COHOMOLOGY OF LINE BUNDLES ON COMPACTIFIED JACOBIANS

D. ARINKIN

Abstract. Let $C$ be an integral projective curve with planar singularities. For the compactified Jacobian $\mathcal{J}$ of $C$, we prove that topologically trivial line bundles on $\mathcal{J}$ are in one-to-one correspondence with line bundles on $C$ (the autoduality conjecture), and compute the cohomology of $\mathcal{J}$ with coefficients in these line bundles. We also show that the natural Fourier-Mukai functor from the derived category of quasi-coherent sheaves on $\mathcal{J}$ (where $\mathcal{J}$ is the Jacobian of $X$) to that of quasi-coherent sheaves on $\mathcal{J}$ is fully faithful.

Introduction

Let $C$ be a smooth irreducible projective curve over a field $k$, and let $J$ be the Jacobian of $C$. As an abelian variety, $J$ is self-dual. More precisely, $J \times J$ carries a natural line bundle (the Poincaré bundle) $P$ that is universal as a family of topologically trivial line bundles on $J$.

The Poincaré bundle defines the Fourier-Mukai functor

$$\mathfrak{g} : D^b(J) \to D^b(J) : F \mapsto R\pi_2^*(p_1^*(F) \otimes P).$$

Here $D^b(J)$ is the derived category of quasi-coherent sheaves on $J$ and $p_1, 2 : J \times J \to J$ are the projections. Mukai ([22]) proved that $\mathfrak{g}$ is an equivalence of categories; the proof uses the formula

$$R\pi_1^*P \simeq O_{\zeta}[-g],$$

where $O_{\zeta}$ is the structure sheaf of the zero element $\zeta \in J$ and $g$ is the genus of $C$. Formula (1) goes back to Mumford (see the proof of the theorem in [23, Section III.13]).

Now suppose that $C$ is a singular curve, which we assume to be projective and integral. The Jacobian $J$ is no longer projective, but it admits a natural compactification $\mathcal{J} \supset J$. By definition, $\mathcal{J}$ is the moduli space of torsion-free sheaves $F$ on $C$ such that $F$ has generic rank one and $\chi(F) = \chi(O_C)$; $J$ is identified with the open subset of locally free sheaves $F$. It is natural to ask whether $\mathcal{J}$ is in some sense self-dual. For instance, one can look for a Poincaré sheaf (or complex of sheaves) $\mathcal{P}$ on $\mathcal{J} \times \mathcal{J}$. One can then ask whether $\mathcal{P}$ is, in some sense, a universal family of sheaves on $J$ and whether the corresponding Fourier-Mukai functor $\mathfrak{F} : D^b(\mathcal{J}) \to D^b(\mathcal{J})$ is an equivalence.

In the case when singularities of $C$ are nodes or cusps, such Poincaré sheaf $\mathcal{P}$ is constructed by E. Esteves and S. Kleiman in [12]; they also prove the universality of $P$. In addition, if $C$ is a singular plane cubic, $\mathfrak{F}$ is known to be an equivalence ([8, 9], also formulated as Theorem 5.2 in [6]).

If singularities of $C$ are more general, constructing the Poincaré sheaf $\mathcal{P}$ on $\mathcal{J} \times \mathcal{J}$ is much harder (see Remark (i) at the end of the introduction). However, it
is easy to construct a Poincaré bundle $P$ on $J \times \overline{J}$. It can then be used to define a Fourier-Mukai transform

$$F : D^b(J) \to D^b(J) : \mathcal{F} \mapsto R\pi_{2,*}(\pi_1^\ast \mathcal{F} \otimes P).$$

Here it is important to work with the derived categories of quasicoherent sheaves, since $\mathcal{F}$ does not preserve coherence.

In this paper, we assume that $C$ is an integral projective curve with planar singularities; the main result is that the formula (1) still holds in this case. This implies that (2) is fully faithful. As a simple corollary, we prove the following autoduality result: $P$ is the universal family of topologically trivial line bundles on $\overline{J}$, so that $J$ is identified with the connected component of the trivial bundle in the moduli space of line bundles on $\overline{J}$. This generalizes the Autoduality Theorem of [11] (see the remark after Theorem C).

Remarks. (i) Suppose that there exists an extension of $P$ to a sheaf $\overline{P}$ on $J \times \overline{J}$ such that the corresponding Fourier-Mukai transform $\overline{\mathcal{F}} : D^b(\overline{J}) \to D^b(\overline{J})$ is an equivalence. After the first version of this paper was completed, such an extension was constructed in [3]. Then (2) is a composition of $\overline{\mathcal{F}}$ and the direct image $j_\ast : D^b(J) \to D^b(\overline{J})$ for the open embedding $j : J \hookrightarrow \overline{J}$. Since $j_\ast$ is fully faithful, so is (2). Thus our result is natural assuming existence of $\overline{\mathcal{F}}$.

(ii) Compactified Jacobians appear as (singular) fibers of the Hitchin fibration for the group $GL(n)$; therefore, our results can be interpreted as a kind of autoduality of the Hitchin fibration. Conversely, some of our results can be derived from a theorem of E. Frenkel and C. Teleman [15] (see Theorem 15). We explore this relation in more details in Section 7.

(iii) Recall that the curve $C$ is assumed to be integral with planar singularities. We assume integrality of $C$ to avoid working with stability conditions for sheaves on $C$. It is likely that our argument works without this assumption if one fixes an ample line bundle on $C$ and defines the compactified Jacobian $\overline{J}$ to be the moduli space of semi-stable torsion-free sheaves of degree zero. Such generalization is natural in view of the previous remark, because some fibers of the Hitchin fibration are compactified Jacobians of non-integral curves.

