On a family of Cuntz-Krieger algebras connected to the rational elliptic curves in projective plane

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Abstract

Let $O_B$ be a family of Cuntz-Krieger algebras given by square matrix $B = (b - 1, 1, b - 2, 1)$, where $b$ is an integer greater or equal to two. It is proved, that there exists a dense self-adjoint subalgebra of $O_B$, which is isomorphic (modulo an ideal) to a twisted homogeneous coordinate ring of the rational elliptic curve $E(\mathbb{Q}) = \{(x, y, z) \in \mathbb{P}^2(\mathbb{C}) | y^2 z = x(x - z)(x - \frac{b - 2}{b + 2}z)\}$.

Key words: rational elliptic curves, Cuntz-Krieger algebras

MSC: 14H52 (elliptic curves); 16R10 (associative algebras); 46L85 (noncommutative topology)

A. Twisted coordinate rings. It was realized since 1950’s, that algebraic geometry can be based on a non-commutative algebra, see e.g. [5] for an introduction. On a very basic level, the following example (from functional analysis) illustrates the idea. If $X$ is a Hausdorff topological space and $C(X)$ the commutative algebra of continuous functions from $X$ to $\mathbb{C}$, then topology of $X$ is determined by algebra $C(X)$ by the Gelfand Duality; in terms of K-theory this can be written as $K_0^\text{top}(X) \cong K_0^\text{alg}(C(X))$. Taking the two-by-two

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matrices with entries in \(C(X)\) one gets an algebra \(C(X) \otimes M_2(\mathbb{C})\); in view of stability of the algebraic K-theory under the tensor products (e.g. \(\mathbb{I}\), §5), it holds \(K^\text{top}_0(X) \cong K^\text{alg}_0(C(X)) \cong K^\text{alg}_0(C(X) \otimes M_2(\mathbb{C}))\). In other words, the topology of \(X\) is defined by algebra \(C(X) \otimes M_2(\mathbb{C})\), which is no longer a commutative algebra. Roughly speaking, in algebraic geometry one replaces \(X\) by a projective variety, \(C(X)\) by its coordinate ring, \(C(X) \otimes M_2(\mathbb{C})\) by a twisted coordinate ring and \(K^\text{top}(X)\) by the quasi-coherent sheaves on \(X\) (although such sheaves are more general than \(K^\text{top}(X)\)) \([2]\), p. 173. We shall give a brief account of this construction for elliptic curves; the details can be found in \([3]\) and \([4]\), pp. 265-268 or \([5]\), p.197.

Let \(k\) be a field of \(\text{char} \ (k) \neq 2\). A four-dimensional Sklyanin algebra \(S_{\alpha,\beta,\gamma}(k)\) is a free \(k\)-algebra on four generators \(x_i\) and six quadratic relations:

\[
\begin{align*}
x_1x_2 - x_2x_1 & = \alpha(x_3x_4 + x_4x_3), \\
x_1x_2 + x_2x_1 & = x_3x_4 - x_4x_3, \\
x_1x_3 - x_3x_1 & = \beta(x_4x_2 + x_2x_4), \\
x_1x_3 + x_3x_1 & = x_4x_2 - x_2x_4, \\
x_1x_4 - x_4x_1 & = \gamma(x_2x_3 + x_3x_2), \\
x_1x_4 + x_4x_1 & = x_2x_3 - x_3x_2,
\end{align*}
\]

(1)

where \(\alpha, \beta, \gamma \in k\) and \(\alpha + \beta + \gamma + \alpha\beta\gamma = 0\). If \(\alpha \notin \{0; \pm 1\}\) then algebra \(S_{\alpha,\beta,\gamma}(k)\) defines a non-singular elliptic curve \(E \subset \mathbb{P}^3(k)\) given by the intersection of two quadrics \(u^2 + v^2 + w^2 + z^2 = \frac{1-\alpha}{1+\alpha}u^2 + \frac{1+\alpha}{1-\alpha}w^2 + z^2 = 0\) together with an automorphism \(\sigma : E \rightarrow E\). We shall use the following isomorphism (see \([3]\) and \([4]\)):

\[
\text{QGr} \ (S_{\alpha,\beta,\gamma}(k) / \Omega) \cong \text{Qcoh} \ (E),
\]

(2)

where \(\text{QGr}\) is a category of the quotient graded modules over the algebra \(S_{\alpha,\beta,\gamma}(k)\) modulo torsion, \(\text{Qcoh}\) a category of the quasi-coherent sheaves on \(E\) and \(\Omega \subset S_{\alpha,\beta,\gamma}(k)\) a two-sided ideal generated by the central elements \(\Omega_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2\) and \(\Omega_2 = x_2^2 + \frac{1+\beta}{1-\gamma}x_3^2 + \frac{1-\beta}{1+\gamma}x_4^2\) (\([3]\), Theorem 2). Therefore, the factor of Sklyanin algebra by the ideal \(\Omega\) is a homogeneous coordinate ring of elliptic curve \(E\), which is called twisted. Clearly, such a ring is no longer a commutative ring.

