1. Introduction

The moduli space of complex structures of a polarized Kähler manifold is of fundamental interest to algebraic geometers and to string theorists. The study of its geometry is greatly enriched by considering the Hodge structure of the underlying manifolds. The moduli space at infinity is particularly interesting because it is related to the degeneration of Kähler manifolds. Even when we know very little about the degeneration itself, by the works of Schmid [27], Steenbrink [28], Cattani-Kaplan-Schmid [4], and many others, we have a good understanding of the degeneration of the corresponding Hodge structures.

The purposes of this paper are twofold: first, based on the work of [27, 28, 4], we prove a Gauss-Bonnet-Chern type theorem in full generality for the Chern-Weil forms of Hodge bundles. That is, the Chern-Weil forms compute the corresponding Chern classes. This settles a long standing problem. Second, we apply the result to Calabi-Yau moduli, and proved the corresponding Gauss-Bonnet-Chern type theorem in the setting of Weil-Petersson geometry. As an application of our results in string theory, we prove that the number of flux vacua of type II string compactified on a Calabi-Yau

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manifold is finite, and their number is bounded by an intrinsic geometric quantity.

Several partial results on Gauss-Bonnet-Chern type theorems have been proved during the last 30 years. If the dimension of the moduli space is 1, then Chern-Weil forms are “good” in the sense of Mumford [25] and thus the Gauss-Bonnet-Chern theorem follows. Cattani-Kaplan-Schmid [4] and Kollár [18] independently obtained the Gauss-Bonnet-Chern type theorem for the first Chern class on a general moduli space. In the setting of Weil-Petersson geometry [22], the rationality of the Weil-Petersson volume was proved independently in [23] and [32], and the rationality of the first Chern class of the Weil-Petersson metric was proved in [23]. If the dimension of the moduli space is 2, then the Gauss-Bonnet-Chern type theorem was proved in [8].

Since in general, moduli spaces are not compact, the major difficulty in the proof of a Gauss-Bonnet-Chern type theorem is to give a good estimate of the Chern-Weil forms at infinity. If the growth of the connections and curvatures were mild, or in the terminology of Mumford [25], the forms are “good”, then by a general theorem in [25], the Gauss-Bonnet-Chern type theorem is valid. However, Chern-Weil forms are not “good” even when the invariant polynomial is linear (corresponding to the case of the first Chern class). In this simplest case some tricks have been used to prove the results in [18,23]. The proof of the general case is far beyond the techniques developed in the above two papers. Unlike the compact case, the “Splitting Principle” for characteristic classes is not valid for moduli space.

The first main result of this paper is Theorem 4.1. Since it is quite technical, it might be a good idea to outline the proof here.

For the sake of simplicity, we may assume that the moduli space $M$ is smooth, and we are dealing with a single Hodge bundle. After passing a finite cover of the manifold, we proved the following result:

$$
\int_M f\left(\frac{\sqrt{-1}}{2\pi} R\right) \in \mathbb{Z},
$$

where $f$ is an invariant polynomial with integer coefficients, and $R$ is the curvature of the Hodge bundle.

By the classical results of Mumford and Viehweg (see [34] for details), we know that the moduli space must be a quasi-projective variety. Let $\overline{M}$ be the compactification of $M$. We assume that $\overline{M}$ is smooth and the divisor $Y = \overline{M}\setminus M$ is of normal crossings. By the nilpotent orbit theorem, after passing a finite cover (See Lemma 4.1 for details), the Hodge bundle extends to $\overline{M}$.

Let $\Gamma^0$ be a smooth connection on the Hodge bundle on $\overline{M}$, then we should prove that

$$
(1.1) \quad \int_M f\left(\frac{\sqrt{-1}}{2\pi} R\right) = \int_{\overline{M}} f\left(\frac{\sqrt{-1}}{2\pi} R^0\right),
$$
where \( R^0 \) is the curvature of \( \Gamma^0 \) (Throughout this paper, we shall use notations like \( \Gamma, R \) to represent the connection or curvature operators as well as the corresponding matrices under a fixed frame). The right hand side of the above equation is an integer by the Gauss-Bonnet-Chern theorem on compact manifold, and thus the theorem follows.

We did prove (1.1). However, the proof is not straightforward. If (1.1) is true for one smooth connection \( \Gamma^0 \), it must be true for any smooth connection. The complicated asymptotic behaviors of the Hodge metric indicated by the \( SL_2 \)-orbit theorem reveals that we have to construct a smooth connection \( \Gamma^0 \) very carefully in order that it matches different growth rates of the Hodge metric in different directions.

To take a closer look at this phenomenon, we let \( \tilde{f} \) be the polarization of \( f \). There are two ways to compare the Chern-Weil forms \( f(R) \) and \( f(R^0) \): the standard way is to write \( f(R) - f(R^0) \) in terms of the Bott-Chern form. The estimation of the Bott-Chern form at infinity of moduli space should be interesting by itself but we were not able to get the appropriate results. This might not be very surprising because of the lack of intrinsic background metrics.

The other way is more elementary. By the linearity of the polarization \( \tilde{f} \), we get

\[
\tilde{f}(R, \cdots, R) - \tilde{f}(R^0, \cdots, R^0) = \sum_j \tilde{f}(R, \cdots, R, \partial_j(\Gamma - \Gamma^0), R^0, \cdots, R^0).
\]

Using integration by parts, we were able to reduce the proof of the theorem to the proof of the uniform (with respect to \( \varepsilon \)) Poincaré boundedness (Definition 4.1) of the form

\[
\partial\varepsilon \wedge \tilde{f}(R, \cdots, R, \Gamma - \Gamma^0, R^0, \cdots, R^0),
\]

where \( \rho_\varepsilon \) is the cut-off function defined in Lemma 4.4 and \( \Gamma \) is the connection of the Hodge metric. The proof of the Poincaré boundedness of (1.2) is the most difficult part of the paper.

The estimate of (1.2) being local, we consider the differential form on the space \( \Delta^a \times \Delta^b \), where \( \Delta \) and \( \Delta^a \) are the unit disk and the punctured unit disk, respectively. Let’s assume that the coordinates on \( \Delta^a \times \Delta^b \) are \( (s_1, \cdots, s_a, w_1, \cdots, w_b) \). The set \( \{s_1 \cdots s_a = 0\} \) is called the infinity of \( \Delta^a \times \Delta^b \). Obviously, \( \Gamma \) and \( R \) are not smooth at infinity. What we know about \( \Gamma \) and \( R \) is from the \( SL_2 \)-orbit theorem of several variables: there are finitely many cones \( C_j \) with \( \Delta^a \times \Delta^b = \bigcup C_j \) such that, after a singular change of frame, the connection and curvature matrices are Poincaré bounded.

More precisely, let the singular frame change be represented by the transition matrix \( e_j \) (which blows up at infinity). Then \( Ad(e_j)\Gamma \) and \( Ad(e_j)R \) are Poincaré bounded. The problem is that, since \( e_j \) blows up at infinity, even though both \( \Gamma^0 \) and \( R^0 \) are smooth, \( Ad(e_j)\Gamma^0 \) or \( Ad(e_j)R^0 \) may not be Poincaré bounded.
We overcame the difficulty by constructing a special smooth connection such that both $\text{Ad}(e_j)^0\Gamma$ and $\text{Ad}(e_j)^0 R^0$ are Poincaré bounded. Lemma 4.8 is the technical heart of the paper addressing this key property. With the construction, by using the invariance of the function $\tilde{f}$, we get

$$\tilde{f}(R, \cdots, \Gamma^0, \cdots, R^0) = \tilde{f}(\text{Ad}(e_j)^0 R, \cdots, \text{Ad}(e_j)(\Gamma - \Gamma^0), \cdots, \text{Ad}(e_j)^0 R^0),$$

which is Poincaré bounded and this finished the proof of the theorem.

The second major result of this paper is Theorem 5.1 in which we prove the convexity property of Chern-Weil forms and give an intrinsic bound of the forms in terms of the generalized Hodge metrics.

The Hodge metric on classifying space was studied by Griffiths and Schmid as early as in [14] (and later by Peters [26]). The curvature properties (with respect to the Hermitian connection) of the Hodge metric were the key to prove the nilpotent and $SL_2$-orbit theorems. In [19], the first author took a further step and proved that the Hodge metric, when restricted to a horizontal slice, must be Kählerian. Moreover, the Kähler metric has negative holomorphic sectional curvatures as well as other good curvature properties. By the Schwarz lemma of Yau [36], the Hodge metric must be Poincaré bounded.

In [11], the results in [19] were generalized to degenerate cases, and the generalized Hodge metrics were defined. Again, the generalized Hodge metrics are Poincaré bounded.

It would be appropriate to make a remark on the Schwarz lemma at this moment. The one dimensional Schwarz lemma (the Kobayashi’s hyperbolicity) was used extensively in [27] and is one of the most important technical tool in that paper. Although the high dimensional Schwarz lemma is not absolutely necessary in the proof of the $SL_2$-orbit theorem of multi-variables, we did need the lemma to get intrinsic estimate. In particular, Theorem 5.1 is not only intrinsic, but also sharper than the estimate directly from [4, (5.22)].

The differential geometry on a Calabi-Yau moduli space is called the Weil-Petersson geometry [22]. It is important in both complex geometry and string theory. On any smooth part of a moduli space, the Weil-Petersson metric can be defined. But on Calabi-Yau moduli space, the Weil-Petersson metric is directly related to the variation of Hodge structure, thanks to a theorem of Tian [30]. In [36] we discovered the relation between our results on Hodge bundles to those in Weil-Petersson geometry. The main result of that section, Theorem 6.3 essentially follows from Theorem 4.1 and Theorem 5.1. However, in Weil-Petersson geometry, we can do more. We feel that Theorem 6.4 has no Hodge theoretic counterpart. There should be one more layer of convexity in the Weil-Petersson geometry (Conjecture 1). By the computation in [20], the conjecture can be interpreted as the property of the

1 The result is different from that of Griffiths-Schmid, because the negativity of the curvature of a submanifold, instead of the whole classifying space, was proved.
underlying Calabi-Yau manifolds. We will include results in this direction in a separate paper.

Our results on Weil-Petersson geometry have many applications in string theory. The geometry of Calabi-Yau moduli and its singularity structure is important in mirror symmetry [24] and in deriving field theoretic limits of string compactifications [16].

In [15], Horne and Moore suggested that finiteness of the volume of a moduli space of string compactifications should be important in early cosmology, so that realistic vacua could be produced with finite probability.

In [1], Ashok and Douglas began the project of counting the number of flux compactifications of the type IIb string on a Calabi-Yau threefold. If our Universe is described by string theory, this is one factor in a count of the number of feasible Universes, which in the type II context appears to be dominant.

In [1, equation (1.5)], a formula for the index counting these supersymmetric vacua was given. Moreover, in Corollary 1.6 and Theorem 1.8 of [9], an estimate for the error term of formula (1.5) of Ashok-Douglas [1] was obtained. By our results, all these numbers are finite. Thus by the theory of Ashok and Douglas, the number of these parallel Universes is finite. Note also [10] which makes similar observations in a number of special cases.

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2. Variation of Hodge structures

Most of the material in this section is standard. It can be found in [12, 13, 14].

Let $Z$ be a compact Kähler manifold of dimension $n$. A $(1, 1)$ form $\omega$ is called a polarization if $[\omega]$ is the first Chern class of an ample line bundle over $Z$. The pair $(Z, \omega)$, or sometimes $Z$ itself, is called a polarized Kähler manifold.

Let $H^k(X, \mathbb{C})$ be the $k$-th cohomology group of $Z$ with $\mathbb{C}$ as the coefficient. Using $\omega$, we can define

$$L : H^k(Z, \mathbb{C}) \to H^{k+2}(Z, \mathbb{C}), \quad [\alpha] \mapsto [\alpha \wedge \omega]$$

to be the multiplication by $[\omega]$ for $k = 0, \cdots, 2n - 2$. Let $P^k(Z, \mathbb{C}) = \ker L^{n-k+1}$ on $H^k(Z, \mathbb{C})$ for $k \leq n$. The group $P^k(Z, \mathbb{C})$ is called the $k$-th primitive cohomology group.

Let $H^k(Z, \mathbb{Z})$ be the $k$-th cohomology group with $\mathbb{Z}$ as the coefficient. Let $H^{p,q}(Z)$ be the complex space of harmonic $(p, q)$ forms. In view of the Hodge decomposition theorem and the two Lefschetz theorems, we make the following definition: let $H^k = P^k(Z, \mathbb{C}) \cap H^k(Z, \mathbb{Z})$ and $H^{p,q} = P^k(Z, \mathbb{C}) \cap H^{p,q}(Z)$ for $p + q = k$. Let $H_\mathbb{R} = H_\mathbb{Z} \otimes \mathbb{R}$ and $H = H_\mathbb{Z} \otimes \mathbb{C}$. Suppose that $S$
is the quadratic form on $H_Z$ induced by the cup product of the cohomology group $H^k(X, \mathbb{C})$. $S$ can be represented by

$$S(\phi, \psi) = (-1)^{k(k-1)/2} \int_Z \phi \wedge \psi \wedge \omega^{n-k}$$

for $\phi, \psi \in H$. By the following definition, the triple $\{H_Z, H^{p,q}, S\}$ defines the so-called polarized Hodge structure.

**Definition 2.1.** A polarized Hodge structure of weight $k$, which is denoted by $\{H_Z, H^{p,q}, S\}$, or $\{H_Z, F^p, S\}$, is given by a lattice $H_Z$, a decomposition $H = \oplus H^{p,q}$ with $p + q = k$ and $H^{p,q} = H^{q,p}$, together with a bilinear form

$$S : H_Z \otimes H_Z \to \mathbb{Z},$$

which is skew-symmetric if $k$ is odd and symmetric if $k$ is even. The bilinear form satisfies the two Hodge-Riemann relations:

1. $S(H^{p,q}, H^{p',q'}) = 0$ unless $p' = k - p, q' = k - q$;
2. $(\sqrt{-1})^{p-q} S(\phi, \phi) > 0$ for any nonzero element $\phi \in H^{p,q}$.

