Learning Mixtures of Spherical Gaussians via Fourier Analysis

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April 14, 2020

Abstract
Suppose that we are given independent, identically distributed samples $x_i$ from a mixture $\mu$ of no more than $k$ of $d$-dimensional spherical gaussian distributions $\mu_i$ with variance 1, such that the minimum $\ell_2$ distance between two distinct centers $y_i$ and $y_j$ is greater than $\sqrt{d}\Delta$ for some $c \leq \Delta$, where $c \in (0,1)$ is a small positive universal constant. We develop a randomized algorithm that learns the centers $y_i$ of the gaussians, to within an $\ell_2$ distance of $\delta < \frac{\Delta\sqrt{d}}{2}$ and the weights $w_i$ to within $cw_{\text{min}}$ with probability greater than $1 - \exp(-k/c)$. The number of samples and the computational time is bounded above by $\text{poly}(k,d,\frac{1}{\delta})$. Such a bound on the sample and computational complexity was previously unknown when $\omega(1) \leq d \leq O(\log k)$. When $d = O(1)$, this follows from work of Regev and Vijayaraghavan. These authors also show that the sample complexity of learning a random mixture of gaussians in a ball of radius $\Theta(\sqrt{d})$ in $d$ dimensions, when $d$ is $\Theta(\log k)$ is at least $\text{poly}(k,\frac{1}{\delta})$, showing that our result is tight in this case.

1 Introduction

Designing algorithms that estimate the parameters of an underlying probability distributions is a central theme in statistical learning theory. An important special instance of this learning problem is the case when the underlying distribution is known to be a finite mixture of Gaussian distributions in $d$-dimensional euclidean space, because such mixtures are popular models for high-dimensional
data clustering and learning mixture of Gaussians in an unsupervised setting has been a topic of intensive research for the last few decades.

In its most explicit form, the underlying problem is as follows: we have access to random samples drawn independently as per some Gaussian mixture $\mu := \omega_1 \mu_1 + \cdots + \omega_k \mu_k$, where $(\omega_1, \cdots, \omega_k)$ is some probability vector, and each $\mu_l$ is a Gaussian density in $\mathbb{R}^d$, with mean $y_l \in \mathbb{R}^d$ and covariance $\Sigma_l$. The main task is to estimate the parameter set $\{(\omega_1, y_1, \Sigma_1), \cdots, (\omega_k, y_k, \Sigma_k)\}$ of the density function $\mu$, within a pre-specified accuracy $\delta$. For the purpose of this paper, we will restrict to the scenario where all the Gaussian components are spherical.

Many of the earlier approaches to this learning problem were based on local search heuristics, e.g. the EM algorithm and $k$-means heuristics, that resulted in weak performance guarantees. In [3], Dasgupta presented the first provably correct algorithm for learning a mixture of Gaussians, with a common unknown covariance matrix, using only polynomially many (in dimension as well as number of components) samples and computation time, under the assumption that the minimum separation between the centers of component Gaussians is at least $\Omega(\text{polylog}(kd)\sqrt{d})$. In a consequent work, Dasgupta and Schulman showed a variation of EM to work with minimum separation only $\text{polylog}(kd)d^{\frac{1}{4}}$, when the components are all spherical. Subsequently, many more algorithms (see [9] and the references therein) with improved sample complexity and/or computational complexity have since been devised, that work on somewhat weaker separation assumption; in particular, the SVD-based algorithm of Vempala and Wang [11] learns a mixture of spherical Gaussians with poly-sized samples and polynomially many steps, under the separation assumption

$$\min_{1 \leq i \neq j \leq k_0} \|y_i - y_j\|_2 \geq C \max\{\sigma_1, \sigma_j\} \left( \left(\min(k, d) \log (dk/\delta)\right)^{\frac{1}{4}} + (\log (dk/\delta))^{\frac{1}{2}} \right)$$

In another line of work, Kalai-Moitra-Valiant [7] and Moitra-Valiant [9], the question of polynomial learnability of arbitrarily separated mixture of Gaussians (and more general families of distributions have been investigated by Belkin and Sinha in [2]). It has been established that, for Gaussian mixture with a fixed number of components, there is an algorithm that runs in polynomial time and uses polynomially many samples to learn the parameters of the mixture to any desired accuracy, with arbitrarily high probability.

In [10], Regev and Vijayaraghavan considered the question of obtaining a lower bound on the separation of centers necessary for the polynomial learnability of Gaussian mixtures. They devised an iterative algorithm for amplifying accuracy of parameter estimation that, given initializer $y_1^0, \cdots, y_{k_0}^0$ and a desired accuracy parameter $\delta > 0$, uses polynomially many samples from the underlying mixture and computation steps to return $y_1', \cdots, y_{k_0}'$, that lie within Hausdorff distance at most $\delta$ from the true centers; for more details, see Theorem 4.1 of the full version of [10]. One of their results establishes that, in constant dimension $d = O(1)$, with minimum separation at least $\Omega(1)$, any uniform mixture
of spherical Gaussians can be learned to desired accuracy $\delta$ (with high probability) in number of samples and computation time depending polynomially on the number of components and $\delta^{-1}$.

In the present paper we consider the question of learning mixture of Gaussians, with minimum separation $\Omega(\sqrt{d})$, in number of samples and computational time that depends polynomially on the ambient dimension $d$, and the number of components $k$. Here is the main theorem:

**Main Theorem:** Given a mixture of $k$ spherical Gaussians in $\mathbb{R}^d$, with identical variances, such that all the mixing weights are in $[c, c^{-1}]$ for some universal constant $c \in (0, 1)$, and the separation of the centers of the components satisfies

$$\min_{1 \leq i \neq j \leq k} ||y_i - y_j||_2 \geq c\sqrt{d},$$

there is an algorithm that recovers the parameters (i.e. the centers and the respective weights) up to accuracy $\delta$ in time $\text{poly}(k, d, \delta^{-1})$.

In particular, we generalize the main upper bound in [10] (see theorem 5.1 of [10]) to arbitrary dimension ($d$) and number of components ($k$).

## 2 Overview

We first observe that if two clusters of centers are very far (i.e. the minimum distance of a center in one cluster is at least $\sqrt{d}$ away from every center of the other cluster), then the samples are unambiguously of one cluster only. This is shown in Lemma 1 allowing us to reduce the question to the case when a certain proximity graph defined on the centers is a connected graph.

In algorithm *LearnMixture*, we use the Johnson-Lindenstrauss lemma and project the data in the ambient space to $d + 1$ carefully chosen subspaces of dimension at most $O(1 + \min(\log k, d))$, and show that if we can separate the Gaussians in these subspaces, the resulting centers can be used to obtain a good estimate of the centers in the original question. Thus the question is further reduced to one where the dimension $d = O(\log k)$.

Next, in Lemma 2 we show that if the number of samples is chosen appropriately, then all the $k_0$ centers are with high probability contained in a union of balls $B_i$ of radius $2\sqrt{d}$ centered around the data points. This allows us to focus our attention to $n$ balls of radius $2\sqrt{d}$.

