New evaluation method for shared firing table of two ammunition types

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Abstract: In this paper, a statistical function is proposed for the test of Shared Firing Table of two ammunition types, regarding the problems of two common evaluation methods of Shared Firing Table in evaluating Shared Firing Table of two ammunition types, based on basic concept of Shared Firing Table and the difference in characteristic values of dispersion ellipses of two ammunition types at drop point, and the maximum allowable value between two dispersion ellipse centers and the criterion of test evaluation are derived from the preset probability value of two ammunition types group dropping into dispersion ellipse overlapping region; through application of relevant sample size calculation formula and OC (Operating Characteristics) curve (OC curve), the relationship between the test efficiency and the minimum test sample size is solved, so as to provide an effective means for the decision and planning of sample size before test and the determination of test efficiency after test; and the validity and practicability of Shared Firing Table evaluation method established herein are verified by comparing the shooting test results of 1 group x 7 shoots, and the firing table of basic howitzer type - explosive warhead bomb.

1. Introduction

The firing table is a table prepared specially for guns, cannons, rocket weapons and other specific launchers and their ammunition type, fuse and zone charge, indicating relationships among angle and range of firing and other ballistic elements. It is the basis for accurate shooting and manufacture of aiming device, and used as basic ballistic data in design of fire control computer. However, preparation of the firing table depends on the firing test, which is very expensive. In practical application, only basic type of similar artillery ammunition (refers to the firstly developed ammunition in multiple types of ammunition used for the same gun) has the firing table, which is basis for subsequent development or improvement, and requires same firing table and fire control system as the basic type. Therefore, the test of Shared Firing Table is an external ballistic application problem frequently encountered in research and development of artillery (rocket) weapon system, the decision and planning in preparation of firing table and combat training of troops.
After preparing the firing table officially for certain type of weapon ammunition, it is necessary to test whether the firing table meets requirements; when the troops find problem of accuracy in use of the shooting table, they need to test the firing table; for the artillery with firing table, applicability of original firing table should be tested after improvement of production process or partial modification of the drawings of ammunition. For multiple types of ammunition launched from the same weapon platform and the ammunition system of same artillery weapon installed on different carrier platforms, they should be tested to determine whether the firing table can be shared.

For a long time, no test method of Shared Firing Table has been reported publicly worldwide. The Z test method provided in literature [1] is adopted generally to determine whether the fire table can be shared by ammunitions. Sometime, “Firing Table Test Method” in literature [2] can be adopted for such test as well. Real purpose of “Firing Table Test Method” is to test whether big error occurs in data processing during preparation of the firing table which leads to error of the firing table prepared, rather than test of Shared Firing Table. The Z test method is too strict, requiring that the tested weapon and ammunition have same average ballistic performance as the basic type (that is, the values shown in the firing table), which is difficult to meet in practice. The long-term test and application indicate that, “Firing Table Test Method” has defects in principle and the threshold value is too loose.

This paper analyzes problems in above test methods of Shared Firing Table specifically, and presents a new test method of Shared Firing Table.

2. Existing test methods of shared firing table and problems

Generally, the Z test method is adopted to test whether certain type of ammunition has same average ballistic performance with the $\mu_0$ value in the firing table of basic ammunition type, and its evaluation criteria is given below:

$$\left| X - \mu_0 \right| \leq z_{\alpha/2} \frac{\sigma_0}{\sqrt{n}}$$

(1)

Wherein: $\mu_0$ is the fixed value shown in the firing table, and $\sigma_0$ is the fixed dispersion value shown in the firing table.

n is test sample size, and $X$ is average of test sample size.

$z_{\alpha/2}$ is the threshold value of standard normal distribution function.

If equation (1) is satisfied, it can be judged that average ballistic performance of the weapon ammunition under test is consistent with the value in firing table of basic ammunition type.

This method is too harsh to be possible in engineering practice. For example: regarding basic ammunition type, even the firing table customized for it cannot reflect its ballistic performance accurately due to errors in the ballistic model, tests for preparation of firing table, data processing, calculation of pneumatic parameters and process of identification, let along the firing table of different ammunition type, which has greater difference from that of basic ammunition type.

