Stability and existence the solution for a coupled system of hybrid fractional differential equation with uniqueness

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1. Introduction

Fractional differential equation (FDEs) models have got a lot of attention from researchers in the last few decades due to helpful and factual natural and physical problems compared to the models of integer order FDEs. Fractional calculus theory has been developed for over 300 years to FDEs are becoming increasingly common. The study of FDEs has become in popularity and importance in several fields of applied science. The motivation behind those interest stems from the important and extensive applications more appropriate in the modeling of various physical and natural phenomena like image processing, fractals theory, fluid dynamics, control system, electromagnetism theorem, control theory ecology, metallurgy, plasma physics, aerodynamics, economics, biology and Newtonian mechanics. For details, see (Abdel-Aty, Khater, Bâleaneu, Khalil, et al., 2020; Anastassiou, 2009; Chu, Khater, & Hamed, 2021; Deimling, 1985; Jafari, Bâleaneu, Khan, & Khan, 2015; Khan & Khan, 2014; Khater, Ali, Khan, Mousa, & Attia, 2021; Khater, Attia, Alodhaibi, & Lu, 2020; Khater, Attia, Park, & Lu, 2020; Kilbas, Srivastava, & Trujillo, 2006; Podlubny, 1999; Qian, Attia, Qiu, Lu, & Khater, 2019; Samko, Kilbas, & Marichev, 1993; Yue, Khater, Attia, & Lu, 2020; Yue, Lu, et al., 2020).

Recently, several from researchers studied a lot applications of differential equations, for example (Abdel-Aty, Khater, Bâleaneu, Abo-Dahab, et al., 2020; Ak, Osman, & Kara, 2020; Ali, Abd El Salam, et al., 2020; Ali, Cattani, Gómez-Aguilar, Bâleaneu, & Osman, 2020; Ameen, Bâleaneu, & Ali, 2020; Arqub, Osman, Abdel-Aty, Mohamed, & Momani, 2020; Bayones et al., 2021; Dhawan, Machado, Brzeziński, & Osman, 2021; Khater, Lu, Attia, & In, 2019; Khater, Park, & Lu, 2020; Kumar, Kumar, Osman, & Samet, 2021; Kumar, Kumar, Samet, Gómez-Aguilar, & Osman, 2020; Park et al., 2020; Raza, Osman, Abdel-Aty, Abdel-Khalene, & Besbes, 2020; Yue, Khater, Mustafa Inc Attia, & Lu, 2020). Stability analysis of FDEs solutions has been extensively studied. It’s crucial in both optimization and numerical solutions of various types of differential equations. Ulam introduced a new concept of stability analysis in the field of stability theory in 1940, and Hyers developed it further in 1941. The stability study of FDEs is another significant feature of the qualitative theory. Since an unstable solution is not useful and does not provide the necessary information within the specified domain, its stability is just as important as its nature. Conversely, Stable solutions often include significant details within the prescribed domain. In this regard, Researchers have studied several stability theories to FDEs such as Lyapunov, exponential and Mittag–Laffer in addition to other important kinds of stability analysis are HU-stability and UH-Rassias for various aspects of couple system of hybrid FDEs for EUS. Here, are some studies of a
couple of systems in FDEs. For example, see these studies (Ahmad & Nieto, 2014; Ahmad & Sivasundaram, 2009; Ali, Samet, Shah, & Khan, 2017; Chasreechai & Sithiwiratham, 2018; CDurs, 2013; Dhage & Jadhav, 2012; Khan, Li, Shah, & Khan, 2017; Mashayekhi & Razzaghi, 2015; Shah, Ali, & Khan, 2016; Shahand & Khan, 2016; Tariboon, Ntouyas, & Suantai, 2017; Yukunthorn, Ahmad, & Ntouyas, 2016; Zhao, Sun, Han, & Li, 2011). We are concerned to study the existence and uniqueness of solutions for a couple of system HFDEs with a p-Laplacian operator because this type is very important in various models. The stability study of FDEs is another significant feature of the qualitative theory. Since an unstable solution is not useful and does not provide the necessary information within the specified domain, its stability is just as important as its nature. Conversely, Stable solutions often include significant details within the prescribed domain. In this regard, researchers have studied several stability theories to FDEs such as Lyapunov, exponential and Mittag–Laffer in addition to other important kinds of stability analysis are HU-stability and UH-Rassias for various aspects of couple system of hybrid FDEs for EUS. Here, are some studies of a couple of systems in FDE. For example, see these studies like Pesticides in Soil and Trees, Chemical Kinetics, Irregular Heartbeats and Chemostats (Ahmad & Nieto, 2009; Bai & Fang, 2004; Hu, Liu, & Liu, 2013; Iqbal, Li, Shah, & Khan, 2017; Khan, Li, Sun, & Khan, 2017; Samina, Shah, & Khan, 2017; Shah & Tunc, 2017; Su, 2009). At first, we introduce some modern and important contributions of researchers for the investigation of EUS of FDEs of different types of FDEs. For example, Abdeljawi and Dahmani (2015) used fractional calculus and fixed point theorems to investigate EUS for fractional coupled systems

\[
\begin{align*}
D^\sigma \mu(t) &= \mathcal{Q}(\zeta, \mu(t), \mu(t)) + \frac{(C^\sigma - 1)}{T^\sigma} \int_0^T \phi_1(\mu(t), \mu(t)) dt, \quad \zeta \in [0, 1], \\
D^\nu \nu(t) &= \mathcal{Q}(\zeta, \nu(t), \nu(t)) + \frac{(C^n - 1)}{T^n} \int_0^T \phi_2(\mu(t), \nu(t)) dt, \quad \zeta \in [0, 1], \\
\sum_{i=1}^m |\mu_0^i(0)| + |\nu_0^i(0)| &= 0, \\
\mu^{n-1}(0) &= 0, \\
\nu^{n-1}(0) &= 0,
\end{align*}
\]

where \(D^\sigma\) and \(D^n\) are caputo derivatives with \(m - 1 < \sigma, \nu < n, m \in \mathbb{N}, \kappa, \delta \in \mathbb{R}, \alpha, \beta, \rho, q \) are real number, \(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Y}_1\) and \(\mathcal{Y}_2\) are functions. Hu et al. (2013) studied BVP for a coupled system of FDEs by using the degree theorem given by

\[
\begin{align*}
\mathcal{D}^\sigma \zeta(s) - \phi(s, \mu(s), \nu(s)) &= 0, \quad s \in (0, 1), \\
\mathcal{D}^\nu \mu(s) - \eta(s, \zeta(s), \zeta(s)) &= 0, \quad s \in (0, 1), \\
\zeta(0) &= \mu(0) = 0, \\
\zeta(1) &= \zeta(1), \quad \mu(1) = \mu(1),
\end{align*}
\]

where \(1 < \alpha < 2, \mathcal{D}^\sigma\) and \(\mathcal{D}^\nu\) are Caputo derivatives and \(\eta, \phi : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}\). Shah and Khan (2016) studied the EUS for a couple system of FDEs by degree theory as follows

