How quark hadron duality in QCD may work.

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Abstract

We pursue the issue of the local quark-hadron duality at high energies in two- and four-dimensional QCD. A mechanism of the dynamical realization of the quark-hadron duality in two-dimensional QCD in the limit of large number of colors, \( N_c \to \infty \), (the 't Hooft model) is investigated. We argue that a similar mechanism of dynamical smearing may be relevant in four-dimensional QCD. Although particular details of our results are model-dependent (especially in the latter case), the general features of the duality implementation conjectured previously get further support.

I. INTRODUCTION

In the recent years the focus of applications of the operator product expansion (OPE) has shifted towards the processes with the essentially Minkowskian kinematics. Perhaps, the most well-known example is the theory of the inclusive decays of heavy flavors (for a review see e.g. Ref. [3]). This fact, as well as the increasingly higher requirements to the accuracy of predictions, puts forward the study of the quark–hadron duality as an urgent task.

A detailed definition of the procedure which goes under the name of the quark–hadron duality (a key element of every calculation referring to Minkowskian quantities) was given in Refs. [2,4]. In a nutshell, a truncated OPE is analytically continued, term by term, from the Euclidean to the Minkowski domain. A smooth quark curve obtained in this way is postulated to coincide at high energies (energy releases) with the actual hadronic cross section.

If duality is formulated in this way, it is perfectly obvious that deviations from duality must exist. In Ref. [3] it was shown that one of the sources is the asymptotic divergence of high orders in the power series. If we knew the leading asymptotic behavior of the high order terms in the power series we could predict the pattern of the duality-violating contributions at high energies. Unfortunately, very little is known about this aspect of OPE, and we have

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1Moreover, usually one deals with the practical version of OPE, see Ref. [4] for further details
to approach the problem from the other side—either by modelling the phenomenon [4] or by studying some general features of the appropriate spectral densities. One can also try to approach the problem purely phenomenologically (for recent attempts see e.g. [5–7]).

An illustrative spectral density, quite instructive in the studies of the issue of the quark-hadron duality, was suggested in [3],

\[ \text{Im}\Pi = \text{Const.} N_c \pi \sum_{n=1}^{\infty} \delta(\mathcal{E} - n) \]  

(1.1)

where

\[ \mathcal{E} = \frac{s}{\Lambda^2}, \]

and we dropped an inessential constant in front of the sum. The color factor \( N_c \) is singled out for convenience. The imaginary part above represents, for positive values of \( s \), a sum of infinitely narrow equidistant resonances, with equal residues. It defines \( \Pi(q^2) \) everywhere in the complex plane \( q^2 \), through the standard dispersion relation, up to an additive constant which can be adjusted arbitrarily. It is not difficult to see that the corresponding correlation function

\[ \Pi(q^2) = -N_c \left[ \psi(\varepsilon) + \frac{1}{\varepsilon} \right] \]  

(1.2)

where \( \psi \) is the logarithmic derivative of Euler’s gamma function, and

\[ \varepsilon = -\frac{q^2}{\Lambda^2} = -\mathcal{E}. \]

In the Minkowski domain \( \mathcal{E} \) is positive, in the Euclidean domain \( \varepsilon \) is positive. Then, the asymptotic expansion of \( \Pi(q^2) \) in deep Euclidean domain is well known,

\[ \Pi(q^2) \to -N_c \left[ \ln \varepsilon + \frac{1}{2\varepsilon} - \sum_{n=1}^{\infty} \frac{B_{2n} \varepsilon^{-2n}}{2n} \right] \]  

(1.3)

where \( B_{2n} \) are the Bernoulli numbers. At large \( n \) they grow factorially, as \( B_{2n} \sim (2n)! \) (see [8], page 23), and are sign alternating.

Although the spectral density (1.1) is admittedly a model, it was argued [3] that a similar factorial growth of the coefficients in the power (condensate) series is a general feature.

The spectral density (1.1) may be relevant in the limit of the large number of colors, \( N_c \to \infty \), when all mesons are infinitely narrow. This limit is not realistic, however, and, moreover, in this limit the local quark-hadron duality, as we defined it, never takes place since even at high energies the hadronic spectral density never becomes smooth. One can smear it by hand, of course, but then deviations from the local duality will be determined not only by the intrinsic hadronic dynamics, as is the case in the real world, but also by details of the smearing procedure—the weight function chosen for smearing, the interval of smearing and so on. In the actual world the smearing occurs dynamically, since at high energies the resonance widths become non-negligible. The limits of \( E \to \infty \) and \( N_c \to \infty \) are not interchangeable.
In this work we study more realistic (dynamically smeared) spectral densities compatible with all general properties of Quantum Chromodynamics. Starting from infinitely narrow resonances, as in Eq. (1.1), we then introduce finite widths, ensuring smooth behavior. Various dynamical regimes leading to specific patterns of duality violations are considered, with a special emphasis on the high-order asymptotics of the power series. Technically, in the first part of the paper the problem of duality is analyzed in the two-dimensional ’t Hooft model [9] (see also [11–13]). We see that there indeed exists the dynamic smearing of the spectral density. This smearing occurs already if one takes into account the fact that the resonance widths are nonzero and calculates the relevant spectral densities in the Breit-Wigner approximation. We see that there are two characteristic scales in the problem. Starting from the first scale the spectral density is well approximated by the oscillations round the average value given by the OPE asymptotics. Starting from the second scale the oscillations become small and local duality is valid.

