Testing quantum adiabaticity with quench echo

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Abstract. Adiabaticity of quantum evolution is important in many settings; one example is adiabatic quantum computation (AQC). Nevertheless, to date, there is no effective method available for testing the adiabaticity of the evolution for the case where the eigenenergies of the driven Hamiltonian are not known. We propose a simple method for checking the adiabaticity of a quantum process for an arbitrary quantum system. We further propose an operational method for finding more efficient protocols that approximate adiabaticity, and suggest a ‘uniformly adiabatic’ quench scheme based on the Kibble–Zurek mechanism for the case where the initial and final Hamiltonians are given. This method should help in implementing AQC and other tasks where preserving the system in the ground state of a time-dependent Hamiltonian is desired.

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1. Introduction

In a quantum quench process, when the Hamiltonian of a quantum system is driven from $H_0$ to $H_1$, interstate excitations of the system usually occur, owing to the non-commutativity of the Hamiltonians at different moments. However, when the quench process is slow enough, the interstate excitations will be suppressed. According to the quantum adiabatic theorem [1], when the condition for quantum adiabatic approximation

$$
\langle \Phi_{\text{ground}}(t) | \frac{d H(t)}{d t} | \Phi_{\text{excited}}(t) \rangle \ll \Delta(t)^2
$$

is satisfied, the system will remain in the ground state—its evolution will be adiabatic—except for some special situations. Here, $H(t)$ is the changing Hamiltonian, and $\Delta(t)$ is the minimal energy gap between the ground state $| \Phi_{\text{ground}}(t) \rangle$ and the first excited state $| \Phi_{\text{excited}}(t) \rangle$ of $H(t)$.

In order to ensure that a quantum system evolves adiabatically, one usually needs to find the energy spectrum (or at least the smallest energy gap $\Delta$) of the driven Hamiltonian. One can then use the quantum adiabatic theorem to choose a proper time scale, so that the conditions for quantum adiabatic approximation are satisfied and the evolution remains adiabatic. Nevertheless, in practice, neither eigenenergies nor eigenstates of a complex quantum many-body system are easy to obtain. This is often the case in implementing adiabatic quantum comutation (AQC) [5] as well as quantum annealing [6, 7]. Hence, one does not have the ingredients to use the quantum adiabatic theorem. One cannot count on the direct comparison between the final state and the instantaneous ground state of the final Hamiltonian, either. Thus, it would be useful to find a reliable method for evaluating the adiabaticity of an evolution under an arbitrary Hamiltonian, especially when one has no idea about the eigenstates and/or eigenenergies of the system (except at the initial moment).

The quench echo method we propose here is one solution to the above problem. It will allow one to evaluate unambiguously the adiabaticity of a process. What is more, it can help one find the efficient ‘uniformly adiabatic’ quench path in the parameter space of the Hamiltonian.

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2 In recent years, the conditions for the adiabatic approximation have become a subject of controversy [2, 26]. The standard criterion is found to be neither sufficient nor necessary [3]. Additional assumptions are required to ensure that the stated conditions are rigorous. See [4] for a clarification of the weakness of the adiabatic condition.
Such ideas may have applications in the implementation of AQC [5]. This paper is organized as follows. In section 2, we introduce the quench echo method and briefly explain its underlying physics. In section 3, we use a simple model to demonstrate the main ideas of the general theory. In section 4, we propose a uniformly adiabatic scheme that is based on the application of the Kibble–Zurek mechanism (KZM) to quantum phase transitions. In section 5, we present the discussions and conclusions.

2. Quench echo

Consider a system described by the Hamiltonian $H(g(t))$, where $g(t)$ is a time-dependent parameter. The system is initially prepared in the ground state of $H(g(t = 0))$. The system evolves under the influence of the driven Hamiltonian, which changes from $H(g(t = 0))$ to $H(g(t = T))$. Our aim is to test the adiabaticity of this evolution, but we know nothing about the eigenenergies and eigenstates of the time-dependent Hamiltonian except at $t = 0$. Hence, we cannot count on the comparison between the evolving state and the instantaneous ground state. Neither can we use the adiabatic theorem. Nevertheless, we can apply a return ‘echo’ quench following the initial quench (from $t = 0$ to $t = T$). That is, one extends the evolution from $t = T$ to $t = 2T$,

$$H(g(t)) = \begin{cases} H(g(t)), & (0 < t < T), \\ H(g(2T - t)), & (T < t < 2T). \end{cases}$$

Reference [8] studied the density of topological defects under such quench echo, but our emphasis is different. The final Hamiltonian is identical to the initial Hamiltonian $H(g(t = 2T)) \equiv H(g(t = 0))$. Hence, we can use the fidelity of the initial state, e.g. the ground state of $H(g(t = 0))$, and the final evolving state as a criterion for the adiabaticity of the evolution.

