The set chromatic numbers of the middle graph of graphs

G R J Eugenio, M J P Ruiz and M A C Tolentino
Department of Mathematics, School of Science and Engineering, Loyola Schools, Ateneo de Manila University, Philippines
E-mail: gerone.eugenio@obf.ateneo.edu, mruiz@ateneo.edu, mtolentino@ateneo.edu

Abstract. For a simple connected graph $G$, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of $G$, where adjacent vertices may be colored the same. The neighborhood color set of a vertex $v$, denoted by $NC(v)$, is the set of colors of the neighbors of $v$. The coloring $c$ is called a set coloring provided that $NC(u) \neq NC(v)$ for every pair of adjacent vertices $u$ and $v$ of $G$. The minimum number of colors needed for a set coloring of $G$ is referred to as the set chromatic number of $G$ and is denoted by $\chi_s(G)$. In this work, the set chromatic number of graphs is studied in relation to the graph operation called middle graph. Our results include the exact set chromatic numbers of the middle graph of cycles, paths, star graphs, double-star graphs, and some trees of height 2. Moreover, we establish the sharpness of some bounds on the set chromatic number of general graphs obtained using this operation. Finally, we develop an algorithm for constructing an optimal set coloring of the middle graph of trees of height 2 under some assumptions.

1. Introduction

A neighbor-distinguishing coloring of a graph $G$ is a vertex or edge coloring that induces a vertex labelling for which any two adjacent vertices in $G$ are assigned distinct labels. The proper vertex coloring is a classic example of a neighbor-distinguishing coloring. Other examples can be found in [2], [3], [5], [6], [7], and [8]. Moreover, while new notions of neighbor-distinguishing colorings are introduced, many researchers have started to focus on the effects of some graph operations on the chromatic numbers associated with these colorings. Examples of such papers are [1], [10], [12], [16], and [22].

In this work, we focus on a neighbor-distinguishing coloring called set coloring in relation to the operation called the middle graph of a graph. For a nontrivial graph $G$, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of $G$ where adjacent vertices maybe assigned the same color. For a set $S \subseteq V(G)$, define the set $c(S)$ of colors assigned to the vertices of $S$ by $c(S) = \{c(v) : v \in S\}$.

Definition 1.1. Let $G$ be a graph.

(i) The neighborhood color set $NC(v)$ of a vertex $v$ is defined as the set containing the colors of all neighbors of $v$; that is, $NC(v) = c(N(v))$.

(ii) The coloring $c$ is called a set coloring if $NC(u) \neq NC(v)$ for every pair of adjacent vertices $u$ and $v$ of $G$. The minimum number of colors required in a set coloring is called the set chromatic number of $G$ and is denoted by $\chi_s(G)$.
It has been established in [4] that $\chi_s(G) \leq \chi(G)$ for any graph $G$ and that $\chi_s(G) \geq 3$ if $G$ contains an odd cycle. The set chromatic number has been studied in relation to graph operations such as join [9, 17], corona [4], and comb product [9]. We now define the middle graph of a graph.

**Definition 1.2** ([11]). For a graph $G$, denote by $V'(G)$ the set of all singletons each containing a vertex of $G$. Then we define the *middle graph* of $G$, denoted by $M(G)$, as the intersection graph $\Omega(F)$ of the set $F = V'(G) \cup E(G)$.

Loosely speaking, we may say that the middle graph of a graph $G$ is the graph $M(G)$ whose vertex set is $V(G) \cup E(G)$ and in which two vertices $u$ and $v$ are adjacent if and only if $u$ and $v$ are adjacent vertices in $G$ or $u \in V(G)$ is incident to $v \in E(G)$.

It has been established in [15] that $\chi(M(G)) = \Delta(G) + 1$ for any graph $G$. The middle graph of a graph has been studied in relation to the following neighbor-distinguishing colorings: harmonious coloring [1], irregular coloring [19, 20], total coloring [14], star coloring [21], equitable coloring [18], and $r$-dynamic vertex coloring [13].

In Section 2, we establish a lower bound for the set chromatic number of the middle graph of a general family of graphs. Furthermore, we establish the sharpness of existing bounds for the set chromatic number of the middle graph of a graph. In Section 3, we determine the set chromatic number of the middle graph of stars, cycles, paths and tadpoles. Finally, in Section 4, we determine the set chromatic number of the middle graph of some trees of height 2 including the double-star graphs. We develop an algorithm for generating an optimal set coloring of the middle graph of any tree of height 2 under some assumptions. Throughout this paper, we denote the set $\{1, 2, ..., k\}$ by $\mathbb{N}_k$.

In some of our results, we will use one particular strategy developed by Chartrand et al. [4] that we summarize as follows. Consider a graph $G$ with a set coloring $c$. Given a clique $S$ of $G$, we may define the maximal subset $X \subseteq S$ such that for each $x \in X$ there is a vertex $y \in X \setminus \{x\}$ for which $c(x) = c(y)$. By analysing this set $X$, we can arrive at some information about the coloring $c$.

