The Standard Model in Noncommutative Geometry
and Morita equivalence

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Abstract

We discuss some properties of the spectral triple \((A_F, H_F, D_F, J_F, \gamma_F)\) describing the internal space in the noncommutative geometry approach to the Standard Model, with \(A_F = \mathbb{C} \oplus H \oplus M_3(\mathbb{C})\). We show that, if we want \(H_F\) to be a Morita equivalence bimodule between \(A_F\) and the associated Clifford algebra, two terms must be added to the Dirac operator; we then study its relation with the orientability condition for a spectral triple. We also illustrate what changes if one considers a spectral triple with a degenerate representation, based on the complex algebra \(B_F = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})\).

1 Introduction

In the spectral action approach to (quantum) field theory, the space of the theory is the product of an ordinary spin manifold with a finite noncommutative space (cf. \cite{11, 23} and references therein). States of the system are represented by unit vectors in \(L^2(M, S) \otimes H\), where \(L^2(M, S)\) are square integrable sections of the spinor bundle \(S \to M\) and \(H\) is a finite-dimensional Hilbert space representing the internal degrees of freedom of a particle. The algebra containing the observables is the tensor product of smooth functions \(C^\infty(M)\) on \(M\) with certain finite dimensional algebra \(A\). More precisely, one has an “almost commutative” geometry described by a product of spectral triples, with Dirac operator constructed from the canonical Dirac operator on \(M\) and certain selfadjoint operator (a Hermitian matrix) \(D\) on \(H\).

A deep algebraic characterization of the space of Dirac spinor fields \(L^2(M, S)\) on a spin manifold is as the Morita equivalence bimodule between \(C^\infty(M)\) and the algebra \(\mathcal{C}\ell(M)\) of sections of the Clifford bundle of \(M\). It is natural to investigate if also the finite-dimensional spectral triple \((A, H, D)\) describes a (noncommutative) spin manifold, and in particular if the elements of \(H\) are in some sense “spinors”. This condition – which we name “property (M)” in Def.\textsuperscript{5} – can be precisely formulated again in terms of Morita equivalence involving \(A\) and certain noncommutative analogue of \(\mathcal{C}\ell(M)\), and is satisfied in some basic examples like e.g. Einstein-Yang-Mills systems.
We investigate the consequence of such a requirement on the finite non-commutative geometry that should describe the Standard Model of elementary particles. We shall show that in order to satisfy such a condition, we are forced to introduce two additional terms in the Dirac operator, and consider a non-standard grading. (One of these terms, the one compatible with the original grading of Chamseddine-Connes \([3, 9, 5, 11]\), was already considered in \([21]\).) In order to get the correct experimental value of the Higgs mass, various modifications of the original model have been proposed: to enlarge the Hilbert space thus introducing new fermions \([22]\); to turn one of the elements of the internal Dirac operator into a field by hand \([4]\) rather than getting it as a fluctuation of the metric; to break (relax) the 1st order condition \([6, 7]\), thus allowing the presence of new terms in the Dirac operator; to enlarge the algebra \([14]\) and use the twisted spectral triple \([15]\) with bounded twisted commutators. In the present paper from the Morita condition and a different grading we get two extra fields (without breaking any of the other conditions). We postpone to future work a discussion of the physical implications and, in particular, how the Higgs mass is modified.

Besides the original model, which is built around the real algebra:

\[ A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) , \tag{1} \]

where \(\mathbb{H}\) denotes the division ring of quaterions, we shall also consider the complex algebra

\[ B_F = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) , \tag{2} \]

which has an interesting interpretation from quantum group theory. Namely, it is the semisimple part \([12]\) of a certain quotient of \(U_q(sl(2))\) for \(q\) a 3rd root of unity. As explained in \([13]\), the dual compact quantum group \(Q\) fits the exact sequence

\[ 1 \rightarrow Q \rightarrow SL_q(2) \rightarrow SL(2, \mathbb{C}) \rightarrow 1 . \]

Recall that \(SL(2, \mathbb{C})\) is a double covering of the restricted Lorentz group. One might argue that trading a commutative space for an almost commutative one, the Lorentz group should be replaced by a compact quantum group covering it, which takes into account the symmetries of the internal space as well. (For preliminary studies of Hopf-algebra symmetries of \(A_F/B_F\) see \([17, 12, 13]\); for compact quantum group symmetries see \([1, 2]\).)

We show that a minimal modification in the representation allows to replace \(A_F\) by \(B_F\) without changing the content of the theory. In particular, at the representation level the complexification \(\pi(A_F)_C\) of \(\pi(A_F)\) is the minimal unitalization of the degenerate representation \(\pi(B_F)\) (the representations, here denoted by the same symbol which we will omit later on, are introduced in \([33]\); adding the identity operator (which commutes with the Dirac operator) does not produce new gauge field.

The plan of the paper is the following. In \([2]\) we review some basic ideas of noncommutative geometry \([8, 16, 19]\), with a view to applications to gauge theory \([11, 23]\). In \([33]\) we review the derivation of the finite spectral triple of the Standard Model and discuss an alternative based on the complex algebra \(B_F\) \([33]\). In \([4]\) we describe the most general Dirac operator satisfying the 1st order condition (which is necessary for the “property (M)” in Def. \([5]\), and in \([45]\) two possible grading operators; the Dirac operator
of Chamseddine-Connes \([9, 5, 11]\) is in \([5, 3]\). In \([6]\) we discuss the natural condition for a spectral triple to be “spin”, based on Morita equivalence, and derive some necessary conditions for this to be satisfied; we show that in order to satisfy these condition one has to introduce two additional terms in the Dirac operator of Chamseddine-Connes, one mixing \(e_R\) with \(\bar{\nu}_R\) and one mixing leptons with quarks. The last term is also necessary in order to have an irreducible spectral triple, cf. \([8]\). In \([7]\) we study the problem of orientability for the modified Dirac operator. We conclude in \([9]\) with few comments about irreducibility of the Pati-Salam model.

2 Mathematical set-up

Let \(M\) be a closed oriented Riemannian manifold, \(C(M)\) and \(C^\infty(M)\) the algebras of complex-valued continuous resp. smooth functions, and \(\mathcal{C}(M)\) the algebra of continuous sections of the bundle of (complexified) Clifford algebras: as a \(C(M)\)-module, it is equivalent to the module of continuous sections of the bundle \(\Lambda^\bullet TM \to M\), but with product defined by the Clifford multiplication. The manifold \(M\) is spin if and only if there exists a Morita equivalence \(\mathcal{C}(M)\)-\(\mathcal{C}(M)\) bimodule \(\Sigma\) (see e.g. §1 of \([24]\)). Such a \(\Sigma\) is automatically projective and finitely generated, hence by Serre-Swan theorem \(\Sigma = \Gamma(S)\) is the module of sections of some complex vector bundle \(S \to M\), the spinor bundle in the conventional picture from differential geometry.

Once we have \(S\), we can introduce the Dirac operator \(\mathcal{D}\), a self-adjoint operator on the Hilbert space \(L^2(M, S)\) of square integrable sections of \(S \to M\) \([24, \S 1.4]\). Let \(\pi\) be the representation of \(C(M)\) on \(L^2(M, S)\) by pointwise multiplication and \(c\) the representation of \(\mathcal{C}(M)\) by Clifford multiplication (see e.g. \([16]\) or \([24]\) for the details). The data

\[
(\mathcal{C}^\infty(M), \pi, L^2(M, S), \mathcal{D})
\]

is the prototypical example of commutative spectral triple, and one can indeed prove under some additional assumptions that any commutative spectral triple comes from such a construction \([10, \text{Thm. 1.2}]\). The spectral triple \(\mathcal{C}(M)\) is \(\mathbb{Z}_2\)-graded if \(M\) is even dimensional.

There is an algebraic characterization for spin manifolds as well: a spin manifold \(M\) is spin if and only if there exists a real structure for the spectral triple \(\mathcal{C}(M)\) (whose definition we recall below in the finite-dimensional case).

Let us observe that, for any \(f \in \mathcal{C}^\infty(M)\), \(i[\mathcal{D}, \pi(f)] = c(df)\) is the operator of Clifford multiplication by \(df\) and such operators generate \(\mathcal{C}(M)\). In the even case, the grading \(\gamma\) belongs to \(\mathcal{C}(M)\).

For later use, we recall the definition of spectral triple in the finite-dimensional case, adapted to our purposes.

**Definition 1.** A finite-dimensional spectral triple \((A, \pi, H, D)\) is given by a complex Hilbert space \(H\), a Hermitian operator \(D\) on \(H\), and a real or complex \(\mathcal{C}^\ast\)-algebra with a faithful \(*\)-representation \(\pi : A \to \text{End}_\mathbb{C}(H)\). The spectral triple is even if \(H\) is \(\mathbb{Z}_2\)-graded, \(\pi(A)\) is even and \(D\) is odd; we denote by \(\gamma\) the grading operator. The spectral triple is real if there is an antilinear isometry \(J\) on \(H\) – called the real structure – satisfying

\[
J^2 = \epsilon \text{id}_H, \quad JD = \epsilon' DJ, \quad J\gamma = \epsilon'' \gamma J \quad (\text{only in the even case})
\]
for some \( \epsilon, \epsilon', \epsilon'' \in \{ \pm 1 \} \), plus the 0th order condition (we omit the representation symbol):
\[
[a, JbJ^{-1}] = 0 \quad \forall \ a, b \in A,
\]
and the 1st order condition:
\[
[[D, a], JbJ^{-1}] = 0 \quad \forall \ a, b \in A.
\] (4)

Note that we assume that \( H \) is complex even when \( A \) is real. We don’t lose generality if we assume that the representation is faithful. The values of \( \epsilon, \epsilon', \epsilon'' \) determine the KO-dimension of the spectral triple (according to the table that is, for example, in [24, §3.8]).

