Finite-Range Corrections to the Thermodynamics of the One-Dimensional Bose Gas

A. Cappellaro\(^1\) and L. Salasnich\(^{1,2}\)

\(^1\)Dipartimento di Fisica e Astronomia "Galileo Galilei" and CNISM, Università di Padova, via Marzolo 8, 35131 Padova, Italy

\(^2\)CNR-INO, via Nello Carrara, 1 - 50019 Sesto Fiorentino, Italy

(Dated: September 30, 2018)

The Lieb-Liniger equation of state accurately describes the zero-temperature universal properties of a dilute one-dimensional Bose gas in terms of the s-wave scattering length. For weakly-interacting bosons we derive non-universal corrections to this equation of state taking into account finite-range effects of the inter-atomic potential. Within the finite-temperature formalism of functional integration we find a beyond-mean-field equation of state which depends on scattering length and range of the interaction potential. Our analytical results, which are obtained performing dimensional regularization of divergent zero-point quantum fluctuations, show that for the one-dimensional Bose gas thermodynamic quantities like pressure and sound velocity are modified by changing the ratio between the effective range and the scattering length.

PACS numbers: 03.70.+k, 05.70.Ce, 67.85.-d

I. INTRODUCTION

For almost eighty years one-dimensional (1D) quantum systems have been subject of intense fascination. From the seminal work of Bethe in 1931 on the Heisenberg model \(^1\), 1D physics was then explored in great detail, providing exact and approximate solutions to a wide variety of systems \(^2\). These 1D physical systems were considered toy models but recent advances in experimental setups, like Josephson nano-junctions \(^3\) and magneto-optical trapping of cold atoms \(^4\), have achieved the realization of 1D quantum fluids. In the case of the 1D atomic Bose gas, Lieb-Lininger (LL) equation provides a reliable description of the zero temperature thermodynamic quantities \(^4\), while numerical results at finite temperature can be extracted from the Yang-Yang theory \(^5\), which reduces to LL one at zero temperature. The resulting thermodynamics is universal, since the only dependence from the inter-atomic potential is via the s-wave scattering length. However, deviations from universality due to the finite-range of interaction potential have been shown to be quite important for a better understanding of the 3D Bose gas \(^6\). Finite-range effects naturally arise by modelling the two-body interactions beyond the unphysical zero-range approximation. As recently shown for the 2D Bose gas \(^7\), since finite-range corrections affect the behavior of quantum and thermal fluctuations, they have to be taken into account also in lower dimensional system, where fluctuations are strongly enhanced (Mermin-Wagner-Hohenberg theorem \(^8\)). Improvements in cold-atom experimental techniques are providing a precision benchmark which makes possible to explore in great details beyond-mean-field effects in low-dimensional systems \(^9, 10, 21, 22\). It was recently shown, both theoretically and experimentally, that quantum fluctuations provides a stabilization mechanism against collapse, as predicted by the mean-field theory, both in dipolar condensate and binary Bose mixture \(^23, 32\). Moreover, by tuning the interaction via Feshbach resonances one can reach regimes where the effective range of interaction is comparable the scattering length: in these situations, a universal GPE-based description is no more reliable, and one has to consider finite-range effects in the thermodynamical and dynamical description of the system.

In this paper we investigate the role of non-universal corrections to the thermodynamics of the 1D Bose gas both at zero and at finite temperature. Within the framework of a local effective action \(^13, 14, 23, 24, 34\), we derive a novel equation of state, where the contributions of quantum and thermal fluctuations crucially depend also on the effective range of the interatomic potential. These corrections are computed at Gaussian (one-loop) level, where non-physical divergences are removed by using dimensional regularization. We show that the Gaussian theory reproduces extremely well the LL equation of state for weak and intermediate couplings. Finite-range effects are analyzed by considering measurable quantities such as the grand potential, the pressure, and the sound velocity. We find that, at fixed temperature and scattering length, these physical quantities can be widely tuned by varying the effective range.

II. EFFECTIVE FIELD THEORY FOR 1D BOSE GASES

The Euclidean Lagrangian density of identical bosonic particles of mass \(m\) and chemical potential \(\mu\) in a 1D configuration is given by \(^35\)

\[
\mathcal{L} = \psi^\dagger(x, \tau) \left[ \frac{i}{\hbar} \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \mu \right] \psi(x, \tau) + \frac{1}{2} \int dx' |\psi(x', \tau)|^2 V(|x - x'|)|\psi(x, \tau)|^2 ,
\]

(1)
where bosons are described by the complex field $\psi(x, \tau)$ and $V(|x - x'|)$ is the two-body interaction potential between atoms.

