Group of L-homeomorphisms and permutation groups

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Abstract. A subgroup $G$ of the symmetric group $S(P)$ of all permutations of a set $P$ is called $L_f$-representable on $P$ if there is an $L$-topology $\delta$ on $P$ with the group of $L$-homeomorphisms of $(P, \delta) = G$. In this paper we study the $L_f$-representability of some subgroups of the symmetric group.

1. Introduction
The problem of representing a permutation group as the the group of homeomorphisms of a topological space was studied in [1, 2, 3, 4, 5]. Johnson T P [6, 7, 8] and Ramachandran P T [9, 10] studied analogous problem in $L$-topological spaces. This paper is a continuation of this problem. The permutation group generated by cycle and some normal subgroups of the symmetric group $S(P)$ on $P$ can be expressed as the $L$-homeomorphism group of an $L$-topological space when $|L| \geq |P|$ [6]. Ramachandran P T [9, 10] studied the representability of the cyclic group generated by a cycle and the group generated by arbitrary product of infinite cycles when $L \neq \{0,1\}$. A subgroup $G$ of the group $S(P)$ of all permutations of a set $P$ is called $L_f$-representable on $P$ if there is an $L$-topology $\delta$ on $P$ with the $L$-homeomorphism group of $(P, \delta) = G$ [11]. In [11], we studied some properties of $L_f$-representable permutation groups and determined $L_f$-representability of dihedral groups. Here we study the $L_f$-representability of semi-regular subgroups of $S(P)$ and alternating group.

2. Preliminaries
Here we give some basic definitions in permutation groups and $L$-topology, which we will be used in this paper. For more details see [12, 13, 14]. Through out this paper $P$ stands for a non empty set and $L$ for an F-lattice.

A bijection of a set $P$ onto $P$ is called a permutation of $P$. The set of all permutations of $P$ forms a group under permutation multiplication. This group is called the symmetric group[12]. We denote the symmetric group by $S(P)$ and $S_n$ to denote the special group...
$S(P)$ when $P = \{1, 2, \ldots, n\}$. A subgroup of $S(P)$ is said to be a permutation group [12]. If $P$ is a finite set with cardinality $n$, then the alternating group $A_n$ is the set of all even permutations in $S_n$. Note that for $n \neq 4$, $A_n$ is the only non trivial proper normal subgroup of $S_n$. If $n = 4$, $S_n$ has another non-trivial normal subgroup \{1, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}.

If $P$ is an infinite set and $g \in S(P)$. Then support of $g$ is defined by $supp(g) = \{p \in P : g(p) \neq p\}$. A permutation having finite support is called a finitary permutation [13]. Let $FS(P) = \{g \in S(P) : g$ is a finitary permutation$\}$ Let $g \in FS(P)$. Then $g$ is a product of finite number of transpositions. A finitary permutation $g$ is said to be even if it can be expressed as a product of even number of transpositions and odd if it can be written as a product of odd number of transpositions. The set $\{g \in FS(P) : g$ is even $\}$ is the alternating group $A(P)[13]$.

Let $P$ be any set. Then a subgroup $H$ of $S(P)$ is semi-regular [12] if any non identity permutation in $H$ has no fixed points. Also $H$ is regular if $H$ is transitive and semi-regular.

Let $L$ be a complete lattice, then an $L$-subset $f$ of $P$ is a function from $P$ to $L$. The set of all $L$-subsets of $P$ is denoted by $L^P$. A completely distributive lattice $L$ with an order reversing involution $h: L \rightarrow L$ is called an $F$-lattice. Let $P$ and $Q$ be two sets and $g : P \rightarrow Q$ be a function. Then for any $L$-set $f$ in $P$, $g(f)$ is an $L$-set in $Q$ defined by $g(f)(q) = \bigvee\{f(p) : p \in P, g(p) = q\}$ if $g^{-1}(q) \neq \emptyset$ and $g(f)(q) = 0$ if $g^{-1}(q) = \emptyset$. For an $L$-set $f'$ in $Q$, we define $g^{-1}(f')(p) = f'(g(p))$ for all $p \in P$.

