Plane Partition Polynomial Asymptotics

Robert P. Boyer and Daniel T. Parry

Abstract. The plane partition polynomial $Q_n(x)$ is the polynomial of degree $n$ whose coefficients count the number of plane partitions of $n$ indexed by their trace. Extending classical work of E.M. Wright, we develop the asymptotics of these polynomials inside the unit disk using the circle method.

1. Introduction

A plane partition $\pi$ of a positive integer $n$ is an array $[\pi_{i,j}]$ of nonnegative integers such that $\sum \pi_{i,j} = n$ while its trace is the sum of the diagonal entries $\sum \pi_{i,i}$. The asymptotics of the plane partition numbers $PL(n)$, the number of all plane partitions of $n$, was found by Wright [8] in 1931 using the circle and saddle point methods. In this paper, we study the asymptotics of polynomial versions $Q_n(x)$ of the plane partitions of $n$ given by

**Definition 1.** Let $Q_n(x)$ be the $n$-th degree polynomial given by $\sum_{k=1}^{n} pp_k(n)x^k$ where $pp_k(n)$ is the number of plane partitions of $n$ with trace $k$. These polynomials have generating function

\[
P(x, u) = \prod_{m=1}^{\infty} \frac{1}{(1 - xu^m)^m} = 1 + \sum_{n=1}^{\infty} Q_n(x)u^n
\]

(see [1, 6]).

In adapting the circle method to develop the asymptotics of these polynomials (see Section 3), we needed to introduce the sequence $\{L_k(x)\}$ of functions to describe the dominant contributions to their asymptotics where

\[
L_k(x) = \frac{1}{k} \sqrt{2Li_3(x^k)}
\]

and $Li_3(x)$ is the trilogarithm function given by $\sum_{n=1}^{\infty} x^n/n^3$.

An important first step in the asymptotic analysis of the polynomials was obtained in our paper [3] where we determined exactly when $\Re L_m(x)$ $(m = 1, 2)$ dominates $\Re L_k(x)$, for $k \neq m$, inside the unit disk (see [3]).

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1991 Mathematics Subject Classification. Primary: 11C08 Secondary: 11M35, 30C55, 30E15

Key words and phrases. Plane partition, polynomials, asymptotics, circle method, trilogarithm, phase.
A primary motivation for us to develop the asymptotics was to find the limiting behavior of the zeros of the plane partition polynomials which is described in detail in our paper [4].

2. Factorization of the Generating Function \( P(x, u) \)

We are interested in the behavior of \( P(x, u) \) in a neighborhood of \( u = e^{2\pi i h/k} \) inside the unit disk \( \mathbb{D} \), where \( h \) and \( k \) are relatively prime. To start we expand the logarithm of the generating function

\[
\ln P(w, u) = \sum_{\ell=1}^{\infty} \frac{x^{\ell}}{\ell} \left( \frac{u^\ell}{1 - u^\ell} \right)^2.
\]

for \( x, u \in \mathbb{D} \). Next with \( u = e^{-w + 2\pi i h/k} \) with \( \Re(w) > 0 \), we introduce two functions

\[
A_{h,k}(x, w) = \sum_{k \nmid \ell} \frac{x^{\ell}}{\ell} e^{-\ell w + 2\pi i \ell h/k}, \quad B_{h,k}(x, w) = \sum_{\ell=1}^{\infty} \frac{x^{k\ell}}{k\ell} \left( 1 - e^{-k\ell w} \right)^2.
\]

Then the generating function decomposes as

\[
\ln[P(x, e^{-w + 2\pi i h/k})] = A_{h,k}(x, x, w) + B_{h,k}(x, w).
\]

Note that when \( k = 1 \), \( A_{h,k}(x, 0) = 0 \) and that

\[
A_{h,k}(0) = -\frac{1}{4} \sum_{k \nmid \ell} \frac{x^{\ell}}{\ell} \csc^2(\pi \ell h/k), \quad k \geq 2.
\]

We need three additional functions \( \Psi_{h,k}(x, w) \), \( \omega_{h,k,n}(x) \), and \( g_{h,k}(x, w) \) where

\[
\Psi_{h,k}(x, w) = \frac{L_{2h}(x^k)}{k^2 w},
\]

\[
\ln[\omega_{h,k,n}(x)] = \begin{cases} \frac{1}{12} \ln(1 - x^k) - A_{h,k}(x, 0) - 2\pi inh/k, & k \geq 2 \\ \frac{1}{12} \ln(1 - x), & k = 1 \end{cases}
\]

\[
g_{h,k}(x, w) = \begin{cases} [A_{h,k}(x, w) - A_{h,k}(x, 0)] + [B_{h,k}(x, w) - \Psi_{h,k}(x, w) - \frac{1}{12k} \ln(1 - x^k)], & k \geq 2 \\ B_{0,1}(x, w) - \Psi_{0,1}(x, w) - \frac{1}{12} \ln(1 - x), & k = 1 \end{cases}
\]

By construction, \( P(x, e^{-w + 2\pi i h/k}) \) admits a factorization as follows.

**Proposition 2.** Let \( h, k, n \) be nonnegative integers such that \( (h, k) = 1 \). If \( \Re(w) > 0 \) and \( x \in \mathbb{D} \), then the generating function \( P(x, e^{-w + 2\pi i h/k}) \) factors as

\[
P(x, e^{-w + 2\pi i h/k}) = \omega_{h,k,n}(x)e^{2\pi i nh/k}e^{\Psi_{h,k}(x, w)}g_{h,k}(x, w),
\]

where \( \omega_{h,k,n}(x) \), \( \Psi_{h,k}(x, w) \) and \( g_{h,k}(x, w) \) are given in equations **4**, **3**, **6**.

