LOCAL STRUCTURE OF IDEAL SHAPES OF KNOTS

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Abstract. Relatively extremal knots are the relative minima of the rope-length functional in $C^1$ topology. They are the relative maxima of thickness (normal injectivity radius) functional on the set curves of fixed length, and they include the ideal knots. We prove that a $C^{1,1}$ relatively extremal knot in $\mathbb{R}^n$ either has constant maximal (generalized) curvature, or its thickness is equal to half of the minimal double critical distance. Our main approach is to show that the shortest curves with bounded curvature and $C^1$ boundary conditions in $\mathbb{R}^n$ contain CLC (circle-line-circle) curves, if they do not have constant maximal curvature.

1. Introduction

In this article, we study the local structure of $C^{1,1}$ relatively extremal knots in $\mathbb{R}^n$ ($n \geq 2$), by using a length minimization problem with bounded curvature and $C^1$ boundary conditions. The thickness of a knotted curve is the radius of the largest tubular neighborhood around the curve without intersections of the normal discs. This is known as the injectivity radius $i(K, \mathbb{R}^n)$ of the normal exponential map of the curve $K$ in the Euclidean space $\mathbb{R}^n$. The ideal knots are the embeddings of $S^1$ into $\mathbb{R}^n$, maximizing $i(K, \mathbb{R}^n)$ in a fixed isotopy (knot) class of fixed length. A relatively extremal knot is a relative maximum of $i(K, \mathbb{R}^n)$ in $C^1$ topology, if the length is fixed.

We prove every result in $\mathbb{R}^n$ ($n \geq 2$) in this article, since our methods are not dependent on dimension. However, all one dimensional knots are trivial in $\mathbb{R}^n$, for $n \neq 3$. Although ideal knots are not interesting for $n \neq 3$, relatively extremal knots, the length minimization with bounded curvature, and some of the local results on curves we obtained may be useful for other purposes.

As noted in [Ka], "...the average shape of knotted polymeric chains in thermal equilibrium is closely related to the ideal representation of the corresponding knot type". "Knotted DNA molecules placed in certain solutions follow paths of random closed walks and the ideal trajectories are good predictors of time averaged properties of knotted polymers" as a biologist referee pointed out to the author. Since the length of the molecule is fixed, this problem becomes the maximization of its thickness within a fixed homotopy class of a knot. The analytical properties of the ideal knots will be tools in research of the physics of knotted polymers.

For simple knots, one has a good idea of the approximation of the ideal shapes by using computers, see [Ka] and [GM]. Gonzales and Maddocks introduced the notion of the global radius of curvature which is another characterization of thickness in

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$\mathbb{R}^3$, and used it on discrete curves to obtain very good approximations of the ideal shapes in [GM]. However, we do not know the exact shape of most of the nontrivial knots (including trefoil knots) in ideal position or the exact value of their thickness. Some estimates of the thickness of ideal knots have been obtained by Diao [Di], Buck and Simon [BS] and Rawdon and Simon [RS] by using results of Freedman, He and Wang [FHW].

Since a positive lower bound on thickness bounds curvature, the completion of this class must include $C^{1,1}$ curves. The extremal cases in $\mathbb{R}^3$ are unlikely to be smooth. Very few ideal knots in $\mathbb{R}^3$ are expected to be $C^2$, and the unknotted standard circles are possibly the only smooth ones. This requires the study of $i(K, \mathbb{R}^n)$ in $C^{1,1}$ category.

In [D], the author proved the following Thickness Formula in the general context and developed the notion of "Geometric Focal Distance, $F_\delta(K)$" by using metric balls, which naturally extends the notion of the focal distance of smooth category to $C^1$ topology. These are also done by Litherland in [L] for $C^{1,1}$-knots in $\mathbb{R}^3$. Nabutovsky, [N] has an extensive study of $C^{1,1}$ hypersurfaces $K$ in $\mathbb{R}^n$ and their injectivity radii. [N] proves the upper semicontinuity of $i(K, \mathbb{R}^n)$ and lower semicontinuity of $\frac{\text{vol}(K)}{i(K, \mathbb{R}^n)^k}$ in $C^1$ topology.

We will use a corollary of the formula for curves in $\mathbb{R}^n$ in Section 4. $i(K, M) = R^O(K, M)$, a rolling ball/bead description of the injectivity radius in $\mathbb{R}^n$, was known by Nabutovsky for hypersurfaces, and by Buck and Simon for $C^2$ curves, [BS]. The rolling ball/bead characterization is our main geometric tool. The notion of the global radius of curvature developed by Gonzales and Maddocks for smooth curves in $\mathbb{R}^3$ defined by using circles passing through 3 points of the curve in [GM] is a different characterization of $i(K, \mathbb{R}^3)$ from $R^O$ due to positioning of the circles and metric balls. $MDC(K)$ is the minimal double critical distance. See Section 2 for the basic definitions.

**GENERAL THICKNESS FORMULA** [D, Theorem 1]

For every complete smooth Riemannian manifold $M^n$ and every compact $C^{1,1}$ submanifold $K^k$ ($\partial K = \emptyset$) of $M$,

$$i(K, M) = R^O(K, M) = \min\{F_\delta(K), \frac{1}{2} MDC(K)\}.$$  

For a $C^{1,1}$ curve $\gamma$, $\gamma''$ exists almost everywhere by Rademacher’s Theorem, [F]. For a $C^{1,1}$ curve $\gamma(s)$ parametrized by the arclength $s$, define the (generalized) curvature $\kappa \gamma(s) = \lim_{x \neq y \to s} \frac{\langle \gamma'(x), \gamma'(y) \rangle}{|x-y|}$. $\kappa$ is defined for all $s$. See Lemmas 1 and 2 below for a proof of $F_\delta(K^1) = F_k(\gamma) = (\sup \kappa \gamma)^{-1} = (\sup ||\gamma''||)^{-1}$ for curves parametrized by arclength in $\mathbb{R}^n$, and [D, Proposition 12] for a similar curvature description of $F_\delta(K^k)$ for higher dimensional $K^k \subset \mathbb{R}^n$.

Given a certain type of knot and a rope of set thickness, finding the exact shape to tie the knot by using the shortest amount of the rope is basically the same as finding the ideal shape of a DNA molecule of fixed length in this knot type in $\mathbb{R}^3$. For any simple $C^{1,1}$ closed curve $\gamma$ in $\mathbb{R}^n$, define the ropelength (see [BS], [L]) or extrinsically isosembloric length to be $\ell_e(\gamma) = \frac{\ell(\gamma)}{R^O(\gamma)}$ where $\ell(\gamma)$ is the length of $\gamma$. A curve $\gamma_0$ is called an ideal (thickest) knot in a knot class $[\theta]$, if $\ell_e$ attains its absolute
minimum over \([\theta] \cap C^{1,1}\) at \(\gamma_0\); and \(\gamma_0\) is called relatively extremal, if \(\ell_e\) attains a relative minimum at \(\gamma_0\) with respect to \(C^1\) topology.

In this article, we study the pieces of relatively extremal knots away from minimal double critical points by using minimization of length with bounded curvature.

**Question.** Given \(p, q, v, w \in \mathbb{R}^n\), with \(|v| = |w| = 1\) and \(\Lambda > 0\). Classify all shortest curves in \(\mathcal{C}(p, q; v, w; \Lambda)\) which is the set of all curves \(\gamma\) between the points \(p\) and \(q\) in \(\mathbb{R}^n\) with \(\gamma'(p) = v, \gamma'(q) = w\) and \(\kappa\gamma \leq \Lambda\).

Even though this looks like an elementary problem, a complete answer is not known yet. This is a minimization problem with a second order differential inequality and \(C^1\) boundary values. There exists a shortest \(C^{1,1}\) curve by Arzela-Ascoli Theorem. The following theorem classifies all cases except the constant maximal curvature case, and also brings out the mathematical difficulties of this problem. The results of this article are proved by using simple geometric methods, in contrast to their analytical nature. We include the proofs of several basic geometric facts for \(C^{1,1}\) curves.

**Theorem 1.** Let \(\gamma : I = [0, L] \to \mathbb{R}^n\) be a shortest curve in \(\mathcal{C}(p, q; v, w; \Lambda)\) parametrized by arclength.

a. If \(\gamma\) does not have constant curvature \(\Lambda\), i.e. \(\kappa\gamma(s_0) \neq \Lambda\) for some \(s_0\), then there exist \(a_0\) and \(b_0\) such that \(s_0 \in [a_0, b_0] \subset [0, L]\) and \(\gamma([a_0, b_0])\) is a CLC(\(\Lambda\)) curve where each circular part has length at least \(\frac{\pi}{\Lambda}\) unless it contains the initial or terminal point of \(\gamma\).

b. If \(R_O(\gamma(I), \mathbb{R}^n) \geq \frac{1}{\Lambda}\) and \(\gamma\) does not have constant curvature \(\Lambda\), then \(\gamma\) is a CLC(\(\Lambda\)) curve.

A CLC(circle-line-circle)(\(\Lambda\)) curve is one circular arc followed by a line segment and then by another circular arc in a \(C^1\) fashion (like two letters J with common straight parts, one hook at each end, and possibly non-coplanar) (like two letters J with common straight parts, one hook at each end, and possibly non-coplanar), where the circular arcs have radius \(1/\Lambda\). If \(p = q\) and \(v = -w\), then the shortest curve with curvature restriction satisfies \(\kappa \equiv \Lambda\) and it is not a CLC-curve. One can construct curves of constant curvature \(\Lambda\) with countably infinite points where the curve is not twice differentiable. We note that the classification of shortest curves in \(\mathcal{C}(p, q; v, w; \Lambda)\) with \(\kappa \equiv \Lambda\) is not a simple matter, and it will be discussed in a different article.

**Theorem 1** tells us that the parts of a relatively extremal knot with the minimal double critical points removed are expected to be CLC curves or overwound, i.e. \(\kappa \equiv \Lambda\). As J. Simon pointed out that there are physical examples (no proofs) of relatively extremal unknots in \(\mathbb{R}^3\), which are not circles, and hence not ideal knots. One can construct similar physical examples for composite knots.