On the other hand, the assumption that $C$ has planar singularities is more important. There are two reasons why the assumption is natural. First of all, $\overline{J}$ is irreducible if and only if the singularities of $C$ are planar (20); so if one drops this assumption, $J$ is no longer dense in $\overline{J}$. Secondly, only compactified Jacobians of curves with planar singularities appear in the Hitchin fibration.

Acknowledgments. I would like to thank R. Bezrukavnikov for stimulating my interest in this subject. This text was influenced by my numerous discussions with T. Pantev, and I am very grateful to him for sharing his ideas. I would also like to thank V. Drinfeld, T. Graber, C. Teleman, J. Starr, and J. Wahl for their remarks and suggestions.

1. Main results

Fix a ground field $k$. For convenience, let us assume that $k$ is algebraically closed. Let $C$ be an integral projective curve over $k$. Denote by $J$ its Jacobian, that is, $J$ is the moduli space of line bundles on $C$ of degree zero. Denote by $\overline{J}$ the compactified Jacobian; in other words, $\overline{J}$ is the moduli space of torsion-free sheaves
on $C$ of generic rank one and degree zero. (For a sheaf $F$ of generic rank one, the degree is $\deg(F) = \chi(F) - \chi(O_C)$.)

Let $P$ be the Poincaré bundle; it is a line bundle on $J \times \overline{J}$. Its fiber over $(L, F) \in (J \times \overline{J})$ equals

$$P_{(L,F)} = \det \Gamma(L \otimes F) \otimes \det \Gamma(O_C) \otimes \det \Gamma(L)^{-1} \otimes \det \Gamma(F)^{-1}.$$  

More explicitly, we can write $L \simeq O(\sum a_i x_i)$ for a divisor $\sum a_i x_i$ supported by the smooth locus of $C$, and then

$$P_{(L,F)} = \bigotimes (F_{x_i})^{a_i}.$$ 

From now on, we assume that $C$ has planar singularities; that is, the tangent space to $C$ at any point is at most two-dimensional. Our main result is the computation of the direct image of $P$:

**Theorem A.**

$$Rp_1_\ast P = \det(H^1(C, O_C)) \otimes O_\zeta[-g].$$

Here $O_\zeta$ is the structure sheaf of the neutral element $\zeta = [O_C] \in J$, and $p_1 : J \times \overline{J} \to J$ is the projection.

Let us now view $P$ as a family of line bundles on $\overline{J}$ parametrized by $J$. For fixed $L \in J$, denote the corresponding line bundle on $\overline{J}$ by $P_L$. In other words, $P_L$ is the restriction of $P$ to $\{L\} \times \overline{J}$. Applying base change, we can use Theorem $A$ to compute cohomology of $P_L$:

**Theorem B.**

(i) If $L \not\in O_C$, then $H^i(\overline{J}, P_L) = 0$ for any $i$;

(ii) If $L = O_C$, then $P_L = O_{\overline{J}}$ and $H^i(\overline{J}, O_{\overline{J}}) = \bigwedge^i H^1(C, O_C)$. (The identification is described more explicitly in Proposition 77.)

Let $\text{Pic}(\overline{J})$ be the moduli space of line bundles on $\overline{J}$. The correspondence $L \mapsto P_L$ can be viewed as a morphism $\rho : J \to \text{Pic}(\overline{J})$. Denote by $\text{Pic}^0(\overline{J}) \subset \text{Pic}(\overline{J})$ the connected component of the identity $[O_{\overline{J}}] \in \text{Pic}(\overline{J})$. In Section 6, we derive the following statement.

**Theorem C.** $\rho$ gives an isomorphism $J \isom \text{Pic}^0(\overline{J})$.

**Remark.** Theorem $C$ answers the question raised in [11]. Following [17], set

$$\text{Pic}^\tau(\overline{J}) = \{ L \in \text{Pic}(\overline{J}) : L^\otimes n \in \text{Pic}^0(\overline{J}) \text{ for some } n > 0 \},$$

$$\text{Pic}^\sigma(\overline{J}) = \{ L \in \text{Pic}(\overline{J}) : L^\otimes n \in \text{Pic}^0(\overline{J}) \text{ for some } n \text{ coprime to } \text{char} k \}$$

(if $\text{char} k = 0$, $\text{Pic}^\sigma(\overline{J}) = \text{Pic}^\tau(\overline{J})$ by definition). The main result of [11] is the Autoduality Theorem, which claims that if all singularities of $C$ are double points, then $\rho : J \isom \text{Pic}^0(\overline{J})$ and $\text{Pic}^0(\overline{J}) = \text{Pic}^\tau(\overline{J})$. Theorem $C$ generalizes the first statement to curves with planar singularities; as for the second statement, we show in Proposition 12 that $\text{Pic}^0(\overline{J}) = \text{Pic}^\sigma(\overline{J})$. We do not know whether $\text{Pic}^\tau(\overline{J})$ and $\text{Pic}^\sigma(\overline{J})$ coincide when $\text{char}(k) > 0$ and $C$ has planar singularities.

Theorem $A$ can be reformulated in terms of the Fourier functor

$$\mathfrak{F} : D^b(J) \to D^b(\overline{J}) : \mathcal{F} \mapsto Rp_2_\ast(p_1^\ast(\mathcal{F}) \otimes P)$$
given by $P$. Recall that $D^b(J)$ stands for the (bounded) derived category of quasi-coherent sheaves on $J$. The functor $\mathfrak{F}$ admits a left adjoint given by $F^\vee: D^b(J) \to D^b(J): F \mapsto \varinjlim (Rp_1,*(p_2^*F \otimes P^{-1}) \otimes \det(H^1(C, O_C)^{-1}[g]$. This formula relies on the computation of the dualizing sheaf on $J$: see Corollary 9.

