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Abstract

We initiate a general structure theory for vertex operator algebras \(V\). We discuss the center and the blocks of \(V\), the Jacobson radical and solvable radical, and local vertex operator algebras. The main consequence of our structure theory is that if \(V\) satisfies some mild conditions, then it is necessarily semilocal, i.e. a direct sum of local vertex operator algebras. 2000MSC: 17B69

1 Introduction

The purpose of this paper is to establish some results concerning the algebraic structure of vertex operator algebras. Much of the existing literature is devoted to the study of simple vertex operator algebras, which are certainly very important. However, we perceive the need to have available a general structure theory for vertex operator algebras. The results in the present paper may serve as a step towards this goal.

It is a well-known heuristic, a consequence of the commutivity and associativity axioms [LL], that vertex operator algebras behave, in some ways, like finite-dimensional commutative associative algebras. It is therefore natural to see to what extent one can develop a structure theory for vertex operator algebras that parallels that of finite-dimensional algebras, which is particularly transparent if the algebra is commutative and associative. This entails the idea of the center and various types of radical ideals of \(V\). The center of \(V\) is defined as follows:

\[
Z(V) = \{ v \in V \mid L(-1)v = 0 \}.
\] (1.1)

It is both a finite-dimensional commutative, associative algebra with respect to the \(-1\) product in \(V\), and a vertex subalgebra of \(V\). The center provides the connection between vertex operator algebras and the classical theory of algebras. We define \(J(V)\) along the lines of one of the definitions for associative algebras (some of the other definitions do not work so well for vertex operator algebras). Thus,

\[
J(V) = \cap M
\] (1.2)

where the intersection ranges over the maximal ideals of \(V\). For obvious reasons we call it the Jacobson radical of \(V\). It is not hard to see that \(J(V)\) is the smallest ideal of \(V\):

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such that the quotient vertex operator algebra \( V/J(V) \) is a semisimple vertex operator algebra. Furthermore it is always true (Theorem 3.6) that \( J(V) \cap Z(V) = J(Z(V)) \). We also introduce nilpotent and solvable ideals and the soluble radical of a vertex operator algebra.

We consider the decomposition of \( V \) into blocks, i.e. the indecomposable direct summands of the adjoint module. These are vertex operator algebras themselves, having the same central charge as \( V \). An important point (Proposition 2.6) is that a vertex operator algebra is indecomposable precisely when its center is a local algebra. These general results lead to the main structural results of the paper. To describe them, let us define a vertex operator algebra to be local in case it has a unique maximal ideal. In this case, \( J(V) \) is the maximal ideal and \( V/J(V) \) is a simple vertex operator algebra. We say that \( V \) is semilocal if \( V \) is (isomorphic to) a direct sum of local vertex operator algebras (which necessarily have the same central charge).

**Theorem 1.** A vertex operator algebra \( V \) is semilocal under either of the following two conditions:

(i) There are no nonzero weight spaces \( V_n \) for \( n < -1 \);
(ii) The Jacobson radical of \( V \) coincides with the soluble radical.

Our approach to the question of semilocality and the structure of local vertex operator algebras under the hypotheses of part (i) of Theorem 1 depend on the nature of the nonassociative algebra structure on the zero weight space \( V_0 \) induced by the \(-1\)st and related products. We shall establish the following result, which readily implies part (i) of Theorem 1.

**Theorem 2.** Suppose that \( V \) is a vertex operator algebra such that \( V_n = 0 \) for \( n < -1 \). Then the following are equivalent:

(a) \( V \) is local;
(b) \( V \) is indecomposable;
(c) \( Z(V) \) is a local algebra;
(d) \( V_0 \) is a commutative, power associative local algebra with respect to the product \( a \ast b = 1/2(a(-1)b + b(-1)a) \).

**Remark 3.** (i) We call a power associative algebra \( A \) local in case the nil radical (largest ideal for which all elements are nilpotent) has codimension 1 and is the unique maximal ideal in \( A \). (See Section 4 for more details.)
(ii) If \( V \) satisfies the stronger condition that \( V_n = 0 \) for \( n < 0 \) then \( V_0 \) is actually a commutative, associative local algebra and the \( \ast \) product in (d) is the usual product \( a \ast b = a(-1)b \). In this case, (d) says that \( V_0 \) is a local algebra in the usual sense.

Our approach to part (ii) of Theorem 1 might appear somewhat different to that of part (i), depending as it does on general properties of the radical ideals. But the
goal in each case is to establish a sort of idempotent lifting theorem (cf. Section 4), and the two approaches eventually converge. This raises the general question:

are all vertex operator algebras semilocal?

This amounts to asking if conditions (a) and (b) in Theorem 2 are always equivalent.

The paper is organized as follows: in Section 2 we develop the ideas of the center and the blocks of a vertex operator algebra \( V \). Our development here is related to parts of Li’s paper [L]. Indeed the center of \( V \), being the kernel of \( L(-1) \), is in a sense dual to the cokernel of \( L(1) \), which is the focus of Li’s paper. In Section 3 we consider various radical ideals, in particular the Jacobson radical of \( V \), while in Section 4 we prove the main Theorems. In the final Section 5 we discuss some examples.

2 The center and the blocks of a vertex operator algebra

We begin with some elementary results about vertex operator algebras \( V \). We consider \( V \) as a \( Z \)-graded space with respect to the usual \( L(0) \)-grading,

\[
V = \bigoplus_{n \in \mathbb{Z}} V_n. \tag{2.1}
\]

As in (1.1), \( Z(V) \) denotes the center of \( V \).

