Asymptotic behavior of a generalized Navier-Stokes-Bardina’s model and applications to related models

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Abstract

We consider here a general theoretical equation on the whole three-dimensional space, which contains as a particular case some relevant equations of the fluid dynamics as the Navier-Stokes-Bardina’s model, the fractional and the classical Navier-Stokes equations with an additional drag/friction term. These equations arise from ocean and atmospheric models. For the general equation, we study first the existence and in some cases the uniqueness of finite energy solutions. Then, we use a general framework to study their long behavior with respect to the weak and the strong topology of the phase space. We thus prove the existence of a weak global attractor and in some cases the existence of a strong global attractor. Moreover, we study some sufficient conditions to insure the weak global attractor becomes a strong global attractor. As a bi-product, we obtain some new results on the long time description of the fractional and classical Navier-Stokes models with a damping term.

Keywords: Navier-Stokes equations; Bardina’s model; Bessel potentials; Weak and strong global attractor.

AMS Classification: 35B40, 35D30.

1 Introduction

The study of the fluid dynamics provides us several evolution models of great importance, among them, the well-known Navier-Stokes equations and some related equations also called the α—models. The α—models have been developed in the mathematical literature as physically relevant approximations of the Navier-Stokes equations see, for instance, the chapter 17 of [16]. A deep comprehension of the long time asymptotics of their finite energy solutions is one of the key questions to a better understanding of these models.

In the previous work [11], we focused on one of these α—models, known as the damped Bardina’s model. This equation arises as a successful model in the study of oceanography, turbulence and some atmospheric models [1, 3, 2, 8, 19]. Mathematically, the damped Bardina’s model writes down as the following equation on the whole space $\mathbb{R}^3$:

\[
\begin{align*}
\partial_t \vec{u} - \nu \Delta \vec{u} + (I_d - \Delta)^{-1} (\text{div}(\vec{u} \otimes \vec{u})) + \nabla p &= \vec{f} - \gamma \vec{u}, \\
\text{div}(\vec{u}) &= 0 \\
\vec{u}(0, \cdot) &= \vec{u}_0.
\end{align*}
\]

(1)

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Here, \( \tilde{u} : [0, +\infty[ \times \mathbb{R}^3 \to \mathbb{R}^3 \) and \( p : [0, +\infty[ \times \mathbb{R}^3 \to \mathbb{R} \) are the velocity of the fluid and the pressure term respectively. The second equation \( \text{div}(\tilde{u}) = 0 \) describes the fluid’s incompressibility. Moreover, the function \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) is the external force, which is assumed as a time independent and divergence free vector field, while the function \( \tilde{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3 \) denotes the initial velocity field at the time \( t = 0 \). Finally, \( 0 < \nu \) is the viscosity parameter.

The main features in the mathematical writing of this model are given by the following key terms. On the one hand, the Bessel potential \((I_d - \Delta)^{-1}\) in the nonlinear term of this equation. This operator was added in the classical Navier-Stokes equation by J. Bardina, J. H. Ferziger, & W. C. Reynolds in [3] as a filtering/averaging operator which allows us to obtain an accurate model describing the large-scale motion of the fluid while filtering or averaging the fluid motion at small scales. On the other hand, the damping term \(-\gamma \tilde{u}\) on the right side of this equation. This drag/friction term \(-\gamma \tilde{u}\), where \( 0 < \gamma \) is the Rayleigh or Ekman friction coefficient, has a relevant physical meaning as it models the bottom friction in ocean models and is the main energy sink in large scale atmospheric models [19].

A remarkable featured of the equation (1) is the existence of a strong global attractor to which all the finite energy solutions arising from any initial datum \( \tilde{u}_0 \in H^1(\mathbb{R}^3) \), with \( \text{div}(\tilde{u}_0) = 0 \), and for any external force \( f \in H^1(\mathbb{R}^3) \), with \( \text{div}(f) = 0 \). Moreover, they verify:

\[
\tilde{u} \in L^\infty_{\text{loc}}\left([0, +\infty[; H^1(\mathbb{R}^3)\right) \cap L^2_{\text{loc}}\left([0, +\infty[; H^2(\mathbb{R}^3)\right), \quad p \in L^2_{\text{loc}}\left([0, +\infty[; H^3(\mathbb{R}^3)\right).
\]

Thereafter, in Theorem 2 of [11] we proved the existence of a strong global attractor for the equation (1) which gives a good comprehension of the long time dynamics of finite energy solutions. For a definition and the main properties of the strong global attractor see Definition 2.4 below. The existence of a strong global attractor is essentially based on the contributions of the two key terms explained above. In what follows, we briefly explain these facts.

- For \( 0 < \gamma \) the damping term \(-\gamma \tilde{u}\) acts as a dissipative term and it allows us to prove the following energy estimate:

\[
\| \tilde{u}(t, \cdot) \|^2_{H^1(\mathbb{R}^3)} \lesssim \| \tilde{u}_0 \|^2_{H^1(\mathbb{R}^3)} e^{-\gamma t} + \frac{1}{\gamma^2} \| f \|^2_{H^1(\mathbb{R}^3)}, \quad 0 \leq t,
\]

which gives us a good control on the quantity \( \| \tilde{u}(t, \cdot) \|^2_{H^1(\mathbb{R}^3)} \) respect to the time variable \( t \). Precisely, due to the term \( \| \tilde{u}_0 \|^2_{H^1(\mathbb{R}^3)} e^{-\gamma t} \), when the time \( t \) is large enough we essentially obtain the inequality

\[
\| \tilde{u}(t, \cdot) \|^2_{H^1(\mathbb{R}^3)} \lesssim \frac{1}{\gamma^2} \| f \|^2_{H^1(\mathbb{R}^3)}, \quad t \to +\infty,
\]

which is one of the key estimates in the proof of Theorem 2 in [11]: when the time goes to infinity, all the finite energy solutions arising from any initial datum \( \tilde{u}_0 \in H^1(\mathbb{R}^3) \) fall inside of the ball in the space \( H^1(\mathbb{R}^3) \) centered at the origin and with ratio \( \frac{1}{\gamma^2} \| f \|^2_{H^1(\mathbb{R}^3)} \). This ball is also called an absorbing set associated for the equation (1) (see Definition 2.2) and is one of the key features of the equation (1) in the existence of a global attractor. See Theorems 4.1 and 4.2 for more details.

Moreover, it worth mentioning in the case when \( \gamma = 0 \), where we obtain the classical Bardina’s model:

\[
\partial_t \tilde{u} - \nu \Delta \tilde{u} + (I_d - \Delta)^{-1} (\text{div}(\tilde{u} \otimes \tilde{u})) + \nabla p = \tilde{f},
\]

in the space-periodic setting of the torus \( \mathbb{T}^3 = [0, L]^3 \), we can use the Poincaré inequality to obtain the following estimate:

\[
\| \tilde{u}(t, \cdot) \|^2_{H^1(\mathbb{T}^3)} \lesssim \| \tilde{u}_0 \|^2_{H^1(\mathbb{T}^3)} e^{-\nu t} + \| f \|^2_{H^1(\mathbb{T}^3)}, \quad 0 \leq t,
\]
for a constant $0 < \eta$ depending on the period $0 < L$. However, in the non-periodic setting of the whole space $\mathbb{R}^3$, due to the lack of the Poincaré inequality, to the best of our knowledge we have the energy estimate:
\[
\|\tilde{u}(t, \cdot)\|^2_{H^1(\mathbb{R}^3)} \lesssim \|\tilde{u}_0\|^2_{H^1(\mathbb{R}^3)} + t \int_0^t \|\hat{\tilde{u}}(\xi, \cdot)\|^2_{H^1(\mathbb{R}^3)} \, ds \leq 0, \quad t \leq t,
\]
where we clearly lose any control when the time goes to infinity. Therefore, from the mathematical point of view, the damping term $-\gamma \tilde{u}$ is useful to compensate the lack of the Poincaré inequality. This damping term was also used in some previous works related to the study of the long time behavior for the Navier-Stokes equations and related models [10, 13, 14]. Let us mention that other (merely technical) damping terms can be also considered for this type of study [5, 15], but we shall consider here the term $-\gamma \tilde{u}$ due to its relevant physical meaning in the Bardina’s model.

- The Bessel potential $(I_d - \Delta)^{-\frac{\alpha}{2}}$ applied to the nonlinear term $\text{div}(\tilde{u} \otimes \tilde{u})$ in the equation (1) has an important regularising effect in the framework of the non-homogeneous Sobolev spaces. This fact allows us to prove the uniqueness of finite energy solutions, which is still an outstanding open question for the classical Navier-Stokes equations. Uniqueness of finite energy solutions is also one of the key ideas in the proof of Theorem 2 in [11]: we are able to define a semigroup $S(t) : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$, where $S(t)\tilde{u}_0 = \tilde{u}(t, \cdot)$ is the unique finite energy solution of (1) arising from the initial datum $\tilde{u}_0$. Moreover, $(S(t))_{t \geq 0}$ is a strongly continuous semigroup acting on the Banach space $H^1(\mathbb{R}^3)$ and then we can use some results of the theory of dynamical systems (see Section 4.2.2 for more details) to prove the existence of a strong global attractor.

The aim of this work is to investigate in a fine way the contribution of the Laplacian operator and the Bessel potential in the long time dynamics of finite energy solutions of the damped Bardina’s model. For this, our key idea is to substitute these operators with their fractional versions:
\[
(-\Delta)^{\frac{\alpha}{2}}, \quad (I_d - \Delta)^{-\frac{\alpha}{2}} \quad 0 < \alpha, \quad 0 \leq \beta,
\]
which are defined in the Fourier variable by the symbols $|\xi|^\alpha$ and $(1 + |\xi|^2)^{-\beta}$ respectively. Therefore, we shall consider the following generalized Navier-Stokes Bardina’s type model on the whole space $\mathbb{R}^3$:
\[
\begin{cases}
\partial_t \tilde{u} + \nu(-\Delta)^{\frac{\alpha}{2}} \tilde{u} + (I_d - \Delta)^{-\frac{\alpha}{2}} \mathbb{P}(\text{div}(\tilde{u} \otimes \tilde{u})) = \tilde{f} - \gamma \tilde{u}, & \text{div}(\tilde{u}) = 0, \quad 0 < \nu, \quad 0 < \gamma, \\
\tilde{u}(0, \cdot) = \tilde{u}_0.
\end{cases}
\]
(3)
Here, the operator $\mathbb{P}$ stands for the Leray’s projector given by $\mathbb{P}(\tilde{\varphi}) = \tilde{\varphi} - \nabla \cdot \frac{1}{\Delta}(\text{div}(\tilde{\varphi}))$. It is worth mentioning we have applied the Leray’s projector to this equation as the pressure term does not play any substantial role in our study.

In this theoretical equation, the dissipative effects of the fractional Laplacian operator are measured by the parameter $0 < \alpha$, while the regularizing effects of the Bessel potential are measured by the parameter $0 \leq \beta$. Therefore, the quantity $0 < \alpha + \beta$ quantifies the total contribution of both dissipative and regularizing effects in the qualitative study of the equation (3). Our main objective is then to find conditions on the quantity $\alpha + \beta$ to prove the existence of finite energy solutions, to study their uniqueness and to study their long time behavior through the notion of the global attractor.

Briefly summarizing, our main results are driven by the parameter $\alpha + \beta$ as follows: for $0 < \alpha + \beta < +\infty$ we construct finite energy solutions of the equation (3); and for $\frac{\alpha}{2} \leq \alpha + \beta < +\infty$ we prove their uniqueness. Thereafter, on the one hand, for the range of values $\frac{3}{2} \leq \alpha + \beta < +\infty$ we are able to adapt some methods used in [11], mainly based on uniqueness of solutions and energy estimates, to prove the existence of a strong global attractor. On the other hand, for the range of values $0 < \alpha + \beta \leq \frac{5}{2}$, where the uniqueness of finite energy solutions is unknown, we introduce the notion of the weak global attractor developed in [6] to describe their long time behavior in a sharp way. Finally, we are also interested in studying the effects of the viscosity parameter $0 < \nu$ and the damping parameter $0 < \gamma$ in the long time behavior. It is thus interesting
to observe that, in the range of values $2 \leq \alpha + \beta \leq \frac{3}{2}$, the weak global attractor becomes a strong global attractor, provided the parameters $\nu$ and $\gamma$ are larger than the size (in a norm to precise) of the external force acting on the equation (3).

It is worth mentioning our results are essentially obtained by sharp energy methods, which make them more interesting from a physical point of view as we only control the naturally energy quantities derived from the equation (3).

The theoretical equation (3) is also of interest as it contains as a particular case some relevant models. When we set $\alpha = \beta = 2$, the equation (3) agrees with the damped Navier-Stokes-Bardina’s model (1); and in this sense the equation (3) is a generalization of the equation (1). On the other hand, for $0 < \alpha$ and $\beta = 0$ we obtain the following damped version of the fractional Navier-Stokes equations:

$$\partial_t \bar{u} + \nu(-\Delta)^\frac{\alpha}{2} \bar{u} + \mathbb{P}(\text{div}(\bar{u} \otimes \bar{u})) = \bar{f} - \gamma \bar{u}, \quad \text{div}(\bar{u}) = 0.$$ 

This equation has recently attired the attention of researchers in the mathematical fluid dynamics to understand the dissipative effects, given by the fractional Laplacian operator, in the study of outstanding open problems in the classical Navier-Stokes equations, for instance, uniqueness and regularity issues of Leray’s weak solutions [7, 9, 18]. Finally, in the particular case when $\alpha = 2$ and $\beta = 0$, the equation (3) deals with the classical damped Navier-Stokes equations:

$$\partial_t \bar{u} - \nu \Delta \bar{u} + \mathbb{P}(\text{div}(\bar{u} \otimes \bar{u})) = \bar{f} - \gamma \bar{u}, \quad \text{div}(\bar{u}) = 0.$$ 

As already explained, in the setting of the whole space $\mathbb{R}^3$, when studying the large time behavior of solutions this equation is an interesting counterpart of the classical (when $\gamma = 0$) Navier-Stokes equations with space-periodic conditions. We refer to [10] and [13] for some interesting previous related works on this equation.

To close this section, let us mention this is the first step in a wider program that aims a deep comprehension on the main features of the global attractor for the equation (3), both in the weak and the strong sense. In forthcoming investigations we study the fractal dimension of the global attractor in terms of the parameters $\alpha, \beta, \gamma$ and $\nu$. Moreover, we are also interested in some convergence properties of the inviscid limit when $\nu \to 0^+$. 

Organization of the paper: In Section 2 we introduce some definition and notation, and moreover, we present and we discuss all our results. Section 3 is devoted to the study of the main features of finite energy solutions of the equation (3), while in Section 4 we focus on their long time asymptotic behavior.

2 Definitions and presentation of the results

For the reader’s convenience, we have divided this section in three parts. In the first part we study the finite energy solutions of the equation (3). Then, in the second part we study their long time behavior and finally, in the third part, we explain how our results apply to the related models (1), (9) and (10).

2.1 Finite energy solutions

From now on, these solutions shall be called the Leray-type solutions as they share the main properties of the well-known Leray’s solutions in the classical Navier-Stokes theory.

Definition 2.1 (Leray-type solution) Let $0 < \alpha$ and $0 \leq \beta$. We shall say that $\bar{u}$ is a Leray-type solution of the equation (3) if:

1. The function $\bar{u}$ belongs to the energy space: $L^\infty_{\text{loc}}([0, +\infty[, H^\alpha(\mathbb{R}^3)) \cap L^2_{\text{loc}}([0, +\infty[, H^{\alpha+\beta}(\mathbb{R}^3))$, and it verifies the equation (3) in the distributional sense.
2. For all $0 \leq t$, the following energy inequality holds:
\[
\|\bar{u}(t, \cdot)\|_{H^\beta_2}^2 \leq \|\bar{u}_0\|_{H^\beta_2}^2 - 2\nu \int_0^t \|(-\Delta)^{\frac{\beta}{2}} (I_d - \Delta)^{\frac{\beta}{2}} \bar{u}(s, \cdot)\|_{L^2}^2 ds + 2 \int_0^t \left( \bar{f}(s, \cdot), \bar{u}(s, \cdot) \right)_{H^\beta_2} ds - 2\gamma \int_0^t \|\bar{u}(s, \cdot)\|_{H^\beta_2}^2 ds,
\]
provided that $\bar{u}_0 \in H^\beta_2(\mathbb{R}^3)$ and $\bar{f} \in L^2_{\text{loc}}([0, +\infty[, H^\beta_2(\mathbb{R}^3))$, and where $(\cdot, \cdot)_{H^\beta_2}$ stands for the usual inner product in the space $H^\beta_2(\mathbb{R}^3)$.

Our first result proves the existence of Leray-type solutions. It is worth emphasizing the existence is insured for all $0 < \alpha$ and $0 \leq \beta$.

**Theorem 2.1** Let $0 < \alpha$ and $0 \leq \beta$. Let $\bar{u}_0 \in H^\beta_2(\mathbb{R}^3)$ be a divergence free initial datum. Moreover, let $\bar{f} \in L^2_{\text{loc}}([0, +\infty[, H^\beta_2(\mathbb{R}^3))$ be a divergence free external force. Then, there exists $\bar{u}$ a Leray-type solution of the equation (3) given in Definition 2.1.

