The structure of continuous rigid functions of
two variables

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September 26, 2011

Abstract

A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called vertically rigid if \( \text{graph}(cf) \) is isometric to \( \text{graph}(f) \) for all \( c \neq 0 \). In [1] we settled Janković’s conjecture by showing that a continuous function \( f : \mathbb{R} \to \mathbb{R} \) is vertically rigid if and only if it is of the form \( a + bx \) or \( a + be^{kx} \) \((a,b,k \in \mathbb{R})\). Now we prove that a continuous function \( f : \mathbb{R}^2 \to \mathbb{R} \) is vertically rigid if and only if after

∗Partially supported by Hungarian Scientific Foundation grant no. 72655.
†Partially supported by Hungarian Scientific Foundation grants no. 49786, 61600, and 72655.
MSC codes: Primary 26A99 Secondary 39B22, 39B52, 39B72, 51M99
Key Words: rigid, functional equation, transformation, exponential
a suitable rotation around the $z$-axis $f(x, y)$ is of the form $a + bx + dy$, $a + s(y)e^{kx}$ or $a + be^{kx} + dy$ ($a, b, d, k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous).

The problem remains open in higher dimensions.

1 Introduction

An easy calculation shows that the exponential function $f(x) = e^x$ has the somewhat ‘paradoxical’ property that $cf$ is a translate of $f$ for every $c > 0$. It is also easy to see that every function of the form $a + be^{kx}$ shares this property. Moreover, for every function of the form $f(x) = a + bx$ the graph of $cf$ is isometric to the graph of $f$. In [2] Cain, Clark and Rose introduced the notion of vertical rigidity, which we now formulate for functions of several variables.

**Definition 1.1** A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **vertically rigid**, if $\text{graph}(cf)$ is isometric to $\text{graph}(f)$ for all $c \in (0, \infty)$. (Clearly, $c \in \mathbb{R} \setminus \{0\}$ would be the same.)

Then D. Janković formulated the following conjecture (see [2]).

**Conjecture 1.2** (D. Janković) A continuous function $f : \mathbb{R} \to \mathbb{R}$ is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ ($a, b, k \in \mathbb{R}$, $k \neq 0$).

This conjecture, and more, was proved in [1].

**Theorem 1.3** Janković’s conjecture holds. (It is actually enough to assume that $f$ is vertically rigid for an uncountable set $C$, see Definition 1.6 below.)

Later C. Richter gave generalisations of this theorem in various directions, see [3].

The main goal of the present paper is to give a complete description of the continuous vertically rigid functions of two variables.

**Theorem 1.4 (Main Theorem)** A continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid if and only if after a suitable rotation around the $z$-axis $f(x, y)$ is of the form $a + bx + dy$, $a + s(y)e^{kx}$ or $a + be^{kx} + dy$ ($a, b, d, k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous).

As these classes look somewhat ad hoc, we do not even have conjectures in higher dimensions.

**Problem 1.5** Characterise the continuous vertically rigid functions of $n$ variables for $n \geq 3$.

In fact, for the proof of the Main Theorem we need the following technical generalisations.
Definition 1.6 If $C$ is a subset of $(0, \infty)$ and $\mathcal{G}$ is a set of isometries of $\mathbb{R}^3$ then we say that $f$ is vertically rigid for a set $C \subset (0, \infty)$ via elements of $\mathcal{G}$ if for every $c \in C$ there exists a $\varphi \in \mathcal{G}$ such that $\varphi(\text{graph}(f)) = \text{graph}(cf)$.

(If we do not mention $C$ or $\mathcal{G}$ then $C$ is $(0, \infty)$ and $\mathcal{G}$ is the set of all isometries.)

Definition 1.7 Let us say that a set $C \subset (0, \infty)$ condensates to $\infty$ if for every $r \in \mathbb{R}$ the set $C \cap (r, \infty)$ is uncountable.

The Main Theorem will immediately follow from the following, in which we just replace $(0, \infty)$ by a set $C$ condensating to $\infty$.

Theorem 1.8 (Main Theorem, technical form) Let $C \subset (0, \infty)$ be a set condensating to $\infty$. Then a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid for $C$ if and only if after a suitable rotation around the $z$-axis $f(x, y)$ is of the form $a + bx + dy$, $a + s(y)e^{kx}$ or $a + be^{kx} + dy$ ($a, b, d, k \in \mathbb{R}, k \neq 0, s : \mathbb{R} \to \mathbb{R}$ continuous).

The structure of the proof will be the following. First we check in Section 2 that functions of the above forms are rigid. (Of course, they are all continuous.) Then we start proving the Main Theorem in more and more general settings. In Section 3 first we show that if all the isometries are horizontal translations then the vertically rigid function $f(x, y)$ is of the form $s(y)e^{kx}$ ($k \in \mathbb{R}, k \neq 0, s : \mathbb{R} \to \mathbb{R}$ continuous). The punchline here is that we can derive a simple functional equation from vertical rigidity (some sort of ‘multiplicativity’, see Lemma 3.5). Then we conclude this section by referring to a completely algebraic proof in [1] showing that if we allow arbitrary translations then $f(x, y)$ is of the form $a + s(y)e^{kx}$ ($a, k \in \mathbb{R}, k \neq 0, s : \mathbb{R} \to \mathbb{R}$ continuous).

Then we start working on the case of general isometries. The central idea is to consider the set $S_f$ of directions of segments connecting pairs of points on $\text{graph}(f)$ (see Definition 4.1). We collect the necessary properties of this set in Section 4. The set $S_f$ has some sort of rigidity in that the transformation $f \mapsto cf$ distorts the shape of it, but the resulting set has to be isometric to the original one (see Definition 5.1 and Remark 5.2). Using these we determine the possible $S_f$’s in Section 5 then in Section 6 we complete the proof by handling these cases using various methods.

