Deterministic Conditions for Subspace Identifiability from Incomplete Sampling

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Abstract
Consider a generic $r$-dimensional subspace of $\mathbb{R}^d$, $r < d$, and suppose that we are only given projections of this subspace onto small subsets of the canonical coordinates. The paper establishes necessary and sufficient deterministic conditions on the subsets for subspace identifiability.

1 Introduction
Consider low-rank matrices of size $d \times N$ with columns from a generic $r$-dimensional subspace $S^*$ of $\mathbb{R}^d$, $r < d$. Suppose that only a specific subset of the entries in these matrices are observed. This situation arises in the so-called matrix completion problem [1]. This paper establishes deterministic conditions on the sampling pattern of entries that guarantee that $S^*$ is the only $r$-dimensional subspace consistent with all such incomplete observations.

It is easy to see that an identifiability condition of this sort can only be possible if at least $r + 1$ entries are observed in each column of the matrices, and so we will assume this bare minimum number of observed entries. Let $\Omega$ be a $d \times N$ binary mask with exactly $r + 1$ nonzero entries per column. Since $\ker S^*$ is $(d-r)$-dimensional, we will see that $N \geq d - r$ is necessary for identifiability. Thus, we will assume $N = d - r$ for the rest of the paper. Let $\omega_i$ denote the $i^{th}$ column of $\Omega$, and $S^*_{\omega_i} \subset \mathbb{R}^{r+1}$ the restriction of $S^*$ to the nonzero coordinates in $\omega_i$.

Let $\text{Gr}(r, \mathbb{R}^d)$ denote the Grassmannian manifold of $r$-dimensional subspaces in $\mathbb{R}^d$. Define $S(S^*, \Omega) \subset \text{Gr}(r, \mathbb{R}^d)$ such that every $S \in S(S^*, \Omega)$ satisfies $S_{\omega_i} = S^*_{\omega_i}$, $\forall i$. In words, $S(S^*, \Omega)$ is the set of all $r$-dimensional subspaces matching $S^*$ on $\Omega$. The main result of this paper is the following theorem, which gives necessary and sufficient conditions on $\Omega$ to guarantee that $S(S^*, \Omega)$ contains no subspace other than $S^*$. 

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Given a matrix, let \( n(\cdot) \) denote its number of columns, and \( m(\cdot) \) the number of its nonzero rows.

**Theorem 1.** For almost every (a.e.) \( S^* \), with respect to the uniform measure over \( \text{Gr}(r, \mathbb{R}^d) \), \( S^* \) is the only subspace in \( \mathcal{S}(S^*, \Omega) \) if and only if for every matrix \( \Omega' \) formed with a subset of the columns in \( \Omega \),

\[
m(\Omega') \geq n(\Omega') + r.
\]

**Example 1.** The following mask, where \( 1 \) denotes a block of all 1’s, and \( I \) the identity matrix, satisfies the conditions of Theorem 1:

\[
\Omega = \begin{pmatrix} 1 \\ I \end{pmatrix} \}
d - r.
\]

2 Proof

For any subspace, matrix or vector that is compatible with a binary vector \( \upsilon \), we will use the subscript \( \upsilon \) to denote its restriction to the nonzero coordinates/rows in \( \upsilon \). For a.e. \( S^* \), \( S^*_{\omega_i} \) is an \( r \)-dimensional subspace of \( \mathbb{R}^{r+1} \), and the kernel of \( S^*_{\omega_i} \) is a 1-dimensional subspace of \( \mathbb{R}^{r+1} \).

**Lemma 1.** Let \( a_{\omega_i} \in \mathbb{R}^{r+1} \) be a nonzero element of ker \( S^*_{\omega_i} \). All entries of \( a_{\omega_i} \) are nonzero for a.e. \( S^* \).

**Proof.** Suppose \( a_{\omega_i} \) has at least one zero entry. Use \( \upsilon \) to denote the binary vector of the nonzero entries of \( a_{\omega_i} \). Since \( a_{\omega_i} \) is orthogonal to \( S^*_{\omega_i} \), for every \( u_{\omega_i} \in S^*_{\omega_i} \) we have that \( a_{\omega_i}^T u_{\omega_i} = a_{\omega_i}^T u = 0 \). Then \( S^*_{\upsilon} \) satisfies

\[
\dim S^*_{\upsilon} \leq \dim \ker a_{\upsilon}^T = \| \upsilon \|_1 - 1 < \| \upsilon \|_1.
\]

Observe that for every binary vector \( \upsilon \) with \( \| \upsilon \|_1 \leq r \), a.e. \( r \)-dimensional subspace \( S \) satisfies \( \dim S_{\upsilon} = \| \upsilon \|_1 \). Thus (2) holds only in a set of measure zero.

