The Coupled Cluster Method in Hamiltonian Lattice Field Theory: SU(2) Glueballs

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The glueball spectrum within the Hamiltonian formulation of lattice gauge theory (without fermions) is calculated for the gauge group SU(2) and for two spatial dimensions. The Hilbert space of gauge-invariant functions of the gauge field is generated by its parallel-transporters on closed paths along the links of the spatial lattice. The coupled cluster method is used to determine the spectrum of the Kogut-Susskind Hamiltonian in a truncated basis. The quality of the description is studied by computing results from various truncations, lattice regularisations and with an improved Hamiltonian. We find consistency for the mass ratio predictions within a scaling region where we obtain good agreement with standard lattice Monte Carlo results.

I. INTRODUCTION AND SURVEY

It is the purpose of this paper to present the results of an attempt to compute the spectrum of a lattice gauge field theory within the Hamiltonian formulation. Our computational framework is the coupled cluster method which has been presented and discussed in detail in Ref. \[14\]. We have applied our method to the lattice version of the SU(2) Yang-Mills theory in two spatial dimensions which is the simplest non-trivial non-abelian lattice gauge field theory. We consider this model whose dynamics is given by a special Kogut-Susskind Hamiltonian as an important test case for controlling the coupled cluster method. Although there is no experiment to compare with, there exist for this model very reliable results for the ground state energy (see Ref. \[15\]) and for glueball masses (see Ref. \[18\]) which allow a critical test of the coupled cluster predictions.

In the past, several attempts have been made to compute glueball masses within the same model. We mention the light-front approach of van de Sande et al. (see Ref. \[13\]) and the maximal tree approach of Bishop et al. (see Ref. \[14\]).

Calculations similar to ours, i.e. treating the Kogut-Susskind Hamiltonian within the coupled cluster method, were undertaken by Llewellyn Smith and Watson (see Ref. \[13\]) and Guo et al. (see Ref. \[14\]). Within the present work we are able to go far beyond the limitations of this earlier work:

- When expanding the characteristic correlation functions within a localized loop basis (see Ref. \[14\]), we can push the expansion up to the 7th order.
- We are able to check the validity of the computation of the characteristic coupled cluster matrix elements by comparing two independent formulations of the local loop space basis expansion (see Ref. \[14\]).
- We can compare the predictions of different lattice regularizations which exist in two spatial dimensions: square, hexagonal and triangular lattice.
- We compute glueballs with all possible angular momentum quantum numbers allowed within the lattice formulation, i.e. $J^P = 0^+, 0^-, 2^+, 2^-, 1^\pm$ for the square lattice and $J^P = 0^+, 0^-, 3^+, 3^-, 1^\pm, 2^\pm$ for the hexagonal or triangular lattice.
- We also test the quality of our results by comparing to those with an improved Hamiltonian.

Taking dimensionless mass ratios we observe a good scaling window (around $\beta = 3$ for the non-improved case). Judging the stability of the results within the scaling window by testing the convergence with the order of the coupled cluster expansion, by comparing the predictions of the different lattice regularisations and the inclusion of improvement, we are able to predict the spectrum of the energetically low-lying glueballs within certain errors (see Fig. \[7\]). Within these errors, our prediction of the glueball spectrum agrees with that of Teper (see Ref. \[18\]). Since our results are very encouraging it is, of course, desirable to extend these computations to more physical models. The SU(2) lattice Yang-Mills theory for $D = 3+1$ dimensions has been worked out (see Ref. \[13\]), but the results are too preliminary to be presented. The extension of our model to the SU(3) gauge group is under way (see Ref. \[14\]). The inclusion of quarks, i.e. the treatment of full QCD, is possible in principle, but technically difficult. First attempts using the coupled cluster

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method for such models exist, but only for the Schwinger model (i.e. the 1+1 dimensional $U(1)$ case) (see Ref. [8]).

Our paper is organized as follows:

Section II defines our computational scheme, especially details of the coupled cluster expansion.

Section III describes some calculational details, especially momentum (only lattice momentum zero is considered within this paper).

Section IV gives some details of how to project localized wave function on good lattice momentum and angular momentum (only lattice momentum zero is considered in Ref. [14]). In summary, the structure is the following:

- The Kogut Susskind wave functions $\Psi(U)$ is the Kogut Susskind wave function (see Ref. [6]).
- The excitation operators $F$ abbreviates “link variable derivatives” of functions $f(U)$:

$$F = F_{\sigma}^{k,\nu} = \sum_{\nu} \tilde{\Pi}_{\nu\sigma} F_{\text{int}}^{k,\nu}$$
$$S = \tilde{\Pi}^0 S_{\text{int}}$$

where $\tilde{\Pi}_{\nu\sigma}^{k}$ is the projection operator on states with vanishing (lattice) momentum and angular momentum-parity $k$ (e.g. $k = J^P = 0^+, 0^-, 2^+, 2^-, 1^\pm$ for the square lattice). $\nu, \sigma$ are magnetic quantum numbers, for more details see Section IV. We abbreviate $\tilde{\Pi}_{00}^\nu = \tilde{\Pi}^0$.

