Random Walk Approach to Simple Evolution Model

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Abstract

The dynamics of the avalanche width in the evolution model is described using a random walk picture. In this approach the critical exponents for avalanche distribution, $\tau$, and avalanche average time, $\gamma$, are found to be the same as in the previous mean field approximation but SOC appear at $\lambda_{\text{critical}} = 2/3$, which is very close to numerical value. A continuous time random walk is studied numerically as a possible way to reconstruct in simpler concepts the evolution model.

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INTRODUCTION

Self Organized Criticality (SOC) theory studies non-equilibrium spatially extended systems which show scale invariance on a large spatial or temporal domain. The phenomena which have these characteristics appear in astrophysics, geophysics, biological evolution, stock market, etc. [1], [2], [6]. The study of this type of problems has emerged through various models; we will concentrate to the Bak & Sneppen model (BS) [2] proposed as explanatory for certain aspects of the biological evolution. Its mathematical simplicity attracted numerical studies [2], [5], [7] and mean-field analytic treatments [3], [4].

The model treats a number of $N$ species interacting on an one dimensional chain (a simple pictures of the food chain). Each species has assigned a scalar parameter, called fitness, with values in $(0,1)$ interval as a measure for the adaptability of species to the ecosystem. The dynamics starts from the site with the smallest fitness: new random values from $(0,1)$ interval are independently attributed to this site and to its two neighbors with a uniform distribution and uncorrelated from previous values then the cycle is repeated for a large number of times. For this system we define a $\lambda$-avalanche ($0 < \lambda < 1$) as the number of steps between two consecutive configuration with all the fitness values greater than $\lambda$. Numerically it has been found that the the stationary state exhibits scaling laws in $N \to \infty$ limit for the correlation function of the active sites and avalanches distribution at $\lambda \approx 0.66$ [2]. We mention that the model shows also critical behavior in the versions with more than one dimensions or with slightly modified dynamics [3], [4].

In this paper we propose a new approximated solution. Our treatment brings in the dynamics of the avalanche width in a mean field approach. In the BS model the the avalanche width is a random variable which shows memory effect; we removed this effect by updating all the sites within an avalanche (see section II for more details), in this way we we kept track of the spatial extension of the avalanches, thing which is not possible in the infinite range approximation [4]. Even the approximation seems too crude we found the value of $\lambda_{critical} = 2/3$, which is very close to the numerical result.
The article organized as follow: Section I presents the general master equation for the fitness distribution and the derivation of the mean field equation found with probabilistic arguments in Ref. [2]. Section II introduces our analytical treatment based on a mapping in a random walk problem. In this approximation we compute exactly the value of \( \lambda_{\text{critical}} \) and the critical exponents \( \tau \) and \( \gamma \) as they were defined in Ref. [3]. Section III presents a way to improve the method introducing a continuous time random walk description, numerically we show the improvement of the critical exponent \( \tau \) keeping \( \lambda_{\text{critical}} \) fixed. The details of calculation are given in the Appendix.

I. THE MASTER EQUATION.

The BS model is completely characterized by the probability \( P(x_1, x_2, \ldots, x_N; t) \) to find the system in the state \( (x_1, x_2, \ldots, x_N) \) at the time \( t \) given the initial distribution at \( t = 0 \). Because there are not memory effects, the evolution of the system is described by the following master equation

\[
P(x_1, x_2, \ldots, x_N; t + 1) = \sum_i \int dx'_i dx'_{i-1} dx'_{i+1} P_{st}(i; x_1, \ldots, x'_{i-1}, x'_i, x'_{i+1}, \ldots, x_N) \times P(x_1, \ldots, x'_{i-1}, x'_i, x'_{i+1}, \ldots, x_N; t),
\]

where periodic boundary conditions were assumed and \( P_{st}(i; x_1, \ldots, x'_{i-1}, x'_i, x'_{i+1}, \ldots, x_N) \) is the probability to have activity at site \( i \) if the system is in the configuration \( (x_1, x'_{i-1}, x'_i, x'_{i+1}, \ldots, x_N) \). For the original one dimensional BS model

\[
P_{st}(i; x_1, x_2, \ldots, x_N) = \prod_{j \neq i} \theta(x_j - x_i),
\]

where \( \theta(x) \) is the step function

\[
\theta(x) = \begin{cases} 
1, & \text{if } x > 0; \\
0, & \text{if } x \leq 0.
\end{cases}
\]

