Semi-analytic Conjugate Gradient Method applied to a simple Inverse Heat Conduction Problem

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Abstract: On the most simple 1-D heat equation, a simple identification problem of heat flux on one side from temperature measurement on the other side is solved with a conjugate gradient method (CGM). What is new in this well known and academic problem is that the CGM can be developed explicitely. This means that the CGM is given with explicit formulae to determine the two determinations of the complex square root of the operator. The classical way to find (2) is to seek firstly \( \theta(x) \) such that \( \sqrt{\omega} \theta - \theta_x = 0 \), \( \theta_z(0) = 0 \), \( \theta_x(1) = 1 \) and secondly \( S(x,t) \) by variable separation and Fourier series such that \( S_t - S_{xx} = 0 \), \( S_x(0,t) = S_x(1,t) = 0 \), \( \theta(x) + S(x,0) = 0 \). Now, let us set \( T(t) = T(0,t) \). This projection defines the operator \( A_t \) by

\[
T = A_t u.
\]

Since the initial condition in (1) is homogeneous, operator \( A_t \) is linear. A classical Inverse Heat Conduction Problem (IHCP) consists to determine \( u_* \) for given \( T_* \), i.e to look for \( A_t^{-1} \):}

\[
u_\ast = A_t^{-1} T_*.
\]

The existence and unicity of \( u_\ast \) can be established in suitable spaces for (1). However, since operator \( A_t \) is infinite dimensional and compact, it is an ill posed problem. It has been shown [Ali95] that with the CGM, which is a descent method applied to the criterion \( J^2(u) = \frac{1}{2} \| T(u) - T_* \|^2 \) and which gives a minimizing sequence \( (u^K)_k \), it can be regularized in a Tichonoff sense with the number \( k \) of iterations as a regularization parameter. Then, under suitable conditions \( C \) on the perturbation of \( A_t \) and the choice of \( u^0 \), there exists a stopping rule \( K = K(C) \) for the CGM that gives \( u^K \) theoretically as closed as wanted to the unique solution \( u_* \). Despite there is no efficient way to ensure that, for a given precision, such conditions \( C \) are satisfied, many applications much more complex than (4) have shown that the CGM can bring a significant improvement. More precisely, introducing the initial influx \( u^0 \), it is frequently observed in applications that for a certain \( K \) the CGM gives a quite satisfying solution to (4), that is to say

\[
u_\ast \approx \text{CGM}_{A_t} u^0, T_* \Rightarrow \Theta^K_{A_t, T_*} (u^0) = u^K
\]

where \( \Theta_{A_t, T_*} \) is the discrete dynamical system underlied to the CGM and \( \Theta^K_{A_t, T_*} = \Theta_{A_t, T_*} \circ \cdots \circ \Theta_{A_t, T_*} \). In this infinite-dimensional framework, despite in some cases explicit solutions to (3) are well known [Zau06], the calculus of (5) is always made via a discretization method needed for solving (3). Consequently, no explicit formulation of \( \Theta_{A_t, T_*} \) is known. In the second section, under a natural approximation, we will present an algebraic formulation of \( \Theta_{A_t, T_*} \). More exactly, \( u^K \) in \( \Theta_{A_t, T_*} \) will be replaced by \( (u^K, p^{k-1}) \) or \( (u^K, p^{k-1}, t^{k-1}) \), to be defined. This will be established for

\[
u_\ast(t) = \sum_{i=1}^m \psi_i q_i(t) \text{ where } q_i(t) = \exp (j\omega_i t), \quad \omega_i \in \mathbb{R}^n \text{ and } \psi_i \in \mathbb{C},
\]

1. INTRODUCTION

Let \( t_f > 0 \), \( \omega \in \mathbb{R}^n \), and \( q(t) = \exp(j\omega t) \) where \( j^2 = -1 \).