The essential point of the coupled cluster method is that it can be shown that the functions $S_{\text{int}}$ and $F_{\text{int}}$ are localized (see Ref. [14]) so that it is plausible to expand these intrinsic functions with respect to a localized, gauge invariant, linked, standard basis $\chi^\alpha$ in the form

$$S_{\text{int}}(U) = \sum_{\alpha} S_{\alpha} \chi^\alpha(U)$$
$$F_{\text{int}}^{k,\nu}(U) = \sum_{\alpha} F_{\alpha}^{k,\nu} \chi^\alpha(U)$$

and to define the approximation to the eigenvalue problem by a truncation of this basis.

Introducing the constant function via $\chi^0 = 1$, the “plaquette function” by putting $\tilde{\Pi}_0^0 \chi^1 = 4V$ and using the strong coupling structure $\sum_{lc} \chi_{lc}^0 = \epsilon_a \chi^a$ which is true for any convenient basis, the coupled cluster equations (1) and (2) become equations for the coefficients $S_{\alpha}$ and $F_{\alpha}^{k,\nu}$.

II. COMPUTATIONAL SCHEME

Within the Hamiltonian formulation, the computational framework of any lattice Yang-Mills theory is given by the Kogut Susskind theory which has been discussed in detail in Ref. [14]. In summary, the structure is the following:

The Kogut Susskind wave functions $\Psi(U) = \Psi(U_1, \ldots, U_N)$ depend on link variables $U_i$ ($l = 1, \ldots, N$) which are elements of the gauge group $SU(N_c)$. $N$ is the number of oriented links in a lattice (with $d$ spatial dimensions). $N$ is related to the finite volume chosen. Our many-body technique allows us to perform the final calculations in the infinite volume limit $N \to \infty$. Our concrete calculations are done for the case $N_c = 2$ and $d = 2$.

The idea of the coupled cluster method is to re-formulate the eigenvalue problem $H\Psi = E\Psi$ as equations for the ground state correlation function $S$ and for the excitation operators $F$, following from the ansatz $\Psi_0(U) = \exp S(U)$ for the ground state and $\Psi(U) = F(U) \exp S(U)$ for excited states.

If $H = 1/(2g)(\sum_{lc} g^2 E_{lc}^2 - 2/g^2 V)$ is the Kogut Susskind Hamiltonian (we use the notations $(lc)$ for link-colour quantum numbers, $E_{lc}$ for the momentum operators conjugate to $U_l$ and $V$ for the plaquette term) this yields the non-linear equation

$$\sum_{lc} (S_{lc} + S_{lc} S_{lc}) - \frac{2}{g^2} V = \frac{2a}{g^2} E_0$$

for $S$ and the linear equation

$$\sum_{lc} (F_{lc} + 2S_{lc} F_{lc}) = \frac{2a}{g^2} (E - E_0) F$$

for the excitation operators $F$. We use the notation that $f_{lc}$ abbreviates “link variable derivatives” of functions $f(U)$:

$$f_{lc} = [E_{lc}, f]$$
$$f_{lc} = [E_{lc}, [E_{lc}, f]]$$
The “order” of the functions in the subspace $H^{\delta,k}$ is defined to be $\delta$. In general, the quantum number $k$ characterizes the different, geometrically independent possibilities for the construction of the plaquette products of the same order whose total number is $n_\delta$.

Up to 7th order we have for our two-dimensional $SU(2)$ case

- $n_\delta = 1, 1, 2, 4, 12, 35, 129, 495$ for the square lattice,
- $n_\delta = 1, 1, 2, 5, 15, 53, 235, 1125$ for the hexagonal lattice and
- $n_\delta = 1, 1, 2, 3, 8, 17, 54, 162$ for the triangular lattice, for $\delta = 0, \ldots, 7$.

The functions $\Lambda_G^{\delta,k}$ are characterized by simple loop patterns exemplified up to 4th order in Tables III and IV. Here, also the dimensions $m_{\delta,k}$ of the corresponding subspaces $H^{\delta,k}$ are given.

Each function $\Lambda_G^{\delta,k}$ ($\delta, k$ fixed) generates an orthogonal basis $\chi^{(\delta,k,r)}$, $r = 1, \ldots, m_{\delta,k}$, of $H^{\delta,k}$, and taking all quantum numbers $(\delta, k, r)$ and the limit $\delta \to \infty$, this basis becomes complete for the expansion of localized functions.

There are two ways to define the basis of $H^{\delta,k}$ explicitly.