**Lemma 2.1.** Let \( v \in V \). Then the following are equivalent:

(a) \( v \in Z(V) \);

(b) The vertex operator for \( v \) is a constant, i.e. \( Y(v, z) = v(-1) \).

**Proof:** If (a) holds then \( L(-1)v = 0 \), and therefore \( (L(-1)v)(n) = 0 \) for all \( n \). Since

\[
(L(-1)v)(n) = -nv(n-1)
\]

it follows that \( v(n-1) = 0 \) for \( n \neq 0 \). Hence, \( Y(v, z) = \sum v(n)z^{-n-1} = v(-1) \). This shows that (a) \( \Rightarrow \) (b). If (b) holds then we have

\[
0 = \frac{d}{dz} Y(v, z) = Y(L(-1)v, z),
\]

so that

\[
0 = Y(L(-1)v, z)1 = L(-1)v + O(z).
\]

Therefore (a) holds. \( \square \)
Lemma 2.2. The following hold:
(a) \( Z(V) \) is annihilated by all Virasoro operators \( L(n) \) with \( n \geq -1 \).
(b) \( Z(V) \subseteq V_0 \) consists of primary states.

Proof: (a) Let \( v \in Z(V) \). For some positive integer \( k \) we have \( L(k)v = 0 \). Then
\[
0 = [L(k), L(-1)]v = (k + 1)L(k - 1)v,
\]
so that also \( L(k - 1)v = 0 \). From this we deduce that \( L(n)v = 0 \) for all \( n \geq 0 \), and in particular (a) holds. Part (b) is a special case of (a), being equivalent to the equalities \( L(2)v = L(1)v = L(0)v = 0 \) for \( v \in Z(V) \).

Lemma 2.3. The \((-1)st\) product \( ab = a(-1)b \) for \( a, b \in Z(V) \) gives \( Z(V) \) the structure of a commutative, associative algebra with identity \( 1 \).

Proof: Let \( a, b \in Z(V) \). Then
\[
L(-1)a(-1)b = [L(-1), a(-1)]b = (L(-1)a)(-1)b = 0.
\]
This shows that \( a(-1)b \in Z(V) \), so that \( Z(V) \) is closed with respect to the product \( a(-1)b \). From skew-commutativity we get
\[
b(-1)a = \sum_{n \geq 0} (-1)^n L(-1)^n a(n - 1)b = a(-1)b,
\]
where we have used Lemma 2.1. So the product is commutative. A similar proof (which we omit) using associativity shows that the product is also associative, and the Lemma follows.

For an element \( v \in Z(V) \) we define the annihilator of \( v \) as
\[
\text{Ann}_V(v) = \text{Ann}(v) = \{ u \in V | v(-1)u = 0 \}.
\]

Lemma 2.4. The following hold:
(a) \([v(-1), Y(u, z)] = 0 \) for any \( v \in Z(V) \), \( u \in V \);
(b) \( \text{Ann}(v) \) is an ideal in \( V \) for any \( v \in Z(V) \).

Proof: Let \( v \in Z(V) \) and \( u \in V \). By the commutativity formula we have
\[
[v(-1), u(n)] = \sum_{m \geq 0} (-1)^m (v(m)u)(m + n - 1) = 0,
\]
the second equality following from Lemma 2.1. So (a) holds. As for (b), suppose that \( a \in \text{Ann}(v) \). Using part (a) we get
\[
v(-1)u(n)a = u(n)v(-1)a = 0.
\]
This shows that $\text{Ann}_V(v)$ is closed under the (right) action of all modes $u(n)$, and hence is an ideal of $V$.

We define the endomorphism ring $\text{End}(V) = \text{Hom}_V(V, V)$ in the usual way, namely

$$\text{End}(V) = \{ \varphi \in \text{End}_C(V, V) \mid \varphi Y(u, z) = Y(u, z)\varphi, \text{all } u \in V \}.$$ 

For $a \in Z(V)$ we define a linear map $\varphi_a : V \to V$ via

$$\varphi_a : v \mapsto a(-1)v. \quad (2.2)$$

Note that as a direct consequence of Lemma 3.4(a), we have

$$\varphi_a \in \text{End}(V). \quad (2.3)$$

**Proposition 2.5.** The map $a \mapsto \varphi_a$ induces a natural isomorphism of rings

$$Z(V) \xrightarrow{\cong} \text{End}(V).$$

**Proof:** Let $\varphi \in \text{End}(V)$ and $u \in V$. Because $\varphi$ commutes with all modes $u(n)$ then

$$\varphi(u) = \varphi(u(-1)1) = u(-1)\varphi(1).$$

This shows that $\varphi$ is uniquely determined by the image of the vacuum state. Furthermore

$$L(-1)\varphi(1) = \varphi L(-1)1 = 0,$$

so that $\varphi(1) \in Z(V)$. On the other hand, for $a \in Z(V)$ we have $\varphi_a(1) = a$, as follows from (2.2). This shows that the map $a \mapsto \varphi_a$ induces a linear isomorphism from $Z(V)$ to $\text{End}(V)$. That it is an isomorphism of rings follows from the identity $(a(-1)b)(-1) = a(-1)b(-1)$ for $a, b \in Z(V)$.