The key in using this factorization for the circle method is the following bound.
Proposition 3. Assume that $0 < |x| < 1$ and $\Re w > 0$. (a) If $\Im w \neq 0$ as well, then there exists $M > 0$ such that

$$|g_{h,k}(x, w)| \leq \frac{2|w|}{1 - |x|} \left[ k^3 + \frac{|x|^{\pi|\Im w|}}{1 - e^{-\pi|\Im w|}} \right] + \frac{1}{1 - |x|^2} \left[ M|w|^2 k + \left( \frac{2}{(1 - e^{-\pi|\Im w|})^3} + 1 \right) |x|^{\pi|\Im w|} \right].$$

(b) If $w$ is real and positive, then there exists $M > 0$ such that

$$|g_{h,k}(x, w)| \leq \frac{2|w|}{1 - |x|} k^3 + \frac{1}{1 - |x|^2} \left[ M|w|^2 k + \left( \frac{2}{(1 - e^{-\pi|\Im w|})^3} + 1 \right) |x|^{\pi|\Im w|} \right].$$

The proof is given in the following two lemmas.

Lemma 4. Let $k \geq 2$ and $\Re w > 0$. (a) If $\Im w \neq 0$, then

$$|A_{h,k}(x, w) - A_{h,k}(x, 0)| \leq \frac{2|w|}{1 - |x|} \left[ k^3 + \frac{|x|^{\pi|\Im w|}}{1 - e^{-\pi|\Im w|}} \right].$$

(b) When $\Im w = 0$, $|A_{h,k}(x, w) - A_{h,k}(x, 0)| \leq \frac{2|w|}{1 - |x|} k^3$.

Proof. (a) For fixed $w$ and $x$, consider the function of $t$, $A_{h,k}(x, wt)$. By the Mean Value Theorem, we find

$$|A_{h,k}(x, w) - A_{h,k}(x, 0)| \leq \sup_{0 < t < 1} \left| \frac{d}{dt} A_{h,k}(x, wt) \right|$$

$$\leq |w| \left| \sum_{k|\ell} x^\ell e^{-\ell wt + 2\pi i\ell/k} \frac{(1 + e^{-\ell wt + 2\pi i\ell/k})}{(1 - e^{-\ell wt + 2\pi i\ell/k})^3} \right|$$

$$\leq 2|w| \sum_{k|\ell} \left| \frac{|x|^\ell}{e^{-\ell wt + 2\pi i\ell/k} - 1} \right|^3.$$

For $0 < t < 1$ and $\ell | k$ such that $|\ell|3w|t < \pi/k$, we have the bound

$$\frac{1}{|e^{-\ell wt + 2\pi i\ell/k} - 1|} \leq \frac{1}{|\sin(2\pi \ell/k - 3wt)|} \leq |\csc(\pi/k)| \leq k/2$$

since $|\sin(y)| \leq |e^{x+iy} - 1|$. On the other hand, if $\ell \geq \frac{\pi}{k|\Im w|}$, we see that

$$\frac{1}{|e^{-\ell wt + 2\pi i\ell/k} - 1|} \leq \frac{1}{1 - e^{-\Re w\ell t}} \leq \frac{1}{1 - e^{-\pi|\Im w|}}.$$
Combining these last two bounds, we can complete the proof:

\[
\left| \frac{d}{dt} A_{\delta, \beta}(x, w) \right| \leq 2 \sum_{k \ell} \frac{|x|^\ell}{(e^{-|\delta\ell|} + 2\pi i |\beta|)^3} \\
\leq \sum_{k \ell, \ell < \pi / |\beta|} 2k^3 |x|^\ell + \frac{2}{(1 - e^{-\pi i k})^3} \sum_{k \ell, \ell \geq \pi / |\beta|} |x|^\ell \\
\leq 2k^3 \frac{1}{1 - |x|} + \frac{2}{(1 - e^{-\pi i k})^3} \frac{|x|^{\ell} |\beta|}{1 - |x|} \\
\leq \frac{2}{1 - |x|} \left[ k^3 + \frac{|x|^{\ell} |\beta|}{1 - e^{-\pi i k}} \right].
\]

(b) Both $|x|^{\ell} |\beta|$ and $e^{-\pi i k}$ go to zero as $\Re w \to 0$, so the bound in part (b) follows from part (a).

**Lemma 5.** Assume that $0 < |x| < 1$ and $\Re w > 0$. There exists $M > 0$ such that

\[(8) \quad \left| B_k(x, w) - \frac{L_k(x^k)}{k^3 w^2} - \frac{1}{12} \ln(1 - x^k) \right| \leq \frac{1}{(1 - |x|)^2} \left[ \frac{2}{(1 - e^{-\Re w |\beta|})^3} + 1 \right] |x|^{\Re w |\beta|}.
\]

**Proof.** We begin by expanding the left-hand side of (8) as a series

\[
\sum_{\ell=1}^{\infty} \frac{x^{k \ell}}{k \ell} \left[ \frac{e^{-w k \ell}}{(e^{-w k \ell} - 1)^2} - \frac{1}{\ell^2 k^2 w^2} + \frac{1}{12} \right].
\]

For $|k \ell| < \pi$, there exists $M > 0$ such that

\[
\left| \frac{e^{-w k \ell}}{(e^{-w k \ell} - 1)^2} - \frac{1}{\ell^2 k^2 w^2} + \frac{1}{12} \right| \leq M|k \ell|^2
\]

since

\[
\lim_{z \to 0} \frac{1}{z^2} \left( \frac{e^z}{(e^z - 1)^2} - \frac{1}{z^2} + \frac{1}{12} \right) = \frac{1}{240}.
\]

