RAINBOW PATHS AND RAINBOW MATCHINGS IN GRAPHS

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ABSTRACT. We prove that if \( n \geq 3 \), then any family of \( 3n - 3 \) sets of matchings of size \( n \) in any graph has a rainbow matching of size \( n \). This improves on a previous result, in [ABC+19], in which \( 3n - 3 \) is replaced by \( 3n - 2 \).

We also prove a “cooperative” generalization: for \( t > 0 \) and \( n \geq 3 \), any \( 3n - 4 + t \) sets of edges, the union of every \( t \) of which contains a matching of size \( n \), have a rainbow matching of size \( n \).

1. Introduction

Given a collection of sets, \( \mathcal{S} = (S_1, \ldots, S_m) \), an \( \mathcal{S} \)-rainbow set is the image of a partial choice function of \( \mathcal{S} \). So, it is a set \( \{x_{i,j}\} \), where \( 1 \leq i_1 < \ldots < i_k \leq m \) and \( x_{i,j} \in S_{i,j} \) \( (j \leq k) \). We call the sets \( S_i \) colors and we say that \( x_{i,j} \) is colored by \( S_{i,j} \) and that \( x_{i,j} \) represents \( S_{i,j} \) in the rainbow set.

Given numbers \( m, n, k \), we write \( (m, n) \to k \) if every \( m \) matchings of size \( n \) in any graph have a rainbow matching of size \( k \), and \( (m, n) \to_B k \) if the same is true in every bipartite graph. Generalizing a result of Drisko [Dri98], the first author and E. Berger proved [AB09]:

Theorem 1.1. \( (2n - 1, n) \to_B n \).

In [ABC+19] it was conjectured that almost the same is true in all graphs:

Conjecture 1.2. \( (2n, n) \to n \). If \( n \) is odd then \( (2n - 1, n) \to n \).

If true, this is reminiscent of the relationship between König’s theorem, stating that \( \chi_e \leq \Delta \) in bipartite graphs, and Vizing’s theorem, \( \chi_e \leq \Delta + 1 \) in general graphs, where \( \chi_e \) is the edge chromatic number, namely the minimal number of matchings covering the edge set of the graph. The conjecture says that there is a price of just 1 for passing from bipartite graphs to general graphs.

Date: April 17, 2020.

The first author was supported by BSF grant no. 2006099, an ISF grant and the Discount Bank Chair at the Technion.
The conjecture is implicit already in \cite{BGS17} - the examples showing sharpness appear there. Another conjecture of the first author and E. Berger, stated in the bipartite case in \cite{AB09}, but presently no counterexample is known also in general graphs, is:

**Conjecture 1.3.** $(n, n) \to n - 1$.

If true, this conjecture implies a famous conjecture of Brualdi-Stein: every $n \times n$ Latin square contains $n - 1$ distinct entries, lying in different rows and different columns. It also implies the first part, $(2n, n) \to n$, of Conjecture 1.2. To see this, assume that $M_1, \ldots, M_{2n}$ are matchings of size $n$ in any graph. Let $V_1, V_2$ be two disjoint copies of the vertex set of the graph, and for $j = 1, 2$ and $i \leq 2n$ let $M_i^j$ be a copy of $M_i$ on $V_j$. Let $N_i = M_1^i \cup M_2^i$ ($i \leq 2n$). If Conjecture 1.3 is true, then the system $(N_1, \ldots, N_{2n})$ has a rainbow matching $N$ of size $2n - 1$. By the pigeonhole principle, $n$ of the edges of $N$ belong to the matchings $M_i^j$ for the same $j$, proving Conjecture 1.2. If indeed this is the reason (in some non-rigorous sense) for the validity of Conjecture 1.2, this may explain the difficulty of the latter, since as mentioned above Conjecture 1.3 belongs to a family of notoriously hard problems.

The best result so far on Conjecture 1.2 is:

**Theorem 1.4.** \cite{ABC+19} $(3n - 2, n) \to n$.

In \cite{HL20} an alternative, topological proof was given for this result. Theorem 1.1 has had more than one proof. Three quite distinct topological proofs were given in \cite{AHJ19, ABKZ18, HL20}. There are also more than one combinatorial proof, but they are similar in spirit. They are based on the following:

**Theorem 1.5.** \cite{AKZ18} If $F$ is a matching of size $k$ in a bipartite graph and $A = (A_1, \ldots, A_{k+1})$ is a family of augmenting $F$-alternating paths then there exists an $A$-rainbow augmenting $F$-alternating path.

For a given matching $F$, we write “$F$-AAP” for “augmenting $F$-alternating path”. Given a family $\mathcal{P}$ of sets of edges, an $F$-alternating path is called $\mathcal{P}$-rainbow if its non-$F$ edges form a $\mathcal{P}$-rainbow set.

To deduce Theorem 1.1, let $H$ be a family of $2n - 1$ matchings of size $n$. Let $F$ be a maximal size rainbow matching, and assume for contradiction that $k := |F| < n$. Then more than $k$ matchings are not represented in $F$. Since each of them is larger than $F$, each of them forms an $F$-AAP. Taking the symmetric difference of $F$ and the rainbow alternating path provided by Theorem 1.5 yields a rainbow matching larger than $F$, a contradiction.

The result $(3n - 2, n) \to n$ follows in a similar way, using the following theorem:
Theorem 1.6. If $F$ is a matching of size $k$ in any graph and $A = (A_1, \ldots, A_{2k+1})$ is a family of $F$-AAPs, then there exists a rainbow $F$-AAP.

This is sharp: in the next section we shall introduce “origamistrips”, that are families of $2|F|$ $F$-AAPs that do not have a rainbow AAP. This means that the strategy that works in the bipartite case cannot bring us close to Conjecture 1.2. But some additional effort can take us one step further, which is the main result of this paper:

Theorem 1.7. $(3n - 3, n) \to n$ for any $n \geq 3$.

