COMPACTIFICATION OF SEMI-SIMPLE LIE GROUPS

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Abstract. We discuss the ‘hd-compactification’ of a semi-simple Lie group to a manifold with corners; it is the real analog of the wonderful compactification of deConcini and Procesi. There is a 1-1 correspondence between the boundary faces of the compactification and conjugacy classes of parabolic subgroups with the boundary face fibering over two copies of the corresponding flag variety with fiber modeled on the (compactification of the) reductive part. On the hd-compactification Harish-Chandra’s Schwartz space is identified with a space of conormal functions of rapid-logarithmic decay relative to square-integrable functions.

Introduction

In this note we describe a systematic compactification procedure for Lie groups. Although semisimple Lie groups are the primary focus of interest for well known ‘iterative’ reasons we work in the setting of real reductive groups with compact centers. The ‘hd-compactification’ of such a group, \( G \), is a compact manifold with corners, \( \widehat{G} \), with interior identified with \( G \) and with additional properties described below. The hd-compactification is closely related to (derivable from) the wonderful compactification of De Concini and Procesi, \([5]\), in the case of complex adjoint groups, and to the (maximal) Satake compactification, and more especially the Oshima compactification, \([10]\), of the homogeneous space \( G/K \) for a maximal compact subgroup \( K \); see also \([6]\).

By a compactification \( \overline{M} \) of a manifold \( M \) without boundary we mean a smooth map \( I : M \rightarrow \overline{M} \) which is a diffeomorphism onto the interior of a compact manifold with corners \( \overline{M} \). Two compactifications are equivalent if there is a diffeomorphism between them intertwining the inclusions into the interiors. As part of the definition we always demand that the boundary hypersurfaces of a compact manifold (by default meaning with corners) are embedded. This implies that each such hypersurface \( H \) has a global boundary defining function \( \rho_H \in C^\infty(M) \) such that \( \rho_H > 0 \) on \( \overline{M} \setminus H \), \( H = \{ \rho = 0 \} \) and \( d\rho_H \neq 0 \) at \( H \). Thus \( d\rho_H \big|_H \) spans the conormal bundle to \( H \). These assumptions also imply that the components of the intersections of the hypersurfaces, the boundary faces, are all embedded and are naturally compact manifolds with corners.

In the category of manifolds with corners the arrows are required to be smooth maps \( f : X' \rightarrow X'' \) in the usual sense that \( f^* C^\infty(X'') \subset C^\infty(X') \) but also they are b-maps meaning that for each boundary hypersurface \( H \) of \( X'' \)

\[
f^* \rho_H = \begin{cases} 0 & \text{or} \\ a \prod_K \rho_K^{n_KH}, & 0 < a \in C^\infty(X') \end{cases}
\]

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The maps we are most interested in are interior b-maps, in which the first option does not occur. The powers, $n_{KH}$, in the product decomposition over the boundary hypersurfaces of $X'$ are necessarily non-negative integers (so the last case is where these integers all vanish).

The Lie algebra of the group of diffeomorphisms of a compact manifold consists of the smooth vector fields tangent to all boundary faces, the b-vector fields. Equivalently these are the smooth vector fields satisfying $V\rho \in \rho C^\infty(X)$ for all boundary defining functions $\rho_H$. The b-vector fields form a Lie algebra, $V_b(X) = C^\infty(X, bTX)$, where $bTX \rightarrow TX$ has null space at each boundary point $p$ of codimension $k$ i.e. lying in the interior of a boundary face $F$ of codimension $k$, a canonically trivial vector space $bN_pF \subset bTM$ which extends to a smooth subbundle over $F$. These spaces are spanned by the vector fields $x_i\partial_{x_i}$ in terms of local coordinates in which the $x_i$ define the boundary hypersurfaces through $p$.

**Definition 1.** An hd-compactification of a real reductive Lie group with compact center is a compact manifold with corners $\overline{G}$ and a diffeomorphism onto the interior $G \hookrightarrow \overline{G}$ such that

(D1) [inversion] Inversion extends to a diffeomorphism of $\overline{G}$.

(D2) [b-normality] The right action of $G$ extends smoothly to $\overline{G}$ with isotropy algebra at each boundary point projecting to span the b-normal space.

(D3) [b-transitivity] The combined action of $G \times G$ on left and right is b-transitive, i.e. has Lie algebra spanning $bT\overline{G}$.

(D4) [minimality] Near the interior of a boundary face of codimension $d$ the span of the Lie algebra for the right action contains vector fields $x_jv_j$ where the $x_j$ are defining functions for the local boundary hypersurfaces and the $v_j$ are locally independent tangent vector fields not themselves in the span of the Lie algebra.

**Main Theorem.** Any real reductive group with compact center has an hd-compactification.

**Conjecture.** The hd-compactification of a real-reductive Lie group with compact center is unique up to equivalence, intertwining the right and left actions, and so defines a functor from these groups and isomorphisms to compact manifolds with corners and diffeomorphisms.

In view of the invariance shown below this hd-compactification extends to spaces which are left and right principal $G$ spaces, reduced to $G$ by fixing a point, for a real reductive group $G$.

One direct consequence of the properties demanded above is that the action, on left or right, of a maximal compact subgroup $K \subset G$ on $\overline{G}$ is necessarily free. Thus $\overline{G}/K$ defines a compactification, essentially the maximal Satake compactification, of $G/K$, see Borel and Ji [3].

Below we give a detailed differential-geometric description of the hd-compactification with subsequent geometric analysis in mind, see Mazzeo an Vasy [9], [8] and Parthasarathy and Ramacher [11]. In particular the boundary faces of $\overline{G}$ are shown to be in 1-1 correspondence with the conjugacy classes of parabolic subgroups of $G$ and hence with the subsets $S \subset D$ of the nodes of the reduced Dynkin diagram. If $F_S$ is the flag variety parameterizing the parabolics associated to $S$ then the
corresponding boundary face of $G$ can be realized as a bundle

$$
M \xrightarrow{F_S} \mathcal{F}_S \times \mathcal{F}_S
$$

where the model fibre is the compactification of the real-reductive group in the Langlands decomposition $\mathcal{F}_S \ni P = M_P A_P N_P$ of an associated parabolic. More geometrically the fibre at $(P', P)$ is the hd-compactification of the space of elements of $G$ such that $M_P g = g M_P$. The isotropy group for the right action of $G$ at a point of $F_S$ is the associated normal subgroup $A_P N_P$.

We show how to assemble a compactification

$$
\overline{G} = G \sqcup \bigcup_{S \subseteq D} F_S
$$

from the unions of the interiors of the boundary faces. To do so we define ‘gluing maps’ from the putative normal bundles to the $F_S$ which have fiber at a point of $F_S$ a partial compactification of the corresponding group $A_P$ and show that this is an hd-compactification.

For analytic purposes it is of prime importance to describe the behavior of the invariant vector fields for the action of $G$ on $\overline{G}$; this can be seen from the root space decomposition as discussed by Knapp [7], Wallach [14] and Varadarajan [13].