Alternatively, the decomposition (2.3) can be described by the filtration $\{F^p\}$ of $H$:

$$0 \subset F^k \subset F^{k-1} \subset \cdots \subset F^0 = H,$$

such that

$$H = F^p \oplus \overline{F^{k-p+1}}, \quad H^{p,q} = F^p \cap \overline{F^q}.$$

In this case, the Hodge-Riemann relations can be written as

1. $S(F^p, F^{k-p+1}) = 0$ for $p = 1, \cdots, k$;
2. $(\sqrt{-1})^{p-q} S(\phi, \phi) > 0$ for any nonzero element $\phi \in H^{p,q}$.

**Definition 2.2.** The dual classifying space $\hat{D}$ for the polarized Hodge structure of weight $k$ is the set of all filtrations

$$0 \subset F^k \subset \cdots \subset F^1 \subset F^0 = H, \quad F^p \oplus \overline{F^{k-p+1}} = H,$$

or the set of all the decompositions

$$\sum H^{p,q} = H, \quad H^{p,q} = \overline{F^{q-p}}, \quad (p + q = k)$$

on which $S$ satisfies the Hodge-Riemann relation 1 or 3 above. The classifying space $D$ is an open set of $\hat{D}$ defined by the Hodge-Riemann relation 2 or 4 above.

$D$ and $\hat{D}$ are dual in the following Lie group theoretic sense: let

$$G_\mathbb{R} = \{\xi \in \text{Hom}(H_\mathbb{R}, H_\mathbb{R}) \mid S(\xi \phi, \xi \psi) = S(\phi, \psi)\}.$$ 

Then $D$ can also be written as the homogeneous space

$$D = G_\mathbb{R} / V,$$
where $V$ is the compact subgroup of $G_\mathbb{R}$ which leaves a fixed Hodge decomposition $\{H^{p,q}\}$ invariant. $G_\mathbb{R}$ is a semisimple real Lie group of noncompact type without compact factors.

Let $M$ be the compact dual of $G_\mathbb{R}$, and let $G_\mathbb{C}$ be the complexification of $G_\mathbb{R}$. Then $V \subset M \cap G_\mathbb{R}$ and

$$\hat{D} = M/V = G_\mathbb{C}/B,$$

where $B$ is some parabolic subgroup of $G_\mathbb{C}$.

Over the classifying space $D$ we have the holomorphic vector bundles $F^k, \cdots, F^1, H$ whose fibers at each point are $F^k, \cdots, F^1, H$, respectively. These bundles are called Hodge bundles.

We identify the holomorphic tangent bundle $T^{(1,0)}(D)$ as a subbundle of $\text{Hom}(H, H)$:

$$T^{(1,0)}(D) \subset \oplus \text{Hom}(F^p, H/F^p) = \oplus \text{Hom}(H^{p,q}, H^{p-r,q+r}),$$

such that the following compatibility condition holds

$$F^p \rightarrow \begin{array}{c} \downarrow \\ H/F^p \end{array} F^{p-1} \leftarrow \begin{array}{c} \downarrow \\ H/F^{p-1} \end{array}.$$

**Definition 2.3.** A subbundle $T_h(D)$ is called the horizontal distribution of $D$, if

$$T_h(D) = \{ \xi \in T^{(1,0)}(D) \mid \xi F^p \subset F^{p-1}, p = 1, \cdots, k \}.$$

**Definition 2.4.** A horizontal slice $\mathcal{M}$ of $D$ is a complex integral submanifold of the distribution $T_h(D)$. A holomorphic map $\phi: \mathcal{M} \rightarrow D$ is called horizontal, if its image is a horizontal slice at the regular values.

$T_h(D)$ is not always integrable. Thus in general, $\dim \mathcal{M} \leq \text{rank } T_h(D)$.

Let $U$ be an open neighborhood of the universal deformation space (Kuranishi space) of a polarized Kähler manifold $Z$. Assume that $U$ is smooth. Then for each $Z'$ near $Z$, we have the isomorphism $H^\ast(Z', \mathbb{C}) = H^\ast(Z, \mathbb{C})$ induced from the differmorphism between $Z$ and $Z'$. Under this isomorphism, $\{H^{p,q}(Z') \cap P^k(Z', \mathbb{C})\}_{p+q=k}$ can be regarded as a point of $D$. The map

$$\phi: U \rightarrow D, \quad Z' \mapsto \{H^{p,q}(Z') \cap P^k(Z', \mathbb{C})\}_{p+q=k}$$

is called the period map.

In general, the universal deformation space allows some singularities so that the monodromy group $\Upsilon$ is not trivial. Thus the period map is actually a map from $U$ to $\Upsilon \setminus D$. Let $\tilde{U}$ be the universal covering space of $U$. Then $\phi$ lifts to $\tilde{\phi}: \tilde{U} \rightarrow D$. We will call both $\phi$ and $\tilde{\phi}$ period maps.

The most important property of the period map is the following [12]:

**Theorem 2.1** (Griffiths). The period map $\phi: U \rightarrow T \setminus D$ or $\tilde{\phi}: \tilde{U} \rightarrow D$ is holomorphic. Furthermore, it is an immersion and is horizontal.
3. Asymptotic behavior of the period map

We don’t know much about the degeneration of a general given family of compact Kähler manifolds. But the asymptotic behaviors of the corresponding Hodge structures were known to Schmid [27], Steenbrink [28], and Cattani-Kaplan-Schmid [4] by their results of nilpotent and $SL_2$-orbit theorems of one and several variables. These theorems are not only deep, but also very long. For the full version of the theorems, we refer to the above papers. In this section, we will only define and discuss what we need for the rest of the paper. Most of the materials of this section can be found in [27] [3] [4].

We first introduce the nilpotent orbit theorem of several variables. Let $f : \mathcal{X} \to \mathcal{W}$ be a family of compact polarized Kähler manifolds. Since the study of degeneration of Hodge structures is local, we assume that $\mathcal{W} = \Delta^a \times \Delta^b$, where $\Delta, \Delta^*$ are the unit disk and the punctured unit disk in the complex plane, respectively. Let $(s_1, \cdots, s_a, w_1, \cdots, w_b)$ be the standard coordinate system of $\mathcal{W}$. Consider the period map $\phi : \Delta^a \times \Delta^b \to \Upsilon \setminus D$, where $\Upsilon$ is the monodromy group. Let $V$ be the upper half plane. Then $V^a \times \Delta^b$ is the universal covering space of $\Delta^a \times \Delta^b$, and we can lift $\phi$ to a map $\tilde{\phi} : V^a \times \Delta^b \to D$.

Let $(z_1, \cdots, z_a, w_1, \cdots, w_b)$ be the coordinates of $V^a \times \Delta^b$ such that $s_j = e^{2\pi i z_j}$ for $1 \leq j \leq a$. Corresponding to each of the first $a$ variables, we choose a monodromy transformation $T_j \in \Upsilon$, so that $\tilde{\phi}(z_1, \cdots, z_j + 1, \cdots, z_a, w_1, \cdots, w_b) = T_j \circ \tilde{\phi}(z_1, \cdots, z_a, w_1, \cdots, w_b)$ holds identically in all variables. $T_j$’s commute with each other. By a theorem of Borel, after passing a finite cover, we may assume that the eigenvalues of $T_j$ are all 1 (we call such $T_j$’s unipotent), so that we can define $N_j = \log T_j$ using the Taylor expansion of the logarithmic function. These $N_j$’s are called nilpotent operators. All $N_j$’s commute with each other.

Let $z = (z_1, \cdots, z_a)$, and $w = (w_1, \cdots, w_b)$. The map

$$\tilde{\psi}(z, w) = \exp\left(- \sum_{j=1}^{a} z_j N_j \right) \circ \tilde{\phi}(z, w)$$

remains invariant under the translation $z_j \mapsto z_j + 1, 1 \leq j \leq a$. It follows that $\tilde{\psi}$ drops to a map $\psi : \Delta^a \times \Delta^b \to \hat{D}$. 

Theorem 3.1 (Nilpotent Orbit Theorem [27]). The map $\psi$ extends holomorphically to $\Delta^{a+b}$. For $w \in \Delta^b$, the point 

$$F(w) = \psi(0, w) \in \hat{D}$$

is left fixed by $T_j$, $1 \leq j \leq a$. For any given number $\eta$ with $0 < \eta < 1$, there exist constants $\alpha, \beta \geq 0$, such that under the restrictions

$$\Im z_j \geq \alpha, 1 \leq j \leq a \quad \text{and} \quad |w_j| \leq \eta, 1 \leq j \leq b,$$

the point $\exp(\sum_{j=1}^{a} z_j N_j) \cdot F(w)$ lies in $D$ and satisfies the inequality

$$d(\exp(\sum_{j=1}^{a} z_j N_j) \cdot F(w), \tilde{\phi}(z, w)) \leq C \sum_{j=1}^{a} (\Im z_j)^{\beta} \exp(-2\pi \Im z_j),$$

where $C$ is a constant and $d$ is the $G_{\mathbb{R}}$ invariant Riemannian distance function on $D$. Finally, the mapping

$$(z, w) \mapsto \exp(\sum_{j=1}^{a} z_j N_j) \cdot F(w)$$

is horizontal.

In complying with the above theorem, we make the following definition

Definition 3.1. A nilpotent orbit is a map $\theta : \mathbb{C}^a \to \hat{D}$ of the form

$$\theta(z) = \exp(\sum_{j=1}^{a} z_j N_j) \cdot F,$$

where

i.) $F \in \hat{D}$;
ii.) $\{N_j\}_{j=1}^{a}$ is a commuting set of nilpotent elements of $g_{\mathbb{R}}$, the Lie algebra of $G_{\mathbb{R}}$;
iii.) $\theta$ is horizontal, that is, $N_j(F^p) \subset F^{p-1}$;
iv.) There exists an $\alpha \in \mathbb{R}$ such that $\theta(z) \in D$ for $\Im(z_j) > \alpha$.

Given a (real) nilpotent endomorphism $N$ of a finite dimensional complex vector space $H$, we consider the monodromy weight filtration $W = W(N)$. This is defined as the unique increasing filtration $(W_j)_{j \in \mathbb{Z}}$ (defined over $\mathbb{R}$) satisfying

i.) $N(W_j) \subset W_{j-2}$;
ii.) For every $l \geq 0$, $N^l : Gr^W_l \to Gr^W_{l-1}$ is an isomorphism.

Assume that $N^{k+1} = 0$. Unless otherwise stated, we will use a shifting

$W_l(N, k) = W_{l+k}(N)$.

Note that $W_l(N, k) = \{0\}$ for $l < 0$ and $W_l(N, k) = H$ for $l \geq 2k$.

\footnote{The inequality is a refinement of [27 (4.12)], which was observed by Deligne. See [4, page 465] for details.}
We have the following abstract Lefschetz decomposition:

$$Gr^W_l = \bigoplus_{j \geq 0} N^j(P_{l+2j}),$$

where the primitive subspaces $P_l \subset Gr^W_l$ are defined by

$$P_l = \ker \{ N^{l+1} | Gr^W_l \to Gr^W_{l-2} \}, \quad \text{if } l \geq 0,$$

$$P_l = \{0\}, \quad \text{if } l < 0.$$

Let $F \in \hat{D}, N \in g_0$ be a nilpotent element such that $N^{k+1} = 0$, and $W$ be an increasing filtration on $H$. We shall say that $(W, F, N)$, or sometimes $(W, F)$, is a polarized mixed Hodge structure, if

i.) $W$ is the monodromy weight filtration of $(N,k)$;

ii.) $(W,F)$ is a mixed Hodge structure; that is, for $l \geq 0$, the filtration induced by $F$ on $Gr^W_l$ is a Hodge structure of weight $l$;

iii.) $N(F^p) \subset F^{p-1}$ for $0 \leq p \leq k$;

iv.) For $l \geq k$, the Hodge structure induced by $F$ on the primitive subspace $P_l \subset Gr^W_l$ is polarized by the bilinear form $S_l$, where $S_l(\cdot, \cdot) = S(\cdot, N^l \cdot)$, and $S$ is a nondegenerate bilinear form on $H$ which is skewsymmetric if $k$ is odd and symmetric if $k$ is even.

A splitting of a mixed Hodge structure $(W, F)$ is a bigrading $H = \oplus J^{p,q}$ such that

$$W_l = \bigoplus_{p+q \leq l} J^{p,q}, \quad F^p = \bigoplus_{r \geq p} J^{r,s}.$$

We have

$$J^{p,q} \equiv J^{q,p}(\text{mod } W_{p+q-1})$$

for any splitting of $(W, F)$. A mixed Hodge structure is called to split over $\mathbb{R}$, if $J^{p,q} = J^{q,p}$. An $(r, r)$-morphism $X$ of $(W,F)$ is compatible with $\{J^{p,q}\}$ if $X(J^{p,q}) \subset J^{p+r,q+r}$. To any mixed Hodge structure $(W, F)$ on $H = H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ we associated the nilpotent algebra

$$L^{-1,-1}_R = L^{-1,-1}_R(W,F) = \{X \in gl(H) | X(J^{p,q}) \subset \bigoplus_{r \leq p-1, s \leq q-1} J^{r,s} \}.$$

The following result is from [4, Proposition 2.20]:

**Proposition 3.1.** Given a mixed Hodge structure $(W,F)$, there exists a unique $\delta \in L^{-1,-1}_R(W,F)$ such that $(W, e^{-i\delta} \cdot F)$ is a mixed Hodge structure which splits over $\mathbb{R}$. Every morphism of $(W,F)$ commutes with $\delta$; thus, the morphisms of $(W,F)$ are precisely those morphisms of $(W, e^{-i\delta} \cdot F)$ which commute with this element.

Now we turn to the several variable cases. Let $N_1, \ldots, N_a$ be a commutative set of nilpotent operators coming from the unipotent monodromy operators $T_1, \ldots, T_a$. Let

$$C = \{\lambda_1 N_1 + \cdots + \lambda_a N_a | \lambda_j > 0, 1 \leq j \leq a\}$$

(3.10)
be the cone of the monodromy operators. The basic fact about the above setting is the following \cite{27,3,4}:

**Theorem 3.2.** Any \(N \in C\) defines the same monodromy weight filtration. Moreover, let \(F\) be defined in \((3.9)\). Then \((W, F)\) defines a mixed Hodge structure, polarized by each \(N \in C\).

We call that the mixed Hodge structure is polarized by \((N_1, \ldots, N_a)\).

