Self-Similar Collapse in Brans-Dicke Theory and Critical Behavior

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Abstract

We use the technique of conformal transformations to generate self-similar collapse in Brans-Dicke theory. We analyze the solutions concerning the critical behavior found recently by Choptuik. The critical exponent associated to the formation of black hole for near critical evolution is obtained. The role of the coupling parameter is discussed.

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Recently, Choptuik\cite{1} has showed numerically the occurrence of critical phenomena in spherically symmetric gravitational collapse of massless scalar field. These critical phenomena are characterized by power-law behavior, scaling relations and a form of universality. In his study, Choptuik analyzed the evolution one-parameter families of initial conditions (parameter $p$, say), and classified the solutions in three classes: (i) subcritical ($p << p_{cr}$), where the final state is a complete dispersion of scalar field resulting in flat spacetime; (ii) supercritical ($p >> p_{cr}$) in which the scalar field is intense enough to trap itself in a black hole; (iii) critical solutions ($p = p_{cr}$), that represents the transition between complete dispersal and black hole formation. The most interesting results merge for near critical ($p \approx p_{cr}$) evolutions due to two important features. First, the mass of formed black hole in this regime is given by $M_{BH} \propto |p - p_{cr}|^{\gamma}$, where $\gamma \approx 0.37$ is conjectured to be universal since it does not depend on the particular initial condition. Second, for near critical evolution a strong field region exists where the scalar field oscillates in a very particular way. Such oscillations are echoes and the scalar field satisfies $\phi(e^{-n \Delta t}, e^{-n \Delta r}) \approx \phi(t, r)$, where $n$ is an integer number and $\Delta \approx 3.4$. It is important to mention that the exact critical solution has an infinite train of echoes in the strong field region.

Quite surprisingly the same results were obtained for axisymmetric collapse of gravitational waves\cite{2} and for the spherically symmetric collapse of radiative fluids\cite{3} exhibiting local self-similarity. Therefore, it is strongly suggested that these critical phenomena are independent of the symmetries as well as the collapsing matter. Analytical solutions describing such features could be extremely useful for their understanding. Unfortunately, this task has proved to be very difficult, and an alternative approach was to suppose continuous self-similarity. In several self-similar exact solutions with ordinary scalar field the critical exponent was found to be 0.5\cite{4}, whereas for the conformally coupled case, 0.21\cite{5}. Maison\cite{6}, analyzing the collapse of perfect fluid, showed the non-universality of the critical exponent in the sense that it depends strongly on the state equation relating the energy density and pressure.

In this paper, our objective is to study analytically the self-similar collapse in Brans-Dicke theory of gravitation (BDT), and its relation with the critical behavior as described above. We are motivated by the renewed interest in the called scalar-tensor\cite{7} theories of gravitation in which the BDT is the most simple. We mention, for instance, the numerical work performed by Scheel et al\cite{8} concerning the collapse of matter in BDT. We will consider here a simplified model where the matter field is absent. Then, we expect to find, if it is present, the role of the coupling parameter $\omega$ on the value of the critical exponent.

BDT is the most simple of the scalar-tensor theories. In such theories the spacetime is characterized by the metric tensor $g_{\mu \nu}$ and by a scalar field coupled to the metric and matter. The action for the BDT is:

$$S = \int d^4 x \sqrt{-g} (\phi R - \frac{\omega}{\phi} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + L_m) \quad (1)$$

where $\phi$ is the scalar field, $\omega$ the coupling constant and $L_m$ is the matter Lagrangian. For a more general scalar-tensor theory $\omega = \omega(\phi)$. The basic characteristic of this theory is the variable gravitational term measured locally as $G = \frac{2\omega+4}{\phi(2\omega+3)}$. According with the most recent experimental tests, $\omega$ has minimum value about 500. As showed by Dicke\cite{9}, it is
possible to express the theory in the Einstein frame. In this frame the action (1) is formally given as General Relativity with ordinary scalar field, which is a suitable redefinition of the scalar field $\phi$. Then, we have:

$$S = \int d^4 x \sqrt{-\tilde{g}} \left( \tilde{R} - \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right)$$  \hspace{1cm} (2)

where $\tilde{R}$ is the scalar of curvature obtained from the metric $\tilde{g}_{\mu\nu}$, $\Phi$ is the ordinary scalar field and we have considered $L_m = 0$. Both actions integrals are equivalent if the following conformal transformation holds:

$$g_{\mu\nu} = \frac{1}{\phi} \tilde{g}_{\mu\nu} ,$$  \hspace{1cm} (3)

and the scalar field $\Phi$ is expressed in terms of $\phi$ by:

$$\Phi = \sqrt{\frac{2 \omega + 3}{2} \ln \phi}$$  \hspace{1cm} (4)

Therefore, from the above equations, it is possible to generate solutions in BDT starting from those solutions for ordinary scalar fields in GR.

