DIMENSION-FREE $L^p$-MAXIMAL INEQUALITIES IN $\mathbb{Z}^{m+1}_{N}$

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Abstract. For $m \geq 2$, let $(\mathbb{Z}^{N}_{m+1}, | \cdot |)$ denote the group equipped with the so-called $l^0$ metric,

$$|y| := |\{1 \leq i \leq N : y(i) \neq 0\}|,$$

and define the $L^1$-normalized indicator of the $r$-ball,

$$\beta_r := \frac{1}{|\{x : |x| \leq r\}|} \mathbf{1}_{\{x : |x| \leq r\}}.$$

We study the $L^p \to L^p$ mapping properties of the maximal operator

$$M_N B f(x) := \sup_{r \leq N} |\beta_r \ast f|$$

acting on functions defined on $\mathbb{Z}^{N}_{m+1}$.

Specifically, we prove that for all $p > 1$, there exist absolute constants $C_{m,p}$ so that

$$\|M_N B f\|_{L^p(\mathbb{Z}^{N}_{m+1})} \leq C_{m,p} \|f\|_{L^p(\mathbb{Z}^{N}_{m+1})}$$

for all $N$. This result may be viewed as an extension of the main theorem of [5] – the existence of dimension-free $L^p$-bounds for $p > 1$ for the spherical maximal function in the hypercube, $\mathbb{Z}^N$. Indeed, our approach is that of [5], which grew out of the arguments of [3], which were in turn motivated by the spectral technique developed in [6] and [7] in the context of pointwise ergodic theorems on general groups.

1. Introduction

In $\mathbb{R}^N$, let

$$M_B^{R^N} f(x) := \sup_{r > 0} \frac{c_N}{r^N} \int_{|y| \leq r} |f(x - y)| \, dy,$$

denote the standard Hardy-Littlewood maximal function, where $c_N^{-1}$ is the volume of the $N$-dimensional Euclidean unit ball.

A celebrated result of Stein and Strömberg [8] in Euclidean harmonic analysis concerns the following dimension-independent bounds:

**Theorem 1.1 (Theorem A of [8]).** For each $p > 1$ there exists a constant $A_p$ independent of $N$ so that

$$\left\|M_B^{R^N} f\right\|_{L^p(\mathbb{R}^N)} \leq A_p \|f\|_{L^p(\mathbb{R}^N)}.$$

In particular, while the maximal operators are themselves dimension-dependent, they are all uniformly bounded in $L^p \to L^p$ operator-norm by the same constant, $A_p$.

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This result was more recently extended by Bourgain [1] to the cubic maximal function
\[ M^R_N f(x) := \sup_{r > 0} \frac{1}{(2r)^N} \int_{y \in Q_r} |f(x - y)| \, dy, \]
where \( Q_r := \{ y = (y(1), \ldots, y(N)) : |y(i)| \leq r \text{ for each } 1 \leq i \leq N \} \) is the cube of side-length 2r centered at the origin.

**Theorem 1.2** (Theorem of [1]). For each \( p > 1 \) there exist constants \( A'_p \) independent of \( N \) so that
\[ \|M^R_N f\|_{L^p(\mathbb{R}^N)} \leq A'_p \|f\|_{L^p(\mathbb{R}^N)}. \]

The purpose of this article is to establish comparable dimension independent bounds in a discrete setting.

Specifically, for \( m \geq 2 \), let \( \mathbb{Z}_m^{N+1} \) denote the group equipped with the so-called \( l^0 \)-metric,
\[ |y| := \{|1 \leq i \leq N : y(i) \neq 0\}|. \]
We also define the \( (L^2\text{-normalized}) \) characters
\[ \chi_S(x) := \frac{1}{(m+1)^{N/2}} \xi_S^x \prod_{i \in S} (\xi_m)^{s(i)x(i)}, \]
where \( \xi_m := e^{2\pi i/(m+1)} \) is a primitive \((m+1)\)th root of unity. Define the Fourier transform
\[ \mathcal{F}f(S) = \hat{f}(S) = \sum_{x \in \mathbb{Z}_m^{N+1}} f(x) \chi_S(x), \]
and the \( L^1\text{-normalized} \) indicator functions of the r-sphere and the r-ball:
\[ \sigma_r := \frac{1}{|S_r|} 1_{S_r}, \]
\[ \beta_r := \frac{1}{|\{ |x| \leq r \}|} 1_{\{ |x| \leq r \}}. \]
are respectively the \( L^1\text{-normalized} \) indicator functions of the r-sphere and r-ball. We adopt the convention that both functions are 0 and the respective spheres and balls are empty for \( r < 0 \).

Motivated by [1] and [8], we will be interested in establishing dimension-independent bounds for the family of maximal functions
\[ M^N_B f(x) := \sup_{r \leq N} |\beta_r * f| \]
acting on functions in \( \mathbb{Z}_m^{N+1} \). Observe that an application of this operator to \(|f|\) yields the Hardy-Littlewood maximal function.

We establish the following theorem:

**Theorem 1.3.** For any \( p > 1 \), and any \( m \geq 2 \), there exists a constant \( C_{p,m} \) so that
\[ \|M^N_B f\|_{L^p(\mathbb{Z}_m^{N+1})} \leq C_{p,m} \|f\|_{L^p(\mathbb{Z}_m^{N+1})}. \]
In particular, the above bounds exist independent of the dimension, \( N \).

A similar problem was studied in the \( m = 1 \) case in [8]:
Theorem 1.4 ([3], Theorem 1). There exists a constant $C_2$ so that for all $N$,
\[
\left\| \sup_{r \leq N} |\sigma_r * f| \right\|_{L^2(\mathbb{Z}_N^2)} \leq C_2 \|f\|_{L^2(\mathbb{Z}_N^2)}.
\]
and later in [5]:

Theorem 1.5 ([5], Theorem 2.2). For any $p > 1$, there exist constants $C_p$ so that
\[
\left\| \sup_{r \leq N} |\sigma_r * f| \right\|_{L^p(\mathbb{Z}_N^2)} \leq C_p \|f\|_{L^p(\mathbb{Z}_N^2)}.
\]

Remark 1.6. Since the spherical maximal function, $\sup_{r \leq N} |\sigma_r * f|$ pointwise dominates $M_B^N$ in $\mathbb{Z}_N^2$, this last result establishes dimension independent bounds in the hypercube, $\mathbb{Z}_N^2$.

The argument is an application of Stein’s method [7], used in extending the well-known Hopf-Dunford-Schwartz maximal theorem for semigroups to more “singular” maximal averages, and breaks into four main steps:

1. By comparison with the noise semigroup from Boolean Analysis [3, §4], [5, §3] the “smoother” maximal function
\[
\sup_K \frac{1}{K + 1} \left| \sum_{k \leq K} \sigma_k * f \right|
\]
is shown to satisfy a dimension-free weak-type $1 - 1$ inequality:
\[
\left\{ \left( \sup_K \frac{1}{K + 1} \left| \sum_{k \leq K} \sigma_k * f \right| > \lambda \right) \right\} \leq C_1 \|f\|_{L^1(\mathbb{Z}_N^2)} \lambda, \lambda \geq 0;
\]

2. The “rougher” maximal function $\sup_{r \leq N} |\sigma_r * f|$ is compared to the “smoother” maximal function in $L^2$ by using Littlewood-Paley theory on the group $\mathbb{Z}_N^{m+1}$. The key tool is an analysis of the (radial) spherical multipliers
\[
\mathcal{F}\sigma_k(S) := \kappa^N_k(S)
\]
the Krawtchouk polynomials, which are introduced and discussed in [3, §2];

3. The “rough” maximal function, $\sup_{r \leq N} |\sigma_r * f|$, is compared to increasingly “rougher” maximal functions in $L^2$. Analysis of the Krawtchouk polynomials are pivotal in these further comparisons;

4. Stein interpolation is used to control $\sup_{r \leq N} |\sigma_r * f|$ in $L^p$, $p > 1$.

Although we are studying the maximal function over balls, our approach is very much that of [5]. Indeed, our analysis will in the main be centered on an appropriately defined spherical maximal function, introduced in §3.

Theorems 1.3 and 1.5 together synthesize a generalization to arbitrary direct sums of finite cyclic groups, which can be viewed as a statement about all finite abelian groups.