We will refer to the idempotent elements of $Z(V)$ as the *idempotents of $V$*. We similarly transfer other standard constructions and language concerning idempotents (for example orthogonal idempotents and primitive idempotents) from $Z(V)$ to $V$. Thus $V$ has a *unique* set of primitive idempotents $e_1, ..., e_n$, and we have

$$e_1 + ... + e_n = 1. \quad (2.4)$$

For $a \in Z(V)$, both image and kernel of $\varphi_a$ are ideals in $V$, indeed the kernel is nothing but the annihilating ideal $\text{Ann}(a)$ (cf. Lemma 2.4(b)). Of course, in general we have $\text{Im}\varphi_a \cong V/\text{Ann}(a)$. But if $e$ is an idempotent of $V$ then there is a splitting

$$V = e(-1)V \oplus (1 - e)(-1)V = \text{Im} \varphi_e \oplus \ker \varphi_e. \quad (2.5)$$
Now set $V^i = e_i(-1)V, 1 \leq i \leq n$. We therefore obtain a decomposition of $V$ into ideals

$$V = V^1 \oplus \ldots \oplus V^n$$

such that $e_i = 1$, is the vacuum state for $V^i$. The uniqueness of the primitive idempotents for $V$ entails the uniqueness of this decomposition. We call the ideals $V^i$ the blocks of $V$, and (2.6) the decomposition of $V$ into blocks.

**Proposition 2.6.** The following are equivalent:

(a) $V$ is indecomposable as a $V$-module.

(b) $\text{End}(V) = Z(V)$ is a local commutative, associative algebra.

**Proof:** We already know (Proposition 2.5) that $\text{End}(V) = Z(V)$ is always a commutative, associative algebra with identity element $1$. Then the equivalence of (a) and (b) has essentially the same proof as for finite-dimensional (associative) algebras. Namely, every idempotent in $\text{End}(V)$ determines a splitting (2.5) into ideals of $V$ as above. Conversely if $V = A \oplus B$ is a splitting into ideals of $V$ then projection onto $A$ is an idempotent in $\text{End}(V)$. We leave further details to the reader.

We have proved most of the next result.

**Theorem 2.7.** (Block decomposition of $V$) A vertex operator algebra $V$ has a unique decomposition (2.6) into blocks $V^1, \ldots, V^n$. The blocks of $V$ are the indecomposable direct summands of the adjoint module $V$. The number of blocks $n$ is equal to the number of primitive idempotents in $Z(V)$.

**Proof:** It only remains to explain why the blocks are indecomposable. Indeed, it is clear that

$$Z(V) = Z(V^1) \oplus \ldots \oplus Z(V^n),$$

and that $e_i$ is the identity element of $Z(V^i)$. Thus, $e_i$ is the unique nonzero idempotent in $Z(V^i)$. So $Z(V^i)$ is a local algebra, and therefore $V^i$ is indecomposable by Proposition 2.6. This completes the proof of Theorem 2.7.

Indecomposable vertex operator algebras arise naturally in representation theory. First some standard definitions: given a vertex operator algebra $V$ and a (nonzero) $V$-module $(M, Y_M)$, the annihilator of $M$ is

$$\text{Ann}(M) = \{ v \in V \mid Y_{M}(v,z)M = 0 \}.$$ 

It is an ideal of $V$. $M$ is called faithful if $\text{Ann}M = 0$. So in general, $M$ is a faithful module over $V/\text{Ann}(M)$.

**Lemma 2.8.** Suppose that $M$ is a faithful, indecomposable $V$-module. Then the following hold:

(a) $V$ is indecomposable;

(b) If $M$ is simple then $Z(V) = C1$. 

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Proof: By a standard argument, since \( M \) is indecomposable then \( \text{Hom}_V(M, M) \) is a local algebra. Because of Lemma 2.4(a) there is a natural injection of rings \( Z(V) \to \text{Hom}_V(M, M) \), and therefore \( Z(V) \) is also a local algebra. Now part (a) follows from Proposition 2.6. If \( M \) is simple then \( \text{Hom}_V(M, M) = \mathbb{C} \) by Schur’s Lemma, so (b) holds.

3 Radical ideals of a vertex operator algebra.

We discuss various types of radical ideals for a vertex operator algebra. They are modelled in a rather obvious way on corresponding ideas from the theory of algebras, both associative and nonassociative. We repeatedly use various elementary properties of ideals \( I \subseteq V \) (cf. [LL]): left ideals are necessarily 2-sided, and they are \( Z \)-graded subspaces of \( V \). That is,

\[
I = \bigoplus_{n \in \mathbb{Z}} I_n, \quad I_n = I \cap V_n.
\]

It is evident that there is a unique ideal \( T = T(V) \subseteq V \) maximal with respect to having the property that the smallest weight of a nonzero state in \( T \) is at least 2. (By Lemma 2.2(b), this is the same as requiring that \( T_0 = T_1 = 0 \).) Note also that \( T(V/T(V)) = 0 \), so that \( T(V) \) is a type of radical, though of a very simple kind. We call \( T \) the trivial radical because it has little effect on the main issues we are concerned with, which center around the weight-space \( V_0 \). Here is a simple example of this idea:

**Lemma 3.1.** Let \( T \) be the trivial radical of the vertex operator algebra \( V \). Then the following are equivalent:

(a) \( V \) is indecomposable;

(b) \( V/T \) is indecomposable.

