PSEUDOMODES FOR NON-SELF-ADJOINT DIRAC OPERATORS

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Abstract. Depending on the behaviour of the complex-valued electromagnetic potential
in the neighbourhood of infinity, pseudomodes of one-dimensional Dirac operators corre-
sponding to large pseudoeigenvalues are constructed. This is a first systematic approach
which goes beyond the standard semi-classical setting. Furthermore, this approach results
in substantial progress in achieving optimal conditions and conclusions as well as in covering
a wide class of previously inaccessible potentials, including superexponential ones.

1. Introduction

1.1. Motivations. Spectral theory of self-adjoint operators has exhibited an enormous de-
velopment since the discovery of quantum mechanics at the beginning of the last century and
it can be regarded as well understood in many respects by now. Recent years have brought
new motivations for considering non-self-adjoint operators, too, notably due to the uncon-
ventional concept of representing physical observables by operators which are merely similar
to self-adjoint ones [3]. This brand-new construct, nicknamed quasi-self-adjoint quantum
mechanics, remained overlooked for almost hundred years and has led to challenging math-
ematical problems which cannot be handled by standard tools and the theory is by far not
complete.

It has been accepted by mathematicians as well as physicists that an appropriate charac-
teristic which conveniently describes the pathological properties of non-self-adjoint operators
is the notion of pseudospectra [29, 12, 18]. Given a positive number \( \varepsilon \), the \( \varepsilon \)-pseudospectrum
\( \sigma_\varepsilon(H) \) of any operator \( H \) in a complex Hilbert space is defined as its spectrum \( \sigma(H) \) en-
riched by those complex points \( \lambda \) (called pseudoeigenvalues) for which there exists a vector
\( \Psi \in D(H) \) (called pseudoeigenvector or pseudomode or quasimode) such that

\[
\| (H - \lambda)\Psi \| < \varepsilon \| \Psi \|. \tag{1.1}
\]

This notion is trivial for self-adjoint (or, more generally, normal) operators, because then
\( \sigma_\varepsilon(H) \) merely coincides with the \( \varepsilon \)-tubular neighbourhood of the spectrum. If \( H \) is non-
normal, however, the pseudospectrum \( \sigma_\varepsilon(H) \) can contain points which lie outside (in fact,
possibly “very far” from) the spectrum \( \sigma(H) \). It turns out that it is the pseudospectrum
which determines the decay of the semigroup generated by \( H \) as well as the behaviour of
the spectrum of \( H \) under small perturbations.

The usefulness of pseudospectra in quasi-self-adjoint quantum mechanics was pointed out
by Siegl and one of the present authors in [27]. Based on the semiclassical construction of
pseudomodes in Davies’ pioneering work [11] (see also [30] and [13]), we proved an abrupt
lack of quasi-self-adjointness for the prominent imaginary cubic oscillator of [4], which stayed
at the advent of the so-called \( \mathcal{PT} \)-symmetric quantum mechanics. Many other Schrödinger
operators with complex-valued potentials were included in the subsequent works [22, 19, 20,
25, 24, 21, 23, 2].

In this series of works, the paper [24] of Siegl and one of the present authors is ex-
ceptional in that it develops a direct construction of large-energy pseudomodes (i.e. those
corresponding in (1.1) to \( |\lambda| \to +\infty \) with \( \varepsilon_\lambda \to 0 \)), which does not require the passage
through semiclassical Schrödinger operators. In fact, the semiclassical setting follows as a
special consequence of [24]. Moreover, the newly developed, general approach of [24] enables
one to cover previously inaccessible potentials such as the exponential and discontinuous ones. What is more, the technique is applicable to other models such as the damped wave equation \[1\].

The objective of the present paper is to extend the method developed in [23] to relativistic quantum mechanics by considering Dirac instead of Schrödinger operators. This extension is both mathematically challenging and physically interesting, because the Dirac equation is not scalar and the external electromagnetic perturbations are allowed to be fundamentally matrix-valued. We also remark that the present model is additionally relevant in the context of graphene materials. What is more, we substantially generalise the method of [23] on a technical level, which enables us to cover previously inaccessible potentials, including superexponential ones. Non-self-adjoint Dirac operators have attracted a lot of attention recently [8, 6, 7, 15, 9, 16, 5, 10], however, we are not aware of any result related to the construction of pseudomodes in $\lambda$-dependent WKB form.

1.2. The model and main results. Following the classical reference [28], we introduce the one-dimensional free Dirac operator by

\[ H_0 := -i\sigma_1 \frac{d}{dx} + m\sigma_3, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

with mass $m \geq 0$. We think of $H_0$ as an operator acting in the Hilbert space $\mathcal{H}$ which is a direct sum of two $L^2(\mathbb{R})$ spaces,

\[ \mathcal{H} := L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : u_1, u_2 \in L^2(\mathbb{R}) \right\}, \]

equipped with the inner product

\[ \langle u, v \rangle := \int_{\mathbb{R}} \sum_{j=1}^{2} u_j(x) \overline{v_j(x)} \, dx. \quad (1.2) \]

Alternatively, we identify $\mathcal{H}$ with $L^2(\mathbb{R})^2$ or $L^2(\mathbb{R}) \otimes \mathbb{C}^2$. It is well known that $H_0$ is self-adjoint if $\mathcal{D}(H_0) := H^1(\mathbb{R})^2$ and one has

\[ \sigma(H_0) = (-\infty, -m] \cup [m, +\infty). \quad (1.3) \]

The free operator $H_0$ is perturbed by a complex matrix-valued potential $V : \mathbb{R} \to \mathbb{C}^{2 \times 2}$. We write

\[ V(x) := \begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{21}(x) & V_{22}(x) \end{pmatrix}, \]

where $x \in \mathbb{R}$, and assume that $V \in L^2_{\text{loc}}(\mathbb{R}) \otimes \mathbb{C}^{2 \times 2}$, meaning that all the components $V_{ij}$ with $i, j \in \{1, 2\}$ are complex-valued (scalar) functions belonging to $L^2_{\text{loc}}(\mathbb{R})$. In the case of real-valued potentials, the special scenario $V_{11} = V_{22}$ and $V_{12} = V_{21} = 0$ (respectively, $V_{11} = V_{22} = 0$ and $V_{12} = V_{21}$) corresponds to purely electric (respectively, purely magnetic) fields. The special case $V_{11} = -V_{22}$ and $V_{12} = V_{21} = 0$ is known as the scalar potential. We keep the same terminology in the general, complex-valued case.

The perturbed operator $H_V$ is introduced as the maximal extension of the operator sum $H_0 + V$, where we denote by the same symbol $V$ the maximal operator of multiplication by the generated function $V$. More specifically,

\[ H_V f := \begin{pmatrix} (-i\partial_x + V_{12}) f_2 + (V_{11} + m) f_1 \\ (-i\partial_x + V_{21}) f_1 + (V_{22} - m) f_2 \end{pmatrix}, \quad (1.4) \]

\[ \mathcal{D}(H_V) := \{ f := (f_1, f_2) \in \mathcal{H} : H_V f \in \mathcal{H} \}. \]

The local integrability conditions imposed on the coefficients of $V$ ensure that all the actions of $H_V$ in (1.4) are well defined in the sense of distributions. By straightforward arguments, it follows that $H_V$ is a closed operator. However, the closedness of $H_V$ is inessential for
our construction of pseudomodes. In fact, many of the constructed pseudomodes belong to $C_0^\infty(\mathbb{R})^2$, so the majority of our results apply also to any (possibly non-closed or trivial) extension of the sum $H_0 + V$ initially defined on $C_0^\infty(\mathbb{R})^2$.

The main purpose of this work is to build a $\lambda$-dependent family $\Psi_\lambda$ such that

$$\| (H_V - \lambda) \Psi_\lambda \| \leq o(1) \| \Psi_\lambda \| \quad \text{as } \lambda \to \infty \text{ in } \Omega \subset \mathbb{C},$$

(1.5)

where $\| \cdot \|$ is the norm associated with (1.2). The principal tool for this construction is the (J)WKB analysis (also known as the Liouville–Green approximation). In addition to investigating the rate of the decay in (1.5), we also address the question of describing the shape of the complex region $\Omega$ depending on $V$. The method used in Section 4 gives us a way to sketch the region $\Omega$ which seems rather optimal, even for low regular potential. It is interesting to observe that the domain $\Omega$ does not depend on the regularity of the potentials when the imaginary parts of $V_{11}$ and $V_{22}$ grow slowly at $+\infty$ such as logarithmic one (see Example 4) or root-type $x^\gamma$ with $\gamma \in (0, 1)$ (see Example 5).

As a foretaste of our main theorems without going into technical details, we present here a very special example of (1.5) for real $\lambda$ and purely electric perturbations. The following theorem is a particular consequence of Theorem 3.10 and Remark 3.12 below. From now on, $\text{Re } u$ and $\text{Im } u$ denote, respectively, the real part and the imaginary part of a function $u : \mathbb{R} \to \mathbb{C}$.

**Theorem 1.1.** Let $v \in W_{\text{loc}}^{2,\infty}(\mathbb{R})$ satisfy

$$\limsup_{x \to -\infty} \text{Im } v(x) < 0, \quad \liminf_{x \to -\infty} \text{Im } v(x) > 0,$$

(1.6)

and set $V_{11} := v =: V_{22}$ and $V_{12} := 0 =: V_{21}$. Assume further that there exist continuous function $f_\pm : I^\pm \to (0, +\infty)$

$$|f_\pm(x)| = \mathcal{O} \left( \int_0^x \text{Im } v(t) \, dt \right) \quad \text{as } x \to +\infty,$$

and, for all $n \in \{1, 2\}$,

$$|v^{(n)}(x)| = \mathcal{O}(f_\pm(x)^n |v(x)|) \quad \text{as } x \to +\infty.$$

Then, there exists a $\lambda$-dependent family $(\psi_\lambda) \subset \mathcal{D}(H_V) \setminus \{0\}$ such that

$$\frac{\| (H_V - \lambda) \Psi_\lambda \|}{\| \Psi_\lambda \|} = o(1) \quad \text{as } \lambda \to \pm \infty.$$

(1.7)

Although the matrix structure of the potential is rather simple, the assumptions of the above theorem allow us to touch a very large class of potentials. For example, the following functions for $|x| \geq 1$ (the middle part of the functions for $|x| \leq 1$ can be adjusted such that we have $v \in W_{\text{loc}}^{2,\infty}(\mathbb{R})$) are covered:

i) Polynomial-like functions $v(x) := |x|^\alpha + i \text{sgn}(x)|x|^{\gamma}$ with $\alpha \in \mathbb{R}, \gamma \geq 0$.

Here we choose $f_\pm(x) := |x|^{-1}$.

ii) Exponential functions $v(x) := e^{\alpha(x)} + i \text{sgn}(x)e^{\gamma(x)}$ with $\alpha \in \mathbb{R}, \gamma \geq 0$.

Here we choose $f_\pm(x) := |x|^{\max(\alpha, \gamma)-1}$.

iii) Superexponential functions $v(x) := i \text{sgn}(x)e^{x}$.

Here we choose $f_\pm(x) := e^x$.

**Remark 1.2.** Condition (1.6) ensures that $H_V$ is “significantly non-normal”. Indeed, if $(H_V)^*$ is the formal adjoint of $H_V$, i.e. $(H_V)^* = H_V^*$, then it is straightforward to verify (at least algebraically) that the normality $H_V(H_V)^* = (H_V)^*H_V$ holds if and only if the
The following identities

\[
\begin{align*}
\text{Im} V_{11} &= \text{Im} V_{22} = \text{constant}, \\
\text{Re} V_{12} &= \text{Re} V_{21}, \\
|\text{Im} V_{12}| &= |\text{Im} V_{21}|, \\
\text{Im} V_{12} + \text{Im} V_{21} &= \text{constant}, \\
(\text{Im} V_{12} + \text{Im} V_{21})(\text{Re} V_{22} - \text{Re} V_{11} - 2m) &= 0,
\end{align*}
\]

hold simultaneously on \( \mathbb{R} \). For matrix-valued potentials \( V \) of more general structures, the asymptotics of the sum of the imaginary parts of the diagonal components \( V_{11} \) and \( V_{22} \) should be considered instead of the imaginary part of \( v \) in (1.6), see condition (3.1) below.

1.3. Comparison between Schrödinger and Dirac pseudomodes. The Dirac setting is richer in that the perturbation \( V \) is a matrix-valued function, while it is just a scalar potential in the Schrödinger case. Let us make a brief comparison of the present results with the Schrödinger situation considered in [23] (see Assumption I below in this article and [23, Ass. I]):

a) Because of the unboundedness of the spectrum of \( H_0 \) both from below and from above, see (1.3), it is not surprising that we are able to construct pseudomodes for \( \lambda \to \pm \infty \), while just the limit \( \lambda \to +\infty \) is relevant in the Schrödinger case.

b) The regularity of the potentials has a direct influence on the decay rates of the problem (1.5) for both the Schrödinger and Dirac cases. The more regular the potential is, the stronger the rate of decay in (1.5) is obtained.

c) A version of the “non-normality condition” is imposed in the Schrödinger case as well, see [23, Cond. (3.1)]. However, as explained in Remark 1.2, an alternative condition needs to be imposed for more general structures of the matrix-valued potentials \( V \) in the Dirac case.

d) The assumption that the real part of the potential is controlled by its imaginary part in [23, Cond. (3.3)] can be completely ignored in the Dirac case. This salient feature due to the Dirac structure is explained in Remark 3.7 below.

e) Moreover, the class of functions whose derivatives are controlled by the functions is significantly extended in our assumptions. For example, the superexponential function \( e^{\nu x} \) of example iii) above does satisfy our assumption, while it is not covered by [23, Cond. (3.2)]. This more general result of the present paper is technically due to the freedom in the choice of the function \( f_{\pm} \) above, while it is fixed to the canonical choice \( f_{\pm}(x) := |x|^\nu \) with some \( \nu \in \mathbb{R} \) in [23].

1.4. Handy notations. Here we summarise some special notations which will appear regularly in the paper:

1) \( \mathbb{N}_k \), with a non-negative integer \( k \), is the set of integers starting from \( k \);
2) \( \mathbb{R}_- := (-\infty, 0) \) and \( \mathbb{R}_+ := (0, +\infty) \);
3) \( f^n \) and \( f^{(n)} \) denotes respectively the power \( n \) and the \( n \)-th derivative of a function \( f : \mathbb{R} \to \mathbb{C} \) with \( n \in \mathbb{N}_0 \);
4) We use the same symbol \( \|\cdot\| \) for \( L^2 \)-norms of both scalar- and vector-valued functions;
5) For two real-valued functions \( a \) and \( b \), we write \( a \lesssim b \) (respectively, \( a \gtrsim b \)) if there exists a constant \( C > 0 \), independent of \( \lambda \) and \( x \) (or any other relevant parameter), such that \( a \leq Cb \) (respectively, \( a \geq Cb \));
6) \( a \approx b \) if \( a \lesssim b \) and \( a \gtrsim b \);
7) \([m, n] := [m, n] \cap \mathbb{Z}\) for all \( m, n \in \mathbb{R} \);
8) \(|j| := j_1 + j_2 \) for all \( j = (j_1, j_2) \in \mathbb{N}_0^2 \).
1.5. Structure of the paper. The paper is organised as follows. In Section 2, we present a general scheme of constructing a pseudomode satisfying (1.5) for the Dirac operator by the WKB method. This scheme is applied to real λ’s in Section 3, while more general complex curves are allowed in Section 4. Many illustrative examples are considered at the end of each of the two sections.

2. WKB construction

2.1. Warming-up. Let us start the scheme of constructing the pseudomode of the Dirac operator (1.4) satisfying (1.5) by searching it in the form

$$\Psi_\lambda := \begin{pmatrix} k_1 u_\lambda \\ k_2 v_\lambda \end{pmatrix}, \quad k_1(x) := \exp \left( -i \int_0^x V_{21}(\tau) \, d\tau \right),$$

$$k_2(x) := \exp \left( -i \int_0^x V_{12}(\tau) \, d\tau \right),$$

(2.1)

where $u_\lambda, v_\lambda$ depending on $\lambda$ will be determined later. This structure of pseudomode allows us to pull out the off-diagonal terms of potential $V$ by the following step

$$(H_V - \lambda)\Psi_\lambda = \begin{pmatrix} -ik_2 \partial_x - ik_2^{(1)} + V_{12} k_2 \\ -ik_1 \partial_x - ik_1^{(1)} + V_{21} k_1 \end{pmatrix} \begin{pmatrix} u_\lambda + (V_{11} + m - \lambda) k_1 u_\lambda \\ u_\lambda + (V_{22} - m - \lambda) k_2 v_\lambda \end{pmatrix} = \begin{pmatrix} k_2 (-i \partial_x) v_\lambda + k_1 (V_{11} + m - \lambda) u_\lambda \\ k_1 (-i \partial_x) u_\lambda + k_2 (V_{22} - m - \lambda) v_\lambda \end{pmatrix}.$$

By letting one of the two components of $(H_V - \lambda)\Psi_\lambda$ be zero, for example the second one, we can compute $v_\lambda$ by $u_\lambda$:

$$v_\lambda = \frac{k_1}{k_2} \frac{(-i \partial_x) u_\lambda}{\lambda + m - V_{22}}.$$  

Here we assume that $\lambda + m - V_{22} \neq 0$ on the support of the pseudomode; this will be ensured by the condition (2.10) later. In this way, we have relaxed the problem to finding $u_\lambda$ such that

$$\frac{\| (H_V - \lambda)\Psi_\lambda \|}{\| \Psi_\lambda \|} = \frac{\| \mathcal{L}_{\lambda,V} u_\lambda \|_{L^2}}{\| k_1 u_\lambda \|_{L^2}^2 + \| k_1^{(1)} \frac{(-i \partial_x) u_\lambda}{\lambda + m - V_{22}} \|_{L^2}^2} = o(1) \quad \text{as } \lambda \to \infty.$$  

Here the differential expression $\mathcal{L}_{\lambda,V}$ is defined by

$$\mathcal{L}_{\lambda,V} := k_2 (-i \partial_x) K_\lambda (-i \partial_x) - k_1 (\lambda - m - V_{11}), \quad K_\lambda := \frac{k_1}{k_2} \frac{1}{\lambda + m - V_{22}}.$$  

Remark 2.1. In [1], in order to construct the pseudomode for a damped wave system, the authors transferred the problem into finding the pseudomode of a quadratic operator having the Schrödinger form. We, meanwhile, will establish the WKB construction directly for the Sturm–Liouville-like operator $\mathcal{L}_{\lambda,V}$ without converting it to the Schrödinger operator.

Let us consider a sufficiently regular and complex-valued function $P : \mathbb{R} \to \mathbb{C}$ which will be determined later in the WKB process. We consider the formal conjugated operator

$$\mathcal{L}_{\lambda,V}^P := e^P \mathcal{L}_{\lambda,V} e^{-P} = \frac{k_1}{\lambda + m - V_{22}} \left[ -\partial_x^2 + \left( 2P^{(1)} - \frac{K_\lambda^{(1)}}{K_\lambda} \right) \partial_x + P^{(2)} + \frac{K_\lambda^{(1)}}{K_\lambda} P^{(1)} - (P^{(1)})^2 - V_\lambda \right],$$

where

$$V_\lambda := (\lambda - m - V_{11})(\lambda + m - V_{22}).$$  

(2.2)
Let us denote
\[ R_\lambda := P^{(2)} + \frac{K^{(1)}_\lambda}{K_\lambda} P^{(1)} - (P^{(1)})^2 - V_\lambda, \quad (2.3) \]
which will play the role of the remainder in the WKB analysis.

We consider \( u_\lambda \) in the form \( u_\lambda := \xi e^{-P} \), where \( \xi \) is a cut-off function whose support is allowed to depend on \( \lambda \); it will be determined later in Section 3.2. Hence, the action of \( \mathcal{L}_{\lambda,V} \) on \( u_\lambda \) can be expressed as
\[
\mathcal{L}_{\lambda,V} u_\lambda = e^{-P} \mathcal{L}^{P}_{\lambda,V} \xi = -\frac{k_1 e^{-P}}{\lambda + m - V_{22}} \xi^{(2)} + \frac{k_1 e^{-P}}{\lambda + m - V_{22}} \left( 2P^{(1)} - \frac{K^{(1)}_\lambda}{K_\lambda} \right) \xi^{(1)} + \frac{k_1 e^{-P} R_\lambda}{\lambda + m - V_{22}} \xi. \quad (2.4)
\]
The WKB strategy is as follows. For each \( n \in \mathbb{N}_0 \), we look for the phase \( P \) in the form
\[
P_{\lambda,n}(x) = \sum_{k=-1}^{n-1} \lambda^{-k} \psi_k(x), \quad (2.5)
\]
where functions \( (\psi_k)_{k \in [-1,n-1]} \) are to be determined by solving ordinary differential equations (ODEs); the number \( n \) will be chosen later depending on the maximal possible order derivative of \( V \). After that, we show that the exponential decay of \( \psi_{-1} \) allows the norm of the first two terms in (2.4) to decay exponentially according to \( \lambda \) (Proposition 3.9) and the norm of the final term to decrease with the rate power of \( \lambda^{-1} \) (Theorem 3.10).

