This paper aims to solve numerically the linearized Korteweg-de Vries equation. We begin by deriving suitable boundary conditions then approximate them using finite difference method. The methodology of derivation, used in this paper, yields to Non-Standard Boundary Conditions (NSBC) that perfectly absorb wave reflections at the boundary. In addition, these NSBC are exact and local in time and space for non necessarily supported initial data and source terms. We finish with numerical examples that show the absorbing quality of these boundary conditions. Further comparisons are made using standard boundary conditions like, Dirichlet, Neumann and a variant of absorbing boundary conditions called discrete artificial ones.

1. Introduction. The Korteweg–de Vries (KdV) equation is a dispersive nonlinear partial differential equation that has attracted attention of both mathematicians and physicists for the study of solitary waves, an elegant kind of waves on the water surface [18]. The first observation and description of a solitary wave was done by John Scott Russel in 1834 [14], over forty years ago, Korteweg and de Vries described models for small amplitude waves in a narrow and shallow channel of water whose solutions are solitary waves in 1895 [9]. One has to say that KdV equation was a particular case of Boussinesq models introduced earlier in 1877 [3]. The KdV equation arises in many physical applications, like acoustic waves and magneto hydrodynamic waves [17]. It can also describe internal waves in the coastal ocean [12], or pressure pulse propagation in blood vessels [10].

Several techniques were used to approximate numerically the Linearized KdV (LKdV) equation with non-reflecting boundary conditions. For example, exact absorbing boundary conditions have been derived in [19], while in [2] discrete artificial boundary conditions were considered. We present, in this paper, a new method to construct exact non-reflecting boundary conditions that we call non-standard ones.

In this paper, we present a numerical approximation of LKdV equation in unbounded domain as well as derivation of NSBC. Hence, describing requires introduction of bounded domain, called artificial, and addition of boundary conditions. The resulting initial boundary value problem has to approximate, as close as possible, the LKdV equation restricted to the artificial domain. We claim that using standard boundary conditions yield some reflections at boundary and distort the
solution on the whole artificial domain. Thus, our main target is to minimize reflections by introducing NSBC. These NSBC were derived, recently, for BBM equation in [1] and have proved to be perfectly absorbing. Numerical examples are presented to evaluate the absorption property and to compare with the use of standard boundary conditions. We accord a special attention to solitary waves. The derivation of NSBC is obtained by introducing, not only one auxiliary function as presented for the BBM equation in a previous paper [1], but two auxiliary functions since KdV is a third order equation while BBM is only second order. An adapted numerical approximation is used based on Crank-Nicholson method for time discretization and finite difference for space derivatives.

This paper is organized as follows, next section 2 deals with the derivation of NSBC for LKdV equation. Then, section 3 presents an approximation to the obtained initial value problem. The third section 4 highlights the numerical study of the above approximation followed by section 5 that concerns several simulations using various initial data and source terms. A final section gives conclusions on obtained results.

2. Formulation of NSBC for the LKdV equation. We consider the following LKdV equation

\[
\begin{align*}
\partial_t u(t, x) + \alpha \partial_{xxx} u(t, x) + \beta \partial_x u(t, x) &= f(t, x) \quad \text{for} \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}, \\
\partial_t v(t, x) &= \partial_x u(t, x) \quad \text{for} \quad x \in \mathbb{R},
\end{align*}
\] (1)

where \(\alpha\) and \(\beta\) represent respectively dispersion and advection coefficients.

This section focuses on the derivation of boundary conditions to add to equation (1) while considered on an artificial domain \([0, T] \times [a, b] \subset \mathbb{R}_+^* \times \mathbb{R}\). For this regard we need two supplementary functions in order to obtain a coupled system that will preserve the dynamic of the starting equation (1). To this aim, let \(u\) be a solution of equation (1), we introduce two functions \(v\) and \(w\) such that,

\[
\begin{align*}
v(t, x) &= \partial_x u(t, x), \\
w(t, x) &= \partial_{xx} u(t, x),
\end{align*}
\] (2, 3)

differentiating, with respect to the space variable, the first line of equation (1) with equation (2) yield the following equation satisfied by \(v\),

\[
\partial_t v(t, x) + \alpha \partial_{xxx} v(t, x) + \beta \partial_x v(t, x) = \partial_x f(t, x),
\]

then differentiating one more time with respect to \(x\) and replacing by equation (3), one can obtain the equation satisfied by \(w\),

\[
\partial_t w(t, x) + \alpha \partial_{xxx} w(t, x) + \beta \partial_x w(t, x) = \partial_{xx} f(t, x),
\]

these two last equations are satisfied for all \((t, x) \in \mathbb{R}_+^* \times \mathbb{R}\), and particularly for \((t, x) \in [0, T] \times [a, b]\).

In other hand, we obtain from the first line of equation (1) with equation (2)

\[
\partial_t u(t, x) + \alpha \partial_x w(t, x) + \beta v(t, x) = f(t, x),
\]

which is also verified in \(\mathbb{R}_+^* \times \mathbb{R}\) but will be specified as boundary condition on \([0, T] \times \{a, b\}\) as well as equations (2, 3).
Therefore, the initial boundary value problem associated to the LKdV equation is given by

\[
\begin{aligned}
\partial_t u + \alpha \partial_{xxx} u + \beta \partial_x u &= f & \text{in } & [0, T] \times [a, b], \\
\partial_t v + \alpha \partial_{xxx} v + \beta \partial_x v &= g & \text{in } & [0, T] \times [a, b], \\
\partial_t w + \alpha \partial_{xxx} w + \beta \partial_x w &= h & \text{in } & [0, T] \times [a, b], \\
\partial_x u &= v & \text{on } & [0, T] \times \{a, b\}, \\
\partial_x v &= w & \text{on } & [0, T] \times \{a, b\}, \\
\partial_x u &= f & \text{on } & [0, T] \times \{a, b\}, \\
u(0, x) &= u_0(x) & \text{on } & [a, b], \\
v(0, x) &= v_0(x) & \text{on } & [a, b], \\
w(0, x) &= w_0(x) & \text{on } & [a, b].
\end{aligned}
\]  

(4)

with

\[\alpha > 0, \quad \beta \in \mathbb{R}, \quad g(t, x) = \partial_x f(t, x), \quad h(t, x) = \partial_{xx} f(t, x), \quad v_0(x) = u_0'(x), \quad w_0(x) = u_0''(x).\]  

(5)

Functions \(f\) and \(u_0\) are the given data of the original equation (1).

We have derived from the LKdV equation (1) posed in \(\mathbb{R}^+ \times \mathbb{R}\) an associated initial boundary value problem in the finite domain \([0, T] \times [a, b]\) represented by (4) with NSBC. Furthermore, we can also prove that system (4) is an exact restriction of (1) in \([0, T] \times [a, b]\). Hence, we announce the following theorem.

**Theorem 2.1.** The initial boundary value problem (4) is equivalent to the initial value problem (1) restricted to \([0, T] \times [a, b]\).

**Proof.** From what precedes, it remains to verify that problem (4) in bounded domain gives the restriction of (1) from unbounded domain to the same bounded one. For this regard, let \((u, v, w)\) be a solution of (4), then our main goal is to prove that \(u_x(t, x) = v(t, x)\) and \(u_{xx} = w(t, x)\) for all \((t, x) \in [0, T] \times [a, b]\). We use the notation \(u_x\), where the subscript \(x\) notifies partial derivative of \(u\) with respect to \(x\).

We denote \(U = u_x - v\) and we have from boundary conditions of (4) that \(U(t, x) = 0\) and \(U_x(t, x) = 0\) at the interface \([0, T] \times \{a, b\}\) and \(U(0, x) = 0\) in \([a, b]\).

The partial differential equation satisfied by \(u_x\) is obtained by differentiating, with respect to \(x\), the first equation of (4) as follows

\[\partial_t (u_x) + \alpha \partial_{xxx} (u_x) + \beta \partial_x (u_x) = \partial_x f \quad (t, x) \in [0, T] \times [a, b].\]  

(6)

Taking the difference between (6) and the second equation of (4), one obtains the following equation of \(U\)

\[\partial_t U + \alpha \partial_{xxx} U + \beta \partial_x U = 0,\]  

(7)

multiplying (7) by \(U\) and integrating over \([a, b]\) yields

\[\frac{1}{2} \frac{d}{dt} \| U \|^2_{L^2} + \alpha \int_a^b U \partial_{xxx} U \, dx + \beta \int_a^b U \partial_x U \, dx = 0,\]

integrating by parts the second term, we get

\[\frac{1}{2} \frac{d}{dt} \| U \|^2_{L^2} - \alpha \int_a^b \partial_x U \partial_x U \, dx + \beta \int_a^b U \partial_x U \, dx = 0,\]

which implies

\[\frac{1}{2} \frac{d}{dt} \| U \|^2_{L^2} - \frac{\alpha}{2} \left[ (U_x)^2 \right]_a^b + \frac{\beta}{2} \left[ U^2 \right]_a^b = 0.\]
using the fact that \( U = U_x = 0 \) at boundary points \( a \) and \( b \), we obtain

\[
\frac{d}{dt} \| U \|_{L^2}^2 = 0.
\]

Hence, the \( L^2 \) norm of \( U \) is a constant function of time, one can writes

\[
\| U \|_{L^2} = \| U(t = 0) \|_{L^2} = 0,
\]

thus

\[
u_x(t, x) = v(t, x) \quad (t, x) \in [0, T] \times [a, b].
\] (8)

Similarly, let \( V = v_x - w \) which obviously satisfies the same interior equation and boundary conditions as \( U \), then we conclude that

\[
v_x(t, x) = w(t, x) \quad (t, x) \in [0, T] \times [a, b].
\] (9)

Thus, getting equations (8) and (9) completes the proof.

