LARGE TIME BEHAVIOR OF SOLUTIONS TO A DISSIPATIVE BOUSSINESQ SYSTEM

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Abstract. In this article we consider the Boussinesq system supplemented with some dissipation terms. These equations model the propagation of a waterwave in shallow water. We prove the existence of a global smooth attractor for the corresponding dynamical system.

1. Introduction

This article is concerned with the long time behavior of the solutions to a damped-forced Boussinesq system that read

\[
\begin{align*}
\eta_t + u_x + (\eta u)_x - \eta_{xx} &= 0, \\
u_t - u_{txx} - u_{xx} + \eta_x + u_x u &= f.
\end{align*}
\]

(1)

Here we have an incompressible fluid on a channel. \(u(t, x)\) is the horizontal velocity at the top of the fluid, \(\eta\) is the fluctuation of the height of the fluid with respect to the rest position that is \(z = \eta(t, x) = 0\), assuming that the bottom of the channel is at \(z = -1\). Observe that in our model we have to ensure that \(\eta(t, x) > -1\) \(\forall t, x\).

Here \(f(x)\) is an external force that does not depend on time and the damping terms are respectively \(-u_{xx}, -\eta_{xx}\). In the conservative case, that read

\[
\begin{align*}
\eta_t + u_x + (\eta u)_x &= 0, \\
u_t - u_{txx} + \eta_x + u_x u &= 0;
\end{align*}
\]

(2)

this system has been introduced by Boussinesq in 1877 to model the fluctuation of a waterwave in shallow water. Other well-known asymptotic models are Korteweg-de Vries equations and Benjamin-Bona-Mahony equation, also

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known as the regularized long wave equation. For these asymptotical models we would like to refer to [5, 19] and to the references therein.

In this article we are interested in the dissipative case. In the case where $f = 0$, the solutions converge to the equilibrium and the issue is to find out the rate of convergence. Following the pioneering work of Amick, Bona and Schonbek [4], this issue has been addressed in the case where $x \in \mathbb{R}^D, D \geq 2$ using the famous Schonbek splitting method [16]. Here we plan to study the dynamical system provided by (2) into the framework of infinite dimensional dynamical system [11, 15, 17]. Our main result states as follows

**Theorem 1.1.** The dynamical system provided by (2) features a compact global attractor into a suitable energy space. Moreover this compact global attractor has finite fractal and Hausdorff dimension.

This result compares with previous results obtained for dissipative KdV equations [7–10] or dissipative BBM equations [2, 18]. This article is organized as follows; in the next section we introduce the mathematical framework that we have chosen to study this dynamical system. In a third section we address the initial value problem for the evolution equation. In a fourth section we prove the existence of a smooth finite dimensional attractor.

2. **Mathematical framework**

2.1. **Initial data.** For the sake of convenience, we are interested in considering periodic boundary conditions. We now consider functions for $x \in [0,1]$ that are 1-periodic. We also assume that $\int_0^1 f(x)dx = 0$ and $f \in L^2(0,1)$. Introducing $w(t,x) = 1 + \eta(t,x)$, we rather use the following system

\[
\begin{aligned}
\begin{cases}
u_t - u_{txx} - u_{xx} + w_x + u_xu &= f &\text{for } (x,t) \in (0,1) \times \mathbb{R}_+, \\
w_t + (wu)_x - w_{xx} &= 0 &\text{for } (x,t) \in (0,1) \times \mathbb{R}_+. 
\end{cases}
\end{aligned}
\] (3)

The natural space for the velocity $u(t,x)$ is then

\[\hat{H}^1_{\text{per}} = \{v \in H^1_{\text{per}}/ \int_0^1 v(x)dx = 0\}.\] (4)

We then assume that $u_0 \in \hat{H}^1_{\text{per}}$.

We now proceed to the assumptions on $w_0(x)$.

The first physical assumption is that

\[\inf w_0(x) > 0.\] (5)

This assumption ensures that the top of the fluid does not hit the bottom of the channel. The second assumption is that

\[\int_0^1 w_0 = 1,\] (6)

that describes that $w_0$ fluctuates around 1 the height at rest of the fluid, and that the fluids has constant volume.
The third assumption is related to the very definition of the entropy for convection equation, see [16].