The connectedness of the knots in Theorems 2 and 3 is not essential, and these theorems are valid on links. The General Thickness Formula does not assume that \(K\) is connected, and Propositions 5-8 do not use connectedness. Proof of Theorem 2 is local and based on the nonexistence of local length decreasing and curvature nonincreasing perturbations by repeated use of Theorem 1. The existence of thickest submanifolds with many components is also discussed in [D]. These proofs can be modified by a simply changing the domain from \(S^1\) to a finite disjoint union of circles and keeping track of which component is worked on.

**Theorem 2.** Let \(\gamma : S^1 \to \mathbb{R}^n\) be a relatively extremal knot, parametrized by arclength such that \(\exists s_0 \in S^1, \kappa\gamma(s_0) = \sup \kappa\gamma\). Then both of the following holds.

i. \(i(\gamma(S^1), \mathbb{R}^n) = R_O(\gamma(S^1), \mathbb{R}^n) = \frac{1}{2}MDC(\gamma(S^1))\).
For any relatively extremal knot $\gamma$ in $\mathbb{R}^n$, whose curvature $\kappa \gamma$ is not identically $R_0(K)^{-1}$, the thickness of $\gamma$ is $\frac{1}{2} MDC(\gamma(S^1))$.

As a consequence we obtain the following.

**Theorem 3.** For any relatively extremal knot $\gamma$ in $\mathbb{R}^n$, whose curvature $\kappa \gamma$ is not identically $R_0(K)^{-1}$, the thickness of $\gamma$ is $\frac{1}{2} MDC(\gamma(S^1))$. Equivalently, if there exists a relatively extremal knot $\gamma$ such that $\frac{1}{2} MDC(K) > R_0(K) = F_k(\gamma)$, then $\gamma$ must have constant generalized curvature, $\kappa \gamma \equiv F_k(\gamma)^{-1}$.

Some of our results on ideal knots overlap with [GM] which studies smooth knots. Proposition 8 and [GM, section 4] obtain line segments away from the maxima of the global radius of curvature $\rho_G$. However, maximal $\rho_G$ does not distinguish between minimal double critical points and maximal curvature points. Hence, we can obtain further conclusions, such as Theorem 3, and they are in a larger class ($C^{1,1}$) than smooth ideal knots. For an ideal knot, Theorem 2 proves that (i) after a line segment, the ideal curve must go through a minimal double critical point before reaching the next line segment, and (ii) if there is a non-linear piece of the ideal knot between a line segment and the next minimal double critical point, then that must be a planar circular arc whose radius is the thickness of the ideal knot.

Basic definitions are given in Section 2, shortest curves with curvature restrictions and proof of Theorem 1 are given in Section 3, and ideal knots and proof of Theorem 2 are given in Section 4.

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## 2. Basic definitions for Thickness Formula

For the generalizations of the following concepts and the Thickness Formula to $C^{1,1}$ submanifolds of Riemannian manifolds, we refer to [D].

**Definition 1.** Let $K \subset \mathbb{R}^n$, be a $C^1$ curve in $\mathbb{R}^n$ and $\gamma(s)$ be a parametrization of $K$ such that $\|\gamma'\| = 1$.

i. The normal bundle to $K$ in $\mathbb{R}^n$ and the tangent bundle of $K$ are

$$NK = \{(p,w) \in \mathbb{R}^n \times \mathbb{R}^n : s \in \text{dom}(\gamma), p = \gamma(s) \text{ and } w \cdot \gamma'(s) = 0\}$$

and

$$TK = \{(p,v) \in \mathbb{R}^n \times \mathbb{R}^n : s \in \text{dom}(\gamma), p = \gamma(s) \text{ and } v = c\gamma'(s), c \in \mathbb{R}\},$$

respectively.

$UNK$ and $UTK$ denote all unit vectors in $NK$ and $TK$, respectively.

$NK_p$ is the collection of all normal vectors of $NK$ at $p$, and the others are defined similarly.

ii. $\exp_p v = p + v : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the exponential map of $\mathbb{R}^n$ and

$$\exp^N_p w = p + w : NK \to \mathbb{R}^n$$

is the normal exponential map of $K$ into $\mathbb{R}^n$.

**Definition 2.** i. For any metric space $X$ with a distance function $d$, $B(p,r) = \{x \in X : d(x,p) < r\}$ and $\overline{B}(p,r) = \{x \in X : d(x,p) \leq r\}$. For $A \subset X$ and $x \in X$ define $d(x,A) = \inf\{d(x,a) : a \in A\}$ and $B(A,r) = \{x \in X : d(x,A) < r\}$. The diameter $d(X)$ of $X$ is defined to be $\sup\{d(x,y) : x,y \in X\}$. If there is ambiguity, we will use $d_X$ and $B(p,r;X)$.
Both denote the length of $\gamma$.

**Definition 3.** Let $K$ be a $C^1$ submanifold of $\mathbb{R}^n$. Define the thickness of $K$ in $\mathbb{R}^n$ or the normal injectivity radius of $\exp^N$ to be

\[ i(K, M) = \sup \{0 \cup \{ r > 0 : \exp^N : \{ v \in NK : \| v \| < r \} \rightarrow M \ is \ one-to-one} \} \]

Equivalently, if $\gamma(s)$ parametrizes $K$, then

\[ r > i(K, M) \iff \exists \gamma(s), \gamma(t), q \in \mathbb{R}^n, \| \gamma(s) - q \| < r, \| \gamma(t) - q \| < r, \ and \ (\gamma(s) - q) \cdot \gamma'(s) = (\gamma(t) - q) \cdot \gamma'(t) = 0 \].

**Definition 4.** Let $K$ be a $C^1$ curve in $\mathbb{R}^n$. For any $v \in UTR_p$ and any $r > 0$, define

i. $O_p(v, r) = \bigcup_{w \in v^+(1)} B(\exp_p rw, r)$, where $v^+(1) = \{ w \in UTR_p : \{ v, w \} = 0 \}$ or equivalently,

\[ O_p(v, r) = \{ x \in R^n : \exists w \in R^n, v \cdot w = 0, \| w \| = 1, \| x - p - rw \| < r \}

ii. $O_p^c(v, r) = R^n - O_p(v, r)$ and $O_p^c(v) = O_p^c(v, 1) = \{ x \in R^n : \forall w \in R^n, v \cdot w = 0, \| w \| = 1, \| x - p - w \| \geq 1 \}$

iii. $O_p(r; K) = O_p(v, r)$ where $v \in UTK_p$

iv. $O(r; K) = \bigcup_{p \in K} O_p(r; K)$

**Definition 5.** Let $K$ be a $C^1$ curve in $\mathbb{R}^n$. Define

i. The ball radius of $K$ in $\mathbb{R}^n$ to be $R_0(K, R^n) = \inf \{ r > 0 : O_p(v; K) \cap K \neq \emptyset \}$

ii. The pointwise geometric focal distance $F_g(p) = \inf \{ r > 0 : p \in O_p(v; K) \cap K \}$ for any $p \in K$, and the geometric focal distance $F_g(K) = \inf_{p \in K} F_g(p)$.

**Definition 6.** Let $K$ be a $C^1$ submanifold of $M$. A pair of points $p$ and $q$ in $K$ are called a double critical pair for $K$, if there is a line segment $\gamma_{pq}$ of positive length between $p$ to $q$, normal to $K$ at both $p$ and $q$. Define the minimal double critical distance

\[ MDC(K) = \inf \{ \| p - q \| : \{ p, q \} \ is \ a \ double \ critical \ pair \ for \ K \} \]

3. **Shortest Curves in $\mathbb{R}^n$ with Curvature Restrictions**

In this section, $\gamma : I \rightarrow \mathbb{R}^n$ denotes a simple $C^1$ curve with $I = [0, L]$, $\| \gamma' \| \neq 0$ and $K = \text{image}(\gamma)$.

**Definition 7.** Let $\{ e_i : i = 1, 2, \ldots, n \}$ be the standard basis in $\mathbb{R}^n$. Let $E_i$, $E_i^+$, $E_i^-$ denote the $e_i$ axis, its positive and negative parts.

**Definition 8.** For $\gamma : I \rightarrow \mathbb{R}^n$, define:

Dilations: $\text{dil}^d_\gamma(s, t) = \frac{\| \gamma'(s) - \gamma'(t) \|}{\ell'_{\gamma}(s)'}$ and $\text{dil}^\alpha_\gamma(s, t) = \frac{\| \gamma'(s) - \gamma'(t) \|}{\ell'_{\gamma}(s)}$ for $s \neq t$

Curvature: $\kappa_\gamma(s) = \limsup_{t,u \to s} \text{dil}^\alpha_\gamma(t, u)$

Lower curvature: $\kappa^-_\gamma(s) = \limsup_{t \to s} \text{dil}^\alpha_\gamma(s, t)$

Analytic focal distance: $F_K(\gamma) = (\sup_{s} \kappa_\gamma(s))^{-1}$
Remark 1. i. Since  \( \lim_{n \to \infty} \frac{\int_{a}^{b} |f(x)|^n dx}{n!} = 1 \) for  \( |w| = 1 \), one obtains the same  \( \kappa \gamma \), if one uses  \( \text{dil}^s \) instead of  \( \text{dil}^a \), provided that  \( \gamma \) is parametrized with respect to the arc length. Same true for  \( \kappa \gamma \).

ii.  \( \kappa \gamma (s) \geq \kappa^{-\gamma} (s), \forall s. \)

iii. If  \( \gamma \in C^{1,1} \) and  \( \|\gamma\| \equiv 1 \), then  \( \|\gamma''(s)\| = \kappa^{-\gamma} (s), \forall s \) a.e.

iv. \( \lim_{n \to s} \kappa \gamma (s_n) = \kappa \gamma (s) \).

