Normalisation for Fitch-Style Modal Calculi

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Fitch-style modal lambda calculi enable programming with necessity modalities in a typed lambda calculus by extending the typing context with a delimiting operator that is denoted by a lock. The addition of locks simplifies the formulation of typing rules for calculi that incorporate different modal axioms, but each variant demands different, tedious and seemingly ad hoc syntactic lemmas to prove normalization. In this work, we take a semantic approach to normalization, called normalization by evaluation (NbE), by leveraging the possible-world semantics of Fitch-style calculi to yield a more modular approach to normalization. We show that NbE models can be constructed for calculi that incorporate the K, T and 4 axioms of modal logic, as suitable instantiations of the possible-world semantics. In addition to existing results that handle β-equivalence, our normalization result also considers η-equivalence for these calculi. Our key results have been mechanized in the proof assistant Agda. Finally, we showcase several consequences of normalization for proving meta-theoretic properties of Fitch-style calculi as well as programming-language applications based on different interpretations of the necessity modality.

CCS Concepts: • Theory of computation → Type theory; Modal and temporal logics; Constructive mathematics.

Additional Key Words and Phrases: Fitch-style lambda calculi, Possible-world semantics, Normalization by Evaluation

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1 INTRODUCTION

In type systems, a modality can be broadly construed as a unary type constructor with certain properties. Type systems with modalities have found a wide range of applications in programming languages to specify properties of a program in its type. In this work, we study typed lambda calculi equipped with a necessity modality (denoted by □) formulated in the so-called Fitch style.

The necessity modality originates from modal logic, where the most basic intuitionistic modal logic IK (for “intuitionistic” and “Kripke”) extends intuitionistic propositional logic with a unary connective □, the necessitation rule (if Γ ⊢ A then Γ ⊢ □A) and the K axiom (□(A ⇒ B) ⇒ □A ⇒ □B). With the addition of further modal axioms T (□A ⇒ A) and 4 (□□A ⇒ □A) to IK, we obtain richer logics IT (adding axiom T), IK4 (adding axiom 4), and IS4 (adding both T and 4). Type systems with necessity modalities based on IK and IS4 have found applications in partial evaluation and staged computation [Davies and Pfenning 1996, 2001], information-flow control [Miyamoto and Igarashi 2004], and recovering purity in an effectful language [Choudhury and Krishnaswami 2020]. While

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type systems based on IT and IK4 do not seem to have any prior known programming applications, they are nevertheless interesting as objects of study that extend IK towards IS4.

Fitch-style modal lambda calculi [Borghuis 1994; Clouston 2018; Martini and Masini 1996] feature necessity modalities in a typed lambda calculus by extending the typing context with a delimiting “lock” operator (denoted by \( \mathcal{L} \)). In this paper, we consider the family of Fitch-style modal lambda calculi that correspond to the logics IK, IT, IK4, and IS4. These calculi extend the simply-typed lambda calculus (STLC) with a type constructor \( \Box \), along with introduction and elimination rules for \( \Box \) types formulated using the \( \mathcal{L} \) operator. For instance, the calculus \( \lambda_{IK} \), which corresponds to the logic IK, extends STLC with Rules \( \Box\text{-Intro} \) and \( \lambda_{IK}/\Box\text{-Elim} \), as summarized in Fig. 1. The rules for \( \lambda \)-abstraction and function application are formulated in the usual way—but note the modified variable rule \( \text{VAR}! \).

![Typing rules for \( \lambda_{IK} \) (omitting \( \lambda \)-abstraction and application)](image)

The equivalence of terms in STLC is extended by Fitch-style calculi with the following rules for \( \Box \) types, where the former states the \( \beta \)- (or computational) equivalence, and the latter states a type-directed \( \eta \)- (or extensional) equivalence.

\[
\begin{align*}
\Box\beta & \quad \text{unbox (box } t \text{) } \sim t \\
\Box\eta & \quad \text{t } \sim \text{ box (unbox } t \text{)}
\end{align*}
\]

We are interested in the problem of normalizing terms with respect to these equivalences. Traditionally, terms in a calculus are normalized by rewriting them using rewrite rules formulated from these equivalences, and a term is said to be in normal form when it cannot be rewritten further. For example, we may formulate a rewrite rule \( \text{unbox (box } t \text{) } \rightarrow \rightarrow t \) by orienting the \( \Box\beta \) equivalence from left to right. This naive approach to formulating a rewrite rule, however, is insufficient for the \( \Box\eta \) rule since normalizing with a rewrite rule \( t \rightarrow \text{ box (unbox } t \text{) } \) (for \( \Gamma \vdash t : \Box A \)) does not terminate as it can be applied infinitely many times. It is presumably for this reason that existing normalization results [Clouston 2018] for some of these calculi only consider \( \beta \)-equivalence.

While it may be possible to carefully formulate a more complex set of rewrite rules that take the context of application into consideration to guarantee termination (as done, for example, by Jay and Ghani [1995] for function and product types), the situation is further complicated for Fitch-style calculi by the fact that we must repeat such syntactic rewriting arguments separately for each calculus under consideration. The calculi \( \lambda_{IT}, \lambda_{IK4}, \) and \( \lambda_{IS4} \) differ from \( \lambda_{IK} \) only in the \( \Box\)-elimination rule, as summarized in Fig. 2. In spite of having identical syntax and term equivalences, each calculus demands different, tedious and seemingly ad hoc syntactic renaming lemmas [Clouston 2018, Lemmas 4.1 and 5.1] to prove normalization.

In this paper, we take a semantic approach to normalization, called normalization by evaluation (NbE) [Berger and Schwichtenberg 1991]. NbE bypasses rewriting entirely, and instead normalizes terms by evaluating them in a suitable semantic model and then reifying values in the model as normal forms. For Fitch-style calculi, NbE can be developed by leveraging their possible-world semantics. To this end, we identify the parameters of the possible-world semantics.
The main idea underlying this paper is that normalization can be achieved in a modular fashion for the calculi under consideration, and show that NbE models can be constructed by instantiating those parameters. The NbE approach exploits the semantic overlap of the Fitch-style calculi in the possible-world semantics and isolates their differences to a specific parameter that determines the modal fragment, thus enabling the reuse of the evaluation machinery and many lemmas proved in the process.

In Section 2, we begin by providing a brief overview of the main idea underlying this paper. We discuss the uniform interpretation of types for four Fitch-style calculi ($\lambda_{\mathit{IK}}$, $\lambda_{\mathit{IT}}$, $\lambda_{\mathit{IK4}}$ and $\lambda_{\mathit{IS4}}$) in possible-world models and outline how NbE models can be constructed as instances. The reification mechanism that enables NbE is performed alike for all four calculi. In Section 3, we construct an NbE model for $\lambda_{\mathit{IK}}$ that yields a correct normalization algorithm, and then show how NbE models can also be constructed for $\lambda_{\mathit{IS4}}$, and for $\lambda_{\mathit{IT}}$ and $\lambda_{\mathit{IK4}}$ by slightly varying the instantiation. The calculi $\lambda_{\mathit{IK}}$ and $\lambda_{\mathit{IS4}}$ and their normalization algorithms have been implemented and verified correct [Valliappan, Ruch, et al. 2022] in the proof assistant AGDA [Abel, Allais, et al. 2005–2021].

NbE models and proofs of normalization in general have several useful consequences for term calculi. In Section 4, we show how NbE models and the accompanying normalization algorithm can be used to prove meta-theoretic properties of Fitch-style calculi including completeness, decidability, and some standard results in modal logic in a constructive manner. In Section 5, we discuss applications of our development to specific interpretations of the necessity modality in programming languages, and show (but do not mechanize) how application-specific properties that typically require semantic intervention can be proved syntactically. We show that properties similar to capability safety, noninterference, and binding-time correctness can be proved syntactically using normal forms of terms.

2 MAIN IDEA

The main idea underlying this paper is that normalization can be achieved in a modular fashion for Fitch-style calculi by constructing NbE models as instances of their possible-world semantics. In this section, we observe that Fitch-style calculi can be interpreted in the possible-world semantics for intuitionistic modal logic with a minor refinement that accommodates the $\square$ operator, and give a brief overview of how we construct NbE models as instances.

Possible-World Semantics. The possible-world semantics for intuitionistic modal logic [Božić and Došen 1984] is parameterized by a frame $F$ and a valuation $V_i$. A frame $F$ is a triple $(W, R_i, R_m)$ that consists of a type $W$ of worlds along with two binary accessibility relations $R_i$ (for “intuitionistic”) and $R_m$ (for “modal”) on worlds that are required to satisfy certain conditions. An element $w : W$ can be thought of as a representation of the “knowledge state” about some “possible world” at a certain point in time. Then, $w R_i w’$ represents an increase in knowledge from $w$ to $w’$, and $w R_m v$ represents a possible passage from $w$ to $v$. A valuation $V_i$, on the other hand, is a family of types $V_i w$ indexed by $w : W$ along with functions $wk_{i,w,w} : V_i w \rightarrow V_i w’$ whenever $w R_i w’$. An element $p : V_i w$ can be thought of as “evidence” for (the knowledge of) the truth of the atomic proposition $i$ at the world $w$. The requirement for functions $wk_{i,w,w’}$ enforces that the knowledge of the truth of $i$ at $w$ is preserved as time moves on to $w’$, and is neither forgotten nor contradicted.

Fig. 2. □-elimination rules for $\lambda_{\mathit{IT}}$, $\lambda_{\mathit{IK4}}$, and $\lambda_{\mathit{IS4}}$. 
by any new evidence learned at $w'$. There are no such requirements on a valuation $V_i$ with respect to the modal accessibility relation $R_m$.

Given a frame $(W, R_l, R_m)$ and a valuation $V_i$, we interpret (object) types $A$ in any Fitch-style calculus as families of (meta) types $\llbracket A \rrbracket_w$ indexed by worlds $w : W$, following the work by Ewald [1986], Fischer-Servi [1981], Plotkin and Stirling [1986], and Simpson [1994] as below:

\[
\begin{align*}
\llbracket t \rrbracket_w &= V_{i,w} \\
\llbracket A \Rightarrow B \rrbracket_w &= \forall w'. w R_l w' \rightarrow \llbracket A \rrbracket_w \rightarrow \llbracket B \rrbracket_{w'} \\
\llbracket \Box A \rrbracket_w &= \forall w'. w R_l w' \rightarrow \forall u. w' R_m v \rightarrow \llbracket A \rrbracket_v
\end{align*}
\]

The nonmodal type formers are interpreted as in the Kripke semantics for intuitionistic propositional logic: the base type $i$ is interpreted using the valuation $V_i$, and function types $A \Rightarrow B$ at $w : W$ are interpreted as families of functions $\llbracket A \rrbracket_w \rightarrow \llbracket B \rrbracket_{w'}$ indexed by $w' : W$ such that $w R_l w'$. Recall that the generalization to families is necessary for the interpretation of function types to be sound.

As for the interpretation of modal types, at $w : W$ the types $\Box A$ are interpreted by families of elements $\llbracket A \rrbracket_v$ indexed by those $v : W$ that are accessible from $w$ via some $w' : W$ such that $w R_l w'$ and $w' R_m v$. In other words, $\Box A$ is true at a world $w$ if $A$ is necessarily true in “the future”, whichever concrete possibility this may turn out to be. We remark that the interpretation of $\Box A$ as $\forall u. w R_m v \rightarrow \llbracket A \rrbracket_v$, as in classical modal logic without the first quantifier $\forall w'$. $w R_l w'$, requires additional conditions [Božič and Došen 1984; Simpson 1994] on frames that (some of) the NbE models we construct do not satisfy.

In order to extend the possible-world semantics of intuitionistic modal logic to Fitch-style calculi, we must also provide an interpretation of contexts and the $\overline{\cdot}$ operator, which is unique to the Fitch style, in particular:

\[
\begin{align*}
\llbracket \cdot \rrbracket_w &= \top \\
\llbracket \Gamma, A \rrbracket_w &= \llbracket \Gamma \rrbracket_w \times \llbracket A \rrbracket_w \\
\llbracket \Gamma, \overline{\cdot} \rrbracket_w &= \sum_u \llbracket \Gamma \rrbracket_u \times u R_m w
\end{align*}
\]

The empty context $\cdot$ and the context extension $\Gamma, A$ of a context $\Gamma$ with a type $A$ are interpreted as in the Kripke semantics for STLC by the terminal family and the Cartesian product of the families $\llbracket \Gamma \rrbracket$ and $\llbracket A \rrbracket$, respectively. While the interpretation of types $\Box A$ can be understood as a statement about the future, the interpretation of contexts $\Gamma, \overline{\cdot}$ can be understood as a dual statement about the past: $\Gamma, \overline{\cdot}$ is true at a world $w$ if $\Gamma$ is true at some world $u$ for which $w$ is a possibility, i.e. $u R_m w$.

With the interpretation of contexts $\Gamma$ and types $A$ as $(W, R_l)$-indexed families $\llbracket \Gamma \rrbracket$ and $\llbracket A \rrbracket$ at hand, the interpretation of terms $t : \Gamma \vdash A$, also known as evaluation, in a possible-world model is given by a function $\llbracket - \rrbracket : \Gamma \vdash A \rightarrow (\forall w. \llbracket \Gamma \rrbracket_w \rightarrow \llbracket A \rrbracket_w)$ as follows. Clouston [2018] shows that the interpretation of STLC in Cartesian closed categories (CCCs) extends to an interpretation of Fitch-style calculi in any CCC equipped with an adjunction by interpreting $\Box$ and $\overline{\cdot}$ by the right and left adjoint as well as box and unbox using the right and left adjoints, respectively. The key idea here is that, correspondingly, the interpretation of terms in the nonmodal fragment of Fitch-style calculi using the familiar CCC structure on $(W, R_l)$-indexed families extends to the modal fragment: the interpretation of $\Box$ in a possible-world model has a left adjoint that is denoted by our interpretation of $\overline{\cdot}$. In summary, the possible-world interpretation of Fitch-style calculi can be given by instantiation of Clouston’s generic interpretation in CCCs equipped with an adjunction.