“Firing Table Test Method” provided in literature [2] was stemmed from the Soviet Union in the 1950s,
and the evaluation formula is as followings:

Calculate

$$\Delta R = \left| \frac{R_T - R_N}{R_T} \right| \times 100(\%)$$

(2)

$$\Delta R_{\text{max}} = U_0 \sqrt{B_T^2 + B_N^2}$$

$$B_T = \sqrt{\frac{B_R^2}{N_1 n_1} + \frac{\varepsilon_R^2}{N_1} + \eta_R^2}$$

$$B_N = \sqrt{\frac{B_R^2}{N_2 n_2} + \frac{\varepsilon_R^2}{N_2}}$$

Wherein: $R_T$ is the firing range shown in the firing table, m;

$R_N$ is the standard firing range for the firing table test, m;

$U_0$ is the test limit, 2.44 for shoot group 1, and 2.91 for shoot group 2 or 3;

$B_T$ is the probability error of the firing range shown in the firing table, m;

$B_N$ is the standardized probability error of shooting for the firing table test;

$B_R$ is total probability error of the firing range, its value is according to table 1;

$\varepsilon_R$ is the error at standardizing date, its value is according to table 1;

$\eta_R$ is system error, valued according to table 1; if the shooting support point is used to test the firing table, 0;

$N_1$ is the shoot group number shown in the firing table;

$n_1$ is the shoot number per group shown in the firing table;

$N_2$ is the shoot group number for the firing table test;

$n_2$ is the shoot number per group for the firing table test.

**Table 1.** Error value of firing table for ground cannon.

| Error  | $V_0 < 400m/s$ | $V_0 > 400m/s$ |
|--------|----------------|-----------------|
|        |                |                 |
| $B_R$ | 0.5 | 0.45 |
|-------|-----|-----|
| $\varepsilon_R$ | 0.46 | 0.36 |
| $\eta_R$ | 0.20 | 0.20 |

Note: The data shown in the table is percentage of firing range

When the test points meet

$$|\Delta R| \leq \Delta R_{\text{max}}$$

(3)

The precision of firing table is deemed as qualified, otherwise, disqualified.

The left of equation (3) is calculated by average test value and the value of firing table, the right of the inequality is calculated by experiment data and formula. They are mismatched, not involving respective dispersion, furthermore, and the test threshold value cannot vary with actual charge and firing angle (only two values: 3 groups x 7 shoots, when initial speed is more than 400m/s, test threshold value $\Delta R_{\text{max}} = 1.111\%$, when the initial speed is less than 400m/s, test threshold value $\Delta R_{\text{max}} = 1.317\%$). For the user, this threshold value is too broad and the ammunition is easy to pass the “firing table test”, so that larger firing error will occurs in application of the firing table for basic type to the non-base type (referring to other ammunition types developed after the first ammunition type), and thus reduces the operational effectiveness of weapon.

In conclusion, the classic Z test is too strict for two ammunition types to satisfy the requirements of Shared Firing Table in evaluation. “Firing Table Test Method” is defective principally and ammunition is easy to pass the test. Therefore, they fail to meet practical needs. Furthermore, the threshold values of the two methods cannot reflect the differences in tactical purpose of different ammunition types. Therefore, the ballistic performance of non-basic ammunition should be somewhat different from the value in the firing table of basic ammunition type according to the engineering practice. The threshold value and the minimum test sample size should be determined according to the requirement to equivalent dropping probability of two ammunition types within overlapping region of dispersion ellipses, and establish a new test evaluation method of Shared Firing Table.