$$F = \left| \langle GS | \hat{T} e^{-i \int_{T}^{2T} H(g(2T - t)) dt} \hat{T} e^{-i \int_{0}^{T} H(g(t)) dt} | GS \rangle \right|^2,$$

where $\hat{T}$ is the time-ordered operator and $|GS\rangle$ is the ground state of $H(g(t = 0))$. When the fidelity $F$ is greater than a threshold value close to unity (e.g. 0.999, so the error tolerance is 0.001), the whole process ($0 < t < 2T$) is adiabatic. This implies that the forward quench process ($0 < t < T$) is also adiabatic. The underlying mechanism for this ‘quench echo’ method is straightforward: except for the eigenstate of the Hamiltonian at the initial moment $H(g(t = 0))$, we do not have any information about its eigenstates at other moments. Hence, we can only quench the Hamiltonian back, so that it goes back to its initial $H(g(t = 0))$, and we can measure the final state (and compare it with the initial state). The quench echo protocol (equation (1)) ensures that the excitation probabilities in the forward quench process and those in the backward quench process are similar (but not identical; see [9, 10]). As a result, when the forward process is adiabatic (no excitations), so is the backward quench process. Otherwise, both the forward and the backward processes are nonadiabatic, and the phase accumulated between transitions (known as the Stückelberg phase) may result in constructive or destructive interference [4, 11]. Usually the excitations in the forward and backward processes cannot cancel each other out (but see [4]). Hence, through this quench echo method, without knowing about the eigenenergies and eigenstates, one is able to evaluate the adiabaticity of an arbitrary evolution in most cases. Nevertheless, in some special cases (e.g. impulse evolution, which is so fast that the state of the system is frozen), the final fidelity is equal to unity, but the process may not be adiabatic.
A solution to this problem is to let the system evolve freely for some time before the backward quench. We will discuss this in detail in the next section. By utilizing quench echo, one can even find a uniformly adiabatic quench protocol for a given Hamiltonian by repeating the above process with different quench time scales.

3. A case study: Ising chain in a transverse magnetic field

We now use a simple model to demonstrate our central ideas: consider the quench dynamics of an Ising model in a transverse magnetic field [12]. The time-dependent Hamiltonian is

$$H(t) = -J \sum_{i=1}^{N} [\sigma_i^x \sigma_{i+1}^x + g(t) \sigma_i^z],$$

(3)

where \( J \) indicates the energy scale; \( \sigma_i^\alpha, \alpha = x, y, z, \) is the Pauli matrix on the \( i \)th lattice site; and \( g(t) \) is the reduced strength of the magnetic field, which varies with time. It is known that for this model there is a finite energy gap \( \Delta_1 \) and the backward quench time scale, \( \tau_{Q} \). An Ising chain in a transverse magnetic field

$$\Delta_1 \approx \frac{2J}{N^2}, \quad \tau_{Q} \approx \frac{T}{N}.$$

(4)

where \( \tau_{Q} \) is the time scale of the quench. The larger the \( \tau_{Q} \), the slower the quench. In the forward quench, the strength of the magnetic field is ramped from \( g = 0 \) to \( g = g_T \) continuously, and in the quench echo, it is ramped back from \( g = g_T \) to \( g = 0 \), where \( g_T \) is the turnaround point. Initially the system is prepared in the ground state of \( H(g = 0) \). When one quenches the system at different rates (by choosing different \( \tau_{Q} \)), the fidelity (2) will be different.

The Hamiltonian of the Ising model (3) can be decoupled into \( N \) independent fermionic modes [12].

$$H(t) = \sum_{k} \Lambda_k(g(t)) \left[ |+(t)\rangle_k \langle+(t)|_k - |-(t)\rangle_k \langle-(t)|_k \right],$$

(5)

where \( |+(t)\rangle_k \) and \( |-(t)\rangle_k \) are the two instantaneous eigenstates of the \( k \) mode. Their corresponding eigenenergies are \( \pm \Lambda_k(g(t)) \), and \( \Lambda(g(t)) = J \sqrt{g(t)^2 - 2g(t) \cos k + 1}. \) Here, \( k = (2s + 1)\pi/N, \ s = 0, 1, 2, \ldots, N/2 - 1 \) is the wave vector, and the number of spins \( N \) is even.

We write the Schrödinger equation

$$i\hbar \frac{d}{dt} |\Phi(t)\rangle = H(t) |\Phi(t)\rangle$$

in the instantaneous eigenbases \( |+(t)\rangle_k \) and \( |-(t)\rangle_k \) of \( H(t) \), where \( |\Phi(t)\rangle = \prod_k \alpha_k(t) |+(t)\rangle_k + \beta_k(t) |-(t)\rangle_k \). For simplicity, we choose \( \hbar = 1 \) hereafter. For both the forward quench \( 0 < t < (g_0 - g_T)\tau_{Q} \) and the backward \( ((g_0 - g_T)\tau_{Q} < t < 2(g_0 - g_T)\tau_{Q}) \) process, the Schrödinger equation can be written as

$$\frac{d}{dt} \left[ \begin{array}{c} \alpha_k(t) \\ \beta_k(t) \end{array} \right] = \left[ \begin{array}{cc} 2\Lambda_k(g(t)) & -iJ^3 \sin k \frac{dg(t)}{\Lambda_k^3(g(t))} \\ iJ^3 \sin k \frac{dg(t)}{2\Lambda_k^2(g(t))} & -2\Lambda_k(g(t)) \end{array} \right] \left[ \begin{array}{c} \alpha_k(t) \\ \beta_k(t) \end{array} \right],$$

(6)
where the initial condition for equation (6) is $\alpha_k(t = 0) = 0$, $\beta_k(t = 0) = 1$. The modulus square of the overlap between the final state of equation (6) and the instantaneous ground state at $g = g_0$ gives the fidelity (2)

$$F = P_{GS}(2(g_0 - g_T)\tau_Q) = \prod_{k>0} |\beta_k(2(g_0 - g_T)\tau_Q)|^2.$$  

(7)

In the following, we will focus on the solution of equation (7). We will consider both the numerical and the analytical results.

