MULTI-POINT PASSAGE PROBABILITIES AND GREEN’S FUNCTIONS FOR SLE$_{8/3}$

O. V. Alekseev

We consider a loop representation of the $O(n)$ model at the critical point. In the case $n = 0$, the model reduces to statistical ensembles of self-avoiding loops, which can be described by Schramm–Loewner evolution (SLE) with $\kappa = 8/3$. In this limit, the $O(n = 0)$ model corresponds to a logarithmic conformal field theory (LCFT) with the central charge $c = 0$. We study the LCFT correlation functions in the upper half-plane containing several twist operators in the bulk and a pair of the $\Phi_{1,2}$ boundary operators. By using a Coulomb gas representation for the correlation functions, we obtain explicit results for the probabilities of the SLE$_{8/3}$ trace to pass in various ways about $N \geq 1$ marked points. When the points approach each other pairwise, the probabilities reduce to multipoint SLE Green’s functions. We propose an explicit representation for the Green’s functions in terms of the correlation functions of the bulk $\Phi_{3,1}$ and boundary $\Phi_{1,2}$ operators.

Keywords: Schramm–Loewner evolution, conformal field theory

DOI: 10.1134/S004057792302006X

1. Introduction

Schramm–Loewner evolution (SLE) provides a common framework to study fractal curves or sets growing into simply connected planar domains $D \subset \mathbb{C}$ [1]. This approach focuses on constructing measures on random curves that occur in such systems. In the simplest setting of SLE from $x_1$ to $x_2$ (such that $x_1, x_2 \in \partial D$), the measure is generated dynamically by evolving the curve starting from one end point. Conventionally, the domain is taken to be the upper half-plane $\mathbb{H} = \{z \in \mathbb{C}: \text{Im } z > 0\}$. Then the curve $\gamma_t$ evolving until the time instant $t$ (or rather its hull $K_t$) is characterized by a conformal map $g_t: \mathbb{H} \setminus K_t \to \mathbb{H}$ normalized such that $g_t(z) \sim z + 2t/z + O(z^{-2})$ as $z \to \infty$. This function, $g_t$, satisfies the Loewner equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z)} - \sqrt{\kappa}B_t,$$

where $B_t$ is standard Brownian motion with zero mean and quadratic variation $[B_t, B_s] = t\delta_{t-s}$. In Loewner equation, the only real parameter $\kappa$ determines geometric properties of SLE curves. These curves are simple paths if $\kappa \leq 4$; when $4 < \kappa \leq 8$, the curves have no self-intersections but can have double points, and for $\kappa \geq 8$ the curves become space-filling [2].

*Chebyshev Laboratory, Department of Mathematics and Mechanics, St. Petersburg State University, Saint-Petersburg, Russia, e-mail: teknoanarchy@gmail.com.

The work is supported by the Russian Science Foundation (grant No. 19-71-30002).

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 214, No. 2, pp. 243–267, February, 2023. Received September 3, 2022. Revised September 3, 2022. Accepted October 11, 2022.
Various geometric observables are useful and important in the SLE theory. One of the simplest SLE observable is the probability $P_L(z)$ that a curve passes to the left of a given point $z \in \mathbb{H}$ [3]:

$$P_L(z) = \frac{1}{2} + \frac{\Gamma(4/\kappa)}{\sqrt{\pi} \Gamma((8 - \kappa)/2\kappa)} \frac{x}{y} 2F_1\left(\frac{1}{2}, \frac{3}{2}; \frac{x^2}{y^2}\right)$$

where $2F_1(a, b; c; x)$ is the hypergeometric function, $\Gamma(x)$ is the gamma function, and $z = x + iy$ is a complex coordinate on the plane. When $\kappa = 8/3$, the probability is simplified to $P_L(z) = 1/2 + x/2|z|$. A similar formula for the two-point left passage probability was predicted by Simmons and Cardy by using conformal field theory (CFT) techniques in the case $\kappa = 8/3$ [4].

We briefly discuss the result of Simmons and Cardy [4]. Their approach uses an intimate relation between SLE$_{\kappa}$ and CFT, which allows studying critical curves using CFT methods [5]. It is well known that many two-dimensional statistical systems, e.g., the $O(n)$ and percolation models, can be mapped to an equivalent loop representation [6]. Various loop ensembles can be conventionally described in terms of SLE [1], [2]. Alternatively, the loop model can be mapped to a height model via Coulomb gas. In the continuum limit, the latter model is described by CFT. Hence, it becomes possible to study loop ensembles in the CFT framework. We briefly describe this relation in Sec. 2.

An essential part of the Simmons–Cardy construction is the identification of the so-called twist operator with the 0-weight Schramm operator for $\kappa = 8/3$. Twist operators at the points $z_i \in \mathbb{H}$ modify statistical weights of the loops that wind in various ways about these points. Roughly speaking, the correlation function containing a single twist operator at a point $z \in \mathbb{H}$ counts the expected number of loops that separate $z$ from the boundary. In Sec. 3.1 we show that the correlation function is closely related to Schramm’s formula (2). In a similar way, the correlation function containing two twist operators in $\mathbb{H}$ can be used to counting the expected number of loops that separate both points from the boundary. In [4], Cardy and Simmons showed that this function is closely related to the two-point left passage probability.

In this paper, we generalize the Simmons–Cardy result to the case of $N \geq 3$ points in $\mathbb{H}$. The system of PDEs that governs the corresponding probabilities is then very difficult to solve directly. To construct the solutions, we use the Coulomb gas formalism [7], [8]. As the result, we obtain explicit expressions for probabilities of the SLE$_{8/3}$ trace to wind in various ways about $N \geq 3$ points in $\mathbb{H}$. Remarkably, this result can be used to study SLE multipoint Green’s functions. Indeed, the probability that the SLE trace passes between the points $z_1, z_2 \in \mathbb{H}$ becomes the one-point SLE Green’s function as the points collapse to one. Similarly, we can expect the 2N-point passage probability to become the N-point Green’s function as the points $z_1, z_2, \ldots, z_{2N}$ collapse pairwise.

The structure of the paper is straightforward. In Sec. 2, we briefly review the $O(n)$ model, which serves as a connection between SLE and CFT. We also introduce the twist and legs operators, and describe their conformal properties. In Sec. 3, we use CFT techniques to calculate probabilities of the SLE$_{8/3}$ trace to wind in various ways about 1, 2, $\ldots$, $N$ marked point in the upper half-plane. In particular, in Sec. 3.6, we obtain a Coulomb gas representation for the $N$-point passage probabilities of SLE curves. Section 4 is devoted to multipoint Green’s functions of SLE$_{8/3}$ curves in the upper half-plane. We obtain explicit expressions for the Green’s functions in terms of the correlation functions of 1/3-weight operators in the bulk, and 1-leg operators on the boundary in a $c = 0$ logarithmic CFT in $\mathbb{H}$. Finally, we draw our conclusions in Sec. 5.

2. The $O(n)$ model, CFT, and SLE

We start with a standard loop representation of the $O(n)$ model with $n$-component spins $s(r_i)$, such that $s^2(r_i) = 1$, on the lattice $\{r_i\}$. The partition function of the model can be written as

$$Z = \text{Tr} \prod_{(ij)} (1 + x s(r_i) \cdot s(r_j)),$$  

(3)
where $\text{Tr} \, s_a(r_i)s_b(r_j) = \delta_{ab}$, the factor $x$ is a parameter of the model, and the product ranges the pairs of nearest neighbors. We can expand the product into a sum of $2^K$ terms, where $K$ is the number of nearest neighbors. Each term is associated with a graph on the lattice in what follows: the bond between $r_i$ and $r_j$ is included in the graph if the factor $xs(r_i) \cdot s(r_j)$ appears in the expansion; only graphs composed of closed loops contribute to the sum.

The partition function takes a particularly simple form if the model is considered on a honeycomb lattice, where loops can visit each site at most once. Each loop contributes a total weight $n$ to the partition function because $\text{Tr} \, s_a(r_i)s_b(r_j) = \delta_{ab}$. Each occupied bond contributes the factor $x$. Hence, the partition function is equivalent to

$$Z = \sum_\Lambda n^N x^L,$$

where the sum is taken over all closed nonintersecting loop configurations $\Lambda$ on the honeycomb lattice, $N$ is the number of loops, and $L$ is the total length of loops in each configuration.

The long loops are suppressed for small values of $x$, such that the model flows to the vacuum under the renormalization group flow. For large values of $x$, the system flows to a fixed point of densely packed loops. At the boundary between these regimes, there exists a critical point at $x = x_c$, $x_c = (2 + 2\sqrt{2-n})^{-1/2}$, for which the mean loop length diverges and the system flows to the dilute fixed point [6]. At this point, the model is supposed to be conformally invariant. Hence, the critical point of the $O(n)$ model can be studied in the CFT framework.

The sum over closed loops in (4) can be decomposed into two parts depending on the orientation of the loops. This can be achieved by inserting the factors $e^{i\pi \chi}$ ($e^{-i\pi \chi}$) at each vertex where the curve turns to the right (left), and summing over two possible orientations of each loop. As a result, each closed loop on the honeycomb lattice contributes the factor $e^{6\pi i \chi} + e^{-6\pi i \chi}$ to the partition function. Because the contribution of each loop must be $n$, we conclude that

$$n = 2 \cos(6\pi \chi).$$

It is well known that the oriented loops can be regarded as level lines of a height variable, say $h(r)$, on the dual lattice. The only requirement is that the height variable change by $\pi$ ($-\pi$) when crossing a loop pointing to the right (left). Hence, there exists a one-to-one correspondence between a given configuration of heights and a unique graph of oriented loops. It can be argued that under the renormalization group flow, the height model flows into a free field theory described by the action

$$S[h(r)] = \frac{g}{4\pi} \int (\partial h(r))^2 \, d^2 r,$$

where $g = 1 - 6\chi/\pi$ (see [9] for the details). Hence, the height model relates the loop model to the CFT describing its continuum limit.

Moreover, it can be shown that the measure on the loops in the scaling limit of the $O(n)$ model at the critical point $x_c$ approaches that of chordal SLE$_\kappa$ for [10]

$$n = 2 \cos \left( \frac{(\kappa - 4)\pi}{\kappa} \right),$$

where $2 < \kappa < 4$ for the dilute phase and $4 < \kappa$ for the dense phase. Hence, the loop model corresponds to a rational CFT with the central charge and conformal weights given by

$$c = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa}, \quad h_{r,s} = \frac{(kr - 4s)^2 - (\kappa - 4)^2}{16\kappa}.$$

Below, we consider only the dilute regime with $2 < \kappa < 4$. 212
We note that the case $\kappa = 8/3$ corresponds to a logarithmic CFT (LCFT) with the central charge $c = 0$. LCFTs are characterized by a logarithmic structure in the operator product expansion, explained by indecomposable representations that occur in fusion of primary operators [11], [12]. In other words, there exist primary operators with degenerate scaling dimension constituting a Jordan block structure.