3. Establishment of evaluation method of Shared Firing Table test

3.1. Definition of Shared Firing Table

Sharing the firing table means that, for basic and non-basic ammunition types, the basic ammunition type has firing table which can reflect its ballistic characteristics sufficiently, such firing table is used to determine the equivalence of basic ammunition type and non-basic ammunition type in hit rate under same ballistic condition, topographic condition and meteorological condition, i.e. the overlapping degree of their dispersion ellipses at drop point (figure 1).
3.2. Mathematical Description of Shared Firing Table
The drop point distribution of the projectile is regarded as a normal distribution generally, and a normal variable is set to $X \sim N(\mu_0, \sigma_0^2)$ for basic type and $Y \sim N(\mu, \sigma^2)$ for non-basic types. If the characteristic quantity of two normal variables meets:

$$|\mu - \mu_0| \leq \delta,$$

$$\left|\frac{\sigma}{\sigma_0} - 1\right| \leq \epsilon; \text{ (where } \delta \text{ and } \epsilon \text{ is minor variable)}$$

The characteristic quantity of two normal matrixes has consistency.

This concept is generalized, and the mathematical description of Shared Firing Table is given below: under same shooting condition, if following requirements are met,

**Distance:**

$$|u_x - u_{x0}| \leq \delta_x, \quad \left|\frac{\sigma_x}{\sigma_{x0}} - 1\right| \leq \epsilon_x$$

**Direction:**

$$|u_z - u_{z0}| \leq \delta_z, \quad \left|\frac{\sigma_z}{\sigma_{z0}} - 1\right| \leq \epsilon_z$$

Wherein: $\delta_x, \epsilon_x, \delta_z$ and $\epsilon_z$ are minor variables.

Then, the basic type of ammunition can share the firing table with the non-basic type of ammunition.

How to test validity of mathematical propositions under above hypothesis is the key to solve the test problem of Shared Firing Table of two ammunition types.

3.3. Equivalent fall probability of Shared Firing Table
As shown in figure 2, the dispersion ellipse area is \( E_A \) for the basic ammunition type A with fall probability \( P_0 \) in a certain area, and \( E_B \) for the non-basic ammunition type B with fall probability \( P \), \( P_0(E_A) \approx 1.0 \), \( P(E_B) \approx 1.0 \), Shared Firing Table is defined by the fall probability \( K \) of projectile in the overlapping region of two dispersion ellipses:

\[
P_0(E_A \cap E_B) \geq K
\]

\[
P(E_A \cap E_B) \geq K
\]

(4)

Wherein: \( 0.5 < K \leq 1.0 \)

![Figure 2. Intersection of two dispersion ellipses.](image)

### 3.4. Construction of statistics function for Shared Firing Table Test and evaluation criteria

#### Mean Test Statistic Function

The normal matrix is set to \( X \sim N_0(\mu_0, \sigma_0^2) \) for the basic type and \( Y \sim N(\mu, \sigma^2) \) for non-basic types. \( \mu_0, \sigma_0 \) are values of ballistic elements and their mean variances shown in the firing table, construct a statistical function

\[
z = \frac{\bar{x} - \mu_0 - (\mu - \mu_0)}{\sigma_0/\sqrt{n}}, \quad \mu - \mu_0 \text{ is replaced with } \delta \text{, then the statistical function is } z \text{ test distribution:}
\]

\[
z = \frac{\bar{x} - \mu_0 - \delta}{\sigma_0/\sqrt{n}}
\]

(5)

Means Test Evaluation Criteria. Given a significant level \( \alpha \), when equation (6) is satisfied, it can be judged that the average trajectory of non-basic type is consistent with that of basic type.

\[
|\bar{x} - \mu_0| \leq \delta + z_\alpha \frac{\sigma_0}{\sqrt{n}}
\]

(6)

#### Variance Test Statistic Function

The statistic function of the tested problem (\( \mu \) unknown) considered for \( \chi^2 \) test

\[
\chi^2 = (n - 1) \frac{\sigma^2}{\sigma_0^2}
\]

(7)

Variance Test Evaluation Criteria. Given significant level \( \alpha \), when equation (8) is satisfied, the
dispersion is consistent.

\[ \chi^2_{1-\alpha/2}(n-1) \leq \chi^2 \leq \chi^2_{\alpha/2}(n-1) \]  

(8)

\( \chi^2 \) test is performed first. After the original hypothesis is established, the Z test is performed. If both original assumptions are true, it can be determined that two ammunition types can share the firing table.