To this end we observe that the flag varieties carry iterated tangent structures. If $P \in \mathcal{F}_S$ is a parabolic subgroup in a fixed conjugacy class then the transitive action of $K$, by conjugation, on $\mathcal{F}_S$ has isotropy group $K \cap P$ at $P$. In the Langlands decomposition $P = M_P A_P N_P$ the Lie algebra $a_P$ of $A_P$ has rank $s = \#(D \setminus S)$. The positive root vectors, joint eigenvectors for the conjugation action of $a_P$ on the Lie algebra span the Lie algebra of $n_P$. The eigen-decomposition of $\mathfrak{k}$ defines a surjective map $\mathfrak{k} \twoheadrightarrow n_P$ with null space $\mathfrak{k} \cap m_P$, the Lie algebra of the isotropy group. The decomposition of $n_P$ into eigenspaces with joint eigenvalues $\alpha \cdot a_P$ thus induces a corresponding filtration

$$
E_\alpha^\alpha \subset T_P F_S.
$$

The action of $K$ extends the filtration to $T^* \mathcal{F}_S$ by subbundles labeled by $s$-multi-indices satisfying

$$
E^\alpha \subset T^* \mathcal{F}_S \forall \alpha \in \mathbb{N}^s, \ E^\alpha = T^* \mathcal{F}_S \text{ for some } \alpha
$$

$$
\alpha \leq \beta \implies E^\alpha \subset E^\beta,
$$

$$
E^\alpha + E^\beta \subset E^{\alpha + \beta},
$$

$$
[V_\alpha, V_\beta] \subset V_{\alpha + \beta}, \ V_\alpha = C^\infty(\mathcal{F}_S; E^\alpha).
$$

The iterated structures on intersecting boundary faces are related in such a way that they induce a Lie algebroid structure $E_R \subset b^* \mathcal{G}$ consisting of the vector fields tangent to all boundary faces, to the fibers of the left fibration to $\mathcal{F}_S$ induced by (1) for each $S$ and with normal vanishing properties corresponding to the $E_\alpha$.

As noted above there is a direct relationship between the wonderful compactification of the adjoint form of a complex semisimple group and the hd-compactification.

**Prop-W.** The real blow up of the exceptional divisors in the wonderful compactification of the adjoint group of a complex semisimple group is an hd-compactification.
At the boundary of the hd-compactification Haar measure takes the form
\[ dg = \rho_*^{-\sigma} \nu_b \]
where \( \nu_b \) is a non-vanishing smooth b-measure, \( \rho_* \) is a vector of defining functions for the boundary hypersurfaces (corresponding to maximal parabolic subgroups, so to the sets \( D \setminus \{ n \} \) for \( n \in D \)) and \( \sigma \) is the multiindex given by the sum of the positive roots.

As already noted in [1] the utility of the hd-compactification is illustrated by the fact that Harish-Chandra’s Schwartz space takes a relatively simple form with respect to the hd-compactification.

**Prop-H-C.** Harish-Chandra’s Schwartz space is the space of conormal functions with respect to the boundary of \( \overline{G} \) of log-rapid decay with respect to \( L^2(G) \).

This was proved for SL(2, \( \mathbb{R} \)) in [1] but the argument is sufficiently general to apply here.

In §1 general properties of groups actions on manifolds with corners are discussed and the characterization of Harish-Chandra’s Schwartz space is recalled from [1]. The special case of SL(2, \( \mathbb{R} \)) is treated in §2 where the hd-compactification is obtained by blow-up of the sphere in the homomorphisms. For the closely related case of SL(\( n, \mathbb{C} \)) both the wonderful and the hd-compactification are obtained by similar methods in §3. The construction in the general case is contained in §4.

1. **Group actions on manifolds with corners**

First consider the smooth action of a compact Lie group on a compact manifold with corners; the fact that the boundaries are ‘one-sided’ imposes constraints on such actions since the isotropy algebra at boundary points must act trivially on the normal bundle to the boundary.

**Lemma 1.** If a compact Lie group acts smoothly on a connected compact manifold with corners and acts trivially on one boundary face then it acts trivially.

**Proof.** This is a consequence of linearizability of compact group actions. Linearizing the action around a point in the interior of a boundary hypersurface shows that the local action is the same as that on the spherical normal bundle. If the action on the hypersurface is trivial then, from the one-sided nature of the boundary, there is no \( \mathbb{Z}_2 \) action between the two normal directions so the action on the normal bundle is also trivial and hence the action is trivial nearby. Again the triviality of the action on an open set implies that the action is trivial on that component of the space. Applying this argument iteratively, triviality on any one boundary face implies triviality on all faces of which it is a boundary hypersurface, and hence triviality on the full manifold. \( \square \)

In particular a smooth action of a connected compact group on a connected compact manifold with corners up the dimension, so including points, is necessarily trivial. This includes simplexes.

Conversely there is a similar inheritance of freedom of an action.

**Proposition 1.** If a compact group acts smoothly on a compact manifold with corners with the action free on the interior then it acts freely.
Proof. Consider the isotropy group at any boundary point, which is always in the interior of some boundary face. By linearizability, the fact that the action is free in the interior implies that the isotropy group must act freely and linearly on the fibres of the inward-pointing part of the spherical normal bundle; since this is a simplex the action must preserve all boundary faces and so is trivial and hence so is the isotropy group and the full action must be free. □

One of the most fundamental requirements that we place on the compactification of a group $G$ as the interior of a compact manifold with corners in Definition 1 is that the left and right actions of $G$ extend smoothly to $\overline{G}$. The existence of such a smooth extension of the action is equivalent to the condition that the left- and right-invariant vector fields extend from the interior to be $b$-vector fields on $\overline{G}$. Thus each action gives a Lie algebra homomorphism

$$\mathfrak{g} \rightarrow \mathcal{V}_b(\overline{G}) = C^\infty(\overline{G}; b^T).$$

A compactification for which the actions extends smoothly is by no means unique since one can make ‘transcendental’ changes to $\overline{G}$ under which the smooth action of the vector fields lifts. Suppose one has such a (weak) compactification. Take a defining function for one of the boundary hypersurfaces, $x$, and replace it by

$$t = \frac{1}{\log x}, \quad x = \exp(-\frac{1}{t}).$$

The manifold with new $C^\infty$ structure generated near this boundary hypersurface by $t$, in addition to $x$ and all other smooth functions, is homeomorphic to the original $\overline{G}$ and maps smoothly back to it. Moreover, a simple computation shows that the tangential vector fields lift to be smooth and tangential on the manifold, since $x\partial_x$ lifts to $t^2\partial_t$. Thus the ‘blown up’ manifold is still a weak compactification but is, in a sense, much larger and certainly not equivalent.

The condition (1) in the definition of an $hd$-compactification in the Introduction prevents this extreme sort of ‘enlargement’.

Definition 2. The action of a Lie group on a compact manifold with corners is $b$-normal if at each boundary point the $b$-normal space is spanned by elements of the Lie algebra of the isotropy group.

