Thermostatistics of a $q$-deformed Relativistic Ideal Fermi Gas

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In this paper, we formulate a $q$-deformed many-body theory for the relativistic Fermi gas and discuss the effects of the deformation parameter $q$ on physical properties of such systems. Since antiparticle excitations appear in the relativistic regime, a suitable treatment to the choice of deformation parameters for both fermions and antifermions must be carefully taken in order to get a consistent theory. By applying this formulation, we further study the thermostatistic properties of a $q$-deformed ideal relativistic Fermi gas. It can be shown that even in the noninteracting scenario, the system exhibits interesting characteristics which are significantly different from ordinary Fermi gases. Explicitly, antiparticles may become dominant due to the shift of chemical potential by the deformation parameter $q$. This may build a solid foundation for the further studies of $q$-deformed relativistic interacting systems.

I. INTRODUCTION

About three decades ago, the studies on exactly solvable models inspired the concepts of quantum groups and the associated algebras[1-5]. Among these quantum algebras, the $q$-deformed algebra obtains a lot of study interests and has become the subject of intensive research works due to its beautiful mathematical structure and rich physical significance. It has found many applications in various topics including the string theory[6], quantum optics[7-11], many-body systems[12] and the nuclear physics[13-21]. Importantly, people have begun to apply it to study more realistic phenomena, such as the explanation of the emissivity of the light fermionic dark matter in the cooling of the supernova SN1987A[22].

Deformed many-particle systems are formed by indistinguishable particles satisfying the $q$-deformed commutative or anti-commutative relations, which form representations of $q$-deformed algebras. They are essentially different from the well-known anyons since the former can exist in arbitrary dimensions while the latter is only meaningful in two-dimensional systems. In some literatures, the $q$-deformed particles are called $q$-particles or simply quons. There have been intensive studies on the ideal many–quon systems[23–32]. In our recent paper[33], we successfully constructed a $q$-deformed many-body theory, the $q$-deformed BCS theory, for the most famous interacting fermionic system, the superconductor. By applying it to the $q$-deformed interacting Fermi gases, we obtained some interesting physical predictions[34].

Until now, most research works focus on the nonrelativistic systems or low energy systems in which no antiparticle excitations onset. It is natural to generalize the current studies to the high energetic/relativistic regime. In this paper, we try to make the first step, i.e. construct a many-body theory for the ideal $q$-deformed relativistic Fermi gas. Future studies on interacting $q$-deformed relativistic Fermi gases must be built based on this foundation. Note both particle and antiparticle excitations have their deformation parameters, we must carefully tune the parameters to get a self-consistent theory. It will be found that even in this noninteracting situation, the $q$-deformed relativistic Fermi gas already possesses many interesting features due to the deformation parameter $q$.

The rest of the paper is organized as follows. In Sec.II, we give a self-consistent deformed relation for operators of $q$-fermions and $q$-antifermions. We further construct a finite temperature many-body theory of $q$-deformed relativistic ideal Fermi gas based on the Green’s function formalism. In Sec.III, we derive the associated equations of states and discuss the thermodynamics. In Sec.IV, we give numerical analysis to give a deep understanding of the characteristics of such systems. The conclusion is summarized in the end.

II. THEORETICAL FRAMEWORK

Throughout this paper, we adopt the natural unit system with $\hbar = e = k_B = 1$ to simply the notations. In a relativistic $q$-deformed many-fermion system, there exist both fermionic and antifermionic excitations. Before starting to formulate a $q$-deformed algebra for these $q$-particles, we need to deal with an important question: how to assign the deformation parameters to $q$-fermions and $q$-antifermions respectively? Or more precisely, what is the relation between these two deformation parameters? It has been shown in [33, 35] that in a $q$-deformed many-particle system, the chemical potential is shifted by the deformation

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parameter. It is known that the chemical potential $\mu$ is defined by the energy change as one extra particle is added to a quantum system containing a large number of particles. Moreover, in a relativistic system, adding an antiparticle is equivalent to removing a particle. Thus, the chemical potential of the corresponding antiparticle is $-\mu$. For $q$-deformed many-fermion systems, the chemical potential $\mu$ of a $q$-fermion is shifted to $\mu + T \ln q$ where $T$ is the temperature[33]. Thus, it is reasonable to infer that the chemical potential of a $q$-antifermion is shifted by the deformation parameter as

$$-\mu - T \ln q = -\mu + T \ln q^{-1}. \quad (1)$$

That is to say, the deformation parameter for a deformed antifermion is $q^{-1}$ if the deformation parameter of the deformed fermion is $q$. Based on this analysis, the deformed canonical (anti)commutation relations can be constructed as follows. We introduce the field operator $a_k$ to denote annihilation operator for the $q$-fermion with momentum $k$, and $b_k^\dagger$ the creation operator for the $q$-antifermion with momentum $k$. We assume both types of operators satisfy the Viswanathan-Parthasarathy-Jagannathan-Chaichian (VPJC) algebra[29], i.e.,

$$a_k a_k^\dagger + q a_k^\dagger a_k = (2\pi)^3 \delta_{kk'}, \quad [\hat{n}_k^- a_k^\dagger] = - a_k \delta_{kk'}, \quad [\hat{n}_k^- a_k^\dagger] = a_k^\dagger \delta_{kk'},$$

$$q b_k b_k^\dagger + b_k^\dagger b_k = (2\pi)^3 \delta_{kk'}, \quad [\hat{n}_k^+ b_k^\dagger] = - b_k \delta_{kk'}, \quad [\hat{n}_k^+ b_k^\dagger] = b_k^\dagger \delta_{kk'},$$