The main idea is that in low dimensions, it is possible to efficiently implement a deconvolution on a mixture of identical spherical Gaussians having standard deviation $\sigma = 1$, and recover a good approximation to a mixture of Gaussians with the same centers and weights, but in which each component now has standard deviation $\Delta/\sqrt{d} < c\Delta/\sqrt{d}$, where $\Delta$ is the minimum separation between two centers. Once this density is available within small $L_\infty$ error, the
local maxima can be approximately obtained using a robust randomized zeroth
order convex optimization method developed in [3] started from all elements of
a sufficiently rich $\ell^\infty$ net on $\bigcup_i B_i$ (which has polynomial size by Lemma [3]),
and the resulting centers are good approximations (i.e. within $c d^{-\frac{5}{2}}$ of the true
centers in Hausdorff distance) of the true centers with high probability. We then
feed these $d^{-\frac{5}{2}}$ approximate centers into the iterative procedure developed by
Regev and Vijayaraghavan in [10] and by Theorem 4.1 of the full version of that
paper, a seed of this quality, suffices to produce in $\text{poly}(k,d,\delta^{-1})$ time, a set
of centers that are within $\delta$ of the true centers in Hausdorff distance, together
with good approximations of the weights.

The deconvolution is achieved by convolving the empirical measure $\mu_e$ with
the Fourier transform of a certain carefully chosen $\hat{\zeta}$. The function $\hat{\zeta}$ is up to
scalar multiplication, the reciprocal of a Gaussian with standard deviation $\sqrt{1 - \Delta^2}$ restricted to a ball of radius $(\sqrt{\log k} + \sqrt{d})\Delta^{-1}$. It follows from
Lemma [3] that the effect of this truncation on the deconvolution process can
be controlled. The pointwise evaluation of the convolution is done using the
Monte Carlo method. The truncation plays an important role, because without
it, the reciprocal of a Gaussian would not be in $L^p$ for any $p \in [1, \infty]$, leading
to difficulties in the Fourier analysis.

3 Preprocessing and Reduction

Suppose that we are given independent, identically distributed samples $x_1,
\ldots, x_n$ from a mixture $\mu$ of no more than $k d$-dimensional spherical gaussian
distributions $\mu_i$ with variance 1, such that the minimum $\ell_2$ distance between
two distinct centers $y_i$ and $y_j$ is greater than $c \Delta$ for some $c \leq \Delta$, where $c$
is a small positive universal constant. Suppose that $\mu := \sum_{l=1}^{k_0} w_l \mu_l$, where $\mu_l$ are $d$-dimensional Gaussians with center at $y_l$, $k_0 \leq k$, and $(w_1, \ldots, w_{k_0})$ is a probability vector such that $w_{\text{min}} := \min_{l \in [k_0]} w_l$ satisfies $w_{\text{min}} \geq \frac{1}{k}$.

Notation 1. For $l \in (1/10)\mathbb{Z} \cap [0,\infty)$, we denote by $C_l \geq 10$ and $c_l \leq 1/10$,
positive universal constants such that $C_l C_l = 1$, and $C_l$ depends on the values
of constants in $\{C_j | j \in (1/10)\mathbb{Z} \cap [0,1]\}$. If $C_l$ is undefined for some index
$l \in (1/10)\mathbb{Z} \cap [0,\infty)$, we set $C_l := 1$.

Let $Y := \{y_1, \ldots, y_{k_0}\}$ be the set of centers of the component gaussians in
the mixture $\mu$:

$$\mu(x) := (2\pi)^{-\frac{d}{2}} \sum_{l=1}^{k_0} w_l \exp \left( -\frac{||x-y||^2}{2} \right)$$

Let us fix $1 - \eta := \frac{\eta_0}{10}$, to be the success probability we will require. This can
be made $1 - \eta_0$ that is arbitrarily close to 1 by the following simple clustering technique in the metric space associated with Hausdorff distance, applied to the outputs of $100(\log \eta_0^{-1})$ runs of the algorithm.
3.0.1 Algorithm Boost

1. Find the median of all the number of centers output by the different runs of the algorithm, and set that to be \( k_0 \).

2. Pick a set of centers \( Y \) (that is the output of one of the runs) which has the property that \( |Y| = k_0 \) and at least half of the runs output a set of centers that is within a hausdorff distance of less than \( \frac{\Delta}{\sqrt{d}} \) to \( Y \). It is readily seen that provided each run succeeds with probability \( 1 - \eta \), this clustering technique succeeds in producing an acceptable set of centers with probability at least \( 1 - \eta_0 \).

Let \( \{(x_1, y(x_1)), \ldots, (x_n, y(x_n))\} \) be a set of \( n \) independent identically distributed random samples from \( \mu \) generated by first sampling the mixture component \( y(x_l) \) with probability \( w_l \) and then picking \( x_l \) from the corresponding gaussian. With probability 1, all the \( x_l \) are distinct, and this is part of the hypothesis in the lemma below.

**Lemma 1.** Let \( G \) be a graph whose vertex set is \( X = \{x_1, \ldots, x_n\} \), in which the vertices \( x_l \) and \( x_j \) are connected by an edge if the \( l_2 \) distance between \( x_l \) and \( x_j \) is less than \( 2\sqrt{3\ln(C_1 dn)} \). Decompose \( G \) into the connected components \( G_1, \ldots, G_r \) of \( G \). Then, the probability that there exist \( l \neq j \) and \( x \in G_l, x' \in G_j \) such that \( y(x) = y(x') \) is less than \( \eta/100 \).

**Proof.** By bounds on the tail of a \( \chi^2 \) random variable with parameter \( d \) (see equation (4.3) of [8]), the probability that \( ||x_l - y(x_l)|| \geq \sqrt{3\ln(C_1 dn)} \) can be bounded from above as follows.

\[
\Pr \left[ ||x - y(x)|| \geq \sqrt{3\ln(C_1 dn)} \right] \leq (3e \ln(C_1 dn))^{\frac{d}{2}} (C_1 dn)^{-\frac{d}{2}} \tag{1}
\]

Union bound yields

\[
\Pr \left[ \exists x_l \text{ such that } ||x_l - y(x_l)|| \geq \sqrt{3\ln(C_1 dn)} \right] \\
\leq n \left( 3e \ln(C_1 dn) \right)^{\frac{d}{2}} (C_1 dn)^{-\frac{d}{2}} \\
\leq \frac{\eta}{100}
\]

By Principal Component Analysis (PCA) it is possible to find a linear subspace \( S_k \) of dimension \( k \), such that all the \( y_l \) are within \( \frac{\sqrt{2\Delta}}{C} \) of \( S_k \) in \( l_2 \) with probability at least \( 1 - \frac{\eta}{100} \) using \( poly(d, k) \) samples and computational time (see Appendix C of [10]). PCA has been used previously in this context (see [11]). We then take the push forward of \( \mu \) via an orthoprojection of \( \mathbb{R}^d \) on to \( S_k \), and work with that instead. This allows us to reduce the dimension \( d \) to \( k \) (if \( d \) started out being greater than \( k \)), while retaining the same \( \Delta \) to within a multiplicative factor of 2.
Remark 1. In what follows, we suppose that all the centers are contained within an origin centric $\ell_2$ ball of radius $R := k \sqrt{d \ln(C_1 dn)}$. This can be ensured by Lemma 7.

3.0.2 Algorithm LearnMixture

1. Let $e_1, \ldots, e_d$ be an orthonormal set of vectors, sampled uniformly at random from the haar measure.

2. Define $\tilde{d} = \min(d, O(\log k))$ dimensional subspace $A_{\tilde{d}}$ to be the span of $e_1, \ldots, e_{\tilde{d}}$. By the Johnson-Lindenstrauss Lemma (see theorem 5.3.1 of [12]), with probability greater than $1 - \frac{\eta}{100}$, the distance between the projections of any two centers is at least $\left(\Delta^2 \right)^{\frac{1}{\tilde{d}}}$.

