Entanglement of topological phase factors

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Abstract. The topological phase factor induced on interfering electrons by external quantum electromagnetic fields has been studied. Two and three electron interference experiments inside distant cavities are considered and the influence of correlated photons on the phase factors is investigated. It is shown that the classical or quantum correlations of the irradiating photons are transferred to the topological phases. The effect is quantified in terms of Weyl functions for the density operators of the photons and illustrated with particular examples. The scheme employs the generalized phase factor as a mechanism for information transfer from the photons to the electric charges. In this sense, the scheme may be useful in the context of flying qubits (corresponding to the photons) and stationary qubits (electrons), and the conversion from one type to the other.

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1. Introduction

The study of phase factors arising in quantum interference has been crucial for the understanding of a wide range of physical phenomena [1]. The Aharonov-Bohm phase factor [2], \( \exp(iq\Phi) \), is acquired by a particle with charge \( q \) in a looping trajectory that encloses a classical magnetostatic flux \( \Phi \). This is true even when the particle moves in entirely field-free regions. The effect has been investigated in relation to transport phenomena in solid state physics [3] and electron coherence in mesoscopic devices [4]. The reciprocal phase factor [5] and the dual counterparts [6, 7] have also been studied and have recently found applications in different contexts, such as topological quantum information processing [8], the quantum Hall effect analogue with neutral atoms [9] and ultra-cold atom technology [10].

The generalized phase factor, \( \exp(iq\hat{\phi}) \), which is induced on a charge \( q \) by a nonclassical electromagnetic field with magnetic flux \( \hat{\phi} \) has also been studied in the literature [11]. In this case the magnetic flux and the induced phase factor are quantum mechanical operators. Consequently the important quantity in terms of interference properties is the expectation value of the phase factor, \( \langle \exp(iq\hat{\phi}) \rangle = \text{Tr}[\rho \exp(iq\hat{\phi})] \), with respect to the density matrix \( \rho \) that describes the external electromagnetic field. This phase factor is topological in the sense that it depends on the number of times an electron winds around the enclosed magnetic flux and it is independent of the electron velocity. The \( \langle \exp(iq\hat{\phi}) \rangle \) is a complex quantity, in general, which is known as the Weyl (or characteristic) function from quantum phase-space studies [12].

Clearly the inherent fluctuations of the external quantum fields bring about the problem of decoherence of the interfering electrons. Solutions have been proposed in relation to this problem using various methods [13]. Here it is assumed that under certain conditions the external photons do not interact with the interfering charges. In particular, it is assumed that the electromagnetic fields that are induced via Faraday’s law by the circulating electrons are negligible in comparison to the external fields, and so there is no back reaction. The inherent noise of the external photons manifests itself as a reduction of the absolute value of the phase factor, \( |\langle \exp(iq\hat{\phi}) \rangle| \), which becomes slightly less than one [14].

Nonclassical electromagnetic fields in various quantum states [15], such as squeezed and number states, have been generated at both optical and microwave frequencies [16]. Quantum mechanically correlated [17] photons have also been produced in the laboratory [18]. It is therefore reasonable to enquire whether we can use certain quantum interference devices, which are sensitive to the external radiation, as detectors of photon correlations. This has indeed been proposed recently using different techniques [19, 20]. In this paper we study photon-induced correlations between electron phase factors, which is the precursory mechanism for the detection of photon entanglement in distant quantum interference devices. It is shown that the phase factors of the electrons in interference experiments, which are initially independent of one another, become correlated when the experiments are irradiated with correlated photons. The setup
considered here may also be useful in the general area of flying and stationary qubits
and their interaction.

The rest of the paper is organized as follows. A possible implementation is analyzed and background material is provided in section 2. The correlations induced by the photons on the phase factors are quantified for the bipartite case in section 3. The problem is approached through examples, that involve classically and quantum mechanically correlated photons in number states and in coherent states, in section 4. This is subsequently generalized to the tripartite case in section 5, where examples are also provided. The results are discussed and conclusions are drawn in section 6.

2. Influence of entangled photons on distant interference experiments

We begin by introducing the setup depicted in figure 1: two interference devices for charged particles, A and B, are placed inside cavities that are far from each other. A source S_{EM} of two-mode nonclassical microwaves sends one mode of frequency \omega_1 into the cavity where A has been placed, and the other mode of frequency \omega_2 into the cavity where B has been placed. It has been shown that in this case the correlation between the two electromagnetic field modes is transferred to the distant quantum interference devices [20]. These devices could be, for example, nanoscale superconducting quantum interference devices (SQUIDs) [19], in which case the interfering particles are Cooper pairs; or simply two-path electron interference devices [20]. In either case the value of the phase factor, which depends on the external electromagnetic fields, influences the measurable physical quantities (in the case of superconducting rings the measurable variable is the current, while in electron interference one measures the intensity of electrons on the interference screen).