Lemma 1. All of the following are equivalent for a  \( C^1 \) curve  \( \gamma : I \to \mathbb{R}^n \) with  \( \|\gamma\| \equiv 1 \), for the same  \( \Lambda \).

i. \( \kappa \gamma (s) \leq \Lambda, \forall s \in I \).

ii. \( \text{dil}^s \gamma (s,t) \leq \Lambda, \forall s, t \in I \).

iii. \( \text{dil}^a \gamma (s,t) \leq \Lambda, \forall s, t \in I \).

iv. \( \|\gamma''(s)\| \leq \Lambda, \forall s \in I \) a.e., and  \( \gamma \) is absolutely continuous.

Proof. (i  \( \Rightarrow \) iii) :  \( \forall s < t < u, \text{dil}^s \gamma (s,u) \leq \max (\text{dil}^s \gamma (s,t), \text{dil}^a \gamma (t,u)) \). Hence, if  \( \text{dil}^a \gamma (s,u) \geq A \) for some  \( s \neq u \) and  \( A \), then there exists  \( s_0 \in [s,u] \) with  \( \kappa \gamma (s_0) \geq A \).

\( (iv \Rightarrow ii) : \|\gamma'(t) - \gamma'(s)\| = \| \int_s^t \gamma''(u) du \| \leq \int_s^t \|\gamma''(u)\| du \leq \Lambda \|t-s\| \), by absolute continuity.

\( (iii \Rightarrow i) \) and \( (ii \Rightarrow iv) \) are obvious, and \( (i \iff iii) \) is by Remark 1.i.

\( \square \)

Definition 9. A curve  \( \gamma : I \to \mathbb{R}^n \) is called to have curvature at most  \( \Lambda \), if  \( \kappa \gamma \leq \Lambda \) on  \( I \). By the previous lemma,  \( \gamma \) must be of class  \( C^{1,1} \).

Definition 10. A  \( C^{1,1} \) curve  \( \gamma : I = [a,b] \to \mathbb{R}^n \) is called a CLC(\( \Lambda \)) curve if there are  \( a \leq c \leq d \leq b \) such that (i)  \( \gamma ([c,d]) \) is a line segment of possibly zero length, and (ii) each of  \( \gamma ([a,c]) \) and  \( \gamma ([d,b]) \) is a planar circular arc of radius  \( \frac{1}{\Lambda} \) and of length in  \( [0,\frac{2\pi}{\Lambda}] \), with the possibility that  \( \gamma \) is not planar.

Proposition 1. Let  \( p \in \mathbb{R}^n \) and  \( v \in \mathbb{R}^n \) with  \( \|v\| = 1 \).

a)  \( \varphi \in O^\prime_p (v), 3w \in \mathbb{R}^n, 3q' \in \partial O^\prime_p (v) \) and a  \( C^1 \) curve  \( \gamma_{pq} \subset \text{span}\{v,w\} \) such that

\( i. \ v \cdot w = 0, \|w\| = 1, \) and

\( ii. \ \gamma_{pq}(t) = \begin{cases} \varphi \sin t + w(1 - \cos t) & \text{if } 0 \leq t \leq t_0 \\ q' + (t-t_0)(q'-q) & \text{if } t_0 \leq t \leq t_1 \end{cases}, \text{where } q' = \gamma_{pq}(t_0) \) and  \( q = \gamma_{pq}(t_1) \), and

\( iii. \ \gamma_{pq} \) is a shortest curve among all the continuous curves  \( \varphi \) from  \( p \) to  \( q \) in  \( O^\prime_p (v) \) with  \( \varphi(t) - p \cdot \varphi > 0 \) for small  \( t > 0 \) and  \( \varphi(0) = p \).

b) If  \( q-p \neq \lambda v, \forall \lambda \), then  \( q' \) and  \( w \) are unique and  \( \gamma_{pq} \) is unique up to parametrization.

Proof. Consider a non-empty set of all rectifiable curves of length  \( \leq L \) satisfying (a.iii). Parametrize each curve  \( \varphi \) by arclength and extend the domain to  \( [0,L] \) by keeping  \( \varphi \) constant after reaching  \( q \) so that  \( \|\varphi(s) - \varphi(t)\| \leq |s-t|, \forall s, t \). This forms a non-empty, bounded and equicontinuous family, and length functional is lower semi-continuous under uniform convergence. By Arzelà-Ascoli Theorem, a shortest curve  \( \gamma_{pq} \) from  \( p \) to  \( q \) in  \( O^\prime_p (v) \) satisfying (a.ii) exists. Also, the proof
below shows how to deform any curve $\varphi$ as in (a.iii) to a shorter curve, where the aim is to reach $\gamma_{pq}$.

It suffices to give the rest of the proof for $p = 0, v = e_1$ and $q \neq 0$. Set $\gamma_{pq} = \gamma$.

**Case 1.** If $q = \lambda e_1, \lambda > 0$, then $\gamma$ is the line segment from 0 to $q$ where $q' = 0$ and $t_0 = 0$. Conversely, if $\gamma$ intersects $E^+_1$ at $q'' \neq 0$, then $q = \lambda e_1$ for some $\lambda > 0$. For, $\gamma$ must be along $E^+_1$ between 0 and $q''$, and then extends uniquely as a geodesic of $R^n$ beyond $q''$.

For any $u \in R^n$, define $w^N = u - (u \cdot e_1)e_1$.

**Case 2.** $\gamma \cap E_1 = \{0\}$. Thus, $q^N \neq 0$ and define $w = q^N / \|q^N\|$. It suffices to give the proof for $w = e_2$. Define $f : R^n - E_1 \to A = \{xe_1 + ye_2 : x, y \in R, y > 0\}$ by $f(u) = (u \cdot e_1)e_1 + \|u^N\|e_2$. $f$ is a length decreasing map:

$$\|f(u) - f(z)\|^2 = \|(u \cdot e_1)e_1 - (z \cdot e_1)e_1\|^2 + \|u^N\|^2 \geq \|(u - z) \cdot e_1\|^2 + \|u - z\|^2$$

and equality holds if and only if $u^N = ce_2$ for some $c > 0$, i.e. $u \in span(e_1, z)$. Reparametrize $\gamma$ with respect to arclength. By following Federer [F], pp. 109, 163-168, we obtain that $\gamma$ is lipschitz, absolutely continuous, $\gamma'$ exists a.e. and

$$\ell(\gamma) = \int_0^{\ell(\gamma)} \|\gamma'(s)\| ds \geq \int_0^{\ell(\gamma)} \|f_*\gamma'(s)\| ds \geq \ell(f(\gamma)).$$

Since $\gamma$ is a shortest curve from 0 to $q$, $\|\gamma'(s)\| = \|f_*\gamma'(s)\|$ and $\gamma'(s) \in span\{e_1, \gamma\}$ for almost all $s \in [0, \ell(\gamma)]$.

$$\gamma^N(s) = \gamma^N(\gamma N(s), \gamma N(s)) = \lambda(s)\gamma^N(s), \text{ for } s \in [0, \ell(\gamma)], \text{ a.e.}$$

By absolute continuity and $\gamma(\ell(\gamma)) = q \in span\{e_1, e_2\}$, one obtains that $\gamma(1) \subset span\{e_1, e_2\}$. This reduces the proof to the $R^2$ case.

**Subcase 2.1.** $\|q - e_2\| = 1$, that is $q \in \partial O^+_p(v)$. Define $g : \{u \in R^2 : \|u - e_2\| \geq 1\} \to \{u \in R^2 : \|u - e_2\| = 1\}$ by $g(u) = e_2 + \frac{u - e_2}{\|u - e_2\|}$. Then, $q$ is a distance decreasing map, $\|g(u) - g(z)\| \leq \|u - z\|$ and equality holds if and only if $\|u - e_2\| = \|z - e_2\| = 1$. Hence, $\ell(\gamma) \geq \ell(g(\gamma))$, and consequently the shortest curve $\gamma$ must lie on the circle $\|u - e_2\| = 1$ between $p$ and $q$, by a proof similar to above with $f$.

**Subcase 2.2.** $\|q - e_2\| > 1$, that is $q \in intO^+_p(v)$. Any component of $\gamma \cap intO^+_p(v)$ is a line segment. Let $\eta$ be the component containing $q$. By case assumption and Case 1, $\eta \cap E^+_1 = \emptyset$. There exists unique $qt$ in $\eta \cap \partial O^+_p(v)$ with $\|q - e_2\| = 1$.

By Case 2.1, $\gamma$ is a union of a segment and a circular arc. If $\gamma$ were not $C^1$ at $qt = \gamma(t_0)$, then for sufficiently small $\varepsilon > 0$, the line segments between $\gamma(t_0 - \varepsilon)$ and $\gamma(t_0 + \varepsilon)$ lie in $O^+_p(v)$ and have length $< 2\varepsilon$, by the first variation. Hence, $\gamma$ is $C^1$, satisfies a.i-iii, in $R^2 = span\{e_1, e_2\}$ and consequently in $R^n$. In Case 2, $q'$ and $w$ are unique and $\gamma_{pq}$ is unique up to parametrization.

**Case 3.** $\gamma \cap E^+_1 \neq \emptyset$. **Subcase 3.1.** $q \in E^+_1 \cap intO^+_p(v)$ and let $\eta$ be the line segment part of $\gamma$ ending at $q$. Obviously, $\eta \not\subset E^+_1$. Choose $q'' \in (\eta - \{q\} - \partial O^+_p(v))$. $\gamma$ restricts the shortest curve from $p$ to $q''$, by Case 2. Hence, $\gamma$ follows a circular...
arc to $q'$ then a segment to $q''$, which must be $\eta$. This proves (a.i-iii). By rotating $\gamma$ around $E_1$, one obtains infinitely many shortest curves $\gamma_a$ satisfying (a.i-iii).

**Subcase 3.2.** Suppose there exists $q'' \in \gamma \cap E_1$ and $q'' \neq q$. Then by following any $\gamma_a \neq \gamma$ from $p$ to $q''$, and $\gamma$ from $q''$ to $q$, creates a shortest curve with a corner within $\text{int}\Omega_T^p(v)$, an open subset of $\mathbb{R}^n$. Hence, Subcase 3.2 does not occur.

