Quantifying Unsharpness of Observables in an Outcome-Independent way

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Abstract
Recently, in the paper (Liu and Luo, Phys. Rev. A 104, 052227, 2021) a very beautiful measure of the unsharpness (fuzziness) of the observables constructed via uncertainty is discussed. This measure does not depend on the values of the outcomes and can measure the intrinsic unsharpness of the observables. In this work, we also quantify the unsharpness of observables in an outcome-independent way. But our approach is operationally motivated and different than the approach of the paper (Liu and Luo, Phys. Rev. A 104, 052227, 2021). In this work, at first, we construct two Lüders instrument-based unsharpness measures and provide the tight upper bounds of those measures. Then we prove the monotonicity of our proposed measures under a class of fuzzifying processes (processes that make the observables more fuzzy). This is consistent with the resource-theoretic framework. Then we relate our approach to the approach of the paper (Liu and Luo, Phys. Rev. A 104, 052227, 2021). Next, we try to construct two instrument-independent unsharpness measures. In particular, we define two instrument-independent unsharpness measures and provide the tight upper bounds of those measures and then we derive the condition for the monotonicity of those measures under a class of fuzzifying processes and prove the monotonicity for dichotomic qubit observables. Then we show that for an unknown measurement, the values of all of these measures can be determined experimentally. Finally, we present the idea of the resource theory of the sharpness of the observables and discuss an example where the sharpness of the observables can be considered as a resource.

1 Introduction

In quantum mechanics, observables are known as positive operator-valued measures (POVMs), mainly of two types- (i) sharp observables (also known as projection-valued measure or PVM) and (ii) unsharp observables (that are not PVM). Quantifying the unsharpness of observables is an interesting research direction to look at. Few works in this direction have been already done
[1, 2, 3, 4, 5, 6, 7, 8, 9]. Recently, in Ref. [9], the unsharpness of observables is quantified using uncertainty. The measure defined in Ref. [9], is outcome-independent.

It is well-known that POVMs provide more advantages than PVMs in different information-theoretic tasks and in that sense, one can consider the unsharpness as a resource (from the information-theoretic task perspective) [10, 11, 12, 13, 14]. But in the above-said case, it is implicitly assumed that one has the ability to perform the measurement of any single observable (i.e., PVMs as well as POVMs) with arbitrary accuracy. But of course, the simultaneous measurement of multiple observables is restricted by the uncertainty principle [34].

Recently, it has been shown in [15] that one requires infinite amount of resources to perform ideal PVMs accurately. Therefore, one can not perform PVMs accurately with finite amount of resources and the more resources one has, the more accurately one can perform a PVM. Therefore, the ability to perform PVMs or equivalently the sharpness of observables can be considered as a resource from the cost perspective. It should be mentioned here that in Section 7, we discuss an example of an information-theoretic task where sharpness can be considered as a resource. Furthermore, it is known that the optimal measurement for minimum-error state discrimination $d$ linearly independent pure states is a projective measurement where $d$ is the dimension of the Hilbert space on which the pure state lies [16]. The above-said discussions suggest that whether sharpness or unsharpness can be considered as a resource, is context-dependent.

The beautiful measure of unsharpness that is proposed in Ref. [9] is based on uncertainty and main mathematically motivated. This fact immediately raises the question: Is it possible to construct an operationally motivated unsharpness measure? We try to tackle this question in this paper. More specific details are provided in the next paragraph.

In this work, we also quantify the unsharpness of the observables in an outcome-independent way. But our approach is different than the approach of Ref. [9] and is mainly operationally motivated. In Ref. [9], the authors derived the measures of the unsharpness of observables via uncertainty. But in this paper, we have constructed our measures of the unsharpness of observables through the repeated measurements of same observable. Now, in practice, one must need a quantum instrument to measure any observable in the lab as quantum instruments are equivalent with measurement models [17, 18, 19, 20]. One of the most well-known types of instruments is the Lüders instrument. It is well-known that any quantum instrument associated with an observable is a quantum postprocessing of the Lüders instrument associated with that observable [20]. This fact makes the Lüders instrument very special. Therefore, it is important to construct the Lüders instrument-based measures. Therefore, we first define two Lüders instrument-based measures. Then we discuss the different properties of these measures. Now, as unsharpness is a property of the observable itself and not the property of an instrument associated with it, one should try to construct instrument-independent unsharpness measure. Therefore, we try to construct two instrument-independent unsharpness measures. We provide a conjecture and if that can be proven, those instrument-independent measures will be consistent with the resource-theoretic framework for qubit observables. Then we discuss that the values of all of these measures can be determined experimentally. Then we provide the justification for taking sharpness as a resource and also present the idea of the resource theory which can be completed in the future.

The rest of this paper is organized as follows. In Section 2, we discuss the preliminaries. From Section 3 we start discussing our main results. In particular, in Section 3.1, we construct two Lüders instrument-based unsharpness measures and provide the tight upper bounds of those measures. In Section 3.2, we prove the monotonicity of the above-said measures under
a class of fuzzifying processes. In Section 3.3, we relate our approach to the approach of Ref. [9]. In the Section 4, we try to construct two instrument-independent unsharpness measures. In particular, in Section 4.1, we define two instrument-independent unsharpness measures and derive the tight upper bounds of those measures. In Section 4.2, we derive the condition for the monotonicity of those measures under a class of fuzzifying processes and prove the monotonicity for dichotomic qubit observables. In Section 5, we show that for an unknown measurement, the values of all of these measures can be determined experimentally. In Section 7, we discuss an example where sharpness can be considered as a resource. In Section 6, we present the idea of the resource theory of the sharpness of the observables. Finally, in Section 8, we summarize our results and discuss the future outlook.

2 Preliminaries

In this section, we discuss the preliminaries.

2.1 Observables

An observable (positive operator-valued measure or POVM) \( A \) acting on the Hilbert space \( \mathcal{H} \) is defined as a set of positive Hermitian matrices i.e., \( A = \{A_i\}_{i=1}^n \) such that \( \sum_i A_i = I_{d \times d} \) where \( d \) is the dimension of the Hilbert space \( \mathcal{H} \). The dimension of the Hilbert space is \( d \) and the set \( \{1,\ldots,n\} \) is called outcome set of \( A \) and is denoted by \( \Omega_A \). Clearly \( A_i \in L^+(\mathcal{H}) \) and \( I_{d \times d} \geq A_i \geq 0 \) for all \( i \in \Omega_A \). Therefore, \( A_i^2 \leq A_i \) for all \( i \in \Omega_A \). If \( A_i^2 = A_i \) holds for all \( i \in \Omega_A \), we call \( A \) as a projection-valued measure (PVM). PVMs are the sharp observable and clearly PVMs are the special cases of POVMs. Clearly, one outcome trivial sharp observable is \( I_{d \times d} \). If there exist at least one \( j \in \Omega_A \) such that \( A_j^2 < A_j \) then the observable \( A \) is not a PVM. This type of observables are unsharp observables [9].

2.2 Quantum Channels

A quantum channel \( \Gamma : S(\mathcal{H}) \rightarrow S(\mathcal{K}) \) is a completely positive trace preserving (CPTP) map where \( S(\mathcal{H}) \) is the state space (i.e., the set of density matrices on the Hilbert space \( \mathcal{H} \)) [21, 22]. For a quantum channel \( \Gamma \), \( \Gamma(\rho) \) can always be written as \( \Gamma(\rho) = \sum_i K_i \rho K_i^\dagger \). This form of \( \Gamma \) is called as the Kraus representation of \( \Gamma \) and \( K_i \)'s are called the Kraus operators of \( \Gamma \). The channel \( \Gamma^* : L(\mathcal{K}) \rightarrow L(\mathcal{H}) \) is called as the dual channel (i.e., in Heisenberg picture) of \( \Gamma : S(\mathcal{H}) \rightarrow S(\mathcal{K}) \), \( \text{Tr}[\Gamma(\rho)X] = \text{Tr}[\rho \Gamma^*(X)] \) holds, where \( L(\mathcal{H}) \) is the set of bounded linear operators on the Hilbert space \( \mathcal{H} \).

A special type of channel is the depolarising channel. A depolarising channel \( \Gamma_d \) is defined as \( \Gamma_d(\rho) = t \rho + (1-t)\frac{I}{d} \) for all \( \rho \in S(\mathcal{H}) \) and \( t \in [-\frac{1}{3},1] \). To simplify the notation we have written \( I_{d \times d} = I \).

2.3 Quantum Instruments

A quantum instrument \( I \) is a set of completely positive (CP) maps \( \{\Phi_i : S(\mathcal{H}) \rightarrow L^+(\mathcal{K})\} \) i.e., \( I = \{\Phi_i\} \) such that \( \Phi = \sum_i \Phi_i \) is a quantum channel where \( L^+(\mathcal{H}) \) is the set of positive bounded linear operators on the Hilbert space \( \mathcal{H} \) [22]. Suppose \( A = \{A_i\} \) is an
observable. A quantum instrument \( I = \{ \Phi_i \} \) is called \( \mathcal{A} \)-compatible instrument if \( \text{Tr}[\Phi_i(\rho)] = \text{Tr}[\rho A_i] \) for all \( i \in \Omega_\mathcal{A} \). Therefore, the observable \( \mathcal{A} \) can be measured using the instrument \( I \).

There exist a special type of of instruments which are known as Lüders instruments. For an observable \( \mathcal{A} = \{ A_j \} \), the \( \mathcal{A} \)-compatible Lüders instrument is defined as \( I^L_\mathcal{A} = \{ \Phi^L_j \}_{j=1}^n \) such that \( \Phi^L_j(\rho) = \sqrt{A_j}\rho\sqrt{A_j} \) for all \( i \in \{ 1, \ldots, n_\mathcal{A} \} \).

### 2.4 Quantifying Unsharpness of Observables via Uncertainty

In this subsection, we briefly discuss the approach of the Ref. [9]. For the complete discussion, readers can check the Ref. [9]. Suppose we have an observable \( \mathcal{A} = \{ A_j \}_{j=1}^n \) acting on the Hilbert space \( \mathcal{H} \). Suppose the exact value of the \( i \) th outcome is \( \alpha_i \). Let \( \alpha \) be a row vector such that \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n_\mathcal{A}}) \). Then \( K_{\alpha}^A \) is defined as \( K_{\alpha}^A = \sum_i \alpha_i^2 A_i \). Similarly, \( K_{\alpha^2}^A \) is defined as \( K_{\alpha^2}^A = \sum_i \alpha_i^2 A_i \). Then a function \( F_\rho(\mathcal{A}, \alpha) \) is introduced as

\[
F_\rho(\mathcal{A}, \alpha) = V_\rho(\mathcal{A}, \alpha) - V(K_{\alpha^2}^A) \tag{1}
\]

where \( V_\rho(\mathcal{A}, \alpha) = \sum_i \text{Tr}[\rho A_i] \alpha_i^2 - (\sum_i \text{Tr}[\rho A_i] \alpha_i)^2 \) is the uncertainty (i.e., variance) of the observable \( \mathcal{A} \) for the state \( \rho \) and \( V(K_{\alpha^2}^A) = \text{Tr}[\rho(K_{\alpha^2}^A)^2] - (\text{Tr}[\rho K_{\alpha^2}^A])^2 \) is the conventional variance.

