Classification of degenerate non-homogeneous hydrodynamic operators

Marta Dell’Atti\textsuperscript{1} Pierandrea Vergallo\textsuperscript{2,3}

\textsuperscript{1} School of Mathematics & Physics, University of Portsmouth marta.dell’atti@port.ac.uk
\textsuperscript{2} Department of Mathematical, Computer, Physical and Earth Sciences University of Messina, V.le F. Stagno D’Alcontres 31, I-98166 Messina, Italy pierandrea.vergallo@unime.it
\textsuperscript{3} Istituto Nazionale di Fisica Nucleare, Sez. Lecce

Abstract
The authors investigate non-homogeneous Hamiltonian operators composed of a first order Dubrovin–Novikov operator and an ultralocal one. The study of such operators turns out to be fundamental for the inverted system of equations associated with a class of Hamiltonian scalar equations. Often, the involved operators are degenerate in the first order term. For this reason a complete classification of the operators with degenerate leading coefficient in systems with 2 and 3 components is presented.

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1 Introduction
The Hamiltonian formalism for Partial Differential Equations (PDEs) is one of the leading tools to study nonlinear systems [24, 6], following the well known developed theory for finite dimensional ones. As shown in [14, 17], Hamiltonian operators link conserved quantities with symmetries of the system, mapping the former onto the latter, and then leading to a deeper investigation of the structure of the solutions. This formalism represents a strong theoretical and practical connection between geometry and mathematical physics [22, 5, 2]. Dubrovin and Novikov introduced differential-geometric Poisson brackets as a natural extension of finite dimensional symplectic structures in traditional Hamiltonian mechanics that turned out to arise in several examples in nonlinear
PDEs. The characterisation of these structures is related to (pseudo-) Riemannian geometry and modern algebraic geometry, especially for 1 + 1 systems (in independent variables $x,t$).

More in general, geometrical methods are well established tools widely used to find solutions to systems, such as the generalised hodograph method introduced by Tsarev [28], valid for strictly hyperbolic systems. The Hamiltonian formalism is also used to discuss the integrability. In particular, finding two compatible Hamiltonian structures is strictly related to the existence of infinitely many commuting symmetries and conservation laws, as proved by Magri [19]. In the context of hydrodynamic type systems, an approach to describe integrability is based on the analysis of geometrical elements in the so called method of hydrodynamic reductions, developed for systems in 2+1 dimensions [10, 9] and then extended to systems in 1 + 1 dimensions with infinitely many components [8].

First, we recall some basic notions concerning the Hamiltonian formalism [22]. Let us consider a system described by $n$ field variables $u^i = u^i(t,x)$, with $i = 1,\ldots,n$ depending on the independent variables $t,x$, and let $u^i_\sigma$ denote the $x$-derivatives of $u^\sigma$-times. A Hamiltonian operator is a linear operator $A_{ij} = a_{ij\sigma} D^\sigma$ such that the associated bracket for functionals $f,g$

$$
\{f,g\} = \int \frac{\delta f}{\delta u^i} A_{ij} \frac{\delta g}{\delta u^j} dx,
$$

(1)
is a Poisson bracket, i.e. it is bilinear and satisfies the properties

$$
\{f,g\} = -\{g,f\},
$$

(2)
skew-symmetry,

$$
\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0,
$$

(3)
Jacobi identity.

An evolutionary system

$$
u_t^i = F^i(x,u,u_\sigma),
$$

(3)
with $u = \{u^i\}^n_{i=1}$ is Hamiltonian if it admits the following representation

$$
u_t^i = F^i(x,u,u_\sigma) = A_{ij} \frac{\delta H}{\delta u^j},
$$

(4)
where $\delta$ is the variational derivative, $H$ is the Hamiltonian functional $H = \int h(u) dx$, written in terms of the Hamiltonian density $h$, and $A$ is a Hamiltonian operator.

In [4, 5] Dubrovin and Novikov present a class of Hamiltonian operators which are homogeneous in the order of derivation and are also known as homogeneous Hamiltonian operators. They prove that first order homogeneous operators of the form

$$
g^{ij} \partial_x + \Gamma^i_k \partial_x^k u_x^k
$$

(5)
with $\det g \neq 0$, are Hamiltonian if and only if $g_{ij} = (g^{ij})^{-1}$ is a flat metric and $\Gamma^i_k = -g^{ij} \Gamma^j_k$, where $\Gamma^i_k$ are Christoffel symbols for the the metric tensor $g$. Operators of this type naturally arise in homogeneous quasilinear systems of first order PDEs, also known as hydrodynamic type systems

$$
u_t^i = v^i(u) u_x^i,
$$

(6)
where $v(u) = (v^i)^{1 \leq i, j \leq n}$ is the coefficient matrix depending on the field variables. Indeed, if (6) is
Hamiltonian with a Dubrovin–Novikov operator (5), it can be expressed as

\[ u_i = v_i(u) u_x = A^{ij} \frac{\partial h}{\partial u^j} = (g^{ij} \partial x + \Gamma^{ij}_k u^k_x) \frac{\partial h}{\partial u^j} = (\nabla^i \nabla_j h) u^i, \]  

(7)

where \( h = h(u) \) is the hydrodynamic Hamiltonian density and \( \nabla_i \) the covariant derivative.