The proof uses a characterization of the extreme negative cases in Theorem 1.6, namely of those families of $2k$ AAPs not having a rainbow AAP.

In the second part of the paper we prove a “cooperative” generalization of the theorem. This means that we are not given matchings, but sets of edges, the union of every $t$ of which ($t$ being a parameter of the result) contains a matching of size $n$. The conclusion is, again, the existence of a rainbow matching of size $n$.

### 1.1. Paths terminology.

We shall use paths extensively. The default assumption is that paths are undirected. Throughout the paper we shall tacitly identify a path with its edge set. If $P = \{v_1v_2, v_2v_3, \ldots, v_{m-1}v_m\}$, $v_1$ and $v_m$ are called the endpoints of $P$ and $v_2, \ldots, v_{m-1}$ are called interior vertices of $P$. We sometimes write $P = v_1v_2 \ldots v_m$. Note that $v_1v_2 \ldots v_m$ and $v_mv_{m-1} \ldots v_1$ are the same path.

For paths $S = s_1s_2 \ldots s_q$ and $T = t_1t_2 \ldots t_q$ (taken in this case in a directed sense) let $ST$ be the walk $s_1s_2 \ldots s_q t_1t_2 \ldots t_q$. In particular, if $x$ is a vertex then $Sx = s_1s_2 \ldots s_qx$ (both notations will be used below only when the resulting walks are paths).

### 2. Badges

**Definition 2.1.** An $m$-origamistrip $OS$ is a graph whose vertex set is \{u_1, \ldots, u_m\} $\cup$ \{v_1, \ldots, v_m\} $\cup$ \{x, y\} (all $u_i, v_i, x, y$ being distinct) and whose edge set is the union of three disjoint matchings: $M = M(OS) = \{e_1 = u_1v_1, \ldots, e_m = u_mv_m\}; A(OS) = \{xu_1, v_1u_2, v_2u_3, \ldots, v_my\};$ and $B(OS) = \{xv_1, u_1v_2, \ldots, u_my\}$.

See Figure 1.

There is no real difference between $A(OS)$ and $B(OS)$, the differentiation is merely for notational convenience. Then $P^A(OS) := A(OS) \cup M$ and $P^B(OS) := B(OS) \cup M$ are $M$-AAPs.
The vertices $x, y$ are called the endpoints of OS, and $u_1, v_1, \ldots, u_m, v_m$ are called the interior vertices. The matching $F$ is called the skeleton of OS.

We will want to refer to the $F$-alternating paths reaching vertices of the origamistrip, to be named below:

**Observation 2.2.** For every interior vertex $v$ of OS there exist an even (counting edges) $F$-alternating path $Q^A_v(OS) \subseteq P^A(OS)$ from an endpoint of OS to $v$ and an even $F$-alternating path $Q^B_v(OS) \subseteq P^B(OS)$ from the other endpoint of OS to $v$. If $x$ is an endpoint of OS, write $Q^n_x(OS)$, or simply $Q^n_x$, for whichever of $\{Q^A_x(OS), Q^B_x(OS)\}$ begins at $x$.

$m$-origamistrips can be used to show the sharpness of Conjecture 1.2, for $n := m + 1$ even.

**Observation 2.3.** Let $m$ be odd, and let OS be an $m$-origamistrip with skeleton $F$. Taking $m$ copies of $A(OS)$, $m$ copies of $B(OS)$ and the matching $F \cup \{xy\}$ provides an example of $2m + 1$ matchings of size $m + 1$, having no rainbow matching of size $m + 1$.

For $m$ odd this, and the “badges” formed from it, to be defined below, are the only examples for sharpness we know. For $m$ even, the example does not work: it has a rainbow matching of size $m + 1$, consisting of the edge $xy$ and pairs $u_2v_{2i-1}, u_{2i-1}v_{2i}$.

The following observation explains the reason why Observation 2.3 is true:
Observation 2.4. Let $OS$ be an $m$-origamistrip with skeleton $F$, and let $Q$ be the collection of paths containing each of $P^A(OS)$ and $P^B(OS)$, each repeated $m$ times. Then there is no $Q$-rainbow $F$-AAP.

Definition 2.5. A badge $B$ is collection of paths, obtained from a multigraph $H$ and an integer weighting $w$ on its edge set, in the following way. For each edge $e = xy$ of $H$ let $OS(e)$ be a $w(e)$-origamistrip with endpoints $x, y$, where all $OS(e)$s have disjoint interior vertex sets. The badge $B$ is then the collection of all paths $P^A(OS(e))$, $P^B(OS(e))$, each repeated $w(e)$ times. Let $M(B) = \bigcup_{e \in E(H)} M(OS(e))$ and $w(B) = \sum_{e \in E(H)} w(e)$. We also call $B$ is a $k$-badge, where $k = w(B)$, and we call $M(B)$ its skeleton.

Thus $B$ consists of $2w(B)$ $M(B)$-alternating paths. For $v \in \bigcup M(B)$ let $OS_v$ be the origamistrip in $B$ that contains $v$.

In the next section, on cooperative versions of the main theorem, we shall use also a more general type of object:

Definition 2.6 (generalized badges). The notion of a badge can be generalized, by allowing the addition and the deletion of edges from $M(B)$ to each $P^A(OS(e))$ and to each $P^B(OS(e))$. The resulting construction is then called a generalized badge.

By Observation 2.4 there is no rainbow $M(B)$-AAP in any origamistrip, hence in all of $B$. We next show that $B$ is edge-maximal with respect to this property, by

Lemma 2.7. Let $P$ be a badge with skeleton $F$. Let $P^+$ be obtained from $P$ by replacing one path $P \in P$ by $P^+ = P \cup \{xy\}$, where $xy \notin P$ and $xy \notin F$. Then $P^+$ has a rainbow $F$-AAP.

