THE HODGE NUMBER $h^{1,1}$ OF IRREGULAR ALGEBRAIC SURFACES

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Abstract. We prove a new inequality for the Hodge number $h^{1,1}$ of irregular complex smooth projective surfaces of general type without irregular pencils of genus $\geq 2$. More specifically we show that if the irregularity $q$ satisfies $q = 2^k + 1$ then $h^{1,1} \geq 4q - 3$. This generalizes results previously known for $q = 3$ and $q = 5$.

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1. Introduction

Let $S$ be a complex smooth projective surface of general type. We let $h^{p,s} = \dim H^{p,s}(S)$ be its $(p,s)$ Hodge number. One has that $h^{1,0}$ is the irregularity $q$ of $S$ whilst $h^{2,0}$ is its geometric genus $p_g$. We say that $S$ has an irregular pencil of genus $b$ if there is a morphism with connected fibres $\pi : S \to B$ over a curve $B$ of genus $b > 0$.

In this paper we prove the following theorem:

Theorem 1.1. If $S$ is a complex smooth projective surface of general type without irregular pencils of genus $\geq 2$ and $q = 2^k + 1$ then $h^{1,1} \geq 4q - 3$.

We note that $h^{1,1} \geq 3q - 2$ for complex smooth projective surfaces of general type. In fact from the Bogomolov-Myiaoka-Yau inequality $c_2 \geq 3\chi$, and the fact that $c_2 = 2 - 4q + 2p_g + h^{1,1}$ one obtains

\begin{equation}
\tag{1}
h^{1,1} \geq p_g + q + 1.
\end{equation}

Then, by $p_g \geq 2q - 4$ and equality holds if and only if $S$ is birational to a product of two curves one of which has genus $2$. In this last case, one has $c_2 = 4\chi$, and $p_g = 2(q - 2)$ hence $h^{1,1} = 4q - 6$. Since in this case also $q \geq 4$, we conclude that always

\begin{equation}
\tag{2}
h^{1,1} \geq 3q - 2.
\end{equation}

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For surfaces $S$ without irregular pencils of genus $\geq 2$, Lazarsfeld and Popa in [11] reproved inequality (2) and also that, if $q$ is odd,

$$h^{1,1} \geq 3q - 1.$$  

If there is an irregular pencil with connected fibres $\pi : S \to B$ over a curve $B$ of genus $b$, the following inequality has been recently proven in [12]:

$$h^{1,1} \geq 2b(q - b) + 2 + \sum (l(F) - 1)$$

where $l(F)$ is the number of the irreducible components of a fiber $F$.

In some instances, for surfaces $S$ without irregular pencils of genus $\geq 2$, inequalities (2) and (3) can be improved:

- in the case $q = 3$, $h^{1,1} \geq 9$ by [3]. No examples of surfaces with $h^{1,1} = 9$ are known; the symmetric product of a curve of genus 3, however, has $h^{1,1} = 10$;
- when $q = 4$, in [5] it is proven that $h^{1,1} \geq 11$. The Schoen surface (see [16] and also [7]) realizes the smallest known value of $h^{1,1} = 12$; the next known example, $h^{1,1} = 17$, is given by the symmetric product of a curve of genus 4;
- for $q = 5$ we have $h^{1,1} \geq 17$ again by [5]. No examples are known for $h^{1,1} < 25$; the Fano surface of the lines of a smooth cubic 3-fold has $h^{1,1} = 25$; the symmetric product of a curve of genus 5 gives $h^{1,1} = 26$.

Theorem 1.1 extends the above cases $q = 3$ and $q = 5$. To prove Theorem 1.1, we use the following linear algebra result to estimate the dimension of the $(1,1)$ part of image of the cup product map $\bigwedge^2 H^1(S, \mathbb{C}) \to H^2(S, \mathbb{C})$:

**Proposition 1.2.** Let $\mathcal{H}$ be the real vector space of $q \times q$ complex hermitian matrices. If $q = 2^k + 1, k \geq 2$ and $L \subset \mathcal{H}$ is a (real) vector subspace such that every $X \in L \setminus \{0\}$ has at least 2 positive and 2 negative eigenvalues then $\dim L \leq q^2 - (4q - 3)$.

Theorem 1.1 does not hold for the Schoen surface which has $q \neq 2^k + 1$. We notice the analogy with the role played by $2$-adic valuations for the estimates of dimensions of spaces of constant rank matrices in [11] and [6].