The $\psi$-function model was originally suggested in Ref. [3] for the heavy-light quark systems. In Ref. [14] it was noted that it was more appropriate for the light-light quark systems because in the heavy-light systems the resonances are not expected to be equidistant. Straightforward quasiclassical estimates yield in this case that the meson energies (measured from the heavy quark mass) asymptotically scale as $\sqrt{n}$. (More elaborated models, e.g. the one of Ref. [13], based on a relativistic linear potential, yield not only the $\sqrt{n}$ law, but also the coefficient in front of $\sqrt{n}$.) In the present paper we further develop the $\psi$-function model adapting it for the light-quark systems.

II. RESONANCE WIDTHS IN T’HOOF MODEL

Our main original contribution to t’Hooft model is the calculation of the first nonvanishing corrections to the widths of the resonances of t’Hooft model. Then, using these widths, we shall find the asymptotic behavior of the polarization operator of two scalar currents. This resonance-saturated polarization operator will be referred to as phenomenological. We will confront it with the truncated power expansion. The difference between these two expressions presents duality violation.

It was shown by t’Hooft that the 2d QCD is exactly solvable in the limit $N_c \to \infty$. The bound state spectrum includes an infinite number of bound states whose masses $m_n$ lie on the almost linear trajectory. The properties of these bound states are described by the t’Hooft equation

$$\mu_n^2 \phi_n(x) = \frac{(\gamma^2 - 1)}{x(x-1)} = \int_0^1 \frac{\phi_n(y)dy}{(x-y)^2}. \tag{2.1}$$

Here the integral is understood as a principal value, The variable $x$ is the momentum fraction of the meson carried by the antiquark (in the infinite momentum frame), while $1-x$ is that of the quark; $\phi_n(x)$ is the wave function of the $n$-th bound state, $\mu_n^2 = m_n^2/\mu^2$

The integral equation (2.1) must be solved with the boundary conditions:

$$\phi_n(x) \to x^\beta(1-x)^\beta \tag{2.2}$$

for $x \to 0, 1$. Here
\[ \pi \beta \cotg(\pi \beta) = 1 - \gamma^2 \]  

(2.3)

and \( \gamma^2 = m_q^2/\mu^2 \), \( m_q \) is the mass of the quark.

Below we shall be interested in the massless case \( \gamma = 0 \) and in the number of flavors equal to 1. In this case the asymptotic behavior of \( \mu^2_n \) \textit{versus} \( n \) is given by equation

\[ \mu^2_n = \pi^2 n \left(1 + O(\ln(n)/n + ...) \right). \]  

(2.4)

As for the wave functions \( \phi_n(x) \), their calculation is quite complicated even numerically, especially in the massless case. The recent calculations [15], exploiting a new improved numerical procedure (spline method), show that for the massless case these wave functions are very accurately approximated by

\[ \phi_n(x) = \sqrt{2} \cos(\pi n x). \]  

(2.5)

Note that the wave functions 2.3 satisfy the proper boundary conditions for the massless case.

Let us now calculate the meson widths. As it was already mentioned, in the limit \( N_c \to \infty \) the bound states in the t’Hooft model are stable, their widths vanish. However, once one takes into account the leading \( 1/N_c \) correction, the resonances begin to decay. In the first order in \( 1/N_c \) expansion there are only two-particle decays \( a \to b + c \). The relevant coupling constants \( g_{abc} \) are given by [11,13,15,12,13]:

\[ g_{abc} = \frac{(g^2 N_c)}{\pi} \sqrt{\frac{\pi}{N_c}(1 - (-1)^{(\sigma_a + \sigma_b + \sigma_c)})(f_{abc}^+ + f_{abc}^-)} \]  

(2.6)