3.1. The Kibble–Zurek mechanism (KZM) and three regimes

Before the quantitative study of the fidelity and its relation to the time scales of quench, we describe the KZM [13, 14] of second-order phase transitions, which provides a qualitative and quantitative understanding of the quench process. KZM describes e.g. the relation between the density of topological defects, which are generated during quenching across a phase transition, and the time scale of quench (see [15] for a good review). KZM was first introduced in the classical phase transitions [13,14] and later generalized to quantum phase transitions [18]. In our study, however, we will not focus on the density of topological defects, but on the adiabaticity of the evolution of the system.

A quantum phase transition is characterized by a vanishing excitation gap $\Delta_1(g(t)) \approx \Delta_0 |g(t) - g_c|^\nu$ and a divergent correlation length $\xi \approx \xi_0 / |g(t) - g_c|^\nu$, where $\nu$ and $\nu$ are the critical exponents, and $\Delta_0$ and $\xi_0$ are constants [12]. We define a dimensionless distance from the critical point $g_c$ by

$$\epsilon(t) = \frac{g(t) - g_c}{g_c}.$$  

(8)

A generic $\epsilon(t)$ can be linearized near the critical point $\epsilon(t) = 0$ as

$$\epsilon(t) \approx -\frac{t}{\tau_Q}.$$  

(9)

There are two interlinked time scales during a quench: the system reaction time is given by the inverse of the gap $\tau(\epsilon(t)) = 1/|\Delta_0| |g(t) - g_c|^\nu$ and the time scale of transition is given by $|g(t) - g_c|^\nu / \frac{d}{dt} |g(t) - g_c|^\nu$. Away from the critical point the reaction time is small in comparison with the time scale of transition and the evolution is adiabatic. Near the critical point, however, the opposite situation occurs and the evolution is approximately impulse (the state of the system is frozen out). The boundary $\hat{t}$ between the two regions is determined by the relation $\tau(\epsilon(t)) = \epsilon / \dot{\epsilon}$; or

$$1/|g(\hat{t}) - g_c|^\nu \sim \frac{d}{dt} |g(\hat{t}) - g_c|^\nu.$$  

(10)

That is, $\hat{t} \sim (\tau_Q)^{\nu}$, which gives $\hat{t} \sim \tau_Q^{\nu/(1+\nu)}$ [14, 18]. For the Ising model, we have $\nu = \nu = 1$, resulting in $\hat{t} \sim \tau_Q^{1/2}$ [18]. According to KZM, when $t \in (-\hat{t}, \hat{t})$, the system will not evolve (the wavefunction will be frozen). Outside this time interval the system will evolve approximately adiabatically.

3 The KZM with nonlinear quench schemes was also discussed recently, see [15]–[17].
For an infinitely large system, the energy gap is vanishingly small at the critical point, and one can always find a $\hat{t}$. According to KZM, this implies that, no matter how slow one quenches the Hamiltonian in an infinite system, the evolution across the critical point can never be adiabatic. For a finite-size system, however, there is a finite energy gap even at the critical point, such as in the model that we are studying. When one quenches the system sufficiently slowly (large $\tau_Q$), $\hat{t}$ approaches very near the critical point where—for a finite system—scalings no longer hold. As a consequence, KZM does not lead to simple scaling, as $\tau(\epsilon(t)) = \epsilon/\dot{\epsilon}|_{\hat{t}}$ leads to a more difficult equation which has to be solved to obtain $\hat{t}$ [19]. Indeed—in accord with the adiabatic theorem—KZM predicts that when $\tau_Q$ is larger than the inverse of the gap, the transition will remain adiabatic throughout. Thus, a finite energy gap allows an adiabatic evolution across the critical point when the Hamiltonian is driven sufficiently slowly. This is the adiabatic quench regime. By contrast, when one quenches the Hamiltonian very fast (small $\tau_Q$), there is a big $\hat{t}$, and there is an approximately impulse regime for $t \in (-\hat{t}, \hat{t})$ when the quench is essentially instantaneous. In this time interval, the system will approximately cease to evolve—its wavefunction will be frozen. This is the so-called impulse regime [18, 20].

When one chooses a time scale of quench $\tau_Q$ between the above two limiting cases, the system will evolve adiabatically when either $t < -\hat{t}$ or $t > \hat{t}$, and will be frozen when $t \in (-\hat{t}, \hat{t})$. We call this regime the intermediate regime. We can summarize the quench behavior as follows: for a finite-size system, when $\tau_Q$ is large enough, the evolution will be adiabatic; when $\tau_Q$ is extremely small, the state of the system will be frozen; when $\tau_Q$ is in between these two limiting cases, the process is in the intermediate regime.