Next a noise operator \( N^A_{\alpha} = K_{\alpha^2}^A - (K_{\alpha^2}^A)^2 \) is introduced. Then it is shown that

\[
F_\rho(\mathcal{A}, \alpha) = \text{Tr}[\rho N^A_{\alpha}] \tag{2}
\]

Clearly, \( F_\rho(\mathcal{A}, \alpha) \geq 0 \). \( F_\rho(\mathcal{A}, \alpha) = 0 \) for all \( \rho \in S(\mathcal{H}) \) iff \( \mathcal{A} \) is a PVM. Now, it is shown in the Ref. [9] that

\[
F_\rho(\mathcal{A}, \alpha) = \alpha F_\rho(\mathcal{A}) \alpha^T \tag{3}
\]

where \( F_\rho(\mathcal{A}) \) is a matrix such that

\[
[F_\rho(\mathcal{A})]_{ij} = \delta_{ij} \text{Tr}[\rho A_i] - \text{Tr}[\rho \frac{A_j A_i + A_i A_j}{2}] \tag{4}
\]

Here \( \delta_{ij} \) is Kronecker delta. Since, \( F_\rho(\mathcal{A}, \alpha) \geq 0 \), \( F_\rho(\mathcal{A}) \geq 0 \) and \( F_\rho(\mathcal{A}) = 0 \) iff \( \mathcal{A} \) is a PVM. This \( F_\rho(\mathcal{A}) \) matrix is independent of \( \alpha \) and very important to construct the the unsharpness measure of an observable \( \mathcal{A} \). Next, the matrix \( \mathcal{F}(\mathcal{A}) \) is defined as \( \mathcal{F}(\mathcal{A}) = F_{\vec{\alpha}}(\mathcal{A}) \). Now it has been mentioned in the Ref. [9] that any unitarily invariant norm of \( \mathcal{F}(\mathcal{A}) \) can quantify of the unsharpness of \( \mathcal{A} \). For simplicity they have taken \( l^1 \) norm \( l_1 \) which is defined as \( ||X||_1 = \sum_{ij} |[X]_{ij}| \) for a matrix \( X \). Therefore, the unsharpness measure of an observable \( \mathcal{A} \) is

\[
\mathcal{F}(\mathcal{A}) = ||\mathcal{F}(\mathcal{A})||_1 \tag{5}
\]

**Remark 1** It should be mentioned here that information extraction through a measurement and the sharpness of the measurement are completely different concept. For example, one outcome trivial observable \( l_{\alpha \beta} \) is sharp, but one can not extract information using it.

Now, note that above-said approach is mathematical and based on uncertainty (standard deviation based). Therefore, one immediate question is whether one can construct an operationally motivated unsharpness measure. In this paper, we try to tackle this question.
3 Lüders Instrument-Based Unsharpness Measures of Observables

From this section, we start to discuss our main results.

3.1 Construction and the Upper Bound of the Lüders Instrument-Based Unsharpness Measure $E^L$

The sharp quantum observables (PVMs) have an interesting property that makes those observables different from the unsharp observables. Next, we discuss this property of PVMs which motivates us to quantify the unsharpness of the observables in the following outcome independent way. Suppose Alice is measuring an observable $A = \{A_i\}$ on a quantum state $\rho \in \mathcal{H}_d$ using the Lüders instrument $L^A = \{\Phi^A_i(\rho)\}$. After obtaining the outcome $i$, the post measurement state will be $\rho_i = \Phi^A_i(\rho)$. The probability of obtaining the outcome $i$ is $p_i = \text{Tr}[\rho A_i]$. Now, after obtaining the outcome $i$ if one more time the same observable $A$ is immediately measured by Alice on this post measurement state $\rho_i$, the probability of again obtaining the same outcome $i$ is

$$p_{ii} = \text{Tr}[\rho_i A_i] = \frac{\text{Tr}[\rho A_i^2]}{\text{Tr}[\rho A_i]}.$$  \hfill (6)

Now if $A$ is PVM $A_i^2 = A_i$ for all $i$. Therefore, $p_{ii} = 1$ if $A$ is a PVM. Therefore, if a PVM $A$ is measured repeatedly on the same system, the outcome will definitely repeat. If $A$ is not a PVM, there exist an outcome $j$ for which $A_j < A_j^2$ and therefore, $p_{jj} < 1$. Therefore, there is a non-zero probability that an unsharp observable will not produce the same outcome on immediate repeated measurements of the same observable. This is a feature of an unsharp observable or equivalently is the evidence of the unsharpness of an observable and is of course an essential difference between a PVM and POVM. This fact motivates us to quantify the unsharpness of the observables in the following outcome-independent way.

In the above experiment, the average probability that any outcome will repeat in the successive measurement of $A$ is

$$P^L(\rho; A) = \sum_i p_i p_{ii} = \sum_i \text{Tr}[\rho A_i^2] = \text{Tr}[\rho \sum_i A_i^2] = \text{Tr}[\rho E^A].$$  \hfill (7)

where $E^A = \sum_i A_i^2$. We will call $E^A$ as $E$-matrix of $A$. Clearly, $E^A$ is a positive Hermitian matrix and $E^A \leq \mathbb{1}$. Now the average probability that a outcome will never repeat is

$$E^L(\rho; A) = 1 - P^L(\rho; A) = \text{Tr}[\rho (\mathbb{1} - E^A)] \leq \|\rho\|_\rho \|\mathbb{1} - E^A\|$$  \hfill (8)

$$\leq \|\mathbb{1} - E^A\|$$  \hfill (9)

where $\|X\|$ denotes the operator norm i.e., the highest eigen value of a Hermitian matrix $X$ and $\|X\|_\rho$ denotes the trace norm of a Hermitian matrix $X$ i.e., $\|X\|_\rho = \text{Tr}[\sqrt{X^* X}]$. In the second last inequality, we have used the fact that if $T \in \mathcal{L}(\mathcal{H})$ is a trace-class (i.e., has a finite trace norm) Hermitian operator and $S \in \mathcal{L}(\mathcal{H})$ is an arbitrary Hermitian operator,
then $\text{Tr}[ST] \leq \|T\|_r \|S\|_r$ [22]. In the last equality, we have used the fact that $\rho$ is Hermitian and $\rho \geq 0$ and therefore, $\|\rho\|_r = \text{Tr}[\rho] = 1$. Next, we show that there exist a quantum state for which the bound written in (9) is achievable. Suppose $|e'_{\text{max}}\rangle$ is the eigen state (i.e., normalised eigen vector) corresponding to the maximum eigen value of $(\mathbb{I} - E^A)$. Then, $\langle e'_{\text{max}}|E^A|e'_{\text{max}}\rangle = \|(\mathbb{I} - E^A)\|$. Taking maximization of the quantity $\mathcal{E}^\ell(\rho; A)$ over all set density matrices $\rho$, we obtain

$$
\mathcal{E}^\ell(\rho; A) = \max_\rho \mathcal{E}^\ell(\rho; A)
= \langle e'_{\text{max}}|E^A|e'_{\text{max}}\rangle
= \|(\mathbb{I} - E^A)\|.
$$

We define $\mathcal{E}^\ell(\rho; A)$ as the Lüders instrument-based unsharpness measure of the observable $A$. Clearly, if $A$ is a PVM, $E^A = \mathbb{I}$ and therefore, $\mathcal{E}^\ell(\rho; A) = 0$. If $A$ is not a PVM then there exists at least one $i$ such that $A_i^2 < A_i$ and therefore, $E^A < \mathbb{I}$ and therefore, $\mathcal{E}^\ell(\rho; A) > 0$. Therefore, $\mathcal{E}^\ell$ is a faithful measure. Clearly, $\mathcal{E}^\ell$ measure is independent of the bijective relabeling of outcomes and of the values of outcomes.

There exist a upper bound for this unsharpness measure $\mathcal{E}^\ell$. Our following lemma states that-

**Lemma 1** For any arbitrary observable $A = \{A_i\}_{i=1}^d$, $\mathcal{E}^\ell(\rho; A) \leq 1 - \frac{1}{n_A}$. This bound is achieved by the observable $T^{n_A} = \{T_i^{n_A} = \frac{1}{n_A}\}^n_{i=1}$.

**Proof** Suppose, $a'^{\text{max}}_{\text{min}} = 1 - a_{\text{min}}$ is the maximum eigen value of $(\mathbb{I} - E^A)$ and $|a'^{\text{max}}_{\text{min}}\rangle$ is the corresponding eigen vector. Therefore, $\|(\mathbb{I} - E^A)\| = a'^{\text{max}}_{\text{min}}$. This implies that $a_{\text{min}}$ is the minimum eigen value of $E^A$ and $|a'^{\text{max}}_{\text{min}}\rangle = |a_{\text{min}}\rangle$ is the corresponding eigen vector. Now suppose, $\{|n\rangle\}$ is the eigen basis of $E^A$. Therefore, for some $n = n', |n\rangle = |a_{\text{min}}\rangle$. Then

$$
a_{\text{min}} = \langle a_{\text{min}}|E^A|a_{\text{min}}\rangle
= \sum_i \langle a_{\text{min}}|A_i^2|a_{\text{min}}\rangle
= \sum_i \sum_{n=1}^d \langle a_{\text{min}}|A_i|n\rangle\langle n|E_i|a_{\text{min}}\rangle
\geq \sum_i \sum_{n=1}^d \langle a_{\text{min}}|A_i|a_{\text{min}}\rangle^2
= \sum_i \langle a_{\text{min}}|A_i|a_{\text{min}}\rangle
$$

where $x_i = |\langle a_{\text{min}}|A_i|a_{\text{min}}\rangle|$. Now we know that

$$
\sum_i x_i = \sum_i |\langle a_{\text{min}}|A_i|a_{\text{min}}\rangle| = 1
$$

as $A_i \geq 0$ for all $i \in \{1, \ldots, n_A\}$ and $\sum_i A_i = \mathbb{I}$. Now we know from the optimization method of Lagrange’s undetermined multipliers that $\sum_{i=1}^n x_i^2$ takes the minimum value subject to condition $\sum_{i=1}^n x_i = 1$ for $x_1 = x_2 = \ldots = x_n = \frac{1}{n}$. Therefore, using this fact, the inequality (11) becomes
This implies that
\[
E^L(A) = \| I - E^A \| \\
= 1 - a_{min} \\
\leq 1 - \frac{1}{n_A}.
\]

Now, for the observable \( T^n_A \),
\[
E^L(T^n_A) = \| I - E^{T^n_A} \| \\
= \| I - \sum_i \frac{1}{n_A} \| \\
= \| I - \frac{1}{n_A} \| \\
= 1 - \frac{1}{n_A}.
\]

We can also define another measure of the unsharpness in a different way. It is to be noted that (8) is linear in \( \rho \). Now let \( \mathcal{R} = \{ \rho_1, ..., \rho_k \} \) is a set of \( k \) states.

