Quantum Theory of Quantum-Hall Smectics

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We propose a quantum stripe (smectic) coupled-Luttinger-liquid model for the anisotropic states which occur in two-dimensional electron systems with high-index partial Landau level filling, $\nu^* = \nu - [\nu]$. Perturbative renormalization group calculations establish that interaction terms neglected in this model are relevant - probably driving the system into an anisotropic Wigner crystal—but for $0.4 \lesssim \nu^* \lesssim 0.6$ only below temperatures which are outside of the experimentally accessible range. We argue that the Hall conductance of the ground state flows toward $[\nu]\epsilon^2/h$ and $(\lfloor\nu\rfloor + 1)\epsilon^2/h$ respectively, on the low and high filling factor sides of this range, consistent with recent observations.

A semiclassical theory of smectic state transport properties, which incorporates Luttinger liquid effects in the evaluation of scattering amplitudes, accounts for the magnitude of the dissipative resistivities at $\nu^* = 1/2$, for their $\nu^*$-dependence, and for the observation of non-linearities of opposite sign in easy and hard direction resistivities.

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I. INTRODUCTION

Recent transport experiments\textsuperscript{6-9} have established a qualitative difference between low-energy states of two-dimensional electron systems with large and small index partially filled Landau levels. For Landau level filling factors $\nu < 4$ (orbital Landau level indices smaller than $N = 2$), isotropic quantum Hall fluid states occur at fractional values of $\nu$. For $N \geq 2$, on the other hand, experiments have discovered regions of strongly anisotropic dissipative transport near half-odd-integer filling factors, bracketed by reentrant integer-quantum-Hall effect regions with Hall conductivities $[\nu]\epsilon^2/h$ and $(\lfloor\nu\rfloor + 1)\epsilon^2/h$. This dependence on $N$ is presumably due to subtle changes in the effective interactions among the electrons of the partially filled Landau level. In this article we describe a theory which accounts qualitatively and often semi-quantitatively for the principle facts uncovered by this series of experiments.

Following Eisenstein and co-workers\textsuperscript{10} we start from the assumption that the true ground state is close to the unidirectional charge-density-wave states proposed for $N \geq 2$ on the basis of Hartree-Fock calculations by Koulakov et al.\textsuperscript{11} and Moessner and Chalker. In Section II we derive a model of coupled one-dimensional chiral Luttinger liquid electron systems for this state. The derivation provides microscopic expressions for the interaction parameters of the model, which are long-range because of the long-range of the underlying Coulomb interaction between electrons. This model neglects small interstripe backscattering terms. In Section III we demonstrate that these terms are technically relevant, but near half filling $(0.4 \lesssim \nu^* \lesssim 0.6)$ only at inaccessibly low temperatures. Outside this range, however, observable Wigner crystal instabilities are predicted. In Section III we present an estimate of the $\nu^*$ dependence of the temperature below which Wigner crystal states are expected to form. Transport physics in the interesting stripe state regime near $\nu^* = 1/2$ is considered in Section IV. We present a semiclassical theory in which Luttinger liquid effects are incorporated in the evaluation of scattering amplitudes and which describes experiments\textsuperscript{6-9} semi-quantitatively. This theory makes a number of parameter free quantitative predictions which are in good accord with observations. In particular, the product of easy and hard direction resistivities in this theory is independent of disorder strength and has a value which agrees well with experiments. Moreover, Luttinger liquid effects lead to a natural explanation of the non-linear transport effects observed experimentally.

Several recent papers\textsuperscript{12-15} have explored the properties of interacting electron systems in higher Landau levels. The basic framework of our theory has much in common with the work of Fradkin and Kivelson\textsuperscript{16} whose approach intriguingly suggests a similarity between the strong correlation physics of quantum Hall and doped Mott insulator systems. These authors have emphasized the intimate relationship (based on shared symmetry properties) between uni-directional charge density wave states and smectic liquid crystal states. We have followed their lead in referring to the anisotropic high Landau level states as quantum-Hall smectics. Both theories identify the electron stripes as one dimensional electron systems and use bosonization techniques to describe the low energy excitations of their left-going and right-going states. The most important difference in our work is that stripe position and shape fluctuations are identified microscopically with the same low energy excitations. They are not separate low-energy degrees of freedom. Our theory can be developed in terms of either standard Luttinger liquid boson fields or equivalently in terms of stripe width and position fields. We find one set of gapless collective modes
for quantum-Hall smectics, which encompasses all of the low energy degrees of freedom. A physical consequence of this difference is that in our theory, the quantum-Hall smectic ground state is always unstable to the formation of either an electron or a hole Wigner crystal, depending on the sign of $1/2 - \nu^r$.

II. QUANTUM SMECTIC MODEL

The smectic state of Hartree-Fock theory is a single-Slater-determinant with alternating occupied and empty guiding-center occupation-numbers stripes as illustrated schematically in Fig. 1. These states spontaneously break translational and rotational symmetry. For large $N$ they tend to have lower energy than isotropic fluid states because the electrostatic energy penalty, which usually thwarts the phase-separation favored by exchange interactions and by electronic correlations, is small when the density wave period is comparable to the cyclotron orbit diameters of index $N$ electrons. We can consider these states to be composed of either electron or hole stripes with right and left going quasiparticles at opposite edges.