Here \( \sigma_a \) is the parity of the \( a \)-th resonance. The constants \( f_{abc}^\pm \) are determined from the following expressions:

\[ f_{abc}^\pm = \frac{1}{1 - \omega_\pm} \int_0^{\omega_\pm} \phi_a(x) \phi_b(x/\omega_\pm) \Phi_c \left( \frac{x - \omega_\pm}{1 - \omega_\pm} \right) dx, \]

\[ - \frac{1}{\omega_\pm} \int_{\omega_\pm}^1 \phi_a(x) \Phi_b(x/\omega_\pm) \phi_c \left( \frac{x - \omega_\pm}{1 - \omega_\pm} \right) dx. \]  

(2.7)

Here \( \omega_\pm \) are two roots of the algebraic equation corresponding to the mass-shell condition,

\[ m_a^2 = \frac{m_b^2}{\omega} + \frac{m_c^2}{(1 - \omega)}. \]  

(2.8)

Two different indices of \( \omega \) correspond to two different modes of the two-dimensional decay: in the rest frame of the resonance \( a \) the resonance \( b \) can go to the right and the resonance \( c \) to the left, and \textit{vice versa}.

The function \( \Phi_a(x) \) is defined as

\[ \Phi_a(x) = \int_0^1 dy \phi_a(y)/(x - y)^2. \]
It is easy to carry the calculation of the couplings $f_{abc}$ analytically (see Ref. [1]) and express them via integral sinus and integral cosinus functions. Using the above expressions for the decay couplings one can readily calculate the resonance widths in the leading $1/N_c$ approximation. They are given by

$$
\Gamma_a = \frac{1}{8m_a^2} \sum_b \sum_c \frac{(m_a^2 + m_b^2 - m_c^2)g_{abc}^2}{\sqrt{I(m_a^2, m_b^2, m_c^2)}},
$$

(2.9)

where $I$ is the standard “triangular” function,

$$
I(x, y, z) = \frac{1}{4}[m_a^2 - (m_b + m_c)^2][m_a^2 - (m_b - m_c)^2].
$$

(2.10)

The sum in Eq. (2.9) runs over all mesons $b$ and $c$ whose masses are lighter than that of $a$.

Details of our calculation of the resonance widths are described in Ref. [1]. The calculation was done numerically using the trial wave functions (2.3). Our task was to establish the asymptotic behavior of the widths, as a function of the excitation number, at large values of $n$, in the leading $1/N_c$ approximation (all widths are proportional to $1/N_c$). After computing the overlap integral we performed the summation over $b$ and $c$. The result for the widths exhibits a remarkable pattern. The widths of the individual levels oscillate near a smooth square-root curve, see Fig. 1. This plot shows the width of the $a$-th state versus $a$, up to $a = 500$. The result of averaging over the interval of 20 resonances is depicted in Fig. 2. We see that the curve of the resonance widths $\Gamma(a) \equiv \Gamma_a$ is very well approximated by the function

$$
\Gamma(a) = \frac{A\mu}{\pi^3 N_c} \sqrt{a} \left(1 + O(1/a)\right),
$$

(2.11)

where $\mu^2 = g^2 N_c / \pi$.

Since the square-root law (2.11) for the (averaged) widths is valid in such a large interval of the excitation numbers and turns out to be so accurate, it seems to be doubtless that this formula could be obtained analytically. This is an interesting question by itself, especially in the four-dimensional QCD. Unfortunately, we were unable to find analytic solution so far.

The numerical value of the constant $A$ is

$$
A \sim 0.44.
$$

(2.12)

Below the above result for the (averaged) widths will be used for determining of the asymptotic behavior of the polarization operator.

Concluding this section let us note that the same square-root was reported previously in Ref. [19]. We failed to reproduce the arguments of this work leading to the square-root law, however. Moreover, what is even more important, the constant $A$ in Ref. [19] is claimed to be proportional to $1/\sqrt{m}$ (!), and, thus, blows up for massless quarks. This poses perplexing questions. The coincidence looks completely accidental.
III. QUARK-HADRON DUALITY IN 2D QCD

Once we had found the resonance widths, we can calculate the polarisation operator in the Breit-Wigner approximation. Here we shall consider the most interesting case of the polarisation operator of the two scalar currents. Let us start from this polarisation operator in the $N_c \to \infty$ limit \[11,18\].

