Wong-Zakai approximation for the dynamics of stochastic evolution equations driven by rough path with Hurst index

\[ H \in \left( \frac{1}{3}, \frac{1}{2} \right)^* \]

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Abstract

In this paper, we obtain the existence of random attractors for a class of evolution equations driven by a geometric fractional Brownian rough path with Hurst index \( H \in \left( \frac{1}{3}, \frac{1}{2} \right] \) and establish the upper semi-continuity of random attractors \( \mathcal{A}_\eta \) for the approximated systems of the evolution equations.

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1 Introduction

In this paper, we consider the existence of random attractors for a class of evolution equations driven by a geometric fractional Brownian rough path (GFBRP) with

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Hurst index $H \in (\frac{1}{3}, \frac{1}{2}]$. There are some results about the existence of random attractors for stochastic (partial) differential equations driven by a fractional Brownian motion (fBm). For additive noise with Hurst index $H \in (0, 1]$, the cohomology method transforms stochastic (partial) differential equations to random differential equations, then deterministic method can be used to construct random attractors. We refer to [20, 27] and references therein. For nonlinear multiplicative noise, Garrido-Atienza, Maslowski, Schmalfuss [13] considered the existence of random attractors of stochastic differential equations (SDEs) driven by a fBm with Hurst index $H > \frac{1}{2}$, where the stochastic integral can be defined by the fractional integral. Gao, Garrido-Atienza, Schmalfuss [10] extended the existence of random attractors of SDEs to a class of evolution equations driven by an infinite dimensional fBm with Hurst index $H > \frac{1}{2}$, they used the modified Hölder space $C^{\beta, \alpha}([0, T], V)$ to overcome semigroup is not Hölder continuity at zero. However, there are a few of works for random attractors of SDEs (or SPDEs) driven by fBm with Hurst index $H \in (\frac{1}{3}, \frac{1}{2})$. Duc [6] proved the existence of random attractors for rough differential equations driven by a Gaussian rough path which corresponding to a Gaussian process with $(\frac{1}{3}, \frac{1}{2}) \ni \nu$ Hölder regularity a.s., and the drift term is locally Lipschitz and satisfies $\langle f(y), y \rangle \leq ||y|| (D_1 - D_2 ||y||)$. In particular, the Doss-Sussmann technique was used to derive the uniform energy estimate, somehow it can be regard as cohomology method.

To the best of our knowledge, for stochastic partial differential equations (SPDEs) driven by multiplicative fBm with $H \in (\frac{1}{3}, \frac{1}{2})$, there are only few works on the solutions of SPDEs can generate a random dynamical system. About the global well-posedness of the solution. Garrido-Atienza, Lu, Schmalfuss [11] considered a class of evolution equations driven by an infinite dimensional fBm with Hurst index $H \in (\frac{1}{3}, \frac{1}{2}]$. They constructed a second order process which depends on the semigroup $S$ by the fractional integral [11], and then the global solution can be obtained. In addition, they used the fractional integral to construct rough paths and consider the global solutions of a class of evolution equations driven by fBm with Hurst index $H \in (\frac{1}{3}, \frac{1}{2}]$ and its random dynamical system in [12]. Hesse and Neamtu [16] constructed the local solutions for a class of rough evolution equation driven by a rough path via establishing rough integral, and then obtained the global solutions [17]. Compared with finite-dimensional case [6, 8], the stochastic integrals in the above works have more complex structure and therefore
it is vital to define the controlled rough paths as finite-dimensional case. Thus, Gerasimovićs, Hocquet, Nilssen [15] established the controlled rough paths over interpolation spaces as finite dimensional case. Using the approach of [15], Hesse and Neamtu [18] constructed the global solutions for SPDEs driven by the finite-dimensional rough paths. We will use this framework to construct the random attractors. In addition, Kuehn and Neamtu [21] also established center manifold for a class of SPDEs driven by a Gaussian rough path under this framework.

Our first aim in this paper is to obtain the existence of random attractors for following rough SPDEs

\[ dy_t = Ay_t dt + F(y_t) dt + G(y_t) dW_t, \]  

(1.1)

where \( W : [0, T] \times \Omega \rightarrow W_t(\omega) \) is a GFBRP, and let \( W(\omega) \) represent a geometric rough path lifted by the path \( \omega \in \Omega \) and the Hurst index \( H \in \left( \frac{1}{3}, \frac{1}{2} \right] \). We will use the technique as [10, 13] to obtain the random attractors of (1.1). Note that the metric dynamical system are ergodic in [10,13] for the case of fBm. There are two strategies to construct the metric dynamical system for GFBRP. One is that the orbits of a fBm generate a metric dynamical systems. Due to the GFBRP is a geometric rough path, namely its second order process \( \hat{W}(\omega) \) as the limit of canonical lift(see, [8, page 16]) a smooth path \( \hat{W}^\eta(\omega) \), and \( \theta_\tau \hat{W}^\eta(\omega) \) is also the canonical lift of the path \( \theta_\tau W^\eta(\omega) \), \( \tau \in \mathbb{R}, \omega \in \Omega \).

Thus, we use the same symbol \( \theta \) for the path component and the second order process in our paper. So it is enough to define the classical Wiener shift, and the metric dynamical system is ergodic [14]. The other one is that GFBRP can be regard as a new tensor-valued stochastic process and its distribution is the transformation of the law of fBm. In [2], the authors introduced the new Wiener shift for tensor-valued paths and rough cocycle, and then to construct a metric dynamical system. Similar to [14], it is not difficult to prove that the metric dynamical system is also ergodic. In our paper, we adopt the first strategy, so we require that the path of sample space \( \Omega \) has a canonical lift.

In our paper, for the argument of the compactness of the absorbing set, we impose the condition that a scale of Banach spaces \( B_\gamma, 0 < \gamma < (\alpha - \sigma) \wedge (1 - \delta) \) compactly embedding into a separable Banach space \( B \). For a class of evolution equations driven by a fractional Brownian motion with Hurst index \( H > \frac{1}{2} \), Gao et al. [10] did not impose the smallness condition for the diffusion term \( G \), but the covariance \( q \) of the
fBm is sufficiently small. Due to the structure of the solution \((y, G(y))\) and the estimate of radius of the absorbing set require temperedness of the stopping times, we impose the condition “\(C_G \leq \mu\)”. But the case \(C_G > \mu\) can be dealt as the case \(C_G \leq \mu\) (see Remark 3.1). For simplicity, we only consider the case \(C_G \leq \mu\). So we construct a sequence of stopping times in the spirit of [10]. In addition, Duc [6] considered rough differential equations driven by a Gaussian rough path, and the similar condition for the diffusion term \(G\) was imposed. For the drift term \(F\), the smallness condition is still necessary.

Hence, under the smallness conditions for \(F\) and the noise, we obtain the existence of random attractor of the system \((1.1)\) with Hurst index \(H \in \left(\frac{1}{3}, \frac{1}{2}\right]\). In addition, the existence of global random attractor of SPDEs with generally multiplicative noise is still open by now, our results can be regarded as first attempt to solve this problem using rough path theory.

Our second aim is to establish the upper semi-continuity for \(A_\eta\), where \(A_\eta\) are the random attractors of the above evolution equations driven by \(W_\eta(\omega)\) which is the smooth and stationary approximation of the GFBRP \(W(\omega)\) in Section 5. For fixed \(\eta \in (0, 1]\), due to the smoothness of \(W_\eta(\omega)\), the rough integral
\[
\int_0^t S(t-r)G(y_\eta) dW_\eta^r = \lim_{\|P(0,t)\| \to 0} \sum_{[u_i, v_i] \in P(0,t)} S(t-u_i) \left[ (G(y_\eta_{u_i})W_{u_i, v_i} + DG(y_\eta_{u_i})G(y_\eta_{u_i})W_{u_i, v_i}) \right]
\]
is just a Young integral
\[
\int_0^t S(t-r)G(y_\eta) dW_\eta^r = \lim_{\|P(0,t)\| \to 0} \sum_{[u_i, v_i] \in P(0,t)} S(t-u_i)G(y_\eta_{u_i})W_{u_i, v_i}.
\]
Then the approximate equation can be regarded as a random partial differential equation. Since the approximate noise is not truly rough [8, page 109], then the Gubinelli derivative is not unique. In order to obtain the existence and uniqueness of the solutions of approximate equations, similar to controlled rough paths, we choose a proper Gubinelli derivative \(G(y_\eta)\), then we can get the existence of the approximate equations in rough paths sense and the uniqueness for \(y_\eta\) can be easily obtained by Lipschitz conditions for coefficients. It means that there is a unique solution for random partial differential equation. That is to say that the random dynamical system for an approximate equation in the framework of rough path coincides with the random dynamical system generated by a random partial differential equation. Moreover, we
concern the limit behavior of $\eta \to 0$ for approximate system, then we regard $W^\eta(\omega)$ as a smooth rough path. Since all moments of the approximated noise $W^\eta$ are uniform bounded, it guarantees that the sufficiently small and uniform covariance $q$ for $W$ and $W^\eta, \eta \in (0, 1]$ can be chosen. Thus, the existence of the $A_\eta$ can be obtained as $A$. Finally, we establish the convergence of the stopping times to get the convergence of absorbing sets, and then get the upper semi-continuity of $A_\eta$ as $[3, 28]$.

Compared with $[3, 28]$, due to uniform convergence of the approximated noise, the random dynamical system $\varphi^\eta$ with respect to $y^\eta$ converge to $\varphi$ with respect to $y$ is uniform for any $t \in [0, \infty)$. For this stationary and smooth approximation, our method can be used to rough noise not just Brownian motion.

The paper is organized as follows: In Section 2, we give some basic knowledge on rough path theory and random dynamical system. In Section 3, we consider the dynamics for non-autonomous dynamical system $\Phi$ which is generated by the solution of (1.1) on stopping times $\{T_i(W(\omega))\}_{i \in \mathbb{Z}}$ and $\varphi$ is generated by the solution of (1.1) on $\mathbb{R}$. Section 4 is devoted to the existence of random attractor under the random settings. In Section 5, we establish the upper semi-continuity for approximated attractors $A_\eta$. In an Appendix, we give some proofs of lemmas and theorems which are used in our paper.

2 Preliminaries

In this section, we will recall some basic concepts in rough path theory and random dynamical system.

Notation: Let $I$ be a compact interval on $\mathbb{R}$. Denote $C(I, V)$ and $C^\alpha(I, V)$ by the $V$-valued continuous function space and the space of $V$-valued $\alpha$-Hölder continuous function space respectively, where $V$ is a Banach space and $\alpha \in (0, 1)$. We use the notation $y_t$ instead of $y(t)$, and $y_{s,t} = y_t - y_s$. Constant $C$ different from line to line and the notion $C(x)$ to indicate the constant $C$ depends on $x$. The simplex $\triangle^n_{[0,T]} := \{0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n-1} \leq t_n \leq T\}$. $\delta \xi_{s,u,t} = \xi_{s,t} - \xi_{s,u} - \xi_{u,t}$ for $(s, u, t) \in \triangle^3$. The space $C^\alpha_2([0, T], B_\gamma)$ of all families such that its element $g : \triangle^2_{[0,T]} \to B_\gamma$ satisfying $\|g\|_{\alpha, \gamma} = \frac{\|g_{(s,t)}\|_{B_\gamma}}{|t-s|^\alpha} < \infty$ and the space $C^\alpha_3([0, T], B_\gamma)$ of all families such that its element $h : \triangle^3_{[0,T]} \to B_\gamma$ satisfying $\|h\|_{\alpha_1, \alpha_2, \gamma} = \frac{\|h_{(s,u,t)}\|_{B_\gamma}}{|t-u|^{\alpha_1}|u-s|^{\alpha_2}} < \infty$. 

5
2.1 Rough path and rough integral

**Definition 2.1** (\(\alpha\)-Hölder rough paths). Let \(\alpha \in (\frac{1}{3}, \frac{1}{2}]\), we call the couple \(W = (W, \mathbb{W})\) a \(d\)-dimensional \(\alpha\)-Hölder rough path on \(I\), if \(W \in C^\alpha(I, \mathbb{R}^d)\) and \(\mathbb{W} \in C^{2\alpha}(\Delta^2_I, \mathbb{R}^d \otimes \mathbb{R}^d)\), and satisfy the Chen’s identity
\[
W_{s,t} - W_{s,u} - W_{u,t} = W_{s,u} \otimes W_{u,t}, \quad s \leq u \leq t \in I.
\]

Let \(C^\alpha(I, \mathbb{R}^d)\) denote the set of all \(\alpha\)-Hölder rough paths. In addition, for any \(W \in C^1(I, \mathbb{R}^d)\), there is a canonical lift \(S(W) := (W, \mathbb{W})\) in \(C^\alpha(I, \mathbb{R}^d)\) defined as
\[
\mathbb{W}_{s,t}^{k,l} = \int_s^t \int_s^r dW^k_r dW^l_r, \quad s < t \in I \text{ and } k, l \in \{1, \ldots, d\}.
\]

We denote the geometric rough space by \(C^\alpha_g(I, \mathbb{R}^d)\), i.e. the closure of the canonical lift \(S(W), W \in C^1(I, \mathbb{R}^d)\).

**Remark 2.1.** The first component \(W\) of \(\alpha\)-Hölder rough path \(W\) is a path on interval \(I\) and it has an \(\alpha\)-Hölder regularity, and the second component is called second order process or Lévy area, and it satisfies an analytic condition, namely \(\sup_{(s,t) \in \Delta^2_I} \frac{|W_{s,t}|}{|t-s|^\alpha} < \infty\). More details on rough path theory we refer to [8].

For two different \(\alpha\)-Hölder rough paths we can compare their distance by defining a metric on rough path space \(C^\alpha(I, \mathbb{R}^d)\) as follows.

**Definition 2.2.** Let \(I \subset \mathbb{R}\) be a compact set and \(W, \tilde{W} \in C^\alpha(I, \mathbb{R}^d), \alpha \in (\frac{1}{3}, \frac{1}{2}]\), we give an (inhomogeneous) \(\alpha\)-Hölder rough path metric
\[
d_{\alpha,I}(W, \tilde{W}) = \sup_{(s,t) \in \Delta^2_I} \frac{|W_{s,t} - \tilde{W}_{s,t}|}{|t-s|}\alpha + \sup_{(s,t) \in \Delta^2_I} \frac{|W_{s,t} - \tilde{W}_{s,t}|}{|t-s|^{2\alpha}}.
\]

**Remark 2.2.** The rough path metric induces an inhomogeneous norm of rough paths,
\[
\|W\|_{\alpha,I} = \|\|W\|\|_{\alpha,I} + \|\|\mathbb{W}\|\|_{2\alpha, \Delta^2_I}.
\]

In order to keep with [18,21], we sometimes use the notation \(\rho_{\alpha,I}(X)\) instead of \(\|X\|_{\alpha,I}\) and more results on this topic can be found in [8].

In order to introduce the controlled rough path for a class of evolution equations, we need the following notion of interpolation spaces.
Definition 2.3 (A monotone family of interpolation spaces). We call a family of separable Banach spaces \((B_\gamma, \|\cdot\|_\gamma)_{\gamma \in \mathbb{R}}\) are monotone interpolation spaces if for \(\gamma_1 \leq \gamma_2\), the space \(B_{\gamma_2} \subset B_{\gamma_1}\) with dense and continuous embedding and have the following interpolation inequality holds for \(\gamma_1 \leq \gamma_2 \leq \gamma_3\) and \(x \in B_{\gamma_3}:
\]
\[\|x\|_{\gamma_2}^{\gamma_3 - \gamma_1} \leq C \|x\|_{\gamma_1}^{\gamma_3 - \gamma_2} \|x\|_{\gamma_3}^{\gamma_2 - \gamma_1}.
\]

Remark 2.3. The above monotone family of interpolation spaces have the following nice properties: If semigroup \(S : [0, T] \rightarrow \mathcal{L}(B_{\gamma_1}, B_{\gamma_1 + 1})\) is such that for each \(x \in B_{\gamma_1 + 1}\) and \(t \in [0, T]\) we have \(\|(S(t) - \text{Id})x\|_{\gamma} \leq C t \|x\|_{\gamma_1 + 1}\) and \(\|S(t)x\|_{\gamma_1 + 1} \leq C t^{-1} \|x\|_{\gamma}\), then for all \(\sigma \in [0, 1]\), we get \(S(t) \in \mathcal{L}(B_{\gamma_1 + \sigma}, B_{\gamma_1 + \sigma})\) and have the following estimates
\]
\[\|(S(t) - \text{Id})x\|_{\gamma} \leq C t^\sigma \|x\|_{\gamma_1 + \sigma},\]  
\[\|S(t)x\|_{\gamma_1 + \sigma} \leq C t^{-\sigma} \|x\|_{\gamma}.
\]

An example for these interpolation spaces is fractional Sobolev spaces (Bessel potential spaces) \(H^s, s \in \mathbb{R}\). The further results on interpolation spaces we refer to [22]. In the following deliberation, we stress that \(\alpha \in (\frac{1}{3}, \frac{1}{2})\) represents the time regularity and \(\gamma\) is the spatial regularity in \(B_\gamma\).

In the following section, we will consider the mild solution of (1.1)
\]
\[y_t = S(t)y_0 + \int_0^t S(t - r)F(y_r)dr + \int_0^t S(t - r)G(y_r)dW_r. \tag{2.3}
\]

It is vital to understand the last integral in the mild solution. To this end, we need to give a proper definition of controlled rough paths. Based on the monotone family of interpolation spaces, we introduce the concept of controlled rough paths which are similar to finite-dimensional case, and it reflects the interaction effect of regularity in time and space.

Definition 2.4 (Controlled rough paths). We call the pair \((y, y')\) is a controlled rough path if
\]
\[\bullet (y, y') \in C([0, T], B_\gamma) \times (C([0, T], B_{\gamma - \alpha}) \cap C^\alpha([0, T], B_{\gamma - 2\alpha})), \text{ the component } y' \text{ is also called Gubinelli derivative as finite-dimensional case.}
\]
\[\bullet (y, y') \text{ satisfies the following identity}
\]
\[R^y_{s,t} = y_{s,t} - y'_s W_{s,t}, \tag{2.4}
\]
where \(R^y_{s,t}\) is called the remainder and it belongs to \(C^\alpha(\Delta_{[0,T]}^2, B_{\gamma - \alpha}) \cap C^{2\alpha}(\Delta_{[0,T]}^2, B_{\gamma - 2\alpha}).\)
We denote the set of all controlled rough paths by $D_{W,\gamma}^{2\alpha}([0,T],B_{\gamma})$ and endowed with the norm as follows

$$
\|y, y'\|_{W,2\alpha,\gamma} = \|y\|_{\infty,\gamma} + \|y'\|_{\infty,\gamma-\alpha} + \|y'\|_{\alpha,\gamma-2\alpha} + \|y\|_{\alpha,\gamma-\alpha} + \|y\|_{2\alpha,\gamma-2\alpha}.
$$

**Remark 2.4.** $D_{W,\gamma}^{2\alpha}$ is a Banach space under the above norm. Furthermore, the $\alpha$-Hölder semi-norm for $y$ does not emerge here. Indeed, using (2.4) for $\theta \in \{\alpha,2\alpha\}$ we have that

$$
\|y\|_{\alpha,\gamma-\theta} \leq \|y'\|_{\infty,\gamma-\theta} \|W\|_{\alpha} + \|y\|_{\alpha,\gamma-\theta}.
$$

The following lemma gives the definition of rough integral and its’ estimate.

**Lemma 2.1** (Lemma 4.5, [15]). Let $W$ be an $\alpha$-Hölder rough path and $(y, y') \in D_{W,\gamma}^{2\alpha}$, then the rough integral

$$
\int_0^t S(t-r)y_r dW_r := \lim_{|\mathcal{P}(0,t)| \to 0} \sum_{[u,v] \in \mathcal{P}(0,t)} S(t-u) \left[ y_u W_{u,v} + y'_u W'_{u,v} \right]
$$

exists in $B_{\gamma-2\alpha}$, where $\mathcal{P}(0,t)$ is a partition of the interval $[0,t]$ and the limit is independent of the choice of the specific partition $\mathcal{P}(0,t)$. In addition, there is an estimate as follows

$$
\left| \int_s^t S(t-r)y_r dW_r - S(t-s)y_s W_{s,t} - S(t-s)y'_s W'_{s,t} \right|_{\gamma-2\alpha+\beta} \leq C_{\mathcal{P}(0,t)} \|y, y'\|_{X,2\alpha,\gamma,\mathcal{P}(0,t)} (t-s)^{3\alpha-\beta},
$$

for all $0 \leq \beta < 3\alpha$ and $0 \leq s < t \leq T$.

### 2.2 Non-autonomous and random dynamical system

The rough path theory in previous subsection provides a framework of pathwise. In this subsection, we give some basic concepts on non-autonomous and random dynamical systems.

Let $\mathbb{T} = \mathbb{R} \text{ or } \mathbb{Z}$, a flow of non-autonomous perturbations $(\Omega, \theta)$ as follows

$$
\theta : \mathbb{T} \times \Omega \to \Omega,
$$

$\theta_{t_1} \circ \theta_{t_2} = \theta_{t_1+t_2}$ for $t_1, t_2 \in \mathbb{T}$ and $\theta_0 = Id_{\Omega}$. Furthermore, we call a mapping $\varphi : \mathbb{T}^+ \times \Omega \times V \to V$ is a non-autonomous dynamical system, if

$$
\varphi(t + \tau, \omega, u_0) = \varphi(t, \theta_{\tau} \omega, \cdot) \circ \varphi(\tau, \omega, u_0),
$$
for \( t, \tau \in \mathbb{T}^+, \omega \in \Omega, u_0 \in V \), where \( V \) is a separable Banach space. For the random settings, we have the following definition.

