SOJOURN RUIN OF A TWO-DIMENSIONAL FRACTIONAL BROWNIAN MOTION RISK PROCESS

GRIGORI JASNOVIDOV

Abstract: This paper derives the asymptotic behavior of

\[ \mathbb{P} \left\{ \int_0^\infty \mathbb{I} \left( B_h(s) - c_1 s > q_1 u, B_h(s) - c_2 s > q_2 u \right) ds > T_u \right\}, \quad u \to \infty, \]

where \( B_h \) is a fractional Brownian motion, \( c_1, c_2, q_1, q_2 > 0, \) \( H \in (0, 1) \), \( T_u \geq 0 \) is a measurable function and \( \mathbb{I}(\cdot) \) is the indicator function.

Key Words: fractional Brownian motion; simultaneous ruin probability; two-dimensional risk processes; sojourn ruin;

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction & Preliminaries

Consider the risk model defined by

\[ R(t) = u + \rho t - X(t), \quad t \geq 0, \]

where \( X(t) \) is a centered Gaussian risk process with a.s. continuous sample paths, \( \rho > 0 \) is the net profit rate and \( u > 0 \) is the initial capital. This model is relevant to insurance and financial applications, see, e.g., [13]. A question of numerous investigations (see [1, 3–8, 10, 12, 14, 15, 19, 22–25]) is the study of the asymptotics of the classical ruin probability

\[ \lambda(u) := \mathbb{P} \{ \exists t \geq 0 : R(t) < 0 \} \]

as \( u \to \infty \) under different levels of generality. It turns out, that only for \( X \) being a Brownian motion (later on BM) \( \lambda(u) \) can be calculated explicitly: if \( X \) is a standard BM, then \( \lambda(u) = e^{-2\rho u}, \) \( u, \rho > 0, \) see [11]. Since it seems impossible to find the exact value of \( \lambda(u) \) in other cases, the approximations of \( \lambda(u) \) as \( u \to \infty \) is dealt with. Some contributions (see, e.g., [9, 17]), extend the classical ruin problem to the so-called sojourn ruin problem, i.e., approximation of the sojourn ruin probability defined by

\[ \mathbb{P} \left\{ \int_0^\infty \mathbb{I}(R(s) < 0) ds > T_u \right\}, \]

where \( T_u \geq 0 \) is a measurable function of \( u \). As in the classical case, only for \( X \) being a BM the probability above can be calculated explicitly, see [9]:

\[ \mathbb{P} \left\{ \int_0^\infty \mathbb{I}(B(s) - cs > u) ds > T \right\} = \left( 2(1 + c^2) \Psi(c\sqrt{T}) - \frac{c\sqrt{2T}}{\sqrt{\pi}} e^{-\frac{c^2T}{2}} \right) e^{-2cu}, \quad c > 0, \ T, u \geq 0, \]
where $\Psi$ is the survival function of a standard Gaussian random variable, $B$ is a standard BM and $\mathbb{I}(\cdot)$ is the indicator function. Motivated by [18] (see also [15, 16]), we study a generalization of the main problem in [18] for the sojourn ruin, i.e., we shall study the asymptotics of

$$C_{T_u}(u) := \mathbb{P}\left\{ \int_0^\infty \mathbb{I}\left(B_H(s) - c_1 s > q_1 u, B_H(s) - c_2 s > q_2 u\right)ds > T_u \right\}, \quad u \to \infty,$$

where $B_H$ is a standard fractional Brownian motion (later on fBM), i.e., a Gaussian process with zero expectation and covariance defined by

$$\text{cov}(B_H(s), B_H(t)) = \frac{|t|^{2H} + |s|^{2H} - |t-s|^{2H}}{2}, \quad t, s \in \mathbb{R}.$$ 

The ruin probability above is of interest for reinsurance models, see [18] and references therein. By the self-similarity of fBM we have

$$C_{T_u}(u) = \mathbb{P}\left\{ \int_0^\infty \mathbb{I}\left(u^H B_H(s) > (c_1 s + q_1) u, u^H B_H(s) > (c_2 s + q_2) u\right)ds > T_u/u \right\} = \mathbb{P}\left\{ \int_0^\infty \mathbb{I}\left(\frac{B_H(s)}{\max(c_1 s + q_1, c_2 s + q_2)} > u^{1-H}\right)ds > T_u/u \right\}. \tag{4}$$

In order to prevent the problem of degenerating to the one-dimensional sojourn problem discussed in [9, 17] (i.e., to impose the denominator in the line above be nonlinear function) we assume that

$$c_1 > c_2, \quad q_2 > q_1. \tag{5}$$

The variance of the process two lines above can achieve its unique maxima only at one of the following points:

$$t_\ast = \frac{q_2 - q_1}{c_1 - c_2}, \quad t_1 = \frac{q_1 H}{(1-H)c_1}, \quad t_2 = \frac{q_2 H}{(1-H)c_2}. \tag{5}$$

From (4) it follows that $t_1 < t_2$; as we shall see later, the order between $t_1, t_2$ and $t_\ast$ determines the asymptotics of $C_{T_u}(u)$ as $u \to \infty$. As mentioned in [5], for the approximation of the one-dimensional Parisian ruin probability we need to control the growth of $T_u$ as $u \to \infty$. As in [5], we impose the following condition:

$$\lim_{u \to \infty} T_u u^{1/H-2} = T \in [0, \infty), \quad H \in (0, 1). \tag{6}$$

Note that $T_u$ satisfying (6) may go to $\infty$ for $H > 1/2$, converges to non-negative limit for $H = 1/2$ and approaches 0 for $H < 1/2$ as $u \to \infty$. We see later on in Proposition 2.2 that the condition above is necessary and it seems very difficult to derive the exact asymptotics of $C_{T_u}(u)$ as $u \to \infty$ without it.