The polarisation operator is given by the correlator

$$\Pi(q^2) = \int d^2 x e^{ixq} < 0\{j(x), j(0)\}|0>$$

Here $j(x)$ is the scalar current:

$$j(x) = \bar{q}(x) q(x)$$

and $q(x)$ is the quark field. It was shown in Ref. \[11\] that this polarisation operator can be calculated explicitly in the $N_c \to \infty$ limit and is given by

$$\Pi(q^2) = \frac{N_c}{\pi} \sum_{n=1}^{n=\infty} \frac{f_n^2}{q^2 - m_n^2 + i\epsilon}$$

(3.2)

It was shown in Ref. \[18\] that in order for eqs. 3.2 and the perturbation theory for QCD2 to be compatible, the coefficients $f_n$ for sufficiently large $n$ must be independent of $n$. Then, taking into account the linear dependence of the mass squared on $n$ one can approximate the polarisation operator 3.1 for sufficiently large $q^2$ by $\psi$-function \[18\]:

$$\Pi(q^2) - \Pi(0) \sim \frac{N_c}{\pi} \psi(-q^2/\mu^2)$$

(3.3)

Here $\lim_{n\to\infty} f_n^2 \sim f^2 = \mu^2 \pi^2$.

Consider now what will happen if we take into account the finite widths of the resonances. We shall calculate the polarisation operator in the Breit-Wigner approximation. In order to do it we shall first find the inverse propagator of the $n$-th bound state:

$$\Pi_n(q^2 + i\epsilon) = q^2 - m_n^2 + \Sigma(q^2 + i\epsilon)$$

(3.4)

Using the procedure described in Ref. \[19\] it is easy to prove (see Ref. \[1\] for details) that

$$\Sigma(q^2 + i\epsilon) \sim \frac{Bq^2}{N_c\pi} \log(-q^2/q^2 - i\epsilon)$$

(3.5)

The latter equation is valid for $q^2 \geq \bar{q}^2$, where $\bar{q}^2$ is some intermediate scale, and $B = A/\pi^4$.

Then one obtains for the polarisation operator

$$\Pi(q^2) = \frac{N_c}{\pi} \sum_n \frac{f_n^2}{q^2 - m_n^2 + \frac{Bq^2}{N_c\pi} \log(-q^2/q^2 - i\epsilon)}$$

(3.6)

(for the nonphysical sheet). Now we can use the known asymptotic behaviour of $m_n^2$ and $f_n^2$ to sum the series 3.6. We immediately obtain:
\[ \Pi(q^2) - \Pi(0) \sim \frac{N_c}{\pi} \psi\left(-\frac{q^2 + \frac{Bq^2}{N_c\pi} \log(q^2/q) + iBq^2/N_c}{\mu^2\pi^2}\right) \] (3.7)

In other words we obtain that the polarisation operator on the unphysical sheet is the psi-function of the complex argument. This is our main result. In the limit \( N_c \to \infty \) we clearly recover eq. (3.3).

We can now study the duality violation and the operator product expansion for the polarisation operator (3.1).

Let us start from the imaginary part. There are two possible patterns of behaviour of the imaginary part of the polarisation operator as a function of \( q^2 \): Euclidean kinematics corresponds to \( q^2 \leq 0 \) and Minkowskian kinematics corresponds to \( q^2 \geq 0 \). Note that in the Euclidean domain the imaginary part of the polarisation operator (3.6) is zero. Indeed, the logarithm in the argument of the \( \psi \) function does not have any imaginary part, and the entire polarisation operator is real. Thus, the imaginary part of the polarisation operator is zero for \( q^2 \leq 0 \) as it must be. Consider now the imaginary part in the Minkowskian domain (i.e. the spectral density). In the limit \( N_c \to \infty \) the imaginary part of the polarisation operator is the sum of the delta-functions, corresponding to the poles of the psi-function:

\[ \text{Im}(\Pi(q^2) - \Pi(0)) \sim \frac{f^2 N_c}{\pi} \sum_n \delta(q^2 - n\mu^2\pi^2). \] (3.8)

However, once we take into account the leading \( 1/N_c \) correction, the situation immediately changes. Instead of the sum of the delta functions, one obtains the oscillations of the imaginary part round the constant value \( N_c/\pi \) with decreasing amplitude. Note that the latter is just the average value of the imaginary part of (3.8) over the interval \( q^2 \). Thus we see the example of dynamic smearing: instead of infinite peaks, QCD dynamically produces decreasing oscillations round average smooth function.

Let us now check that this is indeed the case for the concrete example of our polarisation operator (3.6). In order to find the relevant imaginary part explicitly, we shall make use of the reflection property of \( \psi \)-function:

\[ \psi(z) = \psi(-z) - \pi\text{ctg}(\pi z) - 1/z. \] (3.9)

Then, using eq. (3.9) one obtains

\[
\text{Im}(\Pi(q^2) - \Pi(0)) = \frac{N_c}{\pi} \left\{ \left( \text{Im}\psi((q^2 + \frac{Bq^2}{N_c\pi} \log(q^2/q) + iBq^2/N_c)/(\pi^2\mu^2)) \right) \\
+ \text{Im}\pi\text{ctg}(\pi(q^2 + \frac{Bq^2}{N_c\pi} \log q^2/q^2 + iBq^2/N_c)/(\pi^2\mu^2)) \\
+ \text{Im}\left( \frac{\pi^2\mu^2}{q^2 + \frac{Bq^2}{N_c\pi} \log q^2/q^2 + iBq^2/N_c} \right) \right\}
\] (3.10)

