Knowledge spillover entrepreneurship in an endogenous growth model

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Abstract We present a model that separates entrepreneurship from profit-motivated corporate R&D aimed at improving existing production processes. Our model embeds the core idea of the knowledge spillover theory of entrepreneurship in established knowledge-based growth models by enriching their knowledge spillover structure. Introducing knowledge spillovers drives a wedge between the optimal and market allocation of resources between new knowledge creation and commercialization. We show the first best allocation depends exclusively on the relative strength of knowledge spillovers between them and derive propositions to guide policy that can bring the market equilibrium closer to this optimum.

Keywords Innovation rents · Endogenous growth · R&D · Entrepreneurship · Incentives · Knowledge spillovers

JEL Classifications L26 · O31 · O33 · O41 · R58

1 Introduction

This article is about the role of entrepreneurs and innovation rents in the theory of economic growth. The issue has a long history in the literature on innovation and capitalist dynamics. Smith (1776) already hinted at and Schumpeter was very explicit about the importance of rents in motivating entrepreneurs to innovate—that is commercialize new knowledge. Schumpeter (1911 [1934], 88) suggested that, “economic leadership in particular must hence be distinguished from “invention.” As long as they are not carried into practice, inventions are economically irrelevant…. It is, therefore, not advisable, and it may be misleading, to stress the element of invention as much as many writers do.” For some reason, modern growth theory, however, has not heeded that call and largely focuses on (profit driven) knowledge creation and invention as the key driving forces of economic growth. The entrepreneurship literature, on the other hand, to date has failed to convincingly address this issue. This can be explained by considering the developments in economic thinking on growth and entrepreneurship since Schumpeter.¹

¹ Early entrepreneurship scholars like Kirzner (1973), von Hayek (1937) and Baumol (1991) all followed Schumpeter and considered innovation to be a profit-driven, rent-seeking economic activity while keeping opportunities exogenous. The entrepreneurship literature, however, largely turned away from mainstream equilibrium macro modeling, and the two strands of literature got separated. This article is an attempt to bring them back together.
In mainstream economics, Solow (1956) presented his neoclassical model of capital accumulation-driven growth, where the role of knowledge accumulation and commercialization through entry was eliminated entirely. The sole source of (per capita) growth in his model was capital accumulation, and at the macro-level it was savings that determined the levels of investment and capital accumulation. However, Solow (1957) found capital accumulation explained <13 % of the US 1901–1949 growth and attributed the remaining 87 % to (pure labor augmenting) technical change. This unexplained Solow-residual presented the Holy Grail in growth theory for several decades.

Kennedy (1964) experimented with innovation production functions to endogenize the required Harrod neutrality of that technical change. And even though they focused on bias rather than the rate of accumulation, they were among the first to envision the creation of new technology and the accumulation of knowledge as an economic process, driven by profit. Arrow (1962) argued instead that process innovation was the accidental and costless by-product of production and hypothesized a more or less automatic process of knowledge accumulation through learning by doing. Thereby he was the first to introduce knowledge spillovers and envisioned knowledge accumulation as an externality. His model required no profit or rents to motivate knowledge creation and also left no role for commercialization through new firm entry. But having knowledge accumulate for no apparent and economically sound reason was unsatisfying. Growth was reduced from the “manna from heaven” in Solow to a mere waste product of economic activity. In addition, Arrow’s model also contrasted sharply with observably rising corporate R&D expenditures that suggested knowledge accumulation was a conscious and profitable economic activity in its own right (Schmookler 1966).

Romer (1986, 1990) integrated these two key insights in modern growth theory. First, he assumed a private incentive for generating knowledge by doing industrial R&D. And second he introduced a positive knowledge spillover in the tradition of Arrow (1962), from current to future R&D that would provide the mechanism for endogenous growth. A whole literature has spawned from this powerful combination. But in his model as in most of its offshoots, the rents of innovation typically flow to a specialized R&D sector that auction off the blueprints for innovations to entrepreneurs. Competition among the latter leaves no rents for entrepreneurs, and their role is reduced to merely and mechanically commercializing whatever the R&D sector comes up with. Romer in a sense turned Schumpeter upside down. The entrepreneurs in the Romer-type models are freely available and up in the air. As they compete over new ideas, they will transfer all innovation rents to the knowledge creators, who then generate new ideas in the pursuit of profit. Schumpeter saw this exactly the other way around.

In Schumpeter we have no explanation of where the opportunity set comes from, or how it is expanded, but in Romer (1990) [and even in “Schumpeterian” growth models like Aghion and Howitt (1992)] the Schumpeterian entrepreneur, who commercializes for profit but does not necessarily invent, is missing. These models all assume that knowledge and economic knowledge are the same and that knowledge spillovers are ubiquitous and cost- and frictionless.

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2 Harrod neutrality implies pure labor augmenting technical change. Kennedy (1964) failed in the sense that he replaced the assumption of the Harrod neutral technical change in production by the assumption of a particular type of efficiency improvements in innovation. See Thirlle and Ruttan (1987) for an excellent overview of this literature from Schumpeter up to the modern growth literature.

3 This externality was both intra- and inter-temporal in the sense that the knowledge increased the productivity of all firms equally and increased the productivity of all future firms who would build upon the accumulated knowledge.

4 The bulk of expenditure on R&D is corporate, and large corporations and established firms make up for the bulk of corporate R&D. Private industry accounted for 64 % of industrial R&D in OECD countries according to Science and Technology Indicators in 2004 (Inklaar et al. 2006).

5 See for example Barro and Sala-I-Martin (2004), Aghion and Durlauf (2005), Jones (2006), Acemoglu (2008) and Aghion and Howitt (2011) for surveys.

6 To our knowledge there are no models in the literature that disconnect the decision to enter from the decision to generate knowledge. Earlier papers have made attempts at introducing entrepreneurship in growth models but never made the separation in functions we suggest here. Acs et al. (2005, 2009) and Michelacci (2003) are some of the few that recognize the importance of entrepreneurs but also model innovation as a co-production where entrepreneurs are complementary to R&D. In Acs and Sanders (2012), we have presented an earlier version of the present model in which we introduce both knowledge creation and commercialization as separate economic activities.
Entrepreneurship scholars (e.g., Acs et al. 2004; Braunerhjelm et al. 2010) following Schumpeter see entrepreneurship as the “missing link” in converting knowledge into economically relevant knowledge, and Acs et al. (2005, 2009) advanced the microeconomic foundations of endogenous growth theory by developing a knowledge spillover theory of entrepreneurship (KSTE). Knowledge created by incumbent firms’ R&D results in knowledge spillovers to entrepreneurs who identify and exploit these opportunities for profit. They, however, do not develop a general equilibrium steady-state growth model, making it impossible to analyze the welfare implications.

In this article, we add to this literature by fully specifying and endogenizing a more general knowledge spillover process in a dynamic general equilibrium framework. That is, we present a model in which knowledge generation and commercialization are two separate and costly activities that both need to be properly rewarded for the private sector to engage in them. We then specify and analyze four types of knowledge spillovers. With a richer knowledge spillover structure and R&D and entrepreneurship competing over the same resources, a delicate balance needs to be found. In our model, we will show that the optimal ratio of R&D to entrepreneurship will depend exclusively on the relative strength of knowledge spillovers and policy can improve over the market outcome.

Our article thereby presents a significant theoretical improvement and contribution to the theoretical development of the KSTE. We strengthen its microfoundations by modeling agents’ decisions as optimization problems, we derive general equilibrium positive steady-state growth, and we show the socially optimal versus private equilibrium outcomes. We thus build on the original idea of knowledge spillover entrepreneurship and construct a fully developed endogenous growth model in which we can compare optimal to equilibrium outcomes to formulate propositions on policy interventions.

The rest of the article is organized as follows. Section 2 establishes a few of our key assumptions. Notably, it presents evidence in support of the fact that entrepreneurs are the residual claimants of entrepreneurial rents and that corporate R&D (accidentally) generates a lot of opportunities for entrepreneurs outside the firm. Section 3 then develops our model. Section 4 has the decentralized and centralized equilibrium. Section 5 presents the welfare analysis and policy implications, and the final section concludes.

2 Entrepreneurial rents and knowledge spillovers

To accommodate the key insights from the knowledge spillover theory of entrepreneurship, we have to change the standard growth model in two important respects. First, established models in ideas based growth theory (Jones 2006) typically assume that innovation rents must be used to repay the investors that finance the creation of new knowledge. Instead, we assume entrepreneurs capture the full rents of their innovations. Second, established growth models typically assume all new knowledge created is in fact intended and knowledge spillovers are (therefore) typically from the past to the present, as new ideas build on accumulated knowledge from the past. In addition, we assume that intra-temporal knowledge spillovers exist. In our model, new entry by entrepreneurs is based on intra-temporal knowledge spillovers from corporate R&D to entrepreneurship (upstream knowledge spillovers), but R&D also benefits from new entry and increasing variety in intermediate goods (downstream knowledge spillovers).

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7 In the extant KSTE (Acs et al. 2009 p. 17), the commercialization efficiency hypothesis predicts “the more efficiently incumbents exploit knowledge flows, the smaller the effect of new knowledge on entrepreneurship,” suggesting entrepreneurs can only pick up knowledge that incumbents fail to develop. Plummer and Acs (2012) argue and show, however, that in fact entrepreneurs compete for the best ideas, implying all locally produced knowledge is in principle “in play.” We follow the latter in assuming all ideas for commercially viable new products spill over to the entrepreneurs.

8 It may strike entrepreneurship scholars as odd to develop a general equilibrium model of Schumpeterian entrepreneurship. Schumpeterian entrepreneurs are, after all, upsetting the static Walrasian equilibrium by introducing (radical) innovations. It should be noted, however, that this model is not intended to describe the entrepreneurial process at the micro level but rather models its implications at the macro level. We have to abstract from a lot of micro level heterogeneity and Knightian uncertainty to focus on the macro-level impact of an entrepreneurial process that on average generates a flow of innovations that create growth in a dynamic, steady-state equilibrium.

