Some problems about co-consonance of topological spaces

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Abstract

In this paper, we first prove that the retract of a consonant space (or co-consonant space) is consonant (co-consonant). Using this result, some related results have obtained. Simultaneously, we proved that (1) the co-consonance of the Smyth powerspace $P_S(X)$ implies the co-consonance of $X$ under a necessary condition; (2) the co-consonance of $X$ implies the co-consonance of the smyth powerspace under some conditions; (3) if the lower powerspace $P_H(X)$ is co-consonant, then $X$ is co-consonant; (4) the co-consonance of $X$ implies the co-consonance of the lower powerspace $P_H(X)$ with some sufficient conditions.

Key words: retraction; consonant space; co-consonant space; Smyth powerspace; lower powerspace

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1 Introduction

Given a topological space $X$, there are two topologies on $\mathcal{O}(X)$, let $\mathcal{O}(X)$ be the poset of all open sets of $X$ with inclusion order. In this paper, we consider three kinds of topologies on $\mathcal{O}(X)$. One is the topology $\tau$ that has a base $\{\square Q \mid Q \text{ is compact in } X\}$, where $\square Q = \{V \in \mathcal{O}(X) \mid Q \subseteq V\}$, the second is the Scott topology $\sigma(\mathcal{O}(X))$ and the third is the upper topology $\nu(\mathcal{O}(X))$. It is not difficult to find that $\tau \subseteq \sigma(\mathcal{O}(X))$. A topological space

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X is consonant if $\tau = \sigma(\mathcal{O}(X))$. Equivalently, a space $X$ is consonant if the Upper Kuratowski topology and the co-compact on the set of all closed sets of $X$ are equal (see [13]). Consonant spaces have been researched by many scholars and the more conclusions about consonance can see [1,4,13]. Given a poset $P$, the upper topology $\nu(P)$ on $P$ is a topology that has a subbase $\{P \setminus \downarrow x \mid x \in P\}$. Clearly, $\nu(P) \subseteq \sigma(P)$, where $\sigma(P)$ is the Scott topology on $P$. A topological space is co-consonant if the upper topology and the Scott topology on the open set lattice $\mathcal{O}(X)$ coincide (see [2]). Compared with the consonance, the co-consonance is lack of research. In this paper, we focus our interest on co-consonant spaces.

Retract is an ordinary relation of topological spaces. There are many topological properties which are preserved by retraction, such as sobriety, well-filteredness, d-spaces. Naturally, it arises the following two questions:

(1) Is the retract of a consonant space consonant?

(2) Is the retract of a co-consonant space co-consonant?

In section 3, we give a positive answer. Using this result, some related conclusions are discovered.

Given a topological space $X$, there are two constructions of topological spaces, namely, the lower powerspace $P_H(X)$ and smyth powerspace $P_S(X)$ (see [5,7]). Recently, M. Brecht and T. Kawai proved that $X$ is consonant iff the lower powerspace and Smyth powerspace on $X$ is commute (see [2]). Furthermore, M. Brecht and T. Kawai asked that whether the consonance is preserved by the Smyth powerspace construction. This question is first answered by Z. Lyu, Y. Chen, and X. Jia (see [11]). It has proved in [11] that if $X$ is consonant and for natural number $n$,

$$\Sigma(\prod^n \mathcal{O}(X)) = \prod^n \Sigma(\mathcal{O}(X)),$$

then $P_S(X)$ is consonant. Similarly, Y. Chen, H. Kou and Z. Lyu also showed that if $X$ is consonant and for natural number $n$,

$$\Sigma(\prod^n \mathcal{O}(X)) = \prod^n \Sigma(\mathcal{O}(X)),$$

then $P_H(X)$ is consonant (see [3]). Naturally, it arises the following questions.

(3) Is $X$ co-consonant when $P_S(X)$ is co-consonant?

(4) Is the smyth powerspace $P_S(X)$ of a co-consonant space $X$ co-consonant?

(5) Is the lower powerspace $P_H(X)$ of a co-consonant space $X$ co-consonant?

(6) Is $X$ co-consonant when $P_H(X)$ is co-consonant?
In section 4, we will give some answers for these questions.

2 Preliminaries

Given a poset $P$ and $A \subseteq P$, let

$$\uparrow A = \{x \in P \mid a \leq x \text{ for some } a \in A\}$$

and

$$\downarrow A = \{x \in P \mid x \leq a \text{ for some } a \in A\}.$$  

For every $x \in P$, we write $\downarrow x$ for $\downarrow \{x\}$ and $\uparrow x$ for $\uparrow \{x\}$.

Let $P$ be a poset and $x, y \in P$. We say that $x$ is way below $y$, in symbols $x \ll y$, iff for all directed subsets $D \subseteq P$ for which $\bigvee D$ exists, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. If $\Downarrow a = \{b \in P \mid b \ll a\}$ is directed and $\bigvee \Downarrow a = a$ for all $a \in P$, we call $P$ a continuous poset. A complete lattice $L$ is called a continuous lattice if $L$ is a continuous poset.

Let $P$ be a poset. A subset $U$ of $P$ is Scott open (see [5]) if (i) $U = \uparrow U$ and (ii) for any directed subset $D$, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\bigvee D$ exists. The Scott open sets on $P$ form the Scott topology $\sigma(P)$. The Scott space $(P, \sigma(P))$ will be simply written as $\Sigma(P)$. The upper topology $\upsilon(P)$ on poset $P$ is a topology that has $\{P \setminus \downarrow x \mid x \in P\}$ as a subbase.

For a $T_0$ space $(X, \tau)$, the specialization order $\leq$ on $X$ is defined by $x \leq y$ if and only if $x \in \text{cl}(\{y\})$ (see [5], p. 42). We use $\mathcal{O}(X)$ ($\Gamma(X)$) to denote the lattice of all open(closed) subsets of $X$. Note that for each subset $A \subseteq X$, $\uparrow A$ is equal to the intersection of all the open sets containing $A$ and we say $\uparrow A$ is the saturation of $A$. A subset $A \subseteq X$ is saturated if $A = \uparrow A$.