Constructing NbE Models as Instances. To construct an NbE model for Fitch-style calculi, we must construct a possible-world model with a function $quote : (\forall w. \llbracket \Gamma \rrbracket_w \rightarrow \llbracket A \rrbracket_w) \rightarrow \Gamma \vdash_{NF} A$ that inverts the denotation $(\forall w. \llbracket \Gamma \rrbracket_w \rightarrow \llbracket A \rrbracket_w)$ of a term to a derivation $\Gamma \vdash_{NF} A$ in normal form. The
normal forms for the modal fragment of λIK are defined below, where \( \Gamma \vdash_{NE} A \) denotes a special case of normal forms known as neutral elements.

\[
\begin{array}{c}
\text{Nf/Box-Intro} \\
\Gamma, \Delta \vdash_{NF} t : A \\
\Gamma \vdash_{NF} \text{box } t : \Box A
\end{array}
\]

\[
\begin{array}{c}
\text{\( \lambda_{IK}/\Box\)-Elim} \\
\Gamma \vdash_{NE} t : \Box A \\
\Gamma \vdash_{NE} \text{unbox}_{\lambda_{IK}} t : A \quad \mathfrak{A} \notin \Gamma'
\end{array}
\]

The normal forms for \( \lambda_{IT}, \lambda_{IK4}, \) and \( \lambda_{IS4} \) are defined similarly by varying the elimination rule as in their term typing rules in Fig. 2.

Following the work on NbE for STLC with possible-world\(^1\) models [Coquand 2002], we instantiate the parameters that define possible-world models for Fitch-style calculi as follows: we pick contexts for \( W \), order-preserving embeddings (sometimes called “weakenings”, defined in the next section) \( \Gamma \leq \Gamma' \) for \( \Gamma \), \( \Gamma' \), and neutral derivations \( \Gamma \vdash_{NE} t \) as the valuation \( V_\Gamma \). It remains for us to instantiate the parameter \( R_m \) and show that this model supports the quote function.

The instantiation of the modal parameter \( R_m \) in the possible-world semantics varies for each calculus and captures the difference between them. Recall that the syntaxes of the four calculi only differ in their elimination rules for \( \Box \) types. When viewed through the lens of the possible-world semantics, this difference can be generalized as follows:

\[
\begin{array}{c}
\text{\( \Box\)-Elim} \\
\Delta \vdash t : \Box A \\
\Gamma \vdash \text{unbox } t : A \quad (\Lambda \ll \Gamma)
\end{array}
\]

We generalize the relationship between the context in the premise and the context in the conclusion using a generic modal accessibility relation \( \ll \) between contexts. When viewed as a candidate for instantiating the \( R_m \) relation, this rule states that if \( \Box A \) is derivable in some past world \( \Delta \), then we may derive \( A \) in the current world \( \Gamma \). The various \( \Box\)-elimination rules for Fitch-style calculi can be viewed as instances of this generalized rule, where we define \( \ll \) in accordance with \( \Box\)-elimination rule of the calculus under consideration. For example, for \( \lambda_{IK} \), we observe that the context of the premise in Rule \( \lambda_{IK}/\Box\)-Elim is \( \Gamma \) and that of the conclusion is \( \Gamma, \Delta, \Gamma' \) such that \( \mathfrak{A} \notin \Gamma' \), and thus define \( \Delta/\ll_{\lambda_{IK}} \Gamma \) as \( \exists \Delta' : \mathfrak{A} \notin \Delta' \land \Gamma = \Delta, \Delta'. \) Similarly, we define \( \Delta/\ll_{\lambda_{IS4}} \Gamma \) as \( \exists \Delta' \Gamma = \Delta, \Delta' \) for \( \lambda_{IS4} \), and follow this recipe for \( \lambda_{IT} \) and \( \lambda_{IK4} \). Accordingly, we instantiate the \( R_m \) parameter in the NbE model with the corresponding definition of \( \ll \) in the calculus under consideration.

A key component of implementing the quote function in NbE models is reification, which is implemented by a family of functions \( \text{reify}_{\mathcal{A}} : \forall \Gamma, \llbracket \mathcal{A} \rrbracket_\Gamma \rightarrow \Gamma \vdash_{NF} \mathcal{A} \) indexed by a type \( \mathcal{A} \). While its implementation for the simply-typed fragment follows the standard, for the modal fragment we are required to give an implementation of \( \text{reify}_{\Box A} : \forall \Gamma, \llbracket \Box A \rrbracket_\Gamma \rightarrow \Gamma \vdash_{NF} \Box A \). To reify a value of \( \llbracket \Box A \rrbracket_\Gamma \), we first observe that \( \llbracket \Box A \rrbracket_\Gamma = \forall \Gamma' : \Gamma \leq \Gamma' \rightarrow \forall \Delta, \Gamma' \ll \Delta \rightarrow \llbracket A \rrbracket_\Delta \) by definition of \( \llbracket \Box \rrbracket \) and the instantiations of \( R_i \) with \( \leq \) and \( R_m \) with \( \ll \). By picking \( \Gamma \) for \( \Gamma' \) and \( \mathfrak{A} \) for \( \Delta \), we get \( \llbracket A \rrbracket_\Gamma \mathfrak{A} \) since \( \leq \) is reflexive and it can be shown that \( \Gamma \ll \mathfrak{A} \mathfrak{A} \) holds for the calculi under consideration. By reifying the value \( \llbracket A \rrbracket_\Gamma \mathfrak{A} \) recursively, we get a normal form \( \Gamma, \mathfrak{A} \vdash_{NE} n : A \), which can be used to construct the desired normal form \( \Gamma \vdash_{NF} \text{box } n : \Box A \) using the rule Nf/Box-Intro.

### 3 POSSIBLE-WORLD SEMANTICS AND NbE

In this section, we elaborate on the previous section by defining possible-world models and showing that Fitch-style calculi can be interpreted soundly in these models. Following this, we outline the details of constructing NbE models as instances. We begin with the calculus \( \lambda_{IK} \), and then show how the same results can be achieved for the other calculi.

Before discussing a concrete calculus, we present some of their commonalities.

\(^1\)also called “Kripke” or “Kripke-style”
Types, Contexts and Order-Preserving Embeddings. The grammar of types and typing contexts for Fitch-style is the following.

\[ Ty \quad A ::= \iota | A \Rightarrow B | \Box A \]

\[ Ctx \quad \Gamma ::= \cdot | \Gamma, A | \Gamma, \Box \]

Types are generated by an uninterpreted base type \( \iota \), function types \( A \Rightarrow B \), and modal types \( \Box A \), and typing contexts are “snoc” lists of types and locks.

We define the relation of order-preserving embeddings (OPE) on typing contexts in Fig. 3. An \( OPE \Gamma \leq \Gamma' \) embeds the context \( \Gamma \) into another context \( \Gamma' \) while preserving the order of types and the order and number of locks in \( \Gamma \).

\[ o : \Gamma \leq \Gamma' \]

\[ \text{base} : \cdot \leq \cdot \]

\[ \text{drop} o : \Gamma \leq \Gamma', A \]

\[ \text{keep} o : \Gamma, A \leq \Gamma', A \]

\[ \text{keep} \Box o : \Gamma, \Box \leq \Gamma', \Box \]

Fig. 3. Order-preserving embeddings

3.1 The Calculus \( \lambda_{IK} \)

3.1.1 Terms, Substitutions and Equational Theory. To define the intrinsically-typed syntax and equational theory of \( \lambda_{IK} \), we first define a modal accessibility relation on contexts \( \Delta <_{\lambda_{IK}} \Gamma \), which expresses that context \( \Gamma \) extends \( \Delta, \Box \) to the right without adding locks. Note that \( \Delta <_{\lambda_{IK}} \Gamma \) exactly when \( \exists \Delta'. \Box \not\in \Delta' \land \Gamma = \Delta, \Box, \Delta' \).

\[ \text{nil} : \Gamma <_{\lambda_{IK}} \Gamma, \Box \]

\[ e : \Delta <_{\lambda_{IK}} \Gamma \]

\[ \var e : \Delta <_{\lambda_{IK}} \Gamma, A \]

Fig. 4. Modal accessibility relation on contexts (\( \lambda_{IK} \))

\[ \text{VAR-ZERO} \quad \Gamma, A \vdash_{\text{VAR}} \text{zero} : A \]

\[ \text{VAR-SUCCE} \quad \Gamma \vdash_{\text{VAR}} v : A \]

\[ \Gamma, B \vdash_{\text{VAR}} \text{succ} v : A \]

\[ \text{VAR} \quad \Gamma \vdash_{\text{VAR}} v : A \]

\[ \Gamma + \var v : A \]

\[ \Gamma + \lambda t : A \Rightarrow B \]

\[ \Rightarrow\text{-INTRO} \quad \Gamma + t : A \Rightarrow B \]

\[ \Gamma + u : A \]

\[ \Gamma \vdash \text{app} t u : B \]

\[ \Box\text{-INTRO} \quad \Gamma, \Box \vdash t : A \]

\[ \Gamma \vdash \text{box} t : \Box A \]

\[ \lambda_{IK}/\Box\text{-ELIM} \quad \Delta \vdash t : \Box A \]

\[ e : \Delta <_{\lambda_{IK}} \Gamma \]

\[ \Gamma \vdash \text{unbox}_{\lambda_{IK}} t e : A \]

Fig. 5. Intrinsically-typed terms of \( \lambda_{IK} \)

Fig. 5 presents the intrinsically-typed syntax of \( \lambda_{IK} \). We will use both \( \Gamma \vdash t : A \) and \( t : \Gamma \vdash A \) to say that \( t \) denotes an (intrinsically-typed) term of type \( A \) in context \( \Gamma \), and similarly for substitutions, which will be defined below. Instead of named variables as in Fig. 1, variables are defined using De Bruijn indices in a separate judgement \( \Gamma \vdash_{\text{VAR}} A \). The introduction and elimination rules for function types are like those in STLC, and the introduction rule for the type \( \Box A \) is similar to that of Fig. 1. The elimination rule \( \lambda_{IK}/\Box\text{-ELIM} \) is defined using the modal accessibility relation \( \Delta <_{\lambda_{IK}} \Gamma \) which relates the contexts in the premise and the conclusion, respectively. This relation replaces...
the side condition (\(\mu \not\in \Gamma'\)) in Fig. 1 and other \(\Box\)-elimination rules in Sections 1 and 2. Note that formulating the rule for the term unbox_{\lambda K} \with e : \Delta <_{\lambda K} \Gamma as a second premise is in sharp contrast to Clouston [2018, Fig. 1] where the relation is not mentioned in the term but formulated as the side condition \(\Gamma = \Delta, \mu, \Gamma'\) for some lock-free \(\Gamma'\).

A term \(\Gamma \vdash t : A\) can be weakened, which is a special case of renaming, with an OPE (see Fig. 3) using a function \(wk : \Gamma \leq \Gamma' \rightarrow \Gamma \vdash A \rightarrow \Gamma' \vdash A\). Given an OPE \(o : \Gamma \leq \Gamma'\), renaming the term using \(wk\) yields a term \(\Gamma' \vdash wk o t : A\) in the weaker context \(\Gamma'\). The unit element for \(wk\) is the identity OPE \(id_< : \Gamma \leq \Gamma\), i.e. \(wk id_< t = t\). Renaming arises naturally when evaluating terms and in specifying the equational theory (e.g. in the \(\eta\) rule of function type).

### Fig. 6. Substitutions for \(\lambda_{IK}\)

Substitutions for \(\lambda_{IK}\) are inductively defined in Fig. 6. A judgement \(\Gamma \overset{s}{\vdash} s : \Delta\) denotes a substitution for a context \(\Delta\) in the context \(\Gamma\). Applying a substitution to a term \(\Delta \vdash t : A\), i.e. \(\text{subst } s t : \Gamma \vdash A\), yields a term in the context \(\Gamma\). The substitution \(id_s : \Gamma \overset{s}{\vdash} \Gamma\) denotes the identity substitution, which exists for all \(\Gamma\). As usual, it can be shown that terms are closed under the application of a substitution, and that it preserves the identity, i.e. \(\text{subst } id_s t = t\). Substitutions are also closed under renaming and this operation preserves the identity as well.

The equational theory for \(\lambda_{IK}\), omitting congruence rules, is specified in Fig. 7. As discussed earlier, \(\lambda_{IK}\) extends the usual rules in STLC (Rules \(\Rightarrow \beta\) and \(\Rightarrow \eta\)) with rules for the \(\Box\) type (Rules \(\Box \beta\) and \(\Box \eta\)). The function \(\text{factor} : \Delta <_{\lambda K} \Gamma \rightarrow \Delta, \mu \leq \Gamma\), in Rule \(\Box \beta\), maps an element of the modal accessibility relation \(e : \Delta <_{\lambda K} \Gamma\) to an OPE \(\Delta, \mu \leq \Gamma\). This is possible because the context \(\Gamma\) does not have any lock to the right of \(\Delta, \mu\).