In the case of higher order homogeneous operators of degree \( m \) or \( k \), the Dubrovin–Novikov operator generalizes as

\[ A^{ij} = g^{ij} D^m_x + b^{ij}_k u^k_x D^{m-1}_x + \left( c^{ij}_k u^k_{xx} + c^{ij}_{kl} u^k_x u^l_x \right) D^{m-2}_x + \ldots + \left( d^{ij}_k u^k_{nx} + \ldots + d^{ij}_{k \ldots m} u^k_x \ldots u^m_x \right), \]  

(8)

where the coefficients \( b^{ij}_k, c^{ij}_k, \ldots \) depend on the field variables. In [5], the authors present an extension of homogeneous structures. In particular, they introduce the non-homogeneous Hamiltonian operators as sum of two or more homogeneous ones. A leading example in this context is offered by the Korteweg–De Vries equation

\[ u_t = 6u u_x + u_{xxx}, \]  

(9)

which possesses a Hamiltonian structure through the operator

\[ A = \partial_x^2 + 2u \partial_x + u_x, \]  

(10)

given by the sum of the third order operator \( \partial_x^3 \) and the first order operator \( 2u \partial_x + u_x \).

Following the notation used by Dubrovin and Novikov, if an operator is given by the sum of two homogeneous operators of order \( k \) and \( m \) respectively, we denote the order of the non-homogeneous operator via the sum \( k + m \).

Let us consider the simplest case, with \( k = 1 \) and \( m = 0 \) for the so-called non-homogeneous hydrodynamic type operators \( 1 + 0 \). They naturally arise in non-homogeneous quasilinear systems of first-order PDEs. Let \( C^{ij} = A^{ij} + \omega^{ij} \), where \( A^{ij} \) is homogeneous of order \( 1 \) and \( \omega^{ij} \) is a symplectic structure of order \( 0 \). Then we can easily generalise (7) to systems of this type

\[ u^i = (g^{ij} \partial x + \Gamma^{ij}_k u^k_x + \omega^{ij}_x) \frac{\partial h}{\partial u^j} = (\nabla^i \nabla_j h) u^i + \nabla^i h, \]  

(11)

where \( \nabla^i = \omega^{ij} \nabla_j \) and \( \nabla_i \) is the covariant derivative with respect to the symplectic structure \( \omega^{ij} \).

A remarkable example of a non-homogeneous quasilinear system possessing such a construction is given by the 3-waves equation [22]

\[
\begin{align*}
   u_1 &= -c_1 u^3_x - 2(c_2 - c_3)u^2 u^3_x, \\
   u_2 &= -c_2 u^2_x - 2(c_1 - c_3)u^2 u^3, \\
   u_3 &= -c_3 u^3_x - 2(c_2 - c_1)u u^2 \\
\end{align*}
\]

(12)

that is a Hamiltonian with operator

\[ C^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & -2u^3 & 2u^2 \\ 2u^3 & 0 & 2u^1 \\ -2u^2 & -2u^1 & 0 \end{pmatrix}. \]  

(13)

Finally, following the approach introduced by S. I. Tsarev [27], we will see how non-homogeneous
hydrodynamic operators arise in a class of systems obtained by the inversion of an evolutionary Hamiltonian equation (see Theorem 4.1). Often, the inversion of such equations leads to a degeneration of the leading coefficient $g^{ij}$ in the first order operator. This is a strong motivation for the investigation of degenerate $1+0$ structures.

In this paper, we present a complete classification of Hamiltonian operators for systems in two and three components of the form

$$C^{ij} = g^{ij} \partial_x + b^{ij}_k u^k_x + \omega^{ij},$$  \hspace{1cm} (14)

focusing on the case when the leading coefficient is degenerate (i.e. its rank is lower than the number of components of the system) and some related remarkable examples of systems in $1+1$ dimensions exhibiting this feature. The importance of a deeper study of such operators has been remarked by O. I. Mokhov in [22], who finds Hamiltonian structures of this type in the study of the real reduction of 2-waves interaction system, but also by Dubrovin and Novikov themselves [5].

In section 2, we introduce the conditions for non-homogeneous operators of hydrodynamic type to be Hamiltonian, either with non-degenerate or degenerate assumptions. We establish the connection between such operators and non-homogeneous systems of first order PDEs, introducing the corresponding inverted systems and their associated Hamiltonian structures. In section 3, we show a complete classification, up to diffeomorphisms of the manifold defined by the field variables, of degenerate operators of type $1+0$ for systems with 2 and 3 components. In section 4, we provide several examples with Hamiltonian structures fitting the above mentioned classification, with particular emphasis on inverted Hamiltonian systems.