Proof. If $x, y \notin \bigcup F$, then $xy$ is by itself a rainbow $F$-AAP. So, without loss of generality we may assume that $x \in \bigcup F$. Also without loss of generality, $P = P^A(OS_x)$. Then the $F$-AAP path $Q^B_x(OS_x) \cup \{xy\}$ (see Observation 2.2 for the definition of $Q^B_x(OS_x)$) is $P^+$-rainbow. The reason is that there are enough paths in $B(OS_x)$ to color the edges of $Q^B_x(OS_x)$, while $P^+$ can color $xy$.

So, we may assume that $y \in \bigcup F$ as well.

Case I. $OS_x \neq OS_y$. Let $z$ be an endpoint of $OS_y$ which is not an endpoint of $OS_x$. Then $Q^B_x(OS_x) \cup \{xy\} \cup Q^z_y(OS_y)$ is a $P^+$-rainbow $F$-AAP.

Case II. $OS_x = OS_y$. Then a path of the form $Q^p_x(OS_x) \cup \{xy\} \cup Q^q_y(OS_y)$, where each of $p, q$ is either $A$ or $B$, is a $P^+$-rainbow $F$-AAP. □
This easily implies that adding an edge as a separate singleton set also results in a rainbow $F$-AAP:

**Corollary 2.8.** If $P$ is a badge with skeleton $F$ and $P^+ = P \cup \{\{xy\}\}$ for some $xy \notin F$, then $P^+$ has a rainbow $F$-AAP.

To see this, add $xy$ to any path to which it does not belong, and use the lemma.

The main structural theorem that will be used in the proof of Theorem 1.7 is:

**Theorem 2.9.** Let $F$ be a matching of size $k$ in a graph $G$, and let $P$ be a family of $F$-AAPs. Suppose $P$ has no rainbow $F$-AAP. Then,

1. $|P| \leq 2k$, and
2. if $|P| = 2k$ then $P$ is a $k$-badge.

Part (1) is Theorem 1.6, which as remarked above was proved in [ABC+19]. The proof given here (occupying most of this section) is different - it is done inductively, together with (2).

**Proof.** We shall prove (2), by induction on $k$. This suffices, since (1) is implied by (2). Assuming its negation, namely $|P| > 2k$, let $\mathcal{P}$ be a subset of $P$ of $2k + 1$ paths, let $P \in \mathcal{P}$ and let $B = \mathcal{P} \setminus \{P\}$. Assuming (2), $B$ is a $k$-badge, and by Corollary 2.8 $P \supseteq \mathcal{P} = B \cup \{P\}$ has a rainbow $F$-AAP. So, assuming the induction hypothesis we prove (2).

Remember that we are still within the proof of Theorem 2.9, meaning that we assume throughout that there is no $P$-rainbow $F$-AAP.

Let $U = \bigcup F$.

**Claim 2.10.** There exists an odd $P$-rainbow $F$-alternating cycle, containing a vertex that does not belong to $U$.

**Proof.** Choose $Q \in P$, and let $va$ be its first edge, where $v \notin U$ and $a \in U$. Let $b$ be the vertex matched to $a$ in $F$. If the pair $vb$ lies on a path belonging to $P$, then the triangle $vab$ is an odd rainbow cycle as desired. So, we may assume that $vb$ does not lie on a $P$ path. Let $F' = F \setminus \{ab\}$.

Remove the vertex $a$. Replace every $P \in P \setminus \{Q\}$ by an $F'$-AAP $P'$, as follows.

- If $P$ does not contain $ab$, let $P' = P$.
- If $P$ is a path of the form $P = RabS$, let $P' = bS$.

Let $\mathcal{P}' = \{P' \mid P \in P \setminus \{Q\}\}$. The matching $F'$ is of size $k - 1$, and $|\mathcal{P}'| = 2k - 1 > 2(k - 1)$. By the induction hypothesis for (1), there exists a $\mathcal{P}'$-rainbow $F'$-AAP $R$. $R$ contains $b$, otherwise it is a $P$-rainbow $F$-AAP, contrary to our assumption. Note that $v \in R$,
otherwise $R \cup \{ba, av\}$ is a $\mathcal{P}$-rainbow $F$-AAP. Hence $R \cup \{ba, av\}$ is the desired odd cycle. 

\noindent \textbf{Claim 2.11.} There is no odd $\mathcal{P}$-rainbow $F$-alternating cycle of length larger than 3.

\noindent \textbf{Proof.} Suppose that $C$ is a $\mathcal{P}$-rainbow $F'$-alternating cycle with $|C| = 2q + 1$ for $q > 1$. Contract $C$ to one vertex, and remove from $\mathcal{P}$ the set $\mathcal{P}_C$ of the paths in $\mathcal{P}$ represented by edges of $C$. This results in a matching $F'$ of size $k' := k - q$ and a family $\mathcal{P}'$ of $2k - q - 1$ walks, each of which contains an $F'$-alternating path. Since $q > 1$, we have $2k - q - 1 > 2k'$, so by the induction hypothesis on (1) there exists a $\mathcal{P}'$-rainbow $F'$-AAP $R'$. Note that every vertex in $C$ is reachable from $v$ by an even (possibly empty) $\mathcal{P}_C$-rainbow $F$-AAP. Therefore $R'$ can be extended by adding edges from $C$ to a $\mathcal{P}$-rainbow $F$-AAP, a contradiction. 

Combining the above claims gives a rainbow $F$-alternating triangle $C$, namely $C = \{va, ab, bv\}$ for some vertex $v \not\in U$ and edge $ab \in F$. As before, contract $C$ to a point $w$, and let $F' = F \setminus \{ab\}$. For every $P \in \mathcal{P}$ define a path $P'$ as follows:

- If $P$ does not pass through $a, b$ or $v$, let $P' = P$.
- If $P = P_1v$ and it does not pass through $a$ or $b$, let $P' = P_1w$.
- If $P = P_1abP_2$ or $P_1baP_2$, where $v$ possibly appears in $P_2$ but does not appear in $P_1$, let $P' = P_1w$.

Let $\mathcal{P}'$ be the resulting family of $2kF'$-AAPs.

