MOMENTS OF THE BLOCK OPERATORS IN THE GROUP VON NEUMANN ALGEBRAS

ILWOO CHO

Abstract. In this paper, we will consider the moments of the block operators of the given group von Neumann algebra L(G), where the given group G is a finitely presented discrete group < X : R >, where X is the generator set of G and R is the relation on the set X, as the set of relators. Define the canonical trace tr on L(G) and the W*-probability space (L(G), tr) which is our free probabilistic object of this paper. Define the block operators T_x by T_x = x + x^{-1} of L(G), where x ∈ X. In this paper, we will compute the moments and the moment series of T_x, for x ∈ X. By the computation, we can get that if x_1 and x_2 are generators of presented groups < X_1 : R_1 > and < X_2 : R_2 >, respectively, and (i) if there is n ∈ N such that x_1^n ∈ R_1 and x_2^n ∈ R_2, or (ii) if there is no n_1, n_2 ∈ N such that x_1^n_1 ∈ R_1 and x_2^n_2 ∈ R_2, then the block operators x_1 + x_1^{-1} in L(G_1) and x_2 + x_2^{-1} in L(G_2) are identically distributed.

The group von Neumann algebras are studied recently by various authors. Group von Neumann algebras are interesting objects in Operator Algebra and Free Probability. In this paper, we will consider the moments of certain operators in group von Neumann algebras, where the group is presented by a finite generator set and a finite relation. We will take a presented group < X : R >, where X is the generator set and R is the relation on the group, as the nonempty set of relators. For instance, the symmetric group S_3 can be presented by its generator set X_{S_3} and its relation R_{S_3}, where

\[ X_{S_3} = \{ a, b \} \]

and

\[ R_{S_3} = \{ a^2, b^3, (ab)^2 \} \]

Let H = < X : R > be a presented group and let L(H) be the group von Neumann algebra generated by H, i.e.,

\[ L(H) = \overline{\lambda(H)^\sigma} = \overline{C[H]^\sigma} \]

where \lambda is the left (unitary) representation. If a ∈ L(H) is an operator, then it has the Fourier expansion,

\[ a = \sum_{h \in H} a_h h, \quad \text{for} \quad a_h \in \mathbb{C} \text{ and } h \in H. \]

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In fact, the group element $h$ in the previous expansion are understood as unitary operators $\lambda_h$ on the Hilbert space $l^2(H)$. Recall that

$$x^* = \sum_{h \in H} a_h h^{-1},$$

where $h^{-1} = \lambda_{h^{-1}} = \lambda_h^* = h^*$, on $l^2(H)$, for all $h \in H$. Define the canonical trace $\text{tr}$ on $L(H)$ by

$$\text{tr} (x) = \text{tr} \left( \sum_{h \in H} a_h h \right) = a_{e:H}, \text{ for all } x \in L(H)$$

where $e_H$ is the identity of $H$. Then we can have the $W^*$-probability space $(L(H), \text{tr})$. The main purpose of this paper is to compute the moments of the block operators of $L(H)$. In order to do that we observed the free monoid $X^* = (X \cup X^{-1})'$ of the group $H$ and the corresponding combinatorial forms in $X^*$ of the group elements in $G$. (If $Y$ is an arbitrary set, then $Y'$ is the set of all free words in $Y$, which is called the free monoid of $Y$. The elements in $Y'$ are called the combinatorial forms of $<Y>$, where $<Y>$ is the group generated by $Y$.)

Let $G = <X : R>$ be a presented group with its generator set $X = \{x_1, ..., x_N\}$ and its relation $R = \{r_1, ..., r_M\}$. In Chapter 1, we will consider the free monoid $X^*$ of the group $G$ defined by

$$X^* \overset{\text{def}}{=} \bigcup_{n=0}^{\infty} \left\{ x_{j_1}^{p_1} ... x_{j_n}^{p_n} : (j_1, ..., j_n) \in \{1, ..., N\}^n, (p_1, ..., p_n) \in \{1, -1\}^n \right\},$$

which is the set of all free words of the generator set $X$ and $X^{-1}$. When $n = 0$, the corresponding word is the empty word $\emptyset$. There exists a monoid homomorphism $\pi$ from $X^*$ onto the given group $G$. Notice that, for any $g \in G$, there is a subset $\pi^{-1}(g)$ in $X^*$. The elements $w_g$ in $\pi^{-1}(g)$ are called the combinatorial forms of $g \in G$. We will use the word problem on $X^*$, by computing the moments of the block operators.