9 The distinction and importance of intra-temporal knowledge spillovers in the knowledge spillover theory of entrepreneurship have been pointed out by Acs (2012).
Figure 1 illustrates the key elements in our model. We have intermediate producers on the left using only capital to produce intermediate goods and services for the final goods producers. They then use labor to produce final goods for the consumers all the way on the right. Goods thus flow from left to right. The relevant knowledge generation processes are depicted by the oval shapes. Final goods producers can invest in R&D by hiring “educated labor” to improve their productivity. Developing such process innovations is assumed to also create new opportunities for the intermediate sector. Entrepreneurship in our model then takes the form of new entry in the intermediate sector and combines upstream knowledge spillovers from corporate R&D with accumulated knowledge on business venturing from the past. New entry in turn increases the productivity of corporate R&D through downstream spillovers. The up- and downstream knowledge spillovers are modeled as pure externalities. By assuming R&D competes for the same scarce resources (educated labor in our model) as entrepreneurial entry, we can show that the socially optimal allocation will depend on the relative strength of these spillovers, and policy is justified to bring the market equilibrium closer to the first best. To justify this setup, we establish that (1) entrepreneurs retain most if not all of the innovation rents of their new ventures and (2) up- and downstream intra-temporal knowledge spillovers are plausible and important.

In support of the first key assumption, a review of the empirical literature on entrepreneurial rents suggests that entrepreneurs are the recipient of rents in the economy. However, not all entrepreneurs receive large amounts of rents (Hamilton 2000). They are the residual claimant to whatever rents accrue to a venture, meaning they bear all risks. In fact, it is well established that, on average, all else being equal, an entrepreneur makes less than when working for wages, and the pay-off distribution is very skewed. For the few that are successful, the rents are substantial (Henrekson and Sanandaji 2013).

A casual reading of the Forbes list of the richest Americans confirms that most billionaires are entrepreneurs, while the few others typically inherited their wealth (usually from successful entrepreneurs from the past). In fact, <20% of the richest 400 inherited their wealth. The top five in 2007, William Henry Gates III, Warren Edward Buffett, Sheldon Adelson, Lawrence Joseph Ellison and Paul Gardner Allen, are all entrepreneurs who founded companies that became extremely successful. The industries in which they started vary from software to retail, candy, finance, commodities and oil. Even looking at the richest in technology, we find that most were entrepreneurs and not scientists, innovators not inventors.

In addition, one could argue that even the few scientists and engineers in the list are not there because of their inventions but because of the rather exceptional and rare combination of skills and knowledge to both invent and commercialize their ideas. And although it is hard to prove, we believe they obtained the biggest share of their wealth from the latter. In history there are ample examples of successful inventors that died poor and very few of successful entrepreneurs who did. Therefore, we feel justified in assuming it is the entrepreneurial talent for commercialization, not for invention, that brings financial rewards.
Our second key assumption is that corporate R&D generates opportunities for upstream entrepreneurial entry. The literature offers ample examples of objective opportunities that arose accidentally and ultimately did not benefit or even hurt the firm doing the R&D. Our model below is consistent with a story of upstream firms exploiting opportunities for innovation based on downstream firms’ R&D but also allows for the existence spillovers the other way. Let us first consider the upstream spillovers. Firms develop their own product, and in the course of doing research on their own products, they stumble into opportunities the firm cannot or will not commercialize itself. A new venture is then set up to probe, explore and exploit the new opportunity, more often than not without ‘appropriate’ compensation to the firm of origin. Corporate R&D, therefore, generates opportunities as a pure spillover to potential entrants in the economy.

But it is also not hard to imagine that downstream corporate R&D benefits from these innovations. Once established, these new goods and services give the engineers in the R&D lab more flexibility in optimizing production processes and reducing production costs. We are not aware of systematic empirical research that can confirm or belie our assumption that downstream R&D benefits from more variety in intermediate goods and services, but it is hard to imagine the alternative assumption be true. That is, more upstream variety in intermediates has no or even negative effects on the productivity of downstream R&D. One need only consider the new options and opportunities the availability of new carbon composite materials has generated for those working in car design to see how upstream innovation may have downstream benefits.

3 The model

In this section, we develop our ‘upstream’ model and first consider consumers, then producers, then intermediate producers and finally new entrants. Consumers save and consume a homogeneous final output that is produced using uneducated production labor and a differentiated bundle of intermediate inputs. The intermediates are produced using only raw capital. Educated labor can be used to invest in corporate R&D, increasing the productivity of final goods production, or in entrepreneurial innovation, setting up a new intermediate firm. The decentralized market equilibrium is first derived and then compared to the socially optimal solution in Sect. 4.

3.1 Consumers

The consumer problem below is standard in the literature [see for example Barro and Sala-I-Martin (2004)]. The representative, infinitely lived consumer maximizes the value function:

\[
V_C = \int_0^\infty e^{-\rho t}U(C(t))dt
\]

where \(\rho > 0\) is the subjective discount rate, and \(U(C(t))\) is given by \(log\ C(t)\), the natural log of consumption, \(C(t)\). This value function is maximized subject to the inter-temporal budget constraint:

\[
\dot{B}(t) = r(t)B(t) + w_E(t)L_E^* + w_P(t)L_P^* - C(t)
\]

where \(r(t)\) is the interest rate on the stock of bonds, \(B(t)\), held at time \(t\) and \(w_{E,P}(t)\) are total \((E)ducated and \((P)roduction labor incomes for the representative consumer, respectively. Labor supplies are inelastic at \(L_E^*\) and \(L_P^*\), and wages clear the respective labor markets. Appendix 1 shows that for any constant interest rate consumers will then choose consumption level:

\[
C(t) = r\left(B(0) + \int_0^\infty e^{-\rho t}w_E(t)L_E^*dt + \int_0^\infty e^{-\rho t}w_P(t)L_P^*dt\right) e^{(r-\rho)t}
\]

where \(B(0)\) is the level of initial wealth and the integrals represent the discounted present value of life time educated and production labor incomes. Equation (3) merely implies there is a positive demand for final goods at all times and consumers are willing to save part of their income as long as the return on their savings, invested in bonds, exceeds the discount rate. To endogenize the equilibrium interest and wage levels, we need to specify the supply side.

3.2 Producers

Assume \(m\) producers produce the homogenous final good and maximize their profits by choosing the levels of labor, intermediate goods and R&D labor to
employ, taking as given the price level that we normalize to 1. All final goods-producing firms are assumed to have the same production function:

\[ X_j(t) = A_j(t)^{\gamma} L_P(t)^{\beta} \sum_{i=0}^{n(t)} x_{ij}(t)^{1-\alpha-\beta} \quad \text{with} \quad 0 \leq \alpha + \beta \leq 1 \quad \text{and} \quad 0 \leq \alpha, \beta \leq 1 \]  

(4)

where \( X_j(t) \) is the output of final goods producer \( j \in \{0, m\} \), \( L_P(t) \) is production labor that earns wage \( w_P(t) \) and \( x_{ij}(t) \) is the quantity of intermediate \( i \in \{0, n(t)\} \) bought at price \( x_j(t) \). \( n(t) \) is the number of available varieties of intermediates at time \( t \). Firms can increase their total factor productivity \( A_j(t) \) by hiring educated labor to do R&D. Productivity will then increase according to:

\[ \dot{A}_j(t) = \psi A_j(t)^{1-\gamma} n(t)^{\gamma} L_E(t) \quad \text{with} \quad 0 \leq \gamma \leq 1 \]  

(5)

The presence of \( A_j(t) \) on the right hand side reflects an inter-temporal knowledge spillover. R&D is more productive when a large knowledge base has already been accumulated in the past (but at a decreasing rate). The presence of \( n(t) \) reflects the fact that more variety in intermediates allows the final goods-producing sector to better fine tune the production process and thereby generate more total factor augmenting technical change for a given accumulated knowledge stock and level of R&D employment. We included a productivity parameter, \( \psi \), and have chosen a linear specification in educated labor, \( L_E \), following Romer (1990) and thereby introduced the scale effect. Eliminating it would not affect our key results.\(^{11}\) The firm’s problem is now a dynamic optimization problem due to the R&D investment decision. Dropping time arguments to save on notation, we can write the Hamiltonian for this problem:

\[
H_j = e^{-\gamma} \left( A_j^\alpha L_P^\beta \sum_{i=0}^{n} x_{ij}^{1-\alpha-\beta} - (w_P L_P + w_E L_E) \right) \\
- \sum_{i=0}^{n} \lambda_i x_{ij} + \mu_j (\psi A_j^{1-\gamma} n^\gamma L_E) 
\]  

(6)

The levels of employment in production and R&D and intermediate use are control variables, and the stock of firm specific knowledge is the state variable. Standard

dynamic optimization in Appendix 2 then yields the demand for production labor in firm \( j \):

\[
L^D_{Pj} = \frac{\beta X_j}{w_P} \]  

(7)

where we obtain the standard result that production labor’s share in total income is equal to \( \beta \), the output elasticity of labor in production. Summing Eq. (7) over all firms \( j \) and assuming an exogenous supply of production labor, production wages will have to grow at the same rate as output in equilibrium. The demand for intermediate \( i \) by firm \( j \) is given by:

\[
x_{ij}^D = \frac{X_j^\gamma}{\sum_{i=0}^{n} x_{ij}^\gamma} (1-\beta) X_j \]  

(8)

where again it can be verified that (if all intermediates are sold at the same price as they will be in equilibrium) the income share of intermediates is equal to the output elasticity of intermediate goods. Appendix 2 also shows how to solve the dynamic problem of optimally investing in R&D and shows that the wage for educated labor must be equal to:

\[
\tilde{w}_E = \frac{\alpha \psi}{r - \frac{w_P}{w_E} + \frac{\gamma}{\alpha}} \left( \frac{A}{n} \right)^{-\gamma} X \]  

(9)

Equation (9) shows that the threshold wage for educated labor will grow at the same rate as output, such that relative wages will remain stable, as long as \( A \) over \( n \) is constant.

