On the electromagnetic emission from charged test particles in a five dimensional spacetime

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Abstract. We study the motion of charged particles radially falling in a class of static and electromagnetic-free, five-dimensional Kaluza-Klein backgrounds. Particle dynamics in such spacetimes is explored by an approach à la Papapetrou. The electromagnetic radiation emitted by these particles is studied, outlining the new features emerging in the spectra for the five-dimensional case. A comparison with the dynamics in the four dimensional counterpart, i.e the Schwarzschild background, is performed.

Key words. Kaluza Klein theory–Generalized Schwarzschild solution (GSS)–Test particle orbits–Emission spectra

1 Introduction

Multidimensional theories are considered a great promise of the present physics as possible candidates for the grand unification theory. Great attention is therefore devoted to any available test to probe the validity of these theories (see for example [12] and also [9]). The five-dimensional (5D) Kaluza-Klein (KK) theory unifies electromagnetism and gravitation in a geomet-
rical picture, reformulating the internal U(1) gauge symmetry in a geometrical one by the introduction of a fifth dimension. Translations in this space are a generalization of the gauge transformation (see for instance [13,14,15,16]). In this work we consider a 5D-background, known as Generalized Schwarzschild Solution (GSS) [17,18], solution of the electromagnetic-free KK-equations. We investigate the motion of charged particles radially falling in GSS spacetimes. Electromagnetic (EM) radiation emitted by such particles is studied as a perturbation of the background metric. Spectral observations could be used to test the validity of a multidimensional theory: the presence of an additional dimension could be recognized via the deformation induced on the spectra predicted by the General Relativity (see for example [19]). The paper is organized as follows; in Sec. 2 we discuss some general aspects of the GSS, in Sec. 3 we briefly review some fundamental statements of the KK-particle dynamics, in Sec. 4 EM-perturbations in GSS spacetimes are analyzed. Finally in Sec. 5 concluding remarks follow.

2 Generalized Schwarzschild Solution

The GSS is a family of free-electromagnetic, static 5D-vacuum solutions of KK-equations. Adopting spherical polar coordinates the GSS reads

$$ds^2 = \Delta^k dt^2 - \Delta^{-\epsilon(k-1)} dr^2 - r^2 \Delta^{1-\epsilon(k-1)} d\Omega^2 - \Delta^{-\epsilon} (dx^5)^2$$

(1)

where $d\Omega^2 \equiv (d\theta^2 + \sin^2 \theta d\phi^2)$, $\Delta \equiv (1 - 2M/r)$, $M > 0$ is a constant, $\epsilon = (k^2 - k + 1)^{-1/2}$ and $k$ is a positive dimensionless parameter. On the 4D-counterpart metric (the spacetime section $dx^5 = 0$), the Schwarzschild solution is recovered for $k \to \infty$, where in this limit $M$ is the Schwarzschild mass. The region $k \geq 0$ and $\epsilon \geq 0$ of the metric parameter is generally taken into account to investigate the physical properties of solutions. In [17], for example, is showed how positive density of solution requires $k > 0$ and for positive mass (as measured at infinity) one must have $\epsilon k > 0$. For a review about the constraint on $k$ parameter see [17] and [20]. To test the validity of GSS model, the predictions elaborated within a GSS scenario have been compared with the observed data. Each comparison gives a peculiar estimation for $k$; all these different estimations are based on different tests assumed to prove the model validity. For example, a particular value of $k$, adapted to fit the prediction of the standard gravity tests with experimental data, has been associated to the Sun [17,21]. In [22], experimental constraints on equivalence principle violation in the solar system translate in a $k > 5 \times 10^7$. We could conclude therefore that extra dimensions play thus a negligible role in the solar system.

With Latin capital letters $A$ we label the five-dimensional indices, they run in $\{0, 1, 2, 3, 5\}$, Greek indices $\alpha$ run from 0 to 3, $x^5$ is the angle parameter for the fifth circular dimension. We consider metric of $\{+, -, -, -, -\}$ signature. Here $c = G = 1$. 
dynamics. Meanwhile, by measures of the surface gravitational potential in \cite{23,24}, the Sun seems to be characterized by a $k = 2.12$. On the other hand all the standard tests on light-bending around the Sun, or the perihelion precession of Mercury, constrain $k \gtrsim 14$.