In Chapter 2, we will compute the moments of the block operators $T_x = x + x^{-1}$, for $x \in X$, in the $W^*$-probability space $(L(G), \text{tr})$. In particular, we have that:

(1) Suppose that there is no relator $r_t \in R$ such that $x^k = r_t$, for all $k \in \mathbb{N}$. Then

$$\text{tr} (T_x^m) = \left[ \begin{array}{c} m \\ \frac{m}{2} \end{array} \right], \text{ for all } m \in \mathbb{N},$$

where $\left[ \begin{array}{c} m \\ \frac{m}{2} \end{array} \right] \overset{\text{def}}{=} \begin{cases} \left( \begin{array}{c} m \\ \frac{m}{2} \end{array} \right) = \frac{m!}{(\frac{m}{2})!(\frac{m}{2})!} & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}$.
(2) Suppose that there exist \( r_t \in \mathbb{R} \) and \( n_x \in \mathbb{N} \) such that \( r_t = x^{n_x} \). Then

\[
\text{tr} \left( T_{x^m} \right) = \begin{cases} 
\left[ \frac{m}{2} \right] & \text{if } m < n_j \\
\left( 2k_1 - \left[ \frac{k_1}{2} \right] \right) + \left[ \frac{m}{2} \right] & \text{if } m = k_1n_j + k_2,
\end{cases}
\]

where \( k_1 \in \mathbb{N} \) and \( k_2 \in \mathbb{N} \cup \{0\} \) such that \( 0 \leq k_2 < n_x \).

Notice that if \( G_1 \) and \( G_2 \) are finitely presented groups (not necessarily distinct) and \( x_1 \) and \( x_2 \) are generators of \( G_1 \) and \( G_2 \), respectively, and if there exists \( n \in \mathbb{N} \) such that \( x_1^n \) and \( x_2^n \) are relators of \( G_1 \) and \( G_2 \), respectively, then the block operators \( x_1 + x_1^{-1} \) in \( L(G_1) \) and \( x_2 + x_2^{-1} \) in \( L(G_2) \) are identically distributed. Also, if \( x_1 \) and \( x_2 \) have no relators \( x_1^n \) and \( x_2^k \), for \( n, k \in \mathbb{N} \), then they are also identically distributed.

1. Preliminaries

In this paper, we will compute the moments of certain operators on a group von Neumann algebra with its canonical faithful normal trace. We will restrict our interests to finitely presented group von Neumann algebras. Let \( G = \langle X : R \rangle \) be a presented group, where \( X \) is the finite generator set of the group \( G \) and \( R \) is the relation on \( G \), as the set of all relators. Denote the corresponding group von Neumann algebra by \( L(G) \). Then each operator \( a \in L(G) \) has its Fourier expansion

\[
a = \sum_{g \in G} a_g g, \quad a_g \in \mathbb{C}.
\]

Note that we can regard \( g \) in (1.3) as \( \lambda g \), for all \( g \in G \), where \( \lambda \) is the left regular representation. Remark that \( g^* = g^{-1} \), for all \( g \in G \), in \( L(G) \), and hence each operator \( g \in L(G) \) is unitary. For the group von Neumann algebra \( L(G) \), we can define the canonical trace \( tr \) by

\[
\text{tr}(a) \overset{def}{=} \text{tr} \left( \sum_{g \in G} a_g g \right) = a_{\epsilon G},
\]

for all \( a \in L(G) \), where \( \epsilon_G \) is the identity of the group \( G \).

**Definition 1.1.** Let \( G = \langle X : R \rangle \) be the presented group and \( L(G) \), the corresponding group von Neumann algebra. The algebraic pair \( (L(G), \text{tr}) \) is called the presented group \( W^* \)-probability space, where \( tr \) is the canonical trace given in (1.2).
The operators in \((L(G), tr)\) are called the random variables. Let \(a \in L(G)\). Then the \(n\)-th moments of \(a\) is defined by

\[\text{tr}(a^n), \text{ for all } n \in \mathbb{N}.\]

Now, we have our free probabilistic objects of this paper.

