The Jacobian Conjecture and Partial Differential Equations

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Abstract

The Jacobian conjecture over a field of characteristic zero is considered directly in view of the nonlinear partial differential equations it is associated with. Exploring such partial differential equations, this work obtains broad families of polynomial maps satisfying the conjecture in all dimensions and of arbitrarily high degrees and henceforth affirmatively and constructively settles a reformulated multiply parametrized version of the conjecture in all dimensions.

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1 Introduction

The Jacobian conjecture was first formulated in 1939 by Ott-Heinrich Keller \cite{Keller} which states that a polynomial map $P : \mathbb{F}^n \to \mathbb{F}^n$ where $\mathbb{F}$ is a field of characteristic zero and $n \geq 2$ has a polynomial inverse if the Jacobian $J(P)$ of $P$ is a nonzero constant. This conjecture has not been solved for any $n \geq 2$ and appears in Smale’s list of the eighteen mathematical problems for the new century \cite{Smale}. With a suitable normalization, the polynomial map $P$ may be taken to satisfy the condition $P(0) = 0$ and $DP(0) = I$ such that it has the representation

$$P(x_1, \ldots, x_n) = (x_1 + H_1, \ldots, x_n + H_n), \quad (x_1, \ldots, x_n) \in \mathbb{F}^n,$$

where $H_1, \ldots, H_n$ are polynomials in the variables $x_1, \ldots, x_n$ consisting of terms of degrees at least 2 in nontrivial situations so that the condition imposed on $J(P)$ becomes $J(P) = 1$. Among the notable developments, Wang \cite{Wang} established the conjecture when $H_1, \ldots, H_n$ are all quadratic and Bass, Connell, and Wright \cite{Bass} and Yagzhev \cite{Yagzhev} proved an important reduction theorem which states that the general conjecture amounts to showing that the conjecture is true for the special case when each of $H_1, \ldots, H_n$ is either cubic-homogeneous or zero for all $n$. Subsequently, Druzkowski \cite{Druzkowski} further showed that the cubic-homogeneous reduction of \cite{Bass,Yagzhev} may be assumed to be of the form of cubic-linear type,

$$H_i = (a_{i1}x_1 + \cdots + a_{in}x_n)^3, \quad i = 1, \ldots, n.$$

In \cite{BondtEssen}, Bondt and Essen proved that the conjecture for the case $\mathbb{F} = \mathbb{C}$ may be reduced to showing that the conjecture is true when the Jacobian matrix of the map $H = (H_1, \ldots, H_n)$ is homogeneous,
nilpotent, and symmetric, for all $n \geq 2$. For $n = 2$, Moh \cite{14} established the conjecture when the degrees of $H_1$ and $H_2$ are up to 100. See the survey articles \cite{5,6,13} and monograph \cite{7} and references therein for further results and progress. While these developments were mainly based on ideas and methods of algebra and algebraic geometry, the problem also naturally prompts us to explore its structure in view of partial differential equations directly. With this approach, in the first part of this study, we obtain some new families of polynomial maps satisfying the conjecture, which may be summarized as follows.

**Theorem 1.1** In its simplest form, the Jacobian conjecture amounts to solving an under-determined first-order nonlinear partial differential equation and obtaining relevant polynomial-function solutions which may be used to construct the inverse map from the original polynomial map over $\mathbb{F}^n$ for a field $\mathbb{F}$ of characteristic zero.

(i) When $n = 2$, the equation may be reduced into a homogeneous Monge–Ampère equation over $\mathbb{F}^2$ whose homogeneous solutions give rise to a broad family of solutions of any prescribed degree and the inverse polynomial map may be readily constructed based on an invariance structure of the variables so that the degrees of the map and its inverse coincide.

(ii) When $n = 3$, the equation may be reduced into an under-determined equation of the Monge–Ampère type given in terms of some Hessian determinants involving three unknown functions. For this equation, broad families of polynomial solutions of arbitrary degrees can be constructed among which, one family is of homogeneous type involving two arbitrary functions and a single invariance structure so that the degree of the map and its inverse coincide as in the $n = 2$ situation, and another is not of the homogeneous type but involves two arbitrary functions in composition and obeys a partial invariance structure so that the degree of the inverse map may be as high as twice of that of the original map.

(iii) For any $n \geq 2$, by imposing a full invariance structure for the variables, a broad family of solutions involving $n - 1$ arbitrary functions of any prescribed degrees of a single linear combination of the variables may be constructed explicitly so that the degrees of the map and its inverse coincide.

We note that, if $P : \mathbb{F}^n \to \mathbb{F}^n$ ($n \geq 2$) is a polynomial automorphism (that is, the inverse $P^{-1}$ of $P$ exists which is also a polynomial map), then it has been shown \cite{2,5,15} that there holds the following general bound between the degrees of $P$ and $P^{-1}$:

$$\deg(P^{-1}) \leq (\deg(P))^{n-1}.$$  \hfill (1.3)

In the $n = 2$ situation here, the map $P$ stated in (i) of Theorem 1.1 satisfies $\deg(P) = \deg(P^{-1})$ where $\deg(P)$ may be any positive integer. Besides, the results in (ii) of Theorem 1.1 says $P^{-1}$ may fulfill both $\deg(P^{-1}) = \deg(P)$ and $\deg(P^{-1}) = (\deg(P))^2$ where $\deg(P)$ can be any positive integer as well. In other words, the bound (1.3) is seen to be sharp or realizable when $n = 2, 3$ in all degree number cases, with the solutions obtained here.

From the viewpoint of partial differential equations, the main difficulty with the Jacobian conjecture lies in the fact that the normalized Jacobian equation, $J(P) = 1$, is a single equation, which is vastly underdetermined with mixed nonlinearities involving the derivatives of the unknowns. On the other hand, the reduction, or more precisely, separation, method effectively employed in the
first part of this study impels us to find a natural reduction mechanism for the method. For this purpose, in the second part of this study, we modify the polynomial map (1.1) into

\[
P(x_1, \ldots, x_n) = (\lambda_1 x_1 + H_1, \ldots, \lambda_n x_n + H_n), \quad (x_1, \ldots, x_n) \in \mathbb{F}^n,
\]

where \( \lambda_1, \ldots, \lambda_n \) are parameters. It is clear that if (1.4) has a polynomial inverse for fixed \( \lambda_1, \ldots, \lambda_n \), then

\[
J(P) = \lambda_1 \cdots \lambda_n,
\]

and \( \lambda_1, \ldots, \lambda_n \) satisfy

\[
\lambda_1 \neq 0, \quad \ldots, \quad \lambda_n \neq 0.
\]

Thus, conversely, we ask whether any polynomial solution to (1.5) has a polynomial inverse under the condition (1.6). We refer to this statement as the parametrized Jacobian conjecture and we establish this conjecture affirmatively and constructively. More precisely, we will prove:

**Theorem 1.2** For the multiply parametrized polynomial map (1.4) where \( H_1, \ldots, H_n \) are polynomials comprised of quadratically and more highly powered terms of the variables, all the solutions to the Jacobian equation (1.5) can be schematically and explicitly constructed. Besides, under the condition (1.6), these obtained solutions render the map (1.4) invertible in the category of polynomial maps and the associated inverse maps may also be constructed schematically and explicitly.

In Sections 2–5, we establish Theorem 1.1. In Section 6, we establish Theorem 1.2. Since being able to construct explicitly the associated inverse polynomial maps is of obvious importance in applications, in Section 7, we briefly illustrate an application to cryptography. In Section 8, we summarize the results of this work.

### 2 Two dimensions

First we consider \( n = 2 \) and rewrite (1.1) conveniently as

\[
P(x, y) = (x + f(x, y), y + g(x, y)),
\]

where \( f \) and \( g \) are polynomials in the variables \( x, y \) over \( \mathbb{F} \) consisting of terms of degrees at least 2 in nontrivial situations. Inserting (2.1) into \( J(P) = 1 \) we have

\[
f_x + g_y + J(f, g)(x, y) = 0,
\]

where \( f_x \) (e.g.) denotes the partial derivative of \( f \) with respect to \( x \) and \( J(f, g)(x, y) \) the Jacobian of the map \( (f, g) \) over \( x, y \). That is, \( J(f, g)(x, y) = \frac{\partial(f, g)}{\partial(x, y)} \). This is an underdetermined equation which may be solved by setting

\[
f_x + g_y = 0, \quad J(f, g)(x, y) = 0,
\]

separately, such that the first equation in (2.3) implies that there is a polynomial \( h(x, y) \) serving as a scalar potential of the divergence-free vector field \( (f, g) \) satisfying

\[
f = h_y, \quad g = -h_x.
\]
Inserting (2.4) into the second equation in (2.3) we see that \( h \) satisfies the homogeneous Monge–Ampère equation
\[
\det(D^2 h) = h_{xx} h_{yy} - h_{xy}^2 = 0. \tag{2.5}
\]
Alternatively, if we are only concerned with \( f \) and \( g \) being homogeneous of the same degree, then a degree counting argument applied to (2.2) leads to two separate equations, as given in (2.3), as well. Hence we arrive at (2.5) again.