**Definition 2.** Let \((A, H, D)\) be a spectral triple and \( \Omega^1(A) := \text{Span}\{a[D, b] : a, b \in A\} \). We call \( \mathcal{C}(A)_o \) the complex *-algebra generated by \( A \) and \( \Omega^1 \). In addition, if \((A, H, D, \gamma)\) is even, we define \( \mathcal{C}(A)_e \) as the complex *-algebra generated by \( \mathcal{C}(A)_o \) and \( \gamma \).

This is similar to Definition 3.19 of [20] (\( \mathcal{C}(A)_o \) in their \( \mathcal{C}_D(A) \) in the even case, while in the odd case they double the Hilbert space to get a \( \mathbb{Z}_2 \)-graded algebra). In the even case, one can either include the grading or not in the definition of Clifford algebra. If the orientability condition recalled below is satisfied, \( \gamma \in \mathcal{C}(A)_o \) and then \( \mathcal{C}(A)_e = \mathcal{C}(A)_o \).

There are two versions of the orientability condition for an even spectral triple (the first stronger than the second). Let \( H_*(A, B) \) be the Hochschild homology of \( A \) with coefficients in a bimodule \( B \), \( HH_*(A) = H_n(A, A) \), identify \( a \in A \) with \( \pi(a) \) and the range of the map \( a \mapsto a^\circ := J\pi(a^*)J^{-1} \) with the opposite algebra \( A^\circ \). Then:

**Definition 3.** An even spectral triple is orientable, with global dimension \( \leq n \) \((n \geq 0)\), if there exists \( [c] \in HH_n(A) \), represented by a cycle
\[
c = \sum_{\text{finite}} a_1^0 \otimes a_1^1 \otimes a_2^1 \otimes \ldots \otimes a_n^1
\]
with \( a_j^i \in A \), such that
\[
\pi_D(c) = \sum_{\text{finite}} a_0^1[D, a_1^1][D, a_2^1] \ldots [D, a_n^1] = \gamma.
\]

**Definition 4.** An even real spectral triple is orientable in a weak sense, with global dimension \( \leq n \) \((n \geq 0)\), if there exists \( [c] \in H_n(A, A \otimes A^\circ) \), represented by a cycle
\[
c = \sum_{\text{finite}} (a_0^1 \otimes b_0^1) \otimes a_1^1 \otimes a_2^1 \otimes \ldots \otimes a_n^1
\]
with \( a_j^i \in A \) and \( b_j^0 \in A^\circ \), such that
\[
\pi_D(c) = \sum_{\text{finite}} a_0^1b_0^1[D, a_1^1][D, a_2^1] \ldots [D, a_n^1] = \gamma.
\]

There is an analogous definition for odd spectral triples, with \( \gamma \) replaced by 1.

Set \( \mathcal{C}(A)_o \) for an odd real spectral triple and \( \mathcal{C}(A)_e \) for an even one. The 0th and 1st order conditions imply that \( H \) is a \( \mathcal{C}(A)_o \)-\( A^\circ \) bimodule \((a, [D, a]) \text{ commute with } b^\circ \) for all \( a \in A, b^\circ \in A^\circ \). Inspired by the example 3 we give then the following definition: (which is basically the “condition 5” of [20]):

**Definition 5.** A real spectral triple \((A, H, D, J)\) has the property \((M)\) if \( H \) is a Morita equivalence \( \mathcal{C}(A)_o \)-\( A^\circ \) bimodule.

**Example 6.** If \( H = A, J(a) = a^* \) and \( D = 0 \), the spectral triple has the property \((M)\).
2.1 The gauge group of a real spectral triple

Let \((A, \pi, H, D, J)\) be a real spectral triple (even or odd). In this section, we assume that \(A\) is a unital and \(\pi\) is a unital representation. Let \(U(A)\) be the group of unitary elements of \(A\). Due to the 0th order condition, the map

\[
\rho : u \mapsto uJ u J^{-1}
\]

is a representation of \(U(A) \to \text{Aut}(H)\), which we call adjoint representation.

The gauge group \(G(A)\) of a real spectral triple is

\[
G(A) := \{ u J u^{-1} : u \in U(A) \}.
\]

**Example 7.** In the spectral triple \((M_n(\mathbb{C}), M_n(\mathbb{C}), 0, J)\) of the Einstein Yang-Mills system the algebra acts by left multiplication, \(J(\alpha) = \alpha^*\) is the Hermitian conjugation, and the gauge group is \(G(A) = PU(n)\). This spectral triple has the property \((M)\), cf. Example 4.

2.2 Spectral triples with a degenerate representation

A necessary and sufficient condition for the map \(\rho\) in (5) to send \(U(A)\) into invertible operators is that \(\rho(1) = 1\) (then automatically, \(\rho(u^{-1}) = \rho(u)^{-1}\)). A sufficient condition is that \(\pi\) is a unital representation, that is \(\pi(1) = 1\). For a spectral triple with a degenerate representation, the unit of \(A\) is not the identity operator on \(H\), and (5) is in general not a representation of the unitary group \(U(A)\). Here we explain how to bypass this problem.

Degenerate representations appear for example when one tries to sum a real spectral triple with one which has no real structure. Let \((A, \pi_0, H_0, 0)\) and \((A, \pi_1, H_1, 0, J_1)\) be two finite-dimensional spectral triples, the latter real with \(J_1^2 = 1\), and both with zero Dirac operator. Then we can define a new real spectral triple \((A, \pi, H, 0, J)\) as follows. We set

\[
H := H_0 \oplus H_0 \oplus H_1,
\]

where \(H_0 = (H_0)^*\) is the dual space. We define

\[
\pi(a)(x, y, z) = (0, \pi_0(a)y, \pi_1(a)z), \quad J(x, y, z) = (y^*, x^*, J_1 z),
\]

for all \(x \in H_0, y \in H_0, z \in H_1\). Note that the representation \(\pi\) is degenerate. If we extend \(\pi_0\) and \(\pi_1\) trivially to \(H\) (as zero on \(H_0 \oplus H_1\) resp. \(H_0 \oplus \bar{H}_0\)), then we can simply write:

\[
\pi = \pi_0 + \pi_1.
\]

Since \(\pi\) is degenerate, the map \(u \mapsto \pi(u) J \pi(u) J\) is not a representation of \(U(A)\) in \(\text{Aut}(H)\) (it doesn’t map \(1 \mapsto 1\), and \(u\) into an invertible operator). A unitary representation \(\rho\) of \(U(A)\) on \(H\) is given by

\[
\rho(u) := \pi_0(u) + J \pi_0(u) J + \pi_1(u) J \pi_1(u) J.
\]

Indeed \(\pi_0(1) = \text{id}_{H_0}, \pi_1(1) = \text{id}_{H_1}\) and \(J \pi_0(1) J = \text{id}_{H_0}\). So \(\rho(1) = 1\). Moreover, \(\pi_0, J \pi_0(\cdot) J, \pi_1\) and \(J \pi_1(\cdot) J\) are mutually commuting, hence \(\rho\) is multiplicative, and from \(\rho(u) \rho(u^*) = \rho(u u^*) = 1\) we deduce that the representation is also unitary.
Basically, we are considering the direct sum of three representations of \( \text{U}(A) \): the fundamental associated to \( \bar{\pi}_0 \) and its dual, and the adjoint representation of \( \pi_1 \).

In \( \S 3.3 \) we exhibit a possible choice of the above data, such that \( \text{U}(A) \supset G_{\text{SM}} \) (strictly) contains the gauge group of the Standard Model (modulo a finite subgroup), and \( \rho|_{G_{\text{SM}}} \) gives the correct representation.

3 From particles to algebras

In this section we give a review of the derivation of the data \((A_F, H_F, J_F)\) from physical considerations, and collect at the end few results about the algebra and its commutant that will be useful in the following sections. In some sense, these data reflect the “topology” of the finite noncommutative manifold describing the internal space of the Standard Model, while the Dirac operator encodes the metric properties. In \( \S 3.3 \) we explain how to get the same gauge group from a spectral triple based on the complex algebra \((2)\) with a degenerate representation. For simplicity, we work with only one generation of leptons/quarks.

