Selected synchronous state of the vibration system driven by three homodromy eccentric rotors

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Abstract
In order to verify that the vibration system driven by multi-eccentric rotors has multiple synchronous states, a model of three eccentric rotors horizontal installation of plane motion is established to study the coupling dynamic characteristics. Based on the average method of modified small parameters, the frequency capture equation is constructed to obtain the conditions of synchronous motion and vibration synchronization transmission. According to the synchronous conditions, the stability state of achieving synchronous motion is converted into the solution of balance equation of the synchronous torque. There are multiple solutions to the torque equation because of the non-linear characteristics of the vibration system driven by multi-eccentric rotors. Then the stability condition is used to estimate which of the solutions is stable. After substituting the parameters of the experimental machine into the above method, the curves of the phase difference and its stability coefficients of three eccentric rotors system are obtained numerically. Two experiments show that the selected synchronous state depends on the initial condition and external disturbance, and if the vibration synchronization transmission conditions are satisfied, three eccentric rotors can not only achieve vibration synchronization transmission of one motor with switching off the power but also that of two motors with switching off the power.

Keywords
Synchronization, vibration system, eccentric rotor, synchronous state, phase difference

Introduction
Synchronization is a special phenomenon in nonlinear vibration system, which is also called frequency capture. Since Huygens proposed a description of synchronization of two pendulums, a large number of scholars have been attracted to study synchronization on different objects. With their efforts, a lot of research results have been proposed, and they mainly focus on two aspects of the eccentric rotor (ER, also named exciter) and pendulum. In these results, synchronization of ERs is one of the greatest engineering significance. That is divided into two main aspects, the former is the vibration suppression, such as synchronization of gas turbine engines with multiple shafts, which is not expected to occur because of the destruction of the large amplitude. And the latter is the vibration utilization; on the contrary, it is expected to happen because the system can get the resultant force excited by ERs, which is also the research object of this paper.

The study of synchronization theory of ERs started from Blekhman, who not only gave the general definition and example design of synchronization but also proposed one method of the separation of fast and slow motion to solve a number of synchronous problems. Thereafter, the research team of Wen proposed the average method of small parameters to obtain the synchronous condition and its stability condition of the vibration system. By applying such synchronization theory to engineering, they designed a lot of vibration machines.
In Zhao et al., synchronization of two ERs rotating in the reverse direction of plane motion is proposed to design the line vibration feeder. In Zhang et al., synchronization of two homodromy ERs of plane motion is proposed to design the elliptical vibration screen. In Zhao et al., synchronization of two ERs with the cross-shaft ERs of spatial motion is proposed to design the vertical vibration conveyor. In Chen et al., synchronization of two ERs with the common shaft of spatial motion is proposed to design the vibration mill.

Since synchronization theory of two ERs in the vibration system has been successfully applied in engineering, vibration machine has created significant production benefits. In order to satisfy the demand for large-scale development of industry, scientists began to develop the vibration machine driven by multi-ERs. However, the vibration machine driven by multi-ERs of utilizing vibration synchronization has never successfully been used in engineering. Although the relevant reasons are explained in Zhang et al., the lack of experimental studies leads to the inability to fully understand the dynamic characteristics of the vibration system.

The key to study synchronization theory of the vibration system driven by multi-ERs is that the stable synchronous state of ERs must be comprehensively understood. This paper is an extension of the works in Zhang et al. Based on the method of average method of small parameters, some quantitative analyses about the synchronous state of three ERs in the vibration system will be given in this paper by numerical and experimental ways to further perfectly understand the dynamic characteristics of the vibration system.

In our paper, the vibration system driven by three homodromy ERs horizontally mounted on a single base of plane motion is used as a model to investigate the coupling dynamic characteristics when the system operates in super-resonance (more than three times natural frequency). The paper contains the following components: first, the electromechanical coupling equations of the system is given in “Synchronization characteristic of three ERs” section. Then, the synchronous conditions of synchronous motion and VST are deduced. After that, an analysis method of the synchronous state is proposed. In “Numerical results and discussion” section, numerical discussions corresponding to the theory are given to study the selected synchronous state. In “Experiments” section, two experimental results are presented to verify the effectiveness of the theoretical analysis. Finally, conclusions are presented briefly.

**Synchronization characteristic of three ERs**

*Frequency capture equation*

As shown in Figure 1, the dynamic model of the vibration system consists of a rigid body, a fixed base, springs, and three ERs. The springs connect the rigid body with the base. Three induction motors drive three ERs in the same rotational direction, respectively. The translational motions of the body are $x$ and $y$, and the angular rotation of the body is $\phi$.

Next, synchronization of three ERs in the vibration system will be analyzed by using the average method of modified small parameters. This paper only gives the critical conclusions obtained by the method of modified small parameters, and the detailed derivation process can refer to Zhang et al. In order to better guarantee the readability of this work, the application process of the average method of modified small parameters is given in this paper, as shown in Figure 2.

![Figure 1. The dynamical model of the system.](image-url)
Step 1 in Figure 2, the motion equations of the system are obtained applying Lagrange’s equation. The 
electromechanical coupling equations of the system are given as follows  

\[
\begin{align*}
M\ddot{x} + f_x\dot{x} + k_xx &= \sum_{i=1}^{3} m_i r_i \left( \dot{\phi}_i^2 \cos \phi_i + \dot{\phi}_i \sin \phi_i \right) \\
M\ddot{y} + f_y\dot{y} + k_yy &= \sum_{i=1}^{3} m_i r_i \left( \dot{\phi}_i^2 \sin \phi_i - \dot{\phi}_i \cos \phi_i \right) \\
J_i\dddot{\phi}_i + f_i\dddot{\phi}_i + k_i\phi_i &= \sum_{i=1}^{3} m_i r_i \left[ \dot{\phi}_i^2 \sin(\phi_i - v_i\beta_i) - \dot{\phi}_i \cos(\phi_i - v_i\beta_i) \right] \\
J_i\dddot{\phi}_i + f_i\dddot{\phi}_i &= T_{ei} - m_i r_i \left[ \dot{\phi}_i \cos(\phi_i - v_i\beta_i) + i_i \dot{\phi}_i \cos(\phi_i - v_i\beta_i) + v_i \dot{\phi}_i \dot{\phi}_i \sin(\phi_i - v_i\beta_i) \right], \quad i = 1, 2, 3
\end{align*}
\]

