Analytic approximation of the Tavis–Cummings ground state via projected states

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Abstract
We show that an excellent approximation to the exact quantum solution of the ground state of the Tavis–Cummings model is obtained by means of a semi-classical projected state. This state has an analytical form in terms of model parameters and, in contrast to the exact quantum state, allows for an analytical calculation of the expectation values of field and matter observables, entanglement entropy between field and matter, squeezing parameter and population probability distributions. The fidelity between this projected state and the exact quantum ground state is very close to 1, except for the region of classical phase transitions. We compare the analytical results with those of the exact solution obtained through direct Hamiltonian diagonalization as a function of atomic separation energy and matter–field coupling.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Progress in the technology of trapped-atom lasers and cavity quantum electrodynamics (QED) experiments has caught the interest of researchers again in the basic models that describe the interaction between quantized radiation and atoms. The Dicke model (DM) [1] describes the interaction of a quantized radiation field with a sample of $N$ two-level atoms located within a distance smaller than the wavelength of the radiation. Dicke realized that under certain conditions a gas of radiating molecules shows collective behavior called superradiance. This phenomenon was observed experimentally in optically pumped HF gas [2]. The simplest case $N = 1$ in the rotating wave approximation is known as the Jaynes–Cummings model (JCM) [3]. Many theoretical predictions of this latter model, such as the existence of collapse and revivals in Rabi oscillations [4], the formation of macroscopic quantum states or measures of entanglement associated with spin-squeezed states, have been confirmed, and many experimental studies of Rydberg atoms with very large principal quantum numbers within single-mode cavities have been observed [5]. It is well known that the JCM has served as a guide in understanding several quantum optics phenomena. While this considers the rotating wave approximation in order to discard the non-conserving energy terms, the model is widely applicable. In 1991 it was proposed [6] that by using circularly polarized light together with atomic selection rules the non-conserving energy terms can be eliminated; additionally, the diamagnetic term can be taken into account by properly changing the frequency of the field mode as a function of the coupling interaction strength. Renormalization of the field mode frequency can be understood by making a Bogoliubov transformation on the field part of the Dicke Hamiltonian. This implies that, at least in some cases, the results of the JCM can be valid beyond the rotating wave approximation.

The JCM, which might well be called the standard model of quantum optics, has been studied intensively for more than 40 years. An experimental example is provided
by Aoki et al [7]. When there are many atoms interacting with a single-mode quantized radiation field of one and the same cavity, the exact solution can be obtained by means of the so-called Tavis–Cummings model [8] (TCM) and its recent generalizations [9]. This model predicts the collective $N$-atom interaction strength to be $\gamma_N = \sqrt{N} \gamma_i$, where $\gamma_i$ is the dipole coupling strength of each individual atom $i$. The TCM has also been considered to describe cavity QED with a Bose–Einstein condensate [10]. Exact solutions of the generalized TCM have been found by means of quantum inverse methods, an algebraic procedure based on finding sets of solutions to the Bethe equations [11]. Other algebraic methods solve the eigenvalue problem of the TCM using polynomially deformed algebras [12] where analytical expressions may be found up to third order in a specific perturbation theory.

Since the presence of superradiant phase transitions in the DM was established [13], there have been several contributions simplifying the original computation and others that have also found the presence of phase transitions in generalized DMs [14–16]. In particular, one of the contributions showed the existence of phase transitions in a Dicke Hamiltonian that includes the counter-rotating atom–field interaction, although the necessary condition between the coupling parameter and the field mode frequency was modified [16].

In 1975 it was shown that the superradiant phase transition was due to the absence of the diamagnetic term in the Hamiltonian describing the $N$ two-level atoms interacting with a one-mode electromagnetic field [17]. It established, through the Thomas–Reiche–Kuhn sum rule, that the phase transition cannot be reached because it would place contradictory bounds on the parameters of the model. Other authors [18–20] have established that gauge invariance requires the presence of diamagnetic and counter-rotating terms; moreover, the sum rule can be derived from charge-current conservation. For this reason, a criterion was sought to establish the validity of DMs and determine whether a superradiant phase transition can occur. It was shown that two-level atoms that make electric dipole transitions cannot exhibit superradiant phase transitions, although their result does not rule out phase transitions based on magnetic dipole interactions. In the last decade, the work of Crisp was extended to many two-level atoms, and it was concluded that in the strong coupling regime, in the thermodynamical limit, it is not possible to have superradiant phase transitions [21].

