A \( p \)-th Yamabe equation on graph

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Abstract

Assume \( \alpha \geq p > 1 \). Consider the following \( p \)-th Yamabe equation on a connected finite graph \( G \):

\[
\Delta_p \phi + h \phi^{p-1} = \lambda f \phi^{\alpha-1},
\]

where \( \Delta_p \) is the discrete \( p \)-Laplacian, \( h \) and \( f > 0 \) are fixed real functions defined on all vertices. We show that the above equation always has a positive solution \( \phi \) for some constant \( \lambda \in \mathbb{R} \).

1 Introduction

The well known smooth Yamabe problem asks for the considering of the following smooth Yamabe equation \([1, 5, 6]\)

\[
\Delta \phi + h(x)\phi = \lambda f(x)\phi^{N-1}
\]

on a \( C^\infty \) compact Riemannian manifold \( M \) of dimension \( n \geq 3 \), where \( h(x) \) and \( f(x) \) are \( C^\infty \) functions on \( M \), with \( f(x) \) everywhere strictly positive and \( N = 2n/(n-2) \). The problem is to prove the existence of a real number \( \lambda \) and of a \( C^\infty \) function \( \phi \), everywhere strictly positive, satisfying the above Yamabe equation. In this short paper, we consider the corresponding discrete Yamabe equation

\[
\Delta \phi + h \phi = \lambda \phi^{\alpha-1}, \quad \alpha \geq 2
\]

on a finite graph. More generally, we shall establish the existence results of the following \( p \)-th discrete Yamabe equation

\[
\Delta_p \phi + h \phi^{p-1} = \lambda f \phi^{\alpha-1}
\]

on a finite graph \( G \) with \( \alpha \geq p > 1 \). This work is inspired by Grigor’yan, Lin and Yang’s pioneer paper \([3, 4]\), where they studied similar equations on finite or locally finite graphs.
2 Settings and main results

Let $G = (V, E)$ be a finite graph, where $V$ denotes the vertex set and $E$ denotes the edge set. Fix a vertex measure $\mu : V \to (0, +\infty)$ and an edge measure $\omega : E \to (0, +\infty)$ on $G$. The edge measure $\omega$ is assumed to be symmetric, that is, $\omega_{ij} = \omega_{ji}$ for each edge $i \sim j$.

Denote $C(V)$ as the set of all real functions defined on $V$, then $C(V)$ is a finite dimensional linear space with the usual function additions and scalar multiplications. For any $p > 1$, the $p$-th discrete graph Laplacian $\Delta_p : C(V) \to C(V)$ is

$$\Delta_p f_i = \frac{1}{\mu_i} \sum_{j \sim i} \omega_{ij} |f_j - f_i|^{p-2}(f_j - f_i)$$

for any $f \in C(V)$ and $i \in V$. $\Delta_p$ is a nonlinear operator when $p \neq 2$.

**Theorem 2.1.** Let $G = (V, E)$ be a finite connected graph. Given $h, f \in C(V)$ with $f > 0$. Assume $\alpha \geq p > 1$. Then the following $p$-th Yamabe equation

$$\Delta_p \varphi + h \varphi^{p-1} = \lambda f \varphi^{\alpha-1}$$

(2.1)

on $G$ always has a positive solution $\varphi$ for some constant $\lambda \in \mathbb{R}$.

Taking $p = 2$, we get the following

**Corollary 2.2.** Let $G = (V, E)$ be a finite connected graph. Given $h, f \in C(V)$ with $f > 0$. Assume $\alpha > 2$. Then the following Yamabe equation

$$\Delta \varphi + h \varphi = \lambda f \varphi^{\alpha-1}$$

(2.2)

on $G$ always has a positive solution $\varphi$ for some constant $\lambda \in \mathbb{R}$.

