Solving the hourglass instability problem using rare mesh variation-difference schemes

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Abstract. The hourglass instability effect is characteristic of the Wilkins explicit difference scheme or similar schemes based on two-dimensional 4-node or three-dimensional 8-node finite elements with one integration point in the element. The hourglass effect is absent in schemes with cells in the form of simplexes (triangles in two-dimensional case, tetrahedrons in three-dimensional case). But they have another well-known drawback - slow convergence. One of the authors proposed a rare mesh scheme, in which elements in the form of a tetrahedron are located one at a time in the centers of the cells of a hexahedral grid. This scheme showed the absence "hourglass" effect and other drawbacks with high efficiency. This approach was further developed for solving 2D and 3D problems.

1. Introduction

The hourglass effect [1] is explained by the incompleteness of the systems of grid operators, since the intersection of their kernels is not equal to a function that is identically zero. At one point of integration in a finite element, the first partial derivatives of the function are constant within the element. Thus, in the two-dimensional case, the function is specified by three parameters at 4 nodes in the element, in the three-dimensional case, by four parameters at 8 nodes in the element, that is, the number of parameters is less than the number of nodes, which results in the indicated incompleteness of the systems of grid operators.

Distortions of grids of the "hourglass" type do not carry energy and do not in any way affect the determination of the stress-strain state of the medium. At the same time, the given calculations for computer are carried out on a finite bit grid, they can contribute to the accumulation of errors in floating point calculations. In this case, the displacement field distortions can be quite significant. Formally, the "hourglass" effect cannot be attributed to the typical instability of difference schemes, since it is not accompanied by an exponential increase in the error of the numerical solution. But outwardly, the picture is very similar. To combat this undesirable effect, various authors have proposed a number of techniques. In particular, in the simplest case of a plane problem of the theory of elasticity, in addition to quadrangular cells, triangular ones can be introduced, taking them with a small weight. In this case, it is sufficient to calculate only the spherical components of deformations and stresses on triangular cells. As a result, we get a difference scheme without the "hourglass" effect. The described technique, called the "moment scheme", is described in more detail in [2] in relation to three-dimensional problems of the theory of elasticity and plasticity.
Let’s consider another approach to combat this effect. It is based on the use of linear finite elements in the form of simplexes. As you know, there is no hourglass effect in them. But the use of linear elements in the form of simplexes leads to other well-known disadvantages of numerical schemes, in particular - the effect of overestimated shear stiffness of the element. These results in slow convergence of numerical solutions. In [3-5], an rare mesh scheme was proposed for solving three-dimensional problems of the theory of elasticity, in which elements in the form of a tetrahedron are located one at a time in the centers of the cells of a hexahedral grid. This scheme was investigated in detail, showed the absence of both the "hourglass" effect and other drawbacks with high efficiency.

To solve two-dimensional problems, it was proposed in [6] to take a projection of three-dimensional rare mesh scheme onto a two-dimensional grid, as a result of which an additional grid operator appears in Wilkins-type schemes [7], which approximates a constant bending moment within an element. Thanks to this, the problem of instability of the hourglass is solved. This idea was also developed for a 3D 8-node element.

The application of this approach is also possible when solving other problems of mathematical physics [8].

2. Rare mesh scheme for solving a three-dimensional problem of the theory of elasticity

Let us consider an example of an rare mesh FEM scheme for solving three-dimensional problems of the theory of elasticity [2]. This is a circuit based on a linear finite element in the form of a tetrahedron. Several methods are known for constructing regular grids on tetrahedral cells. With one of them, the cell of the main grid (hexagon) is divided into 5 tetrahedra - 1 at the center and 4 at the edges. If we remove all tetrahedra, except for the central one, we get the required FEM scheme (figure 1). The area is divided into elements as follows: the area is divided into hexagons and then we leave one tetrahedron in them. In this case, not all the nodes of the difference grid are involved in the calculations (see figure 2, where the nodes participating in the calculations are highlighted in bold).

![Figure 1](image-url)  
Figure 1. An element of a rare mesh scheme (a central tetrahedron inside a cube). The other four tetrahedra are not included in the calculations.
Figure 2. Pattern (neighborhood of a regular grid node) of an openwork scheme. The dots mark the nodes of the template.