Leray-type solutions also verify the following energy estimates. As we shall see later, these estimates are of key importance when studying the long time behavior.

**Proposition 2.1** Within the framework of Theorem 2.1, Leray-type solutions of the equation (3) verify the following energy estimates.

1. For $0 < \gamma$ and for all $0 \leq t$ we have:
\[
\|\bar{u}(t, \cdot)\|_{H^\beta_2}^2 \leq e^{-\gamma t} \left( \|\bar{u}_0\|_{H^\beta_2}^2 + \frac{4}{\gamma} \int_0^t e^{\gamma s} \|\bar{f}(s, \cdot)\|_{H^\beta_2}^2 ds \right).
\]

2. For $0 < \alpha$ and $0 < \gamma$ there exists a constant $0 < A_{\alpha, \gamma, \nu} = \frac{\min(\gamma, \nu)}{\max(1, 2^{\frac{\alpha - 2}{2}})}$, such that for all $0 \leq t$ we have:
\[
\|\bar{u}(t, \cdot)\|_{H^\beta_2}^2 + A_{\alpha, \gamma, \nu} \int_0^t \|\bar{u}(s, \cdot)\|_{H^{\alpha + \beta}_2}^2 ds \leq \|\bar{u}_0\|_{H^\beta_2}^2 + 2 \int_0^t \left( \bar{f}(s, \cdot), \bar{u}(s, \cdot) \right)_{H^\beta_2} ds.
\]

3. For $0 < A_{\alpha, \gamma, \nu}$ the constant given above and for all $0 \leq t$ and $0 < T$ we have:
\[
A_{\alpha, \gamma, \nu} \int_t^{t + T} \|\bar{u}(s, \cdot)\|_{H^{\alpha + \beta}_2}^2 ds \leq e^{-\gamma t} \left( \|\bar{u}_0\|_{H^\beta_2}^2 + \frac{4}{\gamma} \int_0^t e^{\gamma s} \|\bar{f}(s, \cdot)\|_{H^\beta_2}^2 ds \right)
\]
\[
+ \frac{1}{A_{\alpha, \gamma, \nu}} \int_t^{t + T} \|\bar{f}(s, \cdot)\|_{H^\beta_2}^2 ds.
\]

The first estimate is a direct consequence of the dissipative effects of the damping term $-\gamma \bar{u}$. In particular, the expression $e^{-\gamma t}$ gives us a very good control in time on the quantity $\|\bar{u}(t, \cdot)\|_{H^\beta_2}^2$ and this fact shall be well exploited later.

In the next result, we study some sufficient conditions on the parameter $\alpha + \beta$ to prove the uniqueness of Leray-type solutions.

**Theorem 2.2** Within the framework of Theorem 2.1, if $\frac{5}{2} \leq \alpha + \beta$ then the equation (3) has a unique Leray-type solution.
The lower bound $\frac{5}{2}$ found in this result corresponds to minimal regularity properties on the Leray-type solutions to handle the nonlinear term in the equation (3). Essentially, this result shows that uniqueness of Leray-type solutions is insured when the total contribution $\alpha + \beta$ of both dissipative and regularising effects (given by the fractional operator and the Bessel potential respectively) is larger than $\frac{5}{2}$.

It is interesting to observe that the value $\frac{5}{2}$ corresponds to the critical one obtained in related studies [7, 17, 18] on the uniqueness of Leray-type solutions of the fractional Navier-Stokes equation (9): by setting $\beta = 0$ we obtain the critical value $\frac{5}{2} \leq \alpha$.

For the range of values $0 < \alpha + \beta < \frac{5}{2}$, the regularity properties of Leray-type solutions naturally given by the energy space $L^\infty_t H^{\beta}_x \cap L^2_t H^{\alpha+\beta}_x$ seems not to be enough to control the nonlinear term. Therefore, their uniqueness in this supercritical range is still an open problem. We refer to Remark 6.11 in Chapter 1 of [17] for a more detailed discussion in the case of the equation (9).

2.2 The asymptotic behavior of Leray-type solutions

We consider $\vec{f} \in H^{\beta}(\mathbb{R}^3)$ (with $0 \leq \beta$) a time-independent external force acting on the evolution equation (3) and we shall study the long time behavior of Leray-type solutions. Before to state our results we need first to precise some notation and definition.

Our first definition concerns the notion of an absorbing set for the evolution equation (3):

**Definition 2.2 (Absorbing set)** A set $B \subset H^{\tilde{\beta}}_x(\mathbb{R}^3)$ is an absorbing set for the equation (3) if for every initial datum $\vec{u}_0 \in H^{\tilde{\beta}}_x(\mathbb{R}^3)$ there exists a time $0 < T = T(\vec{u}_0)$ such that for all $T < t$ all the Leray-type solutions $u(t,x)$ arising from $\vec{u}_0$ verify $\vec{u}(t,\cdot) \in B$.

As a direct consequence of the energy estimate given in first point of Proposition 2.1 we have the following result:

**Proposition 2.2** Let $0 < \gamma$ and $\vec{f} \in H^{\tilde{\beta}}_x(\mathbb{R}^3)$. We define

$$B = \left\{ \vec{u}_0 \in H^{\tilde{\beta}}_x(\mathbb{R}^3) : \|\vec{u}_0\|^2_{H^{\frac{\beta}{2}}} \leq \frac{5}{\gamma^2} \|\vec{f}\|^2_{H^{\frac{\tilde{\beta}}{2}}} \right\}.$$  

Then, $B$ is an absorbing set for the equation (3) in the sense of Definition 2.2.

As we may observe, the absorbing set is defined by the damping parameter $\gamma$ and the external force $\vec{f}$. Moreover, the expression $\frac{1}{\gamma^2}$ clearly shows that this definition only makes sense when $0 < \gamma$, i.e., in the damped case of the equation (3).

The existence of an absorbing set for the equation (3) is one of its key features in the study of the long time behavior of Leray-type solutions. In Definition 2.2 we may observe that all these solutions belong to the set $B$ when the time is large enough, and consequently, the study of their long-time behavior can be restricted to the set $B$.

In what follows, we explain how the absorbing set $B$ is the key tool in our study. The set $B \subset H^{\tilde{\beta}}_x(\mathbb{R}^3)$ can be provided of two topologies: the strong topology and the weak topology inherited from the space $H^{\tilde{\beta}}_x(\mathbb{R}^3)$. Thus, when considering the strong topology the absorbing set $B$ is a topological space with the topology generated by the usual strong distance:

$$d_s(\vec{u}_0, \vec{v}_0) = \|\vec{u}_0 - \vec{v}_0\|_{H^{\frac{\beta}{2}}}, \quad \text{for all} \quad \vec{u}_0, \vec{v}_0 \in B.$$  

On the other hand, as $B$ is a strongly bounded set in $H^{\tilde{\beta}}_x(\mathbb{R}^3)$, it is well-known that $B$ is a compact space when considering the weak topology. Moreover, this weak topology on $B$ is generated by the weak distance
Definition 2.4 (Global attractor) A set $\mathcal{A}$ for the evolution equation (3) is a global attractor when the time goes to infinity the time behavior of Leray-type solutions attains the global attractor. More precisely, by Definition 2.3 we have that from any initial datum $\vec{u}_0 \in \mathcal{B}$ we have that $\vec{u}_0 = \sum_{n \in \mathbb{N}} u_n \vec{e}_n$, where $u_n = (\vec{u}_0, \vec{e}_n)_{H^2}$. Thereafter, the weak distance $d_w$ on $\mathcal{B}$ is given by:

$$d_w(\vec{u}_0, \vec{v}_0) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} \frac{|u_n - v_n|}{1 + |u_n - v_n|}, \quad \text{for all } \vec{u}_0, \vec{v}_0 \in \mathcal{B}. \quad (6)$$

For the sake of simplicity, we shall denote the topological metric space $(\mathcal{B}, d_s)$, where $s$ stands for either $s$ or $w$ in the case of the strong or the weak distances given in (5) and (6) respectively. In this framework, when $s = w$ all the properties of the topological metric space $(\mathcal{B}, d_w)$ will refer as strong properties, while when $s = w$ all the properties of the topological metric space $(\mathcal{B}, d_w)$ will refer as weak properties.

Our next definition is devoted to the notion of an strong and weak attracting set for the evolution equation (3). We recall first that, in the metric space $(\mathcal{B}, d_s)$, for $\mathcal{B} \subset \mathcal{B}$ and $\vec{u}_0 \in \mathcal{B}$ we define by

$$d_s(\vec{u}_0, \mathcal{B}) = \inf_{\vec{v} \in \mathcal{B}} d_s(\vec{u}_0, \vec{v}), \quad (7)$$

the distance of the point $\vec{u}_0$ to the set $\mathcal{B}$. Then, we have:

**Definition 2.3 (Attracting set)** A set $\mathcal{B} \subset \mathcal{B}$ is a $s$-attracting set for the equation (3) if for all initial datum $\vec{u}_0 \in H^2(\mathbb{R}^3)$ and for all $0 < \varepsilon$ there exists $0 < T = T(\vec{u}_0, \varepsilon)$ such that all the Leray-type solutions arising from $\vec{u}_0$ verify $d_s(\vec{u}(t, \cdot), \mathcal{B}) < \varepsilon$, for all $T < t$.

Once we have the notion of the $s$-attracting set, we are able to introduce now the $s$-global attractor for the evolution equation (3).

**Definition 2.4 (Global attractor)** A set $\mathcal{A} \subset \mathcal{B}$ is a $s$-global attractor for the equation (3) if:

1. The set $\mathcal{A}$ is $s$-compact.
2. The set $\mathcal{A}$ is the minimal $s$-attracting set in the sense of Definition 2.3.

As mentioned, the notion of the $s$-global attractor is the key idea in a sharp understanding of the long time behavior of Leray-type solutions. In this definition we focus on third point to remark that when the time goes to infinity the $s$-global attractor attires the Leray-type solutions of the equation (3). More precisely, by Definition 2.3 we have that from any initial datum $\vec{u}_0 \in H^2(\mathbb{R}^3)$ all the arising Leray-type solutions are as close to $\mathcal{A}$ as we want when the time is large enough.

This convergence of Leray-type solutions to the $s$-global attractor is then is measured in terms of the distance $d_s$ given in (7). Thus, when $s = w$ these solutions converge to the strong global attractor $\mathcal{A}_s$ in the strong topology of the metric space $(\mathcal{B}, d_s)$, while when $s = w$ these solutions converge to the weak global attractor in the weak topology of the metric space $(\mathcal{B}, d_w)$. Therefore, the strong or the weak featured of the global attractor is uniquely determined by the type of convergence in terms of the strong or the weak topology respectively.

Of course, when the strong global attractor exists there also exists the weak global attractor and both coincide. However, the reverse property may not be true. We refer to [6] for some examples of simpler evolution equations that possess a weak global attractor, but not a strong global attractor.

Getting back to our evolution equation (3), in our next result, we study the existence of a global attractor and its weak or strong featured.

**Theorem 2.3** The following statements hold:
1. When \( 0 < \alpha + \beta < \frac{5}{2} \), there exists a unique weak \((\bullet = w)\) global attractor \(A_w\) for the equation (3).

2. When \( \frac{5}{2} \leq \alpha + \beta \), there exists a unique strong \((\bullet = s)\) global attractor \(A_s\) for the equation (3).

For the sake of a better exposition of this theorem, first we shall focus on the second point. This result is deeply based on the uniqueness of Leray-type solutions proven in Theorem 2.2. Precisely, uniqueness allows us to define a strongly continuous semigroup \((S(t))_{t \geq 0}\) on the space \(H^\frac{5}{2}(\mathbb{R}^3)\): for \( \tilde{u}_0 \in H^\frac{5}{2}(\mathbb{R}^3) \) and for \( 0 \leq t \) we have \( S(t)\tilde{u}_0 = \tilde{u}(t, \cdot) \), where \( \tilde{u}(t, \cdot) \) is the unique Leray-type solution of the equation (3) arising from \( \tilde{u}_0 \). Uniqueness is also one of the key properties to show that \( (\bullet = w) \) global attractor. In the case when \( \alpha = \beta = 2 \), we recover the Theorem 2 proven in [11].

These ideas are not longer valid in the first point of this theorem, where the uniqueness of Leray-type solutions is unknown, and we are obliged to use a different approach. For all \( \tilde{u}_0 \in \mathcal{B} \), where \( \mathcal{B} \) is the absorbing set given in (4), and for \( 0 \leq t \) we define now the set \( R(t)\tilde{u}_0 \subset H^\frac{5}{2}(\mathbb{R}^3) \) of all the Leray-type solutions \( \tilde{u}(t, \cdot) \) arising from \( \tilde{u}_0 \). As uniqueness is unknown in this case, the family \((R(t))_{t \geq 0}\) does not define a semigroup on \( H^\frac{5}{2}(\mathbb{R}^3) \) and this fact is the main difference respect to the previous case. In this case, the key idea is that the absorbing set \( \mathcal{B} \) (which contains the global attractor when it exists) is a compact topological metric space with the weak distance \( d_w \) given in (6). Thus, we can prove that \((R(t))_{t \geq 0}\) is a weakly uniformly compact family (see Definition 4.1), and then, we are able to apply some interesting results of [6] (see Theorem 4.1) to prove the existence of a weak global attractor.

As explained before, the main difference between the strong global attractor (in the case \( \frac{5}{2} \leq \alpha + \beta \)) and the weak global attractor (in the case \( 0 < \alpha + \beta < \frac{5}{2} \)) essentially bases on the convergence in the strong topology or the weak topology of the space \( H^\frac{5}{2}(\mathbb{R}^3) \) respectively. Nevertheless, both notions give us a good comprehension of the long time dynamics of Leray-type solutions. In particular, the notion of a weak global attractor is not studied in some prior works [4, 12], which consider related equations to (3) in the space-periodic setting.

In the case \( 0 < \alpha + \beta < \frac{5}{2} \), the existence of a strong global attractor is not yet known in all generality. However, in our next result we investigate some sufficient conditions to give a positive answer in this case.

**Theorem 2.4** Let \( \frac{3}{2} < \alpha + \frac{5}{2} \) and \( 2 \leq \alpha + \beta < \frac{5}{2} \). There exists an universal constant \( 0 < C_0 \), which does not depend on any parameter in the equation (3), such that if

\[
\| \tilde{f} \|_{H^\frac{5}{2}} < C_0 \gamma^2 \quad \text{and} \quad \| \tilde{f} \|_{H^\frac{5}{2}} < C_0 \nu^2,
\]

then the weak global attractor \( A_w \) becomes a strong global attractor \( A_s \).

The conditions \( \frac{3}{2} < \alpha + \frac{5}{2} \) and \( 2 \leq \alpha + \beta < \frac{5}{2} \) are required to justify all the computations in the proof, but we think they could be improved to some values in the case \( \alpha + \beta < 2 \). However, these conditions are not too restrictive as they include the case \( \alpha = 2 \) and \( \beta = 0 \) which is of particular interest as we shall explain in the next section. On the other hand, an interesting featured about this result is the condition (8). We may observe here the effects of the parameters \( \gamma \) and \( \nu \) in the long time behavior of the equation (3). More precisely, the existence of a strong global attractor is insured when these parameters are large enough respect to the size of the external force acting on the equation (3).

Let us briefly explain the key idea to prove this result. By assuming the condition (8), we can construct a particular solution \( \tilde{U} \) of the equation (3), which only depends on the spatial variable, also called the stationary solution. Thereafter, always with the assumption (8); and by performing some sharp energy estimates, we prove that \( \{ \tilde{U} \} \) is an strong global attractor of the equation (3). Therefore, by uniqueness we can conclude \( A_w = A_s = \{ \tilde{U} \} \).
It is thus interesting to observe this link between the stationary solution and the long time behavior of Leray-type solutions, provided the condition (8) holds. On the other hand, the existence or not of a strong global attractor in its complementary case (when $\gamma^2 \lesssim \|f\|_{H^{\gamma}}^2$ or $\nu^2 \lesssim \|f\|_{H^{\gamma}}^2$) is far from obvious and it shall be a matter of further investigations.

2.3 Applications to the fractional and classical Navier-Stokes equations

The results obtained for the general equation (3) have as corollaries some new results on the long time behavior of the fractional and classical damped Navier-Stokes equations respectively. In this section, we summarize and we discuss these results.

For the sake of completeness, by setting the parameter $\beta = 0$, we start by recalling the definition of Leray-type solutions for the equation:

$$\partial_t \bar{u} + \nu(-\Delta)^{\frac{\alpha}{2}} \bar{u} + P(\text{div}(\bar{u} \otimes \bar{u})) = \bar{f} - \gamma \bar{u}, \quad \text{div} \bar{u} = 0.$$  \tag{9}

**Definition 2.5** Let $0 < \alpha$. We shall say that $\bar{u}$ is a Leray-type solution of the equation (9) if:

1. The function $\bar{u}$ belongs to the energy space: $L^\infty_{\text{loc}}([0, +\infty[, L^2(\mathbb{R}^3)) \cap L^2_{\text{loc}}([0, +\infty[, \dot{H}^\alpha(\mathbb{R}^3))$, and it verifies the equation (9) in the distributional sense.