The external quantum fields are usually described by the vector potential \hat{A}_i and the electric field \hat{E}_i, which are dual quantum variables. The \hat{A}_i, \hat{E}_i can be transformed into another pair of dual variables by integrating them around a small loop l (that is, ‘small’ in comparison to the wavelength so that the field strengths are locally the same). This operation yields the magnetic flux \hat{\phi} = \oint_l \hat{A}_i dx_i and the electromotive force \hat{V}_{EMF} = \oint_l \hat{E}_i dx_i, respectively. The boson creation and annihilation operators may now be introduced as

\[ \hat{a}^\dagger = \frac{1}{\sqrt{2\xi}}(\hat{\phi} - i\omega^{-1}\hat{V}_{EMF}), \quad \hat{a} = \frac{1}{\sqrt{2\xi}}(\hat{\phi} + i\omega^{-1}\hat{V}_{EMF}) \]  

where \xi is a constant proportional to the area enclosed by l. They obey the usual commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\) (note that we employ units in which the Boltzmann constant, the Planck constant divided by 2\pi, and the speed of light in vacuum are set equal to one, \(k_B = \hbar = c = 1\)). The flux operator is consequently written in the Heisenberg picture as

\[ \hat{\phi}(t) = \exp(it\hat{H})\hat{\phi}(0) \exp(-it\hat{H}) \]

where

\[ H = H_{\text{free}} + H_{\text{int}}, \quad H_{\text{free}} = \omega(\hat{a}^\dagger \hat{a} + 1/2). \]
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Figure 1. Two distant interference devices for charged particles, A and B, are irradiated with nonclassical electromagnetic fields of frequencies $\omega_1$ and $\omega_2$ correspondingly. The electromagnetic fields emanate from a single source $S_{EM}$ and are correlated. It is required that the wavelengths of the fields are $\sim 1\text{mm}$ (microwaves) and that the interference devices have mesoscopic dimensions ($\sim 0.1\mu m$) operating at low temperatures of $10^{-100}\text{mK}$, such that $k_BT \ll \hbar\omega_1, \hbar\omega_2$.

The full Hamiltonian $H$ contains the free electromagnetic field Hamiltonian and an interaction term $H_{int}$, which includes the Hamiltonian of the interfering charges as well. In this paper we assume that the $H_{int}$, which describes the back reaction from the charges to the electromagnetic field, is neglected. In other words it is assumed that the self-induced magnetic flux of the charges is negligible compared to the external flux $\langle \hat{\phi}(t) \rangle$. In this approximation [11, 20] we get

$$\hat{\phi}(t) = \frac{\xi}{\sqrt{2}} \left[ \exp(i\omega t)\hat{a}^\dagger + \exp(-i\omega t)\hat{a} \right].$$

Exponentiating we obtain the phase factor for an electron of charge $e$:

$$\exp\left[ i\hat{\phi}(t) \right] = D\left[ iq \exp(i\omega t) \right], \quad q = \frac{\xi e}{\sqrt{2}}$$

where $q$ is introduced as a scaled electric charge. $D(\lambda) \equiv \exp(\lambda\hat{a}^\dagger - \lambda^*\hat{a})$ is the displacement operator.

Let $\rho_A$ be the density matrix describing the external nonclassical electromagnetic field mode in cavity A. The expectation value of the phase factor is given by the trace of the operator $\exp[\text{i}\hat{\phi}_A(t)]$ with respect to $\rho_A$. It is easily seen that taking the trace we obtain the single mode Weyl function

$$\tilde{W}_A(\lambda_A) \equiv \text{Tr}[\rho_A D(\lambda_A)], \quad \lambda_A = iq \exp(i\omega_1 t).$$

Similarly, the expectation value of the electron phase factor in experiment B is given by the Weyl function

$$\tilde{W}_B(\lambda_B) \equiv \text{Tr}[\rho_B D(\lambda_B)], \quad \lambda_B = iq \exp(i\omega_2 t).$$

It is important to note that these ‘expectation values’ are, in general, complex numbers. The reason for this is that the operator $D(z)$ is not Hermitian, since $D^\dagger(z) = D(-z)$.

