A Non-Archimedean Wave Equation

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Abstract

Let $K$ be a non-Archimedean local field with the normalized absolute value $|\cdot|$. It is shown that a “plane wave” $f(t + \omega_1 x_1 + \cdots + \omega_n x_n)$, where $f$ is a Bruhat-Schwartz complex-valued test function on $K$, $(t, x_1, \ldots, x_n) \in K^{n+1}$, $\max_{1 \leq j \leq n} |\omega_j| = 1$, satisfies, for any $f$, a certain homogeneous pseudo-differential equation, an analog of the classical wave equation. A theory of the Cauchy problem for this equation is developed.

Key words: local field, plane wave, pseudo-differential equation, Cauchy problem

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1 INTRODUCTION

Pseudo-differential equations for complex-valued functions defined on non-Archimedean local fields, in particular the field $\mathbb{Q}_p$ of $p$-adic numbers, are becoming increasingly important, both in view of rich mathematical structures involved in their studies and due to emerging applications; see the surveys in [1, 7, 8, 9, 13, 14, 16].

In most cases, pseudo-differential equations over $\mathbb{Q}_p^n$ with the symbols $|P(\xi_1, \ldots, \xi_n)|_p^\alpha$, $\alpha > 0$, where $P$ is a polynomial, were studied. The class of elliptic operators correspond to such polynomials $P$ that $P(\xi_1, \ldots, \xi_n) \neq 0$ for $(\xi_1, \ldots, \xi_n) \neq 0$. An equation of the Schrödinger type is obtained if there is a distinguished variable, say $\xi_1$, and $P(\xi_1, \ldots, \xi_n) = \xi_1 - r(\xi_2, \ldots, \xi_n)$ where $r$ is a $p$-adic quadratic form. It has been understood also that analogs of parabolic equations are evolution equations with a real time variable and $p$-adic spatial variables (this is connected with probabilistic applications; see [7] and references therein). It seemed natural to interpret the case where $P(\xi_1, \ldots, \xi_n) = \xi_1^2 - r(\xi_2, \ldots, \xi_n)$ with an anisotropic quadratic form.

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As the case of hyperbolic equations. However the results available for this case (see [7]) are quite scant. In particular, no well-posed problems for such equations have been identified.

In this paper we propose an alternative approach. Instead of a formal resemblance in the definition of an equation, we proceed from its properties. Let us call a function \( u(t, x_1, \ldots, x_n) : \mathbb{Q}_p^{n+1} \to \mathbb{C} \) a plane wave, if

\[
u(t, x_1, \ldots, x_n) = f(t + \omega_1 x_1 + \cdots + \omega_n x_n)
\]

where \( f \) belongs to the Bruhat-Schwartz space \( \mathcal{D}(\mathbb{Q}_p) \) of test functions, \( \omega_1, \ldots, \omega_n \in \mathbb{Q}_p \), \( \max_{1 \leq j \leq n} |\omega_j|_p = 1 \) (in fact, we will consider not only \( \mathbb{Q}_p \) but arbitrary non-Archimedean local fields; see below).

We will show that every function (1) is a solution of the equation

\[
D^\alpha_t u - D^{\alpha,n}_x u = 0
\]

where \( D^\alpha \) is Vladimirov’s fractional differentiation operator, that is a pseudo-differential operator with the symbol \(|\xi|^\alpha_p\), while \( D^{\alpha,n} \) is a pseudo-differential operator of \( n \) variables with the symbol \( \max_{1 \leq j \leq n} |\xi_j|^\alpha_p \), \( \alpha > 0 \) is an arbitrary number.

The equation (2) with \( n = 1 \) was mentioned in [15] as an example of the following pathology. Consider the equation for a related fundamental solution \( E \):

\[
D^\alpha_t E - D^{\alpha,n}_x E = \delta
\]

where \( E \) belongs to some class of distributions, on which the operators are defined, with the usual relations between them and the Fourier transform. Then, performing the Fourier transform we obtain the contradictory identity

\[
\tilde{E}(\tau, \xi_1, \ldots, \xi_n) = 1 \text{ where the left-hand side vanishes on the open set}
\]

\[
\left\{ 0 \neq (\tau, \xi_1, \ldots, \xi_n) \in \mathbb{Q}_p^{n+1} : |\tau|_p = \max_{1 \leq j \leq n} |\xi_j|_p \right\}
\]

Therefore the fundamental solution cannot exist, and one cannot expect any reasonable behavior of an inhomogeneous equation associated with (2). On the other hand, the set of solutions of the one-dimensional equation \( D^\alpha_t u = \lambda u \) is infinite-dimensional (see [7, 14]). Thus, at the first sight, the equation (2) does not look as an evolution equation with the “time” variable \( t \).

Nevertheless, in this paper we prove the existence and uniqueness of solutions for some analogs of the Cauchy problem for the equation (2) in the class of radial functions, that is those depending (in the variable \( t \)) only on \(|t|_p\). On this class, the operator \( D^\alpha \) becomes a counterpart of the Caputo-Dzhrbashyan regularized fractional derivative appearing in fractional evolution equations of real analysis (see [4]). Moreover, the above connection with plane waves, together with the inversion formula for the Radon transform, available in the non-Archimedean case too [3], leads to a formula for solutions and an analog of the Huygens principle.
2 Preliminaries

2.1. Local fields. Let $K$ be a non-Archimedean local field, that is a non-discrete totally disconnected locally compact topological field. It is well known that $K$ is isomorphic either to a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers (if $K$ has characteristic 0), or to the field of formal Laurent series with coefficients from a finite field $\mathbb{F}_q$, if $K$ has characteristic $p \neq 0$; in this case $q = p^\nu$, $\nu \in \mathbb{N}$. For a summary of main notions and results regarding local fields see, for example, [7].

Any local field is endowed with an absolute value $| \cdot |$, such that $|x| = 0$ if and only if $x = 0$, $|xy| = |x| \cdot |y|$, $|x + y| \leq \max(|x|, |y|)$. Denote $O = \{x \in K: |x| \leq 1\}$, $P = \{x \in K: |x| < 1\}$, $U = O \setminus P$. $O$ is a subring of $K$ called the ring of integers, $P$ is an ideal in $O$ called the prime ideal; the multiplicative group $U$ is called the group of units. The ideal $P$ contains an element $\beta$, such that $P = \beta O$ (a prime element). The quotient ring $O/P$ is actually a finite field; denote by $q$ its cardinality. We will always assume that the absolute value is normalized, that is $|\beta| = q^{-1}$. The normalized absolute value $| \cdot |$ takes the values $q^N$, $N \in \mathbb{Z}$.

