PERSISTENCE OF INVARIANT TORI FOR ALMOST PERIODICALLY FORCED REVERSIBLE SYSTEMS

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Abstract. In this paper, nearly integrable system under almost periodic perturbations is studied
\[
\begin{align*}
\dot{x} &= \omega_0 + y + f(t, x, y), \\
\dot{y} &= g(t, x, y),
\end{align*}
\]
where \(x \in \mathbb{T}^n, y \in \mathbb{R}^n, \omega_0 \in \mathbb{R}^n\) is the frequency vector, and the perturbations \(f, g\) are real analytic almost periodic functions in \(t\) with the infinite frequency \(\omega = (\cdots, \omega_\lambda, \cdots)_{\lambda \in \mathbb{Z}}\). We also assume that the above system is reversible with respect to the involution \(M_0 : (x, y) \to (-x, y)\). By KAM iterative method, we prove the existence of invariant tori for the above reversible system. As an application, we discuss the existence of almost periodic solutions and the boundedness of all solutions for a second-order nonlinear differential equation.

1. Introduction. In this paper, we are concerned with the persistence of invariant tori in the nearly integrable reversible system
\[
\begin{align*}
\dot{x} &= \omega_0 + y + f(t, x, y), \\
\dot{y} &= g(t, x, y),
\end{align*}
\]
(1.1)
where \(x \in \mathbb{T}^n, y \in D \subset \mathbb{R}^n\) with \(D\) being an open bounded domain, \(\omega_0 \in \mathbb{R}^n\) is the frequency vector, and the perturbations \(f, g\) are real analytic almost periodic functions in \(t\) with the infinite frequency \(\omega = (\cdots, \omega_\lambda, \cdots)_{\lambda \in \mathbb{Z}}\). Moreover, we assume the system is reversible with respect to the involution \(M_0 : (x, y) \to (-x, y)\).

For a function \(H : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m\), the system
\[
\dot{x} = H(x, t)
\]
is called reversible with respect to the involution \(M_* : \mathbb{R}^m \to \mathbb{R}^m\) with \(M_*^2 = I\), if
\[
DM_* \cdot H(M_*x, -t) = -H(x, t).
\]
For the system (1.1), it follows that
\[
f(-t, -x, y) = f(t, x, y), \quad g(-t, -x, y) = -g(t, x, y).
\]
(1.2)
In the system (1.1), the frequency \(\omega = (\cdots, \omega_\lambda, \cdots)_{\lambda \in \mathbb{Z}}\) is a bilateral infinite sequence of rationally independent frequency, that is, any finite segments of \(\omega\) are...
rationally independent. In this paper, we aim to prove the existence of invariant tori for the system (1.1) under a generalized small divisor condition and almost periodic perturbations.

Reversible systems form a class of special conservative systems with involution structure. During the last decades, there has been a lot of progress in KAM theory for reversible systems (see [1, 2, 3, 4, 7, 13, 16, 22, 23, 25, 24, 27, 28]).

In 1970s and 1980s, the existence of invariant tori for system (1.1) with perturbations independent of \( t \) was investigated by Moser [16], Arnold [1] and Sevryuk [22]. After that, some generalizations were made to study the invariant tori for reversible systems. For instance, in [2, 3, 4, 23, 25, 24], the persistence of lower dimensional invariant tori for reversible systems was studied. More recently, the degenerate lower dimensional invariant tori were considered by [7, 13, 27, 28]. There are many developments and generalizations in this field. For more information, the readers can refer to the above works and the references therein. However, there are few results about the existence of invariant tori for reversible systems under almost periodic perturbations.

In recent years, there have been some results about the existence of invariant curves or tori for some mappings and systems under almost periodic perturbations, see [8, 9, 10, 18].

Huang, Li and Liu [9] proved an invariant curve theorem of planar almost periodic twist mappings. As an application, they used the invariant curve theorem to study the existence of almost periodic solutions and the boundedness of all solutions for the following equation:

\[ \ddot{x} + x^3 = f(t), \]

where \( f(t) \) is a real analytic almost periodic function with the frequency \( \omega = (\cdots, \omega_\lambda, \cdots)_{\lambda \in \mathbb{Z}} \).

In [10], Huang, Li and Liu proved a generalized invariant curve theorem and used it to prove the existence of almost periodic solutions and the boundedness of all solutions for an asymmetric equation with almost periodic external force.

In [18], Piao and Zhang established an invariant curve theorem for planar almost periodic reversible mappings. Based on their invariant curve theorem they discussed the persistence of almost periodic solutions and the boundedness of all solutions for the nonlinear oscillator

\[ \ddot{x} + g(x)\dot{x} + \omega^2 x + \varphi(x) = f(t) \]

with \( f(t) \) being a real analytic almost periodic function with the frequency \( \omega = (\cdots, \omega_\lambda, \cdots)_{\lambda \in \mathbb{Z}} \).

More recently, Huang and Li [8] investigated the persistence of invariant tori for Hamiltonian systems under almost periodic perturbations

\[ H = h(y) + f(x, y, t), \]

where \( y \in D \subset \mathbb{R}^n \) with \( D \) being an open bounded domain, \( x \in \mathbb{T}^n \), \( f(x, y, t) \) is a real analytic almost periodic function with the frequency \( \omega = (\cdots, \omega_\lambda, \cdots)_{\lambda \in \mathbb{Z}} \).

As an application, they proved the existence of almost periodic solutions and the boundedness of all solutions for the second-order differential equations with superquadratic potentials depending almost periodically on time.

In this paper, we will focus on studying the existence of invariant tori for reversible system (1.1). We will prove that reversible system (1.1) owns invariant tori provided that the perturbations are small enough and the system is real analytic.
Following that, we will use the invariant tori theorem to study the existence of almost periodic solutions and the boundedness of all solutions for the second-order nonlinear differential equation

$$\ddot{x} + f(x, t) \dot{x} + g(x, t) = 0, \quad (1.3)$$

where

$$f(x, t) = \sum_{i=0}^{e} a_i(t) x^{2i+1}, \quad g(x, t) = x^{2l+1} + \sum_{i=0}^{l-1} b_i(t) x^{2i+1},$$

e, l are nonnegative integers satisfying

$$l \geq 2(e + 1),$$

and the coefficients $a_i(t)$ (for $0 \leq i \leq e$) and $b_i(t)$ (for $0 \leq i \leq l - 1$) are even and real analytic almost periodic functions with the infinite frequency $\omega = (\cdots, \omega, \cdots)$ for some $\omega \in \mathbb{Z}$. In this case, equation (1.3) is neither a Hamiltonian system nor a dissipative system. However, using the KAM theory for reversible system developed in this paper, we can prove the existence of almost periodic solutions and the boundedness of all solutions for equation (1.3).

The boundedness problem for (1.3) has been widely investigated since 1940’s. When $f(x, t) = 0$, equation (1.3) is a Hamiltonian system. In this case, the boundedness problem was investigated by Morris [15], Moser [17], Levi [11], You [29], Yuan [30], Huang and Li [8] and some references therein. When $f(x, t) \neq 0$, Levinson [12], Reuter [20], Graef [6] and Liu and Zanolin [14] studied the boundedness of the solutions. In their results, they considered equation (1.3) when $f, g$ are periodic or quasi-periodic with respect to $t$. In this paper, we will study equation (1.3) when $f, g$ are almost periodic with respect to $t$.

The paper is organized as follows. In section 2, we will give the definitions of real analytic almost periodic functions and their norms, then list some properties of them. We state the main theorem in section 3. The KAM iterative lemma is given in section 4. In section 5, we prove the KAM iterations converge and finish the proof of our main theorem. The existence of almost periodic solutions and the boundedness of all solutions for (1.3) are given in the final section.

2. Preliminaries. In this section, we will recall some basic definitions about the real analytic almost periodic functions. Before that, we first give the definition of real analytic quasi-periodic functions.

**Definition 2.1.** ([26]) A function $f : \mathbb{R} \to \mathbb{R}$ is called real analytic quasi-periodic with the frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_m)$ if there exists a real analytic function

$$F : \theta = (\theta_1, \theta_2, \cdots, \theta_m) \in \mathbb{R}^m \to \mathbb{R}$$

such that $f(t) = F(\omega_1 t, \omega_2 t, \cdots, \omega_m t)$ for all $t \in \mathbb{R}$, where $F$ is $2\pi$-periodic in each variable and bounded in a complex neighborhood $\Pi^m = \{(\theta_1, \theta_2, \cdots, \theta_m) \in \mathbb{C}^m : |\text{Im} \theta_j| \leq r, j = 1, 2, \cdots, m\}$ of $\mathbb{R}^m$ for some $r > 0$. Here we call $F(\theta)$ the shell function of $f(t)$.

We denote $Q(\omega)$ as the set of real analytic quasi-periodic functions with frequencies $\omega = (\omega_1, \omega_2, \cdots, \omega_m)$. Given $f(t) \in Q(\omega)$, the shell function $F(\theta)$ of $f(t)$ admits a Fourier series expansion

$$F(\theta) = \sum_{k \in \mathbb{Z}^m} f_k e^{i(k, \theta)}$$
where \( k = (k_1, k_2, \ldots, k_m) \), \( k_j \) range over all integers and the coefficients \( f_k \) decay exponentially with \( |k| = |k_1| + |k_2| + \cdots + |k_m| \), then \( f(t) \) can be represented as a Fourier series of the type from the definition,

\[
f(t) = \sum_{k \in \mathbb{Z}^m} f_k e^{i\langle k, \omega \rangle t}.
\]

In what follows, we will define the norm for a real analytic quasi-periodic function \( f(t) \) through its corresponding shell function \( F(\theta) \).