### Fig. 7. Equational theory for \(\lambda_{IK}\)

3.1.2 Possible-World Semantics. A possible-world model is defined using the notion of a possible-world frame as below. We work in a constructive type-theoretic metalanguage, and denote the universe of types in this language by Type.

**Definition 1** (Possible-world frame). A frame \(F\) is given by a triple \((W, R_l, R_m)\) consisting of a type \(W : Type\) and two relations \(R_l\) and \(R_m : W \times W \rightarrow Type\) on \(W\) such that the following conditions are satisfied:

- \(R_l\) is reflexive and transitive

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We use the monotonicity lemma to "transport" evaluation of terms. This lemma is well-known as a requirement to give a sound interpretation of the function type in an arbitrary possible-world model, and can be thought of as the semantic generalization of renaming in terms.

**Definition 1** (Monotonicity). In every possible-world model $\mathcal{M}$, for every type $A$ and worlds $w$ and $w'$, we have a function $wk_A : w R_i w' \rightarrow [A]_w \rightarrow [A]_{w'}$. And similarly, for every context $\Gamma$, a function $wk_\Gamma : w R_i w' \rightarrow [\Gamma]_w \rightarrow [\Gamma]_{w'}$.

We evaluate terms in $\lambda_{IK}$ in a possible-world model as follows.

\[
\begin{align*}
\llbracket \varnothing \rrbracket : \Gamma \vdash A & \rightarrow (\forall w. \llbracket \Gamma \rrbracket_w \rightarrow [A]_w) \\
\llbracket \text{var } v \rrbracket : \Gamma \vdash \lambda x. y & \rightarrow \lambda i. \lambda a. (\llbracket t \rrbracket_y (\llbracket t \rrbracket_y) \delta ) \downarrow \llbracket y \rrbracket_y (\llbracket t \rrbracket_y) \delta \downarrow m \\
\llbracket \text{app } t u \rrbracket : \Gamma \vdash \boxdot e e & \rightarrow \llbracket \boxdot e e \rrbracket_y (\llbracket t \rrbracket_y) \delta \downarrow m \\
\llbracket \text{box } t \rrbracket : \Gamma \vdash \boxdot e e & \rightarrow \llbracket \text{box } t \rrbracket_y (\llbracket t \rrbracket_y) \delta \downarrow m \\
\llbracket \text{unbox}_{\lambda_{IK}} e e & \rightarrow \llbracket \text{unbox}_{\lambda_{IK}} e e \rrbracket_y (\llbracket t \rrbracket_y) \delta \downarrow m \\
\end{align*}
\]

where $(\delta, m) = \text{trim}_{\lambda_{IK}} y e$

The evaluation of terms in the simply-typed fragment is standard, and resembles the evaluator of STLC. Variables are interpreted by a lookup function that projects values from an environment, and $\lambda$-abstraction and application are evaluated using their semantic counterparts. To evaluate $\lambda$-abstraction, we must construct a semantic function $\forall w'. w R_i w' \rightarrow [A]_{w'} \rightarrow [B]_{w'}$ using the given term $\Gamma, A \vdash t : B$ and environment $\gamma : \llbracket \Gamma \rrbracket_w$. We achieve this by recursively evaluating $t$ in an environment that extends $\gamma$ appropriately using the semantic arguments $i : w R_i w'$ and $a : [A]_{w'}$. We use the monotonicity lemma to "transport" $[\Gamma]_w$ to $[\Gamma]_{w'}$, and construct an environment of type $[\Gamma]_w \times [A]_{w'}$ for recursively evaluating $t$, which produces the desired result of type $[B]_{w'}$. Application is evaluated by simply recursively evaluating the applied terms and applying them in the semantics with a value $\overline{id} : w R_i w$, which is available since $R_i$ is reflexive.

In the modal fragment, to evaluate the term $\Gamma \vdash \Box t : A \Gamma \vdash \Box t : A \Gamma \vdash \Box t : A$ in the extended

\[
\begin{align*}
\end{align*}
\]
environment \((wk \ i \ y, m) : \llbracket \Gamma, \Box \rrbracket_w\), since \(\llbracket \Gamma, \Box \rrbracket_w = \sum_{w'} \llbracket \Gamma \rrbracket_{w'} \times w' R_m v\). On the other hand, the term \(\Gamma \vdash \text{unbox}_{\lambda_{IK}} t : A\) with \(e : \Delta \prec_{\lambda_{IK}} \Gamma \) and \(\Delta \vdash t : \Box A\), for some \(\Delta\), must be evaluated with an environment \(\gamma : \llbracket \Gamma \rrbracket_w\). To recursively evaluate the term \(\Delta \vdash t : \Box A\), we must first discard the part of the environment \(\gamma\) that substitutes the types in the extension of \(\Delta, \Box\). This is achieved using the function \(\text{trim}_{\lambda_{IK}} : \llbracket \Gamma \rrbracket_w \to \Delta \prec_{\lambda_{IK}} \Gamma \to \llbracket \Delta, \Box \rrbracket_w\) that projects \(\gamma\) to produce an environment \(\delta : \llbracket \Delta \rrbracket_{\gamma'}\) and a value \(m : \alpha' R_m w\). We evaluate \(t\) with \(\delta\) and apply the resulting function of type \(\forall v. v \alpha' \to \forall v. \alpha' R_m w \to \llbracket A \rrbracket_w\) to \id_{\alpha'}\) and \(m\) to return the desired result.

We state the soundness of \(\lambda_{IK}\) with respect to the possible-world semantics before we instantiate it with the NbE model that we will construct in the next subsection. We note that the soundness proof relies on the possible-world models to satisfy coherence conditions that we have omitted from Definitions 1 and 2 but that will be satisfied by the NbE models. Specifically, \(W\) and \(R_i\) together with the transitivity and reflexivity proofs \(\text{trans}_i\) and \(\text{refl}\); for \(R_i\) need to form a category \(\mathcal{W}\), i.e. \(\text{trans}_i\) needs to be associative and \(\text{refl}\); for \(R_i\) needs to be a unit for \(\text{trans}_i\); the proofs of the factorization condition need to satisfy the functoriality laws \(\text{factor}_m (\text{refl}_i) v = \text{refl}_i w, \text{factor}_m \text{refl}_i v = m, \text{factor}_m (\text{trans}_i j) = \text{trans}_i (\text{factor}_m \text{refl}_i), \text{factor}_m \text{trans}_i j = \text{trans}_i (\text{factor}_m \text{refl}_i)\) where \(m' := \text{factor}_m m i : w' R_m \alpha'\) denotes the modal accessibility proof produced by the first factorization of \(m : w R_m v\) and \(i : v R_i \alpha'\); and \(V_i\) together with the monotonicity proof \(wk_i\) needs to form a functor on the category \(\mathcal{W}\), i.e. \(wk_i (\text{refl}_i)\) needs to be equal to the identity function on \(V_i w\) and \(wk_i (\text{trans}_i j)\) needs to be equal to the composite \(wk_j \circ wk_i\).

**Theorem 2.** Let \(M\) be any possible-world model (see Definition 2). If two terms \(t\) and \(u : \Gamma \vdash A\) of \(\lambda_{IK}\) are equivalent (see Fig. 7) then the functions \(\llbracket t \rrbracket\) and \(\llbracket u \rrbracket : \forall w. \llbracket \Gamma \rrbracket_w \to \llbracket A \rrbracket_w\) as determined by \(M\) are equal.

**Proof.** Let \(M\) be a possible-world model with underlying frame \(F = (W, R_i, R_m)\). Denote the category whose objects are worlds \(w : W\) and whose morphisms are proofs \(i : w R_i w'\) by \(C\). The frame \(F\) can be seen as determining an adjunction \(\Box \vdash \square\) on the category of presheaves indexed by the category \(C\), which is moreover well-known to be Cartesian closed. The interpretation \(\llbracket \cdot \rrbracket\) can then be seen as factoring through the categorical semantics described in Clouston [2018, Section 2.3], of which the category of presheaves over \(C\) can then be seen as factoring through the categorical semantics described in Clouston [2018, Theorem 2.8 (Categorical Soundness) and remark below that].

### NbE Model

3.1.3 NbE Model. The normal forms of terms in \(\lambda_{IK}\) are defined along with neutral elements in a mutually recursive fashion by the judgements \(\Gamma \vdash_{\text{nf}} A\) and \(\Gamma \vdash_{\text{ne}} A\), respectively, in Fig. 8. Intuitively, a normal form may be thought of as a value, and a neutral element may be thought of as a “stuck” computation. We extend the standard definition of normal forms and neutral elements in STLC with Rules \(\text{Ne}/\Box\text{-Intro}\) and \(\lambda_{IK}/\text{Ne}/\Box\text{-El}im\).

| \(\text{Ne}/\text{VAR}\) | \(\text{Ne}/\text{Up}\) | \(\text{Ne}/\Rightarrow\text{-Intro}\) | \(\text{Ne}/\Rightarrow\text{-El}im\) |
|-------------------|-------------------|-------------------|-------------------|
| \(\Gamma \vdash_{\text{VAR}} v : A\) | \(\Gamma \vdash_{\text{NE}} n : i\) | \(\Gamma, A \vdash_{\text{NF}} n : B\) | \(\Gamma \vdash_{\text{NFI}} n : A \Rightarrow B\) |
| \(\Gamma \vdash_{\text{NE}} \text{var} v : A\) | \(\Gamma \vdash_{\text{NF}} \text{up} n : i\) | \(\Gamma \vdash_{\text{NF}} \lambda n : A \Rightarrow B\) | \(\Gamma \vdash_{\text{NF}} \text{app} m n : B\) |
| \(\text{Ne}/\Box\text{-Intro}\) | \(\Gamma, \Box \vdash_{\text{NF}} n : A\) | \(\lambda_{IK}/\text{Ne}/\Box\text{-El}im\) | \(\Delta \vdash_{\text{NE}} n : \Box A\) |
| \(\Gamma \vdash_{\text{NF}} \text{box} n : \Box A\) | | | \(\Gamma \vdash_{\text{NF}} \text{unbox}_{\lambda_{IK}} n e : A\) |

Fig. 8. Normal forms and neutral elements in \(\lambda_{IK}\)

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Recall that an NbE model for a given calculus $C$ is a particular kind of model $\mathcal{M}$ that comes equipped with a function $\text{quote} : \mathcal{M}([\Gamma], [A]) \rightarrow \Gamma \vdash_{\text{NF}} A$ satisfying $t \sim \text{quote}[t]$ for all terms $t : \Gamma \vdash A$ where $[\cdot]$ denotes the generic evaluation function for $C$.

Using the relations defined in Figs. 3 and 4, we construct an NbE model for $\lambda_{\text{IR}}$ by instantiating the parameters that define a possible-world model as follows.

- Worlds as contexts: $W = \text{Ctx}$
- Relation $R_i$ as order-preserving embeddings: $\Gamma R_i \Gamma' = \Gamma \leq \Gamma'$
- Relation $R_m$ as extensions of a “locked” context: $\Delta R_m \Gamma = \Delta \triangleleft_{\text{IR}} \Gamma$
- Valuation $V_i$ as neutral elements: $V_i \Gamma = \Gamma \vdash_{\text{NE}} t$

The condition that the valuation must satisfy $wk_A : \Gamma \leq \Gamma' \rightarrow \Gamma \vdash_{\text{NE}} A \rightarrow \Gamma' \vdash_{\text{NE}} A$, for all types $A$, can be shown by induction on the neutral term $\Gamma \vdash_{\text{NE}} A$. To show that this model is indeed a possible-world model, it remains for us to show that the frame conditions are satisfied.

The first frame condition states that OPEs must be reflexive and transitive, which can be shown by structural induction on the context and definition of OPEs, respectively. The second frame condition states that given $\Delta \triangleleft_{\text{IR}} \Gamma$ and $\Gamma \leq \Gamma'$ there is a $\Delta' : \text{Ctx}$ such that $\Delta \leq \Delta'$ and $\Delta' \triangleleft_{\text{IR}} \Gamma'$,

\[
\begin{array}{c}
\Delta' \xrightarrow{\triangleleft_{\text{IR}}} \Gamma' \\
\downarrow \quad \downarrow \\
\Delta \xrightarrow{\triangleleft_{\text{IR}}} \Gamma
\end{array}
\]

which can be shown by constructing a function by simultaneous recursion on OPEs and the modal accessibility relation.

Observe that the instantiation of the monotonicity lemma in the NbE model states that we have the functions $wk_A : [\Gamma][A] \rightarrow [\Gamma][A]$ and $wk_\Delta : [\Gamma][\Delta] \rightarrow [\Gamma][\Delta]$, which allow denotations of types and contexts to be renamed with respect to an OPE.