For a topological space $X$, we shall use $\mathcal{Q}(X)$ to denote the poset of all nonempty compact saturated subsets of $X$ with the reverse inclusion order. The upper Vietoris topology on $\mathcal{Q}(X)$ is the topology that has $\{\Box U \mid U \in \mathcal{O}(X)\}$ as a base, where $\Box U = \{K \in \mathcal{Q}(X) \mid K \subseteq U\}$. The upper Vietoris topological space (or called Smyth powerspace) is denoted by $P_S(X)$. Naturally, $\{\Diamond U \mid U \in \Gamma(X)\}$ is a subbase of the closed sets of $P_S(X)$, where $\Diamond U = \{Q \in \mathcal{Q}(X) \mid Q \cap U \neq \emptyset\}$. Then the specialization order of the upper space $P_S(X)$ is the reverse inclusion order.

Given a topological space $X$, the lower powerspace on $\Gamma(X)$ is the topology that has $\{\Diamond U \mid U \in \mathcal{O}(X)\}$ as a subbase, where $\Diamond U = \{V \in \Gamma(X) \mid U \cap V \neq \emptyset\}$. The lower powerspace is denoted by $P_H(X)$. Then $\{\Box U \mid U \in \Gamma(X)\}$ is a subbase of the closed sets of $P_H(X)$, where $\Box U = \{F \in \Gamma(X) \mid F \subseteq U\}$. One can see that $\Diamond (U \cup V) = (\Diamond U) \cup (\Diamond V)$ for each pair of open sets $U$ and $V$. 

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Let $X$ be a $T_0$ space. A nonempty subset $F$ of $X$ is irreducible, if for any $A, B \in \Gamma(X)$, $F \subseteq A \cup B$ implies $F \subseteq A$ or $F \subseteq B$. For every $x \in X$, $cl\{x\}$ is an irreducible closed set of $X$. A topological space $X$ is called to be sober if every irreducible closed set of $X$ is the closure of an unique singleton set. A topological space $X$ is well-filtered if for each filtered family $\mathcal{F}$ of compact saturated subsets of $X$ and each open set $U$ of $X$, $\bigcap \mathcal{F} \subseteq U$ implies $\mathcal{F} \subseteq U$ for some $F \in \mathcal{F}$. It is well known that every sober space is well-filtered (see [5]). A topological space $X$ is coherent if the intersection of two compact saturated subsets is compact. For every complete lattice $L$, the Scott space $\Sigma(L)$ is well-filtered and coherent (see [10,14]).

3 The retract of consonant and co-consonant

In the following, we will give positive answers to the question 1 and question 2 proposed in introduction.

Definition 3.1 A $T_0$ space $X$ is consonant if for every $\mathcal{F} \in \sigma(\mathcal{O}(X))$ and $U \in \mathcal{F}$, there is $Q \in \mathcal{Q}(X)$ such that $U \in \square Q \subseteq \mathcal{F}$, where $\square Q = \{V \in \mathcal{O}(X) \mid Q \subseteq V\}$.

Example 3.2 (1) Every locally $k_\omega$ space is consonant (see [15]).
(2) A topological space $X$ is locally compact iff $X$ is a core-compact and consonant space (see [3]).
(3) The Sorgenfrey line is not consonant (see [7]).

Definition 3.3 (see [2]) A $T_0$ space $X$ is co-consonant if for every $\mathcal{F} \in \sigma(\mathcal{O}(X))$ and $U \in \mathcal{F}$, there is a finite subset $\mathcal{E} \subseteq \Gamma(X)$ such that $U \in \bigcap \{\lozenge A \mid A \in \mathcal{E}\} \subseteq \mathcal{F}$, where $\lozenge A = \{V \in \mathcal{O}(X) \mid A \cap V \neq \emptyset\}$. 

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Clearly, \( X \) is co-consonant iff the upper topology is agree with the Scott topology on \( \mathcal{O}(X) \).

**Example 3.4** (1) If \( X \) is a quasi-polish space, then \( P_S(X) \) is a co-consonant space (see [2]).

(2) Let \( L \) be the Isbell complete lattice (see [3]). We define \( \mathcal{O}(\hat{L}) = \{ \hat{L} \setminus K \mid K \in \mathcal{Q}(L) \} \), where \( \hat{L} = L \setminus \{ 1_L \} \) and \( 1_L \) is the top element of \( L \). Then \( (L, \mathcal{O}(\hat{L})) \) is a topological space and \( \Sigma(\mathcal{O}(\hat{L})) \) is non-sober (see [3]). By Proposition 2.9 in [10], the upper topology on \( \mathcal{O}(\hat{L}) \) is sober. Hence we can assert that \( \sigma(\mathcal{O}(\hat{L})) \neq \nu((\mathcal{O}(\hat{L}))) \). This implies that \( (L, \mathcal{O}(\hat{L})) \) is not co-consonant.

The Example 3.4(2) illustrates that the following conclusion holds.

**Fact 3.5** Let \( X \) be a topological space. If \( X \) is co-consonant, then \( \Sigma(\mathcal{O}(X)) \) is sober.

Next, we will give two kinds of order spaces which are co-consonant. Given a poset \( P \), the Alexandroff topology \( \alpha(P) \) is the topology consisting of all its upper subsets of \( P \).

**Proposition 3.6** Let \( P \) be a poset. Then the Alexandroff topological space \( (P, \alpha(P)) \) is co-consonant.

**Proof.** It is clear that \( \nu(\alpha(P)) \subseteq \sigma(\alpha(P)) \). Let \( V \in \sigma(\alpha(P)) \) and \( V \in \mathcal{V} \) with \( V \neq \emptyset \). Note that \( V = \bigcup \{ \uparrow F \mid F \subseteq V \text{ is a finite non-empty subset} \} \). It follows from \( V \in \sigma(\alpha(P)) \) that there exists a subset \( F_0 = \{ x_1, x_2, \ldots, x_n \} \) of \( V \) such that \( \uparrow F_0 \in \mathcal{V} \). Clearly \( V \in \bigcap_{1 \leq m \leq n} \alpha(P) \setminus \downarrow (P \setminus (\downarrow x_m)) \). Let \( B \in \bigcap_{1 \leq m \leq n} \alpha(P) \setminus \downarrow (P \setminus (\downarrow x_m)) \). Then \( B \in \alpha(P) \) and \( B \cap \downarrow x_m \neq \emptyset \) for all \( 1 \leq m \leq n \). This means that \( x_m \in B \) for all \( 1 \leq m \leq n \). Since \( \uparrow F_0 = \bigcup_{1 \leq m \leq n} \uparrow x_m \in \mathcal{V} \), we have \( B \in \mathcal{V} \). So \( V \in \bigcap_{1 \leq m \leq n} \alpha(P) \setminus \downarrow (P \setminus (\downarrow x_m)) \subseteq \mathcal{V} \) and thus \( \nu(\alpha(P)) = \sigma(\alpha(P)) \). \( \Box \)