For dilute and ultracold atomic gases the inter-atomic potential is usually approximated by a zero-range potential. In order to analyze the role played by the finite range of the inter-atomic potential, we go beyond the zero-range approximation by considering the following low-momentum expansion

$$\tilde{V}(q) = g_0 + g_2 q^2 + \mathcal{O}(q^4)$$  \hspace{1cm} (2)

of the Fourier transform $\tilde{V}(q)$ of the interaction potential $V(|x|)$. It is relevant to connect the parameters $g_0$ and $g_2$ with measurable quantities like the 1D scattering length $a_s$ and the 1D effective range $r_e$ of the potential $V(|x|)$. In order to establish this connection, we first recall that, from 1D scattering theory, the scattering amplitude for a wavefunction reads, in terms of the phase shift $\delta_0(q)$,

$$f_0(q) = q e^{i\delta_0(q)} \sin \delta_0(q).$$  \hspace{1cm} (3)

The scattering length and the effective of the inter-atomic potential are defined via a low-momentum power expansion of the phase shift, namely

$$q \tan \delta_0(q) = \frac{1}{a_s} + \frac{1}{2} r_e q^2 + \mathcal{O}(q^4).$$  \hspace{1cm} (4)

By supposing that the most relevant scattering processes in the system are the ones described by $f_0(q)$, then we then write the T-matrix as

$$T_0(q) = -\frac{2\hbar^2}{m} f_0(q).$$  \hspace{1cm} (5)

At low momenta, the T-matrix is determined analytically by solving the equation

$$T_0(q) = \left[ \frac{1}{\tilde{V}(q)} - \frac{m}{2\pi \hbar^2} \int \frac{dp}{p^2 - q^2 + i\kappa} \right]^{-1}$$  \hspace{1cm} (6)

By taking $\tilde{V}(q)$ as in Eq. (2), Eq. (6) becomes

$$T_0(q) = \left[ \frac{1}{g_0} - \frac{g_2 q^2}{g_0^2} + \mathcal{O}(q^4) + i m \frac{\hbar^2}{2 \hbar^2 q} \right]^{-1}. $$  \hspace{1cm} (7)

By inserting Eq. (3) in Eq. (5), with Eq. (4) and the identity $e^{i\delta} \sin \delta = 1/(\cot \delta - i)$, an expansion up to $q^2$ also leads us to

$$T_0(q) = -\frac{2\hbar^2}{m} \left[ a_s - \frac{1}{2} r_e a_s^2 q^2 + \mathcal{O}(q^4) - \frac{i}{q} \right]^{-1}. $$  \hspace{1cm} (8)

By matching Eq. (7) and Eq. (8), we finally obtain

$$g_0 = -\frac{2\hbar^2}{ma_s}, \hspace{1cm} g_2 = -\frac{\hbar^2}{m} r_e,$$  \hspace{1cm} (9)

recovering the familiar result for the 1D coupling constant $g_0$ and a remarkably simple formula relating $g_2$ to $r_e$. A similar procedure to get a relationship between $g_2$ and $r_e$ has been used for the 3D Bose gas while in 2D the connection between $g_2$ and $r_e$ is much more complicated due to a logarithmic dependence on momentum in the T-matrix.

### III. THERMODYNAMIC PROPERTIES

All the relevant thermodynamic quantities can be derived from the grand canonical ensemble partition function $Z$ or, equivalently, from the grand potential $\Omega$, which are defined as

$$Z = e^{-\beta \Omega} = \int d[\psi, \psi^*] \exp \left( -\frac{1}{\hbar} S[\psi, \psi^*] \right)$$  \hspace{1cm} (10)

adopting a functional integration formalism. In Eq. (10) the Euclidean action is given by

$$S[\psi, \psi^*] = \int_L dx \int_0^{\beta \hbar} d\tau \mathcal{L}[\psi, \psi^*], $$  \hspace{1cm} (11)

where $\mathcal{L}[\psi, \psi^*]$ is defined in Eq. (11). $L$ is the system size and $\beta \equiv 1/(k_B T)$, with $k_B$ the Boltzmann constant and $T$ the temperature.