Finally, in Section 7 we collect the open questions.

2 Functions of these forms are rigid

Rotation of the graph around the $z$-axis does not affect vertical rigidity, so we can assume that $f$ is of the given form without rotations.

Functions of the form $a + bx + dy$ are clearly vertically rigid.

Let now $f(x, y) = a + s(y)e^{kx}$ ($a, k \in \mathbb{R}, k \neq 0, s : \mathbb{R} \to \mathbb{R}$ continuous). Then $cf(x, y) = f(x + \frac{\log c}{k}, y) + a(c - 1)$, so $f$ is actually vertically rigid via translations in the $xz$-plane.

Before checking the third case we need a lemma.
Lemma 2.1 Let $f(x, y) = g(x) + dy$, where $d > 0$ and let $c > 0$. If we rotate graph$(f)$ around the $x$-axis by the angle $\alpha_c = \arctan(cd) - \arctan(d)$ then the intersection of this rotated graph with the $xy$-plane is the graph of a function of the form $y = -w_{c,d}g(x)$, where $w_{c,d} > 0$ and the map $c \mapsto w_{c,d}$ is strictly monotone on $(0, \infty)$ for every fixed $d > 0$.

Remark 2.2 By rather easy and short elementary geometric considerations one can check that for every fixed $d > 0$ the map $c \mapsto w_{c,d}$ is positive and real analytic. It is also very easy to see geometrically that the limit at $0$ is $\infty$, hence it is not constant, therefore countable-to-one. This would suffice for all our purposes, but these arguments are unfortunately very hard to write down rigorously, so we decided to present a less instructive and longer algebraic proof.

Proof. Using the matrix of the rotation we can write the rotated image of the point of the graph $(x, y_0, g(x) + dy_0)$ as

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \alpha_c & -\sin \alpha_c \\
0 & \sin \alpha_c & \cos \alpha_c
\end{pmatrix}
\begin{pmatrix}
x \\
y_0 \\
g(x) + dy_0
\end{pmatrix}
= \begin{pmatrix}
x \\
y_0 \cos \alpha_c - d \sin \alpha_c - g(x) \sin \alpha_c \\
y_0 \sin \alpha_c + d \cos \alpha_c + g(x) \cos \alpha_c
\end{pmatrix}.
\]

(2.1)

Let us now determine the intersection of the rotated graph with the $xy$-plane. This right hand side of (2.1) is in the $xy$-plane if and only if the third coordinate vanishes, that is, when

\[
y_0 \cos \alpha_c - d \sin \alpha_c - g(x) \sin \alpha_c = 0.
\]

This yields

\[
y_0 = -\frac{\cos \alpha_c}{\sin \alpha_c + d \cos \alpha_c} g(x).
\]

(2.2)

In order to complete the proof of the lemma we have to calculate the $y$-coordinate of the rotated image of the point $(x, y_0, g(x) + dy_0)$, which is the second entry of the right hand side of (2.1). Hence, using (2.2),

\[
y = y_0 \cos \alpha_c - d \sin \alpha_c - g(x) \sin \alpha_c = -\frac{\cos \alpha_c (\cos \alpha_c - d \sin \alpha_c)}{\sin \alpha_c + d \cos \alpha_c} g(x) -
\]

\[
g(x) \sin \alpha_c = -\frac{\cos^2 \alpha_c - d \cos \alpha_c \sin \alpha_c + \sin^2 \alpha_c + d \cos \alpha_c \sin \alpha_c}{\sin \alpha_c + d \cos \alpha_c} g(x) = -\frac{1}{\sin \alpha_c + d \cos \alpha_c} g(x).
\]

Therefore

\[
w_{c,d} = \frac{1}{\sin \alpha_c + d \cos \alpha_c} = \left(\sqrt{d^2 + 1} \left(\sin \alpha_c \frac{1}{\sqrt{d^2 + 1}} + \cos \alpha_c \frac{d}{\sqrt{d^2 + 1}}\right)\right)^{-1}.
\]

Using the identity

\[
\sin \alpha = \frac{\tan \alpha}{\sqrt{\tan^2 \alpha + 1}} \quad (\alpha \in (-\pi/2, \pi/2))
\]

(2.3)
we obtain \( \sin(\arctan(d)) = \frac{d}{\sqrt{d^2 + 1}} \), which easily implies \( \cos(\arctan(d)) = \frac{1}{\sqrt{d^2 + 1}} \).
(\text{Note that } \arctan(d) \in (-\pi/2, \pi/2).) \text{ So}
\[
w_{c,d} = \left( \sqrt{d^2 + 1} \left( \sin \alpha_c \cos(\arctan(d)) + \cos \alpha_c \sin(\arctan(d)) \right) \right)^{-1}.
\]
By the formula \( \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \) and the definition of \( \alpha_c \) this equals
\[
\left( \sqrt{d^2 + 1} \sin(\alpha_c + \arctan(d)) \right)^{-1} = \left( \sqrt{d^2 + 1} \sin(\arctan(cd)) \right)^{-1}.
\]
Applying (2.3) again yields
\[
w_{c,d} = \left( \sqrt{d^2 + 1} \frac{\tan(\arctan(cd))}{\sqrt{\tan^2(\arctan(cd)) + 1}} \right)^{-1} = \sqrt{\frac{1}{d^2 + 1} \left( 1 + \frac{1}{(cd)^2} \right)}.
\]
From this form it is easy to see that this function is positive and strictly monotone on \((0, \infty)\) for every fixed \(d > 0\).