However, the level of employment in R&D cannot yet be determined as this type of labor has one more application in our model. Therefore, we turn to the intermediate producers.

3.3 Intermediate producers

The intermediate sector produces capital goods according to some specific design that is available to one firm only. We assume, however, that there are a lot of varieties, \( n \), available, and new ones are allowed to enter below. One can think of the designs as being codified and protected by a patent, but entrepreneurs often combine tacit knowledge, talent and public information to come up with a commercial opportunity that no one has recognized or successfully exploited before. The key assumption for our model is that every intermediate is produced exclusively by

\(^{11}\) Jones (2006) offers several alternatives to this specification that would not suffer from this problem, but as the issue has no bearing on our purpose, we chose to stick to the Romer specification.
one firm, and intermediate producers can thus engage in monopolistic competition. The producers in this sector are assumed monopolists that set their own price and compete with imperfect substitutes.

By the assumed symmetry in the final goods production function all varieties face the same, iso-elastic demand curve for their variety. Also we assume that the monopolists are price takers in the market for raw capital. This entire structure was copied from Romer (1990). The problem is then identical for every intermediate producer \(i\). They solve a static and standard profit maximization problem given by:

\[
\max_{\pi_i} \pi_i = \chi_i \sum_{j=0}^{m} x_{ij}^D(\chi_i) - rK_i
\]  

(10)

Subject to a simple one-for-one production technology in \(K\). Substituting for demand by Eq. (8) and setting the first derivative with respect to \(\chi_i\) to 0 yield the profit-maximizing price for intermediate \(i\):

\[
\chi_i = \frac{r}{1 - \alpha - \beta}
\]  

(11)

This does not vary over \(i\) (or for a constant interest rate over \(t\)) anymore. So every intermediate producer sets his price equal to this value, and by the demand function all intermediates are demanded in the same quantity. This implies that in equilibrium the stock of raw capital is divided equally among all \(n\) varieties and the capital share in income is given by

\[
rK = (1 - \alpha - \beta)^2 X
\]

whereas the monopoly rents in the intermediate sector are given by:

\[
\pi_i = \frac{(\alpha + \beta)(1 - \alpha - \beta)X}{n}
\]

\[
\sum_{i=0}^{n} \pi_i = (\alpha + \beta)(1 - \alpha - \beta)X
\]  

(12)

These profits accrue to the entrepreneur who organized the intermediate production unit, as no other inputs or fixed (entry) costs are assumed. Monopoly rents are therefore the reward for commercialization, not invention, in our model. But let us now consider the decision to start an intermediate goods producing venture.

3.4 Entry and entrepreneurs

The positive (expected) flow of rents attracts entrants. These entrants cannot enter the existing intermediate varieties’ markets as we assume that these are protected by patents, trade secrets, tacit knowledge or otherwise. However, the existence of these rents and the knowledge that there is a latent demand for new varieties make it attractive to enter with a new intermediate variety. Abstracting from the risks involved in such an undertaking, the value of a new intermediate firm, \(V_E(t)\) that enters at time \(T\), is equal to the expected present value of an incumbent intermediate firm’s remaining flow of rents from \(T\) to infinity (assuming the impact of one additional intermediate on incumbent intermediate firms’ profits is infinitely small).

We then assume an entrepreneur can organize a new production unit to capture this flow of innovation rents. We propose further, as opposed to Romer (1990), that this requires the allocation of educated labor and is therefore costly in terms of wages foregone. Moreover, following Schumpeter, we assume that entrants can pick up ideas for new intermediates free of charge, as a costless knowledge spillover. In our model the spillovers come from downstream final goods producers’ process R&D. One can think of this process as the spin-out of an employee from the final goods producers’ R&D labs, but it is also possible that others pick up on such ideas. A key difference with established models here is that the entrants do not purchase the new idea. We assume the entrepreneurial innovation function is given by:

\[
\dot{n} = \varphi A^\delta n^{1 - \delta} L_{Eh}
\]  

(13)

Here we assume constant returns to entrepreneurial activity. Moreover, we assume that entry is positive in the accumulated knowledge in final goods producers process R&D, \(A\), past entrepreneurial activity proxied by \(n\), and we introduced a productivity parameter, \(\varphi\), which reflects the productivity of an educated worker in starting a new venture.

Appendix 3 shows that we can compute the marginal value product of educated labor in entrepreneurship. Setting that equal to the wage and dropping time arguments to save on notation, we obtain:

\[
\bar{w}_L = \frac{(\alpha + \beta)(1 - \alpha - \beta)\varphi}{r - \frac{\hat{K}}{X} + \frac{n}{n}} \left(\frac{A}{n}\right)^\delta X
\]  

(14)

If the market wage for educated labor exceeds this level, no entry will take place. The opportunity costs are too high. If it falls below this level, all educated labor will switch to entrepreneurial activity. Again we
have a bang-bang equilibrium due to constant returns to LE. Note that this implies that in this a bang-bang equilibrium variety, n, or knowledge, A, increases, causing \( A/n \) to change until the threshold wages equalize. We use this property to first derive the educated labor market equilibrium and then analyze the long-run steady state. Before we turn to the equilibrium, however, it is useful to consider the complete assumed knowledge spillover structure in more detail as it will drive our main results and represents the main deviation from existing models.

3.5 The knowledge spillovers

The knowledge spillovers in our model are introduced in Eqs. (5) and (13). First note that the structure of (5) and (13) is identical. Both equations are linear in educated labor, \( L_{E} \), which is augmented by productivity parameters, \( \psi \) and \( \varphi \), respectively, and a relevant knowledge base in both R&D and entrepreneurship is assumed. Specifically, the relevant knowledge stocks are proxied by a Cobb-Douglas aggregate of the accumulated stocks of R&D knowledge, \( A \), and intermediate variety, \( n \), respectively.

We assumed in (5) that final goods producing firms can invest R&D resources (skilled labor) in developing and streamlining their production process. This, we believe, is the main goal of a lot of corporate R&D. We also assumed that the same amount of R&D labor generates more knowledge, productivity gains and cost reductions when a large variety of intermediate products, \( n \), is available. The underlying idea is that this variety allows for more specialization and helps R&D workers to optimize the production process. Basically, this variety is an aggregate knowledge spillover from entrepreneurship to R&D (in addition to increasing the productivity of final goods production itself). In our model, no single entrepreneur generates knowledge that is directly useful to a specific final goods production process, but rather the collectively provided variety makes both downstream production and R&D activity easier across the board. We shall refer to these knowledge spillovers as downstream (macro-) spillovers. In Eq. (5), the importance of such downstream spillovers is parameterized by \( \gamma \), and these spillovers are pure externalities to corporate R&D.

We also assume that, by working on optimizing final production, the R&D workers come across ideas and opportunities for new intermediates. The actual commercial introduction of these ideas and opportunities, however, requires additional effort of skilled labor per Eq. (13). The change in the number of available intermediates therefore receives a positive intra-temporal spillover from R&D activity that we proxy by \( A \) in Eq. (13). This is referred to as an upstream (micro-)spillover. For these spillovers to arise, every single idea and opportunity needs to be linked to a single entrepreneur who will bring the product to the market. In reality, of course, this matching process is not costless and frictionless (e.g., Michelacci 2003), and a lot can and has been said about this matching process, the additional essential resources required and alternative sources of commercial opportunities. We have chosen to abstract from these complications to keep our focus on the importance of the knowledge spillover structure in our model and also model this upstream spillover as a pure externality to the entrepreneurial sector.

We also assume that both final goods process oriented R&D and intermediate goods introduction benefit from past experience (as in Romer 1990), proxied by the already accumulated stocks of \( A \) and \( n \). These traditional inter-temporal knowledge spillovers are parameterized by \( 1-\gamma \) and \( 1-\delta \), respectively.\(^\text{12}\) By choosing the Cobb-Douglas structure, we assume constant returns to total knowledge accumulation and a substitution elasticity between the two stocks of knowledge that is equal to 1. As empirical evidence is lacking, this assumption can only be justified by intuition and mathematical convenience.\(^\text{13}\) Our model thus features four knowledge spillovers: downstream from entrepreneurship

\(^{12}\) Note that in fact our model would essentially reduce to the Romer (1990) model for \( \delta = 0 \) and \( \gamma = 0 \). The only difference would then be that Romer’s flow of human capital is replaced by a stock of accumulated R&D knowledge in final goods production.

\(^{13}\) We feel it is intuitively plausible that both R&D and entrepreneurship can substitute one’s own experience and accumulated knowledge for intermediate product variety and outside opportunities, respectively. The constant returns to both sources of knowledge retain the basic assumption in Romer (1990) that the returns to knowledge accumulation are constant at the aggregate level. Not imposing that assumption would eliminate the steady state and cause growth in growth as the rate
to corporate R&D, upstream from corporate R&D to entrepreneurship and inter-temporal from past entrepreneurship and R&D to present entrepreneurship and R&D, respectively. It should be noted that only the latter, inter-temporal spillovers from past to present R&D, are internalized in our decentralized model. That is, final goods producing firms will take into account the positive effects their current R&D will have on future R&D in their firms. All other spillovers are (in the absence of enforceable/enforced intellectual property rights) pure externalities. It is entrepreneurs that will create new ventures to realize the upstream spillovers, possibly through spin-out and competition over new ideas. This is what we consider the core hypothesis in the knowledge spillover theory of entrepreneurship (Acs et al. 2005, 2009; Plummer and Acs 2012). Our model enriches that by adding the downstream spillovers entrepreneurs create (and arguably will also help realize) in their efforts to market new intermediate varieties. Our parameters $c$ and $d$ in Eqs. (5) and (13) capture the strength of these spillovers and will turn out to be key drivers of our results.