Small fluctuations in the positions and shapes of the stripes can be described in terms of particle-hole excitations near the stripe edges. The residual interactions, ignored in Hartree-Fock theory, which scatter into these low energy states fall into two classes: “forward” scattering interactions which conserve the number of electrons on each edge of every stripe, and “backward” scattering processes which do not. The latter processes involve large momentum transfer and will be smaller in magnitude (see below). The quantum smectic model described in this section includes forward scattering only. These interactions are bilinear in the $1d$ electron densities associated with the chiral currents at the stripe edges: $\rho_{\alpha}(x)$, with $\alpha = \pm$. As explained in Fig. 1, these densities are proportional to an “elastic” field $u_{\alpha}(x) = \alpha 2 \pi \ell^2 \rho_{\alpha}(x)$ (with $\ell = (\hbar c/eB)^{1/2}$ the magnetic length), which measures the transverse displacement of a stripe edge relative to its presumed equilibrium position, $y_{\nu^r}^0 = a(n \pm \nu^r/2)$. The quadratic Hamiltonian which describes the classical energetics for small fluctuations has the following general form:

$$H_0 = \frac{1}{2 \ell^2} \int_{x,x'} \sum_{n,n'} u_{\alpha n}(x) D_{\alpha \beta}(x-x';n-n') u_{\alpha' \beta}(x')$$

$$= \frac{1}{2 \ell^2} \int_q u_{\alpha}(q) D_{\alpha \beta}(q) u_{\beta}(q),$$  

(1)

where $\int_q \equiv \int d^2 q/(2\pi)^2$. Here the $q_y$ integral is over the interval $(-\pi/a,\pi/a)$ and a high momentum cutoff $\Lambda \sim 1/\ell$ is implicit on $q_x$.

Symmetry considerations further constrain the form of the elastic kernel. In position space the kernel must be real and symmetric so that, $D_{\alpha \beta}(q) = D_{\beta \alpha}^*(q)$.
$D_{\alpha\beta}(-q) = D_{\alpha\beta}^*(q)$. This implies $D_{++}(q) = D_{--}^*(q)$ and $\text{Im}D_{\alpha\alpha}(q) = 0$. Parity invariance (under $x, n, + \leftrightarrow -x, -n, -$), implies moreover $D_{++}(q) = D_{--}(q)$. Thus, the elastic kernel is fully specified by one real function, $D_{++}(q)$, and one complex function, $D_{+-}(q)$. In the gology notation of the 1D electron gas literature, these amplitudes correspond to $g_1$ and $g_2$, respectively. Finally, provided the broken translational and rotational invariance in the smectic occur spontaneously, the classical Hamiltonian must be invariant under: $u_{\alpha\alpha}(x) \rightarrow u_{\alpha\alpha}(x) + \text{const}$ and $\partial_x u_{\alpha\alpha}(x) \rightarrow \partial_x u_{\alpha\alpha}(x) + \text{const}$. This symmetry determines the form of $D(q) = \sum_{\alpha\beta} D_{\alpha\beta}(q)$ at small wavevector:

$$D(q) = K_x q_x^4 + K_y q_y^2 + \ldots,$$

(2)

the characteristic form for smectic elasticity.

A quantum theory of the quantum-Hall smectic is obtained by imposing Kac-Moody commutation relations on the chiral densities:

$$[\rho_{\alpha\alpha}(x), \rho_{\alpha'\beta'}(x')] = \frac{i}{2\pi} \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\partial_x \delta(x - x').$$

(3)

This commutator together with $H_0$ fully specifies the quantum dynamics. Electron operators in the chiral edge modes are related to the 1D densities via the usual bosonic phase fields: $\psi_{\alpha\alpha} \sim e^{i\phi_{\alpha\alpha}(y)}$ with $\rho_{\alpha\alpha} = \alpha\partial_y \phi_{\alpha\alpha}/2\pi$. It is a notable feature of the strong field regime that the Luttinger liquid bosonic fields, $\phi_{n, \pm}(x)$, fully determine the stripe position and shape fluctuations. In terms of the bosonic fields, the local center of the $n^{th}$ stripe is

$$Y_n(x) = an + \frac{u_{n,+}(x) + u_{n,-}(x)}{2} = an + \frac{\ell^2[\partial_x \phi_{n,+}(x) + \partial_x \phi_{n,-}(x)]}{2}$$

and the local width of the $n^{th}$ stripe is

$$W_n(x) = av^* + u_{n,+}(x) - u_{n,-}(x) = av^* + \ell^2[\partial_x \phi_{n,+}(x) - \partial_x \phi_{n,-}(x)].$$

(4)

(5)

We also remark that even though $H_0$ has a quadratic form, there is no limit in which a free Fermion description of the smectic (with $D_{\alpha\beta}(n) \sim \delta_{n,0}\delta_{\alpha\beta}$) is valid. The interactions which are responsible for the broken symmetry play an essential role.

Quantum properties of the smectic can be computed from the imaginary-time action,

$$S_0 = \frac{1}{4\pi} \sum_{x, \alpha, \alpha} i\alpha \partial_x \phi_{n, \alpha} \partial_x \phi_{n, \alpha} + \int_t H_0$$

$$= \frac{1}{2} \int_{x, \omega} \phi_{\alpha}(q, -\omega) M_{\alpha, \beta}(q, \omega) \phi_{\beta}(q, \omega),$$

(6)

where in an obvious matrix notation,

$$M(q, \omega) = (i\omega q_x/2\pi)\sigma^z + (q_x \ell)^2 D(q).$$

(7)

Correlation functions follow from Wick’s theorem and the momentum space correlator $\langle \phi_{\alpha}(q) \phi_{\beta}(q') \rangle$ with

$$M^{-1}(q, \omega) = \sigma^z M(q, -\omega) \sigma^z/detM(q, \omega).$$

(8)

Due to the spontaneous breaking of translational and rotational symmetry in the smectic, one expects gapless Goldstone modes at zero wavevector. The collective mode dispersion is readily obtained by setting $\text{det}M(q, i\omega) = 0$ giving $\omega = v(q)q_x$ with a velocity,

$$v(q) = (2\pi\ell^2)[D_{++}(q) - |D_{--}(q)|^2]^{1/2}.$$ 

(9)