To proceed, we consider the solution of (2.5) of the homogeneous type
\[
h(x, y) = \sigma(\xi), \quad \xi = ax + by, \quad a, b \in \mathbb{F}, \tag{2.6}
\]
as suggested by (1.1)–(1.2), where the arbitrary polynomial function \( \sigma(\xi) \) is taken to be of degree \( m \geq 3 \) or zero. Thus, with the notation \( P(x, y) = (u, v) \) and the relations (2.1) and (2.4), we have
\[
u = x + b\sigma'(\xi), \quad v = y - a\sigma'(\xi), \tag{2.7}
\]
resulting in the invariance condition between the two sets of the variables:
\[
au + bv = ax + by = \xi. \tag{2.8}
\]
This key structure, just unveiled by the Monge–Ampère equation (2.5), will be exploited effectively in all higher dimensions as well. As a consequence of (2.7) and (2.8), we obtain the inverse of the map \( P \) immediately as follows:
\[
x = u - ba\sigma'(\xi), \quad y = v + a\sigma'(\xi), \quad \xi = au + bv. \tag{2.9}
\]
As a by-product, the arbitrariness of the function \( \sigma \) indicates that the solution gives rise to a family of polynomial maps of arbitrarily high degrees.

For later development, we also observe that the second equation in (2.3) implies that \( f \) and \( g \) are functionally dependent. Therefore, if we set \( g = G(f) \) (say), then the first equation in (2.3) leads to
\[
f_x + G'(f)f_y = 0, \tag{2.10}
\]
which has a nontrivial solution of the homogeneous type, \( f = \phi(\xi) \) (\( \xi = ax + by \)), if and only if
\[
g = G(f) = -\frac{a}{b}f, \quad b \neq 0. \tag{2.11}
\]
Thus, it follows that there hold the simplified relations
\[
u = x + \phi(\xi), \quad v = y - \frac{a}{b}\phi(\xi), \quad au + bv = ax + by = \xi, \tag{2.12}
\]
where the invariance relation between the variables again makes the inverse of the map ready to be read off.

We remark that (2.12) is the most general polynomial automorphism of homogeneous type in two dimensions. To see this, we let the map \( P \) be defined by
\[
u = x + \phi(\xi), \quad v = y + \psi(\eta), \quad \xi = ax + by, \quad \eta = cx + dy, \quad a, b, c, d \in \mathbb{F}, \tag{2.13}
\]
where \( \phi(\xi) \) and \( \psi(\eta) \) are polynomials in the variables \( \xi \) and \( \eta \), respectively, of degrees \( l \geq 2 \) and \( m \geq 2 \), satisfying \( \phi(0) = \psi(0) = 0 \). Inserting (2.13) into \( J(P) = 1 \), or \( f = \phi \) and \( g = \psi \) into (2.2), we get
\[
a\phi'(\xi) + d\psi'(\eta) + (ad - bc)\phi'(\xi)\psi'(\eta) = 0. \tag{2.14}
\]
On the other hand, as polynomials in the variables $x, y$, we have
\begin{align}
\deg(a\phi'(\xi) + b\psi'(\eta)) & \leq \max\{l - 1, m - 1\}, \\
\deg(\phi'(\xi)\psi'(\eta)) & = (l - 1) + (m - 1) > \max\{l - 1, m - 1\},
\end{align}
(2.15) (2.16)
since $l, m \geq 2$. In view of (2.14)–(2.16), we arrive at $ad - bc = 0$. In other words, the variables $\xi$ and $\eta$ as given in (2.13) are linearly dependent. Consequently, (2.13) is simplified into the form
\begin{equation}
u = x + \phi(\xi), \quad v = y + \psi(\xi), \quad \xi = ax + by,
\end{equation}
(2.17) which renders $a\phi'(\xi) + b\psi'(\eta) = 0$. Thus (2.12) follows if $b \neq 0$ and $\phi(\xi) = -\frac{b}{a}\psi(\xi)$ if $a \neq 0$. So we have obtained the most general homogeneous solution to the equation $J(P) = 1$ or (2.2).

It can be checked directly that, when the polynomial map (2.1) is of the type $\deg(P) = 1$ or (2.2) leads to the homogeneous form (2.17), or more precisely, (2.12).

One may wonder whether (2.3) is too strong a condition for the splitting of the single equation (2.2). Here we remark that it arises naturally in three dimensions from a reduction consideration. In fact, in this situation, we may consider the polynomial map
\begin{equation}P(x, y, z) = (x + \zeta(z)f(x, y), y + \zeta(z)g(x, y), z),
\end{equation}
(2.18) where $\zeta(z)$ is an arbitrary polynomial of $z$. Then $J(P) = 1$ gives us
\begin{equation}\zeta(z)(f_x + g_y) + \zeta^2(z)(f_xg_y - f_yg_x) = 0,
\end{equation}
(2.19) leading to (2.3) again. Therefore, with the solution (2.6) and notation $P(x, y, z) = (u, v, w)$, the expression (2.7) is updated with
\begin{equation}u = x + b\zeta(z)\sigma'(\xi), \quad v = y - a\zeta(z)\sigma'(\xi), \quad w = z,
\end{equation}
(2.20) so that the invariance property (2.8) still holds. As a consequence, we obtain the inverse map
\begin{equation}x = u - b\zeta(w)\sigma'(\xi), \quad y = v + a\zeta(w)\sigma'(\xi), \quad \xi = au + bv, \quad z = w,
\end{equation}
(2.21) immediately, which extends the formula (2.9).

### 3 Three dimensions

Next we consider $n = 3$ and rewrite (1.1) as
\begin{equation}(u, v, w) = P(x, y, z) = (x + f(x, y, z), y + g(x, y, z), z + h(x, y, z)), \quad (x, y, z) \in \mathbb{F}^3,
\end{equation}
(3.1) where $f, g,$ and $h$ are polynomials in $x, y, z$ with terms of degrees at least 2 in nontrivial situations. Thus the equation $J(P) = 1$ is recast into
\begin{equation}f_x + g_y + h_z + J(f, g)(x, y) + J(g, h)(y, z) + J(f, h)(x, z) + J(f, g, h)(x, y, z) = 0,
\end{equation}
(3.2) which is under-determined as well. If we focus on $f, g,$ and $h$ being homogeneous of the same degree, then (3.2) splits into the coupled system
\begin{align}f_x + g_y + h_z = 0, \quad J(f, g)(x, y) + J(g, h)(y, z) + J(f, h)(x, z) = 0, \quad J(f, g, h)(x, y, z) = 0,
\end{align}
(3.3)
as in Section 2. However, here, we are interested in solutions of more general characteristics.

To proceed, we solve the third equation in (3.3) by setting \( h = H(f, g) \) where \( H \) is a function of the variables \( f \) and \( g \) to be determined. Hinted by the study in Section 2, we seek for solutions of the form

\[
f(x, y, z) = \phi(\xi), \ g(x, y, z) = \psi(\eta), \ \xi = ax + by + cz, \ \eta = px + qy + rz, \ a, b, c, p, q, r \in \mathbb{F}. \quad (3.4)
\]

Inserting (3.4) into the first equation in (3.3), we get

\[
(a + cH_f)\phi'(\xi) + (q + rH_g)\psi'(\eta) = 0.
\]

We are interested in being able to allow \( \phi \) and \( \psi \) to be arbitrary. This leads to \( a + cH_f = 0 \) and \( q + rH_g = 0 \) or

\[
h = H(f, g) = -\frac{a}{c}f - \frac{q}{r}g, \ c, r \neq 0,
\]

which extends (2.11). In view of (3.6), we see that the second equation in (3.3) is equivalent to the equation

\[
(ar - cp)(br - cq) = 0.
\]

Thus either \( ar = cp \) or \( br = cq \). In other words, subject to (3.7), we have solved the Jacobian equation \( J(P) = 1 \) where \( P(x, y, z) = (u, v, w) \) in 3 dimensions with

\[
u = x + \phi(\xi), \ v = y + \psi(\eta), \ w = z - \frac{a}{c}\phi(\xi) - \frac{q}{r}\psi(\eta), \ \xi = ax + by + cz, \ \eta = px + qy + rz. \quad (3.8)
\]

First, suppose

\[
a = \frac{p}{c}, \ b \neq \frac{q}{r}.
\]

Using (3.9), we see that (3.8) gives us

\[
au + bv + cw = ax + by + cz + \left( b - \frac{cq}{r} \right) \psi(\eta) = \xi + \left( b - \frac{cq}{r} \right) \psi(\eta), \quad (3.10)
\]

\[
pu + qv + rw = px + qy + rz = \eta. \quad (3.11)
\]

So the quantity \( \eta \) is seen as an invariant between the two sets of the variables but not \( \xi \). In other words, we achieve a partial invariance. As a consequence of (3.8)–(3.11), we obtain the inverse of the map \( P \) given by

\[
x = u - \phi \left( au + bv + cw - \left[ b - \frac{cq}{r} \right] \psi(\eta) \right), \quad (3.12)
\]

\[
y = v - \psi(\eta), \quad (3.13)
\]

\[
z = w + \frac{a}{c}\phi \left( au + bv + cw - \left[ b - \frac{cq}{r} \right] \psi(\eta) \right) + \frac{q}{r}\psi(\eta), \quad (3.14)
\]

where \( \eta = pu + qv + rw \).