At stationarity, integrating in (1) over \( x_2, \ldots, x_N \) we get easily the following relation:

\[
P_{ac}(1, x_1) + P_{ac}(2, x_1) + P_{ac}(N, x_1) - \frac{3}{N} = 0;
\]
where

$$P_{ac}(i; x_j) = \int dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_N P_{st}(i; x_1, \ldots, x_N) \times P(x_1, \ldots, x_N)$$

is the probability to have activity in site $i$ when in site $j$ the fitness has the value $x_j$. If in equation (3) we try a stationary self-consistent mean field solution of the form $p(x_1, \ldots, x_N) = p(x_1)p(x_2)\ldots p(x_N)$ after some algebra we get

$$\left(1 - \frac{2}{N-1}\right) Q^N(x) + \frac{2N}{N-1} Q(x) + 3x - 3 = 0 \quad (4)$$

with $Q(x) = \int_x^1 p(x') dx'$. Equation (4) was previously obtained in Ref. [3], in the $N \to \infty$ limit one finds $p(x) = 3/2 \quad x \in (\lambda_{\text{critical}}, 1)$ and $p(x) = 0$ when $x \in (0, \lambda_{\text{critical}})$, $\lambda_{\text{critical}} = 1/3$ whereas numerically $\lambda_{\text{critical}} \approx 2/3$ [2]. The statistical independence between the sites in the mean field solution allows the reduction of the problem at an one dimensional random walk on the positive semi-axis where the state $n$ represents the state of the system with $n$ fitness values greater than $\lambda$. The solution developed in Ref. [4] gives the same $\lambda_{\text{critical}}$ as predicted by eq.(4) and the critical exponents $\tau = 3/2$, $\gamma = 1$.

II. THE NEW APPROACH.

We remember here that the size of an $\lambda$ avalanche is the number of steps between two consecutive events with no fitness below the the value $\lambda$, so, it is a quantity characterizing time intervals. For a system of size $N$, with free boundary conditions, we define the avalanche width at a given moment $t$ as the number of sites between the most left species with the fitness less than $\lambda$ and the most right species with the fitness less than $\lambda$, the species between these two sites can have any value of the fitness. This is a quantity which characterize the spatial structure of our system. The width shows memory effects for the originally proposed dynamics, in the spirit of the mean-field approximation we approximate the evolution of the avalanche width with the following dynamics: at every step the species between right
and left extrema are performing uncorrelated movements and we introduce in dynamics the right nearest neighbor of the right extremum species in order to mimic the evolution. The complete randomness makes the movements of the two extrema completely equivalent and for this reason we have chosen to move only in one direction. In origin we also accept the double step.

With this change the avalanche width is a random variable without memory effects on a discreet set of states which now can be extended to the entire non-negative semi-axis with the state zero corresponding to the state with no species below \( \lambda \) and the state \( n \) to a realization of BS model with with \( n \) sites between the most left and the most right sites with the value of their corresponding fitness less than \( \lambda \). The transition matrix of the model has the following form:

\[
p = \begin{pmatrix}
(1 - \lambda)^2 & 2\lambda(1 - \lambda) & \lambda^2 & 0 & 0 & \ldots \\
(1 - \lambda)^2 & 2\lambda(1 - \lambda) & \lambda^2 & 0 & 0 & \ldots \\
(1 - \lambda)^3 & 3\lambda(1 - \lambda)^2 & 2\lambda^2(1 - \lambda) & \lambda^2 & 0 & \ldots \\
(1 - \lambda)^4 & 4\lambda(1 - \lambda)^3 & 3\lambda^2(1 - \lambda)^2 & 2\lambda^2(1 - \lambda) & \lambda^2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\] (5)