### 3.2. Numerical and analytical results

Having obtained a qualitative understanding of the quench dynamics from the above KZM arguments, in the following we will study the Ising model quantitatively, and compare the results with the estimates obtained above. We consider a spin chain with a finite size $N = 50$, and start evolving it from $g_0 = 10$ and let it turn around at $g_T = 0$. There is a finite energy gap for this system at the quantum critical point $g_c = 1$. We choose three different quench time scales $\tau_Q = 150$, $\tau_Q = 35$ and $\tau_Q = 0.004$, which correspond to the adiabatic, intermediate and impulse regimes. The system evolves under the time-dependent Schrödinger equation. We plot the probability $P_{GS}(g)$ in the instantaneous ground state as a function of the controlling parameter $g$ during the quench process in figures 1(a)–(c).

From figure 1(a), it can be seen that when the time scale of the quench is relatively large, the system evolves almost adiabatically in the whole range of the parameter $g_T = 0 < g < g_0 = 10$; $P_{GS}(g)$ is always close to unity (except for a tiny decay and partial revival at the critical point.) So is the fidelity of the quench echo (see figure 1(a)).

When the time scale of the quench is reduced to $\tau_Q = 35$ (see figure 1(b)), the quench dynamics enter the intermediate regime. It can be seen that away from the critical point, the evolution is adiabatic. But near the critical point, the probability in the instantaneous ground state $P_{GS}(g)$ decays sharply and oscillates rapidly. This is due to the interstate transitions at the anti-cross point. Soon after passing through the quantum critical point, the adiabatic evolution resumes.

When the time scale of the quench is further reduced to $\tau_Q = 0.004$ (almost instantaneous quench), the wave function of the system is nearly frozen. Hence, the probability of being in the instantaneous ground state is simply equal to the overlap of the initial state and the instantaneous
Figure 1. Three regimes of the quench dynamics: (a) nearly adiabatic regime, \(\tau_Q = 150\); (b) the intermediate regime \(\tau_Q = 35\); and (c) nearly impulse regime \(\tau_Q = 0.004\). The horizontal axis represents the parameter \(g(t)\), which varies between 0 and 10, and the vertical axis represents the probability of the system being in the instantaneous ground state during the evolution. There is a quantum phase transition at \(g_c = 1\). The red solid line represents the forward quench (from \(g_0 = 10\) to \(g_T = 0\)) and the green dashed line represents the quench echo (from \(g_T = 0\) to \(g_0 = 10\)). In both the nearly adiabatic and nearly impulse regimes, the fidelity at the final moment is close to unity. In the first row (a)–(c), there is no time delay at the turnaround point. In the second row (a’)–(c’), the delay time at \(g_T = 0\) is \(\Delta t = 0.1, 0.2, 0.3\) and 0.4. In the third row (a”)–(c”), the delay time at \(g_T = 0\) is \(\Delta t = 10, 20, 30\) and 40. The number of spins in the Ising chain is \(N = 50\).
Figure 2. The fidelity as a function of the time scale of quench $\tau_Q$. Panels (a)–(d) represent different free evolution times at the turnaround point $g_T = 0$ before the quench echo. Here, $N = 50$, the start point is $g_0 = 10$, and the times of free evolution are chosen to be $\Delta t = 0$, $\Delta t = 0.1$, $\Delta t = 0.3$ and $\Delta t = 0.7$. It can be seen that in the adiabatic regime the fidelity is always equal to unity, but in the impulse regime it is less than unity and varies with the time of free evolution $\Delta t$.

as a function of the quench time $\tau_Q$, instead of $\ln \tau_Q$, we find that there is a regular quasi-periodic oscillation (see figure 3). We obtain an accurate expression of fidelity in the intermediate regime,

$$F \approx \prod_{k>0} e^{\frac{\pi}{2} \tau_Q \sin^2 k} e^{i\phi_k} + \frac{2\pi \tau_Q \sin^2 k}{\Gamma^2(1 - i \tau_Q \sin^2 k)} e^{-\pi \tau_Q \sin^2 k} e^{-i\phi_k},$$  \hspace{1cm} (11)

where $\phi_k = 2\tau_Q [(-g_T + \cos k)^2 + \sin^2 k \ln \sqrt{4\tau_Q (-g_T + \cos k)^2}]$, and $\Gamma(1 - i \tau_Q \sin^2 k)$ is the Gamma function (see the appendix for details of the derivation). From figure 3, it can be seen that the analytical results agree with the numerical simulations and that the fidelity oscillates quasi-periodically with the increase in $\tau_Q$ as expected. This oscillation can actually be observed in figures 1(b) and (c).

Numerical simulations agree with the results obtained from the KZM very well, i.e. they account for three regimes that correspond to different $\tau_Q$. We are especially interested in the first regime—the adiabatic regime. From figures 1(a) and (c), it can be seen that in both the adiabatic regime and the impulse regime, the fidelity is close to unity. In the next subsection, we will introduce a method to eliminate this ‘degeneracy’ of the adiabatic regime and the impulse regime.

