A novel statistical approach to analyze image classification

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Abstract

The recent statistical theory of neural networks focuses on nonparametric denoising problems that treat randomness as additive noise. Variability in image classification datasets does, however, not originate from additive noise but from variation of the shape and other characteristics of the same object across different images. To address this problem, we introduce a tractable model for supervised image classification. While from the function estimation point of view, every pixel in an image is a variable, and large images lead to high-dimensional function recovery tasks suffering from the curse of dimensionality, increasing the number of pixels in the proposed image deformation model enhances the image resolution and makes the object classification problem easier. We introduce and theoretically analyze three approaches. Two methods combine image alignment with a one-nearest neighbor classifier. Under a minimal separation condition, it is shown that perfect classification is possible. The third method fits a convolutional neural network (CNN) to the data. We derive a rate for the misclassification error that depends on the sample size and the complexity of the deformation class. A small empirical study corroborates the theoretical findings on images generated from the MNIST handwritten digit database.

1 Introduction

The developed brain recognizes even highly rotated, deformed and pixelated objects on images within milliseconds. For the same task, convolutional neural networks (CNNs), when trained on large datasets, can achieve even superhuman performance, see Section 5.19 in [37] for more details.

From a machine learning perspective, object recognition is typically framed as a high-dimensional classification problem, where each pixel is treated as an independent variable. The objective of the classification rule is to learn the functional relation between the pixel values of an input image and the corresponding conditional class probabilities or the labels. However, for high-resolution images with many pixels, the domain of the function is a high-dimensional space, leading to slow convergence rates due to the curse of dimensionality. To align the strong empirical performance of CNNs with theoretical guarantees, one common approach is to assume that the true functional relationship between inputs and outputs has a latent low-dimensional structure, see, e.g., [21]. This assumption allows the convergence rate to depend only on this low-dimensional structure, potentially circumventing the curse of dimensionality.
A functional data perspective is to treat images as highly structured objects that can be represented by a bivariate function, where each pixel value corresponds to a local average of the function over its location. From this viewpoint, variations of the same object in different images are interpreted as deformations of a template image, which introduces additional complexity for classification compared to the pixel-wise approach. This concept has been explored in foundational work on pattern recognition. Grenander and Mumford ([15, 30]) distinguish between pure images and deformed images, with the latter being generated from pure images through specific deformations. Since then, several generative models for object deformation on images have been proposed. For instance, [7, 8, 26] study a rich class of local deformations, while [31, 32] extend these models to address more complex deformations such as noise, blur, multi-scale superposition and domain warping. Generative models are becoming increasingly important in fields such as medical image registration or computer vision [40, 2, 20, 48]. While existing work focuses on algorithms that can effectively handle image deformations, statistical modelling and theoretical generalization guarantees are, however, underexplored.

This paper aims to bridge this gap by introducing a simple image deformation model that addresses a fundamental yet rich class of geometric deformations including object positioning, scaling, image brightness and rotations. We focus on a binary image classification setting, where datasets consist of $n$ labeled images of two objects, such as the digits 0 and 4, with each image representing a random deformation of one of these objects. In the case of digits, these deformations can capture natural variations found in, for instance, individual handwriting.

Our statistical analysis differs significantly from the wide range of well-understood classification problems that rely on local smoothing. In these settings, the source of randomness arises because the covariates (or inputs) do not fully determine the class label, requiring classifiers to aggregate training data with similar covariate values to effectively denoise. The resulting convergence rates for standard smoothness classes typically align with those seen in nonparametric regression and suffer from the curse of dimensionality for high-resolution images [5].

In the proposed statistical setting, the randomness occurs due to the different deformations that can arise on images within one class. The objective of the classification rule, therefore, is to remain invariant to these uninformative variations.

We approach this non-standard classification problem by first constructing classifiers exploiting the specific structure of the random object deformation model. These classifiers interpolate the data and can be interpreted as one-nearest neighbor classifiers in a transformed space. At low image resolutions, however, distinguishing between highly similar objects becomes impossible. We prove that if the two objects satisfy a minimal separation condition, that depends on the image resolution, then, the classifiers can perfectly discriminate between the two classes on test data. Interestingly, the sample size $n$ plays a minor role in the analysis; it suffices to observe one training sample for each class. The imposed minimal separation condition is also necessary in the sense that any smaller separation would result in non-identifiability of the classes, making accurate discrimination impossible (see Theorem 3.7).
A key contribution of this work are the misclassification error rates for CNN classifiers, showing that CNNs can adapt to various geometric deformations. As a first result, we prove that for a suitably chosen network architecture, specific parameter assignments in a CNN can effectively discriminate between the two classes. This shows that among all classifiers that are representable by a given CNN architecture, there exist classifiers that are (nearly) invariant with respect to the possible deformations of an object on an image. Based on this and statistical learning techniques, we derive misclassification bounds for CNN classifiers in Theorem 4.1. For specific deformation classes, the obtained rates depend on the sample size and the number of pixels. The dependence on the number of pixels is polynomial and can be attributed to the pixel dependence of the imposed separation condition between the classes. The dependence on the input dimension is thus much more favorable than the curse of dimensionality observed in standard nonparametric convergence rates. The proposed setting has the potential to provide a more refined understanding of phenomena such as overparametrization or the improved performance through data augmentation.

In summary, this paper presents a novel statistical framework for image recognition. It enables to detect new properties of CNN-based image classification but also provides a novel perspective to address high-dimensional classification. The mathematical analysis is non-standard, yielding joint asymptotics with respect to both, the number of pixels and the sample size. The proposed image deformation model should be viewed as a prototype for more complex image deformation models. Potential extensions and future research directions are discussed in Section 6.

The article is structured as follows. In Section 2 we introduce the image deformation model and discuss several common examples within this general framework. Section 3 presents classification approaches tailored to the proposed model, serving as benchmarks. Section 4 introduces and analyzes a CNN-based classifier. In Section 5, we apply the discussed classifiers to the MNIST handwritten digit database, providing empirical support for the theoretical guarantees. We conclude in Section 6 with the discussion of related results and potential extensions.

Notation: For a real number $x$, $\lfloor x \rfloor$ represents the largest integer that is less than or equal to $x$, whereas $\lceil x \rceil$ represents the smallest integer that is greater than or equal to $x$. We denote vectors and matrices by bold letters, e.g., $\mathbf{v} := (v_1, \ldots, v_d)$ and $\mathbf{W} = (W_{i,j})_{i=1,\ldots,m; j=1,\ldots,n}$. As usual, $|\mathbf{v}|_p := (\sum_{i=1}^{d} |v_i|^p)^{1/p}$ and $|\mathbf{v}|_\infty := \max_i |v_i|$. For a matrix $\mathbf{W} = (W_{i,j})_{i=1,\ldots,m; j=1,\ldots,n}$, we define the maximum entry norm as $|\mathbf{W}|_\infty := \max_{i=1,\ldots,m} \max_{j=1,\ldots,n} |W_{i,j}|$. For two sequences $(a_n)_n$ and $(b_n)_n$, we write $a_n \lesssim b_n$ if there exists a constant $C$ such that $a_n \leq C b_n$ for all $n$. For $m \geq 2$ and $a_1, a_2, \ldots, a_m$ we define $a_1 \lor a_2 \lor \cdots \lor a_m = \max\{a_1, a_2, \ldots, a_m\}$ and $a_1 \land a_2 \land \cdots \land a_m = \min\{a_1, a_2, \ldots, a_m\}$. For functions, $\| \cdot \|_{L^p(D)}$ denotes the $L^p$-norm on the domain $D$. When $D = [0,1]^2$, we also write $\| \cdot \|_p$. For a function $A = (a_1, a_2) : \mathbb{R}^2 \to \mathbb{R}^2$, we define $\|A\|_\infty := \max_{i=1,2} \sup_{x \in [0,1]^2} |a_i(x)|$. For $B$ a set, the indicator function is denoted by $\mathbb{1}(x \in B)$, which is 1 if $x \in B$ and 0 otherwise. Since we frequently work with bivariate functions, we write $f(\cdot, \cdot)$ for a function $(x, y) \mapsto f(x, y)$. In this notation, the dots $\cdot$ represent different variables.
2 Image deformation models

In this section, we first discuss a specific case and then introduce the full image deformation model. For any integers \( j, \ell \in \mathbb{Z} \), define

\[
I_{j,\ell} = \left[ \frac{j-1}{d}, \frac{j}{d} \right) \times \left[ \frac{\ell-1}{d}, \frac{\ell}{d} \right),
\]

representing a square with side length \( 1/d \). A \( d \times d \) image with \( d^2 \) pixels, as illustrated in Figure 1, can be expressed as a bivariate function

\[
f : \mathbb{R}^2 \to [0, \infty),
\]

where the grayscale value of the \((j, \ell)\)-th pixel is given by

\[
\overline{f}_{j,\ell} = d^2 \int_{I_{j,\ell}} f(u, v) \, dudv, \quad j, \ell \in \{1, \ldots, d\}, \quad (1)
\]

representing the average intensity of \( f \) on \( I_{j,\ell} \). The pixel value decodes the grayscale with smaller function values corresponding to darker pixels. To deal with image deformations, it is more convenient to define \( f \) on \( \mathbb{R}^2 \) instead of \([0,1]^2\).

The support of a function \( g \) is defined as the set of all \( x \) for which the function value \( g(x) \) is non-zero. Assuming that the images have zero background, all non-negative pixels are considered as part of the object itself. For a function \( f \) representing an image, we refer to \( f \) restricted to its support as the object. The background is defined as the complement of the support, that is, the set of \( x \) with \( f(x) = 0 \).

Next, we explore how simple transformations such as scaling, shifting, and brightness affect the function \( f \). Multiplying the function values by a factor \( \eta > 1 \) brightens the image, making the pixel values appear whiter, while multiplying by \( \eta < 1 \) darkens the image. Shifting the object within the image, either horizontally or vertically, corresponds to translating the function \( f \) by a vector \((\tau, \tau')\), changing the function values to \( f(x - \tau, y - \tau') \). Stretching or shrinking the object along the \( x \)- or \( y \)-axis transforms the function value to \( f(\xi x, \xi' y) \), where \( \xi < 1 \) or \( \xi' < 1 \) stretches the object along the \( x \)- or \( y \)-axis and \( \xi, \xi' > 1 \) shrinks it. Combining these transformations, the function becomes \((x, y) \mapsto \eta f(\xi x - \tau, \xi' y - \tau')\), capturing the effect of brightness, shifts

![Image represented by pixels](image1.png)

Figure 1: Image represented by pixels

![Different deformations of a cat image](image2.png)

Figure 2: Different deformations of a cat image.
and scaling on the image. See Figure 2 for an example of a deformed image of a cat with different scaling, shifting and brightness.

To distinguish between images of different object classes, the underlying idea of the data generating model is to assume that images from different classes correspond to different template functions \( f \). By drawing the parameters \((\eta, \tau, \tau', \xi, \xi')\) randomly, each observed image in the dataset is then a random transformation of one of these classes.

This means we observe \( n \) independently generated pairs, each consisting of a \( d \times d \) image and its corresponding class label. In the case of a supervised binary classification problem, these pairs are denoted by \((X_i, k_i) \in [0, \infty)^{d \times d} \times \{0, 1\}\). Here \( k_i \in \{0, 1\} \) is the \( i \)-th label and the \( i \)-th image is represented by a \( d \times d \) matrix \( X_i = (X_{i,j,l})_{j,l=1,...,d} \) with entries
\[
X_{i,j,l} = d^2 \eta_i Z_{I,j,l} f_{k_i} \left( \xi_i u - \tau_i, \xi'_i v - \tau'_i \right) du dv,
\]
where \( f_0, f_1 \) are the two unknown template functions and \( \eta_i, \xi_i, \xi'_i, \tau_i, \tau'_i \) are unobserved independent random variables. Note that each image consists of \( d^2 \) pixels. The brightness factor \( \eta_i \) is assumed to be positive. Throughout the article, we assume that the template functions \( f_0, f_1 \) are non-negative. This implies that all pixel values \( X_{i,j,l} \) are also non-negative.

In summary, model (2) generates images of the two objects using template functions \( f_0, f_1 \), where the shifts \( \tau, \tau' \), scaling factors \( \xi, \xi' \) and brightness \( \eta \) are all random variables.

To extend model (2), we introduce a more general framework where the random transformations are deformations \( A = (a_1, a_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), belonging to a set \( \mathcal{A} \) of invertible mappings. In the following, we will always assume that \( \mathcal{A} \) contains the identity. In this generalized model, the deformed template function is expressed as
\[
f \circ A (u, v) := f (A(u, v)) = f (a_1(u, v), a_2(u, v)),
\]
with \( A \in \mathcal{A} \). Let \( A_i \) denote the deformation of the \( i \)-th image in the sample. The image can then be represented by a \( d \times d \) matrix \( X_i = (X_{j,l})_{j,l=1,...,d} \) with entries
\[
X_{j,l} = d^2 \eta_i \int_{f_{j,l}} f_{k_i} \circ A_i (u, v) du dv.
\]
As every image consists of observing a randomly deformed function, the framework can be viewed as a functional data analysis model adapted to image classification. For more on classification for functional data, see [34, 46, 36, 19, 11].

Examples of this deformation model include:

**Affine transformations.** Affine transformations have been widely discussed in the fields of image processing and computer vision; see, e.g., [23, 42]. The deformation is of the form
\[
A(u, v) = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \tau \\ \tau' \end{pmatrix},
\]
with real parameters $b_1, \ldots, b_4, \tau, \tau'$. The transformation is invertible if the determinant $b_1 b_4 - b_2 b_3$ is non-zero. We recover model (2) as a special case by choosing deformations $b_1 = \xi, b_4 = \xi', b_2 = b_3 = 0$. Rotated, scaled and translated images $X_i = (X_i^{(j)})_{j=1, \ldots, d}$ with different brightness can be described by composing a scaling in $x$- and $y$-direction with a rotation by an angle $\gamma \in [0, 2\pi)$, that is, choosing

$$
\begin{pmatrix}
  b_1 & b_2 \\
  b_3 & b_4
\end{pmatrix} = \begin{pmatrix}\
  \xi & -\xi' \\
  \xi' & \xi
\end{pmatrix} = \begin{pmatrix}\
  \cos \gamma & -\sin \gamma \\
  \sin \gamma & \cos \gamma
\end{pmatrix} \begin{pmatrix}\xi & 0 \\
  0 & \xi'
\end{pmatrix}.
$$

(5)

**Nonlinear deformations.** Choosing $a_1(u, v) = u - \tau(u, v)$ and $a_2(u, v) = v - \tau'(u, v)$ for bivariate Lipschitz-continuous functions $\tau(u, v)$ and $\tau'(u, v)$ generates a class of nonlinear and local deformations [7, 8, 26]. Of particular interest among nonlinear deformations are wave-like deformations

$$a_1(u, v) = u + \alpha \sin(2\pi v/\lambda), \quad \text{and} \quad a_2(u, v) = v,$$

which are used to model periodic textures, noise patterns, image warping, and spatial distortions such as those caused by lens aberrations [14, 32, 41]. The parameter $\alpha$ describes the amplitude of the wave-like deformation and $\lambda \neq 0$ controls the wavelength.

3 Classification via inverse mappings and image alignment

We construct and analyze classifiers that are specifically tailored to the proposed image deformation models (2) and (3).

3.1 Classification via inverse mappings

Under the general deformation model (3), each image $X = (X_{j,\ell})_{j=1, \ldots, d}$ is generated by

$$X_{j,\ell} = d^2 \eta \int_{I_{j,\ell}} f \circ A(u, v) \, du \, dv,$$

with $f$ the template function and $A$ the invertible transformation modelling the deformation. To define the classifier, we also interpret an image $X$ as a continuous bivariate function on $\mathbb{R}^2$, via

$$X(u, v) := \sum_{j, \ell \in \mathbb{Z}} X_{j,\ell} \mathbb{1}((u, v) \in I_{j,\ell})$$

(6)

for all $(u, v) \in \mathbb{R}^2$, and setting $X_{j,\ell} := 0$ if $j, \ell \notin \{1, \ldots, d\}$. This means that $X$, viewed as a function, assigns to any point within the pixel its corresponding pixel value. As the random deformations $A \in \mathcal{A}$ do not contain information about the class label, a classifier should not depend on these deformations. To achieve this, we consider the set of invertible transformations $\mathcal{A}^{-1} = \{ A^{-1} : A \in \mathcal{A} \}$. For computational feasibility, instead of using $\mathcal{A}^{-1}$ directly, we approximate it by a discretized subset $\mathcal{A}_d^{-1}$, which covers $\mathcal{A}^{-1}$ with balls of radius $1/d$, meaning that for any $A^{-1} \in \mathcal{A}^{-1}$, there exists a transformation $B \in \mathcal{A}_d^{-1}$ such that

$$||A^{-1} - B||_{\infty} \leq \frac{1}{d}.$$

(7)
To construct the classifier, we apply each transformation \( B \in A^{-1} \) to the input of the bivariate image function and obtain
\[
\tilde{T}_{X \circ B} := X(B(u, v)) = \sum_{j,t \in \mathbb{Z}} X_{j,t} \mathbb{I}(B(u, v) \in I_{j,t}) \quad \text{for all } (u, v) \in \mathbb{R}^2. \quad (8)
\]

Given that any possible deformation \( A \) is invertible by assumption, there exists an approximate inverse \( B \in A^{-1} \), such that \( B \approx A^{-1} \). For such a \( B \), \( \tilde{T}_{X \circ B} \) should generate a nearly deformation-free representation of the image. To account for the effects of the image brightness factor \( \eta \), we normalize the pixel values and obtain
\[
T_{X \circ B} := \frac{\tilde{T}_{X \circ B}}{\| \tilde{T}_{X \circ B} \|_2}. \quad (9)
\]

Combining these steps, the proposed classifier \( \hat{k} \) assigns a label to the new image \( X \) by first applying all possible deformations from \( A^{-1} \) to both, the new image and each training image \( X_i \). It then finds the training image whose transformed version \( T_{X_i \circ B_i} \) is in Euclidean distance the closest fit to the transformed version \( T_{X \circ B} \) of the new image. The label of this closest matching training image is assigned to the new image. The classifier is
\[
\hat{k} := k_{\hat{i}}, \quad \text{with } \hat{i} \in \arg \min_{i \in \{1, \ldots, n\}} \min_{B_i, B \in A^{-1}} \| T_{X_i \circ B_i} - T_{X \circ B} \|_2. \quad (10)
\]

It can be interpreted as an one-nearest neighbor estimator in a transformed space.

We now state the assumptions for the statistical analysis. To make the theory tractable, we impose a Lipschitz condition on the template function.

**Assumption 1.** The support of the two template functions \( f_0, f_1 \) is contained in a rectangle \([\beta_{\text{left}}, \beta_{\text{right}}] \times [\beta_{\text{down}}, \beta_{\text{up}}] \subseteq [0, 1]^2\). Additionally, \( f_0, f_1 \) are Lipschitz continuous, in the sense that there exists a positive constant \( C_L \) such that for any real numbers \( u, v, u', v' \),
\[
|f_k(u, v) - f_k(u', v')| \leq C_L \|f_k\|_1 (|u - u'| + |v - v'|), \quad k = 0, 1. \quad (11)
\]

We also impose conditions that avoid that the deformation moves (part of) the object out of the image. Assumption 1 ensures that the support of \( f \) is contained in \([\beta_{\text{left}}, \beta_{\text{right}}] \times [\beta_{\text{down}}, \beta_{\text{up}}] \subseteq [0, 1]^2\). The deformed object remains fully visible, if the support of the deformed function \( f \circ A \) also lies in \([0, 1]^2\). This is the case if \([\beta_{\text{left}}, \beta_{\text{right}}] \times [\beta_{\text{down}}, \beta_{\text{up}}] \subseteq A([0, 1]^2)\).

**Assumption 2.** The class \( A \) contains the identity, for any \( A = (a_1, a_2) \in A \), \([\beta_{\text{left}}, \beta_{\text{right}}] \times [\beta_{\text{down}}, \beta_{\text{up}}] \subseteq A([0, 1]^2)\), and the partial derivatives of the function \( a_1, a_2 \) on \( \mathbb{R}^2 \) exist and are bounded in supremum norm by a constant \( C_A \).

To ensure correct classification, we also need to guarantee that two images from different object classes cannot be represented as transformations of the template function corresponding to the opposite class, as otherwise, distinguishing between the two classes becomes impossible.
To formalize this minimal separation between the two object classes with template function \( f_0, f_1 \), we introduce the separation quantity

\[
D := D(f_0, f_1) \vee D(f_1, f_0), \quad D(f, g) := \inf_{A, A' \in A, a, s, a', s' \in \mathbb{R}} \|a f \circ A(\cdot + s, \cdot + s') - g \circ A'\|_{L^2(\mathbb{R}^2)} / \|g\|_{L^2(\mathbb{R}^2)}.
\]  

(12)

The quantity \( D(f, g) \) measures the normalized minimal \( L^2 \)-distance between any possible deformation of the function \( g \) and any possible deformation of the function \( f \) in the set \( A \).

**Theorem 3.1.** Let \((X, k), (X_1, k_1), \ldots, (X_n, k_n)\) be defined as in (3). Suppose that the labels 0 and 1 occur at least once in the training data, that is, \( \{i : k_i = 0\} \neq \emptyset \) and \( \{i : k_i = 1\} \neq \emptyset \). Assume moreover that \( f_0 \) and \( f_1 \) satisfy Assumption 1 with Lipschitz constant \( C_L \) and that Assumption 2 holds. If \( D > C(C_L, C_A)/d \), where \( D \) is as defined in (12), and \( C(C_L, C_A) \) is a sufficiently large constant only depending on \( C_L, C_A \), then the classifier \( \hat{k} \) defined in (10) will recover the correct label, that is,

\[
\hat{k} = k.
\]

The proof of Theorem 3.1 is deferred to Section A.1. Notably, the result shows that classifier (10) guarantees perfect classification of any given image if the separation quantity satisfies \( D \geq c/d \) for a sufficiently large constant \( c \), and the dataset contains at least one image from each class.

The classifier can account for a wide range of image deformations, but discretization of the entire set of inverse mappings can be computationally demanding. Therefore we consider this classifier more as a theoretical benchmark for the image deformation model (3) rather than an effective method for practical applications. When dealing with specific deformation models, the computational demands can be significantly reduced. For instance, if the deformation class \( A \) is a group, then \( A^{-1} = A \) and we can instead consider the classifier

\[
\hat{k} := k_i, \quad \text{with} \quad \hat{i} \in \arg\min_{i \in \{1, \ldots, n\}} \min_{A \in A_d} \|T_{X_i} - T_{X \circ A}\|_2.
\]  

(13)

In Theorem 3.7, we show that if the set \( A \) is sufficiently rich, the considered separation criterion \( D \gtrsim 1/d \), as defined in (12), is optimal. This means that any smaller bound could result in deformed versions of one template function being representable by a template function of the opposite class, thereby making it impossible to distinguish between the two classes; see Figure 3 for an illustration of two deformed MNIST images with labels 7 and 1 that can be transformed into each other by a rotation, making classification ambiguous.

We now verify the imposed assumptions for specific deformation models. Let \( y_+ := \max\{0, y\} \).

