It is demonstrated that within the framework of the second quantization, the quantum Hamiltonian operator for a free transverse field reveals an alternative set of states satisfying the eigenstate functional equations. The construction is based on extensions of the quadratic form of the transverse Laplace operator, which are used as a source of spherical basis functions with singularity at the origin. This basis naturally replaces the basis of plane or spherical waves, which is used to separate variables with the help of the Fourier transform or transition to spherical coordinates.

**Bibliography:** 15 titles.

**Introduction**

The second quantization approach [1,2] has been the basic mechanism for constructing the quantum field theory since its inception in the first half of the 20th century. Later, in real calculations of scattering matrix elements, the technique of Feynman diagrams, based on the Lagrangian formulation of the classical theory, was widely used. Unlike this technique, one of the advantages of the second (canonical) quantization is that it provides a description in terms of the quantum Hamiltonian operator. In a correctly defined quantum system, the latter object must be a selfadjoint operator in a Hilbert space. The finite-dimensional examples [3,4] show that after renormalization and elimination of singularities, a candidate for the Hamiltonian may turn out to be a symmetric but still not selfadjoint operator representing a free particle on a restricted space of states. Such a candidate can be extended to a selfadjoint operator, however this procedure is ambiguous, since it requires the introduction of an extension parameter (the dimensional transmutation phenomenon in [4,6]). A similar effect, apparently, can be observed for systems of infinite number of harmonic oscillators. We will argue that the quadratic part of the quantum Hamiltonian of a free transverse vector field

\[ H_0 = \mathcal{H}_0 = \int_{\mathbb{R}^3} \left( -\frac{\delta}{\delta A_k(x)} \frac{\delta}{\delta A_j(x)} A_j(x) A_j(x) \right) d^3 x, \quad \partial_k A_k = 0, \]

which appears, for example, in electrodynamics or as a result of renormalization of a gauge theory, is a limiting case of selfadjoint extension of some symmetric operator defined on a restricted set of states. At the same time, generic selfadjoint extensions turn out to depend on an extension parameter and for that reason do not possess scale invariance.

Due to the lack of adequate definition of a scalar product on the space of functionals, that describe states of the stationary picture of the quantum field theory, we will not make strict statements about the selfadjointness or symmetricity. Instead, we provide a sketch for the new vacuum state and its excitations (the Fock space). These states satisfy the equations for “eigenstate” functionals and form a hierarchy of creation and annihilation of particles. It is natural to require that these equations coincide with the functional equations

\[ \mathcal{H}_0 \Phi_{\sigma_n}(A) = \Lambda_{\sigma_n} \Phi_{\sigma_n}(A) \]

for the eigenstates of Hamiltonian \( \mathcal{H}_0 \), but at the same time they could be defined on a set of functionals satisfying other conditions in the vicinity of the “boundary” functions. As

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such boundary points in the configuration space, where the boundary conditions of the new functional space are specified, one can take the functions with singularities of the form
\[ \tilde{A}(x) \sim \frac{A_0}{|x|}, \quad |x| \to 0, \quad \vec{x} \in \mathbb{R}^3. \] (1)

Thus the possible self-adjoint extensions of the theory depend on a certain preferred point in the three-dimensional space associated with localization in the interaction terms. When the self-interaction is “turned on,” such extensions of the Hamiltonian and the states associated with them most likely turn out to be unstable. However, their presence can contribute to the scattering matrix as intermediate states for particles interacting via the transverse field.

For the sake of brevity, we use the following notations for the scalar and vector products:
\[ \vec{A} \cdot \vec{B} = A_j B_j, \quad (\vec{A} \times \vec{B})_n = \epsilon_{njk} A_j B_k, \quad j, k, n = 1, 2, 3, \]
and the summation is always performed over the repeated indices.

1. Finite-dimensional examples with singular interactions

We start with finite-dimensional examples from quantum mechanics in order to generalize some of their properties to infinite-dimensional case. Let
\[ H_\varepsilon = \Delta + \varepsilon \delta(x) = -\frac{\partial^2}{\partial x^2} + \varepsilon \delta(x) \]
be the Hamiltonian of a particle existing in the two- or three-dimensional space and interacting with a \( \delta \)-potential centered at the origin. The Hamiltonian \( H_\varepsilon \) does not have a correct definition in terms of a closed operator in a Hilbert space. However, one can consider the action of \( H_\varepsilon \) on the set of smooth functions decreasing at the origin along with the derivatives. This action corresponds to the symmetric operator
\[ H : H f(\vec{x}) = \Delta f(\vec{x}) = -\frac{\partial^2}{\partial x^2} f(\vec{x}), \]
which, evidently, does not take into account the potential \( \varepsilon \delta(\vec{x}) \). In terms of the spherical coordinates in the two-dimensional space,
\[ \vec{x} = \vec{x}(r, \varphi) = \begin{pmatrix} r \cos \varphi \\ r \sin \varphi \end{pmatrix}, \quad 0 \leq r, \quad 0 \leq \varphi < 2\pi, \]
or in the three-dimensional space,
\[ \vec{x} = \vec{x}(r, \psi, \varphi) = \begin{pmatrix} r \cos \psi \cos \varphi \\ r \cos \psi \sin \varphi \\ r \sin \psi \end{pmatrix}, \quad 0 \leq r, \quad 0 \leq \psi \leq \pi, \quad 0 \leq \varphi < 2\pi, \] (2)
the action of \( H_0 \) has the following form. If the scalar function \( f(\vec{x}) \) is represented in terms of a sum of spherical harmonics \( e^{i\varphi} \) or \( Y_{lm}(\psi, \varphi) \) with coefficients depending on the radial variable,
\[ f_2(\vec{x}) = f_2(\vec{x}(r, \varphi)) = \sum_{0 \leq l} \frac{1}{\sqrt{r}} u_l(r) \frac{e^{i\varphi}}{\sqrt{2\pi}}, \]
\[ f_3(\vec{x}) = f_3(\vec{x}(r, \psi, \varphi)) = \sum_{0 \leq |m| \leq l} \frac{1}{r} u_{lm}(r) Y_{lm}(\psi, \varphi), \]
then the corresponding operation $\Delta$ acts as follows:

$$\Delta f_2(\vec{x}) = \sum_{0 \leq l} \frac{1}{\sqrt{r}} T_{l-\frac{1}{2}} u_l(\vec{r}) e^{il\varphi} \sqrt{2\pi},$$
$$\Delta f_3(\vec{x}) = \sum_{0 \leq |m| \leq l} \frac{1}{r} T_l u_{lm}(\vec{r}) Y_{lm}(\psi, \varphi),$$

where

$$T_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}, \quad T^{-1}_l(r, s) = \frac{1}{2l+1} \left( \frac{s^{l+1}}{r^l} \theta(r-s) + \frac{r^{l+1}}{s^l} \theta(s-r) \right).$$