3.1 The gauge group of the Standard Model

Let

\[
\tilde{G}_{\text{SM}} := \text{U}(1) \times \text{SU}(2) \times \text{SU}(3)
\]

be the usual gauge group of the Standard Model, let \( H \) be the finite-dimensional Hilbert space representing the internal degrees of freedom of elementary fermions. Let us recall what is the representation of \( \tilde{G}_{\text{SM}} \). We have a decomposition \( H = F \oplus F^* \), with \( F^* \) the dual space of \( F \). The vector space \( F \) (for fermions) has basis \((\nu_L, \epsilon_L, u^c_L, d^c_L)_{c=1,2,3}, \nu_R, \epsilon_R, u^c_R, d^c_R)_{c=1,2,3}\) where \( \nu \) stands for neutrino, \( \epsilon \) for electron, \( u^c \) for up-quark and \( d^c \) for down-quark with color \( c = 1, 2, 3 \), \( L, R \) stands for left-handed resp. right-handed. We will use the label \( \uparrow \) for the first particle in each column (neutrino or quark up) and \( \downarrow \) for the second one (electron or quark down). Left-handed doublets carry the fundamental representation of \( \text{SU}(2) \), while right handed particles are \( \text{SU}(2) \)-invariant; in particular, the \( \uparrow \) particle in each doublet has weak isospin \( I_{3,w} = 1/2 \) and the \( \downarrow \) has weak isospin \( I_{3,w} = -1/2 \). The \( \text{SU}(2) \)-singlets have weak isospin \( I_{3,w} = 0 \). Each one of the colour triplets carry the fundamental representation of \( \text{SU}(3) \), the other particles being \( \text{SU}(3) \)-invariant. Each particle carries a 1-dimentional representation \( \lambda \rightarrow \lambda^{3Y_w} \) of \( \text{U}(1) \), where \( Y_w \in \mathbb{Z}^3/3\mathbb{Z} \) is the weak hypercharge; it is computed from the formula \( Q = I_{3,w} + \frac{1}{2} Y_w \) where \( Q \) is the charge of the particle.

The value of \( 3Y_w \) is given by the following table:

| particle \( 3Y_w \) | \( \nu_L, \epsilon_L \) | \( u^c_L, d^c_L \) | \( \nu_R \) | \( \epsilon_R \) | \( u^c_R \) | \( d^c_R \) |
|---|---|---|---|---|---|---|
| \( 3Y_w \) | \(-3\) | \(1\) | \(0\) | \(-6\) | \(4\) | \(-2\) |

The final representation is actually the direct sum of \( n \) copies of \( H = F \oplus F^* \), where \( n \) is the number of generations (\( n = 3 \) according to our current knowledge). For simplicity, the factor taking into account generations will be neglected.
For the computations, it will be convenient to encode the complex vector space \( F \) of dimension 16 as \( F \simeq M_4(\mathbb{C}) \). Namely we arrange the particles in a \( 4 \times 4 \) matrix as follows:

\[
\begin{bmatrix}

\nu_R & u_R^1 & u_R^2 & u_R^3 \\
\bar{e}_R & d_R^1 & d_R^2 & d_R^3 \\
\nu_L & u_L^1 & u_L^2 & u_L^3 \\
\bar{e}_L & d_L^1 & d_L^2 & d_L^3

\end{bmatrix}
\]

We put in the first column leptons, in the other three the quarks according to the colour. In the rows we put in the order: \( \uparrow R, \downarrow R, \uparrow L, \downarrow L \).

Let \( e_{ij} \) the \( 4 \times 4 \) matrix with 1 in position \((i, j)\) and zero everywhere else. Matrices \( \{e_{ij}\}_{i,j=1}^4 \) form an orthonormal basis of \( M_4(\mathbb{C}) \) for the inner product associated to the trace \( \langle a, b \rangle = \text{Tr}(a^*b) \). With this notation, for example, the state associated to the unit vector \( e_{31} \) represents a left handed neutrino.

In the dual representation \( F^* \), one has:

\[
\begin{bmatrix}

\bar{\nu}_R & \bar{e}_R & \bar{\nu}_L & \bar{e}_L \\
\bar{u}_R & \bar{d}_R & \bar{u}_L & \bar{d}_L \\
\bar{u}_R^2 & \bar{d}_R^2 & \bar{u}_L^2 & \bar{d}_L^2 \\
\bar{u}_R^3 & \bar{d}_R^3 & \bar{u}_L^3 & \bar{d}_L^3

\end{bmatrix}
\]

Elements of \( H \) are then of the form \( a \oplus b \) with \( a, b \in M_4(\mathbb{C}) \).

Endomorphisms of \( F \simeq F^* \simeq M_4(\mathbb{C}) \) are given by \( M_4(\mathbb{C}) \otimes M_4(\mathbb{C}) \), where the first factor acts on \( F = M_4(\mathbb{C}) \) via row-by-column multiplication from the left, and the second via row-by-column multiplication from the right. From the weak hypercharge table we get the following representation \( \pi_{SM} \) of \( \tilde{G}_{SM} \) on \( H \):

\[
\pi_{SM}(\lambda, q, m) = \begin{bmatrix}
\lambda^3 & 0 & 0 & 0 \\
0 & \lambda^3 & 0 & 0 \\
0 & 0 & q & 0
\end{bmatrix} \oplus \begin{bmatrix}
\bar{\lambda}^3 & 0 & 0 & 0 \\
0 & \bar{\lambda}^3 & 0 & 0 \\
0 & 0 & 0 & \lambda^m
\end{bmatrix}
\]

for all \( \lambda \in U(1), q \in SU(2) \) and \( m \in SU(3) \). Here the first summand acts on \( F \) and the second one on \( F^* \).

Computing the kernel of \( \pi_{SM} \) one sees that the relevant group is not exactly \( \tilde{G}_{SM} \) but a quotient by a finite subgroup. Let \( \mathbb{Z}_6 := \{\mu \in \mathbb{C} : \mu^6 = 1\} \) the group of 6-th roots of unity. There is an injective morphism of groups:

\[
\mathbb{Z}_6 \ni \mu \mapsto (\mu, \mu^31_2, \mu^41_3) \in \tilde{G}_{SM}.
\]

One easily checks that the image is exactly the kernel of \( \pi_{SM} \), so that there is an exact sequence:

\[
1 \to \mathbb{Z}_6 = \ker \pi_{SM} \to \tilde{G}_{SM} \to \text{Im} \pi_{SM} \to 1
\]
We want to identify $\text{Im} \pi_{SM} = \tilde{G}_{SM}/\mathbb{Z}_6$. Let $G_{SM} := S(U(2) \times U(3))$ be the group of $SU(5)$ matrices of the form

$$\begin{bmatrix}
2 \times 2 \text{ block} & 0 \\
0 & 3 \times 3 \text{ block}
\end{bmatrix}$$

There is a surjective morphism of groups

$$\tilde{G}_{SM} \ni (\lambda, q, m) \mapsto \begin{bmatrix}
\lambda^3 q \\
\bar{\lambda}^2 \bar{m}
\end{bmatrix} \in G_{SM},$$

where $\bar{m} = (m^*)^t$. An element $(\lambda, q, m)$ is in the kernel of the map $\tilde{G}_{SM} \to G_{SM}$ if and only if $q = \bar{\lambda}^3 1_2$ and $m = \bar{\lambda}^2 1_3$. But $\det(q) = \det(m) = \bar{\lambda}^6$ must be 1, hence $\lambda \in \mathbb{Z}_6$ and

$$\text{Im} \pi_{SM} \simeq G_{SM}.$$ 

The representation can be linearized as follows. Let $J : H \to H$ be the antilinear operator $J(a \oplus b) := b^* \oplus a^*$, transforming a particle into its antiparticle. We can write

$$\pi_{SM}(\lambda, q, m) = \tilde{\pi}(\bar{\lambda}, q, \bar{m}) J \tilde{\pi}(\bar{\lambda}, q, \bar{m}) J^{-1}$$

where $\bar{m} := \bar{\lambda} m \in U(3)$, $\bar{\lambda} := \lambda^3$ and

$$\tilde{\pi}(\bar{\lambda}, q, \bar{m}) = \begin{bmatrix}
\bar{\lambda} & 0 & 0 & 0 \\
0 & \bar{\lambda}^* & 0 & 0 \\
0 & 0 & q & 0 \\
0 & 0 & 0 & \bar{m}
\end{bmatrix} \otimes 1 \oplus \begin{bmatrix}
\bar{\lambda} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \bar{m}
\end{bmatrix} \otimes 1$$

The latter can be now extended in an obvious way, by $\mathbb{R}$-linearity, as a representation of the real algebra $A_F$ in $\mathbb{H}$, where we think of quaternions as $M_2(\mathbb{C})$ matrices of the form

$$\begin{bmatrix}
\alpha & \beta \\
-\bar{\beta} & \bar{\alpha}
\end{bmatrix}, \quad \alpha, \beta \in \mathbb{C},$$

so that with this identification $U(\mathbb{H}) = SU(2)$.

### 3.2 The data $(A_F, A_F^0, H_F, J_F)$

With the identifications as in the previous section the Hilbert space becomes $H_F = M_{8 \times 4}(\mathbb{C})$, with elements:

$$v = \begin{bmatrix} v_1 \\
v_2 \end{bmatrix}, \quad v_1, v_2 \in M_4(\mathbb{C}).$$

and inner product $\langle v, w \rangle = \text{Tr}(v^* w)$. Linear operators on $H_F$ are finite sums $L = \sum_i a_i \otimes b_i$, with $a_i \in M_8(\mathbb{C})$ acting via row-by-column multiplication from the left and $b_i \in M_4(\mathbb{C})$ acting via row-by-column multiplication from the right. One easily checks that the adjoint of $L$ is $L^* = \sum_i a_i^* \otimes b_i^*$, with $a_i^*, b_i^*$ denoting Hermitian conjugation.

The real structure $J_F$ is the operator

$$J_F \begin{bmatrix} v_1 \\
v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\
v_1^* \end{bmatrix}.$$ (9)
We identify $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ with the subalgebra of elements $a \otimes 1 \in \text{End}_\mathbb{C}(H_F)$, with $a$ of the form:

\[
a = \begin{bmatrix}
\lambda & 0 & 0 \\
0 & \bar{\lambda} & 0 \\
0 & 0 & q
\end{bmatrix}
\begin{bmatrix}
\lambda & 0 & 0 \\
0 & \bar{\lambda} & 0 \\
0 & 0 & m
\end{bmatrix},
\]

(10)

with $\lambda \in \mathbb{C}$, $q \in \mathbb{H}$ and $m \in M_3(\mathbb{C})$ (with zeros on the off-diagonal blocks).