**Definition 2.5 (Metric dynamical system)**. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, and the flow \( \theta : \mathbb{T} \times \Omega \to \Omega \) satisfies the following conditions:

1. \( \theta_0 = \text{Id}_{\Omega} \);
2. \( \theta_{t+s} = \theta_t \circ \theta_s \);
3. the mapping \( (t, \omega) \mapsto \theta_t \omega \) is \((\mathcal{B}(\mathbb{T}) \times \mathcal{F}, \mathcal{F})\)-measurable, where \( \mathcal{B}(\mathbb{T}) \) and \( \mathcal{F} \) are Borel \( \sigma \)-algebra generated by \( \mathbb{T} \) and \( \Omega \), respectively;
4. \( \theta_t : \Omega \to \Omega \) are \( \mathbb{P} \)-preserving transformations, namely \( \theta_t \mathbb{P} = \mathbb{P} \).

Then the quadruple \( (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{T}}) \) is called a metric dynamical system.

In following consideration, the Wiener shift \( \theta \) acts on a rough path, we need to extend the action of the \( \theta \) from the path \( W \) to rough path \( W \), namely, for any \( \alpha \)-Hölder rough path \( W = (W, \mathcal{W}) \) and \( \tau, s, t \in \mathbb{R} \), the Wiener shift can be changed to \( \theta_\tau W = (\theta_\tau W, \theta_\tau \mathcal{W}) \) by

\[
\theta_\tau W_t = W_{t+\tau} - W_\tau,
\]
\[
\theta_\tau \mathcal{W}_{s,t} = \mathcal{W}_{s+\tau, t+\tau}.
\]

In particular, let \( W_t(\omega) = \omega_t \). Since fBm \( W_t \) has a canonical lift almost surely, then

\[
\theta_\tau W_t(\omega) = W_t(\theta_\tau \omega) = W_{t+\tau}(\omega) - W_\tau(\omega),
\]
\[
\theta_\tau \mathcal{W}_{s,t}(\omega) = \mathcal{W}_{s+\tau, t+\tau}(\omega),
\]

it means that \( \theta_\tau W(\omega) = W(\theta_\tau \omega) \). According to the definition of the rough path, the new Wiener shift \( \theta \) act on a rough path is also a rough path, we give it in a lemma.

**Lemma 2.2 (Lemma 32, [17])**. Let \( T_1, T_2, \tau \in \mathbb{R} \) and \( W = (W, \mathcal{W}) \) be an \( \alpha \)-Hölder rough path on \([T_1, T_2]\) for \( \alpha \in (\frac{1}{3}, \frac{1}{2}) \), then \( \theta_\tau W \) is an \( \alpha \)-Hölder rough path on \([T_1 - \tau, T_2 - \tau]\).

In our paper, the GFBRP is a geometric rough path, it has a stationary and smooth approximation. So compared with Bailleul et al. [2], our sample space is an \( \alpha \)-Hölder path space rather than an \( \alpha \)-Hölder rough path space.
Definition 2.6. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{T}})$ be a metric dynamical system and $V$ be a separable Banach space. We call the mapping $\varphi$ continuous random dynamical system on $V$ if $\varphi$ has the following properties:

1. $\varphi(0, \omega, \cdot) = \text{Id}_V$ for all $\omega \in \Omega$;
2. $\varphi(t + \tau, \omega, x) = \varphi(t, \theta_\tau \omega, \varphi(\tau, \omega, x))$ for all $t, \tau \in \mathbb{T}^+$ and $x \in V$;
3. $\varphi(t, \omega, \cdot) : V \to V$ is continuous for all $t \in \mathbb{T}^+$ and $\omega \in \Omega$;
4. the mapping $\varphi : \mathbb{T}^+ \times \Omega \times V \to V$ is $(\mathcal{B}(\mathbb{T}^+) \otimes \mathcal{F} \otimes \mathcal{B}(V), \mathcal{B}(V))$-measurable.

In order to describe the attractors, we introduce some concepts.

Definition 2.7. Let $\mathcal{D}$ be consisted of a family of nonempty sets $(D(\omega))_{\omega \in \Omega}$, it has a property $P$ and if $\mathcal{D} = (D(\omega))_{\omega \in \Omega} \in \mathcal{D}$ and $\emptyset \neq D' = (D'(\omega))_{\omega \in \Omega} \subset \mathcal{D}$, then $D' \in \mathcal{D}$.

The property $P$: For a given a constant $\nu > 0$, we call $D$ backward exponential growth if there exist a mapping $\omega \to r(\omega) \in \mathbb{R}^+$ such that $\emptyset \neq D(\omega) \in \mathcal{D}^\nu$ is contained in a ball $B_V(0, r(\omega))$ in space $V$ with center 0 and radius $r(\omega)$, and
\[
\lim_{T \ni t \to -\infty} \frac{\log^+ r(\theta_t \omega)}{|t|} < \nu, \quad \forall \omega \in \Omega.
\]

We use notations $\mathcal{D}^\nu_{\mathbb{R}, V}$, $\mathcal{D}^\nu_{\mathbb{Z}, V}$ to stress different time sets $\mathbb{R}$ and $\mathbb{Z}$, respectively.

In random settings, $\emptyset \neq D = D(\omega) \subset V$ equipped with the measurability, namely $\text{dist}(v, D(\omega))$ is a random variable for $\omega \in \Omega, v \in V$, then $D$ is called a random set.

In addition, for a random set $D$, if there exists a random variable $r(\omega) \in \mathbb{R}^+$ such that $D(\omega) \in B_V(0, r(\omega))$ and
\[
\lim_{T \ni t \to \pm \infty} \frac{\log^+ r(\theta_t \omega)}{|t|} = 0, \quad \forall \omega \in \Omega,
\]
then we call the random set $D$ tempered or subexponential growth. We denote the set of random tempered sets by $\hat{\mathcal{D}}$.

Definition 2.8. A family $B = (B(\omega))_\omega \subset V$ is called pullback absorbing set for $\mathcal{D}$ if
\[
\varphi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)) \subset B(\omega), \quad t \geq T(D, \omega)
\]
for any $D \in \mathcal{D}$ and $\omega \in \Omega$, where $T(D, \omega)$ is called absorption time.

Definition 2.9. Let $A = \{A(\omega)\}$ be a random attractor for random dynamical system $\varphi$ if
\( \mathcal{A}(\omega) \) is a compact set for all \( \omega \in \Omega \).

- for any \( \omega \in \Omega \) and \( t \in \mathbb{T}^+ \),
  \[
  \varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t \omega).
  \]

- for any \( D \in \mathcal{D}, \omega \in \Omega \),
  \[
  \lim_{\mathbb{T}^+ \ni t \to \infty} \text{dist}(\varphi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)), \mathcal{A}(\omega)) \to 0.
  \]

We have the following lemma about the existence of pullback/random attractors (see [7, 24]):

**Lemma 2.3.** Let \( \varphi \) be a continuous nonautonomous dynamical system. If \( \varphi \) has a compact pullback absorbing set \( B \in \mathcal{D} \). Then nonautonomous dynamical system \( \varphi \) has a pullback attractor for \( \mathcal{D} \). Furthermore, if \( \varphi \) is a random dynamical system and has a compact pullback absorbing set \( B \) in \( \mathcal{D} \). Then \( \varphi \) has a unique random attractor \( \mathcal{A} = \{ \mathcal{A}(\omega) \} \) for \( \mathcal{D} \) as follows:

\[
\mathcal{A}(\omega) = \bigcap_{s \geq T(B, \omega)} \bigcup_{t \geq s} \varphi(t, \theta_{-t} \omega, B(\theta_{-t} \omega)), \quad \forall \omega \in \Omega.
\]

### 3 Evolution equations driven by an \( \alpha \)-Hölder geometric rough path

In this section, we consider the following rough evolution equations which are defined on the interval \([0, T] \):

\[
dy_t = (Ay_t + F(y_t)) \, dt + G(y_t) \, dW_t, \quad y(0) = y_0 \in \mathcal{B},
\]

where \( W \in C^\alpha_g([0, T], \mathbb{R}^d) \) and \( \mathcal{B} \) is a separable Banach space. In what follows we give some assumptions on equation (3.1):

(A) \(-A\) generates a \( C_0\)-analytical semigroup \( S(t) \) on \( \mathcal{B} \), and the Banach spaces \( \mathcal{B}_\gamma \subset \subset \mathcal{B}, \gamma > 0 \).

(F) The nonlinear drift term \( F : \mathcal{B}_\gamma \to \mathcal{B}_{\gamma - \delta} \) for \( \delta \in [0, 1) \) and it is global Lipschitz. Furthermore, \( \| F(0) \|_{\gamma - \delta} \) and Lipschitz constant \( l_F \) are less than a constant \( \mu \) which is enough small and will be determined later.
(G) Let \( \theta \in \{0, \alpha, 2\alpha\} \) and \( 0 \leq \sigma < \alpha \). The nonlinear diffusion coefficient \( G : \mathcal{B}_{\gamma-\theta} \to \mathcal{B}_{\gamma-\theta} \) is three times Fréchet differentiable with bounded derivatives, i.e. \( \| D^k G \|_{\mathcal{L}(\mathcal{B}_{\gamma-\theta}, \mathcal{B}_{\gamma-\theta})} < \infty \) for \( k \in \{1, 2, 3\} \) and the derivative of \( DG(\cdot)G(\cdot) : \mathcal{B}_{\gamma-\alpha} \to \mathcal{B}_{\gamma-2\alpha-\sigma} \) is also bounded. In addition, we require \( C_G \leq \mu \), where
\[
C_G = \max \left\{ \| D^k G \|_{\mathcal{L}(\mathcal{B}_{\gamma-\theta}, \mathcal{B}_{\gamma-\theta})}, k = 1, 2, 3, \| D(DG(\cdot)G(\cdot)) \|_{\mathcal{L}(\mathcal{B}_{\gamma-\alpha}, \mathcal{B}_{\gamma-2\alpha-\sigma})} \right\}.
\]

**Remark 3.1.** To obtain the compactness of the absorbing set, the condition (A) is necessary. For example, we can consider the operator \( A = \Delta - I \), \(-A\) is a strictly positive and symmetric operator and its inverse operator is compact on \( L^2 \), and the spaces \( \mathcal{B}_\gamma := \mathcal{D}(\mathcal{A})^\gamma \) with norm \( \| \cdot \| := \| (\mathcal{A})^\gamma \cdot \| \), it coincides with the fractional Sobolev space \( H^\gamma, \gamma \in \mathbb{R} \). Then \( \mathcal{B}_\gamma \) compactly embedded into \( \mathcal{B} \) for \( \gamma > 0 \). Thus, let \(-A\) be a strictly and symmetric operator with a compact inverse which is a generator of an analytic exponential decreasing semigroup \( S(t), t \geq 0 \) in our paper. Furthermore, the semigroup \( S(t) \) has the following properties
\[
\| S(t) \|_{\mathcal{L}(\mathcal{B}_\sigma, \mathcal{B}_{\gamma+\sigma})} \leq \frac{C}{t^\sigma} e^{-\lambda t} \quad \text{for} \quad \sigma \geq 0, \gamma \in \mathbb{R}, \quad t \in [0, T],
\]
\[
\| S(t) - I \|_{\mathcal{L}(\mathcal{B}_\sigma, \mathcal{B}_{\gamma})} \leq Ct^\sigma, \quad \text{for} \quad 1 \geq \sigma \geq 0, \gamma \in \mathbb{R}, \quad t \in [0, T],
\]
where the constant \( C > 0 \) depends on \( S \) and \( \lambda > 0 \) is the smallest eigenvalue of \(-A\).

The condition \( C_G \leq \mu \) seems to be strong. However, for the case \( C_G > \mu \), we transform the diffusion term part \( G(y) dW \) to \( \frac{\mu}{C_G} G(y) dW^\mu, W^\mu = \left( \frac{C_G}{\mu} W, \frac{C_G^2}{\mu^2} W \right) \), Lemma 2.1 shows that \( \int_s^t S(t-r)G(y_r) dW_r = \int_s^t S(t-r) \frac{\mu}{C_G} G(y) dW^\mu, s, t \in [0, T] \). Finally, we can choose \( G(y) = g(x)(-\Delta)^\sigma y \) as [15][18], where \( g \) is a smooth function.

Based on the above assumptions, we have the following global existence of the rough evolution equations.

**Theorem 3.1 (Theorem 3.9, [18]).** Let \( T > 0 \), under assumptions (A),(F),(G), \( W = (W, \bar{W}) \in \mathcal{C}^\alpha ([0, T]; \mathbb{R}^d), \alpha \in \left( \frac{1}{2}, \frac{1}{3} \right) \) and \( y_0 \in \mathcal{B}_\gamma, \gamma \geq 0 \). Then there exists a unique global solution \((y, G(y)) \in \mathcal{D}_{W, \gamma}^{2\alpha}(0, T) \) of (3.1).

**Remark 3.2.** [18] required \( \delta \in [2\alpha, 1) \), we don’t need this condition. Indeed, for \( \delta \in [0, 1) \), the estimate in Lemma 3.3 [18] as follows
\[
\left\| \int_0^t S(\cdot - s) F(y_s) ds, 0 \right\|_{W^{2\alpha, \gamma}} \lesssim T^{(1-\delta)(1-2\alpha)} (1 + \| y \|_{\infty, \gamma}),
\]
the Banach fixed point theorem and concatenation argument can be used to get the global solution as [18]. So we omit its’ proof.

12
We consider the mild solution for (3.4) as follows
\[ y_t = S(t)y_0 + \int_0^t S(t - r)F(y_r)dr + \int_0^t S(t - r)G(y_r)dW_r, \quad t \in [0, T]. \] (3.4)

3.1 Non-autonomous dynamical system associated to (3.4)

The system (3.4) can generate a continuous random dynamical system \( \varphi \), and it can be found in [18,21]. Thus, it also generates non-autonomous continuous dynamical system for a given sample space \( \Omega \) which satisfies every \( \omega \in \Omega \) has canonical lift, namely \( \omega \) can generate an \( \frac{1}{2} \geq H > \alpha' \)-Hölder geometric rough path \( W(\omega) = (W(\omega), \mathbb{W}(\omega)) \).

**Lemma 3.1.** Let \( y_0 \in \mathcal{B}_\gamma, \gamma \geq 0 \) and \( W(\omega) = (W(\omega), \mathbb{W}(\omega)) \) be a geometric \( \alpha' \)-rough path for every \( \omega \in \Omega \). Then the system (3.4) can generate a non-autonomous dynamical system \( \varphi \) on \( \mathbb{R}^+ \) with state space \( \mathcal{B}_\gamma \):
\[
\varphi : \mathbb{R}^+ \times \Omega \times \mathcal{B}_\gamma \to \mathcal{B}_\gamma,
\]
\[
\varphi(t, \omega, y_0) = S(t)y_0 + \int_0^t S(t - r)F(y_r)dr + \int_0^t S(t - r)G(y_r)dW_r(\omega). \] (3.5)

In the following section, we shall derive another discrete non-autonomous dynamical system on \( \mathbb{Z}^+ \) and construct a priori estimate of the solution. To this end, we introduce a sequence of stopping times to make the norm of rough path \( W(\omega) \) sufficiently small on each stopping time interval.

Firstly, for some \( \mu \in (0, 1), \alpha \in (0, \alpha') \) and every \( \omega \in \Omega \), we define the stopping times as follows
\[
T(W(\omega)) = \inf\{\tau > 0 : \|W(\omega)\|_{\alpha,[0,\tau]} + \mu\tau^{1-\alpha} \geq \mu\}, \quad (3.6)
\]
\[
\hat{T}(W(\omega)) = \sup\{\tau < 0 : \|W(\omega)\|_{\alpha,[\tau,0]} + \mu|\tau|^{1-\alpha} \geq \mu\}. \quad (3.7)
\]

**Lemma 3.2.** For all \( \omega \in \Omega \), stopping times \( T(W(\omega)) \) and \( \hat{T}(W(\omega)) \) have the following properties:

(i) \( T(W(\omega)), \hat{T}(W(\omega)) \in (0, 1] \);

(ii) \( T(W(\omega)) = -\hat{T}(\theta_{T(W(\omega))}W(\omega)), \quad \hat{T}(W(\omega)) = -T(\theta_{\hat{T}(W(\omega))}W(\omega)). \)

**Proof.** It is clear that \( T(W(\omega)), \hat{T}(W(\omega)) \leq 1 \), it can be easily obtained from the definition (3.6) and (3.7). Since \( W(\omega) \) is an \( \alpha' \)-Hölder geometric rough path for \( \omega \in \Omega \) and \( \alpha < \alpha' \), we have
\[
\|W(\omega)\|_{\alpha,[0,\tau]} \leq \|W(\omega)\|_{\alpha',[0,\tau]}(\tau^{\alpha'-\alpha} + \tau^{2(\alpha'-\alpha)}) \to 0, \quad \text{as} \quad \tau \to 0
\]
Hence, once proving the mapping $\tau \mapsto \|W(\omega)\|_{\alpha,[0,\tau]} + \mu \tau^{1-\alpha}$ is continuous and strictly increasing, then there is a unique $\tilde{\tau}_0$ such that $\|W(\omega)\|_{\alpha,[0,\tilde{\tau}_0]} + \mu \tilde{\tau}_0^{1-\alpha} = \mu$.

For fixed $\tau_0$, we define the following truncated function:

$$W_{s,\tau_0}(\omega) := \begin{cases} W_s(\omega), & s \leq \tau_0, \\ W_{\tau_0}(\omega), & s > \tau_0. \end{cases}$$

Then $W_{\tau_0}(\omega)$ has a canonical lift, since $W_{s,\tau_0}(\omega), s > \tau_0$ is smooth, using Chen's identity we have that

$$W_{s,t}(\omega) := \begin{cases} W_{s,t}(\omega), & s < t \leq \tau_0, \\ W_{s,\tau_0}(\omega), & s \leq \tau_0 \leq t, \\ 0, & \tau_0 \leq s < t. \end{cases}$$

Thus, for $\tau \geq \tau_0$,

$$
\begin{align*}
\|W(\omega)\|_{\alpha,[0,\tau]} - \|W(\omega)\|_{\alpha,[0,\tau_0]} & = \|W(\omega)\|_{\alpha,[0,\tau]} - \|W_{\tau_0}(\omega)\|_{\alpha,[0,\tau]} \\
& \leq \|W(\omega)\|_{\alpha,[\tau_0,\tau]} - \|W_{\tau_0}(\omega)\|_{\alpha,[\tau_0,\tau]} + \|W(\omega)\|_{2\alpha,[0,\tau]} - \|W_{\tau_0}(\omega)\|_{2\alpha,[0,\tau]} \\
& \leq \|W(\omega)\|_{\alpha,[\tau_0,\tau]} + \|W(\omega)\|_{\tau_0} - \|W_{\tau_0}(\omega)\|_{2\alpha,[0,\tau]} \\
& \leq (\tau - \tau_0)^{\alpha' - \alpha} \|W(\omega)\|_{\alpha',[\tau_0,\tau]} + \left(\tau - \tau_0\right)^{2\alpha' - 2\alpha} \|W(\omega)\|_{2\alpha',[\tau_0,\tau]} \\
& \quad + \|W(\omega)\|_{\alpha,[\tau_0,\tau]} \|W(\omega)\|_{\alpha',[\tau_0,\tau]} (\tau - \tau_0)^{\alpha' - \alpha} \left(\tau - \tau_0\right)^{2\alpha' - 2\alpha} \|W(\omega)\|_{2\alpha',[\tau_0,\tau]} \\
& \leq (\tau - \tau_0)^{2\alpha' - 2\alpha} \|W(\omega)\|_{2\alpha',[\tau_0,\tau]} + (\|W(\omega)\|_{\alpha,[0,\tau_0]} + 1) \|W(\omega)\|_{\alpha',[\tau_0,\tau]} (\tau - \tau_0)^{\alpha' - \alpha}.
\end{align*}
$$

Therefore, we have $\lim_{\tau \downarrow \tau_0} \|W(\omega)\|_{\alpha,[0,\tau]} = \|W(\omega)\|_{\alpha,[0,\tau_0]}$. Furthermore, we can obtain similar result for $\tau \uparrow \tau_0$. Thus, the mapping $\tau \mapsto \|W(\omega)\|_{\alpha,[0,\tau]} + \mu \tau^{1-\alpha}$ is strictly increasing and continuous with respect to $\tau$. And the mapping $\tau \mapsto \|W(\omega)\|_{\alpha,[0,\tau]} + \mu |\tau|^{1-\alpha}$ is strictly decreasing. Then we have that

$$
\begin{align*}
\mu & = \|W(\omega)\|_{\alpha,[0,T(W(\omega))]} + \mu \left(T(W(\omega))\right)^{1-\alpha} \\
& = \|\theta_{T(W(\omega))}W(\omega)\|_{\alpha,[T(\theta_{T(W(\omega))}\omega),0]} + \mu \left(-\theta_{T(W(\omega))}W(\omega))\right)^{1-\alpha}
\end{align*}
$$

if and only if $T(W(\omega)) = -\tilde{T}(\theta_{T(W(\omega))}W(\omega))$. It is not difficult to check that the similar property holds for $\tilde{T}(W(\omega))$. \qed
Furthermore, the stopping times have the following monotonicity.