2 Non-homogeneous hydrodynamic operators

In this section we review non-homogeneous operators of hydrodynamic type, as originally introduced in [5] and further investigated in [23, 16].

Non-homogeneous operators of hydrodynamic type are introduced as the natural generalization of homogeneous Hamiltonian operators (8)

$$C^{ij} = g^{ij} \partial_x + b^{ij}_k u^k_x + \omega^{ij},$$  \hspace{1cm} (15)

where $g^{ij}, b^{ij}_k$ and $\omega^{ij}$ depend on the field variables $u$. Then, the underlying non-homogeneous local Poisson structure of hydrodynamic type is defined as

$$\{ u^i(x), u^j(y) \} = g^{ij}(u(x)) \delta_x(x-y) + b^{ij}_k(u(x)) u^k_x \delta(x-y) + \omega^{ij}(u(x)) \delta(x-y).$$  \hspace{1cm} (16)

Notice that operators of type $1+0$ are composed of two homogeneous operators

$$A^{ij} = g^{ij} \partial_x + b^{ij}_k u^k_x, \quad \text{order 1},$$  \hspace{1cm} (17)

$$\omega^{ij}, \quad \text{order 0}.$$  \hspace{1cm} (18)

The conditions for $C^{ij}$ to be Hamiltonian are given by the constraints for each of its homogeneous part to be Hamiltonian (Theorem 2.2 and Theorem 2.1 respectively) and an additional compatibility condition between the two (Theorem 2.3).
We recall that operators of order zero, also known as ultralocal operators, are Hamiltonian if the following conditions are satisfied.

**Theorem 2.1.** [22] The operator \(\omega^{ij}(u)\) is Hamiltonian if and only if it forms a finite-dimensional Poisson structure, i.e. it satisfies the conditions

\[
\omega^{ij}(u) = -\omega^{ji}(u) \quad \text{(skew-symmetry)} \tag{19}
\]
\[
\omega^i \frac{\partial \omega^{jk}}{\partial u^i} + \omega^j \frac{\partial \omega^{ik}}{\partial u^j} + \omega^k \frac{\partial \omega^{ij}}{\partial u^k} = 0, \quad \text{(closedness)}, \tag{20}
\]

with Einstein notation for repeated indices.

In the case of operators of first order the following result holds.

**Theorem 2.2.** [22] The operator \(A^{ij}\) is Hamiltonian if and only if

\[
g^{ij} = g^{ji} \tag{21}
\]
\[
\frac{\partial g^{ij}}{\partial u^k} = b^{ij}_k + b^{ji}_k \tag{22}
\]
\[
g^{is} b^{jk}_s = g^{is} b^{ks}_j = 0 \tag{23}
\]
\[
g^{is} \left( \frac{\partial b^{jr}_k}{\partial u^s} - \frac{\partial b^{jr}_s}{\partial u^k} \right) + b^{ij}_s b^{jr}_k - b^{jr}_s b^{ij}_k = 0 \tag{24}
\]
\[
g^{is} \frac{\partial b^{jr}_k}{\partial u^s} - b^{ij}_s b^{jr}_k - b^{jr}_s b^{ij}_k = g^{is} \frac{\partial b^{jr}_k}{\partial u^s} - b^{ij}_s b^{jr}_k - b^{jr}_s b^{ij}_k, \tag{25}
\]

and

\[
\sum_{(i,j,k)} \left\{ \frac{\partial}{\partial u^q} \left( g^{is} \left( \frac{\partial b^{jr}_k}{\partial u^q} - \frac{\partial b^{jr}_s}{\partial u^k} \right) + b^{ij}_s b^{jr}_k - b^{jr}_s b^{ij}_k \right) + \sum_{(i,j,k)} \left( b^{ij}_s \left( \frac{\partial b^{jr}_k}{\partial u^s} - \frac{\partial b^{jr}_k}{\partial u^k} \right) \right) \right\} = 0 \tag{26}
\]

with the sum over \((i,j,k)\) is on cyclic permutations of the indices.

Let us remark that here there is no assumption about the non-degeneracy properties of metric. The conditions for non-homogeneous operators of hydrodynamic type to be Hamiltonian are shown in the following theorem.