where \(\mathcal{D}^\sigma\) and \(\mathcal{D}^\nu\) are the Caputo derivatives, \(0 < \sigma, \nu \leq 1, \mathcal{Q}_1, \mathcal{Q}_2 \in \mathbb{C}[0, 1] \times \mathbb{R}^2, \) and \(m, n : (1, \mathbb{R}) \rightarrow \mathbb{R}\) are continuous function, \(0 < \epsilon, \mu < 1, \) \(d_1, d_2, d_3, (i = 1, 2)\) are real number. Lei Hu (2018) studied the EUS for \((n - 1, 1)\)-type coupled systems of FDEs by degree theorem, given by

\[
\begin{align*}
\mathcal{D}^\sigma \zeta \nu(C^\sigma)(\nu, \mu, \mu, \nu, \nu) &= 0, \quad \nu \in (0, 1), \\
\mathcal{D}^\nu \mu(\nu) &= \frac{\mathcal{D}^\nu \mu(\nu)}{\mathcal{D}^\nu \nu(\nu)}, \quad \nu \in (0, 1), \\
\mathcal{Y}(0) &= \mu(0) = 0, \\
\mathcal{Y}(1) &= \lambda \int_0^1 \mathcal{Y}(\nu) d\nu, \quad \omega(1) = \mu \int_0^1 \omega(\nu) d\nu,
\end{align*}
\]

where \(\mathcal{D}^\sigma\) and \(\mathcal{D}^\nu\) are Caputo derivatives, \(k - 1 < \sigma, \nu < k, \mathcal{P}, \mathcal{Y} : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}\) are continuous functions, \(\eta, \xi \in (0, 1)\) are constants satisfying \(\lambda \xi \mu = k\). Hu et al. (2013) investigated existence of the solution of BVP for a coupled system of FDEs with p-Laplacian at resonance given by

\[
\begin{align*}
\mathcal{D}^\sigma (\phi_1(\mathcal{D}^\sigma \mathcal{U}(s))) &= \mathcal{G}_1(s, \mathcal{U}(s), \mathcal{D}^\nu \mathcal{U}(s)), \quad s \in (0, 1), \\
\mathcal{D}^\nu (\phi_2(\mathcal{D}^\nu \mathcal{U}(s))) &= \mathcal{G}_2(s, \mathcal{U}(s), \mathcal{D}^\nu \mathcal{U}(s)), \quad s \in (0, 1), \\
\mathcal{D}^\nu (\mathcal{U}(0)) &= \mathcal{D}^\nu (\mathcal{U}(1)) = \mathcal{D}^\nu (\mathcal{D}^\nu \mathcal{U}(1)) = 1,
\end{align*}
\]

where \(\mathcal{D}^\sigma\) and \(\mathcal{D}^\nu\) are Caputo derivative of orders \(0 < \sigma, \nu \leq 1, 1 < \sigma + \nu < 2, 1 < \nu + 2\) and \(\xi, \eta \in [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}\) is continuous. In recent years, the investigation of HU-stability analysis for non-linear FDEs is a topic of hot research. Therefore, Hyers–Ulam type stability plays important roles for FDEs. The HU stability can be defined as an exact solution exist very near to the approximate solution for FDEs with very small error. Recently, nonlinear differential equations with quadratic perturbations have received much attention. This type of differential equations is called hybrid differential equations. The hybrid FDEs have been considered more significant in different scientific fields and taking as special cases of dynamic systems. The first-order hybrid differential equation is investigated by Dhage and Lakshmilkantham (2010). For more detail about hybrid FDEs, see the articles (Băleanu, Agarwal, Khan, & Jafari, 2015; Băleanu, Agarwal, Mohammadi, & Rezapour, 2013; Băleanu & Mustafa, 2010; Băleanu, Mustafa, & Agarwal, 2010a, 2010b; Isaia, 2006; Royden & Fitzpatrick, 1988; Stanova, 2015; Zada, Faisal, & Li, 2017). Inspired from the above-mentioned contributions, in this article, we investigate a coupled system of HFDEs with p-Laplacian by Leray–Schauder and topological degree theorems. In addition to studying some important conditions for the HU-stability of the solution to our suggested problem

\[
\begin{align}
\mathcal{D}^\sigma (\phi_1(\mathcal{D}^\sigma (\mathcal{U}(\zeta, \mathcal{U}(\zeta)))) &= \mathcal{Q}_1(\zeta, \mathcal{U}(\zeta)), \\
\mathcal{D}^\nu (\phi_2(\mathcal{D}^\nu (\mathcal{U}(\zeta, \mathcal{U}(\zeta)))) &= \mathcal{Q}_2(\zeta, \mathcal{U}(\zeta)), \\
\mathcal{D}^\nu (\mathcal{U}(0)) &= \mathcal{D}^\nu (\mathcal{U}(1)) = \mathcal{D}^\nu (\mathcal{D}^\nu \mathcal{U}(1)) = 1,
\end{align}
\]

where \(k - 1 < \sigma, \nu < k, \mathcal{D}^\sigma, \mathcal{D}^\nu\) are Caputo derivatives, \(\mathcal{Q}_1, \mathcal{Q}_2 \in L[0, 1] \) and \(\phi_1(\eta) = \eta|\eta|^{p - 2}\) is the
p-Laplacian and \(\phi_p^{-1} = \phi_q\), such that \(1/p + 1/q = 1\).

To the best of our knowledge, the topological degree and Leray–Schauder theories have not been widely used for the study of EUS for a coupled system of HFDEs with IBCs having orders in \((k-1, k]\) for \(k \geq 3\) involving the p-Laplacian operator. Therefore, we prove EUS and HU-stability for the coupled system with the help of degree theorem as proposed by Deimling (1985). We study three main aspects of the HFDEs with p-Laplacian \((1.1)\), including existence of solution, uniqueness and HU-stability for our proposed problem. To achieve these aims, we’ll use Green functions to convert the problem \((1.1)\) into an integral equation. After that, we’ll use the topological degree approach to prove existence and uniqueness. Our proposed problem will become complex and more generic compared to the problems previously studied before and aforementioned. This work could draw researchers’ attention to the analysis of HU-stability and various other kinds of stability for more complicated problems. We also advise readers that the issue \((1.1)\) has the potential to be investigated further for other aims, such as multiplicity results.

2. Preliminaries

Here, we will offer some notations, proposition, theorems, definitions and Lammas, which has an important role to study and prove this the article.

**Definition 1.** For \(Q(\zeta) : (0, + \infty) \rightarrow \mathbb{R}\), the fractional integral of order \(\sigma > 0\) is defined as follows

\[
\mathcal{I}^\sigma Q(\zeta) = \frac{1}{\Gamma(\sigma)} \int_0^\zeta (\zeta - \eta)^{\sigma - 1} Q(\eta) \, d\eta,
\]

The integral point is defined on \((0, + \infty)\).

**Definition 2.** For \(Q(\zeta) : (0, + \infty) \rightarrow \mathbb{R}\), the Caputo derivative of order \(\sigma > 0\) is defined as follows

\[
\mathcal{D}^\sigma Q(\zeta) = \frac{1}{\Gamma(\sigma - k)} \int_0^\zeta (\zeta - \eta)^{\sigma - k - 1} Q^{(k)}(\eta) \, d\eta,
\]

The integral point is defined on \((0, + \infty)\).