It is easy to see that a sum of the first and the third terms in eq. (3.10) is the smooth function of \( q^2 \) (for positive \( q^2 \)). Numerically these functions are well approximated by the OPE (asymptotics).
Consider now the second term in eq. (3.10). For \( N_c \to \infty \) this term has simple poles in the points where

\[
q^2 = \mu^2 \pi^2 n,
\]
i.e. the imaginary part of this term is just the sum of the delta functions (3.8).

Consider now the case of the large but finite \( N_c \). Then for \( q^2 \geq \pi^2 \mu^2 \) the imaginary part of the polarisation operator (3.10) is well approximated by the imaginary part of the ctg term in eq. (3.10):

\[
\text{Im}(\Pi(q^2) - \Pi(0)) \sim -N_c \frac{\text{sh}(2y)}{\text{ch}(2y) - \cos(2x)}.
\]

This means that for \( y \to \infty \)

\[
\text{Im}\Pi(q^2) \sim -N_c(1 + 2 \exp(-2y) \cos(2x) + ...)
\]

Here \( x = q^2/(\pi \mu^2) + O(1/N_c) \) and \( y = Bq^2/(N_c \pi \mu^2) \).

We can now see the behaviour of the imaginary part of the polarisation operator. For \( q^2 \geq \pi^2 \mu^2 \) this behaviour is well approximated by the ctg term (eq. 3.11). The poles in the imaginary part of eq. 3.11 disappear for arbitrary nonzero \( y \), i.e. for every finite \( N_c \). Instead we have peaks with the amplitude given for \( q^2 \sim \mu^2 \pi^2 \) by \( N_c/(Bq^2) \). The amplitude of these peaks goes to \( \infty \) as \( N_c \to \infty \).

For

\[
q^2 \geq N_c \pi \mu^2/(2B)
\]

the asymptotic behaviour will be given by eq. (3.12). The oscillations decrease exponentially and local duality is established. At these \( q^2 \) one may use the OPE. For \( q^2 \) smaller than the scale given by eq. (3.13) the oscillations dominate and we can speak only about global duality. For \( q^2 \) much larger than this scale one can safely use the OPE. The corresponding inaccuracy decreases exponentially as \( \exp(-2q^2 B/(N_c \pi \mu^2)) \).

Numerically the typical behaviour of the spectral density is depicted in Fig. 2 for \( \mu^2 = 0.01, N_c = 10 \). The function F1 depicts the (normalised) behaviour of the full \( \psi \) function expression for the spectral density, F2-of the ctg term in eq. (3.7) and F3-the exponentially decreasing term F3.

Consider now the full polarisation operator (3.6). For \( q^2 \leq 0 \) the polarisation operator is evidently given by the \( \psi \) function of the positive real argument and is a smooth function with no singularities. It is well approximated by its asymptotic expansion 3.12:

\[
\Pi(q^2) - \Pi(0) = \frac{N_c}{\pi} (\log(-q^2/\mu^2 \pi^2))
\]

\[+ \sum \frac{(\mu \pi)^{4n-4} B_{2n}}{2n} q^{2n} + O(1/N_c))
\]

(3.14)

Here \( B_{2n} \) are the Bernulli numbers. Note that the expansion (3.14) is not Borel summable. Indeed, for large \( n \) the ratio \( a_{n+1}/a_n \), where \( a_n \) is the n-th term of the series (3.14) becomes
\[ \Delta \Pi(q^2) \sim \exp\left( \frac{2q^2}{\mu^2 \pi} \log \frac{-2q^2}{\lambda^2} \right). \]  

Note that it is senseless to continue analytically the exponentially decreasing term \[3.15\] into the Minkowski domain, although it has the multiplier that begins to oscillate upon such continuation. This teaches us that it is extremely dangerous to continue different parts of the polarisation operator analytically from Euclidean to Minkowski domain.

The \(1/N_c\) terms in the Euclidean domain are only small corrections to the \(N_c \to \infty\) expression and can be neglected numerically.

We can now carry the straightforward analysis of the large \(q^2\) behaviour of the asymptotic expansion of the polarisation operator. This is easily done in the same way as it was done for the imaginary part.