At small wavevectors, the mode velocity vanishes: $v^2(q) \sim q_y^2 + q_x^2$. Internal consistency requires that these soft modes do not restore the symmetries assumed to have been broken in the smectic state. To examine this we consider the complex smectic order parameter, $\Phi \sim e^{i\phi_{\alpha\beta}(y)}$, which describes the charge-density order:

$$\delta \rho = Re\Phi e^{i\Phi_y}$$

with $Q = 2\pi/a$ the ordering wavevector. The average $\langle \Phi \rangle$ can be readily computed using the quantum harmonic theory, and at $T = 0$ one finds $\langle \Phi \rangle \sim \exp(-Q^2 I)$ with

$$I \sim \int_q |q_x| D_{++}(q)/v(q).$$

(10)

With $D_{++}$ non-zero at $q = 0$, the integrals converge at small $q$, so that the deBye-Waller factor (e^{-Q^2 I}) and smectic order parameter are non-vanishing. Evidently, these harmonic quantum fluctuations are insufficient to destroy the broken symmetries in the smectic.

The effect of the neglected backscattering interactions, considered in the next Section, depends sensitively on the elastic constants at $q_x = 0$. In this limit the relevant excited states are simply Slater determinants with straight stripe edges displaced from those of the Hartree-Fock theory ground state. By evaluating the expectation value of the microscopic Hamiltonian in a state with arbitrary stripe edge locations we find that

$$D_{\alpha\beta}(q_x = 0, q_y) = \delta_{\alpha\beta}D_0 + \alpha\beta\frac{a}{4\pi^2\ell^2} \sum_n e^{i\Phi_n}\Gamma(y_n^0 - y_0^0),$$

(11)

where the value of the constant $D_0$ is such that $\sum_{\alpha\beta} D_{\alpha\beta}(q = 0) = 0$. Here, $\Gamma(y)$ is the interaction potential between two electrons located in guiding center states a distance $y$ apart:

$$\Gamma(y) = U(0, y/\ell^2) - U(y/\ell^2, 0),$$

(12)

$$U(q, k) = \int \frac{dp}{2\pi} e^{-(q^2 + p^2)\ell^2/2} V_{\text{eff}}(q, p) e^{-ipk}.$$
III. BACK SCATTERING INTERACTIONS

We now consider the “backward” scattering electron interactions, ignored above. The bare matrix elements for these interactions will fall off exponentially with increasing momentum transfer and with increasing separation between the interacting stripes, so we choose here to focus on the smallest momentum transfer. We discuss explicitly only the case of backscattering across electron stripes and across hole stripes, as illustrated schematically in Fig. 1. For a pair of stripes separated by \( ma \), backscattering across an electron stripe can be expressed in a bosonized form,

\[
S_1 = \int_{x,\tau} \sum_{n,m} u_m [\exp (i \theta_{n,m}(x,\tau)) + h.c.],
\]

where

\[
\theta_{n,m} = (\phi_{n,+} - \phi_{n,-}) - (\phi_{n+m,+} - \phi_{n+m,-}).
\]

Hole backscattering takes a similar form. Since the effects of backscattering across electron and hole stripes are equivalent under a particle/hole transformation \( \nu' \leftrightarrow 1 - \nu \) we focus exclusively on the former.

The effects of backscattering can be deduced by implementing a simple renormalization group (RG) scheme. Specifically, we integrate out “fast” boson modes \( \phi \) in a shell, with \( \Lambda/b < |q_x| < \Lambda \) and \( \omega,q_y \) unrestricted, and then rescale \( q_x' = bq_x \) and \( \omega' = b\omega \) leaving \( q_y \) unchanged. With an appropriate rescaling of \( \phi \), this RG transformation leaves the harmonic smectic action, \( S_0 \), invariant. Stability of the smectic fixed point in the presence of backscattering can be tested by considering the lowest order RG flow equation,

\[
\frac{\partial u_m}{\partial t} = (2 - \Delta_m) u_m,
\]

with \( t = \ln b \). Using Eq. (13) and Eq. (8) we find the following expression for the scaling dimension:

\[
\Delta_m = 4 \int_{-\pi}^{\pi} \frac{d(qa)}{2\pi} \sin^2(mqa/2)W(q_x = 0, q).
\]

Here, \( W \) is a “weight” function,

\[
W(q) = \frac{|D_{++}(q) + ReD_{+-}(q)|}{|D_{++}(q)| - |D_{+-}(q)|^2}^{1/2}.
\]

If the scaling dimension \( \Delta_m < 2 \) the smectic phase is unstable. Fortunately, \( \Delta_m \) only depends on the elastic constants at \( q_x = 0 \), so that we can use the microscopic expressions discussed at the end of the previous section for its evaluation.

If the weight function \( W(q_y) \leq 1 \) in Eq. (17) then \( \Delta_m < 2 \) and backscattering is relevant. To understand the dependence of \( W(q_y) \) on filling factor it is useful to consider \( q_y a = 0, \pi \), so that \( D_{++} \) is real and the expression for \( W \) simplifies. For \( q_y = 0 \), smectic elasticity