Let $\delta \subseteq L^P$ and if $\delta$ satisfies (i) $\emptyset, 1 \in \delta$, (ii) $f_1 \wedge f_2 \in \delta$ for all $f_1, f_2 \in \delta$ and (iii) $\forall \mathcal{A} \in \delta$ for all $\mathcal{A} \subseteq \delta$ then $\delta$ is said to be an $L$-topology on $P$ and $(P, \delta)$ is an $L$-topological space or $L$-ts in short. Every element in $\delta$ is called an $L$-open subset of $P$. Let $(P, \delta)$ be an $L$-topological space, $\mathcal{B} \subseteq \delta$. $\mathcal{B}$ is called a base of $\delta$, if $\delta = \{\forall \mathcal{A} : \mathcal{A} \subseteq \mathcal{B}\}$ and $\forall \mathcal{A} \subseteq \delta$ is called a subbase of $\delta$, if the family $\{\wedge \mathcal{B} : \mathcal{B} \in \mathcal{S}^{<\omega} \setminus \{\emptyset\}\}$ is a base, where $\mathcal{S}^{<\omega}$ denote the family of all finite subsets of $\mathcal{S}$.

Let $(P, \delta)$ and $(Q, \delta')$ be any two $L$-ts and $g$ be a function from $(P, \delta)$ to $(Q, \delta')$. Then (i) $g$ is said to be an $L$-continuous map from $P$ to $Q$, if $g^{-1}(f') \in \delta$ for every $f' \in \delta'$ where $g^{-1}(f')$ means $f' \circ h$ and (ii) $g$ is said to be $L$-open if the image of every $L$-open subset of $P$ as an $L$-open one in $Q$. Now $g$ is said to be an $L$-homeomorphism if it is (i) bijective (ii) $L$-continuous and (iii) $L$-open. Thus $g \in S(P)$ is an $L$-homeomorphism of $(P, \delta)$ on to itself if and only if $f \in \delta \Leftrightarrow f \circ g \in \delta$. The set of all $L$-homeomorphisms of an $L$-ts $(P, \delta)$ onto itself is a group under function composition, which is a subgroup of $S(P)$. It is called the group of $L$-homeomorphisms or $L$-homeomorphism group of $(P, \delta)$ and is denoted by $GLH(P, \delta)$.

3. $L_f$-representability of semi-regular permutation groups
In this section we prove that semi-regular permutation groups on $P$ are $L_f$-representable on $P$ provided $|L| \geq |P|$.

**Theorem 3.1.** Let $G$ be a semi-regular permutation group on $P$. If $|L| \geq |P|$, then there exists an $L$-topology $\delta$ on $P$ with $GLH(P, \delta) = G$.

**Proof.** Let $f : P \rightarrow L$ be an $L$-set such that $f$ is one-one and $f$ take the values 0 and 1.
We can define such a one-one function since \(|L| \geq |P|\). By the well-ordering Theorem, well-order the group \(G\) with order relation \(<\). Define \(S = \{f_i = f \circ g_i : g_i \in G\}\). Let \(\delta\) be the \(L\)-topology having the subbase \(S\). Then any element of \(\delta\) is of the form \(\bigvee_{i \in I}(\bigwedge_{j \in J_i} f_j)\) where \(I\) is an index set and \(J_i\) is finite for each \(i \in I\).

**Claim:** \(GLH(P, \delta) = G\)

Let \(g \in G\) and \(f_i \in S\). Then \(g^{-1}(f_i) = f_i \circ g = (f \circ g_i) \circ g = f \circ (g_i \circ g)\). Since \(g, g_i \in G\), \(g_i \circ g\) is in \(G\). Let \(g_i \circ g = g_k\) for some \(k\). Then it follows that \(g^{-1}(f_i) = f \circ g_k = f_k\). Hence \(g^{-1}(f_i)\) is in \(\delta\). By similar arguments we can prove that \(g(f_i) \in \delta\). So \(g\) is an \(L\)-homeomorphism on \(P\). Thus

\[
G \subseteq GLH(P, \delta) \tag{1}
\]

