Simple Closed Analytic Formulas for the Approximation of the Legendre Complete Elliptic Integrals $K(k)$ and $E(k)$ (and their First Derivatives)

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Abstract: Two sets of closed analytic functions are proposed for the approximate calculus of the complete elliptic integrals of the 1st and 2nd kind in the normal form due to Legendre, their expressions having a remarkable simplicity and accuracy. The special usefulness of the newly proposed original formulas consists in that they allow performing the analytic study of variation of the functions in which they appear, using derivatives (they being expressed in terms of elementary functions only, without any special function; this would mean replacing one difficulty by another of the same kind). Comparative tables of the approximate values so obtained and the exact ones, reproduced from special functions tables are given (vs. the elliptic integrals modulus $k$). It is to be noticed that both sets of formulas are given neither by spline nor by regression functions, but by asymptotic expansions, the identity with the exact functions being accomplished for the left domain’s end. As for their simplicity, the formulas in $k / k’$ do not need any mathematical table (are purely algebraic). As for their accuracy, the 2nd set, although more intricate, gives more accurate values than the 1st one and extends itself more closely to the right domain’s end. Legendre complete elliptic integrals of the 1st and 2nd kinds are very useful in Signal Processing and especially in the design of Digital Filters (Legendre Digital Filters).

Key-Words: analytic methods; Legendre complete elliptic integrals of the 1st and 2nd kind, $K(k)$ and $E(k)$; elliptic integral’s complementary modulus $k’$; elliptic integral’s complementary modulus $k_4$; tables of Legendre complete elliptic integrals; approximate formulas Digital Signal Processing Filters

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2020. 1 Introduction – elliptic integrals

There are many interesting domains in pure and applied mathematics where appear one or both complete elliptic integrals of the 1st and 2nd kind in the normal form due to Legendre. The period of oscillations in a vacuum of the simple pendulum, in the dynamics of a constrained heavy particle, is given by a complete elliptic integral of the 1st kind. The length of an ellipse, in the geometry of plane curves, as well as the lift coefficient of a thin delta wing with subsonic leading edges, in supersonic aerodynamics (small perturbations theory), are given by a complete elliptic integral of the 2nd kind. The following relations define these integrals of the 1st and 2nd kind, respectively

\[ K(k) = \int_0^\pi 2^{1/2} (1 - k^2 \sin^2 \phi)^{-1/2} d\phi = \int_0^\pi \left(1 - \frac{k^2}{t^2} \right)^{1/2} dt; \]

\[ E(k) = \int_0^\pi 2^{1/2} (1 - k^2 \sin^2 \phi)^{-1/2} d\phi = \int_0^\pi \left(1 - \frac{k^2}{t^2} \right)^{1/2} dt; \]

$k = \sin \theta \geq 0$ is called modulus. $K(k), E(k)$ are typical elliptic integrals. They do not admit primitive functions (cannot be expressed in terms of elementary functions), being calculated by expanding the integrands into series, integrating term-by-term, and presented vs. $k \in [0, 1]$, or vs. $\theta \in [0, \pi/2]$, in some mathematical tables [1] – [6]. Modern mathematics defines an elliptic integral as any function $f(x)$ which can be expressed in the form $f(x) = \int R(t, P(t)^{1/2}) dt; R$ is a rational function of its two arguments; $P$ is a polynomial of degrees 3 or 4 with no repeated roots; $c$ is a constant. The values given in some special tables allow performing the calculus for a given case (point), but not the analytic study of variation of the functions in which these integrals appear, using the derivatives. In the next chapter two sets (subscripts 0 and 1) of closed analytic functions are given for the approximate calculus of $K(k)$ and $E(k)$.

