ON A CANONICAL LIFT OF ARTIN’S REPRESENTATION TO LOOP BRAID GROUPS

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Abstract. Each pointed topological space has an associated \( \pi \)-module, obtained from action of its first homotopy group on its second homotopy group. For the 3-ball with a trivial link with \( n \)-components removed from its interior, its \( \pi \)-module \( M_n \) is of free type. In this paper we give an injection of the (extended) loop braid group into the group of automorphisms of \( M_n \). We give a topological interpretation of this injection, showing that it is both an extension of Artin’s representation for braid groups and of Dahm’s homomorphism for (extended) loop braid groups.

1. Introduction

A recent paper [BCK+0] showed that discrete higher gauge theory (as discussed generally and informally by many authors, see e.g. [Mar91, §10.2], and compare with continuous higher gauge theory [Pfe03,BS07,BH11,FMP11,SW13] and several others) is well-defined. The technical engine of the construction notably reflects the combinatorial homotopy of Whitehead, Baues, et al. [Bau91]. Relatively simple aspects of the construction such as loop-particle braiding [BFMM18] yield higher generalisations of classical results, for example in low-dimensional topology [FM09]. We discuss one such result, a lifting of Artin’s representation of the braid group [Bir74] to the (extended) loop braid group [Lin08] (see [Dam17], for a survey).

1.1. From mapping class groups to Artin-like representations. Let \( X \) be an oriented topological manifold with boundary \( \partial X \), and \( A \) a possibly empty subset in the interior of \( X \). A self-homeomorphism of the pair \((X,A)\) relative to the boundary, is a self-homeomorphism \( g \) of \( X \) that fixes \( \partial X \) pointwise, \( A \) setwise (i.e. \( g(A) = A \)), and preserves the orientation of \( X \). Two such homeomorphisms \( f_0 \) and \( f_1 \) are \((X,A)\)-isotopic if they can be included in a 1-parameter family \( \{f_t\}_{t \in [0,1]} \) of self-homeomorphisms of \((X,A)\) relative to the boundary, such that the map \( X \times [0,1] \to X \) sending \((x,t)\) to \( f_t(x) \) is continuous. The mapping class group \( \text{MCG}(X,A) \) is the group of \((X,A)\)-isotopy classes of self-homeomorphisms of \((X,A)\). We write \([g]\) for the class of \( g \).

Our convention for the product in \( \text{MCG}(X,A) \) is:

\[
[g][g'] = [g \circ g']
\]

A self-homeomorphism \( g \) of \((X,A)\) relative to the boundary, takes an \( n \)-path \([0,1]^n \rightarrow X\) to an \( n \)-path \([0,1]^n \xrightarrow{g \circ \gamma} X\). In particular \( g \) takes a loop based at a point \(*\) in \( \partial X \) to another such loop. Furthermore a homeomorphism that fixes a set fixes its complement. If \( \gamma \) avoids \( A \) then so does \( g \circ \gamma \). The map \( g \) thus induces an automorphism of \( \pi_1(X \setminus A, *) \) with \(* \in \partial X \), that is:

\[
[g] \mapsto [g \circ \gamma].
\]

Passing from \( g \) to \([g]\), this gives a well-defined group homomorphism

\[
\tau : \text{MCG}(X,A) \rightarrow \text{Aut}(\pi_1(X \setminus A,*)),
\]

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Artin shows \cite{Art47} that this is an injection.

By considering based \(n\)-paths \(([0, 1]^n, \partial[0, 1]^n) \to (X, *)\) we can in principle use other homotopy functors analogously in place of \(\pi_1\), getting representations of mapping class groups this way. In this paper, we will use the homotopy functor which sends a pointed space \((Y, \ast)\) to the triple \(\pi_1(Y, \ast)\) defined as \((\pi_1(Y, \ast), \pi_2(Y, \ast), \triangleright_{\pi_1})\), where \(\triangleright_{\pi_1}\) is the usual action of \(\pi_1\) on \(\pi_2\). The underlying algebraic notion is that of a \(\pi\)-module, see for example \cite{Bau91}, Chapter 1, §4. A \(\pi\)-module is a triple \(G = (G, A, \triangleright)\), where \(G\) is a group, \(A\) is an abelian group, and \(\triangleright\) is a left-action by automorphisms of \(G\) on \(A\); see Subsection 3.2. A morphism of \(\pi\)-modules is a pair of group morphisms that respect the actions.

Artin’s representation of the braid group \cite{Art47}, and Dahm homomorphism for the extended loop braid group \cite{Dah62,Gol81} can be seen as a special case of this construction. Let us recap here these two maps.

1.2. Artin representation and Dahm homomorphism. For \(m \ge 1\), let \(D^m\) be the \(m\)-disc \(\{[0, 1]^m \subset \mathbb{R}^m\}\). Fix \(d_n = \{p_1, \ldots, p_n\} \subset D^2 \setminus \partial D^2\) a set of \(n\) points in the interior of \(D^2\), and \(L_n = C_1 \cup \ldots \cup C_n \subset D^3 \setminus \partial D^3\) a set made out of the union of \(n\) disjoint, unknotted, oriented circles, that form a trivial link with \(n\) components in the interior of \(D^3\).

The braid group \(B_n\) can be defined as the mapping class group of the pair \((D^2, d_n)\) (for a survey, see for instance \cite{BB05}). Analogously the extended loop braid group \(LB_n^{ext}\) is defined as the mapping class group \(\text{MCG}(D^3, L_n)\), meaning the group of self-homeomorphism of the pair \((D^3, L_n)\), relative to the boundary of \(D^3\), up to \((D^3, L_n)\)-isotopy \cite{Dam17}. Note that this definition appears also in \cite{Gol81}, in terms of motion groups. Homeomorphisms \((D^3, L_n) \to (D^3, L_n)\) do not necessarily preserve the orientation of \(L_n\). If we consider the group \(\text{MCG}(D^3, L_n)\) of isotopy classes of those homeomorphisms \((D^3, L_n) \to (D^3, L_n)\) that preserve also the orientation on \(L_n\), the group obtained is the loop braid group, denoted by \(LB_n\). The nomenclature “loop braid groups” is due to Lin \cite{Lin08}.

Let \(F_n\) be the free group of rank \(n\) generated by \(\{x_1, \ldots, x_n\}\). Then we have that \(\pi_1(D^2 \setminus d_n, *)\) is isomorphic to \(F_n\). Thus, in the case of the pair \((D^2, d_n)\), Equation (1.1) becomes:

\[
\theta : B_n \to \text{Aut}(F_n).
\]

Artin shows \cite{Art47} that this is an injection.

We also have that \(\pi_1(D^3 \setminus L_n, *) \cong F_n\), so the map \(\theta : \text{MCG}(D^3, L_n) \to \text{Aut}(\pi_1(D^3 \setminus L_n, *))\) becomes:

\[
\theta : LB_n^{ext} \to \text{Aut}(F_n).
\]

This homomorphism is proven injective in \cite{Dah62} and published in \cite{Gol81} Theorem 5.2.

1.3. Our main result. The space \(D^2 \setminus d_n\) is aspherical, and therefore its homotopy type can be recovered from the fundamental group. However \(D^2 \setminus L_n\) is not aspherical, hence \(\pi_1\) “forgets” more about \(D^3 \setminus L_n\) than it does about \(D^2 \setminus d_n\). In this paper, we work with \(\pi_{1, 2}(D^3 \setminus L_n, *)\), in the intent of retrieving some of the lost homotopical information when passing from \(D^3 \setminus L_n\) to \(\pi_1(D^3 \setminus L_n, *)\). How much homotopical information is retained is discussed in Subsection 4.3.

In particular, we will prove here that \(\pi_{1, 2}(D^3 \setminus L_n, *)\) is a \(\pi\)-module of free type\(^1\). In practice, with this we mean that \(\pi_{1, 2}(D^3 \setminus L_n, *) = (\pi_1(D^3 \setminus L_n, *), \pi_2(D^3 \setminus L_n, *) \triangleright_{\pi_1})\), is isomorphic to \(M_n = \langle F_n, M_n, \triangleright \rangle\), where \(M_n\) is the free \(\mathbb{Z}[F_n]\)-module generated by \(\{K_1, \ldots, K_n\}\). Given a \(\pi\)-module \(G = (G, A, \triangleright)\), a morphism \(f = (f_1, f_2) : M_n \to G\) is therefore determined by the images \(f_1(x_i)\) and \(f_2(K_i)\); see Subsection 3.2.

In this paper we construct an inclusion \(\Theta : LB_n^{ext} \to \text{Aut}(M_n)\).

\(^1\)The nomenclature “of free type” is borrowed from \cite{BHS11} Definition 7.3.13.
1.4. Structure of the paper. In Section 2 we recall some notions that will be used throughout the paper. We also give a topological realisation of Artin’s representation for the braid group, and a hands-on flavour of the lift we construct for the extended loop braid group. In Section 3 we present Dahm and Goldsmith’s lift of Artin’s representation for extended loop braid groups in Theorem 3.15. Then we introduce $\pi$-modules (Definition 3.5) and describe a lift of Artin’s representation for extended loop braid groups in the $\pi$-module $\pi(1,2)(D^3 \setminus L_n, \ast)$ of the 3-ball with a set of $n$ (unlinked and unknotted) circles excised from its interior (Lemma 3.10 and Theorem 3.15). This is our first main result. In Section 4 we formalise the topological construction of the considered representation, in the second main result of this paper (Theorem 4.4).

2. On automorphisms of free groups and beyond

Here we first recall some constructions that are standard, but which will have useful lifts to higher dimensions later.

