Cartan-Weyl basis for Quantum Affine Superalgebra

\[ U_q(\widehat{\mathfrak{osp}(1|2)}) \]

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Abstract
Cartan-Weyl basis for the quantum affine superalgebra \( U_q(\widehat{\mathfrak{osp}(1|2)}) \) is constructed in an explicit form.

1 Preliminaries

1. Affine Lie superalgebra \( B(0,1)^{(1)} \). The affine untwisted Lie superalgebra (or in other words ”Kac-Moody affine Lie superalgebra”) \( B(0,1)^{(1)} \) has the rank 2, and its Dynkin diagram is presented by the picture

\[
\begin{array}{c}
\alpha_0 \\
\bullet \\
\alpha
\end{array}
\]

Fig.1. Dynkin diagram of the Lie superalgebra \( B(0,1)^{(1)} \)

where the dark root \( \alpha \) is odd and the white root \( \alpha_0 \) is even. The corresponding standard \( A = (a_{ij}) \) Cartan matrix has the form:

\[
A = \begin{pmatrix}
2 & -1 \\
-4 & 2
\end{pmatrix}.
\]

(1.1)

The determinant of this matrix is equal to zero: \( \det A = 0 \). Moreover we shall use the symmetric \( A^{sym} = (a^{sym}_{ij})_{i,j=0,1} \) Cartan matrix:

\[
A^{sym} = \begin{pmatrix}
(\alpha_0, \alpha_0) & (\alpha_0, \alpha) \\
(\alpha, \alpha_0) & (\alpha, \alpha)
\end{pmatrix} = \begin{pmatrix}
4(\alpha, \alpha) & -2(\alpha, \alpha) \\
-2(\alpha, \alpha) & (\alpha, \alpha)
\end{pmatrix},
\]

(1.2)

and a non-singular extended symmetric Cartan matrix \( \bar{A}^{sym} = (\bar{a}^{sym}_{ij})_{i,j=-1} \) and also its inverse \( (\bar{A}^{sym})^{-1} = (d_{ij})_{i,j=-1} \):

\[
\bar{A}^{sym} = \begin{pmatrix}
0 & 1 \\
1 & 4(\alpha, \alpha) -2(\alpha, \alpha) \\
0 & -2(\alpha, \alpha)
\end{pmatrix}, \quad (\bar{A}^{sym})^{-1} = \begin{pmatrix}
0 & 1 & 2 \\
1 & 0 & 0 \\
2 & 0 & \frac{1}{(\alpha, \alpha)}
\end{pmatrix}.
\]

(1.3)

In the case of affine Lie (super)algebras it is convenient to number rows and columns of the extended symmetric Cartan matrix \( \bar{A}^{sym} = (\bar{a}^{sym}_{ij})_{i,j=-1} \) by \(-1,0,\ldots, r\) in such way that the first row and column

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are numbered beginning with -1 then 0 and so on. Here we keep this convention. It is also useful to note that the matrix elements of the standard and symmetric matrices are connected by the relation
\[ a_{ij} = \frac{2a_{ij}^{\text{sym}}}{a_{ij}^{\text{km}}} \, . \] (1.4)

The Lie superalgebra \( B(0,1)^{(1)} \) is generated by the Chevalley basis \( \{ h_d, h_\alpha, h_{\alpha_0}, e_{\pm \alpha}, e_{\pm \alpha_0} \} \) with the defining relations

\[ [h_\gamma, h_{\gamma'}] = 0 \quad (\gamma, \gamma' = \alpha_0, \alpha) \, , \] (1.5)

\[ [e_\beta, e_{-\beta'}] = \delta_{\beta,\beta'} h_\beta \quad (\beta, \beta' = \alpha, \alpha_0) \, , \] (1.6)

\[ [h_d, e_{\pm \alpha_0}] = \pm e_{\pm \alpha_0} \, , \quad [h_d, e_{\pm \alpha}] = 0 \, , \] (1.7)