The method used in this work is purely analytic, not needing any numerical procedure, or sophisticated computer programs. There also is a Legendre complete elliptic integral of the 3rd kind. Legendre complete elliptic integrals of the 1st and 2nd kinds are very useful in Signal Processing and especially in the design of Digital Filters (Legendre Digital Filters)

2 The two sets of newly proposed formulas

The complementary modulus is $k’ = (1 - k^2)^{1/2} = \cos \theta$. For the first set the following formulas are proposed:

\[ K_0(k) = \frac{\pi}{\sqrt{1-k^2}} \left[1 - \frac{1}{2} \frac{1+\sqrt{1-k^2}}{\sqrt{1-k^2}} \right] = \pi \left(\frac{1}{\sqrt{k^2}} - \frac{1}{2\sqrt{2} K^{3/4}} \right), \]

\[ K_1(k) = \frac{\pi}{\cos^2 \theta} \left[1 - \frac{1}{2} \frac{1+\cos \theta/2}{\cos \theta} \right] = \frac{\pi}{\cos^2 \theta} \left(1 - \frac{1}{2} \frac{\cos \theta/2}{\cos \theta} \right), \]

\[ E_0(k) = \frac{\pi}{\sqrt{1-k^2}} \left[\frac{3}{2} \frac{1+\sqrt{1-k^2}-1}{\sqrt{1-k^2}} \right] = \frac{3}{4} \left(1 + k’ \right) - \sqrt{k’}, \]

\[ E_1(k) = \frac{\pi}{\cos^2 \theta} \left[\frac{3}{2} \frac{\cos \theta (\theta/2)}{\cos^2 \theta} - 1 \right] = \frac{3}{4} \left(\cos \theta \right) - \sqrt{\cos \theta}. \]
Similarly, for the second set are proposed the formulas:

\[ K_1(k) = \frac{\pi \sqrt{2}}{\sqrt{(1 + k^2)k'}} \left( 1 - \frac{1}{2} \frac{1 + \sqrt{k'}}{4 (1 + k^2)k'} \right), \]

\[ K_1(\theta) = \frac{\pi}{\cos(\theta/2) \cos^2 \theta} \left( 1 - \frac{1}{4} \frac{1 + \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} \cos^2 \theta} \right), \]

\[ E_1(k) = \frac{\pi}{4} \left( \frac{1}{2} (1 + k^2) - \sqrt{\frac{1}{2} (1 + k^4)} \right) - k' K_1(k), \]

\[ E_1(\theta) = \frac{\pi}{4} \left( \frac{1}{2} \cos \theta - 2 \cos \frac{\theta}{2} \frac{1}{\cos \theta} \right) - \cos \theta K_1(\theta). \]

Table 1. Values of the functions K (part one)

| \( k = \sin \theta \) | K(k) | Kd(k) | K(k) |
|------------------------|------|-------|------|
| 0.00000                | 1.5708 | 1.5708 | 1.5708 |
| 0.01745                | 1.5709 | 1.5709 | 1.5709 |
| 0.03490                | 1.5713 | 1.5713 | 1.5713 |
| 0.05234                | 1.5719 | 1.5719 | 1.5719 |
| 0.06976                | 1.5727 | 1.5727 | 1.5727 |
| 0.08716                | 1.5738 | 1.5738 | 1.5738 |
| 0.10453                | 1.5751 | 1.5751 | 1.5751 |
| 0.12187                | 1.5767 | 1.5767 | 1.5767 |
| 0.13917                | 1.5785 | 1.5785 | 1.5785 |
| 0.15643                | 1.5805 | 1.5805 | 1.5805 |
| 0.17365                | 1.5828 | 1.5828 | 1.5828 |
| 0.19081                | 1.5854 | 1.5854 | 1.5854 |
| 0.20791                | 1.5882 | 1.5882 | 1.5882 |
| 0.22495                | 1.5913 | 1.5913 | 1.5913 |
| 0.24192                | 1.5946 | 1.5946 | 1.5946 |
| 0.25882                | 1.5981 | 1.5981 | 1.5981 |
| 0.27564                | 1.6020 | 1.6020 | 1.6020 |
| 0.29237                | 1.6061 | 1.6061 | 1.6061 |
| 0.30902                | 1.6105 | 1.6105 | 1.6105 |
| 0.32557                | 1.6151 | 1.6151 | 1.6151 |
| 0.34202                | 1.6200 | 1.6200 | 1.6200 |
| 0.35837                | 1.6252 | 1.6252 | 1.6252 |
| 0.37461                | 1.6307 | 1.6307 | 1.6307 |
| 0.39073                | 1.6365 | 1.6365 | 1.6365 |
| 0.40674                | 1.6426 | 1.6426 | 1.6426 |
| 0.42262                | 1.6490 | 1.6490 | 1.6490 |
| 0.43837                | 1.6557 | 1.6557 | 1.6557 |
| 0.45399                | 1.6627 | 1.6627 | 1.6627 |
| 0.46947                | 1.6701 | 1.6701 | 1.6701 |
| 0.48481                | 1.6777 | 1.6777 | 1.6777 |
| 0.50000                | 1.6858 | 1.6858 | 1.6858 |
| 0.51504                | 1.6941 | 1.6941 | 1.6941 |
| 0.52992                | 1.7028 | 1.7028 | 1.7028 |
| 0.54464                | 1.7119 | 1.7119 | 1.7119 |
| 0.55919                | 1.7214 | 1.7214 | 1.7214 |
| 0.57358                | 1.7312 | 1.7312 | 1.7312 |
| 0.58779                | 1.7415 | 1.7415 | 1.7415 |
| 0.60182                | 1.7522 | 1.7522 | 1.7522 |
| 0.61566                | 1.7632 | 1.7632 | 1.7632 |
| 0.62932                | 1.7748 | 1.7748 | 1.7748 |
| 0.64279                | 1.7868 | 1.7868 | 1.7868 |