Define \( a_i \) as the vector in \( \mathbb{R}^d \) with the entries of \( a_{\omega_i} \) in the nonzero positions of \( \omega_i \) and zeros elsewhere. Then \( S \subset \ker a_i^T \) for every \( S \in \mathcal{S}(S^*, \Omega) \) and every \( i \). Letting \( A \) be the \( d \times (d-r) \) matrix formed with \( \{ a_i \}_{i=1}^{d-r} \) as columns, we have that \( S \subset \ker A^T \) for every \( S \in \mathcal{S}(S^*, \Omega) \). Note that if \( \dim \ker A^T = r \), then \( \mathcal{S}(S^*, \Omega) \) contains just one element, \( S^* \), which is the identifiability condition of interest. Thus, we will establish conditions on \( \Omega \) guaranteeing that the columns of \( A \) are linearly independent.
Recall that for any matrix $A'$ formed with a subset of the columns in $A$, $n(A')$ denotes the number of columns in $A'$, and $m(A')$ denotes the number of nonzero rows in $A'$.

**Lemma 2 (Independence).** For a.e. $S^*$, the columns of $A$ are linearly dependent if and only if $m(A') < n(A') + r$ for some matrix $A'$ formed with a subset of the columns in $A$.

In order to prove this statement, we will need Lemmas 3 and 4 below.

**Lemma 3.** Let $\ell(A')$ be the number of linearly independent columns in $A'$. Then $m(A') \geq \ell(A') + r$ for a.e. $S^*$.

**Proof.** Let $v$ be the binary vector of nonzero rows of $A'$, and $A'_v$ be the $m(A') \times n(A')$ matrix formed with these nonzero rows.

For a.e. $S^*$, $\dim S^*_v = r$. Since $S^*_v \subset \ker A'_v^T$, $r = \dim S^*_v \leq \dim \ker A'_v^T = m(A') - \ell(A')$.

We say $a_i$ is minimally linearly dependent on $A'$ if $a_i$ is linearly dependent on the columns of $A'$, but linearly independent of every proper subset of the columns in $A'$.

**Lemma 4.** Let $a_i$ be minimally linearly dependent on $A'$. Then $m(A') = n(A') + r$ for a.e. $S^*$.

**Proof.** Let $m = m(A')$, $n = n(A')$, and $\ell = \ell(A')$. If $A'$ has only one column, then by Lemma 1, $m = r + 1$ and the claim holds. If $A'$ has more than one column, define $\beta \in \mathbb{R}^n$ such that

$$A' \beta = a_i. \tag{3}$$

Note that because $a_i$ is minimally linearly dependent on $A'$, all entries in $\beta$ are nonzero. Since the columns of $A'$ are linearly independent, $n = \ell$. Thus, by Lemma 3, $m \geq n + r$. We want to show that $m = n + r$, so suppose for contradiction that $m > n + r$.

We can assume without loss of generality that $A'$ has all its zero rows (if any) in the first positions. In that case, since $a_i$ is linearly dependent on the columns of $A'$, it follows that the nonzero entries of $a_i$ cannot be in the corresponding rows. Thus, without loss of generality, assume that $a_i$ has its first $r$ nonzero entries in the first $r$ nonzero rows of $A'$, and that the last nonzero entry of $a_i$ is 1 (i.e., re-scale $a_i$ if needed), and is located in the last row. Let $\bar{a}_i \in \mathbb{R}^r$ denote
the vector with the first nonzero entries of $a_i$, such that we can write:

$$
\begin{bmatrix}
A' | a_i \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \tilde{a}_i \\
C & 0 & \mathbf{1} \\
B & 0 & \mathbf{1} \\
\end{bmatrix} = \begin{bmatrix}
d - m \\
r \\
m - r - 1 \\
1, \\
\end{bmatrix}
$$

where $C$ and $B$ are submatrices used to denote the blocks of $A'$ corresponding to the partition of $a_i$.

The columns of $B$ are linearly independent. To see this, suppose for contradiction that they are not. This means that there exists some nonzero $\gamma \in \mathbb{R}^n$, such that $B\gamma = 0$. Let $c = A'\gamma$ and note that only the $r$ rows in $c$ corresponding to the block $C$ may be nonzero. Let $\upsilon$ denote the binary vector of these nonzero entries. Since $S^\ast$ is orthogonal to every column of $A'$ and $c$ is a linear combination of the columns in $A'$, it follows that $S^\ast_{\upsilon} \subset \ker c^T_\upsilon$. This implies that $\dim S^\ast_{\upsilon} \leq \dim \ker c^T_\upsilon = \|\upsilon\|_1 - 1$. As in the proof of Lemma 1, this implies that the columns of $B$ are linearly dependent only in a set of measure zero.