\noindent \textbf{Claim 2.12.} Let $X$ be a path containing $av$ and let $Y$ be a path containing $bv$. Then $\mathcal{P}' \setminus \{X', Y'\}$ is a $(k - 1)$-badge.

\noindent \textbf{Proof.} Suppose that there exists a $(\mathcal{P}' \setminus \{X', Y'\})$-rainbow $F'$-AAP $R'$. If $R'$ does not contain $w$, then it is also a $\mathcal{P}$-rainbow $F$-AAP. Otherwise, $R'$ arises from an odd length $\mathcal{P}$-rainbow $F$-AAP $R$, whose final vertex is in $C$.

- If $a$ is an endpoint of $R$, then $R \cup \{ab, bv\}$ is a $\mathcal{P}$-rainbow $F$-AAP, since $bv \in Y$.
- If $b$ is an endpoint of $R$, then $R \cup \{ba, av\}$ is a $\mathcal{P}$-rainbow $F$-AAP, since $av \in X$.
- If $R$ ends in $v$, then $R$ itself is a $\mathcal{P}$-rainbow $F$-AAP.

So, we may assume that there is no such $R'$. The claim then follows from the induction hypothesis on (2). 

We know two such paths $T_a$ and $T_b$ exist because $C = \{va, ab, bv\}$ is a rainbow $F$-alternating triangle. Since there is no $\mathcal{P}$-rainbow $F$-AAP, by Claim 2.12 $\mathcal{P} \setminus \{T'_a, T'_b\}$ forms a $(k - 1)$-badge, to be named $B'$. 

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Case I. There is no origamistrip in $B'$ ending at $w$.

Case Ia. One of $T_a'$, $T_b'$ is a path of length $> 1$. This path then contains an edge not incident to $w$. Corollary 2.8 yields then a $P'$-rainbow $F'$-AAP $R$. Since in the presently considered case $B'$ has no edge incident to $w$, $R$ avoids $w$, so it is also a $P$-rainbow $F$-AAP.

Case Ib. $T_a' = wz$ and $T_b' = wt$ for some $z, t \notin U$, meaning that $T_a = vabz$ and $T_b = vbat$. If $z \neq t$, then $zbat$ is a $P$-rainbow $F$-AAP. If $z = t$, then $\{T_a, T_b\}$ form a 1-origamistrip, which added to $B'$ makes $P$ a $k$-badge.

Case II. There exists an $m$-origamistrip $OS'$ in $B'$ containing $w$ as an endpoint.

Let $E(OS') \cap F = \{u_1v_1, \ldots, u_nv_m\}$ and let $z$ be the other endpoint of $OS'$. Then there are $2m$ paths of $P' \setminus \{T_a', T_b'\}$ contained in $OS'$, say

- $Q_1' = \ldots = Q_m' = P^A(OS') = zu_1v_1u_2v_2\ldots u_nv_mw$, and
- $Q_{m+1}' = \ldots = Q_{2m}' = P^B(OS') = vz_1u_1v_2u_2\ldots v_mu_mw$.

Recall that each $Q_i'$ arose from a path $Q_i \in P$. For $i \leq m$, $Q_i$ contains exactly one of $v_m a$, $v_m b$ and $v_m v$, and for $m + 1 \leq i \leq 2m$, $Q_i$ contains exactly one of $u_m a$, $u_m b$ and $u_m v$.

Case IIa. $ab \notin Q_i$ for all $i \leq 2m$.

In this case, $Q_i'$ is equal to $Q_i$ for all $i$, with $v$ replaced by $w$. Then, replacing $w$ by $v$ in $OS'$ results in an origamistrip $OS$ in $P$. By the induction hypothesis, $P \setminus OS$ is a $(k-m)$-badge, hence $P$ is a $k$-badge.

Case IIb. $ab \in Q_i$ for some $i$.

Without loss of generality, $i \leq m$, and $a$ is closer than $b$ to $z$ on $Q_i$. So $Q_i$ is of the form $zu_1v_1\ldots u_nv_mabR$ for some path $R$.

To deal with this sub-case, the following will be used repeatedly.

Claim 2.13. Let $r$ be one element of $\{a, b\}$, and $s$ the other element.

(i) If $Q_j$ contains $v_m r$ for some $j \leq m$, then

$$T_s = Q_j = zu_1v_1\ldots u_nv_mrs.$$

(ii) If $Q_j$ contains $u_m r$ for some $j \geq m + 1$, then

$$T_s = Q_j = vz_1u_1\ldots v_mu_mrs.$$

Proof. By symmetry, it suffices to show (i). Without loss of generality, we may assume $r = a$ and $s = b$.

We first claim that $Q_j$ ends with $bv$. For otherwise, $Q_j$ contains an edge $by$, where $y \notin V(OS')$. If $y \notin U$, then $vaby$ is a $P$-rainbow $F$-AAP. If $y \in U$, then it belongs to another origamistrip $OS$ in $B'$. So it is possible to continue the path $vaby$ along one of the paths in $OS$.
to get a $\mathcal{P}$-rainbow $F$-AAP which ends at a vertex $v' \notin U$ with $v' \neq v$. To see the rainbow-ness, color $v_a$ by $T_a$, and the edge by by $Q_j$.

Next we show $T_b = Q_j$. By Claim 2.12, $(B' \setminus \{Q_j\}) \cup \{T_b\} = \mathcal{P}' \setminus \{T_a, Q_j\}$ is also a $(k - 1)$-badge, and since it shares all but one $F$-AAPs with $B'$, we must have $T'_b = Q'_j$ (two badges cannot differ in one path, namely they cannot have symmetric difference of size 2). As the paths $T_b$ and $Q_j$ both begin with $vba$, they must be identical. □

Claim 2.14. Let \( \{r, s\} = \{a, b\} \).