Starting with \( n = 0 \) and putting \( P_0 = \lambda \psi_{-1} \) into (2.3), we obtain
\[
R_{\lambda,0} := \lambda \left( \psi_{-1}^{(2)} + \frac{K^{(1)}_\lambda}{K_\lambda} \psi_{-1}^{(1)} \right) - \lambda^2 (\psi_{-1}^{(1)})^2 - V_\lambda = \lambda \left( \psi_{-1}^{(2)} + \frac{K^{(1)}_\lambda}{K_\lambda} \psi_{-1}^{(1)} \right). \quad (2.6)
\]
Here the second equality follows by solving the eikonal equation \(-\lambda^2 (\psi_{-1}^{(1)})^2 - V_\lambda = 0\); in this way, the second order of \( \lambda \) in \( R_{\lambda,0} \) is removed and the final order of \( \lambda \) has been reduced.

We can do the same trick for any \( n \in \mathbb{N}_1 \). Replacing \( P \) by \( P_{\lambda,n} \) in (2.3), we obtain
\[
\left( \sum_{k=-1}^{n-1} \lambda^{-k} \psi_k^{(2)} \right) + \frac{K^{(1)}_\lambda}{K_\lambda} \left( \sum_{k=-1}^{n-1} \lambda^{-k} \psi_k^{(1)} \right) - \lambda^{-1} \left( \sum_{k=-1}^{n-1} \lambda^{-k} \psi_k^{(1)} \right) - V_\lambda = \sum_{\ell=-2}^{n-2} \lambda^{-\ell} \phi_{\ell} + \sum_{\ell=-1}^{n-2} \lambda^{-(n+\ell)} \phi_{n+\ell}.
\]
Here the functions \( \phi_{\ell} \) with \( \ell \in [2(n-1)] \) are naturally defined by grouping together the terms attached with the same order of \( \lambda \), with the exception of \( V_\lambda \) which we include in the leading order term. In detail, the first \( n + 1 \) functions \( \phi_{\ell} \) are expressed by
\[
\lambda^2 : \quad - (\psi_{-1}^{(1)})^2 - \frac{V_\lambda}{\lambda^2} =: \phi_{-2},
\]
\[
\lambda^1 : \quad \psi_{-1}^{(2)} + \frac{K^{(1)}_\lambda}{K_\lambda} \psi_{-1}^{(1)} - 2 \psi_{-1}^{(1)} \psi_0^{(1)} =: \phi_{-1},
\]
\[
\vdots
\]
\[
\lambda^{-\ell} : \quad \psi_{\ell}^{(2)} + \frac{K^{(1)}_\lambda}{K_\lambda} \psi_{\ell}^{(1)} - \sum_{j=-1}^{\ell+1} \psi_j^{(1)} \psi_{\ell-j} =: \phi_{\ell},
\]
\[
\vdots
\]
\[
\lambda^{-(n-2)} : \quad \psi_{n-2}^{(2)} + \frac{K^{(1)}_\lambda}{K_\lambda} \psi_{n-2}^{(1)} - \sum_{j=-1}^{n-1} \psi_j^{(1)} \psi_{n-2-j} =: \phi_{n-2},
\]
and the last $n$ functions $\phi_\ell$ are

$$\lambda^{-(n-1)} : \quad \psi_{n-1}^{(2)} + \frac{K_\lambda^{(1)}}{K_\lambda} \psi_{n-1}^{(1)} - \sum_{j=0}^{n-1} \psi_j^{(1)} \psi_{n-1-j}^{(1)} = : \phi_{n-1},$$

$$\lambda^{-n} : \quad - \sum_{j=1}^{n-1} \psi_j^{(1)} \psi_{n-j}^{(1)} = : \phi_n,$$

$$\vdots$$

$$\lambda^{-(n+\ell)} : \quad - \sum_{j=\ell+1}^{n-1} \psi_j^{(1)} \psi_{n+\ell-j}^{(1)} = : \phi_{n+\ell}, \quad \text{for } \ell \in \{0, n-2\} \text{ if } n \geq 2,$$

$$\vdots$$

$$\lambda^{-2(n-1)} : \quad - \left( \psi_{n-1}^{(1)} \right)^2 = : \phi_{2(n-1)}. \quad (2.7)$$

Requiring $\phi_\ell = 0$ for all $\ell \in \{[-2, n-2]\}$, we obtain $(n+1)$ ODEs which can be solved explicitly to find all $\{\psi_k\}_{k \in [-1, n-1]}$ by a recursion formula

$$\psi_{-1}^{(1)} = \pm i \lambda^{-1} V_\lambda^{1/2},$$

$$\psi_{\ell+1}^{(1)} = \frac{1}{2 \psi_{-1}^{(1)}} \left( \psi_\ell^{(2)} + \frac{K_\lambda^{(1)}}{K_\lambda} \psi_\ell^{(1)} - \sum_{j=0}^\ell \psi_j^{(1)} \psi_{\ell-j}^{(1)} \right). \quad (2.8)$$

After solving these ODEs, the WKB remainder is

$$R_{\lambda,n} := \sum_{\ell=-1}^{n-2} \lambda^{-(n+\ell)} \phi_{n+\ell}, \quad n \in \mathbb{N}_1. \quad (2.9)$$

**Remark 2.2.** Let us make some comments at this stage.

i) The choice of the sign in the definition of $\psi_{-1}^{(1)}$ will be determined by the sign of the sum $\text{Im} V_{11} + \text{Im} V_{22}$ at infinity and the sign of $\lambda$ (see Remark 3.12).

ii) Since $V_\lambda$ is a complex-valued function, the square root appearing in (2.8) is considered as the principal branch of the square root which is defined as, for $z \in \mathbb{C} \setminus (\infty, 0]$,

$$\sqrt{z} = \frac{1}{\sqrt{2}} \left( |z| + \text{Re } z \right)^{1/2} + i \frac{1}{\sqrt{2}} \left( |z| + \text{Re } z \right)^{1/2}. \quad \text{Im } z$$

This principal branch of square root is holomorphic on $\mathbb{C} \setminus (\infty, 0]$. For that reason, the range of $V_\lambda$ needs to stay away from $(-\infty, 0]$ such that the continuity (and/or the differentiability) of $V_\lambda^{1/2}$ is deduced by the continuity (and/or the differentiability) of $V_{11}$ and $V_{22}$. By writing $\lambda := \alpha + i \beta$ ($\alpha, \beta \in \mathbb{R}$), this will be ensured if, by simple argument of product of two complex numbers,

$$(\alpha - m - \text{Re } V_{11}(x))(\alpha + m - \text{Re } V_{22}(x)) > 0, \quad (2.10)$$

for all $x \in \mathbb{R}$. If $\text{Re } V_{11}$ and $\text{Re } V_{22}$ are bounded, by considering $\alpha$ very large, (2.10) is always satisfied. Otherwise, we need to employ the support of $\xi$ so as to make (2.10) happen.

**Remark 2.3** (Beyond semiclassical). It is a common knowledge that the limit of “large energies” in quantum mechanics is related to the “semiclassical limit”. While we indeed consider the spectral parameter $\lambda$ in (1.5) diverging in the complex plane and employ WKB analysis standardly used for semiclassical regimes too (see, e.g., [14] for the Schrödinger
operator or \[17\] for the magnetic Laplacian), there are some important novelties in our approach that we list here:

i) Our spectral parameter \(\lambda \in \mathbb{C}\) not only plays the scaling role as semi-classical parameter \(h \in \mathbb{R}_+\), but also indicates the direction in which the large complex number will belong to the pseudospectrum.

ii) In the original WKB method applied to the spectral problems in the semi-classical regime, the solutions of the eikonal equation and the transports equations are independent of the semi-classical parameter. On the other hand, our solutions depend on the parameter \(\lambda\) and all estimates established for \((\psi_k)_{k \in [-1,n]}\) are necessary to be uniform in \(\lambda\). This makes the present analysis considerably more demanding. However, both the WKB strategies share the same scheme that the eikonal solution plays a dominant part in deciding the decay of the main problem (the problem \([1.5]\) in this case).

iii) While the semi-classical quasimodes always localise, the supports of our pseudomodes can be extended in some cases. Furthermore, the cut-off functions are occasionally needless in our WKB construction.

iv) While semi-classical works deal with smooth potentials, our framework can cover the potentials with low regularity (possibly discontinuous).

In summary, the present work goes beyond standard semiclassical settings. What is more, our approach is more robust in the sense that semiclassical results can be deduced as a consequence of it (cf. \[23\] Ex. 5.4), but not vice versa (without the important developments mentioned above).

2.2. Structure of solutions of the transport equations and the WKB remainder. From now on, we assume that we are dealing with the plus sign in the formula of \(\psi_{-1}\) in \([2.8]\), unless otherwise stated. Let us list some first solutions of the first transport equations to see which structure they are equipped with:

\[
\begin{align*}
\psi_{-1}^{(1)} &= \frac{iV^{1/2}_\lambda}{\lambda}, \\
\psi_0^{(1)} &= \frac{1}{4} \frac{V^{(1)}_\lambda}{V_\lambda} + \frac{1}{2} \frac{K^{(1)}_\lambda}{K_\lambda}, \\
\psi_1^{(1)} &= \frac{-i\lambda}{8V^{1/2}_\lambda} \left( \frac{V^{(2)}_\lambda}{V_\lambda} - \frac{5 (V^{(1)}_\lambda)^2}{4V^2_\lambda} + \frac{2 K^{(2)}_\lambda}{K_\lambda} - \frac{(K^{(1)}_\lambda)^2}{K^2_\lambda} \right), \\
\psi_2^{(1)} &= \frac{-\lambda^2}{16V^3_\lambda} \left[ \frac{V^{(3)}_\lambda}{V_\lambda} - \frac{9V^{(1)}_\lambda V^{(2)}_\lambda}{2V^2_\lambda} + \frac{15 (V^{(1)}_\lambda)^3}{4V^3_\lambda} - \frac{V^{(1)}_\lambda}{V_\lambda} \left( \frac{2 K^{(2)}_\lambda}{K_\lambda} - \frac{(K^{(1)}_\lambda)^2}{K^2_\lambda} \right) \\
&\quad + \frac{2 K^{(3)}_\lambda}{K_\lambda} - \frac{4 K^{(1)}_\lambda K^{(2)}_\lambda}{K^2_\lambda} - \frac{2 (K^{(1)}_\lambda)^3}{K^3_\lambda} \right].
\end{align*}
\]

The remainder when we solve up to \(\psi_{-1}\):

\[
R_{\lambda,0} = \frac{iV^{(1)}_\lambda}{2V^{1/2}_\lambda} + \frac{iK^{(1)}_\lambda}{K_\lambda} V^{1/2}_\lambda.
\]

The remainder when we solve up to \(\psi_0\):

\[
R_{\lambda,1} = \frac{V^{(2)}_\lambda}{4V_\lambda} - \frac{5 (V^{(1)}_\lambda)^2}{16V^2_\lambda} + \frac{K^{(2)}_\lambda}{2K_\lambda} - \frac{(K^{(1)}_\lambda)^2}{4K^2_\lambda}.
\]
The remainder when we solve up to $\psi_1$:

$$R_{\lambda,2} = \frac{-i}{8V_{\lambda}^{1/2}} \left[ \frac{V_{\lambda}^{(3)}}{V_{\lambda}} \cdot \frac{9 V_{\lambda}^{(1)} V_{\lambda}^{(2)}}{2 V_{\lambda}^2} + \frac{15 (V_{\lambda}^{(1)})^3}{4 V_{\lambda}^3} - \frac{V_{\lambda}^{(1)}}{V_{\lambda}} \left( 2 \frac{K_{\lambda}^{(2)}}{K_{\lambda}} - \frac{(K_{\lambda}^{(1)})^2}{K_{\lambda}^2} \right) + 2 \frac{K_{\lambda}^{(3)}}{K_{\lambda}} \right]$$

$$- 4 \frac{K_{\lambda}^{(1)} K_{\lambda}^{(2)}}{K_{\lambda}^2} - 2 \left( \frac{(K_{\lambda}^{(1)})^3}{K_{\lambda}^2} \right) + \frac{1}{64V_{\lambda}} \left( \frac{V_{\lambda}^{(2)}}{V_{\lambda}} - \frac{5 (V_{\lambda}^{(1)})^2}{4 V_{\lambda}^2} + 2 \frac{K_{\lambda}^{(2)}}{K_{\lambda}} - \frac{(K_{\lambda}^{(1)})^2}{K_{\lambda}^2} \right)^2.$$  

For $n \in \mathbb{N}_1$, since the formulae of solutions $(\psi_k)_{k \in [0,n-1]}$ are obtained from the recursion steps (2.8), if we want to write the formula of $\psi_j$, the formulae of all $\psi_\ell$ with $\ell \leq k-1$ need to be explicitly given. It could be a challenging effort to find out the exact formulae for the transport solutions. However, the good news is that these solutions can be estimated without knowing their exact expression, instead a common structure of them is required. This is the content of the following lemma, but first, some notations should be introduced.

**Notation 2.4.** Let $f, g$ be two functions which are assumed to be sufficiently regular so that all appearing derivatives of them exist. For $j = (j_1, j_2) \in \mathbb{N}_0^2$, $r = (r_1, r_2) \in \mathbb{N}_0^2$ and $s = (s_1, s_2) \in \mathbb{N}_1^2$, we employ the following notations

$$D_j^{r,s}(f, g) := \left\{ \sum_{\alpha \in T_j^{r,s}} c_{\alpha} (f^{(1)})^{\alpha_1} \ldots (f^{(s_1)})^{\alpha_{s_1}} (g^{(1)})^{\alpha_{s_1+1}} \ldots (g^{(s_2)})^{\alpha_{s_1+s_2}} : c_{\alpha} \in \mathbb{C} \right\},$$  

(2.11)

where

$$T_j^{r,s} := \left\{ \alpha \in \mathbb{N}_0^{s_1+s_2} : \sum_{p=1}^{s_1} \alpha_p = j_1, \sum_{p=1}^{s_2} \alpha_{s_1+p} = j_2; \sum_{p=1}^{s_1} p \alpha_p = r_1; \sum_{p=1}^{s_2} p \alpha_{s_1+p} = r_2 \right\}.$$  

(2.12)

When $T_j^{r,s} = \emptyset$, we make a convention that $D_j^{r,s}(f, g) = \{0\}$. Thus, if $j_i = 0$ and $r_i \geq 1$ for some $i \in \{1,2\}$, then $D_j^{r,s}(f, g) = \{0\}$.

**Lemma 2.5.** Let $n \in \mathbb{N}_0$, assume that $V_{11}, V_{22} \in W_{\text{loc}}^{n+1,2}(\mathbb{R})$ and $V_{12}, V_{21} \in W_{\text{loc}}^{n,2}(\mathbb{R})$ and $\lambda \in \mathbb{C}$ are such that (2.10) is satisfied. Let $\{\psi_k^{(1)}\}_{k \in [-1,n-1]}$ be a family determined by the formula (2.8). Then their first order derivatives are of the form

$$\psi_k^{(1)} = \lambda_k^{k} V_{\lambda}^{k/2} \sum_{|j|=0}^{k+1} \sum_{|r|=-k+1} d_j^{r,j+(1,1)}(V_{\lambda}, K_{\lambda}) V_{\lambda}^{j_1} K_{\lambda}^{j_2},$$

where $\{r, j\} \subset \mathbb{N}_0^2$ and $d_j^{r,j+(1,1)}(V_{\lambda}, K_{\lambda}) \in D_j^{r,j+(1,1)}(V_{\lambda}, K_{\lambda})$.

For each $k \geq 0$, the maximal possible order derivative of $V_{11}, V_{22}$ in $\psi_k'$ is $k+1$ and the maximal possible order derivative of $V_{12}$ and $V_{21}$ in $\psi_k^{(1)}$ is $k$. Indeed, notice that, from the definition of $V_{\lambda}$ and $K_{\lambda}$, the levels of the derivatives of $V_{\lambda}$ and $K_{\lambda}$ are equal to the levels of the derivatives of $V_{11}$ and $V_{22}$ while larger than the levels of the derivatives of $V_{12}$ and $V_{21}$ by one order. On the other hand, for all $i \in \{1,2\}$ with $j_i \geq 1$ and $r_i \leq k + 1$, we get

$$\max\{r_i - j_i + 1\} \leq k + 1.$$  

The remainders are controlled by the next lemma.

**Lemma 2.6.** Let $n \in \mathbb{N}_0$, assume that $V_{11}, V_{22} \in W_{\text{loc}}^{n+1,2}(\mathbb{R})$ and $V_{12}, V_{21} \in W_{\text{loc}}^{n,2}(\mathbb{R})$ and $\lambda \in \mathbb{C}$ are such that (2.10) is satisfied. For $n = 0$, let $R_{\lambda,0}$ as in (2.6). For $n \geq 1$, let $\{\psi_k^{(1)}\}_{k \in [-1,n-1]}$ be a family determined by the formula (2.8), $\{\phi_k\}_{k \in [-n,2(n-1)]}$ as in (2.7).
and $R_{\lambda,n}$ as in (2.9). Then the maximal possible order derivative of $V_{11}, V_{22}$ is $n + 1$ and of $V_{12}, V_{21}$ is $n$ in $R_{\lambda,n}$. and

$$
  |R_{\lambda,0}| \lesssim \frac{|V_{\lambda}^{(1)}|}{|V_{\lambda}|^{1/2}} + \frac{|K_{\lambda}^{(1)}|}{|K_{\lambda}|},
$$

for $n \geq 1$:

$$
  |R_{\lambda,n}| \lesssim \sum_{\ell=1}^{n-2} \frac{1}{|V_{\lambda}|^{(n+\ell)/2}} \sum_{j=1}^{n+\ell+2} \frac{|d_j^{r(n+1,n+1)}(V_{\lambda}, K_{\lambda})|}{|V_{\lambda}|^{1/2} |K_{\lambda}|^{1/2}},
$$

(2.13)

where $\{r, j\} \subset \mathbb{N}_0^2$ and $d_j^{r(n+1,n+1)}(V_{\lambda}, K_{\lambda}) \in D_j^{r(n+1,n+1)}(V_{\lambda}, K_{\lambda})$.

Remark 2.7. At the end of this section, we want to show that choosing the shape of the pseudomode for the Dirac operator also plays important role technically. From the beginning, if we choose the basic form $\Psi_{\lambda} = (v_{\lambda})$ and insert it to the eigenvalue equation $(H_{V} - \lambda)\Psi_{\lambda} = 0$, we will have to deal WKB with the electromagnetic-like Schrödinger operator

$$
  \tilde{L}_{\lambda,V} = (-i\partial_x + V_{12}) \frac{1}{\lambda + m - V_{22}} (-i\partial_x + V_{21}) - (\lambda - m - V_{11}).
$$

Then its formal conjugated operator is described as follows

$$
  \tilde{L}_{\lambda,V}^P := e^P \tilde{L}_{\lambda,V} e^{-P}
$$

$$
  = \frac{k_1}{\lambda + m - V_{22}} \left[ -\partial_x^2 + \left( 2P^{(1)} - \frac{V_{22}^{(1)}}{\lambda + m - V_{22}} - i(V_{12} + V_{21}) \right) \partial_x 
  + P^{(2)} + \left( \frac{V_{22}^{(1)}}{\lambda + m - V_{22}} + i(V_{12} + V_{21}) \right) P^{(1)} - (P^{(1)})^2 - \tilde{V}_{\lambda} \right],
$$

where

$$
  \tilde{V}_{\lambda} = (\lambda - m - V_{11})(\lambda + m - V_{22}) - V_{12}V_{21} + iV_{21}^{(1)} + iV_{21} \frac{V_{22}^{(1)}}{\lambda + m - V_{22}}.
$$

We see that this form of $\tilde{V}_{\lambda}$ is very complicated to consider its square-root. Furthermore, by solving some firsts transport equations, we recognize that the sum $(V_{12} + V_{21})$ attached with $P^{(1)}$ will destroy the structure of solutions of transport equations. These difficulties make our WKB analysis unusable. Therefore, multiplying $u_{\lambda}$ and $v_{\lambda}$ with respectively, $k_1$ and $k_2$ not only gauges out $V_{12}$ and $V_{21}$, but also allows this method to be workable.