$$a_k b_k + q b_k^\dagger a_k = 0, \quad b_k^\dagger b_k^\dagger + q a_k^\dagger b_k = 0,$$ \hspace{1cm} (2)

and all other anti-commutations vanish. Here $\hat{n}_k^\pm$ is the number counting operator for $q$-deformed fermions/antifermions, and the factor $(2\pi)^3$ is included for the convenience of normalization. Typically, the Hamiltonian of a noninteracting $q$-gas is given by

$$H = \sum_k [(e_k - \mu) \hat{n}_k^- + (e_k + \mu) \hat{n}_k^+], \quad (3)$$

where $m$ is the mass and $e_k = \sqrt{k^2 + m^2}$. $e_k = \mu$ is the energy dispersion of the $q$-deformed fermion/antifermion.

To formulate the many-body theory of a $q$-gas, we introduce the $q$-fermion field $\psi(x)$ in the coordinate space, which is given by Eq.(B1). We also introduce the 4-momentum $k^\alpha = (e_k, k)$, $\sigma^\alpha = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the pauli matrices to express our formulation in a compact form. In the imaginary-time formalism of finite temperature field theory, we define the finite temperature Heisenberg operator $\psi(x) = e^{iHt} \psi(x) e^{-iHt}$ where $x = (\tau, \vec{x})$ with $\tau = i\tau$ being the imaginary time. The field $\psi(x)$ contains both fermionic and anti-fermionic excitations, which can be expanded in the momentum space as

$$\psi(x) = \sum_k \frac{1}{\sqrt{e_k}} \left[ \frac{\sqrt{k^T \cdot \bar{\sigma}} n_k^R}{\sqrt{k^T \cdot \bar{\sigma}} n_k^L} \right] a_k \eta_k^L e^{-ik \cdot x} + \left[ \frac{\sqrt{k^T \cdot \bar{\sigma}} n_k^L}{\sqrt{k^T \cdot \bar{\sigma}} n_k^R} \right] b_k^\dagger \eta_k^R e^{ik \cdot x}, \quad (4)$$

where $k^\alpha = (e_k, \mu, k)$, $e^{\pm \eta_k \cdot \vec{x}} = e^{\pm (\phi_k + \vec{\eta}_k \cdot \vec{x} + \frac{1}{2} (\vec{x} \cdot \vec{x} + x^2)}$, $\eta_k^L$ and $\eta_k^R$ are the two eigenvectors of $\bar{\sigma} \cdot \vec{k}$ with different handedness, and the scalar product between two 4-vectors is given in the Appendix.A. Explicitly, $\vec{\sigma} \cdot \vec{k} \eta_k^{L,R} = \pm |k| \eta_k^{L,R}$ with

$$\eta_k^L = \begin{bmatrix} -\sin \frac{\theta_k}{2} e^{-i\phi_k} \\ \cos \frac{\theta_k}{2} \end{bmatrix}, \quad \eta_k^R = \begin{bmatrix} \cos \frac{\theta_k}{2} e^{i\phi_k} \\ \sin \frac{\theta_k}{2} \end{bmatrix},$$ \hspace{1cm} (5)

where $\theta_k$ and $\phi_k$ are the polar and azimuthal angels of vector $k$. It can be shown that the fermion field (for simplicity, $\psi(x)$ and $\psi(x)$ are still called fermion field hereafter) satisfies the following $q$-deformed anti-commutative relation (for details, please refer to Appendix.B)

$$\psi_a(x) \psi_b^\dagger(x') + q \psi_b^\dagger(x') \psi_a(x) = \delta_{ab} \delta(x-x'). \quad (6)$$

The equation of motion of the fermion field can be obtained by the Heisenberg equation $\frac{\partial \psi(x)}{\partial \tau} = [H, \psi(x)]$. Applying the algebras (2), and plugging in the expressions (3) and (4), we have

$$\gamma^0 \frac{\partial \psi(x)}{\partial \tau} = (i\vec{\gamma} \cdot \nabla - m + \mu \gamma^0) \psi(x), \quad (7)$$

where $\gamma^0$ and $\vec{\gamma} = (\gamma^1, \gamma^2, \gamma^3)^T$ are gamma matrices. Details can be found in the Appendix.C. A Lagrangian of which the Eular’s equation of motion is exactly Eq.(7) is given by

$$L = \int d^4x \bar{\psi}(i\gamma^\mu \partial_{\mu} - m + \mu \gamma^0) \psi, \quad (8)$$
where $\bar{\psi} \equiv \psi^\dagger \gamma^0$. The deformed Green’s function of the fermion field is defined as

$$G_{ab}(x,x') = -(T_x[\psi_a(x)\bar{\psi}_b(x')]) = -\langle \psi_a(x)\bar{\psi}_b(x')\rangle_0 \delta(\tau - \tau') + q\langle \bar{\psi}_b(x')\psi_a(x)\rangle_0 \delta(\tau' - \tau). \tag{9}$$

