A Topological Formulation of the Standard Model

Marco Spaans\textsuperscript{a,1}

\textsuperscript{a}Harvard-Smithsonian Center for Astrophysics, 60 Garden Street MS 51, Cambridge, MA 02138, USA

Abstract

A topological theory for the interactions in Nature is presented. The theory derives from the cyclic properties of the topological manifold $Q = 2T^3 \oplus 3S^1 \times S^2$ which has 23 intrinsic degrees of freedom, discrete $Z_3$ and $Z_2 \times Z_3$ internal groups, an SU(5) gauge group, and an anomalous U(1) symmetry. These properties reproduce the standard model with a stable proton, a natural place for CP violation and doublet-triplet splitting. The equation of motion for the unified theory is derived and leads to a Higgs field. The thermodynamic properties of $Q$ are discussed and yield a consistent amplitude for the cosmic microwave background fluctuations. The manifold $Q$ possesses internal energy scales which are independent of the field theory defined on it, but which constrain the predicted mass hierarchy of such theories. In particular the electron and its neutrino are identified as ground states and their masses are predicted. The correct masses of quarks and the CKM mixing angles can be derived as well from these energy scales if one uses the anomalous U(1) symmetry.

Furthermore, it is shown that if the Planck scale topology of the universe involves loops as fundamental objects, its spatial dimension is equal to three. The existence of the prime manifold $T^3 = S^1 \times S^1 \times S^1$ is then required for a dynamical universe, i.e. a universe which supports forces. Some links with M-theory are pointed out.

Keywords: general relativity — quantum cosmology

PACS: 98.80.Hw, 98.80.Bp, 04.20.Gz, 02.40.-k

1 Introduction

One of the outstanding questions in quantum cosmology and particle physics is the unification of gravity with the electro-weak and strong interactions. Much

\textsuperscript{1} mspaans@cfa.harvard.edu
effort has been devoted in the past years to formulate a purely geometrical and topological theory for both types of interactions\cite{1,2}. Probably best known are the theories involving super-gravity and superstrings\cite{1}. These theories have been shown recently to be unified in M-theory, although the precise formulation of the latter is not known yet. Two outstanding problems in these approaches are compactification down to four dimensions and the existence of a unique vacuum state. This work aims at extending the topological dynamics approach presented in \cite{3} (paperI from here on) to include non-gravitational interactions and to construct a Quantum-gravitational Grand Unified Theory (QGUT).

The main results of paperI are as follows: The properties of space-time topology are governed by homotopically inequivalent loops in the prime manifolds $S^3$, $S^1 \times S^2$, and $T^3 = S^1 \times S^1 \times S^1$. This is the only set which assures Lorentz invariance and the superposition principle as formulated by the Feynman path integral for all times. The dynamics of the theory are determined by the loop creation ($T^\dagger$) and loop annihilation ($T$) operators. On the Planck scale, space-time has the unique structure of a lattice of Planck size three-tori with 4 homotopically inequivalent paths joining where the $T^3$ are connected through three-ball surgery. The existence of this four-fold symmetry leads to SO(n) and SU(n) gauge groups as well as the numerical factor $1/4$ in the expression for black hole entropy. The number of degrees of freedom of the prime manifolds, referred to as the prime quanta, under the action of $O = T T^\dagger + T^\dagger T$, are 1, 3, and 7 for the three-sphere, the handle manifold, and the three-torus, respectively. During the Planck epoch, $S^1 \times S^2$ handles (mini black holes) can attach themselves to the $T^3$ lattice which can lead to interactions between the degrees of freedom of the prime manifolds. The $S^1 \times S^2$ prime manifolds are referred to as charges on the $T^3$ lattice from hereon. Finally, it was found that the cosmological constant $\Lambda$, as the spontaneous creation of mini black holes from the vacuum, is very small and proportional to the number of (macroscopic) black holes at the current epoch. This result follows from the generalization of Mach’s principle in which local topology change is a consequence of global changes in the matter degrees of freedom and vice versa.

This paper is organized as follows. Section 2 presents a derivation of the dimension of the universe and its link with Lorentz invariance as well as the necessity of a $T^3$ Planck scale topology for the existense of forces. Section 3 discusses the construction of a QGUT based on the charged $T^3$ lattice and presents the derivation of the equation of motion. Section 4 discusses the properties of a quantized state space based on the equation of motion and the cyclic properties of $Q$, which reproduces the standard model. Section 5 contains the conclusions.
The Dimension of the Universe, Lorentz Invariance and the Superposition Principle

2.1 Linked Loops and Spatial Dimensionality

In paper I it is shown that a lattice of three-tori \( L(T^3) \) supports many homotopically distinct but otherwise equivalent paths between any two points. This allows for a direct topological implementation of the superposition principle through the Feynman paths integral. In this approach, the spatial dimension of the universe is postulated, and ideally one would want to derive it from first principles.

Clearly, the invariance of the speed of light should hold in any quantum field theory. Since the three-torus bounds a Lorentz four-manifold with \( SL(2;\mathbb{C}) \) spin structure (paper I), i.e. is nuclear (like the handle), the superposition principle on \( L(T^3) \) implies Lorentz invariance in (3+1)-dimensional space-time. Conversely, the loops in the prime manifolds are considered to be fundamental objects and as such the homotopic structure of \( T^3 \) should be unaffected by dynamics. Therefore, in a loop-topological sense, field interactions should correspond to loops which are linked on the junctions of connected three-tori. Such links preserve the underlying loop homotopy of the \( T^3 \) lattice. Note that this implementation of the dynamics is quite different from string theory, where the strings can break and merge.