Theorem D. (i) The composition $\mathfrak{F}^\vee \circ \mathfrak{F}$ is isomorphic to the identity functor. (ii) $F$ is fully faithful.

Proof. The first statement follows from Theorem A by base change. (This is completely analogous to the original argument of [22, Theorem 2.2].) This implies the second statement, because the functors $\mathfrak{F}^\vee$ and $\mathfrak{F}$ are adjoint.

Remark. For simplicity, we considered a single curve $C$ in this section. However, all our results hold for families of curves. Actually, we prove Theorem A for the universal family of curves (Theorem 10); base change then implies that the statement holds for any family, and, in particular, for any single curve.

2. Line bundles on a compactified Jacobian

Proposition 1. Suppose $H^i(J, P_L) \neq 0$ for some $i$. Then $(P_L)|_J \simeq O_J$.

Proof. Let $T \to J$ be the $Gm$-torsor corresponding to $(P_L)|_J$. One easily sees that $T$ is naturally an abelian group that is an extension of $J$ by $Gm$. The action of $J$ on $T$ lifts to an action of $T$ on $P_L$, therefore, $T$ also acts on $H^i(J, P_L)$. Note that $Gm \subset T$ acts via the tautological character.

Let $V \subset H^i(J, P_L)$ be an irreducible $T$-submodule. Since $T$ is commutative, $\dim(V) = 1$. The action of $T$ on $V$ is given by a character $\chi: T \to Gm$. Since $\chi|_{Gm} = id$, we see that $\chi$ gives a splitting $T \simeq Gm \times J$. This implies the statement.

Remark. If $C$ is smooth, Proposition 1 is equivalent to observation (vii) in [23, Section II.8]; however, our proof uses a slightly different idea, which is better adapted to the singular case.

Let $C^0 \subset C$ be the smooth locus of $C$.

Corollary 2. Suppose $H^i(J, P_L) \neq 0$ for some $i$. Then $L|_{C^0} \simeq O_{C^0}$.

Proof. Fix a degree minus one line bundle $\ell$ on $C$. It defines an Abel-Jacobi map $\alpha: C \to J: c \mapsto \ell(c)$. Here $\ell(c)$ can be defined as the sheaf of homomorphisms from the ideal sheaf of $c \in C$ to $\ell$. Notice that $\alpha^*(P_L) \simeq L$ and $\alpha(C^0) \subset J$. Now Proposition 1 completes the proof.

Set $N = \{L \in J: H^i(J, P_L) \neq 0 \text{ for some } i\} \subset J$. Clearly, $N \subset J$ is closed (by the Semicontinuity Theorem), and $N = \text{supp}(Rp_1,*)P$, where $p_1: J \times J \to J$ is the projection (by base change).

Corollary 3. Let $g$ be the (arithmetic) genus of $C$ and $\tilde{g}$ be its geometric genus, that is, the genus of its normalization. Then $\dim(N) \leq (g - \tilde{g})$. 
Proof. Let \( \nu : \tilde{C} \to C \) be the normalization, and let \( J \) be the Jacobian of \( \tilde{C} \). The map \( \nu^* : J \to \tilde{J} \) is smooth and surjective; its fibers have dimension \((g - \tilde{g})\).

Denote by \( \tilde{N} \subset \tilde{J} \) the set of line bundles on \( \tilde{C} \) that are trivial on \( \nu^{-1}(C^0) \subset \tilde{C} \). By Corollary 2 \( \nu^*(N) \subset \tilde{N} \). Now it suffices to note that \( \tilde{N} \) is a countable set. \( \square \)

3. Moduli of curves

Let \( \mathcal{M} = \mathcal{M}_g \) be the moduli stack of integral projective curves \( C \) of genus \( g \) with planar singularities. The following properties of \( \mathcal{M} \) are well known:

**Proposition 4.** \( \mathcal{M} \) is a smooth algebraic stack of finite type; \( \dim(\mathcal{M}) = 3g - 3 \). \( \square \)

**Remark.** Denote by \( \mathcal{C} \) the universal curve over \( \mathcal{M} \); that is, \( \mathcal{C} \) is the moduli stack of pairs \( (C \in \mathcal{M}, c \in C) \). One easily checks that \( \mathcal{C} \) is a smooth stack of dimension \( 3g - 2 \). This is similar to the statement (ii') after Theorem 8.

Consider the normalization \( \tilde{C} \) of a curve \( C \in \mathcal{M} \), and let \( \tilde{g} \) be the genus of \( \tilde{C} \) (that is, the geometric genus of \( C \)). We need some results on the stratification of \( \mathcal{M} \) by geometric genus due to Tessier (26), Diaz and Harris (10), and Laumon (21). Since our settings are somewhat different, we provide the proofs.

Denote by \( \mathcal{M}(\tilde{g}) \subset \mathcal{M} \) the locus of curves \( C \in \mathcal{M} \) of geometric genus \( \tilde{g} \). Note that we view \( \mathcal{M}(\tilde{g}) \) simply as a subset of the set of points of \( \mathcal{M} \), rather than a substack.

**Proposition 5.** \( \mathcal{M}(\tilde{g}) \) is a stratification of \( \mathcal{M} \):

\[
(M(\tilde{g})) \subset \bigcup_{\gamma \leq \tilde{g}} M(\gamma).
\]

In particular, \( \mathcal{M}(\tilde{g}) \subset \mathcal{M} \) is locally closed.