We denote by $A_F' = J_F A_F J_F$ the subalgebra of elements $\text{End}_\mathbb{C}(H_F)$ of the form:

\[
a' = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \otimes \begin{bmatrix}
\lambda & 0 & 0 \\
0 & \bar{\lambda} & 0 \\
0 & 0 & m
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} \otimes \begin{bmatrix}
\lambda & 0 & 0 \\
0 & \bar{\lambda} & 0 \\
0 & 0 & q
\end{bmatrix}.
\]

(On the first factor of each tensor $0 \in M_4(\mathbb{C})$ are the zero and the identity matrix.)

If $A \subset \text{End}_\mathbb{C}(H_F)$ is a real $*$-subalgebra, we denote by $A_C$ the complex linear span of the elements in $A$; note that $A$ and $A_C$ have the same commutant in $\text{End}_\mathbb{C}(H_F)$. The map $a \mapsto a' = J_F \bar{a} J_F$ (here $\bar{a} = (a^*)^t$) gives two isomorphisms $A_F \rightarrow A_F'$ and $(A_F)_{\mathbb{C}} \rightarrow (A_F')_{\mathbb{C}}$.

**Lemma 8.** The commutant of the algebra of elements (10) in $M_8(\mathbb{C})$ is the algebra $C_F$ with elements

\[
\begin{bmatrix}
q_{11} & \alpha & q_{12} \\
\beta_{12} & \beta_{12} & \beta_{22} \\
q_{21} & q_{22} & \delta_{13}
\end{bmatrix},
\]

(11)

where the $\beta$-block is $2 \times 2$, the $\delta$-block is $3 \times 3$, and all other framed blocks are $1 \times 1$ ($\alpha, \beta, \delta \in \mathbb{C}$, $q = (q_{ij}) \in M_2(\mathbb{C})$). All other blocks are zero (zeroes are omitted).

The commutant of $A_F$ in $\text{End}_\mathbb{C}(H)$ is $A_F' = C_F \otimes M_4(\mathbb{C})$.

**Proof.** By direct computation. ■

Note that $A_F' \simeq M_4(\mathbb{C})^{\oplus 3} \oplus M_8(\mathbb{C})$. The map $x \mapsto J_F \bar{x} J_F$ is an isomorphism between $A_F'$ and $(A_F')^\prime$. From this, we get the following result.
Lemma 9. The commutant \((A_F')'\) of \(A_F\) has elements
\[
a \otimes e_{11} + \begin{bmatrix} b & c \end{bmatrix} \otimes e_{22} + \begin{bmatrix} b & d \end{bmatrix} \otimes (e_{33} + e_{44})
\]
with \(a \in M_8(\mathbb{C})\), \(b, c, d \in M_4(\mathbb{C})\).

Lemma 10. \(A_F' \cap (A_F')' \simeq \mathbb{C}^{10} \oplus M_2(\mathbb{C})\).

Proof. The general element in Lemma 9 belongs to \(A_F'\) if \(a \oplus c\) and \(b \oplus d\) belong to \(C_F\). So \(a\) is any element of \(C_F\), which gives a summand \(\mathbb{C}^3 \oplus M_2(\mathbb{C})\); \(b\) is diagonal with two entries equal, and this gives another factor \(\mathbb{C}^3\); \(c\) and \(d\) are diagonal with the last three entries equal, which gives two additional factors \(\mathbb{C}^2\).

Remark 11. It follows from Grassmann formula that \(A_F' + (A_F')'\) has complex dimension 210. The (real) subset of selfadjoint elements has real dimension 210.

We stress that \(A_F' + (A_F')'\) doesn’t denote the algebra generated by the two commutants, but only the vector space sum of \(A_F'\) and \((A_F')'\).

3.3 The data \((B_F, \bar{H}_0, H_1)\)

In this section we explain how to get the same gauge group from a spectral triple based on the complex algebra (2) with a degenerate representation.

Let us put particles into a row vector and a 3 \times 4 matrix as follows
\[
\begin{bmatrix} e_R & d_R^1 & d_R^2 & d_R^3 \end{bmatrix}, \quad \begin{bmatrix} \nu_R & u^1_R & u^2_R & u^3_R \\ \nu_L & u^1_L & u^2_L & u^3_L \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix}
\]
Thus a particle is represented by a vector in \(\mathbb{C}^3 \oplus M_{3 \times 4}(\mathbb{C})\), with inner product given on each summand by \(\langle v, w \rangle = \text{Tr}(v^*w)\). Antiparticles belong to the dual space.

The Hilbert space is then \(H = H_0 \oplus \bar{H}_0 \oplus H_1\), where elements of \(H_0 \simeq \mathbb{C}^4\) are row vectors, elements of \(\bar{H}_0 \simeq \mathbb{C}^4\) are column vectors, and \(H_1\) has elements
\[
\begin{bmatrix} a \\ b \end{bmatrix}, \quad a \in M_{3 \times 4}(\mathbb{C}), b \in M_{4 \times 3}(\mathbb{C})
\]
The real structure \(J\) is the antilinear operator:
\[
J(v \oplus w \oplus \begin{bmatrix} a \\ b \end{bmatrix}) = w^* \oplus v^* \oplus \begin{bmatrix} b^* \\ a^* \end{bmatrix}
\]
We define two unital \(\ast\)-representations \(\bar{\pi}_0 : B_F \to \text{End}_C(\bar{H}_0)\) and \(\pi_1 : B_F \to \text{End}_C(H_1)\) of the algebra \(B_F = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_8(\mathbb{C})\) in (2) as follows
\[
\bar{\pi}_0(\lambda, q, m) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & m \\ 0 & 0 & 0 \end{bmatrix}, \quad \pi_1(\lambda, q, m) = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]
both acting via row-by-column multiplication from the left. Here \( \lambda \in \mathbb{C} \), \( q \in M_2(\mathbb{C}) \) and \( m \in M_3(\mathbb{C}) \) are \( 2 \times 2 \) and \( 3 \times 3 \) blocks, and the off-diagonal \( 3 \times 4 \) and \( 4 \times 3 \) blocks are zero. An (injective) representation \( \rho \) of \( U(A) = U(1) \times U(2) \times U(3) \) is given by (9).

One can check that \( \rho \), composed with the map 
\[
\tilde{G}_{SM} = U(1) \times SU(2) \times SU(3) \xrightarrow{\tilde{\pi}} U(A) \; , \quad (\lambda, q, m) \mapsto (\lambda^6, \lambda^3 q, \lambda^2 m) \, ,
\]
gives the correct representation of \( \tilde{G}_{SM} \) (in particular, each particle has the correct weak hypercharge). The kernel \( \varphi \) is given again by the elements in (7), so that the range of \( \varphi \) is \( G_{SM} \cong \tilde{G}_{SM} / \mathbb{Z}_6 \). The map 
\[
U(A) \supset G_{SM} \ni (\lambda, q, m) \mapsto \left[ \begin{array}{c} a \\ b \end{array} \right] \in S(U(2) \times U(3))
\]
is an isomorphism. We then recover \( G_{SM} \) as the subgroup of \( U(A) \) satisfying the unimodularity condition 
\[
det \pi_0(u) = \det \pi_1(u) = 1 .
\]
The relation with \( A_F \) is as follows. Let \( \{a_i\}_{i=1}^4 \) be the rows of \( a \in M_4(\mathbb{C}) \) and \( \{b_j\}_{j=1}^4 \) the columns of \( b \in M_4(\mathbb{C}) \). With the isometry 
\[
H \ni a_2 \oplus b_2 \oplus \left[ \begin{array}{cccc} a_1 \\ a_3 \\ a_4 \\ b_1 \\ b_3 \\ b_4 \end{array} \right] \mapsto \left[ \begin{array}{c} a \\ b \end{array} \right] \in M_{8 \times 4}(\mathbb{C})
\]
we transform \( J \) in (13) into the real structure \( J_F \) in (9), and \( \pi \) into the representation (denoted by the same symbols):
\[
\pi(\lambda, q, m) = \left[ \begin{array}{cccc} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & m \end{array} \right] \otimes 1 , \quad (14)
\]
where \( \lambda \in \mathbb{C} \), \( q \in M_2(\mathbb{C}) \) and \( m \in M_3(\mathbb{C}) \).

Note that the only difference between the matrix in (14) and the one in (10) is the zero in position \((2,2)\) replacing \( \lambda \). More precisely, the algebra \( (A_F)_\mathbb{C} \) is the minimal unitalization of \( \pi(B_F) \) in \( \text{End}_\mathbb{C}(H_F) \), and \( (A_F)_{\mathbb{C}^2} \) is the unitalization of \( A^\sigma := J_F \pi(B_F) J_F \).

Adding the identity doesn’t change the commutant, nor \( \Omega^1 \). Thus, the results in the next section which we state for the algebra \( A_F \) are valid for \( B_F \) as well.

4 The 1st order condition

In all the derivations of the finite spectral triple for the Standard Model, one assumes that the spectral triple is even with grading given by the chirality operator (cf. §5.1); this gives
KO-dimension 6. In principle, there are several choices to get an even spectral triple, even without changing the Dirac operator used in \cite{3,9,5,11} or the KO-dimension. We will present a possible alternative choice in \S5.2.