2. Moments of The Block Operators in Group von Neumann Algebras

Throughout this chapter, let \(G = \langle X : R \rangle\) be the fixed presented graph with its generator set \(X\) and its relation \(R\)

\[X = \{x_1, ..., x_N\}\]

and

\[R = \{r_1, ..., r_M\},\]

where \(M, N \in \mathbb{N}\). In this chapter, we will compute the moments of block operators \(T_x = x + x^{-1}\), for \(x, x^{-1} \in X\), for \(j = 1, ..., N\).

**Definition 2.1.** Let \(G = \langle X : R \rangle\) be a presented group with its generator set \(X = \{x_1, ..., x_N\}\) and its relation \(R = \{r_1, ..., r_M\}\), where \(M, N \in \mathbb{N}\) and \(r_1, ..., r_M\) are relators, as elements in \(X^*\). The operators \(T_j\) are block operators in the group von Neumann algebra \(L(G)\), if

\[T_j = x_j + x_j^{-1}, \text{ for all } j = 1, ..., N.\]

2.1. Moments of Block Operators.

Let \(Y\) be an arbitrary set. Then we can define a set \(Y'\), consisting of all free words in \(Y\). This set \(Y'\) is called the free set of \(Y\). Let \(G = \langle X : R \rangle\) be the finitely presented group with

\[X = \{x_1, ..., x_N\} \text{ and } R = \{r_1, ..., r_M\}.\]

Define the set \(X^*\) be the free monoid \((X \cup X^{-1})^*\) of the set \(X \cup X^{-1}\). Notice that, there exists the surjective (monoid) homomorphism \(\pi : X^* \rightarrow G\) and, for any group element \(g \in G\), there exist words \(w_g\) in \(X^*\) satisfying that \(\pi(w_g) = g\) in \(G\).

It is easy to see that a corresponding word \(w_g\) of \(g\) is not uniquely determined. We
say that such words $w_g \in \pi^{-1}(g)$ of $g \in G$ are combinatorial forms of $g$. Let $w_g = x_{j_1}^{p_1} \cdots x_{j_n}^{p_n} \in X^*$ be a combinatorial form of $g \in G$, where

$$(j_1, \ldots, j_n) \in \{1, \ldots, N\}^n$$

and $(p_1, \ldots, p_n) \in \{\pm 1\}^n$.

For convenience, we denote the word $x_{j_1}^{-p_1} \cdots x_{j_n}^{-p_n}$ by $w_g^{-1}$. Notice that $w_g^{-1} \in \pi^{-1}(g^{-1})$ in $X^*$.

Now, fix the generators $x_j \in X$ and the block operator $T_j = x_j + x_j^{-1}$. Consider the $n$-th moments $\text{tr} (T_j^m)$ of $T_j$, for all $m \in \mathbb{N}$. It is easy to see that

$$T_j^m = \sum_{(p_1, \ldots, p_m) \in \{1, -1\}^m} (x_j^{p_1} \cdots x_j^{p_m}) = \sum_{(p_1, \ldots, p_m) \in \{1, -1\}^m} x_j^{\sum_{k=1}^{m} p_k},$$

for all $m \in \mathbb{N}$. Notice that each word $x_j^{\sum_{k=1}^{m} p_k} = x_j^{p_1} \cdots x_j^{p_m}$ is regarded as an element in the free monoid $X^*$. i.e., without loss of generality, we can consider the summands of $T_j^m$ as elements in the free monoid $X^*$.

**Lemma 2.1.** Let $T_j = x_j + x_j^{-1}$ be a block operator of the generator $x_j \in X$. If there is no $n_j \in \mathbb{N}$ such that $x_j^{n_j} \in R$, then

$$\text{tr} (T_j^m) = \begin{cases} \binom{m}{2} & \text{if } m \in 2\mathbb{N} \\ 0 & \text{if } m \in 2\mathbb{N} - 1, \end{cases}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, for $n, k \in \mathbb{N}$.

**Proof.** Assume that there is no relator $r_t \in R$ and $n_j \in \mathbb{N}$ such that $r_t = x_j^{n_j}$ in $R \subset X^*$. Then, by (2.1), we have that

$$\text{tr} (T_j^m) = \text{tr} \left( \sum_{(p_1, \ldots, p_m) \in \{1, -1\}^m} x_j^{\sum_{k=1}^{m} p_k} \right)$$

$$= \text{tr} \left( \sum_{(p_1, \ldots, p_m), \sum_{j=1}^{m} p_j = 0} e_G \right)$$

$$= \sum_{(p_1, \ldots, p_m), \sum_{k=1}^{m} p_j = 0} e_G$$

$$= \left| \{(p_1, \ldots, p_m) \in \{1, -1\}^n : \sum_{k=1}^{n} p_j = 0 \} \right|$$
The last equality holds because, to make $\sum_{k=1}^{m} p_k = 0$, the same number of +1's and −1's should be appeared in the sequence $(p_1, ..., p_m)$. It is easy to see that if $m$ is odd, then the set

$$\{(p_1, ..., p_m) \in \{1, -1\}^n : \sum_{k=1}^{n} p_j = 0\}$$

is empty. Therefore, all odd moments of $T_j$ vanish.

