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The enumeration of simultaneous higher-order contacts between plane curves

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Introduction

Two plane algebraic curves are said to have contact of order \( o \) at a common point \( P \) if each curve is smooth at \( P \) and if the intersection number at \( P \) is \( o \). Thus, for example, a contact of order 1 is a transverse intersection, a contact of order 2 (sometimes called an ordinary contact) is a point of tangency; more generally, a contact of order \( o \) is a point at which, in appropriate local coordinates, the Taylor expansions of the curves agree up to order \( o - 1 \).

In [11], Fulton, Kleiman, and MacPherson consider, inter alia, a \( p \)-parameter family of plane curves together with \( p \) individual curves. They compute, in terms of certain "characteristic numbers", the number of members of the family which simultaneously have an ordinary contact with each of the curves. The analysis has two parts, the first of which is a formal calculation of an intersection number. The second part consists in establishing that, under stipulated hypotheses, this intersection number and the characteristic numbers have their intended geometric meanings. (The tendentious description we have just given greatly understates the scope of the contact formula of [11]. The contact formula for plane curves is an implicit special case—see further remarks in the next paragraph.)

In the present paper, by a similar two-part procedure, we derive a higher-order contact formula. We suppose that \( C_1, C_2, \ldots, C_p \) are plane curves, and that \( \mathcal{X} \) is an \( s \)-parameter family of plane curves of fixed degree. We suppose that \( o_1, o_2, \ldots, o_p \) are specified positive integers, with sum equal to \( s + p \). Our formula then counts, under stipulated hypotheses, the number of members of the family which simultaneously have a contact of order \( o_1 \) with \( C_1 \), a contact of order \( o_2 \) with \( C_2, \ldots, \), and a contact of order \( o_p \) with \( C_p \). The formula is both a specialization and a generalization of the

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general contact formula of [11]: a specialization in that we consider only plane curves, a generalization in that it counts contacts of arbitrary order. Our formula, like that of [11], is stated using the formalism of “modules”, a notion which we explain in Section 3.

The principal tool used in the proof, and in defining appropriate higher-order characteristic numbers, is a tower of \( P^1 \)-bundles implicitly introduced by Semple [27] and later reintroduced by Collino [7]. These bundles, which parametrize in a particularly lucid way the higher-order curvilinear data of the plane, are naturally adapted to the study of higher-order contact. They generalize the variety of second-order data of \( P^2 \) studied by J. Roberts and R. Speiser in [23] and [24], and, in turn, are currently being generalized by E. Arrondo, I. Sols, and R. Speiser [4]. (They are developing a theory of “derived triangles” that provides a general approach for obtaining the higher-order data, both curvilinear and higher-dimensional, of any scheme.) The Semple bundles conform to the intuitive heuristic that \((n + 1)\)st-order curvilinear data should fiber over the \( n \)th-order data. Moreover, since the \((n + 1)\)st Semple bundle variety is a \( P^1 \)-bundle over the \( n \)th variety, the intersection rings are straightforward to calculate. Thus the Semple bundles are superior to the canonical parameter space for data, namely the Hilbert scheme \( \text{Hilb}_P P^2 \) of zero-dimensional subschemes of \( P^2 \), in this respect. Indeed, considerable effort has been devoted recently to determining the homology and cohomology of punctual Hilbert schemes and to using these results to develop enumerative applications. (See, for example, [8], [9], [10], [20].) We have not explored the connections between Semple bundles and punctual Hilbert schemes in any detail, however. Nor have we begun to understand the apparently close connection between Semple bundles and certain subschemes of Kleiman’s iterative multiple-point schemes [15].

In the first two sections of this paper we describe the Semple bundle varieties and their intersection rings. In Section 3 we obtain a “proto-contact formula”, and in Section 4 we show that under stipulated hypotheses this formula counts, as intended, the number of simultaneous contacts between a generic family and specified curves. The proof involves a detailed analysis of the relationship between the universal family of plane curves of degree \( d \) and the Semple bundles which should be of independent interest. The fifth section briefly discusses the ingredients of the contact formula, i.e., the “higher-order characteristic numbers”. The final section treats special cases and variants of the contact formula, compares the formula with those of de Jonquières and Fulton-Kleiman-MacPherson, and presents an example.

We assume that all our varieties and schemes are defined over an algebraically closed field of characteristic zero. However, the results pres-
ented remain true with only minor modifications if the characteristic is sufficiently large, provided one recognizes that in positive characteristic our intersection numbers only count weighted numbers of contacts.

1. Semple bundles

We briefly recall here the definition of Semple's bundles of higher-order curvilinear data; for further discussion see [5] or [6]. The inductive construction begins by declaring that $F(0)$ is the projective plane and that the first *Semple bundle variety* $F(1)$ is $\mathbb{P}TF(0)$, the total space of the projectivized tangent bundle. (We use $\mathbf{PE}$ for the variety representing rank 1 subbundles of the vector bundle $E$, rather than for the variety representing rank 1 quotient bundles.) Inductively suppose that $F(n)$ is the projectivization of a rank 2 subbundle of the tangent bundle $TF(n-1)$. The *focal plane* at a point $p \in F(n)$ is defined to be the preimage, via the derivative of the projection $f_n: F(n) \to F(n-1)$, of the line in $T_{f_n(p)}F(n-1)$ represented by $p$. This construction gives rise to a rank 2 *bundle of focal planes*, denoted $\mathcal{F}_n$; we define $F(n+1)$ to be the total space of the projectivization of this bundle. In this manner we obtain a tower of $\mathbb{P}^1$-bundles:

\[
\begin{array}{c}
\vdots \\
\downarrow f_4 \\
F(3) \\
\downarrow f_3 \\
F(2) \\
\downarrow f_2 \\
F(1) \\
\downarrow f_1 \\
F(0)
\end{array}
\]

Suppose that $C$ is a reduced plane curve, and that $p \in C$ is a nonsingular point. The fiber of $F(1)$ over $p$ parametrizes the various tangent directions at $p$, including the tangent direction $p_1$ of $C$. The totality of such tangent directions to $C$ form a curve in $F(1)$; the closure of this curve is called the *first lift* of $C$, and denoted $C(1)$. Now observe that the tangent direction $p_2$ of $C(1)$ at $p_1$ maps, via the derivative of the projection $f_1: F(1) \to F(0)$, to $p_1$, the tangent direction of $C$ at $p$. Hence the *second lift* $C(2)$, i.e., the lift of the lift of $C$, is in fact a curve in $F(2)$. The same argument shows that there are *higher-order lifts* $C(3) \subset F(3)$, $C(4) \subset F(4)$, etc. We call the rational map $\lambda$ from
C to $F(n)$ the lifting map; it is regular away from the singularities of $C$ and a birational map to $C(n)$. We note that the $n$th lift $C(n)$ is just the $n$th blow-up of $C$ at its singular points. However, for our enumerative purposes we need to understand the embedding of $C(n)$ in the Semple bundle variety $F(n)$.

It is sometimes convenient to regard a point of $F(n)$ as an equivalence class of irreducible germs of plane curves. Thus we sometimes say that the irreducible germ of a curve $C$ represents the point $p_n \in F(n)$, meaning that the lift of $C$ passes through $p_n$ above the closed point of the germ.

To illustrate the transparent nature of calculations in these Semple bundles, let us consider a reduced plane curve $C$ defined, in an affine chart with coordinates $x$ and $y$, by $f(x, y) = 0$. Then one can show that on $F(n)$ there is a primary chart isomorphic to affine $(n + 2)$-space, with coordinates $x, y, y', y'', y^{(3)}, \ldots, y^{(n)}$, and that the ideal defining the $n$th lift $C(n)$ includes the sequence of functions $f, f', f'', f^{(3)}, \ldots, f^{(n)}$ obtained by repeated implicit differentiation with respect to $x$. (There is a second primary chart in which the roles of $x$ and $y$ are reversed.)

In general these functions may not generate the ideal defining the lift. Consider, for example, the cuspidal cubic $y^2 = x^3$. Then the first lift is defined in the primary chart by

$$y^2 = x^3, \quad 2yy' = 3x^2, \quad x = \frac{4}{9}(y')^2, \quad \text{and} \quad y = \frac{8}{27}(y'^3). \quad (1)$$

The variety defined by just the first two equations contains a spurious component over the origin. (To obtain the third equation, we square both sides of the second equation, then use the first equation to replace $y^2$; since the curve is singular at the origin it is then legal to cancel $x^3$ from both sides. The fourth equation is obtained in a similar fashion. Clearly the third and fourth equations define an irreducible curve in $\mathbb{A}^3$, and the first two equations are redundant.)

If we continue this example one step further we see that the lift of a curve may leave the primary charts. Indeed, if we implicitly differentiate the third equation of (1), we obtain this equation for $C(2)$:

$$1 = \frac{8}{9}y'y''.$$  

Since the unique point $p_1$ of $C(1)$ over the origin is $(x, y, y') = (0, 0, 0)$, and since $C(2)$ must be complete, the unique point $p_2$ of $C(2)$ over $p_1$ must be the one point over $p_1$ missed by the primary chart, the point which, intuitively, represents infinite curvature. There is one such point on each fiber of $F(2)$ over $F(1)$; taken together, these points form a section of the $\mathbb{P}^1$-bundle which we call the divisor at infinity and denote by $I_2$. 

Here is another characterization of the divisor at infinity. The derivative of $f_1$ at $p_1$,

$$df_1 : T_{p_1}F(1) \to T_{f_1(p_1)}F(0),$$

maps a 3-dimensional vector space to a 2-dimensional vector space. The kernel is one-dimensional, so there is a unique direction annihilated by $df_1$. This direction is represented by a point on the divisor at infinity. More generally, the derivative of $f_n$ at a point $p_n$,

$$df_n : T_{p_n}F(n) \to T_{f_n(p_n)}F(n - 1),$$

likewise has a one-dimensional kernel. Hence there is a divisor at infinity $I_{n+1}$ on $F(n+1)$; intuitively, a point on this divisor represents (in addition to certain lower-order data) infinite curvilinear data of order $n+1$. To avoid clumsy notation, we will also denote by $I_{n+1}$ the pullback of this divisor to any Semple bundle variety above $F(n+1)$ in the tower, and continue to call the pullback a divisor at infinity. Note that the divisors at infinity have normal crossings; in particular, the intersection of two such divisors has codimension two.

The simplest sort of chart meeting a divisor at infinity is a secondary chart. To specify such a chart on $F(n)$—over the affine chart of $\mathbb{P}^2$ with coordinates $x$ and $y$—choose an integer $j$ between 2 and $n$. Then there is a chart isomorphic to affine $(n+2)$-space, with coordinates $x, y, y', y'', \ldots, y^{(j-1)}, x', x'', \ldots, x^{(n-j+1)}$. For $i = 1, \ldots, j - 1$, the coordinate $y^{(i)}$ measures $dy^{(i-1)}/dx$, a ratio of differentials of coordinates on $F(i - 1)$; for $i = 1, \ldots, n - j + 1$, the coordinate $x^{(i)}$ measures $dx^{(i-1)}/dy^{(j-1)}$, a ratio of differentials of coordinates on $F(j + i - 1)$. (See [5] for an explanation of a full system of charts.) If $C$ is a reduced plane curve defined by $f(x, y) = 0$, then the ideal defining the $n$th lift $C(n)$ includes a sequence of functions obtained by repeated implicit differentiation. In the first $j - 1$ of these differentiations we treat $x$ as the independent variable, and denote the derivative $dy/dx$ of $y$ by $y'$, the derivative $dy'/dx$ of $y'$ by $y''$, etc. In the remaining $n - j + 1$ differentiations we treat $y^{(j-1)}$ as the independent variable, and denote the derivative $dx/dy^{(j-1)}$ of $x$ by $x'$, the derivative $dx'/dy^{(j-1)}$ of $x'$ by $x''$, etc. The derivative $dy^{(i)}/dy^{(j-1)}$ of $y^{(i)}$ ($0 \leq i < j - 1$) is obtained by the chain rule:

$$\frac{dy^{(i)}}{dy^{(j-1)}} = \frac{dy^{(i)}}{dx} \frac{dx}{dy^{(j-1)}} = y^{(i+1)}x'.$$

In other words, the defining ideal for $C(n)$ includes the functions
where \(P\) and \(Q\) are the differential operators

\[
P = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots + y^{(j-1)} \frac{\partial}{\partial y^{(j-2)}}
\]

\[
Q = x' P + \frac{\partial}{\partial y^{(j-1)}} + x^{(2)} \frac{\partial}{\partial x} + x^{(3)} \frac{\partial}{\partial x'} + \cdots + x^{(n-j+1)} \frac{\partial}{\partial x^{(n-j)}}.
\]

Away from the singularities of \(C\), the sequence of functions in (2) generates the ideal, but additional generators will be needed to eliminate spurious components over the singularities. In this \(j\)th secondary chart the divisor at infinity \(I_j\) is defined by the vanishing of \(x'\).