Due to the orthonormality of the sets of spherical harmonics, each of the scalar products

$$(f, g)_{\mathbb{R}^2} = \int_{\mathbb{R}^2} \vec{f}(\vec{x}) \cdot d^2 x, \quad (f, g)_{\mathbb{R}^3} = \int_{\mathbb{R}^3} \vec{f}(\vec{x}) \cdot d^3 x,$$

is transferred to the coefficient functions $u_l(\vec{r})$ as a plain scalar product on the half-axis

$$(u, v) = \int_0^\infty u(r) v(r) \, dr.$$

The operators $T_{l-\frac{1}{2}}$ and $T_l$, defined on the set of smooth functions vanishing at the origin along with the derivatives, are essentially selfadjoint with respect to the scalar product (5) for $l \geq 1$. At the same time, the operators $T_{\frac{1}{2}}$ and $T_0$ that act on the latter set are symmetric operators with deficiency indices $(1, 1)$. Their selfadjoint extensions $T_{\frac{1}{2}}^\kappa$, $T_0^\kappa$ have continuous spectrum eigenfunctions of the form

$$v_\lambda(r) = \sqrt{\lambda^r(\alpha_{\nu \lambda} J_0(\lambda r) + \beta_{\nu \lambda} Y_0(\lambda r))}, \quad T_{\frac{1}{2}}^\kappa v_\lambda = \kappa^2 v_\lambda,$$
$$u_\lambda(r) = \alpha_{\nu \lambda} \sin \lambda r + \beta_{\nu \lambda} \cos \lambda r, \quad T_0^\kappa u_\lambda = \kappa^2 u_\lambda,$$
$$\alpha_{\{u, v\}, \lambda} = \alpha_{\{u, v\}, \kappa}, \quad \beta_{\{u, v\}, \lambda} = \beta_{\{u, v\}, \kappa},$$

along with, possibly, some eigenfunctions of the discrete spectrum. The actions of extensions the $T_{\frac{1}{2}}^\kappa$ and $T_0^\kappa$ coincide with the differential operations $T_{l-\frac{1}{2}}$ and $T_0$, respectively.

Therefore, going back to Cartesian coordinates, the symmetric operators $H$ can be extended to selfadjoint operators $H_{\frac{1}{2}}$ and $H_0$ defined on the set of functions satisfying the asymptotic conditions

$$\lim_{r \to 0} \frac{f(\vec{x}(r))}{\ln r} = \kappa \lim_{r' \to 0} \frac{f(\vec{x}(r'))}{\ln r'},$$

or

$$\lim_{r \to 0} r f(\vec{x}(r)) = -\kappa \lim_{r \to 0} (1 + r \frac{\partial}{\partial r}) f(\vec{x}(r)),$$

at the origin (see Eqs. (3.43), (3.44) in [4]). The action of $H_{\frac{1}{2}}^\kappa$, $H_0^\kappa$ is still the sum of squares of the second derivatives $\Delta$ on the corresponding set of functions on $\mathbb{R}^2$ or $\mathbb{R}^3$.

The extensions $H_{\frac{1}{2}}^\kappa$ and $H_0^\kappa$ depend on the parameter $\kappa$, the dimension of which originates from the presence of the dimension of $H$: $[H] = [x]^{-2}$. From a physical point of view, one can say that $H_{\frac{1}{2}}^\kappa$ and $H_0^\kappa$ appear as a result of renormalization of the respective operators $H_\varepsilon$ as $\varepsilon \to 0$. Meanwhile, the singular functions with asymptotics (6), (7) at the origin, emerging in the domains of $H_{\frac{1}{2}}^\kappa$, $H_0^\kappa$, represent the trace of the renormalized singular interaction $\varepsilon \delta(\vec{x})$. In
the case of a particle in two-dimensional space, one has a phenomenon of dimensional trans-
mutation: the dimensionless parameter $\epsilon$ upon renormalization is replaced by a dimensional
parameter $\kappa$ [5, 6].

As another example, one can consider two- and three-dimensional operators of the form

$$\Delta + \frac{\epsilon}{|x|^2} = -\frac{\partial^2}{\partial x_k^2} + \frac{\epsilon}{|x|^2},$$

with $\epsilon$ being a dimensionless parameter. Such operators are closed symmetric operators at
finite $\epsilon$ in some vicinity of zero (in two-dimensional case, $\epsilon$ is positive). When one tries to
construct a function satisfying the eigenstate equation, it is clear that the increase of the di-
vergence by $|x|^{-2}$ coming from the action of the potential cancels out with the divergence from
the action of the Laplacian. Therefore, operator (8) has an alternative basis of locally square-
integrable “eigenfunctions” that behave as $|x|^\eta$ near the origin ($\eta = -\sqrt{\epsilon}$ in two dimensions
and $\eta = -\frac{1}{4}(1 + \sqrt{1 + 4\epsilon})$ in three dimensions), that is, it allows selfadjoint extensions (this by
no means is a mathematically strict explanation of the Frobenius method [7]). One can show
that in the limit $\epsilon \to 0$, these extensions are continuously transformed into the corresponding
operators $H_2^\epsilon$ and $H_3^\epsilon$.

2. THREE-DIMENSIONAL TRANSVERSE FIELD THEORY

From the point of view of the theory of operators in a Hilbert space, the example of the last
section shows that the restriction of the domain of Laplacian $\Delta$ to the set of smooth functions
vanishing at the origin leads to a symmetric operator and the appearance of ambiguity in the
definition of the Hamiltonian of the system. At the same time, the interaction that disappears
as a result of renormalization, serves only as a catalyst for this ambiguity, since it distinguishes
a point in the space. In the present paper, we try to generalize the lessons from the finite-
dimensional example to the case of field theory. We cannot really speak of selfadjointness in the
case of an infinite dimensional configurational space as long as we do not have a possibility
to define the scalar product on a wide enough class of functionals. For this reason, we restrict
ourselves to the description of eigenvectors of such systems after renormalization, and provide
an instructive example of an alternative set of vacuum state and its excitations.

Consider the following Hamiltonian functional of the classical mechanics:

$$\mathcal{H}_c = \iint_{\mathbb{R}^3} E_j(\bar{x}) P_{kj}^\epsilon(\bar{x}', \bar{x}) P_{kj'}^\epsilon(\bar{x}', \bar{y}) E_{j'}(\bar{y}) \, d^3x \, d^3x' \, d^3y$$

$$+ \int_{\mathbb{R}^3} \left((\partial_k A_j(\bar{x}))^2 + \epsilon(A^3(\bar{x}) + \ldots)\right) \, d^3x.$$

where $A_k^a(x)$, $E_k^a(x)$ are the fields of generalized coordinates and their conjugate momenta in
the three-dimensional space, which satisfy the transversality conditions

$$\partial_k A_k^a = 0, \quad \partial_k E_k^a = 0.$$ 

Denote by $\epsilon(A^3 + \ldots)$ the homogeneous terms of dimension $[x]^{-4}$ of higher order in coordinates
$A_k^a$, which also include the interaction. The matrix $P_{kj}^\epsilon$ is a projector from the transverse to
the covariant-transverse field sets,

$$P_{kj}^\epsilon = \delta_{kj} - \partial_k M^{-1}(\partial_j - \epsilon A_j), \quad M = (\partial_j - \epsilon A_j)\partial_j,$$

and $\epsilon$ is a small dimensionless parameter of the theory. The fields $A_k^a(x)$, $E_k^a(x)$ can also have
an internal symmetry index $a$ over which the summation is assumed everywhere. The action
of the covariant derivative (and of all objects that contain it) can be nontrivial in this index,
\[(\partial_k - \epsilon A_k)^a B^b = \partial_k B^a - \epsilon A_k^c t^{abc} B^b.\]
In what follows, we consider only those quadratic terms for which the nontriviality of the action with respect to this index reduces to summation, provided that the matrices $t^{abc}$ are orthogonal for different $c$. In this way, the components corresponding to different values of the upper index of the field $A_k^a(x)$ are separated.