Let now \( f(x,y) = a + be^{kx} + dy \ (a, b, d, k \in \mathbb{R}, k \neq 0) \). Rescaling the graph in a homothetic way does not affect vertical rigidity, so we can consider \( kf \left( \frac{x}{k}, \frac{y}{k} \right) \) and assume \( k = 1 \). We may also assume \( b, d \neq 0 \), otherwise our function is of one of the previous forms. Adding a constant, reflecting the graph about the \( xz \)-plane (needed only if the signs of \( b \) and \( d \) differ), multiplying by a nonzero constant, as well as a translation in the \( x \)-direction do not affect vertical rigidity, so by applying these in this order we can assume that \( a = 0, bd > 0, d = 1, \) and \( b = 1 \).

Hence it suffices to check that \( f(x,y) = e^x + y \) is vertically rigid. Let us fix a \( c > 0 \). In every vertical plane of the form \( \{ x = x_0 \} \) the restriction of \( f \) is a straight line of slope 1. Rotation around the \( x \)-axis by angle \( \alpha_c = \arctan(c) - \pi/4 \) takes all these lines to lines of slope \( c \). By applying Lemma 2.4 with \( g(x) = e^x \) and \( d = 1 \), the intersection of the rotated graph and the \( xy \)-plane is the graph of the function \( y = -w_{c,1}e^x \).

Now, applying a translation in the \( x \)-direction we can obtain a function with still all lines of slope \( c \) but now with intersection with the \( xy \)-plane of the form \( y = -e^x \) (note that \( w_{c,1} > 0 \)). But then we are done, since this function clearly agrees with \( cf \) (The intersection of graph(\( f \)) and the \( xy \)-plane is of the form \( y = -e^x \), and all lines in this graph are of slope 1, hence for graph(\( cf \)) the intersection is still \( y = -e^x \), and all lines are of slope \( c \)). This finishes the proof of vertical rigidity.

### 3 Vertical rigidity via translations

**Theorem 3.1** Let \( C \subset (0, \infty) \) be an uncountable set. Then a continuous function \( f : \mathbb{R}^2 \to \mathbb{R} \) is vertically rigid for \( C \) via horizontal translations if and only if after a suitable rotation around the \( z \)-axis \( f(x,y) \) is of the form \( s(y)e^{kx} \ (k \in \mathbb{R}, k \neq 0, s : \mathbb{R} \to \mathbb{R} \text{ continuous}) \).
We already checked the easy direction in the previous section. Before proving the other direction we need some preparation. We will need the following result, which is Theorem 2.5 in [1].

**Theorem 3.2** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous vertically rigid function for an uncountable set \( C \subset (0, \infty) \) via horizontal translations. Then \( f \) is of the form \( se^{kx} \) \((s \in \mathbb{R}, k \in \mathbb{R} \setminus \{0\})\).

The following lemma will be useful throughout the paper. Sometimes we will use it tacitly. The easy proof is left to the reader.

**Lemma 3.3** Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be vertically rigid for \( c_0 \) via \( \varphi_0 \) and for \( c \) via \( \varphi \). Then \( cf \) is vertically rigid for \( \frac{c}{c_0} \) via \( \varphi \circ \varphi_0^{-1} \).

From now on we will often use the notation \( \vec{x} \) for two-dimensional (and sometimes three-dimensional) vectors.

**Definition 3.4** For a function \( f : \mathbb{R}^2 \to \mathbb{R} \) and a set \( C \subset (0, \infty) \) let \( T_{f,C} \subset \mathbb{R}^2 \) be the additive group generated by the set \( T' = \{ \vec{t} \in \mathbb{R}^2 : \exists c \in C \ \forall \vec{x} \in \mathbb{R}^2 \ f(\vec{x} + \vec{t}) = cf(\vec{x}) \} \). (We will usually simply write \( T \) for \( T_{f,C} \).)

**Lemma 3.5** Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a vertically rigid function for a set \( C \subset (0, \infty) \) via horizontal translations such that \( f(\vec{0}) = 1 \). Then
\[
f(\vec{x} + \vec{t}) = f(\vec{x})f(\vec{t}) \ \forall \vec{x} \in \mathbb{R}^2 \ \forall \vec{t} \in T.
\]
Moreover, \( f(\vec{t}) > 0 \) for every \( \vec{t} \in T \), and \( T' \) is uncountable if so is \( C \).

**Proof.** By assumption, for every \( c \in C \) there exists \( \vec{t}_c \in \mathbb{R}^2 \) such that \( cf(\vec{x}) = f(\vec{x} + \vec{t}_c) \) for every \( \vec{x} \in \mathbb{R}^2 \). Then \( \vec{t}_c \in T' \) for every \( c \in C \).

Since \( T \) is the group generated by \( T' \), every \( \vec{t} \in T \) can be written as \( \vec{t} = \sum_{i=1}^m n_i \vec{t}_i \) \((\vec{t}_i \in T', n_i \in \mathbb{Z}, i = 1, \ldots, m)\) where \( f(\vec{x} + \vec{t}_i) = c_i f(\vec{x}) \) \((\vec{x} \in \mathbb{R}^2, i = 1, \ldots, m)\).

From these we easily get
\[
f(\vec{x} + \vec{t}) = c_T f(\vec{x}), \text{ where } c_T = \prod_{i=1}^m c_i^{n_i}, \ \vec{x} \in \mathbb{R}^2, \ \vec{t} \in T. \tag{3.1}
\]

Note that \( c_T > 0 \) (and also that it is not necessarily a member of \( C \)). It suffices to show that \( c_T = f(\vec{t}) \) for every \( \vec{t} \in T \), but this follows if we substitute \( \vec{x} = \vec{0} \) into \( 3.1 \).