\[ [h_{\alpha_0}, e_{\pm \alpha_0}] = \pm 4(\alpha, \alpha)e_{\pm \alpha_0} \, , \quad [h_{\alpha_0}, e_{\pm \alpha}] = \mp 2(\alpha, \alpha)e_{\pm \alpha} \, , \] (1.8)

\[ [h_\alpha, e_{\pm \alpha}] = \mp 2(\alpha, \alpha)e_{\pm \alpha} \, , \quad [h_\alpha, e_{\pm \alpha}] = \pm (\alpha, \alpha)e_{\pm \alpha} \, , \] (1.9)

\[ [e_{\pm \alpha}, [e_{\pm \alpha}, e_{\pm \alpha}, e_{\pm \alpha}]]] = 0 \, , \quad [e_{\pm \alpha}, e_{\pm \alpha}, e_{\pm \alpha}] = 0 \, . \] (1.10)

Here the supercommutator \([\cdot, \cdot]\) is given by the formula

\[ [a, b] = ab - (-1)^{\deg a \deg b} ba \, , \] (1.12)

where

\[ \deg h_d = \deg h_\alpha = \deg h_{\alpha_0} = \deg e_{\pm \alpha_0} = 0 \, , \quad \deg e_{\pm \alpha} = 1 \, . \] (1.13)

It should be noted that a subalgebra generated by the elements \( h_\alpha, e_{\pm \alpha} \) is isomorphic to \( osp(1|2) \). A subalgebra of \( B(0,1)^{(1)} \) generated by the elements \( h_d, h_{\alpha_0}, h_\alpha \) is called the Cartan subalgebra \( \mathcal{H} \) of affine Lie superalgebra \( B(0,1)^{(1)} \). It is easy to see that the element

\[ h_\delta := h_{\alpha_0} + 2h_\alpha \] (1.14)

is a center of superalgebra \( B(0,1)^{(1)} \), i.e. it commutes with every element of \( B(0,1)^{(1)} \)

\[ [h_\delta, \text{everything}] = 0 \, . \] (1.15)

From (1.14) we have that

\[ h_{\alpha_0} = h_\delta - 2h_\alpha = h_{\delta - 2\alpha} \, . \] (1.16)

Moreover taking (1.7)-(1.9) into account we also obtain that

\[ (\delta, \alpha) = (\delta, \delta) = (d, \alpha) = 0 \, , \quad (\delta, d) = 1 \, . \] (1.17)

Frequently we shall also use the notation \( \delta - 2\alpha \) instead of \( \alpha_0 \). The root \( \delta = \alpha_0 + 2\alpha \) is called the minimal positive imaginary root of \( B(0,1)^{(1)} \).

2. **Current realization of \( B(0,1)^{(1)} \)**. One of remarkable properties of affine Lie algebras is that they admit an exact current or loop realization, i.e. there is an isomorphism between affine Lie algebras and some extension of current algebras over finite-dimensional simple Lie algebras. In particular this property allows easily to obtain the total list of roots and commutation relations of Cartan - Weyl basis for the affine Lie
algebras. This property is also valid for the supercase $B(0,1)^{(1)}$. We briefly remind the current realization (see [1]).

Let $\mathcal{L}(g) := g[u,u^{-1}]$ be a linear space of Laurent polynomials in variable $u$ with coefficients from a simple finite-dimensional contragredient Lie (super)algebra $g$, i.e. any element $x \in \mathcal{L}(g)$ has the form $x = \sum_{n \in \mathbb{Z}} a_n u^n$ where all but a finite number of $a_k \in g$ are $0$. We specify a Lie (super)algebra structure on $\mathcal{L}(g)$ with the (super)bracket $[\cdot, \cdot]_0$ defined as follows

$$[au^n, bu^m]_0 = [a, b] u^{n+m} \quad (a, b \in g; \ n, m \in \mathbb{Z}).$$

(1.18)