**Proof.** Let \( S \) be the stack of birational morphisms \( (\nu : \tilde{C} \to C) \), where \( C \in \mathcal{M} \), and \( \tilde{C} \) is an integral projective curve of genus \( \tilde{g} \) (with arbitrary singularity). Consider the forgetful map

\[
\pi : S \to \mathcal{M} : (\nu : \tilde{C} \to C) \mapsto C.
\]

Clearly,

\[
\pi(S) \subset \bigcup_{\gamma \leq \tilde{g}} M(\gamma).
\]

Therefore, it suffices to show that \( \pi \) is projective.

Let \( S'' \) be the stack of collections \((C, F, s)\), where \( C \in \mathcal{M} \), \( F \) is a torsion-free sheaf on \( C \) of generic rank one and degree \( g - \tilde{g} \), \( s \in H^0(C, F) \). Also, let \( S' \) be the stack of collections \((C, F, s, \mu)\), where \((C, F, s) \in S''\) and \( \mu : F \otimes F \to F \) is such that \( \mu(s \otimes s) = s \). Consider

\[
S \to S' : (\nu : \tilde{C} \to C) \mapsto (C, \nu_*(O_{\tilde{C}}), 1, \mu),
\]

where \( \mu \) is the product on the sheaf of algebras \( \nu_*(O_{\tilde{C}}) \). This identifies \( S \) and \( S' \).

The forgetful map

\[
S' \to S'' : (C, F, s, \mu) \mapsto (C, F, s)
\]

is a closed embedding (essentially because \( \mu \) is uniquely determined by \( \mu(s \otimes s) = s \)). Finally, the map

\[
S'' \to \mathcal{M} : (C, F, s) \mapsto C
\]

is projective. \( \square \)

**Proposition 6.** \( \text{codim}(\mathcal{M}(\tilde{g})) \geq (g - \tilde{g}) \).
Proof. Let $S$ be as in the proof of Proposition 5. Denote by $S^0$ the substack of morphisms $(\nu : \tilde{C} \to C) \in S$ with smooth $\tilde{C}$; clearly, $M(\tilde{g}) = \pi(S^0)$. Therefore, we need to show that $\dim(S^0) \leq 2g + \tilde{g} - 3$.

Consider the morphism

$$\tilde{\pi} : S^0 \to M_{\tilde{g}} : (\nu : \tilde{C} \to C) \mapsto \tilde{C}.$$ 

It suffices to show $\dim(\tilde{\pi}^{-1}(\tilde{C})) \leq 2(g - \tilde{g})$ for any $\tilde{C} \in M_{\tilde{g}}$. Fix $(\nu : \tilde{C} \to C) \in S^0$. Let us prove that the dimension of the tangent space $T_{\nu} \tilde{\pi}^{-1}(\tilde{C})$ to $\tilde{\pi}^{-1}(\tilde{C})$ at this point is at most $2(g - \tilde{g})$.

$T_{\nu} \tilde{\pi}^{-1}(\tilde{C})$ is isomorphic to the space of first-order deformations of $\nu_* O_C$ viewed as a sheaf of subalgebras of $\nu_* O_{\tilde{C}}$. This yields an isomorphism

$$T_{\nu} \tilde{\pi}^{-1}(\tilde{C}) = \{ \text{differentiations } O_C \to \nu_* O_{\tilde{C}}/O_C \} = \hom_{O_C}(\Omega_C, \nu_* O_{\tilde{C}}/O_C).$$

Now it suffices to notice that the fibers of the cotangent sheaf $\Omega_C$ are at most two-dimensional, and that the length of the sky-scraper sheaf $\nu_* O_{\tilde{C}}/O_C$ equals $g - \tilde{g}$. \hfill $\square$

Remark. By looking at nodal curves, one sees that $\text{codim}(M(\tilde{g})) = g - \tilde{g}$.

4. Universal Jacobian

Let $\mathcal{J}$ (resp. $\mathcal{J} \subset \mathcal{J}$) be the relative compactified Jacobian (resp. relative Jacobian) of $C$ over $M$. Here is the precise definition:

**Definition 7.** For a scheme $S$, let $\mathfrak{J}_S$ be the following groupoid:

- Objects of $\mathfrak{J}_S$ are pairs $(C, F)$, where $C \to S$ is a flat family of integral projective curves with planar singularities (that is, $C \in M_S$), and $F$ is a $S$-flat coherent sheaf on $C$ whose restriction to the fibers of $C \to S$ is torsion free of generic rank one and degree zero;
- Morphisms $(C_1, F_1) \to (C_2, F_2)$ are collections

$$\phi : C_1 \to C_2, \ell, \Phi : F_1 \to \phi^*(F_2) \otimes_{O_S} \ell,$$

where $\phi$ is a morphism of $S$-schemes, and $\ell$ is an invertible sheaf on $S$.

As $S$ varies, groupoids $\mathfrak{J}_S$ form a pre-stack; let $\mathcal{J}$ be the stack associated to it. Also, consider pairs $(C, F)$ where $C \in M_S$ and $F$ is a line bundle on $C$ (of degree zero along the fibers of $S \to C$); such pairs form a sub-prestack of $\mathfrak{J}$; let $\mathcal{J} \subset \mathcal{J}$ be the associated stack.

Clearly, $\mathcal{J} \subset \mathcal{J}$ is an open substack. The main properties of these stacks are summarized in the following theorem ([1]):

**Theorem 8** (Altman, Iarrobino, Kleiman). 

(i) $\overline{\mathcal{J}} : \overline{\mathcal{J}} \to \mathcal{M}$ is a projective morphism with irreducible fibers of dimension $g$;

(ii) $\overline{\mathcal{J}}$ is locally a complete intersection;

(iii) The restriction $p : \mathcal{J} \to \mathcal{M}$ is smooth. \hfill $\square$

Remark. By [14, Corollary B.2], (ii) can be strengthened:

(iii') $\mathcal{J}$ is smooth.

