ESTIMATES FOR MITTAG-LEFFLER FUNCTIONS WITH
SMOOTH PHASE DEPENDING ON TWO VARIABLES

AKBAR R.SAFAROV

Abstract. In this paper we consider the problem on estimates for Mittag-Leffler functions with the smooth phase functions of two variables having singularities of type $D^\infty$, $D^\pm$ and $A_r$. The generalisation is that we replace the exponential function with the Mittag-Leffler-type function, to study oscillatory type integrals. We extend results of paper [1] and [2] to two-dimensional integrals with phase having some simple singularities.

CONTENTS

1. Introduction 1
2. Some auxiliary statements 2
3. Proof of the main results 3
References 7

1. INTRODUCTION

The Mittag-Leffler function $E_\alpha(z)$ is named after the great Swedish mathematician Gösta Magnus Mittag-Leffler (1846-1927) who defined it by a power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \quad Re(\alpha) > 0, \quad (1.1)$$

and studied its properties in 1902-1905 in five subsequent notes [3]-[6] in connection with his summation method for divergent series.

A classic generalization of the Mittag-Leffler function, namely the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad Re(\alpha) > 0, \quad (1.2)$$

which was deeply investigated independently by Humbert and Agarwal in 1953 [7]-[9] and by Dzherbashyan in 1954 [10] - [12], see also [13] and the references therein.

In this paper we also consider a special case the generalized Mittag-Leffler function defined as in (1.2) by
\[
E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \alpha > 0, \beta \in \mathbb{R}.
\]

Obviously,
\[
E_{1,1}(x) = e^x. \quad (1.3)
\]

We consider the following integral with phase \(f\) and amplitude \(\psi\)
\[
I_{\alpha,\beta} = \int_{a}^{b} E_{\alpha,\beta}(i\lambda f(x))\psi(x)dx \quad (1.4)
\]
where \(0 < \alpha \leq 1, \beta > 0\) and \(\lambda > 0\).

If \(\alpha = \beta = 1\) in the integral (1.4). The integral \(I_{1,1}\) is called an oscillatory integral. In harmonic analysis estimates, the most important estimates for oscillatory integrals van der Corput lemma [14]. Estimates for oscillatory integrals with polynomial phase can be bound, for instance, in papers [15]-[20]. In the current paper we replace the exponential function with the Mittag-Leffler-type function and study oscillatory type integrals (1.4). In the papers [1] and [2] analogues of the van der Corpute lemmas involving Mittag-Leffler functions for one dimensional integrals have been considered. We extend results of [1] and [2] for two-dimensional integrals with phase having some simple singularities.

The main result of the paper is the following.

**Theorem 1.1.** Let \(-\infty < a < b < \infty\). Assume that the phase function is a homogenous polynomial of third degree with two variables and let \(\psi \in L^p[a, b]^2, 1 < p \leq \infty\). Then for any \(\alpha \in (0, 1), \beta, \lambda \in (0, +\infty)\)
\[
\left| \int_{[a,b]^2} E_{\alpha,\beta}(i\lambda x_1^2 x_2)\psi(x)dx \right| \leq \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{2} - \frac{1}{3p}}}, \quad (1.5)
\]
\[
\left| \int_{[a,b]^2} E_{\alpha,\beta}(i\lambda(x_1^2 x_2 \pm x_3^2))\psi(x)dx \right| \leq \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{2} - \frac{1}{3p}}}, \quad (1.6)
\]
\[
\left| \int_{[a,b]^2} E_{\alpha,\beta}(i\lambda x_1^3)\psi(x)dx \right| \leq \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{2} - \frac{1}{3p}}}, \quad (1.7)
\]
where constant \(C\) depends only on \(p\).

### 2. Some auxiliary statements

First we give ancillary statements. Let us consider homogenous polynomial third degree with two variables.

**Proposition 1** ([21]). A homogenous polynomial of third degree with two variables may be reduced by a R-linear transformation to one of the forms: 1) \(x_1^2 x_2\), 2) \(x_1^2 x_2 \pm x_3^2\), 3) \(x_1^3\), 4) 0.

**Definition 1.** Given \(\mu \in (1, \infty]\), a critical point, equivalent to the critical point of the function \(x_1^2 x_2 \pm x_2^{\mu-1}\) is said to be a critical point of type \(D_\mu^-\), where \(x_2^{\mu-1} \equiv 0\) for \(\mu = \infty\).