![Figure 3: Deformed MNIST images of the digits 7 and 1.](image-url)
Lemma 3.2. Let \([\beta_{\text{left}}, \beta_{\text{right}}] \times [\beta_{\text{down}}, \beta_{\text{up}}] = [1/4, 3/4] \times [1/4, 3/4]\). The class of deformations in (2) with \(1/2 \leq |\xi|, |\xi'| \leq C_A, |\tau|, |\tau'| \leq \ell_s\),

\[-(-\xi)_+ - \frac{1}{4} \leq \tau \leq \xi_+ - \frac{3}{4} \quad \text{and} \quad -(-\xi')_+ - \frac{1}{4} \leq \tau' \leq \xi'_+ - \frac{3}{4},\]  

(14)
satisfies Assumption 2 with constant \(C_A\). Moreover, there exists an \(1/d\)-covering of \(A^{-1}\) with cardinality \(|A^{-1}_d| \asymp d^3\).

Lemma 3.3. The deformation model described in (4) and (5) is

\[A(u, v) = \begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} \xi & 0 \\ 0 & \xi' \end{pmatrix} \left( \begin{array}{c} u \\ v \end{array} \right) - \begin{pmatrix} \tau \\ \tau' \end{pmatrix}.\]

Let \([\beta_{\text{left}}, \beta_{\text{right}}] \times [\beta_{\text{down}}, \beta_{\text{up}}] = [1/4, 3/4] \times [1/4, 3/4]\). The class of deformations with \(1/2 \leq |\xi|, |\xi'| \leq C_A, |\tau|, |\tau'| \leq \ell_s, \gamma \in [0, \pi/2]\), and

\[-(-\xi)_+ - \frac{1}{4}(\cos \gamma + \sin \gamma) \leq \tau \cos \gamma + \tau' \sin \gamma \leq \xi_+ - \frac{3}{4}(\cos \gamma + \sin \gamma),\]

\[-(-\xi')_+ - \frac{1}{4} \cos \gamma + \frac{3}{4} \sin \gamma \leq \tau' \cos \gamma - \tau \sin \gamma \leq \xi'_+ - \frac{3}{4} \cos \gamma + \frac{1}{4} \sin \gamma,\]
satisfies Assumption 2 with constant \(C_A\) and there exists a \(1/d\)-covering of \(A^{-1}\) with cardinality \(|A^{-1}_d| \asymp d^3\).

Consider the non-linear deformations \(A = (a_1, a_2) \in A, a_1(u, v) = h_1(u, v)\) and \(a_2(u, v) = h_2(v)\), where \(h_2(v)\) is strictly monotone with respect to \(v\), and for any fixed \(v \in \mathbb{R}\), \(h_1(u, v)\) is strictly monotone with respect to \(u\). In this case, the transformation is invertible because one can always retrieve \(v\) from \(h_2(v)\) and then \(u\) from \(h_1(u, v)\).

A specific example is the wave-like deformation model discussed in Section 2.

Lemma 3.4. For the class of deformations

\[A(u, v) = (a_1(u, v), a_2(u, v)) = (u + \alpha \sin(2\pi v/\lambda), v),\]

with \(|\lambda| \geq C_{\text{lower}}, |\alpha| \leq \beta_{\text{left}} \wedge (1 - \beta_{\text{right}}), \text{and} \lambda \neq 0\), Assumption 2 holds with

\[C_A = \max\{1, 2\pi |\beta_{\text{left}} \wedge (1 - \beta_{\text{right}})| / C_{\text{lower}}\},\]

and there exists a \(1/d\)-covering \(A^{-1}_d\) of \(A^{-1}\) with cardinality \(|A^{-1}_d| \asymp d^2\).

The proof of Lemma 3.4 is deferred to Appendix A.1. The next lemma considers compositions of deformation classes. This allows to verify Assumption 2 for more involved deformation classes.

Lemma 3.5. Let \(A_1\) and \(A_2\) satisfy Assumption 2 with respective constants \(C_{A_1}\) and \(C_{A_2}\). If for any \(A_1 \in A_1, [0, 1]^2 \subseteq A_1([0, 1]^2)\), then, the composite deformation class \(A_2 \circ A_1 := \{A_2 \circ A_1, A_1 \in A_1, A_2 \in A_2\}\) satisfies Assumption 2 with constant \(2C_{A_1}C_{A_2}\).

The proof of Lemma 3.5 is deferred to Section A.1.
3.2 Classification via image alignment

We now focus on the specific image deformation model (2) that incorporates random scaling, shifts, and brightness. An image $X = (X_{j,\ell})_{j,\ell=1,...,d}$ is then generated by

$$X_{j,\ell} = \eta \int_{I_{j,\ell}} f(\xi u - \tau, \xi' v - \tau') \, du \, dv,$$

with $(\eta, \xi, \xi', \tau, \tau')$ the random deformation parameters. In this setting, one can find a transformation that aligns the images, in the sense that the transformed images are nearly independent of the deformation parameters. We propose a one-nearest-neighbor classifier based on the aligned training and test images. This approach is similar to curve registration in functional data analysis, see for instance [28]. The classifier can be efficiently computed but relies on this specific deformation model.

The first step of the construction is to detect the object within the image by finding the smallest axes-aligned rectangle that contains all non-zero pixel values; see the left image in Figure 4 for an illustration. We refer to this as the rectangular support. To determine the rectangular support, we denote the smallest and largest indices corresponding to the non-zero pixels in the image by

$$j_- := \arg \min \{ j : X_{j,\ell} > 0 \}, \quad j_+ := \arg \max \{ j : X_{j,\ell} > 0 \}$$

and

$$\ell_- := \arg \min \{ \ell : X_{j,\ell} > 0 \}, \quad \ell_+ := \arg \max \{ \ell : X_{j,\ell} > 0 \}.$$  

The rectangular support of the image is then given by the rectangle $[j_-/d, j_+/d] \times [\ell_-/d, \ell_+/d]$. Similarly, we define the rectangular support of a function as the smallest rectangle containing the support. From the definition of the model (2), it follows that the rectangular support of the image $X$ should be close to the rectangular support of the underlying deformed function $f(\xi \cdot - \tau, \xi' \cdot - \tau')$.

We now rescale the rectangular support of the image to the unit square $[0,1]^2$. The line $[0,1] \ni t \mapsto j_- + t(j_+ - j_-)$ starts at $j_-$ and ends for $t = 1$ at $j_+$. We define the rescaled pixel values as

$$Z_X(t, t') := X_{\lfloor j_- + t(j_+ - j_-) \rfloor, \lfloor \ell_- + t'(\ell_+ - \ell_-) \rfloor},$$

with $\lfloor \cdot \rfloor$ the floor function. The function $(t, t') \mapsto Z_X(t, t')$ runs through the pixel values on the rectangular support, now rescaled to the unit square $[0,1]^2$; see the middle image of Figure 4 for a illustration. This rescaling makes $Z_X(t, t')$ approximately invariant to random shifts and scaling of the image, up to smaller-order effects.

To find a quantity that is independent of the brightness $\eta$, we normalize the pixel values by $Z_X/\|Z_X\|_2$. The image alignment transformation is then given by

$$T_X := \frac{Z_X}{\|Z_X\|_2},$$

Figure 4: $X$, $Z_X$, and $T_X$. 

[Image]
see also Figure 4. We study a one-nearest-neighbor classifier \( \hat{k} \) that assigns the label of \( X_i \) from the training set to \( X \), where \( T_{X_i} \) is the closest to \( T_X \), this is,

\[
\hat{k} := k_i, \quad \text{with } \hat{i} \in \arg\min_{i=1,\ldots,n} \|T_X - T_{X_i}\|_2.
\]

This is an interpolating classifier, in the sense that if the new \( X \) coincides with one of the images in the training set \( X_i \), then, \( T_X = T_{X_i}, \hat{i} = i, \) and \( \hat{k} = k_i \).

To study this model, we assume that \( \xi, \xi' \geq 1/2 \). Applying Lemma 3.2 leads to the following assumption on the parameters to ensure full visibility of the objects on the deformed images.

**Assumption 2'**. The support of \( f \) is contained in \([1/4, 3/4]^2\), and the random parameters \((\tau, \tau', \xi, \xi')\) satisfy \( \xi, \xi' \geq 1/2 \),

\[
-1/4 \leq \tau \leq \xi - 3/4 \quad \text{and} \quad -1/4 \leq \tau' \leq \xi' - 3/4.
\]

Assumption 2' indicates that the range of possible shifts \( \tau, \tau' \) increases as \( \xi, \xi' \) become larger. This is reasonable, as larger values of \( \xi, \xi' \) shrink the object. Consequently, larger shifts \( \tau, \tau' \) can be applied without moving the object out of the image.

Under the deformation model (2), we have \( f \circ A(u, v) = f(\xi u - \tau, \xi' v - \tau') \). Based on this, we consider the separation quantity

\[
D = D(f_0, f_1) \lor D(f_1, f_0), \quad \text{with} \quad D(f, g) := \inf_{a,b,c,b',c' \in \mathbb{R}} \|a f(b - c, b' - c') - g\|_{L^2(\mathbb{R}^2)},
\]

that measures the normalized minimal \( L^2 \)-distance between the function \( g \) and any potential deformation of the function \( f \) due to rescaling, shifting, and change in brightness.

**Theorem 3.6**. Let \((X, k), (X_1, k_1), \ldots, (X_n, k_n)\) be defined as in (2). Suppose that the labels 0 and 1 occur at least once in the training data, that is, \( \{i : k_i = 0\} \neq \emptyset \) and \( \{i : k_i = 1\} \neq \emptyset \). Assume moreover that \( f_0 \) and \( f_1 \) satisfy Assumption 1 with Lipschitz constant \( C_L \) and that Assumption 2' holds. Set \( \Xi_n := \max\{1, \xi, \xi', \xi_1, \xi'_1, \ldots, \xi_n, \xi'_n\} \). If \( D > 4K(C_L \lor C_L')\Xi_n^2/d \), with \( D \) as defined in (21), and \( K \) the universal constant in Lemma A.7, then the classifier \( \hat{k} \) as defined in (20) will recover the correct label, that is,

\[
\hat{k} = k.
\]

The proof of Theorem 3.6 is postponed to Section A. The result indicates that, under proper conditions, the classifier accurately identifies the label when the template functions \( f_0 \) and \( f_1 \) are separated by \( \geq 1/d \) in \( L^2 \)-norm, consistently across all conceivable image deformations. This finding aligns with the theoretical performance outlined in Theorem 3.1 for the general classification approach. The advantage of implementing the image alignment approach is that it eliminates the need to discretize the set of inverse mappings, thereby improving computational efficiency.

We further prove a corresponding lower bound, showing that a \( 1/d \)-rate in the separation criterion is necessary. Without this condition, the same could be represented by deformations of both template functions, making classification impossible.
Theorem 3.7. For any \( \tau, \tau', \xi, \xi' \) satisfying Assumption 2', there exist non-negative Lipschitz continuous functions \( f_0, f_1 \) with Lipschitz constants \( C_{f_0} = C_{f_0}(\xi, \xi') \) and \( C_{f_1} = C_{f_1}(\xi, \xi') \) respectively, such that for any \( d \geq 16(\xi \lor \xi') \),

\[
\| f_0 - f_1 \|_{L^2(\mathbb{R}^2)} \gtrsim \frac{1}{d},
\]

and the data generating model (15) can be written as

\[
X_{j,\ell} = \eta \int_{I_{j,\ell}} f_1(\xi u - \tau, \xi' v - \tau') \, du \, dv = \eta \int_{I_{j,\ell}} f_0(\xi u - \tau, \xi' v - \tau') \, du \, dv.
\]

Consequently, the same pixel values are generated under both classes.

The proof of the lower bound indicates that the separation rate \( 1/d \) arises due to the Lipschitz continuity of \( f_0, f_1 \). If, instead, we assume Hölder regularity with index \( \beta \leq 1 \), we expect the lower bound to be of order \( O(d^{-\beta}) \), which we also conjecture to be the optimal separation rate in this case.

Since the deformation model (2) is a specific case of the general model (3), the lower bound derived in Theorem 3.7 also applies to the general deformation model (3). This indicates that the rate \( 1/d \) is indeed necessary to distinguish between the two classes.

In the presence of background noise, finding the rectangular support of the object is hard, as non-zero pixel values in the image may belong to the background. As an alternative one could instead rely on \( t \)-level sets \( \{ x : g(x) > t \} \). Define the \( t \)-rectangular support as the smallest rectangular containing the \( t \)-level set. For non-negative \( g \), the previously introduced rectangular support corresponds to \( t = 0 \). To construct a classifier, we can first normalize the pixel values to eliminate the brightness factor \( \eta \), then follow a similar strategy as in the zero-background case by determining the \( t \)-rectangular support for each image in the dataset. While increasing \( t \) enhances robustness to background noise, it also reduces the \( t \)-rectangular support and causes larger constants in the separation condition between the two classes.

If an image contains multiple non-overlapping objects, we suggest to first apply an image segmentation method (see, e.g., [17, 29]) to isolate each object. The image alignment classifier can then be applied to each segment separately.

The image alignment step in the construction of the classifier leads to a representation of the image that is, up to discretization effects, independent to rescaling and shifting of the object; see Figure 4. While the proposed image alignment transformation is natural and mathematically tractable for this specific deformation model, other transformations could be employed instead, such as Fourier, Radon, and scattering transforms [7, 8, 26].

4 Classification with convolutional neural networks

Convolutional neural networks (CNNs) have achieved remarkable practical success, particularly in the context of image recognition [25, 24, 37, 35]. In this section, we analyse the performance of a CNN-based classifier...
within the framework of our deformation model (3) introduced in Section 2. We begin by introducing a suitable mathematical notation to describe the structure of a CNN. Here we focus on a particular CNN structure and refer to [47, 35] for a broader introduction.

4.1 Convolutional neural networks

We analyse a CNN with a rectifier linear unit (ReLU) activation function and a softmax output layer. Generally, a CNN consists of three components: Convolutional, pooling and fully connected layers. The input of a CNN is a $d \times d$ matrix representing the pixel values of an image. In the convolutional layer, so-called filters (that is, weight matrices of pre-defined size) slide across the image, performing convolutions at each spatial location. Finally an element-wise nonlinear activation function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, in our case the ReLU function, is applied to the outcome of the convolutions, producing the output matrices known as feature maps.

In this work, we consider CNNs with a single convolutional layer followed by one pooling layer. For mathematical simplicity, we introduce a compact notation tailored to our setting and refer to [21, 22] for a general mathematical definition. Recall that the input to the network is an input image represented by a $d \times d$ matrix $X$. For a $d \times d$ matrix $W$, we define its quadratic support $[W]$ as the smallest square sub-matrix of $W$ that contains all its non-zero entries. For instance,

$$[W] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

is the quadratic support of the matrix

$$W = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this context, $[W]$ represents the network filter. To describe the action of the filter on the image, denoted as $[W] * X$, assume that $[W]$ is a filter of size $\ell \in \{1, \ldots, d\}$. We extend the matrix $X$ by padding it with zero matrices on all sides. Specifically, we define the enlarged matrix as

$$X' := \begin{bmatrix} 0_{\ell \times \ell} & 0_{\ell \times d} & 0_{\ell \times \ell} \\ 0_{d \times \ell} & X & 0_{d \times \ell} \\ 0_{\ell \times d} & 0_{\ell \times d} & 0_{\ell \times \ell} \end{bmatrix},$$

where $0_{j \times k}$ denotes a $j \times k$ zero matrix. The $(i,j)$-th patch is defined as the $\ell \times \ell$ submatrix $X'_{i,j} := (X'_{i+a,j+b})_{a,b=0,\ldots,\ell-1}$. We further define $([W] * X)_{i,j}$ as the entry-wise sum of the Hadamard product of $[W]$ and $X'_{i,j}$. Thus, the matrix $[W] * X$ contains all entry-wise sums of the Hadamard product of $[W]$ with all patches. Finally the ReLU activation function $\sigma(x) = \max\{x, 0\}$ is applied element-wise. A feature map can then be expressed as

$$\sigma([W] * X).$$
This extension of the matrix $X$ to $X'$ is a form of zero padding, which ensures that the in-plane dimension of the input remains equal after convolution [18]. A pooling layer is typically applied to the feature map. While max-pooling extracts the maximum value from each patch of the feature map, average-pooling computes the average over each patch. In this work we consider CNNs with global max-pooling in the sense that the max-pooling extracts from every feature map $\sigma([W] \ast X)$ the largest absolute value. The feature map after global max-pooling is then given by

$$O = |\sigma([W] \ast X)|_{\infty}.$$  

For $k$ filters described by the matrices $W_1, \ldots, W_k$, we obtain the $k$ values

$$O_s = |\sigma([W_s] \ast X)|_{\infty}, \quad s = 1, \ldots, k. \quad (22)$$

We denote by $F_C(k)$ the class of all networks computing $k$ outputs of the form (22). The outputs of these network are typically flattened, i.e., transformed into a vector, before several fully connected layers are applied, with the ReLU activation function being the common choice for these layers.

For any vector $v = (v_1, \ldots, v_r)^T$, $y = (y_1, \ldots, y_r)^T \in \mathbb{R}^r$, we define $\sigma_y = (\sigma(y_1 - v_1), \ldots, \sigma(y_r - v_r))^T$. In the context of binary classification, the last layer of the network should extract a two-dimensional probability vector. To achieve this, the softmax function is typically applied

$$\Phi(x_1, x_2) = \left( \frac{e^{x_1}}{e^{x_1} + e^{x_2}}, \frac{e^{x_2}}{e^{x_1} + e^{x_2}} \right). \quad (23)$$

A feedforward neural network with $L$ fully connected hidden layers and width vector $m = (m_0, \ldots, m_{L+1}) \in \mathbb{N}_{L+2}$, where $m_i$ denotes the number of hidden neurons in the $i$-th hidden layer, can then be described by a function $f : \mathbb{R}^{m_0} \to \mathbb{R}^{m_{L+1}}$ with

$$x \mapsto f(x) = \psi W_L \sigma_{v_L} W_{L-1} \sigma_{v_{L-1}} \cdots W_1 \sigma_{v_1} W_0 x,$$

where $W_j$ is a $m_j \times m_{j+1}$ weight matrix, $v_j$ is the bias vector in layer $j$ and $\psi$ is either the identity function $\psi = id$ or the softmax function $\psi = \Phi$. We denote this class of fully connected neural networks by $F_\psi(L, m)$.

We will construct CNN classifiers based on a CNN architecture of the form

$$\mathcal{G}(m) := \left\{ f \circ g : f \in F_\Phi(1 + 2[\log_2 m], (2m, 4m, \ldots, 4m, 2)), g \in F_C(2m) \right\}, \quad (24)$$

with $m$ a positive integer. Given that we only consider one convolutional and one pooling layer, the number of feature maps equals the input dimension of the fully connected subnetwork.

As the filters are applied to all patches of the image, CNNs are translation-invariant, meaning that up to boundary and discretization effects, the CNN classifier does not depend on the values of the shifting parameters $\tau$ and $\tau'$ in the deformation model. For instance, if a cat in an image is moved from the upper left corner to the lower right corner, the convolutional filter will produce, up to discretization effects, the same feature values at potentially different locations within the feature map; see Figure 5 for an illustration.
A shift of the image pixels causes therefore a permutation of the values in the feature map. Since the global max-pooling layer is invariant to permutations, the CNN output is thus invariant under translations.

More challenging for CNNs are varying object sizes and rotation angles. Some authors argue that scale-invariance is not desirable in image classification as classifiers can benefit from scale information of the object [44, 33, 13]. To address this, they propose architectures with different filters to capture different scales. Similarly, [27] employ separate filters for different rotation angles, thereby achieving rotation invariance for texture classification. Another research direction analyses networks with built-in invariances, such as group equivariant convolutional networks [9], which extend CNNs to handle invariances induced by arbitrary groups.

Data augmentation is another effective method to enhance the learning of CNNs in the presence of rotation and scale deformations. Before training, data augmentation applies simple deformations such as rotations and different scaling to the training images and learns a CNN on the augmented dataset consisting of the original and the transformed training samples (see, e.g., [39]). This helps the network to learn invariances of the class labels to deformations of the object. Enlarging the dataset increases, however, the computational cost of training.

In this paper we derive misclassification bounds for CNN classifiers minimizing the training loss. This gives generalization guarantees but does not directly provide insights which features a CNN learns and how it internally copes with rotations and scale information. An essential part of the developed theory shows, however, that CNNs are expressive enough to separate the classes. This is achieved by designing filters in the convolutional layers that test the inputs for specific deformations of the object.
4.2 A misclassification bound for CNNs

We suppose that the training data consist of \( n \) i.i.d. data, generated as follows: For \( \pi \in [0, 1] \) and each \( i \), we draw a label \( k_i \in \{0, 1\} \) from the Bernoulli distribution with success probability \( \pi \). Let \( Q \) be a distribution on the deformation class \( \mathcal{A} \). The \( i \)-th sample is \((X_i, k_i)\), where \( X_i \) is an independent draw from the general model \( (3) \) with template function \( f_{k_i} \), deformation \( A \in \mathcal{A} \) generated from \( Q \) and corresponding class label \( k_i \). The full dataset is denoted by

\[
D_n = ((X_1, k_1), \ldots, (X_n, k_n)).
\]

(25)

In expectation, the dataset consists of \( n(1 - \pi) \) samples from class 0 and \( n\pi \) samples from class 1.

The parameters of a CNN are then fitted to the normalized images

\[
\bar{X}_i = (X_{i,j,\ell})_{j,\ell=1,\ldots,d}, \quad \text{with} \quad X_{i,j,\ell} := \frac{X_{i,j,\ell}}{\sqrt{\sum_{j,\ell=1}^{d} (X_{i,j,\ell})^2}}.
\]

(26)

This normalization can be viewed as pre-processing step. It ensures that the images are invariant to variations of the brightness \( \eta \) and all pixel values lie between 0 and 1. Fitting a neural network with softmax output to the dataset results in estimators for the conditional class probabilities (or a posteriori probabilities).

In this article, we consider fitting a CNN of the form \( (24) \) by minimizing the empirical misclassification error

\[
\hat{p} = (\hat{p}_1, \hat{p}_2) \in \arg \min_{q=(q_1,q_2) \in \mathcal{G}(m)} -\frac{1}{n} \sum_{i=1}^{n} k_i \mathbb{I}\left(q_2(X_i) > \frac{1}{2}\right),
\]

(27)

with \( \mathbb{I}(q_2(X_i) > 1/2) \) the predicted label of the \( i \)-th sample based on the network \( q = (q_1, q_2) \in \mathcal{G}(m) \). The learned network \( \hat{p} \) outputs estimates for the two conditional class probabilities \( p_1(x) = P(k = 0|X = x) \) and \( p_2(x) = P(k = 1|X = x) \). These probabilities sum to one and optimizing over \( q_2 \) suffices.

For a new test image \( \bar{X} \) that has been normalized according to \( (26) \), the classifier \( \hat{k}(\bar{X}) := \mathbb{I}(\hat{p}_2(\bar{X}) > 1/2) \) assigns the label 1 if the estimated probability belonging to class 1 exceeds 1/2 and assigns class label 0 otherwise.