**Definition 2.2.** ([26]) For \( r > 0 \), let \( Q_{r}^\omega \subset Q(\omega) \) be the set of real analytic quasi-periodic functions \( f \) such that the corresponding shell functions \( F(\theta) \) are bounded on the subset \( \Pi_r^m \), with the supremum norm

\[
|f|_r := |F|_r = \sup_{\theta \in \Pi_r^m} |F(\theta)| = \sup_{\theta \in \Pi_r^m} \left| \sum_k f_k e^{i\langle k, \theta \rangle} \right| < +\infty.
\]

In order to give the definitions of real analytic almost periodic functions and their norm, we define analytic functions on some infinite dimensional space and a space structure for a family of subset of \( \mathbb{Z} \) in the following.

**Definition 2.3.** ([5]) Let \( X \) be a complex Banach space. A function \( f : U \subseteq X \rightarrow \mathbb{C} \), where \( U \) is an open subset of \( X \), is called analytic if \( f \) is continuous on \( U \), and for each finite dimensional subspace \( X_1 \) of \( X \), \( f|_{U \cap X_1} \) is analytic in the classical sense as a function of several complex variables.

**Definition 2.4.** ([19]) Assume \( S \) is a family of finite subset of \( \mathbb{Z} \). We say that \( S \) has a spatial structure, if

\[
A, B \in S, A \cap B \neq \emptyset \Rightarrow A \cup B \in S.
\]

A nonnegative set function \([\cdot] : S \rightarrow \mathbb{Z}\) defined on \( S \cap S = \{A \cap B : A, B \in S\} \) is called weight function of \( S \) if

\[
A \subseteq B \Rightarrow [A] \leq [B],
\]

\[
A \cap B \neq \emptyset \Rightarrow [A \cup B] + [A \cap B] \leq [A] + [B].
\]

Throughout this paper, we always use the following weight function

\[
[A] = 1 + \sum_{i \in A} \log^\alpha(1 + |i|),
\]

where \( \alpha > 2 \) is a constant.

Now we are in a position to give the definition for real analytic almost periodic functions. Before we describe the definition, we state some notations for \( k \) which is used in our definition.

For \( k \in \mathbb{Z}^\mathbb{Z} \) with finite support and \( \theta \in \mathbb{C}^\mathbb{Z} \), we define the inner product of \( k \) and \( \theta \) as

\[
\langle k, \theta \rangle = \sum_{\lambda \in \mathbb{Z}} k_\lambda \theta_\lambda.
\]

And the support of \( k \) is defined by

\[
supp k = \{\lambda : k_\lambda \neq 0\}.
\]
Definition 2.5. ([9]) Suppose \( \omega = (\cdots, \omega_\lambda, \cdots)_{\lambda \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z} \) is a bilateral infinite sequence frequency, its segments are rationally independent. A function \( f : \mathbb{R} \to \mathbb{R} \) is called real analytic almost periodic with the frequencies \( \omega \) if there exists a real analytic periodic function

\[
F : \theta = (\cdots, \theta_\lambda, \cdots) \in \mathbb{R}^\mathbb{Z} \to \mathbb{R},
\]

which admits a rapidly converging Fourier series expansion

\[
F(\theta) = \sum_{A \in S} F_A(\theta),
\]

where

\[
F_A(\theta) = \sum_{\text{supp } k \subseteq A} f_k e^{i(k, \theta)},
\]

and \( S \) has a spatial structure with \( \mathbb{Z} = \cup_{A \in S} A \), such that \( f(t) = F(\omega t) \) for all \( t \in \mathbb{R} \), where \( F \) is \( 2\pi \)-periodic in each variable and bounded in a complex neighborhood \( \Pi_r = \{ \theta = (\cdots, \theta_\lambda, \cdots) \in \mathbb{C}^\mathbb{Z} : |\text{Im}\theta|_{\infty} \leq r \} \) for some \( r > 0 \), and \( |\text{Im}\theta|_{\infty} = \sup_{\lambda \in \mathbb{Z}} |\text{Im}\theta_\lambda| \). Here \( F(\theta) \) is called the shell function of \( f(t) \).

From Definition 2.5, we know that \( f(t) \) can be represented as a Fourier series of the following type

\[
f(t) = \sum_{A \in S} f_A(t),
\]

where

\[
f_A(t) = \sum_{\text{supp } k \subseteq A} f_k e^{i(k, \omega)t}.
\]

We denote by \( AP(\omega) \) the set of real analytic almost periodic functions with the frequency \( \omega \) defined by Definition 2.5.

In the following, similar to the quasi-periodic case, we will define the norm of the real analytic almost periodic function \( f(t) \) through its corresponding shell function \( F(\theta) \).

Definition 2.6 ([9]). Let \( AP_r(\omega) \subseteq AP(\omega) \) be the set of real analytic almost periodic functions \( f \) such that the corresponding shell functions \( F \) are real analytic and bounded on the set \( \Pi_r \) with the norm

\[
\|F\|_{m,r} = \sum_{A \in S} |F_A|_r e^{m[A]} = \sum_{A \in S} |f_A|_r e^{m[A]} < +\infty,
\]

where \( m > 0 \) is a constant and

\[
|F_A|_r = \sup_{\theta \in \Pi_r} |F_A(\theta)| = \sup_{\theta \in \Pi_r} \left| \sum_{\text{supp } k \subseteq A} f_k e^{i(k, \theta)} \right| = |f_A|_r.
\]

We define

\[
\|f\|_{m,r} := \|F\|_{m,r}.
\]

In what follows, we present some properties for real analytic almost periodic functions.

Lemma 2.7. ([9]) The set \( AP(\omega) \) has the following properties:

1. Let \( f(t), g(t) \in AP(\omega) \), then \( f(t) \pm g(t), g(t + f(t)) \in AP(\omega) \);
2. Let \( f(t) \in AP(\omega) \), and \( \tau = \beta t + f(t) (\beta + f' > 0, \beta \neq 0) \), then the inverse relation is given by \( t = \beta^{-1} \tau + g(\tau) \) and \( g \in AP(\omega/\beta) \). In particular, if \( \beta = 1 \), then \( g \in AP(\omega) \).
3. Main results. Consider the following nearly integrable system
\[
\begin{align*}
\dot{x} &= \omega_0 + y + f(t, x, y), \\
\dot{y} &= g(t, x, y),
\end{align*}
\] (3.1)
where \( x \in \mathbb{T}^n, y \in D \subset \mathbb{R}^n \) with \( D \) being an open bounded domain, and the perturbations \( f, g \) are real analytic almost periodic functions in \( t \) with the infinite frequency \( \omega = (\cdots, \omega_\lambda, \cdots)_{\lambda \in \mathbb{Z}} \). If we introduce the extended variable \( \theta \in \mathbb{T}^Z \), system (3.1) can be written in the form of an autonomous system as follows
\[
\begin{align*}
\dot{\theta} &= \omega, \\
\dot{x} &= \omega_0 + y + F(\theta, x, y), \\
\dot{y} &= G(\theta, x, y),
\end{align*}
\] (3.2)
where \( F, G \) are the shell functions for the almost periodic functions \( f, g \) with respect to \( t \).

In addition, we assume that system (3.1) is reversible with respect to the involution \( M_0 : (x, y) \to (-x, y) \). That is, the functions \( f, g \) satisfy (1.2).

Equivalently, we have that the new system (3.2) is reversible with respect to the involution \( M : (\theta, x, y) \to (-\theta, -x, y) \), i.e.,
\[
D M \cdot F \circ M = -F,
\]
where \( F \) is the vector field of system (3.2):
\[
F = \omega \frac{\partial}{\partial \theta} + (\omega_0 + y + F) \frac{\partial}{\partial x} + G \frac{\partial}{\partial y}.
\]
That is, the perturbations \( F, G \) satisfy
\[
F(-\theta, -x, y) = F(\theta, x, y), \quad G(-\theta, -x, y) = -G(\theta, x, y).
\] (3.3)

We call a mapping
\[
\Phi : (\theta, \xi, \eta) \to (\theta, x = \xi + U(\theta, \xi, \eta), y = \eta + V(\theta, \xi, \eta)),
\]
compatible with respect to the involution \( M \), if \( \Phi \circ M = M \circ \Phi \), i.e.,
\[
U(-\theta, -\xi, \eta) = -U(\theta, \xi, \eta), \quad V(-\theta, -\xi, \eta) = V(\theta, \xi, \eta).
\]
Compatible transformations can preserve the reversible structure, in other words, they transform reversible systems into reversible systems.

When the perturbations \( F, G \) vanish, the unperturbed system (3.2) has an invariant torus \( T = \mathbb{T}^Z \times \mathbb{T}^n \times \{0\} \) with the frequency \( \hat{\omega} = (\omega, \omega_0) \), carrying an almost periodic flow \( \theta = \omega t, x(t) = x_0 + \omega_0 t \). The purpose of this paper is to prove the persistence of invariant tori for system (3.2) under small perturbations.

In order to state our main result, we first introduce some notations. Let
\[
D_{r,s} = \{(\theta, x, y) : \left| \text{Im} \theta \right|_{\infty} \leq r, \left| \text{Im} x \right| \leq r, \left| y \right| \leq s \}
\]
\[
\subset \mathbb{C}^Z / 2\pi \mathbb{Z}^Z \times \mathbb{C}^n / 2\pi \mathbb{Z}^n \times \mathbb{C}^n
\]
represent complex neighborhoods of \( \mathbb{T}^Z \times \mathbb{T}^n \times \{0\} \), where
\[
\left| \text{Im} \theta \right|_{\infty} = \sup_{\lambda \in \mathbb{Z}} |\text{Im} \theta_\lambda|,
\]
and \(| \cdot |\) stands for the sup-norm of real vectors.
By the definition of the shell function of almost periodic function, the perturbations \( F, G \) admit spatial series expansions

\[
\mathcal{L}(\theta, x, y) = \sum_{A \in S} \mathcal{L}_A(\theta, x, y) = \sum_{A \in S} \sum_{k \subseteq A} \mathcal{L}_{A,k}(x, y)e^{i(k, \theta)}, \quad \mathcal{L} = F, G.
\]

Expanding \( \mathcal{L}_{A,k}(x, y) \) into a Fourier series with respect to \( x \in \mathbb{T}^n \), we have

\[
\mathcal{L}(\theta, x, y) = \sum_{A \in S} \sum_{k \subseteq A, \hat{k} \in \mathbb{Z}^n} \mathcal{L}_{A,k,\hat{k}}(y)e^{i((k, \theta) + \langle \hat{k}, x \rangle)}, \quad \mathcal{L} = F, G,
\]

where \( \hat{k} = (\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_n) \).