The following proposition deals with the energy of \( u \) in the derived system with NSBC.

**Proposition 1** (Non-Conservation). The coupled system (4) does not preserve the conservation property of energy and mass of the original infinite problem (1).

**Proof.** Consider \((u, v, w)\) solution of (4), we have

\[
\partial_t u(t, x) + \alpha \partial_{xxx} u(t, x) + \beta \partial_x u(t, x) = f(t, x),
\]

multiplying by \( u \) and integrating over \([a, b]\) then applying integration by parts gives

\[
\frac{1}{2} \frac{d}{dt} \| u \|_{L^2}^2 + \left[ \alpha u w - \frac{\alpha}{2} v^2 + \frac{\beta}{2} u^2 \right]_a^b = \int_a^b f u \, dx,
\]

a second integration on time over \([0, t]\) yields

\[
\frac{1}{2} \| u(t, .) \|_{L^2}^2 - \| u_0 \|_{L^2}^2 + \int_0^t \left[ \alpha u w - \frac{\alpha}{2} v^2 + \frac{\beta}{2} u^2 \right]_a^b \, dt = \int_0^t \int_a^b f(t, x) u(t, x) \, dx \, dt,
\]

then, taking \( E(t) = \| u(t, .) \|_{L^2(a, b)}^2 \), we obtain

\[
E(t) = E(0) + \int_0^t \left[ \alpha u^2 - 2 \alpha u w - \beta u^2 \right]_a^b \, dt + 2 \int_0^t \int_a^b f(t, x) u(t, x) \, dx \, dt. \] (10)

Moreover, taking equation verified by \( u \) in (1) and integrating over \([a, b]\), we also have

\[
\frac{d}{dt} \int_a^b u \, dx + \left[ \alpha u + \beta u \right]_a^b = \int_a^b f \, dx,
\]

and integrating over time and noting \( M(t) = \int_a^b u(t, x) \, dx \), we get

\[
M(t) = M(0) - \int_0^t \left[ \alpha u + \beta u \right]_a^b \, dt + \int_0^t \int_a^b f(t, x) \, dx \, dt. \] (11)

These both equations show that the energy \( E(t) \) and the mass \( M(t) \) are conserved if and only if the underlying terms are equal to zero. This happens, for example, when \( f = 0 \) and \( a, b \) go to infinity or when considering periodic boundary conditions but in general, especially with NSBC, no conservation can be assured. 

\[\square\]
Applying this approach to derive boundary conditions yields to a weakly coupled system of equations involving three functions \( u, v \) and \( w \), as expressed above. Indeed, the coupling process arises only on the boundary where all functions are mutually dependent and are not in the interior domain. We are interested only on the function \( u \) that is the restriction, to \([0, T] \times [a, b]\), of solution to LKdV equation (1) whose energies are computed in proposition 1. The two remaining functions are helping to get the information on the artificial boundary. In fact, we obtain dynamic boundary condition involving evolutionary term which is not common and needs particular attention in the approximation step. The next subsection presents the numerical approximation and analysis for the weak coupled system (4) based on finite difference method.

3. Centred approximation of LKdV equation with NSBC. In order to approximate the obtained system with non-standard boundary conditions (4), we use the Crank-Nicolson scheme for interior equations and boundary conditions as described below. For given \( N, M \in \mathbb{N}^* \) one defines time and space steps, respectively, \( \tau = \frac{T}{M}, \ h = \frac{b-a}{N} \), and uniform mesh grid points \( t_n = n \tau, \ 0 \leq n \leq M \), \( x_i = a + ih, \ 0 \leq i \leq N \). We denote \( u^n_i \) the approximation of the solution \( u(t, x) \) at time \( t_n \) at point \( x_i \), i.e. \( u^n_i \approx u(t_n, x_i) \). In the same way, we have \( \partial_x u^n_i \approx \partial_x u(t_n, x_i) \), \( \partial_{xxx} u^n_i \approx \partial_{xxx} u(t_n, x_i) \). The semi-discretization in time, of equation (4) by Crank-Nicolson scheme is

\[
\begin{align*}
\frac{u^{n+1} - u^n}{\tau} + \alpha \frac{\partial_{xxx} u^{n+1} + \partial_{xxx} u^n}{2} + \beta \frac{\partial_x u^{n+1} + \partial_x u^n}{2} &= \frac{f^{n+1} + f^n}{2} \\
\frac{v^{n+1} - v^n}{\tau} + \alpha \frac{\partial_{xxx} v^{n+1} + \partial_{xxx} v^n}{2} + \beta \frac{\partial_x v^{n+1} + \partial_x v^n}{2} &= \frac{g^{n+1} + g^n}{2} \\
\frac{w^{n+1} - w^n}{\tau} + \alpha \frac{\partial_{xxx} w^{n+1} + \partial_{xxx} w^n}{2} + \beta \frac{\partial_x w^{n+1} + \partial_x w^n}{2} &= \frac{h^{n+1} + h^n}{2} \\
\partial_x u^{n+1} + \partial_x w^n &= \frac{v^{n+1} + v^n}{2} \\
\partial_x v^{n+1} + \partial_x w^n &= \frac{w^{n+1} + w^n}{2} \\
u^n_0 &= u_0 \\
v^n_0 &= v_0 \\
w^n_0 &= w_0 \\
\end{align*}
\]

For space discretization, we introduce for the third derivative of interior equations operator \( D \) as

\[
Du^n_i := \frac{1}{2h^3} \left( u^{n+2}_{i+2} - 2u^n_{i+1} + 2u^n_{i-1} - u^{n+2}_{i-2} \right) \approx \partial_{xxx} u(t_n, x_i), \ \text{for} \ \ 1 < i < N - 1.
\]

This approximation can not be used for \( i = 1, N - 1 \), unless if we have the values of \( u^n_1 \) and \( u^n_{N-1} \) that approximates, respectively, the solution at \( x = a - h \) and \( x = b + h \) and time \( t = t_n \). This value can be interpolated by Lagrange polynomial [13]. For example, a third order interpolation yields to

\[
\begin{align*}
u^n_1 &= 4u^n_0 - 6u^n_1 + 4u^n_2 \\
u^n_{N-1} &= 4u^n_N - 6u^n_{N-1} + 4u^n_{N-2} - u^n_{N-3}.
\end{align*}
\]
Hence, we approach third derivative \( \partial_{xxx} u_i^n \) for \( i = 1, N - 1 \) by

\[
\mathcal{D} u_i^n = \frac{1}{2h^3} (2u_{i+1}^n - 6u_i^n + 6u_{i-1}^n - 2u_{i-2}^n).
\]  

(14)

\[
\mathcal{D} u_{N-1}^n = \frac{1}{2h^3} (2u_N^n - 6u_{N-1}^n + 6u_{N-2}^n - 2u_{N-3}^n).
\]  

(15)

For first derivative of interior equations, we define operator \( D_\pm \)

\[
D_\pm u_i^n := \frac{1}{h} \left( u_{i+1}^n - u_i^n \right) \approx \partial_x u(t_n, x_i), \text{ for } 0 < i < N, n > 0,
\]

(16)

and for first derivative on boundary

\[
D_+ u_i^n := \frac{1}{h} \left( u_{i+1}^n - u_i^n \right) \approx \partial_x u(t_n, x_i), \text{ for } i = 0, n > 0,
\]

(17)

\[
D_- u_i^n := \frac{1}{h} \left( u_i^n - u_{i-1}^n \right) \approx \partial_x u(t_n, x_i), \text{ for } i = N, n > 0,
\]

(18)

Although, using operators (14 -18), the fully discretization of equation (12) is obtained by the following

\[
\begin{cases}
\frac{u_{i+1}^n - u_i^n}{h} + \frac{\partial u_{i+1}^n + \partial u_i^n}{2} + \beta \frac{D_\pm u_{i+1}^n + D_\pm u_i^n}{2} = \frac{f_{i+1}^n + f_i^n}{2} \\
\frac{v_{i+1}^n - v_i^n}{h} + \frac{\partial v_{i+1}^n + \partial v_i^n}{2} + \beta \frac{D_\pm v_{i+1}^n + D_\pm v_i^n}{2} = \frac{g_{i+1}^n + g_i^n}{2} \\
\frac{w_{i+1}^n - w_i^n}{h} + \frac{\partial w_{i+1}^n + \partial w_i^n}{2} + \beta \frac{D_\pm w_{i+1}^n + D_\pm w_i^n}{2} = \frac{h_{i+1}^n + h_i^n}{2} \\
\end{cases}
\]

(19)

\[
\begin{cases}
\frac{D_+ w_{i+1}^n + D_- w_i^n}{2} + \beta \frac{v_{i+1}^n + v_i^n}{2} + \frac{u_{i+1}^n + u_i^n}{2} = \frac{f_{i+1}^n + f_i^n}{2} \\
\frac{D_+ v_{i+1}^n + D_- v_i^n}{2} + \beta \frac{v_{i+1}^n + v_i^n}{2} + \frac{w_{i+1}^n + w_i^n}{2} = \frac{g_{i+1}^n + g_i^n}{2} \\
\frac{D_+ u_{i+1}^n + D_- u_i^n}{2} + \beta \frac{u_{i+1}^n + u_i^n}{2} + \frac{w_{i+1}^n + w_i^n}{2} = \frac{h_{i+1}^n + h_i^n}{2} \\
\end{cases}
\]