2. For $\bar{u}_0 \in L^2(\mathbb{R}^3)$, $\bar{f} \in L^2(\mathbb{R}^3)$ and for all $0 \leq t$ the following energy inequality holds:

$$\|\bar{u}(t,\cdot)\|_{H^{\frac{\alpha}{2}}}^2 \leq \|\bar{u}_0\|_{H^{\frac{\alpha}{2}}}^2 - 2\nu \int_0^t \|(-\Delta)^{\frac{\alpha}{4}} \bar{u}(s,\cdot)\|_{L^2}^2 ds + 2 \int_0^t \left( \bar{f}(s,\cdot), \bar{u}(s,\cdot) \right)_{L^2} ds - 2\gamma \int_0^t \|\bar{u}(s,\cdot)\|_{L^2}^2 ds.$$

Now, we are able to state the following result on the existence and uniqueness of Leray-type solutions:

**Theorem 2.5 (Corollary of Theorems 2.1 and 2.2)**. For all $0 < \alpha$, there exists $\bar{u}$ a Leray-type solution of the equation (9). Moreover, if $\frac{5}{2} \leq \alpha$ this equation has a unique Leray-type solution.

Thereafter, by the enegy estimates given in Proposition 2.1, the equation (9) has the following absorbing set given in Definition 2.2 (with $\beta = 0$):

$$\mathcal{B}_1 = \left\{ \bar{u}_0 \in L^2(\mathbb{R}^3) : \|\bar{u}_0\|_{L^2}^2 \leq \frac{5}{\gamma^2} \|\bar{f}\|_{L^2}^2 \right\}.$$

As before, the absorbing set is defined by the damping parameter $0 < \gamma$ and the external force acting on the equation (9), which always assumes a time independent function. Moreover, in this case we have $\mathcal{B}_1 \subset L^2(\mathbb{R}^3)$ and consequently all the weak or the strong properties refer to the weak or the strong topology in $L^2(\mathbb{R}^3)$.

In the following theorem, we state the existence of a global attractor for the equation (9) as well as its weak or strong feature.

**Theorem 2.6 (Corollary of Theorems 2.3 and 2.4)**. Within the setting of Definition 2.4, the following statements hold:

1. When $0 < \alpha < \frac{5}{2}$, there exists a unique weak global attractor $\mathcal{A}_w \subset \mathcal{B}_1$ for the equation (9).

2. When $2 \leq \alpha < \frac{5}{2}$, by assuming the condition (8) the weak global attractor $\mathcal{A}_w$ becomes a strong global attractor $\mathcal{A}_s$. 

9
3. When $\frac{5}{2} \leq \alpha$, there exists a unique strong global attractor $A_s \subset B_1$ for the equation (9).

As we may observe, the weak or the strong feature of the global attractor is mainly determined by parameter $\alpha$, which measures the effects of the operator $(-\Delta)^{\frac{\alpha}{2}}$ in the equation (9). However, in the second point, through the assumption (8) we also observe the effects of the parameters $0 < \gamma$ and $0 < \nu$ in the existence of a strong global attractor. Moreover, in the third point, the existence of a strong global attractor is insured in all generality and this fact strongly relates to the uniqueness of Leray-type solutions.

On the other hand, within the general framework of the equation (3), as we have $A_w \subset H^{\frac{\beta}{2}}(\mathbb{R}^3)$ and $A_s \subset H^{\frac{\beta}{2}}(\mathbb{R}^3)$ we remark the regularity properties (in the framework of the Sobolev spaces) of the global attractor are only given by the parameter $0 \leq \beta$.

Finally, in the equation (9) it is worth focusing on the particular case when $\alpha = 2$, which deals with the classical damped Navier-Stokes equations:

$$\partial_t \vec{u} - \nu \Delta \vec{u} + \mathbb{P} (\text{div}(\vec{u} \otimes \vec{u})) = \vec{f} - \gamma \vec{u}, \quad \text{div}(\vec{u}) = 0. \quad (10)$$

By the first point of Theorem 2.6, there exists a unique weak global attractor $A_w \subset B_1$ for this equation. This result can be observed as an interesting counterpart, in the setting of the whole space $\mathbb{R}^3$, of one of the main results proven in [6] for the classical Navier-Stokes equations with space-periodic conditions. Moreover, by the second point of Theorem 2.6, the weak global attractor $A_w$ becomes a strong global attractor $A_s$, provided the condition (8) holds. We think this result could also be adapted to the space-periodic setting to obtain a new sufficient condition to the existence of a strong global attractor, which is unknown in all generality.

To close this section, in the region $(\alpha, \beta) \in ]0, +\infty[ \times ]0, +\infty[$ we graphically represent our results on the existence of a strong global attractor $A_s$ and a weak global attractor $A_w$ of the equation (3) and their main related models: the Bardina’s model (1) represented at the point $(\alpha, \beta) = (2, 2)$, the fractional Navier-Stokes equations (9) represented in the horizontal axis $(\alpha, 0)$; and the classical Navier-Stokes equation (10) represented at the point $(2, 0)$. The red region represents the conditions $\frac{3}{2} < \alpha + \beta$ and $2 \leq \alpha + \beta < \frac{5}{2}$, where the weak global attractor becomes a strong global attractor provided that (8) holds.

3 The Leray-type solutions

3.1 Existence: proof of Theorem 2.1

The proof follows the Leray’s method in the classical framework of the Navier-Stokes equations. See the Section 12.1 of the book [16] for more details. We start by rewriting the first equation in (3) as follows:

$$\partial_t \vec{u} + \left(\gamma I_d + \nu (-\Delta)^{\frac{\alpha}{2}}\right) \vec{u} + (I_d - \Delta)^{\frac{\beta}{2}} \mathbb{P} (\text{div}(\vec{u} \otimes \vec{u})) + \nabla p = \vec{f}, \quad (11)$$

We set now some definition and notation. For $0 < \alpha$ we denote by $p_\alpha(t, x)$ the fundamental solution of the linear equation $\partial_t \vec{u} + \left(\gamma I_d + \nu (-\Delta)^{\frac{\alpha}{2}}\right) \vec{u} = 0$, where for all $0 < t$ we have $p_\alpha(t, x) = \mathcal{F}_x^{-1} \left( e^{-t (\gamma + \nu |\xi|^\alpha)} \right) (x)$.
Proof. We start by proving the point 1. For the first term in the norm \( \| \cdot \| \) be a positive and radial function such that \( \int_{\mathbb{R}^3} \theta(x)dx = 1 \). For a parameter \( 0 < \varepsilon \) we define \( \theta_{\varepsilon}(x) = \frac{1}{2\pi} \theta\left(\frac{x}{\varepsilon}\right) \).

In the first step, for a time \( 0 < T < 1 \) small enough we will solve the following regularized integral problem:

\[
\bar{u}(t, \cdot) = p_{\alpha}(t, \cdot) \ast \bar{u}_0 - \int_0^t p_{\alpha}(t - s, \cdot) \ast \bar{f}(s, \cdot) \, ds \\
- \int_0^t p_{\alpha}(t - s, \cdot) \ast (I_d - \Delta)^{-\beta/2} \mathbb{P} \left( \theta_{\varepsilon} \ast \text{div}((\theta_{\varepsilon} \ast \bar{u}) \otimes (\theta_{\varepsilon} \ast \bar{u})) \right) (s, \cdot) \, ds.
\]

(12)

This problem will be solved in the space Banach space \( E_T = L_\infty^\beta([0, T], H_\varepsilon^\beta(\mathbb{R}^3)) \cap L^2([0, T], H^{\alpha+\beta}_\varepsilon}(\mathbb{R}^3) \) with the natural norm \( \| \cdot \|_T = \| \cdot \|_{L_\infty^\beta H_\varepsilon^\beta} + \| \cdot \|_{L^2 T H^{\alpha+\beta}_\varepsilon}. \) The first and the second term on the right side of (12) are easy to estimate:

**Lemma 3.1** The following estimates hold:

1. \( \| p_{\alpha}(t, \cdot) \ast \bar{u}_0 \|_T \leq C_{\alpha,\gamma,\nu} \| \bar{u}_0 \|_{H_\varepsilon^\beta}. \)

2. \( \| \int_0^t p_{\alpha}(t - s, \cdot) \ast \bar{f}(s, \cdot) \, ds \|_T \leq C_{\alpha,\gamma,\nu} \left( \int_0^T \| \bar{f}(s, \cdot) \|_{H_\varepsilon^\beta}^2 \, ds \right)^{\frac{1}{2}}. \)

**Proof.** We start by proving the point 1. For the first term in the norm \( \| \cdot \|_T \) we directly have the estimate:

\[
\sup_{0 \leq t \leq T} \| p_{\alpha}(t, \cdot) \ast \bar{u}_0 \|_{H_\varepsilon^\beta} \leq C \| \bar{u}_0 \|_{H_\varepsilon^\beta}.
\]

Then, for the second term in the norm \( \| \cdot \|_T \) we have the following estimates:

\[
\int_0^T \| p_{\alpha}(t, \cdot) \ast \bar{u}_0 \|_{H_\varepsilon^{\alpha+\beta}}^2 \, dt = \int_0^T \int_{\mathbb{R}^3} e^{-2\tau (\gamma + \nu |\xi|^2)} \bar{u}_0(\xi)^2 (1 + |\xi|^2)^{\alpha+\beta} \, d\xi \, dt \\
\leq \int_{\mathbb{R}^3} (1 + |\xi|^2)^{\frac{\beta}{2}} \bar{u}_0(\xi)^2 \left( \int_0^\infty e^{-2\tau (\gamma + \nu |\xi|^2)} (1 + |\xi|^2)^{\frac{\beta}{2}} \, dt \right) d\xi \\
\leq C_{\alpha,\gamma,\nu} \| \bar{u}_0 \|_{H_\varepsilon^\beta}^2.
\]

We prove now the point 2. For the first term in the norm \( \| \cdot \|_T \), as \( T < 1 \) we have:

\[
\sup_{0 \leq t \leq T} \| \int_0^t p_{\alpha}(t - s, \cdot) \ast \bar{f}(s, \cdot) \, ds \|_{H_\varepsilon^\beta} \leq \sup_{0 \leq t \leq T} \int_0^t \| p_{\alpha}(t - s, \cdot) \ast \bar{f}(s, \cdot) \|_{H_\varepsilon^\beta} \, ds \\
\leq C_{\alpha,\gamma,\nu} \int_0^T \| \bar{f}(s, \cdot) \|_{H_\varepsilon^\beta}^2 \, ds \leq C_{\alpha,\gamma,\nu} \left( \int_0^T \| \bar{f}(s, \cdot) \|_{H_\varepsilon^\beta}^2 \, ds \right)^{\frac{1}{2}}.
\]

Thereafter, in order to study the second term in the norm \( \| \cdot \|_T \), we denote by \( F_t \theta(\mathbb{R}^3) \) the Fourier transform respect to the time variable \( t \) and to the spatial variable \( x \). Thus, by the Plancherel’s identity we...
write:

\[
\left\| \int_0^t p_\alpha(t-s, \cdot) * \tilde{f} \, ds \right\|_{L_t^7 H_{x_7}^{\alpha+\beta}}
\]

\[
= \left\| (I_d - \Delta)^{\alpha+\beta} \left( \int_0^t p_\alpha(t-s, \cdot) * \tilde{f} \, ds \right) \right\|_{L_t^7 L_x^2}
\]

\[
= \left\| (I_d - \Delta)^{\alpha+\beta} \left( \int_{-\infty}^{+\infty} p_\alpha(t-s, \cdot) * \mathbb{1}_{[0,T]}(s) \tilde{f} \, ds \right) \right\|_{L_t^7 L_x^2}
\]

\[
= \left\| F_{t,x} \left[ (I_d - \Delta)^{\alpha+\beta} \left( \int_{-\infty}^{+\infty} p_\alpha(t-s, \cdot) * \mathbb{1}_{[0,T]}(s) \tilde{f} \, ds \right) \right] \right\|_{L_t^7 L_x^2}
\]

\[
= \left\| \left(1 + |\xi|^2\right)^{\alpha+\beta} \right\|_{L_t^7 H_{x_7}^\alpha}
\]

\[
\left\| F_{t,x} \left[ \mathbb{1}_{[0,T]}(s) \tilde{f} \right] \right\|_{L_t^7 L_x^2}.
\]

Then, as \( \tilde{f} \in (L_t^7)_{L_0} H_{x_7}^\beta \) we have

\[
\left\| \left(1 + |\xi|^2\right)^{\alpha+\beta} \right\|_{L_t^7 L_x^2} = \left\| \left(1 + |\xi|^2\right)^{\alpha+\beta} F_{t,x} \left[ \mathbb{1}_{[0,T]}(s) \tilde{f} \right] \right\|_{L_t^7 L_x^2}
\]

\[
\leq C_{\alpha, \gamma, \nu} \left\| \mathbb{1}_{[0,T]}(s) (I_d - \Delta)^{\alpha+\beta} \tilde{f} \right\|_{L_t^7 L_x^2} \leq C_{\alpha, \gamma, \nu} \left( \int_0^T \left\| \tilde{f}(s, \cdot) \right\|_{H_{x_7}^\beta}^2 \, ds \right)^{1/2}.
\]

We study now the bilinear term \( B_\varepsilon(\tilde{u}, \tilde{u}) \) in (12).

Lemma 3.2 For \( 0 < \varepsilon \) there exists a constant \( 0 < C_{\alpha, \beta, \gamma, \nu}(\varepsilon) \) such that \( \| B_\varepsilon(\tilde{u}, \tilde{u}) \|_T \leq C_{\alpha, \beta, \gamma, \nu}(\varepsilon) T^{\frac{1}{2}} \| u \|_T^2 \).

**Proof.** For the first term in the norm \( \| \cdot \|_T \) we have the following estimate:

\[
\left\| B_\varepsilon(\tilde{u}, \tilde{u}) \right\|_{L_T^\infty H_{x_7}^\beta} \leq C_{\alpha, \beta} \varepsilon^{-\frac{1}{2}} T^\frac{1}{2} \| u \|_T^2.
\]

Indeed, as \( T < 1 \) we write:

\[
\left\| B_\varepsilon(\tilde{u}, \tilde{u}) \right\|_{H_{x_7}^\beta} \leq \int_0^T \left\| (1 + |\xi|^2)^{\alpha+\beta} e^{-(t-s)\gamma + \nu |\xi|^\alpha} (1 + |\xi|^2)^{-\beta} \right\|_{L_x^2} \left\| \mathcal{F}_x \left( \mathcal{F}_x^{-1} \mathcal{F}_x \left( \mathcal{F}_x^{-1} \mathcal{F}_x \left( \left( \theta_\varepsilon * \tilde{u} \right) \circ \left( \theta_\varepsilon * \tilde{u} \right) \right) \right) \right) \right\|_{L_x^2} \, ds
\]

\[
\leq C_{\alpha, \beta, \gamma, \nu} \left\| \mathcal{F}_x \left( \mathcal{F}_x^{-1} \mathcal{F}_x \left( \left( \theta_\varepsilon * \tilde{u} \right) \circ \left( \theta_\varepsilon * \tilde{u} \right) \right) \right) \right\|_{L_x^2} \, ds
\]

\[
\leq C_{\alpha, \beta, \gamma, \nu} \varepsilon^{-1} T^\frac{1}{2} \left( \sup_{0 \leq s \leq T} \left\| \mathcal{F}_x \left( \left( \theta_\varepsilon * \tilde{u} \right) \circ \left( \theta_\varepsilon * \tilde{u} \right) \right) \right\|_{L_x^2} \right).
\]

In order to estimate the term \( \| \mathcal{F}_x \left( \left( \theta_\varepsilon * \tilde{u} \right) \circ \left( \theta_\varepsilon * \tilde{u} \right) \right) \|_{L_x^2} \), we use the Plancherel’s identity, the Hölder and the Young inequalities to write:

\[
\left\| \mathcal{F}_x \left( \left( \theta_\varepsilon * \tilde{u} \right) \circ \left( \theta_\varepsilon * \tilde{u} \right) \right) \right\|_{L_x^2} \leq C \left\| \left( \theta_\varepsilon * \tilde{u} \right) \circ \left( \theta_\varepsilon * \tilde{u} \right) \right\|_{L_x^2} \leq \left\| \theta_\varepsilon * \tilde{u} \right\|_{L_x^2} \left\| \theta_\varepsilon * \tilde{u} \right\|_{L_x^2} \leq C \varepsilon^{-\frac{1}{2}} \| \tilde{u}(s, \cdot) \|_{L_x^2}^2
\]

\[
\leq C \varepsilon^{-\frac{1}{2}} \| u(\cdot, \cdot) \|_{H_{x_7}^\beta}^2.
\]
Hence, the desired estimate follows. On the other hand, for the second term in the norm \( \| \cdot \|_T \) we have the estimate:

\[
\| B_\varepsilon(\bar{u}, \bar{u}) \|_{L^2_t H^\alpha_{x}} \leq C_{\alpha, \beta, \gamma} (1 + \varepsilon^{-1}) T^{\frac{\beta}{2}} \| u \|_{x}^2.
\]

Indeed, by following the same computations performed in the estimate (13) we obtain:

\[
\| B_{\alpha, \beta}(\theta_\varepsilon * \bar{u}, \bar{u}) \|_{L^2_t H^\alpha_{x}} \leq \left( 1 + |\xi|^2 \right)^{\frac{\alpha + \beta}{2}} \left[ F_{\theta, x} \left[ \left( \theta_\varepsilon * \bar{u} \right) \otimes (\theta_\varepsilon * \bar{u}) \right] \right]_{L^2_t L^2_x} \leq \left( 1 + |\xi|^2 \right)^{\frac{\alpha + \beta}{2}} \left[ F_{\theta, x} \left[ \left( \theta_\varepsilon * \bar{u} \right) \otimes (\theta_\varepsilon * \bar{u}) \right] \right]_{L^\infty \alpha, \beta} \leq \left( 1 + |\xi|^2 \right)^{\frac{\alpha + \beta}{2}} \left[ \| \theta_\varepsilon \|_{L^2_t L^2_x} \right] \leq I_1 \times I_2.
\]

In order to estimate the term \( I_1 \) we cut-off in the low frequencies (\( |\xi| < 1 \)) and in the high frequencies (\( 1 \leq |\xi| \)) to obtain:

\[
I_1 \leq C_{\alpha, \beta, \gamma, \nu} + \left\| |\xi|^{-\frac{\alpha + \beta}{2} + 1} |F_{\theta, x}(\theta_\varepsilon)\right\|_{L^\infty(1 \leq |\xi|, \alpha, \beta, \gamma, \nu} \leq C_{\alpha, \beta, \gamma, \nu} + \left\| |\xi| |F_{\theta, x}(\theta_\varepsilon)\right\|_{L^\infty(1 \leq |\xi|, \alpha, \beta, \gamma, \nu} \leq C_{\alpha, \beta, \gamma, \nu} \varepsilon^{-1}.
\]

Moreover, the term \( I_2 \) was already estimated in (14). Then, the desired estimate follows. \( \square \)

Once we have the estimates proven in Lemmas 3.1 and 3.2, for a time \( 0 < T < 1 \) small enough and for \( 0 < \varepsilon \), by the Banach contraction principle we obtain \( \bar{u}_\varepsilon \in E_T \) a local solution of the equation (12).