If $K = \mathbb{Q}_p$ ($p$ is a prime number), that is a completion of the field $\mathbb{Q}$ of rational numbers with respect to the absolute value

$$|x|_p = p^{-\nu} \text{ for } x = p^\nu \frac{m}{n},$$

where $\nu, m, n \in \mathbb{Z}$, and $m, n$ are prime to $p$, then $\beta = p$ ($p$ is seen as an element of $\mathbb{Q}_p$) and $q = p$ (as a natural number).

Returning to a general local field $K$, denote by $S \subset O$ a complete system of representatives of the residue classes from $O/P$. Then any nonzero element $x \in K$ admits the canonical representation in the form of the convergent series

$$x = \beta^{-n} \left( x_0 + x_1 \beta + x_2 \beta^2 + \cdots \right)$$

where $n \in \mathbb{Z}$, $|x| = q^n$, $x_j \in S$, $x_0 \notin P$. For $K = \mathbb{Q}_p$, one may take $S = \{0, 1, \ldots, p - 1\}$.

The additive group of any local field is self-dual, that is if $\chi$ is any non-constant complex-valued additive character of $K$, then any other additive character can be written as $\chi_a(x) = \chi(ax)$, $x \in K$, for some $a \in K$. See [7] for an explicit description of the character $\chi$ used in harmonic analysis on local fields (“the canonical additive character”). In particular, it is assumed that $\chi$ is a rank zero character, that is $\chi(x) \equiv 1$ for $x \in O$, while there exists such an element $x_0 \in K$ that $|x_0| = q$ and $\chi(x_0) \neq 1$.

The above duality is used in the definition of the Fourier transform over $K$. Denoting by $dx$ the Haar measure on the additive group of $K$ (normalized in such a way that the measure of $O$ equals 1) we write

$$\tilde{f}(\xi) = \int_K \chi(x\xi)f(x) \, dx, \quad \xi \in K,$$

where $f$ is a complex-valued function from $L_1(K)$. As usual, the Fourier transform $\mathcal{F}$ can be extended from $L_1(K) \cap L_2(K)$ to a unitary operator on $L_2(K)$. If $\mathcal{F}f = \tilde{f} \in L_1(K)$, we have the inversion formula

$$f(x) = \int_K \chi(-x\xi)\tilde{f}(\xi) \, d\xi.$$
Similarly, if \( f : K^n \to \mathbb{C} \), we write
\[
\tilde{f}(\xi_1, \ldots, \xi_n) = \int_{K^n} \chi(x_1 \xi_1 + \cdots + x_n \xi_n) f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n.
\]
The inversion formula is then
\[
f(x_1, \ldots, x_n) = \int_{K^n} \chi(-x_1 \xi_1 - \cdots - x_n \xi_n) \tilde{f}(\xi_1, \ldots, \xi_n) \, d\xi_1 \cdots d\xi_n.
\]

2.2. Spaces of test functions and distributions. A function \( f : K \to \mathbb{C} \) is called locally constant, if there exists such an integer \( l \) that for any \( x \in K \)
\[
f(x + x') = f(x), \quad \text{if } |x'| \leq q^{-l}.
\]
The smallest number \( l \) with this property is called the exponent of local constancy of the function \( f \).

Denote by \( \mathcal{D}(K) \) the set of all locally constant functions with compact supports. \( \mathcal{D}(K) \) is a vector space over \( \mathbb{C} \). In order to furnish it with a topology, consider a subspace \( \mathcal{D}_N \subset \mathcal{D}(K) \) of functions with supports in the ball
\[
B_N = \{ x \in K : |x| \leq q^N \}, \quad n \in \mathbb{Z},
\]
and the exponents of local constancy \( \leq l \).

The space \( \mathcal{D}_N \) is finite-dimensional; thus it has a natural topology induced from \( \mathbb{C} \). Then we set
\[
\mathcal{D}_N = \lim_{l \to \infty} \mathcal{D}_N^l,
\]
and define the topology in \( \mathcal{D}(K) \) as the inductive limit topology, that is
\[
\mathcal{D}(K) = \lim_{N \to \infty} \mathcal{D}_N.
\]
The strong conjugate space \( \mathcal{D}'(K) \) is called the space of Bruhat-Schwartz distributions.

The operation of the Fourier transform preserves the space \( \mathcal{D}(K) \) or \( \mathcal{D}(K^n) \) (this property contrasts the Archimedean case). Therefore the Fourier transform of a distribution defined, by duality, just as for distributions from \( S'(\mathbb{R}^n) \), acts continuously on \( \mathcal{D}'(K) \), resp. \( \mathcal{D}'(K^n) \). As in the case of \( \mathbb{R}^n \), there exists a well-developed theory of distributions over local fields including such topics as convolution, direct product, homogeneous distributions etc. Note in particular that a function \( |x|^\alpha - 1, \Re \alpha > 0 \), admits an analytic continuation in \( \alpha \) to a meromorphic distribution
\[
\langle |x|^{\alpha-1}, \varphi \rangle = \int_K |x|^{\alpha-1}[\varphi(x) - \varphi(0)] \, dx, \quad \varphi \in \mathcal{D}(K), \quad (4)
\]
\( \Re \alpha > 0 \) (see Sect. VIII in [14] for \( K = \mathbb{Q}_p \). The general case is completely similar). See [2, 5, 7, 14] for further details.
Below we will often use the subspaces of $\mathcal{D}(K^n)$,

$$\Psi(K^n) = \{ \psi \in \mathcal{D}(K^n) : \psi(0) = 0 \} ,$$

$$\Phi(K^n) = \left\{ \varphi \in \mathcal{D}(K^n) : \int_{K^n} \varphi(x) \, d^n x = 0 \right\} ,$$

introduced in [1]. The space $\Phi(K^n)$ is called the Lizorkin space of test functions of the second kind; it is a non-Archimedean counterpart of a space of test functions on $\mathbb{R}^n$ proposed by Lizorkin ([10]; see also [11]). The conjugate space $\Phi'(K^n)$ is called the Lizorkin space of distributions of the second kind. The most important property of these spaces is that the Fourier transform $\mathcal{F}$ is a linear isomorphism from $\Psi(K^n)$ onto $\Phi(K^n)$, thus also from $\Phi'(K^n)$ onto $\Psi'(K^n)$. At the same time, $\mathcal{F}$ can be considered as a linear isomorphism from $\Phi(K^n)$ to $\Psi(K^n)$.