Let denote,

\[
U^n = \begin{pmatrix}
u_0^n \\ \vdots \\ u_N^n \end{pmatrix}, \quad V^n = \begin{pmatrix}v_0^n \\ \vdots \\ v_N^n \end{pmatrix}, \quad W^n = \begin{pmatrix}w_0^n \\ \vdots \\ w_N^n \end{pmatrix}, \quad X^n = \begin{pmatrix}U^n \\ V^n \\ W^n \end{pmatrix}
\]
The matrix that corresponds to the system (19) writes

$$\begin{pmatrix}
-1 & 1 & 0 & \ldots & 0 & -h & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & d_1 & d_2 & d_3 & d_4 & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & c_1 & c_2 & c_3 & -c_2 & -c_1 & 0 & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -d_4 & -d_3 & d_3 & -d_4 & -1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \ldots & 0 & b_1 & 0 & \ldots & 0 & -b_2 & b_2 & 0 & \ldots & 0 \\
0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
$$

$$A = \begin{pmatrix}
-1 & 1 & 0 & \ldots & 0 & -h & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & d_1 & d_2 & d_3 & d_4 & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & c_1 & c_2 & c_3 & -c_2 & -c_1 & 0 & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -d_4 & -d_3 & d_3 & -d_4 & -1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \ldots & 0 & b_1 & 0 & \ldots & 0 & -b_2 & b_2 & 0 & \ldots & 0 \\
0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
$$

and the related source term of the system (19),

$$F^n = \begin{pmatrix}
\mathcal{F}(f_{n+1}^n + f_n^n - d_1 u_0^n + d_3 u_1^n - d_4 u_2^n - d_4 u_3^n) \\
\mathcal{F}(g_{n+1}^n - g_n^n) - c_1 u_{n-4}^n - c_2 u_{n-3}^n + c_3 u_{n-2}^n + c_2 u_{n-1}^n + c_1 u_n^n \\
\mathcal{F}(f_{n-1}^n + f_n^n) + d_4 u_{n-3}^n + d_3 u_{n-2}^n + d_4 u_{n-1}^n + d_4 u_n^n \\
\mathcal{F}(g_{n+1}^n - g_n^n) - c_1 v_{n-4}^n - c_2 v_{n-3}^n + c_3 v_{n-2}^n + c_2 v_{n-1}^n + c_1 v_n^n \\
\mathcal{F}(g_{n-1}^n + g_n^n) + d_4 v_{n-3}^n + d_3 v_{n-2}^n + d_4 v_{n-1}^n + d_4 v_n^n \\
\mathcal{F}(h_{n+1}^n - h_n^n) - c_1 w_{n-4}^n - c_2 w_{n-3}^n + c_3 w_{n-2}^n + c_2 w_{n-1}^n + c_1 w_n^n \\
\mathcal{F}(h_{n-1}^n + h_n^n) + d_4 w_{n-3}^n + d_3 w_{n-2}^n + d_4 w_{n-1}^n + d_4 w_n^n \\
\mathcal{F}(f_{n+1}^n + f_n^n - d_1 v_0^n - b_1 v_0^n + b_2 w_1^n) \\
\mathcal{F}(h_{n+1}^n + h_n^n) - d_1 w_0^n + d_3 w_1^n - d_4 w_2^n - d_4 w_3^n \\
\mathcal{F}(h_{n-1}^n + h_n^n) - c_1 w_{n-4}^n - c_2 w_{n-3}^n + c_3 w_{n-2}^n + c_2 w_{n-1}^n + c_1 w_n^n \\
\mathcal{F}(h_{n-1}^n + h_n^n) + d_4 w_{n-3}^n + d_3 w_{n-2}^n + d_4 w_{n-1}^n + d_4 w_n^n \\
\mathcal{F}(f_{n+1}^n + f_n^n - b_1 v_0^n - b_2 w_1^n + b_2 w_2^n)
\end{pmatrix}
such that, constants from governed equations in interior domain in the matrix $A$ and the source term $F^n$ are
\[
c_1 = -\frac{\alpha \tau}{4h^3}, \quad c_2 = \frac{\alpha \tau}{2h^3}, \quad c_3 = 1,
\]
whereas constants from interpolation are
\[
d_1 = -\frac{\alpha \tau}{2h^3}, \quad d_2 = 1 + 3\frac{\alpha \tau}{2h^3}, \quad d_3 = \frac{\beta \tau}{4h}, \quad d_4 = \frac{\alpha \tau}{2h^3}, \quad d_5 = 1 - 3\frac{\alpha \tau}{2h^3},
\]
and constants from boundary conditions are
\[
b_1 = \frac{\beta \tau}{2}, \quad b_2 = \frac{\alpha \tau}{2h}.
\]

4. **Convergence of the proposed approximation.** We prove that numerical scheme (19) is consistent of order one in space and two in time and is unconditionally stable in the sense of Von Neumann. We announce the following propositions

**Proposition 2 (Existence).** The matrix $A$ is invertible under the condition
\[
\left(\frac{\beta \tau}{h}, \frac{\alpha \tau}{h^3}\right) \in \left\{(x, y) \in \mathbb{R} \times \mathbb{R}^+: \frac{1}{2}x - \frac{2}{3} < y < \min \left\{\frac{2}{5} + \frac{3}{10}x, \frac{2}{7} + \frac{1}{14}x, \frac{2}{3} - \frac{1}{6}x\right\}\right\}
\]

**Proof.** We prove that the matrix $A$ has the same determinant as a strictly dominated diagonal matrix $A$. Gerschgorin theorem [16], implies that $A$ is invertible and then $A$ is invertible too. The total proof is detailed in appendix A.

**Proposition 3 (Consistency).** Suppose that solution $(u, v, w)$ of equation (1) is of class $C^2$ in time and of class $C^5$ in space, then its approximation given in equation (19) is consistent of first order in space and second order in time.

**Proof.** Let $X = (u, v, w)$ be a solution of (1), we denote by $\xi(X)$ the consistency error of the scheme (19) as an approximation of (1). This error $\xi(X)^n$ is evaluated as a vector for $(i, n) \in \{0, \ldots, N\} \times \{0, \ldots, M\}$ such that

- for $0 < i < N$ and $F = (f, g, h)$, we write
  \[
  \xi(X)^n_i = \frac{X(t^{n+1}, x_i) - X(t^n, x_i)}{\tau} + \frac{1}{2}(D_hX(t^{n+1}, x_i) + D_hX(t^n, x_i)) - \frac{1}{2}(F(t^{n+1}, x_i) + F(t^n, x_i)).
  \]
  One has $D_hX(t, x_i) = D_hX(t, \cdot)(x_i)$ and since $X(t, \cdot)$ is of class $C^5$, Taylor expansions give
  \[
  D_hX(t, x_i) = \alpha X(t, x_{i+1}) - 2X(t, x_{i+1}) + \frac{\alpha h^2}{2}X(t, x_{i+1}) + \beta X(t, x_{i+1}) - X(t, x_{i+1}) + O(h^3) = O(h^2).
  \]

and at time $t^{n+1/2} = t^n + \frac{\tau}{2}$, one writes according to Taylor expansions for $X(t^{n+1/2}, x_i)$ that is of class $C^2$,
\[
\frac{X(t^{n+1}, x_i) - X(t^n, x_i)}{\tau} - \partial_t X(t^{n+1/2}, x_i) = O(\tau^2),
\]
with respect to time. For the sake of simplicity, we take the Euclidean norm in a vector space. We prove that the scheme is consistent of order 2 in time and one in space.

\[ \nabla \in M \text{ implies } O(\tau^2 + h^2). \]

- For \( i = 0 \), we denote \( \xi(X)_0^n = (\zeta_1(X), \zeta_2(X), \zeta_3(X)) \), such that using Taylor expansions and the smoothness of \( X \), we have

\[
\begin{align*}
\zeta_1(X) &= D_+ u_0^n - \partial_x u(t^n, x_0) - v_0^n + v(t^n, x_0) \\
&= O(h) \\
\zeta_2(X) &= D_+ v_0^n - \partial_x v(t^n, x_0) - w_0^n + w(t^n, x_0), \\
&= O(h) \\
\zeta_3(X) &= \frac{u(t^{n+1}, x_i) - u(t^n, x_i)}{\tau} - \partial_t u(t^{n+1/2}, x_i) + \alpha(D_+ u_0^{n+1/2} - \partial_x w(t^{n+1/2}, x_0)) \\
&\quad + \beta(v_0^{n+1/2} - v(t^{n+1/2}, x_0)) \\
&= O(\tau^2 + h).
\end{align*}
\]

Thus, \( \xi(X)_0^n = O(\tau^2 + h) \).

- For \( i = N \), in the same way, we denote \( \xi(X)_N^n = (\epsilon_1(X), \epsilon_2(X), \epsilon_3(X)) \), such that

\[
\begin{align*}
\epsilon_1(X) &= D_- u_N^n - \partial_x u(t^n, x_N) - v_N^n + v(t^n, x_N) \\
&= O(h) \\
\epsilon_2(X) &= D_- v_N^n - \partial_x v(t^n, x_N) - w_N^n + w(t^n, x_N), \\
&= O(h) \\
\epsilon_3(X) &= \frac{u(t^{n+1}, x_i) - u(t^n, x_i)}{\tau} - \partial_t u(t^{n+1/2}, x_i) + \alpha(D_- u_N^{n+1/2} - \partial_x w(t^{n+1/2}, x_N)) \\
&\quad + \beta(v_N^{n+1/2} - v(t^{n+1/2}, x_N)) \\
&= O(\tau^2 + h).
\end{align*}
\]

We obtain, \( \xi(X)_N^n = O(\tau^2 + h) \).

where \( Y^{n+1/2} = \frac{1}{2}(Y^{n+1} + Y^n) \).