3. Now, use the low dimensional gaussian learning primitive Algorithm FindSpikes from Subsection 4.1 on $\{\Pi_{\tilde{d}}x_i\}$ that solves the $\tilde{d} + O(1)$ dimensional problem with high probability, if the distance between any two centers is at least $\left(\Delta^2 \right)^{\frac{1}{\tilde{d}}}$. If this fails to produce such a set of centers, go back to 1, else we recover the projections on to $A_{\tilde{d}}$ of the centers $y^{(d)}_1, \ldots, y^{(d)}_k$, to within a hausdorff distance of $\frac{4}{\sqrt{d}}$.

4. For any fixed $l \geq \tilde{d} + 1$, let $A_l$ denote the span of $e_1, \ldots, e_{\tilde{d}}$ augmented with the vector $e_l$. Suppose that we have succeeded in identifying the projections of the centers on to $A_l$ for $\tilde{d} + 1 \leq l \leq d$ to within $\Delta k^{-C_3}$ in $\ell_2$ distance with high probability. Together with the knowledge of $c^{(d)}_1, \ldots, c^{(d)}_k$, this allows us to patch up these projections and give us the centers $y_1, \ldots, y_k$ to within a Hausdorff distance of $\delta$ with high probability.

By the above discussion, it is clear that it suffices to consider the case when

$$d \leq C_{1,5}(\log k).$$

4 The case of $d \leq C_{1,5}(\log k)$

Lemma 2. The following statement holds with probability at least $1 - \frac{n}{100}$: if

$$n \geq 1 + \frac{k}{c} \log \left[\frac{300k}{\eta}\right] \log \left[\frac{300k \log \left[\frac{300k}{\eta}\right]}{\eta}\right],$$

and $x_1, \ldots, x_n$ are independent random $\mu$-samples, then

$$\{y_1, \ldots, y_k\} \subseteq \bigcup_{l \in [n]} B_2(x_l, 2\sqrt{d}).$$

Proof. Recall that $w_{\min} \geq c k^{-1}$. Suppose that $m > k$ random independent $\mu$-samples, denoted as $x_1, \ldots, x_m$, have been picked up. Let $C := C(k)$ be
a positive integer valued function. For any \( l \in [k_0] \), the probability – that \( x_1, \ldots, x_m \) contain no \( \mu_l \)-sample – is at most \( (1 - e^{ck^{-1}})^m \leq e^{-meck^{-1}} \). Thus, the probability – that \( x_1, \ldots, x_m \) contain no sample from at least one Gaussian component in \( \mu \) – is at most \( e^{lnk - meck^{-1}} \). We ensure this probability is at most \( \eta \) by having

\[
m \geq \frac{k}{c} \left[ \log \left( \frac{300Ck}{\eta} \right) \right]
\]

It follows that, if at least

\[
n_0 := \frac{Ck}{c} \left[ \log \left( \frac{300Ck}{\eta} \right) \right]
\]

random independent \( \mu \)-samples were picked up, then with probability at least \( 1 - \frac{\eta}{300} \) all the components were needed to be sampled at least \( C \) times.

Let \( E \) denote the event that \( n_0 \) random independent \( \mu \)-samples contain at least \( C \) points from each Gaussian component. For \( l \in [k_0] \), let \( A_l \) be the event that none of the \( n_0 \) random samples satisfy \( ||x_j - y_l|| < 2\sqrt{d} \). Applying Gaussian isoperimetric inequality (equation (4.3) of [6]) and union bound, we obtain

\[
P \left[ \bigcup_{l \in [k_0]} A_l \bigg| E \right] \leq 2^{\log k - C d}
\]

Thus, letting

\[
C \geq \log \left( \frac{300k}{\eta} \right)
\]

it follows that

\[
P \left[ \bigcap_{l \in [k_0]} A_l^c \bigg| E \right] \geq P \left[ \bigcap_{l \in [k_0]} A_l^c \bigg| E \right] P \left[ E \right]
\]

\[
\geq \left( 1 - \frac{\eta}{300} \right)^2
\]

provided

\[
n \geq \frac{k}{c} \left[ \log \left( \frac{300k}{\eta} \right) \right] \left[ \log \left( \frac{300k \log \left( \frac{300k}{\eta} \right)}{\eta} \right) \right]
\]

Let \( \mathcal{F} \) denote the Fourier transform. For any function \( f \in L^2(\mathbb{R}^d) \), we have \( \hat{f} = \mathcal{F}(f) \), where

\[
\hat{f}(\xi) = \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_{\mathbb{R}^d} f(x)e^{-ix \cdot \xi} dx.
\]
By the Fourier inversion formula,

$$f(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{\mathbb{R}^d} \hat{f}(\xi) e^{i\xi \cdot x} d\xi,$$

where these are not necessarily Lebesgue integrals, but need to be interpreted as improper integrals. Thus,

$$\hat{f}(\xi) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \lim_{R \to \infty} \int_{|x| \leq R} f(x) e^{-i\xi \cdot x} dx,$$

and

$$f(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \lim_{R \to \infty} \int_{|x| \leq R} \hat{f}(\xi) e^{i\xi \cdot x} d\xi,$$

where the limit is taken in the $L^2$ sense.

Let $\gamma(z) := \left(\frac{1}{\sqrt{2\pi}}\right)^d e^{-|z|^2/2}$ denote the standard Gaussian in $d$ dimensions. Then, $\hat{\gamma} = \mathcal{F}(\gamma) = \gamma$. Let $\tilde{\Delta}$ equal $\Delta/C_{3.2}$. Let

$$\hat{s}(w) := \gamma(w \Delta) \mathbb{1}\left(\left(\sqrt{C_{3.5} \ln k} + \sqrt{d}\right) B_2(0, \Delta^{-1})\right),$$

where $B_2(0, \Delta^{-1})$ is the Euclidean ball of radius $\Delta^{-1}$ and $\mathbb{1}(S)$ the indicator function of $S$, and let $s = \mathcal{F}^{-1}(\hat{s})$. Let $\gamma_\sigma(w) := \sigma^{-d} \gamma(\sigma^{-1} w)$ denote the spherical Gaussian whose one dimensional marginals have variance $\sigma^2$.

**Lemma 3.** For all $z \in \mathbb{R}^d$, we have

$$|s(z) - \gamma_{\tilde{\Delta}}(z)| \leq (2\pi \tilde{\Delta}^2)^{-\frac{d}{2}} k^{-\frac{C_{3.5}}{2}}.$$

**Proof.** Let $d\hat{k}^2$ be a $\chi^2$ random variable with $d$ degrees of freedom, i.e., the sum of squares of $d$ independent standard Gaussians.

By equation (4.3) of [6], we know that

$$\mathbb{P} \left[ \hat{k} \geq \sqrt{\frac{C_{3.5} \ln k}{d}} + 1 \right] = \mathbb{P} \left[ d\hat{k}^2 - d \geq C_{3.5} \ln k + 2\sqrt{C_{3.5} d \ln k} \right] \leq \mathbb{P} \left[ d\hat{k}^2 - d \geq C_{3.5} \ln k + 2\sqrt{C_{3.5} d \ln k} \right] \leq \exp \left( -\frac{C_{3.5} \ln k}{2} \right) = k^{-\frac{C_{3.5}}{2}} \tag{8}$$
For \( w \in \text{supp}(\hat{s}) \), one has \( \hat{s}(w) = \hat{\Delta}^{-d}\gamma(1/\hat{\Delta}) \). Therefore,

\[
\|\hat{s} - \hat{\Delta}^{-d}\gamma(1/\hat{\Delta})\|_{L^1} = \int_{|\Delta w| \geq \sqrt{C_3 \ln k} + \sqrt{d}} \gamma(\Delta w) \, dw
\]

\[
= \Delta^{-d} \mathbb{P}\left[ \hat{k} \geq \sqrt{\frac{C_3 \ln k}{d}} + 1 \right]
\]

\[
\leq \Delta^{-d} k^{-\frac{C_3}{2}}
\]

The lemma follows from fact that \( \mathcal{F}^{-1} \) is a bounded operator with operator norm \((2\pi)^{-\frac{d}{2}}\) from \( L^1 \) to \( L^\infty \).