FIG. 2. Interaction matrix element \( \Gamma(y) \) vs. dimensionless separation \( y/\ell = q\ell \) for the case of interactions in a zero-width 2D layer screened by a parallel metallic layer and with Landau level index \( N = 2 \). \( d \) is the distance to the metallic layer and \( \Gamma \) is in units of \( e^2/\ell \approx 200 \text{meV nm} \) for 2D electron systems formed near the surface of a GaAs crystal. The Luttinger model \( g \)-ology parameter which characterizes interactions between stripes separated by \( na \) in the 2D electron layer is \( \sim \Gamma(na) \). \( \Gamma(y) \) is simply related to the elastic constants in terms of which the chiral Luttinger model is developed in the text. Metallic screening layers are sometimes present in experimental samples but are introduced here mainly as a convenience since \( \Gamma \) varies logarithmically with \( y \) at large \( y \) if they are not present and various sums over stripe indices do not converge. The limit \( d \to \infty \) can be taken at the end of the calculation, if appropriate. \( \Gamma \) vanishes for \( y \to 0 \) because its direct and exchange contributions cancel.
Using Eq. (11) and Eq. (12) we have computed in Fig. (5) for the case 

\[ D_{+}^{-}(q_y a = \pi) = \sum_{n} \frac{(-1)^n a}{4\pi^2 l^2} [\Gamma(an + a(1 - \nu^*)) - \Gamma(an + a\nu^*)]. \]  

(19)

Note that \( D_{+}^{-}(q_y a = \pi) \) vanishes, and the weight function equals 1 for \( \nu^* = 1/2 \). Provided \( \Gamma(y) \) is monotonically decreasing with positive curvature for \( y \geq a \), \( D_{+}^{-}(q_y a = \pi) \) will be negative for all \( \nu^* < 1/2 \), implying \( W(q_y a = \pi) < 1 \). If the weight function is monotonic in \( q_y a \), the backscattering interactions will thus be relevant. Using Eq. (13) and Eq. (2) we have computed \( W(q_y) \) for a range of values of \( N, \nu^*, \) and \( d \), and have always found that it is indeed monotonic; the typical behavior is illustrated in Fig. (3). For the sake of definiteness we have ignored the finite width of the ground subband wavefunction in these calculations. Numerically calculated scaling dimensions for \( m = 1, N = 2, \) and \( d = 10l \), are plotted in Fig. (4). For \( \nu^* > 1/2 \), \( W(q_y a = \pi) > 1 \) so that the electron backscattering amplitude scaling dimension increases, eventually crossing above 2, as seen in Fig. (4). The dependence of the calculated scaling dimension on the distance to the screening plane is illustrated in Fig. (5) for the case \( N = 2 \) and \( \nu^* = 0.5 \). As the distance to the screening plane increases the weighting function approaches 1 more rapidly as \( q_y a \) goes from 0 to \( \pi \). However the values at \( q_y a = 0, \pi \) are fixed at 0 and 1 respectively, and the curves are monotonic at all values of \( d \). As a result the scaling dimension is only weakly dependent on \( d \) and the interaction remains relevant for any finite value of \( d \).

The most significant conclusion which follows from this calculation is that for all \( \nu^* \), backscattering across either electron or hole stripes is relevant, and will destabilize the smectic ground state. The ultimate fate of the ground state will presumably depend on the relative magnitudes of the various backscattering interactions. For the interactions considered above the bare coupling constants will fall rapidly with increasing stripe separation \( m \):

\[ u_m \sim U(\nu^*/l^2, ma/l^2), \]  

(20)

so that \( m = 1 \) will dominate. If each electron stripe is viewed as a 1d conductor, this is a 2kF backscattering interaction, which tends to drive charge ordering along the stripe, with wavelength corresponding to the 1d electron spacing. We thus strongly suspect that for \( \nu^* < 1/2 \) the smectic will be unstable to the formation of an electron Wigner crystal, with one electron per unit cell. For large Landau index \( N \), the crystal would be highly anisotropic, compressed along the x-direction, with an aspect ratio proportional to \( N \). For \( \nu^* > 1/2 \), though, backscattering across the hole stripes will dominate, leading to an anisotropic hole Wigner crystal, with one hole per unit cell. In either crystal phase there will, in contrast to the smectic case, be an energy gap, \( E_q \),
FIG. 4. Scaling dimensions for \( m = 1 \) electron and hole \( 2k_F \) scattering and electron and hole impurity scattering vertices (dashed lines) for a range of filling factor near \( \nu^* = 1/2 \). For this calculation the distance to the screening plane was chosen to be \( d = 10\ell \). Electron scattering vertices are an increasing function of filling factor and hole vertices are a decreasing function of filling factor as discussed in the text. The interaction terms are relevant for scaling dimensions smaller than 2 while impurity terms are relevant for scaling dimensions smaller than 1.5. The interedge scattering rate is enhanced at low energies when the impurity interaction scaling dimension is smaller than 1.0. Interaction terms with \( m \) larger than one are more relevant but have bare coupling constants which are smaller by several orders of magnitude. Interaction terms with larger momentum transfers than those discussed here also have much smaller bare coupling constants.

FIG. 5. Dependence of \( m = 1 \) \( \nu^* = 1/2 \) electron and hole backscattering amplitude scaling dimensions \( (\Delta_{1,e} = \Delta_{1,h} = \Delta_1) \) on distance between the two-dimensional electron system and the model’s metallic screening plane. Here \( d \) is in units of the magnetic length \( \ell \). For \( d \to 0 \), \( \Delta \) approaches \( 16/3\pi \), the value which can be calculated analytically for the case of interactions only between nearest neighbor chiral edge modes. As explained in the text, \( \Delta \) increases with \( d \), but only slowly, and is smaller than 2 for arbitrarily large \( d \).
for single particle excitations. Provided the crystalline order is pinned by the boundaries, these Wigner crystal phases should have vanishing dissipative conductivities $\sigma_{xx}$ and $\sigma_{yy}$. However, the hole Wigner crystal will have an extra Landau level edge state. The quantized Hall conductances of electron and hole Wigner crystal states will be, $\sigma_{xy} = [\nu] e^2/h$ and $[\nu + 1] e^2/h$ respectively.