Let \( B = \{1, 2, \cdots, n\} \) and \( \text{supp} \hat{k} \subseteq B \) for all \( \hat{k} \in \mathbb{Z}^n \). Let \( \hat{S} = \{A \times B : A \in S\} \), then \( F, G \) can be represented as spatial series of the following form

\[
\mathcal{L}(\theta, x, y) = \sum_{A \in \hat{S}} \sum_{\text{supp}(k, \hat{k}) \subseteq \hat{A}} \mathcal{L}_{A,k,\hat{k}}(y)e^{i((k, \theta) + \langle \hat{k}, x \rangle)}, \quad \mathcal{L} = F, G, \tag{3.4}
\]

where \( \text{supp}(k, \hat{k}) = \text{supp}k \times \text{supp} \hat{k} \). In addition, we denote

\[
\mathbb{Z}^n_{\hat{S}} := \{(k, \hat{k}) \in \mathbb{Z}^n \times \mathbb{Z}^n : \text{supp}(k, \hat{k}) \subseteq \hat{A}, \hat{A} \in \hat{S}\}
\]

and

\[
[[k, \hat{k}]] = \min_{\text{supp}(k, \hat{k}) \subseteq \hat{A} \in \hat{S}} [\hat{A}].
\]

Now, we are going to give the norm for the functions \( F, G \). Their norm are defined by

\[
\|\mathcal{L}\|_{m,r,s} = \sum_{A \in \hat{S}} \|\mathcal{L}_A\|_{r,s}e^{m[\hat{A}]}, \quad \mathcal{L} = F, G,
\]

where

\[
\|\mathcal{L}_A\|_{r,s} = \sum_{\text{supp}(k, \hat{k}) \subseteq \hat{A}} \|\mathcal{L}_{A,k,\hat{k}}\|_se^{r(|k| + |\hat{k}|)}, \quad \mathcal{L} = F, G,
\]

and the norm \( \| \cdot \|_s \) is the sup-norm over \( |y| \leq s \).

In this paper, we assume the frequency \((\omega, \omega_0)\) satisfies the following generalized “small divisor condition”

\[
|\langle \bar{k}, \omega \rangle + \langle \hat{k}, \omega_0 \rangle| \geq \frac{\gamma}{\triangle(\{((k, \hat{k}))\})\triangle(\{k\} + |\hat{k}|)}, \quad \text{for all } (k, \hat{k}) \in \mathbb{Z}^n \setminus \{0\}, \tag{3.5}
\]

where

\[
|k| = \sum_{\lambda \in \mathbb{Z}} |k_{\lambda}|, \quad |\hat{k}| = \sum_{j=1}^n k_j,
\]

and \( \triangle \) is some fixed approximation function which is defined as follows.

**Definition 3.1.** ([21]) A nondecreasing function \( \triangle : [0, \infty) \to [1, \infty) \) is called an approximation function, if

\[
\frac{\log \triangle(t)}{t} \searrow 0, \quad 0 \leq t \to \infty, \tag{3.6}
\]

and

\[
\int_0^\infty \frac{\log \triangle(t)}{t^2}dt < \infty.
\]

In addition, the normalization \( \triangle(0) = 1 \) is imposed for definiteness.
The frequencies satisfying the nonresonance condition (3.5) exist. Actually, according to the discussions in Pöschel [19] and Huang and Li [8], there exist an approximation function $\Delta$ and a probability measure $\mu$ such that the measure of the set of $(\omega, \omega_0)$ satisfying (3.5) is positive for a suitably small $\gamma$. As the proof is nothing new, we omit it here.

Under the above settings, we state our main result.

**Theorem 3.2.** Assume that $F, G$ admitting spatial expansions as in (3.4), are real analytic on $D_{r,s}$ and satisfy the estimate

$$\|F\|_{m,r,s} + \|G\|_{m,r,s} \leq \frac{\gamma \epsilon_*}{\Psi_0(\mu)^{45} \Psi_0^{15}(\rho) \Psi_1(\rho)^{30}},$$

for some $0 < \mu < m/4$ and $0 < \rho < r/4$, where $\epsilon_*$ is an absolutely positive constant, $\Psi_0(\mu), \Psi_0(\rho), \Psi_1(\rho)$ are defined by (5.1). Then there exists a transformation

$$\Phi : D_{r-4\rho,0} \to D_{r,s},$$

that is real analytic, such that under this coordinate transformation, system (3.2) becomes

$$\dot{\theta} = \omega,$$
$$\dot{x} = \omega_0 + y + O(y^2),$$
$$\dot{y} = O(y^2).$$

Consequently, the perturbed system owns a real analytic invariant torus $T^2 \times T^n \times \{0\}$, carrying an almost periodic flow $\theta = \omega t, x = x_0 + \omega_0 t$.

4. **Iterative lemma.** The proof of Theorem 3.2 is based on KAM approach. That is, we try to find a sequence of change of variables such that under each transformation, the transformed system will be closer to an integrable system than the previous one in a smaller domain. In order to do that, we will prove a KAM iterative lemma in this section. This is accomplished by several subsections. In subsection 4.1, we obtain the homological equations and give some estimates for their solutions. In subsection 4.2, the reversibility of the transformed system is verified. The estimates for the new perturbations are given in subsection 4.3.

In what follows, in order to avoid a fluid of constants we will use the notations “$\preceq$” to represent “$< c$” with constant $c$ independent of iterations. The letters without sub- or superscript $+$ denote the quantities at the $j$th step, and those with sub- or superscript $+$ denote the corresponding ones at $(j + 1)$th step.

Before entering into the details of KAM-constructions we observe that it suffices to consider some normalized value of $\gamma$, say

$$\tilde{\gamma} = 1.$$

Indeed, stretching the time scale by the factor $1/\gamma$, the right hand side of the system are scaled by the same amount, and so is the frequency $(\omega, \omega_0)$. Scaling the variable $y$ by $1/\gamma$ the radius $s$ may also be normalized to some convenient value. We will not do this here.

In the following, we describe one step of KAM iteration in more details, we assume that after $j$ steps, the transformed system is of the following form

$$\begin{cases}
\dot{\theta} = \omega, \\
\dot{x} = \omega_0 + y + F(\theta, x, y), \\
\dot{y} = G(\theta, x, y),
\end{cases}$$

(4.1)
where the functions on the right hand side are defined in the complex domain $D_{r,s}$. Moreover, we assume that the functions $F, G$ are real analytic almost periodic functions and admit spatial series expansions as (3.4). The frequency $(\omega, \omega_0)$ satisfies the condition
\[
|\langle k, \omega \rangle + \langle \bar{k}, \omega_0 \rangle| \geq \frac{\gamma}{\Delta((\delta(k,k)])} \Delta(|k| + |\bar{k}|),
\]
for all $(k, \bar{k}) \in \mathbb{Z}^2 \setminus \{0\}$, with $\gamma = 1$. In addition, we assume that
\[
\|F\|_{m,r,s} + \|G\|_{m,r,s} \leq \epsilon
\]
is sufficiently small.

We are going to find a transformation $\Phi$ of the following form
\[
\begin{align*}
\phi(x, y) &= (x_1, \ldots, x_n, y_1, \ldots, y_n) \\
\phi(x_+, y_+) &= (x_+, y_+ + F_+ + G_+ + \epsilon)
\end{align*}
\]
where $F_+, G_+$ are real analytic functions in a smaller domain $D_{r_+, s_+}$ and have smaller error estimates than $F, G$ in system (4.1).

Now, we define some notations
\[
\alpha = \epsilon^{\frac{1}{r}}, \quad s_+ = \alpha s, \quad \mu = \frac{m - m_+}{4}, \quad \rho = \frac{r - r_+}{4},
\]
with $0 < m_+ < m$ and $0 < r_+ < r$. For $\varrho > 0$, we define
\[
\Gamma_0(\varrho) = \sup_{t \geq 0} \Delta(t) e^{-\varrho t}, \quad \Gamma_1(\varrho) = \sup_{t \geq 0} (1 + t) \Delta(t) e^{-\varrho t}.
\]

4.1. Homological equations. Submitting (4.4) into (4.1), we have
\[
\begin{align*}
(1 + \partial_x v^0 + \partial_x v^1 y_+) \dot{x}_+ + (\partial_\theta v^0 + \partial_\theta v^1 y_+) \dot{\theta} + v^1 y_+ \dot{y}_+ &= \omega_0 + y_+ + v^0 y_+ + F_+ + G_+ + \epsilon, \\
(1 + v^1) \dot{y}_+ + (\partial_x v^0 + \partial_x v^1 y_+) \dot{x}_+ + (\partial_\theta v^0 + \partial_\theta v^1 y_+) \dot{\theta} &= G_+ + \epsilon.
\end{align*}
\]
Substituting (4.5) into (4.7) and omitting the small terms of order $\epsilon^2$, we obtain the following “homological equations”
\[
\begin{align*}
\partial_x v^0 - v^0 &= F_+(\theta, x_+, 0), \\
\partial_x v^0 &= G_+(\theta, x_+, 0), \\
\partial_x v^1 - v^1 &= -\partial_x v^0 + F_+(\theta, x_+, 0), \\
\partial_x v^1 &= -\partial_x v^0 + G_+(\theta, x_+, 0).
\end{align*}
\]
where $\partial_x$ is the familiar linear partial differential operator with constant coefficients on the torus
\[
\partial_x = \omega_0 \cdot \partial_x + \omega \cdot \partial_\theta.
\]
In what follows, we are going to solve the above “homological equations” (4.8)- (4.11) and give some estimates for their solutions. As the solutions of (4.8) and (4.11) rely on \( v^0 \), we will firstly solve the equation (4.9). Setting up the spatial series expansions for \( v^0 \) of the same form as that for \( G(\theta, x_+, 0) \), the linearized equation (4.9) breaks up into component equations

\[
\partial_x v^0 = G_{\tilde{A}}(\theta, x_+, 0),
\]

for \( \tilde{A} \in \tilde{S} \). Since \( \tilde{A} \) is a finite set, the above equation is well-known solvable if the mean value of \( G_{\tilde{A}}(\theta, x_+, 0) \) is zero. Actually, by (3.3), we have

\[
\langle G_{\tilde{A}}(\theta, x_+, 0) \rangle = 0,
\]

where \( \langle \cdot \rangle \) denote the mean value over \( \mathbb{T}^{\tilde{A}} \).