The paper is organized as follows. In Section 2 we explain how hermitian matrices relate to lower bounds for $h^{1,1}$ and in Section 3 we prove Proposition 1.2 and Theorem 1.1. Finally in Section 4 we give some corollaries of Theorem 1.1.

**2. Albanese variety and Hermitian matrices**

Given a complex variety $V$ with a real structure, we denote by $V(\mathbb{R})$ (or by $V_\mathbb{R}$, should the notation become too cumbersome) the locus of its real points.

Given a complex smooth projective irregular surface of general type $S$, let $A$ be the Albanese variety of $S$ and $a : S \to A$ the Albanese mapping. Defining the pull-back map

$$a^* : H^{1,1}(A) \to H^{1,1}(S),$$
denote by $K$ the kernel of $a^*$. Then clearly $h^{1,1} \geq q^2 - \dim K$ and our strategy for proving Theorem [1.1] is to give an upper bound for the dimension of $K$.

Complex conjugation of forms gives a real structure to the vector spaces $H^{1,1}(A)$ and $H^{1,1}(S)$ and $a^*$ is a real map; denote by $a^*_R$ the induced map:

$$a^*_R : H^{1,1}(A)_R \to H^{1,1}(S)_R.$$  

Since $K(\mathbb{R}) = \ker(a^*_R)$ we have $\dim \mathbb{C} K = \dim_R K(R)$. As in [5], we identify $H^{1,1}(A)$ with the vector space $M := M(q, \mathbb{C})$ of $q \times q$ complex matrices and $H^{1,1}(A)_R$ with the real space $\mathcal{H} \subset M$ of the hermitian matrices. Remark that the real structure of $M$ is here given by the involution $B \to \overline{B}$, and $\mathcal{H} = M(\mathbb{R})$. In this way we identify $K(\mathbb{R})$ with a real subspace of $\mathcal{H}$.

**Definition 2.1.** For any $X \in \mathcal{H}$ the minimal inertia of $X$ is the integer $m_X = \min\{n_+, n_-\}$ where $(n_+, n_-)$ is the signature (or inertia) of $X$.

Remark that $m_X = 0$ if and only if $X$ is semidefinite, that for any non-zero real number $\lambda$ we have $m_X = m_X$ and also that the rank of $X$ is $\geq 2m_X$.

The following statement (which obviously generalizes to higher dimensions, see [5]) is an essential ingredient for studying the dimension of $K$:

**Proposition 2.2.** If $S$ has no irrational pencils of genus $\geq 2$ then $m_X > 1$ for any $X \in K(\mathbb{R}) \setminus \{0\}$.

**Proof.** Suppose $\omega = i \sum_{s,k} x_{k,s} dz_k \wedge d\bar{z}_k \in K(\mathbb{R}) \setminus \{0\}$, with $X = (x_{k,s})$ a hermitian matrix. Let as above $(n_+, n_-)$ be the signature of $X$ and assume by contradiction that $m_X = \min\{n_+, n_-\} \leq 1$; note that we may assume without restrictions (by taking $-\omega$, if necessary) that $n_- \leq 1$.

By diagonalizing $X$ we may write $\omega = i(\sum_j \beta_j \wedge \bar{\beta}_j - \lambda \alpha \wedge \bar{\alpha})$ where $\alpha$ is a non zero $(1,0)$-form, $\beta_j$ are independent $(1,0)$-forms on $A$, and $\lambda \in \mathbb{R}$, $\lambda \geq 0$. Note that $\lambda = 0$ gives the case $n_- = 0$.

The pullback via the Albanese map gives an identification of $H^{1,0}(A)$ with $H^{1,0}(S)$ and, by abuse of language, we set $a^*(\gamma) = \gamma$ for any $\gamma \in H^{1,0}(A)$. Since $a^*(\omega) = 0$, by integration on $S$, we obtain:

$$0 = \int_S a^*(\omega) \wedge \alpha \wedge \bar{\alpha} = i \sum_j \int_S \beta_j \wedge \bar{\beta}_j \wedge \alpha \wedge \bar{\alpha}.$$  

Since all the summands have the same sign it follows that

$$0 = i \int_S \beta_j \wedge \bar{\beta}_j \wedge \alpha \wedge \bar{\alpha} = i \int_S \alpha \wedge \beta_j \wedge \alpha \wedge \bar{\beta}_j, \quad \text{for any } j.$$  