To consider the boundary of the monodromy cone \(C\) in \((3.10)\), we need the notation of relative weight filtration. We use the settings in \cite[pp. 505]{4}. Let \(W_0\) be an increasing filtration of \(H\) and let \(N\) be a nilpotent endomorphism of \(H\) which preserves \(W_0^0\). Deligne \cite[1.6.13]{6} has shown that there exists at most one weight filtration \(W = W(N, W^0)\) of \(H\) such that

i) \(N(W_l) \subset W_{l-2}\);
ii) For each \(j, l \geq 0\)

\[
N^l: Gr_{l+j}W_j Gr_{l+j}W_0^0 \to Gr_{l+j}W_0^0
\]

is an isomorphism.

For each set of indices \(I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, a\}\) let \(C_I\) denote the cone spanned by \(N_{i_1}, \ldots, N_{i_r}\). All elements of \(C_I\) define the same weight filtration \(W(C_I)\).

One can prove (cf. \cite[Proposition 4.72]{4}) that given \(I, J \subset \{1, \ldots, a\}\), \(W(C_I \cup J)\) is the weight filtration of any \(T \in C_J\) relative to \(W(C_I)\).

We need the result in \cite{4} on the asymptotic behavior of the Hodge length. We begin with the following setting: by \((3.8)\), we have

\[
\tilde{\phi}(z, w) = \exp\left(\sum z_j N_j\right) \cdot \psi(s, w),
\]

where \(s_j = e^{2\pi iz_j}\). Let \((W, \psi(0, 0))\) be the mixed Hodge structure, polarized by \((N_1, \ldots, N_a)\). By Proposition 3.1, we let \(F = \exp(-i\delta) \cdot \psi(0, 0)\) be the \(\mathbb{R}\)-split mixed Hodge structure associated to \((W, \psi(0, 0))\). Then \(\xi = \exp(-i\delta) \cdot \psi: \Delta^{a+b} \to \hat{D}\) is holomorphic and \(\xi(0) = F \in \hat{D}\). Since the complex orthogonal group \(G_C\) of the flat form \(S\) acts transitively and holomorphically on \(\hat{D}\), we can write

\[
\xi(s, w) = u(s, w) \cdot F
\]

for a \(G_C\)-valued function \(u(s, w)\), holomorphic on \(\Delta^{a+b}\) and such that

\[
u(0, 0) = 1.
\]

We choose a specific lifting of \(u(s, w)\) as follows. The \(\mathbb{R}\)-split mixed Hodge structure \((W, F)\) determines the bigrading \(H = \oplus I^{p,q}\), where \(I^{p,q} = F_p \cap F_q^* \cap W_{p+q}\). Let the corresponding bigrading of the Lie algebra be \(g = \oplus g^{p,q}\). Because \(F_p^* = \oplus_{r \geq p} F_r^\ast\), the Lie algebra of the stabilizer of \(F\) in \(G_C\) is \(g_F = \oplus_{p \geq 0} g^{p,q}\) and has the nilpotent subalgebra \(v = \oplus_{p \geq 0} g^{p,q}\) as a linear complement. A standard argument shows that the map \(V \mapsto \exp V \cdot F\) is
a holomorphic diffeomorphism from $v$ onto a neighborhood of $F$ in $\hat{D}$. We can write
\[ \xi(s, w) = \exp V(s, w) \cdot F \]
for a unique holomorphic $V : \Delta^{a+b} \to v$. The function $\gamma(z, w) = \exp i\delta \cdot \exp \sum z_j N_j \cdot \exp V(s, w)$ takes values in the unipotent subgroup $\exp v$ and we have
\[ (3.11) \quad \tilde{\phi}(z, w) = \gamma(z, w) \cdot F, \quad z \in U^a, w \in \mathcal{C}, \]
where $\mathcal{C}$ is a small neighborhood of $\Delta^b$ at the origin. Let $C_\phi$ be the Weil operator. That is, $C_\phi = (\sqrt{-1})^{p-q}$ on $H^{p,q}$. Let $W^{(j)} = W(N_1, \ldots, N_j) (1 \leq j \leq a)$ be the weight filtration of $(N_1, \ldots, N_j)$. Define
\[ H \cong \oplus Gr^W_1(H), \quad Gr^W_1(H) \overset{\text{def}}{=} Gr^W_{i_1} \cdots Gr^W_{i_a}(H). \]
We recall the following theorem in [4, Theorem 5.21]:

**Theorem 3.3.** Let $(N_1, \ldots, N_a)$ be the nilpotent operators of a variation of polarized Hodge structures over $\Delta^a \times \Delta^b$, given with a specific ordering. If $v \in \bigcap_j W^{(j)}$, and $Gr^W_1(v) \neq 0$, then (and only then)
\[ ||v|| \sim \left( \frac{\log |s_1|}{\log |s_2|} \right)^{l_1/2} \left( \frac{\log |s_2|}{\log |s_3|} \right)^{l_2/2} \cdots (-\log |s_a|)^{l_a/2} \]
on any region of the form
\[ \left\{ (s, w) \in \Delta^{a+1} \times \Delta^b \bigg| \frac{\log |s_1|}{\log |s_2|} > \varepsilon, \cdots, -\log |s_a| > \varepsilon, w \in \mathcal{C} \right\} \]
for any $\varepsilon > 0$ and $\mathcal{C} \subset \Delta^b$ compact, where
\[ ||v||^2 = S(C_\phi \gamma(z, w) \cdot v, \gamma(z, w) \cdot v). \]

\[ \square \]

4. The rationality of Chern-Weil forms on moduli space

Let $\mathcal{M}$ be the moduli space of a polarized Kähler manifold. By the period map $\phi$ defined in [2] the Hodge bundles $F^k, H^k$, and $H^{p,q}$ can be pulled back to holomorphic bundles $\mathcal{F}^k, \mathcal{H}^k$ and $\mathcal{H}^{p,q}$ on $\mathcal{M}$, respectively. These bundles are Hermitian vector bundles with respect to the polarization $S(\cdot, \cdot)$ and the Weil operator $C_\phi$. For the sake of convenience, we still call them (and the bundles $\mathcal{F}^{p,q}$ defined below) Hodge bundles.

Let $p < q$. Consider the holomorphic bundle
\[ \mathcal{F}^{p,q} = \mathcal{F}^p / \mathcal{F}^q. \]
Note that if $q = p + 1$, then $\mathcal{F}^{p,q} = \mathcal{H}^{p,k-p}$.

Let $\pi_{p,q}$ be the orthogonal projection operator on $\mathcal{F}^{p,q}$. That is, let $\Omega \in \mathcal{F}^p$ be a local holomorphic section. Then $\pi_{p,q} \Omega \in \mathcal{F}^p$ and $S(\pi_{p,q} \Omega, \Omega') = 0$ for any $\Omega' \in \mathcal{F}^q$. $\pi_{p,q} \Omega$ defines a holomorphic section of the bundle $\mathcal{F}^p / \mathcal{F}^q$, but
in general, it is not a holomorphic section of the bundle $\mathcal{F}^p$. By abusing the notations, we will use $\pi_{p,q}\Omega$ as the section of both $\mathcal{F}^p$ and $\mathcal{F}^p/\mathcal{F}^q$. Let $\Omega, \Omega'$ be local holomorphic sections of $\mathcal{F}^p$. Then they define holomorphic sections of $\mathcal{F}^p$. The Hodge metric of $\mathcal{F}^p/\mathcal{F}^q$ is defined as

\[
\langle \Omega, \Omega' \rangle = S(\pi_{p,q} \mathcal{C} \phi \Omega, \Omega').
\]

Let $M$ be a quasi-projective submanifold of $M$. For the sake of simplicity, we use $\mathcal{F}^k, \mathcal{H}, \mathcal{H}^{p,q}$, and $\mathcal{F}^{p,q}$ for both the bundles on $M$ and their restrictions on $M$.

The main result of this section is:

**Theorem 4.1.** Let $M$ be a quasi-projective subvariety of the moduli space $M$ of a polarized Kähler manifold. Let $R_{p,q}$ be the curvature tensor of $\mathcal{F}^{p,q}$ with respect to the metric (4.12). Let $f_{p,q}$ be an invariant polynomial of $\text{Hom}(\mathcal{F}^{p,q}, \mathcal{F}^{p,q})$ with rational coefficients. Then for any sequence $(p_1, q_1), \ldots, (p_r, q_r)$, of nonnegative integers, we have

\[
\int_M f_{p_1, q_1}(\frac{-1}{2\pi} R_{p_1, q_1}) \wedge \cdots \wedge f_{p_r, q_r}(\frac{-1}{2\pi} R_{p_r, q_r}) \in \mathbb{Q}.
\]

Moreover, the Chern-Weil form $\prod f_{p,j}(\frac{-1}{2\pi} R_{p,j})$ computes the corresponding Chern class on any compactification of $M$.

**Remark 4.1.** The curvatures of the Hodge bundles blow up at infinity of the moduli space. Thus the precise meaning of (4.13), at this time, is

\[
\lim_{\varepsilon \to 0} \int_M \rho \varepsilon f_{p_1, q_1}(\frac{-1}{2\pi} R_{p_1, q_1}) \wedge \cdots \wedge f_{p_r, q_r}(\frac{-1}{2\pi} R_{p_r, q_r}) \in \mathbb{Q},
\]

where $\rho = \rho_0$ is the cut-off function with compact support in $M$ such that $0 \leq \rho \leq 1$, and $\rho \equiv 1$ outside the $\varepsilon$-neighborhood of the infinity. However, in the next section, we shall prove that the integration converges absolutely so that the result is independent of the choice of the cut-off functions and the expression of (4.13) makes sense.

Let $M_{\text{reg}}$ be the smooth part of $M$. By Hironaka’s theorem, there is a projective manifold $\overline{M}$, called the compactification of $M$, such that $\overline{M}\setminus M_{\text{reg}}$ is a divisor of $\overline{M}$ of normal crossings. In order to study the asymptotic behavior of the curvatures of Hodge bundles, we first need to extend the bundles to the compactification of $M$.

The following lemma is due to Kawamata [17]. For the sake of completeness, we sketch the proof here. Note that in [23], we proved the similar result under the setting of Weil-Petersson geometry.

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3See the footnote on page 26.

4It may be more accurate to say that, up to a finite cover, $\overline{M}$ can be chosen to be a manifold.
Lemma 4.1. By replacing $M$ with $M_{\text{reg}}$, we assume that $M$ is smooth. Moreover, we assume that $\overline{M}$ is smooth and $Y = \overline{M} \setminus M$ is a divisor of normal crossings. Then there is a finite branched cover $\tilde{M}$ of $\overline{M}$ with branched locus $Y$ such that the monodromy operators along $Y$ in $\tilde{M}$ are all unipotent.

Proof. Let $T$ be a monodromy operator along certain irreducible component of the divisor $Y$ which is not unipotent. Let $T = \gamma_s \gamma_u$ be the decomposition of $T$ into its semi-simple part and its unipotent part. By a theorem of Borel, there is an integer $d$ such that $\gamma_s^d = 1$. Let $L$ be an ample line bundle of $\overline{M}$. Let $Y = \sum D_j$ be the decomposition of the divisor $Y$ into irreducible components. We assume that the monodromy operator $T$ is generated by $U \setminus D_1$, where $U$ is a neighborhood of $D_1$. Assume that $t$ is large enough such that the bundle $L^t(\overline{-D_1})$ is very ample. By taking the $d$-th root of the defining section of $L^t(\overline{-D_1})$ we get a variety $M_1$ such that outside a divisor, it is a finite covering space of $M$. $M_1$ may have some singularities. However, we can always remove those divisors containing singularities to get a smooth manifold.

Let $M'$ be a Zariski open set of $M_1$ that is a covering space of $M$. The Hodge bundles can be pulled back to the manifold $M'$. At any neighborhood of $\overline{M} \setminus M$, the transform of $(\overline{M}, M)$ to $(M_1, M')$ is the $d$-branched cover defined by $z_1 \mapsto \sqrt[d]{z_1}$, where $z_1 = 0$ locally defines the divisor $D_1$. Evidently, the monodromy operator $T$ is transformed to $T^d$, which becomes a unipotent operator.

We observe that if $T'$ is a unipotent monodromy operator, then under the transform $(\overline{M}, M)$ to $(M_1, M')$, $T'$ is still unipotent. Since there are only finitely many irreducible components of $Y$, there are only finitely many monodromy operators which are not unipotent. Thus after finitely many transforms, we can get a compact complex manifold $\tilde{M}$ on which all monodromy operators are unipotent.

For the rest of the section, we let $Y$ be the divisor of $\tilde{M}$ of normal crossings such that $\tilde{M} \setminus Y$ is a finite covering of $M$. We also assume that $\tilde{M}$ is covered by finite open coordinate neighborhoods $\{U_\alpha\}_{\alpha=1,\ldots,t}$, and $\{\psi_\alpha\}$ is the partition of unity subordinating to the cover.

Using Lemma 4.1 and the nilpotent orbit theorem (Theorem 3.1), the Hodge bundles $\mathcal{E}^p, \mathcal{H}^{p,q}, \mathcal{H}$, and $\mathcal{E}^{p,q}$ extend to vector bundles over $\tilde{M}$. We use the same notations to denote these (extended) bundles.

The following definition is standard to experts. However, we find the notation quite convenient to use:

Definition 4.1. Let $\Delta, \Delta^*$ be the unit disk and the punctured unit disk of $\mathbb{C}$, respectively. Let $U = \Delta^a \times \Delta^b$ and let the standard coordinate system of $U$ be $(s_1, \ldots, s_a, w_1, \ldots, w_b)$. A differential form on $U$ is called Poincaré

\footnote{During the process, it is possible that some divisors are added. However, along these divisors, the monodromy operators are the identity operator.}
bounded, if its components are bounded under the coframe
\[ \frac{ds_i}{s_i \log |s_i|}, \frac{d\sigma_i}{\sigma_i \log |s_i|}, dw_j, d\bar{w}_j \]
for \( i = 1, \cdots, a, j = 1, \cdots, b \).

The following Lemma is obvious

**Lemma 4.2.** If a form is bounded, then it is Poincaré bounded. If \( \sigma_1, \sigma_2 \) are Poincaré bounded, so is \( \eta_1 \wedge \eta_2 \).