(i) If $T_s = Q_j = zu_1v_1 \ldots u_m v_m rsv$ for some $j \leq m$, then

\[Q_{m+1} = \ldots = Q_{2m} = T_r = zu_1v_1 \ldots v_m u_m svr.\]

(ii) If $T_s = Q_j = zv_1u_1 \ldots v_m u_m rsv$ for some $j \geq m + 1$, then

\[Q_1 = \ldots = Q_m = T_r = zv_1u_1 \ldots u_m v_m svr.\]

Proof. By symmetry it suffices to prove (i), and we assume without loss of generality $r = a$ and $s = b$.

Let $\ell \geq m + 1$. We claim that $Q_\ell$ contains the edge $u_m b$. Recall that $Q_\ell$ ends with $u_m w$. So if $Q_\ell$ does not contain $u_m b$, then it must contain either $u_m v$ or $u_m a$.

- If $Q_\ell$ contains $u_m v$, then the 5-cycle \{\(u_m v, vb, ba, av_m, v_m u_m\)\} is a rainbow odd $F$-alternating cycle (where $vb$ represents $T_b$, $av_m$ represents $Q_j$ and $u_m v$ represents $Q_\ell$). This contradicts Claim 2.11.
- If $Q_\ell$ contains the edge $u_m a$, then by Claim 2.13 we obtain

\[T_b = Q_\ell = zv_1u_1v_2u_2 \ldots v_m u_m abv,\]

but this contradicts $T_b = Q_j$.

Thus $Q_\ell$ must contain $u_m b$ for all $\ell \geq m + 1$. Applying Claim 2.13 to each $\ell$, it follows

\[T_a = Q_{m+1} = \ldots = Q_{2m} = zv_1u_1 \ldots v_m u_m bav.\]

□

Using Claim 2.13 and Claim 2.14, we determine $Q_j$ for every $j$.

Claim 2.15. If either $Q_i$ contains $v_m a$ for some $i \leq m$ or $Q_i$ contains $u_m b$ for some $i \geq m + 1$ then

(a) $Q_1 = \ldots = Q_m = T_b = zv_1u_1 \ldots u_m v_m abv$, and
(b) $Q_{m+1} = \ldots = Q_{2m} = T_a = zv_1u_1 \ldots v_m u_m bav$.

That is, they form an \((m + 1)\)-origamistrip $OS$. 9
Proof. Suppose $Q_i$ contains $v_m a$ for some $i \leq m$. By Claim 2.13, $Q_i = T_b = zu_1v_1 \ldots u_m v_m a b v$.

Since $Q_i = T_b = zu_1v_1 \ldots u_m v_m a b v$, Claim 2.14 gives $Q_{m+1} = \cdots = Q_{2m} = T_a = zv_1u_1 \ldots v_m u_m b a v$.

Applying Claim 2.14 with $Q_{m+1} = T_a = zv_1u_1 \ldots v_m u_m b a v$, we obtain $Q_1 = \cdots = Q_m = T_b = zu_1v_1 \ldots u_m v_m a b v$, as required. □

Here is a summary of what we have done so far. We proved that if $P$ is a set of $2k$ $F$-AAPs, where $F$ is a matching of size $k$, and there does not exist a $P$-rainbow $F$-AAP, then we can assume (applying an induction hypothesis) that there exists a triangle $C = \{va, ab, bv\}$, where $ab \in F$ and $v \notin U := \bigcup F$. We showed that contracting $C$ to a vertex $w$ results in a badge $B'$, containing an origamistrip $OS'$ having $w$ as one of its endpoints. We then showed how to open up $OS'$ to an origamistrip $OS$ containing $ab$, such that $ab$ is the first $F$-edge on both $PA(OS)$ and $PB(OS)$.

Remove $OS$, namely its $m + 1$ $F$-edges and $2m + 2$ $F$-AAPs. This leaves the matching $F \setminus M(OS)$ of size $k - m - 1$, along with $2k - 2m - 2$ $(F \setminus M(OS))$-AAPs, with no rainbow AAP. By the induction hypothesis for (2), these form a $(k - m - 1)$-badge $B$ with skeleton $F \setminus M(OS)$. Add back the origamistrip $OS$ to $B$, whose skeleton $M(OS)$ is disjoint from $F \setminus M(OS)$. This shows that $P$ is also a badge, proving (2). □

3. A COOPERATIVE GENERALIZATION

In Theorem 1.7 the rainbow matching of size $n$ was chosen from a collection of sets of edges, each being itself a matching of size $n$. As in various other results on rainbow sets, there are also “cooperative” versions, in which the assumption is not on each set individually, but on the union of sub-collections. For example, Bárány’s famous colorful Caratheodory theorem [Bár82] was given in [HPT08] a cooperative version, in which the requirement on individual sets, that their cone contains a given vector, is replaced by the condition that the union of every two sets satisfies this requirement. In this section we prove two such results:

(a) A cooperative version of Theorem 1.4 (Theorem 3.2), and
(b) A generalization of Theorem 1.7 valid for $n \geq 3$ (Theorem 3.3).

As before, the core of the proofs will be in results on rainbow paths in networks. Our point of departure is:
Theorem 3.1. Let $F$ be a matching of size $k$ in a graph $G$, let $t$ be a non-negative integer, and let $\mathcal{A} = (A_1, \ldots, A_m)$ be a family of sets of edges, satisfying the condition that the union of any $t + 1$ sets $A_i$ contains an $F$-AAP. Let $J = \{ j \mid A_j \subseteq F \}$. If there does not exist an $\mathcal{A}$-rainbow $F$-AAP then

1. $m = |\mathcal{A}| \leq 2k + t$, and
2. If $|\mathcal{A}| = 2k + t$, then $|J| = t$ and the $2k$ sets $A_j$, $j \notin J$ form a generalized $k$-badge.

See Definition 2.6 of “generalized badges”.

Proof of Theorem 3.1. By induction on $k + t$. Throughout the proof we assume, by negation, that there is no $\mathcal{P}$-rainbow $F$-AAP. If $k = 0$, i.e. if $F = \emptyset$, then both parts of the theorem are obvious since any edge forms a rainbow $F$-AAP. Hence we may assume $k > 0$.