For \( (x,t) \in (0,1) \times (0,t_f) \), let us consider the normalised 1-D heat equation with Newmann boundary conditions:

\[
T_t(x,t) - T_{xx}(x,t) = 0, \quad T(x,0) = 0, \quad \frac{\partial T}{\partial x}(1,t) = q(t) \quad (1)
\]

where \( u(t) = q(t) \) for the moment. The complex solution of this well posed PDE is [Zau06]

\[
T_t(x,t) = q(t) \theta(x) + S(x,t)
\]

where

\[
\begin{align*}
\theta(x) &= \frac{1}{\sqrt{\omega} \sinh (\sqrt{\omega} x)} \\
S(x,t) &= \lambda - 2 \sum_{n=1}^{\infty} (-1)^n \exp (-n^2 \pi^2 \omega t) \cos(n \pi x) \\
&\text{with } \lambda = \frac{1}{\omega}
\end{align*}
\]

(2)

Because \( \cosh \) is even and \( \sinh \) is odd, this formulation is independent of an arbitrary but fixed choice between the two determinations of the complex square root \( \sqrt{\omega} \).

The classical way to find (2) is to seek firstly \( \theta(x) \) such that \( \sqrt{\omega} \theta - \theta_x = 0 \), \( \theta_z(0) = 0 \), \( \theta_x(1) = 1 \) and secondly (1) by variable separation and Fourier series such that \( S_t - S_{xx} = 0 \), \( S_x(0,t) = S_x(1,t) = 0 \), \( \theta(x) + S(x,0) = 0 \). Now, let us set \( T(t) = T(0,t) \). This projection defines the operator \( A_t \) by

\[
T = A_t u.
\]

(3)

Since the initial condition in (1) is homogeneous, operator \( A_t \) is linear. A classical Inverse Heat Conduction Problem (IHCP) consists to determine \( u_* \) for given \( T_* \), i.e to look for \( A_t^{-1} \):
and in the restricted case where \( u^0 (t) = 0 \). This will give a semi-adjoint solution \( u^k \) expanded in the space \( \mathbb{C}^{2K+1} [t_f - t] \oplus Q_m \), where \( Q_m = \text{Vec}_C\{q_i; i = 1, \ldots, m\} \). For lack of space, proofs will be omitted or only sketched, but precise formulae will be given which can be tested on a computer algebra system. Note that if it is known a priori that \( T_e (t) \) is \( 2\pi \)-periodic, for example \( T_e (t) \) is in \( L^2 (0, 2\pi) \) extended eventually to \( (0, t_f) \) by periodicity, it is relevant to look for \( u_e \) with a spectral method, without invoking the CGM.

Last but not least, problem (5) takes its source in the previous work [Prud’98]. These authors tried to give what we call the initialization step of the CGM and to draw what will look like the next steps, without proving the solvability of the general problem. Their motivation was to establish a filtering property of the CGM. If \( (\omega_k)_{k=1, \ldots, m} \) is an increasing sequence of \( \mathbb{R}^+ \), they suggested that the first iterations of the CGM give a good rendering of \( \psi (0, t_f) \). Further iterations are needed to identify \( \psi (t_f) \) and so on for the following terms \( \psi (t) \). Based on the foregoing semi-adjoint solution \( u^k \) and on simulations, this will be discussed in the third section.

**Notations.** In this paper, if \( M \) is a complex matrix then we note \( M^T \) for its transpose and \( M^* \) for its conjugate transpose and therefore \( z^* \) is the conjugate of \( z \) in \( \mathbb{C} \) while if \( A \) is an operator we note \( \overline{A} \) its adjoint. Notations \( \langle \cdot , \cdot \rangle \) and \( \| \cdot \| \) are used for the usual hermitian product and associated norm in \( L^2 (0, t_f) \). For a complex sequence \( (s_0, s_1, \ldots, s_{n}) \) (resp. \( (s_1, \ldots, s_n) \)) the associated column vector of \( M^{n+1,1} (\mathbb{C}) \) (resp. \( M_{n,1} (\mathbb{C}) \)) is noted \( s^{(n)} \) (resp. \( s^{\langle n \rangle} \)). For a double complex sequence \( (s_1, \ldots, s_m), (s_0, s_1, \ldots, s_n) \), the associated column vector of \( M^{m+n+1,1} (\mathbb{C}) \) is noted \( s^{[m,n]} = \left( \begin{array}{c} s^{\langle m \rangle} \\ s^{\langle n \rangle} \end{array} \right) \). Matrix notations \( \sigma^{\langle m \rangle}, \sigma^{\langle n \rangle} \) designate \( \text{diag}(s_1, \ldots, s_m) \in \mathbb{M}_m (\mathbb{C}) \) and \( (s_1 \cdots s_m) \in \mathbb{M}_{1,m} (\mathbb{C}) \) respectively. The null matrix of \( \mathbb{M}_{n,p} (\mathbb{C}) \) is \( 0_{n,p} \). The notation \( \delta_{i,j} = 1 \) if \( i = j \) and \( \delta_{i,j} = 0 \) otherwise is used. For \( n \geq k \geq 0 \) the number of \( k \)-permutations of \( n \) is \( A^{n}_k = \frac{n!}{(n-k)!} \) and the number of \( k \)-combinations of an \( n \)-set is \( C^{n}_k = \frac{n!}{k!(n-k)!} \). For \( t \in [0, t_f] \), we introduce the backward time \( \tau = t - t_f \).

