Abstract. In this paper, we propose a multiphysics finite element method for a nonlinear poroelasticity model. To better describe the processes of deformation and diffusion, we firstly reformulate the nonlinear fluid-solid coupling problem into a fluid-fluid coupling problem by a multiphysics approach. Then we design a fully discrete time-stepping scheme to use multiphysics finite element method with $P_2 - P_1 - P_1$ element pairs for the space variables and backward Euler method for the time variable, and we adopt the Newton iterative method to deal with the nonlinear term. Also, we derive the discrete energy laws and the optimal convergence order error estimates without any assumption on the nonlinear stress-strain relation. Finally, we show some numerical examples to verify the rationality of theoretical analysis and there is no “locking phenomenon”.

Key words. Nonlinear poroelasticity; Stokes equations; finite element methods; error estimates.

1. Introduction. Poroelasticity model is a fluid-solid coupled system at poro scale, which is widely used in various fields such as geophysics, biomechanics, civil engineering, chemical engineering, materials science and so on, one can refer to [3, 13, 14, 20, 21, 25, 26, 28, 34]. There are many kinds of nonlinear poroelasticity, such as the permeability tensor $K(\text{div} \mathbf{u})$ (cf. [4, 5, 9, 10, 15, 27, 36] and the references therein), the nonlinear constitutive stress-strain of solid (cf. [2, 11]) and so on. In practical applications, many problems, such as cables, beams, shells, polymers and metal foams, require a nonlinear stress-strain relation, one can refer to [13, 24, 34, 35] and so on. For linear poroelasticity, Showalter provides the analysis of well-posedness of weak solution to a linear poroelasticity model in [32]. Phillips and Wheeler propose and analyze a continuous-in-time and a discrete-in-time mixed finite element method in [29, 30] which simultaneously approximates the pressure and its gradient along with the displacement vector field, and the authors pointed that there exists “locking phenomenon” by using continuous Galerkin finite method. Feng, Ge and Li in [17, 18] propose a multiphysics finite method for approximating linear poroelasticity model by reformulating the original model, the multiphysics finite element method is a ef-

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School of Mathematics and Statistics, Henan University, Kaifeng 475004, P.R. China
jective approach to study the poroelasticity model and it overcomes the “locking phenomenon”. Based on the idea of [18], Ge and He in [22] prove the growth, coercivity and monotonicity of \( N(\nabla u) \) based on a multiphysics approach without any assumption on the nonlinear stress-strain relation for a nonlinear poroelasticity model with the constitutive relation \( \tilde{\sigma}(u) = \mu \tilde{\varepsilon}(u) + \lambda \text{tr}(\tilde{\varepsilon}(u))I \), where \( \tilde{\varepsilon}(u) = \frac{1}{2}(\nabla u + \nabla^T u + 2\nabla^T u \nabla u) \) is the deformed Green strain tensor. In this paper, we propose a fully discrete multiphysics finite element method for the nonlinear poroelasticity model(cf. [22]) by using the \( P_2 - P_1 - P_1 \) element pairs for space variables and backward Euler method for time variable. Without any assumption on the nonlinear stress-strain relation, we derive the discrete energy estimates and apply Schauder’s fixed point theorem to prove the existence and uniqueness of the numerical solution of the proposed numerical method. And we prove that the time-stepping method has the optimal convergence order. In the numerical tests, we show some numerical examples to verify the theoretical results and there is no “locking phenomenon”. To the best of our knowledge, it is the first time to propose a fully discrete multiphysics finite element method and derive the optimal order error estimate for the nonlinear poroelasticity model.

The remainder of this paper is organized as follows. In Section 2, we introduce the basic results of PDE model. In Section 3, we propose and analyze the coupled and decoupled time stepping methods based on the multiphysics approach. In Section 4, we prove that the time-stepping has the optimal convergence order. In Section 5, we provide some numerical experiments to verify the theoretical results of the proposed approach and methods. Finally, we draw a conclusion to summary the main results of this paper.

2. Basic results of PDE model. In this paper, we consider the following quasi-static poroelasticity model (for the linear and nonlinear poroelasticity model, one can refer to [17,18,29] and [22], respectively):

\[
\begin{align*}
- \text{div} \tilde{\sigma}(u) + \alpha \nabla p &= f \quad \text{in } \Omega_T := \Omega \times (0,T) \subset \mathbb{R}^d \times (0,T), \\
(c_0 p + \alpha \text{div } u)_t + \text{div } v_f &= \phi \quad \text{in } \Omega_T,
\end{align*}
\]

where

\[
\begin{align*}
\tilde{\sigma}(u) &= \mu \tilde{\varepsilon}(u) + \lambda \text{tr}(\tilde{\varepsilon}(u))I, \\
\tilde{\varepsilon}(u) &= \frac{1}{2}(\nabla u + \nabla^T u + 2\nabla^T u \nabla u), \\
v_f &= -\frac{K}{\mu_f}(\nabla p - \rho_f g).
\end{align*}
\]

Here \( \tilde{\varepsilon}(u) \) is known as the deformed Green strain tensor, \( u \) denotes the displacement vector of the solid and \( p \) denotes the pressure of the solvent. \( I \) denotes the
$d \times d$ identity matrix. $f$ is the body force. The permeability tensor $K = K(x)$ is assumed to be symmetric and uniformly positive definite in the sense that there exists positive constants $K_1$ and $K_2$ such that $K_1|\zeta|^2 \leq K(x)\zeta \cdot \zeta \leq K_2|\zeta|^2$ for a.e. $x \in \Omega$ and any $\zeta \in \mathbb{R}^d$; the solvent viscosity $\mu_f$, Biot-Willis constant $\alpha$, and the constrained specific storage coefficient $c_0$. In addition, $\tilde{\sigma}(u)$ is called the (effective) stress tensor. $v_f$ is the volumetric solvent flux and (2.4) is called the well-known Darcy’s law. $\lambda$ and $\mu$ are Lamé constants, $\hat{\sigma}(u,p) := \tilde{\sigma}(u) - \alpha p I$ is the total stress tensor. We assume that $\rho_f \neq 0$, which is a realistic assumption.

To close the above system, the following set of boundary and initial conditions will be considered in this paper:

\begin{align}
\hat{\sigma}(u,p)n &= \tilde{\sigma}(u)n - \alpha pn = f_1 \quad \text{on } \partial \Omega_T := \partial \Omega \times (0,T), \\
v_f \cdot n &= -\frac{K}{\mu_f}(\nabla p - \rho_f g) \cdot n = \phi_1 \quad \text{on } \partial \Omega_T, \\
\mu = \mu_0, \quad p = p_0 \quad \text{in } \Omega \times \{t = 0\}.
\end{align}

Introduce new variables

$$q := \text{div} u, \quad \eta := c_0 p + \alpha q, \quad \xi := \alpha p - \lambda q.$$ 

Denote

$$N(\nabla u) = \tilde{\sigma}(u) - \lambda \text{div} u I,$$

then we have

$$N(\nabla u) = \mu \varepsilon(u) + \mu \nabla^T u \nabla u + \lambda \|\nabla u\|^2_F I.$$ 

Due to the fact of $(\nabla^T u \nabla u, \text{rot} \, v) = 0$, $(\|\nabla u\|^2_F I, \text{rot} \, v) = 0$, so we have

$$(N(\nabla u), \nabla v) = (N(\nabla u), \varepsilon(v)),$$

where $\varepsilon(u) = \frac{1}{2}(\nabla^T u + \nabla u)$.

In some engineering literature, the Lamé constant $\mu$ is also called the shear modulus and denoted by $G$, and $B := \lambda + \frac{2}{3} G$ is called the bulk modulus. $\lambda$, $\mu$ and $B$ are computed from the Young’s modulus $E$ and the Poisson ratio $\nu$ by the following formulas:

$$\lambda = \frac{E \nu}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)}, \quad B = \frac{E}{3(1-2\nu)}.$$ 

It is easy to check that

$$p = \kappa_1 \xi + \kappa_2 \eta, \quad q = \kappa_1 \eta - \kappa_3 \xi,$$
where \( \kappa_1 = \frac{\alpha}{\alpha^2 + \lambda c_0}, \kappa_2 = \frac{\lambda}{\alpha^2 + \lambda c_0}, \kappa_3 = \frac{c_0}{\alpha^2 + \lambda c_0}. \)

Then the problem (2.1)-(2.4) can be rewritten as

\[
\begin{align*}
-\text{div} \mathbf{N} (\nabla \mathbf{u}) + \nabla \mathbf{\xi} &= \mathbf{f} \quad \text{in } \Omega_T, \\
\kappa_3 \mathbf{\xi} + \text{div} \mathbf{u} &= \kappa_1 \eta \quad \text{in } \Omega_T, \\
\eta_t - \frac{1}{\mu_f} \text{div} \left[ \mathbf{K} (\nabla (\kappa_1 \xi + \kappa_2 \eta) - \rho_f \mathbf{g}) \right] &= \phi \quad \text{in } \Omega_T.
\end{align*}
\]

The boundary and initial conditions (2.5)-(2.7) can be rewritten as

\[
\begin{align*}
\tilde{\sigma} (\mathbf{u}) \mathbf{n} - \alpha (\kappa_1 \xi + \kappa_2 \eta) \mathbf{n} &= \mathbf{f}_1 \quad \text{on } \partial \Omega_T := \partial \Omega \times (0, T), \\
- \frac{K}{\mu_f} (\nabla (\kappa_1 \xi + \kappa_2 \eta) - \rho_f \mathbf{g}) \cdot \mathbf{n} &= \phi_1 \quad \text{on } \partial \Omega_T, \\
\mathbf{u} &= \mathbf{u}_0, \quad p = p_0 \quad \text{in } \Omega \times \{t = 0\}.
\end{align*}
\]

In this paper, \( \Omega \subset \mathbb{R}^d \) \((d = 1, 2, 3)\) denotes a bounded polygonal domain with the boundary \( \partial \Omega \). The standard function space notation is adopted in this paper, their precise definitions can be found in [7, 12, 33]. In particular, \( (\cdot, \cdot) \) and \( \langle \cdot, \cdot \rangle \) denote respectively the standard \( L^2 (\Omega) \) and \( L^2 (\partial \Omega) \) inner products. For any Banach space \( B \), we let \( B = [B]^d \), and denote its dual space by \( B' \). In particular, we use \( (\cdot, \cdot)_{\text{dual}} \) to denote the dual product on \( H^1 (\Omega)' \times H^1 (\Omega) \), and \( \| \cdot \|_{L^p (B)} \) is a shorthand notation for \( \| \cdot \|_{L^p (0, T); B} \).

