Stability of the equation of the $p$-Wright affine functions

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Abstract. We prove some stability results for the equation of the $p$-Wright affine functions.

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1. Introduction and preliminaries

Let $0 < p < 1$ be a fixed real number. We say that a function $f$ mapping a real nonempty interval $I$ into the set of reals $\mathbb{R}$ is $p$-Wright convex provided (see, e.g., [4,9,16,22])

$$f(px + (1-p)y) + f((1-p)x + py) \leq f(x) + f(y), \quad x, y \in I.$$ 

If $f$ satisfies the functional equation

$$f(px + (1-p)y) + f((1-p)x + py) = f(x) + f(y), \quad (1.1)$$

then we say that it is $p$-Wright affine (see [4]).

Note that for $p = 1/2$ Eq. (1.1) becomes the Jensen’s functional equation

$$f \left( \frac{x+y}{2} \right) = \frac{f(x) + f(y)}{2}.$$

For $p = 1/3$ Eq. (1.1) takes the form

$$f(2x + y) + f(x + 2y) = f(3x) + f(3y), \quad (1.2)$$

which has been investigated by Najati and Park [18]; in particular, they proved some results on its stability and applied them in the investigation of the generalized $(\sigma, \tau)$-Jordan derivations on Banach algebras. The cases of more arbitrary $p$ were studied in [4,5,15] (see also [9,13]).

We prove some results concerning the Hyers–Ulam stability and superstability of (1.1). For more information and numerous references on the stability
of functional equations we refer to, e.g., [10, 14, 17, 21]; for some examples of various recent outcomes showing new directions in this area of research see, e.g., [3, 7, 8, 11, 12, 19, 20].

The method of the proof of the main result corresponds to some observations in [6, 7, 20] and the main tool in it is a fixed point result that can be derived from [1, Theorem 1] (see also [2, Theorem 2]). To present it we need the following four hypotheses ($\mathbb{R}_+$ denotes the set of nonnegative reals).

(H1) $X$ is a normed space over a field $\mathbb{F} \in \{ \mathbb{R}, \mathbb{C} \}$ ($\mathbb{C}$ denotes the set of complex numbers) and $Y$ is a Banach space.

(H2) $f_1, \ldots, f_k : X \to X$ and $L_1, \ldots, L_k : X \to \mathbb{R}_+$ are given maps.

(H3) $T : Y^X \to Y^X$ is an operator satisfying the inequality

$$\|(T\xi)(x) - (T\mu)(x)\| \leq \sum_{i=1}^{k} L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|$$

for every $\xi, \mu \in Y^X, x \in X$.

(H4) $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ is a linear operator defined by

$$(\Lambda \delta)(x) := \sum_{i=1}^{k} L_i(x) \delta(f_i(x)), \quad \delta \in \mathbb{R}_+^X, x \in X.$$ 

Now we are in a position to present the above mentioned fixed point theorem.

**Theorem 1.1.** Assume that hypotheses (H1)–(H4) are satisfied. Suppose that there are functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$\|(T\varphi)(x) - \varphi(x)\| \leq \varepsilon(x), \quad x \in X, \quad (1.3)$$

and

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) < \infty, \quad x \in X. \quad (1.4)$$

Then there exists a unique fixed point $\psi$ of $T$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \quad x \in X. \quad (1.5)$$

Moreover

$$\psi(x) := \lim_{n \to \infty} (T^n \varphi)(x), \quad x \in X. \quad (1.6)$$
2. Stability

The next theorem is the main result in this paper and concerns the stability of Eq. (1.1); it corresponds in particular to some results in [18].

**Theorem 2.1.** Let (H1) be valid, \( p \in \mathbb{F} \), \( A, k \in (0, \infty) \), \( |p|^k + |1-p|^k < 1 \), and \( g : X \to Y \) satisfy

\[
\|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \\
\leq A(\|x\|^k + \|y\|^k), \quad x, y \in X.
\]  

(2.1)

Then there exists a unique solution \( G : X \to Y \) of Eq. (1.1) such that

\[
\|g(x) - G(x)\| \leq \frac{A\|x\|^k}{1 - |p|^k - |1-p|^k}, \quad x \in X
\]  

(2.2)

and \( G \) is given by:

\[
G(x) := g(0) + \lim_{n \to \infty} (T^n g_0)(x), \quad x \in X,
\]  

(2.3)

where \( g_0 \) and \( T \) are defined by (2.6) and (2.7).