### 2. CONJUGATE GRADIENT METHOD

Following [Alif’95], two kinds of PDEs are to be solved for the CGM:

- The direct one, as it is the case for (1):

\[
\begin{align*}
T_e^k (x, t) - T_{xx^k} (x, t) & = 0, \\
T_e^k (x, 0) & = 0, \\
T_{xx^k} (1, t) & = u (t) \tag{6}
\end{align*}
\]

but in the more general case where \( u (t) = \sum_{i=1}^{m} \rho_i q_i (t) + \sum_{i=0}^{n} r_i x^k x^l \), which is denoted \( u^{[m,n]} (t) \), with \( \rho_i, r_i \in \mathbb{C} \). This PDE will give not only the direct temperature \( T (t) \) \( \Delta = T^{[m,n+1]} (t) = A u^{[m,n]} (t) \) but also what is the called in the literature the sensitivity temperature \( \overline{T} (t) = \Delta \overline{T}^{[m,n+1]} (t) = A p^{[m,n]} (t) \) for the modified gradient or descent direction \( p (t) = p^{[m,n]} (t) \). Here, superscripts \([\cdot, n]\), \([\cdot, n+1]\) anticipate polynomial forms in \( \tau \) of degrees \( n, n+1 \) respectively while superscript \( k \) anticipates the \( k^{th} \) iteration of the CGM.

- The adjoint one which in backward time \( \tau \) takes the form:

\[
\begin{align*}
\overline{T}^k_e (x, \tau) - \overline{T}_{xx^k} (x, \tau) & = 0, \\
\overline{T}^k_e (x, 0) & = 0, \\
\overline{T}_{xx^k} (1, t) & = 0 \tag{7}
\end{align*}
\]

for \( u (t) = \Delta T^{[m,n]} (t) = T^{[m,n]} (t) - T_e (t) = \sum_{i=1}^{m} \sigma_i q_i (t) + \sum_{i=0}^{n} s_i x^l \). In direct time \( t \),
\[
\overline{T}^k_e (x, t_f - t) = T^k_e (x, t)
\]
is the adjoint temperature at the \( k^{th} \) iteration of the CGM. Anticipating the degree \( n \) in \( \tau \), this defines the adjoint operator \( \overline{A} \) by:

\[
\overline{T}^{k, [m,n+1]} (t) = \overline{A} \left( \Delta T^{k, [m,n]} \right) (t)
\]

Next, we are going to see how temperatures \( T^{k, [m,n+1]} \), \( \overline{T}^{k, [m,n+1]} \) and \( T^{[m,n+1]} \) can be calculated. A \( L^2 \) norm calculus will follow. For sake of clarity, iteration number \( k \) will be omitted.

### 2.1 Direct and sensitive PDE

It can be verified that the solution \( T (x, t) \) to (6) has the form:

\[
\begin{align*}
T (x, t) & = \sum_{i=1}^{m} \rho_i R_{i,t} (x, t) + R_2 (x, t), \\
R_{i,t} (x, t) & = q_i (t) \theta_i (x) + S_i (x, t), \\
R_2 (x, t) & = h (x, t) + X (x, t), \\
h & \text{poly} \text{ of degree } 2 (n+1) \text{ for } x
\end{align*}
\]

where \( R_{i,t}, R_2, \theta_i, S_i, h \) are solutions of

\[
\begin{align*}
\rho \theta_i (x) - \theta_i x (x) & = 0, \\
S_i (x, 0) & = 0 \\
S_i (1, t) & = 0, \\
h (0, t) & = 0, \\
H_x (1, t) & = 0, \\
\theta_i (x) + S_i (x, 0) & = 0, \\
h (0, t) + X (x, 0) & = 0.
\end{align*}
\]

with boundary and initial conditions

\[
\begin{align*}
\theta_i (0) & = 0, \\
S_i (0, 0) & = 0 \\
S_i (1, t) & = 0, \\
h (0, t) & = 0, \\
H_x (0, t) & = 0, \\
h (1, t) & = \sum l \tau^l, \\
H_x (1, t) & = 0, \\
\theta_i (x) & = 0, \\
h (0, t) & = 0.
\end{align*}
\]