The formulae for the matrix elements are:

\[
p_{00} = (1 - \lambda)^2, \quad p_{01} = 2\lambda(1 - \lambda), \quad p_{02} = \lambda^2, \quad p_{0j} = 0, \quad j > 2;
\]

\[
p_{j0} = (1 - \lambda)^{j+1}, \quad j \geq 1;
\]

\[
p_{j1} = (j + 1)\lambda(1 - \lambda)^j, \quad j \geq 1;
\]

\[
p_{j2} = j\lambda^2(1 - \lambda)^{j-1}, \quad j \geq 1;
\]

\[
p_{jl} = \begin{cases} 
p_{j+1,i-1} & \text{if } l \geq 2 \text{ and } j \leq l - 1; \\
0 & \text{otherwise}
\end{cases}
\] (6)

with the convention that \( p_{ij} \) is the transition probability from state the \( i \) to the state \( j \) and \( i, j \in \{0, 1, 2, \ldots\} \). The distribution probability of avalanches is the first return probability distribution for this random walk and it can be written as

\[
p(n + 2) = \sum_{i=1}^{\infty} p_{0i} \tilde{P}_i^{(n)} p_{i0} + \sum_{i=1}^{\infty} p_{02} \tilde{P}_2^{(n)} p_{i0}
\] (7)
where $\tilde{p}^{(n)}_{ij}$ is the $i, j$ element of the $n$-th power of the matrix $\tilde{p}$ obtained from the matrix $p$ removing the row and the column zero and it is describing the evolution of the random walk outside of the origin. The first (second) term in the r.h.s. of eq. (7) represents the first return probability, after $n$ steps, when the initial step is single, (double). For a site different from the origin the forward step can only be single, as matrix $p$ shows.

We modify the first two columns of the transition matrix $p$ such that to have the same elements on the diagonals of $\tilde{p}$ matrix. Keeping the closure relation $\sum_j \tilde{p}_{ij} = 1$ we produce the following matrix:

$$p' = \begin{pmatrix}
(1 - \lambda)^2 & 2\lambda(1 - \lambda) & \lambda^2 & 0 & 0 & \ldots \\
(1 - \lambda)^2(1 + 2\lambda) & 2\lambda^2(1 - \lambda) & \lambda^2 & 0 & 0 & \ldots \\
(1 - \lambda)^3(1 + 3\lambda) & 3\lambda^2(1 - \lambda)^2 & 2\lambda^2(1 - \lambda) & \lambda^2 & 0 & \ldots \\
(1 - \lambda)^4(1 + 4\lambda) & 4\lambda^2(1 - \lambda)^3 & 3\lambda^2(1 - \lambda)^2 & 2\lambda^2(1 - \lambda) & \lambda^2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

The asymptotic behavior of the first return time distribution is the same for the both random walks described by the matrices $p$ and $p'$. In fact in eq. (7) we can go a step further developing $\tilde{p}^{(n)}_{1i}$ with respect to site 1 the remaining matrix from the second column and second row is identical with the $\tilde{p}'$ matrix obtained from $p'$ in the same way as $\tilde{p}$ from $p$.

$$p^{(n)}_{1i} = \sum_{n_1 + \ldots + n_j = n} \prod_{l=1}^j \left( \sum_{k=2}^{\infty} \tilde{p}_{12}\tilde{p}^{(n_l)}_{2k} \tilde{p}_{k1} \right) \left( (1 - \delta_{1i})\tilde{p}_{12}\tilde{p}^{(n')}_{2i} + \delta_{1i}\tilde{p}^{(n')}_{11} \right)$$

with $n = n' + n_1 + \ldots n_j$ and $\delta_{ij}$ the Kronecker symbol. The terms in eq. (8) represent multiple returns in the site 1 before the last step to the origin. In the $n \to \infty$ limit the terms with $n'$ and all the $n_j$ bounded but one have the same asymptotic behavior as $\tilde{p}^{(n)}_{1i}$ because they are generated by the same matrix; the other terms will decay exponentially, due to $\tilde{p}_{12}$ factor, or as a power of the leading term when there are two or more unbounded exponents. If $i = 1$ $n'$ has to be bounded to avoid the exponential decay.