Let $n, N_1, \ldots, N_n \in \mathbb{N}$ and let $A_m$ be the group $\mathbb{Z}_m^{N_m}$ for $m \geq 1$. Also let $A$ be the direct sum
\[
A := \bigoplus_{m=1}^n A_m
\]
with the notation

\[ y = \oplus_{m=1}^{n} (y_m^1, \ldots, y_m^{N_m}) \]
equip with the modified \( l^0 \) metric,

\[ |y| := \left| \{(m, j) : 1 \leq m \leq n, 1 \leq j \leq N_m, y_{m,j} \neq 0\} \right|. \]

Put more simply, viewing \( y \) as a \( N_1 + \cdots + N_n \)-tuple in the natural way, \( |y| \) is the number of nonzero components. Let \( \beta_r^A \) be the \( L^1 \)-normalized indicator function of the radius \( r \) ball, i.e.

\[ \beta_r^A(x) = \frac{1}{|\{|x| \leq r\}|} 1_{(|x| \leq r)} \]

and define the operator

\[ M_B^A f(x) = \sup_{r>0} |\beta_r^A * f(x)|, \]

the ball maximal function.

**Theorem 1.7.** For any \( p > 1 \) there exist constants \( C_{p,n} \) such that \( \|M_B^A f\|_{L^p(A)} \leq C_{p,n} \|f\|_{L^p(A)} \). In particular \( C_{p,n} \) has no direct dependence on \( A \).

This result admits a corollary concerning Cayley graphs of finite abelian groups.

**Corollary 1.8.** Let \( A \) be a finite abelian group whose elements have order at most \( d \). Then there exists a generating set of minimal size up to a factor of \( d \) such that the ball maximal operator on the Cayley graph \( \Gamma(A, S) \) satisfies \( L^p \) bounds for all \( p > 1 \) dependent only on \( p \) and \( n \).

**Proof of Corollary 1.8.** Assuming Theorem 1.7. If \( A \) is a finite abelian group with a minimal size generating set with \( s \) elements, by the fundamental theorem of finitely generated abelian groups there exist \( m_1 < \cdots < m_k < \infty \) and \( \tilde{N}_1 + \cdots + \tilde{N}_k = s \) such that we can identify \( A \) with

\[ \oplus_{j=1}^{k} \mathbb{Z}^{\tilde{N}_j}_{m_j+1} \]

We examine the generator set \( S \) of \( s \)-tuples that have exactly one nonzero component. Note that as long as each element of \( A \) has order at most \( d \), \( |S| \leq sd \) so \( S \) is a generating set of minimal size up to a factor of \( d \). Setting \( n := m_k \), we can identify \( \mathbb{Z}^n \) with

\[ \oplus_{m=1}^{n} \mathbb{Z}^{N_m}_{m+1} \]

where in general \( N_m = 0 \) for some values of \( m \). Note that the distance metric on the Cayley graph \( \Gamma(A, S) \) is precisely the \( l^0 \) metric of Theorem 1.7. From here the corollary is a direct application of the theorem. \( \square \)

The structure of the paper is as follows:

- In §2 we reduce the study of \( M_B^A \) to that of an appropriately defined spherical maximal operator;
- In §3 we introduce our “smoothed out” spherical maximal operator, and prove that they satisfy dimension independent weak type 1 – 1 bounds;
- In §4 we review Stein’s semigroup comparison method, and adapt it to our present context; assuming the technical Proposition 4.3 we prove our main Theorem 1.3;
- In §5 we prove Proposition 4.3 and
- In §6 we prove Theorem 1.7, the generalization of Theorem 1.3 to arbitrary direct sums of finite cyclic groups.
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1.2. Notation. Throughout, we denote $c_m := \frac{m}{m+1}$, and we reserve $H(t) := -t \ln t - (1-t) \ln(1-t)$ as the (natural) entropy function.

When clear from context, we will suppress the superscript “$N$” in the definition of our maximal functions. We will also make use of the modified Vinogradov notation. We use $X \lesssim Y$ or $Y \gtrsim X$ to denote the estimate $X \leq CY$ for some constant $C$ which may depend only on $m$ (in general we will suppress dependence on $m$) until §6 when $m$ varies so we allow constants to implicitly depend on $n$ instead. If we need $C$ to depend on a parameter other than $m$ or $n$, we shall indicate this by subscripts, thus for instance $X \lesssim_a Y$ denotes the estimate $X \leq C_a Y$ for some $C_a$ depending on $a$. We use $X \approx Y$ as shorthand for $X \lesssim Y \lesssim X$, and similarly for $X \approx_a Y$.

2. The Reduction from the Maximal Function to the Spherical Maximal Function

We will study $M_B$ in two separate ways:

$$M_B f(x) \leq M_L f + M_G f := \sup_{r \leq c_m N} |\beta_r \ast f| + \sup_{r > c_m N} |\beta_r \ast f|.$$ 

We will handle the local maximal function using the (restricted) spherical maximal function,

$$(2) \quad M f := \sup_{r \leq c_m N} |\sigma_r \ast f|,$$

which will be treated in the remaining sections. In this section, we control the global maximal function, $M_G$.

Remark 2.1. The reason for considering the maximal function over balls is that the comparison technique of §3 breaks for “distant” spheres §4. The following lemma allows us to bring an easy “covering argument” to bear in treating the global maximal function.

In order to use the following lemma as a black box in §6 we include the hypercube case $m = 1$. The computation can be done more directly on the hypercube but for expediency we wrap it into the lemma.

Lemma 2.2. For any $m \geq 1$ and $r \geq c_m N$, $|\{x| \leq r\}| \approx (m+1)^N$.

Proof. Clearly it suffices to show $|\{x| \leq c_m N\}| \approx (m+1)^N$ and, because we can ignore finitely many cases, it is enough to show that the ratio of the two expressions converges to a positive number as $N$ approaches infinity. Let $\mathbb{P}$ be the uniform probability measure on $\mathbb{Z}^N_{m+1}$ and let

$$X_n(x_1, \ldots , x_N) := 1 - \delta_{0,x_n}$$

(Kronecker delta) be the random variable on $\mathbb{Z}^N_{m+1}$ given by the $l^0$ norm of the $n$th component. Then, by definition,

$$\frac{|\{x| \leq c_m N\}|}{(m+1)^N} = \mathbb{P} \left( \sum_{n=1}^N X_n \leq c_m N \right).$$
Computing directly shows that, independent of $n$ and $N$, $X_n$ has density
\[
\frac{1}{m+1} \delta_0 + \frac{m}{m+1} \delta_1
\]
(Dirac delta). Moreover $\{X_n\}^N_{n=1}$ is jointly independent for any $N$ simply because $\mathbb{Z}_m^{N+1}$ is a Cartesian product. Because the mean of each $X_n$ is $c_m$, The central limit theorem implies that
\[
\lim_{N \to \infty} \frac{|\{x \leq c_m N\}|}{(m + 1)^N} = \lim_{N \to \infty} \mathbb{P}\left(\sum_{n=1}^{N} X_n \leq c_m N\right) = \frac{1}{2}.
\]
\[
\square
\]

The following proposition is an easy consequence:

**Proposition 2.3.** The global maximal function is of weak-type $1 - 1$ with constant independent of the dimension. In particular, there exist constants $C_{1,m}$ so that for all $N$,
\[
\|M_G f\|_{L^{1,\infty}(\mathbb{Z}_m^{N+1})} \leq C_{1,m} \|f\|_{L^{1}(\mathbb{Z}_m^{N+1})}.
\]
Moreover $M_G$ satisfies $L^p$ bounds independent of dimension.

**Proof.** If the set $E := \{M_G f > \lambda\}$ is nonempty, then there exists a large $r$-ball, $B = \{|x - y| \leq r\}$, $r > c_m N$, so that
\[
|B| \leq \frac{1}{\lambda} \int_B |f(x)| \, dx.
\]
By Lemma 2.2 we have that
\[
|E| \leq |\mathbb{Z}_m^N| = (m + 1)^N \approx |B|;
\]
substituting appropriately completes the proof of the weak-type bound. Marcinkiewicz interpolation with the trivial $L^{\infty}$ bound then provides the dimension free $L^p$ bounds.
\[
\square
\]

In light of this proposition, it is sufficient to bound the restricted spherical operator, $Mf$.