Conversely assume that \(g \in GLH(P, \delta)\). Then \(g^{-1}(f) \in \delta\). Now \(g^{-1}(f) = f \circ g = \bigvee_{i \in I}(\bigwedge_{j \in J_i} f_j)\). Note that \(f\) takes the value 0 and hence \(f \circ g(p) = 0\) for some \(p \in P\). It follows that for all \(i \in I\), we can find a \(j_i \in J_i\) with \(f_{j_i}(p) = 0\). Now \(f\) takes the value 1 implies that there exists some \(q \in P\) such that \(f \circ g(q) = 1\). This implies that there exists some \(i_0 \in I\) such that \(\bigwedge_{j \in J_{i_0}} f_j(q) = 1\). It follows that \(|J_{i_0}| = 1\) and hence \(j_i = k\) for all \(i \in I\).

\[
f \circ g = \bigvee_{i \in I, j \in J_i}(f_j) = \bigvee_{i \in I \setminus \{i_0\}, j \in J_i}(f_j) \bigvee_{j \in J_{i_0}} f_j = \bigvee_{i \in I \setminus \{i_0\}, j \in J_i} f_k = f_k = f \circ g_k.
\]

Thus we get \(g = g_k\) and hence \(g \in G\). So

\[
GLH(P, \delta) \subseteq G \tag{2}
\]

From Equations 1 and 2, it follows that \(G\) is \(L_f\)-representable on \(P\).

**Corollary 3.2.** *Every regular permutation group on \(P\) is \(L_f\)-representable when \(|L| \geq |P|\).*

**Proof.** We have a regular permutation group is semi-regular. Proof follows from Theorem 3.1. \(\square\)

Assume that \(P\) is a finite set with \(|P| \leq |L|\). Then any subgroup of \(S(P)\), which is transitive and abelian is regular and hence \(L_f\)-representable on \(P\).
4. \textit{L}$_f$-representability of alternating groups

In [6], Johnson T P proved that the alternating group can be represented as $GLH(P, \delta)$ for some L-topology, when $|P| \leq L$. So if $|P| \leq L$, then the alternating group $A(P)$ is $L_f$-representable on $P$.

Here we enquire the $L_f$-representability of $A(P)$ when $|P| > |L|$.

\textbf{Theorem 4.1.} Let $|P| \geq 4$ and $|L| < |P|$. Then the alternating group $A(P)$ is not $L_f$-representable on $P$.

\textit{Proof.} Suppose that $A(P)$ is $L_f$-representable on $P$. Then $A(P) = GLH(P, \delta)$ for some $\delta$ on $P$.

Now we claim that if $(p,q) \circ g \in GLH(P, \delta)$ for every transposition $(p,q)$ in $P$, then $g \in GLH(P, \delta)$. Let $g \notin A(P)$. It follows that $g$ is not an L-homeomorphism on $(P, \delta)$. So we get least one $f \in \delta$ with $f \circ g \notin \delta$ or $f \circ g^{-1} \notin \delta$. Now since $|L| < |P|$ and $f \in \delta$ implies that $f$ is not one to one. So there exist at least two points $\alpha, \beta \in P$ such that $f(\alpha) = f(\beta)$. Suppose $f \circ g \notin \delta$. Since $g$ is a permutation on $P$ and $\alpha, \beta \in P$ gives that there exist $\alpha_1, \beta_1 \in P$ such that $g(\alpha_1) = \alpha$ and $g(\beta_1) = \beta$ and hence $(f \circ g)(\alpha_1) = (f \circ g)(\beta_1)$.

Consider $(\alpha, \beta) \circ g$. Here we prove that $f \circ ((\alpha, \beta) \circ g) = f \circ g$.