In the Appendix we present the computation for the generating function of the avalanches distribution probability (A13). In terms of the generating function $R(z) = \sum_t z^t P(t)$ the average time for the avalanche distribution is
\[
\bar{t} = \left. \frac{dR(\xi)}{dz} \right|_{z=1} = \left. \frac{dR(\xi)}{d\xi} \frac{d\xi}{dz} \right|_{z=1}.
\]

From (A14) we obtain that the average time of an avalanche can be written as

\[
\bar{t} \approx \left| \lambda - \lambda_{\text{critical}} \right|^{-\gamma}
\]

when \(|\lambda - \lambda_{\text{critical}}| \ll 1\), with the critical exponent \(\gamma = 1\) and the critical value of \(\lambda\), \(\lambda_{\text{critical}} = 2/3\). We can also compute the asymptotic behavior of the avalanches probability distribution from the general formula

\[
p(t) = \frac{1}{(t+1)!} \left. \frac{d^t R(z)}{dz^t} \right|_{z=0} = \frac{1}{2\pi i} \oint_{\Gamma} dz \frac{R(z)}{(z-z_0)^{t+1}} \left. \right|_{z_0=0}
\]

where \(\Gamma\) is an integration contour in the complex plane around \(z_0\) which do not circle other poles. For \(\lambda = 2/3\) we found

\[
p(t) \approx t^{-\tau}, \quad t \to \infty,
\]

with \(\tau = 3/2\), and exponential decay for \(\lambda \neq 2/3\). In the previous equations the critical exponent \(\tau = 3/2\) and \(\gamma\) have values as obtained in the mean field solution [4], whereas \(\lambda_{\text{critical}} = 2/3\) is in extremely good agreement with the critical value of \(\lambda\) found in numerical experiments [2], [3], [7].

In language of Markov chain one can say that \(\lambda = 2/3\) is the transition point between persistent states (\(\lambda \leq 2/3\)) and transient states (\(\lambda > 2/3\)) [8]. Nevertheless \(\lambda\) is not a dynamical parameter for BS model, it introduces an ”observational window” for a certain variable which we may choose from the set of statistical variables compatible with the dynamics of the BS model. SOC appears when there is at least one statistical variable with events at all scale lengths. In our approach \(\lambda\)-avalanches are bounded to origin for \(\lambda < 2/3\) and they escape to \(\infty\) for \(\lambda > 2/3\). At \(\lambda = 2/3\) we have the peculiar stationary state in which the average time of avalanches is diverging, therefore there are events on the all time scales.
A significant difference between random walk proposed in the previous section and the BS model consists in the fact that in the latter the system will spend a characteristic number of steps in a given state because the activity can appear between the most left and the most right sites where the fitness is less than $\lambda$. One possible way to improve our approximation is to promote the previous random walk to a continuous time random walk with inhomogeneous waiting time distributions, each waiting time distribution allowing for the persistence of a given size avalanche. The general equation for such a process can be written

$$P_{ik}(t) = \delta_{ik}e^{-c_i t} + \sum_{j=0}^{\infty} \int_{0}^{t} c_i e^{-c_i t} p_{ij} P_{jk}(t-t') dt'$$  \hspace{1cm} (14)

where $P_{ik}(t)$ is the probability density to have the walker in state $k$ at epoch $t$ if at $t = 0$ it was in state $i$, $p_{ij}$ are the elements of the $p'$ matrix \cite{8} and $c_i^{-1}$ is the characteristic waiting time in site $i$ and it represents the average life time for an avalanche of size $i$ in BS model. Intuitively the average time of an avalanche is a function of the average number of sites with fitness less than $\lambda$ which is increasing with the avalanche width; at criticality we propose a behavior $c_i^{-1} \approx i^\chi$ for $i > 0$ and $c_0^{-1} = \alpha$ with $\alpha$ a given constant.

An avalanche is now defined as an off time interval from the origin, whose probability distribution is independent of $\alpha$ \cite{8}. The avalanche distribution function can be expressed as

$$p_{av}(t) = \sum_{n=1}^{\infty} p(n)p_n(t)$$  \hspace{1cm} (15)

where $p(n)$ is the probability of an $n$ steps excursion out of the origin and it is the first return probability distribution for the random walk defined in section II; $p_n(t)dt$ is the probability of the first return to the origin in the interval $t, t + dt$ after $n$ steps. Intuitively we may say that for the $\lambda < 2/3$ the exponential decay of $p(n)$ will prohibit the long time avalanches and the average off time will be finite \cite{8}, while at $\lambda = 2/3$ there is a qualitative change; even $p_n(t)$ is decaying exponentially the scale invariance of $p(n)$ for large $n$ leads to critical behavior.
We have performed numerical simulation for the continuous time random walk at criticality for four scaling law of the parameters $c_i$, $c_i = \lambda_i^\chi$ with $\chi = 1, 1.5, 2, 2.5$. The numerical values for the critical exponent $\tau$ (Table II) decreases monotonically as $\chi$ increases. This behavior is intuitively clear, the avalanches tends to last longer if the characteristics life times are growing faster and $\tau \to 1$ if $\chi \to \infty$.

