Abstract. In this work, we look at the cosmological constraints of four different $f(R)$-gravity models, which include 2 toy models and 2 more realistic models, such as the Starobinsky and Hu-Sawicki models. We use 359 low- and intermediate-redshift Supernovae Type 1A data obtained from the SDSS-II/SNLS3 Joint Light-curve Analysis (JLA). We then develop a Markov Chain Monte Carlo (MCMC) simulation to determine the best-fit for each $f(R)$-gravity model, as well as for the Lambda Cold Dark Matter ($\Lambda$CDM) model, to obtain the cosmological parameters ($\Omega_m$ and $\bar{h}$). We assume a flat universe with negligible radiation. Therefore, the only difference between these models are the dark energy term and the arbitrary free parameters. When doing a statistical analysis on these models (where we used the $\Lambda$CDM model as the "true model"), we found that the Starobinsky model obtained a larger likelihood function than the $\Lambda$CDM model, while still obtaining the cosmological parameters to be $\Omega_m = 0.268^{+0.027}_{-0.024}$ and $\bar{h} = 0.690^{+0.005}_{-0.005}$. We also found a reduced Starobinsky model, that explained the data, as well as being statistically significant. We also found a further 3 models that can explain the data, even though they are not statistically significant, while also finding 3 models that did not explain the data and were statistically rejected.
1 Introduction

Since the Theory of General Relativity (GR) was proposed by Albert Einstein in 1915, it has developed into the accepted theory to explain gravity. GR was not only able to explain gravity in extreme situations, but was also able to reduce back to a Newtonian description of gravity in a weak gravitational field. Due to the ability of GR to explain the expansion of the Universe [1], the Hot Big Bang theory was developed using GR as its mathematical basis. However, in recent times, with ever-improving accuracy in observations, it was discovered that the expansion of the Universe was accelerating [2], which is not in line with GR predictions, and therefore the Hot Big Bang model had to be improved. An unknown pressure force acting out against gravity, dubbed “dark energy” was added to explain why gravity on cosmological scales were not able to slow down the expansion [3].

We can start by using the Einstein-Hilbert action, which tries to extremize the path between two time-like points in spacetime and is given as

\[ A = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} \left[ R + 2(L_m - \Lambda) \right], \]  

(1.1)

with \( A \) being the cosmological constant representing the “dark energy” pressure force, \( L_m \) the standard matter Lagrangian, \( g \) the metric tensor, and the remaining constants representing their accepted usages [4]. If we apply the variational principle on eq. (1.1), we obtain the cosmological field equations, also called the Einstein field equations, as

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \]  

(1.2)

where \( R_{\mu\nu} \) and \( R \) are the Ricci tensor and Ricci scalar respectively, and \( T_{\mu\nu} \) represents the energy-momentum tensor. Furthermore, for simplicity and normalization we assume a geometric unit system, where \( c = \frac{1}{\sqrt{\mu}} = 8\pi G \). From the field equations, we can derive the two
most important cosmological equations, called the Friedmann equations, and is given by \[5\]

\[
H^2(t) = \frac{\rho(t)}{3} - \frac{\kappa}{a^2(t)} + \frac{\Lambda}{3},
\]

\[
\dot{H}(t) = -H(t)^2 - \frac{1}{6} \left[ \rho(t) + 3P(t) \right] + \frac{\Lambda}{3},
\]

(1.3)

where \( H(t) = \frac{\dot{a}(t)}{a(t)} \) is the Hubble parameter with \( a(t) \) the scale factor which is describing the relative size of the Universe at a certain time, \( \rho(t) \) is the energy density, \( P(t) \) is the isotropic pressure, and \( \kappa \) is the 3D (spacial) curvature. Furthermore, to derive these particular Friedmann equations, we had to assume we have a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime metric. The Friedmann equations is an open system, to which we need to related \( \rho \) and \( P \) to close this system. This is possible through the usage of the equation of state parameter

\[
P(t) = \omega \rho(t),
\]

(1.4)

where we assume a perfect fluid with a constant equation of state parameter \( \omega \) [6]. Even though we will be using a constant equation of state parameter, studies that tries to parameterise the equation of state exist for both the \( \Lambda \)CDM model and modified gravity models [7, 8]. This closed system is called the \( \Lambda \)CDM model, which forms part of the improved Big Bang theory to explain the late-time accelerated expansion with dark energy. However, since dark energy is an unknown pressure force, it poses a problem. What is dark energy? It becomes even more difficult to explain dark energy when taking into account that the majority of the Universe (\( \sim 68\% \)) [7] is presumably filled with it.