It is easy to see that the equation (4.12) has a solution

\[
v^0_{\tilde{A}} = \sum_{\text{supp}(k, \tilde{k}) \subseteq \tilde{A}, (k, \tilde{k}) \neq 0} \frac{G_{\tilde{A}, k, \tilde{k}}}{i(\langle k, \omega \rangle + \langle \tilde{k}, \omega_0 \rangle)} e^{i(\langle \theta, k \rangle + \langle \theta, \tilde{k} \rangle)},
\]

where \( G_{\tilde{A}, k, \tilde{k}} \) represent the Fourier coefficients of \( G_{\tilde{A}}(\theta, x_+, 0) \). It remains to determine the mean value of \( v^0_{\tilde{A}} \). The choice of \( \langle v^0_{\tilde{A}} \rangle \) comes from (4.8). Indeed, in order to make the mean value of \( v^0_{\tilde{A}} + F_{\tilde{A}}(\theta, x_+, 0) \) be zero, we take

\[
\langle v^0_{\tilde{A}} \rangle = -\langle F_{\tilde{A}}(\theta, x_+, 0) \rangle.
\]

Having the above preparations, we can get the estimate for \( v^0_{\tilde{A}}(\theta, x_+) \) in a smaller domain \( D(\tilde{r} - \rho, 0) \) with \( 0 < \rho < \tilde{r} \). By the small divisor condition (3.5) with \( \gamma = \bar{\gamma} = 1 \),

\[
\| v^0_{\tilde{A}} \|_{\tilde{r} - \rho, 0} \leq \sum_{\text{supp}(k, \tilde{k}) \subseteq \tilde{A}, (k, \tilde{k}) \neq 0} (|k|) \Delta([[k, \tilde{k}]])(\| G^{(r - \rho)(|k| + |\tilde{k}|)}_{\tilde{A}, k, \tilde{k}} ||(\tilde{r} - \rho, 0)),
\]

\[
\leq \Delta([\tilde{A}]) \Gamma_0(\rho) \| G_{\tilde{A}} \|_{r, 0} + \| F_{\tilde{A}} \|_{r, 0},
\]

where \( \Gamma_0(\rho) \) is defined by (4.6). Here and in the following, we use \( F_{\tilde{A}} \) and \( G_{\tilde{A}} \) represent \( F_{\tilde{A}}(\theta, x_+, 0) \) and \( G_{\tilde{A}}(\theta, x_+, 0) \) respectively. Similarly, for the convenience of later estimates,

\[
\| \partial_x v^0_{\tilde{A}} \|_{\tilde{r} - \rho, 0} \leq \sum_{\text{supp}(k, \tilde{k}) \subseteq \tilde{A}, (k, \tilde{k}) \neq 0} (|k|) \Delta([[k, \tilde{k}]])(\| G^{(r - \rho)(|k| + |\tilde{k}|)}_{\tilde{A}, k, \tilde{k}} ||(\tilde{r} - \rho, 0)),
\]

\[
\leq \Delta([\tilde{A}]) \Gamma_1(\rho) \| G_{\tilde{A}} \|_{r, 0},
\]

and

\[
\sum_{\lambda \in \tilde{A}} \| \partial_{\theta} v^0_{\tilde{A}} \|_{\tilde{r} - \rho, 0} \leq \sum_{\text{supp}(k, \tilde{k}) \subseteq \tilde{A}, (k, \tilde{k}) \neq 0} (|k|) \Delta([[k, \tilde{k}]])(\| G^{(r - \rho)(|k| + |\tilde{k}|)}_{\tilde{A}, k, \tilde{k}} ||(\tilde{r} - \rho, 0)),
\]

\[
\leq \Delta([\tilde{A}]) \Gamma_1(\rho) \| G_{\tilde{A}} \|_{r, 0},
\]

where \( \Gamma_1(\rho) \) is defined by (4.6). By the definition (4.6) of \( \Gamma_0 \) and \( \Gamma_1 \), it is easy to see that \( \Gamma_0, \Gamma_1 \geq 1 \). In addition, from (4.3), one has

\[
\| G \|_{m, r, 0} + \| F \|_{m, r, 0} \leq \| G \|_{m, r, s} + \| F \|_{m, r, s} < \epsilon,
\]
Therefore, putting the spatial components together, we obtain
\[ \|G\|_{m,r,0} + \|F\|_{m,r,0} \leq \frac{1}{s}\left(\|G\|_{m,r,s} + \|F\|_{m,r,s}\right) < \frac{1}{s}\varepsilon. \] (4.13)

Therefore, putting the spatial components together, we obtain
\[ \|v^0\|_{m-\mu,r-\rho,0} \leq \sum_{A \in S} \left\{ \|A\| \Gamma_0(\rho) \|G_A\|_{r,0} + \|F_A\|_{r,0} \right\} e^{(m-\mu)|A|} \]
\[ \leq \Gamma_0(\mu) \Gamma_0(\rho) \|G\|_{m,r,0} + \|F\|_{m,r,0} \]
\[ < \Gamma_0(\mu) \Gamma_0(\rho) \varepsilon, \]
and
\[ \|v^0\|_{m-\mu,r-\rho,0} \leq \sum_{A \in S} \|A\| \Gamma_0(\mu) \Gamma_0(\rho) \|G\|_{m,r,0} \leq \Gamma_0(\mu) \Gamma_0(\rho) \varepsilon, \]
for \(0 < \mu < m\). Thus, we complete the estimates for \(v^0\).

In what follows, we will solve (4.8) and give the estimate for \(v^0\). We use the similar method as \(v^0\) except for the choice of \(u^A\). Actually, we set \(\langle u^A \rangle = 0\).

Thus, we can summarize the estimates for \(u^A\) and its derivatives
\[ \|v^0\|_{m-2\mu,r-2\rho,0} \leq \Gamma_0(\mu) \Gamma_0(\rho) \|F + v^0\|_{m-\mu,r-\rho,0} \leq \Gamma_0^2(\mu) \Gamma_0^2(\rho) \varepsilon, \]
and
\[ \|v^1\|_{m-3\mu,r-3\rho,0} \leq \Gamma_0(\mu) \Gamma_0(\rho) \|v^1\|_{m-3\mu,r-3\rho,0} \]
\[ < \frac{1}{s} \Gamma_0^2(\mu) \Gamma_0(\rho) \Gamma_1(\rho) \varepsilon, \]
\[ \|u^1\|_{m-4\mu,r-4\rho,0} \leq \Gamma_0(\mu) \Gamma_0(\rho) \|u^1\|_{m-3\mu,r-3\rho,0} \]
\[ < \frac{1}{s} \Gamma_0^2(\mu) \Gamma_0^2(\rho) \Gamma_1(\rho) \varepsilon, \]
\[ \|u^1\|_{m-3\mu,r-3\rho,0} \leq \Gamma_0(\mu) \Gamma_0(\rho) \|u^1\|_{m-3\mu,r-3\rho,0} \]
\[ < \frac{1}{s} \Gamma_0^2(\mu) \Gamma_0^2(\rho) \Gamma_1(\rho) \varepsilon, \]
\[ \|\partial_x v^0\|_{m-\mu,r-\rho,0} \leq \sum_{A \in S} \|A\| \Gamma_0(\mu) \Gamma_0(\rho) \|\partial_x v^0\|_{m-\mu,r-\rho,0} \]
\[ < \frac{1}{s} \Gamma_0^2(\mu) \Gamma_0^2(\rho) \Gamma_1(\rho) \varepsilon, \]
\[ \|\partial_x v^1\|_{m-3\mu,r-3\rho,0} \leq \sum_{A \in S} \|A\| \Gamma_0(\mu) \Gamma_0(\rho) \|\partial_x v^1\|_{m-3\mu,r-3\rho,0} \]
\[ < \frac{1}{s} \Gamma_0^2(\mu) \Gamma_0^2(\rho) \Gamma_1(\rho) \varepsilon, \]
and the method of solving equations (4.10) and (4.11) are similar to equations (4.8) and (4.9). Repeating the similar process and applying Cauchy’s estimate, from (4.13) we get
\[ \|v^1\|_{m-3\mu,r-3\rho,0} \leq \Gamma_0(\mu) \Gamma_0(\rho) \|v^1\|_{m-3\mu,r-3\rho,0} \]
\[ + \| - \partial_x v^0 + G^r_y \|_{m-2\mu,r-2\rho,0} \]
\[ < \frac{1}{s} \Gamma_0^2(\mu) \Gamma_0(\rho) \Gamma_1(\rho) \varepsilon, \]
\[ \|u^1\|_{m-4\mu,r-4\rho,0} \leq \Gamma_0(\mu) \Gamma_0(\rho) \|u^1\|_{m-3\mu,r-3\rho,0} \]
\[ < \frac{1}{s} \Gamma_0^2(\mu) \Gamma_0^2(\rho) \Gamma_1(\rho) \varepsilon, \]
\[ \|\partial_x v^1\|_{m-3\mu,r-3\rho,0} \leq \sum_{A \in S} \|A\| \Gamma_0(\mu) \Gamma_0(\rho) \|\partial_x v^1\|_{m-3\mu,r-3\rho,0} \]
\[ < \frac{1}{s} \Gamma_0^2(\mu) \Gamma_0^2(\rho) \Gamma_1(\rho) \varepsilon, \]
and
\[ \left\| \frac{\partial_x u^1}{m-4\mu,r-4\rho,0} \right\| \leq \Gamma_0(\mu)\Gamma_1(\rho) - \partial_x u^0 + F_y + v^1 \left\| m-3\mu,r-3\rho,0 \right\| < \frac{1}{s} \Gamma_0^3(\mu)\Gamma_0(\rho)\Gamma_1^2(\rho)\epsilon. \]