Clearly, (iii') together with (ii) imply (iii).
Remark. The key step in the proof of [19] is Iarrobino’s calculation (see [19]):
\[
\dim(\operatorname{Hilb}_k(\mathbb{k}[x,y])) = k - 1,
\]
where \(\operatorname{Hilb}_k(\mathbb{k}[x,y])\) is the Hilbert scheme of codimension \(k\) ideals in \(\mathbb{k}[x,y]\). For other proofs of [3], see [7], [24, Theorem 1.13] and [5]. Also, J. Rego gives an alternative inductive proof of [1] in [25].

Denote by \(\bar{j}\) the rank \(g\) vector bundle on \(\mathcal{M}\) whose fiber over \(C \in \mathcal{M}\) is \(H^1(C, O_C)\). Alternatively, \(\bar{j}\) can be viewed as the bundle of (commutative) Lie algebras corresponding to the group scheme \(p : \mathcal{J} \to \mathcal{M}\). The relative dualizing sheaf for \(p\) then equals \(\Omega^g_{\mathcal{J}/\mathcal{M}} = p^*(\det(j)^{-1})\). It is easy to find the dualizing sheaf for \(\bar{p} : \overline{\mathcal{J}} \to \overline{\mathcal{M}}:\)

**Corollary 9.** The relative dualizing sheaf \(\omega_{\overline{\mathcal{J}}}\) of \(\overline{\mathcal{J}}\) equals \(\overline{\mathcal{J}}(det(j)^{-1})\).

**Proof.** By Theorem 5, \(\overline{\mathcal{J}}\) is Gorenstein, so \(\omega_{\overline{\mathcal{J}}} = \Omega^g_{\mathcal{J}/\mathcal{M}}\) it suffices to check that \(\operatorname{codim}(\mathcal{J} - \mathcal{J}) \geq 2\). But this is clear because a generic curve \(C \in \mathcal{M}\) is smooth (see Proposition 6). \(\square\)

5. **Proof of Theorem A**

Consider the Poincaré bundle on \(\mathcal{J} \times_\mathcal{M} \overline{\mathcal{J}}\). We still denote it by \(P\).

**Theorem 10.** Let \(p_1 : \mathcal{J} \times_\mathcal{M} \overline{\mathcal{J}} \to \mathcal{J}\) be the projection. Then
\[
R^1p_1_*P = (\Omega^g_{\mathcal{J}/\mathcal{M}})^{-1} \otimes \zeta_*O_C[-g] = \zeta_* \det(j)[-g],
\]
where \(\zeta : \mathcal{M} \to \mathcal{J}\) is the zero section.

**Proof.** Consider the dual \(P^{-1} = \mathcal{H}\text{om}(P, O)\) of \(P\). (Actually \(P^{-1} = (\nu \times \text{id})^*P\), where \(\nu: \mathcal{J} \to \mathcal{J}\) is the involution \(L \mapsto L^{-1}\).) By Corollary 9, the dualizing sheaf of \(p_1\) is isomorphic to \(p_1^!\Omega^g_{\mathcal{J}/\mathcal{M}}\). Therefore,
\[
R^1\mathcal{H}\text{om}(R^1p_1_*P, O_\mathcal{J}) = (R^1p_1_*P^{-1}) \otimes \Omega^g_{\mathcal{J}/\mathcal{M}}[g]
\]
by Serre’s duality.

Combining Corollary 3 and Proposition 6 we see that
\[
\operatorname{codim}(\operatorname{supp}(R^1p_1_*P)) \geq g.
\]
By (10), we see that both \(R^1p_1_*P\) and \(R^1\mathcal{H}\text{om}(R^1p_1_*P, O_\mathcal{J})[-g]\) are concentrated in cohomological degrees from zero to \(g\). It is now easy to see that \(R^1p_1_*P\) is concentrated in cohomological degree \(g\), and that \(R^g p_1_*P\) is a coherent Cohen-Macaulay sheaf of codimension \(g\).

Next, notice that the restriction of \(P\) to \(\zeta(\mathcal{M}) \times_\mathcal{M} \overline{\mathcal{J}}\) is trivial. This provides a map
\[
\zeta^*(R^g p_1_*P) \to R^g \overline{\mathcal{J}}_*O_\mathcal{J}.
\]
By Serre’s duality, \(R^g \mathcal{J}_*O_\mathcal{J} = \det(j)\). Now by adjunction, we obtain a morphism
\[
R^g p_1_*P \to \zeta_* \det(j).
\]
It remains to verify that (17) is an isomorphism. Since (17) is an isomorphism over \(\zeta(\mathcal{M})\) by construction, we need to verify that \(\operatorname{supp}(R^g p_1_*P) = \zeta(\mathcal{M}) \cap \mathcal{J}\).

Let us check that \(\operatorname{supp}(R^g p_1_*P)\) equals \(\zeta(\mathcal{M})\) as a set. As a set, \(\operatorname{supp}(R^g p_1_*P)\) consists of pairs \((C, L) \in \mathcal{J}\) such that the line bundle \(L\) on \(C\) satisfies \(H^g(\overline{\mathcal{J}}, P_L) \neq 0\). In this case, \(H^0(\overline{\mathcal{J}}, P_L^{-1}) \neq 0\) by Serre’s duality. Since \(\overline{\mathcal{J}}\) is irreducible, we see that the line bundle \(P_L^{-1}\) has a subsheaf isomorphic to \(O_\mathcal{J}\). On the other hand, the line
bundles $P^{-1}_L = P_{L^{-1}}$ and $O_\mathcal{F} = P_0$ are algebraically equivalent, and therefore their
Hilbert polynomials coincide. Hence $P_L \simeq O_\mathcal{F}$. Finally, we can restrict $P_L$ to the
image of the Abel-Jacobi map (see the proof of Corollary 2) to obtain $L \simeq O_C$.