To provide a physical interpretation consider that A is a two-path electron interference experiment. With each path we associate a wavefunction for the electrons, for example, $\psi_0$ and $\psi_1$ (let us assume equal splitting among them, for simplicity). It has
been shown elsewhere [20] that the intensity, or number density, of electrons at position \(x \equiv \text{arg}\psi_0 - \text{arg}\psi_1\) on the interference screen of experiment A is given by

\[I_A(x) = \text{Tr} \left[ \rho_A |\psi_0 + \langle \exp(i\hat{\phi}_A)\rangle \psi_1|^2 \right] = 1 + |\tilde{W}_A(\lambda_A)| \cos \{x + \text{arg} [\tilde{W}_A(\lambda_A)] \}. \tag{8}\]

It is clearly seen that the absolute value of the expectation value of the phase factor, \(|\tilde{W}_A(\lambda_A)|\), is the visibility \(\nu \equiv (I_{\text{max}} - I_{\text{min}})/(I_{\text{max}} + I_{\text{min}})\) of the interference. The \(\text{arg}[\tilde{W}_A(\lambda_A)]\) is the phase shift induced on the electrons by the irradiating electromagnetic field.

3. Correlations between electron phase factors

In this section we show how the electron phase factors in distant interference experiments become correlated when they are irradiated with correlated photons. The nature of the correlation between the external photons can be classical or quantum [17, 18] and the aim here is to compare and contrast the two cases. The difference between the two cases is firstly clarified.

The photons of frequencies \(\omega_1\) and \(\omega_2\) are described by density operators \(\rho_A\) and \(\rho_B\), correspondingly. If they are completely independent of each other then the density operator describing the bipartite state is factorizable, i.e., \(\rho_{\text{fac}} = \rho_A \otimes \rho_B\). If they are classically correlated then the bipartite state is described by the separable density operator \(\rho_{\text{sep}} = \sum_k P_k \rho_{A,k} \otimes \rho_{B,k}\), where the \(P_k\) are probabilities that sum up to unity. If the two photons are quantum mechanically correlated then their density operator \(\rho_{\text{ent}}\) is entangled and it can not be cast in the above forms in any way.

The expectation values of the electron phase factors \(\langle \exp(i\hat{\phi}_A)\rangle\) and \(\langle \exp(i\hat{\phi}_B)\rangle\) in the interference experiments A and B are given by the single mode Weyl functions \(\tilde{W}_A(\lambda_A)\) and \(\tilde{W}_B(\lambda_B)\) of equations (6) and (7), correspondingly. It is also possible to measure the product of the electron phase factors in A and B (joint phase factor). The expectation value of this product, \(\langle \exp(i\hat{\phi}_A) \exp(i\hat{\phi}_B)\rangle\), is given by the two-mode Weyl function

\[\tilde{W}_{AB}(\lambda_A, \lambda_B) = \text{Tr}[\rho D(\lambda_A)D(\lambda_B)]. \tag{9}\]

In the case of independent subsystems, which are described by \(\rho_{\text{fac}} = \rho_A \otimes \rho_B\), the \(\tilde{W}_{AB}(\lambda_A, \lambda_B)\) is equal to the product \(\tilde{W}_A(\lambda_A)\tilde{W}_B(\lambda_B)\). However for classically or quantum mechanically correlated subsystems the two-mode Weyl function is not equal to this product of one-mode Weyl functions, in general. This implies that the electron phase factors in A and B are correlated with each other.

In order to quantify the induced correlations between the electron phase factors we define

\[C \equiv \tilde{W}_{AB}(\lambda_A, \lambda_B) - \tilde{W}_A(\lambda_A)\tilde{W}_B(\lambda_B). \tag{10}\]

If the subsystems are not correlated with each other then \(C = 0\). If they are correlated then \(C \neq 0\) (i.e., the real and imaginary parts of \(C\) do not both vanish).
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The experimentally measurable quantities are the visibilities of the electron interferences in \(_A\) (\(|\tilde{W}_A(\lambda_A)\rangle\)) and in \(_B\) (\(|\tilde{W}_B(\lambda_B)\rangle\)); and the corresponding shifts of the interference fringes \(\arg(\tilde{W}_A), \arg(\tilde{W}_B)\). The absolute value (joint visibility) and the argument (joint phase shift) of \(\tilde{W}_{AB}(\lambda_A, \lambda_B)\) have to be measured simultaneously in the two experiments. Alternatively one may use a SQUID ring with a single Josephson junction irradiated with nonclassical electromagnetic fields \([19]\), in which case the \(\tilde{W}_A, \tilde{W}_B\) and the \(\tilde{W}_{AB}\) are calculated from the expectation values of the currents in \(_A\) and \(_B\) (and the product of the currents in both rings). For example, it is known that the current measured in \(_A\) is given by \(I_A = I_{cr} \text{Im}[\tilde{W}_A(\lambda_A)]\), where \(I_{cr}\) is the critical current.

