GRASSMANN PHASE SPACE THEORY

for the

BEC/BCS CROSSOVER

in

COLD FERMIONIC ATOMIC GASES

B. J. Dalton and N. M. Kidwani

Centre for Quantum Technology Theory, Swinburne University of Technology, Melbourne, Victoria 3122, Australia

1 Abstract

Grassmann Phase Space Theory (GPST) is applied to the BEC/BCS crossover in cold fermionic atomic gases and used to determine the evolution (over either time or temperature) of the Quantum Correlation Functions (QCF) that specify: (a) the positions of the spin up and spin down fermionic atoms in a single Cooper pair and (b) the positions of the two spin up and two spin down fermionic atoms in two Cooper pairs. The first of these QCF is relevant to describing the change in size of a Cooper pair, as the fermion-fermion coupling constant is changed through the crossover from a small Cooper pair on the BEC side to a large Cooper pair on the BCS side. The second of these QCF is important for describing the correlations between the positions of the fermionic atoms in two Cooper pairs, which is expected to be small at the BEC or BCS sides of the crossover, but is expected to be significant in the strong interaction unitary regime, where the size of a Cooper pair is comparable to the separation between Cooper pairs. In GPST the QCF are ultimately given via the stochastic average of products of Grassmann stochastic momentum fields, and GPST shows that the stochastic average of the products of Grassmann stochastic momentum fields at a later time (or lower temperature) is related linearly to the stochastic average of the products of Grassmann stochastic momentum fields at an earlier time (or higher temperature), and that the matrix elements involved in the linear relations are all c-numbers. Expressions for these matrix elements corresponding to a small time increment (or a small temperature change) have been obtained analytically, providing the formulae needed for numerical studies of the evolution that are planned for a future publication. Various initial conditions are considered, including those for a non-interacting fermionic gas at zero temperature and a high temperature gas (where the effect of the interactions can be ignored in the initial state). These would be relevant for studying the time evolution of the creation of a Cooper pair when the interaction is switched on.
via Feshbach resonance methods, or the evolution as the temperature is lowered for particular choices of the fermion-fermion coupling constant, such as where the coupling constant is very large - corresponding to the unitary regime. Full derivations of the expressions have been presented in the Appendices.
2 Introduction

Research into ultracold atomic gases [1] has been a major activity since the 1990’s when Bose-Einstein condensation was achieved for bosonic atoms. Non-interacting untrapped bosonic atoms at zero temperature form a BEC, with a macroscopic occupancy of the lowest single particle energy state, since there is no Pauli exclusion principle that forbids multiple occupancy. Since the 2000's ultracold gases of fermionic atoms have been prepared, and here quite different effects occur. Non-interacting untrapped fermionic atoms at zero temperature form a Fermi gas, with each energy state only being occupied by atoms with different spins due to the Pauli exclusion principle. Consequently, energy states are filled up to the Fermi energy $E_F$, whose associated wave number $k_F$ is proportional to the inverse of the average separation between the atoms. In both cases the single particle states are plane waves with momentum $\hbar k$.

When interactions are taken into account there are features in common though – for both types of atom the interaction potential only has a range $O(10^{-9}m)$, whilst the typical separation between atoms in these dilute gases is of $O(5 \times 10^{-7}m)$ making it unlikely that more than two atoms interact at once. The interaction between a pair of atoms determines the scattering length $a$ – which is related to the collision cross section. However, recent advances involving Feshbach resonances have made it possible to modify the scattering length [2] (for example by varying a static magnetic field that is coupled to the atoms so that an unbound state in an open scattering channel coincides with a bound state in a closed channel) so that its value ranges from negative to positive across the resonance – where (if decay processes are ignored) it would become infinite. At this resonance (or unitary) situation the strong interaction regime applies, with different effects to the weak interaction regimes well away from resonance on either side.

One of the interesting phenomena involving interacting ultracold fermionic atomic gases is the BEC/BCS crossover [3], [4], [5], [6]. If the atoms are essentially unconfined (as in a weak trapping potential) and at zero temperature the weak interaction regime differs for bosons and fermions. For the simplest bosonic case of spin 0 atoms the picture is that each atom behaves essentially independently, with the effect of the interatomic potentials being reflected in a mean field based on the average density of all the other atoms. The behaviour is treated using the Gross-Pitaevskii equation [7]. However, for the simplest fermionic case of spin $\frac{1}{2}$ atoms there is the possibility of two atoms with opposite spins pairing up and behaving as a single entity. The behaviour can be described using the BCS-Leggett theory [3], [5], (also a mean field theory). Here though there are two extreme possibilities depending on whether the two-body scattering length $a$ is positive or negative. For $1/(k_Fa) << 0$ superfluid BCS behaviour occurs based on loosely bound Cooper pairs of atoms with opposite spins and momenta (analogous to the Cooper pairs of electrons in metallic superconductors). For $1/(k_Fa) >> 0$ a BEC forms based on tightly bound pairs of atoms with opposite spins and momenta that constitute a spin 0 bosonic molecule. These regimes involve weakly interacting fermions. By using Fesbach
resonance methods, the BEC/BCS crossover from the BEC side to the BCS side can be achieved experimentally, including the intermediate strong interaction (or unitary) regime, where the scattering length becomes very large. The behaviour of the quantum correlation functions (QCF) describing the positions of the fermionic atoms in the Cooper pairs can be measured using methods such as Bragg spectroscopy [9].

However, in terms of presenting a challenge for developing theories of interacting ultracold fermionic atomic gases it is the strong interaction regime (where the scattering length $|a| >> 1/k_F$), that is currently of great interest. Here the pair size is on the order of the interparticle separation $1/k_F$ [3]. There are a number of methods that can be used to treat interacting fermi gases. These include: (a) Diagrammatic methods [10], [11], [12] (b) Field theory techniques [13], [14] (c) Quantum Monte Carlo methods [15], [16], [17], [18] (d) Density functional methods [19] as well as (e) BEC-Leggett theory [8], [3] and (f) Variational approaches [20]. There are many variants within each category. Many experimental studies have also been carried out, for example [21], [22] and references therein. Detailed descriptions of these theoretical methods and how well they agree with experiment are presented in these references and in review articles such as Refs. [3], [4], [5], [6], [8], [9]. Generally speaking the current theories do not always agree quantitatively with experiment, though qualitative agreement occurs. All of these theoretical methods have advantages and disadvantages. BEC-Leggett theory provides a good simple overview of the crossover, but it is based on a variational approach in which the quantum state is assumed to be pairs of opposite spins each described by the same single pair wavefunction. This may not be a good approximation in the unitary regime. For example fig 1 in Ref. [13], shows that BEC-Leggett theory disagrees quantitatively with experiment for thermodynamic quantities such as the chemical potential. Quantum Monte-Carlo methods are also based around assumed choices of variational functions, and in general the fermion sign problem leads to difficulties. To quote Ref. [16] "A typical wavefunction or density matrix of a many-fermion system has a complex nodal structure that prevents an easy stochastic process from driving the system to its ground state or desired excited state. This problem, originating from the Pauli exclusion principle of fermions, is known as the fermion-sign problem, because crossing a node of the wavefunction or density matrix changes its sign, and thus disallows a straightforward weight interpretation of the wavefunction or density matrix." Field theory methods involve considering re-normalisation issues, which makes the approach rather complicated. Diagrammatic methods involve trying to decide which Feynman diagrams are important and which can be neglected. Density functional methods have been mainly used in atomic physics. Even more sophisticated theories such as the beyond Gaussian pair fluctuation theory in Ref [14] does not agree with experiment in the intermediate temperature range $0.1T_F < T < 0.5T_F$ for the unitary fermi gas, though quantitative agreement occurs in other temperature ranges. Hence there is scope for developing other methods for treating the many-body fermionic systems that are involved in ultracold fermionic atomic gases.
Phase space methods (which were developed to treat topics in quantum optics) can also be applied to treat problems in ultracold atomic gases. These methods are described in a number of textbooks [23], [24], [25], [26]. Essentially the quantum density operator is represented by a distribution function (functional) which depends on phase space variables (fields) representing mode (field) annihilation, creation operators, and the average value of physical quantities are then given by phase space integrals (functional integrals) involving the distribution function and with physical quantities replaced by functions of the phase space variables. Several distinct distribution functions have been introduced, based on whether the physical quantities involve normally ordered, symmetrically ordered or anti-normally ordered expressions involving the mode (field) annihilation, creation operators. These are the Glauber-Sudarshan $P$, Wigner $W$ and Husimi $Q$ functions respectively. Often the phase space variables (field) for annihilation, creation operators are related by complex conjugation, but treatments involving double phase space distributions where this does not apply also exist [27]. The evolution of the density operator is mapped onto a Fokker-Planck equation (functional FPE) for the distribution function (functional). It is then common to introduce stochastic phase space variables (fields) which satisfy Ito (or Stratonovich) stochastic differential (field) equations that are equivalent to the FPE (FFPE) and which involve stochastic Wiener increments, such that the phase space integrals that give the average value for physical quantities can then be replaced stochastic averages of functions of the now-stochastic phase space variables (fields). One of the general advantages of phase space theory is that it does not require any presumptions regarding what the actual behaviour of the system will turn out to be. By contrast, BEC-Leggett theory [8], [3] is based on an assumed state vector in which all the fermions are in Cooper pairs.

Since quantum optics involves the physics of photons - which are bosons - it is not surprising that the application of phase space methods in ultracold atomic gases has mainly been for bosonic atoms such as in the treatment of Bose-Einstein Condensates. However, although less common, phase space methods can also be applied to treat fermionic atoms - though the treatment must take into account the fact that the mode (or field) annihilation, creation operators obey anti-commutation rules rather than the commutation rules that apply in the bosonic case. This means that the square of a fermionic mode annihilation or creation operator is zero, so the question then arises as to what type of phase space variable is best suited for the fermionic situation? For bosons the natural choice is to use c-numbers for the phase space variables or fields - since they commute on multiplication and their square is non-zero. For fermions it may seem more natural to use numbers that have properties such as the square being zero and having anti-commutation properties in multiplication - just as for the mode or field operators they represent. Such quantities are known in mathematics as Grassmann numbers, and the first phase space theory for fermions was formulated by Cahill and Glauber in terms of Grassmann phase space variables [28] - Grassmann Phase Space Theory (GPST). One of the first applications followed shortly later [34] and was to momentum correlations between fermions of opposite spins in a 1D trap.
There are however phase space theories for fermions which are based on c-number quantities. One approach involves introducing Gaussian projectors specified by c-number phase space variables [29], where for $n$ modes these are $1 + n(2n - 1)$ variables that determine the $2n \times 2n$ covariance matrix involved in defining the Gaussian projector. Numerical applications [30] of the Gaussian operator approach have been made to the Hubbard model for the case of 16 sites, and to the dissociation of a bosonic system of two fermionic atoms. These cases are all microscopic and involve small numbers of fermions and modes. In the treatment by Rosales et al [31] the density operator is represented by a $Q$ distribution function of the c-number variables, where $Q$ is a positive quantity that can be interpreted as a probability. The $Q$ function is defined on a phase space equivalent to the space of real, anti-symmetric matrices. The approach has recently been applied to Majorana fermions [32], and the Fermi-Hubbard Hamiltonian [33]. Here the mode annihilation $\hat{c}_i$ and creation $\hat{c}_i^\dagger$ operators being replaced by Hermitian Majorana operators $(\hat{c}_i + \hat{c}_i^\dagger)$ and $-i(\hat{c}_i - \hat{c}_i^\dagger)$. In the Gaussian phase space theory approach, Fokker-Planck equations for the distribution function are also replaced by stochastic equations for phase space variables, which in this case parameterise a $2n \times 2n$ covariance matrix.

Further developments of Grassmann phase space theory (GPST) for fermions are set out in following textbook [26] and articles [35], [36], [37]. The same general phase space theory features apply as described previously. It is convenient to represent the quantum density operator for the system by a Grassmann (un-normalised B) distribution function [34] in a phase space, but again the phase space variables replacing the fermion annihilation and creation operators are Grassmann variables. For $n$ modes there are $2n$ phase space variables if a double space is used. Here the projector used in the expression for the density operator is that involving fermion (Bargmann) coherent states, which are given by similar expressions to the boson (Bargmann) coherent states that are used in phase space theories for bosonic systems. Experimentally measurable quantities for topics such as the BEC/BCS crossover are given by quantum correlation functions (QCF), and these can be related theoretically to Grassmann phase space integrals involving the distribution function and with the fermion operators being replaced by Grassmann phase space variables. As previously, evolution equations (over time or temperature) for the density operator lead to Fokker-Planck equations for the distribution function. These in turn are replaced by Ito stochastic equations for stochastic Grassmann phase space variables, and QCF now given by stochastic averages of products of these stochastic variables.

A key point is that the stochastic averages for QCF at a slightly later time (or slightly lower temperature) can be shown to be related linearly to such stochastic averages at an earlier time via matrices that only involve c-numbers. Even though these matrices involve stochastic quantities such as Wiener increments, they are not dependent on Grassmann variables, which enables computations to be carried out without having to represent Grassmann variables on the computer, and this makes Grassmann phase space theory numerically computable. The initial stochastic averages of products are obtained from the initial density
operator. It should also be noted that although the distribution function (functionals) are only quasi-probabilities (since they are Grassmann functions and not c-number functions), the expressions for QCF and the mean values of physical quantities has the same formal structure as in standard probability theory - with the physical quantities given by Grassmann variables being weighted by the (quasi) probability and then integrated over - albeit involving Grassmann integrals rather than c-number integrals. However, in a further development \cite{38} an underlying c-number interpretation of Grassmann phase space representations has been constructed. The related equations for stochastic Grassmann variables have the same form as in Ref. \cite{35}. In the present paper Grassmann phase space theory is used to develop coupled c-number equations for the QCF themselves after the stochastic Grassmann variables have been averaged out.

The utility of Grassmann phase space theory has been successfully tested \cite{39} on a simple fermion system that can be treated exactly by standard methods, namely determining the coherence between two Cooper pair states functions for interacting spin $\frac{1}{2}$ fermions. Here the numerical application of the theory agrees in detail with the analytic expression for the evolution of the coherence. However, the GPST work of Ref.\cite{34} involved a system consisting of many fermions in a 1D harmonic trap, and GPST was used numerically to calculate momentum correlation functions. Hence it is reasonable to apply the Grassmann phase space theory approach on bigger fermionic systems, of which the BEC/BCS crossover topic is an example.

In the present paper, we now apply the Grassmann phase space theory to treating the BEC/BCS crossover - a much more complex system involving many more fermions and modes. The QCF that are studied are those specifying the position of the spin up and spin down fermions in a single Cooper pair, and those specifying the positions of the two spin up and two spin down fermions in two Cooper pairs. We focus particularly on treating how the size of a Cooper pair changes through the crossover (BEC theory \cite{3} indicates it changes smoothly), and on how the correlations between two Cooper pairs change through the crossover (BEC-Leggett theory \cite{3} indicates there is little correlation at the BEC or BCS ends, but strong correlation should occur in the unitary regime). Both time evolution and temperature evolution will be studied, the former from an initial state at near zero temperature for non-interacting fermions with the fermions confined within the Fermi sphere, the latter from an initial state at high temperature where the fermion interactions can be ignored. GPST may be useful in describing the pseudo-gap regime (see Ref \cite{5}) between the transition temperature $T_c$ for superfluid behaviour and the temperature $T^*$ below which Cooper pairs are thought to form, as it directly focuses on the QCF describing the relative positions of spin up and spin down fermions. This QCF should show evidence of "preformed" Cooper pairs in the temperature regime $T_c < T < T^*$.

The present paper sets out the theory for the equations that determine the above QCF, starting with the Ito stochastic equations for stochastic Grassmann phase space variables. The stochastic averages that determine the matrices that relate the stochastic averages for QCF at a later time (or temperature) to the stochastic averages at an earlier time have been carried out analytically, leading
to standard non-stochastic matrices. The latter matrices can then be used to study numerically the evolution of the QCF over a finite time or temperature interval. Such calculations will be presented in a subsequent paper.

The plan of the paper is as follows. In Section 3 we present the basic equations for GPST in the context of the BEC/BCS crossover both for time and temperature evolution, starting from the Hamiltonian, proceeding via the FFPE for the Grassmann distribution functional and ending with the Ito SFE both for stochastic position fields and stochastic momentum fields. In Section 4 we present the general expressions for the QCF for a single Cooper pair and for two Cooper pairs. In Section 5 we present the results for the temperature and time evolution for the QCF for a single Cooper pair, correct to first order and with the stochastic averages carried out analytically. In Section 6 we present the analogous results for the QCF describing two Cooper pairs. Equations for carrying out the time or temperature evolution over a finite interval based on the matrix $M$ obtained in the previous two Sections are set out in Section 7. The introduction of dimensionless variables and numerical issues are discussed. Finally, a Summary and Conclusions from the work is presented in Section 8. Detailed derivations are set out in Appendices 10 and 11.
3 Grassman Phase Space Theory for Fields

In this Section we set out the basic equations for applying Grassmann phase space theory to the topic of the BEC/BCS crossover in ultracold atomic fermi gases. The general methodology and a full description of Grassmann algebra and calculus is set out in the following textbook [26] and articles [35], [36], [37]. We begin by writing the Hamiltonian involved in terms of fermion field operators, then introduce the quantum density operator and its B representation distribution functional involving fermion Grassmann field functions, followed by the two basic equations the density operator satisfies for describing time and temperature evolution. The quantum correlation functions (QCF) relevant to fermion position measurements for the size of and correlation between Cooper pairs are then defined. We set out the functional Fokker-Planck equations (FFPE) for the distribution functional in the two types of evolution, and then introduce the corresponding Ito stochastic field equations that are equivalent to the FFPE. We finally obtain the Ito stochastic equations for momentum fields in the case of untrapped fermions.

3.1 Hamiltonian and Field Operators

We consider a trapped Fermi gas of spin 1/2 fermionic atoms with spin conserving collisions of zero range between pairs of atoms. Here there are two distinct internal states corresponding to spin up and spin down atoms, designated by $\alpha = u(\uparrow), d(\downarrow)$, with $-\alpha$ referring to the opposite spin state. The Fermi gas is isolated from the environment so no relaxation effects are involved. This model was also considered by Plimak et al [34].

The Hamiltonian is written in terms of the field operators $\hat{\Psi}_\alpha(r), \hat{\Psi}_\alpha(r)^\dagger$ as

$$H_f = \int dr \left( \sum_\alpha \frac{\hbar^2}{2m} \nabla \hat{\Psi}_\alpha(r)^\dagger \cdot \nabla \hat{\Psi}_\alpha(r) + \sum_\alpha \hat{\Psi}_\alpha(r)^\dagger V_\alpha \hat{\Psi}_\alpha(r) + \frac{g}{2} \sum_\alpha \hat{\Psi}_\alpha(r)^\dagger \hat{\Psi}_{-\alpha}(r)^\dagger \hat{\Psi}_{-\alpha}(r) \hat{\Psi}_\alpha(r) \right)$$

$$= \hat{K} + \hat{V} + \hat{U}$$

and is the sum of kinetic energy, trap potential energy and collision interaction energy terms. Here $m$ is the mass of the fermionic atom, $V_\alpha$ is the trap potential for spin $\alpha$ and $g$ is the fermion-fermion interaction constant - which can be parameterised in terms of an effective s-wave scattering amplitude $a_s$ via $g = 4\pi a_s \hbar^2/m$. The exact scattering amplitude is given in Ref. [3] (see Eq. (94)). For simplicity $r$ refers to position in 3D. The Hamiltonian commutes with the number operator $\hat{N}$, where

$$\hat{N} = \sum_\alpha \int dr \left( \hat{\Psi}_\alpha(r)^\dagger \hat{\Psi}_\alpha(r) \right)$$
3.2 B Distribution Functional

For fermionic systems with spin components $\alpha$ the field operators $\hat{\Psi}_\alpha(r), \hat{\Psi}_\alpha(r)\dagger$ may be expanded in a complete orthonormal set of single fermion mode functions $\phi_{\alpha i}(r)$ for each component $\alpha$ as

$$\hat{\psi}_\alpha(r) = \sum_i \hat{c}_{\alpha i} \phi_{\alpha i}(r) \quad \hat{\psi}_\alpha(r)\dagger = \sum_i \hat{c}_{\alpha i}^\dagger \phi_{\alpha i}^*(r)$$

where $\hat{c}_{\alpha i}, \hat{c}_{\alpha i}^\dagger$ are fermionic annihilation, creation operators for the mode $\alpha i$. The orthogonality and completeness properties are

$$\int dr \phi_{\alpha i}^*(r)\phi_{\alpha i}(s) = \delta(r - s)$$

In Grassmann phase space field theory a quantum density operator $\hat{\rho}$ can be represented by its $B$ distribution functional $B[\psi(r)]$ in which the field operators $\hat{\Psi}_\alpha(r), \hat{\Psi}_\alpha(r)\dagger$ are represented via Grassmann fields $\psi_{\alpha}(r)$ and $\psi_{\alpha}^*(r)$ respectively, where $\psi(r) \equiv \{\psi_{\alpha}(r), \psi_{\alpha}^+(r)\}$. Here the Grassmann fields are expanded as

$$\psi_{\alpha}(r) = \sum_i g_{\alpha i} \phi_{\alpha i}(r) \quad \psi_{\alpha}^+(r) = \sum_i g_{\alpha i}^+ \phi_{\alpha i}^*(r)$$

where $g_{\alpha i}, g_{\alpha i}^+$ are Grassmann phase space variables for the mode $\alpha i$.