To complete the proof, let us verify that $\text{supp}(R^q_{1,*}P) = \zeta(M)$ as a scheme. Since
$R^q_{1,*}P$ is Cohen-Macaulay of codimension $g$, it suffices to check the claim generically
on $\zeta(M)$. We can thus restrict ourselves to the open substack of smooth curves in $\mathcal{M}$, and the claim reduces to \textbf{[1]}.

$\square$

Remark. The proof is similar to an argument of S. Lysenko (see proof of Theorem 4 in [4]), see also D. Mumford’s proof of the theorem in [23, Section III.13].

Using base change, one easily derives Theorem 11 from Theorem 10.

6. Autoduality

Recall that the morphism $\rho : J \to \text{Pic}_J$ is given by $L \mapsto P_L$. Since the tangent
space to $J$ at $[O_C]$ (resp. to $\text{Pic}(J)$ at $[O_\mathcal{F}]$) equals $H^1(C, O_C)$ (resp. $H^1(J, O_\mathcal{F})$),
the differential of $\rho$ at $[O_C] \in J$ becomes a linear operator
$$d\rho : H^1(C, O_C) \to H^1(J, O_\mathcal{F}).$$

Let us give a more precise form of Theorem B(ii):

Proposition 11. $d\rho$ is an isomorphism, and the (super-commutative) cohomology
algebra $H^\bullet(J, O_\mathcal{F})$ is freely generated by $H^1(J, O_\mathcal{F})$.

Proof. Let $O_\zeta$ be the structure sheaf of the zero $[O_C] \in J$ viewed as a coherent
sheaf on $J$ (it is a sky-scraper sheaf of length one). Note that $\mathfrak{F}(O_\zeta) = O_\mathcal{F}$, where
$\mathfrak{F} : D^b(J) \to D^b(\mathcal{F})$ is the Fourier transform of Theorem 12. Since $\mathfrak{F}$ is fully faithful,
it induces an isomorphism
$$\text{Ext}^\bullet(O_\zeta, O_\zeta) \simeq \text{Ext}^\bullet(O_\mathcal{F}, O_\mathcal{F}) = H^\bullet(\mathcal{F}, O_\mathcal{F}).$$
Finally, $J$ is smooth; therefore, $\text{Ext}^\bullet(O_\zeta, O_\zeta) = \wedge^\bullet H^1(C, O_C)$. $\square$

Let us fix a line bundle $\ell$ on $C$ of degree minus one. It defines an Abel-Jacobi
map $\alpha : C \to \mathcal{F}$, as in the proof of Corollary 2. We then obtain a morphism
$$\alpha^* : \text{Pic}(\mathcal{F}) \to \text{Pic}(C) : L \mapsto \alpha^* L.$$ By construction, $\alpha^*$ is a left inverse of $\rho$ (cf. [11, Proposition 2.2]).

Remark. Injectivity of $d\rho$ follows from the existence of the left inverse. Once injectivity is known, bijectivity follows from the equality
$$\dim H^1(J, O_\mathcal{F}) = \dim H^1(C, O_C) = g.$$

Proof of Theorem 13. $\text{Pic}(\mathcal{F})$ is a group scheme of locally finite type (see [16, Theorem 3.1], [13, Theorem 9.4.8], or [23, Corollary (6.4)]). Set
$$\text{Pic}'(\mathcal{F}) = (\alpha^*)^{-1}(J) = \{ L \in \text{Pic}(\mathcal{F}) : \deg(\alpha^* L) = 0 \}$$
$$K = \ker(\alpha^*) = \{ L \in \text{Pic}(\mathcal{F}) : \alpha^* L \simeq O_C \}.$$ Clearly, $K \subset \text{Pic}'(\mathcal{F})$ is closed, and $\text{Pic}'(\mathcal{F}) \subset \text{Pic}(\mathcal{F})$ is both open and closed. The map
$$J \times K \to \text{Pic}'(\mathcal{F}) : (L_1, L_2) \mapsto \rho(L_1) \cdot L_2$$
is an isomorphism. Bijectivity of $d\rho$ (Proposition 11) implies that $K$ is a disjoint union of points. Therefore, the connected component of identity of $\text{Pic}(\mathcal{J})$ is contained in $\rho(J)$. Now it remains to notice that $J$ is connected. □

**Proposition 12.** $\text{Pic}^{\sigma}(\mathcal{J}) = \text{Pic}^{0}(\mathcal{J})$ (where $\text{Pic}^{\sigma}$ is defined in (4)).

**Proof.** Consider $p : \mathcal{J} \to M$. It is a projective flat morphism with integral fibers (Theorem 8); we can therefore construct the corresponding family of Picard schemes $\text{Pic}(\mathcal{J}/M) \to M$ (see the references in the proof of Theorem C). The family is separated and its fiber over $C \in M$ is $\text{Pic}(\mathcal{J}_C)$.

Let us work in the smooth topology of $M$. Locally, we can choose a degree minus one line bundle $\ell$ on the universal curve $C \to M$. As in the proof of Theorem C, we then introduce a map $\alpha^* : \text{Pic}(\mathcal{J}/M) \to \text{Pic}(C/M)$ and substacks $\text{Pic}^{\sigma}(\mathcal{J}/M) = (\alpha^*)^{-1}(J)$ and $\mathcal{X} = \ker(\alpha^*)$ such that $\text{Pic}^{\sigma}(\mathcal{J}/M) = J \times_M \mathcal{X}$.