Let \( \mu_e \) denote the uniform probability measure on \( \{x_1, \ldots, x_n\} \). Let \( \ast \) denote convolution in \( \mathbb{R}^d \). Note that the Fourier convolution identity is

\[
(2\pi)^{-\frac{d}{2}} \mathcal{F}(f \ast g) = \hat{f} \hat{g}.
\]

Let \( \hat{\zeta} := \hat{\gamma}^{-1} \cdot \hat{s} \), and \( \zeta = \mathcal{F}^{-1}(\hat{\zeta}) \). We will recover the centers and weights of the gaussians from \( \zeta \ast \mu_e = (2\pi)^{\frac{d}{2}} \mathcal{F}^{-1}(\hat{\zeta} \cdot \mu_e) \). The heuristics are as follows.

Let \( \nu \) denote the unique probability measure satisfying \( \gamma \ast \nu = \mu \). Thus

\[
\nu = \sum_{j=1}^{k_0} w_j \delta_{y_j},
\]

where \( \delta_{y_j} \) is a dirac delta supported on \( y_j \). From this it follows, roughly speaking, that inside \( \text{supp}(\hat{s}) \), we get

\[
\hat{\nu}(w) = (2\pi)^{-d} \hat{\gamma}(w)^{-1} \mathbb{E}_{X \sim \mu} [e^{-iX \cdot w}]
\]

\[
\approx (2\pi)^{-\frac{d}{2}} \hat{\gamma}(w)^{-1} \hat{\mu}_e(w)
\]

\[
\Rightarrow \hat{s}(w) \hat{\nu}(w) \approx (2\pi)^{-\frac{d}{2}} \hat{\zeta}(w) \hat{\mu}_e(w)
\]

pointwise, and this should (roughly) yield

\[
\hat{s} \ast \nu \approx (2\pi)^{-\frac{d}{2}} \hat{\zeta} \ast \mu_e
\]

On the other hand, notice that Lemma \[\text{3}\] shows that \( s \ast \nu \approx \gamma \ast \nu \), and because the spikes of \( \nu \) are approximately (up to scaling) the spikes of \( \gamma \ast \nu \), we restrict to learning the spikes of \( \gamma \ast \nu \) by accessing an approximation via \((2\pi)^{-\frac{d}{2}} \hat{\zeta} \ast \mu_e \).

For notational convenience, we write \( \xi_e := (2\pi)^{-\frac{d}{2}} \hat{\zeta} \ast \mu_e \).

In order to formalize this, we proceed as follows. We will employ Hoeffding’s inequality for \( \mathbb{C} \)-valued random variables:
Lemma 4 (Hoeffding). Let $b > 0$. Let $Y_1, \cdots, Y_r$ be independent identically distributed $\mathbb{C}$-valued random variables such that $|Y_j| \leq b^{-1}$ for all $j \in [r]$. Then

$$
P \left[ \frac{1}{r} \sum_{j=1}^r (\mathbb{E}[Y_j] - Y_j) \geq t \right] \leq e^{-c_{0.1} r t^2 b^2}$$

(9)

where $c_{0.1}$ is a universal constant.

We write $B := \text{supp}(\hat{s})$, so that $z \in B$ if and only if $z \in \mathbb{R}^d$ satisfies $||\Delta z|| \leq \sqrt{C_{3.5} \ln k} + \sqrt{d}$.

The following proposition allows us to construct a (random) black box oracle that outputs a good additive approximation of $\nu \star \gamma_\Delta$ at any given point $x$.

Proposition 1. Let $z_1, \cdots, z_m$ be independent, random variables drawn from the uniform (normalized Lebesgue) probability measure on $B$. Let $x_1, \cdots, x_n$ be independent $\mu$-distributed random points. If

$$m \geq C_{3.6} k^{C_{3.6} + C_{3.5}} \ln \left( C_{3.6} k^{C_{3.6} + C_{3.5}} \ln k \right),$$

(10)

then, for any $x \in \mathbb{R}^d$, the random variable

$$f_x := \frac{\text{vol}(B)}{m} \sum_{l \in [m]} e^{i x \cdot z_l} \gamma(z_l) \Delta e^{i ||z_l||^2 / 2} \hat{\mu}(z_l)$$

(11)

satisfies the following inequality with probability at least $1 - 8k^{-C_{3.5}}$:

$$|\langle \gamma_\Delta \star \nu \rangle(x) - \Re f_x| \leq 3k^{-C_{3.5}}$$

where $\Re f_x$ denotes the real part of $f_x$.

Proof. For any $x \in \mathbb{R}^d$, one has

$$\left| (\gamma_\Delta \star \nu)(x) - (2\pi)^{-d} \sum_{j=1}^{k_0} w_j \int_{B} e^{-\frac{\Delta z ||z||^2}{2}} e^{i(x-y_j) \cdot z} \, dz \right|$$

$$\leq (2\pi)^{-d} \int_{\mathbb{R}^d \setminus B} e^{-\frac{\Delta z ||z||^2}{2}} \, dz$$

$$\leq k^{-C_{3.8}}$$

(12)

Hence, it suffices to estimate

$$I_x := (2\pi)^{-d} \sum_{j=1}^{k_0} w_j \int_{B} e^{-\frac{\Delta z ||z||^2}{2}} e^{i(x-y_j) \cdot z} \, dz$$

10
Next, one has

\[ I_x = (2\pi)^{-\frac{d}{2}} \int_B e^{ix \cdot z} \left( \hat{s}(z) \sum_{j=1}^{k_0} w_j e^{-iy_j \cdot z} \right) \, dz \]

\[ = (2\pi)^{-\frac{d}{2}} \text{vol}(B) \mathbb{E}_z e^{ix \cdot z} \left( \hat{s}(z) \sum_{j=1}^{k_0} w_j e^{-iy_j \cdot z} \right) \]

where \( z \) is a sample from the uniform probability distribution on \( B \). For brevity of notation, write

\[ \phi(z) = (2\pi)^{-\frac{d}{2}} \text{vol}(B) \left( \hat{s}(z) \sum_{j=1}^{k_0} w_j e^{-iy_j \cdot z} \right) \]

so that \( I_x = \mathbb{E}_z [e^{ix \cdot z} \phi(z)] \). By Ramanujan’s approximation of \( \Gamma \) (see theorem 1 of [8]) we get

\[ \text{vol}(B) = \Delta^{-d} \left( 1 + \sqrt{\frac{C_{3.5} \ln k}{d}} \right)^d \frac{(d\pi)^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2} + 1\right)} \]

\[ \leq (2\pi)^{\frac{d}{2}} k^{C_{1.5} \ln(2C_{2.5}) + C_{3.5}} \] (13)

so that

\[ |\phi(z)| \leq (2\pi)^{-\frac{d}{2}} \text{vol}(B) |\hat{s}(z)| \]

\[ < k^{C_{1.5} \ln(\frac{C_{3.5}}{2}) + C_{3.5}} \]

\[ \leq k^{2C_{3.5}} \]