In the second step, we shall prove that this solution is global in time. The function \( \bar{u}_\varepsilon \) also solves the regularized equation:

\[
\partial_t \bar{u}_\varepsilon + (\gamma I_d + \nu(-\Delta)^{\frac{\beta}{2}}) \bar{u}_\varepsilon + (I_d - \Delta)^{-\frac{\beta}{2}} \mathbb{P} \left( \theta_\varepsilon * \text{div}((\theta_\varepsilon * \bar{u}_\varepsilon) \otimes (\theta_\varepsilon * \bar{u}_\varepsilon)) \right) = \bar{f},
\]

hence we have

\[
\partial_t (I_d - \Delta)^{\frac{\beta}{2}} \bar{u}_\varepsilon + (I_d - \Delta)^{\frac{\beta}{2}} (\gamma I_d + \nu(-\Delta)^{\frac{\beta}{2}}) \bar{u}_\varepsilon + \mathbb{P}(\theta_\varepsilon * \text{div}((\theta_\varepsilon * \bar{u}_\varepsilon) \otimes (\theta_\varepsilon * \bar{u}_\varepsilon))) = (I_d - \Delta)^{\frac{\beta}{2}} \bar{f},
\]

and then we write

\[
\partial_t (I_d - \Delta)^{\frac{\beta}{2}} \bar{u}_\varepsilon + (I_d - \Delta)^{\frac{\alpha + \beta}{2}} \frac{(\gamma I_d + \nu(-\Delta)^{\frac{\beta}{2}})}{(I_d - \Delta)^{\frac{\beta}{2}}} \bar{u}_\varepsilon + \mathbb{P}(\theta_\varepsilon * \text{div}((\theta_\varepsilon * \bar{u}_\varepsilon) \otimes (\theta_\varepsilon * \bar{u}_\varepsilon))) = (I_d - \Delta)^{\frac{\beta}{2}} \bar{f}.
\]

In this equation we remark the following facts: on the one hand, the pseudo-differential operator of order zero

\[
D(m) = \frac{(\gamma I_d + \nu(-\Delta)^{\frac{\beta}{2}})}{(I_d - \Delta)^{\frac{\beta}{2}}},
\]

is defined by the symbol \( m(\xi) = \frac{\gamma + \nu |\xi|^{\alpha}}{1 + |\xi|^2} \), which, for \( 0 < A_{\alpha, \gamma, \nu} = \frac{\min(\gamma, \nu)}{\max(1, 2^{\frac{\beta}{2}})} \) and \( 0 < B_{\alpha, \gamma, \nu} = \max(1, 2^{\frac{\beta}{2}} - 1, \gamma, \nu) \), one has \( A_{\alpha, \gamma, \nu} \leq |m(\xi)| \leq B_{\alpha, \gamma, \nu} \). On the other hand, due to the convolution product with the test function \( \theta_\varepsilon \), the nonlinear term is regular enough in the setting of the non-homogeneous Sobolev spaces. So we can write

\[
\int_0^T \frac{1}{2} \frac{d}{dt} \| \bar{u}_\varepsilon(t, \cdot) \|_{H^d}^2 + \| D(m^{1/2}) \bar{u}(t, \cdot) \|_{H^{\frac{\alpha + \beta}{2}}}^2 = \left\langle (I_d - \Delta)^{\frac{\beta}{2}} \bar{f}, \bar{u}_\varepsilon(t, \cdot) \right\rangle \right\|_{L^2 \times L^2},
\]
Theorem 2.1 is proven. Moreover, this sequence also converges to $\vec{u}$ back to the equation (16), hence this energy inequality holds for the regularized solution $\vec{u}_\varepsilon$.

In the weak convergence above we can deduce that the sequence $(I-d-\Delta)^{\alpha/2} f, (I-d-\Delta)^{\alpha/2} \vec{u}_\varepsilon(t, \cdot))_{L^2 \times L^2} = I_3$.

Moreover, the last term can be estimated as:

$$I_3 \leq \|f(t, \cdot)\|_{H^2} \|\bar{u}_\varepsilon(t, \cdot)\|_{H^2} \leq \frac{1}{2A_{\alpha,\gamma,\nu}} \|f(t, \cdot)\|^2_{H^2} + \frac{A_{\alpha,\gamma,\nu}}{2} \|\bar{u}_\varepsilon(t, \cdot)\|^2_{H^2},$$

and we integrate on the interval of time $[0, t]$ to get the following control:

$$\|\bar{u}_\varepsilon(t, \cdot)\|^2_{H^2} + \frac{A_{\alpha,\gamma,\nu}}{2} \|\bar{u}(t, \cdot)\|^2_{H^2} \leq \frac{1}{2A_{\alpha,\gamma,\nu}} \|f(t, \cdot)\|^2_{H^2},$$

(18)

which allows us to extend the local solution $\vec{u}_\varepsilon$ to the whole interval of time $[0, +\infty[$.

In the third step, we study the convergence to a weak solution of the equation (11). By the Rellich-Lions lemma (see [16], Theorem 12.1) there exists a sequence of positive numbers $(\varepsilon_n)_{n \in \mathbb{N}}$ and a function $\vec{u} \in L^2_{loc}([0, +\infty[ \times \mathbb{R}^3)$ such that the sequence $\vec{u}_{\varepsilon_n}$ converges to $\vec{u}$ in the strong topology of the space $(L^2_{L^2})_{loc}$. Moreover, this sequence also converges to $\vec{u}$ in the weak-* topology of the spaces $L^{\infty}([0, T], H^{\alpha/2} \mathbb{R}^3)$ and $L^2([0, T], H^{\alpha/2} \mathbb{R}^3)$ for all $0 < T$. We must study the convergence to the nonlinear term $P \div (\vec{u} \otimes \vec{u})$; and for this we shall need the following:

**Lemma 3.3** Let $\vec{u} \in (L^\infty_{loc})_{H^{\alpha/2}} \cap (L^2_{loc})_{H^{\alpha/2}},$. If $0 < \alpha + \beta < 3$ then $\div (\vec{u} \otimes \vec{u}) \in (L^p_{loc})_{H^{\alpha/2}}.$

**Proof.** As we have $\frac{\alpha + \beta}{2} < \frac{3}{2}$ then by the Hardy-Littlewood-Sobolev (HLS) inequalities we obtain $\vec{u} \in (L^\infty_{loc})_{H^{\alpha/2}} \subset (L^2_{loc})_{H^{\alpha/2}} \subset (L^p_{loc})_{H^{\alpha/2}}$, with $p = \frac{6}{3 - (\alpha + \beta)}$. On the other hand, as $0 \leq \beta$ we have $\vec{u} \in (L^\infty_{loc})_{H^{\beta}} \subset (L^\infty_{loc})_{L^2}$. Thus, we can use the Holder inequalities with $\frac{1}{q} = \frac{1}{2} + \frac{1}{p}$, hence we have $q = \frac{6}{6 - (\alpha + \beta)}$, and we are able to write $\|\vec{u} \otimes \vec{u}\|_{(L^p_{loc})_{L^2}} \leq C \|\vec{u}\|_{(L^\infty_{loc})_{L^2}} \|\vec{u}\|_{(L^p_{loc})_{L^2}}$. Finally, by making use again of the (HLS) inequalities we have the embedding $L^q(\mathbb{R}^3) \subset H^{\frac{3q}{q-1}}(\mathbb{R}^3)$, hence we obtain $\vec{u} \otimes \vec{u} \in (L^p_{loc})_{H^{\alpha/2}}$ and consequently we have $\div (\vec{u} \otimes \vec{u}) \in (L^p_{loc})_{H^{\alpha/2}}$.

In the case when $0 < \alpha + \beta < 3$, by this lemma and by (19) the family $(\theta \ast \div((\theta \ast \vec{u}_\varepsilon) \otimes (\theta \ast \vec{u}_\varepsilon))$ is uniformly bounded respect to the parameter $\varepsilon$ in the space $(L^2_{loc})_{H^{\alpha/2}}$. On the other hand, in the case when $3 \leq \alpha + \beta$ we set $0 < \alpha' < \alpha$ and $0 \leq \beta' < \beta$ such that $0 < \alpha' + \beta' < 3$ to obtain that the family $(\theta \ast \div((\theta \ast \vec{u}_\varepsilon) \otimes (\theta \ast \vec{u}_\varepsilon))$ is uniformly bounded in $(L^2_{loc})_{H^{\alpha/2}}$. Consequently, in both cases we obtain the uniformly boundness of the family above in the larger space $(L^2_{loc})_{H^{\alpha/2}}$. From this fact and the convergences above we can deduce that the sequence $(P \div (\theta \varepsilon_n \ast \vec{u}_{\varepsilon_n} \otimes \vec{u}_{\varepsilon_n}))_{\varepsilon_n \in \mathbb{N}}$ converges to $P \div (\vec{u} \otimes \vec{u})$ in the weak-* topology of the space $(L^p_{loc})_{H^{\alpha/2}}$.

In the fourth step, we study the energy inequality given at the second point of Definition 2.1. We get back to the equation (16), hence this energy inequality holds for the regularized solution $\vec{u}_\varepsilon$. Then, by applying classical tools (see the page 354 of the book[16]) this inequality also holds true for the limit $\vec{u}$. Theorem 2.1 is proven.

\[ \]
3.2 Energy estimates: proof of Proposition 2.1

We consider the functions $\tilde{u}_{\varepsilon_n}$, which are solutions of the regularized equation (15). Then we have the identity

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}_{\varepsilon_n}(t, \cdot)\|^2_{H^{2,\alpha}} + \gamma \|\tilde{u}_{\varepsilon_n}(t, \cdot)\|^2_{H^{2,\alpha}} + \nu \left\| (-\Delta)^{\frac{\alpha}{2}} (I_d - \Delta)^{\frac{\beta}{2}} \tilde{u}_{\varepsilon_n}(t, \cdot) \right\|^2_{L^2}$$

$$= \left\langle (I_d - \Delta)^{\frac{\alpha}{2}} \tilde{f}(t, \cdot), (I_d - \Delta)^{\frac{\beta}{2}} \tilde{u}_{\varepsilon_n}(t, \cdot) \right\rangle_{L^2 \times L^2},$$

(20)

hence we can write

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}_{\varepsilon_n}(t, \cdot)\|^2_{H^{2,\alpha}} + \gamma \|\tilde{u}_{\varepsilon_n}(t, \cdot)\|^2_{H^{2,\alpha}} \leq \left\langle (I_d - \Delta)^{\frac{\alpha}{2}} \tilde{f}(t, \cdot), (I_d - \Delta)^{\frac{\beta}{2}} \tilde{u}_{\varepsilon_n}(t, \cdot) \right\rangle_{L^2 \times L^2}$$

$$\leq \|\tilde{f}\|_{H^{2,\beta}} \|\tilde{u}_{\varepsilon_n}(t, \cdot)\|^2_{H^{2,\alpha}} \leq \frac{2}{\gamma} \|\tilde{f}(t, \cdot)\|^2_{H^{2,\alpha}} + \frac{\gamma}{2} \|\tilde{u}_{\varepsilon_n}(t, \cdot)\|^2_{H^{2,\alpha}},$$

and then we obtain

$$\frac{d}{dt} \|\tilde{u}_{\varepsilon_n}(t, \cdot)\|^2_{H^{2,\alpha}} + \gamma \|\tilde{u}_{\varepsilon_n}(t, \cdot)\|^2_{H^{2,\alpha}} \leq \frac{4}{\gamma} \|\tilde{f}(t, \cdot)\|^2_{H^{2,\alpha}}.$$
3.3 Uniqueness: proof of Theorem 2.2

Let \( \tilde{u}_1, \tilde{u}_2 \in (L^\infty_t)_{loc} H^\beta_x \cap (L^2_t)_{loc} H^{\alpha+\beta}_x \) be two Leray-type solutions of the equation (3) with the same external force \( \tilde{f} \in (L^2_t)_{loc} H^\beta_x \) and arising from the initial data \( \tilde{u}_{0,1} \) and \( \tilde{u}_{0,2} \) respectively. We define \( \tilde{w} = \tilde{u}_1 - \tilde{u}_2 \), and then, for the operator \( D(m) \) given in (17), the function \( \tilde{w} \in (L^\infty_t)_{loc} H^\beta_x \cap (L^2_t)_{loc} H^{\alpha+\beta}_x \) solves the following problem:

\[
\begin{aligned}
\partial_t (I_d - \Delta)^{\frac{\beta}{2}} \tilde{w} + (I_d - \Delta)^{\frac{\alpha+\beta}{2}} D(m) \tilde{w} + \mathcal{P} \left( (\tilde{w} \cdot \nabla) \tilde{u}_1 + (\tilde{u}_2 \cdot \nabla) \tilde{w} \right) &= 0, \quad \text{div}(\tilde{w}) = 0, \\
\tilde{w}(0, \cdot) = \tilde{w}_0 &= \tilde{u}_{0,1} - \tilde{u}_{0,2}.
\end{aligned}
\]  

(22)

We shall perform an energy estimate on the solution \( \tilde{w} \). First, we consider the case when \( 0 < \alpha + \beta < 3 \). In this case, by Lemma 3.3 we have \( \mathcal{P} \left( (\tilde{w} \cdot \nabla) \tilde{u}_1 \right) \in (L^2_t)_{loc} H^{\frac{5-(\alpha+\beta)}{2}}_x \) and \( \mathcal{P} \left( (\tilde{u}_2 \cdot \nabla) \tilde{w} \right) \in (L^2_t)_{loc} H^{\frac{5-(\alpha+\beta)}{2}}_x \). Moreover, by assuming \( \frac{5}{2} \leq \alpha + \beta < 3 \) then we get \( \frac{5-(\alpha+\beta)}{2} \leq \frac{\alpha+\beta}{2} \); so we have \( \tilde{w} \in (L^2_t)_{loc} H^{\frac{5-(\alpha+\beta)}{2}}_x \). Then, we can write

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \| \tilde{w}(t, \cdot) \|^{2}_{H^{\beta/2}} + \| D(m^{1/2}) \tilde{w}(t, \cdot) \|^{2}_{H^{\alpha+\beta/2}} + \mathcal{P} \left( (\tilde{w} \cdot \nabla) \tilde{u}_1 \right), \tilde{w} \|_{H^{\frac{5-(\alpha+\beta)}{2}}_x \times H^{\frac{5-(\alpha+\beta)}{2}}_x} = 0.
\end{aligned}
\]  

(23)