2.3. Pseudo-differential operators. The simplest and best studied pseudo-differential operator, acting on complex-valued functions over $K$, is the fractional differentiation operator $D^{\alpha}$, $\alpha > 0$, whose deep investigation was initiated by Vladimirov (see [14]). It is defined as

$$(D^{\alpha}\varphi)(x) = \mathcal{F}^{-1} ||\xi||^\alpha (\mathcal{F}(\varphi))(\xi) (x) , \quad \varphi \in \mathcal{D}(K).$$

Note that $D^{\alpha}$ does not act on the space $\mathcal{D}(K)$, since the function $\xi \mapsto ||\xi||^\alpha$ is not locally constant.

We can assert, for example, that $D^{\alpha} : \mathcal{D}(K) \to L_2(K)$, and the closure of this operator is self-adjoint on $L_2(K)$. On the other hand, $D^{\alpha} : \Phi(K) \to \Phi(K)$ and $D^{\alpha} : \Phi'(K) \to \Phi'(K)$; see [1].

Similarly, for $x = (x_1, \ldots, x_n) \in K^n$, set $\|x\| = \max_{1 \leq j \leq n} |x_j|$. The pseudo-differential operator $D^{\alpha,n} : \mathcal{D}(K^n) \to L_2(K^n)$ is given by the expression

$$(D^{\alpha,n}\varphi)(x) = \mathcal{F}^{-1} \|\xi\|^\alpha (\mathcal{F}(\varphi))(\xi) (x) , \quad \varphi \in \mathcal{D}(K^n).$$

We have $D^{\alpha,n} : \Phi(K^n) \to \Phi(K^n)$ and $D^{\alpha,n} : \Phi'(K^n) \to \Phi'(K^n)$.

An important property of these operators is the possibility to get rid of the Fourier transform and represent them as hyper-singular integral operators. For any $u \in \mathcal{D}(K)$,

$$\left( D^{\alpha} u \right)(x) = \frac{1 - q^\alpha}{1 - q^{-\alpha - 1}} \int_K |y|^{-\alpha - 1} [u(x - y) - u(x)] \, dy ; \quad (5)$$

see [7, 14]. The expression in the right-hand side of (5) makes sense for wider classes of functions, for example, for all bounded locally constant functions.

Similarly, if $u \in \mathcal{D}(K^n)$, then

$$\left( D^{\alpha,n} u \right)(x) = \frac{1 - q^\alpha}{1 - q^{-\alpha - n}} \int_{K^n} \|y\|^{-\alpha - n} [u(x - y) - u(x)] \, d^n y \quad (6)$$

(see [12, 11, 14]).
Lemma 1. If \( u \) is a bounded locally constant function on \( K^n \), then the distribution \( D^{\alpha,n}u \in \Phi'(K^n) \) coincides with the function (6).

Proof. Let \( \varphi \in \Phi(K^n) \). Then

\[
\langle D^{\alpha,n}u, \varphi \rangle = \frac{1 - q^\alpha}{1 - q^{-\alpha - n}} \int_{K^n} u(x) \, d^n x \int_{K^n} \frac{\varphi(y) - \varphi(x)}{\|x - y\|^\alpha + n} \, d^n y.
\]

Let \( \theta > 0 \) be so small that \( u(x) = u(y) \) and \( \varphi(x) = \varphi(y) \) if \( \|x - y\| < \theta \). Denote

\[
C_\alpha = \frac{1 - q^\alpha}{1 - q^{-\alpha - n}} \int_{\|y\| \geq \theta} d y \|y\|^\alpha + n.
\]

Then

\[
\langle D^{\alpha,n}u, \varphi \rangle = \frac{1 - q^\alpha}{1 - q^{-\alpha - n}} \int_{K^n} u(x) \, d^n x \int_{\|x - y\| \geq \theta} \varphi(y) \, d^n y - C_\alpha \int_{K^n} u(x) \varphi(x) \, d^n x
\]

\[
= \frac{1 - q^\alpha}{1 - q^{-\alpha - n}} \int_{K^n} \varphi(y) \, d^n y \int_{\|x - y\| \geq \theta} \frac{u(x) - u(y)}{\|x - y\|^\alpha + n} \, d^n x = \langle \varphi, \psi \rangle
\]

where \( \psi \) is the right-hand side of (6), as desired.

2.4. The Radon transform. Let \( \varphi \in \mathcal{D}(K^n), \ n \geq 2 \). The Radon transform \( \tilde{\varphi}(\xi, s) \), where \( \xi \in K^n, \ \xi \neq 0, \ s \in K \), is defined by the relation

\[
\tilde{\varphi}(\xi, s) = \int_{\xi \cdot x = s} \varphi(x) \, d\omega_{\xi,s}(x)
\]

(see [3]) where \( \omega_{\xi,s} \) is such a measure on the hyperplane \( \xi \cdot x = s \) (we write \( \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n \)) that for any \( \psi \in \mathcal{D}(K^n) \),

\[
\int_{K} \psi(x) \, d\omega_{\xi,s}(x) = \int_{K^n} \psi(x) \, d x.
\]

The function \( \tilde{\varphi} \) possesses the following properties. It is homogeneous of degree -1 in \( \xi \) and \( s \), that is \( \tilde{\varphi}(s\xi, ss) = |s|^{-1} \tilde{\varphi}(\xi, s) \), for any \( s \in K \setminus \{0\} \). Next, \( \tilde{\varphi}(\xi, s) = 0 \), if the expression \( |s| \cdot \|\xi\|^{-1} \) is sufficiently large. The function \( \tilde{\varphi} \) is jointly locally constant in \( \xi \) and \( s \). Finally, the integral \( \int_{K} \tilde{\varphi}(\xi, s) \, d s \) does not depend on \( \xi \). Note that the above properties of a function of \( \xi \) and \( s \) are also sufficient for such a function to be the Radon transform of some function \( \varphi \in \mathcal{D}(K^n) \).