A. The algebraic method

The idea of this procedure is to act on $\Lambda_G^{\delta,k}$ with a suitable set of commuting Casimir operators of a lattice gauge group. Hereby, this lattice gauge group $SU(N_c)^M$ is finite dimensional, where $M(\delta, k)$ is the number of link variables contained in $\Lambda_G^{\delta,k}$.

This yields a generating system for the space $H^{\delta,k}$ and diagonalizing the corresponding Casimir matrices defines the orthogonal basis. The relation to the generating system allows then to compute also the coefficients $c_{\alpha\omega u}^{\beta \gamma \nu}$ (see Ref. [14] for details and examples).

The drawback of this method to construct the basis $\chi^{(\delta,k,r)} = \chi^{(\delta,k)}$ that one has to take care of possible linear dependencies between the generated loop space functions. In previous investigations (see Ref. [12, 13, 14]) this problem was solved by exploiting the Cayley Hamilton relationship between matrices. We have used in this connection a much simpler procedure: If a set $f_1(U), \ldots, f_n(U)$ contains only $m$ ($m \leq n$) independent functions, the matrix $f_i(U^k)$ ($i, k = 1, \ldots, n$) has for suitable fixed variables $(U^1, \ldots, U^n) \in SU(2)$ exactly the rank $m$.

Our experience is that statistically chosen variable sets $(U^1, \ldots, U^n) \in SU(2)$ are suitable in this sense. The linear relation between the functions $f_i$ is then easily constructed and the dependent functions can be eliminated. This works quite well up to the 6th order. For higher orders, however, this method became too unreliable because of increasing numerical errors when checking the linear dependencies in this way.
B. The Clebsch-Gordan method

Each basis function $\chi^\alpha$ generated by the algebraic method can be characterized by a set of angular momentum quantum numbers adjoined to a certain plaquette pattern related to the corresponding function $\Lambda_{G}^{\delta,k}$. For the case of a hexagonal lattice, this pattern consists of adjoining a certain angular momentum quantum number $j_l$ to each link variable occurring in $\Lambda_{G}^{\delta,k}$, the possible values of $j_l$ being given by simple coupling rules. For the square (see Ref. [14]) or the triangular lattice, also “intermediate” quantum numbers have to be taken into account. This can be systemized by replacing the lattice by a certain “net” containing only three-point vertices and by defining the plaquette pattern related to the corresponding function $\Lambda_{G}^{\delta,k}$. For convenience, the set $\{j_l\} = (j_1, j_2, ...)\) will be assumed to be infinite dimensional by putting $j_l = 0$ for all links of the net not occurring in the plaquette pattern related to $\chi^\alpha$.

This makes it possible to work directly with the orthonormal basis $\chi^{(j_l)}$ and to compute the coupled cluster matrix elements with the help of standard group theoretical methods. Generalizing the notion $\{j_l\}$ also to non-standard functions, i.e. putting $T(u)\chi^{(j_l)} = \chi^{(j'_l)}$, the essential formula allowing to compute the coefficients $c_{\alpha\beta\gamma\delta}$ via certain $9j$-symbols is given by

$$c_{\alpha\beta\gamma\delta} = \sum_{\{j_l\}} c(\{j_l\}, \{j'_l\}) \chi^{(j'_l)},$$

where $\delta = \sqrt{2j_l + 1}$.

In Eq. (8) the (in each case finite) sum runs only over those sets $\{j_l\}$ which obey for each $l$ the selection rule $|j_l - j'_l| \leq j'_l \leq j_l + j'_l$ and the condition $j'_l \neq 0$ or $j_l \neq 0$.

In Eq. (8), we have used the notation $\{l(v, 1), l(v, 2), l(v, 3)\}$ for the three links $l$ adjacent to the vertex $v$.

For the application of the coupled cluster equation including truncation one has to know the order of the function $\chi^{(j_l)}$, i.e. to which subspace $H^{\delta,k}$ it belongs - the plaquette pattern of $\{j_l\}$ does not give this uniquely. This order is, however, easily obtained when constructing the basis $\chi^{(j_l)}$ by iterative Clebsch-Gordan coupling, see Ref. [14] for examples.

IV. THE LATTICE EUCLIDEAN GROUP

Any explicit calculation involves the treatmant of the lattice Euclidean group $G_E$ which consist of translations and rotatons, $G_E = GT \otimes GR$, leaving the (infinite) lattice invariant.

A. Lattice Translations

Translations $x \rightarrow x + a$ are given by translation vectors

$$a = a_1 e_1 + a_2 e_2$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. For the other unit vector we have that

$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for a square lattice and $e_2 = \begin{pmatrix} \cos(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) \end{pmatrix}$ for the hexagonal or triangular lattice, respectively.

All distances will be measured in units of the lattice spacing. The numbers $a_1, a_2$ are then integers which are arbitrary for the square or the triangular lattice and obey a simple restriction for the hexagonal lattice (see Ref. [20]).

