n, m))$ and $\mathcal{B}_m(n)$, with probability at least $1 - n^{-f(C)}$, for all $i \in [n]$ we have $|d_i - d| \leq C \sqrt{d \log n}$ where we may take $f(C) = C^2/3 - 1$. Now, apply Lemma 6.2(ii) with $\alpha = 1/2$ and note that $d > \log^2 n$ by assumption to obtain that $\sigma^2(d) \leq 2d$ with probability at least $1 - n^{-\omega}$. Therefore, $d$ satisfies (ii) and (iii) with probability $1 - n^{-h(C)}$, for some function $h(C) \to \infty$ as $C \to \infty$, and $d$ satisfies (i) always. Hence (a) follows.

For (b) we first consider the ratio for adjacent degree sequences (see Step 1 in the template given at the start of Section 4). Let $Q_1$ be the set of sequences $d \in \mathbb{Z}_n$ such that $d - e_a \in \mathcal{D}$ for some $a \in [n]$. Recall that $P_{av}(d)$ denotes the probability that the edge $av$ is present in a graph $G \in \mathcal{G}(d)$ and that

$$R_{ab}(d) = \frac{P_{S'}(d - e_a)}{P_{S'}(d - e_b)}.$$ 

We now present functions $P_{GR}$ and $R_{GR}$ that approximate the probability and ratio functions $P$ and $R$ sufficiently well. For $a, v, b \in [n]$ set

$$P_{av}^{GR}(d) = \frac{d一段时间}{d(n-1)} \left(1 - \frac{(d_a - d)(d_v - d)}{d(n-1-d)}\right),$$

$$R_{ab}^{GR}(d) = \frac{d_a(n-d_b)}{d_b(n-d_a)} \cdot \left(1 + \frac{(d_a - d_b)}{d^2n} \sigma^2(d) \right),$$

where $d$ is the average of $d$ as usual.

Claim 6.4. For $d = d(n) \in \mathcal{D}$ and $a \neq v \in [n]$ we have

$$P_{av}(d) = P_{av}^{GR}(d) (1 + O(\eta_1 + \eta_2)),$$

and uniformly for all $d \in Q_1$ and for all $a, b \in [n]$

$$R_{ab}(d) = R_{ab}^{GR}(d) (1 + O(\eta_1 + \mu \eta_2)),$$

where $\eta_1 = 1/dn + \varepsilon d/n^2$ and $\eta_2 = \varepsilon /n + \varepsilon^3 d^2/n^2$, with $\varepsilon = C \sqrt{(\log n)/d}$.

Note that $\varepsilon$ is simply the upper bound on the relative degree spread implied by (ii). We remark at this point that the proof of the claim only assumes $\log^2 n/n \ll \mu \leq \mu_0$ for some small enough constant $\mu_0$, which holds in the present context since $\mu = o(1)$.

Proof of Claim 6.4. To show that $P$ and $P_{GR}$ (and $R$ and $R_{GR}$) are $(\eta_1 + \eta_2)$-close in the sense of (6.2) and (6.3), we consider the operator $C$ as defined for Corollary 5.2, with $k_0 = 4 \log n$ (which completely specifies $R$). We first show that $k_0$ iterated applications of the operator $C$ to $P$ yield a function that is $(2\mu)^{k_0}$-close to $P$. We then use the “contraction” property of $C$ as expressed in Corollary 5.2 to show that $C^{k_0}(P_{GR})$ and $C^{k_0}(P)$ are $(2\mu)^{k_0}$-close. Finally, we show that $P_{GR}$ and $C^{k_0}(P_{GR})$ are $(\eta_1 + \eta_2)$-close. These three statements then imply Claim 6.4. We now make this argument precise.

Fix $r = 2(k_0 + 1)^2 + 3 = O(\log^2 n)$. Let $\Omega^{(0)}$ be the set of sequences $d \in \mathbb{Z}_n$ that are at $L^1$ distance at most $r$ from a sequence in $\mathcal{D}$. By Lemma 2.4(a) we obtain that $\mathcal{N}(d) > 0$ for all even $d \in \Omega^{(0)}$ where we use that $\mu \geq (\log n)^3/n$ and assumption (ii) to show that the conditions of that
lemma are satisfied. After this, for \( n \) sufficiently large, Lemma 2.3, together with \( \mu < 1/4 \) and conditions (i) and (ii), implies that for all distinct \( a, v \in [n] \)

\[
P_{av}(d) \leq \frac{\mu}{1-\mu}(1+o(1)) < \frac{1}{2} \quad \text{for all } d \in \Omega(0).
\]  

(6.4)

Define \( \Omega(j) \) as in Corollary 5.2 to be the set of sequences \( d \in \Omega(0) \) of \( L_1 \) distance at least \( j + 1 \) from all sequences outside \( \Omega(0) \). Consider any distinct \( a, v, b \in [n] \) and even \( d \in \Omega(2) \). From the conclusion above, \( N(d - e_a - e_v) > 0 \), and we can take \( P_{av}(d - e_a - e_v) < 1 \) by (6.4). Thus \( N_{av}(d) \geq 0 \) by Lemma 2.2, and we have \( N_{av}(d) < N(d) \) since \( P_{av}(d) < 1 \). This tells us that for all even \( d \in \Omega(2k_0+2) \), the hypotheses of Lemma 3.2 are satisfied, and the consequent alternating bounds imply that

\[
P_{av}(d) = \frac{P_{av}(d)\Sigma_{k_0}(P)}{(1 - P_{av}(d - e_a - e_v))(1 + O(2^{k_0}\mu^{k_0}))} = (p_{av}(d))(P)(1 + O(2^{k_0}\mu^{k_0}))
\]

as per definition (5.1). Hence, Proposition 3.1(b) implies, c.f. (5.4), that for all odd \( d \in \Omega(2k_0+3) \) we have \( R_{ab}(d) = R(P)_{ab}(d)(1 + O(2^{k_0}\mu^{k_0})) \). Using this and comparing (5.3) with Proposition 3.1(a), we obtain \( \chi^{(J+2)}(P, C(P)) = O(2^{k_0}\mu^{k_0}) \), where \( C \) and \( \chi \) are defined as for Corollary 5.2, and we let \( J \) denote \( 2k_0 + 2 \).

Repeated applications of Corollary 5.2 give \( \chi^{(sJ+2)}(C^{s-1}(P), C^{s}(P)) = O(2^{k_0}\mu^{k_0+s}) \) for \( s \geq 1 \).