Then, the simple average (i.e., with same probability \( \frac{1}{k} \)) of \( E^L(\rho_i; A) \) over this set \( \mathcal{R} \) is
\[
< E^L(\rho; A) >_{\mathcal{R}} = \frac{1}{k} \sum_{i=1}^{k} E^L(\rho_i; A) \\
= \frac{1}{k} \sum_{i=1}^{k} \text{Tr}[\rho_i(I - E^A)] \\
= \frac{1}{k} \text{Tr}(\sum_{i=1}^{k} \rho_i)(I - E^A)] \\
= \frac{1}{k} \text{Tr}< \rho >_{\mathcal{R}} (I - E^A)]
\]
where \( < \rho >_{\mathcal{R}} = (\sum_{i=1}^{k} \rho_i) \) is the simple average (i.e., with same probability \( \frac{1}{k} \)) of the states over the set \( \mathcal{R} \). Generalising (16) for whole state space \( S(\mathcal{H}) \), we get that
\[
< E^L(\rho; A) >_{S(\mathcal{H})} = \text{Tr}< \rho >_{S(\mathcal{H})} (I - E^A)] \\
= \text{Tr}(\frac{1}{d} (I - E^A)] \\
= E^L(\frac{1}{d}; A).
\]
where \( < \rho >_{S(\mathcal{H})} \) is the simple average of the states over the whole state space \( S(\mathcal{H}) \) and in the second last equality we have used the well-known fact that \( < \rho >_{S(\mathcal{H})} = \frac{1}{d} \). We define the unsharpness measure of \( A \) as
\[
E^L(A) = < E^L(\rho; A) >_{S(\mathcal{H})} \\
= E^L(\frac{1}{d}; A).
\]
Now the lemma below states the upper bound of \( E^L(A) \).
Lemma 2 For any arbitrary observable \( A = \{A_i\}_{i}^{n_A} \), \( \mathcal{E}^{L}(A) \leq 1 - \frac{1}{n_A} \). This bound is achieved by the observable \( T_{n_A} = \{I_{i}^{n_A} = \frac{1}{n_A}\}_{i=1}^{n_A} \).

Proof From the (18), we get

\[
\mathcal{E}^{L}(A) = \mathcal{E}^{L}(\frac{1}{\rho}; A)
\leq \max_{\rho} \mathcal{E}^{L}(\rho; A)
= \mathcal{E}^{L}(A)
\leq 1 - \frac{1}{n_A}.
\] (19)

Now, it is easy to check that \( \mathcal{E}^{L}(T_{n_A}) = 1 - \frac{1}{n_A} \).

These upper bounds provided in Lemma 1 and 2 are important to determine the maximally unsharp observable \( T_{n_A} \).

Remark 2 For an observable \( A = \{A_i\} \), it is very easy to prove that \( \mathcal{E}^{L}(A) = \mathcal{E}^{L}(A^U) \) and \( \mathcal{E}^{L}(A) = \mathcal{E}^{L}(A^U) \) which \( A^U = \{U^T A U\} \). Therefore, \( \mathcal{E}^{e} \) and \( \mathcal{E}^{L}(A) \) does not change if an unitary is acted on the observables in the Heisenberg picture.

3.2 Monotonicity of \( \mathcal{E}^{L} \) and \( \mathcal{E}^{L} \) Under a Class of Fuzzifying Processes

If \( \mathcal{E}^{L} \) is a useful measure of unsharpness (fuzziness), it should be monotonically non-decreasing under the processes which fuzzify the observables (i.e., under the processes that do not increase the sharpness and may decrease it). These processes are called fuzzifying processes. One may intuit that coarse-graining (a process where two or more outcomes are treated as a single one) is a fuzzifying process. We show through the next example that this is not true in general.

Example 1 Consider two observables \( A = \{A_i\}_{i=1}^{3} \) and \( B = \{B_i\}_{i=1}^{2} \) acting on \( \mathcal{H}^{3} \) where \( A_1 = \sum_{i=1} \lambda_{i} |i\rangle \langle 1| + \frac{1}{4} |2\rangle \langle 2| \), \( A_2 = \sum_{i=1} \lambda_{i} |i\rangle \langle 1| + \frac{1}{4} |2\rangle \langle 2| \), \( A_3 = |3\rangle \langle 3| \) and \( B_1 = |1\rangle \langle 1| + |2\rangle \langle 2| \), \( B_2 = |3\rangle \langle 3| \). Clearly, \( A_1 + A_2 = B_1 \) and \( A_3 = B_2 \) and therefore, \( B \) is a coarse-graining of \( A \). But \( B \) is a PVM (i.e., a sharp observable) and \( A \) is not a PVM (i.e., an unsharp observable). Therefore, this example suggests that coarse-graining may increase sharpness of the observables and therefore, it can not be a fuzzifying process.

In above example, \( \mathcal{E}^{L}(A) > 0 \) and \( \mathcal{E}^{L}(B) = 0 \). Therefore, under this kind of classical post-processing of the outcomes \( \mathcal{E}^{L} \) may be decreasing. Therefore, it is not possible to prove that \( \mathcal{E}^{L} \) is monotonically non-decreasing under the classical post-processing of outcomes as it is not a fuzzifying process, in general. It is well-known that classical post-processing of outcomes can not increase the information extraction through a measuring an observable. But Example 1 suggests that it can increase the sharpness. This is an consequence of the fact that unsharpness and information extraction are two inequivalent concepts. Furthermore, one may intuit that the convex combination of observables is a fuzzifying process i.e. if an arbitrary observable \( A \) is convexly combined with any other arbitrary observable, the resulting observable will be more unsharp than \( A \). We will show through the next example that this is also not true, in general.
Example 2 Consider a pair of observables \( \mathcal{A} = \{A_i\} \) and \( \mathcal{B} = \{B_j\} \) acting on \( \mathcal{H}_3 \) where 
\[ A_1 = \frac{1}{2}|1\rangle\langle 1| + \frac{3}{4}|2\rangle\langle 2|, \quad A_2 = \frac{1}{2}|1\rangle\langle 1| + \frac{3}{4}|2\rangle\langle 2|, \quad A_3 = |3\rangle\langle 3| \]
and 
\[ B_1 = |1\rangle\langle 1|, \quad B_2 = |2\rangle\langle 2|, \quad B_3 = |3\rangle\langle 3| \].
We define a observable \( \mathcal{C} = \{C_i\} \) where \( C_i = \lambda A_i + (1-\lambda)B_i \) and \( 0 \leq \lambda \leq 1 \). Clearly, 
\[ C_i = [1 - \frac{i}{4}]|1\rangle\langle 1| + \frac{3}{4}|2\rangle\langle 2|, \quad C_i = \frac{1}{2}|1\rangle\langle 1| + [1 - \frac{i}{4}]|2\rangle\langle 2|, \quad C_i = |3\rangle\langle 3| \]. It can be observed that the sharpness of the observable \( \mathcal{C} \) increases with with the decrement of \( \lambda \) and for \( \lambda = 0 \), \( C_0 = \mathcal{B} \) which is a PVM. Now since, for \( \lambda = 1 \), \( C^1 = \mathcal{A} \), \( C^2 \) is always sharper than \( \mathcal{A} \) for all values of \( \lambda \). It can be easily shown that \( \mathcal{E}^2(\mathcal{A}) \geq \mathcal{E}^2(\mathcal{C}^i) \) for all values of \( \lambda \)’s.

The above example shows that it is also not possible to prove that \( \mathcal{E}^2 \) is monotonically non-decreasing under the convex combination of observables as it is not a fuzzifying process, in general.

Example 1 and Example 2 suggest that it is not an easy task to specify all fuzzifying processes. But one can specify the special classes of fuzzifying processes. One can easily understand that the addition of white noise in the observables is a fuzzifying process. Therefore, we restrict ourselves to this particular class of fuzzifying processes and show that \( \mathcal{E}^2 \) is monotonically non-decreasing under this class of fuzzifying processes in the following theorem.

Theorem 1 Suppose \( \mathcal{A} = \{A_i\}_{i=1}^{n_A} \) is an unsharp version of \( \mathcal{A} = \{A_i\}_{i=1}^{n_A} \) i.e., \( A_i = \lambda A_i + (1-\lambda)\frac{1}{n_A} \) for all \( i \in \{1, \ldots, n_A\} \) where \( 1 \geq \lambda \geq 0 \). Then \( \mathcal{E}^2(\mathcal{A}^i) \geq \mathcal{E}^2(\mathcal{A}) \) for all \( 1 \geq \lambda \geq 0 \).

Proof The \( E \)-matrix of \( \mathcal{A}^i \), is given by

\[
E^{\mathcal{A}^i} = \sum_i (A_i^i)^2 \\
= \sum_i (\lambda A_i + \frac{1-\lambda}{n_A})^2 \\
= \sum_i (\lambda^2 A_i^2 + \frac{2\lambda(1-\lambda)}{n_A}A_i + \frac{(1-\lambda)^2}{n_A}) \\
= \lambda^2 \sum_i A_i^2 + \frac{2\lambda(1-\lambda)}{n_A} \sum_i \lambda A_i \\
= \lambda^2 \sum_i A_i^2 + \frac{(1-\lambda)^2}{n_A} \lambda. \tag{20}
\]

Therefore,

\[
\mathbb{1} - E^{\mathcal{A}^i} = \mathbb{1} - (\lambda^2 \sum_i A_i^2 + \frac{(1-\lambda)^2}{n_A} \lambda) \\
= \lambda^2 (\mathbb{1} - E^{\mathcal{A}}) + (1 - \lambda^2)(1 - \frac{1}{n_A}). \tag{21}
\]

Now, using the properties of the operator norm, we get

\[
\|\mathbb{1} - E^{\mathcal{A}^i}\| \leq \|\lambda^2 (\mathbb{1} - E^{\mathcal{A}}) + (1 - \lambda^2)(1 - \frac{1}{n_A})\| \\
= \lambda^2 \|\mathbb{1} - E^{\mathcal{A}}\| + (1 - \lambda^2)(1 - \frac{1}{n_A})\| \tag{22}
\]

Suppose, \( |e'_{\text{max}}\rangle \) is the eigen state (i.e., normalised eigenvector) of \( (\mathbb{1} - E^{\mathcal{A}}) \) corresponding to the highest eigenvalue of \( (\mathbb{1} - E^{\mathcal{A}}) \). Then \( \|\mathbb{1} - E^{\mathcal{A}}\| = \langle e'_{\text{max}}|(\mathbb{1} - E^{\mathcal{A}})|e'_{\text{max}}\rangle \). Then, using the properties of the operator norm and (21), we get
\[ \|I - E^{A_i}\| \geq \left( e'_{\text{max}} \right) \left( \|I - E^{A_i}\| e'_{\text{max}} \right) \]
\[ = \lambda^2 \left( e'_{\text{max}} \right) \left( \|I - E^{A_i}\| e'_{\text{max}} \right) \]
\[ + (1 - \lambda^2)(1 - \frac{1}{n_\lambda}) \left( \|I - E^{A_i}\| e'_{\text{max}} \right) \]
\[ = \lambda^2 (1 - \lambda^2)(1 - \frac{1}{n_\lambda}). \]

From inequality (22) and inequality (23), we get
\[ \|I - E^{A_i}\| = \lambda^2 (1 - \lambda^2)(1 - \frac{1}{n_\lambda}). \]

Therefore,
\[ E^L(A^i) - E^L(A) = (1 - \lambda^2)(1 - \frac{1}{n_\lambda}) - (1 - \lambda^2)(1 - \frac{1}{n_\lambda}) \]
\[ \geq 0. \]

We have used Lemma 1 to obtain the last inequality. Hence the theorem is proved.