In order to find a connection between the Radon and Fourier transforms (similar to the well-known one for the case of \( \mathbb{R}^n \) [3]), we write

\[
\tilde{\varphi}(s\xi) = \int_{K^n} \varphi(x) \chi(s(s \cdot \xi)) \, d^n x = \int_{K} dr \int_{\xi \cdot x = r} \varphi(x) \chi(sr) \, d\omega_{\xi,r}(x) = \int_{K} \chi(sr) \tilde{\varphi}(\xi, r) \, d r,
\]

where \( \chi \) is the characteristic function of \( K \).
and it follows from the Fourier inversion formula that
\[ \hat{\varphi}(\xi, r) = \int_K \chi(-sr)\hat{\varphi}(s\xi) \, ds. \] (7)

The inversion formula for the non-Archimedean Radon transform is as follows [3]:
\[ \varphi(x) = \frac{1 - q^{n-1}}{(1 - q^{-1})(1 - q^{-n})} \int \langle |s|^{-n}, \hat{\varphi}(\eta, s + \eta \cdot x) \rangle \, d^n\eta, \quad x \in K^n, \] (8)

where the distribution $|s|^{-n}$ is understood in the sense of (4). Substituting (4) into (8) and comparing with (5) we can write the inversion formula in the following form:
\[ \varphi(x) = \frac{1}{1 - q^{-1}} \int \left. \left( D_s^{n-1}\hat{\varphi}(\eta, s + \eta \cdot x) \right) \right|_{s=0} d^n\eta. \] (9)

The identity (9) can be proved directly, by substituting (7) and calculating the integrals.

If $n = 1$, we define the Radon transform by the formula (7). It is easy to check that the inversion formula (9) remains valid for this case too, in the form
\[ \varphi(x) = (1 - q^{-1})^{-1} \int \hat{\varphi}(\eta, \eta x) \, d\eta. \]

3 Radial Eigenfunctions

3.1. $L_2$-solutions. Let $u(x) = \psi(|x|) \in L_2(K)$,
\[ D^\alpha u = \lambda u, \quad \lambda = q^{\alpha N}, \quad N \in \mathbb{Z}, \] (10)
and $u$ is not identically zero.

Let us apply the Fourier transform to both sides of (10). We get
\[ (|\xi|^\alpha - q^{\alpha N}) \tilde{u}(\xi) = 0 \quad \text{for all } \xi \in K. \] (11)

It follows from (11) that the inequality $\tilde{u}(\xi) \neq 0$ is possible only for $|\xi| = q^N$. Since $u$ is a radial function, $\tilde{u}$ also possesses this property [7, 14]. Therefore
\[ \tilde{u}(\xi) = \begin{cases} c, & \text{if } |\xi| = q^N; \\ 0, & \text{if } |\xi| \neq q^N, \quad c \neq 0. \end{cases} \] (12)

By the Fourier inversion and the well-known integration formula (see [7, 14]), we get
\[ u(x) = \begin{cases} cq^N(1 - q^{-1}), & \text{if } |x| \leq q^{-N}; \\ -cq^{N-1}, & \text{if } |x| = q^{-N+1}; \\ 0, & \text{if } |x| > q^{-N+1}. \end{cases} \] (13)
It is easily seen from (12) or (13) that \( u \in \Phi(K) \).

The only radial eigenfunction \( u \) with \( u(0) = 1 \) (an analog of the function \( t \to e^{-\lambda t}, t \in \mathbb{R} \)) corresponds to \( c = q^{-N}(1 - q^{-1})^{-1} \). On the other hand, if \( u(0) = 0 \), then \( c = 0 \).

### 3.2. Generalized solutions

Let us consider solutions \( u \in \Phi'(K) \) of the equation (10). It is natural to call a distribution \( u \in \Phi'(K) \) radial (or spherically symmetric), if, for any \( \omega \in K \), \( |\omega| = 1 \), and any \( \varphi \in \Phi(K) \)

\[
\langle u, \varphi \rangle = \langle u, \varphi \rangle
\]

where \( \varphi(x) = \varphi(x) \). In a similar way, we define a radial distribution from \( \Psi(K) \). It is easy to check that the Fourier transform maps a radial distribution from \( \Phi'(K) \) to a radial distribution from \( \Psi(K) \).

**Proposition 1.** If a radial distribution \( u \in \Phi'(K) \) satisfies the equation (10), then it coincides, for some \( c \in \mathbb{C} \), with the function (13).

**Proof.** By definition of a generalized solution, we have

\[
\langle u, D^\alpha \varphi \rangle = \lambda \langle u, \varphi \rangle \quad \text{for any } \varphi \in \Phi(K).
\]

Writing \( \varphi = \mathcal{F}^{-1} \psi \), \( \psi \in \Psi(K) \), we see that

\[
(D^\alpha \varphi)(x) = \mathcal{F}^{-1}_{|x| \to \lambda |\xi|^\alpha \psi(\xi)}.
\]

The function \( \xi \to |\xi|^\alpha \psi(\xi) \) belongs to \( \Psi(K) \). Therefore, considering \( \mathcal{F} \) as an operator from \( \Phi'(K) \) to \( \Psi'(K) \), we may write

\[
\langle u, D^\alpha \varphi \rangle = \langle \mathcal{F}u, |\xi|^\alpha \psi(\xi) \rangle = \langle |\xi|^\alpha \psi(\xi), \psi(\xi) \rangle,
\]

so that we come to the equality (11) where this time \( \tilde{u} = \mathcal{F}u \in \Phi'(K) \), and the multiplication by \( |\xi|^\alpha - q^{\alpha N} \) is understood in the distribution sense. Thus, for any \( \psi \in \Psi(K) \),

\[
\langle \mathcal{F}u, |\xi|^\alpha - q^{\alpha N} \psi(\xi) \rangle = 0.
\]

On the sphere \( \{ \xi \in K : |\xi| = q^l \}, l \neq N \), the set of functions \( \xi \mapsto (|\xi|^\alpha - q^{\alpha N}) \psi(\xi) \) runs the set of restrictions of all the functions from \( \Psi(K) \). Therefore the restriction of the distribution \( \mathcal{F}u \) to such a sphere equals zero, so that \( \mathcal{F}u \) is concentrated on the sphere \( S_N = \{ \xi \in K : |\xi| = q^N \} \).

The set of restrictions to \( S_N \) of functions from \( \Psi(K) \) coincides with \( \mathcal{D}(S_N) = \lim_{l \to \infty} \mathcal{D}_l(S_N) \) where \( \mathcal{D}_l(S_N) \) is the set of complex-valued functions on \( S_N \) with the exponents of local constancy \( \leq l \). The space \( \mathcal{D}_l(S_N) \) is finite-dimensional; its basis can be constructed from the functions \( \delta_{\sigma_0, \sigma_1, \ldots, \sigma_{N+1}}(t) \) \( (\sigma_0, \sigma_1, \ldots, \sigma_{N+l-1} \in S, \sigma_0 \notin P) \), which equal 1 on elements \( t \in S_N \) with the canonical representations \( t = \beta^{-N}(\sigma_0 + \sigma_1 \beta + \cdots + \sigma_{N+l-1} \beta^{N+l-1}) + O(\beta^{l+1}) \), and 0 on all other \( t \in S_N \).