B. Lattice Rotations

General Euclidean transformation are given by a pair $(b, a)$ mapping $x \rightarrow bx + a$ where $b \in GR$ is a lattice rotation.

For the square lattice, the “point group” $G_R$ has the eight elements $(b_0,\ldots,b_7)$ which may be written in terms of the following $2 \times 2$ matrices

$$b_n = \begin{pmatrix} \cos(n\frac{\pi}{2}) & \sin(n\frac{\pi}{2}) \\ -\sin(n\frac{\pi}{2}) & \cos(n\frac{\pi}{2}) \end{pmatrix}$$

$$b_{n+4} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} b_n, \quad n = 0, \ldots, 3.$$

For the hexagonal or triangular lattice the group $G_R$ has the 12 elements $(b_0,\ldots,b_{11})$ which we write in the form

$$b_n = \begin{pmatrix} \cos(n\frac{\pi}{3}) & \sin(n\frac{\pi}{3}) \\ -\sin(n\frac{\pi}{3}) & \cos(n\frac{\pi}{3}) \end{pmatrix}$$

$$b_{n+6} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} b_n, \quad n = 0, \ldots, 5.$$

C. Irreducible Representations

We now list the irreducible representaions (irreps) of $G_R$ (see e.g. Ref. [14]).

For the square lattice (hexagonal or triangular lattice) we have 5 (6) inequivalent irreducible representations (irreps) $D^{k,4}$ one-dimensional irreps labelled...
The two-dimensional irreps read for the square case $D_{n=0, \ldots, 7}$ for the hexagonal or triangular lattice
for a rotational invariant Hamiltonian. Explicitly, the irreps read for the one-dimensional cases $(n=0, \ldots, 11)$ for the hexagonal or triangular lattice

$D^{A_k}(b_n) = \begin{cases} 1 \\
\text{det}(b_n) \end{cases}$

$D^{B_k}(b_n) = (-1)^n \text{det}(b_n)$

The two-dimensional irreps read for the square case $D^{E_k}(b_n) = b_n$ $(n=0, \ldots, 7)$, and for the hexagonal or triangular case

$D^{E_i}(b_n) = b_n$ $(n=0, \ldots, 11)$

This allows to define projection operators on good momentum $\mathbf{p}$ within the Brillouin zone by

$\Pi_k^p = \sum_a e^{i\mathbf{p}\mathbf{a}} T(0, \mathbf{a})$.

States with fixed lattice angular momentum may be constructed by the operators

$\Pi_{\nu,\sigma}^k = N_k \sum_n D_{\nu,\sigma}^k (b_n^{-1}) T(b_n,0)$,

where the normalization factor $N_k = \dim D^k / |G_R|$ guarantees the projector property

$\Pi_{\nu,\sigma}^k \Pi_{\nu',\sigma'}^k = \Pi_{\nu,\sigma'}^k$.

The index $\sigma$ in Eq. (10) corresponds to a “magnetic” quantum number since we have

$T(b) \Pi_{\nu,\sigma}^k = \sum_{\sigma'} D_{\sigma,\sigma'}^k (b) \Pi_{\nu,\sigma'}^k$.

Thus for the eigenvalue problem of a rotationally invariant Hamiltonian $\sigma$ can be fixed, e.g. to $\sigma = 0$.

Eq. (11) also shows that the operator $\Pi_{\nu,\sigma}^k$ generates the representation $D^k$ twice if it is two-dimensional. Thus the index $\nu$ is not a “good” quantum number, the Hamiltonian will in general mix states will different values of $\nu$.

D. Projections on Lattice Momenta and Angular Momenta

The group $G_E$ acts on the functions $\chi^{(j)}$ via

$T(u)\chi^{(j)} = \chi^{(j')}$

$l' = u^{-1}l$

for $u \in G_E$.

We have studied alternative truncations (see Ref. [20]) which yields actually the simplest set of equations. In this case one puts in the order $\delta$

$\epsilon_{\sigma_3,\sigma_4}^{\alpha_1,\alpha_2} = 0$ for $\delta(\chi^{\alpha_1}) + \delta(\chi^{\alpha_2}) > \delta$.

Since the expansion of the function $S$ starts with $\delta = 1$, our calculations go up to 8th order if the expansion of the correlation functions is pushed to 7th order, as we do within this work.

We have also studied alternative truncations (see Ref. [20]), but we could not find any prescription which was better than Eq. (13).

VI. RESULTS FOR THE GROUND STATE

It is a special feature of the Hamiltonian formulation that it provides both the energy and the wave function of the vacuum state which is the ground state of the Hamiltonian.
The problem is whether this region of couplings contains the value of the glueball mass within errors.