**Proposition 3.7** Let \( P \) be a continuous poset. Then \( \Sigma(P) \) is co-consonant.

**Proof.** Let \( U \in \sigma(\sigma(P)) \) and \( U \in \mathcal{U} \). Since \( P \) is a continuous poset, we have \( U = \bigcup_{u \in U} \uparrow u \). Note that \( S = \bigcup_{1 \leq s \leq m} \uparrow u_s \mid u_s \in U, m \in \mathbb{N} \} \) is directed and \( U = \bigcup S \). As \( U \in \sigma(\sigma(P)) \), there are finitely \( u_1, u_2, \ldots, u_n \in U \) such that \( \bigcup_{1 \leq k \leq n} \uparrow u_k \in \mathcal{U} \). Define a map \( f_n : \prod P \rightarrow \Sigma(\sigma(P)) \) as follows

\[
\forall (x_1, x_2, \ldots, x_n) \in \prod P, f((x_1, x_2, \ldots, x_n)) = \bigcup_{1 \leq k \leq n} \uparrow x_k.
\]

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Clearly, \( \downarrow u_1 \times \downarrow u_2 \times \cdots \times \downarrow u_n \subseteq f_n^{-1}(U) \). Let \( V \in \bigcap_{1 \leq k \leq n} \downarrow u_k \). Then for all \( 1 \leq k \leq n \), \( V \cap \downarrow u_k \neq \emptyset \). This means that \( u_k \in V \). Since \( \bigcup_{1 \leq k \leq n} \uparrow v_k \subseteq V \) and \( U \in \sigma(\sigma(P)) \), we have \( V \in U \). This implies that \( U \in \bigcap_{1 \leq k \leq n} \downarrow u_k \subseteq U \). Thus, \( \Sigma(P) \) is co-consonant. \( \square \)

A topological space \( X \) is a retract of a topological space \( Y \) if there are two continuous maps \( f : X \to Y \) and \( g : Y \to X \) such that \( g \circ f = id_X \).

**Theorem 3.8** Let \( X \) be a T_0 space. If \( X \) is a retract of a consonant space \( Y \), then \( X \) is consonant.

**Proof.** Let \( F \in \sigma(\mathcal{O}(X)) \) and \( U \in F \). Since \( X \) is a retract of \( Y \), there are two continuous maps \( f : X \to Y \) and \( g : Y \to X \) such that \( g \circ f = id_X \).

Define a mapping \( \alpha : \Sigma(\mathcal{O}(Y)) \to \Sigma(\mathcal{O}(X)) \) as follows:

\[
\forall U \in \mathcal{O}(Y), \alpha(U) = f^{-1}(U).
\]

Clearly, \( \alpha \) is continuous. So we have \( \tilde{F} = \alpha^{-1}(F) \in \sigma(\mathcal{O}(Y)) \). Since \( g \circ f = id_X \), \( g \) is a surjection and for every subset \( A \subseteq X \), \( A = f^{-1}(g^{-1}(A)) \). So \( f^{-1}(g^{-1}(U)) = U \) and thus \( g^{-1}(U) \in \tilde{F} \). Since \( Y \) is consonant and \( g^{-1}(U) \in \tilde{F} \), there is \( Q \in \mathcal{Q}(Y) \) such that \( g^{-1}(U) \subseteq \square Q \subseteq \tilde{F} \). Note that for each subset \( B \subseteq X \), \( g(g^{-1}(B)) = B \) since \( g \) is a surjection.

**Claim:** \( U \in \square \uparrow g(Q) \subseteq F \).

Clearly, \( \uparrow g(Q) \in \mathcal{Q}(X) \) and \( \uparrow g(Q) \subseteq \uparrow g(g^{-1}(U)) = U \). Whence, \( U \in \square \uparrow g(Q) \). For each open set \( E \in \square \uparrow g(Q) \), \( g(Q) \subseteq \uparrow g(Q) \subseteq E \). This implies that

\[
Q \subseteq g^{-1}(g(Q)) \subseteq g^{-1}(E).
\]

Therefore, \( g^{-1}(E) \in \square Q \) and hence \( g^{-1}(E) \in \tilde{F} \). This implies that \( E = f^{-1}(g^{-1}(E)) \in F \). This means that \( \square \uparrow g(Q) \subseteq F \). Hence, \( X \) is consonant. \( \square \)

**Theorem 3.9** Let \( X \) be a T_0 space. If \( X \) is a retract of a co-consonant space \( Y \), then \( X \) is co-consonant.

**Proof.** Let \( F \in \sigma(\mathcal{O}(X)) \) and \( U \in F \). Since \( X \) is a retract of \( Y \), there are two continuous mappings \( f : X \to Y \) and \( g : Y \to X \) such that \( g \circ f = id_X \). Similarly, by the proof of Theorem 3.8, \( \tilde{F} = \{ V \in \mathcal{O}(Y) \mid f^{-1}(V) \in F \} \in \sigma(\mathcal{O}(Y)) \) and \( g^{-1}(U) \in \tilde{F} \). By the co-consonance of \( Y \), there is a finite subset \( \mathcal{E} \subseteq \Gamma(Y) \) such that \( g^{-1}(U) \in \cap \{ \uparrow A \mid A \in \mathcal{E} \} \subseteq \tilde{F} \). Let \( \tilde{\mathcal{E}} = \{ f^{-1}(A) \mid A \in \mathcal{E} \} \). It follows form the continuity of \( f \) that \( \tilde{\mathcal{E}} \) is a finite subset of \( \Gamma(X) \).