Next we have an immediate corollary-

**Corollary 1** For any observable \( A = \{A_i\} \), \( E^L(A^{\lambda_2}) \geq E^L(A^{\lambda_1}) \) for all \( 1 \geq \lambda_1 \geq \lambda_2 \geq 0 \).

**Proof** The observable \( A^{\lambda_i} = \{A_i^{\lambda_i} = \lambda_i A_i + (1 - \lambda_i) \frac{I}{n_\lambda}\} \). For notational simplicity we denote all \( A_i^{\lambda_i} \) as \( A_i^{\lambda} \) i.e., \( A_i^{\lambda_i} = A_i^\lambda \) for all \( i \in \{1, \ldots, n_\lambda\} \) and we also denote the observable \( A^{\lambda_i} \) as \( A^\lambda \) i.e., \( A^{\lambda_i} = A^\lambda \). Now the observable \( A^{\lambda_2} = \{A_i^{\lambda_2} = \lambda_2 A_i + (1 - \lambda_2) \frac{I}{n_\lambda}\} \) Suppose \( \gamma = \frac{\lambda_1}{\lambda_2} \). Clearly, \( 1 \geq \gamma \geq 0 \) as \( \lambda_2 \leq \lambda_1 \) and both are positive. Then for all \( i \in \{1, \ldots, n_\lambda\} \),
\[ A_i^{\lambda_2} = \lambda_2 A_i + (1 - \lambda_2) \frac{I}{n_\lambda} \]
\[ = \gamma A_i^\lambda + (1 - \gamma) \frac{I}{n_\lambda} \]
\[ = \gamma A_i^\lambda + (1 - \gamma) \frac{I}{n_\lambda} \]
\[ = A_i^{\gamma} \]

where \( A_i^{\gamma} = \gamma A_i^\lambda + (1 - \gamma) \frac{I}{n_\lambda} \) for all \( i \in \Omega_{A^\lambda} \). Therefore, \( A^{\lambda_2} = A^{\gamma} = \{A_i^{\gamma}\} \). Then using the fact that \( A_i^{\lambda_1} = A^\lambda \) and Theorem 1, we get that \( E^L(A^{\lambda_2}) \geq E^L(A^{\lambda_1}) \). Hence the corollary is proved.

Therefore, \( E^L(A^\lambda) \) is monotonically non-decreasing with decreasing value of \( \lambda \) or equivalently \( E^L \) is monotonically non-increasing with increasing value of \( \lambda \).

Next, we have to prove the monotonicity of \( E^L \) under the addition of white noise. We start with our next theorem.

**Theorem 2** Suppose \( A^\lambda = \{A_i^\lambda\}_{i=1}^{n_\lambda} \) is an unsharp version of \( A = \{A_i\}_{i=1}^{n_\lambda} \) i.e., \( A_i^\lambda = \lambda A_i + (1 - \lambda) \frac{I}{n_\lambda} \) for all \( i \in \{1, \ldots, n_\lambda\} \) where \( 1 \geq \lambda \geq 0 \). Then \( E^L(A^\lambda) \geq E^L(A) \) for all \( 1 \geq \lambda \geq 0 \).
Proof From (18) and (21), we get that
\[
\mathcal{E}^L(A^\lambda) = \mathcal{E}^L(\frac{1}{\lambda}A^\lambda) = \text{Tr}[\frac{\lambda}{d}(1 - E^{A^\lambda})] \\
= \text{Tr}[\frac{\lambda^2}{d}(1 - E^A) + (1 - \lambda^2)(1 - \frac{1}{n_A})] \\
= \lambda^2\text{Tr}[\frac{\lambda}{d}(1 - E^A)] + (1 - \lambda^2)(1 - \frac{1}{n_A}) \\
= \lambda^2\mathcal{E}^L(A) + (1 - \lambda^2)(1 - \frac{1}{n_A})
\] (27)

Therefore,
\[
\mathcal{E}^L(A^\lambda) - \mathcal{E}^L(A) = (\lambda^2 - 1)\mathcal{E}^L(A) + (1 - \lambda^2)(1 - \frac{1}{n_A}) \\
= (1 - \lambda^2)((1 - \frac{1}{n_A}) - \mathcal{E}^L(A)) \\
\geq 0.
\] (28)

We have used Lemma 2 to obtain the last inequality. Hence, the theorem is proved.

Next, we have an immediate corollary

Corollary 2 For any observable \(A = \{A_i\}\), \(\mathcal{E}^L(A^{\lambda_1}) \geq \mathcal{E}^L(A^{\lambda_2})\) for all \(1 \geq \lambda_1 \geq \lambda_2 \geq 0\).

Proof The proof is similar to the proof of Corollary 1. From the (26), we get that \(A^{\lambda_1} = A^{\lambda_2} = \{A_i^{\lambda_1}\}\). Then using the fact that \(A_i^{\lambda_1} = A_i^{\lambda_2}\) and Theorem 2, we get that \(\mathcal{E}^L(A^{\lambda_1}) \geq \mathcal{E}^L(A^{\lambda_2})\). Hence the corollary is proved.

Therefore, \(\mathcal{E}^L(A^\lambda)\) is monotonically non-decreasing with decreasing value of \(\lambda\) or equivalently \(\mathcal{E}^L\) is monotonically non-increasing with increasing value of \(\lambda\).

3.3 Relation of \(\mathcal{E}^L(A)\) and \(\mathcal{E}^L(A)\) with \(F_\rho(A)\)

In this subsection, we relate the approach given in the Ref. [9] (also briefly discussed in Section 2.4) with our approach. More specifically, we relate \(F_\rho(A)\) with \(\mathcal{E}^L(A)\) and \(\mathcal{E}^L(A)\). From the expression of \(F_\rho(A)\) i.e., from the (4), we get that
\[
\text{Tr}[F_\rho(A)] = \sum_i [F_\rho(A)]_{ii} \\
= \sum_i [\text{Tr}[\rho A_i] - \text{Tr}[\rho \frac{(A_i A_i^*)}{2} ]] \\
= \sum_i [\text{Tr}[\rho A_i] - \text{Tr}[\rho A_i^2]] \\
= \text{Tr}[\rho(\mathbb{1} - \sum_i A_i^2)] \\
= \text{Tr}[\rho(\mathbb{1} - E^A)] \\
= \mathcal{E}^L(\rho;A)
\] (29)

Therefore,
\[
\mathcal{E}^L(A) = \max_\rho \mathcal{E}^L(\rho;A) \\
= \max_\rho \text{Tr}[F_\rho(A)].
\] (30)
Now as it is mentioned in the Ref. [9] that $F_{\rho}(A)$ is Hermitian and $F_{\rho}(A) \geq 0$ for any arbitrary observable $A$, we have $\text{Tr}[F_{\rho}(A)] = \|F_{\rho}(A)\|_{tr}$.

$$\mathcal{E}_{L}(A) = \max_{\rho} \|F_{\rho}(A)\|_{tr}. \quad (31)$$

Therefore, through our approach one of the operational meanings of the matrix $F_{\rho}(A)$ can be understood.

Now taking $\rho = \frac{1}{d}$ and from (29), we get that

$$\text{Tr}[F_{\frac{1}{d}}(A)] = \mathcal{E}_{L}(\frac{1}{d}; A) = \mathcal{E}_{L}(A). \quad (32)$$

As $\text{Tr}[F_{\frac{1}{d}}(A)] = \|F_{\frac{1}{d}}(A)\|_{tr}$, we have

$$\mathcal{E}_{L}(A) = \|F_{\frac{1}{d}}(A)\|_{tr}. \quad (33)$$

Now it has been mentioned in the Ref. [9] that any unitarily invariant norm of $\mathcal{F}(A) = F_{\frac{1}{d}}(A)$ can quantify the unsharpness of $A$. Therefore, we can take trace norm of $\mathcal{F}(A)$ as a quantifier of the unsharpness of $A$ [32]. Therefore, $\mathcal{E}_{L}$ measure is consistent with the measure proposed in the Ref. [9].

4 An Attempt to Construct Instrument-Independent Unsharpness Measures

4.1 Construction and the Upper Bound of the Instrument-Independent Unsharpness Measure $\mathcal{E}$

In the previous section, we discussed two Lüders instrument-based unsharpness measures of observables. This discussion raises an immediate question: can one construct an instrument-independent unsharpness measure of observables? We try to answer this question in this section. But, before we start, we provide the following Remark.

**Remark 3** Unlike the Lüders instrument, if a general instrument is used to measure a PVM, then post-measurement states can be arbitrary (depending on the instrument) and will no longer be the eigenstate of the PVM. Therefore, there is no guarantee that the outcome will repeat. Therefore, a general instrument may not help to quantify the unsharpness of the observables in our approach. But still, we are able to show that it is possible to derive two instrument-independent quantities which can at least witness the unsharpness of the observables.