To illustrate the rules for calculating in secondary charts, we continue our example of the cuspidal cubic \(y^2 = x^3\). Implicitly differentiating the third equation of (1) with respect to \(y'\), we obtain

\[
x' = \frac{8}{9} y'.
\]

Here \(x'\) is the coordinate of the secondary chart measuring \(dx/dy'\). (In other words, \(x'\) is the reciprocal of the ordinary second derivative coordinate \(dy'/dx\).) From the equations of (1) one easily sees that the first lift is tangent to the fiber of \(F(1)\) over the origin. Note that, as expected, the second lift hits the divisor at infinity \(I_2\) over the origin.

We note for future reference that, taken together, the primary and secondary charts cover all of the Semple bundle variety except for intersections of two or more divisors at infinity.

By definition \(F(n + 1)\) is a subvariety of \(PTF(n)\), the total space of the projectivized tangent bundle of \(F(n)\). If \(n \geq 2\), then \(PTI_n\), the total space of the projectivized tangent bundle of the divisor at infinity, is likewise a subvariety of \(PTF(n)\); its codimension is 2. Now one can easily verify, e.g., by a calculation in local coordinates, that \(F(n + 1)\) and \(PTI_n\) are transverse. Hence their intersection is a codimension 2 subvariety of \(F(n + 1)\) which we call the locus of tangency to \(I_n\). A point of this locus represents a point of \(I_n\) together with a tangent direction belonging to the fiber of the focal plane at that point. Again to avoid clumsy terminology, we continue to speak of the "locus of tangency" when we ought to say "the pullback through the Semple bundle tower of the locus of tangency". The \(j\)th secondary chart on \(F(n)\) meets only one such locus, namely the locus of tangency to \(I_j\); this locus is defined by the vanishing of
both \( x' \) and \( x'' \). (But if \( j = n \), there is no such locus meeting the secondary chart at all.)

Suppose now that \( \mathcal{X} \) is a family of plane curves, and that its general member is reduced. For each reduced member there is a (rational) lifting map to \( F(n) \); these maps fit together to form a rational map \( \lambda: \mathcal{X} \to F(n) \). We call the closure of \( \lambda(\mathcal{X}) \) the lift of the family, and denote it by \( \mathcal{X}(n) \). By definition, \( \mathcal{X}(n) \) is the union of the graph of \( \lambda \) and a closed subvariety \( \mathcal{I}(n) \), each point of which lies over a singular or nonreduced point of the corresponding member of \( \mathcal{X} \).

2. Intersection rings

Since the variety \( F(n) \) is the projectivization of the rank 2 bundle \( \mathcal{F}_{n-1} \) over \( F(n-1) \), the intersection rings may be determined inductively from standard theory. Specifically, the Chow ring \( A^*(F(n)) \) is an \( A^*(F(n-1)) \)-algebra generated by the tautological class \( \phi_n := c_1(\mathcal{O}_{F(n)}(1)) \), which satisfies a single quadratic relation
\[
\phi_n^2 + c_1(\mathcal{F}_{n-1})\phi_n + c_2(\mathcal{F}_{n-1}) = 0. \tag{4}
\]
(Note: Here and in the sequel, we omit pullbacks of classes when convenient and when no confusion should result.) The base variety \( F(0) = \mathbb{P}^2 \), whose Chow ring is generated by the hyperplane class \( h \), subject to the relation \( h^3 = 0 \). Thus \( A^*(F(n)) \) is generated by \( h, \phi_1, \ldots, \phi_n \) subject to the relations mentioned above.

For the purpose of studying contact between plane curves, however, this description is not desirable. Instead, we provide an equivalent formulation using more geometric classes. In particular, we will eliminate the \( \phi_n \)'s in favor of \( \tilde{h} \), the dual hyperplane class on \( \tilde{\mathbb{P}}^2 \), and the classes \( i_j \) of the divisors at infinity \( I_j \).

**Theorem 1.** For \( n \geq 1 \), the Chow ring \( A^*(F(n)) \) is generated by \( h, \tilde{h}, i_2, \ldots, i_n \) subject to the relations
\[
h^3 = 0, \quad \tilde{h}^3 = 0, \quad h^2 - h\tilde{h} + \tilde{h}^2 = 0,
\]
and, for \( k = 2, \ldots, n \),
\[
i_k^2 = ((2k - 1)h - (k + 1)\tilde{h} - ki_2 - (k - 1)i_3 - \cdots - 3i_{k-1})i_k.
\]

**Proof.** To begin, \( F(1) \) is the incidence correspondence of \( \mathbb{P}^2 \). Thus \( F(1) \subset \mathbb{P}^2 \times \tilde{\mathbb{P}}^2 \) and it is well known that
\[
A^*(F(1)) \cong \frac{\mathbb{Z}[h, \tilde{h}]}{(h^3, \tilde{h}^3, h^2 - h\tilde{h} + \tilde{h}^2)}.
\]
It follows from a change of basis calculation that

\[ \phi_1 = \tilde{h} - 2h. \]  

(5)

To finish the proof, we need to rewrite (4) without using any \( \phi_k \)'s. To do this, we note first that the focal plane bundle \( F_k, k \geq 1 \), fits into the following commutative diagram of exact sequences:

\[
\begin{array}{cccccc}
0 & \longrightarrow & T_{F(k)/F(k-1)} & \longrightarrow & \mathcal{F}_k & \longrightarrow & \mathcal{O}_{F(k)}(-1) & \longrightarrow & 0 \\
& & \text{=} & & \text{=} & & \text{=} & & \\
0 & \longrightarrow & T_{F(k)/F(k-1)} & \longrightarrow & T_{F(k)} & \longrightarrow & f_k^*T_{F(k-1)} & \longrightarrow & 0
\end{array}
\]

(6)

(This is the dual of diagram 1 of [6].) Let

\[ \sigma: \mathcal{O}_{F(k+1)}(-1) \to f_{k+1}^*\mathcal{F}_k \to f_{k+1}^*\mathcal{O}_{F(k)}(-1) \]

be the composite of the tautological map and the pullback to \( F(k+1) \) of the map in the top row of diagram (6). Then \( \sigma \) has zero locus equal to \( PT_{F(k)/F(k-1)} = I_{k+1} \). In addition, \( \sigma \) defines a section of \( \text{Hom}(\mathcal{O}_{F(k+1)}(-1), f_{k+1}^*\mathcal{O}_{F(k)}(-1)) \). Hence

\[
i_{k+1} = [PT_{F(k)/F(k-1)}]
= f_{k+1}^*c_1(\mathcal{O}_{F(k)}(-1)) - c_1(\mathcal{O}_{F(k+1)}(-1))
= \phi_{k+1} - \phi_k.
\]

(7)

Now the Euler sequence

\[ 0 \to \mathcal{O}_{F(k)} \to f_k^*\mathcal{F}_{k-1} \otimes \mathcal{O}_{F(k)}(1) \to T_{F(k)/F(k-1)} \to 0 \]

and the top sequence of (6) together imply that, for \( k \geq 1 \),

\[
c_1(\mathcal{F}_k) = c_1(\mathcal{F}_{k-1}) + \phi_k
\]

\[
c_2(\mathcal{F}_k) = 2c_2(\mathcal{F}_{k-1}) + c_1(\mathcal{F}_{k-1})\phi_k.
\]

(8)

Using (7) to substitute for \( \phi_{k+1} \) in (4) and simplifying, one finds

\[
i_{k+1}\phi_{k+1} + 3\phi_k + c_1(\mathcal{F}_{k-1}) = 0.
\]

(9)

Note that the inductive formula for \( c_1(\mathcal{F}_k) \) in (8) yields the closed-form formula
Hence (9) can be rewritten as

\[ i_{k+1}(i_{k+1} + 3h + \phi_1 + \cdots + \phi_{k-1} + 3\phi_k) = 0. \]

In view of (5) and (7), this is equivalent to, for \( k > 1 \),

\[ i_{k+1}^2 = ((2k + 1)h - (k + 2)\tilde{h} - (k + 1)i_2 - ki_3 - \cdots - 3i_k)i_{k+1}. \]

In [6] we studied the action of the projective general linear group \( PGL(2) \) on the Semple bundles induced from the \( PGL(2) \)-action on \( \mathbb{P}^2 \). Since this action preserves incidence in \( \mathbb{P}^2 \), there is a special orbit \( Z_k \) on each \( F(k) \), \( k \geq 1 \), that is isomorphic to \( F(1) \) and is represented by the germ of a line. (The intuition is that \( Z_k \) measures "zero data" of orders 2 through \( k \).) Since \( Z_k \) is represented by the germ of a smooth curve, it must be disjoint from the divisors at infinity \( I_j \) for all \( j \leq k \). Thus if \( z_k := [Z_k] \), we have

\[ i_j \cdot z_k = 0, \quad j \leq k. \]

Also each \( Z_k \) is a section of the \( \mathbb{P}^1 \)-bundle obtained by restricting the Semple bundle to \( Z_{k-1} \). Hence the intersection of \( Z_k \) with the fiber of \( F(k) \) over a point of \( F(1) \) has degree 1. Thus, for \( k \geq 1 \),

\[ \int_{F(k)} h^2\tilde{h}z_k = \int_{F(k)} h\tilde{h}^2z_k = 1. \]

In a similar manner, it follows that

\[ \int_{F(k)} h^2\tilde{h}z_{j-1}i_j \cdots i_k = \int_{F(k)} h\tilde{h}^2z_{j-1}i_j \cdots i_k = 1, \quad 2 \leq j \leq k. \]

(Note that \( Z_1 \) is all of \( F(1) \), so that \( z_1 = 1 \); in this case the claims above are vacuous or obvious.)

In view of these remarks and the calculation of the Chow ring of \( F(n) \) above, it is not difficult to deduce the following matrix for the intersection
pairing of $A^1(F(n))$ with $A^{n+1}(F(n))$, in which $f_0 = f_1 = 1$ and $f_j$ denotes the $j$th Fibonacci number:

$$
\begin{array}{cccccc}
& h^2i_n & h^2i_3 \cdots i_n & h^2h_i \cdots i_n & h^2h_3i_4 \cdots i_n & h^2h_5i_5 \cdots i_n & \cdots & h^2h_{n-1} \\
\hline
h & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\tilde{h} & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
i_2 & 0 & -3 & 1 & 0 & 0 & \cdots & 0 \\
i_3 & 0 & 5 & -3 & 1 & 0 & \cdots & 0 \\
i_4 & 0 & -8 & 5 & -3 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
i_n & 0 & (-1)^{n+1}f_{n+1} & (-1)^nf_n & (-1)^{n-1}f_{n-1} & (-1)^{n-2}f_{n-2} & \cdots & 1.
\end{array}
$$

3. The proto-contact formula

We associate to each reduced plane curve $C$ a sequence $d, d, \kappa_2, \kappa_3, \ldots$ of (higher-order) characteristic numbers, beginning with the degree and class. Each of the other numbers is defined by

$$
\kappa_j := \int i_j \cap [C(j)],
$$

i.e., as the intersection number of a lift of the curve with a divisor at infinity; this intersection is defined on the $j$th Semple bundle variety, or any Semple bundle variety above it in the tower. Since the lift of a curve cannot hit a divisor at infinity over a nonsingular point, these characteristic numbers count certain sorts of singularities. A point of $C$ over which $C(j)$ meets $I_j$ will be called a $j$th-order cusp, and the corresponding characteristic number $\kappa_j$ will be called the number of $j$th-order cusps on $C$. For example, the singularity at the origin of $y^2 = x^{2j-1}$ is a $j$th-order cusp and contributes 1 to $\kappa_j$. Note that $\kappa_j$ may count with multiplicities; for example, a curve may have a singularity at which distinct branches each contribute to $\kappa_j$. One can easily show that repeated lifting will desingularize a specified curve, hence that only finitely many characteristic numbers are non-zero.