Obviously, we are able to estimate in the region \( \beta \approx 3 \) the value of the glueball mass within errors. We assume the scaling window to be at \( \beta \approx 3 \). This allows us to estimate the glueball spectrum within certain errors by comparing the prediction of the 7th and 8th order at \( \beta = 3 \), see Table VII. Also included are the predictions of the improved Hamiltonian (see Subsection VII B) and of alternative lattices (see Subsection VII C). For the \( 0^- \) sector we have to remark that because the glueball in question lies in the continuum of two (lowest) \( 0^+ \) glueball states the results are not very reliable.

This can be seen having a look on the wave functions of this states. In the \( 0^- \) sector extending the calculations from 6th to 7th and 8th order a new state occurs (see Fig. 6). In the strong coupling limit (\( \beta = 0 \)) it has an eigenvalue of twice the eigenvalue of the \( 0^+ \) glueball and its wave function is exactly the product of two glueball excitation operators: the glueball excitation \( F \) is given by the product of two plaquettes (two-cluster state):

\[
F_{0^+}(\beta = 0) = \begin{pmatrix} 20.25 \% \\ 10.67 \% \\ 10.57 \% \\ 5.86 \% \\ 4.11 \% \\ 4.02 \% \\ \text{rest} \end{pmatrix}, \text{8th order}.
\]

As is the strong coupling limit, the scaling region (\( \beta = 3 \)) is dominated by two-cluster states:

\[
F_{0^-}(\beta = 3.0) : \begin{pmatrix} 20.25 \% \\ 10.67 \% \\ 10.57 \% \\ 5.86 \% \\ 4.11 \% \\ 4.02 \% \\ \text{rest} \end{pmatrix}, \text{8th order}.
\]

This feature of the wave functions appears in the whole \( 0^- \) sector and also in the \( 1^\pm \) sector for the \( 1^\pm \) state. The quality of the result on the square lattice is now tested by comparing to other computational schemes yielding equivalent descriptions in the continuum limit.
Like on the square lattice also on the hexagonal and the triangular lattice the $0^-$ sector is dominated by two-cluster states. In addition the two-cluster states influence also the first and second excitations in the $1^\pm$ and the $3^\pm$ sector on the hexagonal lattice. Nevertheless we perform a “final” estimate of the glueball spectrum in units of the $0^+$ glueball by calculating mean values of the numbers of different orders and different lattices of Table VII, yielding the predictions of Fig. 3 (left values). The errors are deviations of the mean values. We do not take into account read off errors. We see that, indeed, we obtain predictions quite consistent with the Monte Carlo results of Teper, see Ref. [18] (middle and right values in Fig. 3).

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APPENDIX

Within this Appendix we comment on the question whether it is possible to have a scaling region $\beta \approx 3$ if this region starts only with $\beta \geq 4$ within the standard lattice Monte Carlo method (see Ref. [18]). The point is that a comparison to our Hamiltonian results involves a rescaling of the standard “Euclidean” coupling $g_E$ relative to the Hamiltonian coupling $g$. For $g \rightarrow 0$ this is given by (see Ref. [1])

$$
\beta_E = \beta + 0.077 + O(g^2).
$$

There is also rescaling of the Euclidean masses $M_E$ relative to the Hamiltonian masses $M$ (“velocity of light correction”)

$$
M_E = (1 + 0.084g^2 + O(g^4))M.
$$

However, one cannot expect that these formulae hold for $\beta \approx 3$ because the difference between $\beta_E$ and $\beta$ becomes very large in the strong coupling regime itself. The strong coupling expansions are completely different in the Euclidean and Hamiltonian formalisms: the Euclidean expansion has a logarithmic singularity for $\beta_E = 0$ which is not present in the Hamiltonian framework. The rigorous relation between $\beta_E$ and $\beta$ is not known. Consequently for $\beta \approx 3$ a direct comparison of masses to Teper’s results appears not to be possible.

[1] S.L. Altmann, C.J. Bradley, Rev. of Modern Physics, 37 (1965) 33.
TABLES

TABLE I. Construction of the Loop Space Basis: The functions $\Lambda_{\delta}^{\delta,k}$ generating $m_{\delta,k}$-dimensional subspaces are characterized by simple loop patterns. Altogether we have 84 basis elements in the order $\delta \leq 4$ for the square lattice.