For, if $p \in P \setminus \{\alpha_1, \beta_1\}$,

$$f \circ ((\alpha, \beta) \circ g)(p) = f \circ (\alpha, \beta)(g(p)) = f \circ g(p).$$

If $p = \alpha_1$, then

$$f \circ (\alpha, \beta)(\alpha_1) = f \circ (\alpha, \beta)(g(\alpha_1)) = f \circ (\alpha, \beta)(\alpha) = f(\beta) = f(\alpha) = f \circ g(\alpha_1) = f \circ g(p).$$

Similarly if $p = \beta_1$ we get $f \circ ((\alpha, \beta) \circ g)(\beta_1) = f \circ g(\beta_1)$. Thus

$$(f \circ (\alpha, \beta) \circ g)(p) = (f \circ g)(p) \text{ for all } p \in P.$$ 

It follows that there is an $f \in \delta$ such that $f \circ (\alpha, \beta) \circ g \notin \delta$. So we get a transposition $(\alpha, \beta)$ with $(\alpha, \beta) \circ g \notin GLH(P, \delta)$.

Similarly if $f \circ g^{-1} \notin \delta$, then we can prove that $f \circ g^{-1} \circ (\rho, \xi) = f \circ g^{-1}$ where $\alpha = g^{-1}(\rho)$ and $\beta = g^{-1}(\xi)$. Thus in this case also there exists $f \in \delta$ such that $f \circ (g^{-1} \circ (\rho, \xi)) \notin \delta$. Hence $((\rho, \xi) \circ g)^{-1} \notin GLH(P, \delta)$. Here also there is a transposition $((\rho, \xi) \circ g)^{-1}$ such that $((\rho, \xi) \circ g)^{-1} \notin GLH(P, \delta)$. So if $g \notin GLH(P, \delta)$, then there exists a transposition $(p,q)$ such that $(p,q) \circ g \notin GLH(P, \delta)$. It follows that if $(p,q) \circ g \in GLH(P, \delta)$ for every transposition $(p,q)$ in $P$, then $g \in GLH(P, \delta)$, which is a contradiction since $GLH(P, \delta) = A(P)$ and the alternating group $A(P)$ does not satisfy this if $|P| > 3$. So there exist no L-topology $\delta$ on $P$ such that $GLH(P, \delta) = A(P)$. \hfill \Box
5. $L_f$-representability of normal subgroups of $S_n$

Now we investigate the $L_f$-representability of normal subgroups of $S_n$ when $|L| = 3$.

The alternating group $A_n$ is not $L_f$-representable when $|L| < n$. If $n = 4$, then $S_n$ has another normal subgroup and we determine the $L_f$-representability of that subgroup.

If $L = \{0,1\}$, then $L^P$ is isomorphic to the power set of $P$. So if a permutation group $G$ on $P$ is represented as a homeomorphism group of a topological space, then $G$ is $L_f$-representable on $P$. The simplest $F$-lattice other than $L = \{0, 1\}$ is $L = \{0, l, 1\}$ with the order $0 < l < 1$. Also note that any $F$-lattice other than $\{0,1\}$ contains a sublattice isomorphic to $L = \{0,l,1\}$.

Here we use the following Theorem in [10].

**Theorem 5.1.** Let $L$ and $L'$ be two complete and distributive lattices such that $L$ is isomorphic to a sublattice of $L'$. Then if $G$ is a permutation group which can be expressed as $GLH(P, \delta)$ for an $L$-fuzzy topology $\delta_1$ on $P$, then $G$ can also be expressed as $GLH(P, \delta_2)$ for some $L'$-fuzzy topology $\delta_2$ on $P$.

Using Theorem 5.1, we deduce the following.

Let $L$ and $L'$ be two $F$-lattices such that $L$ is isomorphic to a sublattice of $L'$. Then if a permutation group $G$ is $L_f$-representable on an arbitrary set $P$, then $G$ is also $L'_f$-representable on $P$.

**Theorem 5.2.** If $G = \{I, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$, then $G$ is $L_f$-representable on $P = \{1,2,3,4\}$ if and only if $L \neq \{0,1\}$.