Moreover, \( G \) is the unique solution of Eq. (1.1) such that there exists a constant \( M \in (0, \infty) \) with

\[
\|g(x) - G(x)\| \leq M\|x\|^k, \quad x \in X.
\]  

(2.4)

**Proof.** Note that (2.1) with \( y = 0 \) gives

\[
\|g(px) + g((1-p)x) - g(x) - g(0)\| \leq A\|x\|^k, \quad x \in X.
\]  

(2.5)

Write

\[
g_0(x) = g(x) - g(0), \quad x \in X
\]  

(2.6)

and

\[
T \xi(x) := \xi(px) + \xi((1-p)x), \quad x \in X, \xi \in Y^X.
\]  

(2.7)

Then (2.5) implies the inequality

\[
\|g_0(px) + g_0((1-p)x) - g_0(x)\| \leq A\|x\|^k, \quad x \in X,
\]  

which means that

\[
\|T g_0(x) - g_0(x)\| \leq A\|x\|^k, \quad x \in X.
\]  

(2.8)

Further note that (H3) holds with \( k = 2 \), \( f_1(x) = px \), \( f_2(x) = (1-p)x \), \( L_i(x) = 1 \) for \( i = 1, 2 \), \( x \in X \). Define \( \Lambda \) as in (H4). Clearly, with \( \varepsilon(x) := A\|x\|^k \) for \( x \in X \), we have
\[ \varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) \]
\[ \leq A\|x\|^k \sum_{n=0}^{\infty} (|p|^k + |1-p|^k)^n \]
\[ = \frac{A\|x\|^k}{1 - |p|^k - |1-p|^k}, \quad x \in X. \]

Hence, according to Theorem 1.1, there exists a unique solution \( G_0 : X \to Y \) of the equation
\[ G_0(x) = G_0(px) + G_0((1-p)x) \quad (2.9) \]
such that
\[ \|g_0(x) - G_0(x)\| \leq \frac{A\|x\|^k}{1 - |p|^k - |1-p|^k}, \quad x \in X; \]
moreover
\[ G_0(x) := \lim_{n \to \infty} (T^n g_0)(x), \quad x \in X. \]

Now we show that, for every \( x, y \in X, \ n \in \mathbb{N}_0 \) (nonnegative integers),
\[ \|T^n g_0(px + (1-p)y) + T^n g_0((1-p)x + py) - T^n g_0(x) + T^n g_0(y)\| \]
\[ \leq A(|p|^k + |1-p|^k)^n (\|x\|^k + \|y\|^k). \quad (2.10) \]
Clearly, the case \( n = 0 \) is just (2.1). Next, fix \( m \in \mathbb{N}_0 \) and assume that (2.10) holds for every \( x, y \in X \) with \( n = m \). Then
\[ \|T^{m+1} g_0(px + (1-p)y) + T^{m+1} g_0((1-p)x + py) \]
\[ - T^{m+1} g_0(x) - T^{m+1} g_0(y)\| \]
\[ = \|T^m g_0(p(px + (1-p)y)) + T^m g_0((1-p)(px + (1-p)y)) \]
\[ + T^m g_0(p((1-p)x + py)) + T^m g_0((1-p)((1-p)x + py)) \]
\[ - T^m g_0(px) - T^m g_0((1-p)x) - T^m g_0(py) - T^m g_0((1-p)y)\| \]
\[ \leq \|T^m g_0(ppx + (1-p)py)) + T^m g_0((1-p)px + ppy)) \]
\[ - T^m g_0(px) - T^m g_0(py)\| \]
\[ + \|T^m g_0(p(1-p)x + (1-p)(1-p)y)) \]
\[ + T^m g_0((1-p)(1-p)x + p(1-p)y)) \]
\[ - T^m g_0((1-p)x) - T^m g_0((1-p)y)\| \]
\[ \leq A(|p|^k + |1-p|^k)^m (\|px\|^k + \|py\|^k) \]
\[ + (|p|^k + |1-p|^k)^m (\|(1-p)x\|^k + \|(1-p)y\|^k) \]
\[ = (|p|^k + |1-p|^k)^{m+1} (\|x\|^k + \|y\|^k), \quad x, y \in X. \]
Thus, by induction we have shown that (2.10) holds for every \(x,y \in X\) and \(n \in \mathbb{N}_0\). Letting \(n \to \infty\) in (2.10), we obtain that
\[
G_0(px + (1-p)y) + G_0((1-p)x + py) = G_0(x) + G_0(y), \quad x,y \in X.
\]

Write \(G(x) := G_0(x) + g(0)\) for \(x \in X\). Then it is easily seen that
\[
G(px + (1-p)y) + G((1-p)x + py) = G(x) + G(y), \quad x,y \in X
\]
and (2.2) holds. It remains to show the statement concerning the uniqueness of \(G\).