To generate solutions in BDT describing self-similar spacetimes we consider the Roberts’[10] solution. Such a solution has been studied as an analytical model that exhibits critical behavior in scalar field collapse[4]. The Roberts’ solution is given by:

$$d s^2 = -du dv + r^2(u,v) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right)$$  \hspace{1cm} (5)

where $u$ and $v$ are null coordinates, and

$$r^2(u,v) = -\frac{uv}{2} + a_2 u^2 + b_2 v^2 = \frac{1}{16 a_2} f_+(u,v) f_-(u,v)$$  \hspace{1cm} (6)

with:

$$f_\pm (u,v) = 4 a_2 u - (1 \pm \sqrt{1 - 16 a_2 b_2}) v.$$  \hspace{1cm} (7)

The scalar field $\Phi$ is:

$$\Phi = \pm \frac{1}{\sqrt{2}} \ln \left| \frac{f_+(u,v)}{f_-(u,v)} \right|$$  \hspace{1cm} (8)

The collapse situation is properly described if $a_2$ is chosen positive, more specifically, $a_2 = 1/4$, without loss of generality. The solutions for which $a_2 < 0$ corresponds to expanding self-similar cosmological models. The collapse solutions are classified according to three distinct values of the parameter $b_2$: $0 < b_2 < 1/4$, the subcritical regime, $b_2 = 0$, the critical regime and the supercritical $b_2 < 0$. In this last case, there is the formation of a ”black hole” type configuration (Fig. 1(a)), meaning that the formed singularity is enclosed by an apparent horizon, but the spacetime is not asymptotically flat. The critical exponent related to near critical evolution was found to be equal to 0.5.

To obtain the spherically symmetric self-similar collapse in $BDT$, we begin with Eqs. (4) and (8), which yields:
\[ \phi(u, v) = \left( \pm \frac{f_+(u, v)}{f_-(u, v)} \right)^{\beta(\omega)} \tag{9} \]

where \( \beta(\omega) = \frac{1}{\sqrt{2\omega+3}} \). According with Eqs. (3) and (9), we have:

\[ ds^2 = -\phi^{-1} du dv + \Sigma^2(u, v) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \tag{10} \]

where the proper area, \( \Sigma^2(u, v) \), is given by:

\[ \Sigma^2(u, v) = \frac{1}{16a_2} \phi(u, v)^{-1} f_-(u, v) f_+(u, v). \tag{11} \]

The conformal factor, \( \phi \), is regular everywhere with exception of \( f_-(u, v) = 0 \) or \( f_+(u, v) = 0 \), depending of the sign of \( \beta(\omega) \). Indeed these regions represent, in general, singularities. As an important property of conformal transformations, the causal structure of the spacetime is not changed if the transformation is well behaved, which is the present case. We assume \( \omega > 0 \), implying \( |\beta(\omega)| < 1 \) and, as a consequence, the singular regions have zero proper area. Otherwise, for \( \omega < 0 \ (|\beta(\omega)| > 1) \), the singular regions could be characterized by an infinity proper area. Two types of solution emerge from Eq. (9): the one with positive sign inside the parenthesis, and another with negative sign. For the first type, we must have \( f_+(u, v) > 0 \). Hence, we chose \( a_2 = 1/4 \) so that the proper area is given by

\[ \Sigma(u, v)^2 = \frac{1}{4} f_-(u, v)^{1+\beta(\omega)} f_+(u, v)^{1-\beta(\omega)}. \]

For the second type, it is necessary that \( f_+(u, v) < 0 \). Consequently, \( a_2 \) must be negative in order the proper area to be positive as required. However, due to the fact the collapse is not described for such a choice of \( a_2 \), we are not going to study this situation. Nonetheless, \( a_2 \) can be positive only if \( |\beta(\omega)| \) is an even number that is realized only by \( \omega < 0 \).

The apparent horizon is determined locally by:

\[ g^{\mu\nu} \Sigma_{\mu} \Sigma_{\nu} = 0. \tag{12} \]

After substituting Eq. (11) into (12), two relations are obtained:

\[ u_{AH} = (1 + \beta(\omega) \sqrt{1 - 4b_2}) v \tag{13} \]

\[ u_{AH} = \frac{4b_2 v}{1 - \beta(\omega) \sqrt{1 - 4b_2}} \tag{14} \]

where the first equation corresponds to \( \Sigma_u = 0 \) whereas the second \( \Sigma_v = 0 \). Note that in the limit \( \omega \to \infty \), we recover the results provided by Roberts’ solution as expected. The next step is to verify for which cases the apparent horizon is present.