Suppose Alice is using a general $A$-compatible quantum instrument $\mathcal{I}^{A} = \{\Phi_{i}^{A}\}$ on the state $\rho$ to measure an observable $A = \{A_{j}\}$. Then, $q_{i} = \text{Tr}[\Phi_{i}^{A}(\rho)] = \text{Tr}[\rho A_{i}]$ is the probability of getting the outcome $i$ and $\rho'_{i} = \frac{\Phi_{i}^{A}(\rho)}{\text{Tr}[\Phi_{i}^{A}(\rho)]}$ is the post-measurement after obtaining the outcome $i$. Now, after obtaining the outcome $i$ if one more time the observable $A$ is measured by Alice on this post measurement state $\rho'_{i}$, the probability of again obtaining the same outcome $i$ is
The average probability that the outcome will repeat on successive measurements of the observable \( A \) using the instrument \( \mathcal{I}^A \) on the state \( \rho \) is

\[
q_{ii} = \text{Tr}[\rho_i |A_i]\].
\]

(34)

Now the average probability that a outcome will never repeat is

\[
Q(\rho;A) = \max_{\mathcal{I}^A} Q(\rho;A;\mathcal{I}^A) \leq \text{Tr}[\rho \mathcal{X}^A].
\]

(36)

\[
Q(\rho;\mathcal{J}^A) = \sum_i \text{Tr}[\rho A_i] \text{Tr}[a_{\max} |A_i]\]

(38)

\[
Q(\rho;A) = \max_{\mathcal{I}^A} Q(\rho;A;\mathcal{I}^A)
\]

(39)

From inequality (36), (38) and inequality (39) we get

\[
Q(\rho;A) = \text{Tr}[\rho \mathcal{X}^A] = Q(\rho;A;\mathcal{J}^A).
\]

(40)

Therefore, choosing the best instrument \( \mathcal{J}^A \), one can maximize the average probability that the outcome will repeat on successive measurements of the observable \( A \) on the state \( \rho \).
Therefore,
\[ \mathcal{E}(A) = \max_{\rho} \mathcal{E}(\rho; A) \leq ||I - \mathcal{X}^A||. \] (43)

Now suppose \( x'_{\text{max}} \) is the highest eigen value of \( I - \mathcal{X}^A \) and \( |x'_{\text{max}}\rangle \) is the corresponding eigen vector. Therefore, \( \langle x'_{\text{max}} | (I - \mathcal{X}^A)^{\frac{1}{2}} \rangle = \text{Tr}[(x'_{\text{max}} | (I - \mathcal{X}^A)^{\frac{1}{2}}) A | x'_{\text{max}}\rangle] = ||I - \mathcal{X}^A||. \) Then,
\[ \mathcal{E}(A) = \max_{\rho} \mathcal{E}(\rho; A) \geq \mathcal{E}(x'_{\text{max}} | A) = ||I - \mathcal{X}^A||. \] (44)

From inequality (43) and inequality (44) we get
\[ \mathcal{E}(A) = ||I - \mathcal{X}^A||. \] (45)

We define \( \mathcal{E}^A \) as the instrument-independent unsharpness measure of observables. Clearly, \( \mathcal{E} \) measure is independent of the bijective relabeling of outcomes and of the values of outcomes. If \( A \) is a PVM, \( ||A|| = 1 \) for all \( i \in \{1, \ldots, n_A\} \) and \( \mathcal{E}^A = 0 \). Next, we provide a remark on the faithfulness of \( \mathcal{E} \).

**Remark 4** For an observable \( A = \{A_i\}_{i=1}^{n_A} \) acting on the \( d \)-dimensional Hilbert space \( \mathcal{H} \), we know that \( \mathcal{E}(A) = 0 \) only if \( ||A_i|| = 1 \) for all \( i \in \Omega_A \). Let \( A \) be an observable such that \( ||A_i|| = 1 \) for all \( i \in \Omega_A \) and \( |a_{i_{\text{max}}}| \) be the eigenstate (one of the eigen states if the maximum eigen value 1 is degenerate) corresponding to the maximum eigen value 1 for all \( i \in \Omega_A \). Then for any two \( i, j \in \Omega_A \) and \( i \neq j \), suppose that \( \langle a_{i_{\text{max}}} | a_{j_{\text{max}}} \rangle \neq 0 \). Then \( \langle a_{i_{\text{max}}} | A_i + A_j | a_{j_{\text{max}}} \rangle \geq 1 + | \langle a_{i_{\text{max}}} | a_{j_{\text{max}}} \rangle |^2 > 1 \). But Since, \( A_i + A_j - 0 \), \( \langle \psi | A_i + A_j | \psi \rangle \leq 1 \) for all \( |\psi\rangle \in \mathcal{S}(\mathcal{H}) \). Hence, \( \langle a_{i_{\text{max}}} | a_{j_{\text{max}}} \rangle = 0 \) for all \( i \in \Omega_A \). Now as \( \sum_i A_i = I \) and \( \langle a_{i_{\text{max}}} | A_i | a_{j_{\text{max}}} \rangle = 1 \), we have \( \langle a_{i_{\text{max}}} | A_j | a_{j_{\text{max}}} \rangle = 0 \) for any two \( i, j \in \Omega_A \) and \( i \neq j \). Therefore, for all \( j \in \Omega_A \), there exist a \( n_A - 1 \)-dimensional subspace \( \mathcal{K}_j \) of \( \mathcal{H} \) such that for all \( |\psi\rangle \in \mathcal{K}_j \), \( \langle \psi | A_j | \psi \rangle = 0 \) and for 1-dimensional subspace (i.e., for \( |a_{j_{\text{max}}}| \)), \( \langle a_{j_{\text{max}}} | A_j | a_{j_{\text{max}}\rangle = 1 \). Clearly, such construction is not possible for \( n_A > d \) and for \( n_A = d \), such construction implies \( A = \{A_i = |a_{i_{\text{max}}}\rangle \langle a_{i_{\text{max}}} | \} \) is a rank-1 PVM. Therefore, for \( n_A \geq d \), \( \mathcal{E}(A) = 0 \) implies \( A \) is a PVM (sharp observable). Hence, the measure is faithful. Above statement implies that \( \mathcal{E} \) is a faithful measure for all qubit observables (i.e., \( d = 2 \)). For \( n \leq d \), \( \mathcal{E}(A) > 0 \) imply \( A \) is an unsharp observable. But in this case \( \mathcal{E}(A) = 0 \) does not implies \( A \) is a PVM. For example- The qutrit observable \( \mathcal{A}' = \{|1\rangle \langle 1| + \frac{1}{2} |2\rangle \langle 2|, \frac{1}{2} |2\rangle \langle 2| + |3\rangle \langle 3| \} \) is an unsharp observable. But \( \mathcal{E}(A') = 0 \).

Next we calculate the upper bound of \( \mathcal{E} \).

**Lemma 3** For any arbitrary observable \( A = \{A_i\}_{i=1}^{n_A}, \mathcal{E}(A) \leq 1 - \frac{1}{n_A} \). This bound is achieved by the observable \( T^{n_A} = \{I_{n_A} = \frac{1}{n_A}\}_{i=1}^{n_A} \).

**Proof** From the definition of \( \mathcal{E}^{n_A} \), we have
\[
E(\rho; A) = \min_{T^A} E^{\rho; A \cup T^A} \\
\leq E(\rho; A; \mathcal{L}^A)
\] (46)

Taking maximization over \(\rho\) in both side of inequality (46) and from Lemma 1, we get

\[
E^A \leq E'(A) \leq 1 - \frac{1}{n_A}.
\] (47)

Now for the observable \(T^n_A\),

\[
E(T^n_A) = \left\| \mathbb{1} - \Lambda^{T^n_A} \right\| \\
= \left\| \mathbb{1} - \sum_i \frac{1}{n^2_A} \right\| \\
= 1 - \frac{1}{n_A}
\] (48)

Hence, the lemma is proved.

Similar to \(E^L\), we can define another instrument-independent unsharpness measure \(E'\) by taking average of \(E(\rho; A)\) over full state space \(S(\mathcal{H})\). Then

\[
E'(A) = \langle E(\rho; A) \rangle_{S(\mathcal{H})} \\
= \langle \text{Tr}[\rho(\mathbb{1} - \Lambda^A)] \rangle_{S(\mathcal{H})} \\
= \text{Tr}[\rho] \langle \mathbb{1} - \Lambda^A \rangle_{S(\mathcal{H})} \\
= \text{Tr}[\frac{1}{d} (\mathbb{1} - \Lambda^A)] \\
= E(\frac{1}{d}; A).
\] (49)

The statement similar to Remark 3 also holds \(E'\).

Now the lemma below states the upper bound of \(E'(A)\).

**Lemma 4** For any arbitrary observable \(A = \{A_i\}^n_A\), \(E'(A) \leq 1 - \frac{1}{n_A}\). This bound is achieved by the observable \(T^n_A = \{I^n_i = \frac{1}{n_A}\}^n_A\).

**Proof** From the (49), we get

\[
E'(A) = E(\frac{1}{d}; A) \\
\leq \max_{\rho} E(\rho; A) \\
= E(A) \\
\leq 1 - \frac{1}{n_A}.
\] (50)

Now, it is easy to check that \(E'(T^n_A) = 1 - \frac{1}{n_A}\).

The statement similar to Remark 1 also holds for \(E\) and \(E'\).

### 4.2 Monotonicity of \(E\) and \(E'\) Under a Class of Fuzzifying Processes

Since, from Example 1 and Example 2, we get that the coarse-graining and the convex combination of the observables are not the fuzzifying processes, monotonicity of \(E\) cannot be shown. Therefore, here we try to show that under the addition of white noise \(E\)
is monotonically non-decreasing. But unfortunately, it appears that the proof is not so straightforward. Therefore, at first, we derive the condition for the monotonicity of $\mathcal{E}$ under the addition of white noise (i.e., Theorem 3).

**Theorem 3** Suppose $\mathcal{A}^4 = \{A_i^4\}_{i=1}^{n_A}$ is an unsharp version of $\mathcal{A} = \{A_i\}_{i=1}^{n_A}$ i.e., $A_i^4 = \lambda A_i + (1 - \lambda) \frac{1}{n_A}$ for all $i \in \{1, \ldots, n_A\}$ where $1 \geq \lambda \geq 0$. Then $\mathcal{E}(\mathcal{A}^4) \geq \mathcal{E}(\mathcal{A})$ for all $1 \geq \lambda \geq 0$ iff

$$\Sigma_1^A \geq \Sigma_2^A$$  

holds where $\Sigma_1 = \langle x_{min}^4 | \mathcal{A}^4 | x_{min}^4 \rangle$ and $\Sigma_2 = (\Sigma_1^A - \langle x_{min}^4 | \mathcal{A}^4 | x_{min}^4 \rangle)$ where $x_{min}^4$ is the lowest eigen value of $\mathcal{A}^4$ and $|x_{min}^4\rangle$ is the eigen state of $\mathcal{A}^4$ corresponding to the eigen value $x_{min}^4$.