Since \( f \) is not identically zero, \( \vec{t}_c \neq \vec{t}_{c'} \) whenever \( c, c' \in C \) are distinct. Hence \( \{ \vec{t}_c : c \in C \} \) is uncountable, so \( T' \) is uncountable if so is \( C \).

**Proof.** (Thm. 3.1) If \( f \) is identically zero then we are done, so let us assume that this is not the case. The class of continuous vertically rigid functions for some set condensating to \( \infty \) via horizontal translations, as well as the class of functions of the form \( s(y)e^{kx} \) \((k \in \mathbb{R}, k \neq 0, s : \mathbb{R} \to \mathbb{R} \text{ continuous})\) are
Corollary 3.7 Let $C \subset (0, \infty)$ be an uncountable set. Then a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid for $C$ via translations if and only if after a
suitable rotation around the $z$-axis $f(x, y)$ is of the form $a + s(y)e^{kz}$ ($a, k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous).

4 The set $S_f$

Now we start working on the case of arbitrary isometries.

Let $S^2 \subset \mathbb{R}^3$ denote the unit sphere. For a function $f : \mathbb{R}^2 \to \mathbb{R}$ let $S_f$ be the set of directions between pairs of points on the graph of $f$, that is,

$$S_f = \left\{ \frac{p - q}{|p - q|} \in S^2 : p, q \in \text{graph}(f), \ p \neq q \right\}.$$  

Recall that a great circle is a circle line in $\mathbb{R}^3$ of radius 1 centered at the origin. We call it vertical if it passes through the points $(0, 0, \pm 1)$.

Lemma 4.2 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then

1. $-S_f = S_f$ (symmetric about the origin),
2. $(0, 0, \pm 1) \notin S_f,$
3. $S_f$ is connected,
4. every great circle containing $(0, 0, \pm 1)$ intersects $S_f$ in two (symmetric) nonempty arcs,
5. $S^2 \setminus S_f$ has exactly two connected components, one containing $(0, 0, 1)$ and one containing $(0, 0, -1)$.

Proof. (1) Obvious.

(2) Obvious, since $f$ is a function.

(3) graph$(f)$ is homeomorphic to $\mathbb{R}^2$, hence the squared of it minus the (2-dimensional) diagonal is a connected set. Since $S_f$ is the continuous image of this connected set, it is itself connected.

(4) The intersection of $S_f$ with such a great circle corresponds to restricting our attention to distinct pairs of points $(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ so that the segment $[x_1, x_2]$ is parallel to a fixed line $L \subset \mathbb{R}^2$. Now, given two such nondegenerate segments it is easy to move one of them continuously to the other so that along the way it remains nondegenerate and parallel to $L$. This shows that in both halves of the great circle (separated by $(0, 0, \pm 1)$) $S_f$ is pathwise connected, hence it is an arc.

(5) By (4) every point of $S^2 \setminus S_f$ can be connected with an arc of a vertical great circle either to $(0, 0, 1)$ or to $(0, 0, -1)$ in $S^2 \setminus S_f$, hence there are at most two connected components.

Now we show that $(0, 0, 1)$ and $(0, 0, -1)$ are in different ones. It suffices to show that there exists a Jordan curve in $S_f$ so that $(0, 0, 1)$ and $(0, 0, -1)$ are
Lemma 4.4 Let \( h : \mathbb{S}^1 \rightarrow \mathbb{S}^2 \) be defined as above. Then

1. \( h(\vec{x}) \neq (0, 0, -1) \) for every \( \vec{x} \in \mathbb{S}^1 \)

2. \( h \) is lower semicontinuous (in the obvious sense, or equivalently, \( h_3 \) is lower semicontinuous)

3. \( h \) is convex with respect to great circles, that is, if \( h(\vec{x}) \) and \( h(\vec{y}) \) determine a unique nonvertical great circle (i.e. there is a subarc of \( \mathbb{S}^1 \) of length \( < \pi \) connecting \( \vec{x} \) and \( \vec{y} \), and \( h(\vec{x}), h(\vec{y}) \neq (0, 0, 1) \) then on this subarc \( \text{graph}(h) \) is bounded from above by the great circle.

Proof. (1) Obvious by Lemma 4.2 (2) and (4).

(2) We have to check that if \( h_3(\vec{x}) > u \) then the same holds in a neighbourhood of \( \vec{x} \). (Note that essentially \( h_3 \) is defined as a supremum.) Hence \( h_3(\vec{x}) > u \) if and only if there exists a segment \( [\vec{a}, \vec{b}] \subset \mathbb{R}^2 \) parallel to \( \vec{x} \) over which the slope of \( f \) is bigger than \( u \). But then by the continuity of \( f \) the same holds for segments close enough to \( [\vec{a}, \vec{b}] \), in particular to slightly rotated copies, and we are done.
It is easy to see that for every $\vec{v} \in S^1$ the slope of $f$ over a segment parallel to $\vec{v}$ is at most the slope of the vector $h(\vec{v})$. Let $\vec{z} \in S^1$ be an element of the shorter arc connecting $\vec{x}$ and $\vec{y}$ in $S^1$, let $[\vec{a}, \vec{b}] \subset \mathbb{R}^2$ be a segment parallel to $\vec{z}$, and let $P : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the linear map whose graph passes through the origin, $h(\vec{x})$ and $h(\vec{y})$. (Then $\text{graph}(P)$ contains the great circle determined by $h(\vec{x})$ and $h(\vec{y})$. Moreover, the slope of $P$ over any vector parallel to $\vec{x}$ is the slope of $h(\vec{x})$, and similarly for $\vec{y}$.) We have to show that the slope of $f$ between $\vec{a}$ and $\vec{b}$ is at most that of $P$, that is, $f(\vec{b}) - f(\vec{a}) \leq P(\vec{b}) - P(\vec{a})$. Write $\vec{b} - \vec{a} = \alpha \vec{x} + \beta \vec{y}$ for some $\alpha, \beta > 0$. Then by using the definition of $P$ and our first observation for the segments $[\vec{a}, \vec{a} + \alpha \vec{x}]$ and $[\vec{a} + \alpha \vec{x}, \vec{a} + \alpha \vec{x} + \beta \vec{y}]$, which are parallel to $\vec{x}$ and $\vec{y}$, respectively, we get