By the above estimates and the definitions of U and V in (4.4), we can summarize the estimates for U, V and their derivatives
\[ \left\| U \right\|_{m-4\mu,r-4\rho,s} \leq \Gamma_0^3(\mu)\Gamma_0(\rho)\Gamma_1(\rho)\epsilon, \]
\[ \left\| \partial_y U \right\|_{m-4\mu,r-4\rho,s} = \left\| u^1(\theta,x_+) \right\|_{m-4\mu,r-4\rho,s} < \frac{1}{s} \Gamma_0^3(\mu)\Gamma_0(\rho)\Gamma_1(\rho)\epsilon, \]
\[ \left\| \partial_x U \right\|_{m-4\mu,r-4\rho,s} \sum_\lambda \left\| \partial_{\theta\lambda} u \right\|_{m-4\mu,r-4\rho,s} \leq \Gamma_0^3(\mu)\Gamma_0(\rho)\Gamma_1^2(\rho)\epsilon, \]
\[ \left\| V \right\|_{m-3\mu,r-3\rho,s} \leq \Gamma_0^3(\mu)\Gamma_0(\rho)\Gamma_1(\rho)\epsilon, \]
\[ \left\| \partial_y V \right\|_{m-3\mu,r-3\rho,s} = \left\| v^1(\theta,x_+) \right\|_{m-3\mu,r-3\rho,s} < \frac{1}{s} \Gamma_0^3(\mu)\Gamma_0(\rho)\Gamma_1(\rho)\epsilon, \]
\[ \left\| \partial_x V \right\|_{m-3\mu,r-3\rho,s} \sum_\lambda \left\| \partial_{\theta\lambda} V \right\|_{m-3\mu,r-3\rho,s} \leq \Gamma_0^3(\mu)\Gamma_1^2(\rho)\epsilon. \]

4.2. The proof \( \Phi \circ \mathcal{M} = \mathcal{M} \circ \Phi \).

In this section, we will verify the transformation \( \Phi \) commutes with the mapping \( \mathcal{M} : (\theta,x_+,y_+) \rightarrow (-\theta,-x_+,y_+) \). Such a result guarantees that the transformed system (4.5) is also reversible with respect to the mapping \( \mathcal{M} \).

Firstly, we will prove that
\[ v^0(\theta,x_+) = v^0(-\theta,-x_+). \] (4.15)

Indeed, for each \( \hat{A} \in \hat{S} \), by (4.12) we obtain that for every \( (\hat{k}, \hat{k}) \in \hat{A} \) with \( (\hat{k}, \hat{k}) \neq 0 \),
\[ i(\langle \hat{k}, \omega \rangle + \langle \hat{k}, \omega_0 \rangle)v^0_{\hat{A},\hat{k},\hat{k}} = G_{\hat{A},\hat{k},\hat{k}}. \] (4.16)

From the condition (3.3), we have
\[ G_{\hat{A},-\hat{k},-\hat{k}} = G_{\hat{A},\hat{k},\hat{k}}. \]

Replacing \( (\hat{k}, \hat{k}) \) in (4.16) by \( (-\hat{k}, \hat{k}) \), one gets
\[ i((-\hat{k}, \omega) + \langle \hat{k}, \omega_0 \rangle)v^0_{\hat{A},-\hat{k},-\hat{k}} = G_{\hat{A},-\hat{k},-\hat{k}} = G_{\hat{A},\hat{k},\hat{k}}. \]

In other words,
\[ i(\langle \hat{k}, \omega \rangle + \langle \hat{k}, \omega_0 \rangle)v^0_{\hat{A},-\hat{k},-\hat{k}} = G_{\hat{A},\hat{k},\hat{k}}. \]

As \( \langle \hat{k}, \omega \rangle + \langle \hat{k}, \omega_0 \rangle \neq 0 \), we have
\[ v^0_{\hat{A},-\hat{k},-\hat{k}} = v^0_{\hat{A},\hat{k},\hat{k}}. \]

The equation (4.15) follows immediately.

Similarly, one can prove
\[ u^0(\theta,x_+) = -u^0(-\theta,-x_+), \]
\[ u^1(\theta,x_+) = -u^1(-\theta,-x_+), \]
\[ v^1(\theta,x_+) = v^1(-\theta,-x_+). \]

As a consequence, the following equalities hold
\[ U(\theta,x_+,y_+) = -U(-\theta,-x_+,y_+), \quad V(\theta,x_+,y_+) = V(-\theta,-x_+,y_+). \]
Therefore, we finish the proof of the transformation $\Phi$ commuting with the mapping $M$.

4.3. Estimates of new perturbations. To finish one step of KAM iteration, we have to estimate the new perturbation terms $F_+, G_+$ in (4.5). Before that, we need to show the transformation $\Phi$ is well defined, i.e.,

$$\Phi: \mathcal{D}_{r_+} \to \mathcal{D}_{r_+}. \quad (4.17)$$

By the estimates of $U$, $V$, if the following inequalities

$$c\Gamma_0^3(\mu)\Gamma_0^2(\rho)\Gamma_1(\rho)\epsilon < \rho, \quad c\Gamma_0^3(\mu)\Gamma_0(\rho)\Gamma_1(\rho)\epsilon < \frac{1}{4}s$$

hold, then we have (4.17). Here $c$ is a constant independent of iterations and comes from the estimates of $U, V$. More precisely, we have

$$\Phi: \mathcal{D}_{-4\rho, r_+} \to \mathcal{D}_{-3\rho, r_+}.$$ 

In what follows, we will estimate the new error terms $F_+, G_+$ in (4.5). Substituting (4.5) into (4.7) and noting (4.8)-(4.11), we find

$$\dot{x}_+ = \omega_0 + y_+ - \partial_x u^0 \cdot F_+ - \partial_x u^1 y_+(y_+ + F_+) - u^1 G_+$$

$$+ F(\theta, x_+ + U, y_+ + V) - F(\theta, x_+, 0) - F_y(\theta, x_+, 0)y_+,$$

$$\dot{y}_+ = -\partial_x v^0 \cdot F_+ - \partial_x v^1 y_+(y_+ + F_+) - v^1 G_+$$

$$+ G(\theta, x_+ + U, y_+ + V) - G(\theta, x_+, 0) - G_y(\theta, x_+, 0)y_+.$$

It implies that

$$F_+ = -\partial_x u^0 \cdot F_+ - \partial_x u^1 y_+(y_+ + F_+) - u^1 G_+$$

$$+ F(\theta, x_+ + U, y_+ + V) - F(\theta, x_+, 0) - F_y(\theta, x_+, 0)y_+,$$

$$G_+ = -\partial_x v^0 \cdot F_+ - \partial_x v^1 y_+(y_+ + F_+) - v^1 G_+$$

$$+ G(\theta, x_+ + U, y_+ + V) - G(\theta, x_+, 0) - G_y(\theta, x_+, 0)y_+,$$

which is equivalent to

$$\begin{pmatrix}
1 + \partial_x U & \partial_y U \\
\partial_x V & 1 + \partial_y V
\end{pmatrix}
\begin{pmatrix}
F_+ \\
G_+
\end{pmatrix}
= - \begin{pmatrix}
\partial_x u^1 y_+^2 \\
\partial_x v^1 y_+^2
\end{pmatrix}$$

$$+ \begin{pmatrix}
F(\theta, x_+ + U, y_+ + V) - F(\theta, x_+, 0) - F_y(\theta, x_+, 0)y_+ \\
G(\theta, x_+ + U, y_+ + V) - G(\theta, x_+, 0) - G_y(\theta, x_+, 0)y_+
\end{pmatrix}.$$ 

If

$$c\Gamma_0^3(\mu)\Gamma_0(\rho)\Gamma_1(\rho)\epsilon < \frac{1}{2}, \quad c\Gamma_0^3(\mu)\Gamma_0^2(\rho)\Gamma_1(\rho)\epsilon < \frac{1}{2},$$

then the matrix $\begin{pmatrix}
1 + \partial_x U & \partial_y U \\
\partial_x V & 1 + \partial_y V
\end{pmatrix}$ has an inverse, with the norm $< 2$. Here, for a matrix $M = (m_{ij})_{p \times p}$, we define its norm as $\|M\| = \sup_{1 \leq i, j \leq p} |m_{ij}|$.