| \( \theta (^\circ) \) | \( k = \sin \theta \) |
|------------------------|------------------------|
| 0                      | 0.98092                |
| 1                      | 0.98604                |
| 2                      | 0.99011                |
| 3                      | 0.99341                |
| 4                      | 0.99610                |
| 5                      | 0.99832                |
| 6                      | 0.99973                |
| 7                      | 1.00000                |
| 8                      | 1.00000                |
| 9                      | 1.00000                |
| 10                     | 1.00000                |
| 11                     | 1.00000                |
| 12                     | 1.00000                |
| 13                     | 1.00000                |
| 14                     | 1.00000                |
| 15                     | 1.00000                |
| 16                     | 1.00000                |
| 17                     | 1.00000                |
| 18                     | 1.00000                |
| 19                     | 1.00000                |
| 20                     | 1.00000                |
| 21                     | 1.00000                |
| 22                     | 1.00000                |
| 23                     | 1.00000                |
| 24                     | 1.00000                |
| 25                     | 1.00000                |
| 26                     | 1.00000                |
| 27                     | 1.00000                |
| 28                     | 1.00000                |
| 29                     | 1.00000                |
| 30                     | 1.00000                |
| 31                     | 1.00000                |
| 32                     | 1.00000                |
| 33                     | 1.00000                |
| 34                     | 1.00000                |
| 35                     | 1.00000                |
| 36                     | 1.00000                |
| 37                     | 1.00000                |
| 38                     | 1.00000                |
| 39                     | 1.00000                |
| 40                     | 1.00000                |
The values strings in the last two columns of table 1 were canceled when each of the two closed analytic formulas proposed for the approximation of the Legendre complete elliptic integral of the 1st kind $K(k)$ gives too great relative errors ($\geq 4\%$ – also see chapter 3) for being still accepted in the usual mathematical / technical calculus. The same procedure will be applied in case of the next table (no. 2), for the same reason, concerning the accuracy of the values given by each of the other two closed analytic formulas proposed for the approximation of the Legendre complete elliptic integral of the 2nd kind $E(k)$. The accuracy analysis of the two sets of formulas will be performed in the next chapter (no. 3). In chapter 4 some series representations for the exact functions and for both sets of approximation, as well as for their first order derivatives, will be given.