To implement the function $\text{quote}$, we first implement reification and reflection, using two functions $\text{reify}_A : [A]_\Gamma \rightarrow \Gamma \vdash_{\text{NF}} A$ and $\text{reflect}_A : \Gamma \vdash_{\text{NE}} A \rightarrow [A]_\Gamma$, respectively. Reification converts a semantic value to a normal form, while reflection converts a neutral element to a semantic value. They are implemented as follows by induction on the index type $A$.

\[
\begin{align*}
\text{reify}_{A, \Gamma} & : [A]_\Gamma \rightarrow \Gamma \vdash_{\text{NF}} A \\
\text{reify}_{\text{var}, \Gamma} & : n = \text{up} n \\
\text{reify}_{a \Rightarrow b, \Gamma} & : f = \lambda (\text{reify}_{b, (\Gamma, A)} (f (\text{drop id}_\leq \text{fresh}_{A, \Gamma}))) \\
\text{reify}_{\square A, \Gamma} & : b = \text{box} (\text{reify}_{A, (\Gamma, \Box)} (b \text{id}_\leq \text{nil})) \\
\text{reflect}_{A, \Gamma} & : \Gamma \vdash_{\text{NE}} A \rightarrow [A]_\Gamma \\
\text{reflect}_{\text{var}, \Gamma} & : n = n \\
\text{reflect}_{a \Rightarrow b, \Gamma} & : n = \lambda (o : \Gamma \leq \Gamma'). \lambda a. \text{reflect}_{b, \Gamma} (\text{app} (wk_{A \Rightarrow B, \Gamma} n) (\text{reify}_{A, \Gamma, a})) \\
\text{reflect}_{\square A, \Gamma} & : n = \lambda (o : \Gamma \leq \Gamma'). \lambda (e : \Gamma' \triangleleft_{\text{IR}} \Delta). \text{reflect}_{A, \Gamma} (\text{unbox}_{\text{IR}} (wk_{A, \Gamma} o) n) e
\end{align*}
\]

For the function type, we recursively reify the body of the lambda-abstraction by applying the given semantic function $f$ with suitable arguments, which are an OPE drop $\text{id}_\leq : \Gamma \leq \Gamma, A$ and a value $\text{fresh}_{A, \Gamma} = \text{reflect}_{A, (\Gamma, A)} (\text{var} \emptyset) = [A]_{\Gamma, A}$—which is the De Bruijn index equivalent of a fresh variable. Reflection, on the other hand, recursively reflects the application of a neutral $\Gamma \vdash_{\text{NE}} n : A \Rightarrow B$ to the reification of the semantic argument $a : [A]_\Gamma$, for an OPE $o : \Gamma \leq \Gamma'$. Similarly, for the $\Box$ type, we recursively reify the body of box by applying the given semantic function $b : \forall \Gamma. \Gamma \leq \Gamma' \rightarrow \forall \Delta. \Gamma' \triangleleft_{\text{IR}} \Delta \rightarrow [A]_\Delta$ to suitable arguments $\text{id}_\leq : \Gamma \leq \Gamma$ and the
empty context extension \( \text{nil} : \Gamma \prec_{\lambda_{\text{IK}}} \Gamma, \Box \). Reflection also follows a similar pursuit by reflecting the application of the neutral \( \Gamma \vdash_{\text{NE}} n : \Box A \) to the eliminator \text{unbox}.

Equipped with reification, we implement \text{quote} (as seen below), by applying the given denotation of a term, a function \( f : \forall \Delta. [\Gamma]_{\Delta} \rightarrow [A]_{\Delta} \), to the identity environment \( \text{freshEnv}_\Gamma : [\Gamma]_{\Gamma} \), and then reifying the resulting value. The construction of the value \( \text{freshEnv}_\Gamma \) is the De Bruijn index equivalent of generating an environment with fresh variables.

\[
\text{quote} : (\forall \Delta. [\Gamma]_{\Delta} \rightarrow [A]_{\Delta}) \rightarrow \Gamma \vdash_{\text{NF}} A
\]

\[
\text{quote} f = \text{reify}_{\Lambda, \Gamma} (f \text{freshEnv}_\Gamma)
\]

\[
\text{freshEnv}_\Gamma : [\Gamma]_{\Gamma}
\]

\[
\text{freshEnv}_\cdot = ()
\]

\[
\text{freshEnv}_{\Gamma, A} = (\text{wk} (\text{drop id}_{\epsilon}) \text{freshEnv}_\Gamma, \text{freshEnv}_{\Lambda})
\]

\[
\text{freshEnv}_{\Gamma, \Box} = (\text{freshEnv}_\Gamma, \text{nil})
\]

To prove that the function \text{quote} is indeed a retraction of evaluation, we follow the usual logical relations approach. As seen in Fig. 9, we define a relation \( L_A \) indexed by a type \( A \) that relates a term \( \Gamma \vdash t : A \) to its denotation \( [\Gamma]_{\Delta} \) as \( L_A t a \). From a proof of \( L_A t a \), it can be shown that \( t \sim \text{reify}_{\Lambda} a \). This relation is extended to contexts as \( L_{\Delta} \), for some context \( \Delta \), which relates a substitution \( \Gamma \vdash s : \Delta \) to its denotation \( \delta : [\Gamma]_{\Gamma} \) as \( L_{\Delta} s \delta \).

\[
L_{A, \Gamma} : \Gamma \vdash A \rightarrow [A]_{\Gamma} \rightarrow \text{Type}
\]

\[
L_{\text{t}, \Gamma} \quad t n = t \sim \text{quote} n
\]

\[
L_{A \Rightarrow B, \Gamma} \quad t f = \forall \Gamma', o : \delta \rightarrow \Gamma', u, a. L_{A, \Gamma} u a \rightarrow L_{B, \Gamma'} \quad (\text{app} (\text{wk} o t) u) (f o a)
\]

\[
L_{\Box A, \Gamma} \quad t b = \forall \Gamma', o : \delta \rightarrow \Gamma', e : \Gamma' \prec_{\lambda_{\text{IK}}} \delta. L_{\Lambda, \Delta} (\text{unbox}_{\lambda_{\text{IK}}} (\text{wk} o t) e) (b o e)
\]

\[
L_{\Lambda, \Gamma} : \Gamma \vdash s : \Delta \rightarrow [\Delta]_{\Gamma} \rightarrow \text{Type}
\]

\[
L_{\text{t}, \Gamma} \quad \text{empty} \quad (\delta, a) = L_{\Lambda, \Gamma} s \delta \times L_{A, \Gamma} t a
\]

\[
L_{\text{ext}_{s} t} (\delta, a) = L_{\Lambda, \Gamma} s \delta \times L_{A, \Gamma} t a
\]

\[
L_{\text{ext}_{\theta} s} (\delta, e) = L_{\Lambda, \theta} s \delta
\]

Fig. 9. Logical relations for \( \lambda_{\text{IK}} \)

For the logical relations, we then prove the so-called fundamental theorem.

**Proposition 3** (Fundamental theorem). Given a term \( \Delta \vdash t : A \), a substitution \( \Gamma \vdash_{s} s : \Delta \) and a value \( \delta : [\Delta]_{\Gamma} \), if \( L_{\Lambda, \Gamma} s \delta \) then \( L_{\Lambda, \Gamma} (\text{sub} t s) ([t]_{\delta}) \).

We conclude this subsection by stating the normalization theorem for \( \lambda_{\text{IK}} \).

**Proposition 3** entails that \( L_{\Lambda, \Delta} (\text{sub} \text{id}_{s} t) ([t]_{\delta} \text{freshEnv}_{\Delta}) \) for any term \( t \), if we pick \( s \) as the identity substitution \( \text{id}_{s} : \Delta \vdash s : \Delta \), and \( \delta \) as \( \text{freshEnv}_{\Delta} : [\Delta]_{\Delta} \), since they can be shown to be related as \( L_{\Lambda, \Delta} \text{id}_{s} \text{freshEnv}_{\Delta} \). From this it follows that \( \text{sub} \text{id}_{s} t \sim \text{reify}_{\Lambda} ([t]_{\delta} \text{freshEnv}_{\Delta}) \), and further that \( t \sim \text{quote} [t] \) from the definition of \text{quote} and the fact that \( \text{sub} \text{id}_{s} t = t \). As a result, the composite \( \text{norm} = \text{quote} \circ \lfloor - \rfloor \) is adequate, i.e. \( \text{norm} t = \text{norm} t' \) implies \( t \sim t' \).

The soundness of \( \lambda_{\text{IK}} \) with respect to possible-world models (see **Theorem 2**) directly entails \( \text{quote} [t] = \text{quote} [u] : \Gamma \vdash_{\text{NF}} A \) for all terms \( t, u : \Gamma \vdash A \) such that \( \Gamma \vdash t \sim u : A \), which means that \( \text{norm} = \text{quote} \circ \lfloor - \rfloor \) is complete. Note that this terminology might be slightly confusing because it is the soundness of \( \lfloor - \rfloor \) that implies the completeness of \text{norm}. 

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Theorem 4. Let $\mathcal{M}$ denote the possible-world model over the frame given by the relations $\Gamma \leq \Gamma'$ and $\Delta \triangleleft_{\lambda_{IK}} \Gamma$ and the valuation $V,_{\Gamma} = \Gamma \vdash_{NF} t$.

There is a function $\text{quote} : \mathcal{M}(\llbracket \Gamma \rrbracket, \llbracket A \rrbracket) \rightarrow \Gamma \vdash_{NF} A$ such that the composite norm $= \text{quote} \circ \llbracket - \rrbracket : \Gamma \vdash A \rightarrow \Gamma \vdash_{NF} A$ from terms to normal forms of $\lambda_{IK}$ is complete and adequate.

3.2 Extending to the Calculus $\lambda_{IS4}$

3.2.1 Terms, Substitutions and Equational Theory. To define the intrinsically-typed syntax of $\lambda_{IS4}$, we first define the modal accessibility relation on contexts in Fig. 10.

Fig. 10. Modal accessibility relation on contexts ($\lambda_{IS4}$)

If $\Delta \triangleleft_{\lambda_{IS4}} \Gamma$ then $\Gamma$ is an extension of $\Delta$ with as many locks as needed. Note that, in contrast to $\lambda_{IK}$, the modal accessibility relation is both reflexive and transitive. This corresponds to the conditions on the accessibility relation for the logic IS4.

Fig. 11 presents the changes of $\lambda_{IK}$ that yield $\lambda_{IS4}$. The terms are the same as $\lambda_{IK}$ with the exception of Rule $\lambda_{IK}/\Box$-ELIM which now includes the modal accessibility relation for $\lambda_{IS4}$. Similarly, the substitution rule for contexts with locks now refers to $\triangleleft_{\lambda_{IS4}}$.

Fig. 11. Intrinsically-typed terms and substitutions of $\lambda_{IS4}$ (omitting the unchanged rules of Fig. 5)

Fig. 12 presents the equational theory of the modal fragment of $\lambda_{IS4}$. This is a slightly modified version of $\lambda_{IK}$ (cf. Fig. 7) that accommodates the changes to the rule $\lambda_{IS4}/\Box$-ELIM. Unlike before, Rule $\Box$-$\beta$ now performs a substitution to modify the term $\Delta, \Box \vdash t : A$ to a term of type $\Gamma \vdash A$. Note that the result of such a substitution need not yield the same term since substitution may change the context extension of some subterm.

Fig. 12. Equational theory for $\lambda_{IS4}$ (omitting the unchanged rules of Fig. 7)

3.2.2 Possible-World Semantics. Giving possible-world semantics for $\lambda_{IS4}$ requires an additional frame condition on the relation $R,_{m}$: it must be reflexive and transitive. Evaluation proceeds as before, where we use a function $\text{trim}_{\lambda_{IS4}} : \forall W. \llbracket \Gamma \rrbracket,_{W} \rightarrow \Delta \triangleleft_{\lambda_{IS4}} \Gamma \rightarrow \llbracket \Delta, \Box \rrbracket,_{W}$ to manipulate the environment for evaluating $\text{unbox}_{\lambda_{IS4}} t e$, as seen below.

\[
\begin{align*}
\llbracket \text{unbox}_{\lambda_{IS4}} t e \rrbracket & = \llbracket t \rrbracket \delta \text{id}_{\leq} m \\
\text{where } (\delta, m) & = \text{trim}_{\lambda_{IS4}} \gamma e
\end{align*}
\]
Theorem 7. Let \( \lambda_{\text{IS4}} \) denote the possible-world model over the reflexive and transitive frame given by \( \Delta R_m \Delta \). \( \lambda_{\text{IS4}} \) is required when the context extension adds more than one lock. The desired result is of type \( \top \) as arbitrary extensions of a context: \( \Delta, \Gamma \). \( \lambda_{\text{IS4}} \) itself, and using the reflexivity of \( R_m \) to construct a value of type \( w R_m w \). Similarly, the transitivity of \( R_m \) is required when the context extension adds more than one lock.

Analogously to Theorem 2, we state the soundness of \( \lambda_{\text{IS4}} \) with respect to reflexive and transitive possible-world models before we instantiate it with the NbE model that we will construct in the next subsection. In addition to the coherence conditions stated before Theorem 2 the soundness of \( \lambda_{\text{IS4}} \) relies on coherence conditions involving the additional proofs \( \text{refl}_m \) and \( \text{trans}_m \) that a reflexive and transitive modal accessibility relation \( R_m \) must come equipped with. Specifically, \( \text{trans}_m \) also needs to be associative, \( \text{refl}_m \) also needs to be a unit for \( \text{trans}_m \), and the proofs of the factorization condition also need to satisfy the functoriality laws in the modal accessibility argument, i.e. \( \text{factor}_r (\text{refl}_m w) i = i \), \( \text{factor}_r (\text{refl}_m w') i \text{ and } \text{factor}_r (\text{trans}_m n i) = \text{factor}_r (\text{trans}_m n i') \text{ and } \text{factor}_m (\text{trans}_m n) i = \text{trans}_m (\text{factor}_m n i') \) where \( i' \coloneqq \text{factor}_m i : w R_i w' \).

**Proposition 5.** Let \( C \) be a Cartesian closed category equipped with a comonad \( \Box \) that has a left adjoint \( \Box = \Diamond \), then equivalent terms \( t \) and \( u : \Gamma \vdash A \) denote equal morphisms in \( C \).

**Proof.** This is a version of Clouston [2018, Theorem 4.8] for \( \lambda_{\text{IS4}} \) where the side condition of Rule \( \lambda_{\text{IS4}} / \Box \text{-ELIM} \) appears as an argument to the term former unbox and hence idempotency is not imposed on the comonad \( \Box \).