## 3. The “Smooth” Spherical Maximal Function in $\mathbb{Z}_m^{N+1}$

In this section, we prove:

**Proposition 3.1.** The smooth spherical maximal function
\[
M_S f := \sup_{K \leq c_m N} \left| \frac{1}{K+1} \sum_{k \leq K} \sigma_k * f \right|
\]
is of weak-type $1 - 1$, with bound independent of $N$, i.e. there exists some absolute $C_{1,m}$ so that
\[
\|M_S f\|_{L^{1,\infty}(\mathbb{Z}_m^{N+1})} \leq C_{1,m} \|f\|_{L^{1}(\mathbb{Z}_m^{N+1})}.
\]

Following the lead of [3, §4], we do so by comparison with an appropriate “noise” semigroup, which we now introduce.
3.1. The noise semigroup in $\mathbb{Z}^N_{m+1}$. For fixed $0 < p < c_m$ we define a probability measure $\tilde{\mu}_p$ on $\mathbb{Z}_{m+1}$ given by

$$
\tilde{\mu}_p(\{w\}) := \begin{cases} 1 - p & \text{if } w = 0 \\ \frac{p}{m} & \text{otherwise}, \end{cases}
$$

and for $y \in \mathbb{Z}^N_{m+1}$,

$$
\tilde{\mu}_p^N(\{y\}) = \left(\frac{p}{m}\right)^{|y|} (1 - p)^{N - |y|},
$$

where, as above,

$$
|y| := |\{1 \leq i \leq d : y(i) \neq 0\}|
$$

is the $l^0$-metric. We view $\tilde{\mu}_p^N$ alternatively as a measure and a function depending on context.

Consider the (dimension dependent) convolution operator

$$
\tilde{N}_p f(x) := f * \tilde{\mu}_p^N(x) = \int_{\mathbb{Z}^N_{m+1}} f(x + y) \tilde{\mu}_p^N(y)
$$

We denote by $\xi = \xi_m$ a primitive $(m+1)$th root of unity.

Lemma 3.2. For each ($L^2$-normalized) character

$$
\chi_S(x) := \frac{1}{(m+1)^{N/2}} \xi^{S \cdot x} = \frac{1}{(m+1)^{N/2}} \prod_{i=1}^{N} \xi^{S(i)x(i)}
$$

where $S, x \in \mathbb{Z}^N_{m+1}$ and $S \cdot x = \sum_{i=1}^{N} S(i)x(i)$, we have

$$
\tilde{N}_p \chi_S(x) = (1 - p/c_m)^{|S|} \chi_S(x).
$$

Proof. First note that

$$
\chi_S(x + y) = \frac{1}{(m+1)^{N/2}} \xi^{(x+y) \cdot S} = \chi_S(x) \xi^{y \cdot S}
$$

Thus

$$
(3) \quad \tilde{N}_p \chi_S(x) = \chi_S(x) \int_{\mathbb{Z}^N_{m+1}} \xi^{y \cdot S} \tilde{\mu}_p^N(y) = \chi_S(x) \int_{\mathbb{Z}^N_{m+1}} \prod_{i=1}^{N} \xi^{y(i)S(i)} \tilde{\mu}_p^N(y)
$$

However, $\tilde{\mu}_p^N$ is a Cartesian product of $N$ copies of $\tilde{\mu}_p$ so (3) can be written

$$
(4) \quad \tilde{N}_p \chi_S(x) = \chi_S(x) \prod_{i=1}^{N} \int_{\mathbb{Z}_{m+1}} \xi^{yS(i)} \tilde{\mu}_p(y)
$$

If $S(i) = 0$, the integral in (4) evaluates to 1 because the integrand is 1 and $\tilde{\mu}_p$ is a probability measure.
If \( S(i) \neq 0 \),

\[
\int_{\mathbb{Z}^{m+1}} \xi^{S(i)} \tilde{\mu}_p(y) = \tilde{\mu}_p(\{0\}) + \sum_{y=1}^{m}(\xi^{S(i)})^y \tilde{\mu}_p(\{y\}) = (1 - p) + \frac{p}{m} \sum_{y=1}^{m}(\xi^{S(i)})^y = 1 - p - \frac{p}{m} = 1 - \frac{p}{c_m}
\]

Splitting the factors in (4) into those corresponding to 0 and non-0 indices of \( S \), we see

\[
\tilde{\mathcal{N}}_p \chi_S(x) = \chi_S(x) \left[ \prod_{i: S(i) \neq 0} (1 - p/c_m) \right] = \chi_S(x) (1 - p/c_m)^{|S|}
\]

Consequently, with \( p(t) = c_m(1 - e^{-t}) \) and

\[
\mu_t^N(y) := \tilde{\mu}_p(t)(y) \qquad N_t := \tilde{\mathcal{N}}_p(t)
\]

(so \( \tilde{\mathcal{N}}_p = \mathcal{N}_{-\ln(1-p/c_m)} \)), we have

\[
N_t \chi_S(x) = e^{-t|S|} \chi_S(x),
\]

and thus the family of operators \( \{ N_t : t > 0 \} \) form a semigroup, and the maximal operator \( \mathcal{N}_* \) given by

\[
\mathcal{N}_* f := \sup_T \left| \frac{1}{T} \int_0^T N_t f \, dt \right|
\]

is of weak-type \( 1 - 1 \), independent of dimension ([4, Lemma VIII.7.6, pp. 690-691]).

For the sake of comparison with \( M_S \), it will be convenient to reparametrize the semigroup maximal function in terms of \( p \).

**Proposition 3.3.** The maximal function

\[
\tilde{\mathcal{N}}_* f := \sup_{p \leq c_m} \left| \frac{1}{T} \int_0^T \tilde{\mathcal{N}}_p f \, dp \right|
\]

is bounded pointwise by \( \mathcal{N}_* f \). In particular \( \tilde{\mathcal{N}}_* \) is of weak-type \( 1 - 1 \) independent of dimension.

**Proof.** By direct calculation, one verifies – analogous to the proof of [3, Lemma 9] – that the measure

\[
\nu_p := \left\{ \frac{c_m}{P} T e^{-T} 1_{(0, -\ln[1-P/c_m])} \right\} dT + \left\{ \left( \frac{c_m}{P} - 1 \right) \left( -\ln \left[ 1 - \frac{P}{c_m} \right] \right) \right\} \delta_{-\ln[1-P/c_m]}
\]

is bounded by \( \nu_1 \).
has total mass 1. Moreover, noting that the bracketed expression in (5) below equals $\frac{1}{p}1_{p \leq p}$, further computation reveals that

$$
\left| \frac{1}{P} \int_0^P \tilde{\mu}_p^N \, dp \right| = \left| f * \left[ \frac{1}{P} \int_0^P \tilde{\mu}_p^N \, dp \right] \right|
= \left| f * \left[ \int_0^\infty \left( \frac{1}{T} \int_0^T \tilde{\mu}_t^N \, dt \right) \, d\nu_T(T) \right] \right|
= \left| \int_0^\infty \left( \frac{1}{T} \int_0^T N_t f \, dt \right) \, d\nu_T(T) \right|
\leq \int_0^\infty N_s f \, d\nu_T(T)
= N_s f,
$$

from which the result follows. \(\square\)

Finally, we will compare the smooth maximal function with the reparametrized “semigroup” maximal function:

**Proposition 3.4.** For any nonnegative function $f$ we have the pointwise inequality

$$M_S f \lesssim \tilde{N}_s f.
$$

In particular, $M_S$ is of weak-type $1 - 1$, independent of dimension.

**Proof.** We may express

$$
\tilde{\mu}_p^N = \sum_{k=0}^N \frac{(p/m)^k (1 - p)^{N-k} 1_{\mathbb{S}_k}}{m^k \binom{N}{k}}
= \sum_{k=0}^N \binom{N}{k} (p)^k (1 - p)^{N-k} \frac{1}{m^k \binom{N}{k}} 1_{\mathbb{S}_k}
= \sum_{k=0}^N B(N, p, k) \sigma_k
$$

where $B(N, p, k) := \binom{N}{k} (p)^k (1 - p)^{N-k}$. 

