All Link Invariants for Two Dimensional Solutions of Yang-Baxter Equation and Dressings

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Abstract

All polynomial invariants of links for two dimensional solutions of Yang-Baxter equation is constructed by employing Turaev’s method. As a consequence, it is proved that the best invariant so constructed is the Jones polynomial and there exist three solutions connecting to the Alexander polynomial. Invariants for higher dimensional solutions, obtained by the so-called dressings, are also investigated. It is observed that the dressings do not improve link invariant unless some restrictions are put on dressed solutions.

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1 Introduction

Importance of Yang-Baxter equation (YBE) in knot theory has been widely recognized. In the context of knot theory, the YBE appears as a matrix expression of Reidemeister move of type III when we assign a matrix to each crossing of a link diagram. This means that the YBE is an essential element for constructing topological invariants of links. Turaev gave a general method to construct link invariants starting from solutions of YBE [1]. Kauffman gave a more diagrammatic method to construct invariants by interpreting a solution of YBE as vertex weights of state expansion of a given link diagram [2]. Solutions of YBE also appear in statistical mechanics as a Boltzmann weight of exactly solvable models. Motivated by this fact, Akutsu, Wadati and Deguchi developed a general method of constructing link invariants starting from exactly solvable models in statistical mechanics [3-9].

In any aforementioned construction of invariants, we need to know solutions of YBE explicitly. One way of finding solutions of YBE is to use representations of quantum groups (see for example [10]). In general, for each representation of a quantum group, there exist an associated solution of YBE. This method, therefore, can give arbitrary dimensional solutions. However this construction does not exhaust all possible solutions for fixed dimension. Classification of solutions of YBE has not been completed yet except constant case in two dimension. We mean by constant a YBE and its solutions which do not contain spectral parameters. Two dimensional constant solutions of YBE have been classified by Hietarinta [11, 12]. While classification of solutions having spectral parameters was made in [13] where eight-vertex type solutions of YBE in two dimension were studied in connection with solvable two-component models.

In the present work, we investigate all possible link invariants obtained from the two dimensional constant solutions of YBE. We apply Turaev’s construction to the two dimensional solutions obtained by Hietarinta. We anticipated to find new polynomial invariants of links, since two dimensional solutions contain at most three independent parameters. However the present investigation reveals that two dimensional solutions do not produce better invariants than already known ones. It is also shown that three two dimensional $R$-matrices are related to the Alexander polynomial.

The above mentioned results motivate us to study higher dimensional constant solutions. We study higher dimensional solutions obtained not via quantum groups but via the method called dressing [15]. Dressing is a method to construct higher dimensional solutions from known lower dimensional ones. We shall show that dressing do not improve link invariants unless we put some additional conditions on dressing.

This paper is organised as follows. We give a brief review of Turaev’s construction and two dimensional solutions of YBE in the next two sections in order to fix our notations and conventions. Section 4 is devoted to construction and classification of link invariants for two dimensional solutions. We will see that there exist three $R$-matrices from which the Alexander polynomial is obtained. A state model for one of them is discussed also in section 4. Link invariants for higher dimensional solutions obtained by dressings are investigated in section 5. Some concluding remarks are given in section 6.
2 Turaev’s Construction of Link Invariants

Let $V$ be a $N$ dimensional vector space over a field $K$, and $R : V \otimes V \to V \otimes V$ satisfy the YBE of the braid group form

$$ R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}. \quad (2.1) $$

An enhanced Yang-Baxter operator (EYB) is a collection $(R, \mu, \alpha, \beta)$, where $\mu : V \to V$ is a homomorphism and $\alpha, \beta$ are invertible elements of $K$, subject to the conditions

$$ R \circ (\mu \otimes \mu) = (\mu \otimes \mu) \circ R, \quad (2.2) $$

$$ \text{Tr}_2 (R \circ (\mu \otimes \mu)) = \alpha \beta \mu, \quad (2.3) $$

$$ \text{Tr}_2 (R^{-1} \circ (\mu \otimes \mu)) = \alpha^{-1} \beta \mu, \quad (2.4) $$

where $\text{Tr}_2$ is a trace for the second vector space in the tensor product. It is widely known that the $R$-matrix gives rise to a representation of the $n$-strand braid group $B_n$ on $V^\otimes n$.

Let $\{ \sigma_i, i = 1, \ldots, n-1 \}$ be generators of $B_n$. Define an isomorphism $R_i : V^\otimes n \to V^\otimes n$ by

$$ R_i = \text{id}^\otimes (i-1) \otimes R \otimes \text{id}^\otimes (n-i-1). \quad (2.5) $$

Then the representation $(\pi, V^\otimes n)$ of $B_n$ is defined by

$$ \pi(\sigma_{i+1}) = R_{i+1}. \quad (2.6) $$

where $w(\xi)$ is the writhe of $\xi$ defined by $w(\xi) = w(\xi)$.

The invariant $T_S(L)$ has the following properties

(i) If a link $L$ is the disjoint union of two links $L_1$ and $L_2$, then

$$ T_S(L) = T_S(L_1) T_S(L_2). \quad (2.7) $$

(ii) If the $R$-matrix satisfies the relation

$$ \sum_{i=p}^q k_i R_i = 0, \quad k_p, \ldots, k_q \in K \quad (2.8) $$

then the invariants $T_S$ satisfies the skew-type relation

$$ \sum_{i=p}^q k_i \alpha^i T_S(L_i) = 0, \quad (2.9) $$

where $L_i$‘s are links depicted in Figure 1.

(iii) If $S = (R, \mu, \alpha, \beta)$ is an EYB, then $S_1 = (-R, -\mu, \alpha, \beta)$, $S_2 = (R, -\mu, \alpha, -\beta)$ and $S_3 = (R, -\mu, -\alpha, \beta)$ are also EYB. For any $\ell$-component link $L$, the corresponding invariants are related

$$ T_{S_1}(L) = T_{S_2}(L) = T_{S_3}(L) = (-1)\ell T_S(L). \quad (2.10) $$
3 Two Dimensional Solutions of YBE

Two dimensional \((N = 2)\) constant solutions to YBE are extensively studied by Hietarinta by using computer \cite{11, 12}. The classification of the solutions was made up to the invariance of YBE by the following transformations

\[
R \rightarrow R' = \kappa (Q \otimes Q) R (Q \otimes Q)^{-1}, \quad \text{(3.1)}
\]

\[
R_{ij}^{k\ell} \rightarrow R_{ij}^{k\ell} = R_{ij}^{k\ell}, \quad \text{(3.2)}
\]

\[
R_{ij}^{k\ell} \rightarrow R_{ij}^{k\ell} = R_{i+n,j+n}^{k\ell} \quad \text{(indices mod } N), \quad \text{(3.3)}
\]

\[
R_{ij}^{k\ell} \rightarrow R_{ij}^{k\ell} = R_{j+i}^{k\ell}, \quad \text{(3.4)}
\]

where \(\kappa \neq 0 \in \mathbb{K}\) and \(Q\) is a nonsingular \(N \times N\) matrix. These transformations do not affect the invariant \(T_S(L)\) defined in the last section. The proof of this fact is given in Appendix A. It is, therefore, legitimate to investigate link invariants according to the classification of \cite{11, 12}.