As \( \text{div}(\tilde{w}) = 0 \), we have \( \mathcal{P} \left( (\tilde{w} \cdot \nabla) \tilde{w} \right), \tilde{w} \|_{H^{\frac{5-(\alpha+\beta)}{2}}_x \times H^{\frac{5-(\alpha+\beta)}{2}}_x} = 0 \), so it remains to estimate the term \( \mathcal{P} \left( (\tilde{w} \cdot \nabla) \tilde{u}_1 \right), \tilde{w} \|_{H^{\frac{5-(\alpha+\beta)}{2}}_x \times H^{\frac{5-(\alpha+\beta)}{2}}_x} \). More precisely, the following estimate holds:

\[
\left\| \mathcal{P} \left( (\tilde{w} \cdot \nabla) \tilde{u}_1 \right), \tilde{w} \right\|_{H^{\frac{5-(\alpha+\beta)}{2}}_x \times H^{\frac{5-(\alpha+\beta)}{2}}_x} \leq C \| \tilde{w}(t, \cdot) \|_{H^{\beta/2}_x} \| \tilde{u}_1(t, \cdot) \|_{H^{\alpha+\beta/2}_x} \| \tilde{w}(t, \cdot) \|_{H^{\alpha+\beta/2}_x}. 
\]  

(24)

Indeed, we just write

\[
\left\| \mathcal{P} \left( (\tilde{w} \cdot \nabla) \tilde{u}_1 \right), \tilde{w} \right\|_{H^{\frac{5-(\alpha+\beta)}{2}}_x \times H^{\frac{5-(\alpha+\beta)}{2}}_x} \leq \left\| \mathcal{P} \left( (\tilde{w} \cdot \nabla) \tilde{u}_1 \right)(t, \cdot) \right\|_{H^{\frac{5-(\alpha+\beta)}{2}}_x \times H^{\frac{5-(\alpha+\beta)}{2}}_x} \leq \left\| (\tilde{w} \cdot \nabla) \tilde{u}_1 \right\|_{H^{\frac{5-(\alpha+\beta)}{2}}_x} \| \tilde{u}_1(t, \cdot) \|_{H^{\alpha+\beta}_x}. 
\]

Here, the term \( \left\| (\tilde{w} \cdot \nabla) \tilde{u}_1 \right\|_{H^{\frac{5-(\alpha+\beta)}{2}}_x} \) was already estimated in the proof of Lemma 3.3 and we are able to write \( \left\| (\tilde{w} \cdot \nabla) \tilde{u}_1 \right\|_{H^{\frac{5-(\alpha+\beta)}{2}}_x} \leq C \| \tilde{w}(t, \cdot) \|_{H^{\beta/2}_x} \| \tilde{u}_1(t, \cdot) \|_{H^{\alpha+\beta/2}_x} \). With the estimate (24), and by recalling that for \( 0 < A = A_{\alpha,\gamma,\nu} \), we have \( A \leq |m(\xi)| \), we get back to the last identity to write:

\[
\frac{1}{2} \frac{d}{dt} \| \tilde{w}(t, \cdot) \|^{2}_{H^{\beta/2}_x} + A \| \tilde{w}(t, \cdot) \|^{2}_{H^{\alpha+\beta/2}_x} \leq C \| \tilde{w}(t, \cdot) \|_{H^{\beta/2}_x} \| \tilde{u}_1(t, \cdot) \|_{H^{\alpha+\beta/2}_x} \| \tilde{w}(t, \cdot) \|_{H^{\alpha+\beta/2}_x}. 
\]

By applying the Young inequalities for products at the right side we get

\[
\frac{1}{2} \frac{d}{dt} \| \tilde{w}(t, \cdot) \|^{2}_{H^{\beta/2}_x} + A \| \tilde{w}(t, \cdot) \|^{2}_{H^{\alpha+\beta/2}_x} \leq \frac{C^2}{2A} \| \tilde{w}(t, \cdot) \|^{2}_{H^{\beta/2}_x} \| \tilde{u}_1(t, \cdot) \|^{2}_{H^{\alpha+\beta/2}_x} + \frac{A}{2} \| \tilde{w}(t, \cdot) \|^{2}_{H^{\alpha+\beta/2}_x},
\]

hence we obtain:

\[
\frac{d}{dt} \| \tilde{w}(t, \cdot) \|^{2}_{H^{\beta/2}_x} \leq \frac{C^2}{A} \| \tilde{w}(t, \cdot) \|^{2}_{H^{\beta/2}_x} \| \tilde{u}_1(t, \cdot) \|^{2}_{H^{\alpha+\beta/2}_x}. 
\]
From this estimate and the Grönwall inequalities we have
\[
\|\tilde{u}(t,\cdot)\|_{H^\vartheta,\varrho}^2 \leq \|\bar{u}_0\|_{H^\vartheta,\varrho}^2 \exp \left( \frac{C^2}{A} \int_0^t \|\bar{u}_1(s,\cdot)\|_{H^\vartheta,\varrho}^2 \, ds \right).
\]
Moreover, as \( \bar{u}_1 \) verifies the energy inequality given at the second point of Proposition 2.1 then we write:
\[
\|\bar{u}_1(t,\cdot)\|_{H^\vartheta,\varrho}^2 + A \int_0^t \|\bar{u}_1(s,\cdot)\|_{H^\vartheta,\varrho}^2 \, ds \leq \|\bar{u}_{0,1}\|_{H^\vartheta,\varrho}^2 + 2 \int_0^t \|\tilde{f}(s,\cdot)\|_{H^\vartheta,\varrho} \|\bar{u}(s,\cdot)\|_{H^\vartheta,\varrho} \, ds
\]
\[
\leq \|\bar{u}_{0,1}\|_{H^\vartheta,\varrho}^2 + 2 \int_0^t \|\tilde{f}(s,\cdot)\|_{H^\vartheta,\varrho} \|\bar{u}_1(s,\cdot)\|_{H^\vartheta,\varrho} \, ds
\]
\[
\leq \|\bar{u}_{0,1}\|_{H^\vartheta,\varrho}^2 + 2 \int_0^t \|\tilde{f}(s,\cdot)\|_{H^\vartheta,\varrho} \|\bar{u}_1(s,\cdot)\|_{H^\vartheta,\varrho} \, ds + \frac{A}{2} \int_0^t \|\bar{u}_1(s,\cdot)\|_{H^\vartheta,\varrho}^2 \, ds,
\]
and we get
\[
\int_0^t \|\bar{u}_1(s,\cdot)\|_{H^\vartheta,\varrho}^2 + A \int_0^t \|\bar{u}_1(s,\cdot)\|_{H^\vartheta,\varrho}^2 \, ds \leq \|\bar{u}_{0,1}\|_{H^\vartheta,\varrho}^2 + \frac{2}{A} \int_0^t \|\tilde{f}(s,\cdot)\|_{H^\vartheta,\varrho} \, ds.
\]
Consequently we have
\[
\int_0^t \|\bar{u}(s,\cdot)\|_{H^\vartheta,\varrho}^2 \, ds \leq \frac{2}{A} \|\bar{u}_{0,1}\|_{H^\vartheta,\varrho}^2 + \frac{4}{2} \int_0^t \|\tilde{f}(s,\cdot)\|_{H^\vartheta,\varrho} \, ds.
\]
Then, we can write:
\[
\|\tilde{u}(t,\cdot)\|_{H^\vartheta,\varrho}^2 \leq \|\bar{u}_0\|_{H^\vartheta,\varrho}^2 \exp \left( \frac{2C^2}{A^2} \|\bar{u}_{0,1}\|_{H^\vartheta,\varrho}^2 + \frac{4C^2}{A^3} \int_0^t \|\tilde{f}(s,\cdot)\|_{H^\vartheta,\varrho} \, ds \right).
\]

On the other hand, when \( 3 \leq \alpha + \beta \), we set \( 0 \leq \beta' < \beta \) and \( 0 < \alpha' < \alpha \) such that \( \frac{5}{2} \leq \alpha' + \beta' < 3 \). Then, we have \( \tilde{u}, \bar{u}_1, \bar{u}_2 \in (L_t^\infty)_{loc}H^\vartheta_{\varrho,2} \cap (L_t^\infty)_{loc}H^\vartheta_{\varrho,2} \cap (L_t^\infty)_{loc}H^\vartheta_{\varrho,2} \cap \mathbb{P} \left( (\bar{u}_2 \cdot \nabla)\bar{u}_1 \right) \subset (L^\infty_t)_{loc}H^\vartheta_{\varrho,2} \cap (L^\infty_t)_{loc}H^\vartheta_{\varrho,2} \cap (L^\infty_t)_{loc}H^\vartheta_{\varrho,2} \cap \mathbb{P} \left( (\tilde{u}_2 \cdot \nabla)\tilde{u} \right) \). Moreover, by Lemma 3.3 we have \( \mathbb{P} \left( (\tilde{u} \cdot \nabla)\tilde{u}_1 \right) \subset (L^\infty_t)_{loc}H^\vartheta_{\varrho,2} \) and \( \mathbb{P} (\tilde{u}_2 \cdot \nabla)\tilde{u} \subset (L^\infty_t)_{loc}H^\vartheta_{\varrho,2} \). Thus, the inequality (26) also holds true in the case \( 3 \leq \alpha + \beta \).

Summarizing, for \( \frac{5}{2} \leq \alpha + \beta \) we have the inequality (26) from which the uniqueness of Leray-type solutions directly follows. Theorem 2.2 is proven.

### 4 The long time behavior of Leray-type solutions

As explained in Section 2, the notion of absorbing set is of key importance in the studying of the global attractor either the weak and the strong case. Our starting point is then to verify that the set \( \mathcal{B} \) given in (4) is an absorbing set for the equation (3).

#### 4.1 The absorbing set: proof of Proposition 2.2

Let \( \tilde{u}_0 \in H^\vartheta_{\varrho}(\mathbb{R}^3) \) be an initial datum, and moreover, let \( \tilde{u}(t,\cdot) \) be a Leray-type solution of the equation (3) arising from \( \tilde{u}_0 \). By the first point of Proposition 2.1 and as \( \tilde{f} \in H^\vartheta_{\varrho}(\mathbb{R}^3) \) is a time independent function we can write:
\[
\|\tilde{u}(t,\cdot)\|_{H^\vartheta,\varrho}^2 \leq e^{-\gamma t} \|\tilde{u}_0\|_{H^\vartheta,\varrho}^2 + \frac{4}{\gamma^2} \|\tilde{f}\|_{H^\vartheta,\varrho}^2 \left( 1 - e^{-\gamma t} \right) \leq e^{-\gamma t} \|\tilde{u}_0\|_{H^\vartheta,\varrho}^2 + \frac{4}{\gamma^2} \|\tilde{f}\|_{H^\vartheta,\varrho}^2.
\]
Hence, we can set a time \( 0 < T = T \left( \gamma, \|\tilde{u}_0\|_{H^\vartheta,\varrho}, \|\tilde{f}\|_{H^\vartheta,\varrho} \right) \) large enough such that for all \( T < t \) we have the inequality \( e^{-\gamma t} \|\tilde{u}_0\|_{H^\vartheta,\varrho}^2 \leq \frac{1}{\gamma^2} \|\tilde{f}\|_{H^\vartheta,\varrho}^2 \). Consequently, for all \( T < t \) we have \( \tilde{u}(t,\cdot) \in \mathcal{B} \). Proposition 2.2 is proven.
4.2 Proof of Theorem 2.3

4.2.1 The weak global attractor in the case $0 < \alpha + \beta < \frac{5}{2}$

The existence and uniqueness of a weak global attractor $A_w$ for the equation (3) bases on the following previous results that we summarize as follows. First, for the absorbing set $B$ given in (4) and for a time $0 \leq t$ we define the set

$$R(t)B = \{ \tilde{u}(t, \cdot) : \tilde{u} \text{ is a Leray-type solution of (3) arising from } \tilde{u}_0 \in B \} \subset H^\frac{\beta}{2}(\mathbb{R}^3).$$

(28)

As uniqueness of Leray-type solutions is unknown for this range of values of the parameter $\alpha + \beta$, the family $(R(t))_{t \geq 0}$ does not define a semigroup on the space $H^\frac{\beta}{2}(\mathbb{R}^3)$. However, this family enjoys the following property: $R(t_1 + t_2)B \subset R(t_1)R(t_2)B$, for all $0 \leq t_1, t_2$. We introduce now the following:

Definition 4.1 (Weakly uniformly compact family) The family $(R(t))_{t \geq 0}$ is uniformly weakly compact if there exists a time $0 < T$ such that set $\bigcup_{T \leq t} R(t)B$ is relatively compact in $(B, d_w)$, where the distance $d_w$ is given in (6).

Now we can state the following result on the existence of a weak global attractor. For a proof see [6], Theorem 2.11 and Corollary 2.5.

Theorem 4.1 (Existence of a weak global attractor) If the family $(R(t))_{t \geq 0}$ given in (28) is uniformly weak compact in the sense of Definition 4.1, then there exists a unique weak global attractor $A_w$ in the sense of Definition 2.4.

Proof of the first point in Theorem 2.3. By Theorem 4.1, we shall prove that the family $(R(t))_{t \geq 0}$ is uniformly weak compact. Let $0 \leq t$ and let $\tilde{u}(t, \cdot) \in R(t)B$. By definition of the set $R(t)B$ given in (28) we know that $\tilde{u}(t, \cdot)$ is a Leray-type solution of the equation (3) arising from an initial datum $\tilde{u}_0 \in B$. Then, by definition of the absorbing set $B$ given in (4), and moreover, by the estimate (27) we have

$$\| \tilde{u}(t, \cdot) \|^2_{H^\frac{\beta}{2}} \leq e^{-\gamma t} \frac{5}{\gamma^2} + \frac{4}{\gamma^2} \| \tilde{f} \|^2_{H^\frac{\beta}{2}}.$$

We can set a time $0 < T$, which does not depend on $\tilde{u}_0 \in B$, such that for all $T < t$ we have $e^{-\gamma t} \frac{5}{\gamma^2} \leq \frac{1}{\gamma}$. Thus, for all $T < t$ we have $\| \tilde{u}(t, \cdot) \|^2_{H^\frac{\beta}{2}} \leq \frac{5}{\gamma^2} \| \tilde{f} \|^2_{H^\frac{\beta}{2}}$, hence, always by definition of the set $B$, we obtain $\bigcup_{T \leq t} R(t)B \subset B$. Finally, as $(B, d_w)$ is a compact metric space the family $(R(t))_{t \geq 0}$ is then uniformly weak compact in the sense of Definition 4.1. The first point in Theorem 2.3 is now proven. ■

4.2.2 The strong global attractor in the case $\frac{5}{2} \leq \alpha + \beta$

By Theorem 2.2 we can define the semigroup $S(t) : H^\frac{\beta}{2}(\mathbb{R}^3) \to H^\frac{\beta}{2}(\mathbb{R}^3)$ as:

$$S(t)\tilde{u}_0 = \tilde{u}(t, \cdot), \quad 0 \leq t, \quad \tilde{u}_0 \in H^\frac{\beta}{2}(\mathbb{R}^3),$$

(29)

where $\tilde{u}(t, \cdot)$ is the unique Leray-type solution of the equation (3) which arises from $\tilde{u}_0$. Due to the uniqueness of solutions, it is easy to verify that $(S(t))_{t \geq 0}$ is a strongly continuous semigroup on the Hilbert space $H^\frac{\beta}{2}(\mathbb{R}^3)$.

We recall now the following:

Definition 4.2 (Strongly asymptotically compact semigroup) The semigroup $(S(t))_{t \geq 0}$ is strongly asymptotically compact if for any bounded sequence $(\tilde{u}_{0,n})_{n \in \mathbb{N}}$ in $H^\frac{\beta}{2}(\mathbb{R}^3)$, and moreover, for any sequence of times $(t_n)_{n \in \mathbb{N}}$ such that $t_n \to \infty$ when $n \to \infty$, the sequence $(S(t_n)\tilde{u}_{0,n})_{n \in \mathbb{N}}$ is strongly precompact in $H^\frac{\beta}{2}(\mathbb{R}^3)$. 

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Now we are able to state the following theorem on the existence of a strong global attractor. For a proof of this result see [20] and [21].

**Theorem 4.2 (Existence of a strong global attractor)** Assume that:

1. The semigroup \((S(t))_{t \geq 0}\) has a bounded and closed absorbing set \(B \subset H^2(\mathbb{R}^3)\).
2. The semigroup \((S(t))_{t \geq 0}\) is asymptotically compact in the sense of definition above.
3. For every \(0 \leq t\) fixed, the map \(S(t) : B \rightarrow H^2(\mathbb{R}^3)\) is continuous.

Then, the semigroup \((S(t))_{t \geq 0}\) has a unique strong global attractor \(A_s \subset H^2(\mathbb{R}^3)\) given in Definition 2.4. Moreover, the set \(A_s\) is invariant, i.e., for all \(0 \leq t\) we have:

\[
A_s = \left\{ \tilde{u}(t, \cdot) : \tilde{u} \text{ is a Leray-type solution of (3) arising from } \tilde{u}_0 \in A_s \right\}.
\]

**Proof of the second point in Theorem 2.3.** We shall prove that the semigroup \((S(t))_{t \geq 0}\) verify all the assumptions in Theorem 4.2. For the reader’s convenience, we will study each of them separately.