**Proof** The $X$-matrix of $\mathcal{A}^4$ is

$$\mathcal{A}^4 = \sum_i \| \lambda A_i + (1 - \lambda) \frac{1}{n_A} || \lambda A_i + (1 - \lambda) \frac{1}{n_A} \|$$

$$= \sum_i |\lambda||A_i| + (1 - \lambda) \frac{1}{n_A} || \lambda A_i + (1 - \lambda) \frac{1}{n_A} |$$

$$= \lambda^2 \sum_i ||A_i|| A_i + \lambda \sum_i |(\sum_i ||A_i||)| + \sum_i A_i$$

$$= \lambda^2 \mathcal{A}^4 + \frac{\lambda(\sum_i ||A_i||)}{n_A}$$

Therefore,

$$\| - \mathcal{A}^4 \| = \lambda^2 \| - \mathcal{A}^4 \| + (1 - \lambda^2) \|$$

$$= \| - \mathcal{A}^4 \| + \lambda^2 (\sum ||A_i||)$$

$$= (1 - \lambda^2) \| (n_A - 1) + \lambda (n_A - \sum ||A_i||) \|$$

$$+ \lambda^2 \| - \mathcal{A}^4 \|$$

where $\gamma = \gamma(\mathcal{A}, \lambda) = \frac{(1 - \lambda)}{n_A} (n_A - 1) + \lambda (n_A - \sum ||A_i||)$. As $A_i \leq 0$ and therefore, $\sum_i ||A_i|| \leq n_A$, we have $\gamma \geq 0$. Therefore,

$$\mathcal{E}(\mathcal{A}^4) = \| - \mathcal{A}^4 \|$$

$$= \lambda^2 \mathcal{E}(\mathcal{A}) + \gamma.$$  

Now,

$$\mathcal{E}(\mathcal{A}^4) - \mathcal{E}(\mathcal{A}) = \gamma - (1 - \lambda^2) \mathcal{E}(\mathcal{A})$$

$$= \frac{(1 - \lambda)}{n_A} (n_A - 1) + \lambda (n_A - \sum ||A_i||)$$

$$- (1 - \lambda^2) \mathcal{E}(\mathcal{A})$$

$$= (1 - \lambda) (1 - \frac{1}{n_A} - \mathcal{E}(\mathcal{A}))$$

$$+ \lambda (1 - \sum ||A_i|| - \mathcal{E}(\mathcal{A}))$$
Now, \( \mathcal{E}(A) = \| \mathcal{I} - \mathcal{A}^A \| = 1 - x_{\text{min}}^A = 1 - \left< \frac{x_{\text{min}}^A}{n_A} \right| x_{\text{min}}^A \rangle \) where \( x_{\text{min}}^A \) is the lowest eigen value of \( \mathcal{A}^A \) and \( |x_{\text{min}}^A \rangle \) is the eigen state of \( \mathcal{A}^A \) corresponding to the eigen value \( x_{\text{min}}^A \). Then Therefore,

\[
\mathcal{E}(A^A) - \mathcal{E}(A) = (1 - \lambda) ((\left< \frac{x_{\text{min}}^A}{n_A} \right| x_{\text{min}}^A \rangle - \frac{1}{n_A}) \\
+ \lambda (\left< \frac{x_{\text{min}}^A}{n_A} \right| x_{\text{min}}^A \rangle - \frac{\|A\|}{n_A}))
\]

(56)

where \( \Sigma^A = (\left< \frac{x_{\text{min}}^A}{n_A} \right| x_{\text{min}}^A \rangle - \frac{1}{n_A}, \Sigma^A = (\frac{\|A\|}{n_A} - (\left< \frac{x_{\text{min}}^A}{n_A} \right| x_{\text{min}}^A \rangle)) = \frac{\|A\|}{n_A} - x_{\text{min}}^A \) and \( \Sigma^A(\lambda) = [\Sigma^A - \lambda \Sigma^A_2] \).

Now, since \( \mathcal{E}(A) \leq (1 - \frac{1}{n_A}) \|A\| \geq 0. \) Now, There are two following cases-

(I) \( \Sigma^A_2 < 0 \)

In this case, \( \mathcal{E}(A^A) - \mathcal{E}(A) \geq 0 \) always. In this case \( \Sigma^A \geq \Sigma^A_2 \) trivially holds.

(II) \( \Sigma^A_2 \geq 0 \)

In this case, the minimum value of \( \Sigma^A(\lambda) \) (for \( \lambda = 1 \)) is

\[
\Sigma^A_{\text{min}} = [\Sigma^A_2 - \Sigma^A_2] = 2x_{\text{min}}^A - \frac{\|A\|}{n_A} - \frac{1}{n_A}.
\]

Clearly, the condition for \( \mathcal{E}(A^A) - \mathcal{E}(A) \geq 0 \) for all \( \lambda \) is \( \Sigma^A_{\text{min}} \geq 0 \) or equivalently \( \Sigma^A \geq \Sigma^A_2 \).

It appears that the proof of the inequality (51) for arbitrary observable acting on an arbitrary dimensional Hibert space is difficult and therefore, it is difficult to prove that under the addition of white noise \( \mathcal{E} \) is monotonically non-decreasing. Therefore, next, we prove the inequality (51) for the qubit dichotomic observables.

**Proposition 1** For any dichotomic observable \( \mathcal{W} \), \( \Sigma^W \geq \Sigma^W_2 \) and therefore, \( \mathcal{E}(\mathcal{W}^A) \geq \mathcal{E}(\mathcal{W}) \) for all \( 1 \geq \lambda \geq 0. \)

**Proof** Suppose \( \mathcal{W} = \{W_1, W_2\} \) are two qubit dichotomic observables. Clearly \( W_2 = \mathcal{I} - W_1 \).

Let \( W_1 = \omega_1 |\omega_1\rangle \langle \omega_1| + \omega_2 |\omega_2\rangle \langle \omega_2| \). Without the loss of generality, we can choose \( \omega_1 \geq \omega_2 \). Then \( \|W_1\| = \omega_1 \). Now \( W_2 = (1 - \omega_1)|\omega_1\rangle \langle \omega_1| + (1 - \omega_2)|\omega_2\rangle \langle \omega_2| \). Clearly, \( \|W_2\| = (1 - \omega_2) \).

Therefore,

\[
\Sigma^W = \|W_1\|W_1 + \|W_2\|W_2 \\
= \omega_1[|\omega_1\rangle \langle \omega_1| + \omega_2 |\omega_2\rangle \langle \omega_2|] \\
+ (1 - \omega_2)(|\omega_1\rangle \langle \omega_1| + (1 - \omega_2)|\omega_2\rangle \langle \omega_2|) \\
= [\omega_1^2 + (1 - \omega_1)(1 - \omega_2)]|\omega_1\rangle \langle \omega_1| \\
+ [\omega_2 + (1 - \omega_2)^2]|\omega_2\rangle \langle \omega_2| \\
= \omega_1^2 |\omega_1\rangle \langle \omega_1| + \omega_2^2 |\omega_2\rangle \langle \omega_2| \\
\]

(57)

where \( \omega_1' = [\omega_1^2 + (1 - \omega_1)(1 - \omega_2)] \) and \( \omega_2' = [\omega_1\omega_2 + (1 - \omega_2)^2] \). Now

\[
\omega_1' - \omega_2' = [\omega_1^2 + (1 - \omega_1)(1 - \omega_2)] - [\omega_1\omega_2 + (1 - \omega_2)^2] \\
= \omega_1^2 + 1 + \omega_1\omega_2 - \omega_1 - \omega_2 - \omega_1\omega_2 - 1 \\
+ 2\omega_2 - \omega_2^2 \\
= (\omega_1 - \omega_2)(|\omega_1 + \omega_2| - 1). \\
\]

(58)
Therefore, as \( \omega_1 \geq \omega_2 \), we have \( \omega'_1 \geq \omega'_2 \) for \( (\omega_1 + \omega_2) \geq 1 \) and we have \( \omega'_1 \leq \omega'_2 \) for \( (\omega_1 + \omega_2) \leq 1 \).

Therefore, following two cases-

(I) \textbf{For} \((\omega_1 + \omega_2) \geq 1\) -

In this case the minimum eigen value of \( \mathcal{X}^W \) is \( \lambda^{W}_{\min} = \omega'_2 \). Therefore,

\[
\Sigma^{W}_{\min} = \Sigma^{W}_{1} - \Sigma^{W}_{2} = 2x^{W}_{\min} - \frac{1}{2} - \frac{\|W_{1}\| + \|W_{2}\|}{2} \\
= 2[\omega_1 \omega_2 + (1 - \omega_1)(1 - \omega_2)] - \frac{1}{2} - \frac{\omega_1(1 - \omega_2)}{2} \\
= 1 - 4\omega_2 + 2\omega_2^2 + 2\omega_1 \omega_2 - \frac{(\omega_1 - \omega_2)}{2}.
\]

Now from Fig. 1a, we get that that \( \Sigma^{W}_{\min} \geq 0 \) for all \( \omega_1 \) and \( \omega_2 \) satisfying the conditions \( \omega_1 \geq \omega_2 \) and \( \omega_1 + \omega_2 \geq 1 \).

(II) \textbf{For} \((\omega_1 + \omega_2) \leq 1\) -

In this case the minimum eigen value of \( \mathcal{X}^W \) is \( \lambda^{W}_{\min} = \omega'_1 \). Therefore,

\[
\Sigma^{W}_{\min} = \Sigma^{W}_{1} - \Sigma^{W}_{2} = 2x^{W}_{\min} - \frac{1}{2} - \frac{\|W_{1}\| + \|W_{2}\|}{2} \\
= 2[\omega_1^2 + (1 - \omega_1)(1 - \omega_2)] - \frac{1}{2} - \frac{\omega_1(1 - \omega_2)}{2} \\
= 1 + 2\omega_2^2 - 2\omega_1 \omega_2 - 2(\omega_1 + \omega_2) - \frac{(\omega_1 - \omega_2)}{2}.
\]

Now from Fig. 1b, we get that that \( \Sigma^{W}_{\min} \geq 0 \) for all \( \omega_1 \) and \( \omega_2 \) satisfying the conditions \( \omega_1 \geq \omega_2 \) and \( \omega_1 + \omega_2 < 1 \).

Now, we have to prove monotonicity of \( E' \) under the addition of white noise. We start with our next theorem.

**Theorem 4** Suppose \( \mathcal{A}^4 = \{A^4_i\}_{i=1}^{n_A} \) is an unsharp version of \( \mathcal{A} = \{A_i\}_{i=1}^{n_A} \) i.e., \( A^4_i = \lambda A_i + (1 - \lambda) \frac{1}{n_A} \) for all \( i \in \{1, \ldots, n_A\} \) where \( 1 \leq \lambda \leq 0 \). Then \( E'(\mathcal{A}^4) \geq E'(\mathcal{A}) \) for all \( 1 \geq \lambda \geq 0 \) iff

\[
\Sigma_1^{A^4} \geq \Sigma_2^{A^4}
\]

holds where \( \Sigma_1^{A^4} = \frac{1}{d} \text{Tr}[\mathcal{X}^{A^4}] - \frac{1}{n} \) and \( \Sigma_2^{A^4} = \frac{n_A\|A_i\|}{n_A} - \frac{1}{d} \text{Tr}[\mathcal{X}^{A^4}] \).

**Proof** From (53), we get that \((\mathbb{I} - \mathcal{X}^{A^4}) = \lambda^2[\mathbb{I} - \mathcal{X}^{A}] + \gamma \mathbb{1} \) where \( \gamma = \gamma(\mathcal{A}, \lambda) = \frac{(1 - \lambda)}{n_A}[(n_A - 1) + \lambda(n_A - \sum_i\|A_i\|)] \). As \( A_i \leq \mathbb{1} \) and therefore, \( \sum_i\|A_i\| \leq n_A \), we have \( \gamma \geq 0 \). Therefore, from the (49), we get that

\[
E'(\mathcal{A}^4) = \lambda^2 E'(\mathcal{A}) + \gamma.
\]

Therefore,
Fig. 1  Plots of $\Sigma_{\min}^W$ w.r.t. $\omega_1$ and $\omega_2$ for $\omega_1 \geq \omega_2$. These plots show that $\Sigma_{\min}^W \geq 0$ always.