Hietarinta obtained 35 solutions (including zero matrix); one three-parameter, four two-parameter, 15 one-parameter and 14 no-parameter solutions. No-parameter solutions will give numerical invariants which are less powerful than polynomial invariants. Thus we exclude such solutions. The solutions given by singular matrices are also excluded, since inverse matrices are assigned to negative crossings. We give a list of nonsingular solutions excepting no-parameter ones. Entries of the \(R\)-matrices are labelled as follows

\[
R = \begin{pmatrix}
R_{++}^{++} & R_{++}^{+-} & R_{++}^{+\pm} & R_{++}^{-\pm} & R_{++}^{-+} \\
R_{+-}^{++} & R_{+-}^{+-} & R_{+-}^{+\pm} & R_{+-}^{-\pm} & R_{+-}^{-+} \\
R_{-+}^{++} & R_{-+}^{+-} & R_{-+}^{+\pm} & R_{-+}^{-\pm} & R_{-+}^{-+} \\
R_{-\pm}^{++} & R_{-\pm}^{+-} & R_{-\pm}^{+\pm} & R_{-\pm}^{-\pm} & R_{-\pm}^{-+} \\
R_{-\pm}^{++} & R_{-\pm}^{+-} & R_{-\pm}^{+\pm} & R_{-\pm}^{-\pm} & R_{-\pm}^{-+}
\end{pmatrix}. \quad \text{(3.5)}
\]

3 parameters

\[
R_{3,1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & q & 0 \\
0 & p & 0 & 0 \\
0 & 0 & 0 & s
\end{pmatrix}.
\]
2 parameters

\[ R_{2,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & p & 1 - pq & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_{2,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & p & 1 - pq & 0 \\ 0 & 0 & 0 & -pq \end{pmatrix}, \]

\[ R_{2,3} = \begin{pmatrix} 1 & 1 & p & q \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & p \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

1 parameter

\[ R_{1,1} = \begin{pmatrix} 1 + 2q - q^2 & 0 & 0 & 1 - q^2 \\ 0 & 1 - q^2 & 1 + q^2 & 0 \\ 0 & 1 + q^2 & 1 - q^2 & 0 \\ 1 - q^2 & 0 & 0 & 1 - 2q - q^2 \end{pmatrix}, \]

\[ R_{1,2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & q & 0 \\ 0 & 1 & 1 - q & 0 \\ 0 & 0 & 0 & -q \end{pmatrix}, \quad R_{1,3} = \begin{pmatrix} 1 & 1 & -q \\ 0 & 0 & 1 & -q \\ 0 & 1 & 0 & q \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

\[ R_{1,4} = \begin{pmatrix} 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & 0 \end{pmatrix}. \]

4 Link Invariants

4.1 EYB and List of Invariants

In this section, we derive all invariants of links obtained from the \( R \)-matrices listed in the previous section. Key of the derivation is to find an EYB for a given \( R \)-matrix. Our procedure to obtain all possible EYB for a given \( R \)-matrix is summarised as follows.

(i) Regarding the condition (2.3) as a set of equations with \( \mu, \alpha \) and \( \beta \) as unknown variables, solve it with MAPLE. Usually, this gives several solutions containing some variables that are not determined yet.

(ii) For each solution, the undetermined variables are fixed so as to satisfy the conditions (2.2) and (2.4). There are some cases where the conditions require unacceptable results; \( \mu = 0 \), or \( \alpha = 0 \), or \( \beta = 0 \), or all the parameters in the \( R \)-matrix being fixed. These cases are excluded.

The EYB’s so obtained are summarised in Table 1. Note that one \( R \)-matrix gives rise to some inequivalent EYB’s.
Table 1: EYB and link invariants. $\beta$ takes arbitrary value for all cases.

| $R$   | $\mu$                           | $\alpha$ | invariants        | comments       |
|-------|---------------------------------|----------|-------------------|----------------|
| $R_{3,1}$ | diag($\pm \beta, \pm \beta$) | $\pm 1$  | 1 (for knots)     | $s = 1$       |
|       | diag($\pm \beta, \mp \beta$)  | $\pm 1$  | 0 (for knots)     | $s = -1$      |
|       | diag($\pm \beta, 0$)           | $\pm 1$  |                   | $s = \pm 1$   |
|       | diag($0, \pm \beta$)           | $\pm s$  |                   |                |
| $R_{2,1}$ | diag($\pm \sqrt{pq} \beta, \pm \beta/\sqrt{pq}$) | $\pm \frac{1}{\sqrt{pq}}$ | Jones          |                |
|       | diag($\pm \beta, 0$)           | $\pm 1$  |                   |                |
|       | diag($0, \pm \beta$)           | $\pm 1$  |                   |                |
|       | $\left( \begin{array}{cc} \pm \beta & 0 \\ \lambda & 0 \end{array} \right)$ | $\pm 1$  |                   |                |
|       | $\left( \begin{array}{cc} 0 & \lambda \\ 0 & \pm \beta \end{array} \right)$ | $\pm 1$  |                   |                |
| $R_{2,2}$ | diag($\pm \beta/\sqrt{pq}, \mp \beta/\sqrt{pq}$) | $\pm \sqrt{pq}$ | 0 | Alexander      |
|       | diag($\pm \beta, 0$)           | $\pm 1$  |                   |                |
|       | diag($0, \mp \beta$)           | $\pm pq$ |                   |                |
| $R_{2,3}$ | diag($\pm \beta, \pm \beta$) | $\pm 1$  | 2$^p$             | $p = -1$      |
|       | diag($\pm \beta, \mp \beta$)  | $\pm 2q$ | 0 | Alexander      |
|       | diag($\pm \beta/\sqrt{q}, \mp \beta/\sqrt{q}$) | $\pm 2q$ | 0 | Alexander      |
| $R_{1,1}$ | $\pm \frac{\beta}{2} \left( \begin{array}{cc} 1 + q & \sqrt{1 - q^2} \\ \sqrt{1 - q^2} & 1 - q \end{array} \right)$ | $\mp 2$ | 1 |                |
|       | $\pm \frac{\beta}{2} \left( \begin{array}{cc} -1 - q & \sqrt{1 - q^2} \\ \sqrt{1 - q^2} & -1 + q \end{array} \right)$ | $\mp 2$ | 1 |                |
|       | $\pm \frac{\beta}{2q} \left( \begin{array}{cc} 1 + q & \sqrt{1 - q^2} \\ -\sqrt{1 - q^2} & -1 + q \end{array} \right)$ | $\pm 2$ | 1 |                |
|       | $\pm \frac{\beta}{2q} \left( \begin{array}{cc} -1 - q & \sqrt{1 - q^2} \\ -\sqrt{1 - q^2} & 1 - q \end{array} \right)$ | $\mp 2$ | 1 |                |
| $R_{1,2}$ | diag($\pm \beta/\sqrt{q}, \mp \beta/\sqrt{1 + q}$) | $\pm 2$ | 1 |                |
|       | $\left( \begin{array}{ccc} \pm \beta & \pm \beta/\sqrt{1 + q} \\ 0 & 0 \end{array} \right)$ | $\pm 1$ | 1 |                |
|       | $\left( \begin{array}{ccc} \pm \beta & \mp \beta/\sqrt{1 + q} \\ 0 & 0 \end{array} \right)$ | $\pm 1$ | 1 |                |
| $R_{1,3}$ | diag($\pm \beta, \mp (1 + q) \beta$) | $\pm 1$ | 2$^p$ |                |
| $R_{1,4}$ | diag($\pm \beta, \pm \beta$) | $\pm 1$ | 1 (for knots) |                |
In Table 1, the values of link invariants corresponding to each EYB are also shown. Each EYB has two possibilities of signs. However, because of (2.10), the sign difference of EYB causes a difference of overall factor of invariants. The indicated invariants correspond to one of the sign choices. Some details on the invariants are studied in the following.

(i) $R_{3,1}$ with $s = 1$, $\mu = \text{diag}(\beta, \beta)$, $\alpha = 1$.

For this choice of $\mu$ and $\alpha$, $s = 1$ is required to satisfy the conditions of EYB. Thus the $R$-matrix $R = R_{3,1}$ is reduced to two-parameter and it satisfies

$$R^2 - R - pq + pqR^{-1} = 0.$$  \hfill (4.1)

By (2.9), the invariant satisfies the skein-type relation

$$T_S(L_{++}) - T_S(L_+) - pqT_S(L_0) + pqT_S(L_-) = 0,$$  \hfill (4.2)

where $L_{++}, L_+, L_0$ and $L_-$ are links depicted in Figure 2. The value for the unknot $U$ is $T_S(U) = 2$. Furthermore, it turns out that $T_S(K) = 2$ for any knot $K$. To see this, we derive another version of skein-type relation. Multiplying $R$ to (4.1) and doing a sum of it and (4.1), we obtain

$$R^2 - (1 + pq)R + pqR^{-1} = 0.$$  \hfill (4.3)

Thus the invariant satisfies

$$T_S(L_{+++}) = (1 + pq)T_S(L_+) - pqT_S(L_-).$$  \hfill (4.4)

The induction on the number of crossings proves that $T_S(K) = 2$ for any knot $K$. By normalizing $T_S(K)$ by the value for unknot, we obtain $T_S(K) = 1$.

