HIGHER CODIMENSIONAL ALPHA INVARIANTS AND CHARACTERIZATION OF PROJECTIVE SPACES

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Abstract. We generalize the definition of alpha invariant to arbitrary codimension. We also give a lower bound of these alpha invariants for K-semistable \(\mathbb{Q}\)-Fano varieties and show that we can characterize projective spaces among all K-semistable Fano manifolds in terms of higher codimensional alpha invariants. Our results demonstrate the relation between alpha invariants of any codimension and volumes of Fano manifolds in the characterization of projective spaces.

1. Introduction

We work over the complex number field \(\mathbb{C}\). A variety \(X\) is called \(\mathbb{Q}\)-Fano if \(X\) is a normal projective variety with klt singularities such that the anti-canonical divisor \(-K_X\) is an ample \(\mathbb{Q}\)-divisor. A Fano manifold is a smooth \(\mathbb{Q}\)-Fano variety.

It is well-known that a Fano manifold admits a Kähler-Einstein metric if and only if it is K-polystable due to [CDS15a, CDS15b, CDS15c, Tia15]. More generally, we would like to study K-semistable \(\mathbb{Q}\)-Fano varieties. Recent work of Kento Fujita, Yuji Odaka and Chen Jiang shows that among K-semistable Fano manifolds, the projective space \(\mathbb{P}^n\) can be characterized by either of the following two properties:

1. [Fuj18] \((-K_X)^n \geq (n+1)^n\);
2. [FO16, Jia17] \(\alpha(X) \leq \frac{1}{n+1}\).

Here \((-K_X)^n\) is the volume of \(X\), and \(\alpha(X)\) is the alpha invariant of \(X\).

The purpose of this paper is to show that the above two characterizations of projective spaces are special cases of a more general one where cycles of intermediate codimensions are considered.

We first generalize the definition of alpha invariant:

Definition 1.1. Let \(X\) be a \(\mathbb{Q}\)-Fano variety of dimension \(n\). For \(1 \leq k \leq n\), the complete intersection \(\text{ci}(L_1, \ldots, L_k)\) in \(X\) cut out by effective Cartier divisors \(L_1, \ldots, L_k\) is defined to be the scheme-theoretic intersection of \(L_1, \ldots, L_k\) with the expected codimension \(k\). Then we define the alpha invariant of codimension \(k\) for \(X\) to be

\[\alpha^{(k)}(X) := \inf_{r} \left\{ \text{lct} \left(X, \frac{1}{r}Z\right) \mid Z = \text{ci}(L_1, \ldots, L_k), L_1, \ldots, L_k \in \left|-rK_X\right| \right\} .\]

Remark 1.1. When \(k = 1\), the generalized alpha invariant \(\alpha^{(1)}(X)\) is just the usual alpha invariant \(\alpha(X)\). We will use \(\alpha^{(1)}(X)\) to denote the usual alpha invariant for the rest part of the paper.

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Tian proved in [Tia87] that a Fano manifold $X$ of dimension $n$ admits a Kähler-Einstein metric if $\alpha^{(1)}(X) > n/(n + 1)$. Fujita improved the theorem in [Fuj17] by showing that a Fano manifold $X$ of dimension $n$ is K-stable if $\alpha^{(1)}(X) \geq n/(n + 1)$. A recent related result by Stibitz and Zhuang in [SZ18] shows that a birationally superrigid (or more generally log maximal singularity free) Fano variety $X$ is K-stable (resp. K-semistable) if $\alpha^{(1)}(X) > 1/2$ (resp. $\alpha^{(1)}(X) \geq 1/2$). The result is later improved by Zhuang in [Zhu18]. By interpreting their results using higher codimensional alpha invariants, we can give a version of their theorems in terms of $\alpha^{(2)}$ (See Theorem 3.4 in Section 3).

Following the proof of Proposition 3.4 in [Bir16], we can see that the infimum in the definition of $\alpha^{(k)}(X)$ is actually a minimum when $\alpha^{(k)}(X) < 1$ (we refer to Proposition 3.7 in Section 3 for details).

For K-semistable Q-Fano varieties, we can give a lower bound of higher codimensional alpha invariants as our first main result:

**Theorem A.** Let $X$ be a K-semistable Q-Fano variety of dimension $n$. Then

$$\alpha^{(k)}(X) \geq \frac{k}{n + 1}. \quad (1.1)$$

**Remark 1.2.** When $k = 1$, the inequality (1.1) is proved by Fujita and Odaka in [FO16].

It is well-known that $\mathbb{P}^n$ is K-semistable. Therefore by considering the log canonical thresholds of linear subspaces of $\mathbb{P}^n$, together with Theorem A, we know that the equality holds in (1.1) when $X \cong \mathbb{P}^n$. Then we have our second main result about characterization of projective spaces:

**Theorem B.** Let $X$ be a K-semistable Fano manifold of dimension $n$. Consider the following three statements:

1. $X \cong \mathbb{P}^n$;
2. $\alpha^{(k)}(X) = \frac{k}{n + 1}$;
3. $(-K_X)^k$ is rationally equivalent to $lZ'$ for some integer $l \geq (n + 1)^k$ and $Z'$ an integral $(n - k)$-cycle.

We have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Moreover, if we assume that $k$ divides $n$, then (3) $\Rightarrow$ (1) and therefore all three statements are equivalent.

When $k = 1$, Theorem B reduces to the main result in [Jia17] that characterizes projective spaces among all K-semistable Fano manifolds in terms of the alpha invariant. When $k = n$, Theorem B reduces to the following result:

**Theorem 1.2 ([Fuj18]).** Let $X$ be a K-semistable Q-Fano variety of dimension $n$. Then we have $(-K_X)^n \leq (n + 1)^n$. Moreover if $X$ is smooth and $(-K_X)^n = (n + 1)^n$, then we know that $X \cong \mathbb{P}^n$.

In fact Theorem 1.2 is used to prove the last part of Theorem B. Also note that in both [Liu16] and [LZ17], Liu and Zhuang proved a stronger version of Theorem 1.2 without assuming smoothness.

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2. Preliminaries

2.1. Cycles, rational equivalence and numerical equivalence. Let $X$ be a scheme and $Z$ a $k$-dimensional subscheme of $X$. Let $Z_1, \ldots, Z_t$ be the irreducible components of $Z$. Then following the notation of [Ful12], we define the $k$-cycle $[Z]$ of subscheme $Z$ to be

$$[Z] = \sum_{i=1}^{t} a_i[Z_i],$$

where $a_i = l(O_{Z,Z_i})$ is the length of $O_{Z,Z_i}$ as an $O_{Z,Z_i}$-module.

For any two $k$-cycles $Z$ and $W$ on a proper scheme, we write $Z \sim_{rat} W$ if $Z$ is rationally equivalent to $W$; and write $Z \equiv_{num} W$ if $Z$ is numerically equivalent to $W$. We refer to [Ful12] for definitions.

In particular, we know that rational equivalence is the same as linear equivalence for divisors, and we write $D \sim_{lin} D'$ if $D$ is linear equivalent to $D'$. Also on any smooth proper scheme $X$ of dimension $n$, two $k$-cycles $Z$ and $W$ are numerically equivalent if and only if we have the equality of intersection numbers $Z \cdot T = W \cdot T$ for any $(n-k)$-cycle $T$.