**Lemma 1.** Let \(\sigma > 0\) and \(\vartheta \in C(0, 1) \cap L^1(0, 1)\), then \(\mathcal{D}^\sigma \vartheta(\zeta) = Q(\zeta)\), is given by

\[
\vartheta(\zeta) = Q(\zeta) + d_0 + d_1 \zeta + d_2 \zeta^2 + \ldots + d_k \zeta^k,
\]

for some \(d_j \in \mathbb{R}, j = 0, 1, 2, \ldots, k-1\), such that \(k \geq \sigma\).

**Lemma 2.** Let \(\sigma \in (k-1, k]\), \(Q \in C^{k-1}(0, 1)\). Then,

\[
\mathcal{I}^\sigma \mathcal{D}^\sigma Q(\zeta) = Q(\zeta) + d_0 + d_1 \zeta + d_2 \zeta^2 + \ldots + d_k \zeta^k,
\]

where \(d_j \in \mathbb{R}\) for \(i = 0, 1, 2, \ldots, k-1\), consider the space \(d_j \in \mathbb{R}\) for \(i = 0, 1, 2, \ldots, k-1\), is real, continuous functions with topological norm \(\|\vartheta(\zeta)\| = \sup \{\|\vartheta(\zeta)\| : \zeta \in (0, 1)\}\) for \(\vartheta \in \mathcal{U}\). \(\zeta\) represents the class of all bounded mappings in \(\mathcal{U}\).

**Definition 3.** The mapping \(\mathcal{L} : \xi \rightarrow (0, + \infty)\) for Kuratowski measure of non-compactness is defined as

\[
\mathcal{L}(u) = \inf \{a > 0 : u \text{ is a finite cover for sets of diameter } a\},
\]

where \(u \in \mathcal{U}\).

**Definition 4.** Let \(\psi : \mathcal{U} \rightarrow \mathcal{U}\) is a continuous and bounded on mapping \(\psi \in \mathcal{U}\). Then, \(\psi\) is an \(\mathcal{L}\)-Lipschitz, where \(\eta \geq 0\) such that

\[
\mathcal{L}(\psi(u) \leq \eta \xi(u) \text{ } \forall \text{ bounded } u \in \mathcal{U}.
\]

Then, \(\psi\) is a strict \(\mathcal{L}\)-contraction under the condition \(\eta < 1\).

**Definition 5.** The function continuous \(\psi\) is \(\mathcal{L}\)-condensing if \(\mathcal{L}(\psi(u) \leq \mathcal{L}(u), \text{ for all bounded } u \in \mathcal{U}\) such that \(\mathcal{L}(u) > 0\).

Therefore, \(\mathcal{L}(\psi(u)) \geq \xi(u)\) yields \(\xi(u) = 0\).

Further we have \(\psi : \theta \rightarrow \mathcal{U}\) is Lipschitz for \(\eta > 0\), such that

\[
\|\psi(\vartheta) - \psi(\vartheta')\| \leq \eta \|\vartheta - \vartheta'\|, \text{ for all } \vartheta, \vartheta' \in \mathcal{U}.
\]

If \(\eta < 1\), then, \(\psi\) is strict contraction.

**Proposition 1.** The mapping \(\psi\) is \(\mathcal{L}\)-Lipschitz with constant \(\eta = 0\) if and only if \(\psi : \theta \rightarrow \mathcal{U}\) is compact.

**Proposition 2.** The operator \(\psi\) is \(\mathcal{L}\)-Lipschitz for some constant \(\eta\) if and only if \(\psi : \theta \rightarrow \mathcal{U}\) is Lipschitz with constant \(\eta\).

**Theorem 1.** Let \(\psi : \mathcal{U} \rightarrow \mathcal{U}\) is a \(\mathcal{L}\)-contraction and

\[
E = \{v \in \mathcal{U} : \text{ there exist } 0 \leq \mu \leq 1 \text{ such that } v = \mu \psi(v)\}.
\]

If \(E \subseteq v_i(0), \text{ is bounded in } \mathcal{U}\) there exists \(h > 0\) and \(E \subseteq u_i(0)\) with degree

\[
\deg(\mathcal{L} - \mu \psi, v_i(0), 0) = 1, \text{ for every } \mu \in [0, 1].
\]

Then, \(\psi\) has at least one fixed point.

**Lemma 3.** Let \(\phi_p\) be p-Laplacian. Then, we have

\[
(\xi_1) \text{ If } 1 < p \leq 2, q_1, q_2 > 0 \text{ and } |q_1|, |q_2| \geq \rho > 0, \text{ then,}
\]

\[
|\phi_p(q_1) - \phi_p(q_2)| \leq (p-1)p^{(p-2)}|q_1 - q_2|.
\]

\[
(\xi_2) \text{ If } p > 2 \text{ and } |q_1|, |q_2| \leq \rho^*, \text{ then,}
\]

\[
|\phi_p(q_1) - \phi_p(q_2)| \leq (p-1)p^{(p-2)}|q_1 - q_2|.
\]

3. Existence of solution

**Theorem 3.** Let the function \(Q \in C([0, 1])\) be a satisfying integrable\((1.1)\). Then, for \(\lambda, \sigma \in [3, k]\) and integer \(k \geq 4\), then, the solution of

\[
\begin{align*}
\mathcal{D}^\sigma \phi_p(\mathcal{D}^\sigma (\mu(\zeta - Q(\zeta, \mu(\zeta)))))) = -Q_1(\zeta, \mu(\zeta)), \\
\mathcal{D}^{i+1} \phi_p(\mathcal{D}^\sigma (\mu(\zeta - Q(\zeta, \mu(\zeta)))))) |_{\zeta = 0} = 0, \text{ for } i = 2, 3, 4, \ldots, k-1, \\
\mathcal{D}^{i+1} \phi_p(\mathcal{D}^\sigma (\mu(\zeta - Q(\zeta, \mu(\zeta)))))) |_{\zeta = 0} = 0, \\
\mu(0) = 0, \text{ for } j = 2, 3, 4, \ldots, k-1, \\
\mu(\zeta) = Q_2(\zeta, \mu(\zeta)) |_{\zeta = 0}, \text{ } \mu(0) = \frac{1}{\Gamma(\sigma)} \int_0^\zeta (\sigma - \eta)^{\sigma - 1} Q_3(\mu(\eta)) \, d\eta
\end{align*}
\]

(3.1)
is given by integral equation
\[
\mu(\zeta) = \int_0^1 G^\sigma(\zeta, \eta) \phi_q \left( \int_0^1 G^\sigma(\eta, \xi) Q_1(\eta, \mu(\xi)) d\eta \right) d\xi + Q_2(\zeta, \mu(\zeta)) + \frac{1}{\Gamma(\sigma)} \int_0^{a} (a-\eta)^{\sigma-1} Q_2(\eta, \mu(\eta)) d\eta.
\]
(3.2)