(a) Plot of $\Sigma_{\min}^W$ w.r.t. $\omega_1$ and $\omega_2$ satisfying the conditions $\omega_1 \geq \omega_2$ and $\omega_1 + \omega_2 \geq 1$

(b) Plot of $\Sigma_{\min}^W$ w.r.t. $\omega_1$ and $\omega_2$ satisfying the conditions $\omega_1 \geq \omega_2$ and $\omega_1 + \omega_2 < 1$
\[ \mathcal{E}'(A^4) - \mathcal{E}'(A) = \gamma - (1 - \lambda^2) \mathcal{E}(A) \]
\[ = (1 - \lambda)(1 - \frac{1}{n_A} - \mathcal{E}'(A)) \]
\[ + \lambda(1 - \frac{\sum_i |A_i|^2}{n_A} - \mathcal{E}'(A)) \]
\[ = (1 - \lambda)[\Sigma^A_1 - \lambda \Sigma^A_2] \]
\[ = \Sigma^A(\lambda) \]

where \( \Sigma^A_1 = (1 - \frac{1}{n_A} - \mathcal{E}(A)) = (\frac{1}{2} \text{Tr}[X^4] - \frac{1}{n_A}) \), \( \Sigma^A_2 = (\frac{\sum_i |A_i|^2}{n_A} - \frac{1}{2} \text{Tr}[X^4]) \) and \( \Sigma^A(\lambda) = [\Sigma^A_1 - \lambda \Sigma^A_2] \).

Now, since \( \mathcal{E}'(A^4) \leq (1 - \frac{1}{n_A}) \), \( \Sigma^A_1 \geq 0 \). Now, there are two following cases-

(I) **For \( \Sigma^A_2 < 0 $$**

In this case, \( \mathcal{E}'(A^4) - \mathcal{E}'(A) \geq 0 $$ always. In this case \( \Sigma^A_1 \geq \Sigma^A_2 $$ trivially holds.

(II) **For \( \Sigma^A_2 \geq 0 $$**

In this case, the minimum value of \( \Sigma^A(\lambda) $$ (for \( \lambda = 1 $$) is \( \Sigma^A_{\text{min}} = [\Sigma^A_1 - \Sigma^A_2] = 2 \frac{1}{2} \text{Tr}[X^4] - \frac{\sum_i |A_i|^2}{n_A} - \frac{1}{n_A} $$ .

Clearly, the condition for \( \mathcal{E}'(A^4) - \mathcal{E}'(A) \geq 0 $$ for all \( \lambda $$ is \( \Sigma^A_{\text{min}} \geq 0 $$ or equivalently \( \Sigma^A_1 \geq \Sigma^A_2 $$ .

Since, it is difficult to prove inequality (61), we prove it for dichotomic qubit observables. Therefore, our next proposition is

**Proposition 2** For any dichotomic observable \( \mathcal{W} $$, \( \Sigma^W_1 \geq \Sigma^W_2 $$ and therefore, \( \mathcal{E}'(\mathcal{W}^4) \geq \mathcal{E}'(\mathcal{W}) $$ for all \( 1 \geq \lambda \geq 0 $$ .

**Proof** Suppose \( \mathcal{W} = \{W_1, W_2\} $$ are two qubit dichotomic observables. Clearly \( W_2 = 1 - W_1 $$ .

Let \( W_1 = \omega_1 |\omega_1\rangle \langle \omega_1| + \omega_2 |\omega_2\rangle \langle \omega_2| $$ . Without the loss of generality, we can choose \( \omega_1 \geq \omega_2 $$ . Then \( \|W_1\| = \omega_1 $$ . Now \( W_2 = (1 - \omega_1) |\omega_1\rangle \langle \omega_1| + (1 - \omega_2) |\omega_2\rangle \langle \omega_2| $$ . Clearly, \( \|W_2\| = (1 - \omega_2 $$ . Therefore, from (57), we get that

\[ \lambda^{W} = \omega'_1 |\omega_1\rangle \langle \omega_1| + \omega'_2 |\omega_2\rangle \langle \omega_2| \]

where \( \omega'_1 = [\omega_1^2 + (1 - \omega_1)(1 - \omega_2)] $$ and \( \omega'_2 = [\omega_1 \omega_2 + (1 - \omega_2)^2] $$ . Therefore,

\[ \Sigma^{W}_{\text{min}} = 2 \frac{1}{2} \text{Tr}[\lambda^{W}]) - \frac{1}{2} - \frac{\|W_1\| + \|W_2\|}{2} \]
\[ = \omega'_1 + \omega'_2 - \frac{1}{2} - \frac{\omega_1(1 - \omega_2)}{2} \]
\[ = \omega'_1 + \omega_1 \omega_2 + (1 - \omega_2)(2 - \omega_1 - \omega_2) - 1 - \frac{\omega_1 - \omega_2}{2} \]

Figure 2, says that \( \Sigma^{W}_{\text{min}} \geq 0 $$ for \( \omega_1 \geq 1 $$ . Hence, \( \mathcal{E}'(\mathcal{W}^4) \geq \mathcal{E}'(\mathcal{W}) $$ for all \( 1 \geq \lambda \geq 0 $$ .

Therefore, inequality (51) and inequality (61) hold for qubit dichotomic observables. We have searched for examples for which inequality (51) inequality (61) do not hold. But we could not find any such example. Noting these facts, we provide the following conjecture-

**Conjecture 1** For any qubit observable \( A $$, inequality \( \Sigma^A_1 \geq \Sigma^A_2 $$ and inequality \( \Sigma'^A_1 \geq \Sigma'^A_2 $$ hold and therefore, \( \mathcal{E}(A^4) \geq \mathcal{E}(A) $$ and \( \mathcal{E}'(A^4) \geq \mathcal{E}'(A) $$ for all \( 1 \geq \lambda \geq 0 $$ .

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If Conjecture 1 can be proven then two corollaries similar to Corollary 1 and Corollary 2 can also be proven which establishes the monotonicity of $E$ and $E'$ under the addition of white noise.

5 Experimental Determination of the Value of the Unsharpness Measures

Here we show that experimentally, one can determine the value of $E^L(A)$ and $E'^L(A)$ for an unknown qubit observable $A = \{A_i\}$. We show this for the qubit case. Generalization for the higher dimensions is straightforward.

Let $E$-matrix of an unknown qubit observable $A = \{A_i\}$ be $E^A = \begin{bmatrix} a & c^* \\ c & d \end{bmatrix}$ where this matrix is written in $\sigma_z$ basis. Suppose $|\pm, i\rangle$ are the eigen states of $\sigma_i$ corresponding to the eigen values $\pm 1$ for all $i \in \{x, y, z\}$. Suppose we have $n_{i,\pm}$ copies of such states are available to us. On each of these copies, $A$ has been measured twice successively using Lüders instrument. Suppose that for $f_{i,\pm}$ copies outcomes have repeated (i.e., the outcome of the first $A$ measurement and the outcome of the second $A$ measurement are same) Then from (7), we get that the average probability that any outcome will repeat, is

$$P^L(|\pm, i\rangle\langle \pm, i|, A) = \frac{f_{i,\pm}}{n_{i,\pm}}.$$ (66)

Now we know that for large $n_{i,\pm}$, $P^L(|\pm, i\rangle\langle \pm, i|, A) \approx f_{i,\pm}/n_{i,\pm}$. Now $P^L(|+, z\rangle\langle +, z|, A) = a \approx f_{z,+}/n_{z,+}$, $P^L(|-, z\rangle\langle -, z|, A) = b \approx f_{z,-}/n_{z,-}$, $P^L(|\pm, x\rangle\langle \pm, x|, A) = \frac{a+b+2|\text{Re}(c)|}{2} \approx f_{x,\pm}/n_{x,\pm}$, $P^L(|\pm, y\rangle\langle \pm, y|, A) = \frac{a+b+2|\text{Im}(c)|}{2} \approx f_{y,\pm}/n_{y,\pm}$ where $\text{Re}(c)$ is the real part of $c$ and $\text{Im}(c)$ is the...
imaginary part of $c$. From these approximate equalities, we get the following set of approximate equalities-

$$a \approx \frac{f_{1+}}{2n_{1+}}; b \approx \frac{f_{1-}}{2n_{1-}}$$

(67)

$$c \approx \left( \frac{f_{2+}}{4n_{2+}} - \frac{f_{2-}}{4n_{2-}} \right) + i \left( \frac{f_{3+}}{2n_{3+}} - \frac{f_{3-}}{2n_{3-}} \right).$$

(68)

Clearly for $n_{i\pm} \to \infty$, for all $i \in \{x,y,z\}$, above approximate equalities become exact equalities. In this way, if $a, b$ and $c$ are known approximately, then $E^A$ is known approximately. The lowest eigenvalue of $E^A$ is

$$\Lambda^*(A) = \|1 - E^A\| = 1 - \frac{a+b-\sqrt{(a+b)^2 - 4(ab-|c|^2)}}{2}.$$  

Similarly, $\mathcal{E}^E(A) = \frac{1}{2} - \frac{1}{2} \text{Tr}[E^A] = 1 - \frac{a+b}{4}$. Therefore, in this way, it is possible to determine the values of $\mathcal{E}^E(A)$ and $\mathcal{E}^L(A)$ experimentally.

The experimental determination of the values of $\mathcal{E}(A)$ and $\mathcal{E}^E(A)$ is similar as above. Above-said process is similar to the process tomography of POVMs. For the process tomography, one needs to record each $P^l_j = \text{Tr}[\rho_j A_j]$ for each input state $\rho_j$ separately. But here we need to record only $P^E(\rho_j, A) = \text{Tr}[\rho_j E^A]$ for each input state $\rho_j$.

6 An Attempt to Construct the Resource Theory of the Sharpness of the Observables

Quantification of quantum resources and the construction of the resource theory is a very important and interesting direction of research [24]. A few examples of different resource theories are (i) the resource theory of entanglement [24, 25], (ii) the resource theory of coherence [26, 27], (iii) the resource theory of incompatibility [28], (iv) the resource theory of quantum channels [29], (v) the resource theory of quantum thermodynamics [30, 31] etc. We do not claim we construct the complete resource theory here. But we present the idea of the resource theory of the sharpness of the observables here. We take the sharpness of the observables as a resource here. Although it is already argued in the introduction that the sharpness of the observables can be considered as a resource, we provide the following strong reasons behind taking the sharpness of the observables as a resource-

1. Ref. [15] suggests that an ideal PVM has infinite resource costs. Therefore, with finite amount of resource, a PVM can not be performed with arbitrary accuracy. Therefore, this fact suggests that the ability to perform PVMs (i.e., sharp measurements) or equivalently sharpness itself can be considered as a resource.