4. Examples for the bipartite case

In this section we consider particular examples of classically and quantum mechanically correlated two-mode nonclassical electromagnetic fields in number states and in coherent states. The fundamental relations that are necessary for the derivation of the following results have been collected at the end of the paper in appendix A (for the number states) and appendix B (for the coherent states).

4.1. Photons in number states

Consider a two-mode electromagnetic field in the separable state

\[
\rho_{\text{sep}} = \frac{1}{2}(|N_1N_2\rangle\langle N_1N_2| + |N_2N_1\rangle\langle N_2N_1|). \tag{11}
\]

In this case the difference \(C\) of equation (10) is

\[
C_{\text{sep}} = \exp(-q^2)L_{N_1}(q^2)L_{N_2}(q^2) - \frac{1}{4} \exp(-q^2)[L_{N_1}(q^2) + L_{N_2}(q^2)]^2 \tag{12}
\]

where \(L_N\) are Laguerre functions. The \(C_{\text{sep}}\) is time independent; it depends only on the number of photons \(N_1, N_2\). It is clearly seen that \(|C| > 0\) for any number of photons.

On the other hand the entangled number state \(|n\rangle = 2^{-1/2}(|N_1N_2\rangle + |N_2N_1\rangle)\), with density operator

\[
\rho_{\text{ent}} = \rho_{\text{sep}} + \frac{1}{2}(|N_1N_2\rangle\langle N_2N_1| + |N_2N_1\rangle\langle N_1N_2|) \tag{13}
\]

yields

\[
C_{\text{ent}} = C_{\text{sep}} + \exp(-q^2)L_{N_1}^{N_2-N_1}(q^2)L_{N_2}^{N_1-N_2}(q^2) \cos(\Omega t) \tag{14}
\]

which is time dependent and oscillates around the \(C_{\text{sep}}\) with frequency

\[
\Omega = (N_1 - N_2)(\omega_1 - \omega_2). \tag{15}
\]

If there is no detuning between the external electromagnetic fields, in which case \(\omega_1 = \omega_2\), then the \(C_{\text{ent}}\) is constant in time but it is still different from the \(C_{\text{sep}}\). It is noted that for this example the difference \(C\) is purely real in both the separable and entangled cases.
4.2. Photons in coherent states

Consider two coherent states \( |A_1 \rangle \) and \( |A_2 \rangle \) in the classically correlated state
\[
\rho_{\text{sep}} = \frac{1}{2}(|A_1 A_2 \rangle \langle A_1 A_2| + |A_2 A_1 \rangle \langle A_2 A_1|).
\] (16)

In this case the reduced density operators that describe the coherent states propagating in cavities A and B are
\[
\rho_{\text{sep},A} = \rho_{\text{sep},B} = \frac{1}{2}(|A_1 \rangle \langle A_1| + |A_2 \rangle \langle A_2|).
\] (17)

We also consider the entangled state \( |S\rangle = \mathcal{N}(|A_1 A_2 \rangle + |A_2 A_1 \rangle) \) with density operator
\[
\rho_{\text{ent}} = 2\mathcal{N}^2 \rho_{\text{sep}} + \mathcal{N}^2(|A_1 A_2 \rangle \langle A_2 A_1| + |A_2 A_1 \rangle \langle A_1 A_2|)
\] (18)

where the normalization constant, which is such that \( \langle S|S\rangle = 1 \), is given by
\[
\mathcal{N} = \left[2 + 2 \exp\left(-|A_1 - A_2|^2\right)\right]^{-1/2}.
\] (19)

In this case the reduced density operators in A and B are
\[
\rho_{\text{ent},A} = \rho_{\text{ent},B} = \mathcal{N}^2(|A_1 \rangle \langle A_1| + |A_2 \rangle \langle A_2| + \tau_{12}|A_1 \rangle \langle A_2| + \tau_{12}^*|A_2 \rangle \langle A_1|)
\] (20)

where
\[
\tau_{12} = \langle A_1|A_2 \rangle = \exp\left(-\frac{|A_1|^2}{2} - \frac{|A_2|^2}{2} + A_1^* A_2\right).
\] (21)