$$f(\vec{b}) - f(\vec{a}) = f(\vec{a} + \alpha \vec{x} + \beta \vec{y}) - f(\vec{a}) =$$

$$(f(\vec{a} + \alpha \vec{x} + \beta \vec{y}) - f(\vec{a} + \alpha \vec{x})) + (f(\vec{a} + \alpha \vec{x}) - f(\vec{a})) \leq (P(\vec{a} + \alpha \vec{x} + \beta \vec{y}) - P(\vec{a} + \alpha \vec{x})) + (P(\vec{a} + \alpha \vec{x}) - P(\vec{a})) = P(\vec{b}) - P(\vec{a}).$$

\[\square\]

5 Determining the possible $S_f$'s

Definition 5.1 For $c > 0$ let $\psi_c : S^2 \rightarrow S^2$ denote the map that 'deforms $S_f$ according to the map $c \mapsto cf$', that is,

$$\psi_c((x, y, z)) = \frac{(x, y, cz)}{|(x, y, cz)|} \quad ((x, y, z) \in S^2).$$

Remark 5.2 Let $\varphi_c$ be the isometry mapping $\text{graph}(f)$ onto $\text{graph}(cf)$. Every isometry $\varphi$ is of the form $\varphi^{\text{trans}} \circ \varphi^{\text{ort}}$, where $\varphi^{\text{ort}}$ is an orthogonal transformation and $\varphi^{\text{trans}}$ is a translation. Moreover, if $\varphi$ is orientation-preserving then $\varphi^{\text{ort}}$ is a rotation around a line passing through the origin. A key observation is the following: The vertical rigidity of $f$ for $C$ implies that $\psi_c(S_f) = \varphi^{\text{ort}}(S_f)$ for every $c \in C$.

Now we prove the main theorem of this section. For the definition of $h_3$ see the previous section.

Theorem 5.3 Let $C \subset (0, \infty)$ be a set condensating to $\infty$, and let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function vertically rigid for $C$. Then one of the following holds.

- **Case A.** There is a vertical great circle that intersects $S_f$ in only two points.
- **Case B.** $S_f = S^2 \setminus \{(0, 0, 1), (0, 0, -1)\}$.
- **Case C.** There exists an $x_0 \in S^1$ such that $h_3(x_0) = 0$ and $h_3(\vec{x}) = 1$ for every $\vec{x} \neq x_0$, that is, $S_f$ is $S^2$ minus two quarters of a great circle.
• **Case D.** There exists a closed interval $I$ in $\mathbb{S}^1$ with $0 < \text{length}(I) < \pi$ such that $h_3(x) = 0$ if $x \in I$, and $h_3(x) = 1$ if $x \notin I$, that is, $S_f$ is $\mathbb{S}^2$ minus two spherical triangles’.

**Proof.** In this proof the word ‘component’ will refer to the components of $\mathbb{S}^2 \setminus S_f$. We separate two cases according to whether $h_3 \geq 0$ everywhere or not.

First let us suppose that there exists a $\bar{x} \in \mathbb{S}^1$ such that $h_3(\bar{x}) < 0$. This implies that there is a vertical great circle containing two arcs, one in the top component connecting $(0,0,1)$ with $\mathbb{S}^1$ and even crossing it, and an other one (the symmetric pair in the bottom component) running from the ‘South Pole to the Equator’ and even above. But then considering geometrically the action of $\psi_c$ one can easily check that if we choose larger and larger $c$’s (tending to $\infty$) then we obtain that $\psi_c(S_f)$ contains in the two components two symmetrical arcs on the same great circle which are only leaving out two small gaps of length tending to 0. But then by Remark 5.2 $S_f$ also contains two such arcs in the two components on some (not necessarily vertical) great circle, hence the distance of the components is 0.

Let $p_n^c$ and $q_n^c$ be sequences in the top and bottom component, respectively, so that $\text{dist}(p_n^c, q_n^c) \to 0$. By compactness we may assume $p_n^c, q_n^c \to \bar{p} \in \mathbb{S}^2$. We claim that $p_n^c \to \bar{p}$ implies $\bar{p} \neq (0,0,-1)$. (And similarly $q_n^c \to \bar{p}$ implies $\bar{p} \neq (0,0,1)$.) Indeed, let $\bar{x}_n \in \mathbb{S}^1$ be so that $\bar{x}_n$ and $\bar{p}^c$ lay on the same vertical great circle, and similarly, let $\bar{x} \in \mathbb{S}^1$ and $\bar{p}$ lay on the same vertical great circle. Then $\bar{x}_n \to \bar{x}$, and using the fact $h(\bar{x}) \neq (0,0,-1)$ and the lower semicontinuity of $h$ at $\bar{x}$ (Lemma 4.4 (1) and (2)) we are done.