It should be noted that the $\mathbb{Z}_2$-grading function on $\mathcal{L}(g)$ is defined by the condition

$$\deg au^n = \deg a .$$

(1.19)

Let $(\cdot | \cdot)$ be a non-degenerate invariant supersymmetric bilinear $\mathbb{C}$-valued form on $g$. In the case of superalgebra $osp(1|2)$ such form is the Killing form and it is unique up to a constant multiple. We extend this form by linearity to an $\mathbb{C}[u, u^{-1}]$-valued bilinear form $(\cdot | \cdot)_L$ on $\mathcal{L}(g)$ by

$$(au^n|bu^m)_L = (a|b) u^{n+m} \quad (a, b \in g; \ n, m \in \mathbb{Z}).$$

(1.20)

We also define the derivation $\frac{d}{du}$ of the Lie (super)algebra $\mathcal{L}(g)$ by

$$\frac{d}{du}(au^n) = a(au^n)' = nau^{n-1} \quad (a \in g; \ n \in \mathbb{Z}).$$

(1.21)

Now one can define a $\mathbb{C}$-valued 2-cocycle $\psi(\cdot, \cdot)$ on the Lie (super)algebra $\mathcal{L}(g)$ as follows:

$$\psi(x, y) = \text{Res} \left( \frac{d}{du}(x)|y \right)_L .$$

(1.22)

where the linear function (residue) Res is defined by usual way, i.e. $\text{Res}(u) = c_{-1}$ if $p(u) = \sum_k c_k u^k$ ($c_k \in \mathbb{C}$). In particular, for $x = au^n$, $b = bu^m$ ($a, b \in g; \ n, m \in \mathbb{Z}$) we have

$$\psi(au^n, bu^m) := n \delta_{n,-m}(a|b) .$$

(1.23)

Recall that $\mathbb{C}$-valued 2-cocycle on a Lie (super)algebra $\mathcal{A}$ is a bilinear $\mathbb{C}$-valued function $\psi(\cdot, \cdot)$ satisfying two conditions:

$$\psi(x, y) = (-1)^{1+\deg x \deg y} \psi(y, x) ,$$

(1.24)

$$(-1)^{\deg x \deg z} \psi([x, y], z) + (-1)^{\deg z \deg x} \psi([z, y], x) + (-1)^{\deg x \deg y} \psi([z, x], y) = 0 .$$

(1.25)

for any homogeneous elements $x, y, z \in \mathcal{A}$. It is not difficult to check that our function (1.22) satisfies these conditions.

Denote by $\hat{\mathcal{L}}(g)$ the extension of the Lie (super)algebra $\mathcal{L}(g)$ by a 1-dimensional center, associated to the cocycle $\psi$. Explicitly the (super)algebra Lie $\hat{\mathcal{L}}(g)$ is the direct sum of vector spaces:

$$\hat{\mathcal{L}}(g) = \mathcal{L}(g) \oplus \mathbb{C}c .$$

(1.26)

with the (super)bracket defined by

$$[au^n + \lambda c, bu^m + \mu c] = [a, b] u^{n+m} + n \delta_{n,-m} \psi(a|b)c \quad (a, b \in g; \ n, m \in \mathbb{Z}; \ \lambda, \mu \in \mathbb{C}) .$$

(1.27)

Finally we denote by $\hat{\mathcal{L}}(g)$ the Lie (super)algebra which is obtain by adjoint to $\hat{\mathcal{L}}(g)$ a derivation $d$ which acts on $\mathcal{L}(g)$ as $u \frac{d}{du}$ and kills $c$. More explicitly, $\hat{\mathcal{L}}(g)$ is a complex vector space

$$\hat{\mathcal{L}}(g) = \hat{\mathcal{L}}(g) \oplus \mathbb{C}c \oplus \mathbb{C}d .$$