The so-called twist operators considered in [13] play a crucial role in Simmons and Cardy’s study of the probabilities for the SLE trace to wind in various ways about marked points in $\mathbb{H}$. A pair of twist operators changes the weights of all loops that separate them. Because the weight of the loops separating the twist operators is $-n$, the partition function for the loop model in the presence of twist operators takes the form

$$Z = \sum_{\Lambda} (-1)^{N_\Lambda} n^{N_\Lambda} x^{c},$$

where $N_\Lambda$ is the number of loops separating the twist operators. Hence, the twist operators can be used to count loops with weights $-n$ rather than $n$. The scaling dimension of the twist operators can be calculated explicitly [13]. Indeed, these operators correspond to CFT primary fields $\Phi_{2,1}$, and therefore their Kač weight is

$$h_{2,1} = \frac{3\kappa - 8}{16}.$$

The twist operators are spinless, and hence the antiholomorphic dimension coincides with the holomorphic one.

Another set of operators relevant to the SLE/CFT correspondence is given by the so-called boundary $N$-leg operators, which anchor SLE traces to the boundary of the domain. In the Coulomb gas framework, these operators change the boundary conditions by $N$ steps within an $\epsilon$-neighborhood of their insertion, and can be identified with the boundary primary operators $\Phi_{1,N+1}$, with the weights

$$h_{1,N+1} = \frac{N(4 + 2N - \kappa)}{2\kappa}.$$ (10)

By means of the 1-leg boundary operators, the partition function of all SLE traces from $x_1$ to $x_2$ with $x_1, x_2 \in \mathbb{R}$ can be represented as the two-point correlation function of boundary fields,

$$H_0(x_1, x_2) = \langle \Phi_{1,2}(x_1) \Phi_{1,2}(x_2) \rangle_{\mathbb{H}} = (x_2 - x_1)^{-2h_{1,2}},$$

where $\langle \cdots \rangle_{\mathbb{H}}$ denotes the correlation function in $\mathbb{H}$, and we set the normalization constant to 1 by choosing an appropriate normalization of the fields. We note that correlation function (11) is fixed by scale invariance up to the overall normalization.

3. The passage probabilities of the SLE$_{8/3}$ trace

3.1. Anchored correlation functions with a twist operator. It was shown in [4] that the correlation functions of twist operator at a point $z \in \mathbb{H}$ are closely related to SLE left/right passage probabilities around this point. This result can be easily generalized to the case of several points in the upper half-plane. We start this section by rederiving the famous Schramm’s formula for the left/right passage probability of the SLE$_{8/3}$ trace. It is closely related to the correlation function of the boundary 1-leg operators in the presence of the twist defect at the point $z \in \mathbb{H}$ [4],

$$H_1(z, \bar{z}; x_1, x_2) = \langle \Phi_{2,1}(z, \bar{z}) \Phi_{1,2}(x_1) \Phi_{1,2}(x_2) \rangle_{\mathbb{H}},$$

Here and below, we adapt the notation from Ref. [4]
where the bar denotes complex conjugation and $x_1, x_2 \in \mathbb{R}$. We note that this function differs from those in (11) by the inclusion of the spinless twist field $\Phi_{2,1}$. As in the standard CFT approach, the correlation function $H_1(z, \bar{z}, x_1, x_2)$ in $\mathbb{H}$ can be represented as a correlation function in the complex plane $\mathbb{C}$ [14],

$$H_1(z, \bar{z}; x_1, x_2) = \langle \Phi_{2,1}(z)\Phi_{2,1}(z^*)\Phi_{1,2}(x_1)\Phi_{1,2}(x_2) \rangle,$$

subjected to certain constraints on $\mathbb{R}$ specified below. Here, we let $\langle \cdot \rangle$ denote the correlation function in $\mathbb{C}$, and the points $z, z^*$ are treated as independent variables (assuming that $z^* = \bar{z}$ is set at the end of the computation).

CFT methods allow deriving a set of second-order PDEs satisfied by the correlation functions containing degenerate fields $\Phi_{1,2}$ and $\Phi_{2,1}$ [5]. In particular, it can be shown that correlation function (13) satisfies the equations

$$\left[ \frac{3\partial_2^2}{2(1 + 2h_{2,1})} - \frac{h_{2,1}}{(z^* - z)^2} + \frac{\partial_{z^*}}{z^* - z} - \frac{h_{1,2}}{(x_1 - z)^2} + \frac{\partial_{x_1}}{x_1 - z} - \frac{h_{1,2}}{(x_2 - z)^2} + \frac{\partial_{x_2}}{x_2 - z} \right] H_1 = 0,$$

$$\left[ \frac{3\partial_2^2}{2(1 + 2h_{1,2})} - \frac{h_{2,1}}{(z^* - x_1)^2} + \frac{\partial_{z^*}}{z^* - x_1} - \frac{h_{1,2}}{(x - z_1)^2} + \frac{\partial_{z_1}}{z - z_1} - \frac{h_{1,2}}{(x_2 - x_1)^2} + \frac{\partial_{x_2}}{x_2 - x_1} \right] H_1 = 0. \tag{14}$$

These equations have a common solution

$$H_1(z, z^*, x_1, x_2) = (z - z^*)^{-2h_{2,1}}(x_2 - x_1)^{-2h_{1,2}}G_1(\eta), \tag{15}$$

where $G_1(\eta)$ is a function of the cross ratio $\eta$:

$$G_1(\eta) = \frac{2 - \eta}{2\sqrt{1 - \eta}}, \quad \eta = \frac{(z - z^*)(x_2 - x_1)}{(z - x_1)(x_2 - z^*)}.$$ \tag{16}

The function $G_1(\eta)$ has a branch cut from 1 to $\infty$. We note that

$$1 - \eta = \frac{(z^* - x_1)(x_2 - z)}{(z - x_1)(x_2 - z^*)},$$

It is convenient to apply the Möbius transform sending the points $\{z, z^*, x_1, x_2\}$ to $\{z, \bar{z}, 0, \infty\}$. It is then easy to see that the choice of the branch of the square root in Eq. (16) is determined by the argument of $z$. We thus arrive at

$$\frac{H_1(z, \bar{z}; 0, \infty)}{H_0(0, \infty)} = (2 \text{Im} z)^{1 - 3\alpha/8} \frac{\text{Re} z}{|z|}, \tag{17}$$

where $H_0$ is the two-point function of the boundary 1-leg operators (11).

3.2. The Coulomb gas formalism. We note that exact solutions of the system of equations (14) can also be obtained within the Coulomb gas formalism introduced by Dotsenko and Fateev [7], [8]. We briefly recall their construction. In the Dotsenko–Fateev language, the conformal operator, say $\Phi$, can be represented in terms of a vertex operator, i.e., the exponential of a free field,

$$V_\alpha(z) = e^{i\sqrt{2}a \Phi(z)}, \tag{18}$$

where $\Phi(z)$ is the free field specified by the two-point function $\langle \Phi(z)\Phi(w) \rangle = -\ln(z - w)$. The real parameter $\alpha$ is called the charge of the vertex operator. It determines the properties of the vertex operators under conformal transformations and hence the conformal weight of the vertex operator (the exact relation is specified below).
The crucial ingredient of the Dotsenko–Fateev construction is a background charge in the system. The background charge not only modifies the conformal dimensions of the vertex operators but also spoils unitarity except for discrete sets of operators corresponding to minimal models. Conventionally, the background charge is denoted as \(-2\alpha_0\), and then the CFT central charge is given by

\[ c = 1 - 24\alpha_0^2. \] (19)

The background charge modifies the conformal dimension of the vertex operator \(V_\alpha\) to

\[ h_\alpha = \alpha(\alpha - 2\alpha_0). \] (20)

As we have mentioned, the free boson model with a background charge is unitary only for those values of \(\alpha_0\) that correspond to minimal models of CFT. In this case, it is convenient to introduce the parameterization

\[ \alpha_+ = \frac{2}{\sqrt{\kappa}}, \quad \alpha_- = -\frac{\sqrt{\kappa}}{2}, \] (21)

with \(\alpha_+ + \alpha_- = 2\alpha_0\) and \(\alpha_+ \alpha_- = -1\). Then admissible sets of vertex operators \(V_{\alpha_{r,s}}\) can be parameterized as

\[ \alpha_{r,s} = \frac{1}{2}(1 - r)\alpha_- + \frac{1}{2}(1 - s)\alpha_+. \] (22)

The conformal dimension in (20) is invariant under \(\alpha \rightarrow 2\alpha_0 - \alpha\), and hence the vertex operators \(V_\alpha\) and \(V_{2\alpha_0 - \alpha}\) have the same dimension. Therefore, the conformal field \(\Phi_{r,s}\) can be associated with two different vertex operators \(V_{\alpha_{r,s}}\) and \(V_{-r,-s}\), implying that the correlation functions of conformal fields can be evaluated in several different but equivalent ways.

Because the two-point function of the free boson, \(\varphi(z)\), has simple logarithmic form, the correlation function of the vertex operators can be written as

\[ \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\ldots V_{\alpha_n}(z_n) \rangle = \prod_{i<j}(z_i - z_j)^{2\alpha_i\alpha_j} \] (23)

if the neutrality condition

\[ \sum_{i=1}^{n} \alpha_i = 2\alpha_0. \] (24)

is satisfied (otherwise, the correlation function vanishes) [7], [8]. Hence, any multipoint correlation function of vertex operators is nontrivial if and only if the charges satisfy the neutrality condition.

We consider the four-point function of the vertex operators corresponding to the four-point correlation function of primary fields (13). It is impossible to write a product of four vertex operators made of \(V_{1,2}, V_{2,1},\) \(V_{-1,-2}\), and \(V_{-2,-1}\) that satisfies the neutrality condition. To circumvent the problem, we insert a sufficient number of screening charges \(Q^\pm_\gamma\) into the correlation function,

\[ Q^\pm_\gamma = \oint \gamma V_{\pm}(u) du, \] (25)

where \(\gamma\) is a contour in the complex plane, and \(V_{\pm} = V_{\alpha_\pm}\). The screening charges \(Q^\pm_\gamma\), obtained by contour integrating fields of conformal dimension 1, have conformal dimension zero. Therefore, these charges are invariant under conformal maps. Inserting \(Q^\pm_\gamma\) an integer number of times in a correlation function of vertex operators does not affect its conformal properties, but forces the neutrality condition to be satisfied.