2.2. Multiplicity of ideals. Let $X$ be a scheme of dimension $n$, and $Z \subset X$ a closed subscheme corresponding to the ideal sheaf $I_Z \subset \mathcal{O}_X$. Let $V$ be an irreducible component of $Z$ with codimension $k$. Then the Samuel multiplicity of the ideal $I_Z$ in the local ring $\mathcal{O}_{X,V}$, also called the multiplicity of $X$ along $Z$ at the generic point of $V$, is defined as

$$e(I_Z \cdot \mathcal{O}_{X,V}) := \lim_{t \to \infty} \frac{l((\mathcal{O}_{X,V}/I_Z^t \cdot \mathcal{O}_{X,V})}{t^k/k!}.$$

A geometric interpretation of the Samuel multiplicity is given in [Ful12]. Suppose in addition $Z$ is irreducible. Let $\sigma : Y \to X$ be a proper birational morphism such that $\sigma^{-1}I_Z \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$. Then we have the following equality of $(n-k)$-cycles:

$$e(I_Z \cdot \mathcal{O}_{X,V})[V] = (-1)^{k-1} \sigma_* (E^k). \quad (2.1)$$

For the multiplicity of an ideal corresponding to a complete intersection, we have the following property (See Example 4.3.5 of [Ful12] for a proof):

**Proposition 2.1.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $k$ and $a = (x_1, \ldots, x_k)$ an $\mathfrak{m}$-primary ideal. If $R$ is Cohen-Macaulay, then $e(a) = l(R/a)$.

2.3. Log discrepancy, log resolution and log canonical threshold. Let $X$ be a normal variety such that $K_X$ is $\mathbb{Q}$-Cartier and $Z \subset X$ a closed subscheme. Denote by $I_Z$ the ideal sheaf defining $Z$. We say $E$ is a divisor over $X$ if $E$ is a prime divisor on some normal variety $Y$ with a proper birational morphism $\sigma : Y \to X$, and $\sigma(E)$ is called the center of $E$ on $X$. We say that $E$ is exceptional if the center of $E$ on $X$ has dimension smaller than $E$. Pick canonical divisors on $X$ and $Y$ respectively such that $\sigma_* K_Y = K_X$. Then the log discrepancy of $E$ is defined as $A_X(E) = \text{ord}_E(K_Y - \sigma^* K_X) + 1$. For $a > 0$, we say that the pair $(X, aZ)$ has

1. terminal singularities if $A_X(E) - a \text{ord}_E(I_Z) > 1$ for any exceptional prime divisor $E$ over $X$;
2. canonical singularities if $A_X(E) - a \text{ord}_E(I_Z) \geq 1$ for any exceptional prime divisor $E$ over $X$;
3. Kawamata log terminal (klt) singularities if $A_X(E) - a \text{ord}_E(I_Z) > 0$ for any prime divisor $E$ over $X$;
(4) log canonical singularities if $A_X(E) - a \ord_E(I_Z) \geq 0$ for any prime divisor $E$ over $X$.

Let $\mathcal{H}$ be a linear system on $X$ with base scheme $Z$. Singularities of the pair $(X, a\mathcal{H})$ are defined to be the same as the singularities of the pair $(X, aZ)$.

A log resolution of the pair $(X, Z)$ is a proper birational morphism $\sigma : Y \to X$ with the following properties:

1. $Y$ is smooth;
2. the exceptional locus $\text{Exc}(\sigma)$ is of pure codimension 1;
3. $\sigma^{-1}I_Z \cdot O_Y = O_Y(-F)$ and $F + \text{Exc}(\sigma)$ has simple normal crossing support.

In the definition of terminal, canonical, klt and log canonical singularities, it is enough to examine finitely many divisors that are in the support of $F + \text{Exc}(\sigma)$ for a fixed log resolution. We refer to [KM08] and [Laz04] for more details.

Let $X$ have klt singularities and assume $Z$ is non-empty. Then the log canonical threshold of the pair $(X, aZ)$, denoted by either $\text{lct}(X, aZ)$ or $\text{lct}(I_Z^a)$, is a positive number defined as

$$\text{lct}(X, aZ) = \text{lct}(I_Z^a) := \inf_{E} \frac{A_X(E)}{a \ord_E(I_Z)}, \quad (2.2)$$

where the infimum runs through all prime divisors $E$ over $X$. Note that immediately from the definition, the log canonical threshold of $(X, aZ)$ can also be viewed as the largest number $t > 0$ such that $(X, taZ)$ is log canonical. Then by fixing a log resolution $\sigma$ of $(X, Z)$, we see that the infimum in $(2.2)$ is in fact a minimum running through finitely many divisors in the support of $F + \text{Exc}(\sigma)$. In particular, log canonical thresholds of pairs are rational numbers. Also from the definition, for $a > 0$, we have $\text{lct}(X, aZ) := \frac{1}{a} \text{lct}(X, Z)$.

2.4. K-stability. Here we recall the definition of K-stability via normal test configurations and Donaldson-Futaki invariants.

**Definition 2.2.** A (semi-)test configuration for a polarized variety $(X, L)$ contains all of the following data:

1. a proper flat family $\pi : \mathcal{X} \to \mathbb{A}^1$,
2. an equivariant $\mathbb{C}^*$-action on $\pi : \mathcal{X} \to \mathbb{A}^1$, where $\mathbb{C}^*$ acts on $\mathbb{A}^1$ by multiplication in the standard way, and
3. a $\mathbb{C}^*$-equivariant line bundle $\mathcal{L}$ on $\mathcal{X}$ which is $\pi$-relatively (semi-)ample, such that $(\mathcal{X}, \mathcal{L})|_{\pi^{-1}(\mathbb{A}^1\setminus\{0\})}$ is $\mathbb{C}^*$-equivariantly isomorphic to $(X \times (\mathbb{A}^1\setminus\{0\}), L_{\mathbb{A}^1\setminus\{0\}})$, where $L_{\mathbb{A}^1\setminus\{0\}}$ is the pull back of $L$ from $X$ to $X \times (\mathbb{A}^1\setminus\{0\})$ and $(X \times (\mathbb{A}^1\setminus\{0\}), L_{\mathbb{A}^1\setminus\{0\}})$ has trivial $\mathbb{C}^*$-action on the fibers.

For a normal $\mathbb{Q}$-Fano variety of dimension $n$, pick a rational number $r$ such that $rK_X$ is Cartier. Let $(\mathcal{X}, \mathcal{L})$ be a semi-test configuration of $(X, -rK_X)$. We can compactify the test configuration into a flat family $(\mathcal{X}, \mathcal{L})$ over $\mathbb{P}^1$, such that over $\mathbb{P}^1\setminus\{0\}$, the family $(\mathcal{X}, \mathcal{L})$ is $\mathbb{C}^*$-equivariantly isomorphic to $X \times \mathbb{P}^1\setminus\{0\}$ with trivial $\mathbb{C}^*$-action on the fibers. Then we can define the Donaldson-Futaki invariant of $(\mathcal{X}, \mathcal{L})$ to be

$$\text{DF}(\mathcal{X}, \mathcal{L}) := \frac{1}{(n+1)(-K_X)^n} \left( \frac{n}{r^{n+1}} \mathcal{L}^{n+1} + \frac{n+1}{r^n} (\mathcal{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1}) \right) \quad (2.3)$$

**Definition 2.3.** Let $X$ be a $\mathbb{Q}$-Fano variety of dimension $n$. $X$ is said to be

1. K-semistable if $\text{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -rK_X)$.
(2) K-stable if $DF(X, \mathcal{L}) \geq 0$ for any normal test configuration $(X, \mathcal{L})$ of $(X, -rK_X)$, and the equality holds only if $(X, \mathcal{L})$ is $\mathbb{C}^*$-equivariantly isomorphic to the trivial test configuration $(X \times \mathbb{A}^1, -rK_{X \times \mathbb{A}^1/\mathbb{A}^1})$ with trivial $\mathbb{C}^*$-action on the fiber.