We also introduce the function spaces

\[
L^2_0 (\Omega) := \{ q \in L^2 (\Omega); (q, 1) = 0 \}, \quad X := H^1 (\Omega).
\]

Let \( \mathbf{RM} := \{ \mathbf{r} := \mathbf{a} + \mathbf{b} \times \mathbf{x}; \mathbf{a}, \mathbf{b}, \mathbf{x} \in \mathbb{R}^d \} \) denote the space of infinitesimal rigid motions. It is well known (cf. [6, 23, 33]) that \( \mathbf{RM} \) is the kernel of the strain operator \( \varepsilon \), that is, \( \mathbf{r} \in \mathbf{RM} \) if and only if \( \varepsilon (\mathbf{r}) = 0 \). Hence, we have

\[
\varepsilon (\mathbf{r}) = 0, \quad \text{div} \mathbf{r} = 0 \quad \forall \mathbf{r} \in \mathbf{RM}.
\]

Let \( L^2_0 (\partial \Omega) \) and \( H^1_0 (\Omega) \) denote respectively the subspaces of \( L^2 (\partial \Omega) \) and \( H^1 (\Omega) \) which are orthogonal to \( \mathbf{RM} \), that is,

\[
\begin{align*}
H^1_0 (\Omega) := \{ \mathbf{v} \in H^1 (\Omega); (\mathbf{v}, \mathbf{r}) = 0 \ \forall \mathbf{r} \in \mathbf{RM} \}, \\
L^2_0 (\partial \Omega) := \{ \mathbf{g} \in L^2 (\partial \Omega); (\mathbf{g}, \mathbf{r}) = 0 \ \forall \mathbf{r} \in \mathbf{RM} \}.
\end{align*}
\]

Next, we introduce the definition of the weak solution to the problem (2.1)-(2.7) and (2.11)-(2.13) with (2.14)-(2.16) as follows.
Definition 2.1. Let \( \mathbf{u}_0 \in \mathbf{H}^1(\Omega) \), \( \mathbf{f} \in \mathbf{L}^2(\Omega) \), \( \mathbf{f}_1 \in \mathbf{L}^2(\partial \Omega) \), \( p_0 \in \mathbf{L}^2(\Omega) \), \( \phi \in \mathbf{L}^2(\Omega) \), and \( \phi_\Omega \in \mathbf{L}^2(\partial \Omega) \). Assume \( c_0 > 0 \) and \( \langle \mathbf{f}, \mathbf{v} \rangle + \langle \mathbf{f}_1, \mathbf{v} \rangle = 0 \) for any \( \mathbf{v} \in \mathbf{RM} \). Given \( T > 0 \), a tuple \((\mathbf{u}, p)\) with
\[
\mathbf{u} \in L^\infty(0, T; \mathbf{H}_\perp^1(\Omega)), \quad p \in L^\infty(0, T; \mathbf{L}^2(\Omega)) \cap L^2(0, T; \mathbf{H}^1(\Omega)),
\]
\[
p_t, (\text{div} \mathbf{u})_t \in L^2(0, T; \mathbf{H}^1(\Omega)'),
\]
is called a weak solution to the problem (2.1)-(2.7), if there hold for almost every \( t \in [0, T] \)
\[
(\mathbf{N}(\nabla \mathbf{u}), \varepsilon(\mathbf{v})) + \lambda (\text{div} \mathbf{u}, \text{div} \mathbf{v}) - \alpha (p, \text{div} \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle + \langle \mathbf{f}_1, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),
\]
\[
\langle (c_0 + \text{div} \mathbf{u})_t, \varphi \rangle_{\text{dual}} + \frac{1}{\mu_f} (K (\nabla p - \rho_f \mathbf{g}), \nabla \varphi) = \langle \phi, \varphi \rangle + \langle \phi_1, \varphi \rangle \quad \forall \varphi \in \mathbf{H}^1(\Omega),
\]
\[
\mathbf{u}(0) = \mathbf{u}_0, \quad p(0) = p_0.
\]

Definition 2.2. Let \( \mathbf{u}_0 \in \mathbf{H}^1(\Omega) \), \( \mathbf{f} \in \mathbf{L}^2(\Omega) \), \( \mathbf{f}_1 \in \mathbf{L}^2(\partial \Omega) \), \( p_0 \in \mathbf{L}^2(\Omega) \), \( \phi \in \mathbf{L}^2(\Omega) \), and \( \phi_\Omega \in \mathbf{L}^2(\partial \Omega) \). Assume \( c_0 > 0 \) and \( \langle \mathbf{f}, \mathbf{v} \rangle + \langle \mathbf{f}_1, \mathbf{v} \rangle = 0 \) for any \( \mathbf{v} \in \mathbf{RM} \). Given \( T > 0 \), a 5-tuple \((\mathbf{u}, \xi, \eta, p, q)\) with
\[
\mathbf{u} \in L^\infty(0, T; \mathbf{H}_\perp^1(\Omega)), \quad \xi \in L^\infty(0, T; \mathbf{L}^2(\Omega)),
\]
\[
\eta \in L^\infty(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{H}^1(0, T; \mathbf{H}^1(\Omega)'), \quad q \in L^\infty(0, T; \mathbf{L}^2(\Omega)),
\]
\[
p \in L^\infty(0, T; \mathbf{L}^2(\Omega)) \cap L^2(0, T; \mathbf{H}^1(\Omega))
\]
is called a weak solution to the problem (2.11)-(2.13), if there hold for almost every \( t \in [0, T] \)
\[
(\mathbf{N}(\nabla \mathbf{u}), \varepsilon(\mathbf{v})) - \langle \xi, \text{div} \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle + \langle \mathbf{f}_1, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),
\]
\[
\kappa_3 \langle \xi, \varphi \rangle + (\text{div} \mathbf{u}, \varphi) = \kappa_1 \langle \eta, \varphi \rangle \quad \forall \varphi \in \mathbf{L}^2(\Omega),
\]
\[
(\eta_t, \psi)_{\text{dual}} + \frac{1}{\mu_f} (K (\nabla (\kappa_1 \xi + \kappa_2 \eta) - \rho_f \mathbf{g}), \nabla \psi)
\]
\[
= \langle \phi, \psi \rangle + \langle \phi_1, \psi \rangle \quad \forall \psi \in \mathbf{H}^1(\Omega),
\]
\[
p := \kappa_1 \xi + \kappa_2 \eta, \quad q := \kappa_1 \eta - \kappa_3 \xi,
\]
\[
\eta(0) = \eta_0 := c_0 p_0 + \alpha q_0.
\]

Lemma 2.3. There exist positive constants \( C_1, C_2, C_3 \) such that
\[
\|\mathbf{N}(\nabla \mathbf{u})\|_{\mathbf{L}^2(\Omega)} \leq C_1 \|\varepsilon(\mathbf{u})\|_{\mathbf{L}^2(\Omega)},
\]
(2.27) \( (N(\nabla(u)), \varepsilon(u)) \geq C_2 \|\varepsilon(u)\|^2_{L^2(\Omega)} \).

(2.28) \( (N(\nabla(u)) - N(\nabla(v)), \varepsilon(u) - \varepsilon(v)) \geq C_4 \|\varepsilon(u) - \varepsilon(v)\|^2_{L^2(\Omega)} \).

**Lemma 2.4.** There exists positive real number \( C_3 \) such that the following holds:

(2.29) \( \|N(\nabla u) - N(\nabla v)\|_{L^2(\Omega)} \leq C_3 \|\varepsilon(u) - \varepsilon(v)\|_{L^2(\Omega)}. \)

As for the detailed proofs of Lemma 2.3 and Lemma 2.4, one can refer to [22]. About the energy estimates of the weak solution and the well-posedness of the weak solution, one can also refer to [22], here we only list up the main results (see Lemma 2.5, Lemma 2.6 and Theorem 2.7) as follows:

**Lemma 2.5.** There exists a positive constant \( \hat{C}_1 = \hat{C}_1(\|u_0\|_{H^1(\Omega)}, \|p_0\|_{L^2(\Omega)}, \|f\|_{L^2(\Omega)}, \|f_1\|_{L^2(\partial\Omega)}, \|\phi\|_{L^2(\Omega)}, \|\phi_1\|_{L^2(\partial\Omega)}) \) such that

(2.30) \( \sqrt{C_2}\|\varepsilon(u)\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{K_2}\|\eta\|_{L^\infty(0,T;L^2(\Omega))} \leq \hat{C}_1, \)

(2.31) \( \|u\|_{L^\infty(0,T;L^2(\Omega))} \leq \hat{C}_1, \quad \|p\|_{L^\infty(0,T;L^2(\Omega))} \leq \hat{C}_1(\kappa_2^2 + \kappa_1 \kappa_3^{-2}), \)

(2.32) \( \|p\|_{L^2(0,T;L^2(\Omega))} \leq \hat{C}_1, \quad \|\xi\|_{L^2(0,T;L^2(\Omega))} \leq \hat{C}_1 \kappa_1^{-1}(1 + \kappa_2^2). \)

**Lemma 2.6.** Suppose that \( u_0 \) and \( p_0 \) are sufficiently smooth, then there exist positive constants \( \tilde{C}_2 = \tilde{C}_2(\hat{C}_1, \|\nabla p_0\|_{L^2(\Omega)}) \) and \( \tilde{C}_3 = \tilde{C}_3(\hat{C}_1, \hat{C}_2, \|u_0\|_{H^2(\Omega)}, \|p_0\|_{H^2(\Omega)}) \) such that

(2.33) \( \sqrt{\bar{C}_2}\|\varepsilon(u_t)\|_{L^2(0,T;L^2(\Omega))} + \sqrt{\bar{K}_2}\|\eta_t\|_{L^2(0,T;L^2(\Omega))} \leq \hat{C}_2, \)

(2.34) \( \sqrt{\bar{C}_2}\|\varepsilon(u_t)\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\bar{K}_2}\|\eta_t\|_{L^\infty(0,T;L^2(\Omega))} \leq \hat{C}_2, \)

(2.35) \( \|\eta_t\|_{L^2(0,T;L^2(\Omega))} \leq \frac{K_2}{\mu_f} \hat{C}_3. \)
Theorem 2.7. Let \( u_0 \in H^1(\Omega), f \in L^2(\Omega), f_1 \in L^2(\partial\Omega), p_0 \in L^2(\Omega), \phi \in L^2(\Omega) \) and \( \phi_1 \in L^2(\partial\Omega) \). Suppose \( c_0 > 0 \) and \( (f,v) + \langle f_1,v \rangle = 0 \) for any \( v \in RM \). Then there exists a unique solution to the problem (2.1)-(2.7) in the sense of Definition 2.1. Likewise, there exists a unique solution to the problem (2.11)-(2.13) with (2.14)-(2.16) in the sense of Definition 2.2.

3. Fully discrete multiphysics finite element method.

3.1. Formulation of fully discrete finite element method. Let \( T_h \) be a quasi-uniform triangulation or rectangular partition of \( \Omega \) with maximum mesh size \( h \), and \( \bar{\Omega} = \bigcup_{K \in T_h} \bar{K} \). The time interval \([0,T]\) is divided into \( N \) equal intervals, denoted by \([t_{n-1},t_n]\), \( n = 1,2,...,N \), and \( \Delta t = \frac{T}{N} \), then \( t_n = n\Delta t \).

In this work, we use backward Euler method and denote \( d_tv^n := v^n - v^{n-1} \Delta t \).

Also, let \((X_h,M_h)\) be a stable mixed finite element pair, that is, \( X_h \subset H^1(\Omega) \) and \( M_h \subset L^2(\Omega) \) satisfy the inf-sup condition:

\[
\sup_{v_h \in X_h} \frac{(\text{div}v_h,\varphi_h)}{\|v_h\|_{H^1(\Omega)}} \geq \beta_0 \|\varphi_h\|_{L^2(\Omega)} \quad \forall \varphi_h \in M_{0h} := M_h \cap L^2_0(\Omega), \quad \beta_0 > 0.
\]

(3.1)

A number of stable mixed finite element spaces \((X_h,M_h)\) have been known in the literature [8]. A well-known example is the following so-called Taylor-Hood element (cf. [1,8]):

\[
X_h = \{ v_h \in C^0(\Omega); v_h|_K \in P_2(K) \quad \forall K \in T_h \},
M_h = \{ \varphi_h \in C^0(\Omega); \varphi_h|_K \in P_1(K) \quad \forall K \in T_h \}.
\]

Finite element approximation space \( W_h \) for \( \eta \) variable can be chosen independently, any piecewise polynomial space is acceptable provided that \( W_h \supset M_h \), the most convenient choice is \( W_h = M_h \).

Define

\[
V_h := \{ v_h \in X_h; (v_h,r) = 0 \quad \forall r \in RM \},
\]

(3.2)

it is easy to check that \( X_h = V_h \oplus RM \). It was proved in [19] that there holds the following inf-sup condition:

\[
\sup_{v_h \in V_h} \frac{(\text{div}v_h,\varphi_h)}{\|v_h\|_{H^1(\Omega)}} \geq \beta_1 \|\varphi_h\|_{L^2(\Omega)} \quad \forall \varphi_h \in M_{0h}, \quad \beta_1 > 0.
\]

(3.3)

Also, we recall the following inverse inequality for polynomial functions [7,8,12]:

\[
\|\nabla \varphi_h\|_{L^2(K)} \leq c_1 h^{-1} \|\varphi_h\|_{L^2(K)} \quad \forall \varphi_h \in P_r(K), K \in T_h.
\]

(3.4)
Now, we give the fully discrete multiphysics finite element algorithm for the problem \( (2.11) - (2.13) \).

**Multiphysics Finite Element Algorithm (MFEA)**

(i) Compute \( u_h^0 \in V_h \) and \( q_h^0 \in W_h \) by \( u_h^0 = u_0, p_h^0 = p_0 \).

(ii) For \( n = 0, 1, 2, \cdots \), do the following two steps.