(1)

Steps 2 to 4 in Figure 2 are omitted, and step 5 is given directly. The frequency capture equation of three ERs is

\[
A\ddot{\phi} = B\phi + u
\]

(2)

where \(\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3)^T\), \(A\) and \(B\) are 3 \times 3 matrices, \(A = \begin{pmatrix} a_{11} & \dot{\phi}_{12} & \dot{\phi}_{13} \\ \dot{\phi}_{21} & a_{22} & \dot{\phi}_{23} \\ \dot{\phi}_{31} & \dot{\phi}_{32} & a_{33} \end{pmatrix}\), \(B = \begin{pmatrix} b_{11} & Z_{12} & Z_{13} \\ Z_{21} & b_{22} & Z_{23} \\ Z_{31} & Z_{32} & b_{33} \end{pmatrix}\), \(a_{ii} = \eta_i(1 - \eta_i W_{ei}/2)\), \(b_{ii} = \frac{k_i}{m_i r_i^2 \omega_m} + \frac{f_i}{m_i r_i^2} + \omega_m^2 \eta_i^2 \omega_{n0}^2\), \(u_i = \frac{T_{ei}}{m_i r_i^2 \omega_m} + \frac{f_i}{m_i r_i^2} - \chi_{ai} - \chi_{fi}\), \(u = (u_1, u_2, u_3)^T\), \(i = 1, 2, 3\), others are in Appendix 2.

**Synchronous condition**

Step 6 in Figure 2, if the synchronous motion of three ERs is achieved, the disturbing term \(\bar{\nu}_1 = 0\) and \(\bar{\nu}_2 = 0\). Substituting \(\bar{\nu}_1 = 0\) and \(\bar{\nu}_2 = 0\) into equation (2), we obtain \(u = 0\). According to \(u = 0\), we have

\[
T_{0i} = T_{ei} - f_i \omega_{n0} = m_0 r_i^2 \omega_{n0}(\chi_{ai} + \chi_{fi}), \quad i = 1, 2, 3
\]

(3)
When three ERs operate in the synchronous state, the system distributes the load torques of three motors by adjusting the phase differences of three ERs, which are used to balance the difference of the output torque of three motors. Based on equation (3), the difference of the output torque \(T_{12}\) between motor 1 and motor 2, that \(T_{23}\) between motor 2 and motor 3, and that \(T_{13}\) between motor 1 and motor 3 are expressed as follows

\[
T_{12} = T_{01} - T_{02} = T_m[(\eta_1^2 W_{s1} - \eta_2^2 W_{s2}) + \eta_1 \eta_3 W_{s13}\cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{13}) - \eta_2 \eta_3 W_{s23}\cos(2\bar{x}_2 + \theta_{23})
+ 2\eta_1 \eta_3 W_{c12}\sin(2\bar{x}_1 + \theta_{12}) + \eta_1 \eta_3 W_{c13}\sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{13}) - \eta_2 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{23})]
\]

\[
T_{23} = T_{02} - T_{03} = T_m[(\eta_2^2 W_{s2} - \eta_3^2 W_{s3}) + \eta_2 \eta_3 W_{s12}\cos(2\bar{x}_1 + \theta_{12}) - \eta_3 \eta_3 W_{s13}\cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{13})
+ 2\eta_2 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{23}) + \eta_2 \eta_3 W_{c13}\sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{13}) - \eta_3 \eta_3 W_{c12}\sin(2\bar{x}_1 + \theta_{12})]
\]

\[
T_{13} = T_{01} - T_{03} = T_m[(\eta_1^2 W_{s1} - \eta_3^2 W_{s3}) + \eta_1 \eta_2 W_{s12}\cos(2\bar{x}_1 + \theta_{12}) - \eta_2 \eta_3 W_{s23}\cos(2\bar{x}_2 + \theta_{23})
+ \eta_2 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{23}) + 2\eta_1 \eta_3 W_{c13}\sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{13}) + \eta_1 \eta_3 W_{c12}\sin(2\bar{x}_1 + \theta_{12})]
\]

Equation (4) is simplified as

\[
\frac{T_{ij}}{T_m} = (\eta_i^2 W_{s1} - \eta_j^2 W_{s2}) = T_{cij}(\bar{x}_i, \bar{x}_j), \quad ij = 12, 23, 13
\]

where \(T_{cik}(\bar{x}_i, \bar{x}_k)\) are the synchronous torque, \(i, 1 = 2, 2 = k \leq 3\)

\[
T_{c12}(\bar{x}_1, \bar{x}_2) = \eta_1 \eta_3 W_{s13}\cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{13}) - \eta_2 \eta_3 W_{s23}\cos(2\bar{x}_2 + \theta_{23}) + 2\eta_1 \eta_2 W_{c12}\sin(2\bar{x}_1 + \theta_{12})
+ \eta_1 \eta_3 W_{c13}\sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{13}) - \eta_2 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{23})
\]

\[
T_{c23}(\bar{x}_1, \bar{x}_2) = \eta_1 \eta_2 W_{s12}\cos(2\bar{x}_1 + \theta_{12}) - \eta_3 \eta_3 W_{s13}\cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{13}) + 2\eta_1 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{23})
+ \eta_1 \eta_3 W_{c13}\sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{13}) - \eta_2 \eta_3 W_{c12}\sin(2\bar{x}_1 + \theta_{12})
\]

\[
T_{c13}(\bar{x}_1, \bar{x}_3) = T_{c12}(\bar{x}_1, \bar{x}_2) + T_{c23}(\bar{x}_2, \bar{x}_3)
\]

From equation (5), it can be seen that \(T_{cik}(\bar{x}_i, \bar{x}_k)\) are the function of \(\bar{x}_i\) and \(\bar{x}_k\). Only when equation (6) is satisfied, equation (3) can be solved

\[
\left|\frac{T_{ij}}{T_m} - (\eta_i^2 W_{s1} - \eta_j^2 W_{s2})\right| \leq \max[T_{cij}(\bar{x}_i, \bar{x}_j)], \quad ij = 12, 23, 13
\]

In addition, equation (6) is the synchronous condition of three ERs in the vibration system.