The rest of the paper is organized in the following way. In the next section we present the main results of the paper, in Section 3 we give all proofs, while technical calculations are deferred to the Appendix.
2. Main Results

Define for some function $h$ and $K \geq 0$ the sojourn Piterbarg constant by

$$B^K_K = \int \mathbb{P} \left\{ \int_{\mathbb{R}} \left( \sqrt{2B(s) - |s| + h(s)} > x \right) ds > K \right\} e^x dx$$

when the integral above is finite and Berman’s constant by

$$B_{2H}(x) = \lim_{S \to \infty} \frac{1}{S} \int_{\mathbb{R}} \left\{ \int_0^S \mathbb{I}(\sqrt{2B_H(t)} - t^{2H} + z > 0) dt > x \right\} e^{-z} dz, \quad x \geq 0.$$ 

It is known (see, e.g., [9]) that $B_{2H}(x) \in (0, \infty)$ for all $x \geq 0$; we refer to [9] and references therein for the properties of relevant Berman’s constants. Let for $i = 1, 2$

$$D^H = \frac{c_1 t_s + q_1}{t_s^H}, \quad K_H = \frac{\sqrt{2 + \pi^2} \sqrt{\pi}}{\sqrt{H(1 - H)}}, \quad C_H^{(i)} = c_i^{1-H} H^{1-H}, \quad D_i = \frac{c_i^2 (1 - H)^2 - \frac{1}{2}}{2H^2 (1 - 2H)}.$$

Now we are ready to give the asymptotics of $C_{T_u}(u)$ as $u \to \infty$.

**Theorem 2.1.** Assume that (4) holds and $T_u$ satisfies (6).

1) If $t_s \notin (t_1, t_2)$, then as $u \to \infty$

$$C_{T_u}(u) \sim \left(\frac{1}{2}\right)^{i(t_s = t_i)} \times \begin{cases}
(2(1 + c_i^2)T)\Psi(c_i\sqrt{T}) - \frac{c_i\sqrt{T}}{\sqrt{H}} e^{-\frac{c_i^2}{\sqrt{H}}} e^{-2c_i q_i u}, & H = 1/2 \\
K_H B_{2H}(TD_i)(C_H^{(i)} u^{1-H})^{\frac{1}{1-H}} \Psi(C_H^{(i)} u^{1-H}), & H \neq 1/2,
\end{cases}$$

where $i = 1$ if $t_s \leq t_1$ and $i = 2$ if $t_s \geq t_2$.

2) If $t_s \in (t_1, t_2)$ and $\lim_{u \to \infty} T_u u^{2-1/H} = 0$ for $H > 1/2$, then as $u \to \infty$

$$C_{T_u}(u) \sim \Psi(D^H u^{1-H}) \times \begin{cases}
1, & H > 1/2 \\
B^{d}_{T'}, & H = 1/2 \\
B_{2H}(D) A u^{(1-H)(1/H-2)}, & H < 1/2,
\end{cases}$$

where $B^{d}_{T'} \in (0, \infty)$,

$$T' = T \frac{(c_1 q_2 - q_1 c_2)^2}{2(c_1 - c_2)^2}, \quad d(s) = s \frac{c_1 q_2 + c_2 q_1 - 2c_2 q_2}{c_1 q_2 - q_1 c_2} \mathbb{I}(s < 0) + s \frac{2c_1 q_1 - c_1 q_2 - q_1 c_2}{c_1 q_2 - q_1 c_2} \mathbb{I}(s \geq 0)$$

and

$$A = \left( |H(c_1 t_s + q_1) - c_1 t_s|^{-1} + |H(c_2 t_s + q_2) - c_2 t_s|^{-1} \right) \frac{t_s^{H-1}}{2\pi^H}, \quad D = \frac{(c_1 t_s + q_1)^{\frac{1}{2H}}}{2\pi^{\frac{1}{2H}}}.$$ 

Note that if $T = 0$, then the result above reduces to Theorem 3.1 in [18]. As already mentioned in the introduction (6) is a necessary condition for the theorem above. To illustrate situation when it is not satisfied we consider a "simple" scenario with $T_u$ being a positive constant.

**Proposition 2.2.** If $H < 1/2$, $T_u = T > 0$ and $t_s \in (t_1, t_2)$, then

$$C_{T_u}(u) \sim \Psi(D^H u^{1-H}) e^{-C_1 u^{2-4H} - C_2 u^{2(1-3H)}} \leq C_{T_u}(u) \leq (2 + o(1)) \Psi(D^H u^{1-H}) \Psi\left(u^{1-2H} \frac{TD^H}{2H}\right), \quad u \to \infty,$$ 

where $\mathbb{I}(\cdot)$ is the indicator function, $\Psi(x) = x^{1-H} e^{-x}$.
where $\mathcal{C} \in (0, 1)$ is a fixed constant that does not depend on $u$ and

$$\alpha = \frac{T^{2H}}{2\alpha^2}, \quad C_{i, \alpha} = \frac{\alpha i}{i \bar{D}_H^2}, \quad i = 1, 2.$$ 

Note that the lower bound in the proposition above decays to zero exponentially faster than the upper bound as $u \to \infty$.

3. Proofs

First we give the following auxiliary results. As shown, e.g., in Lemma 2.1 in [12]

$$\ Psi(u) \leq \frac{1}{\sqrt{2\pi u}}e^{-u^2/2}, \quad u > 0. \tag{12}$$

Recall that $K_H, D_1$ and $C_H^{(1)}$ are defined in (7). A proof of the proposition below is given in the Appendix.

**Proposition 3.1.** Assume that $T_u$ satisfies (6). Then as $u \to \infty$

$$\mathbb{P}\left\{ \int_0^\infty \mathbb{I}(B_H(t) - c_1 t > q_1 u) dt > T_u \right\} \sim \begin{cases} (2(1 + c_1^2)\Psi(c_1 \sqrt{T}) - \frac{c_1 \sqrt{T}}{\sqrt{2\pi}} e^{-c_1^2 T})e^{-2c_1 q_1 u}, & H = 1/2, \\ K_H B_2 H(T D_1)(C_H^{(1)} u^{1-H})^{-1/2} \Psi(C_H^{(1)} u^{1-H}), & H \neq 1/2. \end{cases}$$

Now we are ready to perform our proofs.