In the next subsection, we obtain solutions of the system of equations (14) by using the Coulomb gas formalism.
3.3. Dotsenko–Fateev representation of the four-point correlation function. We consider the Coulomb gas representation of correlation function (13). In the Dotsenko–Fateev formalism, each conformal field Φ_{r,s} corresponds to a certain vertex operator V_{r,s}. Besides, we insert screening charges inside the correlation function in order to satisfy the neutrality condition (24). Hence, the Coulomb gas representation of the correlation function becomes

\[ H_1(z, z^*; x_1, x_2; \gamma) = N_1(V_{2,1}(z)V_{2,1}(z^*)V_{1,2}(x_1)V_{1,-2}(x_2)Q^{-}), \]  

where the normalization constant \( N_1 \) depends on the integration contour. By using Eqs. (23), (15), we obtain an explicit integral representation of the correlation function (26)

\[ H_1(z, z^*; x_1, x_2; \gamma) = \frac{\Gamma(2 - \kappa/2)}{4\sin^2(\pi \kappa/4)\Gamma^2(1 - \kappa/4)} \frac{\eta^{2h_{2,1}+\kappa/8}}{(z - z^*)^{2h_{2,1}}(x_2 - x_1)^{2h_{1,2}}\sqrt{1 - \eta}} \times \int_{\gamma(0,\eta)} u^{-\kappa/4}(1 - u)(u - \eta)^{-\kappa/4} du, \]  

(27)

where \( h_{r,s} = h_{\alpha_{r,s}} \), the cross-ratio \( \eta \) was introduced in (16), and \( \Gamma(x) \) denotes the gamma function.

We discuss the prefactor in (27). The normalization is closely related to the choice of the integration contour. To guarantee that (27) satisfies the system of equations (14), the integration contour must be closed. Besides, it must surround at least one of the branch points of the integrand. If the powers of the branch points are irrational (as is usually the case), the winding number of the contour around each of the points must be zero in order for the contour to close. The Pochhammer contour \( \gamma(z_1, z_2) \) is the simplest such contour. It is convenient to replace the Pochhammer contour with the path joining the endpoints in accordance with the rule

\[ \int_{\gamma(z_1, z_2)} f(z_1, z_2, \ldots; u) \, du = 4e^{i\pi(\beta_1 - \beta_2)} \sin(\pi \beta_1) \sin(\pi \beta_2) \int_{z_1}^{z_2} f(z_1, z_2, \ldots; u) \, du, \]  

(28)

where \( f(z_1, z_2, \ldots; u) = \prod(u - z_i)^{\beta_i} \) and the monodromy factors \( \beta_1 \) and \( \beta_2 \) are greater than \(-1\). By using (28), we transform the integral in (27) into the integral along a simple curve connecting 0 and \( \eta \). Then the additional multipliers cancel the prefactor.

Now, we can consider the limit of the correlation function (27) as \( z \to z^* \). It is easy to show that

\[ \lim_{z \to z^*} (z - z^*)^{2h_{2,1}} H_1(z, z^*, x_1, x_2; \gamma) = H_0(x_1, x_2) \]  

(29)

if \( \kappa = 8/3 \). Indeed, before shrinking the integration contour that surrounds the point \( u = 0 \) and \( u = \eta \) in (27), we first replace \( u = t\eta \). Then the integral reduces to the standard beta function integral. Thus, as \( z \to x \in \mathbb{R} \), the correlation function containing a twist operator reduces to the correlation function without twist operators. As we explain below, this limit is in excellent agreement with the Simmons–Cardy interpretation of the correlation function \( H_1 \).

We show that the integral representation of correlation function (27) reduces to the previously obtained function (17). Indeed, the integral in (27) decomposes into a sum of two integrals. Each of these integrals can easily be evaluated in terms of the beta function. We then obtain

\[ \frac{H_1(z, z^*; x_1, x_2)}{H_0(x_1, x_2)} = (2 \text{Im } z)^{1-3\kappa/8} \frac{2 - \eta}{2\sqrt{1 - \eta}}, \]  

(30)

where we set \( z^* = \bar{z} \) at the end of the computation. This is exactly the function obtained previously (see Eq. (17)) as the solution of the null-vector constraints (14).
3.4. The 1-point SLE\textsubscript{8/3} passage probabilities. Below, we consider the correlation function (30) for $\kappa = 8/3$. This case is rather tricky because it corresponds to a $c = 0$ logarithmic CFT. On the other hand, the twist operators can then be treated as the 0-weight indicator Schramm’s operators, which can be used to distinguish between different configurations of SLE traces. When $\kappa = 8/3$, we have $n = 0$, and hence the $O(n)$ model describes an ensemble of self-avoiding walks (loops) in $\mathbb{H}$. At $n = 0$, all loops are suppressed, and partition function (4) becomes $Z = 1$. In the presence of 1-leg boundary operators at the points $x_1, x_2$, we only have those configurations with a self-avoiding path that connect the boundary points $x_1$ and $x_2$. The total weight of these configurations is given by the correlation function of the 1-leg operators at the boundary:

$$H_0(x_1, x_2) = \langle \Phi_{1,2}(x_1)\Phi_{1,2}(x_2) \rangle_{\mathbb{H}} = (x_2 - x_1)^{-5/4}. \quad (31)$$

In the presence of the twist operator $\Phi_{2,1}$, partition function (31) can be decomposed into the weights representing possible configurations of self-avoiding anchored paths interacting with a twist defect. These configurations are shown in Fig. 1. The corresponding statistical weights is denoted as $\Pi_{\text{in}}$ or $\Pi_{\text{out}}$ to respectively represent the cases where the SLE trace does or does not separate the twist operator from the interval $[x_1, x_2] \in \mathbb{R}$. The statistical weights $\Pi_{\text{in}}$ and $\Pi_{\text{out}}$ satisfy the system of linear equations

$$\Pi_{\text{in}} - \Pi_{\text{out}} = H_1(z, \bar{z}; x_1, x_2) = (x_2 - x_1)^{-5/4} \frac{2 - \eta}{2\sqrt{1 - \eta}},$$

$$\Pi_{\text{in}} + \Pi_{\text{out}} = H_0(x_1, x_2) = (x_2 - x_1)^{-5/4}, \quad (32)$$

where

$$\eta = \frac{(\bar{z} - z)(x_2 - x_1)}{(z - x_1)(\bar{z} - x_2)} \quad (33)$$

and the coefficients of the linear combinations in front of the weights are determined as follows.\textsuperscript{2} To find the coefficient in front of $\Pi_{\text{in}}$, we send $z, \bar{z} \to x \in [x_1, x_2]$ on both sides of the first equation in (32). In this case, $\eta = \epsilon(x_2 - x_1)/(x - x_1)(x - x_2)$. Then $\Pi_{\text{in}} \to H_0(x_1, x_2)$ and $\Pi_{\text{out}} \to 0$, and the expression in the right-hand side of (32) goes to $H_0(x_1, x_2)$. Therefore, the coefficient in front of $\Pi_{\text{in}}$ equals 1. Next, to find coefficient of $\Pi_{\text{out}}$, we send $z, \bar{z} \to x \in \mathbb{R} \setminus [x_1, x_2]$. In this limit, we have $\Pi_{\text{in}} \to 0$, $\Pi_{\text{out}} \to 1$, while the expression in the right-hand side goes to $-1$, thus justifying the coefficient $-1$ in front of the weight $\Pi_{\text{out}}$.

\textbf{Fig. 1.} Possible configurations of the SLE\textsubscript{8/3} traces in the presence of a marked point $z$ in the upper half plane. The weight $\Pi_{\text{out}}$ ($\Pi_{\text{in}}$) accounts for the cases where the SLE trace separates (does not separate) the twist operator from the interval $[x_1, x_2] \in \mathbb{R}$.

The system of equations (32) determines probabilities for SLE\textsubscript{8/3} traces to wind in various ways about the point $z \in \mathbb{H}$. In particular, the probability that the SLE trace separates the point $z$ from the interval $[0, \infty]$, i.e., the left-crossing probability, $P_L(z)$, is

$$P_L(z) = \frac{\Pi_{\text{out}}}{\Pi_{\text{out}} + \Pi_{\text{in}}} = \frac{1}{2} \frac{H_1(z, \bar{z}; x_1, x_2)}{2H_0(x_1, x_2)} = \frac{1}{2} \frac{\cos(\arg z)}{2}. \quad (34)$$

In the last equality, we set $x_1 = 0$ and $x_2 = \infty$, and take into account that $\eta = 1 - e^{-2i\arg(z)}$.

\textsuperscript{2}These coefficients depend on $\kappa$. However, in the case $\kappa = 8/3$, the coefficients are $\pm 1$. 

217
3.5. The 2-point SLE$_{8/3}$ passage probabilities. We have already noted that the SLE$_{8/3}$ left-crossing probability is determined by the correlation function containing the twist operator $\Phi_{2,1}$ and a pair of the boundary 1-leg operators $\Phi_{1,2}$. Similarly, in the presence of two marked points in the bulk, $z_1, z_2 \in \mathbb{H}$, the passage probabilities are determined by the correlation functions of two twist operators at the points $z_1, z_2$, and a pair of the 1-leg boundary operators at the points $x_1, x_2 \in \mathbb{R}$:

$$ H_2(z_1, \bar{z}_1, z_2, \bar{z}_2; x_1, x_2) = \langle \Phi_{2,1}(z_1, \bar{z}_1)\Phi_{2,1}(z_2, \bar{z}_2)\Phi_{1,2}(x_1)\Phi_{1,2}(x_2) \rangle_{\mathbb{H}}. \quad (35) $$

The correlation function (35) is specified by the boundary conditions, which determine an appropriate linear combination of conformal blocks that contribute to the correlation function.