Consider next the case $t = 0$. By the condition of the theorem, $A_i$ contains an $F$-AAP $P_i$ for every $i \leq m$, so (1) follows by Theorem 2.9. To prove (2), suppose $|\mathcal{A}| = 2k$. Since $P_i \notin F$, we have $|J| = 0$, as required in (2). Since $\{P_1, \ldots, P_{2k}\}$ do not have a rainbow $F$-AAP, they form a $k$-badge by Theorem 2.9. By Corollary 2.8, $A_i \setminus P_i \subseteq F$ for every $i$, meaning that the sets $A_i$ form a generalized badge.

Assume next that $k, t > 0$. It suffices to prove (2), since it implies (1). To see this, assume (2) and suppose $|\mathcal{A}| > 2k + t$. By (2), we have $|J \cap [2k + t]| = t > 0$. Let $i \in J \cap [2k + t]$. Again, by (2), we have $|J \cap ([2k + t + 1] \setminus \{i\})| = t$. Thus $|J| \geq t + 1$, and being contained in $F$, the set $\bigcup_{j \in J} A_j$ does not contain an $F$-AAP, contrary to assumption.

If $|J| = t$, then, for each $i \notin J$, applying the condition of the theorem to $J \cup \{i\}$ yields that $A_i$ contains an $F$-AAP. Let $\mathcal{A}' = \mathcal{A} \setminus \{A_i : i \in J\}$. As in the proof above of the case $t = 0$, Theorem 2.9 and Corollary 2.8 imply the conclusion of the theorem, namely that $\mathcal{A}'$ forms a generalized badge.

Now assume $|J| < t$. If $J \neq \emptyset$, say $A_i \subseteq F$ for some $i$, then the union of any $t$ sets in $\mathcal{A}' := \mathcal{A} \setminus \{A_i\}$ contains an $F$-AAP. Since $|\mathcal{A}'| = 2k + t - 1$ and $|J \setminus \{i\}| < t - 1$, applying the induction hypothesis on $t$ to the family $\mathcal{A}'$, there is a rainbow $F$-AAP. Thus it is sufficient to show that if $A_i \setminus F \neq \emptyset$ for all $i$, then there is necessarily an $\mathcal{A}$-rainbow $F$-AAP. Since $\mathcal{A}$ contains some $F$-AAP, we may assume that $A_1$ contains an edge $va$ where $v \notin \bigcup F$ and $ab \in F$. Let $F' = F \setminus \{ab\}$.

Suppose first that the edge $vb$ belongs to some $A_i$, $i \neq 1$, say to $A_2$. Let $G'$ be the graph obtained from $G$ by contracting $v, a, b$ to one vertex $\omega$. Consider $A_i' = A_i \setminus \{va, vb, ab\}$ as (possibly empty) edge sets in $G'$, where every edge of the form $xy$ for some $x \in \{a, b, v\}, y \notin \{a, b, v\}$ is
replaced by \(wy\). If \(\mathcal{A}'_1 = \{A'_3, \ldots, A'_{2k+t}\}\) has a rainbow \(F'\)-AAP in \(G'\), then it can be extended to a rainbow \(F\)-AAP in \(G\), a contradiction.

So, we may assume that \(\mathcal{A}'_1\) has no rainbow \(F'\)-AAP. Applying the induction hypothesis, and relabelling if necessary, we may assume that \(A'_{2k+1}, \ldots, A'_{2k+t} \subseteq F'\) and that \(\mathcal{B} := \{A'_3, \ldots, A'_{2k}\}\) is a generalized \((k-1)\)-badge.

Let \(i \in \{2k + 1, \ldots, 2k + t\}\). Since \(A'_i \subseteq F' \subset F\), recalling that \(A_i \setminus F \neq \emptyset\), it follows that \(A_i\) contains \(va\) or \(vb\). Without loss of generality, assume \(va \in A_{2k+t}\). Let \(\mathcal{A}'_2 = \{A'_1, A'_3, A'_4, \ldots, A'_{2k+t-1}\} = \mathcal{A}'_1 \cup \{A'_i\} \setminus \{A'_{2k+t}\}\). By the same reasoning as above, \(\mathcal{A}'_2\) does not contain a rainbow \(F'\)-AAP. Then necessarily \(A'_1 \setminus F' = \emptyset\), because \(\mathcal{B} \subseteq \mathcal{A}'_2\) is a \((k-1)\)-generalized badge, and the sets \(A'_i\) not belonging to it are, by assumption, contained in \(F'\). Therefore \(A_1, A_{2k+1}, \ldots, A_{2k+t} \subseteq F \cup \{va, vb\}\). But then these \(t+1\) sets have no \(F\)-AAP, a contradiction.

Thus we may assume that \(vb \notin A_i\) for any \(i > 1\). Let \(G_a = G - a, B_i = A_i \setminus \{a\}\) for each \(i > 1\).

Let \(\mathcal{D} = \{B_2, \ldots, B_{2k+t}\}\). Then the union of any \(t+1\) members of \(\mathcal{D}\) contains an \(F'\)-AAP in \(G_a\). Since

\[
|\mathcal{D}| = 2k + t - 1 > 2(k - 1) + t \quad \text{and} \quad |F'| = k - 1,
\]

by the induction hypothesis, there exists a \(\mathcal{D}\)-rainbow \(F'\)-AAP \(P\) in \(G_a\). If the two endpoints of \(P\) are \(v\) and \(b\), then \(P \cup \{ab, va\}\) is an odd rainbow \(F\)-alternating cycle of length at least 5 in \(G\), and the argument proceeds as in the proof of Claim 2.11: contract the cycle to a vertex \(w\), set aside the represented colour sets, use the inductive hypothesis for (1) to find an AAP \(R\) on what is left, and then use the edges of the odd cycle to make an \(F\)-AAP in \(G\) if \(R\) ends in \(w\). Thus we may assume that at least one of \(v\) and \(b\) is not an endpoint of \(P\). If \(v\) is an endpoint of \(P\) and \(b\) is not, then \(P\) is also an \(\mathcal{A}\)-rainbow \(F\)-AAP in \(G\). If \(b\) is an endpoint of \(P\) and \(v\) is not, then \(P \cup \{ab, va\}\) is an \(\mathcal{A}\) rainbow \(F\)-AAP in \(G\). In all cases we have thus found the desired rainbow \(F\)-AAP for contradiction.