**Lemma 3.3.** For any \( t_1 \leq t_2 \), we have

\[
t_1 + \hat{T}(\theta_{t_1} W(\omega)) \leq t_2 + \hat{T}(\theta_{t_2} W(\omega)).
\]

**Proof.** It’s proof is similar to \cite{10} Lemma 3.5, we omit it. \(\square\)

**Remark 3.3.** In particular, if the \( t_2 + \hat{T}(\theta_{t_2} W(\omega)) \leq t_1 \leq t_2 \). Then repeatedly use the formula in Lemma 3.3, we obtain the following property of stopping times.

\[
t_2 \geq t_1 \geq t_2 + \hat{T}(\theta_{t_2} W(\omega)) \geq t_1 + \hat{T}(\theta_{t_1} W(\omega)) \geq t_1 + \hat{T}(\theta_{t_1 + \hat{T}(\theta_{t_1} W(\omega))} W(\omega)) \geq t_2 + \hat{T}(\theta_{t_2} W(\omega)) + \hat{T}(\theta_{t_1} W(\omega)) W(\omega)) \geq \cdots.
\]

### 3.2 Global attractor for the non-autonomous dynamical system associated to (3.4)

We first establish a priori estimate of the solution to (3.4). To this end, for fixed \( \omega \in \Omega \), we define a sequence of stopping times \((T_i(W(\omega)))_{i \in \mathbb{Z}}\) as follows

\[
T_i(W(\omega)) = \begin{cases} 
0, & i = 0, \\
T_{i-1}(W(\omega)) + T(\theta_{T_{i-1}(W(\omega))} W(\omega)), & i \in \mathbb{N}^+, \\
T_{i+1}(W(\omega)) + \hat{T}(\theta_{T_{i+1}(W(\omega))} W(\omega)), & i \in \mathbb{N}^-.
\end{cases}
\] (3.8)

According to the definition of stopping times, we have the following cocycle property:

\[
T_0(W(\omega)) = 0, \quad T_j(W(\omega)) + T_i(\theta_{T_j(W(\omega))} W(\omega)) = T_{i+j}(W(\omega)), \forall i, j \in \mathbb{Z}.
\] (3.9)

Furthermore, \( T(W(\omega)) = T_1(W(\omega)) \) and \( \hat{T}(W(\omega)) = T_{-1}(W(\omega)).\)

**Lemma 3.4.** For all \( \omega \in \Omega \), let \((y, y')\) be the solution of (3.1) with respect to \( y_0 \in \mathcal{B}_\gamma, \gamma \geq 0 \) and a sequence of stopping times \((T_i(\theta_{T_j(W(\omega))} W(\omega))))_{i \in \mathbb{Z}} \) which are defined by (3.8). Then we have

\[
\|y, y'\|_{W^{2,\alpha,\gamma}[T_{i-1}(\theta_{T_j(W(\omega))} W(\omega)), T_i(\theta_{T_j(W(\omega))})]} = \cdots.
\]
\[
\leq C\mu \sum_{m=1}^{i-1} (1 + \|y, y'\|_{W_{2\alpha,\gamma}[T_{m-1}(\theta T_j W, T_m(\theta T_j W(\omega))])}) e^{-\lambda(T_{i-1}(\theta T_j W(\omega)) - T_m(\theta T_j W(\omega)))} + C\mu (1 + \|y, y'\|_{W_{2\alpha,\gamma}[T_{i-1}(\theta T_j W(\omega)), T_i(\theta T_j W(\omega))])} + Ce^{-\lambda T_{i-1}(\theta T_j W(\omega))} \|y_0\|_\gamma,
\]

where \( T_j = T_j(W(\omega)) \), and \( \lambda > 0 \) is the smallest eigenvalue of \(-A\), the constant \( C \) depends on \( S, \sigma, \alpha \).

The proof of this lemma can be found in the Appendix A. We can define a discrete non-autonomous dynamical system based on the cocycle property of the stopping times \( (T_i(W(\omega)))_{i \in \mathbb{Z}} \) with index set \( \mathbb{Z}^+ \) for \( \omega \in \Omega \), where \( \omega \) as a parameter. We introduce a new shift as follows

\[
\tilde{\theta} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z},
\]

\[
\tilde{\theta}_i, j = i + j \quad i, j \in \mathbb{Z}.
\]

Then we define

\[
\Phi : \mathbb{Z}^+ \times \mathbb{Z} \times \Omega \times \mathcal{B} \rightarrow \mathcal{B},
\]

\[
\Phi(i, j, \omega, y_0) = S(T_i(\theta T_j(W(\omega)))W(\omega)))y_0
\]

\[
+ \int_0^{T_i(\theta T_j(W(\omega)))W(\omega))} S(T_i(\theta T_j(W(\omega)))W(\omega)) - r)F(y_r)dr
\]

\[
+ \int_0^{T_i(\theta T_j(W(\omega)))W(\omega))} S(T_i(\theta T_j(W(\omega)))W(\omega)) - r)G(y_r)d\theta T_j(W(\omega))W_r(\omega)
\]

\[
= \varphi(T_i(\theta T_j(W(\omega)))W(\omega)), \theta T_j(W(\omega))\omega, y_0).
\]

Note that \( \Phi(i, j, \omega, y_0) \) is the solution of \( 3.2 \) driven by rough noise \( \theta T_j(W(\omega))W(\omega) \) at time \( T_i(\theta T_j(W(\omega))W(\omega)) \). In order to construct the absorbing set of the discrete non-autonomous dynamical system \( \Phi \), we need a discrete Grönwall lemma, its proof can be found in [10].

**Lemma 3.5** (Lemma 3.6, [10]). Assume \( \lambda^*, \nu_0, k_0, k_1 < 1, k_2 \) are positive constants and \( \{t_i\}_{i \in \mathbb{Z}^+} \) is a sequence of positive constants with \( t_0 = 0 \), satisfy

\[
t_{i-1} - t_{i-2} \leq -\frac{2}{\lambda^*} \log k_1, \quad \forall i \geq 2.
\]

Suppose that the sequence of positive constants \( \{U_i\}_{i \in \mathbb{N}} \) have the following relation:

\[
U_i \leq k_0\nu_0e^{-\lambda t_{i-1}} + \sum_{m=1}^{i-1} k_1U_m e^{-\lambda (t_{i-1} - t_m)}
\]

16
\[
+ \sum_{m=1}^{i-1} k_2 e^{-\lambda (t_{i-1} - t_m)} + k_2, \quad i = 1, 2, 3, \ldots .
\]

(3.10)

Then \( U_i \) have the following estimates

\[
U_i \leq (k_0 v_0 + k_2) (1 + k_1)^{i-1} e^{-\frac{\lambda}{2} T_{i-1}}
+ \sum_{m=1}^{i-1} 2k_2 (1 + k_1)^{i-1-m} e^{-\frac{\lambda}{2} (t_{i-1} - t_m)}, \quad \forall i = 1, 2, 3, \ldots .
\]

(3.11)

We can specify coefficients which emerge in (3.10). Let \( \mu > 0 \) be a sufficiently small constant such that

\[
k_1(\mu) = \frac{C}{1 - C\mu} < 1 \text{ and } -\frac{2}{\lambda^*} \log k_1(\mu) > 1,
\]

(3.12)

where constant \( C \) emerges in the inequality of Lemma 3.4. Furthermore, choosing

\[
\lambda^* = \lambda, k_0 = \frac{C}{1 - C\mu}, k_2 = k_1(\mu), t_i = T_i(\theta_{T_j W(\omega)} W(\omega)).
\]

Based on the above lemmas we have the following corollary.

**Corollary 3.1.** Let \( y_0 \in \mathcal{B} \) and the constant \( \mu \) satisfies (3.12). Then the discrete non-autonomous dynamical system \( \Phi \) has the following estimate

\[
\| \Phi(i, j, \omega, y_0) \| \leq (1 + k_1)^{i-1} e^{-\frac{\lambda}{2} T_{i-1}} (\theta_{T_j W(\omega)} W(\omega))(k_0 \| y_0 \| + k_2)
+ \sum_{m=1}^{i-1} 2k_2 (1 + k_1)^{i-1-m} e^{-\frac{\lambda}{2} (T_{i-1} - T_{i-m}) (\theta_{T_j W(\omega)} W(\omega) - T_{i-m} (\theta_{T_j W(\omega)} W(\omega)))}.\]

In order to construct the absorbing set of the non-autonomous dynamical system \( \Phi \), we impose the smallness condition for \( \omega \in \Omega \). Firstly, we assume that the stopping times satisfy

\[
1 > \liminf_{i \to -\infty} \frac{|T_i(W(\omega))|}{|i|} \geq d_1 > \frac{2(\log(1 + k_1(\mu)) + \nu)}{\lambda},
\]

(3.13)

where \( \nu \) is parameter of \( \mathcal{D}_{Z, \mathcal{B}}^\nu \) and \( \nu \in [0, \frac{d_1}{2}) \). We also need

\[
\nu + d_1 > 1.
\]

(3.14)

Finally, the subexponential growth condition is imposed on the sequence \( \{ |T(\theta_{T_i W(\omega)} W(\omega))|^{-\gamma} \}_{i \in \mathbb{Z}} \) for \( \omega \in \Omega, \gamma \in (0, (\alpha - \sigma) \wedge (1 - \delta)], \) namely

\[
\lim_{i \to -\infty} \log^+ (|T(\theta_{T_i W(\omega)} W(\omega))|^{-\gamma}) \geq 0.
\]

(3.15)
In the concrete example (fractional Brownian rough path), we shall check these conditions.

For the discrete non-autonomous dynamical system $\Phi$, we consider $D_{\mathbb{Z},B}^\nu$ which element $\{D(i)\}_{i \in \mathbb{Z}} \subset B$ included in a ball with center 0 and radius $r(i)$ and satisfy

$$\limsup_{i \to -\infty} \frac{\log^+ r(i)}{|i|} < \nu.$$

Next, we shall construct an absorbing set $B \in D_{\mathbb{Z},B}^\nu$. Thus, our first step is to determine the absorbing set $B$ with respect to $\Phi$.

**Lemma 3.6.** Assume that (3.12), (3.13), (3.14), (3.15) hold. Then there exists a $D_{\mathbb{Z},B}^\nu$-absorbing set $B$ with center 0 and radius

$$R(i, \omega) = 2 \sum_{m=-\infty}^{0} 2k_2(1 + k_1)^{-m} e^{-\frac{1}{2}T_{m}(\theta_{T_{i-1}(W(\omega))}W(\omega))}.$$  \hfill (3.16)

**Proof.** For any $D \in D_{\mathbb{Z},B}^\nu$, namely the set $D(j, \omega) \in D_{\mathbb{Z},B}^\nu$ and there is a sequence $\{B_B(0, \rho(i, \omega))\}_{i \in \mathbb{Z}} \in D_{\mathbb{Z},B}^\nu$ such that $D(i, \omega) \subset B_B(0, \rho(i, \omega))$. In terms of Corollary 3.1 we know that

$$\sup_{y_0 \in D(j-i, \omega)} \|\Phi(i, j - i, \omega, y_0)\| \leq (1 + k_1)^{i-1} e^{-\frac{1}{2}T_{i-1}(\theta_{T_{j-i}(W(\omega))}W(\omega))} (k_0 \rho(j - i, \omega) + k_2) + \sum_{m=1}^{i-1} 2k_2(1 + k_1)^{i-1-m} e^{-\frac{1}{2}(T_{i-1}(\theta_{T_{j-i}(W(\omega))}W(\omega)) - T_{m}(\theta_{T_{j-i}(W(\omega))}W(\omega)))}. \hfill (3.17)$$

Using the cocycle property of stopping times

$$T_{i-1}(\theta_{T_{j-i}(W(\omega))}W(\omega)) - T_{m}(\theta_{T_{j-i}(W(\omega))}W(\omega)) = T_{i-m-1}(\theta_{T_{j-i+m}(W(\omega))}W(\omega)) = -T_{i+m+1}(\theta_{T_{j-i}(W(\omega))}W(\omega)),$$

then we have

$$\sup_{y_0 \in D(j-i, \omega)} \|\Phi(i, j - i, \omega, y_0)\| \leq (1 + k_1)^{i-1} e^{-\frac{1}{2}T_{i-1}(\theta_{T_{j-i}(W(\omega))}W(\omega))} (k_0 \rho(j - i, \omega) + k_2) + \sum_{m=2-i}^{0} 2k_2(1 + k_1)^{i-1-m} e^{-\frac{1}{2}T_{m}(\theta_{T_{j-i}(W(\omega))}W(\omega))}. \hfill (3.18)$$

The second inequality of (3.13) means that for any $\epsilon > 0$ there exists a number $m(\epsilon) > 0$ such that for $|m| > m(\epsilon)$, we have $|T_{m}(\theta_{T_{j-i}(W(\omega))}W(\omega))| > (d_1 - \epsilon)|m|$. Thus, by the
third inequality of (3.13) and the property of the backward \( \nu \)-exponentially growing of \( \rho(j - i, \omega) \), then

\[
(1 + k_1)^{i-1}e^{\frac{1}{2}T_{i+1}(\theta_{T_{j-1}}(w(\omega))W(\omega))}(k_0\rho(j - i, \omega) + k_2)
\]

\[
\leq C(k_0, k_2)e^{\nu(j-1)e^{(i-1)(\log(1+k_1)+\frac{1}{2}(d_1-\epsilon)+\nu)}}.
\]

Hence, the first term of (3.18) converges to zero as \( i \to \infty \). Similarly, for the second term of (3.18), choosing \( \epsilon \) sufficiently small, we have

\[
(1 + k_1)^{-m}e^{\frac{1}{2}T_{m}(\theta_{T_{j-1}}(w(\omega))W(\omega))} = e^{-m\log(1+k_1)+\frac{T_{m}(\theta_{T_{j-1}}(w(\omega))W(\omega))}{2}}
\]

\[
\leq e^{m(-\log(1+k_1)+\frac{1}{2}(d_1-\epsilon))} \leq e^{m\nu}, \quad m < 0.
\]

Then the second term of (3.18) converges to \( \frac{R(j)}{2} \) as \( i \to \infty \). The attraction of the \( B \) can be easily obtain by the cocycle property of the \( \Phi \). Indeed,

\[
\|\Phi(i + 1, j - i - 1, \omega, y_0)\| = \|\Phi(i, j - i, \omega, \Phi(1, j - i - 1, \omega, y_0))\|.
\]

Thus,

\[
\sup_{y_0 \in D(j-i-1)} \|\Phi(i + 1, j - i - 1, \omega, y_0)\| \leq \sup_{y'_0 \in D(j-i)} \|\Phi(i, j - i, \omega, y'_0)\| < R(j)
\]

for sufficiently enough \( i \in \mathbb{Z}^+ \).

Lemma 3.6 shows that a absorbing set \( B \) in \( \mathcal{B} \). In order to construct pullback attractor for \( \Phi \), we need to illustrate \( B \in \mathcal{D}_0^\nu \mathcal{B} \). We give this fact by the following lemma and its proof can be found in [10 Lemma 3.10].

**Lemma 3.7.** The absorbing sets \( B(i, \omega) \) which are constructed in Lemma 3.6 belong to \( \mathcal{D}_0^\nu \mathcal{B} \).

Based on the above considerations, we construct the pullback attractor of the discrete non-autonomous dynamical system \( \Phi \).

**Theorem 3.2.** Assume \( \omega \in \Omega \) which satisfy the assumptions (3.12), (3.13), (3.14), (3.15). Then the discrete non-autonomous dynamical system \( \Phi(\cdot, \omega) \) has a pullback attractor \( \{A(i, \omega)\}_{i \in \mathbb{Z}} \) on \( \mathcal{D}_0^\nu \mathcal{B} \).

**Proof.** We give the outline of the proof, according to Lemma 2.3 we need to illustrate \( \Phi \) for initial data has continuously dependence(Section 5), and we need to check the
existence of the compact pullback absorbing set. In terms of the absorbing set $B(j, \omega) \in D^\nu_{\mathbb{Z}, B}$, let $T^*(j)$ is absorbing time of $B(j, \omega)$, then $\Phi(T^* - T^* + j - 1, \omega, B(-T^* + j - 1, \omega))$ is also an absorbing set. In order to get the compactness of absorbing sets. Firstly, for any $y_0 \in D(j - 1, \omega) \in D^\nu_{\mathbb{Z}, B}$, the solutions $y_{T^j(\theta_{T-1}W(\omega))}W(\omega) = \Phi(1, j - 1, \omega, y_0)$ have more nicely regularity. Indeed, similar to the computation in Appendix A, we have the following estimate

$$
\|\Phi(1, j - 1, \omega, y_0)\|_\gamma \leq \frac{C}{|T^j(\theta_{T-1}W(\omega))W(\omega)|^\gamma} \|y_0\|
+ C\mu (1 + \|y, y'\|_{W, 2, 0, [0, T^j(\theta_{T-1}W(\omega))W(\omega)]}),
$$

where we have to add the condition $\gamma < (1 - \delta) \land (\alpha - \sigma)$ to guarantee that above inequality holds. In addition, let $i = 1, \gamma = 0$ in Lemma 3.4, then

$$
\|y, y'\|_{W, 2, 0, [0, T^j(\theta_{T-1}W(\omega))W(\omega)]} \leq \frac{C}{1 - C\mu} \|y_0\| + \frac{C\mu}{1 - C\mu}.
$$

Thus, (3.15) shows that $\Phi(1, j - 1, \omega, D(j - 1, \omega)) \in D^\nu_{\mathbb{Z}, B}$. Then we can construct a compact pullback absorbing set $C(j, \omega)$ as follows

$$
C(j, \omega) := \Phi(1, j - 1, \omega, \Phi(T^* - T^* + j - 1, \omega, B(-T^* + j - 1, \omega))).
$$

Note that the compactness of $C(j, \omega)$ can be get from $B_\gamma$ compactly embedded into $B$ and $C(j, \omega)$ is bounded in $B_\gamma$. So

$$
\mathcal{A}(i, \omega) = \cap_{i' \in \mathbb{Z}^+} \cup_{j \geq i'} \Phi(j, i - j, \omega, C(i - j, \omega)), \quad i \in \mathbb{Z}.
$$

3.3 The absorbing set for non-autonomous dynamical system $\varphi$

We shall use the above results to study the dynamical behavior of the non-autonomous dynamical system $\varphi$ in this subsection. Different from the previous subsection, we consider the sample ‘$\omega$’ instead of ‘$i$’ for the continuous dynamical system $\varphi$. Thus, we will try to prove that $\mathcal{A}(\omega) = A(0, \omega)$ is an invariant attracting set in $D^0_{\mathbb{R}, B}$ with respect to $\varphi$, where we say $D(\omega) \in D^0_{\mathbb{R}, B}$ for each $\omega \in \Omega$, if there exists a radius $r(\omega)$ such that $D(\omega) \subset B(0, r(\omega))$ and

$$
\limsup_{\mathbb{R} \ni t \to -\infty} \frac{\log^+ r(\theta_{t\omega})}{|t|} = 0.
$$
We construct the following sets

\[ G(i, \omega) = \bigcup_{t \in (T_{i-1}(W(\omega), T_i(W(\omega)))} D(\theta_i\omega) \]

for \( D \in \mathcal{D}^0_{R,B} \) and \( i \in \mathbb{Z}, \omega \in \Omega \), then \( G(i, \omega) \in \mathcal{D}^0_{Z,B}. \) Indeed, if not, then there exists \( \omega \in \Omega \) and subsequence \( i' := \{i'(i, \omega)\}_{i \in \mathbb{Z}}, i' \in \mathbb{Z}, t' \in (T_{i'-1}(W(\omega)), T_{i'}(W(\omega))) \) such that

\[ \limsup_{i' \to -\infty} \frac{\log^+ \|y_{i'}\|}{|i'|} > 0, \]

for any \( y_{i'} \in D(\theta_{i'}\omega). \) However, we have

\[ \limsup_{i' \to -\infty} \frac{\log^+ \|y_{i'}\|}{|i'|} \leq \limsup_{i' \to -\infty} \frac{\log^+ \|y_{i'}\|}{|T_{i'-1}|} \limsup_{i' \to -\infty} \left( \frac{T_{i'-1}}{|i'|} \right) \]

\[ \leq \limsup_{i' \to -\infty} \frac{\log^+ \|y_{i'}\|}{|T_{i'}|} \limsup_{i' \to -\infty} \left( \frac{T_{i'-1}}{|i'|} \right) = 0, \]

where the last equality we use the fact \( y_{i'} \in D(\theta_{i'}\omega) \) and \( \frac{T_{i'-1}}{|i'|} \leq \frac{|i'-1|}{|i'|} < 2. \)

Before proving that \( \mathcal{A}(\omega) \) is an invariant attracting set of continuous dynamical system \( \varphi \), we need the following lemma.

**Lemma 3.8.** Let \( D \in \mathcal{D}^0_{R,B} \), then the set

\[ E_D(i, \omega) = \bigcup_{t \in [-T(\theta_{T_i}(W(\omega)), W(\omega)))} \bigcup_{t \in D(\theta_t \theta_{T_i}(W(\omega)))W(\omega))} \varphi(t, \theta_{-t \theta_{T_i}(W(\omega))}, y_0) \]

belongs to \( \mathcal{D}^0_{Z,B} \). Furthermore, we can define a set

\[ H_D(i, \omega) = \bigcup_{t \in [0, T(\theta_{T_i}(W(\omega)), W(\omega)))} \bigcup_{t \in D(\theta_{T_i}(W(\omega)))W(\omega))} \varphi(t, \theta_{T_i}(W(\omega))\omega, y_0) \]

in \( \mathcal{D}^0_{Z,B} \) for \( D \in \mathcal{D}^0_{R,B} \).