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Figure 3. Fidelity as a function of the time scale $\tau_Q$. Here, the black dashed line represents the numerical results, whereas the red solid line represents the analytical results (equation (11)). It can be seen that the analytical results (equation (11)) agree well with the numerical results except for the case $\tau_Q \to 0$, where the quench process enters the impulse regime. The length of the spin chain is $N = 50$ and the delay time at the turnaround point is $\Delta t = 0$.

3.3. Free evolution and decay of fidelity

To distinguish the adiabatic and the impulse regime using quench echo, one can let the system evolve freely for some time at the turnaround point before quenching back. A study of the Landau–Zener problem with waiting at the minimum gap has been reported in [22]. It was observed that the waiting influences the excitation probability. Similarly, in our model, the free evolution at the turnaround point leads to a decay in fidelity in the impulse regime, but makes no difference in the adiabatic regime (see figures 1(a′)–(c′) and (a′′)–(c′′)). The reason is straightforward. Let us first consider the adiabatic regime. Because the system is always in its instantaneous ground state, the effect of the free evolution is simply a global phase factor, which does not affect the fidelity (see figures 1(a′), (a′′) and 2(b)–(d). In the impulse regime, the wavefunction before the free evolution is the ground state of the initial Hamiltonian $H(g_0 = 10)$, and alternatively a superposition of the excited and the ground states of $H(g_T = 0)$. The excited and ground states acquire different phase factors during the free evolution. Thus, the wave function acquires relative phase factors in its components and is no longer the ground state of $H(g_T = 0)$, but a superposition of its ground and excited states. Hence, in the impulse regime when one quenches the system back to the initial Hamiltonian $H(g_0 = 10)$, the system will no longer be in its ground state, but in a superposition of the ground state and the excited states. As a result, the fidelity is less than unity (see figures 1(c′), (c′′) and 2(b)–(d). The length of time of the free evolution $\Delta t$ also influences the fidelity. One can analytically calculate the fidelity as a function of the time of free evolution $\Delta t$.

$$\left| \left( g = +\infty \right) \right>_k = \sin \left( \frac{\theta_k}{2} \right) \left| +\left( g_T \right) \right>_k + \cos \left( \frac{\theta_k}{2} \right) \left| -\left( g_T \right) \right>_k ,$$
where $\theta_k = \arctan\left(-\frac{\sin k}{\cos k - g_T}\right)$. After free evolution for $\Delta t$, the wavefunction becomes

$$\sin\left[\frac{\theta_k}{2}\right] e^{-i\Lambda_k(g_T)\Delta t} |+(g_T)\rangle_k + \cos\left[\frac{\theta_k}{2}\right] e^{i\Lambda_k(g_T)\Delta t} |-(g_T)\rangle_k.$$ 

The fidelity can then be calculated as

$$F = \prod_{k>0} \left(1 - \sin^2 k \sin^2 [\Lambda_k(g_T)\Delta t]\right). \quad (12)$$

Note that for a fixed chain size, the value of $\Delta t$ needed to scramble all the relevant phases is relevant to the range of the spectrum of the system or the size of gap $\Lambda_k(g_T)$ of different $k$ at the turnaround point $g_T$. When $\Lambda_k(g_T)$ is very small, i.e. the energy spectrum of the system is concentrated within a very small energy range, so one needs to wait for a long time in order to scramble all the relevant phases: $\Delta t$ is inversely proportional to the energy scale $J$ of $\Lambda_k(g_T)$. For a spin chain of $N = 50$, when the time of free evolution is very short, e.g. $\Delta t = 0.1$, there is a pronounced decay in fidelity in the impulse regime (see figure 2(b)). The analytical result gives $F \approx 0.882$, which agrees with the numerical result. The fidelity decreases with an increase in time of free evolution. The fidelity decays to 0.002 when $\Delta t = 0.7$ (see figure 2(d)). Hence, the quench echo with a free evolution at the turnaround point can distinguish the adiabatic and the impulse regime. Our numerical results confirm our theoretical predictions.

4. Beyond the linear quench

In the above discussion, we focused on the linear quench. One may repeat the above process with different $\tau_Q$ until one finds the smallest $\tau_c^Q$ under which the process is sufficiently adiabatic, for example $F \geq 0.9$. Nevertheless, the linear quench with $\tau_c^Q$ obtained above may waste a lot of time. The reason is obvious: in different regions of the parameter $g$, the energy gaps are different. According to KZM, different energy gaps correspond to different relaxation times $\tau$. For a linear quench protocol, we are treating the whole range of the parameter uniformly, and the relaxation time is determined by the global minimal energy gap. Thus, we waste a lot of time. Usually we want to ensure that the process is not only nearly adiabatic but also as fast as possible. In the following we will consider nonlinear quench.