This gives:

\[
T (t) = \sum_{i=1}^{m} \rho_i q_i (t) \theta_i (0) + S_i (0, t) + h (0, t) + X (0, t).
\]

Functions \( \theta_i \) and \( S_i \) have already been given in (2).

Polynomial \( h \) is sought in the form:

\[
h (x, t) = \sum_{i=0}^{n} h_i (x, t) x^i
\]

It can be shown that \( B_i \) must satisfy:

\[
B_i (\tau) = (-1)^{n+i} \sum_{l=0}^{n+i} A_{2l+1}^{2l+1} \tau^l, \quad i = 0, \ldots, n + 1
\]
However, since leads to: Thus a second complex sequence where the complex sequence admits a calculable inverse which takes the form:

\[ C^{-1} = (-1)^n \left( \frac{d_{j-i}}{A^{2i}_{2i}} \delta_{j \geq i} \right) _{i,j=1,...,n+1} \]

where the complex sequence \( (d_t)_{t \in \mathbb{N}} \) is given by the recurrence relation

\[ d_0 = 1, \quad d_l = \sum_{i=0}^{l-1} \frac{(-1)^{l+i+1}}{(2(l-i)+1)!} d_i, \text{ for } l \geq 1. \]

Thus, \( h(x,t) \) is defined. In particular, we have:

\[ h(0,t) = -\sum_{i=1}^{n} \tau^j \left( \frac{r_{j-i}}{l} + \sum_{j=0}^{n} d_{j-i+1} A^{j-i}_{2j} r_j \right) \cdot \frac{\tau^{n+i} r_n}{n+1} \]

Theoretically, \( H(x,t) \) can then be found by separation of variables and Fourier series from \( h(x,0) + H(x,0) = 0 \) in the same way as for \( S(x,0) \) from \( \theta(x) + S(x,0) = 0 \). This means that \( H(x,t) \) will take the form

\[ H(x,t) = c_0 [-h(x,0)] + \sum_{n=1}^{\infty} \exp(-n^2 \pi^2 t^2) a_n [-h(x,0)] \cos(n \pi x). \]

However, since \( h(x,0) \) has a much more complicated form than \( \theta(x) \) for Fourier series, coefficients \( a_n [-h(x,0)] \) are practically not available. We make the important hypothesis:

**Assumption 1.** Let us suppose \( t \gg 0 \) and that the following approximations are available:

\[ S_i(x,t) \approx \lambda_i, \quad H(x,t) \approx c_0 [-h(x,0)]. \]

From a system point of view, this amounts to neglecting the transient part in front of the steady state part of the PDE response to \( u(t) \). Even for \( c_0 [-h(x,0)] \), the calculus is not trivial. We found:

\[ c_0 [-h(x,0)] = \sum_{j=0}^{n} s_j r_j \]

with \( s_j = \frac{1}{j+1} \left( \frac{t}{t+1} \right)^{j+1} - e_j, \)

\[ e_j = (-1)^j j! \sum_{i=0}^{j-1} (-1)^i \frac{1}{(2(j-i)+3)!} d_i. \]

Thus a second complex sequence \( (e_j)_{j \in \mathbb{N}} \) appears. Solving (6) is achieved and, according to (2), (8), (9), (10), this leads to:

\[ T(t) \approx \sum_{i=1}^{m} \rho_i \left( \mu_i q(t) + \lambda_i \right) + h(0,t) + c_0 [-h(x,0)]. \]