As mentioned, the Penrose diagrams are not altered by the conformal transformation. Therefore, following ref. [4], we classify the solutions as subcritical (\( 0 < b_2 < 1/4 \)), critical

\[ \text{For instance, the scalar of curvature is given by } R = 16 \omega \beta^2 (1 - 4b_2) u v \left( \pm \frac{f_+^{1-2/\beta}}{f_+} \right). \text{ Since } \omega > 0, |\beta| < 1 \text{ meaning that at } f_-(u, v) = 0 \text{ or } f_+(u, v) = 0, \text{ the scalar of curvature diverges. For sake of completeness, only if } |\beta| > 2 \text{ (or } -1.5 < \omega < -1.375) \text{ } R \text{ is finite at } f_-(u, v) = 0 (\beta < 0) \text{ and } f_+(u, v) = 0 (\beta > 0). \]
(b_2 = 0) and supercritical (b_2 < 0). Studying the presence of the apparent horizon in the collapse in BDT the following results are obtained. For b_2 = 0, it is not difficult to show that there is no apparent horizon described either by Eq. (13) or Eq. (14), unless \( \omega < -1 \) that has no interest. The subcritical case (0 < b_2 < 1/4) displays some novelty. Taking \( \beta(\omega) = \frac{1}{\sqrt{4\omega+3}} < 0 \), and after a straight analysis, we conclude that there is no apparent horizon. However, for \( \beta(\omega) = \frac{1}{\sqrt{4\omega+3}} > 0 \), the apparent horizon given by Eq. (14) exists for any value of \( \omega > 0 \) and encloses the timelike singularity. This situation is also found for the collapse of conformally coupled scalar field\(^5\), but has no relevance regarding the critical behavior.

The most interesting case is the supercritical (b_2 < 0). Let us consider first \( \beta(\omega) > 0 \), where, after easy verification, the apparent horizon indicated by the line \( OH \) (Fig. 1(b)) is spacelike and described if by Eq. (14) with \( \omega > \omega^* = -2 b_2 - 1 \) holds. For \( \omega \to \infty \) we recover the apparent horizon given by Roberts’ solution (line \( OR \), Fig. 1(b)). On the other hand if \( \omega = \omega^* \), the horizon becomes the null surface \( v = 0 \). In Fig. 2, we depict the effect of varying \( \omega \) on the apparent horizon. Finally, considering \( \beta(\omega) < 0 \) the apparent horizon is again described by Eq. (14), but with no restriction on possible values of \( \omega \). The effect of varying \( \omega \) from \(-3/2\) to \(\infty\) is to rotate the line \( OH \) from the \( v \)-axis to the line \( OR \) (the apparent horizon of Roberts’ solution) as indicated in Fig. 2.

**Black Hole mass and the Power Law for Near Critical Behavior**

We are going to see the influence of the coupling parameter in the power law associated to the mass of the formed black hole. This last quantity is the effective gravitational mass inside the apparent horizon:

\[
m_{AH} = \frac{1}{2} \sqrt{\left.f_-(u, v)^{1+\beta(\omega)} f_+(u, v)^{1-\beta(\omega)}\right|_{AH}}
\]

Substituting Eq. (14) with \( b_2 < 0 \) into the above equation and considering near critical evolution, \( b_2 \approx 0 \), we have:

\[
m_{AH} \approx \frac{(1 - \beta(\omega))^{1+\beta(\omega)}}{1 - \beta(\omega)} \left[ - (3 - \beta(\omega)) b_2 + 1 - \beta(\omega) \right]^{1-\beta(\omega)} \left( - b_2 \right)^{1+\beta(\omega)} v.
\]

As mentioned previously, self-similar spacetimes fails to be asymptotically flat and black hole configurations have infinite mass when \( v \to \infty \). However, an important point to be stressed is the obtained power-law \( m_{AH} \approx (-b_2)^\gamma \) for near critical evolution. Notice that the exponent \( \gamma \) depends strongly on \( \omega \), or, in another word, of the intensity of which the scalar field couples to the geometry. Then, as showed in Fig. 3, if \( \beta(\omega) > 0 \), we have 0.79 < \( \gamma \) < 0.5 for \( \omega \) varying from 0 to \(\infty\). In the case of \( \beta(\omega) < 0 \), 0.21 < \( \gamma \) < 0.5 for \( \omega \) varying from 0 to \(\infty\). Incidentally, the exact value 0.37 is achieved in this case when \( \omega \approx 5.8 \). A large value of the coupling constant compatible with earlier experimental results, \( \omega \approx 500 \), produces \( \gamma \approx 0.52 \) and 0.48 for \( \beta(\omega) \) positive and negative, respectively.