Introducing \( Q(y) = y \ln y - y + 1 \) that is convex and non negative, we assume that
\[
\int_0^1 Q(w_0(x)) \, dx < +\infty. \tag{7}
\]

**Remark 2.1.** Assume that \( f = 0 \) here and that we are given regular enough solution \((u, w)\) to (2) such that \( w > 0 \) \( \forall x, t \). Multiplying (2) by \((u, 1 + \ln w)\) and summing the two resulting equations, we thus obtain
\[
\frac{d}{dt} \left[ \frac{1}{2} |u|^2_{H^1} + \int_0^1 w \ln w \right] + |u_x|^2_{L^2} + \int_0^1 \frac{w^2}{w} \, dx = 0; \tag{8}
\]
then, using \( \int_0^1 w = 1 \)
\[
\lim_{t \to \infty} \left[ \frac{1}{2} \int_0^1 (u^2 + u_x^2) + \int_0^1 Q(w) \right] = 0
\]
and the fluid converges to the equilibrium \((u, w) = (0, 1)\).

**2.2. Functional Analysis.** We set
\[
\tilde{K} = \{ w(x) > 0; \int_0^1 w(x) \, dx = 1 \text{ and } \int_0^1 Q(w(x)) \, dx < \infty \}; \tag{9}
\]
one may wonder which kind of topology we shall use in \( \tilde{K} \). First of all, we observe that \( \tilde{K} \) is a convex set (this is obvious since the map \( y \to Q(y) \) is a convex function). Furthermore, \( \tilde{K} \) is related to the following Orlicz space (see [1]). Introducing
\[
H(y) = (1 + y) \ln(1 + y) - y, \tag{10}
\]
that is a convex function, we observe that \( H \) and \( Q \) are related by the following inequalities.

**Lemma 2.2.** \( \exists C_0 > 0 \) such that \( \forall y \geq 0 \)
\[
Q(y) - 1 \leq H(y) \leq C_0(Q(y) + 1). \tag{11}
\]

Proof: On the one hand, if \( y \in [0, 1] \), then \( Q(y) \leq 1 - y \leq 1 + H(y) \). If \( y \geq 1 \) then
\[
Q(y) = y \ln y + 1 - y \leq (y + 1) \ln(y + 1) - y + 1.
\]
To establish the reverse inequality, we observe that \( y \ln y \sim \infty (1+y) \ln(y+1) \), then, for large \( y \)'s, say \( y \geq R \), \( H(y) \leq 2(y \ln y - y + 1) \).

For \( y \in [0, R] \), \( H(y) \) is bounded by \( H(R) \). Then the proof of the lemma is completed. \( \Box \)

Therefore \( w > 0 \) belongs to \( \tilde{K} \) iff \( \int_0^1 w = 1 \) and \( w \in L_H \), the Orlicz space, whose norm is defined by
\[
||w||_{L_H} = \inf \{ \lambda > 0, \int_0^1 H(\frac{w}{\lambda})(x) \, dx \leq 1 \} \tag{12}
\]
We now endow \( \tilde{K} \) with the topology of \( L_H \), that is given by the distance
\[
d(w_1, w_2) = ||w_1 - w_2||_{L_H}.
\]

**Remark 2.3.** \( \tilde{K} \) is not a closed subset of \( L_H \), but \( K = \{ w \geq 0; \int_0^1 w = 1 \text{ and } \int_0^1 Q(w) < \infty \} \) is. Consider \( w_n \) in \( K \) such that \( ||w_n - w||_{L_H} \to 0 \).

There exists \( \lambda_n \to 0 \) such that
\[
\int_0^1 H\left( \frac{|w_n - w|}{\lambda_n} \right)(x)dx \leq 2.
\]

(14)

We shall use
\[
H(y) \geq \frac{1}{2}(\sqrt{y + 1} - 1)^2.
\]

(15)

Then, setting \( v_n(x) = \lambda_n^{-1}|w_n(x) - w(x)| \),
\[
\int_0^1 \frac{|w_n - w|}{\lambda_n} = \int_0^1 v_n \leq \int_0^1 (\sqrt{v_n + 1} - 1)^2 + 2 \int_0^1 (\sqrt{v_n + 1} - 1) \leq 4 \int_0^1 H(v_n) + 4\left( \int_0^1 (\sqrt{v_n + 1} - 1)^2 \right)^{\frac{1}{2}} \leq 4(1 + \sqrt{2}).
\]