**Remark 1.** Grigor’yan, Lin and Yang [4] established similar results for the following equation

$$-\Delta u + hu = |u|^{\alpha-2}u, \quad \alpha > 2$$

(2.3)

on a finite graph under the assumption $h > 0$. They show that the above equation always has a positive solution. They also studied the following equation

$$-\Delta_p u + h|u|^{p-2}u = f(x, u), \quad p > 1$$

(2.4)

and established some existence results under certain assumptions of $f(x, u)$. However, it is remarkable that their $\Delta_p$ considered in the equation [2.4] is different with ours when $p \neq 2$. It is also remarkable that our Theorem [2.1] doesn’t require $h > 0$. 

2
3 Proofs of theorem 2.1

3.1 Sobolev embedding

For any $f \in C(V)$, define an integral of $f$ over $V$ with respect to the vertex weight $\mu$ by

$$\int_V f \, d\mu = \sum_{i \in V} \mu_i f_i.$$ 

Set $\text{Vol}(G) = \int_V d\mu$. Similarly, for any function $g$ defined on the edge set $E$, we define an integral of $g$ over $E$ with respect to the edge weight $\omega$ by

$$\int_E g \, d\omega = \sum_{i \sim j} \omega_{ij} g_{ij}.$$ 

Specially, for any $f \in C(V)$,

$$\int_E |\nabla f|^p \, d\omega = \sum_{i \sim j} \omega_{ij} |f_j - f_i|^p,$$

where $|\nabla f|$ is defined on the edge set $E$, and $|\nabla f|_{ij} = |f_j - f_i|$ for each edge $i \sim j$. Next we consider the Sobolev space $W^{1,p}(G)$ on the graph $G$. Define

$$W^{1,p}(G) = \left\{ u \in C(V) : \int_E |\nabla u|^p \, d\omega + \int_V |u|^p \, d\mu < +\infty \right\},$$

and

$$\|u\|_{W^{1,p}(G)} = \left( \int_E |\nabla u|^p \, d\omega + \int_V |u|^p \, d\mu \right)^{\frac{1}{p}}.$$ 

Since $G$ is a finite graph, then $W^{1,p}(G)$ is exactly $C(V)$, a finite dimensional linear space. This implies the following Sobolev embedding:

**Lemma 3.1.** (Sobolev embedding) Let $G = (V, E)$ be a finite graph. The Sobolev space $W^{1,p}(G)$ is pre-compact. Namely, if $\{\varphi_n\}$ is bounded in $W^{1,p}(G)$, then there exists some $\varphi \in W^{1,p}(G)$ such that up to a subsequence, $\varphi_n \to \varphi$ in $W^{1,p}(G)$.

**Remark 2.** The convergence in $W^{1,p}(G)$ is in fact pointwise convergence.

3.2 Proofs step by step

We follow the original approach pioneered by Yamabe [6]. Denote an energy functional

$$I(\varphi) = \left( \int_E |\nabla \varphi|^p \, d\omega - \int_V h \varphi^p \, d\mu \right) \left( \int_V f \varphi^\alpha \, d\mu \right)^{-\frac{p}{\alpha}},$$

(3.1)
where $\varphi \in W^{1,p}(G)$, $\varphi \geq 0$ and $\varphi \not\equiv 0$. Define
\[ \beta = \inf \{ I(\varphi) : \varphi \geq 0, \varphi \not\equiv 0 \}. \tag{3.2} \]
We shall find a solution to (2.1) step by step as follows.