**Claim:** \( U \in \cap \{ \uparrow B \mid B \in \tilde{\mathcal{E}} \} \subseteq F \).
Since \( g^{-1}(U) \in \bigcap \{ \Diamond A \mid A \in \mathcal{E} \} \), \( g^{-1}(U) \cap A \neq \emptyset \) for each \( A \in \mathcal{E} \). Take \( y \in g^{-1}(U) \cap A \). So \( g(y) \in U \) and thus \( y = f(g(y)) \in A \). Equivalently, \( g(y) \in U \cap f^{-1}(A) \). This means that \( U \in \bigcap \{ \Diamond B \mid B \in \mathcal{F} \} \). Take \( W \in \bigcap \{ \Diamond B \mid B \in \mathcal{F} \} \). One can easily check that \( f(x) \in g^{-1}(W) \cap A \). This implies that \( g^{-1}(W) \in \bigcap \{ \Diamond A \mid A \in \mathcal{E} \} \). So \( W = f^{-1}(g^{-1}(W)) \in \mathcal{F} \) and thus \( \bigcap \{ \Diamond B \mid B \in \mathcal{F} \} \subseteq \mathcal{F} \). Therefore \( X \) is co-consonant.

Next, we present some applications of Theorem 3.8 and Theorem 3.9.

Remark 3.10 (1) For a complete lattice \( L \), \( \Sigma L \) is a retract of \( \Sigma(\Gamma(\Sigma L)) \) (see [12]).

(2) Let \( X \) be a \( T_0 \) space. Define mappings \( \phi : P_S(X) \to P_S(P_S(X)) \) and \( \varphi : P_S(P_S(X)) \to P_S(X) \) as follows:
\[
\forall Q \in \mathcal{Q}(X), \phi(Q) = \uparrow_{P_S(X)}\xi(Q)
\]
and
\[
\forall A \in \mathcal{Q}(P_S(X)), \varphi(A) = \bigcup A,
\]
where the mapping \( \xi : X \to P_S(X) \) is defined by \( \xi(x) = \uparrow x \). Then \( \phi \) and \( \varphi \) are continuous such that \( \varphi \circ \phi = \text{id}_{P_S(X)} \) and \( \phi \circ \varphi \geq \text{id}_{P_S(P_S(X))} \). So we can conclude that \( P_S(X) \) is a strong retract of \( P_S(P_S(X)) \).

Corollary 3.11 (1) Let \( L \) be a complete lattice. If \( \Sigma(\Gamma(L)) \) is consonant (co-consonant), \( \Sigma L \) is consonant (co-consonant).

(2) Let \( X \) be a \( T_0 \) space. If \( P_S(P_S(X)) \) is consonant (co-consonant), \( P_S(X) \) is consonant (co-consonant).

Lemma 3.12 (see [13]) Let \( L \) be a complete lattice. Then \( \mathcal{Q}(L) = \mathcal{Q}(\Sigma L) \) is a complete Hetying algebra.

Remark 3.13 Let \( L \) be a complete lattice. Then for any \( \{ Q_i \mid i \in I \} \subseteq \mathcal{Q}(L) \), \( \bigvee_{i \in I} Q_i = \bigcap_{i \in I} Q_i \).

Lemma 3.14 Let \( L \) be a complete lattice. If \( \Sigma L \) is consonant, then \( \Sigma(\mathcal{Q}(L)) \) is a retract of \( \Sigma(\sigma(\sigma(L))) \).

Proof. Define mappings \( f : \Sigma(\mathcal{Q}(L)) \to \Sigma(\sigma(\sigma(L))) \) and \( g : \Sigma(\sigma(\sigma(L))) \to \Sigma(\mathcal{Q}(L)) \) as follows:
\[
\forall Q \in \mathcal{Q}(L), f(Q) = \Box Q,
\]
and
\[
\forall \mathcal{F} \in \sigma(\sigma(L)), g(\mathcal{F}) = \bigcap \mathcal{F}.
\]
Proposition 3.16
Let \( \Sigma L \) be consonant and \( F \in \sigma(\sigma(L)) \), there exists \( K \subseteq Q(L) \) such that \( F = \bigcup \{ \Box K \mid K \in K \} \). Then \( \bigcap F = \bigcap \{ \bigcup \{ \Box K \mid K \in K \} \} = \bigcap \bigcap K \subseteq \bigcap K \). By Remark 3.12, we can see \( \bigcap F \in Q(L) \) and \( g \) is well-defined. Clearly, \( g \circ f = id_{\Sigma Q(L)} \).

Claim: \( f \) and \( g \) are continuous mappings.

Let \( \{ Q_i \mid i \in I \} \subseteq Q(L) \) be a directed subset. By Remark 3.12, \( \bigvee \{ Q_i \mid i \in I \} = \bigcap \{ Q_i \mid i \in I \} \). Then
\[
\begin{align*}
  f(\bigvee \{ Q_i \mid i \in I \}) &= f(\bigcap \{ Q_i \mid i \in I \}) = \Box(\bigcap \{ Q_i \mid i \in I \}).
\end{align*}
\]
Clearly,
\[
\bigvee \{ f(Q_i) \mid i \in I \} = \bigcup \{ \Box Q_i \mid i \in I \} \subseteq f(\bigvee \{ Q_i \mid i \in I \}) = \Box(\bigcap \{ Q_i \mid i \in I \}).
\]
For each \( A \in \Box(\bigcap \{ Q_i \mid i \in I \}) \), equivalently \( \bigcap \{ Q_i \mid i \in I \} \subseteq A \). Since \( \Sigma L \) is well-filtered, there exists \( Q_{i_0} \) such that \( Q_{i_0} \subseteq A \). So \( A \in \Box Q_{i_0} \) and thus
\[
\begin{align*}
  f(\bigvee \{ Q_i \mid i \in I \}) &\subseteq \bigvee \{ f(Q_i) \mid i \in I \}.
\end{align*}
\]
Hence, \( f(\bigvee \{ Q_i \mid i \in I \}) = \bigvee \{ f(Q_i) \mid i \in I \} \). This means that \( f \) is Scott continuous. Let \( \{ F_i \mid i \in I \} \subseteq \sigma(\sigma(L)) \) be a directed subset. Then \( g(\bigvee \{ F_i \mid i \in I \}) = g(\bigcup \{ F_i \mid i \in I \}) = \bigcap \{ \bigcup F_i \mid i \in I \} = \bigcap \bigcap F_i = \bigvee \{ g(F_i) \mid i \in I \} \). So \( g \) is also Scott continuous. Hence, \( \Sigma(Q(L)) \) is a retract of \( \Sigma(\sigma(\sigma(L))) \). \( \square \)

By Theorem 3.8 and Theorem 3.9, we have the following corollary immediately.