Then \( w_n \to w \) in \( L^1 \), and, up to a subsequence extraction \( w_n \to w \) a.e. □

3. The Initial Value Problem

3.1. Main Theorem.

**Theorem 3.1.** Consider the initial data \((u_0, w_0)\) in \( H^1_{\text{per}} \times \tilde{K} \). Then there exists a unique solution for \( (u(t), w(t)) \in C(\mathbb{R}_+; H^1_{\text{per}}) \times C(\mathbb{R}_+; \tilde{K}) \)

that satisfies moreover \( \sqrt{w} \in L^2_{\text{loc}}(\mathbb{R}_+; H^1_{\text{per}}) \).

**Remark 3.2.** In the theorem above, we would like to point out two important facts:

* If \( \inf w_0(x) > 0 \), then, for any \( t, x > 0 \), \( w(t, x) > 0 \), that is physically relevant.

* The dissipative Boussinesq system provides a smoothing effect in the \( w \) variable. In fact \( \sqrt{w} \) belongs \( C(\mathbb{R}_+ - \{0\}; H^1_{\text{per}}) \).

Proof of Theorem 3.1.

Existence: we first regularize the initial data \((u_0^\varepsilon, w_0^\varepsilon)\) to construct smooth solutions \((u^\varepsilon(t), w^\varepsilon(t))\) in \( C([0, T], H^1_{\text{per}} \times H^1_{\text{per}}) \) for instance. We then prove some a priori estimates and finally pass to limit. Since these methods are classical, we just indicate below how to derive the a priori estimates, referring the reader to [3], for details. For the sake of simplicity, we drop the subscript \( \varepsilon \) throughout the proof of the theorem.
First step: we first prove \( w(t, x) \) is positive.
Consider \( \alpha = \inf w_0 > 0 \). Introduce
\[
J(t, x) = \max(0, \alpha - w(t, x)).
\] (16)
Using the Kato’s inequality (see [12, 13]) that reads (in the distribution’s sense)
\[
w_{xx} \text{sgn}(w) \leq |w|_{xx},
\] (17)
we thus obtain
\[
J_t + (uJ)_x - J_{xx} \leq 0.
\] (18)
Then, integrating in \( x \), we have that
\[
\int_0^1 J(t, x) \, dx \leq \int_0^1 J(0, x) \, dx = 0,
\] (19)
and \( w(t, x) \geq \alpha \) a.e.
This result is related to parabolic Harnack’s inequalities, see [6].

Second step: a priori estimate in \( H^1_{per} \times \dot{K} \).
We begin with a technical lemma

**Lemma 3.3.** Consider \( w > 0 \) a smooth periodic function such that \( \int_0^1 w(x) \, dx = 1 \) then
\[
\int_0^1 Q(w) \, dx \leq \left( \int_0^1 \frac{w^2}{w} \right)^{\frac{1}{2}}.
\] (20)
Proof: since \( -\ln \) is convex, \( Q(w) \leq w(w - 1) - w + 1 \). Then, using once more \( \int_0^1 w = 1 \),
\[
\int_0^1 Q(w) \leq \int_0^1 w^2 - 1.
\] (21)
On the other hand, for any \( \varphi \) smooth periodic function, then
\[
\varphi^2(x) \leq 2 \left( \int_0^1 \varphi^2 \right)^{\frac{1}{2}} \left( \int_0^1 \dot{\varphi}^2 \right)^{\frac{1}{2}} + \int_0^1 \varphi^2.
\] (22)
We apply this to \( \varphi = \sqrt{w} \). Then
\[
\int_0^1 w^2 \leq ||w||_{L^\infty} \left( \int_0^1 w \right) \leq \left( \int_0^1 \frac{w_0^2}{w} \right)^{\frac{1}{2}} + 1
\] (23)
this concludes the proof of the lemma □.