On the other hand, for $|k \ell| > \pi$,

\[
\left| \frac{e^{-w k \ell}}{(e^{-w k \ell} - 1)^2} \right| \leq \frac{e^{-\Re w k \ell}}{(e^{-\Re w k \ell} - 1)^2} \leq \frac{2}{(1 - e^{-\Re w |\beta|})^3}.
\]

So we have the bound for $\Re w > 0$

\[
\left| \frac{e^{-w k \ell}}{(e^{-w k \ell} - 1)^2} - \frac{1}{\ell^2 k^2 w^2} + \frac{1}{12} \right| \leq \frac{2}{(1 - e^{-\Re w |\beta|})^3} + 1
\]

The proof is now completed in the same way as Lemma 4.

3. Phases

For convenience, we record a result from our paper [3].

**Definition 6.** Let $\{L_k(x)\}$ be any sequence of complex-valued functions on a domain $D$. The set $R(m)$ is the $m$-th phase (or phase $m$) of $\{L_k(x)\}$ if (1) if $x \in R(m)$, then $\Re L_m(x) > \Re L_k(x)$ for all $k \neq m$ and (2) if $V$ is any open subset of $D$ satisfying (1), then $V \subset R(m)$.
THEOREM 7. (Parry-Boyer [3]) Let $D$ be the punctured open unit disk and let \( \{L_k(x)\} \) be given as in \([3]\). Then $D$ contains exactly two nonempty phases $R(1)$ and $R(2)$ whose union is dense in $D$ and whose common boundary is the level set \( \{x \in D : RL_1(x) = RL_2(x)\} \). It has exactly one real point $x^* \simeq -0.8250030529$ and its closure contains exactly two points $e^{\pm \Theta^*}$ on the unit circle where $\Theta^* \simeq \pm 0.9517031251\pi$. Further $R(2)$ lies in the open left half plane and $R(2) \cap \mathbb{R} = (-\infty, x^*)$.

4. Asymptotics of the Polynomials on the Phases $R(1)$ and $R(2)$

4.1. Introduction. We will adapt the circle method which is usually used to give asymptotics for a sequence $\{c_n\}$ of positive numbers through their generating function $\sum_{n \geq 0} c_n u^n$. With the coefficients $c_n$ replaced with the polynomials $Q_n(x)$, we find that the dominant contribution to their asymptotics depends on the location of $x$ in the unit disk $\mathbb{D}$. The purpose of this section is to show that the subsets of $\mathbb{D}$ where the asymptotics have the same dominant form coincide with the phases of $\{L_k(x)\}$. See Andrews’s classic book [11] Chapter 5 for a thorough discussion of the circle method.

THEOREM 8. Let $R(1)$ and $R(2)$ be the phases of $\{L_k(x)\}$ given in Theorem 7.

(a) Let $x \in X \subset R(1) \setminus [x^*, 0]$ be a compact set, then
\[
Q_n(x) = \sqrt[3]{1 - x^2} \left( \sqrt{\frac{L_1(x)}{6\pi n^{1/3}}} \exp \left( \frac{2}{3} n^{2/3} L_1(x) \right) \left( 1 + O_X \left(n^{-1/3}\right) \right) \right).
\]

(b) Let $x \in X \subset R(2)$ be a compact, then
\[
Q_n(x) = (-1)^n \sqrt[3]{1 - x^2} \left( \sqrt{\frac{L_2(x)}{6\pi n^{1/3}}} \exp \left( \frac{2}{3} n^{2/3} L_2(x) \right) \left( 1 + O_X \left(n^{-1/3}\right) \right) \right).
\]

By the Cauchy integral formula, we have an integral expression for $Q_n(x)$ with integration contour the circle $|u| = e^{-2\pi \alpha}$ where
\[
\alpha = \frac{1}{2\pi n^{1/3}} \Re L_m(x)
\]
and $m = 1$ or $2$. Then we dissect the circular contour relative to a Farey sequence $F_N$ of order $N = [\delta n^{1/3}]$ as follows
\[
Q_n(x) = \frac{1}{2\pi i} \oint_{|u|=e^{-2\pi \alpha}} \frac{P(x, u)}{u^{n+1}} du = \sum_{h/k \in F_N} \int_{h/k}^{h+k'/k'} \frac{P(x, e^{2\pi(-\alpha+i\psi)})}{e^{2\pi n(-\alpha+i\psi)}} d\psi = \sum_{h/k \in F_N} e^{-2\pi inh/k} e^{2\pi n(\alpha - i(\psi - h/k))} P(\psi, e^{-2\pi(\alpha - i(\psi - h/k)) + 2\pi ih/k}) d\psi
\]
where $h' < k' < h''$ are consecutive elements from the Farey sequence $F_N$ in reduced form. By convention, we will always assume that $N > m$.

We now make the change of variables $v = \psi - h/k$ and apply the factorization in Proposition 2 to get
\[
P(x, e^{-2\pi(\alpha - iv) + 2\pi ih/k}) = e^{2\pi inh/k} \omega_{h,k,n}(x) \exp \left( \frac{L_k(x)^3}{8\pi^2(\alpha - iv)^2} + g_{h,k}(x, 2\pi(\alpha - iv)) \right).
\]
We introduce the integral
\[ I_{h,k,n}(x) = \int_{h/k-1/k(k+k')} \exp \left( \frac{L_k(x)^3}{8\pi^2(\alpha - iv)^2} + 2\pi n(\alpha - iv) \right) e^{g_{h,k}(x,2\pi(\alpha - iv))} \, dv. \]
So we may write \( Q_n(x) \) as
\[ Q_n(x) = \sum_{h/k \in P_N} \omega_{h,k,n}(x) I_{h,k,n}(x). \]

Our next goal is to show that \( \omega_{1,m,n}(x) I_{1,m,n}(x) \) is the dominant term in this expansion.