**Corollary 3.15** Let \( L \) be a complete lattice. If \( \Sigma L \) is consonant, then the following statements hold:

1. if \( \Sigma(\sigma(\sigma(L))) \) is consonant, then \( \Sigma(Q(L)) \) is consonant;
2. if \( \Sigma(\sigma(\sigma(L))) \) is co-consonant, then \( \Sigma(Q(L)) \) is co-consonant.

**Proposition 3.16** Let \( X \) and \( Y \) be a pair of \( T_0 \) spaces. If \( X \times Y \) is co-consonant (consonant), then \( X \) is co-consonant (consonant).

**Proof.** Fix a \( y_0 \in Y \), the mapping \( \alpha_{y_0} : X \longrightarrow X \times Y \) is given by
\[
\forall x \in X, \quad \alpha_{y_0}(x) = (x, y_0).
\]
For each \( A \in O(X \times Y) \), \( A = \bigcup_{i \in I} U_i \times V_i \) for some \( \{ U_i \mid i \in I \} \subseteq O(X) \) and

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\{V_i \mid i \in I\} \subseteq \mathcal{O}(Y)$. One can check that

$$\alpha_{y_0}^{-1}(A) = \begin{cases} \bigcup_{i \in I_0} U_i, & I_0 = \{i \in I \mid y_0 \in V_i\} \neq \emptyset, \\ \emptyset, & y_0 \notin \bigcup_{i \in I} V_i. \end{cases}$$

This means that $\alpha_{y_0}$ is a continuous mapping. Clearly, $p \circ \alpha_{y_0} = \text{id}_X$, where $p : X \times Y \longrightarrow X$ is the projection. So we can assert that $X$ is a retract of $X \times Y$. By Theorem 3.8 and Theorem 3.9, $X$ is co-consonant (consonant). 

\section{Co-consonance of powerspaces}

In this section, we will give some answers for those four questions which is proposed in the introduction. Specifically, the answer of question 3 is negative and the partial answers of question 4 and question 5 is given. Furthermore, the answer of question 6 is positive.

Before answering question 3, it is necessary to introduce the concept of strongly compact subsets. A subset $K$ of a topological space $X$ is strongly compact if for each open set $U$, $K \subseteq U$ implies that there is a finite subset $F \subseteq X$ such that $K \subseteq \uparrow F \subseteq U$ (see [2]).

\textbf{Lemma 4.1} \textit{(see [2])} Let $X$ be a co-consonant $T_0$ space. Then every compact subset of $X$ is strongly compact.

In the following, we prove that the converse of Lemma 4.1 is true if $P_S(X)$ is co-consonant.

\textbf{Theorem 4.2} Let $X$ be a $T_0$ space. If $P_S(X)$ is co-consonant and every compact subset of $X$ is strongly compact, then $X$ is co-consonant.

\textbf{Proof.} Define a mapping $\xi : X \longrightarrow P_S(X)$ by $\xi(x) = \uparrow x$, for all $x \in X$. Then $\xi$ is continuous. So the mapping $f : \mathcal{O}(P_S(X)) \longrightarrow \mathcal{O}(X)$ is well defined, where $f(U) = \xi^{-1}(U)$ for all $U \in \mathcal{O}(P_S(X))$. Clearly, $f$ is Scott continuous. For each $\mathcal{H} \in \sigma(\mathcal{O}(X))$ and $U \in \mathcal{H}$. Then $\square U \in f^{-1}(\mathcal{H})$ and

$$f^{-1}(\mathcal{H}) = \{\bigcup_{i \in I} \square U_i \mid f(\bigcup_{i \in I} \square U_i) \in \mathcal{H}\}$$

$$= \bigsqcup_{i \in I} \{x \mid \uparrow x \in \bigcup_{i \in I} \square U_i \in \mathcal{H}\}$$

$$= \bigsqcup_{i \in I} \{x \mid x \in \bigcup_{i \in I} U_i \in \mathcal{H}\}$$

$$= \bigcup_{i \in I} \bigsqcup_{i \in I} U_i \in \mathcal{H}.\}$$
By the continuity of $f$, $f^{-1}(\mathcal{H}) \in \sigma(\mathcal{O}(P_s(X)))$. Since $P_s(X)$ is co-consonant, there is a finite subset $F \subseteq \Gamma(P_s(X))$ such that $\Box U \in \bigcap\{\Diamond F \mid F \in F\} \subseteq f^{-1}(\mathcal{H})$. For each $F \in F$, let $F = \bigcap\{\Diamond V_i \mid i \in I_F\}$, where $\{V_i \mid i \in I_F\} \subseteq \Gamma(X)$. Since $\Box U \in \Diamond F$ for each $F \in F$, there exists a $Q_F \in \mathcal{Q}(X)$ such that $Q_F \subseteq U$ and $Q_F \in F$. By assumption, $Q$ is strongly compact. Then there is a finite subset $N_F$ such that $Q_F \subseteq \uparrow N_F \subseteq U$. Let $\bar{F} = \{\text{cl}(\{x\}) \mid x \in \bigcup_{F \in F} N_F\}$. Clearly, $\bar{F}$ is a finite subset of $\Gamma(X)$.

**claim 1:** $U \in \bigcap\{\Diamond A \mid A \in \bar{F}\}$.

For each $x \in \bigcup_{F \in F} N_F$, $x \in N_{F_0}$ for some $F_0 \in F$. It follows form $\uparrow N_{F_0} \subseteq U$ that $x \in U$. So $U \in \Diamond(\text{cl}(\{x\}))$ and thus $U \in \bigcap\{\Diamond A \mid A \in \bar{F}\}$.