The $n$th contact module of a reduced plane curve $C$ is a certain element of the polynomial algebra over the integers in the following indeterminates:

$$
\begin{align*}
\Lambda_j, & \quad 0 \leq j \\
\Pi_j, & \quad 1 \leq j \\
\Gamma_j, & \quad 2 \leq k \leq j.
\end{align*}
$$
Specifically, it is

\[ m_n(C) := d\Lambda_n + d\Pi_n + (3\tilde{d} + \kappa_2)\Gamma_n^2 + (4\tilde{d} + 3\kappa_2 + \kappa_3)\Gamma_n^3 + (5\tilde{d} + 4\kappa_2 + 3\kappa_3 + \kappa_4)\Gamma_n^4 + \cdots + ((n + 1)\tilde{d} + n\kappa_2 + (n - 1)\kappa_3 + \cdots + 3\kappa_{n-1} + \kappa_n)\Gamma_n^n. \]

In particular the 0th and 1st contact modules are

\[ m_0(C) := d\Lambda_0, \quad \text{and} \quad m_1(C) := d\Lambda_1 + d\Pi_1. \]

If we assign weight \( n \) to each of \( \Lambda_n, \Pi_n, \) and \( \Gamma_n^k \), then the \( n \)th module is homogeneous of weight \( n \).

Now suppose that \( C_1, C_2, \ldots, C_p \) are reduced plane curves. Suppose that \( \mathcal{X} \) is a family of plane curves over \( S \), a parameter space of dimension \( s \); suppose that the general member of the family is reduced. Suppose that \( n_1, n_2, \ldots, n_p \) are specified positive integers, with sum equal to \( s \). Let \( F(n) \) be the product of Semple bundle varieties \( F(n_1) \times F(n_2) \times \cdots \times F(n_p) \), and let \( \pi_1, \pi_2, \ldots, \pi_p \) be the various projections to the factors. The fiber product

\[ \mathcal{X}_S(n) := \mathcal{X}(n_1) \times_S \mathcal{X}(n_2) \times_S \cdots \times_S \mathcal{X}(n_p) \]

is a subvariety of \( F(n) \times S \); its fiber over a point \( s \) of \( S \) is the product of \( p \) lifts of the curve \( X_s \) in the family. Let \( \sigma \) denote the projection of \( F(n) \times S \) onto its first factor.

\[ C_j(n_j) \]

\[ \mathcal{X}(n_1) \times_S \mathcal{X}(n_2) \times_S \cdots \times_S \mathcal{X}(n_p) = \mathcal{X}_S(n) \rightarrow F(n) \times S \xrightarrow{\sigma} F(n) \xrightarrow{\pi_j} F(n_j) \]

We define the proto-contact number of type \( (n_1, n_2, \ldots, n_p) \) by

\[ I := \int_{F(n)} \pi_1^*[C_1(n_1)] \cdot \pi_2^*[C_2(n_2)] \cdot \cdots \cdot \pi_p^*[C_p(n_p)] \cap \sigma_*[\mathcal{X}_S(n)]. \]

**Theorem 2.** The proto-contact number is obtained by multiplying the contact modules \( m_{n_1}(C_1), m_{n_2}(C_2), \ldots, m_{n_p}(C_p) \), evaluating each monomial in the resulting product, and performing the indicated arithmetic. Each monomial in the product is of weight \( s \) and of the form \( y_1 y_2 \cdots y_p \), where each \( y_j \) is either \( \Lambda_{n_j} \) or \( \Pi_{n_j} \) or...
some $\Gamma^k_{nj}$, to evaluate this monomial means to replace it by

$$\int_{F(n)} \pi_1^{\ast}(\tilde{y}_1) \cdot \pi_2^{\ast}(\tilde{y}_2) \cdot \ldots \cdot \pi_p^{\ast}(\tilde{y}_p) \cap \sigma_*[\mathcal{X}_S(n)],$$

(12)

where

$\tilde{y}_j := \begin{cases} h & \text{if } y_j = \Lambda_0, \\ h^j z_{nj} & \text{if } y_j = \Lambda_{nj}, \quad nj > 0, \\ h^j i_{i_1} \cdots i_{i_j} & \text{if } y_j = \Pi_{nj}, \\ h^j h_{z_{k-1}} i_{i_{k+1}} \cdots i_{i_j} & \text{if } y_j = \Gamma_{nj}. \end{cases}$

\textbf{Proof.} Our discussion in Section 2 shows that the set

$\{h^j z_n, h^j i_{i_1} \cdots i_{i_j}, h^j h z_{i_4} \cdots i_{i_j}, h^j h z_{i_5} \cdots i_{i_j}, h^j h z_{i_6} \cdots i_{i_j}, h^j h z_{n-1}\}$

forms a basis for $A^{n+1}(F(n))$, and that the dual basis for $A^1(F(n))$ is

$\{h, h, 3h + i_2, 4h + 3i_2 + 3i_3, 5h + 4i_2 + 3i_3 + i_4, \ldots, (n+1)h + ni_2 + (n-1)i_3 + \cdots + 3i_{n-1} + i_n\}.$

(For $n = 0$ the singleton $\{h\}$ is a self-dual basis.) The characteristic numbers of a reduced plane curve $C$ are determined by

$$d = \int_{F(n)} h \cap [C] = \int_{F(n)} h \cap [C(n)],$$

$$\tilde{d} = \int_{F(n)} \tilde{h} \cap [C(1)] = \int_{F(n)} \tilde{h} \cap [C(n)],$$

$$\kappa_j = \int_{F(n)} i_j \cap [C(j)] = \int_{F(n)} i_j \cap [C(n)] \quad (j \leq n).$$

Hence the rational equivalence class of the $n$th lift of $C$ is given by

$$[C(n)] = dh^2 z_n + \tilde{d}h^2 i_{i_1} \cdots i_{i_j}$$

$$+ (3\tilde{d} + \kappa_2)h^2 h z_{i_4} \cdots i_{i_j}$$

$$+ (4\tilde{d} + 3\kappa_2 + \kappa_3)h^2 h z_{i_5} i_{i_6} \cdots i_{i_j}$$

$$+ (5\tilde{d} + 4\kappa_2 + 3\kappa_3 + \kappa_4)h^2 h z_{i_6} i_{i_7} \cdots i_{i_j}$$

$$+ \cdots$$

$$+ ((n + 1)\tilde{d} + n\kappa_2 + (n - 1)\kappa_3 + \cdots + 3\kappa_{n-1} + \kappa_n)h^2 h z_{n-1}.$$
The enumeration of simultaneous higher-order contacts

(For \( n = 0 \), the class of \( C(0) = C \) is of course \( dh \).) The theorem follows immediately from this formula.

4. The simultaneous contact formula

Suppose that \( C \) and \( X \) are reduced plane curves. A contact (or honest contact) of order \( o \) between them is a point \( x \in C(o - 1) \cap X(o - 1) \) whose image in \( \mathbb{P}^2 \) is a nonsingular point on each curve. Note that, for nonsingular curve germs, the following statements are equivalent:

- There is a contact of order \( o \) between them.
- In appropriate local coordinates, the Taylor expansions agree up to order \( o - 1 \).
- The intersection number is at least \( o \).

We will call a point \( x \in C(o - 1) \cap X(o - 1) \) whose image in \( \mathbb{P}^2 \) is a singular point on \( C \) or \( X \) a false contact.

Next suppose that \( C_1, C_2, \ldots, C_p \) are reduced plane curves. A simultaneous contact of order \( (o_1, o_2, \ldots, o_p) \) between \( X \) and these \( p \) curves is a \( p \)-tuple \((x_1, x_2, \ldots, x_p)\) in which the point \( x_1 \in F(o_1 - 1) \) is a contact of order \( o_1 \) between \( X \) and \( C_1 \), the point \( x_2 \in F(o_2 - 1) \) is a contact of order \( o_2 \) between \( X \) and \( C_2 \), etc.

We say that a plane curve \( C \) has a profound cusp if, for some \( j \), the \( j \)th lift \( C(j) \) meets the intersection of \( I_j \) and the pullback of another divisor at infinity; i.e., \( C \) has a profound cusp at \( P \) if some branch of \( C \) has simultaneously a \( j \)th-order cusp and a cusp of lower order. For example, \( y^3 = x^5 \) has a profound cusp; its lifts meet both \( I_2 \) and \( I_3 \). We say that \( C \) has a flat cusp if the \( j \)th lift of \( C \) meets the locus of tangency to the divisor at infinity \( I_{j-1} \). If \( C(j - 1) \) happens to be nonsingular over \( P \), then \( C \) has a flat cusp if and only if \( C(j - 1) \) is tangent to \( I_{j-1} \). For example, the second lift of \( y^3 = x^4 \) is tangent to \( I_2 \); hence this curve has a flat cusp at the origin.

A family of plane curves of degree \( d \) over a parameter space \( S \) determines, and is determined by, a morphism from \( S \) to the projective space \( \mathbb{P}^{N(d)} \) parametrizing such curves, where

\[
N(d) = \binom{d + 2}{2} - 1.
\]

We call such a family generic if it is obtained from one particular specified family by composing the morphism from \( S \) to \( \mathbb{P}^{N(d)} \) with a generic motion of the projective space. In other words, "proposition \( \mathcal{P} \) is valid for a generic
family" means that if \( \mathcal{X} \) is a family of curves over \( S \) determined by \( \sigma: S \to \mathbb{P}^{N(d)} \), then, for all \( \gamma \) in some Zariski open dense subset of the projective linear group \( PGL(N(d)) \), proposition \( \mathcal{P} \) is valid for the family determined by \( \gamma \circ \sigma \).

**THEOREM 3.** Suppose that \( \mathcal{X} \) is a generic family of degree \( d \) plane curves over \( S \), a parameter space of dimension \( s = o_1 + o_2 + \cdots + o_p - p \). Suppose that each \( o_i > 1 \), and that \( d + 1 \geq \sum_{i=1}^{p} o_i \). Suppose that \( C_1, C_2, \ldots, C_p \) are reduced plane curves, none of which has a profound cusp or a flat cusp. Suppose that these curves have only pairwise transverse intersections. Then the number of simultaneous contacts of order \( (o_1, o_2, \ldots, o_p) \) between some reduced member of \( \mathcal{X} \) and \( C_1, C_2, \ldots, C_p \) is the proto-contact number of type \( (o_1 - 1, o_2 - 1, \ldots, o_p - 1) \).

We have assumed that each \( o_i > 1 \) only to avoid a clumsy exposition. For remarks concerning the omitted cases, see Section 6(c).

To prove Theorem 3, we begin in Lemmas A and B by analyzing a lift of the universal family \( \mathcal{C} \) of degree \( d \) plane curves over \( \mathbb{P}^{N(d)} \); a point of \( \mathcal{C} \) is an ordered pair \((P, C)\) consisting of a degree \( d \) plane curve \( C \) and a point \( P \in C \). Over the affine chart with coordinates \( x \) and \( y \), the hypersurface \( \mathcal{C} \) is defined in \( \mathbb{A}^2 \times \mathbb{P}^{N(d)} \) by

\[
   f(x, y) = \sum_{u + v \leq d} a_{uv} x^u y^v = 0. \tag{13}
\]

Since we wish to study simultaneous contacts, in Lemmas C and D we analyze the fiber product of several lifts of the universal family. Finally, in order to ultimately rule out the possibility that our proto-contact formula counts false contacts, in Lemmas E and F we analyze the higher-order data carried by a singular point of a curve. Our analysis uses, *faute de mieux*, explicit and detailed calculations.

**LEMMA A.** Suppose that \( d \geq n \).

- Except possibly above intersections of two or more divisors at infinity and above the loci of tangency to divisors at infinity, the \( n \)-th lift \( \mathcal{C}(n) \) is smooth over \( F(n) \).
- Over the primary chart with coordinates \( x, y, y', y'', \ldots, y^{(n)} \), the \( n \)-th lift \( \mathcal{C}(n) \) is defined in \( \mathbb{A}^{n+2} \times \mathbb{P}^{N(d)} \) by an ideal generated by the function \( f \) of (13) and the functions \( f', f'', f^{(3)}, \ldots, f^{(n)} \) obtained by repeated implicit differentiation with respect to \( x \). Over each point of the primary chart the matrix of this system of linear equations has rank \( n + 1 \).
- Over a point on the exceptional divisor of the secondary chart with coordinates \( x, y, y', y'', \ldots, y^{(j-1)}, x', x'', \ldots, x^{(n-j+1)} \) (i.e., a point at which \( x' \)
vanishes) the nth lift $\mathcal{C}(n)$ is defined by an ideal generated by the function $f$ of (13) and the sequence of functions (2) obtained by the procedure explained in Section 1. Over this point the matrix of this system of linear equations has rank $n + 1$.

**Proof.** Recall that, taken together, the primary charts cover all of $F(n)$ except divisors at infinity, and that the primary and secondary charts cover all of $F(n)$ except intersections of two or more divisors at infinity. And note that the codimension of $\mathcal{C}(n)$ in $\mathbb{A}^{n+2} \times \mathbb{P}^{N(d)}$ is $n + 1$. Hence the first claim follows from the other two.