Minimizing the empirical risk function with 0-1 loss has been widely studied in classification theory \([1, 6, 45]\). While using this loss function to find a classifier might seem like a natural approach, its optimization is typically NP-hard and computational intractable due to its lack of continuity and convexity. As a result, much of the research has focused on finding convex surrogate loss functions that maintain similar consistency properties and convergence rates as classifiers trained with 0-1 loss (see, e.g., \([49]\) and \([4]\) and the literature cited therein). As the primary focus of our work is to analyze the performance of CNN classifiers under the proposed deformation model, we restrict ourselves to the 0-1 loss. In view of Lemma 1 and Lemma 2 in \([22]\), that relate 0-1 loss and cross entropy loss, we conjecture that our analysis can be extended to networks minimizing other surrogate loss functions such as cross entropy loss.

For the theory, we consider the general deformation model \( (3) \) and impose the following assumption.
Assumption 3. The deformation class $\mathcal{A}$ contains a finite subset $\mathcal{A}_d$ such that for any $A \in \mathcal{A}$, there exists an $A' \in \mathcal{A}_d$ and indices $j, \ell \in \{1, \ldots, d\}$ such that $A'(\cdot + j/d, \cdot + \ell/d)$ satisfies Assumption 2 and

$$\left\| A' \left( \cdot + \frac{j}{d} \cdot + \frac{\ell}{d} \right) - A \right\|_{\infty} \leq \frac{1}{d}. \quad (28)$$

The subset $\mathcal{A}_d$ can be obtained through suitable discretization of the deformation class $\mathcal{A}$. The cardinality of the discretized class $\mathcal{A}_d$ is typically of the order $d^\kappa$ with $\kappa$ the number of free parameters that are not related to the shifts. For instance, in model (2), there are four parameters of which two are controlling the shift and two vary the scaling such that $\mathcal{A}_d \lesssim d^2$. Adding one parameter controlling the rotation, such as in (4) and (5), yields then $|\mathcal{A}_d| \asymp d^3$.

The next result states the main misclassification bound for the CNN-based classifier.

Theorem 4.1. Consider the general deformation model (3) and suppose Assumptions 1-3 hold. Let $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)$ be the estimator in (27) based on the CNN class $\mathcal{G}(|\mathcal{A}_d|)$ defined in (24). Suppose a new datapoint $(X, k)$ is independently drawn from the same distribution as the data in (25). If the separation constant in (12) satisfies $D > \sqrt{\kappa/d}$, where $\kappa$ is a constant that depends only on the constants present in the assumptions, and $|\mathcal{A}_d| \geq d^2 \geq 4$, then for the classifier $\hat{k}(X) = 1(\hat{\beta}_2(X) > 1/2)$, there exists a universal constant $C > 0$ such that

$$\mathbb{P}(\hat{k}(X) \neq k) \leq C \frac{|\mathcal{A}_d|^2}{n} \log^3(|\mathcal{A}_d|) \log n. \quad (29)$$

Here, $\mathbb{P}$ denotes the distribution over all randomness in the data and the new sample $X$. Since the cardinality $|\mathcal{A}_d|$ depends on $d$, the upper bound (29) grows in the image resolution $d$. At first glance, this might seem counterintuitive as a high image resolution typically provides more information about the object and should lead to improved misclassification bounds. However, as $d$ increases, the template functions can become closer in their separation distance (12) making the two objects more similar and thus harder to separate.

The CNN has $2|\mathcal{A}_d|$ filters and of the order of $|\mathcal{A}_d|^2 \log(|\mathcal{A}_d|)$ network parameters. Up to logarithmic factors this means that we obtain a consistent classifier if the sample size is of larger order than the number of network parameters. More generally, Theorem 4.1 implies that for sufficiently large sample sizes the misclassification error of the proposed CNN-based classifier can become arbitrary small. Compared to the image classifiers discussed in Section 3, the CNN-based classifier only requires knowledge of the deformation class for the choice of the architecture.

That the CNN misclassification error can become arbitrary small is in line with the nearly perfect classification results of deep learning for a number of image classification tasks. Interestingly most of the previous statistical analysis for neural networks considers settings with asymptotically non-vanishing prediction error. Those are statistical models where every new image contains randomness that is independent of the training data and can therefore not be predicted by the classifier. To illustrate this, consider the nonparametric regression model $Y_i = f(X_i) + \sigma \varepsilon_i$, $i = 1, \ldots, n$ with fixed noise variance $\sigma^2$. The squared prediction error of
the predictor $\hat{Y} = \hat{f}_n(X)$ for $Y$ is $E(\hat{Y} - Y)^2 = \sigma^2 + E[(\hat{f}_n(X) - f(X))^2]$. This implies that even if we can perfectly learn the function $f$ from the data, the prediction error is still at least $\sigma^2$. Taking a highly suboptimal but consistent estimator $\hat{f}_n$ for $f$, yields a prediction error $\sigma^2 + o(1)$, thereby achieving the lower bound $\sigma^2$ up to a vanishing term. For instance, the one-nearest neighbor classifier does not employ any smoothing and results in suboptimal rates for conditional class probabilities but is optimal for the misclassification error up to a factor of 2, [10]. This shows that optimal estimation of $f$ only affects the second order term of the prediction error. As classification and regression are closely related, the same phenomenon also occurs in classification, whenever the conditional class probabilities lie strictly between 0 and 1. The only possibility to achieve small misclassification error requires that the conditional class probabilities are consistently close to either zero or one. This means that the covariates $X$ contain (nearly) all information about the label $Y$, [5]. The main source of randomness lies then in the sampling of the covariates $X$. An example are the random deformations considered in this work that only affect the covariates but not the labels.

To prove Theorem 4.1, one can decompose the misclassification error into an approximation error and a stochastic error term (see Corollary 5.3 in [6] and Lemma B.8). The stochastic error can be bounded via statistical learning tools such as the Vapnik-Chervonenkis (VC) dimension of the CNN class $\mathcal{G}(|\mathcal{A}_d|)$, see Lemma B.11. The approximation error vanishes as CNNs from the class $\mathcal{G}(|\mathcal{A}_d|)$ with suitably chosen parameters can achieve perfect classification. The result is stated in the following theorem and is proved in Section B.

**Theorem 4.2.** If Assumptions 1-3 hold and the separation quantity $D$ in (21) satisfies $D > \sqrt{\kappa/d}$, with $\kappa$ a constant depending only on the constants that occur in the assumptions, then, for any $(X, k)$ generated from the same distribution as the data in (25), there exists a network $\hat{p} = (\hat{p}_1, \hat{p}_2) \in \mathcal{G}(|\mathcal{A}_d|)$, such that the corresponding classifier $\hat{k}(X) = 1(\hat{p}_2(X) > 1/2)$ satisfies

$$\hat{k}(X) = k, \text{ almost surely.}$$

To ensure the existence of the interpolating classifier in the previous result, the separation quantity $D$ needs to satisfy the lower bound

$$D \gtrsim \frac{1}{\sqrt{d}},$$

which is more restrictive if compared to the lower bound $D \gtrsim 1/d$ imposed on the image classification methods discussed in Section 3. This discrepancy arises from the construction of the proposed CNN architecture. In fact, the same lower bound could be achieved by the CNN-based approach if we would adapt the network architecture respectively. To handle different deformations, the CNN construction uses the $2|\mathcal{A}_d|$ separate filters to test whether the image has been generated by applying any of the deformations in the discretized set $\mathcal{A}_d$ to one of the two possible template images. Consider a given input image that has been generated by deforming the template function of class $k \in \{0, 1\}$ by $A$. In Proposition B.4, we show that the convolutional filter corresponding to the correct class $k$ and the deformation in $\mathcal{A}_d$ that is closest to the true deformation $A$
produces the maximum activation, that is, the highest output value after convolution and global max-pooling; see Figure 6 for an illustration. This requires \( D \geq 1/\sqrt{d} \).

However, we conjecture that the \( 1/d \) lower bound on the separation quantity can be obtained if we instead take \(|A_d|^2\) many convolutional filters, resulting in a CNN architecture with \( O(|A_d|^3 \log(|A_d|)) \) network parameters. The high level idea is to test for all possible differences of the two template functions \( f_0, f_1 \), deformed by \( A_0, A_1 \in A_d \). Those are in total \(|A_d|^2\) such tests that can be computed by the \(|A_d|^2\) convolutional filters. The key step in Theorem 4.2 is to show that for input image generated from \( f = f_0 \) or \( f = f_1 \), we have

\[
\int \frac{\phi(u - s, v - t) f \circ A(u, v) du dv}{\|f \circ A\|_2} + O\left(\frac{1}{d}\right).
\]

Choosing now \( \phi_{A_0, A_1} = (r_1 f_1 \circ A_1 - r_0 f_0 \circ A_0)/\|r_1 f_1 \circ A_1 - r_0 f_0 \circ A_0\|_2 \) with \( r_k := \|f_k \circ A_k\|_2^{-1} \), and ignoring the maximum over \( s, t \) by just considering at the moment \( s = t = 0 \), we find

\[
\int \frac{\phi_{A_0, A_1}(u, v) f_k \circ A_k(u, v) du dv}{\|f_k\|_2} + O\left(\frac{1}{d}\right) = \frac{(r_1 f_1 \circ A_1 - r_0 f_0 \circ A_0) f_k \circ A_k(u, v) du dv}{\|r_1 f_1 \circ A_1 - r_0 f_0 \circ A_0\|_2} + O\left(\frac{1}{d}\right)
\]

\[
= \frac{(-1)^{k+1}}{2} \|r_1 f_1 \circ A_1 - r_0 f_0 \circ A_0\|_2 + O\left(\frac{1}{d}\right).
\]

This indicates that one can discriminate between the two classes under the separation condition \( \|r_1 f_1 \circ A_1 - r_0 f_0 \circ A_0\|_2 \geq c/d \), where \( c \) has to be chosen large enough, such that the \( O(1/d) \) term cannot change the sign of the right hand side of the equation in the previous display. Since this construction requires \(|A_d|^2\) more filters and \( O(|A_d|) \) more network parameters, the misclassification error is then also expected to be slower by a factor \(|A_d|^2\) if compared to (29). The main reason why we are not adopting this approach is the envisioned extension to the multiclass case, where images are generated by \( K \) template functions \( f_0, \ldots, f_K \). For separation condition \( \geq 1/d \) instead of \( \geq 1/\sqrt{d} \), we have to test for all differences between the \( K \) template functions and the increase in the number of network parameter and the loss is then expected to be of order \( |A_d|^{K-1} \).

A consequence of the approximation bound is that under the separation condition in Theorem 4.2, the label can be retrieved from the image.

**Lemma 4.3.** Let Assumptions 1-3 hold. If the separation quantity \( D \) in (21) satisfies \( D \geq \sqrt{\kappa/d} \), where \( \kappa \) is a constant depending only on the constants that occur in the assumptions. If \( (X, k) \) is generated from the same distribution as the data in (25), then, \( k \) can be written as a deterministic function evaluated at \( X \) and we have

\[
p(X) = k(X),
\]

with \( p(x) = P(k = 1|X = x) \) the conditional class probability.

A consequence is that under the imposed conditions

\[
\min_{q: [0,1]^2 \to \{0,1\}} P(q(X) \neq k(X)) = 0.
\]
Figure 6: Effect of different filters on the same image. The global max-pooling layer will generate the largest values for filters that are most similar to the object.

5 Numerical results

This section compares the image alignment classifier (20) with learning outcome from three different CNN architectures. CNN1 consists of one convolutional layer with 28 filters of size 3, followed by one pooling layer with patch size 2 and one fully connected layer with 128 neurons, before applying the softmax layer. CNN2 retains the same architecture as CNN1, but has three convolutional layers with 28, 64, and 128 filters. CNN3 has moreover three fully connected layers with 128 neurons, respectively. All CNNs are trained using the Adam optimizer in Keras (Tensorflow backend) with default learning rate 0.001. [38] compares different CNN architectures on the MNIST dataset [12] and finds minimal changes in test error across different architectures. It is therefore sufficient to limit ourselves to the three settings mentioned above.

The MNIST dataset consists of 60,000 samples of $28 \times 28$ pixel images of handwritten digits from 0 to 9. In our notation, this means that $d = 28$. As we consider binary classification in this article, we select two digit as the two classes, e.g., those with labels 0 and 4. Following the proposed deformation model (2), we randomly select one or more samples from each class and apply random deformations using the Keras ImageDataGenerator. The transformed images are randomly shifted (corresponding to $\tau, \tau'$ in (2)), randomly scaled (corresponding to $\xi, \xi'$ in (2)) and randomly lightened or darkened (corresponding to $\eta$ in (2)). Samples are shown in Figure 7. We consider three different classification task. In the first two tasks we select one sample from each of the two classes and generate the full dataset through random deformations, as described above. In Task 1, we classify the digits 0 and 4 and in Task 2, the digits 1 and 7. Note that, discriminating between 1 and 7 is slightly more challenging due to the similarity of the two digits.

For Task 3, we consider a generalization of our image deformation model. In continental Europe the digit 7 is written as 7 with an additional middle line. To incorporate different variations of the same digit class, a natural extension of the image deformation model (2) is to assume that every sample in a class is represented by a random deformation of one out of multiple template functions. In the binary classification case, we have
Figure 7: Augmented versions of number 1 (first two rows) and number 7 (last two rows) of the MNIST data set

$m_0$ template functions for label 0, denoted by $f_{0,1}, \ldots, f_{0,m_0}$, and $m_1$ template functions for label 1, denoted by $f_{1,1}, \ldots, f_{1,m_1}$. Extending (2), each image is then a $d \times d$ matrix $X_i = (X_{j,\ell}^{(i)})_{j,\ell=1,\ldots,d}$ with entries

$$X_{j,\ell}^{(i)} = \eta_i f_{k_i,r_i} \left( \xi_i \frac{j}{d} - \tau_i, \xi_i' \frac{\ell}{d} - \tau_i' \right),$$

(31)

for $r_i \in \{0, \ldots, m_{k_i}\}$.

In Task 3 we sample from this model with $m_0 = m_1 = 10$, meaning that we draw 10 sample images each of the digit 0 and 4, respectively and apply random deformations using the Keras ImageDataGenerator. Although this model is beyond the developed theory, we still expect that the classifiers should have similar misclassification behavior.

For each of the three tasks all classifiers are trained with $n$ labelled sample images and balanced design, that is, we observe $n/2$ images from each of the two classes. The derived theory suggests that small sample sizes will lead already to negligible (or even vanishing) classification error. As this is the interesting regime for comparison of the methods, we study sample sizes $n \in \{2, 4, 8, 16, 32, 64\}$. The performance of each classifier $\hat{k}$ is evaluated by the empirical misclassification risk (test error)

$$R_N = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}(\hat{k}(X_{n+i}) \neq k_{n+i}),$$

based on test data $(X_{n+1}, k_{n+1}), \ldots, (X_N, k_N)$, that are independently generated from the same distribution.
as the training data. For $N = 100$, Figure 8 reports the median of the test error based on 30 repetitions in each setting.

![Graphs showing misclassification error for different tasks](image)

**Figure 8:** Comparison of trained CNN (orange) and image alignment classifier (IAC) in (20) (blue). Reported is the median of the test error over 30 repetitions for varying size of training sample and the three tasks described in the main text.

As indicated by the theory, the image alignment classifier (20) clearly outperforms the CNN classifiers for all three tasks. With only two training samples in Task 1, the test error of this classifier is already close to zero. By construction, Task 1 to Task 3 are gradually more challenging and lead to increased misclassification errors for both methods. The performance of the CNN classifiers improves as the sample size grows, but the misclassification error remains still large. This is expected as the CNN needs to learn the invariances due to the random deformations.

In Task 3, the training data are generated from random deformations of ten different template images of the digits 0 and 4. The image alignment classifier is more prone to misclassifying a test image if it is a random deformation of a template image that did not occur in the training set. Once the image alignment classifier has seen a randomly deformed sample from each of the ten different template images, it is able to classify the test data perfectly. Interestingly, the performance of the CNN classifiers in Task 3 only deteriorates slightly compared to the previous two tasks. This might be attributed to the fact that the image alignment classifier heavily exploits the specific structure of the image deformation model (2), while CNNs are extremely flexible and can adapt to various structures in the data. While $\text{CNN}^2$ performs best across all three tasks, Figure 8 shows that the choice of the CNN architecture has only a minor influence on the misclassification error.

Based on the empirical results, an interesting open question is to combine both methods.

### 6 Conclusion and extensions

This paper introduces a novel statistical framework for image classification. Instead of treating each pixel as a variable and analysing a nonparametric denoising problem where randomness occurs as additive noise, the proposed deformation framework models the variability of objects within one class as geometric deformations
of template images. The abstract framework includes various linear and nonlinear deformations, including rotations, shifts, and rescaling.

Extensions to background noise and multiple objects have briefly been discussed in Section 3.2. Another relevant extension is the case of partially visible objects. In this scenario relying on global characteristics, such as the full support of the object, seems unreasonable. However, it might still be possible to construct classifiers, that provide similar theoretical guarantees by focusing on local properties instead.

Another potential extension is to incorporate perspective transformations from the computer vision literature [20, 48]. The underlying idea is that images captured from different perspectives can be modeled as

$$\begin{pmatrix}
    \tilde{a}_1(u, v) \\
    \tilde{a}_2(u, v) \\
    w(u, v)
\end{pmatrix} =
\begin{pmatrix}
    h_{11} & h_{12} & h_{13} \\
    h_{21} & h_{22} & h_{23} \\
    h_{31} & h_{32} & h_{33}
\end{pmatrix}
\begin{pmatrix}
    u \\
    v \\
    1
\end{pmatrix},$$

where $h_{ij}$ are the parameters of the non-singular homography matrix and $w$ is the so-called scaling factor. In this framework, $a_1(u, v)$ and $a_2(u, v)$ are obtained by normalizing the output by $w$, namely $a_1(u, v) = \tilde{a}_1(u, v)/w(u, v)$ and $a_2(u, v) = \tilde{a}_2(u, v)/w(u, v)$. Affine transformations can be recovered as a special case by choosing $h_{31} = h_{32} = 0$ and $h_{33} = 1$. To include perspective transformations one needs to relax the partial differentiability imposed in Assumption 2.

[31] and Section 5 in [32] discuss further deformation classes, including those that account for noise and blur, multi-scale superposition, domain warping and interruptions.

Furthermore, various sophisticated image deformation models have been proposed for medical image registration. Given a target image $T$ and a source image $S$, the goal of image registration is to find the transformation from $T$ to $S$ that optimizes (in a suitable sense) the alignment between $T$ and $S$. To study and compare image registration methods, it is essential to construct realistic image deformation models describing the generation of the deformed image $T$ from the template $S$. The survey article [40] classifies these deformation models into several categories, such as ODE/PDE based models, interpolation-based models and knowledge-based models. For instance, a simple ODE based random image deformation model takes $X$ as template/source image and generates a random vector field $u$. This can be achieved by selecting a basis and generating independent random coefficients according to a fixed distribution. Given the vector field $u$, a continuous image deformation $X(t)$ is generated by solving the differential equation $\partial_t X(t) = u(X(t))$ with $X(0) = X$. The randomly deformed image is then $X(1)$. The DARTEL algorithm [2] is a widely recognized approach for image registration within this deformation model. However, a statistical analysis of these methods is still lacking.

To conclude, we emphasize that for enhancing our understanding of image classification methods, such as those based on CNNs, it is crucial to extend the statistical analysis in this work to more general image deformation models. It seems natural to follow a similar program as outlined in this work: Starting with formalizing deformation classes in a statistical model, then designing suitable methods and analyzing how
widely applied procedures such as deep learning perform. While various image deformation models have been proposed in the computer science and pattern recognition literature, analyzing them theoretically from a statistical perspective is challenging. However, addressing these challenges could lead to more effective image classifiers. The interplay between data models and algorithms is in line with Grenander’s claim that pattern analysis and statistical modelling are inseparable problems [15, 32].

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A Proofs for Section 3

A.1 Proofs for general deformation model

Throughout this section, assume that \( f \) is one of the template functions \( f_0, f_1 \).

**Lemma A.1.** If the function \( f \) satisfies Assumption 1 and \( A \in \mathcal{A} \) satisfies Assumption 2, then, \( f \circ A \) is Lipschitz continuous in the sense that for any \( (u, v), (u', v') \in \mathbb{R}^2 \),

\[
|f \circ A(u, v) - f \circ A(u', v')| \leq 2C_A C_L \|f\|_1 (|u - u'| + |v - v'|).
\]

**Proof.** Note that Assumption 2 implies that for any \( (u, v), (u', v') \in \mathbb{R}^2 \) and \( k = 1, 2 \),

\[
|a_k(u, v) - a_k(u', v')| \leq C_A (|u - u'| + |v - v'|).
\]

Together with the Lipschitz continuity of \( f \) this implies that for any \( (u, v), (u', v') \in \mathbb{R}^2 \),

\[
|f \circ A(u, v) - f \circ A(u', v')| = |f(a_1(u, v), a_2(u, v)) - f(a_1(u', v'), a_2(u', v'))| \\
\leq C_L \|f\|_1 (|a_1(u, v) - a_1(u', v')| + |a_2(u, v) - a_2(u', v')|) \\
\leq C_L \|f\|_1 2C_A (|u - u'| + |v - v'|) \\
= 2C_A C_L \|f\|_1 (|u - u'| + |v - v'|).
\]

\( \square \)

**Proposition A.2.** Given Assumptions 1 and 2, let \( A_d^{-1} \) be a covering of \( A^{-1} \) with balls of radius \( 1/d \) satisfying (7). Then, for any \( A^{-1} \in A^{-1} \), there exists \( B_* \in A_d^{-1} \) such that, for any \( (u, v) \in [0, 1]^2 \),

\[
|X(A^{-1}(u, v)) - X(B_*(u, v))| \leq 12\eta C_A C_L \|f\|_1 \frac{1}{d}.
\]
Proof. Write $A^{-1} = (b_1, b_2)$ and $B_* = (b_1^*, b_2^*)$. For any $(u, v) \in [0, 1]^2$, there exist integers $j, \ell$ such that $A^{-1}(u, v) \in I_{j, \ell}$ and integers $j', \ell'$ such that $B_*(u, v) \in I_{j', \ell'}$.

We first deal with the case where both $I_{j, \ell}$ and $I_{j', \ell'}$ are contained in $[0, 1]^2$. By the definition of $A^{-1}$ and the condition (7), we know that for any $A^{-1} \in A^{-1}$, there exists $B_* \in A_*^{-1}$ such that for any $(u, v) \in [0, 1]^2$,

$$|b_1(u, v) - b_1^*(u, v)| \leq \frac{1}{d} \quad \text{and} \quad |b_2(u, v) - b_2^*(u, v)| \leq \frac{1}{d},$$

which implies that

$$|j - j'| \leq 2 \quad \text{and} \quad |\ell - \ell'| \leq 2. \quad (32)$$

As a consequence of (32), for any $(x, y) \in I_{j, \ell}$ and any $(x', y') \in I_{j', \ell'}$,

$$|x - x'| \leq \frac{|j - j'| + 1}{d} \leq \frac{3}{d}, \quad |y - y'| \leq \frac{|\ell - \ell'| + 1}{d} \leq \frac{3}{d}. \quad (33)$$

Under Assumptions 1 and 2, we know from Lemma A.1 that $f \circ A$ is Lipschitz continuous. With Lemma A.1 and (33), we can derive that for any $(x, y) \in I_{j, \ell}$ and any $(x', y') \in I_{j', \ell'}$,

$$|f(A(x, y)) - f(A(x', y'))| \leq 2C_A C_L \|f\|_1 (|x - x'| + |y - y'|) \leq 12C_A C_L \|f\|_1 \frac{1}{d}. \quad (34)$$

Therefore, under Assumptions 1 and 2, for any $(u, v) \in [0, 1]^2$, using (34),

$$|X(A^{-1}(u, v)) - X(B_*(u, v))| = \eta \left| d^2 \int_{I_{j, \ell}} f(A(x, y)) dx dy - d^2 \int_{I_{j', \ell'}} f(A(x, y)) dx dy \right| \leq 12\eta C_A C_L \|f\|_1 \frac{1}{d}.$$

This proves the result in this case.