By Lemma 3.6 below (similar to [3]), for each $K \leq N$ we can choose $P(K) \in (0, c_m]$ that satisfies the favorable pointwise comparison

$$\frac{1}{K+1} \lesssim \frac{1}{P(K)} \int_0^{P(K)} B(N, p, k) \, dp$$

for each $k \leq K$. Thus

$$\sum_{k \leq K} \frac{1}{K+1} \sigma_k \lesssim \sum_{k \leq N} \left( \frac{1}{P(K)} \int_0^{P(K)} B(N, p, k) \, dp \right) \sigma_k$$

(6)

$$\frac{1}{K+1} \sum_{k \leq K} \sigma_k \lesssim \frac{1}{P(K)} \int_0^{P(K)} \tilde{\mu}_p \, dp$$

Noting that all terms in (6) are nonnegative, we observe that for any nonnegative function $f$, we have the pointwise comparison

$$\frac{1}{K+1} \sum_{k \leq K} \sigma_k \ast f \lesssim \left( \frac{1}{P(K)} \int_0^{P(K)} \tilde{\mu}_p \, dp \right) \ast f$$

$$= \left( \frac{1}{P(K)} \int_0^{P(K)} \tilde{N}_p f \, dp \right) \lesssim \tilde{N}_* f$$

where the first equality above is justified as in Proposition 3.3. Taking a supremum over all $K \leq c_mN$ provides the desired pointwise inequality. To prove the weak-type bound, first observe that because $M_S$ is a supremum over convolution operators with nonnegative kernels, we immediately have the pointwise inequality $M_S|f| \geq M_S f$ for an arbitrary function $f$. Thus for any $\|f\|_{L^1(\mathbb{R}_{m+1}^N)} = 1$

$$\|M_S f\|_{L^1(\mathbb{R}_{m+1}^N)} \leq \|M_S|f|\|_{L^1(\mathbb{R}_{m+1}^N)}$$

(7)

$$\lesssim \|\tilde{N}_*|f|\|_{L^1(\mathbb{R}_{m+1}^N)}$$

Simply because $f$ and $|f|$ share $L^1$ norms (i.e. 1), (7) is bounded by the weak 1–1 operator norm of $\tilde{N}_*$. Taking a supremum over all $L^1$ normalized $f$ then proves that $M_S$ inherits the dimension independent weak-type 1–1 bound from $\tilde{N}_*$.

Applying the Marcinkiewicz interpolation theorem with the trivial $L^\infty$ bound yields the desired $L^p$ bounds.

**Corollary 3.5.** The operator $M_S$ satisfies $L^p$ bounds for all $p > 1$ that depend on $p$ and $m$ but are independent of dimension.

All that remains in the section is to prove the key Lemma 3.6

**Lemma 3.6.** For each $0 \leq k \leq K \leq c_mN$, there exists $P(K) \in (0, c_m]$ (independent of $N$ and $k$) such that

$$\frac{1}{K+1} \lesssim \frac{1}{P(K)} \int_0^{P(K)} B(N, p, k) \, dp$$
Proof. We choose the value $P(K)$ as follows:

$$P(K) = \begin{cases} 
\frac{1}{N} & \text{if } K = 0 \\
\min\left(\frac{K+\sqrt{K}}{N}, c_m\right) & \text{if } K > 0.
\end{cases}$$

We observe that whether $P(K)$ is $\frac{1}{N}$, $\frac{K+\sqrt{K}}{N}$, or $c_m$, a quick case-by-case examination reveals $\frac{P(K)}{N+1} \approx \frac{1}{N}$. Thus it suffices to prove

$$\frac{1}{N} \lesssim \int_0^{P(K)} B(N, p, k) \, dp \tag{8}$$

independent of $N$ and $k$. Also note that if $k = 0$ we have

$$\int_0^{P(K)} B(N, p, 0) \, dp \geq \int_0^{1/N} (1 - p)^N \, dp \gtrsim \frac{1}{N}$$

so we can assume $1 \leq k$ (and recall $k \leq K \leq c_m N$). We estimate the right side of (8) from below by

$$\int_{k/N - \sqrt{k}/2N}^{k/N} B(N, p, k) \, dp$$

From there it will suffice to show that for all $p \in [k/N - \sqrt{k}/2N, k/N]$ the inequality $B(N, p, k) \gtrsim 1/\sqrt{k}$ holds. To prove this, we first observe that by a direct application of Stirling’s formula, $B(N, k/N, k) \gtrsim 1/\sqrt{k}$. Then we show that $B(N, p, k)$ maintains this bound for all

$$\frac{k}{N} - \frac{\sqrt{k}}{2N} \leq p \leq \frac{k}{N}$$

as follows:

$$\left| \ln \frac{B(N, k/N, k)}{B(N, p, k)} \right| = \left| \int_p^{k/N} \partial_t \ln B(N, t, k) \, dt \right|$$

$$= \left| \int_p^{k/N} \frac{k - Nt}{t(1-t)} \, dt \right|$$

$$\leq \left( \frac{k}{N} - p \right) \left( \max_{t \in [p,k/N]} \frac{1}{t(1-t)} \right) \left( \max_{t \in [p,k/N]} k - Nt \right)$$

$$\leq \frac{\sqrt{k} N}{k} \frac{1}{\sqrt{k}}$$

Exponentiating, it follows that

$$\frac{B(N, k/N, k)}{B(N, p, k)} \approx 1 \implies B(N, p, k) \gtrsim \frac{1}{\sqrt{k}}.$$

Note that we are limited by the restriction $0 \leq P(K) \leq c_m$, which poses a problem when we are considering “distant” spherical means. If we have $K = N$
then we need $P(N)$ such that

$$\frac{1}{N+1} \lesssim \frac{1}{P(N)} \int_0^{P(N)} B(N, p, k) \, dp$$

$$= \frac{1}{P(N)} \int_0^{P(N)} p^N \, dp$$

$$= \frac{1}{P(N)^{N+1}}$$

$$= \frac{P(N)^N}{N+1}$$

which is impossible given the constraint $P(K) \leq c_m$. A similar problem arises for the other “distant” means, which is why we have considered the restricted spherical maximal function – and the maximal function over balls in the first place.

4. The Comparisons – Stein’s Method

As announced, in this section we adapt the Nevo-Stein spectral machinery to our present context. We prepare to do so in our first subsection:

4.1. Krawtchouk Preliminaries. It is helpful to define the convolution operators:

$$P^k f(x) := f * \sigma_k.$$  

Their discrete derivatives

$$\triangle^0 P^k := P^k,$$

$$\triangle^1 P^k := P^k - P^{k-1},$$

$$\vdots$$

$$\triangle^t P^k := \triangle (\triangle^{t-1} P^k) = \sum_{j \leq t} (-1)^j \binom{t}{j} P^{k-j},$$

$$\vdots$$

and their associated (radial) multipliers

$$\mathcal{F} (\triangle^t P^k) (|S|)$$

will be central to our study.

First, when $|S| = r$ we have [2] §5.3

$$\mathcal{F} P^k (r) = \sum_{j=\max(0,r+k-N)}^{\min(r,k)} (-1)^j \binom{N}{k}^{-1} \binom{r}{j} \binom{N-r}{k-j} m^{-j} =: \kappa^N_r (k),$$

the $k$th (normalized) Krawtchouk polynomial in $\mathbb{Z}^N_{m+1}$. By expanding the binomial coefficients in the expression above, it is easy to see that $\kappa^N_r (k) = \kappa^N_k (r)$ for all $r, k, N$.

The Krawtchouk polynomials have the following useful difference relation:
Lemma 4.1. In $\mathbb{Z}^N_{m+1}$

$$(\mathcal{F}\triangle P^k)(r) = \kappa^N_k(r) - \kappa^N_{k-1}(r) = \kappa^N_r(k) - \kappa^N_r(k-1) = \left(\frac{1}{c_m}\right)^{\binom{N-1}{r-1}}\binom{N}{r}^{-1}\kappa^N_{r-1}(r-1).$$

Proof. Because dimension is not a constant in the lemma, we adopt the notation

$$\mathbb{S}^N_r = \{x \in \mathbb{Z}^N_{m+1} : |x| = r\}$$

$$\sigma^N_r = \frac{1}{|\mathbb{S}^N_r|} \mathbb{1}_{\mathbb{S}^N_r}$$

Letting $y^N_j = (1, \ldots, 1, 0, \ldots, 0)$ with $j$ 1’s and $N-j$ 0’s, we exploit the radiality of $\mathcal{F}\sigma^N_j$ to see

$$\kappa^N_r(k) - \kappa^N_r(k-1) = \left< \sigma^N_r, \xi^x y^N_k - \xi^x y^N_{k-1} \right>$$

$$= \frac{1}{|\mathbb{S}^N_r|} \sum_{x \in \mathbb{S}^N_r} (\xi^{x_1 + \cdots + x_{k-1}})(\xi^{x_k} - 1)$$

$$= \frac{1}{|\mathbb{S}^N_r|} \sum_{x \in \mathbb{S}^N_{r-1}} (\xi^{x_1 + \cdots + x_{k-1}}) \sum_{x_k=1}^m (\xi^{x_k} - 1) \tag{10}$$

The last equality follows from the observation that any summand corresponding to an $x \in \mathbb{S}^N_r$ such that $x_k = 0$ is 0. As in the proof of Lemma 4.2 we have

$$\sum_{x_k=1}^m (\xi^{x_k} - 1) = -(m+1)$$

Rearranging (10) then yields

$$-(m+1) \frac{|\mathbb{S}^N_{r-1}|}{|\mathbb{S}^N_r|} \left< \frac{1}{|\mathbb{S}^N_{r-1}|} \mathbb{1}_{\mathbb{S}^N_{r-1}}, \xi^{x_1 + \cdots + x_{k-1}} \right> = -(m+1) \frac{|\mathbb{S}^N_{r-1}|}{|\mathbb{S}^N_r|} \left< \sigma^N_{r-1}, \xi^x y^{N-1}_{k-1} \right>$$

$$= -\frac{\binom{N-1}{r-1}}{c_m} \frac{1}{\binom{N}{r}} \kappa^{N-1}_{r-1}(k-1)$$

$$= -\frac{\binom{N-1}{r-1}}{c_m} \kappa^{N-1}_{k-1}(r-1) \square$$

Applying Lemma 4.1 $t$ times yields a useful general expression for higher orders differences.