**Step 1:** Solve for \( (u_h^{n+1}, \xi_h^{n+1}, \eta_h^{n+1}) \in V_h \times M_h \times W_h \) such that

\[
(3.5) (\mathcal{N}(\nabla u_h^{n+1}), \varepsilon(v_h)) - (\xi_h^{n+1}, \text{div} v_h) = (f, v_h) + (f_1, v_h) \forall v_h \in V_h,
\]

\[
(3.6) \quad \kappa_3 (\xi_h^{n+1}, \varphi_h) + (\text{div} u_h^{n+1}, \varphi_h) = \kappa_1 (\eta_h^{n+1} + \varphi_h) \quad \forall \varphi_h \in M_h,
\]

\[
(3.7) \quad (d_n \eta_h^{n+1}, \psi_h) + \frac{1}{\mu_f} (K(\nabla (\kappa_1 \xi_h^{n+1} + \kappa_2 \eta_h^{n+1})) - \rho_f g, \nabla \psi_h) = (\phi, \psi_h) + (\phi_1, \psi_h) \quad \forall \psi_h \in W_h,
\]

where \( \theta = 0 \) or 1.

**Step 2:** Update \( p_h^{n+1} \) and \( q_h^{n+1} \) by

\[
(3.8) \quad p_h^{n+1} = \kappa_1 p_h^{n+1} + \kappa_2 \eta_h^{n+1}, \quad q_h^{n+1} = \kappa_1 \eta_h^{n+1} - \kappa_3 \xi_h^{n+1}.
\]

**Remark 3.1.** The Newton’s iterative method to solve the nonlinear Stokes problem \( (3.5) - (3.6) \) when \( \theta = 0 \) is

\[
(3.9) \quad - (\lambda \nabla u_h^n : \nabla u_h^n I, \varepsilon(v_h)) + (f, v_h) + (f_1, v_h) + \kappa_1 (\eta_h^{n+1} + \varphi_h)
\]

\[
\quad = \mu (\varepsilon(u_h^{n+1}), \varepsilon(v_h)) + \mu (\nabla^T u_h^{n+1} \nabla u_h^n, \varepsilon(v_h)) + \mu (\nabla^T u_h^{n+1} \nabla u_h^{n+1}, \varepsilon(v_h))
\]

\[
\quad - \mu (\nabla^T u_h^n \nabla u_h^n, \varepsilon(v_h)) + (2 \lambda \nabla u_h^n : (\nabla u_h^{n+1} - \nabla u_h^n) I, \varepsilon(v_h))
\]

\[
\quad - (\xi_h^{n+1}, \text{div} v_h) + \kappa_3 (\xi_h^{n+1} + \varphi_h) + (\text{div} u_h^{n+1}, \varphi_h).
\]

**Remark 3.2.** As for the multiphysics finite element algorithm, the original pressure is eliminated in the reformulation, which will be helpful to overcome the “locking phenomenon”; the later numerical tests show that our proposed method has no “locking phenomenon”, one can see Section 5.

**3.2. Stability analysis.** The primary goal of this subsection is to derive a discrete energy law which mimics the PDE energy law \( [22] \). Before discussing the stability of (MFEA), we first show that the numerical solution satisfies the following constraints which are fulfilled by the PDE solution.

**Lemma 3.1.** Let \( \{(u_h^n, \xi_h^n, \eta_h^n)\}_{n \geq 0} \) be defined by the (MFEA), then there hold

\[
(3.10) \quad (\eta_h^n, 1) = C_\eta(t_n) \quad \text{for } n = 0, 1, 2, \cdots,
\]
\[ (\xi_h^{n+1}, 1) = C_\xi(t_{n-1+\theta}) \quad \text{for } n = 1 - \theta, 1, 2, \ldots, \]
\[ (u_h^{n+1} \cdot n, 1) = C_u(t_{n-1+\theta}) \quad \text{for } n = 1 - \theta, 1, 2, \ldots. \]

**Proof.** Taking \( \psi_h = 1 \) in (3.7), we have
\[
(\eta_h^{n+1}, 1) = (\phi, 1) + (\phi_1, 1).
\]
Summing (3.13) over \( n \) from 0 to \( \ell \) (\( \geq 0 \)), we get
\[
(\eta_h^{\ell+1}, 1) = (\eta_h^0, 1) + [(\phi, 1) + (\phi_1, 1)] t_{\ell+1} = (\eta_0, 1) + [(\phi, 1) + (\phi_1, 1)] t_{\ell+1} = C_\eta(t_{\ell+1})
\]
for \( \ell = 0, 1, 2, \ldots \), which implies that (3.10) holds.

Taking \( v_h = x \) in (3.5) and \( \varphi_h = 1 \) in (3.6), we get
\[
(N(\nabla u_h^{n+1}), 1) - d(\xi_h^{n+1}, 1) = (f, x) + (f_1, x),
\]
\[
\kappa_3 (\xi_h^{n+1}, 1) + (\text{div } u_h^{n+1}, 1) = \kappa_4 C_\eta(t_{n+\theta}).
\]
Substituting (3.16) into (3.15) we have
\[
(\xi_h^{n+1}, 1) = \frac{1}{d - \kappa_3} \left[ (N(\nabla u_h^{n+1}), 1) + (\text{div } u_h^{n+1}, 1) - \kappa_4 C_\eta(t_{n+\theta}) - (f, x) - (f_1, x) \right].
\]
Hence, by the definition of \( C_\xi(t) \) in [22], we conclude that (3.11) holds for all \( n \geq 1 - \theta \).

Using (3.10), (3.11), (3.16) and Gauss divergence theorem, we deduce that (3.12) holds. The proof is complete. \( \square \)

**Lemma 3.2.** Let \( \{u_h^n, \xi_h^n, \eta_h^n\}_{n \geq 0} \) be defined by the (MFEA), then there holds the following inequality:
\[
J_{h,\theta}^l + S_{h,\theta}^l \leq J_{h,\theta}^0 \quad \text{for } l \geq 1, \theta = 0, 1,
\]
where
\[
J_{h,\theta}^l := \frac{1}{2} \left[ C_2 \|\varepsilon(u_h^{l+1})\|^2_{L^2(\Omega)} + \kappa_2 \|\eta_h^{l+\theta}\|^2_{L^2(\Omega)} + \kappa_3 \|\xi_h^{l+1}\|^2_{L^2(\Omega)} \right. \\
- 2(f, u_h^{l+1}) - 2(\phi_f, u_h^{l+1})] ,
\]
\[
S_{h,\theta}^l := \Delta t \left[ \frac{\Delta t}{2} C_4 \|d\varepsilon(u_h^{n+1})\|^2_{L^2(\Omega)} + \frac{1}{\mu_f} (K \nabla p_h^{n+1} - K p_{f} g, \nabla p_h^{n+1}) \right. \\
+ \kappa_2 \Delta t \|d\eta_h^{n+\theta}\|^2_{L^2(\Omega)} + \frac{\kappa_3}{2} \|d\xi_h^{n+1}\|^2_{L^2(\Omega)} - (\phi, p_h^{n+1}) - (\phi_f, p_h^{n+1}) \right] \\
\]
can rewrite (3.21) as

\[-(1 - \theta)\frac{\kappa_1 \Delta t}{\mu_f} (K_d t \nabla \xi_h^{n+1}, \nabla p_h^{n+1}) - \frac{C_1}{\Delta t} \| \varepsilon(u_h^n) \|_{L^2(\Omega)} \| \varepsilon(u_h^{n+1}) \|_{L^2(\Omega)} \].

**Proof.** (i) when $\theta = 0$, based on (3.6), we can define $\eta_h^{-1}$ by

\[(3.18) \quad \kappa_1(\eta_h^{-1}, \varphi_h) = \kappa_3(\xi_h^n, \varphi_h) + (\nabla \cdot u_h^n, \varphi_h).\]

Setting $v_h = d_t u_h^{n+1}$ in (3.5), we have

\[(3.19) \quad (\nabla(u_h^{n+1}), \varepsilon(d_t u_h^{n+1})) - (\xi_h^{n+1}, \nabla \cdot d_t u_h^{n+1}) = (f, d_t u_h^{n+1}) + \langle f, d_t u_h^{n+1} \rangle.\]

Using (3.19), (2.28), the Cauchy-Schwarz inequality, (2.26) and (2.27), we have

\[(3.20) \quad \frac{1}{2 \Delta t} \left[ C_1 \Delta t^2 \| d_t \varepsilon(u_h^{n+1}) \|^2_{L^2(\Omega)} - 2C_1 \| \varepsilon(u_h^n) \|_{L^2(\Omega)} \| \varepsilon(u_h^{n+1}) \|_{L^2(\Omega)} + \Delta t C_2 d_t \| \varepsilon(u_h^{n+1}) \|^2_{L^2(\Omega)} - (\xi_h^{n+1}, \nabla \cdot d_t u_h^{n+1}) \right]
\leq (f, d_t u_h^{n+1}) + \langle f, d_t u_h^{n+1} \rangle.\]

Setting $\varphi_h = \xi_h^n$ in (3.6), we get

\[(3.21) \quad \kappa_3(d_t \xi_h^n, \xi_h^{n+1}) + (\nabla \cdot d_t u_h^{n+1}, \xi_h^{n+1}) = \kappa_1(d_t 
eta_h^n, \xi_h^{n+1}).\]

Setting $\psi_h = \rho_h^n$ in (3.7) after lowering the super-index from $n + 1$ to $n$ on both sides of (3.8), we get

\[(3.22) \quad (d_t \eta_h^n, \rho_h^{n+1}) + \frac{1}{\mu_f} (K(\nabla(\kappa_1 \xi_h^n + \kappa_2 \eta_h^n) - \rho_f g), \nabla \rho_h^{n+1})
\leq (\phi, \rho_h^{n+1}) + \langle \phi_1, \rho_h^{n+1} \rangle.\]

Using the fact of $(d_t \xi_h^n, \xi_h^{n+1}) = \frac{\Delta t}{2} \| d_t \xi_h^{n+1} \|^2_{L^2(\Omega)} + \frac{1}{2} d_t \| \xi_h^{n+1} \|^2_{L^2(\Omega)}$, we can rewrite (3.21) as

\[(3.23) \quad \frac{\kappa_3 \Delta t}{2} \| d_t \xi_h^{n+1} \|^2_{L^2(\Omega)} + \frac{\kappa_3}{2} d_t \| \xi_h^{n+1} \|^2_{L^2(\Omega)} + (\nabla \cdot d_t u_h^{n+1}, \xi_h^{n+1})
\leq \kappa_1(d_t \eta_h^n, \xi_h^{n+1}).\]

Similarly, we get

\[(3.24) \quad \frac{\kappa_2 \Delta t}{2} \| d_t \eta_h^n \|^2_{L^2(\Omega)} + \frac{\kappa_2}{2} d_t \| \eta_h^n \|^2_{L^2(\Omega)} + \kappa_1(d_t \eta_h^n, \xi_h^{n+1})
+ \frac{1}{\mu_f} (K(\nabla \rho_h^{n+1} - \rho_f g), \nabla \rho_h^{n+1}) - \frac{\kappa_4 \Delta t}{\mu_f}(K d_t \nabla \xi_h^{n+1}, \nabla \rho_h^{n+1})
\leq (\phi, \rho_h^{n+1}) + \langle \phi_1, \rho_h^{n+1} \rangle.\]
Combining (3.20), (3.23) and (3.24), we see that (3.17) holds for \( \theta = 0 \) if \( \Delta t = O(h^2) \).

(ii) When \( \theta = 1 \), setting \( \varphi_h = \xi_h^{n+1} \) in (3.6) and \( \psi_h = p_h^{n+1} \) in (3.7), we get

\[
\frac{\kappa_3 \Delta t}{2} \left\| d_t \xi_h^{n+1} \right\|_{L^2(\Omega)}^2 + \frac{\kappa_3}{2} d_t \left\| \xi_h^{n+1} \right\|_{L^2(\Omega)}^2 + (\nabla \cdot d_t u_h^{n+1}, \xi_h^{n+1})
\]

\[
= \kappa_1 (d_t \eta_h^{n+1}, \xi_h^{n+1}),
\]

(3.25)

\[
\frac{\kappa_2 \Delta t}{2} \left\| d_t \eta_h^{n+1} \right\|_{L^2(\Omega)}^2 + \frac{\kappa_2}{2} d_t \left\| \eta_h^{n+1} \right\|_{L^2(\Omega)}^2 + \kappa_1 (d_t \eta_h^{n+1}, \xi_h^{n+1})
\]

\[
+ \frac{1}{\mu_f} (K(\nabla p_h^{n+1} - \rho_f g), \nabla p_h^{n+1}) = (\phi, p_h^{n+1}) + \left\langle \phi_1, p_h^{n+1} \right\rangle.
\]

Combining (3.20), (3.25) and (3.26), we imply that (3.17) holds. The proof is complete.