Now, we assume that there is a relator $r_t \in R$ such that $r_t = x_{n_j}^m$, for $n_j \in \mathbb{N}$, as a free word in $X^*$. Clearly, the relator $r_t$ is a combinatorial form of $e_G$ (i.e., $\pi(r_t) = e_G$) in the group $G$ and the length $|r_t|$ of $r_t$ in $X^*$ is $n_j$. Also, notice that if $r_t \in R$, then the words $wr_tw^{-1}$ and $wr_t^{-1}w^{-1}$ are also combinatorial forms of $e_G$, for all words $w$ and $w'$ in $X^*$. By (2.1), we have that

$$T_j^m = \sum_{(p_1, ..., p_m) \in \{1, -1\}^m} x_j^{\sum_{k=1}^{m} p_k}.$$  

If $m = n_j$, then

$$T_j^{n_j} = \left( x_j^{n_j} + x_j^{-n_j} \right) + \sum_{(p_1, ..., p_{n_j}) \in \{1, -1\}^{n_j}, (p_1, ..., p_{n_j}) \neq (\pm 1, ..., \pm 1)} x_j^{\sum_{k=1}^{n_j} p_k}$$

(2.2)

$$= 2e_G + \sum_{(p_1, ..., p_{n_j}) \in \{1, -1\}^{n_j}, (p_1, ..., p_{n_j}) \neq (\pm 1, ..., \pm 1)} x_j^{\sum_{k=1}^{n_j} p_k}.$$  

By the above formula (2.2), we can get the following lemma:

**Lemma 2.2.** Let $x_j \in X$ and assume that there exists $t \in \{1, ..., M\}$ such that $r_t = x_{n_j}^m$, for $n_j \in \mathbb{N}$. Then

$$tr(T_j^m) = \begin{cases} \left( \frac{m}{2} \right) & \text{for all even } m < n_j \\ 0 & \text{for all odd } m < n_j \end{cases}$$

and

$$tr(T_j^{n_j}) = \begin{cases} 2 & \text{if } n_j \in 2\mathbb{N} - 1 \\ 2 + \left( \frac{n_j}{2} \right) & \text{if } n_j \in 2\mathbb{N} \end{cases}.$$
Proof. The first formula is trivial, by the previous lemma.

Suppose that \( n_j \) is an odd number in \( \mathbb{N} \). Then, by (2.2), we have that

\[
T_j^{n_j} = 2e_G + \sum_{(p_1, ..., p_{n_j}) \in \{1, -1\}^{n_j}, (p_1, ..., p_{n_j}) \neq (\pm 1, ..., \pm 1)} \sum_{k=1}^{n_j} p_k.
\]

Since \( n_j \) is an odd number, we cannot find the sequence \((p_1, ..., p_{n_j})\) in \( \{\pm 1\}^{n_j} \) satisfying that \( \sum_{k=1}^{n_j} p_k = 0 \). So, we cannot find the \( e_G \)-terms in the summand

\[
\sum_{(p_1, ..., p_{n_j}) \in \{1, -1\}^{n_j}, (p_1, ..., p_{n_j}) \neq (\pm 1, ..., \pm 1)} \sum_{k=1}^{n_j} p_k
\]

of \( T_j^{n_j} \). Thus if \( n_j \) is an odd number, then \( tr(T_j^{n_j}) = 2 \).

