Symplectic manifolds and Hamiltonian dynamical systems

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Abstract. This paper is devoted to the study of symplectic manifolds and their connection with Hamiltonian dynamical systems. We review some properties and operations on these manifolds and see how they intervene when studying the complete integrability of these systems, with detailed proofs. Several explicit calculations for which references are not immediately available are given. These results are exemplified by applications to some Hamiltonian dynamical systems.

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1 Introduction

It is well known that symplectic manifolds play a crucial role in classical mechanics, geometrical optics and thermodynamics and currently has conquered a rich territory, asserting himself as a central branch of differential geometry and topology. In addition to its activity as an independent subject, symplectic manifolds are strongly stimulated by important interactions with many mathematical and physical specialities among others. The aim of this paper is to study some properties of symplectic manifolds and Hamiltonians dynamical systems and to review some operations on these manifolds, with detailed proofs. This paper is organized as follows: the first section is an introduction to the subject. In section 2 we begin by briefly recalling some notions about symplectic vector spaces. Section 3 develops the explicit cal-
calculation of symplectic structures on a differentiable manifold. Section 4 is
devoted to the study of some properties of one-parameter groups of diffeo-
morphisms or flow, Lie derivative, interior product and Cartan’s formula.
We review some interesting properties and operations on differential forms,
with detailed proofs. Section 5 deals with the study of a central theorem of
symplectic geometry namely Darboux’s theorem : the symplectic manifolds
\((M, \omega)\) of dimension \(2m\) are locally isomorphic to \((\mathbb{R}^{2m}, \omega)\). The classic proof
[4] given by Darboux of his theorem is by recurrence on the dimension of
the variety. We give a preview and see another demonstration [40] due to
Weinstein based on a result of Moser [31]. Section 6 contains some technical
statements concerning Hamiltonian vector fields. The latter form a Lie subal-
gebra of the space vector field and we show that the matrix associated with a
Hamiltonian system forms a symplectic structure. Several properties concern-
ing Hamiltonian vector fields, their connection with symplectic manifolds,
Poisson manifolds or Hamiltonian manifolds as well as interesting examples
are studied in section 7. We will see in section 8, how to define a symplectic
structure on the orbit of the coadjoint representation of a Lie group. The re-
mainder is dedicated to the explicit computation of symplectic structures on
adjoint and coadjoint orbits of a Lie group with particular attention given to
the groups \(SO(3)\) and \(SO(4)\). Integrable Hamiltonian systems are nonlinear
ordinary differential equations described by a Hamiltonian function and pos-
sessing sufficiently many independent constants of motion in involution. By
the Arnold-Liouville theorem [2, 4, 9, 24], the regular compact level mani-
folds defined by the intersection of the constants of motion are diffeomorphic
to a real torus on which the motion is quasi-periodic as a consequence of the
following differential geometric fact : a compact and connected \(n\)-dimensional
manifold on which there exist \(n\) vector fields which commute and are inde-
pendent at every point is diffeomorphic to an \(n\)-dimensional real torus and
there is a transformation to so-called action-angle variables, mapping the
flow into a straight line motion on that torus. Outline in section 9 we give
a proof as direct as possible of the Arnold-Liouville theorem and we make
a careful study of its connection with the concept of completely integrable
systems and finally, in section 10, apply it to concrete situations : the prob-
lem of the rotation of a rigid body about a fixed point and Yang-Mills fields
valued in the Lie algebra associated to the Lie group \(SU(2)\).

2 Symplectic Vector Spaces

First, remember that a symplectic vector space \((E, \omega)\) is a vector space
\(E\) over a field equipped with a bilinear form \(\omega : E \times E \rightarrow \mathbb{R}\) which is
alternating (or antisymmetric, i.e, \(\omega(x, y) = -\omega(y, x)\), \(\forall x, y \in E\)) and non
degenerate (i.e., \(\omega(x, y) = 0\), \(\forall y \in E \Rightarrow x = 0\)). The form \(\omega\) is called
symplectic form (or symplectic structure). The dimension of a symplectic
vector space is necessarily even. We show (using a reasoning similar to the Gram-Schmidt orthogonalization process) that any symplectic vector space \((E, \omega)\) has a base \((e_1, ..., e_{2m})\) called symplectic basis (or canonical basis), satisfying the following relations:

\[
\omega(e_{m+i}, e_j) = \delta_{ij}, \quad \omega(e_i, e_j) = \omega(e_{m+i}, e_{m+j}) = 0.
\]

Note that each \(e_{m+i}\) is orthogonal to all base vectors except \(e_i\). In terms of symplectic basic vectors \((e_1, ..., e_{2m})\), the matrix \(\omega_{ij}\) where \(\omega_{ij} \equiv \omega(e_i, e_j)\) has the form

\[
\begin{pmatrix}
\omega_{11} & ... & \omega_{12m} \\
... & ... & ... \\
\omega_{2m1} & ... & \omega_{2m2m}
\end{pmatrix} = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix},
\]

where \(I_m\) denotes the \(m \times m\) unit matrix.

**Example 1** The vector space \(\mathbb{R}^{2m}\) with the form

\[
\omega(x, y) = \sum_{k=1}^{m} (x_{m+k}y_k - x_ky_{m+k}), \quad x \in \mathbb{R}^{2m}, \quad y \in \mathbb{R}^{2m},
\]

is a symplectic vector space. Let \((e_1, ..., e_m)\) be an orthonormal basis of \(\mathbb{R}^m\). Then, \(((e_1, 0), ..., (e_m, 0), (0, e_1), ..., (0, e_m))\) is a symplectic base of \(\mathbb{R}^{2m}\).

Let \((E, \omega)\) be a symplectic vector space and \(F\) a vector subspace of \(E\). Let \(F^\perp\) be the orthogonal (symplectic) of \(F\), i.e., the vector subspace of \(E\) defined by

\[
F^\perp = \{ x \in E : \forall y \in F, \omega(x, y) = 0 \}.
\]

The subspace \(F\) is isotropic if \(F \subset F^\perp\), coisotropic if \(F^\perp \subset F\), Lagrangian if \(F = F^\perp\) and symplectic if \(F \cap F^\perp = \{0\}\). If \(F, F_1\) and \(F_2\) are subspaces of a symplectic space \((E, \omega)\), then

\[
\dim F + \dim F^\perp = \dim E, \quad (F^\perp)^\perp = F,
\]

\[
F_1 \subset F_2 \implies F_2^\perp \subset F_1^\perp,
\]

\[
(F_1 \cap F_2)^\perp = F_1^\perp + F_2^\perp, \quad F_1^\perp \cap F_2^\perp = (F_1 + F_2)^\perp,
\]

\(F\) coisotropic if and only if \(F^\perp\) isotropic and \(F\) Lagrangian if and only if \(F\) isotropic and coisotropic.
3 Symplectic Manifolds

We will define a symplectic structure on a differentiable manifold and study some properties. A symplectic structure (or symplectic form) on an even-dimensional differentiable manifold $M$ is a closed non-degenerate differential $2$-form $\omega$ defined everywhere on $M$. The non-degeneracy condition means that:

$$\forall x \in M, \ \forall \xi \neq 0, \ \exists \eta : \omega(\xi, \eta) \neq 0, (\xi, \eta \in T_x M).$$

The pair $(M, \omega)$ (or simply $M$) is called a symplectic manifold. Hence, at a point $p \in M$, we have a non-degenerate antisymmetric bilinear form on the tangent space $T_p M$, which explains why the dimension of the $M$ manifold is even.

**Example 2** The space $M = \mathbb{R}^{2m}$ with the $2$-form

$$\omega = \sum_{k=1}^{m} dx_k \wedge dy_k,$$

is a symplectic manifold. The vectors

$$\left( \left( \frac{\partial}{\partial x_1} \right)_p , \ldots , \left( \frac{\partial}{\partial x_m} \right)_p \right), \left( \frac{\partial}{\partial y_1} \right)_p , \ldots , \left( \frac{\partial}{\partial y_m} \right)_p , \ p \in M,$$

constitute a symplectic basis of the tangent space $T_p M$. Similarly, space $\mathbb{C}^m$ with the form

$$\omega = \frac{i}{2} \sum_{k=1}^{m} dz_k \wedge d\overline{z}_k,$$

is a symplectic manifold. Note that this form coincides with that of the preceding example by means of the identification $\mathbb{C}^m \simeq \mathbb{R}^{2m}$, $z_k = x_k + iy_k$.

The Riemann surfaces are symplectic manifolds. Other examples of symplectic manifolds which will not be considered here (and for which I refer for example to [2, 26]) are the kählerian manifolds as well as complex projective manifolds. Another important class of symplectic manifolds consists of the coadjoint orbits $\mathcal{O} \subset \mathcal{G}^*$, where $\mathcal{G}$ is the algebra of a Lie group $\mathcal{G}$ and $\mathcal{G}_\mu = \{ \text{Ad}_g^* \mu : g \in \mathcal{G} \}$,

is the orbit of $\mu \in \mathcal{G}^*$ under the coadjoint representation (to see further).

We will see that the cotangent bundle $T^* M$ (that is, the union of all cotangent spaces at $M$) admits a natural symplectic structure. The phase spaces of the Hamiltonian systems studied below are symplectic manifolds and often they are cotangent bundles equipped with the canonical structure.
Theorem 1 Let $M$ be a differentiable manifold of dimension $m$ and let $T^*M$ be its cotangent bundle. Then $T^*M$ possesses in a natural way a symplectic structure and in a local coordinate $(x_1, \ldots, x_m, y_1, \ldots, y_m)$, the form $\omega$ is given by

$$\omega = \sum_{k=1}^{m} dx_k \wedge dy_k.$$ 

Proof. Let $(U, \varphi)$ be a local chart in the neighborhood of $p \in M$,

$$\varphi : U \subset M \rightarrow \mathbb{R}^m, \quad p \mapsto \varphi(p) = \sum_{k=1}^{m} x_k e_k,$$

where $e_k$ are the vectors basis of $\mathbb{R}^m$. Consider the canonical projections $TM \rightarrow M$, and $T(T^*M) \rightarrow T^*M$, of tangent bundles respectively to $M$ and $T^*M$ on their bases. We notice

$$\pi^* : T^*M \rightarrow M,$$

the canonical projection and

$$d\pi^* : T(T^*M) \rightarrow TM,$$

its linear tangent application. We have

$$\varphi^* : T^*M \rightarrow \mathbb{R}^{2m}, \quad \alpha \mapsto \varphi^*(\alpha) = \sum_{k=1}^{m} (x_k e_k + y_k \varepsilon_k),$$

where $\varepsilon_k$ are the basic forms of $T^*\mathbb{R}^m$ and $\alpha$ denotes $\alpha_p \in T^*M$. So, if $\alpha$ is a 1-form on $M$ and $\xi_\alpha$ is a vector tangent to $T^*M$, then

$$d\varphi^* : T(T^*M) \rightarrow T\mathbb{R}^{2m} = \mathbb{R}^{2m}, \quad \xi_\alpha \mapsto d\varphi^*(\xi_\alpha) = \sum_{k=1}^{m} (\beta_k e_k + \gamma_k \varepsilon_k),$$

where $\beta_k, \gamma_k$ are the components of $\xi_\alpha$ in the local chart of $\mathbb{R}^{2m}$. Let

$$\lambda_\alpha(\xi_\alpha) = \alpha(d\pi^*\xi_\alpha) = \alpha(\xi),$$

where $\xi$ is a tangent vector to $M$. Let $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ be a system of local coordinates compatible with a local trivialization of the tangent bundle $T^*M$. Let’s show that :

$$\lambda_\alpha(\xi_\alpha) = \alpha \left( \sum_{k=1}^{m} \beta_k e_k \right) = \sum_{k=1}^{m} (x_k e_k + y_k \varepsilon_k) \left( \sum_{j=1}^{m} \beta_j e_j \right) = \sum_{k=1}^{m} \beta_k y_k.$$


Indeed, remember that if \((x_1, ..., x_m)\) is a system of local coordinates around \(p \in M\), like all \(\alpha \in T^* M\) can be written in the basis \((dx_1, ..., dx_m)\) under the form

\[
\alpha = \sum_{k=1}^{m} \alpha_k dx_k,
\]

then by defining local coordinates \(y_1, ..., y_m\) by \(y_k(\alpha) = y_k, k = 1, ..., m\), the 1-form \(\lambda\) is written

\[
\lambda = \sum_{k=1}^{m} y_k dx_k.
\]

The form \(\lambda\) on the cotangent bundle \(T^* M\) doing correspondence \(\lambda_\alpha\) to \(\alpha\) is called Liouville form. We have

\[
\lambda(\alpha) = \sum_{k=1}^{m} y_k(\alpha) dx_k(\alpha),
\]

\[
\lambda(\alpha)(\xi_\alpha) = \sum_{k=1}^{m} y_k(\alpha) dx_k(\alpha) \left( \sum_{j=1}^{m} \beta_j e_j + \gamma_j e_j \right),
\]

i.e.,

\[
\lambda(\alpha)(\xi_\alpha) = \sum_{k=1}^{m} y_k \beta_k = \lambda_\alpha(\xi_\alpha), \quad \lambda = \sum_{k=1}^{m} y_k dx_k.
\]

The symplectic structure of \(T^* M\) is given by the exterior derivative of \(\lambda\), i.e., the 2-form \(\omega = -d\lambda\). The forms \(\lambda\) and \(\omega\) are called canonical forms on \(T^* M\). We can visualize all this with the help of the following diagram:

\[
\begin{array}{ccc}
\mathbb{R} & \xleftarrow{\lambda} & T^*(T^*M) \\
\downarrow{d\pi^*} & & \downarrow{\varphi^*} \\
T(T^*M) & \rightarrow & T^* M \\
\downarrow{d\pi^*} & & \downarrow{\pi^*} \\
\mathbb{R} & \xrightarrow{\alpha(\xi)} & TM \\
\downarrow{\varphi} & & \downarrow{\pi} \\
\mathbb{R}^m & \rightarrow & M \\
\end{array}
\]

The form \(\omega\) is closed: \(d\omega = 0\) since \(d \circ d = 0\) and it is non degenerate. To show this last property, just note that the form is well defined independently of the chosen coordinates but we can also show it using a direct calculation. Indeed, let \(\xi = (\xi_1, ..., \xi_{2m}) \in T_p M\) and \(\eta = (\eta_1, ..., \eta_{2m}) \in T_p M\). We have

\[
\omega(\xi, \eta) = \sum_{k=1}^{m} dx_k \wedge dy_k(\xi, \eta) = \sum_{k=1}^{m} (dx_k(\xi) dy_k(\eta) - dx_k(\eta) dy_k(\xi)).
\]

Since \(dx_k(\xi) = \xi_{m+k}\) is the \((m+k)\)th-component of \(\xi\) and \(dy_k(\xi) = \xi_k\) is the \(k\)th-component of \(\xi\), then

\[
\omega(\xi, \eta) = \sum_{k=1}^{m} (\xi_{m+k} \eta_k - \eta_{m+k} \xi_k) = (\xi_1 \cdots \xi_{2m}) \left( \begin{array}{cc} O & -I \\ I & O \end{array} \right) \left( \begin{array}{c} \eta_1 \\ \vdots \\ \eta_{2m} \end{array} \right),
\]
with $O$ the null matrix and $I$ the unit matrix of order $m$. Then, for all $x \in M$ and for all $\xi = (\xi_1, ..., \xi_{2m}) \neq 0$, it exists $\eta = (\xi_{m+1}, ..., \xi_{2m}, -\xi_1, ..., -\xi_m)$ such that :

$$\omega(\xi, \eta) = \sum_{k=1}^{m} (\xi_{m+k}^2 - \xi_k^2) \neq 0,$$

because $\xi_k \neq 0$, for any integer $k = 1, ..., 2m$. In the local coordinate system $(x_1, ..., x_m, y_1, ..., y_m)$, this symplectic form is written

$$\omega = \sum_{k=1}^{n} dx_k \wedge dy_k,$$

which completes the proof. □

A manifold $M$, is said to be orientable if there exists on $M$ an atlas such that the Jacobian of any change of chart is strictly positive or if $M$ has a volume form (i.e., a differential form that does not vanish anywhere). For example, $\mathbb{R}^n$ is oriented by the volume form $dx_1 \wedge ... \wedge dx_n$. The circle $S^1$ is oriented by $d\theta$. The torus $T^2 = S^1 \times S^1$ is oriented by the volume form $d\theta \wedge d\varphi$. All holomorphic manifolds are orientable.

**Theorem 2**

a) A closed differential 2-form $\omega$ on a differentiable manifold $M$ of dimension $2m$ is symplectic, if and only if, $\omega^m$ is a volume form.

b) Any symplectic manifold is orientable.

c) Any orientable manifold of dimension two is symplectic. On the other hand in even dimensions larger than 2, this is no longer true.

**Proof.**

a) Indeed, this is due to the fact that the non-degeneracy of $\omega$ is equivalent to the fact that $\omega^m$ is never zero.

b) In a system of symplectic charts $(x_1, ..., x_{2m})$, we have

$$\omega = dx_1 \wedge dx_{m+1} + ... + dx_m \wedge dx_{2m}.$$  

Therefore,

$$\omega^m = dx_1 \wedge dx_{m+1} \wedge ... \wedge dx_m \wedge dx_{2m},$$

$$= (-1)^{m(m-1)/2} dx_1 \wedge dx_2 \wedge ... \wedge dx_{2m},$$

which means that the $2m$-form $\omega^m$ is a volume form on the manifold $M$ and therefore this one is orientable. The orientation associated with the differential form $\omega$ is the canonical orientation of $\mathbb{R}^{2m}$.

c) This results from the fact that any differential 2-form on a 2-manifold is always closed. □

**Theorem 3**

Let $\alpha$ be a differential 1-form on the manifold $M$ and denote by $\alpha^* \lambda$ the reciprocal image of the Liouville form $\lambda$ on the cotangent bundle $T^*M$. Then, we have $\alpha^* \lambda = \alpha$. 

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Proof. Since $\alpha : M \rightarrow T^*M$, we can consider the reciprocal image that we note $\alpha^* : T^*T^*M \rightarrow T^*M$, of $\lambda : T^*M \rightarrow T^*T^*M$ (Liouville form), such that, for any vector $\xi$ tangent to $M$, we have the following relation

$$\alpha^*\lambda(\xi) = \lambda(\alpha)(d\alpha\xi).$$

Since $d\alpha$ is an application $TM \rightarrow TT^*M$, then

$$\alpha^*\lambda(\xi) = \lambda(\alpha)(d\alpha\xi) = \lambda(\alpha)(\alpha(\pi^*d\alpha(\xi))),$$

because $\pi^*\alpha(p) = p$ where $p \in M$ and the result follows. □

A submanifold $N$ of a symplectic manifold $M$ is called Lagrangian if for all $p \in N$, the tangent space $T_pN$ coincides with the following configuration space: $\{\eta \in T_pM : \omega_p(\xi, \eta) = 0, \forall \xi \in T_pN\}$. On this space the 2-form $\sum dx_k \wedge dy_k$ that defines the symplectic structure is identically zero. Lagrangian submanifolds are considered among the most important submanifolds of symplectic manifolds. Note that $\dim N = \frac{1}{2} \dim M$ and that for all vector fields $X, Y$ on $N$, we have $\omega(X, Y) = 0$.

Example 3 If $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ is a local coordinate system on an open $U \subset M$, then the subset of $U$ defined by $y_1 = \cdots = y_m = 0$ is a Lagrangian submanifold of $M$. The submanifold $\alpha(M)$ is Lagrangian in $T^*M$ if and only if the form $\alpha$ is closed because

$$0 = \alpha^*\omega = \alpha^*(-d\lambda) = -d(\alpha^*\lambda) = -d\alpha.$$  

Let $M$ be a differentiable manifold, $T^*M$ its cotangent bundle with the symplectic form $\omega$, and

$$s_\alpha : U \rightarrow T^*M, \quad p \mapsto \alpha(p),$$

a section on an open $U \subset M$. From the local expression of $\omega$ (theorem 1), we deduce that the null section of the bundle $T^*M$ is a Lagrangian submanifold of $T^*M$. If $s_\alpha(U)$ is a Lagrangian submanifold of $T^*M$, then $s_\alpha$ is called Lagrangian section. We have (theorem 3), $s_\alpha^*\lambda = \alpha$, and according to example 3, $s_\alpha(U)$ is a Lagrangian submanifold of $T^*M$ if and only if the form $\alpha$ is closed. Let $(M, \omega)$ and $(N, \eta)$ be two symplectic manifolds of the same dimension and $f : M \rightarrow N$, a differentiable application. We say that $f$ is a symplectic morphism if it preserves the symplectic forms, i.e., $f$ satisfies $f^*\eta = \omega$. When $f$ is a diffeomorphism, we say that $f$ is a symplectic diffeomorphism or $f$ is a symplectomorphism.
**Theorem 4**

a) A symplectic morphism is a local diffeomorphism.

b) A symplectomorphism preserve the orientation.

**Proof**

a) Indeed, the 2-form $\Omega$ being non degenerate then the differential

$$df(p) : T_pM \rightarrow T_pM, \quad p \in M,$$

is a linear isomorphism and according to the local inversion theorem, $f$ is a local diffeomorphism. Another proof is to note that

$$f^*\eta^m = (f^*\eta)^m = \omega^m.$$

The map $f$ has constant rank $2m$ because $\omega^m$ and $\eta^m$ are volume forms on $M$ and $N$ respectively. And the result follows.

b) It is deduced from a) that the symplectic diffeomorphisms or symplectomorphisms preserve the volume form and therefore the orientation. The Jacobian determinant of the transformation is $+1$. □

**Remark 1**

Note that the inverse $f^{-1} : N \rightarrow M$ of a symplectomorphism $f : M \rightarrow N$ is also a symplectomorphism.

Let $(M, \omega)$, $(N, \eta)$ be two symplectic manifolds,

$$pr_1 : M \times N \rightarrow M, \quad pr_2 : M \times N \rightarrow N,$$

the projections of $M \times N$ on its two factors. The two forms $pr_1^*\omega + pr_2^*\eta$ and $pr_1^*\omega - pr_2^*\eta$ on the product manifold $M \times N$, are symplectic forms. Take the case where $\dim M = \dim N = 2m$ and consider a differentiable map $f : M \rightarrow N$, as well as its graph defined by the set

$$A = \{(x, y) \in M \times N : y = f(x)\}.$$

Note that the application $g$ defined by

$$g : M \rightarrow A, \quad x \mapsto (x, f(x)),$$

is a diffeomorphism. We show that $A$ is a $2m$-dimensional Lagrangian submanifold of $(M \times N, pr_1^*\omega + pr_2^*\eta)$ if and only if the reciprocal image of $pr_1^*\omega - pr_2^*\eta$ by the application $g$ is the identically zero form on $M$. Therefore, for the differentiable map $f$ to be a symplectic morphism, it is necessary and sufficient that the graph of $f$ is a Lagrangian submanifold of the product manifold $(M \times N, pr_1^*\omega - pr_2^*\eta)$.

**Theorem 5**

a) Let $f : M \rightarrow M$ be a diffeomorphism. Then, the application $f^* : T^*M \rightarrow T^*M$, is a symplectomorphism.

b) Let $g : T^*M \rightarrow T^*M$ be a diffeomorphism such that : $g^*\lambda = \lambda$. Then, there is a diffeomorphism $f : M \rightarrow M$ such that : $g = f^*$. 

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Proof. a) Let’s show that $f^{**} \omega = \omega$. We have

\[
f^{**} \lambda(\xi) = \lambda(f^*(\alpha))(df^* \xi),
\]

so

\[
f^{**} \lambda(\xi) = \alpha(df^* \xi^* (\xi)),
\]

and therefore,

\[
f^{**} \lambda(\xi) = \alpha(df \circ \pi \circ f^*)(\xi).
\]

Since $f^* \alpha = \alpha f^{-1}(p)$ and $\pi f^* \alpha = f^{-1}(p)$, then

\[
f \circ \pi \circ f^*(\alpha) = p = \pi^* \alpha,
\]

i.e.,

\[
f \circ \pi \circ f^* = \pi^*
\]

and

\[
f^{**} \lambda(\xi) = \alpha(d \pi^* (\xi)) = \lambda(\xi) = \lambda(\xi).
\]

Consequently, $f^{**} \lambda = \lambda$, and $f^{**} \omega = \omega$.

b) Since $g^* \lambda = \lambda$, then

\[
g^* \lambda(\eta) = \lambda(dg \eta) = \omega(\xi, dg \eta) = \lambda(\eta) = \omega(\xi, \eta).
\]

Moreover, we have $g^* \omega = \omega$, hence

\[
\omega(dg \xi, dg \eta) = \omega(\xi, \eta) = \omega(\xi, dg \eta),
\]

\[
\omega(dg \xi - \xi, dg \eta) = 0, \ \forall \eta.
\]

Since the form $\omega$ is non-degenerate, we deduce that $dg \xi = \xi$ and that $g$ preserves the integral curves of $\xi$. On the null section of the tangent bundle (i.e., on the manifold), we have $\xi = 0$ and then $g|_M$ is an application $f : M \rightarrow M$. Let’s show that:

\[
f \circ \pi \circ g = \pi^* = f \circ \pi \circ f^*.
\]

Indeed, taking the differential, we get

\[
(df \circ d \pi \circ dg)(\xi) = df \circ d \pi^*(\xi) = df(\xi_p),
\]

because $dg(\xi) = \xi$ and $\xi_p \equiv d \pi^*(\xi)$, hence,

\[
df \circ d \pi \circ dg(\xi) = \xi_p = d \pi^*(\xi).
\]

Therefore,

\[
df \circ d \pi \circ dg = d \pi^*,
\]

and

\[
f \circ \pi \circ g = \pi^*.
\]

Since $f \circ \pi \circ f^* = \pi^*$ (according to (1)), so $g = f^*$. □
Theorem 6 Let
\[ I : T^*_x M \rightarrow T_x M, \quad \omega^1_\xi \mapsto \xi, \]
be a map defined by the relation
\[ \omega^1_\xi (\eta) = \omega(\eta, \xi), \quad \forall \eta \in T_x M. \]
Then I is an isomorphism generated by the symplectic form \( \omega \).