It is reasonable to assume that the deformed Green’s function has a translational symmetry in spacetime, i.e. $G(x,x') = G(x-x') \equiv G(\tau - \tau', x - x').$ Moreover, it can be shown that the deformed Green’s function has a different periodic property from that of the ordinary fermionic Green’s function$^[33]:$

$$G(-\beta < \tau - \tau' < \beta, x - x') = -qG(\tau - \tau' + \beta, x - x'), \tag{10}$$

where $\beta = 1/T$ is the inverse temperature. This leads to the fact that the fermionic Matsubara frequency gets an extra imaginary part, which can be absorbed into the chemical potential. The details will be shown later. By using the equations of motion (7), we have

$$[\gamma^0 \partial_x G(x,x')]_{ab} = -\delta(\tau - \tau')\gamma^0_{ac}(\bar{\psi}_c(x') + q\psi_c(x'))\gamma^0_{db} - \langle T_x(\gamma^0 \partial_x \psi(x)\bar{\psi}(x'))\rangle_{ab}$$

$$= -\delta_{ab}\delta(\tau - \tau')\delta(x - x') - [(i\bar{\gamma} \cdot \nabla - m + \mu\gamma^0)(T_x(\psi(x)\bar{\psi}(x')))]_{ab}. \tag{11}$$

Rearrange the equation we get

$$[-\gamma^0 \partial_x + i\bar{\gamma} \cdot \nabla - m + \mu\gamma^0]G(x,x') = \delta(x - x')\mathbf{1}_{4\times4} \tag{12}$$

which is the equation of motion for the deformed Green’s function. Using the property (10) and solving Eq.(12) in the momentum space, the solution to the deformed Green’s function is

$$G^{-1}(K) = \left( i\omega_n + \frac{\ln q}{\beta} + \mu \right)\gamma^0 - \bar{\gamma} \cdot K - m, \tag{13}$$

where $K = (i\omega_n, K)$ is the fermionic 4-momentum at finite temperature with $\omega_n$ being the ordinary fermionic Matsubara frequency $\omega_n = (2n + 1)\pi T$ ($n = 0, \pm 1, \pm 2, \ldots$). As we have pointed out, the Matsubara frequency obtains an imaginary part $\frac{\ln q}{\beta}$ in $q = T \ln q$. Thus, the chemical potential $\mu$ is shifted to $\mu + T \ln q$, which agrees with our former discussions at the beginning of the section. Introducing the energy projectors for particle and antiparticle sectors

$$\Lambda_\pm(K) = \frac{1}{2} \left( 1 \pm \frac{\gamma^0(\bar{\gamma} \cdot K + m)}{\epsilon_K} \right), \tag{14}$$

and taking the inverse of Eq.(13), the Green’s function can be expressed as

$$G(K) = \left[ \frac{\Lambda_+(K)}{i\omega_n - (\xi^+_K - \frac{\ln q}{\beta})} + \frac{\Lambda_-(K)}{i\omega_n + (\xi^-_K + \frac{\ln q}{\beta})} \right] \gamma^0, \tag{15}$$

where $\xi^\pm_K = \epsilon_K \pm \mu$. After taking complex continuation as $i\omega_n \to \omega + i0^+$, the two poles of the Green’s function indicate that the energy dispersions of quasi $q$-fermions/quasi $q$-antifermions are $\pm(\epsilon^+_K + T \ln q)$ respectively. However, the energy spectrum of a physical system can not be negative. Since adding an antiparticle is equivalent to removing a particle, the energy dispersion of the quasi $q$-antifermion is, in fact, $\xi^-_K + T \ln q$. Thus, the chemical potential of quasi $q$-fermions/quasi $q$-antifermions are shifted as $\pm(\mu + T \ln q)$ respectively, which indeed coincides with Eq.(1) and the related discussions. Hence, our choice of the deformation parameters is self-consistent.

In the nonrelativistic limit $|K| \ll m$, the energy dispersion of the $q$-fermion is $\xi^+_K \approx \frac{K^2}{2m} - (\mu - m)$, so the nonrelativistic result is recovered and the quantity $\mu - m$ plays the role of $\mu$. Moreover, since $\epsilon_K \approx m$ in the nonrelativistic limit, then $\Lambda_+(K) \approx 1$, $\Lambda_-(K) \approx 0$. Hence the main contribution to the Green’s function comes from the particle sector

$$G(K) \approx \frac{\gamma^0}{i\omega_n + \frac{\ln q}{\beta} - \xi^+_K}, \tag{16}$$

which essentially reproduces the non-relativistic Green’s function$^[36].$

III. EQUATION OF STATES AND THERMODYNAMICS

The most important equation of states is the particle number equation. It is obtained by the ensemble average of the deformed fermion number operator $\langle \psi^\dagger \psi \rangle$, which can be further evaluated by the Green’s function

$$N = \frac{1}{2} \sum_a \int d^3x \langle \psi_a^\dagger(x)\psi_a(x) \rangle = \frac{1}{2} \sum_a \int d^3x \langle \bar{\psi}_a^\dagger(x)\gamma^0_{ab}\psi_b(x) \rangle = \frac{1}{2q} \int d^3x \text{Tr}[G(x,x')\gamma^0].$$
\[\sum_{\text{fermions}} + \sum_{\text{antifermions}} \text{Tr}[G(i\omega_n, \mathbf{k})\gamma^0] = \frac{V}{2q} \sum_{\mathbf{k}} \left[ f(\xi_k^- - T \ln q) + f(-\xi_k^+ - T \ln q) \right],\]

where \( V \) is the volume, and the factor of 1/2 is included to remove the extra degrees of freedom. Omitting an infinitely large number \( \sum_{\mathbf{k}} \frac{1}{q} \), the particle number density \( n \equiv \frac{\beta}{\Omega} \) is given by

\[n = \sum_{\mathbf{k}} \left( \frac{1}{e^{\beta \xi_k^+} + 1} + \frac{1}{e^{\beta \xi_k^-} + 1} \right) \to \sum_{\mathbf{k}} \left( \frac{1}{e^{\beta \xi_k^+} + 1} - \frac{1}{q \beta q e^{\beta \xi_k^+} + 1} \right),\]

which is equal to the difference between the number densities of \( q \)-fermions and \( q \)-antifermions. That is, the quasiparticles give positive contribution while the anti-quasiparticles give negative contribution to the total particle number. This is consistent with the fact that they annihilate each other in a flash of energy when brought together. If \( q = 1 \), our result reduces to the known result of ordinary relativistic Fermi gases [37].