Corollary 4.2. For any $t \leq r, k$

$$(\mathcal{F}\triangle^t P^k)(r) = \left(-\frac{1}{c_m}\right)^t \binom{N-t}{r} \kappa^{N-t}_{r-t}(k-t).$$

Now we define

$$\partial^0\kappa^N_r(k) = \kappa^N_r(k),$$

$$\partial^1\kappa^N_r(k) = \partial^1\kappa^N_r(k) := \kappa^N_r(k) - \kappa^N_r(k-1),$$
and $\partial^t \kappa^N_r (k) := \partial(\partial^{t-1}\kappa^N_r (k))$, provided $t \leq \min\{r, k\}$. Otherwise we set $\partial^t \kappa^N_r (k) = 0$. Using this notation, we may in particular express

$$(\mathcal{F}\Delta^t P^k)(r) \kappa^N_r (k) = \left( - \frac{1}{c_m} \right)^t \frac{t^{N-t}}{r^N} \kappa^N_{r-t} (k-t).$$

The following proposition, whose proof we defer to §5 below, is the key quantitative ingredient needed to anchor the argument:

**Proposition 4.3.** There exists a constant $d$ (dependent only on $m$) such that for all $r, k, N$ we have

$$|\kappa^N_r (r)| \leq e^{-d \frac{rk}{N}}.$$

### 4.2. A Review of Nevo-Stein

In this subsection, we shall regard $N$ as fixed, and (quickly) review the comparison argument of [7] as it relates to our current setting. For a fuller treatment, we refer the reader to [6].

In the last subsection, we introduced the convolution operators $\{P^k\}$. Because they are self-adjoint, positive, norm-one $L^1$- and $L^\infty$-contractions, we may use the following outline from [7], [6]:

With $\lambda = \alpha + i\beta \in \mathbb{C}$, we recall the complex binomial coefficients

$$A^\lambda_n = \frac{(\lambda + 1)(\lambda + 2)\ldots(\lambda + n)}{n!}, \quad A^\lambda_0 := 1, A^\lambda_{-n} := 0.$$

We define the Cesaro means

$$S^\lambda_n f(x) := \sum_{k \leq n} A^\lambda_{n-k} P^k f(x), \quad \lambda \in \mathbb{C},$$

for $n \leq c_m N$ and remark that in the special case that $\lambda = -t - 1$ is a negative integer, we have

$$S^{-t-1}_n f(x) = \Delta^t P^k f(x)$$

by a simple computation using [9]. In particular, because we are only working with $S^\lambda_n$ for $n \leq c_m N$, Corollary [4.2] shows that whenever $t > c_m N$ we have $S^{-t-1}_n f \equiv 0$.

The maximal functions associated to these higher Cesaro means are

$$S^\lambda f(x) := \max_{0 \leq n \leq c_m N} \left| \frac{S^\lambda_n f(x)}{n+1} \right|.$$

The following lemmas are finitary adaptations of the results in [6]; we emphasize that the formal nature of the arguments in [6] allows them to be applied in much greater generality than our current setting.

**Lemma 4.4** ([6], Proof of Lemma 4, pp. 144-145). For $\alpha > 0, \beta \in \mathbb{R}$, there exist positive constants $C_\alpha$ so that

$$S^{-t}_{*} f \leq C_\alpha \epsilon^{3\beta^2} S^0_{*} |f|$$

holds pointwise.

**Lemma 4.5** ([6], Proof of Lemma 5, pp. 145-146). For each nonnegative integer $t$ and each real $\beta$, there exist positive constants $C_t$ so that

$$S^{-t-1}_{*} f \leq C_t \epsilon^{3\beta^2} \left( S^{-1}_{*} f + S^{-t}_{*} f + \cdots + S^{-1}_{*} f \right)$$

holds pointwise.
Lemma 4.6 ([6], Proof of Lemma 5, p. 147). Define

\[ R_t f(x)^2 := \sum_{0 \leq k \leq c_m N} (k + 1)^{2t-1}|S_k^{t-1}f(x)|^2 \]

for any nonnegative integer \( t \). Then there exists a positive constant \( c_t \) so that

\[ S_{r-t}^t f \leq c_t R_t f + 2S_{r-t}^1 f \]

holds pointwise.

Before the proof of Proposition 4.7, we show that it implies Theorem 1.3.

Proposition 4.7. With

\[ R_t f(x)^2 := \sum_{0 \leq k \leq c_m N} (k + 1)^{2t-1}|S_k^{t-1}f(x)|^2 \]

for any nonnegative integer \( t \), there exist constants \( C_{t,m} \) independent of \( N \) so that

\[ \|R_t f\|_{L^2(Z_{m+1}^N)} \leq C_{t,m} \|f\|_{L^2(Z_{m+1}^N)} \]

for all \( N \).

Proof of Theorem 1.3 Assuming Proposition 4.7. First we note that \( S_0^0 \) this is the smooth spherical maximal operator \( M_S \) from Proposition 3.1, while \( S_{r-1}^0 \) is the restricted spherical maximal operator \( M \) from \([2]\) at the beginning of Section 2. Thus, as shown in Section 2, Theorem 1.3 will follow from dimension independent \( L^p \) bounds on \( S_{r-1}^0 \).

By Corollary 3.5, we know that there exist constants \( \{A_{p,m}\}, p > 1 \), so that for each \( N \),

\[ \|S_0^0 f\|_{L^p(Z_{m+1}^N)} \leq A_{p,m} \|f\|_{L^p(Z_{m+1}^N)} \]

where the operators \( \{S_0^0\} \) are \( N \)-dependent, but the bounds are not.

By Lemma 4.4, for each \( \alpha > 0, \beta \in \mathbb{R} \), we therefore have the bound

\[ \|S_0^\alpha + i\beta f\|_{L^p(Z_{m+1}^N)} \leq e^{2\beta^2} A_{p,m} \|f\|_{L^p(Z_{m+1}^N)} \]

independent of \( N \).

By Proposition 4.7 Lemma 4.6 and induction on \( t \), we see that there exist constants \( \{B_{2,m}^t\}, t \geq 1 \) so that for all \( N \),

\[ \|S_{r-t}^t f\|_{L^2(Z_{m+1}^N)} \leq B_{2,m}^t \|f\|_{L^2(Z_{m+1}^N)} \]

By Lemma 4.5 this means that for all \( N \), there exist constants \( \{D_{2,m}^t\} \) so that

\[ \|S_{r-t+i\beta}^t f\|_{L^2(Z_{m+1}^N)} \leq e^{3\beta^2} D_{2,m}^t \|f\|_{L^2(Z_{m+1}^N)} \]

for all \( N \).

The theorem then follows by linearizing the \( S_{r-1}^0 \)-supremum and applying the Stein interpolation theorem as in the conclusion of the proof of [6, Theorem 2, pp. 150-151]. \( \square \)

It remains only to prove Proposition 4.7 which we accomplish in the following subsection.
4.3. Proof of Proposition 4.7

Proof. By Plancherel, it is enough to show that there exists a constant, $C_{t,m}$, independent of $r$ and $N$, so that for all $r$

\[
\sum_{k=0}^{c_mN} (k + 1)^{2t-1} |\mathcal{F} \Delta^t P^k|^2 (r) \leq C_{t,m}
\]

or equivalently

\[
\sum_{k=0}^{c_mN} (k + 1)^{2t-1} |\partial^t \kappa_{r,N}^t (k)|^2 \leq C_{t,m}.
\]

(11)

Ignoring a finite set of cases for fixed $t$, we can assume that $N > 2t$. Vital to the proof is the difference relation

\[
\partial^t \kappa_{r,N}^t (k) = \left( \frac{1}{c_m} \right)^{\frac{t}{r}} \left( \frac{N-t}{r} \right) \kappa_{r-t}^{N-t} (k - t)
\]

from Corollary 4.2 and the upper bound

\[
\kappa_{r-t}^{N-t} (k - t) \leq e^{-d \frac{r}{N-t}(k-t)} 1_{[0, \min(r,k)]}
\]

from Proposition 4.3.