(1.28)
with the (super)brackets defined by
\[
[au^n + \lambda_1 c + \mu_1 d, bu^m + \lambda_2 c + \mu_2 d] = [a, b]u^{n+m} + n\delta_{n,-m}(a|b)c + \mu_1 mbu^m - \mu_2 nau^n. \quad (1.29)
\]
for any \(a, b \in g; \ n, m \in \mathbb{Z}; \ \lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{C} \). The (super)algebra \( \hat{\mathcal{L}}(g) \) is often denoted as \( \hat{g} \) and if it is isomorphic to \( g^{(1)} \) then \( \hat{g} \) is called the current realization of \( g^{(1)} \).

In our case, i.e. when \( g = \mathfrak{osp}(1|2) \), we can show that the superalgebra \( B(0,1)^{(1)} \) is isomorphic to \( \mathfrak{osp}(1|2) \). The isomorphism is arranged as
\[
h_d \mapsto d, \quad h_\delta \mapsto c, \quad h_\alpha \mapsto h_\alpha, \quad (1.30)
\]
\[
e_{\pm \alpha} \mapsto e_{\pm \alpha}, \quad e_{\pm(\delta-2\alpha)} \mapsto \mp \sqrt{\frac{2}{(\alpha, \alpha)}} e_{-2\alpha}u^{\pm1}. \quad (1.31)
\]
This isomorphism immediately allows us to describe the total root system and the root space decomposition of \( B(0,1)^{(1)} \) with respect to the Cartan subalgebra \( \mathcal{H} \) since such description is easy obtained for \( \mathfrak{osp}(1|2) \).

Indeed, the elements \( h_\alpha u^n, e_{\pm \alpha} u^n, e_{\pm 2\alpha} u^n \) for all \( n \in \mathbb{Z} \) are weight vectors with respect to \( \mathcal{H} = \mathbb{C} h_\alpha \oplus \mathbb{C} c \oplus \mathbb{C} d \) and they form a basis in \( \mathcal{L}(\mathfrak{osp}(1|2)) \). Let \( \delta \) be a linear function on \( \mathcal{H} \) defined by \( \delta|_{\mathfrak{e}_{h_\alpha} + \mathfrak{e}_c} = 0 \), \( \delta(d) = 1 \) then the total root system of \( \mathfrak{osp}(1|2) \) has the form
\[
\Delta := \{ n\delta \pm \alpha; n\delta \pm 2\alpha; n\delta (n \neq 0) \mid n \in \mathbb{Z} \} \quad (1.32)
\]
and its root space decomposition can be presented as
\[
\mathfrak{osp}(1|2) = \mathcal{H} \oplus \left( \bigoplus_\gamma \mathfrak{osp}(1|2)_\gamma \right), \quad (1.33)
\]
where
\[
\mathfrak{osp}(1|2)_{n\delta \pm \alpha} = \mathbb{C} e_{\pm \alpha} u^n, \quad \mathfrak{osp}(1|2)_{n\delta \pm 2\alpha} = \mathbb{C} e_{\pm 2\alpha} u^n \quad (n \in \mathbb{Z}), \quad (1.34)
\]
\[
\mathfrak{osp}(1|2)_{n\delta} = \mathbb{C} h_\alpha u^n \quad (n \in \mathbb{Z} \setminus \{0\}). \quad (1.35)
\]
We set
\[
e_{n\delta \pm 2\alpha} = (-1)^n \sqrt{\frac{2}{(\alpha, \alpha)}} e_{\pm 2\alpha} u^n, \quad e_{-n\delta \mp 2\alpha} = \sqrt{\frac{2}{(\alpha, \alpha)}} e_{\mp 2\alpha} u^{-n}, \quad (1.36)
\]
\[
e_{n\delta \pm \alpha} = \pm (-1)^n e_{\pm \alpha} u^n, \quad e_{-n\delta \mp \alpha} = \pm e_{-\alpha} u^{-n}, \quad (1.37)
\]
\[
e_{n\delta} = \frac{(-1)^{n+1}}{\sqrt{(\alpha, \alpha)}} h_\alpha u^n, \quad e_{-n\delta} = \frac{-1}{\sqrt{(\alpha, \alpha)}} h_\alpha u^{-n}. \quad (1.38)
\]
where \( n \in \mathbb{Z}_+ \setminus \{0\} \). The formulas \((1.34)-(1.38)\) are called the current realization of the affine root vectors of \( B(0,1)^{(1)} \). The root vectors \((1.36)-(1.38)\) satisfy the relations \((3.3)-(3.13)\) where \( q = 1 \). In particular we have
\[
[e_{n\delta - 2\alpha}, e_{-n\delta + 2\alpha}] = (-1)^{n+1} h_{n\delta - 2\alpha}, \quad [e_{n\delta + 2\alpha}, e_{-n\delta - 2\alpha}] = (-1)^n h_{n\delta + 2\alpha}, \quad (1.39)
\]
\[
[e_{n\delta - \alpha}, e_{-n\delta + \alpha}] = (-1)^{n+1} h_{n\delta - \alpha}, \quad [e_{n\delta + \alpha}, e_{-n\delta - \alpha}] = (-1)^n h_{n\delta + \alpha},
\]
where \( n \in \mathbb{Z}_+ \setminus \{0\} \).
Now we return to the root system (1.30). The roots \( n\delta \ (n \in \mathbb{Z} \setminus \{0\} \) are called imaginary roots, the remaining ones are called the real roots. Moreover we have \( \Delta = \Delta_0 + \Delta_1 \), where \( \Delta_0 = \{ \pm 2\alpha; \ n\delta \pm 2\alpha, \ n\delta \mid n \in \mathbb{Z} \setminus \{0\} \) is the even root system and \( \Delta_1 = \{ n\delta \pm \alpha \mid n \in \mathbb{Z} \} \) is the odd root system. The positive root system is