**B. Cuntz-Krieger algebras** (\([2]\)). Let \(A\) be a two-by-two matrix with the non-negative integer entries \(a_{ij}\), such that every row and every column of \(A\) is non-zero. The two-dimensional Cuntz-Krieger algebra \(O_A\) is a \(C^*\)-algebra.
of bounded linear operators on a Hilbert space $\mathcal{H}$, which is generated by the partial isometries $s_1$ and $s_2$, and relations:

\[
\begin{align*}
\ell \quad & s_1^* s_1 = a_{11} s_1^* s_1 + a_{12} s_2^* s_2, \\
& s_2^* s_2 = a_{21} s_1^* s_1 + a_{22} s_2^* s_2, \\
& Id = s_1^* s_1 + s_2^* s_2, 
\end{align*}
\]

where $Id$ is the identity operator on $\mathcal{H}$. Sometimes the algebra $O_A$ will be written explicitly as $O_{a_{11}, a_{12}, a_{21}, a_{22}}$. If one defines $x_1 = s_1$, $x_2 = s_1^*$, $x_3 = s_2$, and $x_4 = s_2^*$, then it is easy to see, that $O_A$ contains a dense sub-algebra $O_A^0$, which is a free $C$-algebra on four generators $x_i$ and three quadratic relations:

\[
\begin{align*}
x_2 x_1 &= a_{11} x_1 x_2 + a_{12} x_2 x_4, \\
x_4 x_3 &= a_{21} x_1 x_2 + a_{22} x_3 x_4, \\
1 &= x_1 x_2 + x_3 x_4, 
\end{align*}
\]

and an involution acting by the formula:

\[
x_1^* = x_2, \quad x_3^* = x_4. 
\]

Notice, that equations (4) are invariant of this involution.

**C. Main results.** It is known, that ideal $1$ is stable under involution $5$, if and only if, $\alpha = \alpha, \beta = 1$ and $\gamma = -1$ (lemma 1); the involution turns the Sklyanin algebra $S_{a,1,-1}(\mathbb{C})$ in a $*$-algebra (i.e a self-adjoint algebra). Denote by $I_0$ a non-homogeneous two-sided ideal of $S_{a,1,-1}(\mathbb{C})$ generated by relation $x_1 x_2 + x_3 x_4 = 1$. Let $J_0$ be a two-sided ideal of $O_A$ generated by four relations $x_4 x_2 - x_1 x_3 = x_3 x_1 + x_2 x_4 = x_4 x_1 - x_2 x_3 = x_3 x_2 + x_1 x_4 = 0$. The following theorem and corollary describe a family of Cuntz-Krieger algebras, which can be interpreted as twisted coordinate rings of rational elliptic curves in the projective plane.

**Theorem 1** For every integer $b \geq 2$, there exists a $*$-isomorphism:

\[
S_{b-2,\frac{b}{b+2},1,-1}(\mathbb{C}) / I_0 \cong O_B^0 / J_0, \quad \text{where } B = \begin{pmatrix} b-1 & 1 \\ b-2 & 1 \end{pmatrix}. \tag{6}
\]

**Corollary 1** For every integer $b \geq 2$, there exists a dense self-adjoint sub-algebra of the Cuntz-Krieger algebra $O_B$, which is isomorphic (modulo the ideal $I_0$) to a twisted homogeneous coordinate ring of the rational elliptic curve $E(Q) = \{(x,y,z) \in \mathbb{P}^2(\mathbb{C}) \mid y^2 z = x(x - z)(x - \frac{b-2}{b+2}z)\}$ in the complex projective plane; the curve is non-singular unless $b = 2$. 

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D. Proof of theorem 1. We shall split the proof in a series of lemmas, which have an independent interest.

Lemma 1 The ideal of free algebra $\mathbb{C}\langle x_1, x_2, x_3, x_4 \rangle$ generated by equations (1) is stable under involution (5), if and only if, $\bar{\alpha} = \alpha, \beta = 1$ and $\gamma = -1$.