When three ERs achieve the synchronous motion, if the power of one of three motors is switched off, the system may guarantee the synchronous motion of three ERs by vibration synchronization transmission (VST).\(^{14-17}\) Assuming that the power of motor 3 is switched off, the synchronous velocity of the system is \(\omega_1^s\), and the phase differences are \(2\bar{x}_1^2\) and \(2\bar{x}_2^2\). Based on the synchronous condition, the condition of VST under the case of one motor with power switched off is given as follows

\[
T_{01} = T_{e01} - f_1 \omega_1^s m_0 \omega_1^s \left(\bar{x}_1^{'a1} + \bar{x}_1^{'b1}\right)
\]

\[
T_{02} = T_{e02} - f_2 \omega_2^s m_0 \omega_2^s \left(\bar{x}_2^{'a2} + \bar{x}_2^{'b2}\right)
\]

\[
T_{03} = -f_3 \omega_3^s m_0 \omega_3^s \left(\bar{x}_3^{'a3} + \bar{x}_3^{'b3}\right)
\]

\[
\left|\frac{T_{12}}{T_m} - (\eta_1^2 W_{s1}^* - \eta_2^2 W_{s2}^*)\right| \leq \max[T_{c12}(\bar{x}_1, \bar{x}_2)]
\]

\[
\left|\frac{T_{23}}{T_m} - (\eta_2^2 W_{s2}^* - \eta_3^2 W_{s3}^*)\right| \leq \max[T_{c23}(\bar{x}_2, \bar{x}_3)]
\]
where
\[
T_{e12}(\bar{x}_1, \bar{x}_2) = \eta_1 \eta_3 W_{s13}\cos(2\bar{x}_1' + 2\bar{x}_2' + \theta_{e13}') - \eta_2 \eta_3 W_{s23}\cos(2\bar{x}_2' + \theta_{e23}') \\
+ 2\eta_1 \eta_2 W_{e12}\sin(2\bar{x}_1' + \theta_{e12}') + \eta_1 \eta_3 W_{e13}\sin(2\bar{x}_1' + 2\bar{x}_2' + \theta_{e13}') \\
- \eta_2 \eta_3 W_{e23}\sin(2\bar{x}_2' + \theta_{e23}')
\]
\[
T_{e23}(\bar{x}_2, \bar{x}_3) = \eta_1 \eta_2 W_{e12}\cos(2\bar{x}_1' + \theta_{e12}') - \eta_1 \eta_3 W_{s13}\cos(2\bar{x}_1' + 2\bar{x}_2' + \theta_{e13}') \\
+ 2\eta_2 \eta_3 W_{e23}\sin(2\bar{x}_2' + \theta_{e23}') + \eta_1 \eta_3 W_{e13}\sin(2\bar{x}_1' + 2\bar{x}_2' + \theta_{e13}') - \eta_1 \eta_2 W_{e12}\sin(2\bar{x}_1' + \theta_{e12}')
\]

The same as above, the condition of VST under the case of two motors with power switched off is given as follows
\[
T'_{01} = T'_{e01} - f_1 a''_{m0} = m_0 r^2 a''_{m0}(\chi''_{a1} + \chi''_{a1})
\]
\[
T'_{02} = -f_2 a''_{m0} = m_0 r^2 a''_{m0}(\chi''_{a2} + \chi''_{a2})
\]
\[
T'_{03} = -f_3 a''_{m0} = m_0 r^2 a''_{m0}(\chi''_{a3} + \chi''_{a3})
\]
\[
\frac{|T'_{12}|}{T_m} = |(\eta_1^T W'_{s1} - \eta_2^T W'_{s2})| \leq \max[T'_{e12}(\bar{x}_1, \bar{x}_2)]
\]
\[
\frac{|T'_{23}|}{T_m} = |(\eta_2^T W'_{s2} - \eta_3^T W'_{s3})| \leq \max[T'_{e23}(\bar{x}_1, \bar{x}_2)]
\]

**Stability condition of the synchronous state**

Step 7 in Figure 2, if the system satisfies the synchronous condition, \( u = 0 \), and equation (2) is the generalized system, we have
\[
A'_{\bar{z}} = B'_{\bar{z}}
\]
where \( A' \) and \( B' \) denote that \( A \) and \( B \) instead of \( \bar{z}_1 = \bar{z}_{10}, \bar{z}_2 = \bar{z}_{20} \) and \( \omega_{n0} = \omega_{n0}^* \).

In equation (13), \( A' \) is a symmetrical matrix and \( B' \) is an antisymmetrical one. If \( A' \) is a positive matrix and its elements are positive, they satisfy the generalized Lyapunov equations as follows\(^{20}\)
\[
da''_{ij} > 0, \quad \det(A'), \quad \det(A_i'), \quad i = 1, 2, 3, \quad j = 1, 2, 3
\]
\[
\Gamma B' + B'^T I = -\omega_{n0}\text{diag}(\kappa_{11}, \kappa_{22}, \kappa_{33})
\]
\[
A'^T I = I A' > 0
\]

where \( I \) is the unit matrix.