**Theorem 6.** Let \( M \) be a possible-world model (see Definition 2) such that the modal accessibility relation \( R_m \) is reflexive and transitive. If two terms \( t \) and \( u : \Gamma \vdash A \) of \( \lambda_{\text{IS4}} \) are equivalent (see Fig. 12) then the functions \( \Gamma \vdash t \) and \( \Gamma \vdash u \) as determined by \( M \) are equal.

**Proof.** The right adjoint determined by a reflexive and transitive frame has a comonad structure so that we can conclude by applying Proposition 5.

### 3.2.3 NbE Model

The normal forms of \( \lambda_{\text{IS4}} \) are defined as before, except for the following rule replacing the neutral rule \( \lambda_{\text{IS4}} / \Box \text{-NE} / \Box \text{-ELIM} \).

\[
\begin{array}{c}
\Delta \vdash \text{ne} n : \Box A \\
\Gamma \vdash \text{unbox}_{\lambda_{\text{IS4}}} n e : A
\end{array}
\]

The NbE model construction also proceeds in the same way, where we now pick the relation \( R_m \) as arbitrary extensions of a context: \( \Delta R_m \Gamma = \Delta \). The modal fragment for \( \text{reify} \) and \( \text{reflect} \) are now implemented as follows:

\[
\begin{align*}
\text{reify}_{\Delta, \Gamma} b & = \text{box} (\text{reify}_{A,(\Gamma \Box)} (b \text{id}_\Sigma (\text{lock} \text{nil}))) \\
\text{reflect}_{\Delta, \Gamma} n & = \lambda (\varepsilon : \Gamma \leq \Gamma'). \lambda (e : \Gamma' \vdash_{\lambda_{\text{IS4}}} \Delta). \text{reflect}_{A,\Lambda} (\text{unbox} (wk(\text{lock} \text{nil})))
\end{align*}
\]

**Theorem 7.** Let \( M \) denote the possible-world model over the reflexive and transitive frame given by the relations \( \Gamma \leq \Gamma' \) and \( \Delta \vdash_{\lambda_{\text{IS4}}} \Gamma \) and the valuation \( V_{\text{NP}} = \Gamma \vdash_{\text{NP}} \).

There is a function \( \text{quote} : M (\Gamma, \langle A \rangle) \rightarrow \Gamma \vdash_{\text{NP}} A \) such that the composite \( \text{norm} = \text{quote} \circ \text{quote} \) is complete and adequate.

The proof of this theorem requires us to identify terms by extending the equational theory of \( \lambda_{\text{IS4}} \) with an additional rule. To understand the need for it, consider unboxing a term \( \Gamma \vdash t : \Box A \) into an extended context \( \Gamma, B \) in \( \lambda_{\text{IS4}} \). We may first weaken \( t \) as \( \Gamma, B \vdash wk(\text{drop id}_\Sigma) t : \Box A \) and then apply unbox as \( \Gamma, B \vdash \text{unbox} (wk(\text{lock} \text{nil}))) \) \( n \). However, we may also apply...
unbox on $t$ as $\Gamma, B \vdash \unbox t$ (var nil) : $A$. This weakens the term “explicitly” in the sense that the weakening with $B$ is recorded in the term by the proof var nil of the modal accessibility relation $\Gamma \triangleleft_{\lambda_{IS4}} \Gamma, B$. The two ways of unboxing $\Gamma \vdash t : \Box A$ into the extended context $\Gamma, B$ result in two terms with the same denotation in the possible-world semantics but distinct typing derivations. We wish the two typing derivations unbox $t$ (var nil) and unbox $(\mathit{wk} (\mathit{drop id}_e) t) \text{nil}$ to be identified. For this reason, we extend the equational theory of $\lambda_{IS4}$ with the rule unbox $t (\mathit{trans}_m e e') \sim$ unbox $(\mathit{wk} (\mathit{toOPE}e) t) e'$ for any lock-free extension $e$, which can be converted to a sequence of drops using the function toOPE. Explicit weakening can also be avoided by, instead of extending the equational theory, changing the definition of the modal accessibility relation such that $\Delta \triangleleft_{\lambda_{IS4}} \Gamma$ holds only if $\Gamma = \Delta$ or $\Gamma = \Delta, \mathbb{U}, \Gamma'$ for some $\Gamma'$. Note that the modal accessibility relation for $\lambda_{IK}$, where the issue of explicit weakening does not occur, satisfies this property.

### 3.3 Extending to the Calculi $\lambda_{IT}$ and $\lambda_{IK4}$

The NbE model construction for $\lambda_{IT}$ and $\lambda_{IK4}$ follows a similar pursuit as $\lambda_{IS4}$. We define suitable modal accessibility relations $\triangleleft_{\lambda_{IT}}$ and $\triangleleft_{\lambda_{IK4}}$ as extensions that allow the addition of at most one $\mathbb{U}$, and at least one lock $\mathbb{A}$, respectively. To give possible-world semantics, we require an additional frame condition that the relation $R_m$ be reflexive for $\lambda_{IT}$ and transitive for $\lambda_{IK4}$. For evaluation, we use a function $\mathit{trim}_{\lambda_{IT}} : \mathbb{G} \rightarrow \mathbb{A} \triangleleft_{\lambda_{IT}} \Gamma \rightarrow \mathbb{A} \mathbb{U} \mathbb{G} \mathbb{A}$ for $\lambda_{IT}$, and similarly $\mathit{trim}_{\lambda_{IK4}}$ for $\lambda_{IK4}$. The modification to the neutral rule $\lambda_{IK}/\mathit{NE/\Box-\mathbf{ELIM}}$ is achieved as before in $\lambda_{IS4}$ using the corresponding modal accessibility relations. Unsurprisingly, reification and reflection can also be implemented, thus yielding normalization functions for both $\lambda_{IT}$ and $\lambda_{IK4}$.

### 4 COMPLETENESS, DECIDABILITY AND LOGICAL APPLICATIONS

In this section we record some immediate consequences of the model constructions we presented in the previous section.

**Completeness of the Equational Theory.** As a corollary of the adequacy of an NbE model $\mathcal{N}$, i.e. $\Gamma \vdash t \sim u : A$ whenever $\llbracket t \rrbracket = \llbracket u \rrbracket : \mathcal{N}(\llbracket \Gamma \rrbracket, \llbracket A \rrbracket)$, we obtain completeness of the equational theory with respect to the class of models that the respective NbE model belongs to. Given the NbE models constructed in Subsections 3.1.3 and 3.2.3 this means that the equational theories of $\lambda_{IK}$ and $\lambda_{IS4}$ (cf. Fig. 7) are (sound and) complete with respect to the class of Cartesian closed categories equipped with an adjunction and a right-adjoint comonad, respectively.

**Theorem 8.** Let $t, u : \Gamma \vdash A$ be two terms of $\lambda_{IK}$. If for all Cartesian closed categories $\mathcal{M}$ equipped with an adjunction it is the case that $\llbracket t \rrbracket = \llbracket u \rrbracket : \mathcal{M}(\llbracket \Gamma \rrbracket, \llbracket A \rrbracket)$ then $\Gamma \vdash t \sim u : A$.

**Proof.** Let $\mathcal{M}_0$ be the model we constructed in Subsection 3.1.3. Since $\mathcal{M}_0$ is a Cartesian closed category equipped with an adjunction, by assumption we have $\llbracket t \rrbracket_{\mathcal{M}_0} = \llbracket u \rrbracket_{\mathcal{M}_0}$. And lastly, since $\mathcal{M}_0$ is an NbE model, we have $\Gamma \vdash t \sim \mathit{quote} \llbracket t \rrbracket_{\mathcal{M}_0} = \mathit{quote} \llbracket u \rrbracket_{\mathcal{M}_0} \sim u : A$. □

Note that this statement corresponds to Clouston [2018, Theorem 3.2] but there it is obtained via a term model construction and for the term model to be equipped with an adjunction the calculus needs to be first extended with an internalization of the operation $\mathbb{U}$ on contexts as an operation $\mathbb{P}$ on types.

**Theorem 9.** Let $t, u : \Gamma \vdash A$ be two terms of $\lambda_{IS4}$. If for all Cartesian closed categories $\mathcal{M}$ equipped with a right-adjoint comonad it is the case that $\llbracket t \rrbracket = \llbracket u \rrbracket : \mathcal{M}(\llbracket \Gamma \rrbracket, \llbracket A \rrbracket)$ then $\Gamma \vdash t \sim u : A$.

**Proof.** As for **Theorem 8**. □
This statement corresponds to Clouston [2018, Section 4.4] but there it is proved for an equational theory that identifies terms up to differences in the accessibility proofs and with respect to the class of models where the comonad is idempotent, to which the model of Subsection 3.2.3 does not belong.

Completeness of the Deductive Theory. Using the quotation function of an NbE model \( N \), i.e. \( \text{quote} : N([\Gamma], [A]) \to \Gamma \vdash A \), we obtain completeness of the deductive theory with respect to the class of models that the respective NbE model belongs to. Given the NbE models constructed in Subsections 3.1.3 and 3.2.3 this means that the deductive theories of \( \lambda_{\text{IK}} \) and \( \lambda_{\text{IS4}} \) (cf. Figs. 2 and 5) are (sound and) complete with respect to the class of possible-world models with an arbitrary frame and a reflexive–transitive frame, respectively.

**Theorem 10.** Let \( \Gamma : \text{Ctx} \) be a context and \( A : \text{Ty} \) a type. If for all possible-world models \( M \) it is the case that \( M([\Gamma], [A]) \) is inhabited then there is a term \( t : \Gamma \vdash A \) of \( \lambda_{\text{IK}} \).

**Proof.** Let \( M_0 \) be the model we constructed in Subsection 3.1.3. Since \( M_0 \) is a possible-world model, by assumption we have a morphism \( p : M_0([\Gamma], [A]) \). And lastly, since \( M_0 \) is an NbE model, we have the term \( \text{quote} p : \Gamma \vdash A \).

**Theorem 11.** Let \( \Gamma : \text{Ctx} \) be a context and \( A : \text{Ty} \) a type. If for all possible-world models \( M \) with a reflexive–transitive frame it is the case that \( M([\Gamma], [A]) \) is inhabited then there is a term \( t : \Gamma \vdash A \) of \( \lambda_{\text{IS4}} \).

**Proof.** As for Theorem 10.

Note that the proofs of Theorems 10 and 11 are constructive.

Decidability of the Equational Theory. As a corollary of the completeness and adequacy of an NbE model \( N \), i.e. \( \Gamma \vdash t \sim u : A \) if and only if \( [t] = [u] : N([\Gamma], [A]) \), we obtain decidability of the equational theory from decidability of the equality of normal forms \( n, m : \Gamma \vdash_{\text{nf}} A \). Given the NbE models constructed in Subsections 3.1.3 and 3.2.3 this means that the equational theories of \( \lambda_{\text{IK}} \) and \( \lambda_{\text{IS4}} \) (cf. Fig. 7) are decidable.

To show that any of the following decision problems \( P(x) \) is decidable we give a constructive proof of the proposition \( \forall x. P(x) \lor \neg P(x) \). Such a proof can be understood as the construction of an algorithm \( d \) that takes as input an \( x \) and produces as output a Boolean \( d(x) \), alongside a correctness proof that \( d(x) \) is true if and only if \( P(x) \) holds.

**Theorem 12.** For any two terms \( t, u : \Gamma \vdash A \) of \( \lambda_{\text{IK}} \) the problem whether \( t \sim u \) is decidable.

**Proof.** We first observe that for any two normal forms \( n, m : \Gamma \vdash_{\text{nf}} A \) of \( \lambda_{\text{IK}} \) the problem whether \( n = m \) is decidable by proving \( \forall n, m. n = m \lor n \neq m \) constructively. All the cases of an simultaneous induction on \( n, m : \Gamma \vdash_{\text{nf}} A \) are immediate.

Let \( N \) be the NbE model we constructed in Subsection 3.1.3. Completeness and adequacy of \( N \) imply that we have \( t \sim u \) if and only if \( \text{norm} t = \text{norm} u \) for the function \( \text{norm} : \Gamma \vdash A \to \Gamma \vdash_{\text{nf}} A, t \mapsto \text{quote} [t] \). Now, \( t \sim u \) is decidable because \( \text{norm} t = \text{norm} u \) is decidable by the observation we started with.

**Theorem 13.** For any two terms \( t, u : \Gamma \vdash A \) of \( \lambda_{\text{IS4}} \) the problem whether \( t \sim u \) is decidable.

**Proof.** As for Theorem 12.
**Denecessitation.** The last of the consequences of the NbE model constructions we record is of a less generic flavour than the other three, namely it is an application of normal forms to a basic proof-theoretic result in modal logic.

Using invariance of truth in possible-world models under bisimulation\(^2\) it can be shown that \(\Box A\) is a valid formula of IK (or IS4) if and only if \(A\) is. A completeness theorem then implies the same for provability of \(\Box A\) and \(A\). The statement for proofs in \(\lambda_{IK}\) (and \(\lambda_{IS4}\)) can also be shown by inspection of normal forms as follows.

Firstly, we note that while deduction is not closed under arbitrary context extensions (including locks) it is closed under extensions (including locks) on the left:

**Lemma 14** (cf. Clouston [2018, Lemma A.1]). Let \(\Delta, \Gamma : Ctx\) be arbitrary contexts, both possibly containing locks, and \(A : Ty\) an arbitrary type. There is an operation \(\Gamma \vdash A \rightarrow \Delta, \Gamma \vdash A\) on terms of \(\lambda_{IK}\) (and \(\lambda_{IS4}\)), where \(\Delta, \Gamma\) denotes context concatenation.