(ii) $R_{3,1}$ with $s = -1$, $\mu = \text{diag}(\beta, -\beta)$, $\alpha = 1$.

For this choice of $\mu$ and $\alpha$, $s = -1$ is required to satisfy the conditions of EYB. Thus the $R$-matrix $R = R_{3,1}$ is reduced to two-parameter and it satisfies

$$R^2 - (1 + pq) + pqR^{-2} = 0.$$  \hfill (4.4)
Thus we obtain
\[ T_S(L^+) - (1 + pq)T_S(L_0) + pqT_S(L^-) = 0. \] (4.5)

It turns out that \( T_S(K) = 0 \) for any knot \( K \). One can prove it in a way similar to case (i). The reason of the vanishing invariants is that the invariant for unknot is equal to 0. Even if we regularize the value for unknot being nonvanishing, one can show, by the skein relation, that \( T_S \) for trefoil is the same as the one for unknot. Thus the invariant is less powerful than the Jones or the Alexander polynomial.

(iii) \( R_{3,1}, \mu = \text{diag}(\beta, 0), \alpha = 1 \).

There is no restriction on \( R = R_{3,1} \) in this case. It is easy to verify
\[ R^2 - (1 + s)R + (s - pq) + pq(1 + s)R^{-1} - spqR^{-2} = 0. \] (4.6)

It follows that
\[ T_S(L^+) - (1 + s)T_S(L_0) + (s - pq) + pq(1 + s)T_S(L^-) - spqT_S(L^-) = 0. \] (4.7)

Although the skein-type relation has a complex form, it turns out that \( T_S(L) = 1 \) for any link \( L \). This is a consequence of \( (\mu \otimes \mu)R^\pm = s^\pm(\mu \otimes \mu) \).

(iv) \( R_{3,1}, \mu = \text{diag}(0, \beta), \alpha = s \).

This case is exactly the same as case (iii). The skein-type relation is given by (4.7) and \( T_S(L) = 1 \) for any link \( L \), since \( (\mu \otimes \mu)R^\pm = s^\pm(\mu \otimes \mu) \).

(v) \( R_{2,1}, \mu = \text{diag}(\sqrt{pq}\beta, \beta/\sqrt{pq}), \alpha = 1/\sqrt{pq} \).

The matrix \( R = R_{2,1} \) satisfies the relation
\[ R + (pq - 1) - pqR^{-1} = 0. \] (4.8)

It follows that
\[ (pq)^{-1}T_S(L^+) - pqT_S(L_-) = (1/\sqrt{pq} - \sqrt{pq})T_S(L_0). \] (4.9)

The value for the unknot is \( T_S(U) = \sqrt{pq} + 1/\sqrt{pq} \). The two parameters in \( R \) are combined together to give a polynomial in one variable \( t = pq \). This invariant is the Jones polynomial.

(vi) \( R_{2,1}, \mu = \text{diag}(\beta, 0), \alpha = 1 \).

The skein relation is found from (4.8) and it is given by
\[ \frac{1}{\sqrt{pq}}T_S(L^+) - \sqrt{pq}T_S(L_-) = \left( \frac{1}{\sqrt{pq}} - \sqrt{pq} \right)T_S(L_0). \] (4.10)

Although the skein relation has the form of specialisation of HOMFLY polynomial, the invariant takes the trivial value; \( T_S(L) = 1 \) for any link \( L \). This is due to the same reason as case (iii).
(vii) \( R_{2,1}, \mu = \text{diag}(0, \beta), \alpha = 1. \)
This case is exactly the same as case (vi).

(viii) \( R_{2,1} \) with \( q = 1, \mu = \begin{pmatrix} \beta & 0 \\ \lambda & 0 \end{pmatrix}, \alpha = 1. \)
For this choice of \( \mu \) and \( \alpha \), \( q = 1 \) is required to satisfy the conditions of EYB.
Thus the \( R \)-matrix \( R = R_{2,1} \) is reduced to one-parameter. The matrix \( \mu \) contains an additional free parameter \( \lambda \).
The skein relation is obtained by setting \( q = 1 \) in (4.10).
It turns out again that \( T_S(L) = 1 \) for any link \( L \). This is due to the same reason as case (iii).

(ix) \( R_{2,1} \) with \( q = 1, \mu = \begin{pmatrix} 0 & \lambda \\ 0 & \beta \end{pmatrix}, \alpha = 1. \)
This case is exactly the same as case (viii).

(x) \( R_{2,2}, \mu = \text{diag}(\beta/\sqrt{pq}, -\beta/\sqrt{pq}), \alpha = \sqrt{pq}. \)
The \( R = R_{2,2} \) satisfies the same relation as \( R_{2,1} \)
\[ R + (pq - 1) - pqR^{-1} = 0. \] (4.11)
It follows that the skein relation is
\[ T_S(L_+) - T_S(L_-) = \left( \frac{1}{\sqrt{pq}} - \sqrt{pq} \right) T_S(L_0). \] (4.12)
This is the skein relation of Alexander polynomial. However, it turns out that \( T_S(L) = 0 \) for any link \( L \).
To see this, note that if a link \( L \) is the disjoint union of two links \( L_1 \) and \( L_2 \), then, because of the skein relation (4.12), \( T_S(L) = 0. \) While by (2.7) \( T_S(L) = T_S(L_1)T_S(L_2) = 0. \) This means \( T_S(L_1) = 0 \) or \( T_S(L_2) = 0. \) Thus \( T_S(L) = 0 \) for any \( L. \)

(xi) \( R_{2,2}, \mu = \text{diag}(\beta, 0), \alpha = 1. \)
Because of the same reason as case (iii), \( T_S(L) = 1 \) for any link \( L. \)

(xii) \( R_{2,2}, \mu = \text{diag}(0, \beta), \alpha = -pq. \)
Note that \( (\mu \otimes \mu)R^\pm_1 = - (pq)^\pm_1(\mu \otimes \mu) \) holds in this case. The same argument as case (iv) shows that \( T_S(L) = 1 \) for any link \( L. \)

(xiii) \( R_{2,3} \) with \( p = -1, \mu = \text{diag}(\beta, \beta), \alpha = 1. \)
For this choice of \( \mu \) and \( \alpha \), \( p = -1 \) is required to satisfy the conditions of EYB.
Thus the \( R \)-matrix \( R = R_{2,3} \) is reduced to one-parameter and it satisfies
\[ R^2 - R - 1 + R^{-1} = 0. \] (4.13)
It follows that the invariant satisfies the skein-type relation
\[ T_S(L_{++}) - T_S(L_+) - T_S(L_0) + T_S(L_-) = 0. \] (4.14)
One can see, by the induction on the number of crossings, that $T_S(L) = 2^\ell$ for a $\ell$-component link $L$.

(xiv) $R_{1,1}$, $\mu = \text{diag}(\beta, -\beta)$, $\alpha = 2q$. The matrix $R = R_{1,1}$ satisfies the relation

$$R + 2(q^2 - 1) - 4q^2 R^{-1} = 0.$$  

(4.15)

The skein relation is given by

$$T_S(L+) - T_S(L-) = (q^{-1} - q)T_S(L_0).$$  

(4.16)

This is the skein relation of the Alexander polynomial. However, by the same reason as case (x), it turns out that $T_S(L) = 0$ for any link $L$.

(xv) $R_{1,1}$, $\mu = \frac{\beta}{2} \left( \begin{array}{cc} 1 + q & -\sqrt{1 - q^2} \\ \sqrt{1 - q^2} & 1 - q \end{array} \right)$, $\alpha = 2$.

The skein relation follows from (4.15) and it reads

$$q^{-1}T_S(L+) - qT_S(L-) = (q^{-1} - q)T_S(L_0).$$  

(4.17)

It is easy to see that $(\mu \otimes \mu)R^{\pm 1} = 2^{\pm 1}(\mu \otimes \mu)$. Therefore, by the same reason as case (iv), the invariant takes a single value $T_S(L) = 1$ for any link $L$.

(xvi) $R_{1,1}$, $\mu = \frac{\beta}{2q} \left( \begin{array}{cc} 1 + q & -\sqrt{1 - q^2} \\ \sqrt{1 - q^2} & 1 - q \end{array} \right)$, $\alpha = 2$.