Let $\text{Pic}^{\sigma}(\mathcal{J}/M) \subset \text{Pic}(\mathcal{J}/M)$ be the substack whose fiber over $C \in M$ is $\text{Pic}^{\sigma}(\mathcal{J}_C)$. We have $\text{Pic}^{\sigma}(\mathcal{J}/M) = J \times_M \mathcal{X}^{\sigma}$, where $\mathcal{X}^{\sigma} = \{L \in \mathcal{X} : L^{\otimes n} \simeq O \text{ for some } n \text{ coprime to char } k\}$.

By [17, Theorem 2.5], the map $\text{Pic}(\mathcal{J}/M) \to \text{Pic}(\mathcal{J}/M) : L \mapsto L^{\otimes n}$ is étale for all $n$ coprime to char $k$. Therefore, $\mathcal{X}^{\sigma}$ is étale over $M$.

Finally, the morphism $\mathcal{X}^{\sigma} \to M$ is separated, and over the locus of smooth curves $C \in M$, we have $\text{Pic}^{0}(\mathcal{J}_C) = \text{Pic}^{\sigma}(\mathcal{J}_C)$ by [23 Corollary IV.19.2]. Therefore, $\mathcal{X}^{\sigma}$ is the zero group scheme, and $\text{Pic}^{\sigma}(\mathcal{J}/M) = J$, as required. □

### 7. Fibers of the Hitchin fibration

Recall the construction of the Hitchin fibration [18] (for $GL(n)$). Fix a smooth curve $X$ and an integer $n$.

**Definition 13.** A Higgs bundle is a rank $n$ vector bundle $E$ on $X$ together with a Higgs field $A : E \to E \otimes \Omega_X$.

Given a Higgs bundle $(E, A)$, consider the characteristic polynomial of $A$:

$$\det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n; \quad a_i \in H^0(X, \Omega_X^{\otimes i}).$$

The zero locus of (8) is a curve $C \subset T^*X$: the spectral curve of $A$. Higgs bundle $(E, A)$ gives rise to a coherent sheaf $F$ on $C$; informally, $F$ is the ‘sheaf of co-eigenspaces’: its fiber over a point $(x, \mu) \in T^*X$ is the co-eigenspace $\text{coker}(A(x) - \mu : E_x \to E_x \otimes \Omega_{X,x})$.

Here $x \in X$, $\mu \in \Omega_{X,x}$.

**Proposition 14.**

(i) $F$ is a torsion-free sheaf on $C$ whose fiber at a generic point of $C$ has length equal to the multiplicity of the corresponding component of $C$. In particular, if $C$ is reduced, $F$ is a torsion-free sheaf of generic rank one.
Fix a spectral curve $C$ (that is, fix a polynomial $\mathfrak{g}$). Then $(E, A) \mapsto F$ is a one-to-one correspondence between Higgs bundles with spectral curve $C$ and sheaves $F$ as in $(i)$. □

Given $F$, $E$ is reconstructed as the push-forward of $F$ with respect to $C \to X$. Therefore, $F$ and $E$ have equal Euler characteristics. We have $\chi(O_C) = n^2 \chi(O_X) = n^2 (1 - g)$, where $g$ is the genus of $X$. Hence $\deg(F) = 0$ if and only if $\deg(E) = n(n-1)(1-g)$, where $g$ is the genus of $X$. (Recall that $\deg(F) = \chi(F) - \chi(O_C)$.) Also, note that $(E, A)$ is (semi)stable if and only if $F$ is (semi)stable. If $C$ is integral, $F$ has generic rank one and stability is automatic.

Let $\mathfrak{Higgs}$ be the moduli space of semi-stable Higgs bundles $(E, A)$ with $\text{rk}(E) = n$ and $\text{deg}(E) = n(n-1)(1-g)$. Also, let $\mathfrak{Curves}$ be the space of spectral curves $C \subset T^*X$; explicitly, $\mathfrak{Curves} = \prod_{i=1}^n H^0(X, \Omega^2_X)$.

Finally, let $\mathfrak{Curves}' \subset \mathfrak{Curves}$ be the locus of integral spectral curves $C \subset T^*X$.

The correspondence $(E, A) \mapsto C$ gives a map $h : \mathfrak{Higgs} \to \mathfrak{Curves}$ (the Hitchin fibration). For $C \in \mathfrak{Curves}$, the fiber $h^{-1}(C)$ is the space of Higgs bundles with spectral curve $C$; Proposition 14 identifies $h^{-1}(C)$ with the moduli space of semi-stable coherent sheaves $F$ on $C$ that satisfy Proposition 14(i) and have degree zero. In other words, the fiber is the compactified Jacobian of $C$.

The results of this paper can be applied to integral spectral curves $C \in \mathfrak{Curves}'$. For instance, Theorem B(ii) implies that

$$H^i(h^{-1}(C), O) = \bigwedge^i H^1(C, O_C).$$

Actually, applying the relative version of Theorem B(ii) to the universal family of spectral curves, we obtain an isomorphism

$$R^i h_* O_{\mathfrak{Higgs}} |_{\mathfrak{Curves}'} = \Omega^i_{\mathfrak{Curves}'},$$

where we used the symplectic form on $T^*X$ to identify $H^1(C, O_C)$ with the cotangent space to $C \in \mathfrak{Curves}'$. Recently, E. Frenkel and C. Teleman proved that the isomorphism (9) can be extended to the space of all spectral curves:

**Theorem 15.** There is an isomorphism

$$R^i h_* O_{\mathfrak{Higgs}} = \Omega^i_{\mathfrak{Curves}'}.$$

□

When $i = 0, 1$, Theorem 15 is proved by N. Hitchin ([18, Theorems 6.2 and 6.5]); the general case is announced in [15].