An actual physical example of the Hamiltonian of type (9) is given in the third chapter of the book [8]. In Eq. (2.5) there, the following Hamiltonian density is presented:
\[h = \frac{1}{2}(E_k^a)^2 + \frac{1}{4}(\partial_k A_j^a - \partial_j A_k^a - \epsilon[A_j, A_k]^a)^2,\]  
(11)
where $\vec{A}(\vec{x})$ is the transverse field, and the constraint condition (2.41) is imposed on the conjugate momentum $E_k^a$,
\[(\partial_k - \epsilon A_k)^a E_k^b = 0.\]  
(12)
After splitting the momentum $\vec{E}(\vec{x})$ into its longitudinal and transverse components
\[E_k = E_k^L + E_k^T, \quad \partial_k E_k^T = 0, \quad E_k^L = \partial_k \xi(\vec{x}),\]
condition (12) implies that
\[\xi(\vec{x}) = -M^{-1}(\partial_t - \epsilon A_t) E_t^T, \quad M = (\partial_j - \epsilon A_j) \partial_j,\]
\[E_k = (\delta_{kl} - \partial_k M^{-1}(\partial_t - \epsilon A_t)) E_t^T,\]
and the Hamiltonian density (11), after integrating by parts, is transformed to the following form of Eq. (9):
\[h = \frac{1}{2}((\delta_{kl} - \partial_k M^{-1}(\partial_t - \epsilon A_t)) E_t^T)^2 + \frac{1}{2}(\partial_k A_j^a)^2 + \epsilon \partial_t A_j^a [A_j, A_k]^a + \frac{1}{2} \epsilon^2 ([A_j, A_k]^a)^2.\]
In the coordinate representation, the quantum counterpart $\mathcal{H}_\epsilon$ of the Hamiltonian $\mathcal{H}_0$ acts on the functionals $\Phi(A_j(\vec{x}))$ in accordance with expression (9) with $A_j(\vec{x})$ replaced by the operator of multiplication by $A_j(\vec{x})$, and $E_j(\vec{x})$ replaced by the variation $\delta_A \sqrt{A_j(\vec{x})}$ (we assume that the Planck constant equals one, and do not discuss the ordering of the canonical pairs),
\[\mathcal{H}_\epsilon = -\int_{\mathbb{R}^3} P_{k}^\epsilon(x', \vec{x}) P_{k'}^\epsilon(x', \vec{y}) \frac{\delta}{\delta A_j(\vec{x})} \frac{\delta}{\delta A_{j'}(\vec{y})} d^3 x' d^3 x' d^3 y + \int \left((\partial_k A_j(\vec{x}))^2 + \epsilon (A_j^3(\vec{x}) + \ldots)\right) d^3 x.\]
During the renormalization procedure as $\epsilon \to 0$, the higher order terms $\epsilon (A_j^3 + \ldots)$ disappear, while the projector $P_{k}^\epsilon$ turns into the orthogonal projector onto the transverse component
\[P_{k}^{\epsilon \to 0} P_{k} = \delta_{k} - \partial_k \partial_k \partial_j, \quad P_{k}^{T} P_{k} = P_{k}, \quad P_{k} P_{k} = P_{k}.\]  
(13)
However, in general, there is a difference between the result $\mathcal{H}_{\text{ren \epsilon}}$ of renormalization of $\mathcal{H}_\epsilon$ as $\epsilon \to 0$ and the first term $\mathcal{H}_0$ in the expansion of $\mathcal{H}_\epsilon$ in $\epsilon$ in the vicinity of zero,
\[\mathcal{H}_\epsilon = \mathcal{H}_0 + \sum_{n \geq 1} \frac{\partial^n \mathcal{H}_\epsilon}{\partial \epsilon^n} \bigg|_{\epsilon = 0} \frac{\epsilon^n}{n!}.\]  
(14)
Treatment of divergences in $\mathcal{H}_\epsilon$ requires introducing regularization parameters, which in turn, are related to $\epsilon$ (see the resolvent example in [5] or [6]). As a result, some or even all terms in expansion (14) may become finite even at $\epsilon = 0$ and the renormalized Hamiltonian $\mathcal{H}_{\text{ren \epsilon}}$ is not equal to $\mathcal{H}_0$. Below we assume that the contributions of these finite terms are such that
the final Hamiltonian $\mathcal{H}_{\text{ren}}$ only acquires a new set of eigenstates, while retaining its action in the form of $\mathcal{H}_0$. In the last section, we have already seen how this picture is realized in the finite-dimensional examples.

More specifically, our assumption is that the Hamiltonian $\mathcal{H}_\varepsilon$ may have singularities via the projector $P_{k_j}$ around the boundary functions, which locally behave as $|x|^{-1}$. Homogeneous and interaction terms of higher orders have singularities of the same kind. By analogy with example (8), these two types of singularities may cancel each other, and supply the renormalized quantum Hamiltonian with a domain having new boundary conditions and, accordingly, different spectral properties.

To see this, we consider the action of the quantum Hamiltonian operator $\mathcal{H}_0$ from (14) on the functionals $\Phi(A)$,

$$\mathcal{H}_0 \Phi(A) = -\iint_{\mathbb{R}^3} \frac{\delta}{\delta A_k(x)} P_{k_j}(\vec{x}, \vec{y}) \frac{\delta}{\delta A_j(y)} d^3x d^3y \Phi(A) + Q(A) \Phi(A). \tag{15}$$

Here, $P_{k_j}$ is the projector (13) onto the transverse subspace, and $Q(A)$ is the quadratic form of the Laplace operator $\Delta$,

$$Q(A) = \int_{\mathbb{R}^3} (\partial_k A_j(x))^2 d^3x \tag{16}$$

$$= -\int_{\mathbb{R}^3} A_j(x) \frac{\partial^2}{\partial x_k^2} A_j(x) d^3x = \int_{\mathbb{R}^3} A_j(x) \Delta A_j(x) d^3x. \tag{17}$$

The unnormalized vacuum state of the operator $\mathcal{H}_0$ can be constructed as a Gaussian functional

$$\Phi_0(A) = \exp \left\{ -\frac{1}{2} (A, P\Delta^{1/2} PA) \right\}. \tag{18}$$

Then the $n$-particle excitations (the Fock states) are

$$\Phi_{\sigma_n}(A) = \iint \sigma_n^{j_1\ldots j_n}(\vec{x}_1, \ldots, \vec{x}_n) b_{j_1}(\vec{x}_1) \ldots b_{j_n}(\vec{x}_n) d^3x_1 \ldots d^3x_n \Phi_0(A),$$

where the $\sigma_n$ are some Bose-Einstein symmetric functions, and the $b_j(\vec{x})$ are the creation operators from the corresponding pairs

$$b_j = P_{jk} \left( \frac{\delta}{\delta A_k} - \Delta^{1/2}_{j'} A_{j'} \right), \quad a_j = P_{jk} \left( \frac{\delta}{\delta A_k} + \Delta^{1/2}_{j'} A_{j'} \right)$$

of creation and annihilation operators. In this case, an essential role is played by the fact that the projector $P$ commutes with $\Delta$, and hence with any of its functions, for example, with $\Delta^{1/2}$ ($\Delta$ should be defined as a selfadjoint operator).