The mean-field plus Gaussian approximation is obtained by splitting the field $\psi(x, \tau)$ as

$$\psi(x, \tau) = v + \eta(x, \tau)$$  \hspace{1cm} (12)

and expanding the action $S[\psi, \psi^*]$ around the constant field $v$ up to the quadratic terms in the fluctuations field $\eta$ and $\eta^*$. Within the framework of second quantization, this procedure is closely related to the Bogoliubov approximation, where the quartic hamiltonian is disentangled in quadratic terms. The grand potential is then composed by three different contributions

$$\Omega(\mu, v, T) = \Omega_0(\mu, v) + \Omega_g^{(0)}(\mu, v) + \Omega_g^{(T)}(\mu, v, T). $$  \hspace{1cm} (13)

Assuming a real $v$, $\Omega_0(\mu, v)$ is given by the terms in the action independent from $\eta$ and $\eta^*$, namely

$$\Omega_0(\mu, v) = L \left( -\mu v^2 + \frac{1}{2} g_0 v^4 \right) $$  \hspace{1cm} (14)

The contribution of fluctuations has double nature: first, quantum fluctuations arising a zero temperature, i.e. the zero-point energy of collective excitations

$$\Omega_g^{(0)} = \frac{1}{2} \sum_q E_q(\mu, v) $$  \hspace{1cm} (15)

and then the thermal ones

$$\Omega_g^{(T)} = \beta^{-1} \sum_q \log \left( 1 - e^{-\beta E_q(\mu, v)} \right). $$  \hspace{1cm} (16)
In Eq. (15) and Eq. (16), $E_q(\mu, v)$ is the spectrum of bosonic excitations:

$$E_q(\mu, v) = \left[ \frac{\hbar^2 q^2}{2m} - \mu + v^2[g_0 + \tilde{V}(q)] + v^2 \tilde{V}(q)^2 \right]^{1/2}.$$  

(17)

One can eliminate the dependence on the parameter $v$ by imposing the saddle-point condition on $\Omega_0(\mu, v)$, which reads

$$v = \sqrt{\frac{\mu}{g_0}},$$  

(18)

and make the spectrum in Eq. (17) gapless

$$E_q(\mu) = \sqrt{\frac{\hbar^2 q^2}{2m} \left[ 1 + \chi \mu \right] \frac{\hbar^2 q^2}{2m} + 2\mu},$$  

(19)

where finite range effects are encoded in the parameter

$$\chi = \frac{4m g_0}{\hbar^2 g_0}.$$  

(20)

The grand potential $\Omega(\mu, T)$ is easily related to the pressure through the formula $P = -\Omega/L$ so, by replacing Eq. (18) in Eq. (14), one gets the familiar mean-field result

$$P_0(\mu) = \frac{\mu^2}{2g_0},$$  

(21)

which reproduces the LL solution in the weak-coupling regime. Notice that this mean-field contribution depends only on the zero-range strength $g_0$.

### A. Quantum fluctuations

In the continuum limit the contribution arising from quantum fluctuations is given by

$$P^{(0)}_g(\mu) = -\frac{1}{2} \int_{-\infty}^{+\infty} \frac{dq}{2\pi} E_q(\mu).$$  

(22)

This integral is ultraviolet divergent. In order to regularize it we use dimensional regularization \[11, 12\]: the integral of $E_q(\mu)$ is computed over a $D$-dimensional momentum space, finding

$$P^{(0)}_g = -\frac{S_D(2\tilde{\mu})^{D/2+1}}{4(2\pi)^D} \frac{m}{\hbar^2} B \left( \frac{D+1}{2}, -\frac{D+2}{2} \right),$$  

(23)

where $\tilde{\mu} = \mu/(1 + \chi \mu)$, $S_D = 2\pi^{D/2}/\Gamma(D/2)$ with $\Gamma(x)$ the Euler Gamma function, and $B(x,y)$ the Euler beta function. The dimension $D$ is taken complex such that $B(x,y)$ can be analytically continued and expressed in terms of Gamma functions: $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. Finally, setting $D = 1$ we obtain

$$P^{(0)}_g(\mu) = \frac{2}{3\pi} \sqrt{m} \frac{\mu^{3/2}}{\hbar^2} \frac{1}{1 + \chi \mu}.$$  

(24)

This beyond-mean-field Gaussian contribution depends explicitly only on the finite-range parameter $\chi$, given by Eq. (20).