If $\mathcal{C}$ is a concrete category, then $F: \mathcal{C} \to \text{SET}$ is the forgetful functor. Let $\text{Grp}$ be the category of groups, and define $\text{Aut}(G) \subset \text{Grp}(G,G)$ as the subset of invertible homomorphisms. The left adjoint $F^*: \text{SET} \to \text{Grp}$ takes a set to the free group on that set. We consider sets of form $x_\mathbb{N}$ defined to be $\{x_1, x_2, \ldots, x_n\}$, $n \in \mathbb{N}_0$ as a skeleton in the full subcategory $\text{FINSET}$ of finite sets, and define

\begin{equation}
F_n = F^\circ(\{x_1, \ldots, x_n\}).
\end{equation}

By the adjoint functor property ("freeness") elements of $\text{Grp}(F_n, G)$ are uniquely specified by giving the image of each $x_i$. For example, for $\sigma \in \text{Sym}(n)$, the symmetric group on $\{1, \ldots, n\}$, define $t_\sigma \in \text{Aut}(F_n)$ by:

\begin{equation}
t_\sigma: x_i \mapsto x_{\sigma(i)}.
\end{equation}

A non-automorphism example in $\text{Grp}(F_n, F_n)$ is given by $x_i \mapsto x_1$. We use cycle notation for elements of $\text{Sym}(n)$, thus we have $t_{(12)}(x_i)$ and so on; and define the translation automorphism by $t = t_{(1,2)\ldots(n)}$. We say an automorphism is Coxeter geometric (CG) if it acts trivially on all except at most two adjacent $x_i, x_{i+1}$, and on these it acts to produce elements of $F^\circ(F_1, F_{n})$. We say this action is local at $i, i + 1$.

For example, the subset of automorphisms of form (2.2) are not CG in general. In particular the translation automorphism $t$ is not CG. However the subset forms a subgroup and the subgroup is generated by the CG automorphism $t_{(12)}$ local at 1, 2 and $n - 2$ translates thereof.

2.1. On automorphisms of free groups realised topologically. A topological realisation of $F_n$ is given by $\pi_1(D^2 \setminus d_n, \ast)$. In order to explicitly write down an Artin representation $\theta: B_n \to \text{Aut}(F_n)$, we specify $d_n$ and a free basis for $\pi_1(D^2 \setminus d_n, \ast)$. We take the points $d_n = \{p_1, \ldots, p_n\}$ to be along a horizontal line. We consider paths $\tilde{x}_i$, $i = 1, \ldots, n$ passing clockwise around each $p_i$, as in Figure 1 such that their images intersect only at $\ast \in \partial D^2$. Let $x_i$ be the homotopy class of $\tilde{x}_i$.

Note that $\pi_1(D^2 \setminus d_n, \ast)$ is free on $\{x_1, \ldots, x_n\}$. Also, $x_1 x_2 \ldots x_n$ is the homotopy class of a path that traces the boundary of $D^2$, clockwise, starting and finishing at $\ast$.

Note also that $(D^2 \setminus d_n, \ast)$ strongly deformation retracts into the pointed subspace made of the images of the paths, which is homeomorphic to $\bigvee_{i=1}^n (S^1, \ast)$. Combining with Seifert–van Kampen theorem, this implies that $\pi_1(D^2 \setminus d_n, \ast)$ is freely generated by $\{x_1, \ldots, x_n\}$. The same type of argument will be used when we address the higher case in Section 4 and prove that $\pi_{(1,2)}(D^3 \setminus L_n, \ast)$ is of free type.

Using Equation (1.1), elements of $\text{MCG}(D^2, d_n)$ induce elements of $\text{Aut}(F_n)$. Consider for example the mapping class $\Sigma_1$ indicated by $a$ - $c$ of Figure 2. Note that this corresponds to
the 1, 2 local automorphism $S_1^1: F_n \to F_n$ given by

\begin{align}
    x_1 &\mapsto x_2, \\
    x_2 &\mapsto x_2^{-1} x_1 x_2.
\end{align}

Define $S_1^1: F_n \to F_n$ as the translate of $S_1^1$; an explicit formula is in Equation (3.2). From the topological realisation (or direct calculation) we have:

\begin{equation}
    S_1^1 \circ S_{i+1}^1 = S_{i+1}^1 \circ S_i^1.
\end{equation}

2.2. Towards automorphisms in higher dimension. Our task here is to describe automorphisms of a suitable lift of the free group to “higher dimension”. Here dimension refers to the topological interpretation of $F_n$ as $\pi_1(D^2 \setminus d_n, \ast)$, whose building blocks are paths in the point-punctured disk $D^2 \setminus d_n$.

\begin{figure}[ht]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{a) - c) An exchange of the marked points inducing a homeomorphism of the disk, and the corresponding automorphism of $\pi_1(D^2 \setminus d_n, \ast)$. d) The new paths are expressed in the original basis: thus $x_1$ has morphed to $x_2$ and $x_2$ has morphed to $x_2^{-1} x_1 x_2$. f) The loops following the timelines of the braiding.}
\end{figure}
The lift involves the triple \( \pi_1(D^3 \setminus L_n, *) = (\pi_1(D^3 \setminus L_n, *), \pi_2(D^3 \setminus L_n, *), \triangleright_{\pi_1}) \), where \( \triangleright_{\pi_1} \) denotes the usual action of \( \pi_1 \) on \( \pi_2 \).

In the \( F_n \) case, freeness means that an automorphism is determined by the images of a set of free group generators. In the lift that we consider, the corresponding structure \( \pi_1(D^3 \setminus L_n) \) is not a free group, but a \( \pi \)-module of free type. In particular, as we will prove in Subsection 3.2, \( \pi_2(D^3 \setminus L_n, *) \) is a free \( \mathbb{Z} \)-module. In order to understand the basis we make some preparations.

Given a pointed topological space \((X, *)\), then the action of \( \pi_1(X, *) \) on \( \pi_2(X, *) \), means that \( \pi_2(X, *) \), can naturally be equipped with the structure of \( \mathbb{Z}[\pi_1(X, *)] \)-module. We will give a visual idea of this structure for the case \( X = D^3 \setminus L_n \).

We note that the “balloons and hoops” point of view we present in this subsection for the visual description of \( \pi_1(D^3 \setminus L_n, *) \) is essentially as in [BN15], where the term is coined. This balloons and hoops approach for understanding \( \pi_1(D^3 \setminus L_n, *) \) was also used in [BFMM18] §4.5.1.

Note that there is not a canonical choice of the elements considered to be generating for example in \( \pi_1(D^3 \setminus L_n, *) \). Varying the precise choice of \( L_n \) satisfying the “unlinked circles” characterisation affects what might be considered as a natural choice, and hence affects the construction — albeit only up to isomorphism. Later it will be convenient to work with \( L_n \) a row of circles confined to a single axis of rotational symmetry. Now let \( g \) be a homeomorphism representing a certain class in \( \text{MCG}(D^2, d_n) \). This \( g \) restricts to \((0, 1] \times [0, 1]; \) the omitted edge is the rotation axis, so rotating gives a self-homeomorphism of \((0, 1] \times [0, 1] \times S^1 \). Topologically, \((0, 1] \times [0, 1] \times S^1 \) is \( D^3 \) with the axis removed. The latter homeomorphism thus extends to a self-homeomorphism \( r(g) \) of \( D^3 \) by inserting the constant function on the axis. This construction lifts to a well-defined group homomorphism

\[
r: \text{MCG}(D^2, d_n) \to \text{MCG}(D^3, L_n).
\]

In particular, consider the mapping class \( \Sigma_i \) in \( \text{MCG}(D^2, d_n) \) exchanging consecutive points \( p_i \) and \( p_{i+1} \), as in Figure 3(c) which considers the case \( n = 3 \) and \( i = 1 \). Its image under \( r \), which we also denote \( \Sigma_i \), corresponds to an exchange of circles, as illustrated in Figure 3 for \( n = 2 \) and \( i = 1 \). To the mapping class \( \Sigma_i \in \text{MCG}(D^3, L_n) \) we will call elementary braid permutation.

One new aspect in the higher setting is that there are elements of \( \text{MCG}(D^3, L_n) \) that do not have representatives with the rotational symmetry, such as maps that exchange the circles by taking them off the rotation axis. Note that these induce elements of the automorphism group \( \text{Aut}(\pi_1(D^3 \setminus L_n)) \) of type \( i_{(i,i+1)} \). Also breaking the axial symmetry in this sense, are motions that flip a single circle \( C_i \) onto itself. One can see that this induces an automorphism local at \( i \), given by \( T: x_i \mapsto x_i^{-1} \).

Another new aspect in the higher setting is that \( \pi_2(D^3 \setminus L_n, *) \) is not trivial. An example of a non-trivial element of \( \pi_2(D^3 \setminus L_n, *) \) is the homotopy class of a wrapping square \( K_i^2 : (D^2, \partial D^2) \to (D^3 \setminus L_n, *) \), based at \(*_i \) (a point in the rotating axis) that wraps a single circle \( C_i \), including the disk of which \( C_i \) is boundary, exactly once, see Subfigures d) and e) in Figure 3 for the \( i, n = 2 \) case. Concretely \( K_i^2 \) is a positively oriented parametrisation of a balloon — i.e. a 2-sphere \( b_i \) containing \( C_i \), and no other circle \( C_j \), oriented by an exterior normal. If we connect \(*_i \) to \(* \) by a path \( \gamma_i \) that does not cross the disks spanned by the circles in \( L_n \), and consider the usual
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Figure 3. a), b), c) Rotating the two-punctured disk into $\mathbb{R}^3$. d), e) constructing a sphere (a “balloon”) $b_1$, enclosing $C_1$, by rotating the path $v_1$. The element $K_1 \in \pi_2(D^3 \setminus L_n, \ast)$ is obtained from the homotopy class $K'_1 \in \pi_2(D^3 \setminus L_n, \ast_1)$ of a positively oriented parametrisation $(D^2, \partial D^2) \to (b_1, \ast_1)$, of $b_1$, made into an element $K_1 \in \pi_2(D^3 \setminus L_n, \ast)$ by using the isomorphism $\pi_2(D^3 \setminus L_n, \ast_1) \to \pi_2(D^3 \setminus L_n, \ast)$ derived from the path $\gamma_1$.