We now use Theorem 3.1 to prove a cooperative rainbow matchings result. We shall write \((m, q, n) \rightarrow k\) for the statement “every \(m\) sets of edges in any graph, satisfying the condition that the union of every \(q\) of them contains a matching of size \(n\), have a rainbow matching of size \(k\)”.

**Theorem 3.2.** If \(n \geq 1\) and \(t \geq 0\) then \((3n - 2 + t, t + 1, n) \rightarrow n\).

**Proof.** The proof goes along similar lines to previous arguments. We induct on \(n\). The case \(n = 1\) is simple (the rainbow condition is always satisfied), so we may assume \(n > 1\).
Let $G$ be any graph and $E_1, \ldots, E_{3n-2+t}$ sets of edges in $G$ such that the union of every $t + 1$ of them contains a matching of size $n$. By the induction hypothesis, there exists a rainbow matching $F$ representing $E_j$, $j \in K$ for a set $K \subseteq [3n-2+t]$ of size $n-1$. Then $|[3n-2+t] \setminus K| = 2(n - 1) + t + 1$. For every set $I \subseteq [3n-2+t] \setminus K$ of size $t + 1$ the set $\bigcup_{i \in I} (E_i \cup F)$ contains a matching of size $n$, and hence an $F$-AAP. Hence, by (1) of Theorem 3.1, applied with $k = n - 1$, the family $(E_i \cup F \mid i \in [3n-2+t] \setminus K)$ has a rainbow $F$-AAP $R$. Since $R \setminus F$ and $F \setminus R$ are vertex disjoint matchings, $(R \setminus F) \cup (F \setminus R)$ is a rainbow matching of size $n$.

Putting $t = 0$ yields Theorem 1.4 $((3n - 2, n) \rightarrow n)$. The proof here is different from that in [ABC+19], in which the main weapons were the Edmonds-Gallai decomposition theorem and the Edmonds’ blossom algorithm.

For $n \geq 3$ Theorem 3.2 can be improved, to yield a cooperative version of Theorem 1.7 valid for $n \geq 3$.

**Theorem 3.3.** Let $t \geq 0$ and $n \geq 3$. Then $(3n - 3 + t, t + 1, n) \rightarrow n$.

**Proof.** Let $n \geq 3$, $t \geq 0$. Let $E_1, \ldots, E_{3n-3+t}$ be (possibly empty) edge sets in a graph $G$ such that the union of any $t + 1$ of them contains a matching of size $n$. By Theorem 3.2 we know

$$(3(n - 1) - 2 + t, t + 1, n - 1) \rightarrow n - 1.$$  

Since $3n - 3 + t \geq 3(n - 1) - 2 + t$ we can find a rainbow matching $F$ of size $n - 1$. Without loss of generality, let $F = \{e_1, \ldots, e_{n-1}\}$ where $e_i \in E_i$ for each $i \in [n - 1]$.

Since $\bigcup_{i \in I} E_i$ contains a matching of size $n$ for each $I \subseteq [3n-3+t] \setminus [n-1]$ of size $t + 1$, the set $(\bigcup_{i \in I} E_i) \cup F$ of edges contains an $F$-AAP. If there exists a rainbow $F$-AAP $Q$, then $(Q \setminus F) \cup (F \setminus Q)$ is a rainbow matching of size $n$.

Thus, assuming negation, there is no rainbow $F$-AAP. Then by part (2) of Theorem 3.1, we may assume that $E_{3n-2}, \ldots, E_{3n-3+t} \subseteq F$ and $B = \{E_n, \ldots, E_{3n-3}\}$ forms an $(n - 1)$-generalized badge. For each $j \in [3n-3]$, let $D_j = E_j \cup E_{3n-2} \cup \cdots \cup E_{3n-3+t}$.

Suppose there is a single $(n - 1)$-origamistrip $OS$ in $B$. Let $x$ and $y$ be its endpoints. We may assume that for each $i \leq n - 1$, $e_i$ is the $i$-th edge of $F$ on the path $PA^k(OS)$ starting at $x$. Since $D_1$ contains a matching of size $n$, it must contain an edge not belonging to $F$. We next proceed as in the proof of Lemma 2.7 to use this edge to find a rainbow matching of size $n$. 


Claim 3.4. If some $e \in D_1$ is not incident to any vertex of $OS$, then there is a rainbow matching of size $n$.

Proof. Since there are $n-1$ copies of $P^A(OS)$ there is a rainbow matching $R \subseteq A(OS) = P^A(OS) \setminus F$ of $B$ of size $n-1$. Then $R \cup \{e\}$ is a rainbow matching of size $n$. \qed

Claim 3.5. If some $e \in D_1$ connects a vertex $z$ of $OS$ and a vertex not in $OS$, then there is a rainbow matching of size $n$.

Proof. If $z \in \{x,y\}$, take a rainbow matching $R \subseteq A(OS)$ of $B$ of size $n-1$ that is disjoint from $e$. Then $R \cup \{e\}$ is a rainbow matching of size $n$. Thus we may assume $z \notin \{x,y\}$.

Recalling Observation 2.2 for the definition of $Q^i_z$, let $Q = Q^i_z + e$. Then $Q \setminus F$ is a rainbow matching of size $|Q \cap F| + 1$ for the family \{\(E_i, E_{n+1}, \ldots, E_{n-3}\). Since $Q$ contains $e_1$, $F \setminus Q$ is a rainbow matching for \{\(E_2, \ldots, E_{n-1}\). Thus $(Q \setminus F) \cup (F \setminus Q)$ is a rainbow matching. Clearly, its size is $n$, attaining the desired goal. \qed

By Claim 3.4 and Claim 3.5, we may assume that any edge of $D_1$ has both endpoints in $OS$. Since $D_1$ contains a matching of size $n$ and $OS$ consists of $2n$ vertices, $D_1$ contains a perfect matching on $OS$. In particular, it contains an edge $e$ that is incident to $y$.