This scheme is completely symmetrical, since the tetrahedron is located symmetrically about the center of the cube. The scheme has a second order of approximation on a uniform grid. When solving dynamic problems according to an explicit scheme (or when iteratively solving the system in an implicit scheme), the computational costs for one time layer (one iteration) of this scheme are at least 5 times lower compared to the traditional scheme on tetrahedrons and at least 2 times lower in comparison with the scheme on polylinear hexagonal elements. In the case of using an explicit “cross” scheme, the stability of this scheme is not worse than that of a multilinear hexagonal finite element scheme.

The circuit was tested on a large number of dynamic elastic and elastic-plastic problems. An explicit “cross” scheme was used. The solutions were compared with experimental data and results obtained using the Wilkins scheme, a traditional linear finite element scheme, ANSYS LsDyna. Based on the test results, we can conclude that the rare mesh scheme is not inferior to others in accuracy with greater efficiency of calculations. The gain in computational costs was 2 to 6 times. As an example, let us give the solution of one dynamic problem of the theory of elasticity.

On the problem of rigid loading with different intensities of a prismatic bar with free lateral surfaces, the effect of hourglass instability was investigated. One of the ends of the beam is rigidly pinched, the second acted on a constant in time velocity in the axial direction and transverse displacements were prohibited. The length of the timber is 100 cm, the sides are 6 cm. The material is steel, it was assumed to be elastic. Figures 3.1-3.3 show deformed beam configurations with an image of the field of axial displacements for various types of elements.
Figure 3.1. Wilkins scheme. $V = 0.003 \text{ cm/mks}, t = 1064 \text{ mks}$.

Figure 3.2. Rare mesh scheme. $V = 0.040 \text{ cm/mks}, t = 939.5 \text{ mks}$. 
Figure 3.3. Traditional scheme of linear tetrahedral finite element. $V = 0.050\text{cm/mks}$, $t = 820\text{ mks}$.

From the figures 3.1-3.3 the following conclusions can be drawn.

Wilkins difference scheme begins to lose stability even at relatively small deformations (the "hourglass" effect). In contrast, the tetrahedron scheme and the rare mesh scheme are not subject to such an undesirable effect and give significantly better solutions. The curvature obtained in numerical solutions for large (more than 30%) deformations of the rod for the rare mesh scheme and periodic changes in the thickness for the scheme on tetrahedrons are due to the loss of stability of the rod when the critical Euler load is exceeded. In this case, the solution based on the rare mesh scheme is more consistent with the real physical picture of the buckling of the rod.

3. Momentary schemes based on rare mesh scheme

3.1 Two-dimensional scheme

Let us formulate the dynamic problem of the theory of elasticity as a variational problem. For discretization, we will apply the rare mesh scheme of a linear 4-nodes finite element, described above, in combination with an explicit "cross" scheme. As a result, on a uniform orthogonal grid, we obtain it in the form of a finite-difference scheme

$$
\left( \lambda + G \right) \begin{vmatrix} D_{11}u_1 + D_{12}u_2 + D_{13}u_3 \\ D_{21}u_1 + D_{22}u_2 + D_{23}u_3 \\ D_{31}u_1 + D_{32}u_2 + D_{33}u_3 \end{vmatrix} + \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} + \rho \begin{vmatrix} F_1 \\ F_2 \\ F_3 \end{vmatrix} = \rho D_{0} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}
$$

(1)

similar in form to the Lamé equations system
\[
(\lambda + \mu) \text{grad} \div u + \mu \Delta u + \rho F = \frac{\partial^2 u}{\partial t^2}.
\] (2)

Here \( \lambda, \mu \) are the Lamé parameters, \( \rho \) is the density of the medium, \( F \) is the external mass force, \( u \) is the displacement field, \( D_1, D_2, D_3 \) are grid operators. The operator \( d_m \), approximating the derivative \( \partial^2 \partial t^2 \) has the form: \( d_m f = [f(t + \tau) - 2f(t) + f(t - \tau)] / \tau^2 \), the rest are constructed as follows. Consider basic operators approximating the first partial derivatives in the element \( \{d_m^i \approx \partial / \partial x^m \} : \)

\[
(d_1^+ f)_{ijk} = \frac{1}{2h_1}(f_{i+1,j+1,k+1} + f_{i-1,j+1,k+1} - f_{i,j+1,k} - f_{i,j-1,k+1})
\]
\[
(d_2^+ f)_{ijk} = \frac{1}{2h_2}(f_{i+1,j+1,k-1} + f_{i-1,j+1,k-1} - f_{i,j+1,k} - f_{i,j-1,k-1})
\]
\[
(d_3^+ f)_{ijk} = \frac{1}{2h_3}(f_{i+1,j+1,k+1} + f_{i+1,j+1,k+1} - f_{i,j+1,k} - f_{i,j+1,k+1})
\]