Before defining conformal blocks, it is convenient to recast the correlation function in $\mathbb{H}$ into a correlation function in $\mathbb{C}$. By replacing the antiholomorphic coordinates $\bar{z}_1, \bar{z}_2 \in \mathbb{H}$ by holomorphic coordinates $z^*_1, z^*_2 \in \mathbb{C}$, we consider the 6-point correlation function in the whole complex plane:

$$ H_2(z_1, z^*_1, z_2, z^*_2; x_1, x_2) = \prod_{i=1}^{2} \phi_{2,1}(z_i)\phi_{2,1}(z^*_i)\phi_{1,2}(x_1)\phi_{1,2}(x_2). \quad (36) $$

The conformal symmetry implies that it can be written in the form

$$ H_2(z_1, z^*_1, z_2, z^*_2; x_1, x_2) = \frac{G_2(\eta_1, \eta_2, \eta_3)}{(x_2 - x_1)^{2h_{1,2}}(z_1 - z^*_1)^{2h_{1,2}}(z_2 - z^*_2)^{2h_{1,2}}}. \quad (37) $$

Here, $G_2(\eta_1, \eta_2, \eta_3)$ is a function of the cross-ratios $\eta_1 = \eta(z^*_1)$, $\eta_2 = \eta(z_2)$, and $\eta_3 = \eta(z^*_2)$, where

$$ \eta(s) = \frac{(z_1 - s)(x_1 - x_2)}{(z_1 - x_1)(s - x_2)}. \quad (38) $$

In particular, $\eta(s) = 1 - s/z_1$ when $x_1 = 0$, and $x_2 \to \infty$. We can also consider the correlation function $H_2(0, \eta_1, \eta_2, \eta_3; 1, \infty)$, which (due to conformal symmetry) can be written in the form

$$ H_2(0, \eta_1, \eta_2, \eta_3; 1, \infty) = \frac{G_2(\eta_1, \eta_2, \eta_3)}{\eta_1^{2h_{1,2}}(\eta_2 - \eta_1)^{2h_{1,2}}}. \quad (39) $$

By eliminating the function $G_2$ from Eqs. (35) and (39), we obtain the following relation for the correlation function:

$$ H_2(z_1, z^*_1, z_2, z^*_2; x_1, x_2) = \frac{\eta_1^{2h_{1,2}}(\eta_2 - \eta_1)^{2h_{1,2}}}{(x_2 - x_1)^{2h_{1,2}}(z_1 - z^*_1)^{2h_{1,2}}(z_2 - z^*_2)^{2h_{1,2}}} \times H_2(0, \eta_1, \eta_2, \eta_3; 1, \infty). \quad (40) $$

The modules generated by the fields $\Phi_{1,2}$ and $\Phi_{2,1}$ are degenerate at the second level. Thus, the decoupling of null states implies that the correlation function (35) satisfies six second-order PDEs. Besides, the required solution of these equations must satisfy the factorization condition

$$ \lim_{z_1 \to \infty} \frac{H_2(z_1, \bar{z}_1, z_2, \bar{z}_2, x_1, x_2)}{H_0(x_1, x_2)} = H_1(z_1, z_1, x_1, x_2)H_1(z_2, z_2, x_1, x_2). \quad (41) $$

In [4], Simmons and Cardy found a unique solution of the relevant equations that satisfies condition (41). However, in the case of $N \geq 3$ twist operators in the bulk, the system of null-state PDEs is very difficult to solve directly.
Below, we obtain a Dotsenko–Fateev representation for the correlation function of two twist operators.

Following the guidelines described in Sec. 3.3, we consider a product of vertex operators $V_{2,1}$ and $V_{1,2}$,

$$\mathcal{H}_2(z_1, z_1^*, z_2, z_2^*, x_1, x_2; \gamma_1, \gamma_2) = \left\langle \prod_{i=1}^{2} V_{2,1}(z_i) V_{2,1}(z_i^*) V_{1,2}(x_1) V_{-1,-2}(x_2) Q_{\gamma_1}^- Q_{\gamma_2}^- \right\rangle,$$  \hspace{1cm} (42)

where we inserted two screening charges $Q_i^-$ inside the correlation function in order to satisfy the neutrality condition (24). For convenience, we used the reflection property $\alpha \rightarrow 2\alpha_0 - \alpha$ discussed after Eq. (22), and replaced the vertex operator $V_{1,2}$ with $V_{-1,-2}$.

We refer to the correlation function (42) as a conformal block. It depends on the Pochhammer integration contours $\gamma_1$ and $\gamma_2$, which determine the screening charges (25),

$$\mathcal{H}_2(z_1, z_1^*, z_2, z_2^*, x_1, x_2; \gamma_1, \gamma_2) = \oint_{\gamma_1} du_1 \oint_{\gamma_2} du_2 (u_1 - u_2)^{2\alpha - \alpha_+} \times$$

$$\times \prod_{j=1,2} \prod_{i=1,2} (z_i - u_j)^{2\alpha_1, \alpha_2 - \alpha_+} (u_j - x_1)^{2\alpha_1, \alpha_2 - \alpha_+} (u_j - x_2)^{2\alpha_1, \alpha_2 - \alpha_+}. \hspace{1cm} (43)$$

The correlation function of primary fields (36) is given by an appropriate linear combination of these blocks,

$$H_2(\ldots) = \sum_{i,j} N(\gamma_i, \gamma_j) \mathcal{H}_2(\ldots; \gamma_i, \gamma_j), \hspace{1cm} (44)$$

where the coefficients $N(\gamma_i, \gamma_j)$ depend on the integration contours. In the case of 6-points function (43), there exist 10 natural pairs of the contours $(\gamma_i, \gamma_j)$. However, it can be argued that only one choice of the contours is reasonable, namely, the one where $\gamma_1$ and $\gamma_2$ are simple paths respectively connecting $z_1$ to $z_1^*$ and $z_2$ to $z_2^*$. Below, we give simple argument in support of the statement, which can also be justified by explicit calculations [4].

As proposed by Simmons and Cardy, correlation function (35) is closely connected to the probabilities of the SLE trace to wind in various ways around the points $z_1$ and $z_2$ (see also Eqs. (49) below). We discuss the possible asymptotic behavior of the conformal block as the point $z_1$ approaches the real axis at $x \in (x_1, x_2)$. We recall that $\Phi(z, z^*) = \Phi(z) \Phi(z^*)$, and hence the possible asymptotic behavior of the correlation function is determined by the bulk–boundary fusion $\Phi_{2,1}(z_1) \times \Phi_{2,1}(z_1^*)$ as $z_1, z_1^* \rightarrow x$. In this case, the statistical weight of paths separating $z_1$ from the interval $[x_1, x_2]$ vanishes, and possible SLE configurations are determined by the twist field at $z_2$ only. In other words, the twist field must disappear from the correlation function in the limit $z_1 \rightarrow x$. Therefore, the bulk–boundary fusion $\Phi_{2,1}(z_1) \times \Phi_{2,1}(z_1^*)$ should be realized via the identity channel only (also see Ref. [4] for explicit computations).

The conformal block $\mathcal{H}_2(z_1, z_1^*, z_2, z_2^*, x_1, x_2)$ with the required asymptotic behavior is determined by the integration contour connecting $z_1$ and $z_1^*$. This can easily be shown by inserting the product\(^3\)

$$\sigma(z_1, z_1^*) = \int_{z_1}^{z_1^*} V_{2,1}(z_1) V_{2,1}(z_1^*) V_-(u) \, du \hspace{1cm} (45)$$

in conformal block (26). The choice of the integration contour (from $z_1$ to $z_1^*$) implies that a pair of vertex operators fuse via the identity channel as $z_1$ and $z_1^*$ approach the real axis. Indeed, by fusing $V_{2,1}(z_1) \times V_{2,1}(z_1^*)$ as $z_1, z_1^* \rightarrow x \in \mathbb{R}$ we obtain the vertex operator $V_\alpha(x)$ with the charge $\alpha = 2\alpha_{2,1} = \alpha_{3,1}$. The screening charge $V_-$ in the integrand in (45) merges with the vertex operators if and only if the integration contour contracts to the point $x$. In this case, the total charge of the product $V_{2,1}V_{2,1}V_-$ vanishes, $2\alpha_{2,1} + \alpha_- = 0$, and operator (45) becomes the identity operator as expected. Similarly, it can be argued that the integration contour of the second screening operator in (43) connects the points $z_2$ and $z_2^*$.

In Fig. 2, we show the integration contours (dashed lines) of the conformal block.
Fig. 2. The conformal block $H_2(z_1, z_2^*, z_2, z_2^*; x_1, x_2)$ is shown with respect to the bulk-boundary fusion. The boundary is shown by the solid line. The dashed lines represent the integration paths for two screening charges $Q_{-\gamma}$.

By taking the explicit expression for conformal block (43) into account and applying Möbius transformation (38), we obtain the integral representation of the correlation function

$$H_2(0, \eta_1, \eta_2, \eta_3; 1, \infty) = \prod_{i=1}^{3} \frac{\eta_i^{1/3}}{(1 - \eta_i)^{1/2}} \prod_{i<j} \left(\frac{\eta_i - \eta_j}{\eta_i - \eta_j}^{1/3}\right) \mathcal{J}_2(\eta_1, \eta_2, \eta_3),$$

where $\mathcal{J}_2$ denotes the double contour integral

$$\mathcal{J}_2(\eta_1, \eta_2, \eta_3) = \int_{\eta_2}^{\eta_3} du_1 \int_{0}^{\eta_1} du_2 \left(u_1 - u_2\right)^{4/3} \prod_{i=1}^{2} u_i^{-2/3} \left(1 - u_i\right) \prod_{j=1}^{3} \left(u_i - \eta_j\right)^{-2/3}.$$ 

The overall normalization of the correlation function is chosen such that the correlation function (40) reduces to (30) in the limit $\eta_3 \to \eta_2$. The limit is defined as

$$H_n-1(\ldots, z_i, z_i+2, \ldots) = \lim_{z_i^*_{i+1} \to z_i} (z_i - z_i^*)^{2h_{2;1}} H_n(\ldots, z_i, z_i+1, z_i+2, \ldots),$$

where the explicit dependence of $H_n$ on $\{z_i^*\}$ is not indicated for brevity.

Now, we are ready to determine the probabilities for SLE$_{8/3}$ traces to wind about two marked points in $\mathbb{H}$. In the presence of twist defects, the partition function (31) for SLE from $x_1$ to $x_2$ can be decomposed into the sum of weights depending on the winding of the paths around two points. We label these weights by $\Pi_{12:8}$, $\Pi_{1:2}$, $\Pi_{2:1}$, and $\Pi_{0:12}$, where $\Pi_{ij:kl}$ denotes the weight of the paths that separate the points $z_k$ and $z_l$ from the interval $[x_1, x_2] \in \mathbb{R}$ while the points $z_i$ and $z_j$ remain unseparated (see Fig. 3).

Fig. 3. Possible configurations of SLE$_{8/3}$ trace in the presence of two marked points in the upper half plane. The weight $\Pi_{ij:kl}$ counts the cases where SLE trace does not separate the twist operators at the points $z_i$ and $z_j$ from the interval $[x_1, x_2] \in \mathbb{R}$ while the points $z_k$ and $z_l$ are separated from the interval.

