THE LITTLEWOOD-PALEY $p$th-ORDER MOMENTS IN
THREE-DIMENSIONAL MHD TURBULENCE

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Abstract. In this paper, we consider the Littlewood-Paley $p$th-order ($1 \leq p < \infty$) moments of the three-dimensional MHD periodic equations, which are defined by the infinite-time and space average of $L^p$-norm of velocity and magnetic fields involved in the spectral cut-off operator $\dot{\Delta}_m$. Our results imply that in some cases, $k^{-3}$ is an upper bound at length scale $1/k$. This coincides with the scaling law of many observations on astrophysical systems and simulations in terms of 3D MHD turbulence.

1. Introduction. In many natural phenomena, magnetic fields are closely related to many natural and man-made fluids. The model of magnetohydrodynamics (MHD) system is applied to study the interaction between magnetic fields and moving, conducting flows. More precisely, incompressible magnetohydrodynamical equations read

\[
\begin{aligned}
\partial_t u - \nu \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla P &= f, \\
\partial_t b - \mu \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u &= g, \\
\text{div } u = \text{div } b &= 0,
\end{aligned}
\]

(MHD)

where vector functions $u$ and $b$ represent velocity field and magnetic field respectively. $P$ is thermodynamic pressure. The constants $\nu, \mu > 0$ are kinematic viscosity and magnetic diffusivity. $f$ and $g$ are external force terms, which are smooth time-independent vector fields.

The fact that nearly the dynamics of all non-stationary astrophysical plasma flows feature a diversity of length scales and speed turns out widespread existence of turbulence in the universe. Turbulence acts a pivotal part in MHD system. For instance, the flux tube reconnections in astrophysical MHD cannot be interpreted by small molecular diffusivity, while turbulence theory is an effective tool for it.

Turbulence theory mainly involves statistical properties of fluids, which are usually obtained under some symmetry hypothesis. Therefore, considering MHD system in $\mathbb{R}^3$ can gain maximum symmetry. However, the whole space usually gives

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rise to some difficulties in both mathematical derivations and numerical simulations. Hence researchers often study MHD turbulence with periodic boundary conditions.

In this paper, we consider (MHD) system in a cubic domain $\Omega = [0, L]^3$ and assume that $u$, $b$, $f$, $g$ and $P$ are $\Omega$-periodic functions. This system is invariant under the Galilean transformation: $t = t, x = \tilde{x} + U_0 t$, $u = \tilde{u} + U_0, b = b$. Therefore we can restrict ourselves to consider the solutions $(u, b)$ with $\langle u \rangle = 0$, where $(u) = L^{-3} \int_{\Omega} u \, dx$. A mean magnetic field $B_0 \neq 0$ leads to the presence of Alfvén wave in MHD. Hence researchers often study MHD turbulence with periodic boundary conditions.

Therefore we can restrict ourselves to consider the solutions $(u, b)$ with $\langle u \rangle = 0$, where $(u) = L^{-3} \int_{\Omega} u \, dx$. A mean magnetic field $B_0 \neq 0$ leads to the presence of Alfvén wave. Hence, we drop the tilde for simplicity and the resulting equations become

\[
\begin{align*}
\partial_t u - \nu \Delta u + (u \cdot \nabla) u - (b \cdot \nabla) b - (B_0 \cdot \nabla) b + \nabla P &= f, \\
\partial_t b - \mu \Delta b + (u \cdot \nabla) b - (b \cdot \nabla) u - (B_0 \cdot \nabla) u &= g, \\
\text{div } u &= \text{div } b = 0,
\end{align*}
\]

(MHD')

associated with initial data $u_0$ and $b_0$ satisfying divergence-free, $\Omega$-periodic boundary conditions as well as $\langle u_0 \rangle = \langle b_0 \rangle = 0$. Moreover, $u$, $b$, $f$ and $g$ possess zero spatial average. This model is widely applied to various physical states. For example, solar wind and neutral plasma in fusion confinement devices. Therefore, experimentalists and theoreticians have deep interest in studying the scaling properties of MHD turbulence due to the interaction between the propagating Alfvén modes and dissipative modes.

1.1 Main results. As we know, Leray [20] established global existence theory of weak solutions for the Cauchy problem for the Navier-Stokes equations with initial data $u_0 \in L^2(\mathbb{R}^3)$. And this result was further developed by Hopf [15]. We call these solutions introduced in [15, 20] as Leray-Hopf weak solutions. In this paper, $L^2(\Omega)$ and $H^1(\Omega)$ denote the set of $\Omega$-periodic functions that respectively belong to $L^2(\Omega)$ and $H^1(\Omega)$ with zero spatial average. Following [20], it can be shown that there exists a Leray-Hopf type weak solution of system (MHD') for initial data $(u_0, b_0) \in L^2(\Omega)$.

**Definition 1.1.** Let $(u_0, b_0) \in L^2(\Omega)$ with divergence-free and $(f, g) \in H^{-1}(\Omega)$. $(u, b, P)$ is called a Leray-Hopf type weak solution of system (MHD') if

- $(u, b, P)$ satisfies equations (MHD') in the distribution sense. Moreover, $(u(t), b(t)) \to (u_0, b_0)$ weakly in $(L^2(\Omega))^3$ as $t \to 0$.
- $(u, b) \in (L^\infty_{\text{loc}}(\mathbb{R}^+; (L^2(\Omega))^3) \cap L^2_{\text{loc}}(\mathbb{R}^+; (H^1(\Omega))^3))^2$.
- For a.e. $T > 0$,
  \[
  \|u(\cdot, T)\|^2_{L^2(\Omega)} + \|b(\cdot, T)\|^2_{L^2(\Omega)} + \int_0^T (2\nu \|\nabla u(\cdot, s)\|^2_{L^2(\Omega)} + 2\mu \|\nabla b(\cdot, s)\|^2_{L^2(\Omega)}) \, ds \\
  \leq 2 \int_0^T \langle u(f) \rangle_{H^1, H^{-1}} \, ds + 2 \int_0^T \langle b(g) \rangle_{H^1, H^{-1}} \, ds + \|u_0\|^2_{L^2(\Omega)} + \|b_0\|^2_{L^2(\Omega)}.
  \]

Before giving our main results, we provide some definitions and notations firstly.

**Definition 1.2.** Let $v$ be a scalar $\Omega$-periodic function. The spatial average of $v$ is denoted by

\[
\langle v \rangle \overset{\text{def}}{=} \frac{1}{L^3} \int_{\Omega} v(x) \, dx.
\]
Moreover, for any \(1 \leq p < \infty\) and \(T > 0\), we define finite time and space average of the \(p\)th power of \(v\) as
\[
\langle \langle |v|^p \rangle \rangle_T \overset{\text{def}}{=} \frac{1}{T} \int_0^T \langle |v(s,\cdot)|^p \rangle \, ds,
\]
and the infinite time and space average of the \(p\)th power of \(v\) is denoted by
\[
\langle \langle |v|^p \rangle \rangle \overset{\text{def}}{=} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \langle |v(s,\cdot)|^p \rangle \, ds.
\]
Similarly, for the vector function \(\mathbf{v} = (v_1, v_2, v_3)\), we define
\[
\langle \langle |\mathbf{v}|^p \rangle \rangle_T = \sum_{i=1}^3 \langle \langle |v_i|^p \rangle \rangle_T, \quad \langle \langle |\nabla \mathbf{v}|^2 \rangle \rangle = \sum_{i,j=1}^3 \langle \langle |\partial_x v_i|^2 \rangle \rangle.
\]

Now we give our main results and then provide some comments from a physical perspective of our main results.

**Theorem 1.3.** Let \((\mathbf{u}, \mathbf{b}, P)\) be a Leray-Hopf type weak solution of system (MHD') and the forcing terms \(f, g\) satisfy that \(\Delta \mathbf{m} f = \Delta \mathbf{m} g = 0\) for each \(m \geq k_0\), \(\mu, \nu\) are positive constants. We define \(\mu \land \nu = \min\{\mu, \nu\}\) and \(\mu \lor \nu = \max\{\mu, \nu\}\). Then
\[
\langle \langle |\Delta \mathbf{m} \mathbf{u}|^p \rangle \rangle + \langle \langle |\Delta \mathbf{m} \mathbf{b}|^p \rangle \rangle \leq C(\mu \land \nu)^{-1} 2^{-3m} \left( \langle \langle |\nabla \mathbf{u}|^2 \rangle \rangle + \langle \langle |\nabla \mathbf{b}|^2 \rangle \rangle \right)
\]
holds for every \(m \geq k_0\). Here \(C\) depends on \(\frac{\mu \lor \nu}{\mu \land \nu}\) and \(\phi_0\), a function stemming from localization homogeneous operator \(\tilde{\Delta}_0\).