4 The decentralized equilibrium

4.1 The labor market

The labor market for uneducated production labor is in equilibrium when wages equate supply (given by $L_P^*$) to demand (summing Eq. (7) over all $j$) in production. We obtain:

$$w_P = \frac{\beta X}{L_P^*}$$ (15)

The market for educated labor has three possible states in the short run. If $\bar{w}_E > \tilde{w}_E$ all educated labor will be employed in R&D. By Eqs. (5) and (13), this implies $A/n$ will rise. If $\bar{w}_E < \tilde{w}_E$ instead, all educated labor is allocated to entrepreneurship, and $A/n$ will fall. Only when $\bar{w}_E = \tilde{w}_E$ is the labor market allocation stable at positive levels of both activities. Figure 2 plots the ratio $\bar{w}_E/\tilde{w}_E$ against $A/n$. The above implies that the labor market may clear at any ratio in the short run, but the corresponding allocation of labor over R&D or entrepreneurship implies that we will move towards the point where this ratio equals 1. The model, however, is not yet in steady state. The position of the convex curve still depends on the various growth rates in the model as can be verified in the ratio of Eq. (14) over (9):

$$\frac{\bar{w}_E}{\tilde{w}_E} = \left( \frac{A}{n} \right)^{-\gamma-\delta} \frac{X\psi}{(\alpha+\beta)(1-\alpha-\beta)\rho r - \bar{w}_E/\tilde{w}_E + \gamma n/n}$$ (16)

Out of steady-state equilibrium the labor market will thus ensure that first $A/n$ is at $A/n^*$, but due to the fact that Eq. (16) depends on the growth rates of output, wages, the interest rate and the growth rate of $n$, this $A/n^*$ is not necessarily a long-run stable steady-state ratio. A long-run stable steady state is reached when educated labor is allocated such that the levels of employment in R&D and entrepreneurship eventually reach the level for which $A$ and $n$ grow at the same rate and $A/n$ is at $A/n^*$. We analyze that steady state below.

4.2 The steady state

The model is in a steady-state equilibrium when all variables expand at a constant rate and the labor market allocation is stable. From the arbitrage Eqs. (9) and (14) and the analysis of the labor market above, we know that the latter can only be the case when $A$ and $n$ expand at the same rate and wages grow at the same

Footnote 13 continued

of new knowledge creation would then be positively related to the size of the aggregate knowledge stock.
rate as output. By the production function (4) and the fact that all intermediates are used at level \( K/n \) in equilibrium and using the consumers’ dynamic budget constraint, we know that at a constant interest rate the capital stock will expand at the same rate as output. Output will then grow at the rate:

\[
\frac{\dot{K}}{K} = \frac{\dot{X}}{X} = \frac{\dot{C}}{C} = \frac{\dot{B}}{B} = \frac{\dot{w}_E}{w_E} = \frac{\dot{w}_p}{w_P} = r - \frac{n}{\bar{n}} \left( \frac{2\alpha + \beta}{\alpha + \beta} \right)
\]

This solves the model if we can obtain the steady-state growth rate of \( n \) (and \( A \)). The first steady-state condition follows from rewriting Eq. (16) for the steady state. That ratio is 1 in equilibrium and can be solved for \( A/n \):

\[
A = \left( \frac{\alpha \psi}{(\alpha + \beta)(1 - \alpha - \beta)\psi} \rho + \bar{n}/n \right)^{1/n}
\]

which solves in parameters only for the special case that \( \rho = 0 \). Using the condition that in the steady-state variety expansion equals productivity growth, we can derive a second steady-state relation between entrepreneurial activity and R&D labor using Eqs. (5) and (13) to compute the ratio between total R&D and entry activity for which both variety and the knowledge base of final producers expand at the same rate:

\[
L_{E_j}^{SS} = \frac{L_{E_A}^*/m}{L_{E_A}^*} = \frac{\varphi}{\psi} \left( \frac{A}{n} \right)^{\gamma + \delta}
\]

where \( L_{E_A} = \sum_{j=0}^{m} L_{E_j} = mL_{E_j} \) is the total R&D employment in the final goods producing sector. Using Eq. (18) in (19) we obtain:

\[
L_{E_A}^* = \frac{\alpha m}{(\alpha + \beta)(1 - \alpha - \beta)\rho + \bar{n}/n}
\]

This gives the steady-state ratio of R&D to entrepreneurial activity for which the arbitrage wages and the rate of expansion for \( A \) and \( n \) are equal. It gives the ratio as a function of the rate of expansion for \( n \) and parameters. Using the labor market clearing condition \( L_E = L_{E_A}^* + L_{E_d}^* \), we can compute the steady-state level of entrepreneurial activity, using (20) to eliminate \( L_{E_d}^* \). We thus obtain for entrepreneurial activity in the steady state:

\[
L_{E_A}^{SS} = \frac{\frac{(x + \beta)}{\rho + \bar{n}/n}}{(x + \alpha - \beta\rho + \bar{n}/n) + \alpha(x + \beta)}L_E^*
\]

Plugging into the entry function in Eq. (13), dividing both sides by \( n \) and using (18) then yields:

\[
\frac{n^{SS}}{n} = \frac{L_E^*}{C_1(\rho + \bar{n}/n)^{1/\gamma} + C_2(\rho + \bar{n}/n)^{1/\gamma}}
\]

where two positive real constants, \( C_1, 2. \), were introduced to save on notation. This equation determines the steady-state growth rate. A closed form solution only exists for the special case where \( \rho = 0 \). Even if we cannot compute the analytical closed form solution, however, it is straightforward to establish that the model has a unique steady state. The right hand side of Eq. (22) will increase or decrease monotonically from one positive constant \( C_3 \) to another \( C_4 \) as the growth rate, also on the left hand side, goes from 0 to infinity. This implies that the model has a unique steady state as the 45-degree line intersects the right hand side once and only once in the positive domain. The stability of that steady state is established in Appendix 5.

The comparative statics of this steady-state growth rate can now be evaluated. First we note from Eq. (22) that indeed our model has a scale effect. Increasing total educated labor supply increases the steady-state growth rate of the economy, as more labor is available for R&D and entrepreneurship. Similarly it is easy to establish that growth will be lower in the steady state when the number of final goods producers, \( m \), is larger. The reason is that in that case R&D labor is spread over more labs and the average knowledge accumulation rate will be lower in each final goods producing firm.
parameters. Both do not appear in \(C_2\) and affect \(C_1\) negatively. Higher \(\phi\) and \(\psi\) therefore increase the steady-state growth rate as one would expect. The impact of relative profitability, \((\alpha+\beta)/(1-\omega-\beta)\), is negative on \(C_1\) and positive on \(C_2\) so the net effect is ambiguous and depends on the knowledge spillover and productivity parameters, \(\gamma\) and \(\delta\), to which we turn next.

Parameter \(\delta\) captures the importance of accumulated R\&D knowledge in creating new ventures (upstream spillovers). The effect of increasing this parameter on \(C_1\) and \(C_2\) is strictly negative, implying a higher \(\delta\) will increase \(C_3\) but the effect on \(C_4\) remains ambiguous in general. The effect of a higher \(\gamma\) on \(C_3\) is also strictly positive, but again the effect on \(C_4\) and therefore on the steady state remains ambiguous. This implies it is likely that for low rates of growth (close to 0), the effect of reducing barriers to up- and downstream knowledge spillovers is likely to be positive on steady-state growth, as the offsetting negative effect comes from higher discounting of innovation rents that is not likely to dominate when growth is low.

Finally, the impact of an increase in \(\rho\) depends on the relative size of \(C_3\) to \(C_4\). If \(C_3 > C_4\), the effect of higher \(\rho\) is to increase the steady-state growth rate. As most of these parameters are at best only indirectly affected by policy, however, these comparative statics do not provide a very compelling case for innovation policy. To motivate our policy implications we need to show more formally that a first best allocation would shift resources between R\&D and new firm entry and that policy interventions can bring about such a more optimal allocation.

### 4.3 Welfare analysis and policy implications

To obtain the optimal solution to our model, we can formulate the maximization problem for a hypothetical central planner. We assumed a standard time-separable log linear utility function in a homogenous final good that was given by Eq. (1). This is also the objective function for our central planner, and he can optimize taking only resource and technological constraints into consideration, internalizing all externalities in the model. The first technological constraint is the production function for final output that was given by Eq. (4). Dropping subscript \(j\), using the production function for intermediates, \(x_i(t) = K_i(t)\), and using the fact that the CES aggregate in (4) is maximized by setting \(x_i(t) = K(t)/n(t)\) we can rewrite Eq. (4) as:

\[
X(t) = A(t)^{\alpha}L^\alpha(t)^{\beta}K(t)^{1-\alpha-\beta}n(t)^{\alpha+\beta}
\]

(23)

where it is immediately clear how variety expansion and knowledge accumulation both increase the final output and thereby social welfare. Also note that production labor can be set to \(L_P^*\). Total skilled labor must be allocated between entrepreneurial venturing and R\&D such that \(L_{EA}(t) = L_E^* - L_{EA}(t)\). This reduces the central planner’s maximization problem to a dynamic optimization problem with two control \((C(t)\) and \(L_{EA}(t))\) and three state variables, \((K(t), A(t)\) and \(n(t))\). The state variables follow their respective laws of motion. Raw capital, \(K\), knowledge in final output, \(A\), and the number of intermediate varieties, \(n\), evolve according to:

\[
\dot{K}(t) = X(t) - C(t)
\]

(24)

\[
\dot{A}(t) = \psi A(t)^{1-\gamma}n(t)^{\gamma}L_{EA}(t)
\]

(5’)

\[
\dot{n}(t) = \phi A(t)^{\delta}n(t)^{1-\delta}(L_E^* - L_{EA}(t))
\]

(13’)

where subscript \(j\) in Eq. (5) is dropped as the central planner will concentrate all knowledge accumulation to avoid costly duplication of efforts. Educated labor in Eq. (13) was replaced by \(L_{E}^* - L_{EA}(t)\) to ensure full employment of educated labor. We can now choose the consumption path and allocation of skilled labor to maximize social welfare. Appendix 6 shows one must save, consume and accumulate raw capital in a standard fashion [see, e.g., Barro and Sala-I-Martin (2004)]. In the social optimum, however, we take into account all externalities in the model and consequently allocate educated labor differently between entrepreneurial activity and knowledge accumulation. It is shown in Appendix 6 that educated labor is employed according to:

\[
\frac{L_{EA}}{L_{E}^*} = \frac{\delta}{\gamma}
\]