3. Pseudomodes for $\lambda \to \pm \infty$

Let us recall here the picture of the Schrödinger operators to compare and outline the direction for the Dirac operators simultaneously. It is well known that the spectrum of the free Schrödinger operator (i.e. the Laplacian in $L^2(\mathbb{R})$ with domain being the Sobolev space $H^2(\mathbb{R})$) is given by the set $[0, +\infty)$. In [23], when the pseudoeigenvalue $\lambda$ is real, the pseudomode of the Schrödinger operator with the complex-valued potential are constructed successfully when $\lambda$ is positive and very large. Now, as mentioned in Section 1.2, the spectrum of the free Dirac operator is a set which is symmetric through the origin, see (1.3). Therefore, this evokes that the construction of the pseudomode for the Dirac operator $H_V$ for the positive and negative $\lambda$'s can be established.

This expectation is also supported by looking at the structure of the WKB construction in both cases, especially the solution of eikonal equation which depends on the square root
of $V_\lambda$:

**In the Schrödinger case:** $V_\lambda^{\text{Schrödinger}} := \lambda - V_\text{Schrödinger}$.

**In the Dirac case:** $V_\lambda^{\text{Dirac}} := (\lambda - m - V_{11})(\lambda + m - V_{22})$.

Here, $V_\text{Schrödinger}$ denotes the scalar potential in the Schrödinger operator. Assume that the real part of $V_\text{Schrödinger}$ (respectively, real parts of $V_{11}$ and $V_{22}$) in the Schrödinger (respectively, Dirac) case is (respectively, are) bounded. Then the principal branch of the square root is well-defined only when $\lambda \to +\infty$ in the Schrödinger case, while it is also able to be valid when $\lambda \to -\infty$ in the Dirac case.

However, in this section, unless otherwise stated, we always assume that $\lambda$ is positive. The case of negative $\lambda$ can be considered analogously (see Remark 3.12 below).

### 3.1. General shapes of the potentials.

Let us denote

$$F(x) := \int_0^x (\text{Im} V_{11}(t) + \text{Im} V_{22}(t)) \, dt.$$  

Our main hypothesis reads as follows.

**Assumption I.** Let $N \in \mathbb{N}_0$, assume that $V_{11}, V_{22} \in W^{N+1,\infty}_{\text{loc}}(\mathbb{R})$, $V_{12}, V_{21} \in W^{N,\infty}_{\text{loc}}(\mathbb{R})$ and there exist $a_\pm > 0$, by denoting $I^\pm := \{x \in \mathbb{R}_\pm : |x| > a_\pm \}$, such that

1) the sum of diagonal terms of $V$ has a different asymptotic behaviour at $\pm \infty$:

$$\begin{align*}
\text{Im} V_{11} + \text{Im} V_{22}(x) &\lesssim -1, \quad \forall x \in I^-,
\text{Im} V_{11} + \text{Im} V_{22}(x) &\gtrsim 1, \quad \forall x \in I^+,
\end{align*}$$

and there exist $\mu_\pm \in (0, 1]$ such that

$$\left|\text{Im} V_{11}(x) + \text{Im} V_{22}(x)\right| \geq \mu_\pm \left(\left|\text{Im} V_{11}(x)\right| + \left|\text{Im} V_{22}(x)\right|\right), \quad \forall x \in I^\pm;$$

2) the primitive of the sum of off-diagonal terms $U := \text{Im} V_{12} + \text{Im} V_{21}$ is controlled by $F$ at $\pm \infty$: there exist $\varepsilon_\pm \in \left(0, \frac{\mu_\pm}{2}\right)$ such that

$$\int_0^x U(t) \, dt \leq 2\varepsilon_\pm F(x), \quad \forall x \in I^\pm;$$

3) there exist continuous functions $f_\pm : I^\pm \to \mathbb{R}_+$ such that

$$f_\pm(x) \lesssim F(x), \quad \forall x \in I^\pm,$$

and, for all $i \in \{1, 2\}$,

$$\begin{align*}
\forall n \in [1, N+1], \quad &|V_{ii}^{(n)}(x)| \lesssim f_\pm(x)^n |V_{ii}(x)|, \quad \forall x \in I^\pm, \\
\forall n \in [0, N], \quad &|(V_{21} - V_{12})^{(n)}(x)| \lesssim f_\pm(x)^{n+1}, \quad \forall x \in I^\pm.
\end{align*}$$

Notice that the first condition (3.1) implies that

$$F(x) \gtrsim |x|, \quad \forall |x| \gtrsim 1.$$  

Next lines gather some comments on Assumption I. Let us recall the expression of $\lambda \psi_{-1}^{(1)}$ when $\lambda \in \mathbb{R}$:

$$\text{Re} \left(\lambda \psi_{-1}^{(1)}(x)\right) = \frac{1}{\sqrt{2}} \frac{\text{Im} V_{11}(\lambda + m - \text{Re} V_{22}) + \text{Im} V_{22}(\lambda - m - \text{Re} V_{11})}{\sqrt{|V_\lambda| + |V_\lambda|}}.$$  

We will see later that the shape of the pseudomode depends a lot on $\psi_{-1}$ and the sign of $\text{Im} V_{11} + \text{Im} V_{22}$ (which is attached with very large $\lambda$) will decide the sign for the decay of the pseudomode. The larger the sum is, the faster the pseudomode decreases at infinity (see the proof of Proposition 3.9). Furthermore, by looking at Remark 1.2, the assumption (3.1) also ensures that the operator defined in (1.4) is “significantly non-self-adjoint”.


From the conditions (3.2), we deduce the similarity of the sum of absolute values and the absolute value of the sum of $\Im V_{11}$ and $\Im V_{22}$ in the neighbourhood of infinity:

$$|\Im V_{11}(x) + \Im V_{22}(x)| \approx |\Im V_{11}(x)| + |\Im V_{22}(x)|, \quad \forall x \in I^\pm. \quad (3.9)$$

If $\Im V_{11}$, $\Im V_{22}$ have the same signs at $+\infty$ (or $-\infty$), the condition (3.2) is obviously satisfied. Thus, these conditions guarantee that the opposite signs of $\Im V_{11}$ and $\Im V_{22}$ does not spoil the decay of the quasimode. As for the condition (3.3), it is easy to find $V_{12}$ and $V_{21}$ that can verify this. Indeed, since $F(x)$ is positive at $\pm \infty$, the class of all functions $V_{12}$ and $V_{21}$ such that $U(x) \geq 0$ for all $x \leq -1$ and $U(x) \leq 0$ for all $x \geq 1$ will fulfill (3.3) completely.

The conditions (3.1), (3.2) and (3.3) of Assumption I combine together to ensure the exponential decay of all the terms attached with $\xi$ completely.

3.2. Shapes of the cut-off functions. The role of the cut-off functions in the construction of quasimodes is very important. Not all functions which are created from the WKB method would become the quasimodes for the operator, since most of them do not belong to the domain of the operator. Therefore, the cut-off functions are added to complete this task. Furthermore, as discussed in Remark 2.2, when $V_{11}$ and $V_{22}$ are differentiable, in order to make $V_{1/2}$ well-defined (i.e. non-multi-valued) and differentiable, the condition (2.10) need to be satisfied. It is obvious that (2.10) will be broken if $\Re V_{11}$ or $\Re V_{22}$ is not bounded. Thus, it is necessary to employ a suitable cut-off function whose support allows (2.10) to occur.

Let us denote by $\xi : \mathbb{R} \to [0, 1]$ the cut-off function satisfying the following properties

$$\begin{cases} \xi \in C^\infty(\mathbb{R}), \\ \xi(x) = 1, \quad \forall x \in (-\delta_\lambda^- + \Delta_\lambda^-, \delta_\lambda^+ - \Delta_\lambda^+), \\ \xi(x) = 0, \quad \forall x \in \mathbb{R} \setminus (-\delta_\lambda^-, \delta_\lambda^+), \end{cases} \quad (3.10)$$

where $\delta_\lambda^\pm$ and $\Delta_\lambda^\pm < \delta_\lambda^\pm$ are $\lambda$-dependent positive numbers which will be determined later. Notice that the cut-off $\xi$ can be selected in such a way that

$$\|\xi^{(j)}\|_{L^\infty(\mathbb{R}_\pm)} \lesssim (\Delta_\lambda^\pm)^{-j}, \quad j \in \{1, 2\}. \quad (3.11)$$

To simplify the notation, we define the following sets

$$J_\lambda^- := (-\delta_\lambda^-, 0], \quad J_\lambda^+ := [0, \delta_\lambda^+), \quad \tilde{J}_\lambda^- := (-\delta_\lambda^- + \Delta_\lambda^-, 0], \quad \tilde{J}_\lambda^+ := [0, \delta_\lambda^+ - \Delta_\lambda^+), \quad \text{and} \quad J_\lambda := J_\lambda^- \cup J_\lambda^+.$$

The next lemma is set up to define the boundary of the cut-off functions.

**Lemma 3.1.** Let $a > 0$ and let $g : [a, +\infty) \to [0, +\infty)$ be a continuous function and let $\lambda$ be a positive number; we define

$$\delta(\lambda) := \inf \{x \geq a : g(x) = \lambda\}. \quad (3.12)$$

Then $\delta(\lambda)$ can be infinite (inf $\emptyset = +\infty$), however, when $g$ is unbounded at $+\infty$ and for all sufficiently large $\lambda > 0$, the number $\delta(\lambda)$ is finite and

$$\lim_{\lambda \to +\infty} \delta(\lambda) = +\infty. \quad (3.13)$$

Furthermore, if $\lambda > g(a)$ then

$$g(x) \leq \lambda, \quad \forall x \in [a, \delta(\lambda)]. \quad (3.14)$$
Proof. When \( g \) is unbounded at \(+\infty\) and \( \lambda > \min_{x \geq a} g(x) \), the number \( \delta(\lambda) \) is finite. Given arbitrary \( M > a \), we consider \( \lambda \geq \max_{[a,M]} g(x) + 1 \), then \( \delta(\lambda) \geq M \), thus the unboundedness of \( \delta(\lambda) \) is checked. In order to prove (3.14) under the assumption that \( \lambda > g(a) \), we assume opposite that there exists \( x_0 \in [a, \delta(\lambda)] \) such that \( g(x_0) > \lambda \), then by the intermediate value theorem, there exists \( \tilde{x}_0 \in (a, x_0) \) such that \( g(\tilde{x}_0) = \lambda \). This implies that \( \tilde{x}_0 \geq \delta(\lambda) \) which is a contradiction. \( \square \)

By using Lemma 3.1, we introduce the boundary of the cut-off functions

\[
\delta^\pm_\lambda := \inf \{ x \geq a_\pm : g_\pm(x) = \lambda \} \tag{3.15}
\]

through defining functions \( g_\pm : [a_\pm, +\infty) \rightarrow [0, +\infty) \) as follows

\[
g_\pm(x) := \max \left\{ \frac{1}{\eta} |\text{Re} V_{11} + m|, \frac{1}{\eta} |\text{Re} V_{22} - m|, \frac{1}{\eta} |\text{Im} V_{11} - \text{Im} V_{22}|, f_{\pm}^{1/2} \right\} (\pm x). \tag{3.16}
\]

Here, \( \varepsilon_1, \eta \) are fixed numbers such that \( 0 < \varepsilon_1 < 1 \) and \( 0 < \eta < \min \{ \mu_-, \mu_+ \} \), in which \( \eta \) will be chosen small enough later in Lemma 3.8

Remark 3.2. The continuity of \( g_\pm \) will be given by the continuities of \( V_{11}, V_{22} \) (since they belong to \( W_{1,\infty}^1(\mathbb{R}) \)) and of \( f_\pm \). Note that, when \( g_+ \) is bounded at \(+\infty\), \( i.e. \), all the functions \( \text{Re} V_{11}, \text{Re} V_{22}, |\text{Im} V_{11} - \text{Im} V_{22}| \) and \( f_+ \) are bounded at \(+\infty\), we have \( \delta^+_\lambda = +\infty \) for all sufficiently large \( \lambda > 0 \). In this case, we want to say that \( \xi \) is constant on the positive side, \( i.e. \) \( \xi(x) = 1 \) for all \( x \geq 0 \). This remark is also the same for \( g_- \) for the negative axis. In other words, sometimes we may not need the cut-off functions to localize the pseudomode.

When \( \delta^\pm_\lambda \) is finite, we define

\[
\Delta^\pm_\lambda := \frac{1}{\delta^\pm_\lambda}. \tag{3.17}
\]

Proposition 3.3. There exists \( \lambda_0 > 0 \) such that for all \( \lambda > \lambda_0 \) and for all \( x \in J_\lambda \), we have

i) \[
(1 - \eta)\lambda \leq \lambda - m - \text{Re} V_{11}(x) \leq (1 + \eta)\lambda,
\]

\[
(1 - \eta)\lambda \leq \lambda + m - \text{Re} V_{22}(x) \leq (1 + \eta)\lambda,
\]

\[
|\text{Im} V_{11}(x) - \text{Im} V_{22}(x)| \leq \eta\lambda; \tag{3.18}
\]

ii) \[
|\lambda - m - V_{11}(x)| \approx |\lambda + m - V_{22}(x)|. \tag{3.19}
\]

Proof. In case \( \text{Re} V_{11}, \text{Re} V_{22} \) and \( \text{Im} V_{11} - \text{Im} V_{22} \) are bounded at infinity, it is easy to check the above estimates. Now we assume that the unboundedness of \( \text{Re} V_{11} \) or \( \text{Re} V_{22} \) or \( \text{Im} V_{11} - \text{Im} V_{22} \) at \(+\infty\) occurs. The case of unboundness at the negative infinity is analogous. It follows from the estimate (3.14) that, for all \( x \in J_\lambda^\pm \),

\[
\begin{cases}
|\text{Re} V_{11}(x) + m| \leq \eta\lambda, \\
|\text{Re} V_{22}(x) - m| \leq \eta\lambda, \\
|\text{Im} V_{11}(x) - \text{Im} V_{22}(x)| \leq \eta\lambda.
\end{cases}
\]

Consequently, the three estimates in (3.18) follow. From them, we deduce that

\[
\frac{|\lambda - m - V_{11}(x)|}{|\lambda + m - V_{22}(x)|} \lesssim \frac{|\lambda - m - \text{Re} V_{11}(x)| + |\text{Im} V_{11}|}{|\lambda + m - \text{Re} V_{22}(x)| + |\text{Im} V_{22}|} \leq \frac{|\lambda - m - \text{Re} V_{11}(x)| + |\text{Im} V_{11} - \text{Im} V_{22}| + |\text{Im} V_{22}|}{|\lambda + m - \text{Re} V_{22}(x)| + |\text{Im} V_{22}|} \lesssim \frac{(1 + 2\eta)\lambda + |\text{Im} V_{22}|}{(1 - \eta)\lambda + |\text{Im} V_{22}|} \leq \frac{1 + 2\eta}{1 - \eta}.
\]
Thus, \(|\lambda - m - V_{11}(x)| \lesssim |\lambda + m - V_{22}(x)|\). The other direction is proved analogously, therefore the second estimate (3.19) is verified. □

3.3. Auxiliary steps. The next lemma shows us that \(V_\lambda\) and \(K_\lambda\) inherit the properties of \(V\) in (3.5) and (3.6).

Lemma 3.4. Let \(N \in \mathbb{N}_0\), assume that \(V_{11}, V_{22} \in W^{N+1,\infty}_\text{loc}(\mathbb{R})\) and \(V_{12}, V_{21} \in W^{N,\infty}_\text{loc}(\mathbb{R})\) satisfy the assumptions (3.5) and (3.6). There exists \(\lambda_0 > 0\) such that, for all \(\lambda > \lambda_0\) and for all \(\ell \in [1, N+1]\), we have

- on \(I^\pm \cap J^\pm_\lambda\), \(|V_\lambda^{(\ell)}| \lesssim |V_\lambda| \frac{f^\ell_\pm \max \{|V_{11}|, |V_{22}|\}}{|\lambda + m - V_{22}|}, |K^{(\ell)}_\lambda| \lesssim |K_\lambda| f^{\ell}_\pm,\)

  and

- on \([-a_-, a_+]\), \(|V_\lambda^{(\ell)}| \lesssim |V_\lambda| \frac{f^\ell_\pm}{|\lambda + m - V_{22}|}, |K^{(\ell)}_\lambda| \lesssim |K_\lambda|\).

Furthermore, if \(V_{12} = V_{21}\), we have

- on \(I^\pm \cap J^\pm_\lambda\), \(|K^{(\ell)}_\lambda| \lesssim |K_\lambda| \frac{f^\ell_\pm |V_{22}|}{|\lambda + m - V_{22}|},\)

  on \([-a_-, a_+]\), \(|K^{(\ell)}_\lambda| \lesssim |K_\lambda| \frac{f^\ell_\pm}{|\lambda + m - V_{22}|}.

Proof. We can choose \(\lambda_0 > 0\) satisfying \(\delta^\pm > a_+\) for all \(\lambda > \lambda_0\), thanks to (3.13). Then, \(I^\pm \cap J^\pm_\lambda \neq \emptyset\). From the formula of \(V_\lambda\), the general Leibniz rule for the \(\ell\)-th derivative of the product yields that

\[
(V_\lambda)^{(\ell)} = \sum_{k=0}^{\ell} \binom{\ell}{k} (\lambda - m - V_{11})^{(k)}(\lambda + m - V_{22})^{(\ell-k)}
\]

\[
= -(\lambda - m - V_{11})V_{22}^{(\ell)} + \sum_{k=1}^{\ell-1} \binom{\ell}{k} V_{11}^{(k)}V_{22}^{(\ell-k)} - (\lambda + m - V_{22})V_{11}^{(\ell)}.
\]

From the assumption (3.5), we obtain the estimate on \(I^\pm \cap J^\pm_\lambda\),

\[
\frac{|V_\lambda^{(\ell)}|}{|V_\lambda|} \lesssim \frac{f^\ell_\pm |V_{22}|}{|\lambda + m - V_{22}|} + \sum_{k=1}^{\ell-1} \frac{f^\ell_\pm |V_{11}| |V_{22}|}{|\lambda - m - V_{11}| |\lambda + m - V_{22}|} + \frac{f^\ell_\pm |V_{11}|}{|\lambda - m - V_{11}|}
\]

\[
\lesssim \frac{f^\ell_\pm \max \{|V_{11}|, |V_{22}|\}}{|\lambda + m - V_{22}|},
\]

where in the last step, we used (3.19) and the fact that (with some large \(\lambda_0\),

\[
\frac{|V_{11}|}{|\lambda - m - V_{11}|} \lesssim \frac{|\Re V_{11}| + |\Im V_{11}|}{|\lambda - m - \Re V_{11}| + |\Im V_{11}|} \leq \frac{\eta \lambda + m + |\Im V_{11}|}{(1 - \eta) \lambda + |\Im V_{11}|} \lesssim 1.
\]

Next, we prove the estimate for \(K_\lambda\). Let us recall that

\[
K_\lambda = \frac{1}{\lambda + m - V_{22}} e^x, \quad \text{with } u(x) := -i \int_0^x (V_{21} - V_{12})(\tau) d\tau.
\]

The Leibniz rule also leads us to

\[
(K_\lambda)^{(\ell)} = \sum_{k=0}^{\ell} \binom{\ell}{k} \left(\frac{1}{\lambda + m - V_{22}}\right)^{(k)} (e^x)^{(\ell-k)}.
\]
Proof. Using Faà di Bruno’s formula for the derivative of a composition of two functions (see [26]),

\[
\left( \begin{array}{c}
\frac{1}{\lambda + m - V_{22}}
\end{array} \right)^{(k)} = \frac{1}{\lambda + m - V_{22}} \sum_{1+2\alpha_2+\cdots+k\alpha_k=k} \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_k!} \prod_{j=1}^{k} \left( \frac{V_{22}^{(j)}}{j!(\lambda + m - V_{22})} \right)^{\alpha_j},
\]

\[
(e^{u})^{(\ell-k)} = e^{u} \sum_{1+2\beta_2+\cdots+(\ell-k)\beta_{\ell-k} = \ell-k} \frac{(\ell-k)!}{\beta_1! \beta_2! \cdots \beta_{\ell-k}!} \prod_{j=1}^{\ell-k} \left( \frac{(u^{(j)})^{\beta_j}}{j!} \right),
\]

where \((\alpha_j)_{1 \leq j \leq k}\) and \((\beta_j)_{1 \leq j \leq \ell-k}\) are non-negative integers.

From the assumption (3.5) for \(V_{22}\) and (3.6), we obtain the estimate on \(I^{\pm} \cap J_{\lambda}^{\pm}\),

\[
|K_{\lambda}^{(\ell)}| \lesssim \sum_{k=0}^{\ell} |K_{\lambda}| \left( \sum_{1+2\alpha_2+\cdots+k\alpha_k = k} \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_k!} \prod_{j=1}^{k} \left( \frac{|V_{22}| f_{\pm}^{j}}{\lambda + m - V_{22}} \right)^{\alpha_j} \right)
\times \left( \sum_{1+2\beta_2+\cdots+(\ell-k)\beta_{\ell-k} = \ell-k} \frac{(\ell-k)!}{\beta_1! \beta_2! \cdots \beta_{\ell-k}!} \prod_{j=1}^{\ell-k} \left( \frac{(f_{\pm}^{j})^{\beta_j}}{j!} \right) \right)
\]

(3.21)

\[
|K_{\lambda}^{(\ell)}| \lesssim |K_{\lambda}| \sum_{1+2\alpha_2+\cdots+k\alpha_k = k} \prod_{j=1}^{k} \left( \frac{|V_{22}| f_{\pm}^{j}}{\lambda + m - V_{22}} \right)^{\alpha_j}
\]

\[
\lesssim |K_{\lambda}| \prod_{j=1}^{\ell} \left( \frac{|V_{22}| f_{\pm}^{j}}{\lambda + m - V_{22}} \right)^{\alpha_j}
\]

The last step is to the bound \(\frac{|V_{22}|}{\lambda + m - V_{22}} \leq 1\), and the fact that \(\sum_{j=1}^{\ell} \alpha_j \geq 1\).