We now proceed to the a priori estimates. We multiply (13) by \( (u, 1+\ln w) \). We integrate in \( x \) over \([0, 1]\) the resulting equations and then sum to obtain
\[
\frac{d}{dt} \left[ \int_0^1 (w \ln w - w + 1) \right] + \frac{1}{2} ||u||_{H^1}^2 + \int (uw)_x \ln w
\]
\[
+ \int_0^1 \frac{w^2}{w} + \int_0^1 w_x u + ||u_x||_{L^2}^2 = \int_0^1 fu.
\] (24)
Integrating by part, we observe that
\[
\int_0^1 (uw)_x \ln w + \int_0^1 w_x u = 0.
\] (25)
On the other hand, using Young’s and Poincaré-Wirtinger inequalities, we obtain
\begin{align*}
- \int_0^1 f u + ||u_x||^2_{L^2} & \geq ||u_x||^2_{L^2} - (\pi - \frac{1}{2})||u||^2_{L^2} - \frac{1}{4\pi - 2} ||f||^2_{L^2} \\
& \geq \frac{1}{2} ||u||^2_{H^1} - \frac{1}{4\pi - 2} ||f||^2_{L^2}.
\end{align*}
(26)

We thus obtain,
\begin{align*}
\frac{d}{dt} \left[ \int_0^1 Q(w) + \frac{1}{2} ||u||^2_{H^1} \right] + \frac{1}{2} ||u||^2_{H^1} + \int_0^1 \frac{w^2}{w} \leq \frac{1}{4\pi - 2} ||f||^2_{L^2}.
\end{align*}
(27)

We now infer from (21) and (23) that
\begin{align*}
\int_0^1 Q(w) \leq \left( \int_0^1 \frac{w^2}{w} \right) + \frac{1}{4}
\end{align*}
(28)

We combine this inequality together with (24), we integrate with respect to \( t \) (thanks to the Gronwall lemma), and thus obtain
\begin{align*}
\int_0^1 Q(w(t)) + \frac{1}{2} ||u(t)||^2_{H^1} \leq \frac{1}{4} + \frac{1}{4\pi - 2} ||f||^2_{L^2} + e^{-t} \left( \int_0^1 Q(w_0) + \frac{1}{2} ||u_0||^2_{H^1} \right).
\end{align*}
(29)

Since \( H \) is a convex function, for \( \lambda \geq 1 \) (observe \( H(0) = 0 \)), and due to Lemma 2.2,
\begin{align*}
\int_0^1 H \left( \frac{w}{\lambda} \right) dx \leq \frac{1}{\lambda} \int_0^1 H(w) dx \leq C_0 \left[ 1 + \sup_{\lambda \geq 0} \int_0^1 Q(w(t)) \right] \leq 1,
\end{align*}
(30)

for \( \lambda \) large enough. Then \( w(t) \) remains bounded in the Orlicz space \( L_H \).

We then have established an a priori estimate for \( (u, w) \) in \( L^\infty(R_+; H^1_{per}) \times L^\infty(R_+; L_H) \).

**Remark 3.4.** Actually (27) implies that \( \sqrt{w} \) is a.e. in \( t \) in \( H^1_x \). Since this Sobolev space is an algebra and since we can solve the evolution equation under consideration with initial data \( w \) in \( H^1_x \), this implies that for all \( t > 0 \) \( w \) is in \( H^1_x \) (smoothing effect). We precise this fact below.

Third step: smoothing effect; \( \sqrt{w} \) belongs to \( H^1_x \) for \( t > 0 \).

We set \( v = \sqrt{w} \) that solves
\begin{align*}
v_t - v_{xx} - \frac{v^2}{v} + \frac{1}{2} u_x v + w_x = 0.
\end{align*}
(31)

Multiply (31) by \( -v_{xx} \) and integrate. We then get
\begin{align*}
\frac{1}{2} \frac{d}{dt} ||v_x||^2_{L^2} + ||v_{xx}||^2_{L^2} + \frac{1}{3} \int \frac{v^4}{v^2} = \frac{1}{2} \int u_x w v_{xx} + \int w_x v_{xx} \leq \frac{1}{2} ||v_{xx}||^2_{L^2} + c \left[ ||v||^2_{L^\infty} ||u_x||^2_{L^2} + ||u||^2_{L^\infty} ||v_x||^2_{L^2} \right].
\end{align*}
(32)

Then, since \( ||v||_{L^2} = 1 \),
\begin{align*}
\frac{d}{dt} ||v_x||^2_{L^2} \leq c_1 ||u||_{H^1} ||v||^2_{H^1} \leq c_1 ||u||_1^2 (1 + ||v_x||^2_{L^2}).
\end{align*}
(33)
Due to Gronwall lemma and since $u$ is bounded in $H^1$ we then get the $H^1$ bound on $v = \sqrt{w}$.