**claim 2:** $\bigcap\{\Diamond A \mid A \in \bar{F}\} \subseteq \mathcal{H}$.

Let $V$ be an open set of $X$ with $V \in \bigcap\{\Diamond A \mid A \in \bar{F}\}$. Then for each $x \in \bigcup_{F \in F} V \cap \text{cl}(\{x\}) \neq \emptyset$. This means that $\bigcup_{F \in F} N_F \subseteq V$. Thus for each $F \in F$, $\uparrow N_F \subseteq \Box V$. Since $Q_F \subseteq \uparrow N_F$ and $Q_F \in F$, $\uparrow N_F \in F$. It follows from $\uparrow N_F \subseteq \Box V \cap F$ that $\Box V \subseteq \Diamond F$ for each $F \in F$. Since $\bigcap\{\Diamond F' \mid F' \in F\} \subseteq f^{-1}(\mathcal{H})$, $\Box V \in f^{-1}(\mathcal{H})$. Then there are open sets $\{U_i \mid i \in I\} \subseteq \mathcal{O}(X)$ such that $\Box V = \bigcup_{i \in I} \Box U_i$ and $\bigcup_{i \in I} U_i \in \mathcal{H}$. Note that

$$V = \bigcup_{i \in I} \Box V = \bigcup_{i \in I} (\bigcup_{i \in I} \Box U_i) = \bigcup_{i \in I} (\bigcup_{i \in I} U_i) = \bigcup_{i \in I} U_i.$$

So we have $V \in \mathcal{H}$. Hence the claim 2 is proved. By two claims, we can conclude that $X$ is co-consonant. □

Note that the co-consonance of $P_s(X)$ does not imply that every compact subset of $X$ is strongly compact. Please see the following example.

**Example 4.3** Let $P = \{\infty\} \cup \mathbb{N}$ and $\mathbb{N}$ be the set of all natural numbers. The order on $P$ is given by:

$$\forall x \in P, \, x \leq \infty.$$

Then $X = (P, \nu(P))$ is a quasi-polish space (see [2] Example 3.2). By Example 3.4(1), $P_s(X)$ is co-consonant. It is not difficult to verify that $X$ is a compact space. However the compact subset $P$ is not strongly compact in $X$. So we can assert that $X$ is not co-consonant.

The following conclusion is one of the most important conclusions of this paper. This gives a partial answer of question 4.

**Theorem 4.4** Let $X$ be a co-consonant space. If for every natural number $n$,

$$\Sigma(\prod^n \mathcal{O}(X)) = \prod^n \Sigma(\mathcal{O}(X)),$$

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then $P_S(X)$ is co-consonant.

**Proof.** Let $\mathcal{F} \in \sigma(\mathcal{O}(P_S(X)))$ and $\mathcal{F} \in \mathcal{F}$. Then $\mathcal{F} = \bigcup_{i \in I} \Box U_i$ for some family $\{U_i \in \mathcal{O}(X) \mid i \in I\}$. Since $\mathcal{F}$ is Scott open, there is a finite subset $\{U_k \mid 1 \leq k \leq n\} \subseteq \{U_i \mid i \in I\}$ such that $\bigcup_{1 \leq k \leq n} \Box U_k \in \mathcal{F}$. Define a mapping $\beta_n : \Sigma(\prod(\mathcal{O}(X))) \rightarrow \Sigma(\mathcal{O}(P_S(X)))$ as follows:

$$\beta_n(V_1, V_2, \ldots, V_n) = \bigcup_{i=1}^n \Box V_i.$$ 

Since $\beta_n$ is Scott continuous for each component $V_i$, $\beta_n$ is continuous. As $\Sigma(\prod(\mathcal{O}(X))) = \prod_{n} \Sigma(\mathcal{O}(X))$, $\beta_n$ is also continuous form $\prod_{n} \Sigma(\mathcal{O}(X))$ to $\Sigma(\mathcal{O}(P_S(X)))$. Then there are finitely Scott open sets $\{\mathcal{H}_k \mid 1 \leq k \leq n\} \subseteq \sigma(\mathcal{O}(X))$ such that

$$(U_1, U_2, \ldots, U_n) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n \subseteq \beta_n^{-1}(\mathcal{F}).$$

By the co-consonance of $X$, there is a finite subset $\mathcal{E}_k \subseteq \Gamma(X)$ such that $U_k \in \cap\{\Diamond V \mid V \in \mathcal{E}_k\} \subseteq \mathcal{H}_k$, for each $1 \leq k \leq n$. Let $E_k = \cap\{\Diamond V \mid V \in \mathcal{E}_k\}$. Then $\mathcal{E} = \{E_k \mid 1 \leq k \leq n\}$ is a finite subset of $\Gamma(P_S(X))$.

**Claim 1:** $\bigcup_{1 \leq k \leq n} \Box U_k \in \mathcal{E} = \{\Diamond E_k \mid 1 \leq k \leq n\}$.

For each $1 \leq k \leq n$, $U_k \in \cap\{\Diamond V \mid V \in \mathcal{E}_k\}$. Then for each $V \in \mathcal{E}_k$, $U_k \cap V \neq \emptyset$. Take a $x_{k,V} \in U_k \cap V$. Let $F_k = \uparrow\{x_{k,V} \mid V \in \mathcal{E}_k\}$. By the finiteness of $\mathcal{E}_k$, $F_k$ is compact in $X$. Clearly, $F_k \supseteq \Box U_k \cap E_k$. It follows from $\Box U_k \in \Diamond E_k$ that $\bigcup_{1 \leq k \leq n} \Box U_k \in \Diamond E_k$. So $\bigcup_{1 \leq k \leq n} \Box U_k \in \cap\{\Diamond E_k \mid 1 \leq k \leq n\}$ and thus $\mathcal{F} \in \cap\{\Diamond E_k \mid 1 \leq k \leq n\}$.

**Claim 2:** $\cap\{\Diamond E_k \mid 1 \leq k \leq n\} \subseteq \mathcal{F}$.

For each $\bigcup_{j \in J} \Diamond W_j \in \cap\{\Diamond E_k \mid 1 \leq k \leq n\}$, we have that for each $1 \leq k \leq n$, there is a $Q_k \in \mathcal{Q}(X)$ such that $Q_k \in \bigcup_{j \in J} \Diamond W_j \cap E_k$. Then there exists $j_k \in J$ satisfying $Q_k \in \Box W_{j_k}$. As $Q_k \in E_k$ and $Q_k \in \Box W_{j_k}$, $W_{j_k} \in \cap\{\Diamond V \mid V \in \mathcal{E}_k\}$. This means that $W_{j_k} \in \mathcal{H}_k$ for each $1 \leq k \leq n$. So we have

$$(W_{j_1}, W_{j_2}, \ldots, W_{j_n}) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n \subseteq \beta_n^{-1}(\mathcal{F}).$$

Whence, $\bigcup_{1 \leq k \leq n} \Box W_{j_k} \in \mathcal{F}$. Since $\mathcal{F}$ is an upper set in $\mathcal{O}(P_S(X))$, $\bigcup_{j \in J} \Box W_j \in \mathcal{F}$. Those two claims imply that $P_S(X)$ is co-consonant. \(\square\)

By Lemma 2.1, Lemma 2.2 and Theorem 4.4, the following conclusion holds immediately.