Moreover, the notion of Poincaré boundedness is independent of the choice of coordinates:

**Lemma 4.3.** Let \( M \) be a quasi-projective manifold and let \( \overline{M} \) be its compactification such that \( \overline{M} \setminus M \) is a divisor of normal crossings. Let \( U, U' \) be two neighborhoods of the divisor such that \( U \cap U' \neq \emptyset \). Then a smooth form is Poincaré bounded on any compact subset of \( U \cap U' \cap M \) with respect to the coordinate system of \( U \) if and only if it is Poincaré bounded with respect to that of \( U' \).

**Proof.** We assume that \( U \approx \Delta^a \times \Delta^b \) and \( U' \approx \Delta^{a'} \times \Delta^{b'} \). Let the coordinates of the two neighborhoods be \((s_1, \cdots, s_a, w_1, \cdots, w_b)\) and \((s'_1, \cdots, s'_a, w'_1, \cdots, w'_b)\), respectively. We further assume that the divisor is the zero locus of either \( \{s_1 \cdots s_a = 0\} \) or \( \{s'_1 \cdots s'_a = 0\} \). Assuming \( a \leq a' \), then on \( U \cap U' \), we can rearrange the order of \( s'_j \) such that
\[ s_j = \xi_j s'_j \]
for \( j = 1, \cdots, a \), where \( \xi_j \) are smooth nonzero functions. Since
\[ d \log s_j = d \log s'_j + d \log \xi_j, \]
we concluded that \( \frac{ds_j}{s_j \log |s_j|}, \frac{d\sigma_j}{\sigma_j \log |s_j|} \) are bounded under the coframe \( \frac{ds'_j}{s'_j \log |s'_j|}, \frac{d\sigma_j}{\sigma_j \log |s'_j|} \) for \( j = 1, \cdots, a \), and vice versa. On the other hand, \( s'_j \) for \( j > a \) are nonzero. The lemma is proved.

As the first step of the proof of Theorem 4.1, we need to choose the cut-off function on \( \Delta \) carefully. The following construction of the cut-off function depends on the particular choice of the cover and partition of unity. However, the main feature is that the complex Hessian of the function is of order \( O\left( \frac{1}{\varepsilon^2 \log \frac{1}{\varepsilon}} \right) \), a little bit better than \( O\left( \frac{1}{\varepsilon^2} \right) \).

**Lemma 4.4.** For any \( \varepsilon > 0 \) small enough, there is a smooth real function \( \rho = \rho_\varepsilon \) on \( \Delta \) such that
1. \( 0 \leq \rho \leq 1 \);
2. \( \partial \rho, \overline{\partial} \rho, \) and \( \partial \overline{\partial} \rho \) are Poincaré bounded;
3. The Euclidean measure of \( \text{supp} (\partial \rho) \) goes to zero as \( \varepsilon \to 0 \);
(4) In a neighborhood of $Y$, $\rho \equiv 0$; and $\rho(x_0) = 1$ if the distance of $x_0 \in M$ to $Y$ is greater than $2\varepsilon$.

**Proof.** Rearranging the order of $\{U_\alpha\}$, we may assume that $\{U_1, \cdots, U_s\}$ is an open cover of the divisor $Y$ and
$$(U_{s+1} \cup \cdots \cup U_t) \cap Y = \emptyset.$$
We further assume that $U_\alpha \backslash Y = (\Delta^*)^{a_\alpha} \times \Delta^{b_\alpha}$ and the coordinates are $(s_1^\alpha, \cdots, s_{a_\alpha}^\alpha, w_1^\alpha, \cdots, w_{b_\alpha}^\alpha)$. Assume that locally $Y$ is the zero locus of
$$s_1^\alpha \cdots s_{a_\alpha}^\alpha = 0$$
on each $U_\alpha$. Let $\eta : \mathbb{R} \to \mathbb{R}$, $0 \leq \eta \leq 1$ be a smooth decreasing function defined as
$$\eta(x) = \begin{cases} 0 & x \geq 1 \\ 1 & x \leq 0 \end{cases}.$$

Let
$$(4.14) \quad \eta_\varepsilon(z) = \eta \left( \frac{(\log \frac{1}{|z|})^{-1} - \varepsilon}{\varepsilon} \right),$$
and let
$$\eta_\varepsilon^\alpha(s_1^\alpha, \cdots, s_{a_\alpha}^\alpha) = \prod_{j=1}^{a_\alpha} (1 - \eta_\varepsilon(s_j^\alpha)).$$
Then the function $\rho = \rho_\varepsilon$ is defined as
$$\rho_\varepsilon = \sum_{\alpha=1}^s \psi_\alpha \eta_\varepsilon^\alpha + \sum_{\alpha=s+1}^t \psi_\alpha,$$
where $\{\psi_\alpha\}$ is the fixed partition of unity defined before.

For the above $\rho_\varepsilon$, (1) is trivial. In order to prove (2), by Lemma 4.2 and 4.3, we only need to prove that $\partial \eta_\varepsilon$ and $\partial \partial \eta_\varepsilon$ are Poincaré bounded. By a straightforward computation, we have
$$\partial \eta_\varepsilon = \frac{1}{2\varepsilon} \eta' \frac{d\sigma}{|\sigma|(\log \frac{1}{|z|})^2};$$
$$\partial \partial \eta_\varepsilon = \frac{1}{4\varepsilon^2} \eta'' \left( \frac{dz \wedge d\sigma}{|z|^2(\log \frac{1}{|z|})^4} + \frac{1}{2\varepsilon} \eta' \frac{dz \wedge d\sigma}{|z|^2(\log \frac{1}{|z|})^3}.\right.$$
The above expressions are non-zero only if
$$\varepsilon \leq \left( \log \frac{1}{|z|} \right)^{-1} \leq 2\varepsilon.$$
Thus there is a constant $C$ such that
$$|\partial \eta_\varepsilon| \leq C \left| \frac{d\sigma}{|z|(\log \frac{1}{|z|})} \right|, \quad |\partial \partial \eta_\varepsilon| \leq C \left| \frac{dz \wedge d\sigma}{|z|^2(\log \frac{1}{|z|})^2} \right|,$$
and thus both $\partial \eta_\varepsilon$ and $\partial \partial \eta_\varepsilon$ are Poincaré bounded.
Since (3) is implied by (4), we only prove the latter. Let \( x_0 \in M \). For any \( x_0 \) which is close enough to \( Y \), \( \psi_\alpha = 0 \) for \( \alpha \geq s + 1 \). On the other hand, since \( x_0 \) is close to \( Y \), for any \( 1 \leq \alpha \leq s \), there is an \( s_j^{\alpha} \) which is sufficiently small. Thus \( \eta_\varepsilon(s_j^{\alpha}) = 1 \) and consequently \( \eta_\varepsilon = 0 \). This proves \( \rho_\varepsilon(x_0) = 0 \). If the distance of \( x_0 \) to \( Y \) is at least \( 2\varepsilon \), then there is a constant \( C > 0 \) such that \( |s_j^{\alpha}| \geq C\varepsilon \) for any \( 1 \leq j \leq a_\alpha \) and \( 1 \leq \alpha \leq s \). Since \( \varepsilon \log \varepsilon^{-1} \to 0 \) for \( \varepsilon \) small, we have \( \eta_\varepsilon = 1 \) for \( 1 \leq \alpha \leq s \). Thus we conclude that \( \rho_\varepsilon(x_0) = \sum \psi_\alpha(x_0) = 1 \) and this completes the proof.

By pulling back the Hodge bundles to \( \tilde{M}\setminus Y \), the curvature operator \( R_{p_j,q_j} \) makes sense as the Hom (\( \mathcal{E}^{p_j,q_j} \), \( \mathcal{E}^{p_j,q_j} \))-valued \((1,1)\)-form on \( \tilde{M}\setminus Y \). Since \( \tilde{M}\setminus Y \) is a finite cover of \( M \), there is a positive integer \( \mu \) such that

\[
\int_M f_{p_1,q_1}(\frac{\sqrt{-1}}{2\pi} R_{p_1,q_1}) \wedge \cdots \wedge f_{p_r,q_r}(\frac{\sqrt{-1}}{2\pi} R_{p_r,q_r}) = \mu \int_M f_{p_1,q_1}(\frac{\sqrt{-1}}{2\pi} R_{p_1,q_1}) \wedge \cdots \wedge f_{p_r,q_r}(\frac{\sqrt{-1}}{2\pi} R_{p_r,q_r}).
\]

Let \( c_1^j, \ldots, c_{d_j}^j \) be the elementary invariant polynomials on Hom (\( \mathbb{C}^{d_j}, \mathbb{C}^{d_j} \)), where \( d_j \) is the rank of the vector bundle \( \mathcal{E}^{p_j,q_j} \). Then there are polynomials \( g_j \) such that

\[
f_{p_j,q_j} = g_j(c_1^j, \ldots, c_{d_j}^j)
\]

for \( j = 1, \ldots, r \).

The following theorem implies the main result of this section, Theorem 4.1.

**Theorem 4.2.** Using the same assumptions and notations as in Theorem 4.1 and Lemma 4.4, we have

\[
\lim_{\varepsilon \to 0} \int_M \rho_\varepsilon f_{p_1,q_1}(\frac{\sqrt{-1}}{2\pi} R_{p_1,q_1}) \wedge \cdots \wedge f_{p_r,q_r}(\frac{\sqrt{-1}}{2\pi} R_{p_r,q_r}) \in \mathbb{Z},
\]

if the coefficients of the polynomials \( g_j \) \( (j = 1, \ldots, r) \) are integers.

Let \( n_j \) be the degree of \( f_{p_j,q_j} \). Let \( \tilde{f}_{p_j,q_j} \) be the polarization of \( f_{p_j,q_j} \). That is,

\[
\tilde{f}_{p_j,q_j} : (\mathbb{C}^{d_j} \times \mathbb{C}^{d_j})^{n_j} \to \mathbb{C},
\]

such that

1. \( \tilde{f}_{p_j,q_j}(A_1, \ldots, A_{n_j}) \) is linear with each \( A_l \) \( (1 \leq l \leq n_j) \);
2. \( \tilde{f}_{p_j,q_j} \) is symmetric. That is,

\[
\tilde{f}_{p_j,q_j}(\cdots, A_k, \cdots, A_l, \cdots) = \tilde{f}_{p_j,q_j}(\cdots, A_l, \cdots, A_k, \cdots);
\]
3. \( \tilde{f}_{p_j,q_j}(A, \ldots, A) = f_{p_j,q_j}(A) \).
We let
\[ \mathcal{E} = \bigoplus_{j=1}^{r} \mathcal{F}_{p_j,q_j}^{p_j,q_j}. \]

Then
\[ R = \begin{pmatrix} R_{p_1,q_1} & \cdots & R_{p_r,q_r} \end{pmatrix} \]
is the curvature tensor of \( \mathcal{E} \). For the sake of simplicity, we use \( R_j = R_{p_j,q_j} \), \( f_j = f_{p_j,q_j} \), and \( \tilde{f}_j = \tilde{f}_{p_j,q_j} \) for \( 1 \leq j \leq r \). Let
\[ f(R) = f_1(R_1) \wedge \cdots \wedge f_r(R_r). \]

Let \( \Gamma^0 \) be a smooth \((1,0)\)-type connection of \( \mathcal{E} \) over \( \tilde{M} \) of the form
\[ \Gamma^0 = \begin{pmatrix} \Gamma^0_1 & \cdots & \Gamma^0_r \end{pmatrix}, \]
where \( \Gamma^0_j \) are smooth \((1,0)\)-type connections on \( \mathcal{F}_{p_j,q_j}^{p_j,q_j} \). Let \( R^0 = \overline{\partial} \Gamma^0 \) be the curvature tensor. Since \( \tilde{M} \) is compact, we have
\[ (4.16) \quad \int_{\tilde{M}} f(\sqrt{-1} R^0) \in \mathbb{Z} \]
by the Gauss-Bonnet-Chern theorem on smooth manifold.

We proved the following

**Lemma 4.5.** If there is a smooth connection \( \Gamma^0 \) on \( \tilde{M} \) such that for any \( i \) and \( j \), \( f_j(R_j) \) and
\[ \eta_{i,j} = \overline{\partial} \rho_\varepsilon \wedge \tilde{f}_j(R_j, \cdots, R_j, \Gamma_j - \Gamma^0_j, R^0_j, \cdots, R^0_j) \]
are Poincaré bounded, then Theorem 4.2 is true. Here \( \Gamma_j \) is the connection operator of the Hodge bundle \( \mathcal{F}_{p_j,q_j}^{p_j,q_j} \).

**Proof.** We use the following obvious equality
\[ f_1(R_1) \wedge \cdots \wedge f_r(R_r) = \sum_{j=1}^{r} \left\{ \left( \prod_{l<j} f_l(R_l) \right) \wedge (f_j(R_j) - f_j(R^0_j)) \wedge \left( \prod_{l>j} f_l(R^0_l) \right) \right\}. \]

Noting that \( \Gamma_j - \Gamma^0_j \) is globally defined for any \( j \), we have
\[ (4.17) \quad \lim_{\varepsilon \to 0} \int_{\tilde{M}} \rho_\varepsilon(f(R) - f(R^0)) \]
\[ = -\lim_{\varepsilon \to 0} \int_{\tilde{M}} \sum_{i,j} \eta_{i,j} \wedge \left( \prod_{l<j} f_l(R_l) \right) \wedge \left( \prod_{l>j} f_l(R^0_l) \right). \]
By the assumption and by Lemma 4.2, the integrand of the right hand side of the above equation is Poincaré bounded. Let

\[ \xi ds_1 \wedge \cdots \wedge ds_a \wedge dw_1 \wedge \cdots \wedge dw_b \]

be the \((a + b, a + b)\)-component of the integrand on a general neighborhood \(U = U_\alpha\), where \(\xi\) is a smooth function on \(U\). Then there is a constant \(C\) such that

\[ |\xi| \leq C \prod_{j=1}^{a} \frac{1}{|s_j|^2 (\log \frac{1}{|s_j|})^2}. \]

It is elementary to see that the above function is Euclidean integrable. Since the Euclidean measure of \(\text{supp} (\partial \rho_\varepsilon)\) goes to zero, by the Lebesgue theorem, the right hand side of (4.17) is zero.

The lemma and hence Theorem 4.1 follow from (4.16), the ordinary Gauss-Bonnet-Chern Theorem.