| $\Lambda_{\delta}^{\delta,k}$ | $m_{\delta,k}$ | $\Lambda_{\delta}^{\delta,k}$ | $m_{\delta,k}$ |
|-----------------------------|----------------|-----------------------------|----------------|
| $\Lambda_{G}^{0,1} = 1$    | 1              | $\Lambda_{G}^{4,1} = \blacksquare$ | 1              |
| $\Lambda_{G}^{1,1} = \blacksquare$ | 1              | $\Lambda_{G}^{4,2} = \blacksquare$ | 2              |
| $\Lambda_{G}^{2,1} = \blacksquare$ | 1              | $\Lambda_{G}^{4,3} = \blacksquare$ | 7              |
| $\Lambda_{G}^{3,1} = \blacksquare$ | 1              | $\Lambda_{G}^{4,4} = \blacksquare$ | 3              |
| $\Lambda_{G}^{4,1} = \blacksquare$ | 2              | $\Lambda_{G}^{5,1} = \blacksquare$ | 5              |
| $\Lambda_{G}^{0,2} = \blacksquare$ | 2              | $\Lambda_{G}^{4,6} = \blacksquare$ | 6              |
| $\Lambda_{G}^{2,2} = \blacksquare$ | 2              | $\Lambda_{G}^{4,7} = \blacksquare$ | 4              |
| $\Lambda_{G}^{3,2} = \blacksquare$ | 2              | $\Lambda_{G}^{4,8} = \blacksquare$ | 9              |
| $\Lambda_{G}^{4,1} = \blacksquare$ | 4              | $\Lambda_{G}^{4,9} = \blacksquare$ | 9              |
| $\Lambda_{G}^{0,3} = \blacksquare$ | 3              | $\Lambda_{G}^{4,10} = \blacksquare$ | 10             |
| $\Lambda_{G}^{2,3} = \blacksquare$ | 3              | $\Lambda_{G}^{4,11} = \blacksquare$ | 7              |
| $\Lambda_{G}^{3,3} = \blacksquare$ | 3              | $\Lambda_{G}^{4,12} = \blacksquare$ | 6              |
| $\Lambda_{G}^{4,1} = \blacksquare$ | 3              | $\Lambda_{G}^{4,12} = \blacksquare$ | 6              |

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TABLE II. Construction of the Loop Space Basis: The functions $\Lambda_{G}^{\delta,k}$ generating $m_{\delta,k}$-dimensional subspaces are characterized by simple loop patterns. Altogether we have 94 basis elements in the order $\delta \leq 4$ for the hexagonal lattice.

| $\Lambda_{G}^{\delta,k}$ | $m_{\delta,k}$ | $\Lambda_{G}^{\delta,k}$ | $m_{\delta,k}$ |
|--------------------------|----------------|--------------------------|----------------|
| $\Lambda_{G}^{0,1} = 1$  | 1              | $\Lambda_{G}^{4,1} = \bigcirc$ | 1              |
| $\Lambda_{G}^{1,1} = \bigcirc$ | 1              | $\Lambda_{G}^{4,2} = \bigcirc$ | 2              |
| $\Lambda_{G}^{2,1} = \bigcirc$ | 1              | $\Lambda_{G}^{4,3} = \bigcirc$ | 6              |
| $\Lambda_{G}^{3,1} = \bigcirc$ | 1              | $\Lambda_{G}^{4,4} = \bigcirc$ | 5              |
| $\Lambda_{G}^{4,2} = \bigcirc$ | 2              | $\Lambda_{G}^{4,5} = \bigcirc$ | 3              |
| $\Lambda_{G}^{5,1} = \bigcirc$ | 2              | $\Lambda_{G}^{4,6} = \bigcirc$ | 5              |
| $\Lambda_{G}^{6,1} = \bigcirc$ | 4              | $\Lambda_{G}^{4,7} = \bigcirc$ | 4              |
| $\Lambda_{G}^{7,1} = \bigcirc$ | 4              | $\Lambda_{G}^{4,8} = \bigcirc$ | 4              |
| $\Lambda_{G}^{8,2} = \bigcirc$ | 6              | $\Lambda_{G}^{4,9} = \bigcirc$ | 6              |
| $\Lambda_{G}^{9,3} = \bigcirc$ | 10             | $\Lambda_{G}^{4,10} = \bigcirc$ | 10             |
| $\Lambda_{G}^{10,4} = \bigcirc$ | 7              | $\Lambda_{G}^{4,11} = \bigcirc$ | 7              |
| $\Lambda_{G}^{11,5} = \bigcirc$ | 6              | $\Lambda_{G}^{4,12} = \bigcirc$ | 6              |
| $\Lambda_{G}^{12,6} = \bigcirc$ | 8              | $\Lambda_{G}^{4,13} = \bigcirc$ | 8              |
| $\Lambda_{G}^{13,7} = \bigcirc$ | 4              | $\Lambda_{G}^{4,14} = \bigcirc$ | 4              |
| $\Lambda_{G}^{14,8} = \bigcirc$ | 6              | $\Lambda_{G}^{4,15} = \bigcirc$ | 6              |

TABLE III. Construction of the Loop Space Basis: The functions $\Lambda_{G}^{\delta,k}$ generating $m_{\delta,k}$-dimensional subspaces are characterized by simple loop patterns. Altogether we have 58 basis elements in the order $\delta \leq 4$ for the triangular lattice.