Proof. (i) Let us consider the first two equations (1); this pair is invariant of involution (5). Indeed, by the rules of composition for an involution

\[
\begin{align*}
(x_1 x_2)^* &= x_2^* x_1^* = x_1 x_2, \\
(x_2 x_1)^* &= x_1^* x_2^* = x_2 x_1, \\
(x_3 x_4)^* &= x_4^* x_3^* = x_3 x_4, \\
(x_4 x_3)^* &= x_3^* x_4^* = x_4 x_3.
\end{align*}
\]

Since $\alpha^* = \bar{\alpha} = \alpha$, the first two equation (1) remain invariant of involution (5).

(ii) Let us consider the middle pair of equations (1); by the rules of composition for an involution

\[
\begin{align*}
(x_1 x_3)^* &= x_3^* x_1^* = x_4 x_2, \\
(x_3 x_1)^* &= x_1^* x_3^* = x_2 x_4, \\
(x_2 x_4)^* &= x_4^* x_2^* = x_3 x_1, \\
(x_4 x_2)^* &= x_2^* x_4^* = x_1 x_3.
\end{align*}
\]

One can apply the involution to the first equation $x_1 x_3 - x_3 x_1 = \beta(x_4 x_2 + x_2 x_4)$; then one gets $x_4 x_2 - x_2 x_4 = \bar{\beta}(x_1 x_3 + x_3 x_1)$. But the second equation says that $x_1 x_3 + x_3 x_1 = x_4 x_2 - x_2 x_4$; the last two equations are compatible if and only if $\bar{\beta} = 1$. Thus, $\beta = 1$.

The second equation in involution writes as $x_4 x_2 + x_2 x_4 = x_1 x_3 - x_3 x_1$; the last equation coincides with the first equation for $\beta = 1$.

Therefore, $\beta = 1$ is necessary and sufficient for invariance of the middle pair of equations (1) with respect to involution (5).

(iii) Let us consider the last pair of equations (1); by the rules of composition for an involution

\[
\begin{align*}
(x_1 x_4)^* &= x_4^* x_1^* = x_3 x_2, \\
(x_4 x_1)^* &= x_1^* x_4^* = x_2 x_3, \\
(x_2 x_3)^* &= x_3^* x_2^* = x_4 x_1, \\
(x_3 x_2)^* &= x_2^* x_3^* = x_1 x_4.
\end{align*}
\]
One can apply the involution to the first equation \( x_1x_4 - x_4x_1 = \gamma(x_2x_3 + x_3x_2) \); then one gets \( x_3x_2 - x_2x_3 = \bar{\gamma}(x_4x_1 + x_1x_4) \). But the second equation says that \( x_1x_4 + x_4x_1 = x_2x_3 - x_3x_2 \); the last two equations are compatible if and only if \( \bar{\gamma} = -1 \). Thus, \( \gamma = -1 \).

The second equation in involution writes as \( x_3x_2 + x_2x_3 = x_4x_1 - x_1x_4 \); the last equation coincides with the first equation for \( \gamma = -1 \).

Therefore, \( \gamma = -1 \) is necessary and sufficient for invariance of the last pair of equations \((11)\) with respect to involution \((5)\).

(iv) It remains to verify that condition \( \alpha + \beta + \gamma + \alpha\beta\gamma = 0 \) is satisfied by \( \beta = 1 \) and \( \gamma = -1 \) for any \( \alpha \in k \). Lemma \((11)\) follows. □

**Lemma 2** If \( \alpha \neq 1 \), then the system of equations

\[
\begin{align*}
    x_1x_2 - x_2x_1 & = \alpha(x_3x_4 + x_4x_3), \\
    x_1x_2 + x_2x_1 & = x_3x_4 - x_4x_3
\end{align*}
\]

is equivalent to the system of equations

\[
\begin{align*}
    x_2x_1 & = (b - 1)x_1x_2 + x_3x_4, \\
    x_4x_3 & = (b - 2)x_1x_2 + x_3x_4,
\end{align*}
\]

where \( \alpha = \frac{b - 2}{b + 2} \).

**Proof.** (i) Let us isolate \( x_2x_1 \) and \( x_4x_3 \) in equations \((10)\); for that, we shall write \((10)\) in the form

\[
\begin{align*}
    x_2x_1 + \alpha x_4x_3 & = x_1x_2 - \alpha x_3x_4, \\
    x_2x_1 + x_4x_3 & = -x_1x_2 + x_3x_4.
\end{align*}
\]