(25)

where the latter expression can be compared to Eq. (20) above. It should be noted that relative profitability (the terms in \(x\) and \(\beta\)) and relative effective discounting of private profit flows (the terms involving \(\rho\) and the growth rate of \(n\)) do not affect the optimal allocation of educated labor. Instead, Eq. (25) shows that this optimal allocation is determined by the knowledge spillover parameters only. If \(\delta\), the
upstream spillover from knowledge production to entrepreneurship, is larger, more labor should be allocated to knowledge production. Vice versa, if \( \gamma \), the downstream spillover from variety to knowledge creation, is stronger, more educated labor should be used in entrepreneurship. The corresponding growth rate for \( \dot{A} \) and \( \dot{n} \) is given by:

\[
\frac{\dot{A}}{A} = \frac{\dot{n}}{n} = \frac{(\gamma \psi d \phi \gamma \phi ^{\lambda}) ^{\frac{1}{\lambda}}}{\gamma + \delta} L_E
\]  \hspace{1cm} (26)

that can be compared to (22) above. The expression in Eq. (26) is much simpler and more intuitive. More educated labor allows for a proportionately higher growth rate. In contrast to the result in Eq. (22), however, there is no impact of \( \rho \) and the relative profitability parameters. These drop out because we have optimized over the entire model and did not maximize profits at each individual stage. Higher \( \phi \) and \( \psi \) again increase the steady-state growth rate, as one would expect. Moreover, Eq. (26) now contains additional terms in \( \gamma \) and \( \delta \) that capture the internalized knowledge spillovers between corporate R&D and entrepreneurship in the model. We concluded above that the effects of these parameters on the growth rate are ambiguous in the decentralized steady state and they remain ambiguous in the social optimum. From Eq. (26), we can derive the elasticity of growth with respect to \( \gamma \) and \( \delta \) respectively as:

\[
\frac{\partial (26)}{\partial \gamma} (26) = - \frac{\partial (26)}{\partial \delta} (26) = \frac{\gamma \delta}{(\gamma + \delta)^2} \log \left( \frac{\delta \psi}{\gamma \phi} \right)
\]  \hspace{1cm} (27)

From Eq. (27), we see that the effects have an opposite sign and entirely depend on \( \delta \psi < \gamma \phi \). Steady-state growth in the optimum is positive in \( \gamma \), the downstream spillover from entrepreneurship to knowledge creation, when \( \delta \psi > \gamma \phi \), that is, when the productivity of educated labor in knowledge creation and the spillover to entrepreneurship, is also large. Growth is positive in \( \delta \), the upstream knowledge spillover from R&D to entrepreneurship, when \( \delta \psi < \gamma \phi \), that is, when the productivity of educated labor in entrepreneurship and the downstream knowledge spillovers are strong.

A more intuitive way to compare the optimal and decentralized market equilibrium is to compare Eqs. (25) and (20), which give the optimal and equilibrium relative employment levels in corporate R&D and entrepreneurship, respectively.

\[
\frac{L_{E_A}}{L_{E_h}} = \frac{\delta}{\gamma} \leq \frac{\gamma}{(\alpha + \beta)(1 - \alpha - \beta) \rho + \gamma \bar{n}/n}
\]  \hspace{1cm} (28)

In Eq. (28) we see that the market equilibrium is unlikely to be optimal. Because of the various externalities, however, it is a priori unclear in what direction a policy maker should try to adjust the educated labor allocation. Empirical estimates for the various parameters in our model are not available and not easy to obtain. At this stage we can, however, derive some policy relevant propositions from Eq. (28).

**Proposition 1** In the presence of positive up- and downstream knowledge spillovers (\( \gamma, \delta > 0 \)), all else being equal, a higher output elasticity of accumulated knowledge in final goods production, \( \alpha \) (e.g., more knowledge intensive production processes), makes it more likely that the market equilibrium allocates too many resources to R&D.

**Proposition 2** In the presence of positive up- and downstream knowledge spillovers (\( \gamma, \delta > 0 \)), all else being equal, a higher output elasticity of intermediate diversity, \( n \), in final goods production, \( \alpha + \beta \) (e.g., more technically complex production processes) makes it more likely that the market equilibrium allocates too many resources to R&D.

**Proposition 3** In the presence of positive up- and downstream knowledge spillovers (\( \gamma, \delta > 0 \)), all else being equal, a higher profit margin in intermediate goods production, \( 1 - \alpha - \beta \) (e.g., less competitive intermediate goods producing sector) makes it more likely that the market equilibrium allocates too many resources to entrepreneurship.

**Proposition 4** In the presence of positive up- and downstream knowledge spillovers (\( \gamma, \delta > 0 \)), all else being equal, a higher effective discount rate, \( \rho + \bar{n}/n \) (e.g., a higher real interest rate), makes it more likely that the market equilibrium allocates too many resources to entrepreneurship.

**Proposition 5** In the presence of positive up- and downstream knowledge spillovers (\( \gamma, \delta > 0 \)), all else being equal, a stronger downstream spillover, \( \gamma \), makes it more likely that the market equilibrium allocates too many resources to R&D.
Proposition 6  In the presence of positive up- and downstream knowledge spillovers \((\gamma, \delta > 0)\), all else being equal, a stronger upstream spillover, \(\delta\), makes it more likely that the market equilibrium allocates too many resources to entrepreneurship.

Formal proof for these propositions follows straightforwardly from taking partial derivatives on the right and left hand side of inequality (28). The propositions are also intuitively plausible when we realize that a first best allocation must take into account the four market failures in our model. First, we can avoid the monopoly rents in intermediate production. Second, we can take into account the downstream knowledge spillovers from entrepreneurship to R&D, the upstream knowledge spillover from R&D to entrepreneurship and the inter-temporal knowledge spillover from past to present entrepreneurship. These market failures enter the right hand side of inequality (28), whereas only the knowledge spillover from past to present R&D is internalized in the decentralized market equilibrium. Depending on the relative importance of these market failures, a policy maker can improve on the decentralized market equilibrium by bringing the allocation of resources between entrepreneurship and R&D closer to the ratio on the left hand side. Obviously it would help a great deal if empirical studies could establish reliable estimates of the fundamental parameters on the left hand side directly. As good data on knowledge flows in the economy are, however, still largely unavailable and measurement of innovation inputs and output inevitably remains very indirect, we feel the propositions above represent the best we can do at present.

5 Conclusions

We present a model that features a knowledge generation and commercialization structure that is more in line with the stylized facts on rent appropriation and knowledge spillovers. In our model, entrepreneurs invest resources and capture the rents for commercializing new ideas. They, however, do not produce these ideas. In line with the knowledge spillover theory of entrepreneurship, we model the opportunities as a pure upstream knowledge spillover from incumbent firms’ R&D. These firms do R&D to maintain competitiveness through efficiency improvements on their final output. We add to the knowledge spillover theory of entrepreneurship the notion that downstream R&D will also receive knowledge spillovers from entrepreneurial activity. In addition, we retain the Romer (1990) idea that inter-temporal knowledge spillovers drive growth. In our model, we put Schumpeter’s entrepreneur into Romer’s model of endogenous growth and endogenized both knowledge creation and opportunity commercialization as separate but closely related processes.

The implications of this slightly more generalized model of knowledge spillovers are more than trivial. We have shown that policy can improve on the market equilibrium allocation of skilled labor between R&D and entrepreneurship. It is, however, urgent to obtain reliable empirical estimates of knowledge spillover parameters before policy makers can design policies that unambiguously get us closer to the optimum. In the second, or even third best world in which such estimates are absent, policy should be piecemeal and establish by trial and error the optimal mix between supporting R&D, supporting entrepreneurship and establishing or widening the channels for up- and downstream knowledge spillovers.

Some preliminary conclusions can thus be drawn from our analysis. Mainstream growth theory is right in asserting that knowledge creation is the ultimate source of steady-state growth, but it is wrong in asserting that more knowledge is a sufficient condition or even the most important one to increase welfare. Extending the traditional framework with a richer knowledge spillover structure to accommodate the KSTE has provided the latter with a robust general equilibrium foundation and the former with new policy considerations. Policy makers would be seriously misguided in focusing exclusively on knowledge creation, and although more empirical research is warranted, the existence of up- and downstream knowledge spillovers puts, for example, intellectual property rights protection and non-compete clauses in a different light.

We can think of several extensions to this model that would be worth further investigation. An obvious one is the introduction of risk. Our model is deterministic in both the R&D process and the entrepreneurial process. But R&D is uncertain, and entry is even more risky than ongoing R&D. Modeling these processes as stochastic (following, e.g., Aghion and Howitt 1992)
would lend the model more credibility, although we do not expect to see the results change qualitatively.

Also our model follows the mainstream assumptions on perfect rationality, full information and homogenous preferences, which are clearly at odds with empirical evidence, especially in the entrepreneurship literature. We feel, however, that showing the importance of entrepreneurship, even in a mainstream, general equilibrium setting, only emphasizes the importance of developing more realistic models in those directions.