Next, similarly assume

\[
a \neq \frac{p}{c}, \ b = \frac{q}{r}. \quad (3.15)
\]

Then (3.8) and (3.15) lead to

\[
au + bv + cw = ax + by + cz = \xi, \quad (3.16)
\]

\[
pu + qv + rw = px + qy + rz + \left( p - \frac{ar}{c} \right) \phi(\xi) = \eta + \left( p - \frac{ar}{c} \right) \phi(\xi). \quad (3.17)
\]
Thus $\xi$ is an invariant but not $\eta$. This again realizes a partial invariance and gives rise to the inverse map analogously by the expressions

$$x = u - \phi(\xi), \quad (3.18)$$
$$y = v - \psi\left( p u + q v + r w - \left[ p - \frac{a r}{c} \right] \phi(\xi) \right), \quad (3.19)$$
$$z = w + \frac{a}{c} \phi(\xi) + \frac{q}{r} \psi\left( p u + q v + r w - \left[ p - \frac{a r}{c} \right] \phi(\xi) \right), \quad (3.20)$$

where $\xi = a u + b v + c w$.

It will be of interest to compare the degrees of the map $P$ given by (3.8) and its inverse $P^{-1}$ either given by (3.12)–(3.14) or (3.18)–(3.20), with regard to the general bound (1.3), which are

$$\text{deg}(P) = \max\{\text{deg}(\phi), \text{deg}(\psi)\}; \quad \text{deg}(P^{-1}) = \text{deg}(\phi) \text{deg}(\psi). \quad (3.21)$$

Hence we have

$$\text{deg}(P^{-1}) \leq (\text{deg}(P))^2, \quad (3.22)$$

which is a realization of (1.3) when $n = 3$. Of course, $\text{deg}(P^{-1}) = (\text{deg}(P))^2$ if and only if $\text{deg}(\phi) = \text{deg}(\psi)$ and a wide range of integer combinations in the inequality (3.22) can be achieved concretely by choosing appropriate pair of the generating polynomials, $\phi$ and $\psi$.

It is worth noting that, if both factors in (3.7) vanish, or

$$\frac{a}{c} = \frac{p}{r}, \quad \frac{b}{c} = \frac{q}{r} \quad (3.23)$$

are simultaneously valid, then (3.10) and (3.11) imply that both $\xi$ and $\eta$ are invariant quantities between the two sets of the variables:

$$a u + b v + c w = a x + b y + c z = \xi, \quad (3.24)$$
$$p u + q v + r w = p x + q y + r z = \eta. \quad (3.25)$$

In fact, now $\xi$ and $\eta$ are linearly dependent quantities,

$$r \xi = c \eta. \quad (3.26)$$

In this situation, we may rewrite (3.8) as

$$u = x + \phi(\xi), \quad v = y + \psi(\xi), \quad w = z - \frac{a}{c} \phi(\xi) - \frac{b}{c} \psi(\xi), \quad \xi = a x + b y + c z = a u + b v + c w, \quad (3.27)$$

where $\phi$ and $\psi$ are arbitrary functions of $\xi$, which is a direct 3-dimensional extension of (2.12) for which the inverse is obviously constructed as well:

$$x = u - \phi(\xi), \quad y = v - \psi(\xi), \quad z = w + \frac{a}{c} \phi(\xi) + \frac{b}{c} \psi(\xi), \quad \xi = a u + b v + c w, \quad c \neq 0. \quad (3.28)$$

Of course, we now have $\text{deg}(P) = \text{deg}(P^{-1})$ and the equality in (3.22) never occurs in nontrivial situations where $\min\{\text{deg}(\phi), \text{deg}(\psi)\} \geq 2$.

We emphasize that the polynomial functions $\phi$ and $\psi$ in (3.8) and (3.27) are of arbitrary degrees in particular.

It may be of interest to explore a Monge–Ampère equation type structure, as (2.5) as we did in 2 dimensions for the Jacobian equation (2.2), for (3.2). For this purpose, we note that the first
equation in (3.3) implies that the vector \((f, g, h)\), being divergence free, has a vector potential, \((A, B, C)\), satisfying
\[
(f, g, h) = \text{curl of } (A, B, C) = (C_y - B_z, A_z - C_x, B_x - A_y).
\]  
(3.29)
Hence (3.2) becomes the following second-order nonlinear equation
\[
C_{xy} - B_{xz} C_{yy} - B_{yz} + A_{xz} - C_{xx} A_{yz} - C_{xx} = 0.
\]  
(3.30)
of a kind of the Hessian type.

As another reduction of (3.2), we may set \(h = H(f, g)\) where \(H\) is a prescribed function of \(f\) and \(g\). Hence (3.2) becomes
\[
f_x + g_y + H f_x + H g z + J(f, g)(x, y) + H g J(f, g)(x, z) + H f J(g, f)(y, z) = 0.
\]  
(3.31)
The under-determined equations (3.30) and (3.31) can be reduced further by imposing some appropriate constraints on the unknowns.

4 General dimensions

In the general situation, with the notation
\[
(u_1, \ldots, u_n) = P(x_1, \ldots, x_n) = (x_1 + f_1, \ldots, x_n + f_n), \quad f_{ij} = \frac{\partial f_i}{\partial x_j}, \quad i, j = 1, \ldots, n,
\]  
(4.1)
then it is clear that the Jacobian equation \(J(P) = 1\) or \(\det(I + F) = 1\) where \(F = (f_{ij})\) assumes the form
\[
E_1(F) + E_2(F) + \cdots + E_n(F) = 0,
\]  
(4.2)
where \(E_k(F)\) is the sum of all \(k\) by \(k\) principal minors of the matrix \(F\), \(k = 1, \ldots, n\), such that \(E_1(F) = \text{tr}(F)\) and \(E_n(F) = \det(F)\) (cf. [10]).

We now aim to obtain a family of solutions of (4.2) of our interest that satisfy the Jacobian conjecture and extend what we found earlier in low dimensions.

For such a purpose and suggested by the study in Section 3, we use \((a_{ij})\) to denote an \((n-1)\) by \(n\) matrix in \(\mathbb{F}\) and introduce the variables
\[
\xi_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, n - 1.
\]  
(4.3)
Define
\[
u_j = x_j + f_j, \quad j = 1, \ldots, n,
\]  
(4.4)
where \(f_1, \ldots, f_{n-1}\) are arbitrary polynomials in \(\xi_1, \ldots, \xi_{n-1}\), respectively, but
\[
f_n = \sum_{j=1}^{n-1} b_j f_j(\xi_j),
\]  
(4.5)
where the coefficients \( b_1, \ldots, b_{n-1} \in \mathbb{F} \) are to be determined through the equation (4.2) which due to (4.5) is now slightly reduced into

\[
E_1(F) + E_2(F) + \cdots + E_{n-1}(F) = 0, \tag{4.6}
\]

which is still rather complicated. For simplicity and in view of the study in Section 3, we impose the following full invariance condition between the two sets of variables \( x_1, \ldots, x_n \) and \( u_1, \ldots, u_n \):

\[
\xi_i = \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} a_{ij} u_i, \quad i = 1, \ldots, n-1, \tag{4.7}
\]

so that by virtue of (4.4) we arrive at

\[
\sum_{j=1}^{n} a_{ij} u_j = \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{n-1} a_{ij} f_j + a_{in} f_n
\]

\[
= \xi_i + \sum_{j=1}^{n-1} (a_{ij} + a_{in} b_j) f_j, \quad i = 1, \ldots, n-1, \tag{4.8}
\]

which results in the solution

\[
b_j = -\frac{a_{ij}}{a_{in}}, \quad a_{in} \neq 0, \quad i, j = 1, \ldots, n-1. \tag{4.9}
\]

This solution indicates that the quantities \( \xi_1, \ldots, \xi_{n-1} \) are linearly dependent:

\[
a_{in} \xi_j = a_{jn} \xi_i, \quad i, j = 1, \ldots, n-1, \tag{4.10}
\]

which extends (3.26). Since the functions \( f_1, \ldots, f_{n-1} \) are arbitrary, we may now set

\[
f_i = \phi_i(\xi), \quad i = 1, \ldots, n-1, \quad \xi = \sum_{j=1}^{n} a_j x_j. \tag{4.11}
\]

Hence we obtain the polynomial map \( P \) defined by

\[
u_1 = x_1 + \phi_1(\xi), \quad \ldots, \quad u_{n-1} = x_{n-1} + \phi_{n-1}(\xi), \quad u_n = x_n - \sum_{i=1}^{n-1} \frac{a_i}{a_n} \phi_i(\xi). \tag{4.12}
\]

With (4.12), it is readily checked that the inverse of the polynomial map \( P \) defined in (4.1) is given by

\[
x_1 = u_1 - \phi_1(\xi), \quad \ldots, \quad x_{n-1} = u_{n-1} - \phi_{n-1}(\xi), \quad x_n = u_n + \sum_{i=1}^{n-1} \frac{a_i}{a_n} \phi_i(\xi), \tag{4.13}
\]

where \( \phi_1, \ldots, \phi_{n-1} \) are polynomial functions of the variable \( \xi = a_1 u_1 + \cdots + a_n u_n \). Of course we now have \( \deg(P^{-1}) = \deg(P) \).