As in the standard argument giving bounds on the total distance moved during iterations of a contraction mapping, this gives

\[
\chi^{(k_0J)}(P, C^{k_0}(P)) = O(2^{k_0}\mu^{k_0}).
\]

For \( d \in \Omega(0) \) and all appropriate \( a \) and \( v \), (6.4) implies \( P_{av}(d) < 3\mu/2 \) say (as \( \mu < \mu_0 \)). Condition (ii) of the present theorem, together with the lower bound on \( \mu \) with \( K > 1 \), imply that \( d_i \sim \mu n \) uniformly for all \( i \), so by definition (6.1) we have \( P_{av}(d) \sim \mu \). Hence for large \( n \) we have \( P_{av}(d) = P_{av}(\epsilon d)(1 \pm 1) \) for such \( d \). Applying Lemma 5.1(a) and then (b) consecutively with \( \xi = 1 \) gives \( P_{av}(d) = P_{av}(\epsilon d)(1 + O(\mu)) \) for \( d \in \Omega(J) \). Thus \( \chi^{(J)}(P, P_{av}) = O(\mu) \), and \( k_0 - 1 \) iterated applications of Corollary 5.2 produce

\[
\chi^{(k_0J)}(C^{k_0}(P), C^{k_0}(P_{av})) = O(2^{k_0}\mu^{k_0}) = o(1/dn).
\]

Finally, \( C(P_{av})_{ab}(d) \) can be estimated for all \( d \in \Omega(0) \) as follows. Straightforward expansions give (recalling \( k_0 = 4\log n \))

\( a \) \( R(P_{gr})_{ab}(d) = R_{ab}(d)(1 + O(\eta_1)) \) for all odd \( b \in \Omega(0) \), all \( a, b \in V = [n] \); and

\( b \) \( P(P_{gr}, P_{gr})_{ab}(d) = P_{ab}(d)(1 + O(\eta_2)) \) for all even \( d \in \Omega(0) \), all \( a, v \in V = [n] \), \( a \neq v \).

This is justified in the form of Lemma A.1 in Appendix A, after making two observations. One is that the error term \( \eta_1 \) can be simplified to be \( 1/dn + \varepsilon d/n^2 + \varepsilon^2 d^2/n^2 \) since \( k_0 = 4\log n \), and then \( \varepsilon^2 d^2/n^2 \) can be dropped because \( \varepsilon^3 d = o(1) \). The second is that the error terms \( \eta_1 \) and \( \eta_2 \) are now defined with reference to sequences which are at distance \( O(\log^2 n) \) from our initial sequences in \( \mathcal{D} \) and are thus asymptotically the same as the stated values. (In fact, a better approximation is proved in Lemma 7.1 using computer assistance.) Apply these two statements one after another (c.f. the proof of Lemma 5.1(b)), first using (a) to estimate \( R \) and then using (b) to find that \( \chi^{(0)}(C(P_{gr}), P_{gr}) = O(\eta_1 + \eta_2) \). Then the “total distance” argument used above gives

\[
\chi^{(0)}(P_{gr}, C^{k_0}(P_{gr})) = O(\eta_1 + \eta_2).
\]
Combining the three bounds on $\chi$ above, and using the monotonicity of $\chi^{(s)}$ in $s$, the triangle inequality gives $\chi^{(k_0J+2)}(P, P_{gr}) = O(q_1 + q_2 + 2k_0\mu k_0)$, which implies (6.2) for all even $d \in \Omega(\delta + J)$ since $k_0 = 4\log n$ and $2\mu < 1/e$. Note that $D \in \Omega(\delta + J)$ by choice of $r$. Feeding this into Lemmas 5.1(a) and A.1 gives (6.3) for odd $d \in \Omega(\delta + J + 3)$, which proves the claim. 

Since $\varepsilon = C\sqrt{(\log n)/d}$ and $d < n/\sqrt{\log n}$, the claim gives

$$\frac{P_{S'}(d - e_a)}{P_{S'}(d - e_b)} = R_{ab}(d) = R_{ab}^{gr}(d) \left( 1 + O\left( \frac{1}{dn} + \frac{(\sqrt{d \log n})^3}{n^3} + \frac{\sqrt{d \log n}}{n^2} \right) \right)$$  \hspace{1cm} (6.5)

uniformly for all $d \in Q_1^1$.

We now move to Steps 2 and 3 in the template in Section 4. Let $H(d) = P_{Bm}(d)\tilde{H}(d)$ be the conjectured formula in the right hand side of (b) (without error terms), where

$$\tilde{H}(d) = \exp\left( \frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1 - \mu)^2} \right).$$

Define the probability spaces $S$ and $S'$ exactly as in the proof of Theorem 4.1 with the same underlying set $\Omega$. That is,

$$P_S(d) = H(d)/\sum_{d' \in \Omega} H(d') = \frac{H(d)}{E_{Bm}\tilde{H}}$$

and $S' = D(G(n, m))$. Also define the graph $G$ as before, with vertex set $\mathcal{N} := \Omega$, and with an edge joining each two sequences in $\mathcal{D}$ of the form $d - e_a$ and $d - e_b$ for some $a \neq b$. The $L_1$-distance from a sequence $d \in G$ to the constant sequence $(d, \ldots, d)$ is $\sum_{i} |d_i - d|$, which is at most $n\sqrt{2d}$ by (iii) and Cauchy’s Inequality. Some vertex of $G$ has $L_1$-distance at most $n$ from this constant sequence. It follows that the diameter of $G$ is $r = O(n\sqrt{d})$.

We claim (see Step 4 of the template) that $\mathcal{N}$ has probability at least $1 - \varepsilon_0$ for some suitably chosen $\varepsilon_0$ in both $S$ and $S'$. Note that $P_{S'}(\mathcal{N}) = 1 - n^{-h(C)}$ and $P_{Bm}(\mathcal{N}) = 1 - n^{-h(C)}$ by (a) proved above. Furthermore, if $d \in B_m(n)$ then $\gamma_2(d) = n^{-1}n^{1/2}\sigma^2(d) = \mu(1 - \mu)(1 + O(\xi))$ with probability $1 - o(n^{-\omega})$, where $\xi = \log^2 n/\sqrt{n}$ (this is the more precise implication of Lemma 6.2(ii) applied with $\alpha = \log^2 n/\sqrt{n}$). Thus, for such $d$ in $B_m(n)$, the exponential factor $\tilde{H}(d)$ is $1 + O(\xi)$ with probability $1 - o(n^{-\omega})$. Therefore,

$$E_{Bm}\tilde{H} = 1 + O(\xi)$$  \hspace{1cm} (6.6)

and thus $P_{S'}(\mathcal{N}) = 1 - n^{-h(C)}$. It follows that, as in the proof of Theorem 4.1, we may use $\varepsilon_0 = n^{-1}$ in Lemma 2.1.