It is however known that, for a one-electron atom interacting with a non-quantized electromagnetic field (that is, in the semiclassical approximation), a unitary transformation of the form $U(\vec{r}, t) = \exp(i \vec{\omega} \cdot \vec{r} \cdot \vec{A}(0, t))$ can be performed to eliminate terms depending on the electromagnetic vector potential, giving rise to an electric dipole–field interaction. The same procedure can be applied to the quantized electromagnetic field, the difference being that we are dealing with operators and there is an extra term, due to the commutation relations, that represents dipole–dipole interaction [22]. In 2001, similar arguments on the interaction between radiation and molecular dipole moments were given [23], where a gauge-invariant formulation of molecular electric dipole–photon interactions was used, which allows for superradiant phase transitions as in the Hepp–Lieb formulation. It was shown that the dipole–field interaction is strictly linear in the electric field, with an extra self-interaction term. Thus, there are no quadratic terms, and therefore a physical system displaying superradiant phase transitions should exist. The difference with previous gauge-invariant formulations is due to the time dependence of the unitary transformation.

The considerations above render likely the physical reality of systems presenting superradiant phase transitions, associated with a Hamiltonian of the type used to describe linear dipole–photon interactions. In particular, there are physical systems where interactions between bosonic degrees of freedom with a collective spin [24, 25] are relevant. Then, the recently presented method [26], which allows a simple and elegant determination of the stability properties and phase transitions for a finite number of particles, is of importance.

In this work we obtain an excellent approximation to the superradiant states of the ground state of the Tavis–Cummings Hamiltonian, which admits analytical expressions for both field and matter observables, including the entanglement entropy between field and matter, the squeezing parameter and the population probability distribution. The fitness between this approximation and the exact quantum solution is measured through the evaluation of the fidelity parameter [27].

Section 2 establishes the Tavis–Cummings Hamiltonian in terms of the constant of motion, and shows that the variational tensorial product of coherent states has difficulties in describing, for example, photon number fluctuations. It is argued that restoring the symmetry by projecting the variational state to a given value of the constant of motion, a much better (and analytical) approximation is obtained. This projected state is justified and built. In section 3, by means of the overlap of the projected state, the calculation of expectation values for field and matter observables is done analytically for an arbitrary value of the constant of motion. The entanglement entropy between matter and field, the squeezing parameter of the state and the probability distributions of photons and atoms are also calculated. In section 4 the selection of the appropriate value of the constant of motion $\lambda$ is discussed. Section 5 compares the results of section 3 with the corresponding ones for the exact quantum state, and section 6 draws some conclusions.

2. Projected state

The TCM Hamiltonian for $N$ identical two-level systems (e.g. atoms) immersed in an electromagnetic field is given by

$$H_{\text{TCM}} = \omega_F a^\dagger a + \bar{\omega}_A J_z + \frac{\gamma_N}{\sqrt{N}} (a^\dagger J_- + a J_+) \, ,$$

(1)

where $\omega_F$ is the field frequency, $\bar{\omega}_A$ the atomic energy-level difference and $\gamma_N$ the dipole coupling. The operators $a, a^\dagger$ denote the one-mode annihilation and creation photon operators, $J_z$ the atomic relative population operator and $J_\pm$ the atomic transition operators.

It is immediate that this Hamiltonian commutes with the operator

$$\Lambda = a^\dagger a + J_z.$$  

(2)
It is then convenient to rewrite it by introducing a detuning parameter $\Delta = \omega_\text{F} - \omega_\Lambda$, and by dividing it by $\omega_\text{F}$ (which can be thought of as the natural unit of frequency) and by the total number of particles, having in this way an intensive Hamiltonian operator

$$H = \frac{1}{N} \Lambda - \frac{\Delta}{N} J_z + \frac{\gamma}{\sqrt{N}} (a^\dagger J_- + a J_+) , \quad (3)$$

where $\Delta = 1 - \frac{\omega_\text{F}}{\omega_\Lambda} \equiv 1 - \omega_\Lambda$ and $\gamma = \frac{\gamma}{\omega_\Lambda}$.

Following the variational procedure indicated in [26], it was found that the direct product of coherent states, Heisenberg–Weyl for the photon part $|\alpha\rangle$ [5, 28] and $SU(2)$ or spin for matter $|\chi\rangle$ [29], i.e. $|\alpha, \chi\rangle = |\alpha\rangle \otimes |\chi\rangle$, is a good approximation to the exact quantum solution of the Tavis–Cummings Hamiltonian, in spite of the fact that the overlap of the two states is small. This approximation describes very well the expectation values of the atomic observables and most of the field observables. However, particular difficulties are found in the description of photon observables and most of the field observables. However, the expectation value of the Hamiltonian with respect to the variational state has contributions from all the eigenvalues $\lambda$, taking values greater than 1.

The first of these problems arises because the variational state has contributions from all the eigenvalues $\lambda$ of the operator $\Lambda$, violating its conservation symmetry. It is then natural to propose as ground state, a state with the symmetry restored by projecting the variational state to a given value of $\lambda$, namely, the one that minimizes the classical energy obtained from the variational procedure and approached to the closest integer or half-integer according to whether $N$ is even or odd, respectively. The exact procedure is presented in section 4.