One could actually wonder whether the spectral triple should be even or not: if its KO-dimension doesn’t have to agree with the the metric dimension, we could argue that the parity of the spectral triple doesn’t have to agree with the metric dimension as well. From a physical point of view, the grading seems to play no role in the theory. Moreover, with an odd spectral triple it is possible to satisfy the orientability condition (cf. \S7).

In this section, firstly we describe the most general Dirac operator satisfying the 1st order condition. Then we impose the additional requirement $J_F D_F = D_F J_F$, with the plus sign on the right hand side dictated by the physical content of the theory (the mass terms in the spectral action come from elements commuting with $J_F$). We will show that for any $D_F$ satisfying the 1st order condition, there is one commuting with $J_F$ which gives the same Clifford algebra (so, the condition $J_F D_F = D_F J_F$ does not create any particular problem).

In the next sections, we discuss the issue of the grading, and the property (M) and orientability, both in the presence of grading and without it.

In the rest of the paper, we employ $A_F, A_F^g, H_F, J_F$ as defined in \S3.2 but the same results are valid for the algebra $B_F$ and the representation discussed in \S3.3.

**Proposition 12.** $D_F \in \text{End}_\mathbb{C}(H_F)$ satisfies the 1st order condition (1) if and only if it is of the form

$$D_F = D_0 + D_1$$

where $D_0 \in (A_F^g)'$ and $D_1 \in A_F'$.

**Proof.** Clearly if $D_F = D_0 + D_1$ with $D_0 \in (A_F^g)'$ and $D_1 \in A_F'$, the 1st order condition is satisfied. Indeed for all $a \in A_F$, since $A_F \subset (A_F^g)'$ due to the 0th order condition, the commutator $[D_F, a] = [D_0, a]$ still belongs to $(A_F^g)'$. We now to prove the “only if” part.

Let $D_F = \sum_{ij} D_{ij} \otimes e_{ij}$. We now assume the commutator $[D_F, .]$ maps $A_F$ into $(A_F^g)'$, and prove that $D_F$ must belong to $A_F' + (A_F^g)'$.

Elements of $A_F$ have the form $a \otimes 1$ with $a$ as in \cite{10}, and elements of $(A_F^g)'$ are as in \cite{12}. For $i = 2, 3, 4$ write

$$D_{ij} = \begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix}$$

with $\alpha_i, \beta_i, \gamma_i, \delta_i \in M_4(\mathbb{C})$. Clearly $[D_F, a \otimes 1] = \sum_{ij} [D_{ij}, a] \otimes e_{ij}$ belongs to $(A_F^g)'$ iff:

$$D_{ij} \in C_F \text{ for all } i \neq j; \quad \begin{bmatrix} 0 & \beta_i \\ \gamma_i & 0 \end{bmatrix} \in C_F \text{ for all } i = 2, 3, 4; \quad \begin{bmatrix} 0 & 0 \\ 0 & \delta_i - \delta_j \end{bmatrix} \in C_F \forall i, j = 3, 4.$$

Since $C_F \otimes M_4(\mathbb{C}) = A_F'$ and $D_{11} \otimes e_{11} \in (A_F^g)'$ for any $D_{11} \in M_8(\mathbb{C})$, this proves $D_F \in A_F' + (A_F^g)'$.

It follows from Remark \cite{11} that in $D_F$ up to now we have 210 free parameters. We now impose the commutation with $J_F$. 

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**Proposition 13.** An operator $D_F = D_F^*$ as in Prop. 12 commutes with $J_F$ if and only if it is of the form:

$$D_F = D_0 + J_F D_0 J_F$$

with $D_0 = D_0^* \in (A_F')'$. 

**Proof.** $x \mapsto J_F \bar{x} J_F$ gives a bijection $A_F' \to (A_F')'$. The condition $J_F D_F J_F = D_F$ gives

$$(J_F D_0 J_F - D_1) + (J_F D_1 J_F - D_0) = 0.$$ 

Since the first term is in $A_F'$ and the second in $(A_F')'$, the sum is zero if and only if both $J_F D_0 J_F - D_1$ and $J_F D_1 J_F - D_0$ belong to $A_F' \cap (A_F')'$. Called $D' = D_1 - J_F D_0 J_F$, one has the decomposition

$$D_F = D_0 + J_F D_0 J_F + D'.$$

From $J_F D_F J_F - D_F = J_F D' J_F - D'$ one deduces that $J_F$ and $D'$ must commute. So $D_F = (D_0 + D'/2) + J_F (D_0 + D'/2) J_F$ and we get the decomposition (16), after renaming $D_0 + D'/2 \to D_0$.

Decompose $D_0 = S + iT$ with $S$ and $T$ selfadjoint. Since $J_F$ is antilinear and $J_F = J_F^*$:

$$D_F - D_F^* = 2i(T - J_F T J_F)$$

which must be zero. But this implies

$$D_F = S + J_F SJ_F + i(T - J_F T J_F) = S + J_F SJ_F .$$

Renaming $S \leadsto D_0$ (which now is selfadjoint) we conclude the proof. 

In (16) we now have no more than 112 free real parameters.

**Remark 14.** Note that the $D_1$ term does not contribute to $\mathcal{C}(A_F)$ (it commutes with $A_F$). Then, for any Dirac operator as in Prop. 12 we can find one commuting with $J_F$ (replacing $D_1$ by $J_F D_0 J_F$) without changing the Clifford algebra $\mathcal{C}(A_F)$. In particular, the property (M) puts constrains only on $D_0$.

It is useful to reformulate Prop. 12 and Prop. 13 as follows. Let

$$D_R = (\Upsilon_R e_{51} + \bar{\Upsilon}_R e_{15}) \otimes e_{11} ,$$

with $\Upsilon_R \in \mathbb{C}$. Note that $D_R \in A_F' \cap (A_F')'$ and $J_F D_R = D_R J_F$.

**Proposition 15.** The most general $D_F = D_F^*$ satisfying the 1st order condition is

$$D_F = D_0 + D_1 + D_R$$

where $D_0 = D_0^* \in (A_F')'$ and $D_1 = D_1^* \in A_F'$ have null entry in direction of $e_{15} \otimes e_{11}$ and $e_{51} \otimes e_{11}$, and $D_1 = J_F D_0 J_F$ if $D_F$ and $J_F$ commute.

In this way we isolated all the terms which do not contribute to $\Omega^1$. For any $a \in A_F$ and $D_F$ as in (16), $[D_F, a] = [D_0, a]$. 

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5 The grading operator

With $D_F$ as in Prop. 13 we get KO-dimension 0, 6 or 7. Note that in the even case, replacing $J_F$ by $J_F\gamma_F$ and forgetting about the grading, we transform KO-dimension 0 into KO-dimension 1 and 6 into 5. If we replace $D_F$ by $D_F\gamma_F$ instead, we transform KO-dimension 6 into 1. If we replace both $J_F$ by $J_F\gamma_F$ and $D_F$ by $D_F\gamma_F$, we transform KO-dimension 6 into 3 or into 2 (if we keep the grading). Thus with $D_F$ as in Prop. 13 in principle we can get any KO-dimension except 4.

Here we discuss some particular classes of operators like in Prop. 13, which we get by introducing a grading operator anticommuting with $J_F$ (which means KO-dimension 6, or 1, 2, 3, 5, 7 with the replacements as above and/or forgetting the grading: so every KO-dimension except 0 and 4).

Lemma 16. Let $\gamma_F$ be a grading operator. Any odd Dirac operator satisfying the 1st order condition can be written in the form $D_F = D_0 + D_1 + eD_R$ as in Prop. 13, with both $D_0$ and $D_1$ odd operators and $e = 0$ or 1 depending on the parity of $D_R$.

Proof. From Prop. 12 we can write $D_F = D_0 + D_1 + T_0 + T_1$ where $D_0, T_0 \in (A_F^0)'$, $D_1, T_1 \in A_F$, $D_0, D_1$ are odd and $T_0, T_1$ are even. From

$$\gamma_F D_F \gamma_F + D_F = 2(T_0 + T_1) = 0$$

we deduce $T_0 + T_1 = 0$, so that $D_F = D_0 + D_1$ with both $D_0$ and $D_1$ odd operators. ■

Lemma 17. Let $\gamma_F$ be a grading operator either commuting or anticommuting with $J_F$. Any odd Dirac operator satisfying the 1st order condition and commuting with $J_F$ can be written in the form $D_F = D_0 + J_F D_0 J_F + eD_R$ as in Prop. 13, with $D_0$ an odd operator and $e = 0$ or 1 depending on the parity of $D_R$.

Proof. It follows from Lemma 16. Since $D_1 = J_F D_0 J_F$, the condition $\gamma_F D_0 \gamma_F = -D_0$ implies $\gamma_F D_1 \gamma_F = -D_1$. ■

We now study the form of $D_0$ for Dirac operators of the type described by Lemma 16 or 17 for some natural choices of the grading operator (we don’t care about $D_1$).