Appendix 1: The full dynamic optimization problem of consumers

The Hamiltonian to this problem:

$$H_C = e^{-\mu t} \log C(t) + \mu(t)(r(t)B(t) + w_E(t)L^*_E + w_p(t)L^*_p - C(t))$$

(29)

yields the first order conditions:

$$\frac{\partial H_C}{\partial C(t)} = 0 = e^{-\mu t} - \mu(t)$$

$$\frac{\partial H_C}{\partial B(t)} = -\mu(t)$$

$$\lim_{t \rightarrow \infty} \mu(t)B(t) = 0$$

$$\frac{\partial H_C}{\partial \mu(t)} = \frac{\partial B(t)}{\partial (r(t))} = r(t)B(t) + w_E(t)L^*_E + w_p(t)L^*_p - C(t)$$

(30)

Taking the first two conditions, solving the first for $\mu(t)$, taking the time derivative and substituting into the second yields:

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho$$

(31)

For any constant $r(t) = r$ we then obtain:

$$C(t) = C(0)e^{(r-\rho)t}$$

(32)

Now we can use the third and fourth condition to derive $C(0)$ and express final goods demand in variables that are given to the consumer. First rewrite condition four to:

$$\dot{B}(t) - rB(t) = w_E(t)L^*_E + w_p(t)L^*_p - C(t)$$

(33)

Then multiply both sides with integrating factor $e^{-rt}$ and solve for $C(0)$:

$$\frac{dB(t)}{dt}e^{-rt} - rB(t)e^{-rt} = w_E(t)L^*_Ee^{-rt} + w_p(t)L^*_pe^{-rt}$$

$$-C(t)e^{-rt}\frac{de^{-rt}B(t)}{dt} = w_E(t)L^*_Ee^{-rt} + w_p(t)L^*_pe^{-rt}$$

$$-C(t)e^{-rt}de^{-rt}B(t) = w_E(t)L^*_Ee^{-rt}dt$$

$$+ w_p(t)L^*_p e^{-rt}dt - C(t)e^{-rt}dt \int_0^\infty de^{-rt}B(t)$$

$$= \int_0^\infty w_E(t)L^*_Ee^{-rt}dt + \int_0^\infty w_p(t)L^*_pe^{-rt}dt - \int_0^\infty C(t)e^{-rt}dt$$

(34)

which by using the third (transversality) condition in (30) and the expression for consumption in (32) yields:

$$-B(0) = \int_0^\infty e^{-rt}w_E(t)L^*_Edt + \int_0^\infty e^{-rt}w_p(t)L^*_pdt$$

$$- C(0) \int_0^\infty e^{-rt}dt$$

(35)

such that:

$$C(0) = r \left( B(0) + \int_0^\infty e^{-rt}w_E(t)L^*_Edt + \int_0^\infty e^{-rt}w_p(t)L^*_pdt \right)$$

(36)

To the consumers initial wealth, interest rate and life time wage income are given, so this determines the optimal consumption path:

$$C(t) = r \left( B(0) + \int_0^\infty e^{-rt}w_E(t)L^*_Edt + \int_0^\infty e^{-rt}w_p(t)L^*_pdt \right) e^{(r-\rho)t}$$

(3)

---

15 The assumption of a stable equilibrium interest rate is consistent with a steady-state equilibrium later on but convenient to also make here. The interest rate cannot have a positive or negative growth rate as it would imply bond prices going to 0 or infinity, which is not consistent with rational expectations. It is a very common assumption in the literature. See, for example, Campbell and Mankiw (1989)
Appendix 2: The dynamic optimization problem for the final goods producer

From the Hamiltonian in (6), we can obtain $n + 5$ first order conditions that characterize the profit-maximizing levels of employment in production and R&D and demand for intermediates. For production labor, we have:

$$\frac{\partial H_i}{\partial L_{pj}} = 0 = e^{-\tau} \left( \beta A_j^x L_{pj}^{\beta-1} \sum_{i=0}^{n} x_{ij}^{1-x-\beta} - w_p \right)$$ (37)

which can easily be rewritten into a labor demand function:

$$L_{pj}^D = \left( \frac{\beta A_j^x \sum_{i=0}^{n} x_{ij}^{1-x-\beta}}{w_p} \right)^{\frac{1}{\beta}} = \frac{\beta X_j}{w_p}$$ (38)

This shows that all firms will spend exactly the same share, $\beta$, of output, $X$, on wages.\(^\text{16}\) Summing over all final goods producers, we obtain for total production labor demand:

$$L_{pj}^D = \frac{\beta X_j}{w_p}$$ (7)

such that the total wage sum for production workers is $\beta X$, and labor demand is stable as long as wages and production grow at the same rate in equilibrium.

For intermediates, the firm will choose their levels to satisfy:

$$\frac{\partial H_i}{\partial x_{nij}} = 0 = e^{-\tau} \left( (1 - \alpha - \beta) A_j^x L_{pj}^{\beta-1} x_{ij}^{x-\beta} - \chi_n \right)$$

$$\frac{\partial H_i}{\partial x_{ij}} = 0 = e^{-\tau} \left( (1 - \alpha - \beta) A_j^x L_{pj}^{\beta-1} x_{ij}^{x-\beta} - \chi_1 \right)$$

$$\vdots$$

$$\frac{\partial H_i}{\partial x_{nij}} = 0 = e^{-\tau} \left( (1 - \alpha - \beta) A_j^x L_{pj}^{\beta-1} x_{nij}^{x-\beta} - \chi_n \right)$$ (39)

The $n$ conditions in (39) allow one to derive the demand for intermediate good $i$ in terms of the relative price and quantity of the $n$th intermediate:

$$x_{ij}^D = x_{nij} \frac{\chi_n}{\chi_i} = x_{nij} \frac{Z_n}{Z_i}$$ (40)

Substituting this demand function into the production function and rewriting in terms of total output yields:

$$\sum_{i=0}^{n} x_{ij}^{1-x-\beta} = \frac{\sum_{i=0}^{n} x_{nij}^{1-x-\beta} \chi_n}{\chi_n}$$

$$= x_{nij}^{1-x-\beta} \frac{1-\alpha-\beta}{\chi_n} \sum_{i=0}^{n} \chi_i = \frac{X_j}{A_j^x L_{pj}^{\beta}}$$ (41)

From the $n$th order condition, we also know that for all $i$:

$$A_j^x L_{pj}^{\beta} = x_{nij}^{1-x-\beta} \frac{X_n}{1-\alpha-\beta}$$ (42)

So combining (41) and (42) and solving for $x_{nij}$, we get:

$$x_{nij}^D = \frac{\sum_{i=0}^{n} \chi_i}{1-\alpha-\beta} X_j$$ (43)

And by the symmetry in the production function, this implies that all varieties $i$ have that demand function:

$$x_{ij}^D = \frac{\sum_{i=0}^{n} \chi_i}{1-\alpha-\beta} X_j$$ (8)

Multiplying (8) by $\chi_i$ and summing over all varieties, $i$ shows that total expenditure on intermediates is $(1 - \alpha - \beta)X_j$.\(^\text{17}\)

Together with the result on the wage costs, this implies that the final goods producer $j$ makes an operating profit of $\alpha X_j$. We assume that final goods producers are perfectly symmetric, facing the same input and output prices, $w$, $\chi$, and 1, respectively. As they also use the same production technology, increases in the firm’s level of accumulated knowledge $A_j(t)$ and consequently $X_j(t)$ will cause increases in operating profit. Firms, however, have to invest labor in R&D to increase their $A_j(t)$.

Formally the stock of knowledge is a firm specific state variable, and its optimal path is determined by choosing the optimal level of R&D labor. The final goods producer will increase R&D activity as long as the discounted future benefits of doing so exceed the current labor costs at the margin. As R&D is a deterministic process in our model, the firms can decide to spend on R&D exactly up to that point. The

\(^\text{16}\) As final output is homogeneous and we normalized its price to 1, sales equal production.

\(^\text{17}\) Summing over all final goods producers, $j$ then yields the result that total expenditure on intermediates in the economy is $(1 - \alpha - \beta)X$. 

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solution is formally characterized by two first order conditions, one transversality condition and the law of motion for $A_j$:  

\[
\frac{\partial H_j}{\partial L_{Ej}} = 0 = e^{-\eta} w_E + \mu_j \psi A_j^{1-\gamma} n^\gamma
\]

\[
\frac{\partial H_j}{\partial A_j} = -\mu_j = e^{-\eta} \alpha A_j^{\alpha-1} L_{Ej}^\beta \sum_{i=0}^n x_i \gamma \psi A_j^{1-\gamma} n^\gamma + (1 - \gamma) \mu_j \psi A_j^{1-\gamma} n^\gamma L_{Ej}
\]

\[
\lim_{t \to -\infty} H_j(t) = 0
\]

\[
\frac{\partial H_j}{\partial \mu_j} = \dot{A}_j = \psi A_j^{1-\gamma} n^\gamma L_{Ej}(t)
\]

where the first condition sets the present value of labor costs equal to the present value of the marginal product of R&D labor times the shadow price of a marginal increase in $A_j$, $\mu_j$. Solving for that shadow price yields:

\[
\mu_j = e^{-\eta} \frac{w_E}{\psi A_j^{1-\gamma} n^\gamma}
\]

Then we take the time derivative and set this expression equal to minus the right hand side in the second condition to equate the marginal return on $A_j$ to the shadow price:

\[
\dot{\mu}_j = \left( r - \frac{w_E}{\psi A_j^{1-\gamma} n^\gamma} \right) \dot{A}_j + \frac{w_E}{\psi A_j^{1-\gamma} n^\gamma}
\]

\[
\dot{\mu}_j = e^{-\eta} \frac{\alpha x_i}{A_j} - (1 - \gamma) e^{-\eta} \frac{w_E L_{Ej}}{A_j}
\]

Substituting the law of motion (5) for $\dot{A}_j$ into (46) and solving for $w_E$ yields the wage level at which a positive finite amount of R&D workers will be employed by firm $j$:

\[
\bar{w}_{Ej} = \frac{\alpha \psi}{r - \frac{w_E}{\psi A_j^{1-\gamma} n^\gamma}} \left( \frac{A_j}{n} \right)^{-\gamma} X_j
\]

This wage level represents a horizontal demand function or arbitrage condition. If market wages for R&D labor exceed this threshold, no R&D workers will be employed by firm $j$. If wages fall short, firm $j$ will hire additional R&D workers until all are hired or wages have gone up. This so-called bang-bang equilibrium is a result of the constant returns to R&D labor assumption that we have made. Eq. (47) holds for all firms $j$, and this wage will be equal for all firms $j$ as they are price takers in the market for R&D labor.