IV. CONCLUSIONS

We have proposed a new approach to Bak & Sneppen evolution model based on the dynamics of the avalanche width, the critical exponents are equal to those found previously in the mean field solution [4], in fact they are universal properties of the one dimensional random walk, but the value $\lambda_{\text{critical}} = 2/3$ is very close to the numerical results reported in Ref. [3], [7]. Thereby we believe that the dynamics of the avalanche width is carrying useful information on the critical behavior for this model. The structure of the generating function (A15) is intimately connected with critical behavior, the branch line appearing in $N \to \infty$ limit generates the algebraic decay for the probability distribution of the avalanches. A generating function with a finite number of poles will lead, through formula (12), to an exponential decay of the avalanches probability distribution. The continuous time random walk picture allows for a more carefully analysis of the avalanche structure and it improves the $\tau$ critical exponent keeping $\lambda_{\text{critical}}$ at the same value; it also gives an intuitive decomposition of the the algebraic decay distribution of the avalanches in a convolution of Poisson distributed events.

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APPENDIX:

We present the detailed calculation for the generating function of the avalanches probability distribution. For this propose we shall use the special form of the matrix $\tilde{p}'$ obtained from the matrix $p'$ removing the line and the row with index zero. This matrix has equal diagonal elements and we can write it as a linear combination of one-diagonal matrices $I_i$ defined as follow:

$$ (I_i)_{kl} = \begin{cases} 
\delta_{k+i,l} & i \geq 0, \\
\delta_{k,l+i} & i < 0 
\end{cases} $$  \hspace{1cm} (A1) 

$I_0$ being the identity matrix. From the definitions (A1) we can compute the commutator $T^{(i)} = I_1 I_{-i} - I_{-i} I_1$; for $i > 0$ we have

$$ (T^{(i)})_{kl} = \delta_{i+1,1} $$  \hspace{1cm} (A2) 

and for $ij > 0$ we have the property

$$ I_i I_j = I_{i+j}. $$  \hspace{1cm} (A3) 

The matrix $\tilde{p}'$ can be expressed as:

$$ \tilde{p}' = \lambda^2 I_1 + 2\lambda^2(1-\lambda)I_0 + \sum_{i=1}^{\infty} (i+2)\lambda^2(1-\lambda)^i I_{-i} = \lambda^2 I_1 A \begin{align*} 
\text{(A4)}
\end{align*} $$

where $A = \sum_{i=0}^{\infty} (i+1)(1-\lambda)^i I_{-i}$. Using eq. (A3) it is easy to compute the $n$-th power of this matrix

$$ A^n = \sum_{j=0}^{\infty} \binom{2n+j-1}{j} (1-\lambda)^j I_{-j}. $$  \hspace{1cm} (A5) 

From eq. (A4) we can compute the commutator $T_n = I_1 A_n - A_n I_1$ which has only the first column non zero

$$ (T_n)_{kl} = \binom{2n+k+1}{k} (1-\lambda)^k \delta_{l1} $$  \hspace{1cm} (A6)
consequently,
\[(I_n T_n)_{j1} = \binom{3n + j - 1}{j + n}(1 - \lambda)^{j+n}.\]  
(A7)

All the previous mentioned properties lead us to the relation
\[(I_1 A)^n = I_n A^n - \sum_{i=0}^{n-2} I_i T_i (I_1 A)^{n-i-2}.\]  
(A8)