If \( \lim_{t \to \infty} \bar{z} = 0 \), this generalized system is stable, and equation (13) is also stable. According to Zhang et al., the stability condition of the synchronous state of the vibration system driven by three ERs is equation (14).\(^{20}\)

Based on the stability condition of the synchronous state, the ranges of the stable phase differences are
\[
2\bar{z}_{10} + \theta_{e12} \in \begin{cases} 
(-90^\circ, 90^\circ), & \theta_{e12} \geq 0 \\
(90^\circ, 270^\circ), & \theta_{e12} < 0
\end{cases}
\]
the periodic motion of the motor, the phase differences of three ERs are also periodic change. The change range of differences tend a set of values. In Figure 5(b), the nonlinear equations. We propose a method to judge which solution is stable by applying the stability algorithm can be used. Obviously, it is difficult to estimate the number of solutions due to the multi-solutions of equation (4) is a transcendental equation, only the way of the numerical achieve the synchronous motion. Because equation (4) is a transcendental equation, only the way of the numerical

tances, which represents that the vibration system has two stable synchronous states. When

do not know which solution is stable. Using the method presented in this paper, the stable solutions are given in

state, which reflects the coupling dynamic characteristic of the vibration system.

According to Figure 3, some results are obtained. In Figure 4, all phase differences versus with \( r_{\psi} < 5 \) is enough to explain the synchronization questions. Considering the periodic motion of the motor, the phase differences of three ERs are also periodic change. The change range of the phase difference is selected in one period (-2\( \pi \), 2\( \pi \)). Based on the distribution of the load torque of the vibration system, the problem of the synchronous state is converted into that of solving the equations of the synchronous torque. From the synchronous condition, it is remarked that \( T_{c12}(\bar{x}_1, \bar{x}_2) \) must be equal to \( T_{c23}(\bar{x}_2, \bar{x}_3) \) if three ERs achieve the synchronous motion. Because equation (4) is a transcendental equation, only the way of the numerical algorithm can be used. Obviously, it is difficult to estimate the number of solutions due to the multi-solutions of the nonlinear equations. We propose a method to judge which solution is stable by applying the stability conditions of the synchronous state. The detailed steps are shown in Figure 3.

According to Figure 3, some results are obtained. In Figure 4, all phase differences versus with \( r_{\psi} \) are solved when \( T_{c12}(\bar{x}_1, \bar{x}_2) = T_{c23}(\bar{x}_2, \bar{x}_3) \). From two cases with different \( \beta_3 \), there are really a lot of solutions. However, we do not know which solution is stable. Using the method presented in this paper, the stable solutions are given in Figure 5, which represent the stable synchronous state. In Figure 5(a), \( \beta_3 = 0^\circ \), the range of the phase difference varies greatly at the point \( r_{\psi} = \sqrt{2} \). The point \( r_{\psi} = \sqrt{2} \) is the boundary of the motion state of the system. The motion tendency of the system is close to 120° when \( r_{\psi} < \sqrt{2} \) and the motion tendency of the system is close to 180° when \( r_{\psi} > \sqrt{2} \). Hence, the data jumps at this point. When \( r_{\psi} < \sqrt{2} \), there are two sets of the phase differences, which represents that the vibration system has two stable synchronous states. When \( r_{\psi} > \sqrt{2} \), the phase differences tend a set of values. In Figure 5(b), \( \beta_3 = 40^\circ \), there are two sets of the phase differences when \( 0 < r_{\psi} < 5 \). It is remarked that the movement of the vibration system depends on the phase differences. Since the unbalanced forces of ERs cancel out each other, the vibration system is almost motionless when the phase differences are close to ±2\( \pi \)/3. In other cases, the movement of the vibration system is elliptic motion, of which amplitude is less than that of two ERs. Figure 5 explains that the vibration system has the selected synchronous state, which reflects the coupling dynamic characteristic of the vibration system.

Comparing to Figure 5, the stability coefficients of the synchronous state are shown in Figure 6. When \( 0 < r_{\psi} < 5 \), the coefficients are greater than 0, which proves that the phase differences in Figure 5 are stable.

**Experiments**

In this section, we will verify the validity of the theoretical analysis and the numerical results of the above sections by comparing to the experimental results of a vibration machine. Figure 7 shows the mechanical components of the vibration machine, which consists of three vibration motors: the rigid frame, the springs, and the fixed base. Nos. 0 and 1 represent the positions measured by the acceleration sensors (INV9832-50), and No. 2 represents the reflective stripe used for the photography. The rotational velocities of three motors and the phases of three ERs are measured by the hall sensors (NJK-5002C). All acceleration signals in the vertical and horizontal directions, velocities of motors, and phases of three ERs are collected by the data acquisition equipment (INV306DF), as well

\[
2\bar{x}_{20} + \theta_{c23} \in \begin{cases} 
(90^\circ, 270^\circ), & \theta_{c23} < 0 \\
(0^\circ, 90^\circ), & \theta_{c23} \geq 0
\end{cases}
\]  

(18)

\[
2\bar{x}_{10} + 2\bar{x}_{20} + \theta_{c13} \in \begin{cases} 
(90^\circ, 270^\circ), & \theta_{c13} < 0 \\
(0^\circ, 90^\circ), & \theta_{c13} \geq 0
\end{cases}
\]  

(19)

**Numerical results and discussions**

The synchronous condition and its stability condition have been given in the previous section. This section will present numerical results to propose a dynamic analysis method of the stable synchronous state. Based on the parameters of the test machine (it will be introduced in “Experiments” section), we make an appropriate hypothesis that it is completely symmetric, that is, \( l_1 = l_3, \beta_1 = \pi - \beta_3, \beta_2 = \pi/2, \eta_j = 1 \) and \( v_i = -1, i = 1, 2, 3 \). The actual parameter values of the vibration machine are: \( r \approx 0.05 \text{ m}, m_0 = 3 \text{ kg}, M = 271 \text{ kg}, J_p = 70 \text{ kg} \cdot \text{m}^2, k_x \approx k_y \approx 120 \text{ kN/m}, k_\phi \approx 31 \text{ kN/rad}, f_x \approx f_y \approx 798 \text{ Nms/rad}, \beta_1 \approx 140^\circ, \beta_2 = 90^\circ, \beta_3 \approx 40^\circ, l_1 \approx 0.45 \text{ m}, \) and \( l_2 \approx 0.3 \text{ m} \). The parameters of motors are the same, \( f_i \approx 0.02, n_p = 3, U = 220 \text{ V}, \omega_{m0} \approx 94 \text{ rad/s}, R_c \approx 40 \Omega, R_\psi \approx 12 \Omega, L_x \approx 1213 \text{ mH}, L_y \approx 1222 \text{ mH}, L_m \approx 1116 \text{ mH}. \)