Furthermore, dark energy is not the only problem faced by the \( \Lambda \)CDM model. It has been shown that another accelerated expansion epoch occurred, called the Inflation epoch [9]. This epoch occurred at the early stages of the evolution of the Universe. The Inflation epoch was introduced to solve early-universe problems of GR, such as the horizon problem and the flatness problem [2, 10, 11]. Other arising problems also included the magnetic monopole problem (they are predicted to exist, but none has been found) and the Universe’s matter/anti-matter ratio, which is expected to be equal to one, but is close to zero [11, 12].

Due to the discussed problems, it has been previously suggested that we need to modify our theory of gravity. The modified theories that we will be looking at, aims to explain the late-time accelerated expansion without the inclusion of dark energy. In some of these theories, you may add extra fields or you might assume a higher dimension. We will, specifically, be looking at a higher-order derivative theory, called \( f(R) \)-gravity. For these models, the modification occurs, by changing the Ricci scalar within the Einstein-Hilbert action eq. (1.1) to an arbitrary function of the Ricci scalar, namely \( f(R) \). This change leads to a new set of cosmological field equations, specifically for \( f(R) \) gravity. Re-deriving the cosmological field equations, we obtain [2, 13–15]

\[
f'(R)R_{\mu\nu} + g_{\mu\nu} \Box f'(R) - \nabla_{\mu} \nabla_{\nu} f'(R) - \frac{1}{2} g_{\mu\nu} f(R) = T_{\mu\nu},
\]

(1.5)

where \( \Box = \nabla_\sigma \nabla^\sigma \) is the covariant d’Alembert operator. As done before, we can then re-derive
the Friedman equations (this time for $f(R)$-gravity), and find [4, 9, 16]
\[ H^2(t) = \frac{\rho(t)}{3f'(R)} - \frac{\kappa}{a^2(t)} + \frac{1}{6} \left[ R - \frac{f(R)}{f'(R)} \right] - H \dot{R} \frac{f''(R)}{f'(R)}, \]
\[ \dot{H}(t) = -H^2(t) - \frac{\rho(t)}{3f'(R)} + \frac{f(R)}{6f'(R)} + H \dot{R} \frac{f''(R)}{f'(R)}. \]

2 Supernovae cosmology and MCMC simulations

2.1 Distance modulus

To test eq. (1.6), especially the first one, we will use Supernovae Type 1A data. This class of supernovae is the resultant of a white dwarf (WD) star accreting a low-mass companion star until the accreted outer-layer Hydrogen from the companion star is compressed to a point that the WD explodes [17]. This means that their luminosities are relatively similar to one another, and is therefore regarded as standard candles [18, 19]. Therefore, the measured flux is only dependent on the distance to the particular supernova and not the composition of the WD. We will use redshift $z$ to approximate the distance. By using the distance modulus, we can then test the expansion of the Universe, since the distance to the supernovae is changing. For simplicity, we will assume a flat universe ($\Omega_k = 0$), with a negligible radiation density ($\Omega_r \approx 0$).

To obtain the distance modulus, we need to find the line-of-sight comoving distance ($D_c$) function [4, 20]. We start with the luminosity distance ($D_L$) function, which relates two bolometric quantities, namely the luminosity ($L$) and the flux ($f$) of a distance object, such as a supernova. We can then relate $D_L$ to the proper motion distance ($D_M$) function, by using redshift and obtain
\[ D_L = (1 + z)D_M. \]

However, $D_M$ has different conditions depending on the curvature of the Universe, namely [21]
\[ D_M = \begin{cases} 
D_H \frac{1}{\sqrt{\Omega_k}} \sinh \left( \sqrt{\Omega_k} D_c D_H \right) & \text{if } \Omega_k > 0, \\
D_c & \text{if } \Omega_k = 0, \\
D_H \frac{1}{\sqrt{|\Omega_k|}} \sin \left( \sqrt{|\Omega_k|} D_c D_H \right) & \text{if } \Omega_k < 0.
\end{cases} \]