4.1. Adjusting quench rate to the instantaneous gap

An improved scheme is to divide the whole range of the parameter into many, e.g. $M$, parts with equal length $(g_0 - g_T)/M$, and then apply the above linear quench protocol to these ranges separately to find the uniformly adiabatic quench for each range $\tau_c^Q$, $i = 1, 2, \ldots, M$. We can also use the KZM to find a uniformly adiabatic quench protocol. From the discussion in section 3.1, we know that the transition time scale is given by the absolute value of $(g(t))/d\Delta(g(t))$. Meanwhile, the relaxation time scale is given by $1/\Delta(g(t))$. When the former is many times larger than the latter, the process should be uniformly adiabatic. That is, when the parameter $g(t)$ satisfies the relation

$$\left|\frac{\Delta(g(t))}{d\Delta(g(t))}\right| = \frac{\gamma}{\Delta(g(t))}, \quad (13)$$

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where $\gamma$ is a constant many times larger than unity, e.g. $\gamma = 10$, the process is uniformly adiabatic in the sense that the ratio of two time scales remains a constant. Such a quench scheme is better than the linear quench. The solution to the above ordinary differential equation is

$$\Delta(g(t)) = \frac{1}{\mp(1/\gamma)t + c},$$  \hspace{1cm} (14)

where $\mp$ corresponds to the sign of $\Delta/\dot{\Delta}$ on the left-hand-side of equation (13) being positive or negative and $c$ is a constant of integration. For simplicity, $c$ can be chosen such that at $t = 0$, $\Delta$ in equation (14) is the minimal gap. Now we know exactly the energy gap as a function of the controlling parameter (see figure 4(a))

$$\Delta(g(t)) = 2J\sqrt{1 - 2g(t)\cos\left(\frac{\pi}{N}\right) + g^2(t)}.$$  \hspace{1cm} (15)

Therefore, $c$ can be determined by using $g(t = 0) = \cos(\pi/N)$. Combining equations (14) and (15), we find the following uniformly adiabatic quench protocol (see figure 4(b)),

$$g_{KZ}(t) = \begin{cases} 
\cos\left(\frac{\pi}{N}\right) - \sqrt{-\left(\sin\left(\frac{\pi}{N}\right)\right)^2 + \frac{(\gamma)^2}{4J^2}\left(t + \frac{\gamma}{2J\sin(\pi/N)}\right)^2}, & \left(-\frac{\gamma}{2J\sin(\pi/N)} < t < 0\right), \\
\cos\left(\frac{\pi}{N}\right) + \sqrt{-\left(\sin\left(\frac{\pi}{N}\right)\right)^2 + \frac{(\gamma)^2}{4J^2}\left(-t + \frac{\gamma}{2J\sin(\pi/N)}\right)^2}, & \left(0 < t < \frac{\gamma}{2J\sin(\pi/N)}\right). 
\end{cases},$$  \hspace{1cm} (16)

Figure 4. (a) Energy gap as a function of the controlling parameter $g$ for a finite size chain. Here the solid line indicates the spin chain of $N = 50$ and the dashed line indicates the gap of an infinite chain. (b) The uniformly adiabatic quench protocol associated with the KZM criterion (solid) (16) and associated with the RC criterion (dashed) (19). Here, the ratio $\gamma = 2$ and the energy scale $J = 1/2$. We have chosen the condition $g(t = 0) = \cos(\pi/N)$.

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4 The energy gap of other modes is much larger than the considered mode $k = \pi/N$. So one usually considers only the excitations (if there are any) in the smallest mode $k = \pi/N$ and ignores the excitations in the other modes.
It can be seen that the time required for the whole process (quenching the controlling parameter from \( g = 0 \) to \( g = \infty \)) is given by

\[
\Delta t_{KZ} = \frac{\gamma}{J \sin(\pi/N)} \approx \frac{\gamma N}{J \pi},
\]

or \( \Delta t_{KZ} = \frac{2\gamma}{\Delta_{\text{min}}} \), which is proportional to the chain size \( N \) and the ratio \( \gamma \), and inversely proportional to the minimum energy gap \( \Delta_{\text{min}} = 2J \sin^2(\pi/N) \approx 2J \frac{\pi}{N} \). In linear quench the minimal time required for adiabatic evolution grows with the system size like \( N^2 \) [21]. The quench scheme of equation (16) is obviously better. This agrees with previous studies that ‘nonlinear’ quench can improve the adiabaticity (minimize excitation) [15]–[17]. The energy gap and the protocols for uniformly adiabatic quench (equation (16)) are shown in figure 4.

Note that the criterion for uniformly adiabatic evolution derived from KZM (equation (13)) is similar, but not identical, to the criterion proposed by Roland and Cerf (RC) that was derived from the quantum adiabatic theorem (see equation (17) of [23]). In the RC model, the energy gap is inversely proportional to \( \sqrt{N} \), and the minimum time required is proportional to \( \sqrt{N} \). But in the Ising chain, the energy gap is inversely proportional to \( N \), and the minimum time required is proportional to \( N \). It can be proved that if one uses the RC criterion to evaluate the minimum time required for uniformly adiabatic evolution, the minimum time is also proportional to \( N \).