We immediately see that in the Minkowski domain the polarisation operator is given by the sum of the asymptotic expansion \[3.14\] (continued analytically into the Minkowski domain) and the oscillating part that for \(q^2 \geq N_c \pi \mu^2/(2B)\) is equal to

\[ \Pi^{\text{osc}}(q^2) \sim 2 \frac{N_c}{\pi} \sin\left( \frac{2q^2}{(\pi \mu^2)} \right) \exp\left(-\frac{2Bq^2}{(N_c \pi \mu^2)}\right). \]  

We once again see an oscillating part with the exponentially decreasing amplitude of the oscillations that must be added to the smooth OPE function.

We conclude that there are indeed two characteristic scales in the problem. For \(q^2 \geq \pi^2 \mu^2\) the polarisation operator is dominated by oscillations, and these oscillations begin to decrease exponentially at the scale given by eq. \[3.13\]. Starting from the latter scale the amplitude of the oscillations become smaller than the leading term in the OPE and there is sense to speak about local duality, that establishes itself for \(q^2\) much bigger than the scale of eq. \[3.13\]. The corresponding inaccuracies decrease exponentially. On other hand the global duality holds for all values of \(q^2\).

### IV. OPE AND THE POLARISATION OPERATOR IN 4D QCD.

In this section we shall consider the OPE in 4d QCD. Once again, we consider the polarisation operator of 2 vector currents and calculate it for large absolute values of \(q^2\) for both Euclidean and Minkowski domains.

#### A. Polarisation operator of vector currents in the Veneziano model

Our goal in this subsection will be to study the polarisation operator of two vector currents:

\[ \Pi_{\mu\nu}(q^2) = \int \exp iqx < 0|^T\{j_\mu(x)j_\nu(0)\}|0 > d^4x. \]  

Here \(j_\mu(x)\) is the vector current,
Due to the conservation of vector current the polarisation operator can be represented as

$$
\Pi_{\mu\nu}(q^2) = (q^\mu q^\nu - q^2 g_{\mu\nu}) \Pi(q^2).
$$

The polarisation operator can be calculated in the Euclidean domain using the OPE:

$$
\Pi(q^2) \sim \log -q^2 + a_0/q^4 + \ldots
$$

Usually in order to calculate $\Pi(q^2)$ in the physical, Minkowski domain we just analytically continue eq. 4.4 into the Minkowski domain. In particular, the logarithm acquires the constant imaginary part. Then we get

$$
\Pi(q^2) \sim \pi + O(1/q^2)\ldots
$$

This is not the whole story however. Indeed, we know from our experience with QCD$^2$ that in addition to the smooth constant there may be oscillations.

The question of oscillations is especially interesting since in QCD the imaginary part of the polarisation operator of vector currents is directly related to the cross-section of the $e^+e^-$ annihilation into hadrons.

**B. Polarisation operator and resonance widths**

We have seen already the connection between the resonance masses and widths asymptotics and the asymptotic of the polarisation operator in the Breit-Wigner approximation in 2d QCD. The same connection exists in 4d QCD. The problem is, however, that we do not know how to calculate the relevant resonance masses and widths in a model independent way, like it was done in QCD$^2$. We need to know the masses and widths of the radial excitations with the quantum numbers of $\rho$-meson created by the vector current. In fact, we are interested only in the asymptotic behaviour of these widths and masses. We shall calculate masses of these excitations using the Veneziano model. The relevant radial excitations all have the same spin and lie on the daugter trajectories to the $\rho$ meson trajectory (see Fig. 4):

$$
m_n^2 = n/\alpha'.
$$

Here $\alpha' \sim 1 \text{GeV}^{-2}$.

Although there exists a working model for the resonance masses, that is in agreement with the experimentally observed Regge behaviour, virtually nothing is known about the dependence of the resonance widths on the number $n$ of the excitation. Based on our experience for 2d QCD it is natural to assume the power like behaviour:

$$
\Gamma_n \sim \frac{A}{N_c}(m_n^2)^{1/2+\gamma}.
$$

For $N_c \to \infty$ the widths $\Gamma_n \to 0$. The constant $A$ is some unknown proportionality constant. The parameter $\gamma$ parametrises our ignorance of the 4d resonance widths. The only known
theoretical estimate of $\gamma$ was made by Green and Veneziano [20] on the basis of the string model, and their result is $\gamma = -1/2$. Here we shall parametrise the widths by the apriory arbitrary parameter $\gamma$. Using eq. 4.7 one immediately obtains the $q^2$ dependence of the self energy part of the inverse meson propagator:

$$\text{Im}\Sigma(q^2 + i\epsilon) = \frac{A}{N_c} q^{2(1+\gamma)}. \quad (4.8)$$

Then one can use the dispersion relations to obtain $\Sigma(q^2 + i\epsilon)$. In order to do it we shall need to take the integral

$$I = \frac{Aq^4}{N_c\pi} \int_{s_0}^{\infty} \frac{s^{\gamma-1}}{s' - q^2} ds'. \quad (4.9)$$