Fujita and Li independently developed a criterion of K-semistability which we will use later.

**Definition 2.4** ([Fuj16]). Let $X$ be a $\mathbb{Q}$-Fano variety of dimension $n$, and $F$ a prime divisor over $X$ corresponding to a projective birational morphism $\sigma : Y \to X$. Then we define the $\beta$-invariant of $F$ to be

$$\beta(F) := A_X(F) \text{vol}_X(-K_X) - \int_0^\infty \text{vol}_Y(\sigma^*(-K_X) - xF) \, dx.$$ 

Similarly, we can also define $\beta$-invariants for proper closed subschemes of $X$.

**Definition 2.5** ([Fuj18]). Let $X$ be a $\mathbb{Q}$-Fano variety of dimension $n$ and $Z \subset X$ a subscheme of $X$ defined by the ideal sheaf $I_Z \subset \mathcal{O}_X$. Take a projective birational morphism $\sigma : Y \to X$ that factors through the blow-up of $X$ along $Z$, and write $\sigma^{-1}I_Z \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. Then we define the $\beta$-invariant of $Z$ to be

$$\beta(Z) := \text{lct}(X, Z) \text{vol}_X(-K_X) - \int_0^\infty \text{vol}_Y(\sigma^*(-K_X) - xF) \, dx.$$ 

Note that both $\beta(F)$ and $\beta(Z)$ do not depend on the choice of the birational morphism $\sigma$. Fujita and Li both proved that we can use $\beta$-invariants to characterize K-semistability.

**Theorem 2.6** ([Fuj18, Fuj16, Li17]). Let $X$ be a $\mathbb{Q}$-Fano variety. Then the following are equivalent:

1. $X$ is K-semistable;
2. $\beta(F) \geq 0$ for any divisor $F$ over $X$;
3. $\beta(Z) \geq 0$ for any proper closed subscheme $Z \subseteq X$.

2.5. **Characterizations of Projective Spaces.** We recall some results about characterization of projective spaces among Fano manifolds. They are all in some sense related to the divisibility of the canonical divisor.

**Theorem 2.7** ([KO73]). Let $X$ be a smooth Fano manifold of dimension $n$ and $H$ an ample divisor on $X$. If $-K_X \sim_{\text{lin}} lH$ with $l \geq n + 1$, then $X \cong \mathbb{P}^n$, $l = n + 1$, and $\mathcal{O}_X(H) = \mathcal{O}_{\mathbb{P}^n}(1)$.

**Remark 2.1.** Note that in the theorem it is enough to assume $-K_X \equiv_{\text{num}} lH$. Indeed, if $K_X + lH$ is numerically trivial, then we know that $\chi(K_X + lH) = \chi(\mathcal{O}_X) = 1$. By Kodaira vanishing, we have $h^i(X, K_X + lH) = 0$ for $i > 0$. Therefore $h^0(X, K_X + lH) = 1$, which implies that $K_X + lH \sim_{\text{lin}} 0$.

If we only consider K-semistable Fano manifolds of dimension $n$, we can characterize $\mathbb{P}^n$ by the volume $\text{vol}_X(-K_X) = (-K_X)^n$ instead, which is exactly the second part of Theorem 1.2. Note that the condition $(-K_X)^n = (n + 1)^n$ in Theorem 1.2 can be viewed as the divisibility of the 0-cycle $(-K_X)^n$ by $(n + 1)^n$, comparing to the divisibility of the divisor $-K_X$ by $n + 1$ in Theorem 2.7.

By considering the divisibility of cycles with intermediate codimension, we have the following immediate corollary of Theorem 1.2:
**Corollary 2.8.** Let $X$ be a $K$-semistable Fano manifold of dimension $n$. Suppose $k$ divides $n$. If $(-K_X)^k \equiv_{num} lZ$ for some integer $l \geq (n+1)^k$ and $Z$ an integral $(n-k)$-cycle, then $X \cong \mathbb{P}^n$.

*Proof.* because $k$ divides $n$, we can write
\[ (-K_X)^n = \left((-K_X)^k\right)^{\frac{n}{k}} = (lZ)^{\frac{n}{k}} \geq (n+1)^n. \]
Since $X$ is K-semistable, we know from Theorem 1.2 that $(-K_X)^n = (n+1)^n$ and $X \cong \mathbb{P}^n$. \qed

For any dimension $n$, we at least know that $k$ divides $n$ when $k = 1$ or $k = n$. These are the two cases discussed in Theorem 2.7 and Theorem 1.2 respectively. Along with some mild assumptions if necessary, we expect that the divisibility condition of the cycle $(-K_X)^k$ described in Corollary 2.8 can characterize projective spaces among all $K$-semistable Fano manifolds. More precisely, we would like to ask the following question:

**Question 2.9.** Let $X$ be a $K$-semistable Fano manifold of dimension $n$. If $(-K_X)^k \equiv_{num} lZ$ for some integer $l \geq (n+1)^k$ and $Z$ an integral $(n-k)$-cycle, then is $X$ isomorphic to the projective space $\mathbb{P}^n$?

Corollary 2.8 answers Question 2.9 when $k$ divides $n$. If the answer to Question 2.9 is yes in general, then Theorem B can be improved by stating that for an $n$-dimensional $K$-semistable Fano manifold $X$ and any positive integer $k \leq n$, we have $\alpha(k)(X) = \frac{k}{n+1}$ if and only if $X \cong \mathbb{P}^n$.

3. **Higher Codimensional Alpha Invariants**

In this section, we will discuss some examples and properties of higher codimensional alpha invariants. We first recall the definition of higher codimensional alpha invariants.

**Definition 3.1.** Let $X$ be a $\mathbb{Q}$-Fano variety of dimension $n$. For $1 \leq k \leq n$. Denote by $ci(L_1, \ldots, L_k)$ the complete intersection in $X$ cut out by effective Cartier divisors $L_1, \ldots, L_k$. Then we define the alpha invariant of codimension $k$ for $X$ to be
\[ \alpha(k)(X) := \inf \left\{ \text{lct} \left( \frac{1}{r} Z \right) \mid Z = ci(L_1, \ldots, L_k), L_1, \ldots, L_k \in |-rK_X| \right\}. \]

**Example 3.2.** Let $X$ be a Fano manifold with Fano index $l$. Assume that $-K_X \sim_{lin} lD$ such that the linear system $|D|$ is base point free. Then we know that $\alpha(k)(X) \leq k/l$. Indeed we can take $k$ sufficiently general smooth elements $L_1, \ldots, L_k \in |D|$ such that $Z$ is the transversal intersection of $L_1, \ldots, L_k$. Then $\text{lct}(X, Z) = k$, and therefore $\alpha(k)(X) \leq \text{lct}(X, lZ) = k/l$. This gives an upper bound of higher codimensional alpha invariants for smooth Fano hypersurfaces in the projective spaces. In particular, for the projective space $\mathbb{P}^n$, we know that $\alpha(k)(\mathbb{P}^n) \leq k/(n+1)$. On the other hand, because $\mathbb{P}^n$ is $K$-semistable, we know from Theorem A that $\alpha(k)(\mathbb{P}^n) \geq k/(n+1)$. Consequently we have $\alpha(k)(\mathbb{P}^n) = k/(n+1)$.