**Proof:** Let \( a \in V \) be such that \( a + T \in Z(V/T) \). Without loss we may choose \( a \in V_0 \). Then \( L(-1)(a + T) = T \), so that \( L(-1)a \in T \cap V_1 = 0 \). This shows that we must have \( a \in Z(V) \). Because \( T_0 = 0 \) it follows that the projection \( V \to V/T \) restricts to an isomorphism \( Z(V) \cong Z(V/T) \). Now the Lemma follows immediately from Proposition 2.6.

**Lemma 3.2.** Let \( V \) be a vertex operator algebra such that the trivial radical \( T(V) \) vanishes. Then every nonzero ideal of \( V \) contains a minimal ideal (that is, a nonzero irreducible \( V \)-submodule).

**Proof:** Let \( I \) be a nonzero ideal of \( V \). As \( T(V) = 0 \) then \( I_0 + I_1 \neq 0 \). Since \( I_0 + I_1 \) has finite dimension it is clear that we can find a nonzero ideal \( P \) of \( V \) contained in \( I \) for which \( P_0 + P_1 \) has minimal dimension. Moreover, we may assume that \( P \) is generated
(as $V$-module) by $P_0 + P_1$. We assert that $P$ is a minimal ideal. Indeed, if $Q \subseteq P$ is a non-zero ideal of $V$ then $0 \neq Q_0 + Q_1 \subseteq P_0 + P_1$. Choice of $P$ forces $Q_0 + Q_1 = P_0 + P_1$, so that $P = Q$ since $P$ is generated by $P_0 + P_1$. This completes the proof of the Lemma.

We next consider the Jacobson radical $J(V)$. Let $\mathcal{M}$ be the set of maximal (proper) ideals of $V$. It is evident that $\mathcal{M}$ is non-empty, indeed that every (proper) ideal of $V$ is contained in a maximal ideal. Moreover, an ideal $M$ is maximal if, and only if, $V/M$ is a simple vertex operator algebra. We then define $J(V)$ as in (1.2).

**Lemma 3.3.** Suppose that $I$ is an ideal in $V$ with $I_0 = \{0\}$. Then $I \subseteq J(V)$.

**Proof:** If there is a maximal ideal $M$ in $V$ which does not contain $I$ then $V = M + I$. Then $1 \in V_0 = I_0 + M_0 = M_0 \subseteq M$, contradiction. Lemma 3.3 implies that $T(V) \subseteq J(V)$.

**Proposition 3.4.** The following hold:
(a) If $I$ is an ideal in $V$ such that $V/I$ is semisimple, then $J(V) \subseteq I$;
(b) $V/J(V)$ is semisimple.

**Proof:** Part (a) follows by a standard argument which we omit. As for part (b), let $J = J(V)$ and let the maximal ideals of $V$ be $M^1, M^2, ..., M^k, ...$. A standard argument shows that each intersection $D^k = \bigcap_{i=1}^{k} M^i$ is such that $V/D^k$ is semisimple, so it is enough to show that there is an integer $n$ such that $D^n = J$. Because $V_0$ has finite dimension there is certainly an integer $n$ such that $(D^n)_0 = J_0$. Then in the quotient vertex operator algebra $V/J$ we have $(D^n/J)_0 = 0$. By Lemma 3.3,

$$D^n/J \subseteq J(V/J) = \{0\}.$$  

Thus using part(a) we get $J \subseteq D^n \subseteq J$. The Lemma is proved.

It follows immediately from Proposition 3.4 that we have

**Corollary 3.5.** Let $V$ be a vertex operator algebra. Then $V$ has only a finite number of distinct maximal ideals, say $M^1, ..., M^k$. Moreover $V/J(V)$ is semisimple and has a decomposition

$$V/J(V) = (V^1/J(V)) \oplus ... \oplus (V^k/J(V))$$  

into the direct sum of $k$ simple vertex operator algebras $V^i/J(V)$.

**Lemma 3.6.** Suppose that $V$ is a direct sum of $k$ simple vertex operator algebras $V^1, ..., V^k$. Let $1_i$ be the vacuum vector of $V^i$. Then

$$Z(V) = C_1 1_1 \oplus ... \oplus C_k 1_k$$

is a semisimple algebra of rank $k$ and $\{1_1, ..., 1_k\}$ is the complete set of primitive idempotents of $Z(V)$. 

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Proof: It is clear that $Z(V) = Z(V^1) \oplus ... \oplus Z(V^n)$. Now the present Lemma follows from Lemma 2.8(b). \hfill \Box

**Theorem 3.7.** For any vertex operator algebra $V$, we have

\[ J(V) \cap Z(V) = J(Z(V)). \]

**Proof:** By Theorem 2.7 we may assume that $V$ is indecomposable, and therefore (Proposition 2.6) that $Z = Z(V)$ is a local commutative algebra. Since $V/J$ is semisimple (Proposition 3.4(b)) then $Z(V/J)$ is semisimple by Lemma 3.6. Being a subalgebra of $Z(V/J)$, it follows that $Z + J/J \cong Z/Z \cap J$ is also semisimple, and hence that $J(Z) \subseteq Z \cap J$. Since $J(Z)$ has codimension 1 in $Z$, and since $1 \notin J$, the Theorem follows. \hfill \Box

It is possible to refine Theorem 3.7. In order to do this, we make the following definitions. Let $t \geq 1$ be an integer, and $I$ an ideal in a vertex operator algebra $V$. Set

\[ I^t = \text{subspace of } V \text{ spanned by all states of the form } a_1(n_1)...a_{t-1}(n_{t-1})a_t, \text{ all } a_1,...,a_t \in I, n_1,...,n_t \in Z. \]

It is easy to see that $I^t$ is an ideal of $V$ whenever $I$ is. In this way we obtain two descending sequences of ideals

\[ I \supseteq I^2 \supseteq I^3 \supseteq ... \supseteq I^r \supseteq ... \]

and

\[ I \supseteq I^2 \supseteq (I^2)^2 \supseteq ... \supseteq (I^r)^2 \supseteq ... \]

where we inductively define $I^1 = I$ and $I^{(r)} = (I^{(r-1)})^2$. We say that $I$ is nilpotent in case there is $r \geq 1$ such that $I^r = 0$, and solvable if there is $r \geq 1$ such that $I^{(r)} = 0$. Observe that $I^{(r)} \subseteq I^2$, so that $I$ nilpotent $\Rightarrow$ $I$ solvable.