**Step 1.** $I(\varphi)$ is bounded below for all $\varphi \geq 0$, $\varphi \not\equiv 0$. Hence $\beta \neq -\infty$ and $\beta \in \mathbb{R}$. In fact, it’s easy to see
\[ 0 < \left( \int_V \varphi^\alpha d\mu \right)^{\frac{p}{\alpha}} \leq f_\alpha^\nu \left( \int_V \varphi^\alpha d\mu \right)^{\frac{p}{\alpha}} = f_\alpha^\nu \| \varphi \|_\alpha^p, \]
where $f_\alpha = \max_{i \in V} f_i > 0$. Hence
\[ \left( \int_V \varphi^\alpha d\mu \right)^{\frac{p}{\alpha}} \geq f_\alpha^\nu \| \varphi \|_\alpha^{-p} > 0. \tag{3.3} \]
Similarly, we also have
\[ -\int_V h\varphi^p d\mu \geq (-h)_m \int_V \varphi^p d\mu = (-h)_m \| \varphi \|_p^p, \]
where $(-h)_m = \min_{i \in V} (-h_i)$. Then it follows
\[ \int_E |\nabla \varphi|^p d\omega - \int_V h\varphi^p d\mu \geq (-h)_m \| \varphi \|_p^p. \tag{3.4} \]
By (3.3) and (3.4), we get
\[ I(\varphi) \geq (-h)_m \| \varphi \|_p f_\alpha^{-\frac{p}{\alpha}} \| \varphi \|_\alpha^{-p}, \]
and further
\[ I(\varphi) \geq ((-h)_m \wedge 0) \| \varphi \|_p f_\alpha^{-\frac{p}{\alpha}} \| \varphi \|_\alpha^{-p}, \tag{3.5} \]
where $(-h)_m \wedge 0$ is the minimum of $(-h)_m$ and 0. Since $\alpha \geq p$, then
\[ 0 < \| \varphi \|_p^p \leq \left( \int_V (\varphi^p)^{\frac{\alpha}{p}} d\mu \right)^{\frac{p}{\alpha}} \left( \int_V 1^{\frac{\alpha}{\alpha-p}} d\mu \right)^{\frac{\alpha-p}{\alpha}} = \| \varphi \|_\alpha^p \text{Vol}(G)^{1-\frac{p}{\alpha}}, \tag{3.6} \]
which leads to
\[ 0 < \| \varphi \|_p \| \varphi \|_\alpha^{-p} \leq \text{Vol}(G)^{1-\frac{p}{\alpha}}. \tag{3.7} \]
Thus by (3.5) and (3.7), we obtain
\[ I(\varphi) \geq ((-h)_m \wedge 0) f_\alpha^{-\frac{p}{\alpha}} \text{Vol}(G)^{1-\frac{p}{\alpha}} = C_{\alpha,p,h,f,G}, \tag{3.8} \]
where $C_{\alpha,p,h,f,G} \leq 0$ is a constant depending only on the information of $\alpha$, $p$, $h$, $f$ and $G$. Note that the information of $G$ contains $V$, $E$, $\mu$ and $\omega$. Hence $I(\varphi)$ is bounded below by a universal constant.

**Step 2.** There exists a $\hat{\varphi} \geq 0$, such that $\beta = I(\hat{\varphi})$. To find such $\hat{\varphi}$, we choose $\varphi_n \geq 0$, satisfying

$$
\int_V f \varphi_n^\alpha d\mu = 1
$$

and

$$
I(\varphi_n) \to \beta
$$
as $n \to \infty$. We may well suppose $I(\varphi_n) \leq 1 + \beta$ for all $n$. Note

$$
1 = \int_V f \varphi_n^\alpha d\mu \geq f_m \int_V \varphi_n^\alpha d\mu = f_m \|\varphi_n\|_\alpha^\alpha,
$$

where $f_m = \min_{i \in V} f_i$. Hence

$$
\|\varphi_n\|_\alpha^\alpha \leq f_m^{-\frac{\beta}{\alpha}}. \quad (3.9)
$$

Denote $|h|_M = \max_{i \in V} |h_i|$, then by (3.6) and (3.9), we obtain

$$
\|\varphi_n\|_{W^{1,p}(G)}^p = \int_E |\nabla \varphi|^p d\omega + \int_V |\varphi|^p d\mu
$$

$$
= I(\varphi_n) + \int_V h \varphi_n^\alpha d\mu + \|\varphi_n\|_\alpha^p
$$

$$
\leq 1 + \beta + (1 + |h|_M) \|\varphi_n\|_\alpha^p
$$

$$
\leq 1 + \beta + (1 + |h|_M) \text{Vol}(G)^{1 - \frac{\beta}{\alpha}} \|\varphi_n\|_\alpha^p
$$

$$
\leq 1 + \beta + (1 + |h|_M) \text{Vol}(G)^{1 - \frac{\beta}{\alpha}} f_m^{-\frac{\beta}{\alpha}},
$$

