Entanglement minimization in hadronic scattering with pions

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ABSTRACT: Recent work [1] conjectured that entanglement is minimized in low-energy hadronic scattering processes. It was shown that the minimization of the entanglement power (EP) of the low-energy baryon-baryon $S$-matrix implies novel spin-flavor symmetries that are distinct from large-$N_c$ QCD predictions and are confirmed by high-precision lattice QCD simulations. Here the conjecture of minimal entanglement is investigated for scattering processes involving pions and nucleons. The EP of the $S$-matrix is constructed for the $\pi\pi$ and $\pi N$ systems, and the consequences of minimization of entanglement are discussed and compared with large-$N_c$ QCD expectations.

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1 Introduction

It is of current interest to uncover implications of quantum entanglement for the low-energy interactions of hadrons and nuclei\(^1\). As these interactions are profitably described by effective quantum field theory (EFT), which is an expansion of the relevant effective action in local operators, entanglement may have subtle implications for EFT which are difficult to identify due to its intrinsic non-locality. Ideally entanglement properties reveal themselves as regularities in hadronic data and, possibly, as accidental approximate symmetries. In addition to the non-local nature of entanglement, a difficulty lies with parsing the distinction, if any, between entanglement effects and generic quantum correlations which account for the deviation of QCD path integral configurations from a classical path. For instance, if one assumes that QCD with \(N_c = 3\) is near the large-\(N_c\) limit \([3–6]\), then one might expect that it would be difficult to distinguish between large-\(N_c\) expectations and some fundamental underlying principle that minimizes entanglement independent of the value of \(N_c\). To make this more concrete, consider two local or non-local QCD operators \(O_1\) and \(O_2\). If the vacuum expectation value of the product of these operators obeys the factorization rule \([3–5]\)

\[
\langle O_1 O_2 \rangle = \langle O_1 \rangle \langle O_2 \rangle + O(\epsilon) \tag{1.1}
\]

where \(\epsilon\) is a small number, then the variance of any operator vanishes in the limit \(\epsilon \to 0\). A theory whose operators obey this factorization behaves like a classical theory\(^2\) and therefore has a small parameter \(\epsilon\) which measures quantum effects. Large-\(N_c\) QCD is such a theory, and indeed, at least for a class of QCD operators, one can identify \(\epsilon = 1/N_c\). The factorization property, Eq. (1.1), is then easily deduced from Feynman diagrams involving quarks and gluons and amounts to the dominance of disconnected contributions in the path integral.

On the other hand, one might imagine that the factorization of Eq. (1.1) arises as a property of the path integral, rather than as a property of the local action (as in varying \(N_c\) and taking it large in QCD). It is not \textit{a priori} unlikely that, at least for a class of QCD operators, the path integral minimizes quantum fluctuations via a mechanism that is not currently understood. For instance, starting with QCD defined at short distances, the procedure of sequentially integrating out short distance modes to obtain low-energy hadronic scattering amplitudes may remove highly-entangled states that arise from non-perturbative QCD dynamics, leaving a low-energy EFT that is near a classical trajectory. It is intuitively sensible that the QCD confinement length acts as a

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\(^1\)For a recent review, see Ref. [2].

\(^2\)Ordinarily one identifies the classical theory with the trivial \(\hbar \to 0\) limit. However, Ref. [7] has established a more general criterion for the classical limit.
natural cutoff of entanglement in the low-energy EFT. This notion can be raised to the
conjecture that QCD will minimize the entanglement in low-energy hadronic interac-
tions. Testing this conjecture relies on finding hadronic systems where its consequences
deviate from those implied by large-$N_c$. And the success of the large-$N_c$ approxima-
tion in describing the world renders this task challenging. Evidence in favor of this con-
jecture was found in Ref. [1] in a study of baryon-baryon scattering systems (See also
Refs. [8] and [9]). This work relied both on theoretical arguments and high-precision
lattice QCD simulations of baryon-baryon scattering systems with strangeness. In this
paper, the conjecture of minimal entanglement will be investigated in both $\pi\pi$ and $\pi N$
scattering.

Finding measures of the entanglement due to interaction is both non-trivial and
non-unique. The most fundamental object in the scattering process is the unitary $S$-
matrix. In a scattering process in which the two in-state particles form a product state,
the $S$-matrix will entangle the in-state particles in a manner that is dependent on the
energy of the scattering event. A useful measure of this entanglement is the entan-
glement power (EP) of the $S$-matrix [1, 10, 11]. In the case of nucleon-nucleon ($NN$)
scattering, the EP was found for all momenta below inelastic threshold [1]. However,
the most interesting phenomenological result is at threshold, where the vanishing EP
implies the vanishing of the leading-order spin entangling operator, which in turn im-
plies Wigner $SU(4)$ symmetry [12–14]. As this symmetry is a consequence of large-$N_c$
QCD [15–17], the minimization of entanglement and the large-$N_c$ approximation are
found to be indistinguishable in the two-flavor case. By contrast, in the three-flavor
case, minimization of the entanglement power in baryon-baryon scattering implies an
enhanced $SU(16)$ symmetry which is not necessarily implied by large-$N_c$ and is rea-
лизed in lattice QCD simulations [1, 9]. Given that baryon-baryon scattering exhibits
entanglement minimization, it is of interest to determine whether other low-energy
QCD scattering systems exhibit this property. In investigating the EP of scattering
systems involving pions, once again a crucial difficulty is distinguishing consequences of
entanglement minimization and the large-$N_c$ limit. In the $\pi\pi$ system the implications
of entanglement minimization are found to be indistinguishable from implications of
large-$N_c$. In the $\pi N$ system the implications of entanglement minimization are distinct,
however the absence of an enhanced symmetry limits the predictive power to simple
scaling laws with no smoking-gun predictions.