In fact this still does not prevent ‘blow-up’ indeterminacy, as discussed above, from arising. However, the transformations can no longer be transcendental. Still, the introduction of $s = x^k$ in place of $x$ at a boundary hypersurface, for any positive integer $k$, again defines a smooth map from the ‘resolved’ space which is a homeomorphism and under which the $b$-vector fields lift to smooth $b$-vector fields – so a smooth action lifts to be smooth. In this case the $b$-normal $x\partial_x$ is replaced by $k^{-1}t\partial_t$. Thus $b$-normality is also preserved. In consequence we need to impose the further ‘minimality’ condition, (1), in the definition of an $hd$-compactification, to ensure uniqueness.

The action of $G$ on itself is of course transitive hence so is the combined right and left action of $G \times G$. In (1) we impose a condition which, in a weakened sense, extends this to the compactification.

Definition 3. The action of a Lie group on a manifold with corners is $b$-transitive if its Lie algebra generates the $b$-vector fields as a $C^\infty$ module.
A \( b \)-compactification \( G \to \overline{G} \) is an inclusion as the interior of a compact manifold with corners such that the right and left actions extend smoothly and are \( b \)-normal and the combined action \( G \times G \) is \( b \)-transitive.

Since the interiors of all boundary faces are (by assumption) connected, a \( b \)-transitive action is transitive on these interiors.

**Lemma 2.** For a \( b \)-compactification the isotropy groups of the left and right actions at the interior are conjugate over the interior of any boundary face.

*Proof.* Since the right and left actions commute the orbits through points in the interior of any one boundary face are equivariantly diffeomorphic. \( \square \)

The proof of the analogous result in [1] only uses properties holding for a \( b \)-compactification and (5) so yields Prop-H-C of the Introduction.

### 2. \( \text{SL}(n, \mathbb{R}) \)

Before proceeding to the construction of an \( \text{hd} \)-compactification in general we consider the case of \( \text{SL}(n, \mathbb{R}) \) (and subsequently \( \text{SL}(n, \mathbb{C}) \)). For these standard groups we give an explicit construction of an \( \text{hd} \)-compactification which illustrates and guides the more abstract construction below. For later convenience in place of \( \mathbb{R}^n \) consider an oriented real Euclidean vector space, \( V \), of dimension \( n \). Then consider the inclusions

\[
(2.1) \quad \text{SO}(V) \subset \text{SL}(V) \subset \text{GL}(V) \subset \text{Hom}(V),
\]

where \( \text{Hom}(V) = \text{Hom}(V, V) \) is the space of linear maps. We give \( \text{Hom}(V) \) the usual Hilbert-Schmidt norm \( \|e\| = \text{Tr}(e^*e)^{\frac{1}{2}} \).

Let \( \text{SH}(V) \subset \text{Hom}(V) \) be the unit sphere and consider the map given by (positive) radial scaling

\[
(2.2) \quad \text{SL}(V) \to \text{SI}(V) \subset \text{SH}(V)
\]

where \( \text{SI}(V) = \text{SH}(V) \cap \text{GL}(V) \) is the open subset consisting of the invertible homomorphisms of norm one.

**Lemma 3.** Radial scaling (2.2) is a diffeomorphism onto its range \( \text{SI}_+(V) \subset \text{SI}(V) \), the open subset of \( \text{SH}(V) \) where the determinant is positive.

*Proof.* The inverse of (2.2) is

\[
(2.3) \quad \text{SI}_+(V) \ni e \to e/(\det(e))^{\frac{1}{n}} \in \text{SL}(V)
\]

so this map is a diffeomorphism. \( \square \)

The \( \text{hd} \)-compactification of \( \text{SL}(V) \) is obtained by compactifying the range \( \text{SI}_+(V) \) of (2.2) and this in turn is accomplished through blow-up of \( \text{SH}(V) \). Again for use in inductive arguments below we proceed slightly more generally. Suppose \( W_1 \) and \( W_2 \) are two oriented real Euclidean vector spaces of the same dimension, \( k \). Then consider the unit sphere (always in the Hilbert-Schmidt norm) \( \text{SH}(W_1, W_2) \subset \text{Hom}(W_1, W_2) \). The orthogonal groups \( \text{SO}(W_i) \) act on \( \text{SH}(W_1, W_2) \) on the right and left and both these actions are free on \( \text{SI}(W_1, W_2) \subset \text{SH}(W_1, W_2) \), the open set of invertible homomorphisms. Following [2] these actions may be resolved by successive blow-up of the isotropy types which are the

\[
(2.4) \quad S_q(W_1, W_2) = \{e \in \text{SH}(W_1, W_2); e \text{ has corank } q\}, \ 1 \leq q \leq k-1;
\]
here $q$ is the ‘depth’ of the stratum.

Denote the resolution, (obtained by blow up in order of increasing dimension or decreasing depth),

$$\mathcal{SH}(W_1, W_2) = [\mathcal{SH}(W_1, W_2), S_*].$$

In case $W_1 = W_2 = W$ we set $\mathcal{SH}(W, W) = \mathcal{SH}(W)$ and for $W = \mathbb{R}^k$, $\mathcal{SH}(W) = \mathcal{SH}(k)$, etc.

Diffeomorphisms fixing the center lift under blow-up so the orthogonal actions lift and Euclidean isomorphisms $I_i : W_i \rightarrow C^k$ $i = 1, 2$ result in a commutative square covering the blow-down maps

$$\mathcal{SH}(W_1, W_2) \xrightarrow{\beta} \mathcal{SH}(k)$$

Proposition 2. The resolution (2.5) is a compact manifold with corners up to codimension $k - 1$ with the boundary hypersurface corresponding to $S_q$ fibering over the double Grassmannian with fiber modeled (inductively) by resolved spaces

$$\mathcal{SH}(W_1, W_2) \xrightarrow{\beta} \mathcal{SH}(k)$$

where the fiber above $(U_1, U_2)$ is $\mathcal{SH}(U_1, U_2) \times \mathcal{SH}(U_1^\perp, U_2^\perp)$.

Proof. Consider the deepest stratum $S_{k-1} \subset \mathcal{SH}(W_1, W_2)$ with maximal isotropy group. Each element of $S_{k-1}$ has one-dimensional range and $k - 1$ dimensional null space. So these are determined precisely by the pair of lines $U_1$, the range of the adjoint, and the range $U_2$ and the element itself lies in the two-point space $\mathcal{SH}(U_1, U_2) \ni e$.