All the estimates for \(x \in [-a_-, a_+]\) hold thanks to the boundedness of the appearing derivatives of \(V\) on a compact set.

We use the next lemma to gather all the real parts of the diagonal terms to one group and their imaginary parts to the other group. This allows us to estimate the denominator of \(\text{Re}(\lambda \psi_{11}^{(1)})\) in an easier way. Furthermore, it also tells us that the case \(\text{Im} V_{11} = \text{Im} V_{22}\) is very special.

**Lemma 3.5.** On \(J_{\lambda}\), we have the following inequalities

\[
\sqrt{|V_{\lambda}| + \text{Re} V_{\lambda}} \geq \sqrt{2} \sqrt{(\lambda - m - \text{Re} V_{11})(\lambda + m - \text{Re} V_{22})},
\]

\[
\sqrt{|V_{\lambda}| + \text{Re} V_{\lambda}} \leq \frac{1}{\sqrt{2}} \sqrt{(\text{Im} V_{11} - \text{Im} V_{22})^2 + (2\lambda - \text{Re} V_{11} - \text{Re} V_{22})^2}. \tag{3.22}
\]

**Proof.** Using the Cauchy–Schwarz inequality, we have

\[
|V_{\lambda}| + \text{Re} V_{\lambda} = \sqrt{[\lambda - m - \text{Re} V_{11}]^2 + (\text{Im} V_{11})^2}[\lambda + m - \text{Re} V_{22}]^2 + (\text{Im} V_{22})^2
\]

\[
+ (\lambda + m - \text{Re} V_{22})(\lambda - m - \text{Re} V_{11}) - (\text{Im} V_{11})(\text{Im} V_{22})
\]

\[
\geq |(\lambda + m - \text{Re} V_{22})(\lambda - m - \text{Re} V_{11}) + (\text{Im} V_{11})(\text{Im} V_{22})|
\]

\[
+ (\lambda + m - \text{Re} V_{22})(\lambda - m - \text{Re} V_{11}) - (\text{Im} V_{11})(\text{Im} V_{22})
\]

\[
\geq 2(\lambda + m - \text{Re} V_{22})(\lambda - m - \text{Re} V_{11}).
\]
By an elementary inequality, the modulus of $V_\lambda$ can be bounded from above as follows:

$$|V_\lambda| = \sqrt{[(\lambda - m - \text{Re } V_{11})^2 + (\text{Im } V_{11})^2][(\lambda + m - \text{Re } V_{22})^2 + (\text{Im } V_{22})^2]}$$

$$\leq \frac{1}{2} \left((\lambda - m - \text{Re } V_{11})^2 + (\text{Im } V_{11})^2 + (\lambda + m - \text{Re } V_{22})^2 + (\text{Im } V_{22})^2\right).$$

From this we can deduce successively that

$$|V_\lambda| + \text{Re } V_\lambda \leq \frac{1}{2} \left((\text{Im } V_{11} - \text{Im } V_{22})^2 + (2\lambda - \text{Re } V_{11} - \text{Re } V_{22})^2\right).$$

**Lemma 3.6.** Let Assumption I hold for some $N \in \mathbb{N}_0$. Let $n \in [0, N]$ and $\{\psi_k^{(1)}\}_{k \in [-1, n-1]}$ be determined by (2.8) with the plus sign in the formula of $\psi_{-1}^{(1)}$. There exists $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$,

- on $I^+ \cap J_\lambda^+$,
  $$\text{Re } (\lambda \psi_{-1}^{(1)}) \geq \frac{\mu_+ - \eta}{\sqrt{\eta^2 + (2 + 2\eta)^2}} (\text{Im } V_{11} + \text{Im } V_{22}),$$
- on $I^- \cap J_\lambda^-$,
  $$\text{Re } (\lambda \psi_{-1}^{(1)}) \leq \frac{\mu_- - \eta}{\sqrt{\eta^2 + (2 + 2\eta)^2}} (\text{Im } V_{11} + \text{Im } V_{22}),$$
- on $J_\lambda$,
  $$|\text{Re } (\lambda \psi_{-1}^{(1)})| \lesssim |\text{Im } V_{11}| + |\text{Im } V_{22}|,$$

and for all $k \in [0, n-1]$,

- on $I^+ \cap J_\lambda^+$,
  $$|\lambda^{-k} \psi_k^{(1)}| \lesssim \frac{\rho_{|k|}^{k+1}}{\lambda^k},$$
- on $[-a_-, a_+]$,
  $$|\lambda^{-k} \psi_k^{(1)}| \lesssim \frac{1}{\lambda^k}.$$

**Proof.** Firstly, we prove the lemma for the first two estimates in (3.23). By looking at the formula of $\text{Re } (\lambda \psi_{-1}^{(1)})$ in (3.8) and using assumptions (3.1) and (3.2), we see that the numerator of $\text{Re } (\lambda \psi_{-1}^{(1)})$ has opposite signs at $-\infty$ and $+\infty$. Namely, employing the remark (3.18) and recalling that $\eta < \mu_{\pm}$, we have the following estimate on $I^+ \cap J_\lambda^+$:

$$\text{Im } V_{11}(\lambda + m - \text{Re } V_{22}) + \text{Im } V_{22}(\lambda - m - \text{Re } V_{11})$$

$$= \lambda(\text{Im } V_{11} + \text{Im } V_{22}) + \text{Im } V_{11}(m - \text{Re } V_{22}) - \text{Im } V_{22}(m + \text{Re } V_{11})$$

$$\geq \lambda \mu_+ (|\text{Im } V_{11}| + |\text{Im } V_{22}|) - \eta \lambda (|\text{Im } V_{11}| + |\text{Im } V_{22}|)$$

$$\geq \lambda (\mu_+ - \eta) (|\text{Im } V_{11} + \text{Im } V_{22}|);$$

and similarly, on $I^- \cap J_\lambda^-$:

$$\text{Im } V_{11}(\lambda + m - \text{Re } V_{22}) + \text{Im } V_{22}(\lambda - m - \text{Re } V_{11}) \leq \lambda (\mu_- - \eta) (|\text{Im } V_{11} + \text{Im } V_{22}|).$$

Next, it follows from the upper bound in (3.22) for the denominator of $\text{Re } (\lambda \psi_{-1}^{(1)})$ in (3.8) that, on $I^+ \cap J_\lambda^+$,

$$\text{Re } (\lambda \psi_{-1}^{(1)}) \geq \frac{\lambda (\mu_+ - \eta) (\text{Im } V_{11} + \text{Im } V_{22})}{((\text{Im } V_{11} - \text{Im } V_{22})^2 + (2\lambda - \text{Re } V_{11} - \text{Re } V_{22})^2)^{1/2}}$$

$$\geq \frac{\mu_+ - \eta}{\sqrt{\eta^2 + (2 + 2\eta)^2}} (\text{Im } V_{11} + \text{Im } V_{22}).$$

In the last step of the above expression, we used the estimates (3.18). On $I^- \cap J_\lambda^-$, we do it in the same manner.

Secondly, the final estimate in (3.23) is obtained by using the lower bound in (3.22) for the denominator of $\text{Re } (\lambda \psi_{-1}^{(1)})$ on $J_\lambda$:

$$|\text{Re } (\lambda \psi_{-1}^{(1)})| \lesssim \frac{|\text{Im } V_{11}(\lambda + m - \text{Re } V_{22})| + |\text{Im } V_{22}(\lambda - m - \text{Re } V_{11})|}{(|\lambda + m - \text{Re } V_{22}|(\lambda - m - \text{Re } V_{11})|^{1/2}} \lesssim |\text{Im } V_{11}| + |\text{Im } V_{22}|.$$
Finally, let us prove the first estimate in \((3.24)\) while the second one can be considered in a similar way. For \(x \in I^\pm \cap J_\lambda^\pm\), from Lemma \([3.4]\) we have \(|V_\alpha^{(t)}(x)| \lesssim |V_\lambda(x)|f_+\) and \(|K^{(t)}(x)| \lesssim |K_\lambda(x)|f_+^2\). Here we used the fact implied by \((3.19)\) that, for all \(x \in J_\lambda\),

\[
\max\{|V_{11}(x)|, |V_{22}(x)|\} \lesssim \frac{|V_{11}(x)| + |V_{22}(x)|}{|\lambda + m - V_{22}|} \lesssim \frac{|V_{11}(x)|}{|\lambda + m - V_{11}|} + \frac{|V_{22}(x)|}{|\lambda - m - V_{22}|} \lesssim 1. \quad (3.25)
\]

Thus, applying this to each element \(d_j^{r,r-j+(1,1)}(V_\lambda, K_\lambda)\) on \(I^\pm \cap J_\lambda^\pm\):

\[
|d_j^{r,r-j+(1,1)}(V_\lambda, K_\lambda)| \lesssim \sum_{\alpha \in J_j^{r,r-j+(1,1)}} \left|V_\lambda^{(1)}\right|^{a_1} \cdots \left|\lambda\right|^{a_{r-1-j_1+1}} \left|K_\lambda^{(1)}\right|^{a_{r_1-j_1+2}} \cdots \left|K_\lambda^{(r_2-j_2+1)}\right|^{a_{r_1-j_1+2}} f_+^2
\]

\[
\lesssim \sum_{\alpha \in J_j^{r,r-j+(1,1)}} \left|V_\lambda\right| |K_\lambda|^2 f_+^2,
\]

in which we borrowed the definition of the set \(J_j^r\) in \((2.12)\). The estimate in \((3.24)\) for \(x \in I^\pm \cap J_\lambda^\pm\) follows from the formula of \(\psi_1^{(1)}\) in Lemma \([2.5]\).

\[\square\]

**Remark 3.7.** From the estimates \((3.23)\) and \((3.9)\), it follows that, for all \(x \in I^\pm \cap J_\lambda^\pm\),

\[\text{In the Dirac case: } \text{Re} (\lambda \psi_{-1}^{(1)}(x)) \approx \text{Im} V_{11}(x) + \text{Im} V_{22}(x).\]

The sign of the sum \(\text{Im} V_{11}(x) + \text{Im} V_{22}(x)\) decides the sign of \(\text{Re} (\lambda \psi_{-1}^{(1)})\) in the neighbourhood of infinity.

This is to be compared with the Schrödinger case \([23\text{, Lem. 3.4]}\) where the sign of \(\text{Im} V\) (with scalar \(V\) now) plays this role, more precisely

\[\text{In the Schrödinger case: } \text{Re} (\lambda \psi_{-1}^{(1)}(x)) \approx \lambda^{-\frac{1}{2}} \text{Im} V.\]

In this case, when \(\lambda\) is considered to be large, \(\text{Im} V\) needs to be proportional to (and larger than) \(\lambda^\frac{1}{2}\) near \(\delta_\pm\) such that \(\text{Re} (\lambda \psi_{-1}^{(1)}(x))\) is also large. This was handled in \([23]\) thanks to the definition of \(\delta_\pm\) which is in terms of \(\text{Im} V\). Then \(\text{Re} V\) needs to be controlled by \(\text{Im} V\) such that \(\text{Re} V\) can also be bounded by \(\lambda\), whence the extra condition \([23\text{, Cond. (3.3)}]\).

However, this extra work can be relaxed in the Dirac case thanks to the above form of \(\text{Re} (\lambda \psi_{-1}^{(1)})\). Technically, this can be explained by the product structure of \(V^{\text{Dirac}}_\lambda\) which allows \(\lambda\) to show up in \(\text{Im} V^{\text{Dirac}}\) and therefore it cancels \(\lambda\) appearing in the denominator of \(\text{Re} (\lambda \psi_{-1}^{(1)})\) asymptotically.

Furthermore, the case \(\text{Im} V_{11} = \text{Im} V_{22}\) needs to be taken into account. For example, if this happens on \([0, +\infty)\) (obviously, \(\mu_+ = 1\) will be chosen in this situation), then the first estimate in \((3.23)\) can be taken strictly as follows:

\[
\text{Re} (\lambda \psi_{-1}^{(1)}(x)) = \frac{1}{\sqrt{2}} \frac{\text{Im} V_{11}(2\lambda - \text{Re} V_{22} - \text{Re} V_{11})}{\sqrt{|V_\lambda|} + \text{Re} V_\lambda} \geq \frac{\text{Im} V_{11}(2\lambda - \text{Re} V_{22} - \text{Re} V_{11})}{2\lambda - \text{Re} V_{22} - \text{Re} V_{11}} = \text{Im} V_{11}.
\]

Meanwhile, the constant which turns up at \((3.23)\), \(\frac{2(\mu_+ - \eta)}{\sqrt{\eta^2 + (2+2\eta)^2}}\), is strictly smaller than and close to 1 when \(\eta\) is chosen small enough. However, it does not matter because this constant will be attached with \(1 - o(1)\) as \(\lambda \to +\infty\) when we deal with it in the next lemma.

With the derivatives of \(\psi_k^{(1)}\) given in \((2.8)\), we can determine the primitives \(\psi_k\) uniquely by choosing the initial data \(\psi_k(0) = 0, \forall k \in [-1, n - 1]\).
Lemma 3.8. Let Assumption [I] hold for some $N \in \mathbb{N}_1$. Let $n \in \{1, N]\}$ and $\{\psi_k^{(1)}\}_{k \in \{-1, n-1\}}$ be determined by (2.8) with the plus sign in the formula of $\psi^{(-1)}_k$. Let $P_{\lambda,n}$ defined as in (2.5). There exist $c_1 > 0$, $c_2 > 0$ and $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$ and for all $x \in J_\lambda$,

$$
\exp(-c_1 F(x)) \exp \left( \frac{1}{2} \int_0^x U(t) \, dt \right) \lesssim |k_1(x)| \exp(-P_{\lambda,n}(x)) \lesssim \exp(-c_2 F(x)) .
$$

Furthermore, if $V_{12} = V_{21}$, the statement is also true as $n = 0$, i.e. $N = 0$ is allowed.

Proof. Let us recall that

$$
P_{\lambda,n}(x) = \int_0^x \psi_0^{(1)}(t) \, dt + \sum_{k=1}^{n-1} \int_0^x \lambda^{-k} \psi_k^{(1)}(t) \, dt .
$$

From the formula of $\psi^{(1)}_0$, we observe that

$$
\left| \exp\left(-\int_0^x \psi_0^{(1)}(t) \, dt\right) \right| = \frac{|V_\lambda(0)|^{1/4} |K_\lambda(0)|^{1/2}}{|V_\lambda(x)|^{1/4} |K_\lambda(x)|^{1/2}} .
$$

Then, it follows from the definition of the functions $V_\lambda$, $K_\lambda$ and estimates in (3.19) that

$$
|k_1(x)| \exp(-P_{\lambda,n}(x)) |
$$

$$
= |k_1(x)| \frac{|V_\lambda(0)|^{1/4} |K_\lambda(0)|^{1/2}}{|V_\lambda(x)|^{1/4} |K_\lambda(x)|^{1/2}} \exp \left( - \sum_{k=1}^{n-1} \int_0^x \text{Re} \left( \lambda^{-k} \psi_k^{(1)}(t) \right) \, dt \right) 
$$

$$
= |k_1(x)| k_2(x)|^{1/2} \frac{|\lambda - m - V_{11}(0)|^{1/4} |\lambda + m - V_{22}(x)|^{1/4}}{|\lambda + m - V_{22}(0)|^{1/4} |\lambda - m + V_{11}(x)|^{1/4}} \exp \left( - \sum_{k=1}^{n-1} \int_0^x \text{Re} \left( \lambda^{-k} \psi_k^{(1)}(t) \right) \, dt \right) 
$$

$$
\approx \exp\left(-\int_0^x \text{Re} \left( \lambda \psi^{(-1)}_k(t) \right) \, dt \right) + \frac{1}{2} \int_0^x U(t) \, dt - \sum_{k=1}^{n-1} \int_0^x \text{Re} \left( \lambda^{-k} \psi_k^{(1)}(t) \right) \, dt .
$$

Thanks to the estimate (3.23) and (3.24), $U \in L^\infty_{\text{loc}}(\mathbb{R})$, we have the uniform bound, for all $x \in [0, a_+]$,

$$
\left| \int_0^x \text{Re} \left( \lambda \psi^{(-1)}_k(t) \right) \, dt \right| + \left| \frac{1}{2} \int_0^x U(t) \, dt \right| + \sum_{k=1}^{n-1} \int_0^x \text{Re} \left( \lambda^{-k} \psi_k^{(1)}(t) \right) \, dt \lesssim 1 .
$$

By the estimate (3.23) and (3.24) again, there exists a constant $M > 0$ such that for all $x \in I^+ \cap J_\lambda^+$,

$$
- \int_0^x \text{Re} \left( \lambda \psi^{(-1)}_k(t) \right) \, dt + \frac{1}{2} \int_0^x U(t) \, dt - \sum_{k=1}^{n-1} \int_0^x \text{Re} \left( \lambda^{-k} \psi_k^{(1)}(t) \right) \, dt 
$$

$$
\leq - \frac{\mu_+ - \eta}{\sqrt{\eta^2 + (2 + 2\eta)^2}} F(x) + \frac{1}{2} \int_0^x U(t) \, dt - \sum_{k=1}^{n-1} \int_{a_+}^x \text{Re} \left( \lambda^{-k} \psi_k^{(1)}(t) \right) \, dt + M 
$$

$$
= - \frac{\mu_+ - \eta}{\sqrt{\eta^2 + (2 + 2\eta)^2}} F(x) \left( 1 - \frac{1}{2} \int_0^{a_+} U(t) \, dt + \sum_{k=1}^{n-1} \int_{a_+}^x \text{Re} \left( \lambda^{-k} \psi_k^{(1)}(t) \right) \, dt + M \right) .
$$
To estimate the term with $\mathcal{U}$, we use the condition \([3.3]\). Let $\epsilon_+ \in (0, \frac{\mu_+}{2})$ be the number given in the condition \([3.3]\). We choose $\eta$ very small such that
\[
\frac{\mu_+ - \eta}{\sqrt{\eta^2 + (2 + 2\eta)^2}} > \epsilon_+.
\]

We deduce that, for all $x \in I^+ \cap J_\lambda^+$,
\[
\frac{1}{2} \int_0^x \mathcal{U}(t) \, dt = \frac{\mu_+ - \eta}{\sqrt{\eta^2 + (2 + 2\eta)^2}} F(x) \leq \frac{\epsilon_+}{\mu_+ - \eta} < 1.
\]

To bound the terms with $\psi_k$ for $k \in \{1, n - 1\}$, we apply \([3.24]\) for all $t \in I^+ \cap J_\lambda^+$ and obtain
\[
\left| \sum_{k=1}^{n-1} \text{Re} \left( \lambda^{-k} \psi_k^{(1)}(t) \right) \right| \lesssim \sum_{k=1}^{n-1} \frac{f_+(t)^{k+1}}{\lambda^k} \lesssim \begin{cases} 
\lambda^{-1} & \text{if } f_+ \text{ is bounded at } +\infty, \\
\lambda^{-\epsilon_1} & \text{if } f_+ \text{ is unbounded at } +\infty.
\end{cases}
\]

Indeed, the case $f_+$ is bounded at $+\infty$ is obvious. In contrast, we employ the property \([3.14]\) of $\delta^+_\lambda$ and the definition of function $g$ in \([3.16]\), for all $t \in I^+ \cap J_\lambda^+$,
\[
f_+(t)^{k+1} \lambda^k \leq \frac{\lambda^{k+1}(1+\epsilon_1)}{\lambda^k},
\]
and notice that $\frac{k+1}{2}(1-\epsilon_1) - k \leq -\epsilon_1$ for all $k \geq 1$. By employing \([3.7]\), we have
\[
\left| \sum_{k=1}^{n-1} \int_{a_+}^x \text{Re} \left( \lambda^{-k} \psi_k^{(1)}(t) \right) \, dt \right| = o(1) \quad \text{as } \lambda \to +\infty.
\]

Hence, the second inequality in the statement of this lemma is proved. The first inequality is obtained easily by the final estimate in \([3.23]\), the observation \([3.9]\) and the selected sign of the sum $\text{Im} \, V_1 + \text{Im} \, V_2$ in \([3.3]\) for all $x \in I^+ \cap J_\lambda^+$, $\text{Re} \left( \lambda \psi_1^{(1)} \right) \lesssim \text{Im} \, V_1 + \text{Im} \, V_2$. Finally, combining this with \([3.27]\), we obtain the result.