**Definition 2.** A critical point, equivalent to the critical point of the function \(x_1^{r+1}\), \(r \geq 1\) is said to be a critical point of type \(A_r\).
Proposition 2 ([22]). If $0 < \alpha < 2, \beta$ is an arbitrary real number and $\mu$ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$, then there is $C > 0$ such that
\[
|E_{\alpha, \beta}(z)| \leq \frac{C}{1 + |z|}, \quad z \in \mathbb{C}, \quad \mu \leq |\arg(z)| \leq \pi.
\] (2.1)

Proposition 3. Let $\alpha, \beta > 0$ and $f : [a, b] \to \mathbb{C}$. Then for all $\lambda \in \mathbb{C}$
\[
E_{\alpha, \beta}(i\lambda f(x)) = E_{2\alpha, \beta}(-\lambda^2 f^2(x)) + i\lambda f(x)E_{2\alpha, \beta + \alpha}(-\lambda^2 f^2(x)).
\] (2.2)

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. As for small $\lambda$ the integral (1.4) is clearly bounded, we prove Theorem 1.1 only for $\lambda \geq 1$. Without loss of generality, we can consider the integral on $[0, 1]^2$, otherwise we reduce to this case using a linear transformation. Since we are given a homogenous polynomial of third degree with two variables, Proposition 1 we can represent it one of the following: 1) $x_1^2 x_2$; 2) $x_1^2 x_2 \pm x_2^2$; 3) $x_1^4$. 0. If phase function is $f(x) \equiv 0$ it is clear that integral will be identically zero. So we will consider the other three cases separately.

Using inequalities (2.1) and (2.2) we obtain:
\[
|E_{\alpha, \beta}(i\lambda f(x))| \leq |E_{2\alpha, \beta}(-\lambda^2 f^2(x))| + \lambda|f(x)||E_{2\alpha, \beta + \alpha}(-\lambda^2 f^2(x))|
\]
\[
\leq \frac{C}{1 + \lambda^2 f^2(x)} + \frac{C\lambda|f(x)|}{1 + \lambda^2 f^2(x)} \leq \frac{C(1 + \lambda|f(x)|)}{1 + \lambda^2 f^2(x)} \leq \frac{C}{1 + \lambda|f(x)|}.
\] (3.1)

Case I. First we assume that the phase function has critical point of type $D_\infty$ so that $f(x) = x_1^2 x_2$.

We consider integral (1.4) of the form:
\[
I_{\alpha, \beta} = \int_{[0,1]^2} E_{\alpha, \beta}(i\lambda x_1^2 x_2)\psi(x)dx.
\] (3.2)

We use inequality (3.1) in the integral (3.2) and we obtain:
\[
|I_{\alpha, \beta}| = \left|\int_{[0,1]^2} E_{\alpha, \beta}(i\lambda x_1^2 x_2)\psi(x)dx\right| \leq \int_{[0,1]^2} \left|E_{\alpha, \beta}(i\lambda x_1^2 x_2)|\psi(x)| dx
\]
\[
\leq C \int_0^1 dx_1 \int_0^1 \frac{|\psi(x)|dx_2}{1 + \lambda x_1^2 x_2}.
\] (3.3)

Let $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume first that $p \neq \infty$, so that $q > 1$. Then using the H"older inequality in the inner integral we get
\[
J_{m1} := \int_0^1 \frac{|\psi(x)|dx_2}{1 + \lambda x_1^2 x_2} \leq \left(\int_0^1 |\psi(x)|^p dx_2\right)^\frac{1}{p} \left(\int_0^1 \frac{dx_2}{1 + \lambda x_1^2 x_2^q}\right)^\frac{1}{q}
\]
\[
= \left(\int_0^1 |\psi(x)|^p dx_2\right)^\frac{1}{p} \left(1 - \frac{(1 + \lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\right)^\frac{1}{q}.
\]