4.2. Major arcs. We start by making another change of variables \( z = 2\pi n^{1/3}v \) in \( I_{h,k,n}(x) \) to get
\[ I_{h,k,n}(x) = \frac{1}{2\pi n^{1/3}} \int_{-\frac{2\pi n^{1/3}}{m+n}}^{\frac{2\pi n^{1/3}}{m+n}} \exp \left[ n^{2/3} \left( \frac{L_m(x)^3}{2(\Re L_m(x) - iz)^2 + (\Re L_m(x) - iz)} \right) \right] \times \exp \left[ g_{h,k} \left( x, \frac{1}{n^{1/3}}(\Re L_m(x) - iz) \right) \right] \, dz. \]
We decompose \( I_{h,k,n}(x) \) into the sum \( I'_{h,k,n}(x) + I''_{h,k,n}(x) \) where
\[ I'_{h,k,n}(x) = \frac{1}{2\pi n^{1/3}} \int_{-\frac{2\pi n^{1/3}}{m+n}}^{\frac{2\pi n^{1/3}}{m+n}} \exp \left[ n^{2/3} \left( \frac{L_m(x)^3}{2(\Re L_m(x) - iz)^2 + (\Re L_m(x) - iz)} \right) \right] \, dz. \]
and
\[ I''_{h,k,n}(x) = \frac{1}{2\pi n^{1/3}} \int_{-\frac{2\pi n^{1/3}}{m+n}}^{\frac{2\pi n^{1/3}}{m+n}} \exp \left[ n^{2/3} \left( \frac{L_m(x)^3}{2(\Re L_m(x) - iz)^2 + (\Re L_m(x) - iz)} \right) \right] \times \left\{ \exp \left[ g_{h,k} \left( x, \frac{1}{n^{1/3}}(\Re L_m(x) - iz) \right) \right] - 1 \right\} \, dz. \]
We will obtain an asymptotic expansion of \( I'_{h,m,n}(x) \) using a saddle point expansion. To begin we need a basic inequality.

**Lemma 9.** Let \( \alpha > 0, v \in \mathbb{R}, \) and \( |\arg(L)| \leq \pi/3, \) then
\[ \Re \left( \frac{L^3}{(\alpha - iv)^2} \right) \leq \frac{(\Re L)^3}{\alpha^2}. \]
If \( |\arg(L)| < \pi/3 \) and \( L \neq 0, \) then equality is attained uniquely at \( v = -\Im L. \) If \( |\arg(L)| = \pi/3 \) and \( L \neq 0, \) then equality is attained only at \( v = \pm \Im L. \)

**Lemma 10.** Let \( x \in X \subset D \) be compact. For \( 0 < \delta < \pi/4 \sqrt{2} \), then
(a) for every \( x \in X, \pm \Im L_m(x) \in (-\pi/m\delta, \pi/m\delta) \subset [-\frac{2\pi n^{1/3}}{m+n}, \frac{2\pi n^{1/3}}{m+n}], \)
(b) there exists a constant \( K > 0 \) such that
\[ I'_{h,m,n}(x) = \frac{1}{2\pi n^{1/3}} \int_{-\frac{2\pi n^{1/3}}{m+n}}^{\frac{2\pi n^{1/3}}{m+n}} \exp \left[ n^{2/3} \left( \frac{L_m(x)^3}{2(\Re L_m(x) - iz)^2 + (\Re L_m(x) - iz)} \right) \right] \, dz \]
\[ + \quad O \left( \exp \left( n^{2/3} \frac{3}{2} \Re L_m(x) - K \right) \right). \]
Proof. (a) Since the Farey sequence has order $|\delta n^{1/3}|$, we have

$$\frac{\pi}{m\delta} \leq \frac{2\pi n^{1/3}}{m + m'}, \quad \frac{2\pi n^{1/3}}{m + m'} \leq \frac{2\pi n^{1/3}}{|\delta n^{1/3}|}$$

since $h'/m' < h/m < h''/m''$ are consecutive terms of the Farey sequence. In particular, with $0 < \delta < \pi/\sqrt{2\zeta(3)}$, we find $|\Im L_\nu(x)| \leq \frac{1}{m_2} \sqrt{2\zeta(3)} < \pi/m_1$. 

(b) By Lemma 3 if $|v| \geq \pi/m_1$, there exists $K > 0$ such that for all $x \in X$

$$\Re\left(\frac{L_m(x)^3}{2(\Re L_m(x) - iz)^2 - iz^2} + (\Re L_m(x) - iz)\right) \leq \frac{3}{2} \Re L_m(x) - K.$$ 

We consider the estimates for the integral on the set $J = [-\frac{2\pi n^{1/3}}{m + m'}, \frac{2\pi n^{1/3}}{m + m'}] \setminus (\frac{-\pi}{m\delta}, \frac{\pi}{m\delta})$:

$$\int_J \exp\left[n^{2/3}\left(\frac{L_m(x)^3}{2(\Re L_m(x) - iz)^2} + (\Re L_m(x) - iz)\right)\right] dz = \int_J \exp\left[n^{2/3} \Re\left(\frac{L_m(x)^3}{2(\Re L_m(x) - iz)^2} + (\Re L_m(x) - iz)\right)\right] dz$$

$$\leq \int_J \exp\left[n^{2/3} \left(\frac{3}{2} \Re L_m(x) - K\right)\right] dz = O\left(n^{1/3} \exp\left[n^{2/3} \left(\frac{3}{2} \Re L_m(x) - K\right)\right]\right).$$