**Theorem 2.3.** [23, 22] The operator \((15)\) is Hamiltonian if and only if \(g^{ij} \partial_x + b^{ij}_k u^k\) is Hamiltonian, \(\omega^{ij}\) is Hamiltonian, and the compatibility conditions are satisfied

\[
\Phi^{ij} = \Phi^{ji}, \quad \text{(27)}
\]
\[
\frac{\partial \Phi^{ijk}}{\partial u^r} = \sum_{(i,j,k)} b^{ij}_s \frac{\partial \omega^{jk}}{\partial u^r} + \left( \frac{\partial b^{ij}_s}{\partial u^r} - \frac{\partial b^{ji}_s}{\partial u^r} \right) \omega^{sk}, \quad \text{(28)}
\]

where \(\Phi^{ijk}\) is the \((3,0)\)-tensor

\[
\Phi^{ijk} = g^{is} \frac{\partial \omega^{ik}}{\partial u^r} - b^{ij}_s \omega^{sk} - b^{jk}_s \omega^{is}. \tag{29}
\]
3 Classification for systems in 2 and 3 components

In [26], Savoldi presents a complete classification of degenerate first order homogeneous operators for systems with 2 and 3 components. Starting from these results, in this section we provide a novel complete classification of degenerate operators of type 1+0. In particular, to obtain an explicit form of $\omega^{ij}$ by means of Theorem 2.3 it is sufficient to solve conditions (27) and (28) with fixed tensors $g^{ij}$ and $b_k^{ij}$, giving rise to an overdetermined system of PDEs. In addition, we require the ultralocal operator $\omega^{ij}$ to be Hamiltonian imposing (19) and (20), via Theorem 2.1.

The following computations are carried out via computer algebra methods, implemented in Maple, Reduce and Mathematica. The use of symbolic computation for integrable systems and Hamiltonian structures is itself an ongoing topic of research (see e.g. [17, 31]).

3.1 Systems in $n=2$ components

Let us consider systems with 2 components, with field variables $u, v$. In general, given $n$ the number of components of the hydrodynamic system, in the degenerate case the operator $g^{ij}$ can be classified by its rank, with $\text{rank}(g^{ij}) = m < n$. In the following, we explicit the number of components $n$ for the operator $C_{n,k}^{ij}$ while the index $k$ is used to distinguish between different operators.

For $n = 2$, $\text{rank}(g^{ij}) \in [0,1]$. The only solution for the case $\text{rank}(g^{ij}) = 0$ is the trivial one, then the operator reduces to a symplectic form.

- $\text{rank}(g^{ij}) = 1$. We can construct two different operators, $C_{2,1}^{ij}$ and $C_{2,2}^{ij}$.

\[
C_{2,1}^{ij} = \begin{pmatrix}
\partial_x & 0 \\
0 & f(v) \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & f(v) \\
-f(v) & 0
\end{pmatrix},
\]

\[
C_{2,2}^{ij} = \begin{pmatrix}
\partial_x & 0 \\
0 & f(v) \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & f(v) \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & -v_x \\
-f(v) & u \\
-v_x & 0
\end{pmatrix},
\]

where $f(v)$ is an arbitrary function depending only on the variable $v$.

**Theorem 3.1.** Every degenerate operator of type 1+0 in two components can be mapped either onto an ultralocal Hamiltonian operator or onto one between $C_{2,1}^{ij}$ and $C_{2,2}^{ij}$.

**Proof.** Considering Theorem 2.3, we compute the symplectic structure satisfying (27) and (28) for each degenerate operator of the classification introduced by Savoldi in two components.

3.2 Systems in $n=3$ components

Let us consider the case of systems with three components $u, v, w$, for which the degenerate metric has $\text{rank}(g^{ij}) = 0, 1, 2$. In the following we denote with $f, g, h, l$ arbitrary functions, specifying the explicit dependence on the variables, and with $c$ arbitrary constants.

- $\text{rank}(g^{ij}) = 0$

\[
C_{3,1}^{ij} = \begin{pmatrix}
0 & w_x & 0 \\
-w_x & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & f(u,v,w) & 0 \\
-f(u,v,w) & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
• \( \text{rank} \{ g^{ij} \} = 1 \)

\[
C_{3,2}^{ij} = \begin{pmatrix}
\partial_x & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & f(v, w) & g(v, w) \\
-f(v, w) & 0 & h(v, w) \\
-g(v, w) & -h(v, w) & 0
\end{pmatrix},
\]

where the function \( f(v, w) \) is expressed in terms of the functions \( h(v, w) \) and \( g(v, w) \) as

\[
f(v, w) = h(v, w) \left( l(w) + \int_1^v \frac{g(s, w)\partial_w h(s, w) - h(s, w)\partial_w g(s, w)}{h(s, w)^2} \, ds \right). \tag{34}
\]

\[
C_{3,3}^{ij} = \begin{pmatrix}
\partial_x & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & w_x & 0 \\
-w_x & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & f(v, w) & 0 \\
-f(v, w) & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
C_{3,4}^{ij} = \begin{pmatrix}
\partial_x & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & -w_x & \frac{w_x}{u} \\
0 & 0 & 0 \\
\frac{w_x}{u} & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & \frac{f(v, w)}{u} \\
0 & 0 & 0 \\
\frac{f(v, w)}{u} & 0 & 0
\end{pmatrix},
\]

\[
C_{3,5}^{ij} = \begin{pmatrix}
\partial_x & 0 & 0 \\
0 & -v_x & \frac{w_x}{u} \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & \frac{v_x}{u} & -\frac{w_x}{u} \\
\frac{v_x}{u} & 0 & 0 \\
\frac{w_x}{u} & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & \frac{f(v, w)}{u} & \frac{g(v, w)}{u} \\
\frac{f(v, w)}{u} & 0 & \frac{h(v, w)}{u} \\
\frac{g(v, w)}{u} & \frac{h(v, w)}{u} & 0
\end{pmatrix},
\]

with \( f(v, w) \) given in (34).