Now, assume that \( n_j \) is an even number in \( \mathbb{N} \). Then, again by (2.2), we have that

\[
tr(T_j^{n_j}) = 2 + \sum_{(p_1, ..., p_{n_j}) \in \{1, -1\}^{n_j}, (p_1, ..., p_{n_j}) \neq (\pm 1, ..., \pm 1)} tr\left(\sum_{j=1}^{n_j} x_j^{n_j} p_k\right)
\]

\[
= 2 + \left| \{(p_1, ..., p_{n_j}) \in \{\pm 1\}^{n_j} : \sum_{k=1}^{n_j} p_k = 0\} \right|
\]

\[
= 2 + \left(\frac{n_j}{2}\right).
\]

Now, suppose that \( m > n_j \). There are two cases:

(i) \( m = k_1 n_j + k_2 \), where \( 1 \leq k_2 < n_j \) or

(ii) \( m = kn_j \), for some \( k \in \mathbb{N} \).

Lemma 2.3. Let \( x_j \in X \) and assume that there exists \( t \in \{1, ..., M\} \) such that \( r_t = x_j^{n_j} \), for \( n_j \in \mathbb{N} \). Then

(1) If \( m = kn_j \), for \( k \in \mathbb{N} \), then

\[
tr(T_j^m) = 2^{k} - \left[ \frac{k}{2} \right] + \left[ \frac{kn_j}{m} \right].
\]

(2) If \( m = k_1 n_j + k_2 \), for \( k_1, k_2 \in \mathbb{N} \) and if \( 1 \leq k_2 < n_j \), then

\[
tr(T_j^m) = 2^{k_1} + \left[ \frac{m}{2} \right] - \left[ \frac{k_1}{k_2} \right].
\]
where

\[
\begin{bmatrix} t \\ \frac{t}{2} \end{bmatrix} \overset{\text{def}}{=} \begin{cases} \left( \frac{t}{2} \right) & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd.} \end{cases}
\]

Proof. (1) Let \((p_1, ..., p_m) \in \{\pm 1\}^m\), where \(m = kn_j\) is sufficiently big number in \(\mathbb{N}\), where \(k, n_j \in \mathbb{N}\). Define subsequences

\[
i_+ = \left( \frac{1, \ldots, 1}{n_j\text{-times}} \right) \quad \text{and} \quad i_- = \left( \frac{-1, \ldots, -1}{n_j\text{-times}} \right).
\]

Since \(m = kn_j\), there exists a sequence \(P = (p_1, ..., p_m)\) such that

\[P = (i_{i_1}, ..., i_{i_k}), \text{ for } i_1, ..., i_k \in \{+, -\}.\]

We define the set \(W_j\), consisting of such sequences. i.e.,

\[(2.3) \quad W_j \overset{\text{def}}{=} \{ (i_{i_1}, ..., i_{i_k}) : i_1, ..., i_k \in \{+, -\} \}.
\]

Note that, for \(i_+\), we have \(x_{j_{i_1}} = r_{t_i} \) and, for \(i_-\), we have \(x_{j_{i_1}}^{-n_j} = r_{t_i}^{-1} \) in \(X^*\). Thus, we can get that

\[(2.4) \quad |W_j| = 2^k \quad \text{and} \quad (2.5) \quad \sum_{k=1}^m p_k = pm_j, \text{ where } p = 1 \text{ or } ... \text{ or } k,
\]

for all \((p_1, ..., p_m)\) in \(W_j\). We will define a subset \(W'_j\) of the set \(W_j\) (if exists) by

\[(2.6) \quad W'_j = \left\{ (i_{i_1}, ..., i_{i_k}) : i_{j_{i_1}} = ... = i_{j_{i_2}} = +, \right. \]

\[i_{s_{i_1}} = ... = i_{s_{i_2}} = -, \quad \{i_{j_1}, ..., i_{j_{i_k}}\} \cup \{i_{s_1}, ..., i_{s_{i_k}}\} = \{i_1, ..., i_k\} \right\}
\]

Then \(W'_j \subseteq W_j\) and, since there are same numbers of +’s and -’s, we can have

\[(2.7) \quad |W'_j| = \binom{k}{\frac{k}{2}},
\]

if the nonempty subset \(W'_j\) exists in \(W_j\). (It is easily see that if \(k\) is even, then \(W'_j\) exists in \(W_j\). And if \(k\) is odd, then \(W'_j\) is empty.) Define the subset \(S_0\) of sequences in \(\{\pm 1\}^m\) by
\[ S_0 = \{ (p_1, \ldots, p_m) \in \{ \pm 1 \}^m : \sum_{k=1}^m p_k = 0 \}. \]

Then, by (2.6), \( W'_j \subset S_0 \). In fact,

\[(2.8) \quad S_0 \cup W'_j = S_0 \text{ and } (W_j \setminus W'_j) \cap S_0 = \emptyset.\]