| $\theta$ (°) | $k = \sin \theta$ | $E(k)$ | $E_0(k)$ | $E_1(k)$ |
|-------------|-----------------|--------|-----------|-----------|
| 0           | 0.00000         | 1.5708 | 1.5708    | 1.5708    |
| 1           | 0.01745         | 1.5707 | 1.5707    | 1.5707    |
| 2           | 0.03490         | 1.5703 | 1.5703    | 1.5703    |
| 3           | 0.05234         | 1.5697 | 1.5697    | 1.5697    |
| 4           | 0.06976         | 1.5689 | 1.5689    | 1.5689    |
| 5           | 0.08716         | 1.5678 | 1.5678    | 1.5678    |
| 6           | 0.10453         | 1.5665 | 1.5665    | 1.5665    |
| 7           | 0.12187         | 1.5649 | 1.5649    | 1.5649    |
| 8           | 0.13917         | 1.5632 | 1.5632    | 1.5632    |
| 9           | 0.15643         | 1.5611 | 1.5611    | 1.5611    |
| 10          | 0.17365         | 1.5589 | 1.5589    | 1.5589    |
| 11          | 0.19081         | 1.5564 | 1.5564    | 1.5564    |
| 12          | 0.20791         | 1.5537 | 1.5537    | 1.5537    |
| 13          | 0.22495         | 1.5507 | 1.5507    | 1.5507    |
| 14          | 0.24192         | 1.5476 | 1.5476    | 1.5476    |
| 15          | 0.25882         | 1.5442 | 1.5442    | 1.5442    |
| 16          | 0.27564         | 1.5405 | 1.5405    | 1.5405    |
| 17          | 0.29237         | 1.5367 | 1.5367    | 1.5367    |
| 18          | 0.30902         | 1.5326 | 1.5326    | 1.5326    |
| 19          | 0.32557         | 1.5283 | 1.5283    | 1.5283    |
| 20          | 0.34202         | 1.5238 | 1.5238    | 1.5238    |
| 21          | 0.35837         | 1.5191 | 1.5191    | 1.5191    |
| 22          | 0.37461         | 1.5141 | 1.5141    | 1.5141    |
| 23          | 0.39073         | 1.5090 | 1.5090    | 1.5090    |
| 24          | 0.40674         | 1.5037 | 1.5037    | 1.5037    |
| 25          | 0.42262         | 1.4981 | 1.4981    | 1.4981    |
| 26          | 0.43837         | 1.4924 | 1.4924    | 1.4924    |
| 27          | 0.45399         | 1.4864 | 1.4864    | 1.4864    |
| 28          | 0.46947         | 1.4803 | 1.4803    | 1.4803    |
| 29          | 0.48481         | 1.4740 | 1.4740    | 1.4740    |
| 30          | 0.50000         | 1.4675 | 1.4675    | 1.4675    |
| 31          | 0.51504         | 1.4608 | 1.4608    | 1.4608    |
| 32          | 0.52992         | 1.4539 | 1.4539    | 1.4539    |
| 33          | 0.54464         | 1.4469 | 1.4469    | 1.4469    |
| 34          | 0.55919         | 1.4397 | 1.4397    | 1.4397    |
| 35          | 0.57358         | 1.4323 | 1.4323    | 1.4323    |
| 36          | 0.58779         | 1.4248 | 1.4248    | 1.4248    |
Table 2. Values of the functions E (part two) reproduced from special functions tables [6], as well as values of both Legendre complete elliptic integrals $D$