Where \( G^\sigma(\zeta, \eta), G^\sigma(\eta, \xi) \) are Green functions defined by
\[
G^\sigma(\zeta, \eta) = \begin{cases} \frac{(\zeta-\eta)^{\sigma-1}}{\Gamma(\sigma)} - \frac{(\epsilon-\eta)^{\sigma-2}}{\Gamma(\sigma-1)}, & \eta \leq \zeta \leq \epsilon, \\ \frac{-(\zeta-\eta)^{\sigma-2}}{\Gamma(\sigma-1)}, & \zeta \leq \eta \leq \epsilon, \\ \frac{(\zeta-\eta)^{\sigma-1}}{\Gamma(\sigma)}, & \epsilon \leq \eta \leq \zeta, \end{cases}
\]
(3.3)
\[
G^\sigma(\eta, \xi) = \begin{cases} \frac{-(\eta-\xi)^{\sigma-1}}{\Gamma(\sigma)} + \frac{1}{\Gamma(\lambda)} (\eta-\xi)^{\lambda-1}, & \eta \leq \zeta \leq \xi, \\ \frac{1}{\Gamma(\lambda)} (\eta-\xi)^{\lambda-1}, & \zeta \leq \eta \leq \xi, \\ \frac{-(\eta-\xi)^{\lambda-1}}{\Gamma(\lambda)}, & \xi \leq \eta \leq \zeta, \end{cases}
\]
(3.4)

**Proof.** By applying integral \( I^\sigma \) and using Lemma 2 on problem (3.1) we get
\[
(\phi_p) I^\sigma(D^\sigma(\mu(\zeta) - Q_2(\zeta, \mu(\zeta)))) = \int_0^1 G^\sigma(\zeta, \eta) Q_1(\eta, \mu(\eta)) d\eta + Q_2(\zeta, \mu(\zeta)) + \frac{1}{\Gamma(\sigma)} \int_0^{a} (a-\eta)^{\sigma-1} Q_2(\eta, \mu(\eta)) d\eta
\]
(3.5)

By the conditions \( I^{i=\lambda} [\phi_p) I^\sigma(D^\sigma(\mu(\zeta) - Q_2(\zeta, \mu(\zeta))))] |_{\zeta=0} = 0 \) in (3.5). For \( i = 2, 3, 4, ..., k \), we obtain \( a_2 = a_3 = a_4 = ... = a_k = 0 \), we have
\[
(\phi_p) I^\sigma(D^\sigma(\mu(\zeta) - Q_2(\zeta, \mu(\zeta)))) = \int_0^1 G^\sigma(\zeta, \eta) Q_1(\eta, \mu(\eta)) d\eta + Q_2(\zeta, \mu(\zeta)) + \frac{1}{\Gamma(\sigma)} \int_0^{a} (a-\eta)^{\sigma-1} Q_2(\eta, \mu(\eta)) d\eta
\]
(3.6)

Applying the condition \( D^\sigma(\phi_p) I^\sigma(D^\sigma(\mu(\zeta) - Q_2(\zeta, \mu(\zeta)))) |_{\zeta=0} = 0 \) in the Equation (3.6), we get
\[
a_1 = \frac{\Gamma(\sigma)}{\Gamma(\lambda)} I^{i=\sigma} Q_1(\zeta, \mu(\zeta)) |_{\zeta=0}.
\]
(3.7)

Putting the value \( a_1 \) in (3.6), we get
\[
(\phi_p) I^\sigma(D^\sigma(\mu(\zeta) - Q_2(\zeta, \mu(\zeta)))) = \int_0^1 G^\sigma(\zeta, \eta) Q_1(\eta, \mu(\eta)) d\eta + Q_2(\zeta, \mu(\zeta)) + \frac{1}{\Gamma(\sigma)} \int_0^{a} (a-\eta)^{\sigma-1} Q_2(\eta, \mu(\eta)) d\eta
\]
(3.8)

where \( G^\sigma(\zeta, \eta) \) is a Green function given in (3.4). From (3.8), we further have
\[
(\phi_p) I^\sigma(D^\sigma(\mu(\zeta) - Q_2(\zeta, \mu(\zeta)))) = \int_0^1 G^\sigma(Q_1(\eta, \mu(\eta))) d\eta.
\]
(3.9)

Applying \( \phi_p^{-1} = \phi_q \) on sides of (3.9), we get
\[
(\phi_p) I^\sigma(D^\sigma(\mu(\zeta) - Q_2(\zeta, \mu(\zeta)))) = \int_0^1 G^\sigma(Q_1(\eta, \mu(\eta))) d\eta.
\]
(3.10)

By applying integral \( I^\sigma \) and using Lemma 2 on both sides of (3.11), we get
\[
\mu(\zeta) - Q_2(\zeta, \mu(\zeta)) = I^\sigma(\phi_q) I^1 G^\sigma(Q_1(\eta, \mu(\eta))) d\eta + b_0 + b_1 \zeta + b_2 \zeta^2 + ... + b_{k-1} \zeta^{k-1}.
\]
(3.11)

For the values \( j = 2, 3, 4, ..., k-1 \), by the conditions \( \mu(\zeta)(\eta)|_{\zeta=0} = 0 \) in (3.12), then, \( b_2 = b_3 = b_4 = ... = b_{k-1} = 0 \), we get
\[
\mu(\zeta) = Q_2(\zeta, \mu(\zeta)) + I^\sigma(\phi_q) I^1 H^\sigma(Q_1(\eta, \mu(\eta))) d\eta + b_0 + b_1 \zeta.
\]
(3.13)

Applying the condition \( (\mu(\epsilon)^{\prime} = Q_2(\zeta, \mu(\zeta))) |_{\zeta=\epsilon} \) in the Equation (3.13), we have
\[
b_1 = -I^{\sigma-1}(\phi_q) I^1 G^\sigma(Q_1(\eta, \mu(\eta))) d\eta |_{\zeta=\epsilon}.
\]
(3.14)

By the condition \( \mu(0) = \frac{1}{\Gamma(\sigma)} \int_0^a (a-\eta)^{\sigma-1} Q_2(\eta, \mu(\eta)) d\eta \) in the Equation (3.13), we have
\[
b_0 = \frac{1}{\Gamma(\sigma)} \int_0^a (a-\eta)^{\sigma-1} Q_2(\eta, \mu(\eta)) d\eta.
\]
(3.15)

Putting the values \( b_0, b_1 \) in (3.12), we get
\[
\mu(\zeta) = Q_2(\zeta, \mu(\zeta)) + I^\sigma(\phi_q) I^1 G^\sigma(Q_1(\eta, \mu(\eta))) d\eta + b_0 + b_1 \zeta.
\]
\[\mu(\zeta) = \int_0^1 G^\varphi(\zeta, \vartheta) \phi_q \left( \int_0^1 G^\varphi(\vartheta, \eta) Q_1(\eta, \mu(\eta)) d\eta \right) d\vartheta + Q_2(\zeta, \mu(\zeta)) + \frac{1}{\Gamma(\sigma)} \int_0^\vartheta (\vartheta - \eta)^{\sigma - 1} Q_2(\vartheta, \mu(\vartheta)) d\eta. \]  

(3.16) 

Where \( G^\varphi(\zeta, \vartheta), G^\varphi(\vartheta, \eta) \) are Green functions defined by (3.3), (3.4).