Assume that \( k \) is even. Then

\[
\text{tr} (T^m_j) = \text{tr} \left( \sum_{(p_1, \ldots, p_m) \in \{ \pm 1 \}^m} x^{\sum_{k=1}^m p_k} \right)
\]

\[
= \text{tr} \left( \sum_{(p_1, \ldots, p_m) \in W_j \setminus W'_j} x^{\sum_{k=1}^m p_k} \right)
\]

\[
+ \text{tr} \left( \sum_{(p_1, \ldots, p_m) \in \{ \pm 1 \}^m, (p_1, \ldots, p_m) \in S_0} x^{\sum_{k=1}^m p_k} \right)
\]

\[
= |W_j \setminus W'_j| + \text{tr} \left( \sum_{(p_1, \ldots, p_m) \in \{ \pm 1 \}^m, (p_1, \ldots, p_m) \in S_0} e^G \right)
\]

where \( W_j \) is defined in (2.3), by (2.8)

\[
= \left( 2^k - \left( \frac{k}{2} \right) \right) + |S_0|
\]

\[(2.9) \quad = \begin{cases} \left( 2^k - \left( \frac{k}{2} \right) \right) + \left( \frac{m}{2} \right) & \text{if } m \text{ is even} \\ 2^k - \left( \frac{k}{2} \right) & \text{if } m \text{ is odd} \end{cases},\]

where \( k \in 2\mathbb{N} \) and \( m \in \mathbb{N} \). Now, let’s suppose that \( k \) is an odd number greater than 1 in \( \mathbb{N} \). Then we can have that

\[(2.10) \quad W'_j = \emptyset \quad \text{and} \quad W_j \cap S_0 = \emptyset.\]

Therefore, by (2.10), we have that

\[
\text{tr} (T^m_j) = \text{tr} \left( \sum_{(p_1, \ldots, p_m) \in W_j} x^{\sum_{k=1}^m p_k} \right)
\]
\[ + \text{tr} \left( \sum_{(p_1, \ldots, p_m) \in \{\pm 1\}^m : \sum_{k=1}^m p_k \in S_0} \sum_{k=1}^m p_k \right) \]

\[ = |W_j| + \text{tr} \left( \sum_{(p_1, \ldots, p_m) \in \{\pm 1\}^m : (p_1, \ldots, p_m) \in S_0} e_G \right) \]

\[ = |W_j| + |S_0| \]

(2.11)

\[ = \begin{cases} 2^k + \left( \frac{m}{2} \right) & \text{if } m \text{ is even} \\ 2^k & \text{if } m \text{ is odd.} \end{cases} \]

Now, define a new notation

\[ \left[ \frac{t}{t} \right] \overset{\text{def}}{=} \begin{cases} \left( \frac{t}{t} \right) & \text{if } t \in 2 \mathbb{N} \\ 0 & \text{if } t \in 2 \mathbb{N} - 1. \end{cases} \]

Then the formula (2.9) and (2.11) can be shortened by

\[ \text{tr} \left( T_{kn}^m \right) = 2^k - \left[ \frac{k}{2} \right] + \left[ \frac{kn_j}{2} \right]. \]

(2) Let's assume that \( m = k_1 n_j + k_2 \), where \( n_j \nmid k_2 \) and \( 1 \leq k_2 < n_j \). By (2.1), we have that

\[ T_{kn}^m = \sum_{(p_1, \ldots, p_m) \in \{\pm 1\}^m} x_{\sum_{k=1}^m p_k}. \]

Let's regard the summands \( x_{\sum_{k=1}^m p_k} \) as free words in the free monoid \( X^* \). Then there is a set

\[ S_0 = \{ (p_1, \ldots, p_m) \in \{\pm 1\}^m : \sum_{k=1}^m p_k = 0 \} \]

with its cardinality \( |S_0| = \left[ \frac{m}{2} \right] \). Now, define a set of free words \( W^\prime \) by

\[ W^\prime = \left\{ ((p_{i_1}, \ldots, p_{i_{k_2}}) \sim (i_{i_1}, \ldots, i_{i_{k_2}})) : \begin{array}{c} p_{i_k} \in \{\pm 1\}, \\ i_{k_2} \in \{+, -\}, \end{array} \right\}, \]

where \( i_+ \) and \( i_- \) are defined in (1) and \( \sim \) means the insertion. i.e., the sequence