The estimate in part (b) now follows. \hfill \Box

Proposition 11. (a) Let $x \in X \subset \mathbb{D} \setminus \{x^m \leq 0\}$ be compact. Then

$$I'_{h,m,n}(x) = \frac{1}{\sqrt{2\pi n^{2/3}}} \sqrt{\frac{L_m(x)}{3}} \exp\left(\frac{3}{2} n^{2/3} L_m(x)\right) \left(1 + O_X(n^{-2/3})\right).$$

(b) Let $x \in X \subset \{x^m \leq 0\}$ be compact. Then we have

$$I''_{h,m,n}(x) = \frac{1}{\sqrt{2\pi n^{2/3}}} \left[\sqrt{\frac{L_m(x)}{3}} e^{\frac{3}{2} n^{2/3} L_m(x)}\right] \left(1 + O_X(n^{-2/3})\right)$$

$$+ \frac{1}{\sqrt{2\pi n^{2/3}}} \left[\sqrt{\frac{L_m(x)}{3}} e^{\frac{3}{2} n^{2/3} L_m(x)}\right] \left(1 + O_X(n^{-2/3})\right).$$

Proof. We apply the saddle point method as given in [5], p. 10-11, and note that standard arguments will make the estimates there uniform for $x \in X$. We let $B(z) = \frac{L_m(x)^3}{2(\Re L_m(x) - iz)^2} + (\Re L_m(x) - iz)$.

(a) By the inequality in Lemma 3, $B(z)$ has a unique maximum on $\mathbb{R}$ at $z_0 = -3L_m(x)$ with $B(z_0) = \frac{3}{2}\Re L_m(x) > 0$. Note that uniqueness follows since, for $x \in X$, $L(x) < 0$ if and only if $x < 0$. Now the second derivative $B''(z) = -3/(\Re L_m(x) - iz)^4$ is Lipschitz continuous on $\mathbb{R}$ and $B''(z_0) = -3/L_m(x)$. Hence, by [5], p. 11, we find that as $n \to \infty$

$$\int_{-\infty}^{\infty} e^{n^{2/3} B(z)} dz = e^{n^{2/3} B(z_0)} \sqrt{\frac{\pi}{n^{2/3} - \frac{3}{2} \Re L_m(x)}} \left(1 + O_X(n^{-2/3})\right)$$

which simplifies to the desired expression.
(b) Let \( z_0 = \Re L_m(x) \). Since \( x_m < 0 \), for \( z \in \mathbb{R} \), \( B(z) \) has two equal maxima at \( \pm z_0 \) with \( B(\pm z_0) = \Re L_m(x) \). Since \( |\arg L_m(x)| = \pi/3 \) and \( x \neq 0 \), \( \Re L_m(x) \neq 0 \). Now \( B(-z) = \Re L_m(z) \) for \( z \in \mathbb{R} \) since \( L^3 \) is real. We get desired the equation by applying the saddle point method as in part (a) to \( \int_0^{n/3} e^{n^2/3} B(z) \, dz \) and observing that
\[
\int_{-\infty}^{\infty} e^{n^2/3} B(z) \, dz = \int_{-\infty}^{0} e^{n^2/3} B(z) \, dz.
\]

Note that bounds that come from the above saddle point asymptotics must exclude the interval \([x^*, 0]\) from the phase \( R(1) \). This explains choosing between \( R(1) \) and \( R(1) \setminus [x^*, 0] \) as a region below.

4.3. Estimates for Non-dominant Contributions.

4.3.1. Bounds for the Integrals \( I_{h,m,n}^n(x) \).

**Lemma 12.** Let \( x \in X \subset D \) be compact with \( M_X = \max \{|x| : x \in X\} \). Fix \( \delta > 0 \) so \( 0 < \delta < 1/\sqrt{2\zeta(3)} \) and \( n \geq 1/\delta^3 \). Then

(a) For \( k \in \mathbb{N} \) fixed, then as \( n \to \infty \)
\[
\left| g_{h,k} \left( x, \frac{1}{n^{1/3}} (\Re L_m(x) - iz) \right) \right| \leq O_{X,k} \left( \frac{1}{\delta n^{1/3}} \right).
\]

(b) For \( k \leq [\delta n^{1/3}] \), then as \( n \to \infty \)
\[
\left| g_{h,k} \left( x, \frac{1}{n^{1/3}} (\Re L_m(x) - iz) \right) \right| \leq \frac{2\sqrt{5}}{1 - M_X} \delta^2 n^{2/3} + O_{X} \left( \frac{1}{\delta n^{1/3}} \right).
\]

**Proof.** We will bound the terms in equation (17) individually. For convenience, we work with the variable \( v = 2\pi n^{1/3} \). Recall that \( \sqrt{2\zeta(3)} < 1/\delta \) and \( |v| \leq \frac{1}{\sqrt{2\zeta(3)}} \), where \( N = [\delta n^{1/3}] \) is the order of the Farey fractions. Introduce \( C_X = \max \{1/\Re L_m(x) : x \in X\} \). We start with the easy estimate:
\[
|w| = \left| \Re L_m(x) + i v \right| \leq \frac{1}{n^{1/3}} \sqrt{(2\zeta(3))^{2/3} + \frac{1}{(k\delta)^2}} \leq \frac{\sqrt{5}}{\delta n^{1/3}}
\]
since \( \sqrt{2\zeta(3)} \) is an upper bound for \( \Re L_m(x) \), \( x \in X \).