The canonical form of a density operator for the $B$ distribution is written in terms of Bargmann coherent state projectors $\Lambda[\psi(r)]$ in terms of a Grassmann functional integral as

$$\hat{\rho} = \int \prod_\alpha d\psi^+_{\alpha} d\psi_{\alpha} B[\psi(r)] \Lambda[\psi(r)]$$

where

$$\Lambda[\psi(r)] = |\psi(r)\rangle_B \langle \psi^+(r)^* |_B$$

and the Bargmann states are given by

$$|\psi(r)\rangle_B = \exp(\sum_\alpha \sum_i \hat{c}_{\alpha i}^\dagger g_{\alpha i}) |0\rangle = \prod_\alpha (|0_{\alpha i}\rangle - g_{\alpha i} |1_{\alpha i}\rangle)$$

$$|\psi^+(r)^* \rangle_B = \exp(\sum_\alpha \sum_i \hat{c}_{\alpha i} (g_{\alpha i}^+)^*) |0\rangle = \prod_\alpha ([0_{\alpha i}\rangle - (g_{\alpha i}^+)^* |1_{\alpha i}\rangle)$$

with $|0\rangle$ being the vacuum state with no fermions and $|0_{\alpha i}\rangle, |1_{\alpha i}\rangle$ being one fermion states with 0, 1 fermion in the $\alpha i$ mode.

The Bargmann states are eigenstates for the field operators

$$\hat{\psi}_{\alpha}(r) |\psi(r)\rangle_B = \psi_{\alpha}(r) |\psi(r)\rangle_B \quad B \langle \psi^+(r)^* | \hat{\Psi}_{\alpha}(r)\dagger = B \langle \psi^+(r)^* | \psi_{\alpha}^+(r)$$

and are not normalised to unity. In fact

$$B \langle \chi^+(r)^* | \psi(r) \rangle_B = \exp \sum_\alpha h_{\alpha i}^+ g_{\alpha i}$$
where the Grassman field $\chi^+(r)$ involves phase space amplitudes $h^{+}_{\alpha i}$. The Grassmann functional integral $\int \prod \alpha d\psi^{+}_\alpha d\psi_\alpha = \int d\psi^{+} d\psi$ involving the functional $B[\psi(r)]$ is equivalent to the multiple Grassmann phase space integral over the phase space variables $\prod \alpha dg^{+}_{\alpha 1}dg^{+}_{\alpha 2}...dg^{+}_{\alpha n}...dg_{\alpha 2}dg_{\alpha 1}$ of the distribution function $B(g)$ (where $g \equiv \{g_{\alpha i}, g^{+}_{\alpha i}\}$ which is equivalent to the functional $B[\psi(r)]$.

### 3.3 Evolution of Density Operator

There are two cases of interest. The first is where the density operator describes time evolution in a non-equilibrium system, in which case $\hat{\rho}$ satisfies the Liouville-von Neumann equation

$$\frac{\partial}{\partial t} \hat{\rho} = -\frac{i}{\hbar}[\hat{H}_f, \hat{\rho}]$$  \hspace{1cm} (12)

involving the Hamiltonian $\hat{H}_f$, and the second where the density operator describes temperature evolution in an equilibrium system, in which case the density operator is given by the expression for the canonical ensemble

$$\hat{\rho} = \exp(-\beta \hat{H}_f)/Z$$  \hspace{1cm} (13)

$$Z = Tr \exp(-\beta \hat{H}_f)$$  \hspace{1cm} (14)

Here $Z$ is the partition function and $\beta = 1/(k_B T)$, where $T$ is the temperature. The numerator $\hat{\sigma} = \exp(-\beta \hat{H}_f)$ satisfies a Matsubara equation.

$$\frac{\partial}{\partial \beta} \hat{\sigma} = -\frac{1}{2} [\hat{H}_f, \hat{\sigma}]_+$$  \hspace{1cm} (15)

In both cases $\hat{\rho}$ and $\hat{\sigma}$ can be represented via a $B$ distribution, as in Eq.(7).

### 3.4 Quantum Correlation Functions and B Distribution Functional

Physical quantities of interest can be expressed as quantum correlation functions (referred to as pair-correlation functions in the BEC/BCS crossover community, though QCF in the present paper may involve more than one pair of fermions) involving the field operators $\hat{\Psi}_{\alpha}(r)$, $\hat{\Psi}_{\alpha}^\dagger(r)$ and the density operator $\hat{\rho}$ and $\hat{\sigma}$. In the case of the BEC/.BCS crossover we are interested in the probability of finding a fermion of type $\alpha_1$ at position $r_1$, a fermion of type $\alpha_2$ at position $r_2$, ..., and a fermion of type $\alpha_p$ at position $r_p$. This probability is given by

$$P(\alpha_1 r_1, \alpha_2 r_2 \cdots \alpha_p r_p) = Tr(\hat{\rho} \Lambda(\alpha_1 r_1, \alpha_2 r_2 \cdots \alpha_p r_p))$$  \hspace{1cm} (16)
where \( \Lambda(\alpha_1 r_1, \alpha_2 r_2 \cdots \alpha_p r_p) \) is the projector
\[
\Lambda(\alpha_1 r_1, \alpha_2 r_2 \cdots \alpha_p r_p)
= \hat{\Psi}_{\alpha_1}(r_1) \cdots \hat{\Psi}_{\alpha_p}(r_p) |0\rangle \langle 0| \hat{\Psi}_{\alpha_p}(r_p) \cdots \hat{\Psi}_{\alpha_1}(r_1)
\] (17)

It can be shown (see Ref. [26], see Sects. 7.8.2, 12.4.2) that the required probability is given by a phase space functional integral involving the \( B \) distribution functional. This is an example of a QCF. We have
\[
P(\alpha_1 r_1, \alpha_2 r_2 \cdots \alpha_p r_p)
= \int \int d\psi^+ d\psi \psi_{\alpha_p}(r_p) \cdots \psi_{\alpha_2}(r_2) \hat{\rho} B[\psi(r), \psi^+(r)] \psi_{\alpha_1}(r_1) \cdots \psi_{\alpha_p}^+(r_p)
\] (18)
for both the \( \hat{\rho} \) and \( \hat{\sigma} \) cases.

3.5 Functional Fokker-Planck Equations

The functional Fokker-Planck equation for the \( B \) distribution functional \( B[\psi(r)] \) is derived starting from Eq. (7) via using the correspondence rules in conjunction with functional differentiation rules. We have for both the \( \hat{\rho} \) and \( \hat{\sigma} \) cases
\[
\hat{\rho} \Rightarrow \hat{\Psi}_{\alpha}(r) \hat{\rho} \quad B[\psi] \Rightarrow \psi_{\alpha}(r) B[\psi]
\]
\[
\hat{\rho} \Rightarrow \hat{\rho} \hat{\Psi}_{\alpha}(r) \quad B[\psi] \Rightarrow B[\psi] \left( + \frac{\delta}{\delta \psi_{\alpha}(r)} \right)
\]
\[
\hat{\rho} \Rightarrow \hat{\psi}_{\alpha}^+(r) \hat{\rho} \quad B[\psi] \Rightarrow ( + \frac{\delta}{\delta \psi_{\alpha}(r)} ) B[\psi]
\]
\[
\hat{\rho} \Rightarrow \hat{\rho} \hat{\psi}_{\alpha}^+(r) \quad B[\psi] \Rightarrow B[\psi] \psi_{\alpha}^+(r)
\] (19)

The correspondence rules are derived in Ref. [26] (see Sects. 13.1.2, 13.1.3). The first and fourth are based on the eigenvalue equations in Eq. (10). The functional Fokker-Planck equation for the time evolution of the \( B \) distribution functional \( B[\psi(r)] \) for the density operator \( \hat{\rho} \) due to the Hamiltonian in Eq. (1) is derived in Ref. [26] (see Sects. 1.2). An analogous treatment applies for the temperature evolution of the \( B \) distribution functional \( B[\psi(r)] \) for the density operator numerator \( \hat{\sigma} \) - the evolution being governed by Eq. (15).

3.5.1 Time Evolution Case

In the case of the density operator \( \hat{\rho} \) satisfying the Liouville-von Neumann equation the time evolution of the \( B \) distribution functional is as follows. There are terms corresponding to the kinetic \( (K) \), trap potential \( (V) \) and fermion-fermion interaction \( (U) \) terms in the Hamiltonian. The latter term has been written as a double space integral via the introduction of a Dirac delta function to comply
with the requirements for deriving the Ito stochastic equations \[37\] from the general theory \[26, 35, 36\].

The kinetic energy term is

\[
\left( \frac{\partial}{\partial t} B[\psi(r)] \right)_K = -\frac{i}{\hbar} \int ds \left\{ \left( \frac{-\hbar^2}{2m} \nabla^2 \psi_u(s) \right) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_u(s)} \right\} \\
+ \left\{ \left( \frac{-\hbar^2}{2m} \nabla^2 \psi_d(s) \right) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_d(s)} \right\} \\
- \left\{ \left( \frac{-\hbar^2}{2m} \nabla^2 \psi_u^+(s) \right) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_u^+(s)} \right\} \\
- \left\{ \left( \frac{-\hbar^2}{2m} \nabla^2 \psi_d^+(s) \right) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_d^+(s)} \right\}
\]

(20)

and only contributes to the drift term.

The trap potential energy term is

\[
\left( \frac{\partial}{\partial t} B[\psi(r)] \right)_V = -\frac{i}{\hbar} \int ds \left\{ \left( V_u \psi_u(s) \right) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_u(s)} \right\} + \left\{ \left( V_d \psi_d(s) \right) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_d(s)} \right\} \\
- \left\{ \left( V_u \psi_u^+(s) \right) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_u^+(s)} \right\} - \left\{ \left( V_d \psi_d^+(s) \right) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_d^+(s)} \right\}
\]

(21)

and also only contributes to the drift term.

The fermion-fermion interaction term is

\[
\left( \frac{\partial}{\partial t} B[\psi(r)] \right)_U = -\frac{i \, g}{\hbar} \int ds dr \left\{ \psi_u(s) \delta(r - s) \psi_d(r) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_u(s)} \frac{\nabla \delta}{\delta \psi_d(r)} \right\} \\
+ \left\{ \psi_d(s) \delta(r - s) \psi_u(r) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_d(s)} \frac{\nabla \delta}{\delta \psi_u(r)} \right\} \\
- \left\{ \psi_u^+(s) \delta(r - s) \psi_d^+(r) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_u^+(s)} \frac{\nabla \delta}{\delta \psi_d^+(r)} \right\} \\
- \left\{ \psi_d^+(s) \delta(r - s) \psi_u^+(r) B[\psi(r)] \frac{\nabla \delta}{\delta \psi_d^+(s)} \frac{\nabla \delta}{\delta \psi_u^+(r)} \right\}
\]

(22)
and is the only contribution to the diffusion term.

### 3.5.2 Temperature Evolution Case

In the case of the density operator $\hat{\sigma}$ satisfying the Matsubara equation the temperature evolution of the $B$ distribution functional is as follows.

The kinetic energy term is

$$\left( \frac{\partial}{\partial \beta} B[\psi(r)] \right)_K = -\frac{1}{2} \int ds \left[ \left\{ \left( -\frac{\hbar^2}{2m} \nabla^2 \psi_u(s) B[\psi(r)] \right) \frac{\tilde{\delta}}{\delta \psi_u(s)} \right\} + \left\{ \left( -\frac{\hbar^2}{2m} \nabla^2 \psi_d(s) B[\psi(r)] \right) \frac{\tilde{\delta}}{\delta \psi_d(s)} \right\} ight]$$

and only contributes to the drift term.

The trap potential energy term is

$$\left( \frac{\partial}{\partial \beta} B[\psi(r)] \right)_V = -\frac{1}{2} \int ds \left[ \left\{ (V_u \psi_u(s) B[\psi(r)]) \frac{\tilde{\delta}}{\delta \psi_u(s)} \right\} + \left\{ (V_d \psi_d(s) B[\psi(r)]) \frac{\tilde{\delta}}{\delta \psi_d(s)} \right\} + \left\{ (V_u \psi_u(s) B[\psi(r)]) \frac{\tilde{\delta}}{\delta \psi_u(s)} \right\} + \left\{ (V_d \psi_d(s) B[\psi(r)]) \frac{\tilde{\delta}}{\delta \psi_d(s)} \right\} \right]$$

and also only contributes to the drift term.
The fermion-fermion interaction term is

\[
\left( \frac{\partial}{\partial \beta} B[\psi(r)] \right)_U = -\frac{g}{4} \int \int ds \, dr \left\{ \psi_u(s) \delta(r-s) \psi_d(r) B[\psi(r)] \frac{\varphi}{\delta \psi_u(r)} \frac{\varphi}{\delta \psi_d(s)} \right\} + \left\{ \psi_u(s) \delta(r-s) \psi_u(r) B[\psi(r)] \frac{\varphi}{\delta \psi_u(r)} \frac{\varphi}{\delta \psi_u(s)} \right\} + \left\{ \psi_d(s) \delta(r-s) \psi_d(r) B[\psi(r)] \frac{\varphi}{\delta \psi_d(r)} \frac{\varphi}{\delta \psi_d(s)} \right\} + \left\{ \psi_d(s) \delta(r-s) \psi_u(r) B[\psi(r)] \frac{\varphi}{\delta \psi_u(r)} \frac{\varphi}{\delta \psi_d(s)} \right\} \tag{25}
\]

and is the only contribution to the diffusion term. Note the differences between the Liouville-von Neumann and Matusubara cases. Apart from the overall multiplier being related to $-\frac{i}{\hbar}$ rather than $-\frac{1}{2}$, the pairs of terms involving $\tilde{\psi}_\alpha(r)$ and the pairs of terms involving $\tilde{\psi}_\alpha^+(r)$ have the same sign in the Matsubara case and opposite signs in the other case.

### 3.6 Ito Stochastic Position Field Equations

The Ito stochastic field equations are derived from the FFPE via using the first order functional derivative (or drift) terms to give the classical (or non-noise) terms in the Ito SFE and via using the second order (or diffusion) terms to determine the noise (or fluctuation) terms in the Ito SFE. The latter terms involve Gaussian-Markoff stochastic noise quantities and require the Takagi factorisation [40] of the diffusion matrix.

The basic principle is that the time (or temperature) evolution of the $\alpha \beta$ distribution functional is equivalent to the time (or temperature) evolution of stochastic Grassmann fields $\tilde{\psi}_\alpha(r,t)$, $\tilde{\psi}_\alpha^+(r,t)$ (or $\tilde{\psi}_\alpha(r,\beta)$, $\tilde{\psi}_\alpha^+(r,\beta)$) which now replace the Grassman phase space fields $\psi_\alpha(r)$, $\psi_\alpha^+(r)$, and where the evolution is now described by Ito stochastic field equations. These are chosen so that the stochastic averages of the $\tilde{\psi}_\alpha(r,t)$, $\tilde{\psi}_\alpha^+(r,t)$ (or $\tilde{\psi}_\alpha(r,\beta)$, $\tilde{\psi}_\alpha^+(r,\beta)$) gives the same result as the Grassmann functional integrals involving the $\psi_\alpha(r)$, $\psi_\alpha^+(r)$. In particular, for the time evolution case the joint position probabilities $P(\alpha_1 r_1, \alpha_2 r_2 \cdots \alpha_p r_p)$ are now given by

\[
P(\alpha_1 r_1, \alpha_2 r_2 \cdots \alpha_p r_p) = \text{Tr}(\hat{\Psi}_{\alpha_1}(r_1)^\dagger \cdots \hat{\Psi}_{\alpha_p}(r_p)^\dagger) [0] \langle 0 | \hat{\Psi}_{\alpha_p}(r_p) \cdots \hat{\Psi}_{\alpha_1}(r_1) \rangle.
\]

\[
= \int \int d\psi^+ d\psi \psi_{\alpha_p}(r_p) \cdots \psi_{\alpha_1}(r_1) B[\psi(r), \psi^+(r)] \psi_{\alpha_1}^+(r_1) \cdots \psi_{\alpha_p}^+(r_p) \psi_{\alpha_p}(r_p) \cdots \psi_{\alpha_1}(r_1) \psi_{\alpha_1}^+(r_1) \cdots \psi_{\alpha_p}^+(r_p)
\]
where the bar indicates a stochastic average. The same expressions apply for the temperature evolution case, but with $t$ being replaced by $\beta$.

To set out the FFPE and the related Ito SFE it is useful in this Section to modify the notation for the field functions. The indices $A, B$ are introduced where each takes on the values 1, 2 and where (which $\alpha, \beta$ specifying $u$ or $d$ as before) $\psi_{\alpha1}(s) \equiv \psi_{\alpha}(s)$ and $\psi_{\alpha2}(s) \equiv \psi_{\alpha}^+(s)$. Similarly, $\psi_{\alpha1}(s, t) \equiv \psi_{\alpha}(s, t)$ and $\psi_{\alpha2}(s, t) \equiv \psi_{\alpha}^+(s, t)$ for the stochastic fields.

For the case of time evolution we may write the functional Fokker-Planck equation in the general form

$$\frac{\partial}{\partial t} B[\psi] = -\sum_{\alpha A} \int dr (A_{\alpha A}[\psi(r), r] B[\psi]) \frac{\delta}{\delta \psi_{\alpha A}(r)}$$

$$+ \frac{1}{2} \sum_{\alpha A, \beta B} \int ds dr (D_{\alpha A \beta B}[\psi(s), s; \psi(r), r] B[\psi]) \frac{\delta}{\delta \psi_{\beta B}(r)} \frac{\delta}{\delta \psi_{\alpha A}(s)}$$

and then the Ito stochastic field equations can be written as the sum of a classical and a noise contribution for the change in $\psi_{\alpha A}$ as

$$\delta \bar{\psi}_{\alpha A}(r, t) \equiv \bar{\psi}_{\alpha A}(r, t + \delta t) - \bar{\psi}_{\alpha A}(r, t)$$

$$= A_{\alpha A}[\bar{\psi}(r, t), r] \delta t + \sum_{\alpha A} B_{\alpha A}^A[\bar{\psi}(r, t), r] \delta \omega_{\alpha}(t_+)$$

$$= \left( \delta \bar{\psi}_{\alpha A}(r) \right)_{\text{class}} + \left( \delta \bar{\psi}_{\alpha A}(r) \right)_{\text{noise}}$$

where the matrix $B$ is linked to the diffusion matrix $D$ via Takagi factorisation as (see Ref \[26\], p312)

$$(BB^T)_{\alpha A \beta B} = \sum_{\alpha A} B_{\alpha A}^A \bar{\psi}(s, t), s \] B_{\alpha A}^{\beta B} \bar{\psi}(r, t), r$$

$$= D_{\alpha A \beta B}[\bar{\psi}(s, t), s; \bar{\psi}(r, t), r]$$

The drift and diffusion matrices can be read off from Eqs. \[20\], \[21\] and \[22\].