Thus,
\[
|I_{\alpha, \beta}| \leq C \int_0^1 \left(\int_0^1 |\psi(x)|^p dx_2\right)^\frac{1}{p} \left(1 - \frac{(1 + \lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\right)^\frac{1}{q} dx_1.
\]
Then using again the Hölder inequality in this integral we obtain

$$|I_{\alpha,\beta}| \leq C \left( \int_0^1 \int_0^1 |\psi(x)|^p dx_2 dx_1 \right)^{\frac{1}{p}} \left( \int_0^1 \frac{1 - (1 + \lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2} dx_1 \right)^{\frac{1}{q}} \leq C \|\psi\|_{L^p} \left( \int_0^1 \frac{1 - (1 + \lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2} dx_1 \right)^{\frac{1}{q}}.$$  

Let

$$K := \int_0^1 \frac{1 - (1 + \lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2} dx_1.$$  

Since \((1+\lambda x_1^2)^{1-q} = 1 + O(\lambda x_1^2)\) near \(x_1 = 0\) and \(q > 1\), the integral \(K\) is convergent. To estimate \(K\), first we use the change of variables \(t = \sqrt{\lambda} x_1\) to get

$$K = \frac{1}{(q-1)\sqrt{\lambda}} \int_0^{\sqrt{\lambda}} \frac{1 - (1 + t^2)^{1-q}}{t^2} dt = \frac{1}{(q-1)\sqrt{\lambda}} \int_0^{\sqrt{\lambda}} \frac{1}{t^2(1 + t^2)^{q-1}} dt$$  

$$= \frac{1}{(q-1)\sqrt{\lambda}} \int_0^1 \frac{1}{t^2(1 + t^2)^{q-1}} dt + \frac{1}{(q-1)\sqrt{\lambda}} \int_{\sqrt{\lambda}}^1 \frac{1}{t^2(1 + t^2)^{q-1}} dt =: K_1 + K_2.$$  

Since \(q-1 \leq [q]\), where \([q]\) is the integer part of \(q > 1\), by the Newton’s binomial formula

$$(1 + t^2)^{q-1} \leq (1 + t^2)^{[q]} = 1 + [q]t^2 + \frac{[q][(q-1)]}{2} t^4 + \ldots + t^{2[q]},$$

and hence

$$K_1 = \frac{1}{(q-1)\sqrt{\lambda}} \int_0^1 \frac{1}{t^2(1 + t^2)^{q-1}} dt \leq \frac{C_q}{\sqrt{\lambda}},$$

where

$$C_q := \frac{1}{q-1} \int_0^1 [q] + \frac{[q][(q-1)]}{2} t^2 + \ldots + t^{2[q]-2} dt.$$  

Moreover, since \(\frac{1}{t^2(1 + t^2)^{q-1}} < \frac{1}{t^2}\),

$$K_2 = \frac{1}{(q-1)\sqrt{\lambda}} \int_{\sqrt{\lambda}}^1 \frac{1}{t^2(1 + t^2)^{q-1}} dt \leq \frac{1}{(q-1)\sqrt{\lambda}} \int_{\sqrt{\lambda}}^1 \frac{1}{t^2} dt$$  

$$= \frac{1}{(q-1)\sqrt{\lambda}} \left( 1 - \frac{1}{\sqrt{\lambda}} \right) < \frac{1}{(q-1)\sqrt{\lambda}}.$$  

Hence,

$$K \leq \frac{C'_q}{\sqrt{\lambda}}, \quad C'_q := C_q + \frac{1}{q-1},$$

and

$$|I_{\alpha,\beta}| \leq \frac{C''_q \|\psi\|_{L^p}}{\lambda^{\frac{1}{q'}}},$$

where \(C''_q\) is some coefficient depending only on \(q\), and hence only on \(p\).

Now we consider the case \(q = 1\). Notice that the coefficient \(C''_q \to +\infty\) as \(q \to 1\) and therefore we cannot directly conclude the requires estimate from the one for \(q > 1\).
As \( q = 1 \), we have \( p = \infty \) and \( \psi \in L^\infty \). In view of (3.3), first we estimate inner integral as

\[
|J_{m1}| = \left| \int_0^1 \frac{\psi(x) dx}{1 + \lambda x_1^2 x_2} \right| \leq \sup_{x_2 \in [0,1]} |\psi(x)| \left| \int_0^1 \frac{dx_2}{1 + \lambda x_1^2 x_2} \right| \leq \sup_{x_2 \in [0,1]} |\psi(x)| \left( \ln(1 + \lambda x_1^2) \right) \left| \frac{1}{\lambda x_1^2} \right| \ln(1 + \lambda x_1^2).
\]