This paper is organized as follows. In Section 2, the EP of the $\pi\pi$ $S$-matrix is
considered in detail. After introducing the standard definition and conventions of
the $\pi\pi$ $S$-matrix, the $S$-matrix is formulated in a basis convenient for calculation of
the EP. Explicit expressions are derived for the EP of the first few partial waves in
terms of phase shifts and leading-order expressions in chiral perturbation theory are
provided. Using the highly-accurate Roy-equation solutions for the low-energy phase shifts, the experimental EP for the first few partial waves are given up to inelastic threshold. The consequences of minimizing the EP are considered and compared to large-$N_c$ expectations. In Section 3, the same procedure is repeated for the $\pi N$ $S$-matrix. Finally, Section 4 is a discussion of the possible conclusions that can be drawn from the conjecture of minimal entanglement.

2 The $\pi\pi$ System

There are, of course, several important differences between baryon-baryon and pion-pion scattering. Firstly, with pions there is no notion of spin entanglement. However, isospin (or generally flavor) entanglement is present and can be quantified using the EP and it is not clear that there is any meaningful distinction between these two kinds of entanglement. Indeed, it is straightforward to see that the “spin” entanglement of Ref. [1] can be reformulated as “isospin” entanglement with identical consequences\(^3\). This is no surprise as entanglement is fundamentally a property of a non-product state vector whose existence relies either on an internal or a spacetime symmetry. Secondly, the crucial distinction between baryon-baryon scattering at very low-energies and the scattering of pions is that pion scattering at low-energies is strongly constrained by spontaneous chiral symmetry breaking in QCD. In particular, chiral symmetry implies that low-energy pion scattering on an arbitrary hadronic target is weak. The weak nature of the interaction is due to the smallness of the light-quark masses relative to a characteristic QCD scale. This translates to a chiral suppression of the EP at low-energies. Chiral symmetry breaking at large-$N_c$ does involve enhanced symmetry [6]; for $N$ flavors, the QCD chiral symmetries and their pattern of breaking are enhanced to $U(N) \otimes U(N) \rightarrow U(N)$, as signaled by the presence of a new Goldstone boson, $\eta'$, whose squared mass scales as $1/N_c$. Intuitively, the anomaly, as an intrinsically quantum phenomenon, is a strongly entangling effect which would generally vanish as quantum fluctuations are suppressed. However, this is not assumed as the focus of this paper is two-body scattering which does not access the anomaly.

2.1 $S$-matrix definition

The $S$-matrix is defined as

$$S = 1 + iT$$

where unity, corresponding to no interaction, has been separated out. This then defines the $T$-matrix. The $S$-matrix element for the scattering process $\pi^i \pi^j \rightarrow \pi^k \pi^l$ is then

\(^3\text{At the level of the EFT, this is simply realized via Fierz identities.}\)
given by
\[
\langle \pi^k(p_3) \pi^l(p_4) | S | \pi^i(p_1) \pi^j(p_2) \rangle = \langle \pi^k(p_3) \pi^l(p_4) | \pi^i(p_1) \pi^j(p_2) \rangle \\
+ \langle \pi^k(p_3) \pi^l(p_4) | iT | \pi^i(p_1) \pi^j(p_2) \rangle \tag{2.2}
\]
where \( i, j, k, \) and \( l \) are the isospin indices of the pion states. The projection operators onto states of definite isospin are\(^4\)
\[
P^{0,kl,ij} = \frac{1}{3} \delta^{kl} \delta^{ij},
\]
\[
P^{1,kl,ij} = \frac{1}{2} \left( \delta^{ki} \delta^{lj} - \delta^{li} \delta^{kj} \right),
\]
\[
P^{2,kl,ij} = \frac{1}{2} \left( \delta^{ki} \delta^{lj} + \delta^{li} \delta^{kj} \right) - \frac{1}{3} \delta^{kl} \delta^{ij},
\]
where the subscript indicates the total isospin, \( I \), of the \( \pi \pi \) system. Straightforward field-theoretic manipulations then give
\[
\langle \pi^k(p_3) \pi^l(p_4) | S | \pi^i(p_1) \pi^j(p_2) \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \frac{16\pi}{\sigma(s)} \sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(\cos \theta) S_{\ell}^{kl,ij},
\]
where the \( P_{\ell} \) are the Legendre polynomials, and
\[
\sigma(s) \equiv \sqrt{1 - 4M_\pi^2/s},
\]
with \( s = 4(q^2 + M_\pi^2) \) and \( q \) is the center-of-mass three-momentum of the pions. The focus here will be on the \( S \)-matrices of definite partial wave:
\[
S_{\ell}^{kl,ij} \equiv e^{2i\delta_0^l} P_{0}^{kl,ij} + e^{2i\delta_1^l} P_{1}^{kl,ij} + e^{2i\delta_2^l} P_{2}^{kl,ij},
\]
which satisfy the unitarity constraint
\[
S_{\ell}^{kl,ij} S_{\ell}^{*mn} = \delta^{km} \delta^{ln}.
\]
Since the pions obey Bose statistics, the angular momentum, \( \ell \), is even for the states with \( I = 0 \) or 2 and odd for states with \( I = 1 \).