\(^3\)Below, we refer to $\sigma(z, \bar{z})$ in $\mathbb{H}$ (or, $\sigma(z, z^*)$ in $\mathbb{C}$) as the dressed twist operator, or, for brevity, simply the twist operator at the point $z$. 

220
Following the guidelines in Sec. 3.4, we decompose the correlation functions $H_0$, $H_1$, and $H_2$ in terms of the weights $\Pi_{ij:kl}$ of the possible trace configurations:

$$
\begin{align*}
\Pi_{12:0} + \Pi_{1:2} + \Pi_{2:1} + \Pi_{0:12} &= H_0(x_1, x_2), \\
\Pi_{12:0} + \Pi_{1:2} - \Pi_{2:1} - \Pi_{0:12} &= H_1(z_1, \bar{z}_1; x_1, x_2), \\
\Pi_{12:0} - \Pi_{1:2} + \Pi_{2:1} - \Pi_{0:12} &= H_1(z_2, \bar{z}_2; x_1, x_2), \\
\Pi_{12:0} - \Pi_{1:2} - \Pi_{2:1} + \Pi_{0:12} &= H_2(z_1, \bar{z}_1, z_2, \bar{z}_2; x_1, x_2).
\end{align*}
$$

The coefficients of these linear combinations in front of the weights are determined similarly to the preceding case (see the discussion below Eq. (32)). By solving these equations, we obtain the statistical weights, e.g.,

$$
\Pi_{0:12} = \frac{1}{4} \left[ H_0(x_1, x_2) + H_2(z_1, \bar{z}_1, z_2, \bar{z}_2; x_1, x_2) - \sum_{i=1}^{2} H_1(z_i, \bar{z}_i; x_1, x_2) \right],
$$

and the probabilities of the corresponding events

$$
P_{ij:kl} = \frac{\Pi_{ij:kl}}{\Pi_{12:0} + \Pi_{1:2} + \Pi_{2:1} + \Pi_{0:12}} = \frac{\Pi_{ij:kl}}{H_0(x_1, x_2)}.
$$

In particular, the probability, $P_L(z_1, z_2)$ that the SLE trace passes to the left of both points $z_1, z_2 \in \mathbb{H}$ can be written as

$$
P_L(z_1, z_2) = \frac{1}{2} (P_L(z_1) + P_L(z_2)) + \frac{1}{4} \left( \frac{H_2(z_1, \bar{z}_1, z_2, \bar{z}_2; x_1, x_2)}{H_0(x_1, x_2)} - 1 \right),
$$

where $P_L(z)$ is the SLE$_{8/3}$ one-point left passage probability (34).

### 3.6. The N-point SLE$_{8/3}$ passage probabilities.

In this section, we generalize the results in the preceding sections to the case of $N$ twist operators in the upper half plane. The probabilities of SLE$_{8/3}$ traces to wind about the twist operators are determined by the set of $N$ multipoint correlation functions

$$
H_n(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n; x_1, x_2) = \left\langle \prod_{i=1}^{n} \Phi_{2,1}(z_i, \bar{z}_i) \Phi_{1,2}(x_1) \Phi_{1,2}(x_2) \right\rangle_{\mathbb{H}},
$$

where $n = 0, 1, \ldots, N$.

To obtain the Coulomb gas representation for correlation functions (53), we consider the product of $2n$ vertex operators $V_{2,1}(z_i)$, $V_{2,1}(z_i^*)$, $i = 1, 2, \ldots, n$, and the boundary operators $V_{1,2}(x_1)$ and $V_{-1,-2}(x_2)$ at $x_1, x_2 \in \mathbb{R}$. The corresponding correlation function of the vertex operators requires $n$ screening charges in order to satisfy neutrality condition (24). By inserting $n$ dressed twist operators (45) at the points $z_1, \ldots, z_n$ in the correlation function $H_0 = (V_{1,2}V_{-1,-2})$, we obtain

$$
H_n(z) = \left\langle \prod_{i=1}^{n} \sigma(z_i, z_i^*)V_{1,2}(x_1)V_{-1,-2}(x_2) \right\rangle_{\mathbb{H}}.
$$

We recall that the integration contours that determine the operators $\sigma(z_i, z_i^*)$ are simple paths connecting the points $z_i, z_i^*, i = 1, 2, \ldots, n$, pairwise. In this case, the bulk–boundary fusion of vertex operators $V_{2,1}(z_i) \times V_{2,1}(z_i^*)$ as $z_i^*, z_i \rightarrow x \in \mathbb{R}$, is realized via the identity channel (see the discussion below Eq. (44)).

By using Möbius transformation (38), we obtain the Coulomb gas representation for the correlation function $H_n$,

$$
H_n(z_1, z_1^*, \ldots, z_n, z_n^*; x_1, x_2) = \prod_{i=1}^{2n-1} \eta_i^{1/3}(1 - \eta_i)^{-1/2} \prod_{k<l} (\eta_k - \eta_l)^{1/3} / (x_2 - x_1)^{5/4} \times J_n(\eta_1, \ldots, \eta_{2n-1}),
$$
where we took into account that \( h_{1,2} = 5/8 \) and \( h_{2,1} = 0 \) when \( \kappa = 8/3 \). The function \( \mathcal{J}_n \) is determined by the \( n \)-fold integral
\[
\mathcal{J}_n(\eta_1, \ldots, \eta_{2n-1}) = \int_{0}^{\eta_1} du_1 \int_{\eta_2}^{\eta_3} du_2 \cdots \int_{\eta_{2n-2}}^{\eta_{2n-1}} du_{n} \times \\
\times \prod_{i<j} (u_i - u_j)^{4/3} \prod_{i=1}^{n} u_i^{-2/3} \prod_{k=1}^{2n-1} (1 - u_i) (u_i - \eta_k)^{-2/3}. 
\]

Similarly to the preceding case, the overall normalization of the correlation function is chosen such that the property (48) holds.

Correlation function (53) is closely related to probabilities of the \( \text{SLE}_{8/3} \) trace to wind about the points \( z_1, z_2, \ldots, z_n \in \mathbb{H} \). Indeed, in the presence of \( n \) twist operators, the partition function of the \( \text{SLE}_{8/3} \) curve can be decomposed into the sum of statistical weights of the possible trace configurations.

To label these weights, we introduce the following notation. Let \( I_n = \{1, 2, \ldots, n\} \) be a set of \( n \) integers, and \( I_n \subset I_N \). We decompose the set \( I_n \) into two subsets \( I^+_n \) and \( I^-_n \) such that \( I^+_n \cup I^-_n = I_n \) and \( I^+_n \cap I^-_n = \emptyset \). Besides, we introduce the set of points \( Z_I = \{z_i|i \in I\} \) labeled by integers from the set \( I \). We let \( \Pi_{I^+_n \cap I^-_n} \) denote the weight of the \( \text{SLE}_{8/3} \) traces that separate the points \( Z_{I^+} \) from the interval \( \{x_1, x_2\} \subset \mathbb{R} \) while the points \( Z_{I^-} \) remain unseparated.

With this notation, we can decompose the partition function in terms of the weights as (cf. Eqs. (49))
\[
\sum_{I^+_n \cap I^-_n = I_N} (-1)^{\# I^+_n} \Pi_{I^+_n \cap I^-_n} = H_n(Z_{I^+_n}, Z_{I^-_n}; x_1, x_2), \quad n = 0, 1, \ldots, N. 
\]

Here, the sum is taken over all decompositions of \( I_N \) into two subsets \( I^+_n \) and \( I^-_n \). The coefficients \( (-1)^{\# I^+_n} \) can be obtained by sending \( z_i, z_i^* \to x \in [x_1, x_2] \) \( (x \in \mathbb{R} \setminus [x_1, x_2]) \), as explained in the discussion below Eq. (32). Therefore, we obtain \( 2^N \) linear equations for the \( 2^N \) unknown statistical weights \( \Pi_{I^+_n \cap I^-_n} \).

Equations (57) allow us to determine statistical weights of \( \text{SLE}_{8/3} \) traces from \( x_1 \) to \( x_2 \) to pass to the right of the points \( Z_{I^+} \) and to the left of the points \( Z_{I^-} \) (cf. Eqs. (50)),
\[
\Pi_{I^+_n \cap I^-_n} = 2^{-N} \sum_{n=0}^{N} \sum_{I_n} (-1)^{\# (I^+_n \cap I^-_n)} H_n(Z_{I^+_n}, x_1, x_2), 
\]
and the probability that the \( \text{SLE}_{8/3} \) trace from \( x_1 \) to \( x_2 \) separates the points \( Z_{I^-} \) from the interval \( [x_1, x_2] \) is given by
\[
P_{I^+_n \cap I^-_n} = \frac{\Pi_{I^+_n \cap I^-_n}}{H_0(x_1, x_2)}. 
\]

4. 

**Green’s functions for \( \text{SLE}_{8/3} \)**

4.1. The 1-point \( \text{SLE}_{8/3} \) Green’s function. In this section, we discuss the probability for \( \text{SLE}_{8/3} \) traces to pass in an \( \epsilon \)-neighborhood of the marked point in the upper half-plane. This probability determines a one-point \( \text{SLE}_{8/3} \) Green’s function in what follows. We consider the \( \text{SLE}_{8/3} \) trace from \( x_1 \) to \( x_2 \). The probability \( P\{z < \epsilon; x_1, x_2\} \) that the trace passes in the \( \epsilon \)-neighborhood of the point \( z \in \mathbb{H} \) vanishes with the leading asymptote [2]
\[
\lim_{\epsilon \to 0} \epsilon^{-2/3} P\{z < \epsilon; x_1, x_2\} = c_1 G_{\mathbb{H}}^{\text{SLE}}(z; x_1, x_2), 
\]
where \( c_1 \) is a constant and \( G_{\mathbb{H}}^{\text{SLE}}(z; x_1, x_2) \) is the so-called one-point Green’s function of the SLE trace. We show below that the Green’s function can be written in terms of the correlation function of certain primary operators in a \( c = 0 \) LCFT. Besides, we propose a similar representation for the multipoint Green’s function.
We start with the one-point function (60). It can be obtained as the limit case of the two-point passage probability as the points approach each other. There are two possible trace configurations that contribute to the probability, namely, $\Pi_{1:2}$ and $\Pi_{2:1}$ (see Fig. 3). Hence,

$$
P(z_1, z_2; x_1, x_2) = \frac{\Pi_{1:2} + \Pi_{2:1}}{\Pi_{1:2} + \Pi_{2:1} + \Pi_{0:12}} = \frac{1}{2} \frac{H_2(z_1, \bar{z}_1, z_2, \bar{z}_2; x_1, x_2)}{2H_0(x_1, x_2)}. \quad (61)
$$

We set $z_1 = z + \epsilon \nu / 2$, $z_2 = z - \epsilon \nu / 2$, where $z \in \mathbb{H}$, $\epsilon \ll 1$, $|\nu| = 1$, and consider the series expansion of $P(z_1, z_2; x_1, x_2)$ for small $\epsilon$. The leading term of the series expansion determines the Green’s function of the trace:

$$
\lim_{\epsilon \to 0} \epsilon^{-2/3} P \left( z + \frac{\epsilon \nu}{2}, z - \frac{\epsilon \nu}{2}, x_1, x_2 \right) = c_1 G_1(z; x_1, x_2).
$$

(62)

Because the probability in (61) is determined by the correlation function of primary operators, we can use conformal symmetry to study the series expansion. Namely, we use the so-called *operator product expansion* (OPE) of the primary fields $\Phi_{2:1}(z_1)\Phi_{2:1}(z_2)$ inside the correlation function. This allows obtaining the leading term of the series expansion of the correlation function explicitly. We briefly recall the definition of OPE [5].