In Figure 3, the dynamic analysis method of the stable synchronous state is given as a flowchart. Combining the actual structure of the vibration machine, \( r_{\psi} < 5 \) is enough to explain the synchronization questions. Considering the periodic motion of the motor, the phase differences of three ERs are also periodic change. The change range of the phase difference is selected in one period (-2\( \pi \), 2\( \pi \)). Based on the distribution of the load torque of the vibration system, the problem of the synchronous state is converted into that of solving the equations of the synchronous torque. From the synchronous condition, it is remarked that \( T_{c12} (\bar{x}_1, \bar{x}_2) \) must be equal to \( T_{c23} (\bar{x}_2, \bar{x}_3) \) if three ERs achieve the synchronous motion. Because equation (4) is a transcendental equation, only the way of the numerical algorithm can be used. Obviously, it is difficult to estimate the number of solutions due to the multi-solutions of the nonlinear equations. We propose a method to judge which solution is stable by applying the stability conditions of the synchronous state. The detailed steps are shown in Figure 3.

According to Figure 3, some results are obtained. In Figure 4, all phase differences versus with \( r_{\psi} \) are solved when \( T_{c12} (\bar{x}_1, \bar{x}_2) = T_{c23} (\bar{x}_2, \bar{x}_3) \). From two cases with different \( \beta_3 \), there are really a lot of solutions. However, we do not know which solution is stable. Using the method presented in this paper, the stable solutions are given in Figure 5, which represent the stable synchronous state. In Figure 5(a), \( \beta_3 = 0^\circ \), the range of the phase difference varies greatly at the point \( r_{\psi} = \sqrt{2} \). The point \( r_{\psi} = \sqrt{2} \) is the boundary of the motion state of the system. The motion tendency of the system is close to 120° when \( r_{\psi} < \sqrt{2} \) and the motion tendency of the system is close to 180° when \( r_{\psi} > \sqrt{2} \). Hence, the data jumps at this point. When \( r_{\psi} < \sqrt{2} \), there are two sets of the phase differences, which represents that the vibration system has two stable synchronous states. When \( r_{\psi} > \sqrt{2} \), the phase differences tend a set of values. In Figure 5(b), \( \beta_3 = 40^\circ \), there are two sets of the phase differences when \( 0 < r_{\psi} < 5 \). It is remarked that the movement of the vibration system depends on the phase differences. Since the unbalanced forces of ERs cancel out each other, the vibration system is almost motionless when the phase differences are close to ±2\( \pi \)/3. In other cases, the movement of the vibration system is elliptic motion, of which amplitude is less than that of two ERs. Figure 5 explains that the vibration system has the selected synchronous state, which reflects the coupling dynamic characteristic of the vibration system.

Comparing to Figure 5, the stability coefficients of the synchronous state are shown in Figure 6. When \( 0 < r_{\psi} < 5 \), the coefficients are greater than 0, which proves that the phase differences in Figure 5 are stable.
Figure 3. Flow diagram of the theoretical analysis method of the stable phase difference.

Figure 4. All phase differences versus with \( r_\psi \): (a) \( \beta_3 = 0^\circ \) and (b) \( \beta_3 = 40^\circ \).
as the phases of three ERs are recorded by the high-speed camera (Y3C,4GB/2000 fps), while three motors operate in 45 Hz power supplied by Siemens inverters. The detailed experimental process is shown in Figure 8.

Due to the structural constraints, this machine can only verify the stability ranges of the phase differences of $r_{\psi} < \sqrt{2}$, but it does not affect the qualitative verification of this theoretical method. We will prove the numerical result in Figure 5 by two groups of experiments.

**Figure 5.** Stable phase differences versus with $r_{\psi}$: (a) $\beta_3 = 0^\circ$ and (b) $\beta_3 = 40^\circ$.

**Figure 6.** Stability coefficients of synchronous state versus with $r_{\psi}$: (a) $\beta_3 = 0^\circ$ and (b) $\beta_3 = 40^\circ$.

**Figure 7.** Vibration machine driven by three ERs with the horizontal linear installation.
As shown in Figure 9, $r_\phi \approx 1.13$, the power frequency of three motors is 45 Hz. When the system is operating in the synchronous state, the average value of the synchronous velocity is about 896 r/min, and the phase differences are $2\alpha_1 \approx 123^\circ$, $2\alpha_2 \approx 115^\circ$, and $2\alpha_3 \approx -122^\circ$. When $r_\phi = 1.03$, $2\alpha_1 = 114^\circ$, $2\alpha_2 = 113^\circ$ in Figure 5. By comparing the numerical analysis with the experimental results, it is found that the value of phase differences conforms to the trend of the theoretical analysis in Figure 5(b). There are two reasons for this error. First, the structural parameters are not completely symmetrical due to machining, including the motor parameters. Second, the system parameters we calculated are not accurate, such as $\beta = 40^\circ$ and $M = 271$ kg. The parameters all affect the result of the phase differences. However, the error does not affect the qualitative study of the motion law.

From the change of the phase difference curves in Figure 9(b), it can be seen that the phase differences are unstable before the synchronous state is achieved. Once motors reach the set speed, the synchronous torque of the vibration system plays a regulating role to make the phase difference fast and stable. The mechanism of vibration synchronization is that when the velocities of three motor are close to each other, the vibration system acts the load torque on the fast motor and acts the driving torque on the slow motor, which makes three motors achieve the synchronous motion. We can also judge whether the phase differences are stable from the amplitude curves in Figure 9(c) to 9(f). Figure 9(c) and 9(d) shows the responses of the position for No. 0, and Figure 9(e) and 9(f) shows those of No. 1. Since motor has excellent acceleration performance, there is not much resonance in $x$- and $y$-directions when the vibration system crosses the resonance region.