Thus $S_{k-1}$ is a double cover of the product $\mathbb{P}W_2 \times \mathbb{P}W_1$ of the real projective spaces with fiber $\mathcal{SH}(U_1, U_2)$. By the collar neighbourhood theorem, a neighbourhood of $S_{k-1}$ in $\mathcal{SH}(W_1, W_2)$ fibers over $S_{k-1}$ and can be taken to be form

$$\mathcal{SH}(k - q) \times \mathcal{SH}(q) \xrightarrow{H_q} \text{Gr}(W_1, k - q) \times \text{Gr}(W_2, k - q)$$

Indeed this follows from the left and right polar decomposition of homomorphisms close to $e$. Both polar parts must have only one eigenvalue near 1, hence with smooth eigenspace, and this gives the projection onto $S_{k-1}$ with the remainder in $\text{Hom}(U_1^\perp, U_2^\perp)$. This identifies the normal bundle to $S_{k-1}$ and hence the elements of the unit normal sphere bundle, which is the face $H'_{k-1}$ produced by blow-up of $S_{k-1}$ (before the other blow-ups) as a fiber bundle

$$\mathcal{SH}(k - 1) \xrightarrow{H'_{k-1}} S_{k-1}$$

with fibre $\mathcal{SH}(U_1^\perp, U_2^\perp)$ at $(U_1, U_2)$. 
The naturality of blow-up shows that the subsequent steps in the resolution of $\overline{SH}(W_1, W_2)$ restrict to $H_{k-1}'$ as the corresponding resolution of $\overline{SH}(U_1^+, U_2^+)$. Thus the final hypersurface $H_{k-1}' \subset \overline{SH}(W_1, W_2)$ becomes the fiber bundle

\[
\begin{array}{c}
\text{SH}(k-1) \\
\downarrow \\
S_{k-1}
\end{array} \xrightarrow{\text{H}_{k-1}} \begin{array}{c}
\text{PW}_2 \times \text{PW}_1 \\
\end{array}
\]

with total fiber in the second case $\overline{SH}(U_1, U_2) \times \overline{SH}(U_1^+, U_2^+)$. This is (2.7) for $q = 1$.

The structure of the other faces follows iteratively. Namely after the blow-up of the faces $S_{k-j}$ for $j < q$, the stratum $S_{k-q}$ in (2.7) is resolved to the bundle

\[
\begin{array}{c}
\text{SH}(k-q) \\
\downarrow \\
\text{Gr}(W_2, k-q) \times \text{Gr}(W_1, k-q)
\end{array}
\]

where the fiber is $\overline{SH}(U_1, U_2)$ over the pair $(U_2, U_1)$ in the Grassmannian. The fiber of the normal bundle is precisely $\overline{SH}(U_1^+, U_2^+)$ by essentially the same argument in terms of the two polar decompositions. The subsequent blow-ups resolve the fibers to give (2.7).

We define a compactification of $\text{SL}(V)$ by taking the closure of the image of (2.12):

\[
\overline{\text{SL}}(V) = \text{Closure of } \text{SL}_+(V) \text{ in } \overline{\text{SH}}(V).
\]

In fact this is one of the two components of $\overline{\text{SH}}(V)$; $\overline{\text{SH}}(V)$ is an hd-compactification of $\text{SL}(V)$ and from this we deduce that

**Proposition 3.** The compact manifold in (2.12) is an hd-compactification of $\text{SL}(V)$ for an oriented Euclidean space $V$.

**Proof.** The description of the iterative blow-up allows the blow-down map to be described near a point of (exactly) codimension $r$ in $\overline{\text{SH}}(W_1, W_2)$. This corresponds to a subset

\[
\{q \in \{1, \ldots, k-1\}, \#(q) = r, q_1 < q_2 < \cdots < q_r \leq k-1.
\]

Then the corresponding boundary face $F_{\overline{q}} = H_{q_1} \cap H_{q_2} \cdots \cap H_{q_r}$ fibres over the product of two copies of the flag manifold

\[
\mathcal{F}(W_j, \overline{q}) = \{U_j, q_j\}, \ U_{j,q_r} \subset U_{j,q_{r-1}} \subset \cdots \subset U_{q_1} \subset W_i, \ \dim(U_{j,q_j}) = k - q_j.
\]

We set $q_{r+1} = k$ and $q_0 = 0$ for notational convenience and $U_{j,q_{r+1}} = W_j, U_{j,q_0} = \{0\}$. Then the flag defines a sequence of orthogonal projections,

\[
\pi_{j, \overline{q}} \text{ onto } V_{j, q_j} = U_{j, q_j} \ominus U_{j,q_{j+1}}.
\]

There are $r$ local defining functions $\tau_{q_i} \in [0, \epsilon)$ and an element of $\overline{\text{SH}}(W_1, W_2)$ projecting to this neighbourhood is of the form

\[
\gamma = \sum_{i=1}^{r} (\prod_{i' > i} \tau_{q_{i'}}) \pi_{2,q_i} e_{q_i} \pi_{1,q_i^*}, \ e_{q_i} \in \text{SH}(V_{1,q_i}, V_{2,q_i}).
\]
This point, after radial rescaling, lies in $\overline{SL}(V)$ if all the $\tau_q \geq 0$ and the determinant at $\tau_q = 1$ is positive. From \( (2.23) \) the corresponding point in $\text{SL}(n, \mathbb{R})$, for all $\tau_i > 0$, is

\[
g = a \left( \prod_{i=1}^{r} \tau_i^{d_i/n} \right) \sum_{i=1}^{r} (\prod_{i' > i} \tau_{i',q_i'}) \pi_{2,q_i} e_{q_i} \pi_{1,q_i}.
\]

where $a > 0$ is locally smooth. The inverse is therefore of the form

\[
g^{-1} = a \left( \prod_{i=1}^{r} \tau_i^{(d_i-n)/n} \right) \sum_{i=1}^{r} (\prod_{i' \leq i} \tau_{i,q_i}) \pi_{1,q_i} e_{q_i}^{-1} \pi_{2,q_i}.
\]

From this it follows that inversion is a diffeomorphism on $\overline{SL}(n, \mathbb{R})$. That the right and left actions extend smoothly follows from the construction since these actions are smooth after projection to $SH$ and fix the centers of blow-up, so lift smoothly to $\overline{SL}$.

A point in the flag variety determines a parabolic subgroup although this involves a consistent choice of Weyl chamber. Consider the standard flag with $A$ into a product of half-lines $A$ and this action conjugates the Iwasawa decomposition as well, giving the corresponding groups $A_B$ and $N_B$, the centers of blow-up, so lift smoothly to $\overline{SL}$.

A point in the flag variety determines a parabolic subgroup although this involves a consistent choice of Weyl chamber. Consider the standard flag with $A$ into a product of half-lines $A$ and this action conjugates the Iwasawa decomposition as well, giving the corresponding groups $A_B$ and $N_B$, the centers of blow-up, so lift smoothly to $\overline{SL}$.

\[
\text{SL}(n) = \text{SO}(n)AN
\]

where $A$ is the subgroup of positive diagonal matrices and $N$, corresponding to a choice of positive roots, consists of the upper diagonal matrices with all diagonal entries one. Thus the Lie algebra of $N$ is spanned by the elementary matrices $E_{ij}$ with one non-zero entry, 1 at $(i,j)$ for $j > i$. The Lie algebra, $\mathfrak{a}$, of $A$ consists of the diagonal matrices, with diagonal entries $\alpha_i$ satisfying the trace condition

\[
\sum \alpha_i = 0.
\]

The $E_{ij}$ are joint eigenvectors for the adjoint action of $\mathfrak{a}$ with collective eigenvalues, the positive roots, being the elements $\alpha_i - \alpha_j$ of $\mathfrak{a}'$. The primitive roots are the $\alpha_i - \alpha_{i+1}$ for $1 \leq i \leq n-1$. The dual basis of $\mathfrak{a}$, the coroots, are the diagonal matrices with first $k$ diagonal entries $1 - \frac{k}{n}$ and remaining entries $-\frac{k}{n}$. This decomposes $\mathfrak{a}$ into a sum of one-dimensional spaces, with basis elements, and hence decomposes $A$ into a product of half-lines $A_i$ with coordinates (the coweights) $t_i$ so that the coroots become $t_i \partial_i$. In terms of these coordinates on $A$ an element of $A_i$ is of the form

\[
\text{diag}(t_i^{1-\frac{k}{n}}, \ldots, t_i^{1-\frac{k}{n}}, t_i^{\frac{k}{n}}, \ldots, t_i^{\frac{k}{n}})
\]

Observe that as $t_i \to \infty$ the projection of this 1-parameter group into $\overline{SL}(n, \mathbb{R})$ is smooth up to the boundary in terms of the parameter $\tau_i = t_i^{-1}$, where it meets $H_i$ as a normal vector field with $z_i$ extending to a local boundary defining function.

