A note on ferromagnetism in the Hubbard model on the complete graph

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Abstract

Recently there have appeared some papers which discuss the existence of ferromagnetism in the Hubbard model defined on the complete graph. At least for the special electron number $N_e = N - 1$, where $N$ denotes the number of sites in the lattice, the existence of ferromagnetism in this model was established rigorously some time ago, as special (and indeed the simplest) cases of more general classes of models. Here we explain these implications to clarify the situation, although we believe the implications are straightforward.

We are posting this note to the preprint archive to make it public, but we are not planning to publish it in other forms. This is because we do not think the problem warrants any extra publications, and we believe that the validity of our explanation is evident to the readers.

1 The problem

Since our motivation is explained in the abstract, we start by defining the problem precisely. Let $N$ be a positive integer. We identify our lattice $\Lambda$ with the set of integers $\{1, 2, \ldots, N\}$. For $i \in \Lambda$ and $\sigma = \uparrow, \downarrow$, we let $c_{i,\sigma}^\dagger$, $c_{i,\sigma}$, and $n_{i,\sigma} = c_{i,\sigma}^\dagger c_{i,\sigma}$ the standard creation, annihilation, and number operators, respectively, for an electron at site $i$ with spin $\sigma$. The model under consideration has the Hamiltonian

$$H = t \sum_{\sigma = \uparrow, \downarrow, i \neq j} c_{i,\sigma}^\dagger c_{j,\sigma} + U \sum_{i \in \Lambda} n_{i,\uparrow} n_{i,\downarrow}, \quad (1)$$

where the hopping amplitude satisfies $t > 0$ and the Coulomb interaction $U > 0$. Another important parameter of the model is the electron number $N_e$, which is the eigenvalue of $\sum_{\sigma = \uparrow, \downarrow} \sum_{i \in \Lambda} (n_{i,\uparrow} + n_{i,\downarrow})$.

Among the statements discussed in \cite{1, 2} is the following.

Corollary: In the Hilbert space where the electron number is fixed to $N_e = N - 1$, the ground states of (1) exhibit saturated ferromagnetism and are nondegenerate\textsuperscript{3} apart from the trivial spin degeneracy for any $t > 0$ and $U > 0$.

As we mentioned in the abstract, the above Corollary follows as special (and the easiest) cases of general results in \cite{1} and in \cite{2}. It is also straightforward to get the Corollary from

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\textsuperscript{3} When $N_e < N - 1$, one can easily construct ferromagnetic and non-ferromagnetic ground states.
Nagaoka’s theorem \cite{Nagaoka}. We discuss these three proofs briefly\cite{Review}. See \cite{R1, R2} for the progress of the research in ferromagnetism which followed \cite{R0, R1}.

\section{First proof}

We first review the general result for the Hubbard model on a line graph proved in \cite{R0}. We start from abstract notations. Let $G = (V, E)$ be an abstract graph, where $V$ is the set of vertices (sites) $\alpha, \beta, \ldots \in V$, and $E$ is the set of edges (bonds) which are nothing but pairs of vertices like $\{\alpha, \beta\}$. Given a graph $G$, one can construct the corresponding line graph $L(G) = (V_L, E_L)$ by the following procedure. The set of vertices (sites) $V_L$ (whose elements are denoted as $x, y, \ldots \in V_L$) is taken to be identical to the set $E$. This means that we identify edges in $G$ with the vertices (sites) in $L(G)$ as, for example, $x = \{\alpha, \beta\}$, $y = \{\alpha, \gamma\}$, etc. Next we declare that two vertices $x, y \in V_L$ are adjacent to each other if the corresponding two edges in $E$ share a common vertex in $V$. The $x$ and $y$ in the above example are adjacent to each other since the corresponding edges in $E$ have a common vertex $\alpha$. $E_L$ is the set of edges (bonds) in $L(G)$, which consists of all the adjacent pairs (like $\{x, y\}$) of vertices (sites) in $V_L$. Finally we set $M(G) = |E| - |V| + 1$ if $G$ is bipartite\footnote{G is bipartite if it can be decomposed into two disjoint sublattices as $G = A \cup B$ with the property that any edge in $E$ joins a vertex in $A$ with a vertex in $B$.}, and $M(G) = |E| - |V|$ if $G$ is non-bipartite.