As the initial state in the scattering process is a product state of two pions, each in the \( 3 \)-dimensional (\( I = 1 \) irrep of \( SU(2) \) isospin, it is convenient to work in the

\(^4\)For a detailed construction, see Ref. [18].
direct-product matrix basis. The pion isospin matrices are the three-by-three matrices \( \hat{t}_\alpha \) which satisfy

\[
[\hat{t}_\alpha, \hat{t}_\beta] = i \epsilon_{\alpha\beta\gamma} \hat{t}_\gamma.
\]

In the product Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), the total isospin of the two-pion system is \( \hat{t}_1 \otimes \hat{I}_3 + \hat{I}_3 \otimes \hat{t}_2 \), where \( \hat{I}_3 \) is the three-by-three unit matrix, which implies

\[
\hat{t}_1 \cdot \hat{t}_2 = \frac{1}{2} \left[ I (I + 1) - 4 \right] \hat{I} = \begin{cases} 
-2, & I = 0 \\
-1, & I = 1 \\
1, & I = 2
\end{cases}
\]

where \( \hat{I} = \hat{I}_3 \otimes \hat{I}_3 \) and \( \hat{t}_1 \cdot \hat{t}_2 = \sum_{a=1}^{3} \hat{t}_{1a} \otimes \hat{t}_{2a} \). The \( 9 \times 9 \) dimensionality of the matrix is in correspondence with the dimensionality of the \( SU(2) \) isospin product representation \( 3 \otimes 3 = 1 \oplus 3 \oplus 5 \). There are now three invariants and three observables; one easily finds the \( S \)-matrix in the direct-product matrix basis

\[
\hat{S}_\ell = e^{2i\delta^0} \hat{P}_0 + e^{2i\delta^1} \hat{P}_1 + e^{2i\delta^2} \hat{P}_2,
\]

where the three \( 9 \times 9 \) projection matrices are

\[
\hat{P}_0 = -\frac{1}{3} \left( \hat{I} - (\hat{t}_1 \cdot \hat{t}_2)^2 \right),
\]

\[
\hat{P}_1 = \hat{I} - \frac{1}{2} \left( (\hat{t}_1 \cdot \hat{t}_2) + (\hat{t}_1 \cdot \hat{t}_2)^2 \right),
\]

\[
\hat{P}_2 = \frac{1}{3} \left( \hat{I} + \frac{3}{2} (\hat{t}_1 \cdot \hat{t}_2) + \frac{1}{2} (\hat{t}_1 \cdot \hat{t}_2)^2 \right).
\]

It is readily checked that the \( S \)-matrix is unitary, and using the representation \( (t_\gamma)_{\alpha\beta} = -i \epsilon_{\alpha\beta\gamma} \), it is straightforward to establish equivalence with the component form, Eq. \((2.1)\). The trace is given by \( e^{i2\delta^0} + 3e^{i2\delta^1} + 5e^{i2\delta^2} \) which correctly counts the isospin multiplicity, and is in correspondence with the nine eigenvalues of \( \hat{S} \).

### 2.2 Entanglement power

Consider the \( \ell = 1 \) \( S \)-matrix. As this system can scatter only in the \( I = 1 \) channel, it provides a useful example of how the \( S \)-matrix entangles the initial two-pion state. From Eq. \((2.12)\) one finds

\[
\hat{S}_1 = \frac{1}{2} \left( 1 + e^{i2\delta^0} \right) \hat{I} + \frac{1}{2} \left( 1 - e^{i2\delta^0} \right) \hat{P}_{12}
\]

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where the SWAP operator is given by
\[ \mathcal{P}_{12} = (\hat{t}_1 \cdot \hat{t}_2)^2 + \hat{t}_1 \cdot \hat{t}_2 - 1 . \] (2.17)