**Proof of Theorem 2.1.** Case (1). Assume that $t_* < t_1$. Let

$$V_i(t) = \frac{B_H(t)}{c_i t + q_i} \quad \text{and} \quad \psi_i(T_u, u) = \mathbb{P} \left\{ \int_0^\infty \mathbb{I}(B_H(t) - c_i t > q_i u) ds > T_u \right\}, \quad i = 1, 2.$$ 

For $0 < \varepsilon < t_1 - t_*$ by the self-similarity of fBM we have

$$\psi_1(T_u, u) \geq \mathbb{C}_T(u) \geq \mathbb{P} \left\{ \int_{t_1 - \varepsilon}^{t_1 + \varepsilon} \mathbb{I}(V_1(t) > u^{1-H}, V_2(t) > u^{1-H}) dt > T_u/u \right\} = \mathbb{P} \left\{ \int_{t_1 - \varepsilon}^{t_1 + \varepsilon} \mathbb{I}(V_1(t) > u^{1-H}) dt > T_u/u \right\}.$$ 

We have by Borel-TIS inequality, see [24] (details are in the Appendix)

$$\psi_1(T_u, u) \sim \mathbb{P} \left\{ \int_{t_1 - \varepsilon}^{t_1 + \varepsilon} \mathbb{I}(V_1(t) > u^{1-H}) ds > T_u/u \right\}, \quad u \to \infty \tag{13}$$

implying $\mathbb{C}_T(u) \sim \psi_1(T_u, u)$ as $u \to \infty$. The asymptotics of $\psi_1(T_u, u)$ is given in Proposition 3.1, thus the claim follows.

Assume that $t_* = t_1$. We have

$$\mathbb{P} \left\{ \int_{t_1}^\infty \mathbb{I}(V_1(s) > u^{1-H}) ds > T_u \right\} \leq \mathbb{C}_T(u)$$

$$\leq \mathbb{P} \left\{ \int_{t_1}^\infty \mathbb{I}(V_1(s) > u^{1-H}) ds > T_u \right\} + \mathbb{P} \left\{ \exists t \in [0, t_1] : V_2(t) > u^{1-H} \right\}.$$
From the proof of Theorem 3.1, case (4) in [18] it follows that the second term in the last line above is negligible comparing with the final asymptotics of \( C_T(u) \) given in (8), hence

\[
C_T(u) \sim \mathbb{P} \left\{ \int_{t_1}^{\infty} \mathbb{I}(V_1(s) > u^{1-H}) ds > T \right\}, \quad u \to \infty.
\]

Since \( t_1 \) is the unique maxima of \( \text{Var} \{V_1(t)\} \) from the proof of Theorem 2.1, case i) in [9] we have

\[
\mathbb{P} \left\{ \int_{t_1}^{\infty} \mathbb{I}(V_1(t) > u^{1-H}) dt > T_u/u \right\} \sim \frac{1}{2} \mathbb{P} \left\{ \int_{0}^{\infty} \mathbb{I}(V_1(t) > u^{1-H}) dt > T_u/u \right\} = \frac{1}{2} \mathbb{P} \left\{ \int_{0}^{\infty} \mathbb{I}(B_H(t) - c_1 t > q_1 u) dt > T_u \right\}, \quad u \to \infty.
\]

The asymptotics of the last probability above is given in Proposition 3.1 establishing the claim. Case \( t_* \geq t_2 \) follows by the same arguments.

**Case (2).** Assume that \( H > 1/2 \). We have by Theorem 2.1 in [16] and Theorem 3.1 in [18] with

\[
\mathcal{R}_T(u) = \mathbb{P} \left\{ \exists t \geq 0 : B_H(t) - c_1 t > q_1 u, B_H(t) - c_2 t > q_2 u \right\},
\]

\[
\mathcal{P}_T(u) = \mathbb{P} \left\{ \exists t \geq 0 : \inf_{s \in [t,t+T_u]} (B_H(s) - c_1 s) > q_1 u, \inf_{s \in [t,t+T_u]} (B_H(s) - c_2 s) > q_2 u \right\}
\]

that

\[
\Psi(D_H u^{1-H}) \sim \mathcal{P}_T(u) \leq C_T(u) \leq \mathcal{R}_T(u) \sim \Psi(D_H u^{1-H}), \quad u \to \infty,
\]

and the claim follows.

Assume that \( H = 1/2 \). First let (6) holds with \( T_u = T > 0 \). We have as \( u \to \infty \) and then \( S \to \infty \) (proof is in the Appendix)

\[
C_T(u) \sim \mathbb{P} \left\{ \int_{ut_*-S}^{ut_*+S} \mathbb{I}(B(s) - c_1 s > q_1 u, B(s) - c_2 s > q_2 u) ds > T \right\} =: \kappa_S(u).
\]

Next with \( \phi_u \) the density of \( B(ut_*) \), \( \eta = c_1 t_* + q_1 = c_2 t_* + q_2 \) and \( \eta_* = \eta/t_* - c_2 = q_2/t_* \) we have

\[
\kappa_S(u) = \int_{\mathbb{R}} \mathbb{P} \left\{ \int_{ut_*-S}^{ut_*} \mathbb{I}(B(s) - c_2 s > q_2 u) ds + \int_{ut_*}^{ut_*+S} \mathbb{I}(B(s) - c_1 s > q_1 u) ds > T \big| B(ut_*) = \eta u - x \right\} \phi_u(\eta u - x) dx
\]

\[
= \int_{\mathbb{R}} \mathbb{P} \left\{ \int_{ut_*-S}^{ut_*} \mathbb{I}(B(s) - c_2 s > q_2 u) ds \right. \\
+ \int_{ut_*}^{ut_*+S} \mathbb{I}(B(s) - B(ut_*) - c_1 (s - ut_*) - c_1 ut_* > q_1 u - \eta u + x) ds > T \big| B(ut_*) = \eta u - x \right\} \phi_u(\eta u - x) dx
\]

\[
= \int_{\mathbb{R}} \mathbb{P} \left\{ \int_{ut_*-S}^{ut_*} \mathbb{I}(B(s) - c_2 s > q_2 u) ds + \int_{0}^{S} \mathbb{I}(B_*(s) - c_1 s > x) ds > T \big| B(ut_*) = \eta u - x \right\} \phi_u(\eta u - x) dx
\]
where \( Z_u(t) \) is a Gaussian process with expectation and covariance defined below:

\[
\mathbb{E} \{ Z_u(t) \} = \frac{-x}{ut_*}, \quad \text{cov}(Z_u(s), Z_u(t)) = \frac{st - t}{ut_*}, \quad s \leq t \leq 0.
\]