Of considerable interest is the magnitude of the Wigner crystal gap as a function of $\nu^*$. With knowledge of the dimensionless backscattering interaction, $u$, this gap can be estimated by integrating the RG flow equations. Specifically, under an RG transformation, the energy gap should rescale as,

$$E_g(u) = b^{-1}E_g(b^{2-\Delta} u),$$

with $\Delta = \Delta_1$. When the interaction becomes of order one, $b^{2-\Delta}u = 1$, the energy gap should be roughly equal to the characteristic Coulomb energy, $E_c$, giving,

$$E_g(u) = (U/E_c)^{1/(2-\Delta)} E_c,$$

with $U = u E_c$ the (dimensionful) backscattering strength. The $\nu^*$ dependence of the gap enters both through $U$, which is extremely small for $\nu^*$ near 1/2 because of the long period of the stripe lattice, and the scaling dimension, $\Delta$, which is maximal at $\nu^* = 1/2$. [For $\nu^* > 1/2$ the same applies to backscattering across hole stripes.] Both effects conspire to strongly reduce the gap magnitude near half-filling. Using the above estimates, it is possible to obtain the $\nu^*$ dependence of the gap explicitly. Taking $E_c = 0.3e^2/\ell$, the order of the maximum correlation energy per electron in a partially filled Landau level, the resulting gap for $N = 2$ and $d = 10\ell$ is shown in Fig. 6. Notice that the Wigner crystal gap plummets rapidly to extremely small values near $\nu^* = 1/2$, dropping below the range accessible to dilution fridges over the filling factor range $0.4 \lesssim \nu^* \lesssim 0.6$, where anisotropic transport is observed in low-temperature experiments. In this region the Wigner crystal states will be inaccessible (melted at experimental temperatures), and the anisotropic transport of the smectic phase should be unmasked. Outside of this range, the Wigner crystal will be pinned by even weak impurities, resulting in quantized Hall plateaus. For $\nu^* = 0.3$, the gap values estimated here are typical of those found on the reentrant integer quantum Hall plateaus which bracket the anisotropic transport regimes.

We remark that electron and hole Wigner crystal states are also the ground states in the Hartree-Fock approximation. In that approximation, however, the gaps are orders of magnitude larger $\sim E_c \approx 0.3 e^2/\ell \ell$ over a wide-range of filling factors. The Hartree-Fock approximation is expected to be reasonably accurate for the nearly classical Wigner crystal states which occur in the tails of $N \lesssim 1$ Landau levels. Evidently quantum fluctuations have a larger importance for these $N \geq 2$ crystal states.
IV. ANISOTROPIC TRANSPORT PROPERTIES

Transport near $\nu^* = 1/2$ in the smectic regime will be strongly influenced by impurities, which are in fact necessary to get any transport in the “hard” $y$-direction. The dominant effect will presumably come from impurity scattering across electron or hole stripes, with the latter being the bottleneck when $\nu^* < 1/2$ and the former when $\nu^* > 1/2$. For weak impurity scattering it is possible to examine their effects perturbatively. Consider for example impurity scattering across electron stripes,

$$H_{\text{imp}} = \int x \sum_n \xi_n(x)e^{i(\phi_n-\phi_n') + h.c.},$$  \hspace{1cm} (23)

with $\xi(x)$ a complex random potential. Taking $\xi_n(x)$ to be uncorrelated and Gaussian,

$$[\xi^*_n(x)\xi_n(x')]_{\text{ens}} = D\delta_{nn'}\delta(x-x'),$$ \hspace{1cm} (24)

a simple RG perturbative in the variance $D$ is possible. One finds, $\partial D/\partial t = (3 - 2\Delta_e)D$, with the scaling dimension of the operator $e^{i(\phi_n-\phi_n')}$. Given by,

$$\Delta_e = \int_{-\pi}^{\pi} \frac{d(q)}{2\pi} W(q_*) (q_2 = 0, q).$$ \hspace{1cm} (25)

Here $W$ is the same “weight” function as in Eq. (7). The filling factor dependence of $\Delta_e$ can be understood from considerations similar to those for the backscattering amplitudes detailed in the previous section. For 1D non-interacting electrons $\Delta_e = 1$, so that disorder is relevant and eventually leads to localization. For the smectic we can estimate $\Delta_e$ as a function of filling $\nu^*$; the result of this calculation was included in Fig. (4). At all $\nu^*$ impurity scattering across either electron or hole stripes is more relevant than in the non-interacting electron case.

In the strict zero temperature limit, we thus expect that impurities (aided by interactions) will ultimately drive localization for all $\nu^*$, except right at the $\nu^* = 1/2$ plateau transition. However, samples in which quantum-Hall smectic physics is observed have extremely weak impurity scattering, so that it might be possible to ignore localization effects at accessible temperatures. More specifically, consider the dimensionless disorder strength, $D = 4\alpha/E_c^2$, with $E_c$ the Coulomb energy scale. Provided $D << 1$, there should be a large temperature range over which impurity backscattering can be treated perturbatively and localization effects ignored. To see this, it is convenient to introduce an effective temperature-dependent disorder strength that follows from the RG; $D_{\text{eff}}(T) = (T/E_c)^{2\Delta_e-3} D$, which increases upon cooling. Provided $D_{\text{eff}}(T) < 1$, localization effects should be negligible, and Boltzmann transport should be operative.