Next, we handle the case where neither $I_{j, \ell}$ nor $I_{j', \ell'}$ is contained in $[0, 1]^2$. According to the definition of $X$ in (6), we have

$$|X(A^{-1}(u, v)) - X(B_*(u, v))| = 0,$$

which satisfies the conclusion.

Finally, we consider the case where $I_{j, \ell} \subseteq [0, 1]^2$ but $I_{j', \ell'}$ is not contained in $[0, 1]^2$. If $I_{j', \ell'} \subseteq [0, 1]^2$ but $I_{j, \ell}$ is not contained in $[0, 1]^2$, the proof is the same and therefore omitted. Under Assumption 2, the image is fully visible hence $\int_{I_{j', \ell'}} f(A(x, y)) dx dy = 0$, if $I_{j', \ell'}$ is not contained in $[0, 1]^2$. As Assumptions 1, 2 hold, we similarly derive using Lemma A.1 and (7) that

$$|X(A^{-1}(u, v)) - X(B_*(u, v))| = \eta \left| d^2 \int_{I_{j, \ell}} f(A(x, y)) dx dy - d^2 \int_{I_{j', \ell'}} f(A(x, y)) dx dy \right| \leq 12\eta C_A C_L \|f\|_1 \frac{1}{d},$$

proving the claim also in this case.
Lemma A.3. Consider a generic image of the form (3) and suppose Assumptions 1 and 2 hold. Let $A_d^{-1}$ be a covering of $A^-$ satisfying the condition (7). Then, there exists an inverse mapping $B_* \in A_d^{-1}$ and a universal constant $K > 0,$ such that
\[
\left\| T_{x \circ B_*} - \frac{f}{\|f\|_2} \right\|_2 \leq K \max \{C_A^2 C_L^2, C_A C_L \} \frac{1}{d}.
\]

Proof. In a first step of the proof, we show that there exists an inverse mapping $B_* \in A_d^{-1}$ such that
\[
\left| X(B_*(u,v)) - \eta f(u,v) \right| \leq 16 \eta C_A C_L \|f\|_1 \frac{1}{d}, \quad \text{for all } (u,v) \in [0,1]^2,
\]
where $X(B_*(u,v))$ is as defined in (8).

Recall that $X(u,v)$ is defined as in (6). For any $(u,v) \in [\beta_{\text{left}}, \beta_{\text{right}}] \times [\beta_{\text{down}}, \beta_{\text{up}}]$, we denote $(u_0,v_0) = A^{-1}(u,v)$, representing the corresponding point in the deformed function $f(A(\cdot, \cdot))$. Under Assumption 2, we have $(u_0,v_0) \in [0,1]^2$, which implies that either there exist $j, \ell \in \{1, \ldots, d\}$ such that $(u_0,v_0) \in I_{j,\ell}$ or $(u_0,v_0)$ belongs to the upper or right boundaries of the square $[0,1]^2$. For the latter, the value of $f(A(u_0,v_0))$ must be zero; otherwise, this contradicts Assumption 2, which ensures that the image is fully visible, as $f \circ A$ is Lipschitz continuous according to Lemma A.1. Therefore, given the definition of $X$ in (6), one can simply obtain
\[
\left| X(A^{-1}(u,v)) - \eta f(u,v) \right| = \left| X(u_0,v_0) - \eta f(A(u_0,v_0)) \right| = 0.
\]
We now focus on the situation, where there exist $j, \ell \in \{1, \ldots, d\}$ such that $(u_0,v_0) \in I_{j,\ell}$. With the definition of $X$ in (6), we then compute that
\[
\left| X(A^{-1}(u,v)) - \eta f(u,v) \right| = \left| X(u_0,v_0) - \eta f(u,v) \right| = \eta \left\| d^2 \int_{I_{j,\ell}} f(A(x,y)) \, dx \, dy - f(A(u_0,v_0)) \right\|_1.
\]
For any $(x,y) \in I_{j,\ell}$, due to the Lipschitz continuity of $f$ and $A$, under Assumption 1 and 2, we obtain by applying Lemma A.1 that
\[
\left| f(A(x,y)) - f(A(u_0,v_0)) \right| \leq 2C_A C_L \|f\|_1 (|x-u_0| + |y-v_0|) \leq 4C_A C_L \|f\|_1 \frac{1}{d}.
\]
With (38), we deduce from (37) that
\[
\left| X(A^{-1}(u,v)) - \eta f(u,v) \right| = \eta \left\| d^2 \int_{I_{j,\ell}} f(A(x,y)) \, dx \, dy - f(A(u_0,v_0)) \right\|_1 \leq 4 \eta C_A C_L \|f\|_1 \frac{1}{d}.
\]
Putting (36) and (39) together, we obtain that for all $(u,v) \in [\beta_{\text{left}}, \beta_{\text{right}}] \times [\beta_{\text{down}}, \beta_{\text{up}}]$,
\[
\left| X(A^{-1}(u,v)) - \eta f(u,v) \right| \leq 4 \eta C_A C_L \|f\|_1 \frac{1}{d}.
\]
Now we consider the points outside the support of $f$, namely $(u, v) \in [0, 1]^2 \setminus [\beta_{\text{left}}, \beta_{\text{right}}] \times [\beta_{\text{down}}, \beta_{\text{up}}]$. There are two possible situations. If $(u_0, v_0) = A^{-1}(u, v) \in [0, 1]^2$, then we proceed similarly as above and obtain
\begin{equation}
|X(A^{-1}(u, v)) - \eta f(u, v)| \leq 4\eta C_A C_L \|f\|_1 \frac{1}{d}.
\end{equation}
If $(u_0, v_0) = A^{-1}(u, v) \notin [0, 1]^2$, according to the definition of $X$ as in (6), we have $X(A^{-1}(u, v)) = 0$. By Assumption 1, we have $f(u, v) = 0$ in this case and can deduce
\begin{equation}
|X(A^{-1}(u, v)) - \eta f(u, v)| = 0.
\end{equation}
Combining (40), (41) and (42), we conclude that for any $(u, v) \in [0, 1]^2$,
\begin{equation}
|X(A^{-1}(u, v)) - \eta f(u, v)| \leq 4\eta C_A C_L \|f\|_1 \frac{1}{d}.
\end{equation}
Choosing $B_* \in A^{-1}_d$ which satisfying condition (7), we derive with (43) and Proposition A.2 that for any $(u, v) \in [0, 1]^2$,
\begin{equation}
|X(B_*(u, v)) - \eta f(u, v)| \leq |X(B_*(u, v)) - X(A^{-1}(u, v))| + |X(A^{-1}(u, v)) - \eta f(u, v)|
\leq |X(B_*(u, v)) - X(A^{-1}(u, v))| + 4\eta C_A C_L \|f\|_1 \frac{1}{d}
\leq 16\eta C_A C_L \|f\|_1 \frac{1}{d}.
\end{equation}
In the next step, we show that
\begin{equation}
\|X \circ B_*\|_2 - \eta \|f\|_2 \leq 288\eta \max\{C_A^2 C_L^2, C_A C_L\} \|f\|_1 \frac{1}{d}.
\end{equation}
Using that for real numbers $a, b, a - b = (a^2 - b^2)/(a + b)$, we rewrite
\begin{equation}
\|X \circ B_*\|_2 - \eta \|f\|_2 = \|X \circ B_*\|_2^2 - \eta^2 \|f\|_2^2 \frac{1}{\|X \circ B_*\|_2 + \eta \|f\|_2}
\leq \|X \circ B_*\|_2^2 - \eta^2 \|f\|_2^2 \frac{1}{\eta \|f\|_2}.
\end{equation}
Using (35), we bound the first term in (46) by
\begin{align*}
&\|X \circ B_*\|_2^2 - \eta^2 \|f\|_2^2 \\
&= \int_0^1 \int_0^1 (X(B_*(u, v)) - \eta f(u, v))^2 dudv - \eta^2 \|f\|_2^2 \\
&= \int_0^1 \int_0^1 (X(B_*(u, v)) - \eta f(u, v))^2 dudv + 2\eta \int_0^1 \int_0^1 (X(B_*(u, v)) - \eta f(u, v)) f(u, v)dudv \\
&\quad + \int_0^1 \int_0^1 \eta^2 f^2(u, v)dudv - \eta^2 \|f\|_2^2 \\
&\leq \int_0^1 \int_0^1 (X(B_*(u, v)) - \eta f(u, v))^2 dudv + 2\eta \int_0^1 \int_0^1 X(B_*(u, v)) - \eta f(u, v) |f(u, v)| dudv \\
&\leq \int_0^1 \int_0^1 16\eta^2 C_A^2 C_L^2 \|f\|_1^2 \frac{1}{d^2} dudv + 32\eta^2 \int_0^1 \int_0^1 C_A C_L \|f\|_1 \frac{1}{d} |f(u, v)| dudv
\end{align*}
and hence the case \( \parallel \).

Getting (45) to the first and (35) to the second term and using again proving (45).

By the Cauchy-Schwarz inequality, \( \parallel f \parallel_1 \leq \parallel f \parallel_2 \). Summarizing, (46) is bounded by

\[
\parallel X \circ B_\ast \parallel_2 - \parallel f \parallel_2 \parallel 288 \eta^2 \max\{C_A^2, C_A C_L\} \parallel f \parallel_2 \frac{1}{d} \frac{1}{\parallel f \parallel_2}
\]

proving (45).

We now finish the proof. Using \( T_{X \circ B_\ast} = X \circ B_\ast / \parallel X \circ B_\ast \parallel_2 \) and \( (a + b)^2 \leq 2a^2 + 2b^2 \) for arbitrary real numbers \( a, b \), we bound

\[
\parallel T_{X \circ B_\ast} - \frac{f}{\parallel f \parallel_2} \parallel_2 \leq \parallel T_{X \circ B_\ast} - \frac{X \circ B_\ast}{\parallel f \parallel_2} \parallel_2 + 2 \parallel X \circ B_\ast - \frac{\eta f}{\parallel f \parallel_2} \parallel_2^2 \leq \frac{2}{\parallel f \parallel_2} \parallel f \parallel_2 - \parallel X \circ B_\ast \parallel_2 + \frac{2}{\parallel f \parallel_2} \parallel X \circ B_\ast - \eta f \parallel_2^2.
\]

Applying (45) to the first and (35) to the second term and using again \( \parallel f \parallel_1 \leq \parallel f \parallel_2 \), it follows

\[
\parallel T_{X \circ B_\ast} - \frac{f}{\parallel f \parallel_2} \parallel_2^2 \leq \frac{2}{\parallel f \parallel_2} \frac{288 \eta^2 \max\{C_A^4, C_A^2 C_L^2\} \parallel f \parallel_2 \frac{1}{d^2}} + \frac{2}{\parallel f \parallel_2} \frac{16 \eta^2 C_A^2 \parallel f \parallel_2 \frac{1}{d^2}} \leq K \frac{\max\{C_A^4, C_A^2 C_L^2\} \frac{1}{d^2}}
\]

for a universal constant \( K > 0 \).

**Proof of Theorem 3.1.** The lower bound (48) is symmetric in \( f_0 \) and \( f_1 \). Therefore it is enough to consider the case \( k = 0 \). Thus, the entries of \( X \) are described by the template function \( f_0 : \mathbb{R}^2 \rightarrow \mathbb{R} \). Recall that the image \( X \) is generated through the deformed function \( f_0 \circ A \).

If \( k_i = 0 \), it follows from the triangle inequality and by Lemma A.3 that there exist \( B_\ast^i \in A_0^{-1} \) satisfying

\[
\parallel T_{X \circ B_\ast^i} - T_{X \circ B_\ast} \parallel_2 = \parallel T_{X \circ B_\ast^i} - \frac{f_0}{\parallel f_0 \parallel_2} \parallel_2 + \frac{f_0}{\parallel f_0 \parallel_2} \parallel_2 \leq \parallel T_{X \circ B_\ast^i} - \frac{f_0}{\parallel f_0 \parallel_2} \parallel_2 + \parallel T_{X \circ B_\ast} - \frac{f_0}{\parallel f_0 \parallel_2} \parallel_2 \leq 2K \frac{\max\{C_A^4, C_A^2 C_L^2\} \frac{1}{d^2}}
\]

By the definition of the separation constant \( D \) in (12), for any \( a, b > 0 \),

\[
\parallel a f_1 - b f_0 \parallel_{L^2(\mathbb{R}^2)} = b \left\| \frac{a}{b} f_1 - f_0 \right\|_{L^2(\mathbb{R}^2)} \geq b \parallel f_0 \parallel_{L^2(\mathbb{R}^2)} D(f_1, f_0) = b \parallel f_0 \parallel_{2} D(f_1, f_0)
\]

and hence

\[
\parallel \frac{f_1}{\parallel f_1 \parallel_2} - \frac{f_0}{\parallel f_0 \parallel_2} \parallel_2 = \parallel \frac{f_1}{\parallel f_1 \parallel_2} - \frac{f_0}{\parallel f_0 \parallel_2} \parallel_{L^2(\mathbb{R}^2)}
\]

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where we used the assumption $D > 4K \max\{C_A^2C_L^2, C_AC_L\} / d$ for the last step. For the index $i$ with $k_i = 1$, we use the reverse triangle inequality

\[ |a' - b'| \geq |a - b| - |a' - a| - |b' - b|, \text{ for all } a, b, a', b' \in \mathbb{R}, \]

which together with (49) and Lemma A.3 yields for any $e$

Proof. Let $A$ be the class of affine transformations on $\mathbb{R}^2$, where each $A \in A$ is defined as in (4) with parameters $(b_1, \ldots, b_4, \tau, \tau')$ satisfying $|b_1|, \ldots, |b_4| \leq C_A, |\tau|, |\tau'| \leq \ell_s$, and $|\det(B)| \geq \alpha$, for positive constants $C_A, \ell_s, \alpha > 0$. If we further constraint the class such that $p$ parameters out of the six parameters $b_1, \ldots, b_4, \tau, \tau'$ can vary while the remaining parameters remain constant, then there exists a $1/d$-covering $A_{d}^{-1}$ of $A^{-1}$ with cardinality $|A_{d}^{-1}| \asymp d^p$.

Lemma A.4. Let $A$ be the class of affine transformations on $\mathbb{R}^2$, where each $A \in A$ is defined as in (4) with parameters $(b_1, \ldots, b_4, \tau, \tau')$ satisfying $|b_1|, \ldots, |b_4| \leq C_A, |\tau|, |\tau'| \leq \ell_s$, and $|\det(B)| \geq \alpha$, for positive constants $C_A, \ell_s, \alpha > 0$. If we further constraint the class such that $p$ parameters out of the six parameters $b_1, \ldots, b_4, \tau, \tau'$ can vary while the remaining parameters remain constant, then there exists a $1/d$-covering $A_{d}^{-1}$ of $A^{-1}$ with cardinality $|A_{d}^{-1}| \asymp d^p$.

Proof. Fix a deformation $A \in A$ with parameters $b_1, \ldots, b_4, \tau, \tau'$. If any of the parameters $b_i, \tau,$ or $\tau'$ is fixed, we take $\tilde{b}_i = b_i, \tilde{\tau} = \tau$ and $\tilde{\tau}' = \tau'$. Otherwise, we consider the perturbed parameters $\tilde{b}_1, \ldots, \tilde{b}_4, \tilde{\tau}, \tilde{\tau}'$ such that

\[ \max_{i=1,\ldots,4} |b_i - \tilde{b}_i| \leq \frac{C_b}{d}, \quad |\tau - \tilde{\tau}| \lor |\tau' - \tilde{\tau}'| \leq \frac{C_s}{d}, \quad \det(B) \leq \det(\tilde{B}), \]

and define the perturbed matrices

\[ \tilde{B} = \begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 \\ \tilde{b}_3 & \tilde{b}_4 \end{pmatrix} \quad \text{and} \quad \tilde{B}^{-1} = \frac{1}{\det(\tilde{B})} \begin{pmatrix} \tilde{b}_4 & -\tilde{b}_2 \\ -\tilde{b}_3 & \tilde{b}_1 \end{pmatrix}. \]

Then, the inverse of the perturbed affine transformation is given by

\[ \tilde{A}^{-1}(u, v) = (\tilde{a}_1^{-1}(u, v), \tilde{a}_2^{-1}(u, v)) = \tilde{B}^{-1}((u, v)^T + \tilde{\tau}). \]
It follows that for any \((u, v) \in [0, 1]^2\),
\[
|a_1^{-1}(u, v) - \bar{a}_1^{-1}(u, v)|
\]
\[
= \left| \frac{1}{\det(B)} \left[ b_4(u + \tau) - b_2(v + \tau') \right] \right| - \frac{1}{\det(B)} \left[ \bar{b}_4(u + \bar{\tau}) - \bar{b}_2(v + \bar{\tau}') \right] \right|
\]
\[
\leq \frac{1}{\det(B)} \left| b_4(u + \tau) - b_2(v + \tau') - \bar{b}_4(u + \bar{\tau}) + \bar{b}_2(v + \bar{\tau}') \right| + \left( \frac{1}{\det(B)} - \frac{1}{\det(B)} \right) \cdot |\bar{b}_4(u + \bar{\tau}) - \bar{b}_2(v + \bar{\tau}')|
\]
\[
\leq \frac{2}{\alpha} \left( 1 + \ell_s \frac{C_b}{d} + (C_b + C_A) \frac{C_s}{d} \right) + 2 \left( \frac{1}{\det(B)} - \frac{1}{\det(B)} \right) (C_b + C_A)(1 + C_s + \ell_s).
\]

For the chosen \(\bar{b}_i, i \in \{1, \ldots, 4\},\)
\[
\left| \frac{1}{\det(B)} - \frac{1}{\det(B)} \right| \leq \frac{1}{\alpha^2} \left| \det(B) - \det(B) \right| \leq \frac{2}{\alpha^2} \left( 2C_A + C_b \right) \frac{C_b}{d},
\]
which implies that for any \((u, v) \in [0, 1]^2\),
\[
|a_1^{-1}(u, v) - \bar{a}_1^{-1}(u, v)| \leq \frac{C(\alpha, \ell_s, C_A, C_b, C_s)}{d},
\]
where \(C(\alpha, \ell_s, C_A, C_b, C_s)\) is a numerical constant depending only on \(\alpha, \ell_s, C_A, C_b, C_s\). For any \((u, v) \in [0, 1]^2\), one can similarly derive
\[
|a_2^{-1}(u, v) - \bar{a}_2^{-1}(u, v)| \leq \frac{C(\alpha, \ell_s, C_A, C_b, C_s)}{d},
\]
which together with (50) gives
\[
\|A^{-1} - \bar{A}^{-1}\|_\infty \leq \frac{C(\alpha, \ell_s, C_A, C_b, C_s)}{d}.
\]

With suitable chosen constants \(C_b, C_s\), condition (7) holds and \(|A^{-1}_d| \approx d^\ell\).

Proof of Lemma 3.2. The first claim is a special case of Lemma 3.3. The bound on the covering number follows from Lemma A.4.

Proof of Lemma 3.3. The inverse of the rotation matrix
\[
D_\gamma := \begin{pmatrix}
\cos \gamma & -\sin \gamma \\
\sin \gamma & \cos \gamma
\end{pmatrix}
\]
is \(D_{-\gamma}\). The conditions on the parameter now ensure that \([-(-\xi_+), \xi_+] \times [-(\xi')_+, \xi'_+] \supseteq D_{-\gamma}[1/4, 3/4]^2 - (\tau, \tau')^\top\). To see this, it is enough to check the four vertices of \([1/4, 3/4]^2\).

Now we examine the partial differentiability condition in Assumption 2. Consider any transformation \(A = (a_1, a_2) \in A\) with parameters \(b_1, b_2, b_3, b_4, \tau, \tau'\). Observe that \(a_1(u, v) = b_1 u + b_2 v - \tau\) and \(a_2(u, v) = b_3 u + b_4 v - \tau'\) are differentiable at any \(u \in \mathbb{R}\) with \(|\partial_u a_1(u, v)| \leq |b_1|, |\partial_v a_1(u, v)| \leq |b_2|, |\partial_u a_2(u, v)| \leq |b_3|,\) and \(|\partial_v a_2(u, v)| \leq |b_4|\). Since \(|b_1|, \ldots, |b_4| \leq C_A\), Assumption 2 is satisfied with \(C_A\).

The bound on the covering number can be derived similarly as the one in Lemma A.4, based on the perturbation of the five parameters \(\gamma, \xi, \xi', \tau, \tau'\). This yields \(|A_d^{-1}| \approx d^{\ell}\).
Proof of Lemma 3.4. For \( \lambda \neq 0 \), \( A \) is invertible and \( A^{-1}(u, v) = (u - \alpha \sin(2\pi v/\lambda), v) \). Moreover, if \( |\alpha| \leq \beta_{\text{left}} \wedge (1 - \beta_{\text{right}}) \) and \( \lambda \neq 0 \), then for any point \((u, v) \in [\beta_{\text{left}}, \beta_{\text{right}}] \times [\beta_{\text{down}}, \beta_{\text{up}}] \subseteq [0, 1]^2\),

\[
u - \alpha \sin(2\pi v/\lambda) \leq u + |\alpha| \leq 1 \quad \text{and} \quad \nu - \alpha \sin(2\pi v/\lambda) \geq u - |\alpha| \geq 0,
\]

which implies the full visibility condition in Assumption 2. Consider a fixed \( A \in \mathcal{A} \) associated with the parameters \( \alpha \) and \( \lambda \). Then, \( a_2(\cdot, \cdot) \) satisfies Assumption 2 with constant 1. Moreover, under Assumption 2 with a bounded value of \( \alpha \) and \( |\lambda| \geq C_{\text{lower}} \), the function \( a_1(\cdot, \cdot) \) is partially differentiable at any \( u \in \mathbb{R} \) with

\[
|\frac{\partial a_1}{\partial u}| \leq 1,
\]

and is partially differentiable at any \( v \in \mathbb{R} \) with

\[
|\frac{\partial a_1}{\partial v}| = \left| \frac{2\pi}{\lambda} \cos\left(\frac{2\pi v}{\lambda}\right) \right| \leq \left[ \beta_{\text{left}} \wedge (1 - \beta_{\text{right}}) \right] \frac{2\pi}{C_{\text{lower}}},
\]

which completes the argument.