This case is exactly the same as case (xv).

(xvii) $R_{1,1}$, $\mu = \frac{\beta}{2q} \left( \begin{array}{cc} 1 + q & \sqrt{1 - q^2} \\ \sqrt{1 - q^2} & 1 + q \end{array} \right)$, $\alpha = 2$.

This case is exactly the same as case (xv).

(xviii) $R_{1,1}$, $\mu = \frac{\beta}{2q} \left( \begin{array}{cc} 1 + q & -\sqrt{1 - q^2} \\ \sqrt{1 - q^2} & 1 + q \end{array} \right)$, $\alpha = 2$.

This case is exactly the same as case (xv).

(xix) $R_{1,2}$, $\mu = \text{diag}(\beta/\sqrt{q}, -\beta/\sqrt{q})$, $\alpha = \sqrt{q}$.

The matrix $R = R_{1,2}$ satisfies the relation

$$R + (q - 1) - qR^{-1} = 0.$$  

(4.18)

It follows that the invariant satisfies the skein relation

$$T_S(L+) - T_S(L-) = (\frac{1}{\sqrt{q}} - \sqrt{q})T_S(L_0).$$  

(4.19)

This is the skein relation of the Alexander polynomial. However, by the same reason as case (x), it turns out that $T_S(L) = 0$ for any link $L$. 

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The skein relation follows from (4.18) and it reads
\[
\frac{1}{\sqrt{q}} T_S(L_+) - \sqrt{q} T_S(L_-) = \left( \frac{1}{\sqrt{q}} - \sqrt{q} \right) T_S(L_0). \tag{4.20}
\]

It turns out that \( T_S(L) = 1 \) for any link \( L \), due to the same reason as case (iii).

This case is exactly the same as case (xx).

The matrix \( R = R_{1,3} \) is peculiar, since it satisfies \( R^2 = 1 \). It follows that the invariant has the property \( T_S(L_+) = T_S(L_-) \). The value for the unknot is \( T_S(U) = 2 \). Thus \( T_S(L) = 2^\ell \) for a \( \ell \)-component link \( L \).

The matrix \( R = R_{1,4} \) satisfies the relation
\[
R^2 - R - q^2 + q^2 R^{-1} = 0. \tag{4.21}
\]

It follows that the skein-type relation is given by
\[
T_S(L_{++}) - T_S(L_+) - q^2 T_S(L_0) + q^2 T_S(L_-) = 0. \tag{4.22}
\]

One can show, in a way similar to case (i), that \( T_S(K) = 1 \) for any knot \( K \).

Besides the above listed \( R \)-matrices, there exist four non-singular no-parameter \( R \)-matrices. It is seen that one can construct EYB’s from those four \( R \)-matrices and they give numerical invariants. We do not show them here, because numerical invariants are less powerful than polynomial ones. In summary, we obtained the following theorem.

**Theorem 1** All the non-singular two dimensional solutions of YBE gives rise to EYB’s. The best invariant of links obtained from the EYB’s is the Jones polynomial.

### 4.2 A Yang-Baxter State Model for Alexander Polynomial

From Table 1, it is seen that three \( R \)-matrices, \( R_{2,2} \), \( R_{1,1} \) and \( R_{1,2} \), are related to the Alexander polynomial. However, the Turaev’s construction gives vanishing value for the corresponding invariants. We need to use another method to show that those \( R \)-matrices can be a source of the Alexander polynomial. The case of one-parameter restriction of \( R_{2,2} \) has been studied in some works. With a \( R \)-matrix obtained from \( R_{2,2} \) by a scaling, Jaeger constructed a state model for the Alexander polynomial which can be interpreted.
as an ice-type model [14]. This model was reinterpreted as a Yang-Baxter state model by Kauffman [2]. The supersymmetric aspects of the Alexander polynomial were studied in connection with the supersymmetric counterpart of a one-parameter restriction of $R_{2,2}$ [15, 16]. In [15], the Alexander polynomial is described as a fermionic integral, that is, a state model involving bosonic and fermionic loops. The quantum field theory descriptions of the multi-variable Alexander polynomial based on the WZW model and the Chern-Simons model are given in [16].

These works encourage us to study $R_{1,1}$ and $R_{1,2}$ in the same footing. We note that there is a fundamental difference between $R_{1,1}$, $R_{1,2}$ and $R_{2,2}$ besides the number of parameters. The matrix $R_{2,2}$ is spin preserving, however $R_{1,1}$ and $R_{1,2}$ are not. The $R$-matrices having the following property are said to be spin preserving

$$R_{ab}^{cd} \neq 0, \quad \text{if and only if} \quad a + b = c + d.$$  

Since the Jones polynomial also arises from the spin preserving $U_q(sl_2)$ $R$-matrix, Kauffman said that the Jones and Alexander polynomials have the same footing [2]. In this subsection, we construct a Yang-Baxter state model for $R_{1,2}$ to show the relevance of the $R$-matrix for the Alexander polynomial. The $R_{1,1}$ case will be studied elsewhere.

A Yang-Baxter state models is a combinatorial summation that is well-defined on oriented diagrams. For a given diagram of a link, states are produced based on the $R$-matrix. Each state carries a uniquely determined quantity and their sum is shown to be an invariant. We mention that the first state model for the Alexander polynomial, that is, not Yang-Baxter state model, is given in [17]. The Yang-Baxter state model discussed in [2] is technically distinct from the ones for other invariants. We have to represent links as two-strand tangles, otherwise the invariant vanishes for all links. The reason of this is similar to Turaev’s construction. Naive construction of the state model shows that the value for loops has to be zero. We therefore regularize the unknot by replacing it with an unknotted strand and assign 1 for the strand.

We follow this regularization and investigate the simpler case $R_{1,2}$. Set $R = tR_{1,2}$ and $t = q^{-1/2}$

$$R = \begin{pmatrix}
  t & 0 & 0 & t \\
  0 & 0 & t^{-1} & 0 \\
  0 & t & t^{-1} & 0 \\
  0 & 0 & 0 & -t^{-1}
\end{pmatrix}, $$  

then

$$R - R^{-1} = (t - t^{-1})I,$$  

where $I$ denotes the identity matrix. The $R$-matrix and its inverse are assigned to positive and negative crossings of diagrams, respectively. Thus the positive and negative crossings have the splicings and projections given in Figure 3. The vertex weights for each splicing and projection are indicated below the diagrams. The diagrams with double-line stand for spin non-preserving contributions. Note that the spin change occurs only from negative to positive. Thus the spin changing diagrams never form closed loops, that is, the spin changing vertices have no contributions to states. Let us denote the state sum for a
diagram $L$ by

$$\langle L \rangle = \sum_\sigma \langle L | \sigma \rangle i^{||\sigma||},$$

(4.26)

where $\sigma$ stands for a state and $\langle L | \sigma \rangle$ denotes the product of vertex weights of the state. $||\sigma||$ is the sum of the indices of the loops in $\sigma$ multiplied by the rotation number defined by

$$\text{rot} \left( \begin{array}{c} \circ \end{array} \right) = +1, \quad \text{rot} \left( \begin{array}{c} \circ \end{array} \right) = -1.$$

(4.27)

By the regularization mentioned above, for the unknotted strand

$$\langle \begin{array}{c} - \end{array} \begin{array}{c} + \end{array} \rangle = \langle \begin{array}{c} - \end{array} \begin{array}{c} - \end{array} \rangle + \langle \begin{array}{c} + \end{array} \begin{array}{c} - \end{array} \rangle = 1.$$

(4.28)

It is seen by the splicings and projections given in Figure 3 that the state summation satisfies

$$\langle \begin{array}{c} \times \end{array} \begin{array}{c} \times \end{array} \rangle - \langle \begin{array}{c} \times \end{array} \begin{array}{c} \times \end{array} \rangle = (t - t^{-1}) \langle \begin{array}{c} \times \end{array} \begin{array}{c} \times \end{array} \rangle.$$

(4.29)

Since the spin non-preserving vertices do not contribute to the state summation, one can prove the invariance of $\langle L \rangle$ under the Reidemeister moves II, III essentially the same way as in [2]. It is easy to see the behaviour under the Reidemeister move I

$$\langle \begin{array}{c} - \end{array} \begin{array}{c} \circ \end{array} \rangle = -it \langle \begin{array}{c} - \end{array} \begin{array}{c} - \end{array} \rangle, \quad \langle \begin{array}{c} + \end{array} \begin{array}{c} \circ \end{array} \rangle = it^{-1} \langle \begin{array}{c} + \end{array} \begin{array}{c} - \end{array} \rangle,$$

$$\langle \begin{array}{c} - \end{array} \begin{array}{c} \circ \end{array} \rangle = -it \langle \begin{array}{c} + \end{array} \begin{array}{c} + \end{array} \rangle, \quad \langle \begin{array}{c} + \end{array} \begin{array}{c} \circ \end{array} \rangle = it^{-1} \langle \begin{array}{c} - \end{array} \begin{array}{c} - \end{array} \rangle.$$

(4.30)

We obtain an invariant of links by normalising $\langle L \rangle$ by the rotation number of the diagram $L$. The proof of the following proposition is easy, so it is omitted.