Going back to (4), since the $n$ columns of $B$ are linearly independent and because we are assuming that $m - r > n$, it follows that $B$ has $n$ linearly independent rows. Let $B_1$ denote the $n \times n$ block of $B$ that contains $n$ linearly independent rows, and $B_2$ the $(m - n - r) \times n$ remaining block of $B$.

Notice that the row of $B$ corresponding to the 1 in $a_i$ must belong to $B_1$, since otherwise, we have that $B_1\beta = 0$, with $\beta$ as in (3), which implies that $B_1$ is rank deficient, in contradiction to its construction.

We can further assume without loss of generality that the first nonzero entry of every column of $B$ is 1 (otherwise we may just re-scale each column), and that these nonzero entries are in the first columns (otherwise we may just permute the columns accordingly). We will also let $B_2$ denote all but the first row of
Thus, our matrix is organized as

\[
\begin{bmatrix}
0 & 0 \\
C & \tilde{a}_i \\
1 & 0 \\
\tilde{B}_2 & 0 \\
B_1 & 1 \\
\end{bmatrix}
\]

\[\begin{cases}
0 \\
r \\
1 \\
m - n - r - 1 \geq 0 \\
n - 1
\end{cases}
\]

Now (3) implies \(B_1 \beta = [0 \mid 1]^T\), and since \(B_1\) is full-rank, we may write

\[
\beta = B_1^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

i.e., \(\beta\) is the last column of the inverse of \(B_1\), which is a rational function in the elements of \(B_1\).

Next, let us look back at (3). If \(m > n + r\), then using the additional row \([1 \mid 0]\) of (5) (which does not appear if \(m = n + r\)) we obtain \([1 \mid 0] \beta = 0\). Recall that all the entries of \(\beta\) are nonzero. Thus, the last equation defines the following nonzero rational function in the elements of \(B_1\):

\[
[1 \mid 0] B_1^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.
\]

Equivalently, (6) is a polynomial equation in the elements of \(B_1\), which we will denote as \(f(B_1) = 0\).

Next note that for a.e. \(S^*\), we can write \(S^* = \ker A^{*T}\) for a unique \(A^* \in \mathbb{R}^{d \times (d-r)}\) in column-echelon form:\(^1\)

\[
A^* = \begin{bmatrix}
I \\
D^*
\end{bmatrix}
\]

\[\begin{cases}
d - r \\
r
\end{cases}
\]

On the other hand every \(D^* \in \mathbb{R}^{r \times (d-r)}\) defines a unique \(r\)-dimensional subspace of \(\mathbb{R}^d\), via (7). Thus, we have a bijection between \(\mathbb{R}^{r \times (d-r)}\) and a dense open subset of \(\text{Gr}(r, \mathbb{R}^d)\).

\(^1\)Certain \(S^*\) may not admit this representation, e.g., if \(S^*\) is orthogonal to certain canonical coordinates. However, as discussed in Lemma 1, this is not the case for almost every \(S^*\) in \(\text{Gr}(r, \mathbb{R}^d)\).
Since the columns of \( \mathbf{A}' \) must be linear combinations of the columns of \( \mathbf{A}^* \), the elements of \( \mathbf{B}_1 \) are linear functions in the entries of \( \mathbf{D}^* \). Therefore, we can express \( f(B_1) \) as a nonzero polynomial function \( g \) in the entries of \( \mathbf{D}^* \) and re-write (6) as \( g(\mathbf{D}^*) = 0 \). But we know that \( g(\mathbf{D}^*) \neq 0 \) for almost every \( \mathbf{D}^* \in \mathbb{R}^{r \times (d-r)} \), and hence for almost every \( \mathbf{S}^* \in \text{Gr}(r, \mathbb{R}^d) \). We conclude that almost every subspace in \( \text{Gr}(r, \mathbb{R}^d) \) will not satisfy (6), and thus \( m = n + r \).

We are now ready to present the proofs of Lemma 2 and Theorem 1.

Proof. (Lemma 2) \((\Rightarrow)\) Suppose that column \( \mathbf{a}_i \) in \( \mathbf{A} \) is minimally linearly dependent on the columns in \( \mathbf{A}'' \), a matrix formed with a subset of the columns in \( \mathbf{A} \). By Lemma 4, \( n(\mathbf{A}'') = m(\mathbf{A}'') - r \). Let \( \mathbf{A}' = [\mathbf{A}'' \mid \mathbf{a}_i] \). It is clear that \( m(\mathbf{A}') = m(\mathbf{A}'') \) and \( n(\mathbf{A}') = n(\mathbf{A}'') + 1 \). Thus \( m(\mathbf{A}') < n(\mathbf{A}') + r \), and we have the first implication.