**Point 1.** This point was already satisfied by Proposition 2.2.

**Point 2.** Let \((\tilde{u}_n)_{n \in \mathbb{N}}\) be a bounded sequence in \(H^2(\mathbb{R}^3)\), and moreover, let \((t_n)_{n \in \mathbb{N}}\) be a sequence of positive times such that \(t_n \to +\infty\) when \(n \to +\infty\). We must show that the sequence \((S(t_n)\tilde{u}_n)_{n \in \mathbb{N}}\) is strongly precompact in the space \(H^2(\mathbb{R}^3)\) and for this we shall perform the following energy method: for each \(n \in \mathbb{N}\), we consider the following initial value problem for the equation (3):

\[
\begin{aligned}
\partial_t \tilde{u}_n + (\gamma I_d + \nu(-\Delta)^{\frac{2}{d}}) \tilde{u}_n + (I_d - \Delta)^{-\frac{2}{d}} \mathbb{P} \text{div}(\tilde{u}_n \otimes \tilde{u}_n) &= \tilde{f}, & \text{div}(\tilde{u}_n) &= 0, \\
\tilde{u}_n(-t_n, \cdot) &= \tilde{u}_{0,n}.
\end{aligned}
\]  

(30)

By Theorems 2.1 and 2.2 there exists a unique Leray-type solution \(\tilde{u}_n : [-t_n, +\infty[ \times \mathbb{R}^3 \rightarrow \mathbb{R}^3\). Moreover, by definition of the semigroup \(S(t)\) given in (29), for all \(n \in \mathbb{N}\) we have the identity \(S(t_n)\tilde{u}_{0,n} = \tilde{u}_n(0, \cdot)\). Therefore, we shall prove that the sequence \((\tilde{u}_n(0, \cdot))_{n \in \mathbb{N}}\) is strongly precompact in \(H^2(\mathbb{R}^3)\). Our general strategy is the following: first, we shall prove the existence of an eternal solution associated to the equation (3). We recall that an eternal solution associated to this equation is a function

\[
\tilde{v} \in L^\infty_{\text{loc}}\left([-\infty, +\infty[, H^2(\mathbb{R}^3)\right) \cap L^2_{\text{loc}}\left([-\infty, +\infty[, H^{\alpha+\beta}(\mathbb{R}^3)\right),
\]

(31)

which is a weak solution of the equation

\[
\partial_t \tilde{v} + (\gamma I_d + \nu(-\Delta)^{\frac{2}{d}}) \tilde{v} + (I_d - \Delta)^{-\frac{2}{d}} \mathbb{P} \text{div}(\tilde{v} \otimes \tilde{v}) = \tilde{f}, \quad \text{div}(\tilde{v}) = 0, \quad \text{on } [-\infty, +\infty[ \times \mathbb{R}^3.
\]

(32)

Thereafter, we will show that the sequence \((\tilde{u}_n(0, \cdot))_{n \in \mathbb{N}}\) converges (via a sub-sequence) to \(\tilde{v}(0, \cdot)\) in the strong topology of the space \(H^2(\mathbb{R}^3)\).

Our starting point is then the existence of an eternal solution:

**Proposition 4.1** There exists a function \(\tilde{v}\) which verifies (31) and (32).

**Proof.** This function will be obtained as the limit when \(n \to +\infty\) of the solutions \(\tilde{u}_n : [-t_n, +\infty[ \times \mathbb{R}^3 \rightarrow \mathbb{R}^3\) of the initial value problems (30). By the first point in Proposition 2.1, for all \(n \in \mathbb{N}\) and for all \(-t_n \leq t\) we have

\[
\|\tilde{u}_n(t, \cdot)\|_{H^2}^2 \leq e^{-\gamma(t+t_n)}\|\tilde{u}_{0,n}\|_{H^2}^2 + \frac{4}{\gamma^2} \|\tilde{f}\|_{H^2}^2.
\]
Moreover, as the sequence \((\bar{u}_{0,n})_{n \in \mathbb{N}}\) is bounded in \(H^\frac{\alpha}{2}(\mathbb{R}^3)\), there exists \(0 < R\) such that we can write

\[
\sup_{n \in \mathbb{N}} \sup_{t \geq -t_n} \|\bar{u}_n(t, \cdot)\|^2_{H^\frac{\alpha}{2}} \leq e^{-\gamma(t+t_n)}R^2 + \frac{4}{\gamma^2}\|\bar{f}\|^2_{H^\frac{\alpha}{2}} \leq R^2 + \frac{4}{\gamma^2}\|\bar{f}\|^2_{H^\frac{\alpha}{2}}. \quad (33)
\]

On the other hand, by the third point in Proposition 2.1, where we shall write \(0 < A_{\alpha,\gamma,\nu} = A\), for all \(-t_n \leq t \leq T = 1\) we have

\[
A \int_{t}^{t+1} \|\bar{u}_n(s, \cdot)\|^2_{H^\frac{\alpha}{2}} ds \leq e^{-\gamma(t+t_n)}\|\bar{u}_{0,n}\|^2_{H^\frac{\alpha}{2}} + \left(\frac{4}{\gamma^2} + \frac{1}{A}\right)\|\bar{f}\|^2_{H^\frac{\alpha}{2}},
\]

hence, for the constant \(0 < R\) above we get:

\[
\sup_{n \in \mathbb{N}} \sup_{t \geq -t_n} \left(\int_{t}^{t+1} \|\bar{u}_n(s, \cdot)\|^2_{H^\frac{\alpha}{2}} ds\right) \leq R^2 + \left(\frac{4}{\gamma^2} + \frac{1}{A}\right)\|\bar{f}\|^2_{H^\frac{\alpha}{2}}. \quad (34)
\]

In this fashion, by the estimates (33) and (34) and by the Banach-Alaoglu theorem, there exists \(\bar{v} \in L^\infty_{loc}(\mathbb{R}, H^\frac{\alpha}{2}(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}, H^\frac{\alpha+\beta}{2}(\mathbb{R}^3))\) such that the sequence \((\bar{u}_n)_{n \in \mathbb{N}}\) converges (via a sub-sequence) to \(\bar{v}\) in the weak \(-\star\) topology of the spaces \(L^\infty([-\tau, \tau], H^\frac{\alpha}{2}(\mathbb{R}^3))\) and \(L^2([-\tau, \tau], H^\frac{\alpha+\beta}{2}(\mathbb{R}^3))\), for all \(0 < \tau\). Moreover, as in the proof of Theorem 2.1, by using the Rellich-Lions lemma we obtain that the limit \(\bar{v}\) is a weak solution of the equation (32).

Once we have constructed an eternal solution associated to the equation (3), we will prove now the convergence (via a sub-sequence) of the sequence \((\bar{u}_n(0, \cdot))_{n \in \mathbb{N}}\) to \(\bar{v}(0, \cdot)\) in the strong topology of the space \(H^\frac{\alpha}{2}(\mathbb{R}^3)\). For this we shall need the following identity:

**Lemma 4.1** Let \(\frac{\alpha}{2} \leq \alpha + \beta\). Moreover, let \(D(m)\) be the operator defined in (17). Then, for all \(0 < t\), Leray-type solutions of the equation (3) verify the identity:

\[
\frac{d}{dt} \|\bar{u}(t, \cdot)\|^2_{H^\frac{\alpha}{2}} + 2\left\|D(m^{1/2})\bar{u}(t, \cdot)\right\|^2_{H^\frac{\alpha+\beta}{2}} = 2\langle (I_d - \Delta)^\frac{\alpha}{2} \bar{f}(t, \cdot), (I_d - \Delta)^\frac{\beta}{2} \bar{u}(t, \cdot)\rangle_{L^2 \times L^2}.
\]

Moreover, the eternal solution \(\bar{v}\) of the equation (32) also verifies this identity.

The proof of this lemma follows the same ideas performed to verify the identity (23), so we shall omit this proof. We multiply each term in this identity by \(e^{2t}\), and moreover, we integrate in the interval \([-t_n, 0]\) to get:

\[
\|\bar{u}_n(0, \cdot)\|^2_{H^\frac{\alpha}{2}} - e^{-2t_n}\|\bar{u}_{0,n}\|^2_{H^\frac{\alpha}{2}} - 2\int_{-t_n}^{0} e^{2t} \|\bar{u}_n(t, \cdot)\|^2_{H^\frac{\alpha}{2}} dt + 2\int_{-t_n}^{0} e^{2t} \left\|D(m^{1/2})\bar{u}_n(t, \cdot)\right\|^2_{H^\frac{\alpha+\beta}{2}} dt
\]

\[
= -2\int_{-t_n}^{0} e^{2t} \langle (I_d - \Delta)^\frac{\alpha}{2} \bar{f}_n, (I_d - \Delta)^\frac{\beta}{2} \bar{u}_n(t, \cdot)\rangle_{L^2 \times L^2} dt.
\]

By applying the \(\limsup\) when \(n \to +\infty\) in each term of this identity we obtain:

\[
\limsup_{n \to +\infty} \|\bar{u}_n(0, \cdot)\|^2_{H^\frac{\alpha}{2}} \leq \limsup_{n \to +\infty} e^{-2t_n}\|\bar{u}_{0,n}\|^2_{H^\frac{\alpha}{2}} + \limsup_{n \to +\infty} \left(2\int_{-t_n}^{0} e^{2t} \|\bar{u}_n(t, \cdot)\|^2_{H^\frac{\alpha}{2}} dt\right) + \limsup_{n \to +\infty} \left(-2\int_{-t_n}^{0} e^{2t} \left\|D(m^{1/2})\bar{u}_n(t, \cdot)\right\|^2_{H^\frac{\alpha+\beta}{2}} dt\right) \quad (35)
\]

\[
+ \limsup_{n \to +\infty} \left(2\int_{-t_n}^{0} e^{2t} \langle (I_d - \Delta)^\frac{\alpha}{2} \bar{f}_n, (I_d - \Delta)^\frac{\beta}{2} \bar{u}_n(t, \cdot)\rangle_{L^2 \times L^2} dt\right).
\]
where we must study each term on the right side. For the first term, always by the fact that the sequence \((\tilde{u}_n)_{n\in\mathbb{N}}\) is bounded in \(H^a_2(\mathbb{R}^3)\), we have

\[
\limsup_{n\to\infty} e^{-2t_n} \|\tilde{u}_{0,n}\|_{H^a_2}^2 = 0. \tag{36}
\]

For the second term, by the estimate (33) the sequence \((\tilde{u}_n)_{n\in\mathbb{N}}\) converges to \(\tilde{v}\) in the weak-\(\ast\) topology of the space \(L^\infty_{loc}(\mathbb{R}, H^a_2(\mathbb{R}^3))\); and then it converges in the weak-\(\ast\) topology of the space \(L^2_{loc}(\mathbb{R}, H^a_2(\mathbb{R}^3))\). We thus have:

\[
\liminf_{n\to\infty} \left( 2 \int_{-t_n}^{0} e^{2t} \|\tilde{u}_n(t, \cdot)\|_{H^a_2}^2 \, dt \right) \geq 2 \int_{-\infty}^{0} e^{2t} \|\tilde{v}(t, \cdot)\|_{H^a_2}^2 \, dt,
\]

hence we can write

\[
\liminf_{n\to\infty} \left( 2 \int_{-t_n}^{0} e^{2t} \|\tilde{u}_n(t, \cdot)\|_{H^a_2}^2 \, dt \right) \leq 2 \int_{-\infty}^{0} e^{2t} \|\tilde{v}(t, \cdot)\|_{H^a_2}^2 \, dt.
\]

Similarly, for the third term, by the estimate (34) the sequence \((\tilde{u}_n)_{n\in\mathbb{N}}\) converges to \(\tilde{v}\) in the weak-\(\ast\) topology of the space \(L^2_{loc}(\mathbb{R}, H^{a+\beta}_2(\mathbb{R}^3))\), then we have

\[
\limsup_{n\to\infty} \left( -2 \int_{-t_n}^{0} e^{2t} \|\tilde{u}_n(t, \cdot)\|_{H^{a+\beta}_2}^2 \, dt \right) \leq -2 \int_{-\infty}^{0} e^{2t} \|\tilde{v}(t, \cdot)\|_{H^{a+\beta}_2}^2 \, dt. \tag{37}
\]

Moreover, for the fourth term we obtain

\[
\limsup_{n\to\infty} \left( 2 \int_{-t_n}^{0} e^{2t} \left( \langle (I_d - \Delta)^{\beta/2} f, (I_d - \Delta)^{\beta/2} \tilde{u}_n(t, \cdot) \rangle \right)_{L^2 \times L^2} dt \right) = 2 \int_{-\infty}^{0} e^{2t} \left( \langle (I_d - \Delta)^{\beta/2} f, (I_d - \Delta)^{\beta/2} \tilde{v}(t, \cdot) \rangle \right)_{L^2 \times L^2} dt, \tag{38}
\]

Thus, with these estimates we get back to (35) to write:

\[
\limsup_{n\to\infty} \|\tilde{u}_n(0, \cdot)\|_{H^a_2}^2 \leq 2 \int_{-\infty}^{0} e^{2t} \|\tilde{v}(t, \cdot)\|_{H^a_2}^2 \, dt - 2 \int_{-\infty}^{0} e^{2t} \|D(m^{1/2})\tilde{v}(t, \cdot)\|_{H^{a+\beta}_2}^2 \, dt + 2 \int_{-\infty}^{0} e^{2t} \left( \langle (I_d - \Delta)^{\beta/2} f, (I_d - \Delta)^{\beta/2} \tilde{v}(t, \cdot) \rangle \right)_{L^2 \times L^2} dt = (A).
\]

We shall study now the term \((A)\). By Lemma 4.1 the eternal solution \(\tilde{v}\) of the equation (32) verifies the identity:

\[
\frac{d}{dt} \|\tilde{v}(t, \cdot)\|_{H^a_2}^2 = -2 \|D(m^{1/2})\tilde{v}(t, \cdot)\|_{H^{a+\beta}_2}^2 + 2 \left( \langle (I_d - \Delta)^{\beta/2} f, (I_d - \Delta)^{\beta/2} \tilde{v}(t, \cdot) \rangle \right)_{L^2 \times L^2}.
\]

We multiply each term by \(e^{2t}\), then we integrate in the interval \([-\infty, 0]\) to get:

\[
\|\tilde{v}(0, \cdot)\|_{H^a_2}^2 = \int_{-\infty}^{0} e^{2t} \|\tilde{v}(t, \cdot)\|_{H^a_2}^2 \, dt - 2 \int_{-\infty}^{0} e^{2t} \|D(m^{1/2})\tilde{v}(t, \cdot)\|_{H^{a+\beta}_2}^2 \, dt + 2 \int_{-\infty}^{0} e^{2t} \left( \langle (I_d - \Delta)^{\beta/2} f, (I_d - \Delta)^{\beta/2} \tilde{v}(t, \cdot) \rangle \right)_{L^2 \times L^2} dt = (A).
\]

In this fashion, by the previous estimate we get \(\limsup_{n\to\infty} \|\tilde{u}_n(0, \cdot)\|_{H^a_2}^2 \leq \|\tilde{v}(0, \cdot)\|_{H^a_2}^2\). Moreover, as sequence \((\tilde{u}_n)_{n\in\mathbb{N}}\) converges (via a sub-sequence) to \(\tilde{v}\) in the weak-\(\ast\) topology of the space \(L^\infty(\mathbb{R}, H^a_2(\mathbb{R}^3))\),
we are able to write \( \|\vec{v}(0, \cdot)\|_{H^s_\beta}^2 \leq \liminf_{n \to +\infty} \|\vec{u}_n(0, \cdot)\|_{H^s_\beta}^2 \). We thus obtain the desired strong convergence:
\[
\lim_{n \to +\infty} \|\vec{u}_n(0, \cdot)\|_{H^s_\beta}^2 = \|\vec{v}(0, \cdot)\|_{H^s_\beta}^2.
\]

Point 3. The continuity of the map \( S(t) : B \to H^\frac{s}{2}(\mathbb{R}^3) \) directly follows from the estimate (26), where we have \( \vec{w}(t, \cdot) = \vec{w}_1(t, \cdot) - \vec{w}_2(t, \cdot) = S(t)\vec{u}_{0,1} - S(t)\vec{u}_{0,2} \), and \( \vec{w}(0, \cdot) = \vec{u}_{0,1} - \vec{u}_{0,2} \).