We now move to Step 5 in the template. For $d \in Q_1^1$,

$$\frac{P_{S}(d - e_a)}{P_{S}(d - e_b)} = \frac{H(d - e_a)}{H(d - e_b)} = \frac{d_a(n - d_b)}{d_b(n - d_a)} \exp\left( \frac{(d_a - d_b)\gamma_2}{d^2(1 - \mu')^2} + O\left( \frac{\Delta^2}{(dn)^2} \right) \right)$$  \hspace{1cm} (6.7)

by (2.1) where $\bar{d}$, $\gamma_2$, $\mu'$ and $\Delta$ are the values of $d$, $\gamma_2$, $\mu$ and $\Delta$ defined for (2.1) with respect to $d$. Note that the first term in the exponential in (6.7) is $O\left( \sqrt{\log n/dn^2} \right)$ for $d \in Q_1^1$ by (ii) and (iii) and since $\bar{d} = d(1 + 1/2m)$ or equivalently $\mu' = \mu(1 + 1/2m)$. Thus,

$$\frac{(d_a - d_b)\gamma_2}{d^2(1 - \mu')^2} = \frac{(d_a - d_b)}{(dn)^2} \sum_i (d_i - \bar{d})^2 + O\left( \frac{\sqrt{d \log n}}{n^2} \right),$$

22
and the second term, $\Delta^2/(d_n)^2$, is $O(1/n^2)$ by assumption (ii). We can now infer from the definition of $R^{gr}$ that (6.7) is equivalent to

$$P_S(d - e_a) = P^{gr}_{ab}(d) \left( 1 + O\left( \frac{\sqrt{d\log n}}{n^2} \right) \right).$$

(6.8)

This together with (6.5) gives

$$P_{S'}(d - e_a) = e^{O(\delta)} P_S(d - e_a)$$

with

$$\delta = \frac{1}{dn} + \frac{(\sqrt{d\log n})^3}{n^3} + \frac{\sqrt{d\log n}}{n^2}$$

for $d \in Q_1$. Therefore, by Lemma 2.1 and (6.6),

$$P_{S'}(d) = P_S(d) e^{O(r\delta + \varepsilon_0)}$$

$$= H(d) (1 + O(\xi + r\delta + \varepsilon_0))$$

for all $d \in D$, which proves (b) since $\xi = (\log^2 n)/\sqrt{n}$, $r\delta = O\left( d^{-1/2} + d\sqrt{\log n}/n + d^2(\log n)^{3/2}/n^2 \right)$ and $\varepsilon_0 = 1/n$.  

It is a simple exercise in analysis to see that the theorem implies Conjecture 1.3 in the gap range: the truth of the theorem itself implies a slightly altered version of the theorem’s statement (b), in which $C$ is a function of $n$ that tends (“slowly”) to $\infty$. (The same can be done with the constant in the $O(\cdot)$ if desired.) The fact that $h(C) \to \infty$ then shows that the asymptotic approximation (1.2) holds for the sequences $d$ in a suitable set $R_\eta(n)$. All that remains is to note that the distribution of $m$ in $D(G(n,p))$ is identical to that in $E'_p$, and that the latter restricted $\sum d_i = 2m$ is identical to $B_m(n)$.

**Corollary 6.5.** Conjecture 1.3 holds.

We remark that one can avoid the sharp concentration results that we used, instead employing only variance via Chebyshev’s inequality, at the expense of relaxing the $o(n^{-\omega})$ error in the conjecture to $o(1)$. The result would still be interesting; we leave the details to the reader.

7 A wider range of degrees: proof of Theorem 1.4

In this section we prove Theorem 1.4. Compared with Theorem 6.3, some crucial differences that affect the argument include $\mu$ being permitted to have constant size, the allowable degree spread being $d^\alpha$ for $\alpha > 1/2$, and the transfer of $\sigma^2$ from explicit bounds to a term in the ratio formula.

The proof has the same structure as for Theorem 6.3. The crucial change required is to redefine the approximations, $P^{gr}$ and $R^{gr}$, of the probability and the ratio functions so that the error functions corresponding to $\eta_1$ and $\eta_2$ in Claim 6.4 satisfy $\eta_1 + \mu \eta_2 = o(1/(nd^{\alpha}))$. This error bound is necessary to obtain a final formula with $(1 + o(1))$ error, since with the range of degrees under consideration, the diameter of the graph $G$ of degree sequences (see the proof of Theorem 6.3) is up to $r = O(nd^{\alpha})$. 
To define the approximations, we write $P_{av}^{gr}$ and $R_{av}^{gr}$ parametrised to facilitate identifying negligible terms. For an integer $n$ and real numbers $\varepsilon_a, \varepsilon_v, \mu, \sigma^2$ and $d$ we define the expressions

$$
\pi = \mu(1 + \varepsilon_a)(1 + \varepsilon_v) \left( 1 + \frac{-\mu \varepsilon_a \varepsilon_v + (\varepsilon_a + \varepsilon_v) \sigma^2/dn}{1 - \mu} + \frac{\varepsilon_a + \varepsilon_v}{n - 1} \right),
$$

$$
\rho = \frac{1 + \varepsilon_a}{1 + \varepsilon_b} \cdot \frac{1 - \mu(1 + \varepsilon_b) + 1/n}{1 - \mu(1 + \varepsilon_a) + 1/n} \left( 1 + \frac{(\varepsilon_a - \varepsilon_b) \sigma^2}{(1 - \mu)^2 dn} \right).
$$

When referring to $\pi$ and $\rho$, we list only an initial segment of parameters apart from $n$ containing all those that are different from the ones in the definitions above. So for instance $\pi(x, y)$ stands for $\pi$ with $\varepsilon_a, \varepsilon_v$ replaced by $x, y$. Recall that we consider sequences $d$ of length $n$ and $\mu = d/(n - 1)$. For this section, we define

$$
P_{av}^{gr} = \pi, \quad R_{av}^{gr} = \rho,
$$

where $\varepsilon_i = (d_i - d)/d$ and $\sigma^2 = \sigma^2(d) = \sum(d_i - d)^2/n$, so that $P_{av}^{gr}$ etc. are functions of degree sequences.

Note also that $d$ and $\mu = d/(n - 1)$ were specified in the theorem statement (determined by $m$), but for the following lemma we make a slight abuse and for any sequence $d$ of length $n$ define $\mu = \mu(d) = \frac{1}{2} M_1(d)/\binom{n}{2}$ so that the average is $d = \mu(n - 1)$.

In the following lemma, the parity of $d$ is immaterial, though it will only be applied for odd $d$ in (a) and even $d$ in (b).

**Lemma 7.1.** Let $n$ and $k_0$ be integers and let $1/2 \leq \alpha < 3/5$. Let $A = \binom{[n]}{2}$ and let $d = d(n)$ be a sequence of length $n$ with average $d$ such that $\mu = d/(n - 1) < 1/4$, and assume that for all $1 \leq i \leq n$ we have $|d_i - d| \leq \epsilon d$, where $\epsilon = d^{\alpha - 1} > 0$. Assume that $k_0 = o(\sqrt{d})$ and let $a, v, b \in V = [n]$ such that $a \neq v$. Then

(a) $R(P_{av}^{gr})_{ab}(d) = P_{av}^{gr}(d) \left( 1 + O(\mu \epsilon_4 + (2\mu)^{k_0 - 1}) \right)$,

(b) $P(P_{av}^{gr}, R_{av}^{gr})_{av}(d) = P_{av}^{gr}(d) \left( 1 + O(\mu \epsilon_4) \right)$.