Therefore, the consistency error is given as \( \xi(X) = O(\tau^2 + h) \), which means that the scheme is consistent of order 2 in time and one in space.

**Proposition 4 (Stability).** The scheme (19) is stable under the condition

\[
\tau < \min \left( \frac{h^3}{\alpha + |\beta|h^2 + \left| \frac{3\beta}{2h^2} + \frac{1}{2} \right|}, \frac{4h^3}{35\alpha + 3|\beta|h} \right) \tag{25}
\]

**Proof.** We compute the numerical energy and we prove that it remains bounded with respect to time. For the sake of simplicity, we take \( f_i^n = g_i^n = h_i^n = 0 \) for all \( i = 0, \ldots, N \) and \( n = 0, \ldots, M \). Let \( X^n = (u^n, v^n, w^n) \), we denote by \( ||.|| \) the Euclidean norm in a vector space. We prove that \( ||X^M|| \) is bounded independently on \( M \) in four steps as follows.
**Step 1 : Bound of** $u^n$

We multiply both sides of the system (19) by the vector $(0, u_1^{n+1}, \cdots, u_N^{n+1}, 0; 0, \cdots, 0; u_0^{n+1}, 0, \cdots, 0, u_N^{n+1})$ and we obtain

\[
\begin{aligned}
&d_1 u_0^{n+1} u_1^{n+1} + c_1 u_0^{n+1} u_2^{n+1} + (d_3 + c_2) u_1^{n+1} u_2^{n+1} + (d_4 + c_1) u_1^{n+1} u_3^{n+1} \\
&+ |u_0^{n+1}|^2 + d_2 |u_1^{n+1}|^2 + \sum_{i=2}^{N-2} c_3 |u_i^{n+1}|^2 + d_5 |u_{N-1}^{n+1}|^2 + |u_N^{n+1}|^2 \\
&- (d_3 + c_2) u_{N-1}^{n+1} u_{N-2}^{n+1} - (d_4 + c_1) u_{N-1}^{n+1} u_{N-3}^{n+1} - d_1 u_N^{n+1} u_{N-1}^{n+1} - c_1 u_{N-1}^{n+1} u_{N-2}^{n+1} \\
&= u_0^n u_1^{n+1} - b_1 (v_0^n + v_0^{n+1}) u_2^{n+1} + b_2 (w_0^n + w_0^{n+1}) u_2^{n+1} - b_2 (w_1^n + w_1^{n+1}) u_2^{n+1} \\
&- d_1 u_0^n u_1^{n+1} + d_5 u_0^n u_3^{n+1} + d_3 u_1^n u_2^{n+1} - d_3 u_2^n u_1^{n+1} \\
&+ \sum_{i=2}^{N-2} (c_1 u_{i-1}^{n+1} - c_2 u_{i-1}^{n+1} + c_3 u_i^n u_1^{n+1} + c_2 u_{i-1}^{n+1} u_1^{n+1} + c_1 u_{i+2}^{n+1} u_1^{n+1} \\
&+ d_3 u_{N-3}^{n+1} u_{N-2}^{n+1} + d_3 u_{N-3}^{n+1} u_{N-1}^{n+1} + d_2 u_{N-2}^{n+1} u_{N-1}^{n+1} + d_1 u_{N-1}^{n+1} u_{N-2}^{n+1} + u_N^n u_{N-1}^{n+1} \\
&- b_1 (v_0^n + v_N^n) u_N^{n+1} + b_2 (w_0^n + w_N^n) u_N^{n+1} - b_2 (w_1^n + w_N^n) u_N^{n+1}
\end{aligned}
\]

Using the Young inequality and regrouping terms, we find

\[
\begin{aligned}
\sum_{i=0}^{N} \theta_i^1 |u_i^{n+1}|^2 &\leq \sum_{i=0}^{N} \left( \theta_i^2 |u_i^{n+1}|^2 + \theta_i^3 |u_i^n|^2 \right) \\
&+ |b_1| \left( |v_0^n|^2 + |v_N^n|^2 + |v_0^n|^2 + |v_N^n|^2 \right) \\
&+ |b_2| \left( |w_0^n|^2 + |w_1^n|^2 + |w_{N-1}^{n+1}|^2 + |w_N^{n+1}|^2 \right) \\
&+ |b_2| \left( |w_0^n|^2 + |w_1^n|^2 + |w_{N-1}^{n+1}|^2 + |w_N^{n+1}|^2 \right)
\end{aligned}
\]

(26)

where

\[
\theta_i^1 = \begin{cases} 
1 - \frac{|d_1|}{2} - \frac{|c_1|}{2} & \text{for } i = 0, N \\
\frac{|d_1|}{2} - \frac{|c_2 + d_3|}{2} & \text{for } i = 1 \\
\frac{|c_2|}{2} - \frac{|c_2 + d_3|}{2} & \text{for } i = 2, N - 2 \\
\frac{|c_1 + d_4|}{2} & \text{for } i = 3, N - 3 \\
\frac{c_3}{2} & \text{for } i = 4, \cdots, N - 4 \\
\frac{|d_5|}{2} - \frac{|c_2 + d_3|}{2} & \text{for } i = N - 1
\end{cases}
\]

\[
\theta_i^2 = \begin{cases} 
\frac{1}{2} + \frac{|b_1|}{2} + \frac{|b_2|}{2} & \text{for } i = 0, N \\
\frac{|d_5|}{2} + \frac{|d_1|}{2} + \frac{|d_3|}{2} + |d_4| & \text{for } i = 1 \\
\frac{|c_3|}{2} + |c_1| + |c_2| & \text{for } i = 2, \cdots, N - 2 \\
\frac{|d_2|}{2} + \frac{|d_1|}{2} + |d_4| & \text{for } i = N - 1
\end{cases}
\]
From discretisation parameters (21-23), we get the following bounds

\[
\theta_i^3 = \begin{cases} 
\frac{1}{2} + \frac{|d_1| + |c_1|}{d_3} & \text{for } i = 0, N \\
\frac{1}{2} + \frac{|c_1| + |c_2|}{e_3} & \text{for } i = 1 \\
\frac{2}{e_3} + \frac{|c_1| + |d_4|}{e_3} & \text{for } i = 2, N - 2 \\
\frac{2}{e_3} + \frac{|c_1| + |d_4|}{e_3} & \text{for } i = 3, N - 3 \\
\frac{2}{e_3} + |c_1| + |c_2| & \text{for } i = 4, \ldots, N - 4 \\
\frac{2}{e_3} + |c_1| + |c_2| & \text{for } i = N - 1 
\end{cases}
\]

It follows that for all \( i = 1, \ldots, N \)

\[
\theta_i^3 \leq \theta_i = \begin{cases} 
\frac{1}{2} + \frac{3 \alpha \tau}{8 h^3} + \frac{1}{8 h} \big|\beta \big| \tau + \frac{1}{4} \big|\beta \big| \tau & \text{for } i = 0, N \\
\frac{1}{2} + \frac{3 \alpha \tau}{8 h^3} + \frac{1}{8 h} \big|\beta \big| \tau & \text{for } i = 1, N - 1 \\
\frac{1}{2} + \frac{3 \alpha \tau}{8 h^3} + \frac{1}{8 h} \big|\beta \big| \tau & \text{for } i = 2, N - 2 \\
\frac{1}{2} + \frac{3 \alpha \tau}{8 h^3} + \frac{1}{8 h} \big|\beta \big| \tau & \text{for } i = 3, N - 3 \\
\frac{1}{2} + \frac{3 \alpha \tau}{8 h^3} + \frac{1}{8 h} \big|\beta \big| \tau & \text{for } i = 4, \ldots, N - 4 
\end{cases}
\]

Thus, inequality (26) becomes

\[
\Theta \|u^{n+1}\|^2 - \Theta \|u^n\|^2 \leq |b_1| \left( |v^n_{N+1}|^2 + |v^{N+1}_N|^2 + |v^0_n|^2 + |v^n_N|^2 \right) \\
+ |b_2| \left( |w^{n+1}_N|^2 + |w^{n+1}_N|^2 + |w^{n+1}_{N-1}|^2 + |w^{n+1}_N|^2 \right) \\
+ |b_2| \left( |w^{n+1}_N|^2 + |w^{n+1}_N|^2 + |w^{n+1}_{N-1}|^2 + |w^{n+1}_N|^2 \right)
\]

(28)
then,
\[
\|u^M\|^2 + \left(1 - \frac{\Theta}{2}\right) \sum_{n=1}^{M-1} \|u^n\|^2 \leq \frac{|\beta|^2}{2\Theta} \sum_{n=0}^{M-1} \left(\|v^n\|^2 + \|v^{n+1}\|^2\right)
\]
\[+ \frac{\alpha \tau}{2h^2 \Theta} \sum_{n=0}^{M-1} \left(\|w^n\|^2 + \|w^{n+1}\|^2\right) + \frac{\Theta}{\Theta} \|u^0\|^2.
\]
Thus using boundary conditions \(u_0 = hv_0^N, u_N - u_{N-1} = hv_N^N, v_0 - v_{N} = hw_0\) and \(v_N - v_{N-1} = hw_N\) and if \(\Theta > 0\), yield
\[
\|u^M\|^2 + \left(1 - \frac{\Theta}{2}\right) \sum_{n=1}^{M-1} \|u^n\|^2 \leq \frac{|\beta|^2}{2\Theta} \sum_{n=0}^{M-1} \left(\|u^n\|^2 + \|u^{n+1}\|^2\right)
\]
\[+ \frac{\alpha \tau}{2h^2 \Theta} \sum_{n=0}^{M-1} \left(\|u^n\|^2 + \|u^{n+1}\|^2\right) + \frac{\Theta}{\Theta} \|u^0\|^2.
\]