The particle number density can also be evaluated by taking derivative of the grand partition function \( Z = \text{Tr}(e^{-\beta H}) = \sum_n \langle n | e^{-\beta H} | n \rangle \). Here \( n \) denotes the eigenstates of Fock space. Previously we have concluded that the energy dispersions of quasi-\(-\)fermions/quasi-\(-\)antifermions are \( \xi^\pm \mp T \ln q \) respectively, then it is reasonable to evaluate the trace over the Fock space of quasi-\(-\)fermions/quasi-\(-\)antifermions as

\[Z = \prod_{\mathbf{k}} \left( \sum_{n_k^0=0,1} \sum_{\text{fermions}} e^{-\beta \xi_k^+ - T \ln q n_k^0} |n_k^+\rangle \langle n_k^+| \right) \left( \sum_{n_k^0=0,1} \sum_{\text{antifermions}} e^{-\beta \xi_k^- + T \ln q n_k^0} |n_k^-\rangle \langle n_k^-| \right) \]

\[= \prod_{\mathbf{k}} \left( 1 + q e^{-\beta \xi_k} \right) \left( 1 + q^{-1} e^{-\beta \xi_k} \right),\]

where \( z = e^{\beta \mu} \) is the fugacity. It is straightforward to verify that

\[n = \frac{1}{q} \partial \ln Z \frac{\partial \ln Z}{\partial z},\]

where the inclusion of the deformation factor \( \frac{1}{q} \) is consistent with our definition of the deformed Green’s function. The thermodynamical potential is defined as \( \Omega = -\frac{1}{\beta} \ln Z \). Thus, Eq.(20) is equivalent to the well-known form (also in a deformed form)

\[n = -\frac{1}{q} \frac{\partial \Omega}{\partial \mu} \]

The pressure is given by \( PV = -\Omega \), which leads to another equation of state

\[PV = T \ln Z = -\sum_{\mathbf{k}} \left( \xi_k^- + \xi_k^+ - T \ln \frac{1}{e^{\beta \xi_k} + q} - T \ln \frac{1}{e^{\beta \xi_k} + 1} \right).\]

The entropy can be obtained by taking derivative of the partition function, or thermodynamical potential, as \( S = -\frac{\partial \Omega}{\partial T} \). A straightforward calculation shows

\[S = \sum_{\mathbf{k}} \left( f(\xi_k^- - T \ln q) \ln \left[ \frac{1}{q} f(\xi_k^- - T \ln q) \right] + f(-\xi_k^- + T \ln q) \ln \left[ f(-\xi_k^- + T \ln q) \right] + f(\xi_k^+ + T \ln q) \ln \left[ q f(\xi_k^+ + T \ln q) \right] + f(-\xi_k^+ - T \ln q) \ln \left[ f(-\xi_k^+ - T \ln q) \right] \right).\]

The internal energy is defined as

\[E = -\frac{1}{q} \frac{\partial \ln Z}{\partial \beta} + \mu n = \frac{1}{q} (\Omega + TS) + \mu n.\]

Substitute Eqs.(18), (22) and (23), we get

\[E = \sum_{\mathbf{k}} \left( \frac{\xi_k}{e^{\beta \xi_k} + q} + \frac{1}{q \beta q e^{\beta \xi_k} + 1} \right).\]
where the degenerate ordinary Fermi gas when the Fermi functions are immediately found by calculations, the thermodynamic potential and the internal energy are respectively given by Eqs. (22), (18) and (24) respectively become

\[ PV = -\Omega = \frac{gV}{6\pi^2} \int_m^\infty \, \text{d} \epsilon \frac{e^\beta (\epsilon - \mu) + 1}{\epsilon^2 - m^2 \epsilon + 1}, \]  

(27)

\[ N/V = \frac{gV}{2q\pi^2} \int_m^\infty \, \text{d} \epsilon \frac{\sqrt{\epsilon^2 - m^2 \epsilon} - \sqrt{\epsilon^2 - m^2 \epsilon}}{\epsilon^2 - m^2 \epsilon + 1}, \]  

(28)

and

\[ E = \frac{gV}{2q\pi^2} \int_m^\infty \, \text{d} \epsilon \frac{\sqrt{\epsilon^2 - m^2 \epsilon} + \sqrt{\epsilon^2 - m^2 \epsilon}}{\epsilon^2 - m^2 \epsilon + 1}. \]  

(29)

Due to the relativistic energy dispersion, there is no way to get an explicit equation of state as \( PV = \frac{2}{3} E \) in the case of nonrelativistic ideal Fermi gases. In the zero temperature limit, the \( q \)-deformed fermi distribution shown in Eq.(18) reduces to a step function

\[ \left( \frac{1}{\epsilon^2} - \frac{1}{\epsilon^2} \right)_{\epsilon < \mu} \rightarrow \left\{ \begin{array}{ll} 0 & \epsilon > \mu \\ 1/q & \epsilon < \mu \end{array} \right\} = \theta(\mu - \epsilon). \]  

(30)