Since \( Z_u(t) \) converges to BM in the sense of convergence finite-dimensional distributions for any fixed \( x \in \mathbb{R} \) as \( u \to \infty \) we have (details are in the Appendix)

\[
\int \mathbb{P} \left\{ \int_{-S}^{0} \mathbb{I} \left( Z_u(s) + \eta s > x \right) ds + \int_{0}^{S} \mathbb{I} (B_s(s) - c_1 s > x) ds > T \right\} e^{\frac{ux}{t_*}} \ dx
\]

\[
\sim \int \mathbb{P} \left\{ \int_{-S}^{0} \mathbb{I} \left( B(s) + \eta s > x \right) ds + \int_{0}^{S} \mathbb{I} (B_s(s) - c_1 s > x) ds > T \right\} e^{\frac{ux}{t_*}} \ dx
\]

\[
= K(S).
\]

Since \( \mathbb{P} \{ \exists t > 0 : B(t) - ct > x \} = e^{-2c x}, \ c, x > 0 \) (see, e.g., [11]) we have

\[
K(S) \leq \int_{0}^{\infty} \left( \mathbb{P} \{ \exists s < 0 : B(s) + \eta s > x \} + \mathbb{P} \{ \exists s \geq 0 : B_s(s) - c_1 s > x \} \right) e^{\frac{ux}{t_*}} \ dx + \int_{-\infty}^{0} e^{\frac{ux}{t_*}} \ dx
\]

\[
= \int_{0}^{\infty} \left( e^{-2c + \eta / t_*} x + e^{-2c + \eta / t_*} x \right) dx + t_*/\eta < \infty
\]

provided by \( t_* \in (t_1, t_2) \). Since \( K(S) \) is an increasing function and \( \lim_{S \to \infty} K(S) < \infty \) we have as \( S \to \infty \)

\[
K(S) \to \int \mathbb{P} \left\{ \int_{0}^{\infty} \mathbb{I} \left( B(s) - \eta s > x \right) ds + \int_{0}^{\infty} \mathbb{I} (B_s(s) - c_1 s > x) ds > T \right\} e^{\frac{ux}{t_*}} \ dx
\]

\[
= \frac{t_*}{\eta} \int \mathbb{P} \left\{ \int_{0}^{\infty} \mathbb{I} \left( B(s) - \eta t_* / \eta s > x \right) ds + \int_{0}^{\infty} \mathbb{I} (B_s(s) - c_1 t_* / \eta s > x) ds > \eta^2 T / t_*^2 \right\} e^{x} \ dx
\]

\[
= \frac{t_*}{\eta} \int \mathbb{P} \left\{ \int_{-\infty}^{\infty} \mathbb{I} \left( \sqrt{2} B(s) - |s| + d(s) > x \right) ds > \eta^2 T / 2 t_*^2 \right\} e^{x} \ dx
\]

\[
= \frac{t_*}{\eta} \mathcal{B}^{d}_{T^*} \in (0, \infty),
\]

where \( T^* \) and \( d(s) \) are defined in (10). Finally, combining (16) with the line above we have as \( u \to \infty \) and then \( S \to \infty \)

\[
\kappa_S(u) \sim \mathcal{B}^{d}_{T^*} \Psi (\mathbb{D}^{1/2}/u)
\]

and by (14) the claim follows. If (6) holds with \( T_u = 0 \), then we obtain the claim immediately by Theorem 3.1 in [18] and observation that \( \mathcal{B}^{d}_{0} \) coincides with the corresponding Piterbarg constant introduced in [18].
Now assume that (6) holds with any possible $T_u$. If (6) holds with $T > 0$, then for large $u$ and any $\varepsilon > 0$ it holds, that $C_{(1+\varepsilon)T}(u) \leq C_{T_u}(u) \leq C_{(1-\varepsilon)T}(u)$ and hence

$$(1 + o(1))B^d_{T/(1+\varepsilon)}(\Omega_1) \leq C_{T_u}(u) \leq B^d_{T/(1-\varepsilon)}(\Omega_1) + o(1), \quad u \to \infty.$$ 

By Lemma 4.1 in [9] $B^d_1$ is a continuous function with respect to $x$ and thus letting $\varepsilon \to 0$ we obtain the claim. If (6) holds with $T = 0$, then for large $u$ and any $\varepsilon > 0$ we have

$$B^d_1(\Omega_1) \leq C_{T_u}(u) \leq B^d_0(\Omega_1)$$

and again letting $\varepsilon \to 0$ we obtain the claim by continuity of $B^d_1$.

Assume that $H < 1/2$. First we have with $\delta_u = u^{2H-2}\ln^2 u$ as $u \to \infty$ (proof is in Appendix)

$$C_{T_u}(u) \sim \mathbb{P}\left\{ \int_{ut_*}^{ut_*+u\delta_u} \|B_H(t) - c_2 t > q_2 u\|dt > T_u \right\} + \mathbb{P}\left\{ \int_{ut_*}^{ut_*+u\delta_u} \|B_H(t) - c_1 t > q_1 u\|dt > T_u \right\}$$

(17) $= g_1(u) + g_2(u)$.

Assume that (6) holds with $T > 0$. Using the approach from [9] we have with $\mathbb{I}_a(b) = \mathbb{I}(b > a)$, $a, b \in \mathbb{R}$

$$g_2(u) = \mathbb{P}\left\{ \frac{\delta_u T_u^{-1} u}{0} \mathbb{I}_{M(u)} \left( \frac{B_H(ut_* + tT_u)}{u(q_1 + c_1 t_*) + c_1 tT_u} M(u) \right)dt > 1 \right\}$$

$$= \mathbb{P}\left\{ \frac{\delta_u T_u^{-1} u}{0} \mathbb{I}_{M(u)} (Z_u^{(1)}(t))dt > 1 \right\}$$

$$= \mathbb{P}\left\{ \frac{\delta_u T_u^{-1} uK_1}{0} \mathbb{I}_{M(u)} (Z_u^{(1)}(tK_1^{-1}))dt > K_1 \right\}$$

$$= \mathbb{P}\left\{ \frac{\delta_u T_u^{-1} uK_1}{0} \mathbb{I}_{M(u)} (Z_u^{(2)}(t))dt > K_1 \right\},$$

where

$$K_1 = \frac{T^{1/H}}{2\pi t_*}, \quad M(u) = \inf_{t \in [t_*, \infty)} \frac{u(c_1 t + q_1)}{\text{Var}(B_H(ut))} = \mathbb{D}_H u^{1-H}.$$