Before giving the explicit construction of the connection \(\Gamma^0\), we define the local frame on each \(U = U_\alpha\) (\(1 \leq \alpha \leq s\)). We use the notations in (3.11). Let \(F^p_\infty\) be the limiting Hodge filtration. Then for any basis \(\{\tilde{v}_{p,j}\}\) of \(F^p_\infty\),

\[ \exp(\sqrt{-1}\delta) \exp V(s, w)\tilde{v}_{p,j} \]

gives a local frame of the bundle \(\mathcal{F}^p\). In fact, this is the local frame we use to define the extension of the Hodge bundles. We call such a local frame defined by the nilpotent orbit theorem. Likewise, if \(\{v_{p,q,j}\}\) is a basis of the vector space

\[ \bigoplus_{j=1}^{r} F^p_j / F^d_j, \]

then

\[ \exp(\sqrt{-1}\delta) \exp V(s, w)v_{p,q,j} \]

gives a local frame of the bundle \(\mathcal{E}\), which we also call it defined by the nilpotent orbit theorem.

Now we construct the connection \(\Gamma^0\) explicitly: as before \(U_\alpha \cap Y \neq \emptyset\) if and only if \(1 \leq \alpha \leq s\). On each \(\mathcal{F}^{p_j,q_j}\), if \(1 \leq \alpha \leq s\), let \(\Omega_{\alpha,p_j,q_j,a}\), where \(a = 1, \cdots, d_j\), be the local holomorphic frame of the bundle \(\mathcal{F}^{p_j,q_j}\) defined by the nilpotent orbit theorem; if \(\alpha \geq s + 1\), we let \(\Omega_{\alpha,p_j,q_j,a}\) be an arbitrary holomorphic local frame of \(\mathcal{F}^{p_j,q_j}\). Let

\[ \Omega_\alpha = (\Omega_{\alpha,p_1,q_1,1}, \cdots, \Omega_{\alpha,p_1,q_1,d_1}, \cdots, \Omega_{\alpha,p_r,q_r,1}, \cdots, \Omega_{\alpha,p_r,q_r,d_r}). \]

Then the transition matrices of the vector bundles \(A_{\alpha\beta}\) are holomorphic matrix-valued functions:

\[ \Omega_\alpha = \Omega_\beta A_{\alpha\beta}^\beta \]

on \(U_\alpha \cap U_\beta \neq \emptyset\). Note that \(\{A_{\alpha\beta}\}\) are (block) diagonalized matrix-valued functions. The diagonalization is compatible with respect to the direct sum structure of \(\mathcal{E} = \bigoplus_{j=1}^{r} \mathcal{F}^{p_j,q_j}\).
We define the connection matrix
\[ \Gamma_\alpha^0 = \sum_\gamma \psi_\gamma \partial A_{\alpha \gamma} A_{\alpha \gamma}^{-1}. \]
on \{U_\alpha\}, where \{\psi_\alpha\} is the partition of unity subordinating to the cover \{\{U_\alpha\}\}. Then \(\Gamma_\alpha^0\) and the curvature matrix \(R_\alpha^0\) are all (block) diagonalized.

As a general fact, we have the following result:

**Lemma 4.6.** The collection of matrix-valued \((1,0)\) forms \(\{\Gamma_\alpha^0\}\) defines a smooth \((1,0)\) connection on the vector bundle \(E \to \tilde{M}\).

**Proof.** The compatibility conditions of the transition matrices are
\[ A_{\alpha \gamma} = A_{\alpha \beta} A_{\beta \gamma} \]
on \(U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset\). Thus
\[ \partial A_{\alpha \gamma} = \partial A_{\alpha \beta} A_{\beta \gamma} + A_{\alpha \beta} \partial A_{\beta \gamma}. \]
It follows that
\[ \partial A_{\alpha \gamma} A_{\alpha \gamma}^{-1} = \partial A_{\alpha \beta} A_{\alpha \beta}^{-1} + A_{\alpha \beta} \partial A_{\beta \gamma} A_{\beta \gamma}^{-1} A_{\alpha \beta}^{-1}. \]
Using (4.18), we have
\[ \Gamma_\alpha^0 = \partial A_{\alpha \beta} A_{\alpha \beta}^{-1} + A_{\alpha \beta} \Gamma_\beta^0 A_{\alpha \beta}^{-1} \]
on \(U_\alpha \cap U_\beta \neq \emptyset\). Thus \(\{\Gamma_\alpha^0\}\) defines a smooth connection of \(E\).

The problem to verify the assumptions in Lemma 4.5 is purely local. So we will concentrate on a typical neighborhood \(U = U_\alpha\) and suppress the subscript \(\alpha\) for the sake of simplicity. We assume that \(U \approx \Delta^a \times \Delta^b\). Also, we will use \(\Gamma, \Gamma^0, R, R^0\) to represent the connection and curvature matrices, respectively, under the local frame defined by the nilpotent orbit theorem.

In order to study the properties of the smooth connection \(\Gamma^0\), we use the following notations in [4, §5].

As before, we assume that the local coordinate system of \(U\) is
\[ (s_1, \cdots, s_a, w_1, \cdots, w_b), \]
and \(U \cap Y\) is the zero locus of \(s_1 \cdots s_a = 0\).

We define a cone \(C\) in \(\Delta^a\) by
\[ C = \{ |s_1| \leq |s_2| \leq \cdots \leq |s_a| \mid (s_1, \cdots, s_a) \in \Delta^a \}. \]
Such a cone gives an ordering of \(\{1, \cdots, a\}\). Let
\[ I = \{i_\alpha\}, \quad 1 \leq i_1 < \cdots < i_r = a. \]
Define \(|I| = r\), and

---

6 Since in this paper we use a large set of notations, for simplicity, we will try to keep them the same as in [4]. Thus we have to sacrifice the uniqueness of the notations. For example, \(\alpha\) is used in \(U = U_\alpha\) and also as the subscripts of the set \(I\), etc. It should be clear from the context.
Lemma 4.7. Let $N_1, \ldots, N_a$ be the nilpotent operators. Let $Y_a(u)$ be the semisimple elements corresponding to $X_a$ by the Jacobson-Morozov theorem. Define

$$e(y) = e(t, u) = \exp \left( \sum_a \frac{1}{2} \log y_{i_a} Y_a(u) \right) = \exp \left( \sum_a \frac{1}{2} \log t_a Y_a(u) \right),$$

where $Y_a = Y_1 + \cdots + Y_a$.

Moreover, let

- $\mathcal{A}$: analytic functions of $u \in \mathbb{R}_+^{a-r}$;
- $\mathcal{L}$: Laurent polynomials in $\{t_{i_a}^{1/2}\}$ (or $\{y_{i_a}^{1/2}\}$ with coefficients in $\mathcal{A}$);
- $\mathcal{O}$: pull back to $V^a$ of the ring of holomorphic germs at 0 in $\Delta^a$, via $V^a \to \Delta^a \to \Delta^a$, where $V$ is the upper half plane and $V \to \Delta$ is the covering map $z \mapsto e^{2\pi \sqrt{-1}z}$;
- $\mathcal{L}_b$: polynomials in $\{t_{i_a}^{-1/2}\}$ with coefficients in $\mathcal{A}$;
- $(\mathcal{O} \otimes \mathcal{L})^b$: subring of $(\mathcal{O} \otimes \mathcal{L})$ generated by $\mathcal{O}, \mathcal{L}_b$, and all polynomials of the form $s_j t_{i_1}^{m_1/2} \cdots t_{i_p}^{m_p/2}$ for $j \leq i_{q_l}, m_l \in \mathbb{Z}, l = 1, \ldots, p$;
- $(\mathcal{O} \otimes \mathcal{O} \otimes \mathcal{L})^b$: ring generated by $(\mathcal{O} \otimes \mathcal{L})^b, (\mathcal{O} \otimes \mathcal{O} \otimes \mathcal{L})^b$;
- $\mathcal{R}_{K,L}$: ring of rational expressions $f/g$, $f, g \in (\mathcal{O} \otimes \mathcal{O} \otimes \mathcal{L})^b$, with $g$ bounded away from zero on $(V^a)_{K,L}$ (defined below), where $0 < K < L$ are two positive numbers.

Via $y \to (t, u), \mathbb{R}_+^{a}$ is identified with $\mathbb{R}_+^{a} \times \mathbb{R}_+^{a-r}$. For any $K \ll L$, let

$$\begin{align*}
(\mathbb{R}_+^{a})_{K,L} &= \{y \in \mathbb{R}_+^{a} \mid t_a > L, 1 \leq u_{i_a} \leq K\}, \\
(V^a)_{K,L} &= \{z \in V^a \mid z = x + \sqrt{-1}y, y \in (\mathbb{R}_+^{a})_{K,L}\}, \\
(\Delta^{sa})_{K,L} &= \left\{s \in \Delta^{sa} \mid \frac{\log s_j}{2\pi \sqrt{-1}} \in (V^a)_{K,L}\right\}.
\end{align*}$$

Remark 4.2. The above definition is slightly different from the one in [4, page 509]. We use the notation $(\Delta^{sa})_{K,L}$ instead of $(\Delta^{sa})^f_K$ so that it is a little easier to show the $C^\infty$ convergence when $L \gg K$ in Theorem 3.3. Of course, we do have the $C^\infty$ convergence on $(\Delta^{sa})_{K,K} = (\Delta^{sa})_{K}$. But the proof is well hidden in [4, §5] and more explanations are needed.

With the above settings, we have the following combinatorial lemma:

Lemma 4.7. Let

$$1 = K_{a+1} \leq K_a < K_{a-1} < \cdots < K_0 = +\infty$$
be a sequence such that

\[ K_j > K^a_{j+1} \]

for \( j = 1, \ldots, a \). Let

\[ A_j = \bigcup_{|I|=j} (\Delta^{a_I})_{K^a_{j+1},K_j} \]

for \( j = 1, \ldots, a \), and let

\[ A_0 = \{ s \mid y_a \leq K_1 + 1 \}. \]

Then we have

\[ \bigcup_{j=0}^a A_j \supset C. \]

**Proof.** We let

\[ \xi_1 = \frac{y_1}{y_2}, \ldots, \xi_{a-1} = \frac{y_{a-1}}{y_a}, \xi_a = y_a. \]

Consider \((a + 1)\) intervals

\[ (K_{a+1}, K_a), \ldots, (K_2, K_1), (K_1, K_0(= +\infty)). \]

By the pigeonhole principle, if \( s \not\in A_0 \), then there is an \( 1 < l \leq a + 1 \) such that

\[ \xi_j \not\in [K_l, K_{l-1}] \]

for any \( 1 \leq j \leq a \). Define \( I = \{ i_1 < i_2 < \cdots < i_r = a \} \) such that

\[ \xi_j \leq K_l \quad \text{for} \ j \not\in I; \]

\[ \xi_j > K_{l-1} \quad \text{for} \ j \in I. \]

Note that we must have \( i_r = a \). Then we have

\[ u^j_\alpha = \frac{y_j}{y_i} = \frac{y_j}{y_{j+1}} \cdots \frac{y_{a-1}}{y_a} \leq K^n_l, \]

and

\[ t_\alpha = y_i/y_{i+1} > K_{l-1}. \]

Thus we have \( s \in A_{l-1} \) and the lemma is proved. \( \square \)

In what follows, we will use matrix notations extensively. In particular, for a fixed frame, the Hodge metrics (locally) are represented by matrices, and the change of frames are represented using the matrix notations as well.

Let \( \Omega = \Omega_\alpha \) be the local frame of \( \mathcal{E} \) defined by the nilpotent orbit theorem. Let \( C \) be a fixed cone of \( U \) of the form \((\Delta^{a_I})_{K,L} \times \mathcal{C}\), where \( \mathcal{C} \) is a compact subset of \( \Delta^b \). In [4], pp. 514, for such a cone, there is a basis of \( \{v_{C,j}\} \) of \( \bigoplus_{j=1}^s F^p_{\infty}/F^q_{\infty} \) on the typical fiber \( H \) flagged according to the "limiting split" Hodge filtration \( F \), which we call it the basis of the cone. We let \( \Omega_C \) be the frame of the cone defined by the above basis via the nilpotent orbit theorem, and let \( e \) be the matrix of \( e = e(y) \) of (4.20) under the frame \( \Omega_C \). Then the following is true (cf. (5.19) or "Proof of (5.22)" of [4]):
Theorem 4.3. Let $h$ be the metric matrix of the Hodge bundle $E$ under the basis $\Omega_C$, and let

$$h = e^t k e.$$ 

Then the matrix $k$ and its inverse matrix are bounded on the cone $C$. \hfill \Box

By the definition of the basis on the cone, there is a constant matrix $A_C$ (cf. [4, (5.20)]) such that

$$\Omega = \Omega_C A_C.$$ 

Let $\Gamma_C, R_C$ be the connection and the curvature operators of the Hodge metric under the local frame $\Omega_C$, respectively. Then in Proposition (5.22) of [4], the following was proved

Theorem 4.4. The coefficients of the forms $\text{Ad}((e^{-1} t) \Gamma_C$ and $\text{Ad}((e^{-1} t) R_C$ are Poincaré bounded. \hfill \Box

The key technical lemma of this section is the following:

Lemma 4.8. Let $U' = U_\gamma$ be an open set such that $U' \cap C \neq \emptyset$. Let $A = A_{\alpha \gamma}$. Then on $U' \cap C \neq \emptyset$,

$$\text{Ad}((e^{-1} t) \text{Ad}(A^{-1}_C)(\partial AA^{-1})$$

is Poincaré bounded.

**Proof.** Let $\Omega' = \Omega_\gamma$ be the local frame of $U'$ defined by the nilpotent orbit theorem. Let $C'$ be a fixed cone of $U'$ and let $\Omega'_C$ be the frame of the cone. We assume that $C \cap C' \neq \emptyset$. We just need to prove the assertion of the lemma on $C \cap C'$ because as $C'$ is running over all the cones, the whole $U' \cap C$ will be covered.