Nevertheless we must note here that the metric parameter $k$ characterizes the spacetime external to any astrophysical object described by the selected GSS; the Sun is therefore only one particular astrophysical source to be investigated, but there is one different value of the $k$ parameter associated to one different astrophysical source.

GSS metrics are asymptotically flat. For $r \to 2M$ from the right, $g_{tt} \to 0$ and $g_{55} \to \infty$, therefore the length of the extra dimension increases as well as $r$ approaches $2M$.

The 5D-Kretschmann scalar and the square of the 4D-Ricci tensor are divergent in such a point. At first order in $(1/k)$ for large $k$, namely the quasi-Schwarzschild regime, the 5D-Kretschmann scalar diverges in $r = 2M$ while the square of the 4D-Ricci tensor is identically zero. These solutions represent extended objects. The induced scalar matter has a trace-free energy-momentum tensor $T_{\mu \nu}$ and its gravitational mass $M_g = \int (T^0_0 - T^1_1 - T^2_2 - T^3_3) \sqrt{-g_4} dV_3$ is $M_g = ckM$ at infinity and goes to zero at $r = 2M$, for every $k$, where $g_4$ is the determinant of the 4D-metric and $dV_3$ is the ordinary spatial 3D-volume element. In the Schwarzschild limit however, $M_g = M$ for every $r$, therefore also in $r = 2M$.

Hence, for every finite values of $k$, GSS are naked singularities surrounded by the induced scalar matter, and only in the limit $k \to \infty$ they are black holes with an horizon in $r = 2M$ and vacuum for $r > 2M$. In this work we consider GSS with $k > 1$ as the (5D-vacuum) spacetime, surrounding a compact object with a radius $R > 2M$.

A solution of KK-equations \cite{26}, in the region $r \leq R$, that matches the GSS in $r = R$ was obtained in \cite{26}, by a numerical integration, considering the energy momentum tensor of a 4D-perfect fluid with an equation of state $p = C \rho^b$, where $p$ is the pressure, $\rho$ the density, and the constants $C, b$ are real numbers. The existence of such configuration allows to remove the naked singularity providing physical significance to our analysis. A solution of the form $ds^2_{(5)} = \tilde{f} dt^2 - \tilde{g} dr^2 - \tilde{h} d\Omega^2 - \tilde{q}(dx^5)^2$ is considered, where $(\tilde{f}, \tilde{g}, \tilde{h}, \tilde{q})$ are functions of the $r$ only (see also \cite{27} for an analogue analysis on the hydrostatic equilibrium and stellar structure in in $f(R)$ theories.). For a discussion on the Papapetrou approach to matter and geometry and the properties of the interior solutions we refer to \cite{25,26}.

In Fig.1 the solution with $C = 2/5$, $b = 1.329$ and $k = 5$ is plotted. Metric components $(\tilde{f}, \tilde{g}, \tilde{h}, \tilde{q})$ and the functions $(p, \rho)$ are plotted as function of the radial coordinate $r$. The radius $R$ of the configuration has been numerically evaluated via the condition $\rho(R) = 0$.

We consider the particle dynamics and the fields propagating in a region $r > R \equiv 2(1 + 10^{-a})M$, with $a > 0$ constant. We compare the dynamic in GSS with the dynamic in the Schwarzschild geometry (See \cite{28} for a similar problem applied to the Janis-Newman-Winicour spacetime).
3 Papapetrou approach to particle dynamics

The physical characterization of the considered solution as a real four-dimensional spacetime requires that the process of dimensional reduction be satisfactorily addressed.

The fundamental requirement to make unobservable the fifth dimension is the existence of a closed topology on the extra-dimension, which allows to compactify the spacetime, reducing it to be physically undistinguishable from a four-dimensional one.