**Proposition 2.** Let $\gamma: I = [0, L] \to \mathbb{R}^n$ be with $\kappa \gamma \leq 1$ and $\|\gamma\| = 1$. Then,

a. $\gamma(s) \in \Omega_T^\gamma(\gamma'(a)), \forall a, s \in I$ with $|s - a| \leq \pi$. Also,

$$\langle \gamma(s_0) \in \partial \Omega_T^\gamma(\gamma'(a)) \rangle \forall a, s_0 \in I \text{ with } 0 < |s_0 - a| \leq \pi \text{ if and only if }$$

$\gamma$ is a circular arc of radius 1 in $\partial \Omega_T^\gamma(\gamma'(a))$ between $\gamma(a)$ and $\gamma(s_0)$.

b. If $\|\gamma(0)\| = \|\gamma(L)\| = 1$ and $\|\gamma(a)\| > 1$ for some $a \in [0, L]$, then $L > \pi$.

c. If $\gamma'(a)$ exists and $\|\gamma''(a)\| = 1$, for some $a \in [0, L]$ then

$$\forall R > 1, \exists \varepsilon > 0 \text{ such that } \gamma((a, a + \varepsilon)) \subset B(\gamma(a) + R\gamma''(a), R).$$

**Proof.** The proof follows the following order: (a) for $0 \leq |s - a| \leq \frac{\pi}{2}$, (b) is next, and then (a) for $|s - a| \leq \pi$. (c) is independent.

**(a):** By using an isometry of $\mathbb{R}^n$, reparametrization and symmetry, it suffices to prove this for $a = 0$, $\gamma(0) = 0$, $\gamma'(0) = e_1$ and for $0 \leq s \leq \frac{\pi}{2}$.

$$\gamma'(s) = \alpha(s)e_1 + \beta(s)v(s) \text{ where } \|v(s)\| = 1 \text{ and } v(s) \cdot e_1 = 0, \text{ for } s \in [0, \frac{\pi}{2}].$$

Then, by Lemma 1, $\angle(\gamma(0), \gamma'(s)) \leq s$, $\alpha(s) \geq \cos s$, and $\beta(s) \leq \sin s$, since $\alpha^2 + \beta^2 = 1$.

For any $u \in \mathbb{R}^n$, define $u^N = u - (u \cdot e_1)e_1$.

$$\gamma(s) \cdot e_1 = \int_0^s \gamma'(t) \cdot e_1 dt \geq \sin s$$

$$\|\gamma(s)^N\| \leq \int_0^s |\beta(t)| dt \leq 1 - \cos s$$

For any unit vector $u$ normal to $e_1$,

$$\|u - \gamma(s)\|^2 = (\gamma(s) \cdot e_1)^2 + \|u - \gamma(s)^N\|^2 \geq \sin^2 s + \|u - \|\gamma(s)^N\| u\|^2 \geq 1.$$ 
Hence, $\gamma(s) \in \Omega_0^\gamma(e_1)$ for $0 \leq s \leq \frac{\pi}{2}$. Suppose that $\gamma(s_0) \in \partial \Omega_0^\gamma(e_1)$ for some $s_0 \in (0, \frac{\pi}{2})$. Then, all of the above inequalities become equalities for a fixed $u$ and $\gamma(s)^N$ is parallel to $u$, to conclude $\gamma(s) = (\sin s)e_1 + (1 - \cos s)u$, for $s \in (0, s_0]$.

**(b):** Choose $m \in [0, L]$, such that $\|\gamma(m)\| \geq \|\gamma(s)\|$, $\forall s \in [0, L]$. Since $\gamma(m)$ is a furthest point from $0$, $\gamma'(m) \cdot \gamma(m) = 0$ and $0$ is on the hyperplane through $\gamma(m)$ normal to $-\gamma'(m)$. Choose any point $p \in \partial B(0, 1) \cap \Omega_T^\gamma(-\gamma'(m))$ and a shortest curve $\eta$ in $\Omega_T^\gamma(-\gamma'(m)) = O' \cap \gamma(m)$ from $\gamma(m)$ to $p$, in the opposite direction of $\gamma(\eta)$. By Proposition 1, $\eta$ lies in a 2-plane $X$ through $\gamma(m)$ and $p$, parallel to $\gamma'(m)$ and it is a $C^{1,1}$ curve following a circular arc of length $\theta$ of radius 1 and a line segment to $p$. Let $A$ be the set $\{x \in \mathbb{R}^n : |x \cdot \gamma'(m)| \leq 1\}$ whose boundary consists of two parallel hyperplanes. $\eta \subset A$, since $B(0, 1) \cup \Omega_T^\gamma(-\gamma'(m)) \subset \text{int} \ A$, $\gamma(\eta)$ and $p$ are in $A$. Consequently, $\eta \subset \text{int} \ A \cap X \cap O'$. Since $\gamma(m)$ and $p$ are in different components of $(\text{int} \ A) \cap X \cap O'$, $\eta$ must pass through $\partial A \cap X \cap O'$. This shows that $\theta \geq \frac{\pi}{2}$ and $\ell(\eta) > \frac{\pi}{2}$. Suppose that $m \leq \frac{\pi}{2}$. Then, $\gamma([0, m]) \subset \Omega_T^\gamma(\gamma'(m))$ by part (a.ii) and take $p = \gamma(0) \in \partial B(0, 1)$. This gives us a contradiction: $\frac{\pi}{2} < \ell(\eta) \leq \ell(\gamma([0, m])) = m \leq \frac{\pi}{2}$. Consequently, one must have $\frac{\pi}{2} < m$, and $\frac{\pi}{2} < L - m$ by symmetry.

**(a):** By reparametrization and symmetry, it suffices to prove this for $a = 0$ and for $0 \leq s \leq \pi$. Suppose that $\gamma(b) \in \Omega_T(\gamma'(0), 1)$ for some $b \in [0, \pi] \cap I$. Then $\gamma(b) \in B(q, 1)$ where $q = \gamma(0) + v$ for some unit vector $v$ normal to $\gamma'(0)$. 

There is a unique $c \in [0, b)$ such that $\gamma((c, b]) \subset B(q, 1)$ and $\gamma(c) \in \partial B(q, 1)$. One must have $\gamma([0, b]) \subset B(q, 1)$ by part (b), since $c < \pi$, $\gamma(0)$ and $\gamma(c)$ are in $\partial B(q, 1)$. $\gamma(c)$ is tangent to $\partial B(q, 1)$, since $|\gamma(t) - q|$ has a local maximum at $t = c \neq 0$, and $c = 0$ case is obvious. By part (a; $\frac{c}{2}$), $\gamma$ must stay out of $O_{\gamma(0)}(\gamma'(c), 1) \supset B(q, 1)$ for $c \leq t \leq c + \frac{c}{2}$, which contradicts $\gamma((c, b]) \subset B(q, 1)$. Hence, $\gamma([0, \pi] \cap [0, L]) \cap O_{\gamma(0)}(\gamma'(0), 1) = \emptyset$.

Assume that $\exists s_0 \in I$ with $\gamma(s_0) \in \partial O_{\gamma(0)}(\gamma'(0))$ and $0 < s_0 \leq \pi$. Then, $\gamma(s_0)$ and $\gamma(0) \in \partial B(q, 1)$ where $q = \gamma(0) + v$ for some unit vector $v$ normal to $\gamma'(0)$. By part (b) and the previous paragraph, $\gamma([0, s_0]) \subset B(q, 1) \cap O_{\gamma(0)}(\gamma'(0))$ which is the desired circle.

(c) Let $q = \gamma(a) + R\gamma''(a)$, and define $f(s) = \frac{1}{2} ||\gamma(s) - q||^2$. $f'(s) = \gamma'(s) \cdot (\gamma(s) - q)$ which is lipschitz, and $f'(a) = \gamma'(a) \cdot (-R\gamma''(a)) = 0$, by $||\gamma'|| = 1$.

$f''(s) = \gamma''(s) \cdot (\gamma(s) - q) + \gamma'(s) \cdot \gamma'(s)$ a.e., and $f''(a) = \gamma''(a) \cdot (-R\gamma''(a)) + 1 < 0$. Hence, $\lim_{s \to a} f'(s) - f'(a) < 0$. There exists $\varepsilon > 0$ such that $f'(s) < 0$ and $f(s) < f(a), \forall s \in (a, a + \varepsilon)$. \hfill \qedsymbol

**Example 1.** $\pi$ in part (b) of the previous proposition is sharp. Consider the part of the circle $(x - \varepsilon)^2 + y^2 = 1$ outside the disc $x^2 + y^2 \leq 1$, in $\mathbb{R}^2$, for small $\varepsilon$.

**Lemma 2.** For all $C^{1,1}$ curves $\gamma : I \rightarrow \mathbb{R}^n$, analytic and geometric focal distances are the same: $F_g(\gamma(I)) = F_k(\gamma)$.

**Proof.** Reparametrize $\gamma$ to assume that $||\gamma'(s)|| = 1$. $\frac{1}{F_g(\gamma)} \geq \kappa \gamma$. By Proposition 2(a; $\frac{\pi}{2}$) and rescaling, $\forall p \in \gamma$, $\gamma$ locally avoids $O_p(F_k(\gamma); \gamma)$ near $p$ and $F_k(\gamma) \leq F_g(p) = \inf \{ r > 0 : p \in O_p(r; \gamma) \cap \gamma \}$. Hence, $F_k(\gamma) \leq F_g(\gamma) = \inf_{p \in \gamma} F_g(p)$.

Suppose that $F_k(\gamma) < F_g(\gamma)$, i.e. $\sup_{s \in \gamma} \kappa(\gamma) > \frac{1}{F_g(\gamma)}$. Define

$$A = \left\{ s \in I : \kappa(\gamma(s)) > \frac{1}{F_g(\gamma)} \right\} \text{ and } B = \left\{ s \in I : \kappa''(s) \text{ exists} \right\}.$$ .

$A \neq \emptyset$ and the Lebesgue measure $\mu(B^c) = 0$, where $X^c = I - X$.