Proof. Denote by \( I^{-1} \) the map
\[ I^{-1} : T^*_x M \rightarrow T^*_x M, \quad \xi \mapsto I^{-1} (\xi) \equiv \omega^1_\xi, \]
with
\[ I^{-1}(\xi)(\eta) = \omega^1_\xi (\eta) = \omega(\eta, \xi), \quad \forall \eta \in T_x M. \]
The fact that the form \( \omega \) is bilinear implies that
\[ I^{-1}(\xi_1 + \xi_2)(\eta) = I^{-1}(\xi_1)(\eta) + I^{-1}(\xi_2)(\eta), \quad \forall \eta \in T_x M. \]
Since \( \dim T_x M = \dim T^*_x M \), to show that \( I^{-1} \) is bijective, it suffices to show that is injective. The form \( \omega \) is non-degenerate, it follows that \( \text{Ker} I^{-1} = \{0\} \). Hence \( I^{-1} \) is an isomorphism and consequently \( I \) is also an isomorphism (the inverse of an isomorphism is an isomorphism). \( \square \)

4 Flows, Lie derivative, inner product and Cartan’s formula

Let \( M \) be a differentiable manifold of dimension \( m \). Let \( TM \) be the tangent bundle to \( M \), i.e., the union of spaces tangent to \( M \) at all points \( x \), \( TM = \bigcup_{x \in M} T_x M \). This bundle has a natural structure of differentiable variety of dimension \( 2m \) and it allows us to convey immutably to the manifolds the whole theory of ordinary differential equations. Let \( X : M \rightarrow TM \), be a vector field assumed to be different from the zero vector of \( T_x M \) only on a compact subset \( K \) of the manifold \( M \).

Given a point \( x \in M \), we write \( g^X_t(x) \) (or simply \( g^t(x) \)) the position of \( x \) after a displacement of a duration \( t \in \mathbb{R} \). There is thus an application
\[ g^X_t : M \rightarrow M, \quad t \in \mathbb{R}, \]
which is a diffeomorphism (a one-to-one differentiable mapping with a differentiable inverse), by virtue of the theory of differential equations (see theorem below). The vector field \( X \) generates a one-parameter group of diffeomorphisms \( g^X_t \) on \( M \), i.e., a differentiable application \( (C^\infty) : M \times \mathbb{R} \rightarrow M \), satisfying a group law :
\[ i) \forall t \in \mathbb{R}, \ g^X_t : M \rightarrow M \text{ is a diffeomorphism.} \]
ii) \( \forall t, s \in \mathbb{R}, \ g_t^X s = g_t^X \circ g_s^X \).

The condition ii) means that the mapping \( t \mapsto g_t^X \), is a homomorphism of the additive group \( \mathbb{R} \) into the group of diffeomorphisms of \( M \) in \( M \). It implies that \( g_t^{-1} = (g_t^X)^{-1} \), because \( g_0^X = id_M \) is the identical transformation that leaves every point invariant.

The one-parameter group of diffeomorphisms \( g_t^X \) on \( M \), which we have just described is called a flow and it admits the vector field \( X \) for velocity fields

\[
\frac{d}{dt} g_t^X (x) = X (g_t^X (x)),
\]

with the initial condition : \( g_0^X (x) = x \). Obviously

\[
\frac{d}{dt} g_t^X (x) \bigg|_{t=0} = X(x).
\]

Hence by these formulas \( g_t^X (x) \) is the curve on the manifold that passes through \( x \) and such that the tangent at each point is the vector \( X (g_t^X (x)) \).

We will show how to construct the \( g_t^X \) flow on the manifold \( M \).

**Theorem 7** The vector field \( X \) generates a unique group of diffeomorphisms of the compact manifold \( M \). In addition, every solution of the differential equation

\[
\frac{d}{dt} x(t) = X(x(t)), \quad x \in M
\]

with the initial condition \( x \) (for \( t = 0 \)), can be extended indefinitely. The value of the solution \( g_t^X (x) \) at time \( t \) is differentiable with respect to \( t \) and the initial condition \( x \).

**Proof.** For the construction of \( g_t^X \) for \( t \) small, we proceed as follows: for \( x \) fixed, the differential equation

\[
\frac{d}{dt} g_t^X (x) = X (g_t^X (x)),
\]

function of \( t \) with the initial condition : \( g_0^X (x) = x \), admits a unique solution \( g_t^X \) defined in the neighborhood of the point \( x_0 \) smoothly \((C^\infty)\) depending on the initial condition. Then \( g_t^X \) is locally a diffeomorphism. For each point \( x_0 \in M \), we can find a neighborhood \( U (x_0) \subset M \), a real positive number \( \varepsilon \equiv \varepsilon (x_0) \) such that for all \( t \in ]-\varepsilon, \varepsilon[ \), the differential equation in question with its initial condition has a unique differentiable solution \( g_t^X (x) \) defined in \( U (x_0) \) and satisfying the group relation

\[
g_t^X (x) = g_t^X _s (x) = g_t^X \circ g_s^X (x),
\]

with \( t, s, t + s \in ]-\varepsilon, \varepsilon[ \). Indeed, put \( x_1 = g_t^X (x), \ t \) fixed and consider the solution of the differential equation satisfying in the neighborhood of the
point \( x_0 \) to the initial condition: \( g_{s=0}^X = x_1 \). This solution satisfies the same differential equation and coincides in a point \( g_{t+s}^X(x) = x_1 \), with the function \( g_t^X \). Therefore, by uniqueness of the solution of the differential equation, the two functions are locally equal. Therefore, the application \( g_t^X \) is locally a diffeomorphism. The vector field \( X \) is assumed to be differentiable (of class \( C^\infty \)) and with compact support \( K \). Since \( K \) is compact, then from the open covering \( U(x) \) of \( K \), we can extract a finite sub-covering \( (U_i) \). Let us denote by \( \varepsilon_i \) the numbers \( \varepsilon \) corresponding to \( U_i \) and let

\[
\varepsilon_0 = \inf (\varepsilon_i), \quad g_t^X(x) = x, \quad x \notin K.
\]

The equation in question admits a unique solution \( g_t^X \) on \( M \times [-\varepsilon_0, \varepsilon_0[ \) satisfying the relation of the group above, the inverse of \( g_t^X \) being \( g_{-t}^X \) and therefore \( g_t^X \) is a diffeomorphism for \( t \) sufficiently small. We will now see how to construct \( g_t^X \) for every \( t \in \mathbb{R} \). From what precedes, just construct \( g_t^X \) for \( t \in ]-\infty, -\varepsilon_0[ \cup ]\varepsilon_0, \infty[ \). We will see that the applications \( g_t^X \) are defined according to the multiplication law of the group. Note that \( t \) can be written as

\[
t = k \frac{\varepsilon_0}{2} + r, \quad k \in \mathbb{Z}, \quad r \in \left[ 0, \frac{\varepsilon_0}{2} \right].
\]

Let

\[
g_t^X = \underbrace{g_{\frac{\varepsilon_0}{2}}^X \circ \cdots \circ g_{\frac{\varepsilon_0}{2}}^X}_k, \quad t \in \mathbb{R}^*,
\]

\[
g_t^X = \underbrace{g_{\frac{\varepsilon_0}{2}}^X \circ \cdots \circ g_{\frac{\varepsilon_0}{2}}^X}_k, \quad t \in \mathbb{R}^*.
\]

The diffeomorphisms \( g_{\frac{\varepsilon_0}{2}}^X \) and \( g_{\frac{\varepsilon_0}{2}}^X \) have been defined above. Therefore, for all real \( t \), \( g_t^X \) is a diffeomorphism defined globally on \( M \) and the result is deduced immediately. \( \square \)

With every vector field \( X \) we associate the first-order differential operator \( L_X \). This is the differentiation of functions in the direction of the vector field \( X \). We have

\[
L_X : C^\infty (M) \longrightarrow C^\infty (M), \quad F \mapsto L_X F,
\]

where

\[
L_X F(x) = \left. \frac{d}{dt} F \left( g_t^X (x) \right) \right|_{t=0}, \quad x \in M.
\]

\( C^\infty (M) \) being the set of functions \( F : M \longrightarrow \mathbb{R} \), of class \( C^\infty \). The operator \( L_X \) is linear:

\[
L_X (\alpha_1 F_1 + \alpha_2 F_2) = \alpha_1 L_X F_1 + \alpha_2 L_X F_2,
\]

where \( \alpha_1, \alpha_2 \in \mathbb{R} \), and satisfies the Leibniz formula:

\[
L_X (F_1 F_2) = F_1 L_X F_2 + F_2 L_X F_1.
\]
Since $L_X F(x)$ only depends on the values of $F$ in the neighborhood of $x$, we can apply the operator $L_X$ to the functions defined only in the neighborhood of a point, without the need to extend them to the full variety $M$. Let $(x_1, \ldots, x_m)$ a local coordinate system on $M$. In this system the vector field $X$ has components $f_1, \ldots, f_m$ and the flow $g^X_t$ is defined by a system of differential equations. Therefore, the derivative of the function $F = F(x_1, \ldots, x_m)$ in the direction of $X$ is written

$$L_X F = f_1 \frac{\partial F}{\partial x_1} + \cdots + f_m \frac{\partial F}{\partial x_m}. $$

In other words, in the coordinates $(x_1, \ldots, x_m)$ the operator $L_X$ is written

$$L_X = f_1 \frac{\partial}{\partial x_1} + \cdots + f_m \frac{\partial}{\partial x_m},$$

this is the general form of the first order linear differential operator.

Let $X$ be a vector field on a differentiable manifold $M$. We have shown (theorem 7) that the vector field $X$ generates a unique group of diffeomorphisms $g^X_t$ (that we also note $g_t$) on $M$, solution of the differential equation

$$\frac{d}{dt} g^X_t(p) = X(g^X_t(p)), \quad p \in M,$$

with the initial condition $g^X_0(p) = p$. Let $\omega$ be a $k$-differential form. The Lie derivative of $\omega$ with respect to $X$ is defined by

$$L_X \omega = \left. \frac{d}{dt} g^*_t \omega \right|_{t=0} = \lim_{t \to 0} \frac{g^*_t(\omega(g_t(p))) - \omega(p)}{t}. $$

In general, for $t \neq 0$, we have

$$\left. \frac{d}{dt} g^*_t \omega \right|_{s=0} = \left. \frac{d}{ds} g^*_s \omega \right|_{s=0} = g^*_0(L_X \omega). \quad (2)$$

We can easily verify that for the differential $k$-form $\omega(g_t(p))$ at the point $g_t(p)$, the expression $g^*_t \omega(g_t(p))$ is indeed a differential $k$-form in $p$.

For all $t \in \mathbb{R}$, the application $g_t : \mathbb{R} \rightarrow \mathbb{R}$ being a diffeomorphism then $dg_t$ and $dg_{-t}$ are applications

$$dg_t : T_p M \rightarrow T_{g_t(p)} M,$$

$$dg_{-t} : T_{g_t(p)} M \rightarrow T_p M.$$  

The Lie derivative of a vector field $Y$ in the direction $X$ is defined by

$$L_X Y = \left. \frac{d}{dt} g_{-t} Y \right|_{t=0} = \lim_{t \to 0} \frac{g_{-t}(Y(g_t(p))) - Y(p)}{t}. $$
In general, for $t \neq 0$, we have
\[
\frac{d}{dt} g_{-t}Y = \frac{d}{ds} g_{-t-s}Y \bigg|_{s=0} = g_{-t} \frac{d}{ds} g_{-s}Y \bigg|_{s=0} = g_{-t}(L_Y).
\]

An interesting operation on differential forms is the inner product that is defined as follows: the inner product of a differential $k$-form $\omega$ by a vector field $X$ on a differentiable manifold $M$ is a differential $(k-1)$-form, denoted $i_X \omega$, defined by
\[
(i_X \omega)(X_1, ..., X_{k-1}) = \omega(X, X_1, ..., X_{k-1}),
\]
where $X_1, ..., X_{k-1}$ are vector fields. It is easily shown that if $\omega$ is a differential $k$-form, $\lambda$ a differential form of any degree, $X$ and $Y$ two vector fields, $f$ a linear map and $a$ a constant, then
\[
\begin{align*}
    i_{X+Y} \omega &= i_X \omega + i_Y \omega, & i_{aX} \omega &= a i_X \omega, \\
    i_{X} i_{Y} \omega &= -i_{Y} i_{X} \omega, & i_{X} i_{X} \omega &= 0, \\
    i_{X} (f \omega) &= f (i_{X} \omega), & i_{X} f^{\ast} \omega &= f^{\ast} (i_{fX} \omega), \\
    i_{X} (\omega \wedge \lambda) &= (i_{X} \omega) \wedge \lambda + (-1)^{k} \omega \wedge (i_{X} \lambda).
\end{align*}
\]

**Example 4** Let’s calculate the expression of the inner product in local coordinates. If
\[
X = \sum_{j=1}^{m} X_j(x) \frac{\partial}{\partial x_j},
\]
is the local expression of the vector field on the variety $M$ of dimension $m$ and
\[
\omega = \sum_{i_1 < i_2 < ... < i_k} f_{i_1...i_k} (x) dx_{i_1} \wedge ... \wedge dx_{i_k},
\]
then,
\[
\begin{align*}
    i_X \omega &= \omega(X, \cdot), \\
    &= \sum_{i_1 < i_2 < ... < i_k} \sum_{j=1}^{m} f_{ji_2...i_k} X_j dx_{i_2} \wedge ... \wedge dx_{i_k} \\
    &\quad - \sum_{i_1 < i_2 < ... < i_k} \sum_{j=1}^{m} f_{i_1i_2...i_k} X_j dx_{i_1} \wedge dx_{i_3} \wedge ... \wedge dx_{i_k} \\
    &\quad + \cdots + (-1)^{k-1} \sum_{i_1 < i_2 < ... < i_{k-1}} \sum_{j=1}^{m} f_{i_1i_2...i_{k-1}} X_j dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_{k-1}} \\
    &= k \sum_{i_2 < i_3 < ... < i_k} \sum_{j=1}^{m} f_{j_{i_2...i_k}} X_j dx_{i_2} \wedge ... \wedge dx_{i_k}.
\end{align*}
\]
Hence,

\[ i_{\frac{\partial}{\partial x_j}} \omega = \frac{\partial}{\partial(dx_j)} \omega, \]

where we put \(dx_j\) in first position in \(\omega\).

The following properties often occur when solving practical problems using Lie derivatives.

**Theorem 8**

a) If \(f : M \rightarrow \mathbb{R}\) is a differentiable function, so the Lie derivative of \(f\) is the image of \(X\) by the differential of \(f\),

\[ L_X f = df(X) = X.f. \]

b) \(L_X\) and \(d\) commute, \(L_X \circ d = d \circ L_X\).

c) Let \(X, X_1, ..., X_k\) be vector fields on \(M\) and \(\omega\) a \(k\)-form differential. So

\[(L_X \omega)(X_1, ..., X_k) = L_X(\omega(X_1, ..., X_k)) - \sum_{j=1}^{k} \omega(X_1, ..., L_X X_j, ..., X_k). \]

d) For all differential forms \(\omega\) and \(\lambda\), we have

\[ L_X(\omega \wedge \lambda) = L_X \omega \wedge \lambda + \omega \wedge L_X \lambda. \]

**Proof.**

a) Indeed, we have

\[ L_X f = \frac{d}{dt} g^*_t f \bigg|_{t=0} = \frac{d}{dt} f \circ g_t \bigg|_{t=0} = df \left( \frac{dg_t}{dt} \right) \bigg|_{t=0} = df(X), \]

and (see further theorem 10),

\[ L_X f = i_X df = X.f. \]

b) Indeed, as the differential and the reciprocal image commute, then

\[ d \circ L_X \omega = d \circ \frac{d}{dt} g^*_t \omega \bigg|_{t=0} = \frac{d}{dt} g^*_t \circ d\omega \bigg|_{t=0} = L_X \circ d\omega. \]

c) We have

\[(L_X \omega)(X_1, ..., X_k) = \frac{d}{dt} g^*_t \omega(X_1, ..., X_k) \bigg|_{t=0},
\]

\[ = \frac{d}{dt} \omega(g_t) (dg_t X_1, ..., dg_t X_k) \bigg|_{t=0},
\]

\[ = L_X \omega(g_t) (dg_t X_1, ..., dg_t X_k)_{t=0}
\]

\[ + \sum_{j=1}^{k} \omega(g_t) \left( dg_t X_1, ..., \frac{d}{dt} dg_t X_j, ..., dg_t X_k \right)_{t=0}, \]

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and the result is deduced from the fact that
\[
\frac{d}{dt} dg t X_j \bigg|_{t=0} = - \frac{d}{dt} d g t - X_j \bigg|_{t=0} = -L_X X_j.
\]

d) Just consider \( \omega \) and \( \lambda \) of the form
\[
\omega = f dx_{i_1} \wedge ... \wedge dx_{i_k}, \quad \lambda = g dx_{j_1} \wedge ... \wedge dx_{j_l}.
\]
We have
\[
\omega \wedge \lambda = fg dx_{i_1} \wedge ... \wedge dx_{i_k} \wedge dx_{j_1} \wedge ... \wedge dx_{j_l},
\]
and
\[
L_X(\omega \wedge \lambda)(X_1, ..., X_k, X_{k+1}, ..., X_{k+l}) = (L_X f)g dx_{i_1} \wedge ... \wedge dx_{i_k} \wedge dx_{j_1} \wedge ... \wedge dx_{j_l}(X_1, ..., X_k, X_{k+1}, ..., X_{k+l})
\]
\[
+ f(L_X g)dx_{i_1} \wedge ... \wedge dx_{i_k} \wedge dx_{j_1} \wedge ... \wedge dx_{j_l}(X_1, ..., X_k, X_{k+1}, ..., X_{k+l}),
\]
\[
= (L_X(\omega) \wedge \lambda + \omega \wedge (L_X\lambda))(X_1, ..., X_k, X_{k+1}, ..., X_{k+l}),
\]
and the result follows. \( \square \)

**Theorem 9** Let \( X \) and \( Y \) be two vector fields on \( M \). Then, the Lie derivative of \( L_X Y \) is the Lie bracket \( [X, Y] \).

**Proof.** We have
\[
L_X Y(f) = \lim_{t \to 0} \frac{d g t Y - Y}{t}(f) = \lim_{t \to 0} \frac{d g t - Y}{t}(f),
\]
hence
\[
L_X Y(f) = \lim_{t \to 0} \frac{Y(f) - d g t Y(f)}{t} = \lim_{t \to 0} \frac{Y(f) - Y(f \circ g_t) \circ g_t^{-1}}{t}.
\]
Put \( g_t(x) \equiv g(t, x) \), and apply to \( g(t, x) \) the Taylor formula with integral remainder. So there is \( h(t, x) \) such that :
\[
f(g(t, x)) = f(x) + th(t, x),
\]
with
\[
h(0, x) = \frac{\partial}{\partial t} f(g(t, x))(0, x).
\]
According to the definition of the tangent vector, we have
\[
X(f) = \frac{\partial}{\partial t} f \circ g_t(x)(0, x),
\]
hence, \( h(0, x) = X(f)(x) \). Therefore,

\[
L_X Y(f) = \lim_{t \to 0} \left( \frac{Y(f) - Y(f) \circ g_t^{-1}}{t} - Y(h(t, x)) \circ g_t^{-1} \right),
\]

\[
= \lim_{t \to 0} \left( \frac{(Y(f) \circ g_t - Y(f)) \circ g_t^{-1}}{t} - Y(h(t, x)) \circ g_t^{-1} \right).
\]

Since

\[
\lim_{t \to 0} g_t^{-1}(x) = g_0^{-1}(x) = id.,
\]

we deduce that:

\[
L_X Y(f) = \lim_{t \to 0} \left( \frac{Y(f) \circ g_t - Y(f)}{t} - Y(h(0, x)) \right),
\]

\[
= \frac{\partial}{\partial t} Y(f) \circ g_t(x) - Y(X(f)),
\]

\[
= X(Y(f)) - Y(X(f)),
\]

\[
= [X, Y],
\]

which completes the demonstration. □

We will now establish a fundamental formula for the Lie derivative, which can be used as a definition.

**Theorem 10** Let \( X \) be a vector field on \( M \) and \( \omega \) a differential \( k \)-form. Then

\[
L_X \omega = d(i_X \omega) + i_X (d\omega).
\]

In other words, we have the Cartan homotopy formula

\[
L_X = d \circ i_X + i_X \circ d.
\]

**Proof.** We will reason by induction on the degree \( k \) of the differential form \( \omega \). Let

\[
D_X \equiv d \circ i_X + i_X \circ d.
\]

For a differential 0-form, i.e., a \( f \) function, we have

\[
D_X f = d(i_X f) + i_X (df).
\]

Or \( i_X f = 0 \), hence

\[
d(i_X f) = 0, \quad i_X df = df(X),
\]

and so

\[
D_X f = df(X).
\]

Moreover, we know (theorem 8, a)) that \( L_X f = df(X) = X.f \), so

\[
D_X f = L_X f.
\]
Assume that the formula in question is true for a differential \((k-1)\)-form and is proved to be true for a differential \(k\)-form. Let \(\lambda\) be a differential \((k-1)\)-form and let \(\omega = df \wedge \lambda\), where \(f\) is a function. We have

\[
L_X \omega = L_X (df \wedge \lambda),
= L_X df \wedge \lambda + df \wedge L_X \lambda, \quad \text{(theorem 8, d)},
= dL_X f \wedge \lambda + df \wedge L_X \lambda, \quad \text{(because } L_X df = dL_X f, \text{theorem 8, b)},
= d(df(X)) \wedge \lambda + df \wedge L_X \lambda, \quad \text{(because } L_X f = df(X), \text{theorem 8, a)}.
\]

By hypothesis of recurrence, we have

\[
L_X \lambda = d(i_X \lambda) + i_X (d\lambda).
\]

Or \(i_X df = df(X)\), then

\[
L_X \omega = d(i_X df) \wedge \lambda + df \wedge d(i_X \lambda) + df \wedge i_X (d\lambda).
\]

Moreover, we have

\[
i_X d\omega = i_X d(df \wedge \lambda) = -(i_X df) \wedge d\lambda + df \wedge (i_X (d\lambda)),
\]

and

\[
d(i_X \omega) = di_X (df \wedge \lambda),
= d((i_X df) \wedge \lambda - df \wedge (i_X \lambda)),
= d(i_X df) \wedge \lambda + (i_X df) \wedge d\lambda + df \wedge d(i_X \lambda), \quad \text{(because } d(df) = 0),
\]

hence,

\[
di_X (df \wedge \lambda) + i_X d(df \wedge \lambda) = d(i_X df) \wedge \lambda + df \wedge d(i_X \lambda) + df \wedge i_X (d\lambda).
\]

Comparing this expression with that obtained in (3), we finally obtain

\[
L_X \omega = d(i_X \omega) + i_X (d\omega),
\]

and the theorem is proved. \(\square\)

**Example 5** For a differential form \(\omega\), we have

\[
i_X L_X \omega = L_X i_X \omega.
\]

Indeed, we have

\[
i_X L_X \omega = i_X (di_X \omega) + i_X (i_X d\omega), \quad \text{(theorem 10)}
= i_X (di_X \omega), \quad \text{(because } i_X i_X = 0),
\]

and

\[
L_X i_X \omega = (d \circ i_X + i_X \circ d) i_X \omega = i_X (di_X \omega),
\]

hence, \(i_X L_X \omega - L_X i_X \omega = 0\).
Theorem 11 Let $X$ and $Y$ be two vector fields on $M$ and $\omega$ a differential form. Then,

\[
L_{X+Y}\omega = L_X\omega + L_Y\omega,
\]
\[
L_{fX}\omega = fL_X\omega + df \wedge i_X\omega,
\]

where $f : M \to \mathbb{R}$ is a differentiable function.

Proof. Indeed, just use the theorem 10,

\[
L_{X+Y}\omega = d(i_{X+Y}\omega) + i_{X+Y}(d\omega),
\]
\[
= d(i_X \omega + i_Y \omega) + i_X(d\omega) + i_Y(d\omega),
\]
\[
= d(i_X \omega) + i_X(d\omega) + d(i_Y \omega) + i_Y(d\omega),
\]
\[
= L_X\omega + L_Y\omega.
\]

Similarly, we have

\[
L_{fX}\omega = d(i_{fX}\omega) + i_{fX}(d\omega),
\]
\[
= d(f_i X \omega) + f_i X(d\omega),
\]
\[
= df \wedge i_X\omega + f d(i_X\omega) + f_i X(d\omega),
\]
\[
= df \wedge i_X\omega + fL_X\omega,
\]

which completes the demonstration. □

Example 6 Let’s calculate the expression of the Lie derivative of the differential form

\[
\omega = \sum_{i_1 < \ldots < i_k} f_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k},
\]

in local coordinates. If

\[
X = \sum_{j=1}^m X_j(x) \frac{\partial}{\partial x_j},
\]

is the local expression of the vector field on the $m$-dimensional manifold $M$, then

\[
L_X\omega = \sum_{j=1}^m L_{X_j} \omega = \sum_{j=1}^m \left( dX_j \wedge i_{\frac{\partial}{\partial x_j}}\omega + X_j L_{\frac{\partial}{\partial x_j}}\omega \right).
\]

According to example 4, we know that

\[
i_{\frac{\partial}{\partial x_j}}\omega = \frac{\partial}{\partial(dx_j)}\omega = k \sum_{i_2 < i_3 < \ldots < i_k} f_{j i_2 \ldots i_k} dx_{i_2} \wedge dx_{i_3} \wedge \ldots \wedge dx_{i_k},
\]

hence,

\[
dX_j \wedge i_{\frac{\partial}{\partial x_j}}\omega = k \sum_{i_1 < i_2 < \ldots < i_k} f_{j i_2 \ldots i_k} \frac{\partial X_j}{\partial x_{i_1}} dx_{i_1} \wedge \ldots \wedge dx_{i_k}.
\]
Similarly, using theorem 8, c), we obtain

\[ L_{\nabla_{\gamma}} \omega = \sum_{i_1 < \ldots < i_k} \frac{\partial f_{i_1,\ldots,i_k}}{\partial x_j} dx_{i_1} \wedge \ldots \wedge dx_{i_k}. \]

Since \( \left[ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right] = 0 \), we finally get

\[ L_X \omega = \sum_{i_1 < \ldots < i_k} \sum_{j=1}^m \left( \frac{\partial f_{i_1,\ldots,i_k}}{\partial x_j} X_j + k f_{ij_2,\ldots,i_k} \frac{\partial X_j}{\partial x_{i_1}} \right) dx_{i_1} \wedge \ldots \wedge dx_{i_k}. \]

**Theorem 12** If \( X \) and \( Y \) are two vector fields on \( M \), then

a) \( [L_X, i_Y] = i_{[X,Y]} \).

b) \( [L_X, L_Y] = L_{[X,Y]} \).