**Theorem 1.4.** Let \((\mathbf{u}, \mathbf{b}, P)\) be a Leray-Hopf type weak solution of system (MHD') and the forcing terms \(f, g\) satisfy that \(\Delta \mathbf{m} f = \Delta \mathbf{m} g = 0\) for each \(m \geq k_0\), \(\mu, \nu\) are positive. We define \(\mu \land \nu = \min\{\mu, \nu\}\) and \(\mu \lor \nu = \max\{\mu, \nu\}\). If \(\bar{\varepsilon}_{p+1} < \infty\), then
\[
\langle \langle |\Delta \mathbf{m} \mathbf{u}|^p \rangle \rangle^{\frac{1}{p}} + \langle \langle |\Delta \mathbf{m} \mathbf{b}|^p \rangle \rangle^{\frac{1}{p}} \leq C_{p, \mu, \nu} 2^{-m(\frac{1}{p} + \frac{1}{p'})}(\bar{\varepsilon}_{p+1})^{\frac{1}{p'}}.
\]
where
\begin{enumerate}
  \item For \(p = 1\), \(C_{1, \mu, \nu} = C \left( \frac{\mu \lor \nu}{\mu \land \nu}, \phi_0 \right) (\mu \land \nu)^{-1}\),
  \item For \(p = 2\), \(C_{2, \mu, \nu} = C(\phi_0)(\mu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}})\),
  \item For \(1 < p \neq 2 < \infty\) and \(\mu \land \nu > \frac{1}{4}(\mu \lor \nu)\), \(C_{p, \mu, \nu} = C \left( \frac{3(\mu \lor \nu) - \mu \land \nu}{3(\mu \land \nu) - \mu \lor \nu}, \phi_0 \right) (3(\mu \land \nu) - \mu \lor \nu)^{-\frac{1}{p'}}\).
\end{enumerate}
And
\[
\bar{\varepsilon}_{p+1} = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \|u(\cdot, t)||^p \, dt + \limsup_{T \to \infty} \frac{1}{T} \int_0^T \|b(\cdot, t)||^p \, dt.
\]

**1.2 Comments on the results from a physical perspective.** The classical hydrodynamic turbulence theory was developed by Kolmogorov [17, 18], in which a turbulence velocity field was considered as the composition of eddies depicted by a wide range of spatial scales. When the Reynolds number is large enough, there exist two scales \(\ell_{EI}\) and \(\ell_{DI}\) to differ three spatial scale ranges. \(\ell > \ell_{EI}\) is the energy containing range and \(\ell < \ell_{DI}\) is the dissipation range where viscous effect dominates energy dissipation. For \(\ell \in [\ell_{DI}, \ell_{EI}]\), called inertial range, turbulence can be characterized by some scaling properties. For example, we use the notation \(\delta v = v(x + \ell) - v(x)\) and define structure function of order \(n\) by \(S^n(\ell) = \langle |\delta v|^n \rangle\). We denote the viscosity coefficient by \(\nu\) and the energy dissipation rate per unit
mass by \( \epsilon \). Moreover, there has the following law for the nonintermittent limiting case in inertial range:

\[
S_n^v(\ell) \sim (\ell \epsilon)^{\frac{4}{3}}
\]

(1)

Energy spectrum, an important quantitative tool in studying turbulence theory, is a function with respect to wave numbers \( k \) defined as

\[
E_v(k) = \frac{1}{2} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \int_{|\xi|=k} |\hat{v}(\xi, t)|^2 \, dS(\xi) \, dt,
\]

(2)

where \( \hat{v} \) is the spatial Fourier transform of \( v \). The Kolmogorov-Obukhov energy spectrum law shows that as the Reynolds number tends to infinity,

\[
E_v(k) \sim \epsilon^{\frac{4}{3}} k^{-\frac{4}{3}}
\]

(3)

for \( k_0 \leq k \leq k_d \), where \( k_0 = \ell_E^{-1} \) and \( k_d = \ell_D^{-1} \). Moreover, \( k_d \), the Kolmogorov dissipation wave number, can be normalized by \( \nu^{-\frac{1}{4}} \epsilon^{\frac{1}{4}} \). If we define a characteristic velocity at spatial scale \( k^{-1} \) by

\[
V_k \overset{\text{def}}{=} 2^{1/2} \left( \int_{\frac{k}{2}}^{2k} E_v(m) \, dm \right)^{\frac{1}{2}},
\]

from (3) we infer that

\[
V_k \sim \epsilon^{\frac{1}{4}} k^{-\frac{1}{2}}.
\]

(4)

In MHD turbulence, the totally energy is

\[
E = E^K + E^M := \frac{1}{2} ||u||^2_{L^2(\Omega)} + \frac{1}{2} ||b||^2_{L^2(\Omega)}.
\]

Kinetic-energy spectra \( E_u(k) \) and magnetic-energy spectra \( E_b(k) \) are defined by (2). And energy dissipation rate \( \varepsilon \) per unit mass is defined by

\[
\varepsilon := \nu(\langle |\nabla u|^2 \rangle) + \mu(\langle |\nabla b|^2 \rangle).
\]

We introduce magnetic Prandtl number \( \mathbf{P}_{r_m} := \nu/\mu \). Using the Renormalization Group formalism, Fournier et al. have show that the turbulent viscosity and magnetic diffusivity settle at a ratio close to unity (\( \mathbf{P}_{r_m} \sim 1 \)). \( E^K \gg E^M \) implies the amplification of a magnetic field by plasma turbulence, such as the dynamo problem. Another extreme, \( E^K \ll E^M \), can be encountered in the solar corona. Some researchers are focused on exploring inherent properties of the nonlinear interaction between the turbulent fields, thus the conditions that \( E^K \sim E^M \) and magnetic Prandtl number \( \mathbf{P}_{r_m} = 1 \) are paid much attention to, e.g., [22, 13]. Therefore, we give some comments on physical interpretation of our main results under these conditions.

Considering Alfvén waves generated by a mean magnetic field \( \mathbf{B}_0 \), the IK phenomenology [16, 19] employed a different model in terms of the nonlinear energy flux for studying MHD turbulence. Kraichnan assumed that the energy cascade process should be characterized by another timescale different from the usual Kolmogorov timescale and hence obtained the energy spectrum law \( E_u(k), E_b(k) \sim k^{-\frac{2}{3}} \). However, many observations on astrophysical systems [14, 24] and simulations of 3D MHD system [9, 10] seem to support the K41 model. Basu et al [2] studied the 3D MHD system without additional assumptions on the velocity and magnetic field and they found that the energy spectra are Kolmogorov-like, that is, \( E_u(k), E_b(k) \sim k^{-\frac{2}{3}} \) even in the presence of Alfvén waves. Therefore, it appears to be natural to apply the Kolmogorov energy law and Kolmogorov length. So for
3D MHD turbulence, characteristic velocity and magnetic fields at scale $k^{-\frac{1}{3}}$ should be $\sim k^{-\frac{1}{3}}$. Let us define the Littlewood-Paley $p$th-order characteristic velocity and magnetic field at length scale $k^{-1}(k \in [2^{m-1}, 2^m])$ by

$$U_m^p \defeq \langle (|\Delta_m u|^p) \rangle^\frac{1}{p}, \quad B_m^p \defeq \langle (|\Delta_m b|^p) \rangle^\frac{1}{p}. $$

For magnetic Prandtl number $P_{rm} = 1(\mu = \nu)$, Theorem 1.3 shows that

$$U_m^p + B_m^p \leq C(\phi_0)\epsilon^\mu \mu^{-2}(2^m)^{-3} \leq C(\phi_0)\epsilon^\mu k^{-3}. $$

Using $k_d = \mu^{-\frac{2}{3}} \varepsilon^{-\frac{1}{3}} (\mu^{-2} = k_d^\frac{8}{3} \varepsilon^{-\frac{8}{3}})$, then for $k_0 \leq c_0 k_d \leq k \leq k_d$,

$$U_m^p + B_m^p \leq C(\phi_0)\left(\frac{k_d}{k}\right)^\frac{8}{3} \epsilon^{\frac{8}{3}} k^{\frac{1}{3}} \leq C(\phi_0, c_0)\epsilon^{\frac{8}{3}} k^{\frac{1}{3}} ,$$

which shows that we obtain an upper bound coincident with (4). Moreover, Theorem 1.4 implies that

$$U_m^p + B_m^p \leq C(\phi_0)\mu^{-\frac{1}{3}} k^{\left(\frac{4}{3} + \frac{8}{3}\right)} (\varepsilon_{p+1})^\frac{1}{3}$$

$$\leq C(\phi_0)\left(\frac{k_d}{k}\right)^\frac{8}{3} \epsilon^{\frac{8}{3}} k^{\frac{1}{3}} (\varepsilon_{p+1})^\frac{1}{3}$$

$$= C(\phi_0, c_0)\epsilon^{\frac{8}{3}} (\varepsilon_{p+1})^\frac{1}{3} k^{\frac{1}{3}}.$$  

When the mean magnetic field is zero, numerical simulations and observation data for 3D MHD [3, 4] showed that deviations of the scaling exponents of the structure functions of velocity and the magnetic fields from K41 are similar to those of pure fluid turbulence on intermittency. Therefore, it is reasonable to assume that $S_m^u(\ell), S_m^B(\ell) \sim (\ell)^{\frac{8}{3}}$ (coincident with (1)) in the nonintermittent limiting case. This fact combined with Proposition 2 (a characterization of Besov spaces), shows that it is natural to have the mathematical assumption associated to the law that $\varepsilon_{p+1} < \infty$. Particularly, for $p = 2$, this result is coincided with [11].

For general case, $P_{rm} \sim 1$, we can get similar results from Theorem 1.3 and Theorem 1.4, where constant $C$ depends on $P_{rm}$.

Compared with [11, 23], we study 3D MHD system rather than Navier-Stokes equations. They were focused on the Littlewood-Paley first-order [23] and second-order [11] characteristic velocity and we generalize the results to the Littlewood-Paley $p$th ($1 \leq p < \infty$)-order characteristic velocity and magnetic fields.