**Lemma 3.3.** Let \( \{ (u_h^n, \xi_h^n, \eta_h^n) \}_{n \geq 0} \) be defined by the (MFEA) with \( \theta = 0 \), then there holds the following inequality:

\[
J_{h, \theta}^l + \tilde{S}_{h, \theta}^l \leq J_{h, \theta}^0 \quad \text{for } l \geq 1
\]

provided that \( \Delta t = O(h^2) \). Here

\[
\tilde{S}_{h, \theta}^l := \Delta t \sum_{n=1}^{l} \left[ \frac{\Delta t}{4} C_4 \left\| d_t \varepsilon(u_h^{n+1}) \right\|_{L^2(\Omega)}^2 + \frac{K}{2 \mu_f} \left\| \nabla p_h^{n+1} \right\|_{L^2(\Omega)}^2 - \frac{K}{\mu_f} (\rho_f g, \nabla p_h^{n+1}) + \frac{\kappa_2 \Delta t}{2} \left\| d_t \eta_h^{n+1} \right\|_{L^2(\Omega)}^2 + \frac{\kappa_2 \Delta t}{2} \left\| d_t \varepsilon(u_h^{n+1}) \right\|_{L^2(\Omega)}^2
\]

\[
- (\phi, p_h^{n+1}) - \left\langle \phi_1, p_h^{n+1} \right\rangle - C_1 \left[ \left\| d_t \varepsilon(u_h^{n+1}) \right\|_{L^2(\Omega)} \left\| \varepsilon(u_h^{n+1}) \right\|_{L^2(\Omega)} \right].
\]

**Proof.** Using the Cauchy-Schwarz inequality and inverse inequality (3.4), we get

\[
\frac{K \kappa_1 \Delta t}{\mu_f} (d_t \nabla \xi_h^{n+1}, \nabla p_h^{n+1})
\]

\[
\leq \frac{K \kappa_1^2}{2 \mu_f} \left\| \nabla \xi_h^{n+1} - \nabla \xi_h^n \right\|_{L^2(\Omega)}^2 + \frac{K}{2 \mu_f} \left\| \nabla p_h^{n+1} \right\|_{L^2(\Omega)}^2
\]

\[
\leq \frac{K \kappa_1^2 C_2}{2 \mu_f h^2} \left\| \xi_h^{n+1} - \xi_h^n \right\|_{L^2(\Omega)}^2 + \frac{K}{2 \mu_f} \left\| \nabla p_h^{n+1} \right\|_{L^2(\Omega)}^2.
\]

To bound the first term on the right-hand side of (3.28), we appeal to the inf-sup condition (3.3) and get

\[
\left\| \xi_h^{n+1} - \xi_h^n \right\|_{L^2(\Omega)} \leq \frac{1}{\beta_1} \sup_{v_h \in V_h} \frac{(\nabla \cdot v_h, \xi_h^{n+1} - \xi_h^n)}{\left\| \nabla v_h \right\|_{L^2(\Omega)}}
\]

(3.29)
Following the method of [18] or [17], one can prove that the problem (3.31) has a unique solution.

Substituting (3.29) into (3.28) and combining it with (3.17) imply that (3.27) holds if \( \Delta t \leq \frac{C_3 \Delta t}{2 C_3^2 \kappa_3^2 K c_1} \). The proof is complete. \( \square \)

**Theorem 3.4.** The numerical solution \( \{u_h^{n+1}, \xi_h^{n+1}, \eta_h^{n+1}\}_{n \geq 0} \) of the problem (3.5)–(3.7) exists uniquely.

**Proof.** Given a function \( u_h^n \in V_h \), supposing that \( U \subset V_h \) is a compact and convex subspace, setting \( g(t) := -\mu \nabla^T \nabla u_h^{n+1} \nabla u_h^{n+1} - \lambda \| \nabla u_h^{n+1} \|_{L^2(\Omega)} I(0 \leq t \leq T) \), we have

\[
(3.30) \quad \left\| \mu \nabla^T \nabla u_h^{n+1} \nabla u_h^{n+1} + \lambda \| \nabla u_h^{n+1} \|^2 \right\|_{L^2(\Omega)} \leq \mu \left\| \nabla u_h^{n+1} \right\|^2_{L^2(\Omega)} + \lambda \Delta t^2
\]

which implies that \( g \in L^2(0, T; L^2(\Omega)) \).

Next, we consider the linear problem: find \( (w_h^{n+1}, \xi_h^{n+1}, \eta_h^{n+1}) \in V_h \times M_h \times W_h \) satisfying

\[
(3.31) \quad \mu(\varepsilon(w_h^{n+1}), \varepsilon(v_h)) - (\xi_h^{n+1}, \text{div } v_h) = (g, v_h) \\
+ (f, v_h) + (f_1, v_h), \forall v_h \in V_h,
\]

\[
(3.32) \quad \kappa_3(\xi_h^{n+1}, \varphi_h) + (\text{div } u_h^{n+1}, \varphi_h) = \kappa_1(\eta_h^{n+1}, \varphi_h), \forall \varphi_h \in M_h,
\]

\[
(3.33) \quad (d_t \eta_h^{n+1}, \psi_h) + \frac{1}{\mu_f} (K(\kappa_1 \xi_h^{n+1} + \kappa_2 \eta_h^{n+1})) \\
- \rho_f g, \nabla \psi_h = (\phi, \psi_h) + (\phi_1, \psi_h), \forall \psi_h \in W_h.
\]

As for [2.12]–[2.13], according to the theory of linear parabolic equations (cf. [16]), we know that \( \xi \) and \( \eta \) can be uniquely determined by \( w \), that is, \( \exists \Phi, \Psi \) such that \( \xi_h^{n+1} = \Phi(w_h^{n+1}) \) and \( \eta_h^{n+1} = \Psi(w_h^{n+1}) \). Thus, the problem (3.31)–(3.33) is equivalent to the following problem

\[
\left\{ \begin{array}{l}
Solve \ for \ w_h^{n+1} \in V_h \ such \ that \\
\mu(\varepsilon(w_h^{n+1}), \varepsilon(v_h)) + (\Phi(w_h^{n+1}), \text{div } v_h) = (g, v_h) + (f, v_h) + (f_1, v_h), \forall v_h \in V_h.
\end{array} \right.
\]

Following the method of [18] or [17], one can prove that the problem (3.31) has a unique solution.

Define \( A : V_h \rightarrow \rightarrow V_h \) by \( A[u_h^{n+1}] = w_h^{n+1} \). Similarly, it’s easy to know...
the following problem is equivalent to the problem (3.5)-(3.7)

\[
\begin{cases}
\text{Solve for } u_{h}^{n+1} \in V_h \text{ such that } \\
(N'(\nabla u_{h}^{n+1}), \varepsilon(v_h)) + (\Phi(u_{h}^{n+1}), \text{div } v_h) = (f, v_h) + (f_1, v_h) \quad \forall v_h \in V_h.
\end{cases}
\]

Next, we prove that \( A \) is continuous. To do that, choose \( u_{h}^{n+1}, \tilde{u}_{h}^{n+1} \) and define \( \tilde{w}_{h}^{n+1} = A[u_{h}^{n+1}], \tilde{w}_{h}^{n+1} = A[\tilde{u}_{h}^{n+1}] \) as above. Consequently \( \tilde{w}_{h}^{n+1} \) verifies (3.31)-(3.33) and \( \tilde{w}_{h}^{n+1} \) satisfies a similar identity for \( \tilde{g} = -\mu \nabla \tilde{u}_{h}^{n+1} \nabla \tilde{u}_{h}^{n+1} - \lambda \|\nabla \tilde{u}_{h}^{n+1}\| I \). Using (3.31), Korn’s inequality, Poincaré inequality and Young inequality, we get

\[
\begin{align*}
(3.34) \quad & \mu \|\tilde{w}_{h}^{n+1} - w_{h}^{n+1}\|_{L^2(\Omega)} + c_1^2 (\Phi(\tilde{w}_{h}^{n+1}) - \Phi(w_{h}^{n+1}), \tilde{w}_{h}^{n+1} - w_{h}^{n+1}) \\
& \leq c_1^2 \|\varepsilon(\tilde{w}_{h}^{n+1}) - \varepsilon(w_{h}^{n+1})\|_{L^2(\Omega)} + c_1^2 (\Phi(\tilde{w}_{h}^{n+1}) - \Phi(w_{h}^{n+1}), \tilde{w}_{h}^{n+1} - w_{h}^{n+1}) \\
& = c_1^2 (\varepsilon g - g, \tilde{w}_{h}^{n+1} - w_{h}^{n+1}) \\
& \leq c_1^2 \left[ \epsilon \|\tilde{w}_{h}^{n+1} - w_{h}^{n+1}\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \|g\|_{L^2(\Omega)} \right],
\end{align*}
\]

where \( c_1 \) is a real positive parameter. Choosing \( \epsilon > 0 \) sufficiently small in (3.34), we have

\[
c_1 (\Phi(\tilde{w}_{h}^{n+1}) - \Phi(w_{h}^{n+1}), \tilde{w}_{h}^{n+1} - w_{h}^{n+1}) \leq C_f \|g - g\|_{L^2(\Omega)} \leq C_f \|\tilde{u}_{h}^{n+1} - u_{h}^{n+1}\|_{L^2(\Omega)},
\]

where \( C_f \) is a real positive number.

It is easy to check that

\[
(3.35) \quad \|A[\tilde{u}_{h}^{n+1}] - A[u_{h}^{n+1}]\|_{L^2(\Omega)}^2 \leq \tilde{C}_f \|\tilde{u}_{h}^{n+1} - u_{h}^{n+1}\|_{L^2(\Omega)}^2.
\]

Using (3.35), we get

\[
\|A[\tilde{u}_{h}^{n+1}] - A[u_{h}^{n+1}]\|_{L^2(\Omega)} \leq \sqrt{\tilde{C}_f} \|\tilde{u}_{h}^{n+1} - u_{h}^{n+1}\|_{L^2(\Omega)}.
\]

If \( \tilde{C}_f \) is so small, thus \( A \) is continuous. Since \( U \) is a compact and convex, according to Schauder’s fixed point theorem (cf. [16]), then \( A \) has a fixed point in \( U \).

Next, we prove that the problem (3.5)-(3.7) has the unique solution. Due to \( C_{\eta}(t_n) = (\eta_{h}^{n}, 1) = (\eta_0, 1) + [(\phi, 1) + (\phi_1, 1)] \), using Lemma 3.1 it is easy to check that \( \eta \) is unique.

Assume that \( (u_{h}^{n+1}, \xi_{h}^{n+1}, \eta_{h}^{n+1}) \) and \( (\tilde{u}_{h}^{n+1}, \tilde{\xi}_{h}^{n+1}, \tilde{\eta}_{h}^{n+1}) \) are the two different solution of the problem (3.5)-(3.7).