**Proof.** The proof is to show that for a a differential \( k \)-form \( \omega \), we have

\[ [L_X, i_Y] \omega = i_{[X,Y]} \omega, \]

and

\[ [L_X, L_Y] \omega = L_{[X,Y]} \omega. \]

a) We reason by induction assuming first that \( k = 1 \), that is, \( \omega = df \). We have

\[ [L_X, i_Y] df = \begin{align*}
&= L_X i_Y df - i_Y L_X df, \\
&= L_X(Y.f) - i_Y d L_X f, \quad \text{(because } L_X \circ d = d \circ L_X) \\
&= X.(Y.f) - i_Y d(X.f), \quad \text{(because } L_X f = X.f) \\
&= X.(Y.f) - Y.(X.f), \\
&= [X,Y].f, \\
&= i_{[X,Y]} df.
\end{align*} \]

Suppose the formula in question is true for a \( \omega \) form of degree less than or equal to \( k-1 \). Let \( \lambda \) and \( \theta \) be two forms of degree less than or equal to \( k-1 \), so that \( \omega = \lambda \wedge \theta \) is a form of degree \( k \). We have

\[ [L_X, i_Y] \omega = \begin{align*}
&= L_X i_Y \omega - i_Y L_X \omega, \\
&= \lambda X i_Y \lambda \wedge \theta - i_Y \lambda L_X \lambda \wedge \theta, \\
&= L_X(i_Y \lambda \wedge \theta + (-1)^{\deg \lambda} \lambda \wedge i_Y \theta) - i_Y (L_X \lambda \wedge \theta + \lambda \wedge L_X \theta), \\
&= L_X i_Y \lambda \wedge \theta + i_Y \lambda \wedge L_X \theta + (-1)^{\deg \lambda} L_X \lambda \wedge i_Y \theta \\
&\quad + (-1)^{\deg \lambda} \lambda \wedge L_X i_Y \theta - i_Y L_X \lambda \wedge \theta - (-1)^{\deg \lambda} L_X \lambda \wedge i_Y \theta \\
&\quad - i_Y \lambda \wedge L_X \theta - (-1)^{\deg \lambda} \lambda \wedge i_Y L_X \theta, \\
&= (L_X i_Y \lambda - i_Y L_X \lambda) \wedge \theta + (-1)^{\deg \lambda} \lambda \wedge (L_X i_Y \theta - i_Y L_X \theta), \\
&= i_{[X,Y]} \lambda \wedge \theta + (-1)^{\deg \lambda} \lambda \wedge i_{[X,Y]} \theta, \\
&= i_{[X,Y]} (\lambda \wedge \theta), \\
&= i_{[X,Y]} \omega.
\end{align*} \]
We have
\[
[L_X, L_Y] \omega = L_X L_Y \omega - L_Y L_X \omega,
\]
\[
= L_X dY \omega + L_X i_Y d\omega - dY L_X \omega - i_Y dL_X \omega,
\]
\[
= dL_X i_Y \omega + L_X i_Y d\omega - dY L_X \omega - i_Y dL_X \omega,
\]
\[
= i_{[X,Y]} \omega + L_{[X,Y]} \omega,
\]
and the demonstration ends. □

Example 7 Using the results seen above, we give a quick proof of Poincaré lemma: in the neighborhood of a point of a manifold, any closed differential form is exact. Indeed, consider the differential equation
\[
\dot{x} = X(x) = \frac{x}{t},
\]
as well as its solution \( g_t(x_0) = x_0 t \). The latter is defined in the neighborhood of the point \( x_0 \), depends on \( C^\infty \) of the initial condition and is a one-parameter group of diffeomorphisms. We have
\[
g_0(x_0) = 0, \quad g_1(x_0) = x_0, \quad g_0^* \omega = 0, \quad g_1^* \omega = \omega.
\]
Hence,
\[
\omega = g_1^* \omega - g_0^* \omega = \int_0^1 \frac{d}{dt} g_t^* \omega dt = \int_0^1 g_t^* (L_X \omega) dt \quad (\text{according to (2)}),
\]
and according to the theorem 10 and the fact that \( d\omega = 0 \), we have
\[
\omega = \int_0^1 g_t^* (i_X \omega) dt = \int_0^1 d g_t^* i_X \omega dt, \quad (\text{because } df^* \omega = f^* d\omega).
\]
We can therefore find a differential form \( \lambda \) such that \( \omega = d\lambda \), where
\[
\lambda = \int_0^1 g_t^* i_X \omega dt.
\]

5 The Darboux-Moser-Weinstein theorem

Theorem 13 Let \( \{\omega_t\}, \quad 0 \leq t \leq 1 \), be a family of symplectic forms, differentiable in \( t \). Then, for all \( p \in M \), there exists a neighborhood \( U \) of \( p \) and a function \( g_t : U \rightarrow U \), such that : \( g_0^* = \text{identity} \) et \( g_1^* \omega_t = \omega_0 \).

Proof : Looking for a family of vector fields \( X_t \) on \( U \) such that these fields generate locally a one-parameter group of diffeomorphisms \( g_t \) with
\[
\frac{d}{dt} g_t(p) = X_t(g_t(p)), \quad g_0(p) = p.
\]
First note that the form $\omega_t$ is closed (i.e., $d\omega_t = 0$) as the form $\frac{d}{dt} \omega_t$ (since $\frac{d}{dt} \omega_t = \frac{d}{dt} (d\omega_t) = 0$). Therefore, by deriving the relationship $g_t^* \omega_t = \omega_0$ and using the Cartan homotopy formula (theorem 10) : $L_{X_t} = i_{X_t} d + di_{X_t}$, taking into account that $\omega_t$ depends on time, we obtain the expression

$$
\frac{d}{dt} g_t^* \omega_t = g_t^* \left( \frac{d}{dt} \omega_t + L_{X_t} \omega_t \right) = g_t^* \left( \frac{d}{dt} \omega_t + di_{X_t} \omega_t \right).
$$

By Poincaré’s lemma (in the neighborhood of a point, any closed differential form is exact), the form $\frac{\partial}{\partial t} \omega_t$ is exact in the neighborhood of $p$. In other words, we can find a form $\lambda_t$ such that : $\frac{d}{dt} \omega_t = d\lambda_t$. Hence,

$$
\frac{d}{dt} g_t^* \omega_t = g_t^* d(\lambda_t + i_{X_t} \omega_t). \tag{4}
$$

We want to show that for all $p \in M$, there exists a neighborhood $U$ of $p$ and a function $g_t : U \rightarrow U$, such that : $g_0^* = \text{identity}$ and $g_t^* \omega_t = \omega_0$, therefore such that : $\frac{d}{dt} g_t^* \omega_t = 0$. And by (4), the problem amounts to finding $X_t$ such that : $\lambda_t + i_{X_t} \omega_t = 0$. Since the form $\omega_t$ is non degenerate, then the above equation is solvable with respect to the vector field $X_t$ and defines the family $\{g_t\}$ for $0 \leq t \leq 1$. In local coordinates $(x_k)$ of the $2m$-dimensional manifold $M$, with $(\frac{\partial}{\partial x_k})$ a basis of $TM$ and $(dx_k)$ the dual basis of $(\frac{\partial}{\partial x_k})$, $k = 1, ..., 2m$, we have

$$
\lambda_t = \sum_{k=1}^{2m} \lambda_k(t, x) dx_k,
$$

$$
X_t = \sum_{k=1}^{2m} X_k(t, x) \frac{\partial}{\partial x_k},
$$

$$
\omega_t = \sum_{k,l=1}^{2m} \omega_{k,l}(t, x) dx_k \wedge dx_l,
$$

$$
L_{X_t} \omega_t = 2 \sum_{l=1}^{2m} \left( \sum_{k=1}^{2m} \omega_{k,l} X_k \right) dx_l.
$$

We therefore solve the system of equations in $x_k(t, x)$ according to :

$$
\lambda_l(t, x) + 2 \sum_{k=1}^{2m} \omega_{k,l}(t, x) X_k(t, x) = 0.
$$

The form $\omega_t$ is non degenerate and the matrix $(\omega_k(t, x))$ is nonsingular. Then the above system has a unique solution. This determines the vector field $X_t$ and thus functions $g_t^*$ such that : $g_t^* \omega_t = \omega_0$, which completes the proof. $\square$
Using the above theorem (well known as Moser’s lemma [31]), we give a proof (Weinstein [39, 40]) of the Darboux theorem, which states that every point in a symplectic manifold has a neighborhood with Darboux coordinates. The Darboux theorem plays a central role in symplectic geometry; the symplectic manifolds \((M, \omega)\) of dimension \(2m\) are locally isomorphic to \((\mathbb{R}^{2m}, \omega)\). More precisely, if \((M, \omega)\) is a symplectic manifold of dimension \(2m\), then in the neighborhood of each point of this manifold, there exist local coordinates \((x_1, ..., x_{2m})\) such that:

\[
\omega = \sum_{k=1}^{m} dx_k \wedge dx_{m+k}.
\]

In particular, there is no local invariant in symplectic geometry, analogous to the curvature in Riemannian geometry. The classical proof given by Darboux is by induction on the dimension of the manifold (see below and [4]).

**Theorem 14** Any symplectic form on a manifold \(M\) of dimension \(2m\) is locally diffeomorphic to the standard form on \(\mathbb{R}^{2m}\). In other words, if \((M, \omega)\) is a symplectic manifold of dimension \(2m\), then in the neighborhood of each point of \(M\), there exist local coordinates \((x_1, ..., x_{2m})\) such that:

\[
\omega = \sum_{k=1}^{m} dx_k \wedge dx_{m+k}.
\]

**Proof 1.** Let \(\{\omega_t\}, 0 \leq t \leq 1\), be a family of 2-differential forms which depends differentiably on \(t\) and let

\[
\omega_t = \omega_0 + t(\omega - \omega_0), \quad \omega_0 = \sum_{k=1}^{m} dx_k \wedge d_{m+k},
\]

where \((x_1, ..., x_{2m})\) are local coordinates on \(M\). Note that these 2-forms are closed. At \(p \in M\), we have

\[
\omega_t(p) = \omega_0(p) = \omega(p).
\]

By continuity, we can find a small neighborhood of \(p\) where the form \(\omega_t(p)\) is non degenerate. So the 2-forms \(\omega_t\) are non degenerate in a neighborhood of \(p\) and independent of \(t\) at \(p\). In other words, \(\omega_t\) are symplectic forms and by theorem 13, for all \(p \in M\), there exists a neighborhood \(U\) of \(p\) and a function \(g_t : U \to U\) such that : \(g_t^* = \text{identity}\) and \(g_t^* \omega_t = \omega_0\). Differentiating this relation with respect to \(t\), we obtain (as in the proof of theorem 13),

\[
\frac{d}{dt} g_t^* \omega_t = 0,
\]

\[
g_t^* \left( \frac{d}{dt} \omega_t + L_{X_t} \omega_t \right) = 0,
\]

\[
g_t^* \left( \frac{d}{dt} \omega_t + d_{X_t} \omega_t \right) = 0.
\]

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Therefore,
\[ di_X \omega_t = -\frac{d}{dt} \omega_t, \]
and since the form \( \frac{d}{dt} \omega_t \) is exact in the neighborhood of \( p \) (Poincaré’s lemma), then
\[ di_X \omega_t = d\theta_t, \]
where \( \theta_t \) is a 1-differential form. In addition, \( \omega_t \) being non degenerate, the equation \( i_{X_t} \omega_t = \theta_t \) is solvable and determines uniquely the vector field \( X_t \) depending on \( t \). Note that for \( t = 1 \), \( \omega_1 = \omega \) and for \( t = 0 \), \( \omega_0 = \omega_0 \) and also we can find \( g_t^i \) such that : \( g_t^i \omega = \omega_0 \). Vector fields \( X_t \) generate one-parameter families of diffeomorphisms \( \{g_t\} \), \( 0 \leq t \leq 1 \). In other words, you can make a change of coordinates as :
\[ \omega = \sum_{k=1}^{m} dx_k \wedge d_{m+k}, \]
and the proof is completed.

**Proof 2.** We proceed by induction on \( m \). Suppose the result true for \( m-1 \geq 0 \) and show that it is also for \( m \). Fix \( x \) and let \( x_{m+1} \) be a differentiable function on \( M \) whose differential \( dx_m \) is a nonzero point \( x \). Let \( X \) be the unique differentiable vector field satisfying the relation \( i_X \omega = dx_{m+1} \). As this vector field does not vanish at \( x \), then we can find a function \( x_1 \) in a neighborhood \( U \) of \( x \) such that \( X(x_1) = 1 \). Consider a vector field \( Y \) on \( U \) satisfying the relation \( i_Y \omega = -dx_1 \). Since \( d\omega = 0 \), then \( L_X \omega = L_Y \omega = 0 \), according to the Cartan homotopy formula. Therefore
\[ i_{[X,Y]} \omega = L_X i_Y \omega = L_X(i_Y \omega) - i_Y(L_X \omega) = L_X(-dx_1) = -d(X(x_1)) = 0, \]
from which we have \([X,Y] = 0\), since any point in the form \( \omega \) is of rank equal to \( 2m \). By the Recovery theorem it follows that there exist local coordinates \( x_1, x_{m+1}, z_1, Z_2, \ldots, z_{2m-2} \) on a neighborhood \( U_1 \subset U \) of \( x \) such that : \( X = \frac{\partial}{\partial x_1}, Y = \frac{\partial}{\partial x_{m+1}} \). Consider the differential form
\[ \lambda = \omega - dx_1 \wedge dx_{m+1}. \]
We have \( d\lambda = 0 \) and
\[ i_X \lambda = L_X \lambda = i_Y \lambda = L_Y \lambda = 0. \]
So \( \lambda \) is expressed as a 2-differential form based only on variables \( z_1, z_2, \ldots, z_{2m-2} \). In particular, we have \( \lambda^{m+1} = 0 \). Furthermore, we have
\[ 0 \neq \omega^m = mdx_1 \wedge dx_{m+1} \wedge \lambda^{m-1}. \]
The 2-form $\lambda$ is closed and of maximal rank (rank half) $m - 1$ on an open set of $\mathbb{R}^{2m-2}$. It is therefore sufficient to apply the induction hypothesis to $\lambda$, which completes the proof. □

**Remark 2** If the variety $M$ is compact, connected and

$$\int_M \omega_t = \int_M \omega_0,$$

where $\{\omega_t\}$, $0 \leq t \leq 1$ is a family of volume forms, then one can find a family of diffeomorphisms $g_t : M \rightarrow M$, such that $g_0^* = \text{identity}$ and $g_t^* \omega_t = \omega_0$. Indeed, just use a reasoning similar to theorem 7, provided to replace the Poincaré’s lemma which is local, by the De Rham’s theorem which is global. This means that a volume form $\omega$ on $M$ is exact if and only if $\int_M \omega = 0$.

### 6 Poisson brackets on symplectic manifolds and Hamiltonian systems

As a consequence of the foregoing, the symplectic form $\omega$ induces a Hamiltonian vector field

$$\text{Id}_H : M \rightarrow T_x M, \quad x \mapsto \text{Id}_H(x),$$

where $H : M \rightarrow \mathbb{R}$, is a differentiable function (Hamiltonian). In others words, the differential system defined by

$$\dot{x}(t) = X_H(x(t)) = \text{Id}_H(x),$$

is a Hamiltonian vector field associated to the function $H$. The Hamiltonian vector fields form a Lie subalgebra of the vector field space. The flow $g_t^x$ leaves invariant the symplectic form $\omega$.

**Theorem 15** The matrix that is associated to an Hamiltonian system determine a symplectic structure.

**Proof.** Let $(x_1, \ldots, x_m)$ be a local coordinate system on $M$, ($m = \dim M$). We have

$$\dot{x}(t) = \sum_{k=1}^n \frac{\partial H}{\partial x_k} (dx_k) = \sum_{k=1}^n \frac{\partial H}{\partial x_k} \xi_k, \quad (5)$$

where $I(dx_k) = \xi^k \in T_x M$ is defined such that :

$$\forall \eta \in T_x M, \quad \eta_k = dx_k(\eta) = \omega(\eta, \xi^k), \quad \text{($k^{th}$-component of $\eta$).}$$

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Define \((\eta_1, \ldots, \eta_m)\) and \((\xi^k_1, \ldots, \xi^k_m)\) to be respectively the components of \(\eta\) and \(\xi^k\), then
\[
\eta_k = \sum_{i=1}^{m} \eta_i \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \xi^k_j = (\eta_1, \ldots, \eta_m) J^{-1} \begin{pmatrix} \xi^k_1 \\ \vdots \\ \xi^k_m \end{pmatrix},
\]
where \(J^{-1}\) is the matrix defined by
\[
J^{-1} \equiv \left( \omega \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right)_{1 \leq i, j \leq m}.
\]
Note that this matrix is invertible. Indeed, it suffices to show that the matrix \(J^{-1}\) has maximal rank. Suppose this were not possible, i.e., we assume that \(\text{rank}(J^{-1}) \neq m\). Hence
\[
\sum_{i=1}^{m} a_i \omega \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = 0, \quad \forall 1 \leq j \leq m,
\]
with \(a_i\) not all null and
\[
\omega \left( \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = 0, \quad \forall 1 \leq j \leq m.
\]
In fact, since \(\omega\) is non-degenerate, we have \(\sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i} = 0\). Now \(\left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \right)\) is a basis of \(T_xM\), then \(a_i = 0, \forall i\), contradiction. Since this matrix is invertible, we can search \(\xi^k\) such that :
\[
J^{-1} \begin{pmatrix} \xi^k_1 \\ \vdots \\ \xi^k_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \overset{\text{kth-place}}{\sim} 0 \\ \vdots \\ 0 \end{pmatrix}.
\]
The matrix \(J^{-1}\) is invertible, which implies
\[
\begin{pmatrix} \xi^k_1 \\ \vdots \\ \xi^k_m \end{pmatrix} = J \begin{pmatrix} 0 \\ \vdots \\ 1 \overset{\text{0}}{\sim} 0 \\ \vdots \\ 0 \end{pmatrix},
\]
from which \( \xi^k = (k^{th}\text{-column of } J) \), i.e., \( \xi^k_i = J_{ik}, 1 \leq i \leq m \), and consequently

\[
\xi^k = \sum_{i=1}^{m} J_{ik} \frac{\partial}{\partial x_i}.
\]

It is easily verified that the matrix \( J \) is skew-symmetric. From (3) we deduce that

\[
\dot{x}(t) = \sum_{k=1}^{m} \sum_{i=1}^{m} J_{ik} \frac{\partial}{\partial x_i} = \sum_{i=1}^{m} \left( \sum_{k=1}^{m} J_{ik} \frac{\partial H}{\partial x_k} \right) \frac{\partial}{\partial x_i}.
\]

Writing

\[
\dot{x}(t) = \sum_{i=1}^{m} \frac{dx_i(t)}{dt} \frac{\partial}{\partial x_i},
\]

it is seen that

\[
\dot{x}_i(t) = \sum_{k=1}^{m} J_{ik} \frac{\partial H}{\partial x_k}, \quad 1 \leq i \leq j \leq m
\]

which can be written in more compact form

\[
\dot{x}(t) = J(x) \frac{\partial H}{\partial x},
\]

this is the Hamiltonian vector field associated to the function \( H \). □

Let \((M, \omega)\) be a symplectic manifold. To any pair of differentiable functions \((F, G)\) over \(M\), we associate the function

\[
\{F, G\} = d_u F(X_G) = X_G F(u) = \omega(X_G, X_F),
\]

where \(X_F\) and \(X_G\) are the Hamiltonian vector fields associated with the functions \(F\) and \(G\) respectively. We say that \(\{F, G\}\) is a Poisson bracket (or Poisson structure) of the functions \(F\) and \(G\). It is easily verified that the Poisson bracket on the space \(C^\infty\), i.e., the bilinear application

\[
\{,\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M), \quad (F, G) \mapsto \{F, G\},
\]

defined above (where \(C^\infty(M)\) is the commutative algebra of regular functions on \(M\)) is skew-symmetric \(\{F, G\} = -\{G, F\}\), obeys the Leibniz rule

\[
\{FG, H\} = F\{G, H\} + G\{F, H\},
\]

and satisfies the Jacobi identity

\[
\{\{H, F\}, G\} + \{\{F, G\}, H\} + \{\{G, H\}, F\} = 0.
\]

2. Indeed, since \(\omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = -\omega\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)\), i.e., \(\omega\) is symmetric, it follows that \(J^{-1}\) is skew-symmetric. Then, \(I = J J^{-1} = (J^{-1})^T = J^\top = -J^{-1}.J\) and consequently \(J^\top = J\)
The variety $M$ is called a Poisson manifold or a Hamiltonian variety. The Leibniz formula ensures that the mapping $G \mapsto \{G, F\}$ is a derivation. The antisymmetry and identity of Jacobi ensure that $\{\cdot, \cdot\}$ is a Lie bracket, they provide $C^\infty(M)$ of an infinite-dimensional Lie algebra structure. When this Poisson structure is non-degenerate, we obtain the symplectic structure discussed above.

Consider now $M = \mathbb{R}^n \times \mathbb{R}^n$ and let $p \in M$. By Darboux’s theorem, there exists a local coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in a neighborhood of $p$ such that

$$\{H, F\} = \sum_{i=1}^n \left( \frac{\partial H}{\partial x_i} \frac{\partial F}{\partial y_i} - \frac{\partial H}{\partial y_i} \frac{\partial F}{\partial x_i} \right) = X_H,$$

and

$$X_H F = \{H, F\}, \quad \forall F \in C^\infty(M)$$

The manifold $M$ with the local coordinates $y_1, \ldots, y_n, x_1, \ldots, x_n$ and the above mentioned canonical Poisson bracket is a Poisson manifold. The Hamiltonian systems form a Lie algebra. A nonconstant function $F$ is called an integral (first integral or constant of motion) of $X_F$, if $X_H F = 0$; this means that $F$ is constant on the trajectories of $X_H$. In particular, $H$ is integral. Two functions $F$ and $G$ are said to be in involution or to commute, if $\{F, G\} = 0$. An interesting result is given by the following Poisson theorem:

**Theorem 16** If $F$ and $G$ are two first integrals of a Hamiltonian system, then $\{F, G\}$ is also a first integral.

**Proof.** Jacobi’s identity is written

$$\{\{H, F\}, G\} + \{\{F, G\}, H\} + \{\{G, H\}, F\} = 0,$$

where $H$ is the Hamiltonian. Since $\{H, F\} = \{H, G\} = 0$, then we have $\{\{F, G\}, H\} = 0$, which shows that $\{F, G\}$ is a first integral. □

**Remark 3** If we know two first integrals, we can, according to Poisson’s theorem, find new integrals. But let’s mention that we often fall back on known first integrals or a constant.

Let $M$ and $N$ two differentiable manifolds and $f \in C^\infty(M, N)$. The linear tangent map to $f$ at the point $p$ is the induced mapping between the tangent spaces $T_p M$ and $T_{f(p)} N$, defined by

$$f_* : T_p M \longrightarrow T_{f(p)} N, \quad f_* v(\varphi) = v(\varphi \circ f),$$

where $v \in T_p M$ and $\varphi \in C^\infty(N, \mathbb{R})$. Let $L : TM \longrightarrow \mathbb{R}$ be a differentiable function (Lagrangian) on the tangent bundle $TM$. We say that $(M, L)$ is
invariant under the differentiable application \( g : M \to M \) if for all \( v \in TM \), we have

\[
L(g_* v) = L(v).
\]

The theorem of Noether below, expresses the existence of a first integral associated with a symmetry of the Lagrangian. In other words, each parameter of a group of transformations corresponds to a conserved quantity. One of the consequences of the invariance of the Lagrangian with respect to a group of transformations is the conservation of the generators of the group. For example, the first integral associated with rotation invariance is the kinetic moment. Similarly, the first integral associated with the invariance with respect to the translations is the pulse. The Noether theorem applies to certain classes of theories, described either by a Lagrangian or a Hamiltonian. We will give below the theorem in its original version, which applies to the theories described by a Lagrangian. There is also a version that applies to theories described by a Hamiltonian.

**Theorem 17** If \((M, L)\) is invariant under a parameter group of diffeomorphisms \( g_s : M \to M, s \in \mathbb{R}, g_0 = E \), then the system of Lagrange equations,

\[
\frac{d}{dt} \frac{\partial L}{\partial q} = \frac{\partial L}{\partial q},
\]

corresponding to \( L \) admits a first integral \( I : TM \to \mathbb{R} \) with

\[
I(q, \dot{q}) = \frac{\partial L}{\partial q} \frac{dg_s(q)}{ds} \bigg|_{s=0},
\]

the \( q \) being local coordinates on \( M \).