We first dispose of the boundary case $r = t$, in which case

\[
\kappa_{r-t}^{N-t} (k - t) = \kappa_0^{N-t} (k - t) = 1.
\]

In this instance, we estimate

\[
\sum_{k=0}^{c_mN} (k + 1)^{2t-1} |\partial^t \kappa_{r,N}^t (k)|^2 \leq \sum_{k=1}^{N} k^{2t-1} \left( \frac{(c_m)^t}{N} \right) \left( \frac{N-t}{r} \right) \leq \left( \frac{(N/c_m)^t}{r} \right) 2 \\lesssim t 1,
\]

simply bounding $\left( \frac{N}{r} \right)$ from below by $(N-t)/t \approx t N^t$ because $N > 2t$.

Henceforth, we may assume $r > t$. We note that the summands in (11) are identically zero for all $k < t$. Therefore

\[
\sum_{k=0}^{c_mN} (k + 1)^{2t-1} |\partial^t \kappa_{r,N}^t (k)|^2 \leq \sum_{k=t}^{\infty} (k + 1)^{2t-1} |\partial^t \kappa_{r,N}^t (k)|^2
\]

\[
\lesssim t \sum_{k=t}^{\infty} (k + 1)^{2t-1} \left[ \frac{(N-t)}{r} \right] e^{-d \frac{r}{N-t}(k-t)}
\]

\[
= \left( \frac{(N-t)}{r} \right)^2 \sum_{k=t}^{\infty} (k + 1)^{2t-1} e^{-2d \frac{r}{N-t}(k-t)}
\]

\[
= \left( \frac{(N-t)}{r} \right)^2 \sum_{k=0}^{\infty} (k + (t+1))^{2t-1} e^{-2d \frac{r}{N-t}k}.
\]

We record the following easy lemma concerning infinite series:
Lemma 4.8. For any positive integer \( n \), there exists a constant \( A_n \) such that for all \( |s| < 1 \),

\[
\sum_{k=0}^{\infty} k^n s^k \leq \frac{A_n}{(1 - s)^{n+1}}
\]

Proof. Define the operator

\[
L f(s) := s \frac{df}{ds}(s),
\]

and note that

\[
\sum_{k=0}^{\infty} k^n s^k = L^n \frac{1}{1 - s}
\]

Induction on \( n \) shows that the right side of this equation can be expressed as

\[
\frac{s^n + p_n(s)}{(1 - s)^{n+1}},
\]

where \( p_n(s) := \sum_{j<n} a_j^n s^j \) is a polynomial of degree \( n - 1 \).

In particular, for \( s < 1 \), we may bound

\[
\left| \frac{s^n + p_n(s)}{(1 - s)^{n+1}} \right| \leq \frac{A_n}{(1 - s)^{n+1}},
\]

where we let \( A_n := 1 + \sum_{j<n} |a_j^n| \). \( \square \)

Now, following the lead (and notation) of \([3, \S 4]\), we set

\[
\alpha = \alpha(r) := 2d \frac{r - t}{N - t};
\]

possibly after reducing \( d \), we may assume that \( d < \frac{1}{2} \), so that

\[
|\alpha(r)| \leq 2d \frac{r - t}{N - t} \leq 2d < 1
\]

for all \( r \).

In the following estimate we use the notation \( A_n \) from Lemma 4.8 and \( A^*_t := \max_{n \leq t} A_n \):

\[
\sum_{k=0}^{\infty} (k + (t + 1))^{2t-1} e^{-2d \frac{r - t}{N - t} k} = \sum_{k=0}^{\infty} (k + (t + 1))^{2t-1} e^{-\alpha k}
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{n=0}^{2t-1} \binom{2t-1}{n} k^n (t + 1)^{2t-1-n} \right) e^{-\alpha k}
\]

\[
\leq \sum_{k=0}^{2t-1} \left( \sum_{n=0}^{2t-1} (2t)^{2t} k^n (t + 1)^{2t} \right) e^{-\alpha k}
\]

\[
\leq t \sum_{n=0}^{2t-1} \left( \sum_{k=0}^{\infty} k^n e^{-\alpha k} \right)
\]

\[
\leq \sum_{n=0}^{2t-1} \frac{A_n}{(1 - e^{-\alpha})^{n+1}}
\]

\[
\leq \sum_{n=0}^{2t-1} \frac{A_n}{\alpha^{n+1}}
\]

\[
\leq t \alpha^{-2t}
\]

\[
\leq 2t A^*_t \alpha^{-2t}
\]

\[
\leq \alpha^{-2t},
\]
Note that we used the mean value theorem in passing to the third-to-last line and the fact that $\alpha < 1$ in passing to the second-to-last line.

The upshot is that we may bound

$$\sum_{k=0}^{\infty} (k + (t + 1))^{2t-1} e^{-2\frac{dr}{r-t}t} \lesssim_t \left( \frac{N-t}{r-t} \right)^{2t},$$

so that we have

$$\sum_{k=0}^{c_m N} (k + 1)^{2t-1} |\partial^t \kappa^N_r(k)|^2 \lesssim_t \left[ \left( \frac{(N-t)^t}{r-t} \right) \left( \frac{N-t}{r-t} \right)^t \right]^2.$$

All that remains is to show that for any fixed $t < r$ we have

$$\left( \frac{(N-t)!}{N!} \right) \left( \frac{N-t}{r-t} \right)^t = \left( \frac{(N-t)!}{N!(r-t)!} \right) \left( \frac{N-t}{r-t} \right)^t \lesssim_t 1.$$  \hspace{1cm} (12)

To this end we use the assumption $N > 2t$ to estimate

$$\frac{(N-t)!}{N!} \approx_t N^{-t} \text{ and } (N-t)^t \approx_t N^t,$$

thus bounding the left hand side of (12) (up to a constant depending on $t$) by

$$\left( \frac{r!}{(r-t)!} \right) \left( \frac{1}{r-t} \right)^t$$

If $r \leq 2t$ we bound this crudely by $r! \leq (2t)! \lesssim_t 1$. If $r > 2t$, we use the estimates

$$\frac{r!}{(r-t)!} \approx_t r^t \approx_t (r-t)^t$$

to complete the proof.  \hspace{1cm} \square

5. PROOF OF PROPOSITION 4.3

First we introduce the notation

$$a_j = \binom{N}{k}^{-1} \binom{r}{j} \binom{N-r}{k-j} m^{-j}$$

for the magnitude of the $j$th summand in the full expression for $\kappa^N_k(r)$, which we recall is given by

$$\kappa^N_k(r) = \sum_{j=\max(0,r+k-N)}^{\min(r,k)} (-1)^j \binom{N}{k}^{-1} \binom{r}{j} \binom{N-r}{k-j} m^{-j}. \hspace{1cm} (13)$$

We restate the proposition for the reader’s convenience:

**Proposition 4.3** (restatement). There exists a constant $d$ (dependent only on $m$) such that for all $r, k, N$ we have

$$|\kappa^N_k(r)| \leq e^{-d(rk/N)}.$$

By the symmetry of the Krawtchouk polynomials in $r$ and $k$, without loss of generality $r \leq k$ so the sum (13) will terminate at $r$. The thrust of the proof is to show that the largest summand magnitude in (13) decays exponentially in $rk/N$. So Lemma 5.1 below will prove the proposition.
For the remainder of the section we define
$$\ell := \max(0, r + k - N)$$
to be the lowest index of summation. Also we define $n$ to be the lowest index in the region of summation, i.e. $[\ell, r] \cap \mathbb{Z}$, such that $a_n$ is a maximal summand magnitude. In other words, $n \in \mathbb{Z}$ is minimal subject to the constraints that $\ell \leq n \leq r$ and $a_j \leq a_n$ for all $j \in \mathbb{Z}$ in that range.

**Lemma 5.1.** Each Krawtchouk polynomial is dominated by its maximal summand magnitude. More concretely, $|\kappa_N^k(r)| \leq a_n$.