Operator product expansion determines the behavior of operators as they approach each other. Namely, it suggests that two local operators at nearby points can be approximated by an (infinite) sum of certain local operators (including the so-called *descendant* operators) at one of these points. The form of the OPE can be deduced from global conformal invariance and the form of the two- and three-point functions.\(^4\)

We consider the product of the primary fields $\Phi_{h_i}(z)$ and $\Phi_{h_j}(0)$. In the limit $z \to 0$, it can be replaced with the sum of local operators at the point $z = 0$,

$$
\Phi_{h_i}(z)\Phi_{h_j}(0) = z^{-h_i - h_j} \sum_k C_{ij}^k z^h_k \left( \Phi_{h_k}(0) + \sum_n \beta_{i,j}^{k,\{n\}} z^{\{n\}} \Phi_{h_k}^{(-\{n\})}(0) \right),
$$

(63)

where the coefficients $\beta_{i,j}^{k,\{n\}}$ are fixed by conformal invariance and $\Phi_{h_k}^{(-\{n\})}$ denotes the contribution of the $\{n\}$-level descendant operators,

$$
\Phi_{h_k}^{(-\{n\})} = L_{-n_1} L_{-n_2} \cdots L_{-n_L} \Phi_{h_k},
$$

(64)

where $\{n\} = (n_1, n_2, \ldots, n_L)$ and $|n\{n\}| = |n_1 + n_2 + \cdots + n_L|$. Here, $L_{-n}$ are generators of the Virasoro algebra [15] and $C_{ij}^k$ are the structure constants of the operator algebra. They can be determined by the two- and three-point functions,\(^5\)

$$
C_{ij}^k = \lim_{z \to \infty} |z|^{4h_i} \langle \Phi_{h_i}(z)\Phi_{h_j}(1)\Phi_{h_k}(0) \rangle,
$$

(65)

where the normalization $\langle \Phi_{h_i}(z)\Phi_{h_i}(0) \rangle = z^{-2h_i}$ is assumed. The structure constants are not fixed by conformal invariance. However, they can be shown to satisfy a set of nontrivial relations that express the associativity of the operator algebra and are known as the conformal bootstrap equations [5].

We briefly recall the structure of $c = 0$ LCFT (see [4] for the details).\(^6\) We let $\mathcal{V}_{r,s}$ denote the Verma module generated from the highest vector $|\Phi_{r,s}\rangle$ by acting with linear combinations of the Virasoro generators. In a $c = 0$ LCFT, the vacuum module is indecomposable $\mathcal{M}_{1,1} = \mathcal{V}_{1,1}/\mathcal{V}_{4,1}$. Furthermore, the

---

\(^4\)This form of the OPE is typical for rational CFTs, while in LCFTs the OPE of certain operators can be modified.

\(^5\)In LCFT, certain structure constants become functions containing logarithms.

\(^6\)We note that we follow the notation in [4], and hence the order of Kač indices is reversed compared to [16], [17].
This equation allows rewriting probability (61) in the form where 

$$\mathcal{M}_{2,1} \times \mathcal{M}_{2,1} = \mathcal{M}_{1,1} + \mathcal{M}_{3,1},$$

(66)

where \( \mathcal{M}_{1,1} \) was introduced above and \( \mathcal{M}_{3,1} \) is the irreducible module with the highest weight \( h_{3,1} = 1/3 \). Fusion rule (66) implies the OPE of the \( \Phi_{2,1} \) primary fields at nearby points in the form

$$\lim_{\epsilon \to 0} \Phi_{2,1}(\epsilon) \Phi_{2,1}(0) = \Phi_{1,1}(0) + C_{3,1} \epsilon^{1/3} \Phi_{3,1}(0) + O(\epsilon),$$

(67)

where \( C_{3,1} \) is the OPE coefficient (the structure constant). In this case, we can say that OPE is realized via two channels: the first involves \( \Phi_{1,1} \) and the second involves \( \Phi_{3,1} \). We note that in a CFT defined in a bounded region, the form of the OPE can be modified because of the boundary conditions. As explained below, the case of a boundary \( c = 0 \) LCFT is even more tricky.

We consider the bulk–boundary fusion \( \Phi_{2,1} \Phi_{2,1} \) in correlation function (36). As discussed, the only possible fusion channel is the identity operator. Indeed, by setting \( z = x + \epsilon/2 \) and \( \bar{z} = x - \epsilon/2 \) and shrinking the integration contour connecting these points, we obtain a series expansion in the form \( g_0 + \epsilon g_1 + \epsilon^2 g_2 + \cdots \), where \( g_n \) are certain functions of the coordinates \( \{ z, \bar{z}, x_1, x_2 \} \). By comparing this expansion with OPE (67), we conclude that the bulk–boundary fusion is realized via the identity operator and the second channel, \( \Phi_{3,1} \), is forbidden.

![Fig. 4.](image)

Fig. 4. Two conformal blocks that contribute to the correlation function \( H_2(z_1, \bar{z}_1, z_2, \bar{z}_2; x_1, x_2) \), are shown with respect to the bulk–bulk fusion, \( \Phi_{2,1} \Phi_{2,1} \). The fusion can be realized via two channels, \( \Phi_{1,1} \) and \( \Phi_{3,1} \). Fusion with boundary operators is realized via the identity channel or stress–energy tensor \( T \) only. The other possibilities are forbidden. Rectangles \( [m, n] \) correspond to the fields \( \Phi_{m,n} \) and the boundary operators are connected by double lines.

Further, we consider the bulk–bulk fusion of the fields \( \Phi_{2,1} \Phi_{2,1} \) inside the correlation function as \( z_2 \to z_1, \bar{z}_2 \to \bar{z}_1 \) (see Fig. 4). Explicit calculations shows that both channels \( \Phi_{1,1} \) and \( \Phi_{3,1} \) appear in the OPE in this case [4]. Using (67), we find the series expansion of the correlation function

$$\lim_{\epsilon \to 0} H_2 \left( z + \frac{\epsilon \nu}{2}, z - \frac{\epsilon \nu}{2}, \bar{z} + \frac{\epsilon \nu}{2}, \bar{z} - \frac{\epsilon \nu}{2}; x_1, x_2 \right) = H_0(x_1, x_2) - (C_{3,1})^2 \epsilon^{2/3} \langle \Phi_{3,1}(z, \bar{z}) \Phi_{1,2}(x_1) \Phi_{1,2}(x_2) \rangle_\mathbb{H} + O(\epsilon).$$

(68)

This equation allows rewriting probability (61) in the form

$$\lim_{\epsilon \to 0} \epsilon^{-2/3} P \left( z + \frac{\epsilon \nu}{2}, z - \frac{\epsilon \nu}{2}; x_1, x_2 \right) = (C_{3,1})^2 G_\mathbb{H}(z; x_1, x_2),$$

(69)

where \( G_\mathbb{H}(z; x_1, x_2) \) denotes the correlation function

$$G_\mathbb{H}(z; x_1, x_2) = \frac{\langle \Phi_{3,1}(z, \bar{z}) \Phi_{1,2}(x_1) \Phi_{1,2}(x_2) \rangle_\mathbb{H}}{\langle \Phi_{1,2}(x_1) \Phi_{1,2}(x_2) \rangle_\mathbb{H}}.$$  

(70)

We show below that this function coincides with the one-point SLE$_{8/3}$ Green’s function obtained in Ref. [2].
We emphasize that the proposed representation of the SLE_{3/2} Green’s function in terms of a CFT correlation function is novel (70). Moreover, this result can be easily generalized to the multipoint case, which is almost unachievable by other methods. In this paper, we discuss two main approaches to computing correlation functions. The first approach reduces the problem to a system of PDEs that occur as null-vector decoupling conditions (see, e.g., Eqs. (14)). The second approach leads to the integral representations for the correlation function based on the Dotsenko–Fateev technique. Below, we use this second approach to compute the Green’s function.

4.2. The Coulomb gas representation for the Green’s function. As discussed, correlation functions in CFT can be computed in the framework of the Coulomb gas representation. In particular, the correlation function in the right-hand side of Eq. (70) can be written as a linear combination of conformal blocks:

\[
\langle \Phi_{3,1}(z)\Phi_{3,1}(z^*)\Phi_{1,2}(x_1)\Phi_{1,2}(x_2) \rangle = \sum_{\{\gamma\}} F_1(z, z^*; x_1, x_2; \gamma). \tag{71}
\]

In the Coulomb gas representation, these blocks are correlation functions of vertex operators,

\[
F_1(z, z^*; x_1, x_2; \gamma) = \langle V_{3,1}(z)V_{3,1}(z^*)V_{1,2}(x_1)V_{1,2}(x_2)Q_{\gamma}^+ \rangle, \tag{72}
\]

where we inserted the screening charge \( Q_{\gamma}^+ \) in order to satisfy neutrality condition (24). We note that the same correlation function (71) can be realized via another set of conformal blocks, namely, \( \langle V_{3,1}V_{3,1}V_{1,2}V_{1,2}Q_{\gamma}^+Q_{\gamma}^- \rangle \). In this case, however, two screening charges are required. Thus, it is convenient to proceed with the first representation (72).