\[ ((p_{i_1}, \ldots, p_{i_{k_2}}) \sim (i_{i_1}, \ldots, i_{i_{k_2}})) \]
is the free word in $X^*$ with its length $m = k_1 n_j + k_2$. For example,

$$(p_1) \sim (i_+, i_+)$$

is

$$(i_+, p_1, i_+) \text{ or } (p_1, i_+, i_+) \text{ or } (i_+, i_+, p_1).$$

where $m = 2n_j + 1$. Define the subset $W^j_{k_2}$ of $W^j$ (if exists) by

$$W^j_{k_2} = \{((p_{i_1}, \ldots, p_{i_{k_2}}) \sim (i_{i_1}, \ldots, i_{i_{k_2}})) : \sum_{i=1}^{k_2} p_{i_1} = 0\}.$$ 

Let’s assume that $W^j_{k_2}$ exists. Then we can define the subset $W^j_{k_2}(S_0)$ of $W^j_{k_2}$ by

$$W^j_{k_2}(S_0) = \{((p_{i_1}, \ldots, p_{i_{k_2}}) \sim (i_{i_1}, \ldots, i_{i_{k_2}})) : \sum_{i=1}^{k_2} p_{i_1} = 0, (i_{i_1}, \ldots, i_{i_{k_2}}) \in W^j, (i_{i_1}, \ldots, i_{i_{k_2}}) \in W^j_{k_2} \}\},$$

where $W^j$ is defined in (1). It is easy to see that if both $k_1$ and $k_2$ are even, then $W^j_{k_2}$ exists in $W^j_{k_2}$. Then

\begin{align}
(2.12) & \quad |W^j_{k_2}| = 2^{k_1} + \left[ \frac{k_2}{2} \right], \\
(2.13) & \quad |W^j_{k_2}(S_0)| = \left[ \frac{k_1}{2} \right] + \left[ \frac{k_2}{2} \right], \\
(2.14) & \quad W^j_{k_2}(S_0) \cup S_0 = S_0, \\
(2.15) & \quad |S_0 \setminus W^j_{k_2}(S_0)| = \left[ \frac{m}{2} \right] - \left( \left[ \frac{k_1}{2} \right] + \left[ \frac{k_2}{2} \right] \right)
\end{align}

By (2.12) and (2.15), we can compute that:

\begin{align}
tr \left( T_j^m \right) & = tr \left( \sum_{(p_1, \ldots, p_m) \in \{\pm 1\}^m} x^{\sum_{k=1}^{m} p_m} \right) \\
& = tr \left( \sum_{(p_1, \ldots, p_m) \in W^j_{k_2}} x^{\sum_{k=1}^{m} p_m} \right) \\
& \quad + tr \left( \sum_{(p_1, \ldots, p_m) \in S_0 \setminus W^j_{k_2}(S_0)} x^{\sum_{k=1}^{m} p_m} \right) \\
& = |W^j_{k_2}| + |S_0 \setminus W^j_{k_2}(S_0)|
\end{align}
\[
= \left(2^{k_1} + \left[\frac{k_2}{2}\right]\right) + \left(\left[\frac{m}{m}\right] - \left(\left[\frac{k_1}{2}\right] + \left[\frac{k_2}{2}\right]\right)\right) = \left[\frac{m}{m}\right] - \left[\frac{k_1}{2}\right].
\]

Remark that, by the previous lemma, we have that if \(k = 1\), then,

\[
tr \left(T_j^{n_j}\right) = \left(2^1 - \left[\frac{1}{2}\right]\right) + \left[\frac{n_j}{2}\right] = 2 + \left[\frac{n_j}{2}\right].
\]

By the previous lemmas, we can get the following theorem;

**Theorem 2.4.** Let \(G = \langle X : R \rangle\) be a finitely presented group with its generator set \(X = \{x_1, \ldots, x_N\}\) and its relation \(R = \{r_1, \ldots, r_M\}\). Fix a generator \(x_j \in X\) satisfying that \(r_t = x_j^{n_j}\), for some \(n_j \in \mathbb{N} \setminus \{1\}\), where \(r_t \in R\). Then