Here an implicit premise that \( \epsilon + \mu > 0 \) at \( T \rightarrow 0 \) has been applied. Otherwise, only the second term of Eq.(18) gives non-zero contribution and the number density is accordingly negative. This means the high energetic quasi antiparticle is dominant, which is impossible since all kinetic degrees of freedom become frozen at \( T \rightarrow 0 \). Therefore, the zero temperature behavior of the \( q \)-deformed relativistic ideal Fermi gas is quite similar to the ordinary Fermi gas except the number density is normalized by \( q \) and the energy dispersion is different. Thus, there also exists the Fermi level \( E_F = \mu \) at \( T = 0 \), which is the largest energy that a \( q \)-fermion can possess. Using Eq.(30), the number equation (28) is readily evaluated as

\[ n = \frac{g}{6q\pi^2} (\mu^2 - m^2)^{3/2}. \]  

(31)

This leads to the expression of Fermi level

\[ E_F = \mu(T = 0) = \sqrt{\frac{6q^2\pi^2 n}{g} + m^2} = \sqrt{k_F^2 + m^2}, \]  

(32)

where

\[ k_F = \left( \frac{6q\pi^2 n}{g} \right)^{1/3} = (3q^2\pi^2 n)^{1/3} \]  

is the Fermi momentum. Note the shift \( T \ln q \) of the chemical potential has no effect at \( T = 0 \). Similarly, by straightforward calculations, the thermodynamic potential and the internal energy are respectively given by

\[ PV = \frac{gV}{6\pi^2} \left[ \sqrt{\mu^2 - m^2} \left( -\frac{5m^2}{8} \mu + \frac{\mu^3}{4} \right) + \frac{3}{8} m^4 \ln \left( \frac{\mu}{m} + \sqrt{\frac{\mu^2}{m^2} - 1} \right) \right], \]  

(33)
Similarly, we define the variable \( y \equiv \beta(e + \mu + \frac{\ln a}{\beta}) \). Thus, the second term on the right-hand-side of Eq. (27) is rewritten as

\[
I_2 = \frac{1}{\beta^2} \int_{\beta m + \beta \mu + \ln q}^{\infty} dy \frac{[y - \beta \mu - \ln q]^2 - (\beta m)^2}{e^y + 1}.
\]

Note this integral is exponentially small in the limit \( \beta m + \beta \mu + \ln q \to \infty \), then this term can be safely ignored.

Now, change the variable \( x \to -x \) in the first integral, and apply the identity \((e^{-x} + 1)^{-1} = 1 - (e^{-x} + 1)^{-1}\), we get

\[
I_1 = \frac{1}{\beta^2} \int_{0}^{\infty} dx \frac{(x + \alpha)^2 - b^2}{e^x + 1} + \frac{1}{\beta^2} \int_{0}^{\infty} dx \frac{(x + \alpha)^2 - b^2}{e^x + 1} - \frac{1}{\beta^2} \int_{-\infty}^{-b} dx \frac{[(a - x)^2 - b^2]}{e^x + 1}.
\]

The chemical potential \( \mu \) is always larger than the mass \( m \) at low enough temperatures, then \( a - b \) is extremely large. Thus, the last term of Eq. (38) is exponentially small in this limit. Moreover, the numerator in the second integral can be approximated as

\[
[(x + \alpha)^2 - b^2] - [(a - x)^2 - b^2] \approx 6(a^2 - b^2)x + \cdots, \quad \text{when } a, b \to \infty.
\]

A straightforward calculation gives the asymptotic expansion to \( I_1 \)

\[
I_1 = \left[ \frac{1}{8} \left( \mu + \frac{\ln q}{\beta} \right) \sqrt{\left( \mu + \frac{\ln q}{\beta} \right)^2 - m^2} - \frac{1}{2} \left( \mu + \frac{\ln q}{\beta} \right)^2 - 5m^2 \right] + \frac{3}{8} m^4 \ln \left( \frac{\beta \mu + \ln q}{\beta m} + \sqrt{\left( \frac{\beta \mu + \ln q}{\beta m} \right)^2 - 1} \right) - \frac{\beta \mu + \ln q}{2 \beta} \left( \mu + \frac{\ln q}{\beta} \right)^2 - m^2 \right]^{1/2} + \cdots.
\]

Therefore, the thermodynamic potential is

\[
P V \xrightarrow{T \to 0} \frac{gV}{6\pi^2} I_1
\]

where \( I_1 \) is given by Eq. (40).

### IV. NUMERICAL ANALYSIS

To visualize more properties of the \( q \)-deformed relativistic ideal Fermi gas, we give a quantitative study based on numerical analysis. This can be accomplished by solving the number equation (18). For convenience, we fix the number density and solve \( \mu \) at certain choices of \( T \) and \( q \). Eq. (33) shows \( n = \frac{k_F}{4\pi} \) at \( T = 0 \), i.e. \( n \) depends on \( q \). To give a better comparison between the results at different \( q \)'s, we choose the unit \( k_F \) satisfying \( n = \frac{k_F}{3\pi^2} \), which is the Fermi momentum of an un-deformed relativistic
ideal Fermi gas. For simplicity, we still use the notation \( k_F \) as our unit of momentum, which is different from Eq. (33). The units of temperature and energy are chosen as \( k_B T_F = E_F = \sqrt{k_F^2 + m^2} \). Moreover, to show how relativistic an ideal Fermi gas is, we introduce the ratio \( \zeta = \frac{k_F}{m} \). The system under considering is in the non-relativistic limit if \( \zeta \ll 1 \), in the ultrarelativistic limit if \( \zeta \gg 1 \). Since \( k_F \) is fixed, \( m \) changes as \( \zeta \) varies.