• \( \text{rank} \{ g^{ij} \} = 2 \)

\[
C_{3,6}^{ij} = \begin{pmatrix}
\partial_x & 0 & 0 \\
0 & \partial_x & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & f(w) & g(w) \\
-f(w) & 0 & cg(w) \\
-g(w) & -cg(w) & 0
\end{pmatrix},
\]

\[
C_{3,7}^{ij} = \begin{pmatrix}
\partial_x & 0 & 0 \\
0 & \partial_x & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -w_x & \frac{w_x}{v} \\
0 & \frac{w_x}{v} & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & \frac{cf(w)}{v} \\
0 & 0 & \frac{(1-cu)f(w)}{v} \\
-cf(w) & -\frac{(1-cu)f(w)}{v} & 0
\end{pmatrix},
\]

(39)
classification for degenerate first order operators presented by Savoldi in three components [26].

Theorem 3.2. an ultralocal operator satisfying the closure relation, or onto one among

\[ \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{w}{uw-v} \\ 0 & 0 & \frac{w}{uw-v} \\ \frac{ww}{uw-v} & -\frac{w}{uw-v} & 0 \end{pmatrix} + (1 + w^2)f(w) \begin{pmatrix} 0 & 1 & \frac{w-cv\sqrt{1+w^2}}{uw-v} \\ \frac{1}{1+w^2} & 0 & \frac{1-cu\sqrt{1+w^2}}{uw-v} \\ \frac{-w-cv\sqrt{1+w^2}}{uw-v} & \frac{1-cu\sqrt{1+w^2}}{uw-v} & 0 \end{pmatrix} \]

(40)

\[ C_{3,9}^{ij} = \begin{pmatrix} 0 & \partial_x & 0 \\ \partial_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & f(w) & cg(w) \\ -f(w) & 0 & g(w) \\ -cg(w) & g(w) & 0 \end{pmatrix}, \]

(41)

\[ C_{3,10}^{ij} = \begin{pmatrix} \partial_x & 0 & 0 \\ 0 & \partial_x & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{w}{v} & 0 \\ 0 & 0 & 0 \\ \frac{w}{v} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & f(w) & \frac{h(w)-ug(w)}{v} \\ -f(w) & 0 & g(w) \\ -\frac{h(w)-ug(w)}{v} & g(w) & 0 \end{pmatrix}, \]

(42)

with the additional condition

\[ h(w)g'(w) - g(w)(f(w) + h'(w)) = 0. \]

Remark 3.1. Condition (43) can be explicitly solved with respect to any function among \( f, g \) and \( h \).

\[ C_{3,11}^{ij} = \begin{pmatrix} \partial_x & 0 & 0 \\ 0 & \partial_x & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{w}{uw-v} \\ 0 & 0 & \frac{w}{uw-v} \\ \frac{ww}{uw-v} & \frac{ww}{uw-v} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{ cf(w) }{ \sqrt{w} } & \frac{u w - 2 c \sqrt{w} f(w)}{u w - v} \\ \frac{ - c f(w) }{ \sqrt{w} } & 0 & \frac{ - w ( v - 2 c \sqrt{w} f(w) ) }{u w - v} \\ \frac{ (u w - 2 c \sqrt{w} f(w) ) }{u w - v} & \frac{ w ( v - 2 c \sqrt{w} f(w) ) }{u w - v} & 0 \end{pmatrix} \]

(44)

Theorem 3.2. Every degenerate operator of type 1 + 0 in 3 components can be mapped either onto
an ultralocal operator satisfying the closure relation, or onto one among \( C_{3,k}^{ij} \) with \( k = 1, \ldots, 11 \).

Proof. Imposing the conditions on the operators to be Hamiltonian, we obtain the extension of the
classification for degenerate first order operators presented by Savoldi in three components [26]. □
4 Applications

In this section we present some examples of non-homogeneous quasilinear systems with degenerate 1 + 0 Hamiltonian structure in 2 and 3 components.

Example 4.1 (2-wave interaction system). In [22], the author introduces the real reduction of 2-waves interaction system formulated in terms of the system of hydrodynamic equations in two field variables $u = u(x,t)$ and $v = v(x,t)$

\[
\begin{align*}
  u_t &= a uv \\
  v_t &= av_x + u^2,
\end{align*}
\]

with $a$ constant. The system admits a Hamiltonian formulation, with the operator

\[
C^{ij} = \begin{pmatrix} 0 & 0 \\
0 & \partial_x \end{pmatrix} + \begin{pmatrix} 0 & -u \\
u & 0 \end{pmatrix},
\]

and the Hamiltonian functional

\[H = \frac{1}{2} \int \left( av^2 - u^2 \right) dx.\]

The 1 + 0 operator (46) is degenerate, since the rank of the order 1 term is lower than the number of components of the system. Moreover, by applying the simple exchange of $u$ and $v$, we obtain that the operator found by Mokhov is exactly of type $C^{ij}_{2,1}$ (30).

