THE MONOID OF INJECTIVE AND EXTENSIVE PARTIAL TRANSFORMATIONS OF A CHAIN WITH THREE ELEMENTS IS NONFINITELY BASED

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ABSTRACT. We show that the monoid of all injective and extensive partial transformations of a chain with three elements admits no finite basis of its identities. This completes solving of the finite basis problem for the monoids in the basic frame of partial transformation monoids posed by Volkov.

A monoid $M$ is said to be finitely based if all identities holding in $M$ follow from a finite set of such identities; otherwise $M$ is called nonfinitely based. The finite basis problem for a class of monoids consists in determining which monoids in this class are finitely based and which are not.

Recall that a partial transformation $\alpha$ of $\{1, 2, \ldots, n\}$ is said to be
- total if its domain $\text{dom} \alpha$ is the whole set $\{1, 2, \ldots, n\}$;
- injective if $i \alpha = j \alpha$ implies $i = j$ for all $i, j \in \text{dom} \alpha$;
- order-preserving if $i \leq j$ implies $i \alpha \leq j \alpha$ for all $i, j \in \text{dom} \alpha$;
- extensive if $i \leq i \alpha$ for every $i \in \text{dom} \alpha$.

In the survey [10, Problem 6.3], Volkov posed the finite basis problem for the monoids in the so-called basic frame consisting of the monoids of partial transformations whose elements are characterized by a combination of some of the four fundamental properties of transformations defined above. This basic frame shown in Fig. 1 is the subsemilattice of the meet-semilattice of all semigroups of partial transformations of $\{1, 2, \ldots, n\}$ generated by:
- $T_n$, the monoid of all total transformations;
- $I_n$, the monoid of all injective partial transformations;
- $PO_n$, the monoid of all order-preserving partial transformations;
- $PE_n$, the monoid of all extensive partial transformations.

For $n \leq 2$, all the monoids in the basic frame except $I_2$, $PE_2$, $PO_2$, $POE_2$ and $POI_2$ have at most five elements and so finitely based by Trahtman [9]. For every $n$, the monoid $S_n$ of all total and injective transformations is nothing but the group of all permutations of $\{1, 2, \ldots, n\}$ and so is finitely based by the theorem of Oates and Powell [5]. For $n > 2$, the monoids $I_{n-1}$, $O_n$, $PO_{n-1}$, $POI_{n-1}$ and $T_n$ are nonfinitely based by the general result of M. Sapir [6]. Volkov [11] has solved the finite basis problem for the series $OE_n$: the
monoid $OE_n$ is finitely based if and only if $n \leq 4$. Gol’dberg [1] has shown that, for each $n \geq 1$, the monoids $PEI_n$ and $POEI_n$ satisfy the same identities and that these monoids are nonfinitely based whenever $n \geq 4$. Further, Goldberg [2] has proved the same result for the monoids $E_{n+1}$, $PE_n$ and $POE_n$. Shortly after, Lee [3] has verified that the monoids $E_3$, $PE_2$ and $POE_2$ are finitely based. Finally, a finite basis of identities for the monoids $E_4$, $PE_3$ and $POE_3$ has been found in Li and Luo [4]. So, the finite basis problem remains open only for the following two monoids in the basic frame: the monoid $PEI_3$ of all injective and extensive partial transformations of the set $\{1, 2, 3\}$ and the monoid $POE_3$ of all injective, order-preserving and extensive partial transformations of the same set. The goal of the present note is to solve the finite basis problem for these two monoids, completing the classification of finitely based monoids in the basic frame.

Our main result is the following theorem.

**Theorem.** The monoids $PEI_3$ and $POE_3$ are nonfinitely based.

A description of the identities of the monoid $PEI_3$ found in [1] plays a critical role in the proof of the theorem. To reproduce it, we remember some definitions and notation. We fix a countably infinite set $X$ called *alphabet* and denote by $X^*$ the free monoid over $X$; elements of $X^*$ are called *words*, while elements of $X$ are said to be *variables*. Words unlike variables are written in bold. The alphabet of a word $w$ is denoted by $\text{alph}(w)$. A word $u = x_1x_2 \cdots x_k$, where $x_1, x_2, \ldots, x_k \in X$, is a *scattered subword* of a word $v$ if there are words $v_0, v_1, \ldots, v_{k-1}, v_k \in X^*$ with

\[
v = v_0x_1v_1 \cdots x_kv_{k-1}v_k.
\]

A scattered subword $u = x_1x_2 \cdots x_n$ of a word $v$ is said to be *unambiguously scattered* if $v$ has a unique decomposition of the form (1).

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1Notice that, just recently, O. Sapir and Volkov [8] found out a general reason that forces the monoids $E_{n+1}$, $OE_{n+1}$, $PE_n$, $POE_n$, $PEI_n$ and $POEI_n$ with $n \geq 4$ to be nonfinitely based.
Since the word \( w \) is nonfinitely based, it suffices to find a property \( \phi \) of \( w \).