Next we will give an estimate for the last term of (4.18). Obviously, we have

$$F(\theta, x_+ + U, y_+ + V) - F(\theta, x_+, 0) - F_y(\theta, x_+, 0)y_+ = F^1 + F^2,$$

where

$$F^1 = F(\theta, x_+ + U, y_+ + V) - F(\theta, x_+, y_+),$$

$$F^2 = F(\theta, x_+, y_+) - F(\theta, x_+, 0) - F_y(\theta, x_+, 0)y_+.$$
From Cauchy’s estimate, it follows that
\[ \| F^1 \|_{m_+, r_+, s_+} \lesssim \frac{1}{\rho} \epsilon \cdot \Gamma^3_0(\mu) \Gamma^2_0(\rho) \Gamma_1(\rho) \epsilon + \frac{1}{s} \epsilon \cdot \Gamma^2_0(\mu) \Gamma_0(\rho) \Gamma_1(\rho) \epsilon \]
\[ \lesssim \left( \frac{1}{\rho} + \frac{1}{s} \right) \Gamma^3_0(\mu) \Gamma^2_0(\rho) \Gamma_1(\rho) \epsilon \]
and
\[ \| F^2 \|_{m_+, r_+, s_+} \lesssim \frac{\epsilon}{s^2} (\epsilon s)^2 = \epsilon^2. \]
Hence, we obtain
\[ \| F(\theta, x_+ + U, y_+ + V) - F(\theta, x_+, 0) - F_y(\theta, x_+, 0) y_+ \|_{m_+, r_+, s_+} \]
\[ \lesssim \left( \frac{1}{\rho} + \frac{1}{s} \right) \Gamma^3_0(\mu) \Gamma^2_0(\rho) \Gamma_1(\rho) \epsilon^2 + \epsilon \alpha^2. \]
The same estimate holds for \( G \). In addition, one has
\[ \| \partial_x u^1 y^2_+ \|_{m_+, r_+, s_+}, \quad \| \partial_x u^2 y^2_+ \|_{m_+, r_+, s_+} \lesssim \Gamma^3_0(\mu) \Gamma^2_0(\rho) \Gamma_1(\rho) \epsilon s, \]
Thus we conclude from (4.18) that
\[ \| F_+ \|_{m_+, r_+, s_+} + \| G_+ \|_{m_+, r_+, s_+} \]
\[ \lesssim \Gamma^3_0(\mu) \Gamma^2_0(\rho) \Gamma_1(\rho) \epsilon s + \left( \frac{1}{\rho} + \frac{1}{s} \right) \Gamma^3_0(\mu) \Gamma^2_0(\rho) \Gamma_1(\rho) \epsilon^2 + \epsilon \alpha^2. \]
The above discussions lead to the following lemma.

**Lemma 4.1** (Iterative lemma). Consider a system
\[
\begin{cases}
\dot{\theta} = \omega, \\
\dot{x} = \omega_0 + y + F(\theta, x, y), \\
\dot{y} = G(\theta, x, y),
\end{cases}
\]
where \( F \) and \( G \) are real analytic in the domain \( D_{r,s} \). And system (4.19) is reversible with respect to the mapping \( M : (\theta, x, y) \rightarrow (-\theta, -x, y) \). Assume the frequency \( (\omega, \omega_0) \) satisfies condition (4.2). Moreover, the perturbations \( F \) and \( G \) satisfy the estimate
\[ \| F \|_{m,r,s} + \| G \|_{m,r,s} \leq \epsilon. \]
Then, for \( 0 < \mu = \frac{m-m_+}{4}, \ 0 < \rho = \frac{-r}{4} \), if
\[ c \Gamma^3_0(\mu) \Gamma^2_0(\rho) \Gamma_1(\rho) \epsilon < \rho, \quad c \Gamma^2_0(\mu) \Gamma_0(\rho) \Gamma_1(\rho) \epsilon < \frac{1}{4}, \frac{s}{4}, \]
\[ c \Gamma^3_0(\mu) \Gamma^2_0(\rho) \Gamma_1(\rho) \epsilon < \frac{1}{2}, \quad c \Gamma^2_0(\mu) \Gamma^2_0(\rho) \Gamma_1(\rho) \epsilon < \frac{1}{2}, \]
there exists a transformation \( \Phi \) of the following form
\[
\begin{cases}
x = x_+ + U(\theta, x_+, y_+) = x_+ + u^0(\theta, x_+) + u^1(\theta, x_+) y_+, \\
y = y_+ + V(\theta, x_+, y_+) = y_+ + v^0(\theta, x_+) + v^1(\theta, x_+) y_+,
\end{cases}
\]
which is defined in a smaller domain \( D_{r_+, s_+} \). Under this transformation, system (4.19) becomes the form
\[
\begin{cases}
\dot{\theta} = \omega, \\
\dot{x}_+ = \omega_0 + y_+ + F_+(\theta, x_+, y_+), \\
\dot{y}_+ = G_+(\theta, x_+, y_+),
\end{cases}
\]
(4.20)
where the functions $F_+$ and $G_+$ are real analytic in the domain $D_{r+,s+}$. And the transformed system (4.20) is reversible with respect to the mapping $M : (\theta, x_+, y_+) \to (-\theta, -x_+, y_+)$. Moreover, the following estimates hold:

$$
\|U\|_{m-4\mu, r-4\rho, s} \leq \Gamma_0^\mu(\mu) \Gamma_1^\rho(\rho) \varepsilon, \quad \|V\|_{m-3\mu, r-3\rho, s} \leq \Gamma_0^\mu(\mu) \Gamma_1^\rho(\rho) \varepsilon,
$$

$$
\|F_+\|_{m_+, r_+, s_+} + \|G_+\|_{m_+, r_+, s_+} < \Gamma_0^\mu(\mu) \Gamma_1^\rho(\rho) \varepsilon s + \left(\frac{1}{s} + \frac{1}{\varepsilon}\right) \Gamma_0^\mu(\mu) \Gamma_1^\rho(\rho) \varepsilon^2 + \varepsilon^2 + \varepsilon^2.
$$

5. **Proof of Theorem 3.2.** In this section, we are going to use Iterative lemma 4.1 to prove our main theorem 3.2. We will use the Iterative lemma infinite times to construct a sequence of transformations. Under these transformations, system (3.2) will converge to an integrable system.

Let $\kappa = 4/3$. Given $0 < \mu < m/4$ and $0 < \rho < r/4$, there exist sequences $\mu_0 \geq \mu_1 \geq \cdots > 0$ and $\rho_0 \geq \rho_1 \geq \cdots > 0$ such that

$$
\sum_{\nu=0}^\infty \mu_\nu = \mu, \quad \sum_{\nu=0}^\infty \rho_\nu = \rho, \quad \kappa_\nu = \frac{\kappa - 1}{\kappa_{\nu+1}}.
$$

We denote

$$
\Psi_0(\mu) = \prod_{\nu=0}^\infty \Gamma_0(\mu)^{\kappa_\nu}, \quad \Psi_0(\rho) = \prod_{\nu=0}^\infty \Gamma_0(\rho)^{\kappa_\nu}, \quad \Psi_1(\rho) = \prod_{\nu=0}^\infty \Gamma_1(\rho)^{\kappa_\nu}. \quad (5.1)
$$

Here, the functions $\Psi_0, \Psi_1$ are well defined, see [19].

Furthermore, set

$$
m_j = m - 4 \sum_{\nu=0}^{j-1} \mu_\nu, \quad r_j = r - 4 \sum_{\nu=0}^{j-1} \rho_\nu,
$$

$$
\varepsilon_{j+1} = \varepsilon_j^{\frac{3}{2}}, \quad \kappa_j = \kappa_j^\nu, \quad \alpha_j = \alpha_j^\nu. \quad (5.2)
$$

From the above setting, it is easy to see that the sequences $r_j \to r-4\rho, m_j \to m-4\mu$ as $j \to \infty$. While the sequences $\varepsilon_j, \kappa_j$ tend to zero as $j$ tends to infinity. These sequences define the complex domain $D_j = D_{r_j, s_j}$.

Before giving the proof of main theorem 3.2, we collect some useful facts. The $\kappa_\nu$ satisfy the identities:

$$
\sum_{\nu=0}^\infty \kappa_\nu = 1, \quad \sum_{\nu=0}^\infty \mu_\nu = \frac{1}{\kappa - 1}.
$$

This and the monotonicity of the $\Gamma$-function imply that

$$
\Gamma_0(\mu_j) = \prod_{\nu=j}^\infty \Gamma_0(\mu_j)^{\kappa_\nu} \leq \left(\prod_{\nu=j}^\infty \Gamma_0(\mu)^{\kappa_\nu}\right)^{\kappa_j} \leq \Psi_0(\mu)^{\kappa_j}.
$$

Similarly, we have the estimate

$$
\Gamma_0(\rho_j) \leq \Psi_0(\rho)^{\kappa_j}, \quad \Gamma_1(\rho_j) \leq \Psi_1(\rho)^{\kappa_j}. \quad (5.2)
$$
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Now, we are in a position to prove our main theorem 3.2. When \( \nu = 0 \), we choose \( F_0 = F, G_0 = G, r_0 = r, s_0 = s \) and \( \epsilon_0 = \epsilon \). In this case, by the assumptions of our main theorem 3.2, we have

\[
\|F_0\|_{m_0,r_0,s_0} + \|G_0\|_{m_0,r_0,s_0} \leq \epsilon_0 = \frac{\gamma \epsilon_*}{\Psi_0(\mu) \Psi_0^{15}(\rho) \Psi_1(\rho)^{30}}, \tag{5.3}
\]

where \( \epsilon_* \) is an absolutely positive constant. If we choose \( \Phi_0 = id \), there is nothing to do for \( j = 0 \). Inductively, we suppose there is a sequence of transformations

\[
\Phi_\nu : \mathcal{D}_{r_\nu,s_\nu} \to \mathcal{D}_{r_{\nu-1},s_{\nu-1}}, (1 \leq \nu \leq j)
\]

such that under these change of variables, system (3.2) becomes

\[
\begin{align*}
\dot{\theta} &= \omega, \\
\dot{x}_j &= \omega_0 + x_j + F_j(\theta, x_j, y_j), \\
\dot{y}_j &= G_j(\theta, x_j, y_j).
\end{align*}
\]

Moreover, we have

\[
\|F_j\|_{m_j,r_j,s_j} + \|G_j\|_{m_j,r_j,s_j} \leq \epsilon_j.
\]

In order to verify the corresponding estimates for \((j + 1)\)th step, we need to verify the following inequalities

\[
c\Gamma^3_0(\mu_j) \Gamma_0^2(\rho_j) \Gamma_1(\rho_j) \epsilon_j < \rho_j, \quad c\Gamma^3_0(\mu_j) \Gamma_0(\rho_j) \Gamma_1(\rho_j) \epsilon_j < \frac{1}{4} s_j, \tag{5.4}
\]

\[
c\Gamma^3_0(\mu_j) \Gamma_0(\rho_j) \Gamma_1^2(\rho_j) \epsilon_j < \frac{1}{2}, \quad c\Gamma^3_0(\mu_j) \Gamma_0^2(\rho_j) \Gamma_1(\rho_j) \epsilon_j < \frac{1}{2}, \tag{5.5}
\]

\[
c\Gamma^3_0(\mu_j) \Gamma_0^2(\rho_j) \Gamma_1(\rho_j) \epsilon_j > c \left( \frac{1}{\rho_j} + \frac{1}{s_j} \right) \Gamma^3_0(\mu_j) \Gamma_0^2(\rho_j) \Gamma_1(\rho_j) \epsilon_j^3 + \epsilon_j \alpha_j^2 \leq \epsilon_{j+1}. \tag{5.6}
\]

where \( c \) is an absolutely positive constant and independent of iterations. The first inequality in (5.4) is equivalent to

\[
c \frac{1}{\rho_j} \Gamma^3_0(\mu_j) \Gamma_0^2(\rho_j) \Gamma_1(\rho_j) \epsilon_j < 1.
\]