**Case 1.** $A \cap B = \emptyset$. There exists $s_0 \in A \cap B$ such that $c := ||\gamma''(s_0)|| = \kappa(\gamma(s_0)) > \frac{1}{F_g(\gamma)}$. Choose $r$ such that $\frac{1}{c} < r < F_g(\gamma)$. Let $\eta(s) = c\gamma(s)$, so that $||\eta'(s)|| = 1, \forall s$, and $||\eta''(c\gamma)|| = 1$. By Proposition 2c, $\eta(c\gamma, c\gamma + cr) \subset B(\eta(c\gamma) + cr\gamma''(c\gamma), cr)$ for some $\varepsilon > 0$. Hence, $(c\gamma, c\gamma + \varepsilon) \subset B(\gamma(s_0) + \frac{1}{2\kappa(\gamma)} r, r) \subset O_{\gamma(s_0)}(r; \gamma)$. However this contradicts $r < F_k(\gamma)$ by the definition of $F_g$.

**Case 2.** $A \cap B = \emptyset$. Since $\gamma$ is $C^{1,1}$, $\gamma'$ is absolutely continuous, $\gamma''(s)$ exists almost everywhere by Rademacher's Theorem and $||\gamma''(s)|| = \kappa(\gamma(s)) \leq \frac{1}{F_g(\gamma)}$ a.e. By Lemma 1, $\frac{1}{2\kappa(\gamma)} = \sup_{s \in \gamma} \kappa(\gamma(s)) \leq \frac{1}{2\kappa(\gamma)}$, which contradicts $F_k(\gamma) < F_g(\gamma)$.

Neither of the cases is possible, hence one must have $F_k(\gamma) = F_g(\gamma(I))$. \hfill \qedsymbol

**Definition 11.** Let $p, q \in \mathbb{R}^n$, $v \in UTR^r_{\mathbb{R}^n}$, $w \in UTR^r_{\mathbb{R}^n}$ and $\Lambda > 0$ be given. Define $C(p, q; v, w; \Lambda)$ to be the set of all $C^{1,1}$ curves $\gamma : [0, L] \rightarrow \mathbb{R}^n$ with $\gamma(0) = p, \gamma'(0) = v, \gamma(L) = q$, $\gamma'(L) = w$, $||\gamma'|| = 1$, and $\kappa \gamma \leq \Lambda$, where $L = \ell(\gamma)$ is not fixed on $C$.

**Proposition 3.** There exists a shortest curve in $C(p, q; v, w; \Lambda)$.
Choose any union of disjoint line segments, and \( \kappa_\gamma \) of \( \mathcal{C}, n \), \( h \). Let \( \gamma \in \mathcal{C} \), \( \| \gamma'(s) - \gamma'(t) \| \leq \Lambda |s-t| \), and thus, \( \mathcal{C} \) is \( C^1 \)-equicontinuous. \( \mathcal{C} \) is \( C^1 \)-bounded by \( \| \gamma' \| = 1 \). \( C^0 \)-equicontinuity and boundedness are obvious. By Arzela-Ascoli Theorem, there exists a subsequence of \( \{ \gamma_m \}_{m=1}^\infty \) uniformly converging to \( \gamma_0 \) in \( C^1 \) sense: \( (\gamma_m(s), \gamma_m'(s)) \rightarrow (\gamma_0(s), \gamma_0'(s)) \). \( \gamma_0 \in \mathcal{C} \), since all conditions of \( \mathcal{C} \) are preserved under this convergence and \( \ell(\gamma_m) \rightarrow \ell(\gamma_0) \).

**Proposition 4.** Let \( \gamma : I = [0, L] \rightarrow \mathbb{R}^n \) be a shortest curve in \( \mathcal{C}(p, q; v, w; \Lambda) \). Then, \( \forall s \in I \), \( (\kappa_\gamma(s) = 0 \text{ or } \Lambda) \). \( \kappa_\gamma^{-1}(\Lambda) \) is a closed subset of \( I \), and \( \kappa_\gamma^{-1}(0) \) is countable union of disjoint line segments.

**Proof.** By the upper semi-continuity of \( \kappa_\gamma \), \( \forall \lambda \leq \Lambda \), \( \kappa_\gamma^{-1}([\lambda, \Lambda]) \) is a closed subset of \( I \) and \( J(\lambda) = \kappa_\gamma^{-1}(0, \lambda)) \) is countable union of relatively open intervals in \( I \).

Choose any \( \lambda < \Lambda \) and \( a < b \) in a given component \( J' \) of \( J(\lambda) \).

Suppose that \( \gamma'(a) \neq \gamma'(b) \). Choose any smooth bump function \( h : \mathbb{R} \rightarrow [0, 1] \) such that \( \text{supp}(h) \subset [-1,1] \), \( h(0) = 1 \), and \( \int_{-1}^1 h(s)ds = 1 \). Let \( h_n \) be defined by \( h_n(\frac{a+b}{2}) = 1 \) and \( h_n'(s) = n[h(n(s-a)) - h(n(s-b))] \).

Then,

\[
\lim_{n \to \infty} \int_I h'_n(s)\gamma'(s)ds = \gamma'(b) - \gamma'(a) \neq 0.
\]

Choose and fix \( n \) sufficiently large such that \( \text{supp}(h_n) \subset J' \) and \( -\int_J h'(s)\gamma'(s)ds = V \neq 0 \). Let \( \gamma_\varepsilon(s) = \gamma(s) + \varepsilon V h_n(s) \) be a variation of \( \gamma \). By the First Variation formula, \( [CE, p6], \)

\[
\frac{d}{d\varepsilon} \ell(\gamma_\varepsilon)|_{\varepsilon=0} = \int_I [V h_n(s)]'\gamma'(s)ds = -\|V\|^2 < 0
\]

Hence, for sufficiently small \( \varepsilon \), \( \gamma_\varepsilon \) is strictly shorter than \( \gamma \). For all \( s < t \):

\[
dil d_{\gamma_\varepsilon}(s,t) = \frac{\|\gamma_\varepsilon'(s) - \gamma_\varepsilon'(t)\|}{\ell_\varepsilon(\gamma_\varepsilon)} \leq \frac{\|\gamma'(s) - \gamma'(t)\| + \varepsilon \|V\|}{t-s} = \frac{\|\gamma'(s) - \gamma'(t)\|}{t-s} + \varepsilon C(\|V\|, \sup |h'_n|, \sup |h''_n|)
\]

By Remark 1.i, for sufficiently small \( \varepsilon \), \( \kappa_\varepsilon \leq \frac{\Lambda + \Lambda}{2} < \Lambda \), and \( \gamma_\varepsilon \in \mathcal{C} \). This contradicts the minimality of \( \gamma \). Consequently, \( \gamma' \) is constant on \( J' \), \( \forall \lambda < \Lambda \). \( \gamma(J(\lambda)) \) is a countable union of disjoint line segments, to conclude that \( \gamma(J(\Lambda)) \) is a countable union of disjoint line segments, and \( \kappa_\gamma(J(\Lambda)) \equiv 0 \).

**3.1. Proof of Theorem 1.**

**Proof.** By using dilations of \( \mathbb{R}^n \), one can assume that \( \Lambda = 1 \). We are going to proceed in proving parts (a) and (b) simultaneously, and point out the differences when they are needed. Let \( A = \pi \) for part (a) and \( A = 2\pi \) for part (b). By Proposition 4, there exist maximally chosen \( c \) and \( d \) such that \( s_0 \in [c, d] \subset [0, L] \) and \( \gamma([c, d]) \) is a line segment \( L_0 \).

Assume that \( \gamma([a, b]) \) is a \( CLC(1) \)-curve for \([c, d] \subset [a, b] \subset [0, L] \). We will show that if \( a > 0 \) and \( c - a < A \), then \( \exists \delta > 0 \) such that \( \gamma([a-\delta, b]) \) still is a \( CLC(1) \) curve.
For \( r \in [0, a] \) define \( J_r = \{-\lambda \gamma'(a - r) : \lambda > 0\} \) and \( V_r = O_{\gamma(a-r)}^\epsilon(\gamma'(a - r)) \).

Choose \( d' = d \) when \( \gamma(c, d) \cap J_0 = \emptyset \); otherwise, by \( \gamma(d') \in \gamma((c, d)) \cap J_0 \neq \emptyset \).

Let \( \epsilon = \frac{1}{2} \min(d' - c, A - (c - a)) \), \( c_1 = c + \epsilon \), and \( m = \gamma(c_1) \). \( m \in \text{int}V_0 \), since \( \gamma([a, c]) \) is an arc of a circle of radius 1 and \( \gamma([c, c_1]) \) is a line segment, \( a \leq c < c_1 \), and \( \text{Proposition 2a} \). One obtains that \( \forall r \in [0, \epsilon] \), \( \gamma([a - r, c_1]) \subseteq V_r \). \( \text{Proposition 2a} \) and \( c_1 - (a - r) < c - a + 2\epsilon \leq \pi \) for part (a), and by \( R_O(\gamma) \geq 1 \) for part (b). \( m \in \text{int}V_r \), since \( \gamma([c, c_1]) = L_0 \) is a line segment.

For each fixed \( r \in [0, \epsilon] \), define \( \gamma_r \) to be a shortest curve parametrized by arclength from \( \gamma(a - r) \) to \( m \) within \( V_r \) \textbf{without curvature restrictions}, by using \( \text{Proposition 1} \). \( \gamma_r \) follows a circular arc of radius 1 starting from \( \gamma(a - r) \) along \( \partial V_r \), then a line segment \( L_r \) of positive length until \( m \). In \( \text{Proposition 1} \), \( \gamma_r \) at \( m \) is not controlled.

**Claim 1.** \( \gamma_r'(m) = \gamma'(m) \) for sufficiently small \( r > 0 \).

\( \exists \delta_1 > 0 \) such that \( \forall r \in [0, \delta_1], \ell(L_r) \geq \frac{\epsilon}{2}, d(m, J_r) > 0 \), since \( J_r \) and \( \partial V_r \) change continuously in \( r \), \( \ell(L_0) = \epsilon \) and \( d(m, J_0) > 0 \). For \( r \in [0, \delta_1] \), \( \gamma_r \) is uniquely defined.