It follows immediately that one can exclude spatial dimensions \( \leq 2 \) in the link picture. Furthermore, any two linked loops embedded in a space of dimension \( \geq 4 \) are homeomorphic to two separate loops, i.e. no intersection/interaction need ever occur in the deformation. Because Lorentz invariance implies time dilation of the arbitrary intersection point of two loops, one wants the linkage of two loops to be a topological invariant and to be independent of the particular embedding. That is, the number of interactions should be both a Lorentz scalar as well as an intrinsic property of the underlying topological manifold which describes the dynamics. It follows that the spatial dimension of a relativistic universe must be \( D = 3 \). As a corollary one finds that the occurrence of spin \( 1/2 \) in the universe, a property of the three-torus through the \( SL(2;\mathbb{C}) \) spin structure it induces on the four-manifold it bounds, follows from the superposition principle.

Conversely, the sum of the prime quanta, \( d = 1 + 3 + 7 = 11 \), implies that the kinematic dimension of the universe is eleven. This dimension reflects the number of degrees of freedom in the fundamental constituents ("basis vectors") of space-time, the prime manifolds. That is, \( d \) reflects the different ways in which the loop can exist in space-time on the Planck scale when
direct interactions between three-tori and handle are neglected (paper I). This is elaborated in Section 3.

2.2 The Three-Torus and the Existence of Forces

The existence of the prime manifold $T^3$ should ideally be derived from some fundamental physical principle. To find this physical principle, the nature of gravitation will be investigated. The equivalence principle in general relativity leads to the notion of gravitational accelerations as an expression of space-time curvature. In analogy with Einstein’s thought experiment one can ask how observers can determine the existence of a net force which acts on some object. Obviously, one wants to determine whether the object is accelerating. To do this, one formally needs no less than three distinct spatial measurements of the object’s position. To extend such an experiment to arbitrary small scales, the quantum-mechanical notion of position as an expectation value comes into play. The view is taken here that even when a particle is replaced by a field and a probability amplitude, the concept of a force is equivalent to an interaction between (quantum) fields and the application of Yang-Mills theory then leads to the notion of a field strength or a curvature two-form generated by some gauge potential.

In general relativity, the existence of a force is interpreted as a curvature of space-time. Mathematically, a curvature involves the second derivatives of some metric field. Physically, the required existence of such an object is a powerful constraint. The occurrence of a second derivative of any field requires that field to be defined in three points. These three points need not be infinitesimally close because the coarseness of a derivative only depends on the measurement scale one is interested in. This leads to a natural partition of space into triplets. That is, whether there is a force field or not, its possible existence requires any triplet of points to be viewed as a physical entity.

This yields a formal object $s$ denoted as $s = \{\ldots \} \equiv [123]$, where the three dots indicate the notion of a second derivative defined by three arbitrary points. For interactions to occur and for space-time curvature to be induced by some metric field, the object should be endowed with a topological structure $\sigma$, yielding the dynamical object $S = \sigma(s)$. The structure $\sigma$ follows from the realization that all three points, as well as the paths (partial trajectories) connecting them, are distinct. If the latter were not the case, then on the Planck scale one could only superpose non-interacting trajectories. This then implies

\begin{align}
[1] &\neq [2] \neq [3], \quad (1a) \\
[12] &\neq [23] \neq [13], \quad (1b)
\end{align}
and

\[[ab] = [ba], \quad (2)\]

where the last relation reflects that there is no preferred orientation on \(S\). The paths in relation (1b) should be topologically distinct to satisfy (1a). Clearly, these relations imply the homotopic structure of the three-torus with the loops [12], [23] and [13]. Conversely, they do not specify the shape of \(T^3\) or favor any loop in the three homotopic equivalence classes. Since the force argument is independent of scale, the three-tori should be inter-connected which immediately leads to the \(T^3\) lattice of paperI and the results of section 2.1. Finally, the above argument does not provide any specific numerical value for the Planck scale \(\ell_{\text{Planck}}\). If anything, all finite values, with equal probabilities, are allowed except zero.

## 3 Quantum Gravity, Superstrings and Topological Dynamics

### 3.1 QGUT Phenomenological Preliminaries

In paperI it was shown that interactions between \(T^3\) and \(S^1 \times S^2\) occur as matter traverses the wormholes which are attached to \(L(T^3)\). The three-torus has 3 homotopic branches and it is easy to see that individual branches yield spin 1 particles and pairs of branches lead to spin 1/2[4]. The handle introduces an additional degree of freedom which can increase or decrease the homotopic complexity of the structure to which it is attached, e.g. provide single loop paths through two connected loops or bind two separate loops. Therefore, as a handle connects itself to two singlets or connects two branches of a doublet, where singlets and doublets should be viewed as different physical substructures of \(T^3\), it can effectively convert 1-bosons into 1/2-fermions and vice versa. The aim is now to find a manifold consisting of three-tori and handles, which provides the right number and properties of elementary fermions and bosons.

In paperI it is shown that the lattice junctions support SO(n) and SU(n) symmetry groups because \(L(T^3)\) naturally yields quadratic and quartic interaction terms. Above it is found that the kinematic dimension of the homotopic theory is \(d = 7 + 3 + 1 = 11\). The 1 in this formula corresponds to the homotopically trivial manifold \(S^3\) which is needed in the construction of \(L(T^3)\) and which provides the large scale topology of the universe. The junctions which connect the three-tori correspond to the propagators in a field theory and are homotopically equivalent to line elements. It follows that the dimension of the junction groups which describe the interactions between the propagators on \(L(T^3)\) is 10, i.e. SU(5) and SO(10) which are viable candidates for a GUT.
3.2 QGUT Construction