Let $e = zy$. Suppose first $z \neq x$. Let $Q$ be a rainbow $F$-alternating path of even length from $x$ to $z$ and let $Q' = Q + e$. Then $Q' \setminus F$ is a rainbow matching of size $|Q \cap F| + 1$. Since $Q'$ contains $e_1$, it follows that $(Q' \setminus F) \cup (F \setminus Q')$ is a rainbow matching of size $n$. So, we may assume that $e = zy$. Since $n-1 \geq 2$, there are disjoint edges $e_A \in A(OS)$ and $e_B \in B(OS)$ that connect $e_1$ and $e_2$. Then \{\(e, e_A, e_B, e_3, \ldots, e_{n-1}\) is a rainbow matching of size $n$.

Now suppose there are two different origamistrips in $B$. For each $n \leq i \leq 3n-3$, let $OS_i$ be the origamistrip that contains $D_i \setminus F$.

Claim 3.6. Suppose $D_i \setminus F = A(OS_i)$. Then $D_i \supseteq A(OS_i) \cup (F \setminus M(OS_i))$.

Proof. Clearly, $A(OS_i) \subseteq D_i$. Thus it is sufficient to show that $F \setminus M(OS_i) \subseteq D_i$. By the assumption, $OS_i$ is a $k$-origamistrip for some $k < n-1$, and hence $|D_i \setminus F| = k + 1 < n$. Since $D_i$ contains a matching of size $n$, there are edges of $D_i$ that are not incident to any vertex of $OS_i$. By Corollary 2.8, such edges should be included in $F \setminus OS_i$. Since $|A(OS_i)| = k + 1$ and $|F \setminus M(OS_i)| = n - 1 - k$, it follows that $F \setminus M(OS_i) \subseteq D_i$. \qed

Now take any $e_j \in F$ and let $OS$ be the origamistrip that contains $e_j$. Since $D_j$ contains a matching of size $n$ and $|F| = n-1$, there
exists an edge \( e \in D_j \setminus F \). Let \( B' = B \cup \{\{e\}\} \). By Corollary 2.8, there exists a \( B' \)-rainbow \( F \)-AAP \( Q \) containing \( e \). If \( Q \) contains \( e_j \), then \((Q \setminus F) \cup (F \setminus Q)\) is a rainbow set. Otherwise, \( Q \) is contained in an origamistrip \( OS' \) that is different from \( OS \), and \((Q \setminus F) \cup (F \setminus Q)\) contains both \( e \) and \( e_j \). We will show that \( e_j \) is contained in some \( E_{i_0} \) that did not participate in \( Q \). Then it follows that \((Q \setminus F) \cup (F \setminus Q)\) is a rainbow set.

If \( e \in E_{3n-2} \cup \cdots E_{3n-3+t} \), then we can take \( i_0 \in \{3n-2, \ldots, 3n-3+t\} \). Suppose not. Without loss of generality, assume \( Q = P^A(OS') \). Take any \( i_0 \in \{n, \ldots, 3n-3\} \) so that \( E_{i_0} \setminus F \subset B(OS') \). Then by Claim 3.6, \( D_{i_0} \) contains \( e_j \), implying that \( e_j \in E_{i_0} \), as required.

The following examples show why the condition \( n \geq 3 \) is indeed necessary.

**Example 3.7.** Consider first the case \( n = 1 \). Then \( t \) empty sets vacuously satisfy the condition that any \( t+1 \) of them (satisfy any condition), and they do not have a rainbow matching of size 1.

For \( n = 2 \), let \( G = ([4], \{4\}) = K_4 \) and let \( E_1 = \{12, 34\} \), \( E_2 = \{13, 24\} \), \( E_3 = \{14, 23\} \), and \( E_i = \emptyset \) for all \( 3 < i \leq t + 3 \). Then the union of any \( t+1 \) of \( E_i \)'s contains a matching of size 2, and yet there is no rainbow matching of size 2.

In fact, these are the only examples showing that 3n-3+t colors do not suffice. Let \( t \geq 0 \) and \( F_1, \ldots, F_{t+3} \) be sets of edges of any graph \( G \) such that the union of any \( t+1 \) \( F_i \)'s contains a matching of size 2. Suppose there is no rainbow matching of size 2. We may assume that \( F_1 \neq \emptyset \). Take any edge \( e \in F_1 \). By Theorem 3.1 we may assume that \( F_2 \cup \{e\} \) and \( F_3 \cup \{e\} \) are \( \{e\} \)-AAPs from \( u \) to \( v \) forming a 1-badge, and \( F_i \subseteq \{e\} \) for each \( 4 \leq i \leq t+3 \).

Note that \( F_1 \cup F_4 \cup F_5 \cup \cdots \cup F_{t+3} \) contains a matching of size 2. Since it is not rainbow, this implies the existence of an edge \( f \neq e \) in \( F_1 \). Moreover, \( F_1 = \cdots = F_{t+3} = \emptyset \). If \( f \neq uv \), then there is a rainbow matching of size 2 by Claim 3.4 and Claim 3.5. Thus \( f = uv \), meaning that \( F_1 = \{e, f\} \). Then \( F_2 \) and \( F_3 \) do not contain \( e \): otherwise, \( \{e, f\} \) is a rainbow matching of size 2. This completes the proof.

**Remark 3.8.** A result in [HI20] implies a slightly weaker version of Theorem 3.3. They gave a topological proof of \((3n-3+t, t+1, n) \to n \) for all \( n \geq 1 \) and \( t > 0 \) when at least \( 3n-2 \) sets are nonempty.

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