When the power of motor 3 is switched off at the time of 30 s, the vibration system begins to operate in the state of VST. Due to the strong mechanical characteristics of AC motor, the synchronous velocity changes very little in this state, and it is about 895 r/min. However, the phase differences change greatly, which change from one state to another state, $2\alpha_1 \approx -140^\circ$, $2\alpha_2 \approx -140^\circ$, $2\alpha_3 \approx 80^\circ$. When the vibration system loses its original equilibrium state due to the power interruption, the synchronous torque plays a role in adjusting the phase difference to make the system reach equilibrium. It is worth noting that this new synchronous state chooses another set of the phase differences in Figure 5(b). Because the phase differences change greatly, it is most directly reflected in the response, as shown in Figure 9(c) to 9(f). This experiment proves that the selected synchronous state is affected by external disturbance, and the vibration system driven by three ERs can achieve VST.

In order to study this phenomenon of VST, two of three motors are switched off the power at the time of 60 s. As the power supply of motor 2 is switched off, the vibration system enters the same motion state of VST with different phase differences. Due to only one motor provides energy, the synchronous velocity decreases slightly, and it is about 892 r/min. And the phase differences are $2\alpha_1 \approx -125^\circ$, $2\alpha_2 \approx -144^\circ$, and $2\alpha_3 \approx 91^\circ$. From the two experiments of VST, it is remarked that as long as the condition is satisfied, the motion state of VST can also be achieved even if the powers of two motors are switched off. According to the changes of the phase differences, the existence of the synchronous torque of the vibration system is observed, which reflects the coupling dynamic characteristics of the vibration system.

In order to more easily observe the instantaneous phase of three ERs in the process of the synchronous motion, the high-speed camera is used to capture pictures at 50 Hz. Six pictures in one rotational period are used to verify the accuracy of the data acquisition instrument. Figure 10 shows the first synchronous state of three ERs when
they are simultaneously powered in 45 Hz. The values of the phase difference are basically consistent with the data acquisition instrument. For the calculation of the phase difference, we usually use the form of less than $\pi$, so as to determine which rotor is the fast motor, which rotor is the slow motor. Hence, $2\alpha_1 = 222^\circ$ usually is converted to the form $2\alpha_1 = -138^\circ$, which explains the fast motor is ER2.

As shown in Figure 11, the second group of experiments is obtained in accordance with the experimental scheme and the structural parameters of the first group of experiments. It is remarked that the vibration system selects another synchronous state in the second group of experiments, and the synchronous velocity is about 896 $r/min$, the phase differences are $2\alpha_1 \approx -138^\circ$, $2\alpha_2 \approx -147^\circ$, and $2\alpha_3 \approx 75^\circ$. In two different synchronous states, the responses of the vibration system are also different. These two groups of experiments prove the correctness of the theoretical analysis, and there are two stable synchronous states of the vibration system driven

Figure 9. The first experiment results of the vibration system operating in super resonant, $r_n \approx 1.13$: (a) the velocities of three motors, (b) the phase difference, (c) amplitude of center of mass in $y$-direction, (d) amplitude of center of mass in $x$-direction, (e) amplitude of the farthest position in $y$-direction, and (f) amplitude of the farthest position in $x$-direction.
Figure 10. Phases recorded of first experiment by the high-speed camera.

Figure 11. The second experiment results of the vibration system operating in super resonant, \(n/s = 1.13\): (a) the velocities of three motors, (b) the phase difference, (c) amplitude of center of mass in y-direction, (d) amplitude of center of mass in x-direction, (e) amplitude of the farthest position in y-direction, and (f) amplitude of the farthest position in x-direction.
by three ERs. From the two experiments, it can be seen that the selected synchronous state of the vibration system depends on the initial conditions. In a word, which synchronous state is satisfied first, the system runs in which synchronous state.

When the power of motor 3 is switched off at the time of 30 s, the phase differences do not choose another synchronous state. The phase differences are kept with the same synchronous state in the process of VST. The state of VST is consistent between the time period from 30 to 60 s. Due to only one motor provides energy, the adjustment amplitude of the phase differences is more obvious than the first 60 s. The experimental method of switching off the power of motor can not only verify VST, but also can be regarded as a disturbance to the vibration system. Therefore, it can be concluded that the selected synchronous state of the vibration system depends on the initial condition and the external disturbance. From these two experiments, they show that the synchronous state of the second experiment is easier to satisfy the force balance when the vibration system runs in VST.

Figure 12 shows the second synchronous state of three ERs by using the high-speed camera at 50 Hz to capture pictures. The values of the phase differences are basically consistent with the data acquisition instrument.

To sum up, three ERs run in the out of phase motion of two synchronous states because of the motionless requirement of the vibration system, which causes the motion that is close to the motionless state. The motionless requirement reflects the minimum potential energy principle. Only when all the unbalanced forces cancel out each other can the motionless be achieved, which determines the selection motion tendency of the vibration system. When a primal balance state is broken, the vibration system finds a new balance state to ensure the motionless tendency. Because the vibration system driven by three ERs has two synchronous states, the system prioritizes states that are easier to implement. Hence, the initial condition and the external disturbance are important for the selected synchronous state of the vibration system driven by three ERs.

Conclusions

In this research, synchronization of the vibration system driven by three-ERs of plane motion is investigated. According to the synchronous conditions of the vibration system, a theoretical analysis method is proposed to analyze the synchronous state. The curves of the phase differences and their stability coefficients are obtained by substituting the experimental parameters into this method. By comparing these curves with the experimental data, the correctness and validity of the theoretical analysis method are proved. The results are given as follows:

1. Theoretical analysis explains that the vibration system driven by three-ERs has multiple synchronous states. Experiments also prove that when \( r_w < \sqrt{2} \), the vibration system has two stable synchronous states operating in super resonance. Based on the comprehensive analysis of the experiment and the theory, it can be concluded that the selected synchronous state of the vibration system depends on the initial condition and the external disturbance.

2. If the conditions of VST are satisfied, three ERs can not only achieve VST under the condition of one motor with switching off the power but also that of two motors with switching off the power.