This gives by positivity $\alpha \wedge \beta_j = 0$. Since, by hypothesis, $S$ has no irrational pencils of genus $\geq 2$, the Castelnuovo - de Franchis theorem ([8], see also [4]) gives $\alpha = \beta_j = 0$. Hence $\omega = 0$, a contradiction. \qed
3. Proof of Proposition 3.1 and Theorem 3.1

Let $M$ be the space of $q \times q$ complex matrices and $\mathcal{H} \subset M$ the real space of the hermitian matrices. Let $M_2 \subset M$ be the locus of the matrices of rank $\leq 2$ and $\mathcal{H}_2 = M_2(\mathbb{R}) \subset \mathcal{H}$ be the set of hermitian matrices of rank $\leq 2$.

Consider the projective space $\mathbb{P} := \mathbb{P}(M)$ which is a complex projective space of dimension $N = q^2 - 1$. Let $\mathbb{P}_R \subset \mathbb{P}$ be the real projective space corresponding to $\mathcal{H}$ and $D_2 \subset \mathbb{P}$ the determinantal variety corresponding to $M_2$:

$$D_2 = \{(x) \in \mathbb{P} : x \in M_2\}.$$  

Then we have

$$D_2(\mathbb{R}) = D_2 \cap \mathbb{P}_R = \{(x) \in \mathbb{P}_R : x \in \mathcal{H}_2\};$$

Moreover, $D_2(\mathbb{R})$ is the union of two components $D_1$ and $D_0$, where $D_1$ is the closure of the locus of matrices with signature $(1, 1)$, that is of $\{(x) \in D_2 : m_X = 1\}$, and $D_0 = \{(x) \in D_2 : m_X = 0\}$. Note that the intersection $D_1 \cap D_0$ corresponds to the hermitian matrices of rank $1$.

One has $\dim \mathbb{C} D_2 = \dim \mathbb{R} D_2(\mathbb{R}) = 4q - 5$ and moreover, by [9], and [10]:

**Proposition 3.1.** The degree of $D_2$ is odd if and only if $q = 2^k + 1$.

**Proof.** By [9], the degree of $D_2$ is

$$\deg D_2 = \prod_{j=0}^{q-3} \frac{(q + j)}{(q - 2)} \cdot \frac{(q - 2 + j)}{q - 2} = \prod_{j=0}^{q-3} \frac{(q + j - 1)(q + j)}{(j + 1)(j + 2)}$$

and by [10], sect. 6, this quantity is odd if and only if $q - 2$ and $q - 1$ have disjoint binary expansion, *i.e.* when $q = 2^k + 1$.  

Let $h \in H^1(\mathbb{P}_R, \mathbb{Z}_2)$ be the class of a hyperplane. We recall that the ring of $\mathbb{Z}_2$-cohomology of the real projective space $\mathbb{P}_R$ is generated by $h$ with the relation $h^{N+1} = 0$, that is $H^*(\mathbb{P}_R, \mathbb{Z}_2) \cong \mathbb{Z}_2[h]/h^{N+1}$. It follows that the dual $\mathbb{Z}_2$-cohomology class $[Y]$ of any real algebraic variety $Y \subset \mathbb{P}_R$ of dimension $n$ and odd degree is not trivial, hence equal to $h^{N-n}$.

In particular, for $q = 2^k + 1$, by Proposition 3.1 the class $x = [D_2(\mathbb{R})]$ is

$$x = h^{N-(4q-5)} \in H^{N-(4q-5)}(\mathbb{P}_R, \mathbb{Z}_2).$$

We assume from now on that $q = 2^k + 1$ and $q \geq 5$.

We now take into account the classes of the components of $D_2(\mathbb{R})$:

**Lemma 3.2.** Let $y = [D_0]$ and $z = [D_1]$ in $H^{N-(4q-5)}(\mathbb{P}_R, \mathbb{Z}_2)$. Then $y = 0$ and $z = x$.

**Proof.** Recall that $x = y + z$. We want to prove that $y = 0$. Consider the hyperplane $T$ in $\mathbb{P}_R$ whose points are the trace zero matrices, $T := \{(x) \in \mathbb{P}_R : \text{trace}(x) = 0\}$. Then, since $D_0$ corresponds to semi-definite matrices, $T \cap D_0 = \emptyset$. This implies that the cup-product $[T] \cdot [D_0] = h \cdot y = 0$, hence $y = 0$ and consequently $z = x$.  