The basic request to deal with a closed topology is the periodicity of the metric field on the fifth coordinate, which allows to expand the Einstein-Hilbert action in Fourier series and so to develop the dynamics of the different harmonics. This approach is intrinsically different from the cylindricity condition, due to Kaluza, which prescribes the independence of the metric tensor of the fifth coordinate, directly within the field equations. However, if on one hand we truncate the Einstein-Hilbert action to the zero-order and, on the other hand, we impose by hands the closure of the fifth coordinate in the field equations independent of \( x^5 \), the two approaches formally overlap.

In fact, the variation of the zero-order Lagrangian (after the integration on the extra-coordinate has been performed) coincides (without further restrictions) with the Einstein equations in which the four-dimensional quantities are explicitly outlined. It is just this coincidence of the two approaches in the specified limit which gives to the five-dimensional KK-model \[32\] a privileged character among the other numbers of extra-dimensions (where a non-trivial integration on the extra-dimensions is also required on the field equations too \[16,33,34\]).

In the case of the GSS, the situation is further simplified by the absence of an electromagnetic potential \( g_{\mu\nu} = 0 \), which makes diagonal the metric tensor, reduced to provide a scale factor for each space coordinate. In particular the scalar field \( \phi \), which governs the evolution of the fifth dimension, can be regarded as matter in both the two points of view elucidated above, just in view of the possibility to split the five-dimensional Einstein equations into a four-dimensional set, characterized by a scalar matter source.

Our approach to include matter in the vacuum KK-theory, is based on the analysis in \[29,30,31\], which relays on the idea that, since the fifth dimension is compactified to planckian-like scales, it makes no sense to handle with...
a classical fifth component of the particle velocity and therefore the only viable treatment of matter sources is through the energy-momentum description. Such a point of view leads to adopt a Papapetrou scheme \[35\] to fix the dimensionally reduced equation of motions for fields and macroscopic matter.

Since the Papapetrou procedure is based on the use of the field equation and the analysis in \[29\] makes reference the independence of the metric tensor of \(x^5\), we can claim that the present work relays more on a cylindricity requirement, associated to a closed topology, than on a pure dimensional compactification. Thus, in \[30\] the test particle dynamics in a 5D-KK-scenario has been reformulated assuming that, as a consequence of the unobservable length of the extra dimension, under the Planck scale, test particles are described as localized sources only in the ordinary 4D-spacetime and delocalized ones along the fifth (circular) dimension. The reformulation \textit{a l\`a} Papapetrou of matter and motion is therefore based on the following two assumptions: (1) The metric components and the matter fields do not depend of the extra dimension or \(\partial_5 g_{\alpha\beta} = 0\), this assumption is extended also to the 5D-matter tensor or \(\partial_5 T_{\alpha\beta} = 0\) and, (2) The extra dimension has a circular topology and an unobservable length or \(L_{(5)} \equiv \int d^5x \sqrt{-g_{55}} < 10^{-18}\text{cm}\).

A 5D-energy momentum tensor \(\mathcal{T}^{\Lambda\nu}_{\mu}\) is associated to the generic 5D-matter distribution governed by the conservation law \(\partial_5 \mathcal{T}^{\Lambda\nu}_{\mu} = 0\), where \(\partial_5 \mathcal{T}^{\Lambda\nu}_{\mu} = 0\) (here \(\partial_5\) is the covariant derivative compatible with the 5D-metric).

Performing a multipole expansion of \(\mathcal{T}^{\Lambda\nu}_{\mu}\), centered on a trajectory \(X^\alpha\), at the lowest order the procedure gives the equation of motion for a test particle:

\[
\begin{align*}
  u^\mu (\mathcal{T}^{\nu\mu}_{\mu}) &= \frac{q}{m} F^{\nu\mu} u_\mu + A(u^\mu u_\nu - g^{\nu\mu}) \frac{\partial \phi}{\partial \phi}, \quad (2) \\
  \frac{dm}{ds} &= -A \frac{d\phi}{\phi^3} ds. \quad (3)
\end{align*}
\]

where \(u^\mu \equiv dx^\mu/ds\), \(g_{\mu\nu} u^\mu u^\nu = 1\), \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the Faraday tensor, being \(A_\mu\) the U(1) four-vector potential, and \(\phi \equiv \sqrt{-g_{55}}\) is the extra KK-scalar field, (see \[30\] and references thereby). Coupling factors to these fields are the charge \(q\), coming from the vector current \(\mathcal{J}^{\mu}_{\mu}\), and the scalar charge \(A \equiv u^0 \int dV_3 \sqrt{-g_4} \phi \mathcal{T}^{5\mu}_{\mu}\).