**Remark 3.5.** Actually for $T < +\infty \sqrt{w}$ is in $L^2(0, T; H^1_x)$ and then $w$ in $L^1(0, T; H^1_x)$.

Fourth step: uniqueness

Consider two trajectories $(u_2, w_2)$ and $(u_1, w_1)$ that start from the same initial data. Due to the previous estimates both $(u_2, w_2)$ and $(u_1, w_1)$ remain bounded in $L^1(0, T; L^\infty)$. We set $u = u_2 - u_1, w = w_2 - w_1$ that are solutions to

\[
\begin{aligned}
& \quad u_t - u_{txx} - u_{xx} + w + \frac{1}{2}(u_2^2 - u_1^2)_x = 0, \\
& w_t - w_{xx} + (u_2w_2 - u_1w_1)_x = 0.
\end{aligned}
\] (34)

Then multiply these equations by $(u, w)$ and integrate the resulting equation to obtain

\[
\frac{d}{dt}(\|u\|_{H^1} + \|w\|_{L^2}) \leq C(1 + \max(\|u_2\|_{L^\infty}, \|w_2\|_{L^\infty}, \|u_1\|_{L^\infty}, \|w_1\|_{L^\infty})(\|u\|_{H^1} + \|w\|_{L^2}).
\] (35)

Therefore $\sqrt{w}$ is bounded for large times into $H^1$. Since $H^1$ is an algebra, then $w$ is also bounded for large times in $H^1$. □

4. The Global Attractor

4.1. Existence of the Global Attractor. To begin with we state and prove

**Proposition 4.1.** The semigroup $S(t)$ defined on $\hat{H}^1_{per} \times \hat{K}$ possesses an absorbing set that is bounded in $\hat{H}^1_{per} \times H^1$

**Proof:** The existence of a bounded absorbing set in $\hat{H}^1_{per} \times \hat{K}$ comes from the estimate (29) of the previous section. Let $t_0$ be the entrance time into this absorbing ball. Going back to (33) and applying the Uniform Gronwall Lemma (see Lemma III.1.1 in [17]), we thus obtain that for $t > 0$, for some numerical constant $c$,

\[
t\|(\sqrt{w})_x(t + t_0)\|_{L^2}^2 \leq c(1 + t)(1 + \|f\|_{L^2}^2) \exp(c + c\|f\|_{L^2}^2).
\] (35)

Therefore $\sqrt{w}$ is bounded for large times into $H^1$. Since $H^1$ is an algebra, then $w$ is also bounded for large times in $H^1$. □

**Theorem 4.2.** The semigroup $S(t)$ possesses a global attractor $A$ in $\hat{H}^1_{per} \times L_H$, that is a compact subset of $H^2 \times H^2$.

**Proof:** we introduce the splitting $(u, w) = (u^1, w) + (u^2, 0)$, where $u^1$ satisfies

\[
\begin{aligned}
& u^1_t - u^1_{txx} - u^1_{xx} + w + u_x u = f, \\
u^1(0) = 0,
\end{aligned}
\] (36)
and $u^2$ is solution to
\[
\begin{cases}
  u^2_t - u^2_{txx} - u^2_{xx} = 0 \\
  u^2(0) = u_0.
\end{cases}
\] (37)

We now define the families $\{S_1(t)\}_{t \geq 0}$ and $\{S_2(t)\}_{t \geq 0}$ of maps in $H^1 \times L_H$, where $S_1(t)(u_0, w_0) = (u^1, w)$ and $S_1(t)(u_0, w_0) = (u^2, 0)$.

