Chiral anomalies and Poincaré invariance

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Abstract

I study variations of the fermionic determinant for a nonabelian Dirac fermion with external vector and axial vector sources. I consider different regularizations, leading to different chiral anomalies when the variations are chiral transformations. For these different regularizations, I then consider variations associated with Poincaré transformations. I find that both Lorentz and translational invariance are anomalously violated in general, but that they are respected when the variations of the determinant are regularized to give a Wess-Zumino consistent anomaly (the Bardeen anomaly). If the variations are regularized to give a covariant anomaly, then Poincaré invariance is not respected. Following Manohar in an investigation of Poincaré anomalies in a chiral gauge theory, this gives an alternative way to understand the need for a consistent regularization of the variations of the fermionic determinant.

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1 Introduction

In many approaches to the problem of deriving low energy effective theories of QCD we are required to calculate a variation of the fermionic determinant with respect to a chiral transformation. For example, in chiral perturbation theory we must construct the Lagrangian for the pseudoscalar octet so as to transform in the same way as the underlying QCD – including the chiral anomalies [1].

For the calculation of such a variation it is necessary to choose a regularization, and in general different choices of regularization leads to different expressions for the chiral anomalies. If we consider for example a theory of a nonabelian Dirac fermion coupled to external vector (V) and axial vector (A) sources, it is possible to regularize the chiral variation of the fermionic determinant so that the divergence of the vector current vanishes and the divergence of the axial vector current is equal to the Wess-Zumino consistent chiral anomaly [2]. Another possibility is to have covariant anomalies [3], in which case both the divergence of the current and of the axial current are different from zero. (Note that I am referring to anomalies for the divergence

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of the same axial current for different regularizations, rather than anomalies for the divergence of different currents within the same consistent theory, which is perhaps more conventional.)

A consistent regularization ensures that the anomalies satisfy Wess–Zumino consistency conditions. These consistency conditions are integrability conditions for the variation of the determinant on gauge orbits of chiral gauge transformations \[ \bar{\psi}(D + i\epsilon)\psi \]. When the variation is regularized in another way, like the covariant, the determinant itself is not well defined. For example, it cannot be obtained by integrating variations with respect to external fields.

In this paper I investigate the Poincaré transformation properties of the nonabelian “VA-theory” mentioned above, and I will show that for many other regularizations than the consistent one Poincaré invariance is not respected. Rather, it is broken by anomalies, that is, quantum effects. For example, the theory with covariant regularization is not Poincaré invariant.

This provides an alternative way to understand the need for a consistent regularization when variations of the fermionic determinant is calculated. It may be more useful to have a symmetry argument for this, rather than the perhaps more unfamiliar integrability condition.

An investigation slightly related to the one in this paper was carried out some time ago by Manohar \[ 3 \] (see also \[ 4 \]). He demonstrated that for a theory of a chiral fermion where the chiral anomalies did not cancel, there were also Poincaré anomalies, and he pointed out that this was another way to interpret the need for anomaly cancellation.

It may also be worth pointing out that the Poincaré anomalies in this paper are not directly related to the gravitational anomalies in ref. \[ 7 \]. First of all, the theory is regularized consistently in that paper, and the fermionic determinant in is well defined. Furthermore, the anomalies appear from Feynman diagrams with a number of energy-momentum tensors at the vertices. The Poincaré anomalies here, on the other hand, appear from diagrams with one energy-momentum tensor and a number of vectors and axial vectors (for the translational anomalies), or one angular momentum current and a number of vectors and axial vectors (for the Lorentz anomalies).

The organization of the paper is the following. In sec. 2, I discuss the nonabelian “VA-theory” on the classical level, and in particular the symmetries of the model. In sec. 3, I discuss the regularization of the variations of the fermionic determinant of the VA-theory. The regularization scheme I use is proper time regularization in Minkowski space, which is convenient here because the exact specification of the regularization is controlled by a “regularization operator” \( \tilde{D} \), related to the Dirac operator \( D \). I then discuss various choices for \( \tilde{D} \). In sec. 4, I show that the choice that leads to the Wess–Zumino consistent Bardeen anomaly leads to Lorentz and translational invariance. On the other hand, in sec. 5, I show that the choice that leads to the covariant anomaly is not Poincaré invariant. Sec. 6 is a brief summary.