| $\alpha$  | $E$   | $E'$  | $E''$  | $E'''$  | $E''''$  |
|-----------|-------|-------|--------|---------|----------|
| 41        | 0.65606 | 1.3849 | 1.3849 | 1.3849 | 1.0338    |
| 42        | 0.66913 | 1.3765 | 1.3765 | 1.3765 | 1.0327    |
| 43        | 0.68200 | 1.3680 | 1.3680 | 1.3680 | 1.0315    |
| 44        | 0.69466 | 1.3594 | 1.3594 | 1.3594 | 1.0303    |
| 45        | 0.70711 | 1.3506 | 1.3506 | 1.3506 | 1.0292    |
| 46        | 0.71934 | 1.3419 | 1.3419 | 1.3419 | 1.0280    |
| 47        | 0.73135 | 1.3330 | 1.3330 | 1.3330 | 1.0269    |
| 48        | 0.74314 | 1.3238 | 1.3238 | 1.3238 | 1.0258    |
| 49        | 0.75471 | 1.3147 | 1.3147 | 1.3147 | 1.0247    |
| 50        | 0.76604 | 1.3055 | 1.3055 | 1.3055 | 1.0236    |
| 51        | 0.77715 | 1.2963 | 1.2963 | 1.2963 | 1.0226    |
| 52        | 0.78801 | 1.2872 | 1.2872 | 1.2872 | 1.0215    |
| 53        | 0.79864 | 1.2776 | 1.2776 | 1.2776 | 1.0205    |
| 54        | 0.80902 | 1.2684 | 1.2684 | 1.2684 | 0.9975    |
| 55        | 0.81915 | 1.2590 | 1.2590 | 1.2590 | 0.9946    |
| 56        | 0.82904 | 1.2492 | 1.2492 | 1.2492 | 0.9918    |
| 57        | 0.83867 | 1.2397 | 1.2397 | 1.2397 | 0.9890    |
| 58        | 0.84805 | 1.2201 | 1.2201 | 1.2201 | 0.9863    |
| 59        | 0.85717 | 1.2106 | 1.2106 | 1.2106 | 0.9836    |
| 60        | 0.86603 | 1.2011 | 1.2011 | 1.2011 | 0.9809    |
| 61        | 0.87462 | 1.1915 | 1.1915 | 1.1915 | 0.9782    |
| 62        | 0.88295 | 1.1820 | 1.1820 | 1.1820 | 0.9755    |
| 63        | 0.89101 | 1.1725 | 1.1725 | 1.1725 | 0.9729    |
| 64        | 0.89879 | 1.1630 | 1.1630 | 1.1630 | 0.9703    |
| 65        | 0.90631 | 1.1535 | 1.1535 | 1.1535 | 0.9677    |
| 66        | 0.91355 | 1.1440 | 1.1440 | 1.1440 | 0.9651    |
| 67        | 0.92050 | 1.1345 | 1.1345 | 1.1345 | 0.9626    |
| 68        | 0.92718 | 1.1250 | 1.1250 | 1.1250 | 0.9599    |
| 69        | 0.93358 | 1.1155 | 1.1155 | 1.1155 | 0.9574    |
| 70        | 0.93969 | 1.1060 | 1.1060 | 1.1060 | 0.9549    |
| 71        | 0.94552 | 1.1008 | 1.1008 | 1.1008 | 0.9523    |
| 72        | 0.95082 | 1.0953 | 1.0953 | 1.0953 | 0.9498    |
| 73        | 0.95147 | 1.0908 | 1.0908 | 1.0908 | 0.9474    |
| 74        | 0.95630 | 1.0863 | 1.0863 | 1.0863 | 0.9449    |
| 75        | 0.96126 | 1.0818 | 1.0818 | 1.0818 | 0.9425    |
| 76        | 0.96635 | 1.0773 | 1.0773 | 1.0773 | 0.9401    |
| 77        | 0.97137 | 1.0726 | 1.0726 | 1.0726 | 0.9376    |
| 78        | 0.97630 | 1.0680 | 1.0680 | 1.0680 | 0.9351    |
| 79        | 0.98132 | 1.0635 | 1.0635 | 1.0635 | 0.9326    |
| 80        | 0.98634 | 1.0590 | 1.0590 | 1.0590 | 0.9301    |
| 81        | 0.99136 | 1.0545 | 1.0545 | 1.0545 | 0.9277    |
| 82        | 0.99638 | 1.0500 | 1.0500 | 1.0500 | 0.9253    |
| 83        | 0.99638 | 1.0455 | 1.0455 | 1.0455 | 0.9230    |

In the comparative tables 1 and 2, the 4D (four digit) exact values of both Legendre complete elliptic integrals reproduced from special functions tables [6], as well as their 4D approximate values obtained by applying the two
sets of proposed closed analytic formulas were given (all versus the respective elliptic integrals modulus, \( k = \sin \theta \)). It is to be noticed that both sets of approximate formulas are not given by spline or regression functions, but by asymptotic expansions, the respective expressions having a remarkable simplicity (see, e.g.: the 2\(^{nd}\) form of \( E(k) \) or \( E_k(\theta) \); more, all newly found formulas in \( k / k' \) do not need any mathematical table, being purely algebraic) and accuracy (see table 3). The identity with the exact functions is satisfied for the left end \( k = 0 (\theta = 0^\circ) \) of the domain. As one can see, the 2\(^{nd}\) set of functions \((K_1, E_1)\), although something more intricate, gives more accurate values than the first one \((K_0, E_0)\) and extends itself more closely to the right end \( k = 1 (\theta = 90^\circ) \) of the domain.