Since we have assumed a flat universe, we have $D_M = D_c$. Furthermore, we know that $D_c$ is per definition given as
\[ D_c = \int \frac{c dt}{a} = D_H \int_0^z \frac{dz'}{h(z')}, \]
where $D_H = 3000\bar{h} \frac{km}{Mpc}$ is the Hubble distance and $h(z) = \frac{H(z)}{H_0}$ is the normalized Hubble parameter in terms of redshift. We can then use the definition of the distance modulus (in Mpc), and the above mentioned different distance definitions, and obtain
\[ \mu = m - M = 25 + 5 \times \log_{10} \left( 3000\bar{h}^{-1}(1 + z) \int_0^z \frac{dz'}{h(z')} \right), \]
where $m$ is the apparent magnitude and $M$ is the absolute magnitude of the measured supernova. Now that we have a model, we need the data. We will be using 359 low- and intermediate
redshift supernovae data obtained from the SDSS-II/SNLS3 Joint light-curve Analysis (JLA). In particular, we have 123 supernovae with redshift values between $0.01 < z \leq 0.1$ and 236 supernovae with redshift values between $0.1 < z \leq 1.1$. It is necessary that we used low redshift data, since the late-time acceleration epoch only started at around $z \approx 0.5$ [22, 23]. Furthermore, having data on both sides of $z \approx 0.5$, will enable us to show the transition phase between the decelerating expansion of the Universe epoch (matter-dominated) and the accelerating expansion of the Universe epoch which we are currently experiencing. Furthermore, we will be using the absolute magnitudes of these supernovae for the B-filter, that can be obtained from the research papers [24–26] (also available on NASA’s Extragalactic Database). This entire method is called supernovae cosmology.

2.2 Markov Chain Monte Carlo (MCMC) simulation

To find the best-fitting distance modulus for each model, we will use MCMC simulations. The MCMC simulation is able to search for the most probable free parameter values, given certain physical constraints. In particular, we will be using the Metropolis-Hastings algorithm [27, 28], which starts by calculating the likelihood for each initial chosen free parameter value’s distance modulus. After this the simulation takes a random step for each of the free parameters away from the initial conditions, but still within the boundaries (priors). The MCMC simulation then calculates the likelihood for each possible combination between the initial parameter values and the new parameter values, to find which combination has the largest likelihood of occurring. The simulation then finds an acceptance ratio between the likelihood of the initial parameter values and the new largest likelihood combination value. The acceptance ratio is given by

$$
\alpha(x, x^*) = \min \left\{ 1, \frac{p(x^*)q(x|x^*)}{p(x)q(x^*|x)} \right\},
$$

where $x$ is the current parameter value, $x^*$ is the proposed candidate, $p(x)$ is the invariant probability distribution, and $q(x^*|x)$ is the proposed probability distribution. Therefore, if the acceptance ratio is above 1, the new parameter values are accepted and becomes the current values, while if it is below 1, a chance is created for the second combination to still be accepted in ratio to the probabilities for each combination to occur. If the proposed parameter value is not accepted, the procedure starts over with a new random step away from the initial/current parameter values. This process continues until the simulation converges to the most probable best-fitting parameter values. Since we need a probability distribution, we will, for simplicity assume that the data follows a Gaussian distribution. Therefore, the likelihood function is given as

$$
\mathcal{L}(\hat{\theta}|\text{data}) = \exp \left( -\frac{1}{2} \sum \left[ \frac{(\mu_{\text{data}} - \mu_{\text{theoretical}})^2}{\sigma^2} \right] \right),
$$

where $\mu_{\text{data}}$ is the distance modulus for each data point and $\mu_{\text{theoretical}}$ the model’s predicted distance modulus value at that point. We will use the EMCEE Hammer Python package to execute the MCMC simulation. This package uses different random walkers (in most cases we will use 100 random walkers), each executing the Metropolis-Hasting algorithm and all starting at the same initial parameter values and converging on the most probable parameter values. The last iteration of the algorithm then creates a Gaussian distribution based on each random walker’s ending parameter values. Using the average values for each probability distribution for each parameter value, we will then have on average the best-fitting parameter value and its $1\sigma$-value (error-bars) for each of the free parameters.
2.3 AIC and BIC statistical analysis