The quantity \( C \) of equation (10) has been studied numerically using the relations provided in appendix B. For the separable case we have calculated numerically the \( C_{\text{sep}} = \tilde{W}_{AB,\text{sep}}(\lambda_A, \lambda_B) - \tilde{W}_{A,\text{sep}}(\lambda_A) \tilde{W}_{B,\text{sep}}(\lambda_B) \) and for the entangled case we have calculated the \( C_{\text{ent}} = \tilde{W}_{AB,\text{ent}}(\lambda_A, \lambda_B) - \tilde{W}_{A,\text{ent}}(\lambda_A) \tilde{W}_{B,\text{ent}}(\lambda_B) \). These are complex quantities and therefore in the following we present the results in terms of their absolute values \( |C_{\text{sep}}|, |C_{\text{ent}}| \) and their imaginary parts \( \text{Im}(C_{\text{sep}}), \text{Im}(C_{\text{ent}}) \).

4.3. Numerical results

For the numerical results in this section the values of the microwave frequencies have been set at \( \omega_1 = 1.2 \times 10^{-4} \) and \( \omega_2 = 1.0 \times 10^{-4} \). We have used units in which \( k_B = \hbar = c = 1 \). Other fixed parameters are \( \xi = 1 \) and the dimensionless electric charge \( e = (4\pi/137)^{1/2} \).

We study the entangled number state \( |\beta\rangle = 2^{-1/2}(|01\rangle + |10\rangle) \) and its closest separable state. We therefore let \( N_1 = 1, N_2 = 0 \) in the separable state of equation (11) and the entangled state of equation (13). The corresponding results for the \( |C_{\text{sep}}| \) and the \( |C_{\text{ent}}| \) against \( \Omega t \) have been plotted in figure 2. We note that the \( |C_{\text{sep}}| \), which is time independent, is not zero but it is very small in this case (\( \simeq 5 \times 10^{-4} \)).

In the case of coherent states we study the separable state of equation (16) and the entangled state of equation (18) for the same average number of photons as in the number states, that is, \( |A_1|^2 = N_1 \) and \( |A_2|^2 = N_2 \) (whereas \( \arg A_1 = 0, \arg A_2 = 0 \)).
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Figure 2. $|C_{\text{sep}}|$ (line of stars) corresponding to the separable number states of equation (11) and $|C_{\text{ent}}|$ (continuous line) corresponding to the entangled number states of equation (13) for $N_1 = 1, N_2 = 0$ as a function of $\Omega t$. The frequencies are $\omega_1 = 1.2 \times 10^{-4}$ and $\omega_2 = 10^{-4}$, in units where $k_B = \hbar = c = 1$. Note that $|C_{\text{sep}}|$ is not zero but $5 \times 10^{-4}$.

The results for the $|C_{\text{sep}}|$ and the $|C_{\text{ent}}|$ have been plotted against $\Omega t$ in subplots (a) and (c) of figure 3, correspondingly. We note that in this case $C$ is complex and also $C_{\text{sep}}$ is time dependent (in contrast to the case of number states). In subplots (b) and (d) of figure 3 the imaginary parts, Im($C_{\text{sep}}$) and Im($C_{\text{ent}}$), have been plotted against $\Omega t$.

In figure 2 we see that both the $C_{\text{sep}}$ and the $C_{\text{ent}}$ are nonzero; and that the $C_{\text{ent}}$ is time dependent. In fact this is true for any number of photons $N_1, N_2$ in the separable and entangled states $\rho_{\text{sep}}, \rho_{\text{ent}}$ as we can see from equations (12) and (14). Consequently the electron phase factors become correlated when the interference devices are irradiated with classically correlated ($\rho_{\text{sep}}$) or quantum mechanically correlated ($\rho_{\text{ent}}$) photons in number states. Clearly the quantity $C$ of equation (10) is different for the two cases, which implies that the nature of the correlation between the irradiating photons influences the induced correlation between the topological phase factors. In figure 3 we see that the same general result is true for the case of classically and quantum mechanically correlated photons in coherent states. It is also evident that the correlations between the phase factors are influenced by the quantum noise and statistics of the external photons, by comparison of figures 2 and 3, which correspond to photons in number states and coherent states.