Define \( \mathcal{M}_1^* : \mathfrak{H} \rightarrow \mathfrak{H} \) for \( i = 1, 2 \) by

\[\mathcal{M}_1(\mu(\zeta)) = \left( \int_0^1 G^\varphi(\zeta, \vartheta) \phi_q \left( \int_0^1 G^\varphi(\vartheta, \eta) Q_1(\eta, \mu(\eta)) d\eta \right) d\vartheta \right), \]

\[\mathcal{M}_2(\mu(\zeta)) = Q_2(\zeta, \mu(\zeta)) + \frac{1}{\Gamma(\sigma)} \int_0^\vartheta (\vartheta - \eta)^{\sigma - 1} Q_2(\vartheta, \mu(\vartheta)) d\eta, \]

(3.17) 

by Theorem 3, then, the solution (3.16) is a fixed point \( \mu(\zeta) \) of the operator \( \mathcal{M} \) defined by

\[\mathcal{M}(\mu) = \mathcal{M}_1(\mu) + \mathcal{M}_2(\mu). \]

To proceed further, we need some propositions:

(A1) With \( a_1, T_{\omega} > 0, m_1 \in [0, 1] \) function \( Q_1 \) satisfies

\[\left| Q_1(\zeta, \mu) \right| \leq \phi_p(a_1 ||\mu||^m_1 + T_{\omega}), \]

(A2) There exist real valued constants \( \gamma_{\Omega_1} \), for all \( \nu, \omega \in \mathfrak{H} \)

\[\left| Q_1(\zeta, \nu) - Q_1(\zeta, \omega) \right| \leq \gamma_{\Omega_1} ||\nu - \omega||. \]

(A3) With \( a_2, T_{\omega} > 0, m_1 \in [0, 1] \) function \( Q_2 \) satisfies

\[\left| Q_2(\zeta, \mu) \right| \leq (a_2 ||\mu||^m_1 + T_{\omega}), \]

(A4) There exist real valued constants \( \gamma_{\Omega_2} \), for all \( \nu, \omega \in \mathfrak{H} \)

\[\left| Q_2(\zeta, \nu) - Q_2(\zeta, \omega) \right| \leq \gamma_{\Omega_2} ||\nu - \omega||. \]

For simplicity, we define these symbols:

\[\nabla_1 = \left( \frac{1}{\Gamma(\sigma + 1)} + \frac{\sigma - 1}{\Gamma(\lambda)} \right) - \frac{1}{\Gamma(\lambda)} \frac{\alpha^{\sigma - 1}}{(\lambda - \sigma)\Gamma(\lambda)}, \]

\[\nabla_2 = \left( 1 + \frac{\sigma^\theta}{\Gamma(\sigma + 1)} \right). \]

**Theorem 4.** Under assumption (A1), (A2), the operator \( \mathcal{M}^* : \sigma^* \rightarrow \sigma^* \) is continuous and satisfies

\[||\mathcal{M}^*(\mu)|| \leq \Omega ||\mu||^m_1 + \mathcal{F}_2^*, \]

(3.19) 

for each \( (\mu) \in \Omega, \sigma^* \).

**Proof.** Assume a bounded set \( \Omega = \{(\mu) \in \sigma^* : ||\mu|| \leq r\} \) with sequence \( (\mu_n) \) converging to \( (\mu) \) in \( \Omega \). To show that \( ||\mathcal{M}^*(\mu_n) - \mathcal{M}^*(\mu)|| \rightarrow 0 \) as \( n \rightarrow +\infty \), let us consider

\[|\mathcal{M}_1^*(\mu_n) - \mathcal{M}_1^*(\mu)| = \left| \int_0^1 G^\varphi(\zeta, \vartheta) \phi_q \left( \int_0^1 G^\varphi(\vartheta, \eta) Q_1(\eta, \mu_n(\eta)) d\eta \right) d\vartheta \right| \]

\[\leq \left| \int_0^1 G^\varphi(\zeta, \vartheta) \phi_q \left( \int_0^1 G^\varphi(\vartheta, \eta) Q_1(\eta, \mu(\eta)) d\eta \right) d\vartheta \right| \]

\[+ \left| \int_0^1 G^\varphi(\zeta, \vartheta) \phi_q \left( \int_0^1 G^\varphi(\vartheta, \eta) (Q_1(\eta, \mu_n(\eta)) - Q_1(\eta, \mu(\eta))) d\eta \right) d\vartheta \right| \]

\[\leq (q-1)\rho_1^2 \int_0^1 |G^\varphi(\zeta, \vartheta)| \left| \left| \phi_q \right| \left( \int_0^1 G^\varphi(\vartheta, \eta) Q_1(\eta, \mu_n(\eta)) d\eta \right) \right| d\vartheta \]

(3.20) 

by the estimate (3.20) and continuity of the function \( Q_1 \), we get \( |\mathcal{M}_1^*(\mu_n) - \mathcal{M}_1^*(\mu)| \rightarrow 0 \) as \( n \rightarrow +\infty \). This proves that \( \mathcal{M}^* \) is continuous. To demonstrate the continuity of \( \mathcal{M}^*_2 \), let us consider

\[|\mathcal{M}_2^*(\mu_n) - \mathcal{M}_2^*(\mu)| = \left| Q_2(\zeta, \mu_n(\zeta)) - Q_2(\zeta, \mu(\zeta)) \right| \]

\[\leq \left| Q_2(\zeta, \mu_n(\zeta)) - Q_2(\zeta, \mu(\zeta)) \right| + \left| Q_2(\zeta, \mu(\zeta)) \right| \]

\[\leq \left| Q_2(\zeta, \mu_n(\zeta)) - Q_2(\zeta, \mu(\zeta)) \right| + \left| Q_2(\zeta, \mu(\zeta)) \right| \]

(3.21) 

With the help of Equation (3.21) and from the continuity of the function \( Q_2(\zeta, \mu) : (0, 1) \times R \rightarrow R \), we have \( |\mathcal{M}_2^*(\mu_n) - \mathcal{M}_2^*(\mu)| \rightarrow 0 \) as \( n \rightarrow +\infty \). This means \( \mathcal{M}^*_2 \) is continuous. Thus, from (3.20), (3.21) we get \( \mathcal{M} = \mathcal{M}_1^*(\mu) + \mathcal{M}_2^*(\mu) \) is a continuous.

Now, by (3.20), (3.17) and pseudnonorm (A1), we have

\[|\mathcal{M}_1^*(\mu)| = \left| \int_0^1 G^\varphi(\zeta, \vartheta) \phi_q \left( \int_0^1 G^\varphi(\vartheta, \eta) Q_1(\eta, \mu_n(\eta)) d\eta \right) d\vartheta \right| \]

\[\leq \left| \int_0^1 G^\varphi(\zeta, \vartheta) \phi_q \left( \int_0^1 G^\varphi(\vartheta, \eta) Q_1(\eta, \mu(\eta)) d\eta \right) d\vartheta \right| \]

\[+ \left| \int_0^1 G^\varphi(\zeta, \vartheta) \phi_q \left( \int_0^1 G^\varphi(\vartheta, \eta) (Q_1(\eta, \mu_n(\eta)) - Q_1(\eta, \mu(\eta))) d\eta \right) d\vartheta \right| \]

\[\leq \left( \frac{1}{\Gamma(\sigma + 1)} + \frac{\sigma - 1}{\Gamma(\lambda)} \right) + \frac{\alpha^{\sigma - 1}}{(\lambda - \sigma)\Gamma(\lambda)} \]

\[\leq \nabla_1 \left( \alpha ||\mu||^m_1 + T_{\omega} \right), \]