Eq. (A8) imply the following equation for the generating matrix \(G(z) = \sum_{i=0}^{\infty} (\lambda^2 I_1 A)^i z^i\)
\[G(z) = F(z) - \sum_{i=1}^{\infty} I_i T_i^2 \lambda^{2(i+1)} z^{i+1} G(z)\]  
(A9)

where \(F(z) = \sum_{i=1}^{\infty} I_i A^i z^i\), \(z\) complex number. The sums which are appearing in eq.(A9) can be done in the following way:
\[u_j(z) = \sum_{k=1}^{\infty} (I_k T_k)_{j1} \lambda^{2(k+1)} z^{k+1} = \sum_{k=1}^{\infty} \left( \frac{3k + j - 1}{j + k} \right) (1 - \lambda)^{j+k} \lambda^{2(k+1)} z^{k+1}\]
\[= (1 - \lambda)^{j-1} \xi^3 \sum_{k=1}^{\infty} \frac{1}{(k + j)!} (3k + j - 1) \ldots 2k \xi^{2k-1}\]
\[= (1 - \lambda)^{j-1} \xi^3 \sum_{k=1}^{\infty} \frac{1}{(k + j)!} \frac{d^{k+j}}{d \xi^{k+j}} \xi^{3k+j-1} = (1 - \lambda)^{j-1} \xi^3 \sum_{k=1}^{\infty} \frac{1}{2 \pi i} \oint_{\Gamma} d\eta \frac{\eta^{3k+j-1}}{(\eta - \xi)^{j+k+1}}\]
\[= (1 - \lambda)^{j-1} \xi^3 \frac{1}{2 \pi i} \oint_{\Gamma} d\eta \frac{\eta^{j+2}}{(\eta - \xi)^{j+1} (-\eta^3 + \eta - \xi)}\]  
(A10)

where \(\xi^2 = (1 - \lambda)\lambda^2 z\). We can perform the summation if \(|\eta^3/(\eta - \xi)| < 1\), this set is not empty for \(0 < \xi < 2/3 \sqrt{1/3}\). There is a annulus with inner radius and external radius obtained from the the positive solutions of the equation \((r + \xi)^3 - r = 0\); more than that, one of the roots of the polynomial \(-\eta^3 + \eta - \xi\) is inside of the minimal integration contour \(\Gamma\) for \(0 < \xi < 2/3 \sqrt{1/3}\) and the other two are outside of the maximal integration contour. Using the above mentioned properties of the matrices \(\{I_k\}\) (A3) we can compute the elements of the matrix \(I_n A^n\) which appear in the expression of the generating matrix \(F(z)\):
\[(I_n A^n)_{1j} = \binom{3n - j}{n - j + 1}(1 - \lambda)^{n-j+1}, \quad (I_n A^n)_{21} = \binom{3n}{n+1}(1 - \lambda)^{n+1},\]
\[(I_n A^n)_{2j} = (I_n A^n)_{1j-1} \quad j > 1.\]

Eq.(7) shows that we need to compute only the first two rows in the generating matrices \(G(z)\) and \(F(z)\). The general formula for these matrix elements of \(F(z)\) is
\[ F_{1j}(z) = \delta_{j1} + \sum_{n=j-1}^{\infty} (1 - \lambda)^{n-j+1} \left( \frac{3n - j}{n - j + 1} \right) \lambda^{2n} z^n. \]

The previous series can be summed following the same computational path as in eq. (A10).

If \( j = 1 \) we get

\[ F_{1,1}(\xi) = 1 + \frac{\xi}{2\pi i} \oint_{\Gamma} \frac{\eta^2}{(\eta - \xi)(-\eta^3 + \eta - \xi)}, \]

for \( j > 1 \) we have the following expression:

\[ F_{1j}(\xi) = \frac{\xi}{(1 - \lambda)^{j-1}} \frac{1}{2\pi i} \oint_{\Gamma} \frac{\eta^{2j}}{(-\eta^3 + \eta - \xi)} \]

\[ F_{2j}(z) = F_{1j-1}(z), \quad j > 1, \text{ because } I_n A^n \text{ has equal elements on diagonals, and by direct calculation} \]

\[ F_{21}(z) = \sum_{n=1}^{\infty} \left( \frac{3n}{n+1} \right) (1 - \lambda)^{n+1} z^n = \frac{1 - \lambda}{2\pi i} \oint_{\Gamma} \frac{\eta^3}{(\eta - \xi)(-\eta^3 + \eta - \xi)}. \]