A key parameter in a Boltzmann approach is the impurity scattering rate $\Gamma_e (\Gamma_h)$ across an electron (hole) stripe. Within a simple free-fermion Golden-rule calculation one expects $\Gamma^0 = cDE_c$ (with $c$ an order one constant), which is independent of temperature. But under the RG transformation the scattering rate rescales as,

$$\Gamma_e(D, T) = b^{-1}\Gamma_e(b^{1-2\Delta_e} D, bT).$$ \hspace{1cm} (26)

Running the RG until $bT = E_c$ gives, $\Gamma(D, T) = (T/E_c)\Gamma(D_{\text{eff}}, E_c)$. Using the free-fermion result at $T = E_c$ one has

$$\Gamma_e(T) = cTD_{\text{eff}}(T) = \Gamma^0_e (T/E_c)^{2\Delta_e-2}.$$ \hspace{1cm} (27)

This should be valid provided that $D_{\text{eff}} < 1$. For a non-interacting 1D electron gas $\Delta_e = 1$ so that $\Gamma_e$ is temperature independent. In contrast, Luttinger liquid effects in the quantum Hall smectic give a temperature dependence to the Boltzmann scattering rate - generally increasing upon cooling. Equivalently, the impurity mean free path varies with temperature, in marked contrast to low temperature metallic transport.

In the Boltzmann approach to transport in the quantum Hall smectic that we develop below, quantum interference effects between successive inter-edge impurity backscattering events are ignored. This is valid provided, $\Gamma_e$ is not large compared to $\Gamma_{\phi}$, where $\Gamma_{\phi}$ is the electron phase breaking rate. Within a single chiral edge mode, forward scattering interactions will rapidly dephase an electron. A simple perturbative calculation for the electron self energy is expected to give the form: $\Gamma_{\phi} = c'Tu_f^2$ with $u_f$ a dimensionless forward scattering amplitude and $c'$ of order one. Since $u_f$ is also of order one this implies, $\Gamma_{\phi} = c_eT$. Comparing with Eq. (27) one sees that $\Gamma_e$ is thus legitimate to ignore interference between successive impurity backscattering events provided $D_{\text{eff}}(T)$ is not large compared to one. For temperatures low enough that $D_{\text{eff}}(T)$ is large, quantum interference effects cannot be neglected and one expects an onset of localization (except right at $\nu^* = 1/2$). In strong field, the leading one-loop weak localization effects will not be operative, so that two-loop interference processes will drive the localization.

With this preamble in hand, we proceed to develop a semiclassical Boltzmann transport theory for the quantum Hall smectic phase. We assume that the charge density wave itself is pinned and immobilized by both the edges of the sample and weak impurities which couple to the electrons within the stripes. In this case, collective sliding motion of the charge-density will be absent, and the electrical transport will be dominated by single-particle inter-edge electron tunneling. It is convenient to characterize the non-equilibrium current-carrying state by separate local steady state chemical potentials, $\mu_{\pm} x$, for left and right going electrons in each stripe. Due to the discrete translational symmetry of the smectic, the steady state chemical potential must increase by $eE_y a$ upon translation by one period, with $E_y$ the $y$-component of a (uniform) electric field. Taking the zero of chemical potential as the center of the $n = 0$ electron stripe, we can thus write,

$$\mu_{n, +} = neE_y a + \mu/2,$$

$$\mu_{n, -} = neE_y a - \mu/2.$$ \hspace{1cm} (28)
Here the chemical potential drops across electron and hole stripes are \( \mu \) and \( eE_y a - \mu \) respectively. An electric field \( E_x \) in the \( \hat{x} \) direction, induces a steady flow in momentum space which moves each electron stripe to smaller \( y \) (for \( E_x > 0 \), lowering the chemical potential on right-going edges and raising it on left-going edges. This disequilibrium induces a tunneling current across both electron and hole stripes, which attempts to restore equilibrium:

\[
\dot{\mu}_{n,+} = -eE_x v_F + \frac{\mu_{n,-} - \mu_{n,+}}{\tau_e} + \frac{\mu_{n+1,-} - \mu_{n,+}}{\tau_h},
\]
\[
\dot{\mu}_{n,-} = eE_x v_F + \frac{\mu_{n,+} - \mu_{n,-}}{\tau_e} + \frac{\mu_{n-1,+} - \mu_{n,-}}{\tau_h},
\]

Here, we have introduced inter-edge scattering times, related to the rates above via: \( \Gamma_e = 1/\tau_e \) and \( \Gamma_h = 1/\tau_h \), for tunneling across electron and hole stripes, respectively. The electric field, \( E_x \), induces a drift in the wavevector of the electrons in each chiral edge mode, \( h\vec{k} = -eE_x \). In Eq. (29) \( v_F \) is a “Fermi velocity”, which relates changes in the edge chemical potential to wavevector: \( v_F = \partial \mu / \partial k \). This velocity is determined by the “onsite” piece of the smectic elastic constants as, \( v_F = 2\pi e^2 D_{++} (q_x = 0, n = 0) \).

In the steady state \( \dot{\mu}_{n,+} = 0 \) so that

\[
\mu (\tau_e^{-1} + \tau_h^{-1}) = \frac{eE_y a}{\tau_h} - eE_x v_F,
\]

relating the unknown parameter \( \mu \) to the electric fields. The current in the \( \hat{x} \) direction is due to the imbalance between left-going and right-going electrons in each stripe

\[
I_x = \frac{e^2}{h} \left[ \frac{L_y}{a} \right] (-\mu /c).
\]

In Eq. (31) the contribution from each stripe is given by the familiar expression for the quantum Hall current and the factor in square brackets is the number of electron stripes in a sample with width \( L_y \). The current in the \( \hat{y} \) direction is equal to the tunneling current across the hole stripes:

\[
I_y = \frac{eL_x}{v_F h} \frac{eE_y a - \mu}{\tau_h}.
\]

with \( L_x \) the sample width. The first factor on the right-hand-side of Eq. (33) is the charge per unit energy in a chiral 1D electron system of length \( L_x \).