As the SWAP operator interchanges the pions in the initial two-pion product state, leaving another two-pion product state, it does not entangle. Therefore, the S-matrix has the two obvious non-entangling solutions \( \delta_1^1 = 0 \) (no interaction) and \( \delta_1^1 = \pi/2 \) (at resonance). One measure of S-matrix entanglement would then be the (absolute value squared of the) product of the coefficients of the non-entangling solutions:
\[ \left| \left( 1 + e^{i\delta_1^1} \right) \left( 1 - e^{i\delta_1^1} \right) \right|^2 \sim \sin^2 (2\delta_1^1) . \] (2.18)

A state-independent measure of the entanglement generated by the action of the S-matrix on the initial product state of two free particles is the EP [1, 10, 11]. In order to compute the EP an arbitrary initial product state should be expressed in a general way that allows averaging over a given probability distribution folded with the initial state. Recall that in the NN case, there are two spin states (a qubit) for each nucleon and therefore the most general initial nucleon state involves two complex parameters or four real parameters. Normalization gets rid of one parameter and there is an overall irrelevant phase which finally leaves two real parameters which parameterize the CP\(^1\) manifold, also known as the 2-sphere \( \mathbb{S}^2 \), or the Bloch sphere. Now in the isospin-one case we have three isospin states (a qutrit) which involves three complex parameters. Again normalization and removal of the overall phase reduce this to four real parameters which parameterize the CP\(^2\) manifold [19–21]. Since the \( \pi\pi \) initial state is the product of two isospin-one states, there will be eight parameters to integrate over to get the EP.

There are now two qutrits in the initial state, which live in the Hilbert spaces \( \mathcal{H}_{i,2} \), each spanned by the states \( \{ | -1_i \rangle, | 0_i \rangle, | 1_i \rangle \} \) with \( i = 1, 2 \). It is of interest to determine the EP of a given S-matrix operator, which is a measure of the entanglement of the scattered state averaged over the CP\(^2\) manifolds on which the qutrits live. Consider an arbitrary initial product state of the qutrits
\[ | \Psi \rangle = U (\alpha_1, \beta_1, \mu_1, \nu_1) | \rangle_1 \otimes U (\alpha_2, \beta_2, \mu_2, \nu_2) | \rangle_2 \] (2.19)
with
\[ U (\alpha_{i}, \beta_{i}, \mu_{i}, \nu_{i}) | \rangle_{i} = \cos \beta_{i} \sin \alpha_{i} | -1 \rangle_{i} + e^{i\mu_{i}} \sin \beta_{i} \sin \alpha_{i} | 0 \rangle_{i} + e^{i\nu_{i}} \cos \alpha_{i} | 1 \rangle_{i} , \] (2.20)
where \( 0 \leq \mu_{i}, \nu_{i} < 2\pi \) and \( 0 \leq \alpha_{i}, \beta_{i} \leq \pi/2 \). The geometry of CP\(^2\) is described by the Fubini-Study (FS) line element [19–21]
\[ ds^2_{FS} = d\alpha^2 + \sin^2(\alpha) d\beta^2 + ( \sin^2(\alpha) \sin^2(\beta) - \sin^4(\alpha) \sin^4(\beta) ) d\mu^2 + \sin^2(\alpha) \cos^2(\alpha) d\nu^2 - 2 \sin^2(\alpha) \cos^2(\alpha) \sin^2(\beta) d\mu d\nu . \] (2.21)
Of special interest here is the differential volume element which in these coordinates is
\begin{equation}
    dV_{FS} = \sqrt{\det g_{FS}} \, d\alpha \, d\beta \, d\mu \, d\nu
    = \cos \alpha \cos \beta \sin^3 \alpha \sin \beta \, d\alpha \, d\beta \, d\mu \, d\nu
\end{equation}
and the volume of the \( \mathbb{C}P^2 \) manifold is found to be,
\begin{equation}
    \int dV_{FS} = \frac{\pi^2}{2} .
\end{equation}

The final state of the scattering process is obtained by acting with the unitary \( S \)-matrix of definite angular momentum on the arbitrary initial product state:
\begin{equation}
    | \bar{\Psi} \rangle = \hat{S}_\ell | \Psi \rangle .
\end{equation}
The associated density matrix is
\begin{equation}
    \rho_{1,2} = | \bar{\Psi} \rangle \langle \bar{\Psi} | ,
\end{equation}
and tracing over the states in \( \mathcal{H}_2 \) gives the reduced density matrix
\begin{equation}
    \rho_1 = \text{Tr}_2[\rho_{1,2}] .
\end{equation}
The linear entropy of the \( S \)-matrix is then defined as\(^5\)
\begin{equation}
    E_{\hat{S}_\ell} = 1 - \text{Tr}_1[ (\rho_1)^2 ] .
\end{equation}
Finally, the linear entropy is integrated over the initial \( \mathbb{C}P^2 \) manifolds to form the average, and the entanglement power is
\begin{equation}
    \mathcal{E}(\hat{S}_\ell) = \left( \frac{2}{\pi^2} \right)^2 \left( \prod_{i=1}^{2} \int dV_{FS}^i \right) \mathcal{P} E_{\hat{S}_\ell} ,
\end{equation}
where \( \mathcal{P} \) is a probability distribution which here will be taken to be unity. Evaluating this expression using Eq. (2.12) yields the s-wave \( \pi \pi \) EP:
\begin{equation}
    \mathcal{E}(\hat{S}_0) = \frac{1}{648} (156 - 6 \cos[4\delta_0^0] - 65 \cos[2(\delta_0^0 - \delta_0^2)])
    \right. \\
    \left. - 10 \cos[4(\delta_0^0 - \delta_0^2)] - 60 \cos[4\delta_0^2] - 15 \cos[2(\delta_0^0 + \delta_0^2)]) ,
\end{equation}
and the p-wave \( \pi \pi \) EP:
\begin{equation}
    \mathcal{E}(\hat{S}_1) = \frac{1}{4} \sin^2 (2\delta_1^1) .
\end{equation}