Figure 4. An “exchange” of coaxial circles, where the circle $C_1$ is originally on top. This is a smooth family of embeddings of $S^1 \sqcup S^1$ into $D^3$. It induces a diffeotopy $t \in [0, 1] \mapsto \phi_t \in \text{Homeo}(D^3)$ of $D^3$, relative to the boundary, by the isotopy extension theorem (either before or after applying $\tau$). The end-value $\phi_1 \in \text{Homeo}(D^3)$ of the isotopy is a homeomorphism $(D^3, L_2) \to (D^3, L_2)$. The elementary braid permutation $\Sigma_1 \in \text{MCG}(D^3, L_2)$ is the mapping class of $\phi_1$.

The isomorphism $\pi_2(D^3 \setminus L_n, \ast) \to \pi_2(D^3 \setminus L_n, \ast)$ derived from $\gamma_1$ (see e.g. [Hat02 Page 343]), this gives a non-trivial element $K_i = [\gamma \triangleright K'_i] \in \pi_2(D^3 \setminus L_n, \ast)$.

We argue in Section 3 that $\pi_2(D^3 \setminus L_n, \ast)$ is freely generated, as an abelian group, by the elements $p \triangleright K_i$, where $i \in \{1, \ldots, n\}$ and $p \in \pi_1(D^3 \setminus L_n, \ast)$. Each $p \triangleright K_i$ can be visualised as a hoop $p$, connecting $\ast$ to $\ast$, which is then attached to $K_i$. In Figure 5 we show some examples of elements of $\pi_2(D^3 \setminus L_2, \ast)$. 


A crucial property of the classes $K_1, \ldots, K_n \in \pi_2(D^3\setminus L_n, \ast)$ is that $K_1 + K_2 + \cdots + K_n$ can be represented by a parametrisation $(D^2, \partial D^2) \to (\partial D^3, \ast)$. With these points in mind, we lift consideration of the mapping group action on the fundamental group to an action of the mapping class group $\text{MCG}(D^3, L_n)$ on the triple:

$$\pi_{1,2}(D^3 \setminus L_n, \ast) = (\pi_1(D^3 \setminus L_n, \ast), \pi_2(D^3 \setminus L_n, \ast), \triangleright),$$

where $\triangleright$ denotes the usual action of $\pi_1(D^3 \setminus L_n, \ast)$ on $\pi_2(D^3 \setminus L_n, \ast)$. Since $\pi_2(D^3 \setminus L_n, \ast)$ is abelian, $\pi_{1,2}(D^3 \setminus L_n, \ast)$ is a $\pi$-module, a structure we will define in Section 3.2.

3. The lifted Artin representation

We discussed in Subsection 1.2 that the homotopical information coming from $\pi_2(D^3 \setminus L_n, \ast)$ is not taken into account by Dahm’s lift of Artin’s representation. With this in mind, we proceed to define a lifted Artin representation for extended loop braid groups $LB^n_{\text{ext}}$. The codomain for our representation stores two levels of information, coming from the first and second homotopy groups of $D^3 \setminus L_n$, as well as the action of the first on the former. It will thus be a $\pi$-module. In this Section, we recap some results about extended loop braid groups and $\pi$-modules, which will subsequently be used to define the lifted Artin representation.

3.1. Loop braid groups and extended loop braid groups. We recall [Art47] that the braid group $B_n = \text{MCG}(D^2, d_n)$ is isomorphic to the group defined by generators $\{\sigma_1, \ldots, \sigma_{n-1}\}$, subject to relations:

$$\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i, & \text{for } |i - j| > 1; \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & \text{for } i = 1, \ldots, n-2.
\end{align*}$$

The generator $\sigma_i$ is the image of the mapping class $\Sigma_i$ of $\text{MCG}(D^2, d_n)$ exchanging the $i$th and $i+1$th point.

Passing to the automorphisms of the free group $F_n$ of rank $n$, let $S_1^i : F_n \to F_n$ be defined on the generators $\{x_1, \ldots, x_n\}$ as:

$$S_1^i : \begin{cases} x_i \mapsto x_{i+1}, \\
x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\
x_j \mapsto x_j, & \text{if } j < i \text{ or } j > i + 1 \end{cases}.$$

Note from Figure 2 that Artin’s representation $\theta : B_n \to \text{Aut}(F_n)$ is such that, for $i = 1, \ldots, n-1$, $\Sigma_i \mapsto S_1^i$, and this data determines $\theta$. 

![Figure 5. Elements in $\pi_2(D^3 \setminus L_2, \ast)$. We number the top circle 1. Note that the difference between the second and third figures is given by the path $x_2^{-1}$ around circle $C_2$.](image-url)
Theorem 3.1 ([Art47], for a more recent proof see for instance [Han89] Theorem 5.1). The map \( \theta : B_n \to \text{Aut}(F_n) \) is injective, and an automorphism \( \phi \) of \( F_n \) is in \( \theta(B_n) \) if and only if the following two conditions are satisfied:

1. There exist \( a_1, \ldots, a_n \in F_n \), and a permutation \( \alpha \) of \( \{1, \ldots, n\} \), such that \( \phi(x_i) = a_i x_{\alpha(i)} a_i^{-1} \).
2. \( \phi(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n \).

Note that (2) holds since \( x_1 x_2 \ldots x_n \) is the homotopy class of a path that traces the boundary of \( D^2 \), clockwise, starting and finishing at *. Hence \( x_1 x_2 \ldots x_n \) is left untouched by all elements of \( \text{MCG}(D^2, \partial D^2) \), since they are required to be the identity on \( \partial D^2 \).

Let us move one dimension up, in the realm of extended loop braid groups \( LB_n^{\text{ext}} \), defined as the mapping class groups \( \text{MCG}(D^3, L_n) \), and loop braid groups \( LB_n \), defined as \( \text{MCG}(D^3, \hat{L}_n) \); see Section 1.2. In this case we have the following result [BH13, BWC07, Dam17].

Theorem 3.2. The group \( LB_n \) is isomorphic to the abstract presented group defined by generators \( \{\sigma_i, \rho_i \mid i = 1, \ldots, n-1\} \) subject to relations:

\[
\begin{align*}
(3.3a) \quad & \sigma_i \sigma_j = \sigma_j \sigma_i, & \text{for } |i - j| > 1; \\
(3.3b) \quad & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & \text{for } i = 1, \ldots, n-2; \\
(3.3c) \quad & \rho_i \rho_j = \rho_j \rho_i, & \text{for } |i - j| > 1; \\
(3.3d) \quad & \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, & \text{for } i = 1, \ldots, n-2; \\
(3.3e) \quad & \rho_i^2 = \text{id}, & \text{for } i = 1, \ldots, n-1; \\
(3.3f) \quad & \rho_i \sigma_j = \sigma_j \rho_i, & \text{for } |i - j| > 1; \\
(3.3g) \quad & \rho_{i+1} \rho_i \sigma_{i+1} = \sigma_i \rho_{i+1} \rho_i, & \text{for } i = 1, \ldots, n-2; \\
(3.3h) \quad & \sigma_{i+1} \rho_i \sigma_{i+1} = \rho_i \sigma_{i+1} \sigma_i, & \text{for } i = 1, \ldots, n-2.
\end{align*}
\]

Moreover the group \( LB_n^{\text{ext}} \) is isomorphic to the abstract presented group defined by generators \( \{\sigma_i, \rho_i \mid i = 1, \ldots, n-1\} \cup \{\tau_j \mid j = 1, \ldots, n\} \) subject to relations (3.3a) to (3.3h) above, together with:

\[
\begin{align*}
(3.4a) \quad & \tau_i \tau_j = \tau_j \tau_i, & \text{for } i \neq j; \\
(3.4b) \quad & \tau_i^2 = \text{id}, & \text{for } i = 1, \ldots, n; \\
(3.4c) \quad & \sigma_i \tau_j = \tau_j \sigma_i, & \text{for } |i - j| > 1; \\
(3.4d) \quad & \rho_i \tau_j = \tau_j \rho_i, & \text{for } |i - j| > 1; \\
(3.4e) \quad & \tau_i \rho_i = \rho_i \tau_{i+1}, & \text{for } i = 1, \ldots, n-1; \\
(3.4f) \quad & \tau_i \sigma_i = \sigma_i \tau_{i+1}, & \text{for } i = 1, \ldots, n-1; \\
(3.4g) \quad & \tau_{i+1} \sigma_i = \rho_i \sigma_i \rho_i^{-1} \tau_i, & \text{for } i = 1, \ldots, n-1.
\end{align*}
\]

Let \( \Sigma_i \) denote the mapping class corresponding to the elementary braiding permutation in \( \text{MCG}(D^3, L_n) \). In the coaxial configuration of \( L_n \), \( n = 2 \), we may represent this as in Figure 1. Similarly, let \( \rho_i \) denote the mapping class of the non-braiding permutation; and \( \tau_i \) be the mapping class corresponding to a 180 degrees flip of \( C_i \) with respect to the vertical axis, see Figure 2. The isomorphism between \( LB_n^{\text{ext}} \) and the abstract presented group sends \( \Sigma_i \mapsto \sigma_i, \rho_i \mapsto \rho_i \), and \( \tau_i \mapsto \tau_i \).

Dahm [Dah62] generalised Artin’s representation to general manifolds with a compact submanifold in its interior. Goldsmith published his result for the case of the pair \((\mathbb{R}^3, L_n)\), which gives the same group as the pair \((D^3, L_n)\). Let us recall it:
Theorem 3.3 ([Gol81 Theorem 5.3]). For $n \geq 1$, the map

$$\theta : LB_n \rightarrow \text{Aut}(F_n).$$

is injective. Its image is the subgroup of $\text{Aut}(F_n)$ consisting of automorphisms of the form $x_i \mapsto a_i^{-1}x_{\alpha(i)}^{-1}a_i$, where $\alpha$ is a permutation of $\{1, \ldots, n\}$ and $a_i \in F_n$. Moreover, this subgroup of $\text{Aut}(F_n)$ is generated by the automorphisms $\{S^i_1 \mid i = 1, \ldots, n - 1\}$, $\{R^1_i \mid i = 1, \ldots, n - 1\}$ and $\{T^1_i \mid i = 1, \ldots, n\}$, where $S^i_1$ is as in Equation (3.2), and:

\begin{align}
R^1_i : & \begin{cases} 
  x_i \mapsto x_{i+1}; \\
  x_{i+1} \mapsto x_i; \\
  x_j \mapsto x_j, & \text{for } j \neq i, i + 1.
\end{cases} \\
T^1_i : & \begin{cases} 
  x_i \mapsto x_{i}^{-1}; \\
  x_j \mapsto x_j, & \text{for } j \neq i.
\end{cases}
\end{align}

This case of the map $\theta$ is known in the literature as Dahm’s homomorphism. The generators of $LB_n$ of type $\sigma_i$, $\rho_i$, and $\tau_i$ are respectively sent by Dahm’s homomorphisms to automorphisms $S^i_1$, $R^1_i$ and $T^1_i$.