\[
\Delta_+ := \{ \alpha, 2\alpha; \ n\delta \pm \alpha, n\delta \pm 2\alpha, n\delta \mid n \in \mathbb{Z}_+ \setminus \{0\} \}.
\]

(1.40)

We denote by \( \Delta_+ \) the reduced system of positive roots, which is obtained from \( \Delta_+ \) by removing the double real roots \( 2\alpha, 2n\delta \pm 2\alpha \ (n \in \mathbb{Z}_+ \setminus \{0\} \). It is convenient to present the total root system \( \Delta \) by the picture 3:

![Fig.2. The total root system \( \Delta \) of \( \widehat{\text{osp}}(1|2) \)]

The picture of the reduced root system looks as follows:

![Fig.3. The total reduced root system \( \widetilde{\Delta} \) of \( \widehat{\text{osp}}(1|2) \)]

\section{Defining relations of \( U_q(\widehat{\text{osp}}(1|2)) \)}

The q-deformation of the universal enveloping algebra \( U(\widehat{\text{osp}}(1|2)) \) (or in other words the quantum affine superalgebra \( U_q(\widehat{\text{osp}}(1|2)) \)) is generated by the Chevalley elements \( k_\gamma^\pm := q^{\pm h_\gamma}, \ k_\alpha^\pm := q^{\pm h_\alpha}, \ k_{\delta-2\alpha}^\pm := q^{\pm h_{\delta-2\alpha}}, \ e_{\pm \alpha}, \ e_{(\delta-2\alpha)} \) with the defining relations 3, 4

\[
k_\gamma k_\gamma^{-1} = k_\gamma^{-1} k_\gamma = 1 \quad (\gamma = d, \alpha, \delta-2\alpha),
\]

(2.1)

\[
[k_\gamma^\pm, k_\gamma'^\pm] = 0 \quad (\gamma, \gamma' = d, \alpha, \delta-2\alpha),
\]

