Impact of an elastic sphere with an elastic half space revisited: Numerical analysis based on the method of dimensionality reduction

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An impact of an elastic sphere with an elastic half space under no-slip conditions (infinitely large coefficient of friction) is studied numerically using the method of dimensionality reduction. It is shown that the rebound velocity and angular velocity, written as proper dimensionless variables, are determined by a function of only the ratio of tangential and normal stiffness ("Mindlin-ratio"). The obtained numerical results can be approximated by a simple analytical expression.
variables which are of interest and will be used in the following analysis. After this the impact problem is solved using the method of dimensionality reduction.

Results
Simplified model of the impact with no-slip condition. Classical "rigid body" solution. Let us consider an impact of an elastic sphere with mass $m$ and radius $R$ on an elastic half space, as shown in Fig. 1. Let the moduli of elasticity of the sphere and the half space be $E_1$ and $E_2$, their Poisson’s numbers $v_1$ and $v_2$, and their shear moduli $G_1$ and $G_2$, accordingly. The main notations are illustrated in Fig. 1: The incident velocity of the center of mass of the sphere is $v_0$ and $R$, their Poison’s numbers $v_1$ and $v_2$, the incident angular velocity $\omega_0$, the rebound velocity is $v$ with components $v_x$ and $v_y$, the grazing angle is $\alpha$, and the rebound angle $\beta$.

We first reproduce the classical solution of the impact problem. Let $F_x$ and $F_y$ be the components of contact force acting on the sphere during the impact. The equations of motion of the sphere in the integral form can be written as

\[
m(v_x - v_{x0}) = - \int_0^t F_x(t) \, dt,
\]

\[
m(v_y - v_{y0}) = - \int_0^t F_y(t) \, dt,
\]

\[
I(\omega - \omega_0) = - R \int_0^t F_z(t) \, dt,
\]

where $t$ is the duration of the impact, and $I = (2/5)mr^2$ is the moment of inertia of a homogeneous sphere. Together with the rolling condition for the tangential rebound velocity,

\[v_x + \omega_0 R = 0,
\]

these equations determine unambiguously all kinematic quantities of the sphere after the impact:

\[v_x = \frac{5}{7} v_{x0} - \frac{2}{7} R \omega_0,
\]

\[v_y = \frac{2}{7} \omega_0 - \frac{5}{7} v_{y0} R.
\]

It can be easily seen that the impact is non-elastic, as the energy change during the impact,

\[\Delta E = \frac{m}{2} (v_x^2 - v_{x0}^2) + \frac{I}{2} (\omega^2 - \omega_0^2) = - \frac{m}{7} (v_{x0} + R \omega_0)^2,
\]

is negative. This solution is, however, oversimplified. While equations (1)–(3) are exact (under assumption of very short impact time), the kinematic condition (4) is intrinsically controversial: it cannot be valid during the whole time of impact, and its application to the last moment of impact is an arbitrary and not substantiated assumption. In reality, due to the elasticity of the sphere, the condition (4) will be valid only at one point in time during the impact.

Impact in the case of a constant contact stiffness. In a second step let us take into account the normal and tangential compliance of the contact in a simplified way. The normal and tangential compliance of the contact are changing during the impact due to changing contact configuration. Let us simplify this situation by considering an impact of a rigid sphere having a linear spring in the contact region. This also can be an elastic sphere with a flat patch. Due to the flat the contact stiffness will be constant provided the contact radius does not change considerably during the impact. The considered system and notation are shown in Fig. 2.

Equations of motion can be written as

\[m\ddot{u}_x = - F_N,
\]

\[m\ddot{u}_y = - F_f,
\]

\[I\ddot{\phi} = - F_f R,
\]

where

\[F_N = k_z u_z,
\]

\[F_f = k_x (u_x + R\ddot{\phi})
\]

are the normal and tangential components of the contact force. In the last equation we took into account the fact that the tangential displacement of the contact point is a sum of the displacement of the center of mass and the displacement due to rigid rotation. The solution of the set of Eq. (8)–(12) with the initial conditions $u_x(0) = 0$, $\dot{u}_x(0) = v_{x0}$, $u_y(0) = 0$, $\dot{u}_y(0) = v_{y0}$, $\phi(0) = 0$, $\dot{\phi}(0) = \omega_0$ has the form

\[u_x(t) = \frac{5}{7} v_{x0} - \frac{2}{7} R \omega_0 t + \frac{2}{7} (v_{x0} + R \omega_0) \cos(\omega_2 t),
\]

\[u_y(t) = \frac{2}{7} \omega_0 - \frac{5}{7} v_{y0} R t + \frac{5}{7} (v_{y0} + R \omega_0) \cos(\omega_2 t),
\]

where $\omega_2 = \sqrt{k_z/m}$ and $\omega_2 = \sqrt{7k_x/(2m)}$. The duration of the impact, $t_0$, is determined by the Eq. (13) and is equal to $t_0 = \pi/\omega_2$. The velocities at the last moment of the impact are equal to

\[\dot{u}_x(t_0) = - v_{y0},
\]

\[\dot{u}_y(t_0) = - v_{x0}.
\]
Indeed, for an arbitrary rotationally symmetric body the ratio of stiffness. This suggests that the structure of the relations (20) and (21) are functions of the ratio \( k_x/k_z \) of the normal and tangential contact stiffness. This suggests that the structure of the relations (20) and (21) may be valid for a more general case of contact of any shape. Indeed, for an arbitrary rotationally symmetric body the ratio of differential tangential and normal stiffnesses is constant and equal to Ref. 8 \( k_x/k_z = G*/E* \), where

\[
\frac{1}{E} = \frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2},
\]

\[
\frac{1}{G} = \frac{2-v_1}{4G_1} + \frac{2-v_2}{4G_2}.
\]