Using the lower semicontinuity of $h$ at $\bar{x}$ again (and $p_n^c \to \bar{p}$) we get that $h(\bar{x})$ cannot be above $\bar{p}$. Similarly, $-h(-\bar{x})$ cannot be below $\bar{p}$. But $h(\bar{x})$ is always above $-h(-\bar{x})$, so the only option is $h(\bar{x}) = -h(-\bar{x})$, hence there is a vertical great circle whose intersection with $S_f$ is just a (symmetric) pair of points, so Case A holds, and hence we are done with the first half of the proof.

Now let us assume that $h_3 \geq 0$ everywhere. First we prove that $h_3(\bar{x}) \in \{0,1\}$ for Lebesgue almost every $\bar{x} \in \mathbb{S}^1$. Indeed, fix an arbitrary $c \in C \setminus \{1\}$. By rigidity the (equal) measure of the two components remains the same after applying $\psi_c$. Since $h_3 \geq 0$, the intersection of the top component with the vertical great circle containing an $\bar{x}$ shrinks if $c > 1$ and grows if $c < 1$, unless $h_3(\bar{x}) = 0$ or 1. Hence we are done, since the measure of the top component can be calculated from the lengths of these arcs.

Now we show that $\{\bar{x} : h_3(\bar{x}) = 0\}$ is either empty, or a pair of points of the form $\{\bar{x}_0^c, -\bar{x}_0^c\}$, or a closed interval in $\mathbb{S}^1$ (possibly degenerate or the whole $\mathbb{S}^1$). So we have to show that if $\bar{x}, \bar{y} \in \mathbb{S}^1$ are so that the shorter arc connecting them is shorter than $\pi$, and $h_3(\bar{x}) = h_3(\bar{y}) = 0$ then $h_3(\bar{z}) = 0$ for every $\bar{z}$ in this arc. But $h_3(\bar{z}) \geq 0$ by assumption, and $h_3(\bar{z}) \leq 0$ by the convexity of $h$ applied to $h(\bar{z}) = \bar{x}$ and $h(\bar{y}) = \bar{y}$. The fact that the endpoints are also contained in $\{\bar{x} : h_3(\bar{x}) = 0\}$ easily follows from the semicontinuity.

If $\{\bar{x} : h_3(\bar{x}) = 0\}$ is a symmetrical pair of points or a closed interval of length at least $\pi$ then it is easy to see that Case A holds. Hence we may assume that it is empty, or a singleton, or a closed interval $I$ with $0 < \text{length}(I) < \pi$.  

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Case 1. \( \{ \bar{x} : h_3(\bar{x}) = 0 \} = \emptyset \).

In this case, \( h_3 > 0 \) everywhere, and hence \( h_3 = 1 \) almost everywhere. Therefore one can easily see (using the convexity) that \( h_3 = 1 \) everywhere but possibly at at most two points of the form \( \{ \bar{x}_0, -\bar{x}_0 \} \). We claim that actually \( h_3 = 1 \) everywhere. We know already that \( S_f \) is \( S^2 \) minus two symmetric arcs on the same vertical great circle. The arcs contain \( (0, 0, 1) \) and \( (0, 0, -1) \), respectively, and they do not reach the 'Equator', since \( h_3 > 0 \). Let us fix an arbitrary \( c \in C \setminus \{ 1 \} \). By rigidity the (equal) length of the arcs should not change when applying \( \psi_c \), but it clearly changes, a contradiction.

Hence \( S_f = S^2 \setminus \{ (0, 0, 1), (0, 0, -1) \} \), so Case B holds.

Case 2. \( \{ \bar{x} : h_3(\bar{x}) = 0 \} \) is a singleton.

Let \( \{ \bar{x}_0 \} = \{ \bar{x} : h_3(\bar{x}) = 0 \} \). Similarly as above, \( h_3 = 1 \) almost everywhere. Then convexity easily implies that \( h_3(\bar{x}) = 1 \) whenever \( \bar{x} \neq \{ \bar{x}_0, -\bar{x}_0 \} \). Again similarly, the length of the arcs is unchanged by \( \psi_c \) only if \( h_3(-\bar{x}_0) = 1 \), so \( S_f \) is \( S^2 \) minus two symmetric quarter arcs starting from the 'Poles' on a vertical great circle, so Case C holds.

Case 3. \( \{ \bar{x} : h_3(\bar{x}) = 0 \} \) is a closed interval in \( S^1 \) with \( 0 < \text{length}(I) < \pi \).

Let \( I = \{ \bar{x} : h_3(\bar{x}) = 0 \} \). As \( h_3 = 0 \) or 1 almost everywhere, convexity readily implies that \( h_3 = 1 \) on \( S^1 \setminus I \). Hence \( S_f \) is \( S^2 \) minus two spherical triangles, and Case D holds.

This concludes the proof. \( \square \)

6 The end of the proof

Now we complete the proof of the technical form of the Main Theorem. We repeat the statement here.

**Theorem 6.1** (Main Theorem, technical form) Let \( C \subset (0, \infty) \) be a set condensing to \( \infty \). Then a continuous function \( f : \mathbb{R}^2 \to \mathbb{R} \) is vertically rigid for \( C \) if and only if after a suitable rotation around the \( z \)-axis \( f(x, y) \) is of the form \( a + bx + dy, a + s(y)e^{kx} \) or \( a + be^{kx} + dy \) \((a, b, d, k \in \mathbb{R}, k \neq 0, s : \mathbb{R} \to \mathbb{R} \) continuous\).