3.5. Determination of threshold value \( \delta \)

For the new statistical function equation (5) for test of Shared Firing Table, according to Z function, we can determine allowable range of \( \delta \). According to the equivalent falling probability equation (4) of the two trajectory of basic type and non-basic type, \( K=0.95 \) for grenade, the equivalent hitting probability of two-dimensional plane is decomposed into vertical direction \( P_X \), and horizontal direction \( P_Z \), i.e. \( P = P_X P_Z = 0.95 \), then, \( P_X = P_Z = \sqrt{0.95} = 0.9747 \), thus \( \delta \) value can be derived.

The probability density of one-dimensional normal distribution is shown in figure 3. According to “3\( \sigma \) law”, the probability within \( (\mu-3\sigma, \mu+3\sigma) \) is up to 99.73%. When the dispersion center of two ammunition types is offset by \( |\mu_A - \mu_B| = \sigma \), the interaction of density function of the normal distribution for two ammunition types \( f(x_A \cap x_B) \) is equal to 97.72%, which is close to \( P_X = P_Z = 0.9747 \). Therefore, \( \delta = 1.0\sigma \) should be determined.

![Figure 3. Probability density of normal distribution.](image)

For the two-dimensional plane of the projectile fall point, in vertical direction and horizontal
direction, the location of the dispersion ellipses of two ammunition types is shown in figure 4.

According to figure 4(a) and (b), when two ammunition types have consistent average trajectory, the maximum allowable deviation of the dispersion center is shown in figure 5.

![Figure 4](image1)

(a) Offset by one $\sigma_z$ right and left  (b) Offset by one $\sigma_z$ front and back

**Figure 4.** Different central position of ellipse at same mean variance.

![Figure 5](image2)

**Figure 5.** Schematic diagram for maximum allowable deviation of dispersion center at consistent average trajectory.

In conclusion, $|\mu - \mu_0| \leq \delta = \sigma$ is taken as maximum allowable deviation of dispersion center of two ammunition types for the grenade.

For the illuminating projectile, $K=0.8$, corresponding $\delta = 1.75 \sigma$, for smoke projectile, $K=0.7$, corresponding $\delta = 2.0 \sigma$.

For other cannons and projectiles, the values are determined according to requirements of tactics and technology, the $\delta$ corresponding to $k$ value is shown in table 2 and figure 6.

| $K$  | $\sqrt{K}$ | $Z_\delta (\sigma)$ | $(3\sigma - Z_\delta) (\sigma)$ | $\delta (\sigma)$ |
|------|------------|---------------------|---------------------------------|-------------------|
| 0.95 | 0.9747     | 1.955               | 1.045                           | 1.0               |
| 0.90 | 0.9487     | 1.633               | 1.367                           | 1.37              |
| 0.80 | 0.89443    | 1.250               | 1.750                           | 1.75              |
| 0.70 | 0.83666    | 0.980               | 2.020                           | 2.0               |
| 0.60 | 0.7746     | 0.754               | 2.246                           | 2.25              |
3.6. Determination of minimum test sample size

It can be seen from above analysis that, with the increase in sample size, it is more difficult to establish the discriminant (1) for Z test. Therefore, the minimum sample size, which can meet the test efficiency requirement, should be determined.

When hypothesis testing $H_0$ is $|\mu - \mu_0| \leq \delta$, the minimum sample size can be determined according to Error of the First Kind $\alpha$ (significance level of test), Error of the Second Kind $\beta$ and

$$d = \frac{|\mu - \mu_0|}{\sigma}.$$  

OC curve can be obtained from literature [3], when $H_0$ is $|\mu - \mu_0| \leq \delta$, as shown in figure 7.

Wherein, $n$ is the test sample size;

$$\sigma = \sigma_0$$ is standard deviation.

$$d = \frac{|\mu - \mu_0|}{\sigma}$$
According to specific values of $\alpha$, $\beta$ and $d$, check figure 7, observe the curve passing the point $(0, 1-\alpha)$ and the point $(d, \beta)$ to determine the minimum test sample size $n$.