**Example 3.3.** Cheltsov computed in [Che08] that $\alpha^{(1)}(\mathbb{P}^1 \times \mathbb{P}^1) = 1/2$. However, even in simple examples, it seems hard to compute $\alpha(k)$ when $k \geq 2$. For $\mathbb{P}^1 \times \mathbb{P}^1$, we only know that $2/3 < \alpha^{(2)}(\mathbb{P}^1 \times \mathbb{P}^1) \leq 3/4$. Indeed, we first notice that $-K_{\mathbb{P}^1 \times \mathbb{P}^1}$ is of type $(2,2)$. Let $L_1$ and $L_2$ be two lines of type $(1,0)$ and type $(0,1)$ respectively that are symmetric with respect to each other, and $\Delta$ be the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $L_1 + L_2$ and $\Delta$ are both
linearly equivalent to $-\frac{1}{2}K_{\mathbb{P}^1 \times \mathbb{P}^1}$. Let $Z$ be the complete intersection of $L_1 + L_2$ and $\Delta$, and we have $\text{lct}(X, Z) = 3/2$. Therefore $\alpha^{(2)}(\mathbb{P}^1 \times \mathbb{P}^1) \leq \text{lct}(X, 2Z) = 3/4$. We also know that $\mathbb{P}^1 \times \mathbb{P}^1$ is K-semistable (for example refer to [PW17]). Then by Theorem A and Theorem B, we have $\alpha^{(2)}(\mathbb{P}^1 \times \mathbb{P}^1) > 2/3$.

A first basic property of alpha invariants is that all $\alpha^{(k)}(X)$'s form an increasing sequence in terms of the codimension $k$: $\alpha^{(1)}(X) \leq \alpha^{(2)}(X) \leq \cdots \leq \alpha^{(n)}(X)$. This follows immediately from the definition of alpha invariant and log canonical threshold.

Recent results of [SZ18, Zhu18] can be reinterpreted from the point of view of higher codimensional alpha invariants to give the following result.

**Theorem 3.4.** Let $X$ be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano variety of Picard number 1 and dimension $n$. If

$$\alpha^{(2)}(X) > \frac{n - 1}{n + 1},$$

and

$$\alpha^{(1)}(X) \geq \frac{\alpha^{(2)}(X)}{(n + 1)\alpha^{(2)}(X) - n + 1} \quad \left(\text{resp. } \alpha^{(1)}(X) > \frac{\alpha^{(2)}(X)}{(n + 1)\alpha^{(2)}(X) - n + 1}\right),$$

then $X$ is K-semistable (resp. K-stable).

**Remark 3.1.** The alpha invariant of codimension 2 is related to the notion of log maximal singularity. Recall that a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano variety $X$ of Picard number 1 has a log maximal singularity if there is a movable linear system $\mathcal{H}$ on $X$ such that $\mathcal{H} \equiv_{\text{num}} -rK_X$ and $(X, \frac{1}{r}\mathcal{H})$ is not log canonical. $X$ is called log maximal singularity free if $X$ does not have a log maximal singularity. Note that if a $\mathbb{Q}$-Cartier divisor $L \equiv_{\text{num}} -K_X$, then $L \sim_{\mathbb{Q}} -K_X$ by similar arguments in the smooth case as in Remark 2.1. Therefore we see that the linear system $\frac{1}{r}\mathcal{H}$ we consider in the pair $(X, \frac{1}{r}\mathcal{H})$ is $\mathbb{Q}$-linear equivalent to $-K_X$. Then it follows immediately from the definition that $X$ is log maximal singularity free if and only if $\alpha^{(2)}(X) \geq 1$.

Note that when $\alpha^{(2)}(X) \geq 1$, we have

$$\frac{\alpha^{(2)}(X)}{(n + 1)\alpha^{(2)}(X) - n + 1} \leq \frac{1}{2}.$$ 

Therefore Theorem 3.4 reduces to the following theorem by Stibitz and Zhuang:

**Theorem 3.5** ([SZ18]). Let $X$ be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano variety of Picard number 1. If $X$ is log maximal singularity free and $\alpha^{(1)}(X) \geq 1/2$ (resp. $> 1/2$), then $X$ is K-semistable (resp. K-stable).

Theorem 3.4 is an immediate consequence of the following theorem of Zhuang, which provides a more precise result compared to Theorem 3.5.

**Theorem 3.6** ([Zhu18]). Let $X$ be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano variety of Picard number 1 and dimension $n$. If for every effective divisor $D \sim_{\mathbb{Q}} -K_X$ and every movable linear system $M \sim_{\mathbb{Q}} -K_X$, we have that the pair $(X, \frac{1}{n+1}D + \frac{n-1}{n+1}M)$ is log canonical (resp. klt), then $X$ is K-semistable (resp. K-stable).
Indeed, pick any effective divisor $D \sim_{\mathbb{Q}} -K_X$ and movable linear system $M \sim_{\mathbb{Q}} -K_X$. Then we know that the pairs $(X, \alpha^{(1)}(X)D)$ and $(X, \alpha^{(2)}(X)M)$ are log canonical. We can write $\frac{1}{n+1}D + \frac{n-1}{n+1}M$ as a linear combination of $\alpha^{(1)}(X)D$ and $\alpha^{(2)}(X)M$ as follows:

$$\frac{1}{n+1}D + \frac{n-1}{n+1}M = \frac{1}{(n+1)\alpha^{(1)}(X)}\alpha^{(1)}(X)D + \frac{n-1}{(n+1)\alpha^{(2)}(X)}\alpha^{(2)}(X)M$$

Note that the conditions

$$\alpha^{(2)}(X) > \frac{n-1}{n+1}$$

and

$$\alpha^{(1)}(X) \geq \frac{\alpha^{(2)}(X)}{(n+1)\alpha^{(2)}(X) - n + 1}$$

are equivalent to

$$\frac{1}{(n+1)\alpha^{(1)}(X)} + \frac{n-1}{(n+1)\alpha^{(2)}(X)} \leq 1.$$  \hspace{1cm} (3.1)

Then the pair $(X, \frac{1}{n+1}D + \frac{n-1}{n+1}M)$ is also log canonical and by Theorem 3.6, $X$ is K-semistable. Note that when

$$\alpha^{(1)}(X) > \frac{\alpha^{(2)}(X)}{(n+1)\alpha^{(2)}(X) - n + 1}$$

we will have strict inequality in (3.1) instead. Then $(X, \frac{1}{n+1}D + \frac{n-1}{n+1}M)$ is klt and $X$ is K-stable.