\( \lim_{r \to 0^+} \ell_m(L_r) = 0 \), otherwise one can construct a shortest curve other than \( \gamma \) from \( \gamma(a) \) to \( m \) in \( V_0 \) contradicting \( \text{Proposition 1b} \). \( \exists \delta_2 > 0 \) such that \( \forall r \in [0, \delta_2], \ell_m(L_r) \leq 2 \tan^{-1} \frac{\epsilon}{2} \).

Let \( \delta = \min(\epsilon, \delta_1, \delta_2) \). \( \forall r \in [0, \delta] \), define a curve \( \tilde{\gamma}_r \) which follows \( \gamma \) from \( p \) to \( \gamma(a - r) \), then \( \gamma_r \) from \( \gamma(a - r) \) to \( m \), and \( \gamma \) from \( m \) to \( q \). \( \gamma_r \) is \( C^1 \) at \( \gamma(a - r) \), squeezed by \( O_{\gamma(a-r)}(\gamma'(a - r)) \). Recall that \( \epsilon \leq \frac{1}{2} \tan^{-1} \frac{\epsilon}{2} \) and \( L_0 = \gamma([c, c_1]) \). Define the line segment \( L_0' := \gamma(\{c_1, c_1 + \frac{\epsilon}{2}\}) \).

Suppose that \( \tilde{\gamma}_r \) is not \( C^1 \) at \( m \) for some \( r \in (0, \delta) \), that is \( \ell_m(L_0, L_r) = \pi - \ell_m(L_0', L_r) = \alpha > 0 \). Fix such an \( r \). \( \gamma \cap B(m, \frac{\epsilon}{2}) \) is a union of two segments of length \( \frac{\epsilon}{2} \), joined at \( m \) with an angle of \( \pi - \alpha \), in \( L_0' \cup L_r \). There exists a unique circle \( C \) of radius 1 in the same 2-plane as \( L_0' \cup L_r \), tangent to \( L_r \) at \( p_1 \) and tangent to \( L_0' \) at \( p_2 \) where \( \|p_1 - m\| = \frac{\epsilon}{2}, \alpha \leq 2 \tan^{-1} \frac{\epsilon}{2} \). Let \( \tilde{\gamma} \) be the \( C^1 \) curve obtained from \( \tilde{\gamma}_r \) by replacing \( L_0' \cup L_r \) between \( p_1 \) and \( p_2 \) by the shorter arc of \( C \) between \( p_1 \) and \( p_2 \).

\( \ell(\tilde{\gamma}) < \ell(\tilde{\gamma}_r) \leq \ell(\gamma) \) and \( \kappa \tilde{\gamma} \leq 1 \)

This contradicts the minimality of \( \gamma \) in \( C \). Hence, \( \forall r \in [0, \delta] \), \( \tilde{\gamma}_r \) is \( C^1 \) at \( m \), and \( \ell_m(L_0, L_r) = 0 \). This proves \( \text{Claim 1} \).

For each given \( r \in (0, \delta) \):

1. \( \tilde{\gamma}_r \in C^1 \) and \( \kappa \tilde{\gamma}_r \leq 1 \), hence \( \tilde{\gamma}_r \in C \) and \( \ell(\tilde{\gamma}_r) \geq \ell(\gamma) \).
2. \( \gamma \) and \( \tilde{\gamma}_r \) follow the same path before \( \gamma(a - r) \) as well as after \( m \).
3. \( \gamma([a - r, c_1]) \subseteq V_r \), \( \gamma_r \) is the unique shortest curve from \( \gamma(a - r) \) to \( m \) in \( V_r \), and hence \( \ell(\gamma([a-r, c_1])) \geq \ell(\gamma_r) \).

Consequently, \( \ell(\tilde{\gamma}_r) = \ell(\gamma) \), \( \gamma \) and \( \tilde{\gamma}_r \) are equal up to parametrization, and \( \forall r \in [0, \delta], \gamma([a - r, b]) \) is a \( CLC(1) \)-curve. Obviously, this extends to \( [a - \delta, b] \) and to \( [a - \delta, b + \delta'] \) for some \( \delta' > 0 \), by symmetry when \( b < L \) and \( d - b < A \). One chooses \( a_0 \) and \( b_0 \) maximally so that \( [c, d] \subseteq [a_0, b_0] \subseteq [0, L] \) and \( \gamma([a_0, b_0]) \) is a \( CLC(1) \) curve. It follows from the construction of \( \delta \) that:

Part (a): \( 0 = a_0 \) or \( c - a_0 \geq A = \pi \) and \( (L_0 = b_0 \) or \( d - b_0 \geq A = \pi \))

Part (b): \( 0 = a_0 \) and \( L = b_0 \), since \( c - a_0 = A = 2\pi \) case creates a complete circle through \( \gamma(c) \), which contradicts the minimality of \( \gamma \). \( \square \)
4. RELATIVELY EXTREMAL KNOTS IN $\mathbb{R}^n$

A knot class $[\gamma]$ is a free $C^0$–homotopy class of embeddings of $\gamma : S^1 \to \mathbb{R}^n$. In this section, $\gamma : S^1 \to \mathbb{R}^n$ denotes a simple–$C^1$–closed curve, by identifying $S^1 \cong \mathbb{R} / \mathbb{Z}$ and $K = \text{image}(\gamma)$. In other words, $\gamma(t + L) = \gamma(t)$ and $\gamma'(t + L) = \gamma'(t)$, $\forall t \in \mathbb{R}$ with $\|\gamma'\| \neq 0$ and $\gamma$ is one-to-one on $[0, L)$. Interval notation will be used to describe subsets of $\mathbb{R} / \mathbb{Z}$.

**Definition 12.** For any simple–$C^1$–closed curve $\gamma : S^1 \to \mathbb{R}^n$, one defines the ropelength or extrinsically isometric length to be $\ell_e(\gamma) = \frac{\ell(\gamma)}{\text{vol}(S^1)} = \frac{\text{vol}(\gamma(S^1) \times \mathbb{R}^n)}{\text{vol}(S^1)}$.

**Definition 13.**

i. A simple–$C^1$–closed curve $\gamma_0$ is called an ideal (thickest) knot in $[\theta]$, if $\ell_e(\gamma_0) \leq \ell_e(\gamma)$, $\forall \gamma \in [\theta] \cap C^{1,1}$.

ii. $\gamma_0$ is called relatively extremal, if there exists an open set $U$ in $C^1$ topology such that $\gamma_0 \in U$ and $\ell_e(\gamma_0) \leq \ell_e(\gamma)$, $\forall \gamma \in U \cap [\theta] \cap C^{1,1}$.

We consider two curves $\gamma_1$ and $\gamma_2$ to be geometrically equivalent if there exists an orientation preserving $h : \mathbb{R}^n \to \mathbb{R}^n$, a composition of an isometry and a dilation $(x \to \lambda x, \lambda \neq 0)$ of $\mathbb{R}^n$, such that $h(\gamma_1) = \gamma_2$ up to a reparametrization. On each geometric equivalence class of $C^{1,1}$–closed curves, $\ell_e$ remains constant.

**Theorem 4.** (Thickness Formula) For every simple–$C^{1,1}$–closed curve $\gamma$ in $\mathbb{R}^n$, and $K = \text{image}(\gamma)$, one has $i(K, M) = R_O(K, M) = \min\{F_K(\gamma), \frac{1}{2}\text{MDC}(K)\}$.

**Proof.** See [L], for $n = 3$ case. This is a consequence of Thickness Formula [D, Theorem 1] and Lemma 2.

**Proposition 5.** Let $\{\gamma_m\}_{m=1}^{\infty} : S^1 \to \mathbb{R}^n$ be a sequence uniformly converging to $\gamma_0$ in $C^1$ sense, i.e. $(\gamma_m(s), \gamma'_m(s)) \to (\gamma_0(s), \gamma'_0(s))$ uniformly on $S^1$. Let $K_m = \gamma_m(S^1)$.

i. If $R_O(K_m) \geq r$ for sufficiently large $m$, then $R_O(K_0) \geq r$. Consequently, $\lim\sup_m R_O(K_m) \leq R_O(K_0)$.

ii. If $\lim\inf_m \text{MDC}(K_m) > 0$, then $\lim\inf_m \text{MDC}(K_m) \geq \text{MDC}(K_0)$.

**Proof.** i. Suppose that $R_O(K_0) < r$, for a given $r > 0$. By the definition $R_O$, there exists an $a \in S^1, v \in \mathbb{R}^n$ with $\|v\| = 1$ and $v \cdot \gamma'_0(a) = 0$ such that $B(\gamma_0(a) + rv, r) \cap K_0 \neq \emptyset$. One can find $\gamma_0(b) \in B(\gamma_0(a) + rv, r - \varepsilon)$ for sufficiently small $\varepsilon > 0$. Choose a sequence $\{v_m\}_{m=1}^{\infty} \subset \mathbb{R}^n$ such that $\forall m, \|v_m\| = 1, v_m \cdot \gamma'_m(a) = 0$, and $v_m \to v$. Then for sufficiently large $m$,

$$\|\gamma_m(a) + rv_m - \gamma_m(b)\| < \|\gamma(a) + rv - \gamma_0(b)\| + \varepsilon < r.$$

Hence, $B(\gamma_m(a) + rv_m, r) \cap \gamma_m \neq \emptyset$ and $R_O(K_m) < r$, for sufficiently large $m$, which contradicts the hypothesis. Consequently, $R_O(K_0) \geq r$.

ii. We will use the same indices for subsequences. Let $a = \lim\inf_m \text{MDC}(K_m)$, and choose a subsequence with $a = \lim_m \text{MDC}(K_m)$ and $\text{MDC}(K_m) > 0, \forall m$. By compactness of $K_m$ and positivity of $\text{MDC}(K_m)$, there exists a minimal double critical pair $\{p_m, q_m\}$ for $K_m, \ell(\gamma_{p_m q_m}) = \text{MDC}(K_m), \forall m$. Since $K_0$ is compact and $a > 0$, there exists subsequences $p_m \to p_0 \in K_0, q_m \to q_0 \in K_0$, and $\gamma_{p_m q_m} \to \gamma_{p_0 q_0}$ in $C^1$ sense. Line segments converge to line segments, and normality to $C^1$ curves is preserved under $C^1$ limits. Hence $\{p_m, q_m\}$ is a double critical pair for $K_0$.