3.2.1 The Fundamental Topological Manifold \(Q\)

A manifold \(Q = aT^3 \oplus bS^1 \times S^2\) which is built from three-tori and handles should have an odd number of constituents because only odd sums of nuclear primes bound Lorentz manifolds (paperI). Because the supersymmetry is implemented homotopically, the number of loops in the \(a\) three-tori should be equal to the number of mouths of the \(b\) handles, i.e. \(3a = 2b\). Finally, any solution \(R\) which has \(a_1 = na\) and \(b_1 = nb\) can be considered a multiple of the smallest solution \(aT^3 \oplus bS^1 \times S^2\). Since one wants to construct a lattice \(L(Q)\), the minimal solution is the desired one. Therefore, the coefficients \(a = 2\) and \(b = 3\) result. This requires the direct sum of three handle manifolds and two three-tori. In \(Q\), each handle is connected to two loops and a homotopically supersymmetric manifold can be written as

\[
Q = 2T^3 \oplus 3S^1 \times S^2, \tag{3}
\]

which assures nuclearity and therefore Lorentz invariance. When the density of mini black holes is large enough to support \(Q\), the universe is said to be \(Q\)-supersymmetric. As an aside it is worthwhile to mention that the expression for \(Q\) bears close resemblance to the effective geometric string action (the Chern-Simons form) \(S_g = 2/3A \times A \times A + A \times DA\), for the gauge string field \(A\) (viewed as a homotopic loop \(S^1\)) and the covariant derivative \(D\). In fact, the existence of the \(T^3\) internal space seems to hint at a link with heterotic string duality[1].

3.2.2 The Equation of Motion for Constant Charge

In paperI an elaborate derivation was given for the topological dynamics of space-time in terms of the \(T^3\), \(S^1 \times S^2\), and \(S^3\) prime manifolds when the loop is the fundamental object. The derivation in paperI was based on 1) the existence of homotopic classes of paths in a Lorentz invariant topological manifold as representatives of the linear superposition principle, and 2) the algebra of loop creation and annihilation operators which reflects the fact that the homotopy of space-time is a dynamical quantity and is non-commutative in nature. In order to develop a field theory which involves matter interactions at the Planck energy, the topological object \(Q\) is considered to be fundamental. The equation of motion should then follow from some continuum limit of the loop algebra acting on the four-manifold \(Q \times R\) and the quantum fields defined on it.

The homotopic equation of motion from paperI for a set of prime manifolds
\( T_j \) in the limit of a continuous time variable is given by

\[
\sum_j [T\mathcal{T}, T^{\dagger}\mathcal{T}_j]/\mathcal{T}_j = (TT^{\dagger} + T^{\dagger}T)\mathcal{T}_i.
\] (4)

Because the topology is fixed to be that of \( Q \) now, the interaction between the prime manifolds on the left hand side in Equation (4), must be replaced by the \textit{self-interaction} of a field which lives on \( Q \times R \). Furthermore, the natural limit of the loop creation and annihilation operators is that of two differential operators \( \partial \) and \( \partial^{\dagger} \) with space-time dimension four. \( \partial \) and \( \partial^{\dagger} \) must commute in the continuum limit because \( [T, T^{\dagger}] = 1 \) and the 1 on the right hand side reflects only the discrete nature of the loop algebra. These differential operators should also be conjugate in order to yield a scalar operator of the form \( \square = \partial\partial^{\dagger} = \partial^{\dagger}\partial = \partial_{\mu}\partial^{\mu}, \mu = 1\ldots 4 \). Finally, the right hand side of Equation (4) reflects the fact that the prime manifolds are fundamental objects and that their prime quanta cannot be altered by the topological dynamics of space-time. Now that \( Q \) is the fundamental object and the dynamics of the field it supports are described by the equation of motion, this right hand side should vanish.

This requires the identifications \((T, T^{\dagger} \rightarrow \partial_{\mu}, \partial^{\mu}), \mathcal{T}_i \rightarrow q_{\lambda}\) for the left hand side of Equation (4). This yields for the four vector \( q_{\lambda} \) (see below)

\[
q^{\mu}[q_{\lambda}\square q_{\mu} - q_{\mu}\square q_{\lambda}] = 0,
\] (5)

One finds that the resulting \textit{self-interaction is cubic}, as demanded by the triple loop structure on the three-torus, and linear in the second derivative operators. The possible functional forms of \( q_{\lambda} \) are determined by the quadruplet solutions to the equation of motion, and the cyclic structure of \( Q \) which imposes boundary conditions.

The four-tensor \( q \) is of rank one and its four components correspond to the four currents, i.e. wave amplitudes, which can flow along the homotopically inequivalent paths associated with the \( T^3 \) lattice (paperI). The square of the absolute value of this object gives the probability to find the energy of \( Q \), the dimensional number \( m_{\text{Planck}} \), in the theory, concentrated in some point along any of the four paths through \( Q \). The individual components of \( q_{\lambda} \) yield this probability for each path. To normalize the probability distribution on the underlying lattice of three-tori, one should integrate over the discrete volume \( V_{\text{Planck}} \).

In PaperI it was shown that the \( \text{SL}(2;C) \) gauge group follows naturally from a nuclear manifold. The theory is therefore manifestly Lorentz invariant with \( \partial^{\mu} = \eta^{\mu\nu}\partial_{\nu} \) and \( \square \) the \( \text{d’Alambertian} \). Obviously, the wave function \( q_{\mu} \) is complex-valued and \( q^{\mu} \) is defined as \( \delta^{\mu\nu}q^{\nu} \), to assure a real-valued inner product and therefore positive definite probabilities. A large class of solutions of
(5) is determined by the wave equations $\Box q_\lambda = 0$. This solution is of the form “the boundary of the boundary is zero”, in analogy with the sourceless Maxwell equations. Its solutions are represented by the well-known traveling wave forms. Another class of solutions is determined by the Klein-Gordon equations $\Box q_\lambda + m^2 q_\lambda = 0$. Because these solutions are applicable to a constant charge system, they also hold for the neutral $T^3$ lattice.

3.2.3 Low Energy Behavior

The properties of the resulting QGUT depend on both the charge on the $T^3$ lattice as well as the energy density of the matter degrees of freedom. The former sets the topology of the Planck scale manifold, whereas the latter determines which subgroups of the SU(5) symmetry that lives on the junctions of $L(T^3)$ are realized.