5.1 The standard grading

The grading in [3, 9, 5, 11] (the chirality operator) is:

$$\gamma_F = \begin{bmatrix} 1_2 \\ -1_2 \\ 0_4 \end{bmatrix} \otimes 1 + \begin{bmatrix} 0_4 \\ -1_4 \end{bmatrix} \otimes \begin{bmatrix} 1_2 \\ -1_2 \end{bmatrix}.$$ (17)

It follows from Lemma 9 that any $D_0$ anticommuting with $\gamma_F$ has the form:

$$D_0 = \begin{bmatrix} * & * & * & + & + & + \\ * & * & * & + & + & + \\ * & * & + & + & + & + \\ + & + & + & + & + & + \\ + & + & + & + & + & + \\ + & + & + & + & + & + \\ \end{bmatrix} \otimes e_{11} + \begin{bmatrix} * & + \\ * & * \\ + & * \\ + & + \\ + & + \\ + & + \\ \end{bmatrix} \otimes (1 - e_{11}),$$

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where the asterisks indicate the only positions where one can have non-zero matrix entries. The blue entries (∗) are the ones not present with the non-standard grading of §5.2.

5.2 A non-standard grading

Let

$$\gamma_F = \begin{bmatrix} 1 & -2 & 0_4 \\ -2 & 1 & -4 \\ 0_4 & -4 & 1_3 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix},$$

(18)

This operator assigns opposite parity to chiral leptons and quarks (left resp. right handed leptons have the same parity of right resp. left handed quarks).

Again from Lemma 9, any $D_0$ anticommuting with (18) has the form:

$$D_0 = \begin{bmatrix} * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \otimes e_{11} + \begin{bmatrix} * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \otimes (1 - e_{11})$$

The red entries (∗) are the ones that are not allowed by the standard grading (17).

5.3 Chamseddine-Connes’s Dirac operator

Let

$$D_0 = \begin{bmatrix} \bar{\Upsilon}_\nu & \bar{\Upsilon}_e & \bar{\Omega} \\ \Upsilon_\nu & \Upsilon_e & \Omega \\ \bar{\Omega} & \Omega & \Delta \\ \Delta & \Delta \end{bmatrix} \otimes e_{11} + \begin{bmatrix} \bar{\Upsilon}_u & \bar{\Upsilon}_d \\ \Upsilon_u & \Upsilon_d \\ \Delta & \Delta \end{bmatrix} \otimes (1 - e_{11}),$$

where all $\Upsilon$’s and $\Omega$ are complex numbers and $\Delta \in \mathbb{R}$. The Dirac operator of Chamseddine-Connes [3, 9, 5, 11] is

$$D_F = D_0 + J_F D_0 J_F + D_R$$

with $D_R$ given by (15). $D_0$ as above, and $\Omega = \Delta = 0$. It is compatible with both gradings of previous sections.
6 The property (M)

Suppose $H$ is a finite-dimensional complex Hilbert space and $A, B$ two (real or complex) unital $C^*$-subalgebras of $\text{End}_c(H)$, that commute one with the other. Let $Z(A)$ be the center of $A$ and $Z(B)$ be the center of $B$. Note that

$$A \cap B \subset Z(A) \cap Z(B) \subset A' \cap B'$$

(19) and that $Z(A) = Z(A') = A \cap A'$, and similarly for $B$.

Recall that $H$ is a Morita equivalence $A$-$B^\text{op}$-bimodule iff $A = B'$, which is equivalent to the condition $A' = B$ (by von Neumann Bicommutant Theorem: $A'' = A$ and $B'' = B$ in the finite-dimensional case).

Lemma 18. If $H$ is a Morita equivalence $A$-$B^\text{op}$-bimodule, then the inclusions (19) are equalities.

Proof. It follows trivially from $Z(A) = A \cap A'$ and $A' = B$, and similar for $Z(B)$. ■

Proposition 19.

i) If $D_F$ is any operator of the type described in §5.1, the property (M) is not satisfied.

ii) Let $D_F$ be as in §5.2. If the property (M) holds, then

- each summand in $D_0$ must have at least one red coefficient (⊗) different from zero;
- in each of the first two summands, in both the 1st and 2nd row there must be at least one non-zero element;
- in the first summand: at least one element in the 5th row must be non-zero and at least one element in the upper-right block must be non-zero.

Remark. Let $(D_F, \gamma_F)$ be as in §5.1 or as in §5.2. Then (i) and (ii) holds both for $(A_F, H_F, D_F, J_F)$ thought as an odd spectral triple, or for $(A_F, H_F, D_F, \gamma_F, J_F)$ thought as an even spectral triple (so, regardless of the inclusion of $\gamma_F$ in the definition of $\mathcal{C}(A_F)$).

Proof. We apply Lemma 18 to $A = \mathcal{C}(A_F)_0$, $B = (A_F^0)\cap$ and $H = H_F$. Let $D_0$ be as in §5.1 or §5.2. Note that $A$ is generated by $A_F$ and $[D_0, A_F]$. Moreover, due to the 1st order condition, $A$ and $B$ are mutually commuting.

Any operator $X \in A_F' \cap (A_F')'$ commuting with $D_0$ belongs to $A' \cap B'$ (since it also commute with $[D_0, A_F]$). If we can exhibit such an $X$ and prove that $X \notin Z(B)$, then the Morita condition is not satisfied. Note that $Z(B)$ has elements:

$$\alpha^0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & \beta' \\ 0 & 0 & \beta \end{bmatrix},$$

with $\lambda, \lambda', \alpha, \beta \in \mathbb{C}$. For $D_0$ as in §5.1 the operator $X = e_{55} \otimes (1 - e_{11})$ does the job:

1) it commutes with $D_0$,

2) it belongs to $A_F' = C_F \otimes M_4(\mathbb{C})$ ($e_{55} \in C_F$: take $q_{22} = 1$ and all other coefficients zero in (11)),

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3) it belongs to $\left(A'_F\right)'$ (cf. Lemma 9),
4) and it does not belong to $Z(B)$.

Let now $D_0$ be as in §5.2. Concerning the first summand:

- if all the red terms ($\star$) are zero, then $X = (e_{66} + e_{77} + e_{88}) \otimes e_{11}$ satisfies the conditions (1-4) above;
- if all the elements in the 1st resp. 2nd row are zero (and then also in 1st resp. 2nd column, by hermiticity), then $X = e_{11} \otimes e_{11}$ resp. $X = e_{22} \otimes e_{11}$ satisfies (1-4);
- if all the elements in the 5th row are zero, similarly by hermiticity $X = e_{55} \otimes e_{11}$ satisfies the conditions (1-4) above;
- if all the elements in the upper-right blocks are zero, then $X = (e_{11} + e_{22} + e_{33} + e_{44}) \otimes e_{11}$ satisfies the conditions (1-4) above.

Concerning the second summand:

- if all the elements in the 1st resp. 2nd row are zero (and then also in 1st resp. 2nd column, by hermiticity), then $X = e_{11} \otimes (1 - e_{11})$ resp. $X = e_{22} \otimes (1 - e_{11})$ satisfies the conditions (1-4) above;

Concerning the third resp. fourth summands:

- if all the elements in the 1st row are zero, similarly by hermiticity $X = e_{11} \otimes (1 - e_{11})$ satisfies the conditions (1-4) above;

In all the above cases, $X = x \otimes y$ is the tensor product of two diagonal operators $x, y$. Hence it commutes with both the gradings in §6.1 and §6.2 and using $\mathcal{C}^{\ell}(A_F)_0$ instead of $\mathcal{C}^{\ell}(A_F)_0$ doesn’t change the result.

**Corollary 20.** Let $D_F$ be as in §5.2. If $\Delta = 0$ or $\Omega = 0$, the property $(M)$ is not satisfied.

The operator $D_F$ in §5.3, with $\Omega \neq 0$ and $\Delta \neq 0$, represents a minimal modification of the Dirac operator of $[3, 9, 5, 11]$ which satisfies all the conditions in Prop. 19. We will now show that for such an operator, the Morita condition is satisfied.

### 6.1 Morita with a grading

This section is devoted to prove the following theorem.

**Theorem 21.** Let $\gamma_F$ be as in §5.2, $D_F$ as in §5.3 with all coefficients different from zero, and assume that at least one of the following conditions holds:

1. $\Upsilon_v \neq \pm \Upsilon_u$
2. $\Upsilon_e \neq \pm \Upsilon_d$

Then:

i) the odd spectral triple $(A_F, H_F, D_F, J_F)$ doesn’t have the property $(M)$;

ii) the even spectral triple $(A_F, H_F, D_F, \gamma_F, J_F)$ has the property $(M)$;

iii) $\gamma_F$ does not belong to the algebra generated by $A_F$ and $\Omega^1$.

We need a preliminary lemma. From now on, we assume that the hypothesis of Thm. 21 are satisfied.
Lemma 22. The $A_F$-bimodule $\Omega^1$ is generated by the four elements:

$$\omega_\nu = e_{31} \otimes (Y_\nu e_{11} + Ye_{1}(1 - e_{11})),$$
$$\omega_\varepsilon = e_{42} \otimes (Y_\varepsilon e_{11} + Ye_{d}(1 - e_{11})),$$
$$\xi = e_{52} \otimes e_{11},$$
$$\eta = e_{56} \otimes 1,$$

and their adjoints.