which implies that $\{\varphi_n\}$ is bounded in $W^{1,p}(G)$. Therefore by Lemma 3.1, there exists some $\hat{\varphi} \in C(V)$ such that up to a subsequence, $\varphi_n \to \hat{\varphi}$ in $W^{1,p}(G)$. We may well denote this subsequence as $\varphi_n$. Note $\varphi_n \geq 0$ and $\int_V f \varphi_n^\alpha d\mu = 1$, let $n \to +\infty$, we know $\hat{\varphi} \geq 0$ and $\int_V f \hat{\varphi}_n d\mu = 1$. This implies that $\hat{\varphi} \neq 0$. Since the energy functional $I(\varphi)$ is continuous, we have $\beta = I(\hat{\varphi})$.

**Step 3.** $\hat{\varphi} > 0$.

Calculate the Euler-Lagrange equation of $I(\varphi)$, we get

$$
\frac{d}{dt} \bigg|_{t=0} I(\varphi + t\phi) = -p \left( \int_V f \varphi^\alpha d\mu \right)^{-\frac{p}{\alpha}} \int_V (\Delta_p \varphi + h \varphi^{p-1} - \lambda \varphi \varphi^{\alpha-1}) \phi d\mu,
$$

(3.10)
where
\[ \lambda_\varphi = -\frac{\int_E |\nabla \varphi|^p \, d\omega - \int_V h \varphi^p \, d\mu}{\int_V f \varphi^\alpha \, d\mu} \] (3.11)
for any \( \varphi \geq 0, \varphi \not\equiv 0 \). Thus
\[ \frac{\partial I}{\partial \varphi_i} = -p \mu_i ( \Delta_p \hat{\varphi}_i + h \hat{\varphi}_i^{p-1} - \lambda_\varphi f_i \hat{\varphi}_i^{\alpha-1} ) \left( \int_V f \varphi^\alpha \, d\mu \right)^{-\frac{p}{\alpha}}. \] (3.12)

Note the graph \( G \) is connected, if \( \hat{\varphi} > 0 \) is not satisfied, since \( \hat{\varphi} \geq 0 \) and not identically zero, then there is an edge \( i \sim j \), such that \( \hat{\varphi}_i = 0 \), but \( \hat{\varphi}_j > 0 \). Now look at \( \Delta_p \hat{\varphi}_i \),
\[ \Delta_p \hat{\varphi}_i = \frac{1}{\mu_i} \sum_{k \sim i} \omega_{ik} |\hat{\varphi}_k - \hat{\varphi}_i|^p (\hat{\varphi}_k - \hat{\varphi}_i) > 0. \]
Therefore by (3.12), we have
\[ \frac{\partial I}{\partial \varphi_i} \bigg|_{\varphi = \hat{\varphi}} = -p \mu_i \Delta_p \hat{\varphi}_i \left( \int_V f \varphi^\alpha \, d\mu \right)^{-\frac{p}{\alpha}} < 0. \]
Recall we had proved that \( \hat{\varphi} \) is the minimum value of \( I(\varphi) \), hence there should be
\[ \frac{\partial I}{\partial \varphi_i} \bigg|_{\varphi = \hat{\varphi}} \geq 0, \]
which is a contradiction. Hence \( \hat{\varphi} > 0 \).

**Step 4.** \( \hat{\varphi} \) satisfied the equation (2.1), that is
\[ \Delta_p \hat{\varphi} + h \hat{\varphi}^{p-1} = \lambda_\varphi f \hat{\varphi}^{\alpha-1}, \] (3.13)
where \( \lambda_\varphi \) is defined according to (3.11). Because \( I(\varphi) \) attains its minimum value at \( \hat{\varphi} \), which lies in the interior of \( \{ \varphi \in C(V) : \varphi \geq 0 \} \), so
\[ \frac{d}{dt} \bigg|_{t=0} I(\hat{\varphi} + t\phi) = 0 \]
for all \( \phi \in C(V) \). This leads to (3.13).

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