When $n = 0$, there is no presence of $\psi_0^{(1)}$ in $P_{\lambda,n}$ and thus the integral $\int_{a_+}^x \mathcal{U}(t) \, dt$ will not come out in the above estimates, but the integral $\int_0^x \text{Im} \, V_2(t) \, dt$ appears instead. However, if $V_{12} = V_{21}$, we can perform the proof as we have done above.

The next proposition reveals that the terms attached with the derivatives of the cut-off function $\xi$ decay exponentially as $\lambda \to +\infty$ at a rate controlled by the function $F$.

**Proposition 3.9.** Let Assumption \([1]\) hold for some $N \in \mathbb{N}_1$. Let $n \in \{1, N\}$, \(\{\psi_k^{(1)}\}_{k \in [-1, n - 1]}\) be determined by \([2.8]\) with the plus sign in the formula of $\psi_1^{(1)}$ and $P_{\lambda,n}$ defined as in \([2.5]\). Let $\xi$ be given in \([3.10]\) whose $\delta^+_\lambda$, $g_\pm$ and $\Delta^\pm_\lambda$ is identified by \([3.15]\), \([3.16]\) and \([3.17]\). Let us denote
\[
\kappa(\lambda) := \frac{\| k_1 \exp(-P_{\lambda,n}) \xi^{(2)} \| + \| k_1 \exp(-P_{\lambda,n}) \|}{\lambda + m - V_22} \left( \frac{2P_{\lambda,n}^{(1)} - \frac{\xi^{(1)}}{\xi^{(1)}}}{\lambda,n - \frac{\xi^{(1)}}{\xi^{(1)}}} \right) \xi^{(1)}. \tag{3.28}
\]

Then $\kappa(\lambda) = o(1)$ as $\lambda \to +\infty$. More precisely, there exists $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$,
\[
\kappa(\lambda) = \kappa_-(\lambda) + \kappa_+(\lambda)
\]
where (with some \(d_1, d_2 > 0\))

\[
\kappa_+(\lambda) = \begin{cases} 
0, & \text{if } g_+ \text{ is bounded at } +\infty, \\
O(\exp(-d_1 F(d_2 \delta_\lambda^+))) & \text{otherwise,}
\end{cases}
\]

and

\[
\kappa_-(\lambda) = \begin{cases} 
0, & \text{if } g_- \text{ is bounded at } -\infty, \\
O(\exp(-d_1 F(d_2 \delta_\lambda^-))) & \text{otherwise.}
\end{cases}
\]

Furthermore, if \(V_{12} = V_{21}\), the statement is also true for \(n = 0\), i.e. \(N = 0\) is allowed.

**Proof.** First of all, we want to show that the denominator in (3.28) is bounded from below by a constant not depending on \(\lambda\). Thanks to Lemma 3.8 and the boundedness of \(V_{ij}\) for \(i, j \in \{1, 2\}\) on \([0, 1]\), one has

\[
\int_\mathbb{R} |k_1 \exp(-P_{\lambda,n})|^2 \, dx \gtrsim \int_0^1 \exp \left( -2c_1 F(x) + \int_0^x \mathcal{U}(t) \, dt \right) \, dx \gtrsim 1.
\]

Now, we try to control two terms attached with \(\xi^{(1)}\) and \(\xi^{(2)}\) in the numerator of (3.28). Obviously, the case of \(g_+\) being bounded at \(+\infty\) is trivial, since \(\xi(x) = 1\) on \([0, +\infty)\). The negative case is the same. We just need to care about the remaining situations in which \(\xi\) is a “true” cut-off function. The main idea is to employ the exponential decay in order to limit the growth of polynomials on the support of \(\xi^{(1)}\) and \(\xi^{(2)}\). In detail, applying Lemma 3.8 on the support of \(\xi^{(2)}\) and (3.11), (3.17), we obtain (with some \(c_3, c_4 > 0\))

\[
\left\| \frac{k_1}{\lambda + m - V_{22}} \exp(-P_{\lambda,n})\xi^{(2)} \right\|^2 
\lesssim (\delta_\lambda^-)^4 \int_{J_\lambda^+ \setminus \bar{J}_\lambda^+} \exp(-2c_2 F(x)) \, dx 
+ (\delta_\lambda^+)^4 \int_{J_\lambda^- \setminus \bar{J}_\lambda^-} \exp(-2c_2 F(x)) \, dx
\lesssim (\delta_\lambda^-)^3 \exp(-2c_2 F(-\delta_\lambda^- + \Delta_\lambda^-)) 
+ (\delta_\lambda^+)^3 \exp(-2c_2 F(\delta_\lambda^+ - \Delta_\lambda^+))
\lesssim \exp(-c_3 F(c_4 \delta_-)) + \exp(-c_3 F(c_4 \delta_+)).
\]

In the second inequality, we used the fact that \(F(x)\) increases as \(|x|\) increases. Whereas the observation (3.7) is employed in the third inequality. The term associated with \(\xi^{(1)}\) is estimated in the same way. Just notice that, from the similarity in (3.19), Lemma 3.4, Lemma 3.6 and estimates \(f_\pm\) as in (3.26) for all \(x \in J_\lambda^+ \setminus \bar{J}_\lambda^+\), we have (with some \(c_5 > 0\))

\[
\left\| 2P_{\lambda,n}^{(1)}(x) - \frac{K_\lambda^{(1)}(x)}{K_\lambda(x)} \right\| \frac{k_1(x) \exp(-P_{\lambda,n}(x))}{\lambda + m - V_{22}(x)}
\lesssim \left( |V_\lambda(x)|^{1/2} + \frac{|K_\lambda^{(1)}(x)|}{|K_\lambda(x)|} + \sum_{k=0}^{n-1} |\lambda^{-k} \psi_k^{(1)}(x)| \right) \frac{\exp(-c_2 F(x))}{|\lambda + m - V_{22}|}
\lesssim \left( 1 + \sum_{k=0}^{n-1} \lambda^{-(k+1)} f_\pm(x)^{k+1} \right) \exp(-c_2 F(x))
\lesssim \exp(-c_5 F(x)).
\]

Thus, the desired claim follows. \(\square\)

### 3.4. Main results

Now, we can state our main theorems and their consequences.

The following theorem says that if \(V_{11}, V_{22}\) at least belong to \(W_{loc}^{2,\infty}(\mathbb{R})\) and \(V_{12}, V_{21}\) at least belong to \(W_{loc}^{1,\infty}(\mathbb{R})\) and satisfy Assumption \(\mathfrak{A}\) our WKB solution will become the quasimode for the problem (1.5). When the potential \(V\) is symmetric, the sufficient conditions for the involving spaces of \(V_{11}, V_{22}\) are released to \(W_{loc}^{1,\infty}(\mathbb{R})\) and of \(V_{12}, V_{21}\) are released to \(L_{loc}^{\infty}(\mathbb{R})\) in
some cases. Furthermore, the rate of decay of the estimate when $V$ is symmetric is better in some situations.

**Theorem 3.10.** Let Assumption I hold for some $N \in \mathbb{N}_1$ and set $n = N$. Let $\{\psi_k^{(1)}\}_{k \in [-1,n-1]}$ be determined by (2.8) with the plus sign in the formula of $\psi_1^{(1)}$ and $P_{\lambda,n}$ defined as in (2.5). Let us define

$$
\Psi_{\lambda,n} := \begin{pmatrix} k_1 u_{\lambda,n} \\ k_2 v_{\lambda,n} \end{pmatrix},
$$

where $u_{\lambda,n} := \xi \exp(-P_{\lambda,n})$, $v_{\lambda,n} := \frac{k_1}{k_2} \frac{\partial_x u_{\lambda,n}}{\lambda + m - V_{22}}$.

$k_1$, $k_2$ are functions as in (2.1) and $\xi$ is given in (3.10) whose $\delta_{\lambda}^+$, $\Delta_{\lambda}$ are identified by (3.15) and (3.17). Then,

$$
\frac{\| (H_V - \lambda) \Psi_{\lambda,n} \|}{\| \Psi_{\lambda,n} \|} = o(1), \quad \lambda \to +\infty.
$$

More precisely,

$$
\frac{\| (H_V - \lambda) \Psi_{\lambda,n} \|}{\| \Psi_{\lambda,n} \|} \leq \kappa(\lambda) + \sigma^{(n)},
$$

where $\kappa$ is as in (3.28) and $\sigma^{(n)} = \sigma_-^{(n)} + \sigma_+^{(n)}$ with

$$
\sigma_{\pm}^{(n)} = \begin{cases} O(\lambda^{-n}), & f_{\pm} \text{ is bounded at } \pm \infty, \\
O(\lambda^{-\frac{(n+1)}{2}(n+1)+1}), & f_{\pm} \text{ is unbounded at } \pm \infty.
\end{cases}
$$

**Proof.** Before going to the proof, it is necessary to check that $\Psi_{\lambda,n}$ belongs to the domain of $H_V$. With the choice of $\Psi_{\lambda,n}$ in the statement of the theorem, we have the relation between $H_V$ and $\mathcal{L}_{\lambda,V}$ as follows: $(H_V - \lambda) \Psi_{\lambda,n} = (\mathcal{L}_{\lambda,V} u_{\lambda,n})$. Thus, $\Psi_{\lambda,n} \in \mathcal{D}(H_V)$ if and only if

$$
k_1 u_{\lambda,n} \in L^2(\mathbb{R}), \quad \frac{k_1 (-i \partial_x) u_{\lambda,n}}{\lambda + m - V_{22}} \in L^2(\mathbb{R}), \quad \mathcal{L}_{\lambda,V} u_{\lambda,n} \in L^2(\mathbb{R}).
$$

Obviously, this happens if $\xi$ is a “true” cut-off. The thing that makes us worry, for example, is the case $\delta_{\lambda}^+ = +\infty$ when $g_+$ is bounded at $+\infty$. From the observation (3.7) combined with Lemma 3.8 and mimicking the proof of Proposition 3.9, it yields that, for sufficiently large $x > 0$ (with some $c_3 > 0$),

$$
|k_1(x) u_{\lambda,n}(x)| \lesssim \exp(-c_3 x), \quad \left| \frac{k_1(x)(-i \partial_x) u_{\lambda,n}(x)}{\lambda + m - V_{22}(x)} \right| \lesssim \exp(-c_3 x).
$$

From (2.4), we obtain

$$
|\mathcal{L}_{\lambda,V} u_{\lambda,n}| = \left| \frac{R_{\lambda,n}}{\lambda + m - V_{22}} \right| |k_1 \exp(-P_{\lambda,n})|.
$$

We will see later that $\left| \frac{R_{\lambda,n}(x)}{\lambda + m - V_{22}(x)} \right|$ is uniformly bounded from above. Therefore, the $L^2$-integrability of $\mathcal{L}_{\lambda,V} u_{\lambda,n}$ is ensured by the $L^2$-integrability of $k_1(x) \exp(-P_{\lambda,n}(x))$.

Now, we can come back to prove the statement of the theorem. Let us recall that we need to estimate the quantity

$$
\frac{\| (H_V - \lambda) \Psi_{\lambda,n} \|}{\| \Psi_{\lambda,n} \|} = \sqrt{\| k_1 u_{\lambda,n} \|^2 + \| \frac{k_1}{\lambda + m - V_{22}} (-i \partial_x) u_{\lambda,n} \|^2}.
$$

From (2.4), we obtain

$$
|\mathcal{L}_{\lambda,V} u_{\lambda,n}| = \left| \frac{R_{\lambda,n}}{\lambda + m - V_{22}} \right| |k_1 \exp(-P_{\lambda,n})|.
$$

We will see later that $\left| \frac{R_{\lambda,n}(x)}{\lambda + m - V_{22}(x)} \right|$ is uniformly bounded from above. Therefore, the $L^2$-integrability of $\mathcal{L}_{\lambda,V} u_{\lambda,n}$ is ensured by the $L^2$-integrability of $k_1(x) \exp(-P_{\lambda,n}(x))$.
Using the triangle inequality and \((3.28)\), we obtain
\[
\|\mathcal{L}_{\lambda,n} u_{\lambda,n}\| \leq \left\| \frac{k_1}{\lambda + m - V_{22}} \exp(-P_{\lambda,n})\xi(2) \right\| + \left\| \frac{k_1}{\lambda + m - V_{22}} \left( 2P_{n}^{(1)} - \frac{K_{\lambda}^{(1)}}{K_{\lambda}} \right) \exp(-P_{\lambda,n})\xi(4) \right\|
\]
\[
+ \left\| \frac{R_{\lambda,n}}{\lambda + m - V_{22}} \right\|_{L^\infty(J_\lambda)} \|k_1 u_{\lambda,n}\|,
\]
and thus
\[
\sqrt{\|k_1 u_{\lambda,n}\|^2 + \left\| \frac{k_1}{\lambda + m - V_{22}} (-i\partial_x) u_{\lambda,n} \right\|^2} \leq \kappa(\lambda) + \left\| \frac{R_{\lambda,n}}{\lambda + m - V_{22}} \right\|_{L^\infty(J_\lambda)}.
\]
The estimate of the remainder \((2.13)\) together with Lemma \(3.4\) yield that, for all \(x \in [a_-, a_+],\)
\[
\left| \frac{R_{\lambda,n}(x)}{\lambda + m - V_{22}(x)} \right| \lesssim \lambda^{-n}.
\]
In a similar way, it turns out that, for all \(x \in I^+ \cap J_\lambda^+\),
\[
\left| \frac{R_{\lambda,n}(x)}{\lambda + m - V_{22}(x)} \right| \lesssim \frac{1}{\lambda} \sum_{\ell = -1}^{n-2} \frac{f_\pm(x)^{n+\ell+2}}{V_{\lambda}(x)^{(n+\ell)/2}} \lesssim \begin{cases} \lambda^{-n}, & \text{if } f_\pm \text{ is bounded at } \pm \infty, \\ \lambda^{-\frac{(1+\ell)(n+1)+1}{2}}, & \text{if } f_\pm \text{ is unbounded at } \pm \infty. \end{cases}
\]
To prove the case \(f_\pm\) is unbounded at \(\pm \infty\), we employ the fact \(f_\pm(x)^{\frac{1}{1-n}} \leq \lambda\), which is a consequence of \((3.14)\) and the definition of \(g\) in \((3.16)\).

**Theorem 3.11.** Under the same assumptions and settings as in Theorem \(3.10\) with \(N \in \mathbb{N}_0\) and \(V_{12} = V_{21}\), let \(n = N\). Then,
\[
\frac{\| (H_V - \lambda) \Psi_{\lambda,n} \|}{\| \Psi_{\lambda,n} \|} \leq \kappa(\lambda) + \tau(n)(\lambda), \quad \lambda \to +\infty.
\]
Here, \(\kappa\) is as in \((3.28)\) and \(\tau(n) = \tau_-(n) + \tau_0(n) + \tau_+(n)\) with \(\tau_0 = \lambda^{-(n+1)}\) and
\[
\text{i) if } V_{11} \text{ or } V_{22} \text{ is not bounded at } \pm \infty,
\]
\[
\tau_\pm(n)(\lambda) = \begin{cases} \mathcal{O} \left( \lambda^{-n} \sup_{x \in I^+ \cap J_\lambda^+} \frac{f_\pm(x)^{n+1}}{|\lambda + m - V_{22}(x)|} \max\{|V_{11}|, |V_{22}|\} \right), & \text{if } f_\pm \text{ is bounded at } \pm \infty, \\ \mathcal{O} \left( \lambda^{-\frac{(1+\ell)(n+1)+1}{2}} \right), & \text{if } f_\pm \text{ is unbounded at } \pm \infty; \end{cases}
\]
\[
\text{ii) if } V_{11} \text{ and } V_{22} \text{ are bounded at } \pm \infty
\]
\[
\tau_\pm(n)(\lambda) = \begin{cases} \mathcal{O}(\lambda^{-(n+1)}), & \text{if } f_\pm \text{ is bounded at } \pm \infty, \\ \mathcal{O}(\lambda^{-\frac{(1+\ell)(n+1)}{2}}), & \text{if } f_\pm \text{ is unbounded at } \pm \infty. \end{cases}
\]

**Proof.** Since \(V_{12} = V_{21}\), Lemma \(3.8\) and Proposition \(3.9\) still hold when \(n = 0\). However, we can perform the estimates in Theorem \(3.10\) more strictly. Now, we assume that \(V_{11}\) or \(V_{22}\) is not bounded at \(+\infty\). In detail, from the estimate of the remainder in \((2.13)\) together with Lemma \(3.4\) and for \(n = 0\), we obtain
\[
\left| \frac{R_{\lambda,0}(x)}{\lambda + m - V_{22}(x)} \right| \lesssim \frac{1}{\lambda} \left( \frac{|V_{\lambda}^{(1)}(x)|}{|V_{\lambda}(x)|^{1/2}} + \frac{|K_{\lambda}^{(1)}(x)|}{|K_{\lambda}(x)|} |V_{\lambda}(x)|^{1/2} \right)
\]
\[
\lesssim \begin{cases} \frac{1}{\lambda} f_\pm(x) \max\{|V_{11}|, |V_{22}|\} \frac{|V_{\lambda}(x)|}{|\lambda + m - V_{22}(x)|}, & \forall x \in [0, a_+], \\ \frac{1}{\lambda} f_\pm(x) \max\{|V_{11}|, |V_{22}|\} \frac{|V_{\lambda}(x)|}{|\lambda + m - V_{22}(x)|}, & \forall x \in I^+ \cap J_\lambda^+. \end{cases}
\]
For \( n \geq 1 \), for all \( x \in [0, a_+] \), we have
\[
\left| \frac{R_{\lambda,n}(x)}{\lambda + m - V_{22}(x)} \right| \lesssim \frac{1}{\lambda^{n+1}}.
\]

While for all \( x \in I^+ \cap J_\lambda^+ \),
\[
\left| \frac{R_{\lambda,n}(x)}{\lambda + m - V_{22}(x)} \right| \lesssim \sum_{\ell = -1}^{n-2} \frac{1}{\lambda^{n+\ell+1}} \sum_{|j| = 1}^{n+\ell+2} f_+^{(\ell+1)}(x)^{|j|} \left( \max\{|V_{11}|, |V_{22}|\}(x) \right)^{|j|}.
\]
In the second inequality, we employed (3.25). Notice that, if \( f_+ \) is unbounded, from (3.14) and the definition of \( g \) in (3.16), we have \( f_+(x) \lesssim \lambda^{\frac{n-1}{2}} \) for all \( x \in J_\lambda^+ \) and thus for all \( x \in I^+ \cap J_\lambda^+ \),
\[
\sum_{\ell = -1}^{n-2} \frac{f_+(x)^{\ell+1}}{\lambda^{\ell+1}} \lesssim 1,
\]
in all cases of \( f_+ \). From this, we obtain the estimates in the statement for all \( x \geq 0 \), even in the case \( V_{11} \) and \( V_{22} \) bounded at \( \pm \infty \). The proof for \( x \leq 0 \) is fulfilled in the same way. \( \square \)

**Remark 3.12.** Let us make some comments about the shape of the pseudomodes in connection with the sign of \( \text{Im} V_{11} + \text{Im} V_{22} \) and the sign of \( \lambda \):

i) If \( \lambda > 0 \) and the sum of the diagonal terms of \( V \) changes its sign in the assumption (3.1), i.e.
\[
\begin{align*}
\text{Im} V_{11} + \text{Im} V_{22} & \geq 1 \quad \text{on } I^-; \\
\text{Im} V_{11} + \text{Im} V_{22} & \lesssim -1 \quad \text{on } I^+; 
\end{align*}
\]
then we just need to choose the minus sign in the formula of \( \psi_{-1}^{(1)} \) in (2.8). Then, we have
\[
\text{Re} (\lambda \psi_{-1}^{(1)}) = -\frac{1}{\sqrt{2}} \frac{\text{Im} V_{11}(\lambda + m - \text{Re} V_{22}) + \text{Im} V_{22}(\lambda - m - \text{Re} V_{11})}{\sqrt{|V_\lambda| + \text{Re} V_\lambda}}.
\]
By repeating the procedure when proving (3.25), we have
\[
\begin{align*}
\text{Re} (\lambda \psi_{-1}^{(1)}) & \geq -\frac{\mu_- - \eta}{\sqrt{\eta^2 + (2 + \eta)^2}} (\text{Im} V_{11} + \text{Im} V_{22}) \quad \text{on } I^+ \cap J_\lambda^+; \\
\text{Re} (\lambda \psi_{-1}^{(1)}) & \leq -\frac{\mu_+ - \eta}{\sqrt{\eta^2 + (2 + \eta)^2}} (\text{Im} V_{11} + \text{Im} V_{22}) \quad \text{on } I^- \cap J_\lambda^-.
\end{align*}
\]
Therefore, the function
\[
\tilde{F}(x) := -\int_0^x (\text{Im} V_{11}(t) + \text{Im} V_{22}(t)) \, dt
\]
will play the same role as function \( F \). Although all the other terms \( \{\psi_k^{(1)}\}_{1 \leq k \leq n-1} \) also change their sign, it does not matter because they are all estimated with absolute value. Only the sign of \( \lambda \psi_{-1}^{(1)} \) is crucial. Thus, we still assume the same remaining hypotheses in Assumption [4] but \( F \) is replaced by \( \tilde{F} \) and we have the same outcomes as stating in the above theorems.

ii) Let \( \lambda < 0 \) and Assumption [4] hold. What we need to do is to slightly change \( \lambda \) into \(-\lambda\) in some places such as in Lemma 3.1. We redefine \( \delta_\lambda^+ := \inf \{x \geq a_\pm : g_\pm(x) = -\lambda\} \)
with $g_{\pm}$ being given in the same way as in (3.16). When $(-\lambda)$ is large enough, it follows, as in (3.18), that for all $x \in J_{\lambda}$:

\[
(1 - \eta)(-\lambda) \leq \text{Re} V_{11}(x) + m - \lambda \leq (1 + \eta)(-\lambda),
\]

\[
(1 - \eta)(-\lambda) \leq \text{Re} V_{22}(x) - m - \lambda \leq (1 + \eta)(-\lambda),
\]

\[
|\text{Im} V_{11}(x) - \text{Im} V_{22}(x)| \leq \eta(-\lambda).
\]

In this case, we will choose the minus sign in the formula of $\psi_{1}^{(1)}$ in (2.8) and we obtain the same results as in (3.30). Then, the method of the current section still works for the pseudomode construction and the outcomes of the above theorems are analogous.