In all the above formula the contour \( \Gamma \) is the same as the that one used in eq. (A10) and \( \xi^2 = (1 - \lambda)\lambda^2 z \). Solving eq. (A9) we obtain for the first two rows the solutions in terms of previously computed functions \( u_1(z), u_2(z), F_{1j}(z), F_{2j} \):

\[ G_{1,j}(z) = \frac{F_{1j}(z)}{1 + u_1(z)}, \]

\[ G_{2,j}(z) = F_{2j}(z) - \frac{u_2(z)}{1 + u_1(z)} F_{1j}(z). \]

Residue theorem allow us to compute the generating functions in term of the third solutions of the polynomial \(-\eta^3 + \eta - \xi, \eta_3(\xi)\), that solution which lies inside of the integration contour \( \Gamma \) in the above integrals.

\[ G_{1j}(\xi) = \frac{1}{(1 - \lambda)^{j-1}} \frac{\eta_3(\xi)^{2j}}{\xi^2}, \quad i \geq 1, \quad (A11) \]

\[ G_{2j}(\xi) = -\frac{1}{(1 - \lambda)^{j-2}} \left( \frac{1}{\xi^2} - 2 \right) \frac{\eta_3(\xi)^{2j}}{\xi^2}, \quad j \geq 1; \quad (A12) \]

with \( \xi^2 = \lambda^2 (1 - \lambda) z \) and

\[ \eta_3(\xi) = \frac{1 - i\sqrt{3}}{2^{2/3}(-27\xi + \sqrt{729\xi^2 - 108})^{1/3}} - \frac{(1 + i\sqrt{3})(-27\xi + \sqrt{729\xi^2 - 108})^{1/3}}{6 2^{1/3}}. \quad (A13) \]
From eq. (7) for the avalanches probability distribution one can write the generating function:

\[ R(z) = (1 - \lambda)^2 z + z^2 p_{01} \sum_{i=1}^{\infty} G_{1i}(z)p_{i0} + z^2 p_{02} \sum_{i=1}^{\infty} G_{2i}(z)p_{i0} \]

\[ = (1 - \lambda)^2 z + 2\lambda(1 - \lambda)z^2 \sum_{i=1}^{\infty} \frac{1}{1 + u_1(z)} F_{1i}(z)(1 - \lambda)^i(1 + (i + 1)\lambda) \]

\[ + z^2\lambda^2 \sum_{i=1}^{\infty} (F_{2i}(z) - \frac{u_2(z)}{1 + u_1(z)} F_{1i}(z))(1 - \lambda)^i(1 + (i + 1)\lambda). \]

(A14)

The series which are appearing above can be summed and the closed expression for generating function reads:

\[ R(\xi(z)) = -2\frac{1 - \lambda}{\lambda} \xi^2 + \frac{2(1 - \lambda)}{\lambda^3} \xi^2 \eta_3(\xi)^2 \left(1 + 2\lambda + \frac{\eta_3(\xi)^2}{1 - \eta_3(\xi)^2} \left(1 + \lambda \frac{3 - 2\eta_3(\xi)^2}{1 - \eta_3(\xi)^2}\right)\right) \]

\[ + \frac{1 - \lambda}{\lambda^2} (1 - 2\xi^2) \eta_3(\xi)^2 \left(1 + 2\lambda + (1 + 3\lambda)\eta_3(\xi)^2 + \frac{\eta_3(\xi)^4}{1 - \eta_3(\xi)^2} \left(1 + \lambda^4 - 3\eta_3(\xi)^2\right)\right). \]

(A15)

The study of asymptotic behavior of the avalanches probability distribution is easily obtained by studying the behavior of the derivative of \( G_{1j}(z) \) and \( G_{2j}(z) \) functions. The summation appearing in \( R(z) \) changes only the amplitude for the leading term. It is crucial for the algebraic behavior the existence of the branch line in the complex plane for the generating functions \( G_{ij}(z) \) which we choose on the real axis from \((108/729)\lambda^2(1 - \lambda)\) to \( \infty \).
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### TABLE I. The numerical values of the critical exponent $\tau$ for four values of $\chi$

| $\chi$ | $\tau$     |
|-------|------------|
| 1     | 1.27 ± 0.01|
| 1.5   | 1.19 ± 0.01|
| 2     | 1.14 ± 0.01|
| 2.5   | 1.01 ± 0.01|