**Proof.** We begin by noting that the ratio $a_{j+1}/a_j$ is given by
$$R(j) := \frac{(r - j)(k - j)}{m(j + 1)(j + N - r - k + 1)}.$$
We view $R$ as a function on the real interval $(\ell - 1, r]$ rather than restricting it to the integers. Its key properties for this lemma are
(i) $R(j) \geq 0$,
(ii) $R(j)$ is continuously (strictly) decreasing,
(iii) $R(j)$ approaches $+\infty$ as $j$ approaches $\ell - 1$, and
(iv) $R(r) = 0$.
Property (i) above follows from the fact that all factors in $R(j)$ are nonnegative. Property (ii) is a result of the factors in the numerator diminishing in magnitude and the factors in the denominator growing. Property (iii) follows from property (i) and the fact that $R$ has a pole at $\ell - 1$ while property (iv) is trivial.

By the intermediate value theorem, properties (ii), (iii), and (iv) imply that there exists some $J \in (\ell - 1, r]$ such that $R(J) = 1$. Applying property (ii), we see that
$$j \leq J \implies R(j) \geq 1 \implies a_{j+1} \geq a_j$$
and
$$j \geq J \implies R(j) \leq 1 \implies a_{j+1} \leq a_j.$$ (14)
In particular, this means that $a_{[J]}$ is a maximal summand magnitude. Note that because $R(j)$ is strictly decreasing, $R([J] - 1) > 1$ so $a_{[J]} > a_{[J] - 1}$. Thus $[J]$ must minimal among indices of maximal summand magnitudes, i.e. $n = [J]$.

Finally, we can bound $|\kappa_N^k(r)|$ by splitting it into two monotonic alternating sums, namely
$$\kappa_N^k(r) = \left( \sum_{j=0}^{n} (-1)^j a_j \right) + \left( \sum_{j=n+1}^{r} (-1)^j a_j \right)$$
where the monotonicity is a direct consequence of (14). Note that the second sum above may be empty, but we can ignore this by defining $a_{r+1}$ to be 0.

Because they are monotonic and alternating, the sums are bounded between 0 and their respective largest magnitude summands, namely $\pm a_n$ and $\mp a_{n+1}$. Because these bounds have opposite signs, we can bound $|\kappa_N^k(r)|$ by the maximum of their magnitudes, namely $a_n$. □

To bound $a_n$ we first bound $n$ from below. This technical lemma is largely comprised of algebraic and calculus manipulations.

**Lemma 5.2.** If $n > 0$ and $rk \geq 2Nm$ then $n \geq rk/N$. 
For the sake of clarity we point out that the hypothesis \( rk \geq 2Nm \) proves \( n > 0 \) a posteriori, however it is more efficient to handle the \( n = 0 \) case separately.

**Proof.** We recall from Lemma 5.1 that the ratio \( a_{j+1}/a_j \) is given by

\[
R(j) := \frac{(r-j)(k-j)}{m(j+1)(j+1+N-r-k)}.
\]

To solve the equation \( R(j) = 1 \), we apply the quadratic formula to the quadratic

\[
[m(j+1)(j+1+N-r-k)] - [(r-j)(k-j)] = (m-1)^2 + [2m+Nm-(m-1)(r+k)]j + [m+Nm-rk-km-rk].
\]

This reveals that \( R \) can equal 1 only at the values

\[
j_{\pm} := C \pm \sqrt{C^2 + A}
\]

Where

\[
(15) \quad A := -\frac{4(m-1)(m+Nm-rm-km-rk)}{4(m-1)^2} = \left( \frac{rk-Nm}{m-1} + \frac{rm+km-m}{m-1} \right)
\]

\[
(16) \quad C := -\frac{2m+Nm-(m-1)(r+k)}{2(m-1)} = \left( \frac{r+k}{2} - \frac{m}{m-1} - \frac{Nm}{2(m-1)} \right).
\]

We will show

(I) \( A > 0 \),

(II) \( A \gtrsim rk \), and

(III) \( \sqrt{C^2 + A} - |C| \gtrsim rk/N \).

Item (I) above implies that \( j_- < 0 < j_+ \). We saw in the proof of Lemma 5.1 there exists \( J \in (\ell-1,r) \) such that \( R(J) = 1 \) and \( n = \lceil J \rceil \). It follows that \( J = j_{\pm} \) and, by the assumption \( n > 0 \), that \( J > 0 \). Therefore \( J = j_+ \) simply by default.

Item (II) is the key element in the proof of item (III). Item (III) shows that

\[
n \geq J \gtrsim rk/N
\]

simply because, regardless of the sign of \( C \),

\[
J = C + \sqrt{C^2 + A} \geq \sqrt{C^2 + A} - |C| \gtrsim rk/N.
\]

Therefore all that remains in the lemma is to justify (I), (II), and (III).

**Justification of (I) and (II):**
In light of fact that \( r \) and \( k \) are positive integers, simple arithmetic shows that

\[
\frac{rm+km-m}{m-1} > 0.
\]

and, because \( rk \geq 2Nm \),

\[
\frac{rk-Nm}{m-1} \geq \frac{1}{2(m-1)} \cdot rk.
\]

Adding these two inequalities, the last expression in (15) shows \( A > 0 \) and \( A \gtrsim rk \).

**Justification of (III):**
We split into two cases.
Case 1: If $A > 3C^2$, then
\[
\sqrt{C^2 + A - |C|} \geq A^{1/2} - (A/3)^{1/2} \gtrsim A^{1/2}
\]

We know that $A \gtrsim rk$ and $(rk)^{1/2} \leq N$ by item (II) and the bound $r, k \leq N$ respectively. It follows that
\[
A^{1/2} \gtrsim (rk)^{1/2} = \frac{rk}{(rk)^{1/2}} \geq \frac{rk}{N}.
\]

Case 2: If $A \leq 3C^2$, then we apply the mean value theorem to observe that
\[
\sqrt{C^2 + A - |C|} \geq \left( \inf_{x \in [C^2, C^2 + A]} \frac{1}{2x^{1/2}} \right) A \geq \frac{A}{2(4C^2)^{1/2}} \gtrsim \frac{rk}{N}.
\]

The final inequality follows from the bounds $A \gtrsim rk$ and $|C| \lesssim N$. The former is again item (II) and the latter comes from the fact that each term in the last expression of (16) is bounded in magnitude by 2 or $N$.

Thus, regardless of $A$, $\sqrt{C^2 + A - |C|} \gtrsim \frac{rk}{N}.$

□

From here Proposition 4.3 is fairly straightforward.

Proof of Proposition 4.3. First we use the combinatorial observation
\[
\binom{N}{k} = \left| \left\{ S \subset [N] : |S| = k \right\} \right| \geq \left| \left\{ S \subset [N] : |S| = k, |S \cap [r]| = j \right\} \right| = \binom{r}{j} \binom{N-r}{k-j}
\]

to justify the inequality
\[
a_j = \binom{N}{k}^{-1} \binom{r}{j} \binom{N-r}{k-j} m^{-j} \leq m^{-j},
\]
for all $j$ in the region of summation.

This bound is useful because in order to prove the proposition, it is sufficient to prove $a_n \leq e^{-d(rk/N)}$ by Lemma 5.1. To this end, we split into three cases.

Case 1: The hypotheses of Lemma 5.2 hold, i.e. $n > 0$ and $rk \geq 2mN$. Then there is an index $n$ and a (small) constant $\epsilon > 0$ such that $n \geq \frac{rk}{N}$.

Letting $d := \epsilon \ln m > 0$, this shows $a_n \leq e^{-d(rk/N)}$ by (17).