Remark 3.2. The above theorems are of course related to the well-known result of Tian in [Tia87], which was later improved by Fujita in [Fuj17] stating that a Fano manifold $X$ of dimension $n$ is K-stable if $\alpha^{(1)}(X) \geq n/(n+1)$. Theorem 3.4 does not require smoothness, and the lower bound of $\alpha^{(1)}(X)$ is smaller at the cost of an additional assumption on $\alpha^{(2)}(X)$ and the Picard number.

Following the proof of Proposition 3.4 in [Bir16], we give a proof of the following property of higher codimensional alpha invariants:

**Proposition 3.7.** Let $X$ be a $\mathbb{Q}$-Fano variety. Assume that $\alpha^{(k)}(X) < 1$. Then there exists a subscheme $Z = \text{ci}(L_1, \ldots, L_k)$ such that $L_1, \ldots, L_k \in |-rK_X|$, and $\alpha^{(k)}(X) = \text{lct}(X, \frac{1}{r_i}Z_i)$. In particular, we know that $\alpha^{(k)}(X)$ is a rational number when $\alpha^{(k)}(X) < 1$.

**Proof.** Replacing $X$ with a $\mathbb{Q}$-factorialization, we may assume $X$ is $\mathbb{Q}$ factorial. By definition of $\alpha^{(k)}(X)$, we can find a sequence of subschemes $Z_i$, such that $Z_i = \text{ci}(L_1^{(i)}, \ldots, L_k^{(i)})$, and $L_1^{(i)}, \ldots, L_k^{(i)} \in |-r_iK_X|$, with $t_i = \text{lct}(X, \frac{1}{r_i}Z_i) < 1$ a decreasing sequence with $t := \lim t_i = \alpha^{(k)}(X) < 1$. We may assume $t \neq t_i$ for every $i$.

Now pick $0 \leq H \sim_{\mathbb{Q}} -K_X$ very general such that $(X, H)$ is klt and $(X, H, \frac{1}{r_i}Z_i)$ is log canonical, for any $i$. Let $T_i'$ be a log canonical place of $(X, \frac{1}{r_i}Z_i)$ which is a divisor over $X$ with log discrepancy exactly zero. Then we know from [BCHM10] that there exists a proper birational morphism $\phi_i : X_i' \to X$ that precisely extracts the divisor $T_i'$. Define $H_i'$ and $L_j^{(i)}$ to be the strict transforms of $H_i$ and $L_j^{(i)}$ on $X_i'$ respectively. Note that by the choice of $H$, we have $\phi_i^*H = H_i'$. Therefore we have

$$K_{X_i'} + T_i' + \frac{t_i}{r_i}L_j^{(i)} + (1 - t_i)H_i' = \phi_i^*(K_X + \frac{t_i}{r_i}L_j^{(i)} + (1 - t_i)H) \sim_{\mathbb{Q}} 0.$$
By [BCHM10], we know that \( X_i' \) is a Mori dream space, so run an MMP on
\[- \left( K_{X_i'} + T_i' + (1 - t)H_i' \right),\]
and suppose it terminates at \( X_i'' \). Denote by \( T_i'' \), \( L_j^{(i)} \) and \( H_i'' \) the strict transforms of corresponding divisors via \( X_i' \to X_i'' \). We have \( (X_i'', T_i'' + \frac{t_i}{r_i} L_j^{(i)} + (1 - t_i)H_i'') \log \)
canonical.

Now consider the log canonical pairs \( (X_i'', T_i'' + (1 - t_i)H_i'') \). By ACC of log canonical thresholds [HMX14], we know that \( \{ \lambda_i = \text{lct}(X_i'', T_i'', H_i'') \} \) satisfies ACC. Note that \( \lambda_i \geq 1 - t_i \). Therefore, by replacing with a subsequence, we can assume \( \lambda_i \geq 1 - t \) for all \( i \), and hence the pairs \( (X_i'', T_i'' + (1 - t)H_i'') \) is log canonical for all \( i \).

Next, we want to show the \(( -K_{X_i'} - T_i' - (1 - t)H_i'')\)-MMP ends with a minimal model for all but finitely many \( i \). Assume we instead get a Mori fiber space \( X_i'' \to Z_i'' \) for infinitely many \( i \). Then because we have
\[ K_{X_i''} + T_i'' + \frac{t_i}{r_i} L_j^{(i)} + (1 - t_i)H_i'' \sim_{\mathbb{Q}} 0, \]
and \( L_j^{(i)} \) is nef over \( Z_i'' \), we know that
\[ K_{X_i''} + T_i'' + (1 - t_i)H_i'' = -\frac{t_i}{r_i} L_j^{(i)} \]
is anti-nef over \( Z_i'' \). However we have
\[ K_{X_i''} + T_i'' + (1 - t)H_i'' \]
ample over \( Z_i'' \). Therefore \( K_{X_i''} + T_i'' + (1 - s_i)H_i'' \) are numerically trivial over \( Z_i'' \) for some \( t < s_i \leq t_i \). Then restricting to general fibers of \( X_i'' \to Z_i'' \) for all \( i \), the coefficients of the boundary \( T_i'' + (1 - s_i)H_i \) belong to an infinite set. This contradicts Theorem 1.5 of [HMX14]. Therefore by replacing with a subsequence, we can assume we get a minimal model \( X_i'' \) for all \( i \). Then we know that
\[- \left( K_{X_i''} + T_i'' + (1 - t)H_i'' \right) \]
are semi-ample for all \( i \), and we can find effective divisors \( P_j^{(i)} \) such that
\[ P_j^{(i)} \sim_{\mathbb{R}} \frac{t_i}{r_i} L_j^{(i)} - (t_i - t)H_i'' \sim_{\mathbb{Q}} - \left( K_{X_i''} + T_i'' + (1 - t)H_i'' \right) \]
Pullback \( P_j^{(i)} \) via \( X_i' \to X_i'' \) we get effective divisors \( P_j^{(i)} \) on \( X_i' \) such that
\[ P_j^{(i)} \sim_{\mathbb{R}} \frac{t_i}{r_i} L_j^{(i)} - (t_i - t)H_i' \sim_{\mathbb{Q}} -(K_{X_i'} + T_i' + (1 - t)H_i'), \]
and therefore we get \( P_j^{(i)} = \phi_* P_j^{(i)} \) satisfying
\[ P_j^{(i)} \sim_{\mathbb{R}} \frac{t_i}{r_i} L_j^{(i)} - (t_i - t)H \sim_{\mathbb{Q}} -(K_X + (1 - t)H). \]
If \( t \) is a rational number, then \( P_j^{(i)} \) can be taken to be a \( \mathbb{Q} \)-divisor and so are all \( P_j^{(i)} \)'s. Hence We can pick some integer \( r \) such that \( \frac{r}{t} P_j^{(i)} \) to be Cartier for all \( j = 1, \ldots, k \). Because \( L_1^{(i)}, \ldots, L_k^{(i)} \) intersects properly for any \( i \), by picking sufficiently general \( P_j^{(i)} \) and
sufficiently large $i$, we can get $Z$ to be the complete intersection cut out by $\frac{r}{t}P^{(i)}_1, \ldots, \frac{r}{t}P^{(i)}_k$. Then by construction, the pair $(X, (1 - t)H + \frac{r}{t}Z)$ is not klt at the generic point of the center of $T_i$ on $X$. Therefore $(X, \frac{r}{t}Z)$ is also not klt. Then we have that $\operatorname{lct}(X, \frac{r}{t}Z) \leq t$, and consequently $\operatorname{lct}(X, \frac{1}{r}Z) = t$.