Recently, it has been shown that M-theory unites many of the properties of strings. Although no complete formulation of M-theory exists, it exhibits many encouraging properties. In particular, the manifold $T^3$ plays a fundamental role in the possible explanation of string duality. It is therefore tempting to suggest that for vanishing charge and large energy (above the GUT scale), M-theory results on $L(T^3)$. That is, M-theory corresponds to the neutral quadruplet solutions on the $T^3$ lattice. Clearly, these results hint at underlying properties which remain to be found. The aim of this work will therefore be to use the homotopic properties of $L(T^3)$ and $Q$ to find a formulation of the standard model with particular emphasis on the number and masses of elementary particles.

3.2.4 The Equation of Motion in the Presence of Charge Fluctuations

For large charge densities, $Q_h \approx 1$, $L(T^3)$ is completely interconnected by handles. In this limit $L(T^3) \rightarrow L(Q)$ as the Planck scale manifold for $t \sim t_{\text{Planck}}$. Unlike the three-tori, the mini black holes couple directly to the matter degrees of freedom through the processes of accretion and Hawking radiation. Therefore, even though the charge density is of the order of unity, handles are continuously being created as described in paperI and destroyed through evaporation and merging.

These quantum perturbations in the local number and physical properties of handles lead to the generation of an additional field. This field is envisaged to reflect phase changes in the currents, i.e. the wave amplitudes $q_\lambda$, flowing through $L(T^3)$. The fundamental object to solve for on $L(Q)$ is therefore $\Omega_\lambda \equiv e^{2\pi i \phi} q_\lambda$, with $\phi$ a function of time and position. The real scalar field $\phi$ introduces a modulation of the wave amplitude as a driving force would for a harmonic oscillator. This phase transformation leads to the full QGUT
equation of motion

\[ 4\pi i \partial_\nu \phi[(\partial_\nu q_\lambda)q^\mu q_\mu - (\partial_\nu q_\mu)q^\mu q_\lambda] = q_\lambda q^\mu \Box q_\mu - q^\mu q_\mu \Box q_\lambda, \]  

(6)

with the scalar constraint

\[ q^\mu q_\mu = \text{cst}. \]  

(7)

Due to the continuous creation and destruction of $\Theta$ manifolds, it is possible to travel from a path to any other path. Therefore, from the point of view of the wave amplitudes, the points along a path become indistinguishable, although the homotopy persists. The scalar constraint (7) then expresses the fact that the total probability to find all the mass-energy of $Q$ in some point is the same for all points in $Q$, albeit at the expense of the scalar field $\phi$. This also implies that both on $L(Q)$ and $L(T^3)$ the solutions are identical on every three-torus.

Because the evolution of $\phi$ is driven by the handle degrees of freedom, it follows that on the neutral $T^3$ lattice the field $\phi$ obeys the limiting condition $\partial_\nu \phi = 0$, and is effectively frozen in. In the low charge limit, the field $\phi$ therefore has a constant, Lorentz invariant vacuum expectation value $\langle 0|\phi(x)|0 \rangle = c \neq 0$, defined on the junctions of the $T^3$ lattice. A non-zero vacuum expectation value of $\phi$ requires $\mu^2 < 0$ and leads to spontaneous symmetry breaking, as first suggested by Nambu and co-workers. The additional Poincaré scalar $\phi$ can therefore be identified with a Higgs field, although the number of Higgses is not constrained. In fact, upon freeze-out only the total norm $\phi_l \phi^l$ would be fixed. Furthermore, the constraint (7) no longer applies since $\Theta$ has evaporated.

3.2.5 The Heat Capacity of $Q$

The number of degrees of freedom of $Q$ under the action of the loop algebra is defined as

\[ N_Q = (TT^\dagger + T^\dagger T)(2T^3 \oplus 3S^1 \times S^2) = 23. \]  

(8)

In the loop homotopic approach adopted here, these degrees of freedom are all equivalent and they reflect the different ways in which the loop creation and annihilation operators can act on $Q$ while conserving the homotopic structure of the topological manifold. Any fields which are defined on the non-prime manifold $Q$ are therefore automatically partitioned over these 23 degrees of freedom.

For the neutral submanifold $P = T^3 \oplus T^3$, one has $N_P = 14$. The latent heat associated with the evaporation of the handle triplet $\Theta = 3S^1 \times S^2$ is therefore

\[ H = (N_Q - N_P) m_{\text{Planck}}/N_Q = 9m_{\text{Planck}}/23. \]  

(9)

This number is uniquely determined by the homotopic structure of space-time and the Planck mass. Since both three-tori in the structure $Q$ are equivalent,
the specific heat per three-torus $h$ is given by

$$ h = H/2 = 9 m_{\text{Planck}}/46. \tag{10} $$

### 3.2.6 The QGUT Phase and Large Scale Structure

From the above discussion it follows that the QGUT energy scale corresponds to a size and matter density of the universe where mini black holes are formed rapidly enough to sustain the topological manifold $Q$. During this phase the 23 degrees of freedom of $Q$ are accessible to the wave amplitudes $q_\lambda$ traveling through $L(Q)$. As these currents self-interact, they do so cubically on the three-torus as in Equation (6). The currents carry the mass-energy of the universe and their interactions determine the perturbations in the mass-energy associated with the 23 equivalent degrees of freedom.

The strength of these perturbations in the mass-energy is given by

$$ \frac{\delta \rho}{\rho(Q)} = N_Q^{-3} = 8.2 \times 10^{-5}. \tag{11a} $$

This amplitude is the first non-zero correction term to the average energy in any of the degrees of freedom of $Q$ on purely thermodynamic grounds. PaperI discusses the effective dispersion of a large ensemble of Gaussian perturbations on a lattice of three-tori and this leads to the $1\sigma$ effective amplitude in the mass-energy $E$

$$ \frac{\delta \rho}{\rho(E)} = 3.7 \times 10^{-5}. \tag{11b} $$

For adiabatic perturbations on the horizon scale the fluctuations in the Cosmic Microwave Background temperature is $1/3$ of $\delta \rho/\rho(E)$ and one finds

$$ \delta T/T = \frac{1}{3} \delta \rho/\rho(E) = 1.2 \times 10^{-5}. \tag{12} $$

This amplitude is consistent with recent COBE measurements.