**Proposition 1** Define $\nabla_L = (it^{-1})^{-\text{rot}(L)} \langle L \rangle$, then
(i) $\nabla_L$ is an invariant of links.

(ii) $\nabla_{-} = 1$.

(iii) $\nabla_{L_+} - \nabla_{L_-} = z\nabla_{L_0}$, where $z = t - t^{-1}$.

By this proposition, one can say that $\nabla_L$ is indeed the Alexander polynomial. We note that $\nabla_L$ is independent of the choice of the segment dropped to make a link diagram to a corresponding tangle. This also follows from the proposition 1 (See [2]).

We observed that the spin non-preserving vertex does not contribute to the Alexander polynomial. The legitimacy of this fact is obvious by comparing the spin preserving part of $R_{1,2}$ and $R_{2,2}$.

5 Higher Dimensional Extensions

5.1 Dressings of EYB

We learnt from the results of the last section that two dimensional solutions of YBE never give link invariants better than the Jones polynomial. Even the $R$-matrix contains two or three parameters, obtained invariants are reduced to polynomials in one variable. We thus have to use higher dimensional solutions of YBE to obtain better link invariants. However, classification of all solutions of YBE for $N \geq 3$ is far from completion. Even for $N = 3$, only the partial classification is known [18]. One way to obtain higher dimensional solutions is to use representations of quantum groups. Investigation of link invariants along the line has been done by many authors (see for example [19, 20]). Another way is an embedding of a known $R$-matrix to a larger matrix and then make it solve YBE [18]. Two kinds of such a method, called dressings, are presented in [18]. In this section, we apply the Turaev’s construction to $R$-matrices obtained by dressings and study the constructed link invariants. We will see that improvement of link invariants by dressings is minor. Thus we should find solutions of YBE not by dressings if we want better invariants.

Let us define two kinds of dressings. Let $I = \{1, 2, \cdots, N\}$ be a set of indices and $J$ be a selection of $M(< N)$ numbers from $I$. Let $\tilde{R}$ be a $M$ dimensional solution of YBE and $R$ be a $N$ dimensional dressed solution.

(a) diagonal dressing

$$R_{ijkl} = \begin{cases} \tilde{R}_{ijkl} & i, j, k, \ell \in J \\ s_{ij} \delta_{j}^{k} \delta_{l}^{i} & \text{otherwise} \end{cases} \tag{5.1}$$

is a solution of YBE, if $s_{im}$ satisfies

$$\tilde{R}_{ijkl}(s_{im}s_{jm} - s_{km}s_{ml}) = 0,$$

$$\tilde{R}_{ijkl}(s_{mi}s_{km} - s_{jm}s_{ml}) = 0,$$

$$\tilde{R}_{ijkl}(s_{mi}s_{mk} - s_{mj}s_{mi}) = 0. \tag{5.2}$$

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(b) block dressing

\[
R^{kl}_{ij} = \begin{cases} 
\tilde{R}^{kl}_{ij} & i, j, k, \ell \in J \\
\delta^i_k F^\ell_j & i, \ell \in J, \ j, k \notin J \\
G^k_i \delta^\ell_j & j, k \in J, \ i, \ell \notin J \\
f^{ij}_{ij} \delta^i_j \delta^\ell_j & \text{otherwise}
\end{cases}
\]  \tag{5.3}

is a solution of YBE, if the matrices \(F, G\) satisfy

\[
\begin{align*}
(F \otimes F) \tilde{R} &= \tilde{R} (F \otimes F), & (G \otimes G) \tilde{R} &= \tilde{R} (G \otimes G), \\
(F \otimes 1) \tilde{R} (G \otimes 1) &= (1 \otimes G) \tilde{R} (1 \otimes F), & [F, G] &= 0.
\end{align*}
\]  \tag{5.4}

An important sub-case is \(G = F^{-1}\). In this case, the first equation of (5.4) is sufficient. We note that our definitions of dressings are slightly different from the ones in [18]. One difference comes from that we use a braid group form of YBE. Another is that we put additional constants \(f^{ij}_{ij}\) at a diagonal part of block dressing. The role of the constants will be clear in consideration of EYB.

We now turn to construction of EYB for the dressed \(R\)-matrices. Let \(\tilde{S} = (\tilde{R}, \tilde{\mu}, \alpha, \beta)\) be an EYB for \(\tilde{R}\). Our result for a diagonal dressing is summarised as

**Proposition 2** Let \(R\) be diagonal dressings of \(\tilde{R}\). Then \(S = (R, \mu, \alpha, \beta)\) is an EYB for the following \(\mu\)’s.

1. if there is no further restriction on \(s^{ij}_{ij}\), then

\[
\mu^k_j = \begin{cases} 
\tilde{\mu}^k_j & k, j \in J \\
0 & \text{otherwise}
\end{cases}
\]  \tag{5.5}

is the only possible \(\mu\).

2. if \(s^{kk}_{kk} = \pm \alpha\) for \(k \notin J\) and \(\tilde{\mu}\) is diagonal, then

\[
\mu^k_j = \begin{cases} 
\tilde{\mu}^k_j & k, j \in J \\
\pm \beta \delta^k_j & \text{otherwise}
\end{cases}
\]  \tag{5.6}

While our result for block dressing is as follows.

**Proposition 3** Let \(R\) be a block dressing of \(\tilde{R}\). Then \(S = (R, \mu, \alpha, \beta)\) is an EYB for the following \(\mu\)’s.

1. if there is no further restriction on \(F, G, f^{ij}_{ij}\), then

\[
\mu^k_j = \begin{cases} 
\tilde{\mu}^k_j & k, j \in J \\
0 & \text{otherwise}
\end{cases}
\]  \tag{5.7}

is the only possible \(\mu\).

2. if \([F, \tilde{\mu}] = [G, \tilde{\mu}] = 0\) and \(f^{kk}_{kk} = \pm \alpha\) for \(k \notin J\), then

\[
\mu^k_j = \begin{cases} 
\tilde{\mu}^k_j & k, j \in J \\
\pm \beta \delta^k_j & \text{otherwise}
\end{cases}
\]  \tag{5.8}
Remark. Our main assertion is (5.5) and (5.7). Namely, dressed $R$-matrices allow only trivially dressed EYB’s. We need further restrictions on dressed $R$-matrices if we want to have nontrivial dressed EYB’s. (5.6) and (5.8) are examples of such a case and they may not be the only possibility of dressed EYB’s.

Proof. We here give a proof of both propositions, since the way of proof is similar. We first note that it is easy to verify that the μ’s given in the propositions 2, 3 satisfy the definition of EYB. Therefore, our main concern is the question whether there exist other possible μ’s which make $S$ be an EYB. Let us set
$$
\mu^k_j = \begin{cases} 
\hat{\mu}^k_j & k, j \in \mathcal{J} \\
\mu^k_j & \text{otherwise}
\end{cases}
$$
(5.9)
and study possible values of $t^k_j$. In terms of matrix entries, the conditions of EYB (2.2), (2.3) and (2.4) read
$$
\sum_{k_1, k_2} P^{k_1 j_1} P^{j_2 k_2} \mu_1^{k_1} \mu_2^{j_2} = \sum_{k_1, k_2} \mu_1^{j_1} \mu_2^{j_2} R^{k_1 \ell_2}_{k_1 k_2},
$$
(5.10)
$$
\sum_{k_1, k_2} P^{k_1 j_1} \mu_1^{k_1} \mu_2^{j_2} = \alpha \beta \mu_1^{j_1},
$$
(5.11)
$$
\sum_{k_1, k_2} (R^{-1})^{k_1 j_1} \mu_1^{k_1} \mu_2^{j_2} = \alpha^{-1} \beta \mu_1^{j_1}.
$$
(5.12)

For the sake of clarity, we use Greek letters for indices not belonging to $\mathcal{J}$ throughout the proof.