Below the definitions for the effective test-particle tensor component follow:

\[
\begin{align*}
  \phi \sqrt{-g_4} T^{\mu\nu} &= \int ds m \delta^4 (x - X) u^\mu u^\nu, \quad (6) \\
  ek \phi \sqrt{-g_4} T^\mu_5 &= \int ds q \delta^4 (x - X) u^\mu = \sqrt{-g_4} J^\mu, \quad (7) \\
  \phi \sqrt{-g_4} T_{55} &= \int ds A \delta^4 (x - X) . \quad (8)
\end{align*}
\]

The continuity equation \(\nabla^\mu \mathcal{J}^{\mu}_{\mu} = 0\), derived within the procedure itself, implies that charge \(q\) is conserved, while the requirement \(A \equiv 0\), implies that the mass \(m\) is constant. Here we concern on the simplest scenario where \(A = 0\), because this choice only allows to recover the case of a constant mass for the particles \(m = \text{constant}\) \[31\] \[20\].

For a discussion on the Papapetrou approach to the test particle motion we remind \[29\] \[30\] \[25\]. Therefore, in
such a scenario, particles in the GSS spacetimes follow a geodesic equation:

\[ u^\mu \nabla_\mu u^\nu = 0. \quad (9) \]

in the following the notation \( (4) \) for four dimensional quantity will be dropped. The motion of a radially falling particle is regulated by the \( t \)-component and \( r \)-component of the geodesic equation \( (9) \), with \( \dot{\theta} = \dot{\varphi} = 0 \). The velocity in the \( r \)-direction is \( v_r \equiv r' = dr/dt \). For a particle starting from a point at (spatial) infinity we have:

\[ \frac{dr}{dt} = -\Delta^{1/2}/\sqrt{1 - \Delta^k}. \quad (10) \]

The radius \( \varrho \), at which the radial velocity \( v_r \) starts to decrease, is

\[ \varrho = 2M \left[ 1 - \left( \frac{2k - 1}{3k - 1} \right)^{1/(k(1 - 1))} \right]^{-1}, \quad (11) \]

which in the Schwarzschild limit reduces to \( \varrho = 6M \), otherwise we have \( \varrho = \rho(k) < 6M \). The locally measured radial velocity is \( v_{r_\varrho} \equiv r'_\varrho = dr/\varrho \) where \( dr/dr^\varrho = -\Delta^{1-k}/\sqrt{1 - \Delta^k} \). Therefore we can write:

\[ \frac{dr^\varrho}{dt} = -\sqrt{1 - \Delta^k}. \quad (12) \]

The velocities \( v_r \) and \( v_{r_\varrho} \) converge to zero at (spatial) infinity. The radial velocity \( v_r \) goes to zero in the limit \( r \to 2M \), and the local measured velocity \( v_{r_\varrho} \) goes to \(-1\) in the limit \( r \to 2M \). Light propagation is described by the geodetic equation \( (9) \) with \( d^2s = 0 \).

This equation, for weak fields, has been solved in \[30\], neglecting the fifth component of the particle velocity, to yield a hyperbolic orbit. For \( k > 1 \) the deflection angle lies in the range of values which predicted a possible detectable deviation from the general relativistic values but excludes null deflection and light repulsion. In particular, gravitational redshift parameter \( z \), at first order in \( r/M \), is

\[ z \equiv \frac{\nu_e - \nu_r}{\nu_e} = \epsilon kM \left( \frac{1}{r_r} - \frac{1}{r_e} \right) \quad (13) \]

where \( r_r (\nu_r) \) and \( r_e (\nu_e) \) are the positions of the receiver and emitter (frequencies of the received and emitted wave) respectively. In the Schwarzschild limit this result is in agreement with the general relativistic redshift parameter \( z_{sch} \) as calculated in the Schwarzschild geometry; otherwise, for \( k > 1 \) we have \( z = z(k) < z_{sch} \).