1.3 Main ideas of proof. In [23], for Navier-Stokes equations, there has only one nonlinear term, $\phi \ast (u \cdot \nabla u)$, where $\phi$ is a frequency localization operator. Littlewood-Paley decomposition is an important tool for microlocal analysis, e.g., [7, 25]. Applying Littlewood-Paley decomposition, $u$ can be decomposed into the low frequency part $v_0$ and the high frequency part $w_0$. Then in terms of $\phi \ast (v_0 \cdot \nabla u)$, it can be decomposed by

$$\phi \ast (v_0 \cdot \nabla u) = v_0 \cdot \nabla (\phi \ast u) + [v_0 \cdot \nabla, \phi \ast u].$$

Then for estimating $|\phi \ast u|$, it is natural to estimate $\langle \phi \ast (v_0 \cdot \nabla u)A'(\phi \ast u) \rangle$, where $A(z)$ is a smooth approximation of $|z|$. From the above decomposition, we can get that

$$\langle \phi \ast (v_0 \cdot \nabla u)A'(\phi \ast u) \rangle = \langle v_0 \cdot \nabla (\phi \ast u)A'(\phi \ast u) \rangle + \langle [v_0 \cdot \nabla, \phi \ast u]A'(\phi \ast u) \rangle $$

$$= \langle v_0 \cdot \nabla A(\phi \ast u) \rangle + \langle [v_0 \cdot \nabla, \phi \ast u]A'(\phi \ast u) \rangle.$$
The desired upper bound is involved in $L^2$-norm of the high frequency part $w_0$ and $H^1$-norm of low frequency part $v_0$. Although it is difficult to add first order derivative of $v_0$ in terms of $\langle w_0 \cdot \nabla (\phi \ast u) A' (\phi \ast u) \rangle$, actually this term is vanishing. Thereby the nonlinear term can be dealt with.

However, following this argument, it fails to get our first result when considering system (MHD'). For instance, we decompose $b$ into the low frequency part $(b)^L_0$ and the high frequency part $(b)^H_0$. When tackling $\phi \ast ((b)^L_0 \cdot \nabla b)$, in order to obtain estimates in terms of $(b)^L_0$ with first order derivative, we introduce commutator and this term can be decomposed by

$$\phi \ast ((b)^L_0 \cdot \nabla b) = (b)^L_0 \cdot \nabla (\phi \ast b) + [(b)^L_0 \cdot \nabla, \phi \ast] b.$$ 

For the sake of estimating $\langle \phi \ast u \rangle$, from the above decomposition, we need to consider $\langle (b)^H_0 \cdot \nabla (\phi \ast b) A' (\phi \ast u) \rangle$. This term is not vanishing. Moreover, it is not easy to get upper bound independent of $\Omega$ and involved in $\| \nabla (b)^L_0 \|_{L^2}$ due to low regularity of $A'$.

To overcome these difficulties stemming from some nonlinear terms, we introduce the Elsässer fields $z^\pm = u \pm b$. Then system (MHD') can be written as

$$\begin{cases}
\partial_t z^+ - \frac{\nu + \mu}{2} \Delta z^+ + (z^- - B_0) \cdot \nabla z^+ = \frac{\nu - \mu}{2} \Delta z^- - \nabla P + f + g, \\
\partial_t z^- - \frac{\nu + \mu}{2} \Delta z^- + (z^+ + B_0) \cdot \nabla z^- = \frac{\nu - \mu}{2} \Delta z^+ - \nabla P + f - g, \\
\text{div} z^+ = \text{div} z^- = 0.
\end{cases} (\text{MHD'})$$

This argument that symmetrizes the system has been used in some literatures, e.g., [8]. Problems from nonlinear terms can be addressed for considering system (MHD*). However, compared with Navier-Stokes equations, this system has two additional dissipation terms. So we have to consider how the dissipation terms on the right-hand side of the equalities in Eq. (MHD*) can be absorbed.

When dealing with system (MHD*), if we bound the dissipation terms involved in $\Delta_0$ on the right-hand side by Bernstein’s inequality, only in the condition that $|\nu - \mu| \ll 1$, these terms can be eaten by the ones on the left-hand side. Actually, we find that if frequency of some function $u$ is supported on a ball $B_\delta(\eta)$, not only the lower bound but also the upper bound of $\| \Delta u \|_{L^p}$ can be characterized by $\eta$ and $\delta$ (see Lemma 2.2). In order to use this property, we consider the convolution operator $\Psi_{\omega_j}^\delta$ at the beginning which makes frequency support at a ball. Therefore these coefficients become comparable so that we can obtain upper bound in our first result for any $\nu, \mu > 0$ more than $|\nu - \mu| \ll 1$.

With respect to our second result, we can directly consider system (MHD') for $p = 2$. When estimating the sum of $\langle |\Delta_0 u|^2 \rangle$ and $\langle |\Delta_0 b|^2 \rangle$, these nonlinear terms can be decomposed into two parts by commutators. Relying on a characterization of Besov spaces, the commutators can be bounded by some Besov norm. The remaining nonlinear parts can be cancelled due to the special coupled structure.

However, for $p \neq 2$, the coupled structure of system (MHD') cannot cancel the noncommutator parts in nonlinear terms. So we need to make use of (MHD*), which reduces to deal with commutator terms. To handle these dissipation terms, we need to apply Lemma 2.2, which provides more refined estimates than Bernstein’s inequality. Thus we take the convolution operator $\Psi_{\omega_j}^\delta$ to localize frequency of $z^\pm$ at first.
2. Preliminaries. To begin with, we review briefly the so-called Littlewood-Paley decomposition theory introduced e.g., in [1, 5, 21]. Suppose \( \phi \) be a smooth function with values in \([0, 1]\), where \( \text{supp} \phi \subset \{ \xi \in \mathbb{R}^3 | \frac{1}{2} < |\xi| < 2 \} \). Moreover,
\[
\sum_{j \in \mathbb{Z}} \phi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}, \quad \text{where} \quad \phi_j(\xi) := \phi(2^{-j} \xi).
\]

Let us define the homogeneous localization operators as follows:
\[
\hat{\Delta}_j u = \phi_j(D) u = 2^{3j} \int_{\mathbb{R}^3} \varphi(2^j y) u(x - y) \, dy, \quad \forall j \in \mathbb{Z},
\]
where \( \varphi = \mathcal{F}^{-1} \phi \). For any \( u \in \mathcal{S}' \), we have \( u = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \).

For functions of which Fourier transforms are supported in an annulus, there is a nice property called Bernstein’s inequality in \( \mathbb{R}^3 \) [1, 21]. Following this argument, a similar result can be obtained for \( \Omega \)-periodic functions.

**Proposition 1** ([1, 21] Bernstein’s inequality). Let \( C \) be an annulus and \( B \) be a ball in \( \mathbb{R}^3 \), \( 1 \leq p \leq q \leq \infty \). Then there exists a constant \( C \) such that for any \( k \in \mathbb{N} \) and any \( \Omega \)-periodic function \( u \in L^p(\Omega) \), we have
\[
\sup_{|\alpha| = k} \| \partial^\alpha u \|_{L^q(\Omega)} \leq C \| \lambda^{k+3} (\frac{1}{r} \frac{1}{q} - \frac{1}{p}) \|_{L^p(\Omega)}, \quad \sup \hat{u} \subset \lambda B,
\]
\[
C^{-1} \lambda^{k} \| u \|_{L^p(\Omega)} \leq \sup_{|\alpha| = k} \| \partial^\alpha u \|_{L^p(\Omega)} \leq C \| \lambda^{k+1} \|_{L^p(\Omega)}, \quad \sup \hat{u} \subset \lambda C.
\]

**Definition 2.1** ([1, 21] Homogeneous Besov spaces). Let \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \). The homogeneous Besov space \( B^s_{p,q} \) consists of all \( \Omega \)-periodic tempered distributions \( u \in \mathcal{S}'_\delta \) such that
\[
\| u \|_{B^s_{p,q}(\Omega)} \overset{\text{def}}{=} \left\| \left( 2^{js} \| \hat{\Delta}_j u \|_{L^p(\Omega)} \right)_{j \in \mathbb{Z}} \right\|_{L^q(\mathbb{Z})} < \infty.
\]

Now we present a characterization of Besov spaces with positive indices with regard to finite differences.

**Proposition 2** ([1, 21]). Let \( s \in (0, 1) \), \( p, q \in [1, \infty]^2 \). There exists a constant \( C \) such that for any \( \Omega \)-periodic function \( u \in \mathcal{S}'_\delta \),
\[
C^{-1} \| u \|_{B^s_{p,q}(\Omega)} \leq \left\| \frac{u - \tau_y u}{|y|^s} \right\|_{L^p(\mathbb{R}^3 \setminus \delta \mathbb{Z})} \leq C \| u \|_{B^s_{p,q}(\Omega)},
\]
where \( \tau_y u = u(x - y) \).

**Lemma 2.2.** Let \( u \) be a smooth \( \Omega \)-periodic function satisfying \( \text{supp} \hat{u} \subset B_\delta(\eta) \), where \( 0 < \delta < 1 \), \( \frac{1}{2} < |\eta| < 2 \). We have for every \( 1 \leq p \leq \infty \),
\[
\| \Delta u + |\eta|^2 u \|_{L^p(\Omega)} \leq C \delta \| u \|_{L^p(\Omega)},
\]
where \( C \) is an absolute constant.

**Proof.** In [23], the above result is proved for \( p = 1 \). Actually, one can get the results for \( 1 \leq p \leq \infty \) by following the method in [23] and we omit the details.

**Lemma 2.3** ([23] Commutator estimates). Let \( \phi \) be a Schwartz function and \( u, v \) be smooth \( \Omega \)-periodic functions. Considering the following commutator,
\[
[u, \phi^*] v = u(\phi^* v) - \phi^* (u v),
\]
we have
\[
\langle |u, \phi| v | \rangle \leq C(\phi) \left( \langle |\nabla u|^2 \rangle + \langle |v|^2 \rangle \right),
\]
\[
\langle |u, \phi| \partial_x, v | \rangle \leq C(\phi) \left( \langle |\nabla u|^2 \rangle + \langle |v|^2 \rangle \right).
\]

We provide another commutator estimates, which involve in a characterization of Besov spaces.