**Theorem 3.8.** The following hold for a vertex operator algebra $V$:

(a) $V$ has a unique maximal nilpotent ideal $N(V)$;
(b) $N(V) \subseteq J(V)$;
(c) $Z(V) \cap N(V) = J(Z(V))$.

**Proof:** To prove (a), it is enough to show that the sum $I^1 + I^2$ of two nilpotent ideals $I^1, I^2 \subseteq V$ is again nilpotent. Let $t$ be a positive integer such that $(I^j)^t = 0$, $j = 1, 2$, and choose elements $x_1,...,x_{2t} \in I^1, y_1,...,y_{2t} \in I^2$. It suffices to show that

\[ (x_1+y_1)(n_1)...(x_{2t-1}+y_{t-1})(n_{2t-1})(x_{2t}+y_{2t}) = 0 \]

for all integers $n_1,...,n_{t-1}$. Expanding (3.2) yields a sum of terms, each of which is a product

\[ a_1(n_1)...a_{2t-1}(n_{2t-1})a_{2t} \]

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with each $a_i$ equal to either $x_i$ or $y_i$. So it suffices to show that each expression (3.3) is equal to zero. Now either there are (at least) $t$ indices $i$ for which $a_i = x_i$, or $t$ indices for which $a_i = y_i$. We will assume that the first case holds; if it does not, then the same argument applies with $x_i$ replaced by $y_i$.

Using the relation
\[ a_i(n)a_{i+1}(m) = a_{i+1}(m)a_i(n) + \sum_{j \geq 0} (a_i(j)a_{i+1})(m + n - j) \]
together with the containment $a_i(j)a_{i+1} \in I^1$ whenever $a_i \in I^1$, we see that (3.3) can be reexpressed as a sum of terms of the form
\[ A b_1(m_1) \ldots b_k(m_k)a_{2t} \]  \hspace{1cm} (3.4)
where each $b_t$ is some $x_j; k \geq t - 1$ and either $k \geq t$ or $a_{2t} = x_{2t}$; $A$ is a product of other Fourier modes (of no interest to us). Because $(I^1)^t = 0$, we have $b_1(m_1) \ldots b_k(m_k)a_{2t} = 0$, so that the expression (3.4) vanishes. Therefore so too does (3.3), and part (a) is proved.

Next, it is clear that if $V$ is semisimple then $N(V) = 0$, and that if $I \subseteq V$ is an ideal then $I + N(V)/I \subseteq N(V/I)$. Part (b) is an immediate consequence of these observations.

As for (c), after Theorem 3.7 it is enough to show that the ideal $I$ of $V$ generated by $J' = J(Z(V))$ is nilpotent. Because $J'$ is a nilpotent ideal in $Z(V)$, there is an integer $t$ such that $(J')^t = 0$. We show that $I^t = 0$. Elements of $I^t$ are sums of states of the form
\[ (a_1(-1)b_1)(n_1) \ldots (a_{t-1}(-1)b_{t-1})(n_{t-1})a_t(-1)b_t \]  \hspace{1cm} (3.5)
with $a_1, \ldots, a_t \in J', b_1, \ldots, b_t \in V$ and $n_1, \ldots, n_{t-1} \in Z$ (cf. Lemma 2.1). Indeed, Lemma 2.1 together with associativity also shows that $(a(-1)b)(n) = a(-1)b(n)$ for $a \in J', b \in V$ and $n \in Z$. Then expression (3.5) is equal to
\[ a_1(-1)b_1(n_1) \ldots a_{t-1}(-1)b_{t-1}(n_{t-1})a_t(-1)b_t = b_1(n_1) \ldots b_{t-1}(n_{t-1})b_t(-1)a_1(-1) \ldots a_t(-1) = 0, \]
where we used Lemma 2.4(a) for the first equality. This completes the proof of part (c).

Using similar arguments to the above one can also establish the following (details left to the reader): a vertex operator algebra $V$ has a unique maximal solvable ideal $B(V)$; $B(V/B(V)) = 0$; $B(V) = 0$ if, and only if, $N(V) = 0$; $B(V) \subseteq J(V)$. We have the following containments:
\[ J(Z(V)) \subseteq N(V) \subseteq B(V) \subseteq J(V) \subseteq V. \]
4 Proof of Theorems 1 and 2

We define a local or semilocal vertex operator algebra as in the Introduction. Note that a local vertex operator algebra is indecomposable. From the decomposition into blocks, we see that for $V$ to be semilocal, it is sufficient that every block is a local vertex operator algebra. One is thus led to ask the following question.