Consider the primary chart. Let $M$ denote the $(n + 1) \times (N(d) + 1)$ matrix of partial derivatives of the functions $f, f', f'', f^{(3)}, \ldots, f^{(n)}$ with respect to each of the $a_{uv}$. (Equivalently, since all these functions are linear in the $a_{uv}$'s, $M$ is the matrix of the system of linear equations.) If the rank of $M$ at a point of $\mathcal{C}(n)$ is $n + 1$, then at this point $f, f', f'', f^{(3)}, \ldots, f^{(n)}$ generate the defining ideal, and the projection to $F(n)$ is smooth. Each column of $M$ begins with a monomial in $x$ and $y$, and the other entries are obtained by repeated implicit differentiation:

To see that $M$ has rank $n + 1$ at every point, consider the square submatrix consisting of the partial derivatives of $f, f', f'', \ldots, f^{(n)}$ with respect to $a_{00}, a_{10}, a_{20}, \ldots, a_{n0}$:

$$
\begin{bmatrix}
\frac{\partial f}{\partial a_{uv}} \\
\frac{\partial f'}{\partial a_{uv}} \\
\vdots \\
\frac{\partial f^{(n)}}{\partial a_{uv}}
\end{bmatrix} =
\begin{bmatrix}
x^u y^v \\
x^u y^v \\
\vdots \\
x^u y^v
\end{bmatrix}
\begin{bmatrix}
x^u y^v \\
d(x^u y^v) \\
\vdots \\
d(x^u y^v)
\end{bmatrix}.
$$

To see that $M$ has rank $n + 1$ at every point, consider the square submatrix consisting of the partial derivatives of $f, f', f'', \ldots, f^{(n)}$ with respect to $a_{00}, a_{10}, a_{20}, \ldots, a_{n0}$:

$$
\begin{bmatrix}
1 & x & x^2 & \ldots & x^n \\
0 & 1 & 2x & \ldots & nx^{n-1} \\
0 & 0 & 2 & \ldots & n(n-1)x^{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & n!
\end{bmatrix}
$$

(14)

Now consider the secondary chart. Let $M$ denote the $(n + 1) \times (N(d) + 1)$ matrix of partial derivatives of the functions in (2) with respect to each of the $a_{uv}$. (Once again we may interpret $M$ as the matrix of a system of linear
equations.) Each column of $M$ begins with a monomial in $x$ and $y$, and the 
other entries are obtained by repeated implicit differentiation:

\[
\begin{bmatrix}
x^m y^n \\
P(x^m y^n) \\
\vdots \\
P^{j-1}(x^m y^n) \\
Q P^{j-1}(x^m y^n) \\
\vdots \\
Q^{n-j+1} P^{j-1}(x^m y^n)
\end{bmatrix}.
\]

(15)

If the rank of $M$ at a point of $\mathcal{C}(n)$ is $n + 1$, then at this point the functions in
(2) generate the defining ideal, and the projection to $F(n)$ is smooth.

To complete our analysis of this matrix, we need the following formulas concerning the polynomial ring in the variables $x$, $y$, $y'$, $y''$, ..., $y^{(j-1)}$, $x'$, $x''$, ..., $x^{(n-j+1)}$.

**Lemma B.** (a) For each integer $k \geq j$,

\[
Q^i P^{j-1}(x^k) \equiv \begin{cases} 
0 & \text{if } i < 2(k - j + 1) \\
\text{a positive multiple of } (x^n)^{k-j+1} & \text{if } i = 2(k - j + 1)
\end{cases}
\]

modulo the ideal generated by $x$ and $x'$.

(b) For each pair of positive integers $i$ and $m$,

\[
[Q^i(x^m)]_{x=0} = Q[Q^{i-1}(x^m)]_{x=0} + mx'[Q^{i-1}(x^{m-1})]_{x=0},
\]

where $[ \ ]_{x=0}$ indicates evaluation.

(c) For each nonnegative integer $h \leq j - 1$ and each nonnegative integer $i$,

\[
Q^i(y^{(h)}) = \sum_{b=h}^{j-1} \frac{1}{(b-h)!} y^{(b)}[Q^i(x^{b-h})]_{x=0}
\quad + \quad \text{a polynomial in } x, x', \ldots, x^{(n-j+1)}.
\]

(d) For each nonnegative integer $h \leq j - 1$,

\[
Q^i P^h(y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^i P^h(x^b) \equiv \begin{cases} 
0 & \text{if } i < 2(j - 1 - h) + 1 \\
\text{a positive multiple of } (x^n)^{j-1-h} & \text{if } i = 2(j - 1 - h) + 1
\end{cases}
\]
modulo the ideal generated by $x$ and $x'$.

(e) For each polynomial $\phi$ in $x$ and $y$, for each $h \geq 1$, and for each $i \geq 2$,

$$Q^i P^h(x \cdot \phi) \equiv hQ^i P^{h-1}(\phi) + \sum_{a=0}^{i-2} \binom{i}{a} Q^{a} P^{h}(\phi) \cdot x^{(i-a)}$$

modulo the ideal generated by $x$ and $x'$.

(f) For each nonnegative integer $k$ and each nonnegative integer $h \leq j - 1$,

$$Q^i P^h(x^k y) = \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^j P^h(x^{k+b})$$

$$\equiv \begin{cases} 
0 & \text{if } i < 2(k+j-1-h)+1 \\
\text{a positive multiple of } (x^r)^{k+j-1-h} & \text{if } i = 2(k+j-1-h)+1
\end{cases}$$

modulo the ideal generated by $x$ and $x'$.

(g) For each nonnegative integer $k$,

$$Q^i P^{j-1}(x^k y) = \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^j P^{j-1}(x^{k+b})$$

$$\equiv \begin{cases} 
0 & \text{if } i < 2k+1 \\
\text{a positive multiple of } (x^r)^k & \text{if } i = 2k+1
\end{cases}$$

modulo the ideal generated by $x$ and $x'$.

Proof. To prove statement (a), declare the weight of $x^{(t)}$ to be $2 - t$. Then $Q$ decreases the weight of a polynomial in $x$, $x'$, $\ldots$, $x^{(n-j+1)}$ by 1, so that

$$Q^i P^{j-1}(x^k) = \frac{k!}{(k-j+1)!} Q^i (x^{k-j+1})$$

is a polynomial of degree $k-j+1$ and weight $2(k-j+1)-i$. If $i < 2(k-j+1)$ then the weight is positive, so each term involves either $x$ or $x'$. If $i = 2(k-j+1)$ then the weight is zero, so the only term not involving $x$ or $x'$ is a multiple of $(x^r)^{k-j+1}$. Clearly the coefficient of this term is positive.

Observe that

$$Q^i (x^r) = \sum x^{(e_1)} x^{(e_2)} \ldots x^{(e_m)},$$

where the sum is over all functions $s$ from $\{1, \ldots, i\}$ to $\{1, \ldots, m\}$, and $e_k$ is the
number of elements of \( \{1, \ldots, i\} \) for which the value of \( s \) is \( k \). (The function can be regarded as an instruction to differentiate \( x \cdot x \cdot x \cdots x \) first at factor number \( s(1) \), then at factor number \( s(2) \), etc., thus obtaining one of the \( m' \) terms created by \( i \) applications of the product rule.) The left side of formula (b) is the same sum, taken over the set of surjections. The first term on the right side of (b) is again this sum, now taken over the set of those \( s \) for which the restriction to \( \{1, \ldots, i - 1\} \) is surjective. The second term is the same sum, but taken over the set of those surjective \( s \) for which the restriction to \( \{1, \ldots, i - 1\} \) is not surjective. From these descriptions the equality is clear. (This is essentially a proof of the basic recurrence formula for Stirling numbers of the second kind; cf. formula (23), p. 33 of [28].)

We prove (c) by induction on \( i \). When \( i = 0 \) each term of the sum except the first is divisible by \( x \); in this case the required polynomial in \( x, x', x'', \ldots, x^{(a-j+1)} \) is zero. For the inductive step, apply \( Q \) to both sides of the equation to obtain

\[
Q^{i+1}(y^{(b)}) = \sum_{b=h}^{j-1} \frac{1}{(b-h)!} Q(y^{(b)}[Q^i(x^{b-h})])_{x=0}
+ \sum_{b=h}^{j-1} \frac{1}{(b-h)!} y^{(b)}[Q^i(x^{b-h})]_{x=0}
+ \text{a polynomial in } x, x', \ldots, x^{(a-j+1)}.
\]

(Note that \( Q(x^{(a-j+1)}) = 0 \) by definition of the differential operator \( Q \) in (3).) In the first sum on the right, replace \( b \) by \( c - 1 \); in all terms except the last replace \( Q(y^{(b)}) \) by its chain rule equivalent \( y^{(c)}x' \); and absorb the last term into the polynomial in \( x, x', \ldots, x^{(a-j+1)} \). In each term of the second sum, use formula (b). With these manipulations, we find that

\[
Q^{i+1}(y^{(b)}) = \sum_{c=h+1}^{j} \frac{1}{(c-h-1)!} y^{(c)}x'[Q^i(x^{c-h-1})]_{x=0}
+ \sum_{b=h}^{j-1} \frac{1}{(b-h)!} y^{(b)}[Q^i(x^{b-h})]_{x=0} - (b-h)x'[Q^i(x^{b-h-1})]_{x=0}
+ \text{a polynomial in } x, x', \ldots, x^{(a-j+1)}
= \sum_{b=h}^{j-1} \frac{1}{(b-h)!} y^{(b)}[Q^{i+1}(x^{b-h})]_{x=0}
+ \text{a polynomial in } x, x', \ldots, x^{(a-j+1)}
\]

as required.
To begin the proof of (d), observe that $P^h(x^b)$ vanishes if $b < h$ and equals $(b!/(b - h)!)x^{b-h}$ otherwise. Hence

$$Q^i P^h(y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^i P^h(x^b) = Q^i(y^{(h)}) - \sum_{b=h}^{j-1} \frac{1}{(b-h)!} y^{(b)} Q^i(x^{b-h}).$$  \hspace{1cm} (16)

By statement (c), the right side of (16) is, modulo the principal ideal generated by $x$, a polynomial in $x', x'', \ldots, x^{(a-j+1)}$. Since each term of the summation involves some $y^{(b)}$, the polynomial in question is obtained by expanding $Q^i(y^{(h)})$ and ignoring all terms divisible by $x$ or any $y^{(b)}$. Clearly all the coefficients of this polynomial are nonnegative.

Let us define a $\mathbb{Z} \oplus \mathbb{Z}$ grading by declaring the degree of $x^{(t)}$ to be $(2-t, 1)$ and the degree of $y^{(t)}$ to be $(2(j-t) - 1, j - 1 - t)$. Then $Q$ is a homomorphism of degree $(-1, 0)$. (Note in particular that $y^{(j-1)}$ has degree $(1, 0)$ and that $Q(y^{(j-1)}) = 1$.) Each term in the expansion of $Q^i(y^{(h)})$ has degree

$$(2(j-1-h) + 1 - i, j - 1 - h).$$

If $i < 2(j-1-h) + 1$ then the first component is positive. Hence each monomial in the expansion of $Q^i(y^{(h)})$ involves either $x$ or $x'$ or some $y^{(b)}$. Hence the right side of (16) is contained in the ideal generated by $x$ and $x'$. Similarly, if $i = 2(j-1-h) + 1$ then the first component is zero. Hence each monomial in the expansion of $Q^i(y^{(h)})$ either involves one of the same variables or is a power of $x''$; the second component of the degree tells us that the relevant power is $(x'')^{j-1-h}$. Clearly the coefficient of this term is not zero. Hence the right side of (16) is, modulo the ideal generated by $x$ and $x'$, a positive multiple of $(x'')^{j-1-h}$.

The binomial formula for differential operators tells us that the left side of formula (e) equals

$$\sum_{a=0}^{i} \sum_{b=0}^{h} \binom{i}{a} \binom{h}{b} Q^a P^h(\phi) \cdot Q^{i-a} P^{h-b}(x).$$

Note that $P^{h-b}(x)$ vanishes unless $b$ equals $h$ or $h-1$; that $Q^{i-a} P(x)$ vanishes unless $a = i$; and that $Q^{i-a}(x) = x^{(i-a)}$. Hence

$$Q^i P^h(x \cdot \phi) = hQ^i P^{h-1}(\phi) + \sum_{a=0}^{i} \binom{i}{a} Q^a P^h(\phi) \cdot x^{(i-a)}.$$
Modulo the ideal generated by \( x \) and \( x' \) the last two terms of the sum vanish.