### 4 Electromagnetic radiation emitted by a radially falling particle.

The charged particle motion and the emitted EM radiation will be considered as perturbations of the background metric, therefore we assume that the EM radiation is regulated by the following equation

\[ \partial_\nu \left( \sqrt{-g} \phi F^{\mu\nu} \right) = 4\pi \sqrt{-g} J_\mu, \quad (14) \]

where \( F^{\mu\nu} \) is the Faraday tensor, see \[25\].

The electromagnetic perturbation \( f_{\mu\nu} \) is expanded in (4D)-tensor harmonics \( (Y^{\mu}_{(4)}(\Omega(t))) \). In agreement with the Zerilli notation \[37,38,39\] \( (f^{\mu\nu}) \) denotes the radial (angle-independent) part of the Faraday tensor, while the 4D-current \( J^\mu \) reads

\[ J^\mu = \sqrt{4G/\phi T_5^\mu} = \left( \sqrt{-g}^3 \right)^{-1} \int dsq \delta^4(x - X) u^\mu, \quad (15) \]

where \( q \) denotes the charge of the particle, see Eq. \( (4) \) and \[25,30\].
The harmonic coefficients for the $t$-component and $r$-
component of the current source are respectively:

$$
\psi = -\frac{q}{r^2} \Delta^{2k-1-3/2} \delta(r - R(t)) Y_{lm}^*(\Omega(t)),
$$

$$
\eta = \frac{q}{r^2} \Delta^{-(r/2 + 1)} \frac{dR}{dt} \delta(r - R(t)) Y_{lm}^*(\Omega(t)).
$$

(16)

(17)

The independent electric multipole equations are

$$
i\omega \tilde{f}_{01} + \frac{l(l + 1)}{r^2} \tilde{f}_{12} \Delta^{2k-1-\epsilon} = -4\pi \eta \Delta^{ek+\frac{3}{2}},
$$

(18)

and

$$
i\omega \tilde{f}_{02} \Delta^{3/2 - \epsilon} \partial_r \left( \Delta^{\epsilon} \tilde{f}_{12} \right) + 3 \tilde{f}_{12} \Delta^\epsilon \partial_r \left( \Delta^{\epsilon/2} \right) = 0.
$$

(19)

Via the Eqs. (18,19) and using the homogeneous Maxwell

equation, 

$$
\partial_r \tilde{f}_{02} + i\omega \tilde{f}_{12} - \tilde{f}_{01} = 0,
$$

(20)

according with the Zerilli’s procedure, we find the following

equation for the component $\tilde{f}_{12}$:

$$
\left[ \Delta^{(r-1)/2} \left( \Delta^{(k-1)/2} \tilde{f}_{12} \right) \right]_{r^2} + 3 \left[ \tilde{f}_{12} \Delta^{(2k-1)/2} \left( \Delta^{\epsilon/2} \right) \right]_{r^2} + \tilde{f}_{12} \left( \omega^2 - \frac{l(l + 1)}{r^2} \Delta^{(2k-1)-1} \right) = 4\pi \eta \Delta^{ek+3/2}. 
$$

(21)

Finally we find, from Eq. (21) and Eq. (11), the following

equation for the function $\tilde{f}(\omega, r) \equiv \Delta^{(k-5/4)} \tilde{f}_{12}$

$$
\frac{d^2 \tilde{f}(\omega, r)}{d\omega^2} + \left[ \omega^2 - V_l(r, k) \right] \tilde{f}(\omega, r) = S_l(\omega, r),
$$

(22)

where $dr/dr_\ast = \Delta^{(k-1)/2}$. The source term reads

$$
S_l(\omega, r) = \Delta^{(2k-1)2} q \sqrt{\frac{1}{2} + \frac{t(r)}{r^2} \Delta^{-1+\epsilon/2}},
$$

