WORD PROBLEMS IN ELLIOTT MONOIDS

DANIELE MUNDICI†

to Saunders Mac Lane, in memoriam

ABSTRACT. Algorithmic issues concerning Elliott local semigroups are seldom considered in the literature, although these combinatorial structures completely classify AF algebras. In general, the addition operation of an Elliott local semigroup is partial, but for every AF algebra $\mathcal{B}$ whose Murray-von Neumann order of projections is a lattice, this operation is uniquely extendible to the addition of an involutive monoid $E(\mathcal{B})$. Let $\mathfrak{M}_1$ be the Farey AF algebra introduced by the present author in 1988 and rediscovered by F. Boca in 2008. The freeness properties of the involutive monoid $E(\mathfrak{M}_1)$ yield a natural word problem for every AF algebra $\mathcal{B}$ with singly generated $E(\mathcal{B})$, because $\mathcal{B}$ is automatically a quotient of $\mathfrak{M}_1$. Given two formulas $\phi$ and $\psi$ in the language of involutive monoids, the problem asks to decide whether $\phi$ and $\psi$ code the same equivalence of projections of $\mathcal{B}$. This mimics the classical definition of the word problem of a group presented by generators and relations. We show that the word problem of $\mathfrak{M}_1$ is solvable in polynomial time, and so is the word problem of the Behnke-Leptin algebras $A_{n,k}$, and of the Effros-Shen algebras $\mathcal{F}_\theta$, for $\theta \in \{0, 1\} \setminus \mathbb{Q}$ a real algebraic number, or $\theta = 1/e$. We construct a quotient of $\mathfrak{M}_1$ having a Gödel incomplete word problem, and show that no primitive quotient of $\mathfrak{M}_1$ is Gödel incomplete.

1. Foreword

For quantum physical systems $\mathcal{S}$ with infinitely many degrees of freedom, von Neumann’s uniqueness theorem does not hold in general, and the Hilbert space of $\mathcal{S}$ does not exist. Infinite systems typically occur in quantum statistical mechanics, [17]. Glimm’s UHF algebras yield the standard tool for the algebraization of quantum spin systems [8] [9] [22] [17]. For instance, the Canonical Anticommutation Relation (CAR) algebra provides a mathematical description of the ideal Fermi gas. Eventually, the following general definition emerged: An $AF$ algebra $\mathfrak{B}$ is the norm closure of the union of an ascending sequence of finite dimensional $C^*$-algebras, all with the same unit, [8] [19] [23].

When the Murray-von Neumann order of projections in $\mathfrak{B}$ is a lattice we say that $\mathfrak{B}$ is an $AF\ell$ algebra.

Quite a few classes of AF algebras existing in the literature turn out to be $AF\ell$ algebras, well beyond the most elementary examples given by commutative AF algebras and finite dimensional $C^*$-algebras. An interesting class of $AF\ell$ algebras is given by AF algebras with comparability of projections (in the sense of Murray-von

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Neumann). This includes the CAR algebra and, more generally, all of Glimm’s UHF algebras, [19, 9]. AF algebras with comparability of projections also include the Effros-Shen $C^*$-algebras $\mathcal{F}_\theta$ for irrational $\theta \in [0, 1]$, [19, p.65], which play an interesting role in topological dynamics, [7, 10, 11, 46]. Also the Behncke-Leptin $C^*$-algebras $\mathcal{A}_{m,n}$ with a two-point dual [4] have comparability of projections. Other examples of AF$_{\ell}$ algebras are given by continuous trace AF algebras, liminary $C^*$-algebras with boolean primitive spectrum, [17], (see [38] for an interesting particular case), and AF algebras with a directed set of finite dimensional $*$-subalgebras, [28, 29].

The “universal” AF algebra $\mathfrak{M}$ of [36, §8] is an AF$_{\ell}$ algebra: every AF algebra with comparability of projections is a quotient of $\mathfrak{M}$ by a primitive essential ideal, [36, Corollary 8.7]. Also, every (possibly non-unital) AF algebra may be embedded into a quotient of $\mathfrak{M}$, [36, Remark 8.9]. In this paper we will consider the Farey AF$_{\ell}$ algebra $\mathfrak{M}_1$ introduced in [37]. By [40], every irrational rotation $C^*$-algebra is embeddable into some (Effros-Shen) simple quotient of $\mathfrak{M}_1$, [37, Theorem 3.1 (ii)]. $\mathfrak{M}_1$, in turn, is embeddable into Glimm’s universal UHF algebra, [11, Theorem 1.5]. As shown in [10, 41], the AF algebra $\mathfrak{Q}$ more recently considered by Boca [6] coincides with $\mathfrak{M}_1$. For the many interesting properties and applications of $\mathfrak{M}_1$ see [11, 6, 18, 37, 40, 41].

As we will see in Theorem 2.1, a main property of every AF$_{\ell}$ algebra $\mathfrak{B}$ is that the partial addition in its Elliott local semigroup is canonically extendible to the addition of an involutive monoid $E(\mathfrak{B})$. We are interested in the computational complexity of the word problem of $\mathfrak{B}$, asking whether two formulas in the language of involutive monoids denote the same Murray-von Neumann equivalence class of projections of $\mathfrak{B}$. By Elliott’s classification, this is the equivalent counterpart for $\mathfrak{B}$ of the classical word problem of a group presented by generators and relations, [31].

In Theorem 5.2, we establish the polynomial time complexity of the word problem of $\mathfrak{M}_1$. Since $E(\mathfrak{M}_1)$ is free, the word problem can be naturally posed for all AF$_{\ell}$ algebras $\mathfrak{Q}$ with singly generated $E(\mathfrak{Q})$, because the preservation properties of $K_0$ ensure that $\mathfrak{Q}$ is a quotient of $\mathfrak{M}_1$, just as $E(\mathfrak{Q})$ is a quotient of $E(\mathfrak{M}_1)$. In Theorem 6.1 and Corollary 6.5 we prove the polynomial time complexity of the word problem of Behncke-Leptin $C^*$-algebras $\mathcal{A}_{m,n}$. In Theorem 7.4 and Corollaries 7.5 and 7.6 a large set of Effros-Shen algebras $\mathcal{F}_\theta$ is shown to have a polynomial time word problem. This includes the case when $\theta = 1/e$, or $\theta \in [0, 1] \setminus \mathbb{Q}$ is a real algebraic number. In a final section a quotient of $\mathfrak{M}_1$ is constructed having a Gödel incomplete word problem. It is also shown that Gödel incompleteness does not affect primitive quotients.

Familiarity is required with Elliott’s classification of AF algebras and $K_0$, [19, 20, 21, 23], lattice-ordered abelian groups [24], and the basic properties of the equational class of Elliott involutive monoids arising from AF$_{\ell}$ algebras—better known as MV-algebras, [13, 14, 15, 42]. For the rudiments of diophantine approximation applied to Effros-Shen algebras in this paper, we refer to [27]. For background on Turing machines and algorithmic complexity see [31, 35, 44]. For complexity theoretic issues concerning real numbers we refer to [26] and [34].
2. AF algebras with a Murray-von Neumann lattice of projections

Elliott local semigroup [20], is a ring invariant consisting of equivalence classes of idempotents, together with the partially defined addition of equivalence classes of orthogonal idempotents, two idempotents \( i \) and \( j \) being equivalent iff \( xy = i \) and \( yx = j \) for some \( x \) and \( y \) in the ring.

As explained in [20] Appendix and in [21], for every AF algebra \( \mathfrak{B} \) the invariant is complete. Elements of Elliott local semigroup \( L(\mathfrak{B}) \) boil down to Murray von Neumann equivalence classes of projections, with the partially defined operation \([p] + [q] = [p + q]\) whenever \( p \) and \( q \) are orthogonal.

For any two equivalence classes \([p]\) and \([q]\) in \( L(\mathfrak{B}) \), we write \([p] \leq [q]\) iff \( p \) is equivalent to a subprojection of \( q \). This order is known as the Murray von Neumann order of (projections in) \( \mathfrak{B} \). Elliott’s partially defined addition + is associative, commutative, and monotone: if \([p] \leq [q]\) and \([r] \leq [s]\) then \([p] + [r] \leq [q] + [s]\), whenever \([q] + [s]\) is defined.

Since every AF algebra \( \mathfrak{B} \) in this paper has a unit element, the underlying order of the \( K_0 \)-group of \( \mathfrak{B} \), as well as the unit element \([1_{\mathfrak{B}}]\), will be tacitly understood, whenever no confusion may arise. So we will usually write

\[ K_0(\mathfrak{B}) \] instead of \((K_0(\mathfrak{B}), [1_{\mathfrak{B}}])\) and \((K_0(\mathfrak{B}), K_0^-(\mathfrak{B}), [1_{\mathfrak{B}}])\).

By a unital \( \ell \)-group we mean an abelian lattice ordered group with a distinguished (strong) unit \( u > 0 \). [5] [24].

The theory of AF\(\ell \) algebras is grounded in the following result:

**Theorem 2.1.** Let \( \mathfrak{B} \) be an AF algebra and \( L(\mathfrak{B}) \) its (partially ordered) local semigroup.

(i) [33] Elliott’s partially defined addition + in \( L(\mathfrak{B}) \) has at most one extension to an associative, commutative, monotone operation \( \oplus : L(\mathfrak{B})^2 \to L(\mathfrak{B}) \) satisfying the following condition: For each projection \( p \in \mathfrak{B} \), \([1_{\mathfrak{B}} - p]\) is the smallest element \([q] \in L(\mathfrak{B})\) with \([p] + [q] = [1_{\mathfrak{B}}]\). The unique semigroup \((S(\mathfrak{B}), \oplus)\) expanding the Elliott local semigroup \( L(\mathfrak{B}) \) exists iff \( \mathfrak{B} \) is an AF\(\ell \) algebra.

(ii) [20] [21] Let \( \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \) be AF\(\ell \) algebras. For each \( i = 1, 2 \) let \( \oplus_i \) be the extension of Elliott’s addition given by (i). Then the semigroups \((S(\mathfrak{B}_1), \oplus_1)\) and \((S(\mathfrak{B}_2), \oplus_2)\) are isomorphic iff so are \( \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \).

(iii) [33] For any AF\(\ell \) algebra \( \mathfrak{B} \) the semigroup \((S(\mathfrak{B}), \oplus)\) has the structure of a monoid \((E(\mathfrak{B}), 0, \oplus)\) with an involution operation \([p]^* = [1_{\mathfrak{B}} - p]\). The Murray von Neumann lattice order of equivalence classes of projections \([p], [q] \in \mathfrak{B} \) is definable via the involutive monoidal operations, by writing \([p] \lor [q] = ([p]^* \oplus [q]^*)^* \oplus [q] \) and \([p] \land [q] = ([p]^* \lor [q]^*)^* \) for all \([p], [q] \in E(\mathfrak{B})\).

(iv) [19] [20] [21] [23] For any AF\(\ell \) algebra \( \mathfrak{B} \) the dimension group \( K_0(\mathfrak{B}) \) is a countable unital \( \ell \)-group. All countable unital \( \ell \)-groups arise in this way. Let \( \mathfrak{B} \) and \( \mathfrak{B}' \) be AF\(\ell \) algebras. Then \( K_0(\mathfrak{B}) \) and \( K_0(\mathfrak{B}') \) are isomorphic as unital \( \ell \)-groups iff \( \mathfrak{B} \) and \( \mathfrak{B}' \) are isomorphic.