Let’s define the operators

\[
(d^-_1 f)_{ijk} = \frac{1}{2h_1}(-f_{i+1,j-1,k+1} - f_{i,j-1,k} + f_{i,j-1,k})
\]
\[
(d^-_2 f)_{ijk} = \frac{1}{2h_2}(-f_{i+1,j-1,k+1} - f_{i,j-1,k} + f_{i,j-1,k})
\]
\[
(d^-_3 f)_{ijk} = \frac{1}{2h_3}(-f_{i+1,j-1,k+1} - f_{i,j-1,k} + f_{i,j-1,k})
\]

equal to the adjoint operators to (3), taken with the sign -. Operators \( D_0 \) will be defined as a superposition of operators (3) and (4): \( D_0 = d^+_1 d^-_1 \). \( D_3 = D_{11} + D_{22} + D_{33} \) is the Laplace grid operator.

Consider a plane problem of the elasticity theory (plane deformation). We obtain a two-dimensional problem by projecting a three-dimensional one onto a plane \( x_3 \). Assuming that the three-dimensional computational domain has the form \( \Omega \times (0, \infty) \), where \( \Omega \) is the domain in \( R^2 \), we impose constraints on the solution, taking \( u_3 = 0 \), and the components of the displacement vector \( u_1 \) and \( u_2 \) assuming functions that depend only on \( x^1 \) and \( x^2 \). In this case, the system of equations (2) takes the form

\[
(\lambda + \mu) \left( \frac{\partial^2 u_1}{\partial x^1^2} + \frac{\partial^2 u_2}{\partial x^2^2} \right) + \mu \left( \frac{\partial^2 u_1}{\partial x^1^2} + \frac{\partial^2 u_2}{\partial x^2^2} \right) + \rho F_1 = \rho \frac{\partial^2 u_1}{\partial t^2}
\]
\[
(\lambda + \mu) \left( \frac{\partial^2 u_2}{\partial x^2^2} + \frac{\partial^2 u_2}{\partial x^2^2} \right) + \mu \left( \frac{\partial^2 u_1}{\partial x^1^2} + \frac{\partial^2 u_2}{\partial x^2^2} \right) + \rho F_2 = \rho \frac{\partial^2 u_2}{\partial t^2}
\] (5)

Difference scheme (1) - respectively

\[
(\lambda + \mu)(D_1 u_1 + D_2 u_2) + \mu(D_1 u_1 + D_2 u_2 + D_3 u_1) + \rho F_1 = \rho D_0 u_1
\]
\[
(\lambda + \mu)(D_2 u_1 + D_2 u_2) + \mu(D_1 u_1 + D_2 u_2 + D_3 u_2) + \rho F_2 = \rho D_0 u_2
\] (6)
In (6), the operators $D_{ij}$ are obtained by projecting the operators considered above onto a two-dimensional grid: $D_{ij} = d_i^* d_j^*$, where

$$
(d_i^* f)_y = \frac{1}{2h_1} \left( f_{i+1,j+1} + f_{i,j+1} - f_{i,j} \right)
$$

$$
(d_2^* f)_y = \frac{1}{2h_2} \left( f_{i+1,j} + f_{i,j+1} - f_{i,j} \right)
$$

$$
(d_3^* f)_y = \frac{1}{2h_3} \left( f_{i+1,j} + f_{i,j} - f_{i,j+1} \right)
$$

$$
(d_4^* f)_y = \frac{1}{2h_4} \left( - f_{i-1,j-1} - f_{i-1,j} + f_{i,j} \right)
$$

$$
(d_5^* f)_y = \frac{1}{2h_5} \left( - f_{i-1,j-1} - f_{i,j} + f_{i-1,j} \right)
$$

$$
(d_6^* f)_y = \frac{1}{2h_6} \left( - f_{i-1,j-1} - f_{i,j} + f_{i+1,j} \right)
$$

$$
(d_7^* f)_y = \frac{1}{2h_7} \left( - f_{i-1,j-1} - f_{i,j} + f_{i-1,j-1} \right)
$$

(7)

(compare with (3) and (4)).