**Proof.** We first check \( E_D(i, \omega) \in \mathcal{D}^0_{Z,B} \) for \( D \in \mathcal{D}^0_{R,B} \). For any \( -t \in [T(\theta_{T_i}(W(\omega)), W(\omega)), 0] \) and \( y_0 \in D(\theta_{-t \theta_{T_i}(W(\omega))}W(\omega)) \), by the fact

\[ \|\theta_{-t \theta_{T_i}(W(\omega))}W(\omega))\|_{\alpha, [0, t]} \leq \|\theta_{T_i}(W(\omega))\|_{\alpha, [T(\theta_{T_i}(W(\omega)), W(\omega)), 0]} \leq \mu, \]

similar to Lemma A.1 and A.2 and A.4 and A.5 for \( \gamma = 0 \), we have

\[ \|\varphi(t, \theta_{-t \theta_{T_i}(W(\omega))}(\omega), y_0)\| \leq C\|y_0\| + C\mu(1 + \|y_{i'}\|_{W(2\alpha, 0), [0, T_{i'-1}(W(\omega)), W(\omega))}). \]
Furthermore, let $i = 1, \gamma = 0$ in Lemma 3.4 we get

$$\|y, y'\|_{W, 2, 0, [0, T_i(\theta_{T_{i-1}}(W(\omega)))]} \leq \frac{C}{1 - C\mu} \|y_0\| + \frac{C\mu}{1 - C\mu}. \quad (3.19)$$

Thus, we have

$$\|\varphi(t, \theta_{-t}T_i(W(\omega)), y_0)\| \leq \|y_0\| + C\mu \left( \frac{C}{1 - C\mu} \|y_0\| + \frac{C\mu}{1 - C\mu} + 1 \right). \quad (3.20)$$

In view of $y_0 \in D(\theta_{-t}T_i(W(\omega)))$, hence we obtain

$$\|\varphi(t, \theta_{-t}T_i(W(\omega)), y_0)\| \leq \sup_{y_0 \in G(i, \omega)} C\|y_0\| \left( 1 + \frac{C\mu}{1 - C\mu} \right) + C\mu \left( 1 + \frac{C\mu}{1 - C\mu} + 1 \right).$$

Then $E_D(i, \omega) \in D_{\omega, B}^0$. Similarly, we can prove that $H_D(i, \omega) \in D_{R, B}^\nu$. \qed

**Lemma 3.9.** If conditions in Theorem 3.2 hold and the family of sets $A(\omega) := A(0, \omega)$ which is defined in Theorem 3.2, then $A$ is attracting and invariant with respect to $\varphi$ in $D_{R, B}^\nu$.

**Proof.** Its’ proof is same as [10, Lemma 3.13], we omit it here. \qed

The next lemma shows that the absorbing set $A$ is not a attractor, since it is not an absorbing set on stopping times $\{T_i(W(\omega))\}_{i \in \mathbb{Z}}$.

**Lemma 3.10.** For each $\omega \in \Omega$, let

$$\hat{d} := \limsup_{i \to -\infty} \frac{|T_i(W(\omega))|}{|i|} \leq 1, \quad \tilde{d} := \liminf_{i \to -\infty} \frac{|T_i(W(\omega))|}{|i|} \geq d_1 > 0$$

hold, then we have

- $A \in D_{\omega, B}^\nu / \hat{d}$.
- If $\hat{d} > \tilde{d}$ and $\nu > 0$, then there exists a sample $\omega \in \Omega$ and $D \in D_{R, B}^{\nu / \hat{d}}$ such that the range of mapping $t \to D(\theta_{i}W)$ for $t = T_i(W(\omega)), i \in \mathbb{Z}$ in $D_{\omega, B}^{\nu / \hat{d}}$ but not in $D_{R, B}^{\nu}$;
- Similarly, if $D \in D_{R, B}^0$, then the range of mapping $t \to D(\theta_{i}W)$ for $t = T_i(W(\omega)), i \in \mathbb{Z}$ in $D_{\omega, B}^0$.

**Proof.** For the first conclusion, in view of $A(i, \omega) \in D_{\omega, B}^\nu$. Then we have

$$\limsup_{i \to -\infty} \frac{\log^+(\sup_{y \in A(i, \omega)} |y|)}{|T_i(W(\omega))|} \leq \limsup_{i \to -\infty} \frac{\log^+(\sup_{y \in A(i, \omega)} |y|)}{|i|} \leq \frac{i}{\limsup_{i \to -\infty} |T_i(W(\omega))|} \leq \nu / \tilde{d}.$$
Furthermore, for any $t < 0$, there exists an $i(\omega) \in \mathbb{Z}$ such that $t \in (T_{i-1}(W(\omega)), T_i(W(\omega))]$ and $t = \tau + T_{i-1}(W(\omega))$. Then we have

$$
\limsup_{i \to -\infty} \frac{\log^+ \sup_{y \in \mathcal{A}(i-1,\omega)} \| \varphi(\tau, \theta_{T_{i-1}}(W(\omega)), y) \|}{|i|} \leq \lim_{i \to -\infty} \frac{\log^+ \sup_{y \in \mathcal{A}(i-1,\omega)} \| \varphi(\tau, \theta_{T_{i-1}}(W)\omega, y) \|}{T_{i-1}(W(\omega))} \leq \frac{\nu}{\hat{d}},
$$

where we use the fact $\| \varphi(\tau, \theta_{T_{i-1}}(W(\omega)), y) \| \leq \sup_{y \in \mathcal{A}(i-1,\omega)} C \| y \| \left(1 + \frac{C\mu}{1 + C\mu^2} \right) + C\mu \left(1 + \frac{C\mu}{1 + C\mu^2} \right)$. Hence, $\mathcal{A} \in \mathcal{D}_{\mathbb{R}^d \mathbb{B}}$. The proof of the second property and the third property are similar, we only give the proof for the second conclusion. We can construct $D \in \mathcal{D}_{\mathbb{R}^d \mathbb{B}}$ such that $D(\theta_{T_{i'}}(W(\omega))) = \{0\}$ for $t \neq T_i(W(\omega))$, furthermore, for $t = T_{i'}(W(\omega))$, choosing $D(\theta_{T_{i'}}(W(\omega))) = \{y_{i'}(i,\omega)\}_{i \in \mathbb{Z}}$, where $i'$ is a subsequence of $i$ such that

$$
\limsup_{i \to -\infty} \frac{|T_i(W(\omega))|}{|i|} = \lim_{i' \to -\infty} \frac{|T_{i'}(W(\omega))|}{|i'|},
$$

and for a sufficiently small $\epsilon > 0$,

$$
\lim_{i' \to -\infty} \frac{|y_{i'}|}{T_{i'}(W(\omega))} = \frac{\nu}{\hat{d}} - \epsilon.
$$

Then, it is easy to check

$$
\limsup_{i \to -\infty} \frac{\log^+(\sup_{y \in D(i,\omega)} |y|)}{|i|} = (\frac{\nu}{\hat{d}} - \epsilon)\hat{d} \leq \nu \hat{d}/\hat{d}.
$$

4 Random attractors for RPDEs driven by GFBRP

In this section, we consider the random settings for RPDEs, namely the sample space $\Omega$ equipped with a measurable structure. Firstly, we introduce a metric dynamical system to describe the evolution of the noise. In addition, we shall prove that the non-autonomous dynamical system $\varphi$ is a random dynamical system under the random settings and it has a random attractor. To this end, we first define the metric dynamical system. fBm has the following form: we call a stochastic process $W(t)$ a $d$-dimensional fBm with Hurst index $H \in (0,1)$, if each component $W^i(t), i = 1, \cdots, d$ is an independent, identically distributed, centered Gaussian process and has the following covariance

$$
R(W^i(t), W^j(s)) := E(W^i(t)W^j(s)) = \frac{\beta^2}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H})
$$

23
for \( t, s \in \mathbb{R} \), and we require \( q > 0 \) sufficiently small in following settings. Firstly, we define a quadruple \((C_0(\mathbb{R}, \mathbb{R}^d), \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}^d)), \mathbb{P}, \theta)\), where \( C_0(\mathbb{R}, \mathbb{R}^d) \) denotes the set of all \( \mathbb{R}^d \)-values continuous functions which elements are zero at zero, it is endowed with the compact open topology and \( \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}^d)) \) is a Borel \( \sigma \)-algebra which is generated by \( C_0(\mathbb{R}, \mathbb{R}^d) \). \( \mathbb{P} \) is the law of fBm and \( \theta \) is the Wiener shift which is introduced in previous section. Then the classical results \([14, 23]\) tell us that \((\Omega_1, \mathcal{G}_0, \mathbb{P}, \theta)\) is an ergodic metric dynamical system.

Secondly, according to Kolmogorov’s continuous criterion, there exists a full measure set \( \Omega_1 \subset C_0(\mathbb{R}, \mathbb{R}^d) \) such that its’ elements have \( \alpha’ \)-Hölder continuous paths for every \( \frac{1}{2} < \alpha < \alpha’ < H \leq \frac{1}{2} \) on any interval \([-T, T]\). In particular, \( \Omega_1 \in \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}^d)) \) and it is \((\theta)_t \in \mathbb{R} \)-invariant (see \([3]\)). Furthermore, by Corollary \([5, \text{Section 5}]\), there exists a \((\theta)_t \in \mathbb{R} \)-invariant full set \( \Omega_2 \in \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}^d)) \) such that each \( \omega \in \Omega_2 \) has the canonical lift, namely it’s second order process as the limit of the canonical lift \( C^1 \)-smooth paths, we call \( S \) a canonical lift mapping, if \( S : C^1(\mathbb{R}, \mathbb{R}^d) \rightarrow G^2(\mathbb{R}^d), S(x) = \left(1, x_s t, \int_s^t x_{s r} d x_r\right) \), where \( G^2(\mathbb{R}^d) \) is the free step-2 nilpotent group over \( \mathbb{R}^d \)(see \([8, \text{Page 23}]\)).

Thirdly, according to the denseness property and countability of rational numbers, the continuity of \( \|W^n(\omega)\|_{\alpha’} \) for \( \eta \) and ergodic theorem \([11, \text{p538}]\), there exists a \((\theta)_t \in \mathbb{R} \)-invariant full set \( \Omega_3 \in \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}^d)) \) such that

\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \left( \sup_{r \in [0, 1]} \frac{\|\theta_{t+r}W(\omega)\|_{\alpha’[-1, t]} + \mu}{\mu} \right)^{\frac{1}{\alpha’ - \eta}} ds = E_\mathbb{P} \left( \sup_{r \in [0, 1]} \frac{\|\theta_{t+r}W\|_{\alpha’[-1, 0]} + \mu}{\mu} \right)^{\frac{1}{\alpha’ - \eta}} := d_1^{-1} < \infty, \tag{4.1}
\]

and for any \( \eta \in (0, 1] \), there exists a \( d^0 \) such that \((4.1)\) holds for \( W^\eta \) which is defined in Section 5, it approximates to \( W \). Moreover, due to \([4, \text{Lemma 3.3}]\), \( d^0 > 0 \) have a uniform lower bound for given \( q \). And under the assumption \( q \) sufficiently small, \( d_1 \) and \( d^0 \) shall close sufficiently to 1.

Finally, since GFBP \( W \) and \( W^\eta \) can be extended to any finite interval \([0, T]\), based on the \([4, \text{Theorem 3.1}]\), we know that \( \|W\|_{\alpha’[-t, t]} \) and \( \|W^\eta\|_{\alpha’[-t, t]} \) have any finite order moment, then by Proposition 4.13 \([1]\) and the similar method in step 3, we have a \((\theta)_t \in \mathbb{R} \)-invariant set \( \Omega_4 \in \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}^d)) \) of full measure such that the mapping \( t \rightarrow \sup_{r \in [0, 1]} \|\theta_{t+r}W(\omega)\|_{\alpha’[-1, 0]} \) and \( \sup_{r \in [0, 1]} \|\theta_{t+r}W^\eta(\omega)\|_{\alpha’[-1, 0]} \) are sublinear growth. Hence, let \( \Omega = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4 \), \( \mathcal{F} \) is a \( \sigma \)-algebra of \( \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}^d)) \) with respect to \( \Omega \), the measure \( \mathbb{P} \) is the restriction of the distribution of fBm over \( \mathcal{F} \).
Hence, we construct a new ergodic metric dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, \theta_t)\).

Now, we shall to illustrate that the non-autonomous dynamical system \(\varphi\) is a random dynamical system in random settings.

**Lemma 4.1.** The non-autonomous dynamical system \(\varphi\) is a random dynamical system and the stopping times \(T(W)\) and \(\hat{T}(W)\) are measurable.

**Proof.** \(\varphi\) is random dynamical system, it has been proved in [18,21]. The measurability can be obtained easily, by the Wong-Zakai approximation of the noise in \(\alpha'-\text{H"{o}lder}\) rough path, we know that the mapping \(\omega \mapsto \|W(\omega)\|_{\alpha',[0,r]}\) is continuous, thus it is measurable. Then \(T(W)\) is measurable. Furthermore, the measurability of \(\hat{T}(W)\) can be obtained in the same way. \(\square\)

In order to construct the inequality of (3.13), we need the following lemma.

**Lemma 4.2.** Let random variable \(N(\omega) \in \mathbb{N}\) is the number of stopping times \((T_i(W(\omega)))_{i \in \mathbb{Z}}\) in \([-1,0]\). Then for every \(\omega \in \Omega\), we have

\[
N(\omega) \leq \left(\frac{\|W(\omega)\|_{\alpha',[-1,0]} + \mu}{\mu}\right)^{\frac{1}{\alpha - \alpha'}} , \quad \text{for} \quad \frac{1}{3} < \alpha < \alpha' < H \leq \frac{1}{2}.
\]

**Proof.** In view of \(W(\omega) \in C^\alpha_g\) and \(|t-s| \leq |\hat{T}(W(\omega))| \leq 1\), we have

\[
\mu = \sup_{\hat{T}(W(\omega)) \leq s < t \leq 0} \left(\frac{|W_t(\omega) - W_s(\omega)|}{|t-s|^\alpha} + \frac{|W_{s,t}(\omega)|}{|t-s|^{2\alpha}}\right) + \mu(-\hat{T}(W(\omega)))^{1-\alpha}
\]

\[
\leq \left(\sup_{\hat{T}(X) \leq s < t \leq 0} \left(\frac{|W_t(\omega) - W_s(\omega)|}{|t-s|^\alpha} + \frac{|W_{s,t}(\omega)|}{|t-s|^{2\alpha}}\right) + \mu(-\hat{T}(W(\omega)))^{1-\alpha'}\right) (-\hat{T}(W(\omega)))^{\alpha' - \alpha}
\]

\[
\leq (\|W(\omega)\|_{\alpha',[-1,0]} + \mu)(-\hat{T}(W(\omega)))^{\alpha' - \alpha}.
\]

Thus, we obtain

\[
|\hat{T}(W(\omega))| \geq \left(\frac{\mu}{\|W(\omega)\|_{\alpha',[-1,0]} + \mu}\right)^{\frac{1}{\alpha' - \alpha}}.
\]

Let \(i > 0\) be the largest integer such that \(T_{i-1}(W(\omega)) \leq -1\). Then, let \(N(\omega) = i + 1\) and using the property of cocycle for the stopping times and the above estimate, we can get

\[
1 \geq |T_{-N(\omega)}(W(\omega))| = \sum_{j=0}^{N(\omega)-1} |\hat{T}(\theta_{T_{-j}(W(\omega))} W(\omega))| \geq N(\omega) \left(\frac{\mu}{\|W(\omega)\|_{\alpha',[-1,0]} + \mu}\right)^{\frac{1}{\alpha' - \alpha}}.
\]

Hence, we complete the proof of this lemma. \(\square\)
Lemma 4.3. We have the following relation holds

\[ \liminf_{i \to -\infty} \frac{|T_i(W(\omega))|}{|i|} = \liminf_{i \to -\infty} \frac{T_i(W(\omega))}{i} \geq d_1 > 0 \]

for \( \omega \in \Omega \). Furthermore, \( |T(\theta T_i(W(\omega))W(\omega))|^{-\gamma}, \omega \in \Omega, \gamma \in (0, (\alpha - \sigma) \land (1 - \delta)] \) is subexponentially growing on \( \Omega \) for \( i \to -\infty \).

Proof. The proof of this lemma can be found in [10, Lemma 4.5]. \( \square \)

The constant \( \mu \) and \( \nu \) can be chosen as we determine \( d_1 \). Now we conclude the random attractor to finish this section, and its proof is same as [10, Theorem 4.6].

Theorem 4.1. The pullback attractor \( A \) which is stated in Lemma 3.9 for non-autonomous system \( \varphi \) is random attractor and it attracts random tempered sets in \( \hat{D} \).

Remark 4.1. According to [10, Lemma 2.1], \( \nu \)-exponentially growing random sets is also subexponentially growing. Thus, similar to the first item of Lemma 3.10, \( B(\omega) = B(0, \omega) \) is subexponentially growing on \( \mathbb{R} \). Furthermore, for any \( t > 0 \), we can find a \( i^*(\omega) \in \mathbb{Z}^- \) such that \( -t \in (T_{i^*-1}(W(\omega)), T_{i^*}(W(\omega))] \), then for any \( D \in D_{\mathbb{R}, B}^0 \), by the property of cocycle and stopping times we have

\[
\varphi(t, \theta_{-t}X, D(\theta_{-t}\omega))
= \varphi(T_{i^*}(\theta_{T_{i^*}}W(\omega)), \theta_{T_{i^*}}\omega, \varphi(t - T_{i^*}(\theta_{T_{i^*}}W(\omega)), \theta_{-t}\omega, D(\theta_{-t}\omega)))
= \varphi(T_{i^*}(\theta_{T_{i^*}}W(\omega)), \theta_{T_{i^*}}\omega, \varphi(t - T_{i^*}(\theta_{T_{i^*}}W(\omega)), \theta_{-t-T_{i^*}}\theta_{T_{i^*}}\omega, D(\theta_{-t-T_{i^*}}\theta_{T_{i^*}}\omega)))
= \Phi(-i^*, i^*, \omega, \varphi(t - T_{i^*}(\theta_{T_{i^*}}W(\omega)), \theta_{-t-T_{i^*}}\theta_{T_{i^*}}\omega, D(\theta_{-t-T_{i^*}}\theta_{T_{i^*}}\omega)))
\subset \Phi(-i^*, i^*, \omega, E_D(i^*, \omega)) \subset B(\omega).
\]

In addition, similar to Theorem 3.2, there exists a compact absorbing set

\[ C(\omega) := \Phi(1, -1, \omega, \Phi(T, -T - 1, \omega, B(\theta_{-T-1}\omega))) \subset B(\omega), \]

where \( T \) is the absorption time corresponding to \( B \). Random attractors \( A(\omega) \) are also given by

\[
A(\omega) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \varphi(t, \theta_{-t}\omega, C(\theta_{-t}\omega)) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)).
\]
5 Wong-Zakai approximation for evolution equation driven by GFBRP

5.1 Wong-Zakai approximation for GFBRP

In this subsection, we introduce a smooth and stationary approximation scheme. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be probability space which is defined in previous section. \((\theta_t)_{t \in \mathbb{R}}\) is a Wiener shift and \(\theta_t W(\omega) = W_{t+} - W_t = \omega_{t+} - \omega_t\) for any \(\omega \in \Omega\). As [25] and [26] did, for any \(\eta \in (0, 1)\), we define a random variable \(G_\eta : \Omega \rightarrow \mathbb{R}^d\)

\[ G_\eta(\omega) = \frac{1}{\eta} \omega_{\eta}. \]

Then we have

\[ G_\eta(\theta_t \omega) = \frac{1}{\eta} (\omega_{t+\eta} - \omega_t). \]

By the properties of fBm, \(G_\eta(\theta_t \omega)\) is a stochastic process with normal distribution and stationary increments. Let \(W^n(t, \omega) := \int_0^t G_\eta(\theta_s \omega) ds\). \(W^n(t, \omega)\) can be viewed as an approximation of the fractional Brownian motion. Furthermore, we denote by

\[ \mathbb{W}^n(\omega)_{s,t} := \int_s^t W^n(\cdot, \omega)_{s,r} \otimes dW^n(r, \omega) \]

as a smooth second order process, it is well defined as Riemann-Stieljes integral. In addition, we constructed the convergence between \(W^n(\omega) = (W^n(\cdot, \omega), \mathbb{W}^n(\omega))\) and \(W(\omega) = (W(\omega), \mathbb{W}(\omega))\) in [4].

Lemma 5.1 (Theorem 3.2, [4]). Let \(W = (W, \mathbb{W})\) be the canonical lift of the fractional Brownian motion and \(W^n = (W^n, \mathbb{W}^n)\) as the approximation of \(W\). Then we have

\[ \rho_{\alpha', [-T, T]}(W, W^n) \rightarrow 0, \quad \text{as} \quad \eta \rightarrow 0 \]

for any \(T > 0\), \(\alpha' \in \left( \frac{1}{3}, \frac{1}{2} \right)\). Furthermore, the convergence takes place for all \(\omega\) in an \(\theta\)-invariant set \(\Omega'\) of full measure.

Corollary 5.1. Let \(W = (W, \mathbb{W})\) be the canonical lift of the fractional Brownian motion and \(W^n = (W^n, \mathbb{W}^n)\) as the approximation of \(W\). Then we have

\[ \rho_{\alpha' \cdot [T_j(W), T_{j+1}(W)]}(W, W^n) \rightarrow 0, \quad \text{as} \quad \eta \rightarrow 0 \]

27
for any \( j \in \mathbb{Z}, \alpha' \in \left(\frac{1}{3}, \frac{1}{2}\right) \). Furthermore, the convergence takes place for all \( \omega \) in an \( \theta \)-invariant set \( \Omega'' \) of full measure and the convergence is uniform for \( j \in \mathbb{Z} \).