We also know by the production function in (4) and Eqs. (7) and (8) that $X_j$ is continuous and strictly proportional in $A_j$. Thus, we obtain the result that at any point in time there is a unique level of $A_j$ that all firms hiring R&D labor must attain. The mechanism is that the firms with $A_j = A_{\text{max}}$ also have the highest threshold wage for R&D. They will thus bid up production wages to this threshold level and employ a positive amount of R&D. Their level of $A$ will then rise according to (5), those with $A_j < A_{\text{max}}$ will not hire any R&D, and their $A_j$ remains stable. The rise in $A_{\text{max}}$ pushes up the threshold and thereby the production wage. In any equilibrium with R&D, only those firms that have $A_j = A_{\text{max}}$ can stay in the race, whereas others are forced to bring down their production employment levels to 0. If we assume therefore that all firms start from the same initial level of $A_j(0) = A_0$, the above implies that $A_j(t) = A_{\text{max}}(t) = A(t)$ for all $j$, and we obtain for (47) (dropping time arguments):

\[
\bar{w}_E = \frac{\alpha \psi}{r - \frac{w_E}{\psi A_j^{1-\gamma} n^\gamma}} \left( \frac{A}{n} \right)^{-\gamma} X
\]

If we could establish the total number of R&D workers, $L_{EA}$, at this point, we could put $L_{EA}/m$ into (5) to derive the optimal growth rate of

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18 Time arguments have been included in the transversality condition as the limit is taken for time to infinity.

19 It can be shown that the right hand side of (B11) is actually positive in $A_j$ when the optimal amounts of labor and intermediates have been employed. In that case, output in (4) substituting for labor and intermediates by (8) and (47) equals:

\[
X_j = A_j \left( \frac{\bar{w}_E}{\psi} \right)^{\frac{\gamma}{\gamma - 1}} \left( \frac{A_j}{n} \right)^{-\gamma} X_j
\]

where $\bar{w}_E$ represents the average price for intermediates. Plugging this expression in the threshold wage in (47) and solving for the wage yields an expression that is positive and concave in $A_j$.

20 Taken literally, this result may strike one as unrealistic, and it yields the undesirable result that initial levels of production knowledge have to be exactly equal. At this point, however, it is worth noting that, for example, uncertainty in the R&D process and fixed costs have been assumed away. In real life, the uncertainty in R&D outcomes would create a bandwidth, not a precise level for the threshold wage, and fixed costs would cause firms to actually exit when employment levels fall below a critical level. Then the prediction is that a group of technology leaders will be able to survive in the market, where they must “run to stand still,” and a shakeout will cause firms with less than efficient production processes to exit in the transition to the steady state. Such processes are well known in the empirical literature on industrial dynamics. They are not present in our model as they complicate but do not change the key results.
\[ A_j(t) = A^{\max}(t) = A(t) \]. The starting condition \[ A_j(0) = A_0 \] and the law of motion in (5) then determine the optimal path for \( A(t) \), and the transversality condition helps to solve for \( \mu_j(t) \). This shadow price and the exact optimal path for \( A(t) \) are not very relevant for our purpose here.

### Appendix 3: The intermediate producers and entrants

To close the model, we need to model the behavior of the intermediate producers and the potential new entrants. Intermediate producers are monopolists so they will set prices to maximize profits. They solve:

\[
\max_{\pi_i} \pi_i = \lambda_t \sum_{j=0}^{m} x_{ij}^D(\lambda_i) - rK_i
\]

Subject to their production function and demand for intermediates:

\[
x_i = K_i
\]

\[
x_{ij}^D = \frac{1}{\lambda_i^{1/3}} \left( 1 - \alpha - \beta \right) X_j
\]

Subject substitution of these constraints into the profit function and setting the first derivative with respect to \( \lambda(i) \) to 0 yields the profit maximizing price for intermediate \( i \):

\[
\lambda_i = \frac{r}{1 - \alpha - \beta}
\]

which does not vary over \( i \) anymore. So every intermediate producer sets his price equal to this value, and by the demand function all intermediates are demanded in the same quantity. This implies that in equilibrium the stock of raw capital is divided equally among all \( n \) varieties, and the capital share in income is given by \( rK = (1 - \alpha - \beta)^2X \), whereas the monopoly rents in the intermediate sector are given by:

\[
\pi_i = \frac{\left( \alpha + \beta \right) \left( 1 - \alpha - \beta \right) X}{n}
\]

\[
\sum_{i=0}^{n} \pi_i = \left( \alpha + \beta \right) \left( 1 - \alpha - \beta \right) X
\]

As new firms are assumed infinitely small, the marginal entrant will also enjoy this flow of rents forever and the value of entry given by the discounted flow of rents from entry at time \( T \) to infinity. Using (12), this is given by:

\[
V_E(T) = \int_{T}^{\infty} e^{-\gamma \pi_i(t)} dt
\]

\[
= \left( \alpha + \beta \right) \left( 1 - \alpha - \beta \right) \int_{T}^{\infty} e^{-\gamma X(t)/n(t)} dt
\]

Using the entry function in (13) and equating discounted future marginal rent income to marginal (opportunity) costs at the time of entry at time \( T \), we can derive the entry arbitrage equation:

\[
\frac{\partial n(T)}{\partial L_1(t)} V_E(T) = \left( \alpha + \beta \right) \left( 1 - \alpha - \beta \right) \phi(A(T)) \delta n(T)^{1-\delta} \int_{T}^{\infty} e^{-\gamma X(t)/n(t)} dt = \tilde{w}_E(T)
\]

And as this trade-off is identical for entrants over time, we can replace \( T \) by \( t \), and this equation can be rewritten into the arbitrage condition for entrepreneurial educated labor if we assume that at entry entrepreneurs expect that output and variety will expand at a constant rate (as they will in steady state):

\[
\tilde{w}_E = \left( \alpha + \beta \right) \left( 1 - \alpha - \beta \right) \phi \left( \frac{A}{n} \right)^{\delta} \frac{X}{r - \frac{X}{n} + \frac{n}{nATE}}
\]

### Appendix 4: The steady state

From the production function (5) and substituting \( K/n \) for all \( x_i \), we obtain for the growth rate of final output, \( X \):

\[
\dot{X} = \frac{\dot{X}}{X} = \frac{\dot{A}}{A} + \left( \alpha + \beta \right) \frac{n}{n} + (1 - \alpha - \beta) \frac{\dot{K}}{K}
\]

Using the budget constraint for the consumer, we know that wage and capital income must grow at the same rate as output and consumption in steady state. Wages will grow at the same rate by Eqs. (9), (14) and (15), so total labor income grows at the common rate. For consumption to grow at that rate, so must capital income at a constant interest rate. This implies raw capital will have to expand at the common rate. We

\[21\text{ Such that } X(t) = X(T)e^{\bar{X}/X} \text{ and } n(t) = n(T)e^{\bar{n}/n} \]
can then rewrite (51) to:
\[
\frac{\dot{X}}{X} = \frac{\alpha \dot{A} + \dot{n}}{\alpha + \beta A} \quad (52)
\]

And as a stable labor allocation requires a constant ratio \(A/n\), the steady-state growth rates will be equal to (16):
\[
\frac{\dot{K}}{K} = \frac{\dot{X}}{X} = \frac{\dot{C}}{C} = \frac{\dot{B}}{B} = \frac{\dot{w}_E}{w_E} = \frac{\dot{w}_p}{w_p} = \frac{r - \rho}{n} = \frac{\dot{n}}{n} \left( \frac{2\alpha + \beta}{\alpha + \beta} \right)
\quad (17)
\]

Appendix 5: The uniqueness and stability of the steady state

We can show the uniqueness of the steady-state equilibrium by investigating the properties of the right hand side of Eq. (22):
\[
\frac{n^{SS}}{n} = \frac{L^*_E}{C_1 + C_2}
\]
\[
C_1 = \frac{\psi^{1/\gamma} q^{2/\gamma}}{(\alpha + \beta)(1 - \alpha - \beta)}
\]
\[
C_2 = \frac{m}{(\alpha + \beta)(1 - \alpha - \beta)}
\]

For \(C_1 > 0\) and \(C_2 > 0\), we can calculate the right hand side of Eq. (22) in the limit for \(\dot{n}/n \to \infty\) and obtain a positive constant:
\[
\lim_{\dot{n}/n \to \infty} \text{RHS} = \frac{L^*_E}{C_1 + C_2} > 0
\quad (53)
\]

For growth rates of 0, the right hand side of (22) simplifies to:
\[
\lim_{\dot{n}/n \to \infty} \text{RHS} = \frac{L^*_E}{C_1 \gamma^{-1/\gamma} + C_2 \gamma^{1/\gamma}} > 0
\quad (54)
\]

The right hand side of Eq. (22) will increase or decrease monotonically from one positive constant \(C_3 = \frac{L^*_E}{C_1 + C_2}\) to another \(C_4 = \frac{L^*_E}{C_1 \gamma^{-1/\gamma} + C_2 \gamma^{1/\gamma}}\) as the growth rate, also on the left hand side, goes from 0 to infinity. This implies there is a single steady state in the model. As it is not a priory clear that \(C_3 < C_4\), we can draw the right and left hand side of Eq. (22) in the two panels in Fig. 3.

The arrows indicate that in both cases the steady state is stable if the growth rate of \(n\) increases when the right hand side of Eq. (22) exceeds the left hand side. We can establish this by considering the alternative. If the steady state is not stable, this implies that variety expansion would go to 0 or infinity out of steady state. Both cannot be an equilibrium outcome. Variety expansion can only be zero when all educated labor is allocated to R&D, in which case \(A/n\) would inevitably rise and educated labor would switch to entrepreneurship as soon as this drives the wage ratio in Eq. (16) above 1, turning the growth rate of \(n\) positive. Likewise, the rate of variety expansion cannot go to infinity for any finite level of employment in entrepreneurship unless the ratio \(A/n\) goes to infinity. Consequently, \(A\) would have to outgrow \(n\), and its growth rate would have to rise faster than that of \(n\), in which case Eq. (16) would inevitably drop below 1 and (all) educated labor would reallocate to R&D. This implies there is a unique and stable steady-state growth rate of \(n\) for which (22) holds. We cannot derive the transitional dynamics toward the steady state because we have assumed constant returns to educated labor in both entrepreneurship and R&D. This implies that the labor market equilibrium is a bang-bang equilibrium, and this discontinuity prevents us from analytically solving for out of equilibrium dynamics. We have also assumed perfect foresight and abstracted from uncertainty, such that the effective discount rates are assumed to immediately adjust to their long-run equilibrium values (which depend on the rates of innovation). If instead we had assumed that, more realistically, these effective discount rates are set and updated if proven wrong, then convergence toward the single steady state would also be driven by that process. That assumption, however, would needlessly complicate our model in the steady state.