If $t$ is not rational, by the same argument as in the final step of the proof for Proposition 3.4 in [Bir16], we can pick some rational number $a_i < t$ that is sufficiently close to $t$ such that we can perturb the divisor $P^{(i)}_j$ to get effective $\mathbb{Q}$-divisors

$$R^{(i)}_j \sim_{\mathbb{Q}} \frac{t_i}{r_i} L^{(i)}_j - (t_i - a_i)H'_i.$$ Let $R^{(i)}_j := \phi_{i*} R^{(i)}_j$. Then

$$R^{(i)}_j \sim_{\mathbb{Q}} -a_iK_X,$$

and $\frac{r_i}{a_i} R^{(i)}_1, \ldots, \frac{r_i}{a_i} R^{(i)}_k$ still intersects properly for sufficiently divisible $r$. Let $V$ be resulting complete intersection. Then the pair $(X, \frac{r}{t}V)$ is not klt and hence $\operatorname{lct}(X, \frac{1}{r}V) \leq a_i < t$, which is a contradiction.

4. PROOF OF THEOREM A

In this section, we prove Theorem A which gives a lower bound of higher codimensional alpha invariants for K-semistable $\mathbb{Q}$-Fano varieties. We first state a lemma that will be used in later computation.

Lemma 4.1. Let $X$ be a normal projective variety of dimension $n$ with klt singularities, and $L$ an ample divisor on $X$. Let $Z$ be a complete intersection of $X$ cut out by $k$ elements $L_1, \ldots, L_k$ in the linear system $|L|$ with ideal sheaf $I_Z \subset \mathcal{O}_X$. Let $\sigma : Y \to X$ be any proper birational morphism that factors through the blow-up of $X$ along $Z$, and write $\sigma^{-1} I_Z \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. Let $\mathcal{H}$ be the sub linear system of $|L|$ with base ideal $I_Z$, and $\sigma^* \mathcal{H} = |M| + F$ be the decomposition into the moving part and the fixed part of the linear system $\sigma^* \mathcal{H}$. Then we have

$$\sigma_*(M^{i-1} \cdot F) = \begin{cases} [Z], & i = k; \\ 0, & i \neq k, \end{cases}$$

and in particular,

$$\sigma^*L^{n-i} \cdot (M^{i-1} \cdot F) = \begin{cases} \deg_L(Z) = L^n, & i = k; \\ 0, & i \neq k. \end{cases}$$

Proof. The linear system $|M|$ corresponds to the global sections of $\mathcal{O}_Y(\sigma^*L - F)$. Because $I \cdot \mathcal{O}_X(L)$ is generated by global sections, so is its pull-back $\mathcal{O}_Y(\sigma^*L - F)$. Since the linear system $|M|$ is base point free, by Bertini’s theorem, we can make the intersection involving $|M|$ to be transversal.

Note that

$$\deg_L(Z) = L^n = \sum_{i=1}^n \sigma^* L^{n-i} \cdot (M^{i-1} F) + M^n.$$ Both $\sigma^* L$ and $M$ are base point free, so for any $i$, we know that $\sigma^* L^{n-i} \cdot (M^{i-1} F) \geq 0$ and $M^n \geq 0$. Therefore, we only need to show $[Z] = \sigma_*(M^{k-1} F)$.

We write $[Z] = \sum a_i [Z_i]$ with $a_i = l(O_{Z_i}, Z_i)$, where $Z_i$’s are irreducible components of $Z$. Pick one $Z_i$ and localize at $Z_i$. Let $U = \text{Spec } O_{X,Z_i}$. Because $U$ is affine, we know that $L$ restricted on $U$ is linear equivalent to zero. Therefore $\sigma^* L$ is also linear equivalent to zero.
over $U$. Then over $U$, we have that $M \sim_{lin} -F$, and $\sigma_*(M^{k-1}F) = (-1)^{k-1}\sigma_*(F^k) = e(I_Z \cdot \mathcal{O}_{X,Z_i})[Z_i]$ by formula (2.1). Note that $X$ is klt. Then in particular $X$ has rational singularities and thus is Cohen-Macaulay. Therefore we can apply Proposition 2.1 to the complete intersection $Z$ so that we know $e(I_Z \cdot \mathcal{O}_{X,Z_i})$ can be computed by the length of $\mathcal{O}_{X,Z_i}/(I_Z \cdot \mathcal{O}_{X,Z_i}) = \mathcal{O}_{Z,Z_i}$. Therefore we have $e(I_Z \cdot \mathcal{O}_{X,Z_i}) = a_i$. Now by localizing at all $Z_i$’s, we see that $\sigma_*(M^{k-1}F) = \sum a_i[Z_i] = [Z]$.

Proof of Theorem A. Let $Z = \text{ci}(L_1, \ldots, L_k)$ where $L_1, \ldots, L_k \in |-rK_X|$. Assume $\sigma : Y \to X$ is a proper birational morphism that factors through the blow-up $X$ along $Z$. Write $\sigma^{-1}I_Z \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some Cartier divisor $F$ on $Y$. Let $\epsilon(Z, -K_X)$ be the Seshadri constant of $Z$ with respect to $-K_X$. By the construction of $Z$ we know that $I_Z \cdot \mathcal{O}_X(-rK_X)$ is globally generated. Hence $\epsilon(Z, -K_X) \geq 1/r$. In particular for $x < 1/r$, we know that $\sigma^*(-K_X) - xF$ is the pullback of an ample line bundle on the blow-up of $X$ along $Z$, and hence nef and big. Then for $0 < x < 1/r$, we have

$$\text{vol}_Y(\sigma^*(-K_X) - xF) = (\sigma^*(-K_X) - xF)^n.$$  

Applying Theorem 2.6 to $Z$, we have

$$\beta(X, Z) := \text{lct}(X, Z) \text{vol}_X(-K_X) - \int_0^\infty \text{vol}_Y(\sigma^*(-K_X) - xF)dx \geq 0.$$  

Consequently, we know that

$$\text{lct}(X, Z) \geq \frac{1}{\text{vol}_X(-K_X)} \int_0^\infty \text{vol}_Y(\sigma^*(-K_X) - xF)dx$$

$$\geq \frac{1}{\text{vol}_X(-K_X)} \int_0^{\frac{1}{r}} \text{vol}_Y(\sigma^*(-K_X) - xF)dx$$

$$= \frac{1}{\text{vol}_X(-K_X)} \int_0^{\frac{1}{r}} (\sigma^*(-K_X) - xF)^n dx$$

$$= \frac{1}{r(-rK_X)^n} \int_0^{1} (\sigma^*(-rK_X) - xF)^n dx.$$  \hspace{1cm} (4.1)

Now consider the following integral

$$\int_0^1 \left( \sum_{i=0}^{n-k} \frac{n-k-i}{k-1} (1-x)^{n-k-i} x^k \right) dx.$$  

Integrate by parts $k$ times, we get that

$$\int_0^1 \left( \sum_{i=0}^{n-k} \frac{n-k-i}{k-1} (1-x)^{n-k-i} x^k \right) dx$$

$$= \frac{1}{n-i(n-i+1)}$$

$$= 1 - \frac{k}{n+1}.$$  

Therefore, using formula (4.2) in Lemma 4.2 below, we have

$$\int_0^1 (\sigma^*(-rK_X) - xF)^n dx = (-rK_X)^n \cdot \frac{k}{n+1}.$$
Then by (4.1) we know that \( \text{lct}(X, \frac{1}{r}Z) = r \text{lct}(X, Z) \geq \frac{k}{n + 1} \) for any choice of \( Z \), which implies that \( \alpha^{(k)}(X) \geq \frac{k}{n + 1} \). □

We close this section with the following lemma which we have used in the proof above. The lemma also includes formula (4.3) which will be used in the next section for the proof of Theorem B.