By projecting the state to one (as yet arbitrary) value of $\lambda$, we obtain

$$|\psi\rangle = \begin{cases} 
0 \otimes |j, -j\rangle, & \omega_\Lambda > \gamma^2; \\
\sum_{v=\text{max}[0, \lambda-j]}^{\lambda+j} \binom{2j}{j+\lambda-v} \frac{1}{2\sqrt{(j+\lambda-v)!v!}} |\alpha\rangle \otimes |j, \lambda-v\rangle, & |\omega_\Lambda| \leq \gamma^2; \\
\times e^{-2i\varphi} \frac{\xi}{\sqrt{(j+\lambda-v)!v!}} |\alpha\rangle \otimes |j, \lambda-v\rangle, & \omega_\Lambda < -\gamma^2. 
\end{cases} \quad (4)$$

In this expression we have used $j = N/2$, and for $|\omega_\Lambda| \leq \gamma^2$ the state is not normalized, with

$$\zeta = -\frac{\sqrt{N} \gamma}{2} \left( 1 + \frac{\omega_\Lambda}{\gamma^2} \right). \quad (5)$$

It has the advantage that it only contains one eigenvalue of the constant of motion, it is an analytic solution, it reproduces well the matter and field observables (cf figure 1 (right) and note the scale difference with figure 1 (left)), and the overlap with the exact quantum solution is very close to one (see below).

Conditions on the parameters $\omega_\Lambda$ and $\gamma$ in equation (4) are associated with the existence of a separatrix in the model, which defines the phase transitions in the semiclassical solution [26].

To build the projected state equation (4), we use the semiclassical procedure, which consists of calculation of the expectation value of the Hamiltonian with respect to the tensorial product of coherent states $|\alpha\rangle \otimes |\chi\rangle$, determination of the minima and use of the catastrophe formalism [30] to find the stability properties [26]. We find that the critical points for a minimum are given by

$$q_c = -\sqrt{j} \gamma \sin \theta_c \cos \phi_c, \quad p_c = \sqrt{j} \gamma \sin \theta_c \sin \phi_c,$$

where we have written $\alpha = \frac{1}{\sqrt{2}} (q + ip)$ in terms of the expectation values of the quadratures of the field, and $\chi = e^{i\phi} \tan \left( \frac{j}{2} \right)$, with $(\theta, \phi)$ denoting a point in the unit Bloch sphere; at their critical values $\theta_c$, $\phi_c$, we have [26]

$$\begin{cases} 
\theta_c = 0, & E_0 = -\frac{N \omega_\Lambda}{2}, \quad \lambda_c = -j, \quad \text{for } \omega_\Lambda > \gamma^2; \\
\theta_c = \pi, & E_0 = \frac{N \omega_\Lambda}{2}, \quad \lambda_c = j, \quad \text{for } \omega_\Lambda < -\gamma^2; \\
\theta_c = \arccos \left( \frac{\omega_\Lambda}{\gamma^2} \right), & E_0 = -\frac{N \omega_\Lambda^2 + \gamma^4}{4 \gamma^2}, \quad \lambda_c = j - \frac{\omega_\Lambda (\omega_\Lambda + 2) + \gamma^4}{2 \gamma^2}, \quad \text{for } |\omega_\Lambda| < \gamma^2.
\end{cases} \quad (6)$$

This expression shows the minima critical points, the energy $E_0$, the constant of motion $\lambda_c$ and the conditions in parameter...
space to guarantee that they constitute an energy minimum. The energy surface of the TCM is φ-unstable, for which reason φc can be taken arbitrarily.

The expression for the trial state \(|\alpha|^a \otimes |\chi\rangle\) that minimizes the energy surface takes the following form [26]:

North Pole (\(\omega_A > \gamma^2\)): \(|\psi_{np}\rangle = |0\rangle \otimes |j, -j\rangle\), \((7)\)

South Pole (\(\omega_A < -\gamma^2\)): \(|\psi_{np}\rangle = |0\rangle \otimes |j, j\rangle\), \((8)\)

Parallels (\(|\omega_A| < \gamma^2\)): \(|\psi_{par}\rangle = \sum_{m=-j}^{+j} \sum_{\nu=0}^{\infty} A_{m, \nu} |v\rangle \otimes |j, \lambda = \nu\rangle\), \((9)\)

where we have defined the expansion coefficients

\[ A_{m, \nu} = \left( \frac{2j}{j+m} \right)^{1/2} \exp \left\{ -j \gamma^2 \frac{\nu^2}{4} \right\} \left( i^{j+m-\nu} \frac{\omega_A}{\lambda} \right)^{(j+m+\nu)/2} \left( 1 + \frac{\omega_A}{\lambda} \right)^{(j+m+\nu)/2} \]