Example 4.2 (Sinh-Gordon equation). Let us consider the Sinh-Gordon equation

\[q_{\tau \xi} = \sinh q.\]

Applying the change of variables $q = 2 \log u$, we have

\[
\left( 2 \frac{u_x}{u} \right) = \frac{1}{2} \left( u^2 - \frac{1}{u^2} \right).
\]

Finally, introducing $v = 2 u_x / u$ and considering the light-cone coordinates $\tau = t, \xi = t - x$

\[
\begin{align*}
  u_t &= \frac{1}{2} uv \\
  v_t &= v_x + \frac{1}{2} \left( u^2 - \frac{1}{u^2} \right),
\end{align*}
\]

we show that the system is Hamiltonian with the non-homogeneous hydrodynamic operator (46) with the exchange $u \leftrightarrow v$, $f(u) = 1/2u$, and Hamiltonian density

\[h(u,v) = \frac{1}{2} \left( v^2 - u^2 + \frac{1}{u^2} \right).\]

4.1 Inverted Hamiltonian systems

In this section we show the connection between degenerate operators of type 1 + 0 and scalar equations possessing a local Hamiltonian structure.

Let us briefly recall that the momentum of a Hamiltonian equation $u_t = A^{ij} \delta H/\delta u^j$ is a functional
defining a $\tau$-translation

$$u^i_\tau = A^{ij} \frac{\delta P}{\delta u^j}, \quad i = 1, \ldots, n,$$

with $P = \int p(u, u_\sigma) \, dx$.  \hfill (52)

In [27] Tsarev proves that under the inversion of the independent variables $x$ and $t$, the Hamiltonian property is preserved by the system. It is well known that the momentum is a conserved quantity in a Hamiltonian system, hence there exists $q(u, u_\sigma)$ such that $p_t = q_x$. Then, one can choose $H' = \int q(u, u_\sigma) \, dt$ as the Hamiltonian functional of the inverted system.

Non-homogeneous operators of hydrodynamic type are related to the study of scalar evolutionary equations possessing a Hamiltonian local structure. Indeed, by introducing the new set of variables

$$u^1 = u, \quad u^2 = u_x, \quad u^3 = u_{xx}, \ldots,$$

(53)

it is in some cases possible to write an equivalent non-homogeneous quasilinear system that can be seen as evolutionary with respect to the independent variable $x$, obtaining the inverted system.

**Remark 4.1.** Let us observe that every invertible system of order $k$ has the following form

$$u_t = F_1(u, u_x, \ldots, u_{(k-1)x}) + F_2(u, u_x, \ldots, u_{(k-1)x}) u_k,$$

(54)

where $F_1, F_2$ are arbitrary functions. Note that this is the case of KdV and many other examples in nonlinear phenomena. Indeed, considering the lower derivatives as parameters, we need the system to be linear in $u_{kx}$ in order to conserve linearity in $u_i$ once inverted.

The following result gives an explicit connection between non-homogeneous hydrodynamic operators and inverted systems.

**Proposition 4.1.** Let us consider the evolutionary system as in (54) endowed with a local Hamiltonian structure and a momentum density $p$ in (52) depending on $u^i, i \leq k$ only. Then, if the inverted system in the set of variables (53) admits a local Hamiltonian structure, this is given in terms of a non-homogeneous operator of hydrodynamic type.

**Proof.** We observe the following

$$q_x = p_t = p_a(u, u_\sigma) u_t = p_a(u, u_\sigma) F(u, u_\sigma), \quad \sigma \leq k,$$

(55)

where $p_t$ is of order $\leq k$, at most equal to the order of the equation, and so is $q_x$. Hence, $q(u, u_\sigma)$ is of order at most $k - 1$. This implies that the Hamiltonian $H' = \int q(u^1, \ldots, u^{k-1}) \, dt$ is of hydrodynamic type for the inverted system in the new variables. In [27, 15], it has been proved that the Hamiltonian property is preserved after a change of dependent variables and an inversion of $t$ and $x$. Then, the inverted system is quasilinear of first order and already Hamiltonian. The operator $B^{ij}$ in

$$u^i_\tau = B^{ij} \frac{\delta H'}{\delta u^j},$$

(56)

being local, must be of type $1 + 0$, i.e. a non-homogeneous operator of hydrodynamic type.