Now consider the identity matrix $I$ and the point $I = \langle I \rangle \in \mathbb{P}_\mathbb{R} \subset \mathbb{P}$. We let $C_2$ be the cone over $D$ with vertex $\mathbb{P}$. Then $C_2$ is a variety of (complex) dimension $4q - 4$ invariant under conjugation in $\mathbb{P}$; its real locus $C_2(\mathbb{R})$ is the union $C_1 \cup C_0$ of two real cones with vertices at $I$ over, respectively, $D_1$ and $D_0$. It is easy to verify that any $\langle X \rangle \in C_1$ satisfies $m_X \leq 1$.

Furthermore, the degree of $C_2$ equals the degree of $D$ and therefore it is odd, for $q = 2^k + 1$.

**Lemma 3.3.** The cohomology class $[C_1] \in H^{N-4q-4}(\mathbb{P}_\mathbb{R}, \mathbb{Z}_2)$ is non trivial, that is $[C_1] \neq 0$. In particular $C_1$ intersects any real projective subspace of codimension $4q - 4$.

**Proof.** Write $[C_2(\mathbb{R})] = [C_0] + [C_1]$. Since the degree of $C_2$ is odd one has $[C_2(\mathbb{R})] \neq 0$. We note that the cohomology class associated to a cone over a cycle is zero if and only if the cycle is zero and so $[C_0] = 0$. 

**Proof of Proposition 4.2.** Let $L \subset \mathcal{H}$ be a vector subspace such that every $X \in L \setminus \{0\}$ has minimal inertia $m_X > 1$. Denote by $L \subset \mathbb{P}_\mathbb{R}$ its associate projective space and assume by contradiction that $c = \text{codim} L < 4q - 3$. It follows that $h^c \cdot [C_2] \neq 0$ and therefore $L \cap C_1 \neq \emptyset$ by Lemma 3.3. This is a contradiction since for every $\langle X \rangle \in C_1$ we have $m_X \leq 1$. 

**Proof of Theorem 4.1.** The statement follows immediately from the discussion in Section 2, Propositions 2.2 and Proposition 1.2. 

### 4. Applications

Theorem 1.1 has some applications. The first one concerns the still mysterious surfaces satisfying $p_g = 2q - 3$ (see [13]).

**Proposition 4.1.** Let $S$ be a minimal surface of general type without irregular pencils of genus $\geq 2$ and satisfying $p_g = 2q - 3$. If $q = 2^k + 1, k \geq 3$, then the linear system $|K_S|$ has a fixed part.

**Proof.** One has $\chi(S) = q - 2$ and so by Theorem 1.1 $K_S^2 < 8\chi(S)$. The result then follows by [13] Theorem 1.2 (cf. also [2]). 

**Remark 4.2.** The only known examples of surfaces of general type without irregular pencils of genus $\geq 2$ and satisfying $p_g = 2q - 3$ are the symmetric product of a genus 3 curve and the Schoen surface ([16]). By [15] no such surfaces exist for $q = 5$ and by [13] for $q \geq 6$ such surfaces will always have birational canonical map.

We can obtain also a lower bound for $h^{1,1}$ even when $q \neq 2^k + 1$.

**Proposition 4.3.** Let $S$ be a surface of general type without irregular pencils of genus $\geq 2$. If $q = 2^k + 1 + \epsilon, 0 < \epsilon < 2^k$, then $h^{1,1} \geq 4q - 3 - 4\epsilon$.

**Proof.** Keep the notation of Section 2 and take a decomposition $\mathcal{H} = V \oplus W$, with $V = \{(x_{i,j}) \in \mathcal{H} : x_{i,j} = 0, \max(i,j) > 2^k + 1\}$ and $W$ any complementary subspace; since $V$ is naturally identified with the space of $(q - \epsilon) \times (q - \epsilon)$ hermitian matrices and $K(\mathbb{R}) \subset \mathcal{H}$, then $\dim K(\mathbb{R}) \leq \dim (K(\mathbb{R}) \cap V) + \dim W \leq [(q - \epsilon)^2 - (4(q - \epsilon) - 3)] + [q^2 - (q - \epsilon)^2] = q^2 - (4q - 3 - 4\epsilon)$. 

\[\square\]
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