The other Borel subgroups $B \subset \text{SL}(n, \mathbb{R})$ are conjugate, under the action of $\text{SO}(n)$, to this basic choice, so the flag variety is identified

\[
\mathcal{F}(\mathbb{R}^n, \{1, \ldots, n-1\}) = \text{SL}(n)/\text{SO}(n),
\]

and this action conjugates the Iwasawa decomposition as well, giving the corresponding groups $A_B$ and $N_B$ replete with their root space decompositions.

From this we conclude that in the decomposition \( (2.15) \) near the ‘Borel’ face, of the maximal codimension $n - 1$, the parameters $\tau_i$ correspond (at least to first
order at the boundary) to the action on the right by $A_2$ or on the left by $A_2$, on the fiber of the boundary above $(B, B') \in \text{SL}(n)/\text{SO}(n) \times \text{SL}(n)/\text{SO}(n)$. This certainly shows the b-normality of the extended action since it persists nearby and then extends to all boundary points using the left and right actions of $\text{SO}(n)$.

The b-transversality also follows directly from this analysis. At the maximal codimension face the Lie algebra of the right action projects surjectively to the tangent space of the right flag variety (2.21) and the left action projects surjectively to the left factor. The b-normal space is in the range of either action and the fiber is discrete. By smoothness, the surjectivity of the map from $\text{SL}(n) \times g \times g$ to $b^T \text{SL}(n)$ extends to an open neighborhood. Every orbit of the $G \times G$ action intersects such an open neighborhood of this face so b-transitivity extends to the whole space. In fact this is also clear from the analysis below of the behavior near a general boundary face.

Finally, it remains to show that the minimality condition holds. Again it suffices to compute the behavior of the Lie algebra for the right action near one point in the boundary face of maximal codimension and then extend the result using the action of $G \times G$. Choosing the base point to be the standard flag in (2.16) with the $e_i = 1$ in this case the action of the element $E_{ij} \in \mathfrak{n}$, $j > 1$, on the right acts on the flag on the right as \( \frac{1}{2}(E_{ij} - E_{ji}) \in K \). The infinitesimal action of $\mathfrak{n}$ on the right also shifts the left flag in the same way, in this case after conjugation by $\gamma$. This the right action of $E_{ij}$ projects on the left flat

\[
(2.22) \quad \frac{1}{2} \left( \prod_{k \geq j - 1} \tau_k \right) \kappa_{ij}, \quad \kappa_{ij} = (E_{ij} - E_{ji}), \quad j > i.
\]

It follows that nearby the projection of the right action of the Lie algebra onto the left flag manifold vanishes at the boundary and there is an element (with $j = i + 1$) vanishing simply on a given hypersurface. This the minimality condition is also satisfied.

From the analysis of the action on the right of the Lie algebra leading to (2.22) it follows that this compactification has the properties discussed in the Introduction. Thus on the flag variety corresponding to Borel subgroups, $\mathcal{F} = \text{SO}(n)/Z$, the tangent bundle is stratified by one-dimensional subbundles $E^\alpha$ where $\alpha$ runs over the $(n-1)$-multiindices with entries 1 forming an interval and all others vanishing. The commutators of these subbundles are contained in the sums unless the two intervals are non-overlapping but contiguous, in which case the commutator, modulo the sum, span the bundle corresponding to the union.

3. $\text{SL}(n, \mathbb{C})$

For the complex group, $\text{SL}(n, \mathbb{C})$, there are two distinct generalization of the compactification via the resolution of the projective image as carried out above. In the first we proceed by passing systematically to the complex category and in the second by proceeding by direct analogy. The first approach yields the wonderful compactification of the adjoint group $\text{SL}(n, \mathbb{C})/Z$ where the center is the multiplicative group of $n$th roots of unity. The second approach gives an $\text{hd}$-compactification.

Consider an Hermitian vector space $V$ of complex dimension $n$ then in place of (2.1)

\[
(3.1) \quad \text{SU}(V) \subset \text{SL}(V) \subset \text{GL}(V) \subset \text{hom}(V)
\]
where the last three spaces are complex. Let $\text{Ph}(V) = (\text{hom}(V) \setminus \{0\})/\mathbb{C}^\times$ be the projective space of homomorphisms. Projection gives

$$\text{(3.2)} \quad \text{SL}(V) \longrightarrow \text{PI}(V) \subset \text{Ph}(V)$$

where the center is mapped to the image of the identity and the range, $\text{PI}(V)$ is an open dense subset of $\text{Ph}(V)$. The complement $\text{Ph}(V) \setminus \text{PI}(V)$ is the image of the non-vanishing, non-invertible, matrices and so is again stratified by corank,

$$\text{(3.3)} \quad \text{Ph}(V) \setminus \text{PI}(V) = \bigcup_{1 \leq k < n} S_k.$$ 

These are all complex submanifolds, although only $S_{n-1}$ is compact. As in the real case they are the isotropy types for the action of $\text{SU}(V)/\mathbb{Z}$ which acts freely on $\text{PI}(V)$.

Again in this case we may blow these submanifolds up, iteratively, in the complex sense, in order of increasing dimension to produce the resolved space which we denote tentatively as

$$\text{(3.4)} \quad \overline{\text{Ph}}_{\mathbb{C}P}(V) = [\text{Ph}(V); S_*]_{\mathbb{C}}.$$

To see that this is well-defined as a compact complex manifold note that an element of $S_{n-1}$ is the projective image of a complex homomorphism of rank one, so

$$\text{(3.5)} \quad S_{n-1} = \mathbb{F}V \times \mathbb{F}V$$

where the first point is the orthocomplement of the null space and the second the range. The normal bundle to $S_{n-1}$ can then be identified with the bundle with fibre at a point the homomorphisms from the null space to the orthocomplement. Blowing up replaces $S_{n-1}$ be its projectivized normal bundle $D'_{n-1}$

$$\text{(3.6)} \quad \overline{\text{Ph}}(\mathbb{C}^{n-1}) \longrightarrow D'_{n-1} \quad \downarrow \quad \mathbb{F}V \times \mathbb{F}V.$$