| $\Lambda_{G}^{\delta,k}$ | $m_{\delta,k}$ | $\Lambda_{G}^{\delta,k}$ | $m_{\delta,k}$ |
|--------------------------|----------------|--------------------------|----------------|
| $\Lambda_{G}^{0,1} = 1$  | 1              | $\Lambda_{G}^{4,1} = \bigtriangledown$ | 1              |
| $\Lambda_{G}^{1,1} = \bigtriangle$ | 1              | $\Lambda_{G}^{4,2} = \bigtriangledown$ | 2              |
| $\Lambda_{G}^{2,1} = \bigtriangle$ | 1              | $\Lambda_{G}^{4,3} = \bigtriangledown$ | 7              |
| $\Lambda_{G}^{3,1} = \bigtriangle$ | 1              | $\Lambda_{G}^{4,4} = \bigtriangledown$ | 3              |
| $\Lambda_{G}^{4,2} = \bigtriangle$ | 2              | $\Lambda_{G}^{4,5} = \bigtriangledown$ | 6              |
| $\Lambda_{G}^{5,1} = \bigtriangle$ | 2              | $\Lambda_{G}^{4,6} = \bigtriangledown$ | 9              |
| $\Lambda_{G}^{6,2} = \bigtriangle$ | 4              | $\Lambda_{G}^{4,7} = \bigtriangledown$ | 10             |
| $\Lambda_{G}^{7,1} = \bigtriangle$ | 1              | $\Lambda_{G}^{4,8} = \bigtriangledown$ | 8              |
| $\Lambda_{G}^{8,2} = \bigtriangle$ | 2              | $\Lambda_{G}^{4,9} = \bigtriangledown$ | 8              |

TABLE IV. Lattice Angular Momenta: The number of basis elements $\chi^\alpha$ for the different spins on the square lattice.

| $\chi^\alpha$ | 1 | 2 | 5 | 15 | 84 | 557 | 4942 | 47751 |
|----------------|---|---|---|----|----|-----|------|-------|
| $\Pi^0 \chi$   | 1 | 2 | 5 | 15 | 84 | 557 | 4942 | 47751 |
| $\Pi^0 \chi$   | 0 | 0 | 0 | 1  | 31 | 377 | 4220 | 45132 |
| $\Pi^2 \chi$   | 0 | 0 | 0 | 4  | 42 | 425 | 4386 | 45790 |
| $\Pi^2 \chi$   | 0 | 0 | 0 | 8  | 79 | 874 | 8841 | 92340 |

TABLE V. Lattice Angular Momenta: The number of basis elements $\chi^\alpha$ for the different spins on the hexagonal lattice.

| $\chi^\alpha$ | 1 | 2 | 5 | 17 | 94 | 677 | 6430 | 67036 |
|----------------|---|---|---|----|----|------|------|-------|
| $\Pi^0 \chi$   | 1 | 2 | 5 | 17 | 94 | 677 | 6430 | 67036 |
| $\Pi^0 \chi$   | 0 | 0 | 0 | 1  | 35 | 464 | 5613 | 63947 |
| $\Pi^2 \chi$   | 0 | 0 | 0 | 4  | 43 | 512 | 5728 | 64573 |
| $\Pi^2 \chi$   | 0 | 0 | 0 | 6  | 52 | 573 | 5989 | 65854 |
| $\Pi^2 \chi$   | 0 | 2 | 12| 120| 1128| 12008| 130924|

TABLE VI. Lattice Angular Momenta: The number of basis elements $\chi^\alpha$ for the different spins on the triangular lattice.

| $\chi^\alpha$ | 1 | 2 | 5 | 12 | 58 | 341 | 2958 | 27885 |
|----------------|---|---|---|----|----|-----|------|-------|
| $\Pi^0 \chi$   | 1 | 2 | 5 | 12 | 58 | 341 | 2958 | 27885 |
| $\Pi^0 \chi$   | 0 | 0 | 0 | 1  | 21 | 249 | 2601 | 26857 |
| $\Pi^2 \chi$   | 0 | 0 | 0 | 1  | 12 | 225 | 2603 | 26859 |
| $\Pi^2 \chi$   | 0 | 1 | 2 | 9  | 41 | 324 | 2807 | 27734 |
| $\Pi^2 \chi$   | 0 | 0 | 0 | 7  | 56 | 562 | 5393 | 54550 |
| $\Pi^2 \chi$   | 0 | 2 | 9 | 71 | 577 | 5538 | 54695 |
TABLE VII. The mass ratios of various glueball states $J^P$ relative to the $0^+$ state for the square, the square improved, the hexagonal and the triangular lattice in 7th and 8th order in the scaling region.