Using (3.6) and (3.7), we obtain

\[
(3.36) \quad (N(\nabla u_{h}^{n+1}) - N(\nabla \tilde{u}_{h}^{n+1}), \varepsilon(v_h)) - (\xi_{h}^{n+1} - \tilde{\xi}_{h}^{n+1}, \nabla \cdot v_h) = 0 \quad \forall v_h \in V_h,
\]
\( (3.37) \quad \kappa_3 (\zeta_{h}^{n+1} - \tilde{\zeta}_{h}^{n+1}, \varphi_h) + (\nabla \cdot u_h^{n+1} - \nabla \cdot \tilde{u}_h^{n+1}, \varphi_h) = 0 \quad \forall \varphi_h \in W_h. \)

Adding (3.33) and (3.37), letting \( v_h = u_h^{n+1} - \tilde{u}_h^{n+1}, \varphi_h = \zeta_{h}^{n+1} - \tilde{\zeta}_{h}^{n+1}, \) using (2.28), we have

\( (4.3) \quad 0 \leq C_4 \| \varepsilon(u_h^{n+1}) - \varepsilon(\tilde{u}_h^{n+1}) \|_{L^2(\Omega)}^2 + \kappa_3 \left\| \zeta_{h}^{n+1} - \tilde{\zeta}_{h}^{n+1} \right\|_{L^2(\Omega)}^2 = 0. \)

Using (3.38) and the initial value \( u_0, \) we obtain

\( u_h^{n+1} = \tilde{u}_h^{n+1}, \quad \zeta_{h}^{n+1} = \tilde{\zeta}_{h}^{n+1}. \)

Since \( p_h^{n+1} = \kappa_1 \zeta_{h}^{n+1} + \kappa_2 \eta_{h}^{n+1}, q_h^{n+1} = \kappa_1 \eta_{h}^{n+1} - \kappa_3 \zeta_{h}^{n+1}, \) so we have

\( p_h^{n+1} = \tilde{p}_h^{n+1}, \quad q_h^{n+1} = \tilde{q}_h^{n+1}. \)

Hence, the assumption is false, so the problem (3.5)-(3.7) has a unique weak solution. The proof is complete. \( \square \)

4. Error estimates. To derive the optimal order error estimates of the fully discrete multiphysics finite element method, for any \( \varphi \in L^2(\Omega), \) we firstly define \( L^2(\Omega) \)-projection operators \( Q_h : L^2(\Omega) \rightarrow X_h^k \) by

\( (4.1) \quad (Q_h \varphi, \psi_h) = (\varphi, \psi_h) \quad \forall \psi_h \in X_h^k, \)

where \( X_h^k := \{ \psi_h \in C^0; \psi_h|_E \in P_k(E) \forall E \in T_h \}. \)

Next, for any \( \varphi \in H^1(\Omega), \) we define its elliptic projection \( S_h : H^1(\Omega) \rightarrow X_h^k \) by

\( (4.2) \quad (K \nabla S_h \varphi, \nabla \varphi_h) = (K \nabla \varphi, \nabla \varphi_h) \quad \forall \varphi_h \in X_h^k, \)
\( (4.3) \quad (S_h \varphi, 1) = (\varphi, 1). \)

Finally, for any \( v \in H^1(\Omega), \) we define its elliptic projection \( R_h : H^1(\Omega) \rightarrow V_h^k \) by

\( (4.4) \quad (\varepsilon(R_h v), \varepsilon(w_h)) = (\varepsilon(v), \varepsilon(w_h)) \quad \forall w_h \in V_h^k, \)

where \( V_h^k := \{ v_h \in C^0; v_h|_K \in P_k(K), (v_h, r) = 0 \forall r \in RM \}, k \) is the degree of the piecewise polynomial on \( K. \) From [7], we know that \( Q_h, S_h \) and \( R_h \) satisfy

\( (4.5) \quad \| Q_h \varphi - \varphi \|_{L^2(\Omega)} + h \| \nabla (Q_h \varphi - \varphi) \|_{L^2(\Omega)} \leq C h^{s+1} \| \varphi \|_{H^{s+1}(\Omega)} \quad \forall \varphi \in H^{s+1}(\Omega), \quad 0 \leq s \leq k, \)
\( (4.6) \quad \| S_h \varphi - \varphi \|_{L^2(\Omega)} + h \| \nabla (S_h \varphi - \varphi) \|_{L^2(\Omega)} \leq C h^{s+1} \| \varphi \|_{H^{s+1}(\Omega)} \quad \forall \varphi \in H^{s+1}(\Omega), \quad 0 \leq s \leq k, \)
To derive the error estimates, we introduce the following notations

\[
E_u^n = u(t_n) - u_h^n, \quad E_\xi^n = \xi(t_n) - \xi_h^n, \quad E_\eta^n = \eta(t_n) - \eta_h^n, \\
E_p^n = p(t_n) - p_h^n, \quad E_q^n = q(t_n) - q_h^n.
\]

It is easy to check out

\[
E_p^n = \kappa_1 E_\xi^n + \kappa_2 E_\eta^n, \quad E_q^n = \kappa_3 E_\xi^n + \kappa_1 E_\eta^n.
\]

Also, we denote

\[
E_u^n = u(t_n) - R_h(u(t_n)) + R_h(u(t_n)) - u_h^n := Y_u^n + Z_u^n, \\
E_\xi^n = \xi(t_n) - S_h(\xi(t_n)) + S_h(\xi(t_n)) - \xi_h^n := Y_\xi^n + Z_\xi^n, \\
E_\eta^n = \eta(t_n) - S_h(\eta(t_n)) + S_h(\eta(t_n)) - \eta_h^n := Y_\eta^n + Z_\eta^n, \\
E_p^n = p(t_n) - S_h(p(t_n)) + S_h(p(t_n)) - p_h^n := Y_p^n + Z_p^n, \\
E_\xi^n = \xi(t_n) - Q_h(\xi(t_n)) + Q_h(\xi(t_n)) - \xi_h^n := F_\xi^n + G_\xi^n, \\
E_\eta^n = \eta(t_n) - Q_h(\eta(t_n)) + Q_h(\eta(t_n)) - \eta_h^n := F_\eta^n + G_\eta^n, \\
E_p^n = p(t_n) - Q_h(p(t_n)) + S_h(p(t_n)) - p_h^n := F_p^n + G_p^n.
\]

**Lemma 4.1.** Let \(\{(u_h^n, \xi_h^n, \eta_h^n)\}_{n \geq 0}\) be generated by the (MFEA) and \(Y_u^n, Z_u^n, Y_\xi^n, Z_\xi^n, Y_\eta^n\) and \(Z_u^n\) be defined above. Then there holds

\[
E_h^n + \Delta t \sum_{n=1}^{l} C_4 \|\varepsilon(Z_u^{n+1})\|_{L^2(\Omega)}^2 + \Delta t \sum_{n=1}^{l} \left[ \frac{K}{\mu_f} \|\nabla \hat{Z}_p^{n+1}, \nabla \hat{Z}_p^{n+1}\| \right] \\
+ \frac{\kappa_2 \Delta t}{2} \|d_t G_\xi^{n+\theta}\|^2_{L^2(\Omega)} + \frac{\kappa_3 \Delta t}{2} \|d_t G_\eta^{n+\theta}\|^2_{L^2(\Omega)} \\
\leq E_h^0 + \Delta t \sum_{n=1}^{l} \left[ (F_\xi^{n+1}, \nabla \cdot Z_u^{n+1}) - (\nabla \cdot d_t Z_u^{n+1}, G_\xi^{n+1}) \right] \\
+ \Delta t \sum_{n=1}^{l} \left[ (G_\xi^{n+1}, \nabla \cdot Z_u^{n+1}) - (\nabla \cdot d_t Y_u^{n+1}, G_\xi^{n+1}) \right] \\
+ \kappa_1 (1-\theta)(\Delta t)^2 \sum_{n=1}^{l} (d_t^2 \eta(t_n+1), G_\xi^{n+1}) + \Delta t \sum_{n=1}^{l} (R_h^{n+\theta}, \hat{Z}_p^{n+1}) \\
+ \Delta t \sum_{n=1}^{l} (N(\nabla R_h u(t_n+1)) - N(\nabla u(t_n+1)) \cdot \varepsilon(Z_u^{n+1}))
\]
\[ \begin{align*}
+ (1 - \theta)(\Delta t)^2 \sum_{n=1}^{l} \frac{\kappa_1}{\mu_f} (Kd_t \nabla Z_{n+1}^p, \nabla \hat{Z}_{p}^{n+1}) \\
+ \Delta t \sum_{n=1}^{l} (d_t G_{n+\theta}^\eta, \hat{Y}_{p}^{n+1} - F_{p}^{n+1}),
\end{align*} \]

where \[ \hat{Z}_{p}^{n+1} := F_{p}^{n+1} - Y_{p}^{n+1} + \kappa_1 G_{\xi}^{n+1} + \kappa_2 G_{\eta}^{n+\theta}, \]
\[ E_h := \frac{1}{2} \left[ \kappa_2 \left\| G_{\eta}^{n+\theta} \right\|_{L^2(\Omega)}^2 + \kappa_3 \left\| G_{\xi}^{n+1} \right\|_{L^2(\Omega)}^2 \right], \]
\[ R_{h}^{n+1} := -\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (s - t_n) \eta_{tt}(s) ds. \]

Proof. Subtracting (3.5) from (2.21), (3.6) from (2.22), (3.7) from (2.23), respectively, we get

\[ \begin{align*}
(4.10) \quad & \mathcal{N}(\nabla u(t_{n+1})) - \mathcal{N}(\nabla u_{n+1}^h), \varepsilon(v_h)) - (E_{\xi}^{n+1}, \nabla \cdot v_h) = 0 \quad \forall v_h \in V_h, \\
(4.11) \quad & \kappa_3 (E_{\xi}^{n+1}, \varphi_h) + (\nabla \cdot E_{u}^{n+1}, \varphi_h) = \kappa_1 (E_{\eta}^{n+\theta}, \varphi_h) \\
& \quad + \kappa_1 (1 - \theta) \Delta t (d_t \eta(t_{n+1}), \varphi_h) \quad \forall \varphi_h \in M_h, \\
(4.12) \quad & (d_t E_{\eta}^{n+\theta}, \psi_h) + \frac{1}{\mu_f} (K(\nabla E_{\xi}^{n+1}, \nabla \psi_h) \\
& \quad - (1 - \theta) \frac{\kappa_1 \Delta t}{\mu_f} (Kd_t \nabla E_{\xi}^{n+1}, \nabla \psi_h) = (R_{h}^{n+\theta}, \psi_h) \quad \forall \psi_h \in W_h, \\
(4.13) \quad & E_{\eta}^{0} = 0, E_{\xi}^{0} = 0, E_{\eta}^{-1} = 0.
\end{align*} \]

Using the definitions of the projection operators \( Q_h, S_h, R_h \), we have

\[ \begin{align*}
(4.14) \quad & \mathcal{N}(\nabla u(t_{n+1})) - \mathcal{N}(\nabla u_{h}^{n+1}), \varepsilon(v_h)) - (G_{\xi}^{n+1}, \nabla \cdot v_h) \\
& \quad = (F_{\xi}^{n+1}, \nabla \cdot v_h) \quad \forall v_h \in V_h, \\
(4.15) \quad & \kappa_3 (G_{\xi}^{n+1}, \varphi_h) + (\nabla \cdot Z_{u}^{n+1}, \varphi_h) = \kappa_1 (G_{\eta}^{n+\theta}, \varphi_h) \\
& \quad - (\nabla \cdot Y_{u}^{n+1}, \varphi_h) + \kappa_1 (1 - \theta) \Delta t (d_t \eta(t_{n+1}), \varphi_h) \quad \forall \varphi_h \in M_h, \\
(4.16) \quad & (d_t G_{\eta}^{n+\theta}, \psi_h) + \frac{1}{\mu_f} (K(\nabla Z_{p}^{n+1}, \nabla \psi_h) - (1 - \theta) \frac{\kappa_1 \Delta t}{\mu_f} (Kd_t \nabla E_{\xi}^{n+1}, \nabla \psi_h) \\
& \quad = (R_{h}^{n+\theta}, \psi_h) \quad \forall \psi_h \in W_h.
\end{align*} \]

Setting \( v_h = Z_{u}^{n+1} \) in (4.14), we have

\[ \begin{align*}
(4.17) \quad & \mathcal{N}(\nabla u(t_{n+1})) - \mathcal{N}(\nabla u_{h}^{n+1}), \varepsilon(Z_{u}^{n+1})) - (G_{\xi}^{n+1}, \nabla \cdot Z_{u}^{n+1}) \\
& \quad = (F_{\xi}^{n+1}, \nabla \cdot Z_{u}^{n+1}).
\end{align*} \]
Using (4.17), we get

\begin{align}
(4.18) \quad & (\mathcal{N}(\nabla R_h \mathbf{u}(t_{n+1})) - \mathcal{N}(\nabla \mathbf{u}_h^{n+1}), \varepsilon(Z_u^{n+1})) = (G_\xi^{n+1}, \nabla \cdot Z_u^{n+1}) \\
& + (\mathcal{N}(\nabla R_h \mathbf{u}(t_{n+1})) - \mathcal{N}(\nabla \mathbf{u}(t_{n+1})), \varepsilon(Z_u^{n+1})) + (F_\xi^{n+1}, \nabla \cdot Z_u^{n+1}).
\end{align}