At this point, we are able to apply Theorem 4.2 to deduce that the semigroup \( (S(t))_{t \geq 0} \) has a unique strong global attractor \( \mathcal{A}_s \). The second point of Theorem 2.3 is now proven. ■

4.3 Proof of Theorem 2.4

Our starting point is the construction of time independent solutions of the equation (3). These solutions shall be denoted by \( \vec{U}(x) \) and as they only depend on the spatial variable they formally solve the following elliptic problem:
\[
\nu(-\Delta)^\frac{s}{2}\vec{U} + (I_d - \Delta)^{-\frac{s}{2}}\mathbb{P}\text{div}(\vec{U} \otimes \vec{U}) = \vec{f} - \gamma\vec{U}, \quad \text{div}(\vec{U}) = 0. \tag{39}
\]

**Proposition 4.2** Let \( 2 \leq \alpha + \beta < \frac{5}{2} \). Moreover, let \( 0 < \nu \) and \( 0 < \gamma \) be such that the condition (8) is verified. Then, there exists \( \vec{U} \in H^\frac{s+\beta}{2}(\mathbb{R}^3) \) a solution of the equation (39). Moreover, for the constant \( 0 < A_{\alpha,\gamma,\nu} \) given in the second point of Proposition 2.1, the function \( \vec{U} \) is the unique solution verifying the following energy control:
\[
\|\vec{U}\|_{H^\frac{s+\beta}{2}} \leq \frac{2}{A_{\alpha,\gamma,\nu}} \|\vec{f}\|_{H^\frac{s}{2}}. \tag{40}
\]

**Proof.** The equation (39) can be formally rewritten as the following fixed point problem
\[
\vec{U} = D(m^{-1})(I_d - \Delta)^{-\frac{s+\beta}{2}}\mathbb{P}\text{div}(\vec{U} \otimes \vec{U}) + D(m^{-1})(I_d - \Delta)^{-\frac{s}{2}}\vec{f}, \tag{41}
\]
where \( D(m^{-1}) \) is the inverse operator of the operator \( D(m) \) given in (17). We recall that \( D(m^{-1}) \) is defined by the inverse function of the symbol \( m(\xi) \).

We shall use the Picard’s fixed point argument to construct a solution \( \vec{U} \in H^\frac{s+\beta}{2}(\mathbb{R}^3) \) of this problem. For the constant \( 0 < A_{\alpha,\gamma,\nu} = A \), the nonlinear term on the right side of (41) is estimated as follows:
\[
\left\| D(m^{-1})(I_d - \Delta)^{-\frac{s+\beta}{2}}\mathbb{P}\text{div}(\vec{U} \otimes \vec{U}) \right\|_{H^\frac{s+\beta}{2}} \leq C \ A^{-1} \left\| \vec{U} \right\|^2_{H^\frac{s+\beta}{2}}. \tag{42}
\]

Indeed, we write
\[
\left\| D(m^{-1})(I_d - \Delta)^{-\frac{s+\beta}{2}}\mathbb{P}\text{div}(\vec{U} \otimes \vec{U}) \right\|_{H^\frac{s+\beta}{2}} \leq C \left\| (1 + |\xi|^2)^{-\frac{s+\beta}{2}} m^{-1}(\xi) (1 + |\xi|^2)^{-\frac{s+\beta}{2}} \mathcal{F}_x(\vec{U} \otimes \vec{U}) \right\|_{L^2} \leq C \left\| (1 + |\xi|^2)^{-\frac{s+\beta}{2}} m^{-1}(\xi) (1 + |\xi|^2)^{-\frac{s+\beta}{2}} \mathcal{F}_x(\vec{U} \otimes \vec{U}) \right\|_{L^2} \leq C \left\| m^{-1}(\xi) (1 + |\xi|^2)^{-\frac{s+\beta}{2} + \frac{1}{2}} \mathcal{F}_x(\vec{U} \otimes \vec{U}) \right\|_{L^2}.
\]

In the last expression, we remark first that as \( A < |m(\xi)| \) then \( |m^{-1}(\xi)| \leq A^{-1} \). Moreover, as \( 2 \leq \alpha + \beta \) we have \( -\frac{s+\beta}{4} + \frac{1}{2} \leq 0 \). We thus get
\[
C \left\| m^{-1}(\xi) (1 + |\xi|^2)^{-\frac{s+\beta}{2} + \frac{1}{2}} \mathcal{F}_x(\vec{U} \otimes \vec{U}) \right\|_{L^2} \leq C \ A^{-1} \left\| \vec{U} \otimes \vec{U} \right\|_{L^2}.
\]

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Therefore, by the product laws in the non homogeneous Sobolev spaces, and moreover, always by the fact that \( 2 \leq \alpha + \beta \), we can write
\[ C A^{-1} \| \tilde{U} \|_{L^2} \leq C A^{-1} \| \tilde{U} \|_{H^\alpha} \| \tilde{U} \|_{H^\beta} \leq C A^{-1} \| \tilde{U} \|_{H^{\alpha+\beta}} \| \tilde{U} \|_{H^{\alpha+\beta}}. \]

On the other hand, the second term on the right side of (41) is estimated as follows:
\[ \left\| D(m^{-1})(I_d - \Delta)^{-\frac{\beta}{2}} f \right\|_{H^{\alpha+\beta}} \leq A^{-1} \left\| f \right\|_{H^\alpha}. \]

Indeed, we write
\[ \left\| D(m^{-1})(I_d - \Delta)^{-\frac{\beta}{2}} f \right\|_{H^{\alpha+\beta}} = \left\| (1 + |\xi|^2)^{\frac{\alpha+\beta}{2}} m^{-1}(\xi) (1 + |\xi|^2)^{-\frac{\beta}{2}} \mathcal{F}_x (\tilde{f}) \right\|_{L^2} \]

\[ = \left\| (1 + |\xi|^2)^{-\frac{\beta}{2}} m^{-1}(\xi) (1 + |\xi|^2)^{\frac{\alpha}{2}} \mathcal{F}_x (\tilde{f}) \right\|_{L^2} \leq A^{-1} \left\| f \right\|_{H^\alpha}. \]

Once we have the estimates (42) and (43), in order to apply the Picard’s fixed argument we need to satisfy the following condition
\[ 4 C A^{-2} \left\| f \right\|_{H^\alpha} < 1. \]

As \( \alpha + \beta < \frac{5}{2} \) we have \( \alpha < \frac{5}{2} \) and then we get \( A^{-2} = \frac{\max(1,2^{\alpha-2})}{\min(\gamma,\nu,\rho)} \leq \frac{2^5}{\min(\gamma,\nu,\rho)} \). Then, we can write
\[ 4 C A^{-2} \left\| f \right\|_{H^\alpha} < 1. \]

Thereafter, in the assumption (8) we set the constant \( C_0 \) such that \( \frac{1}{4 C^2 2^5} \leq C_0 \) and we have \( 4 C^2 2^5 \left\| f \right\|_{H^\alpha} < 1 \). In this fashion, by the Picard’s fixed argument there exists \( \tilde{U} \in H^{\alpha+\beta}(\mathbb{R}^2) \) a solution of the equation (41), which is the unique solution verifying (40). Proposition 4.2 is proven.

Now the have the following:

**Proposition 4.3** Let \( \frac{3}{2} < \alpha + \beta < \frac{5}{2} \) and let \( 0 < \nu \) and \( 0 < \gamma \) be such that the condition (8) is verified. Moreover, let \( \tilde{U} \in H^{\alpha+\beta}(\mathbb{R}^2) \) be the unique solution of the equation (39) verifying (40). Then, the set \( \mathcal{A} = \{ \tilde{U} \} \) is an strong global attractor of the equation (3).

**Proof.** We must prove that \( \mathcal{A} = \{ \tilde{U} \} \) verifies the first and the second point in Definition 2.4 (with \( \bullet = s \)).

As the first point is evident, we shall focus on the second point and we shall prove that \( \mathcal{A} = \{ \tilde{U} \} \) is a strong attracting set in the sense of Definition 2.3. Let \( \vec{u}_0 \in H^{\alpha+\beta}(\mathbb{R}^2) \) be an initial datum and let \( \vec{u} \) be a Leray-type solution of (3) arising from \( \vec{u}_0 \). Then, we shall prove the following inequality:
\[ \| \vec{u}(t, \cdot) - \tilde{U} \|^2_{H^\alpha} \leq \| \vec{u}_0 - \tilde{U} \|^2_{H^\alpha} e^{-2\gamma t}, \quad 0 \leq t. \]

We define \( \vec{w}(t, \cdot) = \vec{u}(t, \cdot) - \tilde{U} \). This function solves the equation:
\[ \partial_t \vec{w} + \nu(-\Delta)^{\frac{\beta}{2}} \vec{w} + (I_d - \Delta)^{-\frac{\beta}{2}} \mathbb{P} \left( \text{div}(\vec{u} \otimes \vec{u}) - \text{div}(\tilde{U} \otimes \tilde{U}) \right) + \gamma \vec{w} = 0, \quad \text{div}(\vec{w}) = 0, \]

but, to carry out the computations we shall perform below, it is convenient to write
\[ \text{div}(\vec{u} \otimes \vec{u}) - \text{div}(\tilde{U} \otimes \tilde{U}) = \text{div}(\vec{w} \otimes \vec{w}) + \text{div}(\vec{w} \otimes \tilde{U}) + \text{div}(\tilde{U} \otimes \vec{w}). \]

Therefore, the function \( \vec{w} \) verifies the equation:
\[ \partial_t \vec{w} + \nu(-\Delta)^{\frac{\beta}{2}} \vec{w} + (I_d - \Delta)^{-\frac{\beta}{2}} \mathbb{P} \left( \text{div}(\vec{w} \otimes \vec{w}) + \text{div}(\vec{w} \otimes \tilde{U}) + \text{div}(\tilde{U} \otimes \vec{w}) \right) + \gamma \vec{w} = 0, \quad \text{div}(\vec{w}) = 0. \]

The inequality (44) will be obtained by energy estimates on the function \( \vec{w} \). Our starting point is then the identity
\[ \| \vec{w}(t, \cdot) \|^2_{H^\alpha} = \| \vec{u}(t, \cdot) \|^2_{H^\alpha} - 2 \left( \vec{u}(t, \cdot), \tilde{U} \right)_{H^\alpha} + \| \tilde{U} \|^2_{H^\alpha}. \]
where $(\cdot, \cdot)_{H^2_\beta}$ denotes the standard inner product in the space $H^2_\beta(\mathbb{R}^3)$. To study the second term on the right side, for the sake of simplicity, we shall denote by $J^\beta$ the Bessel potential $(I_d - \Delta)^{\frac{\beta}{2}}$. We thus have the identity
\[
-2\left(\tilde{u}(t, \cdot), \tilde{U}\right)_{H^2_\beta} = -2(\tilde{u}_0, \tilde{U})_{H^2_\beta} - 2\int_0^t \left(\partial_t J^\beta \tilde{u}(s, \cdot), \tilde{U}\right)_{H^{-\frac{1}{2}} \times H^2} ds.
\]
Indeed, this identity bases on the following technical lemmas. Their proofs are straightforward and they will be done in the Appendix A.

**Lemma 4.2** Let $\tilde{U} \in H^{\alpha+\beta}(\mathbb{R}^3)$ be the solution of the equation (39) given by Proposition 4.2. Then, we have $\tilde{U} \in H^{\alpha+\frac{\beta}{2}}(\mathbb{R}^3)$.

**Lemma 4.3** Let $2 \leq \alpha + \beta < \frac{5}{2}$ and let $\tilde{u}$ be a Leray-type solution of the equation (3). Then, we have $\partial_t J^\beta \tilde{u} \in (L^2_t)_{loc} H^{-\frac{4}{3}}_x$.

By Lemma 4.2 and by the assumption $\frac{3}{2} < \alpha + \frac{\beta}{2}$ we have $\tilde{U} \in H^2_\beta(\mathbb{R}^3)$. Moreover, by Lemma 4.3 and as $\tilde{U}$ is a time independent function we have $\partial_t J^\beta \tilde{u} = \partial_t J^\beta \tilde{u} \in (L^2_t)_{loc} H^{-\frac{2}{3}}_x$. In this fashion, the term $\int_0^t \left(\partial_t J^\beta \tilde{u}(s, \cdot), \tilde{U}\right)_{H^{-\frac{1}{2}} \times H^2} ds$ is well defined for all $0 \leq t$. Thus, for $a.e. \ 0 \leq s$ we can write
\[
\partial_t \left(\tilde{u}(s, \cdot), \tilde{U}\right)_{H^2_\beta} = \partial_t \left(J^\beta \tilde{u}(s, \cdot), J^\beta \tilde{U} \right)_{L^2} = \partial_t \left(J^\beta \tilde{u}(s, \cdot), \tilde{U}\right)_{H^{-\frac{1}{2}} \times H^2} = \left(\partial_t J^\beta \tilde{u}(s, \cdot), \tilde{U}\right)_{H^{-\frac{1}{2}} \times H^2}.
\]
Then, by integrating on the interval of time $[0, t]$ we obtain the desired identity. We substitute this identity in the second term of the identity (46) to get
\[
\|\tilde{u}(t, \cdot)\|_{H^2_\beta} = \|\tilde{u}(t, \cdot)\|_{H^2_\beta} - 2(\tilde{u}_0, \tilde{U})_{H^2_\beta} - 2\int_0^t \left(\partial_t J^\beta \tilde{u}(s, \cdot), \tilde{U}\right)_{H^{-\frac{1}{2}} \times H^2} ds + \|\tilde{U}\|_{H^\frac{1}{2}}.
\]
Here, we substitute the first term on the right side by the estimate given in the third point in Proposition 2.1. Moreover, we rearrange the terms to write:
\[
\|\tilde{u}(t, \cdot)\|_{H^2_\beta}^2 \leq \|\tilde{u}_0\|_{H^2_\beta}^2 - 2(\tilde{u}_0, \tilde{U})_{H^2_\beta} + \|\tilde{U}\|_{H^\frac{1}{2}}^2 - 2\nu \int_0^t \|(-\Delta)^{\frac{\beta}{2}} J^\beta \tilde{u}(s, \cdot)\|_{L^2}^2 ds
\]
\[+ 2\int_0^t \left(\tilde{f}(\tilde{u}(s, \cdot))_{H^\frac{1}{2}}, \tilde{U}\right)_{H^\beta} ds - 2\gamma \int_0^t \|\tilde{u}(s, \cdot)\|_{H^2_\beta}^2 ds - 2 \int_0^t \left(\partial_t J^\beta \tilde{u}(s, \cdot), \tilde{U}\right)_{H^{-\frac{1}{2}} \times H^2} ds.
\]
Thereafter, we must study the terms $(A)$, $(B)$ and $(C)$ separately. The term $(A)$ is easy to study and we have $(A) = \|\tilde{u}_0 - \tilde{U}\|_{H^\beta}^2$. Next, in order to study the term $(B)$, we remark first that by the equation (39) we can write
\[
J^\beta \tilde{f} = \nu(-\Delta)^{\frac{\beta}{2}} J^\beta \tilde{U} + \mathbb{P} \text{div}(\tilde{U} \otimes \tilde{U}) + \gamma J^\beta \tilde{U},
\]
and then we have
\[
(B) = 2 \int_0^t \left\langle J^\beta \tilde{f}, \tilde{u}(s, \cdot) \right\rangle_{H^{-\frac{\alpha}{2}} \times H^\beta} ds \\
= 2 \int_0^t \left\langle \nu(-\Delta)^{\frac{\nu}{2}} J^\beta \tilde{U} + \mathcal{P} \text{ div}(\tilde{U} \otimes \tilde{U}) + \gamma J^\beta \tilde{U}, \tilde{u}(s, \cdot) \right\rangle_{H^{-\frac{\alpha}{2}} \times H^\beta} ds \\
= 2 \nu \int_0^t \left( (-\Delta)^{\frac{\nu}{2}} J^\beta \tilde{U}, (-\Delta)^{\frac{\nu}{2}} J^\beta \tilde{u}(s, \cdot) \right)_{L^2} ds + 2 \int_0^t \left\langle \text{div}(\tilde{U} \otimes \tilde{U}), \tilde{u}(s, \cdot) \right\rangle_{H^{-\frac{\alpha}{2}} \times H^\beta} ds \\
+ 2 \gamma \int_0^t \left( \tilde{U}, \tilde{u}(s, \cdot) \right)_{H^\beta} ds.
\]
Finally, to study the term \((C)\), we remark now that by the equation \((45)\) we can write
\[
\partial_t J^\beta \tilde{w} = -\nu(-\Delta)^{\frac{\nu}{2}} J^\beta \tilde{w} - \mathcal{P} \left( \text{div}(\tilde{w} \otimes \tilde{U}) + \text{div}(\tilde{U} \otimes \tilde{w}) + \text{div}(\tilde{U} \otimes \tilde{w}) - \gamma J^\beta \tilde{w}. \right.
\]
Therefore, we obtain
\[
(C) = -2 \nu \int_0^t \left( (-\Delta)^{\frac{\nu}{2}} J^\beta \tilde{w}(s, \cdot), \tilde{U} \right)_{H^{-\frac{\alpha}{2}} \times H^\beta} ds - 2 \int_0^t \left\langle \text{div}(\tilde{w} \otimes \tilde{w}), \tilde{U} \right\rangle_{H^{-\frac{\alpha}{2}} \times H^\beta} ds \\
- 2 \int_0^t \left\langle \text{div}(\tilde{U} \otimes \tilde{w}), \tilde{U} \right\rangle_{H^{-\frac{\alpha}{2}} \times H^\beta} ds - 2 \int_0^t \left( \tilde{w}(s, \cdot), \tilde{U} \right)_{H^\beta} ds.
\]
In this identity, in the first term on the right side we directly write
\[
2 \nu \int_0^t \left( (-\Delta)^{\frac{\nu}{2}} J^\beta \tilde{w}(s, \cdot), (-\Delta)^{\frac{\nu}{2}} J^\beta \tilde{U} \right)_{L^2} ds = -2 \nu \int_0^t \left( (-\Delta)^{\frac{\nu}{2}} J^\beta \tilde{w}(s, \cdot), (-\Delta)^{\frac{\nu}{2}} J^\beta \tilde{U} \right)_{L^2} ds.
\]
Thereafter, to study the second and the third term on the right side, we have the following remarks. On the one hand, by Lemma 4.2 we have \(\tilde{U} \in H^{\alpha+\frac{\nu}{2}}(\mathbb{R}^3)\), and moreover, as we have \(\frac{\alpha}{2} < \frac{\nu}{2}\), by the Sobolev embeddings we get \(\tilde{U} \in L^\infty(\mathbb{R}^3)\). Then, as \(\tilde{w} \in (L^2_{4\nu})_{\text{loc}}L^2_{\nu}\) (recall that \(\tilde{w}\) belongs to the energy space) we have \(\tilde{w} \otimes \tilde{U} \in (L^2_{4\nu})L^2_{\nu}\) and consequently \(\text{div}(\tilde{w} \otimes \tilde{U}) \in (L^2_{4\nu})_{\text{loc}}H^{-1}\). On the other hand, we recall that \(\tilde{w} \in (L^2_{4\nu})_{\text{loc}}H^{\alpha+\frac{\nu}{2}}\), and moreover, as we have \(2 \leq \alpha + \beta\) we get \(\tilde{w} \in (L^2_{4\nu})_{\text{loc}}H^{1}\). With these remarks and by integrating by parts we are able to write:
\[
-2 \int_0^t \left\langle \text{div}(\tilde{w} \otimes \tilde{w}), \tilde{U} \right\rangle_{H^{-\frac{\alpha}{2}} \times H^\beta} ds = 2 \int_0^t \left\langle (\tilde{w} \cdot \tilde{\nabla}) \tilde{U}, \tilde{w} \right\rangle_{H^{-1} \times H^1} ds,
\]
and
\[
-2 \int_0^t \left\langle \text{div}(\tilde{U} \otimes \tilde{w}), \tilde{U} \right\rangle_{H^{-\frac{\alpha}{2}} \times H^\beta} ds = 2 \int_0^t \left\langle (\tilde{U} \cdot \tilde{\nabla}) \tilde{U}, \tilde{w} \right\rangle_{H^{-1} \times H^1} ds.
\]
Finally, as \(\text{div}(\tilde{U}) = \text{div}(\tilde{w}) = 0\), in the fourth term on the right side we have
\[
-2 \int_0^t \left\langle \text{div}(\tilde{U} \otimes \tilde{w}), \tilde{U} \right\rangle_{H^{-\frac{\alpha}{2}} \times H^\beta} ds = 0.
\]
In this fashion, the term \((C)\) writes down as follows:
\[
(C) = -2 \nu \int_0^t \left( (-\Delta)^{\frac{\nu}{2}} J^\beta \tilde{w}(s, \cdot), (-\Delta)^{\frac{\nu}{2}} J^\beta \tilde{U} \right)_{L^2} ds + 2 \int_0^t \left\langle (\tilde{w} \cdot \tilde{\nabla}) \tilde{U}, \tilde{w} \right\rangle_{H^{-1} \times H^1} ds \\
+ 2 \int_0^t \left\langle (\tilde{U} \cdot \tilde{\nabla}) \tilde{U}, \tilde{w} \right\rangle_{H^{-1} \times H^1} ds - 2 \gamma \int_0^t \left( \tilde{w}(s, \cdot), \tilde{U} \right)_{H^\beta} ds.
\]
Once we have these identities for the terms (A), (B) and (C), we get back to the inequality (47) to write