Suppose that $V$ is an indecomposable vertex operator algebra. Is it true that $V$ is local? (4.1)

An affirmative answer would reduce questions about general vertex operator algebras to questions about local vertex operator algebras. Note that by Proposition 2.6, (4.1) is essentially the same as the question of lifting idempotents of vertex operator algebras (cf. the discussion prior to (2.4)), in analogy with the familiar result from the theory of associative algebras.

Suppose that $e \in V$ is an idempotent in $V/J(V)$. (4.2)

Can $e$ be lifted to an idempotent in $V$?

We now turn our attention to the proof of Theorems 1 and 2, which will show that the answers to (4.1) and (4.2) are affirmative in case the appropriate conditions are satisfied. Our approach to the first part of Theorem 1 involves the structure of the nonassociative algebra on $V_0$ induced by the $-1$ product, which contains $Z(V)$ as a subalgebra. With (4.2) in mind, it comes down to showing that all idempotents in $V_0$ are contained in $Z(V)$. We begin by developing the necessary background concerning power associative algebras. Albert developed this theory in several papers (cf. [A1], [A2], [S]). It fits well into the vertex operator algebra formalism.

A finite-dimensional nonassociative (unital) algebra $A$ is called power associative if each element $a \in A$ has the property that all powers of $a$ associate. Thus, the subalgebra of $A$ generated by $a$ is associative and commutative. By [A1], it is sufficient that the identities

$$aa^2 = a^2a, \quad a(aa^2) = a^2a$$

hold. If $A$ is power associative, one may unambiguously refer to nilpotent elements ($a^n = 0$ for some $n \geq 1$). An ideal $I \subseteq A$ is nil in case every element of $I$ is nilpotent. Then $A$ has a unique maximal nil ideal $N$ (the nil radical), and the quotient $A/N$ has trivial nil radical. We call $A$ a local power associative algebra in case its nil radical is the unique maximal ideal of $A$ and has codimension 1.

Proposition 4.1. Suppose that $A$ is a commutative, power associative unital algebra with the property that 1 is the only nonzero idempotent in $A$. Then $A$ is a local algebra.
Proof: If \( A \) is also assumed to be simple, Albert proved ([A2, Theorem 9]) that \( A = C1 \). So in this case the nil radical is trivial and we are done. So we may assume that there is a nonzero proper ideal \( I \subseteq A \).

Choose \( a \in I \). The subalgebra \( \langle a \rangle \) generated by \( a \) is associative, and hence is either nilpotent or contains a nonzero idempotent. In the latter case, the hypothesis of the Theorem shows that \( 1 \in \langle a \rangle \subseteq I \). But then \( I = A \), a contradiction. Therefore \( a \) is nilpotent. This shows that \( I \) is a nil ideal, and hence contained in the nil radical \( N \) of \( A \).

We assert that the hypotheses of the Theorem hold in the quotient algebra \( A/I \). Once this is established, the Theorem follows easily by induction on \( \dim A \). Now \( A/I \) is certainly commutative and power associative, so to prove the assertion it suffices to show that \( 1 + I \) is the only nonzero idempotent in \( A/I \). Let \( e + I \) be a nonzero idempotent in \( A/I \). It is apparent that the subalgebra \( \langle e \rangle \) of \( A \) generated by \( e \) cannot be nilpotent. As before, this leads to \( 1 \in \langle e \rangle \), and even \( \langle e \rangle = C1 \oplus J(\langle e \rangle) \). So \( e = \alpha 1 + x \) for some scalar \( \alpha \in C \) and where \( x \in J(\langle e \rangle) \) is nilpotent. Then \( e^2 - e = (\alpha^2 - \alpha)1 + y \) with \( y = x^2 + (2\alpha - 1)x \) nilpotent. But \( e^2 - e \in I \) and hence is itself nilpotent. It follows that \( \alpha^2 = \alpha = 0 \) or \( 1 \). As \( e \) is not nilpotent then the case \( \alpha = 0 \) is impossible, so \( \alpha = 1 \) and

\[
e = 1 + x; \quad y = x^2 + x \in I.
\]

Let \( n \geq 1 \) be the smallest positive integer satisfying \( x^n \in I \). Because \( x \) is nilpotent, such an \( n \) certainly exists. If \( n \geq 2 \) then

\[
I = x^n + I = x^{n-2}x^2 + I = -x^{n-2}x + I,
\]

where we used the second equality in (4.4). Then \( x^{n-1} \in I \), contradicting the definition of \( n \). We conclude that \( n = 1 \). Then \( x \in I \) and \( e + I = 1 + I \), as we see from the first equality in (4.4). This completes the proof of the Proposition.

For the remainder of this Section we fix \( V \) to be a vertex operator algebra satisfying the truncation condition

\[
V_n = 0 \text{ for } n \leq -2.
\]

Proposition 4.2. With respect to the \(-1\) operation, \( V_0 \) is a power associative algebra. Moreover if \( a \in V_0 \) then

\[
[a(m),a(n)] = 0 \text{ for all integers } m, n.
\]

Proof: Pick \( a \in V_0 \). In addition to (4.6) we have to establish the two identities (4.3), which amount to the following:

\[
a(-1)^2a = (a(-1)a)(-1)a;
\]

\[
a(-1)^3a = (a(-1)a)(-1)a(-1)a.
\]
Now we have \( a(n)a \in V_{-n-1} \), and in particular
\[
a(n)a = 0 \text{ for } n \geq 1
\]  
(4.8)
thanks to (4.5). By skew-symmetry we also have
\[
a(0)a = -a(0)a - \sum_{n \geq 1} (-1)^n \frac{L(-1)^n}{n!} a(n)a.
\]
Then by (4.8) we also get \( a(0)a = 0 \). So in fact \( a(n)a = 0 \) for all \( n \geq 0 \), and (4.6) is a consequence of this together with the commutivity axiom. With (4.6) in hand, the proof of (4.7) is a straightforward application of the associativity axiom. We omit the details.