1. if \(m < n_j\), then \(tr \left(T_j^m\right) = \left[\frac{m}{m}\right]\).
2. if \(m = kn_j\), for \(k \in \mathbb{N}\), then

\[
tr \left(T_j^m\right) = \left(2^k - \left[\frac{k}{2}\right]\right) + \left[\frac{m}{m}\right].
\]
3. if \(m = k_1n_j + k_2\), for \(k_1 \in \mathbb{N}\) and \(1 \leq k_2 < n_j\), then

\[
tr \left(T_j^m\right) = \left(2^{k_1} - \left[\frac{k_1}{2}\right]\right) + \left[\frac{m}{m}\right].
\]

□

We will finish this chapter with the following remark;

**Remark 2.1.** Let \(G = \langle X : R \rangle\) be a finitely presented group with its generator set \(X = \{x_1, \ldots, x_N\}\) and the relation \(R = \{r_1, \ldots, r_M\}\) and let’s fix a generator \(x_j \in X\) and the corresponding block operator \(T_j = x_j + x_j^{-1}\). Suppose there exist \(r_t \in R\) and \(n_j \in \mathbb{N}\) such that \(r_t = x_j^{n_j}\). Then, by (2) and (3) of the previous theorem,

\[
tr \left(T_j^m\right) = \begin{cases} 
\left[\frac{m}{m}\right] & \text{if } m < n_j \\
\left(2^{k_1} - \left[\frac{k_1}{2}\right]\right) + \left[\frac{m}{m}\right] & \text{if } m \geq n_j,
\end{cases}
\]
where \( m = k_1 n_j + k_2 \), for \( k_1 \in \mathbb{N} \) and \( k_2 \in \mathbb{N} \cup \{0\} \). □

### 2.2. Identically Distributedness.

By the previous section, we have that if \( G = \langle X : R \rangle \) is a finitely presented group and if \( x \in X \), then

(i) if there exists \( n \in \mathbb{N} \) such that \( x^n = r \), for \( r \in R \), then

\[
\begin{cases}
\left[ \frac{m}{m_j} \right] & \text{if } m < n_j \\
\left( 2^{k_1} - \left[ \frac{k_1}{k_2} \right] \right) + \left[ \frac{m}{m_j} \right] & \text{if } m \geq n_j,
\end{cases}
\]

where \( m = k_1 n_j + k_2 \), for \( k_1 \in \mathbb{N} \) and \( k_2 \in \mathbb{N} \cup \{0\} \), and

(ii) if there is no \( n \in \mathbb{N} \) such that \( r = x^n \), for all \( r \in R \), then

\[
tr(T^m) = \left[ \frac{m}{m_j} \right], \text{ for all } m \in \mathbb{N}.
\]

The above formula directly proves the following theorem:

**Theorem 2.5.** Let \( G_i = \langle X_i : R_i \rangle \) be finitely presented groups, for \( i = 1, 2 \), and assume that \( x_i \in X_i \) are generators of \( G_i \), for \( i = 1, 2 \). If there are relators \( r_i \in R_i \) and \( n \in \mathbb{N} \) such that \( r_i = x_i^n \), for all \( i = 1, 2 \), then the block operators \( (x_1 + x_1^{-1}) \) of the group von Neumann algebras \( L(G_i) \), \( i = 1, 2 \), are identically distributed.

**Proof.** By (i) and (ii) in the previous paragraph, case by case, we can get that

\[
tr(T_1^m) = tr(T_2^m), \text{ for all } m \in \mathbb{N}.
\]

In the above theorem, \( G_1 \) and \( G_2 \) are not necessarily distinct. i.e., suppose that we have a finitely presented group \( G = \langle X : R \rangle \) and its generators \( x_1, x_2 \) in \( X \) satisfying that \( x_1^n \) and \( x_2^n \) are relators in \( R \). Then the block operators \( (x_1 + x_1^{-1}) \) and \( (x_2 + x_2^{-1}) \) are identically distributed in the group von Neumann algebra, \( L(G) \).

The following theorem is proved, similarly:
Theorem 2.6. Let $G_i = < X_i : R_i >$ be finitely presented groups, for $i = 1, 2$, and assume that $x_i \in X_i$ are generators of $G_i$, for $i = 1, 2$. Suppose that there is no numbers $n_1, n_2 \in \mathbb{N}$ such that $x_i^{n_i} \in R_i$, for $i = 1, 2$, then the block operators $x_i + x_i^{-1}$ in $L(G_i)$ are identically distributed. $\square$

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(A. One and A. Two) Author OneTwo address line 1, Author OneTwo address line 2
E-mail address, A. One: aone@aoneinst.edu