For variance $\sigma^2_{Z_{u}^{(2)}(t)}$ and correlation $r_{Z_{u}^{(2)}(s, t)}$ of $Z_u^{(2)}$ for $t, s \in [0, \delta_u T_u^{-1} uK_1]$ it holds, that as $u \to \infty$

$$1 - \sigma^2_{Z_{u}^{(2)}(t)} = \frac{2^{1/2}H^{1-1/H} |q_1 H - (1 - H)c_1 t_*| u^{1-1/H}}{(q_1 + c_1 t_*)^2} + O(t^2 u^{2(1-1/H)})$$

$$1 - r_{Z_{u}^{(2)}(s, t)} = \mathbb{D}_H^{-1/2} u^{2H-2} |t - s|^{2H} + O(u^{2H-2} |t - s|^{2H} \delta_u).$$

Now we apply Theorem 2.1 in [9]. All conditions of the theorem are fulfilled with parameters

$$\omega(x) = x, \quad \overline{\omega}(x) = x, \quad \beta = 1, \quad g(u) = \frac{2^{1/2}H^{1-1/H} |q_1 H - (1 - H)c_1 t_*| u^{1-1/H}}{(q_1 + c_1 t_*)^2}$$

$$\eta_{\omega}(t) = B_H(t), \quad \sigma^2_{\overline{\omega}}(t) = t^{2H}, \quad \Delta(u) = 1, \quad \varphi = 1,$$
\[ n(u) = D_H u^{1-H}, \quad a_1(u) = 0, \quad a_2(u) = \delta_u T_u^{-1} u K_1, \quad \gamma = 0, \quad x_1 = 0, \quad x_2 = \infty, \quad y_1 = 0, \quad y_2 = \infty, \quad x = K_1, \]
\[ \theta(u) = u^{(1/H-2)(1-H)} H^{-1+1/H} |q_1 H - (1 - H)c_1 t_s|^{-1} t_s^{2 - \frac{1}{2m}}, \]
and thus as \( u \to \infty \)
\[ g_2(u) = \mathbb{P} \left\{ \int_0^\infty \mathbb{I}(Z_u^{(2)}(t)) dt > K_1 \right\} \sim \mathcal{B}_{2H} \left( \frac{T D_H^{1/H}}{2 \pi t_s^2} u (\frac{1}{\pi^2} - 2)(1-H) \right) \frac{t_s^2 D_H^{1+1/H}}{2 \pi |q_2 H - (1 - H)c_2 t_s|} \Psi(D_H u^{1-H}). \]
Similarly we obtain
\[ g_1(u) \sim \mathcal{B}_{2H} \left( \frac{T D_H^{1/H}}{2 \pi t_s^2} u (\frac{1}{\pi^2} - 2)(1-H) \right) \frac{t_s^2 D_H^{1+1/H}}{2 \pi |q_2 H - (1 - H)c_2 t_s|} \Psi(D_H u^{1-H}), \quad u \to \infty \]
and the claim follows if in (6) \( T > 0 \). Now let (6) holds with \( T = 0 \). Since \( \mathcal{P}_{T_u}(u) \leq \mathcal{C}_{T_u}(u) \leq \mathcal{R}_{T_u}(u) \) we obtain the claim by Theorem 2.1 in [16] and Theorem 3.1 in [18].

**Proof of Proposition 2.2.** The proof of this proposition is the same as the proof of Proposition 2.2 in [16], thus we refer to [16] for the proof.

### 4. Appendix

**Proof of (13).** To establish the claim we need to show that
\[ \mathbb{P} \left\{ \int_{[0,\infty) \setminus [t_1 - \varepsilon, t_1 + \varepsilon]} \mathbb{I}(V_1(s) > u^{1-H}) \, ds > T_u/u \right\} = o(\psi_1(T_u,u)), \quad u \to \infty. \]

Applying Borell-TIS inequality (see, e.g., [24]) we have as \( u \to \infty \)
\[ \mathbb{P} \left\{ \int_{[0,\infty) \setminus [t_1 - \varepsilon, t_1 + \varepsilon]} \mathbb{I}(V_1(s) > u^{1-H}) \, ds > T_u/u \right\} \leq \mathbb{P} \left\{ \exists t \in [0,\infty) \setminus [t_1 - \varepsilon, t_1 + \varepsilon] : V_1(t) > u^{1-H} \right\} \]
\[ \leq e^{-\frac{(u^{1-H} - M)^2}{2m^2}}, \]
where
\[ M = \mathbb{E} \left\{ \sup_{\exists t \in [0,\infty) \setminus [t_1 - \varepsilon, t_1 + \varepsilon]} V_1(t) \right\} < \infty, \quad m^2 = \max_{\exists t \in [0,\infty) \setminus [t_1 - \varepsilon, t_1 + \varepsilon]} \text{Var}(V_1(t)). \]

Since \( \text{Var}(V_1(t)) \) achieves its unique maxima at \( t_1 \) we obtain by (12) that
\[ e^{-\frac{(u^{1-H} - M)^2}{2m^2}} = o(\mathbb{P} \left\{ V_1(t_1) > u^{1-H} \right\}), \quad u \to \infty \]
and the claim follows from the asymptotics of \( \psi_1(T_u,u) \) given in Proposition 3.1.

**Proof of (14).** To prove the claim it is enough to show that as \( u \to \infty \) and then \( S \to \infty \)
\[ \mathbb{P} \left\{ \int_{[0,\infty) \setminus [ut_* - S, ut_* + S]} \mathbb{I}(B(t) - c_1 t > q_1 u, B(t) - c_2 t > q_2 u) \, dt > T \right\} = o(C_{T_u}(u)), \quad u \to \infty. \]
We have that the probability above does not exceed
\[ \mathbb{P} \left\{ \exists t \in [0,\infty) \setminus [ut_* - S, ut_* + S] : B(t) - c_1 t > q_1 u, B(t) - c_2 t > q_2 u \right\}.