**Proof.** By Theorem 5.3 it suffices to consider Cases A-D.

**Case A.** There is a vertical great circle that intersects \( S_f \) in only two points.

We may assume using a suitable rotation around the \( z \)-axis that the vertical great circle is in the \( yz \)-plane, hence \( f(x, y) \) is of the form \( g(x) + dy \). The continuity of \( f \) implies that \( g \) is also continuous.

**Subcase A1.** \( d = 0 \).

Let \( c \in C \) be fixed, and let \( \varphi_c \) be the corresponding isometry. The graph of \( cf \) is invariant under translations parallel to the \( y \)-axis. As the same holds for \( f \), by rigidity, \( cf \) is also invariant under translations parallel to the \( \varphi_c \)-image of the \( y \)-axis. If these two directions are nonparallel, then \( \text{graph}(cf) \) is a plane, and hence so is \( \text{graph}(f) \), so we are done since \( f(x, y) \) is of the form \( a + bx \) (note that there is no ‘\(+dy\)’ since \( f \) does not depend on \( y \)). Therefore we may assume
that all lines parallel to the $y$-axis are taken to lines parallel to the $y$-axis, but then all planes parallel to the $xz$-plane are taken to planes parallel to the $xz$-plane. But this shows (by considering the intersections of the graphs with the $xz$-plane) that $g$ is vertically rigid for $c$, hence by Theorem 1.3 $g(x)$ is of the form $a + bx$ or $a + be^{kx}$ ($a, b, k \in \mathbb{R}$, $k \neq 0$), and we are done.

**Subcase A2.** $d \neq 0$.

We may assume that $d > 0$, since otherwise we may consider $-f$.

For every $c \in C$ let $\varphi_c$ be the corresponding isometry. We claim that we may assume that all these are orientation-preserving. If $\{c \in C : \varphi_c$ is orientation-preserving$\}$ condensates to $\infty$ then we are done by shrinking $C$, otherwise we may assume that they are all orientation-reversing (note that if we split $C$ into two pieces then at least one of them still condensates to $\infty$). Let us fix a $c_0 \in C$ and consider $c_0f$ instead of $f$. By Lemma 4.3 this function is rigid for an uncountable set with all isometries orientation-preserving, and if it is of the desired form then so is $f$, so we are done.

We may assume $1 \notin C$. Let us fix a $c \in C$. Similarly as in the previous subcase, we may assume that lines parallel to $(0, 1, d)$ are taken to lines parallel to $(0, 1, cd)$ as follows. The special form of $f$ implies that graph$(f)$ is invariant under translations in the $(0, 1, d)$-direction, hence graph$(cf)$ is invariant under translations in the $(0, 1, cd)$-direction, moreover, by rigidity, graph$(cf)$ is also invariant under translations parallel to the $\varphi_c$-image of the lines of direction $(0, 1, d)$. If these two latter directions do not coincide then graph$(cf)$ is a plane, and we are done.

Therefore the image of every line parallel to $(0, 1, d)$ is a line parallel to $(0, 1, cd)$ under the orientation-preserving isometry $\varphi_c$. As in Remark 5.2 write $\varphi_c = \varphi_c^{\text{trans}} \circ \varphi_c^{\text{ort}}$, where $\varphi_c^{\text{ort}}$ is a rotation about a line containing the origin and $\varphi_c^{\text{trans}}$ is a translation. Since the translation does not affect directions, the rotation $\varphi_c^{\text{ort}}$ takes the direction $(0, 1, d)$ to the nonparallel direction $(0, 1, cd)$ ($d \neq 0$), therefore the axis of the rotation has to be orthogonal to the plane spanned by these two directions. Hence the axis has to be the $x$-axis. Moreover, the angle of the rotation is easily seen to be $\arctan(cd) - \arctan(d)$.

We now show that we may assume that $\varphi_c^{\text{trans}}$ is a horizontal translation. Decompose the translation as $\varphi_c^{\text{trans}} = \varphi_c^{\text{ort}} \circ \varphi_c^{\text{ort}}$, where $\varphi_c^{\text{ort}}$ is a horizontal translation and $\varphi_c^{\text{ort}}$ is a translation in the $(0, 1, cd)$-direction. Since $\varphi_c^{\text{ort}}$(graph$(f)$) is invariant under translations in the $(0, 1, cd)$-direction, so is $\varphi_c^{\text{ort}} \circ \varphi_c^{\text{ort}}$(graph$(f)$), hence

$$\varphi_c^{\text{ort}} \circ \varphi_c^{\text{ort}}$(graph$(f)$) = \varphi_c^{\text{ort}} \circ \varphi_c^{\text{ort}}$(graph$(f)$) = \varphi_c$(graph$(f)$) = graph$(cf)$, 

so we can assume $\varphi_c = \varphi_c^{\text{ort}} \circ \varphi_c^{\text{ort}}$, and we are done.

We will now complete the proof of this subcase by showing that the function $-\frac{1}{2}g$ is rigid for an uncountable set. Indeed, this suffices by Theorem 1.3 and by the special form of $f$.