We prove (f) by induction on \( k \). The base case is statement (d). For the inductive step, apply statement (e) to each term on the left side of (f). We find that, modulo the ideal generated by \( x \) and \( x' \),

\[
Q^i P^h(x^k y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^i P^h(x^{k+b})
\]

\[
\equiv h \left[ Q^{i-1} P^{h-1}(x^{k-1} y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^{i-1} P^{h-1}(x^{k+b-1}) \right]
\]

\[
+ \sum_{a=0}^{i-2} \binom{i}{a} \left[ Q^a P^h(x^{k-1} y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^a P^h(x^{k+b-1}) \right] x^{i-a}.
\]

Apply the inductive hypothesis to both bracketed expressions. If \( i < 2(k + j - 1 - h) + 1 \) then they both vanish. If \( i = 2(k + j - 1 - h) + 1 \) then the first term yields a positive multiple of \((x^\prime)^{k+j-1-h}\), as does the last term \((a = i - 2)\) in the sum. Hence \( Q^i P^h(x^k y) \) is likewise a positive multiple of \((x^\prime)^{k+j-1-h}\).

Statement (g) is the special case \( h = j - 1 \) of statement (f).

We now return to the proof of Lemma A. Recall that the first row of the matrix \( M \) consists of all monomials of degree at most \( d \) in \( x \) and \( y \), and that a typical column is shown in (15). We wish to show that this matrix has rank \( n + 1 \) except possibly above the locus of tangency to the divisor at infinity, i.e., whenever \( x' \neq 0 \) or \( x'' \neq 0 \). There are compatible actions of the projective general linear group \( PGL(2) \) on the universal family \( \mathcal{C} \) and on \( F(1) \), the incidence correspondence of points and lines in the plane. Hence we may assume that we are studying a point of \( \mathcal{C}(n) \) over the point \( x = y = y' = 0 \) of \( F(1) \). We may also assume that \( x' = 0 \), i.e., that the point lies over the divisor at infinity, since we have already dealt with other points when we examined the primary charts.

Consider the square submatrix of \( M \) obtained by extracting the columns headed by these \( n + 1 \) monomials:

\[ 1, x, x^2, \ldots, x^{j-1}, \]

\[ y, x^j, xy, x^{j+1}, x^2 y, x^{j+2}, x^3 y, \ldots. \]

It is of the form

\[
\begin{bmatrix}
A & C \\
B & D
\end{bmatrix},
\]
where the upper left $j \times j$ submatrix $A$ is upper triangular, with diagonal entries $0!, 1!, 2!, 3!, 4!$, etc. The lower left $(n - j + 1) \times j$ submatrix $B$ is a zero matrix. The upper right $j \times (n - j + 1)$ submatrix $C$ is irrelevant. We look at the lower right $(n - j + 1) \times (n - j + 1)$ matrix $D$ modulo the ideal generated by $x$ and $x'$. Statement (a) of Lemma B then says that columns 2, 4, 6, etc.—corresponding to the monomials $x^j, x^{j+1}, x^{j+2},$ etc.—are zero above the diagonal, and have positive multiples of powers of $x''$ on the diagonal. Statement (g) of the same lemma says that if one modifies columns 1, 3, 5, etc.—corresponding to the monomials $y, xy, x^2y$, etc.—by adding suitable linear combinations of previous columns, then one again obtains, modulo the ideal, columns which are zero above the diagonal, and have positive multiples of powers of $x''$ on the diagonal. Hence, modulo the ideal, the submatrix $D$ is nonsingular whenever $x'' \neq 0$. Hence the full square matrix is likewise nonzero, and $M$ is of maximal rank $n + 1$.  

We now consider $\mathcal{C}_{p\text{nd}}(n)$, the fiber product over $\mathbf{P}^{N(d)}$ of the lifts of $\mathcal{C}$ to $F(n_1), F(n_2), \ldots, F(n_p)$. We assume that each $n_i$ is positive. Let $F(n)^o$ denote the open subvariety of $F(n) = F(n_1) \times F(n_2) \times \cdots \times F(n_p)$ obtained by removing the following points:

- each point $(P_1, P_2, \ldots, P_p)$ for which some $P_i$ lies on the intersection of two divisors at infinity;
- each point for which some $P_i$ lies on a locus of tangency to a divisor at infinity;
- each point for which some $P_i$ and some $P_j (i \neq j)$ lie over the same point of $\mathbf{P}^2$ (i.e., each point lying over a large diagonal of $\mathbf{P}^2 \times \mathbf{P}^2 \times \cdots \times \mathbf{P}^2$).

Let $\mathcal{C}_{p\text{nd}}(n)^o$ denote the inverse image of $F(n)^o$ in $\mathcal{C}_{p\text{nd}}(n)$.

**Lemma C.** Suppose that $d \geq p - 1 + \Sigma_{i=1}^p n_i$. Then the morphism $\mathcal{C}_{p\text{nd}}(n)^o \to F(n)^o$ is smooth, with relative dimension $N(d) - (p + \Sigma_{i=1}^p n_i)$.

Let $F(n)^+$ denote the open subvariety of $F(n)$ obtained by removing the following points:

- each point $(P_1, P_2, \ldots, P_p)$ for which some $P_i$ lies on the intersection of two divisors at infinity;
- each point for which some $P_i$ lies on a locus of tangency to a divisor at infinity;
- each point for which some $P_i$ and some $P_j (i \neq j)$ lie over the same point of $F(1)$ (i.e., each point lying over a large diagonal of $F(1) \times F(1) \times \cdots \times F(1)$);
- each point for which some $P_i, P_j,$ and $P_k (i, j, k$ distinct) all lie over the same point of $\mathbf{P}^2$;
each point for which some $P_i$ and some $P_j (i \neq j)$ lie over the same point of $\mathbb{P}^2$, and one of them lies on a divisor at infinity.

Note that $F(n)^+$ is larger than $F(n)^\circ$, since a point $(P_1, P_2, \ldots, P_p)$ of $F(n)^+$ may project to a point $(Q_1, Q_2, \ldots, Q_p)$ of $\mathbb{P}^2 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^2$ for which $Q_1, Q_2,$ etc., are not all distinct. Let us stratify $F(n)^+$ by declaring a point $(P_1, P_2, \ldots, P_p)$ to be in $F(n)^q$ if it projects to a point $(Q_1, Q_2, \ldots, Q_p)$ of $\mathbb{P}^2 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^2$ for which the number of distinct points among $Q_1, Q_2,$ etc., is exactly $q$. The possible values for $q$ are $\lceil \frac{p}{2} \rceil, \lceil \frac{p}{2} \rceil + 1, \ldots, p$; the stratum $F(n)^p = F(n)^\circ$ is open and dense. Let $\mathcal{C}_{p^{\text{per}}}(n)^q$ denote the inverse image of $F(n)^q$ in $\mathcal{C}_{p^{\text{per}}}(n)$.

**Lemma D.** Suppose that $d \geq p - 1 + \sum_{i=1}^p n_i$. Then for each $q$ (from $\lceil \frac{p}{2} \rceil$ to $p$) the morphism $\mathcal{C}_{p^{\text{per}}}(n)^q \to F(n)^q$ is smooth, with relative dimension $N(d) - (q + \sum_{i=1}^p n_i)$.

**Proof of Lemmas C and D.** Lemma C is a special case of Lemma D.

Over each point $(P_1, P_2, \ldots, P_p)$ of $F(n)^q$ the fiber of $\mathcal{C}_{p^{\text{per}}}(n)^q$ is a subspace of $\mathbb{P}^{N(d)}$ of codimension at most $q + \sum_{i=1}^p n_i$, as can be seen by a naive count of the number of imposed conditions. Hence to prove Lemma D it suffices to show that this fiber is a linear subspace of $\mathbb{P}^{N(d)}$ of codimension at least $q + \sum_{i=1}^p n_i$. This fiber is the intersection of the fiber of $\mathcal{C}(n_1)$ over $P_1$, the fiber of $\mathcal{C}(n_2)$ over $P_2$, etc. Lemma A therefore provides an explicit set of generators for its defining ideal in $\mathbb{P}^{N(d)}$. These generators include, for each $P_i$ not on a divisor at infinity, the functions

\[ f_i(x_i, y_i), f'_i(x_i, y_i, y'_i), \ldots, f^{(m_i)}_i(x_i, y_i, y'_i, \ldots, y'^{(m_i)}_i) \quad (17) \]

(where $x_i, y_i,$ etc., are coordinates on a primary chart of $F(n_i)$ containing $P_i$, and $f_i$ generates the ideal of $\mathcal{C}$) together with, for each $P_i$ on a single divisor at infinity $I_j$, the functions

\[
\begin{align*}
f_i(x_i, y_i), \quad P f_i(x_i, y_i, y'_i), \ldots, \\
P^{j_i-1} f_i(x_i, y_i, y'_i, \ldots, y'^{(j_i-1)}_i), \quad Q^{j_i-1} f_i(x_i, y_i, y'_i, \ldots, y'^{(j_i-1)}_i, x_i'), \ldots, \\
Q^{n_i-j_i+1} P^{j_i-1} f_i(x_i, y_i, y'_i, \ldots, y'^{(j_i-1)}_i, x'_i, \ldots, x'^{(n_i-j_i+1)}_i)
\end{align*}
\]

(where $x_i, y_i,$ etc., are coordinates on a secondary chart of $F(n_i)$ containing $P_i$). Each of these generators is a linear function in the $a_m$'s; thus the fiber of $\mathcal{C}_{p^{\text{per}}}(n)$ over $(P_1, P_2, \ldots, P_p)$ is defined by a system of $p + \sum n_i$ linear equations.

Let $M_n$ denote the matrix of this system.

Note that it suffices to prove the lemma in case $d = p - 1 + \sum_{i=1}^p n_i$, since increasing $d$ will only enlarge $M_n$ by adding more columns.
We will use induction on \( q \); the case \( q = 0 \) is vacuous. The inductive step has two cases. Suppose first that the image of \( P_1 \) in \( \mathbb{P}^2 \) is distinct from the images of \( P_2, P_3, \ldots, P_p \). Let \( n - n_1 \) denote \((n_2, n_3, \ldots, n_p)\). On \( \mathbb{P}^2 \) choose an affine chart, with coordinates \( x \) and \( y \), so that the chart contains the images of \( P_2, P_3, \ldots, P_p \) and so that the image of \( P_1 \) is the point at infinity on the \( y \)-axis. Around each \( P_i (i \neq 1) \) choose, as appropriate, a primary or secondary chart on \( F(n_i) \) whose first two coordinates are \( x_i = x \) and \( y_i = y \). (Choose a secondary chart only if the point lies on a divisor at infinity.) Let \( x_i = x/y \) and \( y_i = 1/y \); then the image of \( P_1 \) is at the origin of the \( \mathbb{A}^2 \) with coordinates \( x_1 \) and \( y_1 \). Note that over this affine chart the defining equation for the universal family is
\[
\sum_{u + v \leq d} a_{u,v} x_1^{u} y_1^{d-(u+v)} = 0. \tag{18}
\]

Around \( P_1 \) choose, as appropriate, a primary or secondary chart on \( F(n_1) \) whose first two coordinates are \( x_1 \) and \( y_1 \).

We may assume that the matrix \( M_n \) has been arranged so that the initial columns correspond to those \( a_{u,v} \)'s for which \( u + v = d - 1 - n_1 \). Then \( M_n \) takes the form
\[
\begin{bmatrix}
M_{n-n_1} & C \\
B & D
\end{bmatrix}.
\]

By the inductive hypothesis the rank of the upper left submatrix is at least \( q - 1 + \sum_{l=2}^{p} n_l \). The \((1 + n_1) \times (1 + N(d - 1 - n_1))\) submatrix \( B \) records the partial derivatives of the functions defining the fiber of \( \mathcal{C}(n_1) \) over \( P_1 \) with respect to the \( a_{u,v} \)'s just mentioned. In (18) the coefficient of each such \( a_{u,v} \) is divisible by \( y_1^{n_1+1} \). If we implicitly differentiate this monomial \( n_1 \) or fewer times, as we do when applying the differential operator \( P \) or \( Q \), the result is still divisible by \( y_1 \). Hence all the entries of \( B \) vanish at \( P_1 \). Submatrix \( C \) is irrelevant. Matrix \( D \) contains the submatrix \( M_{n_1} \); by Lemma A its rank is \( 1 + n_1 \). Hence the rank of \( M_n \) is at least \( q + \sum_{l=1}^{p} n_l \).