One would usually perform this sum by selecting a value of ν, making the sum over m and then proceeding with the following value of ν, etc. until we reach some type of convergence. In the TCM model \(\lambda = m + \nu\) is a conserved quantity. By replacing m by \(\lambda - \nu\), we can write [26]

\[ |\psi_{par}\rangle = \sum_{\lambda=-j}^{\infty} \sum_{\nu=0}^{\lambda+j} A_{\lambda-\nu, \nu} |v\rangle \otimes |j, \lambda = \nu\rangle. \]

The eigenstates of the North Pole (7), South Pole (8) and Parallel region (10), for a given \(\lambda\), justify the projected state established in (4) with the unnormalized state for \(|\omega_A| \leq \gamma^2\) defined by

\[ |\xi; j, \lambda\rangle \equiv |\xi\rangle = \sum_{\nu=\max[0, \lambda-j]}^{\lambda+j} \left( \frac{2j}{j+\lambda-\nu} \right)^{1/2} e^{-2j \nu \phi} \left( \frac{\xi^\nu}{\nu!} \right) |v\rangle \otimes |j, \lambda = \nu\rangle, \]

where we have retained only terms depending on the number of photons \(\nu\) from the coefficient \(A_{\lambda-\nu, \nu}\).

To determine the expectation values of the observables of the system, in the Parallels region, it is useful to calculate the overlap

\[ \langle \xi' | \xi \rangle = \sum_{\nu=\max[0, \lambda-j]}^{\lambda+j} \left( \frac{2j}{j+\lambda-\nu} \right)^{\nu} \frac{1}{\nu!}. \]

One can usually express this summation in the form of a hypergeometric confluent function, we obtain

\[ \langle \xi' | \xi \rangle = \binom{j}{\nu} (-\xi')^\nu, \quad |\lambda| \leq j, \]

where \(L_n^\nu(x)\) denotes the associated Laguerre polynomials, which are defined as [31]

\[ L_n^\nu(x) = \frac{x^{-\nu} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n), \quad n \geq 0, \]

From expression (4) for the projected state, it is immediate that the probability of finding \(v\) photons depends on the relative values of \(\omega_A\) and \(\gamma\), i.e.,

\[ P_v = \frac{1}{\langle \xi | \xi \rangle} \xi^{2\nu} j^{2\nu} \left( \frac{2j}{j+\lambda-\nu} \right), \]

for \(|\omega_A| \leq \gamma^2\), while \(P_v = \delta_{v, 0}\) outside that region. The probability of finding \(n_\xi\) excited atoms can be obtained from the previous expression by replacing \(v \rightarrow \lambda + j - n_\xi\).

Defining \(\eta \equiv \xi^2\), and the overlap of the projected state by \(Y \equiv \langle \xi | \xi \rangle\), we can calculate the expectation value of the photon number operator \(\hat{n} = a^\dagger a\) and its corresponding fluctuations squared \((\Delta \hat{n})^2\):

\[ \langle \hat{n} \rangle = \sum_{\nu=\max[0, \lambda-j]}^{\lambda+j} v P_v = \frac{d}{d\eta} \ln Y, \]

\[ (\Delta \hat{n})^2 = \left( \frac{d}{d\eta} \right)^2 \ln Y. \]

Note that for \(\lambda = -j\) the overlap is \(Y = 1\). On the other hand, for \(\lambda = j\) the overlap is \(Y = L_{2j}^\nu(-\xi^2)\), which only equals 1 when \(\omega_A = -\gamma^2\). At the North and South Poles, therefore, the expectation values for \(\hat{n}\) and \((\Delta \hat{n})^2\) can be obtained by assuming \(Y = 1\) there and the expressions (15) and (16) to be valid for all values of the parameters \(\gamma\) and \(\omega_A\). Analytic expressions for \(\langle \hat{n} \rangle\) and \((\Delta \hat{n})^2\) in the Parallels region are given in the appendix.

The expectation values of other matter and field observables can be determined in terms of the equations above.

The quadrature components of the electromagnetic field are given by

\[ \hat{q} = \frac{1}{\sqrt{2}} (a^\dagger + a) \]

\[ \hat{p} = \frac{1}{\sqrt{2} i} (a^\dagger - a). \]
as they change the value of $\lambda$ for any state, their expectation values with respect to the projected state are equal to zero. Their corresponding fluctuations are therefore

$$\langle \hat{q}^2 \rangle = \langle \hat{p}^2 \rangle = \langle \hat{n} \rangle + \frac{1}{2}. \quad (17)$$

For the expectation values of matter observables, we have $\langle J_i \rangle = \langle J^i \rangle = 0$ for the same reason as above, and others take the form

$$\langle J_z \rangle = \lambda - \langle \hat{n} \rangle, \quad (18)$$

$$\langle (\Delta J_z)^2 \rangle = \langle (\Delta \hat{n})^2 \rangle, \quad (19)$$

$$\langle J^2 \rangle = \langle J^2 \rangle = \frac{1}{2} j (j+1) - \frac{1}{2} (\lambda - \langle \hat{n} \rangle)^2. \quad (20)$$