**Proposition 3.** Let \( \phi \) be a Schwartz function and \( p \in (1, \infty) \). \( u \), \( v \) and \( w \) are \( \Omega \)-periodic vector functions and \( u \) is divergence-free. Then we get
\[
\int_{\Omega} \left| u \cdot \nabla \phi \right| |v| \, dx \leq C(\phi) \left\| u \right\|_{B^0_{p, \infty}(\Omega)} \left\| v \right\|_{B^0_{p, \infty}(\Omega)},
\]
\[
\int_{\Omega} \left| u \cdot \nabla \phi \right| |\Delta_0 w|^{p-2} \Delta_0 w \, dx \leq C(\phi) \left\| u \right\|_{B^0_{p+1, \infty}(\Omega)} \left\| v \right\|_{B^0_{p+1, \infty}(\Omega)} \left\| w \right\|_{B^{p-1}_{p+1, \infty}(\Omega)}^{p-1}.
\]

**Proof.** We just prove the second inequality. The first one can be obtained by a similar argument. Thanks to \( \text{div} \ u = 0 \), the commutator can be written as
\[
[u, \nabla, \phi] v = \sum_{i=1}^{3} \int_{\mathbb{R}^3} \left[ u_i(x) \phi(y) \partial_x (v(x - y)) - \phi(y) u_i(x - y) (\partial_x v)(x - y) \right] \, dy
\]
\[
= \sum_{i=1}^{3} \int_{\mathbb{R}^3} \left[ \phi(y) u_i(x - y) \partial_y (v(x - y)) - u_i(x) \phi(y) \partial_y (v(x - y)) \right] \, dy
\]
\[
= \sum_{i=1}^{3} \int_{\mathbb{R}^3} \left( u_i(x) - u_i(x - y) \right) \partial_y \phi(y) (v(x - y) - v(x)) \, dy.
\]

Therefore, we get that
\[
\int_{\Omega} \left| u \cdot \nabla \phi \right| |\Delta_0 w|^{p-2} \Delta_0 w \, dx
\]
\[
= \int_{\mathbb{R}^3} \int_{\Omega} \partial_y \phi(y) |y|^{\frac{3}{2}} \left| \frac{u_i(x) - u_i(x - y)}{|y|^{\frac{1}{2}}} \right| \frac{v(x - y) - v(x)}{|y|^{\frac{1}{2}}} \, dx \, dy
\]
\[
\leq \sum_{i=1}^{3} \left\| \left| u_i(x) - u_i(x - y) \right| \right\|_{L^{p+1}(\Omega)} \left\| \left| v(x - y) - v(x) \right| \right\|_{L^{p+1}(\Omega)} \left\| \left( \frac{v(x - y) - v(x)}{|y|^{\frac{1}{2}}} \right) \right\|_{L^{\infty}(y)}
\]
\[
\times \left\| \Delta_0 w \right\|_{L^{p-1}(\Omega)}^{p-1} \int_{\mathbb{R}^3} \left| \partial_y \phi(y) \right| |y|^{\frac{3}{2}} \, dy
\]
\[
\leq C(\phi) \left\| u \right\|_{B^{0}_{p+1, \infty}(\Omega)} \left\| v \right\|_{B^{0}_{p+1, \infty}(\Omega)} \left\| w \right\|_{B^{p-1}_{p+1, \infty}(\Omega)}^{p-1},
\]
where the last inequality holds true by virtue of Proposition 2. \( \square \)

3. **Proof of Theorem 1.3.** Let \( \mathcal{C} \) be the annulus \( \{ \xi ? \mathbb{R}^3 \mid \frac{1}{2} < |\xi| < 2 \} \). There exists the following partition of unity: for any function \( g \),
\[
\hat{\Delta}_0 g = \sum_j \psi^\delta_{\omega_j} \hat{\Delta}_0 g := \sum_j \psi^\delta_{\omega_j} \hat{\Delta}_0 g,
\]
where \( \{ \psi^\delta_{\omega_j} \} \) is a finite family of Schwartz functions and \( \hat{\psi}^\delta_{\omega_j} \) is supported on the ball \( B_0(\omega_j) \).

Without loss of generality, let \( \nu \geq \mu \) and we consider system \( (\text{MHD}^*) \). To begin with, we give the following marks. \( z^+ \) and \( z^- \) denote the \( k \)th coordinate of \( z^+ \)
and \( z^- \) respectively. We decompose \( z^+ \) into two parts: \( \ell \)-low frequency and \( \ell \)-high frequency. More precisely, \( z^+ = (z^+)_{\ell}^+ + (z^+)_{\ell}^H \), where

\[
(z^+)_{\ell}^+ := \sum_{m \leq \ell - 1} \hat{\Delta}_m z^+ \quad \text{and} \quad (z^+)_{\ell}^H := \sum_{m \geq \ell} \hat{\Delta}_m z^+ .
\]

Similarly, we define

\[
(z^-)_{\ell}^+ := \sum_{m \leq \ell - 1} \hat{\Delta}_m z^- \quad \text{and} \quad (z^-)_{\ell}^H := \sum_{m \geq \ell} \hat{\Delta}_m z^- .
\]

Taking operators \( \hat{\Delta}_0 \), the Leray projection operator \( \mathbb{P} \) and \( \Psi_{\omega_j}^\delta \) on Eq.\((\text{MHD}^*)\), we have

\[
\begin{cases}
\partial_t \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^+ - \frac{\nu + \mu}{2} \Delta \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^+ - \mathbf{B}_0 \cdot \nabla \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^+ \\
\quad = \frac{\nu - \mu}{2} \Delta \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^- - \mathbb{P} \Psi_{\omega_j}^\delta \hat{\Delta}_0 \left( z^- \cdot \nabla \right) z^+ + \mathbb{P} \Psi_{\omega_j}^\delta \hat{\Delta}_0 (f + g),
\end{cases} \quad (\text{MHD}_*)
\]

Indeed, weak solution \((z^+, z^-)\) satisfies the above equations not only in the distribution sense, but also in the a.e. sense. Due to \( \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^\pm \in H^s \) for any \( s > 0 \), and

\[
(z^+, z^-) \in \left( L_{\text{loc}}^\infty \left( \mathbb{R}^+; (L^2(\Omega))^3 \right) \cap L_{\text{loc}}^2 \left( \mathbb{R}^+; (\dot{H}^1(\Omega))^3 \right) \right)^2 ,
\]

we deduce that

\[
\frac{\nu + \mu}{2} \Delta \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^+ + \mathbf{B}_0 \cdot \nabla \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^+ + \frac{\nu - \mu}{2} \Delta \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^-
\]

\[
el L_{\text{loc}}^\infty \left( \mathbb{R}^+; (L^2(\Omega))^3 \right) \subset L_{\text{loc}}^1 \left( \mathbb{R}^+; (L^1(\Omega))^3 \right) ,
\]

and

\[
\mathbb{P} \Psi_{\omega_j}^\delta \hat{\Delta}_0 \left( z^+ \cdot \nabla \right) z^+ + \mathbb{P} \Psi_{\omega_j}^\delta \hat{\Delta}_0 (f + g)
\]

\[
el L_{\text{loc}}^2 \left( \mathbb{R}^+; (L^1(\Omega))^3 \right) + L_{\text{loc}}^\infty \left( \mathbb{R}^+; (L^1(\Omega))^3 \right) \subset L_{\text{loc}}^1 \left( \mathbb{R}^+; (L^1(\Omega))^3 \right) .
\]

Hence \( \partial_t \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^+ \in L_{\text{loc}}^1 \left( \mathbb{R}^+; (L^1(\Omega))^3 \right) \) and Eq.\((\text{MHD}_*)_1\) holds in the a.e. sense. Therefore, system \((\text{MHD}_*)\) also holds in the a.e. sense. By decomposition \( z^- = (z^-)_{\ell}^+ + (z^-)_{\ell}^H \), we write each component of equation \((\text{MHD}_*)_1\) as follows:

\[
\begin{aligned}
\partial_t \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^+_k - \frac{\nu + \mu}{2} \Delta \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^+_k - \mathbf{B}_0 \cdot \nabla \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^+_k \\
- \frac{\nu - \mu}{2} \Delta \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^-_k - \left( (z^-)_{\ell}^H \cdot \nabla \right) \Psi_{\omega_j}^\delta \hat{\Delta}_0 z^+_k + \left( (z^-)_{\ell}^H \cdot \nabla, \mathbb{P} \Psi_{\omega_j}^\delta \hat{\Delta}_0 \right) z^+_k \\
- \mathbb{P} \Psi_{\omega_j}^\delta \hat{\Delta}_0 \left( (z^-)_{\ell}^H \cdot \nabla \right) z^-_k + \mathbb{P} \Psi_{\omega_j}^\delta \hat{\Delta}_0 (f_k + g_k).
\end{aligned}
\]