**Proof.** Assume that $L \neq \{0,1\}$. Let $L = \{1, l, 0\}$ and $\delta$ be the $L$-topology having base

$$\mathcal{B} = \{h_1, h_2, h_3, h_4\}$$

where $h_i$, $i = 1, 2, 3, 4$ are $L$-subsets of $P$ defined as follows

$$h_1(1) = l, h_1(2) = 0, h_1(3) = l, h_1(4) = 0$$

$$h_2(1) = 0, h_2(2) = l, h_2(3) = 0, h_2(4) = l$$

$$h_3(1) = 1, h_3(2) = l, h_3(3) = l, h_3(4) = 1$$

$$h_4(1) = l, h_4(2) = 1, h_4(3) = 1, h_4(4) = l$$

Now we claim that $GLH(P, \delta) = G$. Reader can easily check that every element $g$ of $G$ is an $L$-homeomorphism of $(P, \delta)$ onto itself. Hence $G \subseteq GLH(P, \delta)$.

Suppose $g$ is an $L$-homeomorphism on $P$. So $g^{-1}(f)$ and $g(f)$ are in $\delta$ for all $f \in \delta$. Now consider $g^{-1}(h_1)$ and $g^{-1}(h_2)$. Then either $g^{-1}(h_1) = h_1$ and $g^{-1}(h_2) = h_2$ or $g^{-1}(h_1) = h_2$ and $g^{-1}(h_2) = h_1$.

**Case 1:** $g^{-1}(h_1) = h_1$ and $g^{-1}(h_2) = h_2$

That is $h_1 \circ g = h_1$ and $h_2 \circ g = h_2$. This gives that $g(1) = 1$ or $3$ and $g(2) = 2$ or $4$. If $g(1) = 1$, then $g(3) = 3$. Suppose $g(2) = 4$. So $g(4) = 2$. Hence $g = (2, 4)$. Then $h_2 \circ g \neq h_3$ or $h_4$. Hence $g$ is not an $L$-homeomorphism on $P$, which is a contradiction. So if $g(1) = 1$, then $g = I$.

Now suppose $g(1) = 3$, then $g(3) = 1$. Suppose $g(2) = 2$. So $g(4) = 4$. Hence $g = (1, 3)$. Then $h_3 \circ g \neq h_3$ or $h_4$. Hence $g$ is not an L-homeomorphism, which is also a contradiction. So if $g(1) = 3$, then $g = (1, 3)(2, 4)$. 


Case 2: \( g^{-1}(h_1) = h_2 \) and \( g^{-1}(h_2) = h_1 \)

In this case \( g(1) = 2 \) or 4 and \( g(2) = 3 \) or 1. If \( g(1) = 2 \), then \( g(3) = 4 \). Suppose \( g(2) = 3 \). So \( g(4) = 1 \). Hence \( g = (1, 2, 3, 4) \). Then \( h_3 \circ g \neq h_3 \) or \( h_4 \). Hence \( g \) is not an L-homeomorphism, which is a contradiction. So if \( g(1) = 2 \), then \( g = (1, 2)(3, 4) \).

Now suppose \( g(1) = 4 \), then \( g(3) = 2 \). Suppose \( g(2) = 1 \). So \( g(4) = 3 \). Hence \( g = (1, 4, 3, 2) \). Then \( h_3 \circ g \neq h_3 \) or \( h_4 \). Hence \( g \) is not an L-homeomorphism, which is a contradiction. So if \( g(1) = 4 \), then \( g = (1, 4)(2, 3) \).

So if \( g \) is an L-homeomorphism on \( P \), then \( g \in G \). Thus \( GLH(P, \delta) \subseteq G \). Then \( GLH(P, \delta) = H \).

Let \( L = \{0, 1\} \), the crisp case. Ramachandran P T proved that there exists no topology \( \tau \) on \( P \) with the homeomorphism group of \( (P, \tau) = G \). So if \( G \) is \( L_f \)-representable, then \( L \neq \{0, 1\} \). This completes the proof.

**Remark 5.3.** Johnson T P[6] proved that the above group \( G \) is \( L_f \)-representable if \( |L| \geq 4 \). Here we get \( G \) is \( L_f \)-representable on \( P \) if \( L \neq \{0, 1\} \).

### Acknowledgments

The suggestions and guidance from Dr. Ramachandran P T, Former Professor, University of Calicut during the preparation of this paper are acknowledged.

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