To test whether or not these \( f(R) \)-gravity models are able to explain the data, we will use the Akaike Information Criterion (AIC) and the Bayesian/Schwarz Information Criterion (BIC) selections [29]. These selections will be able to indicate whether or not a particular \( f(R) \)-gravity model is able to explain the data, while also comparing it to a “true model” (in this case the \( \Lambda \)CDM model) [30]. These selection criteria uses the likelihood of the best-fitting model to determine if they statistically explain the data, as well as taking into account the amount of free parameters the model uses. This is important, since a model that has more free parameters than the “true model” can fit the data with a more precisely, since it has more freedom in changing the shape of the function, but might be not be as valuable as the “true model”. Since we choose a Gaussian distribution for our data, the AIC and BIC selections are given as

\[
\begin{align*}
\text{AIC} &= \sum_n \left[ \frac{(\mu_{\text{data}} - \mu_{\text{theoretical}})^2}{\sigma^2} \right] + 2K , \\
\text{BIC} &= \sum_n \left[ \frac{(\mu_{\text{data}} - \mu_{\text{theoretical}})^2}{\sigma^2} \right] + K \log(n) ,
\end{align*}
\]

where the first terms on the R.H.S. are known to be the \( \chi^2 \)-values, \( K \) the amount of free parameters and \( n \) the amount of data points. Therefore, we can also determine the reduced \( \chi^2 \)-value to give us an indicate on the goodness of fit for each model. Furthermore, the particular AIC and BIC values can be any positive values, to which we can not make a conclusion. Therefore, we need to compare the AIC and BIC values of the “true model” and the \( f(R) \)-gravity models and find the difference between them. This difference can then be compared the the Jeffrey’s scale in order to be able to make conclusions about the model. It must be noted that this scale is not an exclusive scale and should be handled with care [31]. The Jeffrey’s scale ranges are:

\[
\begin{align*}
\Delta IC &\leq 2 \quad \text{substantial support} , \\
4 \leq \Delta IC &\leq 7 \quad \text{less support w.r.t. ‘true model’} , \\
\Delta IC &> 10 \quad \text{model has no observational support} .
\end{align*}
\]

3 Results

3.1 The \( \Lambda \)CDM model

We will use the \( \Lambda \)CDM model to calibrate our MCMC simulation, as well as using it as our “true mode” to which we can compare the \( f(R) \)-gravity models against to find if they are viable alternative gravity models. Using the assumptions made in section 2.1, we can obtain the normalized Friedmann equation for the \( \Lambda \)CDM model in terms of redshift as

\[
h(z) = \sqrt{\Omega_m(1 + z)^3 + 1 - \Omega_m} ,
\]

where we made the substitution \( \Omega_\Lambda = 1 - \Omega_m \) [32, 33]. To execute the MCMC simulation for the \( \Lambda \)CDM model, we need to combine eq. (2.4) and eq. (3.1), to which we obtain the MCMC simulation results in figure 1.

From figure 1, we can confirm that the MCMC simulation works, even though the predicted cosmological parameter values are not within 1\( \sigma \) from the Planck2018 results. This
discrepancy between the Planck results that were determined on the Cosmic Microwave Background (CMB) radiation data, and the results from previous Supernovae Type Ia observations have been showed to exist [11, 34]. Surprisingly, this discrepancy is actually not even limited to only these two methods for calculating the Hubble constant. In the paper by [35], they show that different experiments result in different $H_0$ values, with all the local measurements, such as eclipsing binaries in the Large Magellanic clouds or Cepheid stars within the Milky way, tending to result in higher values for the Hubble constant while the CMB results gave the smallest $H_0$ values. This poses the question: Why do we obtain different values when trying to measure $H_0$, through early-time or late-time data? As of yet, there are no convincing explanations for this discrepancy. This particular set of predicted cosmological parameter values on supernovae data from our MCMC simulation are in line with the expected supernovae results for the $\Lambda$CDM model. The only reason for showing the Planck results is just to remind us that we did use the Planck results to make our assumptions, as well as also being able to show the discrepancy between the supernovae and the CMB results.