More precisely, with \( T^{[m,n+1]}(t) = \sum_{i=1}^{m} \alpha_i q_i(t) + \sum_{j=0}^{n+1} a_j t^j \) and \( u^{[m,n]}(t) = \sum_{i=1}^{m} \rho_i q_i(t) + \sum_{j=0}^{n} r_j t^j \), this relation is equivalent to

\[
\begin{aligned}
&\alpha_i = \rho_i \mu_i, \quad i = 1, \ldots, m \\
&a_0 = \sum_{i=1}^{m} \rho_i \lambda_i + c_0 [-h(x,0)] + \sum_{j=0}^{n+1} a_j t^j \\
&= \sum_{i=1}^{m} \rho_i \lambda_i + c_0 [-h(x,0)] + \sum_{j=0}^{n+1} a_j t^j \\
&= \frac{1}{k} r_{k-1} - \sum_{k=0}^{n+1} d_k A_{l-k} - l \quad \text{for } 1 \leq k \leq n \\
&a_k = -\frac{1}{k} r_k - l \quad \text{for } 1 \leq k \leq n \\
a_{n+1} = -\frac{1}{n+1} r_n.
\end{aligned}
\]

Using notations of the introduction as for example

\[ T^{[m,n+1]} = \left( a^{[m,n]} \right) Q^{[m,n]} = \left( \rho^{[m,n]} \right), \]

this in turn can be put under the matrix form

\[ T^{[m,n+1]} = \left( \begin{array}{c}
a^{[m,n]} \\
a^{[m,n]+1}
\end{array} \right) = \left( \begin{array}{c}
\left( \begin{array}{c}
a^{[m,n]} \\
a^{[m,n]+1}
\end{array} \right) = \left( \begin{array}{c} a^{[m,n]} \\
a^{[m,n]+1}
\end{array} \right)
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\left( \begin{array}{c}
a^{[m,n]} \\
a^{[m,n]+1}
\end{array} \right) = \left( \begin{array}{c} a^{[m,n]} \\
a^{[m,n]+1}
\end{array} \right)
\end{array} \right)
\]

that is to say \( T^{[m,n+1]} = A^{[m,n]} Q^{[m,n]} \)
\[
\begin{align*}
\left\{ \begin{array}{l}
f_j = \frac{1}{A_{2j+1}} - \left(2e_{j+1} + (-1)^j d_{j+1}\right) j!
\end{array}\right.
\end{align*}
\]

Relation (14) admits in turn the matrix form :

\[
\left(\begin{array}{c}
\bar{a}[n+1] \\
-\Delta T[n]
\end{array}\right) = \left(\begin{array}{c}
\bar{a}[n] \\
-\Delta T[n]
\end{array}\right) - \left(\begin{array}{c}
l[n] \\
f[n]
\end{array}\right)
\]

that is to say \( \bar{T}[n+1] = \bar{T}[n] - \Delta T[n] \)

where \( \bar{T}[n] = (f, f_1 \cdots f_n) \in M_{1,n+1}(Q) \), \( E[n] = (\eta_{ij})_{i,j=1,\ldots,n+1} \in M_{n+1}(Q) \)

\[
\left\{ \begin{array}{l}
\eta_{jj} = \frac{1}{l} \quad \text{for } 1 \leq j \leq n + 1 \\
\eta_{ij} = \frac{1}{l} \quad \text{for } 2 \leq i + 1 \leq j \leq n + 1 \\
\eta_{ij} = 0 \quad \text{for } 0 \leq i < j \leq n + 1.
\end{array}\right.
\]

### 3.3 Particular \(L^2\) norms

For \( \langle u, v \rangle = \int_0^T u(t)v(t) \, dt \) one can easily compute :

\[
\left\langle (t_f - t)^k, (t_f - t)^l \right\rangle = \frac{f^{k+l+1}}{k+l+1}
\]

\[
\left\langle (t_f - t)^k, q(t) \right\rangle = \lambda_i \left( \frac{\xi_{ii} + \delta_{i1} \sum_{k=0}^{l-1} (-\lambda_i)^k A_{2k+1} t_f^{l-k}}{P_{ii}} \right)
\]

with

\[
\left\{ \begin{array}{l}
\epsilon_{ii} = (-\lambda_i)^k \frac{1}{l} \left(1 - \exp \left(\frac{\omega_i t_f}{l}\right)\right) \\
\xi_{ii} = (-\lambda_i)^k \frac{1}{l} \left(1 - \exp \left(\frac{\omega_i t_f}{l}\right)\right) \\
P_{ii} = \epsilon_{ii} + \delta_{i1} \sum_{k=0}^{l-1} (-\lambda_i)^k A_{2k+1} t_f^{l-k}, \quad P_{0i} = \xi_{0i}
\end{array}\right.
\]

which satisfies \( P_{ii} = t_f^l - \lambda_i t_f P_{l-1} \) if \( l \geq 1 \).