(v) [36] Theorem 3.9] Up to isomorphism, the map \( \mathfrak{B} \to (E(\mathfrak{B}), 0, \oplus) \) is a one-one correspondence between AF\(\ell \) algebras and countable abelian monoids with a unary operation * satisfying the equations:

\[ x^{**} = x, \quad 0^* \oplus x = 0^*, \quad \text{and} \quad (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x. \]
Let $\Gamma$ be the categorical equivalence between unital $\ell$-groups and the class of involutive monoids satisfying these equations. Then the Elliott involutive monoid $(E(\mathfrak{B}), 0^*, \oplus)$ is isomorphic to $\Gamma(K_0(\mathfrak{B}))$.

3. The Farey AF algebra $\mathfrak{M}_1$ of $\mathfrak{M}_1$ (also see [6] and [11, 41])

**First construction**, [37]. For each $m = 1, 2, \ldots$ let $\nu(m) = 1 + 2^{m-1}$. The sequence $A_1, A_2, \ldots$ of $(0, 1)$-matrices with $\nu(m + 1)$ rows and $\nu(m)$ columns is defined by:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \ldots$$

where for each $j = 1, \ldots, 2^{m-1}$ the $2j$th row of $A_m$ has a 1 in positions $j, j + 1$ and is 0 elsewhere, the $(2j - 1)$th row has a 1 in position $j$ and is 0 elsewhere, and the last row has all 0 except a final 1. The map $\zeta_m : x \in \mathbb{Z}^{\nu(m)} \mapsto A_mx \in \mathbb{Z}^{\nu(m+1)}$ is a one-one homomorphism of the free abelian group $\mathbb{Z}^{\nu(m)}$ into $\mathbb{Z}^{\nu(m+1)}$, which preserves the product ordering of these groups. Equipping the simplicial group $\mathbb{Z}^{\nu(t)}$ with the (strong) order unit $u_t = (1, 1)$, the element $u_t = A_tA_{t-1} \cdots A_1u$ is an order unit in $\mathbb{Z}^{\nu(t)}$, $t = 1, 2, \ldots$ Following [37] pp.31,33], we denote by $M_1$, or $(M_1, 1)$ for greater clarity, the unital $\ell$-group of all continuous piecewise linear functions $f : [0, 1] \to \mathbb{R}$ such that each linear piece of $f$ has integer coefficients. $M_1$ is equipped with the pointwise operations $+, -, \wedge, \lor$ of $\mathbb{R}$, and with the constant function 1 as its distinguished order unit. The number of linear pieces of every element of $M_1$ is finite. Throughout, the adjective “linear” is to be understood in the affine sense. In [37, 3.3] it is proved that $\lim((\mathbb{Z}^{\nu(m)}, u_m), A_m) = (M_1, 1)$, where limits are taken in the category of partially ordered groups with order unit. [24]. In the light of Elliott’s classification [20, 21] and further $K_0$-theoretic developments, [23, 12], (see Theorem 2.1(iv)) the AF algebra $\mathfrak{M}_1$ was defined in [37, p.33] by writing

$$(K_0(\mathfrak{M}_1), [1_{\mathfrak{M}_1}]) \cong (M_1, 1).$$

From the unital positive homomorphisms $A_m$ one may construct a Bratteli diagram of $\mathfrak{M}_1$ following [19, Chapter 2].

**Second construction**, [6]. In the Bratteli diagram of Figure 1 the two top (depth 0) vertices $v_1$ and $v_2$ have a label 1, $\lambda(v_1) = \lambda(v_2) = 1$. There are $n(d) = 2^d + 1$ vertices at depth $d$, $(d = 0, 1, \ldots)$. The label $\lambda(v)$ of any vertex $v$ at depth $d = 1, 2, \ldots$ is the sum of the labels of the vertices at depth $d - 1$ connected to $v$ by an edge. Identifying each vertex $v$ with the matrix algebra $M_{\lambda(v)}$, for each $d = 0, 1, 2, \ldots$, the finite-dimensional C*-algebra $\mathfrak{M}_d$ is defined as the direct sum of the $n(d)$ matrix algebras at depth $d$. The edges from depth $d$ to depth $d + 1$
Figure 1. The Bratteli diagram of $\mathfrak{A}$ [6, p. 977].

then determine a unital $*$-homomorphism $\phi_d: \mathfrak{A}_d \to \mathfrak{A}_{d+1}$, following the standard interpretation of Bratteli diagrams, [5], [19, Chapter 2]. The AF algebra $\mathfrak{A}$ is defined as $\mathfrak{A} = \lim(\mathfrak{A}_d, \phi_d)$.

**Theorem 3.1.** [40, p.329], [41, Theorem 1.1] $\mathfrak{A} = \mathfrak{M}_1$.

The Farey algebra $\mathfrak{M}_1$ has many interesting properties and applications, [1], [6], [18], [37], [40], [41].

**Theorem 3.2.** $\mathfrak{M}_1$ has the following properties:

(a) The Elliott involutive monoid $E(\mathfrak{M}_1) = (E(\mathfrak{M}_1), 0^*, \oplus)$ is singly generated.

(b) For any two generators $[p]$ and $[q]$ of $E(\mathfrak{M}_1)$, the map $[p] \mapsto [q]$ uniquely extends to an automorphism of $E(\mathfrak{M}_1)$.

(c) If $[p]$ is a generator of $E(\mathfrak{M}_1)$, the only other generator is $[1_{\mathfrak{M}_1} - p]$.

**Proof.** (a) By Theorem 2.1(v), the involutive monoids $E(\mathfrak{M}_1)$ and $\Gamma(M_1, 1)$ are isomorphic. By definition of the $\Gamma$ functor, [36, p.20], the underlying set of $\Gamma(M_1, 1)$ is the unit interval of $(M_1, 1)$. Therefore, $\Gamma(M_1, 1)$ consists of the $[0, 1]$-valued functions in $(M_1, 1)$, equipped with the pointwise operations $f^* = 1 - f$ and $f \oplus g = (f + g) \wedge 1$. Equivalently, $\Gamma(M_1, 1)$ is the involutive monoid of all piecewise linear continuous $[0, 1]$-valued functions $f$ defined on $[0, 1]$, such that every linear piece of $f$ has integer coefficients. These are known as one-variable McNaughton functions. When $\Gamma(M_1, 1)$ is considered independently of $(M_1, 1)$, the notation $\mathcal{M}([0, 1])$ is customary. So

$$E(\mathfrak{M}_1) \cong \mathcal{M}([0, 1]).$$

As a consequence of the one-variable fragment of McNaughton’s theorem [30], the involutive monoid $\mathcal{M}([0, 1])$ is generated by the identity function $\text{id}_{[0, 1]}$ on $[0, 1]$. Therefore, its isomorphic copy $E(\mathfrak{M}_1)$ is singly generated.

(b) By (1), we can equivalently argue for $\mathcal{M}([0, 1])$ in place of $E(\mathfrak{M}_1)$. As we have seen in (a), $\Gamma(M_1, 1)$ is the involutive monoid $\mathcal{M}([0, 1])$ of all McNaughton
functions $f: [0, 1] \to [0, 1]$. Abelian monoids with a unary operation $*$ satisfying the equations $x^{**} = x$, $0^* = x = 0^*$, and $(x^* + y)^* + y = (y^* + x)^* + x$ of Theorem 2.1(v) are known as MV algebras. [13] [15] [12]. McNaughton’s theorem [30] shows that $\mathcal{M}([0, 1])$ is the free one-generator MV algebra. See [15] Corollary 3.2.8 for a direct proof of the one-variable case.

By the standard MV algebra we mean the MV algebra obtained by equipping the unit real interval with the involutive monoid operations $x^* = 1 - x$ and $x + y = \min(1, x + y)$. Following [15] [12], the standard MV-algebra will be denoted $[0, 1]$.

Chang’s completeness theorem [14] states that if an equation $\sigma$ fails in some MV algebra then $\sigma$ fails in the standard MV algebra $[0, 1]$. Since the operations $*$ and $\oplus$ are continuous, the proof of Chang’s theorem given in [15] Theorem 2.5.3 actually shows that $\sigma$ also fails in the subalgebra $[0, 1] \cap \mathbb{Q}$. Let $\{r_1, \ldots, r_m\}$ be rationals coded by the terms occurring in $\sigma$. Let $d$ be their least common denominator. It follows that $\sigma$ fails in the subalgebra of $[0, 1] \cap \mathbb{Q}$, whose universe is $\{0, 1/d, \ldots, (d - 1)/d, 1\}$.

Thus the free MV algebra $\mathcal{M}([0, 1])$ satisfies the hypothesis of the Jónsson-Tarski theorem, [25] Theorem 2 and remarks on pp.95 and 98. The theorem of the theorem implies that any generator $g$ of $\mathcal{M}([0, 1])$ is a free generator. By definition of a free generating set, if $g'$ is any (automatically free) generator of $\mathcal{M}([0, 1])$, the map $g \mapsto g'$ uniquely extends to an automorphism of $\mathcal{M}([0, 1])$. Going back to $E(\mathfrak{M}_1) \cong \mathcal{M}([0, 1])$ we have the desired conclusion.

(c) Again, we will argue for $\mathcal{M}([0, 1])$ in place of $E(\mathfrak{M}_1)$. In view of (b), it suffices to show that $\mathcal{M}([0, 1])$ has precisely two automorphisms. Suppose $\alpha$ is an automorphism of $\mathcal{M}([0, 1])$ different from the identity. Let $j = \alpha(id_{[0,1]})$. By (b), $j \neq id_{[0,1]}$. Trivially, $j$ is a generator of $\mathcal{M}([0, 1])$, and is one-one, because so is $id_{[0,1]}$. Since $j$ has integer coefficients then $\{j(0), j(1)\} = \{0, 1\}$. From the continuity of $j$ it follows that $j$ is onto $[0, 1]$. For all $0 < z < 1$, both one-sided derivatives $\partial j(z)/\partial x^\pm$ exist, because $j$ is piecewise linear. Neither derivative can be zero, for otherwise $j$ would not be one-one. Since all linear pieces of $j$ have integer coefficients, $\partial j(z)/\partial x^\pm$ is a nonzero integer. Therefore, the one-one function $j$ has only one linear piece. Since $j$ is different from $id_{[0,1]}$, then for all $x \in [0, 1]$, $\partial j(z)/\partial x^+ = -1$, whence $j = 1 - id_{[0,1]}$. Since $j$ is a generator of $\mathcal{M}([0, 1])$, for each $g \in \mathcal{M}([0, 1])$, $\alpha(g)$ is the composite function $g \circ j$, whence $\alpha(g)(x) = g(1 - x)$ for all $x \in [0, 1]$. This is the only possibility for the automorphism $\alpha$ of $\mathcal{M}([0, 1])$, other than the identity automorphism.

4. The primitive quotients of $\mathfrak{M}_1$

For later use, in this section we provide background material for the study of the word problem of primitive quotients of $\mathfrak{M}_1$.

**Theorem 4.1.** The complete list of prime ideals of the Elliott involutive monoid $E(\mathfrak{M}_1) = \mathcal{M}([0, 1])$ is as follows:

(i) For each $\xi \in [0, 1]$, the maximal ideal $\mathfrak{i}_\xi = \{f \in \mathcal{M}([0, 1]) \mid f(\xi) = 0\}$.

(ii) $\mathfrak{i}_0^+ = \{f \in \mathcal{M}([0, 1]) \mid f(0) = 0 \text{ and } \partial f(0)/\partial x^+ = 0\}$.