Let \( \Phi(z) \) be a smooth approximation of \(|z|\). More precisely,
Multiplying Eq. (7) by $\Phi_i^j$, integrating with respect to spatial variable and taking space-average, thanks to Lemma \ref{lemma2}, we get
\[
\frac{\partial}{\partial t} (\Psi_\omega^m \Delta_0 z^+_k) - \frac{\nu + \mu}{2} \langle \Delta (\Psi_\omega^m \Delta_0 z^+_k) \rangle \Phi_i^j \Phi_i^j (\Psi_\omega^m \Delta_0 z^+_k) \leq \frac{\nu - \mu}{2} \langle (\Delta \Phi_i^j \Delta_0 z^+_k) \rangle + \langle (\Delta \Phi_i^j \Delta_0 z^+_k) \rangle + \langle (\Phi_i^j \Delta_0 z^+_k) \rangle + \langle (\Phi_i^j \Delta_0 z^+_k) \rangle \leq I + II + III + C(\Phi_i^j \Delta_0 f) + C(\Phi_i^j \Delta_0 g).
\]
Applying Lemma \ref{lemma2} enables us to bound $I$ as follows:
\[
I \leq \frac{\nu - \mu}{2} \langle |\omega_j|^2 + C\delta \rangle \langle |\Psi_\omega^m \Delta_0 z^+_k| \rangle.
\]
Throughout the paper, we employ Einstein summation. Thanks to Lemma \ref{lemma2}, we can bound $II$ as
\[
II \leq \langle (\Delta \Phi_i^j \Delta_0 z^+_k) \rangle + \langle (\Delta \Phi_i^j \Delta_0 z^+_k) \rangle + C(\Phi_i^j \Delta_0 f) + C(\Phi_i^j \Delta_0 g).
\]
Owing to $\div(z^-)^0 = 0$, it yields that
\[
III \leq \langle (\Delta \Phi_i^j \Delta_0 z^+_k) \rangle + \langle (\Delta \Phi_i^j \Delta_0 z^+_k) \rangle + C(\Phi_i^j \Delta_0 f) + C(\Phi_i^j \Delta_0 g).
\]
Plugging estimates (10)-(12) into (9), letting $t \to \infty$, we have
\[
\frac{d}{dt} \langle |\Psi_\omega^m \Delta_0 z^+_k| \rangle - \frac{\nu + \mu}{2} \langle \Delta (\Psi_\omega^m \Delta_0 z^+_k) \rangle \leq \frac{\nu + \mu}{2} \langle (\Delta \Phi_i^j \Delta_0 z^+_k) \rangle + \langle (\Delta \Phi_i^j \Delta_0 z^+_k) \rangle + C(\Phi_i^j \Delta_0 f) + C(\Phi_i^j \Delta_0 g).
\]
Taking advantage of Lemma \ref{lemma2}, we have
\[
\frac{d}{dt} \langle |\Psi_\omega^m \Delta_0 z^+_k| \rangle + \frac{\nu + \mu}{2} \langle (\Delta \Phi_i^j \Delta_0 z^+_k) \rangle \leq \frac{\nu + \mu}{2} \langle (\Delta \Phi_i^j \Delta_0 z^+_k) \rangle + \langle (\Delta \Phi_i^j \Delta_0 z^+_k) \rangle + C(\Phi_i^j \Delta_0 f) + C(\Phi_i^j \Delta_0 g).
\]
Therefore from (13), it yields that
\[
\frac{d}{dt} \langle |\Psi_\omega^m \Delta_0 z^+_k| \rangle + \frac{\nu + \mu}{2} \langle (\omega_j)^2 - C\delta \rangle \langle |\Psi_\omega^m \Delta_0 z^+_k| \rangle \leq \frac{\nu + \mu}{2} \langle (\omega_j)^2 + C\delta \rangle \langle |\Psi_\omega^m \Delta_0 z^+_k| \rangle + C(\Phi_i^j \Delta_0 f) + C(\Phi_i^j \Delta_0 g).
\]
Following the above argument, we get similar estimates on $z_k^{-}$ from Eq. (14). We omit details and give the result directly.

$$\frac{d}{dt}\langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^{-} | \rangle + \frac{\nu + \mu}{2} (|\omega_j|^2 - C\delta) \langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^{-} | \rangle$$

$$\leq \frac{\nu - \mu}{2} (|\omega_j|^2 + C\delta) \langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^{-} | \rangle + C(\psi_{\omega_j}^{\delta}, \varphi_0)(\langle |\Delta_0 f | \rangle + \langle |\Delta_0 g | \rangle)$$

$$+ C(\psi_{\omega_j}^{\delta}, \varphi_0)(\langle |\nabla(z^+)_0^T | \rangle^2 + \langle |(z^+)_0^T | \rangle^2 + \langle |\nabla(z^+)_0^T | \rangle^2 + \langle |(z^+)_0^T | \rangle^2).$$

(15)

Combining (14) and (15) together, it turns out that

$$\frac{d}{dt}\langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^+ | \rangle + \frac{\nu + \mu}{2} |\omega_j|^2 \langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^+ | \rangle + \langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^- | \rangle \langle |\Delta_0 f | \rangle + \langle |\Delta_0 g | \rangle$$

$$\leq \langle |\nabla(x)^{T} | \rangle^2 + \langle |(z^+)_0^T | \rangle^2 + \langle |\nabla(x^+)_0^T | \rangle^2 + \langle |(z^+)_0^T | \rangle^2.$$ (16)

Hence integrating in terms of $t$ from 0 to $T$ and taking time-average, we obtain that

$$\frac{1}{T} \langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^+ (T) | \rangle + \frac{1}{T} \langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^- (T) | \rangle + \frac{\nu + \mu}{2} |\omega_j|^2 \langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^+ | \rangle_T + \frac{\nu + \mu}{2} |\omega_j|^2 \langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^- | \rangle_T$$

$$\leq C(\psi_{\omega_j}^{\delta}, \varphi_0)(\langle |\nabla(x)^{T} | \rangle^2 + \langle |(z^+)_0^T | \rangle^2 + \langle |\nabla(x^+)_0^T | \rangle^2 + \langle |(z^+)_0^T | \rangle^2) + \langle |\Delta_0 f | \rangle_T + \langle |\Delta_0 g | \rangle_T + \frac{1}{T} \langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^+ (0) | \rangle_T + \frac{1}{T} \langle |\Psi_{\omega_j}^{\delta} \Delta_0 z_k^- (0) | \rangle_T.$$ (17)

By virtue of the unity decomposition (6), summing in $j$ and $k$ yields that

$$\frac{1}{T} \langle |\Delta_0 z^+ (T) | \rangle + \frac{1}{T} \langle |\Delta_0 z^- (T) | \rangle + \frac{\nu + \mu}{2} (|\omega_j|^2 + C\delta) \langle |\Delta_0 z^+ | \rangle_T + \frac{\nu + \mu}{2} (|\omega_j|^2 + C\delta) \langle |\Delta_0 z^- | \rangle_T$$

$$\leq C(\psi_{\omega_j}^{\delta}, \varphi_0)(\langle |\nabla(x)^{T} | \rangle^2 + \langle |(z^+)_0^T | \rangle^2 + \langle |\nabla(x^+)_0^T | \rangle^2 + \langle |(z^+)_0^T | \rangle^2) + \langle |\Delta_0 f | \rangle_T + \langle |\Delta_0 g | \rangle_T + \frac{1}{T} \langle |\Delta_0 z^+ (0) | \rangle_T + \frac{1}{T} \langle |\Delta_0 z^- (0) | \rangle_T.$$ (18)

As the function $\varphi_0$ is fixed, the set of $\{\psi_{\omega_j}^{\delta}\}$ is finite and the number of it depends on $\delta$. From (16), we conclude that constant $C$ is dependent on $\frac{\nu + \mu}{2}$ and $\varphi_0$.

Following the arguments in [23], we take the following change of variables:

$$x = 2^{-m} \tilde{x}, \quad t = 2^{-2m} \tilde{t}, \quad z^+ = 2^m \tilde{z}^+, \quad z^- = 2^m \tilde{z}^-$$

$$P = 2^m \tilde{P}, \quad f = 2^{3m} \tilde{f}, \quad g = 2^{3m} \tilde{g}.$$ (19)

As the $k$th localization operator $\varphi_k$ satisfies $\varphi_k(x) = 2^{3k} \varphi_0(2^k x)$, it is clear that

$$\varphi_k(x) = \varphi_k(2^{-m} \tilde{x}) = 2^{3k} \varphi_0(2^{k-m} \tilde{x}) = 2^{3m} \varphi_{k-m}(\tilde{x}).$$
Based on the above equality and the change of variables, we have

\[ \Delta_k z^+ = \int_{\mathbb{R}^3} \varphi_k(y) z^+(x - y) \, dy \]
\[ = \int_{\mathbb{R}^3} 2^{3m} \varphi_{k-m}(y) 2^m z^+(\tilde{x} - \tilde{y}) 2^{-3m} \, d\tilde{y} = 2^m \Delta_{k-m} \tilde{z}^+. \]

Similarly, we deduce that

\[ \Delta_k z^- = 2^m \Delta_{k-m} \tilde{z}^-, \quad \Delta_k f = 2^m \Delta_{k-m} \tilde{f}, \quad \Delta_k g = 2^m \Delta_{k-m} \tilde{g}. \]

We thus infer that

\[ \Delta_0 z^+ = 2^{-m} \Delta_m z^+, \quad \Delta_0 z^- = 2^{-m} \Delta_m z^-, \quad \Delta_0 f = 2^{-m} \Delta_m f, \quad \Delta_0 g = 2^{-m} \Delta_m g. \]

and

\[ (\tilde{z}^+)^0 = \sum_{k \leq -1} \Delta_k z^+ = 2^{-m} \sum_{k \leq -1} \Delta_{k+m} z^+ = 2^{-m}(z^+)^L_m, \]
\[ (\tilde{z}^+)^H_0 = 2^{-m}(z^+)^H_m, \quad (\tilde{z}^-)^H_0 = 2^{-m}(z^-)^H_m, \quad (\tilde{z}^-)^L_0 = 2^{-m}(z^-)^L_m, \]
\[ \partial_{\bar{x}}(\tilde{z}^+)^L_0 = 2^{-m} \partial_{\bar{x}}(z^+)^L_m, \quad \partial_{\bar{x}}(\tilde{z}^-)^L_0 = 2^{-m} \partial_{\bar{x}}(z^-)^L_m. \]