**Proof.** The first integral \( I \) is independent of the choice of local coordinates \( q \) over \( M \) and so we can just consider the case \( M = \mathbb{R}^n \). Let

\[
f : \mathbb{R} \to M, \quad t \mapsto q = f(t),
\]

be a solution of the system of Lagrange equations above. By hypothesis, \( g_s \) leaves \( L \) invariant, so

\[
g_s \circ f : \mathbb{R} \to M, \quad t \mapsto g_s \circ f(t),
\]

also satisfies the system of Lagrange equations. We translate the solution \( f(t) \) considering the application

\[
F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n, \quad (s, t) \mapsto q = g_s(f(t)).
\]

The fact that \( g_s \) leaves invariant \( L \) implies that:

\[
0 = \frac{\partial L(F, \dot{F})}{\partial s} = \frac{\partial L}{\partial q} \frac{\partial F}{\partial s} + \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{F}}{\partial s},
\]
i.e.,
\[
\frac{\partial L}{\partial q} \frac{\partial q}{\partial s} + \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial s} = 0.
\] (6)

Since \( F \) is also a solution of the system of Lagrange equations, i.e.,
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial q}(F(s,t), \dot{F}(s,t)) \right) = \frac{\partial L}{\partial \dot{q}}(F(s,t), F(s,t)) ,
\]
so noting that :
\[
\frac{\partial \dot{q}}{\partial s} = \frac{d}{dt} \frac{\partial q}{\partial s} ,
\]
and equation (6) is written in the form
\[
0 = \frac{\partial q}{\partial s} \frac{d}{dt} \left( \frac{\partial L}{\partial q}(F(s,t), \dot{F}(s,t)) \right) + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \frac{\partial \dot{q}}{\partial s} ,
\]
which completes the proof of the theorem. □

We now give the following definition of the Poisson bracket :
\[
\{ F, G \} = \left< \frac{\partial F}{\partial x_i} J \frac{\partial G}{\partial x_j} \right> = \sum_{i,j} J_{ij} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} .
\]

We will look for conditions on the matrix \( J \) for Jacobi’s identity to be satisfied. This is the purpose of the following theorem :

**Theorem 18** The matrix \( J \) satisfies the Jacobi identity, if
\[
\sum_{k=1}^{2n} \left( J_{kj} \frac{\partial J_{li}}{\partial x_k} + J_{ki} \frac{\partial J_{lj}}{\partial x_k} + J_{kl} \frac{\partial J_{ij}}{\partial x_k} \right) = 0, \quad \forall 1 \leq i, j, l \leq 2n.
\]

**Proof.** Consider the Jacobi identity :
\[
\{ \{ H, F \}, G \} + \{ \{ F, G \}, H \} + \{ \{ G, H \}, F \} = 0.
\]

We have
\[
\{ \{ H, F \}, G \} = \left< \frac{\partial \{ H, F \}}{\partial x} J \frac{\partial G}{\partial x} \right> = \sum_{k,l} J_{kl} \frac{\partial \{ H, F \}}{\partial x_k} \frac{\partial G}{\partial x_l} ,
\]
hence
\[
\{ \{ H, F \}, G \} = \sum_{k,l} \sum_{i,j} J_{kl} \frac{\partial J_{ij}}{\partial x_k} \frac{\partial H}{\partial x_i} \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial x_l} + \sum_{k,l} \sum_{i,j} J_{kl} J_{ij} \frac{\partial^2 H}{\partial x_k \partial x_l} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} + \sum_{k,l} \sum_{i,j} J_{kl} J_{ij} \frac{\partial H}{\partial x_k} \frac{\partial^2 F}{\partial x_i \partial x_j} \frac{\partial G}{\partial x_l} ,
\]

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By symmetry, we have immediately \( \{ \{ F, G \}, H \} \) and \( \{ \{ G, H \}, F \} \). Then

\[
\{ \{ H, F \}, G \} + \{ \{ F, G \}, H \} + \{ \{ G, H \}, F \}
\]

\[
= \sum_{k,l} \sum_{i,j} J_{kl} \frac{\partial J_{ij}}{\partial x_k} \frac{\partial H}{\partial x_i} \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial x_l}
\]

\[
+ \sum_{k,l} \sum_{i,j} J_{kl} J_{ij} \frac{\partial^2 H}{\partial x_k \partial x_i} \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial x_l}
\]  

(7)

\[
+ \sum_{k,l} \sum_{i,j} J_{kl} J_{ij} \frac{\partial H}{\partial x_i} \frac{\partial^2 F}{\partial x_j \partial x_l} \frac{\partial G}{\partial x_k}
\]

(8)

\[
+ \sum_{k,l} \sum_{i,j} J_{kl} J_{ij} \frac{\partial J_{ij}}{\partial x_k} \frac{\partial G}{\partial x_i} \frac{\partial H}{\partial x_j} \frac{\partial F}{\partial x_l}
\]

\[
+ \sum_{k,l} \sum_{i,j} J_{kl} J_{ij} \frac{\partial^2 G}{\partial x_k \partial x_i} \frac{\partial H}{\partial x_j} \frac{\partial F}{\partial x_l}
\]  

(9)

\[
+ \sum_{k,l} \sum_{i,j} J_{kl} J_{ij} \frac{\partial G}{\partial x_i} \frac{\partial^2 H}{\partial x_j \partial x_l} \frac{\partial F}{\partial x_k}
\]

(10)

\[
+ \sum_{k,l} \sum_{i,j} J_{kl} J_{ij} \frac{\partial J_{ij}}{\partial x_k} \frac{\partial G}{\partial x_i} \frac{\partial H}{\partial x_j} \frac{\partial F}{\partial x_l}
\]

\[
+ \sum_{k,l} \sum_{i,j} J_{kl} J_{ij} \frac{\partial^2 F}{\partial x_k \partial x_i} \frac{\partial G}{\partial x_j} \frac{\partial H}{\partial x_l}
\]  

(11)

\[
+ \sum_{k,l} \sum_{i,j} J_{kl} J_{ij} \frac{\partial F}{\partial x_i} \frac{\partial^2 G}{\partial x_j \partial x_l} \frac{\partial H}{\partial x_k}
\]  

(12)

Notice that the indices \( i, j, k \) and \( l \) play a symmetric role. Applying in the term (10) the permutation \( i \leftarrow l, j \leftarrow k, k \leftarrow i, l \leftarrow j \), and add the term (7), with the understanding that \( J_{lk} = -J_{kl} \), we get

\[
\sum_{k,l} \sum_{i,j} (J_{ij} J_{lk} + J_{kl} J_{ij}) \frac{\partial G}{\partial x_i} \frac{\partial^2 H}{\partial x_j \partial x_k} \frac{\partial F}{\partial x_l} = 0,
\]

as a consequence of the Schwarz’s lemma. Again applying in the term (11) the permutation \( i \leftarrow k, j \leftarrow l, k \leftarrow j, l \leftarrow i \), and add the term (8), yields

\[
\sum_{k,l} \sum_{i,j} (J_{ij} J_{kl} + J_{kl} J_{ij}) \frac{\partial^2 F}{\partial x_j \partial x_k} \frac{\partial G}{\partial x_l} \frac{\partial H}{\partial x_i} = 0.
\]

By the same argument as above, applying in the term (12) the permutation \( i \leftarrow l, j \leftarrow k, k \leftarrow i, l \leftarrow j \), and add the term (9), we obtain

\[
\sum_{k,l} \sum_{i,j} (J_{ij} J_{lk} + J_{kl} J_{ij}) \frac{\partial F}{\partial x_i} \frac{\partial^2 G}{\partial x_j \partial x_l} \frac{\partial H}{\partial x_k} = 0.
\]
and thus
\[
\{\{H, F\}, G\} + \{\{F, G\}, H\} + \{\{G, H\}, F\} \\
= \sum_{k,l} \sum_{i,j} J_{kli} \frac{\partial H}{\partial x_k} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} 
\]
(13)
\[
\sum_{k,l} \sum_{i,j} J_{kli} \frac{\partial H}{\partial x_k} \frac{\partial G}{\partial x_i} \frac{\partial F}{\partial x_j} 
\]
(14)
\[
\sum_{k,l} \sum_{i,j} J_{kli} \frac{\partial F}{\partial x_k} \frac{\partial G}{\partial x_i} \frac{\partial H}{\partial x_j} 
\]

Under permuting the indices \(i \leftarrow l, j \leftarrow i, k \leftarrow k, l \leftarrow j\), for (13) and \(i \leftarrow j, j \leftarrow l, k \leftarrow k, l \leftarrow i\), for (14), we obtain the following:
\[
\{\{H, F\}, G\} + \{\{F, G\}, H\} + \{\{G, H\}, F\} = \sum_{i,j,l} \left[ \sum_{k} \left( J_{kij} \frac{\partial G}{\partial x_k} + J_{kji} \frac{\partial H}{\partial x_k} + J_{kli} \frac{\partial J_{ilj}}{\partial x_k} \right) \right] \frac{\partial H}{\partial x_i} \frac{\partial F}{\partial x_j}.
\]

Since the Jacobi identity must be identically zero, then the expression to prove follows immediately, ending the proof of theorem. □

Consequently, we have a complete characterization of Hamiltonian vector field
\[
\dot{x}(t) = X_H(x(t)) = J \frac{\partial H}{\partial x}, \quad x \in M,
\]
(15)

where \(H : M \rightarrow \mathbb{R}\), is the Hamiltonian and \(J = J(x)\) is a skew-symmetric matrix, for which the corresponding Poisson bracket satisfies the Jacobi identity:
\[
\{\{H, F\}, G\} + \{\{F, G\}, H\} + \{\{G, H\}, F\} = 0,
\]
with
\[
\{H, F\} = \left( \frac{\partial H}{\partial x}, \frac{\partial F}{\partial x} \right) = \sum_{i,j} J_{ij} \frac{\partial H}{\partial x_i} \frac{\partial F}{\partial x_j}, \quad \text{(Poisson bracket)}.
\]

7 Examples

Example 8 An important special case is when
\[
J = \begin{pmatrix} O & -I \\ I & O \end{pmatrix},
\]
where \(I\) is the \(n \times n\) identity matrix. The condition on \(J\) is trivially satisfied. Indeed, here the matrix \(J\) do not depend on the variable \(x\) and we have
\[
\{H, F\} = \sum_{i=1}^{2n} \frac{\partial H}{\partial x_i} \sum_{j=1}^{2n} J_{ij} \frac{\partial F}{\partial x_j} = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial x_{n+i}} \frac{\partial F}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial F}{\partial x_{n+i}} \right).
\]

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Moreover, equations (15) are transformed into

\[ \dot{q}_1 = \frac{\partial H}{\partial p_1}, \ldots, \dot{q}_n = \frac{\partial H}{\partial p_n}, \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \ldots, \dot{p}_n = -\frac{\partial H}{\partial q_n}, \]

where \( q_1 = x_1, \ldots, q_n = x_n, p_1 = x_{n+1}, \ldots, p_n = x_{2n} \). These are exactly the well known differential equations of classical mechanics in canonical form. They show that it suffices to know the Hamiltonian function \( H \) to determine the equations of motion. They are often interpreted by considering that the variables \( p_k \) and \( q_k \) are the coordinates of a point that moves in a space with \( 2n \) dimensions, called phase space. The flow associated with the system above obviously leaves invariant each hypersurface of constant energy \( H = c \). The Hamilton equations above, can still be written in the form

\[ \dot{q}_i = \{H, q_i\} = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \{H, p_i\} = -\frac{\partial H}{\partial q_i}, \]

where \( 1 \leq i \leq n \). Note that the functions \( 1, q_i, p_i \) \( (1 \leq i \leq n) \), verify the following commutation relations:

\[ \{q_i, q_j\} = \{p_i, p_j\} = \{q_i, 1\} = \{p_i, 1\} = 0, \quad \{p_i, q_j\} = \delta_{ij}, \quad 1 \leq i, j \leq n. \]

These functions constitute a basis of a real Lie algebra (Heisenberg algebra), of dimension \( 2n + 1 \).

**Example 9** The Hénon-Heiles differential equations are defined by

\[ \begin{align*}
\dot{y}_1 &= x_1, \quad \dot{x}_1 = -Ay_1 - 2y_1y_2, \\
\dot{y}_2 &= x_2, \quad \dot{x}_2 = -By_2 - y_1^2 - \varepsilon y_2^2,
\end{align*} \]

where \( A, B, \varepsilon \) are constants. The above equations can be rewritten as a Hamiltonian vector field

\[ \dot{x} = J \frac{\partial H}{\partial x}, \quad x = (y_1, y_2, x_1, x_2)^\top, \]

where

\[ H = \frac{1}{2}(x_1^2 + x_2^2 + Ay_1^2 + By_2^2) + y_1^2y_2 + \frac{\varepsilon}{3}y_2^3, \quad (\text{Hamiltonian}) \]

and \( J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \), is the matrix associated with the vector field.

**Example 10** The Euler equations of the rotation motion of a solid around a fixed point, taken as the origin of the reference bound to the solid, when no external force is applied to the system, can be written in the form:

\[ \begin{align*}
\dot{m}_1 &= (\lambda_3 - \lambda_2)m_2m_3, \\
\dot{m}_2 &= (\lambda_1 - \lambda_3)m_1m_3, \\
\dot{m}_3 &= (\lambda_2 - \lambda_1)m_1m_2,
\end{align*} \]
where \((m_1, m_2, m_3)\) is the angular momentum of the solid and \(\lambda_i = J_i^{-1}\), 
\(I_1, I_2\) et \(I_3\) being moments of inertia. These equations can be written in the form of a Hamiltonian vector field:
\[
\dot{x} = J \frac{\partial H}{\partial x}, \quad x = (m_1, m_2, m_3)^T,
\]
with
\[
H = \frac{1}{2} \left( \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 \right), \quad \text{(Hamiltonian)}
\]
To determine the matrix \(J = (J_{ij})_{1 \leq i, j \leq 3}\), we proceed as follows: since \(J\) is antisymmetric, then obviously \(J_{ii} = 0\), \(J_{ij} = -J_{ji}\), \(1 \leq i, j \leq 3\), hence
\[
J = \begin{pmatrix}
0 & J_{12} & J_{13} \\
-J_{12} & 0 & J_{23} \\
-J_{13} & -J_{23} & 0
\end{pmatrix}.
\]
Therefore,
\[
\begin{pmatrix}
\dot{m}_1 \\
\dot{m}_2 \\
\dot{m}_3
\end{pmatrix} = \begin{pmatrix}
0 & J_{12} & J_{13} \\
-J_{12} & 0 & J_{23} \\
-J_{13} & -J_{23} & 0
\end{pmatrix} \begin{pmatrix}
\lambda_1 m_1 \\
\lambda_2 m_2 \\
\lambda_3 m_3
\end{pmatrix}, \quad (16)
\]
\[
= \begin{pmatrix}
(\lambda_3 - \lambda_2) m_2 m_3 \\
(\lambda_1 - \lambda_3) m_1 m_3 \\
(\lambda_2 - \lambda_1) m_1 m_2
\end{pmatrix}. \quad (17)
\]
Comparing (16) and (17), we deduce that: \(J_{12} = -m_3\), \(J_{13} = m_2\) and \(J_{23} = -m_1\). Finally,
\[
J = \begin{pmatrix}
0 & -m_3 & m_2 \\
m_3 & 0 & -m_1 \\
-m_2 & m_1 & 0
\end{pmatrix} \in so(3),
\]
is the matrix of the Hamiltonian vector field. It is easy to verify that it satisfies the Jacobi identity or according to theorem 18, to the formula:
\[
\sum_{k=1}^{3} \left( J_{kj} \frac{\partial J_{li}}{\partial m_k} + J_{ki} \frac{\partial J_{lj}}{\partial m_k} + J_{kl} \frac{\partial J_{ij}}{\partial m_k} \right) = 0, \quad \forall 1 \leq i, j, l \leq 3.
\]

**Example 11** The equations of the geodesic flow on the group \(SO(4)\) can be written in the form:
\[
\begin{align*}
\dot{x}_1 &= (\lambda_3 - \lambda_2) x_2 x_3 + (\lambda_6 - \lambda_5) x_5 x_6, \\
\dot{x}_2 &= (\lambda_1 - \lambda_3) x_1 x_3 + (\lambda_4 - \lambda_6) x_4 x_6, \\
\dot{x}_3 &= (\lambda_2 - \lambda_1) x_1 x_2 + (\lambda_5 - \lambda_4) x_4 x_5, \\
\dot{x}_4 &= (\lambda_3 - \lambda_5) x_3 x_5 + (\lambda_6 - \lambda_2) x_2 x_6, \\
\dot{x}_5 &= (\lambda_4 - \lambda_3) x_3 x_4 + (\lambda_1 - \lambda_6) x_1 x_6, \\
\dot{x}_6 &= (\lambda_2 - \lambda_4) x_2 x_4 + (\lambda_5 - \lambda_1) x_1 x_5,
\end{align*}
\]
where $\lambda_1, \ldots, \lambda_6$ are constants. These equations can be written in the form of a Hamiltonian vector field. We have

$$\dot{x}(t) = J \frac{\partial H}{\partial x}, \quad x \in \mathbb{R}^6,$$

with

$$H = \frac{1}{2} (\lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_6 x_6^2).$$

By proceeding in a similar way to the previous example, we obtain

$$J = \begin{pmatrix} 0 & -x_3 & x_2 & 0 & -x_6 & x_5 \\ x_3 & 0 & -x_1 & x_6 & 0 & -x_4 \\ -x_2 & x_1 & 0 & -x_5 & x_4 & 0 \\ 0 & -x_6 & x_5 & 0 & -x_3 & x_2 \\ x_6 & 0 & -x_4 & x_3 & 0 & -x_1 \\ -x_5 & x_4 & 0 & -x_2 & x_1 & 0 \end{pmatrix}.$$

**Example 12** The movement of the Kowalewski spinning top is governed by the following equations (see subsection 10.1 for more information):

$$\dot{m} = m \wedge \lambda m + \gamma \wedge l,$$

$$\dot{\gamma} = \gamma \wedge \lambda m,$$

where $m, \gamma$ and $l$ denote respectively the angular momentum, the direction cosine of the $z$ axis (fixed in space), the center of gravity which can be reduced to $l = (1, 0, 0)$ and $\lambda m = \left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{m_3}{2}\right)$. These equations can be written in the form of a Hamiltonian vector field. The system above is written in the form of a Hamiltonian vector field

$$\dot{x} = J \frac{\partial H}{\partial x}, \quad x = (m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3)^\top,$$

with

$$H = \frac{1}{2} (m_1^2 + m_2^2) + m_3^2 + 2\gamma_1,$$

the Hamiltonian and

$$J = \begin{pmatrix} 0 & -m_3 & m_2 & 0 & -\gamma_3 & \gamma_2 \\ m_3 & 0 & -m_1 & \gamma_3 & 0 & -\gamma_1 \\ -m_2 & m_1 & 0 & -\gamma_2 & \gamma_1 & 0 \\ 0 & -\gamma_3 & \gamma_2 & 0 & 0 & 0 \\ \gamma_3 & 0 & -\gamma_1 & 0 & 0 & 0 \\ -\gamma_2 & \gamma_1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Example 13** The motion of a solid in a perfect fluid is described using the Kirchhoff equations:

$$\dot{p} = p \wedge \frac{\partial H}{\partial l}, \quad \dot{l} = p \wedge \frac{\partial H}{\partial p} + l \wedge \frac{\partial H}{\partial l},$$

(19)
where \( p = (p_1, p_2, p_3) \in \mathbb{R}^3 \), \( l = (l_1, l_2, l_3) \in \mathbb{R}^3 \) and \( H \) the Hamiltonian. The problem of this movement is a limit case of the geodesic flow on \( \text{SO}(4) \). In the case of Clebsch, we have

\[
H = \frac{1}{2} \sum_{k=1}^{3} (a_k p_k^2 + b_k l_k^2),
\]

with the condition:

\[
\frac{a_2 - a_3}{b_1} + \frac{a_3 - a_1}{b_2} + \frac{a_1 - a_2}{b_3} = 0.
\]

The system (19) is written in the form of a Hamiltonian vector field:

\[
\dot{x} = J \frac{\partial H}{\partial x}, \quad x = (p_1, p_2, p_3, l_1, l_2, l_3)^\top,
\]

where

\[
J = \begin{pmatrix}
0 & 0 & 0 & 0 & -p_3 & p_2 \\
0 & 0 & 0 & p_3 & 0 & -p_1 \\
0 & 0 & 0 & -p_2 & p_1 & 0 \\
0 & -p_3 & p_2 & 0 & -l_3 & l_2 \\
p_3 & 0 & -p_1 & l_3 & 0 & -l_1 \\
-p_2 & p_1 & 0 & -l_2 & l_1 & 0
\end{pmatrix}.
\]

**Example 14**

a) Let

\[
\frac{df}{dt} = \sum_{k=1}^{n} \left( \frac{\partial f}{\partial p_k} \dot{p}_k + \frac{\partial f}{\partial q_k} \dot{q}_k \right) + \frac{\partial f}{\partial t},
\]

be the total derivative of a function \( f(p, q, t) \) with respect to \( t \). We will determine a necessary and sufficient condition for \( f \) to be a first integral of a system described by a Hamiltonian \( H \). Taking into account Hamilton’s equations, we obtain the expression

\[
\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.
\]

We deduce that \( f \) is a first integral of a system described by a Hamiltonian \( H(p, q, t) \) explicitly dependent on \( t \) if and only if

\[
\{f, H\} + \frac{\partial f}{\partial t} = 0, \tag{20}
\]

and obviously if \( f \) does not depend explicitly on \( t \), we have \( \{f, H\} = 0 \).

**Example 15**

Consider a Hamiltonian

\[
H = \frac{1}{2m} (p_1^2 + p_2^2 + p_3^2) + V(r, t), \quad r = \sqrt{q_1^2 + q_2^2 + q_3^2},
\]

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describing the motion of a particle having a mass \( m \) and immersed into a potential \( V(r, t) \). We will determine three first integrals of the system described by this Hamiltonian. The two components of kinetic moment are equal to

\[
H_1 = q_2 p_3 - q_3 p_2, \quad H_2 = q_3 p_1 - q_1 p_3.
\]

They are obviously first integrals. According to Poisson’s theorem 16, we have

\[
\{H_1, H_2\} = q_1 p_2 - q_2 p_1 = H_3
\]

which shows that \( H_3 \) is also a first integral. Note also that :

\[
\{H_3, H_1\} = H_2, \quad \{H_2, H_3\} = H_1.
\]

If in a system two components of kinetic moment are first integrals, then the third component is also a first integral.

**Example 16** We have already seen that in a conservative system, the Hamiltonian \( H(p, q) \) is a first integral. We will show that if \( F(p, q, t) \) denotes another first integral explicitly dependent on \( t \), then \( \frac{\partial F}{\partial t} \) is also a first integral. We will apply this result to the case of the Hamiltonian of the harmonic oscillator :

\[
H = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} q^2.
\]

According to the Poisson theorem 10, \( \{F, H\} \) is also a first integral. Therefore,

\[
\frac{\partial F}{\partial t} = -\{F, H\},
\]

is a first integral under (20). Similarly, we have

\[
\left\{ \frac{\partial F}{\partial t}, H \right\} + \frac{\partial^2 F}{\partial t^2} = 0,
\]

which shows that

\[
\frac{\partial^2 F}{\partial t^2} = -\left\{ \frac{\partial F}{\partial t}, H \right\},
\]

is also a first integral. And similarly, we show that \( \frac{\partial^2 F}{\partial t^2} \) is a first integral. For the Hamiltonian of the harmonic oscillator, we easily check that

\[
F = q \cos \omega t - \frac{1}{m\omega} p \sin \omega t,
\]

and

\[
\frac{\partial F}{\partial t} = -\omega q \sin \omega t - \frac{1}{m\omega} p \cos \omega t,
\]

are first integrals of the Hamiltonian system associated with \( H \).
8 Coadjoint orbits and their symplectic structures

We will first define the adjoint and coadjoint orbits of a Lie group. Let $G$ be a Lie group and $g$ an element of $G$. The Lie group $G$ operates on itself by left translation:

$$L_g : G \to G, \quad h \mapsto gh,$$

and by right translation:

$$R_g : G \to G, \quad h \mapsto hg.$$

By virtue of the associative law of the group, we have

$$L_g L_h = L_{gh}, \quad R_g R_h = R_{hg}, \quad L_g^{-1} = L_g^{-1}, \quad R_g^{-1} = R_g^{-1}.$$

In particular, the applications $R_g$ and $L_g$ are diffeomorphisms of $G$. Also, because of associativity, $R_g$ and $L_g$ commute. Consider

$$R_g^{-1}L_g : G \to G, \quad h \mapsto ghg^{-1},$$

the automorphism of the group $G$. It leaves the unit $e$ of the group $G$ fixed, i.e.,

$$R_g^{-1}L_g(e) = geg^{-1} = e.$$

We can define the adjoint representation of the group $G$ as the derivative of $R_g^{-1}L_g$ in the unit $e$, that is, the induced application of tangent spaces as follows

$$Ad_g : G \to \mathcal{G}, \quad \xi \mapsto \left. \frac{d}{dt} R_g^{-1}L_g(e^t\xi) \right|_{t=0},$$

where $\mathcal{G} = T_eG$ is the Lie algebra of the $G$ group; it is the tangent space at $G$ in its unit $e$. This definition has a meaning because $R_g^{-1}L_g(e^t\xi)$ is a curve in $G$ and passes through the identity in $t = 0$. Therefore, $g\xi g^{-1} \in \mathcal{G}$.