Denote

\[
c_{\sigma_0, \sigma_1, \ldots, \sigma_{N+l-1}} = \langle \mathcal{F}u, \delta_{\sigma_0, \sigma_1, \ldots, \sigma_{N+l-1}} \rangle.
\]

The ratio of any two elements \( \beta^{-N}(\sigma_0 + \sigma_1 \beta + \cdots + \sigma_{N+l-1} \beta^{N+l-1}) \) belongs to the group of units \( U \). The transformation of one of the functions \( \delta_{\sigma_0, \sigma_1, \ldots, \sigma_{N+l+1}} \) into another (with the same
l) is implemented by the multiplication of the argument by the appropriate ratio. Since $\mathcal{F}u$ is a radial distribution, we find that $c_{\sigma_0, \sigma_1, \ldots, \sigma_{N+l-1}}$ depends only on $l$, say

$$c_{\sigma_0, \sigma_1, \ldots, \sigma_{N+l-1}} = c'_{l-1}.$$  

At the same time,

$$\sum_{\sigma_{N+l} \in S} \delta_{\sigma_0, \sigma_1, \ldots, \sigma_{N+l}, \sigma_{N+l}} = \delta_{\sigma_0, \sigma_1, \ldots, \sigma_{N+l-1}},$$

whence $c'_{l-1} = qc'_l$ and $c'_l = c'_0q^{-l}$, $c'_0 \in \mathbb{C}$. Thus, we have found that

$$\langle \mathcal{F}u, \delta_{\sigma_0, \sigma_1, \ldots, \sigma_{N+l}} \rangle = c'_0 q^{-l}$$

for all $l$.

Meanwhile, the integral

$$\int_{|t| = q^{N}} \delta_{\sigma_0, \sigma_1, \ldots, \sigma_{N+l}}(t) \, dt$$

equals $q^{N-(N+l)-1} = q^{-l-1}$ (see Sect. IV in [14]). Together with (14), this shows that the restriction of the distribution $\mathcal{F}u$ to the sphere $S_N$ equals a constant; outside $S_N$, $\mathcal{F}u$ equals 0. Thus, $\mathcal{F}u$ has the form (12), so that $u$ coincides with the function (13). □

4 Plane Waves

Following a classical pattern we call a function

$$F(t, x) = f(t + \omega_1 x_1 + \cdots + \omega_n x_n),$$

(15)

$t \in K$, $(x_1, \ldots, x_n) \in K^n$, where $\| (\omega_1, \ldots, \omega_n) \| = 1$, $f \in \mathcal{D}(K)$, a non-Archimedean plane wave.

Proposition 2. For any $\alpha > 0$, a non-Archimedean plane wave (15) satisfies the equation

$$D_t^\alpha F - D_{x}^{\alpha, n} F = 0.$$  

(16)

Proof. Suppose that $n \geq 2$ (in the case $n = 1$ the validity of (16) is checked in a straightforward way). Let us compute $D_{x}^{\alpha, n} F$. By the definition of $D_{x}^{\alpha, n} F$,

$$(D_{x}^{\alpha, n} F)(t, x) = \frac{1 - q^\alpha}{1 - q^{-n-a}} \int_{K^n} \left( \max_j |y_j| \right)^{-n-a}

\times \left[ f \left( t + \sum_{j=1}^{n} \omega_j x_j - \sum_{j=1}^{n} \omega_j y_j \right) - f \left( t + \sum_{j=1}^{n} \omega_j x_j \right) \right] dy_1 \ldots dy_n.$$
Since \( \max_j |\omega_j| = 1 \), we can choose an index \( j_0 \) in such a way that \( |\omega_{j_0}| = 1 \). Suppose for simplicity that \( |\omega_1| = 1 \). Let us perform the change of variables

\[
\eta_1 = \sum_{j=1}^n \omega_j y_j, \quad \eta_2 = y_2, \ldots, \eta_n = y_n.
\]

Obviously, \( \max_j |\eta_j| \leq \max_j |y_j| \). On the other hand,

\[
y_1 = \frac{1}{\omega_1} (\eta_1 - \omega_2 \eta_2 - \cdots - \omega_n \eta_n),
\]

whence \( \max_j |y_j| \leq \max_j |\eta_j| \), so that

\[
\max_j |y_j| = \max_j |\eta_j|.
\]

The Jacobian of the transformation \((y_1, \ldots, y_n) \mapsto (\eta_1, \ldots, \eta_n)\) equals

\[
\begin{vmatrix}
\omega_1 & \omega_2 & \cdots & \omega_n \\
0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1
\end{vmatrix}
\]

and belongs to \( U \). We have

\[
(D_x^{\alpha,n} F)(t, x) = \frac{1 - q^\alpha}{1 - q^{-n-\alpha}} \int_{K^n} \left( \max_j |\eta_j| \right)^{-n-\alpha} \\
\times \left[ f \left( t + \sum_{j=1}^n \omega_j x_j - \eta_1 \right) - f \left( t + \sum_{j=1}^n \omega_j x_j \right) \right] d\eta_1 \ldots d\eta_n \\
= \frac{1 - q^\alpha}{1 - q^{-n-\alpha}} \int_K \left[ f \left( t + \sum_{j=1}^n \omega_j x_j - \eta_1 \right) - f \left( t + \sum_{j=1}^n \omega_j x_j \right) \right] d\eta_1 \\
\times \int_{K^{n-1}} \left( \max_{1 \leq j \leq n} |\eta_j| \right)^{-n-\alpha} d\eta_2 \ldots d\eta_n.
\]

In order to compute the integral over \( K^{n-1} \), we write it in the form

\[
\int_{K^{n-1}} \left( \max_{1 \leq j \leq n} |\eta_j| \right)^{-n-\alpha} d\eta_2 \ldots d\eta_n = I_1 + I_2,
\]

\[
I_1 = \int_{\max_{2 \leq j \leq n} |\eta_j| < |\eta_1|} |\eta_1|^{-n-\alpha} d\eta_2 \ldots d\eta_n,
\]
\[
I_2 = \int_{\max_{2 \leq j \leq n} |\eta_j| \geq |\eta_1|} \left( \max_{2 \leq j \leq n} |\eta_j| \right)^{-n-\alpha} \, d\eta_2 \ldots d\eta_n.
\]