We can argue that the above result as well as the one obtained for conformally coupled scalar field rise doubts regarding the universality of the value 0.37. In these models, however, there a serious weakness which is the assumption of continuous self-similarity, and contrary...
to the problem treated by Choptuik, the spacetime is not asymptotically flat. Koike and Mishima\cite{11} studying the collapse of a thin massive shell coupled with an outgoing null fluid resulting in an asymptotically flat model showed that the critical exponent is not universal. On the other hand, from the large amount of works dealing with self-similar collapse in connection to cosmic censorship hypothesis, a cutoff is introduced to add an asymptotically flat region. This procedure is equivalent to truncate the spacetime at some, say, \( v = v_0 \). Hence, we can match the self-similar spacetime with Schwarzschild or outgoing Vaidya solutions, where in the later case it models the radiation that escapes during the collapse. In general, a null shell characterized by mass and surface pressure\cite{12} would be present for such a junction. In this way, the local trapped mass, represented by the mass function at the apparent horizon, is given by the mass of the shell and the Brans-Dicke scalar field mass. A reasonable condition to be imposed is that the shell has negligible mass and pressure, which could be seem as natural from the physical point of view. If this hypothesis is satisfied, Eq. 16 is rewritten as:

\[
m_{AH} \sim \frac{(1 - \beta(\omega))^{1+\beta(\omega)/2}}{1 - \beta(\omega)} [-(3 - \beta(\omega)) b_2 + 1 - \beta(\omega)]^{1+\beta(\omega)/2} (-b_2)^{1+\beta(\omega)/2} v_0.
\]

We expect that, depending on the asymptotically flat spacetime to be joined, the conditions \( m_{shell} \approx 0 \) and \( P_{shell} \approx 0 \) at the apparent horizon, render some relation between \( v_0 \) and \( b_2 \). Then, it will be possible to estimate, in this simplified model, the value of the critical exponent. We are investigating these issues and the results will be given elsewhere\cite{13}.

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References

[1] M. W. Choptuik, Phys. Rev. Lett., 70, 9 (1993).
[2] A. M. Abrahams and C. R. Evans, Phys. Rev. Lett., 70, 2 (1993).
[3] C. R. Evans and J. S. Coleman, Phys. Rev. Lett., 72, 1782 (1994).
[4] P. R. Brady, Class. Quantum Grav., 11, 1255 (1995); Y. Oshiro, K. Nakamura and A. Tomimatsu, Prog. Theor. Phys. 91, 1265 (1994).
[5] H. P. de Oliveira and E. S. Cheb-Terrab, Class. Quantum Grav., in press.
[6] D. Maison, Non-Universality of Critical Behavior in Spherically Symmetric Gravitational Collapse, gr-qc/9504008 (1995).
[7] T. Damour and K. Nordtredt, Phys. Rev. Lett., 70, 2217 (1993).
[8] M. A. Scheel, S. L. Shapiro and S. Teukolsky, Phys. Rev. D51, 4028 (1995); Phys. Rev. D51, 4236 (1995).
[9] R. H. Dicke, Phys. Rev. 125, 2163 (1962). see also J. D. Barrow and K. Maeda, Nucl. Phys. B, 294 (1990).
[10] M. D. Roberts, Gen. Rel. Grav., 21, 907 (1989).

[11] T. Koike and T. Mishima, Phys. Rev. D51, 4045 (1995).

[12] For the formalism of junction conditions for null shells, see C. Barrabes and W. Israel, Phys. Rev. D43, 1129 (1991).

[13] A. Wang and H. P. de Oliveira (in preparation).

- Fig. 1(a) "Black hole" type configuration. The spacetime is not asymptotically flat and the mass of the black hole grows without bound as $v \to \infty$. At $v = 0$ we match smoothly the scalar field solution with the Minkowski spacetime. The lines indicated by $R$ and $H$ represent the apparent horizon of the Roberts and the Brans-Dicke solutions, respectively.

- Fig. 1(b) The lines $OR$ and $OH$ represent the apparent horizon for the Brans-Dicke (case $\beta(\omega) > 0$) and Roberts solutions, respectively. The effect of varying $\omega$ from $\omega_*$ to $\infty$ is to rotate $OH$ in the counterclockwise sense. For $\omega = \infty$ both lines coincide.

- Fig. 2 For the case $\beta(\omega) < 0$, the apparent horizon behaves in a different manner as compared to the previous case. As far as $\omega$ grows, the line $OH$ rotates in the clockwise sense and tends to $OR$ for $\omega \to \infty$.

- Fig. 3 Behavior of the critical exponent $\gamma(\omega) = \frac{1+\beta(\omega)}{2}$. 