In determining the matrix $B$ the delta function $\delta(r - s) = \frac{1}{V} \sum q \exp iq \cdot (r - s)$ expressed in terms of box normalised plane waves, with $V = L^3$ and (for 3D) $q \equiv \{q_x, q_y, q_z \}$ - where $q_x, q_y, q_z$ are all integers - is used in the diffusion matrix. The details for determining the matrix $B$ are set out in Appendix \[10\].

The Wiener increments $\delta \omega_{\alpha}(t_+)$ are defined in terms of Gaussian-Markoff random noise terms $\Gamma_{\alpha}(t)$ as

$$\delta \omega_{\alpha}(t_+) = \int_{t}^{t + \delta t} dt_1 \Gamma_{\alpha}(t_1)$$

Properties of the Gaussian-Markoff random noise terms are set out in Appendix \[10\]. From these results we can then show that the Wiener increments have the
following properties

\[ \begin{align*}
\frac{\delta \omega_a(t)}{\delta \omega_a(t) \delta \omega_b(t)} &= 0 \\
\frac{\delta \omega_a(t) \delta \omega_b(t) \delta \omega_c(t)}{\delta \omega_a(t) \delta \omega_b(t) \delta \omega_c(t) \delta \omega_d(t)} &= \delta_{ab} \delta t \\
\frac{\delta \omega_a(t) \delta \omega_b(t) \delta \omega_c(t) \delta \omega_d(t)}{\delta \omega_a(t) \delta \omega_b(t) \delta \omega_c(t) \delta \omega_d(t)} &= 0 \\
\end{align*} \]

(32)

showing that the stochastic averages of a single \( \delta \omega_a \) is zero and the stochastic average of the product of two \( \delta \omega_a \)'s is zero if they are different and equal to \( \delta t \) if they are the same. In addition, the stochastic averages of products of odd numbers of \( \delta \omega_a \) are zero and stochastic averages of products of even numbers of \( \delta \omega_a \) are the sums of products of stochastic averages of pairs of \( \delta \omega_a \).

For the case of temperature evolution, the same forms apply but with \( t \) being replaced by \( \beta \).

### 3.6.1 Time Evolution Case

For the time evolution case the stochastic fields at time \( t + \delta t \) can be related to the stochastic fields at time \( t \) as follows. We have now reverted to the \( u, d, u^+, d^+ \) notation.

We have (see Sect. 11.2 for details)

\[ \begin{align*}
\psi_u(s, t+\delta t) \\
\psi_d(s, t+\delta t) \\
\psi_u^+(s, t+\delta t) \\
\psi_d^+(s, t+\delta t) \\
\end{align*} \]

\[ \begin{pmatrix}
\Theta_{u,u}(\delta t) & \Theta_{u,d}(\delta t) & 0 & 0 \\
\Theta_{d,u}(\delta t) & \Theta_{d,d}(\delta t) & 0 & 0 \\
0 & 0 & \Theta_{u,u}^+(\delta t) & \Theta_{u,d}^+(\delta t) \\
0 & 0 & \Theta_{d,u}^+(\delta t) & \Theta_{d,d}^+(\delta t) \\
\end{pmatrix} = \begin{pmatrix}
\psi_u(s, t) \\
\psi_d(s, t) \\
\psi_u^+(s, t) \\
\psi_d^+(s, t) \\
\end{pmatrix} \]

(33)
where the $\Theta$ quantities are

\[
[\Theta(\delta t)] = \begin{bmatrix}
1 + \frac{i}{\hbar} \left\{-\frac{\hbar^2}{2m} \nabla^2 + V_u\right\} \delta t \\
+ \lambda \sum_q \exp(-iqs) \left\{\delta \tilde{\omega}^q_{u,d} - i\delta \tilde{\omega}^q_{d,u}\right\} \\
+ \lambda \sum_q \exp(+iqs) \left\{\delta \tilde{\omega}^q_{u,d} + i\delta \tilde{\omega}^q_{d,u}\right\}
\end{bmatrix}
\]

\[
[\Theta^+(\delta t)] = \begin{bmatrix}
1 - \frac{i}{\hbar} \left\{-\frac{\hbar^2}{2m} \nabla^2 + V_u\right\} \delta t \\
+ \lambda \sum_q \exp(-iqs) \left\{-\delta \tilde{\omega}^q_{u+,d+} + i\delta \tilde{\omega}^q_{d+,u+}\right\} \\
+ \lambda \sum_q \exp(+iqs) \left\{-\delta \tilde{\omega}^q_{u+,d+} + i\delta \tilde{\omega}^q_{d+,u+}\right\} \\
1 - \frac{i}{\hbar} \left\{-\frac{\hbar^2}{2m} \nabla^2 + V_d\right\} \delta t
\end{bmatrix}
\]

(34)

where $\lambda = \sqrt{-\frac{ig}{2\hbar v}}$. These quantities involve the Laplacian, the trap potentials, the time interval $\delta t$ as well as four stochastic Wiener increments $\delta \tilde{\omega}^q_{u,d}$, $\delta \tilde{\omega}^q_{d,u}$, $\delta \tilde{\omega}^q_{u+,d+}$ and $\delta \tilde{\omega}^q_{d+,u+}$ for each $q$. The latter satisfy conditions analogous to (32). Note the sum over $q$ and that there is no change in the position coordinate $s$. This is a consequence of the zero range interaction.

### 3.6.2 Temperature Evolution Case

For the temperature evolution case the stochastic fields at temperature $\beta + \delta \beta$ can be related to the stochastic fields at temperature $\beta$ as follows. We have also now reverted to the $u, d, u^+, d^+$ notation. We have (see Sect. 10.3 for details)

\[
\begin{align*}
\psi_u(s, \beta + \delta \beta) \\
\psi_d(s, \beta + \delta \beta) \\
\psi^+_u(s, \beta + \delta \beta) \\
\psi^+_d(s, \beta + \delta \beta)
\end{align*}
\]

\[
\begin{bmatrix}
\Theta_{u,u}(\delta \beta) & \Theta_{u,d}(\delta \beta) & 0 & 0 \\
\Theta_{d,u}(\delta \beta) & \Theta_{d,d}(\delta \beta) & 0 & 0 \\
0 & 0 & \Theta^+_{u,u}(\delta \beta) & \Theta^+_{u,d}(\delta \beta) \\
0 & 0 & \Theta^+_{d,u}(\delta \beta) & \Theta^+_{d,d}(\delta \beta)
\end{bmatrix}
\]

(35)
where the $\Theta$ quantities are

\[
\begin{align*}
[\Theta(\delta\beta)] &= \begin{bmatrix}
1 + \frac{1}{2} \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V_u \right\} \delta\beta \\
+\eta \sum_q \exp(-iqs) \left\{ \delta \tilde{\omega}^q_{u,d} + i\delta \tilde{\omega}^q_{d,u} \right\} \\
+\eta \sum_q \exp(+iqs) \left\{ \delta \tilde{\omega}^q_{u,d} - i\delta \tilde{\omega}^q_{d,u} \right\}
\end{bmatrix} \\
[\Theta^+(\delta\beta)] &= \begin{bmatrix}
1 + \frac{1}{2} \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V_u \right\} \delta\beta \\
+\eta \sum_q \exp(-iqs) \left\{ \delta \tilde{\omega}^q_{u,d} + i\delta \tilde{\omega}^q_{d,u} \right\} \\
+\eta \sum_q \exp(+iqs) \left\{ \delta \tilde{\omega}^q_{u,d} - i\delta \tilde{\omega}^q_{d,u} \right\}
\end{bmatrix}
\end{align*}
\]

(36)

where $\eta = \sqrt{-gV}$. These quantities involve the Laplacian, the trap potentials, the temperature interval $\delta\beta$ as well as four stochastic Wiener increments $\delta \tilde{\omega}^q_{u,d}$, $\delta \tilde{\omega}^q_{d,u}$, $\delta \tilde{\omega}^{q+}_{u,d}$ and $\delta \tilde{\omega}^{q+}_{d,u}$ for each $q$. The latter satisfy conditions analogous to (32). Note the sum over $q$ and that there is no change in the position coordinate $s$. This is a consequence of the zero range interaction. Note the sign differences for both the classical and noise terms between the time and temperature evolution cases. There is also a change in the constants for these terms.

### 3.7 Ito Stochastic Momentum Field Equations

For the free field situation where $V_u = V_d = 0$, it is convenient to introduce stochastic momentum fields $\tilde{\phi}_\alpha(k), \tilde{\phi}^+_\alpha(k)$, which are related to the stochastic position fields $\tilde{\psi}_\alpha(s), \tilde{\psi}^+_\alpha(s)$ via spatial Fourier transforms involving box normalised plane waves $\exp(ik \cdot s)/\sqrt{V}$. The wave numbers $k$ are discrete and given by $k = \{k_x, k_y, k_z\} = \{2\pi n / L\} - \{k_x, k_y, k_z\}$, where $k_x, k_y, k_z$ are all integers and $V = L^3$. For ease of notation, we leave the $t$ or $\beta$ dependence implicit.

\[
\begin{align*}
\tilde{\psi}_\alpha(s) &= \frac{1}{\sqrt{V}} \sum_k \exp(ik \cdot s) \tilde{\phi}_\alpha(k) \\
\tilde{\psi}^+_\alpha(s) &= \frac{1}{\sqrt{V}} \sum_k \exp(-ik \cdot s) \tilde{\phi}^+_\alpha(k)
\end{align*}
\]

(37)

The inverse transformation is given as Eq. [94] in Appendix [10]. The use of stochastic momentum fields enables the $\nabla^2$ terms to be replaced by $k^2$ when the stochastic position fields are replaced by stochastic momentum fields in Eqs. [43] and [45].
3.7.1 Time Evolution Case - Momentum Fields

For the time evolution case the stochastic momentum fields at time $t + \delta t$ can be related to the stochastic momentum fields at time $t$ as follows. To obtain this result we substitute the expressions (37) for the stochastic position fields into Eq. (33).

We have

\[
\begin{pmatrix}
\tilde{\phi}_u(k, t + \delta t) \\
\tilde{\phi}_d(k, t + \delta t) \\
\tilde{\phi}_u^+(k, t + \delta t) \\
\tilde{\phi}_d(k, t + \delta t)
\end{pmatrix} =
\begin{pmatrix}
F_{u,u}(k, l; q, \delta t) & F_{u,d}(k, l; q, \delta t) & 0 & 0 \\
F_{d,u}(k, l; q, \delta t) & F_{d,d}(k, l; q, \delta t) & 0 & 0 \\
0 & 0 & F_{u,u}^+(k, l; q, \delta t) & F_{u,d}^+(k, l; q, \delta t) \\
0 & 0 & F_{d,u}^+(k, l; q, \delta t) & F_{d,d}^+(k, l; q, \delta t)
\end{pmatrix}
\times
\begin{pmatrix}
\tilde{\phi}_u(l, t) \\
\tilde{\phi}_d(l, t) \\
\tilde{\phi}_u^+(l, t) \\
\tilde{\phi}_d^+(l, t)
\end{pmatrix}
\]  

(38)

where the $F$ quantities are

\[
\begin{align*}
[F(k, l; q, \delta t)] &= \\
&= \begin{pmatrix}
\delta_{q,0} \delta_{k,l} \left( 1 + \frac{i}{\hbar} \frac{\hbar^2 \mathbf{k}^2}{2m} \delta t \right) + \lambda \delta_{(k-q),l} \left( \delta \tilde{\omega}^q_{u,d} + i \delta \tilde{\omega}^q_{d,u} \right) \\
+ \lambda \delta_{(k+q),l} \left( \delta \tilde{\omega}^q_{u,d} - i \delta \tilde{\omega}^q_{d,u} \right) \delta_{q,0} \delta_{k,l} \left( 1 + \frac{i}{\hbar} \frac{\hbar^2 \mathbf{k}^2}{2m} \delta t \right)
\end{pmatrix}
\end{align*}
\]

\[
[F^+(k, l; q, \delta t)] &= \\
&= \begin{pmatrix}
\delta_{q,0} \delta_{k,l} \left( 1 - \frac{i}{\hbar} \frac{\hbar^2 \mathbf{k}^2}{2m} \delta t \right) + \lambda \delta_{(k+q),l} \left( \delta \tilde{\omega}^q_{u+,d+} + i \delta \tilde{\omega}^q_{d+,u+} \right) \\
+ \lambda \delta_{(k-q),l} \left( -\delta \tilde{\omega}^q_{u+,d+} + i \delta \tilde{\omega}^q_{d+,u+} \right) \delta_{q,0} \delta_{k,l} \left( 1 - \frac{i}{\hbar} \frac{\hbar^2 \mathbf{k}^2}{2m} \delta t \right)
\end{pmatrix}
\]  

(39)

where $\lambda = \sqrt{-\frac{i g}{2 \hbar V}}$. In each row of the right side of Eq. (39) the sum over repeated indices $q, l$ is assumed.

3.7.2 Temperature Evolution Case - Momentum Fields

For the temperature evolution case the stochastic momentum fields at temperature $\beta + \delta \beta$ can be related to the stochastic momentum fields at temperature $\beta$ as follows. To obtain this result we substitute the expressions (37) for the stochastic position fields into Eq. (33).
We have

\[
\begin{bmatrix}
\phi_u(k, \beta + \delta \beta) \\
\phi_d(k, \beta + \delta \beta) \\
\phi^+_u(k, \beta + \delta \beta) \\
\phi^+_d(k, \beta + \delta \beta)
\end{bmatrix}
= 
\begin{bmatrix}
F_{u,u}(k, l; q, \delta \beta) & F_{u,d}(k, l; q, \delta \beta) & 0 & 0 \\
F_{d,u}(k, l; q, \delta \beta) & F_{d,d}(k, l; q, \delta \beta) & 0 & 0 \\
0 & 0 & F^*_{u,u}(k, l; q, \delta \beta) & F^*_{u,d}(k, l; q, \delta \beta) \\
0 & 0 & F^*_{d,u}(k, l; q, \delta \beta) & F^*_{d,d}(k, l; q, \delta \beta)
\end{bmatrix}
\times
\begin{bmatrix}
\phi_u(l, \beta) \\
\phi_d(l, \beta) \\
\phi^+_u(l, \beta) \\
\phi^+_d(l, \beta)
\end{bmatrix}
\]

(40)

where the \( F \) quantities are

\[
\begin{bmatrix}
[F(k, l; q, \delta \beta)] \\
[F^+(k, l; q, \delta \beta)]
\end{bmatrix}
= 
\begin{bmatrix}
\delta_{q,0} \delta_{k,l} \left( 1 + \frac{1}{2} \frac{\hbar^2 k^2}{2m} \right) \delta \beta & \eta \eta \delta_{(k-q),l} \left\{ \delta \omega^q_{u,d} + i \delta \omega^q_{d,u} \right\} \\
\eta \eta \delta_{(k+q),l} \left\{ \delta \omega^q_{u,d} - i \delta \omega^q_{d,u} \right\} & \delta_{q,0} \delta_{k,l} \left( 1 + \frac{1}{2} \frac{\hbar^2 k^2}{2m} \right) \delta \beta
\end{bmatrix}
\]

(41)

where \( \eta = \sqrt{-g} \). In each row of the right side of Eq.(40) the sum over repeated indices \( q, l \) is assumed.
4 Quantum Correlation Functions for Cooper Pairs

In this Section we derive the expressions for the QCF’s involving in determining the size of a single Cooper pair and determining the correlation between the positions of the fermions in two Cooper pairs.

For the case of a single Cooper pair where the spin down fermion is at position \( r_1 \) and the spin up fermion is at position \( r_2 \) the QCF that can be used to describe the size of the Cooper pair is

\[
X(\text{dr}_1, \text{ur}_2) = \text{Tr}(\hat{\rho} \hat{\Psi}_u(r_2) \hat{\Psi}_d(r_1) \hat{\Psi}_u(r_2)) \tag{42}
\]

For the case of two Cooper pairs where the spin down fermions are at positions \( r_1, r_2 \) and the spin up fermions are at positions \( r_3, r_4 \) the QCF that can be used to describe the correlation of the positions of the fermions in the two Cooper pairs is

\[
X(\text{dr}_1, \text{dr}_2, \text{ur}_3, \text{ur}_4) = \text{Tr}(\hat{\rho} \hat{\Psi}_u(r_4) \hat{\Psi}_u(r_3) \hat{\Psi}_d(r_2) \hat{\Psi}_d(r_1) \hat{\Psi}_d(r_3) \hat{\Psi}_u(r_4)) \tag{43}
\]

4.1 One Cooper Pair: QCF

For the case of time evolution the stochastic expressions for \( X(\text{dr}_1, \text{ur}_2) \) can be written in terms of stochastic position or momentum fields as

\[
X(\text{dr}_1, \text{ur}_2; t) = \Psi_d(r_1, t) \Psi_u(r_2, t) \Psi_u^+(r_2, t) \Psi_d^+(r_1, t) \tag{44}
\]

\[
= \frac{1}{(\sqrt{V})^4} \sum_{k_1, k_2, k_3, k_4} \exp i\{(k_1 - k_4) \cdot r_1\} \exp i\{(k_2 - k_3) \cdot r_2\} \times \phi_d(k_1, t) \phi_u(k_2, t) \phi_u^+(k_3, t) \phi_d^+(k_4, t) \tag{45}
\]

The case of temperature evolution leads to the same forms with \( t \) replaced by \( \beta \) in Eq. (44) and in Eq. (45), along with the additional factor \( 1/Z \) on the right side of Eq. (45) to allow for \( \hat{\rho} \) being replaced by \( \hat{\sigma} \) when temperature evolution is involved.

The momentum stochastic correlation function of interest is

\[
X(\text{dk}_1, \text{uk}_2, \text{uk}_3, \text{uk}_4) = \phi_d(k_1) \phi_u(k_2) \phi_u^+(k_3) \phi_d^+(k_4) \tag{46}
\]

where we have left the \( t \) or \( \beta \) dependences implicit to shorten the notation.
4.2 Two Cooper Pairs: QCF

For the case of time evolution the stochastic expressions for $X(\{dr_1, dr_2, ur_3, ur_4\})$ can be written in terms of stochastic position or momentum fields as

$$X(\{dr_1, dr_2, ur_3, ur_4; t\})$$

$$= \frac{1}{(\sqrt{V})^8} \sum_{k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8} \exp i\{(k_1 - k_8) \cdot r_1\} \exp i\{(k_2 - k_7) \cdot r_2\} \exp i\{(k_3 - k_6) \cdot r_3\} \exp i\{(k_4 - k_5) \cdot r_4\} \hat{\phi_d}(k_1,t) \hat{\phi_d}(k_2,t) \hat{\phi_u}(k_3,t) \hat{\phi_u}(k_4,t) \hat{\phi_u}(k_5,t) \hat{\phi_u}(k_6,t) \hat{\phi_d}(k_7,t) \hat{\phi_d}(k_8,t)$$

(47)

The case of temperature evolution leads to the same forms with $t$ replaced by $\beta$ in Eq. (47) and in Eq. (48), along with the additional factor $1/Z$ on the right side of Eq. (48) to allow for $\hat{\rho}$ being replaced by $\hat{\sigma}$ when temperature evolution is involved.