**Proof.** By recursion on terms.

And, secondly, we note that also a converse of this lemma holds by inspection of normal forms:

**Lemma 15.** Let \(\Delta, \Gamma : Ctx\) be arbitrary contexts, both possibly containing locks, \(A : Ty\) an arbitrary type and \(t : \Delta, \Gamma \vdash A\) a term of \(\lambda_{IK}\) (or \(\lambda_{IS4}\)) in the concatenated context \(\Delta, \Gamma\) that does not mention any variables from \(\Delta\), then there is a term \(t' : \Gamma \vdash A\) of \(\lambda_{IK}\) (or \(\lambda_{IS4}\), respectively).

**Proof.** Since normalization (see Theorems 4 and 7) does not introduce new free variables it suffices to prove the statement for terms in normal form. We do so by induction on normal forms \(n : \Delta, \Gamma \vdash \text{nf}\ A\) (see Fig. 8). The only nonimmediate step is for \(n\) of the form unbox \(n' e\) for some neutral element \(n' : \Delta' \vdash \Box A\) and \(\Delta' < \Delta \leq \Delta, \Gamma\). But in that case the induction hypothesis says that we have a neutral element \(n'' : \Gamma \vdash \Box A\), which is impossible.

Note that some form of normalization seems to be needed in the proof of Lemma 15. More specifically, the “strengthening” of a term of the form unbox \(t e\) from the context \(\cdot, \Box, \cdot\) to the empty context \(\cdot\) cannot possibly result in a term of the form unbox \(t' e'\) because there is no context \(\Gamma\) such that \(\Gamma < \cdot\) in \(\lambda_{IK}\). As an example, consider the term unbox (box (\(\lambda\ x\ x\))) nil, which needs to be strengthened to \(\lambda\ x. x\).

With these two lemmas at hand we are ready to prove denecessitation through normalization:

**Theorem 16.** Let \(A : Ty\) be an arbitrary type. There is a term \(t : \cdot \vdash A\) of \(\lambda_{IK}\) (or \(\lambda_{IS4}\)) if and only if there is a term \(u : \cdot \vdash \Box A\) of \(\lambda_{IK}\) (or \(\lambda_{IS4}\), respectively), where \(\cdot : Ctx\) denotes the empty context.

**Proof.** From a term \(t : \cdot \vdash A\) we can construct a term \(t' : \cdot, \Box \vdash A\) using Lemma 14 and thus the term \(u = \text{box} t' : \cdot \vdash \Box A\).

In the other direction, from a term \(u : \cdot \vdash \Box A\) we obtain a normal form \(u' = \text{norm} u : \cdot \vdash \text{nf} \Box A\) using Theorems 4 and 7. By inspection of normal forms (see Fig. 8) we know that \(u'\) must be of the form box \(v\) for some normal form \(v : \cdot, \Box \vdash \text{nf} A\), from which we obtain a term \(t : \cdot \vdash A\) using Lemma 15 since the context \(\cdot, \Box\) does not declare any variables that could have been mentioned in \(v\).

This concludes this section on some consequences of the model constructions presented in this paper. Note that the consequences we recorded are completely independent of the concrete model construction. To wit, the two completeness theorems follow from the mere existence of an NbE

\(^2\)Invariance of truth under bisimulation says that if \(w\) and \(v\) are two bisimilar worlds in two possible-world models \(M_0\) and \(M_1\), respectively, then for all formulas \(A\) it is the case that \(\llbracket A \rrbracket_w\) holds in \(M_0\) if and only if \(\llbracket A \rrbracket_v\) does in \(M_1\).
Normalization for Fitch-Style Modal Calculi

Theorem 10.2: In a Fitch-style modal calculus, every derivation can be transformed into a normal form by a finitary sequence of normalization steps. This theorem is a consequence of the fact that every derivation can be extended to a derivation in the full calculus with application-specific primitives, and that such derivations can be factored into normal forms.

5 PROGRAMMING-LANGUAGE APPLICATIONS

In this section, we discuss some implications of normalization for Fitch-style calculi for specific interpretations of the necessity modality in the context of programming languages. In particular, we show how normalization can be used to prove properties about program calculi by leveraging the shape of normal forms of terms. We extend the term calculi presented earlier with application-specific primitives, ensure that the extended calculi are in fact normalizing, and then use this result to prove properties such as capability safety, noninterference, and binding-time correctness.

Note that we do not mechanize these results in Agda and do not prove these properties in their full generality, but only illustrate special cases. Although possible, proving the general properties requires further technical development that obscures the main idea underlying the use of normal forms for simplifying these proofs.

5.1 Capability Safety

Choudhury and Krishnaswami [2020] present a modal type system based on IS4 for a programming language with implicit effects in the style of ML [Milner et al. 1990] and the computational lambda calculus [Moggi 1989]. In this language, programs need access to capabilities to perform effects. For instance, a primitive for printing a string requires a capability as an argument in addition to the string to be printed. Crucially, capabilities cannot be introduced within the language, and must be obtained either from the global context (called ambient capabilities) or as a function argument.

Let us denote the type of capabilities by \( \text{Cap} \). Passing a printing capability \( c \) to a function of type \( \text{Cap} \Rightarrow \text{Unit} \) in a language that uses capabilities to print yields a program that either (1) does not print, (2) prints using only the capability \( c \), or (3) prints using ambient capabilities (and possibly \( c \)). A program that at most uses the capabilities that it is passed explicitly, as in the cases 1 and 2, is said to be capability safe. To identify such programs, Choudhury and Krishnaswami [2020] introduce a comonadic modality \( \Box \) to capture capability safety. Their type system is loosely based on the dual-context calculus for IS4 [Kavvos 2020; Pfenning and Davies 2001]. A term of type \( \Box A \) is enforced to be capability safe by making the introduction rule for \( \Box \) “brutally” remove all capabilities from the typing context. As a result, programs with the type \( \Box (\text{Cap} \Rightarrow \text{Unit}) \) are denied ambient capabilities and thus guaranteed to behave like the cases 1 and 2.

Choudhury and Krishnaswami [2020] characterize capability safety precisely using their capability space model. A capability space \( (X, w_X) \) is a set \( X \) and a weight relation \( w_X \) that assigns sets of capabilities to every member in \( X \). In this model, they define a comonad that restricts the underlying set of a capability space to those elements that are only related to the empty set of capabilities. This comonad has a left adjoint that replaces the weight relation of a capability space by the relation that relates every element to the empty set of capabilities. This adjunction suggests that capability spaces are a model of \( \lambda_{\text{IS4}} \) and we may thus use \( \lambda_{\text{IS4}} \) to write programs that support reasoning about capability safety.

In this subsection, we present a calculus \( \lambda_{\text{IS4}} + \text{Moggi}^{\text{Cap}} \) that extends \( \lambda_{\text{IS4}} \) with a capability type and a monad for printing effects. We extend the normalization algorithm for \( \lambda_{\text{IS4}} \) to \( \lambda_{\text{IS4}} + \text{Moggi}^{\text{Cap}} \) and show that the resulting normal forms can be used to prove a kind of capability safety. In contrast to the language presented by Choudhury and Krishnaswami [2020], \( \lambda_{\text{IS4}} + \text{Moggi}^{\text{Cap}} \) models a language where effects are explicit in the type of a term. Languages with explicit effects, such as Haskell [Augustsson et al. 1990] (with the IO monad) or PureScript [Freeman 2013] (with the Effect monad), can also benefit from a mechanism for capability safety, and we begin with an example in a hypothetical extension of PureScript to illustrate this.
Example in PureScript. Let us consider web development in PureScript. A web application may consist of a mashup of several components, e.g. social media, news feed, or chat, provided by untrusted sources. A component is a function of type 

\[
\text{type Component} = \text{Element} \rightarrow \text{Effect Unit}
\]

that takes as a parameter the DOM element where the component will be rendered. For the correct functioning of the web application, it is important that components do not interfere with each other in malicious ways. For example, a malicious component (of Bob) could illegitimately overwrite a DOM element (of Alice):

```haskell
evilBob :: Component
evilBob e = do
  w <- window
  doc <- document w
  aliceE <- getElementById "alice.app" doc
  setTextContent "Alice has been hacked!" aliceE
```

The issue here is that Bob has unrestricted access to the function \(\text{window} :: \text{Effect Window}\), and is able to obtain the DOM using \(\text{document} :: \text{Window} \rightarrow \text{Effect DOM}\) and overwrite an element that belongs to Alice. Capabilities can be leveraged to restrict the access to \(\text{window}\). We can achieve this by extending PureScript with a type \(\text{WindowCap}\), a type constructor \(\square\) that works similarly to Choudhury and Krishnaswami’s \(\square\), and replacing the function \(\text{window}\) with a function \(\text{window}' :: \text{WindowCap} \rightarrow \text{Effect Window}\) that requires an additional capability argument. By making \(\text{WindowCap}\) an ambient capability that is available globally, all existing programs retain their unrestricted access to retrieve a window as before. The difference now, however, is that we can selectively restrict some programs and limit their access to \(\text{WindowCap}\) using \(\square\). We can define a variant of the type Component as:

\[
\text{type Component}' = \square (\text{Element} \rightarrow \text{Effect Unit})
\]

By requiring Bob to write a component of the type \(\text{Component}'\), we are ensured that Bob cannot overwrite an element that belongs to Alice. This is because the \(\square\) type constructor used to define Component' disallows access to all ambient capabilities (including WindowCap), and thus restricts Bob to only using the given \(\text{Element}\) argument. In particular, the program evilBob cannot be reproduced with the type Component' since the substitute function \(\text{window}'\) requires a capability that is neither available as an argument nor as an ambient capability.
We say that a program \( \Gamma \). The interpretation of the other primitive types also follows a standard pursuit [Valliappan, Russo, · form For any context \( \Delta \), a \( t : T A \) \( \Gamma \vdash t \sim t \) \( (\text{return} (\text{var zero})) \)

\[
\begin{align*}
T-\beta & \quad \Gamma \vdash t : A \quad \Gamma, A \vdash u : T B \\
& \quad \Gamma \vdash \text{let} (\text{return} t) u \sim \text{subst} (\text{ext id}_t) u
\end{align*}
\]

\[
\begin{align*}
T-\gamma & \quad \Gamma \vdash t_1 : A \quad \Gamma, A \vdash t_2 : T B \\
& \quad \Gamma, B \vdash t_3 : T C \\
& \quad \Gamma \vdash \text{let} (\text{let} t_1 t_2) t_3 \sim \text{let} t_1 (\text{let} t_2 (\text{wk} (\text{keep} (\text{drop id}_t)) t_3))
\end{align*}
\]

Fig. 14. Equational theory for \( \lambda_{\text{IS4}} + \text{Moggi}^\text{Cap} \) (omitting the unchanged rules of Figs. 7 and 12)

\[
\begin{align*}
\text{Nf/T-Intro} & \quad \Gamma \vdash_m n : A \\
& \quad \Gamma \vdash \text{return } m : T A
\end{align*}
\]

\[
\begin{align*}
\text{Nf/T-Elim} & \quad \Gamma \vdash_n n : T A \\
& \quad \Gamma, A \vdash_m n : T B \\
& \quad \Gamma \vdash \text{let } m n : T B
\end{align*}
\]

\[
\begin{align*}
\text{Nf/Unit-Intro} & \quad \Gamma \vdash_n \text{Unit} : \text{Unit} \\
& \quad \Gamma \vdash \text{up } n : \text{Cap}
\end{align*}
\]

\[
\begin{align*}
\text{Nf/Up-Cap} & \quad \Gamma \vdash_m n : \text{Cap} \\
& \quad \Gamma \vdash \text{up } n : \text{String}
\end{align*}
\]

\[
\begin{align*}
\text{Nf/Up-String} & \quad \Gamma \vdash_m n : \text{String} \\
& \quad \Gamma \vdash \text{up } n : \text{String}
\end{align*}
\]

\[
\begin{align*}
\text{Nf/String-Lit} & \quad \Gamma \vdash_m \text{str}_s : \text{String} \\
& \quad s \in \text{String}
\end{align*}
\]

\[
\begin{align*}
\text{Nf/T-Print} & \quad \Gamma \vdash_m c : \text{Cap} \\
& \quad \Gamma \vdash_s \text{String} \\
& \quad \Gamma \vdash \text{Unit} m : T A \\
& \quad \Gamma \vdash \text{let } (\text{print } \text{c} \text{s}) m : T A
\end{align*}
\]

Fig. 15. Normal forms of \( \lambda_{\text{IS4}} + \text{Moggi}^\text{Cap} \) (omitting the unchanged normal forms of \( \lambda_{\text{IS4}} \))

**Extension with a Capability and a Monad.** We extend \( \lambda_{\text{IS4}} \) with a monad for printing based on Moggi’s monadic metalanguage [Moggi 1991]. We introduce a type \( T A \) that denotes a monadic computation that can print before returning a value of type \( A \), a type \( \text{Cap} \) for capabilities, and a type \( \text{String} \) for strings. Fig. 13 summarizes the terms that correspond to this extension. The term construct print is used for printing. The equational theory of \( \lambda_{\text{IS4}} + \text{Moggi}^\text{Cap} \) and the corresponding normal forms are summarized in Fig. 14 and Fig. 15, respectively.