(a) diagonal dressing

Let us first study (5.10). We pick up the choices of indices by which (5.10) is quadratic in $t^k_j$. If $j_1, j_2, \ell_1, \ell_2 \notin \mathcal{J}$, then
$$
t^{\alpha_1 \beta_1}_{\alpha_2 \beta_2} (s_{\alpha_1 \alpha_2} - s_{\beta_1 \beta_2}) = 0.
$$
Since this is true for arbitrary $s_{\alpha_1 \alpha_2}$, $t^{\alpha}_{\beta}$ must be diagonal: $t^{\alpha}_{\beta} = t^{\alpha}_{\alpha} \delta_{\alpha \beta}$. If $j_2, \ell_2 \notin \mathcal{J}$ and $j_1, \ell_1 \in \mathcal{J}$, then
$$
t^{\ell \beta}_{\alpha} t^{\beta}_{\alpha} (s_{\alpha j} - s_{\ell \beta}) = 0.
$$
It follows that we need to set
$$
t^{\ell}_{\alpha} = 0 \quad \text{or} \quad t^{\beta}_{\ell} = 0.
$$
(5.13)
If $\ell_1, j_2 \notin \mathcal{J}$ and $j_1, \ell_2 \in \mathcal{J}$, then
$$
t^{\beta_1 \beta_2}_{\alpha_2 j_1} (s_{\alpha_2 j_1} - s_{\beta_1 \beta_2}) = 0.
$$
It follows that
$$
t^{\ell}_{\alpha} = 0 \quad \text{or} \quad t^{\beta}_{\ell} = 0.
$$
(5.14)
If $j_1, j_2 \notin \mathcal{J}$ and $\ell_1 \in \mathcal{J}$, then
$$
t^{\beta_1 \beta_2}_{\alpha_2 \alpha_1} (s_{\alpha_1 \alpha_2} - s_{\beta_1 \beta_2}) = 0.
$$
Thus

\[ t_\alpha^\ell = 0 \quad \text{or} \quad t_\alpha = 0. \quad (5.15) \]

Other choices of indices not belonging to \( J \) also give one of the requirements (5.13)-(5.15). We therefore have three possibilities:

(i) \( t_\alpha^\ell = t_\alpha = 0 \),  
(ii) \( t_\alpha = t_\beta^j = 0 \),  
(iii) \( t_\alpha = t_\beta^j = 0 \). \quad (5.16)

Note that these are not sufficient conditions for (5.10). Indeed, there are other relations which must be hold to make (5.10) true. Those are automatically satisfied when we take into account (5.11) and (5.12).

Case (i) : Consider the case of \( j_1, \ell_1 \notin J \) for (5.11) and (5.12), we obtain

\[ \delta_\nu^\sigma \ t_\nu \ (s^{\pm 1})_{\nu \nu} t_\nu - \alpha^{\pm 1} \beta = 0. \]

It follows that \( t_\nu = 0 \) or \( t_\nu = \pm \beta \), \( s_{\nu \nu} = \pm \alpha \). The choice of \( t_\nu = 0 \) corresponds to (5.3). While for \( t_\nu = \pm \beta \neq 0 \), we have to return to (5.10) and consider the case of \( j_2, \ell_1 \notin J \) and \( j_1, \ell_2 \in J \). For this case, (5.10) reads

\[ t_\alpha \tilde{\mu}_j (s_{\alpha j} - s_{\alpha \ell}) = 0. \]

Since this is true for any \( s_{\alpha j}, s_{\alpha \ell}, \tilde{\mu}_j \) must be diagonal.

Case (ii) : Consider the case of \( j_1 \notin J, \ell_1 \in J \) for (5.11) and (5.12), we obtain

\[ \sum_{j \in J} (s_{j \nu} \tilde{\mu}_j - \delta_j^\alpha \delta_{\nu \beta}) t_\nu^j = 0, \quad \sum_{j \in J} ((s^{-1})_{\nu j} \tilde{\mu}_j - \delta_j^\alpha \delta_{\nu \beta}) t_\nu^j = 0. \quad (5.17) \]

For a fixed value of \( \nu \), these are regarded as sets of linear equations in \( t_\nu^j \). To have nonvanishing \( t_\nu^j \), the coefficient matrices have to be singular. This requirement puts a restriction on \( s_{j \nu} \) and \( s_{\nu j} \). Thus we have \( t_\nu^j = 0 \), if there is no further restriction on \( s_{ji} \).

On the other hand, if one finds \( s_{j \nu} \) and \( s_{ij} \) that makes the coefficient matrices singular, there exist other \( \mu_j \) which is not mentioned in Proposition 2. However, it may be difficult to satisfy both requirements of singular coefficient matrices and (5.2).

Case (iii) : By considering the case of \( \ell_1 \notin J \) and \( j_1 \in J \) for (5.11) and (5.12), we obtain the same relation as (5.17) in which \( s_{j \nu} \) and \( s_{\nu j} \) are exchanged. We thus follow the same discussion as Case (ii).

We have completed the proof of Proposition 2.

(b) block dressing

Let us first study (5.10). If \( j_1, j_2, \ell_1, \ell_2 \notin J \), then

\[ t_{\alpha_1}^{\beta_1} t_{\alpha_2}^{\beta_2} (f_{\alpha_1 \alpha_2} - f_{\beta_1 \beta_2}) = 0. \]

Thus \( t_\beta^j \) must be diagonal : \( t_\beta^j = t_\beta^j \delta_\alpha^\gamma \). If \( j_2, \ell_1 \notin J \) and \( j_1, \ell_2 \in J \), then

\[ t_\beta^j [F, \tilde{\mu}_j] = 0. \]
It follows that
\[ t_{\alpha} = 0 \quad \text{or} \quad [F, \tilde{\mu}] = 0. \]  
(5.18)

If \( j_1, \ell_2 \notin J \) and \( j_2, \ell_1 \in J \), then
\[ t_{\alpha} \beta [G, \tilde{\mu}]_{j} = 0. \]

It follows that
\[ t_{\alpha} = 0 \quad \text{or} \quad [G, \tilde{\mu}] = 0. \]  
(5.19)

Thus we have two possibilities:

(i) \( t_{\alpha} = 0 \),

(ii) \([F, \tilde{\mu}] = [G, \tilde{\mu}] = 0.\)

We note again that these are not sufficient conditions for (5.10). Others are automatically satisfied when we take into account (5.11) and (5.12).