**Corollary 4.5** Let $X$ be a co-consonant space. If $X$ is core-compact or $\Sigma(\mathcal{O}(X))$
is first-countable, then \( P_s(X) \) is co-consonant.

The next conclusion demonstrates that co-consonance of original space is necessary for the co-consonance of lower powerspace. Thus we give a positive answer of question 6.

**Theorem 4.6** Let \( X \) be a \( T_0 \) space. If \( P_H(X) \) is co-consonant, then \( X \) is co-consonant.

**Proof.** Define a mapping \( \xi : X \rightarrow P_H(X) \) by \( \xi(x) = cl(\{x\}) \), for all \( x \in X \). Then \( \xi \) is a topological embedding. So the mapping \( \eta : \mathcal{O}(P_H(X)) \rightarrow \mathcal{O}(X) \) is well defined, where \( \eta(U) = \xi^{-1}(U) \) for all \( U \in \mathcal{O}(P_H(X)) \). Clearly, \( \eta \) is Scott continuous. Let \( \mathcal{F} \in \sigma(\mathcal{O}(X)) \) and \( U \in \mathcal{F} \). By the definition of mapping \( \eta \), we have

\[
\eta^{-1}(\mathcal{F}) = \{ U \in \mathcal{O}(P_H(X)) \mid \xi^{-1}(U) \in \mathcal{F} \}
\]

\[
= \{ \bigcup_{i \in I} \left( \bigcap_{j \in J_i} \diamond U_j \right) \mid \{ x \mid cl(\{x\}) \in \bigcup_{i \in I} \left( \bigcap_{j \in J_i} \diamond U_j \right) \in \mathcal{F} \}
\]

\[
= \{ \bigcup_{i \in I} \left( \bigcap_{j \in J_i} \diamond U_j \right) \mid \{ x \mid \exists i_0 \in I, x \in \bigcap_{j \in J_{i_0}} U_j \} \in \mathcal{F} \}
\]

\[
= \{ \bigcup_{i \in I} \left( \bigcap_{j \in J_i} \diamond U_j \right) \mid \bigcup_{i \in I} \left( \bigcap_{j \in J_i} U_j \right) \in \mathcal{F} \},
\]

where \( J_i \) is finite for each \( i \in I \) and \( \{ U_j \mid j \in J_i \} \subseteq \mathcal{O}(X) \). Then \( \eta^{-1}(\mathcal{F}) \in \sigma(\mathcal{O}(P_H(X))) \) and \( \diamond U \in \eta^{-1}(\mathcal{F}) \). By the co-consonance of \( P_H(X) \), there is a finite subset \( \mathcal{E} \subseteq \Gamma(P_H(X)) \) such that

\[
\diamond U \in \bigcap \{ \diamond A \mid A \in \mathcal{E} \} \subseteq \eta^{-1}(\mathcal{F}).
\]

Then for each \( A \in \mathcal{E} \), \( \diamond U \cap A \neq \emptyset \). So there exists a nonempty closed \( F_A \subseteq X \) satisfying \( F_A \in \diamond U \cap A \). Let \( \tilde{\mathcal{E}} = \{ F_A \mid A \in \mathcal{E} \} \). Then \( \tilde{\mathcal{E}} \) is finite subset of \( \Gamma(X) \). Clearly, \( U \in \bigcap \{ \diamond F_A \mid A \in \mathcal{E} \} \). Let \( V \) be an open set with \( V \in \bigcap \{ \diamond F_A \mid A \in \mathcal{E} \} \). Then for each \( A \in \mathcal{E} \), \( V \cap F_A \neq \emptyset \). This means that \( F_A \in \diamond V \cap A \). Hence, \( \diamond V \in \bigcap \{ \diamond A \mid A \in \mathcal{E} \} \subseteq \eta^{-1}(\mathcal{F}) \). Whence, \( V = \eta(\diamond V) \in \mathcal{F} \). This implies that \( U \in \bigcap \{ \diamond B \mid B \in \tilde{\mathcal{E}} \} \subseteq \mathcal{F} \). Therefore, \( X \) is co-consonant. \( \square \)

Before answering question 5, it is necessary to introduce the following concept.

**Definition 4.7** A topological space \( X \) is called intersection-compatible if for every pair of open sets \( U, V \in \mathcal{O}(X) \) and a closed set \( W \in \Gamma(X) \), \( U \cap W \neq \emptyset \) and \( V \cap W \neq \emptyset \) implies \( U \cap V \cap W \neq \emptyset \).

**Remark 4.8** If every closed set of a topological space \( X \) is irreducible, then \( X \) is intersection-compatible. In particular, the Scott spaces of chains are intersection-compatible.
Theorem 4.9 Let $X$ be a co-consonant and intersection-compatible space. If for every natural number $n$,

$$\Sigma(\prod^n \mathcal{O}(X)) = \prod^n \Sigma(\mathcal{O}(X)),$$

then $P_H(X)$ is co-consonant.

Proof. Let $F \in \sigma(\mathcal{O}(P_H(X)))$ and $J \in \mathcal{F}$. Then $\mathcal{F} = \bigcup \{ \bigcap \mathcal{O}U_j \}$, where $J_i$ is finite for each $i \in I$ and $\{ U_j \mid j \in J_i, i \in I \} \subseteq \mathcal{O}(X)$. As $\mathcal{F}$ is Scott open, there exists a finite subset $F_0 \subseteq I$ such that $\bigcup \{ \bigcap \mathcal{O}U_j \} \in \mathcal{F}$. For convenience, let $F_0 = \{1, 2, \ldots, n_0\}$.