**Proof.** Firstly, similar to [41 Theorem 3.1] or [9 Theorem 4.5], we can prove that

\[
|\rho_{\alpha',[s,t]}(W,W^n)|_{L^\infty_{\mathbb{F}}} \leq C(q',\alpha'',H)\eta^{H-\alpha''} \quad \text{for any } [s,t] \subset [n,n+2], n \in \mathbb{Z},\]

where \( \alpha'' > \alpha' \) and \( q' \geq 2 \) such that \( \alpha'' - \frac{1}{q'} > \frac{1}{3} \). In particular, compared with [41] and [9], the constant \( C(q',\alpha'',H) \) is uniform for each \( n \in \mathbb{Z} \). Secondly, similar to [41 Theorem 3.2] or [9 Theorem 4.6], we have a sequence \( (\eta_n)_{n \in \mathbb{N}} \) that converges sufficiently fast to zero as \( i \to \infty \) such that \( \lim_{i \to \infty} \rho_{\alpha',[n,n+2]}(W,W^n) = 0 \) for every \( n \in \mathbb{Z} \) and \( \alpha' \in \left(\frac{1}{3}, \frac{1}{2}\right) \) and these convergence take place almost surely, in an \( \theta \)-invariant set \( \Omega'' \) of full measure. Furthermore, since we obtain the uniform estimate of \( |\rho_{\alpha',[s,t]}(W,W^n)|_{L^\infty_{\mathbb{F}}} \), then the convergence is uniform for \( n \in \mathbb{Z} \). Finally, for any \( \omega \in \Omega'' \) and \([T_j(W(\omega)),T_{j+1}(W(\omega))]\), \( j \in \mathbb{Z} \), there exists a \( n_j \in \mathbb{Z} \) such that \([T_j(W(\omega)),T_{j+1}(W(\omega))] \subset [n(j),n(j) + 2] \), similar to the pathwise analysis of [41 Theorem 3.2], we can prove that \( \lim_{\eta \to 0} \rho_{\alpha',[T_j(W(\omega)),T_{j+1}(W(\omega))]}(W^n, W^n) = 0 \), and based on the property of stopping times, i.e. \( |T_{j+1}(W(\omega)) - T_j(W(\omega))| \leq 1 \) and \( \|W(\omega)\|_{\alpha',[T_j(W(\omega)),T_{j+1}(W(\omega))]} \leq \mu \leq 1 \), this convergence is also uniform for \( j \in \mathbb{Z} \). Thus, for any \( \omega \in \Omega'' \), there exists a subsequence \( \{\eta_n\}_{n \in \mathbb{N}} \subset (0,1) \) as before, such that

\[
\rho_{\alpha',[T_j(W(\omega)),T_{j+1}(W(\omega))]}(W^n(\omega),W(\omega)) \leq \rho_{\alpha',[T_j(W(\omega)),T_{j+1}(W(\omega))]}(W^n(\omega),W^n(\omega)) + \rho_{\alpha',[T_j(W(\omega)),T_{j+1}(W(\omega))]}(W^n(\omega),W(\omega)).
\]

Due to \([T_j(W(\omega)),T_{j+1}(W(\omega))] \subset [n(j),n(j) + 2] \), we complete the proof. \( \square \)

### 5.2 Wong-Zakai approximation of the stopping times

**Lemma 5.2.** Assume the following conditions hold

\[
\|W^n(\omega)\|_{\alpha,[0,T(W^n(\omega))]} + \mu(T(W^n(\omega)))^{1-\alpha} = \mu,
\]

\[
\|W^n(\omega)\|_{\alpha,[T(W^n(\omega)),0]} + \mu(T(W^n(\omega)))^{1-\alpha} = \mu
\]

for all \( \omega \in \Omega \), then we have

\[
\lim_{\eta \to 0} T(W^n(\omega)) = T(W(\omega)), \quad \lim_{\eta \to 0} \hat{T}(W^n(\omega)) = \hat{T}(W(\omega)).
\]

**Proof.** Suppose \( \lim_{\eta \to 0} T(W^n(\omega)) \neq T(W(\omega)), \omega \in \Omega \). In view of \( |T(W^n(\omega))| \leq 1 \), then there exists a sequence \( \{\eta_n\}_{n \in \mathbb{N}} \) such that \( \eta_n \to 0, n \to \infty \) and

\[
\lim_{n \to \infty} T(W^{n}(\omega)) = \tau \neq T(W(\omega)).
\]
By Lemma 5.1 we have
\[
\lim_{n \to \infty} \| W^n(\omega) \|_{\alpha,[0,T]} = \| W(\omega) \|_{\alpha,[0,T]}.
\]
Then for any \( \epsilon > 0 \), there is an integer \( N_1(\tau, \epsilon) \) such that for \( n > N_1(\tau, \epsilon) \) we have
\[
\| W^n(\omega) \|_{\alpha,[0,T]} - \| W(\omega) \|_{\alpha,[0,T]} \leq \frac{\epsilon}{2}.
\]
According to the proof of the Lemma 3.2 we know that the mapping \( t \mapsto \| \cdot \|_{\alpha,[0,t]} \) is continuous, then we have
\[
\| W^n(\omega) \|_{\alpha,[0,T(W^n(\omega))]} - \| W^n(\omega) \|_{\alpha,[0,T]} \leq C|T(W^n(\omega)) - \tau|.
\]
Then for any \( \epsilon > 0 \), there is an integer \( N_2(\tau, \epsilon) \) such that for \( n > N_2(\tau, \epsilon) \) we have
\[
|T(W^n(\omega)) - \tau| \leq \frac{\epsilon}{2C}.
\]
Let \( N = \max\{N_1(\tau, \epsilon), N_2(\tau, \epsilon)\} \), thus for \( n > N \) we obtain
\[
\| W^n(\omega) \|_{\alpha,[0,T(W^n(\omega))]} - \| W(\omega) \|_{\alpha,[0,T]} \leq \| W^n(\omega) \|_{\alpha,[0,T(W^n(\omega))]} - \| W^n(\omega) \|_{\alpha,[0,T]} + \| W^n(\omega) \|_{\alpha,[0,T]} - \| W(\omega) \|_{\alpha,[0,T]} \leq \epsilon.
\]
Hence,
\[
\mu = \lim_{n \to \infty} (\| W^n(\omega) \|_{\alpha,0,T(W^n(\omega))} + \mu(T(W^n(\omega)))^{1-\alpha}) = \| W(\omega) \|_{\alpha,[0,T]} + \mu^{1-\alpha}.
\]
Since the function \( t \mapsto \| \cdot \|_{\alpha,0,t} \) is continuously increasing, it obviously contradicts the definition of stopping times.

**Remark 5.1.** Similar to this lemma, we can prove \( \lim_{\eta \to 0} T(\theta_T(W^n(\omega)))W^n(\omega) = T(\theta_T(W(\omega)))W(\omega) \), and based on this inequality and Lemma 5.2 by induction we obtain
\[
\lim_{\eta \to 0} T_i(\theta_T(W^n(\omega)))W^n(\omega) = T_i(\theta_T(W(\omega)))W(\omega), \quad i, j \in \mathbb{Z}.
\]

### 5.3 Wong-Zakai approximation of the solution

As we described in the introduction, it is not difficult to prove the existence and uniqueness of the approximate system
\[
dy^n = Ay^n dt + F(y^n) dt + G(y^n) dW^n, \quad y^n(0) = y_0^n \in \mathcal{B}.
\]
(5.1)

Indeed, applying the method of reference [18] to construct the global solution for a given Gubinelli derivative \( G(y^n) \). We have the following theorem:
Lemma 5.3. To this end, we need some lemmas.

Definition 5.1. Let \( \frac{1}{2} > \alpha > \frac{1}{3} \) and \( W, \tilde{W} \in C^\alpha([0,T];\mathbb{R}^d), I \subset [0,T] \). For any \( (y,y') \in D^{2\alpha}_{W,\gamma}([0,T]) \) and \( (z,z') \in D^{2\alpha}_{\tilde{W},\gamma}([0,T]) \), we define the distance \( d_{2\alpha,\gamma,I}(y,z) \) as follows

\[
d_{2\alpha,\gamma,I}(y,z) = \|y - z\|_{\infty,\gamma,I} + \|y' - z'\|_{\infty,\gamma - \alpha,I} + \|y' - z'\|_{\alpha,\gamma - 2\alpha,I}
\]

\[+ \|R^y - R^z\|_{\alpha,\gamma - \alpha,\Delta_I} + \|R^y - R^z\|_{2\alpha,\gamma - 2\alpha,\Delta_I}.
\]

Our next work is to construct the Wong-Zakai approximation of the solution on stopping time intervals. To this end, we need some lemmas.

Lemma 5.3. Let \( W, \tilde{W} \in C^\alpha([0,T],\mathbb{R}^d) \) for \( \frac{1}{2} > \alpha > \frac{1}{3} \), and \( (y,y') \in D^{2\alpha}_{W,\gamma}, (z,z') \in D^{2\alpha}_{\tilde{W},\gamma} \). Then

\[
\int_0^t S(t-s)y_s dW_s - \int_0^t S(t-s)z_s d\tilde{W}_s = \lim_{\|P(0,\cdot)\| \to 0} \sum_{[u,v] \in P} S(t-u)(y_u W_{u,v} + y'_u \tilde{W}_{u,v} - z_u \tilde{W}_{u,v} - z'_u \tilde{W}_{u,v})
\]

exists in \( B_{\gamma - 2\alpha} \). Moreover, for \( 0 \leq \beta < 3\alpha \) the above integral has the following estimate

\[
\left\| \int_s^t S(t-r)y_r dW_r - \int_s^t S(t-r)z_r d\tilde{W}_r - S(t-s)(y_s W_{s,t} + y'_s \tilde{W}_{s,t} - z_s \tilde{W}_{s,t} - z'_s \tilde{W}_{s,t}) \right\|_{\gamma - 2\alpha + \beta} 
\]

\[
\leq \left( d_{2\alpha,[s,t]}(W,\tilde{W}) \|y - y'\|_{W,2\alpha,\gamma,[s,t]} + d_{2\alpha,\gamma,[s,t]}(y,z)d_{\alpha,[s,t]}(\tilde{W},0) \right)(t-s)^{3\alpha - \beta}.
\]

Proof. The basic idea of the proof is multiplicative Sewing Lemma [15, Theorem 4.1].

Let \( \xi_{s,t} = y_s W_{s,t} + y'_s \tilde{W}_{s,t} - z_s \tilde{W}_{s,t} - z'_s \tilde{W}_{s,t} \), we need to show that \( \xi \in Z^\alpha_{\gamma} \), where the space \( Z^\alpha_{\gamma} \) consists of double index element \( \xi = (\xi_{s,t}) \in C^\alpha_2(B_\gamma) + C^\alpha_2(B_{\gamma - \alpha}) \) with the property \( \delta \xi \in C^\alpha_3(B_{\gamma - 2\alpha}) + C^\alpha_3(B_{\gamma - 2\alpha}) \). That is to say, there exists \( \xi^1, \xi^2 \) and \( h^1, h^2 \) with

\[
\xi_{s,t} = \xi^1_{s,t} + \xi^2_{s,t}, \quad (s,t) \in \Delta_2,
\]

\[
\delta \xi_{s,u,t} = h^1_{s,u,t} + h^2_{s,u,t}, \quad (s,u,t) \in \Delta_3.
\]
such that \( \| \xi^1 \|_{\alpha,\gamma} + \| \xi^2 \|_{2\alpha,\gamma-\alpha} + \| h^1 \|_{2\alpha,\gamma-\alpha} + \| h^2 \|_{\alpha,2\alpha,\gamma-2\alpha} < \infty \), and \( Z_\gamma^\alpha \) is equipped with the norm

\[
\| \xi \|_{Z_\gamma^\alpha} = \inf_{\xi^1,\xi^2,h^1,h^2} ( \| \xi^1 \|_{\alpha,\gamma} + \| \xi^2 \|_{2\alpha,\gamma-\alpha} + \| h^1 \|_{2\alpha,\gamma-\alpha} + \| h^2 \|_{\alpha,2\alpha,\gamma-2\alpha} ).
\]

It is clearly that \( \xi \in C^\alpha_2(B_{\gamma}) + C^2_2(B_{\gamma-\alpha}) \) and

\[
\| \xi \|_{C^\alpha_2(B_{\gamma}) + C^2_2(B_{\gamma-\alpha})} \leq \left( \| y \|_{\alpha,\gamma} \| W - \tilde{W} \|_{\alpha} + \| y - z \|_{\alpha,\gamma} \| \tilde{W} \|_{\alpha} \right) + \left( \| y' \|_{\alpha,\gamma-\alpha} \| W - \tilde{W} \|_{2\alpha} + \| y' - z' \|_{\alpha,\gamma-\alpha} \| \tilde{W} \|_{2\alpha} \right).
\]

Furthermore, In view of Chen’s identity we have that

\[
\delta_{s,u,t} = \left(-y'_{s,u} \tilde{W}_{u,t} + z'_{s,u} \tilde{W}_{u,t} \right) + \left(-R^y_{s,u} W_{u,t} + R^z_{s,u} \tilde{W}_{u,t} \right).
\]

Invoking this identity we obtain

\[
\| \delta_{s,u,t} \| \leq \| W - \tilde{W} \|_{2\alpha} \| y' \|_{\alpha,\gamma-2\alpha} (t-u)^{2\alpha} (u-s) + \| y' - z' \|_{\alpha,\gamma-2\alpha} \| \tilde{W} \|_{2\alpha} (t-u)^{2\alpha} (u-s) + \| R^y - R^z \|_{2\alpha} \| W - \tilde{W} \|_{\alpha} (t-u)^{\alpha} (u-s)^2.
\]

Then \( \delta \in C^\alpha_2(B_{\gamma-2\alpha}) + C^2_2(B_{\gamma-2\alpha}) \) and

\[
\| \delta \|_{C^\alpha_2(B_{\gamma-2\alpha}) + C^2_2(B_{\gamma-2\alpha})} \leq \| W - \tilde{W} \|_{2\alpha} \| y' \|_{\alpha,\gamma-2\alpha} + \| y' - z' \|_{\alpha,\gamma-2\alpha} \| \tilde{W} \|_{2\alpha} + \| R^y - R^z \|_{2\alpha} \| W - \tilde{W} \|_{\alpha} + \| R^y - R^z \|_{2\alpha} \| \tilde{W} \|_{\alpha}.
\]

Hence, we have

\[
\| \xi \|_{Z_\gamma^\alpha} \leq \| y, y' \|_{W;2\alpha,\gamma} d_\alpha(W, \tilde{W}) + d_{2\alpha,\gamma}(y, z)d_\alpha(\tilde{W}, 0).
\]

Finally, the multiplicative Sewing Lemma [15, Theorem 4.1] shows that

\[
\left\| \int_s^t S(t-s)y_sdW_s - \int_s^t S(t-s)z_sdW_s - S(t-s)(y_sW_{s,t} + y'_sW_{s,t} - z_sW_{s,t} + z'_s\tilde{W}_{s,t}) \right\|_{\gamma-2\alpha+\beta} \leq \left( d_{\alpha,[s,t]}(W, \tilde{W}) \| y, y' \|_{W;2\alpha,\gamma,[s,t]} + d_{2\alpha,\gamma,[s,t]}(y, z)d_{\alpha,[s,t]}(\tilde{W}, 0) \right) (t-s)^{3\alpha-\beta}.
\]

\[ \Box \]

**Lemma 5.4.** Let \((y, y') \in D^{2\alpha}_{W,\gamma}([0, T]), (z, z') \in D^{2\alpha}_{W,\gamma}([0, T])\) and \(W, \tilde{W} \in C([0, T], \mathbb{R}^d)\) for \( \frac{1}{2} > \alpha > \frac{1}{3} \). Then controlled rough paths \((G(y), DG(y)G(y)) \in D^{2\alpha}_{W,\gamma}([0, T])\) and \((G(z), DG(z)G(z)) \in D^{2\alpha}_{W,\gamma}([0, T])\) on each stopping time interval \(I_m := [T_{m-1}(W), T_m(W)]\) \(1 \leq m \leq i - 1, i \geq 1\) have the following estimate

\[
d^{2\alpha,\gamma-i,m}(G(y), G(z)) \leq C \mu d^{2\alpha,\gamma,i,m}(y, z)(\| z, z' \|_{W;2\alpha,\gamma,i,m} + \| y, y' \|_{W;2\alpha,\gamma,i,m} + 1)^2
\]

31
Furthermore, analogy with Corollary 3.1 and Lemma 3.6, we obtain
\[ R = C(\|z\|_2 + \|z'\|_{\tilde{W},2\alpha,\gamma,\eta_{\alpha,m}} + \|y, y'\|_{\tilde{W},2\alpha,\gamma,\eta_{\alpha,m}} + 1)^2 d_{\alpha,m}(\tilde{W}, \tilde{W}) \times (\|W\|_{\alpha,m} + \|\tilde{W}\|_{\alpha,m} + 1). \]

The proof of this lemma is too lengthy, we omit it here, it can be found in the Appendix [3].

**Lemma 5.5.** Let \((y^n, G(y^n))\) be the solution of (5.1) driven by \(\theta_{-T}W^n\) with initial data \(y^n_0 \in \mathcal{B}\). Then the norm of controlled rough path \((y^n, G(y^n))\) on stopping times interval \(I_i(T) := [T_{i-1}(\theta_{-T}W), T_i(\theta_{-T}W)]\), \(i \in \mathbb{Z}^+, T > 0\) is bounded for sufficiently small \(\eta\).

**Proof.** By the Corollary 5.1 and (3.13), for any \(\frac{1}{C} 2^{\frac{1 - \nu - 1}{\nu}} \leq \mu > \epsilon > 0\), there exists a \(\eta'()\) sufficiently small, such that
\[ \|\theta_{-T}W^n(\omega)\|_{\alpha,I_i(T)} - \|\theta_{-T}W(\omega)\|_{\alpha,I_i(T)} \leq \epsilon, \quad \eta < \eta'; \]
then
\[ \|\theta_{-T}W^n(\omega)\|_{\alpha,I_i(T)} \leq \mu + \epsilon. \quad (5.2) \]

Similar to Lemma 3.4, we have
\[
\|y^n, (y^n)'\|_{W,2\alpha,0,[T_{i-1}(\theta_{-T}W(\omega)), T_i(\theta_{-T}W(\omega))]} \leq C(\mu + \epsilon) \sum_{m=1}^{i-1} \left( e^{-\lambda(T_{i-1}(\theta_{-T}W(\omega)) - T_m(\theta_{-T}W(\omega)))} \right) \times (1 + \|y^n, (y^n)'\|_{W,2\alpha,0,[T_{m-1}(\theta_{T_j}W(\omega)), T_m(\theta_{T_j}W(\omega))])}) + C(\mu + \epsilon) \left( 1 + \|y^n, (y^n)'\|_{W,2\alpha,0,[T_{i-1}(\theta_{-T}W(\omega)), T_i(\theta_{-T}W(\omega))]} \right) + C e^{-\lambda T_{i-1}(\theta_{-T}W(\omega))}\|y^n\|. 
\]
Furthermore, analogy with Corollary 3.1 and Lemma 3.6, we obtain
\[ \|y^n, (y^n)'\|_{W,2\alpha,0,[T_{i-1}(\theta_{-T}W(\omega)), T_i(\theta_{-T}W(\omega))]} \leq R(0, \theta_{-T}\omega) + o(\epsilon). \quad (5.3) \]

Due to \(R(0, \theta_{-T}\omega)\) is bounded, so \(\|y^n, (y^n)'\|_{W,2\alpha,\gamma,[T_{i-1}(\theta_{-T}W(\omega)), T_i(\theta_{-T}W(\omega))]}\) is bounded.

\[ \square \]
Based on the above results, we could construct the Wong-Zakai approximation of the solution on some stopping time intervals. Now, we consider the following mild solutions:

\[
y_t = S(t)y_0 + \int_0^t S(t-r)F(y_r)dr + \int_0^t S(t-r)G(y_r)d\theta W_r \tag{5.4}
\]

and

\[
y^n_\eta = S(t)y^n_0 + \int_0^t S(t-r)F(y^n_\eta_r)dr + \int_0^t S(t-r)G(y^n_\eta_r)d\theta \omega - T W_\eta \tag{5.5}
\]

**Theorem 5.2.** Let \((y, y')\) be the solution of equation (5.4) with initial data \(y_0 \in B\) and \((y^n, (y^n)^')\) be the solution of equation (5.5) with initial data \(y^n_0 \in B\), where a sequence of stopping times \((T_i(\theta - T W(\omega)))\) is defined above. If \(\|y^n_\eta - y_0\| \to 0\) as \(\eta \to 0\), then

\[
d_{2\alpha,0,T_i(T)}(y, y^n) \to 0, \quad \forall i \in \mathbb{Z}^+, T > 0.
\]

Here, for fixed \(\omega \in \Omega\), the convergence is uniform for each \(i\).

Its proof can be found in Appendix C.

### 5.4 The upper semi-continuity of random attractor

In this subsection, we shall construct the upper semi-continuity for the random attractor \(A_\eta\) of the approximate system \(\varphi_\eta\) which is generated by the solutions \(y^n\).

Since the approximated noise can be regarded as a smooth rough path. For the metric dynamical system in section 4, \(W^n\) has the same properties with \(W\), so those lemmas for \(W\) also hold for \(W^n\). Then the existence of \(A_\eta\) can be obtained as \(A\), we do not claim it any more. For the upper semicontinuity of attractor \(A_\eta\), we refer to [328].

**Theorem 5.3.** For any \(\omega \in \Omega\), we have

\[
\lim_{\eta \to 0} \text{dist}(A_\eta, A) = 0.
\]

**Proof.** For any \(\{x_{\eta_n}\}_{n \in \mathbb{Z}^+} \subseteq B\), and \(x_{\eta_n} \to x \in B\), \(t \geq 0\), we have that

\[
\|\varphi_{\eta_n}(t, \theta_{-t} W^{\eta_n} (\omega), x_{\eta_n}) - \varphi(t, \theta_{-t} W(\omega), x)\|
\]
Thus, \( \bigcup \mathcal{B} \) is precompact. Remark 4.1 implies that we obtain the convergence of random dynamical systems \( \mathcal{B} \) for simplicity, we omit \( \omega \) in \( W(\omega) \) and let \( I_i = [T_{i-1}(\theta T), T_i(\theta T)] \), \( i, j \in \mathbb{Z} \), we shall use the cocycle property of stopping times and the length \( |I_{i,j}| \) of \( I_{i,j}, i, j \in \mathbb{Z} \) less than 1 to prove the next lemmas.