(iii) $\mathfrak{i}_1^- = \{f \in \mathcal{M}([0, 1]) \mid f(1) = 0 \text{ and } \partial f(1)/\partial x^- = 0\}$.

(iv) For every rational $0 < a/b < 1$, $\mathfrak{i}^\pm_{a/b} = \{f \in \mathcal{M}([0, 1]) \mid f(a/b) = 0 \text{ and } \partial f(a/b)/\partial x^\pm = 0\}$.
Lemma 4.2. Let \( \mathcal{B} \) be an AF algebra, and \( (G, u) = K_0(\mathcal{B}) \) its associated unital dimension group. \([19, 23]\).

(i) For every ideal \( \mathfrak{J} \) of \( \mathcal{B} \) we have the isomorphism \( K_0(\mathcal{B}/\mathfrak{J}) \cong K_0(\mathcal{B})/K_0(\mathfrak{J}) \), where \( K_0(\mathfrak{J}) \) is the image of \( \mathfrak{J} \) in \( K_0(\mathcal{B}) \).

(ii) The map \( \mathfrak{J} \mapsto K_0(\mathfrak{J}) \) is an isomorphism of the lattice of ideals of \( \mathcal{B} \) onto the lattice of ideals of the unital dimension group \( K_0(\mathcal{B}) \).

(iii) If \( \mathcal{B} \) is an AF\( \ell \) algebra, the map \( \mathfrak{J} \mapsto \Gamma(K_0(\mathfrak{J})) \) is an isomorphism of the lattice of ideals of \( \mathcal{B} \) onto the lattice of ideals (=kernels of homomorphisms) of \( E(\mathcal{B}) \), where \( \Gamma(K_0(\mathfrak{J})) \) is the intersection of \( K_0(\mathfrak{J}) \) with the unit interval \( [0, u] \) of the unital dimension group \( (G, u) \).

(iv) For any AF\( \ell \) algebra \( \mathcal{B} \) and ideal \( \mathfrak{J} \) of \( \mathcal{B} \), the quotient \( \mathcal{B}/\mathfrak{J} \) is an AF\( \ell \) algebra and \( E(\mathcal{B}/\mathfrak{J}) \cong E(\mathcal{B})/\Gamma(K_0(\mathfrak{J})) \).

Proof. (i) \([21\text{ p.34]} \) or \([19\text{ Corollary 9.2]} \). (ii) \([23\text{ p.196]} \). (iii) Use (ii) together with \([15\text{ Theorem 7.2.2]} \). (iv) Use (i) together with \([15\text{ Theorem 7.2.4]} \). \( \square \)

Corollary 4.3. The complete list of prime ideals of the unital \( \ell \)-group \( M_1 \) is as follows:

(i) For each \( \xi \in [0, 1] \), the maximal ideal \( J_\xi = \{ f \in M_1 \mid f(\xi) = 0 \} \).

(ii) \( J_\xi^+ = \{ f \in J_0 \mid \partial f(0)/\partial x^+ = 0 \} \).

(iii) \( J_\xi^- = \{ f \in J_1 \mid \partial f(1)/\partial x^- = 0 \} \).

(iv) For each rational \( 0 < a/b < 1 \), \( J_{a/b}^\pm = \{ f \in J_{a/b} \mid \partial f(a/b)/\partial x^\pm = 0 \} \).

Proof. Let \( (G, u) \) be a unital \( \ell \)-group, with its associated MV-algebra \( A = \Gamma(G, u) \). By \([13\text{ Theorem 7.2.2]} \), the map

\[ i \in \text{ set of ideals of } A \mapsto \text{ smallest ideal of } (G, u) \text{ containing } i \]

is a lattice isomorphism of the set of ideals of \( A \) onto the set of ideals of \( (G, u) \). The inverse map sends every ideal \( J \) of \( (G, u) \) into the intersection of \( J \) with the unit interval \( [0, u] \) of \( (G, u) \). Now apply Theorem 4.1 \( \square \)

With reference to Lemma 4.2 for every AF algebra \( \mathcal{B} \), let \( \mathfrak{id} = \mathfrak{id}_\mathcal{B} \) be the isomorphism of the lattice of ideals of the dimension group \( K_0(\mathcal{B}) \) onto the lattice of ideals of \( \mathcal{B} \). Thus for every ideal \( \mathfrak{J} \) of \( \mathcal{B} \),

\[ \mathfrak{id}^{-1}(\mathfrak{J}) = K_0(\mathfrak{J}) = \text{ the image of } \mathfrak{J} \text{ in } K_0(\mathcal{B}). \quad (2) \]

From Corollary 4.3 we immediately obtain the classification of the set \( \text{Prim } \mathfrak{M}_1 \) of primitive ideals of \( \mathfrak{M}_1 \) independently re-obtained in \([6\text{ Proposition 7]} \):

Corollary 4.4. The complete list of the set \( \text{Prim } \mathfrak{M}_1 \) of primitive ideals of \( \mathfrak{M}_1 \) is as follows:

(i) For each \( \xi \in [0, 1] \), \( \mathfrak{J}_\xi = \mathfrak{id}(J_\xi) \).

(ii) \( \mathfrak{J}_0^+ = \mathfrak{id}(J_0^+) \).
we choose, once and for all, a projection \( \mathcal{J} \).

(4) For each rational \( 0 < a/b < 1 \), \( \mathcal{J}_{a/b} = \text{id}(J_{a/b}^\pm) \).

Corollaries 4.3 and 4.4 in combination with Lemma 4.2(i)-(ii) yield:

**Corollary 4.5.** Let \( \cong \) denote unital \( \ell \)-group isomorphism, we have:

(i) \( K_0(\mathcal{M}_1/\mathcal{J}_{p/q}) \cong (M_1,1)/J_{p/q} \cong (\mathbb{Z},q) \), for every rational \( p/q \in [0,1] \).

(ii) Let \( \mathbb{Z}_{\leq x} \) denote the free abelian group \( \mathbb{Z}^2 \) equipped with the lexicographic ordering: \((x,y) \geq 0 \) iff either \( x > 0 \) or \((x = 0 \text{ and } y \geq 0) \). Then

\[
K_0(\mathcal{M}_1/\mathcal{J}_{p/q}^+) \cong (M_1,1)/J_{p/q}^+ \cong (\mathbb{Z}_{\leq x},(1,0)) \cong (M_1,1)/J_{p/q} \cong K_0(\mathcal{M}_1/\mathcal{J}_{p/q}^-).
\]

(iii) For any rational \( p/q \in \{x \in \mathbb{R} \mid 0 < x < 1\} \), written in its lowest terms, let \( p^{-1} \in \{1,\ldots,q-1\} \) be the inverse of \( p \) modulo \( q \). Then

\[
K_0(\mathcal{M}_1/\mathcal{J}_{p/q}^+) \cong (M_1,1)/J_{p/q}^+ \cong (\mathbb{Z}_{\leq x},(q,-p^{-1})).
\]

(iv) \( K_0(\mathcal{M}_1/\mathcal{J}_{p/q}^-) \cong (M_1,1)/J_{p/q}^- \cong (\mathbb{Z}_{\leq x},(q,p^{-1})).
\]

**Corollary 4.6.** Proof of Theorem 3.1 For every every irrational \( \theta \in [0,1] \), let \( \mathcal{J}_\theta \) denote the Effros-Shen algebra of \( \{19\} \). Then

\[
K_0(\mathcal{M}_1/\mathcal{J}_\theta) \cong K_0(\mathcal{M}_1)/K_0(\mathcal{J}_\theta) \cong (M_1,1)/J_\theta \cong (\mathbb{Z} + \mathbb{Z}\theta) \cong K_0(\mathcal{J}_\theta). \quad (3)
\]

**Proof.** Combine [3] with the preservation properties of \( K_0 \), Lemma 4.2(i)-(ii). \( \square \)

**Corollary 4.7.** Proposition 4] With the notation of Corollary 4.4 the primitive quotients of \( \mathcal{M}_1 \) are as follows:

(i) For each \( \theta \in [0,1) \setminus \mathbb{Q} \), \( \mathcal{M}_1/\mathcal{J}_\theta = \mathcal{J}_\theta \).

(ii) \( \mathcal{M}_1/\mathcal{J}_0 = \mathcal{M}_1/\mathcal{J}_1 = \mathbb{C} \). For any rational \( 0 < p/q < 1 \) (written in its lowest terms), \( \mathcal{M}_1/\mathcal{J}_{p/q} = \mathbb{M}_q \).

(iii) For every separable Hilbert space \( L \), let \( \mathbb{K}(L) \) denote the \( \mathbb{C}^* \)-algebra of all compact operators on \( L \). Further, let \( H_1 \) stand for an infinite-dimensional separable Hilbert space. Then \( \mathcal{M}_1/\mathcal{J}_0 = \mathcal{M}_1/\mathcal{J}_1 \) is the Behnke-Leptin (unital) \( \mathbb{AF} \) algebra \( \mathcal{A}_{0,1} \) of operators on the Hilbert space \( H = \mathbb{C} \oplus H_1 \otimes \mathbb{C} \) generated by \( \mathbb{K}(H) \) and \( 1_{H_1} \otimes \mathbb{K}(\mathbb{C}) \), (see [4] p.330-331).

(iv) For any rational \( 0 < p/q < 1 \), letting \( k \equiv -p^{-1} \) mod \( q \), \( k \in \{1,\ldots,q-1\} \), \( \mathcal{M}_1/\mathcal{J}_{p/q}^+ \) is the Behnke-Leptin \( \mathbb{AF} \) algebra \( \mathcal{A}_{k,q} \) of operators on the Hilbert space \( H = \mathbb{C}_k \oplus H_1 \otimes \mathbb{C}^q \) generated by \( \mathbb{K}(H) \) and \( 1_{H_1} \otimes \mathbb{K}(\mathbb{C}^q) \).

(v) For any rational \( 0 < p/q < 1 \), letting \( h \equiv p^{-1} \) mod \( q \), \( h \in \{1,\ldots,q-1\} \), \( \mathcal{M}_1/\mathcal{J}_{p/q}^- \) is the Behnke-Leptin \( \mathbb{AF} \) algebra \( \mathcal{A}_{h,q} \).

By Theorem 3.2 the involutive monoid \( E(\mathcal{M}_1) \) has two possible generators. Let us choose, once and for all, a projection \( p \in \mathcal{M}_1 \) such that \( [p] \) generates \( E(\mathcal{M}_1) \). Then \( [p] \) freely generates \( E(\mathcal{M}_1) \). Since the McNaughton function \( \text{id}_{[0,1]} \) is a free generator of \( \mathcal{M}([0,1]) \cong E(\mathcal{M}_1) \), there is precisely one isomorphism \( \omega \) of \( \mathcal{M}([0,1]) \) onto \( E(\mathcal{M}_1) \) mapping \( \omega(\text{id}_{[0,1]}) \) into \( [p] \). In symbols,

\[
\omega: f \in \mathcal{M}([0,1]) \cong \omega(f) \in E(\mathcal{M}_1), \quad \omega(\text{id}_{[0,1]}) = [p]. \quad (4)
\]
Every ideal of $E(\mathfrak{M}_1)$ is the $\omega$-image of a uniquely determined ideal $i$ of $\mathcal{M}([0, 1])$ in such a way that we have the isomorphism

$$[q]/\omega(i) \in E(\mathfrak{M}_1)/\omega(i) \cong \omega^{-1}([q])/i \in \mathcal{M}([0, 1])/i, \text{ for } \forall [q] \in E(\mathfrak{M}_1). \quad (5)$$

**Proposition 4.8.** As in Theorem 4.1(i), for any irrational $\theta$ in $[0, 1]$, let the ideal $\iota_\theta$ of $\mathcal{M}([0, 1])$ be defined by $\iota_\theta = \{ f \in \mathcal{M}([0, 1]) \mid f(\theta) = 0 \}$.