From the proof of Theorem 3.1 in [18], Case (3) and the final asymptotics of $C_{T_u}(u)$ given in (9) it follows that the expression above equals $o(C_{T_u}(u))$, as $u \to \infty$ and then $S \to \infty$.

\textbf{Proof of (16).} Define

$$G(u,x) = \mathbb{P}\left\{ \int_{-S}^{0} \mathbb{I}(Z_u(s) + \eta_s s > x) \, ds + \int_{0}^{S} \mathbb{I}(B_*(s) - c_1 s > x) \, ds > T \right\}.$$ 

First we show that

$$\int_{\mathbb{R}} G(u,x) e^{\frac{nu}{t^\tau}} \frac{x^2}{2ut^\tau} \, dx = \int_{-M}^{M} G(u,x) e^{\frac{nu}{t^\tau}} \, dx + A_{M,u},$$

where $A_{M,u} \to 0$ as $u \to \infty$ and then $M \to \infty$. We have

$$|A_{M,u}| = |\int_{\mathbb{R}} G(u,x) e^{\frac{nu}{t^\tau}} \frac{x^2}{2ut^\tau} \, dx - \int_{-M}^{M} G(u,x) e^{\frac{nu}{t^\tau}} \, dx|$$

$$\leq |\int_{-M}^{M} G(u,x) (e^{\frac{nu}{t^\tau}} \frac{x^2}{2ut^\tau} - e^{\frac{nu}{t^\tau}}) \, dx| + \int_{|x|>M} G(u,x) e^{\frac{nu}{t^\tau}} \, dx =: |I_1| + I_2.$$

Since the variance of $Z_u$ (see (15)) converges to those of BM we have by Borell-TIS inequality for $x > 0$, large $u$ and some $C > 0$

$$G(u,x) \leq \mathbb{P}\{ \exists t \in [-S,0] : (Z_u(t) + \eta_s t) > x \} + \mathbb{P}\{ \exists t \in [0,S] : (B_*(t) - c_1 t) > x \}$$

$$\leq \mathbb{P}\{ \exists t \in [-S,0] : (Z_u(t) - \mathbb{E}\{Z_u(t)\}) > x \} + \mathbb{P}\{ \exists t \in [0,S] : B_*(t) > x \} \leq e^{-x^2/C}.$$ 

Let $u > M^4$. For $x \in [-M,M]$ it holds, that $1 - e^{-\frac{x^2}{2ut^\tau}} \leq \frac{x^2}{2ut^\tau} \leq \frac{1}{M}$ and hence for $u > M^4$ by (19) we have as $M \to \infty$

$$|I_1| \leq \int_{-M}^{0} e^{\frac{nu}{t^\tau}} (1 - e^{-\frac{x^2}{2ut^\tau}}) \, dx + \int_{0}^{M} e^{-x^2/C + \frac{nu}{2ut^\tau}} (1 - e^{-\frac{x^2}{2ut^\tau}}) \, dx \leq \frac{1}{M} \left( \int_{-\infty}^{0} e^{\frac{nu}{t^\tau}} + \int_{0}^{\infty} e^{-x^2/C + \frac{nu}{t^\tau}} \right) \to 0.$$

For $I_2$ we have

$$I_2 \leq \int_{-\infty}^{-M} e^{\frac{nu}{t^\tau}} \, dx + \int_{M}^{\infty} e^{-x^2/C} e^{\frac{nu}{t^\tau}} \, dx \to 0, \quad M \to \infty,$$

hence (18) holds. Next we show that

$$G(u,x) \to \mathbb{P}\left\{ \int_{-S}^{0} \mathbb{I}(B(s) + \eta_s s > x) \, ds + \int_{0}^{S} \mathbb{I}(B_*(s) - c_1 s > x) \, ds > T \right\}, \quad u \to \infty$$

that is equivalent with

$$\lim_{u \to \infty} \mathbb{P}\left\{ \int_{-S}^{S} \mathbb{I}(X_u(s) > x) \, ds > T \right\} = \mathbb{P}\left\{ \int_{-S}^{S} \mathbb{I}(B(s) + k(s) > x) \, ds > T \right\},$$

where $k(s) = \mathbb{I}(s < 0)\eta_s s - \mathbb{I}(s \geq 0)c_1 s$ and

$$X_u(t) = (Z_u(t) + \eta_t t)\mathbb{I}(t < 0) + (B_*(t) - c_1 t)\mathbb{I}(t \geq 0).$$
We have for large $u$

\[
\mathbb{E}\{ (X_u(t) - X_u(s))^2 \} = \begin{cases} 
|t - s| + |t - s|^2 & t, s \geq 0 \\
\frac{(s-t)^2}{ut^*} + |t - s| + \frac{x^2(t-s)^2}{ut^*} - \frac{2x(t-s)^2\eta_x}{ut^*} + \eta_x^2(t-s)^2 & t, s \leq 0 \\
|t - s| - \frac{x^2}{ut^*} + \frac{x^2s^2}{ut^*} - \frac{2x(s\eta_x+c1)}{ut^*} + (\eta_x s + c1)^2 & s < 0 < t
\end{cases}
\]

implying for all $u$ large enough, some $C > 0$ and $t, s \in [-S, S + T]$ that

\[
\mathbb{E}\{ (X_u(t) - X_u(s))^2 \} \leq C|t - s|.
\]

Next, by Proposition 9.2.4 in [24] the family $X_u(t), u > 0, t \in [-S, S + T]$ is tight in $\mathcal{B}(C([-S, S + T]))$ (Borel $\sigma$-algebra in the space of the continuous functions on $[-S, S + T]$ generated by the cylindric sets).