5. Examples for the tripartite case

In this section we consider three electron interference devices of mesoscopic dimensions that are placed inside distant microwave cavities. The interference experiments A, B, and C are irradiated with nonclassical electromagnetic fields of frequencies $\omega_1, \omega_2,$ and
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Figure 3. (a) $|C_{\text{sep}}|$ and (b) Im($C_{\text{sep}}$) corresponding to the separable coherent states of equation (14); (c) $|C_{\text{ent}}|$ and (d) Im($C_{\text{ent}}$) corresponding to the entangled coherent states of equation (18) for $A_1 = 1, A_2 = 0$ as a function of $\Omega t$. The frequencies are $\omega_1 = 1.2 \times 10^{-4}$ and $\omega_2 = 10^{-4}$, in units where $k_B = \hbar = c = 1$.

The phase factor acquired by the interfering electrons in $A$ is given by $\tilde{W}_A(\lambda_A)$ of equation (6) and the phase factor in $B$ is given by $\tilde{W}_B(\lambda_B)$ of equation (7). Similarly the phase factor in $C$ is obtained from $\tilde{W}_C(\lambda_C) = \text{Tr}[\rho_C D(\lambda_C)]$, where $\lambda_C = i q \exp(i \omega_3 t)$. We can also measure the product of the three phase factors, which is given by the three mode Weyl function

$$\tilde{W}_{ABC}(\lambda_A, \lambda_B, \lambda_C) = \text{Tr}[\rho D(\lambda_A) D(\lambda_B) D(\lambda_C)].$$

The tripartite correlations between the electron phase factors can be quantified with a straightforward generalization of the quantity $C$ of equation (11), that is, in this case we define

$$C \equiv \tilde{W}_{ABC}(\lambda_A, \lambda_B, \lambda_C) - \tilde{W}_A(\lambda_A)\tilde{W}_B(\lambda_B)\tilde{W}_C(\lambda_C).$$

If the phase factors are not correlated then $C = 0$. If they are correlated then $|C| > 0$ (and also possibly $\text{Im}(C) \neq 0$).
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5.1. Photons in number states

As an example of tripartite number states consider the separable state
\[ \rho_{sep} = \frac{1}{2}(|N_1N_2N_3\rangle\langle N_1N_2N_3| + |N_2N_3N_1\rangle\langle N_2N_3N_1|) \] (24)
and the entangled state
\[ |n_{tri}\rangle = 2^{-1/2}(|N_1N_2N_3\rangle + |N_2N_3N_1\rangle) \]
with density operator
\[ \rho_{ent} = \rho_{sep} + \frac{1}{2}(|N_1N_2N_3\rangle\langle N_2N_3N_1| + |N_2N_3N_1\rangle\langle N_1N_2N_3|). \] (25)

The results for the three mode Weyl function of equation (22) corresponding to the separable and entangled number states are straightforward, albeit lengthy. Only the numerical calculations are presented in terms of time \( \Omega' t \), where \( \Omega' \) has replaced \( \Omega \) of equation (15), which was valid for the bipartite case. In particular it is not hard to show that the difference between the separable and entangled Weyl functions includes a time dependent term of frequency \( \Omega' \), which is given by
\[ \tilde{W}_{ABC,ent} - \tilde{W}_{ABC,sep} \propto \text{Re}(\langle N_1|D(\lambda_A)|N_2\rangle\langle N_2|D(\lambda_B)|N_3\rangle\langle N_3|D(\lambda_C)|N_1\rangle). \] (26)
From this term we obtain the appropriate frequency for the tripartite case, namely
\[ \Omega' = N_1(\omega_3 - \omega_1) + N_2(\omega_1 - \omega_2) + N_3(\omega_2 - \omega_3). \] (27)

5.2. Photons in coherent states

We consider the separable coherent state
\[ \rho_{sep} = \frac{1}{2}(|A_1A_2A_3\rangle\langle A_1A_2A_3| + |A_2A_3A_1\rangle\langle A_2A_3A_1|). \] (28)
In this case the reduced density operators are

\[ \rho_{\text{sep}, A} = 2^{-1}(|A_1\rangle\langle A_1| + |A_2\rangle\langle A_2|) \]
\[ \rho_{\text{sep}, B} = 2^{-1}(|A_2\rangle\langle A_2| + |A_3\rangle\langle A_3|) \]
\[ \rho_{\text{sep}, C} = 2^{-1}(|A_3\rangle\langle A_3| + |A_1\rangle\langle A_1|). \]