(23)

where $t(r)$ is a solution of the radial geodesic equation or

$$
\frac{dt(r)}{dr} = -\frac{\Delta^{-k}}{\sqrt{\Delta^{-\epsilon}(1 - \Delta^{ek-2})}},
$$

(24)

written as function of $r$ and $\gamma \equiv \frac{1}{\sqrt{1 - v_\infty^2}}$, where $v_\infty$ is the
(radial) particle velocity at spatial infinity. The potential

$$
V_l(r, k) = \frac{l(l + 1) \Delta^{2k-\epsilon-1}}{r^2} + V_0(r, k),
$$

(25)

where

$$
V_0(r, k) \equiv \frac{3e}{4} \frac{M^2 \Delta^{2k-\epsilon-2}}{r^4} \left[ 4 \left(\epsilon k + 1 - \frac{r}{M} \right) + \epsilon \right].
$$

(26)

(see ref. [37][38][39] for current use of the notation). In the Schwarzschild
limit $V_0(r, k) = 0$ and

$$
V_l(r, k) \equiv V_l(r) = \frac{l(l + 1) \Delta}{r^2}
$$

(27)

We approached the Sturm-Liouville problem for the function
$\tilde{f}(\omega, r)$ in Eq. (22) using the Green’s functions method.

We first solved the associated homogeneous equation (obtained
from Eq. (22) imposing $S_l(\omega, r) = 0$). We used the
solutions of this equation to build the Green’s function $G(r, r')$ of the problem. Since we are interested in the EM
spectra as seen by an observer located at (spatial) infinity
we can focus on the outgoing component only of the EM
radiation. We require that the outgoing radiation is purely
oscillating at infinity. The energy spectra at (spatial) infinity
reads

$$
\frac{dE}{d\omega} = \sum_l \frac{dE_l}{d\omega} = \sum_l \frac{l(l + 1)}{2\pi} \left| A_l^{out}(\omega) \right|^2,
$$

(28)

where $\omega \geq 0$ and where $A_l^{out}$ is a function of $\omega$ only. The
homogeneous equation is a “Schrodinger-like” equation
for the function $\tilde{f}(\omega, r^\ast(r))$ with a potential $V_l(r^\ast(r), k)$.

Asymptotically, for large value of $r$, the potential $V_l(r, k)$
decreases to zero.

The values of $V_l(R, k)M^2$ in $R \equiv (2 + 10^{-a})M$ for
different values of $a$, and $k$ are listed in Table 1.
Table 1: The function $V_l(R,k)M^2$ calculated at $l = 1$, and $R \equiv (2 + 10^{-a})M$, for different values of $k$ and $a = \{-6, -10, -50, -80\}$.

| $a$      | -6  | -10 | -50 | -80 |
|----------|-----|-----|-----|-----|
| $k=5$    | 0.0098 | 0.01367 | 0.3777 | 4.553 |
| $k=10$   | 0.0016 | 0.0018 | 0.0038 | 0.0067 |

At first we investigate the problem in a quasi-Schwarzschild regime. Therefore we expand Eq.22 at first order in $1/k$ around $1/k = 0$, neglecting higher order terms. The potential $V_l(r,k)$ vanishes as $r$ approaches $2M$. We look for a solution $f_l$, of the homogeneous equation (22) with the following asymptotical behavior

$$f_l \to e^{-i\omega r^*} \text{ as } r^* \to -\infty$$

$$f_l \to B_l(\omega)e^{+i\omega r^*} + C_l(\omega)e^{-i\omega r^*} \text{ as } r^* \to +\infty$$

where, because we are taking the limit of large $k$, $r^* = r + 2M\log[r/(2M) - 1]$ and $B_l(\omega)$ and $C_l(\omega)$ are functions of the Fourier frequency $\omega$, (see for example [37,38,39,40]).