The \(L^2\) norm of \( T[m,n] \) represented by \( \bar{T}[m,n] \) is then given by \( \| T[m,n] \|_2^2 = \sum T[m,n] \Pi[m,n] T[m,n] \) where \( \Pi[m,n] \in M_{m,n+1}(Q[t_f]) \) is defined by :

\[
\Pi[m,n] = \left( \begin{array}{c}
M[1:m] \\
p[1:m] n[1:m]
\end{array}\right)
\]

with

\[
\left\{ \begin{array}{l}
p[n] = \left( \begin{array}{c}
p[1:m] n[1:m]
\end{array}\right), \quad p[n]^{[i,j]} = \left( P_{ii} \right)_{i,j=1,\ldots,n+1} \in M_{n+1}(Q[t_f]) \\
M[1:m] = \sum_{i=1}^n \sum_{j=1}^n M_{ij} = \sum_{i=1}^n \sum_{j=1}^n M_{ij} = \sum_{i=1}^n \sum_{j=1}^n M_{ij}
\end{array}\right.
\]

where

\[
m_{ij} = \left\{ \begin{array}{l}
1 \quad \text{if } i = j \\
\frac{1}{\omega_j - \omega_i} \left(1 - \exp \left(-\frac{j_f (\omega_j - \omega_i)}{l}\right)\right) \quad \text{if } i \neq j.
\end{array}\right.
\]

### 2.4 Algorithm

In this academic study, the measured temperature \( T_s(t) = \sum_{i=1}^m \psi_i(t) + \lambda_i \) at \( x = 0 \) has been simulated by the response of PDE (6) to \( u_s(t) = \sum_{i=1}^m \psi_i(t) \) under approximation 1. The criterion to be minimized is

\[
J_k^2 \equiv J_2 (u_k) = \frac{1}{2} \| \Delta T (u_k) \|^2.
\]

Following [Alif:95] and using vectors \( T^{[k,m,n]}, \Pi^{[k,m,n]} \), \( \Delta T^{[k,m,n]} \), \( \sum T^{[k,m,n]} \) instead of function \( T^k, \Pi^k, \Delta T^k, \) \( u^k \) according to notations of the introduction, and so on for matrices \( A^{[m,n]} , \Pi^{[m,n]} \) instead of \( A, \Pi, \Pi \|^2 \) according to notations (13), (15), (16), we can explicit below the CGM where the degree \( n \equiv n (k) \) in backward time \( r \) is specified according to \( k \):

1. **Initial step**

   - Choose \( U^0 \), that is to say fixe \( U^0 = 0 \) in our case ; consequently, \( J_0^k = J_0^0 \) is simply \( T^0 = 0 \);
   - Calculate \( \Delta T^0 = T^0 - T^0(m,0) = -T^0 \) and \( J_0^2 = \frac{1}{2} \Delta T^2 \) and set \( \beta_0^2 = J_0^0 \);
   - Calculate \( \beta_0^0 = \sum T^0 = \sum T^0(m,0) = \sum T^0 \Pi^0, \Pi^0 \)
   - Calculate \( \beta_0^0 = \sum T^0 \Pi^0, \Pi^0 \Pi^0 \)

2. **Hereditary step**

   - Let suppose that for \( k \geq 1, U^{[k,m,2k-1]} \), \( \sum U^{[k,m,2k-1]} \), \( \sum U^{[k,m,2k-1]} \) are given ;
   - Calculate \( T^{[k,m,2k]} = \sum T^{[k,m,2k-1]} \Pi^{[k,m,2k-1]} \)
   - Calculate \( \Delta T^{[k,m,2k]} = T^{[k,m,2k]} - T^{[k,m,0]} \) and \( J_k^2 = \frac{1}{2} \Delta T^{[k,m,2k]} \Pi^{[k,m,2k]} \Delta T^{[k,m,2k]} \Pi^{[k,m,2k]} \)
   - Calculate \( \gamma = \sum T^{[k,m,2k+1]} \)
   - Calculate \( \gamma \) given by

\[
\gamma = \sum T^{[k,m,2k+1]} + \gamma \left( \sum T^{[k,m,2k+1]} \right)
\]

and deduce

\[
\sum T^{[k,m,2k+1]} = \sum T^{[k,m,2k+1]} + \gamma \left( \sum T^{[k,m,2k+1]} \right)
\]

3. **Recurrence principle step**

   - Let \( k < K \), replace \( k \) by \( k + 1 \) and return to step 2.