### 3 The accuracy of the two sets of formulas

Let us define the following relative error functions:

\[ e_{K_0}(K) = K_0(K)/K(k) - 1; \quad e_{E_0}(E) = E_0(E)/E(k) - 1, \]

for both sets of approximation of the 1\(^{st}\) kind integral and

\[ e_{K_1}(K) = K_1(K)/K(k) - 1; \quad e_{E_1}(E) = E_1(E)/E(k) - 1, \]

for both sets of approximation of the 2\(^{nd}\) kind integral. Their values are given in the table 3, being expressed in thousandths (‰). These errors were calculated for the 1\(^{st}\) set \((K_0, E_0)\) only in the field \( \theta \in [54^\circ, 71^\circ] \) of the domain, with an increment of 1\(^{\circ}\), while the 2\(^{nd}\) set \((K_1, E_1)\) only in the field \( \theta \in [84^\circ.2, 88^\circ.2] \), with an increment of 0.2\(^{\circ}\), like in the above tables 1 and 2.

| \( \theta (^\circ) \) | \( K_0(\%) \) | \( K_1(\%) \) | \( E_0(\%) \) | \( E_1(\%) \) |
|---------------------|----------|----------|----------|----------|
| 54                  | 0.80902  | -0.250   | +0.255   |          |
| 55                  | 0.81915  | -0.272   | +0.243   |          |
| 56                  | 0.82904  | -0.353   | +0.293   |          |
| 57                  | 0.83867  | -0.420   | +0.334   |          |
| 58                  | 0.84805  | -0.497   | +0.454   |          |
| 59                  | 0.85717  | -0.558   | +0.502   |          |
| 60                  | 0.86603  | -0.669   | +0.566   |          |
| 61                  | 0.87462  | -0.799   | +0.742   |          |
| 62                  | 0.88295  | -0.961   | +0.874   |          |
| 63                  | 0.89101  | -1.118   | +0.973   |          |
| 64                  | 0.89879  | -1.366   | +1.135   |          |
| 65                  | 0.90631  | -1.619   | +1.377   |          |
| 66                  | 0.91355  | -1.918   | +1.627   |          |
| 67                  | 0.92050  | -2.299   | +1.900   |          |
| 68                  | 0.92718  | -2.709   | +2.215   |          |
| 69                  | 0.93358  | -3.253   | +2.573   |          |
| 70                  | 0.93699  | -3.907   | +2.959   |          |
| 71                  | 0.94552  | -4.642   | +3.525   |          |

4.8 0.99588  -   -0.369   -   +0.607  
85.2 0.99649  -   -0.451   -   +0.705  
85.4 0.99678  -   -0.500   -   +0.748  

The relative errors strings are stopped for values \( \geq 4 \)‰.

### 4 Comparative series representations

Expanding into power series, one obtains for the complete elliptic integrals the set of representations below (5) – (7):

\[ K(k) = \frac{\pi}{2} \left[ 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)^2}{2 \cdot 4 \cdots 2n} \left( \frac{k}{4} \right)^{2n} \right] ; \]

\[ E(k) = \frac{\pi}{2} \left[ 1 - \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)^2}{2 \cdot 4 \cdots 2n} \left( \frac{k}{4} \right)^{2n} \right] ; \]

Proceeding in the same manner, we get for the 1\(^{st}\) set the (most inaccurate) of approximate functions the expansions

\[ K_0(k) = \frac{\pi}{2} \left[ 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)^2}{2 \cdot 4 \cdots 2n} \left( \frac{k}{4} \right)^{2n} \right] ; \]

\[ E_0(k) = \frac{\pi}{2} \left[ 1 - \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)^2}{2 \cdot 4 \cdots 2n} \left( \frac{k}{4} \right)^{2n} \right] ; \]

for the 2\(^{nd}\) set being practically identical with the exact ones

\[ K_1(k) = \frac{\pi}{2} \left[ 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)^2}{2 \cdot 4 \cdots 2n} \left( \frac{k}{4} \right)^{2n} \right] ; \]

\[ E_1(k) = \frac{\pi}{2} \left[ 1 - \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)^2}{2 \cdot 4 \cdots 2n} \left( \frac{k}{4} \right)^{2n} \right] ; \]

The difference with respect to the expansions of the
exact functions begins at the terms in $k^8$ for the 1st set of approximation, and at the terms in $k^{16}$ for the 2nd set. For the 1st order derivatives of the exact functions we get:

$$\frac{dK(k)}{dk} = -\frac{E(k)}{k}K(k) = \frac{\pi}{k} \left(\frac{1}{4} k^2 + \frac{75}{64} k^4 + \frac{1225}{1024} k^6 \right);$$
$$\frac{dE(k)}{dk} = \frac{E(k)}{k} - \frac{K(k)}{k} = -\frac{\pi}{8} \left(\frac{1}{8} k^2 + \frac{15}{64} k^4 + \frac{175}{1024} k^6 \right).$$