Finally, the expectation values for the transition operators $a^\dagger J_z$ and $a J_z$ are given by

$$\langle a^\dagger J_z \rangle = \langle a J_z \rangle = \xi \left( j + \lambda - \langle \hat{n} \rangle \right). \quad (21)$$

### 4. Determination of the constant of motion

The expectation value of the Hamiltonian with respect to the projected state for an arbitrary value of $\lambda$ is

$$\langle \hat{H} \rangle = \frac{\langle \xi | \hat{H} | \xi \rangle}{|\xi|} = \frac{\lambda}{2j} (1 - \Delta) + \frac{2\gamma}{(2j)^{3/2}} (j + \lambda) \xi + \frac{1}{2j} \left[ \Delta - \frac{2\gamma}{\sqrt{2j}} \xi \right] \langle \hat{n} \rangle. \quad (22)$$

When $\lambda = -j$ we have $\langle \hat{n} \rangle = 0$, and this expression simplifies to $\langle \hat{H} \rangle = -\frac{j}{2} (1 - \Delta)$. By substituting the expression of the expectation value of $\hat{n}$ given in equation (31) of the appendix, we obtain the energy surface

$$\mathcal{H}(\xi) = \frac{\lambda + j \Delta}{2j} \left[ \Delta - \frac{2\gamma}{\sqrt{2j}} \xi \right] \times \begin{cases} \frac{L_{j+1}^\lambda}{L_{j+1}^\lambda (-\xi^2)} / \frac{L_{j+1}^{-\lambda}}{L_{j+1}^{-\lambda} (-\xi^2)}, & j+1 \leq \lambda \leq j, \\ \frac{\lambda+j}{2j} L_{j+1}^\lambda (-\xi^2) / L_{2j}^\lambda (-\xi^2), & \lambda \geq j \end{cases} \quad (23)$$

and $\mathcal{H}(\xi) = \langle \hat{H} \rangle$.

We calculate this expression as a function of $\Delta$ and $\gamma$, and for all possible values of $\lambda$ with $\Delta \in [-2, 4]$ and $\gamma \in [-5, 5]$, which allows us to choose the value of $\lambda$ for which the energy is minimum and onto which the coherent state is to be projected. By noting that the classical value $\lambda_\text{c}$ of the constant of motion, given in equation (6), when rounded to the nearest integer or half-integer, never differs by more than one unit from its exact quantum counterpart, the minimizing procedure simplifies to testing for $|\lambda - \lambda_\text{c}| \leq 1$, and approximating $\lambda$ to the nearest integer (if $N$ is even) or half-integer (if $N$ is odd). The minimum energy so obtained is plotted in figure 2, while the resulting constant of motion and its contour levels are shown in figure 3.

While similar, the behavior for $\Delta > 1$ appears to be different than that for $\Delta \leq 1$. The energy surface shows a fictitious phase transition near $\gamma \approx 0.7$ for $\Delta > 1$ (cf the right of figure 2) that does not exist for $\Delta \leq 1$. In fact, the regime $\Delta > 1$ appears only when $\omega_A < 0$. One may visualize physical situations where this may occur: if the atoms are immersed in an external magnetic field and the energy levels appear as a Zeeman splitting of spectral lines, one may think of continuously tuning the field intensity until a sign reversal is obtained, thus interchanging the excited and base levels. It will be seen, however, that the exact quantum solution does not show this transition (cf figure 6). The case $\Delta = 1$ (or equivalently $\omega_A = 0$) is, nevertheless, a special and boundary case: the Hamiltonian in equation (1) simplifies considerably, the atomic levels are degenerate, the quantum energy spectrum (see section 5) is also highly degenerate and the poles (both North and South) contract to a single point. In fact (cf equation (6)), the North Pole is no longer a minimum critical region, but a saddle point.

From the expressions derived for the observables in section 3, their expectation values can also be plotted as functions of interaction strength $\gamma$ and detuning parameter $\Delta$. It will be seen that their behavior will be inherited from that of $\lambda_{\text{min}}$ itself. As an example, we show $\langle J_1 \rangle$ in figure 4.

There is no field squeezing since, from the quadrature components, $\langle \hat{q} \rangle = \langle \hat{p} \rangle = 0$ and $\langle \hat{q}^2 \rangle = \langle \hat{n} \rangle + \frac{1}{2} = \langle \hat{p}^2 \rangle$. However, for the matter squeezing coefficient we have [32]

$$\xi = \sqrt{\frac{2(\Delta J^2_1)}{j}}, \quad (24)$$
where $\lambda$ is a component of $\vec{J}$ transverse to $\langle \vec{J} \rangle$; as $\langle \vec{J} \rangle = (J_z)\hat{e}_z$, we take $J_z$ to be this component, and show $\xi$ in figure 5 (left). $\xi$ gives us information on how good the trial projected state is at approximating the exact solution of the eigenvalue problem of the TCM Hamiltonian, and this comparison will be made in section 5.