A Proof of Lemma 3.4

In this subsection, for simplicity, we omit \( \omega \) in \( W(\omega) \) and let \( I_{i,j} = [T_{i-1}(\theta T), T_i(\theta T)] \), \( i, j \in \mathbb{Z} \), we shall use the cocycle property of stopping times and the length \( |I_{i,j}| \) of \( I_{i,j}, i, j \in \mathbb{Z} \) less than 1 to prove the next lemmas.
Lemma A.1. Let $y_0 \in \mathcal{B}_γ, γ ≥ 0$. Then $(S(·)y_0, 0) \in \mathcal{D}_{W,γ, I_{i,j}}^{2α}$ and

$$∥S(·)y_0, 0∥_{W,2α,γ, I_{i,j}} ≤ CE^{−λI_{i−1}(θr α)}∥y_0∥_γ,$$  \hspace{1cm} (A.1)

where the constant $C$ only depends on the semigroup $S$.

Proof. Its proof is same as [18] Lemma 3.2], we can use the estimates (3.2)-(3.3) instead of (2.1)-(2.2) to prove this lemma. Thus, compared with [18] Lemma 3.2, $e^{−λI_{i−1}(θr W)}$ will emerge here. We omit the details here. □

Lemma A.2. Let $(y, y') \in \mathcal{D}_{W,γ}^{2α}$ solve the equation (3.4) on $[0, T]$. Then $\left(\int_0^t S(t−r)F(y_r)dr, 0\right)_{t ∈ I_{i,j}} \in \mathcal{D}_{W,γ, I_{i,j}}^{2α}$ and satisfies the following bounds

$$\left\|\int_0^t S(·−r)F(y_r)dr, 0\right\|_{W,2α,γ, I_{i,j}} ≤ Cμ \sum_{m=1}^{i−1} e^{−λ(T_{i−1}(θr W)−T_m(θr W))}(1 + ∥y∥_∞,γ, I_{m,j}) + Cμ(1 + ∥y∥_∞,γ, I_{i,j}).$$

Proof. Since the Gubinelli derivative of the deterministic integral is zero, then

$$\left\|\int_0^t S(·−r)F(y_r)dr, 0\right\|_{W,2α,γ} = \left\|\int_0^t S(·−r)F(y_r)dr\right\|_{∞,γ} + \left\|\int_0^t S(·−r)F(y_r)dr\right\|_{α,γ−α} + \left\|\int_0^t S(·−r)F(y_r)dr\right\|_{2α,γ−2α}.$$

Thanks to the additivity of the deterministic integral, for $t ∈ I_{i,j}$ the following splitting holds true

$$\int_0^t S(t−r)F(y_r)dr = \sum_{m=1}^{i−1} \int_{I_{m,j}} S(t−r)F(y_r)dr + \int_{T_{i−1}(θr W)}^t S(t−r)F(y_r)dr.$$

Similar to [18] Lemma 3.3, by assumption (F), $|I_{i,j}| ≤ 1$ and (3.2)-(3.3) we have

$$\left\|\int_{T_{i−1}(θr W)}^t S(·−r)F(y_r)dr, 0\right\|_{W,2α,γ} ≤ Cμ(1 + ∥y∥_∞).$$ \hspace{1cm} (A.2)

For $s < t ∈ I_{i,j}$, let

$$s' := s − T_{i−1}(θr W), \hspace{1cm} t' := t − T_{i−1}(θr W).$$

Then, $s' < t' ∈ [0, T(θr_{i+j−1} W)]$ and $T(θr_{i+j−1} W) ≤ 1$. Using the change of variable we obtain

$$\int_{I_{m,j}} S(t−r)F(y_r)dr = \int_0^{T(θr_{m,j−1} W)} S(t−r−T_{m−1}(θr W))F(y_r+T_{m−1}(θr W))dr.$$
= S(t' + T_i-1(θ_{T_j}W) - T_m(θ_{T_j}W)) \int_0^{T(θ_{T_m+j-1}W)} S(T(θ_{T_m+j-1}W) - r)F(y_r + T_{m-1}(θ_{T_j}W))dr.

Thus, by (A.2)-(A.5) and assumption (F) we obtain

\[
\sum_{m=1}^{i-1} \left\| \int_{T_{m-1}(θ_{T_j}W)}^{T(θ_{T_m+j-1}W)} S(t - r)F(y_r)dr \right\|_{L(β,γ)} \leq \sum_{m=1}^{i-1} \left\| S(t' + T_i-1(θ_{T_j}W) - T_m(θ_{T_j}W)) \right\|_{L(β,γ)} 
\times \left\| \int_0^{T(θ_{T_m+j-1}W)} S(T(θ_{T_m+j-1}W) - r)F(y_r + T_{m-1}(θ_{T_j}W))dr \right\|_{L(β,γ)} 
\leq Cμ \sum_{m=1}^{i-1} e^{-λ(T_i-1(θ_{T_j}W) - T_m(θ_{T_j}W))} (1 + \|y\|_{∞,γ,I_{m,j}}).
\] (A.3)

For θ ∈ {α, 2α}, m ∈ {1, · · · , i − 1}, by the change of variable, (A.2)-(A.3) and assumption (F) we have that

\[
\left\| \int_{I_{m,j}} S(t - r)F(y_r)dr - \int_{I_{m,j}} S(s - r)F(y_r)dr \right\|_{L(β,γ)} 
= \left\| (S(t' - s') - Id) S(s' + T_i-1(θ_{T_j}W) - T_m(θ_{T_j}W)) \right\|_{L(β,γ)} 
\times \left\| \int_0^{T(θ_{T_m+j-1}W)} S(T(θ_{T_m+j-1}W) - r)F(y_r + T_{m-1}(θ_{T_j}W))dr \right\|_{L(β,γ)} 
\leq C|t - s| e^{-λ(T_i-1(θ_{T_j}W) - T_m(θ_{T_j}W))} 
\times \left\| \int_0^{T(θ_{T_m+j-1}W)} S(T(θ_{T_m+j-1}W) - r)F(y_r + T_{m-1}(θ_{T_j}W))dr \right\|_{L(β,γ)} 
\leq Cμ|t - s| e^{-λ(T_i-1(θ_{T_j}W) - T_m(θ_{T_j}W))} (1 + \|y\|_{∞,γ,I_{m,j}}). \] (A.4)

Thus, (A.3)+(A.4) show that

\[
\left\| \int_{T_i-1(θ_{T_j}W)}^{T(θ_{T_m+j-1}W)} S(t - r)F(y_r)dr, 0 \right\|_{W_2(α,γ,I_{i,j})} 
\leq Cμ \sum_{i=1}^{i-1} e^{-λ(T_i-1(θ_{T_j}W) - T_m(θ_{T_j}W))} (1 + \|y\|_{∞,γ,I_{m,j}}). \] (A.5)

Furthermore, combining with (A.2), (A.5) we obtain

\[
\left\| \int_0^T S(· - r)F(y_r)dr, 0 \right\|_{W_2(α,γ,I_{i,j})} 
\leq Cμ \sum_{m=1}^{i-1} e^{-λ(T_i-1(θ_{T_j}W) - T_m(θ_{T_j}W))} (1 + \|y\|_{∞,γ,I_{m,j}}) + Cμ(1 + \|y\|_{∞,γ,I_{i,j}}).
\] (A.6)
where the above constant $C > 0$ only depends on $S$. \hfill \Box

**Remark A.1.** For $\int_{T_{i-1}(\theta_j W)} S(\cdot - r)F(y_r)dr$, we do not require that the condition $\delta \in [2\alpha, 1)$, our estimate is different from [18]. Indeed, for any $s < t \in I_{i,j}$ we have

$$
\int_{T_{i-1}(\theta_j W)}^t S(t - r)F(y_r)dr - \int_{T_{i-1}(\theta_j W)}^s S(s - r)F(y_r)dr
$$

$$
= \int_s^t S(t - r)F(y_r)dr + (S(t - s) - Id) \int_{T_{i-1}(\theta_j W)}^s S(s - r)F(y_r)dr,
$$

using this identity for $\theta \in \{\alpha, 2\alpha\}$, we obtain

$$
\left\| \int_s^t S(t - r)F(y_r)dr \right\|_{\gamma - \theta} \leq C \mu \int_s^t (t - r)^{(\theta - \delta)\wedge 0} (1 + \|y\|_{\infty, \gamma, I_{i,j}})dr
$$

$$
\leq C \mu (t - s)^{(1 + \theta - \delta)\wedge 1} (1 + \|y\|_{\infty, \gamma, I_{i,j}}),
$$

and

$$
\left\| (S(t - s) - Id) \int_{T_{i-1}(\theta_j W)}^s S(s - r)F(y_r)dr \right\|_{\gamma - \theta}
$$

$$
\leq C (t - s)^{\theta} \int_{T_{i-1}(\theta_j W)}^s S(s - r)F(y_r)dr
$$

$$
\leq C \mu (t - s)^{\theta} |I_{i,j}|^{1 - \delta} (1 + \|y\|_{\infty, \gamma, I_{i,j}}) \leq C \mu (t - s)^{\theta} (1 + \|y\|_{\infty, \gamma, I_{i,j}}).
$$

Thus,

$$
\left\| \int_{T_{i-1}(\theta_j W)}^t S(\cdot - r)F(y_r)dr \right\|_{\gamma - \theta, I_{i,j}} \leq C \mu (1 + \|y\|_{\infty, \gamma, I_{i,j}})(1 + |I_{i,j}|^{(1 - \delta)\wedge (1 - \theta)})
$$

$$
\leq C \mu (1 + \|y\|_{\infty, \gamma, I_{i,j}}).
$$

**Lemma A.3.** Let $G$ satisfy assumption (G) and $(y, y')$ solves the equation (3.1) on $[0, T]$. Then $(G(y), DG(y)G(y)) \in \mathcal{D}_{W, \gamma - \sigma, I_{i,j}}^{2\alpha}$ and have the following estimate

$$
\|G(y), DG(y)G(y)\|_{W, 2\alpha, \gamma - \sigma, I_{i,j}} \leq C \mu (\rho_{2\alpha, I_{i,j}}(W) + 1)^2 \|y, y'\|_{W, 2\alpha, \gamma, I_{i,j}} + 1)
$$

$$
\leq C \mu (\|y, y'\|_{W, 2\alpha, \gamma, I_{i,j}} + 1).
$$

**Proof.** Its proof is same as [18] Lemma 3.5], we need only to explicitly write the noise $W$ and $\mu$ into the inequality in [18] Lemma 3.5 and use the assumption (G) and the norm of the rough noise less than $\mu \in (0, 1)$ on stopping times interval $I_{i,j}$ to complete the proof. \hfill \Box
Lemma A.5. Let \( t \) for any \( \sigma \) \( \in \mathcal{D}_{W,\gamma},[0,T] \) and \( z = \int_s^t S(\cdot - r)y_r dW_r \), then
\[
\left( \int_0^t S(\cdot - r)y_r dW_r, y \right) \in \mathcal{D}_{W,\gamma},[0,T]
\]
and have the following estimates
\[
\left| \int_s^t S(t - r)y_r dW_r \right|_{\gamma + \sigma} \leq C(t - s)^{\alpha - \sigma} \| y, y' \|_{W,2\alpha,\gamma},[s,t] \rho_{\alpha,[s,t]}(W),
\]
\[
\| z, z' \|_{W,2\alpha,\gamma + \sigma,[s,t]} \leq C(\| y_s |_{\gamma} + | y'_s |_{\gamma - \alpha} \rho_{\alpha,[s,t]}(W)) + (t - s)^{\alpha - \sigma} (\rho_{\alpha,[s,t]}(W) + 1) \| y, y' \|_{W,2\alpha,\gamma},[s,t])
\]
\[
\leq C(1 + (t - s)^{\alpha - \sigma})(\rho_{\alpha,[s,t]}(W) + 1) \| y, y' \|_{W,2\alpha,\gamma},[s,t],
\]
for all \( 0 \leq \sigma < \alpha, 0 \leq s < t \leq T \), where the above constant \( C \) depends on \( \alpha, \sigma \).

Remark A.2. The proof of this lemma can be found in [18, Lemma 3.4], where the additivity of the rough integral has been used. In fact, since the semigroup \( S \) is a continuous linear operator on \( \mathcal{B}_{\gamma - 2\alpha} \), according to Lemma 2.7 we define
\[
\int_{0}^{t_1} S(t - r) y_r dW_r := \lim_{|\mathcal{P}(0,t_1)| \to 0} \left[ \sum_{[u,v] \in \mathcal{P}(0,t_1)} S(t - u)(y_u W_{u,v} + y'_u W_{u,v}) \right]
\]
\[
= S(t - t_1) \lim_{|\mathcal{P}(0,t_1)| \to 0} \left[ \sum_{[u,v] \in \mathcal{P}(0,t_1)} S(t_1 - u)(y_u W_{u,v} + y'_u W_{u,v}) \right]
\]
\[
= S(t - t_1) \int_0^{t_1} S(t_1 - r)y_r dW_r
\]
for any \( t_1 \in (0,t) \). Let \( \mathcal{P}(0,t) = \mathcal{P}(0,t_1) \cup \mathcal{P}(t_1,t) \), then we have that
\[
\int_0^t S(t - r) y_r dW_r = \lim_{|\mathcal{P}(0,t)| \to 0} \left[ \sum_{[u,v] \in \mathcal{P}(0,t)} S(t - u)(y_u W_{u,v} + y'_u W_{u,v}) \right]
\]
\[
= \lim_{|\mathcal{P}(0,t_1)| \to 0} \left[ \sum_{[u,v] \in \mathcal{P}(0,t_1)} S(t - u)(y_u W_{u,v} + y'_u W_{u,v}) \right]
\]
\[
+ \lim_{|\mathcal{P}(t_1,t)| \to 0} \left[ \sum_{[u,v] \in \mathcal{P}(t_1,t)} S(t - u)(y_u W_{u,v} + y'_u W_{u,v}) \right]
\]
\[
= \int_0^{t_1} S(t - r)y_r dW_r + \int_{t_1}^t S(t - r)y_r dW_r.
\]

Lemma A.4. For any \( T > 0 \), let \( (y, y') \in \mathcal{D}_{\alpha}^{2\alpha},[0,T] \) and \( z = \int_s^t S(\cdot - r)y_r dW_r \), then
\[
\left( \int_0^t S(\cdot - r)y_r dW_r, y \right) \in \mathcal{D}_{W,\gamma},[0,T]
\]
and have the following estimates
\[
\left| \int_s^t S(t - r)y_r dW_r \right|_{\gamma + \sigma} \leq C(t - s)^{\alpha - \sigma} \| y, y' \|_{W,2\alpha,\gamma},[s,t] \rho_{\alpha,[s,t]}(W),
\]
\[
\| z, z' \|_{W,2\alpha,\gamma + \sigma,[s,t]} \leq C(\| y_s |_{\gamma} + | y'_s |_{\gamma - \alpha} \rho_{\alpha,[s,t]}(W)) + (t - s)^{\alpha - \sigma} (\rho_{\alpha,[s,t]}(W) + 1) \| y, y' \|_{W,2\alpha,\gamma},[s,t])
\]
\[
\leq C(1 + (t - s)^{\alpha - \sigma})(\rho_{\alpha,[s,t]}(W) + 1) \| y, y' \|_{W,2\alpha,\gamma},[s,t],
\]
for all \( 0 \leq \sigma < \alpha, 0 \leq s < t \leq T \), where the above constant \( C \) depends on \( \alpha, \sigma \).
Proof. Similar to Lemma A.2, for any \( s < t \in I_{i,j} \), by the additivity of the rough integral, we have

\[
\int_{0}^{t} S(t-r)G(y_r)d\theta_T^r W_r = \sum_{m=1}^{i-1} \int_{I_{m,j}} S(t-r)G(y_r)d\theta_T^r W_r + \int_{T_{i-1}(\theta_T W)}^{t} S(t-r)G(y_r)d\theta_T^r W_r
\]

\[
= \sum_{m=1}^{i-1} \int_{0}^{T(T_{m-1}(\theta_T W))} S(t-r-T_{m-1}(\theta_T W))G(y_r+T_{m-1}(\theta_T W))d\theta_T^{m-1}(W)W_r
\]

\[
+ \int_{T_{i-1}(\theta_T W)}^{t} S(t-r)G(y_r)d\theta_T^r W_r
\]

\[
= \sum_{m=1}^{i-1} \left[ S(t+T_{i-1}(\theta_T W)) - T_{m}(\theta_T W) \right] \int_{0}^{T(T_{m-1}(\theta_T W))} S(T(T_{m-1}(\theta_T W)) - r)
\]

\[
G(y_r+T_{m-1}(\theta_T W))d\theta_T^{m-1}(W)W_r + \int_{T_{i-1}(\theta_T W)}^{t} S(t-r)G(y_r)d\theta_T^r W_r. \tag{A.7}
\]

Using Lemma A.3, A.4 and \( |I_{i,j}| \leq 1, \rho_{\alpha,i,0}(W) \leq \mu \leq 1 \), we obtain

\[
\left\| \int_{T_{i-1}(\theta_T W)}^{t} S(t-r)G(y_r)d\theta_T^r W_r, G(y) \right\|_{W,2\alpha,\gamma,I_{i,j}}
\]

\[
\leq C(1 + |I_{i,j}|^{\alpha-\sigma}(\rho_{\alpha,i,0}(W) + 1)||G(y), DG(y)G(y)||_{W,2\alpha,\gamma-\sigma,I_{i,j}}
\]

\[
\leq C\mu(\|y, y\|_{W,2\alpha,\gamma,I_{i,j}} + 1). \tag{A.8}
\]

Note that the Gubelli derivative of rough integral \( \int_{t_1}^{t_2} S(-r)G(y_r)dW_r \) on \( I_{i,j} \) is zero. Indeed, for any interval \([t_1,t_2]\) and \( t > s > t_2 \),

\[
\int_{t_1}^{t_2} S(t-r)G(y_r)d\theta_T^r W_r - \int_{t_1}^{t_2} S(s-r)G(y_r)d\theta_T^r W_r
\]

\[
= (S(t-t_1) - S(s-t_2)) \int_{t_1}^{t_2} S(t_2-r)G(y_r)d\theta_T^r W_r,
\]

and (3.2), (3.3) show that \( \int_{t_1}^{t_2} S(-r)G(y_r)dW_r \in C^\alpha(I_{i,j},\mathcal{B}_{\gamma-\alpha}) \cap C^{2\alpha}(I_{i,j},\mathcal{B}_{\gamma-2\alpha}) \). Thus, by Definition 2.4, we know that 0 is the Gubelli derivative of the rough integral \( \int_{t_1}^{t_2} S(-r)G(y_r)dW_r \) on \( I_{i,j} \). Then

\[
\left\| \sum_{m=1}^{i-1} \int_{I_{m,j}} S(-r)G(y_r)d\theta_T^r W_r, 0 \right\|_{W,2\alpha,\gamma,I_{i,j}} \leq \left\| \sum_{m=1}^{i-1} \int_{I_{m,j}} S(-r)G(y_r)d\theta_T^r W_r \right\|_{\infty,\gamma,I_{i,j}}
\]

\[
+ \left\| \sum_{m=1}^{i-1} \int_{I_{m,j}} S(-r)G(y_r)d\theta_T^r W_r \right\|_{\alpha,\gamma-\alpha,I_{i,j}}
\]

\[
+ \left\| \sum_{m=1}^{i-1} \int_{I_{m,j}} S(-r)G(y_r)d\theta_T^r W_r \right\|_{2\alpha,\gamma-2\alpha,I_{i,j}}.
\]

39
By (A.7), Lemma [A.3, A.4] \( |I_{m,j}| = T(T_{m-1}(\theta T_j W)) \leq 1 \), \( \rho_{\alpha,I_{m,0}}(W) \leq \mu \leq 1 \), \( m = 1, \cdots, i - 1 \), we get

\[
\left\| \sum_{m=1}^{i-1} \int_{I_{m,j}} S(\cdot - r)G(y_r)dW_r \right\|_{\infty,\gamma,I_{i,j}} \\
\leq \sum_{i=1}^{i-1} C_m e^{-\lambda(T_{i-1}^{(1)}(\theta T_j W) - T_{m}^{(1)}(\theta T_j W))} \left( \|y_r y'_r\|_{W;2\alpha,\gamma,[T_{m-1}^{(1)}(\theta T_j W),T_{m}^{(1)}(\theta T_j W)]} + 1 \right). \tag{A.9}
\]

For a given \( \theta \in \{\alpha,2\alpha\} \) and \( s < t \in I_{i,j} \), similar to Lemma [A.2] using the same computation as (A.9), we have that

\[
\left\| \sum_{m=1}^{i-1} \left( \int_{I_{m,j}} S(t - r)G(y_r)d\theta T_j W_r - \int_{I_{m,j}} S(s - r)G(y_r)d\theta T_j W_r \right) \right\|_{\gamma - \theta,I_{i,j}} \\
\leq \sum_{m=1}^{i-1} \|S(t') - S(s')\|_{\mathcal{L}(B_{s,B_{s}},\gamma)} \|S(s' + T_{i-1}^{(1)}(\theta T_j W) - T_{m}^{(1)}(\theta T_j W))\|_{\mathcal{L}(B_{s})} \\
\times \left\| \int_{0}^{T(T_{m-1}^{(1)}(\theta T_j W))} S(T(T_{m-1}^{(1)}(\theta T_j W)) - r)G(y_r + T_{m-1}^{(1)}(\theta T_j W))d\theta T_{m-1}^{(1)}(\theta T_j W)W_r \right\|_{\gamma} \\
\leq \sum_{m=1}^{i-1} C_m (t' - s')^\theta e^{-\lambda(T_{i-1}^{(1)}(\theta T_j W) - T_{m}^{(1)}(\theta T_j W))} (1 + \|y_r y'_r\|_{W;2\alpha,\gamma,[T_{m-1}^{(1)}(\theta T_j W),T_{m}^{(1)}(\theta T_j W)]}). \tag{A.10}
\]

Since \( t' - s' = t - s \), thus,

\[
\left\| \sum_{m=1}^{i-1} \int_{I_{m,j}} S(\cdot - r)G(y_r)d\theta T_j W_r \right\|_{\theta,\gamma - \theta,I_{i,j}} \\
\leq \sum_{m=1}^{i-1} C_m e^{-\lambda(T_{i-1}^{(1)}(\theta T_j W) - T_{m}^{(1)}(\theta T_j W))} (1 + \|y_r y'_r\|_{W;2\alpha,\gamma,[T_{m-1}^{(1)}(\theta T_j W),T_{m}^{(1)}(\theta T_j W)]}). \tag{A.10}
\]

It follows from (A.8)-(A.10), one completes the proof of this lemma. 