**Theorem 19** For any element $\xi \in \mathcal{G}$, we have

$$Ad_g(\xi) = g\xi g^{-1}, \quad g \in G,$$

and

$$Ad_{gh} = Ad_g Ad_h.$$

The application $Ad_g$ is an algebra homomorphism, i.e.,

$$Ad_g[\xi, \eta] = [Ad_g \xi, Ad_g \eta], \quad (\xi, \eta \in \mathcal{G}).$$

**Proof.** We have

$$Ad_g(\xi) = \left. \frac{d}{dt} R_g^{-1}L_g(e^t\xi) \right|_{t=0} = \left. \frac{d}{dt} ge^t\xi g^{-1} \right|_{t=0},$$

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hence
\[
Ad_g(\xi) = \frac{d}{dt} g \left( \sum_{n=0}^{\infty} \frac{t^n \xi^n}{n!} \right) g^{-1} igg|_{t=0},
\]
\[
= \frac{d}{dt} \sum_{n=0}^{\infty} \frac{t^n g \xi^n g^{-1}}{n!} \bigg|_{t=0},
\]
\[
= \frac{d}{dt} \sum_{n=0}^{\infty} \frac{t^n g \xi^{-1} \cdot g \xi g^{-1} \cdot \ldots \cdot g \xi^{-1}}{n!} \bigg|_{t=0},
\]
\[
= \sum_{n=0}^{\infty} \frac{t^n (g \xi^{-1})^n}{n!} \bigg|_{t=0},
\]
and finally
\[
Ad_g(\xi) = \left. \frac{d}{dt} e^{t(g \xi^{-1})} \right|_{t=0} = g \xi g^{-1}.
\]
We easily check that :
\[
Ad_{gh} = Ad_g \cdot Ad_h.
\]
Indeed, we have
\[
Ad_{gh}(\xi) = gh \xi (gh)^{-1} = gh \xi h^{-1} g^{-1},
\]
\[
Ad_g \cdot Ad_h(\xi) = Ad_g (h \xi h^{-1}) = gh \xi h^{-1} g^{-1}.
\]
We have
\[
Ad_g [\xi, \eta] = Ad_g (\xi \eta - \eta \xi),
\]
\[
= g (\xi \eta - \eta \xi) g^{-1},
\]
\[
= g \xi \eta g^{-1} - g \eta \xi g^{-1},
\]
\[
= g \xi g^{-1} \cdot g \eta g^{-1} - g \eta g^{-1} \cdot g \xi g^{-1},
\]
\[
= [g \xi g^{-1} \cdot g \eta g^{-1}],
\]
\[
= [Ad_g \xi, Ad_g \eta],
\]
which completes the demonstration. □

The adjoint orbit of \( \xi \) is defined by
\[
O_G(\xi) = \{ Ad_g(\xi) : g \in G \} \subset G.
\]

Now consider the function
\[
Ad : G \rightarrow \text{End}(\mathcal{G}), \quad g \mapsto Ad(g) \equiv Ad_g,
\]
where \( \text{End}(\mathcal{G}) \) is the space of the linear operators on the algebra \( \mathcal{G} \). The application \( Ad \) is differentiable and its derivative \( Ad_{\xi \epsilon} \) in the unit of the
group $G$ is a linear map from the algebra $T_eG = \mathcal{G}$ to the vector space $T_I\text{End}(\mathcal{G}) = \text{End}(\mathcal{G})$. This application will be noted

$$ad \equiv Ad_{se}: \mathcal{G} \rightarrow \text{End}(\mathcal{G}), \quad \xi \mapsto ad\xi = \frac{d}{dt}Ad_{g(t)} \bigg|_{t=0},$$

where $g(t)$ is a one-parameter group with $\frac{d}{dt}g(t) \bigg|_{t=0} = \xi$ and $g(0) = e$.

**Theorem 20** Let $\xi \in \mathcal{G}$ et $\eta \in \text{End}(\mathcal{G})$. By setting $ad\xi \equiv Ad_{se}(\xi)$, then

$$ad\xi(\eta) = [\xi, \eta].$$

**Proof.** We have

$$ad\xi(\eta) = Ad_{se}(\xi)(\eta),$$

$$= \frac{d}{dt}Ad_{g(t)}(\eta) \bigg|_{t=0},$$

$$= \frac{d}{dt}(g(t)\eta g^{-1}(t)) \bigg|_{t=0},$$

$$= \dot{g}(t)\eta g^{-1}(t) \bigg|_{t=0} - g(t)\eta g^{-1}(t)\dot{g}(t)g^{-1}(t) \bigg|_{t=0},$$

$$= \dot{g}(0)\eta - \eta\dot{g}(0),$$

$$= \xi\eta - \eta\xi,$$

$$= [\xi, \eta],$$

which completes the proof. □

Let $T^*_gG$ be the cotangent space to the group $G$ at $g$; it is the dual to the tangent space $T_gG$. Then an element $\zeta \in T^*_gG$ is a linear form on $T_gG$ and its value on $\eta \in T_gG$ will be denoted by,

$$\zeta(\eta) \equiv \langle \zeta, \eta \rangle.$$

Let $G^* = T^*_eG$ be the dual vector space to the Lie algebra $\mathcal{G}$; it is the cotangent space to the group $G$ in its unit $e$. The transpose operators $Ad^*_g: G^* \rightarrow G^*$, where $g$ runs through the Lie group $G$ are defined by

$$\langle Ad^*_g(\zeta), \eta \rangle = \langle \zeta, Ad_g\eta \rangle, \quad \zeta \in G^*, \quad \eta \in \mathcal{G}.$$

$Ad^*_g$ is called coadjoint representation of the Lie group $G$. The coadjoint orbit (also called Kostant-Kirillov orbit) is defined at the point $x \in G^*$ by

$$O^*_x = \{Ad^*_g(x) : g \in G\} \subset G^*.$$

**Theorem 21** The transpose operators $Ad^*_g$ form a representation of the Lie group $G$, i.e., they satisfy the relations : $Ad^*_{gh} = Ad^*_{h}Ad^*_g$. 41
Proof. Indeed, let $\zeta \in G^*$, $\eta \in G$. We have

$$\langle Ad_{gh}^*(\zeta), \eta \rangle = \langle \zeta, Ad_{gh}(\eta) \rangle = \langle \zeta, Ad_h Ad_g(\eta) \rangle,$$

hence,

$$\langle Ad_{gh}^*(\zeta), \eta \rangle = \langle Ad_g^*(\zeta), Ad_h(\eta) \rangle = \langle Ad_h^*, Ad_g^*(\zeta), \eta \rangle,$$

which completes the demonstration. □

Consider the map

$$Ad^*: G \longrightarrow \text{End}(G^*), \quad g \longmapsto Ad^*(g) \equiv Ad^*_g,$$

and its derivative in the unity of the group

$$ad^* \equiv (Ad^*)_e : G \longrightarrow \text{End}(G^*), \quad \xi \longmapsto ad^*_\xi.$$

**Theorem 22** By setting

$$\langle ad^*_\xi(\zeta), \eta \rangle = \langle \zeta, [\xi, \eta] \rangle = \langle \{\xi, \zeta\}, \eta \rangle,$$

where

$$\{,\} : G \times G^* \longrightarrow G^*, \quad (\xi, \zeta) \longmapsto \{\xi, \zeta\}, \quad (\xi, \eta \in G, \zeta \in G^*),$$

then

$$ad^*_\xi(\zeta) = \{\xi, \zeta\}.$$

**Proof.** We have

$$\langle ad^*_\xi(\zeta), \eta \rangle = \langle (Ad^*)_e(\zeta), \eta \rangle = \left\langle \frac{d}{dt} Ad_e^* \left( e^{t\xi} \right) \bigg|_{t=0}, \eta \right\rangle,$$

with $e^{t\xi}|_{t=0} = e$ and $\frac{d}{dt} e^{t\xi}|_{t=0} = \xi$. Hence,

$$\langle ad^*_\xi(\zeta), \eta \rangle = \frac{d}{dt} \langle Ad^* e^{t\xi}(\zeta), \eta \rangle \bigg|_{t=0},$$

$$\quad = \frac{d}{dt} \langle \zeta, Ad_e^*(\eta) \rangle \bigg|_{t=0},$$

$$\quad = \left\langle \zeta, \frac{d}{dt} Ad_e^*(\eta) \bigg|_{t=0} \right\rangle,$$

$$\quad = \langle \zeta, ad^*_\xi(\eta) \rangle,$$

$$\quad = \langle \zeta, [\xi, \eta] \rangle,$$

$$\quad = \langle \{\xi, \zeta\}, \eta \rangle,$$

which completes the proof. □

We will show below, how to find the adjoint orbit and the coadjoint orbit in the case of the group $SO(n)$. Recall that $SO(n)$ is the special orthogonal group.
group of order \( n \), that is, the set of matrices \( X \) of order \( n \times n \) such that:

\[
X^T X = I \quad \text{(or } X^{-1} = X^T \text{)} \quad \text{and det } X = 1.
\]

\( SO(n) \) is a Lie group. The tangent space to the identity of the group \( SO(n) \), which is denoted \( so(n) \), consists of the antisymmetric matrices of order \( n \times n \), i.e., matrices \( A \) such that:

\[
\text{the commutator of two antisymmetric matrices is still an antisymmetric matrix (if } A, B \in so(n), \text{ then } [A, B] = AB - BA \in so(n)).
\]

This product defines a Lie algebra structure on \( so(n) \); it is the Lie algebra of the group \( SO(n) \). In addition, we have

\[
\dot{X} = AX \quad \text{with } A \in so(n)
\]

and therefore the tangent space to the identity of \( SO(n) \) is

\[
T_I SO(n) = so(n).
\]

Let \( R_Y^{-1} L_Y : SO(n) \to SO(n), \quad X \mapsto YXY^{-1}, \quad Y \in SO(n), \)

be the automorphism interior of the group \( SO(n) \). When looking for the coadjoint orbit, we have to use the following obvious lemma:

**Lemma 1** The Lie algebra \( so(n) \) with the commutator \( [\cdot, \cdot] \) of matrix is isomorphic to the space \( \mathbb{R}^{\frac{n(n-1)}{2}} \) with the vector product \( \wedge \). The isomorphism is given by

\[
a \wedge b \mapsto [A, B] = AB - BA,
\]

where \( a, b \in \mathbb{R}^{\frac{n(n-1)}{2}} \) and \( A, B \in so(n) \).

**Theorem 23** The orbit of the adjoint representation of the group \( SO(n) \) is

\[
\mathcal{O}_{SO(n)}(A) = \{ YAY^{-1} : Y \in SO(n) \}, \quad A \in so(n).
\]

Let \( A \in so(n) \). The coadjoint orbit of the group \( SO(n) \) is

\[
\mathcal{O}^*_{SO(n)}(A) = \{ Y^{-1}AY : Y \in SO(n) \};
\]

\[
= \{ C \in so(n) : C = Y^{-1}AY, \text{ spectrum of } C = \text{ spectrum of } A \}.
\]

With the notation of theorem 22, we have

\[
\{ A, B \} = [B, A], \quad (A, B \in so(n)).
\]

**Proof.** Let \( Y \in SO(n), \ A \in so(n). \) By definition, the adjoint representation of the group \( SO(n) \) is

\[
Ad_Y : so(n) \to so(n), \quad A \mapsto YAY^{-1}.
\]

We have

\[
(YAY^{-1})^T = (Y^{-1})^T A^T Y^T = -YAY^T = -YAY^{-1}.
\]

So \( YAY^{-1} \in so(n) \). Let

\[
Ad : SO(n) \to \text{End}(so(n)), \quad Y \mapsto Ad_Y,
\]

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where $Ad_Y(A) = YAY^{-1}$, $A \in \text{so}(n)$, and let

$$ad : \text{so}(n) \rightarrow \text{End}(\text{so}(n)), \quad \dot{Y}(0) \mapsto ad_Y(0),$$

with

$$ad_Y(0) \cdot = [\dot{Y}(0), \cdot] : \text{so}(n) \rightarrow \text{so}(n), \quad A \mapsto [\dot{Y}(0), A],$$

where $Y(t)$ is a curve in $SO(n)$ with $Y(0) = I$. Since $(\mathbb{R}^{n \times n})^* \simeq \mathbb{R}^{n \times n}$, then according to the previous lemma, we also have the isomorphism $(\text{so}(n))^* \simeq \text{so}(n)$. We can therefore define $Ad^*$ by $Ad^*_Y : \text{so}(n) \rightarrow \text{so}(n)$, with

$$\langle Ad^*_Y(A), B \rangle = \langle A, YBY^{-1} \rangle, \quad (A, B \in \text{so}(n)),$$

i.e.,

$$\langle Ad^*_Y(A), B \rangle = -\frac{1}{2} \text{tr}(AYBY^{-1}) = -\frac{1}{2} \text{tr}(Y^{-1}AYB) = \langle Y^{-1}AY, B \rangle,$$

hence,

$$Ad^*_Y(A) = Y^{-1}AY.$$ 

We easily check that $Y^{-1}AY \in \text{so}(n)$. Indeed, we have

$$(Y^{-1}AY)^\top = Y^\top A^\top (Y^{-1})^\top = -Y^{-1}AY,$$

because $Y \in SO(n)$ and $A \in \text{so}(n)$. Then

$$O^*_{SO(n)}(A) = \{Y^{-1}AY : Y \in SO(n)\},$$

that we can write in the form

$$O^*_{SO(n)}(A) = \{C \in \text{so}(n) : \exists Y \in SO(n), C = Y^{-1}AY\}.$$

Note that $\text{det}(C - \lambda I) = \text{det}(A - \lambda I)$. Then the matrices $C$ and $A$ have the same characteristic polynomial, and consequently they have the same spectrum.

$$O^*_{SO(n)}(A) = \{C \in \text{so}(n) : C = Y^{-1}AY, \text{spectrum of } C = \text{spectrum of } A\}.$$ 

Now apply theorem 16 to the case of the group $SO(n)$. Let’s go back to the linear form knowing that $(\text{so}(n))^* = \text{so}(n)$,

$$\{,\} : \text{so}(n) \times \text{so}(n) \rightarrow \text{so}(n), \quad (A, B) \mapsto \{A, B\},$$ 

as well as the applications

$$Ad^* : SO(n) \rightarrow \text{End}(\text{so}(n)), \quad Y \mapsto Ad^*_Y(B) = Y^{-1}BY, \quad B \in \text{so}(n),$$

$$ad^* : \text{so}(n) \rightarrow \text{End}(\text{so}(n)), \quad A \mapsto ad^*_A,$$
where
\[ (ad_a^*(B), C) = \langle B, [A, C] \rangle = \{A, B\}, C \rangle. \]
We have
\[ \langle \{A, B\}, C \rangle = \langle B, [A, C] \rangle = -\frac{1}{2} \text{tr}(B, [A, C]) = -\frac{1}{2} \text{tr}(BAC - BCA), \]
hence,
\[ \langle \{A, B\}, C \rangle = -\frac{1}{2} \text{tr}([B, A], C) = \langle [B, A], C \rangle. \]
Then \( \{A, B\} = [B, A] \), and the theorem is proved. □

We will see how to define a symplectic structure on the coadjoint orbit with an application in the case of the groups \( SO(3) \) and \( SO(4) \). Let \( x \in \mathcal{G}^* \), \( \xi \) the tangent vector in \( x \) to the orbit. Since \( \mathcal{G}^* \) is a vector space, then obviously \( \xi \in T_x \mathcal{G}^* = \mathcal{G}^* \). Let’s remember that \( O^*_{\mathcal{G}}(x) = \{ Ad_g^*(x) : g \in \mathcal{G} \} \subset \mathcal{G}^* \).

For \( x \in \mathcal{O}^*_{\mathcal{G}}(x) \), there exists \( g \in \mathcal{G} \) such that : \( x = Ad_g^* \). Let \( a \in \mathcal{G} \) and \( e^{ta} \) be a group with a parameter in \( \mathcal{G} \) with \( e^{ta}|_{t=0} = g \) and \( \frac{d}{dt} Ad^*_{e^{ta}}(x)|_{t=0} = \xi \). Since
\[ \frac{d}{dt} Ad^*_{e^{ta}}(x)|_{t=0} \equiv ad^*_a(x) = \{a, x\}, \]
therefore the vector \( \xi \) can be represented as the velocity vector of the motion of \( x \) under the action of a group \( e^{ta}, a \in \mathcal{G} \). In other words, any vector \( \xi \) tangent to the orbit \( \mathcal{O}^*_{\mathcal{G}}(x) \) is expressed as a function of \( a \in \mathcal{G} \) by
\[ \xi = \{a, x\}, \quad a \in \mathcal{G}, \quad x \in \mathcal{G}^*. \]

Therefore, we can determine the value of a 2-form \( \Omega \) on the orbit \( \mathcal{O}^*_{\mathcal{G}}(x) \) as follows : let \( \xi_1 \) and \( \xi_2 \) be two vectors tangent to the orbit of \( x \). From the above, we have
\[ \xi_1 = \{a_1, x\}, \quad \xi_2 = \{a_2, x\}, \quad (a_1, a_2 \in \mathcal{G}), \quad x \in \mathcal{G}^*. \]
We can easily verify that the differential 2-form
\[ \Omega(\xi_1, \xi_2)(x) = \langle x, [a_1, a_2] \rangle, \quad a_1, a_2 \in \mathcal{G}, \quad x \in \mathcal{G}^*, \]
on \( \mathcal{O}^*_{\mathcal{G}}(x) \) is well defined; its value does not depend on the choice of \( a_1, a_2 \). It is antisymmetric, non-degenerate and closed. To determine the symplectic structure on \( \mathcal{O}^*_{SO(3)}(X) \), we proceed as follows : according to (22), we have
\[ \Omega(\xi_1, \xi_2)(X) = \langle X, [A, B] \rangle, \]
where \( A, B \in so(3), \quad X \in (so(3))^* = so(3) \) and according to (21),
\[ \xi_1 = \{A, X\}, \quad \xi_2 = \{B, X\}, \]
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are two tangent vectors to the orbit in X or what the same according to theorem 23, $\xi_1 = [X, A]$, $\xi_2 = [X, B]$. Using the isomorphism between $(so(3), [\cdot, \cdot])$ and $(\mathbb{R}^3, \wedge)$, we also have $\xi_1 = x \wedge a$, $\xi_2 = x \wedge b$, with
\begin{align*}
\Omega(\xi_1, \xi_2)(x) &= \langle x, a \wedge b \rangle.
\end{align*}

According to theorem 23, the coadjoint orbit of $SO(3)$ is
\begin{align*}
O^*_SO(3)(A) = \{ C \in so(3) : C = Y^{-1}AY, \text{ spectrum of } C = \text{ spectrum of } A \},
\end{align*}
where $A \in so(3)$ et $Y \in SO(3)$. Let’s determine the spectrum of the matrix
\begin{align*}
A = \begin{pmatrix}
0 & -a_3 & a_2 \\
-a_3 & 0 & -a_1 \\
a_2 & a_1 & 0
\end{pmatrix} \in so(3).
\end{align*}

We have
\begin{align*}
\det(A - \lambda I) &= -\lambda(\lambda^2 + a_1^2 + a_2^2 + a_3^2) = 0,
\end{align*}

hence, $\lambda = 0$ and $\lambda = \pm i \sqrt{a_1^2 + a_2^2 + a_3^2}$. Then
\begin{align*}
O^*_SO(3)(A) = \{ C \in so(3) : c_1^2 + c_2^2 + c_3^2 = r^2 \},
\end{align*}
with
\begin{align*}
C = \begin{pmatrix}
0 & -c_3 & c_2 \\
c_3 & 0 & -c_1 \\
-c_2 & c_1 & 0
\end{pmatrix} \in so(3),
\end{align*}

and $r^2 = a_1^2 + a_2^2 + a_3^2$. Since the algebra $so(3)$ is isomorphic to $\mathbb{R}^3$, we deduce that the orbit $O^*_SO(3)(A)$ is isomorphic to a sphere $S^2$ of radius $r$. Like vectors $\xi_1$, $\xi_2$ belong to the tangent plane $T_XO^*_SO(3)$ to $X$, they also belong to the tangent plane $T_xS^2$ in $x$. Let
\begin{align*}
S^2 = \{ (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = r^2 \},
\end{align*}
be the sphere of radius $r$, then the plane tangent to this sphere in $x$ of coordinates $(x_1, x_2, x_3)$ is
\begin{align*}
T_xS^2 &= \{ (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1x_1 + y_2x_2 + y_3x_3 = 0 \}, \\
&= \left\{ \left( y_1, y_2, -\frac{y_1x_1 + y_2x_2}{x_3} \right) \right\}.
\end{align*}

Let $z = (z_1, z_2, z_3) \in T_xS^2$ and determine $a = (a_1, a_2, a_3)$ such that $x \wedge a = z$. The latter is equivalent to the system
\begin{align*}
\begin{pmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
a_2 & a_1 & 0
\end{pmatrix} \begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix} &= \begin{pmatrix} z_1 \\
z_2 \\
z_1x_1 + z_2x_2 \\
x_3
\end{pmatrix},
\end{align*}

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whose solution is
\[ a = \left( \frac{x_1 a_3 + x_2}{x_3}, \frac{x_2 a_3 - z_1}{x_3}, a_3 \right), \quad a_3 \in \mathbb{R}. \]

Since the symplectic form on \( S^2 \) that one wants to determine is intrinsic, i.e., does not depend on the choice of local coordinates, one can choose as local coordinates \( x_1, x_2 \) and the same reasoning will be valid for the other cases, i.e., \( x_2, x_3 \) and \( x_3, x_1 \). So we will calculate \( a \) and \( b \) relative to the basis \( \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) \) of \( T_x S^2 \) with
\[
\frac{\partial}{\partial x_1} = \left( 1, 0, \frac{x_1}{x_3} \right), \quad \frac{\partial}{\partial x_2} = \left( 0, 1, \frac{-x_2}{x_3} \right).
\]

We have
\[
a = (a_1, a_2, a_3) = \left( \frac{x_1 b_3 + 1}{x_3}, \frac{x_2 b_3}{x_3}, b_3 \right),
\]
and
\[
a \wedge b = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) = \left( -\frac{b_3}{x_3}, \frac{a_3}{x_3}, \frac{x_1 b_3 - x_2 a_3 + 1}{x_3^2} \right).
\]

Therefore,
\[
\Omega \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) = (x, a \wedge b) = \frac{1}{x_3},
\]
consequently
\[
\Omega = \frac{dx_1 \wedge dx_2}{x_3}.
\]

The symplectic form being intrinsic, we will finally have
\[
\Omega = \frac{dx_1 \wedge dx_2}{x_3} = \frac{dx_2 \wedge dx_3}{x_1} = \frac{dx_3 \wedge dx_1}{x_2}.
\]

**Example 17** The symplectic structure obtained here is equivalent to that associated with the system (17). Indeed, we know that
\[
J^{-1} = \left( \omega \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right)_{i,j=1,2},
\]
so the matrix associated with the form
\[
\Omega = \frac{dx_1 \wedge dx_2}{x_3},
\]
is
\[
\begin{pmatrix} 0 & -x_3 \\ x_3 & 0 \end{pmatrix}. \]
Let’s show that there is equivalence between
\[
\dot{x}(t) = J \frac{\partial H}{\partial x}, \quad \text{where} \quad \begin{cases} x = (m_1, m_2, m_3)^T, \\ H = \frac{1}{2} (\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2), \\ J = \begin{pmatrix} m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix}, \end{cases}
\]

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and
\[ \dot{x}(t) = J \frac{\partial H}{\partial x}, \]
where
\[ \mathbf{x} = (m_1, m_2, m_3)^T, \quad H = H(m_1, m_2, m_3), \quad J = \begin{pmatrix} 0 & -m_3 \\ m_3 & 0 \end{pmatrix}. \]

Indeed, we have
\[ \dot{m}_1 = -m_3 \frac{\partial H}{\partial m_2} = -m_3 \left( \frac{\partial H}{\partial m_2} + \frac{\partial H}{\partial m_3} \frac{\partial m_3}{\partial m_2} \right), \]
and
\[ \dot{m}_2 = m_3 \frac{\partial H}{\partial m_1} = m_3 \left( \frac{\partial H}{\partial m_1} + \frac{\partial H}{\partial m_3} \frac{\partial m_3}{\partial m_1} \right). \]

According to example 10, we have
\[ dm_3 = -\frac{m_1 dm_1 + m_2 dm_2}{m_3}, \]
hence
\[ \frac{dm_3}{dm_2} = -\frac{m_2}{m_3}, \quad \frac{dm_3}{dm_1} = -\frac{m_1}{m_3}. \]

Therefore, we have
\[ \begin{align*}
\dot{m}_1 &= (\lambda_3 - \lambda_2) m_2 m_3, \\
\dot{m}_2 &= (\lambda_1 - \lambda_3) m_1 m_3,
\end{align*} \]
and the result follows.

**Example 18** To determine the symplectic structure on the coadjoint orbit of the Lie group \( SO(4) \), we can follow the same method as in the previous case but the calculation is longer. On the other hand, one can easily obtain the result by using a geometric approach by observing that \( so(4) \) breaks down into two copies of \( so(3) \) and that the generic orbits are a product of two spheres. More precisely, from \( SO(4) = SO(3) \otimes SO(3) \), it is more interesting to consider the coordinates \((x_1, x_2, x_3), (x_4, x_5, x_6)\) with
\[ (x_1, x_2, x_3) \oplus (x_4, x_5, x_6) \in so(4) \cong so(3) \oplus so(3). \]

We obtain
\[ \Omega = -x_3 dx_1 \wedge dx_2 - x_6 dx_1 \wedge dx_5 + x_6 dx_2 \wedge dx_4 - x_3 dx_4 \wedge dx_5. \]
9 Arnold-Liouville theorem and completely integrable systems

The so-called Arnold-Liouville theorem [4] play a crucial role in the study of the integrability of Hamiltonian systems; the regular compact level manifolds defined by the intersection of the constants of motion are diffeomorphic to a real torus on which the motion is quasi-periodic as a consequence of the following purely differential geometric fact: a compact and connected \( n \)-dimensional manifold on which there exist \( n \) vector fields which commute and are independent at every point is diffeomorphic to an \( n \)-dimensional real torus and each vector field will define a linear flow there. Consider the Hamiltonian system (15) associated with the function \( H \) (Hamiltonian) on a \( 2n \)-dimensional symplectic manifold \( M \).

**Theorem 24** Let \( H_1 = H, H_2, ..., H_n, \) be \( n \) first integrals on a \( 2n \)-dimensional symplectic manifold \( M \) that are functionally independent, i.e.,

\[
dH_1 \wedge ... \wedge dH_n \neq 0,
\]

and pairwise in involution, i.e.,

\[
\{H_i, H_j\} = 0, \quad 1 \leq i, j \leq n.
\]

For generic \( c = (c_1, ..., c_n) \in \mathbb{R}^n \), the level set

\[
M_c = \bigcap_{i=1}^{n} \{x \in M : H_i(x) = c_i\},
\]

will be an \( n \)-manifold. If \( M_c \) is compact and connected, it is diffeomorphic to an \( n \)-dimensional torus

\[
T^n = \mathbb{R}^n / \text{lattice} = \{(\varphi_1, ..., \varphi_n) \mod 2\pi\}.
\]

The flows \( g_{t}^{X_1}(x), ..., g_{t}^{X_n}(x) \) defined by the vector fields \( X_{H_1}, ..., X_{H_n} \), are straight-line motions on \( T^n \) and determine on \( T^n \) a quasi-periodic motion, i.e., in angular coordinates \( \varphi_1, ..., \varphi_n \), we have

\[
\dot{\varphi}_i = \omega_i(c), \quad \omega_i(c) = \text{constants}, \quad \varphi_i(t) = \varphi_i(0) + \omega_i t.
\]

The equations (15) of the problem are integrable by quadratures.