If \( z_1, \ldots, z_m \) are independent (Lebesgue) uniformly distributed points in \( B \), then by Hoeffding’s inequality (9), one has

\[ \mathbb{P} \left[ I_x - \frac{1}{m} \sum_{l=1}^{m} e^{ix \cdot z_l} \phi(z_l) \geq k^{-2C_{3.5}} \right] \leq k^{-2C_{3.5}} \] (14)

if we set

\[ m = C_{0.1} C_{3.5} k^{8C_{3.5} + \frac{2C_{3.5} C_{7.5}}{2} + 2C_{1.5} \ln(2C_{2.5}) + C_{1.5} \ln(2\pi) \ln k} \] (15)

Recalling that \( ||\hat{\nu}||_{L^\infty}, ||\hat{\mu}||_{L^\infty} \leq (2\pi)^{-\frac{d}{2}} \), and

\[ \hat{\nu}(z) = \mathbb{E}_{X \approx \hat{\mu}} \left( (2\pi)^{-\frac{d}{2}} e^{-iX \cdot z + \frac{||z||^2}{2}} \right) = \mathbb{E} \left( e^{-\frac{||z||^2}{2}} \hat{\mu}(z) \right) , \]

\[ \hat{\mu}(z) = \mathbb{E}_{Y \approx \hat{\nu}} \left( (2\pi)^{-\frac{d}{2}} e^{-iY \cdot z + \frac{||z||^2}{2}} \right) = \mathbb{E} \left( e^{-\frac{||z||^2}{2}} \hat{\nu}(z) \right) , \]
for any fixed $z \in B$, applying Hoeffding’s inequality (9) once again, we obtain
\[
P \left[ \left| \hat{\nu}(z) - e^{\frac{||z||^2}{2}} \hat{\mu}_e(z) \right| > \frac{k^{-C_{3.5}}}{\text{vol}(B)} \right] < \exp \left( -\frac{c_{0,1} n}{(2\pi)^{-d/2} k^{2C_{3.5} + C_{3.5}} \text{vol}^2(B)} \right)
\]
In particular, letting
\[
C_{3.6} \geq C_{0,1} C_{3.5} + C_{1.5} \left( \ln(2\pi) + 2 \ln \left( \frac{2C_{3.2}}{e} \right) \right) + C_{3.5} \left( 8 + \frac{2C_{3.2}^2}{e^2} \right)
\]
and
\[
n \geq C_{3.6} k^{C_{3.6} + C_{3.5}} \ln \left( C_{3.6} k^{C_{3.6} + C_{3.5}} \ln k \right)
\]
\[
m = C_{3.6} k^{C_{3.6}} \ln k
\]
one has
\[
P \left[ \left| \hat{\nu}(z) - e^{\frac{||z||^2}{2}} \hat{\mu}_e(z) \right| > \frac{k^{-C_{3.5}}}{\text{vol}(B)} \right] < m^{-1} k^{-C_{3.5}}.
\] (17)

By an application of the union bound, equations (12), (14), and (17) give us this proposition. \(\Box\)

We note that, for any $x \in \mathbb{R}^d$, one has
\[
\xi_e(x) = \text{vol}(B) \mathbb{E}_z \left[ \hat{s}(z) e^{\frac{||z||^2}{2}} e^{ix \cdot z} \right]
\]
Writing $\xi(z) := \hat{s}(z) e^{\frac{||z||^2}{2}} e^{ix \cdot z}$, one has
\[
P \left[ |f_x - \xi_e(x)| > k^{-C_{3.5}} \right] = P \left[ \frac{\text{vol}(B)}{m} \left| \sum_{l \in [m]} \xi(z_l) - \mathbb{E} \left[ \xi(z_l) \right] \right| > k^{-C_{3.5}} \right]
\]
Using (14) and Hoeffding’s inequality (9), we get
\[
P \left[ |f_x - \xi_e(x)| > k^{-C_{3.5}} \right] \leq k^{-C_{3.5}}
\] (18)

In the following algorithm $\text{FindSpikes}$, at any point
\[
z \in \bigcup_i B_2(x_i, 2\sqrt{d}),
\]
we shall therefore have access to the random variable $f_x$ in $\mathbb{C}$, such that
\[
P \left[ |f_x - (\gamma \ast \nu)(z)| < k^{-C_4} \right] > 1 - k^{-C_4}.
\]
As established by proposition above, these $f_z$ can be constructed using polynomially many samples and computational steps, in such a way that for any subset 
\[ \{z_1, \ldots, z_m\} \subseteq \mathbb{R}^d, \]
\{f_{z_1}, \ldots, f_{z_m}\} is a set of independent random variables.

For the next part we will employ the efficient zeroth order stochastic concave maximization algorithm, devised in Belloni et al (3). We denote this algorithm as $A_0$. In $d$-dimensional Euclidean space, the algorithm returns an $\epsilon$-approximate maxima of an $d^{-1}\epsilon$-approximate $t$-Lipschitz concave function, and the number of function evaluations used depends polynomially on $d, \epsilon$, and $\log t$. The performance guarantee of algorithm $A_0$ is summarized in the following:

**Fact 1** (Belloni-Liang-Narayanan-Rakhlin). Suppose that $B \subseteq \mathbb{R}^d$ is a convex subset, and $\chi, \psi : B \to \mathbb{R}$ are functions satisfying
\[ \sup_{z \in B} |\chi(z) - \psi(z)| \leq \frac{\epsilon}{d} \]
Suppose that $\psi$ is concave and $t$-Lipschitz; then algorithm $A_0$ returns a point $q \in B$ satisfying
\[ \chi(q) \geq \max_{z \in B} \chi(z) - \epsilon, \]
and uses $O(d^8\epsilon^{-2}\log t)$ computation steps. □

In the following, we use $\bar{d}$ instead of $d$ in order to keep the notation consistent with Algorithm LearnMixture. Let $\text{diam}(Q_\ell)$ denote the $\ell_2$ diameter of $Q_\ell$.

4.1 Algorithm $\text{FindSpikes}$.

Let $\text{count}_{\text{max}} = C_4.5k$.

1. While $\text{count} \leq \text{count}_{\text{max}}$, do the following:
   (a) For each point
   \[ \ell \in \left( \frac{\bar{d}}{1000} \sqrt{\frac{\bar{d}}{C_{1.5} \log k}} \right) \mathbb{Z}^d \cap \bigcup_i B_2(x_i, 2\sqrt{d}), \]
   let $Q_\ell$ be the ball of radius \( \left( \frac{\bar{d}}{400 \sqrt{C_{1.5} \log k}} \right) \), centered at $\ell$.
   (b) Use an efficient zeroth order stochastic concave maximization subroutine (see Fact 1) that produces a point $q_\ell$ in $Q_\ell$ at which $(2\pi)^{-\frac{d}{2}}(\zeta \ast \mu_\epsilon)(z)$ is within $k^{-C_4\epsilon}$ of the maximum of $(2\pi)^{-\frac{d}{2}}(\zeta \ast \mu_\epsilon)$ restricted to $Q_\ell$.  

13
(c) Create a sequence \( L = (q_{\ell_1}, \ldots, q_{\ell_k}) \) that consists of all those \( q_{\ell} \), such that \((2\pi)^{-\frac{d}{2}}(\zeta * \mu_e)(q_{\ell}) > (\frac{w_{\min}}{2})\gamma_\Delta(0)\), and
\[ \|\ell - q_{\ell}\|_2 < \text{diam}(Q_{\ell})/4. \]
If \( k_1 < 1 \), return “ERROR”.