For future work, it will be worth it to try and find a best-fitting model using different datasets, such as $H(z)$ and BAO [36], to see how the different datasets lead to different contributions from the matter and dark energy densities of the Universe. This change will

Figure 1. The MCMC simulation results for the $\Lambda$CDM-model’s eq. (3.1) cosmological free parameters ($\Omega_m$ and $h$), with “true” values (blue lines: $\Omega_m = 0.315$ and $h = 0.674$) provided by the Planck 2018 collaboration data release [7].
Figure 2. The ΛCDM model’s best-fit to the Supernovae Type 1A data (L.H.S. panel), with the cosmological parameter values as determined by the MCMC simulation result figure 1. Furthermore, the residuals between the model’s predicted distance modulus values and the actual data points are also shown (R.H.S. panel).

then lead to different predicted bounds on the $f(R)$-gravity models, such as shown in [33, 37], where they found that the predicted matter density in the Universe changes, when combining the different datasets. They also showed that it leads to different predicted arbitrary free parameter values.

We can then use the MCMC results to make a plot of the best-fitting ΛCDM model on the supernovae data. This is shown in figure 2. From figure 2, we can once again confirm that the MCMC calibration was done correctly, since the ΛCDM model does fit the data with quite a high accuracy, as well as not having an over- or under-estimation at various different redshifts.

3.2 $f(R)$-gravity model results

We can now advance to the testing of various $f(R)$-gravity models. We will use two toy models, namely $f(R) = \beta R^n$ and $f(R) = \alpha R + \beta R^n$ [4], as well as two realistic models, namely the Starobinsky and Hu-Sawicki models, which are given by [38–40]

\[
f(R) = R + \beta R_c \left[ \left( \frac{R}{R_c} \right)^2 - 1 \right]^{\frac{-n}{n-1}},
\]

\[
f(R) = R - \alpha R_c \left[ \frac{(\frac{R}{R_c})^n}{1 + (\frac{R}{R_c})^n} \right],
\]

(3.2)

respectively, with $\alpha$, $\beta$ and $n$ being the arbitrary free parameters and $R_c$ parametrises the curvature scale. For each model, different analytical constraints on these parameters is discussed in more detail in the papers by [2, 4, 15, 38–40]. We also used the effective cosmological constant term ($\Lambda \equiv \frac{\beta R^n}{R^2}$) to mimic dark energy, to allow us to solve these realistic models [33]. Even though only 4 models are listed, we ended up with 8 different models that we have tested, since we found that except for the first toy model, the models become analytically unsolvable. Therefore, we assumed fixed $n$-values for the second toy model, to which we found four different solvable models. We then tried this approach for the two realistic models
and were unsuccessful in this approach. This led us to incorporate a numerical optimization method into the MCMC simulation to find an approximated $H^2$ value at a particular $z$-value. Using this method, we were then able to build a solution map for different approximated $H^2$-values at different redshift values between $0 \leq z \leq z'$. Using the solution map, we were then able to numerically integrate over $z$ using the Simpson integration rule. From here on out the MCMC simulation were able to calculate the approximated distance modulus value for each supernova. Due to the resolution of the numerical methods, we found that for the Starobinsky model, 3 of the free parameters, did not effect the outcome of the predicted model. This led us to also try to fit a reduced version of the Starobinsky model.

According to [11], we also need to use the cosmographic series terms, namely the deceleration parameter and the jerk parameter, to help solve the degeneracy among the models the different Friedmann equations. Althought, we had to find these parameters in terms of the redshift and not as a function of time. We decided to use the parametrisations for these cosmographic parameters as given in [41]. They defined the deceleration parameter as

$$q_z = q_0 + q_1 \frac{z}{1+z},$$

while the jerk parameter was given as a function of the deceleration parameter

$$j(q) = q(z) [2q(z) + 1] + \frac{q_1}{1+z},$$

where $q_0$ is the current deceleration parameter value and $q_1$ is correction. By using all of the above mentioned actions, we are now able to solve and find the best-fitting function for each of the different $f(R)$-gravity models. Due to the limitations of space we will only present the models that seemed to be able to explain the supernovae data to an extend\(^1\). Starting in the order that were given above, our first model to show promise is the second toy model, where we assumed $n = 0$. Therefore, we have $f(R) = \alpha R + \beta$. The best-fitting model on the supernovae data is shown in figure 3.