The quantum operator $\mathcal{H}_0$ intermixes the functionals $\Phi_{\sigma_n}$. However, it is easily seen that it leaves $n$-particle subspaces invariant. For further diagonalization, it is necessary to pass to spectral representation of the operator $\Delta$, which we do below in the framework of a more general approach. The main idea of this approach is to construct an alternative vacuum state via a method which by analogy with the second quantization can be called the method of second selfadjoint extensions.

### 2.1. Method of second selfadjoint extensions.

When the quantum Hamiltonian has the form Eq. (15), and the positive closed quadratic form $Q(A)$ admits nontrivial extensions, there is a natural way to construct an alternative set of "eigenstates" of the operator $\mathcal{H}_0$. 481
In general, a closed semibounded quadratic form $Q(A)$ can be defined with the help of a closed operator $S$, symmetric or selfadjoint with respect to the scalar product $(\cdot, \cdot)$, via the natural formula

$$Q(A) = (A, SA) = (SA, A).$$

Here, the domain $\mathcal{D}_S$ of the operator $S$ is contained in the domain $\mathcal{D}_Q$ of the form $Q$, and the latter, in general, differs from the former quite significantly. (One can see that in order for the field $A$ from (17) to fall into the domain of $\Delta$, it must be twice differentiable, while the existence of the integral in (16) requires only the first derivative of $A$.) As long as $Q(A)$ is semibounded, the symmetric operator $S$ has selfadjoint extensions $S_\kappa$. One of them, the Friedrichs extension [9], also defines the form $Q$, and the others (semibounded extensions) define other quadratic forms $Q_\kappa$ (for general information on quadratic forms, see [11, Sec. VIII.6]). In some cases (including a large number of simple examples), these quadratic forms are extensions of the original form

$$Q \subset Q_\kappa,$$

that is, the domain of $Q$ is contained in the closure of the domain of $Q_\kappa$,

$$\mathcal{D}_Q \subset \overline{\mathcal{D}}_{Q_\kappa},$$

and for all vectors $A$ from $\mathcal{D}_Q$, we have

$$Q(A) = Q_\kappa(A), \quad A \in \mathcal{D}_Q.$$ 

In particular, [12] gives spherically symmetric extensions of the quadratic form (16),

$$Q_\kappa(A) = \lim_{r \to 0} \left( \int_{\mathbb{R}^3 \setminus B_r} \left| \frac{\partial A_j}{\partial x_k} \right|^2 d^3 x - \left( \frac{5}{3r^2} + \frac{44}{27} \kappa \right) \int_{\partial B_r} |\vec{A}(\vec{x})|^2 d^2 s \right), \quad (19)$$

for transverse vectors $\vec{A}(\vec{x})$ with respect to the scalar product

$$(\vec{A}, \vec{B})_{\mathbb{R}^3} = \int_{\mathbb{R}^3} \vec{A}_j(\vec{x}) \vec{B}_j(\vec{x}) d^3 x.$$ 

Here, $B_r$ denotes a ball of radius $r$, centered at any preferred point. For all real-valued vector fields that are regular at this point (in what follows, we take it as the origin), the value of the form $Q_\kappa$ obviously equals the value of form (16),

$$Q_\kappa(A) = Q(A) = \int_{\mathbb{R}^3} \left( \frac{\partial A_j}{\partial x_k} \right)^2 d^3 x.$$

But the domain of the form $Q_\kappa$ also includes fields with singularities of type (1) for three transverse components of angular momentum $l = 1$. This happens because for such fields, the singularities of the order $r^{-1}$ in the volume integral in (19) cancel out with singularities of the integral over the boundary of the ball $B_r$. Notably, the domains of all nontrivial extensions $Q_\kappa$ coincide and do not depend on $\kappa$. In addition, the coefficient at the dimensional parameter $\kappa$ in equation (19) can be taken arbitrary, the value $\frac{44}{27}$ is chosen to match the boundary condition (29), which is introduced later.

Next, we note that since the singular fields of the form (1) are inadmissible for higher order terms of the Hamiltonian $\mathcal{H}_\varepsilon$, we can require that the basic relations for the “eigenfunctionals” $\Phi_{\sigma_n}(A)$ of the operator $\mathcal{H}_{\text{ren}}$,

$$\mathcal{H}_{\text{ren}} \Phi_{\sigma_n}(A) = \Lambda_{\sigma_n} \Phi_{\sigma_n}(A),$$

hold on the domain of the quadratic form $Q(A)$ only. But these relations also hold on such a domain for the quantum operator of the form $Q_\kappa(A)$ in place of the form $Q(A)$. When
the square root is extracted and substituted into a Gaussian integral of the form (18), the form $Q_k(A)$, yields a fundamentally different vacuum state and a different set of excitations corresponding to a different operator $\mathcal{H}_{\text{ren}} \neq \mathcal{H}_0$. It follows that the operator $\mathcal{H}_0$ is a selfadjoint extension of some symmetric operator defined on a set of functionals rapidly vanishing near boundary functions with singularities of type (1). This symmetric operator also admits other extensions $\mathcal{H}_{\text{ren}}$, i.e., those the “eigenstates” for which are constructed with the help of the quadratic form $Q_k(A)$.

It should be noted that instead of (19) within the framework of the method under consideration, one can use any other (possible) extension of form (16); here we considered a ready-made example from [12]. For a more detailed study, we turn to spherical coordinates and select from the field variables a subspace of angular momentum $l = 1$.