In summary, from the two remarks above, our scheme suggests that the sign of the solution $\psi_{1}^{(1)}$ of the eikonal equation should be chosen as in the following table:

|       | $\text{Im} V_{11} + \text{Im} V_{22} \lesssim -1$ on $I_-$ | $\text{Im} V_{11} + \text{Im} V_{22} \gtrsim 1$ on $I_+$ |
|-------|-------------------------------------------------|----------------------------------|
| $\lambda > 0$ | +                                           | -                                |
| $\lambda < 0$  | -                                           | +                                |

Table 1. The sign in the formula of $\psi_{1}^{(1)}$ in (2.8).

3.5. Applications. We consider some special examples of the matrix-valued potentials $V$ which satisfy Assumption I.

**Example 1.** Let us list some smooth potentials $V$ defined on $\mathbb{R}$ such that $V_{12} = V_{21} = u$ and Assumption I holds true. From that, we can apply Theorem 3.11.

1) $V_{11}$ and $V_{22}$ are bounded at $\pm \infty$:

\[
V(x) := \begin{pmatrix} i \frac{x}{\sqrt{x^2 + 1}} & u(x) \\ u(x) & 0 \end{pmatrix},
\]

where $u$ is some smooth function on $\mathbb{R}$ such that, with $\varepsilon \in (0, \frac{1}{2})$,

\[
\int_{0}^{x} \text{Im} u(t) \, dt \leq \varepsilon F(x) = \varepsilon(\sqrt{x^2 + 1} - 1), \quad \forall |x| \gtrsim 1.
\]

Here we choose $\mu_{\pm} = 1$, $f_{\pm}(x) = |x|^{-1}$ for $|x| \gtrsim 1$. Since $g_{\pm}$ are bounded both at $-\infty$ and $+\infty$, the cut-off function is not needed for the pseudomodes construction. For all $n \in \mathbb{N}_0$, there exists $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$,

\[
\frac{\|(H - \lambda)\Psi_{\lambda,n}\|}{\|\Psi_{\lambda,n}\|} \lesssim \lambda^{-(n+1)}.
\]

2) $V_{11}$ is bounded but $V_{22}$ is unbounded at $\pm \infty$:

\[
V(x) := \begin{pmatrix} i \frac{x}{\sqrt{x^2 + 1}} & u(x) \\ u(x) & i \ln(x + \sqrt{x^2 + 1}) \end{pmatrix},
\]

where $u$ is any smooth function on $\mathbb{R}$ such that, with $\varepsilon \in (0, \frac{1}{2})$,

\[
\int_{0}^{x} \text{Im} u(t) \, dt \leq \varepsilon F(x) = \varepsilon x \ln(x + \sqrt{x^2 + 1}), \quad \forall |x| \gtrsim 1.
\]

Here we choose $\mu_{\pm} = 1$, $f_{\pm}(x) = |x|^{-1}$ for $|x| \gtrsim 1$. Following (3.16) and (3.15), for $\lambda > 0$ large enough, the boundary of the cut-off can be computed approximately as
\[ \delta_\lambda = \delta_\lambda^+ \approx \sinh(\eta \lambda) \approx e^{\eta \lambda}, \text{ and thus with some } c > 0, \kappa(\lambda) = \mathcal{O}\left(\exp(-ce^{\eta \lambda})\right). \]

It implies that, for all \( n \in \mathbb{N}_0 \), there exists \( \lambda_0 > 0 \) such that, for all \( \lambda > \lambda_0 \), we have

\[ \frac{\|(H_V - \lambda)\Psi_{\lambda,n}\|}{\|\Psi_{\lambda,n}\|} \lesssim \lambda^{-(n+1)}. \]

3) \( V_{11} \) is bounded at \(-\infty\) but unbounded at \(+\infty\) while \( V_{22} \) is on the contrary:

\[ V(x) := \begin{pmatrix} i e^x \\ 0 \\ u \\ -i e^{-x} \end{pmatrix}, \]

where \( u \) is any smooth function on \( \mathbb{R} \) such that, with \( \varepsilon \in (0, \frac{1}{2}) \),

\[ \int_0^x \Im u(t) \, dt \leq \varepsilon F(x) = \varepsilon \cosh(x), \quad \forall |x| \geq 1. \]

In this situation, we make a choice \( \mu_\pm = \mu \) with some \( \mu \in (2\varepsilon, 1) \), \( f_\pm(x) = 1 \) for \( |x| \geq 1 \). From (3.16) and (3.15) and for \( \lambda > 0 \) large enough, we obtain

\[ \delta_\lambda^- = \delta_\lambda^+ = \arcsinh(\eta \lambda) = \ln \left( \eta \lambda + \sqrt{(\eta \lambda)^2 + 1} \right), \]

and thus with some \( c > 0 \), \( \kappa(\lambda) = \mathcal{O}\left(\exp(-c\lambda^{d_2})\right) \). It implies that, for all \( n \in \mathbb{N}_1 \), there exists \( \lambda_0 > 0 \) such that, for all \( \lambda > \lambda_0 \),

\[ \frac{\|(H_V - \lambda)\Psi_{\lambda,n}\|}{\|\Psi_{\lambda,n}\|} \lesssim \lambda^{-n}. \]

**Example 2** (Polynomial-like diagonal terms). Let us take a look at the potential \( V \) satisfying Assumption II with \( f_\pm(x) = |x|^{-1} \), \( V_{12} = V_{21} = 0 \) (for simplicity) and

\[ |\Re V_{ii}| \approx |x|^{\alpha_{ii}}, \quad |\Im V_{ii}| \approx |x|^{\beta_{ii}}, \quad \forall |x| \geq 1, \]

with \( \alpha_{ii}, \beta_{ii} \in \mathbb{R} \), for \( i \in \{1, 2\} \). It is necessary to assume that max \( \{\beta_{11}, \beta_{22}\} \geq 0 \) such that the sum of the imaginary parts of the diagonal terms \( \Im V_{11} + \Im V_{22} \) satisfies the condition (3.11). Theorem 3.10 provides us with the fact that \( n = 1 \) (i.e., we need \( V_{11}, V_{22} \in W^{2,2}_{\text{loc}}(\mathbb{R}) \) and \( V_{11}, V_{22} \in W^{1,2}_{\text{loc}}(\mathbb{R}) \)) is enough to treat all kinds of potentials satisfying Assumption II. However, we would like to see what type of potential that \( n = 0 \) can be treated and how fast the decay is when the potential is more regular. For that purposes, let us consider two cases in Theorem 3.11.

Case 1: \(|V_{11}| \) and \(|V_{22}| \) are bounded at \( \pm \infty \). This happens if and only if \( \omega := \max_{i \in \{1, 2\}} \{\alpha_{ii}, \beta_{ii}\} = 0 \). The application of Theorem 3.11 yields that, for all \( n \geq 0 \),

\[ \frac{\|(H_V - \lambda)\Psi_{\lambda,n}\|}{\|\Psi_{\lambda,n}\|} = \mathcal{O}(\lambda^{-(n+1)}) \quad \text{as } \lambda \to +\infty. \]

Furthermore, in this case, the pseudomodes globally localise on \( \mathbb{R} \) without being attached with cut-off functions.

Case 2: \(|V_{11}| \) or \(|V_{22}| \) is not bounded at \( \pm \infty \) (i.e., \( \omega > 0 \)). We consider two smaller cases:

i) \( \Re V_{11} \) and \( \Re V_{22} \) and \( \Im V_{11} - \Im V_{22} \) are bounded at \( \pm \infty \). We do not use cut-off in this situation and for all \( n \geq 0 \),

\[ \frac{\|(H_V - \lambda)\Psi_{\lambda,n}\|}{\|\Psi_{\lambda,n}\|} = \begin{cases} \mathcal{O}\left(\lambda^{-(n+1)}\right), & \omega \leq n + 1, \\ \mathcal{O}\left(\lambda^{-n}\right), & \omega > n + 1, \end{cases} \]

as \( \lambda \to +\infty \). In the case \( \omega > n + 1 \), we employed (3.25).
ii) $\text{Re} V_{11}$ or $\text{Re} V_{22}$ or $\text{Im} V_{11} - \text{Im} V_{22}$ is unbounded at $\pm \infty$. Then, the possible maximum order of $g_{\pm}$ denoted by $\tilde{\omega}$ is $\omega$, i.e. $\tilde{\omega} \leq \omega$. Of course, $\tilde{\omega} > 0$ and thus we can compute $\delta^+ = \delta^+ = \delta \approx \lambda^{\frac{2}{\omega}}$. Applying Theorem 3.11 again, it results that, for all $n \geq 1$,

$$\frac{\| (H_V - \lambda)^n \Psi_{\lambda,n} \|}{\| \Psi_{\lambda,n} \|} = \begin{cases} O \left( \lambda^{-(n+1)} \right), & \omega \leq n + 1, \\ O \left( \lambda^{-(n+1)+\frac{\omega - 1}{2}} \right), & \omega > n + 1, \end{cases}$$

as $\lambda \to +\infty$. We see that in the second case, $n = 0$ can cover all the potentials such that $\omega - 1 < \tilde{\omega} \leq \omega$. For example, $n = 0$ can treat

a) $V = \left( \begin{array}{cc} x^2 + ix & 0 \\ 0 & 0 \end{array} \right)$, with $\omega = \tilde{\omega} = 2$.

b) $V = \left( \begin{array}{cc} i(x + |x|^2) & 0 \\ 0 & ix \end{array} \right)$, with $\omega = 1$ and $\tilde{\omega} = \frac{1}{2}$.

The same thing happens as in the Schrödinger case: the pseudomode with $n = 1$ is sufficient to treat all polynomial-like potential (even the case $V_{12} \neq V_{21}$, see Theorem 3.10). The pseudomode associated with $n = 0$ suffices for potentials growing not faster than linearly.

**Example 3** (Exponential potentials). Consider following potentials $V$ satisfying Assumption. Since we would like to apply Theorem 3.11, we will assume further that $V_{12} = V_{21} = 0$ for sake of simplicity.

1) $V_{11}$ and $V_{22}$ are bounded at $\pm \infty$ with $N \geq 0$:

$$V(x) := \left( \begin{array}{cc} \text{sgn}(x) e^{x|\alpha_1|} & 0 \\ 0 & \text{sgn}(x) e^{x|\alpha_2|} \end{array} \right), \quad \forall |x| \gtrsim 1,$$

with $\alpha_1, \alpha_2 \geq 0$. We choose $f_\pm(x) = 1$ for $|x| \gtrsim 1$. The cut-off function is not needed in this case, i.e., $\chi \equiv 1$. From Proposition 3.9 and Theorem 3.11 we have,

$$\frac{\| (H_V - \lambda)^n \Psi_{\lambda,N} \|}{\| \Psi_{\lambda,N} \|} = O \left( \lambda^{-(N+1)} \right) \quad \text{as } \lambda \to +\infty.$$

2) $V_{11}$ is unbounded at $\pm \infty$ while $V_{22}$ is bounded at $\pm \infty$ with $N \geq 1$, moreover, they oscillate on $\mathbb{R}$:

$$V(x) := \left( \begin{array}{cc} |x|^{\alpha} + \text{sgn}(x) e^{\sin(x)} & 0 \\ 0 & \text{sgn}(x) e^{\cos(x)} \end{array} \right), \quad \forall |x| \gtrsim 1,$$

with $\alpha > 0$. We choose $f_\pm(x) = 1$ for $|x| \gtrsim 1$. From (3.15) and (3.16), we have $\delta^- = \delta^+ = \delta \approx \lambda^{\frac{2}{\omega}}$. Using (3.7) we obtain $\kappa \lesssim \exp(-c\lambda^{\frac{2}{\omega}})$. Theorem 3.11 gives us

$$\frac{\| (H_V - \lambda)^n \Psi_{\lambda,N} \|}{\| \Psi_{\lambda,N} \|} = O \left( \lambda^{-N} \right) \quad \text{as } \lambda \to +\infty.$$

3) $V_{11}$ and $V_{22}$ are unbounded at $\pm \infty$ with $N \geq 1$:

$$V(x) := \left( \begin{array}{cc} \text{sgn}(x) e^{x|\alpha_1|} & 0 \\ 0 & \text{sgn}(x) e^{x|\alpha_2|} \end{array} \right), \quad \forall |x| \gtrsim 1,$$

with $\alpha_1, \alpha_2 > 0$. We choose $f_\pm(x) = |x|^{-1}$ where $\omega := \max \{ \alpha_1, \alpha_2 \}$.

i) If $| \text{Im} V_{11} - \text{Im} V_{22} |$ is bounded at $\pm \infty$, i.e. $\alpha_1 = \alpha_2 = \omega$, from (3.15) and (3.16), we consider two situations:

a) if $0 < \omega \leq 1$, $g_{\pm}$ are bounded at $\pm \infty$. From Proposition 3.9, $\kappa(\lambda) = 0$.

b) if $\omega > 1$, $\delta^\pm = \lambda^{\frac{\omega - 1}{\omega - 1}}$. Thanks to (3.7), $\kappa(\lambda) \lesssim \exp(-c\lambda^{\frac{\omega - 1}{\omega - 1}})$ with some $c > 0$. 

ii) If $|\text{Im} V_{11} - \text{Im} V_{22}|$ is unbounded at $\pm \infty$, by (3.15) and (3.16) and for sufficiently large $\lambda > 0, \delta_\lambda \approx (\ln(\lambda))^{\frac{1}{2}}$. From the definition of $F$, we can obtain the estimate $F(x) \gtrsim |x|^\omega$ for $|x| \gtrsim 1$. Thus, by Proposition 3.9 there exists a constant $c > 0$ such that $\kappa(\lambda) \lesssim \lambda^{-c \ln \lambda}$.

Finally, from Theorem 3.11, we have

$$\frac{\|(H_V - \lambda)\Psi_{\lambda,N}\|}{\|\Psi_{\lambda,N}\|} = \begin{cases} O(\lambda^{-N}), & \omega \leq 1, \\ O(\lambda^{-\frac{1+\epsilon_1}{2}(N+1)+1}), & \omega > 1, \end{cases}$$

as $\lambda \to +\infty$.

4) Superexponential functions:

$$V(x) := \begin{pmatrix} ie^{\sinh(x)} & 0 \\ 0 & -ie^{-\sinh(x)} \end{pmatrix}, \quad \forall x \in \mathbb{R}.$$  

In this example, we choose $f_{\pm}(x) = \cosh(x)$ for $|x| \gtrsim 1$. From (3.15) and (3.16), we can compute, with some constant $C > 0$, that

$$\delta_\lambda = \delta_\lambda^+ = \arcsinh \left( \arccosh \left( \frac{\eta \lambda}{2} \right) \right) \geq C \ln(\ln(\lambda)),$$

when $\lambda > 0$ large enough. Clearly, we have the following rough estimate

$$F(x) = \int_0^{x} 2 \sinh(\sinh(t)) \, dt \gtrsim e^{\frac{2}{2\pi} |x|}, \quad \forall |x| \gtrsim 1,$$

where $d_2 > 0$ is the number appearing in Proposition 3.9. Thus, we obtain $F(\pm d_2 \delta_\lambda^+) \gtrsim \ln(\lambda)^2$ and there exists a constant $c > 0$ such that $\kappa(\lambda) = O(\lambda^{-c \ln \lambda})$. From Theorem 3.10 we have

$$\frac{\|(H_V - \lambda)\Psi_{\lambda,N}\|}{\|\Psi_{\lambda,N}\|} = O(\lambda^{-\frac{1+\epsilon_1}{2}(N+1)+1})$$

as $\lambda \to +\infty$.

4. Pseudomodes for large general pseudoeigenvalues

In this section, we want to construct the pseudomode corresponding to a complex pseudoeigenvalue

$$\lambda = \alpha + i\beta, \quad \text{where} \quad \alpha, \beta \in \mathbb{R}.$$  

The interesting part is that the shape of the pseudospectral region is also revealed in the process of the construction. If the large real part of $\lambda$ played a decisive role in the decaying estimation in the previous section, the imaginary part $\beta$ will take on this role in this section. We shall pay attention to the class of potentials whose $\text{Im} V_{11}$ and $\text{Im} V_{22}$ are identical, i.e. $\text{Im} V_{11} = \text{Im} V_{22} = \mathcal{V}$, and increasing on $\mathbb{R}_+$. The WKB analysis will be performed around a turning point $x_\beta > 0$ which is defined by the equation

$$\mathcal{V}(x_\beta) = \beta. \quad (4.1)$$

Since the non-zero part of the pseudomode will live completely in $\mathbb{R}_+$, it will be more convenient to consider the operators on $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$, instead of $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. The application of the results for the class of operators on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is easily obtained by the trivial extension of the pseudomode of the operators on $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$.

4.1. Allowable shapes of the potentials. Since we do not have to bound the derivatives of components of $V$ on a fixed compact set as in Section 3 the whole space of them can be enlarged to $L^2_{loc}(\mathbb{R}_+)$, instead of $L^2_{loc}(\mathbb{R}_+)$. In order that the turning point $x_\beta$ is uniquely determined, we will assume that $\mathcal{V}$ is strictly monotone for sufficiently large $x > 0$. The assumptions (3.5) and (3.6) are kept the same in this section, so that we can control the transport solutions. To be more specific, we make the following hypothesis.
Assumption II. Let $N \in \mathbb{N}_1$ and $\nu \geq -1$, assume that $V_{11}, V_{22} \in W^{N+1,2}_{\text{loc}}(\mathbb{R}_+)$ and $V_{12}, V_{21} \in W^{N,2}_{\text{loc}}(\mathbb{R}_+)$ satisfy the subsequent conditions:

1) the imaginary parts of $V_{11}$ and $V_{22}$ are equal, i.e. $\text{Im} V_{11} = \text{Im} V_{22} =: \mathcal{V}$ satisfy
\[
\lim_{x \to +\infty} \mathcal{V}(x) = +\infty,
\]
and there exist $\varepsilon_1 > 0$ such that, for all $x \geq 1$,
\[
\mathcal{V}^{(1)}(x) \gtrless \begin{cases} 
\mathcal{V}(x)^\frac{1}{2} x^{\frac{1}{2} \nu + \varepsilon_1}, \\
|\mathcal{V}^{(2)}(x)| x^{-\nu}, 
\end{cases}
\]
2) the sum $\mathcal{U} := \text{Im} V_{12} + \text{Im} V_{21}$ is controlled above by $\mathcal{V}^{(1)}$:
\[
|\mathcal{U}(x)| = o(x^{-\nu} \mathcal{V}^{(1)}(x)) \quad \text{as } x \to +\infty;
\]
3) the derivatives of $V_{ii}$ are controlled by $V_{ii}$, for $i \in \{1, 2\}$,
\[
\forall n \in [[1, N + 1]], \quad |V_{ii}^{(n)}(x)| = \mathcal{O}(x^{n\nu} |V_{ii}(x)|) \quad \text{as } x \to +\infty,
\]
and the differences between $V_{12}$ and $V_{21}$ (their derivatives) are controlled by polynomials
\[
\forall n \in [[0, N]], \quad |(V_{21} - V_{12})^{(n)}(x)| = \mathcal{O}(x^{(n+1)\nu}) \quad \text{as } x \to +\infty.
\]

Comparing with Assumption I, although there are more conditions for the imaginary parts of $\mathcal{V}$ of the diagonal terms, the class of admissible potentials is still very large. Furthermore, our assumption allows to cover functions $\mathcal{V}$ which grow slowly at $+\infty$ such as logarithmic ones. This is interesting because the analogous hypothesis [23, Ass. 5.2] for Schrödinger operators does not allow for this kind of functions.