Let $e'$ be the matrix under the frame $\Omega'_C$. Let $A_{C'}$ be the constant matrix defined as

$$\Omega' = \Omega'_C A_{C'}.$$ 

Then we have

$$\Omega_C = \Omega'_C A_{C'} A^t (A^{-1}_C)^t.$$ 

We let $B = A^{-1}_C A A_{C'}$ and let $h'$ be the metric matrix of $\Omega'_C$. Then $\Omega_C = \Omega'_C B$. Thus

$$\Omega_C \Omega_C = B h' B^t.$$ 

(4.22) 

It follows that

$$\partial hh^{-1} = \partial BB^{-1} + \text{Ad}(B)(\partial h'(h')^{-1}).$$

(4.23) 

Since $A_C, A_{C'}$ are constant matrices, we have

$$\partial BB^{-1} = \text{Ad}(A^{-1}_C)(\partial AA^{-1}).$$
By Theorem 4.4 from (4.23), we see that in order to prove the lemma, we only need to prove that

$$Ad((e^{-1})^t)Ad(B)\partial h'(h')^{-1} = D(Ad((e')^{-1}))(\partial h'(h')^{-1})D^{-1}$$

is Poincaré bounded, where $D = (e^{-1})^tB(e')^t$. Using Theorem 4.4 again, we only need to prove that $D$ and $D^{-1}$ are bounded. To prove this, we observe that if $h = e^tk\bar{e}$, and if $h' = (e')^tk'\bar{e'}$, then by (4.22)

$$k = Dk'\bar{D}.$$  

By Theorem 4.3, $k$, $k'$ and their inverse matrices are bounded. Since both $k$ and $k'$ are positive definite, $D$ and $D^{-1}$ must be bounded. 

**Corollary 4.1.** Let $\Gamma^C_0 = Ad(A^{-1}_C)^0$ and $R^C_0 = Ad(A^{-1}_C) R^0$ be the connection (note that $A_C$ is a constant matrix) and the curvature matrices under the frame $\Omega_C$. Then $Ad((e^{-1})^t)(\Gamma^C_0)$ and $Ad((e^{-1})^t)(R^C_0)$ are Poincaré bounded.

**Proof.** We have

$$\Gamma^C_0 = Ad(A^{-1}_C)((\sum_\gamma \psi_\gamma \partial A_{\alpha\gamma}A^{-1}_{\alpha\gamma})) = \sum_\gamma \psi_\gamma Ad(A^{-1}_C)(\partial A_{\alpha\gamma}A^{-1}_{\alpha\gamma}).$$

By the above lemma, for each $\gamma$, $Ad((e^{-1})^t)Ad(A^{-1}_C)(\partial A_{\alpha\gamma}A^{-1}_{\alpha\gamma})$ is Poincaré bounded. Since $\{U_\alpha\}$ is a locally finite cover, the conclusion on $\Gamma^C_0$ follows. The result on $R^C_0$ follows from a similar formula:

$$Ad((e^{-1})^t)R^C_0 = \sum_\gamma \bar{\partial} \psi_\gamma Ad((e^{-1})^t)Ad(A^{-1}_C)(\partial A_{\alpha\gamma}A^{-1}_{\alpha\gamma}).$$

**Proof of Theorem 4.2.** We only need to verify the assumptions in Lemma 4.5. We mention for one more times that all the matrix-valued functions or forms we have constructed so far are (block) diagonalized with respect to the direct sum structure of the bundle $E$. This fact allows us to suppress the index $j$ in Lemma 4.5 when we fix the $j$.

On any cone $C$, by the invariance of $f = f_j$ and $\tilde{f} = \tilde{f}_j$, we have

$$f(R) = f(Ad((e^{-1})^t)(R_C)).$$
and
\[\overline{\partial} \rho \wedge \tilde{f}(R, \cdots, R, \Gamma - \Gamma^0, R^0, \cdots, R^0)\]
\[= \overline{\partial} \rho \wedge \tilde{f}(R_C, \cdots, R_C, \Gamma_C - \Gamma^0_C, R^0_C, \cdots, R^0_C)\]
\[= \overline{\partial} \rho \wedge \tilde{f}(Ad((e^{-1})^t)(R_C), \cdots, Ad((e^{-1})^t)(R_C), Ad((e^{-1})^t)(\Gamma_C - \Gamma^0_C), Ad((e^{-1})^t)(R^0_C), \cdots, Ad((e^{-1})^t)(R^0_C)).\]

By Theorem 4.4, \(Ad((e^{-1})^t)(R_C)\) and \(Ad((e^{-1})^t)(\Gamma_C)\) are Poincaré bounded. On the other side, by Corollary 4.1, \(Ad((e^{-1})^t)(R^0_C)\) and \(Ad((e^{-1})^t)(\Gamma^0_C)\) are also Poincaré bounded. Thus the left hand sides of (4.24) and (4.25) are Poincaré bounded. By Lemma 4.5, this implies Theorem 4.2 (hence Theorem 4.1).

□

For the moduli space \(\mathcal{M}\) itself, we have

**Corollary 4.2.** Using the same notations as in Theorem 4.1, we have

\[\int_{\mathcal{M}} f_{p_1,q_1}(\sqrt{-1}/2\pi R_{p_1,q_1}) \wedge \cdots \wedge f_{p_r,q_r}(\sqrt{-1}/2\pi R_{p_r,q_r}) \in \mathbb{Q}.\]

**Proof.** By the theorem of Viehweg [34], \(\mathcal{M}\) is a quasi-projective variety. The corollary follows from Theorem 4.1.

□

5. **The Generalized Hodge Metrics**

In this section, we shall show that the form in (4.13) is absolutely integrable. The result can be proved using Proposition (5.22) in [4], or by the argument in the last section. However, we provide a proof here which is elementary (avoid using the \(SL_2\)-orbit theorem) and sharper. More importantly, we give the intrinsic upper bound of the integrals, which can be regarded as Chern number inequalities on moduli space.

In the first part of this section, we recall some notations and results in [11].

Let \(M\) be a complex manifold of dimensional \(m\). Suppose that \(M\) is the parameter space of a family of polarized compact Kähler manifolds \(\pi : \mathcal{X} \rightarrow M\). By the functorial property, the Hodge bundles \(\mathcal{H}, \mathcal{F}^p, \mathcal{H}^{p,q}\) and \(\mathcal{F}^{p,q}\) on \(M\) can be defined as the pull-back of the Hodge bundles from the classifying spaces. The bundles can also be identified to the relative cohomology groups as follows

\[\mathcal{H}^{p,q} = PR^q\pi_*\Omega^p_{\mathcal{X}/M}, \quad \mathcal{F}^p = \mathcal{H}^{p+q,0} \oplus \cdots \oplus \mathcal{H}^{p,q}\]

for \(p, q \geq 0\), where \(\Omega^p_{\mathcal{X}/M}\) is the sheaf of relative holomorphic \((p,0)\) forms on \(\mathcal{X}\). In particular, \(\mathcal{H} = PR^k\pi_*\mathcal{C}\). Let \(Z_x = \pi^{-1}(x)\) for \(x \in M\). Assume that \(\dim Z_x = n\). The Kodaira-Spencer map \(T_x(M) \rightarrow H^1(Z_x, T^{(1,0)}Z_x)\) gives the bundle map

\[\tilde{\partial} \frac{\partial}{\partial t_i} : \mathcal{H}^{p,q} \rightarrow PR^k\pi_*\mathcal{C}/\mathcal{H}^{p,q}\]
for $0 \leq k \leq n$ by differentiation, where $PR^k \pi_*(\mathbb{C})$ is the primitive part of $R^k \pi_*(\mathbb{C})$, and $(t_1, \ldots, t_m)$ is a local holomorphic coordinate system at $x$. The map induces the natural bundle map (compare to (2.7)):

$$T^{(1,0)}(M) \rightarrow \bigoplus_{p+q=k} \text{Hom}(\mathcal{H}^{p,q}, PR^k \pi_*(\mathbb{C})/\mathcal{H}^{p,q}).$$

We make the following definition of the generalized Hodge metrics:

**Definition 5.1.** Assume that $0 \leq k \leq n$. Let $h_{PH_k}$ be the pull back of the natural Hermitian metric on the bundle $\bigoplus_{p+q=k} \text{Hom}(\mathcal{H}^{p,q} \rightarrow PR^k \pi_*(\mathbb{C})/\mathcal{H}^{p,q})$ to $T^{(1,0)}(M)$. We use $\omega_{PH_k}$ to denote the corresponding Kähler forms. In complying to the Lefschetz decomposition theorem, we define

$$\omega_{H_k} = \omega_{PH_k} + \omega_{PH_{k-2}} + \cdots.$$  

We call both $\omega_{H_k}$ and $\omega_{PH_k}$ the generalized Hodge metrics.

**Remark 5.1.** The above construction is a generalization of the Hodge metric defined by the first author [19, 21]. In fact, it is proved in [22] that

$$\omega_{PH^n} = \omega_H,$$

the latter being the Hodge metric defined in [19]. Alternatively, the generalized Hodge metrics can also be defined as the restriction of the invariant Hermitian metrics on the corresponding classifying spaces.

Because of the possible degeneration of the action (5.27), a generalized Hodge metric is only semi-positive definite; hence, it is a pseudo-metric. Nevertheless, it enjoys the similar “curvature” properties of the Hodge metric. To elaborate this, we recall the following results in [11, Proposition 2.8]:

**Proposition 5.1.** Let $c_1(E)$ be the Ricci form of a vector bundle $E$. Then we have

$$\omega_{PH_k} = \sum_{0 \leq p \leq k} pc_1(\mathcal{H}^{p,k-p}),$$

and

$$\omega_{H_k} = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{p \leq k-2\lfloor \frac{k}{2} \rfloor + 2l} pc_1(\mathcal{H}^{p,k-2\lfloor \frac{k}{2} \rfloor + 2l-p})$$

for $k \leq n$.

---

7The Hodge metrics are generally referred to the natural Hermitian metrics on Hodge bundles. After [19, 21], the metrics in Definition 5.1 are also called the Hodge metrics or the generalized Hodge metrics, because they have the similar curvature properties as the original ones on Hodge bundles.

8That is, if $h_{PH_k} = (h_{PH_k})_{i\bar{j}} dt_i \otimes d\bar{t}_j$, then $\omega_{PH_k} = \sqrt{-1} 2\pi (h_{PH_k})_{i\bar{j}} dt_i \wedge d\bar{t}_j$. 
Proof. For the sake of completeness, we include the proof here. Fixing a $k \leq n$, we let
\[ F^p_k = H^{k,0} \oplus \cdots \oplus H^{p,k-p} \]
for $p = 0, \cdots, k$. Thus for $q = k - p$,
\[ H^{p,q} = \frac{F^p_k}{F^p_{k+1}}. \]
In terms of the curvatures, we have
\[ c_1(H^{p,q}) = c_1(F^p_k) - c_1(F^{p+1}_k). \]
By the Abel summation formula, we have
\[ \sum_{0 \leq p \leq k} pc_1(H^{p,k-p}) = c_1(F^k_k) + \cdots + c_1(F^1_k) + c_1(F^0_k). \]
Each $F^p_k$ is a sub-bundle of the flat bundle $F^0_k = PR^k \pi_* \mathbb{C}$. Let $t_1, \cdots, t_m$ be the local holomorphic coordinate of $M$ and let the bundle map
\[ \frac{\partial}{\partial t_j} : F^p_k \to F^0_k / F^p_k, 1 \leq j \leq m \]
be represented by
\[ \frac{\partial \Omega_\alpha}{\partial t_j} = b_{j\alpha\mu} T_\mu, \]
where $\Omega_\alpha$ and $T_\mu$ are the basis of $F^p_k$ and $F^0_k / F^p_k$, respectively. Then the first Chern class can be written as
\[ c_1(F^p_k) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j,\alpha,\mu} b_{i\alpha\mu} b_{j\alpha\mu} dt_i \wedge dt_j \]
for $0 \leq p \leq k$. By the definition of the generalized Hodge metrics, (5.29) follows from (5.32) and (5.33). (5.30) follows from (5.29) and the Lefschetz decomposition theorem. The proof is completed.

Corollary 5.1. Using the above notations, we have
\[ d\omega_{PH^k} = d\omega_{H^k} = 0. \]
In particular, if the generalized Hodge metric is positive definite, then it defines a Kähler metric.

Let $g_p$ be the Hodge metric on $H^{p,q}$. In what follows, we shall compute the curvatures of $g_p$ and compare them with the generalized Hodge metrics. For the sake of simplicity, we assume that $H^{k+1,-1} = H^{-1,k+1} = 0$ and $F^{k+1}_k = 0, F^{-1}_k = F^0_k$. Fix $k \leq n, p \leq k$ and $q = k - p$. Let $\{\Omega_{p,i}\}, i = 1, \cdots, h^{p,q} = \text{rank} H^{p,q}$ be a local holomorphic frame of $H^{p,q}$.

\[ \text{This is essentially due to [14]. See also [12, page 34].} \]
Definition 5.2. Let \((t_1, \cdots, t_m)\) be a holomorphic local coordinate system at a point of \(M\). We define \(\nabla_\alpha \Omega_{p,i} \in H^{p-1,q+1}\) to be the projection of \(\partial_\alpha \Omega_{p,i} = \frac{\partial}{\partial t_\alpha} \Omega_{p,i}\) to \(H^{p-1,q+1}\) with respect to the bilinear form \(S(\ , \ )\).

For simplicity, we shall use \((\ , \ )\) in stead of the bilinear form \(S\). With the above notation, 
\begin{equation}
(g_p)_{ij} = \langle \Omega_{p,i}, \Omega_{p,j} \rangle = (\sqrt{-1})^{p-q}(\Omega_{p,i}, \Omega_{p,j})
\end{equation}
is the Hermitian metric matrix of \(\mathcal{H}^{p,q}\) for \(p_0, \cdots, k\). Using (5.33), we can write the generalized Hodge metric in local coordinates as follows (cf. [11]):

Proposition 5.2. For fixed \(k\), the generalized Hodge metric matrix under the local coordinate system \((t_1, \cdots, t_m)\) can be written as
\begin{equation}
 h_{\alpha\beta} = \sum_{p=0}^{k}(\sqrt{-1})^{p-q-2} (g_p^{-1})_{\alpha\beta}(\nabla_\beta \Omega_{p,i}, \nabla_\alpha \Omega_{p,j}),
\end{equation}

where \((g_p^{-1})_{\alpha\beta}\) is the inverse of \((g_p)_{ij}\).