Combining (2.28) and (4.18), we have

\begin{align}
(4.19) \quad & (\mathcal{N}(\nabla R_h \mathbf{u}(t_{n+1})) - \mathcal{N}(\nabla \mathbf{u}_h^{n+1}), \varepsilon(Z_u^{n+1})) \\
& = (\mathcal{N}(\nabla R_h \mathbf{u}(t_{n+1}))), - \mathcal{N}(\nabla \mathbf{u}(t_{n+1})), \varepsilon(\mathbf{u}_h^{n+1}) - \varepsilon(\mathbf{u}_h^{n+1})) \\
& \geq C_4 \|\varepsilon(R_h \mathbf{u}(t_{n+1})), - \mathcal{N}(\nabla \mathbf{u}(t_{n+1})), \varepsilon(\mathbf{u}_h^{n+1})\|^2_{L^2(\Omega)} = C_4 \|\varepsilon(Z_u^{n+1})\|^2_{L^2(\Omega)}.
\end{align}

Using (4.18) and (4.19), we have

\begin{align}
(4.20) \quad & C_4 \|\varepsilon(Z_u^{n+1})\|^2_{L^2(\Omega)} \leq (\mathcal{N}(\nabla R_h \mathbf{u}(t_{n+1}))), - \mathcal{N}(\nabla \mathbf{u}(t_{n+1})), \varepsilon(Z_u^{n+1})) \\
& + (G_\xi^{n+1}, \nabla \cdot Z_u^{n+1}) + (F_\xi^{n+1}, \nabla \cdot Z_u^{n+1}).
\end{align}

Setting $\varphi_h = G_\xi^{n+1}$ after applying the difference operator $d_t$ to (4.15), we get

\begin{align}
(4.21) \quad & \kappa_3 (d_t G_\xi^{n+1}, G_\xi^{n+1}) + (\nabla \cdot (d_t Z_u^{n+1}), G_\xi^{n+1}) = \kappa_1 (d_t G_\eta^{n+1}, G_\xi^{n+1}) \\
& - (\nabla \cdot (d_t Y_u^{n+1}), G_\xi^{n+1}) + \kappa_1 (1 - \theta) \Delta t (d_t^2 \eta(t_{n+1}), G_\xi^{n+1}).
\end{align}

Setting $\psi_h = \hat{Z}_p^{n+1} = F_p^{n+1} - Y_p^{n+1} + \kappa_1 G_\eta^{n+1} + \kappa_2 G_\eta^{n+1}$, we obtain

\begin{align}
(4.22) \quad & (d_t G_\eta^{n+1}, \hat{Z}_p^{n+1}) - (1 - \theta) \frac{\kappa_1 \Delta t}{\mu_f} \left( K d_t \nabla Z_u^{n+1}, \nabla \hat{Z}_p^{n+1} \right) \\
& + \frac{1}{\mu_f} (K (\nabla \hat{Z}_p^{n+1}, \nabla \hat{Z}_p^{n+1}) = (R_h^{n+1}, \hat{Z}_p^{n+1}),
\end{align}

adding (4.20), (4.21) and (4.22), applying the summation operator $\Delta t \sum_{n=1}^l$ to both sides, we get (4.9). The proof is complete. \[ \square \]

**Theorem 4.2.** Let $\{(\mathbf{u}_h^n, \xi_h^n, \eta_h^n)\}_{n \geq 0}$ be defined by the (MFEA), then there holds

\begin{align}
(4.23) \quad & \max_{0 \leq n \leq l} \left[ \sqrt{C_4} \|\varepsilon(Z_u^{n+1})\|_{L^2(\Omega)} + \sqrt{\kappa_3} \|G_\eta^{n+1}\|_{L^2(\Omega)} + \sqrt{\kappa_3} \|G_\xi^{n+1}\|_{L^2(\Omega)} \right] \\
& + \left[ \frac{\Delta t}{\mu_f} \sum_{n=1}^l \frac{K}{\mu_f} \|\nabla \hat{Z}_p^{n+1}\|_{L^2(\Omega)} \right] \leq \hat{C}_1(T) \Delta t + \hat{C}_2(T) h^2
\end{align}

provided that $\Delta t = O(h^2)$ when $\theta = 0$ and $\Delta t > 0$ when $\theta = 1$. Here

\begin{align}
(4.24) \quad & \hat{C}_1(T) = \hat{C} \|\eta_h\|_{L^2((0,T);L^2(\Omega))} + \hat{C} \|\eta_h\|_{L^2((0,T);H^1(\Omega))}, \\
(4.25) \quad & \hat{C}_2(T) = \hat{C} \|\xi_t\|_{L^2((0,T);L^2(\Omega))} + \hat{C} \|\xi_t\|_{L^2((0,T);H^1(\Omega))}
\end{align}
where

\[ 18 \]

ZHIHAO GE, WENLONG HE

\( \| \nabla \cdot \mathbf{u}_i \|_{L^2((0,T);H^2(\Omega))} \). 

\textbf{Proof.} Using (4.9) and the fact of \( Z_0^0 = 0, Z_0^0 = 0 \) and \( Z_{\xi}^{-1} = 0 \), we have

\begin{align*}
(4.26) \quad & \mathcal{E}_h^t + \Delta t \sum_{n=1}^l C_4 \| \varepsilon(Z_{u}^{n+1}) \|^2_{L^2(\Omega)} + \Delta t \sum_{n=1}^l \left[ \frac{K}{\mu_f} (\nabla \hat{Z}_p^{n+1}, \nabla \hat{Z}_\xi^{n+1}) \right. \\
& \quad + \frac{\kappa_2 \Delta t}{2} \| d_t G_{\eta}^{n+\theta} \|^2_{L^2(\Omega)} + \frac{\kappa_3 \Delta t}{2} \| d_t G_{\xi}^{n+1} \|^2_{L^2(\Omega)} \\
& \quad \leq \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6,
\end{align*}

where

\[ \Phi_1 = \Delta t \sum_{n=1}^l \left[ (F_{\xi}^{n+1}, \nabla \cdot Z_{u}^{n+1}) - (\nabla \cdot d_t Z_{u}^{n+1}, G_{\xi}^{n+1}) \right], \]

\[ \Phi_2 = \Delta t \sum_{n=1}^l \left[ (G_{\xi}^{n+1}, \nabla \cdot Z_{u}^{n+1}) - (\nabla \cdot d_t Y_{u}^{n+1}, G_{\xi}^{n+1}) \right], \]

\[ \Phi_3 = \kappa_1 (1 - \theta) (\Delta t)^2 \sum_{n=1}^l (d_t^2 \eta(t_{n+1}), G_{\xi}^{n+1}), \]

\[ \Phi_4 = \Delta t \sum_{n=1}^l \left( R_{h}^{n+\theta}, \hat{Z}_p^{n+1} \right), \]

\[ \Phi_5 = \Delta t \sum_{n=1}^l \left( \mathcal{N}(\nabla R_h \mathbf{u}(t_{n+1})) - \mathcal{N}(\nabla \mathbf{u}(t_{n+1})), \varepsilon(Z_{u}^{n+1}) \right), \]

\[ \Phi_6 = (1 - \theta) (\Delta t)^2 \sum_{n=1}^l \frac{\kappa_1}{\mu_f} (K d_t \nabla Z_{\xi}^{n+1}, \nabla \hat{Z}_p^{n+1}), \]

\[ \Phi_7 = \Delta t \sum_{n=1}^l \left( d_t G_{\eta}^{n+\theta}, Y_{p}^{n+1} - F_{p}^{n+1} \right). \]

Next, we estimate each term on the right-hand of (4.26). For \( \Phi_1 \), using Korn’s inequality, Cauchy-Schwarz inequality, Young inequality and the inequality of \( \| \nabla \cdot \mathbf{w} \|_{L^2(\Omega)} \leq c_k \| \varepsilon(\mathbf{w}) \|_{L^2(\Omega)} \) for all \( \mathbf{w} \in H^1_\eta(\Omega) \), we obtain

\begin{align*}
(4.27) \quad & \Phi_1 = \Delta t \sum_{n=1}^l \left[ (F_{\xi}^{n+1}, \nabla \cdot Z_{u}^{n+1}) - (\nabla \cdot d_t Z_{u}^{n+1}, G_{\xi}^{n+1}) \right] \\
& \quad \leq \frac{\Delta t}{2} \sum_{n=1}^l \left[ \| F_{\xi}^{n+1} \|^2_{L^2(\Omega)} + c_k \| \varepsilon(Z_{u}^{n+1}) \|^2_{L^2(\Omega)} + c_k \| d_t \varepsilon(Z_{u}^{n+1}) \|^2_{L^2(\Omega)} \\
& \quad + \| G_{\xi}^{n+1} \|^2_{L^2(\Omega)} \right].
\end{align*}
Similarly, using Korn’s inequality, the Cauchy-Schwarz inequality and Young inequality for \( \Phi_2 \), we get

\[
\Phi_2 = \Delta t \sum_{n=1}^{l} \left[ (G_{\xi}^{n+1}, \nabla \cdot Z_{u}^{n+1}) - (\nabla \cdot d_{t} Y_{u}^{n+1}, G_{\xi}^{n+1}) \right]
\]

\[
\leq \frac{\Delta t}{2} \sum_{n=1}^{l} \left[ \|G_{\xi}^{n+1}\|_{L^2(\Omega)}^2 + c_k \|\varepsilon(Z_{u}^{n+1})\|_{L^2(\Omega)}^2 + \|\nabla \cdot (d_{t} Y_{u}^{n+1})\|_{L^2(\Omega)}^2 
+ \|G_{\xi}^{n+1}\|_{L^2(\Omega)}^2 \right].
\]

When \( \theta = 0 \), using the integration by parts and \( d_{t} \eta(t_0) = 0 \), we get

\[
\Phi_3 = \kappa_1 (\Delta t)^2 \sum_{n=1}^{l} (d_{t}^2 \eta(t_{n+1}), G_{\xi}^{n+1})
\]

\[
= \kappa_1 (\Delta t)^2 \left[ \frac{1}{\Delta t} (d_{t} \eta(t_{t+1}), G_{\xi}^{t+1}) - \sum_{n=1}^{l} (d_{t} \eta(t_{n+1}), d_{t} G_{\xi}^{n+1}) \right].
\]

Using the Cauchy-Schwarz inequality, Young inequality and \( (3.3) \), we have

\[
\frac{1}{\Delta t} (d_{t} \eta(t_{t+1}), G_{\xi}^{t+1}) \leq \frac{1}{\Delta t} \|d_{t} \eta(t_{t+1})\|_{L^2(\Omega)} \|G_{\xi}^{t+1}\|_{L^2(\Omega)}
\]

\[
\leq \frac{1}{\Delta t} \|\eta_h\|_{L^2(t_{t}, t_{t+1}; \Omega)} \cdot \frac{1}{\beta_1} \sup_{v_h \in V_h} \left[ \frac{(\mathcal{N}(\nabla u(t_{t+1})), \nabla(v_h)) - (F_{\xi}^{t+1}, \nabla \cdot v_h)}{\|\nabla v_h\|_{L^2(\Omega)}} \right]
\]

\[
\leq \frac{1}{\beta_1 \Delta t} \|\eta_h\|_{L^2(t_{t}, t_{t+1}; \Omega)} \left[ C_3 \|\varepsilon(Z_{u}^{t+1})\|_{L^2(\Omega)} + C_3 \|\varepsilon(Y_{u}^{t+1})\|_{L^2(\Omega)}
+ c_k \|F_{\xi}^{t+1}\|_{L^2(\Omega)} \right]
\]

\[
\leq \frac{3}{\beta_1^2} \|\eta_h\|_{L^2(t_{t}, t_{t+1}; \Omega)}^2 + \frac{C_3^2}{4 \Delta t^2} \|\varepsilon(Z_{u}^{t+1})\|_{L^2(\Omega)}^2 + \frac{C_3^2}{4 \Delta t^2} \|\varepsilon(Y_{u}^{t+1})\|_{L^2(\Omega)}^2
+ \frac{c_k^2}{4 \Delta t^2} \|F_{\xi}^{t+1}\|_{L^2(\Omega)}^2.
\]