It is interesting that this particular model is able to explain the data, since this model resembles the \(\Lambda\)CDM model. By this we mean that $f(R) = R - 2\Lambda$, is exactly the same as the \(\Lambda\)CDM model [33]. The second model that were able to explain the supernovae data, is also part of the second toy model group, where we fixed $n = 2$. This particular model $f(R) = \alpha R + \beta R^2$ is also one of the original models developed by Starobinsky to explain the early time expansion. Furthermore, this model obtained a positive and a negative solution. We will be showing the negative solution. The best-fitting model on the supernovae data is shown in figure 4.

Similar to the first model, the second $f(R)$-gravity model, is also able to explain the data with no over- or under-estimations. The last three models that were able to explain the data, was the Starobinsky (with its reduced version) and the Hu-Sawicki model. These were solved using the numerical method. The first one is the Starobinsky model, which actually obtained a larger likelihood probability prediction than the \(\Lambda\)CDM model, as well as being our overall best-fitting $f(R)$-gravity model. The best-fitting Starobinsky model is shown in figure 5.

From figure 5, it is clear that the Starobinsky model fits the data with a high precision. Furthermore, we can also assume that this model is quite stable, since the error bars on this model, just like the \(\Lambda\)CDM model is very small, therefore the MCMC simulation is certain that the predicted best-fit for this model is correct. Next-up is the reduced Starobinsky

\(^1\)We will also not be showing the MCMC results, since they can take up to an entire page, but we will give the predicted cosmological parameter values in the caption for each of the model’s best-fitting plots.
Figure 3. The second toy model (with $n = 0$) fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.317^{+0.061}_{-0.101}$ and $H_0 = 71.5^{+2.8}_{-2.2} \text{ km/s/Mpc}$, while the arbitrary free parameters were calculated to be $\alpha = 1.202^{+0.397}_{-0.392}$ and $\beta = -5.265^{+1.698}_{-1.315}$.

Figure 4. The second toy model (with $n = 2$ - negative solution) fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.249^{+0.102}_{-0.101}$, $H_0 = 63.8^{+1.6}_{-2.7} \text{ km/s/Mpc}$, $q_0 = -0.575^{+0.040}_{-0.046}$, and $q_1 = -0.633^{+0.049}_{-0.040}$, while the arbitrary free parameters were calculated to be $\alpha = 19.642^{+2.967}_{-1.753}$ and $\beta = 0.903^{+0.070}_{-0.107}$.

To reduce this model, we fixed the correctional deceleration parameter to be $q_1 = 0$ (based on the Starobinsky model results). We also fixed $\beta = 1$ and $n = 1$, after we saw that their error bars are large, but did not translate to large errors in the best-fitting Starobinsky model. Even though this model did not find the accuracy of its counterpart, it was still the third best-fitting model (including the $\Lambda$CDM model) that we found. The results for this reduced Starobinsky model is shown in figure 6.

Due to the fewer free parameters in the reduced Starobinsky model, we can see in figure 6 that this model is less stable compared to the original model. Therefore, a small change in one of the remaining parameters, can result in a completely different predicted model. It is this fact makes the $\Lambda$CDM model interesting, since it only has 2 free parameters and were
Figure 5. The Starobinsky model fitted to the Supernovae Type Ia data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.268^{+0.027}_{-0.024}$, $H_0 = 69.0^{+0.5}_{-0.6}$ km s$^{-1}$ Mpc$^{-1}$, $q_0 = -0.512^{+0.328}_{-0.265}$ and $q_1 = 0.037^{+0.991}_{-1.050}$, while the arbitrary free parameters were calculated to be $\beta = 5.284^{+3.191}_{-2.981}$ and $n = 4.567^{+3.346}_{-2.899}$.