\(^5\)Note that this is related to the (exponential of the) second Rényi entropy.
Notice that this matches the intuitive construction which led to Eq. (2.18). The EPs have the non-entangling solutions:

\[
\delta_0^0 = \delta_0^2 = 0, \frac{\pi}{2}, \\
\delta_1^1 = 0, \frac{\pi}{2}.
\]  

(2.31)  

(2.32)

Therefore, in the s-wave, entanglement minimization implies that both isospins are either non-interacting or at resonance, while in the p-wave, entanglement minimization implies that the \(I = 1\) channel is either non-interacting or at resonance. As no \(I = 2\) resonances are observed in nature (and their coupling to pions is suppressed in large-\(N_c\) QCD [22]), the s-wave EP has a single minimum corresponding to no interaction. By contrast, the \(I = 1\) channel will exhibit minima of both types. It is worth considering the EP of a simple resonance model. Consider the unitary \(S\)-matrix:

\[
\hat{S}_1 = \frac{s - m_1^2 - im_1 \Gamma_1}{s - m_1^2 + im_1 \Gamma_1},
\]  

(2.33)

where \(m_1 (\Gamma_1)\) are the mass (width) of the resonance. The EP is

\[
\mathcal{E}(\hat{S}_1) = \left( \frac{m_1 \Gamma_1 (s - m_1^2)}{(m_1 \Gamma_1)^2 + (s - m_1^2)^2} \right)^2,
\]  

(2.34)

which vanishes on resonance at \(s = m_1^2\) and has maxima at \(s = m_1 (m_1 \pm \Gamma_1)\). It is clear that the minimum corresponds to \(\hat{S} \propto P_{12}\). As the \(\rho\)-resonance dominates the \(I = 1\) channel at energies below 1 GeV, the EP in nature will be approximately of this form.

The \(\pi \pi\) phase shifts are the most accurately known of all hadronic \(S\)-matrices as the Roy equation constraints [23] come very close to a complete determination of the phase shifts [24, 25]. In Fig. (1) the EPs for the first few partial waves are plotted using the Roy equation determinations of the \(S\)-matrix.

### 2.3 Chiral perturbation theory

Near threshold, the phase shift can be expressed in the effective range expansion as

\[
\delta^I(s) = \frac{1}{2} \sin^{-1}\{2\sigma(s)q^{2\ell} (a^I_\ell + \mathcal{O}(q^2))\},
\]  

(2.35)

where the scattering lengths, \(a^I_\ell\), relevant to s-wave and p-wave scattering, are given at leading order in chiral perturbation theory by [26, 27]

\[
a_0^0 = \frac{7M^2}{32\pi F^2_\pi}, \quad a_0^2 = \frac{M^2}{16\pi F^2_\pi}, \quad a_1^1 = \frac{1}{24\pi F^2_\pi},
\]  

(2.36)
As expected, this vanishes when $\epsilon = 1/N_c$ and $n = 1/2$ [3–5]. Evidently the implications of vanishing entanglement for the $\pi\pi$ $S$-matrix are indistinguishable from large-$N_c$ expectations\footnote{We also studied the effect of explicit chiral symmetry breaking on the entanglement power by varying the coefficients of operators with insertions of the quark mass matrix in the effective action. No evidence of a connection between chiral symmetry breaking and the entanglement power was found. This aligns with large-$N_c$ expectations as the meson masses are independent of $N_c$. For an example of a relationship between entanglement and chiral symmetry breaking see [28].}.\hfill

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Entanglement power of the $\pi\pi$ $S$-matrix for $\ell = 0, 1$ taken from Roy equation determinations (the bands represent an estimate of the uncertainties [24, 25]) of the $\pi\pi$ phase shifts.}
\end{figure}
3 The $\pi N$ System

As baryons are formed from $N_c$ quarks, the baryon masses and axial matrix elements grow with $N_c$. The unitarity of the $S$-matrix then places powerful constraints on baryon properties via large-$N_c$ consistency conditions [29–32]. At leading order in the large-$N_c$ expansion this yields predictions that are equivalent in the two (three) flavor case to $SU(4)$ ($SU(6)$) spin-flavor symmetry which place the ground-state baryon spin states in the $20$ ($56$) dimensional irrep together with the delta (baryon decuplet). Therefore, the large-$N_c$ limit not only predicts an enhanced symmetry but also alters the definition of a baryon in a fundamental way. Moreover, any sensible effective theory of $\pi N$ scattering in the large-$N_c$ limit must include the delta resonance as an explicit degree of freedom. In what follows, the consequences of entanglement minimization of the low-energy $S$-matrix are considered for $N_c = 3$ QCD.