(i) There is a unique embedding $\eta_\theta$ of the quotient $\mathcal{M}([0, 1])/\iota_\theta$ into the standard MV algebra $[0, 1]$ introduced in the proof of Theorem 3.2(ii).

(ii) There is a unique embedding $\iota_\theta$ of $E(\mathfrak{M}_1)/\omega(\iota_\theta)$ into $[0, 1]$.

(iii) The projection $p \in \mathfrak{M}_1$ has the following property: For each irrational $\theta \in [0, 1]$, $\iota_\theta([p]/\omega(\iota_\theta)) = \theta$.

**Proof.** (i) Since $\iota_\theta$ is a maximal ideal of $\mathcal{M}([0, 1])$ then $\mathcal{M}([0, 1])/\iota_\theta$ is simple, i.e., $\{0\}$ is its only ideal. By [42] Theorem 4.16(i), there is precisely one embedding $\eta_\theta$ of $\mathcal{M}([0, 1])/\iota_\theta$ into $[0, 1]$. In more detail, let $\text{gen}(\theta)$ denote the MV subalgebra of $[0, 1]$ generated by $\theta$. Then $\eta_\theta$ is an isomorphism of $\mathcal{M}([0, 1])/\iota_\theta$ onto $\text{gen}(\theta)$.

(ii) By (5), $E(\mathfrak{M}_1)/\omega(\iota_\theta)$ is isomorphic to $\mathcal{M}([0, 1])/\iota_\theta$. Now apply (i).

(iii) By [41][43], it is enough to prove that the unique embedding

$$\eta_\theta : f/\iota_\theta \in \mathcal{M}([0, 1])/\iota_\theta \mapsto \eta_\theta(f/\iota_\theta) \in [0, 1]$$

satisfies $\eta_\theta([\text{id}_{[0, 1]}]/\iota_\theta) = \theta$. The proof of (i) allows us to identify the MV-algebras $\text{gen}(\theta)$ and $\mathcal{M}([0, 1])/\iota_\theta$. It is well known that the only automorphism of the MV algebra $\text{gen}(\theta)$ is identity, (see [42] Theorem 4.16(i), or apply Hölder’s theorem, [35]). Two McNaughton functions $f, g \in \mathcal{M}([0, 1])$ satisfy $f/\iota_\theta = g/\iota_\theta$ iff they have the same value at $\theta$. Thus the embedding $\eta_\theta$ acts on $f/\iota_\theta$ as evaluation of $f$ at $\theta$, in symbols, $\eta_\theta(f/\iota_\theta) = f(\theta)$ for all $f \in \mathcal{M}([0, 1])$. In particular, $\eta_\theta([\text{id}_{[0, 1]}]/\iota_\theta) = \text{id}_{[0, 1]}(\theta) = \theta$, as desired.

---

5. The word problem of the Farey algebra $\mathfrak{M}_1$

In view of Theorem 4.1(i), the word problem of the AF$\ell$ algebra $\mathfrak{M}_1$ can be defined mimicking the classical definition of the word problem for a group presented by generators and relations, [31] §VIII, 1.7. Thus, given two formulas $\phi$ and $\psi$ in the language of involutive monoids, the word problem asks to mechanically decide whether $\phi$ and $\psi$ code the same equivalence of projections of $\mathfrak{M}_1$.

In the next section we will similarly define the word problem of every AF$\ell$-algebra $\mathfrak{B}$ whose Elliott involutive monoid is singly generated. We will be able to do so because the freeness properties of $E(\mathfrak{M}_1)$ ensure that $E(\mathfrak{B})$ is a quotient of $E(\mathfrak{M}_1)$. The main aim of this paper is to assess the algorithmic complexity the word problem of various quotients of $\mathfrak{M}_1$.

**Syntax/Semantics of the word problem.** To get to work on (Turing) computational issues like word problems, a modicum of syntactical machinery must be introduced. To this purpose, first of all, we enrich the language (= set of constant and operation symbols) $0, *, +$ of one-generator involutive monoids by adding a new constant symbol $X$ for the generator. Throughout, by a *formula* we mean a string of symbols over the alphabet $\{0, X, *, +, \}, \{\}$, given by the following inductive definition: $0$ and $X$ are formulas; if $\phi$ and $\psi$ are formulas, then so are $\phi^*$ and $(\phi \oplus \psi)$. It is a tedious exercise to verify that this definition ensures “unique readability” of every formula...
\( \phi \), i.e., its mechanical unique decomposability into immediate subformulas, unless \( \phi \) is a constant symbol.

Every formula \( \phi \) in the language \( 0, X, \ast, \oplus \) codes precisely one McNaughton function \( f_\phi \) by the following inductive definition on the number of operation symbols in subformulas of \( \phi \): \( f_0 \) is the zero function on \([0, 1]\), and \( f_X \) is id\([0, 1]\). Inductively, \( f_{\phi \ast} = 1 - f_\phi = f_\phi^* \), and \( f_{(\phi \oplus \chi)} = \min(1, f_\phi + f_\chi) = f_\phi \oplus f_\chi \). This definition makes sense because of the unique readability of formulas. We say that \( f_\phi \) is the McNaughton function of \( \psi \).

Figure 2 shows the McNaughton function of the formula

\[
(((X \oplus X)^* \oplus X) \oplus (X \oplus X)^*)^* \oplus (X \oplus X)^*)^* \in E(M_1). \]

via the isomorphism \( \omega \) of (4), every formula \( \psi \) simultaneously codes the Murray-von Neumann equivalence class of projections \( \omega(f_\psi) \) of \( M_1 \).

**Definition 5.1.** The word problem of \( M_1 \) is as follows:

**INSTANCE** : Formulas \( \phi \) and \( \psi \) in the language of \( 0, X, \ast, \oplus \).

**QUESTION** : Are \( \omega(f_\phi) \) and \( \omega(f_\psi) \) the same equivalence class of projections of \( M_1 \)?

Should the symbol \( X \) be interpreted as the other generator of \( E(M_1) \), Theorem 3.2 ensures that every question in the resulting word problem has precisely the same answer as in the original word problem defined above.

For any formula \( \phi \) in the language \( 0, X, \ast, \oplus \) we let \( ||\phi|| \) = the number of occurrences of symbols in \( \phi \).

**Theorem 5.2.** The word problem of \( M_1 \) is solvable in polynomial time. In other words, there is a polynomial \( T : \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\} \) and a Turing machine
which, having in input any pair of formulas \((\phi, \psi)\), decides whether they code the same equivalence class of projections of \(M_1\), in a number of steps \(T(||\phi|| + ||\psi||)\).

\[ \phi \]

\[ \psi \]

\[ \sigma \]

\[ \alpha \]

\[ \beta \]

\[ \rho \]

\[ \partial \]

\[ \partial x^+ \]

\[ \partial x^- \]

\[ \partial \psi \]

\[ \partial \phi \]

\[ \partial \psi \]

\[ \partial \phi \]

\[ ||\sigma|| < ||\sigma'|| \leq ||\rho||. \]

Proof. In view of the isomorphism \([4]\), we may replace \(E(M_1)\) by \(M([0,1])\) and focus on the complexity of the equivalent problem of checking whether the McNaughton functions \(f_\phi\) and \(f_\psi\) coincide. We first make the following:

**Claim 1:** Let \(\rho\) be a formula. For all \(0 < z < 1\) both one-sided derivatives \(\partial \rho(z)/\partial x^+\) (exist and) satisfy \(|\partial \rho(z)/\partial x^+| \leq ||\rho||\).

Existence immediately follows from piecewise linearity. Since every linear piece of every \(g \in M([0,1])\) has integer coefficients, both one-sided derivatives of \(g\) at \(z\) have integer values. For the formulas 0 and \(X\) the claim is trivially true. Proceeding inductively on the number of operation symbols in subformulas \(\sigma\) of \(\rho\), we can write:

\[ \left| \frac{\partial \sigma^*(z)}{\partial x^+} \right| = \left| \frac{\partial \sigma(z)}{\partial x^+} \right| \leq ||\sigma|| < ||\sigma'|| \leq ||\rho||. \]

On the other hand, whenever \(\sigma\) has the form \((\alpha \oplus \beta)\), we can write:

\[ \frac{\partial \sigma(z)}{\partial x^+} = \frac{\partial \alpha(z)}{\partial x^+} + \frac{\partial \beta(z)}{\partial x^+}, \]

unless \(f_\alpha(z) \oplus f_\beta(z) = 1\), in which case

\[ \frac{\partial \sigma(z)}{\partial x^+} = \min \left(0, \frac{\partial \alpha(z)}{\partial x^+} + \frac{\partial \beta(z)}{\partial x^+} \right). \]

In conclusion,

\[ \left| \frac{\partial \sigma(z)}{\partial x^+} \right| \leq \left| \frac{\partial \alpha(z)}{\partial x^+} + \frac{\partial \beta(z)}{\partial x^+} \right| \leq \left| \frac{\partial \alpha(z)}{\partial x^+} \right| + \left| \frac{\partial \beta(z)}{\partial x^+} \right| \leq ||\alpha|| + ||\beta|| < ||\sigma|| \leq ||\rho||. \]

A similar result holds for \(\partial \sigma(z)/\partial x^-\). Our first claim is settled.

**Claim 2:** Let \(\phi\) and \(\psi\) be formulas. If \(f_\phi \neq f_\psi\), there is a rational \(r = c/d\) such that \(f_\phi(r) \neq f_\psi(r)\) and \(0 < d \leq 14(||\phi|| + ||\psi||)\).

As a matter of fact, let \(h = |f_\phi - f_\psi|\). Then a tedious but straightforward calculation \([13] 1.2.4\) yields

\[ h = (f_\phi^* \oplus f_\psi)^* \oplus (f_\phi^* \oplus f_\psi)^* = f_(\phi^* \oplus \psi)^* \oplus (\psi^* \oplus \phi)^* \]

showing that \(h\) is a member of \(M([0,1])\). If \(h\) is the constant 1 we are done. Otherwise, the maximum value of \(h\) is attained at some point \(r \in [0,1]\) of \(h\) which is the unique solution of a linear equation \(d'x + c' = d''x + c''\) with integer coefficients \(c', c'', d', d''\). (If \(r \in \{0,1\}\) the proof is trivial.) The denominator \(d\) of \(r\) satisfies

\[ 0 < d \leq |d' - d''| \leq |d'| + |d''|. \]

By definition, \(|d'|, |d''| \leq |\partial h(r)/\partial x^+|\). By Claim 1,

\[ d \leq |d'| + |d''| \leq 2 \left| \frac{\partial h(r)}{\partial x^+} \right| \leq 2 \left| ((\phi^* \oplus \psi)^* \oplus (\psi^* \oplus \phi)^*) \right|. \]

Now

\[ 2 \left| ((\phi^* \oplus \psi)^* \oplus (\psi^* \oplus \phi)^*) \right| = 2(2(||\phi|| + ||\psi||) + 10) \leq 2(2(||\phi|| + ||\psi||) + 5(||\phi|| + ||\psi||)) \]

which settles our second claim.