**Remarks.** (i) In [18], N. Hitchin works with the Hitchin fibration for the group $SL(2)$, but his argument can be used to compute $R^i h_* O_{\mathfrak{Higgs}}$ for arbitrary $n$ (still assuming $i = 0, 1$). Actually, essentially the same argument computes $R^i \mathfrak{p}_* O_{\mathfrak{M}}$ for $i = 0, 1$. (Recall that $\mathfrak{p} : \mathfrak{M} \to \mathfrak{M}$ is the universal compactified Jacobian over the moduli stack of curves $\mathfrak{M}$.)

(ii) In [15], Theorem 15 is stated for the Hitchin fibration for arbitrary group, not just $GL(n)$. 
One can derive some of our results from Theorem 15, at least for integral curves $C$ that appear as spectral curves of the Hitchin fibration. Indeed, for such $C \in SCurves'$, Theorem 15 implies Theorem B(ii). In turn, this implies Theorem B. Also, one can easily derive from Theorem B(ii) that the isomorphism of Theorem A exists on some neighborhood $U$ of $\zeta \in J$, so Theorem B(i) holds for $L \in U$. Similarly, we see that $P$ defines a fully faithful Fourier-Mukai transform from $D^b(U)$ to $D^b(J)$.

References

[1] A. B. Altman, A. Iarrobino, and S. L. Kleiman. Irreducibility of the compactified Jacobian. In Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pages 1–12. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[2] A. B. Altman and S. L. Kleiman. Compactifying the Picard scheme. Adv. in Math., 35(1):50–112, 1980.

[3] D. Arinkin. Autoduality of compactified Jacobians for curves with plane singularities. arXiv:1001.3868.

[4] D. Arinkin. Orthogonality of natural sheaves on moduli stacks of SL(2)-bundles with connections on $P_1$ minus 4 points. Selecta Math. (N.S.), 7(2):213–239, 2001.

[5] V. Baranovsky. The variety of pairs of commuting nilpotent matrices is irreducible. Transform. Groups, 6(1):3–8, 2001.

[6] D. Ben-Zvi and T. Nevins. From solitons to many-body systems. Pure Appl. Math. Q., 4(2, part 1):319–361, 2008.

[7] J. Briançon. Description de $Hilb^nC\{x, y\}$. Invent. Math., 41(1):45–89, 1977.

[8] I. Burban and B. Kreussler. Fourier-Mukai transforms and semi-stable sheaves on nodal Weierstraß cubics. J. Reine Angew. Math., 584:45–82, 2005.

[9] I. Burban and B. Kreussler. On a relative Fourier-Mukai transform on genus one fibrations. Manuscripta Math., 120(3):283–306, 2006.

[10] S. Diaz and J. Harris. Ideals associated to deformations of singular plane curves. Trans. Amer. Math. Soc., 309(2):433–468, 1988.

[11] E. Esteves, M. Gagné, and S. Kleiman. Autoduality of the compactified Jacobian. J. London Math. Soc. (2), 65(3):591–610, 2002.

[12] E. Esteves and S. Kleiman. The compactified Picard scheme of the compactified Jacobian. Adv. Math., 198(2):484–503, 2005.

[13] B. Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure, and A. Vistoli. Fundamental algebraic geometry, volume 123 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005. Grothendieck’s FPGA explained.

[14] B. Fantechi, L. Göttsche, and D. van Straten. Euler number of the compactified Jacobian and multiplicity of rational curves. J. Algebraic Geom., 8(1):115–133, 1999.

[15] E. Frenkel and C. Teleman. Some remarks on the geometric Langlands correspondence, 2007. Talk by C. Teleman at the Workshop on Homological Mirror Symmetry and Applications II at IAS.

[16] A. Grothendieck. Technique de descente et théorèmes d’existence en géométrie algébrique. V. Les schémas de Picard: théorèmes d’existence. In Séminaire de Géométrie Algébrique de Bourbaki, volume 7, exp. no. 232, pages 143–161. Soc. Math. France, Paris, 1995.

[17] A. Grothendieck. Technique de descente et théorèmes d’existence en géométrie algébrique. VI. Les schémas de Picard: propriétés générales. In Séminaire Bourbaki, volume 7, exp. no. 236, pages 221–243. Soc. Math. France, Paris, 1995.

[18] N. Hitchin. Stable bundles and integrable systems. Duke Math. J., 54(1):91–114, 1987.

[19] A. A. Iarrobino. Punctual Hilbert schemes. Mem. Amer. Math. Soc., 10(188):vi+112, 1977.

[20] H. Kleppe and S. L. Kleiman. Reducibility of the compactified Jacobian. Compositio Math., 43(2):277–280, 1981.

[21] G. Laumon. Fibres de Springer et jacobienne compactifiées. In Algebraic geometry and number theory, volume 253 of Progr. Math., pages 515–563. Birkhäuser Boston, Boston, MA, 2006.

[22] S. Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. Nagoya Math. J., 81:153–175, 1981.
[23] D. Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[24] H. Nakajima. *Lectures on Hilbert schemes of points on surfaces*, volume 18 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.

[25] C. J. Rego. The compactified Jacobian. *Ann. Sci. École Norm. Sup. (4)*, 13(2):211–223, 1980.

[26] B. Teissier. Résolution simultanée: I - Familles de courbes. In M. Demazure, H. C. Pinkham, and B. Teissier, editors, *Séminaire sur les Singularités des Surfaces*, volume 777 of *Lecture Notes in Mathematics*, pages viii+339, Berlin, 1980. Springer.