The momentum stochastic correlation function of interest is

$$X(\{dk_1, dk_2, uk_3, uk_4, uk_5, uk_6, d^+k_7, d^+k_8\})$$

$$= \hat{\phi_d}(k_1) \hat{\phi_d}(k_2) \hat{\phi_u}(k_3) \hat{\phi_u}(k_4) \hat{\phi_u}(k_5) \hat{\phi_u}(k_6) \hat{\phi_d}(k_7) \hat{\phi_d}(k_8)$$

(49)

where we have left the $t$ or $\beta$ dependences implicit to shorten the notation.

4.3 Number of Fermions - Partition Function Factor

The mean number of fermions can be obtained by taking the mean of the number operator $\hat{N}$. The mean equals $N$, the fixed number of fermions. By expanding the position field operators in terms of plane waves and carrying out the space integrals we find that for temperature evolution

$$\langle \hat{N} \rangle = N$$

$$= \frac{1}{Z} \sum_{\alpha} \int dr \text{Tr} \hat{\Psi}_\alpha(r) \hat{\sigma} \hat{\Psi}_\alpha(r)$$

$$= \frac{1}{Z} \sum_{\alpha} \sum_k \hat{\phi}_\alpha(k, \beta) \hat{\phi}_\alpha^+(k, \beta)$$

(50)

showing there is a relationship between the partition function $Z$, the number of fermions $N$ and QCF of the form $X(\alpha k, \alpha k) = \hat{\phi}_\alpha(k, \beta) \hat{\phi}_\alpha^+(k, \beta)$. In applying the theory to temperature evolution we have

$$\frac{1}{Z} = \frac{N}{\langle \sum_\alpha \sum_k X(\alpha k, \alpha k) \rangle}$$

(51)
so the $1/Z$ factors in the results for $X(d r_1, u r_2)$ and $X(d r_1, d r_2, u r_3, u r_4)$ may be replaced by the last expression. Hence for example Eq.(45) gives

$$X(d r_1, u r_2; \beta) \frac{1}{N} = \left\{ \frac{1}{(\sqrt{V})^4} \sum_{k_1,k_2,k_3,k_4} \exp i\{(k_1 - k_4) \cdot r_1\} \exp i\{(k_2 - k_3) \cdot r_2\} \times \bar{\phi}_d(k_1, \beta) \bar{\phi}_u(k_2, \beta) \bar{\phi}_u^+(k_3, \beta) \bar{\phi}_d^+(k_4, \beta) \right\} \div \left\{ \sum_{\alpha} \sum_{k} \bar{\phi}_\alpha(k, \beta) \bar{\phi}_\alpha^+(k, \beta) \right\}$$

so only the stochastic momentum QCF are involved.
5 Single Cooper Pair - First Order Changes to QCF

In this Section we will derive expressions for the first order changes in the QCF for a single Cooper pair based on stochastic momentum fields by carrying out the stochastic averages analytically. The results will be correct to order $\delta t$ or $\delta \beta$ for time or temperature evolution respectively.

5.1 One Cooper Pair - Time Evolution

The first order change in the QCF $\phi_d(k_1) \phi_u(k_2) \phi_u^+(k_3) \phi_d^+(k_4)$ due to time evolution between $t$ and $t+\delta t$ is derived in Appendix 11 and the result is

$$\delta \phi_d(k_1,t+\delta t) \phi_u(k_2,t+\delta t) \phi_u^+(k_3,t+\delta t) \phi_d^+(k_4,t+\delta t)$$

$$= \sum_{l_1,l_2,l_3,l_4} \delta_{k_1,l_1} \delta_{k_2,l_2} \delta_{k_3,l_3} \delta_{k_4,l_4} \left( 1 + i \frac{\hbar}{2m} \right) 2 \lambda^2 \sum_{l_1,l_2,l_3,l_4} \{ \delta_{(k_{1}+k_{2}),l_1+l_2} \delta_{(k_{3}+k_{4}),l_3+l_4} \} \delta t \times \phi_d(l_1,t) \phi_u(l_2,t) \phi_u^+(l_3,t) \phi_d^+(l_4,t)$$

$$+ 2 \lambda^2 \sum_{l_1,l_2,l_3,l_4} \{ \delta_{(k_{3}+k_{4}),l_3+l_4} \delta_{(k_{1}+k_{2}),l_1+l_2} \} \delta t \times \phi_d(l_3,t) \phi_u(l_4,t) \phi_u^+(l_3,t) \phi_d^+(l_4,t)$$

$$- 2 \lambda^2 \sum_{l_1,l_2,l_3,l_4} \{ \delta_{(k_{1}+k_{2}),l_1+l_2} \delta_{(k_{3}+k_{4}),l_3+l_4} \} \delta t \times \phi_d(l_1,t) \phi_u(l_2,t) \phi_u^+(l_3,t) \phi_d^+(l_4,t)$$

where $\lambda = \sqrt{\frac{\mu}{2\hbar}}$. The key feature of this last result is that the

$$\phi_d(k_1) \phi_u(k_2) \phi_u^+(k_3) \phi_d^+(k_4)$$

at time $t+\delta t$ depend linearly on the $\phi_d(l_1) \phi_u(l_2) \phi_u^+(l_3) \phi_d^+(l_4)$ at time $t$, and the change in these QCF is proportional to $\delta t$. Furthermore, the QCF are c-numbers so their time dependence can be calculated on a computer. The Grassmann variables are no longer present. The behaviour depends of course on the initial conditions for $\phi_d(k_1) \phi_u(k_2) \phi_u^+(k_3) \phi_d^+(k_4)$ at $t=0$, and this will be discussed below.

5.2 Initial Condition - Time Evolution from t=0

A natural initial condition at $t=0$ would be to assume the fermionic atoms were non-interacting, and to turn on the interaction via Feshbach resonance methods. The choice of the interaction constant could span the range for the BEC/BCS crossover. If the $N$ fermionic atoms are non-interacting the initial state could be that for zero temperature, in which case all plane wave modes up to the Fermi surface would be occupied by one spin down atom and one spin
up atom. The density operator is based on a pure state $|\Psi\rangle$

$$
\hat{\rho} = |\Psi\rangle \langle \Psi|
$$

$$
|\Psi\rangle = \prod_{k,\,|k| \leq k_F} |1_{ku}\rangle \otimes \prod_{k,\,|k| \leq k_F} |1_{kd}\rangle
$$

(54)

where $k_F = (3\pi^2 N/V)^{1/3}$ is the radius of the Fermi sphere.

Hence

$$
\hat{\Phi}_d(k_1) \hat{\Phi}_u(k_2) |\Psi\rangle = 0 \quad k_1, k_2 \text{ outside FS }
$$

(55)

so that for $t = 0$

$$
\overline{\phi_d(l_1) \phi_u(k_2) \phi_u^+(k_3) \phi_d^+(l_4)} = \text{Tr} (\hat{\Phi}_d(k_1) \hat{\Phi}_u(k_2) \hat{\rho} \hat{\Phi}_u(k_3)^\dagger \hat{\Phi}_d(k_4)^\dagger).
$$

(56)

Thus from Eq. (55) the non-zero $\overline{\phi_d(l_1) \phi_u(k_2) \phi_u^+(k_3) \phi_d^+(l_4)}$ at $t = \delta t$ that are present in first order must originate from $\hat{\Phi}_d(l_1, 0) \hat{\rho} \hat{\Phi}_u(l_2, 0) \hat{\Phi}_u^+(l_3, \delta t) \hat{\Phi}_d(l_4, \delta t)$

where $k_3 + k_4 = l_2 + l_1$ from the first $2\lambda^2$ term and $\overline{\phi_d(l_1, \delta t) \phi_u(k_2, \delta t) \phi_u^+(l_3, \delta t) \phi_d(l_4, \delta t)}$ where $k_1 + k_2 = l_1 + l_2$ from the second $2\lambda^2$ term These restrictions represent momentum conservation.

A specific case of interest is where $l_1 = l_4 = +k$ and $l_2 = l_3 = -k$, where $k$ is on the Fermi sphere. The term $\overline{\phi_d(+k, \delta t) \phi_u(-k, \delta t) \phi_u^+(+k, \delta t) \phi_d(-k, \delta t)}$

could originate from the first $2\lambda^2 \times \overline{\phi_d(+k, 0) \phi_u(-k, 0) \phi_u^+(+k, 0) \phi_d(-k, 0)}$ but would not originate from the second $2\lambda^2 \times \overline{\phi_d(+k, 0) \phi_u(-k, 0) \phi_u^+(+k, 0) \phi_d(-k, 0)}$ term Hence there would be no cancelation of the contributions. On the other hand, for the case where $l_1 = +k$, $l_2 = -k$, $l_3 = -k$ and $l_4 = +k$ where $k$ is on the Fermi sphere, the term $\overline{\phi_d(+k, \delta t) \phi_u(-k, \delta t) \phi_u^+(+k, \delta t) \phi_d(-k, \delta t)}$ could originate from the first $2\lambda^2 \times \overline{\phi_d(+k, 0) \phi_u(-k, 0) \phi_u^+(+k, 0) \phi_d(-k, 0)}$ and could also originate from the second $2\lambda^2 \times \overline{\phi_d(+k, 0) \phi_u(-k, 0) \phi_u^+(+k, 0) \phi_d(-k, 0)}$ term. However in this case the two contributions cancel out.

### 5.3 One Cooper Pair - Temperature Evolution

The first order change in the QCF $\overline{\phi_d(k_1) \phi_u(k_2) \phi_u^+(k_3) \phi_d^+(k_4)}$ due to temperature evolution between $\beta$ and $\beta + \delta \beta$ is derived in Appendix 11 and the result...
is
\[
\begin{align*}
&\sum_{l_1, l_2, l_3, l_4} \delta_{l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8} \delta_{k_1, l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8} \delta_{k_2, l_2, l_3, l_4, l_5, l_6, l_7, l_8} \delta_{k_3, l_3, l_4, l_5, l_6, l_7, l_8} \delta_{k_4, l_4, l_5, l_6, l_7, l_8} \delta_{k_5, l_5, l_6, l_7, l_8} \delta_{k_6, l_6, l_7, l_8} \delta_{k_7, l_7, l_8} \delta_{k_8, l_8} \frac{1}{N} \sum_{\lambda} |\lambda\rangle \langle \lambda| \\
&\approx \frac{N!}{(2n)^N} \sum_{\lambda} |\lambda\rangle \langle \lambda|
\end{align*}
\]

where \( \eta = \sqrt{\frac{\hbar^2}{2m}} \). Similar comments to the time evolution case apply here as well. The initial conditions are discussed below.

### 5.4 Initial Condition - Temperature Evolution from \( \beta = 0 \)

A natural initial condition at \( \beta = 0 \) would be to assume the fermionic atoms were non-interacting, and to turn on the interaction via Feshbach resonance methods. The choice of the interaction constant could span the range for the BEC/BCS crossover. If the \( N \) fermionic atoms are non-interacting the initial state could be that for a very high temperature, in which case a large number of plane wave modes could be occupied, both inside and outside the Fermi surface by at most one spin down atom and one spin up atom. The density operator is based on a mixed state. Suppose we consider the case where there are \( n \) different modes \( k \), where \( |k| \) would range from being \( \ll k_F \) to being \( \gg k_F \). The one-fermion states are \( |1_{ku}\rangle \) or \( |1_{kd}\rangle \) if occupied, and \( |0_{ku}\rangle \) or \( |0_{kd}\rangle \) if unoccupied. So a typical \( N \) fermion state will be \( |\lambda\rangle = \prod_k |\mu_{ku}\rangle |\nu_{kd}\rangle \) where \( \mu, \nu = 0, 1 \) only and with \( \lambda \equiv \{\mu_{ku}, \nu_{kd}\} \). The number of distinct states \( |\lambda\rangle \) is given by the number of ways of choosing \( N \) occupied one fermion states out of the set of \( 2n \) possible one fermion states, and this is given by \( N = 2nC_N = 2n!/[((N)!/(2n - N)!)] \approx (2n)^N/N \) since we have \( n \gg N \). At high temperatures each of these states \( |\lambda\rangle \) is equally probable, so the density operator is given by

\[
\hat{\rho} = \frac{1}{N} \sum_{\lambda} |\lambda\rangle \langle \lambda|
\]

\[
\approx \frac{N!}{(2n)^N} \sum_{\lambda} |\lambda\rangle \langle \lambda|
\]

(58)
We then want to consider the quantity
\[
\text{Tr}(\hat{\Phi}_d(k_1) \hat{\Phi}_u(k_2) \rho \hat{\Phi}_u(k_3) \hat{\Phi}_d(k_4))
\]
so to get a non-zero contribution for a given \( \lambda \) we need \( \hat{\Phi}_d(k_1) \hat{\Phi}_u(k_2) |\lambda\rangle \neq 0 \) and \( \hat{\Phi}_d(k_4) \hat{\Phi}_u(k_3) |\lambda\rangle \neq 0 \). Noting that \( \langle \hat{\Phi}_u(k) | 1_{k\alpha} \rangle = |0_{k\alpha}\rangle \) and \( \hat{\Phi}_u(k) |0_{k\alpha}\rangle = 0 \) we see that to get a non-zero contribution for a given \( \lambda \) we require \( k_1 = k_4 \) and \( k_2 = k_3 \) which restricts \( \lambda \) to be of the form \( |\lambda^\#\rangle = |1_{k_1d}\rangle |1_{k_2u}\rangle \prod_{k \neq k_1, k_2} |\mu_{k\alpha}\rangle |\nu_{kd}\rangle \).

Similarly to the previous situation, there are \( N^\# = 2^{n-2} C_{N-2} \approx (2n)^{(N-2)}/(N-2)! \) different states \( |\lambda^\#\rangle \) (since this equals the number of ways of choosing \( N-2 \) occupied one fermion states from \( 2n-2 \) possible remaining one fermion states). In this case we have
\[
\text{Tr}(\hat{\Phi}_d(k_1) \hat{\Phi}_u(k_2) \rho \hat{\Phi}_u(k_3) \hat{\Phi}_d(k_4))
\]
\[
= \frac{1}{N} \delta_{k_1,k_4} \delta_{k_2,k_3} \sum_{\lambda^\#} \text{Tr}(|0_{k_1d}\rangle |0_{k_2u}\rangle \left( \prod_{k \neq k_1, k_2} |\mu_{k\alpha}\rangle |\nu_{kd}\rangle \otimes |\mu_{k\alpha}\rangle |\nu_{kd}\rangle \right) |0_{k_1d}\rangle |0_{k_2u}\rangle )
\]
\[
= \frac{1}{N} \delta_{k_1,k_4} \delta_{k_2,k_3} \sum_{\lambda^\#} 1 = \frac{N^\#}{N} \delta_{k_1,k_4} \delta_{k_2,k_3}
\]
\[
\approx \frac{1}{4} \left( \frac{N}{n} \right)^2 \delta_{k_1,k_4} \delta_{k_2,k_3} \quad (60)
\]
Writing \( \tilde{\sigma} = Z \rho \) then see that at \( \beta = 0 \)
\[
\tilde{\sigma}_d(k_1) \tilde{\sigma}_u(k_2) \tilde{\sigma}_u(k_3) \tilde{\sigma}_d(k_4)
\]
\[
= \text{Tr}(\hat{\Phi}_d(k_1) \hat{\Phi}_u(k_2) \tilde{\sigma} \hat{\Phi}_u(k_3) \hat{\Phi}_d(k_4))
\]
\[
\approx \frac{Z}{4} \left( \frac{N}{n} \right)^2 \delta_{k_1,k_4} \delta_{k_2,k_3} \quad (61)
\]
Thus from Eq.\(57\) the non-zero \( \tilde{\sigma}_d(k_1) \tilde{\sigma}_u(k_2) \tilde{\sigma}_u(k_3) \tilde{\sigma}_d(k_4) \) at \( \beta = \delta \beta \) that are present in first order must originate from \( \tilde{\sigma}_d(l_1,0) \tilde{\sigma}_u(l_2,0) \tilde{\sigma}_u(l_2,0) \tilde{\sigma}_d(l_1,0) \approx \frac{Z}{4} \left( \frac{N}{n} \right)^2 \) with \( l_1, l_2 \) not restricted to the Fermi sphere. These are \( \tilde{\sigma}_d(l_1, \delta \beta) \tilde{\sigma}_u(l_2, \delta \beta) \tilde{\sigma}_u(l_3, \delta \beta) \tilde{\sigma}_d(l_4, \delta \beta) \) where \( k_3 + k_4 = l_2 + l_1 \) from the first \( 2n^2 \) term and \( \tilde{\sigma}_d(l_1, \delta \beta) \tilde{\sigma}_u(l_2, \delta \beta) \tilde{\sigma}_u(l_2, \delta \beta) \tilde{\sigma}_d(l_1, \delta \beta) \) where \( k_1 + k_2 = l_1 + l_2 \) from the second \( 2n^2 \) term. These restrictions represent momentum conservation.

A specific case of interest is where \( l_1 = l_4 = +k \) and \( l_2 = l_3 = -k \), where \( k \) need not be on the Fermi sphere. The term \( \tilde{\sigma}_d(+k, \delta \beta) \tilde{\sigma}_u(-k, \delta \beta) \tilde{\sigma}_u(+k, \delta \beta) \tilde{\sigma}_d(-k, \delta \beta) \) could originate from the first \( 2n^2 \times \tilde{\sigma}_d(+k, 0) \tilde{\sigma}_u(-k, 0) \tilde{\sigma}_u(-k, 0) \tilde{\sigma}_d(+k, 0) \) term.
but would not originate from the second $2\eta^2 \times \tilde{\phi}_d (+k, 0) \tilde{\phi}_u (-k, 0) \tilde{\phi}_d^+ (-k, 0) \tilde{\phi}_d^+ (+k, 0)$ term. Hence there would be no cancelation of the contributions. On the other hand, for the case where $l_1 = +k$, $l_2 = -k$, $l_3 = -k$ and $l_4 = +k$ where $k$ need not be on the Fermi sphere, the term $\tilde{\phi}_d (+k, \delta\beta) \tilde{\phi}_u (-k, \delta\beta) \tilde{\phi}_d^+ (-k, \delta\beta) \tilde{\phi}_d^+ (+k, \delta\beta)$ could originate from the first $2\eta^2 \times \tilde{\phi}_d (+k, 0) \tilde{\phi}_u (-k, 0) \tilde{\phi}_d^+ (-k, 0) \tilde{\phi}_d^+ (+k, 0)$ term and could also originate from the second $2\eta^2 \times \tilde{\phi}_d (+k, 0) \tilde{\phi}_u (-k, 0) \tilde{\phi}_d^+ (-k, 0) \tilde{\phi}_d^+ (+k, 0)$ term. However in this temperature evolution case the two contributions do not cancel out - in contrast to the time evolution case.
6 Two Cooper Pairs - First Order Changes to QCF

In this Section we will derive expressions for the first order changes in the QCF for two Cooper pairs based on stochastic momentum fields by carrying out the stochastic averages analytically. The results will be correct to order $\delta t$ or $\delta \beta$ for time or temperature evolution respectively.