For comparison, we present a Wilkins-type scheme

$$
(\lambda + \mu) (D_1 u_1 + D_2 u_2) + \mu (D_1 u_1 + D_2 u_2) + \rho F_1 = \rho D_n u_1
$$

$$
(\lambda + \mu) (D_2 u_1 + D_2 u_2) + \mu (D_1 u_1 + D_2 u_2) + \rho F_2 = \rho D_n u_2
$$

(8)

Scheme (6) contains an adjustable parameter $h_3$, which can be selected based on numerical experiments. When $h_3 = \infty$ we get from (6) scheme (8), which does not take into account the moment components in the element.

Scheme (6) as a special case includes Wilkins' scheme (8) and on a square grid it can coincide with the scheme of a bilinear finite element with an appropriate choice of the parameter $h_3$ (see [6]). In this case, the latter has disadvantages: scheme (8) - the effect of hourglass instability, and the scheme of bilinear finite element - worse stability.

3.2 Three-dimensional scheme

Developing this approach, we construct an 8-node FEM scheme for solving 3D elasticity theory problems, which is close to the traditional scheme of a multilinear 8-node finite element. To do this, we will build a hypothetical seven-dimensional rare mesh scheme and then project it onto a three-dimensional space. Let us show that 8 vertices can be chosen in a 7-dimensional unit cube so that they are the vertices of a regular simplex and that the centers of the cube and the simplex coincide. An example is a set of vertices
Consider a hypothetical problem of the three-dimensional problem of the theory of elasticity was obtained. Note that sets of the first three coordinates of vectors form the set of vertices of the unit cube in $\mathbb{R}^3$.

Relationship between stresses and deformations is established on the basis of the "Hooke's law"

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{ij} + 2\mu \varepsilon_{ij}, \quad i, j = 1, \ldots, 7$$

As a result, the seven-dimensional equations of motion in displacements will be written in the form (2), where grad, div and Laplace operator are defined in $\mathbb{R}^7$. Rare mesh scheme on a uniform orthogonal grid $x_i = x_0^i + h_i t_i$, $\ldots, x_7 = x_0^7 + h_7 t_7$ will take the form:

$$\begin{bmatrix}
D_{11} u_1 + \ldots + D_{17} u_7 \\
D_{21} u_1 + \ldots + D_{27} u_7 \\
D_{31} u_1 + \ldots + D_{37} u_7 \\
D_{41} u_1 + \ldots + D_{47} u_7 \\
D_{51} u_1 + \ldots + D_{57} u_7 \\
D_{61} u_1 + \ldots + D_{67} u_7 \\
D_{71} u_1 + \ldots + D_{77} u_7
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
u_7
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
\lambda + \mu \\
\lambda + \mu \\
\lambda + \mu \\
\lambda + \mu
\end{bmatrix}
$$

Assuming at all nodes $u_4 = u_5 = u_6 = u_7 = 0$, we get after projection to $\mathbb{R}^3$ a variational-difference scheme in the form:

$$\begin{bmatrix}
D_{11} u_1 + D_{12} u_2 + D_{13} u_3 \\
D_{21} u_1 + D_{22} u_2 + D_{23} u_3 + \mu D_4 u_2 + \mu (D_{44} + D_{45} + D_{46} + D_{47}) u_2 \\
D_{31} u_1 + D_{32} u_2 + D_{33} u_3 + \mu D_{44} u_3 + \mu (D_{44} + D_{45} + D_{46} + D_{47}) u_3
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}
$$

In (8), operators $D_j = \frac{1}{2}(d_j^+ d_j^- + d_j^- d_j^+)$, $D_\lambda = D_{11} + D_{22} + D_{33}$ are expressed in terms of basic operators similarly to the case considered above.

Thus, as a result of projection in $\mathbb{R}^3$ an rare mesh scheme, a 4-parameter family of numerical FEM schemes for solving a three-dimensional problem of the theory of elasticity was obtained. The parameters $h_4, h_5, h_6, h_7$ can be adjusted by changing the influence of the moment components in the element.

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

All edges of the simplex are equal. Note that sets of the first three coordinates of vectors form the set of vertices of the unit cube in $\mathbb{R}^3$.
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