(2.2)

\[
k_\gamma e_{\pm \beta} k_\gamma^{-1} = q^{\pm (\gamma, \beta)} e_{\pm \beta} \quad (\beta = \alpha, \delta-2\alpha; \ \gamma = d, \beta),
\]

(2.3)

\[
e_{\beta} e_{-\beta'} = \delta_{\beta,\beta'} \frac{k_\beta - k_\beta^{-1}}{q-q^{-1}} \quad (\beta, \beta' = \alpha, \delta-2\alpha),
\]

(2.4)

\[
[e_{\pm \alpha}, [e_{\pm \alpha}, [e_{\pm \alpha}, [e_{\pm \alpha}, e_{(\delta-2\alpha)}]]]]_q q_q q = 0,
\]

(2.5)
where the $q$-bracket $[ \cdot, \cdot ]_q$ means the following $q$-supercommutator
\[ [e_{\beta}, e_{\beta'}]_q = e_{\beta} e_{\beta'} - (-1)^{\vartheta(\beta)\vartheta(\beta')}_q e_{\beta'} e_{\beta}, \]
\[ (2.7) \]

where the following notation is used
\[ \vartheta(\beta) := \deg e_{\beta}. \]
\[ (2.8) \]

A Hopf structure of the quantum superalgebra $U_q(\widehat{\mathfrak{osp}}(1|2))$ is given by the following formulas for a comultiplication $\Delta_q$, an antipode $S_q$ and a counit $\epsilon_q$:
\[ \Delta_q(k^\pm_1) = k^\pm_1 \otimes k^\pm_1, \quad \Delta_q(e_{-\beta}) = e_{-\beta} \otimes 1 + k^{-1}_{-\beta} \otimes e_{-\beta}, \]
\[ (2.9) \]
\[ \Delta_q(e_{-\beta}) = e_{-\beta} \otimes k_{-\beta} + 1 \otimes e_{-\beta}, \]
\[ (2.10) \]
\[ S_q(k^\pm_1) = k^\pm_1, \quad S_q(e_{-\beta}) = -k_{-\beta} e_{-\beta}, \quad S_q(e_{-\beta}) = -e_{-\beta} k^{-1}_{-\beta}, \]
\[ (2.11) \]
\[ \epsilon_q(k^\pm_1) = 1, \quad \epsilon_q(e_{\pm\beta}) = 0, \]
\[ (2.12) \]

where $\beta = \alpha, \delta - 2\alpha; \gamma = d, \alpha, \delta - 2\alpha.$

### 3 Cartan-Weyl basis for $U_q(\widehat{\mathfrak{osp}}(1|2))$

A general scheme for construction of Cartan-Weyl basis for quantized Lie algebras and superalgebras was proposed in [3]. The scheme was applied in detail at first for quantized finite-dimensional Lie (super)algebras [4] and then to quantized non-twisted affine algebras [6].

This procedure is based on a notion of “normal ordering” for the reduced positive root system. For affine Lie (super)algebras this notation was formulated in [5] (see also [3, 7, 8]). In our case the reduced positive system has only two normal orderings:
\[ \alpha, \delta + 2\alpha, \delta + \alpha, 3\delta + 2\alpha, 2\delta + \alpha, \ldots, \infty \delta + \alpha, (2\infty + 1)\delta + 2\alpha, (\infty + 1)\delta + \alpha, \delta, 2\delta, \ldots, \]
\[ (3.1) \]
\[ \infty \delta, (\infty + 1)n\delta - \alpha, (2\infty + 1)\delta - 2\alpha, \infty \delta - \alpha, \ldots, 2\delta - \alpha, 3\delta - 2\alpha, \delta - \alpha, \delta - 2\alpha, \]
\[ \delta - 2\alpha, \delta - \alpha, 3\delta - 2\alpha, 2\delta - \alpha, \ldots, \infty \delta - \alpha, (2\infty + 1)\delta - 2\alpha, (\infty + 1)n\delta - \alpha, \delta, 2\delta, \ldots, \]
\[ (3.2) \]

The first normal ordering (3.1) corresponds to “clockwise” ordering of positive roots on Fig.3 starting from root $\alpha$ to root $\delta - 2\alpha$. The inverse normal ordering (3.2) corresponds to “anticlockwise” ordering for the positive roots when we move from $\delta - 2\alpha$ to $\alpha$.