The problem was approached using numerical integration. Adopting a Runge-Kutta method, we first integrated the homogeneous equation for $k = 1000$ to obtain an evaluation of the function $f_l$ with the required boundary condition (29). Numerical integration started at $r = M(2 + 10^{-6})$. It is stopped at large values of $r$ as well as the integral in (31) converges. Boundary condition (30) is then used to recover a value of $C_l$. Once the values of the parameters $(\gamma, l)$ are fixed, the integration has been performed for many fixed values of $\omega$. The spectra profile, Fig.2 coincides with that emitted in the Schwarzschild background [40].

4.2 2

We integrated Eq.22 numerically for $r > R = M(2 + 10^{-6})$ and for $k = 10, 5$. The potential is plotted in Fig. 3. The homogeneous equation is solved imposing that the solution is ingoing at $R$ and a combination of “incident” and “transmitted” waves at spatial infinity. The $\omega$-function $A_l^{out}$ is therefore the integral

$$A_l^{out}(\omega) = \frac{1}{2i\omega C_l(\omega)} \int_R^\infty f_l S_l dr,$$  (31)

where

$$S_l(\omega, r) = 2q\sqrt{\frac{1}{2} + \frac{e^{i\omega r}(r)}{r^2}} - \Delta^{(k-1)-1}. \quad (32)$$

Spectra profiles are plotted in Fig. 3. A general feature of the emitted spectra for $\gamma \geq 2$ for the Schwarzschild’s case is to grow up to a critical value (corresponding to a critical frequency $\omega$ for a fixed
Fig. 2: EM-energy spectra ($k = 1000$) for the $l$-multipole, for particle falling from infinity into a GSS-background.

Fig. 4: Electromagnetic energy spectra for particle falling into a GSS-background with $k = 10$ is plotted for different $l$-multipoles and different values of $\gamma$.

Fig. 5: Electromagnetic energy spectra for particle falling into a GSS-background with $k = 5$ is plotted for different $l$-multipoles and different values of $\gamma$.

multipole) and then rapidly flow down to zero (see [40] and also Fig. 2 in this section). For the GSS case we found that, for a fixed $\gamma$, the spectra profile coincides with that of Schwarzschild’s case as well as $k$ is sufficiently large ($k > 5$). However there is an increase of the energy emitted rate per frequency $dE/d\omega$, at fixed value of $\omega$ and fixed multipole $l$. There are significant discrepancies with the Schwarzschild’s case for $k \leq 5$ where, for low frequencies, a peak in the spectra profile appears also at large $\gamma$. 
5 Conclusions

In principle the relic of an extra dimension could be recognized in the Universe as a deformation of the EM-emission spectra from compact astrophysical sources compared with that allowed by general relativistic calculations. The spectral analysis could provide therefore a suitable test to constrain the multidimensional theories. We argue in particular that the comparison with the observed data on the spectral emission by collapsing compact object, that could lead at last in principle to the formation of a black hole could get light on the nature of the singularity and the possible presence of the small extra dimension, for a discussion about the possible constraint of the KK-theory by comparing with strong emission astrophysical processes see for example [1,6,10,12,40,43,44].

On the other side, a non trivial question for any theories dealing with one or more extra-dimensions, arises on the non-observation of such an extra-space. The most natural answer to this phenomenological question consists in the assumption of the existence of a closed extra-space, having a volume much smaller that the smallest observed scale in the present day experiment of high-energy physics, say $O(10^{-18}\text{cm})$. In the standard Kaluza-Klein approach, the necessity to reproduce the proper value of the fundamental coupling constants (in five dimension the fine structure constant) implies that the extra dimensions be compactified to very small values, two orders of magnitude above the Planck scale. Therefore, the request for a Planckian-like compactification scale can not be completely escaped and we have to deal with some of the fundamental paradigms proposed for the Kaluza-Klein scenario [16,17]. We point out that the condition of compactification, introduced by Klein, consists in the assumptions that the extra dimension is small and has a closed topology, form this hypothesis we get to a periodicity an the possibility to perform a Fourier decompositions, see for example [17]. On the other hand, the cylindricity hypothesis assumes metric components do not depend on the fifth coordinate or $\partial_5 g_{ab} = 0$. In this work we have considered a class of static solutions of 5D-KK-equations independent by the fifth extra-dimension (GSS). For the observer at radial infinity, the extra-dimension is not observable because it is on a steady Planckian-like scale, and for the observers closer to the star surface its dimension increases according to the increasing behavior of the scale factor associated to the extra-dimension, namely the scalar field $\phi(r)$. The scale factor varies only for few orders of magnitude from infinity to the star surface, so that the phenomenology of the Kaluza-Klein picture is not affected by features of our model. Indeed, it is just the different value that the scale factor (the scalar field) takes near the star, that is responsible for the observability of a deviation in the spectrum of the emitted radiation by the infalling particle, with respect to the ordinary four-dimensional case.