**Proof.** For (a), using (5.4) to evaluate $R(P_{av}^{gr})_{ab}(d)$, we estimate the expression $bad(a, b, d - e_b)$ for which, in turn, we need to estimate $\sum p_{av}(d - e_b)$, where the sum is over all $v \in W$ such that both $av$ and $bv$ are allowable (see (5.2)), and for $p_{av}$ we must use (5.1) with $p$ replaced by $P_{av}^{gr}$. Setting $d' = d - e_b$, this version of $p_{av}$ is given by

$$
p_{av}(d') = \frac{P_{av}^{gr}(d') \cdot P_{bv}^{gr}(d' - e_a - e_v)}{1 - P_{av}^{gr}(d' - e_a - e_v)} \times \left( \sum_{k=1}^{k_0} (-1)^{k-1} \frac{P_{bv}^{gr}(d' - j(e_a + e_v))}{P_{av}^{gr}(d' - e_a - e_v)} \right) \left( \prod_{j=1}^{k-1} \frac{P_{av}^{gr}(d' - j(e_a + e_v))}{1 - P_{av}^{gr}(d' - (j + 1)(e_a + e_v))} \right). \tag{7.1}
$$

For $1 \leq j \leq k_0$ consider the sequence $d' - j(e_a + e_v)$. When we apply formulae inductively to this sequence, $\mu$ becomes $\mu' = \mu + O(j/n^2)$ since $d$ changes by $O(j/n)$. Therefore, the variables $\varepsilon_a$, $\varepsilon_v$, and $\varepsilon_b$ change to $\varepsilon'_a = \varepsilon_a - j/d + O(j/\mu n^2)$, $\varepsilon'_v = \varepsilon_v - j/d + O(j/\mu n^2)$, (recalling that $j/\mu n^2 = j/(\mu n^2)$) and, not following the same pattern, $\varepsilon'_b = \varepsilon_b - 1/d + O(j/\mu n^2)$, respectively. (Note that this takes into account that $\varepsilon'_a$, $\varepsilon'_v$, and $\varepsilon'_b$ are defined with respect to the average degree of $d' - j(e_a + e_v)$.) Furthermore, $\sigma^2(d' - j(e_a + e_v)) = O((j \max |d_i - d| + j^2)/n) = O(j \epsilon d/n)$, and similarly, for $\sigma^2(d - j e_v)$, where we use that $j \leq k_0 = O(\sqrt{d}) = O(\epsilon d)$. Hence,

$$
P_{av}^{gr}(d' - j(e_a + e_v)) = \pi(\varepsilon'_a, \varepsilon'_v, \mu', (\sigma^2)') = \pi(\varepsilon_a - j/d, \varepsilon_v - j/d) (1 + O(j/\mu n^2)), \tag{7.2}
$$

and
and similarly for $P_{b,v}^\xi(d' - j(e_a + e_v))$. It follows that with
\[
A_j = \frac{\pi(\varepsilon_b - 1/d, \varepsilon_v - (j + 1)/d)}{\pi(\varepsilon_b - 1/d, \varepsilon_v - j/d)}, \quad B_j = -\frac{\pi(\varepsilon_a - j/d, \varepsilon_v - j/d)}{1 - \pi(\varepsilon_a - (j + 1)/d, \varepsilon_v - (j + 1)/d)},
\]
for $0 \leq j \leq k_0$, we have that
\[
p_{avb}(d') = \frac{\pi \cdot \pi(\varepsilon_b - 1/d, \varepsilon_v - 1/d)}{1 - \pi(\varepsilon_a - 1/d, \varepsilon_v - 1/d)} \left( O(1/\mu n^2) + 1 + \sum_{k=2}^{k_0} \prod_{j=1}^{k-1} (1 + O(j/\mu n^2)) A_j B_j \right) . \tag{7.3}
\]
Note that both, $A_j$ and $B_j$ are rational functions in $j$, that $A_j = 1 + o(1)$ and $B_j = -\mu(1 + o(1))$ for all $0 \leq j \leq k_0$, and that every occurrence of $j$ comes with a factor of $1/d$. Hence, expanding in powers of $j$ (here and in the following computations we used the algebraic manipulation package Maple) gives $A_j B_j = \hat{c}_0 + \hat{c}_1 j + O(j^2 \mu/d^2)$, where $\hat{c}_0$ and $\hat{c}_1$ are suitable functions independent of $j$. Then
\[
\prod_{j=1}^{k} A_j B_j = \hat{c}_0^k \left( 1 + \frac{\hat{c}_1}{\hat{c}_0} \left( \frac{k}{2} \right) + O(k^3/d^2) \right),
\]
since $\hat{c}_0 = \mu(1 + o(1))$. Noticing that $|A_i B_i| < 2\mu$ (for large $n$) as $\pi \sim \mu < 2/7$ say, we see that the $j/\mu n^2$ error term in (7.3) can be absorbed into the additive $1/\mu n^2$, and also that
\[
1 + \sum_{k=1}^{k_0} \prod_{j=1}^{k} A_j B_j = \sum_{k=0}^{k_0} \hat{c}_0^k + \sum_{k=1}^{k_0} \binom{k}{2} \hat{c}_1 \hat{c}_0^{k-1} + O \left( \sum_{k=1}^{k_0} (2\mu)^k k^3/d^2 \right)
\]
\[
= \frac{1}{1 - \hat{c}_0} + \frac{\hat{c}_1}{(1 - \hat{c}_0)^3} + O \left( (2\mu)^{k_0} + \mu/d^2 \right) . \tag{7.4}
\]
We thus obtain
\[
p_{avb}(d') = \frac{\pi \cdot \pi(\varepsilon_b - 1/d, \varepsilon_v - 1/d)}{1 - \pi(\varepsilon_a - 1/d, \varepsilon_v - 1/d)} \cdot F \left( 1 + O \left( \frac{\mu}{d^2} + \frac{1}{\mu n^2} + (2\mu)^{k_0} \right) \right)
\]
from (7.3), where $F = (1 - \hat{c}_0)^{-1} + \hat{c}_1/(1 - \hat{c}_0)^3$.

Consider expanding this expression for $p_{avb}(d')$ ignoring terms of order $\varepsilon^4$, and hence, since $\varepsilon^2 \geq 1/d$, also ignoring $1/d^2$ and $\varepsilon^2/d$. A convenient way to do this is to make substitutions $\varepsilon = y_1 \varepsilon_v$, $1/d = y_1^2/d$, $\mu = y_2 \mu$, $1/n = y_1^2 y_2/n$, and so on, where $y_1$ represents a parameter of size $O(\varepsilon)$ and $y_2$ of size $O(\mu)$. Then expand about $y_1 = 0$. We note by inspection that in such an expansion of $\pi$, each term has a factor $y_2$ associated with $\mu$, and consequently each term $cy_1$ has size $O(\varepsilon^4)$ relative to $\pi$ (or $\mu$). Hence, by the nature of their definitions via $\pi$, the expansions of $\hat{c}_0$ and $\hat{c}_1$ give terms of at most these same relative sizes. Finally, $p_{avb}(d')$ is of order $\mu^2$, and the terms in its expansion hence have the corresponding upper bound $O(\mu^2 \varepsilon^4)$ on their absolute sizes. We also note that in expanding a rational function about a nonsingular point, the error in the Taylor expansion is bounded by a multiple of the least significant terms omitted. This avoids any need to bound higher derivatives explicitly. In this way, expanding after these $\{y_1, y_2\}$ substitutions, and noting that $1/n = O(\mu \varepsilon^2)$, we obtain
\[
p_{avb}(d') = J + O(\xi),
\]
\[
\xi = \mu^2 \varepsilon^4 + (2\mu)^{k_0},
\]
where $J$ is a polynomial of degree 3 in $y_1$. (Unfortunately $J$ is too large to write here.) Next, removing the ‘sizing’ variables $y_i$ from $J$ by setting them equal to 1, and then expanding the result about $\varepsilon_v = 0$ and retaining all terms of total degree at most 3 in $\varepsilon_v$, we get
\[
p_{avb}(d') = c_0 + c_1 \varepsilon_v + c_2 \varepsilon_v^2 + O(\xi),
\]
25
where the functions $c_0$, $c_1$, and $c_2$ are independent of $\varepsilon_v$, with $c_0$ and $c_1$ linear in $1/d$. (By calculation, the third order term turns out to be 0. The fourth order truncation errors are absorbed into $\xi$ by a simple argument, similar to the one above, that the same is true when expanding the original $p_{avb}(d')$ in powers of $\varepsilon_v$.) Then considering the second summation in $bad(a, b, d - e_b)$ and examining which terms are excluded from the summation, we find that this summation can be written as