Remark 3.4. A consequence of Theorem 3.3 is that $LB_n$ injects into $\text{Aut}(F_n)$ and its image is isomorphic to the group of automorphisms of the form $x_i \mapsto a_i^{-1}x_{\alpha(i)}^{-1}a_i$, where $\alpha$ is a permutation and $a_i \in F_n$. This group is called the group of basis conjugating automorphisms of $\text{Aut}(F_n)$, and is generated by the automorphisms $\{S^i_1 \mid i = 1, \ldots, n - 1\}$ and $\{R^1_i \mid i = 1, \ldots, n - 1\}$.

From now on we will focus on the extended loop braid group, keeping in mind that through Remark 3.4 consequences of this work can be drawn on loop braid groups.

Dahm’s homomorphism can be constructed with the same classical construction of Artin’s representation in terms of mapping classes. Briefly, choose a base-point $*$ in the boundary of $D^3$. Isotopies of maps $f : (D^3, L_n) \rightarrow (D^3, L_n)$ do not move the base-point. If $[f] \in LB_n$ then $[f]$ yields a pointed-homotopy class $\Theta([f]) : (D^3 \setminus L_n, *) \rightarrow (D^3 \setminus L_n, *)$. Let

$$\theta'([f]) : \pi_1(D^3 \setminus L_n, *) \rightarrow \pi_1(D^3 \setminus L_n, *)$$

be the induced map on fundamental groups. Algebraic topological considerations give that $\theta([f])$ coincides with $\theta'([f])$.

3.2. The category of $\pi$-modules.

Definition 3.5. A $\pi$-module is given by a triple $G = (G, A, \triangleright)$, where $G$ is a group, $A$ is an abelian group, and $\triangleright$ is a left-action by automorphisms of $G$ on $A$. Given two $\pi$-modules $G = (G, A, \triangleright)$ and $G' = (G', A', \triangleright')$, a morphism $f = (f^1, f^2) : G \rightarrow G'$ is a pair $(f^1 : G \rightarrow G', f^2 : A \rightarrow A')$ of homomorphisms that preserve actions: for each $g \in G$ and $a \in A$, we have $f^2(g \triangleright a) = f^1(g) \triangleright' f^2(a)$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6}
\caption{a) Pictorial idea of the mapping class $\Sigma_i$. b) The mapping class $\rho_i$. c) The mapping class $\tau_i$.}
\end{figure}
Given two composable \( \pi \)-module morphisms, \( f = (f^1, f^2) \) and \( h = (h^1, h^2) \), then our convention will be that \( (f^1, f^2)(h^1, h^2) = (f^1 \circ h^1, f^2 \circ h^2) \). It is easy to see that the set of \( \pi \)-modules and their morphisms form a category. We write \( \text{Agg} \) for the category of \( \pi \)-modules, since \( \pi \)-modules are abelian group–group pairs. The category of \( \pi \)-modules was also used in [Bau91, Chapter I, \S 1 (1.7)]. Confer also for example [MA74, \S 8], and [Sie93] where they are termed “second homotopy modules”.

Consider the forgetful functor \( \mathcal{U} : \text{Agg} \to \mathcal{SET} \times \mathcal{SET} \) that takes \( G = (G, A, \star) \) to the pair \( (G, A) \), made from the underlying sets of the groups \( G \) and \( A \). Recall \( F_n = F^n([1, 2, \ldots, n]) \) from [2.1]; the adjoint \( F^a \) takes \( \{[1, 2, \ldots, n], [1, 2, \ldots, n] \} \) to a \( \pi \)-module that we construct next.

**Lemma 3.6.** A morphism \( f = (f^1, f^2) : G \to G' \) of \( \pi \)-modules is invertible if and only if both \( f^1 \) and \( f^2 \) are invertible homomorphisms.

**Proof.** It follows from the definition of a morphism of \( \pi \)-modules.

We denote the group of invertible morphisms \( G \to G \) by \( \text{Aut}(G) \).

**Definition 3.7.** Let \( \mathbb{Z}[G] \) denote the group ring of group \( G \). Let \( M^G_n \) be the free \( \mathbb{Z}[G] \)-module on the symbols \( \{K_1, \ldots, K_n\} \), hence

\[
M^G_n = \mathbb{Z}[G] \{K_1, \ldots, K_n\} \cong \bigoplus_{i=1}^n \mathbb{Z}[G]K_i \cong \bigoplus_{g \in G} \bigoplus_{i=1}^n \mathbb{Z}(g, K_i)
\]

equipped with the diagonal action of \( \mathbb{Z}[G] \). We define \( \mathcal{M}^G_n \) to be the \( \pi \)-module \( (G, M^G_n, \star) \). The action of \( G \) on \( M^G_n \) is induced by the \( \mathbb{Z}[G] \)-module structure. Define

\[
M_n = M^F_n, \quad \mathcal{M}_n = \mathcal{M}^F_n \quad \text{and} \quad \mathcal{M}_n = \mathcal{M}^F_n.
\]

The following proposition describes how morphisms \( \mathcal{M}_n \to \mathcal{G} \) can be uniquely specified by their value on the generators \( x_i \) and \( K_i \), for \( i = 1, \ldots, n \). This result is also used, implicitly, in [BFMM18, \S 4.5.1].

**Proposition 3.8.** Let \( G = (G, A, \star) \) be a \( \pi \)-module. There is a canonical one-to-one correspondence between morphisms \( f = (f^1, f^2) : \mathcal{M}^n \to \mathcal{G} \) and pairs of tuples \( (g_1, \ldots, g_m) \in G^m \) and \( (a_1, \ldots, a_n) \in A^n \). The correspondence is such that: \( g_i = f^1(x_i) \) and \( a_i = f^2(K_i) \).

**Proof.** Let us consider a pair of tuples \( (g_1, \ldots, g_m) \in G^m \) and \( (a_1, \ldots, a_n) \in A^n \). We associate to this pair a morphism \( f = (f^1, f^2) \) as follows. We take \( f^1 \) to be the unique homomorphism \( F_m \to G \) such that \( f^1(x_i) = g_i \). As an abelian group, \( M^G_m \) is free on the set of pairs \( (g, K_i) \), where \( g \in F_m \) and \( i = 1, \ldots, n \). We define \( f^2 : M^G_m \to A \) to be the unique homomorphism such that \( f^2(g, K_i) = f^1(g) \star a_i \). Compatibility with group actions is preserved by construction.

**Remark 3.9.** Because of Proposition 3.8 we say that \( \mathcal{M}^n \), and in particular \( \mathcal{M}_n \), is a \( \pi \)-module of free type. Indeed consider the forgetful functor \( \mathcal{U} : \text{Agg} \to \mathcal{SET} \times \mathcal{SET} \) that takes \( G = (G, A, \star) \) to \( (G, A) \). This functor has a left adjoint \( F^a \), where given a pair \((X, Y)\) of sets, \( F^n(X, Y) = (F(X), \mathbb{Z}[F(X)]Y, \star) \), \( F(X) \) is the free group on \( X \), and \( \mathbb{Z}[F(X)]Y \) is the free \( \mathbb{Z}[F(X)] \)-module on \( Y \), with the obvious action of \( F(X) \). Hence:

\[
\mathcal{M}^n \cong F^a \{1, \ldots, m\}, \{1, \ldots, n\}.
\]

We say “of free type” instead of “free”, as \( \mathcal{M}_n \) does not arise from a left adjoint of a forgetful functor from \( \text{Agg} \) to \( \mathcal{SET} \).

We observed in Section 2.2 that \( \pi_1(D^2 \setminus d_n, \star) \) is a free group \( F^n([1, \ldots, x_n]) \). As we will establish in Lemma 1.2 et seq., we have \( \pi_1(D^3 \setminus L_2, \star) \cong F^a \{1, 2\}, \{1, 2\} \). Then the automorphism of the \( \pi \)-module induced by a homeomorphism of \( (D^3, L_2) \), relative to the boundary, is prescribed by giving:
(i) the effect on the $x_i$ generators of $\pi_1(D_3 \setminus L_2, \ast)$;
(ii) the effect on the $K_j$ generators of $\pi_2(D_3 \setminus L_2, \ast)$ as a $\mathbb{Z}[\pi_1]$-module.

In particular we have the following.

**Lemma 3.10.** Recall the elementary braid permutation $\Sigma_1 \in \text{MCG}(D_3, L_2)$ illustrated in Figure 4 for $n = 2$. The image $\Sigma$ of $\Sigma_1 \in \text{MCG}(D_3, L_2)$ in $\text{Aut}(\pi_1(D_3 \setminus L_2, \ast))$ is given by:

$$
\Sigma(x_2) = x_2^{-1}x_1x_2, \quad \Sigma(x_1) = x_2, \\
\Sigma(K_2) = x_2^{-1} \triangleright K_1
$$

(3.8)

The image of $K_1$ is determined by these since $\Sigma$ fixes $K_1 + K_2$.

**Remark 3.11.** This result, and proof, is also used in [BFMM18, §4.5.1], where a dual form of the lifted Artin representation was treated, in the context of biracks derived from $\pi$-modules.

**Proof.** (Sketch) A proof can be deduced from the observation of Figures 5 and 6. It only remains to verify the last claim. Consider the representatives of $K_1 + K_2$ sketched in Figure 7. The last one is evidently not moved by $\Sigma_1$. We will go back to these ideas later in Section 4. □

![Figure 7. Representatives of the element $K_1 + K_2$.](image)

Let us now treat this construction from a more algebraic viewpoint.