## 2 Classical symmetries of the VA-theory

The nonabelian VA-theory is chosen as a “typical” theory of a Dirac fermion, and suitable for illustration of the main points. It is given by the Lagrangian

\[
\mathcal{L} = \bar{\psi}(D + i\epsilon)\psi, \quad D = i\bar{\theta} - \bar{V} - \bar{A}\gamma_5 - \mu.
\]  

Here, \( V_\mu = V_\mu^a t^a \) and \( A_\mu = A_\mu^a t^a \) are external vector and axial vector sources, respectively, in the Lie algebra of some group with generators \( t^a \). I have added a small mass \( \mu \) as an infrared regulator, as well as an \( i\epsilon \). I will usually suppress both of these.

Nothing in our discussion of this model is not well known, until we get to the part where we consider the Poincaré transformations in the quantum theory. Note, however, that I have chosen to not treat the external sources as gauge fields, since gauge invariance is not important for our considerations.
The symmetries of this theory on the classical level include global phase rotations and chiral rotations, translations and Lorentz transformations, provided we assign the appropriate transformations rules to the external fields $V_\mu$ and $A_\mu$. Thus we have phase rotations

$$
\begin{align*}
\psi &\rightarrow e^{i\alpha} \psi, \\
\bar{\psi} &\rightarrow \bar{\psi} e^{-i\alpha}, \\
V_\mu + \gamma_5 A_\mu &\rightarrow e^{i\alpha} (V_\mu + \gamma_5 A_\mu) e^{-i\alpha},
\end{align*}
$$

with $\alpha = \alpha^a t^a$, chiral rotations

$$
\begin{align*}
\psi &\rightarrow e^{i\beta\gamma_5} \psi, \\
\bar{\psi} &\rightarrow \bar{\psi} e^{i\beta\gamma_5}, \\
V_\mu + \gamma_5 A_\mu &\rightarrow e^{-i\beta\gamma_5} (V_\mu + \gamma_5 A_\mu) e^{i\beta\gamma_5},
\end{align*}
$$

with $\beta = \beta^a t^a$, translations

$$
\begin{align*}
\psi &\rightarrow e^{ia_\mu P_\mu} \psi, \\
\bar{\psi} &\rightarrow \bar{\psi} e^{-ia_\mu P_\mu}, \\
V_\mu &\rightarrow e^{ia_\mu P_\mu} V_\mu e^{-ia_\mu P_\mu}, \\
A_\mu &\rightarrow e^{ia_\mu P_\mu} A_\mu e^{-ia_\mu P_\mu},
\end{align*}
$$

where $P_\mu \equiv i\partial_\mu$ are the generators, and Lorentz transformations

$$
\begin{align*}
\psi &\rightarrow e^{\frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}} \psi, \\
\bar{\psi} &\rightarrow \bar{\psi} e^{-\frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}}, \\
V_\mu &\rightarrow e^{\frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}} V_\mu e^{-\frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}}, \\
A_\mu &\rightarrow e^{\frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}} A_\mu e^{-\frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}},
\end{align*}
$$

with generators $J_{\mu\nu} \equiv \frac{1}{2} \sigma_{\mu\nu} + (x_\mu i\partial_\nu - x_\nu i\partial_\mu) \equiv S_{\mu\nu} + L_{\mu\nu}$. This list shows how the transformation rules for the external sources are chosen. Note in particular that the phase and chiral rotations are chosen for convenience to be global, since gauge invariance is not relevant here.

Since these transformations are all symmetries, it means that the Lagrangian transforms under an infinitesimal transformation by

$$
\delta L = \partial_\mu \Lambda^\mu + \Delta
$$

with $\Delta = 0$. We shall see that in the quantum theory, for regularizations that are not consistent, this is not automatically true: Quantum effects lead to $\Delta \neq 0$ for, e.g., the covariant regularization – also for the Poincaré transformations. Therefore, in that case the Poincaré transformations are no longer symmetries of the theory.

Finally, we may derive the classical conservation equations

$$
\begin{align*}
\partial_\mu J^{a\mu} &= 0, & J^{a\mu} &\equiv \bar{\psi} \gamma_\mu t^a \psi, \\
\partial_\mu J^{5a\mu} &= 0, & J^{5a\mu} &\equiv \bar{\psi} \gamma_\mu \gamma_5 t^a \psi,
\end{align*}
$$

where $J_\mu$ and $J^{5\mu}$ are the fermion current and axial current.
3 The quantum theory

The quantum VA-theory is given by the path integral

$$Z = \int D\psi D\bar{\psi} e^{i\int d^4x \bar{\psi}D\psi} \equiv \text{Det}D,$$

which also formally defines the fermionic determinant. The effective action $W$ ($Z \equiv e^{iW}$) is given by

$$W = -i\text{Tr} \ln D.\quad (9)$$

These expressions are formal and need regularization to become well defined. However, we are interested in the variations of the fermionic determinant under some infinitesimal transformations. We should therefore consider such variations and regularize these instead.