Note that \( \phi_1, \ldots, \phi_{n-1} \), in nontrivial situations, consist of terms of degrees at least 2 of the variable \( \xi \), which are arbitrary otherwise. Since \( DP(0) = I \), we automatically get \( J(P) = 1 \). In particular, \( (f_1, \ldots, f_{n-1}, f_n) \) so constructed is a solution to the Jacobian equation (4.2) such that the associated polynomial map \( P \) given in (4.1) satisfies the Jacobian conjecture.

Further reductions to (4.2) may be carried out along the lines shown in Section 3 which are omitted here.
5 Return to four dimensions

The method of Section 3 may be used to obtain solutions involving partial invariance as well in higher dimensions but the computation become rather complicated. As an illustration, in this section, we apply such a method further to consider the four-dimensional situation.

For this purpose, consider the variables given in (4.3)–(4.5) with \( n = 4 \). So we have

\[
\xi_i = \sum_{j=1}^{4} a_{ij} x_j, \quad u_i = x_i + f_i(\xi_i), \quad i = 1, 2, 3; \quad u_4 = x_4 + f_4, \quad f_4 = \sum_{i=1}^{3} b_i f_i(\xi_i).
\]

To solve (4.6), we split it into three separate equations:

\[
E_1 = 0, \quad E_2 = 0, \quad E_3 = 0.
\]

From (5.1), we have

\[
f_{ij} = f_i'(\xi_i) a_{ij}, \quad i = 1, 2, 3; \quad j = 1, 2, 3, 4; \quad f_{4j} = \sum_{i=1}^{3} a_{ij} b_i f_i'(\xi_i), \quad j = 1, 2, 3, 4,
\]

which give us the expressions

\[
E_1 = (a_{11} + a_{14} b_1) f_1' + (a_{22} + a_{24} b_2) f_2' + (a_{33} + a_{34} b_3) f_3',
\]

\[
E_2 = \begin{vmatrix}
    a_{11} & a_{12} & f_1' f_2' + b_1 & a_{11} & a_{13} & a_{13} \\
    a_{21} & a_{22} & f_1' f_2' + b_2 & a_{22} & a_{23} & a_{23} \\
    a_{31} & a_{32} & a_{33} & f_2' f_3' + b_3 & a_{31} & a_{34} \\
\end{vmatrix}
\]

\[
E_3 = \begin{vmatrix}
    a_{11} & a_{12} & a_{13} & a_{13} \\
    a_{21} & a_{22} & a_{23} & a_{23} \\
    a_{31} & a_{32} & a_{33} & a_{33} \\
\end{vmatrix} + \begin{vmatrix}
    a_{11} & a_{12} & a_{13} & a_{13} \\
    a_{21} & a_{22} & a_{23} & a_{23} \\
    a_{31} & a_{32} & a_{33} & a_{33} \\
\end{vmatrix} + \begin{vmatrix}
    a_{11} & a_{12} & a_{13} & a_{13} \\
    a_{21} & a_{22} & a_{23} & a_{23} \\
    a_{31} & a_{32} & a_{33} & a_{33} \\
\end{vmatrix}
\]

Since the functions \( f_1, f_2, f_3 \) are arbitrary, we are led by (5.2) and (5.4)–(5.6) to the equations

\[
a_{11} + a_{14} b_1 = 0, \quad a_{22} + a_{24} b_2 = 0, \quad a_{33} + a_{34} b_3 = 0,
\]

\[
a_{11} a_{12} + b_1 a_{22} a_{24} + b_2 a_{11} a_{14} a_{21} a_{24} = 0,
\]

\[
a_{11} a_{13} + b_1 a_{33} a_{34} + b_3 a_{11} a_{14} a_{31} a_{34} = 0,
\]

\[
a_{22} a_{23} + b_2 a_{33} a_{34} + b_3 a_{22} a_{24} a_{32} a_{34} = 0,
\]

\[
a_{11} a_{12} a_{13} + b_1 a_{22} a_{23} a_{24} + b_2 a_{11} a_{13} a_{14} a_{21} a_{23} a_{24} + b_3 a_{11} a_{12} a_{14} a_{31} a_{32} a_{34} = 0.
\]

To satisfy (5.7), we assume \( a_{14}, a_{24}, a_{34} \neq 0 \) to obtain

\[
b_1 = -\frac{a_{11}}{a_{14}}, \quad b_2 = -\frac{a_{22}}{a_{24}}, \quad b_3 = -\frac{a_{33}}{a_{34}}.
\]
Inserting (5.12) into (5.8)–(5.10), we have

\begin{align*}
(a_{12}a_{24} - a_{14}a_{22})(a_{11}a_{24} - a_{14}a_{21}) &= 0, \quad (5.13) \\
(a_{13}a_{34} - a_{14}a_{33})(a_{11}a_{34} - a_{14}a_{31}) &= 0, \quad (5.14) \\
(a_{23}a_{34} - a_{24}a_{33})(a_{22}a_{34} - a_{24}a_{32}) &= 0. \quad (5.15)
\end{align*}

These equations extend (3.7). To fulfill (5.13)–(5.15) in a minimal manner, we impose, for example,

\begin{align*}
a_{12}a_{24} &= a_{14}a_{22}, \quad a_{13}a_{34} = a_{14}a_{33}, \quad a_{23}a_{34} = a_{24}a_{33}, \quad (5.16) \\
a_{11}a_{24} \neq a_{14}a_{21}, \quad a_{11}a_{34} \neq a_{14}a_{31}, \quad a_{22}a_{34} \neq a_{24}a_{32}. \quad (5.17)
\end{align*}

It can be checked that, with (5.12) and (5.16), the equation (5.11) is automatically satisfied. So we have thus obtained a solution to the differential equation of the problem.

With such a solution, we can construct the desired polynomial automorphism in four dimensions. To this goal, we again explore certain invariance structure as considered earlier. First, using (5.1) and then (5.12) and (5.16), we have

\[
\sum_{j=1}^{4} a_{1j}u_j = \sum_{j=1}^{4} a_{1j}x_j + (a_{12} + a_{14}b_2)f_2 + (a_{13} + a_{14}b_3)f_3 = \xi_1 + \left(a_{12} - \frac{a_{14}a_{22}}{a_{24}}\right)f_2 + \left(a_{13} - \frac{a_{14}a_{33}}{a_{34}}\right)f_3 = \xi_1.
\]

That is, \( \xi_1 \) is an invariant. Similarly, we have

\[
\sum_{j=1}^{4} a_{2j}u_j = \xi_2 + \left(a_{21} - \frac{a_{24}a_{11}}{a_{14}}\right)f_1, \quad (5.19)
\]

\[
\sum_{j=1}^{4} a_{3j}u_j = \xi_3 + \left(a_{31} - \frac{a_{34}a_{11}}{a_{14}}\right)f_1 + \left(a_{32} - \frac{a_{34}a_{22}}{a_{24}}\right)f_2, \quad (5.20)
\]

indicating that \( \xi_2, \xi_3 \) are not invariants due to (5.17). From these results, we immediately obtain the inverse map given by

\[
x_i = u_i - f_i(\xi_i), \quad i = 1, 2, 3, \quad x_4 = u_4 + \frac{a_{11}}{a_{14}}f_1(\xi_1) + \frac{a_{22}}{a_{24}}f_2(\xi_2) + \frac{a_{33}}{a_{34}}f_3(\xi_3), \quad (5.21)
\]

where now

\[
\xi_1 = \sum_{j=1}^{4} a_{1j}u_j, \quad (5.22)
\]

\[
\xi_2 = \sum_{j=1}^{4} a_{2j}u_j - \left(a_{21} - \frac{a_{24}a_{11}}{a_{14}}\right)f_1(\xi_1), \quad (5.23)
\]