Remark 4.1. We have the following helpful properties

i) As discussed in [23, Sec. 3], when $\nu < -1$ the condition (4.6) immediately implies that $V_{11}$ and $V_{22}$ are bounded. Thus the rising of $\mathcal{V}$ in (4.2) needs to go along with the condition $\nu \geq -1$. Furthermore, when $\nu \geq -1$, we can deduce from the condition (4.4) that, for large enough $x > 0$ and every $|h| \leq \frac{x - \nu}{2}$,
\[
\mathcal{V}^{(1)}(x + h) \approx \mathcal{V}^{(1)}(x).
\]

In other word, the values of $\mathcal{V}^{(1)}$ can be comparable up to a constant. The proof of (4.8) can be found in [23], but for the reader’s convenience, we recall the proof in a simpler way, for $\nu > -1$,

\[
\left| \ln \left( \frac{|\mathcal{V}^{(1)}(x + h)|}{|\mathcal{V}^{(1)}(x)|} \right) \right| = \left| \int_x^{x+h} \frac{\mathcal{V}^{(2)}(t)}{\mathcal{V}^{(1)}(t)} \, dt \right| \lesssim \begin{cases} 
\int_x^{x+h} |t|^\nu \, dt & h \geq 0, \\
\int_x^{x+h} |t|^\nu \, dt & h \leq 0, \\
\frac{h(x + h)^\nu}{\mathcal{V}^{(1)}(x)} & h \geq 0, \nu \geq 0, \\
\frac{hx^\nu}{\mathcal{V}^{(1)}(x)} & h \geq 0, \nu < 0, \\
\frac{(-h)x^\nu}{\mathcal{V}^{(1)}(x)} & h \leq 0, \nu \geq 0, \\
\frac{(-h)(x + h)^\nu}{\mathcal{V}^{(1)}(x)} & h \leq 0, \nu < 0, 
\end{cases} \lesssim 1.
\]

In the last inequality, we used the observation that, for all $|h| \leq \frac{x - \nu}{2}$ and for $x > 0$,
\[
\frac{x}{2} \leq x + h \leq \frac{3x}{2}.
\]

The case $\nu = -1$ is treated similarly.
ii) From the assumption \((4.2)\) and \((4.3)\), we deduce that
\[
\mathcal{V}(x) \gtrsim x^{\nu+2\varepsilon_1}, \quad \forall x \gtrsim 1. \tag{4.10}
\]
Indeed, we just need to look at the case \(\nu + 2\varepsilon_1 > 0\) (then \(\frac{3}{2}\nu + \varepsilon_1 + 1 > \nu + 1 \geq 0\)). For some fix large \(x_0 > 0\) and for \(x \geq x_0\), we have,
\[
\int_{x_0}^{x} \frac{\mathcal{V}(t)}{t^{\nu+2\varepsilon_1}} \, dt \gtrsim \int_{x_0}^{x} t^{\frac{3}{2}\nu+\varepsilon_1} \, dt \Rightarrow \mathcal{V}(x)^{\frac{1}{2}} - \mathcal{V}(x_0)^{\frac{1}{2}} \gtrsim x^{\frac{3}{2}\nu+\varepsilon_1+1} - x_0^{\frac{3}{2}\nu+\varepsilon_1+1},
\]
and thus \((4.10)\) is obtained when \(x\) is considered to be large.

The cut-off in this case is constructed such that the pseudomode lives around the turning point \(x_\beta\). Namely, we arrange
\[
\xi \in C_0^\infty(\mathbb{R}_+), \quad 0 \leq \xi \leq 1,
\]
\[
\xi(x) = 1, \quad \forall x \in (x_\beta - \delta_\beta, x_\beta + \delta_\beta) =: J', \tag{4.11}
\]
\[
\xi(x) = 0, \quad \forall x \notin (x_\beta - \delta_\beta, x_\beta + \delta_\beta) =: J_\beta,
\]
with
\[
\delta_\beta := \frac{x_\beta^{-\nu}}{2}. \tag{4.12}
\]
We see that if \(\nu < 0\), the support of the pseudomode is able to be extended on \(\mathbb{R}_+\) when \(\beta \to +\infty\). As the WKB construction for the real pseudoeigenvalue, for each \(n \in \mathbb{N}_0\) and each \(\lambda \in \mathbb{C}\), the pseudomode has the form
\[
\Psi_{\lambda,n} := \begin{pmatrix} k_1 u_{\lambda,n} \\ k_2 v_{\lambda,n} \end{pmatrix}, \tag{4.13}
\]
where
\[
k_1(x) := \exp\left(-i \int_{x_\beta}^{x} V_{21}(\tau) \, d\tau\right) \quad \text{and} \quad k_2(x) := \exp\left(-i \int_{x_\beta}^{x} V_{12}(\tau) \, d\tau\right),
\]
\[
u_{\lambda,n} := \xi \exp(-P_{\lambda,n}) \quad \text{and} \quad u_{\lambda,n} = \frac{k_1}{k_2} \frac{\partial_x u_{\lambda,n}}{\lambda + m - V_{22}},
\]
\[
P_{\lambda,n}(x) = \sum_{k=-1}^{n-1} \int_{x_\beta}^{x} \lambda^{-k} \psi^{(1)}_k(t) \, dt,
\]
with \(\xi\) given in \((4.11)\), \((\psi^{(1)}_k)_{k \in [-1,n-1]}\) given in \((2.8)\).

4.2. Main results. Now, we can state our main theorem in the setting of this section.

**Theorem 4.2.** Let Assumption \([\text{II}]\) holds. Assume that there exists a \((\beta\)-dependent\) \(\alpha\) such that the following conditions hold as \(\beta \to +\infty\), for all \(x \in J_\beta\),
\[
(\alpha - m - \text{Re} V_{11}(x)) (\alpha + m - \text{Re} V_{22}(x)) > 0,
\]
\[
|\alpha - m - \text{Re} V_{11}(x)| \approx |\alpha|,
\]
\[
|\alpha + m - \text{Re} V_{22}(x)| \approx |\alpha|,
\]
and
\[
(\beta x_\beta)^{\frac{3}{2}} \lesssim |\alpha|. \tag{4.15}
\]
Let \(\{\psi^{(1)}_k\}_{k \in [-1,N-1]}\) be determined by \((2.8)\) and \(\Psi_{\lambda,N}\) defined as in \((4.13)\). We choose
i) the plus sign in the formula of \(\psi^{(1)}_{-1}\) in \((2.8)\) if \(\alpha - m - \text{Re} V_{11} > 0\),
ii) the minus sign in the formula of \(\psi^{(1)}_{-1}\) in \((2.8)\) if \(\alpha - m - \text{Re} V_{11} < 0\).
Then, for every $c \in (0, 1)$, there exists $\beta_0 > 0$ such that, for all $\beta > \beta_0$,
\[
\frac{\| (H_V - \lambda) \Psi_{\lambda,N} \|}{\| \Psi_{\lambda,N} \|} \leq \kappa(\beta, c) + \sigma^{(N)}(\beta),
\]
in which
\[
\kappa(\beta, c) := \exp \left( -c F \left( \alpha, x_\beta - \frac{\delta_\beta}{2} \right) \right) + \exp \left( -c F \left( \alpha, x_\beta + \frac{\delta_\beta}{2} \right) \right) = o(1),
\]
\[
\sigma^{(N)}(\beta) := \sum_{\ell = -1}^{N-2} \frac{x_\beta^{N+\ell+2\nu}}{|\alpha|^{N+\ell+1}} \left( 1 + \frac{\beta}{|\alpha|} \right)^{N+\ell+2},
\]
with
\[
F(x_\beta, x) := \int_{x_\beta}^x [V(t) - \beta] \, dt.
\]

The first condition in (4.14) is exactly the condition (2.10) that allows the regularity of pseudomodes be inherited from the regularity of the potential through the principal square root of $V_\lambda$. The last two conditions in (4.14) are inspired from the first ones in (3.18) for the real case of $\lambda$. In order to restrain the wild growth of $(\psi_k^{(1)})_{k \in \llbracket 0, N-1 \rrbracket}$, we require (4.15). Through the statement of the Theorem 4.2, it not only indicates the existence of the pseudomode, but also gives us a way to sketch the pseudospectrum around the infinity by looking for the admissible $\alpha$ (see Subsection 4.5). The quantity $\kappa(\beta, c)$ always has an exponential decay at the rate control by the general function $F(x_\beta, x)$. This is also an improvement upon [23, Thm. 5.1] whose rate is only controlled by some polynomial.

4.3. Intermediate steps. On the way to prove our main results, some useful lemmata are designed similar to lemmata used in Subsection 3.3.

**Lemma 4.3.** Let the assumptions of Theorem 4.2 hold. There exists $\beta_0 > 0$ such that, for all $\beta > \beta_0$, for all $\ell \in [1, N+1]$ and for all $x \in J_\beta$,
\[
|V_\lambda^{(\ell)}(x)| \lesssim |V_\lambda(x)| \left( 1 + \frac{\beta}{|\alpha|} \right)^\ell x^{\ell \nu},
\]
\[
|K_\lambda^{(\ell)}(x)| \lesssim |K_\lambda(x)| \left( 1 + \frac{\beta}{|\alpha|} \right)^\ell x^{\ell \nu}.
\]

**Proof.** The proof of this lemma repeats again the procedure of the proof of Lemma 3.4 with the observation that, thanks to the condition (4.14),
\[
\frac{|V_{11}|}{\lambda - m - V_{11}} \lesssim \frac{|\alpha - m - \text{Re} V_{11}| + |\beta - \text{Im} V_{11}| + |\alpha| + \beta + m}{\lambda - m - \text{Re} V_{11} + |\beta - \text{Im} V_{11}|} \lesssim 1 + \frac{\beta}{|\alpha|}.
\]
Correspondingly, we also have
\[
\frac{|V_{22}|}{\lambda + m - V_{22}} \lesssim 1 + \frac{\beta}{|\alpha|}.
\]
Applying these estimates to (3.20) and (3.21), the claims follow directly. Especially, in (3.21), we notice that $\sum_{j=1}^k \alpha_j \leq \sum_{j=1}^k j \alpha_j = k \leq \ell$. □

**Lemma 4.4.** Let the assumptions of Theorem 4.2 hold and $\{\psi_k^{(1)}\}_{k \in \llbracket 0, N-1 \rrbracket}$ be determined by (2.8). There exists $\beta_0 > 0$ such that, for all $\beta > \beta_0$ and for all $x \in J_\beta$, we have
\[
F(x_\beta, x) \leq \int_{x_\beta}^x \text{Re}(\lambda \psi_{-1}^{(1)}(t)) \, dt \lesssim F(x_\beta, x),
\]
and for all $k \in \llbracket 0, N-1 \rrbracket$,
\[
|\lambda^{-k} \psi_k^{(1)}(x)| \lesssim \left( 1 + \frac{\beta}{|\alpha|} \right)^{k+1} \frac{x_\beta^{(k+1)\nu}}{|\alpha|^k}.
\]
Proof. By working as in (3.22), we also have the inequalities for dealing with the denominator of \( \text{Re}(\lambda \psi^{(1)}_1) \),

\[
\sqrt{|V_\lambda| + \text{Re} V_\lambda} \geq \sqrt{2} \sqrt{(\alpha - m - \text{Re} V_{11})(\alpha + m - \text{Re} V_{22})},
\]

\[
\sqrt{|V_\lambda| + \text{Re} V_\lambda} \leq \frac{1}{\sqrt{2}} \sqrt{(|\text{Im} V_{11} - \text{Im} V_{22}|^2 + (2\alpha - \text{Re} V_{11} - \text{Re} V_{22})^2).}
\]  \hspace{1cm} (4.16)

Notice that, there are two cases:

i) If \( \alpha - m - \text{Re} V_{11} > 0 \) on \( J_\beta \), then the formula of \( \text{Re}(\lambda \psi^{(1)}_1) \) has the expression

\[
\text{Re}(\lambda \psi^{(1)}_1) = \frac{1}{\sqrt{2}} \frac{(\mathcal{V} - \beta)(2\alpha - \text{Re} V_{11} - \text{Re} V_{22})}{\sqrt{|V_\lambda| + \text{Re} V_\lambda}}.
\]

ii) If \( \alpha - m - \text{Re} V_{11} < 0 \) on \( J_\beta \), then the formula of \( \text{Re}(\lambda \psi^{(1)}_1) \) has the expression

\[
\text{Re}(\lambda \psi^{(1)}_1) = -\frac{1}{\sqrt{2}} \frac{(\mathcal{V} - \beta)(2\alpha - \text{Re} V_{11} - \text{Re} V_{22})}{\sqrt{|V_\lambda| + \text{Re} V_\lambda}}.
\]

By using the second estimate in (4.16) and observing the sign of the term \( \mathcal{V}(x) - \beta \) on the left and on the right of \( x_\beta \) on \( J_\beta \), it implies that, for all \( x \in J_\beta \),

\[
\int_{x_\beta}^x \text{Re} (\lambda \psi^{(1)}_1(t)) \, dt \geq \int_{x_\beta}^x [\mathcal{V}(x) - \beta] \, dt.
\]

The rest upper bound for the integral of \( \text{Re}(\lambda \psi^{(1)}_1) \) on \( J_\beta \) is given by the first estimate in (4.16) and (4.14).

For each \( k \geq 0 \), we establish the proof for \( \lambda^{-k} \psi^{(1)}_k \) as in the proof of Lemma 3.6. Then, Lemma 4.3 implies that, for all \( x \in J_\beta \),

\[
|\lambda^{-k} \psi^{(1)}_k(x)| \lesssim \left(1 + \frac{\beta}{|\alpha|}\right)^{k+1} \sup_{x \in J_\beta} \frac{x^{(k+1)\nu}}{|V_\lambda(x)|^{k/2}} \lesssim \left(1 + \frac{\beta}{|\alpha|}\right)^{k+1} \frac{x^{k\nu}}{|\alpha|^k}.
\]

Here, the last inequality is given by (4.14) and (4.9).

For all \( x \in J_\beta \), by changing variable twice in integrals, we have

\[
F(x_\beta, x) = (x - x_\beta)^2 \int_0^1 \int_0^1 \xi \mathcal{V}^{(1)}(x_\beta + \tau \xi(x - x_\beta)) \, d\tau d\xi.
\]

From (4.14), it yields that, for all \( x \in J_\beta \),

\[
F(x_\beta, x) \approx \mathcal{V}^{(1)}(x_\beta)(x - x_\beta)^2.
\]  \hspace{1cm} (4.17)

Using this approximation, we obtain the following lemma.

**Lemma 4.5.** Let the assumptions of Theorem 4.2 hold and let \( \{\psi_1^{(1)}\}_{k \in [-1, N-1]} \) be determined by (2.8). Then

i) for every \( \varepsilon \in (0, 1) \) and for every \( \eta \in (0, 1) \), there exists \( \beta_0 > 0 \) such that, for all \( \beta > \beta_0 \) and for all \( x \in J_\beta \setminus (x_\beta - \eta \delta_\beta, x_\beta + \eta \delta_\beta) \),

\[
|k_1(x) \exp(-P_{\lambda,N}(x))| \lesssim \exp \left(-\frac{\beta}{1 - \varepsilon}F(x_\beta, x)\right);
\]

ii) there exists \( C > 0 \) such that for every \( \eta \in (0, 1) \), there exists \( \beta_0 > 0 \) such that, for all \( \beta > \beta_0 \) and for all \( x \in J_\beta \setminus (x_\beta - \eta \delta_\beta, x_\beta + \eta \delta_\beta) \),

\[
|k_1(x) \exp(-P_{\lambda,N}(x))| \gtrsim \exp \left(-CF(x_\beta, x)\right).
\]
Proof. From the equality of two imaginary parts $\text{Im} V_{11} = \text{Im} V_{11}$, we have

$$|\lambda - m - V_{11}(x)| \approx |\lambda + m - V_{22}(x)|, \quad \forall x \in J_\beta.$$  \hspace{1cm} (4.18)

Indeed, from the choice of $\alpha$ in \[4.14\], it turns out that

$$|\lambda - m - V_{11}(x)| \lesssim |\alpha - m - \text{Re} V_{11}| + |\beta - \nu| \lesssim \mathcal{O}(|\alpha|) + |\beta - \nu| \lesssim 1,$$  \hspace{1cm} (4.19)

and the other direction is similar. Therefore, as in the proof of Lemma \[3.8\], we obtain the following approximation on $J_\beta$:

$$|k_1(x) \exp(-P_{\lambda,N}(x))| \approx \exp \left( \frac{1}{2} \int_{x_\beta}^x U(t) \, dt - \sum_{k=-1}^{N-1} \int_{x_\beta}^x \text{Re} (\lambda^{-k} \psi_k^{(1)}(t)) \, dt \right).$$

Let $\eta \in (0, 1)$. Using Lemma \[4.4\] and condition \[4.5\], one obtains, when $x_\beta$ large enough and for all $x \in J_\beta \setminus (x_\beta - \eta \delta_\beta, x_\beta + \eta \delta_\beta)$,

$$\left| \frac{\int_{x_\beta}^x U(t) \, dt}{\int_{x_\beta}^x \text{Re} (\lambda^{\psi_{\lambda_1}^{(1)}}(t)) \, dt} \right| \lesssim \frac{|x - x_\beta| o(x_\beta^{-\nu} \nu^{(1)}(x_\beta))}{|x - x_\beta|^2 \nu^{(1)}(x_\beta)} \lesssim \frac{1}{\eta} o(x_\beta^{-\nu} \nu^{(1)}(x_\beta)) = o(1).$$

By employing Lemma \[4.4\] for $k \in \mathbb{Z}$ and for all $x \in J_\beta \setminus (x_\beta - \eta \delta_\beta, x_\beta + \eta \delta_\beta)$, we have

$$\int_{x_\beta}^x \lambda^{-k} \psi_k^{(1)}(t) \, dt \approx \frac{|x - x_\beta| \left( 1 + \frac{\beta}{|\alpha|} \right)^{k+1} x_\beta^{(k+1)\nu}}{|x - x_\beta|^2 \nu^{(1)}(x_\beta)} \lesssim \frac{1}{\eta} \frac{x_\beta^{-\nu} \nu^{(1)}(x_\beta)}{\nu^{(1)}(x_\beta)} = o(1).$$

Here, in case $|\alpha| \leq \beta$, we have used \[4.15\] and in other case, we have used \[4.10\]. The conclusion of the lemma is obviously deduced from Lemma \[4.4\].