Inserting Eq. (33) in Eq. (31) and Eq. (32) to eliminate \( \mu \) gives the desired expressions for the conductivity matrix,

\[
\sigma_{xx} = \frac{e^2}{h} \frac{v_F \tau_e \tau_h}{a (\tau_e + \tau_h)},
\]
\[
\sigma_{yy} = \frac{e^2}{h} \frac{a}{v_F (\tau_e + \tau_h)},
\]
\[
\sigma_{yx} = -\sigma_{xy} = \frac{e^2}{h} \left( [\nu] + \frac{\tau_e}{\tau_e + \tau_h} \right),
\]

Inverting the conductivity matrix gives the following expressions for the resistivities:

\[
\rho_{\text{easy}} = \frac{h}{e^2} \frac{1}{\tau_e ([\nu] + 1)^2 + \tau_h [\nu]^2} \frac{a}{v_F},
\]
\[
\rho_{\text{hard}} = \frac{h}{e^2} \frac{1}{\tau_e ([\nu] + 1)^2 + \tau_h [\nu]^2} \frac{a}{v_F \tau_h},
\]
\[
\rho_{\text{hall}} = \frac{h}{e^2} \frac{1}{\tau_e ([\nu] + 1)^2 + \tau_h [\nu]^2} (\nu + [\nu] \tau_h),
\]

where \( \rho_{\text{easy}} = \rho_{yx} \), \( \rho_{\text{hard}} = \rho_{yy} \), and \( \rho_{\text{hall}} = \rho_{xy} \).

Eq. (34) relates the dissipative and Hall resistivities to the two scattering rates, \( \Gamma_e \) and \( \Gamma_h \). The dependencies on temperature and filling factor \( \nu^* \) enter through these scattering rates, in the form established above:

\[
\Gamma_e = \frac{1}{\tau_e} = \Gamma_e^{(0)} (k_B T / E_c)^2 \Delta_e^{-2},
\]
\[
\Gamma_h = \frac{1}{\tau_h} = \Gamma_h^{(0)} (k_B T / E_c)^2 \Delta_h^{-2}.
\]

Here, the free-fermion scattering rates across electron and hole stripes, \( \Gamma_e^{(0)} \) and \( \Gamma_h^{(0)} \), depend on the impurity scattering strength (and \( \nu^* \)) but not the temperature. \( E_c \) is the Coulomb energy scale which serves as a high energy cutoff. The scaling dimensions, \( \Delta_e \) and \( \Delta_h \), depend sensitively on \( \nu^* \) as shown in Fig. 3. As we shall see these equations describe much of the phenomenology of transport in quantum Hall stripe states.

Remarking, for \( \nu^* = 1/2 \) this theory makes two parameter free quantitative predictions:

\[
\rho_{\text{easy}} \rho_{\text{hard}} = (h/e^2)^2 \frac{1}{\frac{1}{([\nu] + 1)^2 + [\nu]^2}},
\]

and

\[
\rho_{\text{hall}} = \frac{h}{e^2} \frac{2[\nu] + 1}{([\nu] + 1)^2 + [\nu]^2}.
\]

Notice that the scattering times have completely dropped out of these expressions! Interestingly, the Hall resistivity at \( \nu = [\nu] + 1/2 \) is predicted to deviate noticeably from (the classical value) \( (h/e^2) / ([\nu] + 1/2) \). The most extensive experimental data is for \( [\nu] = 4 \). In this case, the value predicted for the product of \( \rho_{\text{easy}} \) and \( \rho_{\text{hard}} \) appears to agree with the published data to within better than a factor of two, provided one accounts for the particular current paths appropriate for the sample geometry. Experimental verification of the predicted \( [\nu] \) dependence of this product would help establish the efficacy of this transport theory.

At \( \nu^* = 1/2 \), Eq. (34) predicts a weak temperature dependence of the dissipative resistivities. Specifically, due to Luttinger liquid effects which drive an enhancement of the inter-edge scattering rate upon cooling (since
creases with increasing $1/\nu$. If $\nu^* < 1/2$, the electron stripes are narrower than the hole stripes and a free-fermion evaluation of the relaxation times would give $\tau_H > \tau_e$. Since $\Delta_e$ decreases and $\Delta_h$ increases with increasing $1/2 - \nu^*$, the relaxation rate ratio is expected to increase beyond its free-fermion value at lower temperatures. For $\tau_H \gg \tau_e$, we have that $ho_{\text{hard}} = (\hbar/e^2)(v_F \tau_e/a)/|\nu|^2$. Since $\tau_e$ decreases ever more rapidly upon cooling for larger $1/2 - \nu^*$, the hard resistivity is expected to be large at experimental temperatures only over a narrow interval surrounding $\nu^* = 1/2$.

Backscattering interactions ignored in this Boltzmann transport theory will only tend to enhance this effect, acting in concert with impurity scattering.

In the same regime of filling factor, with $\tau_H \gg \tau_e$, the Hall resistivity approaches $(\hbar/e^2)/|\nu|$. Moreover, one has $\rho_{\text{easy}} = (\hbar/e^2)/|\nu|^2(v_F \tau_H/a)$ in this limit. Thus, $\rho_{\text{easy}}$ also decreases with $1/2 - \nu^*$, because of both bare matrix element and scaling dimension tendencies. Interestingly, for $\nu^* \lesssim 0.4$ the scaling dimension for scattering across hole stripes, $\Delta_h$, becomes larger than one (see Fig. 4). This implies that $\Gamma_h$ actually decreases upon cooling in this regime, strongly enhancing the one-dimensional nature of the electron stripes and driving localization. But for $\nu^* \lesssim 0.4$ one really must include the strong effects of electron (backscattering) interactions which drive the Wigner crystal instability - again, this acts in concert with impurity affects. Upon cooling within this Wigner crystal regime, the dissipative resistivities should rapidly vanish leaving a quantized Hall resistance. It follows from particle-hole symmetry that $\tau_e(\nu^*) = \tau_h(1 - \nu^*)$ so similar conclusions can be reached for transport properties at $\nu^* > 1/2$.