For any \( A \in \mathcal{A} \), let \( \lambda_* \) and \( \alpha_* \) be the true parameter with \( |\lambda_*| \geq C_{\text{lower}} \) and \( |\alpha_*| \leq \beta_{\text{left}} \wedge (1 - \beta_{\text{right}}) \). Taking \( \bar{\alpha} \) and \( \bar{\lambda} \) such that

\[
|\bar{\alpha} - \alpha_*| \leq \frac{C_{\alpha}}{d} \quad \text{and} \quad |\bar{\alpha}| \leq |\alpha_*|,
\]

\[
|\bar{\lambda} - \lambda_*| \leq \frac{C_{\lambda}}{d} \quad \text{and} \quad |\bar{\lambda}| \geq |\lambda_*|,
\]

one can derive for \( \tilde{A}^{-1}(u,v) = (\bar{a}_1(u,v), \bar{a}_2(u,v)) = (u - \bar{\alpha} \sin\left(\frac{2\pi v}{\bar{\lambda}}\right), v) \) with \((u,v) \in [0,1]^2\),

\[
|\bar{a}_1(u,v) - a_1(u,v)| = \left|\nu - \bar{\alpha} \sin\left(\frac{2\pi v}{\bar{\lambda}}\right) - u + \alpha_* \sin\left(\frac{2\pi v}{\lambda_*}\right)\right|
\]

\[
\leq \alpha_* \sin\left(\frac{2\pi v}{\lambda_*}\right) - \alpha_* \sin\left(\frac{2\pi v}{\lambda}\right) + \left|\alpha_* \sin\left(\frac{2\pi v}{\lambda}\right) - \bar{\alpha} \sin\left(\frac{2\pi v}{\bar{\lambda}}\right)\right|
\]

\[
\leq |\alpha_*| \left|\sin\left(\frac{2\pi v}{\lambda_*}\right) - \sin\left(\frac{2\pi v}{\lambda}\right)\right| + |\alpha_* - \bar{\alpha}|
\]

\[
\leq 2\pi |\alpha_*| \left|\frac{1}{\lambda_*} - \frac{1}{\lambda}\right| + |\alpha_* - \bar{\alpha}|
\]

\[
\leq 2\pi \left|\beta_{\text{left}} \wedge (1 - \beta_{\text{right}})\right| \frac{C_{\lambda}}{d C_{\text{lower}}} + \frac{C_{\alpha}}{d}
\]

\[
\leq \frac{C(\beta_{\text{left}}, \beta_{\text{right}}, C_{\text{lower}}, C_{\lambda}, C_{\alpha})}{d}.
\]

where \( C(\beta_{\text{left}}, \beta_{\text{right}}, C_{\text{lower}}, C_{\lambda}, C_{\alpha}) = 2\pi \left|\beta_{\text{left}} \wedge (1 - \beta_{\text{right}})\right| C_{\lambda}/C_{\text{lower}}^2 + C_{\alpha} \). This implies that \( \mathcal{A}_d^{-1} \) can be constructed by discretizing the parameters \( \lambda \) and \( \alpha \), namely \( |\mathcal{A}_d^{-1}| \propto d^2 \).

Proof of Lemma 3.5. For any \( A \in \mathcal{A}_2 \circ \mathcal{A}_1 \), according to the definition of \( \mathcal{A}_2 \circ \mathcal{A}_1 \), there exist \( A_1 \in \mathcal{A}_1 \) and \( A_2 \in \mathcal{A}_2 \) such that \( A = A_2 \circ A_1 \). For any real numbers \( u, v \), denote \( A_1(u,v) = (a_1(u,v), b_1(u,v)) \) and \( A_2(u,v) = (a_2(u,v), b_2(u,v)) \). Consequently, by writing \( A(u,v) = (a(u,v), b(u,v)) \), we have \( a(u,v) = a_2(a_1(u,v), b_1(u,v)) \) and \( b(u,v) = b_2(a_1(u,v), b_1(u,v)) \). If both \( A_1 \) and \( A_2 \) satisfy Assumption 2, we can differentiate the composite function at any \( u \in \mathbb{R} \) by the chain rule and derive

\[
|\frac{\partial a}{\partial u}| = \left| \frac{\partial a_2}{\partial a_1} \cdot \frac{\partial a_1}{\partial u} + \frac{\partial a_2}{\partial b_1} \cdot \frac{\partial b_1}{\partial u} \right| \leq 2C_{A_1}C_{A_2}.
\]
Similarly, we can show at any \( v \in \mathbb{R} \),
\[
\left| \frac{\partial a}{\partial v} \right| \leq 2C_{A_1}C_{A_2},
\]
which completes the proof.

### A.2 Proofs for classification via image alignment

We will frequently use the following notation. For any given function \( f \), denote
\[
\alpha_f := \arg \max \{ u : f(t, v) = 0 \text{ for all } t \leq u \}, \quad \alpha_f^+ := \arg \min \{ u : f(t, v) = 0 \text{ for all } t \geq u \},
\]
\[
\beta_f := \arg \max \{ v : f(u, t) = 0 \text{ for all } t \leq v \}, \quad \beta_f^+ := \arg \min \{ v : f(u, t) = 0 \text{ for all } t \geq v \}.
\]
The rectangular support of \( f \) is then given by \([\alpha_f^-, \alpha_f^+] \times [\beta_f^-, \beta_f^+]\).

**Lemma A.5.** Let \( j_\pm, \ell_\pm \) be as defined in (16) and (17). If \( f \) is a continuous function, then
\[
\alpha_f^- < \xi j_- - \tau \leq \alpha_f^+ \leq \xi j_+ + \frac{\xi}{d},
\]
and
\[
\beta_f^- < \xi' \ell_- - \tau' \leq \beta_f^+ \leq \xi' \ell_+ - \tau' < \beta_f^+ + \frac{\xi'}{d}.
\]

**Proof.** We only prove the inequalities \( \alpha_f^- < \xi j_- - \tau \leq \alpha_f^+ \leq \xi j_+ + \frac{\xi}{d} \). All the remaining inequalities will follow using the same arguments.

Fix \( \tau, \tau', \xi, \xi' \) and set \( \alpha^- := \alpha_f^-(\xi, -\tau, \xi', -\tau') \). First, we show that
\[
\alpha^- < \frac{j_-}{d} \leq \alpha^- + \frac{1}{d}. \tag{51}
\]
It is easy to observe that \( j_-/d > \alpha^- \) based on the definitions of \( j_- \) and \( \alpha^- \). We now assume that \( j_-/d > \alpha^- + 1/d \). Denote the support of \((u, v) \mapsto \eta f(\xi u - \tau, \xi' v - \tau')\) by \( C \). Let \((\alpha^-, v(\alpha^-))\) be one of the points located on the boundary of \( C \). Due to the continuity of \( f \) and the assumption that \((j_--1)/d > \alpha^- \), there exists a \( j_0 \leq j_- \) and a small neighborhood \( U_{\alpha^-} \) of the point \((\alpha^-, v(\alpha^-))\) satisfying \( U_{\alpha^-} \subseteq [(j_0 - 2)/d, (j_0 - 1)/d] \times [(\ell - 1)/d, \ell/d] \) for some \( \ell \) such that
\[
X_{j_0-1, \ell} \geq \eta \int_{U_{\alpha^-}} d^2 f(\xi u - \tau, \xi' v - \tau') dudv > 0. \tag{52}
\]
This contradicts that by definition \( j_- \) is the smallest integer \( j \) satisfying \( X_{j, \ell} > 0 \), proving (51).

The definition of \( \alpha^- \), and \( \xi > 0 \) yield
\[
\xi \alpha^- - \tau = \alpha_f^- \tag{53}
\]
which together with (51) completes the proof.

Set
\[
\Delta_f := \alpha_f^+ - \alpha_f^- \quad \text{and} \quad \Delta_f' := \beta_f^+ - \beta_f^- \tag{54}
\]
for the width and the height of the rectangular support of \( f \).
Lemma A.7. Consider a generic image of the form (15). Assume that the support of \( f \) is contained in \([1/4, 3/4]^2\), and satisfies the Lipschitz property (11) for some constant \( C_L \). Let \( T_X \) be as defined in (19) and \( h \) be the function \((t, t') \mapsto h(t, t') := f(\Delta ft + \alpha_f, \Delta f't' + \beta_f)\). Then there exists a universal constant \( K > 0 \), such that

\[
\left\| T_X - \frac{\sqrt{\Delta f \Delta f'}}{\|f\|_2} h \right\|_2 \leq K(C_L \lor C_L^2)(\xi \lor \xi' \lor 1)^{1/2} \frac{1}{d}.
\]

Proof. In a first step of the proof, we show that

\[
|Z_X(t, t') - \eta h(t, t')| \leq 10\eta C_L \|f\|_1(\xi \lor \xi')^{1/2} \frac{1}{d}, \quad \text{for all } t, t' \in [0, 1], \tag{55}
\]

where \( Z_X(t, t') \) is as defined in (18).

Fix \( t, t' \in [0, 1] \) and recall that \( j_-, j_+, \ell_-, \) and \( \ell_+ \) are defined as in (16). Define \( j_* := [j_- + t(j_+ - j_-)] \) and \( \ell_* := [\ell_- + t'(\ell_+ - \ell_-)] \), where the dependence of \( j_* \) on \( j_- \) and \( j_+ \) on \( \ell_- \) and \( \ell_+ \) has been suppressed. For any \( t, t' \in [0, 1] \),

\[
|Z_X(t, t') - \eta h(t, t')| \leq |X_{j_*,\ell_*} - \eta h(t, t')|
\leq \eta \int_{I_{j_*,\ell_*}} d^2 f(\xi u - \tau, \xi' v - \tau') du dv - f(\Delta ft + \alpha_f, \Delta f't' + \beta_f) \right|.
\]

Since \( f \) satisfies the Lipschitz condition (11), we can bound this further observing that for any \((u, v) \in I_{j_*,\ell_*},

\[
|f(\xi u - \tau, \xi' v - \tau') - f(\Delta ft + \alpha_f, \Delta f't' + \beta_f)| \leq C_L \|f\|_1 \left| \left| (\xi u - \tau - (\Delta ft + \alpha_f)) \right| + \left| (\xi' v - \tau') - (\Delta f't' + \beta_f) \right| \right|.
\]

Proposition A.6. If both \( f \) and \( f(\Delta \cdot + \delta, \Delta' \cdot + \delta') \) for some \( \Delta, \Delta', \delta, \delta' \in \mathbb{R} \) have their support contained in \([0, 1]^2\), then,

\[
\|f(\Delta \cdot + \delta, \Delta' \cdot + \delta')\|_p^p = \frac{\|f\|_p^p}{|\Delta \Delta'|}, \quad \text{for all } p \in \mathbb{N} \setminus \{0\}.
\]

Proof. This follow from,

\[
\|f(\Delta \cdot + \delta, \Delta' \cdot + \delta')\|_p^p = \int_{[0,1]^2} |f(\Delta u + \delta, \Delta' v + \delta')|^p du dv 
= \int_{\mathbb{R}^2} |f(\Delta u + \delta, \Delta' v + \delta')|^p du dv
= \frac{1}{|\Delta \Delta'|} \int_{\mathbb{R}^2} |f(x, y)|^p dx dy
= \frac{1}{|\Delta \Delta'|} \int_{[0,1]^2} |f(x, y)|^p dx dy
= \frac{\|f\|_p^p}{|\Delta \Delta'|}.
\]
Since \( u \in [(j_* - 1)/d, j_*/d] = [(|j_* + t(j_* - j_*)| - 1)/d, |j_* + t(j_* - j_*)|]/d \), we have
\[
\frac{j_* + t(j_* - j_*) - 2}{d} \leq \frac{|j_* + t(j_* - j_*)| - 1}{d} \leq u \leq \frac{|j_* + t(j_* - j_*)|}{d} \leq \frac{j_* + t(j_* - j_*)}{d},
\]
hence, together with Lemma A.5, we obtain
\[
\left| (\xi u - \tau) - (\Delta f t + \alpha_j) \right| \leq \left| \left( \frac{j_* + t(j_* - j_*)}{d} - \tau \right) - \left( \Delta f t + \alpha_j \right) \right| + \frac{2\xi}{d} \leq \left| \left( \frac{j_* - j_-}{d} - \tau \right) - \alpha_j \right| \leq \left( \frac{j_* - j_-}{d} - \tau \right) - \alpha_j^+ \leq 5\xi \frac{1}{d}. \tag{58}
\]

Similarly, for any \( v \in [(\ell_* - 1)/d, \ell_*/d] \),
\[
\left| (\xi' v - \tau') - (\Delta f' t' + \beta_j) \right| \leq \frac{5\xi'}{d}. \tag{59}
\]

Plugging (58) and (59) into (57) yields that for any \((u,v) \in I_{j,*} \times I_{\ell,*} \),
\[
\left| f(\xi u - \tau, \xi' v - \tau') - f(\Delta f t + \alpha_j, \Delta f' t' + \beta_j) \right| \leq 10C_L \| f \|_1 (\xi \vee \xi') \frac{1}{d}. \tag{60}
\]

Combined with (56) and the fact that \( t, t' \in [0,1] \) was arbitrary, this implies (55).

In the next step, we show that for some universal constant \( C_1 > 0 \),
\[
\left\| Z_{\mathbf{X}} - \frac{\| f \|_2}{\sqrt{D_f D'_f}} \right\|_2 \leq C_1 \eta(C_L \vee C_2^2) (\xi \vee \xi') \frac{1}{d}. \tag{61}
\]
Using that for real numbers \( a, b \), \( a - b = (a^2 - b^2)/(a + b) \), we rewrite
\[
\left\| Z_{\mathbf{X}} - \frac{\| f \|_2}{\sqrt{D_f D'_f}} \right\|_2 \leq \left\| Z_{\mathbf{X}} \right\|_2 - \frac{\| f \|_2^2 \eta^2}{\Delta_f \Delta'_f} \frac{1}{\left\| Z_{\mathbf{X}} \right\|_2} \leq \left\| Z_{\mathbf{X}} \right\|_2 - \frac{\| f \|_2^2 \eta^2}{\Delta_f \Delta'_f} \frac{1}{\left\| f \right\|_2^2 \eta^2}. \tag{62}
\]
Since \( h(t, t') = f(\Delta_f t + \alpha_j, \Delta_f' t' + \beta_j) \) and the support of \( h \) is contained in \([0,1]^2\), we have, according to Proposition A.6, for \( p = 1, 2 \), \( \| h \|_p^2 = \| f \|_p^2 / (\Delta_f \Delta'_f) \). Also employing (55) we bound the first term by
\[
\left\| Z_{\mathbf{X}} \right\|_2 - \frac{\| f \|_2^2 \eta^2}{\Delta_f \Delta'_f} = \int_0^1 \int_0^1 (Z_{\mathbf{X}}(t, t') - \eta h(t, t') + \eta h(t, t')^2) \, dt \, dt' - \frac{\| f \|_2^2 \eta^2}{\Delta_f \Delta'_f}
\]
\[
= \int_0^1 \int_0^1 (Z_{\mathbf{X}}(t, t') - \eta h(t, t'))^2 \, dt \, dt' + 2\eta \int_0^1 \int_0^1 (Z_{\mathbf{X}}(t, t') - \eta h(t, t')) h(t, t') \, dt \, dt'
\]
\[
+ \int_0^1 \int_0^1 \eta^2 h^2(t, t') \, dt \, dt' - \frac{\| f \|_2^2 \eta^2}{\Delta_f \Delta'_f}
\]
\[
\leq \int_0^1 \int_0^1 |Z_{\mathbf{X}}(t, t') - \eta h(t, t')|^2 \, dt \, dt' + 2\eta \int_0^1 \int_0^1 |Z_{\mathbf{X}}(t, t') - \eta h(t, t')| |h(t, t')| \, dt \, dt'
\]
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\[
\leq \int_0^1 \int_0^1 100\eta^2 C_L^2 \|f\|_2^2 (\xi \vee \xi')^2 \frac{1}{d^2} dtdt' + 2 \int_0^1 \int_0^1 100\eta^2 C_L \|f\|_1 (\xi \vee \xi') \frac{1}{d} |h(t, t')| dtdt'
= 100\eta^2 C_L^2 \|f\|_2^2 (\xi \vee \xi')^2 \frac{1}{d^2} + 20\eta^2 C_L \frac{\|f\|_1^2}{\Delta_f \Delta'_f} (\xi \vee \xi') \frac{1}{d}
\leq C_1 \eta^2 (C_L \vee C_L^2) (\xi \vee \xi' \vee 1)^2 \frac{\|f\|_1^2}{\Delta_f \Delta'_f} \frac{1}{d^2},
\]

where \(C_1 > 0\) is a sufficiently large universal constant. By the Cauchy-Schwarz inequality, \(\|f\|_1 \leq \|f\|_2\).

Summarizing, (62) is bounded by
\[
\left\| Z_X - \frac{\|f\|_{2\eta}}{\sqrt{\Delta_f \Delta'_f}} \right\|_2 \leq C_1 \eta^2 (C_L \vee C_L^2) (\xi \vee \xi' \vee 1)^2 \frac{\|f\|_1^2}{\Delta_f \Delta'_f} \frac{1}{d^2},
\]

proving (61).

We now finish the proof. Using \(T_X = Z_X/\|Z_X\|_2\) and that \((a + b)^2 \leq 2a^2 + 2b^2\) for arbitrary real numbers \(a, b\), we bound
\[
\left\| T_X - \frac{\sqrt{\Delta_f \Delta'_f}}{\|f\|_{2\eta}} h \right\|_2^2 \leq 2 \left\| T_X - \frac{\sqrt{\Delta_f \Delta'_f}}{\|f\|_{2\eta}} Z_X \right\|_2^2 + 2 \left\| \frac{\sqrt{\Delta_f \Delta'_f}}{\|f\|_{2\eta}} Z_X - \frac{\sqrt{\Delta_f \Delta'_f \eta h}}{\|f\|_{2\eta}} \right\|_2^2
\leq 2 \frac{\Delta_f \Delta'_f}{\|f\|_{2\eta}^2} \left\| Z_X \right\|_2^2 + \frac{2 \Delta_f \Delta'_f}{\|f\|_{2\eta}^2} \|Z_X - \eta h\|_2^2.
\]

Applying (61) to the first and (55) to the second term and using again \(\|f\|_1 \leq \|f\|_2\), as well as \(\Delta_f, \Delta'_f \leq 1\), it follows
\[
\left\| T_X - \frac{\sqrt{\Delta_f \Delta'_f}}{\|f\|_{2\eta}} h \right\|_2^2 \leq \frac{2 \Delta_f \Delta'_f}{\|f\|_{2\eta}^2} (C_1)^2 \eta^2 (C_L \vee C_L^2) (\xi \vee \xi' \vee 1)^4 \frac{\|f\|_1^2}{\Delta_f \Delta'_f} \frac{1}{d^2}
+ \frac{2 \Delta_f \Delta'_f}{\|f\|_{2\eta}^2} 100\eta^2 C_L^2 \|f\|_2^2 (\xi \vee \xi')^2 \frac{1}{d^2}
\leq K^2 (C_L \vee C_L^2) (\xi \vee \xi' \vee 1)^4 \frac{1}{d^2},
\]

for a universal constant \(K > 0\). \(\square\)

**Lemma A.8.** For two functions \(h, g : \mathbb{R}^2 \to \{0, \infty\}\),
\[
\inf_{\eta, \xi, \xi', t, t', \bar{\xi}, \bar{\xi}' \in \mathbb{R}, \bar{\eta}, \bar{\xi}, \bar{\xi}' \in \mathbb{R} \setminus \{0\}} \frac{\sqrt{|\bar{\xi}|}}{|\bar{\eta}|} \left\| \frac{\sqrt{|\xi|}}{|\eta|} \left( g(\bar{\xi} \cdot + t, \xi' \cdot + t') - \bar{\eta} h(\bar{\xi} \cdot + \bar{t}, \bar{\xi}' \cdot + \bar{t}') \right) \right\|_{L^2(\mathbb{R}^2)}
\geq \inf_{a, b, c, b', c' \in \mathbb{R}} \left\| a g(b \cdot + c, b' \cdot + c') - h \right\|_{L^2(\mathbb{R}^2)}.
\]

**Proof.** For arbitrary \(\eta, \xi, \xi', t, t', \bar{\xi}, \bar{\xi}' \in \mathbb{R}, \bar{\eta}, \bar{\xi}, \bar{\xi}' \in \mathbb{R} \setminus \{0\}\), substitution gives
\[
\int_{\mathbb{R}^2} \left( g(\xi u + t, \xi' v + t') - \bar{\eta} h(\bar{\xi} u + \bar{t}, \bar{\xi}' v + \bar{t}') \right)^2 du dv
\]
\[
\leq \int_{\mathbb{R}^2} \left( g(\xi u + t, \xi' v + t') - \bar{\eta} h(\bar{\xi} u + \bar{t}, \bar{\xi}' v + \bar{t}') \right)^2 du dv
\]
\[= \int_{\mathbb{R}^2} \left( \eta g \left( \frac{\xi}{\xi'} (x - \bar{t}) + t, \frac{\xi'}{\xi} (y - \bar{t}') + t' \right) - \tilde{\eta} h(x,y) \right) \frac{1}{|\xi|} \, dxdy\]
\[\geq \tilde{\eta}^2 \int_{\mathbb{R}^2} \left( \frac{\eta g}{\xi} \left( \frac{\xi}{\xi'} (x - \bar{t}) + t, \frac{\xi'}{\xi} (y - \bar{t}') + t' \right) - h(x,y) \right) \frac{1}{|\xi|} \, dxdy\]
\[\geq \frac{\tilde{\eta}^2}{|\xi|} \inf_{a,b,c,b',c' \in \mathbb{R}} \left\| ag \left( b \cdot + c, b' \cdot + c' \right) - h \right\|_{L^2(\mathbb{R}^2)}^2.
\]

**Proof of Theorem 3.6.** The statement is symmetric in \( f_0 \) and \( f_1 \). Therefore it is enough to consider the case with label \( k = 0 \). Set \( \Delta f_k := \alpha^+_{f_k} - \alpha^-_{f_k}, \Delta f'_k := \beta^+_{f_k} - \beta^-_{f_k}, \) and
\[h_k := f_k (\Delta f_k \cdot + \alpha^-_{f_k}, \Delta f'_k \cdot + \beta^-_{f_k}).\]

Since \( k = 0 \), the entries of \( X \) and \( Z_X \) are described by the template function \( f_0 : \mathbb{R}^2 \to \mathbb{R} \). For an arbitrary \( Z_{X_i}, i \in \{1, \ldots, n\} \) the corresponding template function is \( f_{k_i} : \mathbb{R}^2 \to \mathbb{R} \). If \( k_i = 0 \), it follows by Lemma A.7 and the triangle inequality that
\[\| T_{X_i} - T_X \|_2 = \left\| T_{X_i} - \sqrt{\frac{\Delta f_0 \Delta f'_0}{\| f_0 \|_2}} h_{f_0} + \sqrt{\frac{\Delta f_0 \Delta f'_0}{\| f_0 \|_2}} h_{f_0} - T_X \right\|_2 \leq \left\| T_{X_i} - \sqrt{\frac{\Delta f_0 \Delta f'_0}{\| f_0 \|_2}} h_{f_0} \right\|_2 + \left\| \sqrt{\frac{\Delta f_0 \Delta f'_0}{\| f_0 \|_2}} h_{f_0} - T_X \right\|_2 \leq K(C_L \lor C_L^2) \Xi_{\mathbb{R}^n}^2 \Xi_{\mathbb{R}^d}^1 + K(C_L \lor C_L^2) \Xi_{\mathbb{R}^n}^2 \Xi_{\mathbb{R}^d}^1 \leq 2K(C_L \lor C_L^2) \Xi_{\mathbb{R}^n}^2 \Xi_{\mathbb{R}^d}^1. \quad (63)\]

The support of the function \( h_{f_0} \) is contained in \([0, 1]^2\). Recall the definition of the separation constant \( D \) in (21). Applying Lemma A.8 twice by assigning to \((h, \tilde{\eta}, \tilde{\xi}, \tilde{\xi}')\) the values \((h_{f_0}, \sqrt{\Delta f_0 \Delta f'_0 / \| f_0 \|_2}, \Delta f_0, \Delta f'_0)\) and \((h_{f_1}, \sqrt{\Delta f_1 \Delta f'_1 / \| f_1 \|_2}, \Delta f_1, \Delta f'_1)\) gives
\[\left\| \sqrt{\frac{\Delta f_0 \Delta f'_0}{\| f_0 \|_2}} h_{f_1} - \sqrt{\frac{\Delta f_0 \Delta f'_0}{\| f_0 \|_2}} h_{f_0} \right\|_2 \geq \inf_{a,b,c,b',c' \in \mathbb{R}} \| a f_1 (b \cdot + c, b' \cdot + c') - f_0 \|_{L^2(\mathbb{R}^2)} \left\| f_0 \right\|_2 \| a f_0 (b \cdot + c, b' \cdot + c') - f_1 \|_{L^2(\mathbb{R}^2)} \left\| f_0 \right\|_2 \geq \frac{4K(C_L \lor C_L^2) \Xi_{\mathbb{R}^n}^2}{d}, \quad (64)\]
where we used the assumption \( D > 4K(C_L \lor C_L^2) \Xi_{\mathbb{R}^n}^2 / d \) for the last step. For an \( i \) with \( k_i = 1 \), we use the reverse triangle inequality
\[|a' - b'| \geq |a - b| - |a' - a| - |b' - b|, \quad \text{for all } a, b, a', b' \in \mathbb{R},\]
inequality (64) and Lemma A.7 to bound
\[
\|T_{x_i} - T_x\|_2^2 = \left\| T_{x_i} - \frac{\sqrt{\Delta f_1 \Delta f_1'}}{\|f_1\|_2} h_{f_1} + \frac{\Delta f_1 \Delta f_1'}{\|f_1\|_2} h_{f_1} - \frac{\sqrt{\Delta f_0 \Delta f_0'}}{\|f_0\|_2} h_{f_0} + \frac{\Delta f_0 \Delta f_0'}{\|f_0\|_2} h_{f_0} - T_x \right\|_2^2
\geq \frac{\|\Delta f_1 \Delta f_1'\|}{\|f_1\|_2} h_{f_1} - \frac{\|\Delta f_0 \Delta f_0'\|}{\|f_0\|_2} h_{f_0} - \left\| T_{x_i} - \frac{\Delta f_1 \Delta f_1'}{\|f_1\|_2} h_{f_1} \right\|_2 - \left\| T_x - \frac{\Delta f_0 \Delta f_0'}{\|f_0\|_2} h_{f_0} \right\|_2
\geq 4K(C_L \lor C_L')\Xi_n^2 - K(C_L \lor C_L')\Xi_n^2 - K(C_L \lor C_L')\Xi_n^2
\geq 2K(C_L \lor C_L')\Xi_n^2.
\]
Combining this with (63), we conclude that
\[
\hat{i} \in \arg \min_{i=1,\ldots,n} \|T_{x_i} - T_x\|_2
\]
holds for some \(i\) with \(k_i = 0\) implying \(\hat{k} = 0\). Since \(k = 0\), this shows the assertion \(\hat{k} = k\).