**Proof.** We will use the syntactic method (see [10, Section 4] or [7, Fact 2.1]). To show that \( \phi \) for any \( n \geq 1 \) if \( x \geq y \).

\[
\phi \quad \text{in less than} \quad x \geq y \quad \text{if} \quad c \in \text{alphabet} \quad \text{then} \quad \phi \quad \text{in less than} \quad x \geq y \quad \text{if} \quad c \in \text{alphabet}.
\]

Now consider an arbitrary word \( w \) such that \( u_n \approx v \) holds in \( M \). Since \( t \) is an unambiguously scattered subword of \( u_n \), Proposition 1 implies that \( w = w_1 t \).

Further, \( y_1 = \text{unambiguously scattered in} \quad u_n \). Now Proposition 1 applies again, yielding that \( w_1 = x^{y_1} y_2 \).

Now consider the unambiguously scattered subword \( y_1 \) of \( u_n \). In view of Proposition 1, we have \( w_2 = y_2 w_3 \).

Continuing this process, we finally arrive at the decomposition \( w = x^{y_1} y_2 \).

Since the word \( w \) is arbitrary, it follows that

\[
\begin{align*}
\phi & = x^{y_1} y_2 \quad \text{and} \quad \phi \quad \text{in less than} \quad x^{y_1} y_2 \quad \text{if} \quad c \in \text{alphabet} \quad \text{then} \quad \phi \quad \text{in less than} \quad x^{y_1} y_2 \quad \text{if} \quad c \in \text{alphabet}.
\end{align*}
\]
with \( m \geq 1, \ell \geq 0 \) and \( k, m + \ell \geq 2 \).

Arguing by contradiction, suppose that there is an occurrences of \( x \) between \( 2v'y_1 \) and \( 1t \) in \( v \), or in other words \( \ell > 0 \). There exists \( a \in \text{alph}(t) \) such that the image of some occurrence of \( a \) in \( t \) under \( \varphi \) contains \( (m+1)v'^x \). In view of Proposition \( 1 \) \( \text{alph}(s) = \text{alph}(t) \). Then, by \( 2 \), we have \( y_1, t \notin \text{alph}(\varphi(a)) \) because \( y_1x \) and \( xt \) are not factors of \( u \). Hence \( \varphi(a) \) is a power of \( x \). Further, there is \( b \in \text{alph}(t) = \text{alph}(s) \) such that the image of some occurrence of \( b \) in \( t \) under \( \varphi \) contains \( 1v't \). Clearly, the variables \( a \) and \( b \) are distinct. Since the variable \( t \) occurs exactly once in \( v \) by \( 2 \), the variable \( b \) must occur exactly once in \( t \). Finally, since the identity \( s \approx t \) is in less than \( n \) variables, there is \( b' \in \text{alph}(t) = \text{alph}(s) \) such that the image of some occurrence of \( b' \) in \( t \) under \( \varphi \) contains the factor \( 1v'y_1v'y_{j+1} \) for some \( j < n \). The fact that \( \varphi(a) \) is a power of \( x \) implies that \( b' \neq a \), whence \( b' \neq b \). In view of \( 2 \), the word \( v \) contains the unique factor of the form \( y_jv'y_{j+1} \). Hence the variable \( b' \) must occur exactly once in \( t \). Then \( bb' \) is an unambiguously scattered subword of \( t \). According to Proposition \( 1 \) \( bb' \) must be unambiguously scattered in \( s \) as well and both the variables \( b \) and \( b' \) occur exactly once in \( s \). By the choice of the variable \( a \), some occurrence of this variable lies between \( 1b' \) and \( 1b \) in \( t \). Then some occurrence \( a \) must lie between \( 1b' \) and \( 1b \) in \( s \) by Proposition \( 1 \) contradicting \( 2 \).

**Proof of the theorem.** Clearly, \( \text{alph}(u_n) = \text{alph}(v_n) \) for each \( n \geq 1 \). Further, it is routine to check that \( \{t\} \) and \( \{y_1^2, y_2^2, \ldots, y_n^2, tx\} \) are the sets of unambiguously scattered subwords of length 1 and 2, respectively, in both \( u_n \) and \( v_n \). This fact together with Proposition \( 1 \) imply that the monoid \( PEI_3 \) satisfies the identity \( u_n \approx v_n \) for any \( n \geq 1 \). Now Proposition \( 2 \) apples, yielding that the monoid \( PEI_3 \) is nonfinitely based. By \( 1 \) Proposition \( 2 \), the monoids \( POEI_3 \) and \( PEI_3 \) have the same identities. The proof is thus complete. \( \square \)

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