In Figure 1, we plot the chemical potential vs the temperature at several different \( q \)'s for both \( q \)-deformed nonrelativistic and relativistic Fermi gases. Panel (a) shows the result for a \( q \)-deformed nonrelativistic ideal Fermi gas, of which the number equation is [33]

\[
\beta n = \sum_k \frac{1}{e^{\beta \mu} + q}.
\]  

Panel (b) shows a comparative study of the characteristics of a \( q \)-deformed relativistic ideal Fermi gas with \( \zeta = 1.0 \). The typical deformation parameters \( q = 0.5 \) (black), \( 1.0 \) (red), and \( 1.5 \) (blue) are chosen for comparisons. In the nonrelativistic case, obviously \( \mu = E_F \) at \( T = 0 \) for ordinary ideal Fermi gas (\( q = 1.0 \)). When \( q \) becomes larger (smaller) than 1.0, \( \mu(T = 0) \) becomes larger (smaller) than the ordinary result \( E_F \). This is not surprising since \( \mu(T = 0) = \frac{(6\pi^2 n^2)^{2/3}}{(2m)} \) where \( n \) is a fixed value. At high temperatures, the chemical potential becomes negative in all three situations according to Fig. 1(a). Hence the high-temperature behaviors of \( q \)-deformed nonrelativistic ideal gases are quite similar to its ordinary counterpart. This is because the properties of quantum gases become “classical” if no relativistic effects are considered, i.e. all quantum effects are smeared by thermal fluctuations, which is independent of what type of quantum algebra is imposed.

In Fig. 1(b) we give the sketch of the results in the moderate relativistic regime with \( \zeta = 1.0 \) as a direct comparison. The low-temperature property is similar to that of Fig. 1(a) except \( \mu(T = 0) \neq E_F \) when \( q = 1.0 \). This is because the energy dispersion has changed if a Fermi gas becomes relativistic. However, high-temperature behavior is significantly different. This does make sense since more antiparticles are excited due to the high energetic thermal fluctuation at higher temperature. Thus, the quantum
effect is still dominant at high temperatures. In the ordinary situation with \( q = 1.0 \), \( \mu \) approaches 0 but is still positive. As we have pointed out previously, for a relativistic gas, \( \mu - m \) plays the role of nonrelativistic chemical potential \( \mu_{\text{nr}} \) if \( |k| \) is small. Hence, approximately \( \mu \approx \mu_{\text{nr}} + m \), which is always positive. Interestingly, when \( q > 1.0 \) (\( < 1.0 \)), \( \mu \) becomes negative (positive) at high temperatures. The reason is outlined as follows. In the deformed situations with \( q \neq 1.0 \), it is \( \mu + T \ln q \) that plays the role of the ordinary chemical potential. To clarify this more clearly, we define the shifted chemical potential \( \mu' = \mu + T \ln q \), which equals to the ordinary chemical potential at \( q = 1.0 \). If \( \mu' \) behaves like the ordinary chemical potential at \( q \neq 1.0 \), then the second term of \( \mu = \mu' - T \ln q \), i.e. \( -T \ln q \), gives significant effects at high temperatures and makes the asymptotic behavior of \( \mu \) the same as in Fig.1(b).

Now, we need to make sure that \( \mu' \) does behave as expected in the relativistic regimes, and the result is outlined in Fig.2. The situations with \( q = 0.5 \) (\( < 1.0 \)) and \( q = 2.0 \) (\( > 1.0 \)) are respectively shown in panels (a) and (b). \( \zeta = 0.1, 0.5, 1 \) and 2 are chosen to denote the regimes from weak to strong relativistic limits. All \( \mu' \)'s have a similar shape to the ordinary chemical potential shown in Fig.1(b). The only explicit difference is the value of \( \mu'(T = 0) \), which depends on the ratio \( \zeta \). In the weak relativistic regime with \( \zeta = 0.1 \), both panels indicate \( \mu'(T = 0) \approx m/\zeta \) since all kinetic degrees of freedom are frozen at \( T = 0 \).

Since the deformation parameter \( q \) has a nontrivial effect on the chemical potential, it will, in turn, produce an implication on the particle number. A quick inspection to Fig.1(b) reveals the fact that \( \mu \) has a minimum at some finite temperature when \( q = 0.5 \). Thus, it might be easier to create antiparticles within the neighborhood of that point. Moreover, if a system is in a deeper relativistic regime, it is also easier to produce antiparticles, which gives a negative contribution to the total particle number. Hence, we consider a more extreme situation with \( q = 10 \) (ultrarelativistic) and \( T = 10T_F \) (high temperature). We plot \( N \) vs \( \mu \) at different \( q \)’s. As expected, if we take an extreme choice of \( q \), i.e. \( q = 0.1 \), the total particle number becomes negative in some region of \( \mu \). That is, there are more antiparticles than particles. This is essentially different from ordinary noninteracting Fermi gases. It might be possible in future studies to use a simple theory of \( q \)-gas to simulate the phase of the very early universe by adjusting the deformation parameter \( q \).

![Figure 3](image-url)  
**Figure 3:** (Color online). Particle number as a function of \( \mu \) at \( \zeta = 10 \) and \( T = 10T_F \). The black, red, blue and pink solid lines correspond to the situations with \( q = 0.1, 0.5, 1.0 \) and 2.0 respectively.