Next by using that \( 1/(1 - e^{-t}) < 1 + 1/t \) for \( t > 0 \), we can show that
\[
\frac{1}{1 - e^{-\pi n^{1/3}|w|}} \leq 1 + \frac{\sqrt{5}C}{\pi \delta}, \quad \frac{1}{1 - e^{-\pi n^{1/3}|x|}} \leq 1 + \frac{\sqrt{5}C_X}{\pi \delta}.
\]

Using \( e^{-t} \leq 27/(et)^3 \) for \( t > 0 \), we obtain
\[
|w| \leq \frac{27}{(2\pi n^{1/3} \delta \ln M_X)^3}, \quad |x| \leq \frac{-27}{(\pi \delta n^{1/3} \ln M_X)^3}.
\]

Hence, \( g_{h,k} \left( x, \frac{\Re L_m(x)}{n^{1/3}} - iv \right) \) is bounded above by
\[
\frac{2\sqrt{5}}{\delta n^{1/3}(1 - M_X)} \left[ \frac{1}{k^3} + \frac{-27}{(2\pi \epsilon n^{1/3} \delta \ln M_X)^3} \cdot \left( 1 + \frac{2C_X}{\pi \delta} \right) \right] + \frac{1}{(1 - M_X)^2} \left[ \frac{5Mk}{\delta^2 n^{2/3}} + 2 \left( 1 + \frac{\sqrt{5}C_X}{\pi \delta} \right)^3 + 1 \right] \cdot \frac{-27}{(\pi \epsilon n^{1/3} \ln M_X)^3}.
\]
We rewrite this upper bound as a polynomial in $k$ with constants $A$ and $B$ independent of $\delta, k$, and $n$:

\[(12) \frac{2 \sqrt{\pi}}{\delta n^{1/3}} (1 - M_X) k^3 + \frac{1}{(1 - M_X)^2} \frac{5 M}{\delta^2 n^{2/3}} k + A \frac{1}{\delta^5 n^{4/3}} + B \frac{1}{\delta^6 n}.
\]

With $k$ fixed, we obtain part (a) since the bound in (12) is $O_X(1/\delta n^{1/3})$; while if we replace $k$ with $\delta n^{1/3}$, we get the bound in part (b). \(\square\)

**Lemma 13.** Let $x \in X$ where $X$ is a compact subset of either $R(1) \setminus [x^*, 0]$ or $R(2)$. Then $I''_{h,m,n}(x) = O_{X, \delta}(I'_{h,m,n}(x)/n^{1/3})$, $m = 1, 2$.

**Proof.** By the definition of $I''_{h,m,n}(x)$ given in subsection 4.2, it is enough to observe that there exists a positive constant $K_{X,m,\delta}$ such that

\[
\left| e^{g_{n,m}(x, n^{1/3} (RL_m(x) - iz))} \right| \leq K_{X,m,\delta} \frac{1}{n^{1/3}}
\]

by the above lemma. \(\square\)

**4.3.2. Bounds for Minor Arcs.**

**Lemma 14.** Let $x \in X$ where $X$ is a compact subset of unit disk. Then

\[
|\omega_{h,k,n}(x)| \leq 2^{1/12} \exp \left( \frac{k^2}{16 h^2} Li_3(M_X) \right) \leq 2^{1/12} \exp \left( \frac{k^2}{16} (\zeta(3) - \ln(1 - M_X)) \right)
\]

where $M_X = \max\{|x| : x \in X\}$ and $\omega_{h,k,n}(x)$ is given in (3).

**Proof.** We bound each component separately in the definition of $\omega_{h,k,n}(x)$. We start with the first factor

\[
\left| \exp \left( \frac{1}{12k} \ln(1 - x^k) \right) \right| = \exp \left( \frac{1}{12k} \Re \ln(1 - x^{k}) \right) \leq 2^{1/12k}.
\]

Next we obtain an intermediate bound

\[
\left| \exp \left( -\frac{1}{4} \sum_{\ell \not\mid k} \frac{x^\ell}{\ell} \csc^2(\pi h \ell / k) \right) \right| \leq \exp \left( -\frac{1}{4} \sum_{\ell \not\mid k} \frac{|x|^\ell}{\ell} \csc^2(\pi h \ell / k) \right).
\]

Break up the above series into two sums with the sets of indices

\[ J_1 = \{ \ell : \ell \not\mid k, h \ell / k \text{ mod } 1 < 1/2 \}, \quad J_2 = \{ \ell \not\mid k, 1/2 \geq h \ell / k \text{ mod } 1 \}. \]

and use the elementary estimates $\sin \theta \geq 2\theta/\pi$, for $0 \leq \theta \leq \pi/2$, and $\geq 2(1 - \theta/\pi)$ for $\pi/2 \leq \theta \leq \pi$ to obtain the two bounds below to complete the proof:

\[
\exp \left( \frac{1}{4} \sum_{\ell \in J_1} \frac{|x|^\ell}{\ell} \csc^2(\pi h \ell / k) \right) \leq \exp \left( \frac{k^2}{16h^2} Li_3(M_X) \right),
\]

\[
\exp \left( \frac{1}{4} \sum_{\ell \in J_2} \frac{|x|^\ell}{\ell} \csc^2(\pi h \ell / k) \right) \leq \exp \left( -\frac{k^2}{16} \ln(1 - M_X) \right).
\]

\(\square\)
LEMMA 15. Let $x \in X$ where $X$ is a compact subset of $R(m)$, $m = 1, 2$. There exists a positive constant $a_X$ such that

$$\Re \left[ \frac{L_k(x)^3}{8\pi^2(\alpha - iv)^2} + 2\pi n(\alpha - iv) \right] \leq n^{2/3} \left( \frac{3}{2} \Re L_m(x) + a_X \right), \quad k \neq m, \quad x \in X,$$

where $\alpha$ is given in equation (9).