### 2.2. Vector spherical harmonics and separation of variables.

Using the scalar spherical functions $Y_{lm}(\psi, \varphi)$, we introduce three vector spherical harmonics (VSH) [13]:

- $\vec{Y}_{lm}(\Omega) = \frac{\vec{r}}{r} Y_{lm}, \quad 0 \leq l, \quad |m| \leq l,$
- $\vec{\Psi}_{lm}(\Omega) = \vec{r}^{-1} r \partial Y_{lm}, \quad 1 \leq l, \quad |m| \leq l, \quad \vec{l} = \sqrt{l(l+1)},$
- $\vec{\Phi}_{lm}(\Omega) = \vec{r}^{-1} (\vec{r} \times \vec{\partial}) Y_{lm}, \quad 1 \leq l, \quad |m| \leq l,$

which are functions of angular variables $\Omega = (\psi, \varphi)$. These functions are mutually orthogonal and normalized in terms of integration over sphere,

$$
\int_{S^2} \bar{Y}_{lm}(\Omega) \tilde{Y}_{lm'}(\Omega) \, d\Omega = 0, \quad \int_{S^2} \bar{Y}_{lm}(\Omega) \bar{Y}_{lm'}(\Omega) \, d\Omega = \delta_{ll'} \delta_{mm'},$
$$
\int_{S^2} \bar{\Psi}_{lm}(\Omega) \tilde{Y}_{lm'}(\Omega) \, d\Omega = 0, \quad \int_{S^2} \bar{\Psi}_{lm}(\Omega) \bar{\Psi}_{lm'}(\Omega) \, d\Omega = \delta_{ll'} \delta_{mm'},$
$$
\int_{S^2} \bar{\Phi}_{lm}(\Omega) \tilde{Y}_{lm'}(\Omega) \, d\Omega = 0, \quad \int_{S^2} \bar{\Phi}_{lm}(\Omega) \bar{\Phi}_{lm'}(\Omega) \, d\Omega = \delta_{ll'} \delta_{mm'}.$$

The vector spherical harmonics enable one to represent a vector function $\vec{A}(\vec{r})$ uniquely in the form of three sums,

$$
\vec{A}(\vec{r}) = \sum_{0 \leq |m| \leq l} \bar{y}_{lm}(r) \bar{Y}_{lm} + \sum_{l, m} \bar{\chi}_{lm}(r) \bar{\Psi}_{lm} + \sum_{l, m} \bar{w}_{lm}(r) \bar{\Phi}_{lm}. \quad (20)
$$

For brevity, we will further mean that the summation over the indices $l, m$ is always taken in the range $1 \leq l, \quad |m| \leq l$, unless otherwise stated. For each component of expansion (20), the following separation of variable takes place under the action of the Laplacian $\Delta$:

$$
\Delta (z(r) \bar{Z}_{lm}) = -\frac{1}{r^2} \partial_r r^2 \partial_r z(r) \bar{Z}_{lm} + \frac{z(r)}{r^2} \Delta_{\Omega} \bar{Z}_{lm}, \quad \bar{Z} = \bar{Y}, \bar{\Psi}, \bar{\Phi}.
$$

The action of the spherical Laplacian $\Delta_{\Omega}$ on the VSH is off-diagonal (for $l \geq 1$) but with the above normalization it turns out to be symmetric,

$$
\Delta_{\Omega} \bar{Y}_{lm} = (2 + \vec{l}^2) \bar{Y}_{lm} - 2 \vec{l} \bar{\Psi}_{lm},
$$
$$
\Delta_{\Omega} \bar{\Psi}_{lm} = -2 \vec{l} \bar{Y}_{lm} + \vec{l}^2 \bar{\Psi}_{lm},
$$
$$
\Delta_{\Omega} \bar{\Phi}_{lm} = \vec{l}^2 \bar{\Phi}_{lm}.
$$
If the transversality condition (10) is imposed on the vector function $\vec{A}(\vec{x})$, then it is parameterized not by three, as in (20), but by two sets of functions $u_{lm}(r)$, $w_{lm}(r)$,

$$\vec{A}(\vec{x}) = \sum_{l,m} \left( \frac{\tilde{u}_{lm}}{r^2} \mathbf{Y}_{lm} + \frac{w_{lm}}{r} \hat{\phi}_{lm} \right).$$

(21)

The first two terms under the sum are not transverse individually but, when combined they become as follows:

$$\vec{\partial} \cdot \left( \tilde{u}_{lm} \mathbf{Y}_{lm} + \frac{w_{lm}}{r} \hat{\phi}_{lm} \right) = \vec{Y}_{lm} \left( \left( \frac{u_{lm}}{r^2} - \frac{2u_{lm}}{r^3} \frac{\vec{x}}{r} + \frac{u_{lm}}{r^2} \vec{\partial} \cdot \frac{\vec{x}}{r} \right) + \vec{I} - \tilde{u}_{lm} \vec{\partial} \cdot \vec{Y}_{lm} = 0. \right.$$ 

(22)

The action of the quadratic form of the Laplace operator on the transverse field $\vec{A}(\vec{x})$ written in terms of new variables $u_{lm}(r)$, $w_{lm}(r)$ takes the following form (see the corresponding equations in [14]):

$$Q(\vec{A}) = \int_{\mathbb{R}^3} A_j(\vec{x}) \Delta A_j(\vec{x}) \, d^3x = \sum_{l,m} \{u_{lm}, \tilde{T}_l u_{lm}\} + \sum_{l,m} \{w_{lm}, \tilde{T}_l w_{lm}\},$$

where $\{\cdot, \cdot\}_l$ is the scalar product inherited from $\mathbb{R}^3$,

$$\langle u, v \rangle_l = \int_0^\infty \left( u'(r) v'(r) + \frac{l(l+1)}{r^2} u(r) v(r) \right) \, dr, \quad u(0) = v(0) = 0,$$

(23)

while the radial part of the Laplace operator $\tilde{T}_l$ and the scalar product $(\cdot, \cdot)$ have been defined in (3) and (5). Here, we assume for the time being that the functions $u_{lm}(r)$, $w_{lm}(r)$ are smooth enough and rapidly decrease at the origin. A surprising fact significantly simplifying calculations is that the product (23) can be defined as a sesquilinear form of the operation $\tilde{T}_l$ with respect to the scalar product $(\cdot, \cdot)$,

$$\langle u, v \rangle = \int_0^\infty \frac{u'(r)}{v'(r)} \left( - \frac{d^2}{dr^2} v(r) + \frac{l(l+1)}{r^2} u(r) v(r) \right) \, dr = (u, T_l v).$$

(24)

In order to avoid confusion between the differential operation $T_l$ arising from the scalar product and the radial part of the Laplace operator, here and below we denote the latter by $\tilde{T}_l$.

The kinetic term of the Hamiltonian (15) can be rewritten as follows:

$$- \iint_{\mathbb{R}^3} \frac{\delta}{\delta A_k(\vec{x})} P_{kj}(\vec{x}, \vec{y}) \frac{\delta}{\delta A_j(\vec{x})} \, d^3x \, d^3y$$

$$\quad = - \iint \left( \frac{\delta}{\delta w_{lm}(r')} \frac{\delta}{\delta A_k(\vec{x})} + \frac{\delta}{\delta u_{lm}(r')} \frac{\delta}{\delta A_k(\vec{x})} \right) P_{kj}(\vec{x}, \vec{y})$$

$$\times \left( \frac{\delta u_{lm}(r)}{\delta A_j(\vec{y})} + \frac{\delta w_{lm}(r)}{\delta A_j(\vec{y})} \frac{\delta}{\delta u_{lm}(r)} \right) \, dr \, dr' \, d^3x \, d^3y.$$ 

(25)

In order for the projector $P_{kj}(\vec{x}, \vec{y})$,

$$P(\vec{x}, \vec{y}) = \sum_{l,m} \left( \tilde{T}_l T^{-1}(s, r) \tilde{T}_l \mathbf{Y}_{lm}(\Omega) + \frac{\partial}{\partial s} \mathbf{Y}_{lm}(\Omega') \right)$$

$$\quad + \sum_{l,m} \mathbf{Y}_{lm}(\Omega) \delta(s - r) \mathbf{Y}_{lm}(\Omega') \, r^{-1}, \quad \vec{x} = (s, \Omega), \quad \vec{y} = (r, \Omega'),$$