V. CONCLUSION

In summary, we construct a self-consistent \( q \)-deformed algebra for a quantum system containing both particles and antiparticles. Based on this algebra, we further formulate the finite-temperature many-body theory for the \( q \)-deformed relativistic ideal Fermi gas. By applying this formalism, we study the thermodynamic properties of this system. Interestingly, the \( q \)-deformed relativistic Fermi gas presents significantly different features from ordinary Fermi gases even in the noninteracting scenario. The deformation parameter has a notable effect on the chemical potential of both particle and antiparticle excitations, which induces the reversion of particle/antiparticle numbers when \( q \) is very small in the ultrarelativistic limit. These findings will lay a solid ground for future studies on the \( q \)-deformed relativistic interacting Fermi gas.

This work was supported by the National Natural Science Foundation of China (Grant No. 11674051).

Appendix A: Conventions of Notations

In this paper we adopt the Weyl or chiral representation of the gamma matrices,
\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix}, \quad \gamma_S = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \]  

(A1)

Define the 4-vector pauli matrices as
\[ \sigma^\mu = (1, \vec{\sigma}), \quad \vec{\sigma}^\mu = (1, -\vec{\sigma}), \]  

(A2)

then
\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}. \]  

(A3)

The scalar between two 4-vectors, for example, \( k^\mu \) and \( \sigma^\mu \), is defined as \( k \cdot \sigma = \epsilon_k - \vec{k} \cdot \vec{\sigma} \), while \( i k \cdot x = i \epsilon_k t - i \vec{k} \cdot \vec{x} = \epsilon_k \tau - i \vec{k} \cdot \vec{x} \).

**Appendix B: q-deformed Anti-commutative Relation**

Note the time-independent fermion field can be expanded as
\[ \psi(x) = \sum_k \frac{1}{\sqrt{\epsilon_k}} \left\{ \begin{pmatrix} \sqrt{k \cdot \sigma} \eta^L_k \\ \sqrt{k \cdot \sigma} \eta^R_k \end{pmatrix} a_k e^{i k \cdot x} + \begin{pmatrix} \sqrt{k \cdot \sigma} \eta^R_k \\ -\sqrt{k \cdot \sigma} \eta^L_k \end{pmatrix} b_k e^{-i k \cdot x} \right\}, \]
\[ \psi^\dagger(x) = \sum_k \frac{1}{\sqrt{\epsilon_k}} \left\{ \begin{pmatrix} \eta^L_k \sqrt{k \cdot \sigma} & \eta^R_k \sqrt{k \cdot \sigma} \end{pmatrix} a_k^\dagger e^{i k \cdot x} + \begin{pmatrix} \eta^R_k \sqrt{k \cdot \sigma} & -\eta^L_k \sqrt{k \cdot \sigma} \end{pmatrix} b_k^\dagger e^{-i k \cdot x} \right\}. \]  

(B1)

By applying these expressions and the algebras (2), it can be shown that
\[ \psi_a(x)\psi_b^\dagger(x') + q\psi_b(x')\psi_a(x) \]
\[ = \sum_k \left( \frac{2\pi}{\epsilon_k} \right)^3 \left( \frac{(e_k + |k|)\eta^L_k}{m_n^L \eta^L_k} + \frac{m_n^R \eta^R_k}{(e_k - |k|)\eta^L_k} \right) \left( e^{ik(x-x')} + \frac{(e_k - |k|)\eta^L_k}{m_n^L \eta^L_k} + \frac{m_n^R \eta^R_k}{(e_k + |k|)\eta^L_k} \right) e^{-ik(x-x')} \],

(B2)

where we have applied the fact \( (k \cdot \sigma)\eta^L_k = (e_k + |k|)\eta^L_k \) and \( (k \cdot \sigma)\eta^R_k = (e_k - |k|)\eta^L_k \). When the vector changes as \( k \rightarrow -k \), the polar and azimuthal angles satisfy \( \theta_{-k} = \pi - \theta_k \) and \( \phi_{-k} = \pi + \phi_k \). Note
\[ \eta^L_k \eta^L_k = \begin{pmatrix} \sin^2 \frac{\theta_k}{2} - \sin \frac{\theta_k}{2} \cos \frac{\theta_k}{2} e^{-i\phi_k} \\ -\sin \frac{\theta_k}{2} \cos \frac{\theta_k}{2} e^{i\phi_k} \end{pmatrix}, \]
\[ \eta^R_k \eta^R_k = \begin{pmatrix} \cos^2 \frac{\theta_k}{2} \sin \frac{\theta_k}{2} \cos \frac{\theta_k}{2} e^{-i\phi_k} \\ \sin^2 \frac{\theta_k}{2} \cos \frac{\theta_k}{2} e^{i\phi_k} \end{pmatrix}. \]  

(B3)

Hence we have \( \eta^L_k \eta^L_k = \eta^R_k \eta^R_k \) and \( \eta^L_k \eta^L_k \eta^L_k + \eta^R_k \eta^R_k \eta^R_k = 1_{2\times 2} \). Therefore, by changing the variable as \( k \rightarrow -k \) in the second term of Eq.(B2), we have
\[ \psi_a(x)\psi_b^\dagger(x') + q\psi_b(x')\psi_a(x) = (2\pi)^3 \sum_k \left( \frac{\eta^L_k \eta^L_k}{m_n^L \eta^L_k} \right) e^{ik(x-x')} = (2\pi)^3 \sum_k (1_{2\times 2} \otimes \eta^L_k \eta^L_k)_{ab} e^{ik(x-x')} . \]  

(B4)