Thus

\[
|I_{\alpha,\beta}| \leq \int_0^1 \frac{\sup_{x_2 \in [0,1]} |\psi(x)| \ln(1 + \lambda x_1^2)}{\lambda x_1^2} dx_1 \leq C \|\psi\|_{L^\infty} \int_0^1 \frac{\ln(1 + \lambda x_1^2)}{\lambda x_1^2} dx_1.
\]

We use change variables as \( \lambda x_1^2 = y \) in the last integral and get

\[
|I_{\alpha,\beta}| \leq \frac{C \|\psi\|_{L^\infty}}{\lambda} \int_0^y \frac{\ln(1 + y)}{y} dy \leq \frac{C \|\psi\|_{L^\infty}}{\lambda} \int_0^\infty \frac{\ln(1 + y)}{y} dy.
\]

Note that the last integral converges. Instead, using integration by parts we obtain

\[
\int_0^\infty \frac{\ln(1 + y)}{y} dy = - \lim_{N_1 \to 0, N_2 \to \infty} \frac{2 \ln(1 + y)}{y^\frac{1}{2}} \bigg|_{N_2}^{N_1} + \int_0^\infty \frac{2dy}{(1 + y)y^\frac{1}{2}} = \int_0^\infty \frac{4dy}{1 + y} = 4 \arctan y\bigg|_0^\infty = 2\pi.
\]

Thus from (3.4) we get

\[
|I_{\alpha,\beta}| \leq \frac{C \|\psi\|_{L^\infty}}{\lambda}.
\]

**Case II.** Assume that the phase function has critical point of type \( D_3^\pm \) so that

\( f(x) = x_1^2 x_2 \pm x_2^3 \). We estimate integral (1.4) when the phase function has critical point of type \( D_4^\pm \) and the case \( D_4^- \) can be done similarly.

We consider the integral

\[
I_{\alpha,\beta} = \int_{[0,1]^2} E_{\alpha,\beta}(i\lambda(x_1^2 x_2 + x_2^3)) \psi(x) dx.
\]

Using inequality (3.1) for the integral (3.5) we get

\[
|I_{\alpha,\beta}| = \left| \int_{[0,1]^2} E_{\alpha,\beta}(i\lambda(x_1^2 x_2 + x_2^3)) \psi(x) dx \right| \leq \int_{[0,1]^2} \left| E_{\alpha,\beta}(i\lambda(x_1^2 x_2 + x_2^3)) \right| |\psi(x)| dx \leq \int_0^1 dx_1 \int_0^1 \frac{|\psi(x)| dx_1}{1 + \lambda(x_1^2 x_2 + x_2^3)} = \int_0^1 dx_1 \int_0^1 \frac{|\psi(x)| dx_1}{1 + \lambda x_1^2 + \lambda x_2^2}
\]

We use Hölder inequality for the last inner integral we obtain

\[
|J_{m2}| := \int_0^1 \frac{|\psi(x)| dx_1}{1 + \lambda x_1^2 + \lambda x_2 x_2^2} \leq \left( \int_0^1 |\psi(x)|^p dx_1 \right)^\frac{1}{p} \left( \int_0^1 \frac{dx_1}{1 + \lambda x_1^2 + \lambda x_2 x_2^2} \right)^\frac{1}{q}.
\]

Then using again the Hölder inequality for this integral we establish
\[ |I_{\alpha,\beta}| \leq \left( \int_0^1 \int_0^1 |\psi(x)|^p dx_1 dx_2 \right)^{\frac{1}{p}} \left( \int_0^1 \int_0^1 \frac{dx_1}{|1 + \lambda x_2^3 + \lambda x_2 x_1^2|^q} dx_2 \right)^{\frac{1}{q}}. \]