As \( \Gamma_0(\rho_j) < \rho_j \) and (5.2), we have

\[
c \frac{1}{\rho_j} \Gamma^3_0(\mu_j) \Gamma_0^2(\rho_j) \Gamma_1(\rho_j) \epsilon_j \leq c \left( \Psi_0^3(\mu) \Psi_0^2(\rho) \Psi_1(\rho)^{\frac{1}{3}} \right)^{\epsilon_j^3}.
\]

Thus, we can choose a proper \( \epsilon_* \) in (5.3) such that the first inequality of (5.4) holds. The other inequalities of (5.4) and (5.5) can be verified by a similar method. Actually, by the choice of \( \epsilon_0 \) in (5.3), we can find a proper \( \epsilon_* \) such that

\[
c \frac{1}{\rho_j} \Gamma^3_0(\mu_j) \Gamma_0(\rho_j) \Gamma_1(\rho_j) \epsilon_j^{\frac{1}{3}} \leq c \left( \Psi_0^3(\mu) \Psi_0^2(\rho) \Psi_1(\rho)^{\frac{1}{3}} \right)^{\epsilon_j^{\frac{1}{3}}}, \tag{5.7}
\]

For the last inequality (5.6), we have the estimate

\[
c \Gamma^3_0(\mu_j) \Gamma_0(\rho_j) \Gamma_1^2(\rho_j) \epsilon_j^3 + c \Gamma^3_0(\mu_j) \Gamma_0^2(\rho_j) \Gamma_1(\rho_j) \left( \frac{1}{\rho_j} \epsilon_j^2 + \epsilon_j^3 \right) + \epsilon_j^5,
\]

\[
\leq \left\{ c \Gamma^3_0(\mu_j) \Gamma_0(\rho_j) \Gamma_1^2(\rho_j) \epsilon_j^3 + c \Gamma^3_0(\mu_j) \Gamma_0^2(\rho_j) \Gamma_1(\rho_j) \left( \frac{1}{\rho_j} \epsilon_j^2 + \epsilon_j^3 \right) + \epsilon_j^5 \right\} \epsilon_{j+1}.
\]

Similar to the above discussions and (5.7), the coefficient of \( \epsilon_{j+1} \) in the last inequality can be made less than 1. With such a choice, inequality (5.6) follows immediately.
That is, we can find a sequence of $\mathcal{M}$-commute transformations
\[ \Phi^{j+1} = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{j+1} : D_{r_{j+1}, s_{j+1}} \to D_{\tau_0, \sigma_0}. \]
And under $\Phi^{j+1}$, system (3.2) is changed into the system
\[
\begin{cases}
\dot{\theta} = \omega, \\
\dot{x}_{j+1} = \omega_0 + y_{j+1} + F_{j+1}(\theta, x_{j+1}, y_{j+1}), \\
\dot{y}_{j+1} = G_{j+1}(\theta, x_{j+1}, y_{j+1}),
\end{cases}
\tag{5.8}
\]
which is defined in the domain $D_{r_{j+1}, s_{j+1}}$. Moreover, we have the estimate
\[
\|F_{j+1}\|_{m_{j+1}, r_{j+1}, s_{j+1}} + \|G_{j+1}\|_{m_{j+1}, r_{j+1}, s_{j+1}} \leq \epsilon_{j+1}.
\]
In what follows, we are going to show the transformations $\Phi^{j+1} = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{j+1}$ converge. From estimate (5.7), it follows that
\[
\epsilon_1 \frac{1}{\nu_0} \Gamma_0^{\nu_0}(\mu_0) \Gamma_0^{\nu_0}(\rho_0) \epsilon_5 < \epsilon^2.
\tag{5.9}
\]
Then, by estimates (4.14) and (5.9) we have
\[
\|D\Phi_{\nu}(\Phi_{j+1} \circ \cdots \circ \Phi_{j+1})\|_{m_{j+1}, r_{j+1}, s_{j+1}} \leq 1 + \epsilon^2,
\]
where $D$ represents Jacobi matrix with respect to the variables $x, y$. It implies that
\[
\|D\Phi^{\nu}\|_{m_{\nu}, r_{\nu}, s_{\nu}} = \|D\Phi_0(\Phi_1 \circ \cdots \circ \Phi_{\nu})\|_{m_{\nu}, r_{\nu}, s_{\nu}} \leq \prod_{\nu \geq 0} (1 + \epsilon^2) < 2.
\]
Hence,
\[
\|\Phi^{\nu} - \Phi^{\nu-1}\|_{m_{\nu}, r_{\nu}, s_{\nu}} = \|\Phi^{\nu-1}(\Phi^{\nu}) - \Phi^{\nu-1}\|_{m_{\nu}, r_{\nu}, s_{\nu}} \\
\leq \|D\Phi^{\nu-1}\|_{m_{\nu}, r_{\nu}, s_{\nu}} \cdot \|\Phi^{\nu} - \Phi^{\nu-1}\|_{m_{\nu}, r_{\nu}, s_{\nu}} \\
\leq \epsilon^2.
\]
Denoting $D_* = \cap_{\nu \geq 0} D_{r_{\nu}, s_{\nu}} = D_{r-4\rho, 0}$, the mapping
\[
\Phi^{j+1} = \Phi^0 + \sum_{\nu = 1}^{j+1} (\Phi^{\nu} - \Phi^{\nu-1}),
\]
converges uniformly on $D_*$ to a mapping $\Phi^*$. Let $\phi^t$ and $\phi^t_{j+1}$ be the flows of (3.2) and (5.8), respectively. Then
\[
\phi^t \circ \Phi^{j+1} = \Phi^{j+1} \circ \phi^t_{j+1}.
\]
The convergence of $\Phi^{j+1}$ yields
\[
\phi^t \circ \Phi^* = \Phi^* \circ \phi^t \infty
\]
on $D_*$, where $\Phi^* = \lim_{j \to \infty} \Phi^{j+1}$ and $\phi^t \infty$ is the flow of the system
\[
\begin{cases}
\dot{\theta} = \omega, \\
\dot{x} = \omega_0 + y + O(y^2), \\
\dot{y} = O(y^2),
\end{cases}
\]
Hence, the torus $\Phi^*(T^2 \times T^n \times \{0\})$ with the frequency $\tilde{\omega} = (\omega, \omega_0)$ is invariant. This proves Theorem 3.2.
6. Application. In this section, we will apply Theorem 3.2 to the nonlinear differential equation of the second order,
\[ \ddot{x} + f(x, t)\dot{x} + g(x, t) = 0 \quad (6.1) \]
where
\[ f(x, t) = \sum_{i=0}^{e} a_i(t)x^{2i+1}, \quad g(x, t) = x^{2l+1} + \sum_{i=0}^{l-1} b_i(t)x^{2i+1}, \]
e, l are nonnegative integers satisfying
\[ l \geq 2(e + 1), \]
and the coefficients \( a_i(t) \) (for \( 0 \leq i \leq e \)) and \( b_i(t) \) (for \( 0 \leq i \leq l - 1 \)) are even and real analytic almost periodic functions with the infinite frequency \( \omega = (\cdots, \omega_\lambda, \cdots) \) and admit the spatial series expansions similar to (2.1). In this case, equation (6.1) is not a Hamiltonian system any more. There is no KAM theory can apply to equation (6.1). However, using Theorem 3.2, we can prove the existence of almost periodic solutions and the boundedness of all solutions for equation (6.1).

Firstly, we rescale equation (6.1) into a differential equation with small perturbations. Let \( u = \epsilon x, \tau = \epsilon^{-l} t \). Then equation (6.1) becomes
\[ \ddot{u} + f(u, \tau)\dot{u} + g(u, \tau) = 0 \quad (6.2) \]
where
\[ f(u, \tau) = \epsilon \sum_{i=0}^{e} \epsilon^{l-2(i+1)}a_i(\tau)u^{2i+1}, \quad g(u, \tau) = u^{2l+1} + \epsilon \sum_{i=0}^{l-1} \epsilon^{2(l-i)-1}b_i(\tau)u^{2i+1}, \]
where \( a_i(\tau) \) (for \( 0 \leq i \leq e \)) and \( b_i(\tau) \) (for \( 0 \leq i \leq l - 1 \)) are even and real analytic almost periodic functions with the frequency \( \bar{\omega} = \epsilon^l \omega \). In what follows, we will still use \( t \) instead of \( \tau \) without causing any confusion.

Equivalently, differential equation (6.2) can be rewritten in the following form
\[ \begin{cases} \dot{u} = v, \\ \dot{v} = -f(u, \tau)v + g(u, \tau). \end{cases} \quad (6.3) \]
By the odd assumptions of \( f \) and \( g \) in \( u \) and the evenness in \( t \), we have that system (6.2) is reversible with respect to the involution \( (u, v) \rightarrow (-u, v) \).