\[
\Sigma_{bad} := \sum_{v \in A(a) \cap A(b)} p_{avb}(d')
\]

\[
= \sum_{v \in \mathcal{A}(a) \cap \mathcal{A}(b)} (c_0 + c_1\varepsilon_v + c_2\varepsilon_v^2 + O(1))
\]

\[
= nc_0 + nc_2\sigma^2/d^2 - 2c_0 - c_1(\varepsilon_a + \varepsilon_b) - c_2(\varepsilon_a^2 + \varepsilon_b^2) + O(n\varepsilon).
\]

Noting that $\mathcal{A}(a) \cap \mathcal{A}(b) = \{b\}$, we can write $bad(a, b, d - e_b)$ in (5.4), by using (5.2), as

\[
bad(a, b, d - e_b) = \frac{1}{d_a} \left( \Sigma_{bad} + \pi(\varepsilon_a, \varepsilon_b - d^{-1}) + O(1/n^2) \right),
\]

where $d_a = (1 + \varepsilon_a)\mu(n (1 - 1/n)$ and where the $O(1/n^2)$ term captures the fact that we use $\mu = \mu(d)$, $\sigma^2(d)$, and $\sigma^2(d)$, respectively, instead of $\mu(d')$, $\sigma^2(d')$, and $\sigma^2(d')$, in the formula for $\pi$. Note that the error term from $\Sigma_{bad}$ produces an error term of size $O(\xi/\mu)$ in $bad(a, b, d - e_b)$ since $n/d_a \sim 1/\mu$, and note also that $\xi/\mu$ dominates the other error term of order $O(1/n^2)$. Substituting the above expression, stripped of its error terms, into

\[
\mathcal{R}(P_{ab}(d)) = 1 \cdot \left( 1 + \varepsilon_a \right) (1 - \mu(n - 1/n)) - 1
\]

and simplifying gives a rational function $\hat{F}$. That is, $\mathcal{R}(P_{ab}(d))/\rho - 1 = \hat{F} + O(\xi/\mu)$. After inserting the size variables $y_1$ and $y_2$ into $\hat{F}$ as specified above, and simplifying, we find it has $y_2$ as a factor (of multiplicity 1), and its denominator is nonzero at $y_1 = 0$. Then expanding the expression in powers of $y_1$ shows that $\hat{F} = O(y_1^4)$. Along with the extra factor $y_2$, this implies $\hat{F} = O(y_1^4)$. Thus, part (a) follows since $\mu/d \sim 1/n$.

To prove part (b) note that, analogous to (7.2), if $d'$ is the sequence $d - e_v$ we also have

\[
R_{ab}(d') = \rho(\varepsilon_a', \varepsilon_b, \mu', (\sigma^2)', d') = \rho \cdot \left( 1 + O(1/\mu n^2) \right).
\]

Therefore, by definition (5.3),

\[
P(\mathcal{P}_{av}, R_{av}(d)) = d_v \left( \sum_{be\mathcal{A}(v)} R_{ba}(d - e_v) \frac{1 - P_{ba}(d - e_v)}{1 - P_{ba}(d - e_v)} \right)^{-1}
\]

\[
= d_v \left( \sum_{be\mathcal{A}(v)} \rho(\varepsilon_b, \varepsilon_a) \cdot \frac{1 - \pi(\varepsilon_b - 1/d, \varepsilon_v - 1/d)}{1 - \pi(\varepsilon_a - 1/d, \varepsilon_v - 1/d)} \left( 1 + O\left( \frac{1}{\mu n^2} \right) \right) \right)^{-1}.
\]

By expanding in $\varepsilon_b$ we obtain

\[
\rho(\varepsilon_b, \varepsilon_a) \cdot \frac{1 - \pi(\varepsilon_b - 1/d, \varepsilon_v - 1/d)}{1 - \pi(\varepsilon_a - 1/d, \varepsilon_v - 1/d)} = K + O(\varepsilon^3)
\]

where $K$ is a polynomial in $\varepsilon_b$ of degree at most 3. Calculations using the size variables $y_1$ and $y_2$ as above show that $K = k_0 + k_1 \varepsilon_b + k_2 \varepsilon_b^2 + O(\varepsilon^3)$ for some $k_i$ independent of $\varepsilon_b$. Also, recall
that we have \( \mathcal{A}(v) = [n] \setminus \{v\} \). So the main summation over \( b \) in (7.5) can be evaluated (noting that \( \sum_b \varepsilon_b^2 = n\sigma^2/d^2 \)) as
\[
nk_0 + \frac{n\sigma^2k_2}{d^2} - k_0 - k_1\varepsilon_v - k_2\varepsilon_v^2
\]
with relative error \( O(\varepsilon^4) \), noting that \( K \) has constant order, where we use that \( \sum_{b \in [n]} \varepsilon_b = 0 \). Using the size variables \( y_1 \) and \( y_2 \) as described above, we then find that \( \mathbb{P}(\mathcal{P}^{gr}, \mathcal{R}^{gr})_{av}(d) = \pi(1 + O(\mu\varepsilon^4)) \), with the extra factor \( \mu \) arising in the error term in the same way as for \( \mathcal{R} \) in part (a). Part (b) follows.

To prove Theorem 1.4, we set \( \mathfrak{M} = \mathfrak{D} \) and then follow the proof of Theorem 6.3, referring to the set \( \mathfrak{D} \) within it as \( \mathfrak{D}' \). Since \( \mathfrak{D}' \subseteq \mathfrak{D} \), Theorem 6.3(a) implies \( \mathbb{P}(\mathfrak{M}) = 1 - o(1) \) in both models \( \mathcal{B}_m(n) \) and \( \mathcal{D}(\mathcal{G}(n, m)) \). Actually, in the current setting we permit higher values of \( \mu \), which is now only bounded above by a small constant, but the earlier proof applies equally well for this extended range.