In a small neighbourhood of this exceptional divisor the resolved manifold is a bundle over $\mathbb{F}V \times \mathbb{F}V$ with fiber a neighborhood of the zero section of the blow-up of the origin in the homomorphism bundle from null space to orthocomplement of range. The closures of the other strata meet this bundle in the corresponding strata modeled on $S_j(\overline{\text{Ph}}(\mathbb{C}^{n-1}))$. Proceeding by induction as in the real case these lower depth strata can be blown up in order and replace the divisor (3.6) by its resolution giving the bundle

$$\text{(3.7)} \quad \overline{\text{Ph}}(\mathbb{C}^{n-1}) \longrightarrow D_{n-1} \quad \downarrow \quad \mathbb{F}V \times \mathbb{F}V.$$
It follows that the full resolution is possible with the stratum $S_j$ replaced by a divisor which is a complex bundle over the product of two copies of the Grassmannian
\begin{equation}
\mathbb{P}h(C^j) \times \mathbb{P}h(C^{n-j}) \to D_j \\
\text{Gr}(V, n-j) \times \text{Gr}(V, n-j).
\end{equation}

**Proposition 4.** This resolution, $\mathbb{P}h(V)$, of $\mathbb{P}h(V)$ is the wonderful compactification of the adjoint group $SL(n,C)/Z$.

This is consequence of the characterization of the wonderful compactification as the unique regular compactification with a single compact orbit, see Uma [12] and Brion [4] and references therein.

Real and complex blow-up of a complex submanifold are related in a simple way, namely the real blow-up is the complex blow-up followed by the real blow up of the resulting divisor, as a real submanifold of codimension two. For normally intersecting divisors this carries over iteratively – the real blow-ups can all be performed after the complex ones because of transversality.

Now Prop-W in the Introduction can be stated more explicitly.

**Proposition 5.** The blow-up, in the real sense, all of the divisors in (3.4) gives an $hd$-compactification of the adjoint group $SL(n,C)/Z$:
\begin{equation}
SL(n,C)/Z = [\mathbb{P}h V; S_*]_R.
\end{equation}

Rather than prove this immediately we examine the compactification of $SL(n,C)$ obtained by following the procedure for $SL(n,R)$ more closely. That is, first project radially into the sphere in the Hermitian vector space
\begin{equation}
SL(V;C) \to Sh(V).
\end{equation}
Again this is a diffeomorphism but now onto the smooth hypersurface, $Sh_+(V)$, in $Sh(V)$ where the determinant is positive.

The actions of SU($V$) on $Sh(V)$, left and right, fix $Sh_+(V)$ and act freely on it. The isotropy types for the action on $Sh(V)$ are again the submanifolds $S_q \subset Sh(V)$ of corank $q$. We define a resolution of $SL(V)$ by taking the closure in the (real) resolution
\begin{equation}
\overline{SL(V)} \subset [Sh(V), S_*].
\end{equation}

The determinant is well-defined and non-zero on $SI(V) \subset SH(V)$ and we define a normalized version by
\begin{equation}
\widehat{\det} : SI(V) \ni g \mapsto \frac{\det g}{|\det g|} \in \mathbb{T}.
\end{equation}

Now the SU($V$) actions extend to actions of U($V$) with the extended actions free on SI($V$). The isotropy types are the same as for the SU($V$) action, i.e. the $S_q$. Thus in the case $W_1 = W_2 = V$ the resolution (3.11) also resolves the U($V$) actions, which therefore become free on $\overline{SI(V)}$. The normalized determinant factors through the action of U($V$),
\begin{equation}
\widehat{\det}(ug) = \det(u)\widehat{\det}(g)
\end{equation}
and it follows from this that \( \hat{\text{det}} \) extends smoothly to \( \overline{SH}(V) \) and has non-vanishing differential everywhere. In fact the differential must be independent of all conormals as well, so the level surfaces are well-defined p-submanifolds of \( \overline{SH}(V) \).

Thus the definition (3.11) does lead to a compact manifold with corners, with extended actions of \( \text{SL}(V) \), since the initial action preserves the \( S_q \).

That this is an hd-compactification now follows as in the real case discussed above. Moreover there is a simple relation between the real and complex cases:

**Proposition 6.** The closure of \( \text{SL}(n, \mathbb{R}) \) in the compactification \( \overline{\text{SL}}(n, \mathbb{C}) \) is a p-submanifold which is an hd-compactification.

4. DIRECT CONSTRUCTION

In the construction of an hd-compactification of \( G \), a real reductive group with compact center, we use the root space decomposition, see for example [7], which we proceed to summarize briefly. First we make the choice of a ‘real Cartan subgroup’ of \( G \), a maximal product of multiplicative half-lines, \( A \) with Lie algebra \( \mathfrak{a} \) and compatible Cartan involution. The adjoint action of \( \mathfrak{a} \) on the full Lie algebra, \( \mathfrak{g} \), is symmetric and decomposes \( \mathfrak{g} \) into the joint null space and a direct sum of eigenspaces with corresponding joint eigenvectors, the roots, in \( \mathfrak{a}^* \). The choice of a positive subspace, \( R_+ \), of the roots induces an Iwasawa decomposition

\[
G = KAN
\]

where \( K \) is a maximal compact subgroup and \( N \) is the unipotent group generated by the span of the strictly positive root (eigen)_vectors. This also induces a decomposition

\[
A = \prod_{i \in D} A_i, \quad \tau_i : A_i \simeq \mathbb{R}^+, \quad \#(D) = r,
\]

as a product of half-lines labeled by the nodes, \( D \), of the Dynkin diagram. The parameters in the factors which may be identified with the coweights. Thus \( \mathfrak{a} \) is spanned by the \( \tau_i \partial_{\alpha_i} \) and each of the root vectors spanning the Lie algebra \( \mathfrak{n} \) of \( N \) has non-negative homogeneity under the adjoint action \( A_i \) (with at least one positive exponent). This decomposes \( \mathfrak{n} \) as a graded algebra under the conjugation action of the \( A_i \):

\[
\text{Ad}(a)n = \tau^n n, \quad n \in \mathfrak{n}_\alpha, \quad a \in A, \quad n = \bigoplus_{\alpha \in R_+} \mathfrak{n}_\alpha
\]

\[
\mathfrak{n}_{\alpha + \beta} = [\mathfrak{n}_\alpha, \mathfrak{n}_\beta] \text{ if } \alpha, \beta, \alpha + \beta \in R_+.
\]

These choices correspond to a particular minimal parabolic subgroup

\[
P = MAN
\]

where in this case the real-reductive part is compact. The other minimal parabolics are conjugate to this one, under the action of \( K \), and so have a similar decomposition (4.3) and (4.4).

Each conjugacy classes of parabolic subgroups correspond to a subset \( S \subset D \) where the minimal case corresponds to the empty set. To \( S \) we associate the group

\[
A_S = \prod_{i \in S} A_i \subset A.
\]
The normalizer of $A_S$ in $G$ contains a maximal real-reductive subgroup $M_S$ with compact center fixed by the Cartan involution. The corresponding parabolic subgroup has Langlands decomposition

\begin{equation}
P_S = M_S A_S N_S
\end{equation}

where the unipotent group $N_S$ has Lie algebra

\begin{equation}
n_S = \bigoplus_{\gamma \in R^+_S} n_\gamma.
\end{equation}

Here $R^+_S$ is the subset of the positive roots which have a positive value on one of the basis elements of the Lie algebra of $A_S$. This induces an action of $A_S$ of the form (4.3) for these restricted roots.