**Lemma 4.2.** Under the setting of Lemma 4.1, we have the following two formulas for the polynomial \((\sigma^*L - xF)^n\):

\[
(\sigma^*L - xF)^n = L^n \left( 1 - \sum_{i=0}^{n-k} \binom{n-1-i}{k-1} (1-x)^{n-k-i} x^k \right) \tag{4.2}
\]

\[
(\sigma^*L - xF)^n = L^n \left( 1 + \sum_{i=k}^{n} (-1)^{i+k-1} \binom{n}{i} \binom{i-1}{k-1} x^i \right). \tag{4.3}
\]

**Proof.** First of all, we consider the following expression

\[
L^n - (\sigma^*L - xF)^n = xF \cdot \left( \sum_{i=0}^{n-1} \sigma^*L^i (\sigma^*L - xF)^{n-1-i} \right)
= xF \cdot \left( \sum_{i=0}^{n-1} \sigma^*L^i \left( (1-x)\sigma^*L + xM \right)^{n-1-i} \right)
= xF \cdot \left( \sum_{i=0}^{n-k} \sigma^*L^i \left( \frac{n-1-i}{k-1} \right) (1-x)^{n-k-i} x^k \cdot \sigma^*L^{n-k-i} \cdot M^{k-1} \right)
= L^n \cdot \left( \sum_{i=0}^{n-k} \binom{n-1-i}{k-1} (1-x)^{n-k-i} x^k \right),
\]

where we used Lemma 4.1 in the last 2 equalities. Then we get formula (4.2):

\[
(\sigma^*L - xF)^n = L^n \left( 1 - \sum_{i=0}^{n-k} \binom{n-1-i}{k-1} (1-x)^{n-k-i} x^k \right).
\]

On the other hand, expand the intersection number \((\sigma^*L - xF)^n\), we have

\[
(\sigma^*L - xF)^n = L^n + \sum_{i=1}^{n} (-1)^i \binom{n}{i} (\sigma^*L^{n-i} \cdot F^i) x^i.
\]

Note that by Lemma 4.1, for \( i \geq k \),

\[
\sigma^*L^{n-i} \cdot F^i = F \cdot \sigma^*L^{n-i} (\sigma^*L - M)^{i-1}
= F \cdot \sigma^*L^{n-i} \left( (-1)^{k-1} \binom{i-1}{k-1} \sigma^*L^{i-k} \cdot M^{k-1} \right)
= (-1)^{k-1} \binom{i-1}{k-1} L^n,
\]
and \(\sigma^*L^{n-i} \cdot F^i = 0\) if \(i < k\). Therefore, we get formula (4.3):

\[
(\sigma^*L - xF)^n = L^n \left( 1 + \sum_{i=k}^{n} (-1)^{i+k-1} \binom{n}{i} \binom{i-1}{k-1} x^i \right).
\]

\square

5. Proof of Theorem B

We first prove the following proposition, which gives the implication (2) \(\Rightarrow\) (3) in Theorem B:

**Proposition 5.1.** Let \(X\) be a smooth \(K\)-semistable Fano variety of dimension \(n\). If 
\[
\alpha^{(k)}(X) = \frac{k}{n+1},
\]
then \((-K_X)^k \sim_{rat} lZ'\) for some integer \(l \geq (n+1)^k\) and \(Z'\) an integral \((n-k)\)-cycle.

**Proof.** First of all, by Proposition 3.7, we can take a subscheme \(Z = ci(L_1, \ldots, L_k)\) such that \(L_1, \ldots, L_k \in |-rK_X|\) and \(\text{let}(X, Z) = \frac{k}{(n+1)^k}\).

Let \(\sigma : Y \to X\) be a log resolution of \((X, Z)\), with \(\sigma^{-1}I_Z : \mathcal{O}_Y = \mathcal{O}_Y(-F)\) for some Cartier divisor \(F\) on \(Y\). Write \(F = \sum a_i E_i\), where \(a_i = \text{ord}_{E_i}(Z)\). Because \(X\) is smooth, we have

\[
\text{lct}(X, Z) = \min_i \frac{A_X(E_i)}{a_i}.
\]

Let \(E\) be a divisor that computes \(\text{let}(X, Z)\) and \(a = \text{ord}_E(Z)\). We want to show first that the center of \(E\) on \(X\) has the same dimension as \(Z\). By Theorem 2.6, we know that \(\beta(E) \geq 0\), so

\[
\frac{A_X(E)}{a} \geq \frac{1}{a(-K_X)^n} \int_0^\infty \text{vol}_Y(\sigma^*(-K_X) - xE)dx.
\]

Then because all the equality holds in (4.1), we have

\[
\frac{1}{(-K_X)^n} \int_0^\infty \text{vol}_Y(\sigma^*(-K_X) - xF)dx = \text{let}(X, Z) = \frac{A_X(E)}{a}.
\]

Consequently,

\[
\frac{1}{(-K_X)^n} \int_0^\infty \text{vol}_Y(\sigma^*(-K_X) - xF)dx \geq \frac{1}{a(-K_X)^n} \int_0^\infty \text{vol}_Y(\sigma^*(-K_X) - xE)dx.
\]

By a change of variable for the integral on the RHS of the above inequality, we get

\[
\frac{1}{(-K_X)^n} \int_0^\infty \text{vol}_Y(\sigma^*(-K_X) - xF)dx \geq \frac{1}{(-K_X)^n} \int_0^\infty \text{vol}_Y(\sigma^*(-K_X) - xaE)dx.
\]

Also, because \(F \geq aE\), for all \(x\) we have

\[
\text{vol}_Y(\sigma^*(-K_X) - xF) \leq \text{vol}_Y(\sigma^*(-K_X) - xaE).
\]

Then there is an equality of volumes:

\[
\text{vol}_Y(\sigma^*(-K_X) - xF) = \text{vol}_Y(\sigma^*(-K_X) - xaE),
\]

(5.1)

for all \(x\).