3.1 $S$-matrix definition

The $S$-matrix element for the scattering process, $\pi^a(q_1)N(p_1) \rightarrow \pi^b(q_2)N(p_2)$, is given by

$$
\langle \pi^b(q_2)N(p_2)|S|\pi^a(q_1)N(p_1)\rangle = \langle \pi^b(q_2)N(p_2)|\pi^a(q_1)N(p_1)\rangle + \langle \pi^b(q_2)N(p_2)|iT|\pi^a(q_1)N(p_1)\rangle,
$$

(3.1)

where $a$ and $b$ label the isospin of the pion. The $T$ matrix element in the center-of-mass system (cms) for the process may be parameterized as [33]

$$
T_{\pi N}^{ba} = \left(\frac{E + m}{2m}\right) \left\{ \delta^{ba} \left[ g^+(\omega, t) + i\vec{\sigma} \cdot (\vec{q}_2 \times \vec{q}_1) h^+(\omega, t) \right] + i\epsilon^{abc}\tau^c \left[ g^-(\omega, t) + i\vec{\sigma} \cdot (\vec{q}_2 \times \vec{q}_1) h^-(\omega, t) \right] \right\}
$$

(3.2)

where $E$ is the nucleon energy, $\omega$ is the pion energy, $m$ is the nucleon mass and $t = (q_1 - q_2)^2$ is the square of the momentum transfer. The $\sigma(\tau)$ matrices act on the spin(isospin) of the incoming nucleon. This decomposition reduces the scattering problem to calculating $g^\pm$, the isoscalar/isovector non-spin-flip amplitude and $h^\pm$, the isoscalar/isovector spin-flip amplitude. The amplitude can be further projected onto partial waves by integrating against $P_\ell$, the relevant Legendre polynomial:

$$
f_{\ell \pm}^{\pm}(s) = \frac{E + m}{16\pi\sqrt{s}} \int_{-1}^{+1} dz \left[ g^\pm P_\ell(z) + \vec{q}^2 h^\pm(P_{\ell \pm 1}(z) - zP_\ell(z)) \right].
$$

(3.3)

Here $z = \cos \theta$ is the cosine of the scattering angle, $s$ is the cms energy squared and $\vec{q}^2 = \vec{q}_1^2 = \vec{q}_2^2$. The subscript $\pm$ on the partial wave amplitude indicates the total
angular momentum \( J = \ell \pm s \). The amplitudes in the total isospin \( I = \frac{1}{2} \) and \( I = \frac{3}{2} \) can be recovered via the identification:

\[
f_{\ell \pm}^\frac{1}{2} = f_{\ell \pm}^+ + 2f_{\ell \pm}^- \quad , \quad f_{\ell \pm}^\frac{3}{2} = f_{\ell \pm}^+ - f_{\ell \pm}^-. \tag{3.4}
\]

Below inelastic threshold the scattering amplitude is related to a unitary S-matrix by

\[
S_{\ell \pm}^I(s) = 1 + 2i|q|^f_{\ell \pm}(s) \quad , \quad S_{\ell \pm}^I(s)S_{\ell \pm}^I(s)^\dagger = 1 \tag{3.5}
\]

and the S-matrix can be parameterized in terms of phase shifts,

\[
S_{\ell \pm}^I(s) = e^{2i\delta_{\ell \pm}(s)}. \tag{3.6}
\]

For a more detailed derivation of the \( \pi N \) S-matrix see \cite{33–36}. Scattering in a given partial wave and total angular momentum channel leads to a S-matrix which acts on the product Hilbert space of the nucleon and pion isospin, \( \mathcal{H}_{\pi} \otimes \mathcal{H}_N \). The S-matrix can then be written in terms of total isospin projection operators

\[
\hat{S}_{\ell \pm} = e^{2i\delta_{\ell \pm}^{3/2}} \hat{P}_{3/2} + e^{2i\delta_{\ell \pm}^{1/2}} \hat{P}_{1/2} \tag{3.7}
\]

where the 6 \times 6 projection matrices are

\[
\hat{P}_{3/2} = \frac{2}{3} \left( \mathbb{1} + \hat{t}_\pi \cdot \hat{t}_N \right),
\]
\[
\hat{P}_{1/2} = \frac{1}{3} \left( \mathbb{1} - 2(\hat{t}_\pi \cdot \hat{t}_N) \right). \tag{3.8}
\]

The operators \( \hat{t}_N \) and \( \hat{t}_\pi \) are in the 2 and 3 dimensional representations of \( SU(2) \) isospin respectively and \( \hat{t}_\pi \cdot \hat{t}_N = \sum_{a=1}^{3} \hat{t}_a^{\pi} \otimes \hat{t}_N^{a} \).