It is well known (see, for example, [12]) that
\[
\int_{\max_{2 \leq j \leq n} |\eta_j| = q^k} \, d\eta_2 \ldots d\eta_n = q^{(n-1)k} \left( 1 - q^{-n+1} \right).
\]

Suppose that \( |\eta_1| = q^\nu, \nu \in \mathbb{Z} \). Then
\[
I_1 = |\eta_1|^{-n-\alpha} \sum_{k=-\infty}^{\nu-1} q^{(n-1)k} \left( 1 - q^{-n+1} \right) = |\eta_1|^{-n-\alpha} q^{(n-1)(\nu-1)} = q^{-n+1}|\eta_1|^{-\alpha-1},
\]
\[
I_2 = \sum_{k=\nu}^{\infty} q^{-k(n+\alpha)} q^{(n-1)k} \left( 1 - q^{-n+1} \right) \left( 1 - q^{-n+1} \right) \sum_{k=\nu}^{\infty} q^{-k(n+\alpha)} = \frac{1 - q^{-n+1}}{1 - q^{-n+1}} |\eta_1|^{-\alpha-1},
\]
so that
\[
\int_{K^{n-1}} \left( \max_{1 \leq j \leq n} |\eta_j| \right)^{-n-\alpha} \, d\eta_2 \ldots d\eta_n = \frac{1 - q^{-n-\alpha}}{1 - q^{-n+1}} |\eta_1|^{-\alpha-1}.
\]

Therefore
\[
(D^{\alpha,n}_x F) (t, x) = \frac{1 - q^x}{1 - q^{-\alpha-1}} \int_{K} |\eta_1|^{-\alpha-1} \left[ f \left( t + \sum_{j=1}^{n} \omega_j x_j - \eta_1 \right) - f \left( t + \sum_{j=1}^{n} \omega_j x_j \right) \right] \, d\eta_1 = (D^{\alpha}_t F) (t, x),
\]
which means that \( F \) satisfies the equation (16).

5 Cauchy Problems

5.1. Applications of the Radon transform. Let \( \varphi \in \mathcal{D}(K^n) \). We will look for a solution \( F(t, x) \) of the equation (16) satisfying the initial condition
\[
F(0, x) = \varphi(x), \quad x \in K^n,
\]
or the modified initial condition
\[
(D^{\alpha,n-1}_t F) (0, x) = \varphi(x), \quad x \in K^n.
\]

Of course, the conditions (17) and (18) coincide if \( n = 1 \).

Let \( \tilde{\varphi}(\xi, s) \) be the Radon transform of the initial function \( \varphi \). Denote
\[
\Gamma(t, x, u) = \tilde{\varphi}(u, t + u \cdot x), \quad t \in K, \ x, u \in K^n, \ ||u|| = 1.
\]
Let us consider the functions

\[ F_1(t, x) = (1 - q^{-1})^{-1} \int_{\|u\|=1} (D_t^{n-1} \Gamma) (t, x, u) \, d^n u, \]

\[ F_2(t, x) = (1 - q^{-1})^{-1} \int_{\|u\|=1} \Gamma(t, x, u) \, d^n u. \]

**Theorem 1.** The functions \( F_1(t, x) \) and \( F_2(t, x) \) are radial in \( t \), jointly locally constant in \( (t, x) \), bounded solutions of the Cauchy problem (16), (17) and the modified Cauchy problem (16), (18) respectively.

**Proof.** It follows from the identity (7) that \( \tilde{\varphi}(\xi, r) \) belongs to \( \mathcal{D}(K) \) in \( r \) uniformly with respect to \( \xi \in K^n, \|\xi\|=1 \) -- there exists a compact set in \( K \), outside which \( \tilde{\varphi}(\xi, \cdot) \) vanishes, for all the above \( \xi \), and \( \tilde{\varphi}(\xi, r + r') = \tilde{\varphi}(\xi, r) \) if \( |r'| \leq q^{-l} \) where \( l \) does not depend on \( \xi \). This means that \( \Gamma \) is locally constant in \( t, x \), uniformly with respect to \( u \in K^n \) with \( \|u\|=1 \). In addition, \( \Gamma \) and \( D_t^{n-1} \Gamma \) are bounded, uniformly with respect to \( u \). These properties make it possible to change the order of integration while \( D_t^{\alpha} \Gamma \) and \( D_{x,n}^{\alpha} \Gamma \) are computed. Then Proposition 2 shows that \( F_1 \) and \( F_2 \) satisfy the equation (16). The initial conditions are satisfied due to the Radon inversion formula (9).

In order to check that \( F_2(t, x) \) is radial in \( t \), we notice that

\[ \tilde{\varphi}(\omega \xi, \omega s) = \tilde{\varphi}(\xi, s), \quad |\omega| = 1, \]

by virtue of the homogeneity property of \( \tilde{\varphi} \). Therefore \( \Gamma(\omega t, x, u) = \tilde{\varphi}(u, \omega t + u \cdot x) = \tilde{\varphi}(\omega^{-1} u, t + (\omega^{-1} u) \cdot x) = \Gamma(t, x, \omega^{-1} u), \) so that

\[ F_2(\omega t, x) = (1 - q^{-1})^{-1} \int_{\|u\|=1} \Gamma(t, x, \omega^{-1} u) \, d^n u = F_2(t, x). \]

Since the operator \( D_t^{n-1} \) commutes with the operator \( f(t) \mapsto f(\omega t), |\omega| = 1 \), we find also that \( F_1 \) is radial in \( t \).

Let us study the solution \( F_2(t, x) \) of the modified Cauchy problem (16), (18) in a little greater detail. Using the connection (7) between the Fourier and Radon transforms we get that

\[ \int_{\|u\|=1} \Gamma(t, x, u) \, d^n u = \int_K \chi(-st) \, ds \int_{\|u\|=1} \chi(-s(u \cdot x)) \tilde{\varphi}(su) \, d^n u. \quad (19) \]

Next,

\[ \int_{\|u\|=1} \chi(-s(u \cdot x)) \tilde{\varphi}(su) \, d^n u = \int_{K^n} \varphi(y) \, d^n y \int_{\|u\|=1} \chi(s(u \cdot (y - x))) \, d^n u. \]
By the well-known integration formula (see, for example, [12]),

\[
\int_{\|u\|=1} \chi(s(u \cdot (y - x))) \, d^n u = \begin{cases} 
1 - q^{-n}, & \text{if } |s| \cdot \|y - x\| \leq 1; \\
-q^{-n}, & \text{if } |s| \cdot \|y - x\| = q; \\
0, & \text{if } |s| \cdot \|y - x\| > q,
\end{cases}
\]

so that

\[
\int_{\|u\|=1} \chi(-s(u \cdot x)) \tilde{\varphi}(su) \, d^n u = (1 - q^{-n}) \int_{\|y-x\| \leq |s|^{-1}} \varphi(y) \, d^n y - q^{-n} \int_{\|y-x\|=q|s|^{-1}} \varphi(y) \, d^n y. \tag{20}
\]

**Proposition 3.** Suppose that \(\varphi(x) = 0\) for \(\|x\| > q^N\), and \(\varphi(y) = \varphi(x)\) if \(\|y - x\| \leq q^{-\nu}\), \(\nu, N \in \mathbb{N}\). Then \(F_2(t + t', x) = F_2(t, x)\), if \(|t'| \leq q^{-\nu}\), and \(F_2(t, x) = 0\) for \(|t| > q^{N+1}\).