\[
\xi_3 = \sum_{j=1}^{4} a_{3j}u_j - \left(a_{31} - \frac{a_{34}a_{11}}{a_{14}}\right)f_1(\xi_1) - \left(a_{32} - \frac{a_{34}a_{22}}{a_{24}}\right)f_2(\xi_2), \quad (5.24)
\]

iteratively in terms of the variables \( u_i \)'s. From the construction, it is clear that

\[
\deg(P) = \max\{\deg(f_1), \deg(f_2), \deg(f_3)\}, \quad (5.25)
\]

\[
\deg(P^{-1}) = \deg(f_1) \deg(f_2) \deg(f_3). \quad (5.26)
\]
Furthermore, the three nonminimal cases are also worth describing. One is when replacing (5.17) with
\[ a_{11}a_{24} = a_{14}a_{21}, \quad a_{11}a_{34} \neq a_{14}a_{31}, \quad a_{22}a_{34} \neq a_{24}a_{32}. \] (5.27)
Using (5.27) in (5.19) and (5.20), we see that the quantity \( \xi_2 \) becomes an additional invariant such that we update (5.23) into
\[ \xi_2 = \sum_{j=1}^{4} a_{2j}u_j, \] (5.28)
but \( \xi_3 \) remains intact as a non-invariant. Hence the inverse map (5.21) is defined by (5.22), (5.28), and (5.24), such that
\[ \deg(P^{-1}) = \deg(f_3) \max\{\deg(f_1), \deg(f_2)\}. \] (5.29)

Another case is when replacing (5.27) with
\[ a_{11}a_{24} = a_{14}a_{21}, \quad a_{11}a_{34} = a_{14}a_{31}, \quad a_{22}a_{34} \neq a_{24}a_{32}, \] (5.30)
for example. Although the quantity \( \xi_3 \) is still a non-invariant, it is reduced into
\[ \xi_3 = \sum_{j=1}^{4} a_{3j}u_j - \left( a_{32} - \frac{a_{34}a_{22}}{a_{24}} \right) f_2(\xi_2), \] (5.31)
such that the inverse map defined by (5.21) with (5.22), (5.28), and (5.31) satisfies
\[ \deg(P^{-1}) = \max\{\deg(f_1), \deg(f_2)\} \deg(f_3)\}. \] (5.32)

The last case is when replacing (5.30) with all equalities,
\[ a_{11}a_{24} = a_{14}a_{21}, \quad a_{11}a_{34} = a_{14}a_{31}, \quad a_{22}a_{34} = a_{24}a_{32}, \] (5.33)
which finally renders the quantity \( \xi_3 \) an invariant as well,
\[ \xi_3 = \sum_{j=1}^{4} a_{3j}u_j = \sum_{j=1}^{4} a_{3j}x_j, \] (5.34)
such that the inverse map (5.21) is defined by (5.22), (5.28), and (5.34), with
\[ \deg(P^{-1}) = \max\{\deg(f_1), \deg(f_2), \deg(f_3)\} = \deg(P). \] (5.35)

In fact, this last situation with a full set of invariant variables is contained in the general situation treated in Section 4.

The degree formulas (5.26), (5.29), (5.32), and (5.35) are refined realizations of the general upper bound (1.3) when \( n = 4 \).

In summary of the study of this section, we state

**Theorem 5.1** In the situation of four dimensions, the nonlinear first-order differential equation (4.2) with \( n = 4 \) has a family of polynomial solutions depending on three arbitrary polynomial functions, \( f_1, f_2, f_3 \), of the variables \( \xi_1, \xi_2, \xi_3 \) given as in (5.1), respectively, where the coefficients \( a_{ij} \)'s satisfy \( a_{14}, a_{24}, a_{34} \neq 0 \) and the combined conditions (5.12)–(5.15) for the existence of such solutions which give rise to the inverse polynomial map explicitly through the expressions (5.21) so that one, two, or all three of the quantities \( \xi_1, \xi_2, \xi_3 \) appear as invariants relating the two sets of the variables \( x_i \)'s and \( u_i \)'s.
(i) When one of the quantities $\xi_1, \xi_2, \xi_3$ is an invariant, the degree of the inverse map is the highest possible given by (5.26).

(ii) When two of the quantities $\xi_1, \xi_2, \xi_3$ are invariants, say $\xi_1$ and $\xi_2$, the degree of the inverse map is lowered to either (5.29) or (5.32), depending on the details of the residual non-invariant variable $\xi_3$ as described.

(iii) When all the quantities $\xi_1, \xi_2, \xi_3$ are invariants, the map and its inverse are described by the polynomials of the same structures as described, and in particular, the map and its inverse have the same arbitrarily prescribed degrees as stated in (5.35).

The most transparent situation is when $f_1, f_2, f_3$ are of the same degree. With this assumption, the results (5.26), (5.29), (5.32), and (5.35) read

\[ \text{deg}(P^{-1}) = (\text{deg}(P))^3, \quad \text{deg}(P^{-1}) = (\text{deg}(P))^2, \quad \text{deg}(P^{-1}) = \text{deg}(P), \quad (5.36) \]

respectively, arranged in a descending order.

6 The Jacobian problem with parametrization

The separation reductions from (2.2) into (2.3), (3.2) into (3.3), and (4.6) (with $n = 4$) into (5.2) prompt us to pursue a systematic and natural mechanism to split the original Jacobian equation, $J(P) = 1$, in order to render the problem effectively and schematically solvable, which is what we aim to do next. In fact, it is clear that the Jacobian problem allows the following normalized reformulation

\[ (u_1, \ldots, u_n) = P(x_1, \ldots, x_n) = (\lambda_1 x_1 + f_1, \ldots, \lambda_n x_n + f_n), \quad (6.1) \]

generalizing (4.1), where $\lambda_1, \ldots, \lambda_n$ are nonzero scalars, and $f_1, \ldots, f_n$ are polynomials containing quadratic and higher power terms in $x_1, \ldots, x_n$. In this situation, we have

\[ J(P) = \det((\lambda_i \delta_{ij} + f_{ij})), \quad f_{ij} = \frac{\partial f_i}{\partial x_j}, \quad i, j = 1, \ldots, n. \quad (6.2) \]

The condition on $f_1, \ldots, f_n$ implies that $f_{ij}$ all vanish at $x_1 = 0, \ldots, x_n = 0$, which leads to the equation

\[ \det((\lambda_i \delta_{ij} + f_{ij})) = \lambda_1 \cdots \lambda_n, \quad (6.3) \]

for the Jacobian problem. In order to explore this equation, we rewrite its left-hand side as

\[ \det((\lambda_i \delta_{ij} + f_{ij})) = C_0 + \sum_{k=1}^{n-1} \sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}, i_1 < \cdots < i_k} C_{i_1 \ldots i_k} \lambda_{i_1} \cdots \lambda_{i_k} + \lambda_1 \cdots \lambda_n, \quad (6.4) \]

where

\[ C_0 = \det((\lambda_i \delta_{ij} + f_{ij})|_{\lambda_1 = \cdots = \lambda_n = 0}) = E_n(F) = \det(F), \quad (6.5) \]

\[ C_{i_1 \ldots i_k} = \det(M_{i_1 \ldots i_k})|_{\lambda_1 = \cdots = \lambda_{n-k} = 0, \{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}}, \quad (6.6) \]

where $M_{i_1 \ldots i_k}$ is the minor matrix of the Jacobian matrix given in (6.2) with the $i_1, \ldots, i_k$th rows and columns being deleted. In fact, (6.5) is obvious and (6.6) may be obtained from differentiating (6.3).
and (6.4) with respect to \( \lambda_{i_1}, \ldots, \lambda_{i_k} \) and then setting \( \lambda_{j_1}, \ldots, \lambda_{j_{n-k}} \) to zero, where \( \{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\} \). As a consequence, we have

\[
C_{i_1, \ldots, i_k} = \det(F_{i_1, \ldots, i_k}), \quad \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}, \quad i_1 < \cdots < i_k, \quad k = 1, \ldots, n - 1, \tag{6.7}
\]

where \( F_{i_1, \ldots, i_k} \) is the minor matrix of the matrix \( F = (f_{ij}) \) with the \( i_1, \ldots, i_k \)th rows and columns being deleted.

In view of (6.3) and (6.4), with (6.5)–(6.7), we arrive at

\[
\det(F) = 0, \quad \det(F_{i_1, \ldots, i_k}) = 0, \quad \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}, \quad i_1 < \cdots < i_k, \quad k = 1, \ldots, n - 1. \tag{6.8}
\]

It is clear that there are a total number of

\[
\sum_{k=0}^{n-1} \binom{n}{k} = 2^n - 1 \tag{6.9}
\]
equations in (6.7), in contrast with a single equation in the original Jacobian problem. In other words, the reformulated Jacobian problem enjoys a much tighter structure for the determination of its solution, and thus, may hopefully be resolved thoroughly.