Finally, the entanglement entropy $S_E$ is zero for coherent states since these are expressed as product states. In [26], in order to calculate a useful expression for $S_E$, we first traced over the field and then again over one of the matter modes. When a projected state is used, it is no longer a product state and we may trace over the matter (or equivalently over the field) to obtain

$$S_E = - \sum_\nu P_\nu \log P_\nu,$$

where $P_\nu$ is already normalized, as given in equation (14). This is shown in figure 5 (right). Its behavior is very close to that found in [26].

5. Projected coherent state versus exact solution

The exact solution for the ground state in the TCM was presented in [26], where the natural basis $|\nu\rangle \otimes |j, \lambda - \nu\rangle$ was used (natural because the Hamiltonian has $\Lambda$ as a constant of motion):

$$|\psi_{gs}\rangle = \frac{1}{\lambda + j} \sum_{\nu = \max[0, \lambda - j]}^{\lambda + j} c_\nu |\nu\rangle \otimes |j, \lambda - \nu\rangle.$$ (26)

Analytical solutions for values of $\lambda$ up to $\lambda = -\frac{N}{2} + 4$ were also given, both at resonance ($\Delta = 0$) [33] and away from resonance ($\Delta \neq 0$) [26]. For greater values, numerical solutions were obtained.

When the exact solution is compared with that obtained from the projected state, we find that the quantum phase transitions are exactly reproduced, as shown in figure 6, even for a large number of atoms ($N = 20$) and when away from resonance. The straight lines in the lower left figure correspond to the energy of the ground state for different values of the constant of motion $\lambda$, starting from $-10$ (0 photons) and up to $-6$ (4 photons). In each case the eigenstate is a mixture of 0 to $\lambda + \frac{N}{2}$ photons. The projected state approximation is stunningly close; so much so, that both graphs (quantum and projected state) practically lie on top of each other and are indistinguishable. Whereas the projected state has a constant value of the ground state energy inside the poles (North ($|0\rangle \otimes |j, -j\rangle$) and South ($|0\rangle \otimes |j, j\rangle$)), the exact quantum solution does not when $\Delta > 1$ ($\omega_A < 0$); the solution therefore is very precise for a detuning of $\Delta = 0.2$, as shown on the left side of the figure, but fails slightly near the boundary of the poles for $\Delta = 1.5$, as shown on the lower right.
Figure 5. Squeezing parameter $\xi$ for matter (left), and entanglement entropy $S_E$ between field and matter (right), for the variational ground state, as a function of interaction strength $\gamma$ and detuning parameter $\Delta$, for $N = 20$ atoms.

Figure 6. Ground state energy per particle of the projected state solution compared with the exact quantum solution, as a function of interaction strength $\gamma$ for $N = 20$ atoms. The plots on the left correspond to a detuning parameter $\Delta = 0$, while those on the right correspond to $\Delta = 1.5$ (and therefore $\omega_A < 0$). The bottom plots are close-ups of those on top. Note that both solutions lie on top of each other and are indistinguishable, except for the close-up for $\Delta = 1.5$.

The case $\Delta = 1$ is a particular boundary, as mentioned in the previous section, since it simplifies the Hamiltonian (3) making it independent of the inversion population operator $J_z$. In this case, if $\gamma = 0$ also, we have a dense degeneracy of the Hamiltonian eigenvalues: matter does not see the radiation field, so the degeneracy is that of the atoms themselves, i.e. $2j + 1$. Any small deviation of $\gamma$ away from zero unfolds the energy levels, as shown in figure 7. In this case our method for choosing the $\lambda$-value onto which to project fails, as $2j + 1$ values yield the same energy. We shall, in what follows, choose values for $\Delta$ above and below 1 to exemplify the behavior of observables.

The constant of motion $\lambda_{\min}$ is shown in figure 8 as a function of interaction parameter $\gamma$ for both the semiclassical projected and the quantum cases. Its discrete behavior is inherited by all the quantum observables of interest. The slope observed in the $\lambda$-steps is given precisely by the difference in energy between the absorbed photons and the atom’s energy level separation. This is a consequence of the system being away from resonance ($\Delta \neq 0$). When in resonance, the steps are horizontal. The plots at the left show $\lambda$ for a detuning of $\Delta = 0.2$, while those at the right show it for $\Delta = 1.5$. Note (lower right plot) that even when $\omega_A < 0$, the projected state solution tries to reproduce the non-constant
Figure 7. Energy spectrum for $\Delta = 1$. When $\gamma = 0$ we have a $(2j + 1)$ fold degeneracy (the left plot shows the first two such levels), which unfolds as soon as $\gamma$ is perturbed away from zero. Spectra for $\gamma = 0, 0.1, 0.2$ and 0.3 are shown.