4.4. Proofs of the main results.

Proof of Theorem \[4.2\] Fix $c \in (0, 1)$ and consider $\kappa(\beta, c)$ defined in the statement of the theorem. We start the proof by showing that

$$\left\| \frac{k_1 \exp(-P_{\lambda,N})}{\lambda + m - V_{22}} \xi^{(2)} \right\|_{L^2(\mathbb{R}_+)} + \left\| \frac{k_1 \exp(-P_{\lambda,N})}{\lambda + m - V_{22}} \left( 2 P_{\lambda,N}^{(1)} - \frac{K_{\lambda,N}(1)}{K_{\lambda}} \right) \xi^{(1)} \right\|_{L^2(\mathbb{R}_+)} \leq \kappa(\beta, c).$$  \hspace{1cm} (4.20)

By choosing $\varepsilon$ in Lemma \[4.5\] sufficiently small such that $1 - \varepsilon > c$, we can write $1 - \varepsilon = c + \tilde{c}$, for some $\tilde{c} > 0$. The plan is to use the upper bound of $k_1 \exp(-P_{\lambda,N})$ in Lemma \[4.5\] to
control the terms in the numerator of (4.20) and employ the lower bound of \( k_1 \exp (-P_{\lambda,N}) \) for the denominator. We start with the term attached with \( \xi^{(2)} \):

\[
\left\| \frac{k_1 \exp(-P_{\lambda,N}) \xi^{(2)}}{\lambda + m - V_{22}} \right\|_{L^2(\mathbb{R}^+)}^2 \leq \frac{\delta^4}{|\alpha|^2} \left( \int_{x_{\beta} - \delta_{\beta}}^{x_{\beta} + \delta_{\beta}} \exp(-2(1 - \varepsilon)F(x_\beta, x)) \, dx + \int_{x_{\beta} + \delta_{\beta}}^{x_{\beta} + \delta_{\beta}} \exp(-2(1 - \varepsilon)F(x_\beta, x)) \, dx \right)
\]

\[
\lesssim x_{\beta}^{2\nu} \left( \exp\left(-2(1 - \varepsilon)F(x_\beta, x_{\beta} - \frac{\delta_{\beta}}{2})\right) \right) + \exp\left(-2(1 - \varepsilon)F(x_\beta, x_{\beta} + \frac{\delta_{\beta}}{2})\right).
\]

In the second inequality, we used (4.15) and the fact that \( F(x_{\beta}, x) \) is increasing as \( x \) goes far from \( x_{\beta} \). Furthermore, all appearing polynomial terms will be restrained, with some positive constants \( C_1, C_2 > 0 \), by

\[
\max \left\{ \exp\left(-\tilde{c}F(x_{\beta}, x_{\beta} - \frac{\delta_{\beta}}{2})\right), \exp\left(-\tilde{c}F(x_{\beta}, x_{\beta} + \frac{\delta_{\beta}}{2})\right) \right\} \leq \exp\left(-C_1 V^{(1)}(x_{\beta}) x_{\beta}^{-2\nu}\right) \leq \exp\left(-C_2 x_{\beta}^{-2\nu}\right),
\]

which follows directly from (4.15), the definition of \( \delta_{\beta} \) in (4.12) and the condition (4.3) and (4.10). Thus, we have

\[
\left\| \frac{k_1 \exp(-P_{\lambda,N}) \xi^{(2)}}{\lambda + m - V_{22}} \right\|_{L^2(\mathbb{R}^+)}^2 \lesssim \exp\left(-\left(c + \tilde{c}\right)F(x_{\beta}, x_{\beta} - \frac{\delta_{\beta}}{2})\right) + \exp\left(-\left(c + \tilde{c}\right)F(x_{\beta}, x_{\beta} + \frac{\delta_{\beta}}{2})\right).
\]

The second term in the numerator is bounded in the same manner. In detail, we look at the expression

\[
2P^{(1)}_{\lambda,N} - \frac{K^{(1)}_\lambda}{K_\lambda} = 2i V^{1/2}_\lambda + 2 \sum_{k=0}^{N-1} \lambda^{-k} \psi^{(1)}_{k} \left( \frac{\alpha}{\beta} \right)^{k+1} x_{\beta}^{(k+1)\nu} \lesssim \begin{cases} 
\left( \frac{x_{\beta}^{\nu}}{\beta} \right)^{k+1}, & \text{if } |\alpha| > \beta, \\
\left( \frac{\beta x_{\beta}^{\nu}}{|\alpha|^2} \right)^{k+1}, & \text{if } |\alpha| \leq \beta.
\end{cases}
\]

The term related to \( K_\lambda \) is estimated as same as \( \psi^{(1)}_0 \). Then, from (4.18), we have

\[
\frac{1}{|\lambda + m - V_{22}|} \left| 2P^{(1)}_{\lambda,N} - \frac{K^{(1)}_{\lambda}}{K_{\lambda}} \right| \lesssim \frac{|V_\lambda(x)|^2}{|\lambda + m - V_{22}|} \lesssim 1.
\]
Therefore, we also obtain the same estimate as $(4.22)$ for the term attached with $\xi^{(1)}$ in $(4.20)$. We consider $\eta > 0$ in Lemma 4.5 such that $2\eta < \frac{1}{2}$, then it leads to

$$\|k_1 \exp(-P_{\lambda,N})\xi\|_{L^2(\mathbb{R}_+)}^2 \gtrsim \int_{x_\beta + \eta \delta_\beta}^{x_\beta + 2\eta \delta_\beta} \exp \left(-CF(x_\beta, x)\right) \, dx \gtrsim \eta \delta_\beta \exp \left(-2CF \left(x_\beta, x_\beta + 2\eta \delta_\beta\right)\right) .$$

Using $(4.17)$ and choosing $\eta$ small enough, we have

$$(c + \tilde{c}) F \left(x_\beta, x_\beta - \frac{\delta_\beta}{2}\right) - CF \left(x_\beta, x_\beta + 2\eta \delta_\beta\right) \geq (1 - \mathcal{O}(\eta^2)) (c + \tilde{c}) F \left(x_\beta, x_\beta - \frac{\delta_\beta}{2}\right)$$

and similarly

$$(c + \tilde{c}) F \left(x_\beta, x_\beta + \frac{\delta_\beta}{2}\right) - CF \left(x_\beta, x_\beta + 2\eta \delta_\beta\right) \geq (1 - \mathcal{O}(\eta^2)) (c + \tilde{c}) F \left(x_\beta, x_\beta + \frac{\delta_\beta}{2}\right) .$$

Then, $(4.20)$ follows by choosing $\eta$ sufficiently small and $(4.21)$.

Finally, we estimate the remainder $(2.13)$ by using Lemma 4.3. □

4.5. Applications. Let us list here some examples which are direct consequences of Theorem 4.2. We will see that the shape of the pseudospectrum depends not only on the type of the potentials, but also on their regularity.

Example 4. First of all, we want to consider a kind of logarithmic potential on $\mathbb{R}_+$:

$$V(x) := \left(\begin{array}{cc} i \ln(x) & u(x) \\ u(x) & i \ln(x) \end{array}\right), \quad (4.23)$$

where $u \in W^{N,2}_{\text{loc}}(\mathbb{R}_+)$, with $N \in N_1$, is such that $|u(x)| = o(1)$ as $x \to +\infty$. Then all conditions of Assumption II are satisfied with $\nu = -1$ and any $\varepsilon_1 \in (0, \frac{1}{2})$. Given $\beta > 0$, then $x_\beta > 0$ is determined by the relation $x_\beta = e^\beta$. About the quantity $\kappa(\beta, c)$ for $c \in (0, 1)$, on account of the estimate $(4.21)$, it decays in a superexponential way independent of the choice of $\alpha$, with some constant $C > 0$, $\kappa(\beta, c) = \mathcal{O} \left(\exp(-C e^\beta)\right)$. Since $\text{Re} V_{11} = \text{Re} V_{22} = 0$, the condition $(4.14)$ of Theorem 4.2 are clearly satisfied if and only if

$$|\alpha| > m. \quad (4.24)$$

While $(4.15)$ is assured if and only if

$$|\alpha| \geq \beta \frac{1}{2} \exp \left(-\frac{\beta}{2}\right). \quad (4.25)$$

Therefore, we consider two cases:

a) If $m = 0$, by the choice $(4.25)$, there exist a number $\beta_0 > 0$ and a family $(\Psi_{\lambda,N})_{\lambda \in \Omega}$ whose supports are contained in $\mathbb{R}_+$ such that

$$\frac{\|(H_V - \lambda)\Psi_{\lambda,N}\|}{\|\Psi_{\lambda,N}\|} = \begin{cases} \mathcal{O} \left(\beta^{-N} \exp(-(N + 1)\beta)\right), & \text{if } |\alpha| > \beta, \\
\mathcal{O} \left(\beta^{\frac{1}{2}} \exp \left(-\frac{\beta}{2}\right)\right), & \text{if } |\alpha| \leq \beta, \end{cases}$$

where

$$\Omega := \left\{ \alpha + i\beta \in \mathbb{C} : \beta > \beta_0 \text{ and } |\alpha| \geq \beta \frac{1}{2} \exp \left(-\frac{\beta}{2}\right) \right\}. \quad (4.26)$$

b) If $m > 0$, by the choice $(4.24)$, there exist a number $\beta_0 > 0$ and a family $(\Psi_{\lambda,N})_{\lambda \in \Omega}$ whose supports are contained in $\mathbb{R}_+$ such that

$$\frac{\|(H_V - \lambda)\Psi_{\lambda,N}\|}{\|\Psi_{\lambda,N}\|} = \begin{cases} \mathcal{O} \left(\beta^{-N} \exp(-(N + 1)\beta)\right), & \text{if } |\alpha| > \beta, \\
\mathcal{O} \left(\beta^{2N} \exp(-(N + 1)\beta)\right), & \text{if } |\alpha| \leq \beta, \end{cases}$$
where
\[
\Omega := \{ \alpha + i\beta \in \mathbb{C} : \beta > \beta_0 \text{ and } |\alpha| > m \}.
\] (4.27)

From the definition of \(\Omega\), we see that the pseudospectral region contains even points which stay very close to the line \(\alpha = 0\) when \(m = 0\) and when \(\beta\) large enough (see Figure 1).

\[\text{Figure 1. Illustration of the shapes of } \Omega \text{ (in cyan color) with the logarithmic potential } V \text{ given in (4.23) in two cases:}\]

(A) \(m = 0\): The “wine decanter” curve is the graph of \(|\alpha| = \beta^{\frac{1}{2}} \exp \left(-\frac{\beta}{2}\right)\);
(B) \(m > 0\): The vertical lines are the graphs of \(|\alpha| = m\).

**Example 5.** Next, we want to study the polynomial-like potential on \(\mathbb{R}_+\) in the following form
\[
V(x) = \begin{pmatrix} ix^\gamma & v(x) \\ v(x) & ix^\gamma \end{pmatrix},
\] (4.28)
where \(\gamma > 0, v \in W^{N,2}_{\text{loc}}(\mathbb{R}_+)\) with \(N \in \mathbb{N}\) such that \(|v(x)| = o(x^\gamma)\) for \(x \to +\infty\). Then all the conditions of Assumption II are satisfied with \(\nu = -1\) and any \(\varepsilon_1 \in (0, \frac{3+1}{2})\). Given \(\beta > 0\), then \(x_\beta > 0\) is determined, see (4.1), by \(x_\beta = \beta^{\frac{1}{2}}\gamma\). From the estimate (4.21), for any \(c \in (0,1)\), the quantity \(\kappa(\beta, c)\) has an exponentially decay, with some \(C > 0\),
\[
\kappa(\beta, c) = \mathcal{O} \left( \exp \left(-C\beta^{\frac{2+1}{\gamma}} \right) \right).
\]
Since \(\text{Re} V_{11} = \text{Re} V_{22} = 0\), the constraints in (4.14) imposed on \(\alpha\) are satisfied if and only if \(|\alpha| > m\). (4.29)

We can compute directly the left-hand side of (4.15) as a function of \(\beta\):
\[
(\beta x_\beta^\nu)^{\frac{1}{2}} = \beta^{\frac{1}{2}} \gamma - 1.
\]
We consider two cases:

i) If \(\gamma \geq 1\), we may take \(\alpha\) as (with \(\varepsilon > 0\))
\[
|\alpha| \gtrsim \beta^{\frac{1}{2} - \frac{1}{\gamma} + \varepsilon}.
\]

Next, we are concerned about how small \(\varepsilon\) can be chosen such that we have the decay of \(\sigma^N(\beta)\). For \(\beta > 0\) large enough, we have
\[
\sigma^{(N)}(\beta) \lesssim \begin{cases} 
\beta^{-(1+\frac{1}{2})N - \frac{1}{4}}, & \text{if } |\alpha| > \beta, \\
\beta^{-\varepsilon(2N+1)+\frac{1}{2} - \frac{1}{\gamma}}, & \text{if } |\alpha| \leq \beta.
\end{cases}
\]

In order to have a decay for \(\sigma^{(N)}(\beta)\) as \(\beta \to +\infty\), we choose
\[
\varepsilon = \frac{1}{4N + 2} \frac{\gamma}{\gamma} + \eta, \quad \text{with } \eta > 0.
\]
In summary, for any \( \eta > 0 \), there exist \( \beta_0 > 0 \) and a family \( (\Psi_{\lambda,N})_{\lambda \in \Omega} \) such that

\[
\frac{\| (H_V - \lambda) \Psi_{\lambda,N} \|}{\| \Psi_{\lambda,N} \|} = \begin{cases} 
\mathcal{O} \left( \beta^{-\frac{1}{2}} N^{-\frac{1}{2}} \right), & \text{if } |\alpha| > \beta, \\
\mathcal{O} \left( \beta^{-\frac{1}{2}} N^{-\frac{1}{2}} \right), & \text{if } |\alpha| \leq \beta,
\end{cases}
\]

has the decay at the polynomial rate and the pseudospectral region \( \Omega \) is defined as

\[
\Omega := \left\{ \alpha + i\beta \in \mathbb{C} : \beta > \beta_0 \text{ and } |\alpha| \geq \beta \left( \frac{1}{2} + \frac{1}{m+1} \right) \right\}.
\] (4.30)

ii) If \( 0 < \gamma < 1 \), the condition \( (4.15) \) is equivalent to

\[
|\alpha| \geq \beta^{\frac{2}{1-\gamma}}.
\] (4.31)

Depending on the value of \( m \), we can compare two conditions \( (4.29) \) and \( (4.31) \) as \( \beta \to +\infty \). We have two cases as below.

(a) If \( m = 0 \), by the choice \( (4.31) \), there exists \( \beta_0 > 0 \) such that, for all \( \lambda \) belonging to the set

\[
\Omega = \left\{ \alpha + i\beta \in \mathbb{C} : \beta > \beta_0 \text{ and } |\alpha| > \beta^{\frac{1}{1-\gamma}} \right\},
\] (4.32)

our problem has the decay

\[
\frac{\| (H_V - \lambda) \Psi_{\lambda,N} \|}{\| \Psi_{\lambda,N} \|} = \begin{cases} 
\mathcal{O} \left( \beta^{-\frac{1}{2}} N^{-\frac{1}{2}} \right), & \text{if } |\alpha| > \beta, \\
\mathcal{O} \left( \beta^{-\frac{1}{2}} N^{-\frac{1}{2}} \right), & \text{if } |\alpha| \leq \beta.
\end{cases}
\]

(b) If \( m > 0 \), by the choice \( (4.29) \), there exists \( \beta_0 > 0 \) such that, for all \( \lambda \) belonging to the set

\[
\Omega = \{ \alpha + i\beta \in \mathbb{C} : \beta > \beta_0 \text{ and } |\alpha| > m \},
\] (4.33)

our problem has the decay

\[
\frac{\| (H_V - \lambda) \Psi_{\lambda,N} \|}{\| \Psi_{\lambda,N} \|} = \begin{cases} 
\mathcal{O} \left( \beta^{-\frac{1}{2}} N^{-\frac{1}{2}} \right), & \text{if } |\alpha| > \beta, \\
\mathcal{O} \left( \beta^{-\frac{1}{2}} N^{-\frac{1}{2}} \right), & \text{if } |\alpha| \leq \beta.
\end{cases}
\]

The reader is also invited to compare our results with the application of the same method for Schrödinger operators with the polynomial potential \( V(x) := ix^\gamma \) with \( \gamma \geq 1 \) in [23, Ex. 5.3]. We see that the pseudospectra of the Dirac operators are larger than those of the Schrödinger operators. While the outcome \( \alpha \) is kept between two curves in the Schrödinger case, the outcome \( \alpha \) in the Dirac case is just bounded from below by a curve. Technically, this can be explained by the appearance of \( \lambda \) in the denominator of the estimate \( \text{Re} \left( \lambda \psi^{(1)}_\lambda \right) \) for the Schrödinger operator, in which the above bound of \( \alpha \) is employed (to be clear, [23, Est. (5.9)]). Furthermore, [23, Ex. 5.3] only investigates the case \( \gamma \geq 1 \), while ours produce the results for even \( 0 < \gamma < 1 \). Finally, the decay of the problem in [23] is attained only when \( N \) large enough, while our method gives us the decay even for small \( N \). In Figure 2, we see some representatives for the shape of \( \Omega \) corresponding to the power \( \gamma \) and the value of \( m \) (when \( 0 < \gamma < 1 \)). For the faster growing of the polynomial, the pseudospectrum region \( \Omega \) stands further away the axis \( \alpha = 0 \).

**Example 6.** The next example that we want to study is the potential whose \( \text{Im} \, V_{11} = \text{Im} \, V_{22} \) is an exponential function:

\[
V(x) := \begin{pmatrix} ie^{x\gamma} & u(x) \\ u(x) & ie^{x\gamma} \end{pmatrix},
\] (4.34)

where \( \gamma > 0 \), \( u \in W^{N,2}_{\text{loc}}(\mathbb{R}_+) \), with \( N \in \mathbb{N}_1 \), such that \( |u(x)| = o(e^{x\gamma}) \) as \( x \to +\infty \). All conditions of Assumption [11] are satisfied with \( \nu = \gamma - 1 \) and any \( \varepsilon_1 > 0 \). Given \( \beta > 1 \), the
\( \gamma = 2 \).

\( \gamma = \frac{1}{2} \) and \( m = 0 \).

\( \gamma = \frac{1}{2} \) and \( m > 0 \).

Figure 2. Illustrations of the shapes of \( \Omega \) (in cyan color) associated with the potential \( V \) given in (4.28) corresponding to \( \gamma \) and the value of \( m \). The magenta curves are, respectively,

(A) \( \gamma = 2 \): \(|\alpha| = \beta^{\frac{1}{2}} + \frac{1}{3} \), here we took \( \eta \) in (4.30) such that \( \frac{1}{8n+4} + \eta = \frac{1}{3} \),

(B) \( \gamma = \frac{1}{2} \) and \( m = 0 \): \(|\alpha| = \beta^{\frac{1}{2}} \),

(C) \( \gamma = \frac{1}{2} \) and \( m > 0 \): \(|\alpha| = m \).

turning point \( x_\beta > 0 \) is determined by the relation \( x_\beta = \ln(\beta)^{\frac{1}{2}} \). For any \( c \in (0, 1) \), thanks to (4.21), there is some \( C > 0 \) such that

\[ \kappa(\beta, c) = O\left(\exp\left(-C\beta \ln(\beta)^{\frac{1}{2}-1}\right)\right). \]

Again, since \( \text{Re} V_{11} = \text{Re} V_{22} = 0 \) the conditions in (4.14) are equivalent to the fact (4.29). We may take \( \alpha \) that is, with some \( \varepsilon > 0 \), \(|\alpha| \gtrsim \beta^{\frac{1}{2}} + \frac{1}{2} \ln(\beta)^{\frac{1}{2}} \), such that the condition (4.31) is satisfied. Under this choice of \( \alpha \), we can bound above \( \sigma^{(N)}(\beta) \)

\[ \sigma^{(N)}(\beta) \leq \begin{cases} 
\beta^{-N} \ln(\beta)^{\frac{2(\varepsilon-1)}{\gamma}N}, & \text{if } |\alpha| > \beta \text{ and } \gamma \geq 1, \\
\beta^{-N} \ln(\beta)^{\frac{\gamma-1}{\gamma}(N+1)}, & \text{if } |\alpha| > \beta \text{ and } \gamma < 1, \\
\beta^\frac{1}{\gamma-\varepsilon}(2N+1) \ln(\beta)^{\frac{1}{\gamma-\varepsilon}} & \text{if } |\alpha| \leq \beta.
\end{cases} \]

In order to get the decay of \( \sigma^{(N)}(\beta) \) as \( \beta \to +\infty \), we choose

\[ \varepsilon = \frac{1}{4N+2} + \eta, \quad \text{with } \eta > 0. \]

In conclusion: for any \( \eta > 0 \), there exist \( \beta_0 > 0 \) and a family \( (\Psi_{\lambda,N})_{\lambda \in \Omega} \) such that

\[ \frac{\| (H_V - \lambda) \Psi_{\lambda,N} \|}{\| \Psi_{\lambda,N} \|} = \begin{cases} 
O\left(\beta^{-N} \ln(\beta)^{\frac{2(\varepsilon-1)}{\gamma}N}\right), & \text{if } |\alpha| > \beta \text{ and } \gamma \geq 1, \\
O\left(\beta^{-N} \ln(\beta)^{\frac{\gamma-1}{\gamma}(N+1)}\right), & \text{if } |\alpha| > \beta \text{ and } \gamma < 1, \\
O\left(\beta^{-\eta(2N+1)} \ln(\beta)^{\frac{1}{\gamma-\varepsilon}}\right), & \text{if } |\alpha| \leq \beta,
\end{cases} \]

where

\[ \Omega := \left\{ \alpha + i \beta \in \mathbb{C} : \beta > \beta_0 \text{ and } |\alpha| \gtrsim \beta^{\frac{1}{2}} + \frac{1}{4N+2} + \eta \ln(\beta)^{\frac{1}{\gamma-\varepsilon}} \right\}. \] (4.35)

In Figure 3, some sketches are created for the imagination of the pseudospectral region \( \Omega \) in the exponential cases. As in the polynomial cases, the higher \( \gamma \) is, the further \( \Omega \) stays away from the axis \( \alpha = 0 \).

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Figure 3. Illustrations of the shapes of Ω (in cyan color) associated with the potential $V$ given in (4.34) corresponding to $γ$. Here we assume that $N$ is large enough such that we can take $η$ small enough satisfying \( \frac{1}{4N^2} + η = \frac{1}{10} \) on the right-hand side of (4.35). The magenta curves are, respectively,

(A) $γ = 2$: $|α| = β^{\frac{1}{2} + \frac{1}{N}} \ln(β)^{\frac{1}{2}}$;
(B) $γ = 1$: $|α| = β^{\frac{1}{2} + \frac{1}{N}}$;
(C) $γ = \frac{1}{2}$: $|α| = β^{\frac{1}{2} + \frac{1}{N}} \ln(β)^{-\frac{1}{2}}$.

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