The closer is the surface of the star to the radial value corresponding in vacuum to the naked singularity, the stronger will be the modification that the emitted spectrum is expected to outline. In fact, it is just the variation of some order of magnitude in the compactification scale of the fifth dimension that is observed at radial infinity.
as a deformation of the spectrum, since if the scalar field would be equal to unity everywhere, the solution would reduce to an ordinary Schwarzschild profile and the morphology of the emission would be unaltered.

We studied the charged particle motion and the EM-radiation emitted by 4D-pointlike charged particle freely falling in the selected backgrounds. We have considered the particles and the fields as perturbations of the background metric. The EM-emission spectra, as observed at infinity, is found.

The spectra profiles, in the GSS metrics here analyzed, are in good agreement with the emitted spectra calculated in the Schwarzschild case as long as \( k > 5 \). Therefore the effects of the presence of an extradimension in the framework of the KK-theory should be evident, from the point of view of the EM emission spectra only in those spacetime characterized by \( k < 5 \).

For a fixed value of \( \omega \) and a fixed multipole \( l \) an increase of the energy emitted rate per frequency \( dE/d\omega \) is noted. Significant discrepancies respect to the Schwarzschild case appear for \( k \leq 5 \) where, for low frequencies, a peak in the spectra profile appears also at large \( \gamma \).

The development of the present analysis suggests to explore the EM emission by charged particle in the 5D-vacuum spacetime. This analysis should be interesting for many reasons and in particular for the exploration of the naked singularity features of these vacuum solutions. The main problem we could recognize in the treatment of these configurations from the point of view of the EM emission spectra is faced upon the effective potential in Eq. 22. The different physical situations are clearly distinguished by the boundary conditions to be imposed on the Cauchy problem, (see also [15]).

In the vacuum spacetime the effective potential, for each finite value of the \( k \)-parameter explodes on the ring \( r = 2M \), meanwhile in the Schwarzschild case, reached on the 4D-spacetime section and in the limit of large \( k \), it is zero on the horizon \( 2M \). The boundary condition, and therefore the spectra profile obviously remark this different nature of the circle \( r = 2M \). In the Schwarzschild limit case in the vacuum GSS, see Section [11], we impose the ingoing boundary condition at the horizon, in the vacuum GSS background with \( k \) finite we attend to impose on the infinite wall of potential on the singularity \( r = 2M \) the wave function \( f_l \) annihilates.

A remarkable feature of this process is therefore the contrast between the two vacuum cases not resolvable a part in the limit \( k \to \infty \). On the other hand if we compare the dynamics of timelike particles and even photons this breakdowns of these two physical situations does not appear. We finally observe that, this idea that we can measure at a very large distance from a compact object, features accounting for its different status of the extra dimension scale, \( \phi \), in the five-dimensional Kaluza-Klein scenario, is the just the core of the idea we are proposing by the present calculation.

Thus, the analysis of the motions in the vacuum spacetime in [20,31] within the Papapetrou approach should be therefore completed by the analysis of the emission processes induced by charged particles in the (vacuum) GSS.
background. The comparison in particular of the electromagnetic emission spectra due to a freely falling charged test particle, a in the GSS and in Schwarzschild background could get light on physics around the naked singular ring $r = 2M$ ([41],[42] and [43],[44]). The solution of the equations governing the emission process should strongly depend on the boundary conditions imposed on the equations. The comparison between the electromagnetic emission by radially falling charged particles in the GSS and stellar case could be strongly discriminant.

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