Next suppose that the image of \( P_1 \) in \( \mathbb{P}^2 \) coincides with the image of some \( P_i(i \neq 1) \). Without loss of generality we may assume that \( P_1 \) and \( P_2 \) project to the same point of \( \mathbb{P}^2 \), but that this point is distinct from the images of \( P_3, P_4, \ldots, P_p \). We may also assume that \( n_1 \leq n_2 \). Furthermore the images of \( P_1 \) and \( P_2 \) in \( F(1) \) are distinct points, and neither \( P_1 \) nor \( P_2 \) is on a divisor at infinity. As in the previous case, we choose on \( \mathbb{P}^2 \) an affine chart, with coordinates \( x \) and \( y \), so that the chart contains the images of \( P_3, P_4, \ldots, P_p \) and so that the common image of \( P_1 \) and \( P_2 \) is the point at infinity on the \( y \)-axis.
Around each $P_i$ ($i > 2$) we choose, as appropriate, a primary or secondary chart on $F(n_i)$ whose first two coordinates are $x_i = x$ and $y_i = y$. Let $x_1 = x_2 = x/y$ and $y_1 = y_2 = 1/y$; then the common image of $P_1$ and $P_2$ is at the origin of the $A^2$ with coordinates $x_1$ and $y_1$. Without loss of generality we may assume that the image of $P_1$ in $F(1)$ represents the tangent direction of the $y_1$-axis. Hence we may use the primary chart with coordinates $y_1, x_1, x'_1$, etc. on $F(n_1)$ and the primary chart with coordinates $x_2, y_2, y'_2$, etc. on $F(n_2)$.

We may assume that the matrix $M_n$ has been arranged so that the initial columns correspond to those $a_{uv}$'s for which $u + v \leq d - 2 - (n_1 + n_2)$. Then $M_n$ takes the form

$$
\begin{bmatrix}
M_{n-n_1-n_2} & C \\
B & D
\end{bmatrix}.
$$

By the inductive hypothesis the rank of the upper left submatrix is at least $q - 1 + \sum_{i=3}^p n_i$. The $(2 + n_1 + n_2) \times (1 + N(d - 2 - n_1 - n_2))$ submatrix $B$ records the partial derivatives of the functions defining the fiber of $C(n_1)$ over $P_1$ and the fiber of $C(n_2)$ over $P_2$, with respect to the $a_{uv}$'s just mentioned. In (18) the coefficient of each such $a_{uv}$ is divisible by $y_1^{n_1 + n_2 + 2}$. If we implicitly differentiate this monomial at most $n_2 = \max\{n_1, n_2\}$ times, the result is still divisible by $y_1$. Hence all the entries of $B$ vanish at $(P_1, P_2)$. Submatrix $C$ is again irrelevant. Matrix $D$ contains, after reordering the columns, the following submatrix:

$$
\begin{bmatrix}
1 & y_1 & x_1 & y_1^2 & x_1^2 & \cdots & y_1^{n_1} & x_1^{n_1} & x_1^{n_1+1} & \cdots & x_1^{n_1^2} \\
0 & 1 & x'_1 & 2y_1 & 2x_1 & x'_1 & \cdots & n_1y_1^{n_1-1} & \cdots & \cdots \\
0 & 0 & x''_1 & 2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & x_1^{(n_1)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & y_2 & x_2 & y_2^2 & x_2^2 & \cdots & y_2^{n_2} & x_2^{n_2} & x_2^{n_2+1} & \cdots & x_2^{n_2^2} \\
0 & y_2 & 1 & 2y_2 & 2x_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & y_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & y_2^{(n_2)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}.
$$

We can then rearrange the rows:
Evaluating at $x_1 = y_1 = x'_1 = x_2 = y_2 = 0$, we obtain

\[
\begin{bmatrix}
1 & y_1 & x_1 & y_1^2 & x_1^2 & \cdots & y_1^{n_1} & x_1^{n_1} & x_1^{n_1+1} & \cdots & x_1^{n_2} \\
1 & y_2 & x_2 & y_2^2 & x_2^2 & \cdots & y_2^{n_1} & x_2^{n_1} & x_2^{n_1+1} & \cdots & x_2^{n_2} \\
0 & 1 & x'_1 & 2y_1 & 2x_1x'_1 & \cdots & n_1y_1^{n_1-1} & \cdots & \cdots & \cdots & \cdots \\
0 & y_2' & 1 & 2y_2y_2' & 2x_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & x_1^{(2)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & x_1^{(n_1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & y_2^{(n_1)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & y_2^{(n_1+1)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & y_2^{(n_2)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

which clearly has rank $n_1 + n_2 + 1$. Hence at $(P_1, P_2)$ the rank of $D$ is at least $n_1 + n_2 + 1$, and the rank of $M_n$ is at least $q + \sum_{i=1}^{n_1} n_i$. \[\square\]

By definition $\mathcal{C}(n)$ is the union of $\mathcal{L}(\mathcal{C})$ (where $\mathcal{L}$ is the lifting map $\mathcal{C} \rightarrow F(n)$) and a closed subvariety $\mathcal{S}(n)$, each point of which lies over a singular or nonreduced point of some member of $\mathcal{C}$. Since the codimension of $\mathcal{C}(n)$ in $\mathbb{A}^{n+2} \times \mathbb{P}^{N(d)}$ is $n + 1$, the codimension of $\mathcal{S}(n)$ is at least $n + 2$.

**Lemma E.** Suppose that $d \geq n$. Suppose that $P$ is a point of $F(n)$ which is
not on any divisor at infinity. Then the fiber of \( S(n) \) over \( P \) has codimension at least \( n + 2 \) in \( \mathbb{P}^{N(d)} \).

**Proof.** It suffices to check over the primary chart of \( F(n) \) with coordinates \( x, y, y', y'', y^{(3)}, \ldots, y^{(n)} \). Over this chart the defining ideal of \( S(n) \) contains the functions \( f, f', f'', f^{(3)}, \ldots, f^{(n)} \) defining \( G(n) \) (obtained from the function \( f \) of (13) by repeated implicit differentiation) and the partial derivative \( \partial f / \partial y \). Let \( M \) denote the \((n + 2) \times (N(d) + 1)\) matrix of partial derivatives of these functions with respect to each of the \( a_{uv} \). (Once again we may interpret \( M \) as the matrix of a linear system.) The square submatrix consisting of the partials with respect to \( a_{00}, a_{10}, a_{20}, \ldots, a_{n0}, \) and \( a_{01} \) is nonsingular at each point:

\[
\begin{bmatrix}
1 & x & x^2 & \cdots & x^n & y \\
0 & 1 & 2x & \cdots & nx^{n-1} & y' \\
0 & 0 & 2 & \cdots & n(n-1)x^{n-2} & y'' \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & n! & y^{(n)} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\]  

(19)

Hence at each point the functions \( f, f', f'', f^{(3)}, \ldots, f^{(n)} \), and \( \partial f / \partial y \) define a variety of codimension \( n + 2 \) which is smooth over the primary chart of \( F(n) \). \( \square \)

Let \( S(n) \) denote the fiber product

\[
S(n_1) \times G(n_2) \times \cdots \times G(n_p).
\]

Let \( F(n)_{\text{finite}} \) denote the open subvariety of \( F(n) \) obtained by removing, from each factor, all divisors at infinity; let \( S(n)_{\text{finite}} \) denote its inverse image in \( S(n) \). Let

\[
S(n)^{\circ} = G_{\text{finite}}^{\circ} \cap S(n)_{\text{finite}}.
\]

**Lemma F.** Suppose that \( d \geq p - 1 + \Sigma_{i=1}^l n_i \). Then each fiber of the morphism \( S(n)^{\circ} \to F(n)^{\circ} \cap F(n)_{\text{finite}} \) has codimension at least \( 1 + p + \Sigma_{i=1}^l n_i \).

**Proof.** The proof is similar to that of Lemma D. In this instance, we need only to work in primary charts. The defining ideal for the fiber of \( S(n) \) over a point \((P_1, P_2, \ldots, P_p)\) of \( F(n)_{\text{finite}} \) includes the functions of (17) and the partial derivative \( (\partial f / \partial y_j)(x_1, y_1) \). Each of these functions is linear in the \( a_{uv} \)’s; let \( M \) denote the matrix of this linear system. As in the proof of
Lemma D, we note that it suffices to prove the lemma in case
d = p - 1 + \Sigma_{i=1}^{p} n_i.

If \((P_1, P_2, \ldots, P_p)\) is in \(\mathcal{S}(n)^o\), then the images in \(\mathbb{P}^2\) of these \(p\) points are all distinct. By Lemma D, the fiber of \(C_{p^\mathbb{P}^2}(n_2, n_3, \ldots, n_p)\) over \((P_2, P_3, \ldots, P_p)\) has codimension \(p - 1 + \Sigma_{i=2}^{p} n_i\). By Lemma E, the fiber of \(\mathcal{S}(n_1)\) over \(P_1\) has codimension at least \(n_1 + 2\). Since the image of \(P_1\) is distinct from the images of \(P_2\) through \(P_p\), we may argue as in the first case of the inductive step in the proof of Lemma D. By an appropriate choice of affine charts, in particular, by choosing a chart on \(\mathbb{P}^2\) containing the images of \(P_2, P_3, \ldots, P_p\) and so that the image of \(P_1\) is the point at infinity on the \(y\)-axis, we see that \(M\) takes the form

\[
\begin{bmatrix}
M_{n-n_i} & C \\
B & D
\end{bmatrix},
\]

where \(M_{n-n_i}\) has rank \(p - 1 + \Sigma_{i=2}^{p} n_i\), \(B\) is a zero matrix, and \(D\) has rank at least \(n_1 + 2\). Hence \(M\) has rank at least \(1 + p + \Sigma_{i=1}^{p} n_i\). \(\square\)

**Proof of Theorem 3.** Let 

\[n = (n_1, n_2, \ldots, n_p) = (o_1 - 1, o_2 - 1, \ldots, o_p - 1).\]

By Lemma D, each morphism \(C_{p^\mathbb{P}^2}(n)^q \rightarrow F(n)^q\) is smooth.

Let us assume for the moment that \(S\) is nonsingular. Let \(\tau: S \rightarrow \mathbb{P}^{N(d)}\) be the morphism determined by the family \(\mathcal{S}\). Then the morphism

\[
S \times PGL(N(d)) \rightarrow \mathbb{P}^{N(d)}
\]

\[\gamma \mapsto \gamma \cdot \tau(s)\] (20)

is smooth. (This assertion is justified in the course of the proof of Theorem 2 in [14].) Hence the projection

\[
C_{p^\mathbb{P}^2}(n)^q \times_{p^\mathbb{P}^2} (S \times PGL(N(d))) \rightarrow C_{p^\mathbb{P}^2}(n)^q,
\]

which is obtained from (20) by base extension, is smooth. Composing it with the smooth morphism \(C_{p^\mathbb{P}^2}(n)^q \rightarrow F(n)^q\), we obtain

\[
C_{p^\mathbb{P}^2}(n)^q \times_{p^\mathbb{P}^2} (S \times PGL(N(d))) \rightarrow F(n)^q,
\]

which fits into the following diagram. (The vertical morphism on the left is the composite of two projections; the one on the right is inclusion.)
The fiber of $\mathcal{C}_{\mathcal{P}^m}^q(n) \times (S \times PGL(N(d)))$ over a point $y \in PGL(N(d))$ is the fiber product $\mathcal{C}_{\mathcal{P}^m}^q(n) \times \mathcal{P}_y$, where $S$ is regarded as a $\mathcal{P}^r$-variety via the morphism $\gamma \circ \tau$, i.e., via $\tau$ followed by translation by $\gamma$. Kleiman’s transversality lemma (Lemma 1 of [14]) tells us that for generic $\gamma$ the map $\mathcal{C}_{\mathcal{P}^m}^q(n) \times \mathcal{P}_y \rightarrow F(n)^q$ is transverse to $C_1(n_1) \times \cdots \times C_p(n_p) \cap F(n)^q$. Our assumption that the family $\mathcal{F}$ is generic means that we may assume $\gamma$ is the identity.