The Proposition 4.1 justifies a deeper investigation of such operators, for which KdV offers a leading example, as follows. We emphasise the previous theorem does not guarantee that the operator is in general non-degenerate.
Example 4.3 (KdV equation - I). Let us consider the KdV equation
\[ u_t = 6u u_x + u_{xxx}, \] (57)
which is widely known to be Hamiltonian. Inverting the equation, we obtain the evolutionary system with respect to \( x \) in three components \( u^1(x, t), u^2(x, t), u^3(x, t) \) defined as \( u = u^1, u_x = u^2, u_{xxx} = u^3 \), yielding the following non-homogeneous system of hydrodynamic type
\[
\begin{pmatrix}
  u^1_x = u^2 \\
  u^2_x = u^3 \\
  u^3_x = u^1 + 6u^1 u^2
\end{pmatrix}. \tag{58}
\]
This system is Hamiltonian with the following non-homogeneous hydrodynamic type operator \([27]\)
\[
C^{ij} = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 1
\end{pmatrix} \partial_i + \begin{pmatrix}
  0 & 1 & 0 \\
  -1 & 0 & 6u^1 \\
  0 & -6u^1 & 0
\end{pmatrix}, \tag{59}
\]
with the leading coefficient \( g^{ij} \) being degenerate. It is easy to see that applying the change of variables \( u^1 = w \), we obtain again the operator (33), where \( g(v, w) = 0, f(v, w) = -6w, l(w) = 6w \) and \( h(v, w) = -1 \).

Example 4.4 (KdV equation - II). Mokhov [21] finds a transformation of variables (also known as local quadratic unimodular change)
\[ u^1 = \frac{w^1 - w^3}{\sqrt{2}}, \quad u^2 = w^2, \quad u^3 = \frac{w^1 + w^3}{\sqrt{2}} + \left( w^1 - w^3 \right)^2, \] (60)
such that the KdV equation reads as
\[
\begin{pmatrix}
  w^1_x = -\frac{1}{2} \left( w^1 - w^3 \right)_t + w^2 \left( w^1 - w^3 \right) + \frac{1}{\sqrt{2}} w^2 \\
  w^2_x = \left( w^1 - w^3 \right)^2 + \frac{1}{\sqrt{2}} \left( w^1 + w^3 \right) \\
  w^3_x = -\frac{1}{2} \left( w^1 - w^3 \right)_t + w^2 \left( w^1 - w^3 \right) - \frac{1}{\sqrt{2}} w^2
\end{pmatrix}. \tag{61}
\]
After this local change, the KdV is a bi-Hamiltonian system with respect to two non-homogeneous operators \( 1 + 0 \) of hydrodynamic type, one of these being the operator
\[
C^{ij} = \frac{1}{2} \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
  1 & 0 & 1
\end{pmatrix} \partial_t + \begin{pmatrix}
  0 & w^1 - w^3 + 1/\sqrt{2} & 0 \\
  w^3 - w^1 - 1/\sqrt{2} & 0 & w^1 - w^3 + 1/\sqrt{2} \\
  0 & w^3 - w^1 + 1/\sqrt{2} & 0
\end{pmatrix}, \tag{62}
\]
which is degenerate, since \( \text{rank}(g^{ij}) = 1 \). The Hamiltonian given in terms of the new variables is
\[
H = \int \left( (w^1)^2 - (w^2)^2 - (w^3)^2 \right) dx. \tag{63}
\]
To show that the obtain operator is indeed one of the classified above, we consider a new change of
variables
\[ w^1 = \frac{\bar{a}^1 - \bar{a}^3}{\sqrt{2}} , \quad w^2 = \bar{a}^2 , \quad w^3 = \frac{\bar{a}^3 - \bar{a}^1}{\sqrt{2}} . \] (64)

The degenerate first order operator is written with \( \bar{g} = d\bar{a}^1 \otimes d\bar{a}^1 \) as the leading coefficient and the skew-symmetric 2-form \( \bar{\omega} = -\sqrt{2}\bar{a}^3 \left( d\bar{a}^1 \wedge d\bar{a}^2 - d\bar{a}^2 \wedge d\bar{a}^3 \right) \). The operator (62) is of type \( C_{22} \) in three components showed in equation (33). In particular,
\[ g(v, w) = 0 , \quad f(v, w) = l(w)h(v, w) , \quad l(w) = -1 , \quad h(v, w) = \sqrt{2}w . \] (65)

Example 4.5 (Generalised KdV equation). Let us consider the generalised KdV equation
\[ u_t + 3(n + 1)u^n u_x + u_{xxx} = 0 \] (66)
where \( n \) is a positive integer. Then, introducing the variables: \( u^1 = u, u^2 = u_x \) and \( u^3 = u_{xx} \) the equation reads as a quasilinear system of first order PDEs
\[
\begin{align*}
\dot{u}^1 &= u^2, \\
\dot{u}^2 &= u^3, \\
\dot{u}^3 &= -u^1 - 3(n + 1)(u^1)^n u^2
\end{align*}
\] (67)