Changing the variables as \( x_1 = \left( \frac{1 + \lambda x_2^3}{\lambda x_2} \right)^{\frac{1}{3}} t \) we get

\[ |I_{\alpha,\beta}| \leq \|\psi\|_{L^p} \left( \int_0^1 \int_0^1 \frac{dx_1 dx_2}{|1 + \lambda x_2^3 + \lambda x_2 x_1^2|^q} \right)^{\frac{1}{q}} \]

\[ = \|\psi\|_{L^p} \left( \int_0^1 \frac{(1 + \lambda x_2^3)^{\frac{1}{p} - q}}{(\lambda x_2)^{\frac{1}{p}}} dx_2 \int_0^A \frac{dt}{(1 + t^2)^q} \right)^{\frac{1}{q}} \]

where \( A = \left( \frac{\lambda x_2}{1 + \lambda x_2^3} \right)^{\frac{1}{3}} \) and \( \int_0^A \frac{dt}{(1 + t^2)^q} < C \) as \( A \to \infty \). Thus,

\[ |I_{\alpha,\beta}| \leq C \|\psi\|_{L^p} \left( \int_0^1 \frac{(1 + \lambda x_2^3)^{\frac{1}{p} - q}}{(\lambda x_2)^{\frac{1}{p}}} dx_2 \right)^{\frac{1}{q}}. \]

Replacing \( x_2 \) by \( \lambda^{-\frac{1}{3}} \tau \) and using \( \frac{1}{q} = 1 - \frac{1}{p} \) we get

\[ |I_{\alpha,\beta}| \leq \frac{C \|\psi\|_{L^p}}{\lambda^{\frac{1}{3} - \frac{1}{pq}}} \left( \int_0^{\frac{\lambda^{\frac{1}{3}}}{\tau^{\frac{1}{3}(\tau^3 + 1)^{q-\frac{1}{2}}}}} \frac{d\tau}{\tau^{\frac{1}{3}(\tau^3 + 1)^{q-\frac{1}{2}}}} \right)^{\frac{1}{q}}. \]

Since the last integral is convergent,

\[ |I_{\alpha,\beta}| \leq \frac{C \|\psi\|_{L^p}}{\lambda^{\frac{1}{3} - \frac{1}{pq}}}. \]

**Case III.** Assume that the phase function has critical point of type \( A_2 \) so that \( f(x) = x_1^3 \). We estimate the integral (1.4) with phase function \( f(x) = x_1^3 \).

\[ |I_{\alpha,\beta}| \leq \int_0^1 \int_0^1 |E_{\alpha,\beta}(i\lambda x_1^3)| |\psi(x)| dx_1 dx_2. \]

First we use inequality (3.1) for the last inner integral we obtain

\[ |J_{in3}| := \int_0^1 |\psi(x)| dx_1. \]

Then we use Hölder inequality for the last integral \( I_{\alpha,\beta} \) twice and we get:

\[ |I_{\alpha,\beta}| \leq \left( \int_0^1 \int_0^1 |\psi(x)|^p dx_1 dx_2 \right)^{\frac{1}{p}} \left( \int_0^1 \int_0^1 \frac{dx_1}{|1 + \lambda x_2^3|^q} dx_2 \right)^{\frac{1}{q}}. \]

Replacing \( \lambda^{-\frac{1}{3}} x_1 \) by \( t \) in the above inequality, we obtain

\[ |I_{\alpha,\beta}| \leq \frac{C \|\psi\|_{L^p}}{\lambda^{\frac{1}{3} - \frac{1}{pq}}} \left( \int_0^{\frac{\lambda^{\frac{1}{3}}}{1 + t^3}} \frac{dt}{(1 + t^3)^q} \right)^{\frac{1}{q}} \leq \frac{C \|\psi\|_{L^p}}{\lambda^{\frac{1}{3} - \frac{1}{pq}}} \left( \int_0^\infty \frac{dt}{(1 + t^3)^q} \right)^{\frac{1}{q}}. \]
Since $\frac{1}{p} + \frac{1}{q} = 1$ and the last integral converges,

$$|I_{\alpha,\beta}| \leq \frac{C \|\psi\|_{L^p}}{\lambda^{\frac{3}{3} - \frac{1}{p}}}.$$

The proof is complete.

Remark. If $\alpha = \beta = 1$ in the integral (1.4) integral called oscillatory integral and theorem holds for it.

Declaration of competing interest

This work does not have any conflicts of interest.

Acknowledgements. This paper was supported by ”El-yurt umidi” Foundation of Uzbekistan and partially supported in parts by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations and by the Methusalem programme of the Ghent University Special Research Fund (BOF) (Grant number 01M01021). The author thanks to Prof. M.Ruzhansky for proposing the problem and constant attention to the work and also thanks the referees for numerous suggestions which greatly helped to improve the exposition.