Again the conjugation action of the maximal compact subgroup in the initial Iwasawa decomposition induces similar decompositions of all the conjugate parabolics $P = M_P A_P N_P$ independent of the indeterminacy in conjugation.

Let $\mathcal{F}_S$ be the flag variety of parabolic subgroups conjugate to $P_S$. For a pair $(P', P) \in \mathcal{F}_S \times \mathcal{F}_S$ consider the space

\begin{equation}
F(P, P') = \{ g \in K \cdot M_P; M_P g = g M_P \}.
\end{equation}

**Lemma 4.** For each $S \subset D$, the spaces $F(P', P)$ have principal $M_P$ actions on the right and $M_{P'}$ actions on the left and form a smooth bundle with total space $F_S$

\begin{equation}
\begin{array}{ccc}
M_S & \longrightarrow & F_S \\
\downarrow & & \downarrow \\
\mathcal{F}_S \times \mathcal{F}_S & & \\
\end{array}
\end{equation}

and fiber modeled on $M_S$.

**Proof.** The action of $K$ on $F_S$, by conjugation on $G$, is transitive with isotropy group at $P \in \mathcal{F}_S$ the subgroup $K \cap M_P$. It follows that if $k \in K$ and $k P k^{-1} = P'$ then $F(P, P') = k M_P$ from which the result follows. $\square$

The properties of the putative compactification functor imply that it extends to principal spaces such as $F(P, P')$ so, proceeding inductively over dimension or real rank, we can define the closed boundary face corresponding to each conjugacy class of parabolics in $G$ as the fiberwise compactification of the smooth bundle (4.9) with fiber the compactification of $F(P', P) \simeq M_S$

\begin{equation}
\begin{array}{ccc}
\overline{M}_S & \longrightarrow & \overline{F}_S \\
\downarrow & & \downarrow \\
\mathcal{F}_S \times \mathcal{F}_S & & \\
\end{array}
\end{equation}

We define a partial compactification

\begin{equation}
A_S \hookrightarrow \overline{A}_S
\end{equation}

of each $A_S$ by adding a point at infinity to each factor, with smooth structure given by the parameter $\tau_i^{-1}$ where $\tau_i$ is the multiplicative variable in (4.2). Thus $\overline{A}_S$ is a product of $\#(D \setminus S)$ half-closed intervals. The conjugation action of $K$ induces
corresponding compactifications \( \overline{A}_P \) of each \( A_P \) and so defines a bundle over \( \mathcal{F}_S \) and hence another level on the fiber bundles (4.10):

\[
\begin{array}{c}
\overline{A}_S \longrightarrow N_+ \mathcal{F}_S \\
\downarrow \quad \downarrow \\
\overline{M}_S \longrightarrow \overline{F}_S \\
\downarrow \quad \downarrow \\
\mathcal{F}_S \times \mathcal{F}_S.
\end{array}
\]

The fibre of \( N_+ \mathcal{F}_S \) is \( \overline{A}_P \) over \((P, P')\) but may also be naturally identified with \( \overline{A}_{P'} \).

The presumptuous notation \( N_+ \mathcal{F}_S \) presages the identification with the inward-point normal bundle to \( \mathcal{F}_S \) as a boundary face of \( \overline{G} \).

To construct \( \overline{G} \) as a smooth space we define gluing maps, the most fundamental one (recalling that we proceed by induction) is for the minimal face corresponding to (4.12) with \( S = \emptyset \); in this case we drop the subscript. Then \( M \) is already compact. The partially compactified fibre at each point \((B, B') \in \mathcal{F} \times \mathcal{F}\) has a point of maximal codimension, corresponding to passage to infinity in all factors \( A_i \subset A \). This defines a section, realizing the zero section of \( N_+ \mathcal{F} \); the bundle \( \mathcal{F} \) in this case is already compact.

**Proposition 7.** The map

\[
\gamma : N_+ \mathcal{F} \ni (m, a) \mapsto ma \in G, \ m \in \mathcal{F}(P, P'), \ (P', P) \in \mathcal{F} \times \mathcal{F}
\]

is smooth and if \( \overline{O} \subset N_+ \mathcal{F} \) is a small (relatively) open neighborhood of the zero section of \( N_+ \mathcal{F} \), corresponding to minimal parabolic subgroups, then \( \gamma \) is a diffeomorphism of \( \overline{O} \), the intersection of \( \overline{O} \) with the interior of \( N_+ \mathcal{F} \) as a manifold with corners, onto an open subset of \( G \).

In fact we can take \( \overline{O} \) to be invariant under the left and right actions of \( K \).

**Proof.** The map is well-defined and smooth. In the discussion above we have chosen a base minimal parabolic and this induces an identification \( \mathcal{F} = K/Z \). The total space of the bundle \( \mathcal{F} \), the boundary face here is \( K \times Z \) and the map becomes

\[
g = k_1ak_2, \ k_i \in K, \ a \in A
\]

which descends to \( K \times Z \) \( K \times A \). This is the Cartan decomposition. In the open set \( O \), where all the components of \( a \) are large, the decomposition is unique, up to the indeterminacy in the center. The factors may be chosen smoothly, locally, which shows (4.13) to be a smooth bijection onto its open image with smooth local inverses, i.e. a diffeomorphism. \( \square \)

To complete the construction of \( \overline{G} \) we extend the map (4.14) to the bundles corresponding to the other parabolics and then to similar maps between these bundles. For each \( S \subset D \) the total space of \( \mathcal{F}_S \) may be identified with

\[
\mathcal{F}_S \simeq K \times K_S M_S \times K_S K, \ K_S = K \cap M_S, P_S = M_S A_S N_S.
\]

Then consider the smooth map

\[
\gamma_S : K \times M_S \times K \times A_S \ni (k_1, m, k_2, a) \mapsto k_1mak_2 \in G.
\]

This descends to \( \mathcal{F}_S \), using (4.15).
Theorem 1. The total spaces of the fibrations (4.10) form the boundary stratification of an hd-compactification (4.17)

$$G = G \sqcup \bigcup_{S \subseteq D} \mathcal{F}_S$$

where the $C^\infty$ structure, near each boundary face, is fixed by the gluing maps (4.16).

Proof. As noted above, we proceed by induction over the dimension of $G$, allowing for real-reductive groups with compact centers. In particular this means that we take as given the compactification in (4.10), and hence all the $\mathcal{F}_S$ are well-defined when we come to construct $G$.