**Step 2 : Bound of \(v^n\)**

This time, we multiply both sides of the system (19) by the vector \((0, \ldots, 0; v_1^{n+1}, \ldots, v_{N-1}^{n+1}, 0, 0, \ldots, 0)\) and we infer
\[
d_1 v_0^{n+1} v_1^{n+1} + c_1 v_0^{n+1} v_2^{n+1} + (d_3 + c_2) v_1^{n+1} v_2^{n+1} + (d_4 + c_1) v_1^{n+1} v_3^{n+1}
\]
\[+ d_2 \left|v_1^{n+1}\right|^2 + \sum_{i=2}^{N-2} c_3 \left|v_i^{n+1}\right|^2 + d_5 \left|v_N^{n+1}\right|^2
\]
\[- (d_3 + c_2) v_{N-1}^{n+1} v_{N-2}^{n+1} - (d_4 + c_1) v_{N-1}^{n+1} v_{N-3}^{n+1} - d_1 v_N^{n+1} v_{N-1}^{n+1} - c_1 v_{N-1}^{n+1} v_{N-2}^{n+1}
\]
\[= -d_4 v_0^{n+1} + d_5 v_0^{n+1} - d_3 v_2^{n+1} v_1^{n+1} - d_4 v_3^{n+1} v_2^{n+1}
\]
\[+ \sum_{i=2}^{N-2} (-c_1 v_{i-1}^{n+1} v_i^{n+1} - c_2 v_{i-1}^{n-1} v_i^{n+1} + c_3 v_{i}^{n+1} v_{i+1}^{n+1} + c_4 v_{i+1}^{n+1} v_{i+1}^{n+1} + c_5 v_{i+1}^{n+1} v_{i+1}^{n+1} + c_6 v_{i+1}^{n+1} v_{i+1}^{n+1})
\]
\[+ d_4 v_{N-3}^{n+1} v_{N-2}^{n+1} + d_3 v_{N-2}^{n+1} v_{N-1}^{n+1} + d_2 v_{N-1}^{n+1} v_{N-1}^{n+1} + d_1 v_{N}^{n+1} v_{N}^{n+1}
\]

Young inequality gives the following
\[
\sum_{i=1}^{N-1} \lambda_i^1 \left|v_i^{n+1}\right|^2 \leq \sum_{i=1}^{N-1} \left(\lambda_i^2 \left|v_i^{n+1}\right|^2 + \lambda_i^3 \left|v_i^n\right|^2\right)
\]
\[+ \frac{|d_1| + |c_1|}{2} \left(\left|v_0^n\right|^2 + \left|v_N^n\right|^2 + \left|v_0^{n+1}\right|^2 + \left|v_N^{n+1}\right|^2\right)
\]
where
\[
\lambda_i^1 = \begin{cases} 
  d_2 - \frac{|d_1|}{2} - \frac{|c_2 + d_3|}{2} - \frac{|c_1 + d_4|}{2} & \text{for } i = 1 \\
  c_3 - \frac{|c_2 + d_3|}{2} & \text{for } i = 2, N - 2 \\
  c_3 - \frac{|c_1 + d_4|}{2} & \text{for } i = 3, N - 3 \\
  c_3 & \text{for } i = 4, \ldots, N - 4 \\
  d_5 - \frac{|d_1|}{2} - \frac{|c_2 + d_3|}{2} - \frac{|c_1 + d_4|}{2} & \text{for } i = N - 1
\end{cases}
\]
Step 3 : Bound of $w^n$

Since $w$ satisfies the same equation as $v$ at the interior discretization points and if
\[ \Theta > 0, \text{ we obtain} \]
\[ \| u^M \|^2 + \left( 1 - \frac{\Theta}{\Theta_{\min}} \right) \sum_{n=1}^{M-1} \| w^n \|^2 \leq \frac{|d_1| + |c_1|}{2h^2 \Theta} \sum_{n=0}^{M-1} \left( \| u^n \|^2 + \| u^{n+1} \|^2 \right) + \frac{\Theta}{\Theta_{\min}} \| w^0 \|^2 \]

(32)

**Step 4 : Bound of \( X^n \)**

The euclidean norm of the vector \( X^n \) writes \( \| X^n \|^2 = \| u^n \|^2 + \| v^n \|^2 + \| w^n \|^2 \). Taking the sum of equations (30-32) and assuming \( \gamma_M > 0 \) and \( \Theta > 0 \), we infer

\[ \| X^M \|^2 \leq \frac{\zeta_M}{\gamma_M} \sum_{n=1}^{M-1} z_n + \frac{\delta_M}{\gamma_M} z_0 \]

(33)

where
\[
\gamma_M = 1 - \frac{|d_1| + |c_1|}{2h \Theta} - \frac{|d_1| + |c_1|}{2h^2 \Theta} - \frac{|\beta| \tau}{2h \Theta} - \frac{\alpha \tau}{2h^3 \Theta}, \\
\delta_M = \frac{3}{\Theta} + \frac{|d_1| + |c_1|}{2h \Theta} + \frac{|d_1| + |c_1|}{2h^2 \Theta} + \frac{|\beta| \tau}{2h \Theta} + \frac{\alpha \tau}{2h^3 \Theta}, \\
\zeta_M = \frac{3}{\Theta} - 1 + \frac{|d_1| + |c_1|}{h \Theta} + \frac{|d_1| + |c_1|}{h^2 \Theta} + \frac{|\beta| \tau}{h \Theta} + \frac{\alpha \tau}{h^3 \Theta}.
\]

Then using the discrete Gronwall’s theorem[4], inequality (33) becomes

\[ \| X^M \|^2 \leq \frac{\delta_M}{\gamma_M} z_0 \exp \left( \sum_{n=1}^{M-1} \frac{\xi_M}{\gamma_M} \right) \leq \left[ \frac{\delta_M}{\gamma_M} \exp \left( \frac{M \zeta_M}{\gamma_M} \right) \right] z_0 \]

(34)

Henceforward, under conditions \( \gamma_M > 0 \) and \( \Theta > 0 \), we have as \( M \) goes to infinity, \( \tau = \frac{T}{\tau} \) goes to zero. Thus, all sequences \( (\delta_M)_M \), \( (\gamma_M)_M \) and \( \left( \frac{M \zeta_M}{\gamma_M} \right)_M \) are convergent for fixed \( h \). Therefore, the sequence \( \left( \| X^M \|^2 \right)_M \) is also convergent and bounded. In conclusion, the scheme (19) is stable if \( \gamma_M > 0 \) and \( \Theta > 0 \). We have,

\[
\left\{ \begin{array}{ll}
\gamma_M > 0 \\
\Theta > 0
\end{array} \right. \iff \left\{ \begin{array}{l}
\frac{|d_1| + |c_1|}{2h \Theta} + \frac{|d_1| + |c_1|}{2h^2 \Theta} + \frac{|\beta| \tau}{2h \Theta} + \frac{\alpha \tau}{2h^3 \Theta} < 1 \\
\frac{3}{\Theta} + \frac{3 |\beta| \tau}{h} + \frac{3 |\beta| \tau}{h^3} < \frac{1}{2}
\end{array} \right.
\]

\[
\iff \left\{ \begin{array}{l}
\frac{\tau}{2h^3} \left( \alpha + |\beta| h^2 + 2 \frac{3 \alpha}{4h^3} + \frac{\beta}{4} (h + 1) \right) < 1 \\
\frac{\tau}{4h^3} (35 \alpha + 3 |\beta| h) < 1
\end{array} \right.
\]

then,

\[
\left\{ \begin{array}{l}
\tau < \frac{h^3}{\alpha + |\beta| h^2 + 2 \frac{3 \alpha}{4h^3} + \frac{\beta}{4} (h + 1)} \\
\tau < \frac{4h^3}{35 \alpha + 3 |\beta| h}
\end{array} \right.
\]

this means that the scheme is stable if \( \tau < \min \left( \frac{h^3}{\alpha + |\beta| h^2 + 2 \frac{3 \alpha}{4h^3} + \frac{\beta}{4} (h + 1)}, \frac{4h^3}{35 \alpha + 3 |\beta| h} \right) \). We claim that this condition ensures inversibility of the matrix \( A \).

\[ \Box \]

The matrix \( A \) involved in the linear system to be solved for \( X = (u, v, w) \) is inverted as sparse matrix using LU decomposition. The results of this resolution are presented in the next subsection for several values of initial data, source term and parameters \( \alpha \) and \( \beta \).
5. Numerical results. This section presents numerical results of implementing equation (19) on three examples, two of them for the homogeneous equation for various values of $\alpha$, $\beta$ and initial data either supported for $u_0(x) = e^{-x^2}$ or $u_0(x) = e^{-8(x-5)^2} \sin(\frac{50\pi}{T} x)$ and non supported initial data by taking $u_0(x) = c + e^{-(x+2)^2}$ for $c \in \{0.5, 1, 2\}$. The third example shows numerical approximation of the non homogeneous equation with non supported source term of the form $f(t, x) = \cos(\omega t)(e^{-(x-a)^2} + e^{-(x-b)^2})$ with $\omega \in \{5, 10, 20, 30, 40\}$.

To evaluate the efficiency of the boundary conditions, we compare the approximated solution with a reference one. The reference solution is obtained by solving numerically the equation (4) in a sufficiently large domain $[0, T] \times [a, b]$ such that $a_i = m \times a$ and $b_i = m \times b$ for a given integer $m > 2$ using Dirichlet boundary conditions. Hence the reference solution is taken as the restriction to the domain $[0, T] \times [a, b]$ of this numerical solution. The value of $m$ is chosen so that the reflections, due to the use of Dirichlet boundary conditions, do not occur in the domain of interest $[a, b]$, generally, we take $m = 20$ in our numerical tests.