### 4 The Standard Model from the Symmetries of $Q$

With the fundamental manifold and the equation of motion which lives on it in place, one now needs to isolate the symmetry properties of $Q$ and solutions to Equations (5) and (6) in order to arrive at a description of the standard model with as few free parameters as possible.

#### 4.1 Preliminaries
4.1.1 Discrete Groups Generated by $P$ and $\Theta$

The handle manifolds which are created in quantum fluctuations always form triplets on $L(T^3)$. Therefore, the effective action $s^3$, of the triplet as a whole, on any excitation of $Q$ obeys

$$s^3 = 1. \quad (13)$$

That is, a round trip along the manifold $\Theta$ necessarily picks up three phases, which should add up to $2\pi$ since the loop algebra satisfies $[T, T^\dagger] = 1$. Because all 3 handles are identical, this implies a $Z_3$ invariance for the individual quantum fields in the theory defined on $Q$ with angles $\theta_i = \{0, \pm 2\pi/3\}$. From the same arguments it follows that the submanifold $P = T^3 \oplus T^3$ in $Q$ generates a $Z_2 \times Z_3$ symmetry because one can distinguish neither the three loops in a three-torus nor a $T^3$ contained in $P$. Also, the $Z_3$ groups belongs to $SU(3)$ because the latter consists of elements like (13).

4.1.2 $T^3$ Junctions and U(1) Factors

The important distinction between $T^3$ and $L(T^3)$, or $L(Q)$ for that matter, is the presence of junctions which connect the individual three-tori through three-ball surgery and create a lattice. In paperI it was suggested that the presence of a lattice facilitates a geometric description of gravitational effects because the three-dimensional junctions can bend according to some curvature tensor. Indeed, on scales much larger than $\ell_{\text{Planck}}$ the neutral lattice $L(T^3)$ appears as a smooth manifold. On the Planck scale on the other hand, the existence of junctions between the three-tori generates an additional U(1) symmetry. As one patches the individual three-tori together, there is an arbitrary rotation, or twist, one can perform without changing the topological properties of the manifold. Obviously, there is only one twist per $Q$ manifold, i.e. per junction, but each individual three-torus in the lattice formally has six of them. These additional U(1) factors have been proposed as a possible resolution of the doublet-triplet splitting problem[5]. From the discussion above it follows that the existence of the $Z_2 \times Z_3$ cyclic group through $P$ and the absence (presence) of an anomalous U(1) sector are equivalent properties of the $Q$ ($T^3$) lattice.

4.2 Construction of the Standard Model

With these preliminaries in mind, the aim now is to reproduce the features of the standard in the low energy and low charge limit. The philosophy is to group the 23 homotopic degrees of freedom of $Q$ into different equivalence classes under the action of the scalar operators $A \equiv TT^\dagger$ and $B \equiv T^\dagger T$ on $P$ and $\Theta$.  

11
### 4.2.1 Equivalence Classes and Broken $Q$-Supersymmetry

The homotopic properties of $Q$, the SU(5) gauge group on the junction and the $Z_3$ and $Z_2 \times Z_3$ cyclic symmetries of $\Theta$ and $P$ will lead to specific particle sectors. In this, the photon and graviton are not viewed as being generated through the homotopic structure of $Q$, but result from the junction degrees of freedom, i.e. the U(1) twist and GL(4) curvature of $L(T^3)$. Furthermore, there is a symmetry breaking Higgs field which lives on the junction and can become massive, leading to Higgs bosons.

The number of degrees of freedom $N_Q$ is the eigenvalue of the operator $O = TT^\dagger + T^\dagger T \equiv A + B$ acting on $Q$. There is a natural division of the 23 degrees of freedom under $AQ = 14Q$ and $BQ = 9Q$. Furthermore, the decomposition $OQ = O(P \oplus \Theta)$ has the same distribution of degrees of freedom under $A$ and $B$ and leads to the further divisions

$$AQ = OP = (8 + 6)P,$$

with eight gluons and six quarks and

$$BQ = O\Theta = (3 + 6)\Theta,$$

with six leptons and three vector bosons. This identification follows from the fact that the junction potential on $P$ supports the symmetry group SU(5). The $P$ and $\Theta$ sectors decompose $Q$ and are therefore associated with subgroups of SU(5). These subgroups can only contain SU($n \leq 5$) and U(1) because of the junction potential and twist. For SU(5) = SU(3)$\times$SU(2)$\times$U(1) these constraints are satisfied. Since SU(3) contains 8 field particles and SU(2) only 3, the identification of $P$ with QCD and $\Theta$ with the electro-weak interaction is immediate.

A priori both fermionic and bosonic sectors exist for these equivalence classes. That is, because the form of $Q$ is motivated by Lorentz invariance and $Q$-supersymmetry, the identified equivalence classes can be both fermionic and bosonic in nature. When $Q$-supersymmetry is broken the structure $L(T^3)$ with the three-torus as fundamental Planck scale object assures Lorentz invariance. Subsequently, interactions are mediated by field particles which travel along the 6 junctions surrounding any $T^3$. Furthermore, it is the discrete three-torus with its seven degrees of freedom under the operator $O$ which supports a particle and its field quanta. Any field dynamics on $L(T^3)$ therefore requires the interaction of two field particles on a three-torus. If these field particles are fermionic, this violates the Pauli exclusion principle.