### 3.2 Entanglement power

The entanglement power of the \( \pi N \) S-matrix can be computed in a similar manner as for the \( \pi \pi \) EP. The incoming separable state now maps to a point on the product manifold, \( \mathbb{C}P^2 \times S^2 \). The construction of the reduced density matrix follows the same steps as in section 2.2 and the entanglement power is found to be,

\[
\mathcal{E}(\hat{S}_{\ell \pm}) = \left( \frac{2}{\pi^2} \frac{1}{4\pi} \right) \left( \int dV_{FS} d\Omega \right) \mathcal{P} E_{\hat{S}_{\ell \pm}} = \frac{8}{243} \left[ 17 + 10 \cos \left( 2(\delta_{\ell \pm}^{3/2} - \delta_{\ell \pm}^{1/2}) \right) \right] \sin^2 \left( \delta_{\ell \pm}^{3/2} - \delta_{\ell \pm}^{1/2} \right) \tag{3.9}
\]

where \( \mathcal{P} \) has been taken to be 1. Note that the two particles are now distinguishable and so scattering in each partial wave is no longer constrained by Bose/Fermi statistics. It
follows that the $S$-matrix is only non-entangling when it is proportional to the identity which occurs when,

$$\delta_{\ell\pm}^{3/2} = \delta_{\ell\pm}^{1/2}. \quad (3.10)$$

Notice that the EP allows for interesting local minima when the difference in $I = 3/2$ and $I = 1/2$ phase shifts is $\pi/2$. The $\pi N$ phase shifts are determined very accurately by the Roy-Steiner equations up to a center-of-mass energy of 1.38 GeV [37] and the entanglement power for the first couple partial waves is shown in Fig. (2). There is a local minimum near the delta resonance position in the p-wave due to the rapid change of the $I = 3/2$ phase shift.

### 3.3 Chiral perturbation theory

Near threshold the phase shifts can be determined by the scattering lengths through the effective range expansion,

$$\delta_{\ell\pm}^I = \cot^{-1}\left\{ \frac{1}{|q|^2 + 1} \left( \frac{1}{a_{\ell\pm}^I} + O(q^2) \right) \right\}. \quad (3.11)$$

This leads to the threshold form of the entanglement power,

$$\mathcal{E}(\hat{S}_{\ell\pm}) = \frac{8}{9} \left( a_{\ell\pm}^{3/2} - a_{\ell\pm}^{1/2} \right)^2 q^{2+4I}. \quad (3.12)$$
which can only vanish if \( a_0^\pm \) = \( a_0^\pm \). The scattering lengths at leading order in heavy-baryon chiral perturbation theory, including the delta, are given by [33, 38],

\[
\begin{align*}
a_0^+ &= \frac{2M_\pi m}{8\pi(m + M_\pi)F_\pi^2}, \quad a_0^- = \frac{-M_\pi m}{8\pi(m + M_\pi)F_\pi^2} \\
a_1^+ &= -\frac{m(9g_A^2\Delta + 9g_A^2M_\pi - 8g_{\pi N}\Delta M_\pi)}{544F_\pi^2M_\pi(\Delta + M_\pi)(m + M_\pi)}, \quad a_1^- = -\frac{m(9g_A^2\Delta + 9g_A^2M_\pi - 8g_{\pi N}\Delta M_\pi)}{216\pi F_\pi^2M_\pi(\Delta + M_\pi)(m + M_\pi)} \\
a_1^{\pm} &= \frac{m(-3g_A^2\Delta + 3g_A^2M_\pi + 8g_{\pi N}\Delta M_\pi)}{72\pi F_\pi^2M_\pi(\Delta - M_\pi)(m + M_\pi)}, \quad a_1^+ = \frac{m(-3g_A^2\Delta^2 - 2g_{\pi N}\Delta M_\pi\Delta + 3g_A^2M_\pi^2)}{36\pi F_\pi^2M_\pi\left(M_\pi^2 - \Delta^2\right)(m + M_\pi)}
\end{align*}
\]

where \( \Delta = m_\Delta - m_N \) is the delta-nucleon mass splitting. The corresponding EPs near threshold are,

\[
\begin{align*}
\mathcal{E}(\hat{S}_{0^+}) &= \frac{m^2M_\pi^2}{8\pi^2F_\pi^4(m + M_\pi)^2}\bar{q}^2 \\
\mathcal{E}(\hat{S}_{1^-}) &= \frac{m^2(9g_A^2\Delta + 9g_A^2M_\pi - 8g_{\pi N}\Delta M_\pi)^2}{5832\pi^2f_\pi^4M_\pi^2(\Delta + M_\pi)^2(m + M_\pi)^2}\bar{q}^6 \\
\mathcal{E}(\hat{S}_{1^+}) &= \frac{m^2(-9g_A^2\Delta^2 + 4g_{\pi N}\Delta M_\pi + (9g_A^2 + 8g_{\pi N}\Delta)M_\pi^2)^2}{5832\pi^2f_\pi^4M_\pi^2(M_\pi^2 - \Delta^2)^2(m + M_\pi)^2}\bar{q}^6
\end{align*}
\]

Once again the only non-entangling solution consistent with chiral symmetry is no interaction, with the same scaling of \( F_\pi \) as found in \( \pi\pi \) scattering. Unlike the large-\( N_c \) limit, there is no reason to expect an enhancement of the axial couplings, which in that case gives rise to the contracted spin-flavor symmetries [29].