The next result contains a key calculation.

**Proposition 4.3.** \( Z(V) \) contains every idempotent of \( V_0 \).

**Proof:** We begin with the observation that the idempotents and the involutorial units of \( V_0 \) (i.e., those \( u \in V_0 \) satisfying \( u(-1)u = 1 \)) span the same subspace. This is a simple fact about any unital nonassociative algebra, and we omit the proof. In view of this, in order to prove the Proposition it suffices to show that \( Z(V) \) contains every involutorial unit \( u \in V_0 \). Fix such an element \( u \). We must show that \( L(-1)u = 0 \).

Since \( L(-1)1 = 0 \) then
\[
0 = u(-1)L(-1)u(-1)u
= u(-1)([L(-1), u(-1)]u + u(-1)L(-1)u(-1)1)
= u(-1)(u(-2)u + u(-1)[L(-1), u(-1)] 1)
= u(-1)(u(-2)u + u(-1)u(-2) 1)
= 2u(-1)u(-2)u
= 2u(-2)1
= 2(L(-1)u)(-1)1
= 2L(-1)u,
\]
where we have made use of (4.6).

We turn to the proof of Theorem 2. That (a) \( \Rightarrow \) (b) is clear, while (b) \( \Leftrightarrow \) (c) is nothing but Proposition 2.6. If (c) holds then by the previous Proposition, \( 1 \) is the unique nonzero idempotent in \( V_0 \). Let \( V_0^+ \) be the algebra obtained from \( V_0 \) by redefining the product to be \( a * b = 1/2(a(-1)b + b(-1)a) \). Then \( V_0^+ \) is a unital commutative algebra, and it is power associative because \( V_0 \) is. Furthermore, the idempotents of \( V_0 \) and \( V_0^+ \) are the same, whence \( 1 \) is also the unique nonzero idempotent of \( V_0^+ \). By Proposition 4.1, \( V_0^+ \) is a local algebra, and so (d) holds.
To complete the proof of Theorem 2, we must show that (d) ⇒ (a). Suppose then, that \( V_0^+ \) is a local power associative algebra. Any proper ideal \( I \subseteq V_0 \) is an ideal in \( V_0^+ \) and therefore contained in the nil radical of \( V_0^+ \). In particular, \( V_0^+ / I \) is again a local power associative algebra with the identity element the only nonzero idempotent. It follows from this that \( V_0 / I \) also has a unique nonzero idempotent.

Set \( J = J(V) \). Being semisimple, \( V/J \) is the sum of \( n \geq 1 \) simple vertex operator algebras. Choose states \( e_i \in V_0 \) such that \( e_i + J = 1 \) \((1 \leq i \leq n)\) are the vacuum states of the simple components of \( V/J \). Now \( J_0 \) is an ideal of \( V_0 \) and evidently the \( e_i \) map onto distinct idempotents of \( V_0/J_0 \). After the last paragraph, we conclude that in fact \( n = 1 \), that is \( V \) is a local vertex operator algebra. This completes the proof of Theorem 2.

Theorem 1(i) is now an easy consequence of what we have already established. Namely, since \( V \) satisfies the condition (4.5) it is clear that the blocks \( V^i \) of \( V \) have the same property. Since the blocks are indecomposable, Theorem 2 informs us that they are local. This completes the proof of Theorem 1(i).

We turn to the proof of the second part of Theorem 1. So we are assuming that the vertex operator algebra \( V \) is such that the Jacobson radical coincides with the solvable radical. We proceed, using induction on the dimension of \( V_0 + V_1 \), to show that \( V \) is semilocal. The decomposition into blocks shows that we may assume without loss that \( V \) is indecomposable. Thus, \( Z(V) \) is a local algebra by Proposition 2.6, and we must show that \( V \) is a local vertex operator algebra. If \( J = J(V) \) is zero then the result is clear, so we may take \( J \neq 0 \). We may also assume that \( T(V) = 0 \) by Lemma 3.1, in which case \( J \) contains a minimal ideal \( N \) by Lemma 3.2. Since \( J \) is solvable then \( N \) is necessarily nilpotent, indeed \( N^2 = 0 \).

Set \( A = \text{Ann}_V(N) \). Then \( A \) is an ideal of \( V \) which contains \( N \), and \( V/A \) is indecomposable by Lemma 2.8. On the other hand, by induction we know that \( V/N \) is semilocal. Let

\[
V/N = U_1/N \oplus \ldots \oplus U_n/N
\]

be the decomposition of \( V/N \) into blocks. Thus, each \( U_i/N \) is a local vertex operator algebra. Pick states \( a_1, ..., a_n \in V_0 \) such that \( a_i + N \) is the vacuum state of \( U_i/N \). We claim that all but one of the states \( a_1, ..., a_n \) lie in \( A \). If \( n = 1 \) this is clear, so assume that \( n \geq 2 \) and that neither \( a = a_1 \) nor \( b = a_2 \) lie in \( A \). In this case both \( a \) and \( b \) map onto the vacuum of \( V/A \) (because it is indecomposable) and therefore \( a - b \in A \). So \( a - b \in A \cap (U_1 + U_2) \). Since each \( U_i \) is local it follows that \( U_1 + U_2 = J(U_1 + U_2) + A \cap (U_1 + U_2) \), so that \( a, b \in U_1 + U_2 = A \cap (U_1 + U_2) \subseteq A \), a contradiction. So indeed we may assume \( a_1, ..., a_{n-1} \) are contained in \( A \). Now if \( n = 1 \) then there is nothing to prove, so we may assume without loss that \( n \geq 2 \) and \( a = a_1 \in A \). We will show that this leads to a contradiction.