With infinitesimal parameters, the phase and chiral rotations of the fermions induce a change in the Dirac operator

$$D \to e^{-i\alpha + i\gamma_5} De^{i\alpha + i\gamma_5}$$

$$\equiv D + i(D\alpha - \alpha D) + i(D\beta\gamma_5 + \beta\gamma_5 D)$$

$$\equiv D + \delta D,\quad (10)$$

which in turn induces a change in $W$:

$$\delta W = -i\text{Tr}\delta D \frac{1}{D}.\quad (11)$$

The Jacobian $J$ is then determined by the requirement that the path integral $Z$ is unchanged under a change of variables:

$$Z = Je^{iW + i\delta W} = e^{iW}.$$

Defining now a Lagrangian by $J \equiv \exp(i\int d^4x L_J)$, we have

$$\int d^4x L_J = -\delta W = i\text{Tr}\delta D \frac{1}{D}.\quad (13)$$

$L_J$ thus contains the anomalies, $G_\alpha^a$ and $G_\beta^a$,

$$L_J \equiv \alpha^a G_\alpha^a + \beta^a G_\beta^a,\quad (14)$$

and is therefore a quantity of interest.

When a regularization is specified, both the transformations (2) and (3) are in general seen to be broken symmetries. We can then find the conservation equations for the currents:

$$\partial_\mu J_\mu^{a\alpha} = -G_\alpha^a,$$

$$\partial_\mu J_5^{a\mu} = -G_\beta^a.$$

Since we know what the $G$’s look like for different regularization schemes, at least the consistent and covariant ones – see e.g. [4], we will use eqs. (15) later (sec. 5 and 6) to identify what scheme we are in.

Again note that, in the terminology used in this paper, the currents are always defined in the same way in terms of the fermions, while the anomalies vary with the regularization scheme. This is a difference from the usual discussion of anomalies [3], where the regularization...
is assumed to be consistent while different definitions of the currents are considered, each giving different expressions for the anomalies.

Let us also record that for the Poincaré transformations, the infinitesimal variations of the Dirac operator $D$ are

$$\delta D = i(Da_\mu P^\mu - a_\mu P^\mu D)$$

for the translations, and

$$\delta D = i(D\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}D)$$

for the Lorentz transformations.

4 Proper time regularization

The regularization scheme I will use is proper time regularization in Minkowski space [8]. Traditionally proper time regularization is used in connection with the Euclidean formalism, see e.g. the review [4]. However, it turns out to be a great advantage to work in Minkowski space, when different regularizations – consistent, covariant, etc. – are discussed, since in that case the various prescriptions are conveniently controlled by the choice of a “regularization operator” $\tilde{D}$, related to the Dirac operator $D$. This will be discussed in the next sections; see also [8]. There is also no need to perform analytical continuations of the fields and transformations in Minkowski space. But other regularization schemes should of course be possible.

We can introduce a proper time integral and the operator $\tilde{D}$ in the following way:

$$\int d^4x\mathcal{L}_J = i\text{Tr}\delta D \frac{1}{D}$$

$$= i\text{Tr}\delta D\tilde{D} \frac{1}{\tilde{D}}$$

$$= \int_\Lambda^\infty ds\text{Tr}\delta D\tilde{D}e^{is(D\tilde{D}+ic)}$$

The operator $\tilde{D}$ is a priori arbitrary, except that it must be chosen to give the right $\epsilon$-prescription. This will then ensure convergence at the upper integration limit. For the lower integration limit the cutoff $\Lambda$ is introduced, which is to be taken to infinity at the end of the calculation. When the appropriate choice for $\tilde{D}$ is made, $\mathcal{L}_J$ will be regular and well defined.