To conclude, it suffices to prove:
Claim 3: The problem whether for some rational \( r \in [0, 1] \) with denominator \( 0 < d \leq 14(||\phi|| + ||\psi||) \) the function \( h \) vanishes at \( r \), has polytime complexity with respect to \( ||\phi|| + ||\psi|| \).

As a matter of fact, by Claim 2, the number \( N \) of these rational numbers \( r \in [0, 1] \) satisfies \( N < (14(||\phi|| + ||\psi||))^2 \). By (5), for any such \( r \), we must compute the value \( f_\sigma(r) \) for all subformulas \( \sigma \) of \(((\phi^* + \psi)^* + (\psi^* + \phi)^*)^*\). This is done proceeding by induction on the number of operation symbols in \( \sigma \). The number of these subformulas is bounded above by a linear function of \( ||\phi|| + ||\psi|| \). Since both \( * \) and \( + \) are polytime operations, it easily follows that computing the value \( f_\sigma(r) \) takes polynomial time in \( ||\phi|| + ||\psi|| \), whence so does the computation of \( h(r) \).

Having thus settled Claim 3, we have completed the proof that the word problem of \( \mathfrak{M}_1 \) has polytime complexity. \( \square \)

6. The word problem of Behnke-Leptin primitive quotients of \( \mathfrak{M}_1 \)

The freeness properties of the Farey algebra \( \mathfrak{M}_1 \) yield a natural definition of the word problem of every AF\(^\dagger \) algebra \( \Omega \) whose Elliott involutive monoid \( E(\Omega) \) is singly generated. Since \( E(\Omega) \) is the quotient of the free algebra \( E(\mathfrak{M}_1) \) by some ideal \( \mathfrak{i} \) of \( E(\mathfrak{M}_1) \), then by Lemma 4.2 \( \Omega \) is the quotient of \( \mathfrak{M}_1 \) by the ideal \( \mathfrak{I} \) corresponding to \( \mathfrak{i} \) via the functors \( K_0 \) and \( \Gamma \). Once \( E(\mathfrak{M}_1) \) is equipped with the generator \( [p] \) introduced in Proposition 4.3(iii), \( E(\Omega) \) is automatically endowed with the generator \( [p/\mathfrak{I}] \in E(\mathfrak{M}_1/\mathfrak{I}) \).

The word problem of \( \Omega \) asks whether two arbitrary formulas \( \phi, \psi \) in the language \( 0, X, ^*, + \) code the same equivalence class of projections in \( \Omega \), with the symbol \( X \) coding the generator \( [p/\mathfrak{I}] \) of the involutive monoid \( E(\mathfrak{M}_1/\mathfrak{I}) \cong E(\mathfrak{M}_1)/\mathfrak{i} = E(\Omega) \).

When \( \Omega = \mathfrak{M}_1 \) we recover Definition 5.1.

In [21, pp.36-37] Elliott constructs the local semigroup of the Behnke-Leptin (unital) separable AF algebra \( \mathcal{A}_{0,1} \) with a two point dual, [4]. Since \( \mathcal{A}_{0,1} \) has comparability in the sense of Murray-von Neumann, \( \mathcal{A}_{0,1} \) is an AF\(^\dagger \) algebra. Further, Elliott’s analysis in [21] shows that the involutive monoid \( E(\mathcal{A}_{0,1}) \) is singly generated. We then have:

**Theorem 6.1.** The word problem of the Behnke-Leptin \( C^* \)-algebra \( \mathcal{A}_{0,1} \) has polynomial time complexity.

**Proof.** As shown in [21, p. 36], \( K_0(\mathcal{A}_{0,1}) \) is the lexicographic product \( Z + \text{lex} Z \) of two copies of \( Z \), and the image of the unit element of \( \mathcal{A}_{0,1} \) in \( K_0 \) is the order unit \( (1, 0) \) of \( Z + \text{lex} Z \). By (1), we may argue for \( M([0, 1]) \) instead of its isomorphic copy \( E(\mathfrak{M}_1) \). By Theorem 2.1(v) and Corollaries 4.3(ii)-4.7(ii), \( E(\mathcal{A}_{0,1}) \) is (isomorphic to) the quotient of \( M([0, 1]) \) by the ideal \( i_{\mathfrak{M}_1} \) of all McNaughton functions that vanish on a right open neighborhood of 0. Two elements \( f, g \in M([0, 1]) \) are mapped into the same element of \( E(\mathcal{A}_{0,1}) \) by the quotient map \( l \in M([0, 1]) \mapsto l/\mathfrak{i}_{\mathfrak{M}_1} \in M([0, 1])/\mathfrak{i}_{\mathfrak{M}_1} \) iff they have equal value in an open right neighborhood of 0. By (5), the image under the quotient map of the generator \( [p] \) of \( E(\mathfrak{M}_1) \) may be identified with the element \( i_{\mathfrak{M}_1}/\mathfrak{i}_{\mathfrak{M}_1} \) of \( M([0, 1])/\mathfrak{i}_{\mathfrak{M}_1} \) where, as the reader will recall, \( i_{\mathfrak{M}_1} \) denotes the identity function over \([0, 1]\). The element \( i_{\mathfrak{M}_1}/\mathfrak{i}_{\mathfrak{M}_1} \) is known as the germ of \( i_{\mathfrak{M}_1} \) at 0. Every \( f \in M([0, 1]) \) is similarly transformed by the quotient map into its germ at 0. Piecewise linearity ensures that any \( f, g \in M([0, 1]) \) are identified by the quotient map iff they have the same germ at 0 iff they coincide on
some open right neighborhood of 0, iff

\[ f(0) = g(0) \quad \text{and} \quad \frac{\partial f(0)}{\partial x^+} = \frac{\partial g(0)}{\partial x^+}. \]

(Compare with the proof of Claim 1 in Theorem 5.2)

Thus to evaluate the complexity of the problem whether two formulas \( \phi, \psi \) satisfy \( f_{\phi}/j_0^+ = f_{\psi}/j_0^+ \) in \( \mathcal{M}([0,1])/j_0^+ \), we consider the equivalent problem of deciding whether \( f_{\phi} \) and \( f_{\psi} \) have the same value at 0 and the same partial derivative at 0 along \( x^+ \).

We will prove the existence of a polynomial \( P \) such that both problems are decidable in less than \( P(||\phi|| + ||\psi||) \) steps, for any \( \phi \) and \( \psi \). We proceed by induction on the number of operation symbols in subformulas \( \sigma \) of \( \phi \) and \( \psi \).

Since \( f_\sigma \) is piecewise linear with integer coefficients, then \( f_\sigma(0) \in \{0,1\} \). There are only polynomially few subformulas \( \sigma \) of \( \phi \) and \( \psi \). Since both \( \ast \) and \( \oplus \) are polytime operations, it easily follows that the computation of the value \( f_{\sigma}(0) \) takes polynomial time in \( ||\phi|| + ||\psi|| \). Thus the problem whether \( f_{\phi}(0) = f_{\psi}(0) \) is polytime.
By induction on the number of operation symbols, we next evaluate the one-sided derivatives
\[ \frac{\partial f_\phi(0)}{\partial x^+} \quad \text{and} \quad \frac{\partial f_\psi(0)}{\partial x^+} \]
as follows: First of all, the derivative of the generator \( \text{id}_{[0,1]} = f_X \) of \( \mathcal{M}([0,1]) \) is equal to 1, and the derivative of the zero element \( f_0 = 0 \). Proceeding inductively, whenever the operation \( \ast \) is applied to a subformula \( \sigma \) to yield \( \sigma^\ast \), the one-sided derivative flips from \( d \) to \( -d \). Whenever a subformula \( \sigma \) arises as \( (\alpha \odot \beta) \), the one-sided derivatives of \( \sigma \) are evaluated by cases and using (6)-(7) in the proof of Claim 1 in Theorem 5.2. All these derivatives have integer values \( \leq \max(||\phi||, ||\psi||) \), which can be computed in polynomial time. Thus the computation of the one-sided derivatives of \( f_\phi \) and \( f_\psi \) at 0 requires polynomial time in \( ||\phi|| + ||\psi|| \).

We have just shown the existence of the desired polynomial \( P \) such that in less than \( P(||\phi|| + ||\psi||) \) steps it is decidable if two formulas \( \phi, \psi \) satisfy \( f_\phi(0)/J^+_0 = f_\psi(0)/J^-_0 \).

As a consequence, the word problem of \( \mathcal{A}_{0,1} \) has polynomial complexity. \( \square \)

**Remark 6.2.** \( E(\mathcal{A}_{0,1}) \) is isomorphic to the MV algebra \( C \) introduced by Chang in [13, p.474].

**Corollary 6.3.** The word problems of the Behnke-Leptin algebras \( \mathcal{M}_1/\mathcal{J}^+_m/n \) and \( \mathcal{M}_1/\mathcal{J}^-_m/n \) have polynomial time complexity. (Notation of Corollary 4.7.)

**Proof.** A routine variant of the proof of Theorem 6.1. For every formula \( \psi \), one has now to compute the value of \( f_\psi \) and both its one-sided derivatives at \( m/n \). \( \square \)

Figure 3 shows the Elliott involutive monoid of a typical Behnke-Leptin algebra \( \mathcal{M}_1/\mathcal{J}^+_m/n \).

7. **The word problem of Effros-Shen primitive quotients of \( \mathcal{M}_1 \)**

Following [19, p. 65], for each irrational \( \theta \in [0,1] \) the Effros-Shen AF algebra \( \mathfrak{B}_\theta \) is defined by \( K_0(\mathfrak{B}_\theta) \cong (\mathbb{Z} + \mathbb{Z}, \theta, 1) \), the latter denoting the additive subgroup of \( \mathbb{R} \) generated by \( \theta \) and 1, with the natural ordering and 1 as the unit.

It is well known that Effros-Shen algebras are related to continued fractions, [10] [11], [19 §5]. The elementary theory of diophantine approximation is also a key tool for our results on the word problem of \( \mathfrak{B}_\theta \).

We prepare:

**Lemma 7.1.** If \( n \geq 2 \) is even and \( h/k \) is a rational (with \( k > 0 \)) satisfying \( p_n/q_n < h/k < p_{n+1}/q_{n+1} \) then \( k > q_{n+1} > q_k \).

**Proof.** By [27 I, §1, Theorem 2], the integer \( 2 \times 2 \) matrix whose rows are given by \( (p_n, q_n) \) and \( (p_{n+1}, q_{n+1}) \) is unimodular. Passing to homogeneous integer coordinates in \( \mathbb{Z}^2 \), let us consider the integer vector \( (h, k) \). Our hypothesis about \( h/k \) lying in the interior of the interval \( [p_n/q_n, p_{n+1}/q_{n+1}] \) amounts to saying that \( (h, k) \) lies in the interior of the positive span \( \{\lambda(p_n, q_n) + \mu(p_{n+1}, q_{n+1}) \in \mathbb{R}^2 \mid 0 \leq \lambda, \mu \in \mathbb{R} \} \) of \( (p_n, q_n) \) and \( (p_{n+1}, q_{n+1}) \) in \( \mathbb{R}^2 \). On the other hand, \( (h, k) \) lies outside the half-open parallelogram \( \mathcal{P} = \{x \in \mathbb{R}^2 \mid x = \lambda(p_n, q_n) + \mu(p_{n+1}, q_{n+1}) \leq 0, 0 \leq \mu < 1 \) \}. Indeed, \( \mathcal{P} \) does not contain any integer point other than the origin, because \( \{(p_n, q_n), (p_{n+1}, q_{n+1})\} \) is a free generating set of the free abelian group \( \mathbb{Z}^2 \).