**Definition 3.12.** Define $\mathcal{M}$ as the monoidal category with object class $N_0$ and $\mathcal{M}(n, n) = \text{Agg}(\mathcal{M}_n, \mathcal{M}_n)$ with category composition as in $\text{Agg}$, and $\mathcal{M}(n, m)$ empty otherwise; and monoidal composition given by $a \otimes b = a + b$ on objects, and on morphisms as induced by the map $\{1, \ldots, n\} \sqcup \{1, \ldots, m\} \rightarrow \{1, \ldots, n + m\}$ applied “diagonally” to the indices on pairs $x_i, K_i$. Note that the permutation $(12)$ on $\{1, 2\}$ extends to make this a symmetric monoidal category.

**Definition 3.13.** Recall Proposition 3.8. Define $\mathcal{S}, \mathcal{R} \in \text{Agg}(\mathcal{M}_2, \mathcal{M}_2)$ and $\mathcal{T} \in \text{Agg}(\mathcal{M}_1, \mathcal{M}_1)$ to be the uniquely defined morphisms of $\pi$-modules such that:

$$
\mathcal{S}^1(x_j) = \begin{cases} 
    x_2, & \text{if } j = 1, \\
    x_2^{-1}x_1x_2, & \text{if } j = 2,
\end{cases}
$$

(3.9a)

$$
\mathcal{S}^2(K_j) = \begin{cases} 
    K_1 + K_2 - x_2^{-1} \triangleright K_1, & \text{if } j = 1, \\
    x_2^{-1} \triangleright K_1, & \text{if } j = 2,
\end{cases}
$$

(3.9b)
The claim is true for

\[ \mathcal{R}^1(x_j) = \begin{cases} x_2, & \text{if } j = 1, \\ x_1, & \text{if } j = 2, \end{cases} \]

\[ \mathcal{R}^2(K_j) = \begin{cases} K_2, & \text{if } j = 1, \\ K_1, & \text{if } j = 2, \end{cases} \]

\[ \mathcal{T}^1(x_1) = x_1^{-1}, \quad \mathcal{T}^2(K_j) = K_j. \]

Define \( S_i, T_i \in \text{Agg}(M_n, M_n) \), for each \( i = 1, \ldots, n - 1 \), and \( R_i \in \text{Agg}(M_n, M_n) \), for each \( i = 1, \ldots, n \), as:

\[ S_i = 1^{i-1} \otimes S \otimes 1^{n-i-1}, \quad R_i = 1^{i-1} \otimes R \otimes 1^{n-i-1}, \quad T_i = 1^{i-1} \otimes T \otimes 1^{n-i}. \]

Explicitly:

\[ S_i^1(x_j) = \begin{cases} x_j, & \text{if } j < i \text{ and } j > i + 1, \\ x_{i+1}, & \text{if } j = i, \\ x\sub{i+1}^{-1}x_i x_{i+1}, & \text{if } j = i + 1, \end{cases} \]

\[ S_i^2(K_j) = \begin{cases} K_j, & \text{if } j < i \text{ and } j > i + 1, \\ K_1 + K_{i+1} - x\sub{i+1}^{-1}K_1 \triangleright K_i, & \text{if } j = i, \\ x\sub{i+1}^{-1} \triangleright K_i, & \text{if } j = i + 1; \end{cases} \]

\[ R_i^1(x_j) = \begin{cases} x_j, & \text{if } j < i \text{ and } j > i + 1, \\ x_{i+1}, & \text{if } j = i, \\ x_i, & \text{if } j = i + 1, \end{cases} \]

\[ R_i^2(K_j) = \begin{cases} K_j, & \text{if } j < i \text{ and } j > i + 1, \\ K_{i+1}, & \text{if } j = i, \\ K_i, & \text{if } j = i + 1; \end{cases} \]

\[ T_i^1(x_j) = \begin{cases} x_j, & \text{if } j \neq i, \\ x_i^{-1}, & \text{if } j = i, \end{cases} \quad \text{and} \quad T_i^2(K_j) = K_j. \]

Remark that \( T_i^2 : M_n \to M_n \) is not the identity. For example \( T_i^2(x_i \triangleright K_j) = x_i^{-1} \triangleright K_j \).

**Lemma 3.14.** These homomorphisms of \( \pi \)-modules are invertible.

**Proof.** The claim is true for \( R_i \) and the \( T_i \), because \( R_i^2 = \text{id} \) and \( T_i^2 = \text{id} \). Explicit calculation shows that the inverse of \( S_i \) is given by \( S_i = (S_i^1, S_i^2) \) below:

\[ S_i^1(x_j) = \begin{cases} x_j x_{i+1} x_i^{-1}, & \text{if } j = i, \\ x_i, & \text{if } j = i + 1, \\ x_j, & \text{if } j \neq i, i + 1, \end{cases} \]

\[ S_i^2(K_j) = \begin{cases} x_j \triangleright K_{i+1}, & \text{if } j = i, \\ K_i + K_{i+1} - x_j \triangleright K_{i+1}, & \text{if } j = i + 1, \\ K_j, & \text{if } j \neq i, i + 1. \end{cases} \]

For the geometrical idea behind these calculations, see Lemma [3.10].
The following is the first main result of this paper.

**Theorem 3.15.** There exists a unique group homomorphism $\Theta : LB_n^{\text{ext}} \to \text{Aut}(\mathcal{M}_n)$, such that:

$$\Sigma_i \mapsto S_i, \quad \rho_i \mapsto R_i, \quad \tau_j \mapsto T_j,$$

where $i = 1, \ldots, n - 1$, $j = 1, \ldots, n$, and $\Sigma_i$, $\rho_i$ and $\tau_j$ are the generators of $LB_n^{\text{ext}}$ as a mapping class group, as in Figure 6. Furthermore, $\Theta : LB_n^{\text{ext}} \to \text{Aut}(\mathcal{M}_n)$ is injective.

To $\Theta : LB_n^{\text{ext}} \to \text{Aut}(\mathcal{M}_n)$, we call the lifted Artin representation.

**Proof.** These images are sufficient to determine a homomorphism by Theorem 3.2 and the comments just after. In Appendix A we check relations by explicit calculations.

Let $\mathcal{G} = (G, A, \overset{\sim}{})$ be a $\pi$-module. Then we have a morphism $F^1 : \text{Aut}(\mathcal{G}) \to \text{Aut}(G)$, such that $F^1(f^1, f^2) = f^1$. The injectivity of $\Theta$ follows from the fact that $F^1 \circ \Theta$ coincides with Dahm’s homomorphism $LB_n^{\text{ext}} \to \text{Aut}(F_n)$, see Theorem 3.3.

A feature of Artin’s representation, is that it provides a characterisation of $\text{braid automorphisms}$ of $F_n$, as recalled in Theorem 3.1. Also in the case of extended loop braid groups, Goldsmith gives a characterisation elements of $\text{Aut}(F_n)$ that are images of elements of $LB_n^{\text{ext}}$, Theorem 3.3. It is thus natural to ask the following question.

**Open Problem 3.16.** Determine the image of $\Theta : LB_n^{\text{ext}} \to \text{Aut}(\mathcal{M}_n)$.

**Remark 3.17.** Note that for any $(f^1, f^2) \in \Theta(LB_n^{\text{ext}})$ it holds that:

$$f^2(K_1 + \cdots + K_n) = K_1 + \cdots + K_n.$$  \hspace{1cm} (3.16)

This is an analogue for condition 2 in Theorem 3.1. As we will see in Remark 4.6, Equation (3.16) holds since $K_1 + K_2 + \cdots + K_n$ can be represented by a parametrisation $(D^2, \partial D^2) \to (\partial D^2, *)$, and hence is left untouched by all elements of $\text{MCG}(D^2, L_n)$, given that these mapping classes are the identity in $\partial D^2$.

**Remark 3.18.** A dual version of the lifted Artin representation, which considers finite dimensional representations of the loop braid group derived from biracks derived from $\pi$-modules, appeared in [BFMM18]. The underlying invariants of welded knots were initially defined in [KFM08].

4. A **topological interpretation of the lifted Artin representation**

Let $X$ be a path-connected space. A path $\gamma : [0, 1] \to X$ canonically induces isomorphisms $\pi_2(X, \gamma(1)) \to \pi_2(X, \gamma(0))$, and also $\pi_1(X, \gamma(1)) \to \pi_1(X, \gamma(0))$; see e.g. [Hat02, Page 341]. Let $* \in X$ be a base-point. If we consider closed paths, starting and ending in $*$, this descends to an action of $\pi_1(X,*)$ on $\pi_2(X,*)$, by automorphisms; see e.g. [Hat02, page 342]. We denote by $\pi_{(1,2)}(X,*)$ the $\pi$-module obtained from $\pi_1(X,*)$ acting on $\pi_2(X,*)$ in this way. We call $\pi_{(1,2)}(X,*)$ the fundamental $\pi$-module of $(X,*)$ (confer with [Bau91, Chapter I, §1]). We let $\text{TOP}_*$ be the category of pointed topological spaces, and pointed maps. Define also $\text{TOP}_*/\overset{\sim}{\pi}$ as the category with object the pointed spaces, and morphisms $(X,*) \to (Y,*)$ to be pointed maps $(X,*) \to (Y,*)$, considered up to pointed homotopy. The fundamental $\pi$-module extends to functors $\pi_{(1,2)} : \text{TOP}_* \to \text{Agg}$ and $\pi_{(1,2)} : \text{TOP}_*/\overset{\sim}{\pi} \to \text{Agg}$. Given a pointed map $f : (X,*) \to (Y,*)$, the induced map of $\pi$-modules is denoted by $\pi_{(1,2)}(f) : \pi_{(1,2)}(X,*) \to \pi_{(1,2)}(Y,*)$. This is independent of the representative of the pointed homotopy class of $f$. Given a pointed homotopy class of pointed maps $[f] : (X,*) \to (Y,*)$, the induced morphism in $\text{Agg}$ is also denoted by $\pi_{(1,2)}([f]) : \pi_{(1,2)}(X,*) \to \pi_{(1,2)}(Y,*)$. Note that $\pi_{(1,2)}([f]) = \pi_{(1,2)}(f)$. 
4.1. Some bouquets and their fundamental \( \pi \)-modules.