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to act on transverse functions as a unit operator, we choose the following parametrization for new variables \((u_{lm}, w_{lm})\) in terms of \(\vec{A}\):

\[
w_{lm}(r) = r \int d\Omega \, \overline{F}_{lm}(\Omega) \cdot \vec{A}(r, \Omega) = \frac{1}{r} \int d^3x \, \delta(r - s) \overline{F}_{lm}(\Omega) \cdot \vec{A}(\vec{x}),
\]

\[
u_{lm}(r) = \int ds \, T_{l}^{-1}(r, s) \int d\Omega \left( \overline{\Psi}_{lm}(\Omega) - \frac{\partial}{\partial s} \Psi_{lm}(\Omega) \right) \cdot \vec{A}(s, \Omega)
= \int d^3x \left( \frac{i}{s^2} T_{l}^{-1}(r, s) \overline{\Psi}_{lm}(\Omega) + \frac{1}{s} \left( \frac{\partial}{\partial s} T_{l}^{-1}(r, s) \right) \overline{\Psi}_{lm}(\Omega) \right) \cdot \vec{A}(\vec{x}),
\]

where again \(\vec{x} = \vec{x}(s, \Omega)\), and \(T_{l}^{-1}\) comes from (4). It is not difficult to see that these expressions restore the fields \(u_{lm}(r), w_{lm}(r)\) from \(\vec{A}(\vec{x})\) represented in the form (21), and, at the same time, they are zeroed on any longitudinal component

\[
\vec{A}^L(\vec{x}) = \sum_{0 \leq |m| \leq l} \left( v_{lm}(r) \overline{\Psi}_{lm}(\Omega) + \frac{i}{r} \nu_{lm}(r) \overline{\Psi}_{lm}(\Omega) \right) = \delta \sum_{0 \leq |m| \leq l} v_{lm}(r) \Psi_{lm}(\Omega).
\]

Let us calculate the variations \(\frac{\delta w_{lm}}{\delta A}, \frac{\delta u_{lm}}{\delta A}\) from (26), (27) and substitute them into (25). We find

\[
- \int dr \, dr' \, d^3x \left( \frac{\delta}{\delta w_{lm}(r)} \overline{\Psi}_{lm}(\Omega) \right) \delta(r' - s) \frac{\delta}{r'} \overline{\Psi}_{lm}(\Omega) - \frac{\delta}{\delta w_{lm}(r)} \right) \frac{\delta}{\delta \overline{w}_{lm}(r)}
+ \frac{\delta}{\delta \overline{w}_{lm}(r')} \left( \frac{s}{s^2} T_{l}^{-1}(r, s) \overline{\Psi}_{lm}(\Omega) + \frac{1}{s} \left( \frac{\partial}{\partial s} T_{l}^{-1}(r, s) \right) \overline{\Psi}_{lm}(\Omega) \right) \right) \cdot \vec{A}(\vec{x}),
\]

where we immediately dropped the cross terms between \(u\) and \(w\), which vanish due to the orthogonality of the VSH. In the last term, the action of \(T_{l}^{-1}\) on \(T_{l}\) produces a \(\delta\)-function, as a result of which one integration is removed. It should be noted here that the appearance of the coefficient \(T_{l}^{-1}(r, s)\) in the square of conjugate “momenta” (i.e., in the kinetic part of the Hamiltonian) is quite natural if the “coordinate” variable \(u_{l}(r)\) is measured by the scalar product (24) involving operation \(T_{l}\).

Summing up the kinetic and potential parts, we obtain the following expression for the Hamiltonian (15) in terms of the new variables:

\[
\mathcal{H}_0 = \sum_{l,m} \left( - \int_0^\infty dr \, \frac{\delta}{\delta u_{lm}(r)} \frac{\delta}{\delta w_{lm}(r)} + (w_{lm}, \bar{T}_l w_{lm}) \right) + \sum_{l,m} \left( - \int_0^\infty dr \, dr' \, \frac{\delta}{\delta u_{lm}(r')} \frac{\delta}{\delta u_{lm}(r)} \right) \bar{T}_l^{-1}(r, r) \frac{\delta}{\delta \bar{u}_{lm}(r)} + (u_{lm}, \bar{T}_l u_{lm}) \right) + \bar{T}_l u_{lm}) \right) l.
\]

As expected, the variables \(w_{lm}, u_{lm}\) are separated for all \(l\) and \(m\), and the vacuum states and excitations of this Hamiltonian can be thought in the form of the products of states of the
Hamiltonians,
\[
\mathcal{H}_{lm} = -\int_0^\infty dr dr' \frac{\delta}{\delta u_{lm}(r')} T^{-1}_l(r', r) \frac{\delta}{\delta u_{lm}(r)} + \langle u_{lm}, \hat{T} u_{lm} \rangle t
\]
and
\[
\mathcal{H}'_{lm} = -\int_0^\infty dr \frac{\delta}{\delta w_{lm}(r)} \frac{\delta}{\delta w_{lm}(r)} + \langle w_{lm}, \hat{T} w_{lm} \rangle.
\]

The Hamiltonians \( \mathcal{H}'_{lm} \) are the operators obtained by passing to spherical coordinates and separating the variables in a Hamiltonian for a free scalar field. For \( l \geq 1 \), their eigenvectors are apparently uniquely determined, and so we do not consider them in detail, and then focus on the operators \( \mathcal{H}_{lm} \) only.