Consider \((12)\) as a linear system of equations relatively \( x_2x_1 \) and \( x_4x_3 \); since \( \alpha \neq 1 \), it has a unique solution

\[
\begin{align*}
    x_2x_1 & = \frac{1}{1 - \alpha} \begin{vmatrix} x_1x_2 - \alpha x_3x_4 & \alpha \\ -x_1x_2 + x_3x_4 & 1 \end{vmatrix} = \frac{1 + \alpha}{1 - \alpha} x_1x_2 - \frac{2\alpha}{1 - \alpha} x_3x_4, \\
    x_4x_3 & = \frac{1}{1 - \alpha} \begin{vmatrix} 1 & x_1x_2 - \alpha x_3x_4 \\ 1 & -x_1x_2 + x_3x_4 \end{vmatrix} = \frac{-2}{1 - \alpha} x_1x_2 + \frac{1 + \alpha}{1 - \alpha} x_3x_4.
\end{align*}
\]

(ii) Let us substitute \( \alpha = \frac{b - 2}{b + 2} \) in \((13)\); then one arrives at the following system of equations (given in the matrix form)

\[
\begin{pmatrix} x_2x_1 \\ x_4x_3 \end{pmatrix} = \begin{pmatrix} \frac{b}{2} & 1 - \frac{b}{2} \\ -1 - \frac{b}{2} & \frac{b}{2} \end{pmatrix} \begin{pmatrix} x_1x_2 \\ x_3x_4 \end{pmatrix}.
\]

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It is verified directly, that
\[
\begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{b}{2} & 1 - \frac{b}{2} \\
-1 - \frac{b}{2} & \frac{b}{2}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-2 & 1
\end{pmatrix}
= \begin{pmatrix}
b - 1 & 1 \\
b - 2 & 1
\end{pmatrix},
\]
(i.e. matrix (14) is similar to the matrix of equation (11)). Lemma 2 is proved. □

**Lemma 3** If \( b \geq 2 \) is an integer, then there exists a ∗-isomorphism
\[
\mathfrak{F}_{b-\frac{2}{b+2}}, 1, -1(\mathbb{C}) / \mathcal{I}_0 \cong \mathcal{O}_{b-1, 1, b-2, 1}^0 / \mathcal{J}_0;
\]
the isomorphism is given by identification of generators \( x_i \) of the respective algebras.

**Proof.** Since \( b \) is integer number, one gets \( \alpha = \frac{b-2}{b+2} \) is a rational number. In particular, \( \alpha \) is real, i.e. \( \bar{\alpha} = \alpha \); thus, by lemma 1, algebra \( \mathfrak{F}_{b-\frac{2}{b+2}}, 1, -1(\mathbb{C}) \) is a self-adjoint Sklyanin algebra.

Recall, that ideal \( \mathcal{I}_0 \) is generated by relation
\[
x_1x_2 + x_3x_4 = 1,
\]
while ideal \( \mathcal{J}_0 \) is generated by the system of relations
\[
\begin{cases}
x_1x_3 = x_4x_2, \\
x_3x_1 = -x_2x_4, \\
x_1x_4 = -x_3x_2, \\
x_4x_1 = x_2x_3.
\end{cases}
\]

Notice, that ideals \( \mathcal{I}_0 \) and \( \mathcal{J}_0 \) are stable under involution (5).

By lemma 2 the first pair of equations in the system (1) with \( \alpha = \frac{b-2}{b+2} \) coincides with the first pair of equations in the system (14) with \( a_{11} = b - 1, a_{12} = 1, a_{21} = b - 2 \) and \( a_{22} = 1 \). Thus, if one complements system (1) with equation (17) and system (14) with the system of equations (18), then one obtains the required ∗-isomorphism (16). Lemma 3 is proved. □

**Theorem 1** follows from lemma 3. □

**Remark 1** Ideals \( \mathcal{I}_0 \) and \( \mathcal{J}_0 \) do not depend on “modulus” \( b \) of the Sklyanin algebra \( \mathfrak{F}_{b-\frac{2}{b+2}}, 1, -1(\mathbb{C}) \); therefore, algebra \( \mathcal{O}_{b-1, 1, b-2, 1}^0 \) can be viewed as a twisted coordinate ring of elliptic curve \( \mathcal{E} \subset \mathbb{P}^3(\mathbb{C}) \).
E. Proof of corollary

We shall split the proof in a series of lemmas, starting with the following elementary

**Lemma 4** The restriction, \( \Omega_0 \), of the central ideal \( \Omega \) (see item A) to the case \( \beta = -\gamma = 1 \) is invariant under involution (5).