. As usual we calculate this integral for negative $q^2$ and then analytically continue to positive $q^2$. Making substitution $s'/(-q^2) = v/(1-v)$ we obtain that

$$I = \frac{A(-q^2)^{(1+\gamma)}}{N_c\pi} \left( B(\gamma, 1-\gamma) - \frac{1}{\gamma} (x/(1+x))^\gamma F(\gamma, \gamma, 1+\gamma, x/(1+x)) \right). \quad (4.10)$$

Note that for the $\gamma \to 0$ one obtains

$$\Sigma(q^2) = \frac{A}{N_c\pi} q^2 \log \left( -q^2/s_0 \right). \quad (4.11)$$

Finally for integer negative $\gamma$ one needs the dispersion relation without substractions:

$$\Sigma(q^2) = \frac{A}{N_c\pi} q^2 \frac{1}{(\alpha'q^2)^\gamma} \log(-q^2/s_0). \quad (4.12)$$

Note that for each case $\Sigma(q^2)$ is real for $q^2 \leq 0$ and acquires the imaginary part for $q^2 \geq 0$. Now one can immediately obtain the polarisation operator:

$$\Pi(q^2) = \sum_n \frac{f_n^2}{q^2 - n/\alpha' + \Sigma(q^2 + i\epsilon)} = f^2 \alpha' \psi(-q^2 + \Sigma(q^2 + i\epsilon)\alpha') \quad (4.13)$$

Here we used the fact that, analogously to QCD, in order for the sum over resonances to be consistent with the leading asymptotic behaviour, $f_n^2$ must be independent of $n$ for large $n$.

For $q^2 \leq 0$ (Euclidean domain) the polarisation operator is well approximated by OPE:

$$\Pi(q^2) \sim f^2 \alpha' \log(-q^2\alpha') + +O(1/q^2, 1/N_c)\ldots \quad (4.14)$$
The fact that we evidently get here the terms of the order $1/q^2$ reminds us that some of our results are model dependent. It is evident, that by changing $n$ dependence, i.e. taking into account that $f_n$ are constant only for large $n$, one can also change the behaviour of subleading terms in OPE. We do not write these terms explicitly due to their model dependence.

Consider now the $q^2 \geq 0$ case -Minkowski domain. We can analytically continue eq. (4.14) into that domain. The polarisation operator evidently acquires the imaginary part:

$$\text{Im}\Pi(q^2) \sim (f^2\alpha')\pi + ...$$ (4.15)

All the functions that we obtain by analytical continuation of the OPE are evidently smooth. In addition, in the Minkowski kynematics one gets the oscillating part. For sufficiently large $q^2$ and nonpositive $\gamma$,

$$\text{Im}\Pi(q^2) \sim \pi\alpha' f^2 \text{Im ctg}(\pi(q^2\alpha' + iy))$$ (4.16)

Here

$$y = Aq^2\alpha'/N_c$$

for $\gamma = 0$ and

$$y = A|q^2|^{1+\gamma}/(N_c\sin(\pi\gamma))$$ (4.17)

for noninteger $\gamma$. For integer negative $\gamma$

$$y = (A/N_c)q^2 \frac{1}{(\alpha'q^2)^{|\gamma|}}$$ (4.18)

Note that for negative $\gamma \leq -1$ one gets nondecreasing oscillations. This shows that in real QCD one must have the constraint

$$\gamma \geq -1.$$ (4.19)

Moreover, we also see that

$$\gamma \leq 0$$

Indeed, we have seen that if $\gamma$ is positive, than there appears the term in $\Sigma$ that leads to the rise of the real part of the polarisation operator quicker than $q^2$ (although this term is suppressed by $1/N_c$). This will lead to the curving upwards of Regge trajectories that looks quite unlikely at present.

Numerically, the behaviour of the polarisation operator and its imaginary parts are depicted in Figs.3 for the sample value $A/N_c=0.3$ and $\gamma = -1/2$ ($\gamma = 0$ behaviour is very similar to the one given by Fig.2). In these figures we depict as $F0$ the imaginary part of the $\psi$ function, as $F1$ the ctg approximation, and as $F2$ the exponentially decreasing asymptotic oscillations. Already starting from $q^2 \geq 1/\alpha'$ the full polarisation operator is relatively well approximated by ctg. Starting from the scale

$$q^2 \geq (N_c/(A\alpha'))^{1/(1+\gamma)}$$ (4.20)

the full polarisation operator operator is represented by the OPE plus exponentially decreasing asymptotics. For scales much larger than the one given by eq. (4.20) the local duality is restored and deviations from it are exponential.
V. CONCLUSION

We now can answer the basic questions asked in the introduction. First, how the local quark-hadron duality works, and, second when one can use the OPE.