3. When \( r_w < \sqrt{2} \), the absolute values of the phase differences between adjacent ERs are close to 120° in two groups of experiments. It can be concluded that the resultant force of the vibration system using the method of vibration synchronization cannot be increased under any kind of synchronous states of three ERs. Therefore,
the synchronous motion of three ERs with the same phase in the vibration system can only be achieved by other synchronous methods, such as forced synchronization and controlled synchronization.

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Appendix

Notation

\[ f_x, f_y, f_\psi = (f_x f_y^2 + f_y f_x^2)/2 \]
\[ J_i = m_i r_i^2 \]
\[ k_x, k_y, k_\psi = (k_x f_y^2 + k_y f_x^2)/2 \]
\[ l_i \]
\[ l_\psi = \sqrt{J_\psi}/M \]
\[ m_0 = (m_1 + m_2 + m_3)/3 \]
\[ M = m + m_1 + m_2 + m_3 \]

\( f_i \) the damping coefficient of the motor axis, \( i = 1, 2, 3 \)
\( f_x, f_y, f_\psi \) the damping of springs in \( x-, y-, \) and \( \psi-\)direction
\( J_i \) the moment of inertia of ERs, \( i = 1, 2, 3 \)
\( k_x, k_y, k_\psi \) the stiffness of springs in \( x-, y-, \) and \( \psi-\)direction
\( l_i \) the distance between \( o_i \) and \( o, i = 1, 2, 3 \)
\( l_\psi \) the distance between the position of spring and \( o \)
\( m_0 \) the mass of the rigid body
\( m_i \) the masses of ERs, \( i = 1, 2, 3 \).
\( m_0 \) the average mass of three ERs
\( o \) the mass center of the system
\( o_i \) the rotational centers of ERs, \( i = 1, 2, 3 \)
\( r \) the eccentric radius of three ERs
\( r_m = m_0 \)
\( r_{\psi i} = l_i/l_\psi \) the distance ratio between \( l_i \) and \( l_\psi, i = 1, 2, 3 \)
\( T_{ei} \) the electromagnetic torques of the motor, \( i = 1, 2, 3 \)
\( T_{ei} \) the electromagnetic torques of the motor operating steadily at the angular velocity \( \omega_{ei0}, i = 1, 2, 3 \)
\( T_{ei}(\dot{x}_i, \dot{z}_i) \) the synchronous torques, \( i = 1, 2, i < k \leq 3 \)
\( T_m = m_0 r^2 \omega_{m0}^2/2 \) the kinetic energy of ER
\( \omega_{ei} \) the actual output torques of the motor, \( i = 1, 2, 3 \)
\( \omega_{ei0} \) the average angular velocity of ERs
\( \alpha_i \) the phase difference (the difference in phase between two ERs with the same velocity is called the phase difference) \( \phi_1 - \phi_2 = 2x_1, \phi_2 - \phi_3 = 2x_2 \)
\( \beta_i \) the angle between line \( o_i o \) and \( x-\)axis, \( i = 1, 2, 3 \)
\( \gamma_i = \arctan(2\zeta_i / \omega_i) \) the phase angle, \( j = x, y, \psi \)
\( \eta_i = m_i/m_0 \) the mass ratio of ER \( i \) to the standard one, \( i = 1, 2, 3 \)
\( \varphi = (\phi_1 + \phi_2 + \phi_3)/3 \) the average phase of three ERs
\( \mu_j = 1 - (\omega_j/\omega_{m0})^2 \) the frequency ratio, \( j = x, y, \psi \)
\( \nu_i = -1 \) the clockwise rotation of ER\( i-i \)
\( \omega_j = \sqrt{k_j/M} \) the natural frequency of the system in \( j-\)direction, \( j = x, y \)
\( \omega_{\psi j} = \sqrt{k_\psi/J_\psi} \) the natural frequency of the system in \( \psi-\)direction
\( \xi_j = f_j/(2\sqrt{MK_j}) \) the damping ratio of the system in \( j-\)direction, \( j = x, y \)
\( \xi_\psi = f_\psi/(2\sqrt{J_\psi k_\psi}) \) the damping ratio of the system in \( \psi-\)direction
\( (\epsilon) = d^2dt \)

Note: \( \chi, \chi', W, \Gamma, \Pi \) and \( \theta \) with the different subscripts are dimensionless parameters, and their expressions are shown in Appendix 2.

Appendix 1

The electromagnetic torques are\(^{17}\)

\[ T_{ei} = T_{ei0} - k_{ei0} \dot{\epsilon} \]

(20)
where \( T_{r0i} = \frac{k(\omega_r - \omega_0)}{a_2 \omega_0 + i \omega_0 + a_0} \), \( k = \frac{k_{l0} + 2k_{l1} \omega_r \omega_0 + (a_2 \omega_0 + i \omega_0 + a_0) a_0}{(a_2 \omega_0 + i \omega_0 + a_0)^2} \), \( k = \frac{3a_2 \omega_0 L_2^2}{R_f R_c}, \ T_s = \frac{L_s}{R_c}, \ T_r = \frac{L_r}{R_c}, \ \sigma = 1 - \frac{L_s^2}{L_r}, \ \omega_0 = \eta_p \omega_m 0, \ a_0 = (1 - \omega_0^2 T_r T_c \sigma)^2 + \omega_0^2 (T_r + T_s)^2, \ a_1 = 2(1 - \omega_0^2 T_r T_c \sigma) \omega_m 0 T_r T_c \sigma - 2 \omega_m 0 (T_r + T_s) T_r, \ a_2 = T_r^2 + \omega_0^2 T_r^2 T_c^2 \sigma^2, \ i = 1, 2, 3. \ \eta_p \) is the number of pole pairs, \( L_s \) is the stator inductance, \( L_r \) is the rotor inductance, \( L_m \) is the mutual inductance, \( R_s \) is the stator resistance, \( R_r \) is the rotor resistance, \( \omega_m 0 \) is the synchronous electric angular velocity and \( \omega \) is the rotor electric angular velocity, and \( U \) is the phase voltage.