Since \( a + N \) is the vacuum state of \( U_1/N \) then \( a(-1)a = a + n \) for some \( n \in N_0 \). Because \( a \in A \) then \( a(k)N = 0 \) for all integers \( k \), and from this together with \( N^2 = 0 \)
it follows easily that \((a + n)(-1)(a + n) = a + n\). So without loss we may assume that \(a(-1)a = a\). We will show that \(a \in Z(V)\). In this case \(a\) is a nonzero idempotent in \(Z(V)\) and hence \(a = 1\) because \(Z(V)\) is a local algebra. Then \(1 \in A\) and therefore annihilates \(N\), contradiction.

The strategy used to establish \(a \in Z(V)\) is similar to that used in the proof of Proposition 4.3. First note that since \(a + N\) is the vacuum of \(U^1/N\) then \(a(k)V \subseteq N\) for \(k \neq -1\). In particular, because \(a \in A\) we see that \(a(-1)\) annihilates \(a(k)a\) and \((a(k)a)(l)\) annihilates \(a\) for \(k \neq -1\) and all \(l\). From this we conclude that

\[
\begin{align*}
\begin{align*}
-2 \sum_{i \geq 0} (a_i)a\big((-3 + i)a
\end{align*}
\end{align*}
\]

Similarly we get

\[
a(-2)a = a(-2)a(-1)a
\]

\[
= a(-1)a(-2)a + [a(-2), a(-1)]a
\]

\[
= a(-1)a(-2)a + \sum_{i \geq 0} \binom{-2}{i} a(i)a\big((-3 + i)a
\]

\[
= 0.
\]

(4.9)

Similarly we get

\[
a(-1)a(-2)1 = 0.
\]

(4.10)

Now apply the operator \(L(-1)\) to both sides of the equality \(a(-1)a = a\) to see that

\[
L(-1)a = L(-1)a(-1)a
\]

\[
= [L(-1), a(-1)]a + a(-1)L(-1)a
\]

\[
= a(-2)a + a(-1)a(-2)a
\]

\[
= 0,
\]

where we used (4.9) and (4.10). This shows that indeed \(a \in Z(V)\), and the proof of the second part of Theorem 1 is complete.

\[\square\]

5 Examples

We discuss some examples of Jacobson radicals and local vertex operator algebras. Simple vertex operator algebras are obvious examples of local vertex operator algebras.

Example 1: Semidirect products.
Suppose that \(M\) is a \(Z\)-graded module over a vertex operator algebra \(V\). In [L], Li describes how to construct the semidirect product

\[
W = V \oplus M,
\]

(5.1)

where in particular \(V\) is a vertex operator subalgebra of \(W\), \(M\) is an ideal, and \(Y_W(u, z)v = 0\) for all \(u, v \in M\). So \(M\) is a nilpotent ideal of \(W\), and it is evident
that if $V$ is semisimple then $M = J(W) = N(W)$. In particular, if $V$ is simple then $W$ is a local vertex operator algebra.

**Example 2:** Let $V_L$ be the lattice vertex operator algebra associated with the root lattice of type $A_1$ spanned by a single positive root $\alpha$, say. Let $M(1) \subseteq V_L$ be the Heisenberg vertex operator subalgebra. We have the decomposition into simple $M(1)$-modules

$$V_L = \bigoplus_{n \in \mathbb{Z}} M(1) \otimes \mathbb{C} e^{n\alpha}.$$ 

Set

$$U = \bigoplus_{n \geq 0} M(1) \otimes \mathbb{C} e^{n\alpha}.$$ 

Then $V_L$ is simple and $U$ is a subvertex operator algebra of $V_L$ which is also a local vertex operator algebra with Jacobson radical

$$J(U) = \bigoplus_{n > 0} M(1) \otimes \mathbb{C} e^{n\alpha}.$$ 

This shows that the Jacobson radical is not *functorial*. That is, if $f : V_1 \rightarrow V_2$ is a morphism of vertex operator algebras, then $f$ does *not* necessarily induce a morphism $f : J(V_1) \rightarrow J(V_2)$.

**Example 3:** Virasoro vertex operator algebras.
Consider the Virasoro vertex operator algebra $V_c$ of central charge $c$ defined as the quotient of the Verma module over the Virasoro algebra $Vir$ of highest weight zero and central charge $c$ by the submodule generated by $L(-1)\mathbf{1}$. (See [FZ], [W] for more details.) If we choose $c$ to be in the *discrete series* (loc. cit.) then $V_c$ has a submodule $M \neq 0$ such that

$$V_c/M = L_c$$

is the simple Virasoro vertex operator algebra $L_c$. Clearly $V_c$ is a local vertex operator algebra and $J(V_c) = M$. Note that a decomposition $V_c = M \oplus L_c$ with $L_c$ a vertex operator subalgebra is impossible, because $V_c$ is generated by $\mathbf{1}$ as a module over $Vir$.

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