Similarly
\[ \psi_a(x)\psi_b^\dagger(x') + q\psi_b(x')\psi_a(x) = (2\pi)^3 \sum_k \left( \frac{\eta^R_k \eta^R_k}{m_n^R \eta^R_k} \right) e^{-ik(x-x')} = (2\pi)^3 \sum_k (1_{2\times 2} \otimes \eta^R_k \eta^R_k)_{ab} e^{-ik(x-x')} . \]  

(B5)
Then evaluate \((\text{B4})+(\text{B5})\), we get

\[
\psi_a(x)\psi_b^\dagger(x') + g\psi_b^\dagger(x')\psi_a(x) = (2\pi)^3 \sum_k \left\{ (1_{2\times 2} \otimes 1_{2\times 2})_{ab} \cos(k \cdot (x - x')) + i \left[ 1_{2\times 2} \otimes (\eta^L_k \eta^R_k - \eta^R_k \eta^L_k) \right]_{ab} \sin(k \cdot (x - x')) \right\} \\
= (2\pi)^3 \sum_k \delta_{ab} e^{i k \cdot (x - x')} + e^{-i k \cdot (x - x')} \overline{e}^{2\pi} \delta e^{i \theta_k} = 0.
\]

where in the second line we have applied the fact that every element of \(\eta^L_k \eta^L_k - \eta^R_k \eta^R_k\) contains \(e^{i \theta_k}\) and \(\int_{-\pi}^{\pi} d\theta_k e^{i \theta_k} = 0\).

Appendix C: Equation of Motion of the Fermion Field

Applying the Heisenberg equation and the algebras \((2)\), we have

\[
\gamma^0 \frac{\partial \psi(x)}{\partial \tau} - (i \vec{\gamma} \cdot \nabla - m + \mu \gamma^0) \psi(x) \\
= \sum_k \frac{1}{\sqrt{e_k}} \gamma^0 \left\{ \left[ \sqrt{k \cdot \sigma \eta^L_k} + \sqrt{k \cdot \bar{\sigma} \eta^R_k} \right] (\epsilon_k - \mu) a_k e^{-ik \cdot x} + \left[ -\sqrt{k \cdot \sigma \eta^L_k} - \sqrt{k \cdot \bar{\sigma} \eta^R_k} \right] (\epsilon_k + \mu) b_k e^{ik \cdot x} \right\} \\
- \sum_k \frac{1}{\sqrt{e_k}} \left\{ (-\vec{\gamma} \cdot k - m + \mu \gamma^0) a_k e^{-ik \cdot x} + (\vec{\gamma} \cdot k - m + \mu \gamma^0) b_k e^{ik \cdot x} \right\} \\
= \sum_k \frac{1}{\sqrt{e_k}} \left\{ \left[ \sqrt{k \cdot \sigma \eta^L_k} - \sqrt{k \cdot \bar{\sigma} \eta^R_k} \right] a_k e^{-ik \cdot x} + \left[ \sqrt{k \cdot \bar{\sigma} \eta^L_k} + \sqrt{k \cdot \sigma \eta^R_k} \right] b_k e^{ik \cdot x} \right\} \\
= 0,
\]

where we have applied the fact \((k \cdot \sigma)(k \cdot \bar{\sigma}) = (k \cdot \bar{\sigma})(k \cdot \sigma) = m^2\). This leads to the equation of motion \((7)\) for \(\eta\)-deformed fermionic field.

Appendix D: Expression of the Green’s Function

It is easy to verify the following properties of the energy projectors

\[
\Lambda^2_+(k) = \Lambda_+(k), \quad \Lambda_+(k)\Lambda_-(k) = \Lambda_-(k)\Lambda_+(k) = 0
\]

\[
\Lambda_+(k) + \Lambda_-(k) = 1_{4\times 4}, \quad \Lambda_+(k) - \Lambda_-(k) = \frac{1}{\epsilon_k} \gamma^0 (\vec{\gamma} \cdot k + m).
\]

The gamma matrices satisfy

\[
(a + by^0 + c\vec{\gamma} \cdot k + d\gamma^0 \vec{\gamma} \cdot k)^{-1} = \frac{-a + by^0 + c\vec{\gamma} \cdot k + d\gamma^0 \vec{\gamma} \cdot k}{b^2 - a^2 + (d^2 - c^2)|k|^2}.
\]

By using this equality we have

\[
G(K) = \left[ (i\omega_n + \frac{\ln q}{\beta} + \mu)\gamma^0 - \vec{\gamma} \cdot k - m \right]^{-1} \\
= \frac{(i\omega_n + \frac{\ln q}{\beta} + \mu)\gamma^0 - \vec{\gamma} \cdot k + m}{(i\omega_n + \frac{\ln q}{\beta} + \mu)^2 - m^2 - |k|^2} \\
= \frac{[((i\omega_n + \frac{\ln q}{\beta} + \xi_k^+)\Lambda_+(k) + (i\omega_n + \frac{\ln q}{\beta} - \xi_k^-)\Lambda_-(k))]\gamma^0}{(i\omega_n + \frac{\ln q}{\beta} + \xi_k^+)((i\omega_n + \frac{\ln q}{\beta} - \xi_k^-))}.
\]
\[ \Lambda_+^\dagger(k) \left( \frac{\Lambda_-(k)}{i\omega_n + \frac{\ln q}{\beta} + \xi_k} + \frac{\Lambda_-(k)}{i\omega_n + \frac{\ln q}{\beta} + \xi_k} \right) \\nu_0^0 \]  

(D4)