Let \(R_p\) be the curvature operator of \(g_p\). Then by [12, page 34], we have
\begin{equation}
(R_p)_{ij\gamma\delta} = (\sqrt{-1})^{p-q}(\nabla_\gamma \Omega_{p,i}, \nabla_\delta \Omega_{p,j}) - (\sqrt{-1})^{p-q}(\overline{\partial_\gamma \Omega_{p,i}}, \overline{\partial_\delta \Omega_{p,j}}).
\end{equation}

By the Cauchy inequality, the first term of the right hand side of the above is no more than \(\sqrt{h_{\gamma\gamma}h_{\delta\delta}}\). It can be proved (cf. [11] Lemma A.5) that the operator \(\nabla\) is dual to the operator \(\overline{\partial}\). Thus the norm of the two operators must be the same and consequently, the second term of the above is also no more than \(\sqrt{h_{\gamma\gamma}h_{\delta\delta}}\). Thus we have
\begin{equation}
|\langle (R_p)_{ij\gamma\delta}, \rangle| \leq 2 \sqrt{h_{\gamma\gamma}h_{\delta\delta}}
\end{equation}
for any \(i, j\).

Now we turn to the proof of the absolute integrability and Chern number inequalities.

Theorem 5.1. Let \(c_\alpha(g_p)\) be the \(\alpha\)-th Chern-Weil form of the Hodge bundle \(\mathcal{H}^{p,q}\) with respect to the metric \(g_p\). Then we have
\begin{equation}
|c_{\alpha_1}(g_{p_1}) \wedge \cdots \wedge c_{\alpha_r}(g_{p_r})| \leq 2^m \omega^m,
\end{equation}
where
\[ \omega = \sum \omega_{H^k}, \]
and \(\sum \alpha_j = m\). Similarly, we have
\begin{equation}
|c_\alpha(g_p) \wedge \omega_{PH^k}^{m-\alpha}| \leq 2^m \omega_{PH^k}^m
\end{equation}
for \(k = p + q\).
Proof. Let $\omega_0$ be any Kähler metric of $M$. Then by Corollary 5.1, for any $\varepsilon > 0$, $\omega + \varepsilon \omega_0$ is a Kähler metric. Suppose $(t_1, \cdots, t_m)$ is a holomorphic normal coordinate system at $x_0 \in M$. Let $(h_{\alpha\beta})$ be the metric matrix of $\omega$ under this coordinate system. Then $h_{\gamma\gamma} \leq 1$ for any $\gamma$. We first consider a general Chern-Weil form $c_\alpha(g_p)$. For fixed $\gamma, \delta$, by (5.37), we have

\begin{equation}
|\langle R_p \rangle_{ij\gamma\delta} | \leq 2 \sqrt{h_{\gamma\gamma} h_{\delta\delta}} \leq 2.
\end{equation}

Let

\begin{equation}
\langle R \rangle_{ij} = \langle R_p \rangle_{ij\gamma\delta} dt_\gamma \wedge dt_\delta.
\end{equation}

Then by definition,

\begin{equation}
c_\alpha(g_p) = \left( \frac{\sqrt{-1}}{2\pi} \right)^\alpha \frac{(-1)^\alpha}{\alpha!} \sum_{\tau \in S_\alpha} \text{sgn}(\tau) R_{i_1 \tau(1)} \wedge \cdots \wedge R_{i_\alpha \tau(\alpha)}.
\end{equation}

where $S_\alpha$ is the symmetric group on the set $\{1, 2, \cdots, \alpha\}$. Let

$$dt_I = dt_{i_1} \wedge \cdots \wedge dt_{i_\alpha}.$$ 

Using (5.40), if we write

\begin{equation}
c_\alpha(g_p) = \left( \frac{1}{2\pi} \right)^\alpha \frac{1}{\alpha!} \sum a_I dI \wedge dJ,
\end{equation}

then we have

\begin{equation}
|a_I| \leq 2^\alpha.
\end{equation}

Since the number of partition of the set $\{1, \cdots, m\}$ into subsets of $\alpha_1, \cdots, \alpha_r$ elements, respectively, is

$$\frac{m!}{(\alpha_1)! \cdots (\alpha_r)!},$$

using (5.41), we have

\begin{equation}
\left| \frac{c_{\alpha_1}(g_{p_1}) \wedge \cdots \wedge c_{\alpha_r}(g_{p_r})}{(\omega + \varepsilon \omega_0)^m} \right| \leq 2^m,
\end{equation}

and (5.38) follows by taking $\varepsilon \to 0$. Similarly, we get (5.39). The theorem is proved.

\begin{corollary}[Chern number inequalities] Using the same notations as above, we have

$$\int_M c_{\alpha_1}(g_{p_1}) \wedge \cdots \wedge c_{\alpha_r}(g_{p_r}) \leq 2^m \int_M \omega^m;$$

and

$$\int_M c_\alpha(g_p) \wedge \omega^{m-\alpha} \leq 2^\alpha \int_M \omega^{m}_{PH^k}.$$ 
\end{corollary}
Now we state the following result, which can be viewed as a degenerate version of Yau’s Schwarz lemma \[36\]. The result here is a slight generalization of [11, Theorem A.1] because \(\tau\) doesn’t have to be the Poincaré metric.

**Theorem 5.2.** Let \(\tau = \frac{\sqrt{-1}}{2\pi} \tau_{\alpha\overline{\beta}} dt^\alpha \wedge d\overline{t}^\beta\) be a Kähler metric on \(M\) such that

1. \(\tau\) is a complete metric;
2. The Ricci curvature of \(\tau\) has a lower bound.

Then there is a constant \(C\), depending only on the dimension of \(M\) and the lower bound of the Ricci curvature of \(\tau\), such that

\[\omega_{Hk} \leq C\tau.\]

**Proof.** Let \(\xi\) be the smooth function defined by \(\xi = \tau^{\alpha\overline{\beta}} h_{\alpha\overline{\beta}}\), where \(\tau^{\alpha\overline{\beta}}\) is the inverse matrix of \(\{\tau_{\alpha\overline{\beta}}\}\). Then by the Bochner type formula in [11, Appendix A], there is a constant \(C > 0\), depending only on the dimension of \(M\) and the lower bound of the Ricci curvature of \(\tau\), such that

\[\Delta \xi \geq \frac{1}{C} \xi^2 - C\xi.\]

Using the generalized maximum principle [5] (see also [31]), \(\xi \leq C^2\) is a bounded function. This completes the proof.

A typical choice of the metric \(\tau\) is the so-called Poincaré metric whose Kähler form is denoted as \(\omega_P\). At any point \(x_0 \in M\), there is a neighborhood \(U\) of \(p\) such that \(U \cap M\) can be identified as \(\Delta^a \times \Delta^b\). The metric \(\omega_P\) on \(\Delta^a \times \Delta^b\) is asymptotic to the Poincaré metric

\[\omega_P \sim \frac{\sqrt{-1}}{2\pi} \left( \sum_{i=1}^{a} \frac{ds_i \wedge d\overline{s}_i}{|s_i|^2 (\log |s_i|)^2} + \sum_{i=a+1}^{a+b} dw_i \wedge d\overline{w}_i \right).\]

See [23, Section 5] for the detailed constructions.

In our terminology, Theorem A.1 of [11] can be rephrased as

**Corollary 5.3.** The generalized Hodge metrics are Poincaré bounded. In particular, the Hodge volume is finite, hence (4.13) is absolutely integrable.

**Proof.** By a straightforward computation, the volume of the Poincaré metric is finite. This proves the corollary.

The above corollary implies the result in [4 (5.23)]

**Corollary 5.4.** The Chern-Weil forms extend to a current on the compactification \(\overline{M}\) of \(M\).
6. Chern classes on Calabi-Yau moduli

In this section, we assume that $Z$ is a polarized Calabi-Yau manifold of dimension $n$ and $\mathcal{M}$ is the moduli space of $Z$ (the Calabi-Yau moduli). For a Calabi-Yau manifold, the Hodge structure of weight $n$ is the most important one. Let $D$ be the classifying space defined in Definition 2.2 corresponding to the weight $n$, and let $\phi : \mathcal{M} \to D$ be the period map. By [2], the map is an immersion on the smooth part of $\mathcal{M}$.

**Definition 6.1.** Let $Z$ be a polarized Calabi-Yau manifold with the Ricci flat Kähler metric $\mu$ whose Kähler form defines the polarization. Let $X, Y \in H^1(Z, T^{(1,0)}Z)$. Define the $L^2$ inner product by

$$(X, Y) = \frac{1}{n!} \int_Z (X, Y) \mu^n.$$

For a Calabi-Yau manifold, the Kodaira-Spencer map:

$$T_Z \mathcal{M} \to H^1(Z, T^{(1,0)}Z)$$

is an isomorphism. Thus the above inner product defines a metric on the smooth part of $\mathcal{M}$. The metric happens to be Kählerian, and is called the Weil-Petersson metric of $\mathcal{M}$.

Let $\mathcal{L}_n$ be the first Hodge bundle on $\mathcal{M}$. It is a line bundle because $\dim H^{n,0}(Z) = 1$ for Calabi-Yau manifolds. By Griffiths [12, page 34] and [2], we know that

$c_1(\mathcal{L}_n) > 0$.

The crucial result we are going to use is the following [30]:

**Theorem 6.1** (Tian). On the smooth part of $\mathcal{M}$, the Kähler form of the Weil-Petersson metric $\omega_{WP}$ is $c_1(\mathcal{L}_n)$. More precisely, let $\Omega$ be a holomorphic local section of $\mathcal{L}_n$, then

$$\omega_{WP} = -\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log(\Omega, \overline{\Omega}).$$

The Weil-Petersson metric is defined on the smooth part of the Calabi-Yau moduli. Though in general, a moduli space may have singularities, by the theorem of Tian [30] (see also Todorov [33]), we know that the Kuranish space of a Calabi-Yau manifold is smooth. As a result, a Calabi-Yau moduli space must be a complex orbifold, and the Weil-Petersson metric is a Kähler orbifold metric.

By Theorem 6.1, we know that the Weil-Petersson metric depends only on the variation of Hodge structure. Moreover, we shall see that the curvature of the Weil-Petersson metric is also directly related to the variation of Hodge structure, which allows us to make use of Theorem 4.1 and Theorem 5.1 in the previous sections.
We cite the formulas of Strominger [29] (for the Calabi-Yau moduli of a Calabi-Yau threefold) and Wang [35] (for the general case) on the curvature tensor of the Weil-Petersson metric.

**Theorem 6.2.** On the Calabi-Yau moduli of a polarized Calabi-Yau $n$-fold, let $\Omega$ be a local holomorphic section of $\mathbb{F}^n$. Then

$$R(\omega_{WP})_{\alpha\beta\gamma\delta} = g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\delta}g_{\gamma\beta} - \frac{(\nabla_\gamma \nabla_\beta \Omega, \nabla_\delta \nabla_\alpha \Omega)}{(\Omega, \Omega)}$$

for any $1 \leq \alpha, \beta, \gamma, \delta \leq m$, where $m = \dim \mathcal{M}$; $g_{\alpha\beta}$ is the metric matrix of the Weil-Petersson metric; and $\nabla$ is the connection defined in Definition 5.2. Moreover, In the case of moduli space of Calabi-Yau threefolds, the curvature can be represented in terms of the Yukawa coupling $\{F_{\alpha\beta\gamma}\}$:

$$R(\omega_{WP})_{\alpha\beta\gamma\delta} = g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\delta}g_{\gamma\beta} - \frac{1}{(|(\Omega, \Omega)|^2)} g_{\xi\eta} F_{\alpha\gamma\xi} F_{\beta\delta\eta}. $$

Based on the above formulas, in [22, Theorem 5.3], it was proved that the Weil-Petersson curvature is $L^1$ (with respect to the Weil-Petersson metric). In what follows, we shall greatly generalize the result. By relating the curvature tensor of the Weil-Petersson metric to that of the Hodge bundles, we prove the corresponding Gauss-Bonnet-Chern theorem for Calabi-Yau moduli.

For the sake of simplicity, we write $R = R(\omega_{WP})$. Consider the bundle $\mathbb{F}^{n-1}/\mathbb{F}^n = H^{n-1,1}$. By (5.36), the curvature tensor $R'$ of $H^{n-1,1}$ can be written as

$$ (R')_{i\gamma\beta\delta} = ((-1)^{n-2} \left( (\nabla_\gamma \Omega_i, \nabla_\delta \Omega_j) - (\bar{\Omega}_j \Omega_i, \bar{\Omega}_\gamma \bar{\Omega}_\delta) \right),$$

where $\{\Omega_1, \cdots, \Omega_n\}$ is a local frame of $H^{n-1,1}$.

Since $\mathcal{M}$ is the Calabi-Yau moduli, we have the Kodaira-Spencer isomorphism $T\mathcal{M} = H^{n-1,1}$. Let $\Omega$ be a local holomorphic section of the line bundle $\mathbb{F}^n$ and let $(t_1, \cdots, t_m)$ be a holomorphic local coordinate system of $\mathcal{M}$. By [2], we can choose $\{\Omega_i\}$ as $\{\partial_i \Omega\}$, or $\{\nabla_i \Omega\}$. The latter are not holomorphic on $\mathbb{F}^n$ but are holomorphic on $H^{n-1,1}$. We claim that

$$ (R')_{i\gamma\beta\delta} = g_{\gamma\delta} \nabla_i \Omega.$$ 

To see this, we first observe that

$$\nabla_i \Omega = \partial_i \Omega - K_i \Omega,$$

where $K_i = \partial_i \log(\Omega, \Omega)$ are local functions. The choices of local functions $K_i$ are made so that $\nabla_i \Omega$ is orthogonal to $\Omega$. Thus (6.44) follows from Theorem 6.1. By the curvature formula (6.43), we have

$$ (R')_{i\gamma\beta\delta} = ((-1)^{n-2} \left( (\nabla_\gamma \nabla_i \Omega, \nabla_\delta \nabla_j \Omega) - g_{\gamma\delta} g_{\gamma\delta} (\Omega, \Omega) \right).$$

Using the above result, we get the following
Lemma 6.1. Under the Kodaira-Spencer identification, we have

\[ R = I\omega_{WP} + (\sqrt{-1})^n R', \]

where \( I \) is the identity map in \( \text{Hom}(TM, TM) \).