Using (23) to evaluate the product of vertex operators, we arrive at the integral representation for the conformal block

\[
F_1(z, z^*; x_1, x_2; \gamma) = \frac{\Gamma(4/\kappa - 2)}{4\sin(8\pi/\kappa)\sin(4\pi/\kappa)\Gamma(8/\kappa - 3)\Gamma(1 - 4/\kappa)} \times
\]

\[
\times \left( \frac{z - z^*}{2i} \right)^{-2h_{3,1}} \left( \frac{x_2 - x_1}{1 - \eta} \right)^{-2h_{1,2}} \oint_{\gamma} du u^{8/\kappa - 4}(1 - u)^{-4/\kappa}(u - \eta)^2, \tag{73}
\]

where \( \eta \) denotes the cross-ratio introduced in (16), and we included the factor \( (2i)^{2h_{3,1}} \) for further convenience. The normalization of the conformal block represented by the product of gamma functions is consistent with (27). The Pochhammer contour \( \gamma \) surrounds the branching points \( u = 0 \) and \( u = 1 \) of the integrand. The point \( u = \eta \) is not a branch point. Hence, any Pochhammer contour surrounding this point is shrinkable.

We give an explicit expression for correlation function (73) for general values of \( \kappa \). It allows obtaining a well-defined limit of the correlation function as \( \kappa \to 8/3 \). Indeed, by replacing the Pochhammer contour with the path connecting the points 0 and 1, we cancel the factor \( 4\sin(8\pi/\kappa)\sin(4\pi/\kappa) \). The remaining integral reduces to the sum of beta functions,

\[
\int_0^1 du u^{8/\kappa - 4}(1 - u)^{-4/\kappa}(u - \eta)^2 =
\]

\[
= B\left( \frac{8}{\kappa} - 1, 1 - \frac{4}{\kappa} \right) + 2\eta B\left( \frac{8}{\kappa} - 2, 1 - \frac{4}{\kappa} \right) - \eta^2 B\left( \frac{8}{\kappa} - 3, 1 - \frac{4}{\kappa} \right), \tag{74}
\]

where \( B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y) \) is the beta function. Using (74) and evaluating the limit of the quotient in the right-hand side of Eq. (73) as \( \kappa \to 8/3 \), we obtain

\[
F_1(z, z^*; x_1, x_2) = \left( \frac{z - z^*}{2i} \right)^{-2/3} \left( \frac{x_2 - x_1}{1 - \eta} \right)^{-5/4} \eta^2 \tag{75}
\]

\[\text{as discussed, it is convenient to study correlation functions in } \mathbb{C} \text{ rather than } \mathbb{H}. \text{ Hence, we replace } (z, \bar{z}) \to (z, z^*).\]
where we set \( z^* = \bar{z} \) at the end of the computation. Substituting (75) in (70), we find the Green’s function

\[
G_H(z; x_1, x_2) = (\text{Im} \, z)^{-2/3} \frac{\eta^2}{1 - \eta}.
\]

Finally, it is convenient to consider SLE traces from 0 to \( \infty \), with \( \eta = 1 - e^{-2i \arg(z)} \) and

\[
G_H(z; 0, \infty) = (\text{Im} \, z)^{-2/3} \sin^2(\arg(z)).
\]

This expression for the SLE\(_{8/3} \) Green’s function was obtained in [2] by totally different methods. In this paper, we propose a representation for the Green’s function in terms of a correlation function involving the \( \Phi_{3,1} \) operator (70). The main advantage of our approach is that it also allows computing the multipoint Green’s functions (at least in the integral form). We address this point in detail in Sec. 4.3.

4.3. The two-point Green’s function. In this section, we consider the two-point SLE\(_{8/3} \) Green’s function. We use the results in the preceding sections, where the probabilities for SLE traces to wind in various ways about 4 marked points \( z_1, z_2, z_3, \) and \( z_4 \) were obtained. In particular, the probability that the curve passes between the points \( z_1, z_2 \) and \( z_3, z_4 \) is given by the normalized linear combination of the corresponding trace configurations

\[
P(z_1, z_2, z_4, z_4; x_1, x_2) = \frac{\Pi_{13;24} + \Pi_{14;23} + \Pi_{23;14} + \Pi_{24;13}}{H_0(x_1, x_2)} = \frac{1}{4} - \frac{H_2(z_1, z_2; x_1, x_2)}{4H_0(x_1, x_2)} - \frac{H_2(z_3, z_4; x_1, x_2)}{4H_0(x_1, x_2)} + \frac{H_2(z_1, z_2, z_3, z_4; x_1, x_2)}{4H_0(x_1, x_2)}.
\]

Here, \( H_n(z_1, \ldots, z_n; x_1, x_2) \) is the \( n \)-point correlation function in the upper half-plane (53). We set

\[
\begin{align*}
z_1 &= z + \frac{\epsilon \nu}{2}, \\
z_2 &= z - \frac{\epsilon \nu}{2}, \\
z_3 &= w + \frac{\delta \mu}{2}, \\
z_4 &= w - \frac{\delta \mu}{2},
\end{align*}
\]

where \( z, w \in \mathbb{H} \), \( \epsilon, \delta \ll 1, |\nu|, |\mu| = 1 \), and consider the series expansion of the probability (78) in the limit \( \epsilon, \delta \to 0 \). The leading term in the small \( \epsilon \) and \( \delta \) expansion determines the two-point SLE\(_{8/3} \) Green’s function

\[
\lim_{\epsilon, \delta \to 0} \epsilon^{-2/3} \delta^{-2/3} P\left(z + \frac{\epsilon \nu}{2}, z - \frac{\epsilon \nu}{2}, w + \frac{\delta \mu}{2}, w - \frac{\delta \mu}{2}; x_1, x_2\right) = c_2 G_H(z, w; x_1, x_2),
\]

where \( c_2 \) is a constant.

It follows from (78) and (80) that the two-point Green’s function is determined by series expansions of the 4-point and 6-point functions \( H_2 \) and \( H_4 \) as the points \( z_1, z_2 \in \mathbb{H} \) and \( z_3, z_4 \in \mathbb{H} \) pairwise collapse. The series expansion of \( H_2 \) was obtained in Sec. 4.1 (see Eq. (68)). Therefore, we need to study the asymptotic behavior of the 6-point function

\[
H_4(z_1, z_2, z_3, z_4; x_1, x_2) = \left\langle \prod_{i=1}^{4} \Phi_{2,1}(z_i, \bar{z}_i) \Phi_{1,2}(x_1) \Phi_{1,2}(x_2) \right\rangle_{\mathbb{H}}.
\]

The leading-order terms of the small-\( \epsilon, \delta \) expansion of this function are specified by possible channels of the fusion \( \mathcal{M}_{2,1} \times \mathcal{M}_{2,1} \). As discussed, in the case of bulk–bulk fusion, we can use OPE (67). We then find

\[
H_4(z_1, z_2, z_3, z_4; x_1, x_2) = H_0(x_1, x_2) - \epsilon^2/3(C_{3,1})^2(\Phi_{3,1}(z, \bar{z}) \Phi_{1,2}(x_1) \Phi_{1,2}(x_2))_{\mathbb{H}} - \delta^2/3(C_{3,1})^2(\Phi_{3,1}(w, \bar{w}) \Phi_{1,2}(x_1) \Phi_{1,2}(x_2))_{\mathbb{H}} + \epsilon^2/3 \delta^2/3(C_{3,1})^3(\Phi_{3,1}(z, \bar{z}) \Phi_{3,1}(w, \bar{w}) \Phi_{1,2}(x_1) \Phi_{1,2}(x_2))_{\mathbb{H}} + O(\epsilon) + O(\delta).
\]

\(^8\)We let the function \( H_n(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n; x_1, x_2) \) be denoted as \( H_n(z_1, \ldots, z_n; x_1, x_2) \) for brevity.
Substituting this expansion in (78) and using (68), we obtain the following representation for the two-point Green’s function (80) in terms of a $c = 0$ LCFT correlation function:

$$G_H(z, w; x_1, x_2) = \frac{\langle \Phi_{3,1}(z, \bar{z}) \Phi_{3,1}(w, \bar{w}) \Phi_{1,2}(x_1) \Phi_{1,2}(x_2) \rangle_H}{\langle \Phi_{1,2}(x_1) \Phi_{1,2}(x_2) \rangle_H}. \quad (83)$$

We note that the probability of the SLE$_{8/3}$ trace to pass via two points (80) must satisfy the following property: the two-point Green’s function reduces to the one-point function (62) when the points $z$ and $w$ collapse to one, namely,

$$\lim_{\epsilon \to 0} \epsilon^{2/3} G_H \left( z + \frac{\epsilon \nu}{2}, z - \frac{\epsilon \nu}{2}, x_1, x_2 \right) = c_* G_H(z; x_1, x_2), \quad (84)$$

where $\epsilon \ll 1$, $|\nu| = 1$, and $c_*$ is a constant. We also note that the result for the two-point Green’s function (83) can easily be generalized to the case of $N$ marked points in the upper half-plane. We propose an explicit relation between the multipoint SLE$_{8/3}$ Green’s function and the multipoint correlation functions of primary operators:

$$G_H(\{z_i\}_{i=1}^N; x_1, x_2) = \frac{\langle \prod_{i=1}^N \Phi_{3,1}(z_i, \bar{z}_i) \Phi_{1,2}(x_1) \Phi_{1,2}(x_2) \rangle_H}{\langle \Phi_{1,2}(x_1) \Phi_{1,2}(x_2) \rangle_H}. \quad (85)$$

4.4. The Coulomb gas representation of the Green’s function. We finally address the Coulomb gas representation for the two-point Green’s function in a somewhat heuristic manner. Similarly to Eq. (71), the correlation function in the right-hand side of (83) can be written as a linear combination of conformal blocks,

$$\mathcal{H}_2(\beta, \gamma, \delta; x_1, x_2; \gamma_1, \gamma_2, \gamma_3, \gamma_4) = \left\langle \prod_{i=1}^N V_{3,1}(z) V_{3,1}(z^*) V_{3,1}(w) V_{3,1}(w^*) V_{1,2}(x_1) V_{-1,2}(x_2) \prod_{i=1}^4 Q_{\gamma_i}^{-1} \right\rangle. \quad (86)$$

Here, we used four screening charges $Q_{\gamma_i}^{-1}$ in order to satisfy the neutrality condition (24).