It is easy to check that \((\bar{x}, \bar{t}, \tilde{z}^+, \tilde{z}^-, \tilde{f}, \tilde{g})\) satisfies system (MHD) on \([0, 2^{-m} L]^3 \times [0, 2^{2m} T].\) Applying \((z^+, z^-, \tilde{f}, \tilde{g})\) to (17), from (19) and (20), we infer that

\[ \frac{\mu}{8} 2^{-m}(\|\Delta_m z^+\|_T) + \frac{\mu}{8} 2^{-m}(\|\Delta_m z^-\|_T) \leq C \left( \frac{\mu}{\nu}, \varphi_0 \right) \left( 2^{-4m}(\|\nabla(z^+)^L_m\|_T^2) + (\|\nabla(z^-)^L_m\|_T^2) \right) + 2^{-2m}(\|\Delta_m f\|_T^2) + 2^{-3m}(\|\Delta_m g\|_T^2) \]
\[ + \frac{2^{-3m}}{T} ( \|\Delta_m z^+(0)\| + 2^{-3m} (\|\Delta_m z^-(0)\|) \right). \]

The above inequality holds for any \(m \in \mathbb{Z}.\) In particular, for any \(m \geq k_0,\) due to \(\Delta_m f = \Delta_m g = 0,\) we immediately get

\[ \mu 2^{-m}(\|\Delta_m z^+\|_T) + \mu 2^{-m}(\|\Delta_m z^-\|_T) \leq C \left( \frac{\mu}{\nu}, \varphi_0 \right) \left( 2^{-4m}(\|\nabla(z^+)^L_m\|_T^2) + 2^{-2m}(\|\nabla(z^-)^H_m\|_T^2) + 2^{-4m}(\|\nabla(z^-)^L_m\|_T^2) \right) \]
\[ + 2^{-2m}(\|\Delta_m f\|_T^2) + \frac{2^{-3m}}{T} (\|\Delta_m z^+(0)\| + 2^{-3m} (\|\Delta_m z^-(0)\|) \right). \]

Multiplying the above inequality by \(2^{4m},\) taking the lim sup in \(T\) on both sides of the resulting inequality, it yields that

\[ \mu 2^{3m}(\|\Delta_m z^+\|_T) + \mu 2^{3m}(\|\Delta_m z^-\|_T) \leq C \left( \frac{\mu}{\nu}, \varphi_0 \right) \left( 2^{-2m}(\|\nabla(z^+)^L_m\|_T^2) + 2^{2m}(\|\nabla(z^-)^H_m\|_T^2) + 2^{2m}(\|\nabla(z^-)^L_m\|_T^2) \right) \]
\[ + C \left( \frac{\mu}{\nu}, \varphi_0 \right) \left( 2^{-2m}(\|\nabla(z^+)^L_m\|_T^2) + 2^{2m}(\|\nabla(z^-)^H_m\|_T^2) + 2^{2m}(\|\nabla(z^-)^L_m\|_T^2) \right) \]
\[ \leq C \left( \frac{\mu}{\nu}, \varphi_0 \right) \left( (\|\nabla(z^+)^L_m\|_T^2) + (\|\nabla(z^-)^H_m\|_T^2) + (\|\nabla(z^-)^L_m\|_T^2) \right). \]

Hence we complete the proof of Theorem 1.3.
4. Proof of Theorem 1.4.

4.1. The case of \( p = 2 \). When considering the Littlewood-Paley second-order characteristic velocity and magnetic fields, we are able to deal with nonlinear terms in system \((\text{MHD}')\) by using the coupling structure. Hence we directly study equations \((\text{MHD}')\), from which we can get

\[
\begin{align*}
\partial_t \Delta_0 u - \nu \Delta_0^2 u + \Delta_0 ((u \cdot \nabla) u) &- \Delta_0 ((b \cdot \nabla) b) - (B_0 \cdot \nabla) \Delta_0 b + \nabla \Delta_0 P = \Delta_0 f, \\
\partial_t \Delta_0 b - \mu \Delta_0^2 b + \Delta_0 ((u \cdot \nabla) b) - (B_0 \cdot \nabla) \Delta_0 u = \Delta_0 g, \quad (21)
\end{align*}
\]

Multiplying by \( \Delta_0 u \) on both sides of Eq.\((21)\) and integrating over \( \Omega \), thanks to \( \text{div} \, u = 0 \), we obtain that

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_0 u \|^2_{L^2(\Omega)} + c \nu \| \Delta_0 u \|^2_{L^2(\Omega)} \leq \int_{\Omega} [u \cdot \nabla, \Delta_0] u \Delta_0 u \, dx - \int_{\Omega} [b \cdot \nabla, \Delta_0] b \Delta_0 u \, dx + \int_{\Omega} (b \cdot \nabla \Delta_0) b \Delta_0 u \, dx \quad (22)
\]

\[
+ \int_{\Omega} (B_0 \cdot \nabla) \Delta_0 b \Delta_0 u \, dx + \int_{\Omega} \Delta_0 f \Delta_0 u \, dx,
\]

where the second term on the left-hand side of the above inequality is obtained by Proposition 1. For obtaining similar positive estimates, readers can also refer to [6]. Similarly, multiplying Eq.\((21)\) by \( \Delta_0 b \) and integrating over \( \Omega \), it yields that

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_0 b \|^2_{L^2(\Omega)} + c \mu \| \Delta_0 b \|^2_{L^2(\Omega)} \leq \int_{\Omega} [u \cdot \nabla, \Delta_0] \Delta_0 b \Delta_0 u \, dx - \int_{\Omega} [b \cdot \nabla, \Delta_0] u \Delta_0 b \, dx + \int_{\Omega} (b \cdot \nabla \Delta_0) u \Delta_0 b \, dx \quad (23)
\]

\[
+ \int_{\Omega} (B_0 \cdot \nabla) \Delta_0 b \Delta_0 u \, dx + \int_{\Omega} \Delta_0 g \Delta_0 b \, dx.
\]

Noting that

\[
\int_{\Omega} (b \cdot \nabla \Delta_0) b \Delta_0 u \, dx + \int_{\Omega} (b \cdot \nabla \Delta_0) u \Delta_0 b \, dx = \int_{\Omega} b_i (\partial_x \Delta_0 b_j) \Delta_0 u_j \, dx + \int_{\Omega} b_i (\partial_x \Delta_0 u_j) \Delta_0 b_j \, dx
\]

\[
= - \int_{\Omega} b_i \Delta_0 b_j (\partial_x \Delta_0 u_j) \, dx + \int_{\Omega} b_i (\partial_x \Delta_0 u_j) \Delta_0 b_j \, dx = 0,
\]

and

\[
\int_{\Omega} (B_0 \cdot \nabla) \Delta_0 b \Delta_0 u \, dx + \int_{\Omega} (B_0 \cdot \nabla) \Delta_0 u \Delta_0 b \, dx
\]

\[
= \int_{\Omega} (B_0)_{ij} (\partial_x \Delta_0 b_j) \Delta_0 u_j \, dx + \int_{\Omega} (B_0)_{ij} (\partial_x \Delta_0 u_j) \Delta_0 b_j \, dx
\]

\[
= - \int_{\Omega} (B_0)_{ij} \Delta_0 b_j (\partial_x \Delta_0 u_j) \, dx + \int_{\Omega} (B_0)_{ij} (\partial_x \Delta_0 u_j) \Delta_0 b_j \, dx = 0,
\]