Thus \( k \geq q_{n+1} + q_k > q_{n+1} > q_k \). \( \square \)
From \( q_n \geq 2q_{n-2} \) we get \( q_{2n} \geq 2^n q_0 = 2^n \) and \( q_{2n+1} \geq 2^n q_1 \geq 2^n \), whence 
\( q_n \geq 2^{(n-1)/2} \) for all \( n = 2, 3, 4, \ldots \). The following alternative estimates, however, will simplify our calculations in Theorem 7.4 with no loss in its content.

**Lemma 7.2.** Let \( p_n/q_n \) be the \( n \)th principal convergent of \( \theta \). We then have:

(i) \( q_0 = 1 \leq q_1 \) and \( q_n > (3/2)^{n-2} \) for all \( n \geq 2 \).

(ii) For all \( m = 1, 2, \ldots \) and even \( n > \log_2 8m \) we have \( q_n > 2m \).

**Proof.** (i) We have \( q_0 = 1 \) by definition. Necessarily, \( q_1 \geq 1 \). Thus by (27 I §1, Theorem 1), \( q_2 \geq q_1 + q_0 > 1 = (3/2)^{2-2} \). Proceeding inductively, for all \( n \geq 2 \), we have 
\( q_n \geq q_{n-1} + q_{n-2} > (3/2)^{n-3} + (3/2)^{n-4} > (3/2)^2 \cdot (3/2)^{n-4} = (3/2)^{n-2}, \)
as desired.

(ii) By (i), \( n > \log_2 8m \Rightarrow n > 2 + \frac{\log_2 2m}{\log_2 3/2} \Rightarrow (3/2)^{n-2} > 2m \Rightarrow q_n > 2m \).

**Lemma 7.3.** Let \( a_0, a_1, \ldots \) denote the partial quotients of \( \theta \). For each \( n = 2, 3, \ldots \) let \( a_n = \max(a_0, \ldots, a_n) \). Then \( q_n < (2a_n q_1)^n \).

**Proof.** By (27 I §1, p.2], \( q_n = a_n q_{n-1} + q_{n-2} < q_{n-1} (a_n + 1) \leq 2a_n q_{n-1} \). So, in particular, \( q_2 < 2a_2 q_1 < (2a_2 q_1)^2 \). Inductively, \( q_n < 2a_n q_{n-1} < 2a_n (2a_{n-1} q_1)^{n-1} \leq (2a_n q_1)(2a_n q_1)^{n-1} = (2a_n q_1)^n \).

A brute force counting argument shows that uncountably many Effros-Shen AF algebras do not have a decidable word problem. On the other hand, the following result yields a large class of Effros-Shen algebras having a polynomial time word problem:

**Theorem 7.4.** Let \( \theta \in [0, 1) \setminus \mathbb{Q} \). Suppose there is a real \( \kappa > 0 \) such that for every \( n = 0, 1, \ldots \) the sequence \( \{a_0, \ldots, a_n\} \) of partial quotients of \( \theta \) is computable (as a finite list of binary integers) in less than \( 2^{\kappa n} \) steps. Then the word problem of \( \mathfrak{M}_\theta \) has polynomial time complexity.

**Proof.** By (4), we may argue for \( \mathcal{M}([0, 1]) \) in place of \( E(2\mathbb{R}) \). By Lemma 4.2 letting \( \text{gen}(\theta) \) denote the MV subalgebra of the standard MV-algebra generated by \( \theta \), we may write
\[
E(\mathfrak{M}_\theta) = \Gamma(\mathbb{Z} + \mathbb{Z}\theta, 1) = \text{gen}(\theta) = \mathcal{M}([0, 1]) / I_\theta.
\]
By (12) Theorem 4.16, \( \text{gen}(\theta) \) is uniquely isomorphic to the quotient of \( \mathcal{M}([0, 1]) \) by the maximal ideal \( I_\theta \) of all McNaughton functions of \( \mathcal{M}([0, 1]) \) vanishing at \( \theta \). The irrationality of \( \theta \) ensures that every \( f \in \mathcal{M}([0, 1]) \) vanishing at \( \theta \) also vanishes over an open neighborhood of \( \theta \). This is so because every linear piece of \( f \) has integer coefficients. As in the proof of Proposition 4.8 the quotient map \( f \in \mathcal{M}([0, 1]) \mapsto f/1_\theta \in \mathcal{M}([0, 1]) / I_\theta \) boils down to evaluating \( f \) at \( \theta \).

We must prove the existence of a polynomial \( T \) such that for arbitrary formulas \( \phi \) and \( \psi \), the problem whether \( \phi \) and \( \psi \) code the same element of \( E(\mathfrak{M}_\theta) \) is decidable in less than \( T(||\phi|| + ||\psi||) \) steps.

---

1 i.e., some Turing machine over input \( 0 \ldots 0 \), \( n \) zeros, outputs the list of binary numbers \( [a_0, \ldots, a_n] \) in a number of steps bounded above by some exponential function of the input length \( n \). Numerical inputs in unary notation are customary in recursive analysis and in discrete complexity theory, [20].
By (8), the problem amounts to checking whether the function \(|f_\phi - f_\psi| \in \mathcal{M}([0,1])| \) belongs to \( \mathcal{M}_1 \). In turn, this problem is equivalent to checking whether the function \(|f_\phi - f_\psi| \) vanishes at \( \theta \). Let \( \delta > 0 \) be the smallest integer such that
\[
\delta < 2^{|\rho|}. 
\]

The same verification in the proof of Claim 2 in Theorem 5.2 shows that \( f_\delta = |f_\phi - f_\psi| \). The quotient map sends \( f_\delta \in \mathcal{M}([0,1]) \) into the real number \( f_\delta(\theta) \in [0,1] \). Observe that \( f_\delta \) belongs to \( \mathcal{M}_1 \) iff \( f_\delta(\theta) = 0 \).

So, to conclude the proof it is sufficient to prove that there is a polynomial \( T' \) such that for every formula \( \rho \), the problem whether \( f_\rho(\theta) = 0 \) can be solved in less than \( T'(||\rho||) \) steps.

Let the integer \( n_\rho \) be defined by
\[
n_\rho = 2 \cdot \lceil \log_2 8||\rho|| \rceil = 2 \cdot \text{the smallest integer } \geq \log_2 8||\rho||. 
\]
By Lemma 7.2(ii),
\[
q_{n_\rho} > 2||\rho||. 
\]
Claim 1: \( f_\rho \) is linear over the interval \( I_\rho = [p_{n_\rho}/q_{n_\rho}, p_{n_\rho+1}/q_{n_\rho+1}] \).

Otherwise (absurdum hypothesis), the piecewise linear function \( f_\rho \) has a non-differentiability point \( x \) lying in the interior of \( I_\rho \). Since the linear pieces of \( f_\rho \) are restrictions of linear polynomials with integer coefficients, any such \( x \) is the unique solution of an equation of the form \( d'x + c' = d''x + c'' \) for integers \( c', c'', d', d'' \). Thus \( x \) is rational, and its denominator \( d \) satisfies the inequalities
\[
0 < d \leq |d' - d''| \leq |d'| + |d''|. 
\]
Arguing inductively on the number of operation symbols in subformulas of \( \rho \) as in the proof of Theorem 5.2 it follows that the one-sided derivatives \( d' \) and \( d'' \) of \( f_\rho \) at \( x \) are bounded above by the number ||\( \rho || \) of symbols of \( \rho \). Thus \( d \leq 2||\rho|| \).

By Lemma 7.1, the rational \( x \) satisfies \( d > q_{n_\rho+1} > q_{n_\rho} \). By (11), \( d > 2||\rho|| \), a contradiction that settles our first claim.

Since \( f_\rho \) is piecewise linear and \( f_\rho \geq 0 \), from our claim we have: \( f_\rho(\theta) = 0 \) iff \( f \) vanishes over \( I_\rho \) iff \( f_\rho(p_{n_\rho}/q_{n_\rho}) = 0 = f_\rho(p_{n_\rho+1}/q_{n_\rho+1}) \). Therefore, to complete the proof of the theorem there remains to be proved that the values
\[
v = f_\rho(p_{n_\rho}/q_{n_\rho}), \quad w = f_\rho(p_{n_\rho+1}/q_{n_\rho+1})
\]
are computable in polynomial time. To this purpose, we first settle the following:

Claim 2: The time (=number of Turing machine steps) \( t \) needed to compute the sequence of binary integers \([a_0, \ldots, a_{n_\rho}] \) is bounded above by a polynomial in the length of \( \rho \).

As a matter of fact,
\[
t < 2^{2n_\rho}, \quad \text{by hypothesis}
\]
\[
< (2^{2(1+\log_2(8||\rho||))})^\kappa, \quad \text{by (10)}
\]
\[
= 4^\kappa \cdot (8||\rho||)^{2\kappa}.
\]

Claim 3: The two sequences of binary integers \( p_0, \ldots, p_{n_\rho} \) and \( q_0, \ldots, q_{n_\rho} \) are computable in polynomial time.
As a matter of fact, letting \(|j|_2\) denote the length of the integer \(j\) written in binary notation, for some constant \(c\) and polynomial \(P\) we have
\[
|q_n|_2 < n_p \log_2 2q_1a_n, \text{ by Lemma 7.3} \\
< (2 + 2 \log_2 8|\rho|) \cdot (c + |a_n|_2), \text{ by (10)} \\
< P(|\rho|), \text{ a fortiori, by Claim 2.}
\]
Now, the binary integer \(q_n\) is computed via a number \(n_p\) (logarithmic in \(|\rho|\)) of applications of the recursive formulas [27, §1, p.2], involving additions and multiplications of the binary integers \(|a_n|_2, |q_i|_2, |p_i|_2\) as in the euclidean algorithm. It follows that the computation of \(|q_n|_2\) takes polynomial time (say \(P^3(|\rho|))\), and so does the computation of \(|p_n|_2\). Our third claim is thus settled.

For every subformula \(\sigma\) of \(\rho\), arguing inductively on the number of operation symbols of \(\sigma\), it follows that the computation of the rational number \(\operatorname{val}(\sigma)\) is bounded above by \(|\rho|\). At each step, we have the following two alternatives: Either \(\sigma\) is transformed into \(\sigma^*\), whence \(f_\sigma(p_{n_p}/q_{n_q}) = 1 - f_\sigma(p_{n_p}/q_{n_q})\). Or else, \(\sigma = (\alpha \oplus \beta)\), in which case we argue by cases using formulas \([6,7]\) as in Theorem 5.2 Claim 1. Similarly, the value \(w = f_\rho(p_{n_q+1}/q_{n_q+1})\) is computable in polytime.

We have just shown the existence of the desired polynomial \(T'\) such that for every formula \(\rho\), the problem whether \(f_\rho(\theta) = 0\) is decidable in less than \(T'(\log|\rho|)\) steps.

As a consequence, the word problem of \(\mathfrak{F}_\theta\) has polynomial time complexity. \(\square\)

**Corollary 7.5.** The word problem of each Effros-Shen algebra \(\mathfrak{F}_\theta\), for \(\theta\) a quadratic irrational, or \(\theta = 1/e\), has polynomial time complexity.