### 2.3. Extensions of the quadratic form of operator \( \hat{T}_1 \).

It was shown in [12] that the operator \( \hat{T}_1 \) in the scalar product \( \langle \cdot, \cdot \rangle_1 \) is a symmetric operator with deficiency indices \((1,1)\). This operator has nontrivial selfadjoint extensions that act as the mixed expressions
\[
\hat{T}_{1\kappa} u(r) = T_1 u(r) - \frac{2}{r} u'(0) = -\frac{d^2 u(r)}{dr^2} + \frac{2}{r^2} u(r) - \frac{2}{r} u'(0)
\]
on the domains
\[
\mathcal{D}_\kappa = \{ u(r) : \langle u, u \rangle_1 < \infty, \langle \hat{T}_{1\kappa} u, \hat{T}_{1\kappa} u \rangle_1 < \infty, 3u'''(0) = 4u'(0) \},
\]
The operators \( \hat{T}_{1\kappa} \) have a single-valued continuous spectrum occupying the nonnegative semi-axis, which corresponds to the “eigenfunctions” (the kernel of the spectral transformation)
\[
p^\kappa_{1\lambda}(r) = \frac{2r}{\sqrt{2\pi} \lambda^2} \frac{d}{dr} \frac{1}{r} (\cos(\zeta + \lambda r) - \cos \zeta),
\]
where the phase shift \( \zeta \) is defined by
\[
e^{2i\zeta} = \frac{\lambda - i\kappa}{\lambda + i\kappa}.
\]
For \( \kappa < 0 \), the operator \( \hat{T}_{1\kappa} \) has the eigenvalue \(-\kappa\) of multiplicity one (the discrete spectrum) and the eigenfunction
\[
q(r) = q_\kappa(r) = \sqrt{\frac{2}{\kappa^3}} \left( \kappa e^{\kappa r} + \frac{1 - e^{\kappa r}}{r} \right).
\]
The set \( \{ p^\kappa_{1\lambda}, q \} \) satisfies the orthogonality conditions
\[
\langle p^\kappa_{1\lambda}, p^\kappa_{1\mu} \rangle_1 = \delta(\lambda - \mu), \quad \langle p^\kappa_{1\lambda}, q \rangle_1 = 0, \quad \langle q, q \rangle_1 = 1,
\]
and completeness,
\[
\int_0^\infty p^\kappa_{1\lambda}(r) T_{1\kappa} p^\kappa_{1\lambda}(s) d\lambda + q(r) T_{1\kappa} q(s) \big|_{\kappa < 0} = \delta(r - s),
\]
where the index \( s \) of the differential operation \( T_{1\kappa} \) emphasizes that it acts on the variable \( s \). The operators \( \hat{T}_{1\kappa} \) generate extensions \( \langle u, \hat{T}_{1\kappa} u \rangle_1 \) of the quadratic forms from the potential parts of the Hamiltonians \( \mathcal{H}_{lm} \). The original form \( \langle u, \hat{T}_1 u \rangle_1 \) is defined on the set of twice differentiable functions vanishing at zero together with the first derivative
\[
\mathcal{W}_0^2 = \{ u(r) : \langle u, u \rangle_1 < \infty, \langle u, \hat{T}_1 u \rangle_1 < \infty, u(0) = u'(0) = 0 \},
\]
which corresponds to the differentiable fields
\[
\tilde{A}(\tilde{x}) = \sqrt{2} \frac{u_{1m}(r)}{r^2} \tilde{Y}_{1m}(\psi, \varphi) + \frac{u'_{1m}(r)}{r} \tilde{\psi}_{1m}(\psi, \varphi)
\]  
regular at the origin. The extended forms \( \langle u, \tilde{T}_1 \rangle_1 \) are defined on the set of functions with arbitrary bounded value of the derivative at the origin,
\[
W_1^2 = \{ u(r) : \langle u, u \rangle_1 < \infty, \langle u, \tilde{T}_1 \rangle_1 < \infty, u(0) = 0 \}.
\]
Obviously, the latter form equals the former one on set \( W_0^2 \),
\[
\langle u, \tilde{T}_1 u \rangle_1 = \langle u, \tilde{T}_1 u \rangle_1, \quad u \in W_0^2,
\]
as long as the last term in (28) vanishes. We do not present a symmetric limit expression for the extended form (19), and immediately write down the spectral expansion
\[
\langle u, \tilde{T}_1 u \rangle_1 = \iint_0^\infty Q_\kappa(r, s) T_1^s u(r) T_1^s u(s) \, dr \, ds,
\]
where
\[
Q_\kappa(r, s) = \int_0^\infty p^\kappa_\lambda (r) p^\kappa_\lambda (s) \lambda^2 d\lambda - \kappa^2 T_1^s q(r) T_1^s q(s) \big|_{\kappa<0},
\]
and the second summand exists only for \( \kappa < 0 \).

To conclude this subsection, we note that the form \( \langle u, \tilde{T}_1 u \rangle_1 \) is, in fact, a special case of the form \( \langle u, \tilde{T}_1 u \rangle_1 \) corresponding to \( \kappa = \infty \) (i.e., the form of the Friedrichs or maximal extension of the symmetric operator \( T_1 \)). In terms of the spectral properties of these forms, one can observe that the spherical Bessel function
\[
p^\kappa_\lambda (r) = \frac{2r}{\sqrt{2\pi \lambda^2}} \frac{d}{dr} \frac{1}{\sin \lambda r},
\]
appearing in the parametrization of the nonsingular transverse field (30), is a limit case of the function \( p^\kappa_\lambda \),
\[
p^\kappa_\lambda (r) = \lim_{\kappa \to \infty} \frac{2r}{\sqrt{2\pi \lambda^2}} \frac{d}{dr} \frac{1}{\sin (\lambda r)} (\cos (\zeta + \lambda r) - \cos \zeta), \quad \zeta (\kappa) \to -\frac{\pi}{2}.
\]

2.4. **Hamiltonian eigenstates at \( l = 1 \).** The spectral expansion obtained in the previous subsection allows us to write the Gaussian functional \( \phi_0^\kappa (u) \) for the extended quantum operator
\[
\mathcal{H}_1 = -\iint_0^\infty dr ds \left( \frac{\delta}{\delta u(s)} T_1^{-1} (s, r) \frac{\delta}{\delta u(r)} + \langle u, \tilde{T}_1 u \rangle_1 \right)
\]
as the exponent of the integral operator
\[
\phi_0^\kappa (u) = \exp \left\{ -\frac{1}{2} \iint_0^\infty Q_\kappa^{\frac{1}{2}} (r, s) T_1^s u(r) T_1^s u(s) \, dr \, ds \right\},
\]
where
\[
Q_\kappa^{\frac{1}{2}} (r, s) = \int p^\kappa_\lambda (r) p^\kappa_\lambda (s) \lambda \, d\lambda - i\kappa q(r) q(s) \big|_{\kappa<0}.
\]
In this expression we have specially taken out the differential operations \( T^l_1, T^r_1 \) in order to obtain a more smooth kernel \( Q^\frac{1}{2}_x \). It is not difficult to see that the functional \( \phi_0 \) satisfies the equation

\[
\mathcal{H}_{lm}^\kappa \phi_0^\kappa(u) = \Lambda_0^\kappa \phi_0^\kappa(u), \quad \Lambda_0^\kappa = \int_0^\infty T^l_1 Q^\frac{1}{2}_x (r, r')|_{r=r'} dr
\]

for some infinite eigenvalue \( \Lambda_0^\kappa \). In order to diagonalize the operator \( \mathcal{H}_{lm}^\kappa \), we pass to the spectral representation of the quadratic form, i.e., we make the substitution

\[
\hat{u}(\lambda) = \int_0^\infty p^l_1(r) T^l_1 u(r) dr, \quad \hat{u}_d = \int_0^\infty q(r) T^l_1 u(r) |_{\kappa<0}
\]

(note that all the functions here are real). Then

\[
\mathcal{H}_{lm}^\kappa = \int \left( -\frac{\delta}{\delta \hat{u}(\lambda)} \frac{\delta}{\delta \hat{u}(\lambda)} + \lambda^2 \hat{u}^2(\lambda) \right) d\lambda - \kappa^2 \hat{u}_d^2 |_{\kappa<0}.
\]

This quantum Hamiltonian is associated to the creation and annihilation operators

\[
\hat{b}(\lambda) = \lambda \hat{u}(\lambda) - \frac{\delta}{\delta \hat{u}(\lambda)}, \quad \hat{a}(\lambda) = \lambda \hat{u}(\lambda) + \frac{\delta}{\delta \hat{u}(\lambda)}
\]

and the vacuum state

\[
\hat{\phi}_0(\hat{u}) = \phi_0(u(\hat{u})) = \exp \left\{ -\frac{1}{2} \int_0^\infty \hat{u}^2(\lambda) d\lambda + \frac{i\kappa^2}{2} \hat{u}_d^2 |_{\kappa<0} \right\}.
\]

The \( n \)-particle eigenstates are constructed as the integrals

\[
\hat{\phi}_{\sigma_n}(\hat{u}) = \int \sigma(\lambda_1, \ldots, \lambda_n) \hat{b}(\lambda_1) \ldots \hat{b}(\lambda_n) d\lambda_1 \ldots d\lambda_n \hat{\phi}_0(\hat{u}), \quad (31)
\]

with Bose–Einstein coefficients \( \sigma(\lambda_1, \ldots, \sigma_{\lambda_n}) \) and, moreover, there are states related to the excitations of the discrete spectrum for \( \kappa < 0 \).