Proof of Theorem 3.7. We consider \(f_0(x,y) = (1/4 - |1/2 - x| - |1/2 - y|)_+\), whose support is contained in \([1/4, 3/4]^2\), and its rectangular support exactly matches \([1/4, 3/4]^2\). We now show that \(f_0\) satisfies (11) with Lipschitz constant \(C_{f_0} = 96\). To verify this, observe that \(|f_0(x,y) - f_0(x',y')| \leq (|x - x'| + |y - y'|)\). Thus, (11) holds for any \(C_{f_0} \geq 1/\|f_0\|_1\). Using the definition of \(f_0\), we compute
\[
\|f_0\|_1 = \int_{[0,1]^2} |f_0(x,y)| \, dx \, dy = \int_{[0,1]^2} f_0(x,y) \, dx \, dy = \frac{1}{96}.
\]
Hence the Lipschitz condition is satisfied with \(C_{f_0} = 96\). Consider the template function \(f_0\) has been deformed as \(f_{0,\tau,\tau',\xi,\xi'} := f_0(\xi \cdot -\tau, \xi' \cdot -\tau')\). A generic image \(X = (X_{j,l})_{j,l=1,\ldots,d}\) based on \(f_{0,\tau,\tau',\xi,\xi'}\) is described as
\[
X_{j,l} = \eta \int_{\Omega \times \Omega} d^2 f_{0,\tau,\tau',\xi,\xi'}(x,y) \, dx \, dy.
\]
Next, for any random deformation parameters \(\tau, \tau'\) and \(\xi, \xi'\) satisfying Assumption 2', we construct a local perturbation function \(g\) on \(f_{0,\tau,\tau',\xi,\xi'}\). Notice that the square \([3/8, 5/8]^2\) is contained in the support of \(f_0\). Taking into account the random re-scaling and shifting, the square \(I_c = [(3/8 + \tau)/\xi, (5/8 + \tau)/\xi] \times [(3/8 + \tau')/\xi', (5/8 + \tau')/\xi']\) is contained in the support of \(f_{0,\tau,\tau',\xi,\xi'}\). We shall build the perturbation of \(f_{0,\tau,\tau',\xi,\xi'}\) on \(I_c\). More precisely, let
\[
\ell^* := \left[ \left( \frac{3}{8\xi} + \frac{\tau}{\xi} \right) d \right], \quad \ell'^* := \left[ \left( \frac{5}{8\xi} + \frac{\tau}{\xi} \right) d \right],
\]
and
\[
\ell^* := \left[ \left( \frac{3}{8\xi} + \frac{\tau'}{\xi'} d \right] \right], \quad \ell'^* := \left[ \left( \frac{5}{8\xi'} + \frac{\tau'}{\xi'} d \right] \right],
\]
which are the approximated grid location for \(I_c\). Observe that \([\ell^- / d, \ell'^/d] \times [\ell'^/d, \ell'^/d] \subseteq I_c\). Moreover, provided \(d \geq 16(\xi \lor \xi')\), one can derive
\[
\ell^* - \ell^+ \geq \frac{1}{8\xi} d - \frac{3}{8\xi} d - 2 = \frac{d}{4\xi} - 2 \geq \frac{d}{8\xi}.
\]

and

\[ \ell^+_* - \ell^-_* \geq \left( \frac{5}{8\ell^2} + \frac{\tau^1}{\xi^2} \right) d - \left( \frac{3}{8\ell^2} + \frac{\tau^1}{\xi^2} \right) d - 2 = \frac{d}{4\ell^2} - 2 \geq \frac{d}{8\ell^2} \]  \hspace{1cm} (67)

and thus, \([j^-_*/d, j^+_*/d] \times [\ell^-_*/d, \ell^+_*/d]\) is not empty. Set \(I := \{j^-_* + 1, \ldots, j^*_+\}\) and \(I' := \{\ell^-_* + 1, \ldots, \ell^*_+\}\). For any \(i \in \mathbb{N}\), let \(a^-_i := (i - 3/4)/d\), \(a^+_i := (i - 1/4)/d\). For \(j \in I, \ell \in I'\), define the following functions

\[ S^{-}_{j\ell} (x, y) := \left( \frac{1}{4d} - |x - a^-_j| - |y - a^-_\ell| \right)_+, \quad S^{-}_{j\ell} (x, y) := \left( \frac{1}{4d} - |x - a^-_j| - |y - a^+_\ell| \right)_+, \]

and

\[ S^{+}_{j\ell} (x, y) := \left( \frac{1}{4d} - |x - a^+_j| - |y - a^-_\ell| \right)_+, \quad S^{+}_{j\ell} (x, y) := \left( \frac{1}{4d} - |x - a^+_j| - |y - a^+_\ell| \right)_+. \]

Realizations of these functions are shown in Figure 9. The support of \(S^{-}_{j\ell}\), \(S^+_{j\ell}\), \(S^{-}_{j\ell}\), and \(S^+_{j\ell}\) is contained in \([(j - 1)/d, (j - 1/2)/d] \times [(\ell - 1)/d, (\ell - 1/2)/d], [(j - 1)/d, (j - 1/2)/d] \times [(\ell - 1)/d, \ell/d], [(j - 1/2)/d, j/d] \times [(\ell - 1)/d, (\ell - 1/2)/d], \) and \([(j - 1/2)/d, j/d] \times [(\ell - 1/2)/d, \ell/d]\) respectively. The supports of any two functions among \(S^{-}_{j\ell}\), \(S^+_{j\ell}\), \(S^{-}_{j\ell}\), and \(S^+_{j\ell}\) are disjoint. For \(j \in I, \ell \in I'\), set

\[ S_{j\ell}(x, y) := S^{-}_{j\ell}(x, y) - S^+_{j\ell}(x, y) = S^{-}_{j\ell}(x, y) + S^+_{j\ell}(x, y). \]

The support of the function \(S_{j\ell}\) is contained in \([(j - 1)/d, j/d] \times [(\ell - 1)/d, \ell/d]\). For \((j, \ell) \neq (j', \ell')\), \(S_{j,\ell}\) and \(S_{j',\ell'}\) have disjoint support. An example of the function \(S_{j\ell}\) is shown in Figure 10.

Consider the perturbation

\[ g(x, y) := \sum_{j \in I, \ell \in I'} S_{j\ell}(x, y). \]

By construction, the support of \(g\) is contained in \(I_c\). Moreover, on each pixel \(I_{j,\ell} \subseteq I_c\), the support of \(S_{j\ell}\) has been divided into four regions according to the supports of \(S^{-}_{j\ell}\), \(S^+_{j\ell}\), \(S^{-}_{j\ell}\), and \(S^+_{j\ell}\) and for any \(j, \ell \in \{1, \ldots, d\}\),

\[ d^2 \int_{I_{j,\ell}} g(x, y) dxdy = d^2 \int_{I_{j,\ell}} S_{j\ell}(x, y) dxdy = 0. \]  \hspace{1cm} (68)
Now we consider the new function

\[ f_{1, \tau, \tau', \xi, \xi'}(x, y) := f_{0, \tau, \tau', \xi, \xi'}(x, y) + g(x, y). \]

Recall that the function \( f_{0, \tau, \tau', \xi, \xi'} \) takes positive values on the interior of \( I_c \), and the function \( g \) is a small perturbation function defined on the interior of \( I_c \). One can verify that with \( d \) being large enough, the function \( f_{1, \tau, \tau', \xi, \xi'} \) takes non-negative values on \([0, 1]^2\), and therefore, so does \( f_1 := f_{1, \tau, \tau', \xi, \xi'}(1/\xi(\cdot + \tau), 1/\xi'(\cdot + \tau')) \).

Similarly to \( f_0 \), the rectangular support of \( f_1 \) is \([1/4, 3/4]^2\), as the perturbation \( g \) does not alter the function values at the boundary.

In the following, we check that \( f_1 \) satisfies the Lipschitz condition in (11). We first compute

\[
\int_{I_{j,\ell}} (S_{j\ell}(x, y))^2 \, dx \, dy = \int_{I_{j,\ell}} \left( S_{j\ell}^-(x, y) \right)^2 + \left( S_{j\ell}^+(x, y) \right)^2 \, dx \, dy
\]

\[
= 4 \int_{I_{j,\ell}} \left( S_{j\ell}^-(x, y) \right)^2 \, dx \, dy
\]

\[
= 4 \int_{(j-1/2)/d}^{(j+1/2)/d} \int_{(\ell-1/2)/d}^{(\ell+1/2)/d} \left( \frac{1}{4d} - |x - a_j^-| - |y - a_{\ell}^-| \right)^2 \, dy \, dx.
\]

Substituting \( u = 2d(x - a_j^-) + 1/2 \) and \( v = 2d(y - a_{\ell}^-) + 1/2 \), we further obtain

\[
\int_{I_{j,\ell}} (S_{j\ell}(x, y))^2 \, dx \, dy = \frac{4}{4d^2} \int_0^1 \int_0^1 \left[ \left( \frac{1}{4d} - \frac{1}{2d} \left| u - \frac{1}{2} \right| - \frac{1}{2d} \left| v - \frac{1}{2} \right| \right)_+ \right]^2 \, du \, dv
\]

\[
= \frac{1}{4d^2} \int_0^1 \int_0^1 \left[ \frac{1}{2} - \left| u - \frac{1}{2} \right| - \left| v - \frac{1}{2} \right| \right]_+ \right]^2 \, du \, dv
\]

\[
= \frac{1}{192d^2}.
\]

Thus, for a sufficiently large \( d \) in the sense that \( d \geq 16(\xi \vee \xi') \), using (66) and (67), we have

\[
\|g\|_{L^2([0,1]^2)}^2 = \|g\|^2 = \sum_{j, \ell = 1}^{d} \int_{I_{j,\ell}} (S_{j\ell}(x, y))^2 \, dx \, dy = \frac{(j_1^+ - j_1^-)(\ell_1^+ - \ell_1^-)}{192d^2} \geq \frac{1}{8^3 \cdot 192\xi \xi'} \frac{1}{d^2}.
\]  

(69)
Similarly, we deduce that

\[ \|g\|_1 = \sum_{j, \ell = 1}^{d} \left( \int_{I_{j, \ell}} |S_{j\ell}(x, y)| \, dx \, dy \right) \]

\[ = \sum_{j, \ell = 1}^{d} \left[ \frac{4}{4d^2} \int_{0}^{1} \int_{0}^{1} \left( \frac{1}{4d} - \frac{1}{2d} \left| u - \frac{1}{2} \right| - \frac{1}{2d} \left| v - \frac{1}{2} \right| \right) + \, dudv \right] \]

\[ = \sum_{j, \ell = 1}^{d} \left[ \frac{1}{2d^3} \int_{0}^{1} \int_{0}^{1} \left( \frac{1}{2} - \left| u - \frac{1}{2} \right| - \left| v - \frac{1}{2} \right| \right) + \, dudv \right] \]

\[ = \frac{(j_1^+ - j_1^-)(\ell_1^+ - \ell_1^-)}{24d^3} \]

\[ \leq \frac{1}{384 \xi \xi'} d, \]  

(70)

where the last step follows since \( j_1^+, j_1^-, \ell_1^+, \ell_1^- \) implies \( j_1^+ - j_1^- \leq d/(4\xi) \) and \( \ell_1^+ - \ell_1^- \leq d/(4\xi') \).

For any \((x, y), (x', y') \in [0, 1]^2\),

\[ |f_{1, \tau, \tau', \xi', \epsilon'}(x, y) - f_{1, \tau, \tau', \xi, \epsilon'}(x', y')| \leq |f_{0, \tau, \tau', \xi, \epsilon'}(x, y) - f_{0, \tau, \tau', \xi', \epsilon'}(x', y')| + |g(x, y) - g(x', y')| \]

\[ = |f_0(\xi x - \tau, \xi' y - \tau') - f_0(\xi x' - \tau, \xi' y' - \tau')| + |g(x, y) - g(x', y')| \]

\[ \leq (\xi \vee \xi')(|x - x'| + |y - y'|) + 2(|x - x'| + |y - y'|) \]

\[ = ((\xi \vee \xi') + 2)(|x - x'| + |y - y'|). \]  

(71)

As a consequence of the triangle inequality, (65), and (70),

\[ \|f_{1, \tau, \tau', \xi, \epsilon'}\|_1 \geq \|f_{0, \tau, \tau', \xi, \epsilon'}\|_1 - \|g\|_1 = \|f_0\|_1 - \|g\|_1 \geq \frac{1}{96 \xi \xi'} - \frac{1}{384 \xi \xi'} \frac{1}{d} \geq \frac{1}{128 \xi}, \]

which, together with (71), implies that for the constant \( C_L := 128 \xi \xi' ([\xi \vee \xi'] + 2) \), \( f_{1, \tau, \tau', \xi, \epsilon'} \) satisfies the Lipschitz condition in (11) with \( C_L \). Hence, \( f_1 \) satisfies the Lipschitz condition with constant \( C_{f_1} := C_L/(\xi \xi') = 128 ([\xi \vee \xi'] + 2) \).

Meanwhile, according to (69), we have for any \( d \geq 16(\xi \vee \xi') \),

\[ \|f_1 - f_0\|_{L^2(\mathbb{R}^2)} = \xi \xi' \|f_{1, \tau, \tau', \xi, \epsilon'} - f_{0, \tau, \tau', \xi, \epsilon'}\|_{L^2(\mathbb{R}^2)} = \xi \xi' \|g\|_{L^2(\mathbb{R}^2)} \geq \frac{\sqrt{\xi \xi'}}{111 d}. \]

Due to (68), for any \( j, \ell = 1, \ldots, d \),

\[ X_{j, \ell} = \eta \int_{I_{j, \ell}} d^2 f_{0, \tau, \tau', \xi, \epsilon'}(x, y) \, dx \, dy \]

\[ = \eta \int_{I_{j, \ell}} d^2 \left[ f_{0, \tau, \tau', \xi, \epsilon'}(x, y) + g(x, y) \right] \, dx \, dy \]

\[ = \eta \int_{I_{j, \ell}} d^2 f_{1, \tau, \tau', \xi, \epsilon'}(x, y) \, dx \, dy. \]

This implies that both template functions \( f_0, f_1 \) generate the same image \( X = (X_{j, \ell})_{j, \ell} \) under the same (but random) deformation parameters. It is impossible to infer the label from \( X \).  

40
B Proofs for Section 4

Recall that $[W]$ denotes the quadratic support of the matrix $W$.

Lemma B.1. If $W = (W_{i,j})_{i,j=1,\ldots,d}$ and $X = (X_{i,j})_{i,j=1,\ldots,d}$ are matrices with non-negative entries, then,

$$|\sigma([W] \star X)|_\infty = \max_{r,s \in \mathbb{Z}^d} \sum_{i,j=1}^d W_{i+r,j+s} X_{i,j},$$

where $W_{k,\ell} := 0$ whenever $k \land \ell \leq 0$ or $k \lor \ell > d$.

Proof. Due to the fact that all entries of $W$ and $X$ are non-negative, it follows $\sigma([W] \star X) = [W] \star X$. Assume that $[W]$ is of size $\ell \times \ell$. As $|\cdot|_\infty$ extracts the largest value of $[W] \star X$ and each entry is the entrywise sum of the Hadamard product of $[W]$ and a $\ell \times \ell$ sub-matrix of $X'$, we rewrite

$$|W \star X|_\infty = \max_{u,v \in \mathbb{Z}^\ell} \sum_{i,j=1}^\ell [W]_{i,j} \cdot X'_{i+u,j+v},$$

where $X'_{k,\ell} = X_{k,\ell}$ for $k, \ell \in \{1, \ldots, d\}$ and $X'_{k,\ell} := 0$ whenever $k \land \ell \leq 0$ or $k \lor \ell > d$. By definition of the quadratic support, there exist $R, S \in \{0, \ldots, d-\ell\}$ such that $[W]_{a,b} = W_{a+R,b+S}$, for all $a, b \in \{1, \ldots, \ell\}$. Using that $[W]_{i,j} := 0$ whenever $i \land j \leq 0$ or $i \lor j > \ell$, we rewrite

$$\max_{r,s \in \mathbb{Z}} \sum_{i,j=1}^d W_{i+r,j+s} X_{i,j} = \max_{r,s \in \mathbb{Z}} \sum_{i,j=\mathbb{Z}} [W]_{i+r,j+s} X'_{i,j},$$

$$= \max_{r,s \in \mathbb{Z}} \sum_{i',j' \in \mathbb{Z}} [W]_{i',j'} X'_{i'-r,j'-s+S}$$

$$= \max_{u,v \in \mathbb{Z}} \sum_{i,j=\mathbb{Z}} [W]_{i,j} X'_{i+u,j+v}$$

$$= \max_{u,v \in \mathbb{Z}} \sum_{i,j=1}^\ell [W]_{i,j} X'_{i+u,j+v}$$

$$= |\sigma([W] \star X)|_\infty,$$

proving the assertion. \qed

To prove Theorem 4.2, we need the following auxiliary results. To formulate these results, it is convenient to first define the discrete $L^2$-inner product for functions $g, h : [0, 1]^2 \to \mathbb{R}$ by

$$\langle h, g \rangle_{2,d} := \frac{1}{d^2} \sum_{j,\ell=1}^d \overline{h}_{j,\ell} \overline{g}_{j,\ell},$$

with $\overline{h}$ and $\overline{g}$ the pixel values defined in (1). The corresponding norm is then

$$\|g\|_{2,d} := \sqrt{\langle g, g \rangle_{2,d}}.$$  \hspace{1cm} (73)

The next lemma provides a bound for the approximation error of Riemann sums.
Lemma B.2. For functions $h, g : \mathbb{R}^2 \to [0, \infty)$ satisfying the Lipschitz condition (11) with constant $C_L$, we have

(i) \[
\left| \langle h, g \rangle_{2,d} - \int_{[0,1]^2} h(u, v)g(u, v) \, du dv \right| \leq \frac{4}{d^2} \|g\|_1 \|h\|_1 \left( C_L + C_L^2 \frac{1}{d} \right),
\]

(ii) \[
\left| \frac{1}{\|h\|_{2,d}} - \frac{1}{\|h\|_2} \right| \leq \frac{4(C_L + C_L^2 / d)}{d \|h\|_{2,d}},
\]

(iii) \[
\left| \frac{1}{\|h\|_{2,d} \|g\|_{2,d}} - \frac{1}{\|h\|_2 \|g\|_2} \right| \leq \frac{8}{\|h\|_{2,d} \|g\|_{2,d}} \left( \frac{C_L}{d} + \frac{3C_L^2}{d^2} + \frac{4C_L^3}{d^3} + \frac{2C_L^4}{d^4} \right),
\]

(iv) if $\operatorname{supp} h \subseteq [0, 1]^2$, then, for any integers $r, s$,

\[
\left| \frac{\langle h(-r/d, -s/d), g \rangle_{2,d}}{\|h\|_{2,d} \|g\|_{2,d}} - \int_{[0,1]^2} \frac{h(u - r/d, v - s/d)g(u, v)}{\|h\|_2 \|g\|_2} \, du dv \right| \leq \frac{16C_L}{d} \left( 1 + \frac{C_L}{d} \right)^3.
\]

Proof. (i): By the Lipschitz property, we obtain for any $j, \ell = 1, \ldots, d$ and any $(u, v) \in I_{j, \ell} = [(j-1)/d, j/d) \times [(\ell-1)/d, \ell/d)$, that \[
\left| \overline{g}_{j, \ell} - g(u, v) \right| \leq \frac{2C_L \|g\|_1}{d}
\]
and \[
\left| \overline{h}_{j, \ell} - h(u, v) \right| \leq \frac{2C_L \|h\|_1}{d}.
\]