(3.22) 

from (3.18) and pseudnonorm (A1), we get

\[|\mathcal{M}_2^*(\mu)| = \left| Q_2(\zeta, \mu) \right| + \frac{1}{\Gamma(\sigma)} \int_0^\vartheta (\vartheta - \eta)^{\sigma - 1} Q_2(\vartheta, \mu(\vartheta)) d\eta \]

\[\leq \left| Q_2(\zeta, \mu) \right| + \frac{1}{\Gamma(\sigma)} \int_0^\vartheta (\vartheta - \eta)^{\sigma - 1} Q_2(\vartheta, \mu(\vartheta)) d\eta \]

\[\leq \left( \alpha ||\mu||^m_1 + \frac{1}{\Gamma(\sigma)} \right) \left( \alpha ||\mu||^m_1 + T_{\omega} \right) \]

\[\leq \nabla_2 \left( \alpha ||\mu||^m_1 + T_{\omega} \right), \]

(3.23)
From (3.28) and (3.29), we obtain
\[
\mathcal{M}(\mu)(\xi) \leq \nabla_1(\frac{1}{2}(a_1||\mu||^2 + T^*_Q)) + \nabla_2(\frac{1}{2}(a_2||\mu||^2 + T^*_Q)) \\
\leq \Omega ||\mu||^2 + F^*_2.
\]
(3.24)

This completes the proof. \(\square\)

**Theorem 5.** With supposition \((A_1)\), the operator \(\mathcal{M}^*: \sigma^* \to \sigma^*\) is compact and \(\xi^*\)-Lipschitz with constant zero.

**Proof.** By using Theorem 4, we conclude that \(\mathcal{M}^*: \omega \to \omega\) is bounded. Next, by supposition \((A_1)\), Lemma 3.1 and Equation (3.17), for any \(\zeta_1, \zeta_2 \in [0, 1]\), we get
\[
|\mathcal{M}_{\xi}^*(\mu(\zeta_1)) - \mathcal{M}_{\xi}^*(\mu(\zeta_2))| \\
= \left| \int_0^1 \mathcal{G}^*(\zeta_1, \theta)\phi_q(\int_0^1 \mathcal{G}^*(\theta, \eta) Q_1(\eta, \mu(\eta))d\eta)d\theta \\
- \int_0^1 \mathcal{G}^*(\zeta_2, \theta)\phi_q(\int_0^1 \mathcal{G}^*(\theta, \eta) Q_1(\eta, \mu(\eta))d\eta)d\theta \right| \\
\leq \left| \int_0^1 \mathcal{G}^*(\zeta_1, \theta) - \mathcal{G}^*(\zeta_2, \theta)\phi_q(\int_0^1 \mathcal{G}^*(\theta, \eta) Q_1(\eta, \mu(\eta))d\eta)d\theta \right| \\
\times \left| \int_0^1 \mathcal{G}^*(\theta, \eta) Q_1(\eta, \mu(\eta))d\eta \right|d\theta \\
\leq \left| \frac{1 - \zeta_2}{\Gamma(\sigma + 1)} \right| + \left| \frac{1 - \zeta_2}{\Gamma(\sigma + 1)} \right|^{\sigma-1} \left| \frac{1}{\Gamma(\sigma + 1)} \right|^{\sigma-1} (a||\mu||^2 + T^*_Q) \times \left| \frac{1}{\Gamma(\lambda + 1)} \right| + \left| \frac{1}{\Gamma(\lambda + 1)} \right|^{\sigma-1} (a||\mu||^2 + T^*_Q).
\]
(3.25)

As \(\zeta_1 \to \zeta_2\), the right-hand side of (3.25) approaches zero. Thus, the operator \(\mathcal{M}^*_\xi\) is an equi-continuous on \(S\). By Arela–Ascoli theorem implies that \(\mathcal{M}^*_\xi(S)\) is compact. Subsequently, \(S\) is \(\xi^*\)-Lipschitz with constant zero.

**Theorem 6.** Under the suppositions \((A_1)\) and \((A_2)\) and \(\Omega < 1\). The HFDEs (1.1) has positive solution and the set containing solutions of the HFDEs is bounded in \(\sigma^*\).

**Proof.** For EUS of the HFDEs for problem (1.1), with the help of theorem 1. Let us consider that \(q = \{\mu \in \sigma^*: \text{there exist } \Omega \in [0, 1], \text{ where } \mu = \Omega(\mu) \}\). To prove to that \(q\) is bounded, we assume that \(\mu \in q\), such that \(\|\mu\| = N \to \infty\), form theorem 4, we get
\[
\|\mu\| = \|\Omega(\mu)\| \leq \|\chi(\mu)\| \leq \|\mathcal{M}^*_\xi(\mu)\| \\
+ \|\mathcal{M}^*_2(\mu)\| \leq F^*_2 + \Omega ||\mu||^2.
\]
(3.26)
as \(\|\mu\| = N\), then, (3.24) implies that
\[
\|\mu\| \leq \Omega ||\mu||^2 + F^*_2, \\
1 \leq \Omega ||\mu||^2 + F^*_2, \\
1 \leq \frac{1}{N!} F^*_2 \to 0, \text{ as } N \to \infty.
\]

This is a contradiction. In the end, \(\|\mu\| < \infty\) which means that \(q\) is bounded and by theorem 1, the \(q\) has at least one solution of our problem (1.1). Thus, the solution \(\phi\) of the HFDEs (1.1) is bounded. \(\square\)

**Theorem 7.** Let suppositions \((A_1)\) and \((A_2)\) hold. Then, the HFDEs (1.1) has a unique positive solution provided \(\kappa^* < 1\) such that
\[
\kappa_1 = (q-1)\rho^{\sigma-2}\left(\frac{1}{\Gamma(\sigma + 1)} + \frac{\epsilon^{\sigma-1}}{\Gamma(\sigma)}\right) \\
\times \left(\frac{1}{\Gamma(\lambda + 1)} + \frac{\epsilon^{\lambda-\sigma}}{(\lambda-\sigma)\Gamma(\lambda)}\right)^{\gamma^*} \Omega_l, \\
\kappa_2 = (1 + \frac{\rho^\sigma}{\Gamma(\sigma + 1)})^{\gamma^*} \Omega_l, \\
\kappa^* = \kappa_1 + \kappa_2.
\]

**Proof.** From (3.17), suppositions \((A_1)\) and Lemma 3, for any \(\zeta_1, \zeta_2 \in [0, 1]\), we have
\[
|\mathcal{M}_{\xi}^*(\mu(\zeta_1)) - \mathcal{M}_{\xi}^*(\mu(\zeta_2))| \\
= \left| \int_0^1 \mathcal{G}^*(\zeta_1, \theta)\phi_q(\int_0^1 \mathcal{G}^*(\theta, \eta) Q_1(\eta, \mu(\eta))d\eta)d\theta \\
- \int_0^1 \mathcal{G}^*(\zeta_2, \theta)\phi_q(\int_0^1 \mathcal{G}^*(\theta, \eta) Q_1(\eta, \mu(\eta))d\eta)d\theta \right| \\
\leq \left| \int_0^1 \mathcal{G}^*(\zeta_1, \theta) - \mathcal{G}^*(\zeta_2, \theta)\phi_q(\int_0^1 \mathcal{G}^*(\theta, \eta) Q_1(\eta, \mu(\eta))d\eta)d\theta \right| \\
\times \left| \int_0^1 \mathcal{G}^*(\theta, \eta) Q_1(\eta, \mu(\eta))d\eta \right|d\theta \\
\leq \left| \frac{1 - \zeta_2}{\Gamma(\sigma + 1)} \right| + \left| \frac{1 - \zeta_2}{\Gamma(\sigma + 1)} \right|^{\sigma-1} \left| \frac{1}{\Gamma(\sigma + 1)} \right|^{\sigma-1} (a||\mu||^2 + T^*_Q) \times \left| \frac{1}{\Gamma(\lambda + 1)} \right| + \left| \frac{1}{\Gamma(\lambda + 1)} \right|^{\sigma-1} (a||\mu||^2 + T^*_Q).
\]
(3.25)