**Proof.** 1) Let us first show that a compact and connected \( n \)-dimensional manifold \( M \) on which there exist \( m \) differential (of class \( C^\infty \)) vector fields \( X_1, ..., X_m \) which commute and are independent at every point is diffeomorphic to an \( m \)-dimensional real torus:

\[
T^m = \mathbb{R}^m / \text{lattice} = \{(\varphi_1, ..., \varphi_m) \mod 2\pi\}.
\]
Let us define the application

\[ g : \mathbb{R}^m \rightarrow M, \quad (t_1, \ldots, t_m) \mapsto g(t_1, \ldots, t_m), \]

where

\[ g(t_1, \ldots, t_m) = g_{t_1}^{X_1} \circ \cdots \circ g_{t_m}^{X_m}(x) = g_{t_m}^{X_m} \circ \cdots \circ g_{t_1}^{X_1}(x), \quad x \in M. \]

a) The application \( g \) is a local diffeomorphism. Indeed, let

\[ g_r \equiv g|_U : U \rightarrow M, \quad (t_1, \ldots, t_m) \mapsto g_r(t_1, \ldots, t_m) = g_{t_m}^{X_m} \circ \cdots \circ g_{t_1}^{X_1}(x), \]

be the restriction of \( g \) on a neighborhood \( U \) of \((0, \ldots, 0)\) in \( \mathbb{R}^m \) with \( x = g_r(0, \ldots, 0) \). Let us show that the map \( g_r \) is differentiable (of class \( C^\infty \)). We have

\[ \frac{\partial}{\partial t_1} g_{t_1}^{X_1} = X_1(x) = (\dot{x}_1, \ldots, \dot{x}_m), \]

with

\[ \dot{x}_1 = f_1(x_1, \ldots, x_m), \ldots, \dot{x}_m = f_m(x_1, \ldots, x_m), \]

where \( f_1, \ldots, f_m : M \rightarrow \mathbb{R} \) are functions on \( M \). Similarly, we have

\[
\begin{align*}
\frac{\partial^2}{\partial t_1^2} g_{t_1}^{X_1} &= (\ddot{x}_1, \ldots, \ddot{x}_m) = \left( \sum_{k=1}^{m} \frac{\partial f_1}{\partial x_k} \dot{x}_k, \ldots, \sum_{k=1}^{m} \frac{\partial f_m}{\partial x_k} \dot{x}_k \right), \\
\frac{\partial^3}{\partial t_1^3} g_{t_1}^{X_1} &= (\dddot{x}_1, \ldots, \dddot{x}_m), \\
&= \left( \sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\partial^2 f_1}{\partial x_k \partial x_l} \ddot{x}_k \dot{x}_l + \frac{\partial f_1}{\partial x_k} \ddot{x}_k, \ldots, \sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\partial^2 f_m}{\partial x_k \partial x_l} \ddot{x}_k \dot{x}_l + \frac{\partial f_m}{\partial x_k} \ddot{x}_k \right),
\end{align*}
\]

e tc. All these expressions have a meaning because by hypothesis all the functions \( f_1, \ldots, f_m \) are \( C^\infty \). A similar reasoning shows that \( g_{t_2}^{X_2}, \ldots, g_{t_m}^{X_m} \) are also \( C^\infty \). Since the composite of functions \( C^\infty \) is \( C^\infty \), we deduce that \( g_r(t_1, \ldots, t_m) \) is \( C^\infty \). Let us show that the Jacobian matrix of \( g_r \) in \((0, \ldots, 0)\) is invertible. Consider

\[ g_r(t_1, \ldots, t_m) \equiv (G_1(t_1, \ldots, t_m), \ldots, G_m(t_1, \ldots, t_m)). \]

We have

\[ \det \begin{pmatrix} \frac{\partial G_1}{\partial t_1} & \cdots & \frac{\partial G_m}{\partial t_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_1}{\partial t_m} & \cdots & \frac{\partial G_m}{\partial t_m} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial}{\partial t_1} g_{t_1}^{X_1} \circ \cdots \circ g_{t_m}^{X_m}(x) \\ \vdots \\ \frac{\partial}{\partial t_m} g_{t_m}^{X_m} \circ \cdots \circ g_{t_1}^{X_1}(x) \end{pmatrix} \neq 0, \]

because the vector fields \( X_1, \ldots, X_m \) are linearly independent at each point of \( M \). According to the local inversion theorem, there exists a sufficiently small neighborhood \( V \subset U \) of \((0, \ldots, 0)\) and a neighborhood \( W \) of \( x \) such that \( g_r \)
induces a bijection of $V$ on $W$ whose inverse $g_r^{-1} : W \to V$, is $C^\infty$. In other words, $g_r$ is a diffeomorphism of $V$ over $g_r(V)$. This result is local because even if the above Jacobian matrix is invertible for any $(t_1, ..., t_m)$, then the inverse "global" of $g_r$ does not necessarily exist.

b) The application $g$ is surjective. Indeed, let $(t_1, ..., t_m) \in \mathbb{R}^m$ such that:

$$g(t_1, ..., t_m) = g_{t_m}^{X_m} \circ \cdots \circ g_{t_1}^{X_1}(x) = y \in M.$$  

We showed in the part a) that $g$ is a local diffeomorphism. So for every point $x_1$ contained in a neighborhood of $x$, there exists $(t_1, ..., t_m) \in \mathbb{R}^m$ such that:

$$g_{t_m}^{X_m} \circ \cdots \circ g_{t_1}^{X_1}(x) = x_1.$$  

Since the variety $M$ is connected, we can connect the point $x$ to the point $y$ by a curve $\mathcal{C}$. Let $B_1$ be an open ball in $M$ containing the point $x_1$. This ball exists since $M$ is compact. Let $x_2 \in \mathcal{C}$ such that $x_2$ be contained in the ball $B_1$. We reason as before, the map $g$ being a local diffeomorphism, then there exists $(t_1', ..., t_m') \in \mathbb{R}^m$ such that:

$$\left(g_{t_m'}^{X_m}\right)' \circ \cdots \circ \left(g_{t_1'}^{X_1}\right)'(x_1) = x_2.$$  

Hence,

$$x_2 = \left(g_{t_m'}^{X_m}\right)' + t_m \circ \cdots \circ \left(g_{t_1'}^{X_1}\right)' + t_1(x).$$  

Similarly, let $B_2$ be an open ball in $M$ containing the point $x_2$ and let $x_3 \in \mathcal{C}$ such that $x_3$ be either contained in the ball $B_2$. Since the application $g$ is a local diffeomorphism, then there exists $(t_1'', ..., t_m'') \in \mathbb{R}^m$ such that:

$$\left(g_{t_m''}^{X_m}\right)'' \circ \cdots \circ \left(g_{t_1''}^{X_1}\right)''(x_2) = x_3.$$  

So

$$x_3 = \left(g_{t_m''}^{X_m}\right)'' + t_m'' \circ \cdots \circ \left(g_{t_1''}^{X_1}\right)'' + t_1'' + t_1(x).$$  

Continuing this way, we show (after a finite number $k$ of steps) the existence of a point $(t_1^{(k-1)}, ..., t_m^{(k-1)}) \in \mathbb{R}^m$, such that:

$$\left(g_{t_m^{(k-1)}}^{X_m}\right)^{(k-1)} \circ \cdots \circ \left(g_{t_1^{(k-1)}}^{X_1}\right)^{(k-1)}(x_{k-1}) = x_k,$$

where $x_k \in \mathcal{C}$, $x_k$ is contained in an open ball $B_{k-1}$ of $M$, with $B_{k-1} \ni x_{k-1}$. Therefore, for $k$ finite, we have

$$x_k = \left(g_{t_m^{(k-1)}}^{X_m}\right)^{(k-1)} + t_m^{(k-2)} + \cdots + t_m + t_m \circ \cdots \circ \left(g_{t_1^{(k-1)}}^{X_1}\right)^{(k-1)} + t_1^{(k-2)} + \cdots + t_1 + t_1(x).$$  

This construction shows that in a finite number $k$ of steps, we can cover the curve $\mathcal{C}$ connecting the point $x$ to the point $y$ by neighborhoods of $x$;
the point $y$ playing the role of $x_k$. Note that the application $g$ can not be injective. In fact, if $g$ is injective, we would have, according to part a), a bijection between a compact $M$ and a noncompact $\mathbb{R}^m$, which is absurd.

c) The stationary group

$$\Lambda = \left\{ (t_1, ..., t_m) \in \mathbb{R}^m : g(t_1, ..., t_m) = g_{t_m}^{X_m} \circ \cdots \circ g_{t_1}^{X_1}(x) = x \right\},$$

is a discrete subgroup of $\mathbb{R}^m$ independent of point $x \in M$. Indeed, let us first note that $\Lambda \neq \emptyset$ because $(0, ..., 0) \in \Lambda$. Let $(t_1, ..., t_m) \in \Lambda$, $(t'_1, ..., t'_m) \in \Lambda$. We have

$$g(t_1, ..., t_m) = g(t'_1, ..., t'_m) = x.$$

As the vector fields $X_1, ..., X_m$ are commutative, then

$$g(t_1 + t'_1, ..., t_m + t'_m) = g(t_1, ..., t_m + t'_m) \circ \cdots \circ g_{t_1 + t'_1}^{X_1}(x),$$

and

$$g(-t_1, ..., -t_m) = g_{-t_m}^{X_m} \circ \cdots \circ g_{-t_1}^{X_1}(x),$$

Hence $(t_1 + t'_1, ..., t_m + t'_m) \in \Lambda$ and $(-t_1, ..., -t_m) \in \Lambda$. Therefore $\Lambda$ is stable for addition, the inverse of $(t_1, ..., t_m)$ is $(-t_1, ..., -t_m)$ and consequently $\Lambda$ is a subgroup of $\mathbb{R}^m$. We show that $\Lambda$ is independent of $x$. Let

$$\Lambda' = \left\{ (t'_1, ..., t'_m) \in \mathbb{R}^m : g(t'_1, ..., t'_m) = g_{t'_m}^{X_m} \circ \cdots \circ g_{t'_1}^{X_1}(y) = y \right\}.$$

By surjectivity, one can find $(s_1, ..., s_m) \in \mathbb{R}^m$ such that :

$$g_{s_m}^{X_m} \circ \cdots \circ g_{s_1}^{X_1}(x) = y.$$

Let $(t'_1, ..., t'_m) \in \Lambda'$. We have

$$g_{t'_m}^{X_m} \circ \cdots \circ g_{t'_1}^{X_1}(y) = y,$$

$$g_{s_m}^{X_m} \circ \cdots \circ g_{s_1}^{X_1}(x) = g_{t'_m}^{X_m} \circ \cdots \circ g_{t'_1}^{X_1}(y),$$

$$g_{-s_m + t'_m + s_m}^{X_m} \circ \cdots \circ g_{-s_1 + t'_1 + s_1}^{X_1}(x) = x,$$

$$g_{t'_m}^{X_m} \circ \cdots \circ g_{t'_1}^{X_1}(x) = x.$$
Therefore, \((t'_1, ..., t'_m) \in \Lambda\) and therefore \(\Lambda\) does not depend on \(x\). To show that \(\Lambda\) is discrete, we consider a neighborhood \(V\) sufficient small of the point \((0, ..., 0)\) and a neighborhood \(W\) of the point \(x\). From a), the application \(g\) is a local diffeomorphism, so \(g : V \rightarrow W\), is bijective and consequently no point of \(W \setminus \{(0, ..., 0)\}\) is sent on \(x\); the points of the subgroup \(\Lambda\) have no accumulation point in \(\mathbb{R}^m\).

d) The variety \(M\) is diffeomorphic to a \(m\)-dimensional real torus. Indeed, let

\[
T^k \times \mathbb{R}^{m-k} = \{(\varphi_1, ..., \varphi_k; u_1, ..., u_{m-k})\}, \quad (\varphi_1, ..., \varphi_k) \mod 2\pi
\]

be the direct product of \(k\) circles and \(m - k\) straight lines and consider the application

\[
\pi : \mathbb{R}^m \rightarrow T^k \times \mathbb{R}^{m-k},
\]

defined by

\[
\pi(\varphi_1, ..., \varphi_k; u_1, ..., u_{m-k}) = ((\varphi_1, ..., \varphi_k) \mod 2\pi; (u_1, ..., u_{m-k})).
\]

The points \(f_1, ..., f_k \in \mathbb{R}^m\) where each \(f_i\) has the coordinates

\[
\varphi_i = 2\pi, \quad \varphi_j = 0, \quad u_1 = \cdots = u_{m-k} = 0,
\]

are sent in 0 by this application. Let us first note that the stationary group \(\Lambda\) (see point c) can be written in the form

\[
\Lambda = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_k, \quad 1 \leq k \leq m,
\]

where \(e_1, \ldots, e_m\) are linearly independent vectors. Indeed, to fix ideas, let us take \(m = 2\), i.e.,

\[
\Lambda = \{ (t_1, t_2) \in \mathbb{R}^2 : g(t_1, t_2) = g_{t_2}^{X_2} \circ g_{t_1}^{X_1}(x) = x \}.
\]

Three cases are possible: i) \(\Lambda = \{0\}\), ii) \(\Lambda = \mathbb{Z}e_1\), iii) \(\Lambda = \mathbb{Z}e_1 + \mathbb{Z}e_2\).

The first case is to be rejected because we have a diffeomorphism between a non-compact \(\mathbb{Z}^2/\{0\}\) and a compact \(M\), which is absurd. The second case \(\mathbb{Z}^2 / \mathbb{Z}e_1\) (a cylinder) is also to be rejected for the same reasons as in the first case. It remains the last case, which is valid because \(\mathbb{Z}^2 / \mathbb{Z}e_1 + \mathbb{Z}e_2\) is a 2-dimensional torus. In general, for every discrete subgroup of \(\mathbb{R}^m\), there exist \(k\) linearly independent vectors such that this group is the set of all their integer linear combinations. Let \(e_1, \ldots, e_k \in \Lambda \subset \mathbb{R}^m\) be generators of the stationary group \(\Lambda\). We now apply the vector space \(\mathbb{R}^m\) in a surjective way over space vector \(\mathbb{R}^m = \{(t_1, ..., t_m)\}\) such that the vectors \(f_i\) are transformed into \(e_i\). Let \(h : \mathbb{R}^m \rightarrow \mathbb{R}^m\) be such an isomorphism and notice that \(\mathbb{R}^m\) determines charts of \(T^k \times \mathbb{R}^{m-k}\) (respectively of the variety \(M\)). The application \(h\) determines a diffeomorphism

\[
\tilde{h} : T^k \times \mathbb{R}^{m-k} \rightarrow M,
\]
and since by hypothesis $M$ is compact, then $k = m$ and consequently $M$ is a $m$-dimensional torus. Let’s check this out in more detail. Since $\Lambda$ is the kernel of $g$, there exists a canonical surjection

$$\tilde{h} : \mathbb{R}^m/\Lambda \to M, \quad [(t_1, ..., t_m)] \mapsto \tilde{h} [(t_1, ..., t_m)] = g_{t_m} X^m \circ \cdots \circ g_{t_1} X^1 (x).$$

Indeed, let $(t_1, ..., t_m)$ et $(s_1, ..., s_m)$ such that:

$$\tilde{h} [(t_1, ..., t_m)] = \tilde{h} [(s_1, ..., s_m)].$$

We have

$$g_{t_m} X^m \circ \cdots \circ g_{t_1} X^1 (x) = g_{s_m} X^m \circ \cdots \circ g_{s_1} X^1 (x),$$

hence

$$g_{s_1} X^1 \circ \cdots \circ g_{s_m} X^m \circ g_{t_m} X^m \circ \cdots \circ g_{t_1} X^1 (x)$$

$$= g_{s_1} X^1 \circ \cdots \circ g_{s_m} X^m \circ g_{s_m} X^m \circ \cdots \circ g_{s_1} X^1 (x),$$

$$= g_{s_1} X^1 \circ \cdots \circ g_{s_m} X^m \circ g_{s_m} X^m \circ \cdots \circ g_{s_1} X^1 (x),$$

$$\vdots$$

$$= g_{s_1} X^1 \circ g_{s_1} X^1 (x),$$

$$= x.$$

Since $X_1, ..., X_m$ are commutative, then

$$g_{t_m-s_m} X^m \circ \cdots \circ g_{t_1-s_1} X^1 (x) = x.$$

Consequently, we have

$$[(t_1 - s_1, ..., t_m - s_m)] = 0, \quad [(t_1, ..., t_m) - (s_1, ..., s_m)] = 0,$$

and

$$[(t_1, ..., t_m)] = [(s_1, ..., s_m)].$$

So $\tilde{h}$ is a diffeomorphism and the proof of part 1) is complete.

2) By hypothesis the variety $M_c$ is compact and connected. Therefore, from the part 1), it is enough to show that $M_c$ is differentiable, of dimension $n$ and that it is equipped with $n$ commutative vectors fields. The differentiability of this variety arises from the implicit function theorem since the vectors $J^{\partial H_1}/\partial x, \ldots, J^{\partial H_n}/\partial x$ are assumed to be independent. As $m = 2n$, then the first integrals $H_i(x_1, ..., x_{2n})$ are functions of the variables $x_1, ..., x_n, x_{n+1}, ..., x_{2n}$. Therefore,

$$\dim \{ x \in M : H_i = c_i \} = 2n - 1,$$

and

$$\dim (\{ x \in M : H_i = c_i \} \cap \{ x \in M : H_j = c_j \}) = 2n - 2, i \neq j,$$
and so \( \dim M_c = n \). Let \( X_i \) and \( X_j \), \( 1 \leq i, j \leq n \), be differentiable (\( C^\infty \)) vector fields on \( M \), so on the variety \( M_c \) also. Let us define the differential operator \( L_X \) by

\[
L_X : C^\infty (M_c) \rightarrow C^\infty (M_c), \quad F \mapsto L_X F,
\]

such that:

\[
L_X F(x) = \left. \frac{d}{dt} F(g^X_t(x)) \right|_{t=0}, \quad x \in M_c.
\]

We have

\[
L_{X_i} F = \{ F, H_i \}, \quad L_{X_j} L_{X_i} F = \{ \{ F, H_i \}, H_j \},
\]

and

\[
L_{X_i} L_{X_j} F - L_{X_j} L_{X_i} F = \{ \{ F, H_j \}, H_i \} - \{ \{ F, H_i \}, H_j \} = -\{ \{ H_j, F \}, H_i \} - \{ \{ F, H_i \}, H_j \} = \{ \{ H_i, H_j \}, F \},
\]

according to the identity of Jacobi. Since \( H_i \) and \( H_j \) are in involution, then \([L_{X_i}, L_{X_j}] = 0\). The construction of the angular coordinates \( \varphi_1, ..., \varphi_m \) mod. \( 2\pi \) on the variety \( M \) is obviously valid on the invariant variety \( M_c \). Note that

\[
(\varphi_1, ..., \varphi_m) = h^{-1}(t_1, ..., t_m),
\]

and that the angular coordinates \( \varphi_1, ..., \varphi_m \) vary uniformly under the action of the Hamiltonian flow \( H \), i.e.,

\[
\frac{d\varphi_k}{dt} = \{ H_i, \varphi_k \} = \omega_i(c), \quad \omega_i(c) = \text{constants}.
\]

In other words, the motion is quasi-periodic on the invariant torus \( M_c \). Finally, to show that the equations of the problem are integrable by quadratures as well as several information about the variables called action-angle, one will consult with profit [4]. The demonstration of theorem ends. □

If we restrict ourselves to an invariant open set, we can always assume that the fibers of \( M_c \) (where \( c \) is a regular value) are connected. The torii obtained in the theorem are Lagrangian sub-varieties. If \( M_c \) is not compact but the flow of each of the vector fields \( X_{H_b} \) is complete on \( M_c \) (a vector field is called complete if every one of its flow curves exist for all time), then \( M_c \) is diffeomorphic to a cylinder \( \mathbb{R}^k \times T^{n-k} \) under which the vector fields \( X_{H_b} \) are mapped to linear vector fields.

**Example 19** The rank of the matrix \( J \) is even. Indeed, let \( \lambda \) be the eigenvalue associated with the eigenvector \( Z \). We have

\[
JZ = \lambda Z, \quad Z \neq 0,
\]

55
and
\[ Z^* J Z = \lambda Z^* Z, \quad Z^* = \overline{Z^\top}, \]
where \( \lambda = \frac{Z^* J Z}{Z^* Z}. \) Since \( J = J \) and \( J^\top = -J, \) then
\[ Z^* J Z = Z^\top J Z = Z^\top J Z = (Z^\top J Z)^\top = Z^* J^\top Z = -Z^* J Z, \]
which implies that \( Z^* J Z \) is either zero or imaginary pure. Since \( Z^* Z \) is real, it follows that all the eigenvalues of \( J \) are either null or imaginary pure. Now \( J Z = \overline{\lambda Z}, \) so if \( \lambda \) is an eigenvalue, then \( \overline{\lambda} \) is also an eigenvalue. Consequently, the eigenvalues (non-zero) of \( J \) come in pairs, hence the result.

As a consequence, we obtain the concept of complete integrability of a Hamiltonian system (15) with \( x \in M = \mathbb{R}^m. \) For the sake of clarity, we shall distinguish two cases:

a) Case 1: \( \det J \neq 0. \) The rank of the matrix \( J \) is even (example 19), \( m = 2n. \) A Hamiltonian system (15), \( x \in M = \mathbb{R}^m, \) is completely integrable or Liouville-integrable if there exist \( n \) firsts integrals \( H_1 = H, H_2, \ldots, H_n \) in involution, i.e., \( \{H_k, H_l\} = 0, \) \( 1 \leq k, l \leq n, \) with linearly independent gradients, i.e., \( dH_1 \wedge \ldots \wedge dH_n \neq 0. \) For generic \( c = (c_1, \ldots, c_n) \) the level set
\[ M_c = \bigcap_{i=1}^n \{x \in M : H_i(x) = c_i, \ c_i \in \mathbb{R}\}, \]
will be an \( n \)-manifold. By the Arnold-Liouville theorem, if \( M_c \) is compact and connected, it is diffeomorphic to an \( n \)-dimensional torus \( \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \) and each vector field will define a linear flow there. In some open neighborhood of the torus there are coordinates \( s_1, \ldots, s_n, \varphi_1, \ldots, \varphi_n \) in which \( \omega \) takes the form
\[ \omega = \sum_{k=1}^n ds_k \wedge d\varphi_k. \]
Here the functions \( s_k \) (called action-variables) give coordinates in the direction transverse to the torus and can be expressed functionally in terms of the first integrals \( H_k. \) The functions \( \varphi_k \) (called angle-variables) give standard angular coordinates on the torus, and every vector field \( X_{H_k} \) can be written in the form
\[ \dot{\varphi}_k = h_k (s_1, \ldots, s_n), \]
that is, its integral trajectories define a conditionally-periodic motion on the torus. In a neighborhood of the torus the Hamiltonian vector field \( X_{H_k} \) take the following form
\[ \dot{s}_k = 0, \quad \dot{\varphi}_k = h_k (s_1, \ldots, s_n), \]
and can be solved by quadratures.
b) Case 2: \( \det J = 0 \). We reduce the problem to \( m = 2n + k \) and we look for \( k \) Casimir functions \( H_{n+1}, \ldots, H_{n+k} \), leading to identically zero Hamiltonian vector fields

\[
j \frac{\partial H_{n+i}}{\partial x} = 0, \quad 1 \leq i \leq k.
\]

In other words, the system is Hamiltonian on a generic symplectic manifold

\[
\bigcap_{i=n+1}^{n+k} \{ x \in \mathbb{R}^m : H_i(x) = c_i \},
\]

of dimension \( m - k = 2n \). If for most values of \( c_i \in \mathbb{R} \), the invariant manifolds

\[
\bigcap_{i=1}^{n+k} \{ x \in \mathbb{R}^m : H_i(x) = c_i \},
\]

are compact and connected, then they are \( n \)-dimensional tori \( T^n = \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \) by the Arnold-Liouville theorem and the Hamiltonian flow is linear in angular coordinates of the torus.