We choose the normal ordering (3.1) and in accordance with the procedure [3, 4, 6] we put
\[ e_{\delta - \alpha} := \frac{1}{\sqrt{a}} [e_{\alpha}, e_{\delta - 2\alpha}]_q, \]
\[ e_{\delta} := \frac{1}{\sqrt{b}} [e_{\alpha}, e_{\delta - 2\alpha}]_q, \]
\[ e_{n\delta + \alpha} := \frac{1}{\sqrt{b}} [e_{(n-1)\delta + \alpha}, e_{\delta}] = e_{(n+1)\delta - \alpha} := \frac{1}{\sqrt{b}} [e_{\delta}, e_{n\delta - \alpha}], \]
\[ e_{n\delta - \alpha} := \frac{1}{\sqrt{b}} [e_{\delta}, e_{n\delta - \alpha}], \quad e_{(n+1)\delta + \alpha} := \frac{1}{\sqrt{b}} [e_{n\delta + \alpha}, e_{\delta}], \]
\[ (3.3) \]
\[ (3.4) \]
\[ (3.5) \]
\[ (3.6) \]
\[
\epsilon'_{n\delta} = \frac{1}{\sqrt{b}} [\epsilon_{n\delta + \alpha}, \epsilon_{n\delta - \alpha}]_q , \quad \epsilon'_{n\delta} = \frac{1}{\sqrt{b}} [\epsilon_{-n\delta + \alpha}, \epsilon_{-n\delta - \alpha}]_{q^{-1}} ,
\] (3.7)

\[
\epsilon_{(2n-1)\delta + 2\alpha} := \frac{1}{\sqrt{a}} [\epsilon_{(n-1)\delta + \alpha}, \epsilon_{n\delta + \alpha}]_q ,
\] (3.8)

\[
\epsilon_{-(2n-1)\delta - 2\alpha} := \frac{1}{\sqrt{a}} [\epsilon_{-(n-1)\delta - \alpha}, \epsilon_{-(n-1)\delta + \alpha}]_{q^{-1}} ,
\] (3.9)

\[
\epsilon_{(2n+1)\delta - 2\alpha} := \frac{1}{\sqrt{a}} [\epsilon_{(n+1)\delta - \alpha}, \epsilon_{n\delta - \alpha}]_q ,
\] (3.10)

\[
\epsilon_{-(2n+1)\delta + 2\alpha} := \frac{1}{\sqrt{a}} [\epsilon_{-(n+1)\delta + \alpha}, \epsilon_{-(n+1)\delta + \alpha}]_{q^{-1}} ,
\] (3.11)

for \( n = 1, 2, \ldots \), where \( a = [2(\alpha, \alpha)] \) and \( b = [2(\alpha, \alpha)] - [(\alpha, \alpha)] \).

The constructed set of the root vectors \([3.3]-[3.9]\) (together with the Chevalley basis) is called a q-analog Cartan-Weyl basis of \( U(\widehat{\text{oosp}(1|2)}) \) or a Cartan-Weyl basis of the quantum algebra \( U_q(\widehat{\text{oosp}(1|2)}) \). Now we present the commutation relations for the vectors of this basis.