(d) Form a subsequence \( M = (\ell_{m_1}, \ldots, \ell_{m_{k_2}}) \) of \((\ell_1, \ldots, \ell_{k_1})\) by the following iterative procedure:

i. Let \( m_1 = 1 \). Set \( M := \{\ell_1\} \), and \( j = 1 \).

ii. While \( j \leq k_1 \):
   A. if \( \ell_{j+1} \) is not contained in \( B(\ell_j', \sqrt{d}\sqrt{\Delta}/2) \) for any \( j' \leq j \), append \( \ell_{j+1} \) to \( M \).
   B. Increment \( j \).

(e) Pass \( \{(q_{\ell_{mj}})\}_{j \in [k_2]} \) to algorithm \( \text{Boost} \). Increment \( \text{count} \).

2. Pass the output of \( \text{Boost} \) obtained by processing \( \text{count}_{\text{max}} \) copies of \( M \), to the iterative algorithm of Regev and Vijayaraghavan [10], which will correctly output the centers and weights to the desired accuracy \((\delta/k\) and \( w_{\min}/C\) resp.) with the required probability \( 1 - \exp(-k/c) \).

4.2 Analysis of \( \text{FindSpikes} \)

The following lemma shows that the number of cubes in step (a) of \( \text{FindSpikes} \) that need to be considered is polynomial in \( k \).

**Lemma 5.** For any \( x \in \mathbb{R}^d \),
\[
\left| \left( \frac{\Delta}{1000} \sqrt{\frac{d}{C_{1.5}\log k}} \right) \mathbb{Z}^d \cap B_2(x, 2\sqrt{d}) \right| \leq k^{C_7}. \tag{19}
\]

**Proof.** Set \( r \) to \( \sqrt{d} \). We observe that the number of lattice cubes of side length \( cr/\sqrt{\log k} \) centered at a point \( x \) belonging to \((\frac{cr}{\sqrt{\log k}}) \mathbb{Z}^d \) that intersect a ball \( B \) of radius \( r \) in \( d \) dimensions is less or equal to to the number of lattice points inside a concentric ball \( B' \) of dimension \( d \) of radius \( r + \frac{c\sqrt{d}}{\sqrt{\log k}} \). Every lattice cube of side length \( cr/\sqrt{\log k} \) centered at a lattice point belonging to \((\frac{cr}{\sqrt{\log k}}) \mathbb{Z}^d \cap B' \) is contained in the ball \( B'' \) with center \( x \) and radius \( r + \frac{2c\sqrt{d}}{\sqrt{\log k}} \). By volumetric considerations, \( |(\frac{cr}{\sqrt{\log k}}) \mathbb{Z}^d \cap B''| \) is therefore bounded above by \( \frac{\text{vol}(B'')}{(\frac{cr}{\sqrt{\log k}})^d} \).

We write \( v_d := \text{vol}(B_2(0, \sqrt{d})) \). By Ramanujan’s approximation of \( \Gamma \) (see Theorem 1 of [3]) we get
\[
v_d \leq \frac{1}{\sqrt{\pi}} \left( \frac{2\pi}{e} \right)^{\frac{d}{2}} d^{\frac{d}{2}}.
\]
This tells us that
\[
\frac{\text{vol}(B')}{(cr/\sqrt{\log k})^d} \leq \left( \frac{C\sqrt{\log k}}{\sqrt{d}} \right)^d
\leq \left( \frac{C\sqrt{\log k}}{\sqrt{d}} \right) \left( \frac{C\sqrt{d\log k}}{\sqrt{d}} \right)
\leq \left( e^{1/e} \right) \left( \frac{C\sqrt{d\log k}}{\sqrt{d}} \right)
\leq k^{C_7}.
\]

We have used the fact that for \( \alpha > 0, \alpha^{\alpha-1} \) is maximized when \( \alpha = e \), a fact easily verified using calculus.

We will need some results on the structure of \( \gamma_{\Delta} \ast \nu \).

**Definition 1.** Let \( B \subseteq \mathbb{R}^d \) be a convex set, and \( \xi > 0 \). A continuous function \( \phi : B \to \mathbb{R}_+ \) is said to be \( \xi \)-approximately log-concave if there exists a continuous function \( \psi : B \to \mathbb{R}_+ \) such that \( \log \psi \) is concave, and \( \| \log \phi - \log \psi \|_{L^\infty(B)} \leq \xi \). We say \( \phi \) is approximately log-concave if it is \( \xi \)-approximately log-concave for some \( \xi > 0 \).

By Theorem 4.1 of the full version of [10], it suffices to approximate the centers of the Gaussians to within \( cd^{-5} \), and then pass on these approximate centers to an iterative algorithm developed in that paper. To this end, if \( \nu_j := w_j \delta_{y_j} \), it suffices to have, in the vicinity of \( y_j \), access to an \( L^\infty \) approximation of \( \log(\gamma_{\Delta} \ast \nu_j) \) to within an additive \( c \sqrt{\log k} \). This is achieved by Lemma 6.

**Lemma 6.** If \( q_\ell \in B(y_j, \text{diam}(Q_\ell)) \) for some \( j \in [k_0] \), then the restriction of \( \gamma_{\Delta} \ast \nu \) to \( Q_\ell \) is approximately log-concave. Specifically, for \( \nu_j := w_j \delta_{y_j} \), one has
\[
0 \leq \log(\gamma_{\Delta} \ast \nu)(x) - \log(\gamma_{\Delta} \ast \nu_j)(x) \leq \left( 10 \right)^{31} \frac{e^{1/e} \sqrt{d}}{C_{3,2}^2 \cdot 2^{-15} \Delta}
\]

**Proof.** Fix \( x \in Q_\ell \), and write \( a_r := x - y_r \) for \( r \in [k_0] \). One has
\[
||a_j|| \leq \frac{1.1 \Delta \sqrt{d}}{100 C_{3,2}} \sqrt{d} \frac{d}{C_{1,5} \log k} + \frac{\Delta \sqrt{d}}{200 C_{3,2}} \sqrt{d} \frac{d}{C_{1,5} \log k} \leq \frac{3.2 \Delta \sqrt{d}}{200 C_{3,2}}
\]
and
\[
||a_r - a_s|| \geq \Delta \sqrt{d}
\]
if \( r \neq s \). Rewriting \( \Delta = C_{3,2} \bar{\Delta} \), one has
\[
(2\pi \bar{\Delta}^2)^{d/2} ||(\gamma_{\Delta} \ast \nu)(x) - (\gamma_{\Delta} \ast \nu_j)(x)|| = \sum_{r \in [k_0] \setminus \{j\}} w_r e^{-\frac{||a_r||^2}{2\Delta^2}}
\]
\[
= \sum_{m=1}^{\infty} \sum_{r \in [k_0] \setminus \{j\}} w_r e^{-\frac{C_{3,2}^2 ||a_r||^2}{2\Delta^2}}
\]
\[
\frac{\Delta}{2 \sqrt{d}} \leq ||a_r|| < \frac{\Delta}{2 \sqrt{d}}
\]
We write $p_m := |S_m|$ where

$$S_m := \left\{ r \in [k_0] : \frac{\Delta}{2} \leq ||a_r||^{\frac{1}{2}} < \frac{m+1}{2} \Delta \right\}$$