Case 2: $n > 0$ and $rk < 2mN$. Because $m \geq 2$ and $n \geq 1$ by assumption, (17) provides the inequality $a_n \leq 1/2$. Moreover, the assumption $rk < 2mN$ implies that
\[
e^{-2m} \leq e^{-rk/N}.
\]

Then we simply decrease $d$ to a small enough (positive) number that $1/2 \leq e^{-2md}$, to achieve the desired bound
\[
a_n \leq 1/2 \leq e^{-2md} \leq e^{-d(rk/N)}.
\]
Case 3: \( n = 0 \). We assume \( r > 0 \) because otherwise the entire proposition is trivial. Also, because

\[
\max(0, r + k - N) \leq n = 0,
\]

we know that \( r + k \leq N \) so the factors below are all well defined. Then we bound as follows:

\[
a_0 = \binom{N}{k}^{-1} \binom{N - r}{k} \\
= \prod_{j=0}^{k-1} \frac{N - r - j}{N - j} \\
\leq \left( \frac{N - r}{N} \right)^k \\
= \left( 1 - \frac{r}{N} \right)^{\frac{r}{N} k} \\
\leq e^{-\frac{r}{N} k}
\]

Because we are free to assume \( d \leq 1 \), this completes the proof of proposition 4.3. \( \square \)

6. Generalization to Arbitrary Direct Sums of Finite Cyclic Groups

We adopt the notation that for a given direct sum \( G \), \( B_x^G \) is the ball around the identity of radius \( r \) in the \( l^0 \) metric and the analogous notation \( S_x^G \) for spheres. For convenience we write \( C_m \) in place of \( c_m N_m \). Also we note that if we wish to find bounds for a product of Cartesian powers of non-consecutive cyclic groups, we are free to set \( N_m = 0 \) for any index we wish to skip (all estimates below will hold trivially in this case) as in Corollary 1.8. As with \( m \) in the earlier sections, we will generally suppress \( n \) dependence throughout the section with the understanding that all constants may depend on \( n \).

First we split \( M \) into a local and global piece as in §2 based on the cutoff radius \( \tilde{N} := \sum_{m=1}^{n} C_m \). That is,

\[
M_B(x) \leq M_L f(x) + M_G f(x) := \sup_{r \leq \tilde{N}} |\beta_r * f(x)| + \sup_{r \geq \tilde{N}} |\beta_r * f(x)|,
\]

**Lemma 6.1.** \( M_G \) satisfies a weak-type \( 1 - 1 \) bound and strong \( L^p \) bounds for \( p > 1 \) depending only on \( p \) and \( n \).

**Proof.** We bound \( M_G \) bluntly. If \( r \geq \tilde{N} \) then the ball \( B_r^A \) contains the rectangular set

\[
\oplus_{m=1}^{n} B_{c_m}^A.
\]
Therefore, using Lemma \ref{lem:maximal-in-ineq}

\[ |B^A_r| \geq \prod_{m=1}^{n} |B^A_{\tilde{C}_m}| \]
\[ \approx \prod_{m=1}^{n} (m+1)^{N_m} \]
\[ = |A|. \]

The implied constant is simply the product of the implied constants from Lemma \ref{lem:maximal-in-ineq}. Then if \( E := \{ M_G f(x) > \lambda \} \) is non-empty there exists a ball \( B \) in \( A \) with radius at least \( \tilde{N} \) (in particular \( |B| \approx |A| \)) such that
\[ |E| \leq |A| \approx |B| < \frac{1}{\lambda} \int_B |f(x)| \, dx. \]

Marcinkiewicz interpolation of this weak-type \( 1 \rightarrow 1 \) bound (which depends only on \( n \)) with the trivial \( L^\infty \) bound yields the desired \( L^p \) bounds for \( M_G \). \qed

**Lemma 6.2.** Consider the collection of all groups \( A' \) given by

\[ A' := \oplus_{m=1}^{n} \mathbb{Z}_{N_m+1} \]

where \( N_m = 0 \) for at least one index \( 1 \leq m \leq n \). Suppose that the ball maximal operator \( M_B^A \) satisfies \( L^p \) bounds for \( p > 1 \) dependent only on \( p \) and \( n \) (in particular the bounds are uniform on \( A' \) of this form). Then \( M_L^A \) satisfies \( L^p \) bounds for \( p > 1 \) dependent only on \( p \) and \( n \).

**Proof.** Define

\[ A'_m := \oplus_{1 \leq k \leq n} \mathbb{Z}_{N_k+1}. \]

For all \( m \), these sets have the properties that (i) their ball maximal operators satisfy the bounds in the statement of the lemma and (ii) \( A = A_m \oplus A'_m \). Note that for any \( r \leq \tilde{N} \) and any \( x \in B^A_r \), the pigeonhole principle forces \( x \) to have the property that at least one of its summands (the \( A_m \) summand for some \( m \)) lies within the critical radius \( C_m \) of \( 0_{A_m} \). Explicitly, \( x \) must belong to at least one set of the form
\[ R_m = R'_m := \begin{cases} B^A_{C_m} \oplus B^A_{r-C_m} & \text{if } r > C_m \\ B^A_{r} \oplus 0_{A'_m} & \text{if } r \leq C_m. \end{cases} \]

Heuristically, \( R_m \) is a subset of \( A \) for which the \( m \)th summand (viewing \( A \) as a sum of \( A_1, \ldots, A_n \)) is not very large and the remaining summands may be distributed arbitrarily subject to the constraint that they are collectively small enough for \( R_m \) to remain in the radius \( r \) ball. In the degenerate case where \( r \leq C_m \), the set \( R_m \) still serves the same purpose but must be modified directly to remain in \( B^A_r \).

This pigeonhole observation reveals a useful consequence of the reduction to \( M_L^A \). For all \( r \),
\[
\frac{1}{|B_r^A|} \int_{B_r^A} |f(x + x')| dx' \leq \frac{1}{|B_r^A|} \sum_{m=1}^n \int_{R_m} |f(x + x')| dx' \\
\leq \sum_{m=1}^n \frac{1}{|R_m|} \int_{R_m} |f(x + x')| dx'.
\]

(18)

The final inequality was simply a result of the containment \(R_m \subset B_r^A\). From here the strategy is to bound the summands in (18) pointwise (independent of \(r\)) by maximal functions satisfying \(L^p\) bounds depending only on \(p\) and \(n\), which proves the entire lemma.

Now we fix \(m\) and decompose \(R_m\) cylindrically based on a spherical partition of the \(m\)th summand, which we have forced to contain only small spheres (small meaning below the critical radius \(C_m\)). Explicitly, \(R_m\) can be viewed as the following disjoint union:

\[
\bigsqcup_{t=0}^{\min(C_{m,r})} S_A^m \oplus B^{A_m}_{\max(0,r-C_m)}.
\]

For brevity we designate the notation

\[
\Sigma_t = \Sigma_t^m := S_t^m \oplus B^{A_m}_{\max(0,r-C_m)}.
\]

Then \(|R_m|^{-1} R_m\) is a convex combination of the functions \(\{|\Sigma_t|^{-1} \Sigma_t\}_{t=1}^C_m\) so we have the bound

\[
\frac{1}{|R_m|} \int_{R_m} |f(x + x')| dx \leq \sup_t \frac{1}{|\Sigma_t|} \int_{\Sigma_t} |f(x + x')| dx.
\]

Notice that the average on the left hand side is nothing but a tensor product of a ball average in the \(A_m\) summand and a spherical average in the \(A_m\) summand. Therefore it is bounded pointwise by the image of \(f\) under the tensor product of the \(A_m\) ball maximal operator on the \(A_m\) summand and the \(A_m\) restricted spherical maximal operator on the \(A_m\) summand. In other words if

\[
M_1 = M_{S_m}^{A_m} \otimes \mathrm{id}^{A_m} \\
M_2 = \mathrm{id}^{A_m} \otimes M_{B_m}^{A_m}
\]

where \(M_{S_m}^{A_m}\) is the spherical maximal operator on \(A_m\) restricted to spheres of radius at most \(C_m\), then

\[
\frac{1}{|R_m|} \int_{R_m} |f(x + x')| dx \leq M_1 \circ M_2 f(x)
\]

From the proof of Theorem 1.3 on page 15 we know that \(M_{S_m}^{A_m}\) satisfies \(L^p\) bounds for any \(m > 1\). If \(m = 1\), the bound comes directly from Krause’s [5, Theorem 2.2]. Moreover, by the assumption in the lemma statement, \(M_{B_m}^{A_m}\) satisfies \(L^p\) bounds. Thus, as tensor products of \(L^p\) bounded operators, \(M_1\) and \(M_2\) (which do not depend on \(r\)) satisfy \(L^p\) bounds where all aforementioned bounds are for \(p > 1\) and depend only on \(p\) and \(n\). Then, as mentioned after (18), this proves the desired bound on \(M_{A,L}^m\). \[\square\]

Now we can quickly prove the desired bound on \(M_{B}^{A_m}\).
Proof of Theorem 1.7. By [5, Theorem 2.2] we have the desired bound for $M_B^A$ when $n = 1$. From here the proof is a simple inductive argument. Suppose we have the desired $M_B^A$ bound for a given $n$. Then Lemma [6.2] proves that $M_B^A$ satisfies the desired bounds in the $n+1$ case. By Lemma [6.1] this is enough to prove the desired bound for $M_B^A$ in the $n+1$ case. The entire result then follows by induction. 

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