**Proof.** By Theorem 6.1, we know that

\[ g_{ij} = -\partial_i \partial_j \log(\Omega, \Omega). \]

By a straightforward computation, we have

\[ g_{ij} = -\left( \nabla_i \Omega, \nabla_j \Omega \right)(\Omega, \Omega). \]

The above formula gives the relation between the Weil-Petersson metric and the Hodge metric on \( H^{n-1,1} \). If we choose frames such that at a point, \( (\sqrt{-1})^n(\Omega, \Omega) = 1 \) and \( g_{ij} = \delta_{ij} \), then the metric of \( H^{n-1,1} \) is also the identity matrix at that point. The lemma thus follows from (6.45) and Theorem 6.2.

The main result of this section is the following

**Theorem 6.3.** Let \( \gamma \) be a rational number, and let \( f \) be an invariant polynomial on \( \text{Hom}(TM, TM) \) with rational coefficients. Let \( R_{WP} \) be the curvature tensor of the Weil-Petersson metric. Then we have

\[ \int_M f(R_{WP} + \gamma I\omega_{WP}) \in \mathbb{Q}, \]

where \( I \) is the identity map on \( \text{Hom}(TM, TM) \). Moreover, we have

\[ |c_{\alpha_1}(\omega_{WP}) \wedge \cdots c_{\alpha_r}(\omega_{WP}) \wedge \omega^{\alpha_0}_{WP}| \leq 2^{m-\alpha_0} \omega_H^m \]

for \( \sum_{j=0}^r \alpha_j = m \), where \( \omega_H = \omega_{WP} \) is the Hodge metric.

**Proof.** The equation (6.46) follows from Theorem 4.1 and Lemma 6.1. The equation (6.47) essentially follows from Theorem 5.1. In what follows, we directly prove (6.47) in order to get the desired constant.

First we choose a coordinate system at \( x_0 \in M \) so that

\[ g_{ij}(x_0) = \delta_{ij}. \]

Let \( R = (R^i_j) \), and let

\[ R^i_j = \sum_{k,l} R^i_{k,l} dt_k \wedge dt_l, \]

where \( R^i_{k,l} = g^{\overline{ir}} R_{\overline{ijkl}}. \) Then the \( \alpha \)-th Chern class is given by

\[ c_{\alpha}(\omega_{WP}) = \left( \frac{\sqrt{-1}}{2\pi} \right)^\alpha \frac{(-1)^\alpha}{\alpha!} \sum_{\tau \in S_\alpha} \text{sgn}(\tau) R^i_{\tau(1)} \wedge \cdots \wedge R^i_{\tau(\alpha)}, \]

where \( S_\alpha \) is the symmetric group on the set \( \{1, 2, \ldots, \alpha\} \).

We define
\[ h'_{ij} = \delta_{ij} + \sum_{l} \langle \nabla_i \nabla_l \Omega, \nabla_j \nabla_l \Omega \rangle. \]

Then \((h'_{ij})\) defines a Kähler metric \(\omega'\).

By Proposition 5.2, we have
\[ \omega' \leq \omega_H. \]

Thus in order to prove the theorem, we only need to prove that
\[ |c_{\alpha_1}(\omega_{WP}) \wedge \cdots \wedge c_{\alpha_r}(\omega_{WP}) \wedge \omega_{WP}^0| \leq 2^{m-\alpha_0} (\omega')^m. \]

Let \(A_{ij} = \sum_k \langle \nabla_i \nabla_k \Omega, \nabla_j \nabla_k \Omega \rangle \). Since the matrix \((A_{ij})\) is Hermitian, after a suitable unitary change of basis, we can assume
\[ A_{ij}(x_0) = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \]

Since \((A_{ij}(x_0))\) is semi-positive definite, \(\lambda_i \geq 0\), and we have
\[ h'_{ij}(x_0) = \delta_{ij}(1 + \lambda_i). \]

For fixed \(i, j, k, l\), by the Cauchy inequality, we have
\[
|R_{ijkl}^{\prime}| \leq |\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - \langle \nabla_i \nabla_k \Omega, \nabla_j \nabla_l \Omega \rangle| \\
\leq 2 + \sqrt{\langle \nabla_i \nabla_k \Omega, \nabla_j \nabla_l \Omega \rangle} \\
\leq 2 + \sqrt{\lambda_k \lambda_l} \leq 2\sqrt{(1 + \lambda_k)(1 + \lambda_l)}.
\]

So we get
\[
|R_{i_{1}i_{2}i_{3}}^{\prime} \cdots R_{j_{1}j_{2}j_{3}}^{\prime}| \leq 2^\alpha \prod_{j=1}^{\alpha} \left( \sqrt{(1 + \lambda_{k_j})(1 + \lambda_{l_j})} \right).
\]

We note that all \(k_j\)’s (and also \(l_j\)’s) have to be different. Thus we have
\[
\prod_{j} \sqrt{(1 + \lambda_{k_j})(1 + \lambda_{l_j})} \leq \det(h_{\alpha',\alpha}).
\]

Using the same method as in the proof of Theorem 5.1, we get \([6.47]\). \(\square\)

**Corollary 6.1** (Chern number inequalities). Let \(\alpha_0, \cdots, \alpha_r \geq 0\) be nonnegative numbers such that \(\sum_{j=0}^{r} \tau_j = m\). Then
\[ c_{\alpha_1}(\omega_{WP}) \wedge \cdots \wedge c_{\alpha_r}(\omega_{WP}) \wedge \omega_{WP}^0 \]

is absolutely integrable. Moreover, we have the following Chern number inequalities
\[
\int_M c_{\alpha_1}(\omega_{WP}) \wedge \cdots \wedge c_{\alpha_r}(\omega_{WP}) \wedge \omega_{WP}^0 \leq 2^{m-\alpha_0} \int_M \omega_H^m.
\]

\(^{10}\) The metric is equivalent to the partial Hodge metric in [22, Section 4]. But we don’t need this fact here.
Proof. Since the Hodge volume is finite by [22, Theorem 5.2], the corollary follows. □

Theorem 6.3 is the counterpart of Theorem 4.1 and Theorem 5.1 in the Weil-Petersson geometry. In what follows we shall show that for Calabi-Yau moduli, we can do better than Hodge theory provides. Since for application, moduli space of a Calabi-Yau threefold is the most interesting one, for the rest of the paper, we will assume that $\mathcal{M}$ is the Calabi-Yau moduli of a Calabi-Yau threefold.

Recall that for the moduli space of a Calabi-Yau threefold, there is a simple relation between the Hodge metric and the Weil-Petersson metric [21]:

$$\omega_H = (m + 3)\omega_{WP} + \text{Ric}(\omega_{WP}).$$

By the computation of [20], since $\omega_H$ should be of Poincaré like at infinity, it would be natural to conjecture the following

**Conjecture 1.** The curvature of the Hodge metric $\omega_H$ is bounded.

In this direction, we can prove the following

**Theorem 6.4.** The curvature of the Hodge metric is $L^1$ with respect to the Hodge metric.

**Remark 6.1.** In [19, 21], we have proved that the Ricci curvature of the Hodge metric is bounded from above by a negative number. Thus Theorem 6.4 implies that the Hodge volume is finite, a result proved in [22]. On the other hand, a slight modification of our proof also gives the Gauss-Bonnet-Chern theorem for the first Chern class of the Hodge metric.

$$\int_{\mathcal{M}} c_1(\omega_H) \wedge \omega_H^{m-1} \in \mathbb{Q}.$$

Before proving the theorem, we first give the following technical

**Lemma 6.2.** Let $U$ be a small neighborhood of $\mathbb{C}^m$ at the origin. Assume that $(s_1, \cdots, s_a, w_1, \cdots, w_b)$ are the local coordinates $(a + b = m)$ similar to those in Lemma 4.4. Let $\omega_P$ be the Poincaré metric

$$\omega_P = \frac{\sqrt{-1}}{2\pi} \left( \sum_{i=1}^a \frac{ds_i \wedge ds_i}{|s_i|^2 (\log |s_i|)^2} + \sum_{i=a+1}^{a+b} dw_i \wedge dw_i \right).$$

Let

$$\phi_\varepsilon(s_1, \cdots, s_a) = \prod_{j=1}^a (1 - \phi_\varepsilon(s_j)),$$

where $\phi_\varepsilon$ is defined in (4.14). Then we have

$$\int_{U \cap \text{supp } \phi_\varepsilon} \left( \sum_{j=1}^a \log |s_j| \right)^m \omega_P^m \leq C,$$

where $C$ is independent to $\varepsilon > 0$. 
Proof. We observe that the support of $\partial \overline{\partial} \phi_\varepsilon$ is the union
\[ \bigcup_{j=1}^{a} \{ e^{-\frac{1}{2\varepsilon}} \leq |s_j| \leq e^{-\frac{1}{2\varepsilon}} \} . \]
We assume that $U \subset \{ |s_j| \leq \frac{1}{2}, |w_j| \leq \frac{1}{2} \}$. Then the expression is less than a constant times
\[ \sum_{j=1}^{a} \left( \int_{e^{-\frac{1}{2\varepsilon}}}^{1} \frac{1}{|s_j| \log \frac{1}{|s_j|}} d|s_j| \prod_{i \neq j} \int_{0}^{1} \frac{1}{|s_i| \log \frac{1}{|s_i|}} d|s_i| \right) \leq C. \]

\[ \square \]

Proof of Theorem 6.4 By [21], we know that the norm of the curvature is bounded by the Ricci curvature of the Hodge metric. Thus we just need to prove that
\[ \int_{M} c_1(\omega_H) \wedge \omega_{-1}^{m} \leq C. \]
The proof depends on the analysis of the curvature and the Hodge metric itself at infinity. Thus as before, we assume that $M$ can be compactified to be a compact manifold $\overline{M}$ by a divisor $Y$ of normal crossings.

By (6.51) and Theorem 6.2, we know that $|c_1(\omega_{WP})| \leq \omega_{H}$. Thus we have
\[ \int_{M} c_1(\omega_{WP}) \wedge \omega_{-1}^{m} \leq C \]
by Corollary 5.3. Combining the above two inequalities, we need to prove that
\[ \frac{\sqrt{-1}}{2\pi} \int_{M} \partial \overline{\partial} \log \frac{\omega_{WP}^{m}}{\omega_{WP}^{m}} \wedge \omega_{-1}^{m} \leq C. \]
Let $\rho_\varepsilon$ be the cut-off function defined in Lemma 4.4. Using integration by parts, we concluded that the following inequality implies Theorem 6.4
\[ \frac{\sqrt{-1}}{2\pi} \int_{M} \log \frac{\omega_{H}^{m}}{\omega_{WP}^{m}} \wedge \partial \overline{\partial} \rho_\varepsilon \wedge \omega_{-1}^{m} \leq C, \]
where $C$ is independent to $\varepsilon$.

Let $V_\varepsilon$ be the support of the form $\partial \overline{\partial} \rho_\varepsilon$. By the properties of $\rho_\varepsilon$, up to a constant, we have
\[ \frac{\sqrt{-1}}{2\pi} \int_{M} \log \frac{\omega_{H}^{m}}{\omega_{WP}^{m}} \wedge \partial \overline{\partial} \rho_\varepsilon \wedge \omega_{-1}^{m} \leq \int_{V_\varepsilon} \log \frac{\omega_{H}^{m}}{\omega_{WP}^{m}} \wedge \omega_{P} \wedge \omega_{H}^{m}. \]
By Corollary 5.3 again, we know that up to a constant, we have
\[ \omega_{H} \leq \omega_{P}. \]
Thus in order to prove the theorem, we only need to prove that
\[ \int_{V_\varepsilon} \log \frac{\omega_{P}^{m}}{\omega_{WP}^{m}} \wedge \omega_{P}^{m} \leq C. \]
The problem being local, we may consider a neighborhood $U$ in Lemma 6.2. By [23, Lemma 6.4], there is a local function $f$ such that
\[
\omega_{WP}^m = \left( \prod_{j=1}^a |s_j|^{p_j} \right) f ds_1 \wedge \cdots \wedge ds_a \wedge dw_1 \wedge \cdots \wedge dw_b,
\]
and $\log f$ is integrable with respect to the metric $\omega_P$. By the above equation, we have
\[
\frac{\omega_P^m}{\omega_{WP}^m} = \prod_{j=1}^a (\log \frac{1}{|s_j|})^2 \prod_{j=1}^a |s_j|^{p_j+2} f.
\]
Since $\log \log \frac{1}{|s_j|}$ is integrable with respect to $\omega_P$, the theorem follows from Lemma 6.2 and the fact that $\log f$ is also integrable with respect to $\omega_P$.

\[\square\]

Remark 6.2. Theorem 6.4 is a generalization of the fact that the Hodge metrics are Poincaré bounded. At the moment, it seems less well motivated. However, we suspect that for the Calabi-Yau moduli, we can prove one more layer of estimate provided by the variation of Hodge structure. Such an estimate will reflect the inner structure of a Calabi-Yau manifold (cf. [20] and Conjecture [1]). Thus this theorem may point the way for future research.

We end our paper with the following physics applications. First, by Theorem 6.4, the volume of the Weil-Petersson metric is finite (cf. [22]). The result has the following string theoretical implications (cf. [7]).

In [1] and [9], the index of all supersymmetric vacua was given. In [1, eq. (1.5)], the index is given by
\[
I_{\text{vac}}(L \leq L_{\text{max}}) = \text{const.} \int_{M \times H} \det(-R_{WP} - \omega_{WP}),
\]
where $M$ is the Calabi-Yau moduli and $H$ is the moduli space of elliptic curves. In [9, Theorem 1.8], the following strengthened result of the above was given.

**Theorem 6.5.** Let $K$ be a compact subset of $M$ with piecewise smooth boundary. Then
\[
\text{Ind}_{\chi_K}(L) = \text{const.}(L^{2m}) \left[ \int_K \text{c}_m(T^*(M) \otimes F^3) + O(L^{-1/2}) \right].
\]

By Theorem 6.3, we have

**Theorem 6.6.** The indices $I_{\text{vac}}$ and $\text{Ind}_{\chi_K}$ are all finite. Moreover, $\text{Ind}_{\chi_K}$ is bounded from above uniformly with respect to $K$. They are all bounded, up to an absolute constant, by the Hodge volume of the Calabi-Yau moduli.

\[\square\]
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