As follows from (15), $Z_u(t)$ converges to $B(t)$ in the sense of convergence finite-dimensional distributions as $u \to \infty, t \in [-S, S + T]$. Thus, by Theorems 4 and 5 in Chapter 5 in [2] the tightness and convergence of finite-dimensional distributions imply weak convergence

\[
X_u(t) \Rightarrow B(t) + k(t) =: W(t), \quad t \in [-S, S + T].
\]

By Theorem 11 (Skorohod), Chapter 5 in [2] there exists a probability space $\Omega$, where all random processes have the same distributions, while weak convergence becomes convergence almost sure. Thus, we assume that $X_u(t) \to W(t)$ a.s. as $u \to \infty$ as elements of $C[-S, S]$ space with the uniform metric. We prove that for all $x \in \mathbb{R}$

\[
\mathbb{P}\left\{ \lim_{u \to \infty} \int_{-S}^{S} \mathbb{I}(X_u(t) > x)dt = \int_{-S}^{S} \mathbb{I}(W(t) > x)dt \right\} = 1. \tag{20}
\]

Fix $x \in \mathbb{R}$. We shall show that as $u \to \infty$ with probability 1

\[
\mu_\Lambda\{t \in [-S, S]: X_u(\omega, t) > x > W(\omega, t)\} + \mu_\Lambda\{t \in [-S, S]: W(\omega, t) > x > X_u(\omega, t)\} \to 0, \quad \tag{21}
\]

where $\mu_\Lambda$ is the Lebesgue measure. Since for any fixed $\varepsilon > 0$ for large $u$ and $t \in [-S, S]$ with probability one $|W(t) - X_u(t)| < \varepsilon$ we have that

\[
\mu_\Lambda\{t \in [-S, S]: X_u(\omega, t) > x > W(\omega, t)\} + \mu_\Lambda\{t \in [-S, S]: W(\omega, t) > x > X_u(\omega, t)\} \\
\leq \mu_\Lambda\{t \in [-S, S]: W(\omega, t) \in [-\varepsilon + x, \varepsilon + x]\}.
\]

Thus, (21) holds if

\[
\mathbb{P}\left\{ \lim_{\varepsilon \to 0} \mu_\Lambda\{t \in [-S, S]: W(t) \in [-\varepsilon + x, x + \varepsilon]\} = 0 \right\} = 1. \tag{22}
\]

Consider the subset $\Omega_* \subset \Omega$ consisting of all $\omega_*$ such that

\[
\lim_{\varepsilon \to 0} \mu_\Lambda\{t \in [-S, S]: W(\omega_*, t) \in [-\varepsilon + x, x + \varepsilon]\} > 0.
\]

Then for each $\omega_*$ there exists the set $A(\omega_*) \subset [-S, S]$ such that $\mu_\Lambda\{A(\omega_*)\} > 0$ and for $t \in A(\omega_*)$ it holds, that $W(\omega_*, t) = x$. Thus,

\[
\mathbb{P}\{\Omega_*\} = \mathbb{P}\{\mu_\Lambda\{t \in [-S, S]: W(t) = x\} > 0\}.
\]
the right side of the equation above equals 0 by Lemma 4.2 in [20]. Hence we conclude that (22) holds, consequently (21) and (20) are true. Since convergence almost sure implies convergence in distribution we have by (20) that for any fixed \( x \in \mathbb{R} \)
\[
\lim_{u \to \infty} \mathbb{P} \left\{ \int_{-S}^{S} \mathbb{I}(X_u(t) > x) dt > T \right\} = \mathbb{P} \left\{ \int_{-S}^{S} \mathbb{I}(W(t) > x) dt > T \right\}.
\]
By the dominated convergence theorem we obtain
\[
\int_{-M}^{M} G(u, x)e^{\frac{u}{T}} dx \to \int_{-M}^{M} \mathbb{P} \left\{ \int_{-S}^{S} \mathbb{I}(B(s) + \eta_s s > x) ds \right\} ds + \mathbb{I}(B(s) - c_1 s > x) ds > T \right\} e^{\frac{u}{T}} dx, \quad u \to \infty.
\]
Thus, the claim follows from the line above and (18). \( \square \)

**Proof of (17).** We have by the proof of Theorem 3.1 in [18], Case (3) and the final asymptotics of \( C_{T_u}(u) \) given in (9)
\[
\mathbb{P} \left\{ \int_{[0,\infty) \setminus [ut_\star - u\delta_u, ut_\star + u\delta_u]} \mathbb{I}(B_H(t) - c_1 t > q_1 u, B_H(t) - c_2 t > q_2 u) dt > T_u \right\}
\leq \mathbb{P} \{ \exists t \in [0,\infty) \setminus [ut_\star - u\delta_u, ut_\star + u\delta_u] : B_H(t) - c_1 t > q_1 u, B_H(t) - c_2 t > q_2 u \}
= o(C_{T_u}(u)), \quad u \to \infty
\]
and hence
\[
\mathbb{P} \left\{ \int_{[ut_\star - u\delta_u, ut_\star + u\delta_u]} \mathbb{I}(B_H(t) - c_1 t > q_1 u, B_H(t) - c_2 t > q_2 u) dt > T_u \right\} \sim C_{T_u}(u), \quad u \to \infty.
\]
The last probability above is equivalent with \( g_1(u) + g_2(u) \) as \( u \to \infty \), this observation follows from the application of the double-sum method, see the proofs of Theorem 3.1, Case (3) \( H < 1/2 \) in [18] and Theorem 2.1 in [9] case i). \( \square \)

**Proof of Proposition 3.1.** If \( H = 1/2 \), then an equality takes place, see [9], Eq. [5]. Assume from now on that \( H \neq 1/2 \). First let (6) holds with \( T > 0 \). We have for \( c > 0 \) with \( \tilde{M}(u) = u^{1-H} \frac{e^{H}}{(1-H)^{1-H}} \) (recall, \( \mathbb{I}_a(b) = \mathbb{I}(b > a) \), \( a, b \in \mathbb{R} \))
\[
h_{T_u}(u) := \mathbb{P} \left\{ \int_{0}^{\infty} \mathbb{I}(B_H(t) - ct > u) dt > T_u \right\}
= \mathbb{P} \left\{ u(u^{1-H} \frac{e^{2(1-H)^2-H}}{2\pi^{1-H} H^2}) \int_{0}^{\infty} \tilde{M}(u) \frac{B_H(tu) \tilde{M}(u)}{u(1+ct)} dt > T u^{1-H} \frac{e^{2(1-H)^2-H}}{2\pi^{1-H} H^2} \right\}.
\]
Next we apply Theorem 3.1 in [9] to calculate the asymptotics of the last probability above as \( u \to \infty \). For the parameters in the notation therein we have
\[
\alpha_0 = \alpha_\infty = H, \quad \sigma(t) = t^H, \quad \frac{\sigma(t)}{\sqrt{\sigma(t)}} = t^{\frac{1}{2}}, \quad t^* = \frac{H}{c(1-H)}, \quad A = \frac{e^H}{H^{1-H}(1-H)^{1-H}}, \quad x = T \frac{e^{2(1-H)^2-H}}{2\pi^{1-H} H^2}
\]
\[ B = \frac{c^{2+H}(1-H)^{2+H}}{H^{H+1}}, \quad M(u) = u^{1-H} \frac{c^H}{(1-H)^{1-H} H^H}, \quad v(u) = u^{\frac{1}{1-H}} \frac{e^{2(1-H)^{2-\frac{1}{H}}}}{2^{\frac{1}{H}} H^2}. \]