We define the following errors

\[
err(u) := \| (u - u_{ref})(t, x) \|_\infty = \sup_{(t,x) \in [0,T] \times [a,b]} |u(t, x) - u_{ref}(t, x)|, \quad (35)
\]

\[
RelErr_u(t) = \frac{\| u(t, .) - u_{ref}(t, .) \|_2}{\| u_{ref}(t, .) \|_2}, \quad (36)
\]

\[
e_\infty(u) := \sup_{t \in [0,T]} RelErr_u(t), \quad (37)
\]

\[
e_2(u) := \| RelErr_u \|_2, \quad (38)
\]

where, $u(t, \cdot)$ stands for the function $x \mapsto u(t, x)$. These errors are approximated by

\[
err(u) \approx \max_{0 \leq n \leq M} \max_{1 \leq i \leq N} |u^n_i - (u_{ref})^n_i|, \quad (39)
\]

\[
\| q \|_2 \approx \sqrt{\frac{N-1}{h} \sum_{i=0}^{N-1} q^2(x_i)}. \quad (40)
\]

We also plot the evolution in time of the infinite error defined by

\[
R(t) = \sup_{x \in [a,b]} |u_{nsbc}(t, x) - u_{ref}(t, x)|, \quad (40)
\]

We propose a comparison between the use of NSBC derived above and the use of standard Dirichlet and Neumann BC and a kind of non standard BC called Discrete Artificial Boundary Conditions DABC described in [2], whose solution is denoted here $u_{dabc}$. Hence, we denote $u_{nsbc}$, $u_{dir}$ and $u_{neu}$ solutions using NSBC, Dirichlet and Neumann BC respectively.

5.1. First example in the homogeneous case. In this example, we consider the following parameters

\[
\alpha = 1, \quad \beta = 0, \quad T = 4, \quad a = -6, \quad b = 6,
\]

and the following initial data

\[
u_0(x) = e^{-x^2}. \quad (41)
\]

In table (1), we list errors with respect to time and space steps for different BC considered as explained before. We remark that NBC is far to be used as BC for the LKdV equation and blow up on simulations. DBC is not a good approximation
as well, while NSBC and DABC are more accurate to approach the LKdV equation. Stability sufficient condition (25) is also verified. Furthermore, we observe that error using NSBC is $O(h + \tau^2)$ for $\alpha = 1$ as proved in proposition 3 and is one ordered in space step as well as DABC(RCN) while DABC(CCN) is second ordered, see [2]. Next table proves the same when taking long time simulations and varying the values of the parameters $\alpha$ and $\beta$. Figures 1 and 2 show instantaneous and geophones for the largest time $T = 6$, we see the dispersion of the wave to the left side when the maximum value goes away between $t = 1$ and 2, see the left of figure 2. At this time, energies begin to decay as seen in figure 4. Then, the error increases when the wave reaches the right side around $t = 4$, see figure 2, where no instabilities have been revealed.

| $u$    | $\tau$ | $h$  | $e_{rr}(u)$ | $e_2(u)$ | $e_{\infty}(u)$ |
|--------|--------|------|-------------|----------|-----------------|
| $u_{a, b, c}$ | $7.8 \times 10^{-3}$ | $10^{-1}$ | $1.54 \times 10^{-2}$ | $1.62 \times 10^{-2}$ | $1.7 \times 10^{-2}$ |
| $u_{d, b, c}(RCN)$ | $7.8 \times 10^{-3}$ | $10^{-1}$ | $-$ | $\approx 5 \times 10^{-3}$ | $\approx 10^{-1}$ |
| $u_{d, b, c}(CCN)$ | $7.8 \times 10^{-3}$ | $10^{-1}$ | $-$ | $\approx 10^{-3}$ | $\approx 10^{-4}$ |
| $u_{d, b, c}$ | $7.8 \times 10^{-3}$ | $10^{-1}$ | $5.52 \times 10^{-1}$ | $5.17 \times 10^{-1}$ | $4.05 \times 10^{-1}$ |
| $u_{a, b, c}$ | $3.9 \times 10^{-3}$ | $10^{-1}$ | $2.44 \times 10^{-2}$ | $1.74 \times 10^{-2}$ | $2.59 \times 10^{-2}$ |
| $u_{d, b, c}(RCN)$ | $3.9 \times 10^{-3}$ | $10^{-1}$ | $-$ | $\approx 5 \times 10^{-3}$ | $\approx 10^{-1}$ |
| $u_{d, b, c}(CCN)$ | $3.9 \times 10^{-3}$ | $10^{-1}$ | $-$ | $\approx 10^{-3}$ | $\approx 5 \times 10^{-4}$ |
| $u_{d, b, c}$ | $3.9 \times 10^{-3}$ | $10^{-1}$ | $6.5 \times 10^{-3}$ | $9.4 \times 10^{-3}$ | $9.7 \times 10^{-3}$ |
| $u_{a, b, c}$ | $4.88 \times 10^{-4}$ | $10^{-1}$ | $1.32 \times 10^{-2}$ | $1.52 \times 10^{-2}$ | $1.63 \times 10^{-2}$ |
| $u_{d, b, c}(RCN)$ | $4.88 \times 10^{-4}$ | $10^{-1}$ | $-$ | $\approx 10^{-2}$ | $\approx 10^{-3}$ |
| $u_{d, b, c}(CCN)$ | $4.88 \times 10^{-4}$ | $10^{-1}$ | $-$ | $\approx 10^{-3}$ | $\approx 10^{-4}$ |
| $u_{a, b, c}$ | $2.44 \times 10^{-4}$ | $10^{-1}$ | $1.32 \times 10^{-2}$ | $1.52 \times 10^{-2}$ | $1.63 \times 10^{-2}$ |
| $u_{d, b, c}(RCN)$ | $2.44 \times 10^{-4}$ | $10^{-1}$ | $-$ | $\approx 10^{-2}$ | $\approx 10^{-3}$ |
| $u_{d, b, c}(CCN)$ | $2.44 \times 10^{-4}$ | $10^{-1}$ | $-$ | $\approx 10^{-3}$ | $\approx 10^{-4}$ |

Table 1. Comparison of Errors for the first example using various BC, $\tau$ and $h$.

5.2. Second example in the homogeneous case. The second example concerns

$$\alpha = 1, \quad \beta = 1, \quad T = 16.384 \times 10^{-4}, \quad a = 0, \quad b = 10$$

and a wave packet initial datum,

$$u_0(x) = e^{-8(x-5)^2} \sin\left(\frac{50\pi}{4}x\right).$$

(42)

Table 3 endorses the fact that the error using NSBC is majorized by $h + \tau^2$. We see in figure 5 the wave packet crossing the left boundary. We also remark in figure 6 that the infinite error is reached when the highest amplitude arrives to the left boundary which happens around $t = 1$ as observed in figure 7. Energy is constant until this time and decreases to zero when the wave leaves the domain of calculation $[a, b]$ according to proposition 1 and seen in figure 8. This shows the absorbing efficiency of the proposed NSBC.
| $T$ | $\alpha$ | $\beta$ | $\tau$ | $h$ | $err$ | $e_2$ | $e_\infty$ |
|-----|---------|---------|-------|-----|-------|-------|----------|
| 1   | 1       | 0       | $10^{-4}$ | $6.8 \times 10^{-2}$ | $3.3 \times 10^{-3}$ | $4.6 \times 10^{-3}$ |
| 1   | 1       | 0       | $10^{-4}$ | $6.5 \times 10^{-2}$ | $3.2 \times 10^{-3}$ | $4.3 \times 10^{-3}$ |
| 1   | 1       | 0       | $10^{-4}$ | $6.7 \times 10^{-2}$ | $3 \times 10^{-3}$ | $4.2 \times 10^{-3}$ |
| 1   | 1       | 0       | $5 \times 10^{-2}$ | $7.4 \times 10^{-2}$ | $3.6 \times 10^{-3}$ | $5.1 \times 10^{-3}$ |
| 2   | 1       | 0       | $10^{-4}$ | $8.1 \times 10^{-2}$ | $6.2 \times 10^{-3}$ | $6.3 \times 10^{-3}$ |
| 2   | 1       | 0       | $10^{-4}$ | $6.9 \times 10^{-2}$ | $5.8 \times 10^{-3}$ | $5.6 \times 10^{-3}$ |
| 2   | 1       | 0       | $5 \times 10^{-2}$ | $6.4 \times 10^{-2}$ | $3 \times 10^{-3}$ | $4.2 \times 10^{-3}$ |
| 3   | 1       | 0       | $10^{-4}$ | $8.2 \times 10^{-2}$ | $1.02 \times 10^{-2}$ | $10^{-2}$ |
| 3   | 1       | 0       | $5 \times 10^{-2}$ | $-6.4 \times 10^{-3}$ | $-3 \times 10^{-3}$ | $-4.2 \times 10^{-3}$ |
| 4   | 1       | 0       | $10^{-4}$ | $1.55 \times 10^{-2}$ | $1.82 \times 10^{-2}$ | $1.92 \times 10^{-2}$ |
| 4   | 1       | 0       | $10^{-4}$ | $1.33 \times 10^{-2}$ | $1.43 \times 10^{-2}$ | $1.72 \times 10^{-2}$ |
| 5   | 1       | 0       | $10^{-4}$ | $1.82 \times 10^{-2}$ | $3.04 \times 10^{-2}$ | $2.5 \times 10^{-2}$ |
| 5   | 1       | 0       | $10^{-4}$ | $1.75 \times 10^{-2}$ | $2.94 \times 10^{-2}$ | $2.4 \times 10^{-2}$ |
| 6   | 1       | 0       | $10^{-4}$ | $1.84 \times 10^{-2}$ | $3.76 \times 10^{-2}$ | $2.61 \times 10^{-2}$ |
| 6   | 1       | 0       | $10^{-4}$ | $1.77 \times 10^{-2}$ | $3.66 \times 10^{-2}$ | $2.59 \times 10^{-2}$ |
| 4   | 2       | -2      | $10^{-4}$ | $1.52 \times 10^{-2}$ | $7.6 \times 10^{-3}$ | $1.67 \times 10^{-2}$ |
| 4   | 2       | -2      | $10^{-4}$ | $8.8 \times 10^{-3}$ | $4.5 \times 10^{-3}$ | $5.9 \times 10^{-3}$ |
| 4   | 0.5     | -0.5    | $10^{-4}$ | $8.04 \times 10^{-4}$ | $7.56 \times 10^{-4}$ | $7.89 \times 10^{-4}$ |
| 4   | 1       | -4      | $10^{-4}$ | $1.59 \times 10^{-2}$ | $1.05 \times 10^{-2}$ | $1.78 \times 10^{-2}$ |
| 4   | 1       | -4      | $10^{-4}$ | $9 \times 10^{-3}$ | $3.8 \times 10^{-3}$ | $5.9 \times 10^{-3}$ |
| 4   | 8       | 2       | $10^{-4}$ | $5.25 \times 10^{-2}$ | $4.18 \times 10^{-2}$ | $7.14 \times 10^{-2}$ |
| 4   | 8       | 2       | $10^{-4}$ | $2.31 \times 10^{-2}$ | $1.82 \times 10^{-2}$ | $3.29 \times 10^{-2}$ |
| 4   | 0       | 1       | $10^{-4}$ | $4.82 \times 10^{-4}$ | $6.86 \times 10^{-5}$ | $1.76 \times 10^{-4}$ |
| 4   | 0       | 1       | $5 \times 10^{-2}$ | $1.38 \times 10^{-4}$ | $1.87 \times 10^{-5}$ | $4.9 \times 10^{-5}$ |