First step: we prove that $u^1$ is bounded in $H^2$. For this we multiply (36) $-u^1_{xx}$ and integrate between 0 and 1 to obtain
\[
\frac{1}{2} \frac{d}{dt} ||u^1_x||^2_{H^1} + ||u^1_{xx}||^2_{L^2} = - \int f u^1_{xx} + \int w_x u^1_{xx} + \int u u_{xx} u^1_{xx},
\]
due to Young and Cauchy-Schwarz inequalities, then
\[
\frac{d}{dt} ||u^1_x||^2_{H^1} + ||u^1_{xx}||^2_{L^2} \leq c \left( ||f||^2_{L^2} + ||w_x||^2_{L^2} + ||u_x||^2_{L^2} ||u||^2_{L^\infty} \right),
\] (38)
due to Proposition 4.1, (29), Gronwall and Poincaré inequalities, we obtain that $u^1$ remains in a bounded set of $H^2$ for large times ($t > t_0$ the entrance time into the absorbing ball).

On the other hand, it is an exercise to prove that $u^2(t) \to 0$ strongly in $H^1$ when $t \to \infty$. (39)

Then $S_1(t)(u_0, w_0)$ is bounded in $H^2 \times H^1$ then compact in $H^1 \times L_H$ and $S_2(t)(u_0, w_0) \to 0$ in $H^1 \times L_H$ uniformly on bounded sets.

Then from Theorem I.1.1 in [T] we have the existence of a global attractor $\mathcal{A}$ in $H^1 \times L_H$, that is moreover a bounded set in $H^2 \times H^1$.

We now prove that for a trajectory $(u, w)$ in the global attractor, then $w$ remains bounded in $H^2$. For that purpose, multiply the second equation in (3) by $w_{4x}$ and integrate by parts to obtain
\[
\frac{d}{dt} ||w_{xx}||^2_{L^2} + ||w_{xxx}||^2_{L^2} = \int_0^1 u_{xx} w_{w_{3x}} + 2 \int_0^1 u_x w_x w_{3x} - \frac{1}{2} \int_0^1 u_x w_{2x}^2
\]
\[
\leq c ||u||^2_{H^2} ||w||^2_{H^1} + \frac{1}{2} ||w_{xxx}||^2_{L^2}.
\] (40)

Then the results follows promptly. It remains to prove that the global attractor, that is bounded in $H^2 \times H^2$, is in fact a compact subset of this space. This can be performed by the Energy Equation Method of [14] that is a suitable adaptation of the famous J. Ball argument. This is standard and will not be reproduced here; we refer the reader to [3] for details. □

4.2. Dimension of the attractor. In this section we are going to prove that the global attractor $\mathcal{A}$ has a finite dimension in $E = H^1 \times \{ w \in L^2; \int_0^1 w = 1 \}$. $E$ is an affine space whose associated vector space is $E =$
$\hat{H}^1 \times \hat{L}^2$. To begin with, we need a result on the differentiability of the semigroup $S(t)$ on the global attractor. Consider the non-autonomous linearized system

\[
\begin{cases}
v_t - v_{txx} - v_{xx} + h_x + (uv)_x &= 0 \\
h_t + (uh + vw)_x - h_{xx} &= 0
\end{cases}
\]  

(41)

where $(u(t), w(t)) = S(t)(u_0, w_0)$, $(u_0, w_0) \in \mathcal{E}$, is a trajectory solution of \([3]\) and $(v_0, h_0) \in E$. Actually the linear mapping $DS(t)(u_0, w_0)(v_0, h_0) = (v(t), h(t))$ is the uniform differential of $S(t)$ as stated below.

**Theorem 4.3.** The non-autonomous PDE \([41]\) provides a well posed initial value problem in $E$. Moreover for $T > 0$, $(v_0, h_0) \in E$, $(u_0, w_0) \in A$, $t \leq T$ there exists a constant $C = C(T)$ such that

\[
||S(t)(u_0 + v_0, w_0 + h_0) - S(t)(u_0, w_0) - DS(t)(u_0, w_0)(v_0, h_0)||_E \leq C(T)||h(t)||_E^\delta
\]  

(42)

where $1 < \delta < 2$.

Proof: to prove that the initial value problem is well-posed is standard and then omitted. Consider the solutions $(u_1(t), w_1(t)) = S(t)(u_0, w_0)$, $(u_2(t), w_2(t)) = S(t)(u_0 + v_0, w_0 + h_0)$ and $(v(t), h(t)) = (DS(t)(u_0, w_0))(v_0, h_0)$.