and hence we obtain

\[ h_{T_u}(u) \sim K_H B_2H(TD)(C_H u^{1-H})^{\frac{1}{H}-1} \Psi(C_H u^{1-H}), \quad u \to \infty, \tag{23} \]

where

\[ C_H = \frac{c^H}{H^H (1-H)^{1-H}} \quad \text{and} \quad D = 2^{-\frac{1}{H}} c^2 H^{-2}(1-H)^{-2-1/H}. \]

Assume that (6) holds with \( T = 0 \). For \( \varepsilon > 0 \) for all large \( u \) we have \( h_{\varepsilon u^{1/H-2}}(u) \leq h_{T_u}(u) \leq h_0(u) \) and thus

\[ K_H B_2(H)(\varepsilon D)(C_H u^{1-H})^{\frac{1}{H}-1} \Psi(C_H u^{1-H}) \leq h_{T_u}(u) \leq K_H B_2H(0)(C_H u^{1-H})^{\frac{1}{H}-1}. \]

Since \( B_2H(\cdot) \) is a continuous function (Lemma 4.1 in [9]) letting \( \varepsilon \to 0 \) we obtain (23) for any \( T_u \) satisfying (6). Replacing in (23) \( u \) and \( c \) by \( q_1 u \) and \( c_1 \) we obtain the claim. \( \square \)

References

[1] Bai, L. (2018). Asymptotics of Parisian ruin of Brownian motion risk model over an infinite-time horizon. *Scand. Actuar. J.*, (6):514–528.

[2] Bylinskii, A. and Shiryaev, A. (2005). *Theory of Stochastic Processes (in Russian)*.

[3] Dębicki, K. and Sikora, G. (2011). Finite time asymptotics of fluid and ruin models: multiplexed fractional Brownian motions case. *Appl. Math. (Warsaw)*, 38(1):107–116.

[4] Dębicki, K., Hashorva, E., and Ji, L. (2015). Parisian ruin of self-similar Gaussian risk processes. *J. Appl. Probab.*, 52(3):688–702.

[5] Dębicki, K., Hashorva, E., and Ji, L. (2016). Parisian ruin over a finite-time horizon. *Sci. China Math.*, 59(3):557–572.

[6] Dębicki, K., Hashorva, E., Ji, L., and Rolski, T. (2018). Extremal behaviour of hitting a cone by correlated Brownian motion with drift. *Accepted for publication in Stoch. Proc. Appl.*

[7] Dębicki, K., Hashorva, E., and Liu, P. (2017). Extremes of \( \gamma \)-reflected Gaussian process with stationary increments. *ESAIM Probab. Stat.*, 21:495–535.

[8] Dębicki, K. and Liu, P. (2016). Extremes of stationary Gaussian storage models. *Extremes*, 19(2):273–302.

[9] Dębicki, K., Liu, P., and Michna, Z. (2020). Sojourn times of Gaussian processes with trend. *J. Theoret. Probab.*, 33(4):2119–2166.

[10] Debicki, K. (1991). Asymptotics of supremum of scaled Brownian motion.

[11] Dębicki, K. and Mandjes, M. (2015). *Queues and Lévy fluctuation theory*. Springer.

[12] Dieker, A. B. (2005). Extremes of Gaussian processes over an infinite horizon. *Stochastic Process. Appl.*, 115(2):207–248.

[13] Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling extremal events*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin. For insurance and finance.

[14] Jasnovidov, G. (2020). Approximation of ruin probability and ruin time in discrete Brownian risk models. *Scand. Actuar. J.*, (8):718–735.
[15] Jasnovidov, G. (2021). Simultaneous ruin probability for two-dimensional fractional Brownian motion risk process over discrete grid. *in press, Lithuanian Mathematical Journal.*

[16] Jasnovidov, G. and Shemendyuk, A. (2021). Parisian ruin for insurer and reinsurer under quota-share treaty. *arXiv:2103.03213.*

[17] Ji, L. (2020). On the cumulative Parisian ruin of multi-dimensional Brownian motion risk models. *Scand. Actuar. J.*, (9):819–842.

[18] Ji, L. and Robert, S. (2018). Ruin problem of a two-dimensional fractional Brownian motion risk process. *Stoch. Models*, 34(1):73–97.

[19] Kozik, I. A. and Piterbarg, V. I. (2018). High excursions of Gaussian nonstationary processes in discrete time. *Fundam. Prikl. Mat.*, 22(2):159–169.

[20] Kriukov, N. (2020). Parisian & cumulative Parisian ruin probability for two-dimensional Brownian risk model. *arXiv:2001.09302.*

[21] Pickands, III, J. (1969). Upcrossing probabilities for stationary Gaussian processes. *Trans. Amer. Math. Soc.*, 145:51–73.

[22] Piterbarg, V., Popivoda, G., and Stamatovic, S. (2017). Extremes of Gaussian processes with a smooth random trend. *Filomat*, 31(8):2267–2279.

[23] Piterbarg, V. I. (1996). *Asymptotic methods in the theory of Gaussian processes and fields*, volume 148 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI.

[24] Piterbarg, V. I. (2015). *Twenty Lectures About Gaussian Processes*. Atlantic Financial Press London New York.

[25] Piterbarg, V. I. and Fatalov, V. R. (1995). The Laplace method for probability measures in Banach spaces. *Uspekhi Mat. Nauk*, 50(6(306)):57–150.

Grigori Jasnovidov, Department of Actuarial Science, University of Lausanne, UNIL-Dorigny, 1015 Lausanne, Switzerland

*Email address: griga1995@yandex.ru*