References

[1] M.Ruzhansky, B.Torebek Van der Corput lemmas for Mittag-Leffler functions, arXiv:2002.07492.
[2] M.Ruzhansky, B.Torebek, Van der Corput lemmas for Mittag-Leffler functions. II. $\alpha$—directions Bull. Sci. Math., 171 (2021), 103016, 23pp.
[3] M.G.Mittag-Leffler, Sur l’intégrale de Laplace-Abel. C.R. Acad. Sci. Paris 135, (1902), 937-939.
[4] M.G.Mittag-Leffler, Une généralisation de l’intégrale de Laplace-Abel. Comp. Rend. Acad. Sci. Paris 136, (1903), 537–539.
[5] M.G.Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$. Comp. Rend. Acad. Sci. Paris, 137, (1903) 554–558.
[6] M.G.Mittag-Leffler, Sopra la funzione $E_\alpha(x)$. Rend. Acc. Lincei, 5, no. 13, (1904), 3–5.
[7] P.Humbert, Quelques résultats relatifs à la fonction de Mittag-Leffler. C.R. Acad. Sci. Paris 236, (1953), 1467–1468 .
[8] R.P.Agarwal, A propos d’une note de M.Pierre Humbert. C.R. Acad. Sci. Paris 236, (1953), 2031–2032.
[9] P.Humbert, R.P.Agarwal, Sur la fonction de Mittag-Leffler et quelconques de ses généralisations. Bull. Sci. Math. Ser.II, no.77,(1953), 180–185.
[10] M.M.Dzherbashyan, On the asymptotic expansion of a function of Mittag-Leffler type, Akad.Nauk Armjan.SSR Doklady. 19, (1954), 65–72 (Russian).
[11] M.M.Dzherbashyan, On integral representation of functions continuous on given rays (generalization of the Fourier integrals), Izvestija Akad.Nauk SSSR Ser.Mat. 18, (1954), 427–448 (Russian).
[12] M.M.Dzherbashyan, On Abelian summation of the eneralized integral transform, Akad.Nauk Armjan.SSR Izvestija, fiz-mat. estest. techn.nauki. 7 (1954), no. 6, 1–26 (Russian).
[13] Rudolf Gorenflo, Anatoly Kilbas, Francesco Mainardi, Sergei Rogosin, Mittag-Leffler functions, related topics and applications Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg (2014).
[14] K.G. Van der Corput. Zur Methode der stationaren phase Compositio Math., 1, (1934), 15-38.
[15] G.I. Arkhipov, A.A. Karatsuba, V.N. Chubarikov, Theory of multiple trigonometric sums., Moscow, Nauka, (1987), p. 357.
[16] A.N.Varchenko Newton polyhedra and estimation of oscillating integrals Functional Analysis and Its Applications 10, (1976), 175–196.
[17] J. Duistermaat Oscillatory integrals Lagrange immersions and unifoldings of singularities Comm. Pure.Appl.Math., 27, no.2, (1974), 207–281.
[18] V.N.Karpushkin Uniform estimates for oscillatory integrals with parabolic or hyperbolic phase Proceedings of the I.G.Petrovsky Seminar, 9, (1983), 3–39 (Russian).
[19] A. Safarov, Invariant estimates of two-dimensional oscillatory integrals Math. Notes. 104, (2018), 293–302.
[20] A. Safarov, On invariant estimates for oscillatory integrals with polynomial phase, J. Sib. Fed. Univ. Math. Phys. 9 (2016), 102–107.
[21] V.I.Arnold, S.M.Gusein-Zade, A.N.Varchenko Singularities of Differentiable Maps, Birkhauser, Boston Basel · Stuttgart 1985.
[22] I.Podlubny, Fractional Differensial Equations, Academic Press, New York 1999.

Akbar R. Safarov

Uzbekistan Academy of Sciences V.I.Romanovskiy Institute of Mathematics
Olmaizar district, University 46, Tashkent, Uzbekistan
Samarkand State University
Department Mathematics, 15 University Boulevard
Samarkand, 140104, Uzbekistan

Email address: safarov-akbar@mail.ru