Figure 8. Constant of motion $\lambda_{\min}$ of the projected state solution compared with the exact quantum solution, as a function of interaction strength $\gamma$ for $N = 20$ atoms. Note, once again, that both graphs practically lie on top of each other and are indistinguishable. The plots on the left correspond to a detuning parameter $\Delta = 0.2$, while those on the right correspond to $\Delta = 1.5$ (and therefore $\omega_A < 0$). The bottom plots are close-ups of those on top.

Figure 9. Expectation value for $J_z$ and its dispersion $(\Delta J_z)^2$ of the projected state solution compared with the exact quantum solution, as functions of interaction strength $\gamma$ for $N = 20$ atoms. Both graphs practically lie on top of each other (cf the scale for $(\Delta J_z)^2$). The plots correspond to a detuning parameter $\Delta = 0.2$.

behavior near the pole boundary. Otherwise the graphs are indistinguishable.

The photon number fluctuations $(\Delta \hat{n})^2$, as noted in the introduction, very well resemble the projected state solution to the TCM (cf figure 1 (right)). The differences are minute. The same trend is found for the dispersion in $J_z$, as is to be expected, since $\langle J_z \rangle = \lambda - \langle \hat{n} \rangle$, $\langle J_z^2 \rangle = \langle \hat{n}^2 \rangle - 2\lambda \langle \hat{n} \rangle + \lambda^2$ and therefore $\langle \Delta J_z \rangle^2 = (\Delta \hat{n})^2$. Both the expectation value for $J_z$
Figure 10. Squeezing parameter $\xi$ of the projected state solution compared with the exact quantum solution, as functions of interaction strength $\gamma$ for $N = 20$ atoms. The plots correspond to detuning parameters $\Delta = 0.2$ (left) and $\Delta = 1.5$ (right). The round shape of the pole is badly approximated at its boundary, when $\Delta > 1$, while the approximation is exact for $\Delta < 1$.

Figure 11. Entanglement entropy $S_E$ of the projected state solution compared with the exact quantum solution, as functions of interaction strength $\gamma$ for $N = 20$ atoms. The plots correspond to detuning parameters $\Delta = 0.2$ (left) and $\Delta = 1.5$ (right). The round shape of the pole is badly approximated at its boundary, when $\Delta > 1$, while the approximation is excellent for $\Delta < 1$.

and its dispersion are plotted in figure 9 as functions of interaction strength $\gamma$, slightly away from resonance ($\Delta = 0.2$). Note the scale on the ordinate axis for $(\Delta J)^2$.

As a signature of the goodness of our trial projected state in reproducing the exact quantum solution, one may use the behavior of the squeezing spin coefficient $\xi$ as given by equation (24). Even though it greatly exceeds the value of 1 away from the poles (cf figure 10), thus suggesting a small overlap with the quantum state, both results lie exactly on top of each other. Once again, for a large detuning making $\omega_A < 0$, the round shape of the pole is badly approximated at its boundary. For $\Delta < 1$, however, the approximation is exact.

By taking the trace with respect to the field (matter) states, the reduced density matrix takes the form

$$\rho_{\text{matter}} = \sum_{n=0}^{\min(\lambda + \frac{4}{2}, N)} |c_{\lambda + \frac{2}{2} - n}|^2 |N - n, n\rangle\langle N - n, n|,$$

where $c_{\lambda + \frac{2}{2} - n}$ (or $c_{\nu}$) is determined from the Hamiltonian diagonalization. As the reduced density matrix of matter is diagonal, the matter–field entanglement entropy equals that between atoms occupying the hyperfine levels

$$S_E = -\sum_{n=0}^{\min(\lambda + \frac{4}{2}, N)} |c_{\lambda + \frac{2}{2} - n}|^2 \ln |c_{\lambda + \frac{2}{2} - n}|^2,$$

in both cases. This may be compared with that for the projected state, equation (25), to give the result shown in figure 12.
figure 11. The same behavior of excellent approximation for 
\( \Delta < 1 \), and poor approximation at the south pole boundary 
when \( \Delta > 1 \), is obtained.

6. Discussion and conclusions

Our approximation is not exact. Several important observables 
such as \( \xi \), \( S_E \), (\( \Delta q \))^2, (\( \hat{n} \)) and (\( \Delta \hat{n} \))^2 are symmetric under 
a reflection of about \( \Delta = 1 \) for the projected state, while 
this cannot be true for any observable coming from the 
Hamiltonian (3). As an illustration of this we show (\( \Delta \hat{n} \))^2/N^2 
as a function of \( \Delta \), for both the projected state (light curve) 
and the quantum state (dark curve), in figure 12, for \( \gamma = 0.75 \). 
While the projected state has a constant value of 0 at both 
poles, the quantum state decays to 0 asymptotically as it enters 
the South Pole.