**Proposition 3.1** The root vectors \( \epsilon_{\pm(n\delta + \alpha)} \), \( \epsilon_{\pm(n\delta - \alpha)} \), \( \epsilon_{\pm((2n-1)\delta - \alpha)} \), and \( \epsilon_{\pm((2n+1)\delta - \alpha)} \) satisfy the following commutation relations

\[
[\epsilon_{n\delta + \alpha}, \epsilon_{n\delta - \alpha}] = (-1)^n \frac{k_{n\delta + \alpha} - k_{n\delta - \alpha}}{q - q^{-1}} ,
\] (3.12)

\[
[\epsilon_{n\delta - \alpha}, \epsilon_{n\delta + \alpha}] = (-1)^{n-1} \frac{k_{n\delta - \alpha} - k_{n\delta + \alpha}}{q - q^{-1}} ,
\] (3.13)

\[
[\epsilon_{(2n-1)\delta + 2\alpha}, \epsilon_{-(2n-1)\delta - 2\alpha}] = \frac{k_{(2n-1)\delta + 2\alpha} - k_{(2n-1)\delta - 2\alpha}}{q - q^{-1}} ,
\] (3.14)

\[
[\epsilon_{(2n+1)\delta - 2\alpha}, \epsilon_{-(2n+1)\delta + 2\alpha}] = \frac{k_{(2n+1)\delta - 2\alpha} - k_{(2n+1)\delta + 2\alpha}}{q - q^{-1}} ,
\] (3.15)

where \( n = 1, 2, \ldots \).

The sector with imaginary root vectors requires the redefinitions of the generators \([11.7]\). We introduce the new imaginary roots vectors \( \epsilon_{\pm n\delta} \) by the following (Schur) relations (see \([5], [6] [8]\)):

\[
\epsilon'_{n\delta} = \sum_{p_1 + 2p_2 + \ldots + np_n = n} \frac{(q - q^{-1})\sum_{p_i=1}^{p_i-1} (e\delta)^{p_1} \ldots (e\delta_{n\delta})^{p_n}}{p_1! \ldots p_n!} .
\] (3.16)

In terms of generating functions

\[
\mathcal{E}'(u) := (q - q^{-1}) \sum_{n \geq 1} e'_{n\delta} u^{-n} ,
\] (3.17)

\[
\mathcal{E}(u) := (q - q^{-1}) \sum_{n \geq 1} e_{n\delta} u^{-n}
\] (3.18)

the relation \([3.16]\) may be rewritten in the form

\[
\mathcal{E}'(u) = -1 + \exp \mathcal{E}(u)
\] (3.19)

or

\[
\mathcal{E}(u) = \ln(1 + \mathcal{E}'(u)) .
\] (3.20)
This provides the formula inverse to (3.16)

\[ e_{n\delta} = \sum_{p_1 + 2p_2 + \ldots + np_n = n} \frac{(q^{-1} - q)\sum_{i=1}^{n} p_i - 1)!}{p_1! \ldots p_n!} e'_{p_1} \ldots e'_{p_n}^n. \] (3.21)

The root vectors of negative roots are obtained by the Cartan conjugation (*):

\[ e_{-n\delta} = (e_{n\delta})^*. \] (3.22)

**Proposition 3.2** The root vectors \( e_{\pm n\delta} \) satisfy the following commutation relation

\[ [e_{n\delta}, e_{-m\delta}] = \delta_{nm} \frac{(-1)^n}{nb} \left( q^{n(\alpha,\alpha)} + q^{n(\alpha,\alpha)} - 1 \right) [n(\alpha,\alpha)] \frac{k_{n\delta} - k_{n\delta}^{-1}}{q - q^{-1}}, \] (3.23)

where \( n, m = 1, 2, \ldots \).

**Remark.** The root vectors \( e_{\pm n\delta}' \) do not satisfy the relation (3.18), i.e. \([e_{n\delta}, e_{-m\delta}] \neq 0 \) for \( n \neq m \).

## 4 Final Remarks

In this note we restricted ourselves to the description of Cartan-Weyl basis. Related results (\(q\)-Weyl group, automorphisms introducing real forms, universal \( R \)-matrix, classical \( r \)-matrix) will be published in our subsequent publication.

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