**Table 2.** Errors of the first example for homogeneous LKDV equation with NSBC for long time simulation and various values of $\alpha$ and $\beta$.

**Figure 1.** Instantaneous of $u_{ref}(red)$ and $u_{nsbc}(blue)$ for the first example with $\tau = \frac{6}{64}$ and $h = 10^{-1}$. 
Figure 2. Geophones of $u_{ref}(\text{red})$ and $u_{nsbc}(\text{blue})$ for the first example with $\tau = \frac{\nu}{a^2}$ and $h = 10^{-1}$ at different positions.

Figure 3. Evolution of logarithm of error with respect to time and space for the first example with $\tau = \frac{\nu}{a^2}$ and $h = 10^{-1}$ for the first example.

5.3. Non-compactly supported data.

5.3.1. Third example: Non-compactly supported initial data. We note that deriving NSBC do not need to take supported initial data and source term as almost supposed
for absorbing boundary conditions like in [2] and [19]. To evaluate numerically this advantage, we take 
\[ \alpha = 1, \beta = 0, b = -a = 6 \text{ and } T = 1 \] and we consider at first \( f = 0 \) and non-homogeneous initial data
\[ u_0(x) = c + e^{-(x+2)^2} \] (43)

Since the dispersion goes to the left, we have centered initial data on \(-2\) so that it reaches the boundary faster and we show in table 4 errors with respect to the parameter \( c \). We see that errors follow the same properties as using supported initial data and steal as \( O(h + \tau^2) \). Figures 9 and 10 prove that no reflections occur as the wave travels out the left boundary. It is clear from figure 11 that the infinite error is reached around \( t = 0.3 \) and before which energies in figure 12 remain constant and decrease until \( t = 0.4 \) then increase due to the boundary values of the underlying terms in the proof of proposition 1.
5.3.2. Fourth example: Non-compactly supported source term. We consider here supported initial datum $u_0(x) = e^{-x^2}$ and non-supported source term that highly oscillates at boundaries $a$ and $b$

$$f(t, x) = \cos(\omega t)(e^{-(x-a)^2} + e^{-(x-b)^2})$$

We show in table 5 that errors follow the same properties as using supported initial data even for large values of $\omega$. Geophones in figure 13 shows oscillations on both boundaries and permits to see the efficiency of NSBC also on the right boundary.
Figure 7. Evolution of logarithm of error with respect to time and space for the second example with $\tau = 3.2 \times 10^{-6}$ and $h = 2 \times 10^{-3}$.

Figure 8. Energies $E(t)$ and $M(t)$ evolution in time of $u_{nsbc}$ (left) and $u_{ref}$ (right) for the second example with $\tau = 3.2 \times 10^{-6}$ and $h = 2 \times 10^{-3}$. 
5.3.3. Fifth example: Non-compactly supported initial data and source term. This last example concerns non-supported initial data (43) with \(c = 2\) and non-supported source term (44) for \(\omega = 40\). We present in table 6 errors for \(\tau = 1.56 \times 10^{-2}\) and \(h = 10^{-1}\) at different positions.
Figure 11. Evolution of logarithm of error with respect to time and space for the third example using (43) for $c = 1$ with $\tau = 1.56 \times 10^{-2}$ and $h = 10^{-1}$.

Figure 12. Energies $E(t)$ and $M(t)$ evolution in time of $u_{nsbc}$ (left) and $u_{ref}$ (right) third example using (43) for $c = 1$ with $\tau = 1.56 \times 10^{-2}$ and $h = 10^{-1}$. 
Table 5. Errors of the third example using (44) for various values of $\omega$.

| $\omega$ | $\tau$ | $\varepsilon_{1\tau}$ | $\varepsilon_{2}$ | $\varepsilon_{\infty}$ |
|----------|--------|------------------------|------------------|----------------------|
| 5        | 1      | $6.22 \times 10^{-2}$  | $6.92 \times 10^{-2}$ | $8.48 \times 10^{-2}$ |
| 10       | 1      | $3 \times 10^{-2}$    | $4.89 \times 10^{-2}$ | $5.63 \times 10^{-2}$ |
| 20       | 1      | $2.23 \times 10^{-2}$ | $1.67 \times 10^{-2}$ | $2.63 \times 10^{-2}$ |
| 30       | 1      | $5.2 \times 10^{-3}$  | $1.02 \times 10^{-2}$ | $1.36 \times 10^{-2}$ |
| 40       | 1      | $9.6 \times 10^{-3}$  | $5.7 \times 10^{-3}$  | $8.7 \times 10^{-3}$  |

Figure 13. Geophones of $u_{ref}$ (red) and $u_{nsbc}$ (blue) for the third example using (44) for $\omega = 30$ with $\tau = 1.56 \times 10^{-2}$ and $h = 10^{-1}$ at different positions.

$h = 10^{-1}$. We remark that initial data and source term do not affect the accuracy of NSBC which is also observed with instantaneous and geophones in figures 14 and 15. The maximum of the errors comes when the highest amplitude of the wave traverses the left boundary at $t = 0.3$ as showed in figure 16. Energies in figure 17 remains constant to the time when the wave begin traveling the left boundary and decrease for few moment then increase slightly imitating the same phenomenon as figure 12.

Table 6. Errors of the third example using (44) and (43).

| $\omega$ | $c$   | $\varepsilon_{1\tau}$ | $\varepsilon_{2}$ | $\varepsilon_{\infty}$ |
|----------|-------|------------------------|------------------|----------------------|
| 40       | 2     | $8.5 \times 10^{-3}$  | $8.89 \times 10^{-4}$ | $1.1 \times 10^{-3}$ |

6. Conclusion. We utilized a novel method to derive absorbing boundary conditions for the LKdV equation, named NSBC. An approximation of the resulting system is described by finite difference method and a sufficient condition of stability is proved and approved in numerical tests. Geophone plots assign the robustness of the proposed approach and the transparency of NSBC. We have also considered the cases of non homogeneous equation with high oscillations at boundary when no reflections have been noticed. We claim that NSBC are relatively easy to implement than other known absorbing boundary conditions. The advantage arises on locality and exactness of the boundary condition. One can remark that stability condition
Figure 14. Instantaneous of $u_{ref}$ (red) and $u_{nsbc}$ (blue) for the fifth example using (43) for $c = 2$ and (44) for $\omega = 40$ with $\tau = 3.2 \times 10^{-6}$ and $h = 4 \times 10^{-4}$.

Figure 15. Geophones of $u_{ref}$ (red) and $u_{nsbc}$ (blue) for the fifth example using (43) for $c = 2$ and (44) for $\omega = 40$ with $\tau = 1.56 \times 10^{-2}$ and $h = 10^{-1}$ at different positions.
Figure 16. Evolution of logarithm of error with respect to time and space for the fifth example using using (43) for $c = 2$ and (44) for $\omega = 40$ with $\tau = 1.56 \times 10^{-2}$ and $h = 10^{-1}$.

Figure 17. Energies $E(t)$ and $M(t)$ evolution in time of $u_{\text{nsbc}}$ (left) and $u_{\text{ref}}$ (right) fifth example using using (43) for $c = 2$ and (44) for $\omega = 40$ with $\tau = 1.56 \times 10^{-2}$ and $h = 10^{-1}$. 
is affected by interpolation added to complete approximation of third derivative. This may be improved modifying interpolation order or adding other functions in formulation process of deriving NSBC. we will dig deeper into this point in the future as well as deriving NSBC for the nonlinear KdV equation.