The zero-temperature number density $n$ of the system is immediately derived from

$$n = \frac{\partial}{\partial \mu} \left( P_0(\mu) + P^{(0)}_g(\mu) \right)$$  

(25)

and the corresponding speed of sound reads

$$c_s = \sqrt{\frac{n \partial \mu}{m \partial n}}.$$  

(26)

![FIG. 1: Upper panel: Zero-temperature sound velocity in units of the Fermi velocity $v_F = \hbar n/m$ as a function of the adimensional zero-range interaction parameter $\gamma = (mg_0)/(\hbar n)$ \[44, 45\]. We compare our Gaussian sound velocity $c_s^{(\text{G})}$ (solid line), with the exact solution of LL equation \[44\] (dashed line), and the mean-field result $c_s^{(\text{MF})}$ (dotted line). Lower panel: Gaussian sound velocity for different values of the adimensional parameter $\alpha = \hbar^2 \chi/(m a_s^2) = 4\gamma/\alpha_s$, which encodes the role played by the finite range of the inter-atomic potential.](image-url)
for the universal case \((\chi = 0 \text{ in Eq. (21)})\). We characterize the zero-range interaction strength by the parameter \(\gamma = (m g_0) / (\hbar^2 n) \) \cite{32 44}. The figure shows that our Gaussian theory improves the mean-field result, \(c_s^{(mf)}(\gamma) = \hbar n \sqrt{\gamma} / m\), reliable only in the weak-coupling limit (dotted line). Quite remarkably, the range of applicability of our Gaussian theory is up to \(\gamma \simeq 10\). Only the very strong coupling (Tonks-Girardeau) regime of impenetrable bosons is not captured by the Gaussian theory.

In the lower panel of Fig. 1 we plot the behavior of our Gaussian sound velocity for different values of the finite-range adimensional parameter \(\alpha = (\hbar^2 \chi) / (m a_s^2) = 4 r_e / a_s\). The figure clearly shows that the inclusion of finite range effects deeply affects the zero-temperature sound velocity: at fixed zero-range strength the sound velocity becomes larger with a positive effective range, while the opposite happens for a negative effective range. As previously stated, enhanced quantum fluctuations play a crucial role in low-dimensional atomic quantum gases \cite{23 26}, since they cannot be controlled by lowering the temperature. Thus, the usual mean-field scheme has to be modified in such a way to include Gaussian fluctuations in the thermodynamic and dynamical description provided respectively by the grand potential \(\Omega(\mu)\) and the Gross-Pitaevskii equation (GPE).

It has been shown that these corrections deeply affects the stability of 2D and 1D Bose-Bose mixtures, enabling a transition from a homogeneous ground state to liquid-like self-bound states \cite{24 30}. Moreover, while a universal GPE-based approach is reliable only when \(r_e \ll a_s \ll d\), with \(d\) being the average distance between atoms, it has been shown \cite{17 19 21} that one can greatly enhance finite-range effects by means of Fano-Feshbach resonances. In regimes where \(r_s \approx a_s\), deviation from universalities is relevant, and every dynamical picture of the system has to take into account not only quantum fluctuations, but also finite-range corrections modelled around Eq. (24). In the present experiments one can tune the scattering length \(a_s\) and in turn the adimensional parameter \(\alpha = 4 r_e / a_s\) of our theory, because the effective range \(r_e\) remains practically fixed and quite small (\(\simeq 10^{-10}\) m). The nontrivial behavior of the sound velocity \(c_s\) shown in the lower panel of Fig. 1 can be observed experimentally, at fixed \(a_s\) and \(r_e\), by varying the number density \(n\) and consequently the adimensional interaction parameter \(\gamma\). Our approach can be seen as complementary to the one in \cite{46}, where collective excitations of the 1D Bose gas are studied within a hydrodynamic framework based on a GP-like equation. Differently from the usual mean-field approach, they still consider a zero-range two-body interaction, but the chemical potential is modelled around the LL exact solution.