According to literature [4], the following formula is used to calculate the minimum test sample size:

$$n = \frac{(z_{1-\alpha} + z_{1-\beta})^2}{d^2}$$  \hspace{1cm} (6)

Wherein: $z$ - the standard normal distribution function;

$\alpha$ - the significance level of test, i.e. Error of the First Kind;

$\beta$ - Error of the Second Kind;

$$d = \frac{|\mu - \mu_0|}{\sigma} (\sigma = \sigma_0)$$

The sample size $n$ can be calculated by equation (6) at different combination of $\alpha$, $\beta$ and $d$ values, when Shared Firing Table is tested, $\alpha=0.05$ is taken generally, corresponding sample size is shown in table 3.

| $d$ | 1-$\beta$ | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 | 0.95 | 0.99 |
|-----|----------|------|------|------|------|------|------|------|
| 0.1 |          | 271  | 361  | 471  | 619  | 857  | 1083 | 1578 |
| 0.2 |          | 68   | 91   | 118  | 155  | 215  | 271  | 395  |
| 0.4 |          | 17   | 23   | 30   | 39   | 54   | 68   | 99   |
| 0.6 |          | 8    | 11   | 14   | 18   | 24   | 31   | 44   |
| 0.8 |          | 5    | 6    | 8    | 10   | 14   | 17   | 25   |
| 1.0 |          | 3    | 4    | 5    | 7    | 9    | 11   | 16   |
| 1.2 |          | 2    | 3    | 4    | 5    | 6    | 8    | 11   |
| 1.4 |          | 2    | 2    | 3    | 4    | 5    | 6    | 9    |
| 1.6 |          | 2    | 2    | 2    | 3    | 4    | 5    | 7    |
| 1.8 |          | 1    | 2    | 2    | 2    | 3    | 4    | 5    |
| 2.0 |          | 1    | 1    | 1    | 2    | 3    | 3    | 4    |
| 3.0 |          | 1    | 1    | 1    | 1    | 2    | 2    |      |

The sample size determined in figure 7 and table 3 is same. In actual application, they can be used jointly for mutual verification.

Literature [3] provides OC Curve for $\chi^2$ test when $H_0$ is $\sigma = \sigma_0$, as shown in Figure 8.
According to specific values of \( \alpha, \beta \) and \( \lambda \), check figure 8, observe the curve passing the point \((0, 1-\alpha)\) and the point \((\lambda, \beta)\) to determine the minimum test sample size \( n \).

It is obvious that, with same \( \alpha \) and \( \beta \), the sample size necessary for Mean Ballistic Performance Test and Dispersion Consistency Test is different significantly, and the test sample size for \( \chi^2 \) test of Dispersion Consistency is much higher. Therefore, main goal is to ensure the efficiency of Mean Value Test in determination of test sample size.

### 3.7. Test Efficiency

In the test of Shared Firing Table, \( \alpha = 5\% \) is taken generally, according to \( \alpha \), the sample size \( n \),

\[
d = \frac{|x - \mu_0|}{\sigma_0}, \quad \lambda = \frac{\bar{\sigma}}{\sigma_0},
\]

Error of the Second Kind \( \beta \) is obtained from figure 7 and figure 8 respectively, the test efficiency is \( 1-\beta \).

OC curve is instructive to the decision and planning of sample size before the test, and can be used as a basis for determining evaluation effect \( (1-\beta) \) of Shared Firing Table after the test.

### 4. Instances of Application

The smoke projectile of a howitzer type is fired at 590 mil firing angle. Its firing range is shown in table 4 after standardization. The firing table of explosive warhead bomb of such howitzer is available now, the firing range and its mean variance at 590 mil firing angle are 17248m and 102.6m respectively. Try to judge whether the firing table of explosive warhead bomb can be applied to the smoke projectile at this firing angle, and whether the test meet the requirement of significance level (Error of the First Kind) \( \alpha=0.05 \) and Error of the Second Kind \( \beta\leq0.1 \).
Table 4. Standard firing range of smoke projectile of a howitzer type.

| SN of Projectile | Firing Range (m) |
|------------------|------------------|
| 1                | 17098.8          |
| 2                | 17049.4          |
| 3                | 17080.5          |
| 4                | 17079.8          |
| 5                | 16943.0          |
| 6                | 16970.3          |
| 7                | 17072.0          |