For the 2d QCD one can treat these two questions in a model independent way (the deviations from the Breit-Wigner approximation can be estimated and are actually irrelevant for the problem at hand). We found the polarisation operator of two scalar currents and studied its duality properties.

We see that in the Minkowski domain there are two characteristic scales in the problem. The first scale is $\Lambda_1 = \mu \pi$, the second is given by $\Lambda_2^2 = N_c \pi \mu^2/(2B)$ For the momentum $q^2$ between these two scales the oscillations dominate, and only global duality works. For the $q^2$, larger than $\Lambda_2$ the local duality works. The deviations from local duality are only exponential. Consequently, for scales $q^2 \geq \Lambda_2^2$ one can use the local duality to calculate physical quantities due to the exponential decrease of the amplitude of the oscillations.

We thus can use the OPE to calculate global averages, i.e. to calculate some integrals over the interval of $q^2$ much bigger than the distance between the resonances for almost all values of $q^2$. However the use of OPE based on local duality is much more restricted, and we can say now that there is a dynamically generated scale in the problem starting from which the local duality is valid.

Moreover, we have seen that in the Minkowski domain there are always corrections to OPE, in the best case ($q^2 \geq \Lambda_2^2$ these corrections decrease exponentially.)

The next question is what can be learned for 4d QCD. Here we have seen that if Veneziano model and power like behaviour of the resonance widths are indeed true, the local duality is established for large $N_c$ in the same way as in 2d QCD. Once again, using this time Veneziano model, we see there two relevant scales are $1/\alpha'$ and $(N_c/(A\alpha'))^{1/(1+\gamma)}$ given by eq. [4.20]. Between these scales the spectral density is dominated by oscillations with decreasing amplitudes. For the $q^2$ larger than these scales the spectral density behaves like the OPE (analytically continued, term by term, as discussed in introduction) plus the exponentially decreasing oscillating term, that decreases like $\exp(-2A\alpha'q^{2(1+\gamma)})$ where A is some proportionality constant.

We can compare our results with the predictions of instanton vacuum model [4]. We see, similar to that model that there are three characteristic areas of momenta squared, divided by two scales. The deviations from duality (i.e. corrections to the OPE) decrease exponentially for the scales larger than the biggest of these two scales, like in Ref. [4].
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FIG. 1. Comparison of the decay width with the trial function. The oscillating curves are widths $\Gamma_a$ of resonances $a$'s in (a), and $\Gamma_a$ averaged over 20 resonances in (b), while the smooth curve is $\Gamma_a = 0.442a^{0.5}$. (All widths are given in the units $\mu/\pi^3N_c$)
FIG. 2. Polarisation functions in 2d QCD. Here: (a) $F_0 = \text{Im} \frac{1}{\pi \mu^2} \psi(-q^2 + \frac{Bq^2}{N_c \pi} \log(q^2/q^2) + iBq^2/N_c)/(\mu^2 \pi^2)$, (b) $F_1 = \text{Im} \frac{1}{\pi \mu^2} \text{ctg} \pi(q^2 + \frac{Bq^2}{N_c \pi} \log q^2/q^2 + iBq^2/N_c)/(\pi^2 \mu^2))$, (c) $F_2 = -\frac{1}{\pi \mu^2}(1 + 2 \exp(-2Bq^2/\pi N_c \mu^2) \cos(2q^2/(\pi \mu^2)))$, and (d) for $F_1/F_0$, (e) for $F_2/F_0$. We take $N_c = 10$, $B = 4.4 \times 10^{-3}$ and $\mu = 0.01$. We use here obvious units of dimension.
FIG. 3. Polarisation functions in 4d QCD for $\gamma = -0.5$ and $A/N_c = 0.3$. (a) $F_0 = \text{Im} \psi(-q^2 + A/(N_c\pi)q^2(1+\gamma)\text{ctg}(\pi \gamma) + iA/(N_c\pi)q^2(1+\gamma))$, (b) $F_1 = \pi \text{Im} \text{ctg}(\pi q^2 + A/(N_c\pi)q^2(1+\gamma)\text{ctg}(\pi \gamma) + iA/(N_c\pi)q^2(1+\gamma))$, (c) $F_2 = -\pi(1 + \exp(-2A/(N_c\pi)q^2(1+\gamma))\cos(2\pi(q^2 + A/(N_c\pi)q^2(1+\gamma)\text{ctg}(\pi \gamma))))$, and (d) for $F_1/F_0$, (e) for $F_2/F_0$. We use here obvious units of dimension.