A good measure of the distance between quantum 
mechanical states is given by the fidelity parameter; for pure 
quantum states it measures their distinguishability in the sense 
of statistical distance [22]. As an illustration of this we show 
\( (\Delta \hat{n})^2/N^2 \) as a function of \( \Delta \), for the projected state 
(light curve) and the quantum state (dark curve), in figure 12, for \( \gamma = 0.75 \). 
While the projected state has a constant value of 0 at both 
poles, the quantum state decays to 0 asymptotically as it enters 
the South Pole.

A good measure of the distance between quantum 
mechanical states is given by the fidelity parameter; for pure 
quantum states it measures their distinguishability in the sense 
of statistical distance [27], but it is customary to use fidelity 
as a transition probability regardless of whether the states are 
true or not. Measuring the fidelity between our projected state 
and the exact quantum ground states,

\[
F = \left| \langle \psi_{\text{proj}} | \psi_{\text{gs}} \rangle \right|^2, \tag{30}
\]
gives a result very close to 1, except in the region of classical 
phase transitions. Figure 13 (left) shows the result as a 
function of \( \gamma \) for a detuning parameter of \( \Delta = 0.2 \). \( F = 1 \) 
inside the North Pole and drops to \( F = 0.996 \) when crossing 
the separatrix into the Parallels region, only to approach its 
value of 1 again as |\( \gamma | \) continues to increase. We have seen 
that the South Pole is less well represented by the projected 
state for \( \Delta > 1 \); the behavior of \( F \) as a function of \( \Delta \) near 
the South Pole is shown in figure 13 (right) for \( \gamma = 0.75 \), for 
which value the South Pole lies at \( \Delta > 1.5 \). It is seen that 
even in this case \( F \) drops to approximately 0.6 and quickly 
recovers.

It has been emphasized that the DM presents quantum 
phase transitions when \( \omega_{\text{DM}} = \sqrt{\omega_{\text{ch}}}/2 \). Although difficult to 
satisfy for optical systems, some proposals to overcome the 
experimental problems have been reported (see [34] and 
references therein).

More recently [35], the Tavis–Cummings Hamiltonian 
has been physically realized for artificial atoms in the form 
of superconducting q-bits at fixed positions and coupled to 
a resonant cavity mode. The predicted \( \sqrt{N} \) behavior for 
the collective \( N \)-atom interaction strength of the model is 
observed experimentally in good agreement. It is argued that 
the presented approach may enable novel investigations of 
superradiant and subradiant states of articial atoms.

In this work, we have shown that an excellent 
approximation to the exact quantum solution of the ground 
state of the TCM is obtained by means of a semi-classical 
projected state. This state has an analytical form in terms 
of the model parameters and allows for the analytical 
calculation of the expectation values of field and matter 
observables, entanglement entropy between field and matter, 
and squeezing parameter. In our discussions we have taken 
the picture of two-level atoms interacting with a single-mode 
electromagnetic field, but by re-interpreting the parameters 
the results are useful to describe, for example, the superradiant 
properties of condensed matter phases, or linear ion traps 
proposed for quantum computation.

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Appendix A. Expectation values of field operators

The expectation value of the photon number operator can be 
found analytically using overlap (13), with \( \zeta' = \zeta \), to obtain 

\[
\langle \hat{n} \rangle = \left\{ \begin{array}{ll}
(j + \lambda) - 2j L^{j \lambda}_{j-1}(-\eta) / L^{j \lambda}_{j-1}(-\eta), & -j + 1 \leq \lambda \leq j, \\
(\lambda + j) - (\lambda + j) L^{j \lambda}_{j-1}(-\eta) / L^{j \lambda}_{j-1}(-\eta), & \lambda \geq j.
\end{array} \right. \tag{31}
\]
In the same way we find that the squared fluctuations are

\[
(\Delta n)^2 =
\begin{align*}
2 j & \left( L_{j^{\lambda} - 1}(x) / L_{j^{\lambda}}(x) + (2j - 1) \right. \\
& \times L_{j^{\lambda} - 2}(x) / L_{j^{\lambda}}(x) \\
& - 2 j \left[ L_{j^{\lambda} - 1}(x) / L_{j^{\lambda}}(x) \right]^2, \\
& \left. \lambda + j \right) \left( L_{2j}^{\lambda - j}(x) / L_{2j}^\lambda(x) + (\lambda + j - 1) \right. \\
& \times L_{2j - 2}(x) / L_{2j}^\lambda(x) \\
& - (\lambda + j) \left[ L_{2j}^{\lambda - j}(x) / L_{2j}^\lambda(x) \right]^2, \quad \lambda \geq j.
\end{align*}
\]

(32)

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