Thus, only bosonic field particles can carry the strong and electro-weak force, and satisfy the Pauli exclusion principle on $L(T^3)$, after the handle triplets have evaporated. Therefore, even though the principle of supersymmetry is
necessary to identify the fundamental manifold $Q$, the requirement of Lorentz invariance leads to $L(T^3)$ and forces field particles to be bosonic. As such, it determines the way in which $Q$-supersymmetry is broken when the handle triplet evaporates. The origin of the Pauli exclusion principle follows from the homotopic structure of $T^3$ and the fact that a spin $1/2$ particle requires two loops on a three-torus for its support. For two identical fermions this implies the general relation (see also §2.2 above)

\[ [ac()][cb] = [ca()][ab], \quad (16a) \]

which yields

\[ [cb] = [ab]. \quad (16b) \]

This indicates that because one $S^1$ loop is a part of both fermions, the other two are collapsed to one. The consequence is that the three-torus becomes indistinguishable from the prime manifold $S^1 \times R_1$, with $R_1$ a Riemann surface of genus one. This manifold is not nuclear and therefore breaks Lorentz invariance. It is straightforward to verify that the definition of two identical bosons preserves the homotopy of $T^3$ because they involve disjunct loops.

### 4.2.2 Interpretation of the Equation of Motion

Equations (5), (6) and (7) describe the quantum-mechanical interactions of matter in full, i.e. including quantum gravity. The boundary conditions for the solutions should follow from the cyclic properties of $L(T^3)$. The topology of $T^3$ requires the solutions $O_\lambda(x, y, z, t)$ to be periodic on a cube of size $L$ for every time $t$,

\[ O_\lambda(x, y, z, t) = O_\lambda(x + L, y + L, z + L, t). \quad (17) \]

The handle triplet then requires a solution which limits to $\phi = 1/3$ when $Q$-supersymmetry is broken. This phase relation is consistent with the fact that the equation of motion (6) is invariant under the global transformation $\phi \rightarrow \phi + \alpha$. This freedom is fixed by the underlying topology of $Q$. The initial conditions at $t = 0$ for the solutions of (6) can then be taken as $q_\lambda(0) = \text{cst}$, derivatives $\partial_\nu q_\lambda(0)$ given by the Planck energy and $\phi(0) = 0$.

For constant values of $\phi$, i.e. $Q_h \sim 0$, the general solution of (6) can naturally limit to the Klein-Gordon equations $(\Box + m^2)q_\lambda = 0$, which support massive particles. This situation applies when $Q$-supersymmetry is broken. In this phase, the statistics of the mass-energy distribution is represented by the absolute value $q^\mu q_\mu$ of the wave modes. To follow the evolution of the matter degrees of freedom during this epoch, one should solve Equation (5) with the QGUT end solution of (6) as initial conditions. Once, the GUT is broken at some energy (see below) the Einstein equation describes the later time evolution of the mass-energy distribution.
A question which can be assessed is the nature of the statistics of mass-energy fluctuations after the GUT is broken. For this, the solution space of Equation (5) needs to be investigated. Finally, the cubic periodicity of the solutions to the equation of motion on $L(T^3)$ does not lead to a global $T^3$ topology at the present epoch. The Planck size branches of the three-torus are only accessible to sufficiently energetic particles and the global appearance of the universe is therefore dominated by the $S^3$ junctions (paperI).

4.2.3 Symmetry Groups, the $\mu$ Problem and Proton Stability

The homotopic theory as it stands identifies $SU(5)$ and $SO(10)$ as gauge groups and $Z_3$ and $Z_2 \times Z_3$ as additional discrete groups, but leaves room for possible (and in fact necessary) extensions of the minimal standard model. A fundamental problem in supersymmetric GUT is the doublet-triplet splitting problem which results from the unavoidable mixing of Higgs doublets $H, \bar{H}$ with their colored triplet partners $T, \bar{T}$. This also leads to an unacceptably rapid proton decay. In [6] it was suggested that there is no need for the heavy triplet if its Yukawa coupling constant is strongly suppressed with respect to the one of the doublet. This mechanism requires an $SO(10)$ invariant operator with tensor indices $i, k$

$$\frac{Y_{\alpha,\beta}}{M_{GUT}}10_i45_{ik}16^\alpha\gamma_k16^\beta,$$  \hspace{1cm} (18)

in which $16^\alpha (\alpha = 1..3)$ are three families of matter fermions, $10_i (i = 1..10)$ is the multiplet with $H, \bar{H} (i = 7..10)$ and $T, \bar{T} (i = 1..6)$. The 45 is the GUT Higgs in the adjoint presentation of $SO(10)$, $Y_{\alpha,\beta}$ is the coupling constant matrix, and the $\gamma_i$ denote the matrices of the $SO(10)$ Clifford algebra. To realize this effective operator, the 10-plet must transform under the symmetry group $Z_2 \times Z_3$ such that the 10-plet does not couple to the GUT Higgses and the 10-plet is allowed to interact with $16^\alpha$ only in combination with the 45-plet[6].

A possible resolution of the $\mu$ problem in the standard model thus naturally involves the cyclic group $Z_2 \times Z_3$ associated with $P$. In addition, the submanifold $\Theta$ generates $Z_3$. If one introduces a light gauge singlet superfield $N$, then the triple interactions on the individual three-tori fix the associated superpotential to be of the form

$$W = \lambda_1 N10^2 + \lambda_2 N^3.$$  \hspace{1cm} (19)

This potential is invariant under $Z_2 \times Z_3$ because the singlet $N$ has an $Z_2$ invariance on $P$. Both $N$ and 10 do not transform under $Z_3$ due to $\Theta$, and therefore they decouple from the heavy GUT Higgs fields. This provides a natural resolution of the doublet-triplet problem in terms of the homotopy
of space-time as already anticipated in [6], if one accepts the existence of an additional gauge singlet. In fact, the existence of this object is a direct consequence of the anomalous U(1) symmetry which is hidden in Q and emerges when Q-supersymmetry is broken to L(T³) (see §4.1.2 above). Because the triplets have no coupling at all, it follows that the proton is essentially stable even if the decoupled triplet is as light as its doublet partner. That is, its decay rate is suppressed by a factor which is no larger than \((M_W/M_{GUT})^2\), with \(M_W\) the mass of the weak scale. Specific models based on SO(10) are constructed in [6], and the existence of long-lived \(T, \bar{T}\) supermultiplets in the 100 GeV-TeV mass range is discussed there as well.