Then $(p, q) = (u_2, w_2) - (u_1, w_1) - (v, h)$ satisfies the system

\[
\begin{cases}
p_t - px_{xx} - p_{xx} + q_x + (\frac{1}{2} v^2 + vm + u_1p + \frac{1}{2} p^2)_x &= 0 \\
q_t - qx_{xx} + (pq + ph + vq + hv + qu_1 + w_1p)_x &= 0
\end{cases}
\]  

(43)

We shall use in the sequel that $\int_0^1 p = \int_0^1 q = 0$ and then $||p||_{H^1}$ and $||p_x||_{L^2}$ define equivalent norms. Multiply \([42]\) by $(p, q)$ and integrate to obtain (due to straightforward computations)

\[
\frac{1}{2} \frac{d}{dt}[||q||_{L^2}^2 + ||p||_{H^1}^2] + ||q_x||_{L^2}^2 + ||p_x||_{L^2}^2 =
\]

\[- \int q_x p + \frac{1}{2} \int (p + v)^2 p_x + \int u_1pp_x + \int (pq + ph + vq + vh + w_1p + qu_1)q_x
\]

\[
\leq (||v||_{H^1} + ||h||_{L^2})||v||_{H^1} [||q_x||_{L^2}^2 + ||p_x||_{L^2}^2] +
\]

\[(1 + ||u_1||_{H^1} + ||v||_{H^1} + ||w_1||_{L^2} + ||h||_{L^2}) [||q_x||_{L^2}^2 + ||p_x||_{L^2}^2] + [||p||_{L^2} ||q_x||_{L^2}^2].
\]

We thus obtain, using the bounds on the attractor and the local in time bounds on $(v, h)$

\[
\frac{d}{dt} [||p, q||_E^2] \leq K_1(T)||p, q||_E^2 + K_2(T)||(v_0, h_0)||_{L^2}^4 + K_3(T)||p, q||_E^2.
\]  

(44)
Consider a given interval of time \([0, T]\). Set 
\[\varepsilon^2 = K_2(T)||v_0, h_0||_E^2\] 
that is small. Then 
\[\phi(t) = \exp(-tK_1(T)||p, q||_E^2(t)\) satisfies the ODE

\[
\dot{\phi} \leq K\phi^2 + \varepsilon^2,
\]

supplemented with \(\phi(0) = 0\). Then 
\[E(t) \leq 2\varepsilon\] if \(\varepsilon\) is small enough. \(\square\)

We now give the main result of this section

**Theorem 4.4.** The fractal and Hausdorff dimension in \(E\) of the attractor \(A\) are finite.

Proof: set \(\xi = (u, w)\), \(\beta = (v, h)\). Now we study the operators \(DS(t)\xi_0\) that contracts the m-dimensional volumes in \(E\). Let \(\beta_0^1, ..., \beta_0^m\) in \(E\). We study the following quantities

\[
G_m = ||\beta^1(t) \land ... \land \beta^m||_E^2 = \text{det}_{1 \leq i, j \leq m}(\beta^i(t), \beta^j(t))_E,
\]

where \(\beta^i(t) = (DS(t)\xi_0)\beta_0^i\). The Gram determinant \(G_m\) represents the volume of m-dimensional polyhedron defined by the vectors \(\beta^1(t), ..., \beta^m(t)\). We will show that for sufficiently large \(m\) this determinant decays exponentially as \(t \to \infty\).

We consider \(\beta(t) = (DS(t)\xi_0)\beta_0\) solution of (41), we multiply by \(\beta = (v, h)\) and integrate to obtain

\[
\frac{1}{2} \frac{d}{dt}||\beta||_E^2 + ||v_x||_{L^2}^2 + ||h_x||_{L^2}^2 = \int_0^1 uh h_x + \int_0^1 vh v_x + \int_0^1 uv v_x + \int_0^1 hv h_x.
\]

(47)

Recall that \((u, w)\) is a trajectory that belongs to the global attractor. Introduce \(M = c(1 + ||f||_{L^2}^2)\) that is the \(H^1\) bound for \(u\) in the attractor (see (29)). We do not want to use estimates that involve \(H^2\) norms of \(u\) as \(35\).