We first consider the case in two dimensions so that (2.1) assumes the form

\[
P(x, y) = (\alpha x + f(x, y), \beta y + g(x, y)), \tag{6.10}
\]

where \( \alpha, \beta \) are parameters. Hence (6.8) gives rise to \( 2^2 - 1 = 3 \) equations

\[
f_x = 0, \quad g_y = 0, \quad f_x g_y - f_y g_x = 0. \tag{6.11}
\]

The first two equations indicate that \( f \) and \( g \) depend on \( y \) and \( x \) only, respectively. So the last equation in (6.11) is simply \( f_y g_x = 0 \). That is \( g = 0 \) or \( f = 0 \). Therefore we obtain two general solutions to (6.11):

\[
f = f(y), \quad g = 0; \quad f = 0, \quad g = g(x). \tag{6.12}
\]

From these solutions and (6.10), we find under the condition \( \alpha \neq 0, \beta \neq 0 \) the corresponding polynomial maps and their inverses to be

\[
u = \alpha x + f(y), \quad v = \beta y; \quad x = \frac{1}{\alpha} \left( u - f \left( \frac{v}{\beta} \right) \right), \quad y = \frac{v}{\beta}, \tag{6.13}
\]

\[
u = \alpha x, \quad v = \beta y + g(x); \quad x = \frac{u}{\alpha}, \quad y = \frac{1}{\beta} \left( v - g \left( \frac{u}{\alpha} \right) \right), \tag{6.14}
\]

respectively. We note that (6.11) is a system of 3 equations which is under-determined for the 4 unknowns \( f_x, f_y, g_x, g_y \) and its general solution (6.12) contains 1 arbitrary function which accounts for the underlying extra degree of freedom, \( 4 - 3 = 1 \).

We next consider the case in three dimensions so that (3.1) becomes

\[
u, v, w = P(x, y, z) = (\alpha x + f(x, y, z), \beta y + g(x, y, z), \gamma z + h(x, y, z)). \tag{6.15}
\]

Applying (6.8) to (6.15), we obtain \( 2^3 - 1 = 7 \) equations

\[
f_x = 0, \quad g_y = 0, \quad h_z = 0, \tag{6.16}
\]

\[
f_y g_x = 0, \quad f_z h_x = 0, \quad g_z h_y = 0, \tag{6.17}
\]

\[
f_z g_x h_y + f_y g_z h_x = 0, \tag{6.18}
\]

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subsequently and iteratively. The full family of general solutions to this system may be described
with various combinations of the valid choices of \( f, g, h \). In fact, (6.16) indicates that \( f = f(y, z), g = g(x, z), h = h(x, y) \). That is, the three functions \( f, g, h \) can only depend on up to two variables
among \( x, y, z \). Assume that \( h \) depends on \( x, y \). Then \( h_x \neq 0, h_y \neq 0 \). Thus (6.17) indicates that
\( f_z = g_z = 0 \). So \( f, g \) can only depend on at most one variable nontrivially. We may pick \( g \) to be
such a polynomial so that \( g \) depends on \( x \). From \( g_x \neq 0 \), we obtain from the first equation in (6.17)
that \( f_y = 0 \). That is, \( f = 0 \). As a result, the equation (6.18) is automatically satisfied. Hence we
are lead to \( f = 0, g = g(x) \), and \( h = h(x, y) \), which give rise to the polynomial automorphism (the
polynomial map and its polynomial inverse):

\[
\begin{align*}
    u &= \alpha x, \quad v = \beta y + g(x), \quad w = \gamma z + h(x, y); \\
    x &= \frac{u}{\alpha}, \quad y = \frac{1}{\beta} \left( v - g \left( \frac{u}{\alpha} \right) \right), \quad z = \frac{1}{\gamma} \left( w - h \left( \frac{u}{\alpha}, \frac{1}{\beta} \left( v - g \left( \frac{u}{\alpha} \right) \right) \right) \right),
\end{align*}
\]

(6.19)

subject to \( \alpha \neq 0, \beta \neq 0, \gamma \neq 0 \). Now assume \( h \) depends only on either \( x \) or \( y \), say \( x \). Then \( h_x \neq 0 \)
and the second equation in (6.17) gives us \( f_z = 0 \). Using these in (6.18), we have \( f_y g_z = 0 \). If \( f_y = 0 \), then \( f = 0 \), and we arrive at the solution \( f = 0, g = g(x, z), h = h(x) \). If \( f_y \neq 0 \), then \( g_z = 0 \) and
\( g_x = 0 \) from the first equation in (6.17). Thus we arrive at the solution \( f = f(y), g = 0, h = h(x) \).
Then assume that \( h \) is independent of \( x, y \). Hence \( h = 0 \) and we have \( f_y g_x = 0 \) from (6.17) as
the only remaining nontrivial equation. If \( f_y = 0 \), then \( f = f(z) \) and \( g = g(x, z) \). If \( g_x = 0 \), then
\( f = f(y, z) \) and \( g = g(z) \). All of these solutions are of the type (6.19) after a variable relabeling.

Thus we conclude that one of the three polynomials \( f, g, h \) must be trivial in general, and, modulo
symmetry in the choice of variables, (6.19) is the general polynomial solution of the equations
(6.16)–(6.18).

In the case of four dimensions, we consider the polynomial map \((u_1, \ldots, u_4) = P(x_1, \ldots, x_4)\)
given by

\[
u_i = \lambda_i x_i + f_i(x_1, \ldots, x_4), \quad i = 1, \ldots, 4.
\]

(6.21)

With (6.21), the equations (6.8) become

\[
\begin{align*}
    f_{ii} &= 0, \quad (6.22) \\
    f_{ij} f_{ji} &= 0, \quad (6.23) \\
    f_{ij} f_{ki} f_{kj} + f_{ji} f_{ik} f_{kj} &= 0, \quad (6.24)
\end{align*}
\]

where \( i, j, k = 1, \ldots, 4 \), and

\[
f_{12}(f_{24} f_{31} f_{43} + f_{23} f_{34} f_{41}) + f_{13}(f_{24} f_{32} f_{41} + f_{21} f_{34} f_{42}) + f_{14}(f_{21} f_{32} f_{43} + f_{23} f_{31} f_{42}) = 0. \quad (6.25)
\]

There are 4 equations in (6.22), 6 in (6.23), and 4 in (6.24), totaling \( 15 = 2^4 - 1 \) equations in
the coupled system (6.22)–(6.25). To search for the solution, we see from (6.22) that each \( f_i \)
is independent of \( x_i \). Furthermore, from (6.23), we have \( f_{12} f_{21} = f_{13} f_{31} = f_{14} f_{41} = 0 \), which may be
satisfied with setting \( f_{1i} = 0 \) for all \( i \). That is, \( f_1 = 0 \). This condition also renders (6.25). The rest
of (6.23) are

\[
f_{23} f_{32} = 0, \quad f_{24} f_{42} = 0, \quad f_{34} f_{43} = 0,
\]

(6.26)

which may be fulfilled with \( f_2 = f_2(x_1) \) and \( f_3 = f_3(x_1, x_2) \). With these results, it is readily
examined that (6.24) also follows automatically. This indicates that the equations (6.22)–(6.25)
are not all independent ones and that \( f_4 = f_4(x_1, x_2, x_3) \) is arbitrary. Consequently, we arrive at the following general solution of (6.22)–(6.25):

\[
  f_1 = 0, \quad f_2 = f_2(x_1), \quad f_3 = f_3(x_1, x_2), \quad f_4 = f_4(x_1, x_2, x_3),
\]

which gives rise to the polynomial automorphism

\[
  u_1 = \lambda_1 x_1, \quad u_2 = \lambda_2 x_2 + f_2(x_1), \quad u_3 = \lambda_3 x_3 + f_3(x_1, x_2), \quad u_4 = \lambda_4 x_4 + f_4(x_1, x_2, x_3); \quad (6.28)
\]

\[
  x_1 = \frac{u_1}{\lambda_1}, \quad x_2 = \frac{1}{\lambda_2} \left( u_2 - f_2 \left( \frac{u_1}{\lambda_1} \right) \right), \quad x_3 = \frac{1}{\lambda_3} \left( u_3 - f_3 \left( \frac{u_1}{\lambda_1}, \frac{1}{\lambda_2} \left( u_2 - f_2 \left( \frac{u_1}{\lambda_1} \right) \right) \right) \right), \quad x_4 = \frac{1}{\lambda_4} \left( u_4 - f_4 \left( \frac{u_1}{\lambda_1}, \frac{1}{\lambda_2} \left( u_2 - f_2 \left( \frac{u_1}{\lambda_1} \right) \right), \frac{1}{\lambda_3} \left( u_3 - f_3 \left( \frac{u_1}{\lambda_1}, \frac{1}{\lambda_2} \left( u_2 - f_2 \left( \frac{u_1}{\lambda_1} \right) \right) \right) \right) \right),
\]

(6.29)

Solutions associated with \( f_2 = 0, f_3 = 0, \) and \( f_4 = 0, \) respectively, are similarly constructed.