We discuss possible choices for the integration contours in the right-hand side of (86). We recall that contours determine the conformal blocks that contribute to the correlation function. The structure of the boundary $c = 0$ LCFT imposes strong constraints on the admissible blocks. It was argued in [4] that the theory must contain two logarithmic partners of the stress–energy tensor, $\Phi_{5,1}$ and $\Phi_{1,3}$, with $h_{5,1} = h_{1,3} = 2$. However, these fields have different logarithmic couplings and cannot therefore appear in the theory simultaneously. Simmons and Cardy suggested that both fields can coexist if $\Phi_{5,1}$ appears in the bulk while $\Phi_{1,3}$ only on the boundary. This conclusion imposes strong constraints on the bulk–boundary fusion: the bulk operators fuse to the boundary through the identity and the stress–energy tensor only (see Fig. 4).

We introduce the following choice for integration contours that satisfy the Simmons–Cardy proposal. We suppose that the first two contours $\gamma_1$ and $\gamma_2$ surround the boundary operators $V_{1,2}(x_1)$ and $V_{-1,2}(x_2)$ (see Fig. 5). By fusing these operators as $x_2 \to x_1$ and shrinking the integration contours to a point in this process, we obtain the operator $V_{\alpha}(x_1)$ with the charge $\alpha = 2\alpha_0 + 2\alpha_-$, whence $h_{2\alpha_0+2\alpha_-} = 2$. This fusion of vertex operators is in agreement with the fusion of modules in the $c = 0$ boundary LCFT, namely,

$$\mathcal{M}_{1,2} \times \mathcal{M}_{1,2} = \mathcal{I}_{1,3}, \quad (87)$$

where the staggered module $\mathcal{I}_{1,3}$ represents two channels. The logarithmic channel includes a contribution from the identity channel, the stress–energy tensor, and the logarithmic partner to the stress–energy tensor $\Phi_{1,3}$. The regular channel only contains contributions from $T$ and its descendants. Both channels have the conformal dimension 2.
Fig. 5. Possible choices of the integration contours for the conformal block (86). The dashed lines $\gamma_i$, $i = 1, 2, 3, 4$, represent the integration contours and the solid line shows the boundary.

We consider the remaining contours $\gamma_3$ and $\gamma_4$ in the conformal block (86). By requiring these contours to be symmetric with respect to the points $z, z^*, w,$ and $w^*$, we consider two possibilities shown in Fig. 5: the contours connect the points $(z, w)$ and $(z^*, w^*)$ (Fig. 5a) and the contours connect $(z, z^*)$ and $(w, w^*)$ (Fig. 5b). As discussed, the contours determine possible fusion channels that contribute to the OPE of the field $\Phi_{3,1}$ and $\Phi_{3,1}$. The fusion of the corresponding modules has the form [16]

$$M_{3,1} \times M_{3,1} = M_{3,1} + \mathcal{I}_{5,1},$$

(88)

where $\mathcal{I}_{5,1}$ is the staggered module structurally described by the exact sequence

$$0 \to M_{1,1} \to \mathcal{I}_{5,1} \to M_{5,1} \to 0.$$ 

We note that $\mathcal{I}_{5,1}$ is not a highest weight module. It is generated by the state $|\Phi_{5,1}\rangle$ with $h_{5,1} = 2$, and the field $\Phi_{5,1}$ is a Jordan partner of the stress–energy tensor, $L_0|\Phi_{5,1}\rangle = 2|\Phi_{5,1}\rangle + L_{-2}|0\rangle$, and $L_2|\Phi_{5,1}\rangle = -(5/8)|0\rangle$.

Now, using the fusion rules in (88), we discuss possible blocks that contribute to correlation function (86) (see Fig. 5). In Fig. 5b, the bulk–boundary fusion $V_{3,1}(z)V_{3,1}(z^*)$ as $z, z^* \to x \in \mathbb{R}$ results in the vertex operator $V_0(x)$ with the conformal dimension $h_{2\alpha_3,1+\alpha_-} = 1/3$. It represents the boundary field $\Phi_{3,1}(x)$. However, we have already noted that the bulk operators fuse to the boundary through the identity and the stress–energy tensor only. Therefore, the conformal block shown in Fig. 5b is forbidden.

In Fig. 5a, the bulk–bulk fusion $V_{3,1}(z)V_{3,1}(w)$ as $w \to z$ results in the operator $V_{2\alpha_3,1+\alpha_-}(z)$ with $h_{2\alpha_3,1+\alpha_-} = 1/3$. It corresponds to the bulk field $\Phi_{3,1}(z)$ that generates the module $M_{3,1}$ in the right-hand side of (88). This fusion channel is permitted in the $c = 0$ boundary LCFT. We adduce another argument in favor of the conformal block shown in Fig. 5a. In the Sec. 4.3, we introduced the limit property of the two-point Green’s function (84): as the points $z$ and $w$ collapse, the two-point function reduces to the one-point function. This is only possible when the integration contours shrink to a points as $z \to w$ (and $z^* \to w^*$).

The above reasoning suggests the following Coulomb gas representation of the two-point Green’s function: it is determined by the conformal block shown in Fig. 5a. By computing the correlation function of vertex operators, we arrive at the two-point Green’s function in the form

$$G_{\text{coul}}(z, w; x_1, x_2) = \frac{\eta_1^{2/3}(\eta_3 - \eta_2)^{2/3}}{(z_1 - z_1)^{2/3}(z_2 - z_2)^{2/3}} \prod_{i=1}^{3} \frac{\eta_i^{4/3}}{1 - \eta_i} \prod_{j<i} (\eta_i - \eta_j)^{4/3} \mathcal{I}_3(\eta_1, \eta_2, \eta_3),$$

(89)

where $\eta_i = \eta(s_i)$ with $i = 1, 2, 3$ are the cross-ratios (38) of the points $s = \{z_1, z_2, z_2\}$, and $\mathcal{I}_3$ denotes
the 4-fold integral

\[ I_3(\eta_1, \eta_2, \eta_3) = \int_1^\infty du_1 \int_1^\infty du_2 \int_{\eta_1}^{\eta_2} du_3 \int_{\eta_1}^{\eta_2} du_4 \times \]

\[ \times \prod_{i<j} (u_i - u_j)^{4/3} \prod_{i=1}^4 (u_i - 1) u_i^{-2/3} \prod_{j=1}^3 (u_i - \eta_j)^{-2/3}. \]  

(90)

Here, the integration contours are obtained from the contours shown in Fig. 5a by using the Möbius transformation (38).

5. Conclusion and discussion

We summarize the main results in this paper. We considered the loop representation of the \( O(n) \) model and studied multipoint correlation functions of the twist operators \( \Phi_{2,1} \) in the bulk and two 1-leg operators \( \Phi_{1,2} \) on the boundary on the upper half-plane. We used the Dotsenko–Fateev method to obtain the Coulomb gas representation of the \( N \)-point correlation functions of the twist operators. The correlation functions can written explicitly in terms of multifold contour integrals.

We next discussed the case \( n = 0 \) representing self-avoiding loops in the \( O(n) \) model. By following the Simmons–Cardy construction, we studied the multipoint correlation functions with the probabilities of the \( \text{SLE}_{8/3} \) trace to wind in various ways about \( N \geq 1 \) points in the upper half-plane. Further, we obtained explicit expressions for the multipoint passage probabilities as linear combinations of the Coulomb gas integrals.

We proposed a straightforward method to calculate the multipoint Green’s functions of the \( \text{SLE}_{8/3} \) trace. By pairwise collapsing \( 2N \) marked points, we expressed the multipoint passage probabilities for the \( \text{SLE} \) trace in terms of the \( N \)-point Green’s functions. We showed that the Green’s functions can be written in terms of the correlation functions in a \( c = 0 \) boundary LCFT containing the bulk operators \( \Phi_{3,1} \) and a pair of the boundary 1-leg operators \( \Phi_{1,2} \). In the simplest case, our construction leads to the well-known result for the one-point Green’s function. By using heuristic arguments, we proposed an explicit representation for the two-point Green’s function. We plan to elaborate on this result elsewhere.

Conflicts of interest. The author declares no conflicts of interest.

REFERENCES

1. O. Schramm, “Scaling limits of loop-erased random walks and uniform spanning trees,” Israel J. Math., 118, 221–288 (2000).
2. S. Rohde and O. Schramm, “Basic properties of SLE,” Ann. Math., 161, 883–924 (2005).
3. O. Schramm, “A percolation formula,” Electron. Commun. Probab., 6, 115–120 (2001).
4. J. J.H. Simmons and J. Cardy, “Twist operator correlation functions in \( O(n) \) loop models,” J. Phys. A: Math. Theor., 42, 235001, 20 pp. (2009).
5. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, “Infinite conformal symmetry in two-dimensional quantum field theory,” Nucl. Phys. B, 241, 333–380 (1984).
6. B. Nienhuis, “Critical behavior of two-dimensional spin models and charge asymmetry in the Coulomb gas,” J. Statist. Phys., 34, 731–761 (1984).
7. Vl.S. Dotsenko and V. A. Fateev, “Conformal algebra and multipoint correlation functions in 2D statistical models,” Nucl. Phys. B, 240, 312–348 (1984).
8. Vl.S. Dotsenko and V. A. Fateev, “Four-point correlation functions and the operator algebra in 2D conformal invariant theories with central charge \( C \leq 1 \),” Nucl. Phys. B, 251, 691–734 (1985).
9. J. Kondev, “Liouville field theory of fluctuating loops,” Phys. Rev. Lett., 78, 4320–4323 (1997).
10. W. Kager and B. Nienhuı̈s, “A guide to stochastic Löwner evolution and its applications,” *J. Statist. Phys.*, 115, 1149–1229 (2004).
11. V. Gurarie, “Logarithmic operators and logarithmic conformal field theories,” *J. Phys. A: Math. Theor.*, 46, 494003, 18 pp. (2013).
12. H. Eberle and M. Flohr, “Notes on generalised nullvectors in logarithmic CFT,” *Nucl. Phys. B*, 741, 441–466 (2006).
13. A. Gamsa and J. Cardy, “Correlation functions of twist operators applied to single self-avoiding loops,” *J. Phys. A: Math. Gen.*, 39, 12983–13003 (2006).
14. J. Cardy, “Conformal invariance and surface critical behavior,” *Nucl. Phys. B*, 240, 514–532 (1984).
15. M. A. Virasoro, “Subsidiary conditions and ghosts in dual-resonance models,” *Phys. Rev. D*, 1, 2933–2936 (1970).
16. P. Mathieu and D. Ridout, “From percolation to logarithmic conformal field theory,” *Phys. Lett. B*, 657, 120–129 (2007).
17. P. Mathieu and D. Ridout, “Logarithmic $M(2,p)$ minimal models, their logarithmic couplings, and duality,” *Nucl. Phys. B*, 801, 268–295 (2008).