Using this repeatedly, the triangle inequality gives

\[
\left| \overline{h}_{j, \ell} \overline{g}_{j, \ell} - h(u, v) g(u, v) \right|
\leq \left| \overline{h}_{j, \ell} \overline{g}_{j, \ell} - \overline{h}_{j, \ell} g(u, v) \right| + \left| \overline{h}_{j, \ell} g(u, v) - h(u, v) g(u, v) \right|
\leq \left| \overline{h}_{j, \ell} \right| C_L \|g\|_1 \frac{2}{d} + \left| g(u, v) \right| C_L \left| h \right|_1 \frac{2}{d}
\leq \left| h(u, v) \right| C_L \|g\|_1 \frac{2}{d} + C_L^2 \|g\|_1 \|h\|_1 \frac{4}{d^2} + \left| g(u, v) \right| C_L \|h\|_1 \frac{2}{d},
\]

which yields

\[
\int_{I_{j, \ell}} \left| \overline{h}_{j, \ell} \overline{g}_{j, \ell} - h(u, v) g(u, v) \right| \, du dv
\leq C_L \|g\|_1 \frac{2}{d} \int_{I_{j, \ell}} h(u, v) \, du dv + \frac{4}{d^2} C_L^2 \|g\|_1 \|h\|_1 + C_L \left| h \right|_1 \frac{2}{d} \int_{I_{j, \ell}} g(u, v) \, du dv.
\]

Rewriting

\[
\frac{1}{d^2} \sum_{j, \ell} \overline{h}_{j, \ell} \overline{g}_{j, \ell} = \sum_{j, \ell} \int_{I_{j, \ell}} \overline{h}_{j, \ell} \overline{g}_{j, \ell} \, du dv
\]

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Assumption 2, we have for all $A$

Proposition B.3. For any function $f$ that satisfies Assumption 1 and any deformation class $A$ that satisfies Assumptions 2, we have for all $A \in \mathcal{A}$,

$$\|f \circ A\|_1 \leq 2C_A C_L \|f\|_1.$$
and
\[ \| f \circ A \|_2 \leq 2\sqrt{2}C_A C_L \| f \|_1. \]

Proof. Under Assumption 1 and 2, the support of \( f \circ A \) is contained in \([0, 1]^2\), which implies \( f \circ A(0, 0) = 0 \) due to the continuity of \( f \) and \( A \). Given that Assumption 1 and 2 hold, applying Lemma A.1, we obtain
\[
\| f \circ A \|_1 = \int_{[0,1]^2} f \circ A(u, v) \, dudv \\
\leq \int_{[0,1]^2} |f \circ A(u, v) - f \circ A(0, 0)| \, dudv \\
\leq \int_{[0,1]^2} 2C_A C_L \| f \|_1 (u + v) \, dudv \\
= 2C_A C_L \| f \|_1.
\]

Now we consider the bound for \( \| f \circ A \|_2 \). Again, we use the property that under Assumption 1 and 2, \( f \circ A(0, 0) = 0 \). Applying Lemma A.1 yields
\[
\| f \circ A \|_2^2 = \int_{[0,1]^2} [f \circ A(u, v)]^2 \, dudv \\
= \int_{[0,1]^2} \left| f \circ A(u, v) (f \circ A(u, v) - f \circ A(0, 0)) \right| \, dudv \\
\leq \int_{[0,1]^2} |f \circ A(u, v)| 2C_A C_L \| f \|_1 (u + v) \, dudv \\
\leq 4C_A C_L \| f \|_1 \int_{[0,1]^2} |f \circ A(u, v)| \, dudv \\
= 4C_A C_L \| f \|_1 \| f \circ A \|_1 \\
\leq \left( 2\sqrt{2}C_A C_L \| f \|_1 \right)^2,
\]
where the last inequality uses (75). Taking square roots on both sides, we conclude that \( \| f \circ A \|_2 \leq 2\sqrt{2}C_A C_L \| f \|_1. \) \hfill \( \square \)

In the next result, we demonstrate that for any image generated through deformation of the template function \( g \), with a suitably designed filter function based on \( g \), the output provided by the CNN layers is always greater than some quantity of order \( 1 - O(1/d) \). Moreover, for any filter constructed using the template function \( f \), the output given by the CNN layers is always smaller than some quantity depending on the separation quantity between \( f \) and \( g \). For any deformation \( A \), let \( X_{g \circ A} := (X_{g \circ A,j,\ell})_{j,\ell=1,...,d} \) with
\[
X_{g \circ A,j,\ell} := \frac{g \circ A_{j,\ell}}{d\| g \circ A \|_{2,d}}
\]
with \( g \circ A_{j,\ell} \) being the average intensity of \( g \circ A \) on \( I_{j,\ell} \), as defined in (1) and \( w_{f \circ A} := (w_{f \circ A,j,\ell})_{j,\ell=1,...,d} \) where
\[
w_{f \circ A,j,\ell} := \frac{f \circ A_{j,\ell}}{d\| f \circ A \|_{2,d}}.
\]
Proposition B.4. Let \( f, g \) be two non-negative functions satisfying Assumption 1 with Lipschitz constant \( C_L \) and suppose that Assumptions 2, 3 hold for the deformation set \( A \). Then, there are constants \( C_1(C_L, C_A) \) and \( C_2(C_L, C_A) \) such that

\[
(i) \quad \max_{A' \in A_d} \left| \sigma(\mathbf{w}_{gA'} \star \mathbf{x}_{gA}) \right|_\infty \geq 1 - \frac{C_1(C_L, C_A)}{d}, \tag{76}
\]

(ii) and with \( D(f, g) \) as defined in (21),

\[
\max_{A' \in A_d} \left| \sigma(\mathbf{w}_{fA' \star \mathbf{x}_{gA}}) \right|_\infty \leq 1 - \frac{D^2(f, g) \vee D^2(g, f)}{16C_A^2C_L^2} + \frac{C_2(C_L, C_A)}{d}. \tag{77}
\]

Proof. We first prove (i). For any \( A \in A \), we pick a deformation \( A' \in A_d \) satisfying (28). According to Lemma B.1, we deduce that

\[
\sigma(\mathbf{w}_{gA'} \star \mathbf{x}_{gA}) = \max_{r_{j',r} \in Z} \sum_{j, j', \ell, \ell' = 1}^{d} 1_{1 \leq j + r_{j'} \leq d, 1 \leq \ell + r_{\ell'} \leq d} \frac{g \circ A'_{j+r_{j'}, \ell+r_{\ell'}}}{\|g \circ A'\|_{2,d} \cdot \|g \circ A\|_{2,d}} - \frac{g \circ A_{j, \ell}}{\|g \circ A\|_{2,d} \cdot \|g \circ A\|_{2,d}}
\]

\[
= \left\langle g \circ A' \left( \cdot + \frac{\ell_0}{d} \cdot + \frac{\ell_0}{d} \right), g \circ A \right\rangle_{2,d} \geq 1 - \frac{C_1(C_L, C_A)}{d} \tag{78}
\]

where the second equality follows from the fact that the support of \( g \circ A' \left( \cdot + \frac{j_0}{d}, \cdot + \ell_0/d \right) \) is contained in \([0,1]^2\), provided that Assumption 3 holds. To derive (76), it is sufficient to show

\[
\left\langle g \circ A' \left( \cdot + \frac{\ell_0}{d}, \cdot + \frac{\ell_0}{d} \right), g \circ A \right\rangle_{2,d} \geq 1 - \frac{C_1(C_L, C_A)}{d}.
\]

In the next step, we show that under the provided conditions, the functions \( g \circ A \) and \( g \circ A' \) satisfy (11) with constant \( 4C_A^3C_L \). Observe that for any \( g \) satisfying Assumption 1, and any \( A \) fulfilling Assumption 2, 3, we can derive

\[
\|g\|_1 = \int_{[0,1]^2} g(x, y) \, dx \, dy = \int_{[0,1]^2} g \circ A(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv \leq 2C_A^2 \|g \circ A\|_1. \tag{79}
\]

Using (79), applying Lemma A.1 yields for any real numbers \( u, u', v, v' \),

\[
|g \circ A(u, v) - g \circ A(u', v')| \leq 2C_AC_L \|g\|_1(|u - u'| + |v - v'|)
\]

\[
\leq 4C_A^3C_L \|g \circ A\|_1(|u - u'| + |v - v'|),
\]

which validates the claim. Applying Lemma B.2 (iv) with \( h = g \circ A' \) and \( g = g \circ A \) allows us to bound

\[
\frac{\left\langle g \circ A' \left( \cdot + \frac{\ell_0}{d}, \cdot + \frac{\ell_0}{d} \right), g \circ A \right\rangle_{2,d}}{\|g \circ A'\|_{2,d} \|g \circ A\|_{2,d}} \geq \int_{[0,1]^2} g \circ A' \left( u + \frac{\ell_0}{d}, v + \frac{\ell_0}{d} \right) g \circ A(u, v) \, du \, dv \geq 16C_LC_A^3 \left( 1 + \frac{4C_LC_A^3}{d} \right)^3. \tag{80}
\]
Now we derive a lower bound for the first summand in (80). Under Assumptions 1 and 3, one can derive
\[ \left\lVert g \circ A \left( \cdot + \frac{j_0}{d}, \cdot + \frac{\ell_0}{d} \right) - g \circ A \right\lVert_\infty \leq \frac{2C_L \|g\|_1}{d}. \]  
(81)

Let \( h_1, h_2 : [0, 1]^2 \to \mathbb{R} \) be two non-negative bounded functions. Using the triangle inequality,
\[ \|h_1 h_2\|_1 \geq \|h_1\|^2_2 - \|h_1(h_1 - h_2)\|_1 \geq \|h_1\|^2_2 - \|h_1 - h_2\|_\infty \|h_1\|_1. \]  
(82)

Applying this inequality with \( h_1 = g \circ A \) and \( h_2 = g \circ A' (\cdot + j_0/d, \cdot + \ell_0/d) \) yields
\[
\int_{[0,1]^2} g \circ A \left( u + \frac{j_0}{d}, v + \frac{\ell_0}{d} \right) g \circ A(u,v) \, du dv \geq \|g \circ A\|^2_2 - \frac{2C_L \|g\|_1 \|g \circ A\|_1}{d} \\
\geq \|g \circ A\|^2_2 - \frac{4C_L C_A^2 \|g \circ A\|_1 \|g \circ A'\|_1 \|g \circ A\|_1}{d}, \]  
(83)

where the first inequality follows from (81), and the second one uses (79). By the Cauchy-Schwarz inequality, \( \|g \circ A\|_1 \leq \|g \circ A\|_2 \), for all \( A \in \mathcal{A} \), which, together with (83), implies that
\[
\frac{\int_{[0,1]^2} g \circ A' \left( u + \frac{j_0}{d}, v + \frac{\ell_0}{d} \right) g \circ A(u,v) \, du dv}{\|g \circ A'\|_2 \|g \circ A\|_2} \geq \frac{\|g \circ A\|^2_2}{\|g \circ A'\|^2_2} - \frac{4C_L C_A^2 \|g \circ A\|_1 \|g \circ A'\|_1 \|g \circ A\|_1}{d}. \]  
(84)

Next, we proceed by bounding the term \( \|g \circ A\|^2_2 \|g \circ A'\|_2 \). Let \( h_1, h_2 : [0, 1]^2 \to \mathbb{R} \) be two non-negative bounded functions. Interchanging the role of \( h_1 \) and \( h_2 \) in (82) gives \( \|h_1 h_2\|_1 \geq \|h_2\|^2_2 - \|h_1 - h_2\|_\infty \|h_2\|_1 \). Using triangle inequality, we also obtain \( \|h_1\|^2_2 \geq \|h_2\|^2_2 - \|h_1(h_2 - h_1)\|_1 \geq \|h_1 h_2\|_1 - \|h_1\|_1 \|h_2 - h_1\|_\infty \). Combining the previous two inequalities gives \( \|h_1\|^2_2 \geq \|h_2\|^2_2 - (\|h_1\|_1 + \|h_2\|_1) \|h_1 - h_2\|_\infty / \|h_2\|^2_2 \). Applying this inequality with \( h_1 = g \circ A \) and \( h_2 = g \circ A' (\cdot + j_0/d, \cdot + \ell_0/d) \) as well as using (81) yields
\[
\frac{\|g \circ A\|^2_2}{\|g \circ A'\|^2_2} = \frac{\|g \circ A\|^2_2}{\|g \circ A'\|^2_2} \geq 1 - \frac{\|g \circ A\|_1 + \|g \circ A' (\cdot + \frac{j_0}{d}, \cdot + \frac{\ell_0}{d})\|_1}{\|g \circ A'\|^2_2} \cdot \frac{2C_L \|g\|_1}{d} \\
= 1 - \frac{\|g \circ A\|_1 + \|g \circ A'\|_1}{\|g \circ A'\|^2_2} \cdot \frac{2C_L \|g\|_1}{d} \\
\geq 1 - \frac{4C_L C_A^2 \|g \circ A'\|_1}{d} \left( \frac{\|g \circ A\|_1}{\|g \circ A'\|_1} + 1 \right), \]  
(85)

where the second equality follows from Assumptions 2 and 3, the second-to-last inequality comes from (79), and the last inequality is obtained using \( \|g \circ A'\|_1 \leq \|g \circ A'\|_2 \). Moreover,
\[
\frac{\|g \circ A\|_1}{\|g \circ A'\|_1} = 1 + \frac{\int_{[0,1]^2} g \circ A(u,v) - g \circ A' (u + \frac{j_0}{d}, v + \frac{\ell_0}{d}) \, du dv}{\|g \circ A\|_1}, \]
\[ \leq 1 + \frac{\|g \circ A' (\cdot + \frac{t_0}{2}, \cdots + \frac{t_d}{2}) - g \circ A\|_\infty}{\|g \circ A'\|_1} \]
\[ \leq 1 + \frac{2C_L\|g\|_1}{d} \frac{1}{\|g \circ A'\|_1} \]
\[ \leq 1 + \frac{4C_L C_A^2}{d}, \quad (86) \]

where the second-to-last inequality comes from (81), and the last inequality is due to (79). By plugging (86) into (85), we deduce that
\[ \frac{\|g \circ A\|^2_2}{\|g \circ A'\|^2_2} \geq 1 - \frac{4C_L C_A^2}{d} \left( 2 + \frac{4C_L C_A^2}{d} \right) = 1 - \frac{8C_L C_A^2}{d} \left( 1 + \frac{2C_L C_A^2}{d} \right), \]

which implies that
\[ \frac{\|g \circ A\|_2}{\|g \circ A'\|_2} \geq 1 - \frac{8C_L C_A^2}{d} \left( 1 + \frac{2C_L C_A^2}{d} \right). \quad (87) \]

Combining (80), (84) and (87), we obtain
\[ \frac{\langle g \circ A', g \circ A (\cdot + \frac{t_0}{2}, \cdots + \frac{t_d}{2}) \rangle_{2,d}}{\|g \circ A'\|_{2,d} \|g \circ A\|_{2,d}} \geq \frac{\|g \circ A\|_2}{\|g \circ A'\|_2} - \frac{4C_L C_A^2}{d} \frac{d^2}{d^2} - \frac{64C_L C_A^3}{d} \left( 1 + \frac{4C_L C_A^3}{d} \right)^3 \]
\[ \geq 1 - \frac{16C_L^2 C_A^4}{d^2} - \frac{12C_L C_A^2}{d} - \frac{64C_L C_A^3}{d} \left( 1 + \frac{4C_L C_A^3}{d} \right)^3 \]
\[ \geq 1 - \frac{C_1(C_L, C_A)}{d}, \quad (88) \]

where \( C_1(C_L, C_A) \) is a universal constant depending only on \( C_L \) and \( C_A \). This proves (i).

Next we proceed to prove (ii). With Lemma B.1 and the non-negativity of \( f \), we rewrite
\[ |\sigma([w_{f,O' L}] \star \mathbf{X}_{g \circ A})|_\infty = \max_{r,r' \in \{-d, \ldots, d\}} \frac{\sum_{j} \sum_{\ell} \int f \circ A'_{j+r, \ell+r'} \cdot \frac{1}{1 \leq j+r \leq d \leq \ell \leq r' \leq d} \cdot \frac{g \circ A_{j, \ell}}{d^2 \|f \circ A'\|_{2,d} \|g \circ A\|_{2,d}}}{\|f \circ A'\|_{2,d} \|g \circ A\|_{2,d}} \]
\[ \leq \max_{r,r' \in \{-d, \ldots, d\}} \frac{\langle f \circ A' (\cdot + \frac{r}{2}, \cdots + \frac{r_d}{2}, \cdots) \rangle_{2,d}}{\|f \circ A'\|^2_{2,d} \|g \circ A\|^2_{2,d}} \geq 1 + \frac{4C_L C_A^3}{d} \left( 1 + \frac{4C_L C_A^3}{d} \right)^3 \quad (89) \]

By Proposition B.3 and the fact that \( \|g\|_2 \geq \|g\|_1 \),
\[ \inf_{u,s,s' \in \mathbb{R}, A \in \mathcal{A}} \frac{\|a f \circ A'(\cdot + s, \cdots + s') - g \circ A\|_{L^2(\mathbb{R}^2)}}{\|g\|^2_2} \]
\[ \leq 4C_L C_A^2 \inf_{u,s,s' \in \mathbb{R}, A \in \mathcal{A}} \frac{\|a f \circ A'(\cdot + s, \cdots + s') - g \circ A\|_{L^2(\mathbb{R}^2)}}{\|g \circ A\|^2_2} \]

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This requires (Lemma B.5. There exists a network $\text{Max}^r \in \mathcal{F}_{id}(1 + 2\lceil \log_2 r \rceil, (r, 2r, \ldots, 2r, 1))$, such that $\text{Max}^r \in [0, 1]$, $\text{Max}^r(\mathbf{x}) = \max\{x_1, \ldots, x_r\}$ for all $\mathbf{x} = (x_1, \ldots, x_r) \in [0, 1]^r$, and all network parameters are bounded in absolute value by 1.

Proof. Due to the identity $\max\{y, z\} = ((y - z)_+ + z)_+$ that holds for all $y, z \in [0, 1]$, one can compute $\max\{y, z\}$ by a network $\text{Max}(y, z)$ with two hidden layers and width vector $(2, 2, 1, 1)$. This network construction involves five non-zero weights.

In a second step we now describe the construction of the $\text{Max}^r$ network. Let $q = \lceil \log_2 r \rceil$. In the first hidden layer the network computes

$$\text{(91)}$$

This requires $r$ network parameters. Next we apply the network $\text{Max}(y, z)$ from above to the pairs $(x_1, x_2), (x_3, x_4), \ldots, (0, 0)$ in order to compute

$$\text{(Max}(x_1, x_2), \text{Max}(x_3, x_4), \ldots, \text{Max}(0, 0)) \in [0, 1]^{2^q-1}.$$ 

This reduces the length of the vector by a factor two. By consecutively pairing neighboring entries and applying $\text{Max}$, the procedure is continued until there is only one entry left. Together with the layer (91), the resulting network $\text{Max}^r$ has $2q + 1$ hidden layers. It can be realized by taking width vector $(r, 2r, 2r, \ldots, 2r, 1)$. 

The integral in the last step can be restricted to $[0, 1]^2$ because by the assumptions, the support of the function $g \circ A$ is contained in $[0, 1]^2$. Rewriting the previous inequality gives

$$\int_{[0,1]^2} \frac{f \circ A'(u + \frac{r}{d}, v + \frac{r'}{d})}{\|f \circ A'\|_2 \|g \circ A\|_2} \leq 1 - \frac{\inf_{g, s, s' \in \mathcal{F}_{id}, A, A' \in A} \|a \circ A'(- + s, \cdot + s') - g \circ A\|^2_{L^2(\mathbb{R}^2)}}{16C_A^4C_L^2.}$$

with $D(f, g)$ as in (12). By interchanging the role of $g$ and $f$ in (90), we finally get the upper bound $1 - (D^2(f,g) + D^2(g,f))/ [16C_A^4C_L^2]$ in the previous inequality. Together with (89), the asserted inequality in (ii) follows. 

The next lemma shows how one can compute the maximum of a $r$-dimensional vector with a fully connected neural network.
We have $\text{Max}(y, z) = \max\{y, z\}$ and thus also $\text{Max}^r(x_1, \ldots, x_r) = \max\{x_1, \ldots, x_r\}$, proving the assertion.

One can reduce the number of required layers to $1 + \lceil \log_2 r \rceil$ on the cost of a more involved proof.

Proof of Theorem 4.2. In the first step of the proof, we explain the construction of the CNN. By assumption, we can define for every $A \in \mathcal{A}_d$ and for any of the classes $k \in \{0, 1\}$, a matrix $w_{f_k \circ A} = (w_{f_k \circ A, j, \ell})_{j, \ell}$ with entries

$$w_{f_k \circ A, j, \ell} = \frac{f_k \circ A, j, \ell}{\|f_k \circ A\|_2}.$$  

The corresponding filter is then defined as the quadratic support $[w_{f_k \circ A}]$ of the matrix $w_{f_k \circ A}$. Since we have $|\mathcal{A}_d|$ possible choices for the elements of the subset $\mathcal{A}_d$ and two different template functions $f_0$ and $f_1$, this results in at most $2|\mathcal{A}_d|$ different filters. Since each filter corresponds to a feature map $\sigma([w_{f_k \circ A}] \star \mathbf{X})$, we also have at most $2|\mathcal{A}_d|$ feature maps. Among those, half of them correspond to class zero and the other half to class one.

Now, a max-pooling layer is applied to the output of each filter map. As explained before, in our framework the max-pooling layer extracts the signal with the largest absolute value. Application of the max-pooling layer thus yields a network with outputs

$$O_{k, A} = |\sigma([w_{f_k \circ A}] \star \mathbf{X})|_\infty.$$  

In the last step of the network construction, we take several fully connected layers, that extract on the one hand the largest value of $O_{0, A}$ and on the other hand the largest value of $O_{1, A}$. Applying two networks $\text{Max}^r$ from Lemma B.5 and with $r = |\mathcal{A}_d|$ in parallel leads to a network with two outputs

$$\left(\max\{O_{0, A} : A \in \mathcal{A}_d\}, \max\{O_{1, A} : A \in \mathcal{A}_d\}\right).$$  

By Lemma B.5, the two parallelized $\text{Max}^r$ networks are in the network class $\mathcal{F}_{id}(1 + 2\lceil \log_2 r \rceil, (2r, 4r, \ldots, 4r, 2))$ with $r = |\mathcal{A}_d|$.

In the last step the softmax function, $\Phi(x_1, x_2) = (e^{x_1}/(e^{x_1} + e^{x_2}), e^{x_2}/(e^{x_1} + e^{x_2}))$ is applied. This guarantees that the output of the network is a probability vector over the two classes 0 and 1. The whole network construction is contained in the CNN class $G(r)$ that has been introduced in (24).