Figure 6. The reduced Starobinsky model fitted to the Supernovae Type Ia data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.266^{+0.026}_{-0.024}$, $H_0 = 69.4^{+0.6}_{-1.0}$ km s$^{-1}$ Mpc$^{-1}$, and $q_0 = -0.697^{+0.173}_{-0.138}$.

still predicting a best-fit model with small errors. Lastly, we have the Hu-Sawicki model, which to our surprise did not fair as well or even better than the Starobinsky model, but were still able to explain the supernovae data. The best-fitting Hu-Sawicki model results on the supernovae data is shown in figure 7.

From figure 7, we can see that even though the Hu-Sawicki model did fit the data, the error region is just as large as the first $f(R)$-gravity model (that we presented in this article) and that was only a toy model. This, however, might be an effect of the resolution of the numerical methods, since the Hu-Sawicki model used 7 free parameter, therefore the optimization approximations might have struggled within the MCMC simulation. This is why we kept this model within the group, since it might still be a viable model. The last three
The Hu-Sawicki model fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.238^{+0.043}_{-0.040}$, $H_0 = 73.7^{+9.0}_{-4.9} \text{ km/s/Mpc}$, $q_0 = -0.486^{+0.300}_{-0.285}$, and $q_1 = -0.036^{+1.018}_{-0.968}$, while the arbitrary free parameters were calculated to be $\alpha = 5.196^{+2.222}_{-2.073}$, $\beta = 6.923^{+2.120}_{-2.732}$, and $n = 2.262^{+0.800}_{-0.724}$.

**Figure 7.** The Hu-Sawicki model fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.238^{+0.043}_{-0.040}$, $H_0 = 73.7^{+9.0}_{-4.9} \text{ km/s/Mpc}$, $q_0 = -0.486^{+0.300}_{-0.285}$, and $q_1 = -0.036^{+1.018}_{-0.968}$, while the arbitrary free parameters were calculated to be $\alpha = 5.196^{+2.222}_{-2.073}$, $\beta = 6.923^{+2.120}_{-2.732}$, and $n = 2.262^{+0.800}_{-0.724}$.

| Model              | $\mathcal{L}(\hat{\theta}|\text{data})$ | $\chi^2$  | Red. $\chi^2$ | AIC   | $|\Delta\text{AIC}|$ | BIC   | $|\Delta\text{BIC}|$ |
|--------------------|------------------------------------------|-----------|---------------|-------|----------------------|-------|----------------------|
| Starobinsky       | -120.7052                                | 241.4105  | 0.6839        | 253.4105 | 7.9939            | 276.7104 | 23.5272 |
| $\Lambda$ CDM     | -120.7083                                | 241.4166  | 0.6762        | 245.4166 | 0                  | 253.1832 | 0       |
| Starobinsky red.  | -122.4442                                | 244.8885  | 0.6879        | 250.8885 | 5.4719            | 262.5385 | 9.3553  |
| $\alpha R + \beta$| -131.2518                                | 262.5037  | 0.7394        | 270.5037 | 25.0871           | 286.0370 | 32.8538 |
| Hu-Sawicki        | -140.1668                                | 280.3336  | 0.7964        | 294.3336 | 48.9170           | 321.5169 | 68.3336 |
| $\alpha R + \beta R^2$ | -155.0369                              | 310.0738  | 0.8784        | 322.0738 | 76.6572           | 345.3737 | 92.1905 |
| $\beta R^n$       | -175.0105                                | 350.0211  | 0.9916        | 362.0211 | 116.6045          | 385.3210 | 132.1378 |
| $\alpha R + \beta \sqrt{R}$ | -347.0748                               | 694.1496  | 1.9664        | 706.1496 | 460.7330          | 729.4496 | 476.2664 |
| $\alpha R + \beta R$ | -488.3049                               | 976.6099  | 2.7510        | 984.6099 | 739.1933          | 1000.1432 | 746.9600 |

**Table 1.** The best fit for each test model, including the $\Lambda$ CDM model. The models are listed in the order from the largest likelihood function value $\mathcal{L}(\hat{\theta}|\text{data})$ to the smallest likelihood of being viable. The reduced $\chi^2$-values are given as an indication of the goodness of fit for a particular model. The $\Lambda$ CDM model is chosen as the “true” model. The models that we tested, namely the first toy model and the second toy model with $n = \frac{1}{2}$ and $n = 1$, obtained best-fitting models that were not able to explain the data.