6.1 Two Cooper Pairs - Time Evolution

For the case of time evolution the QCF for two Cooper pairs at time $t + \delta t$ is related to that at time $t$ via

$$
\begin{align*}
\mathcal{Q}_{\text{StochAver}} & = \sum_{l_1} \sum_{l_2} \sum_{l_3} \sum_{l_4} \sum_{l_5} \sum_{l_6} \sum_{l_7} \sum_{l_8} \\
& \times \left[ (\delta_{k_1,l_1} \delta_{k_2,l_2} \delta_{k_3,l_3} \delta_{k_4,l_4} \delta_{k_5,l_5} \delta_{k_6,l_6} \delta_{k_7,l_7} \delta_{k_8,l_8}) \\
& \quad + (-2\lambda^2) (\delta_{k_{1},l_{1}} \delta_{k_{2},l_{2}} \delta_{k_{3},l_{3}} \delta_{k_{4},l_{4}} \delta_{k_{5},l_{5}} \delta_{k_{6},l_{6}} \delta_{k_{7},l_{7}} \delta_{k_{8},l_{8}}) \\
& \quad \times \left( \delta_{k_{a},l_{a}} \delta_{k_{b},l_{b}} \delta_{k_{c},l_{c}} \delta_{k_{d},l_{d}} \delta_{k_{e},l_{e}} \delta_{k_{f},l_{f}} \delta_{k_{g},l_{g}} \delta_{k_{h},l_{h}} \right) \\
& \quad + (2\lambda^2) (\delta_{k_{1},l_{1}} \delta_{k_{2},l_{2}} \delta_{k_{3},l_{3}} \delta_{k_{4},l_{4}} \delta_{k_{5},l_{5}} \delta_{k_{6},l_{6}} \delta_{k_{7},l_{7}} \delta_{k_{8},l_{8}}) \\
& \quad \times \left( \delta_{k_{a},l_{a}} \delta_{k_{b},l_{b}} \delta_{k_{c},l_{c}} \delta_{k_{d},l_{d}} \delta_{k_{e},l_{e}} \delta_{k_{f},l_{f}} \delta_{k_{g},l_{g}} \delta_{k_{h},l_{h}} \right) \\
& \quad \times \left[ \overline{\phi}_d(l_1,t) \overline{\phi}_d(l_2,t) \overline{\phi}_u(l_3,t) \overline{\phi}_u(l_4,t) \overline{\phi}_d(l_5,t) \overline{\phi}_u(l_6,t) \overline{\phi}_d(l_7,t) \overline{\phi}_d(l_8,t) \right]
\end{align*}
$$

where $\lambda = \sqrt{\frac{\delta q}{2m}}$. Thus the first order change in the QCF for two Cooper pairs depends linearly on $\delta t$. The derivation is given in Sect. 11.3. Note the signs for the $2\lambda^2$ terms are opposite and that half of the kinetic energy terms $\hbar^2 k_i^2$ have opposite signs. Similar initial conditions to the single Cooper pair time evolution apply.
where \( \eta = \sqrt{\frac{k}{m}} \). Thus the first order change in the QCF for two Cooper pairs depends linearly on \( \delta \beta \). The derivation is similar to the case of time evolution. Note the signs for the \( 2\eta^2 \) terms are now the same and that all the kinetic energy terms \( \frac{k^2 k^2}{2m} \) also have the same sign. Similar initial conditions to the single Cooper pair temperature evolution also apply.
7 Solution of Evolution Equations for QCF

The evolution equations for the QCF can be written in terms of the definitions in Eqs. (10) and (19) for the one Cooper pair QCF \(X(dk_1, uk_2, u^+ k_3, d^+ k_4)\) and the two Cooper pair QCF \(X(dk_1, dk_2, uk_3, uk_4, u^+ k_5, u^+ k_6, d^+ k_7, d^+ k_8)\) in the forms

\[
\frac{\partial}{\partial \xi} X(dk_1, uk_2, u^+ k_3, d^+ k_4) = \sum_{l_1 l_2 l_3 l_4} M(k_1, k_2, k_3, k_4; l_1, l_2, l_3, l_4) X(dl_1, ul_2, u^+ l_3, d^+ l_4) \tag{64}
\]

\[
\frac{\partial}{\partial \xi} X(dk_1, dk_2, uk_3, uk_4, u^+ k_5, u^+ k_6, d^+ k_7, d^+ k_8) = \sum_{l_1 l_2 l_3 l_4} M(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8; l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8) \times X(dl_1, dl_2, ul_3, ul_4, u^+ l_5, u^+ l_6, d^+ l_7, d^+ l_8) \tag{65}
\]

where \(\xi\) is either \(t\) or \(\beta\) for time or temperature evolution, and the matrices \(M\) can be read off from Eq. (53), (57) or (52). The elements of \(M\) are designated \(M_{\{k\};\{l\}}\), where for one Cooper pair \(\{k\} \equiv \{k_1, k_2, k_3, k_4\}; \{l\} \equiv \{l_1, l_2, l_3, l_4\}\), whilst for two Cooper pairs \(\{k\} \equiv \{k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8\}; \{l\} \equiv \{l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8\}\). Clearly the sets \(\{k\}, \{l\}\) respectively define the rows, columns of \(M\).

For the cases of time evolution it can be seen by inspection that \(M = iH\), where \(H\) is a real, symmetric matrix - \(H_{\{k\};\{l\}} = H_{\{l\};\{k\}}\). Note that \(\lambda^2 = \frac{\lambda_1}{4\pi^4}\). The time evolution equations are of the form

\[
\frac{\partial}{\partial t} X(t) = iH X(t) \tag{66}
\]

and can be solved using the normalised, orthogonal column eigenvectors \(x_i\) of \(H\). Thus with real eigenvalues \(\lambda_i\)

\[
H x_i = \lambda_i x_i \quad x_i^T x_j = \delta_{ij} \tag{67}
\]

the solution is of the form

\[
X(t) = \sum_i c_i(0) \exp(i \lambda_i t) x_i \tag{68}
\]

\[
c_i(0) = x_i^T X(0) \tag{69}
\]

where the initial condition is specified by the coefficients \(c_i(0)\) which are determined from the initial column vector \(X(0)\).

For the cases of temperature evolution it can be seen by inspection that \(M = J\), where \(J\) is a real, symmetric matrix - \(J_{\{k\};\{l\}} = J_{\{l\};\{k\}}\). Note that \(\eta^2 = \frac{\pi^2}{4\pi^4}\). The temperature evolution equations are of the form

\[
\frac{\partial}{\partial \beta} X(\beta) = J X(\beta) \tag{70}
\]

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and can be solved using the normalised, orthogonal column eigenvectors $y_i$ of $J$. Thus with real eigenvalues $\theta_i$

$$ J y_i = \theta_i y_i \quad y_i^T y_j = \delta_{ij} $$

the solution is of the form

$$ X(\beta) = \sum_i c_i(0) \exp(\theta_i \beta) y_i $$

(72)

$$ c_i(0) = y_i^T X(0) $$

(73)

where the initial condition is specified by the coefficients $c_i(0)$ which are determined from the initial column vector $X(0)$. This method of solution requires first calculating the matrix $M$ and the initial column vector $X(0)$, then determining the eigenvectors and eigenvalues of $H$ or $J$. The alternative is a step-by-step development of the QCF using Eq.(53), (57) or (62).

7.1 Dimensionless Variables

It is first convenient to introduce dimensionless variables. The wave vector $k_F$, energy $E_F$ and temperature $T_F$ associated with the Fermi surface for non-interacting fermions are all related to the number density $N/V$ as follows:

$$ k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3} $$

(74)

$$ E_F = \frac{h^2 k_F^2}{2m} = k_B T_F $$

(75)

This leads to scales $t_F$ for time, $\beta_F$ for the inverse temperature $1/(k_B T)$ and $k_F$ for the wave vector given by

$$ t_F = \frac{2\pi h}{E_F} \quad t = T t_F $$

(76)

$$ \beta_F = \frac{1}{k_B T_F} = \frac{1}{E_F} \quad \beta = B \beta_F $$

(77)

$$ k = K k_F $$

(78)

so $T$ is the dimensionless time, $B$ is dimensionless inverse temperature and $K$ is the dimensionless wave vector.

As the coupling constant $g$ is given by

$$ g = \frac{4\pi a_s \hbar^2}{m} $$

(79)

we can write the quantities $2\lambda^2$ and $2\eta^2$ that occur in the time evolution and temperature evolution equations as

$$ 2\lambda^2 = \frac{(-ig)}{\hbar V} = -i \frac{16}{3} \frac{1}{N t_F} (a_s k_F) $$

(80)

$$ 2\eta^2 = \frac{(-g)}{2V} = \frac{4}{3\pi N \beta_F} (a_s k_F) $$

(81)
Hence we see that using \( \frac{2\hbar^2}{m} = K^2 \frac{2\pi}{t_F} = K^2 \frac{1}{\beta F} \) the time and temperature evolution equations Eqs. (53), (57), (62) and (63) can be written in terms of dimensionless time \( T \), dimensionless inverse temperature \( B \) and dimensionless wave vectors \( K, L \). The dimensionless quantities \( N \) and \( (a_s k_F) \) play the role of parameters. The unitary case is when \( 1/(a_s k_F) = 0 \), corresponding to \( a_s \to \infty \).

### 7.2 Numerical Issues

In the simplest situation where the BEC/BCS crossover is for a 1D system, we can estimate the size of the matrices \( J \) or \( H \) in Section 7 for both the one and two Cooper pair QCF calculations in terms of the number of modes \( n \) that need to be taken into account. As mentioned previously, not all occupied modes are expected to be important - as it is likely that only \( k \) values near the Fermi surface will be affected. Thus \( n \ll N \), where \( N \) is the total number of fermions. For the one Cooper pair QCF the quantities \( k_1, k_2, k_3, k_4, l_1, l_2, l_3, l_4 \) each take on \( n \) values, so the matrices \( J \) or \( H \) will involve \( n^8 \) elements. For the two Cooper pair QCF the quantities \( k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8 \) each take on \( n \) values, so the matrices \( J \) or \( H \) will involve \( n^{16} \) elements. Fortunately, due to the presence of many Kronecker deltas in the formulae for the matrix elements, most elements are zero so both \( J \) or \( H \) are very sparse matrices - which means that after the non-zero elements have been identified once, the calculation can start by putting all elements as zero and only evaluate those that are non-zero. Actually, because of selection rules such as \( k_1 = l_1 \), \( k_2 = l_2 \), \( k_3 + k_4 = l_3 + l_4 \) that apply in the one Cooper pair case, the number of non-zero elements are of order \( n^5 \) rather than \( n^8 \). Similarly, in the two Cooper pair case the number of non-zero elements is of order \( n^9 \) rather than \( n^{16} \). Nevertheless, these can still be large numbers even though each sparse matrix can have its rows and columns reordered via a unitary transformation to place all the non-zero elements in the top-left corner before determining the eigenvalues and orthogonal eigenvectors (most of the eigenvalues will then be zero corresponding to the reordered matrix having most rows and columns zero). These eigenvectors could then be transformed back via the inverse transformation to give the eigenvectors of the original sparse matrix. However, for \( n = 10 \) the one Cooper pair case would involve calculating of order \( 10^5 \) non-zero elements, though for the two Cooper pair case of order \( 10^9 \) elements would be required. The evolution over time or temperature is then determined from the eigenvalues and eigenvectors at just the final time or temperature. Such calculations should be feasible on supercomputers.

By comparison, if the Gaussian operator fermion phase space theory were applied, the one Cooper pair case would be treatable via stochastic equations involving the \( 2n \times 2n \) covariance matrix - involving \( 4n^2 \) elements. As these elements each involves stochastic Wiener increments, the covariance matrix would be recalculated at each time or temperature step, so if there are \( \nu \) steps, the total number of covariance matrix elements calculated would be \( 4\nu n^2 \). However, as the evolution has to be carried out for a number \( \xi \) stochastic trajectories,
the total number of covariance matrix elements calculated would be $4\xi\nu n^2$. For $n = 10$, $\nu = 100$ and $\xi = 100$ this is of order $10^6$, which is much the same order as for GPST. So there is no obvious reason why the Gaussian operator fermion phase space theory is faster than GPST.
8 Summary

In the present paper it has been shown how Grassmann Phase Space Theory (GSPT) can be applied to the BEC/BCS crossover in cold fermionic atomic gases and used to determine the evolution (over either time or temperature) of the Quantum Correlation Functions (QCF) that specify: (a) the positions of the spin up and spin down fermionic atoms in a single Cooper pair and (b) the positions of the two spin up and two spin down fermionic atoms in two Cooper pairs. The first of these QCF is relevant to describing the change in size of a Cooper pair, as the fermion-fermion coupling constant is changed through the crossover from a small Cooper pair on the BEC side to a large Cooper pair on the BCS side. The second of these QCF is important for describing the correlations between the positions of the fermionic atoms in two Cooper pairs, which is expected to be small at the BEC or BCS sides of the crossover, but is expected to be significant in the strong interaction unitary regime, where the size of a Cooper pair is comparable to the separation between Cooper pairs. In GSPT the QCF are given as the stochastic average of products of Grassman stochastic position fields, which are related via Fourier transforms to the stochastic average of products of Grassman stochastic momentum fields. Using the no-correlation theorem, GSPT shows that the stochastic average of the products of Grassman stochastic momentum fields at a later time (or lower temperature) is related linearly to the stochastic average of the products of Grassman stochastic momentum fields at an earlier time (or higher temperature), and furthermore that the matrix elements involved in the linear relations are all c-numbers. Expressions for these matrix elements corresponding to a small time increment (or a small temperature change) have been obtained analytically, providing the formulae needed for numerical studies of the evolution that are planned for a future publication. The initial conditions envisaged include those for a non-interacting fermionic gas at zero temperature to study the time evolution of the creation of a Cooper pair when the interaction is switched on via Feshbach resonance methods. Other initial conditions described include a high temperature gas (where the effect of the interactions can be ignored in the initial state), and where the evolution will be studied as the temperature is lowered until either a BEC state with small Cooper pairs or a BCS state with large Cooper pairs forms, depending on the fermion-fermion coupling constant. The behaviour for the case where the coupling constant is very large (the unitary regime) would be of particular interest. Full derivations of the expressions have been presented in the Appendices.

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10 Appendix A - Details for Grassmann Field Theory

In this Appendix some details for the Grassmann field theory Section are provided.

10.1 Gaussian-Markoff Noise Terms

The basic stochastic average properties of the Gaussian-Markoff random noise terms are

\[
\begin{align*}
\Gamma_a(t_1) &= 0 \\
\Gamma_a(t_1)\Gamma_b(t_2) &= \delta_{ab}\delta(t_1 - t_2) \\
\Gamma_a(t_1)\Gamma_b(t_2)\Gamma_c(t_3) &= 0 \\
\Gamma_a(t_1)\Gamma_b(t_2)\Gamma_c(t_3)\Gamma_d(t_4) &= \Gamma_a(t_1)\Gamma_b(t_2)\Gamma_c(t_3)\Gamma_d(t_4) + \Gamma_a(t_1)\Gamma_c(t_3)\Gamma_b(t_2)\Gamma_d(t_4) + \Gamma_a(t_1)\Gamma_d(t_4)\Gamma_b(t_2)\Gamma_c(t_3)
\end{align*}
\]

showing that the stochastic averages of a single \( \Gamma \) is zero and the stochastic average of the product of two \( \Gamma \)'s is zero if they are different and delta function correlated in the time difference if they are the same. In addition, the stochastic averages of products of odd numbers of \( \Gamma \) are zero and stochastic averages of products of even numbers of \( \Gamma \) are the sums of products of stochastic averages of pairs of \( \Gamma \).

10.2 Determining the B Matrix - Time Evolution Case

Using the notation in Section 3.6 and after substituting for \( \delta(r - s) \), the non-zero elements of the diffusion matrix in Eq. (22) are given by

\[
\begin{align*}
D_{u_1,d_1}(\psi(s),\psi(r)) &= -\frac{ig}{\hbar V} \sum_q \theta_{u_1}^q(s) \theta_{d_1}^q(r) \\
D_{d_1,u_1}(\psi(s),\psi(r)) &= -\frac{ig}{\hbar V} \sum_q \theta_{d_1}^q(s) \theta_{u_1}^q(r) \\
D_{u_2,d_2}(\psi(s),\psi(r)) &= +\frac{ig}{\hbar V} \sum_q \theta_{u_2}^q(s) \theta_{d_2}^q(r) \\
D_{d_2,u_2}(\psi(s),\psi(r)) &= +\frac{ig}{\hbar V} \sum_q \theta_{d_2}^q(s) \theta_{u_2}^q(r)
\end{align*}
\]

where

\[
\begin{align*}
\theta_{d_1}^q(s) &= \psi_{d_1}(s) \exp(+iqs) & \theta_{u_1}^q(s) &= \psi_{u_1}(s) \exp(-iqs) \\
\theta_{d_2}^q(s) &= \psi_{d_2}(s) \exp(-iqs) & \theta_{u_2}^q(s) &= \psi_{u_2}(s) \exp(+iqs)
\end{align*}
\]
We write the diffusion matrix in the form

\[
D_{\alpha \beta} [\psi(s, t), s; \psi(r, t), r] = \sum_q \sum_{\gamma \delta} M_{\alpha \gamma \beta \delta} \theta_{\gamma}^q(s) \theta_{\delta}^q(r)
\]  

(85)

where the symmetric matrix \(M\) is

\[
[M] = -\frac{ig}{\hbar V} \begin{pmatrix}
(\alpha A, \gamma C) \downarrow \downarrow (\beta B, \delta D) & u_1, d_1 & d_1, u_1 & u_2, d_2 & d_2, u_2 \\
\downarrow & 0 & 1 & 0 & 0 \\
\downarrow & 0 & 0 & 0 & -1 \\
\end{pmatrix}
\]  

(86)

Using Takagi \[40\] factorisation \(M = KK^T\) the matrix \(M\) is given by

\[
M_{\alpha \gamma \beta \delta} = \sum_{\xi} K_{\alpha \gamma \gamma \xi} K_{\beta \delta \xi \delta}
\]  

(87)

where

\[
[K] = \sqrt{-\frac{ig}{\hbar V}} \begin{pmatrix}
(\alpha A, \gamma C) \downarrow (\xi) & u_1, d_1 & d_1, u_1 & u_2, d_2 & d_2, u_2 \\
\downarrow & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
\downarrow & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
\downarrow & 0 & 0 & -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \\
\end{pmatrix}
\]  

(88)

Thus we have

\[
D_{\alpha \beta} [\psi(s, t), s; \psi(r, t), r] = \sum_q \sum_{\xi} \left( \sum_{\gamma} K_{\alpha \gamma \gamma \xi} \theta_{\gamma}^q(s) \right) \left( \sum_{\delta} K_{\beta \delta \xi \delta} \theta_{\delta}^q(r) \right)
\]  

(89)

where

\[
B_{\alpha \xi} [\psi(s, t), s] B_{\beta \xi} [\psi(r, t), r] = \sum_{\gamma} K_{\alpha \gamma \gamma \xi} \theta_{\gamma}^q(s)
\]  

(90)

Hence we have written the diffusion matrix in the form \(D = BB^T\) as required to determine the Ito SFE.

On substituting for \(K_{\alpha \gamma \gamma \xi}\) and \(\theta_{\gamma}^q(s)\) from Eqs. \[88\] and \[84\] we obtain the Ito SFE in the form given in Eq. \[86\].
10.3 Determining the B Matrix - Temperature Evolution Case

The temperature evolution case is treated similarly to the time evolution case in the previous Section.

However in this case the diffusion matrix is

\[
D_{u1,d1}(\psi(s), \psi(r)) = -\frac{g}{2V} \sum_q \theta_{u1}^q(s) \theta_{d1}^q(r)
\]

\[
D_{d1,u1}(\psi(s), \psi(r)) = -\frac{g}{2V} \sum_q \theta_{d1}^q(s) \theta_{u1}^q(r)
\]

\[
D_{u2,d2}(\psi(s), \psi(r)) = -\frac{g}{2V} \sum_q \theta_{u2}^q(s) \theta_{d2}^q(r)
\]

\[
D_{d2,u2}(\psi(s), \psi(r)) = -\frac{g}{2V} \sum_q \theta_{d2}^q(s) \theta_{u2}^q(r)
\]

(91)

and the \( M \) matrix is

\[
[M] = -\frac{g}{2V} \begin{bmatrix}
\alpha A, \gamma C & \downarrow (\beta B, \delta D) & \rightarrow & u1, d1 & d1, u1 & u2, d2 & d2, u2 \\
1, d1 & 0 & 0 & 0 & 0 & 0 & 0 \\
d1, u1 & 1 & 0 & 0 & 0 & 0 & 0 \\
u2, d2 & 0 & 0 & 0 & 1 & 0 & 0 \\
d2, u2 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

(92)

which results in the \( K \) matrix

\[
[K] = \sqrt{-\frac{g}{2V}} \begin{bmatrix}
\alpha A, \gamma C & \downarrow (\xi) & \rightarrow & u1, d1 & d1, u1 & u2, d2 & d2, u2 \\
u1, d1 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\
d1, u1 & \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
u2, d2 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 \\
d2, u2 & 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0
\end{bmatrix}
\]

(93)

On substituting for \( K_{\alpha A, \gamma C \xi} \) and \( \theta_{\gamma C}^q(s) \) from Eqs. (93) and (84) we obtain the Ito SFE in the form given in Eq. (33).