\[
\|\mathbf{u}(t, \cdot)\|_{H^\gamma_x}^2 \leq \|\mathbf{u}_0 - \mathbf{U}\|_{H^\gamma_x}^2 - 2\nu \int_0^t \|(-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u}(s, \cdot)\|_{L^2}^2 ds \\
+ 2\nu \int_0^t \left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{U}, (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u}(s, \cdot) \right)_{L^2} ds + 2\nu \int_0^t \left( \langle \text{div}(\mathbf{U} \otimes \mathbf{U}), \mathbf{u}(s, \cdot) \rangle \right)_{H^{-\alpha}_{\gamma} \times H^\alpha_{\gamma}} ds \\
+ 2\gamma \int_0^t \left( \mathbf{U}, \mathbf{u}(s, \cdot) \right)_{H^\alpha_{\gamma}} ds - 2\gamma \int_0^t \|\mathbf{u}(s, \cdot)\|_{H^\alpha_{\gamma}}^2 ds \\
+ 2\nu \int_0^t \left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u}(s, \cdot), (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{U} \right)_{L^2} ds - 2\nu \int_0^t \left( \langle \mathbf{w} \cdot \nabla \rangle \mathbf{U}, \mathbf{w} \right)_{H^{-1} \times H^1} ds \\
- 2\nu \int_0^t \left( \langle \mathbf{w} \cdot \nabla \rangle \mathbf{U}, \mathbf{w} \right)_{H^{-1} \times H^1} ds. 
\]

By rearranging again the terms on the right side we obtain:

\[
\|\mathbf{u}(t, \cdot)\|_{H^\gamma_x}^2 \leq \|\mathbf{u}_0 - \mathbf{U}\|_{H^\gamma_x}^2 \\
- 2\nu \int_0^t \left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u}(s, \cdot) \right)_{L^2}^2 ds + 2\nu \int_0^t \left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{U}, (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u}(s, \cdot) \right)_{L^2} ds \\
+ 2\nu \int_0^t \left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u}(s, \cdot), (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{U} \right)_{L^2} ds \\
+ 2\gamma \int_0^t \left( \mathbf{U}, \mathbf{u}(s, \cdot) \right)_{H^\alpha_{\gamma}} ds - 2\gamma \int_0^t \|\mathbf{u}(s, \cdot)\|_{H^\alpha_{\gamma}}^2 ds + 2\gamma \int_0^t \left( \mathbf{w}(s, \cdot), \mathbf{U} \right)_{H^\alpha_{\gamma}} ds. 
\]

In this inequality, we remark now the following facts. First, we have:

\[
- 2\nu \int_0^t \left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u}(s, \cdot) \right)_{L^2}^2 ds + 2\nu \int_0^t \left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{U}, (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u}(s, \cdot) \right)_{L^2} ds \\
+ 2\nu \int_0^t \left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u}(s, \cdot), (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{U} \right)_{L^2} ds \leq -2 \min(\gamma, \nu) \int_0^t \|\mathbf{w}(s, \cdot)\|_{H^2_{\alpha + \beta}}. 
\]

Indeed, as \( \mathbf{w} = \mathbf{u} - \mathbf{U} \) we write:

\[
- 2\nu \left\|\left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u} \right) \right\|^2_{L^2} + 2\nu \left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{U}, (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u} \right)_{L^2} \\
+ 2\nu \left\|\left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u}, (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{U} \right) \right\|^2_{L^2} \\
= -2\nu \left\|\left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u} \right) \right\|^2_{L^2} + 4\nu \left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{U}, (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u} \right)_{L^2} \\
+ 2\nu \left\|\left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{U} \right) \right\|^2_{L^2} \\
= -2\nu \left\|\left( (-\Delta)\frac{q}{2} J^{\alpha}_{\beta} \mathbf{u} \right) \right\|^2_{L^2} \leq -2\nu \|\mathbf{w}\|_{H^2_{\alpha + \beta}}^2 \leq -2\gamma \|\mathbf{w}\|_{H^2_{\alpha + \beta}} \leq -2 \min(\gamma, \nu) \|\mathbf{w}\|_{H^2_{\alpha + \beta}}. 
\]

Thereafter, always by the identity \( \mathbf{w} = \mathbf{u} - \mathbf{U} \) we obtain:

\[
2\nu \int_0^t \langle \text{div}(\mathbf{U} \otimes \mathbf{U}), \mathbf{u}(s, \cdot) \rangle_{H^{-\alpha}_{\gamma} \times H^\alpha_{\gamma}} ds - 2\nu \int_0^t \left( \langle \mathbf{w} \cdot \nabla \rangle \mathbf{U}, \mathbf{w} \right)_{H^{-1} \times H^1} ds = 0, \\
2\gamma \int_0^t \left( \mathbf{U}, \mathbf{u}(s, \cdot) \right)_{H^\alpha_{\gamma}} ds - 2\gamma \int_0^t \|\mathbf{u}(s, \cdot)\|_{H^\alpha_{\gamma}}^2 ds + 2\gamma \int_0^t \left( \mathbf{w}(s, \cdot), \mathbf{U} \right)_{H^\alpha_{\gamma}} ds = -2\gamma \int_0^t \|\mathbf{w}(s, \cdot)\|_{H^\alpha_{\gamma}}^2 ds. 
\]
Finally, the following estimate holds:

\[-2 \int_0^t \left\langle (\bar{w} \cdot \nabla) \bar{U}, \bar{w} \right\rangle_{H^{-1} \times H^1} ds \leq C \| \bar{U} \|_{L^3} \int_0^t \| \bar{w}(s, \cdot) \|^2_{H^{\frac{\alpha+\beta}{2}}} ds. \tag{51} \]

Indeed, by the Hölder inequalities, the Hardy-Littlewood-Sobolev inequalities, and moreover, as $2 \leq \alpha + \beta$ we can write:

\[
\left| 2 \int_0^t \left\langle (\bar{w} \cdot \nabla) \bar{U}, \bar{w} \right\rangle_{H^{-1} \times H^1} ds \right| \leq 2 \int_0^t \left\| (\bar{w}(s, \cdot) \cdot \nabla) \bar{U} \right\|_{H^{-1} \times H^1} ds \leq 2 \int_0^t \| \text{div}(\bar{U} \otimes \bar{w}(s, \cdot)) \|_{H^{-1}} \| \bar{w}(s, \cdot) \|_{H^1} ds
\]

\[
\leq 2 \int_0^t \| \bar{U} \otimes \bar{w}(s, \cdot) \|_{L^2} \| \bar{w}(s, \cdot) \|_{H^1} ds \leq 2 \int_0^t \| \bar{U} \|_{L^3} \| \bar{w}(s, \cdot) \|_{L^6} \| \bar{w}(s, \cdot) \|_{H^1} ds
\]

\[
\leq C \| \bar{U} \|_{L^3} \int_0^t \| \bar{w}(s, \cdot) \|_{H^1} \| \bar{w}(s, \cdot) \|_{H^1} ds \leq C \| \bar{U} \|_{L^3} \int_0^t \| \bar{w}(s, \cdot) \|_{H^{\frac{\alpha+\beta}{2}}} ds.
\]

Once we have the estimates (49), (50) and (51), we get back to the inequality (48) to write:

\[
\| \bar{w}(t, \cdot) \|^2_{H^\frac{\alpha}{2}} \leq \| \bar{u}_0 - \bar{U} \|^2_{H^\frac{\alpha}{2}} + (-2 \min(\gamma, \nu) + C \| \bar{U} \|_{L^3}) \int_0^t \| \bar{w}(s, \cdot) \|_{H^{\frac{\alpha+\beta}{2}}}^2 ds - 2\gamma \int_0^t \| \bar{w}(s, \cdot) \|^2_{H^\frac{\beta}{2}} ds.
\]

In the second term on the right side, as we have assumed (8) then we have $(-2 \min(\gamma, \nu) + C \| \bar{U} \|_{L^3}) \leq 0$. Indeed, by the interpolation inequalities, the Hardy-Littlewood-Sobolev inequalities and as $2 \leq \alpha + \beta$, we are able to write:

\[
C \| \bar{U} \|_{L^3} \leq C \| \bar{U} \|_{L^2}^{\frac{1}{2}} \| \bar{U} \|_{L^6}^{\frac{1}{2}} \leq C \| \bar{U} \|_{L^2}^{\frac{1}{2}} \| \bar{U} \|_{H^1}^{\frac{1}{2}} \leq C \| \bar{U} \|_{H^1} \leq C \| \bar{U} \|_{H^{\frac{\alpha+\beta}{2}}}
\]

Thereafter, by the estimate (40) and as $A_{\alpha, \gamma, \nu}^{-1} \leq \frac{C}{\min(\gamma, \nu)}$ (where $0 < C$ is a numerical constant), we have:

\[
C \| \bar{U} \|_{H^{\frac{\alpha+\beta}{2}}} \leq \frac{C}{A_{\alpha, \gamma, \nu}} \| \bar{f} \|_{H^\frac{\beta}{2}} \leq \frac{C}{\min(\gamma, \nu)} \| \bar{f} \|_{H^\frac{\beta}{2}}.
\]

Finally, in the estimate (8) we set the constant $\frac{2}{\beta} \leq C_0$ to obtain \(\frac{C}{\min(\gamma, \nu)} \| \bar{f} \|_{H^\frac{\beta}{2}} \leq 2 \min(\gamma, \nu)\).

Once $(-2 \min(\gamma, \nu) + C \| \bar{U} \|_{L^3}) \leq 0$, we can write $\| \bar{w}(t, \cdot) \|^2_{H^\frac{\alpha}{2}} \leq \| \bar{u}_0 - \bar{U} \|^2_{H^\frac{\alpha}{2}} - 2\gamma \int_0^t \| \bar{w}(s, \cdot) \|^2_{H^\frac{\beta}{2}} ds$. Hence, by applying the Grönwall inequalities we obtain the desired estimate (44).

By the estimate (44) we directly obtain that $\mathcal{A} = \{ \bar{U} \}$ is an strong absorbing set for the equation (3) in the sense of Definition 4. Proposition 4.3 is now proven. ■

By Proposition 4.3 and by the uniqueness of the global attractor we have $\mathcal{A}_w = \mathcal{A}_s = \{ \bar{U} \}$. Theorem 2.4 is proven. ■

A Appendix

**Proof of Lemma 4.2.** This proof is straightforward: we write $\bar{U}$ as the solution of the fixed point problem (41). In this last identity, as $\bar{f} \in H^\frac{\beta}{2}(\mathbb{R}^3)$ we have $D(m^{-1})(I_d - \Delta)^\frac{\beta}{2} \bar{f} \in H^{\alpha+\frac{\beta}{2}}(\mathbb{R}^3)$. Moreover, as $\bar{U} \in H^{\frac{\alpha+\beta}{2}}(\mathbb{R}^3)$ (with $1 \leq \frac{\alpha+\beta}{2}$) by the product laws in the Sobolev spaces and by an iterative argument we obtain $D(m^{-1})(I_d - \Delta)^{-\frac{\alpha+\beta}{2}} \text{div}(\bar{U} \otimes \bar{U}) \in H^{\alpha+\frac{\beta}{2}}(\mathbb{R}^3)$. We thus have $\bar{U} \in H^{\alpha+\frac{\beta}{2}}(\mathbb{R}^3)$. It is worth
emphasizing this gain of regularity of $\vec{U}$ is sharp in the sense that the term $D(m^{-1})(I_d - \Delta)^{\frac{\alpha}{2}} \vec{f}$ only belongs to the space $H^{\alpha+\frac{\beta}{2}}(\mathbb{R}^3)$, provided that $\vec{f} \in H^{\frac{\beta}{2}}(\mathbb{R}^3)$.

**Proof of Lemma 4.3.** As $\vec{u}$ is a Leray-type solution of the equation (3), we can write:

$$\partial_t J_\beta \vec{u} = -\nu (-\Delta)^{\frac{\alpha}{2}} J_\beta \vec{u} - \gamma J_\beta \vec{u} - \mathbb{P}(\text{div}(\vec{u} \otimes \vec{u})) + J_\beta \vec{f},$$

where each term on the right side belong to the space $(L^2_t)_{\text{loc}} H^{\frac{\alpha+\beta}{2}}_x$. Indeed, as $\vec{u} \in (L^2_t)_{\text{loc}} H^{\alpha+\beta}_x$, and moreover, as $\alpha + \beta < \frac{5}{2}$, we have $-\nu (-\Delta)^{\frac{\alpha}{2}} J_\beta \vec{u} \in (L^2_t)_{\text{loc}} H^{-\frac{\alpha+\beta}{2}}_x \subset (L^2_t)_{\text{loc}} H^{-\frac{\alpha+\beta}{2}}_x$. We also have $-\gamma J_\beta \vec{u} \in (L^2_t)_{\text{loc}} H^{-\frac{\alpha+\beta}{2}}_x \subset (L^2_t)_{\text{loc}} H^{-\frac{\alpha+\beta}{2}}_x$. Thereafter, as $2 \leq \alpha + \beta < \frac{5}{2}$ by Lemma 3.3 we have $-\mathbb{P}(\text{div}(\vec{u} \otimes \vec{u})) \in (L^2_t)_{\text{loc}} H^{-\frac{\alpha+\beta}{2}}_x \subset (L^2_t)_{\text{loc}} H^{-\frac{\alpha+\beta}{2}}_x$. Finally, as $\vec{f} \in H^{\frac{\beta}{2}}(\mathbb{R}^3)$, and moreover, as $\beta \leq 3$ (since we have $\beta < \alpha + \beta < \frac{5}{2}$), we have $J_\beta \vec{f} \in (L^2_t)_{\text{loc}} H^{-\frac{\alpha+\beta}{2}}_x \subset (L^2_t)_{\text{loc}} H^{-\frac{\alpha+\beta}{2}}_x$. ■

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