Case (i): Consider the case of \( j_1 \notin J \) and \( \ell_1 \in J \) for (5.11) and (5.12), then we obtain
\[
\sum_{j, k \in J} (G_j \tilde{\mu}_k - \delta_{j}^{1} \alpha \beta) t_{\nu} = 0, \\
\sum_{j, k \in J} ((F^{-1})_j k \tilde{\mu}_k - \delta_{j}^{1} \alpha^{-1} \beta) t_{\nu} = 0.
\]  
(5.21)

For a fixed value of \( \nu \), these are regarded as sets of linear equations in \( t_{\nu} \). To have nonvanishing \( t_{\nu} \), the coefficient matrices have to be singular. This requirement puts a restriction on \( F, G \). Thus we have \( t_{\nu} = 0 \), if there is no further restriction on \( F, G \). Next we consider the case of \( j_1 \in J \) and \( \ell_1 \notin J \) for (5.11) and (5.12):
\[
\sum_{j_2, k_1, k_2 \in J} \tilde{R}_{j_2}^{k_1 k_2} \tilde{\mu}_{k_2}^{j_2} = \alpha \beta t_{\nu} , \\
\sum_{j_2, k_1, k_2 \in J} ((\tilde{R}^{-1})_{j_2}^{k_1 k_2} \tilde{\mu}_{k_2}^{j_2} = \alpha^{-1} \beta t_{\nu}.
\]  
(5.22)

To convert the equations in (5.22) to ones containing \( F \) or \( G \), we go back to (5.10). If \( \ell_1 \notin J \) and \( j_1, j_2, \ell_2 \in J \), then we have
\[
\sum_{k_1, k_2 \in J} \tilde{R}_{j_1, j_2}^{k_1 k_2} \tilde{\mu}_{k_2}^{j_2} = \sum_{k \in J} \tilde{\mu}^{j_2} \tilde{\mu}_{k}^{j_2} F_{k}^{j_2}.
\]

Taking a sum over \( j_2 = \ell_2 \) and using the first equation of (5.22), we obtain
\[
\sum_{j, k \in J} (F_{j}^{k} \tilde{\mu}_{j_1}^{k} - \delta_{j_1}^{1} \alpha \beta) t_{\nu} = 0.
\]  
(5.23)

Similarly, by considering the case \( \ell_2 \notin J \), \( j_1, j_2, \ell_1 \in J \), and taking into account the second equation in (5.22), we obtain
\[
\sum_{j, k \in J} ((G^{-1})_{j}^{k} \tilde{\mu}_{j_1}^{k} - \delta_{j_1}^{1} \alpha^{-1} \beta) t_{\nu} = 0.
\]  
(5.24)

For a fixed value of \( \nu \), (5.23) and (5.24) are regarded as sets of linear equations in \( t_{\nu} \). Thus the same discussion as before concludes that \( t_{\nu} = 0 \) if there is no further restriction on \( F \) and \( G \). If we allow to put restrictions on \( F \) and \( G \), there is a possibility of the existence of nonvanishing \( t_{\nu} \) and \( t_{\nu} \) which is not mentioned in Proposition 3. However,
it may be difficult to satisfy the above requirements of singular coefficient matrices and \((5.4)\) simultaneously.

Case (ii) : Assume that \(j_2, \ell_1, \ell_2 \notin \mathcal{J}\) and \(j_1 \in \mathcal{J}\) for \((5.10)\), then

\[
\sum_{k \in \mathcal{J}} F_{j}^{k} t_{k}^\beta - \sum_{k \in \mathcal{J}} f_{\beta_{1} \beta_{2}} t_{j}^\beta = 0.
\]

While by assuming that \(j_1, j_2, \ell_2 \notin \mathcal{J}\) and \(\ell_1 \in \mathcal{J}\), we obtain

\[
\sum_{k \in \mathcal{J}} G_{k}^{\ell} t_{\ell_{2}} - \sum_{k \in \mathcal{J}} f_{\alpha_{1} \alpha_{2}} t_{\alpha_{2}} = 0.
\]

Since \(t_{\alpha} \neq 0\), the following relations are required to make \((5.10)\) true

\[
\sum_{k \in \mathcal{J}} (F_{j}^{k} - f_{\beta_{1} \beta_{2}} \delta_{j}^{k}) t_{k}^{\beta_{1}} = 0, \quad \sum_{k \in \mathcal{J}} (G_{k}^{\ell} - f_{\alpha_{1} \alpha_{2}} \delta_{\ell_{2}}^{k}) t_{\alpha_{2}} = 0,
\]

\((5.25)\)

We set \(t_{\beta} = t_{\alpha} = 0\) to obtain a simple nontrivial EYB. For nonvanishing \(t_{\beta}, t_{\alpha}\), we have to put further relations between \(F\) and \(f\) to satisfy \((5.25)\). Under the condition of vanishing \(t_{k}, t_{\alpha}\), consider the case of \(j_1, \ell_1 \notin \mathcal{J}\) for \((5.11)\) and \((5.12)\)

\[
t_{\alpha_{1}} (f_{\alpha_{1} t_{\alpha_{1}} - \alpha \beta}) = 0, \quad t_{\alpha_{1}} (f_{-1 \alpha_{1} t_{\alpha_{1}} - \alpha^{-1} \beta}) = 0.
\]

\((5.26)\)

Since \(t_{\alpha} \neq 0\), we obtain

\[
f_{\alpha_{1} t_{\alpha_{1}}} = \pm \alpha.
\]

\((5.27)\)

Substituting this into \((5.20)\), we obtain \(t_{\alpha} = \pm \beta\).

This completes the proof of Proposition 3.

5.2 Examples of Dressed Invariants

In this section, we give some examples of invariants obtained from dressings of two-dimensional EYB’s given in Table 1.

The cases of \((5.3)\) and \((5.7)\) are trivial, i.e. no new invariant is obtained. To see this, note that \(\sum_{k_{1}, k_{2}} R_{j_{1}, j_{2}}^{k_{1}, k_{2}} \mu_{k_{1}} \mu_{k_{2}}^{\ell_{2}} = 0\) if at least one of \(\ell_{1}, \ell_{2}\) is not in the index set \(\mathcal{J}\). It follows that \(\text{Tr}(\pi(\xi) \circ \mu^{\otimes n}) = \text{Tr}(\tilde{\pi}(\xi) \circ \tilde{\mu}^{\otimes n})\), where \(\pi(\sigma_{i}^{\pm 1}) = R_{i}^{\pm 1}\) and \(\tilde{\pi}(\sigma_{i}^{\pm 1}) = \tilde{R}_{i}^{\pm 1}\). Thus \(T_{S}(L) = T_{\tilde{S}}(L)\).

We, therefore, have to consider the cases of \((5.6)\) and \((5.8)\) to obtain nontrivially dressed invariants. We first study extensions to three-dimensional \(R\)-matrices. Let \(\mathcal{I} = \{1, 2, 3\}\) and \(\mathcal{J} = \{1, 3\}\). Throughout this section, we set \(\beta = 1\). Note that all the EYB’s given in Table 1 allows only diagonal dressings, since the conditions \([F, \tilde{\mu}] = [G, \tilde{\mu}] = 0\) for \(\tilde{\mu}\) in Table 1 require that \(F\) and \(G\) are diagonal. This is a special case of diagonal dressing.

(i) \(\tilde{R} = R_{2,1}, \tilde{\mu} = \text{diag}(\sqrt{pq}, 1/\sqrt{pq}), \alpha = 1/\sqrt{pq}\).
Table 2: 3D diagonal dressing of $R_{21}$. Invariant for knots. $t = pq$.

| knots | braid | Jones polynomial | dressed invariant |
|-------|-------|------------------|-------------------|
| $0_1$ | $\sigma_1$ | 1 | 1 |
| $3_1$ | $\sigma_1^3$ | $t^{-1/2}(1 + t^{-1/2})$ | $t^{-1/2}(1 - t + t^2 + t^4 + t^6)$ |
| $4_1$ | $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ | $t^{-2}(1 - t^{1/2} + t^{1/2} + t + t^3 - t^{1/2} + t^4)$ | $t^{-2}(1 - t + t^2 - t^3 - t^4) + 2st^{1/2}$ |
| $5_1$ | $\sigma_1^5$ | $t^{1/2}(1 - t^{1/2} + 2t^{1/2} - 2t^2 + t^{1/2} + t^3 - t^{1/2} + t^4 + t + t^{1/2})$ | $t^{1/2}(1 - t + t^2 - t^3 - t^4)$ |
| $5_2$ | $\sigma_2^3\sigma_1^{-1}\sigma_2\sigma_1^2$ | $t^{-1/2}(1 + t^{-1/2})$ | $t^{-1/2}(1 + t - t^2 + t^3 + t^4 + t^5 - t^{1/2})$ |

This is the case of $T_3(L)$ being Jones polynomial. A diagonal dressing for $\tilde{S}$ contains three additional parameters. We set

$$s_{12} = by, \quad s_{23} = ay, \quad s_{21} = a, \quad s_{32} = b, \quad s_{22} = \frac{1}{\sqrt{pq}}.$$ (5.28)

The dressed invariants for some knots and links are shown in Table 2 and Table 3, respectively. The Jones polynomial computed from $R_{21}$ is also shown in the tables for comparison. It is observed that the dressing makes Jones polynomial more complex and the additional parameters appear only in links.