It is interesting to compare the two criteria for uniformly adiabatic evolution of the Ising chain. In the following, we will first solve the equation of the quench protocol for uniformly adiabatic evolution \( g_{RC}(t) \) associated with the RC criterion and then simulate the dynamic evolution of the Ising chain with both \( g_{RC}(t) \) and \( g_{KZ}(t) \). We will fix the time of the quench process, and compare the fidelity of the two protocols. The RC criterion (see equation (17) of [23]) is

\[
\left| \frac{d}{dt} |g(t) - g_c| \right| = \frac{1}{\gamma'} \Delta^2(g(t)),
\]

where \( \gamma' \) is the ratio between the two time scales. Obviously, when \( N \to \infty \), \( \Delta(g) = |g(t) - g_c| \) is valid for arbitrary \( g \). In this respect, the two criteria, equations (13) and (18), are equivalent. Nevertheless, when \( N \) is finite, the two criteria differ slightly because the gap \( \Delta(g) \) deviates from \( |g(t) - g_c| \) near the critical point (see equation (15) of [23]). As a result, there is a small discrepancy in the quench protocols \( g_{RC}(t) \) and \( g_{KZ}(t) \) associated with the two criteria, especially when \( g(t) \) is close to \( g_c \).

By substituting equation (15) into (18), we obtain the quench protocol,

\[
g_{RC}(t) = \cos\left(\frac{\pi}{N}\right) + \sin\left(\frac{\pi}{N}\right) \tan\left(\frac{2J \sin(\frac{\pi}{N})}{\gamma'} t\right), \quad \left( -\frac{\gamma'N}{4J} < t < \frac{\gamma'N}{4J} \right).
\]

We plot the solution \( g_{RC}(t) \) along with \( g_{KZ}(t) \) in figure 4(b). There is a ‘kink’ at the anti-cross point of the energy levels in \( g_{KZ}(t) \) associated with KZM, but there is none in \( g_{RC}(t) \) associated with the RC criterion (see figure 4(b) and the inset). Although there is a singularity (divergent time derivative of \( g \)) in \( g_{KZ}(t) \) at \( t = 0 \), the time interval of this region is vanishingly small. As a result, the total change in \( g \) in this singular region is very small (see figure 4(b)), and so the eigenstates of \( H(t) \) do not change significantly within it. This is in the same spirit as quantum fidelity [24], where fidelity susceptibility diverges at a quantum critical point but the fidelity is nonzero, indicating that the ground state does not change significantly. Hence, the ‘kind’ at \( t = 0 \)
Figure 5. Comparison of the RC criterion and the KZM criterion. The horizontal axis indicates time, and the vertical axis depicts the instantaneous fidelity. The solid (dashed) line describes the fidelity as a function of time $t$ associated with the RC (KZM) criterion. Left panel: the evolution starts from the ground state at $t = -\frac{\gamma N}{2J\pi}$ ($g = -\infty$). Right panel: the evolution starts from the ground state at $t = -\frac{1}{3}\frac{\gamma N}{2J\pi}$.

will not lead to a lot of excitations. Our simulation verifies this point. Similar to equation (17), we obtain the time required for the uniformly adiabatic evolution

$$\Delta t_{RC} = \frac{\gamma'N}{2J}. \quad (20)$$

Comparing equations (17) and (20), we find that when $\gamma' = \frac{2}{\pi}\gamma$, the times required for the two criteria are equal. In the following, we will simulate the dynamics of the Ising chain under the two quench protocols, equations (16) and (19). Substituting equations (16) and (19) into equation (6), one obtains the instantaneous fidelity as a function of time $F = |\beta(t)|^2$ associated with the two criteria. We plot the fidelity as a function of time in figure 5. When one chooses different initial conditions, the fidelity as a function of time differs a lot. In the left panel of figure 5, we plot the fidelity as a function of time quenching from $t = -\frac{\gamma N}{2J\sin(\pi/N)}$ ($g = -\infty$). The fidelity associated with the RC criterion decays when the system is near the anti-crossing point, and then revives. But the fidelity associated with KZM does not change much and remains close to unity all the time. At $t = \frac{\gamma N}{2J\sin(\pi/N)}$ ($g = \infty$), the fidelity associated with the RC criterion is a bit higher than that associated with KZM. In the right panel of figure 5, we plot the fidelity as a function of time starting from $t = -\frac{1}{3}\frac{\gamma N}{2J\sin(\pi/N)}$. In this case, the fidelity associated with the RC criterion oscillates rapidly and finally reaches a stable value around 0.85. By contrast, the fidelity associated with KZM does not oscillate and remains very close to unity. From the above facts, we conclude that in some cases, the RC criterion is better than the KZM criterion, but in some other cases, it is worse. Hence, we cannot say which criterion is definitely better, but KZM provides new insights into the conditions for uniformly adiabatic evolution.

One is tempted at this point to undertake a variational study in search of optimal quenches. While such a study is beyond the scope of this paper, we note that in practical applications (e.g. adiabatic quantum computing) optimization would involve not just varying rate but (as it was done in figure 5) also the starting and final points of the quenches that can be brought closer to
Figure 6. Final magnetization of the Ising chain as a function of the time scale of quench $\tau_Q$. All the parameters are the same as those in figure 2(a). One can see that the spin chain is not very far away from equilibrium except when the time scale of quench is in the range $\tau_Q \in (10^{-1}, 1)$. Left panel: delay time $\Delta t = 0$; right panel: delay time $\Delta t = 0.7$.

the ‘critical point’. Resulting errors can be detected and the correct result can be ascertained by repeating the computation many times.