Next set \( k_0 = 4 \log n \) to define the operators \( \mathcal{R} \) etc. Then the proof of Claim 6.4 (which assumed only the upper bound \( 1/4 \) on \( \mu \)) applies, with adjustment to the error terms \( \eta_i \) using Lemma 7.1 in place of Lemma A.1, to show that
\[
\mathcal{P}_{av}(d) = \mathcal{P}^{gr}_{av}(d) \left( 1 + O(\mu\varepsilon^4) \right),
\]
\[
\mathcal{R}_{ab}(d) = \mathcal{R}^{gr}_{ab}(d) \left( 1 + O(\mu\varepsilon^4) \right),
\]
uniformly for all appropriate \( d \in \mathfrak{D} \), and \( a, v \) and \( b \). Using the latter together with the definition of \( \rho \), we find in place of (6.5) (with the same definitions of \( \mathcal{S} \) and \( \mathcal{S}' \)) that
\[
\frac{\mathcal{P}_{\mathcal{S}'}(d - e_a)}{\mathcal{P}_{\mathcal{S}'}(d - e_b)} = \mathcal{R}^{gr}_{ab}(d) \left( 1 + O \left( \mu\varepsilon^4 + 1/n^2 \right) \right).
\]

Then for the independent binomial probability space \( \mathcal{S} \), (6.7) is only altered by the \( \gamma_2 \) term cancelling with the \( \sigma^2 \) term in the definition of \( \pi \). Thus we obtain (6.8) with error term reduced to \( O(1/n^2) \).

In this application of Lemma 2.1, the diameter of the auxiliary graph \( G \) is \( r = O(nd^\alpha) \) and \( \delta = \mu\varepsilon^4 + 1/n^2 \) from (7.8). We may again use \( \varepsilon_0 = 1/n \). The result is
\[
\mathcal{P}_{\mathcal{S}'}(d) = \mathcal{P}_{\mathcal{S}}(d)e^{O(rd + \varepsilon_0)} = H(d) \left( 1 + O \left( \xi + r\delta + \varepsilon_0 \right) \right),
\]
with \( \xi = (\log^2 n)/\sqrt{n} \) entering as before. The theorem follows, since \( d^{5\alpha - 3} \geq d^\alpha/n \). 

8 Concluding remarks

In the main result, Theorem 1.4, the upper bound \( \mu_0 \) on the density \( \mu \) can probably be set equal to any constant less than \( 1/2 \), at the expense of only slight changes to the proof. Since other results cover this range of density, we do not follow this line any further. On the other hand, various aspects of the proofs can be improved with some straightforward work, to obtain a wider range of degrees and smaller error terms, and we plan to pursue this elsewhere.

The approach of Sections 3 and 5 can be applied to other problems. We plan to apply the method in order to prove related binomial-based models for the degree sequences of random bipartite graphs, loopless directed graphs, and hypergraphs. The approach can also be used to make a major advance in asymptotic enumeration of Latin rectangles.
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A Approximating the operator fixed points

Here we prove the estimates (a) and (b) inside the proof of Claim 6.4. Recall the definitions of $P^{gr}$ and $R^{gr}$ in (6.1), and of $\sigma^2(d)$. Recall also that $\mathcal{O}$ is a set of sequences of length $n$ satisfying $|d_i - d| \leq C\sqrt{d \log n}$ for all $i \in [n]$ (for some constant $C$) and that $\sigma^2(d) \leq 2d$ by the assumptions of Theorem 6.3. Furthermore, recall that we set $k_0 = 4\log n$ and that $\Omega^{(0)}$ is the set of sequences $d \in \mathbb{Z}_n$ that are at $L^1$ distance $O(\log^2 n)$ from a sequence in $\mathcal{O}$. Therefore, (a) and (b) follow from the following lemma with $\varepsilon = (C + o(1))\sqrt{\log n/d}$.

Lemma A.1. Let $k_0$ and $n$ be integers and let $A = \{[n]\}$. Let $d$ be a sequence of length $n$ with average $d$ such that $d/(n - 1) < 1/4$ and $\sigma^2(d) = O(d)$, and assume that $k_0 = o(\sqrt{d})$ and that for all $1 \leq i \leq n$ we have $|d_i - d| \leq \varepsilon d$, where $\varepsilon = \varepsilon(n) > 0$ is bounded above by a sufficiently small constant. Let $a, v, b \in V = [n]$ such that $a \neq v$. Then

(a) $\mathcal{R}(P^{gr})_{ab}(d) = R^{gr}_{ab}(d)(1 + O(\eta_1))$ if $dn$ is odd, 
(b) $\mathcal{P}(P^{gr}, R^{gr})_{av}(d) = P^{gr}_{av}(d)(1 + O(\eta_2))$ if $dn$ is even, 

where $\eta_1 = 1/dn + \varepsilon d/n^2 + \varepsilon^2 d^2/n^2 + (2d/n)^{k_0}$ and $\eta_2 = \varepsilon/n + \varepsilon^2 d^2/n^2$. 

Proof. In order that it is self-contained, the wording of this proof contains a certain amount of overlap with parts of the proof of Lemma 7.1. We first reparametrise $P^{gr}$ and $R^{gr}$ to facilitate identifying negligible terms. Write $\mu = \mu(d)$ for the “density” $d/(n - 1)$ of a graph with degree sequence $d$, and note that, by assumption, $\mu \leq 1/4$. Define the sequence $(\varepsilon_1, \ldots, \varepsilon_n)$ of relative deviations from the average degree, that is $\varepsilon_a = (d_a - d)/d$ for $1 \leq a \leq n$. Note that $|\varepsilon_a| \leq \varepsilon$ for each $a$. Set $\sigma^2 = \sigma^2(d)$ and

\[
\pi(\varepsilon_a, \varepsilon_v, \mu) = \mu(1 + \varepsilon_a)(1 + \varepsilon_v) \cdot \left(1 - \frac{\varepsilon_a \varepsilon_v \mu}{1 - \mu}\right),
\]
\[
\rho(\varepsilon_a, \varepsilon_b, \mu, \sigma^2, d, n) = \frac{1 + \varepsilon_a}{1 + \varepsilon_b} \cdot \frac{1 - \mu(1 + \varepsilon_b)}{1 - \mu(1 + \varepsilon_a)} \cdot \left(1 + (\varepsilon_a - \varepsilon_b)\frac{\sigma^2}{dn}\right)
\]

so that $P^{gr}_{av}(d) = \pi$ and $R^{gr}_{ab}(d) = \rho \cdot (1 + O(1/n^2))$. (When referring to $\pi$ and $\rho$, we list only an initial segment of parameters containing all those that are different from the ones in the definitions above.)