\((\Leftarrow)\) Suppose there exists an \( \mathbf{A}' \) with \( m(\mathbf{A}') < n(\mathbf{A}') + r \). By Lemma 3, \( n(\mathbf{A}') > \ell(\mathbf{A}') \), which implies \( \mathbf{A}' \), and hence \( \mathbf{A} \), has a linearly dependent column.

Proof. (Theorem 1) Lemma 1 shows that for a.e. \( S^* \), the \((j, i)^{th}\) entry of \( \mathbf{A} \) is nonzero if and only if the \((j, i)^{th}\) entry of \( \mathbf{\Omega} \) is nonzero.

\((\Rightarrow)\) Suppose there exists an \( \mathbf{\Omega}' \) such that \( m(\mathbf{\Omega}') < n(\mathbf{\Omega}') + r \). Then \( m(\mathbf{A}') < n(\mathbf{A}') + r \) for some \( \mathbf{A}' \). Lemma 2 implies that the columns of \( \mathbf{A}' \), and hence \( \mathbf{A} \), are not linearly independent. This implies \( \dim \ker \mathbf{A}^T > r \).

\((\Leftarrow)\) Suppose every \( \mathbf{\Omega}' \) satisfies \( m(\mathbf{\Omega}') \geq n(\mathbf{\Omega}') + r \). Then \( m(\mathbf{A}') < n(\mathbf{A}') + r \) for every \( \mathbf{A}' \), including \( \mathbf{A} \). Therefore, by Lemma 2, \( \mathbf{A} \) has \( d - r \) linearly independent columns, hence \( \dim \ker \mathbf{A}^T = r \).

References

[1] Emmanuel Candès and Benjamin Recht, “Exact Matrix Completion Via Convex Optimization,” in Foundations of Computational Mathematics, 2009, vol. 9, pp. 717–772.
### List of symbols

| Symbol | Description | pp. |
|--------|-------------|-----|
| $A$    | $d \times (d - r)$ matrix with $\{a_i\}_{i=1}^{d-r}$ as its columns | 2   |
| $A'$   | Matrix formed with a subset of the columns in $A$. | 3   |
| $a_{\omega_i}$ | Vector in $\mathbb{R}^{r+1}$ orthogonal to $S^*_{\omega_i}$. | 2   |
| $a_i$  | Vector in $\mathbb{R}^d$ with the entries of $a_{\omega_i}$ in the nonzero positions of $\omega_i$. | 2   |
| a.e.   | Almost every with respect to the uniform measure over $Gr(r, \mathbb{R}^d)$. | 2   |
| $d$    | Ambient dimension. | 1   |
| $Gr(r, \mathbb{R}^d)$ | Grassmannian manifold of $r$-dimensional subspaces in $\mathbb{R}^d$. | 1   |
| $i$    | Used to index vectors. In general, $i \in \{1, \ldots, d - r\}$. | 1   |
| $\ell(\cdot)$ | Number of linearly independent columns in $\cdot$. | 3   |
| $m(\cdot)$ | Number of nonzero rows in $\cdot$. | 2   |
| $n(\cdot)$ | Number of columns in $\cdot$. | 2   |
| $N$    | Number of columns in $\Omega$ and in $A$, $N = d - r$. | 1   |
| $\Omega$ | $d \times (d - r)$ mask of observed entries with $r + 1$ nonzero entries per column. | 1   |
| $\Omega'$ | Mask formed with a subset of the columns in $\Omega$. | 1   |
| $\omega_i$ | $i$th column of $\Omega$. | 2   |
| $\nu$  | Arbitrary binary vector. | 2   |
| $r$    | Dimension of $S^*$. | 1   |
| $S$    | $r$-dimensional subspace. | 1   |
| $S_{\omega_i}$ | Subspace of $\mathbb{R}^{r+1}$. The restriction of $S$ to $\omega_i$. | 2   |
| $S^*$  | Subspace consistent with the incomplete observations. | 1   |
| $S^*_{\omega_i}$ | Subspace of $\mathbb{R}^{r+1}$. The restriction of $S^*$ to $\omega_i$. | 1   |
| $\nu_{\omega_i}$ | The restriction of $\nu$ to $\omega_i$. | 2   |
| $S(S^*, \Omega)$ | Set of all $r$-dimensional subspaces that agree with $S^*$ on $\Omega$. | 1   |