4.2. Gauging the distance from the adiabatic quench

Figure 2 indicates that when the time scale of quench $\tau_Q$ is in the range $\tau_Q \in (10^{-1}, 1)$, the fidelity is almost equal to zero. However, this does not reveal how far the quench is from adiabaticity. For example, when only one out of many modes gets excited, the fidelity will decay to nearly zero due to the orthogonality of one mode. But, in a sense, the system is still close to the ground state, as all but one excited state are empty. In this sense, fidelity is not a good criterion for measuring how far away the quench is from the adiabatic evolution.

A better gauge of the distance of a quench process from adiabaticity may be obtained using other variables, such as magnetization per site along the direction of the external magnetic field

$$m = \frac{1}{N} \sum_{i=1}^{N} \langle \sigma^z_i \rangle = \frac{1}{N} \sum_{k=1}^{N} 2 |\beta_k ((g_0 - 2g_T)\tau_Q) |^2 - 1. \quad (21)$$

When the magnetic field is large, the ground state corresponds to $m = 1$. We plot the final magnetization as a function of $\tau_Q$ in figure 6. In the range of $\tau_Q \in (10^{-1}, 1)$, the fidelity is vanishingly small, but magnetization per site is still large. This indicates that the system is not very far away from the instantaneous ground state. Moreover, when one delays for some time at the turnaround point, the magnetization per site of impulse regime will decrease, but that of adiabatic regime will not (see figure 6). This is similar to fidelity and agrees with our intuition. Last but not least, the magnetization is experimentally easier to measure than fidelity, and it has been used as a tool to study adiabaticity of quantum dynamics in [26].

One can also use the kink density $\frac{1}{2} \sum_i (1 - \langle \sigma_i \sigma_{i+1} \rangle)$ [21] as a measure of the distance of the system from adiabaticity. The relation between the fidelity and density of defects has been studied in [27]. Other variables, such as residual energy [28], can also be used to gauge the distance from the adiabatic quench. Such obvious measures of how far the quench is from...
adiabaticity work well in the one-dimensional Ising model, but finding their useful analogues in other situations (e.g. adiabatic quantum computing) may not be easy.

5. Summary and conclusion

We have proposed a strategy to test the adiabaticity without knowing either the eigenstates or the eigenenergies of the Hamiltonian. Instead of having to find the gap of the Hamiltonian and then using the quantum adiabatic approximation to evaluate the adiabaticity, one can use a quench echo to evaluate the adiabaticity of an evolution. The underlying mechanism is that when the time scale of quench is large in comparison with the inverse of the energy gap, both the forward and backward evolutions are adiabatic. As a result, the fidelity of the initial state and the final state is close to unity. Otherwise, the evolution is not adiabatic, and the fidelity is less than unity. The method for testing the adiabaticity of an evolution presented in this paper is universally valid. It does not depend on the model or the validity of the conditions for adiabatic approximation. We have further proposed a method for finding the uniformly adiabatic quench protocol based on KZM, and discussed the problem of gauging how non-adiabatic is a quench. Given the importance of adiabaticity in various applications, we believe that our results will be broadly applicable and may be useful in experimental applications.

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Appendix. Fidelity in the intermediate regime

From [21, 25], we know that in the wave vector \( k \) representation, the Schrödinger equation for the forward quench can be rewritten as Landau–Zener-type equations (see equation (6) for a comparison),

\[
\frac{d}{dt'} \begin{bmatrix} v_k(t') \\ u_k(t') \end{bmatrix} = \frac{1}{2} \begin{bmatrix} t' \\ -t' \end{bmatrix} \begin{bmatrix} \frac{1}{\tau_Q} & 1 \\ 1 & -\frac{1}{\tau'_Q} \end{bmatrix} \begin{bmatrix} v_k(t') \\ u_k(t') \end{bmatrix},
\]

(A.1)

where \( t' = 4\tau_Q \sin k[-g(t') + \cos k] \) and \( \tau'_Q = 4\tau_Q \sin^2 k \). This equation can be solved in terms of Weber functions. The initial conditions are \( v_k(t' = -\infty) = 0 \) and \( u_k(t' = -\infty) = 1 \). The solution for this equation is [25]

\[
u_k(t) \approx e^{-\pi \tau_Q \sin^2 k} e^{i \tau_Q (-g + \cos k)^2 + \sin^2 k \ln \sqrt{4\tau_Q (-g + \cos k)^2}} \]

(A.2)

\[
v_k(t) \approx \sqrt{\frac{2\pi \tau_Q \sin^2 k}{\Gamma(1-i\tau_Q \sin^2 k)}} e^{-[(\pi \tau_Q \sin^2 k)/2]} e^{-i \tau_Q (-g + \cos k)^2 + \sin^2 k \ln \sqrt{4\tau_Q (-g + \cos k)^2}}.
\]
Similarly, we can obtain the solution for the quench echo. Combining the forward and quench echo processes, we find the solution of fidelity (11). It is worth pointing out that the above solution is only good for the turnaround point far away from the critical point $g_T \ll 1$ (for example, $g_T = 0.5$).

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