For (a), using (5.4) to evaluate $\mathcal{R}(P^{gr})_{ab}(d)$, we estimate the expression $bad(a, b, d - e_b)$ (and $bad(b, a, d - e_a)$ respectively) for which, in turn, we need to estimate $\sum p_{ab}(d - e_b)$, where the sum is over all $v \in [n]$ such that both $av$ and $bv$ are allowable (see (5.2)), and for $p_{ab}$ we must use (5.1) with $p$ replaced by $P^{gr}$. Setting $d' = d - e_b$, this version of $p_{ab}$ is given by

\[
p_{ab}(d') = \frac{P^{gr}_{av}(d') \cdot P^{gr}_{bv}(d' - e_a - e_v)}{1 - P^{gr}_{av}(d' - e_a - e_v)} \times \left(\sum_{k=1}^{k_0} (-1)^{k-1} P^{gr}_{bv}(d' - k(e_a + e_v)) \prod_{j=1}^{k-1} \frac{P^{gr}_{av}(d' - j(e_a + e_v))}{1 - P^{gr}_{av}(d' - (j + 1)(e_a + e_v))}\right). \tag{A.1}
\]

For $1 \leq j \leq k_0 + 1$ consider the sequence $d' - j(e_a + e_v)$. For this sequence, $\mu$ becomes $\mu' = \mu + O(j/n^2)$ since $d$ changes by $O(j/n)$. Therefore, the variables $\varepsilon_a, \varepsilon_v,$ and $\varepsilon_b$ change to $\varepsilon'_a = \varepsilon_a - j/d + O(j/\mu n^2)$, $\varepsilon'_v = \varepsilon_v - j/d + O(j/\mu n^2)$, and, not following the same pattern, $\varepsilon'_b = \varepsilon_b - 1/d + O(j/\mu n^2)$, respectively. (Here and in the following, the bare symbols $\mu, d, \varepsilon_a$ and so on are defined with respect
to the original sequence $d$, whilst $\varepsilon'_a$, $\varepsilon'_v$, and $\varepsilon'_b$ are defined with respect to the average degree of $d' - j(e_a + e_v)$. Hence,

$$P_{av}^g(d' - j(e_a + e_v)) = \pi(\varepsilon'_a, \varepsilon'_v, \mu') = \pi(\varepsilon_a - j/d, \varepsilon_v - j/d, \mu)(1 + O(j/\mu n^2)), \quad (A.2)$$

and similarly for $P_{bv}^g(d' - j(e_a + e_v))$. It follows that, with $\delta = 1/d$ and

$$A_j = \frac{\pi(\varepsilon_a - j\delta, \varepsilon_v - j\delta)}{1 - \pi(\varepsilon_a - \delta, \varepsilon_v - \delta)}, \quad B_j = \frac{\pi(\varepsilon_a - j\delta, \varepsilon_v - j\delta)}{1 - \pi(\varepsilon_a - (j + 1)\delta, \varepsilon_v - (j + 1)\delta)},$$

for $0 \leq j \leq k_0$, we have

$$p_{avb}(d') = \frac{\pi \cdot \pi(\varepsilon_a - \delta, \varepsilon_v - \delta)}{1 - \pi(\varepsilon_a - \delta, \varepsilon_v - \delta)} \left(1 + \sum_{k=2}^{k_0} \prod_{j=1}^{k} (1 + O(j/\mu n^2)) A_j B_j\right). \quad (A.3)$$

It is straightforward to check from the definition of $\pi$ that $A_j = A_0(1 + O(j^2\delta^2))$, using $j\delta \leq k_0\delta \to 0$ (as $k_0^2/d \to 0$) and $\varepsilon$ less than 1/2 say. Similarly, $B_j = B_0(1 + \tilde{c}_1 j + O(j^2\delta^2))$ where $\tilde{c}_1 = \frac{2\delta}{(1-\mu)} + O(\varepsilon\delta) = o(1)$. Set $\tilde{c}_0 = A_0 B_0 \sim \mu$. By expanding the logarithm of the product of these expansions we find

$$\prod_{j=1}^{k} A_j B_j = \tilde{c}_0^k \left(1 + \tilde{c}_1 \left(\frac{k}{2}\right) + O(k^3\delta^2)\right).$$

Noticing that $|A_j B_j| < 2\mu$ (for large $n$) as $\pi \sim \mu < 2/7$ say, we see that the $j/\mu n^2$ error term in (A.3) can be absorbed into the additive $1/\mu n^2$, and also that

$$1 + \sum_{k=1}^{k_0} \prod_{j=1}^{k} A_j B_j = \sum_{k=0}^{k_0} \tilde{c}_0^k + \sum_{k=1}^{k_0} \left(\frac{k}{2}\right) \tilde{c}_1 \tilde{c}_0^k + O\left(\sum_{k=1}^{k_0} (2\mu)^k k^3\delta^2\right)$$

$$= \frac{1}{1 - \tilde{c}_0} + \frac{\tilde{c}_0 \tilde{c}_1}{(1 - \tilde{c}_0)^3} + O\left((2\mu)^{k_0} + \mu \delta^2\right). \quad (A.4)$$

Simple manipulations show that $A_0 = 1 - \delta + O(\delta^2 + \varepsilon\delta)$ and then

$$\frac{1}{1 - \tilde{c}_0} = \frac{1 - \pi(\varepsilon_a - \delta, \varepsilon_v - \delta)}{1 - \pi(\varepsilon_a - \delta, \varepsilon_v - \delta) + \pi \cdot A_0}$$

$$= \frac{1 - \pi(\varepsilon_a - \delta, \varepsilon_v - \delta)}{1 + \mu \delta + O(\mu \delta(\varepsilon + \delta))}, \quad (A.5)$$

and

$$\frac{\tilde{c}_0 \tilde{c}_1}{(1 - \tilde{c}_0)^2} = 2\mu \delta + O(\mu \delta(\varepsilon + \delta)). \quad (A.6)$$

Combining (A.3)–(A.6) and ignoring the $j/\mu n^2$ term as argued above, we obtain

$$p_{avb}(d') = \pi \cdot \pi(\varepsilon_a - \delta, \varepsilon_v - \delta) \left(1 + \mu \delta + O(\xi)\right), \quad (A.7)$$
where \( \xi = (2\mu)k_0 + \mu \delta^2 + \mu \epsilon \delta \). Since \( \mathcal{A}(a) \cap \mathcal{A}(b) = [n]\setminus\{a, b\} \), this gives

\[
\frac{1}{d_a} \sum_{v \in \mathcal{A}(a) \cap \mathcal{A}(b)} p_{avb}(d') = \frac{1}{d_a} \sum_{1 \leq v \leq n, v \neq a, b} \pi \cdot \pi(v_b - \delta, \epsilon_v - \delta) (1 + \mu \delta + O(\xi))
\]

\[
= \mu \frac{(1 + \epsilon_b - \delta)}{n - 1} (1 + \mu \delta + O(\xi)) \sum_{1 \leq v \leq n, v \neq a, b} \left(1 + \epsilon_v\right) \left(1 + \epsilon_v - \delta\right) \left(1 - \frac{\epsilon_a \epsilon_v}{1 - \mu}\right)^2
\]

\[
= \mu \frac{(1 + \epsilon_b - \delta)}{n - 1} (1 + \mu \delta + O(\xi)) \sum_{1 \leq v \leq n, v \neq a, b} \left(1 - \delta + c_1 \epsilon_v + (1 + O(\epsilon \mu)) \epsilon_v^2 + O(\epsilon^4 \mu)\right), 
\]

where \( c_1 = c_1(\delta, \epsilon_a, \mu) \) is some suitable function independent of \( \epsilon_v \) that satisfies \( c_1 = O(1) \). Note that \( \sum_{v=1}^{n} \epsilon_v = \sum_{v=1}^{n} (d - d_v)/d = 0 \) and \( \sum_{v=1}^{n} \epsilon_v^2 = \sigma^2/(d \mu) \), by definition of \( \sigma^2 \). Hence

\[
(1 + \mu \delta + O(\xi)) \sum_{1 \leq v \leq n, v \neq a, b} \left(1 - \delta + c_1 \epsilon_v + (1 + O(\epsilon \mu)) \epsilon_v^2 + O(\epsilon^4 \mu)\right)
\]

\[
= (1 + \mu \delta) \left((n - 2)(1 - \delta) - c_1(\epsilon_a + \epsilon_b) + (1 + O(\epsilon \mu)) \left(\frac{\sigma^2}{d \mu} - \epsilon_a^2 - \epsilon_b^2\right)\right) + O(n \epsilon^4 \mu + n \xi).
\]