As \(\zeta_1 \to \zeta_2\), the right-hand side of (3.25) approaches zero. Thus, the operator \(\mathcal{M}^*_\xi\) is an equi-continuous on \(S\). By Arela–Ascoli theorem implies that \(\mathcal{M}^*_\xi(S)\) is compact. Subsequently, \(S\) is \(\xi^*\)-Lipschitz with constant zero.
\[ |M^c_2(\mu) - M^c_2(\mu^-)| \]
\[ = |Q_2(\zeta, \mu(\zeta)) - Q_2(\zeta, \mu^-) + \frac{1}{\Gamma(\sigma)} \int_0^1 (a-\theta)^{\sigma-1} Q_2(\theta, \mu(\theta)) \, d\theta - \frac{1}{\Gamma(\sigma)} \int_0^1 (a-\theta)^{\sigma-1} Q_2(\theta, \mu^-) \, d\theta| \]
\[ \leq |Q_2(\zeta, \mu(\zeta)) - Q_2(\zeta, \mu^-)| + \frac{1}{\Gamma(\sigma)} \int_0^1 (a-\theta)^{\sigma-1} \times \left| Q_2(\theta, \mu(\theta)) - Q_2(\theta, \mu^-) \right| d\theta \]
\[ \leq \gamma_0 |\mu(\zeta) - \mu^-| + \frac{1}{\Gamma(\sigma)} \int_0^1 (a-\theta)^{\sigma-1} Q_2(\theta, \mu(\theta)) - Q_2(\theta, \mu^-) \, d\theta \]
\[ \leq (1 + \frac{a^\sigma}{\Gamma(\sigma + 1)}) \gamma_0 |\mu(\zeta) - \mu^-| \]
\[ = \kappa_0 |\mu(\zeta) - \mu^-|, \quad (3.28) \]

with the help of (3.26), (3.27), we get

\[ |M(\mu(\zeta)) - M(\mu^-)| \]
\[ \leq (q-1) \rho^{\sigma-2} \frac{1}{\Gamma(\sigma + 1)} + \frac{e^{\sigma-1}}{\Gamma(\sigma)} \]
\[ \times \left( \frac{1}{\Gamma(\lambda + 1)} + \frac{a^{\lambda-\sigma}}{(\lambda - \sigma)\Gamma(\lambda)} \right)^2 |\mu(\zeta) - \mu^-| \]
\[ + (1 + \frac{a^\sigma}{\Gamma(\sigma + 1)}) \gamma_0 |\mu(\zeta) - \mu^-| \]
\[ = \kappa |\mu(\zeta) - \mu^-| + \kappa_2 |\mu(\zeta) - \mu^-| = \kappa^* |\mu(\zeta) - \mu^-|. \quad (3.29) \]

with \( \kappa^* < 1 \), the contraction implies that \( M^* \) has a unique fixed point. Consequently, the coupled system (1.1) of HFDEs has a unique positive solution.

4. Hyers–Ulam stability of system

Here, we will study HU-stability for nonlinear coupled system of HFDEs with p-Laplacian operator and integral IBCs for the solution.

**Definition 6.** The coupled system integral Equations (3.17) and (3.18) are HU-stable if there exist positive constants \( W^* \) satisfies these conditions:

For every \( \beta > 0 \), if

\[ |\mu(\zeta) - Q_2(\zeta, \mu(\zeta)) - \int_0^1 G^*(\zeta, \theta) \phi_q \left( \int_0^1 G^*(\theta, \eta) Q_2(\eta, \mu(\eta)) \, d\eta \right) d\theta \]
\[ - \frac{1}{\Gamma(\sigma)} \int_0^1 (a-\theta)^{\sigma-1} Q_2(\mu(\eta)) \, d\theta | \leq \beta, \quad (4.1) \]

then, there exists a \( (\mu^*(\zeta)) \), satisfying

\[ \mu^*(\zeta) = Q_2(\zeta, \mu^*(\zeta)) + \int_0^1 G^*(\zeta, \theta) \phi_q \left( \int_0^1 G^*(\theta, \eta) Q_2(\eta, \mu^*(\eta)) \, d\eta \right) d\theta \]
\[ \times \left( \frac{1}{\Gamma(\sigma + 1)} + \frac{a^\sigma}{\Gamma(\sigma + 1)} \right)^2 |\mu^*(\zeta) - \mu^*(\zeta^-)| \]
\[ \leq (q-1) \rho^{\sigma-2} \frac{1}{\Gamma(\sigma + 1)} + \frac{e^{\sigma-1}}{\Gamma(\sigma)} \]
\[ \times \left( \frac{1}{\Gamma(\lambda + 1)} + \frac{a^{\lambda-\sigma}}{(\lambda - \sigma)\Gamma(\lambda)} \right)^2 |\mu(\zeta) - \mu^-| \]
\[ \leq (q-1) \rho^{\sigma-2} \frac{1}{\Gamma(\sigma + 1)} + \frac{e^{\sigma-1}}{\Gamma(\sigma)} \]
\[ \times \left( \frac{1}{\Gamma(\lambda + 1)} + \frac{a^{\lambda-\sigma}}{(\lambda - \sigma)\Gamma(\lambda)} \right)^2 |\mu^*(\zeta) - \mu^*(\zeta^-)| \]
\[ = \kappa |\mu^*(\zeta) - \mu^*(\zeta^-)| \leq W^* \beta. \quad (4.3) \]

**Theorem 8.** With the suppositions \( (A_1) \) and \( (A_2) \) hold, the HFDEs with p-Laplacian (1.1) is HU-stable.

**Proof.** From Theory (7) and definition (6), let \( \mu(\zeta) \) be a solution of the system (3.17), (3.18) and \( \mu^*(\zeta) \) be an approximation achieves (4.2). Then, we get