PROOF. In subsection 4.2, we saw that

$$\frac{1}{n^{2/3}} \Re \left[ \frac{L_k(x)^3}{8\pi^2(\alpha - iv)^2} + 2\pi n(\alpha - iv) \right] = \Re \left[ \frac{L_k(x)^3}{2(\Re L_m(x) - 2\pi i n^{1/3}v)^2} + (\Re L_m(x) - 2\pi i n^{1/3}v) \right].$$

By Lemma 9, we have the strict bound for $x \in X$ and $k \neq m$

$$\Re \left[ \frac{L_k(x)^3}{2(\Re L_m(x) - 2\pi i n^{1/3}v)^2} + (\Re L_m(x) - 2\pi i n^{1/3}v) \right] \leq \frac{(\Re L_k(x))^3}{2(\Re L_m(x))^2} + \Re L_m(x) < \frac{3}{2} \Re L_m(x).$$

Hence the difference, for $k \neq m$, has a positive minimum $a_{X,k}$ on $X$:

$$\frac{3}{2} \Re L_m(x) - \Re \left[ \frac{L_k(x)^3}{2(\Re L_m(x) - 2\pi i n^{1/3}v)^2} + (\Re L_m(x) - 2\pi i n^{1/3}v) \right] \geq a_{X,k} > 0, \quad x \in X,$$

by compactness. Consider $a_{X,k}$ as a sequence, then it converges to the minimum of $\frac{3}{2} \Re L_m(x)$ on $X$ which is positive. Hence, $\inf \{a_{X,k} : k \neq m\}$ is attained for some index, say $k_0 \neq m$. In particular, we have

$$a_{X,k} \geq a_{X,k_0} > 0, \quad k \neq m.$$

For simplicity, we write $a_X$ for $a_{X,k_0}$. The inequality in the lemma follows. \(\square\)

4.4. Conclusion of proof of Theorem 8

LEMMA 16. Let $x \in X$ be a compact subset of $R(m)$, $m = 1, 2$. Then there exists $\delta_0 > 0$ and $\eta > 0$ such that

$$\left| \sum_{h,k \in K_n, k \neq m} \omega_{h,k,n}(x) I_{h,k,n}(x) \right| \leq 2^{1/12} \exp \left( \frac{3}{2} \Re L_m(x) - \eta n^{2/3} + O_{X,\delta_0}(1) \right)$$

PROOF. We need to estimate $I_{h,k,n}(x)$ where $k \neq m$:

$$I_{h,k,n}(x) = \omega_{h,k,n}(x) \int_{k(k+h,\pi)}^{k(k+h,\pi,\alpha)} \exp \left( \frac{L_k(x)^3}{8\pi^2(\alpha - iv)^2} + 2\pi n(\alpha - iv) \right) \exp(g_{h,k}(x, 2\pi(\alpha - iv))) \, dv.$$

We begin with the bounds

$$\left| \int_{k(k+h,\pi,\alpha)}^{k(k+h,\pi)} \exp \left( \frac{L_k(x)^3}{8\pi^2(\alpha - iv)^2} + 2\pi n(\alpha - iv) \right) \exp(g_{h,k}(x, 2\pi(\alpha - iv))) \, dv \right| \leq \int_{k(k+h,\pi)}^{k(k+h,\pi,\alpha)} \left| \exp \left( \frac{L_k(x)^3}{8\pi^2(\alpha - iv)^2} + 2\pi n(\alpha - iv) \right) \right| \left| \exp(g_{h,k}(x, 2\pi(\alpha - iv))) \right| \, dv \leq \int_{k(k+h,\pi)}^{k(k+h,\pi,\alpha)} \exp \left( -a_X n^{2/3} + \frac{3}{2} n^{2/3} \Re L_m(x) \right) \left| \exp \left( \frac{2\sqrt{2}}{1 - M_X} n^{2/3} + o_{X,\delta}(1) \right) \right| \, dv$$
where we used Lemma \[12\]. Hence a full bound for the sum over \( h/k \in F_N, k \neq m \) is

\[
2^{1/12} \exp \left( -a_{X,\delta} n^{2/3} + \frac{3}{2} n^{2/3} R L_m(x) + \frac{2 \sqrt{2}}{1 - M_X} \delta^2 n^{2/3} \right) \\
+ \delta^2 n^{2/3} \left( \frac{\zeta(3)}{16} - \frac{\ln(1 - M_X)}{16} n^{2/3} + O_X,\delta(1) \right) \left( \frac{1}{k(k + k')} + \frac{1}{k(k + k'')} \right)
\]

where the last factor is the length of the interval of integration. Note that this bound holds for any \( h/k \in F_N \) with \( k \neq m \).