The partial order on the $S \subseteq D$ by inclusion induces a partial order on the $F_S$ which corresponds to inclusion as boundary faces. Consider two subsets $S_1 \subsetneq S_2 \subsetneq D$ – so excluding the interior which corresponds to $D$. Set $M_i = M_{S_i}$ etc, so in particular $K_i = K \cap M_i$. The partially compactified ‘normal’ spaces are products (4.18)

$$A_1 = A_2 \times A_{12}.$$ 

There is a corresponding ‘lifted’ gluing map (4.19)

$$\gamma_{12} : K \times K_2 \times M_1 \times K \times A_1 \ni (k', k'_1, m, k''_1, a_{1a_1}) \mapsto (k', k'_1 m a_{12} k''_1, k'', a_2)$$

which descends to a map (4.20)

$$N_+ F_1 = K \times K_1 \times M_1 \times K \times A_1 \longrightarrow K \times K_2 \times M_2 \times K_2 \times A_2 = N_+ F_2.$$ 

Since these are proper boundary faces, corresponding to lower dimensional groups $M_S$, the inductive hypothesis means that the maps extend to smooth maps on the compactified spaces (4.21)

$$\gamma_{12} : N_+ F_1 \longrightarrow N_+ F_2.$$ 

Moreover the gluing maps (4.16), $\gamma_i$ to $G$ for the two boundary faces factor (4.22)

$$\gamma_2 = \gamma_1 \circ \gamma_{12}.$$ 

If $U_S \subset M_S$ is an open subset with compact closure and $\overline{U}_S$ is a sufficiently small, relatively open neighborhood of the point at infinity in $\overline{A}_S$, corresponding to the zero section of the normal bundle, then the map $\gamma_S$ restricted to the image of the set $K \times U_S \times K \times O_S$ is a diffeomorphism onto its range in $G$.

Now, starting from the face of maximal codimension, we can successively choose such subsets which together cover the union of the boundary faces in the sense that for all $S$ (4.23)

$$M_S \subset U_S \cup \bigcup_{S' \subseteq S} \gamma_{SS'}(K \times M_{S'} K \times K \times O_{S'S'}).$$

These choices give the $C^\infty$ structure on $\overline{G}$. Namely all of the maps $\gamma_S$ and $\gamma_{SS'}$ are diffeomorphisms when restricted to these small domains. The conditions (4.22) mean that the maps on the components of the covering of $M_S$ in (4.23), $\gamma_S$ on $U_S$ and the composite map $\gamma_S \gamma_{SS'}$ on the other parts are consistent on overlaps. Each of these maps identifies an open neighborhood of the boundary face $F_S$ with a corresponding open subset of the inward-pointing normal bundle $N_+ F_S$ with smoothness on the overlaps, thus making $\overline{G}$ into a manifold with corners.
To see that $\overline{G}$ is an hd-compactification of $G$ we check the conditions in Definition 1. First observe the effect of inversion on the image of one of the gluing maps
\begin{equation}
(k_1 m k_2)^{-1} = k_2^{-1} m^{-1} a' k_1^{-1}.
\end{equation}
Here $(P, P')$ are reversed, $m^{-1} \in F(P', P) = F(P', P_-)$ and $a'$ is a point near infinity in $A_P$ where $P_-$ is the opposite parabolic, which corresponds to inverting $A_P$ and changing to the negatives of the roots. So this does indeed extend to a diffeomorphism.

Next consider the right action of $g \in G$ on the image of some $\gamma_S$. The action of $K$ extends smoothly (and freely) up to the boundary by conjugating the 'incoming' parabolic $P$ to $P''$ and over this action on $\mathcal{F}_S$ mapping $A_P$ to $A_{P''}$ and $F(P, P')$ to $F(P'', P')$. So it suffices to consider the action of $P$ near a boundary fiber over $(P', P)$. In the Langlands decomposition the action of $M_P$ on $F(P, P')$ is smooth, free and transitive and near the identity in $A_P$ the action factors through that on $A_P$, in particular fixing the boundary. So it remains to check the action of the unipotent group $N_P$. Since $a \in A_P$ acts on $N_P$ by conjugation, as in (4.3),
\begin{equation}
an = n_a a, \ n_a \text{ smooth on } A_P
\end{equation}
and with $n_a$ vanishing when $a$ is at the point at infinity. Thus $N_P$ fixes the boundary fibers above $(P, P')$ for all $P'$. The smoothness of the action follows. The b-normality of the action follows from the fact that at a boundary fiber above $(P, P')$ the action of $A_P$ is through scaling on the normal fiber.

This discussion shows that over the boundary face $F_S \rightarrow \mathcal{F}_S \times \mathcal{F}_S$ the right action fixes the left factor of $\mathcal{F}_S$ in the base and acts transitively on the fibers. Since inversion conjugates the right to left action and reverses the factors in the base it follows that the action of $G \times G$ is b-transitive.

Finally, that the minimality condition holds can be seen from the local homogeneity in (4.3). \hfill \square

References

[1] Pierre Albin, Panagiotis Dimakis, and Richard Melrose, Compactification of $sl(2)$, 2018.
[2] Pierre Albin and Richard Melrose, Resolution of smooth group actions, Spectral theory and geometric analysis, Contemp. Math., vol. 535, Amer. Math. Soc., Providence, RI, 2011, pp. 1–26. MR 2560748 (2012j:58008)
[3] Armand Borel and Lizhen Ji, Compactifications of symmetric and locally symmetric spaces, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 2006. MR 2189882
[4] Michel Brion, Spherical varieties an introduction, pp. 11–26, Birkhäuser Boston, Boston, MA, 1989.
[5] Corrado De Concini and Claudio Procesi, Complete symmetric varieties, Invariant theory (Montecatini, 1982), Lecture Notes in Math., vol. 996, Springer, Berlin, 1983, pp. 1–44. MR 718125
[6] Yves Guivarc'h, Lizhen Ji, and John C. Taylor, Compactifications of symmetric spaces, Progress in Mathematics, vol. 156, Birkhäuser Basel, 1998.
[7] Anthony W. Knapp, Representation theory of semisimple groups, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001, An overview based on examples, Reprint of the 1986 original. MR 1880691
[8] Rafe Mazzeo and András Vasy, Analytic continuation of the resolvent of the Laplacian on $SL(3)/SO(3)$, Amer. J. Math. 126 (2004), no. 4, 821–844. MR 2075483
[9] , Analytic continuation of the resolvent of the Laplacian on symmetric spaces of noncompact type, J. Funct. Anal. 228 (2005), no. 2, 311–368. MR 2175410
[10] T. Oshima, *A realization of riemannian symmetric spaces*, J. Math. Soc. Japan **30** (1978), no. 1, 117132.

[11] Aprameyan Parthasarathy and Pablo Ramacher, *Integral operators on the Oshima compactification of a Riemannian symmetric space*, J. Funct. Anal. **267** (2014), no. 4, 919–962. MR 3217053

[12] V. Uma, *K-theory of regular compactification bundles*, 2018.

[13] Veeravalli S. Varadarajan, *An introduction to harmonic analysis on semisimple Lie groups*, Cambridge Studies in Advanced Mathematics, vol. 16, Cambridge University Press, Cambridge, 1999, Corrected reprint of the 1989 original. MR 1725738

[14] Nolan R. Wallach, *Real reductive groups. I.*, Pure and Applied Mathematics, vol. 132, Academic Press, Inc., Boston, MA, 1988. MR 929683

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