For this CNN, we now derive a bound for the approximation error. Denote by $A_* \in \mathcal{A}$ the true deformation for the generic image $\mathbf{X} = (X_{j, \ell})_{j, \ell=1,\ldots,d}$, namely,

$$X_{j, \ell} = d^2 \eta \int_{l_{j, \ell}} f_k(A_*(u, v)) \, dudv.$$  

As a first case, assume that $k = 0$ and thus, $f_0$ is the corresponding template function. By assumption, the conditions of Proposition B.4 are satisfied and we conclude that there exist $A' \in \mathcal{A}_d$ and a corresponding filter $w_{f_0 \circ A'}$ such that

$$|\sigma([w_{f_0 \circ A'}] \star \mathbf{X})|_\infty \geq 1 - \frac{C_1(C_L, C_A)}{d}.$$  

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Proposition B.4 (ii) further shows that all feature maps based on the template function \(f_1\) are bounded by

\[
\max_{A' \in \mathcal{A}_d} |\sigma([w_{f_1 \circ A'}] \ast \mathbf{X})|_\infty \leq 1 - \frac{D^2}{16C^2_\mathcal{A}C^2_L} + \frac{C_2(C_L, C_\mathcal{A})}{d},
\]

with \(D\) as in (12). This, in turn, means that the two outputs \((z_0, z_1)\) of the network (92) can be bounded by

\[
z_0 \geq 1 - \frac{C_1(C_L, C_\mathcal{A})}{d} \quad \text{and} \quad z_1 \leq 1 - \frac{D^2}{16C^2_\mathcal{A}C^2_L} + \frac{C_2(C_L, C_\mathcal{A})}{d}.
\]

As the softmax function \(\Phi\) is applied to the network output, we have \(p = \Phi(z_1, z_2)\) and

\[
p_1(\mathbf{X}) = \frac{e^{z_0}}{e^{z_0} + e^{z_1}}, \quad p_2(\mathbf{X}) = \frac{e^{z_1}}{e^{z_0} + e^{z_1}}.
\]

Set \(\kappa := 16C^2_\mathcal{A}C^2_L [C_1(C_L, C_\mathcal{A}) + C_2(C_L, C_\mathcal{A})]\). Provided \(D^2 > \kappa/d\), we deduce that, \(p_1(\mathbf{X}) > p_2(\mathbf{X})\) hence \(1 - p_2(\mathbf{X}) > 1/2 = 0\). This shows the case when \(k = 0\). If \(k = 1\), the argumentation is completely analogous.

Proof of Lemma 4.3. For such values of \(D\), Theorem 4.2 shows that there exists a function \(p = (p_1, p_2)\) belonging to \(G(|\mathcal{A}_d|)\) such that \(1 - p_2(\mathbf{X}) > 1/2 = k(\mathbf{X})\). This shows that \(k\) can be written as a deterministic function evaluated at \(\mathbf{X}\). To see that \(k(\mathbf{X})\) equals the conditional class probability \(p(\mathbf{X})\), observe that

\[
p(\mathbf{X}) = \mathbf{P}(k = 1|\mathbf{X}) = \mathbf{E}[1(k(\mathbf{X}) = 1)|\mathbf{X}] = 1(k(\mathbf{X}) = 1) \mathbf{E}[1|\mathbf{X}] = 1(k(\mathbf{X}) = 1) = k(\mathbf{X}).
\]

The following oracle inequality decomposes the excess misclassification probability of the estimator defined as in (27) into two terms, namely a term measuring the complexity of the function class and a term measuring the approximation power. The complexity is measured via the VC-dimension (Vapnik-Chervonenkis-Dimension). We first introduce the VC-classes of sets as follows.

**Definition B.6.** Let \(\mathcal{A}\) be a class of subsets of \(\mathcal{X} \subseteq \mathbb{R}^d\) with \(\mathcal{A} \neq \emptyset\). Then the VC dimension (Vapnik-Chervonenkis-Dimension) \(V_\mathcal{A}\) of \(\mathcal{A}\) is

\[
V_\mathcal{A} := \sup\{m \in \mathbb{N} : S(\mathcal{A}, m) = 2^m\},
\]

where \(S(\mathcal{A}, m) := \max_{\{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subseteq \mathbb{R}^d} |\{\mathcal{A} \cap \{\mathbf{x}_1, \ldots, \mathbf{x}_m\} : A \in \mathcal{A}\}|\) denotes the \(m\)-th shatter coefficient.

Besides VC-classes of sets, we also use the concept of VC-classes of functions.

**Definition B.7.** Recall that an (open) subgraph of a function \(f : \mathcal{X} \rightarrow \mathbb{R}\) in \(\mathcal{F}\) is the subset of \(\mathcal{X} \times \mathbb{R}\) given by

\[
\mathcal{S}_f = \{(x, t) \in \mathcal{X} \times \mathbb{R} : t < f(x)\}.
\]

A collection \(\mathcal{F}\) of measurable functions on \(\mathcal{X}\) is called a VC-subgraph class, or just a VC-class, with dimension not larger than \(V\) if, \(\mathcal{F}^+ := \{\mathcal{S}_f : f \in \mathcal{F}\}\) is a VC-class of set in \(\mathcal{X} \times \mathbb{R}\) with dimension \(V_\mathcal{F}^+\) not larger than \(V\).
Now, we present the oracle inequality, upon which we will base the proof of Theorem 4.1.

**Lemma B.8** (Corollary 5.3 in [6]). Assume that \((X_1, k_1), \ldots, (X_n, k_n)\) are i.i.d. copies of a random vector \((X, k) \in \mathbb{R}^d \times \{0, 1\}\). Let \(\tilde{g}_n\) be the classifier

\[
\tilde{g}_n \in \arg \min_{g \in C} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(g(X_i) \neq k_i),
\]

based on some function class \(C \subseteq \{g : [0, 1]^d \to \{0, 1\}\} \) with finite VC dimension \(V\). Then, there exists a universal constant \(C_1\) such that, for any positive integer \(n\), with probability at least \(1 - \delta\),

\[
P\{\tilde{g}_n(X) \neq k \mid D_n\} - \inf_{g \in C} P\{g(X) \neq k\} \leq C_1 \left( \inf_{g \in C} P\{g(X) \neq k\} \frac{V \log n + \log \frac{1}{\delta}}{n} + \frac{V \log n + \log \frac{1}{\delta}}{n} \right).
\]

If the \(\Phi\) in the class \(\mathcal{G}(m)\), as defined in (24), is replaced by the identity \(id\), we denote the resulting class by \(\mathcal{G}_{id}(m)\). Let

\[
\mathcal{H}_\rho(m) := \{\rho \circ (f_1 - f_2) : (f_1, f_2) \in \mathcal{G}_{id}(m)\}
\]

with

\[
\rho(z) = \frac{1}{1 + e^z}.
\]

Denote \(\mathcal{C}(m) = \{x \mapsto \mathbf{1}(p(x) > 1/2), \ p \in \mathcal{H}_\rho(m)\}\). For the proof, it is important to note that

\[
\arg \min_{p = (p_1, p_2) \in \mathcal{G}(m)} -\frac{1}{n} \sum_{i=1}^{n} k_i \mathbf{1}(p_2(X_i) > \frac{1}{2}) = \arg \min_{f \in \mathcal{C}(m)} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(f(X_i) \neq k_i).
\]

(94)

Indeed, in the minimization problem, only the second entry of the vector \((p_1, p_2)\) is considered. By definition, \(p_2\) is of the form

\[
\frac{e^{z_2}}{e^{z_1} + e^{z_2}} = \frac{1}{1 + e^{z_1 - z_2}} = \rho(z_2 - z_1),
\]

with \((z_1, z_2) \in \mathcal{G}_{id}(m)\). Hence \((p_1, p_2) \in \mathcal{G}(m) \iff p_2 \in \mathcal{H}_\rho(m)\). With (94) it is enough to analyze the VC dimension of the function class \(\mathcal{H}_\rho(m)\). To do this, we use the following auxiliary results.

**Lemma B.9.** Let \(\mathcal{F}\) be a family of real functions on \(\mathbb{R}^m\), and let \(g : \mathbb{R} \to \mathbb{R}\) be a fixed non-decreasing function. Define the class \(\mathcal{G} = \{g \circ f : f \in \mathcal{F}\}\). Recall that \(\mathcal{G}^+ = \{(x, t) \in \mathbb{R}^m \times \mathbb{R} : t < h(x)\} : h \in \mathcal{G}\) and \(\mathcal{F}^+ = \{(x, t) \in \mathbb{R}^m \times \mathbb{R} : t < h(x)\} : h \in \mathcal{F}\). Then

\[
V_{\mathcal{G}^+} \leq V_{\mathcal{F}^+}.
\]

**Proof.** See Lemma 16.3 in [16].

**Lemma B.10.** If \(\mathcal{H}\) is a class of VC-subgraph real-valued measurable functions on \(\mathbb{R}^m\) with VC-dimension \(V_{\mathcal{H}^+}\), then, the function class

\[
\mathcal{C} = \left\{x \mapsto \mathbf{1} \left( f(x) > \frac{1}{2} \right) : f \in \mathcal{H} \right\}
\]

is VC-subgraph on \(\mathbb{R}^m\) with a dimension not greater than \(V_{\mathcal{H}^+}\).
Proof. According to Proposition 2.1 of [3], the class $\mathcal{H}$ is weakly VC-major with a dimension no larger than $V_{\mathcal{H}^+}$. This means that the collection of subsets

$$\mathcal{A}_\mathcal{H} = \left\{ x \in \mathbb{R}^m \text{ such that } f(x) > \frac{1}{2} \right\} : f \in \mathcal{H},$$

is a VC-class of subsets of $\mathbb{R}^m$ with a dimension not larger than $V_{\mathcal{H}^+}$. Due to $C = \{ 1_A, A \in \mathcal{A}_\mathcal{H} \}$, the conclusion follows from the property of VC-classes of functions (see, for instance, page 275 of [43]).

Lemma B.11. Recall that $\mathcal{H}_\rho(m)$ is defined as in (93) and $\mathcal{H}_\rho^+(m)$ denotes the class of subgraphs associated with the function class $\mathcal{H}_\rho(m)$. Let $m \geq d^2 \geq 4$. There exists a universal constant $C_2 > 0$ such that

$$V_{\mathcal{H}_\rho^+(m)} \leq C_2 m^2 \log^3 m.$$

Proof. Define

$$\mathcal{H}_\rho^+(m) = \left\{ \rho \circ (f \circ g) : f \in \mathcal{F}_{\text{id}}(1, (2, 2, 1)), g \in \mathcal{G}_{\text{id}}(m) \right\}.$$

The identity $z_1 - z_2 = \sigma(z_1 - z_2) - \sigma(z_2 - z_1) \in \mathcal{F}_{\text{id}}(1, (2, 2, 1))$ shows that the class $\mathcal{H}_\rho(m)$ defined in (93) is a subset, that is,

$$\mathcal{H}_\rho(m) \subseteq \mathcal{H}_\rho^+(m). \quad (95)$$

It follows that $V_{\mathcal{H}_\rho^+(m)} \leq V_{\mathcal{H}_\rho^+(m)}$. It is thus enough to derive a VC dimension bound for the larger function class. Now $\rho$ is a fixed non-decreasing function. Applying Lemma B.9 yields

$$V_{\mathcal{H}_\rho^+(m)} \leq V_{\mathcal{F}_{\text{id}}(1, (2, 2, 1)) \circ \mathcal{G}_{\text{id}}(m)^+}. \quad (96)$$

Now one can rewrite

$$\mathcal{F}_{\text{id}}(1, (2, 2, 1)) \circ \mathcal{G}_{\text{id}}(m) = \left\{ f \circ g : f \in \mathcal{F}_{\text{id}}(3 + 2 \lceil \log_2 m \rceil, (2m, 4m, \ldots, 4m, 2, 2, 1)), g \in \mathcal{F}_C(2m) \right\}$$

$$= : \mathcal{G}_{\text{id}}^+(m). \quad (97)$$

In the following we omit the dependence on $m$ in the function class $\mathcal{G}_{\text{id}}^+ := \mathcal{G}_{\text{id}}^+(m)$. To bound $V_{\mathcal{G}_{\text{id}}^+}$, we apply Lemma 11 in [21]. In their notation they prove on p.44 in [21] the bound

$$2(L_1 + L_2 + 1)W 4 \log_2 \left( 2e(L_1 + L_2 + 1)k_{\max} d_1 d_2 \right).$$

In the notation and for the specific architecture that we consider, this becomes

$$V_{\mathcal{G}_{\text{id}}^+} \leq 8(L + 3)W \log_2 \left( 2e(2m)(L + 3)(4m) d^2 \right) \quad (98)$$

with $W$ the total number of network parameters in the CNN and $L = 3 + \lceil \log_2 m \rceil$ the number of hidden layers.

As each filter has at most $d^2$ weights, the convolutional layer has at most $md^2$ many weights. Recall that there are $3 + \lceil \log_2 m \rceil$ hidden layers in the fully connected part and that the width vector is
(2m, 4m, ..., 4m, 2, 2, 1). The $3 + [\log_2 m]$ weight matrices have all at most $16m^2$ parameters. Moreover there are $[\log_2 m]4m + 4$ biases. This means that the overall number of network parameters $W$ for a CNN $g \in G'_{id}^+$ can be bounded by

$$W \leq md^2 + (4 + \log_2 m)20m^2 \leq 61m^2 \log_2 m,$$

using $m \geq d^2 \geq 4$ for the last step. Since also $L + 3 \leq 7 + \log_2 m \leq 5\log_2 m$, (98) can then be bounded as follows $V_{G'_{id}^+} \leq 40 \cdot 61m^2 \log_2 m \log_2 (80emd^2 \log_2 m) \leq C_2m^2 \log^3 m$ for a universal constant $C_2 > 0$. Together with (96) and (97), the result follows.

Proof of Theorem 4.1. The proof is based on the application of Lemma B.8. Let $m := |A_d|$. Lemma B.11 applied to $m = |A_d|$ yields that for $|A_d| \geq d^2 \geq 4$,

$$V_{\mathcal{H}_m^+} \leq C_2|A_d|^2 \log^3(|A_d|). \quad (99)$$

For $D > \sqrt{\kappa/d}$, Theorem 4.2 implies existence of a network $\hat{p} = (\hat{p}_1, \hat{p}_2) \in G(|A_d|)$, such that the corresponding classifier $\hat{k}(X) = 1(\hat{p}_2(X) > 1/2)$ satisfies $\hat{k}(X) = k$, almost surely. Thus,

$$\inf_{p=(p_1,p_2)\in G(m)} P(\hat{k}(X) \neq k) = 0. \quad (100)$$

We now apply Lemma B.8. Due to (100), the first term in the upper bound disappears and we obtain that for any $|A_d| \geq d^2 \geq 4$, with probability at least $1 - \delta$,

$$P(\hat{k}(X) \neq k|\mathcal{D}_n) \leq C_1V_{\mathcal{H}_m^+}\frac{\log n + \log \frac{1}{\delta}}{n}.$$ 

For a random variable $Z$, writing $Z_+ = \max\{Z, 0\}$ yields $E[Z] \leq E[Z_+] = \int_0^{\infty} P(Z_+ > t) dt = \int_0^{\infty} P(Z > t) dt$. Thus integration with respect to $\delta$ gives

$$\int_0^{\infty} P(\hat{k}(X) \neq k|\mathcal{D}_n) - \log n \leq \int_0^{\infty} P\left(\frac{n}{C_1V_{\mathcal{H}_m^+}} P(\hat{k}(X) \neq k|\mathcal{D}_n) - \log n > t\right) dt$$

$$= \int_0^{\infty} P\left(P(\hat{k}(X) \neq k|\mathcal{D}_n) > C_1V_{\mathcal{H}_m^+}\frac{\log n + t}{n}\right) dt$$

$$\leq \int_0^{\infty} e^{-t} dt$$

$$= 1,$$

which together with (99) implies that for $C = C_1C_2 > 0$,

$$P(\hat{k}(X) \neq k) = E_{\mathcal{D}_n}\left[P(\hat{k}(X) \neq k|\mathcal{D}_n)\right] \leq \frac{C_1|A_d|^2 \log^3(|A_d|)(\log n + 1)}{n},$$

with $P$ the distribution over all randomness in the data and the new sample $X$. The assertion then follows. \[\square\]
References

[1] S. Arlot and P. L. Bartlett, *Margin-adaptive model selection in statistical learning*, Bernoulli, 17 (2011), pp. 687–713.

[2] J. Ashburner, *A fast diffeomorphic image registration algorithm*, NeuroImage, 38 (2007), pp. 95–113.

[3] Y. Baraud, *Bounding the expectation of the supremum of an empirical process over a (weak) VC-major class*, Electron. J. Statist., 10 (2016), pp. 1709–1728.

[4] P. L. Bartlett, M. I. Jordan, and J. D. McAuliffe, *Convexity, classification, and risk bounds*, J. Am. Stat. Assoc., 101 (2006), pp. 138–156.

[5] T. Bos and J. Schmidt-Hieber, *Convergence rates of deep ReLU networks for multiclass classification*, Electron. J. Stat., 16 (2022), pp. 2724–2773.

[6] S. Boucheron, O. Bousquet, and G. Lugosi, *Theory of classification: a survey of some recent advances*, ESAIM Probab. Statist., 9 (2010), pp. 323–375.

[7] J. Bruna and S. Mallat, *Classification with scattering operators*, in CVPR, 2011, pp. 1561–1566.

[8] J. Bruna and S. Mallat, *Invariant scattering convolution networks*, IEEE Trans. Pattern Anal. Mach. Intell., 35 (2013), pp. 1872–1886.

[9] T. Cohen and M. Welling, *Group equivariant convolutional networks*, in International conference on machine learning, PMLR, 2016, pp. 2990–2999.

[10] T. Cover and P. Hart, *Nearest neighbor pattern classification*, IEEE Trans. Inf. Theory, 13 (1967), pp. 21–27.

[11] A. Delaigle and P. Hall, *Achieving near perfect classification for functional data*, J. R. Stat. Soc. Ser. B. Stat. Methodol., 74 (2012), pp. 267–286.

[12] L. Deng, *The MNIST database of handwritten digit images for machine learning research*, IEEE Signal Process Mag., 29 (2012), pp. 141–142.

[13] J. Gluckman, *Scale variant image pyramids*, in Computer Vision and Pattern Recognition, 2006.

[14] R. C. Gonzalez and R. E. Woods, *Digital Image Processing*, Pearson Education Inc., 2008.

[15] U. Grenander, *Lectures in Pattern Theory I, II and III: Pattern Analysis, Pattern Synthesis and Regular Structures*, Springer-Verlag, Heidelberg-New York, 1976-1981.

[16] L. Györfi, M. Kohler, A. Krzyśak, and H. Walk, *A Distribution-Free Theory of Nonparametric Regression*, Springer Series in Statistics, Springer, 2002.
[17] R. M. Haralick and L. G. Shapiro, *Image segmentation techniques*, Computer Vision, Graphics, and Image Processing, 29 (1985), pp. 100–132.

[18] M. Hashemi, *Enlarging smaller images before inputting into convolutional neural network: zero-padding vs. interpolation*, J. Big Data, 6 (2019).

[19] J. Jacques and C. Preda, *Functional data clustering: a survey*, Advances in Data Analysis and Classification, 8 (2014), pp. 231–255.

[20] N. Khatri, A. Dasgupta, Y. Shen, X. Zhong, and F. Y. Shih, *Perspective transformation layer*, in CSCI 2022, 2022, pp. 1395–1401.

[21] M. Kohler, A. Krzyżak, and B. Walter, *On the rate of convergence of image classifiers based on convolutional neural networks*, Ann. Inst. Statist. Math., 74 (2022), pp. 1085–1108.

[22] M. Kohler and S. Langer, *Statistical theory for image classification using deep convolutional neural network with cross-entropy loss under the hierarchical max-pooling model*, Journal of Statistical Planning and Inference, 234 (2025).

[23] S. Korman, D. Reichman, G. Tsur, and S. Avidan, *Fast-match: Fast affine template matching*, in CVPR, 2013, pp. 2331–2338.

[24] A. Krizhevsky, I. Sutskever, and G. E. Hinton, *Imagenet classification with deep convolutional neural networks*, Commun. ACM, 60 (2017), pp. 84–90.

[25] Y. LeCun, Y. Bengio, and G. Hinton, *Deep learning*, Nature, 521 (2015), pp. 436–444.

[26] S. Mallat, *Group invariant scattering*, Comm. Pure Appl. Math., 65 (2012), pp. 1331–1398.

[27] D. Marcos, M. Volpi, and D. Tuia, *Learning rotation invariant convolutional filters for texture classification*, in 2016 23rd International Conference on Pattern Recognition (ICPR), 2016, pp. 2012–2017.

[28] J. S. Marron, J. O. Ramsay, L. M. Sangalli, and A. Srivastava, *Functional data analysis of amplitude and phase variation*, Statist. Sci., 30 (2015), pp. 468–484.

[29] S. Minaee, Y. Y. Boykov, F. Porikli, A. J. Plaza, N. Kehtarnavaz, and D. Terzopoulos, *Image segmentation using deep learning: A survey*, IEEE Trans. Pattern Anal. Mach. Intell., (2021), pp. 1–1.

[30] D. Mumford, *Perception as Bayesian Inference*, Cambridge University Press, 1996, ch. Pattern theory: A unifying perspective, pp. 25–62.

[31] D. Mumford, *The statistical description of visual signals*, unpublished manuscript, 2000.
[32] D. Mumford and A. Desolneux, *Pattern theory*, Applying Mathematics, A K Peters, Ltd., Natick, MA, 2010.

[33] D. Park, D. Ramanan, and C. C. Fowlkes, *Multiresolution models for object detection*, in ECCV, 2010.

[34] J. O. Ramsay and B. W. Silverman, *Functional data analysis*, Springer Series in Statistics, Springer, New York, second ed., 2005.

[35] W. Rawat and Z. Wang, *Deep convolutional neural networks for image classification: A comprehensive review*, Neural Comput., 29 (2017), pp. 2352–2449.

[36] F. Rossi and N. Villa, *Support vector machine for functional data classification*, Neurocomputing, 69 (2006), pp. 730–742. New Issues in Neurocomputing: 13th European Symposium on Artificial Neural Networks.

[37] J. Schmidhuber, *Deep learning in neural networks: An overview*, Neural Netw., 61 (2015), pp. 85–117.

[38] X. Shi, V. De-Silva, Y. Aslan, E. Ekmeckioglu, and A. Kondoz, *Evaluating the learning procedure of CNNs through a sequence of prognostic tests utilising information theoretical measures*, Entropy (Basel), 24 (2021), p. 67.

[39] P. Simard, D. Steinkraus, and J. Platt, *Best practices for convolutional neural networks applied to visual document analysis*, in Seventh International Conference on Document Analysis and Recognition, 2003. Proceedings., 2003, pp. 958–963.

[40] A. Sotiras, C. Davatzikos, and N. Paragios, *Deformable medical image registration: A survey*, IEEE Transactions on Medical Imaging, 32 (2013), pp. 1153–1190.

[41] R. Szeliski, *Computer Vision: Algorithms and Applications.*, Springer-Verlag, 2010.

[42] P. Tarasiuk and M. Pryczek, *Geometric transformations embedded into convolutional neural networks*, J Appl Comput Sci., 24 (2016), p. 33–48.

[43] A. W. v. d. Vaart, *Asymptotic Statistics*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 1998.

[44] N. van Noord and E. Postma, *Learning scale-variant and scale-invariant features for deep image classification*, Pattern Recognit., 61 (2017), pp. 583 – 592.

[45] V. N. Vapnik, *Statistical Learning Theory*, Wiley-Interscience, 1998.

[46] J.-L. Wang, J.-M. Chiou, and H.-G. Müllner, *Functional data analysis*, Annual Review of Statistics and Its Application, 3 (2016), pp. 257–295.
[47] R. Yamashita, M. Nishio, R. K. G. Do, and K. Togashi, *Convolutional neural networks: an overview and application in radiology*, Insights into Imaging, 9 (2018), pp. 611–629.

[48] X. Yan, J. Yang, E. Yumer, Y. Guo, and H. Lee, *Perspective transformer nets: Learning single-view 3d object reconstruction without 3d supervision*, in NeurIPS 2016, 2016, pp. 1704–1712.

[49] T. Zhang, *Statistical behavior and consistency of classification methods based on convex risk minimization*, Ann. Statist., 32 (2004), pp. 56–85.