| $J^P$ | square | square impr. | hexagonal | triangular |
|-------|--------|-------------|-----------|------------|
| $\beta$ = 3 | $\delta = 7$ | $\beta$ = 1.7 | $\delta = 7$ | $\beta$ = 3 | $\delta = 8$ | $\beta$ = 3 | $\delta = 8$ |
| $\delta = 8$ |
| $0^+$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $0^{++}$ | – | 1.47 | 1.43 | 1.41 | 1.51 | 1.42 | – | 1.76 |
| $0^-$ | 1.78$^a$ | 2.00$^a$ | 1.82$^a$ | 2.01$^a$ | 1.65$^a$ | 2.00$^a$ | 2.13$^a$ | 2.26$^a$ |
| $0^{--}$ | 2.01$^a$ | 2.06$^a$ | 2.06$^a$ | 2.07$^a$ | 2.08$^a$ | 2.15$^a$ | 2.18$^a$ | 2.32$^a$ |
| $0^{---}$ | 2.30$^a$ | 2.20$^a$ | 2.34$^a$ | 2.23$^a$ | 2.25$^a$ | 2.20$^a$ | 2.53$^a$ | 2.53$^a$ |
| $1^\pm$ | 2.11 | 2.10 | 2.14 | 2.10 | 2.17 | 2.15 | 2.07$^b$ | 2.04$^b$ |
| $1^{++}$ | 2.23 | 2.22$^a$ | 2.27 | 2.25 | 2.26 | 2.17$^a$ | 2.20$^b$ | 2.12$^b$ |
| $1^{+++}$ | 2.27 | 2.23 | 2.30 | 2.26 | 2.26 | 2.19$^a$ | 2.39 | 2.35 |
| $2^+$ | 1.56 | 1.61 | 1.57 | 1.60 |
| $2^{++}$ | 1.77 | 1.87 | 1.78 | 1.83 |
| $2^{+++}$ | 1.78 | – | 1.82 | – |
| $2^-$ | 1.68 | 1.66 | 1.69 | 1.66 |
| $2^{--}$ | 1.77 | – | 1.81 | – |
| $2^{---}$ | 2.11 | – | 2.08 | – |

$^a$predominantly 2-cluster states  
$^b$level crossing
FIG. 1. Ground state energy density as a function of the inverse coupling $\beta = \frac{1}{g^2}$ up to 8th order. For comparison: Strong coupling expansion SCE [7] and Green’s function Monte-Carlo GFMC method [8].

FIG. 2. Excited $0^+$ state energy as a function of the inverse coupling $\beta = \frac{1}{g^2}$ up to 8th order without and with improvement. For comparison: Strong coupling expansion SCE and Extrapolation of the strong coupling series ESCE with Shafer and integrated differential approximants [7].
FIG. 3. Mass ratios of the $J^P = 0^−$, $J^P = 2^+$, $J^P = 2^−$ and the $J^P = 1^±$ state relative to the $0^+$ state as a function of the inverse coupling $\beta = \frac{4}{g^2}$ in 8th order without and with improvement. For comparison: Standard Monte Carlo SMC results [18] for $\xi = 1$ (only scaling window result). $\xi$ is the anisotropic factor $\xi = \frac{a_s}{a_t}$, $a_s(t) =$ lattice spacing in space (time) direction.

FIG. 4. Mass ratios of the $J^P = 2^−$ state relative to the $0^+$ state as a function of the inverse coupling $\beta = \frac{4}{g^2}$ in 6th, 7th and 8th order to show the convergence with the order. For comparison: Standard Monte Carlo SMC results [18] for $\xi = 1$ (only scaling window result). $\xi$ is the anisotropic factor $\xi = \frac{a_s}{a_t}$, $a_s(t) =$ lattice spacing in space (time) direction.

FIG. 5. Mass ratios of the first excitations $J^P = 0^{+*}$ and $J^P = 2^{+*}$ glueball states in 8th order. For comparison: Standard Monte Carlo SMC results [18] for $\xi = 1$ (only scaling window result). $\xi$ is the anisotropic factor $\xi = \frac{a_s}{a_t}$, $a_s(t) =$ lattice spacing in space (time) direction.
FIG. 6. Mass ratios of the $J^P = 0^-$ state relative to the $0^+$ state as a function of the inverse coupling $\beta = \frac{g^2}{\pi}$ in 6th, 7th and 8th order. This sector is dominated by 2-cluster states. For comparison: Standard Monte Carlo SMC results [18] for $\xi = 1$ (only scaling window result). $\xi$ is the anisotropic factor $\xi = \frac{a_s}{a_t}$, $a_s(t) =$ lattice spacing in space (time) direction.

FIG. 7. Final estimate of the glueball spectrum in units of the $0^+$ glueball. The bars in the left column are mean values of the numbers in Table VI. The shaded area reflects the estimated error. For comparison: Standard Monte Carlo SMC results [18] for $\xi = 1$ in the middle column and for $\xi = 5$ in the right column, where $\xi$ is the anisotropic factor $\xi = \frac{a_s}{a_t}$, with $a_s(t) =$ lattice spacing in space (time) direction.