We now show that the solutions of the above structure are all the solutions possible.

In fact, we first consider \( f_{12} = 0, f_{13} \neq 0, f_{14} = 0. \) Thus \( f_1 = f_1(x_3) \) and \( f_{31} = 0. \) The rest of (6.23) are again given by (6.26). We may choose \( f_3 = f_3(x_2) \) and \( f_4 = f_4(x_1, x_3) \) to fulfill these equations which lead to \( f_{23} = f_{34} = 0. \) Inserting these results into (6.25), we get \( f_{13} f_{24} f_{32} f_{41} = 0 \) or \( f_{24} f_{32} f_{41} = 0. \) We may choose \( f_{24} = 0 \) for example. Then \( f_2 = f_2(x_1). \) On the other hand, from (6.24), we have \( f_{13} f_{41} f_{3k} = 0, \) which gives us \( f_{21} f_{32} = 0. \) If \( f_{21} = 0, \) then \( f_2 = 0; \) if \( f_{32} = 0, \) then \( f_3 = 0. \) Either case has been covered already as discussed. We next consider \( f_{12} = 0, f_{13} \neq 0, f_{14} \neq 0. \) Thus \( f_1 = f_1(x_3, x_4) \) and \( f_{31} = f_{41} = 0. \) So the rest of the equations in (6.23) are still given by (6.26). If \( f_{23} \neq 0, \) then \( f_{32} = 0. \) If \( f_{42} = f_{43} = 0, \) then \( f_4 = 0 \) and we return to a familiar case. If \( f_{42} = 0 \) but \( f_{43} \neq 0, \) then \( f_{34} = 0. \) Hence \( f_3 = 0 \) which is again a familiar case.

The solution (6.27) allows a direct generalization to any \( n \)-dimensional situation with

\[
  f_1 = 0, \quad f_2 = f_2(x_1), \quad f_3 = f_3(x_1, x_2), \quad \ldots, \quad f_n = f_n(x_1, \ldots, x_{n-1}),
\]

(6.30)

because the polynomial map given by

\[
  u_1 = \lambda_1 x_1, \quad u_2 = \lambda_2 x_2 + f_2(x_1), \quad \ldots, \quad u_n = \lambda_n x_n + f_n(x_1, \ldots, x_{n-1}),
\]

(6.31)

automatically satisfies the parametrized Jacobian equation, or

\[
  \det \begin{pmatrix}
    \lambda_1 & 0 & \cdots & \cdots & 0 \\
    f_{21} & \lambda_2 & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & 0 \\
    f_{n1} & f_{n2} & \cdots & f_{n,n-1} & \lambda_n
  \end{pmatrix} = \lambda_1 \lambda_2 \cdots \lambda_n.
\]

(6.32)

The inverse of (6.31) is easily constructed as done in (6.29) in four dimensions and thus omitted.

We now pursue a complete solution of the parametrized Jacobian equation (6.3) following the ideas obtained in the above low-dimensional situations, in a more effective manner.

First, the \( n \)-dimensional form of the equation (6.22) indicates that \( f_i \) is independent of \( x_i \) for \( i = 1, \ldots, n. \) Assume that \( x_1 \) (say) is a variable that enjoys a maximal presence in the problem so
that all }f_i\text{'s } (i \geq 2) \text{ depend on } x_1. \text{ The } n\text{-dimensional form of the equation (6.23) implies that if } f_i \text{ depends on } x_j \text{ then } f_j \text{ is independent of } x_i, i, j = 1, \ldots, n, i \neq j. \text{ Thus, with }
\begin{align*}
  f_{21} &\neq 0, \ldots, f_{n1} \neq 0,
\end{align*}
we have } f_{12} = \cdots = f_{1n} = 0. \text{ Hence } f_1 \text{ and } f_2 \text{ do not depend on } x_2. \text{ Assume that } x_2 \text{ enjoys a maximal presence in the problem so that all } f_i\text{'s } (i \geq 3) \text{ depend on } x_2. \text{ Thus, with }
\begin{align*}
  f_{32} &\neq 0, \ldots, f_{n2} \neq 0,
\end{align*}
we have } f_{23} = \cdots = f_{2n} = 0. \text{ Carrying out this procedure to the } (n - 1)\text{th step, we arrive at the conclusion that the polynomials } f_1, \ldots, f_{n-1} \text{ do not depend on } x_{n-1}. \text{ So we are left with assuming that } f_n \text{ depends on } x_{n-1} \text{ to achieve a maximal presence of the variable } x_{n-1} \text{ in the set of the polynomials. This gives us } f_{n,n-1} = 0. \text{ Hence } f_{n-1,n} = 0. \text{ Consequently, this procedure leads us to (6.32) and the solution (6.31), constructively and systematically.}

Alternatively, if we assume the polynomial function } f_1 \text{ enjoys the maximal presence of the variables, then } f_1 = f_1(x_2, \ldots, x_n). \text{ This assumption leads to } f_{12} \neq 0, \ldots, f_{1n} \neq 0 \text{ and } f_{21} = \cdots = f_{n1} = 0. \text{ Subsequently, assume } f_2 \text{ enjoys the maximal presence of the rest of the variables. Since } f_{21} = 0, \text{ we have } f_2 = f_2(x_3, \ldots, x_n). \text{ Following this process, we eventually have } f_{n-1} = f_{n-1}(x_n) \text{ and } f_n = 0. \text{ That is,}
\begin{align*}
  f_1 = f_1(x_2, x_3, \ldots, x_n), \quad f_2 = f_2(x_3, \ldots, x_n), \quad \ldots, \quad f_{n-1} = f_{n-1}(x_n), \quad f_n = 0. \quad (6.35)
\end{align*}

This result differs from (6.30) only by a relabeling of the variables, of course.

This result in fact also includes all the degenerate cases. For example, assume only } f_{12} \neq 0, \ldots, f_{1,n-1} \neq 0 \text{ but } f_{1n} = 0. \text{ That is, } x_n \text{ does not make an explicit presence in } f_1. \text{ Then we have }
\begin{align*}
  f_{21} = 0, \quad \ldots, \quad f_{n-1,1} = 0, \quad (6.36)
\end{align*}
and we have either } f_{n1} = 0 \text{ or } f_{n1} \neq 0. \text{ In the former case, the equation (6.3) becomes }
\begin{align*}
  \det \begin{pmatrix}
    \lambda_2 & f_{23} & \cdots & f_{2n} \\
    f_{32} & \cdots & \cdots & f_{3n} \\
    \vdots & \ddots & \ddots & \vdots \\
    f_{n-1,2} & \cdots & \cdots & f_{n-1,n} \\
    f_{n2} & f_{n3} & \cdots & \lambda_n
  \end{pmatrix} = \lambda_2 \cdots \lambda_n. \quad (6.37)
\end{align*}

Using induction, we see that, with a relabeling of the variables if necessary, the general solution of (6.37), including all degenerate cases, reads
\begin{align*}
  f_2 = f_2(x_3, x_4, \ldots, x_n), \quad f_3 = f_2(x_4, \ldots, x_n), \quad \ldots, \quad f_{n-1} = f_{n-1}(x_n), \quad f_n = 0, \quad (6.38)
\end{align*}
following (6.35). In view of (6.36), } f_{n1} = 0, \text{ and (6.37), we obtain } f_1 = f_1(x_2, \ldots, x_{n-1}), \text{ which is consistent with (6.35). In the latter case, that is, } f_{n1} \neq 0, \text{ then the } n\text{-dimensional version of (6.24) with } i = n, j = 1 \text{ gives us }
\begin{align*}
  f_{kn}f_{1k} = 0, \quad k = 1, \ldots, n. \quad (6.39)
\end{align*}
However, the assumption } f_{1k} \neq 0 \text{ for } k = 2, \ldots, n-1 \text{ implies } f_{kn} = 0 \text{ for } k = 2, \ldots, n-1. \text{ Therefore } f_1, \ldots, f_{n-1}, f_n \text{ are independent of } x_n \text{ and the problem is actually reduced into an } (n-1)\text{-dimensional problem. By induction, we conclude that the solution structure (6.35) again appears with the absence of an explicit dependence on } x_n.
It should be noted that the general solution (6.31) is not of the homogeneous type studied earlier. Besides, with fixed \(\lambda_1, \ldots, \lambda_n\), say \(\lambda_1 = \cdots = \lambda_n = \lambda\), the result (6.31) gives rise to a new family of solutions to the classical Jacobian equation \(J(P) = 1\), say, as well.

In summary, we state the results of the study of this section as follows.