In Fig. 7, we plot the $\rho_{\text{easy}}$ and $\rho_{\text{hard}}$ filling factor dependencies predicted by this model for $|\nu| = 4$ and $\delta$-correlated disorder model. Here we have taken $T = 0.003\Delta_e/\ell$ and $E_c = 0.3\Delta_e/\ell$. The disorder strength was chosen to give $\rho_{\text{hard}}/\rho_{\text{easy}} = 10$ at $\nu^* = 0.5$. Notice that $\rho_{\text{easy}}$ has a maximum at $\nu^* = 1/2$, not the minimum seen in many experiments. Within the Boltzmann theory this feature depends on the details of the disorder model used; models with only small angle scattering at zero magnetic field tend to give electron relaxation times which decrease and hole relaxation times which increase more rapidly with $1/2 - \nu^*$, changing the shape of these curves. In addition, current-path corrections might be essential in producing the apparent $\rho_{\text{easy}}$ minimum at $\nu^* = 1/2$ in experiments. No plausible disorder model in this theory gives $\rho_{\text{hard}}$ results which drop to zero as strongly with increasing $1/2 - \nu^*$ as in experiments; we believe that backscattering interactions and localization, both effects neglected here, are playing an important role in driving the resistivities to small values away from $\nu^* = 1/2$.

The transport theory described above can be readily generalized to account for the non-linear transport features, which are present experimentally - notably at $\nu^* = 1/2$. The voltage drop across a stripe, given by $V_y = a E_y/2$ at $\nu^* = 1/2$, can be readily incorporated into the RG scaling approach for the scattering rates. Not surprisingly, the resulting dependence on voltage is the same as that on temperature obtained above:

$$\Gamma_e \sim \Gamma_e^{(0)}(V_y/E_c)^{2\Delta_e-2}$$
$$\Gamma_h \sim \Gamma_h^{(0)}(V_y/E_c)^{2\Delta_h-2}. \tag{38}$$

This expression is valid in low temperature or high voltage regions, with $k_B T \ll V_y$. The non-linear differential resistivity in the hard direction can now be obtained by using the expression for the tunneling current across stripes from Eq. 14:

$$I_y = (e^2/\hbar)\Gamma_e V_y L_z/v_F.$$  One thereby obtains,

$$\frac{\partial V_y}{\partial I_y} \sim I_y^\alpha, \tag{39}$$

with an exponent $\alpha = 2(1 - \Delta_e)/(2\Delta_e - 1)$. Using the value of $\Delta_e = 0.756$ calculated from theory at $\nu^* = 1/2$ for the case $|\nu| = 4$ and $d = 10\ell$ (see Fig. 4), gives the estimate $\alpha = 0.93$. Calculations for models with more remote screening planes will give larger values for $\Delta_e$ (but always smaller than one as explained above) and smaller
positive values for $\alpha$. Notice that a positive exponent implies an enhancement of the hard axis resistivity when driven non-linear - consistent with the experimental findings in Ref. 1. This increase in resistivity is due to a voltage suppression of the correlation induced interlayer tunneling enhancement - and as such is a rather direct experimental indication of non-trivial Luttinger liquid correlations of the chiral edge channels.

A similar calculation can be performed for non-linearities in the easy axis current. At $\nu^* = 1/2$, $V_x \propto I_y I_x$ where $I_x$ is the easy-direction current. In this case $V_y \propto \nu I_x$ is the Hall voltage. It follows that

$$\frac{\partial V_x}{\partial I_x} \sim I_x^\beta$$

with an exponent $\beta = 2(\Delta_v - 1)$ For $N = 2$ and $d = 10\ell$, we obtain $\beta = -0.48$; smaller negative values are found for models with more remote screening planes. In this theory the easy-direction resistivity is suppressed when driven non-linear with current, a property which is also consistent with experimental findings. We emphasize that these power laws hold only in the low-temperature or high-voltage limits; a careful comparison of the theoretical dependence on voltage to temperature ratio with experiment could provide a strong test of this transport theory.

V. SUMMARY

Recent experiments have established a consistent set of transport properties for high-mobility two-dimensional electron systems with high orbital index ($N \geq 2$) partially filled Landau levels which differ from those in the low orbital index ($N \leq 1$) fractional quantum Hall regime. At large $N$, the dissipative resistivities are large, strongly anisotropic, and non-linear for $0.4 \lesssim \nu - [\nu] \lesssim 0.6$ within each Landau level. This anisotropic transport regime is bracketed by regions of reentrant integer quantum Hall plateaus. In this paper we have presented a theory which is able to account for most features of these experiments. The theory starts from the unidirectional charge-density-wave (smectic) state of Hartree-Fock theory, in which the electrons reside in periodically spaced stripes with a spontaneously chosen orientation. Forward and backscattering interactions, neglected in the Hartree-Fock theory, are included by retaining the low energy electron excitations at the stripe edges. These form a set of coupled 1D chiral modes, easily described with bosonization techniques. We find that: i) for smectic states in quantum Hall systems, the chiral boson degrees of freedom coincide with stripe position and width degrees of freedom; ii) backscattering interactions which drive the system toward electron or hole Wigner crystal states are always relevant, but only below inaccessibly low temperatures in the anisotropic transport regime; and iii) a semiclassical Boltzmann transport theory for the smectic state is able to account for the magnitude of the anisotropic dissipative resistivities and for the sign of the non-linearities which appear at higher transport currents.

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