We now count dimensions. By Lemma D,

$$\dim(\mathcal{C}_{\mathcal{P}^m}^q(n)) = \dim F(n)^q + N(d) - \left(q + \sum_{i=1}^{p} n_i\right).$$

Since the morphism $\tau$ is generic,

$$\dim\left(\mathcal{C}_{\mathcal{P}^m}^q(n) \times S\right) = \dim F(n)^q - \left(q + \sum_{i=1}^{p} n_i\right) + s = \dim F(n)^q - q.$$

To specify a point of $C_1(n_1) \times \cdots \times C_p(n_p) \cap F(n)^q$, one can first specify $p - q$ points lying on the intersection of two of the $p$ curves $C_1, \ldots, C_p$, then specify $2q - p$ additional points, each of which lies on one of the curves; there may be further choices involved in specifying higher-order data, but these choices cannot contribute to the dimension count:

$$\dim(C_1(n_1) \times \cdots \times C_p(n_p) \cap F(n)^q) = 2q - p.$$
Note that $\mathcal{X}_S(n)$, the fiber product over $S$ of the various lifts of $\mathcal{X}$, is a subvariety of $\mathcal{C}_{p\omega}(n) \times_{p\omega} S$, and that it has the same dimension. Hence the morphism $\sigma: \mathcal{X}_S(n) \to F(n)$ (the restriction of the projection of $F(n) \times S$ onto its first factor) is also transverse to $C_1(n_1) \times C_2(n_2) \times \cdots \times C_p(n_p)$. Each intersection between $\sigma(\mathcal{X}_S(n))$ and the $p$-fold product is a $p$-tuple $(x_1, x_2, \ldots, x_p)$ in which $x_1$ is a contact or a false contact between $C_1$ and some member of $\mathcal{X}$, and $x_2$ is a contact or a false contact between $C_2$ and the same member of $\mathcal{X}$, etc. The number of such intersections is

$$\sum_{F(n)} [C_1(n_1) \times C_2(n_2) \times \cdots \times C_p(n_p)] \cdot \sigma_*[\mathcal{X}_S(n)],$$

which is equal to the proto-contact number as defined by (11) in Section 3.

We must show that each point of intersection is a $p$-tuple $(x_1, x_2, \ldots, x_p)$ of honest contacts rather than false ones. Consider $C_1(n_1)_{\text{sing}}$, the (zero-dimensional) subvariety of $C_1(n_1)$ lying over the singularities of $C_1$. The dimension of $C_1(n_1)_{\text{sing}} \times C_2(n_2) \times \cdots \times C_p(n_p)$ is $p - 1$. Hence by Kleiman's transversality lemma the image of $\mathcal{C}_{p\omega}(n) \times_{p\omega} S$ in $F(n)$ is disjoint from this $p$-fold product. A fortiori, the image of $\mathcal{X}_S(n)$ is disjoint from this $p$-fold product. Hence $x_1$ does not lie over a singular point of $C_1$. Note this implies that $x_1$ does not lie over a divisor at infinity on $F(n_1)$. Similarly one sees that $x_2$ does not lie over a singular point of $C_2$, and hence that $x_2$ does not lie over a divisor at infinity on $F(n_2)$, etc. Lemma F, together with the appropriate dimension count, shows that the image of $\mathcal{X}(n)_{\text{sing}} \times_{p\omega} S$ is disjoint from $C_1(n_1) \times \cdots \times C_p(n_p)$. As we have already seen, enlarging $\mathcal{X}(n)$ to $\mathcal{X}(n)_{\text{finite}}$ creates no intersections. Hence for every point of intersection $(x_1, x_2, \ldots, x_p)$ between the image of $\mathcal{X}_S(n)$ and the $p$-fold product, $x_1$ lies over a nonsingular point of the relevant member of $\mathcal{X}$. Similarly one sees that $x_2$ lies over a nonsingular point of the relevant member of $\mathcal{X}$, etc.

If $S$ is singular, we can apply our argument above to the singular locus $S_{\text{sing}}$. The morphism

$$S_{\text{sing}} \times \text{PGL}(N(d)) \to \mathbb{P}^N(d)$$

is now flat rather than smooth. (Again, this assertion is justified by Kleiman in the proof of his Theorem 2.) Kleiman's lemma now tells us, after the appropriate dimension count, that the image of $\mathcal{X}_{S_{\text{sing}}}(n)$ in $F(n)$ is disjoint from $C_1(n_1) \times \cdots \times C_p(n_p)$. Hence, for a generic family $\mathcal{X}$, the members over $S_{\text{sing}}$ have no simultaneous contacts with $C_1, C_2, \ldots, C_p$. Thus we may assume, as we did, that $S$ is nonsingular. \qed
5. The higher-order characteristic numbers of a family of plane curves

At the beginning of Section 3 we defined the higher-order characteristic numbers of a plane curve. The degree and class are well known. Each of the other numbers counts, perhaps with multiplicities, the number of cusps of a specified order. These numbers are readily calculated, either implicitly or from a parametrization of the curve; in fact one needs only a local or even formal parametrization at each singular point. (See the algorithm presented in the proof of Proposition 3.9 of [5].)

For a family of curves, the numbers defined by (12) are likewise called characteristic numbers. For an s-parameter family $\mathcal{X}$, this formula associates such a number to each monomial of weight $s$ in the indeterminates of (10). To denote this number we use the monomial itself, with lowercase Greek letters replacing their uppercase counterparts.

Note that $\Lambda_0$ has weight 0. If $M$ is a monomial of weight $s$ with associated characteristic number $m$, then it is easy to show that the characteristic number associated to $\Lambda_0 M$ is $dm$, where $d$ is the degree of the general member of $\mathcal{X}$. In the remainder of this section we consider monomials not involving $\Lambda_0$.

If $M$ is a monomial in $\Lambda_1$ and $\Pi_1$, then we call its associated characteristic number ordinary. It is well known that, under mild hypotheses on $\mathcal{X}$, the characteristic number $(\lambda_1)^r (\pi_1)^{s-r}$ is the number of members tangent to $r$ specified general lines and passing through $s-r$ specified general points. These characteristic numbers, especially those of plane cubics, have been the subject of numerous investigations, both classical and contemporary. (For example, see [1], [2], [3], [16], [17], [18], [19], [22], [25], [31].) If $M$ also involves the indeterminate $\Gamma_2^2$, then it should perhaps still be called "ordinary", since its definition uses only the notion of ordinary contact. For example if $\mathcal{X}$ is a general two-parameter family then

$$\gamma_2^2 = \int_{F(1)} \Gamma_2^2 \cap \sigma_* [\mathcal{X}(1)] = \int_{F(1)} h^2 h \cap \sigma_* [\mathcal{X}(1)]$$

is the number of members tangent to a specified line at a specified point. This characteristic number does not, however, appear in classical ordinary contact formulas. It does appear in Schubert's formula for triple contacts between a two-parameter family and a single specified curve. (See [6], [24], [26], and Section 6(a) of the present paper.)

If $M$ involves other indeterminates, then the associated number $m$ is called a higher-order characteristic number. For example, a two-parameter
family has six characteristic numbers. The ordinary characteristic numbers are $(\lambda_1)^2$, $\lambda_1 \pi_1$, $(\pi_1)^2$, and $\gamma_2^2$. The higher-order characteristic numbers are

$$\lambda_2 = \int_{F(2)} \Lambda_2 \cap \sigma_*[\mathcal{X}(2)] = \int_{F(2)} h^2 z_2 \cap \sigma_*[\mathcal{X}(2)];$$

and

$$\pi_2 = \int_{F(2)} \Pi_2 \cap \sigma_*[\mathcal{X}(2)] = \int_{F(2)} h^2 i_2 \cap \sigma_*[\mathcal{X}(2)].$$

Theorem 3 tells us that, for a generic family of curves of degree $\geq 2$, $\lambda_2$ is the number of triple contacts between a member of $\mathcal{X}$ and a specified line. (More generally, $\lambda_s$ is, for a generic $s$-parameter family $\mathcal{X}$ of curves of degree at least $s$, the number of contacts of order $s + 1$ between a member of $\mathcal{X}$ and a specified line.) One can show that $\pi_2$ is the number of members of $\mathcal{X}$ having a cusp at a specified point (See [6], Theorem A1. Note that a "dual" characteristic number involving flexes rather than cusps is considered in [26] and [24].)

The definition

$$\pi_s = \int_{F(s)} h^2 i_2 i_3 \cdots i_s \cap \sigma_*[\mathcal{X}(s)]$$

suggests that this characteristic number measures, for a generic $s$-parameter family $\mathcal{X}$ of curves of sufficiently large degree, the number of member curves of $\mathcal{X}$ whose lift, at a specified point, meets the divisors at infinity $I_2, I_3, \ldots, I_s$. (One might call such a point a "super-profound cusp".) To justify this interpretation would involve an appeal to Kleiman's transversality lemma, to rule out contributions to the intersection number created by the way in which curvilinear data specializes at a singular or nonreduced member of the family. But our crucial Lemma A fails in precisely this "super-profound" situation, and we cannot use Kleiman's lemma. We suspect that the naive interpretation of $\pi_s$ is incorrect, and that one cannot even guarantee that the intersection in question can be made proper.

Similar difficulties plague the interpretation of other characteristic numbers. The number of characteristic numbers also grows rather quickly with the number of parameters. For example, a five-parameter family possesses 70 characteristic numbers.
6. Variations, further remarks, and an example

(a) Contacts between a family and a single curve

When \( p = 1 \), the special case of the contact formula says that the number of contacts of order \( o = n + 1 \) between a specified curve \( C \) (with no profound cusp) and a generic \( n \)-parameter family \( \mathcal{X} \) is

\[
\frac{1}{F(n)} \left[ C(n) \cap \sigma_*[\mathcal{X}(n)] \right] = d\lambda_n + d\pi_n + (3d + \kappa_2)\gamma_n^2 + \ldots + ((n + 1)d + n\kappa_2 + \ldots + 3\kappa_{n-1} + \kappa_n)\gamma_n^n
\]

where

\[
\lambda_n = \int h^2 z_n \cap \sigma_*[\mathcal{X}(n)],
\]

\[
\pi_n = \int h^2 i_2 \cdots i_n \cap \sigma_*[\mathcal{X}(n)],
\]

\[
\gamma_n^k = \int h^2 h_{k-1} i_{k+1} i_{k+2} \cdots i_n \cap \sigma_*[\mathcal{X}(n)], \quad k = 2, \ldots, n.
\]

Note that this formula is essentially implicit in the definition of the contact module \( m_n(C) \).

(b) Simultaneous contact with nonsingular curves

Suppose that the curves \( C_1, \ldots, C_p \) are all nonsingular. Then for each curve the characteristic numbers \( \kappa_1, \kappa_2, \ldots \) vanish and (by the Plücker formula) the class is \( d = d(d - 1) \). In this case the contact formula tells us that the number of simultaneous contacts of order \( (o_1, \ldots, o_p) = (n_1 + 1, \ldots, n_p + 1) \) between \( C_1, \ldots, C_p \) and some member of a \( \Sigma n_j \)-parameter family \( \mathcal{X} \) is obtained from

\[
\prod_{j=1}^{p} \left[ d_j \Lambda_{n_j} + d_j(d_j - 1) \left( \Pi_{n_j} + \sum_{k=2}^{n_j} (k + 1)\Gamma_{n_j}^k \right) \right]
\]

by the expansion and evaluation of monomials described in Theorem 2.

This contact formula is valid under the assumptions of Theorem 3, that is, when \( \mathcal{X} \) is a generic family of plane curves of sufficiently high degree. There is, however, a certain freedom in choosing hypotheses. We could assume, for example, that each individual curve \( C_j \) is generic (hence nonsingular) of degree \( d_j \geq o_j - 1 \), but make no assumption whatever about the family (except, of course, that the generic member is a reduced
curve). Then, if the base field is either of characteristic zero or of characteristic at least \(\max\{o_1, \ldots, o_p\}\), formula (21) counts (again, after expansion and evaluation of monomials) the number of simultaneous contacts. Moreover, if each \(d_j \geq o_j\), then all contacts are of order exactly \((o_1, \ldots, o_p)\) (that is, none of the contacts with any of the \(C_j\)'s is of higher order). The proof is essentially that of Proposition 2.5 of [5], which treats the case \(p = 1\).

(c) Bézout's Theorem

As we remarked in Section 4, Theorem 3 is valid even if we drop the requirement that each specified order of contact \(o_j\) be at least 2. Suppose, for example, that \(C\) and \(C_1, C_2, \ldots, C_p\) are reduced plane curves of degrees \(d, d_1, d_2, \ldots, d_p\) respectively. Then our contact formula says that the number of simultaneous intersections between \(C\) and \(C_1, C_2, \ldots, C_p\) is \(d^p d_1 d_2 \cdots d_p\). This is just a silly formulation of Bézout's Theorem. Indeed, a simultaneous intersection is a \(p\)-tuple \((x_1, x_2, \ldots, x_p)\) in which the point \(x_1\) lies in the intersection of \(C\) and \(C_1\), the point \(x_2\) lies in the intersection of \(C\) and \(C_2\), etc. There are \(dd_1\) possibilities for \(x_1\), together with \(dd_2\) possibilities for \(x_2\), etc. The other cases omitted from our statement of Theorem 3 are of the same ilk.