The Littlewood-Paley pth-order moments...
hence, adding (22) and (23) implies that
\[
\frac{1}{2} \frac{d}{dt}(\|\hat{\Delta}_0 u\|_{L^2(\Omega)}^2 + \|\hat{\Delta}_0 b\|_{L^2(\Omega)}^2) + c\nu \|\hat{\Delta}_0 u\|_{L^2(\Omega)}^2 + c\mu \|\hat{\Delta}_0 b\|_{L^2(\Omega)}^2 \leq \frac{1}{\nu} \int_\Omega \|u \cdot \nabla \hat{\Delta}_0 u\| d\Omega - \int_\Omega [b \cdot \nabla \hat{\Delta}_0 b \hat{\Delta}_0 u] d\Omega + \int_\Omega \hat{f} \hat{\Delta}_0 u d\Omega
\]
\[
+ \int_\Omega [u \cdot \nabla \hat{\Delta}_0 b \hat{\Delta}_0 b] d\Omega - \int_\Omega [b \cdot \nabla \hat{\Delta}_0 u] u d\Omega + \int_\Omega \hat{g} \hat{\Delta}_0 b d\Omega.
\]
Applying Proposition 3, we can deduce from (24) that
\[
\frac{1}{2} \frac{d}{dt}(\|\hat{\Delta}_0 u\|_{L^2(\Omega)}^2 + \|\hat{\Delta}_0 b\|_{L^2(\Omega)}^2) + c\nu \|\hat{\Delta}_0 u\|_{L^2(\Omega)}^2 + c\mu \|\hat{\Delta}_0 b\|_{L^2(\Omega)}^2 \leq C(\varphi_0)(\|u\|_{B^{3/2}_{2,2}(\Omega)}^3 + \|b\|_{B^{3/2}_{2,2}(\Omega)}^3) + \frac{C}{\nu} \|\hat{\Delta}_0 f\|_{L^2(\Omega)}^2 + \frac{c\nu}{2} \|\hat{\Delta}_0 u\|_{L^2(\Omega)}^2
\]
\[
+ \frac{C}{\mu} \|\hat{\Delta}_0 g\|_{L^2(\Omega)}^2 + c\mu \|\hat{\Delta}_0 b\|_{L^2(\Omega)}^2.
\]
Simplifying the above inequality, integrating with respect to time variable and taking the time-space average for the resulting inequality, we get that
\[
\frac{1}{T}(\|\hat{\Delta}_0 u(T)\|^2 + \|\hat{\Delta}_0 b(T)\|^2) + c\nu(\|\hat{\Delta}_0 u\|^2) + c\mu(\|\hat{\Delta}_0 b\|^2)
\]
\[
\leq C(\varphi_0)\left(\int_0^T \|u\|_{B^{3/2}_{2,2}(\Omega)}^3 + \|b\|_{B^{3/2}_{2,2}(\Omega)}^3 \right) dt + \frac{C}{\nu} (\|\hat{\Delta}_0 f\|^2) + \frac{c\nu}{2} (\|\hat{\Delta}_0 u\|^2)
\]
\[
+ \frac{C}{\mu} (\|\hat{\Delta}_0 g\|^2) + \frac{1}{T}(\|\hat{\Delta}_0 u(0)\|^2) + \frac{1}{T}(\|\hat{\Delta}_0 b(0)\|^2).
\]
By taking the change of variables (23), we thus infer from the above inequality that
\[
\frac{c\nu}{T} 2^{-2m}(\|\hat{\Delta}_0 u\|^2) + c\mu 2^{-2m}(\|\hat{\Delta}_0 b\|^2)
\]
\[
\leq C(\varphi_0)\frac{1}{T}\int_\Omega \left(\|\tilde{u}(\cdot, \tilde{t})\|_{B^{3/2}_{2,2}(\Omega)}^3 + \|\tilde{b}(\cdot, \tilde{t})\|_{B^{3/2}_{2,2}(\Omega)}^3 \right) d\tilde{t} + \frac{C2^{-6m}}{\mu} (\|\hat{\Delta}_0 g\|^2) + \frac{C2^{-4m}}{\nu} (\|\hat{\Delta}_0 f\|^2)
\]
\[
+ \frac{C2^{-4m}}{T} (\|\hat{\Delta}_0 u(0)\|^2) + \frac{C2^{-4m}}{T} (\|\hat{\Delta}_0 b(0)\|^2).
\]
An easy computation yields that
\[
\frac{1}{T}\int_0^T \|\tilde{u}(\cdot, \tilde{t})\|_{B^{3/2}_{2,2}(\Omega)}^3 d\tilde{t} = \frac{1}{T}\int_0^T \sup_j 2^j \int_\Omega |\hat{\Delta}_j \tilde{u}(x, \tilde{t})|^3 d\Omega d\tilde{t}
\]
\[
= \frac{2^{5m}}{T}\int_0^T \sup_j 2^{3m} |\hat{\Delta}_j u(x, t)|^3 d\Omega d\tilde{t} = \frac{2^{-4m}}{T}\int_0^T \sup_j 2^{j+m} \int_\Omega |\hat{\Delta}_j u(x, t)|^3 d\Omega dt
\]
\[
= \frac{2^{-4m}}{T}\int_0^T \|u(\cdot, t)\|_{B^{3/2}_{2,2}(\Omega)}^3 dt.
\]
For each $m \geq k_0$, we have $\hat{\Delta}_m f = 0, \hat{\Delta}_m g = 0$. Therefore, multiplying inequality (26) by $2^{2m}$ and then taking the lim sup in $T$, we obtain

$$\nu \langle \langle |\Delta_m u|^2 \rangle \rangle + \mu \langle \langle |\Delta_m b|^2 \rangle \rangle \leq C(\varphi_0)2^{-2m}(\limsup_{T \to \infty} \frac{1}{T L^3} \int_0^T \|u(\cdot, t)\|_B^3 \, dt) + \limsup_{T \to \infty} \frac{1}{T L^3} \int_0^T \|b(\cdot, t)\|_B^3 \, dt$$

$$:= 2^{-2m} C(\varphi_0) \tilde{e}_3.$$

4.2. The case of $p \neq 2$. We cannot make use of the coupling structure to cancel difficult terms. Hence, we turn to consider system (MHD$^*$). With no loss of generality, let us assume $\nu \geq \mu$ and we are focused on $p > 1$ firstly. Multiplying Eq.(MHD$_+$)$_1$ by $|\psi_\omega^\delta \Delta_0 z^+|^p \Delta_0^2 \Delta z^+$ and then integrating with respect to spatial variable, we obtain

$$\frac{d}{dt}\|\psi_\omega^\delta \Delta_0 z^+\|_{L^p(\Omega)} - \frac{\nu + \mu}{2} \int_\Omega \Delta(\psi_\omega^\delta \Delta_0 z^+) |\psi_\omega^\delta \Delta_0 z^+|^p \Delta_0^2 \Delta z^+ \, dx$$

$$\leq \frac{\nu - \mu}{2} \int_\Omega \Delta(\psi_\omega^\delta \Delta_0 z^-) |\psi_\omega^\delta \Delta_0 z^+|^p \Delta_0^2 \Delta z^+ \, dx$$

$$+ \int_\Omega [z^+ \cdot \nabla \Psi \psi_\omega^\delta \Delta_0 z^+] |\psi_\omega^\delta \Delta_0 z^+|^p \Delta_0^2 \Delta z^+ \, dx$$

$$+ (\|\psi_\omega^\delta \Delta_0 f\|_{L^p(\Omega)} + \|\psi_\omega^\delta \Delta_0 g\|_{L^p(\Omega)}) \|\psi_\omega^\delta \Delta_0 z^+\|_{L^p(\Omega)}^{-1}.$$ (27)

Using Lemma 2.2 enables us to estimate the second term on the left-hand side of the above equality as follows:

$$- \frac{\nu + \mu}{2} \int_\Omega (\Delta \psi_\omega^\delta \Delta_0 z^+) |\psi_\omega^\delta \Delta_0 z^+|^p \Delta_0^2 \Delta z^+ \, dx$$

$$= \frac{\nu + \mu}{2} \int_\Omega (-\Delta \psi_\omega^\delta \Delta_0 z^- - |\omega_j|^2 |\psi_\omega^\delta \Delta_0 z^+|) |\psi_\omega^\delta \Delta_0 z^+|^p \Delta_0^2 \Delta z^+ \, dx$$

$$+ \frac{(\nu + \mu) |\omega_j|^2}{2} \|\psi_\omega^\delta \Delta_0 z^+\|_{L^p}^p$$

$$\geq \frac{(\nu + \mu)}{2} (|\omega_j|^2 - C \delta) \|\psi_\omega^\delta \Delta_0 z^+\|_{L^p}^p.$$

Similarly, for the first term on the right-hand side of equality (27), we get

$$\frac{\nu - \mu}{2} \int_\Omega (\Delta \psi_\omega^\delta \Delta_0 z^-) |\psi_\omega^\delta \Delta_0 z^+|^p \Delta_0^2 \Delta z^+ \, dx$$

$$\leq \frac{\nu - \mu}{2} (|\omega_j|^2 + C \delta) \|\psi_\omega^\delta \Delta_0 z^-\|_{L^p(\Omega)} \|\psi_\omega^\delta \Delta_0 z^+\|_{L^p(\Omega)}^{-1}.$$ (27)

By virtue of Proposition 3, the second term on the right-hand side of equality (27) can be bounded as

$$\int_\Omega [z^+ \cdot \nabla \Psi \psi_\omega^\delta \Delta_0 z^+] |\psi_\omega^\delta \Delta_0 z^+|^p \Delta_0^2 \Delta z^+ \, dx$$

$$\leq C(\psi_\omega^\delta, \varphi_0) \|z^+\|_{B_{p+1, \infty}^1(\Omega)} \|z^+\|_{B_{p+1, \infty}^1(\Omega)}.$$
Applying the above three estimates, it follows from (27) that

\[
\frac{d}{dt} \| \Psi^{\delta} \Delta_t \dot{z}^+ \|_{L^p_t(\Omega)}^p + \frac{\nu + \mu}{2} \| \omega_j \|_2^2 \| \Psi^{\delta} \Delta_t \dot{z}^+ \|_{L^p_t(\Omega)}^p \\
\leq \frac{\nu - \mu}{2} \| \omega_j \|^2 \| \Psi^{\delta} \Delta_t \dot{z}^- \|_{L^p_t(\Omega)}^p \\
+ C(\psi^{\delta}, \varphi_0) \| z^+ \|_{L^{\frac{5}{3}+1}(\Omega)} \| z^- \|_{L^{\frac{1}{3}+1}(\Omega)} \\
+ (\| \Psi^{\delta} \Delta_t f \|_{L^p_t(\Omega)} + \| \Psi^{\delta} \Delta_t g \|_{L^p_t(\Omega)}) \| \Psi^{\delta} \Delta_t \dot{z}^+ \|_{L^p_t(\Omega)}^{p-1},
\]

Similarly, from Eq. (MHD), we can obtain that

\[
\frac{d}{dt} \| \Psi^{\delta} \Delta_t \dot{z}^- \|_{L^p_t(\Omega)}^p + \frac{\nu + \mu}{2} \| \omega_j \|_2^2 \| \Psi^{\delta} \Delta_t \dot{z}^- \|_{L^p_t(\Omega)}^p \\
\leq \frac{\nu - \mu}{2} \| \omega_j \|^2 \| \Psi^{\delta} \Delta_t \dot{z}^+ \|_{L^p_t(\Omega)}^p \\
+ C(\psi^{\delta}, \varphi_0) \| z^+ \|_{L^{\frac{5}{3}+1}(\Omega)} \| z^- \|_{L^{\frac{1}{3}+1}(\Omega)} \\
+ (\| \Psi^{\delta} \Delta_t f \|_{L^p_t(\Omega)} + \| \Psi^{\delta} \Delta_t g \|_{L^p_t(\Omega)}) \| \Psi^{\delta} \Delta_t \dot{z}^- \|_{L^p_t(\Omega)}^{p-1}.
\]