Now let \(a_m = \sigma_*\mathcal{O}_Y(-mE)\) for any integer \(m\). Then \(a_m\) defines a subscheme of \(X\) the support of which is \(\sigma(E)\). Combining the conclusions from [Bla16, Theorem 1.3, Theorem]
1.4, Proposition 1.5], we see that the graded sequence of ideals \( a_\bullet \) is finitely generated. In addition, suppose \( a_\bullet \) is generated in degree up to \( m \). Then we have that
\[
\pi : W := Bl_{a_m}X \to X
\]
is the blow-up of \( X \) along \( a_m \), and \( \pi^{-1}a_m \cdot \mathcal{O}_W = \mathcal{O}_W(-mE_W) \), where \( E_W \) is the prime exceptional divisor on \( W \) that induces the same divisorial valuation as \( E \) on \( Y \).

By replacing \( Y \) with a common log resolution of \( Y \) and \( W = Bl_{a_m}X \), we may assume that \( \sigma : Y \to X \) factors through the blow-up \( \pi : W \to X \), and \( \sigma^{-1}a_m \cdot \mathcal{O}_Y = \mathcal{O}(-mD) \) for some divisor \( D \) on \( Y \). Note that \( D \) is the pullback of \( aE_W \) from \( W \) to \( Y \) and satisfies the relation
\[
aE \leq D \leq F.
\]
Now that we have \((5.1)\), we get the following equality:
\[
\text{vol}_Y(\sigma^*(-K_X) - xF) = \text{vol}_Y(\sigma^*(-K_X) - xD) = \text{vol}_Y(\sigma^*(-K_X) - xaE).
\]
Since \( \sigma^*(-K_X) - xD \) is the pullback of \( \pi^*(-K_X) - xaE_W \) from \( W \) to \( Y \), we have the equality of the volumes:
\[
\text{vol}_Y(\sigma^*(-K_X) - xF) = \text{vol}_W(\pi^*(-K_X) - xaE_W). \tag{5.2}
\]
Assume \( \sigma(E) = \pi(E_W) \) is of codimension \( s \) in \( X \). Now on the left hand side of \((5.2)\), for \( x < 1/r \) we know from formula \((4.3)\) that
\[
\text{vol}_Y(\sigma^*(-K_X) - xF) = (-K_X)^n \cdot \left(1 + \sum_{i=k}^{n}(n)_{i+k-1} \binom{\frac{n}{i}}{i-1} \binom{\frac{i}{k-1}}{x^i}\right). \tag{5.3}
\]
On the right hand side of \((5.2)\), for sufficiently small \( x \), the divisor \( \pi^*(-K_X) - xaE_W \) is ample on the blow-up \( W \). Therefore we have
\[
\text{vol}_W(\pi^*(-K_X) - xaE_W) = (\pi^*(-K_X) - xaE_W)^n.
\]
Expanding the intersection number, we get a polynomial in terms of \( x \). Because \( \dim \pi(E_W) = n - s \), we have \( (\pi^*(-K_X))^i E_W^{n-i} = 0 \) when \( n - s < i < n \). Therefore
\[
(\pi^*(-K_X) - xaE_W)^n = (-K_X)^n + (-1)^s \binom{n}{s} (\pi^*(-K_X))^{n-s}a_E^s x^s + O(x^{s+1}). \tag{5.4}
\]
Compare the above two polynomials on the right hand side of \((5.3)\) and \((5.4)\). We see that \( k = s \), so \( \dim \sigma(E) = n - k = \dim Z \). Therefore we know that the center of \( E \) on \( X \) is an irreducible component of \( Z \).

Next, using formula \((4.3)\) for \( \text{vol}(\pi^*(-rK_X) - xF) \), we see that the coefficient of \( x^k \) in \( \text{vol}(\sigma^*(-rK_X) - xF) \) is \(-(-rK_X)^n \binom{n}{k}\). If we compare it with the coefficient of \( x^k \) in the expansion of the polynomial \( \text{vol}(\pi^*(-rK_X) - xaE_W) = (\pi^*(-rK_X) - xaE_W)^n \), we get that
\[
(-rK_X)^n = a^k (1)^{k-1} (\pi^*(-rK_X)^{n-k} E_W^k). \tag{5.5}
\]
Write \([Z] = \sum_i a_i[Z_i]\). Pick any \( Z_i \) to be an irreducible component of \( Z \) that is not the center of \( E_W \). Localizing at \( Z_i \), we see that on the left-hand side of \((5.5)\), we get \( a_i \deg_{rK_X}(Z_i) \). However on the right-hand side of \((5.5)\) it is zero, which is a contradiction. Therefore \( Z \) is irreducible.
Suppose $Z'$ is the support of $Z$. Then $Z'$ is also the center of $E_W$ on $X$. We have the following equality of intersection numbers
\[
(−1)^{k−1}π^∗(−rK_X)^{n−k}E_W^k = \deg(−rK_X)Z' \frac{1}{m^k}e(a_m \cdot O_{X,Z'})
\]
for sufficiently divisible $m$ by (2.1). Together with (5.5), we have
\[
(−rK_X)^n = \deg(−rK_X)Z' \frac{a_k}{m^k}e(a_m \cdot O_{X,Z'}).
\]
Note that $\frac{AX(E)}{a} = \lct(X, Z) = \frac{k}{(n+1)r}$, so we have $a = \frac{AX(E)(n+1)x}{k}$. Consequently,
\[
(−K_X)^n = \deg(−K_X)Z' \left(\frac{n+1}{k}\right)^k \frac{AX(E)^k}{m^k}e(a_m \cdot O_{X,Z'}).
\]
(5.6)

The log discrepancy of $E$ doesn’t change after we localize at $Z'$ because $Z'$ is the center of $E$. Working on $\Spec O_{X,Z'}$, we next want to show that
\[
\frac{AX(E)^k}{m^k}e(a_m \cdot O_{X,Z'}) \geq k^k.
\]
Note that by the definition of log canonical threshold and $a_m = \sigma_\ast O_Y(−mE)$, we have
\[
\lct(a_m \cdot O_{X,Z'}) \leq \frac{AX(E)}{\ord_E(a_m \cdot O_{X,Z'})} = \frac{AX(E)}{m}.
\]
Since $\Spec O_{X,Z'}$ is smooth, we know that
\[
\frac{AX(E)^k}{m^k}e(a_m \cdot O_{X,Z'}) \geq \lct(a_m \cdot O_{X,Z'})^k(e(a_m \cdot O_{X,Z'}) \geq k^k;
\]
where the last inequality follows from [dFEM04]. Therefore from (5.6) we have the following inequality:
\[
(−K_X)^n \geq (n+1)^k(−K_X)^{n−k}Z',
\]
or equivalently,
\[
(−K_X)^{n−k}[−(−K_X)^k − (n+1)^k Z'] \geq 0.
\]
Since $(−K_X)^k \sim_{rat} \frac{1}{r}Z$, we may assume $(−K_X)^k \sim_{rat} lZ'$ for some integer $l$. Then by the ampleness of $−K_X$, we know that $\ l \geq (n+1)^k$. This finishes the proof of Proposition 5.1. □

Proof of Theorem B. We have already seen in Example 3.2 that $\alpha^{(k)}(\mathbb{P}^n) = k/(n+1)$, which gives (1) $\Rightarrow$ (2). We also have (2) $\Rightarrow$ (3) by Proposition 5.1. Finally by Corollary 2.8, we know that when $k$ divides $n$, we have (3) $\Rightarrow$ (1) in Theorem B. □

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