4.2.4 The Fermion Mass Hierarchy in the Standard Model

Another fundamental problem which requires a resolution in GUT is the specific form of the fermion mass hierarchy. The popular approach is to use the anomalous U(1) gauge symmetry as a horizontal symmetry[5]. The motivation is that the anomalous U(1) symmetry breaking scale is given by

\[
\frac{\sqrt{\xi}}{M_{\text{Planck}}} \sim 0.1 - 0.01,
\]

with \(M_{\text{Planck}} = 2 \times 10^{18}\) Gev the reduced Planck mass and \(\xi\) the Fayet-Iliopoulos term[5]. This ratio is of the order of the fermion mass ratios in neighbouring families. The precise value of \(<N>=\sqrt{\xi}/q\), with \(q\) the anomalous U(1) charge, depends on the value of \(\text{tr}Q = \text{tr}Q_{\text{obs}} + \text{tr}Q_{\text{hid}}\), with contributions from the observable and hidden matter singlet[5]. A second mass scale corresponds to \(M_{GUT} = <\phi>\), which marks the energy at which the SU(5) theory is broken down to the SU(3)×SU(2)×U(1) symmetry of the standard model. Obviously, both energy scales should have a common origin.

When the handles evaporate, the expectation value of \(\phi\) becomes 1/3 which breaks the SU(5) symmetry. Since \(L(T^3)\) is the central object for a grand unified theory, the GUT energy scale should correspond to 1/3 times the specific energy per three-torus of a degree of freedom in \(Q\). That is, the breaking of a GUT corresponds to the fact that the energy associated with the fundamental loop, which generates a non-trivial homotopy, is no longer available to excite the loop creation and annihilation operators. At this point, the wave amplitudes \(q_\lambda\) can no longer be influenced by Equation (5). Because the 23 degrees of freedom on \(Q\) are equivalent one expects approximate equipartition of energy for the possible wave amplitudes on \(Q\). Therefore, one finds \(M_{GUT} = 1/3m_{\text{Planck}}/2N_Q \approx 8.8 \times 10^{16}\) GeV, with \(m_{\text{Planck}} = 1/\sqrt{G} = 1.22 \times 10^{19}\) GeV in units with \(\hbar = c = 1\). Above it was shown that the latent heat per \(T^3\) associated with breaking of \(Q\)-supersymmetry, equals \(h = 9/46m_{\text{Planck}}\). Therefore, one finds for the singlet superfield \(<N>=1/3h \approx 8.0 \times 10^{17}\) GeV. Note that this expectation value is again purely topological. It should be emphasized
that the presence of the U(1) factors and the various energy scales are unique predictions of the model, but that a number of field theoretic implementations are possible.

Because \( N \) is an SU(5) singlet, one normally finds identical mass matrices for the charged leptons and the down quarks. Observationally, there is about a factor of 3 splitting between the down quark and lepton masses in the same family. In the homotopic theory presented here, the electro-weak and QCD sector are associated with \( \Theta \) and \( P \), respectively. Because the degrees of freedom of each sector are in equipartition with one another for \( Q_h \sim 1 \), the mass separation \( \rho = M(\Theta)/M(P) \) on \( L(T^3) \) is given by the eigenvalues of the topological operator \( O \). One finds \( \rho = 9/14 \approx 0.64 \) for the ratio of the degrees of freedom.

The unique ratio \( \delta = \langle N \rangle /m_{\text{Planck}} \approx 6.5 \times 10^{-2} \) then yields \( \epsilon = \rho \delta \approx 4.2 \times 10^{-2} \) which reproduces the observed pattern of quark masses in a supersymmetric SU(5) model as in [5] with the anomalous U(1) symmetry of \( L(T^3) \), according to

\[
\begin{align*}
m_t : m_c : m_u & \sim 1 : \epsilon^2 : \epsilon^4 \quad (21a) \\
m_b : m_s : m_d & \sim 1 : \epsilon : \epsilon^2 \quad (21b)
\end{align*}
\]

with the CKM mixing angles

\[
\sigma_{12}, \sigma_{23} \sim \epsilon, \quad s_{13} \sim \epsilon^2. \quad (22)
\]

The observed value of \( \text{tr}Q \) is consistent with this result for \( \epsilon \) if there is a significant negative contribution from the hidden sector[5].

4.2.5 The Absolute Scale of the Mass Hierarchy

The mass scalings have been identified above for a particular implementation of the anomalous U(1) symmetry, but no absolute scale has been determined. It will now be shown that, just like the energy scales of the fields generated by \( Q \) are uniquely determined by the homotopic theory, so is the low mass end of the lepton mass hierarchy. That is, ground state energies can be identified for the electron and neutrino in the electro-weak sector, because \( Q \)-supersymmetry is broken by the evaporation of \( \Theta \).

Before \( Q \)-supersymmetry is broken, the 23 degrees of freedom of \( Q \) are in equilibrium and the configuration space \( \mathcal{C} \) of possible (equivalent) states on \( Q \) consists of

\[
X_Q = 23! \approx 2.6 \times 10^{22} \quad (23)
\]

elements. These states are associated with particles created after symmetry breaking and their mass is thus \( m_{\text{Planck}}/X_Q \). Because \( Q \)-supersymmetry is broken when the handles evaporate, the particle in question belongs to the
electro-weak $\Theta$ sector. It should be charged because $Q$ contains a U(1) sector (twist), and be a ground state since $Q$ generates the maximal size configuration space. It must therefore fix the low part of the charged lepton spectrum. It follows that

$$m_0^e = m_{\text{Planck}}/X_Q \approx 0.47 \text{ MeV} \quad (24)$$

determines the electron mass. This estimate agrees with observations at the 8\% level.