Solution:

It can be calculated from the data in table 4 that, $\bar{x}=17042.0$, $\sigma=60.6$

According to known $\mu_0 = 17248.0$, $\sigma_0 = 102.6$, significance level $\alpha=0.05$, check $\chi^2$ distribution table, it can be derived that $\chi^2_{0.05/2}(6) = 14.449$, and $\chi^2_{1.0.05/2}(6) = 1.237$

$$\chi^2_{1.0.05/2}(6) < \chi^2 = (n-1) \frac{\sigma^2}{\sigma_0^2} = 2.093 < \chi^2_{0.05/2}(6)$$

The dispersion of the smoke projectile is consistent with that specified in the firing table.

4.1. Application of classical Z test to the test

$$|\bar{x} - \mu_0| = 206.0 > z_{0.05/2} \frac{\sigma_0}{\sqrt{n}} = 1.960 \times 102.6/\sqrt{7} = 76.0$$

It can be judged that the firing range is inconsistent with that specified in the firing table, the firing table of explosive warhead bomb of the towed artillery cannot be shared.

4.2. Application of “Firing Table Test Method” shown in literature [2] to the test

$$|\Delta R| = \left| \frac{R_T - R_N}{R_T} \right| = \left| \frac{\bar{x} - \mu_0}{\mu_0} \right| = \frac{206.0}{17248} = 0.11.943\% > \Delta R_{max} = 11.11\%$$

It can be judged that the firing range is inconsistent with that specified in the firing table, the firing table of explosive warhead bomb of the towed artillery cannot be shared.

4.3. Application of the method established herein to the test

For the smoke projectile, $\delta=2\sigma$

$$|\bar{x} - \mu_0| = 206.0$$
\[2\delta + z_{0.05} \frac{\sigma_0}{\sqrt{n}} = 2\sigma_0 + 1.645 \frac{\sigma_0}{\sqrt{7}} = 2 \times 102.6 + 1.645 \times \frac{102.6}{\sqrt{7}} = 269.0\]

It is obvious that \(|x - \mu_0| \leq \delta + z_0 \frac{\sigma_0}{\sqrt{n}}\) is satisfied.

It can be judged that the firing range is consistent with that specified in the firing table, the firing table of explosive warhead bomb of the towed artillery can be shared.

According to \(\alpha=0.05\), \(d = \frac{|x - \mu_0|}{\sigma_0} = 2.008\), \(n=7\), it can be obtained from figure 7 that \(\beta=0.0\), and test efficiency \(1-\beta=1.0\). Therefore, above test meets the technical requirement of test.

It can be seen from the instances that, for the smoke projectile, threshold value of the method established herein is higher, by which it can be judged that the firing range is consistent with that specified in the firing table. However, when the Z test method or “Firing Table Test Method” shown in literature [2] is applied, the conclusion is that the firing range is inconsistent with that specified in the firing table. Therefore, this method can relax the threshold value of evaluation criteria according to tactical hitting requirements of different bomb types, so it is more universal in practical engineering application.

The method of Shared Firing Table for direction is identical to the criterion of Shared Firing Table for the firing range.

After two ammunition types are tested in the method established herein at each zone charge and each firing angle and all tests support the sharing of the firing table, it can be judged that two ammunition types can share the firing table.

5. Conclusion
A set of systematic and complete test method of Shared Firing Table is established herein, which can be used to test whether two ammunition types with different parents can share firing table, overcome the limitation of Z test in evaluation of Shared Firing Table and has broader engineering application characteristics. OC Curve is introduced into the evaluation method to clearly reflect relationship among the test efficiency \(\alpha\), \(\beta\) and the allowable difference \(\delta\) of average ballistic performance and the sample size \(n\), and thus solve the problem that the fixed sample size of 3 groups x 5 shoots, 3 groups x 7 shoots, or 3 groups x 9 shoots in traditional test cannot reflect the test efficiency. The threshold value in the evaluation criteria can reflect the difference of different bomb types in tactical use. Therefore, the test evaluation method of Shared Firing Table established herein is more scientific and practical.

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