It is known that (66) is Hamiltonian with the operator \( \partial_x \). The Hamiltonian structure is conserved also when the scalar equation is transformed into a system, i.e. (67) has Hamiltonian structure with the operator
\[
C^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -3(n + 1)(u^1)^{n-1} \\ 0 & 3(n + 1)(u^1)^{n-1} & 0 \end{pmatrix}
\] (68)

and the Hamiltonian functional
\[ H = \int \left( 3(u^1)^{n+1} - u^1 u^3 + \frac{(u^2)^2}{2} \right) dx . \] (69)

The operator (68) is \( C_{3,2} \) (33) where \( g(u^1, u^2) = 0, f(u^1, u^2) = 1, l(u^1) = 3(n + 1)(u^1)^{n-1} \) and \( h(u^1, u^2) = -1 \), when the exchange of \( u^1 \) and \( u^3 \) is applied.

Remark 4.2. Let us observe that for \( n > 2 \) the generalised KdV equation is not integrable, even if it is Hamiltonian as proved in the previous example. We emphasise that this feature is more general than the integrability property.

We finally present two examples violating the hypothesis of locality, either in terms of the momentum or in the operator for the inverted Hamiltonian structure.

Example 4.6. We consider the so called linearised KdV equation
\[ u_t = u_{xxx} . \] (70)
for which the inverted system is easily given in the new variables by
\[
\begin{align*}
  u_1^x &= u^2 \\
  u_2^x &= u^3 \\
  u_3^x &= u_1^x
\end{align*}
\] (71)

The associated momentum is given in terms of the density \( p(u) = \partial_x^2 u \). Here again, it is not possible to write the resulting system with Hamiltonian operator of type \( 1 + 0 \) [27].

**Example 4.7.** We consider the Harry-Dym equation [16, 25]

\[
u_t = \left( \frac{1}{\sqrt{u}} \right)_{xxx} = -\frac{15}{8}u^{-7/2}(u_x)^3 + \frac{9}{4}u^{-3/2}u_xu_{xx} - \frac{1}{2}u^{-3/2}u_{xxx},
\] (72)

admitting the Hamiltonian structures
\[
\begin{align*}
u_t &= A_1 \frac{\delta H_1}{\delta u} = -\frac{1}{2} \partial_x^3 \frac{\delta H_1}{\delta u}, \\
u_t &= A_2 \frac{\delta H_2}{\delta u} = -(2u\partial_x - u_x) \frac{\delta H_2}{\delta u}. 
\end{align*}
\] (73)

Introducing the variables \( u_x = u^2, u_{xx} = u^3 \), the inverted system is
\[
\begin{align*}
u_1^x &= u^2 \\
u_2^x &= u^3 \\
u_3^x &= -2(u^{1/2})u_1^x - \frac{15}{4}(u^1)^{-2}(u^2)^3 + \frac{9}{2}(u^1)^{1/2}u^2u^3
\end{align*}
\] (74)

The momentum \( P \) associated with the operator \( A_2 \) is
\[
u_x = -(2u\partial_x - u_x) \frac{\delta P}{\delta u}, \quad P = \int p(u)\,dx = -\int u\,dx,
\] (75)

and following the above procedure the Hamiltonian \( H' \) as a functional in the new variables is
\[
H' = -\int \left( \frac{3}{4}(u^1)^{-5/2}(u^2)^2 - \frac{1}{2}(u^1)^{-3/2}u^3 \right)\,dx.
\] (76)

With this Hamiltonian it is not possible to build a local operator of the form \( 1 + 0 \) for the inverted system, hence this operator will be non-local.

As a further perspective, it would be interesting to investigate the properties of the class of systems admitting this representation once inverted.

## 5 Conclusions

The study of non-homogeneous quasilinear systems of first order PDEs is an ongoing research topic in integrable systems and Hamiltonian PDEs. To the authors' knowledge, a general criterion to discuss integrability for this kind of systems is not currently known, unlike the homogeneous systems [28]. This paper represents a first step towards the investigation of integrability of non-homogeneous systems, focusing on the Hamiltonian property. The study of possible bi-Hamiltonian structures associated with these type will be the subject of a future paper.
Non-homogeneous operators of order \( k + m \) play an important role in nonlinear phenomena and their investigation represents another interesting topic \([18, 16, 7, 13, 20]\). Even in the simplest case where \( k = 1 \), and \( m = 0 \) we showed how the conditions for the operator to be Hamiltonian lead to a specific form, this being exactly solvable. Higher order operators require a more general study, especially for what concerns the degenerate case.

As future perspectives, the authors emphasise the necessity to further investigate the integrability of non-homogeneous quasilinear systems, the compatibility conditions between systems and operators in the sense of \([29]\), but also their associated geometric structure, following the lead of the homogeneous case, where both operators and systems are linked to projective algebraic geometry \([1, 11, 12, 30]\) and differential Riemannian geometry. Also, the discrete analogous of non-homogeneous operators were introduced by Dubrovin in \([3]\), letting the classification method described in this paper suitable for discrete operators as well.

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