We also consider the entangled state \( |S_{\text{tri}}\rangle = \mathcal{N}'(|A_1A_2A_3| + |A_2A_3A_1\rangle) \) with density operator

\[ \rho_{\text{ent}} = 2\mathcal{N}'^2 \rho_{\text{sep}} + \mathcal{N}'^2(|A_1A_2A_3\rangle\langle A_2A_3A_1| + |A_2A_3A_1\rangle\langle A_1A_2A_3|) \]

where the normalization constant is given by

\[ \mathcal{N}' = [2 + 2\text{Re}(\tau_{12} + \tau_{23} + \tau_{31})]^{-1/2} \]

for \( \tau_{ij} = \langle A_i|A_j\rangle = \exp(-|A_i|^2/2 - |A_j|^2/2 + A_i^*A_j) \) as in equation (21), for example. In this case the reduced density operators are

\[ \rho_{\text{ent}, A} = \mathcal{N}'^2(2\rho_{\text{sep}, A} + \tau_{12}\tau_{32}|A_1\rangle\langle A_2| + \tau_{12}\tau_{32}^*|A_2\rangle\langle A_1|) \]
\[ \rho_{\text{ent}, B} = \mathcal{N}'^2(2\rho_{\text{sep}, B} + \tau_{21}\tau_{13}|A_3\rangle\langle A_2| + \tau_{21}\tau_{13}^*|A_2\rangle\langle A_3|) \]
\[ \rho_{\text{ent}, C} = \mathcal{N}'^2(2\rho_{\text{sep}, C} + \tau_{12}\tau_{23}|A_3\rangle\langle A_1| + \tau_{12}\tau_{23}^*|A_1\rangle\langle A_3|). \]

5.3. Numerical results

For the numerical results in this section the photon frequencies are \( \omega_1 = 1.2 \times 10^{-4}, \omega_2 = 1.1 \times 10^{-4}, \omega_3 = 1.0 \times 10^{-4} \) in units where \( k_B = \hbar = c = 1 \), and \( \xi = 1 \).
We study the entangled tripartite state $|\beta_{\text{tri}}\rangle = 2^{-1/2}(|012\rangle + |120\rangle)$ and its closest separable state. We therefore let $N_1 = 0, N_2 = 1,$ and $N_3 = 2$ in the separable number state of equation (24) and the entangled number state of equation (25). The corresponding results for the $|C_{\text{sep}}|$ and the $|C_{\text{ent}}|$ against $\Omega' t$ in the case of tripartite number states have been plotted in figure 4. In this case both the $|C_{\text{sep}}|$ and the $|C_{\text{ent}}|$ are very small, but in principle measurable.

In the case of coherent states we study the separable state of equation (28) and the entangled state of equation (30) for the same average number of photons as in the number states, therefore we let $A_1 = 0, A_2 = 1, A_3 = \sqrt{2}$. The $|C_{\text{sep}}|$ and the $|C_{\text{ent}}|$ have been plotted against $\Omega' t$ in (a) and (c) of figure 5, correspondingly. In subplots (b) and (d) of figure 5 the corresponding imaginary parts, $\text{Im}(C_{\text{sep}})$ and $\text{Im}(C_{\text{ent}})$, have been plotted against $\Omega' t$.

In figure 6 we show the $|C_{\text{ent}} - C_{\text{sep}}|$ for coherent states with $A_1 = 0, A_2 = 1, A_3 = \sqrt{2}$ as a function of $\Omega' t$. The solid line corresponds to the tripartite case, where the $\rho_{\text{sep}}$ and $\rho_{\text{ent}}$ are given by equations (28) and (30), respectively. The line of circles corresponds to the bipartite case, where the $\rho_{\text{sep}}$ and $\rho_{\text{ent}}$ are given by equations (16) and (18), respectively. It is clearly seen that in both cases there is a significant difference between the $C_{\text{sep}}$ and the $C_{\text{ent}}$. It is also seen that the absolute value of $C_{\text{ent}} - C_{\text{sep}}$ for the tripartite case is an order of magnitude greater than in the bipartite case. Therefore the quantum part of $C$ does not diminish as the photon correlations are distributed to more than two electron interference devices.
6. Discussion

It has been recognized that geometrical and topological phases \([1, 2, 5, 6]\) could be harnessed for the purposes of inherently fault-tolerant quantum computation \([8, 22]\). It has also been known for some time that the quantum mechanical correlations of physical states are a useful resource for quantum information processing \([17]\). The aim of this paper has been to study the photon-induced correlations of topological phase factors for charged particles in distant interference experiments. It has been shown that the classical or quantum correlations of the irradiating photons are transferred to the phase factors of the circulating electrons. This mechanism may allow for the detection of photon entanglement using nanoscale electronic devices \([19, 20]\).