**Theorem 6.1** In any \(n \geq 2\) dimensions, the general solution to the Jacobian equation \(J(P) = \lambda_1 \cdots \lambda_n\) associated with the polynomial map (6.1) parametrized by the parameters \(\lambda_1, \ldots, \lambda_n\) is given by the expression (6.30), up to a relabeling of the variables. For specific values of these parameters, this map possesses a polynomial inverse if and only if the Jacobian of the map is nonvanishing which is characterized by \(\lambda_1 \neq 0, \ldots, \lambda_n \neq 0\). In this situation, the inverse map is iteratively given by

\[
x_1 = \frac{u_1}{\lambda_1}, \quad x_2 = \frac{1}{\lambda_2}(u_2 - f_2(x_1)), \quad \ldots, \quad x_n = \frac{1}{\lambda_n}(u_n - f_n(x_1, \ldots, x_{n-1})),
\]

schematically and explicitly. In particular, this construction provides a new family of solutions to the non-parametrized (classical) Jacobian problem, which are of non-homogeneous type in general.

From (6.40), it is clear that the degree of the inverse map \(P^{-1}\) satisfies the upper bound

\[
\deg(P^{-1}) \leq \deg(f_1) \cdots \deg(f_{n-1}) \leq \deg(P)^{n-1},
\]

which reconfirms the bound (1.3) again.

Of independent interest is the situation when the multiple parameters \(\lambda_1, \ldots, \lambda_n\) in (6.1) are combined into a single one, \(\lambda_1 = \cdots = \lambda_n = \lambda\). Then the Jacobian equation (6.3) becomes

\[
\det(\lambda I + F) = \lambda^n, \quad F = (f_{ij}),
\]

such that the left-hand side of this equation is the characteristic polynomial of the matrix \(-F\). As a consequence, we are led from (6.42) to (10):

\[
E_1(F)\lambda^{n-1} + E_2(F)\lambda^{n-2} + \cdots + E_{n-1}(F)\lambda + E_n(F) = 0,
\]

or equivalently,

\[
E_i(F) = 0, \quad i = 1, \ldots, n.
\]

That is, the presence of the parameter \(\lambda\) renders a natural splitting of the equation (4.2) into \(n\) subequations, listed in (6.44).

When \(n = 2, 3, 4\), (6.44) becomes (2.3), (3.3), (5.2) (along with \(E_4 = 0\)), respectively, whose solutions have been constructed in Sections 2, 3, 5. In those situations, the equations are imposed in order to reduce the original Jacobian equation. In the present situation, these equations are natural consequences of the presence of the free parameter \(\lambda\). However, unlike in the multiply parametrized situation, the singly parametrized system of the equations, (6.44), defies a complete solution even in the bottom-dimensional situation, \(n = 2\), which explains the associated difficulties in the original problem transparently.

At this juncture, it will be interesting to work out a concrete example as a direct illustration of our study. For simplicity, we consider the two-dimensional case (2.1) with

\[
f(x, y) = \sum_{i+j=k, k=2,3} a_{ij}x^iy^j, \quad g(x, y) = \sum_{i+j=k, k=2,3} b_{ij}x^iy^j,
\]

(6.45)
Substituting (6.45) into the full two-dimensional Jacobian equation (2.2) and solving for coefficients, we get four families of solutions. The two simple ones are given by

\[ f(x, y) = ay^2 + by^3, \quad g(x, y) = 0; \quad f(x, y) = 0, \quad g(x, y) = ax^2 + bx^3, \quad (6.46) \]

where \( a, b \) are arbitrary scalars. These solutions are special cases contained already in (6.12). The other two families of solutions are given by

\[ f(x, y) = a\xi^2 - \frac{b^2}{9c}\xi^3, \quad g(x, y) = -\frac{ab}{3c}\xi^2 + \frac{b^3}{27c^2}\xi^3, \quad \xi = x + \frac{3c}{b}y, \quad b \neq 0, c \neq 0, \quad (6.47) \]

\[ f(x, y) = \frac{\eta^2}{a}, \quad g(x, y) = \frac{b\eta^2}{a^2}, \quad \eta = bx - ay, \quad a \neq 0, \quad (6.48) \]

in terms of arbitrary scalars, \( a, b, c \), again. In these families of solutions, the variables \( \xi \) and \( \eta \) are invariant variables, through \( u = x + f, v = y + g \). That is, we have

\[ \xi = x + \frac{3c}{b}y = u + \frac{3c}{b}v, \quad \eta = bx - ay = bu - av, \quad (6.49) \]

respectively, enabling immediate constructions of inverse polynomial maps, as a consequence. These solutions are special cases of what obtained in Section 2. It is also clear that all the solutions here are “divergence free”, satisfying the equation \( f_x + g_y = 0 \), and thus the homogeneous Monge–Ampère equation as well.

7 Application to cryptography

Due to their analytic simplicity, invertible polynomial maps are of obvious interest and importance in applications. For example, we consider an application of the construction here to cryptography [9,12] in which a polynomial automorphism can be used to serve as a cryptographic key to encrypt and decrypt messages. To see how, we note that although our construction of inverse maps in various dimensions is suggested by the Jacobian problem and the associated nonlinear partial differential equations, the detailed fine structures of such maps are quite general. In other words, the final formulas obtained of the inverse maps by-pass the original technical assumptions and formulation of the problem. For example, the construction works equally well when the underlying field \( F \) is of a nonzero characteristic or replaced by a commutative ring with a unity, say \( R \). As a concrete example, assume that a plaintext is arranged in the form of a sequence of vectors in \( R^n \), denoted by \( x = (x_1, \ldots, x_n) \). As a consequence of the construction comprised of (4.11) and (4.12), we may formulate the (nonlinear and polynomial) encryption map \( u = (u_1, \ldots, u_n) \equiv E(x) \in R^n \) with

\[ u_1 = x_1 + \phi_1(\xi), \quad \ldots, \quad u_{n-1} = x_{n-1} + \phi_{n-1}(\xi), \quad u_n = x_n - \sum_{i=1}^{n-1} a_i \phi_i(\xi), \quad (7.1) \]

\[ \xi = \sum_{i=1}^{n-1} a_i x_i + x_n, \quad (7.2) \]

where \( a_1, \ldots, a_{n-1} \in R \) are arbitrary ring elements and \( \phi_1(\xi), \ldots, \phi_{n-1}(\xi) \) are arbitrary polynomial functions of \( \xi \). In this way, we come up with a ciphertext. Note that, we choose \( a_n = 1 \) in (4.11) in order to avoid division by it in (4.12). Therefore, by (4.13), we get the decryption map \( x = D(u) \)
with

\[ x_1 = u_1 - \phi_1(\xi), \ldots, \ x_{n-1} = u_{n-1} - \phi_{n-1}(\xi), \ x_n = u_n + \sum_{i=1}^{n-1} a_i \phi_i(\xi), \]  
(7.3)

\[ \xi = \sum_{i=1}^{n-1} a_i u_i + u_n. \] 
(7.4)

The invariance of the quantity \( \xi \) relating the plaintext and ciphertext variable vectors, \( x \) and \( u \), respectively, is self-evident. The pair of maps, \( E, D : \mathbb{R}^n \to \mathbb{R}^n \), collectively given in (7.1)–(7.4), is seen to provide a cryptographic key. Moreover, to achieve further enhanced or elevated security, we may also use the polynomial automorphisms with partial invariance properties as constructed in Sections 3 and 5, or those without any invariance structure presented in Section 6.

8 Summary

In this work, the Jacobian conjecture is directly studied by way of the partial differential equations it prompts. These equations are first order, nonlinear, and expressed in terms of the sum of all principal minors of the Jacobian matrix of the nonlinear part of the polynomial map. In two dimensions, the equation may be reduced into a homogeneous Monge–Ampère equation whose solutions uncover an invariance structure between the variables which may be exploited effectively beyond two dimensions. As a consequence, in any general \( n \) dimension \( (n \geq 2) \), solutions depending on \( n-1 \) arbitrary polynomial functions are obtained, which give rise to polynomial maps of arbitrarily high degrees, satisfying the Jacobian conjecture. The maps may or may not be homogeneous, depending on the choice of these arbitrary functions. Interestingly and practically, for these maps, the inverse maps are of similar structures and can be constructed immediately using the established invariance property, either partial or full, relating the two sets of variables in consideration. More importantly, it is demonstrated that the Jacobian problem enjoys a multiply parametrized reformulation which splits the Jacobian equation into a system of coupled equations that may be integrated completely and explicitly such that the nonzero Jacobian condition is necessary and sufficient to ensure that the polynomial map is invertible and possesses a polynomial inverse map. This inverse map may be constructed schematically and explicitly in which no use is made of any invariance property. This result affirmatively settles such a multiply parametrized Jacobian conjecture in all dimensions.

Data availability statement: The data that support the findings of this study are available within the article.

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