For the neutral submanifold $P$ one has $X_P = 14!$, which fixes the number of neutral substates on $L(T^3)$ after $Q$-supersymmetry breaking, and yields the extended configuration space $\tilde{\mathcal{C}}$ with $X_QX_P \approx 2.3 \times 10^{33}$ elements. The energy of the neutral (neutrino) ground state of $\tilde{\mathcal{C}}$ thus follows from

$$m_0^{\nu_e} = m_{\text{Planck}}/(X_QX_P) \approx 5.4 \times 10^{-6} \text{ eV}. \quad (25)$$

Because the neutrinos have no charge, they cannot be distinguished on $P$, unlike the charged leptons which couple non-trivially to the U(1) sector on the junction of $P$. The photon and graviton are massless because they do not depend on the homotopy of space-time. The masses of the vector bosons are determined by the breaking of the SU(2) symmetry which requires the direct intervention of a Higgs field.

### 4.2.6 Higher Order Corrections to the Mass Hierarchy

Higher order quantum corrections on $L(T^3)$ occur, which correspond to the cubic interactions of the particle wave modes in their respective homotopic equivalence classes. If $s_Q$ describes the dispersion of the probability distribution on $Q$, then one can ask with what accuracy $\mathcal{A}$ the properties of $P$ can be determined, given it has a dispersion $s_P$. The uncertainty relation then yields $s_Q = \mathcal{A}s_P$. This question is relevant to the uncertainty of the energy levels of $\tilde{\mathcal{C}}$ since the evaporating mini black holes occupy only a part of the total number of degrees of freedom on $Q$. The relative uncertainty in the ground state energies is therefore $\mathcal{A} = (14/23)^3 = 0.23$. From a thermodynamic point of view this correction reflects a shift in the zero point energy. Still, the topological degrees of freedom set an absolute lower limit to the energy level and the shift can only be upwards. That is, the uncertainty implies a lack of information or smaller entropy and therefore a higher excited state. Clearly, given an energy $m_{\text{Planck}}$, the entropy of the universe can be maximized by allowing for as many occupied states as possible. The accuracy $\mathcal{A}$ is consistent with the underestimate for the electron mass. Even though the statistics of the specific realizations of universes is not known, one can give the confident limit on the electron mass

$$0.47 < m_e < 0.58 \text{ MeV}, \quad (26)$$

which is in good agreement with the measured value of 0.51 MeV.
4.2.7 CP Violation

There is one final global degree of freedom on $L(T^3)$: the lattice junction in a pair of three-tori can vibrate. The energy $F$ of the excitation is given by

$$F = \frac{m_{\text{Planck}} - H}{X_Q X_P} \approx 3.3 \times 10^{-6} \text{ eV}. \quad (27)$$

The denominator reflects the total number of configurations in $\bar{C}$. Until the mini black holes evaporate and the latent heat $H$ is released, a typical energy of $m_{\text{Planck}} - H$ is confined to the internal degrees of freedom of $L(T^3)$. This introduces an additional energy state in $\bar{C}$ which cannot be massive because it is below all particle thresholds, and is therefore associated here with the lattice itself.

To excite this internal degree of freedom one requires a system of two neutral particles which are in resonance. That is, (1) the particles should not be independent because of a common decay route. This allows them to form a mixture with two superposition states. (2) The difference in self-energy between these superposition states should be larger than $F$ to excite the vibration but smaller than the lowest particle ground state in $\bar{C}$. It is well known that the difference in weak self-energy determined by the superposition states

$$K_S \leftrightarrow 2\pi \leftrightarrow K_S \quad (28)$$

and

$$K_L \leftrightarrow 3\pi \leftrightarrow K_L \quad (29)$$

for the decay of the neutral $K^0$ and $\bar{K}^0$ mesons, is extremely small and equal to $f \approx 3.5 \times 10^{-6} \text{ eV}$. Indeed, $f > F$ but below particle threshold. It is this small asymmetry which allows the CP (but not CPT) violating $K$ meson decay processes to occur through $K_L \rightarrow K_S \rightarrow 2\pi$.

5 Conclusion

It has been shown that the requirement of Lorentz invariance follows from the superposition principle and leads to a spatial dimension of the universe equal to three. The presence of forces has been found to imply the physical necessity of the three-torus as the fundamental Planck scale object. Together with mini black holes these three-tori can form a fundamental manifold which is Lorentz invariant and provides a natural mechanism for symmetry breaking through black hole evaporation. An equation of motion has been derived for a QGUT on this supersymmetric manifold $Q = 2T^3 \oplus 3S^1 \times S^2$, which naturally leads to a Higgs field. The manifold $Q$ contains the necessary symmetry groups for the standard model with a stable proton, CP violation and doublet-triplet
splitting, and possesses intrinsic energy scales which reproduce the cosmic microwave background fluctuations. These properties are independent of possible extensions of the standard model, which will allow tighter constraints to be placed on GUTs.

The author is indebted to G. van Naeltwijck van Diosne, J.A.A. Berendse-Vogels, W.G. Berendse and M.A.R. Bremer for valuable assistance.

References

[1] S. Kaku, Introduction to Superstrings (Springer-Verlag, 1990).

[2] S.W. Hawking and A. Strominger 1991 in Quantum Cosmology and Baby Universes, eds. S. Coleman, J.B. Hartle, T. Piran & S. Weinberg, (World Scientific, 1991) p. 245, p. 272.

[3] M. Spaans, Nucl. Phys. B 492 (1997) 526.

[4] J.A. Wheeler in Quantum Cosmology, eds. L.Z. Fang & R. Ruffini, (World Scientific, 1987) p. 27.

[5] Z. Berezhiani and Z. Tavartkiladze, Phys. Lett. B 396 (1997) 150.

[6] G. Dvali, Phys. Lett. B 372 (1996) 113.