**Proof.** By (19) and (20),

\[
F_2(t, x) = (1 - q^{-1})^{-1} \int_{K} \chi(-st) R(s, x) \, ds \tag{21}
\]

where

\[
R(s, x) = (1 - q^{-n}) \int_{\|y-x\| \leq |s|^{-1}} \varphi(y) \, d^n y - q^{-n} \int_{\|y-x\|=q|s|^{-1}} \varphi(y) \, d^n y. \tag{22}
\]

If \(|s| \geq q^{\nu+1}\), then

\[
R(s, x) = \varphi(x) \left( (1 - q^{-n}) \int_{\|y\| \leq |s|^{-1}} d^n y - q^{-n} \int_{\|y\|=q|s|^{-1}} d^n y \right)
\]

\[
= \varphi(x) |s|^{-n} \left[ (1 - q^{-n}) - q^{-n} \cdot q^n (1 - q^{-n}) \right] = 0,
\]

so that

\[
F_2(t, x) = (1 - q^{-1})^{-1} \int_{|s| \leq q^{\nu}} \chi(-st) R(s, x) \, ds,
\]

which implies the required local constancy in \(t\).

Let \(|t| > q^{N+1}\). Then there exists such an element \(s_0 \in K\), \(|s_0| = q^{-N-1}\), that \(\chi(s_0 t) \neq 1\). If \(\|x\| \leq q^N\), then \(\varphi(y) = 0\) for \(\|y - x\| > q^N\). Therefore for \(|s| < q^{-N}\) the second summand in the right-hand side of (22) equals zero, while the domain of integration in the first summand can be fixed as \(\{y \in K^n : \|y - x\| \leq q^N\}\), if \(|s| < q^{-N}\). Therefore \(R(s, x)\) is constant in \(s\) on the set \(\{s \in K : |s| < q^{-N}\}\), which implies the equality \(R(s + s_0, x) = R(s, x)\) for all the values of \(s\). Making in (21) the change of variables \(s = s' + s_0\) we come to the identity \(F_2(t, x) = \chi(s_0 t)F_2(t, x)\), which yields the required equality \(F_2(t, x) = 0\). □

Note that the local constancy of \(F_2\) in \(t\) may be interpreted as a counterpart of the finite domain of dependence for a classical wave equation: if the initial function \(\varphi\) is such that
\( \varphi(x) = 0 \) outside some compact set \( C \subset K^n \), then \( F_2(t,z) = 0 \) for \( z \in K^n \setminus C \), at least on some neighbourhood of the origin \( t = 0 \). Meanwhile, the fact that \( F_2(t,z) \) vanishes, as \( |t| \) becomes big enough (for a given \( \|x\| \)), resembles the Huygens principle, the existence of the trailing edge of a wave.

### 5.2. A uniqueness theorem. Here we consider the uniqueness problem in the class of generalized solutions, radial in \( t \).

Denote by \( \Phi'(K,\Phi'(K^n)) \) the set of distributions over the test function space \( \Phi(K) \), with values in \( \Phi'(K^n) \).

**Theorem 2.** Let \( F \in \Phi'(K,\Phi'(K^n)) \) be a generalized solution of the equation (16), that is

\[
\langle \langle F, D_t^α \varphi_1 \rangle, \varphi_2 \rangle = \langle \langle F, \varphi_1 \rangle, D_x^{α,n} \varphi_2 \rangle
\]

for any \( \varphi_1 \in \Phi(K), \varphi_2 \in \Phi(K^n) \). If \( F \) is radial in \( t \), then \( F \in \mathcal{D}(K, \Phi'(K^n)) \). If, in addition, \( F(0, x) = 0 \) or \( D_t^{α-1} F \) \((0, x) = 0 \), then \( F(t, x) \equiv 0 \).

**Proof.** Denote by \( \tilde{F}(t, \cdot) \) the Fourier transform of \( F \) in the variable \( x \); as usual, we abuse the notation slightly, writing a distribution in the variable \( t \) as a function of \( t \). For any \( \psi \in \Psi(K^n) \) we have

\[
D_t^α \langle \tilde{F}(t, \cdot), \psi \rangle = \langle \|\xi\|^α \tilde{F}(t, \xi), \psi(\xi) \rangle.
\]

If \( \text{supp} \psi \subset S_N = \{ \xi \in K^n : \|\xi\| = q^N \}, N \in \mathbb{N} \), then

\[
D_t^α \langle \tilde{F}(t, \cdot), \psi \rangle = q^{αN} \langle \tilde{F}(t, \cdot), \psi \rangle.
\]

By Proposition 1, the function \( \langle \tilde{F}(t, \cdot), \psi \rangle \) has the form (13) with \( t \) substituted for \( x \) and some \( c \in \mathbb{C} \). If \( \psi \in \Psi(K^n) \), then \( \psi \) is a sum of a finite number of functions supported on spheres \( S_N \). Taking, in particular, \( \psi = \tilde{\varphi}, \varphi \in \Phi(K^n) \), we find that \( \langle F(t, \cdot), \varphi \rangle \) belongs to \( \mathcal{D}(K) \) in the variable \( t \), for any \( \varphi \in \Phi(K^n) \).

If \( F(0, \cdot) = 0 \), then also \( \tilde{F}(0, \cdot) = 0 \). If \( \psi \in \Psi(K^n) \), \( \text{supp} \psi \subset S_N \), then, as we have seen, \( \langle \tilde{F}(t, \cdot), \psi \rangle \) has the form (13), and the assumption \( \tilde{F}(0, \cdot) = 0 \) implies the equality \( c = 0 \), whence \( \langle \tilde{F}(t, \cdot), \psi \rangle = 0 \), and \( \tilde{F}(t, \cdot) = 0 \) (since \( \psi \) and \( N \) are arbitrary), and \( F(t, \cdot) = 0 \).