**Proof.** This follows from a straightforward application of the foregoing theorem, recalling that both \(1/e\) and any quadratic irrational satisfy the hypothesis therein. Specifically, for each quadratic irrational one may use Lagrange theorem, [27, Theorem 3, IV, §1, p.57]. For Euler’s constant \(e\), the proof follows by direct inspection of Euler’s continued fraction for \(e\), [27, V, §2, p.74] together with Serret’s theorem [27, Theorem 8, I, §3, p.14]. \(\square\)

Fix an irrational \(\theta\) and consider the following decision problem:

**INSTANCE** : Integers \(q > 0\) and \(p\), written in binary notation.

**QUESTION** : Does the inequality \(p/q \leq \theta\) hold ?

We say that \(\theta\) has an exponential time computable rational left cut if the above problem can be decided by a Turing machine in a number of steps bounded above by an exponential function of the length \(|(p,q)|\) of the input pair of binary integers \((p,q)\). In the literature on recursive analysis, one similarly defines “dyadic” left cuts, [26, Definition 2.7 and Remark, p.48].

**Corollary 7.6.** For any irrational \(\theta \in [0,1]\) having an exponential time computable left cut, the Effros-Shen algebra \(\mathfrak{F}_\theta\) has a polynomial time word problem.

**Proof.** Two formulas \(\phi\) and \(\psi\) code the same element of \(\mathfrak{F}_\theta\) iff the identity \(f_\phi(\theta) = f_\psi(\theta)\) holds in \(M([0,1])\) iff \(f_\delta(\theta) = 0\), with \(\delta\) the formula \(((\phi^* + \psi)^* \oplus (\psi^* + \phi)^*)\).

Indeed, as the reader will recall, \(f_5 = |f_\phi - f_\psi|\). It is sufficient to show the existence
of a polynomial $T$ such that for any formula $\rho$, the problem of checking whether $f_\rho(\theta) = 0$ can be solved in less than $T(||\rho||)$ Turing steps.

To this purpose, let

$$0 = r_0 < r_1 < \cdots < r_k = 1,\ r_i = n_i/d_i \in [0,1] \cap \mathbb{Q},\ 0 < d_i \leq 2||\rho||,(i = 0, \ldots, k)$$

be the Farey sequence of all rational points $r_i \in [0,1]$ of denominator $\leq 2||\rho||$.

Since every rational $x$ with $r_i < x < r_{i+1}$ has a denominator $d > 2||\rho||$, we claim that

$$f_\rho$$

is linear over each interval $[r_i, r_{i+1})$. (12)

For otherwise (absurdum hypothesis), $f_\rho$ has a non-differentiability point $x$ lying in the interior of $[r_i, r_{i+1}]$. Any such $x$ is the unique solution of an equation of the form $d'x + c' = d''x + c''$ for suitable integers $c', c'', d', d''$. Thus $x$ is rational, and its denominator $d$ satisfies the inequalities $0 < d \leq |d' - d''| \leq |d'| + |d''|$. As in the proof of Theorem 5.2 (Claim 1), the values $d'$ and $d''$ of the one-sided derivatives of $f_\rho$ at $x$ are bounded above by $||\rho||$. Thus $d \leq 2||\rho||$, a contradiction.

There exists a Turing machine $T$ and a (cubic) polynomial $T'$ with the following properties: Over any input $\rho$, $T$ outputs the sequence of intervals $[r_i, r_{i+1}]$ in less than $T''(||\rho||)$ steps. To this purpose, $T$ first computes the Farey mediant $1/2$ of 0 and 1, and then, inductively, $T$ inserts all possible Farey mediants of denominator $3, 4, \ldots, 2||\rho|| - 1, 2||\rho||$, (as pairs of integers in binary notation) until the desired Farey sequence is completed.

For each $i = 1, \ldots, k - 1$, let $L_i$ be the length of the quadruple of binary integers $n_i, d_i, n_{i+1}, d_{i+1}$ denoting the rational interval $[r_i, r_{i+1}]$. Then for some constant $\lambda > 0$, we can write $L_i < \lambda \log_2 ||\rho||$. Arguing inductively on subformulas of $\rho$ as in the proof of Theorem 7.4 we may assume that for some polynomial $T''$, machine $T$ outputs the list of (automatically rational) values $f_\rho(r_i), (i = 0, \ldots, k)$ in less than $T''(||\rho||)$ steps. Each $f_\rho(r_i)$ is a pair of binary integers.

Since $\theta$ has an exponential time computable rational left cut, $T$ detects the only interval $[r_{i*}, r_{i*+1}]$ with $\theta \in [r_{i*}, r_{i*+1}]$ in exponential time with respect to $L_i$, and hence, in polynomial time with respect to $||\rho||$.

We have thus shown the existence of the desired polynomial $T$, having the property that $T$ checks whether $f_\rho(r_{i*}) = f_\rho(r_{i*+1}) = 0$ in less than $T(||\rho||)$ steps. By (12), this condition is equivalent to $f_\rho(\theta) = 0$.

Thus the word problem of $S_\theta$ has polytime complexity. \hfill \Box

The following result strengthens the first statement of Corollary 7.5.

**Corollary 7.7.** Let $\theta \in [0,1] \setminus \mathbb{Q}$ be a real algebraic integer. Then the word problem of $S_\theta$ has polytime complexity.

**Proof.** There exists a one-variable polynomial $l: \mathbb{R} \to \mathbb{R}$ with integer coefficients, together with a rational interval $[r', r''] \subseteq [0,1]$ such that $\theta$ is the only root of $l$ lying in $[r', r'']$. We say that $(l, [r', r''])$ is the interval representation of $\theta$, [34] p.327.

Without loss of generality, we may also assume

$$\frac{dl(\theta)}{dx} > 0.$$  \hfill (13)

There exists a Turing machine $T$ and a constant $\kappa > 0$ such for any input pair $(p, q)$ of binary integers, $T$ decides whether $p/q < \theta$ in less than $2^{(\kappa(||p/q||))}$ steps: in view of (13), this amounts to checking whether $p/q < r'$ or else $p/q \in [r', r'']$, and $l(p/q) < 0$. (Since the data needed for the representation $(l, [r', r''])$ are assumed
8. Quotients of $\mathcal{M}_1$ and Gödel incomplete word problems

Differently from the earlier sections, in this section familiarity is required with first order logic (Gödel completeness and incompleteness, [31, 35]) and Lukasiewicz infinite-valued sentential logic (decidability, Lindenbaum algebras, [15, 42]).

As we have seen, many primitive quotients of the Farey AF algebra $\mathcal{M}_1$ existing in the literature have a polytime word problem. On the other hand, since there are only countably many Turing machines, already uncountably many Effros-Shen quotients $\mathfrak{F}_\theta$ have an undecidable word problem.

An interesting intermediate class of structures is given by quotients $\mathfrak{B} = \mathcal{M}_1/I$ having a Gödel incomplete word problem. This means that the set of pairs of formulas $\langle \phi, \psi \rangle$ that code the same equivalence class of projections of $\mathfrak{B}$ is recursively enumerable and undecidable.

The following result shows that primitive quotients of $\mathcal{M}_1$ are immune to Gödel incompleteness.

**Theorem 8.1.** Let $\mathfrak{P}$ be a primitive ideal of $\mathcal{M}_1$.

(a) If $\mathfrak{P}$ is a maximal ideal of $\mathcal{M}_1$ such that $\mathcal{M}_1/\mathfrak{P}$ is finite-dimensional, then the word problem of $\mathcal{M}_1/\mathfrak{P}$ is solvable in polynomial time.

(b) If $\mathfrak{P}$ is a maximal ideal of $\mathcal{M}_1$ such that $\mathcal{M}_1/\mathfrak{P}$ is not finite-dimensional, and the word problem of $\mathcal{M}_1/\mathfrak{P}$ is recursively enumerable, then it is decidable.

(c) If $\mathfrak{P}$ is not a maximal ideal of $\mathcal{M}_1$, the word problem of $\mathcal{M}_1/\mathfrak{P}$ is solvable in polynomial time.

**Proof.** We will work with $\mathcal{M}([0,1])$ in place of its isomorphic copy $E(\mathcal{M}_1)$.

(a) By Corollaries 4.5(i) and 4.7(ii), $\mathfrak{P} = \mathfrak{J}_{p/q}$ for some rational $p/q \in [0,1]$, and $E(\mathcal{M}_1/\mathfrak{J}_{p/q})$ is finite. Then, trivially, the word problem of $\mathcal{M}_1/\mathfrak{J}_{p/q}$ is solvable in polynomial time.

(b)-(c) By Lemma 4.2(iii), the map

$$\Gamma \circ K_0 : \mathfrak{J} \in \{\text{ideals of } \mathcal{M}_1\} \mapsto i \in \{\text{ideals of } \mathcal{M}([0,1])\}$$

is a lattice isomorphism. Since $\mathfrak{P}$ is primitive, $p$ is prime (meaning that $\mathcal{M}([0,1])/p$ is totally ordered). Recalling the one-one correspondence $\xi \in [0,1] \mapsto i_\xi$ in Theorem 4.1, let $\theta$ be the only point of $[0,1]$ such that the maximal ideal $j_\theta$ contains $p$. By Theorem 4.1, at most two ideals contain $p$, namely $j_\theta = \{f \in \mathcal{M}([0,1]) \mid f(\theta) = 0\}$ and $p$ itself.

**Case 1.** $p = j_\theta$.

In view of the lattice isomorphism $\mathfrak{J} \mapsto i$ in (14), $\mathfrak{P}$ is a maximal ideal of $\mathcal{M}_1$ and $\mathcal{M}_1/\mathfrak{P}$ is simple. By Lemma 4.2(iv), the MV-algebra $\Gamma(K_0(\mathcal{M}_1/\mathfrak{P}))$ may be identified with $\mathcal{M}([0,1])/p$. With reference to [15, §§4.4-4.6] or [22, §§1.5, 2.5], there is a theory (i.e., a set $\Theta$ of formulas containing all consequences of any finite subset of $\Theta$), consisting of one-variable formulas in Lukasiewicz logic, such that $\mathcal{M}([0,1])/p$ is the Lindenbaum algebra of $\Theta$, the algebra of equivalence classes of
formulas modulo $\Theta$. From the assumed recursive enumerability of the word problem of $M_1/P$ it follows that $\Theta$ is recursively enumerable. By [36, Theorem 6.1], the simplicity of $M_1/P$ entails the decidability of the problem whether a formula $\omega$ belongs to $\Theta$. By definition of Lindenbaum algebra, for every formula $\omega$ of the form $\phi \leftrightarrow \psi$ we have: $\omega$ belongs to $\Theta$ iff $f_\phi/p$ coincides with $f_\psi/p$ in the quotient $\text{MV-algebra } M([0,1])/p = \Gamma(K_0(M_1/P))$. Thus the word problem of $M_1/P$ is decidable. We have proved (b).

Case 2. $p \neq j_\theta$. By the classification Theorem 4.1, $\theta$ must be a rational number in $[0,1]$ and $p$ either coincides with the ideal $j_\theta^+ = \{ f \in M([0,1]) | f = 0 \text{ on a right neighborhood of } \theta \}$ or with $j_\theta^- = \{ f \in M([0,1]) | f = 0 \text{ on a left neighborhood of } \theta \}$. By Corollary 6.3, the word problem of both involutive monoids $M([0,1])/j_\theta^+$ and $M([0,1])/j_\theta^-$ has polytime complexity.

Having thus proved (c), the proof is complete. □

**Corollary 8.2.** Let $B$ be a quotient of $M_1$ with finitely many maximal ideals, and with a recursively enumerable word problem. Then the word problem of $B$ is decidable.