**Definition 4.1.** Fix, from now on, base points \( * \in S^1 \) and \( * \in S^2 \), hence \( (S^1, *) \) and \( (S^2, *) \) will be well-pointed. Let \( n \) be a positive integer. For \( i = 1, \ldots, n \), let \( S^1_i \) be a copy of the circle \( S^1 \subset \mathbb{R}^2 \), oriented counterclockwise, and \( S^2_i \) a copy of the 2-sphere \( S^2 \subset \mathbb{R}^3 \), oriented by the exterior normal. We define the following pointed spaces, with the obvious CW-decompositions, where \( * \) is the unique 0-cell:

\[
(\mathcal{H}_n, *) = \bigvee_{i=1}^n ((S^1_i, *) \vee (S^2_i, *)), \quad (\mathcal{F}_n, *) = \bigvee_{i=1}^n (S^1_i, *), \quad (\mathcal{J}_n, *) = \bigvee_{i=1}^n (S^2_i, *).
\]

Let \( A_i \in \pi_2(\mathcal{H}_n, *) \) be given by the homotopy class of a positively oriented characteristic map \( (D^2, \partial D^2) \to (S^2, *) \subset (\mathcal{H}_n, *) \) of the 2-cell \( S^2_i \) of \( \mathcal{H}_n \), for \( i = 1, \ldots, n \). Let \( x_i \in \pi_1(\mathcal{H}_n, *) \) be given by the homotopy class of a positively oriented characteristic map \( ([0,1], \{0,1\}) \to (S^1_i, *) \subset (\mathcal{H}_n, *) \) of the 1-cell \( S^1_i \) of \( \mathcal{H}_n \), for \( i = 1, \ldots, n \).

By the Seifert–van Kampen theorem, the group \( \pi_1(\mathcal{H}_n, *) \) is free on the \( x_1, \ldots, x_n \in \pi_1(\mathcal{H}_n, *) \). Hence \( \pi_1(\mathcal{H}_n, *) \cong F_n \) in (2.1) canonically. This is because, since spaces are well-pointed, we have canonical isomorphisms of groups (see e.g. [Hat02, Theorem 1.20 and Example 1.21]):

\[
\pi_1(\mathcal{H}_n, *) = \pi_1 \left( \bigvee_{i=1}^n ((S^1_i, *) \vee (S^2_i, *)) \right) \cong \bigvee_{i=1}^n \pi_1 \left( (S^1_i, *) \vee (S^2_i, *) \right) \cong \bigvee_{i=1}^n (S^1_i, *) \cong F_n.
\]

In particular, the inclusion \( (\mathcal{F}_n, *) \to (\mathcal{H}_n, *) \) canonically induces an isomorphism of groups \( F_n \cong \pi_1(\mathcal{F}_n, *) \cong \pi_1(\mathcal{H}_n, *) \cong F_n \).

We now determine \( \pi_2(\mathcal{H}_n, *) \) as a \( \pi_1(\mathcal{H}_n, *) \)-module.

**Lemma 4.2.** As an abelian group, \( \pi_2(\mathcal{H}_n, *) \) is freely generated by the \( g \circ A_i \), where \( g \in \pi_1(\mathcal{H}_n, *) \cong \pi_1(\mathcal{F}_n, *) \cong F_n \), and \( i = 1, \ldots, n \). We have an isomorphism sending \( g \circ A_i \) to \( (g, K_i) \):

\[
\pi_2(\mathcal{H}_n, *) \to M_n,
\]

where we recall from Definition 3.7 that:

\[
M_n = \mathbb{Z}[F_n]\{ K_1, \ldots, K_n \} \cong \bigoplus_{i=1}^n \mathbb{Z}[F_n]K_i \cong \bigoplus_{g \in F_n} \bigoplus_{i=1}^n \mathbb{Z}(g, K_i).
\]

In particular, we have a canonical isomorphism of \( \pi \)-modules:

\[
\pi_{(1,2)}(\mathcal{H}_n, *) \cong M_n.
\]

**Proof.** We extend [Hat02, Example 4.27], which deals with case \( n = 1 \). The crucial ideas of the following argument are also in [BFMM15, §4.5.1].

Let \( q : (\mathcal{H}_n, \hat{*}) \to (\mathcal{H}_n, *) \) be the universal cover of \( \mathcal{H}_n \), where we fixed a \( \hat{*} \in q^{-1}(*) \). Lifting elements of \( \pi_2(\mathcal{H}_n, *) \) to elements of \( \pi_2(\mathcal{H}_n, *) \) yields an isomorphism:

\[
\Psi : \pi_2(\mathcal{H}_n, *) \to \pi_2(\mathcal{H}_n, \hat{*}),
\]

see e.g. [Hat02, Proposition 4.1]. Let \( \overline{q} : (\mathcal{F}_n, \hat{*}) \to (\mathcal{F}_n, \hat{*}) \) be the universal cover of \( \mathcal{F}_n \), where we fixed \( \hat{*} \in q^{-1}(*) \). The crucial observation, as in [Hat02, Example 4.27], is that \( \mathcal{H}_n \) is obtained from \( \mathcal{F}_n \) by attaching a copy of \( (\mathcal{J}_n, \hat{*}) = \bigvee_{i=1}^n (S^1_i, 
\]

Covering space theory gives a one-to-one correspondence \( \pi_1(\mathcal{H}_n, *) \to q^{-1}(*) \cong \overline{q}^{-1}(*) \). We choose the convention sending \( g \in \pi_1(\mathcal{H}_n, *) \to \hat{*} \circ g \), where on the right-hand-side we considered the monodromy right-action of \( \pi_1(\mathcal{H}_n, *) \cong \pi_1(\mathcal{F}_n, *) \) on \( q^{-1}(*) \cong \overline{q}^{-1}(*) \).
For each $g \in \pi_1(\mathcal{H}_n, \star)$, consider a copy of $(\mathcal{J}_{n,g}, \star_g)$ of $(\mathcal{J}_n, \star)$. We have a pushout diagram:

$$
\begin{array}{ccc}
\bigcup_{g \in \pi_1(\mathcal{H}_n, \star)} \{ \star \ast g \} & \xrightarrow{\star_n} & \bigcup_{g \in \pi_1(\mathcal{H}_n, \star)} \mathcal{J}_{n,g} \\
\downarrow & & \downarrow \\
\mathcal{F}_n & \xrightarrow{\iota_n} & \mathcal{H}_n
\end{array}
$$

in the category of topological spaces. Here the top horizontal arrow is induced by $\ast \ast g \mapsto \ast_g$. In particular, the top horizontal arrow is a cofibration. Consequently $\iota_n: \mathcal{F}_n \to \mathcal{H}_n$ is a cofibration; see e.g. [May99, Page 44]. Since $\mathcal{F}_n$ is contractible, we hence have a homotopy equivalence $\tilde{\mathcal{H}}_n \to \mathcal{H}_n$, obtained by collapsing $\mathcal{F}_n$ to a point: call it $\ast$; see e.g. [Hat02, Proposition 0.17]. Now note that $(\mathcal{H}_n/\mathcal{F}_n, \ast)$ is canonically homeomorphic to $\bigvee_{g \in \pi_1(\mathcal{H}_n, \star)} \bigvee_{i=1}^n (S^2_{g,i}, \ast)$, where $S^2_{g,i} = S^2$.

Therefore we have a canonical isomorphism $\pi_2(\mathcal{H}_n, \ast) \cong \pi_2\big(\bigvee_{g \in \pi_1(\mathcal{H}_n, \star)} \bigvee_{i=1}^n (S^2_{g,i}, \ast)\big)$. By [Hat02, Example 4.26], $\pi_2\big(\bigvee_{g \in \pi_1(\mathcal{H}_n, \star)} \bigvee_{i=1}^n (S^2_{g,i}, \ast)\big)$ is the free abelian group on the homotopy classes $A_{g,i}$, yielded by the inclusions $(S^2_{g,i}, \ast) \cong (S^2_{g,i}, \ast) \to \bigvee_{g \in \pi_1(\mathcal{H}_n, \star)} \bigvee_{i=1}^n (S^2_{g,i}, \ast)$. Therefore we have:

$$\pi_2(\mathcal{H}_n, \ast) \cong \bigoplus_{g \in \pi_1(\mathcal{H}_n, \star)} \bigoplus_{i=1}^n \mathbb{Z}(g, K_i)$$

canonically, where $\mathbb{Z}(g, K_i) \cong \mathbb{Z}$, and its positive generator is identified with

$$A_{g,i} \in \pi_2\big(\bigvee_{g \in \pi_1(\mathcal{H}_n, \star)} \bigvee_{i=1}^n (S^2_{g,i}, \ast)\big) \cong \pi_2(\mathcal{H}_n, \ast).$$

With our conventions, the positive generator of $\mathbb{Z}(g, K_i)$ corresponds to $\Psi(g \triangleright A_i) \in \pi_2(\mathcal{H}_n, \ast)$, for each $g \in \pi_1(\mathcal{H}_n, \star)$ and each $i \in \{1, \ldots, n\}$. Given that $\Psi: \pi_2(\mathcal{H}_n, \ast) \to \pi_2(\mathcal{H}_n, \ast)$ is an isomorphism, and that $\pi_1(\mathcal{H}_n, \star) \cong F_n$, this completes the proof.