\[ |\mu(\zeta) - \mu^*(\zeta)| \]
\[ = |\left( \int_0^1 G^*(\zeta, \theta) \phi_q \left( \int_0^1 G^*(\theta, \eta) Q_2(\eta, \mu(\eta)) \, d\eta \right) d\theta \]
\[ + \frac{1}{\Gamma(\sigma)} \int_0^1 (a-\theta)^{\sigma-1} Q_2(\mu(\eta)) \, d\theta \]
\[ - \int_0^1 G^*(\zeta, \theta) \phi_q \left( \int_0^1 G^*(\theta, \eta) Q_2(\eta, \mu^*(\eta)) \, d\eta \right) d\theta \]
\[ - \frac{1}{\Gamma(\sigma)} \int_0^1 (a-\theta)^{\sigma-1} Q_2(\mu^*(\eta)) \, d\theta - Q_2(\zeta, \mu^*(\zeta)) | \]
\[ \leq \left( \frac{1}{\Gamma(\sigma + 1)} + \frac{a^\sigma}{\Gamma(\sigma + 1)} \right)^2 |\mu(\zeta) - \mu^-| \]
\[ + \frac{1}{\Gamma(\sigma)} \int_0^1 (a-\theta)^{\sigma-1} Q_2(\mu(\eta)) \, d\theta \]
\[ - \frac{1}{\Gamma(\sigma)} \int_0^1 (a-\theta)^{\sigma-1} Q_2(\mu^*(\eta)) \, d\theta - Q_2(\zeta, \mu^*(\zeta)) | \]
\[ \leq (q-1) \rho^{\sigma-2} \frac{1}{\Gamma(\sigma + 1)} + \frac{e^{\sigma-1}}{\Gamma(\sigma)} \]
\[ \times \left( \frac{1}{\Gamma(\lambda + 1)} + \frac{a^{\lambda-\sigma}}{(\lambda - \sigma)\Gamma(\lambda)} \right)^2 |\mu(\zeta) - \mu^*(\zeta)| \]
\[ + (q-1) \rho^{\sigma-2} \frac{1}{\Gamma(\sigma + 1)} + \frac{e^{\sigma-1}}{\Gamma(\sigma)} \]
\[ \times \left( \frac{1}{\Gamma(\lambda + 1)} + \frac{a^{\lambda-\sigma}}{(\lambda - \sigma)\Gamma(\lambda)} \right)^2 |\mu^*(\zeta) - \mu^*(\zeta^-)| \]
\[ \leq (q-1) \rho^{\sigma-2} \frac{1}{\Gamma(\sigma + 1)} + \frac{e^{\sigma-1}}{\Gamma(\sigma)} \]
\[ \times \left( \frac{1}{\Gamma(\lambda + 1)} + \frac{a^{\lambda-\sigma}}{(\lambda - \sigma)\Gamma(\lambda)} \right)^2 |\mu(\zeta) - \mu^-| \]
\[ \leq \left( \frac{1}{\Gamma(\sigma + 1)} + \frac{a^\sigma}{\Gamma(\sigma + 1)} \right)^2 |\mu(\zeta) - \mu^-| \]

5. Illustrative example

In the part we will offer an example to prove our results of the proposed problem in sections 3 and 4.
Example 1. Let the coupled system HFDE with p-Laplacian as following:

\[
\begin{aligned}
&\frac{c^p D_{\lambda}^\alpha \left( \phi_p \left( D_{\lambda}^\alpha \left( \mu \zeta - Q_2 \left( \zeta, \mu (\zeta) \right) \right) \right) \right)}{\Gamma (\sigma + 1)} = -Q_1 \left( \zeta, \mu (\zeta) \right), \\
&\left. I^{\sigma - \frac{1}{2}} \left( \phi_p \left( D_{\lambda}^\alpha \left( \mu (\zeta) - Q_2 \left( \zeta, \mu (\zeta) \right) \right) \right) \right) \right|_{z=0} = 0, \quad \text{for} \quad i = 2, 3, 4, \ldots, k-1, \\
&\left. \mu (0) = 0, \quad \text{for} \quad i = 1, 2, 3, 4, \ldots, k-1, \\
&\mu (\varepsilon) = Q_2 \left( \zeta, \mu (\zeta) \right) \right|_{z=\varepsilon}, \quad \mu (0) = \frac{1}{\Gamma (\sigma)} \int_0^\varepsilon (\sigma - \eta)^{\sigma - 1} Q_2 \left( \zeta, \mu (\zeta) \right) \eta d\eta.
\end{aligned}
\]  

(5.1)

where \( \zeta \in [0, 1], p = 5, r = \frac{3}{2}, \sigma = \frac{1}{2}, \lambda = \frac{7}{15}, Q_1(\zeta, \mu (\zeta)) = -25/17 + 1/15 \sin (\nu), Q_2(\zeta, \nu (\zeta)) = 26/16 + 1/13 \cos (\mu), \) which implies \( \gamma_1^* = \gamma_2^* = \frac{1}{7}, \zeta = \frac{15}{17}, \epsilon = \frac{1}{8}, \sigma = 0.5. \) By simple mathematical computations, we get

\[
\begin{aligned}
\kappa_1 &= (q-1)p^{\alpha-2} \frac{1}{\Gamma (\sigma + 1)} \times \frac{e^{\sigma - \frac{1}{2}}}{\Gamma \left( \frac{1}{2} \right)} \frac{1}{\Gamma \left( \frac{3}{2} + \frac{1}{2} \right)} + \frac{\frac{1}{6}}{\Gamma (\frac{3}{2})}, \\
\kappa_2 &= (1 + \frac{1}{6} \frac{3}{2} \frac{1}{2} + \frac{1}{6}) \frac{1}{\Gamma (\frac{5}{2})}, \\
\kappa^* &= \kappa_1 + \kappa_2 \\
&= \frac{4}{3} \frac{3}{2} \frac{1}{2} \frac{1}{2} + \frac{1}{6} \frac{1}{2} \frac{1}{2} + \frac{1}{6} \frac{6}{5} \frac{1}{2} \frac{1}{2} \\
&+ \frac{1}{6} + \frac{1}{5} \frac{1}{2} \frac{1}{2} \\
&= 0.329317 < 1.
\end{aligned}
\]

(5.2)

By Theory 7 and Equation (5.2), we deduce that (5.1) has a unique positive solution. Thus, the conditions of Theory 8 can be verified simply. Similarly, the coupled system of HFDEs (5.1) is HU-stable.

To verify our work, we have drawn an example in the following two cases.

Case 1: In the first case, we appointed one value to the derivative \( \lambda \) while the derivative \( \sigma \) calculated at \( i \) value, where \( i = 10.5, \ldots, 12. \) In Figure 1, we have drawn the relationship between \( \sigma_i \) and value \( \kappa^* \) shown in (5.2).

Case 2: In the first case, we appointed one value to the derivative \( \sigma \) while the derivative \( \lambda \) calculated at \( i \) value, where \( i = 3.5, \ldots, 5. \) In Figure 2, we have drawn the relationship between \( \lambda_i \) and value \( \kappa^* \) shown in (5.2).

Through the drawing. Note that whenever we changed derivatives values \( \lambda, \sigma \), the value of \( \kappa^* \) is less than 1. We conclude from example with drawing that the solution is unique and stable.

6. Conclusion The article

In this literature, we have investigated three aspects to a coupled system HFDEs (1.1) involving Caputos derivative with \( \phi_p \) Laplacian through using degree theorem and analysis on Banach space functional. For these purposes, we first transformed our problem (1.1) into integral equations by Green functions. Then, we proved existence of the solution with the help
of the Leray–Schauder and topological degree theorems. We proved the uniqueness of the solution by using the Banach principle. As well as, we used the Hyers–Ulam technique to prove stability. To verify the validity of existing results, we included an example as an application of our findings using Mathematica. Additional objectives In the future, we intend to look into the outcomes of its multiplicity. Also, we plan to study the existence and uniqueness of solution to this problem by different derivatives like Atangana–Baleanu–Caputo and Riemann–Liouville derivatives of various orders.

Authors’ contributions
This article was wrote by both authors. They have all read the final draft and given their approval.

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