Proof. A linear basis of $(A_F)_C$ is given by the elements:

$$X_{ij} := e_{ij} \otimes 1 \quad \text{with} \quad i, j = 3, 4,$$
$$Z_{kl} := e_{kl} \otimes 1 \quad \text{with} \quad k, l = 6, 7, 8,$$
$$Y := e_{22} \otimes 1,$$
$$T := (e_{11} + e_{55}) \otimes 1.$$

For any projection $p^2 = p = p^*$, the commutator $[D_F, p] = [D_F, p^2] = p[D_F, p] + [D_F, p]p$ is a linear combination of $p[D_F, p]$ and its adjoint $-[D_F, p]p$. Hence $X_{33}[D_F, X_{33}]$ and $X_{44}[D_F, X_{44}]$ can be taken as generators, instead of $[D_F, X_{33}]$ and $[D_F, X_{44}]$. An explicit computation gives:

$$-X_{33}[D_F, X_{33}] = \omega_\nu,$$
$$-X_{44}[D_F, X_{44}] = \omega_\varepsilon.$$

Note that $[D_F, X_{33}]$ is also the adjoint of $[D_F, X_{43}]$, and

$$[D_F, X_{43}] = (X_{34}\xi - X_{43}\xi)$$

is still generated by $\omega_\nu, \omega_\varepsilon$ and adjoints. Next

$$[D_F, Y]Y = \omega_\varepsilon + \Omega_\xi,$$
$$[D_F, Z_{66}]Z_{66} = \Delta_\eta.$$

Since $\Omega, \Delta \neq 0$, this proves that $\xi, \eta \in \Omega^1$.

Furthermore $[D_F, Z_{6k}] = \Delta_\eta Z_{6k}$ and $[D_F, Z_{k6}] = -[D_F, Z_{6k}]^*$ are combinations of $\eta$ and $\eta^*$ for all $k = 7, 8$, and $[D_F, Z_{jk}] = 0$ if $j, k = 7, 8$. Finally

$$-T[D_F, T] = \omega_\varepsilon^* + \Omega_\xi^* + \Delta_\eta,$$

proving that the elements $\omega_\nu, \omega_\varepsilon, \xi, \eta$ and their adjoints are a generating family for $\Omega^1$. \qed

Proof of Theorem 22. We now prove that: (i) $\mathcal{C}(A_F)^\gamma \supseteq (A_F^o)_C$ (it is strictly greater), i.e. the property (M) is not satisfied. (ii) $\mathcal{C}(A_F)^\gamma = (A_F^o)_C$, i.e. the property (M) is satisfied. As a corollary, $\mathcal{C}(A_F)^\gamma \neq \mathcal{C}(A_F)^\gamma$, so $\gamma_F$ does not belong to the algebra generated by $A_F$ and $\Omega^1$.

$\mathcal{C}(A_F)^\gamma$ is given by the set of elements in Lemma 8 that commute with the generators in Lemma 22. A tensor $\sum x_{ij} \otimes e_{ij}$, with each $x_{ij}$ as in (11), commutes with $\eta$ and $\eta^*$ iff $q_{12} = q_{21} = 0$ and $q_{22} = \delta$. Hence, the most general $\phi \in A_F^\gamma$ commuting with $\eta, \eta^*$ is:

$$\phi = e_{11} \otimes a + e_{22} \otimes b + (e_{33} + e_{44}) \otimes c + \left(\sum_{i=5}^{8} e_{ii}\right) \otimes d$$

with $a, b, c, d \in M_4(C)$ arbitrary matrices. Its commutator with $\xi$ and $\xi^*$ vanishes iff

$$de_{11} = e_{11}b,$$
$$e_{11}d = be_{11}.$$  \hspace{1cm} (20)

Its commutator with $\omega_\nu, \omega_\varepsilon$ and their adjoints vanishes iff:

$$Ea = cE, \quad aE = Ec, \quad Fb = cF, \quad bF = Fc.$$  \hspace{1cm} (21)
where
\[
E := \begin{bmatrix}
\Upsilon_\nu & 0 & 0 & 0 \\
0 & 0 & \Upsilon_u1_3 & 0 \\
0 & 0 & 0 & \Upsilon_d1_3 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad F := \begin{bmatrix}
\Upsilon_e & 0 & 0 & 0 \\
0 & 0 & \Upsilon_d1_3 & 0 \\
0 & 0 & 0 & \Upsilon_d1_3 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
are invertible by hypothesis. It follows from (21) that \( c \) commutes with both \( E^2 \) and \( F^2 \). If the hypothesis of Theorem 21 are satisfied, at least one of the matrices \( E^2, F^2 \) is not proportional to the identity. Its commutation with \( c \) implies that
\[
c := \begin{bmatrix}
\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & m \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
for some \( \lambda \in \mathbb{C} \) and \( m \in M_3(\mathbb{C}) \). But then \( c \) commutes with \( E \) and \( F \) as well, and it follows from (21) that \( a = E^{-1}cE = c \) and \( b = F^{-1}cF = c \). Now, \( b \) commutes with \( e_{11} \) as well, and from (20) we get
\[
d := \begin{bmatrix}
\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & m' \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
with the same \( \lambda \) as before, and with \( m' \in M_3(\mathbb{C}) \). Thus, \( \mathcal{C}(A_F)'_o \) has elements
\[
\phi = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix}
\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & m \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix}
\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & m' \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad (22)
\]
with \( \lambda \in \mathbb{C} \) and \( m, m' \in M_3(\mathbb{C}) \), and is strictly greater than \( (A_F^p)'_o \).

Imposing the extra condition \([\phi, \gamma_F] = 0\), we reduce one \( M_3(\mathbb{C}) \) to \( \mathbb{C} \oplus M_2(\mathbb{C}) \). Indeed \([\phi, \gamma_F] = 0 \) iff \( d \) commutes with the matrix
\[
\begin{bmatrix} 1 & \lambda & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix}
12 & & & \\
& -12 & & \\
& & & \\
& & &
\end{bmatrix},
\]
i.e. \( m' \) belongs to \( \mathbb{C} \oplus M_2(\mathbb{C}) \subset M_3(\mathbb{C}) \). This proves that \( \mathcal{C}(A_F)'_o = (A_F^p)'_o \).

### 6.2 Morita without the grading

Let \( D_0 \) be as in §5.3 and \( \tilde{D}_F = \tilde{D}_0 + J_F\tilde{D}_0J_F + D_R \), with
\[
\tilde{D}_0 := D_0 + \Gamma(e_{57} + e_{75}) \otimes e_{22}.
\]
Notice that this is still of the type described in §5.2. Here we have three additional parameters \( \Omega \in \mathbb{C} \) and \( \Delta, \Gamma \in \mathbb{R} \) with respect to §5 [5] [11].

With \( \gamma_F \) as in §5.2 we get a even real spectral triple \( (A_F, H_F, \tilde{D}_F, \gamma_F, J_F) \) of KO-dimension 6. This has the advantage that \( \gamma_F \in \mathcal{C}(A_F)'_o \).
Theorem 23. Let $\Upsilon_{\nu}, \Upsilon_{e}, \Upsilon_{u}, \Upsilon_{d}, \Omega, \Delta, \Gamma$ be all different from zero, and at least one of the following two conditions satisfied:

1. $\Upsilon_{\nu} \neq \pm \Upsilon_{u}$,
2. $\Upsilon_{e} \neq \pm \Upsilon_{d}$.

Then $(A_F, H_F, \tilde{D}_F, \gamma_F, J_F)$ has the property (M). Moreover, $\gamma_F \in \mathcal{C}(A_F)_{or}$ and so $\mathcal{C}(A_F)_{or} = \mathcal{C}(A_F)_{o}$.

Lemma 24. If $\Omega, \Delta, \Gamma \neq 0$, the $A_F$-bimodule $\Omega^1$ is generated by the elements in Lemma 22 plus the element

$$\zeta = e_{57} \otimes e_{22}$$

and its adjoint.

Proof. Repeating the proof of Lemma 22, the only change is:

$$[D_F, Z_{77}]Z_{77} = \Gamma \zeta, \quad [D_F, Z_{78}] = \Gamma \zeta Z_{78}, \quad -T[D_F, T] = \omega^*_\nu + \Omega \xi + \Delta \eta + \Gamma \zeta,$$

and $[D_F, Z_{87}] = -[D_F, Z_{78}]^*$. ■

Proof of Theorem 23. $\mathcal{C}(A_F)'_o$ now is the set of elements $\phi$ in (22) which in addition commute with $\zeta$. But $[\phi, \zeta] = 0$ iff $\eta' \in \mathbb{C} \oplus M_2(\mathbb{C}) \subset M_3(\mathbb{C})$, so $\mathcal{C}(A_F)'_o = (A^F_F)'_C$ and the property (M) holds. By taking the commutant we get $\mathcal{C}(A_F)_{or} = (A^F_F)'_o$. But $\gamma_F$ commutes with $A^F_F$, hence $\gamma_F \in \mathcal{C}(A_F)_{or}$ and $\mathcal{C}(A_F)_{or} = \mathcal{C}(A_F)_{o}$. ■

7 Orientability

A study of the orientability condition for finite-dimensional spectral triples was done in [18], under the assumption that the global dimension is zero (i.e. only 0-cycles $c$ were taken into account). For an odd spectral triple, the orientability condition in dimension 0 is trivial since 1 is always the image of $c = 1 \in A_F$ resp. $c = 1 \otimes 1 \in A_F \otimes A^F_F$ (and every 0-chain is a 0-cycle). In the even case:

Proposition 25. Let $\gamma_F$ be as in (17) or (18). Then, there is no 0-cycle $c$ satisfying the orientability condition in Def. 3 or 4.