**Appendix 2**

\[
\dot{\phi}_i = (1 + \epsilon_i) \omega \phi_0, \quad i = 1, 2, 3
\]

\[
\chi_{f1} = \omega_m 0 \left[ \eta_1^2 W_{s1} + \eta_1 \eta_2 W_{s12} \cos(2\bar{x}_1 + \theta_{s12}) + \eta_1 \eta_3 W_{s13} \cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{s13}) \right] / 2
\]

\[
\chi_{f2} = \omega_m 0 \left[ \eta_2^2 W_{s2} + \eta_1 \eta_2 W_{s12} \cos(2\bar{x}_1 + \theta_{s12}) + \eta_2 \eta_3 W_{s23} \cos(2\bar{x}_2 + \theta_{s23}) \right] / 2
\]

\[
\chi_{f3} = \omega_m 0 \left[ \eta_3^2 W_{s3} + \eta_2 \eta_3 W_{s23} \cos(2\bar{x}_2 + \theta_{s23}) + \eta_1 \eta_3 W_{s13} \cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{s13}) \right] / 2
\]

\[
\chi_{d1} = \omega_m 0 \left[ \eta_1 \eta_2 W_{e12} \sin(2\bar{x}_1 + \theta_{e12}) + \eta_1 \eta_3 W_{e13} \sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{e13}) \right] / 2
\]

\[
\chi_{d2} = \omega_m 0 \left[ -\eta_1 \eta_2 W_{e12} \sin(2\bar{x}_1 + \theta_{e12}) + \eta_2 \eta_3 W_{e23} \sin(2\bar{x}_2 + \theta_{e23}) \right] / 2
\]

\[
\chi_{d3} = \omega_m 0 \left[ -\eta_2 \eta_3 W_{e23} \sin(2\bar{x}_2 + \theta_{e23}) - \eta_1 \eta_3 W_{e13} \sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{e13}) \right] / 2
\]

\[
\chi'_{ii} = -\eta_i^2 W_{ei} / 2, \quad i = 1, 2, 3
\]

\[
\chi_{di} = \omega_m 0 \eta_i^2 W_{si}, \quad i = 1, 2, 3
\]

\[
\chi'_{12} = \chi'_{21} = \eta_1 \eta_2 W_{e12} \cos(2\bar{x}_1 + \theta_{e12}) / 2
\]

\[
\chi'_{13} = \chi'_{31} = \eta_1 \eta_3 W_{e13} \cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{e13}) / 2
\]

\[
\chi'_{23} = \chi'_{32} = \eta_2 \eta_3 W_{e23} \cos(2\bar{x}_2 + \theta_{e23}) / 2
\]

\[
\chi_{12} = -\chi_{21} = \omega_m 0 \eta_1 \eta_2 W_{e12} \sin(2\bar{x}_1 + \theta_{e12})
\]

\[
\chi_{13} = -\chi_{31} = \omega_m 0 \eta_1 \eta_3 W_{e13} \sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{e13})
\]

\[
\chi_{23} = -\chi_{32} = \omega_m 0 \eta_2 \eta_3 W_{e23} \sin(2\bar{x}_2 + \theta_{e23})
\]

\[
W_{si} = r_m \left( \frac{\sin^2 \gamma_x}{\mu_x} + \frac{\sin^2 \gamma_y}{\mu_y} + \frac{r_{\phi}^2 \sin^2 \phi_\phi}{\mu_\phi} \right)
\]

\[
W_{ei} = r_m \left( \frac{\cos^2 \gamma_x}{\mu_x} + \frac{\cos^2 \gamma_y}{\mu_y} + \frac{r_{\phi}^2 \cos^2 \phi_\phi}{\mu_\phi} \right)
\]

\[
\Gamma_{iik} = r_m \left[ \nu_i \nu_k \frac{\sin \gamma_x}{\mu_x} + \sin \gamma_y \frac{r_{\phi} \phi_\phi \sin \phi_\phi}{\mu_\phi} \cos(\beta_j - \nu_i \nu_k \beta_k) \right]
\]
\[
\Pi_{sik} = r_m \left[ v_i \frac{r_{\phi i} r_{\phi k} \sin \gamma}{\mu_{\psi}} \cos (\beta_i - v_i v_k \beta_k) \right], \quad i = 1, 2, \quad i < k \leq 3 \quad (39)
\]

\[
\Gamma_{cik} = -r_m \left[ v_i v_k \frac{\cos \gamma_x}{\mu_x} + \frac{r_{\phi i} r_{\phi k} \cos \gamma_{\phi}}{\mu_{\phi}} \cos (\beta_i - v_i v_k \beta_k) \right] \quad (40)
\]

\[
\Pi_{cik} = r_m \left[ v_i \frac{r_{\phi i} r_{\phi k} \cos \gamma_{\phi}}{\mu_{\phi}} \sin (\beta_i - v_i v_k \beta_k) \right], \quad i = 1, 2, \quad i < k \leq 3 \quad (41)
\]

\[
W_{sik} = \sqrt{\Gamma_{sik}^2 + \Pi_{sik}^2}, \quad i = 1, 2, \quad i < k \leq 3 \quad (42)
\]

\[
W_{cik} = \sqrt{\Gamma_{cik}^2 + \Pi_{cik}^2}, \quad i = 1, 2, \quad i < k \leq 3 \quad (43)
\]

\[
\theta_{cik} = \begin{cases} 
\arctan\left(\frac{\Pi_{cik}}{\Gamma_{cik}}\right), & \Gamma_{cik} \geq 0, \\
\pi + \arctan\left(\frac{\Pi_{cik}}{\Gamma_{cik}}\right), & \Gamma_{cik} < 0, \end{cases}, \quad i = 1, 2, \quad i < k \leq 3 \quad (44)
\]

\[
\theta_{sik} = \begin{cases} 
\arctan\left(\frac{-\Pi_{sik}}{\Gamma_{sik}}\right), & \Gamma_{sik} \geq 0, \\
\pi + \arctan\left(\frac{-\Pi_{sik}}{\Gamma_{sik}}\right), & \Gamma_{sik} < 0, \end{cases}, \quad i = 1, 2, \quad i < k \leq 3 \quad (45)
\]