$$\text{MDC}(K_0) \leq \ell(\gamma_{p_0 q_0}) = \lim_m \ell(\gamma_{p_m q_m}) = \lim_m \text{MDC}(K_m) = a.$$  

□
Definition 14. Let $\gamma : S^1 \to \mathbb{R}^n$ be a simple--$C^{1,:1}=$ closed curve in $\mathbb{R}^n$, with $\gamma' \neq 0$. Define

i. $I_c = \{x \in S^1 : \exists y \in S^1 \text{ such that } \|\gamma(x) - \gamma(y)\| = MDC(\gamma) \text{ and } (\gamma(x) - \gamma(y)) \cdot \gamma'(x) = 0\}$ and $K_c = \gamma_c = \gamma(I_c)$

ii. $I_z = \{x \in S^1 : \kappa(\gamma)(x) = 0\}$ and $K_z = \gamma_z = \gamma(I_z)$

iii. $I_{mx} = \{x \in S^1 : \kappa(\gamma)(x) = 1/RO(\gamma)\}$ and $K_{mx} = \gamma_{mx} = \gamma(I_{mx})$

iv. $I_0 = \{x \in S^1 : 0 < \kappa(\gamma)(x) < 1/RO(\gamma)\}$ and $K_0 = \gamma_0 = \gamma(I_0)$

Remark 2. $K_c$ and $K_{mx}$ are closed subsets of $K$. This is obvious for $K_c$ by the continuity of $\gamma'$. See the proof of Proposition 8, for $K_{mx}$.

Proposition 6. For any knot class $[\theta]$, $\exists \gamma_0 \in [\theta] \cap C^{1,:1}$ such that

i. $\forall \gamma \in [\theta] \cap C^{1,:1}$, $0 < \ell_\epsilon(\gamma_0) \leq \ell_\epsilon(\gamma)$, and hence

ii. $\forall \gamma \in [\theta] \cap C^{1,:1}$, $(\ell(\gamma_0) \leq \ell(\gamma) \implies RO(\gamma_0) \geq RO(\gamma))$.

Proof. Let $T_0 = \{\gamma \in [\theta] \cap C^{1,:1} : (\gamma(0) = 0, \|\gamma\| \equiv 1, \ell(\gamma) = 1 \text{ and } RO(\gamma) \geq x\}$. Every geometric equivalence class of $C^{1,:1}=$ closed curves has a representative in $T_0$. Choose any $\gamma_0 \in [\theta] \cap C^{\infty} \cap T_0$ and set $RO(\gamma_0) = A$.

By the Thickness Formula, $0 < A \leq M := \sup\{RO(\gamma) : \gamma \in T_A\} \leq \frac{1}{2}$, $\forall \gamma \in T_0, MDC(\gamma) \leq \frac{1}{2}$. Choose any $\gamma_1 \in [\theta] \cap C^{\infty} \cap T_0$ and set $RO(\gamma_1) = A$. Consequently, $T_A$ is $C^1$-equicontinuous and $C^1$-bounded: $\|\gamma\| \equiv 1$. $C^0$-equicontinuity and boundedness is obvious.

For all $k \in \mathbb{N}^+$, there exists $\gamma_k \in T_A$ such that $RO(\gamma_k) \geq M - \frac{1}{k}$. By Arzela-Ascoli Theorem, there exists a subsequence of $\{\gamma_m\}_{m=1}^\infty$ uniformly converging to $\gamma_0$ in $C^1$ sense: $(\gamma_m(s), \gamma'_m(s)) \to (\gamma_0(s), \gamma'_0(s))$. By Proposition 5, $RO(\gamma_0) \geq M$. The rest is straightforward: $\gamma_0 \in T_M \subset T_A \subset T_0$ and $RO(\gamma_0) = M$.

Finally, all curves in the geometric equivalence class of $\gamma_0$ are $\ell_\epsilon$-minimizers in $[\theta] \cap C^{1,:1}$. □

Proposition 7. Let $\{\gamma_m\}_{m=1}^\infty : S^1 \to \mathbb{R}^n$ be a sequence uniformly converging to $\gamma$ in $C^1$ sense, $K = \gamma(S^1)$ and $K_m = \gamma_m(S^1)$ satisfying

a. $3C < \infty, \forall m, \sup \kappa \gamma_m \leq C$, and

b. $\exists$ compact $A \subset S^1$ such that $\{s \in S^1 : \gamma_m(s) \neq \gamma(s)\} \subset A, \forall m$.

Then both of the following hold.

i. If $A \cap I_c = \emptyset$, then $\exists m_1 \forall m \geq m_1, (MDC(K_m) \geq MDC(K))$.

ii. If $F_k(\gamma) < \frac{1}{2} MDC(K)$ and $(F_k(\gamma_m) \geq F_k(\gamma)), \forall m,

\text{ then } \exists m_1 \forall m \geq m_1, (RO(K_m) \geq RO(K))$.

Proof. All subsequences will be denoted by the same index $m$. The critical pairs will be identified from the domain $S^1$.

i. Suppose there exists a sequence $\gamma_m$ such that $\forall m (MDC(K_m) < MDC(K))$. For all $m$, there exists a minimal double critical pair $\{x_m, y_m\}$ in $S^1$ for $\gamma_m$, $MDC(K_m) = |\gamma_m(x_m) - \gamma_m(y_m)| < MDC(K)$. Then, $\forall m (d_{S^1}(x_m, y_m) \geq \frac{\pi}{2})$ by Proposition 2b. There exist subsequences $x_m \to x_0, y_m \to y_0$ and $d_{S^1}(x_0, y_0) \geq \frac{\pi}{2}$. By the uniform convergence of $\gamma'_m \to \gamma'$, $\{x_0, y_0\}$ is a double critical pair for $\gamma$.

$MDC(K) \leq |\gamma(x_0) - \gamma(y_0)| = \lim_m |\gamma_m(x_m) - \gamma_m(y_m)| = \lim_m MDC(K_m) \leq MDC(K)$

Hence, $\{x_0, y_0\}$ is a minimal double critical pair for $\gamma$ and $\{x_0, y_0\} \subset I_c$. Since $I_c$ and $A$ are disjoint compact subsets of $S^1$, the subsequences $\{x_m\}_{m=1}^\infty$ and $\{y_m\}_{m=1}^\infty$
can be taken in $S^1 - A$. \forall m, \{x_m, y_m\}$ is a double critical pair for $\gamma$, since $\gamma_m = \gamma$ on $S^1 - A$.

$$MDC(K) \leq |\gamma(x_m) - \gamma(y_m)| = |\gamma_m(x_m) - \gamma_m(y_m)| = MDC(K_m)$$

which contradicts the initial assumption. Consequently, there does not exist any subsequence $\gamma_m$ such that $\forall m, (MDC(K_m) < MDC(K))$, proving (i).

ii. $MDC(K) > 2F_k(\gamma) = 2R_O(K)$ and $\forall m, (F_k(\gamma_m) \geq F_k(\gamma))$. Suppose that there exists a subsequence $\gamma_m$ such that $\forall m (MDC(K_m) < 2R_O(K))$. For all $m$, there exists a minimal double critical pair $\{x_m, y_m\}$ in $S^1$ for $\gamma_m$. $MDC(K_m) = |\gamma_m(x_m) - \gamma_m(y_m)| < 2R_O(K)$. Then as in part (i), $\forall m (d_{S_1}(x_m, y_m) \geq \frac{\pi}{6})$ and by taking subsequences $x_m \to x_0$, $y_m \to y_0$, $d_{S_1}(x_0, y_0) \geq \frac{\pi}{6}$, one obtains a double critical pair $\{x_0, y_0\}$ for $\gamma$.

$$MDC(K) \leq |\gamma(x_0) - \gamma(y_0)| = \lim_{m \to \infty} |\gamma_m(x_m) - \gamma_m(y_m)| = \lim_{m \to \infty} MDC(K_m) \leq 2R_O(K)$$

which contradicts the hypothesis. Hence, $\exists m_1 \forall m \geq m_1, (MDC(K_m) \geq 2R_O(K))$, to conclude that

$$R_O(K_m) = \min \left( F_k(\gamma_m), \frac{1}{2} MDC(K_m) \right) \geq \min (F_k(\gamma), R_O(K)) = R_O(K).$$

\[ \square \]

**Proposition 8.** (Also see [GM, p11, 12] for another version for smooth ideal knots.) Let $\gamma$ be a relatively extremal knot.

i. If $MDC(K) = 2R_O(K)$, then $K - (K_c \cup K_{mz})$ is a countable union of open ended line segments, and hence $I_c \subset I_e$.

ii. If $MDC(K) > 2R_O(K)$, then $K - K_{mz}$ is a countable union of open ended line segments.

**Remark 3.** Theorem 2 shows that $K - K_{mz}$ is actually empty when $MDC(K) > 2R_O(K)$.

**Proof.** Let $U$ be an open set in $C^1$ topology such that $\gamma \in U$ and $\ell_e(\gamma) \leq \ell_e(\eta), \forall \eta \in U \cap [\emptyset] \cap C^{1,1}$.

i. Let $\Lambda = \sup \kappa \gamma$. As in the proof of Proposition 4, for all $\lambda \leq \Lambda$, $\kappa \gamma^{-1}([0, \lambda]) - I_e$ is countable union of relatively open intervals in $S^1(= R/LZ)$. Choose any $\lambda < \Lambda$ and a closed interval $[a, b]$ contained in a component of $\kappa \gamma^{-1}([0, \lambda]) - I_e$. By repeating the proof of Proposition 4, if $\gamma[a, b]$ is not a line segment, then there exists a length decreasing variation $\gamma_\varepsilon(s) = \gamma(s) + \varepsilon V\eta, n(s)$ supported in $[a, b]$.