Applying the previous two exact relations and using the four definitions from chapter 2 one gets the expansions:

$$\left[ \begin{array}{c} \frac{dK(k)}{dk} \\ \frac{dE(k)}{dk} \end{array} \right] = \frac{\pi}{4} \left( \begin{array}{c} 1 + \frac{9}{8} k^2 + \frac{75}{64} k^4 + \frac{1225}{1024} k^6 \\ 1 + \frac{3}{8} k^2 + \frac{15}{64} k^4 + \frac{174.25}{1024} k^6 \end{array} \right);$$

for the 1st set of approximate functions, and respectively:

$$\left[ \begin{array}{c} \frac{dK(k)}{dk} \\ \frac{dE(k)}{dk} \end{array} \right] = \frac{\pi}{4} \left( \begin{array}{c} 1 + \frac{9}{8} k^2 + \frac{75}{64} k^4 + \frac{1225}{1024} k^6 + \frac{19845}{16384} k^8 \\ 1 + \frac{3}{8} k^2 + \frac{15}{64} k^4 + \frac{174.25}{1024} k^6 + \frac{33554432}{16384} k^8 \end{array} \right);$$

$$\left[ \begin{array}{c} \frac{dK(k)}{dk} \\ \frac{dE(k)}{dk} \end{array} \right] = \frac{\pi}{4} \left( \begin{array}{c} 1 + \frac{9}{8} k^2 + \frac{75}{64} k^4 + \frac{1225}{1024} k^6 + \frac{19845}{16384} k^8 + \frac{1084576}{131072} \\ 1 + \frac{3}{8} k^2 + \frac{15}{64} k^4 + \frac{174.25}{1024} k^6 + \frac{33554432}{16384} \end{array} \right).$$

for the 2nd set of approximate functions. The difference with respect to the expansions of the 1st order derivatives of the exact functions begins at the terms in $k^7$ for the 1st set of approximation, and at the terms in $k^{15}$ for the 2nd set, being much smaller than that for the expansions of the respective sets of approximate functions. One can also easily find the analytic expressions and series representations for the 2nd derivatives of all K, K$_0$, E, E$_0$.

### 5 Graphic comparison

The variation curves of both Legendre complete elliptic integrals, as well as that of the two sets of newly proposed closed analytic functions are graphically represented in the comparative figures 1 and 2, all versus the angle $\theta$ expressed in sexagesimal degrees and given by $\theta = \sin^{-1}k$. In both figures the exact functions K$(k)$, E$(k)$ were represented by solid (continuous) black lines, the 1st set of approximation K$_0$(k), E$_0$(k) by dashed black lines, and the 2nd set of approximation K$_6$(k), E$_6$(k) by solid red lines, resp.

### 6 Conclusions

As for their simplicity, the formulas in $k/k'$ do not need any mathematical table (are purely algebraic). As for their accuracy, in current mathematical / technical applications, it must use the 1st set until $\theta = 70.5^\circ$ ($k = 0.94264$) only, and for a better accuracy or a greater upper limit of the validity domain, to use the 2nd set, but until $\theta = 88^\circ.2$ ($k = 0.99951$). Legendre complete elliptic integrals of the 1st and 2nd kinds are very useful in Signal Processing and especially in the design of Digital Filters (Legendre Digital Filters)

### 7 Notes; other methods; future research

With an appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the three Legendre canonical forms (of the 1st, 2nd & 3rd kind). Without the comparative tables 1 and 2, the errors table becoming so table 1, this work was published previously in a proceedings volume (scientific bulletin), in Romanian [8]. For the first English version of this work see [9]. Approximations for the complete elliptic integrals based on the trapezoidal-type numerical integration formulas discussed in [10], are developed in [11], [12] (a mixed numerical-analytic method). Newer formulas (using $\Gamma$ function – not an elementary, but a special one, like K & E, even if these formulas are the most accurate) are in [13], [14]; as stated in their abstracts, the works [9], [13] do not have the same goal. Notable special functions suitable for applying such an approximate method of calculation (like in [9]) are: Si(x), Ci(x), Er(x), li(x).
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