### 3.3 Statistical analysis

We are now able to do a statistical analysis on all the different $f(R)$-gravity models, to firstly find their goodness of fit, and secondly to determine whether they are statistically viable alternative models to explain the expansion of the Universe. Using all of the criteria from section 2.3, we can set-up table 1.

From table 1, we see that the two Starobinsky models obtained likelihood function values that are close of even better than the $\Lambda$ CDM model, and only obtained a percentage deviation on the goodness of fit of $\approx 1.14\%$ and $\approx 1.73\%$ respectively. However, based on the goodness of fit from the reduced $\chi^2$, the $\Lambda$ CDM model still fits the supernovae data better than the two
Starobinsky models. The other 3 models that were shown in the previous section, can still be considered good fits, since their $\chi^2$-values are still relatively close to the $\Lambda$CDM model, with the weakest fit (second toy model with $n = 2$) between these 5 models having an $\approx 30\%$ deviation on the “true model’s” goodness of fit. For the last three models, this percentage deviation, based on the goodness of fit, increases exponentially.

From the criteria selection, only the two Starobinsky models were deemed viable, with both obtaining a category 2 status for the AIC: "less support w.r.t. ‘true model’". However, only the reduced Starobinsky model obtained the category 2 status for the BIC, with the rest all being statistically rejected, even though some were able to fit the data.

As for future work, we can as mentioned try to test these models on other dataset. One example of this can be found in the study by [42], where they tested the Hu-Sawicki model on a combination of supernovae and cosmic chronometer data, as well as combining those results with gravitational waves. It not only managed to provide a more strict constrain on the Hu-Sawicki model, but also obtained cosmological parameter values that were closer to the CMB data. Whereas our results found that the Hu-Sawicki model is not as accurate as the Starobinsky model.

4 Conclusions

In this work, we looked at how GR can be used to explain the expansion of the Universe through the usage of the Friedmann equations. This particular set of Friedmann equations, called the $\Lambda$CDM model, had to include the dark energy term to explain the late-time acceleration of the expansion. We then discussed how this model introduces problems due to an early-time acceleration, as well as posing the dark energy problem since it is an unknown pressure force. We then discussed possible alternative modifications to the GR model, which are able to explain the accelerated late-time expansion of the Universe with the exclusion of dark energy. One of these alternative theories is called $f(R)$-gravity.

Following the $f(R)$-gravity model’s theory, we looked at how we will be able to find a best-fitting model for different $f(R)$ models. This led us to developed a MCMC simulation to fit the distance modulus for each $f(R)$ model to Supernovae Type 1A data and find the cosmology parameters ($\Omega_m$ and $\bar{h}$). We used the $\Lambda$CDM model to determine whether or not the MCMC simulation was correctly set-up. We also used the $\Lambda$CDM as a “true model" to compare the $f(R)$-gravity models to it.

We found 5 different $f(R)$-gravity models that were able to explain the data. This includes the models $f(R) = \alpha R + \beta R^n$ (where $n = 0, 2$), the Starobinsky model (as well as a reduced version) and the Hu-Sawicki model. The Starobinsky model obtained a larger likelihood of occurring than the $\Lambda$CDM model, however had a slightly worse goodness of fit, with a deviation of $\approx 1.14\%$ w.r.t to the $\Lambda$CDM model. The Starobinsky model was only given a category 2 on the Jeffery’s scale for the AIC selection, while being statistically rejected by the BIC selection. The reduced Starobinsky had a smaller likelihood of occurring, and a slightly worse fit with a $\approx 1.73\%$ deviation w.r.t the $\Lambda$CDM model. This model though was the only model to receive a category 2 status on both the AIC and BIC selections. Therefore, its the only model that fits the data and have some statistical significance.

The other three models were able to fit to the data, but were statistically rejected. These models might still be viable if we are to test them on other dataset, as indicated by the gravitational wave study. A further 3 models that we investigated were not able to explain
the data and were subsequently statistically rejected. Therefore, we constrained the viability of some of these $f(R)$-gravity models with cosmological data.

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