(4.31)

\[
\sum_{n=1}^{l} (d_{t} \eta(t_{n+1}), d_{t} G_{\xi}^{n+1}) \leq \sum_{n=1}^{l} \|d_{t} \eta(t_{n+1})\|_{L^2(\Omega)} \|d_{t} G_{\xi}^{n+1}\|_{L^2(\Omega)}
\]

\[
\leq \sum_{n=1}^{l} \|d_{t} \eta(t_{n+1})\|_{L^2(\Omega)} \cdot \frac{1}{\beta_1} \sup_{v_h \in V_h} \left[ \frac{(d_{t} N(\nabla u(t_{n+1})), - d_{t} N(\nabla u_{h}^{n+1}), \varepsilon(v_h))}{\|\nabla v_h\|_{L^2(\Omega)}} - (d_{t} F_{\xi}^{n+1}, \nabla \cdot v_h) \right]
\]

\[
\leq \sum_{n=1}^{l} \frac{1}{\beta_1} \|d_{t} \eta(t_{n+1})\|_{L^2(\Omega)} \left[ C_3 \|d_{t} \varepsilon(Z_{u}^{n+1})\|_{L^2(\Omega)} + C_3 \|d_{t} \varepsilon(Y_{u}^{n+1})\|_{L^2(\Omega)} \right].
\]
and (2.29), we have

where we used the fact that

The term of $\Phi_4$ can be bounded by

\[
\frac{\Delta t}{3} \sum_{n=1}^{l} \left( R_{h}^{n+\theta}, \hat{Z}_{p}^{n+1} \right) \leq \frac{\Delta t}{3} \sum_{n=1}^{l} \left| R_{h}^{n+\theta} \right|_{H^1(\Omega)}^{2/3} \left| \nabla \hat{Z}_{p}^{n+1} \right|_{L^2(\Omega)}^{2/3} \]  
\[
\leq \frac{\Delta t}{3} \sum_{n=1}^{l} \left( \frac{K}{4\mu_f} \left| \nabla \hat{Z}_{p}^{n+1} \right|_{L^2(\Omega)}^{2/3} + \frac{\mu_f \Delta t}{K} \left| R_{h}^{n+\theta} \right|_{H^1(\Omega)}^{2/3} \right) \]  
\[
\leq \frac{\Delta t}{3} \sum_{n=1}^{l} \left( \frac{K}{4\mu_f} \left| \nabla \hat{Z}_{p}^{n+1} \right|_{L^2(\Omega)}^{2/3} + \frac{\mu_f \Delta t}{3K} \left| \eta \right|_{H^1((t_{n}, t_{n+1}), H^1(\Omega))}^{2/3} \right), \]

where we used the fact that

\[
\left| R_{h}^{n+\theta} \right|_{H^1(\Omega)}^{2/3} \leq \frac{\Delta t}{3} \int_{t_{n}}^{t_{n+1}} \left| \eta \right|_{H^1(\Omega)}^{2/3} dt.
\]

As for the term of $\Phi_5$, using the Cauchy-Schwarz inequality, Young inequality and (2.29), we have

\[
\frac{\Delta t}{3} \sum_{n=1}^{l} \left( N(\nabla R_{h}u(t_{n+1})) - N(\nabla u(t_{n+1})), \varepsilon(Z_{u}^{n+1}) \right) \]  
\[
\leq \frac{\Delta t}{3} \sum_{n=1}^{l} \left( N(\nabla R_{h}u(t_{n+1})) - N(\nabla u(t_{n+1})), \varepsilon(Z_{u}^{n+1}) \right)_{L^2(\Omega)} \]  
\[
\leq \frac{\Delta t}{3} \sum_{n=1}^{l} \left( C_{3} \varepsilon(R_{h}u(t_{n+1})) - \varepsilon(u(t_{n+1})), \varepsilon(Z_{u}^{n+1}) \right)_{L^2(\Omega)} \]  
\[
\leq \frac{\Delta t}{3} \sum_{n=1}^{l} \left( C_{3} \varepsilon(R_{h}u(t_{n+1})) - \varepsilon(u(t_{n+1})), \varepsilon(Z_{u}^{n+1}) \right)_{L^2(\Omega)}^{2/3} + \left( \varepsilon(Z_{u}^{n+1}) \right)_{L^2(\Omega)}^{2/3} \]  
\[
= \frac{\Delta t}{3} \sum_{n=1}^{l} \left( C_{3} \varepsilon(Y_{u}^{n+1}), \varepsilon(Z_{u}^{n+1}) \right)_{L^2(\Omega)}^{2/3} + \left( \varepsilon(Z_{u}^{n+1}) \right)_{L^2(\Omega)}^{2/3}. \]

As for $\Phi_6$, using the inverse inequality (3.4), Cauchy-Schwarz inequality, Young inequality and inf-sup condition, we have

\[
\Phi_6 = \left( \Delta t \right)^2 \sum_{n=1}^{l} \left( K d_{t} \nabla Z_{\xi}^{n+1}, \nabla \hat{Z}_{p}^{n+1} \right) \]  
\[
\leq \left( \Delta t \right)^2 \sum_{n=1}^{l} \left( c_{1} h^{-1} K \varepsilon(u(t_{n+1})), \varepsilon(Z_{u}^{n+1}) \right)_{L^2(\Omega)}^{2/3} + \left( \varepsilon(Z_{u}^{n+1}) \right)_{L^2(\Omega)}^{2/3}. \]
Substituting (4.27)-(4.35) into (4.26), we have

\[
(4.36)
\]

Using the Cauchy-Schwarz inequality and Young inequality, we get

\[
(4.35) \quad \Phi_7 = \Delta t \sum_{n=1}^{l} (d_t G_{n+\theta}^\eta, Y_{n+1}^p - F_{n+1}^p)
\]

Substituting (4.27)-(4.35) into (4.26), we have

\[
(4.36) \quad \frac{1}{2} \left[ \kappa_2 \| G_{n+\theta}^\eta \|^2_{L^2(\Omega)} + \kappa_3 \| G_{n+\xi}^l \|^2_{L^2(\Omega)} \right] + \Delta t \sum_{n=1}^{l} \left[ C_4 \| (Z_{n+1}^\eta) \|^2_{L^2(\Omega)} + \Delta t \sum_{n=1}^{l} \left[ \frac{K}{\mu_f} \| (\nabla \hat{Z}_{n+1}^\eta, \nabla \hat{Z}_{n+1}^\xi) \|_{L^2(\Omega)} \right] \right]
\]

\[
\leq \frac{\Delta t}{2} \sum_{n=1}^{l} \left[ \| F_{n+1}^\eta \|^2_{L^2(\Omega)} + \| F_{n+1}^\xi \|^2_{L^2(\Omega)} \right] + \Delta t \sum_{n=1}^{l} \left[ \| (Z_{n+1}^\eta) \|^2_{L^2(\Omega)} + c_k \| (Z_{n+1}^\xi) \|^2_{L^2(\Omega)} \right] + \Delta t \sum_{n=1}^{l} \left[ \| (Z_{n+1}^u) \|^2_{L^2(\Omega)} + \| (Z_{n+1}^a) \|^2_{L^2(\Omega)} \right] + \frac{3k_1(\Delta t)^2}{\beta^2} \| \eta_t \|^2_{L^2((t_i,t_{i+1});\Omega)}
\]
provided that $\Delta t \leq \frac{h^2 \beta_2^2 \mu C_4}{4K \kappa_1^2 \alpha} = \Delta t^*\mu C_4$ when $\theta = 0$ or $\Delta t \geq 0$ when $\theta = 1$. Hence, we deduce that (4.23) holds. The proof is complete. $\square$

**Theorem 4.3.** The solution of the (MFEA) satisfies the following error
estimates:

\[
\max_{0 \leq n \leq N} \left[ \sqrt{C_1} \left\| \nabla (u(t_{n+1}) - u^n_h) \right\|_{L^2(\Omega)} + \sqrt{C_2} \left\| \eta(t_{n+1}) - \eta^n_h \right\|_{L^2(\Omega)} \right.
\]
\[
+ \sqrt{C_3} \left\| \xi(t_{n+1}) - \xi^n_h \right\|_{L^2(\Omega)} \right] \leq \tilde{C}_1(T) \Delta t + \tilde{C}_2(T) h^2,
\]
\[
(4.39) \quad \left( \Delta t \sum_{n=0}^{N} K \left\| \nabla p(t_{n+1}) - \nabla p^n_h \right\|_{L^2(\Omega)}^2 \right)^{1/2} \leq \tilde{C}_1(T) \Delta t + \tilde{C}_2(T) h
\]
provided that \( \Delta t = O(h^2) \) when \( \theta = 0 \) and \( \Delta t \geq 0 \) when \( \theta = 1 \). Here \( \tilde{C}_1(T) = \tilde{C}_1(T) + \|\xi\|_{L^\infty([0,T];H^1(\Omega))} + \|\eta\|_{L^\infty([0,T];H^1(\Omega))} + \|\nabla u\|_{L^\infty([0,T];H^1(\Omega))} \).

Proof. The above estimates follow immediately from an application of the triangle inequality on

\[
u(t_n) - u^n_h = Y^n_u + Z^n_u, \quad \xi(t_n) - \xi^n_h = Y^n_\xi + Z^n_\xi = F^n_\xi + G^n_\xi, \]
\[
\eta(t_n) - \eta^n_h = Y^n_\eta + Z^n_\eta = F^n_\eta + G^n_\eta, \quad p(t_n) - p^n_h = Y^n_p + Z^n_p = F^n_p + G^n_p.
\]

and appealing to (4.5), (4.6), (4.7) and Theorem 4.2. The proof is complete. \( \square \)

5. Numerical tests.

Test 1. Let \( \Omega = [0,1] \times [0,1], \Gamma_1 = \{(1, x_2); 0 \leq x_2 \leq 1\}, \Gamma_2 = \{(x_1, 0); 0 \leq x_1 \leq 1\}, \Gamma_3 = \{(0, x_2); 0 \leq x_2 \leq 1\}, \Gamma_4 = \{(x_1, 1); 0 \leq x_1 \leq 1\}, \) and \( T = 1 \). The source functions are as follows:

\[
f = -(\lambda + \mu)t(1,1)^T - 2(\mu + \lambda)t^2(x_1, x_2)^T + \alpha t e^{x_1 + x_2}(1,1)^T, \]
\[
\phi = c_0 e^{x_1 + x_2} - \frac{2K}{\mu_f} t e^{x_1 + x_2} + \alpha(x_1 + x_2),
\]
and the boundary and initial conditions are

\[
p = t e^{x_1 + x_2} \quad \text{on } \partial \Omega_T, \]
\[
u_1 = \frac{1}{2} x_1^2 t \quad \text{on } \Gamma_j \times (0, T), j = 1, 3, \]
\[
u_2 = \frac{1}{2} x_2^2 t \quad \text{on } \Gamma_j \times (0, T), j = 2, 4, \]
\[
\sigma \nu - \alpha \nu = f_1 \quad \text{on } \partial \Omega_T, \]
\[
u(x, 0) = 0, \quad p(x, 0) = 0 \quad \text{in } \Omega,
\]

where

\[
f_1(x, t) = \lambda(x_1 + x_2)(n_1, n_2)^T t + \mu t(n_1, n_2)^T + \mu t^2(x_1^2 n_1, x_2^2 n_2)^T
\]
\[ +\lambda t^2(x_1^2 + x_2^2)(n_1, n_2)^T - \alpha(n_1, n_2)^T te^{x_1 + x_2}. \]

The exact solution of this problem is

\[ u(x, t) = \frac{t}{2}(x_1^2, x_2^2)^T, \quad p(x, t) = te^{x_1 + x_2}. \]

Table 5.1: Values of parameters

| Parameters | Description                          | Values  |
|------------|--------------------------------------|---------|
| \(\nu\)    | Poisson ratio                        | 0.25    |
| \(\alpha\) | Biot-Willis constant                 | 1e-5    |
| \(E\)      | Young’s modulus                      | 0.25    |
| \(\lambda\)| Lamé constant                        | 0.1     |
| \(K\)      | Permeability tensor                  | (1e-3) I|
| \(\mu\)    | Lamé constant                        | 0.1     |
| \(c_0\)    | Constrained specific storage coefficient | 2      |

Table 5.2: Spatial errors and convergence rates of \(u\)

| \(h\)   | \(\|u - u_h\|_{L^2}\) CR | \(\|u - u_h\|_{H^1}\) CR |
|---------|---------------------------|---------------------------|
| \(h = 1/3\) | 1.5015e-6 1.9107e-5     |                           |
| \(h = 1/6\) | 2.3515e-7 2.675 3.3561e-6 2.509 |                   |
| \(h = 1/12\) | 6.3182e-8 1.896 7.2141e-7 2.218 |                   |
| \(h = 1/24\) | 1.4724e-8 2.101 1.976e-7 1.868 |                   |

Table 5.3: Spatial errors and convergence rates of \(p\)

| \(h\)   | \(\|p - p_h\|_{L^2}\) CR | \(\|p - p_h\|_{H^1}\) CR |
|---------|---------------------------|---------------------------|
| \(h = 1/3\) | 0.032683 0.43705     |                           |
| \(h = 1/6\) | 0.008109 2.011 0.21766 1.005 |                   |
| \(h = 1/12\) | 0.002016 2.008 0.10872 1.002 |                   |
| \(h = 1/24\) | 0.000497 2.022 0.05435 1.0004 |                   |
Figure 5.1: The numerical pressure $p^{n+1}_h$ at the terminal time $T$ with the parameters of Table 5.1.