We can use the expression for $\delta D$ from eq. (10) to perform the proper time integral and write $\int d^4x\mathcal{L}_J$ in a Fujikawa-like form [6, 10]:

$$\int d^4x\mathcal{L}_J = \int_1/\Lambda^2 ds\text{Tr}i(D\alpha - \alpha D + D\beta\gamma_5 + \beta\gamma_5 D)\tilde{D}e^{is\tilde{D}D}$$

$$= -\text{Tr}\alpha(e^{i\tilde{D}D/\Lambda^2} - e^{i\tilde{D}D/\Lambda^2}) - \text{Tr}\beta\gamma_5(e^{i\tilde{D}D/\Lambda^2} + e^{i\tilde{D}D/\Lambda^2})$$

Here I have used the cyclicity of the trace, the identity $\tilde{D}e^{isD\tilde{D}} = DDe^{isD\tilde{D}}$, and the fact that only the lower limit of the integral contributes due to the implicit presence of the $\epsilon$. Similarly, for the Lorentz transformations and translations we have

$$\int d^4x\mathcal{L}_J = -\text{Tr}\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}(e^{i\tilde{D}D/\Lambda^2} - e^{i\tilde{D}D/\Lambda^2})$$

$$-\text{Tr}a_\mu P^\mu(e^{i\tilde{D}D/\Lambda^2} - e^{i\tilde{D}D/\Lambda^2})$$

To proceed from here it is necessary to choose a $\tilde{D}$. 

5
5 Consistent regularization

Let us make the choice

\[ \tilde{D} = (i\gamma_5)D(i\gamma_5) \]
\[ = i\partial - V - iA\gamma_5 + \mu. \]  

(21)

Using the cyclicity of the trace and the fact that \( \gamma_5 \) commutes with \( \alpha \), we have

\[ \int d^4x \mathcal{L}_J = -2 \text{Tr} \beta \gamma_5 e^{i\tilde{D}/\Lambda^2}. \]

(22)

The terms proportional to \( \alpha \) in eq. (19) have thus canceled out.

We can find out which kind of chiral anomaly this choice of \( \tilde{D} \) leads to. After a standard calculation we get (see e.g. [11])

\[ \mathcal{L}_J = \frac{1}{4\pi^2} \text{tr} \left[ \epsilon_{\mu\nu\rho\sigma} \left( \frac{1}{4} F_{\mu\nu}^{\rho\sigma} F_{\rho\sigma}^{\mu\nu} + \frac{1}{12} F_{A}^{\mu\nu} F_{A}^{\rho\sigma} \right. \right. \]
\[ \left. \left. - \frac{2i}{3} iA^\mu A^\nu F_{\rho\sigma}^{\mu\nu} F_{\rho\sigma}^{\mu\nu} - \frac{2}{3} iA^\mu F_{\rho\sigma}^{\mu\nu} A^\sigma - \frac{2}{3} iF_{\rho\sigma}^{\mu\nu} A^\rho A^\sigma \right. \right. \]
\[ \left. \left. - \frac{8}{3} iA^\mu A^\nu A^\rho A^\sigma \right) \right] + \frac{1}{3} \{ D_\mu^V A_\nu, A^\nu \} + \frac{1}{3} \{ D_\mu^V A_\nu, A^2 \} + \frac{2}{3} i [D_\mu^V F_{\mu\nu}^V, A_\nu] \]
\[ - \frac{1}{6} [F_{\mu\nu}^{\mu\nu}, F_{\mu\nu}^{\mu\nu} + \frac{1}{4} D_\mu^V D_\nu^V A^\mu - 2A_\mu D_\nu^V A^\nu A^\mu] \]

(23)

Here \( F_{\mu\nu}^{\mu\nu} \) and \( F_{\mu\nu}^{\mu\nu} \) are the Bardeen tensors,

\[ F_{\mu\nu}^{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu + i[V_\mu, V_\nu] + i[A_\mu, A_\nu], \]
\[ F_{\mu\nu}^{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i[V_\mu, A_\nu] + i[A_\mu, V_\nu], \]

(24)

and \( D_\mu^V \equiv \partial_\mu + i[V_\mu, \cdot] \). The terms proportional to the \( \epsilon \)-tensor is the familiar Bardeen anomaly [12], while the other terms are of even intrinsic parity [13]. The Bardeen anomaly is known to be consistent [2], hence the regularization resulting from this choice for \( \tilde{D} \) is a consistent regularization.

(Actually, this is not the whole story: Within the proper time scheme one also gets a term proportional to \( \Lambda^2 \) (and terms proportional to \( 1/\Lambda^2, 1/\Lambda^4, \ldots \)) in addition to the \( \Lambda \)-independent terms in eq. (23). It is necessary to somehow remove this term if we intend to take \( \Lambda \) to infinity at the end of the calculation. This can be done by the Pauli–Villars inspired regularization described in [11]; see also [8]. This implies a “theorem” that we can simply drop all \( \Lambda \)-dependent terms.)