Let us denote the $xy$-plane by $\{z = 0\}$ and consider the intersection of both sides of the equation $\varphi_c$(graph$(f)$) = graph$(cf)$ with $\{z = 0\}$. On the one hand, $\{z = 0\} \cap \varphi_c$(graph$(f)$) = $\{z = 0\} \cap \varphi_c^{\text{ort}}$(graph$(f)$) = $\varphi_c^{\text{ort}}$(\{z =
\(0) \cap \varphi^{ort}(\text{graph}(f)) = \varphi^{\overrightarrow{v}}(\text{graph}(-w_{c,d}g)) = \varphi^{\overrightarrow{v}}(\text{graph}((w_{c,d}d)(-\frac{1}{2}g)))\), where we used the fact that \(\varphi^{\overrightarrow{v}}\) is horizontal and Lemma \ref{lemma:horizontal}. On the other hand, it is easy to see that \(\{z = 0\} \cap \text{graph}(cf) = \text{graph}(-\frac{1}{2}g)\). Therefore, \(\text{graph}(-\frac{1}{2}g) = \varphi^{\overrightarrow{v}}(\text{graph}((w_{c,d}d)(-\frac{1}{2}g)))\) and hence \(-\frac{1}{2}g\) is rigid for \(w_{c,d}d\) for every \(c > 0\). The map \(c \mapsto w_{c,d}d\) is strictly monotone for every fixed \(d\), hence the range of \(C\) is uncountable. So \(-\frac{1}{2}g\) is rigid for an uncountable set, and we are done.

**Case B.** \(S_f = \mathbb{S}^2 \setminus \{(0,0,1),(0,0,-1)\}\).

So \(S_f\) is invariant under every \(\psi_c\), and hence so is under every \(\varphi^{ort}_c\). Then clearly \(\varphi^{ort}_c((0,0,1)) = (0,0,1)\) or \(\varphi^{ort}_c((0,0,1)) = (0,0,-1)\) for every \(c \in C\). By the same argument as above we can assume that the former holds for every \(c \in C\). Using the argument again we can assume that all \(\varphi_c's\) are orientation-preserving. But then each of these is a rotation around the \(z\)-axis followed by a translation, in other words, an orientation-preserving transformation in the \(xy\)-plane followed by a translation in the \(z\)-direction. An orientation-preserving transformation in the plane is either a translation or a rotation. If it is a translation for every \(c\) then we are done by Corollary \ref{corollary:translation}. So let us assume that there exists a \(c\) such that \(\varphi_c\) is a proper rotation around \(\vec{x} \in \mathbb{R}^2\) followed by a vertical translation. We claim that then \(f\) is constant, which will contradict that \(S_f\) is nearly the full sphere, finishing the proof of this case. We will actually show that \(f\) is constant on every closed disc \(B(\vec{x},R)\) centered at \(\vec{x}\). Indeed, consider \(\max_{B(\vec{x},R)} f - \min_{B(\vec{x},R)} f\). This is unchanged by the rotation around \(\vec{x}\) as well as by the vertical translation, hence by \(\varphi_c\). But the map \(f \mapsto cf\) multiplies this amount by \(c \neq 1\), so the only option is \(\max_{B(\vec{x},R)} f - \min_{B(\vec{x},R)} f = 0\), and we are done.

**Case C.** There exists an \(\vec{x}_0 \in \mathbb{S}^1\) such that \(h_3(\vec{x}_0) = 0\) and \(h_3(\vec{x}) = 1\) for every \(\vec{x} \neq \vec{x}_0\), that is, \(S_f\) is \(\mathbb{S}^2\) minus two quarters of a great circle.

So \(S_f\) is invariant under every \(\psi_c\), and hence so is under every \(\varphi^{ort}_c\). Hence \(\varphi^{ort}_c\) maps \((0,0,1)\) to one of the four endpoints of the two arcs. Therefore we can assume by splitting \(C\) into four pieces according to the image of \((0,0,1)\) and applying Lemma \ref{lemma:fixed_point} that \((0,0,1)\) is a fixed point of every \(\varphi^{ort}_c\). But then the two arcs are also fixed, and actually \(\varphi^{ort}_c\) is the identity. Hence every \(\varphi_c\) is a translation, and we are done by Corollary \ref{corollary:translation}.

**Case D.** There exists a closed interval \(I \in \mathbb{S}^1\) with \(0 < \text{length}(I) < \pi\) such that \(h_3(\vec{x}) = 0\) if \(\vec{x} \in I\) and \(h_3(\vec{x}) = 1\) if \(\vec{x} \notin I\), that is, \(S_f\) is \(\mathbb{S}^2\) minus two spherical triangles.

As \(S_f\) is invariant under every \(\varphi^{ort}_c\), vertices of the triangles are mapped to vertices. Hence we may assume (by splitting \(C\) into six pieces) that \((0,0,1)\) is fixed. But then the triangles are also fixed sets, and every \(\varphi^{ort}_c\) is the identity, so we are done as in the previous case.

This finishes the proof of the Main Theorem. \(\square\)
7 Open questions

**Question 7.1** In the Main Theorem can we relax the assumption of continuity to Lebesgue measurability, Baire measurability, Borel measurability, Baire class one, separate continuity or at least one point of continuity?

**Question 7.2** Which notion of largeness of $C$ suffices for the various results of this paper? For example, does the Main Theorem hold if we only assume that $C$ contains three elements that pairwise generate dense multiplicative subgroups of $(0, \infty)$?

**Remark 7.3** It was shown in [3] that two such elements suffice for the analogous one-variable result. However, two independent elements are not enough here, since if $g$ is vertically rigid for $c_1$ via a translation and $h$ is vertically rigid for $c_2$ via a translation then $f(x, y) = g(x)h(y)$ is vertically rigid for both.

Moreover, the main point in that proof in [3] is to replace ‘splitting $C$’ by alternative arguments, and we were unable to do so here.

The following question is rather vague.

**Question 7.4** Let us call a set $H \subset S^2$ rigid if $\psi_c(H)$ is isometric to $H$ for every $c > 0$. Is there a simple description of rigid sets? Or if we assume some regularity?

And finally, the most intriguing problem.

**Question 7.5** What can we say if there are more than two variables?

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