Next, if a function \( u(t) \) has a form (13), then its Fourier transform has a form (12), and it is easy to find \( (D_t^{α-1} u)(t) \):

\[
(D_t^{α-1} u)(t) = \begin{cases} 
  c(1 - q^{-1})q^{Nn}, & \text{if } |t| \leq q^{-N}; \\
  -cq^{Nn-1}, & \text{if } |t| = q^{-N+1}; \\
  0, & \text{if } |t| > q^{-N+1}. 
\end{cases}
\]

Repeating the above arguments, we find that the equality \( (D_t^{α-1} F)(0, x) \) implies \( F(t, x) \equiv 0 \).

It follows from Lemma 1 that bounded locally constant solutions of the equation (16) are generalized solutions of the class considered in Theorem 2. Therefore the solutions of the Cauchy problems constructed in Theorem 1 are unique in the class of radial in \( t \), bounded
locally constant functions. It is natural to see such solutions as \textit{classical} solutions of the non-Archimedean wave equation (16).

\subsection*{5.3. Representation of solutions}
Suppose that \( \varphi \in \Phi(K^n) \). We will look for a solution belonging to \( \Phi(K) \) and radial in \( t \), for each \( x \in K^n \), and belonging to \( \Phi(K^n) \) in \( x \), for each \( t \in K \). In this framework, we may use the Fourier transform, only we should not forget to check that the resulting solution indeed satisfies the above requirements.

Let us consider the modified Cauchy problem (16), (18). Suppose that \( n \geq 2 \). Performing the Fourier transform in \( x \) we come to the problem

\begin{equation}
D_t^n \tilde{F}_2(t, \xi) - ||\xi||^n \tilde{F}_2(t, \xi) = 0, \\
\left( D_t^{n-1} \tilde{F}_2 \right)(0, \xi) = \tilde{\varphi}(\xi).
\end{equation}

As we have seen,

\[ \tilde{F}_2(t, \xi) = \begin{cases} 
  c(\xi)(1-q^{-1})||\xi||, & \text{if } |t| \leq ||\xi||^{-1}; \\
  -c(\xi)q^{-1}||\xi||, & \text{if } |t| = q||\xi||^{-1}; \\
  0, & \text{if } |t| > q||\xi||^{-1},
\end{cases} \]

where \( c(\xi) \in \mathbb{C}, c(0) = 0 \); note that for \( \xi = 0 \) it follows from (23) that \( \tilde{F}_2(t, 0) \) is a constant which must equal zero by our assumption that \( F_2 \in \Phi(K) \) in \( t \).

Computing \( D_t^{n-1} \tilde{F}_2 \) as above (see the proof of Theorem 2) we find that

\[ \left( D_t^{n-1} \tilde{F}_2 \right)(t, \xi) = \begin{cases} 
  c(\xi)(1-q^{-1})||\xi||^n, & \text{if } |t| \leq ||\xi||^{-1}; \\
  -c(\xi)q^{-1}||\xi||^n, & \text{if } |t| = q||\xi||^{-1}; \\
  0, & \text{if } |t| > q||\xi||^{-1},
\end{cases} \]

We find from the initial condition (24) that \( c(\xi) = (1-q^{-1})||\xi||^{-n} \tilde{\varphi}(\xi) \), and come to the expression

\[ \tilde{F}_2(t, \xi) = ||\xi||^{-n+1}b(t\xi)\tilde{\varphi}(\xi) \quad (25) \]

where

\[ b(z) = \begin{cases} 
  1, & \text{if } ||z|| \leq 1; \\
  -1/q, & \text{if } ||z|| = q; \\
  0, & \text{if } ||z|| > q.
\end{cases} \]

Since \( \tilde{\varphi} \in \Psi(K^n) \), it vanishes on a neighbourhood of the origin, and it follows from (25) that \( \tilde{F}_2 \in \Psi(K^n) \) in \( \xi \), so that \( F_2 \in \Phi(K^n) \) in \( x \). In addition, \( F_2, \tilde{F}_2 \in D(K) \) in \( t \), uniformly with respect to \( x \) (in the sense of support and local constancy), which permits to interchange operations in different variables. On the other hand, calculating the Fourier transforms we obtain from (25) the following representation of the solution \( F_2(t, x) \):

\[ F_2(t, x) = (A * B_t * \varphi)(x) \quad (26) \]

where the convolution is taken with respect to \( x \), \( A(x) = \frac{1-q^{-n+1}}{1-q^{-1}}||x||^{-1}, B_t(x) = |t|^{-n}b(t^{-1}x), \)

\[ \tilde{b}(\zeta) = \begin{cases} 
  q^{-n}, & \text{if } ||\zeta|| \leq q^{-1}; \\
  q, & \text{if } ||\zeta|| = 1; \\
  0, & \text{if } ||\zeta|| > 1.
\]
The representation (26) makes it possible, for example, to investigate the dependence \( \varphi \mapsto F_2(t, \cdot) \) with respect to the \( L_\kappa \)-norms (for a fixed \( t \in K \)), \( 1 < \kappa < \infty \).

Note that
\[
\|B_t\|_{L_1(K^n)} = |t|^{-n} \int_{K^n} |\tilde{\beta}(t^{-1}x)| \, d^n x = \int_{K^n} |\tilde{\beta}(x)| \, d^n x,
\]
and the Young inequality, together with the commutativity of convolution, gives
\[
\|F_2(t, \cdot)\|_{L_\kappa} \leq C \|A \ast \varphi\|_{L_\kappa}
\]
where \( C \) does not depend on \( t \). Applying a result regarding the Riesz potentials from \([12]\), we find that for \( 1 < \kappa < \frac{n}{n-1} \),
\[
\|F_2(t, \cdot)\|_{L_\lambda} \leq C' \|\varphi\|_{L_\kappa}
\]
where \( \lambda = \frac{n\kappa}{n - \kappa(n-1)} \), and \( C' \) does not depend on \( t \).

For the Cauchy problem (16), (17) (including the case \( n = 1 \)), we have \( F_1(t, x) = (B_t \ast \varphi)(x) \), so that
\[
\|F_1(t, \cdot)\|_{L_\kappa} \leq C \|\varphi\|_{L_\kappa}
\]
for any \( \kappa \in (1, \infty) \), with a constant \( C \) independent of \( t \).

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