It is easy to see that not only the terms proportional to \( \alpha \) in eq. (19) cancel out, but also the terms proportional to \( a_\mu \) and \( \omega_{\mu\nu} \) in eq. (20). Hence the quantity \( \Delta \) for the Lorentz transformations and translations vanishes for this regularization, and the theory is Poincaré invariant, as expected.

6 Covariant regularization

Let us now instead make the choice

\[ \tilde{D} = D^c \equiv -CD^TC^{-1} \]
\[ = i\partial - V + A\gamma_5 + \mu. \]

(25)
This is the “charge conjugate” of $D$; $C$ is the charge conjugation matrix and transposition is with respect to the Dirac matrix structure.

It is now no longer true that terms proportional to $\alpha$ in eq. (19) automatically cancel. For a combination of a phase rotation with parameter $\alpha = \alpha^a t^a$ and chiral phase rotation with parameter $\beta = \beta^a t^a$ we get

$$L_J = \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr} \left[ \alpha (F_V^{\mu\nu} F_A^{\rho\sigma} + F_A^{\mu\nu} F_V^{\rho\sigma}) + \beta (F_V^{\mu\nu} F_V^{\rho\sigma} + F_A^{\mu\nu} F_A^{\rho\sigma}) \right].$$

(26)

This is the expression for the covariant anomaly [4].

The presence of a term proportional to $\alpha$ means that the current is not conserved. This is well known for a theory that is regularized to have a covariant anomaly [4, 1].

This form of $\tilde{D}$ is similar to that of $D^\dagger$ in the Euclidean formulation, where also the sign of $\not{A}\gamma_5$ is changed relative to $i\not{\partial} - \not{V}$. In the Euclidean case the operator $D^\dagger$ has a special status, since it is used for the construction of the positive operators $DD^\dagger$ and $D^\dagger D$. The positivity of these operators will then ensure the convergence of the upper limit of the proper time integral, instead of the $i\epsilon$ which has the same effect in Minkowski space. For this reason the Euclidean formalism produces “naturally” the covariant anomaly, and it takes a considerable amount of work to produce other anomalies, such as the consistent one [4].

Within this regularization, we can now calculate the Jacobian that corresponds to the translations and Lorentz transformations. The procedure, which also includes some nonstandard elements, is essentially the one described in ref. [8], where it was used in a slightly different context. The result is

$$L_\omega = \frac{1}{8\pi^2} \omega_{\mu\nu} \text{tr} \left[ i \Box \tilde{F}_A^{\mu\nu} + \frac{1}{6} i \partial_\rho [V^{\rho}, \tilde{F}_A^{\mu\nu}] + \frac{1}{6} i \partial_\rho [A^{\rho}, \tilde{F}_V^{\mu\nu}] + x^\mu (2V^{\nu} F_V \tilde{F}_A + A^{\nu} F_V \tilde{F}_V) \right]$$

(27)

and

$$L_a = \frac{1}{8\pi^2} a_\mu \text{tr} (2V^{\nu} F_V \tilde{F}_A + A^{\nu} F_V \tilde{F}_V).$$

(28)

Even for $\omega_{\mu\nu}$ and $a_\mu$ constants, there are terms in eqs. (27) and (28) that are not total derivatives, i.e. of the form $\partial_\mu A^\mu$. In other words, $\Delta \neq 0$ for both Lorentz transformations and translations. Thus, as advertized, Poincaré symmetry is violated in this case.

7 Conclusion

We have seen that in the nonabelian $VA$-theory considered in this paper, Poincaré symmetry survives quantization for a regularization that leads to the consistent Bardeen form of the chiral anomaly, while it is anomalously broken for a regularization that leads to the covariant chiral anomaly. Of course many other choices for the regularization operator $\tilde{D}$ can be made. (Indeed many other choices than proper time regularization can be made for the scheme itself.) In general Poincaré symmetry is violated since eq. (20) does not vanish for general choices for $\tilde{D}$.

Manohar, who investigated Poincaré anomalies in a chiral gauge theory, pointed out that chiral anomalies and Poincaré anomalies cancel simultaneously, since both are proportional to the $\delta$-symbols. A similar result seems to be true here, i.e. that $\Delta \neq 0$ for the Poincaré transformations whenever the $\alpha$’s cancel out from eq. (14). The nonvanishing terms in $\Delta$ come from the derivative operators in $P_\mu$ and $J_{\mu\nu}$. 

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