The above algorithm defines the CGM discrete dynamical system (5) for \( u^0 = 0 \) by :

\[
\sum T^{[k,m,2k+1]} = \sum T^{[k,m,2k+1]} + \gamma \left( \sum T^{[k,m,2k+1]} \right)
\]
\[ \Theta_{A_{f_j}} : (M_{m+2k}(C))^3 \rightarrow (M_{m+2k+2}(C))^3 \]
\[ \left\{ \begin{array}{c}
\Psi_{k+1}, [2k+1], \Psi_{k+1}, [2k+1], \Psi_{k+1}, [2k+1]
\end{array} \right. \]

3. FILTERING PROPERTY OF THE CGM

3.1 A mitigation property of the CGM

The descent memory coefficient \( \gamma^k = \frac{\|T^k\|^2}{\|T^1\|^2} \) admits the
alternative determination \( \gamma^k = \frac{(A_{p^k-1} - AT^m)}{\|A_{p^k-1}\|^2} \) [Alif]. Consequently, in the foregoing algorithm \( \Theta_{A_{f_j}} (u^k, p^{k-1}, T^{k-1}) \) can be substituted by \( \Theta_{A_{f_j}} (u^k, p^{k-1}, T^{k-1}) \).

Let us consider \( u_* = \sum_{i=1}^m \psi_i q_i \), with \( q_i(t) = \exp(j\omega_i t) \) and \( \psi_i \in C^* \), the flux to be identified on the right side of the rode. Applying our CGM to such \( u_* \) leads us to look at sequences \( u^k, p^{k-1} \) in the form \( u^k = Q^k + D^k, p^{k-1} = P^k-1 + C^k \) with \( Q^k = \sum_{i=1}^m \psi_i q_i \), \( P^k-1 = \sum_{i=1}^m \psi_i q_i \), \( D^k, C^k \) polynomials terms in \( C[\tau] \). Two observations can be done. The first one is that \( \mu_i = \mu_i^2 \psi_i \) as defined in (2) is an even function of \( \omega_i \), quickly decreasing towards 0 on \( \mathbb{R}_+ \). The second one, based on simulations (as defined in (2) can be stated as follows:

**Assumption 2.** Terms \( D^k \) in \( u^k \) and \( C^{k-1} \) in \( D^{k-1} \) can be roughly neglected in front of \( Q^k \) and \( P^{k-1} \) respectively.

The following result can then be proved:

**Lemma 1.** For \( \gamma \in \mathbb{R} \) set \( f_\gamma(x) = 1 + \gamma x \). We have thus, for example, \( f_{\gamma^2} \circ f_{\gamma^1}(1) = 1 + \gamma^2 (1 + \gamma^1) \). Let \( \sigma^k \) be defined by

\[ \sigma^k = \sum_{i=0}^k \beta^i + \beta^i \gamma^1 + \sum_{j=2}^k \beta^j \gamma^1 \left( f_{\gamma^1-1} \circ f_{\gamma^1-2} \circ \cdots \circ f_{\gamma^1} \right) (1) \]

which is always positive. Under assumption 2, we have

\[ \psi_{i+1} = \psi_i \left( |\mu_i|^2 \sigma^k + o \left( |\mu_i|^2 \right) \right) \]

(19)

Note that \( \sigma^k \) is independent from \( i \). If \( k = K \) is the rank at which the CGM is stopped, the product \( \psi_i |\mu_i|^2 \sigma^k \) is to be compared with the expected value \( \psi_i \). At most one index \( i \), or two indexes \( i \) and \( j \) if \( \omega_i = -\omega_j \), can give a value of \( |\mu_i|^2 \sigma^k \) closed to one. The filtering property of the CGM does not occur in the sense expected by authors [Prud:98] but rather as a mitigation property.