We bound the fourth term in the r.h.s of (47) by \(\frac{1}{4}||h_x||_{L^2}^2 + ||v||_{L^2}^2\). The third term can be bounded as follows

\[
|\int_0^1 uv v_x| \leq \frac{1}{2}||u||_{H^1}||v||_{L^2}||v||_{L^\infty} \leq c||u||_{H^1}||v||_{L^2}^{3/2}||v_x||_{L^2}^{1/2} \leq \frac{1}{4}||v_x||_{L^2}^2 + cM^{4/3}||v||_{L^2}^2.
\]

We now proceed to the first term as follows

\[
|\int_0^1 uh h_x| \leq \frac{1}{2}||u||_{H^1}||h||_{L^4}^2 \leq c||u||_{H^1}||h||_{H^{-1}}^{3/4}||h_x||_{L^2}^{5/4} \leq \frac{1}{4}||h_x||_{L^2}^2 + cM^{8/3}||h||_{H^{-1}}^2.
\]

For the second term, we have
\[
\int_0^1 v w h_x | \leq \frac{1}{4} ||h_x||_L^2 + ||w||_L^2 ||v||_L^2 \leq \frac{1}{4} ||h_x||_L^2 + \frac{1}{8} ||v_x||_L^2 + c ||w||_L^2 ||v||_L^2.
\]

To go further, we need a new estimate on \(w\) that reads

**Lemma 4.5.** For any \((u, w)\) in \(A\), then \(||w(t)||_L^2 \leq c(1 + M^2)\).

Proof: for a given trajectory in the attractor multiply the second equation by \(w\) and integrate to obtain

\[
\frac{d}{dt} ||w||_L^2 + 2 ||w_x||_L^2 = 2 \int_0^1 u w w_x dx \leq ||w_x||_L^2 + ||u||_L^\infty ||w||_L^\infty; \tag{48}
\]

here we have used that \(\int_0^1 w = 1\). We then infer from (48) that, using Poincaré-Wirtinger inequality,

\[
\int_0^1 (w - 1)^2 = ||w||_L^2 - 1 \leq ||w_x||_L^2, \tag{49}
\]

that

\[
\frac{d}{dt} ||w||_L^2 + \frac{1}{4} ||w||_L^2 \leq c(1 + M^4). \tag{50}
\]

Then the classical Gronwall lemma leads to the result. \(\square\)

We then have

\[
\frac{1}{2} \frac{d}{dt} ||\beta||_E^2 + ||\beta||_E^2 = c \left( (1 + M^8)||\beta||_L^2 \right). \tag{51}
\]

We introduce the Gram determinant

\[
G_m(t) = \det_{1 \leq i, j \leq m} \left( \Lambda(\beta^i(t), \beta^j(t)) \right)_E,
\]

where \(\Lambda(a, b) = \frac{||a + b||_E^2 - ||a - b||_E^2}{4}\), and that represents the \(m\)-dimensional volume. Then we can proceed as in [7, 17] to establish that

\[
\frac{dG_m}{dt} + mG_m \leq c(1 + M^8) \left( \sum_{l=1}^m \max_{A \subset \mathbb{R}^m, \dim A = l \in \mathbb{A}, v \neq 0} \min_{||\beta||_L^2 = 1} \frac{||\beta||_L^2 \times H^{-1}}{||\beta||_E^2} \right) G_m. \tag{52}
\]

Since the eigenvalues of the Laplace periodic operator are \(4\pi^2k^2\) each of multiplicity 2, then

\[
\sum_{l=1}^m \max_{A \subset \mathbb{R}^m, \dim A = l \in \mathbb{A}, v \neq 0} \min_{||\beta||_L^2 = 1} \frac{||\beta||_L^2 \times H^{-1}}{||\beta||_E^2} \sim 2\pi^2 \sum_{k=1}^{m/2} (2\pi k)^{-2} \leq \frac{1}{12}. \tag{53}
\]

Therefore for \(m \geq c(1 + M^8)\) the \(m\)-dimensional volume \(G_m\) decays and the attractor has finite dimension. \(\square\)

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