In particular, we have considered the one-mode Weyl functions of equations (6) and (7) for the density operators \(\rho_A\) and \(\rho_B\) of the photons propagating in the distant cavities \(A\) and \(B\). They yield the expectation values of the electron phase factors in the two interference experiments. These can be measured experimentally through the visibility and the phase shift of the interference fringes. We have also considered the two-mode Weyl function of equation (9) for the bipartite state \(\rho\). This yields the joint phase factor in both experiments. Using these Weyl functions we have defined the difference \(C\) of equation (10), which vanishes only for independent subsystems. Considering suitable examples of classically and quantum mechanically correlated photons in number and in coherent states, we have shown that \(C\) does not vanish and that therefore the electron phase factors are correlated. We have also shown that the value of \(C\) depends on the quantum noise and statistics of the external photons (figures 2 and 3). Further work is required in order to distinguish between classical and quantum mechanical correlations using the proposed setup. One possibility would be to derive a Bell-type inequality for the two-mode Weyl function, which is obeyed in the separable case, but it is violated in the entangled case.

It has also been shown that the same general result applies to the tripartite case. In this case the joint phase factor is measured in three distant electron interference experiments and its expectation value is given by the three mode Weyl function of equation (22). The difference \(C\) is in this case replaced by that of equation (23). Numerical results have been presented in figures 4-6 for several examples of classically and quantum mechanically correlated number states and coherent states.

In conclusion we have shown that it is possible to entangle the topological phase factors of interfering electrons that are irradiated with nonclassical electromagnetic fields. In future work it would be very interesting to derive similar results on the photon-induced entanglement of geometric phases acquired by spin-1/2 particles \([23]\), or Cooper pairs in mesoscopic Josephson junctions \([24]\), for example. In the last few years there has been a lot of work on the role of entanglement in mesoscopic devices \([25]\). The setup discussed in this paper may be useful in the production of entangled electric charges in a normal conductor or a superconductor using topological phases that are induced by external photons. This is within the realm of current experimental techniques, whereby
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a nanoscale Josephson device can be controlled with a single microwave photon [26].

Appendix A. Relations for number states

The following relation yields the matrix elements of the displacement operator in the number state basis [27]:

$$\langle m|D(z)|n \rangle = \left( \frac{n!}{m!} \right)^{1/2} z^{m-n} \exp \left( -\frac{|z|^2}{2} \right) L^n_m (|z|^2). \quad (A.1)$$

Using this it can easily be shown that

$$\tilde{W}_A(\lambda_A) = \tilde{W}_B(\lambda_B) = 2^{-1} \exp(-q^2/2)[L_{N_1}(q^2) + L_{N_2}(q^2)] \quad (A.2)$$

for the $\rho_{\text{sep}}$ of equation (11) and the $\rho_{\text{ent}}$ of equation (13). The two-mode Weyl function of equation (9) for the $\rho_{\text{sep}}$ is

$$\tilde{W}_{AB,\text{sep}}(\lambda_A, \lambda_B) = \exp(-q^2)L_{N_1}(q^2)L_{N_2}(q^2). \quad (A.3)$$

However for the $\rho_{\text{ent}}$ we have

$$\tilde{W}_{AB,\text{ent}}(\lambda_A, \lambda_B) = \tilde{W}_{AB,\text{sep}}(\lambda_A, \lambda_B) + \exp(-q^2)L_{N_1}^{N_2-N_1}(q^2)L_{N_2}^{N_1-N_2}(q^2) \cos(\Omega t). \quad (A.4)$$

Appendix B. Relations for coherent states

In the coherent states basis we have

$$\langle A|D(z)|B \rangle = \langle 0|D(-A + z + B)|0 \rangle \exp(\chi) \quad (B.1)$$

where the $\langle 0|D(-A + z + B)|0 \rangle$ can be calculated with the help of equation (A.1) and the phase $\chi$ is given by

$$\chi = \frac{1}{2}(-Az^* + A^*z - AB^* + A^*B - z^*B + zB^*) \quad (B.2)$$

for any complex numbers $A, B, z$.

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