10.4 Stochastic Momentum Fields

The inverse transformation between the stochastic position and momentum fields is

\[
\tilde{\phi}_\alpha(k) = \frac{1}{\sqrt{V}} \int ds \ exp(-ik \cdot s) \tilde{\psi}_\alpha(s)
\]

\[
\tilde{\phi}_\alpha^+(k) = \frac{1}{\sqrt{V}} \int ds \ exp(+ik \cdot s) \tilde{\psi}_\alpha^+(s)
\]

(94)

where the \( t \) or \( \beta \) dependence is implicit.
11 Appendix B - Stochastic and QCF Equations

In this Appendix the details are set out of applying stochastic averaging to give first order changes in the QCF describing the size of one Cooper pair and the position correlations for two Cooper pairs.

11.1 One Cooper Pair - Time Evolution

For the case of time evolution the QCF at time $t + \delta t$ is related to that at time $t$ via

$$
\bar{\phi}_d(k_1, t + \delta t) \bar{\phi}_u (k_2, t + \delta t) \bar{\phi}_u^+(k_3, t + \delta t) \bar{\phi}_d^+(k_4, t + \delta t) = \sum_{l_1, q_1} (F_{d,d}(k_1, l_1; q_1, \delta t) \bar{\phi}_d(l_1, t) + F_{d,u}(k_1, l_1; q_1, \delta t) \bar{\phi}_u(l_1, t))
$$

$$
\times \sum_{l_2, q_2} (F_{u,d}(k_2, l_2; q_2, \delta t) \bar{\phi}_d(l_2, t) + F_{u,u}(k_2, l_2; q_2, \delta t) \bar{\phi}_u(l_2, t))
$$

$$
\times \sum_{l_3, q_3} (F_{d,+}(k_3, l_3; q_3, \delta t) \bar{\phi}_d^+(l_3, t) + F_{d,+}(k_3, l_3; q_3, \delta t) \bar{\phi}_d^+(l_3, t))
$$

$$
\times \sum_{l_4, q_4} (F_{d,+}(k_4, l_4; q_4, \delta t) \bar{\phi}_u^+(l_4, t) + F_{d,+}(k_4, l_4; q_4, \delta t) \bar{\phi}_u^+(l_4, t))
$$

after substituting from Eq. (38). Apart from the sums over $l_1, q_1, ..., l_4, q_4$ there are 16 terms for which the stochastic averages have to be evaluated. If the two terms in each of the first 2 factors are listed as 1, 2 and the two terms in each of the second 2 factors are listed as $a, b$ then the 16 terms can be listed as $11aa, 12aa, 21aa, 22aa, 11ab, 12ab, 21ab, 22ab, ..., 22bb$.

In Ito stochastic equations we then use the factorisation result that depends on the $F_\ast, F_\ast^+$ involving times later than $t$ (see Eq. (9.43) in Ref. [26])

$$
\begin{align*}
F_{\alpha, \beta}(k_1, l_1; q_1, \delta t) F_{\gamma, \delta}(k_2, l_2; q_2, \delta t) F_{\epsilon, \eta}^+ (k_3, l_3; q_3, \delta t) F_{\theta, \kappa}^+(k_4, l_4; q_4, \delta t) & = \sum_{l_1, q_1} (F_{d,d}(k_1, l_1; q_1, \delta t) \bar{\phi}_d(l_1, t)) \sum_{l_2, q_2} (F_{u,d}(k_2, l_2; q_2, \delta t) \bar{\phi}_d(l_2, t))
\times \sum_{l_3, q_3} (F_{d,+}(k_3, l_3; q_3, \delta t) \bar{\phi}_d^+(l_3, t)) \sum_{l_4, q_4} (F_{d,+}(k_4, l_4; q_4, \delta t) \bar{\phi}_d^+(l_4, t))
\end{align*}
$$

We wish to evaluate the various $F_{\alpha, \beta}(k_1, l_1; q_1, \delta t) F_{\gamma, \delta}(k_2, l_2; q_2, \delta t) F_{\epsilon, \eta}^+ (k_3, l_3; q_3, \delta t) F_{\theta, \kappa}^+(k_4, l_4; q_4, \delta t)$ correct to order $\delta t$. We note that the $F_{d,d}, F_{d,u}, F_{d,d}^+, F_{d,u}^+$ factors are non-stochastic and linear in $\delta t$. Also the $F_{d,u}, F_{d,u}, F_{d,u}^+, F_{d,u}^+$ factors are stochastic and linear in the stochastic Wiener increments $\delta \omega_{a,d}, \delta \omega_{a,u}, \delta \omega_{a,d,u}, \delta \omega_{a,d,u}$. From Eq. (32) the stochastic average of a factor involving an odd number 1, 3 of the $\delta \omega_{a,d}, \delta \omega_{a,u}, \delta \omega_{a,d,+, u}^+, \delta \omega_{a,d,+, u}^+$ with any non-stochastic factor is zero, if there are 2 of these increments the stochastic average is proportional to $\delta t$ and if there
are 4 of these increments the stochastic average is proportional to $\delta t^2$ (apart from cases where there is zero Kroneck delta $\delta_{ab}$ involved). For the case of two stochastic Wiener increments we note from Eq. (92) that the non-zero stochastic averages of the products are

$$
\frac{\delta \omega_{a,d}^2}{\delta \omega_{u,d}^2} = \frac{\delta \omega_{d,a}^2}{\delta \omega_{u,a}^2} = \delta t
$$

(97)

With these considerations in mind, the non-zero cases are as follows:

$$
12ba : \quad F_{d,a}(k_1, l_1; q_1, \delta t) F_{u,u}(k_2, l_2; q_2, \delta t) F_{u,u}^+(k_3, l_3; q_3, \delta t) F_{d,a}^+(k_4, l_4; q_4, \delta t)
$$

$$
= \delta_{q_1,0} \delta_{k_1, l_1} \left( 1 + \frac{i}{\hbar} \left( \frac{h^2 k_2^2}{2m} \right) \delta t \right) \delta_{q_2,0} \delta_{k_2, l_2} \left( 1 + \frac{i}{\hbar} \left( \frac{h^2 k_3^2}{2m} \right) \delta t \right)
$$

$$
\times \delta_{q_3,0} \delta_{k_3, l_3} \left( 1 - \frac{i}{\hbar} \left( \frac{h^2 k_4^2}{2m} \right) \delta t \right) \delta_{q_4,0} \delta_{k_4, l_4} \left( 1 - \frac{i}{\hbar} \left( \frac{h^2 k_4^2}{2m} \right) \delta t \right)
$$

$$
\approx \delta_{q_1,0} \delta_{k_1, l_1} \delta_{q_2,0} \delta_{k_2, l_2} \delta_{q_3,0} \delta_{k_3, l_3} \delta_{q_4,0} \delta_{k_4, l_4}
$$

$$
\times \left( 1 + \frac{i}{\hbar} \left( \frac{h^2 k_2^2}{2m} + \frac{h^2 k_3^2}{2m} - \frac{h^2 k_4^2}{2m} \right) \delta t \right)
$$

(98)

where only terms linear in $\delta t$ have been retained.

$$
12ab : \quad F_{d,a}(k_1, l_1; q_1, \delta t) F_{u,u}(k_2, l_2; q_2, \delta t) F_{u,u}^+(k_3, l_3; q_3, \delta t) F_{d,a}^+(k_4, l_4; q_4, \delta t)
$$

$$
= \delta_{q_1,0} \delta_{k_1, l_1} \left( 1 + \frac{i}{\hbar} \left( \frac{h^2 k_2^2}{2m} \right) \delta t \right) \delta_{q_2,0} \delta_{k_2, l_2} \left( 1 + \frac{i}{\hbar} \left( \frac{h^2 k_3^2}{2m} \right) \delta t \right)
$$

$$
\times \lambda \delta_{(k_3+q_3), l_3} \left\{ \delta_{\omega_{u+,d+}^q, d+} + i \delta_{\omega_{d+,u+}^q, d+} \right\} \lambda \delta_{(k_4-q_4), l_4} \left\{ -\delta_{\omega_{u+,d+}^q, d+} + i \delta_{\omega_{d+,u+}^q, d+} \right\}
$$

$$
= \delta_{q_1,0} \delta_{k_1, l_1} \delta_{q_2,0} \delta_{k_2, l_2} \lambda \delta_{(k_3+q_3), l_3} \lambda \delta_{(k_4-q_4), l_4}
$$

$$
\times \left( 1 + \frac{i}{\hbar} \left( \frac{h^2 k_2^2}{2m} \right) \delta t \right) \left( 1 + \frac{i}{\hbar} \left( \frac{h^2 k_3^2}{2m} \right) \delta t \right)
$$

$$
\times \left\{ -\delta_{\omega_{u+,d+}^q, d+} - \delta_{\omega_{d+,u+}^q, d+} \delta t \right\}
$$

$$
\approx -2 \lambda^2 \left\{ \delta_{q_1,0} \delta_{k_1, l_1} \delta_{q_2,0} \delta_{k_2, l_2} \delta_{(k_3+q_3), l_3} \delta_{(k_4-q_4), l_4} \delta_{q_3, q_4} \right\} \delta t
$$

(99)

where the Wiener increment properties in Eq. (92) has been used and only terms
linear in $\delta t$ have been retained.

\[
21ba : \quad \frac{F_{d,u}(k_1; l_1; q_1, \delta t) F_{u,d}(k_2, l_2; q_2, \delta t) F_{u,u}^+(k_3, l_3; q_3, \delta t) F_{d,u}^+(k_4, l_4; q_4, \delta t)}{
= (\pm \lambda \delta_{(k_1+q_1), l_1} \left\{ \delta_\omega_{u,d} - i \delta_\omega_{d,u} \right\}) (\pm \lambda \delta_{(k_2-q_2), l_2} \left\{ \delta_\omega_{u,d} + i \delta_\omega_{d,u} \right\})
\times \delta_{q_3,0} \delta_{k_3, l_3} \left( 1 - i \left\{ \frac{\hbar^2 k_3^2}{2m} \right\} \delta t \right) \delta_{q_4,0} \delta_{k_4, l_4} \left( 1 - i \left\{ \frac{\hbar^2 k_4^2}{2m} \right\} \delta t \right)
= \lambda \delta_{(k_1+q_1), l_1} \lambda \delta_{(k_2-q_2), l_2} \delta_{q_3,0} \delta_{k_3, l_3} \delta_{q_4,0} \delta_{k_4, l_4}
\times \left( 1 - i \left\{ \frac{\hbar^2 k_3^2}{2m} \right\} \delta t \right) \left( 1 - i \left\{ \frac{\hbar^2 k_4^2}{2m} \right\} \delta t \right)
\times \{ \delta_{q_1, q_2} \delta t + \delta_{q_1, q_2} \delta t \}
\approx 2\lambda^2 \left\{ \delta_{(k_1+q_1), l_1} \delta_{(k_2-q_2), l_2} \delta_{q_3,0} \delta_{k_3, l_3} \delta_{q_4,0} \delta_{k_4, l_4} \delta_{q_1, q_2} \right\} \delta t
\] (100)

where only terms linear in $\delta t$ have been retained.

Substituting the results in Eqs. (98), (99) and (100) into Eq. (95) and performing the sums over $q_1, q_2, q_3, q_4$ gives

\[
\phi_d(k_1, t + \delta t) \phi_u(k_2, t + \delta t) \phi_u^+(k_3, t + \delta t) \phi_d^+(k_4, t + \delta t)
= \sum_{l_1 l_2 l_3 l_4} \delta_{k_1, l_1} \delta_{k_2, l_2} \delta_{k_3, l_3} \delta_{k_4, l_4} \left( 1 + i \left\{ \frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m} - \frac{\hbar^2 k_3^2}{2m} - \frac{\hbar^2 k_4^2}{2m} \right\} \delta t \right)
\times \phi_d(l_1, t) \phi_u(l_2, t) \phi_u^+(l_3, t) \phi_d^+(l_4, t)
-2\lambda^2 \sum_{l_1 l_2 l_3 l_4} \left\{ \delta_{k_1+l_1, l_1} \delta_{k_2+l_2, l_2} \delta_{k_3+l_3, l_3} \delta_{k_4+l_4, l_4} \right\} \delta t \times \phi_d(l_1, t) \phi_u(l_2, t) \phi_d^+(l_3, t) \phi_u^+(l_4, t)
+2\lambda^2 \sum_{l_1 l_2 l_3 l_4} \left\{ \delta_{k_1+k_2, l_1+l_2} \delta_{k_3+k_4, l_3+l_4} \right\} \delta t \times \phi_u(l_1, t) \phi_d(l_2, t) \phi_u^+(l_3, t) \phi_d^+(l_4, t)
= \sum_{l_1 l_2 l_3 l_4} \delta_{k_1, l_1} \delta_{k_2, l_2} \delta_{k_3, l_3} \delta_{k_4, l_4} \left( 1 + i \left\{ \frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m} - \frac{\hbar^2 k_3^2}{2m} - \frac{\hbar^2 k_4^2}{2m} \right\} \delta t \right)
\times \phi_d(l_1, t) \phi_u(l_2, t) \phi_u^+(l_3, t) \phi_d^+(l_4, t)
+2\lambda^2 \sum_{l_1 l_2 l_3 l_4} \left\{ \delta_{k_1+l_1, l_1} \delta_{k_2+l_2, l_2} \delta_{k_3+k_4, l_3+l_4} \right\} \delta t \times \phi_d(l_1, t) \phi_u(l_2, t) \phi_u^+(l_3, t) \phi_d^+(l_4, t)
-2\lambda^2 \sum_{l_1 l_2 l_3 l_4} \left\{ \delta_{k_1+k_2, l_1+l_2} \delta_{k_3+k_4, l_3+l_4} \right\} \delta t \times \phi_d(l_1, t) \phi_u(l_2, t) \phi_u^+(l_3, t) \phi_d^+(l_4, t)
\] (101)

where we have used $\sum_{q_1 q_2} \delta_{(k_3+q_3), l_3} \delta_{(k_4-q_4), l_4} \delta_{q_3, q_4} = \delta_{(k_3+k_4), (l_3+l_4)}$ and $\sum_{q_1} \delta_{q_1, 0} = 1$. Also, in the second last line we have interchanged $l_3$ and $l_4$ and used the anti-commuting feature of the Grassmann stochastic momentum fields $\phi_d^+(l_3, t)$ and $\phi_u^+(l_4, t)$ to place these in the opposite order in the stochastic average.
\[ \phi_d(l_1, t)\phi_u(l_2, t)\phi_d^+(l_3, t)\phi_u^+(l_4, t), \text{ with similar steps for the last line.} \]

### 11.2 One Cooper Pair - Temperature Evolution

For the case of temperature evolution the QCF at time \(\beta + \delta \beta\) is related to that at time \(t\) via

\[
\left(\begin{array}{c}
\phi_d^+(k_1, \beta + \delta \beta)\phi_u(k_2, \beta + \delta \beta)\phi_u^+(k_3, \beta + \delta \beta)\phi_d^+(k_4, \beta + \delta \beta) \\sum_{l_1, q_1} (F_{d,d}(k_1, l_1; q_1, \beta)\phi_d(l_1, \beta) + F_{d,u}(k_1, l_1; q_1, \beta)\phi_u(l_1, \beta)) \times \sum_{l_2, q_2} (F_{d,d}(k_2, l_2; q_2, \beta)\phi_d(l_2, \beta) + F_{d,u}(k_2, l_2; q_2, \beta)\phi_u(l_2, \beta)) \times \sum_{l_3, q_3} (F_{u,u}(k_3, l_3; q_3, \beta)\phi_u(l_3, \beta) + F_{d,u}(k_3, l_3; q_3, \beta)\phi_u(l_3, \beta)) \times \sum_{l_4, q_4} (F_{d,u}(k_4, l_4; q_4, \beta)\phi_d(l_4, \beta) + F_{d,u}(k_4, l_4; q_4, \beta)\phi_u(l_4, \beta))\end{array}\right) \]

\[ \text{StochAver} (102) \]

after substituting from Eq. (101). Apart from the sums over \(l_1, q_1, ..., l_4, q_4\) there are 16 terms for which the stochastic averages have to be evaluated. If the two terms in each of the first 2 factors are listed as 1 and the two terms in each of the second 2 factors are listed as \(a, b\) then the 16 terms can be listed as \(11aa, 12aa, 21aa, 22aa, 11ab, 12ab, 21ab, 22ab, ..., 22bb\).

In Ito stochastic equations we then use the factorisation result that depends on the \(F, F^+\) involving times later than \(t\) (see Eq. (9.43) in Ref. [26])

\[
\left(\begin{array}{c}
F_{\alpha, \beta}(k_a, l_a; q_1, \delta \beta)\phi_\beta(l_1, \beta) F_{\gamma, \delta}(k_b, l_b; q_2, \delta \beta)\phi_\delta(l_2, \beta) \\
\times F_{\epsilon, \eta}(k_c, l_c; q_3, \delta \beta)\phi_\eta(l_3, \beta) F_{\theta, \kappa}(k_d, l_d; q_4, \delta \beta)\phi_\kappa(l_4, \beta) \\
\end{array}\right) \text{StochAver} \]

\[
= F_{\alpha, \beta}(k_a, l_a; q_1, \delta \beta) F_{\gamma, \delta}(k_b, l_b; q_2, \delta \beta) F_{\epsilon, \eta}(k_c, l_c; q_3, \delta \beta) F_{\theta, \kappa}(k_d, l_d; q_4, \delta \beta) \times \phi_\beta(l_1, \beta) \phi_\delta(l_2, \beta) \phi_\eta(l_3, \beta) \phi_\kappa(l_4, \beta) \]

\[ \text{StochAver} (103) \]

We wish to evaluate the various

\[ F_{\alpha, \beta}(k_a, l_a; q_1, \delta \beta) F_{\gamma, \delta}(k_b, l_b; q_2, \delta \beta) F_{\epsilon, \eta}(k_c, l_c; q_3, \delta \beta) F_{\theta, \kappa}(k_d, l_d; q_4, \delta \beta) \]

correct to order \(\delta \beta\). We note that the \(F_{d,d}, F_{u,u}, F_{d,u}, F_{d,d}^+, F_{u,u}^+, F_{d,u}^+\) factors are non-stochastic and linear in \(\delta \beta\). Also the \(F_{d,u}, F_{u,d}, F_{d,u}^+, F_{u,d}^+\) factors are stochastic and linear in the stochastic Wiener increments \(\delta \omega_{u,d}^I, \delta \omega_{d,u}^I, \delta \omega_{u,d+}^I, \delta \omega_{d,u+}^I\). From Eq. (112) the stochastic average of a factor involving an odd number 1, 3 of the \(\delta \omega_{u,d}^I, \delta \omega_{d,u}^I, \delta \omega_{u,d+}^I, \delta \omega_{d,u+}^I\) with any non-stochastic factor is zero, if there are 2 of these increments the stochastic average is proportional to \(\delta \beta\) and if there are 4 of these increments the stochastic average is proportional to \(\delta \beta^2\) (apart from cases where there is zero Kroneck delta \(\delta_{ab}\) involved). For the case of two stochastic Wiener increments we note from Eq. (112) that the non-zero
stochastic averages of the products are 

$$ \frac{\delta \omega_{u,d}^{q} \delta \omega_{u,d}^{d}}{\delta \omega_{u,d}^{q} \delta \omega_{u,d}^{d} \delta \omega_{u,d}^{d}} = \frac{\delta \omega_{u,d}^{q} \delta \omega_{u,d}^{d}}{\delta \omega_{u,d}^{q} \delta \omega_{u,d}^{d} \delta \omega_{u,d}^{d}} = \delta \beta \quad (104) $$