In Refs. 22 and 23 it was shown by numerical simulations that this is valid even for randomly rough fractal surfaces. We thus may anticipate that the Eq. (20) and (21) are valid for both regular forms and rough profiles, while the exact dependence may be replaced by another, shape dependent function. We arrive at the hypothesis that in the general case the relations (20) and (21) have to be replaced by

\[
\frac{7 v_x - \bar{v}_x}{2 V} = \frac{7 R(\omega - \bar{\omega})}{5 V} = \frac{P(\gamma)}{V},
\]

\[
\frac{7}{m(v_{00} + R\omega_0)^2} = -1 + (P(\gamma))^2,
\]

where

\[
\gamma = \sqrt{\frac{7}{2E*}}.
\]

In the next Section, we will prove this hypothesis by numerical simulations and find the form of the function \( P(\gamma) \).

**Impact of a sphere: results of modeling.** In the present work, the equations of motion (8)–(10) were solved by the Euler integration procedure. The results for the energy change during the impact as a function of the parameter \( \gamma = \sqrt{(7/2)(G^*/E^*)} \) are presented in Fig. 3, where the dimensionless variables \( \frac{7 v_x - \bar{v}_x}{2 V} \), \( \frac{7 R(\omega - \bar{\omega})}{5 V} \), and \( \frac{7\Delta E}{m(v_{00} + R\omega_0)^2} \) are plotted as a function of the parameter \( \gamma \) defined by Eq. (27). Note that if the bodies have equal Poisson numbers \( v_1 = v_2 = v \), then \( G^*/E^* = 2(1 - v)(2 - v) \). From the thermodynamic stability criterion, it follows that Ref. 24 \( -1 < v \leq 1/2 \). Thus, for isotropic bodies, \( 2/3 < G^*/E^* < 4/3 \) which corresponds to \( 1.52 < \gamma < 2.16 \).
However, for anisotropic (e.g., orthotropic) media, the effective ratio $G^*/E^*$ can be in a wider range than given by this Equation. We therefore present results outside the region (28) as well.

Fig. 4 is a magnified representation of the most practically relevant range of the variable $\gamma$. In this range the numerically determined function $P(\gamma)$ can be approximated with

$$P(\gamma) = -1 + 2 \cdot \exp(-a \gamma) \cdot \cos^2[k(\gamma - b)]$$

(29)

with $a = 0.195, b = 0.061$, and $k = 1.685$. For practically important cases of $\nu = 1/3$ we get $P \approx 0.20$, and for incompressible media ($\nu = 1/2$), $P \approx -0.09$.

**Discussion**

In the present paper, we used the method of dimensionality reduction in the area of its exact applicability (contact of axis-symmetric bodies) to simulate an impact of an elastic sphere on an elastic half-space. The main result of the study is the proof of the hypothesis (25) as well as numerical determination of the function $P(\gamma)$ appearing in this equation. This function is presented in Fig. 3. A simple analytical approximation (29) was found for this function. We investigated a much wider range of the ratios $G^*/E^*$ than would be relevant for isotropic elastic bodies. As for anisotropic bodies (e.g., media with orthotropic elasticity), this ratio can in principle have arbitrary values. The suggested method can be generalized straightforwardly to impacts of bodies of different form (not necessarily spherical), impact with adhesion, contacts with dry friction or impact of viscoelastic bodies.

**Methods**

For simulation of normal and tangential contact during the impact we use the method of dimensionality reduction (MDR). In the framework of the MDR, two preliminary relations of the equivalent one-dimensional system will reproduce those of the three-dimensional system.

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**Acknowledgments**

We are grateful to E. Willert for helpful discussions and help in analytical calculations. This work is supported in part by COST Action MP1303, the Ministry of Education of the Russian Federation, and by the Deutsche Forschungsgemeinschaft.
Author contributions
V.L.P. and I.A.L. developed the model and the numerical algorithm, I.A.L. carried out computer simulations and prepared figures, V.L.P. wrote the main manuscript text. Both authors reviewed the manuscript.

Additional information
Competing financial interests: The authors declare no competing financial interests.
How to cite this article: Lyashenko, I.A. & Popov, V.L. Impact of an elastic sphere with an elastic half space revisited: Numerical analysis based on the method of dimensionality reduction. Sci. Rep. 5, 8479; DOI:10.1038/srep08479 (2015).

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