**Proof.** A routine refinement of the proof of the above theorem. □

**Theorem 8.3.** A quotient $\Omega$ of $M_1$ with a Gödel incomplete word problem can be explicitly constructed.

**Proof.** We refer to [31, 35] for all unexplained notions and results of first order logic needed for the proof. By Corollary 8.2, $\Omega$ necessarily has infinitely many maximal ideals. A fortiori, $\Omega$ is not a primitive quotient of $M_1$. Let $G$ be the set of Gödel numbers of all first-order tautologies (= universally true sentences) in the first order language of one binary relation symbol $\in$. We can safely assume

$$0, 1 \notin G. \quad (15)$$

On the one hand, Gödel's completeness theorem implies that $G$ is a recursively enumerable subset of $\{0, 1, \ldots \}$. On the other hand, Gödel's incompleteness theorem implies that $G$ is undecidable, because there are undecidable finitely axiomatizable first-order theories more powerful than Peano arithmetic in the language consisting of one binary relation symbol $\in$.

We define the functions $b_0, b_1, \ldots : [0,1] \to [0,1]$ as follows: $b_0$ is the constant 0 function over $[0,1]$. The graph of $b_1$ has two linear pieces, one joining the origin with $(1/2, 0)$, the other joining $(1/2, 0)$ with $(1,1)$. The graph of $b_2$ has three linear pieces, the first joining the origin with $(1/3, 0)$, the second joining $(1/3, 0)$ with $(1/2, 1/2)$, and the third joining $(1/2, 1/2)$ with $(1,0)$. For each $k = 3, 4, \ldots$, the graph of the hat-shaped function $b_k$ has four linear pieces: two pieces lie on the $x$-axis. The left oblique piece joins the points $(1/(k+1),0)$ and $(1/k,1/k)$. The right oblique piece joins the points $(1/k,1/k)$ and $(1/(k-1),0)$. Figure 4 shows the hats $b_1, \ldots, b_5$.

**Claim 1.** There is a Turing machine which, over input $k = 0, 1, \ldots$, outputs a one-variable formula $\beta(k)$ such that $f_{\beta(k)} = b_k$.
Figure 4. The hat functions $b_k$ for $k = 1, \ldots, 5$. Each linear piece of $b_k$ lies on a line whose equation has integer coefficients.

If $k = 0$, the formula $(X \oplus X^*)^*$ will do. For $k = 1$, the formula $\beta(1) = (X^* \oplus X^*)^*$ satisfies $f_{\beta(1)} = b_1$. If $k = 2, 3, \ldots$, the integer $2 \times 2$ matrices with rows $(1, k + 1)$ and $(1, 1)$ is unimodular. The same holds for the matrix with rows $(1, k)$ and $(1, 1)$. Since $b_k(1/k) = 1/k$, a straightforward verification shows that the two oblique linear pieces of the graph of $b_k$ are restrictions of linear polynomials with integer coefficients. McNaughton’s theorem [30] then merely ensures the existence of $\beta(k)$. To prove that $\beta(k)$ is computable from $k$, in [2, Proposition 2.7] an effective method is given to compute a formula $\epsilon_{m,n}$ which, for all integers $m, n$, codes the McNaughton function $\epsilon_{m,n}(x) = 0 \lor (1 \land (mx + n))$, $x \in [0, 1]$. The desired formula $\beta(k)$ is now effectively computed as the pointwise min of two suitably chosen formulas $\epsilon_{m,n}$, recalling that, by Theorem 2.1(iii), the $\land$ operation is definable from the involutive monoidal operations $^*$ and $\oplus$ of $\mathcal{M}([0, 1])$. Our first claim is settled.

Next we fix, once and for all, a recursive enumeration $\eta$ of $G$. In other words, $\eta: \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ is recursively enumerable and its range coincides with $G$. Then for each $t \in \{0, 1, \ldots\}$ we define the formula $\gamma(t)$ by

$$\gamma(t) = \beta(\eta(0)) \lor \beta(\eta(1)) \lor \cdots \lor \beta(\eta(t-1)) \lor \beta(\eta(t)).$$

(16)

Since the underlying lattice order is definable in terms of the $^*$ and $\oplus$ operations, and $G$ is recursively enumerable, then so is the sequence of formulas $\gamma(0), \gamma(1), \ldots$.

For each $t = 0, 1, \ldots$, let the McNaughton function $g_t \in \mathcal{M}([0, 1])$ be defined by

$$g_t = f_{\gamma(t)} = f_{\beta(\eta(0))} \lor f_{\beta(\eta(1))} \lor \cdots \lor f_{\beta(\eta(t-1))} \lor f_{\beta(\eta(t))}.$$  

(17)
The functions $g_t$ form an increasing sequence $g_0 \leq g_1 \leq \ldots$. Let $g$ denote the ideal of $M([0,1])$ generated by $\{g_0, g_1, \ldots\}$. For $n = 1, 2, \ldots$ and $l \in M([0,1])$ let us write

$$n \cdot l = l \oplus \cdots \oplus l.$$  \hfill (18)

By definition of $g$, for any $f \in M([0,1])$ we then have

$$f \in g \iff t.g_t \geq f \quad \text{for some} \ t = 1, 2, \ldots,$$

because the sequence $g_0, g_1, \ldots$ is ascending.

**Claim 2.** The set $\Omega$ of formulas $\gamma$ such that $f_\gamma$ belongs to $g$ is recursively enumerable.

It is easy to verify that for all $x, y \in [0,1]$, $x \leq y$ iff $x^* \oplus y = 1$. A tedious but straightforward verification (15 p.9) shows that a formula $\chi$ satisfies $f_\chi \in g$ iff $\chi^* \oplus t.\gamma(t)$ is a tautology in Łukasiewicz logic for some $t$. Since the set of tautologies in Łukasiewicz logic is decidable, (15 Theorem 4.6.10), (42, 18.3), using the recursive enumerability of the formulas $\gamma(t)$ we can enumerate all tautologies of the form $\chi^* \oplus t.\gamma(t)$, for some $t$ and $\chi$, and extract from these tautologies the desired set $\Omega$. Our second claim is settled.

**Claim 3.** The set $\Omega$ is undecidable.

By way of contradiction, suppose $\Omega$ is decidable. For any integer $k \geq 2$, we compute the one-variable formula $\beta(k)$ satisfying $f_{\beta(k)} = b_k$ as in the proof of Claim 1. If $k \in G$ then $b_k \leq g_k$, whence $b_k \in g$. Conversely, suppose $k \notin G$. By (17), every $g_t$ vanishes at $1/k$, but $b_k(1/k) = 1/k$. As a consequence, for no $t = 1, 2, \ldots$ we may have $t.g_t \geq b_k$, whence $b_k \notin g$. We have shown:

$$\beta(k) \in \Omega \iff b_k \in g$$

$$\quad \quad \iff t.g_t \geq b_k \quad \text{for some} \ t$$

$$\quad \quad \iff k \in G, \quad (\text{by definition of} \ b_k \text{and} g_t).$$

By our absurdum hypothesis, membership of $\beta(k)$ in $\Omega$ is decidable. It follows that for $k = 2, 3, \ldots$ (whence for all $k$, by (15) we may effectively decide whether $k$ belongs to $G$, thus contradicting the undecidability of $G$. Our third claim is settled.

To conclude the proof of the theorem, let $\mathfrak{G}$ be the ideal of $M_1$ corresponding to $g$ via the functors $K_0,$ and $\Gamma$. Let the quotient $\mathcal{A}F\ell$ algebra $\Omega$ be defined by

$$\Omega = M_1/\mathfrak{G}.$$  \hfill (19)

By Lemma 4.2(iv), we can write

$$E(M_1/\mathfrak{G}) = M([0,1])/g.$$  \hfill (19)

As in (9), for any two formulas $\phi$ and $\psi$ in the language $0, X^*, \oplus$ of involutive monoids, let $\delta$ be the formula

$$((\phi^* \oplus \psi)^* \oplus (\psi^* \oplus \phi)^*).$$  \hfill (19)

Then $\phi$ and $\psi$ code the same element of $M([0,1])/g$ iff $|f_\phi - f_\psi|$ belongs to $g$ iff $f_\delta \in g$. From Claim 2 it follows that the set $\Delta$ of pairs of formulas $(\phi, \psi)$ coding the same element of $M([0,1])/g$ is recursively enumerable. From Claim 3 it follows that $\Delta$ is undecidable, because formulas of the form (19) code all possible McNaughton functions (letting $\psi = 0$ in (19)).
In conclusion, the word problem of $\mathcal{M}([0,1])/g$ is G"odel incomplete, whence so
is the word problem of the AF$\ell$ algebra $Q$. □

**Remark 8.4.** The map $n \mapsto n$th hexadecimal digit of $\pi$ is Turing computable,
e.g., via the Bailey-Borwein-Plouffe formula [3].

$$\pi = \sum_{k=0}^{\infty} \left[ \frac{1}{16^k} \left( \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right].$$

As a consequence, the rational left cut of $\pi - 3$ is decidable. As a matter of fact, there is a Turing machine $T$ which, over input a pair of binary integers $p,q$ with $q > p > 0$ decides in a finite number of steps whether $p/q < \pi - 3$. To this purpose, $T$ lists the hexadecimal digits $h_1, h_2, \ldots$ of $p/q$ along with the hexadecimal digits $k_1, k_2, \ldots$ of $\pi - 3$. Let $m$ be the smallest integer such that $h_m \neq k_m$. Then $p/q$ lies in the left cut of $\pi - 3$ iff $h_m < k_m$.

A routine variant of the argument in the proof of Corollary 7.6 now shows that the word problem of the Effros-Shen algebra $\mathfrak{F}_{\pi - 3}$ is decidable.

**Problem 8.5.** What is the computational complexity of the word problem of $\mathfrak{F}_{\pi - 3}$?

**9. CONCLUDING REMARKS**

Further results on G"odel incompleteness in the frame of Elliott’s local semigroups are proved in [36, Example 6.4] and [45, Theorem 2.3]. Decidability and undecidability results for isomorphism problems of AF$\ell$ algebras presented by Bratteli diagrams are the subject matter of [10]-[11] and [39].

Throughout this paper, the categorical equivalence $\Gamma$ of [36] between unital $\ell$-groups and MV-algebras has been combined with Grothendieck $K_0$ functor ([19, 23]), to investigate the computational complexity of the word problem of several AF$\ell$ algebras. The natural deductive-algorithmic properties of the Lukasiewicz sentential calculus (whose semantics is provided by MV algebras, [13, 14]) have played a key role in the final section.

| FROM | TO |
|------|----|
| Finitely presented MV-algebras | Rational polyhedra, [33] |
| Locally finite MV-algebras | Multisets, [16] |
| MV-algebras with the map $(x_1, x_2, \ldots) \mapsto \sum_{k=1}^{\infty} x_k/2^k$ | Compact spaces, [32] |

**Table 1.** Three contravariant categorical equivalences for MV-algebras.

Other (contravariant) categorical equivalences, also known as “dualities”, between classes of MV-algebras and various classes of mathematical structures are summarized in Table 1, showing that the scope of MV-algebras goes beyond Lukasiewicz logic, unital $\ell$ groups, and AF$\ell$ algebras.

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(D. Mundici) Department of Mathematics and Computer Science “Ulisse Dini”, University of Florence, viale Morgagni 67/A, 50134 Florence, Italy
E-mail address: mundici@math.unifi.it