Since $||a_r - a_s|| \geq \Delta \bar{d}$, we can put disjoint balls of radius $0.5 \Delta \bar{d}$ around each center. Thus, letting $\omega_{\bar{d}}$ be the volume of unit ball in $\mathbb{R}^d$, one has

$$\omega_{\bar{d}} \left( \frac{\Delta \bar{d}}{2} \right)^d p_m \leq \omega_{\bar{d}} \left( \frac{\Delta \bar{d}}{2} \right)^d \left( (m+2)^d - (m-1)^d \right)$$

$$\Rightarrow p_m \leq \left( (m+2)^d - (m-1)^d \right)$$

$$< (m+2)^d$$

which gives

$$\left( 2 \pi \Delta^2 \right)^{\frac{d}{2}} \begin{array}{l} |(\gamma_{\bar{d}} * \nu)(x) - (\gamma_{\bar{d}} * \nu_j)(x)| \\ \geq \sum_{m=1}^{\infty} \\ \sum_{r \in [k_0] \setminus \left\{ j \right\}} \\ \frac{\Delta}{2} \leq ||a_r||^{\frac{1}{2}} < \frac{m+1}{2} \Delta \end{array} w_r e^{-\frac{c^2 ||a_r||^2}{4 \Delta^2}}$$

$$\leq (ck)^{-1} \sum_{m=1}^{\infty} e^{-\frac{d C^2 m^2}{16} (m+2) - 8\ln(m+2))}$$

$$\leq (ck)^{-1} \sum_{m=1}^{\infty} e^{-\frac{d C^2 m^2}{16}}$$

We use $e^{-x} < \frac{x^{15}}{2^m}$ and $\sum m^{-30} < 10$ to obtain

$$\left( 2 \pi \Delta^2 \right)^{\frac{d}{2}} \begin{array}{l} |(\gamma_{\bar{d}} * \nu)(x) - (\gamma_{\bar{d}} * \nu_j)(x)| \\ \leq (ck)^{-1} \sum_{m=1}^{\infty} e^{-\frac{d c^2 m^2}{16}} \end{array}$$

$$\leq \frac{(100)^{15} e^{ck d 3^{15} C^{30} \times 3.2}}{ck d 3^{15} C^{30} \times 3.2}$$

$$= \frac{(10)^{31}}{ck d 3^{15} C^{30} \times 3.2}$$

Note that $\gamma_{\bar{d}} * \nu_j$ is log-concave. Moreover, for any $x \in Q_l$, one has

$$\begin{array}{l} (\gamma_{\bar{d}} * \nu_j)(x) \\ \geq \frac{c}{k(2\pi \Delta^2)^{\frac{d}{2}}} e^{-\frac{d C^2 m^2}{16}} \end{array}$$

$$\geq \frac{c}{k(2\pi \Delta^2)^{\frac{d}{2}}} e^{-\frac{d C^2 m^2}{16}}$$

16
so that
\[
\left| \frac{(\gamma_\Delta \ast \nu)(x)}{(\gamma_\Delta \ast \nu_j)(x)} - 1 \right| \leq \frac{1}{(2\pi \Delta^2)^{\frac{D}{2}}} \frac{(10)^{31}}{ckd_{15}C_{3,2}^{20}} \frac{k(2\pi \Delta^2)^{\frac{D}{2}}}{C_{3,2}^{2}} e^{\frac{1}{C_{3,2}^{2}}}
\]

\[
\leq \frac{(10)^{31} e^{\frac{1}{C_{3,2}^{2}}}}{C_{3,2}^{3}d_{15}^{2}}
\]

This gives
\[
0 \leq \log(\gamma_\Delta \ast \nu)(x) - \log(\gamma_\Delta \ast \nu_j)(x)
\]
\[
= \log \left(1 + \frac{(10)^{31} e^{\frac{1}{C_{3,2}^{2}}}}{C_{3,2}^{3}d_{15}^{2}}\right)
\]
\[
\leq \frac{(10)^{31} e^{\frac{1}{C_{3,2}^{2}}}}{C_{3,2}^{3}d_{15}^{2}} \tag{20}
\]

Remark 2. Note that, by remark (1) and convexity of \( Q_\ell \), if

\[ q_\ell \in B \left(y_j, \left(\frac{\tilde{\Delta} d}{200} \sqrt{\frac{1}{C_{1,5} \log k}} \right) \right) \]

then (with high probability) \( \log(\gamma_\Delta \ast \nu_j) \) is \( t \)-Lipschitz for

\[ t \leq \frac{(ck \sqrt{\ln(C_1 d n)} + C_{3,2}) \sqrt{C_{1,5} \log k}}{C_{3,2}^{3}d_{15}^{2}} \]

The following lemma shows that every true spike \( \gamma_\Delta \ast \nu \) corresponding to a center gets detected by \( \text{FindSpike} \).

Lemma 7. If \( q_\ell \in B \left(y_j, \left(\frac{\tilde{\Delta} d}{200} \sqrt{\frac{1}{C_{1,5} \log k}} \right) \right) \) for some \( j \in [k_0] \), then

\[ (\gamma_\Delta \ast \nu)(q_\ell) - k^{-C_{3,5}} \geq \frac{w_{\min}}{2} \gamma_\Delta(0). \tag{21} \]

Proof. By lemma (10) the restriction to \( Q_\ell \) of \( \gamma_\Delta \ast \nu \) is approximately log-concave. The output \( q_\ell \) of stochastic optimization algorithm \( \mathfrak{A}_0 \) satisfies

\[ |\log m_\ell - \log(\gamma_\Delta \ast \nu)(q_\ell)| \leq \frac{1}{C_{3,7}d_{15}^{2}} \]

where \( m_\ell := \max\{(\gamma_\Delta \ast \nu)(x) : x \in Q_\ell\} \). Equivalently,

\[ m_\ell - (\gamma_\Delta \ast \nu)(q_\ell) \leq (\gamma_\Delta \ast \nu)(q_\ell) \left(1 - e^{-\frac{1}{C_{3,7}d_{15}^{2}}} \right) \]
\[
\leq \frac{(\gamma_\Delta \ast \nu)(q_\ell)}{C_{3,7}d_{15}^{2}}
\]
\[
\Rightarrow (\gamma_\Delta \ast \nu)(q_\ell) \geq m_\ell - \frac{(\gamma_\Delta \ast \nu)(q_\ell)}{C_{3,7}d_{15}^{2}}
\]

which proves the lemma.
The next lemma shows that there are no false spikes in \( \gamma \ast \nu \).

**Lemma 8.** If \((\gamma \ast \nu)(q) + k^{-C_{3.5}} \geq (\frac{w_{\text{min}}}{2})\gamma_{\Delta}(0)\), then there exists some \( j \in [k_0] \) such that \( q \in B(y_j, \frac{\sqrt{\pi} \Delta}{6 \sqrt{\nu}}) \).

**Proof.** The arguments are similar to that in lemma 6 above. Suppose that \( q \notin \bigcup_{j \in [k_0]} B(y_j, \frac{\sqrt{\pi} \Delta}{6 \sqrt{\nu}}) \), and write \( a_j := y_j - q \). For \( m \geq 1 \), let \( p_m := |S_m| \) where

\[
S_m := \{ r \in [k_0] : \left( \frac{m}{2} + \frac{1}{5 \sqrt{C_{3.2}}} \right) \Delta \sqrt{d} \leq ||a_r|| < \left( \frac{m+1}{2} + \frac{1}{5 \sqrt{C_{3.2}}} \right) \Delta \sqrt{d} \}
\]

One has
\[
(\gamma \ast \nu)(q) = \sum_{j \in [k_0]} w_j e^{-\frac{||a_j||^2}{2 \Delta^2}}
\]
\[
= \gamma_{\Delta}(0) \left( \sum_{0 \leq ||\frac{a_j}{\Delta \sqrt{d}}|| - \frac{1}{5 \sqrt{C_{3.2}}} < \frac{1}{2}} w_j e^{-\frac{||a_j||^2}{2 \Delta^2}} + \sum_{m=1}^{\infty} \sum_{j \in S_m} w_j e^{-\frac{||a_j||^2}{2 \Delta^2}} \right)
\]