(d) The formula of Fulton, Kleiman, and MacPherson

The overlap of our contact formula and that of Fulton-Kleiman-MacPherson [11] is the following formula:

The number of simultaneous ordinary contacts between the members of a \(p\)-parameter family \(\mathcal{X}\) of plane curves and \(p\) specified plane curves \(C_1, \ldots, C_p\) (of degree \(d_1\) and class \(\tilde{d}_1\), \ldots, \(C_p\) of degree \(d_p\) and class \(\tilde{d}_p\)) is obtained from the product of modules

\[
\prod_{j=1}^{p} (d_j \Lambda + \tilde{d}_j \Pi)
\]

by expansion and evaluation of monomials. To evaluate \(\Lambda^r \Pi^{p-r}\) means to replace it by the (ordinary) characteristic number \(\lambda^r \pi^{p-r}\), the number of members of \(\mathcal{X}\) tangent to \(r\) specified general lines and passing through \(p-r\) specified general points.

Fulton et al. assume that the individual curves \(C_1, C_2, \ldots, C_p\) are in general position, whereas our hypotheses involve genericity assumptions about the family \(\mathcal{X}\) of curves, as well as different, but mild, conditions on \(C_1, \ldots, C_p\). (See (f) below for further comments along these lines.)
(e) The formula of de Jonquières

The classical formula of de Jonquières [13] gives the number of plane curves of degree $d$ making contacts of orders $o_1, o_2, \ldots, o_p$ with a given curve $C$ and which pass through $k$ points, where $k = d(d + 3)/2 + p - \Sigma_j o_j$. It is tempting to use our set-up to obtain this formula by calculating

$$I := \int_{F(n)} \pi_1^*(h^2) \cdot \ldots \cdot \pi_k^*(h^2) \cdot \pi_{k+1}^*[C(n_1)] \cdot \ldots \cdot \pi_{k+p}^*[C(n_p)] \cap \sigma_*[\mathcal{G}_{p^{(0)}}(n)].$$

where now

$$F(n) := \mathbf{P}^2 \times \cdots \times \mathbf{P}^2 \times F(n_1) \times \cdots \times F(n_p),$$

$$\mathcal{G}_{p^{(0)}}(n) := \mathcal{G} \times \cdots \times \mathcal{G} \times \mathcal{G}(n_1) \times \cdots \times \mathcal{G}(n_p),$$

each $n_j = o_j - 1$, and $\mathcal{G}$ denotes the universal family of degree $d$ plane curves. However, Theorem 3 itself will never apply to the universal family. We have $\dim \mathcal{G}_{p^{(0)}}(n) = N(d) + p + k$ and $\dim F(n) = 2k + p + \Sigma_j o_j$ so that the map $\mathcal{G}_{p^{(0)}}(n) \to F(n)$ is smooth only if $N(d) - k > \Sigma_j o_j$ while we must have $N(d) - k = \Sigma_j o_j - p$ for the proto-contact formula to be valid.

Thus our formula represents an approach to problems of higher-order contact distinct from that provided by the formula of de Jonquières. Note that de Jonquières's formula can lose its enumerative significance when, for example, members of the family have nonreduced components of sufficiently high multiplicity, whereas the hypotheses of Theorem 3 guarantee validity of our formula for suitable families of plane curves of fixed degree. (See [11] p. 184ff for further discussion of the enumerative significance of de Jonquières's formula.) Moreover, our results are in keeping with the spirit of Hilbert's 15th problem which asks "to establish rigorously and with an exact determination of the limits of their validity those geometrical numbers..." [12]. Finally, Fulton, Kleiman, and MacPherson [11] note that "it was observed long ago that de Jonquières's formula yields via symbolic multiplication a formula in the case of several given curves." The proto-contact formula of Theorem 2 involves just such a general multiplication.

Other contemporary treatments of de Jonquières's formula include [21] and [29]. In these cases, the problem is formulated in terms of systems of divisors of degree $d$ cut out on the fixed curve $C$ by a family $\mathcal{X}$ of curves. The formula so produced counts the number of such divisors on $C$ that
have coalesced in a prescribed manner. In particular, the support of such
degenerate divisors could include points that are singular either on $C$ or on
the relevant member of $\mathcal{A}$. In our language, such divisors represent "false
contacts" and are not counted by our formula. (See Example 3.8, pp.
503–504 of [5] for an illustration of this phenomenon.)

(f) An example: triple contact with two curves

In this case we have two specified curves $C$ and $D$, and seek the number of
simultaneous contacts of order $(3,3)$ with members of a 4-parameter family
$\mathcal{A}$. If $C$ and $D$ intersect transversely and have no profound or flat
cusps, and if $\mathcal{A}$ is a generic family of curves of degree $d \geq 5$, then Theorems 2 and
3 show that the desired number

$$I = \int_{\mathcal{A}(2) \times \mathcal{A}(2)} \pi^* \langle C(2) \rangle \cdot \pi^* \langle D(2) \rangle \cap \sigma_* [\mathcal{A}(2, 2)]$$

is obtained from the product of modules

$$(d_C \Lambda_2 + \tilde{d}_C \Pi_2 + (3\tilde{d}_C + \kappa_{2C})\Gamma_2^2)(d_D \Lambda_2 + \tilde{d}_D \Pi_2 + (3\tilde{d}_D + \kappa_{2D})\Gamma_2^2),$$

where $d_C$, $\tilde{d}_C$, $\kappa_{2C}$ are the characteristic numbers of $C$, etc. Explicitly, the
number of simultaneous contacts is

$$d_C d_D (\lambda_2^2 + (d_C \tilde{d}_D + d_D \tilde{d}_C) \lambda_2 \pi_2$$
$$+ (d_C (3\tilde{d}_D + \kappa_{2D}) + d_D (3\tilde{d}_C + \kappa_{2C})) \lambda_2 \gamma_2^2 + \tilde{d}_C \tilde{d}_D (\pi_2)^2$$
$$+ (d_C (3\tilde{d}_D + \kappa_{2D}) + (d_C (3\tilde{d}_C + \kappa_{2C})) \pi_2 \gamma_2^2 + (3\tilde{d}_C + \kappa_{2C})(3\tilde{d}_D + \kappa_{2D}) (\gamma_2^2)^2).$$

Another approach can be taken to establishing the validity of this
contact formula. In [6] and [24], the $PGL(2)$-orbits of $F(2)$ are identified.
There are three of them: the dense orbit $\mathcal{O}(-)$, each point of which is
represented by the germ of a nonsingular curve without a flex, the special
orbit $\mathcal{O}(0) = Z_2$, each point of which is represented by the germ of a line,
and the divisor at infinity $\mathcal{O}(\infty) = I_2$, each point of which is represented by
the germ of an ordinary cusp. Thus the action of $PGL(2) \times PGL(2)$ on
$F(2) \times F(2)$ has nine orbits:
If $C$, $D$, and $X$ are suitably transverse, i.e., if $X_{s}(2,2)$ is mapped transversely to $C(2) \times D(2)$ by $\sigma$ and the intersection $(C(2) \times D(2)) \cap \sigma(X_{s}(2,2))$ is contained in the dense orbit $\mathcal{O}(-) \times \mathcal{O}(-)$, then we can argue, as in the proof of Theorem A2 of [6], that the proto-contact number (22) correctly counts the number of simultaneous contacts. In particular, we have the following result.

**Theorem 4.** Suppose that the curves $C$ and $D$ contain no lines, and that the general member of $\mathcal{X}$ contains no line. If $C$, $D$, and $\mathcal{X}$ are in general position with respect to the action of $\text{PGL}(2) \times \text{PGL}(2)$ on $F(2) \times F(2)$, then (22) counts the number of simultaneous contacts of order $(3, 3)$ between $C$ and $D$ and the members of $\mathcal{X}$.

**Proof.** We will show that, for each nondense orbit $\mathcal{O}$ listed above, we have

$$\dim((C(2) \times D(2)) \cap \mathcal{O}) + \dim(\mathcal{X}_{s}(2,2) \cap (\mathcal{O} \times S)) < \dim \mathcal{O}. \tag{23}$$

Since $C$, $D$, and $\mathcal{X}$ are in general position with respect to the action, transversality theory [14] then tells us that the intersection

$$(C(2) \times D(2)) \cap \sigma(\mathcal{X}_{s}(2,2)) \cap \mathcal{O}$$

is empty. Therefore the intersection of $\sigma(\mathcal{X}_{s}(2,2))$ and $C(2) \times D(2)$ is transverse, with all intersections occurring in the dense orbit $\mathcal{O}(-) \times \mathcal{O}(-)$. By definition $\mathcal{X}_{s}(2,2) = \mathcal{X}(2) \times_{s} \mathcal{X}(2)$, where $\mathcal{X}(2)$ is the closure of the graph of a function defined on a dense subset of $\mathcal{X}$, and $C(2) \times D(2)$ is the closure of the graph of a function defined on a dense subset of $C \times D$. By a general position argument, all intersections between $\sigma(\mathcal{X}_{s}(2,2))$ and $C(2) \times D(2)$ are intersections between the graphs of these functions. Hence each intersection point corresponds to a pair of points $(c, d)$, in which $c$ is a nonsingular point of both $C$ and some member $X_{s}$ of the family, and $d$ is a nonsingular point of both $D$ and $X_{s}$. The second-order data of $C$ and $X_{s}$ at $c$ are identical, as are the second-order data of $D$ and $X_{s}$ at $d$. Hence each intersection point is a simultaneous triple contact.
To see that (23) holds for the nondense orbits, note that the intersection of $C(2)$ or $D(2)$ with the divisor at infinity is finite. Thus

$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(\infty) \times \mathcal{O}(\infty))) \leq 1$$
$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(\infty) \times \mathcal{O}(\infty))) \leq 0.$$

If $C$ and $D$ contain no lines, then $C(2) \cap \mathcal{O}$ and $D(2) \cap \mathcal{O}$ are each finite. Hence

$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(\infty) \times \mathcal{O}(0))) \leq 1$$
$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(0) \times \mathcal{O}(0))) \leq 0$$
$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(\infty) \times \mathcal{O}(\infty))) \leq 0.$$

The dimension of $\mathcal{X}_S(2,2)$ is 6, and, since the general member of $\mathcal{X}$ is reduced (hence generically smooth), we obtain

$$\dim(\mathcal{X}_S(2,2) \cap (\mathcal{O}(\infty) \times \mathcal{O}(\infty) \times S)) \leq 5$$
$$\dim(\mathcal{X}_S(2,2) \cap (\mathcal{O}(\infty) \times \mathcal{O}(\infty) \times S)) \leq 5$$
$$\dim(\mathcal{X}_S(2,2) \cap (\mathcal{O}(\infty) \times \mathcal{O}(\infty) \times S)) \leq 5$$
$$\dim(\mathcal{X}_S(2,2) \cap (\mathcal{O}(0) \times \mathcal{O}(\infty) \times S)) \leq 5$$
$$\dim(\mathcal{X}_S(2,2) \cap (\mathcal{O}(\infty) \times \mathcal{O}(0) \times S)) \leq 5.$$

If the general member of $\mathcal{X}$ contains no line, then likewise

$$\dim(\mathcal{X}_S(2,2) \cap (\mathcal{O}(\infty) \times \mathcal{O}(0) \times S)) \leq 5$$
$$\dim(\mathcal{X}_S(2,2) \cap (\mathcal{O}(\infty) \times \mathcal{O}(\infty) \times S)) \leq 5$$
$$\dim(\mathcal{X}_S(2,2) \cap (\mathcal{O}(\infty) \times \mathcal{O}(\infty) \times S)) \leq 5.$$

Thus (23) is true in each of the eight cases.

Theorem 4 has a clear advantage over Theorem 3 in that the hypotheses are readily verified. Thus results like Theorem 4 appear to be highly desirable. To establish such a result, however, one needs an understanding of the $PGL(2)$-orbit structure of $F(n)$, which becomes increasingly complicated as $n$ grows. For example, there are 8 orbits on $F(3)$ and 21 orbits...
on $F(4)$. (For a derivation of the first number, see Theorem 2 of [6]. The second number was first obtained by Oberlin College student Dan Frankowski by Mathematica calculations [30] and later confirmed by us.) Even worse, since $\text{PGL}(2)$ has dimension 8, there are infinitely many orbits on $F(n)$, none of them dense, when $n \geq 7$. (In fact, the authors, along with Oberlin College students Ian Robertson and Susan Sierra, have found that there are infinitely many orbits on $F(6)$.) Hence theorems such as Theorem 4 cannot exist for simultaneous contacts of arbitrary order. This is why Theorem 3 is stated as it is, and appears to be the best possible result of its type.

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