**Example 20** The simple pendulum and the harmonic oscillator are trivially integrable systems (any 2-dimensional Hamiltonian system where the set of non-fixed points is dense, is integrable. Let \( T^* \mathbb{R}^n \) with coordinates \( q_1, \ldots, q_n, p_1, \ldots, p_n \). The system corresponding to the Hamiltonian of the harmonic oscillator is integrable

\[
H = \frac{1}{2} \sum_{j=1}^{n} (p_j^2 + \lambda_j q_j^2).
\]

The Hamiltonian structure is defined by the Poisson bracket

\[
\{ F, H \} = \sum_{j=1}^{n} \left( \frac{\partial F}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial H}{\partial q_j} \right).
\]

The Hamiltonian field corresponding to \( H \) is written explicitly

\[
\dot{q}_j = p_j, \quad \dot{p}_j = -2\lambda_j q_j, \quad j = 1, \ldots, n
\]

and admits the following first \( n \) integral :

\[
H_j = \frac{1}{2} p_j^2 + \lambda_j q_j^2, \quad 1 \leq j \leq n.
\]

The latter are independent, in involution and the system in question is integrable.
10 Rotation of a solid body about a fixed point and $SU(2)$ Yang-Mills equations

10.1 The problem of the rotation of a solid body about a fixed point

One of the most fundamental problems of mechanics is the study of the motion of rotation of a solid body around a fixed point. The differential equations of this problem are written in the form

$$
\dot{M} = M \wedge \Omega + \mu g \Gamma \wedge L, \quad (24)
$$

$$
\dot{\Gamma} = \Gamma \wedge \Omega,
$$

where $\wedge$ is the vector product in $\mathbb{R}^3$, $M = (m_1, m_2, m_3)$ the angular momentum of the solid, $\Omega = (\frac{m_1}{I_1}, \frac{m_2}{I_2}, \frac{m_3}{I_3})$ the angular velocity, $I_1, I_2$ and $I_3$, moments of inertia, $\Gamma = (\gamma_1, \gamma_2, \gamma_3)$ the unitary vertical vector, $\mu$ the mass of the solid, $g$ the acceleration of gravity, and finally, $L = (l_1, l_2, l_3)$ the unit vector originating from the fixed point and directed towards the center of gravity; all these vectors are considered in a mobile system whose coordinates are fixed to the main axes of inertia. The configuration space of a solid with a fixed point is the group of rotations $SO(3)$. This is generated by the one-parameter subgroup of rotations

$$
A_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos t & -\sin t \\
0 & \sin t & \cos t
\end{pmatrix},
$$

$$
A_2 = \begin{pmatrix}
\cos t & 0 & \sin t \\
0 & 1 & 0 \\
-\sin t & 0 & \cos t
\end{pmatrix},
$$

$$
A_3 = \begin{pmatrix}
\cos t & -\sin t & 0 \\
\sin t & \cos t & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Recall that this is the group of $n \times n$ orthogonal matrices $A$ and the motion of this solid is described by a curve on this group. The angular velocity space of all rotations (the set of derivatives $\dot{A}(t)|_{t=0}$ of the differentiable curves in $SO(3)$ passing through the identity in $t = 0 : A(0) = I$) is the Lie algebra of the group $SO(3)$; it is the algebra $so(3)$ of the $3 \times 3$ antisymmetric matrices. This algebra is generated as a vector space by the matrices

$$
e_1 = \dot{A}_1(t)|_{t=0} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix},
$$

$$
e_2 = \dot{A}_2(t)|_{t=0} = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}.
$$
\[
e_3 = \dot{A}_3(t)\big|_{t=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

which verify the commutation relations:
\[
[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.
\]

We will use the fact that if we identify \(so(3)\) to \(\mathbb{R}^3\) by sending \((e_1, e_2, e_3)\) on the canonical basis of \(\mathbb{R}^3\), the bracket of \(so(3)\) corresponds to the vector product. In other words, consider the application
\[
\mathbb{R}^3 \rightarrow so(3), \quad a = (a_1, a_2, a_3) \mapsto A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix},
\]

which defines an isomorphism between Lie algebras \((\mathbb{R}^3, \wedge)\) and \((so(3), [\cdot, \cdot])\) where
\[
a \wedge b \mapsto [A, B] = AB - BA.
\]

By using this isomorphism, the system (24) can be rewritten in the form
\[
\begin{align*}
\dot{M} &= [M, \Omega] + \mu g \ [\Gamma, L], \\
\dot{\Gamma} &= [\Gamma, \Omega],
\end{align*}
\]

where
\[
M = (M_{ij})_{1 \leq i, j \leq 3} \equiv \sum_{i=1}^{3} m_i e_i \equiv \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \in so(3),
\]
\[
\Omega = (\Omega_{ij})_{1 \leq i, j \leq 3} \equiv \sum_{i=1}^{3} \omega_i e_i \equiv \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in so(3),
\]
\[
\Gamma = (\gamma_{ij})_{1 \leq i, j \leq 3} \equiv \sum_{i=1}^{3} \gamma_i e_i \equiv \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix} \in so(3),
\]

and
\[
L = \begin{pmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{pmatrix} \in so(3).
\]

Taking into account that \(M = I \Omega\), then the above equations (25) become
\[
\begin{align*}
\dot{M} &= [M, \Lambda M] + \mu g \ [\Gamma, L], \\
\dot{\Gamma} &= [\Gamma, \Lambda M],
\end{align*}
\]

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where
\[ \Lambda M = \equiv \sum_{i=1}^{3} \lambda_i m_i e_i \equiv \begin{pmatrix} 0 & -\lambda_3 m_3 & \lambda_2 m_2 \\ \lambda_3 m_3 & 0 & -\lambda_1 m_1 \\ -\lambda_2 m_2 & \lambda_1 m_1 & 0 \end{pmatrix} \in so(3), \quad \lambda_i \equiv \frac{1}{I_i} \]

The resolution of this problem was analyzed first by Euler [10] and in 1758, he published the equations (case \( \mu = 0 \)) which carry his name. Euler’s equations were integrated by Jacobi [19] in terms of elliptic functions and around 1851, Poinsot [34] gave them a remarkable geometric interpretation. Before, around 1815 Lagrange [21] found another case (\( I_1 = I_2, l_1 = l_2 = 0 \)) of integrability, that subsequently Poisson has examined at length thereafter. The problem continued to attract mathematicians but for a long time no new results could be obtained. It was then around 1888-1989 that a memoir [20], of the highest interest, appears containing a new case (\( I_1 = I_2 = 2I_3, l_3 = 0 \)) of integrability discovered by Kowalewski. For this remarkable work, Kowalewski was awarded the Bordin Prize of the Paris Academy of Sciences. In fact, although Kowalewski’s work is quite important, it is not at all clear why there would be no other new cases of integrability. This was to be the starting point of a series of fierce research on the question of the existence of new cases of integrability. Moreover, among the remarkable results obtained by Poincaré [33] with the aid of the periodic solutions of the equations of dynamics, we find the following (around 1891) : in order to exist in the motion of a solid body around of a fixed point, an algebraic first integral not being reduced to a combination of the classical integrals, it is necessary that the ellipsoid of inertia relative to the point of suspension is of revolution. In 1896, R. Liouville (not to be confused with Joseph Liouville, well known in complex analysis) also competed for the Bordin prize, presented a paper [30] indicating necessary and sufficient conditions (\( I_3 = 0, 2I_3/I_1 = \text{integer} \)) of existence of a fourth algebraic integral. These conditions have been reproduced in most conventional treatises (eg Whittaker [43]) and in scientific journals. And it was not until the year 1906, when Husson [18], working under the direction of Appell and Painlevé, discovered an erroneous demonstration in the work of Liouville. Indeed, paragraphs I and III of Liouville’s dissertation devoted to the search for the necessary conditions seem at first satisfactory, but a more careful study shows that the demonstrations are at least insufficient and that it is impossible to accept conclusions. In fact, although the conditions found by Liouville are necessary, they can not be deduced from the calculations indicated and these conditions are not sufficient. And it was Husson who first solved completely the question of looking for new cases of integrability. Inspired by Poincaré’s research on the problem of the three bodies and Painlevé on the generalization of Bruns’s theorem, Husson demonstrated that any algebraic integral is a combination of classical integrals except in the cases of Euler, Lagrange and Kowalewski. Moreover, the question of the existence of analytic integrals has been studied rigorously
by Ziglin [45, 46] and Holmes-Marsden [96]. Towards the end of this subsection, we will mention some special cases: cases of Hesse-Appel’rot [16, 3], Goryachev-Chaplygin [12, 8] and Bobylev-Steklov [6, 36].

In the case of the Euler rigid body motion, we have \( l_1 = l_2 = l_3 = 0 \), that is, the fixed point is its center of gravity. The Euler rigid body motion [10] (also called Euler-Poinsot motion [34] of the solid) expresses the free motion of a rigid body around a fixed point. Then the motion of the body is governed by \( \dot{M} = [M, \Lambda M] \), and is explicitly given by

\[
\begin{align*}
\dot{m}_1 &= (\lambda_3 - \lambda_2) m_2 m_3, \\
\dot{m}_2 &= (\lambda_1 - \lambda_3) m_1 m_3, \\
\dot{m}_3 &= (\lambda_2 - \lambda_1) m_1 m_2,
\end{align*}
\]

and (see example 10) can be written as a Hamiltonian vector field

\[
\dot{x} = J \frac{\partial H}{\partial x}, \quad x = (m_1, m_2, m_3)^T,
\]

with the Hamiltonian

\[
H = \frac{1}{2} (\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2),
\]

and

\[
J = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \in so(3).
\]

We have \( \det J = 0 \), so \( m = 2n + k \) and \( m - k = rk J \). Here \( m = 3 \) and \( rk J = 2 \), then \( n = k = 1 \). The system (27) has beside the energy \( H_1 = H \), a trivial invariant \( H_2 \), i.e., such that: \( J \frac{\partial H_2}{\partial x} = 0 \), or

\[
\begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H_2}{\partial m_1} \\ \frac{\partial H_2}{\partial m_2} \\ \frac{\partial H_2}{\partial m_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\]

implying

\[
\frac{\partial H_2}{\partial m_1} = m_1, \quad \frac{\partial H_2}{\partial m_2} = m_2, \quad \frac{\partial H_2}{\partial m_3} = m_3,
\]

and

\[
H_2 = \frac{1}{2} (m_1^2 + m_2^2 + m_3^2).
\]

The system evolves on the intersection of the sphere \( H_1 = c_1 \) and the ellipsoid \( H_2 = c_2 \). In \( \mathbb{R}^3 \), this intersection will be isomorphic to two circles \( \left( \frac{m_1}{\sqrt{c_1}} < c_1 < \frac{m_3}{\sqrt{c_1}} \right) \). According to Arnold-Liouville’s theorem, we have:
Theorem 25 The system (27) is completely integrable and the vector $J \frac{\partial H}{\partial x}$ gives a flow on a variety:

$$\bigcap_{i=1}^{2} \{ x \in \mathbb{R}^3 : H_i(x) = c_i \}, \quad (\text{for generic } c_i \in \mathbb{R}),$$
diffeomorphic to a real torus of dimension 1, that is to say a circle.

Let us now turn to explicit resolution. We shall show that the problem can be integrated in terms of elliptic functions, as Euler discovered using his then newly invented theory of elliptic integrals. We have just seen that the system in question admits two first quadratic integrals:

$$H_1 = \frac{1}{2} (\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2),$$
$$H_2 = \frac{1}{2} (m_1^2 + m_2^2 + m_3^2).$$

We’ll assume that $\lambda_1, \lambda_2, \lambda_3$ are all different from zero (that is, the solid is not reduced to a point and is not focused on a straight line either). Under these conditions, $H_1 = 0$ implies $m_1 = m_2 = m_3 = 0$ and so $H_2 = 0$; the solid is at rest. We dismiss this trivial case and now assume that $H_1 \neq 0$ et $H_2 \neq 0$. When $\lambda_1 = \lambda_2 = \lambda_3$, the equations (27) obviously show that $m_1$, $m_2$ and $m_3$ are constants. Suppose for example that $\lambda_1 = \lambda_2$, the equations (27) are then written

$$\dot{m}_1 = (\lambda_3 - \lambda_1) m_2 m_3, \quad \dot{m}_2 = (\lambda_1 - \lambda_3) m_1 m_3, \quad \dot{m}_3 = 0.$$

We deduce then that $m_3 = \text{constante} \equiv A$ and

$$\dot{m}_1 = A (\lambda_3 - \lambda_1) m_2, \quad \dot{m}_2 = A (\lambda_1 - \lambda_3) m_1.$$

Note that

$$(m_1 + im_2) = iA(\lambda_1 - \lambda_3)(m_1 + im_2),$$

we obtain

$$m_1 + im_2 = Ce^{iA(\lambda_1 - \lambda_3)t},$$

where $C$ is a constant and so

$$m_1 = C \cos A(\lambda_1 - \lambda_3)t, \quad m_2 = C \sin A(\lambda_1 - \lambda_3)t.$$

The integration of Euler’s equations is delicate in the general case where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are all different; the solutions are expressed in this case using elliptic functions. In the following we will suppose that $\lambda_1$, $\lambda_2$ and $\lambda_3$ are all different and we discard the other trivial cases which pose no difficulty.
for solving the equations in question. To fix the ideas we will assume in the following that: \( \lambda_1 > \lambda_2 > \lambda_3 \). Geometrically, the equations
\[
\begin{align*}
\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 &= 2H_1, \\
m_1^2 + m_2^2 + m_3^2 &= 2H_2 = r^2,
\end{align*}
\]
respectively represent the equations of the surface of a half axis ellipsoid: \( \sqrt{\frac{2H_1}{\lambda_1}} \) (half big axis), \( \sqrt{\frac{2H_1}{\lambda_2}} \) (middle half axis), \( \sqrt{\frac{2H_1}{\lambda_3}} \) (half small axis), and a sphere of radius \( r \). So the movement of the solid takes place on the intersection of an ellipsoid with a sphere. This intersection makes sense because by comparing (28) to (29), we see that \( \frac{2H_1}{\lambda_1} < r^2 < \frac{2H_1}{\lambda_3} \), which means geometrically that the radius of the sphere (29) is between the smallest and largest of the half-axes of the ellipsoid (28). To study the shape of the intersection curves of the ellipsoid (28) with the sphere (29), set \( H_1 > 0 \) and let the radius \( r \) vary. Like \( \lambda_1 > \lambda_2 > \lambda_3 \), the semi-axes of the ellipsoid will be \( \frac{2H_1}{\lambda_1} > \frac{2H_1}{\lambda_2} > \frac{2H_1}{\lambda_3} \). If the radius \( r \) of the sphere is less than the half-axis \( \frac{2H_1}{\lambda_3} \) or greater than the half-axis \( \frac{2H_1}{\lambda_1} \), then the intersection in question is empty (and no real movement corresponds to these values of \( H_1 \) and \( r \)). When the radius \( r \) equals \( \frac{2H_1}{\lambda_3} \), then the intersection is composed of two points. When the radius \( r \) increases \( \left( \frac{2H_1}{\lambda_1} < r < \frac{2H_1}{\lambda_3} \right) \), we obtain two curves around the ends of the half minor axis. Likewise if \( r = \frac{2H_1}{\lambda_1} \), we get both ends of the semi-major axis and if \( r \) is slightly smaller than \( \frac{2H_1}{\lambda_3} \), we get two closed curves near these ends. Finally, if \( r = \frac{2H_1}{\lambda_2} \) then the intersection in question consists of two circles.

**Theorem 26** Euler’s differential equations (27) are integrated by means of Jacobi’s elliptic functions.

*Proof.* From the first integrals (28) and (29), we express \( m_1 \) and \( m_3 \) as a function of \( m_2 \). These expressions are then introduced into the second equation of the system (27) to obtain a differential equation in \( m_2 \) and \( \frac{dm_2}{dt} \) only. In more detail, the following relationships are easily obtained from (28) and (29):
\[
\begin{align*}
m_1^2 &= \frac{2H_1 - r^2\lambda_3 - (\lambda_2 - \lambda_3) m_2^2}{\lambda_1 - \lambda_3}, \\
m_3^2 &= \frac{r^2\lambda_1 - 2H_1 - (\lambda_1 - \lambda_2) m_2^2}{\lambda_1 - \lambda_3}.
\end{align*}
\]
By substituting these expressions in the second equation of the system (27), we obtain
\[
\frac{dm_2}{dt} = \sqrt{(2H_1 - r^2\lambda_3 - (\lambda_2 - \lambda_3) m_2^2)(r^2\lambda_1 - 2H_1 - (\lambda_1 - \lambda_2) m_2^2)}.
\]
By integrating this equation, we obtain a function \( t(m_2) \) in the form of an elliptic integral. To reduce this to the standard form, we can assume that \( r^2 > \frac{2H_1}{\lambda_2} \) (otherwise, it is enough to invert the indices 1 and 3 in all the previous formulas). We rewrite the previous equation, in the form

\[
\frac{dm_2}{\sqrt{(2H_1 - r^2\lambda_3)(r^2\lambda_1 - 2H_1)}} = \sqrt{1 - \frac{\lambda_2 - \lambda_3}{2H_1 - r^2\lambda_3}m_2^2}(1 - \frac{\lambda_1 - \lambda_2}{r^2\lambda_1 - 2H_1}m_2^2).
\]

By setting

\[
\tau = t\sqrt{(\lambda_2 - \lambda_3)(r^2\lambda_1 - 2H_1)}, \quad s = m_2\sqrt{\frac{\lambda_2 - \lambda_3}{2H_1 - r^2\lambda_3}},
\]

we obtain

\[
\frac{ds}{d\tau} = \sqrt{(1 - s^2)}\left(1 - \frac{(\lambda_1 - \lambda_2)(2H_1 - r^2\lambda_3)}{(\lambda_2 - \lambda_3)(r^2\lambda_1 - 2H_1)s^2}\right),
\]

which suggests choosing elliptic functions as a module

\[
k^2 = \frac{(\lambda_1 - \lambda_2)(2H_1 - r^2\lambda_3)}{\lambda_2 - \lambda_3}(r^2\lambda_1 - 2H_1).
\]

Inequalities \( \lambda_1 > \lambda_2 > \lambda_3, \frac{2H_1}{\lambda_1} < r^2 < \frac{2H_1}{\lambda_3} \) and \( r^2 > \frac{2H_1}{\lambda_2} \) show that \( 0 < k^2 < 1 \). So we get

\[
\frac{ds}{d\tau} = \sqrt{(1 - s^2)(1 - k^2s^2)}.
\]

This equation admits the solution (we choose the origin of the times such that \( m_2 = 0 \) for \( t = 0 \)) :

\[
\tau = \int_0^s \frac{ds}{\sqrt{(1 - s^2)(1 - k^2s^2)}}.
\]

It is the integral of a holomorphic differential on an elliptic curve :

\[
\mathcal{E} : w^2 = (1 - s^2)(1 - k^2s^2).
\]

The inverse function \( s(\tau) \) is one of Jacobi’s elliptic functions : \( s = \text{sn} \tau \), which determines \( m_2 \) as,

\[
m_2 = \sqrt{\frac{2H_1 - r^2\lambda_3}{\lambda_2 - \lambda_3}} \cdot \text{sn} \tau.
\]

According to the equalities (30) and (31), we know that the functions \( m_1 \) and \( m_3 \) are expressed algebraically as a function of \( m_2 \), so

\[
m_1 = \sqrt{\frac{2H_1 - r^2\lambda_3}{\lambda_1 - \lambda_3}} \cdot \sqrt{1 - \text{sn}^2 \tau},
\]

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Given the definition of the other two elliptical functions:

\[ cn \tau = \sqrt{1 - sn^2 \tau}, \quad dn \tau = \sqrt{1 - k^2 sn^2 \tau}, \]

and the fact that

\[ \tau = t \sqrt{(\lambda_2 - \lambda_3)(r^2 \lambda_1 - 2H_1)}, \]

we finally get the following explicit formulas:

\[
\begin{align*}
m_1 &= \sqrt{\frac{2H_1 - r^2 \lambda_3}{\lambda_1 - \lambda_3}} cn(t \sqrt{(\lambda_2 - \lambda_3)(r^2 \lambda_1 - 2H_1)}), \\
m_2 &= \sqrt{\frac{2H_1 - r^2 \lambda_3}{\lambda_2 - \lambda_3}} sn(t \sqrt{(\lambda_2 - \lambda_3)(r^2 \lambda_1 - 2H_1)}), \\
m_3 &= \sqrt{\frac{r^2 \lambda_1 - 2H_1}{\lambda_1 - \lambda_3}} dn(t \sqrt{(\lambda_2 - \lambda_3)(r^2 \lambda_1 - 2H_1)}).
\end{align*}
\]

In other words, the integration of the Euler equations is done by means of elliptic Jacobi functions and the proof is complete.

**Remark 4** Note that for \( \lambda_1 = \lambda_2 \), we have \( k^2 = 0 \). In this case, the elliptical functions \( sn, cn, dn \) are reduced respectively to functions \( \sin \tau, \cos \tau, 1 \).

From the system (32), we easily obtain the expressions

\[
\begin{align*}
m_1 &= \sqrt{\frac{2H_1 - r^2 \lambda_3}{\lambda_1 - \lambda_3}} \cos \sqrt{(\lambda_1 - \lambda_3)(r^2 \lambda_1 - 2H_1)}t, \\
m_2 &= \sqrt{\frac{2H_1 - r^2 \lambda_3}{\lambda_2 - \lambda_3}} \sin \sqrt{(\lambda_1 - \lambda_3)(r^2 \lambda_1 - 2H_1)}t, \\
m_3 &= \sqrt{\frac{r^2 \lambda_1 - 2H_1}{\lambda_1 - \lambda_3}}.
\end{align*}
\]

We find the solutions established previously where the constants \( A \) and \( B \) are \( A = \sqrt{\frac{r^2 \lambda_1 - 2H_1}{\lambda_1 - \lambda_3}} \) and \( C = \sqrt{\frac{2H_1 - r^2 \lambda_3}{\lambda_1 - \lambda_3}} \).

In the case of the Lagrange top, we have \( I_1 = I_2, l_1 = l_2 = 0 \), i.e., the Lagrange top [21] is a rigid body, in which two moments of inertia are the same and the center of gravity lies on the symmetry axis. In other words, the Lagrange top is a symmetric top with a constant vertical gravitational force acting on its center of mass and leaving the base point of its body symmetry axis fixed. As in the case of Euler, we show that in this case also the problem
is solved by elliptic integrals. Or what amounts to the same, the integration is done using elliptic functions.

The Kowalewski top [20] is special symmetric top with a unique ratio of the moments of inertia satisfy the relation: \( I_1 = I_2 = 2I_3, \ l_3 = 0 \); in which two moments of inertia are equal, the third is half as large, and the center of gravity is located in the plane perpendicular to the symmetry axis (parallel to the plane of the two equal points). Moreover, we may choose \( l_2 = 0, \ \mu gl_1 = l \) and \( I_3 = 1 \). After the substitution \( t \to 2t \) the system (26) is written explicitly in the form

\[
\begin{align*}
\dot{m}_1 &= m_2 m_3, \\
\dot{m}_2 &= -m_1 m_3 + 2\gamma_3, \\
m_3 &= -2\gamma_2, \\
\dot{\gamma}_1 &= 2m_3 \gamma_2 - m_2 \gamma_3, \\
\dot{\gamma}_2 &= m_1 \gamma_3 - 2m_3 \gamma_1, \\
\dot{\gamma}_3 &= m_2 \gamma_1 - m_1 \gamma_2.
\end{align*}
\]  

(33)

These equations are written in the form (see example 12) of a Hamiltonian vector field

\[ \dot{x} = J \frac{\partial H}{\partial x}, \quad x = (m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3)^\top, \]

where

\[ H = \frac{1}{2} (m_1^2 + m_2^2) + m_3^2 + 2\gamma_1, \]

is the Hamiltonian and

\[ J = \begin{pmatrix}
0 & -m_3 & m_2 & 0 & -\gamma_3 & \gamma_2 \\
-m_3 & 0 & -m_1 & \gamma_3 & 0 & -\gamma_1 \\
m_2 & m_1 & 0 & -\gamma_2 & \gamma_1 & 0 \\
0 & -\gamma_3 & \gamma_2 & 0 & 0 & 0 \\
\gamma_3 & 0 & -\gamma_1 & 0 & 0 & 0 \\
-\gamma_2 & \gamma_1 & 0 & 0 & 0 & 0
\end{pmatrix}. \]

The above system admits four first integrals:

\[
\begin{align*}
H_1 &\equiv H, \\
H_2 &= m_1 \gamma_1 + m_2 \gamma_2 + m_3 \gamma_3, \\
H_3 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2, \\
H_4 &= \left( \frac{m_1 + im_2}{2} \right)^2 - (\gamma_1 + i\gamma_2) \left( \frac{m_1 - im_2}{2} \right)^2 - (\gamma_1 - i\gamma_2).
\end{align*}
\]  

(34)

A second flow commuting with the first flow is regulated by the equations:

\[ \dot{x} = J \frac{\partial H_4}{\partial x}, \quad x = (m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3)^\top. \]
The first integrals $H_1$ and $H_4$ are in involution,

$$\{H_1, H_4\} = \left\langle \frac{\partial H_1}{\partial x}, J \frac{\partial H_4}{\partial x} \right\rangle = 0,$$

while $H_2$ and $H_3$ are trivial,

$$J \frac{\partial H_2}{\partial x} = J \frac{\partial H_3}{\partial x} = 0.$$

Let $\mathcal{A}$ be the complex affine variety defined by the intersection of the constants of the motion

$$\mathcal{A} = \bigcap_{k=1}^4 \{ x : H_k(x) = c_k \}, \quad (35)$$

where $c = (c_1, c_2, c_3 = 1, c_4)$ is not a critical value. We will explain how the affine variety $\mathcal{A}$ and vector-fields behave after the quotient by some natural involution on $\mathcal{A}$ and how these vector-fields become well defined when we take Kowalewski’s variables. We show that these variables are naturally related to the so-called Euler’s differential equations and can be seen as the addition-formula for the Weierstrass elliptic function. In the theorem below (for further information, see also [20, 22]), we will use with Kowalewski the following notations: $c_1 = 6h_1$, $c_2 = 2h_2$ et $c_4 = k^2$.

**Theorem 27** a) Let

$$(m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3) \mapsto (x_1, x_2, m_3, y_1, y_2, \gamma_3),$$

be a birationally map on the variety $\mathcal{A}(35)$ where $x_1$, $x_2$, $y_1$, $y_2$ are defined as

$$x_1 = \frac{1}{2}(m_1 + im_2),$$

$$x_2 = \frac{1}{2}(m_1 - im_2), \quad (36)$$

$$y_1 = x_2^2 - (\gamma_1 + i\gamma_2),$$

$$y_2 = x_2^2 - (\gamma_1 - i\gamma_2).$$

Then, the quotient $K \equiv A/\sigma$ by the involution

$$\sigma : M_c \longrightarrow M_c (x_1, x_2, m_3, y_1, y_2, \gamma_3) \mapsto (x_1, x_2, -m_3, y_1, y_2, -\gamma_3), \quad (37)$$

is a Kummer surface

$$K : \left\{ \begin{array}{l}
y_1y_2 = k^2, \\
y_1R(x_2) + y_2R(x_1) + R_1(x_1, x_2) + k^2(x_1 - x_2)^2 = 0, \end{array} \right. \quad (38)$$

where

$$R(x) = -x^4 + 6h_1x^2 - 4h_2x + 1 - k^2, \quad (39)$$
is a polynomial of degree 4 in \( x \) and

\[
R_1(x_1, x_2) = -6h_1x_1^2x_2^2 + 4h_2x_1x_2(x_1 + x_2) - (1 - k^2)(x_1 + x_2)^2 + 6h_1(1 - k^2) - 4h_2^2,
\]

is another polynomial of degree 2 in \( x_1, x_2 \). The ramification points of \( A \) on \( K \) are given by the 8 fixed points of the involution \( \sigma \).

b) The surface \( K \) is a double cover of plane \((x_1, x_2)\), ramified along two elliptic curves intersecting exactly each other at the 8 fixed points of the involution \( \sigma \). These curves give rise to the Euler differential equation

\[
\frac{\dot{x}_1}{\sqrt{R(x_1)}} \pm \frac{\dot{x}_2}{\sqrt{R(x_2)}} = 0,
\]

to which are connected the variables of Kowalewski

\[
s_1 = \frac{R(x_1, x_2) - \sqrt{R(x_1)}\sqrt{R(x_2)}}{(x_1 - x_2)^2} + 3h_1,
\]

\[
s_2 = \frac{R(x_1, x_2) + \sqrt{R(x_1)}\sqrt{R(x_2)}}{(x_1 - x_2)^2} + 3h_1,
\]

where

\[
R(x_1, x_2) \equiv -x_1^2x_2^2 + 6h_1x_1x_2 - 2h_2(x_1 + x_2) + 1 - k^2,
\]

and can be seen as addition formulas for the Weierstrass elliptic function.

c) In terms of the variables \( s_1 \) and \( s_2 \), the system of differential equations (33) is reduced to the system

\[
\frac{ds_1}{\sqrt{P_5(s_1)}} \pm \frac{ds_2}{\sqrt{P_5(s_2)}} = 0,
\]

\[
s_1 ds_1 \pm s_2 ds_2 = dt,
\]

where \( P_5(s) \) is a fifth-degree polynomial and the problem can be integrated in terms of genus two hyperelliptic functions.