4.2. **Fundamental $\pi$-module for $(D^3 \setminus L_n)$.** Recall that $L_n = \cup_{i=1}^n C_i$ is an unlinked union of unlinked circles in the interior of $D^3$. It is now convenient to consider that all circles are confined to the vertical plane $\{y = 1/2\}$. We take each $C_i$ to be a circle centred at $(i/n + 1), 1/2, 1/2)$, with radius $1/2n$, and oriented clockwise, for the point of view of an observer sitting in the $y = 0$ plane. Furthermore, we consider a base-point located in the $\{z = 0\}$ plane. Analogously to the 2-dimensional case in Subsection 2.1 we have:

**Proposition 4.3.** Let $\ast$ be a base point for $D^3 \setminus L_n$ contained in $\partial D^3 \cap \{z = 0\}$. The spaces $(D^3 \setminus L_n, \ast)$ and $(\mathcal{H}_n, \ast)$ are pointed homotopic. Moreover, we have a deformation retraction from $(D^3 \setminus L_n, \ast)$ onto a certain homeomorphic image of $(\mathcal{H}_n, \ast)$. Here $(\mathcal{H}_n, \ast)$ embeds in $(D^3 \setminus L_n, \ast)$ in a way where each oriented circle $S^1_C$ encircles the oriented circle $C_i \subset D^3$, forming a Hopf link, with positive linking number, and each oriented $S^1_C$ embeds as a positively oriented “balloon” (i.e. a 2-sphere, oriented by the exterior normal $\tilde{n}_i$) containing $C_i$, and no other circle, see Figure 8.

From the previous lemma, it follows that the inclusion $(\mathcal{H}_n, \ast) \to \pi_2(D^3 \setminus L_n, \ast)$ induces an isomorphism of $\pi$-modules $\pi_1(\mathcal{H}_n, \ast) \cong \pi_1(D^3 \setminus L_n, \ast)$. Hence $\pi_1(\mathcal{H}_n, \ast) \cong \mathcal{M}_n$. Let $f$ be a self-homeomorphism of the pair $(D^3, L_n)$, relative to the boundary. Isotopies of such homeomorphisms are also considered to be relative to the boundary. Hence each element $[f] \in \text{MCG}(D^3, L_n) = \text{LB}_n^{\text{ext}}$ yields, by restricting to $D^3 \setminus L_n$, a pointed homotopy class of pointed maps $\Upsilon([f]): (D^3 \setminus L_n, \ast) \to (D^3 \setminus L_n, \ast)$; here recall that we imposed $\ast \in \partial D^3$. The
induced map of fundamental $\pi$-modules:

$$
\pi_{(1,2)}(\Theta([f])) : \pi_{(1,2)}(D^3 \setminus L_n, \ast) \cong M_n \rightarrow \pi_{(1,2)}(D^3 \setminus L_n, \ast) \cong M_n
$$

is denoted $\Theta'(|[f]|) : \mathcal{M}_n \rightarrow \mathcal{M}_n$. The functoriality of $\pi_{(1,2)} : \text{TOP}_\ast \rightarrow \text{Agg}$ gives that $\Theta'([f][g]) = \Theta'(|[f] \circ g|) = \Theta'([f]) \circ \Theta'([g])$, for each $[f], [g] \in \text{MCG}(D^3, L_n)$. We thus have a group homomorphism $\Theta' : LB_n^\text{ext} = \text{MCG}(D^3, L_n) \rightarrow \text{Aut}(\mathcal{M}_n)$. The following theorem gives the topological interpretation of the lifted Artin representation.

**Theorem 4.4.** The homomorphism $\Theta' : LB_n^\text{ext} \rightarrow \text{Aut}(\mathcal{M}_n)$ coincides with the homomorphism $\Theta : LB_n^\text{ext} \rightarrow \text{Aut}(\mathcal{M}_n)$ in Theorem 3.15. In particular $\Theta'$ is injective.

The main ideas of the following proof are also in [BFMM18, §4.5.1].

**Proof. (Sketch)** We recall what each generator $g$ of the extended loop braid group in Theorem 3.2 incarnates geometrically. Let $\text{Homeo}(D^3)$ be the group of homeomorphisms $D^3 \rightarrow D^3$, relative to the boundary. Consider an ambient isotopy $t \in [0, 1] \mapsto \phi^g_t \in \text{Homeo}(D^3)$ of $D^3$, for each generator $g$ of $LB_n^\text{ext}$, as in Figure 9. Concretely $t \mapsto \phi^g_t$ is obtained by applying the isotopy extension theorem, as in [Hir76, Chapter 8.1.3 Theorem], to the smooth isotopies outlined in Figure 9. Each ambient isotopy $t \mapsto \phi^g_t$ is relative to the boundary of $D^3$, and satisfies $\phi^g_0 = \text{id}_{D^3}$. Moreover, note that $\phi^g_t$ is a homeomorphism $(D^3, L_n) \rightarrow (D^3, L_n)$. The elements of $\text{MCG}(D^3, L_n)$ corresponding to each of the generators $g$ of $LB_n^\text{ext}$ in Theorem 3.2 are obtained by evaluating the ambient isotopies $\phi^g_t$ at $t = 1$. Our motions are the ones in [Dam17, §3], with the interval $[0, 1]$ taken with reversed extremes due to different conventions for the product in $\text{MCG}(D^3, L_n)$.

![Figure 8. Embedding of $(S^1, \ast) \vee (S^2, \ast)$ inside $(D^3 \setminus L_n, \ast)$.](image)

![Figure 9. Each ambient isotopy $t \mapsto \phi^g_t$ is relative to the boundary of $D^3$, and satisfies $\phi^g_0 = \text{id}_{D^3}$. Each ambient isotopy $t \mapsto \phi^g_t$ is relative to the boundary of $D^3$, and satisfies $\phi^g_0 = \text{id}_{D^3}$.](image)

Recall $\pi_{(1,2)}(D^3, L_n) \cong \mathcal{M}_n = (F_n, M_n, \ast)$, where $M_n$ is the free $\mathbb{Z}[F_n]$-module generated by $\{K_1, \ldots, K_n\}$; see Definition 3.7. Our conventions for the generators $x_j \in \pi_1(D^3 \setminus L_n, \ast)$ and $K_j \in \pi_2(D^3 \setminus L_n, \ast)$, where $j = 1, \ldots, n$, are depicted on the left-hand side of Figure 10. In particular, $K_j$ is obtained from a positively oriented parametrisation $(D^2, \partial D^2) \rightarrow (D^3 \setminus L_n, \ast_j)$ of a “balloon” (i.e. a 2-sphere, oriented by an exterior normal), based in $\ast_j$, as shown, containing the circle $C_j$ only; acted on by the path $\gamma_j$, a straight line connecting $\ast$ and $\ast_j$, in order to
obtain an element of \( \pi_2(D^3 \setminus L_n, \ast) \). This point of view is as in [BN15], [BFMM18 §4.5.1] and [FM09 §2.1.3]. Also, \( x_j \) is given by the loop in Figure 10, which is then conjugated by the path \( \gamma_j \), in order to yield an element of \( \pi_1(D^3 \setminus L_n, \ast) \). By applying the deformation retraction in Proposition 4.3, we hence obtain the generators of \( \pi_{(1,2)}(H_n, \ast) \) as in Lemma 4.2.

\[
\begin{align*}
C_j & \quad K_j \\
C_{j+1} & \quad K_{j+1} \\
\gamma_j & \quad \gamma_{j+1}
\end{align*}
\]

Figure 10. On the left of the figure: our conventions for the generators \( x_j \in \pi_1(D^3 \setminus L_n, \ast) \) and \( K_j \in \pi_2(D^3 \setminus L_n, \ast) \), in the vicinity of the circle \( C_j \). On the right of the figure: the result of acting with \( \Sigma_i \) on \( x_{i+1} \) and on \( K_{i+1} \).

Let us see how the generators of \( \text{MCG}(D^3, L_n) \) in Theorem 3.2 act on the \( x_i \)'s and the \( K_j \)’s. We use the same notation to denote \( g \in LB_{\text{ext}}^\ast \) and the map \( \Theta'(g) : M_n \to M_n \).

Whenever \( |i - j| > 1 \), nothing happens when we apply \( \Sigma_i \) and \( \rho_i \) to \( x_j \) and to \( K_j \). Also, \( \rho_i \) is such that:

\[ \rho_i : \quad x_i \mapsto x_{i+1}, \quad x_{i+1} \mapsto x_i, \quad K_i \mapsto K_{i+1}, \quad K_{i+1} \mapsto K_i. \]

Hence \( \rho_i \) coincides with \( R_i \) in (3.10a) and (3.10b).

If we apply \( \Sigma_i \) to \( x_j \) we get \( x_{i+1} \). The right-hand-side of Figure 10 indicates what happens to \( x_{i+1} \) and \( K_{i+1} \) when we apply \( \Sigma_i \). Hence:

\[ \Sigma_i : \quad x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \quad K_{i+1} \mapsto x_{i+1}^{-1} \triangleright K_i. \]

Let us now determine \( \Sigma_i(K_{i+1}) \). Note that when the circle \( C_{i+1} \) goes inside the circle \( C_j \) it “drags” the balloon representing the class \( K_i \in \pi_2(D^3 \setminus L_n, \ast) \). However \( \Sigma_i(K_i + K_{i+1}) = K_i + K_{i+1} \).

This is because \( K_i + K_{i+1} \in \pi_2(D^3 \setminus L_n, \ast) \) can be seen as being represented by a bigger balloon \( K_{i+1} \) containing \( C_i \cup C_{i+1} \), see Figure 11. Moreover, the isotopy \( t \in [0, 1] \mapsto \phi_t^\circ \) from which we define \( \Sigma_i = \phi_t^\circ \) can be chosen to happen inside the bigger balloon \( K_{i+1} \), hence not moving \( K_{i+1} \).

\[
\begin{align*}
C_i & \quad C_{i+1} \\
K_{i+1}
\end{align*}
\]

Figure 11. The balloon \( K_{i+1} \) representing \( K_i + K_{i+1} \in \pi_2(D^3 \setminus L_n, \ast) \).