Now, we consider the auxiliary system
\[ \begin{cases} \dot{u} = v, \\ \dot{v} = -u^{2l+1}, \end{cases} \quad (6.4) \]
which is \( t \)-independent Hamiltonian system; that is
\[ \begin{cases} \dot{u} = \frac{\partial h}{\partial v}(u, v), \\ \dot{v} = -\frac{\partial h}{\partial u}(u, v), \end{cases} \]
with the Hamiltonian function
\[ h(u, v) = \frac{1}{2} v^2 + \frac{1}{2l+2} u^{2l+2}. \]
It is obvious that (6.2) is a perturbation of the above integrable Hamiltonian system. Clearly, \( h \) is positive on \( \mathbb{R}^2 \setminus \{0\} \). Note that each level line \( h(u, v) = E > 0 \) is a closed orbit of system (6.4). Hence, all the solutions of (6.4) are periodic.

Suppose \( (S(t), C(t)) \) is the solution of (6.4) satisfying the initial condition
\[ (S(0), C(0)) = (0, 1). \]
Let $T_0$ be the minimal period, which is a positive constant. Then the functions $S(t)$ and $C(t)$ satisfy the following properties:

1. $S(t + T_0) = S(t)$, $C(t + T_0) = C(t)$ with $S(0) = 0$, $C(0) = 1$;
2. $\dot{S}(t) = C(t)$, $\dot{C}(t) = -S^{2l+1}(t)$;
3. $S^{2l+2}(t) + (l + 1)C^2(t) = l + 1$;
4. $S(-t) = -S(t)$, $C(-t) = C(t)$.

The action and the angle variables are now defined by the mapping $\Psi : \mathbb{R}^+ \times T \to \mathbb{R}^2 \setminus \{0\}$: \((\xi, \eta) \to (u, v), \text{ where } \xi > 0 \text{ and } \eta \text{ are given by the formula}

\[
\Psi : \begin{cases}
  u = c_1^2 \xi^\alpha S(\frac{\eta T_0}{2\pi}), \\
v = c_1^2 \xi^\beta C(\frac{\eta T_0}{2\pi}),
\end{cases}
\]

with

\[
\alpha = \frac{1}{l + 2}, \quad \beta = 1 - \alpha, \quad c_1 = \frac{2\pi}{\beta T_0}.
\]

In the new coordinates \((\xi, \eta)\), system (6.3) takes the form

\[
\begin{cases}
  \dot{\eta} = \beta c_1^2 \xi^{2\beta - 1} + f_1(\xi, \eta, t), \\
  \dot{\xi} = f_2(\xi, \eta, t),
\end{cases}
\]

(6.5)

where

\[
f_1(\xi, \eta, t) = \alpha c_1 \cdot \epsilon \sum_{i=0}^{2l-1} \epsilon^{2(i+1)} c_1^{2(i+1)} \xi^{2(i+1)} \alpha S^{2i+2} \left(\frac{\eta T_0}{2\pi}\right) C \left(\frac{\eta T_0}{2\pi}\right) a_i(t) \\
+ \alpha c_1 \xi^{\alpha - 1} \cdot \epsilon \sum_{i=0}^{l-1} \epsilon^{2(i-l)-1} c_1^{2(i+1)} \xi^{2(i+1)} \alpha S^{2i+2} \left(\frac{\eta T_0}{2\pi}\right) b_i(t),
\]

\[
f_2(\xi, \eta, t) = -\frac{T_0}{2\pi} c_1 \cdot \epsilon \sum_{i=0}^{2l-1} \epsilon^{2(i+1)} c_1^{2(i+1)} \xi^{2(i+1)} \alpha S^{2i+1} \left(\frac{\eta T_0}{2\pi}\right) C^2 \left(\frac{\eta T_0}{2\pi}\right) a_i(t) \\
- \frac{T_0}{2\pi} c_1 \xi^{\alpha - 1} \cdot \epsilon \sum_{i=0}^{l-1} \epsilon^{2(i-l)-1} c_1^{2(i+1)} \xi^{2(i+1)} \alpha S^{2i+1} \left(\frac{\eta T_0}{2\pi}\right) C \left(\frac{\eta T_0}{2\pi}\right) b_i(t).
\]

Note that the function $a_i(t)$, $b_i(t)$, $C \left(\frac{\eta T_0}{2\pi}\right)$ and $S \left(\frac{\eta T_0}{2\pi}\right)$ satisfy

\[
a_i(-t) = a_i(t), \quad b_i(-t) = b_i(t),
\]

\[
S \left(\frac{-\eta T_0}{2\pi}\right) = -S \left(\frac{\eta T_0}{2\pi}\right), \quad C \left(\frac{-\eta T_0}{2\pi}\right) = C \left(\frac{\eta T_0}{2\pi}\right),
\]

then the system is reversible with respect to the mapping $G : (\xi, \eta) \to (\xi, -\eta)$.

After introducing the additional variable $\theta \in \mathbb{T}^\mathbb{Z}$, the system can be written in the form of an autonomous system as follows

\[
\begin{align*}
  \dot{\theta} &= \bar{\omega}, \\
  \dot{\eta} &= \beta c_1^2 \xi^{2\beta - 1} + F_1(\xi, \eta, \theta), \\
  \dot{\xi} &= F_2(\xi, \eta, \theta),
\end{align*}
\]

where $F_1(\xi, \eta, \theta)$, $F_2(\xi, \eta, \theta)$ are the shell functions for the almost periodic functions $f_1(\xi, \eta, t)$, $f_2(\xi, \eta, t)$ with respect to $t$.

Let $[q_1, q_2] \subset \mathbb{R}^+$ be any bounded interval without 0, not depending on $\epsilon$. For any $\xi_0 \in [q_1, q_2]$, we denote

\[
\xi = \xi_0 + Q^{-1} I, \quad Q = \beta c_1^2 (2\beta - 1) \xi_0^{2\beta - 2}
\]
and do Taylor expansion for $\beta c_1^{2\beta} \xi^{2\beta-1}$ at $\xi_0$. Then, we have

$$\begin{cases}
\dot{\theta} = \bar{\omega}, \\
\dot{\eta} = \beta c_1^{2\beta} \xi_0^{2\beta-1} + I + \bar{F}_1(I, \eta, \theta), \\
I = \bar{F}_2(I, \eta, \theta),
\end{cases}$$

(6.6)

where

$$\bar{F}_1(\xi, \eta, \theta) = \beta c_1^{2\beta} (\xi_0 + Q^{-1}I)^{2\beta-1} - \beta c_1^{2\beta} \xi_0^{2\beta-1} - I$$

$$+ F_1(\xi_0 + Q^{-1}I, \eta, \theta),$$

$$= O(|I|^2) + F_1(\xi_0 + Q^{-1}I, \eta, \theta),$$

$$\bar{F}_2(I, \eta, \theta) = QF_2(\xi_0 + Q^{-1}I, \eta, \theta).$$

Denote $\omega_0 = \beta c_1^{2\beta} \xi_0^{2\beta-1}$, for any $\xi_0 \in [q_1, q_2]$, we get

$$\frac{\partial \omega_0}{\partial \xi_0} = \beta c_1^{2\beta} (2\beta - 1)\xi_0^{2\beta-2} \neq 0.$$

We therefore single out the subsets $O_\gamma \subset O := [\beta c_1^{2\beta} q_1^{2\beta-1}, \beta c_1^{2\beta} q_2^{2\beta-1}]$ which consists of $\omega_0$ satisfying

$$|\langle k, \epsilon^l \omega \rangle + \langle \hat{k}, \omega_0 \rangle| \geq \frac{\gamma}{\Delta(|\{k, \hat{k}\}|\Delta(|k| + |\hat{k}|)}$$

with some fixed $\omega$ has positive Lebesgue measure provided that $\gamma$ is small by the measure estimate in Pöschel [19] and Huang and Li [8].

Thus, system (6.6) can be written in the form

$$\begin{cases}
\dot{\theta} = \bar{\omega}, \\
\dot{\eta} = \omega_0 + I + \bar{F}_1(I, \eta, \theta; \omega_0), \\
I = \bar{F}_2(I, \eta, \theta; \omega_0).
\end{cases}$$

(6.7)

And $\bar{F}_1, \bar{F}_2$ are periodic in $\theta, \eta$ with period $2\pi$ and real analytic in $(\theta, \eta, I) \in T^2 \times T \times \mathbb{R}$. Then, there exist $r > 0$ such that the functions $\bar{F}_1, \bar{F}_2$ admit analytic extension in the complex neighborhood $\{\theta, \eta : |\text{Im } \theta| < r, |\text{Im } \eta| < r\}$ of $T^2 \times T$.

Taking $s = \epsilon^{l/2}$, there exists $C_0$, depending on $\epsilon, l, T_0, r$, but not on $\epsilon$, such that for any $|\text{Im } \theta| < r, |\text{Im } \eta| < r, |I| < s$, we have $\|\bar{F}_1\|_{m,r,s}, \|\bar{F}_2\|_{m,r,s} \leq C_0 \epsilon$. Without losing of generality, we can assume that $\|\bar{F}_1\|_{m,r,s}, \|\bar{F}_2\|_{m,r,s} \leq \epsilon$.

And we have the following theorem.

**Theorem 6.1.** Every solution of equation (6.1) is bounded. Moreover, equation (6.1) has infinitely many almost periodic solutions.

**Proof.** If we let $F = \bar{F}_1, G = \bar{F}_2$ and

$$0 < \epsilon < \frac{\gamma \epsilon_*}{\Psi_0(\mu)^{15} \Psi_0(\rho)^{15} \Psi_1(\rho)^{30}}$$

with $0 < \mu < \frac{\pi}{4}$ and $0 < \rho < \frac{\pi}{4}$, and $\epsilon_*$ is an absolutely positive constant, $\Psi_0(\mu), \Psi_0(\rho), \Psi_1(\rho)$ are defined by (5.1), then the assumptions of Theorem 3.2 are met. Applying Theorem 3.2 to system (6.7) with $n = 1$, we obtain an invariant torus for system (6.6) with frequency $(\epsilon^l \omega, \omega_0)$ for each frequency $\omega_0 \in O_\gamma$. And each torus produces a family of almost periodic solutions with the frequency $(\epsilon^l \omega, \omega_0)$. This implies that all solutions are bounded for all time. Consequently, system (6.1) has infinitely many almost periodic solutions as well as the boundedness of solutions. \qed
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