We define the Hubbard model on the line graph $L(G)$. With each site $x \in V_L$, we associate fermion operator $c_{x,\sigma}$, and consider the Hamiltonian

$$H = t \sum_{\langle x,y \rangle \in E_L} \sum_{\sigma = \uparrow, \downarrow} c_{x,\sigma}^\dagger c_{y,\sigma} + U \sum_{x \in V_L} n_{x,\uparrow} n_{x,\downarrow}. \quad (2)$$

Then the main result of \cite{R0} is the following.

\textbf{Theorem 1:} Suppose that the graph $G$ is twofold connected\footnote{A graph is twofold connected if and only if one cannot make it disconnected by a removal of a single vertex.}. Then in the Hilbert space where the electron number is fixed to $N_e = M(G)$, the ground states of (2) exhibit saturated ferromagnetism and are nondegenerate apart from the trivial spin degeneracy for any $t > 0$ and $U > 0$.

This theorem applies to the Hubbard model defined on various line graphs, a typical one being the kagomé lattice.

By construction, a general line graph is a graph that consists of complete graphs connected at the vertices such that two complete graphs have some vertices in common. Thus the complete graph is the most trivial line graph. It can be constructed taking graph $G$ which has only two vertices, and has $N$ edges joining them. More precisely we set $V = \{\alpha, \beta\}$, and $E$ to be the set consisting of $N$ identical copies of the edge $\{\alpha, \beta\}$. The corresponding $V_L$
consists of \( N \) sites, and any pair of sites are adjacent with each other. Since \( G \) is bipartite, we get \( M(G) = |E| - |V| + 1 = N - 2 + 1 = N - 1 \). Thus Theorem 1 precisely reduces to the Corollary in Section I.

The proof of the Corollary in [2] is based on a theorem in [3], which is nothing but a generalization of the above Theorem 1. The structure of the proof in [3] is the same as in [1].

3 Second proof

In [3], a class of Hubbard models with ferromagnetism which is stable against the change of electron density was discussed. It was noted in the Remark on pages 355 and 356 that ferromagnetism can be established for more general models defined on lattices with certain cell structure

A trivial (and the least interesting) version of this model is the one consisting of a single cell. Let us reproduce it here (in a slightly generalized form) for the readers' convenience. Let \( \Lambda = \{1, 2, \ldots, N\} \) be the set of \( N \) sites. For \( i = 1, 2, \ldots, N \), we let \( \lambda_i \) be an arbitrary nonvanishing (complex) quantity. We take the Hamiltonian

\[
H = t \sum_{\sigma = \uparrow, \downarrow} \left( \sum_{i \in \Lambda} \lambda_i c_{i,\sigma} \right)^\dagger \left( \sum_{i \in \Lambda} \lambda_i c_{i,\sigma} \right) + U \sum_{i \in \Lambda} n_{i,\uparrow} n_{i,\downarrow}.
\]

Then we have

**Theorem 3:** In the Hilbert space where the electron number is fixed to \( N_e = N - 1 \), the ground states of (3) exhibit saturated ferromagnetism and are nondegenerate apart from the trivial spin degeneracy for any \( t > 0 \) and \( U > 0 \).

This reduces to the Corollary in Section I if we set \( \lambda_i = 1 \) for all \( i \).

4 Third proof

There is another straightforward proof that makes use of the wellknown Nagaoka’s theorem [4]. The theorem (in its most general form) applies to the \( U \to \infty \) limit of the Hamiltonian (1), and states that the ground states exhibit ferromagnetism and are nondegenerate apart from the trivial spin degeneracy.

One can easily check that the state

\[
\Phi^\uparrow = \left( \prod_{i=1}^{N-1} (c_{i,\uparrow}^\dagger - c_{N,\uparrow}^\dagger) \right) \Phi_{\text{vac}}
\]
minimizes both the hopping part and the interaction part of the Hamiltonian $H$ when $t > 0$. Therefore $\Phi^\uparrow$ is a ground state of $H$ for any $U \geq 0$, and the only nontrivial task in the proof of the Corollary in Section I is to show that there are no other ground states. Given Nagaoka’s theorem, this is easy.

Assume that, for some $U > 0$, there is a ground state which is not an SU(2) rotation of $\Phi^\uparrow$. Since the ground state energy does not depend on $U$, it must remain to be ground state for any larger $U$. But this contradicts to the Nagaoka’s theorem.

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