4 Discussion

In QCD the number of colors, \( N_c \), is a parameter that appears in the action and in some sense acts as a knob that dials the amount of quantum correlation in the hadronic \( S \)-matrix. The simplifications, counting rules and enhanced symmetries implied by the large-\( N_c \) approximation have proved highly successful in explaining regularity in the hadronic spectrum. Recent work in Ref. [1] has conjectured that, independent of the value of \( N_c \), quantum entanglement is minimized in hadronic \( S \)-matrices. Verifying this conjecture relies on finding consequences of the conjecture that are distinct from large-\( N_c \) predictions, and indeed this has been found to be the case in baryon-baryon scattering. In particular, minimization of entanglement near threshold leads to enhanced symmetry that is verified by lattice QCD simulations. Here this conjecture has been considered for \( \pi\pi \) and \( \pi N \) scattering. As shown long ago by Weinberg, the
scattering of soft pions off any target is completely determined by chiral symmetry \[39\] and is weak at low energies. Here it has been found that the only $\pi\pi$ or $\pi N$ $S$-matrix, consistent with the low energy theorems, that does not entangle isospin is the identity i.e. no scattering. In the context of chiral perturbation theory this corresponds to $F_\pi$ being large when entanglement is minimized, consistent with large $N_c$ scaling. Unlike in the large $N_c$ limit, entanglement minimization of the $S$-matrix says nothing about the scaling of the baryon masses and axial couplings and therefore implies no new symmetries in the $\pi N$ sector. Because of the weakness of pion processes implied by chiral symmetry, it may be the case that only systems without external Goldstone bosons (like $NN$) give non-trivial constraints from entanglement minimization. Considering general meson-nucleon scattering, it is clear that scalar-isoscalar mesons have no spin or isospin to entangle. Insofar as resonance saturation is effective, entanglement minimization would then predict the contribution to baryon-baryon scattering from the exchange of non scalar-isoscalar resonances to sum together to give an equal contribution to all spin-isospin channels \[40\]. This would then naturally lead to the $SU(16)$ symmetry seen in the three flavor baryon sector \[1\].

Techniques which make use of entanglement minimization to select out physically-relevant states and operators from an exponentially large space have a long history. For instance, tensor methods and DMRG crucially rely on the fact that ground states of reasonable Hamiltonians often exhibit much less entanglement than a typical state \[41\]. In nuclear physics it has recently been shown that entanglement is a useful guiding principle when constructing many-body wave-functions of atomic nuclei \[42\]. The use of entanglement minimization to constrain hadronic $S$-matrices in other contexts has also been investigated recently. The authors of \[43, 44\] have considered entanglement minimization as an ingredient in an effort to revive the $S$-matrix bootstrap program. When applied to the $\pi\pi$ $S$-matrix a correspondence is found between minima of entanglement and linear Regge trajectories. This is an intriguing prospect and may be related to the observation made here that, at least in p-wave $\pi\pi$ scattering, the entanglement power has a zero at resonance. Outside of hadronic physics, it was shown recently that the minimization of spin entanglement in scattering due to the exchange of gravitons picks out parameters which correspond to minimally coupled gravity \[45\].

An interesting and directly related line of inquiry is the connection between entanglement and renormalization group (RG) flow. As a zeroth order observation, macroscopic objects are distinguishable from their surroundings despite being coupled to the environment. Therefore, in some sense classical objects behave like coherent quantum states whose entropy does not increase when they interact with an open quantum system \[46\]. This “motivates” the idea that at large scales a fixed point of the entanglement entropy is reached. From an RG point of view this may be manifest in the
entanglement structure between different momentum modes of fields. Work has been
done on computing the momentum space entanglement in both scattering events and
between regions of the ground state of a quantum field theory [47–49]. It is speculated
that the RG flow of parameters in an effective action is driven by entanglement between
the IR and UV. Along a similar vein, recent work has employed numerical methods to
study the ground state entanglement structure between disjoint regions of massless free
scalar field theory [50, 51]. In tension with the tenets of EFT, it was found that long
distance entanglement gets most of its support from short distance field modes. With
this in mind it may be possible that EFT, with its insensitivity to physics at the cutoff,
is not the best framework with which to study entanglement. There is clearly much to
explore on the relationship between entanglement, RG flow and EFT.

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