On the other hand, noting that for (5.2) in this case \( \mathcal{A}(a) \cap \mathcal{A}(b) = \{b\} \), (so the first summation only has one term), and recalling that \( p_{ab} = P_{ab}^{gr} \), we compute (dealing with the shift in \( \mu \) as in (A.2)) that

\[
p_{ab}(d') = \frac{\pi(\epsilon_a, \epsilon_b - \delta)}{\mu(n - 1)(1 + \epsilon_a)} \left(1 + O\left(\frac{1}{\mu n^2}\right)\right) \frac{1 + \epsilon_b}{n - 1} \left(1 + O\left(\frac{\epsilon^2 \mu + \frac{1}{\mu n^2}}{}\right)\right).
\]

Thus, from (5.2),

\[
\text{bad}(a, b, d') = \frac{1 + \epsilon_b}{n - 1} + \frac{\mu (1 + \epsilon_b - \delta)}{n - 1} (1 + \mu \delta) \left((n - 1)(1 - \delta) - 1 + \frac{\sigma^2}{d \mu}\right)
\]

\[
= \mu(1 + \epsilon_b) + \frac{\sigma^2}{d n}(1 + \epsilon_b) - \frac{1}{n} + O\left(\frac{\epsilon \mu}{n} + \epsilon^4 \mu^2 + \mu k_0\right), 
\]

where we use that \( \sigma^2 = O(d) \) and that \( \delta = 1/d = 1/(\mu(n - 1)) \) and note some non-trivial cancellations. The analogous formula is obtained for bad \((b, a, d - e_a)\) by swapping indices. Hence

\[
\mathcal{R}(P_{ab}^{gr})(d) = \frac{d_a}{d_b} \cdot \frac{1 - \text{bad}(a, b, d - e_b)}{1 - \text{bad}(b, a, d - e_a)}
\]

\[
= \frac{1 + \epsilon_a}{1 + \epsilon_b} \cdot \frac{1 - \mu (1 + \epsilon_b)}{1 - \mu (1 + \epsilon_a)} \cdot \left(1 + (\epsilon_a - \epsilon_b) \frac{\sigma^2}{d n} + O\left(\frac{\epsilon \mu}{n} + \epsilon^4 \mu^2 + \mu k_0\right)\right)
\]

\[
= R_{ab}^{gr}(d) + O\left(\frac{1}{\mu n^2} + \frac{\epsilon \mu}{n} + \epsilon^4 \mu^2 + \mu k_0\right).
\]

This proves part (a) of the lemma.
To prove part (b) note that, analogous to (A.2), if \( d' \) is the sequence \( d - e_v \) we also have

\[
R_{ab}^{gr}(d') = \rho(\varepsilon'_a, \varepsilon'_b, \mu', (\sigma^2)'', d') = \rho \cdot (1 + O(1/\mu n^2)).
\]

(In particular, \((\sigma^2)' - \sigma^2 = O(\max |d'_i - d_i|/n) = O(\varepsilon d/n)\). Therefore, by definition (5.3),

\[
\mathcal{P}(P^{gr}, R^{gr})_{av}(d) = d_v \left( \sum_{b \in [n] \setminus \{v\}} \frac{R_{ba}^{gr}(d - e_v)}{1 - \frac{P_{ba}^{gr}(d - e_b - e_v)}{P_{av}^{gr}(d - e_a - e_v)}} \right)^{-1}
\]

\[
\quad = d_v \left( \sum_{b \in [n] \setminus \{v\}} \rho \cdot \frac{1 - \pi (\varepsilon_b - \delta, \varepsilon_v - \delta)}{1 - \pi (\varepsilon_a - \delta, \varepsilon_v - \delta)} \left( 1 + O \left( \frac{1}{\mu n^2} \right) \right) \right)^{-1}
\]

\[
\quad = d_v (1 + \varepsilon_a) \frac{1 - \pi (\varepsilon_b - \delta, \varepsilon_v - \delta)}{1 - \mu (1 + \varepsilon_a)} \left( 1 + O \left( \frac{\varepsilon}{n} + \frac{1}{\mu n^2} \right) \right)
\]

\[
\quad \times \left( \sum_{b \in [n] \setminus \{v\}} (1 + \varepsilon_b) \frac{1 - \pi (\varepsilon_b - \delta, \varepsilon_v - \delta)}{1 - \mu (1 + \varepsilon_b)} \right)^{-1}, \quad \text{(A.10)}
\]

where we used \( \sigma^2 = O(d) \) and \( \pi(\cdot, \cdot) = O(\mu) \). Straightforward calculations show that for \( c \in \{a, b\} \),

\[
\frac{1 - \pi (\varepsilon_c - \delta, \varepsilon_v - \delta)}{1 - \mu (1 + \varepsilon_c)} = 1 + \frac{\mu}{1 - \mu} (2\delta - \varepsilon_v - \varepsilon_v\varepsilon_c) + O \left( \frac{\varepsilon}{n} + \varepsilon^2 \mu^2 + \mu \delta^2 \right)
\]

\[
= \left( 1 - \mu \varepsilon_c \varepsilon_v \right) \left( 1 - \frac{\mu (\varepsilon_v - 2\delta)}{1 - \mu} \right) + O \left( \frac{\varepsilon}{n} + \varepsilon^2 \mu^2 + \mu \delta^2 \right). \quad \text{(A.11)}
\]

Note that \( \mu \delta^2 = O(1/\mu n^2) \). Therefore, (A.10) is equivalent to

\[
\mathcal{P}(P^{gr}, R^{gr})_{av}(d) = d_v (1 + \varepsilon_a) \left( 1 - \frac{\varepsilon_a \varepsilon_v \mu}{1 - \mu} \right) \left( 1 + O \left( \frac{\varepsilon}{n} + \frac{1}{\mu n^2} + \varepsilon^2 \mu^2 \right) \right) \sum_{b \in \mathbb{N}} (1 + \varepsilon_b) \left( 1 - \frac{\varepsilon_b \varepsilon_v \mu}{1 - \mu} \right)
\]

\[
= \mu (1 + \varepsilon_v) (1 + \varepsilon_a) \left( 1 - \frac{\varepsilon_a \varepsilon_v \mu}{1 - \mu} \right) \left( 1 + O \left( \frac{\varepsilon}{n} + \frac{1}{\mu n^2} + \varepsilon^2 \mu^2 \right) \right)
\]

\[
= \pi \cdot \left( 1 + O \left( \frac{\varepsilon}{n} + \frac{1}{\mu n^2} + \varepsilon^2 \mu^2 \right) \right),
\]

where in the second equality we use \( \sum_b \varepsilon_b = 0 \) and \( \sum_b \varepsilon_b^2 = O(1/\mu) \).