Appendix A. (Proof of proposition 2). Let $h$ and $\tau$ be the space and time steps respectively, we denote $\theta_{\alpha}^{h} = \frac{\alpha h}{h}$ and $\theta_{\beta}^{h} = \frac{\beta h}{h}$. Take the matrix (20) and make the following elementary combinations on its columns $c_{2} \leftarrow c_{2} + c_{1}$, $c_{n-1} \leftarrow c_{n-1} + c_{n}$, $c_{n+2} \leftarrow c_{n+1} + c_{n+2}$, $c_{2n-1} \leftarrow c_{2n-1} + c_{2n}$, $c_{2n+2} \leftarrow c_{2n+1} + c_{2n+2}$ and $c_{3n-1} \leftarrow c_{3n-1} + c_{3n}$. This implies that the matrix (20) is similar to the following matrix

$$
\hat{A} = \begin{pmatrix}
\hat{A}_1 & -hB & 0 \\
0 & \hat{A}_1 & -hB \\
B & b_1B & \hat{A}_2
\end{pmatrix}
$$

where,

$$
B = \begin{pmatrix}
1 & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 1 & 1
\end{pmatrix}
$$

$$
\hat{A}_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
d_1 & d_1 + d_2 & d_3 & d_4 & \ddots & \vdots & \vdots \\
c_1 & c_1 + c_2 & c_3 & -c_2 & -c_1 & \ddots & \vdots \\
0 & c_1 & c_2 & c_3 & -c_2 & -c_1 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & -d_4 & -d_3 & d_5 - d_1 & -d_1
\end{pmatrix}
$$

$$
\hat{A}_2 = \begin{pmatrix}
-b_2 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
d_1 & d_1 + d_2 & d_3 & d_4 & \ddots & \vdots & \vdots \\
c_1 & c_1 + c_2 & c_3 & -c_2 & -c_1 & \ddots & \vdots \\
0 & c_1 & c_2 & c_3 & -c_2 & -c_1 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & -d_4 & -d_3 & d_5 - d_1 & -d_1
\end{pmatrix}
$$
We seek, then, conditions under which the matrix $\tilde{A}$ has a dominant diagonal. For the sake of simplicity, we denote $n = N + 1$, where we remind that $h = \frac{\alpha}{2\tau}$.

- In the lines 1, $n$, $n + 1$ and $2n$ of the matrix $\tilde{A}$, the diagonal element is strictly dominant for $h$ sufficiently small.
- While in lines $2n + 1$ and $3n$ a sufficient condition to have a dominant diagonal is
  \[
  \frac{\alpha}{2h} - \frac{2}{\tau} \geq \beta
  \]  

(45)

- In line $2$, $n + 2$ and $n + 3$, we have to ensure the following

\[
\begin{cases}
\text{if } d_1 \leq 0 \text{ then } & \\
\text{if } d_1 \geq 0 \text{ then }
\begin{cases}
\text{if } 1 - \frac{1}{4} \theta^r_{\beta} + \frac{1}{2} \theta^r_{\alpha} \geq 0 \text{ then } & \\
\text{if } 1 - \frac{1}{4} \theta^r_{\beta} + \frac{1}{2} \theta^r_{\alpha} \leq 0 \text{ then } & \\
\text{if } 1 - \frac{1}{4} \theta^r_{\beta} + \frac{1}{2} \theta^r_{\alpha} \geq 0 \text{ then } & \\
\text{if } 1 - \frac{1}{4} \theta^r_{\beta} + \frac{1}{2} \theta^r_{\alpha} \leq 0 \text{ then } & \\
\end{cases}
\end{cases}
\]

Combining all these cases gives

\[
(\theta^r_{\beta}, \theta^r_{\alpha}) \in R_2 := \left\{ (x, y) \in \mathbb{R} \times \mathbb{R}^+ : \frac{1}{2} x - \frac{2}{3} \leq y \leq \min \left\{ -\frac{1}{6}, x + \frac{1}{2}, x + 2 \right\} \right\}
\]

which corresponds to the polygon $(FGHI)$ where $F = (-4, 0)$, $G = (\frac{4}{3}, 0)$, $H = (2, \frac{1}{2})$ and $I = (-2, 1)$ (see Figure 18).

- In lines $n - 1$, $2n - 1$ and $3n - 1$, the assertion $|d_5 - d_1| \geq |d_1| + |d_3| + |d_4|$ implies that

\[
\begin{cases}
\text{if } d_5 - d_1 \geq 0 \text{ then } & \\
\text{if } d_5 - d_1 \leq 0 \text{ then }
\begin{cases}
\text{if } d_3 \geq 0 \text{ and } \frac{1}{4} \theta^r_{\beta} + \frac{1}{2} \theta^r_{\alpha} \leq 1 \text{ then } & \\
\text{or } & \\
\text{if } d_3 \leq 0 \text{ and } \frac{1}{4} \theta^r_{\beta} + \frac{7}{2} \theta^r_{\alpha} \leq 1 \text{ then } & \\
\text{if } d_3 \geq 0 \text{ and } \frac{3}{4} \theta^r_{\beta} + 3 \theta^r_{\alpha} \geq 1 \text{ then } & \\
\text{or } & \\
\text{if } d_3 \leq 0 \text{ and } -\frac{3}{2} \theta^r_{\alpha} \geq 1 \text{ then } & \\
\end{cases}
\end{cases}
\]

That is

\[
(\theta^r_{\beta}, \theta^r_{\alpha}) \in R_3 := \left\{ (x, y) \in \mathbb{R} \times \mathbb{R}^+ : y \leq \min \left\{ \frac{2}{5} + \frac{3}{10} x, \frac{2}{7} + \frac{1}{14} x, 2 - \frac{1}{2} x \right\} \right\}
\]

This is a polygon denoted $(AKLM$, where $A = (\frac{2}{3}, 0)$, $K = (4, 0)$, $L = (3, \frac{1}{2})$ and $I = (\frac{1}{2}, \frac{1}{4})$ (see Figure 18).
• In lines 3, $n + 3$, $2n + 3$, $n - 3$, $2n - 3$ and $3n - 3$

\[
\begin{align*}
\text{if } c_1 + c_2 \geq 0 \text{ then } & \quad \begin{cases} 
  c_2 \geq 0 \text{ and } -\frac{1}{2} \theta_{\beta} + \frac{5}{4} \theta_{\alpha} h \leq 1 \\
  \text{or} \\
  c_2 \leq 0 \text{ and } \frac{1}{3} \theta_{\alpha} \leq 1 \\
  c_2 \geq 0 \text{ and } \frac{2}{4} \theta_{\alpha} h \leq 1 \\
  \text{or} \\
  c_2 \leq 0 \text{ and } \frac{1}{2} \theta_{\beta} + \frac{3}{2} \theta_{\alpha} h \leq 1
\end{cases} \\
\text{if } c_1 + c_2 \leq 0 \text{ then } & \quad \begin{cases} 
  c_2 \geq 0 \text{ and } 3 \theta_{\tau,h} \alpha \leq 1 \\
  \text{or} \\
  c_2 \leq 0 \text{ and } 1 \theta_{\beta} h \alpha \leq 1
\end{cases}
\end{align*}
\]

Then, the combination of these cases allows to write

\[
(\theta_{\beta}^{r,h}, \theta_{\alpha}^{r,h}) \in R_3 := \left\{ (x, y) \in \mathbb{R} \times \mathbb{R}^+: 2x - 4 \leq y \leq \min\left\{ \frac{4}{3}, \frac{4}{5} + \frac{2}{5} \right\} \right\}
\]

which is the polygon \((ACED)\) where \(A = \left(\frac{-4}{3}, 0\right)\), \(C = (2, 0)\), \(E = \left(\frac{8}{3}, \frac{4}{3}\right)\) and \(D = \left(\frac{4}{3}, \frac{4}{3}\right)\) (see Figure 18).

• Remaining lines give

\[
\begin{align*}
\text{if } c_1 + c_2 \geq 0 \text{ then } & \quad \begin{cases} 
  -1 \frac{\beta \tau}{h} + \frac{3 \alpha \tau}{2 h^3} \leq 1 & \text{if } c_2 \geq 0 \\
  \frac{1 \beta \tau}{h} - \frac{1 \alpha \tau}{2 h^3} \leq 1 & \text{if } c_2 \leq 0
\end{cases} \\
\text{if } c_1 + c_2 \leq 0 \text{ then } & \quad \begin{cases} 
  c_2 \geq 0 \text{ and } 3 \theta_{\tau,h} \alpha \leq 1 \\
  \text{or} \\
  c_2 \leq 0 \text{ and } 1 \theta_{\beta} h \alpha \leq 1
\end{cases}
\end{align*}
\]

Thus,

\[
(\theta_{\beta}^{r,h}, \theta_{\alpha}^{r,h}) \in R_4 := \left\{ (x, y) \in \mathbb{R} \times \mathbb{R}^+: x - 2 \leq y \leq \frac{2}{3} + \frac{1}{3} x \right\}
\]

that corresponds to the triangle \((ACB)\) where \(A = \left(\frac{-4}{3}, 0\right)\), \(C = (2, 0)\) and \(B = (4, 2)\).

Thus a sufficient condition to have dominant diagonal is that \((\theta_{\beta}^{r,h}, \theta_{\alpha}^{r,h}) \in R\) such that

\[
\bigcap_{1 \leq i \leq 4} R_i := \left\{ (x, y) \in \mathbb{R} \times \mathbb{R}^+: \frac{1}{2} x - \frac{2}{3} \leq y \leq \min\left\{ \frac{2}{5} + \frac{3}{10} x, \frac{2}{7} + \frac{1}{14} x, \frac{2}{3} - \frac{1}{6} x \right\} \right\}
\]

that is the polygon \((JGHNM)\) plotted in figure 18.
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