**Proof.** Now, we have all tools to prove Lemma 3.4. Since \((y, y') \) solves (3.4), then

\[
\|y, y'\|_{W;2\alpha,\gamma,I_{i,j}} \leq \|S(\cdot)\|_{0,W;2\alpha,\gamma,I_{i,j}} + \left\| \int_{0}^{\cdot} S(\cdot - r)F(y_r)dr \right\|_{W;2\alpha,\gamma,I_{i,j}} \\
+ \left\| \int_{0}^{\cdot} S(\cdot - r)G(y_r)d\theta T_j W_r, G(y) \right\|_{W;2\alpha,\gamma,I_{i,j}}.
\]

Therefore, using Lemma [A.1, A.2, A.5] we complete the proof of Lemma 3.4. 

40
B Proof of Lemma 5.4

Proof. According to Definition[5.1] we have

\[ d_{2\alpha,\gamma-\sigma,I_m}(G(y), G(z)) = \left\| G(y) - G(z) \right\|_{\infty,\gamma-\sigma,I_m} + \left\| DG(y)G(y) - DG(z)G(z) \right\|_{\infty,\gamma-\sigma-\alpha,I_m} + \left\| DG(y)G(y) - DG(z)G(z) \right\|_{\alpha,\gamma-\sigma-2\alpha,I_m} + \left\| R^G(y) - R^G(z) \right\|_{\alpha,\gamma-\sigma-\alpha,I_m} + \left\| R^G(y) - R^G(z) \right\|_{2\alpha,\gamma-\sigma-2\alpha,I_m}. \]

By assumption (G), it is easy to get

\[ \left\| G(y) - G(z) \right\|_{\infty,\gamma-\sigma,I_m} \leq \mu \left\| y - z \right\|_{\infty,\gamma,I_m}. \]

Similarly, we have

\[ \left\| DG(y)G(y) - DG(z)G(z) \right\|_{\infty,\gamma-\sigma-\alpha} \leq C\mu \left\| y - z \right\|_{\infty,\gamma,I_m} \left( \left\| y, y' \right\|_{W,2\alpha,\gamma,I_m} + 1 \right). \]

For any \( s < t \in I_m \),

\[ \left\| DG(y_t)G(y_t) - DG(z_t)G(z_t) - DG(y_s)G(y_s) + DG(z_s)G(z_s) \right\|_{\gamma-\sigma-2\alpha} \]

\[ \leq \left\| DG(y_t) - DG(z_t) - DG(y_s) + DG(z_s) \right\|_{\mathcal{L}(B_{\gamma-2\alpha,B_{\gamma-\sigma-2\alpha})}} \times \left\| G(y_t) + G(z_t) + G(y_s) + G(z_s) \right\|_{\gamma-2\alpha} \]

\[ + \left\| DG(y_t) + DG(z_t) - DG(y_s) - DG(z_s) \right\|_{\mathcal{L}(B_{\gamma-2\alpha,B_{\gamma-\sigma-2\alpha})}} \times \left\| G(y_t) + G(z_t) + G(y_s) + G(z_s) \right\|_{\gamma-2\alpha} \]

\[ + \left\| DG(y_t) - DG(z_t) + DG(y_s) - DG(z_s) \right\|_{\mathcal{L}(B_{\gamma-2\alpha,B_{\gamma-\sigma-2\alpha})}} \times \left\| G(y_t) + G(z_t) - G(y_s) - G(z_s) \right\|_{\gamma-2\alpha} \]

\[ + \left\| DG(y_t) + DG(z_t) + DG(y_s) + DG(z_s) \right\|_{\mathcal{L}(B_{\gamma-2\alpha,B_{\gamma-\sigma-2\alpha})}} \times \left\| G(y_t) - G(z_t) - G(y_s) + G(z_s) \right\|_{\gamma-2\alpha} \]

\[ := J_1 + J_2 + J_3 + J_4. \]

We deal with these four terms separately. For the first one, using the Mean Value Theorem twice we have that

\[ \left\| DG(y_t) - DG(z_t) - DG(y_s) + DG(z_s) \right\|_{\mathcal{L}(B_{\gamma-2\alpha,B_{\gamma-\sigma-2\alpha})}} \]

\[ \leq \left\| D^2G \right\|_{\mathcal{L}(B^2_{\gamma-2\alpha,B_{\gamma-\sigma-2\alpha})}} \left\| y, t - z, t \right\|_{\gamma-2\alpha} + \left\| D^3G \right\|_{\mathcal{L}(B^3_{\gamma-2\alpha,B_{\gamma-\sigma-2\alpha})}} \times \left\| y - z \right\|_{\infty,\gamma-2\alpha,I_m} \left\| y, t \right\|_{\gamma-2\alpha} \]

\[ \leq C\mu \left( \left\| y, t \right\|_{\gamma-2\alpha} + \left\| y - z \right\|_{\infty,\gamma-2\alpha,I_m} \left\| y, t \right\|_{\gamma-2\alpha} \right). \quad (B.1) \]
Invoking identities $y_{s,t} = y'_s W_{s,t} + R^{y}_{s,t}$ and $z_{s,t} = z'_s \tilde{W}_{s,t} + R^{z}_{s,t}$, we have

$$
\|y_{s,t} - z_{s,t}\|_{\gamma - 2\alpha} = \|y'_s W_{s,t} + R^{y}_{s,t} - z'_s \tilde{W}_{s,t} - R^{z}_{s,t}\|_{\gamma - 2\alpha}
\leq \|y'_s (W_{s,t} - \tilde{W}_{s,t})\|_{\gamma - 2\alpha} + \|(y'_s - z'_s) \tilde{W}_{s,t}\|_{\gamma - 2\alpha} + \|R^{y}_{s,t} - R^{z}_{s,t}\|_{\gamma - 2\alpha}
\leq \left(\|y'_s\|_{\gamma - 2\alpha} \|W - \tilde{W}\|_{\alpha, I_m} + \|R^{y} - R^{z}\|_{\alpha, \gamma - 2\alpha, I_m} \right)
+ \|y'_s - z'_s\|_{\gamma - 2\alpha} \|\tilde{W}\|_{\alpha, I_m} (t - s)^{\alpha}
\leq C \left(\|y\|_{\infty, \gamma - \alpha, I_m} \|W - \tilde{W}\|_{\alpha, I_m} + \|R^{y} - R^{z}\|_{\alpha, \gamma - \alpha, I_m} \right)
+ \|y' - z'\|_{\infty, \gamma - \alpha, I_m} \|\tilde{W}\|_{\alpha, I_m} (t - s)^{\alpha}.
$$

(B.2)

Thus,

$$
\|y - z\|_{\alpha, \gamma - 2\alpha, I_m} \leq C \left(\|y\|_{\alpha, I_m} + \|y\|_{x, 2\alpha, \gamma, I_m} d_{\alpha, I_m}(W, \tilde{W}) \right) (1 + \|\tilde{W}\|_{\alpha, I_m}).
$$

(B.3)

Similarly,

$$
\|y\|_{\alpha, \gamma - 2\alpha} \leq C \|y\|_{x, 2\alpha, \gamma, I_m} (1 + \|W\|_{\alpha, I_m}).
$$

(B.4)

Combining (B.1), (B.3), (B.4), we get

$$
\|DG(y_t) - DG(z_t) - DG(y_s) + DG(z_s)\|_{\mathcal{L}(s_{\gamma - 2\alpha}, s_{\gamma - \alpha - 2\alpha})}
\leq C \mu \left(\|y\|_{x, 2\alpha, \gamma, I_m} d_{\alpha, I_m}(W, \tilde{W}) \right) (t - s)^{\alpha}.
$$

(B.5)

Then, by (B.5) we obtain

$$
J_1 \leq C \mu \left(\|y\|_{x, 2\alpha, \gamma, I_m} d_{\alpha, I_m}(W, \tilde{W}) \right)
\times \left(\|y\|_{x, 2\alpha, \gamma, I_m} d_{\alpha, I_m}(W, \tilde{W}) \right) (t - s)^{\alpha}.
$$

(B.6)
For the second term, using (B.4) for \( y, z \) we get
\[
J_2 \leq C \mu d_{2\alpha, \gamma, I_m}(y, z) \left( \|y, y'\|_{W;2\alpha, \gamma, I_m} + \|z, z'\|_{\tilde{W};2\alpha, \gamma, I_m} \right) \\
\times (1 + \|W\|_{\alpha, I_m} + \|\tilde{W}\|_{\alpha, I_m})(t - s)^\alpha. \tag{B.7}
\]

For the third term, we have
\[
J_3 \leq C \mu d_{2\alpha, \gamma, I_m}(y, z) \left( \|y, y'\|_{W;2\alpha, \gamma, I_m} + \|z, z'\|_{\tilde{W};2\alpha, \gamma, I_m} \right) (t - s)^\alpha. \tag{B.8}
\]

Finally, we can easily get the estimate of \( J_4 \) as follows
\[
J_4 \leq C \mu d_{2\alpha, \gamma, I_m}(y, z)(t - s)^\alpha. \tag{B.9}
\]

Combining the above estimates, we have
\[
\|DG(y)G(y) - DG(z)G(z)\|_{\alpha, \gamma - \sigma - 2\alpha} \\
\leq C \mu d_{2\alpha, \gamma, I_m}(y, z) \left( \|y, y'\|_{W;2\alpha, \gamma, I_m} + \|z, z'\|_{\tilde{W};2\alpha, \gamma, I_m} + 1 \right)^2 \\
\times (1 + \|W\|_{\alpha, I_m} + \|\tilde{W}\|_{\alpha, I_m}) \\
+ C \mu \left( \|y, y'\|_{W;2\alpha, \gamma, I_m} + \|z, z'\|_{\tilde{W};2\alpha, \gamma, I_m} + 1 \right)^2 d_{\alpha, I_m}(W, \tilde{W}). \tag{B.10}
\]

Finally, we deal with the remainder term. For any controlled rough path \((y, y') \in D_{W;\gamma, \gamma}^{2\alpha}\), Lemma A.3 shows that \((G(y), DG(y)G(y)) \in D_{\tilde{W};\gamma - \sigma}^{2\alpha}\). Thus,
\[
R_{s,t}^{G(y)} = G(y_t) - G(y_s) - DG(y_s)G(y_s)W_{s,t} \\
= \int_0^1 DG(y_s + \eta(y_t - y_s))(y_t - y_s)d\eta - DG(y_s)G(y_s)W_{s,t} \\
= \int_0^1 DG(y_s + \eta(y_t - y_s))R_{s,t}^{y, \eta}d\eta + \int_0^1 DG(y_s + \eta(y_t - y_s)) - DG(y_s)d\eta G(y_s)W_{s,t}.
\]

For \( \theta \in \{\alpha, 2\alpha\}, s < t \in I_m \), using the above inequality for \( y, z \) we have
\[
\|R_{s,t}^{G(y) - R_{s,t}^{G(z)}\|_{\gamma - \theta - \sigma} \leq \left\| \int_0^1 DG(y_s + \eta y_{s,t})R_{s,t}^{y, \eta}d\eta - \int_0^1 DG(z_s + \eta z_{s,t})R_{s,t}^{z, \eta}d\eta \right\|_{\gamma - \sigma - \theta} \\
+ \left\| \int_0^1 DG(y_s + \eta y_{s,t}) - DG(y_s)d\eta G(y_s)W_{s,t} \\
- \int_0^1 DG(z_s + \eta z_{s,t}) + DG(z_s)d\eta G(z_s)\tilde{W}_{s,t} \right\|_{\gamma - \theta - \sigma} \\
\leq \left\| \int_0^1 DG(y_s + \eta y_{s,t})(R_{s,t}^{y} - R_{s,t}^{z})d\eta \right\|_{\gamma - \theta - \sigma}.
\]

43
Similarly, we can deal with the second term, we obtain

\[ II \leq \|D^2G\|_{\mathcal{L}(B_{\gamma-\theta},B_{\gamma-\theta})} \|y - z\|_{\infty,\gamma-\theta,I_m} \|R^2\|_{\theta,\gamma-\theta,I_m}(t - s)^\theta \]

For the first term, we have

\[ I \leq \|DG\|_{\mathcal{L}(B_{\gamma-\theta},B_{\gamma-\theta})} \|R\| - R^2\|_{\theta,\gamma-\theta}(t - s)^\theta \]

\[ \leq \mu d_{2\alpha,\gamma,I_m}(y, z)(t - s)^\theta. \tag{B.12} \]

Firstly, as \( \theta = \alpha \), for the third term we have

\[ III \leq C \|DG\|_{\mathcal{L}(B_{\gamma-\theta},B_{\gamma-\theta})} \|G(y_s)W_{s,t} - G(z_s)\tilde{W}_{s,t}\|_{\gamma-\alpha} \]

\[ \leq C \mu \|y, y^{'}\|_{W,2\alpha,\gamma,I_m} \|d_{2\alpha,\gamma,I_m}(W, \tilde{W})\|_{\alpha,I_m} (t - s)^{2\alpha}. \tag{B.14} \]

Secondly, for \( \theta = 2\alpha \), by (B.14) we obtain

\[ III \leq \|D^2G\|_{\mathcal{L}(B_{\gamma-2\alpha},B_{\gamma-2\alpha})} \|y_s, t\|_{\gamma-2\alpha} \|G(y_s)W_{s,t} - G(z_s)\tilde{W}_{s,t}\|_{\gamma-2\alpha} \]

\[ \leq C \mu \|y, y^{'}\|_{W,2\alpha,\gamma,I_m} (1 + \|W\|_{\alpha,I_m}) \|y, y^{'}\|_{W,2\alpha,\gamma,I_m} d_{2\alpha,\gamma,I_m}(W, \tilde{W}) \]

\[ + \|y, y^{'}\|_{W,2\alpha,\gamma,I_m} (1 + \|W\|_{\alpha,I_m}) \|\tilde{W}\|_{\alpha,I_m} d_{2\alpha,\gamma,I_m}(y, z) \]

\[ \leq C \mu \left[ \|y, y^{'}\|_{W,2\alpha,\gamma,I_m} (1 + \|W\|_{\alpha,I_m}) \|\tilde{W}\|_{\alpha,I_m} d_{2\alpha,\gamma,I_m}(y, z) \right] (t - s)^{2\alpha}. \tag{B.15} \]

Similarly, we can deal with IV. For \( \theta = \alpha \), we have

\[ IV \leq C \|D^2G\|_{\mathcal{L}(B_{\gamma-\alpha},B_{\gamma-\alpha})} \|y - z\|_{\infty,\gamma-\alpha,I_m} \|G(z_s)\tilde{W}_{s,t}\|_{\gamma-\alpha} \]

\[ := I + II + III + IV. \]
Furthermore, we obtain
\[
\leq C\mu \|y - z\|_{\infty, \gamma, I_m} \|z, z'|_{\tilde{W}, 2\alpha, \gamma, I_m} \|\tilde{W}\|_{\alpha, I_m} (t - s)^\alpha
\]
\[
\leq C\mu d_{2\alpha, \gamma, I_m}(y, z) \|z, z'|_{\tilde{W}, 2\alpha, \gamma, I_m} \|\tilde{W}\|_{\alpha, I_m} (t - s)^\alpha. \tag{B.16}
\]

For \(\theta = 2\alpha\), by \([B.3]-[B.4]\) we have
\[
IV \leq \left[\|D^2 G\|_{L(B_{\gamma - 2\alpha}^2, B_{\gamma - 2\alpha})} \|y_{s,t} - z_{s,t}\|_{\gamma - 2\alpha} + C\|D^3 G\|_{L(B_{\gamma - 2\alpha}^3, B_{\gamma - 2\alpha})} \right]
\leq C\mu \left[\left(2\alpha, \gamma, I_m \right) \|y, y'|_{\tilde{W}, 2\alpha, \gamma, I_m} (1 + \|W\|_{\alpha, I_m}) \right]
\leq C\mu \left[\left(2\alpha, \gamma, I_m \right) \|y, y'|_{\tilde{W}, 2\alpha, \gamma, I_m} + 1\right]^2
\times (\|W\|_{\alpha, I_m} + \|\tilde{W}\|_{\alpha, I_m} + 1)
\leq C\mu \left[\left(2\alpha, \gamma, I_m \right) \|y, y'|_{\tilde{W}, 2\alpha, \gamma, I_m} + 1\right]^2 d_{\alpha, I_m}(W, \tilde{W}) (t - s)^{2\alpha}. \tag{B.17}
\]

Then, by the above results we get
\[
\|R^y - R^z\|_{\alpha, \gamma - \alpha, I_m} + \|R^y - R^z\|_{2\alpha, \gamma - 2\alpha, I_m}
\leq C\mu d_{2\alpha, \gamma, I_m}(y, z) \left(\|z, z'|_{\tilde{W}, 2\alpha, \gamma, I_m} + \|y, y'|_{W, 2\alpha, \gamma, I_m} + 1\right)^2
\times (\|W\|_{\alpha, I_m} + \|\tilde{W}\|_{\alpha, I_m} + 1)
\leq C\mu \left(\|z, z'|_{\tilde{W}, 2\alpha, \gamma, I_m} + \|y, y'|_{W, 2\alpha, \gamma, I_m} + 1\right)^2 d_{\alpha, I_m}(W, \tilde{W})
\times (\|W\|_{\alpha, I_m} + \|\tilde{W}\|_{\alpha, I_m} + 1). \tag{B.18}
\]

Furthermore, we obtain
\[
d_{2\alpha, \gamma - \sigma, I_m}(G(y), G(z)) \leq C\mu d_{2\alpha, \gamma, I_m}(y, z) \left(\|z, z'|_{\tilde{W}, 2\alpha, \gamma, I_m} + \|y, y'|_{W, 2\alpha, \gamma, I_m} + 1\right)^2
\times (\|W\|_{\alpha, I_m} + \|\tilde{W}\|_{\alpha, I_m} + 1)
\leq C\mu \left(\|z, z'|_{\tilde{W}, 2\alpha, \gamma, I_m} + \|y, y'|_{W, 2\alpha, \gamma, I_m} + 1\right)^2 d_{\alpha, I_m}(W, \tilde{W})
\times (\|W\|_{\alpha, I_m} + \|\tilde{W}\|_{\alpha, I_m} + 1).
\]
C Proof of Theorem 5.2

Proof. By the definition (5.1) we have that

$$d_{2\alpha,0,I_i(T)}(S(\cdot)y_0, S(\cdot)y_0^n) = \|S(\cdot)(y_0 - y_0^n)\|_{\infty,0,I_i(T)} + \|S(\cdot)(y_0 - y_0^n)\|_{\alpha,-\alpha,I_i(T)}$$

$$+ \|S(\cdot)(y_0 - y_0^n)\|_{2\alpha,-2\alpha,I_i(T)}.$$  \(\text{(C.1)}\)

Clearly,

$$\|S(\cdot)(y_0 - y_0^n)\|_{\infty,0,I_i(T)} \leq Ce^{-\lambda T_{i-1}(\theta - r)W(\omega)}\|y_0 - y_0^n\|$$  \(\text{(C.2)}\)

In addition, for \(\theta = \{\alpha, 2\alpha\}\) we have

$$\|S(\cdot)(y_0 - y_0^n)\|_{\theta,0} \leq Ce^{-\lambda T_{i-1}(\theta - r)W(\omega)}\|y_0 - y_0^n\|.$$

Then, (C.2) and (C.3) show that

$$d_{2\alpha,0,I_i(T)}(S(\cdot)y_0, S(\cdot)y_0^n) \leq Ce^{-\lambda T_{i-1}(\theta - r)W(\omega)}\|y_0 - y_0^n\|.$$  \(\text{(C.4)}\)

For the drift term we need to compute

$$d_{2\alpha,0,I_i(T)} \left( \int_0^\infty S(\cdot - r)F(y_r)dr, \int_0^\infty S(\cdot - r)F(y_r^n)dr \right)$$

$$= \left\| \int_0^\infty S(\cdot - r) (F(y_r) - F(y_r^n)) dr \right\|_{\infty,0,I_i(T)}$$

$$+ \left\| \int_0^\infty S(\cdot - r) (F(y_r) - F(y_r^n)) dr \right\|_{\alpha,-\alpha,I_i(T)}$$

$$+ \left\| \int_0^\infty S(\cdot - r) (F(y_r) - F(y_r^n)) dr \right\|_{2\alpha,-2\alpha,I_i(T)}.$$  \(\text{(C.5)}\)

The estimates of these three terms are similar to Lemma (A.2) and Remark (A.1) we only use the property of Lipshcitz of \(F\) rather than linear growth. We give directly its result as follows

$$d_{2\alpha,0,I_i(T)} \left( \int_0^\infty S(\cdot - r)F(y_r)dr, \int_0^\infty S(\cdot - r)F(y_r^n)dr \right)$$

$$\leq C\mu \sum_{m=1}^{i-1} e^{-\lambda(T_{i-1}(\theta - r)X) - T_m(\theta - r)X)}\|y - y^n\|_{\infty,0,I_m(T)}$$

$$+ C\mu\|y - y^n\|_{\infty,0,I_i(T)}.$$  \(\text{(C.6)}\)
Finally, denote $T_m := T_m(\theta_T W(\omega)), m \in \{1, \cdots, i\}$. Similar to the drift term, for any $t \in I_i(T)$ we have

\[
\int_0^t S(t - r)G(y_r) d\theta_T W_r(\omega) - \int_0^t S(t - r)G(y^n_r) d\theta_T W^n_r(\omega) = \sum_{m=1}^{i-1} \left( \int_{T_{m-1}}^{T_m} S(t - r)G(y_r) d\theta_T W_r(\omega) - \int_{T_{m-1}}^{T_m} S(t - r)G(y^n_r) d\theta_T W^n_r(\omega) \right) + \int_{T_{i-1}}^{t} S(t - r)G(y_r) d\theta_T W_r(\omega) - \int_{T_{i-1}}^{t} S(t - r)G(y^n_r) d\theta_T W^n_r(\omega). \tag{C.7}
\]