### B. Thermal fluctuations

The contribution of thermal fluctuations is obtained from Eq. (10). In the continuum limit

\[
P_g^{(T)}(\mu, T) = \frac{1}{\pi \beta} \int_0^{+\infty} \frac{d \pi}{d \xi} \left( \frac{1}{e^{\pi \beta E_q} - 1} \right).
\]

By changing the integration variable to \(x = \beta E_q(\mu)\), one gets

\[
P_g^{(T)}(\mu, T) = \frac{1}{\beta} \int_0^{+\infty} \frac{dx}{\sqrt{2 m \mu}} \left( \frac{1}{e^{2 \beta E_q} - 1} \right).
\]

where

\[
q(x) = \sqrt{\frac{2 m \mu}{\hbar^2 (1 + \chi \mu)}} \left[ -1 + \sqrt{1 + \left( \frac{1 + \chi \mu}{\mu^2} \right) (k_B T)^2} \right].
\]

After having inserted Eq. (28) in Eq. (29), an expansion at low temperatures finally leads us to the following results

\[
P_g^{(T)}(\mu, T) = \frac{\pi}{6} \sqrt{\frac{m}{\hbar^2 \mu}} (k_B T)^2 \left[ 1 - \frac{\pi^2}{20} \frac{1 + \chi \mu}{\mu^2} (k_B T)^2 \right].
\]

Therefore, by recalling Eq. (28) and Eq. (29), the grand canonical ensemble pressure is given by

\[
P(\mu, T) = P_0(\mu) + P_g^{(0)}(\mu) + P_g^{(T)}(\mu, T).
\]

In Fig. 2 we report the pressure \(P(\mu, T)\) as a function of temperature \(T\) at fixed chemical potential \(\mu\) for three values of the finite-range adimensional parameter \(\alpha = 4 r_e / a_s\), obtained with the low-temperature analytical formula \cite{42}. At finite temperature non-universal effects slightly increase as ratio \((k_B T / \mu)^4\) grows. This is explained by recalling that the details of the inter-atomic potential become more relevant when atoms scatter at higher energy.
We have derived a non-universal equation of state for the 1D Bose gas providing a truthful connection between the scattering parameters and the coupling constants of the interaction potential. The non-universal 1D equation of state is quite different with respect to the 3D [14] and 2D [24] ones, and its quantum fluctuations crucially depend on the scattering length and the dimensional ratio $r_e/a_s$, where $r_e$ is the 1D effective range of the inter-atomic potential. Our beyond-mean-field description of the 1D Bose gas can help the understanding of a wide range of problems. For instance, the role of quantum fluctuations, thermal fluctuations and finite-range corrections in the stability and structural properties of topological defects (dark solitons) and localized self-bound states (i.e. bright solitons).

**Acknowledgments.** The authors acknowledge for partial support the 2016 BIRD project ”Superfluid properties of Fermi gases in optical potentials” of the University of Padova. The authors thank Giacomo Bighin and Flavio Toigo for enlightening discussions.