Table 3: 3D diagonal dressing of $R_{21}$. Invariant for links. $s = aby$. The dressed invariant is not normalised by the value of unknot: $t^{1/2} + 1 + t^{-1/2}$.

| links | braids | Jones polynomial | dressed invariant |
|-------|-------|------------------|-------------------|
| $2^2_1$ | $\sigma_1^2$ | $t^{1/2}(1 + t^{-1/2})$ | $2 + t + t^2 + t^3 + 2st^{1/2}(1 + t)$ |
| $4^2_1$ | $\sigma_1^4$ | $t^{3/2}(1 + t^2 - t^4 + t^8)$ | $1 + t + t^2 + t^3 + t^4 + 2s^2t^{1/2}(1 + t)$ |
| $5^2_1$ | $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ | $-t^{-1/2}(1 - t^{-1/2} + t^{-3} - 2t^3 - t^3 - t^{-1/2} - 2t^{1/2})$ | $-t^{-1/2}(1 - t + t^2 - t^3 + t^4 + 2t^{1/2})$ |
| $6^2_1$ | $\sigma_1^6$ | $t^{5/2}(1 + t^2 - t^4 + t^6 - t^7 + t^9)$ | $1 + t + t^2 + t^3 + t^4 + 2s^2t^{1/2}(1 + t)$ |
| $6^2_2$ | $\sigma_2^3\sigma_2^{-1}\sigma_2\sigma_1^{-1}$ | $t^{3/2}(1 - t + 2t^2 - 2t^3 + 2t^4 - t^5 + t^9)$ | $1 + t + t^2 + t^3 + 2s^2t^{1/2}(1 + t)$ |
| $6^2_2$ | $\sigma_2^3\sigma_1^{-1}\sigma_2\sigma_1^{-1}$ | $t^{-3/2}(1 - 2t + 2t^3 - 2t^4 + 2t^5 - t^7 + t^9)$ | $t^{-3/2}(1 - t + 2t^2 - 2t^3 + 2t^4 + 2t^5 + t^9)$ |

(ii) $\tilde{R} = R_{22}, \quad \mu = \text{diag}(1/\sqrt{pq}, -1/\sqrt{pq}), \quad \alpha = \sqrt{pq}.$
This is the case in which all the invariants equal to zero and have the skein relation of Alexander polynomial. A diagonal dressing for $\tilde{S}$ contains three additional parameters. We set

$$s_{12} = a, \quad s_{23} = b, \quad s_{21} = by, \quad s_{32} = ay, \quad s_{22} = \sqrt{pq}. \quad (5.29)$$

The dressed invariants take the same value for all knots and links in Table 2 and Table 3: $T_{\tilde{S}}(L) = 1$.

Next we study a four dimensional extension of $R_{2,2}$. Let $I = \{1, 2, 3, 4\}$ and $J = \{1, 3\}$. $\tilde{S}$ is the same as case (ii). We consider a diagonal dressing and this case allows us to have ten additional parameters. We set

$$s_{12} = a, \quad s_{14} = c, \quad s_{23} = b, \quad s_{24} = h, \quad s_{34} = d, \quad s_{21} = by, \quad s_{41} = dw, \quad s_{32} = ay, \quad s_{42} = g, \quad s_{43} = cw, \quad s_{22} = s_{44} = \sqrt{pq}. \quad (5.30)$$

The dressed invariants for all knots given in Table 2 take the same value: $T_{\tilde{S}}(L) = 1$. While there is a slight improvement for links shown in Table 4.

Table 4: 4D diagonal of $R_{2,2}$. Invariant for links. $t = pq$, $s = hg$.

| link   | $2\text{a}$ | $4\text{a}^*(4)$ | $4\text{a}^*(4)$ | $5\text{a}$ |
|--------|-------------|------------------|------------------|-------------|
| invariant | $1 + st^{-1}$ | $1 + (st^{-1})^2$ | $1 + (st^{-1})^2$ | $2$         |

6 Concluding Remarks

We have seen that two dimensional solutions of YBE produce no link invariants better than the Jones polynomial. An interesting observation is that when a solution contains two parameters, those parameters are combined to give one parameter polynomial invariant. This means that we have to use higher dimensional solutions to obtain more powerful polynomial invariants. One way of constructing higher dimensional solutions of YBE is dressings of known lower dimensional ones. We learnt via some examples that improvement of invariants by dressing of two dimensional solutions is minor. This does not mean that dressings are useless to find powerful invariants. Dressings of higher dimensional $R$-matrices have a possibility to produce better invariants. A problem of two dimensional $R$-matrices is that they allow only diagonal dressings. Block dressings will appear if we start with a higher dimensional $R$-matrix and we anticipate to obtain better invariants, since block dressings give more complexity to EYB.

Another interesting result of the present work is that there exist three two dimensional $R$-matrices which produce the Alexander polynomial. The spin preserving one ($R_{2,2}$) is a well-known $R$-matrix which has a connection to the Alexander polynomial. We have observed that two spin non-preserving $R$-matrices also have a connection to the Alexander polynomial. Spin non-preserving part of one of them ($R_{1,2}$) does not contribute to computation of the polynomial, while other’s ($R_{1,1}$) does. It would be an interesting problem to construct a state model for $R_{1,1}$, since a role of spin non-preserving part will be clear in the computation of state models. It will be a future work.
Appendix

This Appendix is devoted to show the following fact.

**Proposition 4** Let $S = (R, \mu, \alpha, \beta)$ be an EYB and $R'$ be a solution of YBE obtained from $R$ by one of the transformations (3.1) - (3.4). Then $S' = (R', \mu', \alpha', \beta)$ is also an EYB for the following choice of $\mu'$, $\alpha'$ and $T_{S'}(L) = T_{S}(L)$.

1. $\mu' = \kappa Q\mu Q^{-1}$, $\alpha' = \kappa \alpha$ for (3.1)
2. $\mu'^{i}_{k} = \mu^{i}_{k}$, $\alpha' = \alpha$ for (3.2)
3. $\mu'^{i}_{k} = \mu^{i+k+n}_{k+n}$, $\alpha' = \alpha$ for (3.3)
4. $\mu' = \mu$, $\alpha' = \alpha$ for (3.4)

**Proof.** It is straightforward and easy to verify that $S'$ of (1)-(3) satisfy the definition of EYB. Case (4) requires a bit care. The lhs of (2.3) and (2.4) for $S'$ of (4) are reduced to

$$\text{Tr}_{2}(R'^{\pm 1} \circ (\mu' \otimes \mu')) = \text{Tr}_{1}(R^{\pm 1} \circ (\mu \otimes \mu))$$

(6.1)

It can be shown that one can change $\text{Tr}_{2}$ to $\text{Tr}_{1}$ in the definition of EYB and $T_{S}(L)$ defined by (2.7) is invariant again. We assume for case (4) that $S = (R, \mu, \alpha, \beta)$ is EYB defined with $\text{Tr}_{1}$. Then, one can show that $S'$ of case (4) becomes EYB with $\text{Tr}_{2}$.

We now turn to the proof of $T_{S'}(L) = T_{S}(L)$. Case (3) is obvious, since $T_{S}(L)$ is obtained by summing over all indices. Case (2) is an exchange of upper indices and lower ones. This corresponds to vertical flip of the braid $\xi \in B_{n}$ whose closure is isotopic to $L$. Thus $T_{S'}(\xi)$ is equal to $T_{S}(\eta)$, where $\eta \in B_{n}$ is the braid obtained by flipping $\xi$ vertically. Since $\bar{\eta} = \bar{\xi}$, we obtain $T_{S'}(L) = T_{S}(L)$. Similarly, case (4) is corresponding to horizontal flipping of a braid. For case (1), it is easy to see

$$T_{S'}(L) = \alpha'^{-w(\xi)} \beta^{-n} \kappa^{w(\xi)} \text{Tr}(Q^{\otimes n} \circ \pi(\xi) \circ \mu^{\otimes n} \circ (Q^{-1})^{\otimes n}),$$

where $\pi(\sigma^{\pm 1}_{i}) = R^{\pm 1}_{i}$. It follows immediately that $T_{S'}(L) = T_{S}(L)$.

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