**Proof.** a) Using the change of variables (36), with \( t \to it \), equations (33) and (34) become

\[
\dot{x}_1 = m_3x_1 - \gamma_3,
\]

\[
\dot{x}_2 = -m_3x_2 + \gamma_3,
\]

\[
\dot{m}_3 = -x_1^2 + y_1 + x_2^2 - y_2,
\]

\[
\dot{y}_1 = 2m_3y_1,
\]

\[
\dot{y}_2 = -2m_3y_1,
\]

\[
\dot{\gamma}_3 = x_1(x_2^2 - y_2) - x_2(x_1^2 - y_1),
\]

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and
\[
\begin{align*}
y_1y_2 &= k^2, \\
m_3^2 &= 6h_1 + y_1 + y_2 - (x_1 + x_2)^2, \\
m_3\gamma_3 &= 2h_2 + x_1y_2 + x_2y_1 - x_1x_2(x_1 + x_2), \\
\gamma_3^2 &= 1 - k^2 + x_1^2y_2 + x_2^2y_1 - x_1^2x_2^2. \\
\end{align*}
\]

It’s obvious that \(\sigma(37)\) is an automorphism of \(\mathcal{A}\), of order two. The quotient \(\mathcal{A}/\sigma\) by the involution \(\sigma\) is a Kummer \(K\) defined by (38). The variety \(\mathcal{A}\) is a double cover of the surface \(K\) branched over the fixed points of the involution \(\sigma\). To find them, we substitute \(m_3 = \gamma_3 = 0\) in the system (34), to wit
\[y_1y_2 = k^2, \quad y_1 + y_2 = (x_1 + x_2)^2 - 6h_1, \quad x_2y_1 + x_1y_2 = x_1x_2(x_1 + x_2) - 2h_2, \quad x_2^2y_1 + x_1^2y_2 = x_1^2x_2^2 + k^2 - 1.
\]

Away from the \(x_1^2 = x_2^2\), we may solve (45) and (47) in \(y_1\) and \(y_2\) and substitute into the equations (44) and (46); one then finds two curves in \(x_1\) and \(x_2\) whose equations are
\[\begin{align*}
R(x_1, x_2) &= -x_1^2x_2^2 + 6h_1x_1x_2 - 2h_2(x_1 + x_2) + 1 - k^2 = 0, \\
S(x_1, x_2) &= (x_1^4 + 2x_1^3x_2 - 6h_1x_1^2 + 1 - k^2) (x_1^4 + 2x_1^3x_2 - 6h_1x_1^2 + 1 - k^2) + k^2(x_1^2 - x_2^2)^2 = 0.
\end{align*}
\]

These curves intersect at the zeroes of the resultant \(\text{Res}(R, S)\) of \(R, S\):
\[\text{Res}(R, S)_{x_2} = x_2^2 (x_1^4 + 6h_1x_1^2 + k^2 - 1)^2 P_8(x_1),
\]
where \(P_8(x_1)\) is a monic polynomial of degree 8. Since the root \(x_1\) must be excluded (it indeed implies that the leading terms of \(R\) and \(S\) vanish), the possible intersections of the curve \(R\) and \(S\) will be,

(i) at the roots of
\[x_1^4 + 6h_1x_1^2 + k^2 - 1 = 0,
\]

this is unacceptable, because then one checks that the common roots of \(R\) and \(S\) would have the property that \(x_1^2 = x_2^2\), which was excluded.

(ii) at the roots of \(P_8(x_1) = 0\); there, for generic \(k\) and \(h\), \(x_1^2 \neq x_2^2\).

Finally, we must analyze the case \(x_1^2 = x_2^2\) for which one checks that (44)....(47) has no common roots. Consequently the involution \(\sigma\) has 8 fixed points on the affine variety \(\mathcal{A}\). Clearly the vector field (42) vanishes at the fixed points of the involution \(\sigma\).
b) From equations (38), we deduce

\[ y_1 = \frac{-1}{2R(x_2)} (R_1(x_1, x_2) + k^2(x_1 - x_2)^2 + \Delta), \]

\[ y_2 = \frac{-1}{2R(x_1)} (R_1(x_1, x_2) + k^2(x_1 - x_2)^2 - \Delta), \]

where

\[ \Delta^2 = (R_1(x_1, x_2) + k^2(x_1 - x_2)^2)^2 - 4k^2R(x_1)R(x_2) \equiv P(x_1, x_2). \]

Therefore, the surface \( K \) is a double cover of \( \mathbb{C}^2 \), ramified along the curve \( C : P(x_1, x_2) = 0 \). This equation is reducible and can be written as the product

\[ P(x_1, x_2) = P_1(x_1, x_2)P_2(x_1, x_2), \]

of two symmetric polynomials (in \( x_1, x_2 \)) of degree two in each one of the variables \( x_1, x_2 \), i.e.,

\[ P_1(x_1, x_2) = a(x_1)x_2^2 + 2b(x_1)x_2 - c(x_1) = a(x_2)x_1^2 + 2b(x_2)x_1 - c(x_2), \]

where

\[ a(x) = -2(k + 3h_1)x^2 + 4h_2x - 1, \]

\[ b(x) = 2h_2x^2 + (2k(k + 3h_1) - 1)x - 2h_2k, \]

\[ c(x) = x^2 + 4h_2kx + 2(k - 1)(k + 3h_1) + 4h_2^2, \]

while the polynomial \( P_2(x_1, x_2) \) is obtained from \( P_1(x_1, x_2) \) after replacing \( k \) with \( -k \). Note that the curve \( C_1 : P_1(x_1, x_2) = 0 \), is elliptic :

\[ x_1 = \frac{-b(x_2) \pm \sqrt{2(k + 3h_1) - 4h_2^2\sqrt{R(x_2)}}}{a(x_2)}, \]

\[ x_2 = \frac{-b(x_1) \pm \sqrt{2(k + 3h_1) - 4h_2^2\sqrt{R(x_1)}}}{a(x_1)}, \]

where \( R(x) \) is given by (39). Similarly, the curve \( C_2 : P_2(x_1, x_2) = 0 \), is elliptic and we notice that the two curves \( C_1 \) and \( C_2 \) intersect exactly at the 8 fixed points of involution \( \sigma \), because (see (48)) :

\[ \text{Res}(P_1, P_2)_{x_2} = 16k^2P_8(x_1). \]

Differentiating the symmetric equation \( P_1(x_1, x_2) = 0 \) (or \( P_2(x_1, x_2) = 0 \)) with regard to \( t \), one finds

\[ \pm 2\sqrt{2(k + 3h_1) - 4h_2^2\sqrt{R(x_2)}} \dot{x}_1 \pm 2\sqrt{2(k + 3h_1) - 4h_2^2\sqrt{R(x_1)}} \dot{x}_2 = 0. \]
Hence
\[ \frac{\dot{x}_1}{\sqrt{R(x_1)}} \pm \frac{\dot{x}_2}{\sqrt{R(x_2)}} = 0. \] (49)

Since \( R(x_1) \) and \( R(x_2) \) are two polynomials of the fourth degree in \( x_1 \) and \( x_2 \) respectively and having the same coefficients, then (49) is the so-called Euler’s equation. The reader is referred to Halphen [15] and Weil [38] for this theory that we summarize here as follows: let

\[ F(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4, \]

be a polynomial of the fourth degree. The general integral of Euler’s equation
\[ \frac{\dot{x}}{\sqrt{F(x)}} \pm \frac{\dot{y}}{\sqrt{F(y)}} = 0, \]
can be written in two different ways:

\[ F_1(x, y) + 2sF(x, y) - s^2(x - y)^2 = 0, \]

where
\[ F(x, y) = a_0x^2y^2 + 2a_1xy(x + y) + 3a_2(x^2 + y^2) + 2a_3(x + y) + a_4, \]

and
\[ F_1(x, y) = \frac{F(x)F(y) - F^2(x, y)}{(x - y)^2}, \]

or in an irrational form
\[ \frac{F(x, y) + \sqrt{F(x)}\sqrt{F(y)}}{(x - y)^2} = s, \]

which can be seen as the addition-formula for the Weierstrass elliptic function
\[ 2\wp(u + v) = \frac{(\wp(u) + \wp(v))(2\wp(u)\wp(v) - g_2 - g_3 - \wp(u)\wp'(v))}{(\wp(u) + \wp(v))^2}, \]
\[ \wp'(u) = \left(\frac{d\wp}{du}\right)^2 = 4\wp^3 - g_2\wp - g_3, \]
\[ \wp(u) = x, \wp(v) = y, F(x) = 4x^3 - g_2x - g_3, \wp'(u) = F(x), \wp'(v) = F(y), \]
\[ 2\wp(u + v) = s \text{ and } g_2, g_3 \text{ are constants. We apply these facts to Kowalewski’s problem with } F(x) = R(x), F(x_1, x_2) = R(x_1, x_2) + 3h_1(x_1 - x_2)^2 \text{, and } a_0 = -1, a_1 = 0, a_2 = h_1, a_3 = -h_2, a_4 = 1 - k^2 \text{ and } s = k + 3h_1. \] So the polynomial \( P_1(x_1, x_2) \) which can also be regarded as a solution of (49), can also be written as
\[ R_1(x_1, x_2) + 2sR(x_1, x_2) - s^2(x_1 - x_2)^2 = 0, \]
where \( R_1(x_1, x_2) \) is given by (40) and has the form

\[
R_1(x_1, x_2) = \frac{R(x_1)R(x_2) - R^2(x_1, x_2)}{(x_1 - x_2)^2}.
\]

Remember that \( R(x_1, x_2) \) is given by (41). The solution of (49) can also be expressed

\[
R(x_1, x_2) \mp \sqrt{R(x_1)R(x_2)} (x_1 - x_2)^2 + 3h_1 = s. \tag{50}
\]

c) Let us carry out the calculations, assuming the polynomial \( R(x) \) reduced to the form \( 4x^3 - g_2x - g_3 \) and call \( s_1 \) (resp. \( s_2 \)) the relation (50) with the sign - (resp. +). Now, outside the branch locus of \( K \) over \( \mathbb{C}^2 \), the equation (49) is not identically zero and may be written in the form

\[
\begin{align*}
\frac{\dot{x}_1}{\sqrt{R(x_1)}} + \frac{\dot{x}_2}{\sqrt{R(x_2)}} &= \frac{s_1}{\sqrt{4s_1^2 - g_2s_1 - g_3}} \neq 0, \tag{51} \\
\frac{\dot{x}_1}{\sqrt{R(x_1)}} - \frac{\dot{x}_2}{\sqrt{R(x_2)}} &= \frac{s_2}{\sqrt{4s_2^2 - g_2s_2 - g_3}} \neq 0.
\end{align*}
\]

where \( g_2 = k^2 - 1 + 3h_1^2 \) and \( g_3 = h_1(k^2 - 1 - h_1^2) + h_2^2 \). After some algebraic manipulation we deduce from (43),

\[
\begin{align*}
(m_3x_1 - \gamma_3)^2 &= R(x_1) + (x_1 - x_2)^2y_1, \\
(m_3x_2 - \gamma_3)^2 &= R(x_2) + (x_1 - x_2)^2y_2, \\
(m_3x_1 - \gamma_3)(m_3x_2 - \gamma_3) &= R(x_1, x_2),
\end{align*}
\]

and from (42),

\[
\begin{align*}
\dot{x}_1^2 &= R(x_1) + (x_1 - x_2)^2y_1, \\
\dot{x}_2^2 &= R(x_2) + (x_1 - x_2)^2y_2.
\end{align*}
\]

This together with (38) and (51) implies that

\[
\frac{s_1^2}{4s_1^2 - g_2s_1 - g_3} = \left( \frac{\dot{x}_1}{\sqrt{R(x_1)}} + \frac{\dot{x}_2}{\sqrt{R(x_2)}} \right)^2 = \left( \frac{(x_1 - x_2)^2}{R(x_1)R(x_2)} \left[ \frac{R(x_1, x_2) - \sqrt{R(x_1)R(x_2)}}{(x_1 - x_2)^2} \right] - k^2 \right),
\]

\[
= 4\frac{(s_1 - 3h_1)^2 - k^2}{(s_1 - s_2)}.
\]

In the same way, we find

\[
\frac{s_2^2}{4s_2^2 - g_2s_2 - g_3} = \left( \frac{\dot{x}_1}{\sqrt{R(x_1)}} - \frac{\dot{x}_2}{\sqrt{R(x_2)}} \right)^2 = 4\frac{(s_2 - 3h_1)^2 - k^2}{(s_2 - s_1)}.
\]

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In terms of the variables $s_1$ and $s_2$, the system (33) becomes
\begin{align*}
\frac{\dot{s}_1}{\sqrt{P(s_1)}} + \frac{\dot{s}_2}{\sqrt{P(s_2)}} &= 0, \\
\frac{s_1\dot{s}_1}{\sqrt{P(s_1)}} + \frac{s_2\dot{s}_2}{\sqrt{P(s_2)}} &= i,
\end{align*}
where
\[P_5(s) = ((s - 3h_1)^2 - k^2)(4s^3 - g_2s - g_3),\]
is a polynomial of degree 5. As known, such integrals are called hyperelliptic integrals and the problem can be integrated in terms of genus two hyperelliptic functions of time. More precisely, these equations are integrable by the transformation of Abel
\[\mathcal{H} \to \text{Jac}(\mathcal{H}) = \mathbb{C}^2/\Lambda, \quad p \mapsto \left(\int_{p_0}^p \theta_1, \int_{p_0}^p \theta_2\right),\]
where $\mathcal{H}$ is the hyperelliptic curve of genus 2 associated with equation $w^2 = P_5(s)$, $\Lambda$ is the lattice generated by the vectors $n_1 + \Omega_H n_2$, $(n_1, n_2) \in \mathbb{Z}^2$, $\Omega_H$ is the matrix of periods of $\mathcal{H}$, $(\theta_1, \theta_2)$ is a basis of holomorphic differentials on $\mathcal{H}$, i.e.,
\[\theta_1 = \frac{ds}{\sqrt{P_5(s)}}, \quad \theta_2 = \frac{sds}{\sqrt{P_5(s)}},\]
and $p_0$ is a fixed point on $\mathcal{H}$. The theorem is thus proved. □

We mentioned previously that any algebraic integral of equations (24) is a combination of classical integrals except in the cases of Euler, Lagrange and Kowalewski and that there could therefore be no first algebraic integral other than those highlighted in these three cases. In addition, there are a few special cases:

- The case of Hesse-Appel’rot [16, 3]:
\[l_2 = 0, \quad l_1\sqrt{I_1(I_2 - I_3)} + l_3\sqrt{I_3(I_1 - I_2)} = 0.\]
In this case, equation $l_1m_1 + l_3m_3 = 0$ represents a particular first integral obtained by Hesse and the integration is carried out using elliptic functions.

- The case of Goryachev-Chaplygin [12, 8]: $I_1 = I_2 = 4I_3, l_2 = l_3 = 0$. In this case, the system (22) admits the first integral
\[\lambda_3m_3(\lambda_1^2 + m_1^2 + \lambda_2^2m_2^2) + \mu g\lambda_1\lambda_3m_1\gamma_3 = g, \quad \lambda_i = \frac{1}{I_i}, i = 1, 2, 3\]
and integration is carried out using hyperelliptic functions of genus 2.

- The case of Bobylev-Steklov [6, 36]: $I_2 = 2I_1, l_1 = l_3 = 0$. The integration of the equations in this case is easy, using elliptic functions.
10.2 Yang-Mills field with gauge group $SU(2)$

We begin by introducing some notions related to Yang-Mills field [44] with $SU(2)$ as gauge group. For general considerations and the details of certain notions, one can consult for example [9]. Consider the special unitary group $SU(2)$ of degree 2, i.e., the set of $2 \times 2$ unitary matrices with determinant 1. This is a real Lie group of dimension three. It is compact, simply connected, simple and semi-simple. The group $SU(2)$ is isomorphic to the group of quaternions of norm one and is diffeomorphic to the 3-sphere $S^3$. It is well known that the quaternions represent the rotations in 3-dimensional space and hence there exists a surjective homomorphism of $SU(2)$ on the rotation group $SO(3)$ whose kernel is $\{I, -I\}$ (the identical application and its opposite). Recall also that $SU(2)$ is identical to one of the symmetry spinor groups, $Spin(3)$, that enables a spinor presentation of rotations. The Lie algebra $su(2)$ corresponding to $SU(2)$ consists of the $2 \times 2$ antihermitian complex matrices with null trace, the standard commutator serving as a Lie bracket. It is a real algebra. The algebra $su(2)$ is isomorphic to the Lie algebra $so(3)$.

We consider the Yang-Mills field $F_{kl}$ as a vector field with values in the algebra $su(2)$. It is a local expression of the gauge field or connection defining the covariant derivative of $F_{kl}$ in the adjoint representation of $su(2)$. To determine this expression, note that each Lorentz component of the Yang-Mills field develops on a basis $(\sigma_1, \sigma_2, \sigma_3)$ de $su(2)$, $A_k = A_\alpha^k \sigma_\alpha$, $\alpha = 1, 2, 3$, $k = 1, 2, 3, 4$, where the $\sigma_\alpha$ are the matrices of Pauli

$$
\sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

These matrices are often used in quantum mechanics to represent the spin of particles. Moreover, the group $SU(2)$ is associated with gauge symmetry in the description of the weak or weak force interaction (one of the four fundamental forces of nature) and is therefore of particular importance in the physics of particles. In fact, it was only in the late 1960s that the importance of the Yang-Mills equations became apparent, especially when the concept of the gauge fields was defined as the one of the four fundamental physical interactions (gravitational, electromagnetic, weak and strong interactions).

The dynamics of the Yang-Mills theory is determined by the Lagrangian density

$$
\mathcal{L} = -\frac{1}{2} Tr \{ F_{kl} F^{kl} \}, \quad 1 \leq k, l \leq 4
$$

where

$$
F_{kl} = \frac{\partial A_l}{\partial \tau_k} - \frac{\partial A_k}{\partial \tau_l} + [A_k, A_l],
$$

is the expression of the anti-symmetric Faraday tensors with values in $su(2)$. These tensors are not invariant under gauge transformations. On the other
hand, we verify that $Tr\{F_{kl}F^{kl}\}$ is actually gauge invariant. The trace relates to the internal space $su(2)$. The equations of the motion are given by

$$D_k F^{kl} = \frac{\partial F^{kl}}{\partial \tau_k} + [A_k, F^{kl}] = 0, \quad F^{kl}, A_k \in su(2), \quad 1 \leq k, l \leq 4,$$

with $D_k$ the covariant derivative in the adjoint representation of the algebra $su(2)$ and in which $[A_k, F^{kl}]$ is the crochet of the two fields in $su(2)$. The Yang-Mills theory extends the principle of gauge invariance of electromagnetism to other groups of continuous Lie transformations. Thus the $F^{kl}$ tensor generalizes the electromagnetic field and the Yang-Mills equations are the non-commutative generalization of the Maxwell equations. The self-dual Yang-Mills (SDYM) equations is an universal system for which some reductions include all classical tops from Euler to Kowalewski (0+1-dimensions), KdV, Nonlinear Schrödinger, Sine-Gordon, Toda lattice and N-waves equations (1+1-dimensions), KP and D-S equations (2+1-dimensions). There is a vast literature devoted to the study of the above equations (see monograph [35] for many references concerning both theoretical and practical results).

We are interested here in the field of homogeneous double-component field. In this case, we have $\frac{\partial A_k}{\partial \tau_k} = 0, (k \neq 1)$, $A_1 = A_2 = 0$, $A_3 = n_1 U_1 \in su(2)$, $A_4 = n_2 U_2 \in su(2)$, where $n_i$ are $su(2)$-generators (i.e., they satisfy commutation relations : $n_1 = [n_2, [n_1, n_2]], n_2 = [n_1, [n_2, n_1]])$. The system becomes

$$\frac{\partial^2 U_1}{\partial t^2} + U_1 U_2^2 = 0,$$

$$\frac{\partial^2 U_2}{\partial t^2} + U_1 U_2^2 = 0,$$

with $t = \tau_1$. By setting $U_1 = q_1$, $U_2 = q_2$, $\frac{\partial U_1}{\partial t} = p_1$, $\frac{\partial U_2}{\partial t} = p_2$, Yang-Mills equations are reduced to Hamiltonian system

$$\dot{x} = J \frac{\partial H}{\partial x}, \quad x = (q_1, q_2, p_1, p_2)^T, \quad J = \begin{pmatrix} O & -I \\ I & O \end{pmatrix},$$

where

$$H = \frac{1}{2} \left( p_1^2 + p_2^2 + q_1^2 q_2^2 \right),$$

is the Hamiltonian. Note that the symplectic transformation :

$$p_1 = \frac{\sqrt{2}}{2} (x_1 + x_2), \quad p_2 = \frac{\sqrt{2}}{2} (x_1 - x_2),$$

$$q_1 = \frac{1}{2} \left( \sqrt{2} \right)^3 (y_1 + iy_2), \quad q_2 = \frac{1}{2} \left( \sqrt{2} \right)^3 (y_1 - iy_2),$$

takes this Hamiltonian into

$$H = \frac{1}{2} \left( x_1^2 + x_2^2 \right) + \frac{1}{4} \left( y_1^2 + y_2^2 \right)^2.$$
The Hamiltonian dynamical system associated with \( H \) is written

\[
\begin{align*}
\dot{y}_1 &= x_1, \\
\dot{y}_2 &= x_2, \\
\dot{x}_1 &= -(y_1^2 + y_2^2) y_1, \\
\dot{x}_2 &= -(y_1^2 + y_2^2) y_2.
\end{align*}
\]

These equations give a vector field on \( \mathbb{R}^4 \). The existence of a second independent first integral in involution with \( H_1 \equiv H \), is enough for the system to be completely integrable. The above differential system implies

\[
\begin{align*}
\ddot{y}_1 + (y_1^2 + y_2^2) y_1 &= 0, \\
\ddot{y}_2 + (y_1^2 + y_2^2) y_2 &= 0.
\end{align*}
\]

Obviously, the moment : \( H_2 = x_1 y_2 - x_2 y_1 \), is a first integral, \( H_1 \) and \( H_2 \) are in involution \( \{ H_1, H_2 \} = 0 \) and \( H_2 \) determines with \( H_1 \) an integrable system. Let

\[
\mathcal{M}_c = \{ x \equiv (y_1, y_2, x_1, x_2) \in \mathbb{R}^4 : H_1(x) = c_1, H_2(x) = c_2 \},
\]

be the invariant surface (where \( c = (c_1, c_2) \) is not a critical value). Substituting \( y_1 = r \cos \theta, y_2 = r \sin \theta \), in equations

\[
\begin{align*}
H_1 &= \frac{1}{2} (x_1^2 + x_2^2) + \frac{1}{4} (y_1^2 + y_2^2)^2 = c_1, \\
H_2 &= x_1 y_2 - x_2 y_1 = c_2,
\end{align*}
\]

we obtain

\[
\frac{1}{2} \left( \dot{r}^2 + (r \dot{\theta})^2 \right) + \frac{1}{4} r^2 = c_1, \quad r^2 \dot{\theta} = -c_2.
\]

Hence

\[
(r \dot{r})^2 + \frac{1}{2} r^4 - 2c_1 r^2 + c_2^2 = 0,
\]

and

\[
w^2 + P(z) = 0,
\]

where \( w \equiv r \dot{r}, z \equiv r^2 \), and \( P(z) = \frac{1}{2} z^3 - 2c_1 z + c_2^2 \). The polynomial \( P(z) \) is of degree 3, the Riemann surface \( C \)

\[
C = \{ (w, z) : w^2 + P(z) = 0 \},
\]

is of genus \( g = 1 \) (an elliptic curve). We thus have a single holomorphic differential

\[
\omega = \frac{dz}{\sqrt{P(z)}},
\]

and the linearization occurs on the elliptic curve \( C \). Although the variety \( \mathcal{M}_c \) has dimension 2, here we have a reduction of dimension 1 and, consequently, we get the following result :
Theorem 28 The differential system (52) is completely linearized on the Jacobian variety of \( C \), i.e. on the elliptic curve \( C(53) \).

Remark 5 Further information and methods for resolving the Yang-Mills system will be found for example in [35, 25] and references therein.

Conclusion: We do not consider here the solution techniques based on the important notion of dynamical systems that are algebraically completely integrable (the survey of these results, as well as, the extensive list of references can be found in [2, 27]). Let’s just mention that the concept of algebraic complete integrability is quite effective in small dimensions and has the advantage to lead to global results, unlike the existing criteria for real analytic integrability, which, at this stage are perturbation results (in fact, the perturbation techniques developed in that context are of a totally different nature). However, besides the fact that many Hamiltonian dynamical integrable systems possess this structure, another motivation for its study is due to the fact that algebraic completely integrable systems come up systematically whenever you study the isospectral deformation of some linear operator containing a rational indeterminate (indeed a theorem of Adler-Kostant-Symes [2] applied to Kac-Moody algebras provides such systems which, by a theorem of van Moerbeke-Mumford [2] are algebraic completely integrable). In recent years, other important results have been obtained following studies on the KP and KdV hierarchies (we refer the interested reader for example to [24] for an exposition and a survey of the results in this field, as well as, a list of references). In fact, many problems related to algebraic geometry, combinatorics, probabilities and quantum gauge theory,..., have been solved explicitly by methods inspired by techniques from the study of dynamical integrable systems.

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