With these considerations in mind, the non-zero cases are as follows:

$$ 12ba : \quad F_{d,u}(k_1, l_1; q_1, \delta \beta) F_{u,u}(k_2, l_2; q_2, \delta \beta) F_{u,u}(k_3, l_3; q_3, \delta \beta) F_{d,u}(k_4, l_4; q_4, \delta \beta) = \delta_{q_1,0} \delta_{k_1,l_1} \left( 1 + \frac{1}{2} \left( \frac{h^2 k_1^2}{2m} \right) \delta \beta \right) \delta_{q_2,0} \delta_{k_2,l_2} \left( 1 + \frac{1}{2} \left( \frac{h^2 k_2^2}{2m} \right) \delta \beta \right) \times \delta_{q_3,0} \delta_{k_3,l_3} \left( 1 + \frac{1}{2} \left( \frac{h^2 k_3^2}{2m} \right) \delta \beta \right) \delta_{q_4,0} \delta_{k_4,l_4} \left( 1 + \frac{1}{2} \left( \frac{h^2 k_4^2}{2m} \right) \delta \beta \right) \approx \delta_{q_1,0} \delta_{k_1,l_1} \delta_{q_2,0} \delta_{k_2,l_2} \delta_{q_3,0} \delta_{k_3,l_3} \delta_{q_4,0} \delta_{k_4,l_4} \times \left( 1 + \frac{1}{2} \left( \frac{h^2 k_1^2}{2m} + \frac{h^2 k_2^2}{2m} + \frac{h^2 k_3^2}{2m} + \frac{h^2 k_4^2}{2m} \right) \delta \beta \right) \quad (105) $$

where only terms linear in $\delta t$ have been retained.

$$ 12ab : \quad F_{d,u}(k_1, l_1; q_1, \delta \beta) F_{u,u}(k_2, l_2; q_2, \delta \beta) F_{u,u}(k_3, l_3; q_3, \delta \beta) F_{d,u}(k_4, l_4; q_4, \delta \beta) = \delta_{q_1,0} \delta_{k_1,l_1} \left( 1 + \frac{1}{2} \left( \frac{h^2 k_1^2}{2m} \right) \delta \beta \right) \delta_{q_2,0} \delta_{k_2,l_2} \left( 1 + \frac{1}{2} \left( \frac{h^2 k_2^2}{2m} \right) \delta \beta \right) \times \eta \delta_{k_3,q_3} \delta_{k_4,q_4} \delta \beta \approx +2n^2 \{ \delta_{q_1,0} \delta_{k_1,l_1} \delta_{q_2,0} \delta_{k_2,l_2} \delta_{k_3,q_3} \delta_{k_4,q_4} \} \delta \beta \quad (106) $$

where the Wiener increment properties in Eq. (62) has been used and only terms linear in $\delta t$ have been retained.

$$ 21ba : \quad F_{d,u}(k_1, l_1; q_1, \delta \beta) F_{u,u}(k_2, l_2; q_2, \delta \beta) F_{u,u}(k_3, l_3; q_3, \delta \beta) F_{d,u}(k_4, l_4; q_4, \delta \beta) = \left( + \eta \delta_{k_1,q_1} \right) \left( + \eta \delta_{k_2,q_2} \right) \left( + \eta \delta_{k_3,q_3} \right) \left( + \eta \delta_{k_4,q_4} \right) \times \delta_{q_1,0} \delta_{k_1,l_1} \left( 1 + \frac{1}{2} \left( \frac{h^2 k_1^2}{2m} \right) \delta \beta \right) \delta_{q_2,0} \delta_{k_2,l_2} \left( 1 + \frac{1}{2} \left( \frac{h^2 k_2^2}{2m} \right) \delta \beta \right) \times \delta_{q_3,0} \delta_{k_3,l_3} \left( 1 + \frac{1}{2} \left( \frac{h^2 k_3^2}{2m} \right) \delta \beta \right) \delta_{q_4,0} \delta_{k_4,l_4} \left( 1 + \frac{1}{2} \left( \frac{h^2 k_4^2}{2m} \right) \delta \beta \right) \times \{ \delta_{q_1,q_2} \delta \beta + \delta_{q_1,q_2} \delta \beta \} \approx 2n^2 \{ \delta_{k_1,q_1} \delta_{k_2,q_2} \delta_{k_3,q_3} \delta_{k_4,q_4} \} \delta \beta \quad (107) $$

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where only terms linear in $\delta \beta$ have been retained.

Substituting the results in Eqs. (105), (106) and (107) into Eq. (102) and performing the sums over $q_1, q_2, q_3, q_4$ gives

$$
\sum_{l_1 l_2 l_3 l_4} \delta_{k_1, l_1} \delta_{k_2, l_2} \delta_{k_3, l_3} \delta_{k_4, l_4} \left( 1 + \frac{1}{2} \left( \frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m} + \frac{\hbar^2 k_3^2}{2m} + \frac{\hbar^2 k_4^2}{2m} \right) \delta \beta \right) 
$$

$$
\times \phi_d(l_1, \beta) \phi_u(l_2, \beta) \phi_u(l_3, \beta) \phi_d(l_4, \beta)
$$

$$
+ 2\eta^2 \sum_{l_1 l_2 l_3 l_4} \{ \delta_{k_1, l_1} \delta_{k_2, l_2} \delta_{l_3 + l_4, k_3 + k_4} \} \delta \beta \times \phi_d(l_1, \beta) \phi_u(l_2, \beta) \phi_d(l_3, \beta) \phi_u(l_4, \beta)
$$

$$
+ 2\eta^2 \sum_{l_1 l_2 l_3 l_4} \{ \delta_{l_1 + l_3, k_1 + k_2} \delta_{k_1, l_1} \delta_{k_2, l_2} \delta_{k_3, l_3} \delta_{k_4, l_4} \} \delta \beta \times \phi_u(l_1, \beta) \phi_d(l_2, \beta) \phi_u(l_3, \beta) \phi_d(l_4, \beta)
$$

$$
= \sum_{l_1 l_2 l_3 l_4} \delta_{k_1, l_1} \delta_{k_2, l_2} \delta_{k_3, l_3} \delta_{k_4, l_4} \left( 1 + \frac{1}{2} \left( \frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m} + \frac{\hbar^2 k_3^2}{2m} + \frac{\hbar^2 k_4^2}{2m} \right) \delta \beta \right)
$$

$$
\times \phi_d(l_1, \beta) \phi_u(l_2, \beta) \phi_u(l_3, \beta) \phi_d(l_4, \beta)
$$

$$
- 2\eta^2 \sum_{l_1 l_2 l_3 l_4} \{ \delta_{k_1, l_1} \delta_{k_2, l_2} \delta_{l_3 + l_4, k_3 + k_4} \} \delta \beta \times \phi_u(l_1, \beta) \phi_d(l_2, \beta) \phi_u(l_3, \beta) \phi_d(l_4, \beta)
$$

$$
- 2\eta^2 \sum_{l_1 l_2 l_3 l_4} \{ \delta_{l_1 + l_3, k_1 + k_2} \delta_{k_1, l_1} \delta_{k_2, l_2} \delta_{k_3, l_3} \delta_{k_4, l_4} \} \delta \beta \times \phi_d(l_1, \beta) \phi_u(l_2, \beta) \phi_u(l_3, \beta) \phi_d(l_4, \beta)
$$

(108)

where we have used $\sum_{q_1, q_4} \delta_{l_3, l_4} \delta_{l_3 - q_4, l_4} = \delta_{l_3 + q_4, l_3 + l_4}$ and $\sum_{q_1} = 1$. Also, in the second last line we have interchanged $l_3$ and $l_4$ and used the anti-commuting feature of the Grassmann stochastic momentum fields $\bar{\phi}_d(l_3, \beta)$ and $\bar{\phi}_u(l_4, \beta)$ to place these in the opposite order in the stochastic average $\bar{\phi}_d(l_1, \beta) \bar{\phi}_u(l_2, \beta) \bar{\phi}_d(l_3, \beta) \bar{\phi}_u(l_4, \beta)$, with similar steps for the last line.
11.3 Two Cooper Pairs - Time Evolution

For the case of time evolution the QCF for two Cooper pairs at time $t + \delta t$ is related to that at time $t$ via

$$
\begin{align*}
&\left[ \begin{array}{c}
\tilde{\phi}_d(k_1, t + \delta t) \tilde{\phi}_d(k_2, t + \delta t) \tilde{\phi}_u(k_3, t + \delta t) \tilde{\phi}_u(k_4, t + \delta t) \\
\times \tilde{\phi}_u(k_5, t + \delta t) \tilde{\phi}_u(k_6, t + \delta t) \tilde{\phi}_d(k_7, t + \delta t) \tilde{\phi}_d(k_8, t + \delta t)
\end{array} \right]_{\text{StochAver}} \\
&\times \sum_{l_1,q_1} (F_{d,d}(k_1,l_1;q_1,\delta t) \tilde{\phi}_d(l_1,t) + F_{d,u}(k_1,l_1;q_1,\delta t) \tilde{\phi}_u(l_1,t)) \\
&\times \sum_{l_2,q_2} (F_{d,d}(k_2,l_2;q_2,\delta t) \tilde{\phi}_d(l_2,t) + F_{d,u}(k_2,l_2;q_2,\delta t) \tilde{\phi}_u(l_2,t)) \\
&\times \sum_{l_3,q_3} (F_{u,d}(k_3,l_3;q_3,\delta t) \tilde{\phi}_d(l_3,t) + F_{u,u}(k_3,l_3;q_3,\delta t) \tilde{\phi}_u(l_3,t)) \\
&\times \sum_{l_4,q_4} (F_{u,d}(k_4,l_4;q_4,\delta t) \tilde{\phi}_d(l_4,t) + F_{u,u}(k_4,l_4;q_4,\delta t) \tilde{\phi}_u(l_4,t)) \\
&= \sum_{l_5,q_5} (F_{u,d}(k_5,l_5;q_5,\delta t) \tilde{\phi}_d(l_5,t) + F_{u,u}(k_5,l_5;q_5,\delta t) \tilde{\phi}_u(l_5,t)) \\
&\times \sum_{l_6,q_6} (F_{u,d}(k_6,l_6;q_6,\delta t) \tilde{\phi}_d(l_6,t) + F_{u,u}(k_6,l_6;q_6,\delta t) \tilde{\phi}_u(l_6,t)) \\
&\times \sum_{l_7,q_7} (F_{d,d}(k_7,l_7;q_7,\delta t) \tilde{\phi}_d(l_7,t) + F_{d,u}(k_7,l_7;q_7,\delta t) \tilde{\phi}_u(l_7,t)) \\
&\times \sum_{l_8,q_8} (F_{d,d}(k_8,l_8;q_8,\delta t) \tilde{\phi}_d(l_8,t) + F_{d,u}(k_8,l_8;q_8,\delta t) \tilde{\phi}_u(l_8,t))
\end{align*}
$$

(109)

after substituting from Eq. (38). Apart from the sums over $l_1,q_1, \ldots, l_8,q_8$ there are 256 terms for which the stochastic averages have to be evaluated. If the two terms in each of the first 4 factors are listed as 1, 2 and the two terms in each of the second 4 factors are listed as $a, b$ then the 256 terms can be listed as

1111aaaa, 1211aaaa, 2111aaaa, 2211aaaa, ..., 2222aaaa,
1111abaa, 1211abaa, 2111abaa, 2211abaa, ..., 2222abaa, ..., 1111bbbb, ..., 2222bbbb.

In Ito stochastic equations we then use the factorisation result that depends
on the $F$, $F^+$ involving times later than $t$ (see Eq. (9.43) in Ref. 26)

\[
\begin{align*}
&\begin{bmatrix}
F_{\alpha,\beta}(k_a,l_a;q_a,\delta t) \tilde{\phi}_\beta(l_a,t) F_{\gamma,\delta}(k_b,l_b;q_b,\delta t) \tilde{\phi}_\delta(l_b,t) \\
\times F_{\epsilon,\eta}(k_c,l_c;q_c,\delta t) \tilde{\phi}_\eta(l_c,t) F_{\theta,\rho}(k_d,l_d;q_d,\delta t) \tilde{\phi}_\rho(l_d,t) \\
\times F_{\zeta,\tau}^+(k_f,l_f;q_f,\delta t) \tilde{\phi}_\tau^+(l_f,t) F_{\zeta,\tau}^-(k_f,l_f;q_f,\delta t) \tilde{\phi}_\tau^+(l_f,t) \\
\times F_{\nu,\mu}^+(k_g,l_g;q_g,\delta t) \tilde{\phi}_\mu^+(l_g,t) F_{\omega,\chi}^+(k_h,l_h;q_h,\delta t) \tilde{\phi}_\chi^+(l_h,t) \\
\end{bmatrix}_{\text{StochAver}} \\
&\times \tilde{\phi}_\beta(l_a,t) \tilde{\phi}_\gamma(l_b,t) \tilde{\phi}_\eta(l_c,t) \tilde{\phi}_\xi(l_d,t) \tilde{\phi}_\tau^+(l_f,t) \tilde{\phi}_\mu^+(l_g,t) \tilde{\phi}_\chi^+(l_h,t) \\
&\times \delta_{\kappa,\delta t}\delta_{\lambda,\delta t} = +2\lambda^2 \delta_{q_a,q_b} \delta_{(k_a+q_a),l_a} \delta_{(k_b-q_b),l_b} \delta t \\
&\times \delta_{\kappa,\delta t}\delta_{\lambda,\delta t}^+ = -2\lambda^2 \delta_{q_a,q_b} \delta_{(k_a+q_a),l_a} \delta_{(k_b-q_b),l_b} \delta\delta t \\
\end{align*}
\] (110)

We wish to evaluate the various $F_{\alpha,\beta}(k_a,l_a;q_a,\delta t) F_{\gamma,\delta}(k_b,l_b;q_b,\delta t)$ correct to order $\delta t$. We note that the $F_{d,u}, F_{u,d}, F_{d+d,u}^+, F_{u+d,u}^+$ factors are non-stochastic and linear in $\delta t$. Also the $F_{d,u}, F_{u,d}, F_{d+d,u}^+, F_{u+d,u}^+$ factors are stochastic and linear in the stochastic Wiener increments $\delta\omega_{u,d}^+, \delta\omega_{d,u}^+, \delta\omega_{d+d,u}^+, \delta\omega_{u+d,u}^+$ from Eq. (32) the stochastic average of a factor involving an odd number 1, 3, 5, 7 of the $\delta\omega_{u,d}^+, \delta\omega_{d,u}^+, \delta\omega_{d+d,u}^+, \delta\omega_{u+d,u}^+$ with any non-stochastic factor is zero, if there are 2 of these increments the stochastic average is proportional to $\delta t$, if there are 4 of these increments the stochastic average is proportional to $\delta^2 t$ and if there are 6 the stochastic average is proportional to $\delta^3 t$ (apart from cases where there is zero Kroneck delta $\delta_{ab}$ involved). For the case of two stochastic Wiener increments we note from Eq. (32) that the non-zero stochastic averages of the products of $F's$ and $F^+'s$ are

\[
\begin{align*}
F_{d,u}(k_a,l_a;q_a,\delta t) F_{d,u}(k_b,l_b;q_b,\delta t) &= +2\lambda^2 \delta_{q_a,q_b} \delta_{(k_a+q_a),l_a} \delta_{(k_b-q_b),l_b} \delta t \\
F_{u,d}(k_a,l_a;q_a,\delta t) F_{d,u}(k_b,l_b;q_b,\delta t) &= -2\lambda^2 \delta_{q_a,q_b} \delta_{(k_a+q_a),l_a} \delta_{(k_b-q_b),l_b} \delta\delta t \\
\end{align*}
\] (111)

As we are looking for contributions that are only linear in $\delta t$, the only terms that will result in this have:

(1) Only non-stochastic $F's$ from the first 4 factors and only non-stochastic $F^+'s$ from the second 4 factors. There is only one term - 1122bbaa.

(2) Only non-stochastic $F's$ from the first 4 factors and only one stochastic $F^+$ and one stochastic $F^+$ from the second 4 factors. There are four terms - 1122abba, 1122abab, 1122baba, 1122baab.

(3) Only one stochastic $F_{d,u}$ and one stochastic $F_{d,u}$ from the first 4 factors and only non-stochastic $F^+$ from the second 4 factors. There are four terms - 1212bbba, 1212bbba, 2112bbaa, 2112bbab.

With these considerations in mind, the non-zero cases are as follows:
For the terms of type (1):

$$
\begin{align*}
1122bb & : \left[ F_{d,d}(k_1, l_1; q_1, \delta t) F_{d,d}(k_2, l_2; q_2, \delta t) F_{u,u}(k_3, l_3; q_3, \delta t) F_{u,u}(k_4, l_4; q_4, \delta t) \\
& \times F_{u,u}^+(k_5, l_5; q_5, \delta t) F_{u,u}^+(k_6, l_6; q_6, \delta t) F_{d,d}^+(k_7, l_7; q_7, \delta t) F_{d,d}^+(k_8, l_8; q_8, \delta t) \right]_{\text{StochAver}} \\
& = \delta_{q_1,0} \delta_{k_1,l_1} \left( 1 + \frac{i}{\hbar} \left( \frac{\hbar^2 k_1^2}{2m} \right) \delta t \right) \delta_{q_2,0} \delta_{k_2,l_2} \left( 1 + \frac{i}{\hbar} \left( \frac{\hbar^2 k_2^2}{2m} \right) \delta t \right) \\
& \times \delta_{q_3,0} \delta_{k_3,l_3} \left( 1 + \frac{i}{\hbar} \left( \frac{\hbar^2 k_3^2}{2m} \right) \delta t \right) \delta_{q_4,0} \delta_{k_4,l_4} \left( 1 + \frac{i}{\hbar} \left( \frac{\hbar^2 k_4^2}{2m} \right) \delta t \right) \\
& \times \delta_{q_5,0} \delta_{k_5,l_5} \left( 1 - \frac{i}{\hbar} \left( \frac{\hbar^2 k_5^2}{2m} \right) \delta t \right) \delta_{q_6,0} \delta_{k_6,l_6} \left( 1 - \frac{i}{\hbar} \left( \frac{\hbar^2 k_6^2}{2m} \right) \delta t \right) \\
& \times \delta_{q_7,0} \delta_{k_7,l_7} \left( 1 - \frac{i}{\hbar} \left( \frac{\hbar^2 k_7^2}{2m} \right) \delta t \right) \delta_{q_8,0} \delta_{k_8,l_8} \left( 1 - \frac{i}{\hbar} \left( \frac{\hbar^2 k_8^2}{2m} \right) \delta t \right) \\
& \approx \delta_{q_1,0} \delta_{k_1,l_1} \delta_{q_2,0} \delta_{k_2,l_2} \delta_{q_3,0} \delta_{k_3,l_3} \delta_{q_4,0} \delta_{k_4,l_4} \\
& \times \delta_{q_5,0} \delta_{k_5,l_5} \delta_{q_6,0} \delta_{k_6,l_6} \delta_{q_7,0} \delta_{k_7,l_7} \delta_{q_8,0} \delta_{k_8,l_8} \\
& \times \left( 1 + \frac{i}{\hbar} \left( \frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m} + \frac{\hbar^2 k_3^2}{2m} + \frac{\hbar^2 k_4^2}{2m} + \frac{\hbar^2 k_5^2}{2m} + \frac{\hbar^2 k_6^2}{2m} + \frac{\hbar^2 k_7^2}{2m} + \frac{\hbar^2 k_8^2}{2m} \right) \delta t \right)
\end{align*}
$$

(112)

where only terms linear in $\delta t$ have been retained.