2. In practice, it is very difficult to get rid of the interaction between the system and the environment. The interaction between the system and the environment disturbs the quantum state of a system or equivalently one can say that due to the interaction between the system and the environment, an effective channel $\Lambda$ acts on the system state. In Heisenberg picture, this channel acts on the observable $A$, which we want to measure, as $\Lambda^*(A) = \{\Lambda^*(A_j)\}$. Depending on the type of the interaction $\Lambda$ can convert a sharp observable into an unsharp observable. For an example- if $\Lambda = \Lambda^d_\rho$ is depolarising channel i.e., $\Lambda(\rho) = \Gamma^d(\rho) = \rho + (1 - t) \frac{1}{d}$ and $A = \{|a_i\rangle\langle a_i|\}$ is a rank one PVM, then
\[ \Lambda^*(A) = \Gamma^*_d(A) = \{ \Gamma^*_d(A_i) = tA_i + (1 - t)\frac{1}{d} \}. \]

Therefore, for a given value of \( t < 1 \), the effective observable i.e. \( \Gamma^*_d(A) \) is not a PVM and therefore, in this case it is impossible to perform a PVM accurately. Therefore, given the type of interaction, it may not be possible to perform a PVM with arbitrary accuracy. Therefore, to perform a PVM in a lab, one needs to make proper arrangements in the lab to get rid of such interactions between the system and the environment which prevents one to perform the desired PVM with arbitrary accuracy. Therefore, this fact also suggests that the ability to perform PVMs (i.e., sharp measurements) or equivalently sharpness of the observables itself can be considered as a resource.

3. There exist several information-theoretic tasks which cannot be performed perfectly without the sharp observables. For example- the full set of orthogonal states (which forms a basis) can be distinguished perfectly only with certain PVMs (assuming minimally sufficient observables). Furthermore, the optimal measurement for minimum-error state discrimination \( d \) linearly independent pure states is a projective measurement where \( d \) is the dimension of the Hilbert space on which the pure state lies [16, 39]. Therefore, this fact also suggests that the ability to perform PVMs (i.e., sharp measurements) or equivalently the sharpness itself can be considered as a resource. One more example is discussed in Section 7.

Now we state the different elements of the resource theory of the sharpness of the observables below-

1. **The resource** - The sharpness of the observables.
2. **The free operation** - The fuzzifying processes. For example- a class of fuzzifying processes is the addition of white noise.
3. **The resource measure** - We know that unsharpness is the opposite of the sharpness. Therefore, as sharpness is monotonically non-increasing under fuzzifying processes, the unsharpness is monotonically non-decreasing under fuzzifying processes. Since, from Theorem 1 and Corollary 1, we get that \( \mathcal{E}^L \) is monotonically non-decreasing under the addition of white noise, \( \mathcal{E}^L \) can be a possible measure of unsharpness. The higher value of \( \mathcal{E}^L \) corresponds to less sharpness (i.e., less resource). Similarly, from Theorem 2 and Corollary 2, we get that \( \mathcal{E}^L \) can be a possible measure of unsharpness. It is to be noted that if the Conjecture 1 can be proven then \( \mathcal{E} \) and \( \mathcal{E}^L \) also can be an unsharpness measure for qubit observables consistent with the resource-theoretic framework.
4. **Most resourceful measurements** - The sharp measurements (PVMs).
5. **Free measurements** - Given the number of outcomes \( n \), the observable \( T^n = \{ I_i^n = \frac{1}{n} \}_{i=1}^n \) is a free measurement (most unsharp).
6. **Example of an information-theoretic task that requires the resource** - Sharp measurements are required in the perfect discrimination of the full set of orthogonal states (Assuming available observables are minimally sufficient), more generally, in the optimal minimum-error state discrimination \( d \) linearly-independent pure states where \( d \) is the dimension of the Hilbert space on which the pure state lies [16, 39]. One more example is discussed in Section 7.

Now a complete resource theory can be constructed only if all the fuzzifying processes are specified which is out of the scope of the present work. One point should be mentioned that the above-said resource theory is completely different the resource theory of quantum
uncomplexity which is presented in Ref. [33] and the fuzzy operations which are discussed in Ref. [33] are quite different than our idea of fuzzifying processes.

7 Sharpness of the Observables as a Resource in Quantum Communication Using Quantum Switch: An Example

Quantum switch is described by a circuit that simulates indefinite causal order among multiple quantum operations on a quantum system [35, 36, 37]. The simplest example of quantum switch is described by a circuit that simulates indefinite causal order among two above-said quantum channels $\Lambda_A$ and $\Lambda_B$. Suppose, we have a control qubit such that if control qubit is in $|0\rangle$, $\Lambda_A$ will act before $\Lambda_B$ on the system and if control qubit is in $|1\rangle$, $\Lambda_B$ will act before $\Lambda_A$ on the system. Here, $\{0\rangle, |1\rangle\}$ is the eigen basis of $\sigma_z$. Now, if the initial state of the control qubit is $\omega = |\phi\rangle\langle \phi |$ where $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha, \beta \neq 0$ then indefinite causal order is established between two above-said quantum channels. Now, one can choose $\Lambda_A = \Lambda_B = \Lambda$.

It has been discussed in [38] that in this case the final joint state of system and control is

$$S_{\Lambda,\Lambda,\omega}(\rho) = \frac{1}{4} \sum_{x,y} \left( [K_x, K_y] \rho [K_x, K_y] \right)^\dagger \otimes \omega$$

(69)

where $\{K_x\}$ are the Krauss operators of $\Lambda$, $\omega$ is initial state of the control qubit, $C_{+\Lambda,\Lambda}(\rho) = \sum_{x,y} [K_x, K_y] \rho [K_x, K_y] ^\dagger$ and $C_{-\Lambda,\Lambda}(\rho) = [K_x, K_y] \rho [K_x, K_y] ^\dagger$. Clearly, $C_{+\Lambda,\Lambda}(\rho)$ and $C_{-\Lambda,\Lambda}(\rho)$ are CP maps.

Now if the initial quantum channel $\Lambda$ is a qubit pauli channel i.e., $\Lambda(\rho) = p_0 \rho + p_x \sigma_x \rho + p_y \sigma_y \rho + p_z \sigma_z \rho$ then it is shown in [38] that $C_{+\Lambda,\Lambda}(\rho) = \left(p_0^2 + p_x^2 + p_y^2 + p_z^2 \right) \rho + 2p_0(p_x \sigma_x \rho p_x + p_y \sigma_y \rho p_y + p_z \sigma_z \rho p_z)$ and $C_{-\Lambda,\Lambda}(\rho) = 2p_0(p_x \sigma_x \rho p_x + p_y \sigma_y \rho p_y + p_z \sigma_z \rho p_z) + 2p_0 \sigma_x \rho p_x + 2p_0 \sigma_y \rho p_y + 2p_0 \sigma_z \rho p_z) = p_0 \Gamma_{\Lambda}(\rho)$ where $p_+ = 1 - 2(p_y \sigma_y + p_z \sigma_z)$, $p_- = 1 - p_+$. Clearly, $\Gamma_{\Lambda}(\rho) = \frac{C_{+\Lambda,\Lambda}(\rho)}{p_0}$ and $\Gamma_{\Lambda}(\rho) = \frac{C_{-\Lambda,\Lambda}(\rho)}{p_0}$ are quantum channels. Now consider the quantum channel $\Lambda_p = \frac{1}{2} \sigma_x \rho \sigma_x + \frac{1}{2} \sigma_y \rho \sigma_y$. It can be shown easily from the above-said expressions that the final joint state of the system and control after switch operation with initial control state $\omega = |+\rangle\langle +|$ is

$$S_{\Lambda_p,\Lambda_p,\omega}(\rho) = \frac{1}{2} \rho \otimes |+\rangle\langle +| + \frac{1}{2} \sigma_z \rho \sigma_z \otimes |-\rangle\langle -|$$

(70)

where $\{|+, |-\rangle\}$ is the eigen basis of $\sigma_z$.

Now, Alice has enough resource to implement PVM $\{|+\rangle\langle +|, |-\rangle\langle -|\}$. If she gets outcome $+$, she does nothing i.e., she implements identity channels and if she gets outcome $-$, she implements unitary channel $\sigma_z$. Then final state of the system becomes $\rho$ which is same as initial state. Therefore, effective channel implemented on the system an identity channel. This is how perfect communication is established through a noisy channel using quantum switch. The above-said fact is already discussed in [38].

But if Alice does not have enough resource to implement PVM $\mathcal{A} = \{|+\rangle\langle +|, |-\rangle\langle -|\}$ (i.e., $\sigma_z$ measurement) perfect communication can not established through above-said procedure. Suppose Alice has enough resource to implement POVM $\mathcal{A}' = \{A'_+ = t |+\rangle\langle +| + (1-t)^\frac{1}{2}, A'_- = t |\rangle\langle -| + (1-t)^\frac{1}{2}\}$ (which is an unsharp version of above-said PVM) for some value of $0 \leq t < 1$. Clearly, $(1 - E^{\mathcal{A}'}) = \frac{(1-t)^\frac{1}{2}}{2}$ and

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\( E^L(A^t) = \| 1 - E^{A^t} \| = \frac{(1-t^2)}{2} \) and \( E^U(A^t) = \text{Tr} \left[ \frac{1}{2} (1 - E^{A^t}) \right] = \frac{(1-t^2)}{2} \). Both unsharpness measure decreases with increment of \( t \). Then if she implements above-said procedure i.e., if she gets outcome + , she does nothing i.e., she implements identity channels and if she gets outcome −, she implements unitary channel \( \sigma_z \), the effective channel implemented on the system will be

\[
C_{\text{eff}}(\rho) = \text{Tr}[|+\rangle\langle+|A^t_+|^2\rho + \text{Tr}[|-\rangle\langle-|A^t_-|^2\rho \sigma_z + \text{Tr}[|+\rangle\langle+|A^t_-|^2\rho \sigma_z + \text{Tr}[|-\rangle\langle-|A^t_+|^2\rho \sigma_z \sigma_z]
\]

\[= \frac{(1+t)}{2} \rho + \frac{(1-t)}{2} \sigma_z \rho \sigma_z
\]

which is no longer an identity channel and therefore, perfect communication is not established. For, larger value of \( t \), implemented POVM will be more close to the above-said PVM. From (71), we get for larger value of \( t \), effective channel is more close to identity channel.

### 8 Conclusion

In this work, at first, we have constructed two Lüders instrument-based unsharpness measures and provided the tight upper bounds of those measures. Then we have proved the monotonicity of the above-said measures under a class of fuzzifying processes (i.e., the addition of white noise). This is consistent with the resource-theoretic framework. We have also discussed the fact that these measures do not change if a unitary is acted on the observables in the Heisenberg picture. Then we have related our approach to the approach of the Ref. [9]. Next, we have tried to construct tried instrument-independent unsharpness measures. In particular, we have defined two instrument-independent unsharpness measures and provided the tight upper bounds of those measures and then we have derived the condition for the monotonicity of those measures under a class of fuzzifying processes and proved the monotonicity for dichotomic qubit observables. Then we have shown that for an unknown measurement, the values of all of these measures can be determined experimentally. Finally, we have presented the idea of the resource theory of the sharpness of the observables and discussed an example where the sharpness of the observables can be considered as a resource.

It would be interesting to prove Conjecture 1 in the future. It would be also interesting to construct a complete resource theory of the sharpness of the observables in the future.

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