We obtain \( \Sigma_i(K_{i+1}) = K_{i+1} \). Therefore:

\[ \Sigma_i(K_i) + \Sigma_i(K_{i+1}) = \Sigma_i(K_i + K_{i+1}) = K_i + K_{i+1}, \]

and

\[ \Sigma_i(K_i) = K_i + K_{i+1} - \Sigma_i(K_{i+1}) = K_i + K_{i+1} - x_{i+1}^{-1} \triangleright K_{i+1}. \]
Hence $\Sigma_i$ coincides with $S_i$ in (3.9a) and (3.9b). That $\tau_i$s coincide with $T_i$s in (3.11a) can be seen in a similar way. Note that $\tau_i(1) = K_1$, since the isotopy yielding $K_1$ can be chosen to happen inside the balloon representing $K_1$. \hfill $\square$

**Remark 4.5.** The restriction of the representation $\Theta : LB_n^{ext} \to Aut(M_n)$ to the loop braid group $LB_n$, as well as its topological interpretation, are also discussed in [BFMM18 §4.5.1].

**Remark 4.6.** In Remark 3.17, we promised an interpretation of the property $\Theta(g)^2(K_1 + \cdots + K_n) = K_1 + \cdots + K_n$, for each $g \in \text{MCG}(D^3, L_n)$. Observe that, given our conventions, $M = K_1 + \cdots + K_n$ is homotopic to the element of $\pi_2(D^3 \setminus L_n, *)$ yielded by the inclusion of $(\partial D^3, *)$ inside $(D^3 \setminus L_n, *)$. Since all elements of $\text{MCG}(D^3, L_n)$ restrict to the identity over $\partial D^3$, it follows that $\Theta(g)^2(M) = M$, for all $g \in \text{MCG}(D^3, L_n) = LB_n^{ext}$.

4.3. **Remarks on connections to Baues’ combinatorial homotopy.** Given a pointed space $(X, *)$, we denote by $\mathcal{E}(X, *)$ the group of pointed homotopy equivalences $(X, *) \to (X, *)$, up to pointed homotopy. If $g \in \partial D^3$, we have a group homomorphism $\Upsilon : LB_n^{ext} \to \mathcal{E}(D^3 \setminus L_n, *)$, obtained by restricting $f : (D^3, L_n) \to (D^3, L_n)$. The fundamental $\pi$-module functor $\text{TOP}_* \cong \text{Agg}$ gives another group homomorphism $\mathcal{E}(D^3 \setminus L_n, *) \to \text{Aut}(\pi_1(L_n, *)) \cong \text{Aut}(M_n)$. Composing with $\Upsilon$, this yields a group homomorphism $\Theta' : LB_n \to \text{Aut}(M_n)$, which by Theorem 4.4 coincides with $\Theta : LB_n \to \text{Aut}(M_n)$.

Note that taking into account our results, it follows from elementary algebraic topological techniques, or as a particular case of [Bau91 Theorem III(7.1)] or [Bau98 Corollary VI(3.5)], that $\mathcal{E}(D^3 \setminus L_n, *) \cong \mathcal{E}(H_n, *) \cong \text{Aut}(M_n)$. This is a generalisation of the well known fact that $\mathcal{E}(D^2 \setminus d_n, *) \cong \mathcal{E}(F_n, *) \cong \text{Aut}(F_n)$. Hence in this $(D^3 \setminus L_n, *)$ case, $\pi$-modules retain all of the homotopy theoretical information necessary to combinatorially model the group $\mathcal{E}(D^3 \setminus L_n, *)$. Therefore, the lifted Artin representation $\Theta : \text{MCG}(D^3, L_n) \to \text{Aut}(M_n)$ completely models the pointed homotopy type of the restriction of a mapping class in $\text{MCG}(D^3, L_n)$ to $D^3 \setminus L_n$.

**Appendix A. Calculations**

In this Appendix we explicitly verify that relations from Theorem 3.15 hold. It will be enough to consider the case $n = 3$. In order to reduce indices, with respect to Theorem 3.15 we replace $\{x_1, x_2, x_3\}$ with $\{x, y, z\}$ and $\{K_1, K_2, K_3\}$ with $\{K, L, M\}$. We will only show computation for generators $\{K, L, M\}$, since on $\{x, y, z\}$ our representation coincides with Dahn’s homomorphism.

- $S_1 \circ S_2 \circ S_1 = S_2 \circ S_1 \circ S_2$.
  Applying $S_1 \circ S_2 \circ S_1$ one gets
  \[
  \begin{align*}
  K &\mapsto K + L - y^{-1} \triangleright K \mapsto K + L + M - z^{-1} \triangleright L - z^{-1} \triangleright K \\
  &\mapsto K + L - y^{-1} \triangleright K + y^{-1} \triangleright K + M - z^{-1} \triangleright y^{-1} \triangleright K - z^{-1} \triangleright (K + L - y^{-1} \triangleright K) \\
  &\mapsto K + L + M - z^{-1} \triangleright (K + L), \\
  L &\mapsto y^{-1} \triangleright K \mapsto z^{-1} \triangleright K \mapsto z^{-1} \triangleright (K + L - y^{-1} \triangleright K), \\
  M &\mapsto z^{-1} \triangleright L \mapsto z^{-1} \triangleright y^{-1} \triangleright K
  \end{align*}
  \]
  while applying $S_2 \circ S_2 \circ S_2$ one gets
  \[
  \begin{align*}
  K &\mapsto K + L - y^{-1} \triangleright K \mapsto K + L + M - z^{-1} \triangleright L - z^{-1} \triangleright K, \\
  L &\mapsto L + M - z^{-1} \triangleright L \mapsto y^{-1} \triangleright K + M - z^{-1} \triangleright y^{-1} \triangleright K \\
  &\mapsto z^{-1} \triangleright K + z^{-1} \triangleright L - (z^{-1}y^{-1}z) \triangleright z^{-1} \triangleright K \\
  M &\mapsto z^{-1} \triangleright L \mapsto z^{-1} \triangleright y^{-1} \triangleright K \mapsto (z^{-1}y^{-1}z) \triangleright z^{-1} \triangleright K.
  \end{align*}
  \]
- $R_2 \circ R_1 \circ S_2 = S_1 \circ R_2 \circ R_1$ Applying $R_2 \circ R_1 \circ S_2$ one gets
\[
\begin{align*}
K &\mapsto K \mapsto L \mapsto M \\
L &\mapsto L + M - z^{-1} \triangleright L \mapsto K + M - z^{-1} \triangleright K \mapsto K + L - y^{-1} \triangleright K \\
M &\mapsto z^{-1} \triangleright L \mapsto z^{-1} \triangleright K \mapsto y^{-1} \triangleright K
\end{align*}
\]

while applying \( S_1 \circ R_2 \circ R_1 \) one gets

\[
\begin{align*}
K &\mapsto L \mapsto M \mapsto M \\
L &\mapsto K \mapsto K + L - y^{-1} \triangleright K \\
M &\mapsto M \mapsto L \mapsto y^{-1} \triangleright K.
\end{align*}
\]

- \( S_2 \circ S_1 = R_1 \circ S_2 \circ S_1 \)

Applying \( S_2 \circ S_1 \circ R_2 \) one gets

\[
\begin{align*}
K &\mapsto K \mapsto K + L - y^{-1} \triangleright K \mapsto K + L + M - z^{-1} \triangleright L - z^{-1} \triangleright K \\
L &\mapsto M \mapsto M \mapsto z^{-1} \triangleright L \\
M &\mapsto L \mapsto y^{-1} \triangleright K \mapsto z^{-1} \triangleright K
\end{align*}
\]

while applying \( R_1 \circ S_2 \circ S_1 \) one gets

\[
\begin{align*}
K &\mapsto K + L - y^{-1} \triangleright K \mapsto K + L + M - z^{-1} \triangleright L - z^{-1} \triangleright K \\
&\quad \mapsto L + K + M - z^{-1} \triangleright K - z^{-1} \triangleright L \\
L &\mapsto y^{-1} \triangleright K \mapsto z^{-1} \triangleright K \mapsto z^{-1} \triangleright L \\
M &\mapsto M \mapsto z^{-1} \triangleright L \mapsto z^{-1} \triangleright K.
\end{align*}
\]

- \( T_1 \circ R_1 = R_1 \circ T_2 \) Applying \( T_1 \circ R_1 \) one gets

\[
\begin{align*}
K &\mapsto L \mapsto L \\
L &\mapsto K \mapsto K \\
M &\mapsto M \mapsto M
\end{align*}
\]

while applying \( R_1 \circ T_2 \) one gets

\[
\begin{align*}
K &\mapsto K \mapsto L \\
L &\mapsto L \mapsto K \\
M &\mapsto M \mapsto M.
\end{align*}
\]

- \( T_1 \circ S_1 = S_1 \circ T_2 \) Applying \( T_1 \circ S_1 \) one gets

\[
\begin{align*}
K &\mapsto K + L - y^{-1} \triangleright K \mapsto K + L + y^{-1} \triangleright K \\
L &\mapsto y^{-1} \triangleright K \mapsto y^{-1} \triangleright K \\
M &\mapsto M \mapsto M
\end{align*}
\]

while applying \( S_1 \circ T_2 \) one gets

\[
\begin{align*}
K &\mapsto K + L - y^{-1} \triangleright K \mapsto K + L + y^{-1} \triangleright K \\
L &\mapsto y^{-1} \triangleright K \mapsto y^{-1} \triangleright K \\
M &\mapsto M \mapsto M.
\end{align*}
\]

- \( T_2 \circ S_1 = R_1 \circ S_1^{-1} \circ R_1 \circ T_1 \) Applying \( T_2 \circ S_1 \) one gets

\[
\begin{align*}
K &\mapsto K + L - y^{-1} \triangleright K \mapsto K + L - y \triangleright K \\
L &\mapsto y^{-1} \triangleright K \mapsto y \triangleright K \\
M &\mapsto M \mapsto M
\end{align*}
\]

while applying \( R_1 \circ S_1^{-1} \circ R_1 \circ T_1 \) one gets
\[
\begin{align*}
K \mapsto L & \mapsto K + L - x \triangleright L \mapsto L + K - y \triangleright K \\
L \mapsto K & \mapsto x \triangleright L \mapsto y \triangleright K \\
M \mapsto M & \mapsto M \mapsto M 
\end{align*}
\]

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