To extend the NbE model of \( \lambda_{\text{IS4}} \) with an interpretation for the monad, we use the standard techniques used for normalizing computational effects [Ahman and Staton 2013; Filinski 2001]. The interpretation of the other primitive types also follows a standard pursuit [Valliappan, Russo, et al. 2021]: we interpret \( \text{Cap} \) by neutrals of type \( \text{Cap} \) and \( \text{String} \) by the disjoint union of \( \text{String} \) and neutrals of type \( \text{String} \). The difference in their interpretation is caused by the fact that there is no introduction form for the type \( \text{Cap} \).

**Proving Capability Safety.** Programs that lack access to capabilities are necessarily capability safe. We say that a program \( \Gamma \vdash p : A \) is trivially capability safe if there is a program \( \Delta \vdash p' : A \) such that \( \Gamma \vdash p \sim \text{leftConcat}_c p' : A \), where \( \text{leftConcat}_c : \forall A, A \vdash A \rightarrow \Gamma, A \vdash A \) can be defined similarly to the operation given by Lemma 14 for \( \lambda_{\text{IS4}} \). First, we prove an auxiliary lemma about normal forms with a capability in context.

**Lemma 17.** For any context \( \Gamma \), type \( A \) and normal form \( c : \text{Cap}, \Gamma, \Delta \vdash_m n : A \) there is a normal form \( \Delta, \Gamma \vdash_m n' : A \) such that \( n = \text{leftConcat}_c n' \).
Proof. We prove the statement for both normal forms and neutral elements by mutual induction. The only nonimmediate case is when the neutral is of the form \( c : \text{Cap}, \Box, \Gamma \vdash_{\text{NE}} \text{unbox} \ n \ e : A \) for some \( n : \Delta \vdash_{\text{NE}} \Box A \) and \( e : \Delta \vdash_{\lambda_{\text{IS4}}} c : \text{Cap}, \Box, \Gamma \). We observe that there are no neutral elements of type \( \Box A \) in context \( c : \text{Cap} \) and that hence \( \Delta \) must contain the leftmost lock in \( c : \text{Cap}, \Box, \Gamma \). Thus, this case also holds by induction hypothesis. \( \square \)

Now, we observe that all terms \( c : \text{Cap} \vdash t : \Box A \) are trivially capability safe. By normalization, we have that \( c : \text{Cap} \vdash t \sim_{\text{norm}} \Box A \). Given the definition of normal forms of \( \lambda_{\text{IS4}} + \text{Moggi}^{\text{Cap}} \), \( \text{norm} \) must be box \( n \) for some normal form \( c : \text{Cap}, \Box, \Gamma \vdash_{\text{NF}} n : A \). By Lemma 17, there is a normal form \( \Box, \vdash_{\text{nf}} n' : A \) such that \( n = \text{leftConcat} \cdot \text{Cap} \cdot n' \). Since the operation \( \text{leftConcat} \) commutes with box, i.e. \( \text{leftConcat} \cdot \text{Cap} (\text{box} n') = \text{box} (\text{leftConcat} \cdot \text{Cap} n') \), we also have that \( t \sim \text{box} n = \text{leftConcat} \cdot \text{Cap} (\text{box} n') \). As a result, \( t \) must be trivially capability safe.

A consequence of this observation is that any term \( c : \text{Cap} \vdash t : \Box (T \text{Unit}) \) is trivially capability safe. This means that \( t \) does not print since it could not possibly do so without a capability. Going further, we can also observe that \( t \sim \text{box} \) (return unit) : \( \Box (T \text{Unit}) \), since the only normal form of type \( T \text{Unit} \) in the empty context is \( \cdot \vdash_{\text{nf}} \text{return} \text{unit} : T \text{Unit} \). Note that this argument (and the one above) readily adapts to a vector of capabilities \( \vec{c} \) in context as opposed to a single capability \( c \).

### 5.2 Information-Flow Control

Information-flow control (IFC) [Sabelfeld and Myers 2003] is a technique used to protect the confidentiality of data in a program by tracking the flow of information within the program.

In type-based static IFC [e.g. Abadi et al. 1999; Russo et al. 2008; Shikuma and Igarashi 2008] types are used to associate values with confidentiality levels such as secret or public. The type system ensures that secret inputs do not interfere with public outputs, enforcing a security policy that is typically formalized as a kind of noninterference property [Goguen and Meseguer 1982].

Noninterference is proved by reasoning about the semantic behaviour of a program. Tomé Cortiñas and Valliappan [2019] present a proof technique that uses normalization for showing noninterference for a static IFC calculus based on Moggi’s monadic metalanguage [Moggi 1991]. This technique exploits the insight that normal forms represent equivalence classes of terms identified by their semantics, and thus reasoning about normal forms of terms (as opposed to terms themselves) vastly reduces the set of programs that we must take into consideration. Having developed normalization for Fitch-style calculi, we can leverage this technique to prove noninterference.

In this subsection, we extend \( \lambda_{\text{IK}} \) with Booleans (denoted \( \lambda_{\text{IK}} + \text{Bool} \)), extend the NbE model of \( \lambda_{\text{IK}} \) to \( \lambda_{\text{IK}} + \text{Bool} \), and illustrate the technique of Tomé Cortiñas and Valliappan on \( \lambda_{\text{IK}} + \text{Bool} \) for proving noninterference. We interpret the type \( \Box A \) as a secret of type \( A \), and other types as public.

**Extension with Booleans.** Noninterference can be better appreciated in the presence of a type whose values are distinguishable by an external observer. To this extent, we extend \( \lambda_{\text{IK}} \) with a type \( \text{Bool} \) and corresponding introduction and elimination forms—as described in Fig. 16.

```
| Ty   | A, B ::= \ldots | Bool |
|------|----------------|------|
| \text{Bool-Intro-true} | \text{Bool-Intro-false} | \text{Bool-Elim} |
| \Gamma \vdash \text{true} : \text{Bool} | \Gamma \vdash \text{false} : \text{Bool} | \Gamma, \Gamma' \vdash b : \text{Bool} |

\text{Gamma-Elim} \quad \Gamma, \Gamma' \vdash t_1 : A \quad \Gamma, \Gamma' \vdash t_2 : A \\
\Gamma, \Gamma' \vdash \text{ifte} b t_1 t_2 : A
```

Fig. 16. Types, contexts and intrinsically-typed terms of \( \lambda_{\text{IK}} + \text{Bool} \) (omitting the unchanged rules of Fig. 5)
We modify the usual elimination rule for Bool by allowing the context of the conclusion ifte b t₁ t₂ and branches t₁ and t₂ in the rule Bool-ELIM to extend the context of the scrutinee b. This modification (following Clouston [2018, Fig. 2]) enables the following commuting conversion, which is required to ensure that terms can be fully normalized and normal forms enjoy the subformula property:

\[
\begin{array}{c}
\Delta \vdash b : \text{Bool} \\
\Delta, \Delta' \vdash t_1 : \Box A \\
\Delta, \Delta' \vdash t_2 : \Box A \\
e : \Delta, \Delta' < \Gamma \\
\Gamma \vdash \text{unbox (ifte } b \ t_1 \ t_2) \ e \sim \text{ifte } b \ (\text{unbox } t_1 \ e) \ (\text{unbox } t_2 \ e)
\end{array}
\]

A commuting conversion is required as usual for every other elimination rule, including the rule ⇒-ELIM. These are however standard and thus omitted here.

We extend the equational theory of λIK to λIK+Bool by adding the usual rules ifte true t₁ t₂ ~ t₁, ifte false t₁ t₂ ~ t₂, and t ~ ifte t true false for terms t of type Bool. The normal forms of λIK+Bool include those of λIK in addition to the following.

\[
\begin{array}{c|c}
\text{NF/Intro-true} & \text{NF/Intro-false} \\
\Gamma \vdash_{\text{NF}} \text{true} : \text{Bool} & \Gamma \vdash_{\text{NF}} \text{false} : \text{Bool} \\
\hline
\text{NF/Bool-ELIM} & \text{NF/Bool-ELIM} \\
\Gamma \vdash_{\text{NF}} n : \text{Bool} & \Gamma, \Gamma' \vdash_{\text{NF}} m_1 : A \\
\Gamma, \Gamma' \vdash_{\text{NF}} m_2 : A \\
\Rightarrow \\
\Gamma, \Gamma' \vdash_{\text{NF}} \text{ifte } n \ m_1 \ m_2 : A
\end{array}
\]

Observe that a neutral of type Bool is not immediately in normal form, and must be expanded as ifte n true false. This is unlike neutrals of the type i, which are in normal form by Rule NF/Up. To extend the NbE model of λIK with Booleans, we leverage the interpretation of sum types used by Abel and Sattler [2019], who attribute their idea to Altenkirch and Uustalu [2004]. This interpretation readily supports commuting conversions, and a minor refinement that reflects the change to the rule Bool-ELIM yields a reifiable interpretation for Booleans in λIK+Bool.

Proving Noninterference. A program · ⊢ f : A ⇒ Bool is noninterferent if it is the case that · ⊢ app f s₁ ~ app f s₂ : Bool for any two secrets · ⊢ s₁, s₂ : A. By instantiating A to Bool, we can show that any program · ⊢ f : □ Bool ⇒ Bool is noninterferent and thus cannot leak a secret Boolean argument. In λIK+Bool, the type system ensures that data of type □ A type can only influence (or flow to) data of type □ B, thus all programs of type □ Bool ⇒ Bool must be noninterferent. To show this, we analyze the possible normal forms of f and observe that they must be equivalent to a constant function, such as λ x. true or λ x. false, which evidently does not use its input argument x and is thus noninterferent.

In detail, normal forms of type □ Bool ⇒ Bool must have the shape λ x. m, for some normal form ·, □ Bool ⊢ nf m : Bool. If m is either true or false, then λ x. m must be a constant function and we are done. Otherwise, it must be some normal form ·, □ Bool ⊢ nf ifte n m₁ m₂ : Bool with a neutral n : Bool either in context · or in context ·, □ Bool. Such a neutral could either be of shape unbox n’ or app n”’ m’ for some neutrals n’ and n”. However, this is impossible, since the context of the neutral unbox n’ must contain a lock, and neither the context · nor the context ·, □ Bool do. The existence of n”’ can also be similarly dismissed by appealing to the definition of neutrals.

Discussion. Observe that not all Fitch-style calculi are well-suited for interpreting the type □ A as a secret, because noninterference might not hold. In λIS4, the term λ x. unbox x : □ A ⇒ A (axiom T) is well-typed but leaks the secret x, thus breaking noninterference. The validity of the interpretation of □ A as a secret depends on the calculus under consideration and the axioms it exhibits.

5.3 Partial Evaluation

Davies and Pfenning [1996, 2001] present a modal type system for staged computation based on IS4. In their system, the type □ A represents code of type A that is to be executed at a later stage, and the axioms of IS4 correspond to operations that manipulate code. The axiom K : □ (A ⇒ B) ⇒ (□ A ⇒ □ B) corresponds to substituting code in code, T : □ A ⇒ A to evaluating code, and 4 : □ A ⇒ □ □ A to
further delaying the execution of code to a subsequent stage. A desired property of this type system is that code must only depend on code, and thus the term $\lambda x : A. \text{box} \ x$ must be ill-typed.

Although $\lambda_{\text{IS4}}$ exhibits the desired properties of a type system for staging, its equational theory in Fig. 12 does not reflect the semantics of staged computation. For example, the result of normalizing the term $\text{box} \ (2 * \text{unbox} \ (\text{box} \ 3))$ in $\lambda_{\text{IS4}}$ extended with natural number literals and multiplication is $\text{box} \ 6$. While the result expected from reducing it in accordance with Davies and Pfenning’s operational semantics is box $(2 * 3)$. The equational theory of Fitch-style calculi in general do not take into account the occurrence of a term (such as the literal $3$) under box, while this is crucial for Davies and Pfenning’s semantics. We return to this discussion at the end of this subsection.

If we restrict our attention to a special case of staged computation in partial evaluation [Jones et al. 1993], however, the semantics of Fitch-style calculi are better suited. In the context of partial evaluation, the type $\square A$ represents a dynamic computation of type $A$ that must be executed at runtime, and other types represent static computations. Static and dynamic are also known as binding-time annotations, and they are used by a partial evaluator to evaluate all static computations.

In the term $\text{box} \ (2 * \text{unbox} \ (\text{box} \ 3))$, we consider the literal $3$ to be annotated as dynamic since it occurs under box. The construct unbox strips this annotation and brings it back to static. The multiplication of static subterms $2$ and $\text{unbox} \ (\text{box} \ 3)$ is however considered annotated dynamic since it itself occurs under box. As a result, a partial evaluator that respects these annotations does not perform the multiplication and specializes the term to box $(2 * 3)$—which matches the result of evaluating with Davies and Pfenning’s staging semantics. Observe that the same partial evaluator would specialize the expression $2 * \text{unbox} \ (\text{box} \ 3)$ to $6$ since the multiplication does not occur under box and is thus considered to be annotated static.

The goal of a partial evaluator is to optimize runtime execution of a program by eagerly evaluating as many static computations as possible and yielding an optimal dynamic program. The term $\text{box} \ 6$ is more optimized than the term box $(2 * 3)$ since the evidently static multiplication has also been evaluated. Normalization in a Fitch-style calculus yields the former result, and the gain in optimality can be seen as a form of binding-time improvement [Jones et al. 1993] that is performed automatically during normalization.