*Proof.* Indeed, \( \Omega_0 \) is generated by the equations

\[
\begin{align*}
x_1^2 + x_4^2 &= 0, \\
x_2^2 + x_3^2 &= 0.
\end{align*}
\]

(19)

It is easy to see, that involution (5) interchanges the two equations leaving the ideal \( \Omega \) invariant. □

**Lemma 5** If \( \alpha \) is a real number different from 0 and 1, then the algebra \( S_{\alpha, 1, -1}(C) / \Omega_0 \) is the coordinate ring of a non-singular elliptic curve \( E(C) = \{(x, y, z) \in \mathbb{P}^2(C) \mid y^2z = x(x-z)(x-\alpha z)\} \).

*Proof.* Recall, that the Sklyanin algebra \( S_{\alpha, 1, -1}(C) \) defines an elliptic curve \( E \subset \mathbb{P}^3(C) \) given by the intersection of two quadrics ([4], p.267):

\[
\begin{align*}
(1-\alpha)v^2 + (1+\alpha)w^2 + 2z^2 &= 0, \\
u^2 + v^2 + w^2 + z^2 &= 0.
\end{align*}
\]

(20)

We shall pass in (20) from variables \((u, v, w, z)\) to the new variables \((X, Y, Z, T)\) given by the formulas

\[
\begin{align*}
u^2 &= T^2, \\
v^2 &= \frac{1}{2}Y^2 - \frac{1}{2}Z^2 - T^2, \\
w^2 &= X^2 + \frac{1}{2}Y^2 - \frac{1}{2}Z^2 - T^2, \\
z^2 &= Z^2.
\end{align*}
\]

(21)

Then equations (20) take the form

\[
\begin{align*}
\alpha X^2 + Z^2 - T^2 &= 0, \\
X^2 + Y^2 - T^2 &= 0.
\end{align*}
\]

(22)

Let us consider another (polynomial) transformation \((x, y) \mapsto (X, Y, Z, T)\) given by the formulas

\[
\begin{align*}
X &= -2y, \\
Y &= x^2 - 1 + \alpha, \\
Z &= x^2 + 2(1-\alpha)x + 1 - \alpha, \\
T &= x^2 + 2x + 1 - \alpha.
\end{align*}
\]

(23)
Then both of the equations (22) give us the equation 

\[ y^2 = x(x+1)(x+1-\alpha), \]

which after a shift \( x' = x + 1 \) takes the canonical form

\[ y^2 = x(x-1)(x-\alpha). \tag{24} \]

Using projective transformation \( x = \frac{x'}{z'}, y = \frac{y'}{z'} \) in (24), one gets the homogeneous equation of elliptic curve \( E \):

\[ y^2z = x(x-z)(x-\alpha z). \tag{25} \]

Lemma 5 follows. □

**Lemma 6** If \( b \geq 2 \) is an integer, then there exists a dense self-adjoint sub-algebra of the Cuntz-Krieger algebra \( O_{b-1,1,b-2,1} \), which is isomorphic (modulo ideal \( I_0 \)) to a twisted coordinate ring of the rational elliptic curve \( E(\mathbb{Q}) = \{(x,y,z) \in \mathbb{P}^2(\mathbb{C}) \mid y^2z = x(x-z)(x-\frac{b-2}{b+2}z)\}; the curve is non-singular unless \( b = 2 \).

**Proof.** Let us assume \( \alpha = \frac{b-2}{b+2} \) in lemma 5, then dividing both sides of (16) by the self-adjoint ideal \( \Omega_0 \), one obtains

\[ \Omega_0 \setminus \mathcal{S}_{\frac{b-2}{b+2}, 1, -1}(\mathbb{C}) / \mathcal{I}_0 \cong \Omega_0 \setminus O^0_{b-1,1,b-2,1} / J_0. \tag{26} \]

The RHS of (26) is a sub-algebra of the Cuntz-Krieger algebra \( O_{b-1,1,b-2,1} \); such an algebra is self-adjoint, since the ideals \( J_0 \) and \( \Omega_0 \) are invariant of the involution \( \mathcal{I}_0 \).

On the other hand, if \( b \neq 2 \), the algebra \( \Omega_0 \setminus O^0_{b-1,1,b-2,1} / J_0 \) is isomorphic to the factor (by the ideal \( I_0 \)) of the coordinate ring \( \mathcal{S}_{\frac{b-2}{b+2}, 1, -1}(\mathbb{C}) / \Omega_0 \) of the non-singular curve \( E(\mathbb{Q}) = \{(x,y) \in \mathbb{P}^2(\mathbb{C}) \mid y^2z = x(x-z)(x-\frac{b-2}{b+2}z)\} \). It is easy to see, that the curve \( E(\mathbb{Q}) \) is singular if and only if \( b = 2 \). Lemma 6 is proved. □

Corollary 1 follows from lemma 6. □

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