Since \( ||a_r - a_s|| \geq \Delta d \), we can put disjoint balls of radius \( 0.5 \Delta \sqrt{d} \) around each center. Thus,
\[
p_m \leq 2^d \left( \left( \frac{m+2}{2} + \frac{1}{5 \sqrt{C_{3.2}}} \right)^d - \left( \frac{m-1}{2} + \frac{1}{5 \sqrt{C_{3.2}}} \right)^d \right)
\]
\[
< \left( m + 2 \right)^d
\]
which gives
\[
(\gamma \ast \nu)(q) = \gamma_{\Delta}(0) \left( \sum_{0 \leq ||\frac{a_j}{\Delta \sqrt{d}}|| - \frac{1}{5 \sqrt{C_{3.2}}} < \frac{1}{2}} w_j e^{-\frac{||a_j||^2}{2 \Delta^2}} + \sum_{m=1}^{\infty} \sum_{j \in S_m} w_j e^{-\frac{||a_j||^2}{2 \Delta^2}} \right)
\]
\[
\leq \frac{C \gamma_{\Delta}(0)}{k} \left( e^{-\frac{C_{3.2} d}{50}} + \sum_{m=1}^{\infty} e^{-\frac{4 C_{3.2} m^2}{160}} \right)
\]
\[
< \frac{C \gamma_{\Delta}(0)}{k} \left( e^{-\frac{C_{3.2} d}{50}} + \frac{2000}{d^2 C_{3.2}^2} \right)
\]

Thus, for \( C_{3.2} \) sufficiently large, the inequality
\[
(\gamma \ast \nu)(q) + k^{-C_{3.5}} < \left( \frac{w_{\text{min}}}{2} \right) \gamma_{\Delta}(0).
\]
Lemma 9. If there exists some $j \in [k_0]$ such that $q_\ell \in B\left(y_j, \sqrt[\frac{\bar{\Delta}_\ell}{5}}\right)$, and $|q_\ell - \ell| < \text{diam}(Q_\ell)/4$, then with high probability, there exists some $j \in [k_0]$ such that $q_\ell \in B\left(y_j, c\bar{d}^{-\frac{\bar{\Delta}}{2}}\right)$.

Proof. If $x \in B\left(y_j, \sqrt[\frac{\bar{\Delta}_\ell}{5}}\right)$, by Lemma 3

$$0 \leq \ln(\gamma \ast \nu)(x) - \ln(\gamma \ast \nu_j)(x) \leq \frac{1}{K\bar{d}^5},$$

where $K$ is an absolute constant that can be made arbitrarily large. We note that $\ln(\gamma \ast \nu_j)(x) = a - \left(\frac{1}{2\bar{\Delta}}\right)\|x - y_j\|^2$, for some constant $a$. This implies that if

$$|\ln(\gamma \ast \nu)(q_\ell) - \sup_x \ln(\gamma \ast \nu)(x)| < \frac{2}{K\bar{d}^5},$$

then $q_\ell \in B\left(y_j, c\bar{d}^{-\frac{\bar{\Delta}}{2}}\right)$. However, $|\ln(\gamma \ast \nu)(q_\ell) - \sup_x \ln(\gamma \ast \nu)(x)|$ is indeed less than $\frac{2}{K\bar{d}^5}$ by Proposition 1 and Fact 1. Noting that $\bar{\Delta} < c$, this completes the proof of this lemma. □

The following proposition shows that every spike extracted out of $\gamma \ast \nu$ by $\text{FindSpikes}$, is within $\delta/k$ of some $y_i$.

Proposition 2. With probability at least $1 - \exp(-k/c)$, the following is true. The hausdorff distance between $\{y_1, \ldots, y_{k_0}\}$ and $\{\ell_{m_1}, \ldots, \ell_{m_k}\}$ (which is the output of Boost in step 2. of $\text{FindSpikes}$ above) is less than $\delta/k$.

Proof. Consider the sequence $L$ in 1(c) of $\text{FindSpikes}$. However, we know from the statements in Lemma 2, Lemma 3 and Lemma 4 that

1. If $q_\ell \in \bigcup_{j \in [k_0]} B\left(y_j, \left(\frac{\bar{\Delta}_\ell}{200\sqrt[\frac{\bar{\Delta}_\ell}{5}}\right)\right)$, then $(\nu \ast \gamma \ast \Delta)(q_\ell) - k^{-C_{3.5}} \geq \left(\frac{w_{\min}}{2}\right)\gamma \ast \Delta(0)$.

2. If $(\gamma \ast \nu)(q_\ell) + k^{-C_{3.5}} \geq \left(\frac{w_{\min}}{2}\right)\gamma \ast \Delta(0)$, then there exists some $j \in [k_0]$ such that $q_\ell \in B\left(y_j, \sqrt[\frac{\bar{\Delta}_\ell}{5}}\right)$.

3. If there exists some $j \in [k_0]$ such that $q_\ell \in B\left(y_j, \sqrt[\frac{\bar{\Delta}_\ell}{5}}\right)$, and (as is true from step 1(c) of $\text{FindSpikes}$), $|q_\ell - \ell| < \text{diam}(Q_\ell)/4$, then with probability at least $1 - \exp(-\ell/c)$, there exists some $j \in [k_0]$ such that $q_\ell \in B\left(y_j, c\bar{d}^{-\frac{\bar{\Delta}}{2}}\right)$.

We have shown that $L$ is, with high probability, contained in $\bigcup_{j \in [k_0]} B(y_j, c\bar{d}^{-5/2})$. Lastly, we observe that for each $i \in [k_0]$ there must exist a $q_\ell$ such that $q_\ell \in B\left(y_j, c\bar{d}^{-\frac{\bar{\Delta}}{2}}\right)$, by the exhaustive choice of starting points. This proposition now follows from Theorem 4.1 of the full version of [10]. □
5 Conclusion and open problems

We developed a randomized algorithm that learns the centers $y_i$ of standard gaussians, to within an $\ell_2$ distance of $\delta < \frac{\Delta}{\sqrt{d}}$ and the weights $w_i$ to within $cw_{\min}$ with high probability when the minimum separation between two centers is at least $\sqrt{d}\Delta$, where $\Delta$, where $\Delta$ is larger than an arbitrary universal constant in $(0, 1)$. The number of samples and the computational time is bounded above by $\text{poly}(k, d, \frac{1}{\delta})$. Such a polynomial bound on the sample and computational complexity was previously unknown when $d \geq \omega(1)$. There is a matching lower bound due to Regev and Vijayaraghavan [10] on the sample complexity of learning a random mixture of gaussians in a ball of radius $\Theta(\sqrt{d})$ in $d$ dimensions, when $d$ is $\Theta(\log k)$. It remains open whether (as raised in [10]) $\text{poly}(k, d, 1/\delta)$ upper bounds on computational complexity of this task can be obtained when the minimum separation between two centers in $\Omega(\sqrt{\log k})$ in general, although when $d \leq O(\log k)$, this follows from our results. It would also be interesting to extend the results of this paper to mixtures of spherical gaussians whose variances are not necessarily equal.

Acknowledgments

HN was partially supported by a Ramanujan Fellowship and NSF award no. 1620102. Both authors acknowledge the support of DAE project no. 12-R&D-TFR-5.01-0500. This research was supported in part by the International Centre for Theoretical Sciences (ICTS) during a visit for the program - Statistical Physics of Machine Learning (Code: ICTS/SPMML2020/01).

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