We assume \( \frac{\nu}{3} < \mu \leq \nu \), relying on Young’s inequality, we thus deduce that

\[
\frac{d}{dt} \left( \| \Psi^{\delta} \Delta_t \dot{z}^+ \|_{L^p_t(\Omega)}^p + \| \Psi^{\delta} \Delta_t \dot{z}^- \|_{L^p_t(\Omega)}^p \right) \\
+ \frac{3\mu - \nu}{2} \| \omega_j \|_2^2 \left( \| \Psi^{\delta} \Delta_t \dot{z}^+ \|_{L^p_t(\Omega)}^p + \| \Psi^{\delta} \Delta_t \dot{z}^- \|_{L^p_t(\Omega)}^p \right) \\
\leq \frac{3\nu - \mu}{2} C(\psi^{\delta}, \varphi_0) \| z^+ \|_{L^{\frac{5}{3}+1}} \| z^- \|_{L^{\frac{1}{3}+1}} \\
+ 2 \left( \frac{32}{3\mu - \nu} \right)^{p-1} \left( \| \Delta_t f \|_{L^p_t(\Omega)}^p + \| \Delta_t g \|_{L^p_t(\Omega)}^p \right) \\
+ \frac{3\mu - \nu}{16} \left( \| \Psi^{\delta} \Delta_t \dot{z}^+ \|_{L^p_t(\Omega)}^p + \| \Psi^{\delta} \Delta_t \dot{z}^- \|_{L^p_t(\Omega)}^p \right).
\]

As \( \frac{1}{4} < |\omega_j|^2 < 4 \), choosing \( \delta \) such that \( \delta < \min\{ \frac{(3\nu - \mu)}{16(3\mu - \nu)}, 1 \} \), we have

\[
\frac{3\nu - \mu}{2} C \delta \leq \frac{3\mu - \nu}{8} \frac{1}{4} < \frac{3\mu - \nu}{8} |\omega_j|^2 \quad \text{and} \quad \frac{3\mu - \nu}{16} < \frac{3\mu - \nu}{4} |\omega_j|^2.
\]

Simplifying the above inequality and integrating in terms of time variable, taking

\[
\frac{1}{T} \int_0^T \left( \| \Delta_t \dot{z}^+ \|^p + \| \Delta_t \dot{z}^- \|^p \right) \| \Delta_0 \dot{z}^+ \|_T^p \left( \| \Delta_t \dot{z}^+ \|^p \right) \| \Delta_t \dot{z}^- \|_T^p \left( \| \Delta_t \dot{z}^- \|^p \right) \| \Delta_0 \dot{z}^+ \|_T^p + \| \Delta_0 \dot{z}^- \|_T^p \| \Delta_0 \dot{z}^- \|_T^p
\]

\[
\leq C \left( \frac{3\nu - \mu}{3\mu - \nu}, \varphi_0 \right) \left( \frac{1}{T} \int_0^T \left( \| z^+ \|_{L_t^{p+1}} B_{p+1}^{\frac{5}{3}+1}(\Omega) \| z^- \|_{L_t^{p+1}} B_{p+1}^{\frac{1}{3}+1}(\Omega) \right) + \| \Delta_0 \dot{z}^+ \|_T \left( \| \Delta_0 \dot{z}^+ \|^p \right) \| \Delta_0 \dot{z}^- \|_T \left( \| \Delta_0 \dot{z}^- \|^p \right) \| \Delta_0 \dot{z}^+ \|_T + \| \Delta_0 \dot{z}^- \|_T \left( \| \Delta_0 \dot{z}^- \|^p \right) \| \Delta_0 \dot{z}^+ \|_T \right).
\]

(28)
Taking the change of variable (18), (28) also holds for \( \tilde{z}^+, \tilde{z}^-, \tilde{f}, \tilde{g} \) on \( \varOmega \times \tilde{T} \), where \( \varOmega = [0, 2^{-m}L]^3 \) and \( \tilde{T} = 2^{m}T \). An easy computation yields that

\[
\frac{1}{T|\varOmega|} \| \tilde{z}^- \|_{L_{p+1}^{\frac{1}{p+1}} B_{p+1, \infty}^\frac{1}{2} (\varOmega)}^{p+1} = \frac{1}{T|\varOmega|} \int_{\varOmega} \int_{\tilde{T}} \sup_{j} 2^{\frac{j}{2(p+1)}} |\tilde{\Delta}_j \tilde{z}^+|^{p+1} \, dx \, dt
\]

\[
= \frac{1}{T|\varOmega|} \int_{\varOmega} \int_{\tilde{T}} \sup_{j} 2^{\frac{j}{2(p+1)}} 2^{-m(p+1)} |\tilde{\Delta}_{j+m} \tilde{z}^+|^{p+1} \, dx \, dt
\]

\[
= \frac{2^{-\frac{4}{2}m(p+1)}}{TL^3} \| \tilde{z}^+ \|_{L_{p+1}^{\frac{1}{p+1}} B_{p+1, \infty}^\frac{1}{2} (\varOmega)}^{p+1}.
\]

Therefore, we infer from (28) that

\[
\frac{3\mu - \nu}{32} - 2^{-mp} \left( \langle |\tilde{\Delta}_m \tilde{z}^+|^p \rangle_T + \langle |\tilde{\Delta}_m \tilde{z}^-|^p \rangle_T \right)
\]

\[
\leq C \left( \frac{3\nu - \mu}{3\mu - \nu} \varphi_0 \right) \frac{2^{-\frac{4}{2}m(p+1)}}{TL^3} \left( \| \tilde{z}^+ \|_{L_{p+1}^{\frac{1}{p+1}} B_{p+1, \infty}^\frac{1}{2} (\varOmega)}^{p+1} + \| \tilde{z}^- \|_{L_{p+1}^{\frac{1}{p+1}} B_{p+1, \infty}^\frac{1}{2} (\varOmega)}^{p+1} \right)
\]

\[
+ 2^{-3m} \langle |\tilde{\Delta}_m \tilde{z}^- (0)|^p \rangle_T + 2^{-3m} \langle |\tilde{\Delta}_m \tilde{z}^+ (0)|^p \rangle_T
\]

\[
+ 2 \left( \frac{32}{3\mu - \nu} \right)^{p-1} 2^{-3mp} \langle \langle |\tilde{\Delta}_m f|^p \rangle_T + \langle |\tilde{\Delta}_m g|^p \rangle_T \rangle.
\]

As every \( m \geq k_0 \), \( \tilde{\Delta}_m \tilde{f} = \tilde{\Delta}_m \tilde{g} = 0 \). Hence, multiplying (29) by \( 2^{mp} \), then taking the lim sup in \( T \), we obtain that

\[
\frac{3\mu - \nu}{32} - 2^{-mp} \left( \langle |\tilde{\Delta}_m \tilde{z}^+|^p \rangle_T + \langle |\tilde{\Delta}_m \tilde{z}^-|^p \rangle_T \right)
\]

\[
\leq C \left( \frac{3\nu - \mu}{3\mu - \nu} \varphi_0 \right) \frac{2^{-\frac{4}{2}m(p+1)}}{TL^3} \left( \limsup_{T \to \infty} \frac{1}{T|\varOmega|} \| \tilde{z}^+ \|_{L_{p+1}^{\frac{1}{p+1}} B_{p+1, \infty}^\frac{1}{2} (\varOmega)}^{p+1}
\]

\[
+ \limsup_{T \to \infty} \frac{1}{T|\varOmega|} \| \tilde{z}^- \|_{L_{p+1}^{\frac{1}{p+1}} B_{p+1, \infty}^\frac{1}{2} (\varOmega)}^{p+1} \right),
\]

which implies that

\[
\langle |\tilde{\Delta}_m \tilde{z}^+|^p \rangle_T^{\frac{1}{p}} + \langle |\tilde{\Delta}_m \tilde{z}^-|^p \rangle_T^{\frac{1}{p}}
\]

\[
\leq C \left( \frac{3\nu - \mu}{3\mu - \nu} \varphi_0 \right) 2^{-m(\frac{4}{2} + \frac{1}{p})} (3\mu - \nu)^{-\frac{1}{p}} \left( \limsup_{T \to \infty} \frac{1}{T|\varOmega|} \| \tilde{z}^+ \|_{L_{p+1}^{\frac{1}{p+1}} B_{p+1, \infty}^\frac{1}{2} (\varOmega)}^{p+1}
\]

\[
+ \limsup_{T \to \infty} \frac{1}{T|\varOmega|} \| \tilde{z}^- \|_{L_{p+1}^{\frac{1}{p+1}} B_{p+1, \infty}^\frac{1}{2} (\varOmega)}^{p+1} \right)^{\frac{1}{p}}.
\]

With respect to \( p = 1 \), we multiply equation (7) by \( \tilde{\Phi}_\delta' (\tilde{\psi}_{\omega_j} \tilde{\Delta}_0 \tilde{z}_k^\pm) \) for estimating \( \langle |\tilde{\Delta}_m \tilde{z}^+|^1 \rangle \). The proof of the remaining parts of estimates in the case of \( p = 1 \) proceeds similarly and we omit it. It can be checked that for \( p = 1 \), constant \( C \) depends on \( \frac{\mu}{\nu} \) and \( \varphi_0 \).

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