Let $\mathcal{S}_1 = \{ \bigcup_{1 \leq k \leq n_0} U_{i_k} \mid i_k \in J_k \}$. Let $n = |\mathcal{S}_1|$ and $\mathcal{S}_1 = \{ U_1, U_2, \ldots, U_n \} \subseteq \mathcal{O}(X)$. Then

$$\bigcup_{i \in F_0} \bigcap_{j \in J_i} \mathcal{O}U_j = \bigcap_{U \in \mathcal{S}_1} \mathcal{O}U = \bigcap_{1 \leq k \leq n} \mathcal{O}U_k,$$

and $\bigcap_{1 \leq k \leq n} \mathcal{O}U_k \in \mathcal{F}$. Define a mapping $\beta_n : \Sigma(\prod^n \mathcal{O}(X)) \rightarrow \Sigma(\mathcal{O}(P_H(X)))$ as follows:

$$\beta_n(V_1, V_2, \ldots, V_n) = \bigcap_{i=1}^n \mathcal{O}V_i.$$

Since $\beta_n$ is Scott continuous for each component $V_i$, $\beta_n$ is continuous. As $\Sigma(\prod^n \mathcal{O}(X)) = \prod^n \Sigma(\mathcal{O}(X))$, $\beta_n$ is also continuous form $\prod^n \Sigma(\mathcal{O}(X))$ to $\Sigma(\mathcal{O}(P_H(X)))$.

Then there are finitely Scott open sets $\{ \mathcal{H}_k \mid 1 \leq k \leq n \} \subseteq \sigma(\mathcal{O}(X))$ such that

$$(U_1, U_2, \ldots, U_n) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n \subseteq \beta_n^{-1}(\mathcal{F}).$$

By the co-consonance of $X$, there is a finite subset $\mathcal{F}_k \subseteq \Gamma(X)$ such that $U_k \in \bigcap \{ \diamond A \mid A \in \mathcal{F}_k \} \subseteq \mathcal{H}_k$, for each $1 \leq k \leq n$. Let $\mathcal{E}_1 = \{ \square(\bigcup_{j=1}^n A_j) \mid A_j \in \mathcal{F}_j, 1 \leq j \leq n \}$. Clearly, $\mathcal{E}_1$ is a finite subset of $\Gamma(P_H(X))$.

claim 1: $\bigcap_{j=1}^n \mathcal{O}U_j \in \{ \diamond M \mid M \in \mathcal{E}_1 \}$.

For each $M \in \mathcal{E}_1$, $M = \square(\bigcup_{j=1}^n A_j)$ and for every $1 \leq j \leq n$, $A_j \in \mathcal{F}_j$. Since $U_j \in \bigcap \{ \diamond A \mid A \in \mathcal{F}_j \}$ for all $1 \leq j \leq n$, then $U_j \cap A_j \neq \emptyset$. Take $x_j \in U_j \cap A_j$.

Then $\bigcup_{j=1}^n cl(\{x_j\}) \in (\bigcap_{j=1}^n \mathcal{O}U_j) \cap (\bigcap_{j=1}^n A_j)$. So $\bigcap_{j=1}^n \mathcal{O}U_j \in \{ \diamond M \mid M \in \mathcal{F} \}$ and thus $\mathcal{F} \in \bigcap \{ \diamond M \mid M \in \mathcal{E}_1 \}$.

claim 2: $\bigcap \{ \diamond M \mid M \in \mathcal{E}_1 \} \subseteq \mathcal{F}$.
For each \( A = \bigcup_{s \in S} \left( \bigcap_{t \in T_s} (\bigcap_{V_t} s \triangleright V_t) \right) \in \cap \{ \Diamond M \mid M \in E_1 \} \), where \( T_s \) is finite for each \( s \in S \) and \( \{ V_t \mid t \in T_s \} \subseteq \mathcal{O}(X) \). Similarly, since \( \cap \{ \Diamond M \mid M \in E_1 \} \) is Scott open in \( \mathcal{O}(P_H(X)) \), there are finite open sets \( V_1, V_2, \ldots, V_m \) of \( X \) such that \( \bigcap_{i=1}^{m} \triangleright V_i \subseteq A \) and \( \bigcap_{i=1}^{m} \triangleright V_i \in \cap \{ \Diamond M \mid M \in E_1 \} \). Suppose that there is a \( V_r (1 \leq r \leq m) \) such that \( V_r \not\in \cap \{ \Diamond A \mid A \in F_k \} \) for each \( 1 \leq k \leq n \). Then for each \( 1 \leq k \leq n \), there exists \( B_k \in F_k \) satisfying \( B_k \cap V_r = \emptyset \).

Let \( M_0 = \Box \left( \bigcup_{k=1}^{n} B_k \right) \). Clearly, \( M_0 \in E_1 \) and \( \bigcap_{i=1}^{m} \triangleright V_i \subseteq A \) and \( \bigcap_{i=1}^{m} \triangleright V_i \in \Diamond M_0 \). So there is a \( F \in \Gamma(X) \) such that \( F \subseteq \bigcup_{k=1}^{n} B_k \) and \( F \cap V_r \neq \emptyset \). This contradicts with \( B_k \cap V_r = \emptyset \), for each \( 1 \leq k \leq n \). Then we can see that for each \( 1 \leq i \leq m \), there is a \( 1 \leq k \leq n \) such that \( V_i \in \cap \{ \Diamond A \mid A \in F_k \} \). For each \( 1 \leq k \leq n \), let \( G_k = U_k \cap \{ V_i \mid 1 \leq i \leq m, V_i \in \cap \{ \Diamond A \mid A \in F_k \} \} \). Since \( X \) is a intersection-compatible space, \( G_k \in \cap \{ \Diamond A \mid A \in F_k \} \) for each \( 1 \leq k \leq n \) and \( \bigcap_{k=1}^{n} \triangleright G_k \subseteq \bigcap_{i=1}^{m} \triangleright V_i \). Then we have

\[
(G_1, G_2, \ldots, G_n) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n \subseteq \beta_{n-1}^1(F).
\]

Whence, \( \bigcap_{k=1}^{n} \triangleright G_k \in \mathcal{F} \). Since \( \mathcal{F} \) is an upper set, it follows from \( \bigcap_{k=1}^{n} \triangleright G_k \subseteq \bigcap_{i=1}^{m} \triangleright V_i \subseteq A \) that \( A \in \mathcal{F} \). So the claim 2 is proved.

These two claims are enough for the proof. \( \Box \)

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