Figure 5.2: Arrow plot of the computed displacement $u$ with the parameters of Table 5.1.
### Table 5.4: Values of parameters

| Parameters | Description                  | Values |
|------------|------------------------------|--------|
| $\nu$      | Poisson ratio                | 0.25   |
| $\alpha$   | Biot-Willis constant         | 1e-5   |
| $E$        | Young's modulus              | 2500   |
| $\lambda$  | Lamé constant               | 1e3    |
| $K$        | Permeability tensor          | (1e-3)I|
| $\mu$      | Lamé constant               | 1e3    |
| $c_0$      | Constrained specific storage coefficient | 1      |

### Table 5.5: Spatial errors and convergence rates of $u$

| $h$   | $\|u - u_h\|_{L^2}$ | CR      | $\|u - u_h\|_{H^1}$ | CR      |
|-------|----------------------|---------|----------------------|---------|
| $h = 1/3$ | 1.2573e-6          |         | 1.8951e-5            |         |
| $h = 1/6$  | 1.2922e-7          | 3.2824  | 3.2945-6             | 2.5241  |
| $h = 1/12$ | 2.7945e-8          | 2.2092  | 6.9895e-7            | 2.2368  |
| $h = 1/24$ | 3.2025-9           | 3.1253  | 1.9216e-7            | 1.8629  |

### Table 5.6: Spatial errors and convergence rates of $p$

| $h$   | $\|p - p_h\|_{L^2}$ | CR      | $\|p - p_h\|_{H^1}$ | CR      |
|-------|----------------------|---------|----------------------|---------|
| $h = 1/3$ | 0.024431            |         | 0.45261              |         |
| $h = 1/6$  | 0.0049599           | 2.3003  | 0.22246              | 1.0247  |
| $h = 1/12$ | 0.0010675           | 2.2161  | 0.10974              | 1.0195  |
| $h = 1/24$ | 0.00024727          | 2.1101  | 0.054506             | 1.0096  |
Figure 5.3: The numerical pressure $p_{h}^{n+1}$ at the terminal time $T$ with the parameters of Table 5.4.

Figure 5.4: Exact solution of pressure $p$ at the terminal time $T$ of Test 1.
Figure 5.5: Arrow plot of the computed displacement $u$ with the parameters of Table 5.4.

Table 5.2 and Table 5.3 display the error of displacement $u$ and the pressure $p$ with $L^2(\Omega)$-norm and $H^1(\Omega)$-norm at the terminal time $T$ with the parameters of Table 5.1, which are consistent with the theoretical result. Table 5.5 and Table 5.6 display the error of displacement $u$ and the pressure $p$ with $L^2(\Omega)$-norm and $H^1(\Omega)$-norm at the terminal time $T$ with the parameters of Table 5.4. It is easy to find that there is no “locking phenomenon”.

Figure 5.1 and Figure 5.3 show the numerical solution of pressure $p_n^{n+1}$ at the terminal time $T$ due to the difference of parameters between Table 5.1 and Table 5.4. Figure 5.4 shows the analytical solution of pressure $p_n^{n+1}$ at the terminal time $T$. Figure 5.2 and Figure 5.5 show the arrow plot of the computed displacement $u$ corresponding to the parameters of Table 5.1 and Table 5.4, respectively.

**Test 2.** Let $\Omega = [0, 1] \times [0, 1]$, $\Gamma_1 = \{(1, x_2); 0 \leq x_2 \leq 1\}$, $\Gamma_2 = \{(x_1, 0); 0 \leq x_1 \leq 1\}$, $\Gamma_3 = \{(0, x_2); 0 \leq x_2 \leq 1\}$, $\Gamma_4 = \{(x_1, 1); 0 \leq x_1 \leq 1\}$, and $T = 1$. The source functions are as follows:

$$f = -(\lambda + \mu)t^2(1, 1)^T - 2(\mu + \lambda)t^4(x_1, x_2)^T + \alpha \cos(x_1 + x_2)e^t(1, 1)^T,$$

$$\phi = \left(c_0 + \frac{2K}{\mu_f}\right) \sin(x_1 + x_2)e^t + 2t\alpha(x_1 + x_2).$$

and the boundary and initial conditions are as follows:

$$p = \sin(x_1 + x_2)e^t \quad \text{on} \ \partial \Omega_T,$$

$$u_1 = \frac{1}{2}x_1^2t^2 \quad \text{on} \ \Gamma_j \times (0, T), \ j = 1, 3,$$
u_2 = \frac{1}{2} x_2^2 t^2 \quad \text{on } \Gamma_j \times (0, T), j = 2, 4,

\sigma u - \alpha p \mathbf{n} = \mathbf{f}_1, \quad \text{on } \partial \Omega_T,

u(x, 0) = 0, \quad p(x, 0) = \sin(x_1 + x_2) \quad \text{in } \Omega.

where

f_1(x, t) = \lambda (x_1 + x_2)(n_1, n_2)^T t^2 + \mu t^2 (x_1 n_1, x_2 n_2)^T + \mu t^4 (x_1^2 n_1, x_2^2 n_2)^T + \lambda t^4 (x_1^2 + x_2^2)(n_1, n_2)^T - \alpha \sin(x_1 + x_2)(n_1, n_2)^T e^t.

The exact solution of this problem is

u(x, t) = \frac{t^2}{2} (x_1^2, x_2^2)^T, \quad p(x, t) = \sin(x_1 + x_2)e^t.

| Parameters | Description | Values |
|------------|-------------|--------|
| \nu        | Poisson ratio | 0.00495 |
| \alpha     | Biot-Willis constant | 1e-4 |
| \mu        | Lamé constant | 0.1 |
| K          | Permeability tensor | 0.1 I |
| \lambda    | Lamé constant | 10 |
| \mu        | Lamé constant | 10 |
| \nu_0      | Constrained specific storage coefficient | 20 |

Table 5.8: Spatial errors and convergence rates of \( u \)

| h        | \| u - u_h \|_{L^2} | CR | \| u - u_h \|_{H^1} | CR |
|----------|------------------|----|------------------|----|
| h = 1/3  | 1.1698e-7        |     | 4.0858e-7        |     |
| h = 1/6  | 3.0566e-8        | 1.9363 | 1.0727e-7        | 1.9294 |
| h = 1/12 | 7.7309e-9        | 1.9832 | 2.7139e-8        | 1.9828 |
| h = 1/24 | 1.9451e-9        | 1.9908 | 6.8316e-9        | 1.9901 |
Table 5.9: Spatial errors and convergence rates of $p$

| $h$       | $\|p - p_h\|_{L^2}$ | CR   | $\|p - p_h\|_{H^1}$ | CR   |
|-----------|----------------------|------|----------------------|------|
| $h = 1/3$ | 0.021791             | 0.29681 |
| $h = 1/6$ | 0.005449             | 1.9997 | 0.14871             | 0.9970 |
| $h = 1/12$| 0.001368             | 1.9939 | 0.07439             | 0.9993 |
| $h = 1/24$| 0.000349             | 1.9708 | 0.03720             | 0.9998 |

Figure 5.6: The numerical pressure $p_h^{n+1}$ at the terminal time $T$ with the parameters of Table 5.7

Figure 5.7: Arrow plot of the computed displacement $\mathbf{u}$ with the parameters of Table 5.7
Table 5.10: Values of parameters

| Parameters | Description               | Values |
|------------|---------------------------|--------|
| $\nu$      | Poisson ratio             | 0.25   |
| $\alpha$   | Biot-Willis constant      | 1e-4   |
| $E$        | Young’s modulus           | 2500   |
| $\lambda$  | Lamé constant             | 1e3    |
| $K$        | Permeability tensor       | (0.1) I |
| $\mu$      | Lamé constant             | 1e3    |
| $c_0$      | Constrained specific storage coefficient | 0.01 |

Table 5.11: Spatial errors and convergence rates of $u$

| $h$     | $\|u - u_h\|_{L^2}$ | CR | $\|u - u_h\|_{H^1}$ | CR |
|---------|---------------------|----|---------------------|----|
| $h = 1/3$ | 1.2573e-6           |    | 1.8951e-5           |    |
| $h = 1/6$ | 1.2917e-7           | 3.2830 | 3.2945e-7           | 2.5241 |
| $h = 1/12$ | 2.7942e-8           | 2.2088 | 6.9895e-7           | 2.2368 |
| $h = 1/24$ | 3.2000e-9           | 1.9216e-7 | 1.8629 |

Table 5.12: Spatial errors and convergence rates of $p$

| $h$     | $\|p - p_h\|_{L^2}$ | CR | $\|p - p_h\|_{H^1}$ | CR |
|---------|---------------------|----|---------------------|----|
| $h = 1/3$ | 0.0218              |    | 0.2968              |    |
| $h = 1/6$ | 0.0054              | 2.0133 | 0.1487              | 0.9971 |
| $h = 1/12$ | 0.0014              | 1.9475 | 0.0744              | 0.9990 |
| $h = 1/24$ | 0.00033949          | 2.0440 | 0.0372              | 1    |

Figure 5.8: The numerical pressure $p_h^{n+1}$ at the terminal time $T$ with the parameters of Table 5.10
Table 5.8 and Table 5.9 display the error of displacement $u$ and the pressure $p$ with $L^2(\Omega)$-norm and $H^1(\Omega)$-norm at the terminal time $T$ with the parameters of Table 5.7, which are consistent with the theoretical result. Table 5.11 and Table 5.12 display the error of displacement $u$ and the pressure $p$ with $L^2(\Omega)$-norm and $H^1(\Omega)$-norm at the terminal time $T$ with the parameters of Table 5.10, which show that our method overcomes the “locking phenomenon”.

Figure 5.6 and Figure 5.8 show the numerical solution of pressure $p_{n+1}^h$ at the terminal time $T$ due to the difference of parameters between Table 5.7 and Table 5.10. Figure 5.9 shows the analytical solution of pressure $p_{n+1}^h$ at the terminal time.
Figure 5.7 and Figure 5.10 show the arrow plot of the computed displacement \( u \) corresponding to the parameters of Table 5.7 and Table 5.10, respectively.

6. Conclusion. In this paper, we propose a multiphysics finite element method and analyze the optimal error convergence order for a nonlinear poroelasticity model. Firstly, we reformulate the nonlinear fluid-solid coupling problem into a fluid-fluid coupling problem by a multiphysics approach. Secondly, we design a fully discrete time-stepping scheme to use multiphysics finite element method with \( P_2 - P_1 - P_1 \) element pairs for the space variables and backward Euler method for the time variable, and we adopt the Newton iterative method to deal with the nonlinear term. Also, we derive the discrete energy laws and the optimal convergence order error estimates without any assumption on the nonlinear stress-strain relation. Finally, we show some numerical examples to verify the rationality of theoretical analysis and there is no “locking phenomenon”. To the best of our knowledge, the proposed fully discrete multiphysics finite element method for the nonlinear poroelasticity model is completely new.

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