3.2 Simulations

Three configurations were tested with the CGM algorithm (17) under Maple 18. For same \( t_f = 10 \) and \( q_1 = \exp(\pi t) \), \( q_2 = \exp(2\pi t) \), \( q_3 = \exp(5\pi t) \), they are:

(i) \( u_* (t) = \text{Im} (q_1(t)) = \sin(\pi t) \);
(ii) \( u_* (t) = \text{Im} (5q_1 + 2q_2 + q_3) = 5 \sin(\pi t) + 2 \sin(2\pi t) + \sin(5\pi t) \);
(iii) \( u_* (t) = \text{Im} (q_1 + 2q_2 + 5q_3) = \sin(\pi t) + 2 \sin(2\pi t) + 5 \sin(5\pi t) \).

Results are shown on figures 1, 2 and 3 obtained in about 200s for each one. The upper-left sub-figure gives the measured temperature \( T_1(t) \) (dash line) and the calculated temperature \( T_K(t) \) (solid line) at \( x = 0 \), the upper-right sub-figure gives the true flux \( u_1(t) \) (dash line) and the calculated flux \( u_K(t) \) at \( x = 1 \), the lower-left subfigure gives the iterated residual criterion \( J_{\theta}^2 \), the lower-right sub-figure gives the true flux \( u_1(t) \) (dash line) and the polynomial part \( D^k \) of the calculated flux \( u_K(t) \). The regularization parameter \( K \) has been chosen to minimize both \( J_{\theta}^2 = \frac{1}{2} \|A_{u^k} - T_{K}^2\|^2 \) and \( \Delta u_k = \|u_k - u_*\|^2 \).

In this three cases the polynomial parts \( D^k \) compared to the exponential part \( K \) has a low contribution to the total heat flux \( u_k \). This is measured by rates \( \|D^k\| \) more or less close to zero by upper value, see the first line of table (20). Lemma 1 is illustrated by \( (|\mu_i|^2 \sigma^K)_{i\in{1,2,3}} \) given in the second line of table (20). Its validity, linked to the rightness of the approximation of \( |\psi_i|^2 \) by \( (|\mu_i|^2 \sigma^K)_{i\in{1,2,3}} \) can be appreciated by comparison of the second and third lines of table (20). This third line tests the ability of the CGM to recover a sequence of increasing frequencies.

| (\omega_i) | (\psi_i) | (\psi_i)^2 |
|------------|-------------|-------------|
| (\pi)      | (\pi, 2\pi, 5\pi) | (\pi, 2\pi, 5\pi) |
| (5, 1, 2)  | (1, 2, 5)    | (1, 2, 5)    |

(20)

With a ratio \( |\psi_i|^2 \) between 0.97 and 1.09 (fourth line of table (20)), we can see that the CGM gives a good rendering of the lower frequency \( \omega_i = \pi \). This is not the case for upper frequencies \( \omega_i \), \( i = 1, 2 \) for which a ratio \( |\psi_i|^2 \) less than 0.3 shows that the identification by the CGM is not null but quite poor. In cases (ii) and (iii), it is interesting to calculate the mitigation ratios \( |\psi_i|^2 |\psi_i|^2 \):

| mitigation ratios | (\psi_i) | (\psi_i)^2 |
|-------------------|--------|--------|
| (\psi_i) \| (\psi_i)^2 | 0.25 | 0.057 |
| (\psi_i) \| (\psi_i)^2 | 0.27 | 0.058 |

This mitigation ratio is a decreasing function of the frequency which is, in this application, quite independent of the magnitudes \( |\psi_i| \) for higher iteration than \( K \) does not allow to hope a better recovery of the higher frequencies by the CGM.

4. CONCLUSION

Two new results have been presented here for the CGM applied to a simple IHCP. The first one is explicit formulæ
Fig. 1. $\text{Im}(u_s(t)) = \sin(\pi t)$

Fig. 2. $\text{Im}(u_s(t)) = 5\sin(\pi t) + 2\sin(2\pi t) + \sin(5\pi t)$

Observations done on simulations suggest a low contribution of the polynomial part of the solution to the inverse problem compared to its exponential part. Further investigations should be done with the transfer function [Zwa:04] of (1) to quantify this assumption.

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given in (13), (15), (16), (17) to approach a solution for $0 \ll t \ll t_f$ under assumption 1. The second one with lemma 1 is a mitigation property of the CGM which doesn’t allow to recover high frequencies as well as low frequencies.