Proof. It is enough to show that $\gamma_F$ in not in the algebra generated by $A_F$ and $A^F_F$. For this, it is enough to find an operator $X$ commuting with both $A_F$ and $A^F_F$, but not with $\gamma_F$; $X := e_{15} \otimes e_{11}$ (one of the green asterisks of $D_0$ in §5.1 and in §5.2) does the job: it anticommutes with both the gradings (17) and (18), and belongs to $A^F_F \cap (A^F_F)'$. ■

In order to satisfy in dimension 0 the orientability condition of Def. 3 or 4 one should choose a grading which does not anticommute with $e_{15} \otimes e_{11}$. But this would force the $\Upsilon_R$ (green term) in the Dirac operator to be zero, in order to have a even spectral triple (a Dirac operator anticommuting with the grading).
8 Irreducibility

We say that a real spectral triple \((A, H, D, J)\) is \emph{irreducible} if there is no proper subspace of \(H\), other than \(\{0\}\), which carries a subrepresentation of \(A\) and is stable under \(D, J\) and (in the even case) \(\gamma\). Equivalently, it is irreducible if there is no non-trivial projection \(p = p^* = p^2 \in \text{End}_C(H)\) (so, other than 0 and 1), which commute with \(A, D, J\) and \(\gamma\) in the even case [16, Def. 11.2].

If \(D_F\) is the operator in §5.3, and \(\gamma_F\) one of the gradings in (17) or (18). If \(\Delta = 0\) (and possibly \(\Omega \neq 0\)), then \((A_F, H_F, D_F, \gamma_F, J_F)\) is clearly reducible. Take:

\[
p = \left(\sum_{i=1}^{4} e_{ii}\right) \otimes e_{11} + e_{55} \otimes 1
\]

the operator projecting on the subspace of \(H_F\) containing only leptons. It clearly commutes with \(A_F, D_F, \gamma_F\) and \(J_F\).

In order to have irreducibility, we need in \(D_F\) a term mixing leptons and quarks.

**Proposition 26.** The even spectral triple of Theorem 21 and the odd spectral triple of Theorem 23 are both irreducible.

**Proof.** For the even spectral triple of Theorem 21 if \(p\) is a projection commuting with \(A_F, D_F, \gamma_F, J_F\). Then it belongs to \(\mathcal{C}(A'_F) = A'_F\) (property (M)). Similarly, for the odd spectral triple of Theorem 23 one proves that \(p\) must belong to \(\mathcal{C}(A'_F) = (A'_F)_C\). But it also commutes with \(A'_F = J_F A_F J_F\), so it belongs to the center of \((A'_F)_C\). Hence:

\[
p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta_{13} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & \delta_{12} \end{bmatrix},
\]

with \(\lambda, \lambda', \beta, \delta \in \mathbb{C}\). Since

\[
J_F p J_F = \left\{\lambda(e_{11} + e_{55}) + \lambda' e_{22} + \delta(e_{33} + e_{44}) + \beta(e_{55} + e_{66} + e_{77})\right\} \otimes 1,
\]

if \(p\) commutes with \(J_F\) it is proportional to the identity, hence \(p = 0\) or \(p = 1\). \(\blacksquare\)

Let us mention that other inequivalent definitions of irreducibility can be used. For example, the one adopted in §18.3 of [11] says that a real spectral triple is irreducible if \(H\) carries an irreducible representation of \(A\) and \(J\). Such a condition is stronger than the one used by us, and is the condition leading to the algebra \(M_2(\mathbb{H}) + M_4(\mathbb{C})\). This is later reduced to \(A_F\) (which allows to introduce a grading and leads to the original Dirac operator of [3, 9, 5, 11]), thus losing the irreducibility property. In the next section, we discuss the intermediate algebra \(A^{\text{ev}}\) of the Pati-Salam model.

9 On the Pati-Salam model

The Pati-Salam model is a grand unified theory with gauge group \(\text{Spin}(4) \times \text{Spin}(6) \simeq \text{SU}(2) \times \text{SU}(2) \times \text{SU}(4)\). The relevant algebra is now \(A^{\text{ev}} = \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})\), which we
identify with the subalgebra of elements $a \otimes 1 \in \text{End}_C(H_F)$, with $a$ of the form:

$$a = \begin{bmatrix} x & \cdot & \cdots & \cdot \\ \cdot & y & \cdot & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & m \end{bmatrix},$$

with $x, y \in \mathbb{H}$ (and we think of them as $2 \times 2$ complex matrices) and $m \in M_4(\mathbb{C})$. All off-diagonal blocks are zero.

The data $(A^\text{ev}, H_F, J_F, \gamma_F, D_F)$, with $D_F$ as in §5.3 and $\gamma_F$ as in (17), satisfies all the conditions of a real spectral triple except for the 1st order condition [6, 7] (and then the property (M) cannot be satisfied). On the other hand, it is a simple check to verify that irreducibility, in the stronger sense of §18.3 of [11] (so, without $\gamma_F$ and $D_F$) is satisfied.

Lemma 27. The commutant $(A^\text{ev})'$ has elements $\sum a \otimes b$ with $b \in M_4(\mathbb{C})$ arbitrary and $a \in M_8(\mathbb{C})$ of the form

$$a = \begin{bmatrix} \alpha_{12} & \cdot & \cdots & \cdot \\ \cdot & \beta_{12} & \cdot & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \delta_{14} \end{bmatrix},$$

where the $\alpha$ and $\beta$-blocks are $2 \times 2$, the $\delta$-block is $4 \times 4$, and $\alpha, \beta, \delta \in \mathbb{C}$.

Proof. By direct computation. ■

It follows from previous lemma that any $p$ commuting with $A^\text{ev}$ has the form

$$p = (e_{11} + e_{22}) \otimes \alpha + (e_{33} + e_{44}) \otimes \beta + \left( \sum_{i=5}^{8} e_{ii} \right) \otimes \delta,$$

where now $\alpha, \beta, \delta \in M_4(\mathbb{C})$ three projections. Since (in $4 \times 4$ blocks):

$$J_F p J_F = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \end{bmatrix} \otimes (e_{11} + e_{22}) + \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \end{bmatrix} \otimes (e_{33} + e_{44}) + \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix} \otimes 1,$$

we deduce that $p$ commutes with $J_F$ if and only if $\alpha = \beta = \delta$ are proportional to the identity, and then $p = 0$ or $p = 1$ is a trivial projection.

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References

[1] J. Bhowmick, F. D’Andrea and L. Dąbrowski, Quantum Isometries of the finite noncommutative geometry of the Standard Model, Commun. Math. Phys. 307 (2011), 101–131.
[2] J. Bhowmick, F. D’Andrea, B. Das and L. Dąbrowski, *Quantum gauge symmetries in Noncommutative Geometry*, J. Noncomm. Geom. 8 (2014), 433–471.

[3] A.H. Chamseddine and A. Connes, *The Spectral Action Principle*, Commun. Math. Phys. 186 (1997), 731–750.

[4] A.H. Chamseddine and A. Connes, *Resilience of the Spectral Standard Model*, JHEP 1209 (2012) 104.

[5] A.H. Chamseddine, A. Connes and M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. 11 (2007), 991–1090.

[6] A.H. Chamseddine, A. Connes and W.D. van Suijlekom, *Inner Fluctuations in Noncommutative Geometry without the first order condition*, J. Geom. Phys. 73 (2013), 222–234.

[7] A.H. Chamseddine, A. Connes and W.D. van Suijlekom, *Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification*, JHEP 1311 (2013), 132.

[8] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[9] A. Connes, *Noncommutative geometry and the Standard Model with neutrino mixing*, JHEP 11 (2006), 081.

[10] A. Connes, *On the spectral characterization of manifolds*, J. Noncommut. Geom. 7 (2013), 1–82.

[11] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*, Colloquium Publications, vol. 55, AMS, 2008.

[12] R. Coquereaux, *On the finite dimensional quantum group $M_3 \oplus (M_{21}(\mathbb{A}^2))_0*$, Lett. Math. Phys. 42 (1997), 309–328.

[13] L. Dąbrowski, F. Nesti and P. Siniscalco, *A Finite Quantum Symmetry of $M(3, \mathbb{C})$*, Int. J. Mod. Phys. A13 (1998), 4147–4162.

[14] A. Devastato, F. Lizzi and P. Martinetti, *Grand Symmetry, Spectral Action, and the Higgs mass*, JHEP 1401 (2014), 042.

[15] A. Devastato and P. Martinetti, *Twisted spectral triple for the Standard Model and spontaneous breaking of the Grand Symmetry*,

[16] J.M. Gracia-Bondía, J.C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, Boston, 2001.

[17] D. Kastler, *Regular and adjoint representation of $SL_q(2)$ at third root of unit*, CPT internal report (1995).

[18] T. Krajewski, *Classification of Finite Spectral Triples*, J. Geom. Phys. 28 (1998), 1–30.

[19] G. Landi, *An introduction to noncommutative spaces and their geometries*, Springer, 2002.

[20] S. Lord, A. Rennie and J.C. Várilly, *Riemannian manifolds in noncommutative geometry*, J. Geom. Phys. 62 (2012), 1611–1638.

[21] M. Paschke, F. Scheck and A. Sitarz, *Can (noncommutative) geometry accommodate leptoquarks?*, Phys. Rev. D59 (1999), 035003.

[22] C.A. Stephan, *New Scalar Fields in Noncommutative Geometry*, Phys. Rev. D79 (2009), 065013.

[23] W.D. van Suijlekom, *Noncommutative Geometry and Particle Physics*, Springer, 2015.

[24] J.C. Várilly, *An introduction to noncommutative geometry*, EMS Lect. Ser. in Math., 2006.