[1] H. Bethe, Z. Physik 71, 205 (1931).
[2] T. Giamarchi, *Quantum Physics in One Dimension*, (Clarendon Press, Oxford, 2003).
[3] E. Chow, P. Del Curry, and D. R. Haviland, Phys. Rev. Lett. 81, 204 (1998).
[4] D. B. Haviland, K. Andersen, and P. Agren, J. Low Temp. Phys. 118, 733 (2000).
[5] B. Paredes, A. Widera, V. Murg, O. Mandel, S. Folling, I. Cirac, G. V. Shlyapnikov, T. W. Hansch, and I. Bloch, Nature (London) 429, 277 (2004).
[6] T. Kinoshita, T. Wenger, and D. S. Weiss, Science 305, 1125 (2004).
[7] E. Haller, M. Gustavsson, M. J. Mark, J. G. Danzl, R. Hart, G. Pupillo, and H.-C. Nagerl, Science 325, 1224 (2009).
[8] E. Haller, R. Hart, M. J. Mark, J. G. Danzl, L. Reichsoller, M. Gustavsson, M. Dalmonte, G. Pupillo, and H.-C. Nagerl, Nature (London) 466, 597 (2010).
[9] E. H. Lieb and W. Lininger, Phys. Rev. 130, 1605 (1963).
[10] E. H. Lieb and W. Lininger, Phys. Rev. 130, 1616 (1963).
[11] C. N. Yang and C. P. Yang, J. Math. Phys. 10, 1115 (1969).
[12] C. N. Yang and C. P. Yang, Phys. Rev. A 2, 154 (1970).
[13] A. Parola, L. Salasnich, and L. Reatto, Phys. Rev. A 57, R3180 (1998).
[14] E. Braaten, H.-W. Hammer, and S. Hermanns, Phys. Rev. A 63, 063609 (2001).
[15] R. Roth and H. Feldmeier, Phys. Rev. A 64, 043603 (2001).
[16] H. Fu, Y. Wang, and B. Gao, Phys. Rev. A 67, 053612 (2003).
[17] J. J. Garcia-Ripoll, V. V. Konotop, B. A. Malomed, and V. M. Perez-Garcia, Mathematics and Computers in Simulation 62, 21 (2003).
[18] J. O. Andersen, Rev. Mod. Phys. 76, 599 (2004).
[19] A. Collin, P. Massignan, and C. J. Pethick, Phys. Rev. A 75, 013615 (2007).
[20] N. T. Zinner and M. Thogersen, Phys. Rev. A 80, 023607 (2009); M. Thogersen, N. T. Zinner, and A. S. Jensen, Phys. Rev. A 80, 043625 (2009).
[21] H. Veksler, S. Fishman, and W. Ketterle, Phys. Rev. A 90, 023620 (2014).
[22] F. Sgarlata, G. Mazzarella, and L. Salasnich, J. Phys. B: At. Mol. Opt. Phys. 48, 115301 (2015).
[23] A. Cappellaro and L. Salasnich, Phys. Rev. A 95, 033627 (2017).
[24] L. Salasnich, Phys. Rev. Lett. 118, 130402 (2017).
[25] N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966).
[26] P. C. Hohenberg, Phys. Rev. 158, 383 (1967).
[27] P. A. Murthy, I. Boettcher, L. Bayha, M. Holzmann, D. Kedar, M. Neidig, M. G. Ries, A. N. Wenz, G. Zrn and S. Jochim, Phys. Rev. Lett. 115, 010401 (2015).
[28] S. Murmann, F. Deuretzbacher, G. Zurn, J. Björn, S. M. Reimann, L. Santos, T. Lompe, and S. Jochim, Phys. Rev. Lett. 115, 215301 (2015).
[29] D.S. Petrov, Phys. Rev. Lett. 115, 155302 (2015).
[30] D.S. Petrov, Phys. Rev. Lett. 117, 100401 (2016).
[31] M. Schmitt, M. Wenzel, F. Bittcher, I. Ferrier-Barbut and Tilman Pfau, Nature 539, 259 (2016).
[32] L. Chomaz, S. Baier, D. Petter, M. J. Mark, F. Wichtler, L. Santos, F. Ferlaino, Phys. Rev. X 6, 041039 (2017).
[33] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney, L. Tarruell, arXiv:1708.07806 (2017).
[34] E. Braaten and A. Nieto, Eur. Phys. J. B 11, 143 (1999).
[35] A. Altland and B. Simons, *Condensed Matter Field Theory* (Cambridge University Press, Cambridge, England, 2006).
[36] V. E. Barlette, M. M. Leite, and S. Adhikari, Eur. J. Phys. 21, 435-440 (2000).
[37] H. T. C. Stoof, D. B. M. Dickerscheid, and K. Gubbels, *Ultracold Quantum Fields* (Springer, Dordrecht, 2009).
[38] M. Olshanii, Phys. Rev. Lett. 81, 938 (1998).
[39] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac and M. Rigol, Rev. Mod. Phys. 83, 1405 (2011).
[40] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, Boston, 1971).
[41] G. ’t Hooft and J. M. G. Veltman, Nucl. Phys. B 44, 189 (1972).
[42] D. R. Phillips, S. R. Beane and T. D. Cohen, Annals Phys. 263, 255-275 (1998).
[43] L. Salasnich and F. Toigo, Phys. Rep. 640, 1 (2016).
[44] G. De Rosi, G. E. Astrakharchik, and S. Stringari, Phys. Rev. A 96, 013613 (2017).
[45] T. Kaminaka and M. Wadati, Phys. Lett. A 375, 2460 (2011).
[46] S. Choi, V. Dunjko, Z.D. Zhang, M. Olshanii, Phys. Rev. Lett. 115, 115302 (2015).