There exists a sufficiently small $\varepsilon_1 > 0$ such that $\forall \varepsilon, 0 < \varepsilon \leq \varepsilon_1$, one has

1. $\gamma_\varepsilon$ and $\gamma$ belong to the same knot class and $\gamma_\varepsilon \in U$.
2. $\ell(\gamma_\varepsilon) < \ell(\gamma)$, (proof of Proposition 4)
3. $\kappa \gamma_\varepsilon \leq \Lambda$ and hence $F_k(\gamma_\varepsilon) \geq F_k(\gamma)$, (proof of Proposition 4), and
4. $MDC(K_{\gamma_\varepsilon}) \geq MDC(K)$ (Proposition 7(i) and $[a, b] \cap I_e = \emptyset$).

By the Thickness Formula, one obtains $R_O(K_{\gamma_\varepsilon}) \geq R_O(K)$ and $\ell_e(\gamma_{\gamma_\varepsilon}) = \frac{\ell(\gamma_{\gamma_\varepsilon})}{R_O(K_{\gamma_\varepsilon})} = \ell_e(\gamma)$ which is in contradiction with the hypothesis. Hence, $\gamma[a, b]$ must be a line segment. Consequently, $I_0 - I_e = \emptyset$.

ii. $MDC(K) > 2R_O(K) = 2F_k(\gamma)$. The proof is essentially the same as in (i), with the following modifications. $[a, b]$ is taken in any component of $\kappa \gamma^{-1}([0, \lambda])$, thus $[a, b] \cap I_e(\gamma)$ may not be empty. 1-3 above hold. To conclude $R_O(K_{\gamma_\varepsilon}) \geq R_O(K)$,
one uses Proposition 7(ii). In this case, $I_b = \emptyset$ and $K - K_{mx}$ is a countable union of open ended line segments.

4.1. Proof of Theorem 2.

Proof. Let $U$ be an open set in $C^1$ topology such that $\gamma \in U$ and $\ell_c(\gamma) \leq \ell_c(\eta)$, $\forall \eta \in U \cap [\emptyset] \cap C^{1,1}$. We prove part (ii) first.

By Proposition 8, there exist maximally chosen $a, b$ such that $\gamma([a, b])$ is an open ended line segment, $\gamma_0 \in (a, b)$ and $(a, b) \cap I_c = \emptyset$. If $b \in I_c$, then take $d = b$, to finish the positive direction. If $b \notin I_c$, proceed as follows. Assume that $\gamma([s_0, b + \varepsilon])$ is a $CLC(F_k(\gamma)^{-1})$-curve (in fact, line segment followed by circular arc) such that $0 \leq \varepsilon < \pi F_k(\gamma)$ and $[s_0, b + \varepsilon] \cap I_c = \emptyset$. We will show that the same is true for some $\varepsilon_1 > \varepsilon$. We point out that a priori $\gamma([s_0, b + \varepsilon])$ is not known to be a shortest curve in a certain $C$, replacing it with a shortest curve may create a knot outside $U$ or the knot class of $\gamma$, and this shortest curve may not have a point of zero curvature.

Let $\{b_m\}_{m=1}^\infty$ be a sequence and $A > 0$ be such that
1. $b_{m+1} < b_m$, $\forall m \in \mathbb{N}^+$,
2. $b_m \to b_0 = b + \varepsilon$,
3. $\varepsilon < b_m - b \leq A \leq \pi F_k(\gamma)$, and
4. $[s_0, b + A] \cap I_c = \emptyset$.

Since $\gamma(s_0) \in int O'_{\gamma(b_0)}(\gamma(b_0))$, $\gamma(s_0) \in int O'_{\gamma(b_m)}(\gamma(b_m))$ for sufficiently large $m \geq m_0$. Let $f_m(s)$ be the unique shortest curve parametrized by arclength in $C(\gamma(s_0), \gamma(b_m); \gamma(s_0), \gamma'(b_m); F_k(\gamma)^{-1})$ by Proposition 3, such that $f_m(s_0) = \gamma(s_0)$ and $f_m(c_m) = \gamma(b_m)$. Extend $f_m$ to $[s_0, b + A]$ in a $C^1$ fashion beyond $\gamma(b_m)$ by $\gamma(s - c_m + b_m) = f_m(s)$. 

$\forall m, \kappa f_m \leq F_k(\gamma)^{-1}, \|f_m\| = 1$ and $f_m(s_0) = \gamma(s_0)$. Hence, the sequence $\{f_m\}_{m=1}^\infty$ is $C^1$ equicontinuous and bounded. By Arzela-Ascoli Theorem, there exists a convergent subsequence (which we will denote by the same subindices $m$) $f_m \to f_0$ uniformly in $C^1$ topology. By the construction above, $f_0$ follows $\gamma$ past $\gamma(b_0)$ and $f_0(c_0) = \gamma(b_0)$ for some $c_0$.

\[ c_m - s_0 \leq b_m - s_0 \]
\[ \lim_{m} \sup \ c_m \leq b_0 \]
\[ c_0 \leq b_0 \]
\[ f_0(s_0) = \gamma(s_0) \text{ and } f_0(c_0) = \gamma(b_0) \]
\[ f_0 \in C^1 \text{ and } f_0'(c_0) = \gamma'(b_0) \]
\[ \kappa f_0 \leq F_k(\gamma)^{-1} \]

By Proposition 1, $\gamma$ which is a line segment followed by a circular arc is the unique shortest curve satisfying the last 3 conditions. Consequently, $b_0 = c_0$ and $f_0 = \gamma$ on $[s_0, b + A]$.

Let $\gamma_m$ be the curve obtained from $\gamma$ by replacing $\gamma([s_0, b_m])$ by $f_m([s_0, c_m])$. Reparametrize $\gamma_m$ (not necessarily with respect to arclength) so that $\gamma_m(s) = \gamma(s)$ for $s \notin [s_0, b + A]$ and $\gamma_m \to \gamma$ in $C^1$ sense on $S^1$, which is possible since $2m - s_0 \to 1$. For sufficiently large $m \geq m_1$,
1. $\gamma_m$ and $\gamma$ belong to the same knot class and $\gamma_m \in U$.
2. $\{s: \gamma_m(s) \neq \gamma(s)\} \subset [s_0, b + A]$ which is disjoint from $I_c$.
3. $F_k(\gamma_m) \geq F_k(\gamma)$, by construction of $f_m$. 
4. \(MDC(K_m) \geq MDC(K)\), by Proposition 7(i).
5. \(R_O(K_m) \geq R_O(K)\), by Thickness Formula.
6. \(\ell_c(\gamma_m) \geq \ell_c(\gamma)\), since \(\gamma\) is relatively extremal and (1).
7. \(\ell(\gamma_m) \geq \ell(\gamma)\) by (5), (6) and the definition of \(\ell_c\).
8. \(\ell(\gamma_m) \leq \ell(\gamma)\) by construction of \(f_m\) and \(\gamma_m\).
9. \(f_m|_{[s_0, c_m]}\) and \(\gamma|_{[s_0, b_m]}\) have the same minimal length in
   \(C(\gamma(s_0), \gamma(b_m); \gamma'(s_0), \gamma'(b_m); F_k(\gamma)^{-1})\).
10. \(\gamma|_{[s_0, b_m]}\) is a \(CLC(F_k(\gamma)^{-1})\)-curve, by Theorem 1 and \(k\gamma(s_0) = 0\).

   We proved that if \(\gamma|_{[s_0, b + \varepsilon]}\) is a \(CLC(F_k(\gamma)^{-1})\)-curve (line segment followed by circular arc) such that \(0 \leq \varepsilon < \pi F_k(\gamma)\) and \([s_0, b + \varepsilon] \cap I_c = \emptyset\), then there exists \(\varepsilon_1 = b_m - b > b_0 - b = \varepsilon\) such that \(\gamma|_{[s_0, b + \varepsilon_1]}\) is a \(CLC(F_k(\gamma)^{-1})\)-curve. In fact, \(\gamma|_{[s_0, b + \varepsilon_1]}\) must be one line segment followed by one circular arc by the definition of \(CLC\) and the shape of \(\gamma|_{[s_0, b + \varepsilon]}\).

   Hence, \(\varepsilon_0 := \max\{\delta : \gamma|_{[s_0, b + \delta]}\) is a \(CLC(F_k(\gamma)^{-1})\) - curve\} and \(\delta = b + \varepsilon_0 \in I_c\). The proof is the same for the opposite direction before \(a\).

   i. Suppose that \(\frac{1}{2}MDC(K) > R_O(K) = F_k(\gamma)\). One proceeds as in proof of part (ii), omitting all conditions about avoiding \(I_c\). Use Proposition 8(ii), to obtain the line segment \(\gamma|[a, b]\). Even though \(MDC(K_m) \geq MDC(K)\) may not be valid by Proposition 7(i), \(R_O(K_m) \geq R_O(K)\) is valid by Proposition 7(ii). This shows that \(\gamma|[b, b + \pi F_k(\gamma)]\) is a \(CLC(F_k(\gamma)^{-1})\) - curve, even passing through MDC-points. \(\gamma(b)\) and \(\gamma(b + \pi F_k(\gamma))\) is an antipodal pair of a circle of radius \(F_k(\gamma)\), forming a double critical pair. This shows that \(MDC(K) \leq ||\gamma(b) - \gamma(b + \pi F_k(\gamma)|| = 2F_k(\gamma)\) which is contrary to the hypothesis. Consequently, the case of \(\frac{1}{2}MDC(K) > R_O(K) = F_k(\gamma)\) with \(\exists a \in S^1, k\gamma(a) < \sup \kappa\gamma\) is vacuous.

**Remark 4.** We do not know any example or the existence of any ideal knot \(\gamma\) with \(R_O(K) = F_k(\gamma) < \frac{1}{2}MDC(K)\) and constant generalized curvature \(\kappa\gamma \equiv R_O(K)^{-1}\).

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