Appendix 6: The central planner

The problem is a dynamic optimization problem with two control and three state variables that can be solved by constructing a Hamiltonian and deriving the first order conditions for that problem. The Hamiltonian is given by:
\[
H_C = e^{-\eta t} \log C(t) + \lambda(t) \left( A(t)^2 L^*_p(t) K(t)^{1-2-\beta} \right.
\]
\[
+ \mu(t) \left( \psi A(t)^{-\gamma} n(t)^{1-\beta} L_{EA}(t) \right)
\]
\[
+ \nu(t) \left( \phi A(t)^{1-n(t)^{1-\beta} (L^*_E - L_{EA})} \right)
\quad (55)
\]
In this setup, the optimal rate of capital accumulation is computed in a standard fashion. To save on notation, we will drop time arguments where this causes no confusion. Taking the first derivative of the Hamiltonian with respect to consumption and setting equal to zero yields an expression for the shadow price of consumption in terms of utility. This is given by:

\[
\lambda = \frac{e^{-\rho t}}{C} \tag{56}
\]

Taking the derivative of the Hamiltonian with respect to capital, \( K \), and setting it equal to minus the change in the shadow price of consumption yields the optimal consumption path:

\[
\dot{\lambda} = \frac{(1 - \alpha - \beta)X}{K} \lambda \tag{57}
\]

Solving this first order differential equation, using (56), we obtain:

\[
\frac{\dot{C}}{C} = gC = \frac{(1 - \alpha - \beta)X}{K} - \rho \tag{58}
\]

which shows that the central planner will save, invest and consume to keep the growth rate of consumption equal to the marginal product of capital minus the discount rate. This is a standard result in the literature [see, e.g., Barro and Sala-I-Martin (2004)] that our model replicates. The central planner invests up to the point where the marginal loss of utility due to lower immediate consumption is exactly offset by the additional discounted marginal utility increase due to higher future consumption (cf. Eq. 31 above).

The other first order conditions are more interesting. With respect to the control variable \( L_{EA} \), we obtain that the central planner equates their marginal contribution to discounted utility in the two alternative activities for skilled labor, invention and commercialization:

\[
A^{1 - \gamma} n^{\mu} = \phi A^{\delta} n^{1 - \delta} \tag{59}
\]

where \( \mu \) and \( \nu \) represent the shadow prices of the marginal process; an increase in \( A \), and product; an increase in \( n \), innovation, respectively. It will be convenient to solve (59) for \( \mu \) and compute the growth rate of this shadow price in the optimum as:

\[
\frac{\dot{\mu}}{\mu} = \dot{\nu} + (1 - \gamma - \delta)(\dot{n}/n - \dot{A}/A) \tag{60}
\]

The first order conditions with respect to \( A \) are given by:

\[
\frac{\partial H_C}{\partial A} = -\dot{\mu} = \frac{\alpha}{A} X - \mu(1 - \gamma) \frac{\dot{A}}{A} + \nu \frac{\dot{n}}{A} + \delta \frac{\dot{n}}{n} \tag{61}
\]

Rearranging this can be rewritten as:

\[
\frac{\dot{\mu}}{\mu} = -\frac{\alpha}{A} \frac{X}{\mu} - \frac{\delta}{A} \frac{\dot{n}}{n} - (1 - \gamma) \frac{\dot{A}}{A} \tag{62}
\]

The first order condition with respect to \( n \) is given by:
\[ \frac{\partial H_C}{\partial n} = -\dot{v} = (\alpha + \beta) X + \mu \gamma n + \mu \dot{A} + v(1 - \delta) \dot{n} \quad (63) \]

This expression can be rewritten into:

\[ \frac{\dot{v}}{v} = - (\alpha + \beta) \frac{X \dot{X}}{n} - (1 - \delta) \frac{\dot{n}}{n} - \gamma \frac{A \mu \dot{A}}{n v A} \quad (64) \]

When we equate Eq. (62) to (60) and substituting for the implied ratio between \( \mu \) and \( v \) in Eq. (59), we obtain:

\[ \frac{\dot{v}}{v} = -\delta \frac{\dot{A}}{\dot{A}} - \frac{X \dot{X}}{A \mu} - \left(1 - \gamma - \delta \frac{\psi (A)}{\phi (n)} \right) \frac{\dot{n}}{n} \quad (65) \]

Equating this to Eq. (64), taking all terms including \( X \) to the left hand side, rearranging and using the laws of motion to eliminate \( \dot{A} \) and \( \dot{n} \) from the right hand side allows us to write:

\[ \lambda(X(\alpha + \beta)A/\mu v n) = \delta \psi \left( \frac{A}{\gamma} \right) - \gamma \phi \left( \frac{A}{n} \right) \quad (66) \]

We can show that a steady state exists for \( \frac{\dot{A}}{\dot{A}} = \frac{\dot{n}}{n} \) and \( \dot{X} = \frac{\dot{X}}{X} = \left(1 - \alpha - \beta \right) \frac{\dot{X}}{X} - \phi \) by guessing it exists, and then show it is indeed a steady-state equilibrium that satisfies the first order and transversality conditions:

\[ \lim_{t \to \infty} \lambda(t) K(t) = 0 \]

\[ \lim_{t \to \infty} \mu(t) A(t) = 0 \quad (67) \]

\[ \lim_{t \to \infty} v(t) n(t) = 0 \]

By Eq. (56) and the “guess” that the growth rate of \( K \) is the same as that of \( C \), we can derive that the product \( \lambda(t) K(t) \) will grow at \( -\rho \) in the steady state. As this is negative, the first transversality condition is satisfied.

By Eq. (60) we can immediately derive that if \( \frac{\dot{A}}{\dot{A}} = \frac{\dot{n}}{n} \), the shadow prices for \( A \) and \( n \) also grow at the same rate, and we obtain \( \frac{\dot{n}}{n} = \frac{\dot{v}}{v} \). Therefore, the growth rate of the products \( \mu(t) A(t) \) and \( v(t) n(t) \) are equal and have to be negative to satisfy the transversality conditions, such that:

\[ \frac{\dot{n}}{n} = \frac{\dot{v}}{v} > \frac{\dot{A}}{A} = \frac{\dot{X}}{X} \quad (68) \]

Taking the growth rate of the left hand side in Eq. (66) and using \( \frac{\dot{X}}{X} = \frac{\dot{A}}{A} \), the left hand side growth rate is given by:

\[ \text{LHS} = -\frac{\dot{n}}{n} - \frac{\dot{v}}{v} - \rho \quad (69) \]

If this growth rate is positive, the left hand side will go to infinity in the steady state. As the right hand side of Eq. (66) is a function of \( \frac{\dot{A}}{A} \) and parameters only, we know that in that case \( \frac{\dot{A}}{A} \) should go to infinity (or zero depending on \( \gamma, \delta, \phi \) and \( \psi \)) as well. As that cannot be a steady state, we assume the growth rate in (69) is negative. With the constraint in inequality (68), this implies \( \frac{\dot{A}}{A} < -\frac{\dot{n}}{n} < \frac{\dot{X}}{X} + \rho \) and we know the right hand side of Eq. (66) will be zero in the steady state. This allows us to obtain for the steady state:

\[ \frac{A^{SS}}{n} = \left(\frac{\delta \psi}{\gamma \phi}\right)^{\frac{1}{\gamma - \delta}} \quad (70) \]

By Eqs. (5') and (13') and \( \frac{\dot{A}}{A} = \frac{\dot{n}}{n} \), we can also derive that in steady state the share of total educated labor allocated to R&D is given by:

\[ \frac{L_E^{SS}}{L} = \frac{\phi}{\phi + \psi \left(\frac{A}{n}\right)^{\gamma - \delta}} \quad (71) \]

And using Eqs. (71) and (70) in (5') and (13'), we can also derive the steady-state rate of knowledge accumulation and variety expansion:

\[ \frac{\dot{A}^{SS}}{A} = \frac{n^{SS}}{n} = \left(\frac{\gamma - \delta}{\gamma + \delta}\right)^{\frac{1}{\gamma - \delta}} \frac{L_E^{SS}}{L} \quad (72) \]

Taking growth rates of Eq. (23) and solving for growth in output using \( \frac{\dot{X}}{X} = \frac{\dot{K}}{K} \) yields:

\[ \frac{X^{SS}}{K} = \frac{\dot{X}^{SS}}{\dot{K}} = \frac{\dot{C}^{SS}}{\dot{C}} = \frac{2\alpha + \beta A^{SS}}{\alpha + \beta A} \]

\[ \frac{X}{K} = \frac{\dot{X}}{\dot{K}} = \frac{\dot{C}}{\dot{C}} = \frac{2\alpha + \beta (\gamma - \delta)(\phi + \psi)}{\alpha + \beta (\gamma + \delta)} \frac{L_E^{SS}}{L} \quad (73) \]

and plugging this into the utility function yields lifetime utility at:

\[ U^{CP} = \left(\frac{2\alpha + \beta}{(\gamma - \delta)} \frac{L_E^{SS}}{L} + (\alpha + \beta)(\gamma + \delta)\rho \log C_0\right) \frac{(\alpha + \beta)(\gamma + \delta)^2}{(\alpha + \beta)^2} \quad (74) \]

where \( C_0 \) is the initial level of consumption that is chosen to exhaust total discounted lifetime production (cf. Eq. 36).
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