Let \( \delta_0 > 0 \) be chosen so that

\[
(13)
\]

\[
a_X - \frac{2 \sqrt{2}}{1 - M_X} \delta_0 - \frac{\zeta(3)}{16} \delta_0 + \frac{\ln(1 - M_X)}{16} \delta_0 > 0.
\]

Set \( \eta = a_X - \frac{2 \sqrt{2}}{1 - M_X} \delta_0 - \frac{\zeta(3)}{16} \delta_0 + \frac{\ln(1 - M_X)}{16} \delta_0 > 0 \) so we now write new full bound as

\[
2^{1/12} \exp \left( \left( \frac{3}{2} R L_m(x) - \eta \right)n^{2/3} + O_X,\delta_0(1) \right).
\]

The bound for the contributions over all the minor arcs is

\[
\left| \sum_{h/k \in F_N, k \neq m} \omega_{h,k,n}(x) I_{h,k,n}(x) \right| \leq 2^{1/12} \exp \left( \left( \frac{3}{2} R L_m(x) - \eta \right)n^{2/3} + O_X,\delta_0(1) \right)
\]

since the sum of the lengths of all the minor arcs is less than 1. The proof is complete. \( \square \)

**PROOF.** We now complete the proof of Theorem \[10\]. Choose the order \( N \) of the Farey fractions to be \( \lfloor \delta_0 n^{1/3} \rfloor \) where \( \delta_0 \) is given in Lemma \[16\]. By equation \[(11),\] we write

\[
Q_n(x) = \omega_{1,m,n}(x) I_{1,m,n}(x) + \sum_{h/k \in F_N, k \neq m} \omega_{h,k,n}(x) I_{h,k,n}(x)
\]

By Proposition \[11\] and Lemma \[13\] we see that

\[
\omega_{1,m,n}(x) I_{1,m,n}(x) = \omega_{1,m,n}(x) \sqrt{\frac{L_m(x)}{6 \pi n^{2/3}}} \exp \left( \frac{3}{2} n^{2/3} L_m(x) \right) \left( 1 + O_X(n^{-1/3}) \right)
\]

which dominates the contribution of the sum over \( h/k \in F_N, k \neq m \) by Lemma \[16\]. Finally, we observe that

\[
\omega_{1,1,n}(x) = \sqrt[3]{1 - x}, \quad \omega_{1,2,n}(x) = (-1)^n \sqrt[3]{1 - x^2} \sqrt[6]{\frac{1 - x}{1 + x}}
\]

to complete the proof. \( \square \)

### 5. Asymptotics on the Boundaries of the Phases \( R(1) \) and \( R(2) \)

Let \( x \in X \subset \{ x : \Re L_1(x) = \Re L_2(x), x \neq x^* \} \) be compact. Then the choice of \( \alpha \) in equation \[(19)\] must satisfy

\[
\alpha = \frac{1}{2 \pi n^{1/3}} \Re L_1(x) = \frac{1}{2 \pi n^{1/3}} \Re L_2(x)
\]

so both terms \( \omega_{1,1,n} I_{1,1,n}(x) \) and \( \omega_{1,2,n} I_{1,2,n}(x) \) will contribute to the dominant asymptotics of \( Q_n(x) \). A similar modification must be used for the asymptotics on the compact subsets of \( (x^*, 0) \). In this case, the two terms are \( \omega_{1,1,n} I_{1,1,n}(x) \) and its
complex conjugate which combine to give an oscillatory term. Further simplification occurs because the argument of \( L_1(x) \) is constant on \((x^*, 0)\). We record the results:

Theorem 17. Let \( x^* \) be the negative real number given in Theorem 7
(a) Let \( x \in X \subset (x^*, 0) \) be compact. Then

\[
Q_n(x) = 2^{7/6} \sqrt[6]{1-x} \frac{1}{\sqrt{6\pi n^{4/3}}} |L_{3}(x)|^{1/6} \exp \left( \frac{3}{4} \sqrt[3]{2n^{2/3}} |L_{3}(x)|^{1/3} \right) \\
\times \left( \cos \left( \frac{3\sqrt[3]{3}}{4} \sqrt[3]{2n^{2/3}} |L_{3}(x)|^{1/3} + \frac{\pi}{6} \right) + O_X(n^{-1/3}) \right).
\]

(b) Let \( x \in X \subset \{ x : \Re L_1(x) = \Re L_2(x), x \neq x^* \} \) be compact. Then

\[
Q_n(x) = 2^{7/6} \sqrt[6]{1-x} \sqrt[4]{\frac{L_{1}(x)}{6\pi n^{4/3}}} \exp \left( \frac{3}{4} n^{2/3} L_{1}(x) \right) \left( 1 + O_X(n^{-1/3}) \right) \\
+ (-1)^n \sqrt[6]{1-x} \sqrt[4]{\frac{L_{2}(x)}{6\pi n^{4/3}}} \exp \left( \frac{3}{4} n^{2/3} L_{2}(x) \right) \left( 1 + O_X(n^{-1/3}) \right).
\]

Note that in part (a) above, we are abusing notation since this relation holds only if \( \Re \omega_{1,1,n}(x)I_{1,1,n}(x) \) is nonzero. On the other hand, if \( \Re(\omega_{1,1,n}(x)I_{1,1,n}(x)) = 0 \) holds, \( Q_n(x) \) reduces to \( O(\omega_{1,1,n}(x)I_{1,1,n}(x)n^{-1/3}) \).

Finally, we remark that the asymptotics of \( Q_n(x) \) over \((-1, 1)\) has three separate regimes: \((-1, x^*)\), \((x^*, 0)\), and \((0, 1)\). On \((-1, x^*)\) and \((0, 1)\), the polynomials have exponential growth while on \((x^*, 0)\) the polynomials are oscillating. In [4], this asymptotic was found to give good approximations to the zeros of \( Q_n(x) \) in that interval. The partition polynomials built from the usual partition numbers and studied in [2] do not have this oscillatory behavior.

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Department of Mathematics, Drexel University USA
E-mail address: rboyer@drexel.edu

Department of Mathematics, Drexel University USA
E-mail address: dtp23@drexel.edu