Let $t' = t - T_{i-1}, s' = s - T_{i-1}$. For any $m \in \{1, 2, \cdots, i - 1\}$, by Lemma A.3, Lemma 5.3 and Lemma 5.4 for $\gamma = 0$, we have that

\[
\left\| \int_{T_{m-1}}^{T_m} S(t - r)G(y_r) d\theta_T W_r(\omega) - \int_{T_{m-1}}^{T_m} S(t - r)G(y^n_r) d\theta_T W^n_r(\omega) \right\| \leq C e^{-\lambda(T_{m-1} - T_{m})} \left\| \int_{0}^{T_{m-1} - T_{m}} S(T(\theta_{T_{m-1} - T} W(\omega))) (G(y_{T_{m-1} - T} W_{T_{m-1} - T}(\omega)) - G(y^n_{T_{m-1} - T} W^n_{T_{m-1} - T}(\omega))) d\theta_{T_{m-1} - T} W_{T_{m-1} - T} \right\| + \left\| S(T(\theta_{T_{m-1} - T} W(\omega))) \left( DG(y_{T_{m-1} - T} W_{T_{m-1} - T}(\omega)) - DG(y^n_{T_{m-1} - T} W^n_{T_{m-1} - T}(\omega)) \right) \right\| + \left( d_{T_{m-1} - T} \right) \left( \theta_T W(\omega), \theta_T W^n(\omega) \right) \|G(y)\|_\delta \|DG(y)\|_\delta \right\|_{W^{2\alpha - \sigma, I_m(T)}} + \left( d_{2\alpha - \sigma, I_m(T)} \right) \left( G(y), G(y^n) \right) d_{T_{m-1} - T} \left( \theta_T W^n(\omega), 0 \right) \|T(\theta_{T_{m-1} - T} W(\omega))\|^\alpha - \sigma \leq C e^{-\lambda(T_{i-1} - T_{m})} \left\| S(T(\theta_{T_{m-1} - T} W(\omega))) \right\|_{L^{2}(\delta - \sigma, \delta)} \left\| G(y_{T_{m-1} - T} W_{T_{m-1} - T}(\omega)) - G(y^n_{T_{m-1} - T} W^n_{T_{m-1} - T}(\omega)) \right\|_{-\sigma} + \left\| DG(y_{T_{m-1} - T} W_{T_{m-1} - T}(\omega)) - DG(y^n_{T_{m-1} - T} W^n_{T_{m-1} - T}(\omega)) \right\|_{-\sigma - \alpha}.
\]
\[ \times \|S(T(\theta_{T_{m-1}} W(\omega))\|_{L(\mathcal{B}_{\alpha-m}, \mathcal{B})} + \left( d_{\alpha, I_m(T)}(\theta_{T} W(\omega), \theta_{T} W^{\eta}(\omega))\|G(y), DG(y)G(y)\|_{W_{2\alpha,-\sigma}, I_m(T)} \\
+ d_{2\alpha,-\sigma, I_m(I_m(T))}((G(y), G(y^\eta))d_{\alpha, I_m(I_m(T))}(\theta_{T} W^{\eta}(\omega), 0)\|T(\theta_{T_{m-1}} W(\omega))\|^{\alpha-\sigma}) \right) \]
\leq C e^{-\lambda(T_{m-1} - T_m)} \left( d_{\alpha, I_m(T)}(\theta_{T} W(\omega), \theta_{T} W^{\eta}(\omega))\|G(y), DG(y)G(y)\|_{x_{2\alpha,-\sigma}, I_m(T)} \\
+ d_{2\alpha,-\sigma, I_m(I_m(T))}((G(y), G(y^\eta))d_{\alpha, I_m(I_m(T))}(\theta_{T} W^{\eta}(\omega), 0))\|T(\theta_{T_{m-1}} W(\omega))\|^{\alpha-\sigma}) \right)
\leq C \mu e^{-\lambda(T_{m-1} - T_m)}(\|y, y'\|_{W_{2\alpha,0}, I_m(T)} + \|y^\eta, (y^\eta)\|_{W_{\alpha,0}, I_m(T)} + 1)^2 \\
\times (1 + \|\theta_{T} W(\omega)\|_{\alpha, I_m(T)} + \|\theta_{T} W^{\eta}(\omega)\|_{\alpha, I_m(T)}^2) \\
\times \left( d_{\alpha, I_m(T)}(\theta_{T} W(\omega), \theta_{T} W^{\eta}(\omega)) + d_{2\alpha,0, I_m(I_m(T))}(y, y^\eta) \right). \tag{C.8}

Furthermore, we have
\[ \left\| \int_{T_{i-1}}^{t} S(t-r)G(y_r)d\theta_{T} W(\omega) - \int_{T_{i-1}}^{t} S(t-r)G(y_r^\eta)d\theta_{T} W^{\eta}(\omega) \right\| \\
\leq C \mu (\|y, y'\|_{W_{2\alpha,0}, I_m(T)} + \|y^\eta, (y^\eta)\|_{W_{\alpha,0}, I_m(T)} + 1)^2 \\
\times (1 + \|\theta_{T} W(\omega)\|_{\alpha, I_m(T)} + \|\theta_{T} W^{\eta}(\omega)\|_{\alpha, I_m(T)}^2) \\
\times \left( d_{\alpha, I_m(T)}(\theta_{T} W(\omega), \theta_{T} W^{\eta}(\omega)) + d_{2\alpha,0, I_m(I_m(T))}(y, y^\eta) \right). \tag{C.9}

Combining \text{(C.8)} and \text{(C.9)}, we obtain
\[ \left\| \int_{0}^{t} S(t-r)G(y_r)d\theta_{T} W(\omega) - \int_{0}^{t} S(t-r)G(y_r^\eta)d\theta_{T} W^{\eta}(\omega) \right\|_{\infty,0,I_{i}(T)} \]
\leq \sum_{m=1}^{i-1} C \mu e^{-\lambda(T_{m-1} - T_m)}(\|y, y'\|_{W_{2\alpha,0}, I_m(T)} + \|y^\eta, (y^\eta)\|_{W_{\alpha,0}, I_m(T)} + 1)^2 \\
\times (1 + \|\theta_{T} W(\omega)\|_{\alpha, I_m(T)} + \|\theta_{T} W^{\eta}(\omega)\|_{\alpha, I_m(T)}^2) \\
\times \left( d_{\alpha, I_m(T)}(\theta_{T} W(\omega), \theta_{T} W^{\eta}(\omega)) + d_{2\alpha,0, I_m(I_m(T))}(y, y^\eta) \right) \\
+ C \mu \left( \|y, y'\|_{W_{2\alpha,0}, I_{i}(T)} + \|y^\eta, (y^\eta)\|_{W_{\alpha,0}, I_{i}(T)} + 1)^2 \\
\times (1 + \|\theta_{T} W(\omega)\|_{\alpha, I_{i}(T)} + \|\theta_{T} W^{\eta}(\omega)\|_{\alpha, I_{i}(T)}^2) \\
\times \left( d_{\alpha, I_{i}(T)}(\theta_{T} W(\omega), \theta_{T} W^{\eta}(\omega)) + d_{2\alpha,0, I_{i}(I_{i}(T))}(y, y^\eta) \right) \right). \tag{C.10}

The estimation of the term \(\|G(y) - G(y^\eta)\|_{\infty, -\alpha}\) is similar to \text{(A.8)}, and using Lemma \text{5.4}, we have
\[ \|G(y) - G(y^\eta)\|_{\infty, -\alpha} \leq C(1 + (T_{i}(\theta_{T} W(\omega)) - T_{i-1}(\theta_{T} W(\omega))^{\alpha-\sigma}))d_{2\alpha,-\sigma, I_{i}(T)}(G(y), G(y^\eta)) \]
\[ \leq C(1 + \|\theta_{T} W(\omega)\|_{\alpha, I_{i}(T)} + \|\theta_{T} W^{\eta}(\omega)\|_{\alpha, I_{i}(T)}^2) \\
\times \left( d_{\alpha, I_{i}(T)}(\theta_{T} W(\omega), \theta_{T} W^{\eta}(\omega)) + d_{2\alpha,0, I_{i}(I_{i}(T))}(y, y^\eta) \right). \tag{C.11}

The above estimates hold for all \(\alpha, \beta, \sigma > 0\), where \(\alpha, \beta, \sigma > 0\) are the same as those in Lemma \text{5.4}.
\[
\begin{align*}
\leq C\mu(\|y, y'\|_{W,2a,0,I_i(T)} + \|y^n, (y^n)'\|_{W^n,2a,0,I_i(T)} + 1)^2 \\
\times (1 + \|\theta^{-T}W_\omega\|_{\alpha,I_i(T)} + \|\theta^{-T}W^n_\omega\|_{\alpha,I_i(T)})^2 \\
\times \left(d_{\alpha,I_i(T)}(\theta^{-T}W_\omega, \theta^{-T}W^n_\omega) + d_{2a,0,I_i(T)}(y, y^n)\right).
\end{align*}
\]

(C.11)

The estimation of \(\|G(y) - G(y^n)\|_{\alpha,-2a}\) is also similar to \(\|G(y)\|_{\alpha,-2a}\), namely \([A.8]\).

Thus, by Lemma 5.4 we have that

\[
\begin{align*}
\|G(y) - G(y^n)\|_{\alpha,-2a} &\leq C\mu(\|y, y'\|_{W,2a,0,I_i(T)} + \|y^n, (y^n)'\|_{W^n,2a,0,I_i(T)} + 1)^2 \\
\times (1 + \|\theta^{-T}W_\omega\|_{\alpha,I_i(T)} + \|\theta^{-T}W^n_\omega\|_{\alpha,I_i(T)})^2 \\
\times \left(d_{\alpha,I_i(T)}(\theta^{-T}W_\omega, \theta^{-T}W^n_\omega) + d_{2a,0,I_i(T)}(y, y^n)\right).
\end{align*}
\]

(C.12)

Finally, we consider the Hölder seminorm of the remainder terms of the stochastic convolution \(Z_t = \int_0^t S(t-r)G(y_r)d\theta^{-T}W_r(\omega)\) and \(Z^n_t = \int_0^t S(t-r)G(y^n_r)d\theta^{-T}W^n_r(\omega)\), namely \(\|R^Z - R^n_Z\|_{\theta}\). For any \(s < t \in I_i(T)\), we have

\[
\begin{align*}
R^Z_{s,t} - R^n_Z_{s,t} &= \int_s^t S(t-r)G(y_r)d\theta^{-T}W_r(\omega) - \int_s^t S(t-r)G(y^n_r)d\theta^{-T}W^n_r(\omega) \\
&= -S(t-s)(G(y_s)\theta^{-T}W_{s,t}(\omega) + DG(y_s)G(y_s)\theta^{-T}\mathcal{W}_{s,t}(\omega) \\
&- G(y^n_s)\theta^{-T}W^n_{s,t}(\omega) - DG(y^n_s)G(y^n_s)\theta^{-T}\mathcal{W}^n_{s,t}(\omega)) \\
&+ (S(t-s) - 1) \left(G(y_s)\theta^{-T}W_{s,t}(\omega) - G(y^n_s)\theta^{-T}W^n_{s,t}(\omega)\right) \\
&+ S(t-s)(DG(y_s)G(y_s)\theta^{-T}\mathcal{W}_{s,t}(\omega) - DG(y^n_s)G(y^n_s)\theta^{-T}\mathcal{W}^n_{s,t}(\omega)) \\
&+ (S(t-s) - 1) \left(\int_0^s S(s-r)G(y_r)d\theta^{-T}W_r(\omega) \\
&- \int_0^s S(s-r)G(y^n_r)d\theta^{-T}W^n_r(\omega)\right).
\end{align*}
\]

(C.13)

By Lemma 5.3, A.3, 5.4 we obtain

\[
\begin{align*}
\left\|\int_s^t S(t-r)G(y_r)d\theta^{-T}W_r(\omega) - \int_s^t S(t-r)G(y^n_r)d\theta^{-T}W^n_r(\omega) \\
- S(t-s)(G(y_s)\theta^{-T}W_{s,t}(\omega) + DG(y_s)G(y_s)\theta^{-T}\mathcal{W}_{s,t}(\omega) \\
- G(y^n_s)\theta^{-T}W^n_{s,t}(\omega) - DG(y^n_s)G(y^n_s)\theta^{-T}\mathcal{W}^n_{s,t}(\omega))\right\|_{\theta} \\
\leq \left(d_{\alpha,I_i(T)}(\theta^{-T}W_\omega, \theta^{-T}W^n_\omega)\right)\|G(y), DG(y)G(y)\|_{W,2a,-\sigma,I_i(T)} \\
+ d_{2a,-\sigma, I_i(T)}(G(y), G(y^n))\|\theta^{-T}W^n_\omega\|_{\alpha,I_i(T)}(t-s)^{\alpha+\theta-\sigma} \\
\leq C\mu(\|y, y'\|_{W,2a,0,I_i(T)} + \|y^n, (y^n)'\|_{W^n,2a,0,I_i(T)} + 1)^2
\end{align*}
\]
\begin{align*}
\times (1 + \|\theta_T W(\omega)\|_{\alpha, I_t(T)} + \|\theta_T W^n(\omega)\|_{\alpha, I_t(T)})^2 \\
\times \left(d_{\alpha, I_t(T)}(\theta_T W(\omega), \theta_T W^n(\omega)) + d_{2\alpha, 0, I_t(T)}(y, y^n)\right)(t-s)^{\alpha+\theta-\sigma}.
\end{align*}

By (3.3), Lemma A.3, we have

\begin{align*}
\| (S(t-s) - I_d) \right) (G(y_s)\theta_T W_{s,t}(\omega) - G(y^n_s)\theta_T W^n_{s,t}(\omega))\|_{-\theta} \\
\leq \| S(t-s) - I_d \right) \mathcal{L}(S_{-\theta})\| \| G(y_s)\theta_T W_{s,t}(\omega) - G(y^n_s)\theta_T W^n_{s,t}(\omega)\|_{-\sigma} \\
\leq C|t-s|^{\theta+\alpha-\sigma}\left[\| G(y), DG(y)G(y)\|_{W_{2\alpha,-\sigma, I_t(T)}(\theta_T W(\omega), \theta_T W^n(\omega))} \\
+ d_{2\alpha,-\sigma, I_t(T)}d(G(y), G(y^n))d_{\alpha, I_t(T)}(\theta_T W^n(\omega), 0)\right]
\end{align*}

Using (3.2), Lemma A.3, we get

\begin{align*}
\| S(t-s)(D(y_s)G(y_s)\theta_T W_{s,t}(\omega) - DG(y^n_s)G(y^n_s)\theta_T W^n_{s,t}(\omega))\|_{-\theta} \\
\leq C(t-s)^{\theta-2\alpha}\| DG(y_s)G(y_s)\theta_T W_{s,t}(\omega) - DG(y^n_s)G(y^n_s)\theta_T W^n_{s,t}(\omega)\|_{-2\alpha} \\
\leq C(t-s)^{\theta}\left(\| DG(y_s)G(y_s)\|_{-2\alpha}d_{\alpha, I_t(T)}(\theta_T W(\omega), \theta_T W^n(\omega)) \\
+ d_{\alpha, I_t(T)}(\theta_T W^n(\omega), 0)\| DG(y_s)G(y_s) - DG(y^n_s)G(y^n_s)\|_{-2\alpha}\right)
\end{align*}

For the last term (C.13), by (C.8), (C.10), we obtain

\begin{align*}
\left\| (S(t-s) - I_d) \left( \int_0^t S(s-r)G(y_r)dr \theta_T W_r(\omega) - \int_0^t S(s-r)G(y^n_r)dr \theta_T W^n_r(\omega) \right) \right\|_{-\theta} \\
\leq (t-s)^{\theta}\left( \sum_{m=1}^{i-1} C\mu e^{-\lambda(T_{i-1}-T_m)}(\| y, y' \|_{W_{2\alpha, 0, I_m(T)}(y^n, y^n')} + 1) \\
\times (1 + \| \theta_T W(\omega) \|_{\alpha, I_m(T)} + \| \theta_T W^n(\omega) \|_{\alpha, I_m(T)})^2 \\
\times \left(d_{\alpha, I_m(T)}(\theta_T W(\omega), \theta_T W^n(\omega)) + d_{2\alpha, 0, I_m(T)}(y, y^n)\right)\right)
\end{align*}
Furthermore, in terms of (C.4), (C.6), (C.19), we have

\[ \| \theta_T W (\omega) \|_{\alpha,l(T)} + \| \theta_T W^\eta (\omega) \|_{\alpha,l(T)} \]

\[ \times \left( d_{\alpha,l(T)} (\theta_T W (\omega), \theta_T W^\eta (\omega)) + d_{2\alpha,0,l(T)} (y, y^\eta) \right) \].

(C.17)

Combining (C.13)-(C.17), we have

\[ \| R^\eta - R^\eta \|_{\theta,-l(T)} \leq \sum_{m=1}^{i-1} C\mu \left[ e^{-\lambda(T_{i-1}-T_m)} \left( \| y, y' \|_{W,2\alpha,0,l(T)} + \| y^\eta, (y^\eta)' \|_{W^\eta,2\alpha,0,l(T)} + 1 \right)^2 \right. \]

\[ \times \left( \| \theta_T W (\omega) \|_{\alpha,l(T)} + \| \theta_T W^\eta (\omega) \|_{\alpha,l(T)} \right)^2 \]

\[ \times \left( d_{\alpha,l(T)} (\theta_T W (\omega), \theta_T W^\eta (\omega)) + d_{2\alpha,0,l(T)} (y, y^\eta) \right) \]

Thus,

\[ d_{2\alpha,0,l(T)} \left( \int_0^1 S(-r) G(y_r) d\theta_T W (\omega), \int_0^1 S(-r) G(y^\eta_r) d\theta_T W^\eta (\omega) \right) \]

\[ \leq \sum_{m=1}^{i-1} C\mu \left[ e^{-\lambda(T_{i-1}-T_m)} \left( \| y, y' \|_{W,2\alpha,0,l(M(T))} + \| y^\delta, (y^\delta)' \|_{W^\delta,2\alpha,0,l(T)} + 1 \right)^2 \right. \]

\[ \times \left( \| \theta_T W (\omega) \|_{\alpha,l(T)} + \| \theta_T W^\eta (\omega) \|_{\alpha,l(T)} \right)^2 \]

\[ \times \left( d_{\alpha,l(T)} (\theta_T W (\omega), \theta_T W^\eta (\omega)) + d_{2\alpha,0,l(T)} (y, y^\eta) \right) \]

(C.18)

Furthermore, in terms of (C.4), (C.6), (C.19), we have

\[ d_{2\alpha,0,l(T)} (y, y^\eta) \leq \sum_{m=1}^{i-1} C\mu \left[ e^{-\lambda(T_{i-1}-T_m)} \left( \| y, y' \|_{W,2\alpha,0,l(T)} + \| y^\eta, (y^\eta)' \|_{W^\eta,2\alpha,0,l(T)} + 1 \right)^2 \right. \]

\[ \times \left( \| \theta_T W (\omega) \|_{\alpha,l(T)} + \| \theta_T W^\eta (\omega) \|_{\alpha,l(T)} \right)^2 \]

\[ \times \left( d_{\alpha,l(T)} (\theta_T W (\omega), \theta_T W^\eta (\omega)) + d_{2\alpha,0,l(T)} (y, y^\eta) \right) \]

51
Then by the Corollary 5.1, there exists $\eta$ we have

$$\frac{1}{2} \leq \left( \frac{1}{2} \right)^2$$

where the inequality is uniform for $C > 0$

In order to estimate the above inequality, we use the property of stopping times

$$d_{\alpha, I(T)}(\theta-T W(\omega), \theta-T W^\eta(\omega)) + d_{\alpha, 0, I(T)}(y, y^\eta)$$

and the discrete Grönwall inequality [19], i.e. for the non-negative sequences $\{y_n\}$ and $\{g_n\}$ which satisfy

$$y_n \leq C + \sum_{j=0}^{n-1} g_j y_j,$$

where $C > 0$. Then

$$y_n \leq C \prod_{j=0}^{n-1} (1 + g_j).$$

Then by the Corollary 5.1 there exists $\eta(\epsilon)$ for any $\epsilon > 0$, such that $\eta < \eta(\epsilon)$

$$d_{\alpha, I_m(T)}(\theta-T W(\omega), \theta-T W^\eta(\omega)) < \epsilon, \ m \in \mathbb{Z}$$

where the inequality is uniform for $m$. Furthermore, by the property of stopping times we have

$$\sum_{m=1}^{i-1} e^{-\lambda(T_{i-1} - T_m)} = \sum_{m=1}^{i-1} e^{\lambda T_{i-1} - T_m} D(T_{i-1} - T_m)$$
\[ \begin{align*}
&= \sum_{m=2-i}^{0} e^{\lambda T_{m}(\theta_{\tau_{i-1}} - \tau_{\omega})} \\
&\leq \sum_{m=-\infty}^{0} e^{m \lambda (d-\epsilon)} \leq C.
\end{align*} \]

Finally, we claim that \[ \prod_{m=1}^{i-1} (1 + \frac{C \mu}{1 - \epsilon} e^{-m \lambda (d-\epsilon)}) \] is uniformly finite for \( i \). Then we can obtain

\[ d_{2\alpha,0,I,1}(T)(y, y') \to 0, \quad \eta \to 0. \]

\[ \square \]

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