For the terms of type (2):

$$
\begin{align*}
1122ab & : \left[ F_{d,d}(k_1, l_1; q_1, \delta t) F_{d,d}(k_2, l_2; q_2, \delta t) F_{u,u}(k_3, l_3; q_3, \delta t) F_{u,u}(k_4, l_4; q_4, \delta t) \\
& \times F_{u,u}^+(k_5, l_5; q_5, \delta t) F_{u,u}^+(k_6, l_6; q_6, \delta t) F_{d,d}^+(k_7, l_7; q_7, \delta t) F_{d,d}^+(k_8, l_8; q_8, \delta t) \right]_{\text{StochAver}} \\
& = \delta_{q_1,0} \delta_{k_1,l_1} \left( 1 + \frac{i}{\hbar} \left( \frac{\hbar^2 k_1^2}{2m} \right) \delta t \right) \delta_{q_2,0} \delta_{k_2,l_2} \left( 1 + \frac{i}{\hbar} \left( \frac{\hbar^2 k_2^2}{2m} \right) \delta t \right) \\
& \times \delta_{q_3,0} \delta_{k_3,l_3} \left( 1 + \frac{i}{\hbar} \left( \frac{\hbar^2 k_3^2}{2m} \right) \delta t \right) \delta_{q_4,0} \delta_{k_4,l_4} \left( 1 + \frac{i}{\hbar} \left( \frac{\hbar^2 k_4^2}{2m} \right) \delta t \right) \\
& \times \delta_{q_5,0} \delta_{k_5,l_5} \left( 1 - \frac{i}{\hbar} \left( \frac{\hbar^2 k_5^2}{2m} \right) \delta t \right) \delta_{q_6,0} \delta_{k_6,l_6} \left( 1 - \frac{i}{\hbar} \left( \frac{\hbar^2 k_6^2}{2m} \right) \delta t \right) \\
& \times \left( -2\lambda^2 \right) \delta_{q_7,q_7} \delta_{(k_5+q_5),l_5} \delta_{(k_7-q_7),l_7} \delta t \\
& \approx \left( -2\lambda^2 \right) \\
& \times \delta_{q_1,0} \delta_{k_1,l_1} \delta_{q_2,0} \delta_{k_2,l_2} \delta_{q_3,0} \delta_{k_3,l_3} \delta_{q_4,0} \delta_{k_4,l_4} \\
& \times \delta_{q_5,0} \delta_{k_5,l_5} \delta_{q_6,0} \delta_{k_6,l_6} \delta_{q_7,q_7} \delta_{(k_5+q_5),l_5} \delta_{(k_7-q_7),l_7} \delta t
\end{align*}
$$

(113)
\[
1122abab : \left[ F_{d,d}(k_1, l_1; q_1, \delta t) F_{d,d}(k_2, l_2; q_2, \delta t) F_{u,u}(k_3, l_3; q_3, \delta t) F_{u,u}(k_4, l_4; q_4, \delta t) \\
\times F_{u,u}^+(k_5, l_5; q_5, \delta t) F_{u,u}^+(k_6, l_6; q_6, \delta t) F_{d,d}^+(k_7, l_7; q_7, \delta t) F_{d,d}^+(k_8, l_8; q_8, \delta t) \right]_{\text{StochAver}} \\
= \delta_{q_1,0} \delta_{k_1,l_1} \left( 1 + \frac{i}{\hbar} \left\{ \frac{h^2 k_1^2}{2m} \right\} \delta t \right) \delta_{q_2,0} \delta_{k_2,l_2} \left( 1 + \frac{i}{\hbar} \left\{ \frac{h^2 k_2^2}{2m} \right\} \delta t \right) \\
\times \delta_{q_3,0} \delta_{k_3,l_3} \left( 1 + \frac{i}{\hbar} \left\{ \frac{h^2 k_3^2}{2m} \right\} \delta t \right) \delta_{q_4,0} \delta_{k_4,l_4} \left( 1 + \frac{i}{\hbar} \left\{ \frac{h^2 k_4^2}{2m} \right\} \delta t \right) \\
\times \delta_{q_5,0} \delta_{k_5,l_5} \left( 1 - \frac{i}{\hbar} \left\{ \frac{h^2 k_5^2}{2m} \right\} \delta t \right) \delta_{q_6,0} \delta_{k_6,l_6} \left( 1 - \frac{i}{\hbar} \left\{ \frac{h^2 k_6^2}{2m} \right\} \delta t \right) \\
\times ( -2\lambda^2 ) \delta_{q_5,q_6} \delta_{(k_5-k_6),l_5} \delta_{(k_8+q_6),l_6} \delta t \\
\approx ( -2\lambda^2 ) \\
\times \delta_{q_1,0} \delta_{k_1,l_1} \delta_{q_2,0} \delta_{k_2,l_2} \delta_{q_3,0} \delta_{k_3,l_3} \delta_{q_4,0} \delta_{k_4,l_4} \\
\times \delta_{q_5,0} \delta_{k_5,l_5} \delta_{q_6,0} \delta_{k_6,l_6} \delta_{q_5,q_6} \delta_{(k_5+q_5),l_5} \delta_{(k_8-q_6),l_6} \delta t
\]

\[
1122baba : \left[ F_{d,d}(k_1, l_1; q_1, \delta t) F_{d,d}(k_2, l_2; q_2, \delta t) F_{u,u}(k_3, l_3; q_3, \delta t) F_{u,u}(k_4, l_4; q_4, \delta t) \\
\times F_{u,u}^+(k_5, l_5; q_5, \delta t) F_{u,u}^+(k_6, l_6; q_6, \delta t) F_{d,d}^+(k_7, l_7; q_7, \delta t) F_{d,d}^+(k_8, l_8; q_8, \delta t) \right]_{\text{StochAver}} \\
= \delta_{q_1,0} \delta_{k_1,l_1} \left( 1 + \frac{i}{\hbar} \left\{ \frac{h^2 k_1^2}{2m} \right\} \delta t \right) \delta_{q_2,0} \delta_{k_2,l_2} \left( 1 + \frac{i}{\hbar} \left\{ \frac{h^2 k_2^2}{2m} \right\} \delta t \right) \\
\times \delta_{q_3,0} \delta_{k_3,l_3} \left( 1 + \frac{i}{\hbar} \left\{ \frac{h^2 k_3^2}{2m} \right\} \delta t \right) \delta_{q_4,0} \delta_{k_4,l_4} \left( 1 + \frac{i}{\hbar} \left\{ \frac{h^2 k_4^2}{2m} \right\} \delta t \right) \\
\times \delta_{q_5,0} \delta_{k_5,l_5} \left( 1 - \frac{i}{\hbar} \left\{ \frac{h^2 k_5^2}{2m} \right\} \delta t \right) \delta_{q_6,0} \delta_{k_6,l_6} \left( 1 - \frac{i}{\hbar} \left\{ \frac{h^2 k_6^2}{2m} \right\} \delta t \right) \\
\times ( -2\lambda^2 ) \delta_{q_5,q_7} \delta_{(k_5+q_7),l_6} \delta_{(k_5-q_7),l_7} \delta t \\
\approx ( -2\lambda^2 ) \\
\times \delta_{q_1,0} \delta_{k_1,l_1} \delta_{q_2,0} \delta_{k_2,l_2} \delta_{q_3,0} \delta_{k_3,l_3} \delta_{q_4,0} \delta_{k_4,l_4} \\
\times \delta_{q_5,0} \delta_{k_5,l_5} \delta_{q_6,0} \delta_{k_6,l_6} \delta_{q_5,q_7} \delta_{(k_5+q_7),l_6} \delta_{(k_5-q_7),l_7} \delta t
\]

\[(114) \quad (115) \quad (116)\]
\[
\frac{\partial}{\partial t} \rho(t) = \sum_{l_1,l_2} \left[ F_{d,d}(k_1, l_1; q_1, \delta t) F_{d,u}(k_2, l_2; q_2, \delta t) F_{u,u}(k_3, l_3; q_3, \delta t) F_{u,d}(k_4, l_4; q_4, \delta t) \right]_{\text{StochAver}}
\]

where only terms linear in \(\delta t\) have been retained.

For terms of type (3):

\[
\frac{\partial}{\partial t} \rho(t) = \sum_{l_1,l_2} \left[ F_{d,d}(k_1, l_1; q_1, \delta t) F_{d,u}(k_2, l_2; q_2, \delta t) F_{u,u}(k_3, l_3; q_3, \delta t) F_{u,d}(k_4, l_4; q_4, \delta t) \right]_{\text{StochAver}}
\]

(117)
\[ 1212baa: \quad \left[ F_{d,d}(k_1, l_1; q_1, \delta t)F_{d,a}(k_2, l_2; q_2, \delta t)F_{a,d}(k_3, l_3; q_3, \delta t)F_{a,a}(k_4, l_4; q_4, \delta t) \right. \\
\left. \times F_{a,a}^+(k_5, l_5; q_5, \delta t) F_{a,a}^+(k_6, l_6; q_6, \delta t) F_{d,a}^+(k_7, l_7; q_7, \delta t) F_{d,a}^+(k_8, l_8; q_8, \delta t) \right]_{\text{StochAver}} \\
= \delta_{q_1,0} \delta_{k_1,l_1} \left( 1 + \frac{i}{\hbar} \left( \frac{h^2k_1^2}{2m} \right) \delta t \right) \delta_{q_4,0} \delta_{k_4,l_4} \left( 1 + \frac{i}{\hbar} \left( \frac{h^2k_4^2}{2m} \right) \delta t \right) \\
\times \delta_{q_5,0} \delta_{k_5,l_5} \left( 1 - \frac{i}{\hbar} \left( \frac{h^2k_5^2}{2m} \right) \delta t \right) \delta_{q_6,0} \delta_{k_6,l_6} \left( 1 - \frac{i}{\hbar} \left( \frac{h^2k_6^2}{2m} \right) \delta t \right) \\
\times \delta_{q_7,0} \delta_{k_7,l_7} \left( 1 - \frac{i}{\hbar} \left( \frac{h^2k_7^2}{2m} \right) \delta t \right) \delta_{q_8,0} \delta_{k_8,l_8} \left( 1 - \frac{i}{\hbar} \left( \frac{h^2k_8^2}{2m} \right) \delta t \right) \\
\times \left( +2\lambda^2 \right) \delta_{q_2,q_3} \delta_{(k_2+q_2),l_2} \delta_{(k_3-q_3),l_3} \delta \tau \\
\approx \left( +2\lambda^2 \right) \\
\left. \times \delta_{q_1,0} \delta_{k_1,l_1} \delta_{q_4,0} \delta_{k_4,l_4} \delta_{q_5,0} \delta_{k_5,l_6} \delta_{q_6,0} \delta_{k_5,l_6} \right)
\right) \\
\right) \\
(119) \\

\[ 2121baa: \quad \left[ F_{d,u}(k_1, l_1; q_1, \delta t)F_{d,d}(k_2, l_2; q_2, \delta t)F_{u,a}(k_3, l_3; q_3, \delta t)F_{a,d}(k_4, l_4; q_4, \delta t) \right. \\
\left. \times F_{a,a}^+(k_5, l_5; q_5, \delta t) F_{a,a}^+(k_6, l_6; q_6, \delta t) F_{d,a}^+(k_7, l_7; q_7, \delta t) F_{d,a}^+(k_8, l_8; q_8, \delta t) \right]_{\text{StochAver}} \\
= \delta_{q_2,0} \delta_{k_2,l_2} \left( 1 + \frac{i}{\hbar} \left( \frac{h^2k_2^2}{2m} \right) \delta t \right) \delta_{q_3,0} \delta_{k_3,l_3} \left( 1 + \frac{i}{\hbar} \left( \frac{h^2k_3^2}{2m} \right) \delta t \right) \\
\times \delta_{q_5,0} \delta_{k_5,l_5} \left( 1 - \frac{i}{\hbar} \left( \frac{h^2k_5^2}{2m} \right) \delta t \right) \delta_{q_6,0} \delta_{k_6,l_6} \left( 1 - \frac{i}{\hbar} \left( \frac{h^2k_6^2}{2m} \right) \delta t \right) \\
\times \delta_{q_7,0} \delta_{k_7,l_7} \left( 1 - \frac{i}{\hbar} \left( \frac{h^2k_7^2}{2m} \right) \delta t \right) \delta_{q_8,0} \delta_{k_8,l_8} \left( 1 - \frac{i}{\hbar} \left( \frac{h^2k_8^2}{2m} \right) \delta t \right) \\
\times \left( +2\lambda^2 \right) \delta_{q_2,q_4} \delta_{(k_2+q_4),l_4} \delta_{(k_3-q_3),l_3} \delta \tau \\
\approx \left( +2\lambda^2 \right) \\
\left. \times \delta_{q_2,0} \delta_{k_2,l_2} \delta_{q_3,0} \delta_{k_3,l_3} \delta_{q_5,0} \delta_{k_5,l_6} \delta_{q_6,0} \delta_{k_5,l_6} \right)
\right) \\
\right) \\
(120) \]
\[ \begin{align*}
\text{StochAver} & = \\
&= \left[ F_{d,u}(k_1, l_1; q_1, \delta t) F_{d,d}(k_2, l_2; q_2, \delta t) F_{u,d}(k_3, l_3; q_3, \delta t) + \\
&\times F_{u,u}(k_4, l_4; q_4, \delta t) F_{d,d}(k_5, l_5; q_5, \delta t) F_{u,u}(k_6, l_6; q_6, \delta t) F_{d,d}(k_7, l_7; q_7, \delta t) + \\
&\times F_{d,d}(k_8, l_8; q_8, \delta t) \right] + 2\lambda^2 \delta q_1, q_3 \delta (k_1 + q_1), l_1 \delta (k_3 - q_3), l_3 \delta t
\end{align*} \]

where the Wiener increment properties in Eq. (32) has been used and only terms linear in \( \delta t \) have been retained.
Substituting the 9 results in Eqs. (112), (113), ..., (121) into Eq. (110) gives

\[
\begin{bmatrix}
\tilde{\phi}_d(k_1, t + \delta t) \tilde{\phi}_d(k_2, t + \delta t) \tilde{\phi}_d(k_3, t + \delta t) \tilde{\phi}_d(k_4, t + \delta t) \\
\times \phi_+^u(k_5, t + \delta t) \phi_+^u(k_6, t + \delta t) \phi_+^d(k_7, t + \delta t) \phi_+^d(k_8, t + \delta t)
\end{bmatrix}
\times \left[ \sum_{l_1, q_1} \sum_{l_2, q_2} \sum_{l_3, q_3} \sum_{l_4, q_4} \sum_{l_5, q_5} \sum_{l_6, q_6} \sum_{l_7, q_7} \sum_{l_8, q_8} \sum_{l_9, q_9} \sum_{l_10, q_10} \right]
\left[ \delta_{q_1, 0} \delta_{k_1, l_1} \delta_{q_2, 0} \delta_{k_2, l_2} \delta_{q_3, 0} \delta_{k_3, l_3} \delta_{q_4, 0} \delta_{k_4, l_4} \times \delta_{q_5, 0} \delta_{k_5, l_5} \delta_{q_6, 0} \delta_{k_6, l_6} \delta_{q_7, 0} \delta_{k_7, l_7} \delta_{q_8, 0} \delta_{k_8, l_8} \right]
\left[ \frac{h^2 k_1^2}{2m} + \frac{h^2 k_2^2}{2m} + \frac{h^2 k_3^2}{2m} - h^2 k_4^2 - \frac{h^2 k_5^2}{2m} - \frac{h^2 k_6^2}{2m} - \frac{h^2 k_7^2}{2m} - \frac{h^2 k_8^2}{2m} \right] \delta t
\]
(122)
and performing the sums over \(q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8\) gives

\[
\sum_{l_1} \sum_{l_2} \sum_{l_3} \sum_{l_4} \sum_{l_5} \sum_{l_6} \sum_{l_7} \sum_{l_8} \left( \begin{array}{c}
\widehat{\phi}_d(k_1, t + \delta t) \widehat{\phi}_d(k_2, t + \delta t) \widehat{\phi}_u(k_3, t + \delta t) \widehat{\phi}_u(k_4, t + \delta t) \\
\times \phi_u^+(k_5, t + \delta t) \phi_u^+(k_6, t + \delta t) \phi_u^+(k_7, t + \delta t) \phi_u^+(k_8, t + \delta t)
\end{array} \right)
\]

\[
\times \left( 1 + \frac{\delta}{2m} \left\{ \frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m} + \frac{\hbar^2 k_3^2}{2m} + \frac{\hbar^2 k_4^2}{2m} - \frac{\hbar^2 k_5^2}{2m} - \frac{\hbar^2 k_6^2}{2m} \right\} \delta t \right)
\]

\[
\times \phi_d(l_1, t) \phi_d(l_2, t) \phi_u(l_3, t) \phi_u(l_4, t) \phi_u(l_5, t) \phi_u(l_6, t) \phi_u(l_7, t) \phi_u(l_8, t)
\]

\[
\phi_u^+(l_9, t) \phi_u^+(l_{10}, t) \phi_u^+(l_{11}, t) \phi_u^+(l_{12}, t)
\]

\[
= \sum_{q_1, q_2} \sum_{q_3, q_4} \sum_{q_5, q_6} \sum_{q_7, q_8} \sum_{l_1} \sum_{l_2} \sum_{l_3} \sum_{l_4} \sum_{l_5} \sum_{l_6} \sum_{l_7} \sum_{l_8} \left( \begin{array}{c}
\delta_{(k_3+q_3),l_3} \delta_{(k_4-q_4),l_4} \delta_{q_3, q_4} = \delta_{(k_3+q_3),l_3} + \delta_{(k_4-q_4),l_4}
\end{array} \right)
\]

\[
(123)
\]

where we have used \(\sum_{q_1, q_4} \delta_{(k_3+q_3),l_3} \delta_{(k_4-q_4),l_4} \delta_{q_3, q_4} = \delta_{(k_3+k_4),l_3+l_4}\) and \(\sum_{q_1} \delta_{q_1,0} = 1\).

As can be seen, all the QCF involved contain two stochastic momentum fields for each of \(\widehat{\phi}_d, \widehat{\phi}_u, \phi_u^+, \phi_u^+\) so by re-arranging these Grassmann fields using their anti-commutation properties and then re-ordering the indices \(l_1, l_2, ..., l_8\) and introducing the notation \(X(d, k_1, d, k_2, u, k_3, u, k_4, u^+ k_5, u^+ k_6, d^+ k_7, d^+ k_8)\) from

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Eq. (49) we have

\[ X(d \, k_1, d \, k_2, u \, k_3, u \, k_4, u^+ \, k_5, u^+ \, k_6, d^+ \, k_7, d^+ \, k_8)_{t+\delta t} \]

\[ = \sum_{l_1} \sum_{l_2} \sum_{l_3} \sum_{l_4} \sum_{l_5} \sum_{l_6} \sum_{l_7} \sum_{l_8} \]

\[ \times \left( 1 + \frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m} + \frac{\hbar^2 k_3^2}{2m} - \frac{\hbar^2 k_4^2}{2m} - \frac{\hbar^2 k_5^2}{2m} + \frac{\hbar^2 k_6^2}{2m} \right) \delta t \]

\[ \times \left[ \delta l_1 \, \delta l_2 \, \delta l_3 \, \delta l_4 \, \delta l_5 \, \delta l_6 \, \delta l_7 \, \delta l_8 \times \delta \right] \]

\[ + \left( -2\lambda^2 \right) \delta l_1 \, \delta l_2 \, \delta l_3 \, \delta l_4 \, \delta l_5 \, \delta l_6 \, \delta l_7 \, \delta l_8 \]

\[ \times \left( \delta k_1 \, \delta k_2 \, \delta k_3 \, \delta k_4 \, \delta k_5 \, \delta k_6 \, \delta k_7 \, \delta k_8 \right) \]

\[ \times X(d \, l_1, d \, l_2, u \, l_3, u \, l_4, u^+ \, l_5, u^+ \, l_6, d^+ \, l_7, d^+ \, l_8)_{t} \]

Thus the first order change in the QCF for two Cooper pairs depends linearly on \( \delta t \).
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