### 2.5. Eigenstates of the quantum Hamiltonian of a free transverse field.

The eigenstates of the quantum Hamiltonian \( \mathcal{H}^\kappa_{\text{ren}} \) in which its extension (19) is involved instead of the quadratic form, are constructed as products of eigenstates of the operators \( \mathcal{H}_{lm}^\kappa \) with \( 1 \leq l, |m| \leq l \), \( \mathcal{H}_{lm} \) with \( 2 \leq l, |m| \leq l \), and \( \mathcal{H}^\kappa_{lm} \). To diagonalize the first two sets of operators, one can use the standard spectral transformation

\[
\hat{u}_{lm}(\lambda) = \int_0^\infty p_{l\lambda}(r) T^l_1 u_{lm}(r) dr, \quad \hat{w}_{lm}(\lambda) = \int_0^\infty \lambda p_{l\lambda}(r) w_{lm}(r) dr,
\]

where \( p_{l\lambda}(r) \) is a kind of the spherical Bessel function

\[
p_{l\lambda}(r) = \frac{2^{l+1}}{\sqrt{2\pi} \lambda^l} \left( \frac{d}{dr} \right)^l \sin \lambda r.
\]

The corresponding creation and annihilation operators as well as the vacuum states are as follows:

\[
\hat{t}_{lm}(\lambda) = \lambda \hat{u}_{lm}(\lambda) - \frac{\delta}{\delta \hat{u}_{lm}(\lambda)}, \quad \hat{\alpha}_{lm}(\lambda) = \lambda \hat{u}_{lm}(\lambda) + \frac{\delta}{\delta \hat{u}_{lm}(\lambda)}
\]

\[
\hat{b}_{lm}(\lambda) = \lambda \hat{w}_{lm}(\lambda) - \frac{\delta}{\delta \hat{w}_{lm}(\lambda)}, \quad \hat{\alpha}'_{lm}(\lambda) = \lambda \hat{w}_{lm}(\lambda) + \frac{\delta}{\delta \hat{w}_{lm}(\lambda)}.
\]
\[ \tilde{\phi}_0(\tilde{u}_{lm}) = \exp \left\{ -\frac{1}{2} \int_0^\infty \tilde{u}_{lm}^2(\lambda) \lambda \, d\lambda \right\}, \]
\[ \tilde{\phi}_0'(\tilde{w}_{lm}) = \exp \left\{ -\frac{1}{2} \int_0^\infty \tilde{w}_{lm}^2(\lambda) \lambda \, d\lambda \right\}. \]

The diagonalization of the Hamiltonian $\mathcal{H}_m$ via the transformation
\[ \tilde{u}_{lm}(\lambda) = \int_0^\infty p_{\lambda}^m(r) T_1 u_{lm}(r) \, dr, \]
was described in the previous subsection. It should be noted here that in a spherically nonsymmetric case, the coefficients $\kappa$ can be different for the components corresponding to different values $m$ of the third component of the angular momentum.

In terms of the variables $\tilde{u}_{lm}$, $\tilde{w}_{lm}$, we find the resulting Hamiltonian
\[ \tilde{\mathcal{H}}_{\text{ren}} = \sum_{-1 \leq m \leq 1} \tilde{\mathcal{H}}_{lm}^m + \sum_{2 \leq |m| \leq l} \tilde{\mathcal{H}}_{lm} + \sum_{1 \leq |m| \leq l} \tilde{\mathcal{H}}_{lm}^l, \]
and the vacuum state
\[ \Phi_0 = \prod_{-1 \leq m \leq 1} \phi_{1m}(\tilde{u}_{1m}) \times \prod_{2 \leq |m| \leq l} \phi_{lm}(\tilde{u}_{lm}) \times \prod_{l,m} \phi_{lm}(\tilde{w}_{lm}), \]
while the $n$-particle states are obtained from Eq. (31) by replacing the creation operator $\tilde{b}(\lambda)$ with operator $c(\lambda)$ and $\Phi_0$ with $\tilde{\mathcal{H}}_{\text{ren}}$, $\Phi_0'$,
\[ \Phi_{\sigma_n}(\{\tilde{u}\}) = \int \sigma_n(\lambda_1, \ldots, \lambda_n) c(\lambda_1) \ldots c(\lambda_n) d\lambda_1 \ldots d\lambda_n \tilde{\mathcal{H}}_{\text{ren}}(\{\tilde{u}\}), \]
where $c(\lambda)$ can take any of the values $\tilde{b}_{lm}(\lambda)$ or $\tilde{b}_{lm}^l(\lambda)$.

3. Conclusion and discussion

We have constructed a system of states satisfying the eigenstate equations for a quantum Hamiltonian operator of a free transverse field. The resulting sets generally depend on a selected point in space and do not possess scale invariance (i.e., they depend on a dimensional parameter). Our construction essentially used the properties of extensions of the quadratic form of the Laplace operator which appears in the potential term of the Hamiltonian. These extensions can be written in the invariant form (19) which, like the transversality condition, does not imply transition to spherical coordinates or the use of any distinguished functional parametrization of type (21). Here a natural question arises about the possibility of generalizing the form (19) to the case of two or more distinguished points in the space
\[ Q_{\{\kappa\}}(A) = \lim_{r \to 0} \left( \int_{\mathbb{R}^3 \setminus \{B_r \}} \left( \frac{\partial A_k}{\partial x_j} \right)^2 d^3 x - \sum_{n=1}^N \left( \frac{5}{3r} + \kappa_n \right) \int_{\partial B_{r,n}} |\bar{A}(\tilde{x})|^2 d^2 s \right), \]
namely, does such a form satisfy the conditions of Theorem VIII.15 in [11], does it have the corresponding selfadjoint operator, and if it does, can a spectral representation be found for the latter? Important results in this direction were obtained in [15], but their application to the physical model still needs to be improved.

Another important remark is that, apparently, the representation of a physical object (a field mediating an interaction) in the form of a vector function in three-dimensional space
is not a correct way to describe the problem. While two functions with singularities at two
different points can represent a single physical object at different moments in time, they cannot
be expressed in terms of a common basis, i.e., they do not have a common representation in
terms of a single orthogonal set. And thus, there is a significant obstacle in the description of
the possible dynamics of the system.

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