In this subsection, we extend $\lambda_{\text{IK}}$ with natural number literals and multiplication (denoted $\lambda_{\text{IK}}+N$), and extend the NbE model of $\lambda_{\text{IK}}$ to $\lambda_{\text{IK}}+N$. We use $\lambda_{\text{IK}}$ as the base calculus since the other axioms are not needed in the context of partial evaluation [Davies and Pfenning 1996, 2001]. The resulting normalization function yields an optimal partial evaluator for $\lambda_{\text{IK}}+N$. In partial evaluation, as with staging in general, we desire that a term $\lambda x : N. \text{box} \ x$ be disallowed, since a runtime execution of a dynamic computation must not have a static dependency. While this term is already ill-typed in $\lambda_{\text{IK}}+N$, we prove a kind of binding-time correctness property for $\lambda_{\text{IK}}+N$ that implies that no term equivalent to $\lambda x : N. \text{box} \ x$ can exist.

**Extension with Natural Number Literals and Multiplication.** We extend $\lambda_{\text{IK}}$ with a type $N$, a construct lift for including natural number literals, and an operation $*$ for multiplying terms of type $N$—as described in Fig. 17.

\[
\begin{align*}
\text{Ty} & \quad A, B ::= \ldots \mid N \\
\text{N-Lift} & \quad \frac{\Gamma \vdash \text{lift} \ k : N}{\Gamma \vdash k \in \mathbb{N}} \\
\text{N-Mul} & \quad \frac{\Gamma \vdash t_1 : N \quad \Gamma \vdash t_2 : N}{\Gamma \vdash t_1 * t_2 : N}
\end{align*}
\]

**Fig. 17.** Types, contexts, intrinsically-typed terms of $\lambda_{\text{IK}}+N$ (omitting the unchanged rules of Fig. 5)
We extend the equational theory of $\lambda_{IK}$ with some rules such as $\text{lift } k_1 \ast \text{lift } k_2 \sim \text{lift } (k_1 \ast k_2)$ (for natural numbers $k_1$ and $k_2$), $\text{lift } 0 \ast t \sim \text{lift } 0$, $t \sim \text{lift } 1 \ast t$, $t \sim \text{lift } k \sim \text{lift } k \ast t$, etc. The normal forms of $\lambda_{IK} + N$ include those of $\lambda_{IK}$ in addition to the following.

\[
\begin{array}{c|c}
\text{NF/N}_1 & \text{NF/N}_2 \\
\hline
\Gamma \vdash_{\text{NF}} \text{lift } 0 : N & \Gamma \vdash_{\text{NE}} n_1 : N \quad \ldots \quad \Gamma \vdash_{\text{NE}} n_j : N \\
\hline
& \Gamma \vdash_{\text{NF}} \text{lift } k \ast n_1 \ast \ldots \ast n_j : N \quad k \in \mathbb{N} \setminus \{0\}
\end{array}
\]

The normal form $\text{lift } k \ast n_1 \ast \ldots \ast n_j$ denotes a multiplication of a nonzero literal with a sequence of neutrals of type $N$, which can possibly be empty. The term box $(2 \ast \text{unbox } (\text{box } 3))$ from earlier can be represented in $\lambda_{IK} + N$ as box $(\text{lift } 2 \ast \text{unbox } (\text{box } (\text{lift } 3)))$, and its normal form as box $(\text{lift } 6)$. To extend the NbE model for $\lambda_{IK}$ to natural number literals and multiplication, we use the interpretation presented by Valliappan, Russo, et al. [2021] for normalizing arithmetic expressions. Omitting the rule $\text{lift } 0 \ast t \sim \text{lift } 0$, this interpretation also resembles the one constructed systematically in the framework of Yallop et al. [2018] for commutative monoids.

**Proving Binding-Time Correctness.** Binding-time correctness for a term $\cdot \vdash f : N \Rightarrow \Box N$ can be stated similar to noninterference: it must be the case that $\cdot \vdash \text{app } f \ u_1 \sim \text{app } f \ u_2 : \Box N$ for any two arguments $\cdot \vdash u_1, u_2 : N$. The satisfaction of this property implies that no well-typed term equivalent to $\lambda x : N. \text{box } x$ exists, since applying it to different arguments would yield different results. As before with noninterference, we can prove this property by case analysis on the possible normal forms of $f$. A normal form of $f$ is either of the form $\lambda x. \text{box } (\text{lift } 0)$ or $\lambda x. \text{box } (\text{lift } k \ast n_1 \ast \ldots \ast n_j)$ for some natural number $k$ and neutrals $n_1, \ldots, n_j$ of type $N$ in context $\cdot, N, \Box$. In the former case, we are done immediately since $\lambda x. \text{box } (\text{lift } 0)$ is a constant function that evidently satisfies the desired criterion. In the latter case, we observe by induction that no such neutrals $n_i$ exist, and hence $f$ must be equivalent to the function $\lambda x. \text{box } (\text{lift } k)$, which is also constant.

As a part of binding-time correctness, we may also desire that nonconstant terms $\Box A \Rightarrow A$ like $\lambda x : \Box A. \text{unbox } x$ be disallowed since a static computation must not have a dynamic dependency. This can also be shown by following an argument similar to the proof of noninterference in Subsection 5.2.

**Discussion.** The operational semantics for staged computation is given by Davies and Pfenning via translation to a dual-context calculus for IS4, where evaluation under the introduction rule box for $\Box$ is disallowed. While it is possible to implement a normalization function for $\lambda_{IS4}$ that does not normalize under box, this then misses certain reductions that are enabled by the translation. For instance, the term box $(2 \ast \text{unbox } (\text{box } 3))$ is already in normal form if we simply disallow normalization under box, while the translation ensures the reduction of unbox (box 3) by reducing the term to box $(2 \ast 3)$. This mismatch, in addition to the lack of a model for their system, makes the applicability of Fitch-style calculi for staged computation unclear.

## 6 RELATED AND FURTHER WORK

**Fitch-Style Calculi.** Fitch-style modal type systems [Borghini 1994; Martini and Masini 1996] adapt the proof methods of Fitch-style natural deduction systems for modal logic. In a Fitch-style natural deduction system, to eliminate a formula $\Box A$, we open a so-called strict subordinate proof and apply an “import” rule to produce a formula $A$. Fitch-style lambda calculi achieve a similar effect, for example in $\lambda_{IK}$, by adding a $\Box$ to the context. To introduce a formula $\Box A$, on the other hand, we close a strict subordinate proof, and apply an “export” rule to a formula $A$—which corresponds to removing a $\Box$ from the context. In the possible-world reading, adding a $\Box$ corresponds to travelling to a future world, and removing it corresponds to returning to the original world.
The Fitch-style calculus \( \lambda_{IK} \) was presented for the logic IK by Borghuis [1994] and Martini and Masini [1996], and later investigated further by Clouston [2018]. Clouston showed that \( \Box \) can be interpreted as the left adjoint of \( \Box \), and proves a completeness result for a term calculus that extends \( \lambda_{IK} \) with a type former \( \diamond \) that internalizes \( \Box \). The extended term calculus is, however, somewhat unsatisfactory since the normal forms do not enjoy the subformula property. Normalization was also considered by Clouston, but only with Rule \( \Box - \beta \) and not Rule \( \Box - \eta \). The normalization result presented here considers both rules, and the corresponding completeness result achieved using the NbE model does not require the extension of \( \lambda_{IK} \) with \( \diamond \). The decidability result that follows for the complete equational theory of \( \lambda_{IK} \) also appears to have been an open problem prior to our work.

For the logic IS4, there appear to be several possible formulations of a Fitch-style calculus, where the difference has to do with the definition of the rule \( \lambda_{IS4} \Box - \text{ELIM} \). One possibility is to define \( \text{unbox} \beta \) by explicitly recording the context extension as a part of the term former. Davies and Pfenning [1996, 2001] present such a system where they annotate the term former unbox as \( \text{unbox}_n \) to denote the number of \( \Box \)s. Another possibility is to define \( \text{unbox} \) without any explicit annotations, thus leaving it ambiguous and to be inferred from a specific typing derivation. Such a system is presented by Clouston [2018], and also discussed by Davies and Pfenning. In either formulation terms of type \( \Box A \Rightarrow A \) (axiom T) and \( \Box A \Rightarrow \Box A \) (axiom 4) that satisfy the comonad laws are derivable. As a result, both formulations exhibit the logical equivalence \( \Box A \Rightarrow \Box A \). The primary difference lies in whether this logical equivalence can also be shown to be an isomorphism, i.e. whether the semantics of the modality \( \Box \) is a comonad which is also idempotent. In Clouston’s categorical semantics the modality \( \Box \) is interpreted by an idempotent comonad. The \( \lambda_{IS4} \) calculus presented here falls under the former category, where we record the extension explicitly using a premise instead of an annotation.

Gratzer, Sterling, et al. [2019] present yet another possibility that reformulates the system for IS4 in Clouston [2018]. They further extend it with dependent types, and also prove a normalization result using NbE with respect to an equational theory that includes both Rule \( \Box - \beta \) and Rule \( \Box - \eta \). Although their approach is semantic in the sense of using NbE, their semantic domain has a very syntactic flavour [Gratzer, Sterling, et al. 2019, Section 3.2] that obscures the elegant possible-world interpretation. For example, it is unclear as to how their NbE algorithm can be adapted to minor variations in the syntax such as in \( \lambda_{IK}, \lambda_{IK4} \) and \( \lambda_{IT} \)—a solution to which is at the very core of our pursuit. This difference also has to do with the fact that they are interested in NbE for type-checking (also called “untyped” or “defunctionalized” NbE), while we are interested in NbE for well-typed terms (and thus “typed” NbE), which is better suited for studying the underlying models. Furthermore, we also avoid several complications that arise in accommodating dependent types in a Fitch-style calculus, which is the main goal of their work.

Davies and Pfenning present their calculus for IS4 using a stack of contexts, which they call “Kripke-style”, as opposed to the single Fitch-style context with a first-class delimiting operator \( \Box \). The elimination rule \( \text{unbox}_n \) for \( \Box \) in the Kripke-style calculus for IS4 is indexed by an arbitrary natural number \( n \) specifying the number of stack frames the rule adds to the context stack of its premise. This index \( n \) corresponds to the modal accessibility premise of the Fitch-style unbox rule presented in Fig. 11. As in the Fitch-style presentation, Kripke-style calculi corresponding to the other logics IK, IT and IK4 can be recovered by restricting the natural numbers \( n \) for which the \( \text{unbox}_n \) rule is available. Hu and Pientka [2022] present a normalization by evaluation proof for the Kripke-style calculi for all four logics IK, IT, IK4, and IS4. Their solution has a syntactic flavour similar to Gratzer, Sterling, et al. [2019] and also does not leverage the possible-world semantics. Furthermore, their proof is given for a single parametric system that encompasses the modal logics of interest, which need not be possible when we consider further modal axioms such as \( R : A \Rightarrow \Box A \).
**Possible-World Semantics for Fitch-Style Calculi.** Given that Fitch-style natural deduction for modal logic has itself been motivated by possible-world semantics, it is only natural that Fitch-style calculi can also be given possible-world semantics. It appears to be roughly understood that the $\square$ operator models some notion of a past world, but this has not been—to the best of our knowledge—made precise with a concrete definition that is supported by a soundness and completeness result. As noted earlier, this requires a minor refinement of the frame conditions that define possible-world models for intuitionistic modal logic given by Božić and Došen [1984].

**Dual-Context Calculi.** Dual-context calculi [Davies and Pfenning 1996, 2001; Kavvos 2020; Pfennig and Davies 2001] provide an alternative approach to programming with the necessity modality using judgements of the form $\Delta; \Gamma \vdash A$ where $\Delta$ is thought of as the modal context and $\Gamma$ as the usual (or “local”) one. As opposed to a “direct” eliminator as in Fitch-style calculi, dual-context calculi feature a pattern-matching eliminator formulated as a let-construct. The let-construct allows a type $\square A$ to be eliminated into an arbitrary type $C$, which induces an array of commuting conversions in the equational theory to attain normal forms that obey the subformula property. Furthermore, the inclusion of an $\eta$-law for the $\square$ type former complicates the ability to produce a unique normal form. Normalization (and, more specifically, NbE) for a pattern-matching eliminator—while certainly achievable—is a much more tedious endeavour, as evident from the work on normalizing sum types [Abel and Sattler 2019; Altenkirch, Dybjer, et al. 2001; Lindley 2007], which suffer from a similar problem. Presumably for this reason, none of the existing normalization results for dual-context calculi consider the $\eta$-law. The possible-world semantics of dual-context calculi is also less apparent, and it is unclear how NbE models can be constructed as instances of that semantics.

**Multimodal Type Theory (MTT).** Gratzer, Kavvos, et al. [2020] present a multimodal dependent type theory that for every choice of mode theory yields a dependent type theory with multiple interacting modalities. In contrast to Fitch-style calculi, their system features a variable rule that controls the use of variables of modal type in context. Further, the elimination rule for modal types is formulated in the style of the let-construct for dual-context calculi. In a recent result, Gratzer [2021] proves normalization for multimodal type theory. In spite of the generality of multimodal type theory, it is worth noting that the normalization problem for Fitch-style calculi, when considering the full equational theory, is not a special case of normalization for multimodal type theory.

**Further Modal Axioms.** The possible-world semantics and NbE models presented here only consider the logics IK, IT, IK4 and IS4. We wonder if it would be possible to extend the ideas presented here to further modal axioms such as $R : A \Rightarrow \square A$ and $GL : \square(\square A \Rightarrow A) \Rightarrow \square A$, especially considering that the calculi may differ in more than just the elimination rule for the $\square$ type.

**DATA AVAILABILITY STATEMENT**
The Agda mechanization [Valliappan, Ruch, et al. 2022] of the calculi $\lambda_{IK}$ and $\lambda_{IS4}$ and their normalization algorithms are available in the Zenodo repository.

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