So suppose that \(M_0 \in (0, \infty)\) and \(G_1 : X \to Y\) is a solution to (1.1) with
\[
\|g(x) - G_1(x)\| \leq M_0\|x\|^k, \quad x \in X.
\]

Note that \(G(0) = g(0) = G_1(0)\),
\[
G_1(px) + G_1((1-p)x) = G_1(x) + G_1(0), \quad x \in X, \quad (2.11)
\]
\[
G(px) + G((1-p)x) = G(x) + G(0), \quad x \in X, \quad (2.12)
\]
and, by (2.2),
\[
\|G(x) - G_1(x)\| \leq \frac{(M + A)\|x\|^k}{1 - |p|^k - |1-p|^k} \leq (M + A)\|x\|^k \sum_{n=0}^{\infty} (|p|^k + |1-p|^k)^n, \quad x \in X, \quad (2.13)
\]
where \(M := M_0(1 - |p|^k - |1-p|^k)\).

We show that, for each \(j \in \mathbb{N}_0\),
\[
\|G(x) - G_1(x)\| \leq (M + A)\|x\|^k \sum_{n=0}^{\infty} (|p|^k + |1-p|^k)^n, \quad x \in X. \quad (2.14)
\]
The case \(j = 0\) is exactly (2.13). So fix \(l \in \mathbb{N}_0\) and assume that (2.14) holds for \(j = l\). Then, in view of (2.11) and (2.12),
\[
\|G(x) - G_1(x)\| = \|G(px) + G((1-p)x) - G_1(px) - G_1((1-p)x)\|
\leq \|G(px) - G_1(px)\| + \|G((1-p)x) - G_1((1-p)x)\|
\leq (M + A)(\|px\|^k + (1-p)x\|^k) \sum_{n=l}^{\infty} (|p|^k + |1-p|^k)^n
\leq (M + A)\|x\|^k \sum_{n=l+1}^{\infty} (|p|^k + |1-p|^k)^n, \quad x \in X.
\]
Thus we have shown (2.14). Now, letting \(j \to \infty\) in (2.14) we get \(G_1 = G\). \(\square\)
3. A complementary observation on superstability

The following very simple observation on the superstability of Eq. (1.1) complements Theorem 2.1.

**Theorem 3.1.** Let (H1) be valid, \( p \in \mathbb{F}, A, k \in (0, \infty), |p|^{2k} + |1 - p|^{2k} < 1, \) and \( g : X \to Y \) satisfy

\[
\|g(px + (1 - p)y) + g((1 - p)x + py) - g(x) - g(y)\| \leq A\|x\|^k\|y\|^k \tag{3.1}
\]

for every \( x, y \in X. \) Then \( g \) is a solution to (1.1).

**Proof.** Note that (3.1) with \( y = 0 \) gives

\[
g(x) = g(px) + g((1 - p)x) - g(0), \quad x \in X. \tag{3.2}
\]

We show that, for every \( x, y \in X, n \in \mathbb{N}_0, \)

\[
\|g(px + (1 - p)y) + g((1 - p)x + py) - g(x) - g(y)\|
\leq A(|p|^{2k} + |1 - p|^{2k})^n \|x\|^k\|y\|^k. \tag{3.3}
\]

Clearly the case \( n = 0 \) is just (3.1). Next, fix \( m \in \mathbb{N}_0 \) and assume that (3.3) holds for every \( x, y \in X \) with \( n = m. \) Then, by (3.2),

\[
\|g(px + (1 - p)y) + g((1 - p)x + py) - g(x) - g(y)\|
= \|g(p(px + (1 - p)y)) + g((1 - p)(px + (1 - p)y))
+ g(p((1 - p)x + py)) + g((1 - p)((1 - p)x + py))
- g(px) - g((1 - p)x) - g(py) - g((1 - p)y)\|
\leq A(|p|^{2k} + |1 - p|^{2k})^m \|px\|^k\|py\|^k
+ A(|p|^{2k} + |1 - p|^{2k})^m \|(1 - p)x\|^k\|(1 - p)y\|^k
= A(|p|^{2k} + |1 - p|^{2k})^{(m+1)} \|x\|^k\|y\|^k
\]

for every \( x, y \in X. \)

Thus, by induction we have shown that (3.3) holds for every \( x, y \in X \) and \( n \in \mathbb{N}_0. \) Letting \( n \to \infty \) in (3.3), we obtain that \( g \) is a solution to (1.1), because \( |p|^k + |1 - p|^k < 1. \)

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