2. **Bounds for the set chromatic number of the middle graph of graphs**

We begin by proving the following lemma, which provides a lower bound for the set chromatic number of the middle graph of a general family of graphs.

**Lemma 2.1.** Let $G$ be any graph and let $W \subseteq V(G)$ such that each vertex in $W$ has at least one pendant neighbor and at least one nonpendant neighbor. For $v \in W$, we denote by $S(v)$ the set containing all pendant neighbors of $v$. If $W \neq \emptyset$, then

$$\chi_s(M(G)) \geq \max\{|S(v)| + 1 : v \in W\}.$$  

*Proof.* Suppose $W = \{v_1, v_2, \ldots, v_{|W|}\}$. For each $i \in \{1, 2, \ldots, |W|\}$, we denote by

- $Q_i$, the set of all nonpendant neighbors (in $G$) of $v_i$,
- $T_i$, the set of all vertices in $M(G)$ introduced by subdividing the edges of the form $\{v_i, q\}$ where $q \in Q_i$,
- $S_i$ the set $S(v_i)$,
- $R_i$ the set of all vertices in $M(G)$ introduced by subdividing the edges of the form $\{v_i, s\}$ where $s \in S_i$.

Moreover, without loss of generality, we assume that $\max\{|S(v)| : v \in W\} = |S(v_1)| = |S_1|$ and we let $H$ be the clique $K_{|S_1|+2}$ of $M(G)$ formed by $v_1$, one vertex $t_1$ in $T_1$, and all the vertices in $R_1$. (See Figure 1.)

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Suppose that $\chi_s(M(G)) = k \leq |S_1|$. Let there be given a set coloring that uses exactly $k$ colors. Permuting colors if necessary, we can have a set $k$-coloring $c : V(M(G)) \to \mathbb{N}_k$ such that $c(V(H)) = \mathbb{N}_l$ where $l \leq k$. Let $X$ be the subset of $V(H)$ such that for all $x \in X$, there exists $y \in X \setminus \{x\}$ for which $c(x) = c(y)$. Since $c$ uses exactly $k$ colors and $k \leq |S_1| < |S_1| + 2 = |H|$, we have $|X| \geq 3$. Since other vertices in $V(H) \setminus X$ receive unique colors, we have $|S_1| + 2 - |X| + 1 \leq l$, which implies that $|S_1| + 2 \leq l + |X| - 1$. For all $x \in X$, we have

(i) $NC(x) = N_{1}(x) \cup c[N(t_1) \setminus (V(H) \setminus \{t_1\})]$, if $x = t_1 \in X$, or

(ii) $NC(x) = N_{1}(x) \cup \{c(s)\} \cup c[T_1 \setminus \{t_1\}]$ where $s \in S_1 \cap N(x)$ and $c(s) \notin N_{1}$, or

(iii) $NC(x) = N_{1}(x) \cup c(T_1 \setminus \{t_1\})$.

Hence, assuming these neighborhood color sets are all different, the number of possible neighborhood color sets for vertices in $X$ is at most $k - l + 2$ and we must have $k - l + 2 \geq |X|$ since vertices in $X$ must have different neighborhood color sets. But this implies $k \geq l + |X| - 1 - 1 \geq |S_1| + 1$ which is a contradiction. Therefore, $\chi_s(M(G)) \geq |S_1| + 1$. □

It has been established in [4] that $\chi_s(G) \geq 1 + \lfloor \log_2 \omega(G) \rfloor$ for any graph $G$ and that this lower bound is sharp. Combining the facts that $\chi(M(G)) = \Delta(G) + 1$, $\chi_s(G) \leq \chi(G)$, and $\omega(M(G)) = \Delta(G) + 1$, we get the following proposition.

**Proposition 2.2.** For any graph $G$,

$$1 + \lfloor \log_2(\Delta(G) + 1) \rfloor \leq \chi_s(M(G)) \leq \Delta(G) + 1$$

The sharpness of the upper bound follows from Observation 3.2 in Section 3. On the other hand, the sharpness of the lower bound is established in the following theorem.

**Theorem 2.3.** For every positive integer $m$, there is a graph $G$ with $\Delta(G) = m$ and $\chi_s(M(G)) = 1 + \lfloor \log_2(\Delta(G) + 1) \rfloor$.

**Proof.** We construct a graph $G$ with $\Delta(G) = m$ and $\chi_s(M(G)) = 1 + \lfloor \log_2(\Delta(G) + 1) \rfloor$ by the following. Let $1 + \lfloor \log_2(m+1) \rfloor = k$. We start with the star graph $K_{1,m}$ with vertices $v_0, v_1, ..., v_m$ and edges $v_1v_0, v_2v_0, ..., v_mv_0$. Now, consider the set $\mathbb{N}_k \setminus \{1\} = \{2, 3, ..., k\}$. Denote by $S_1, S_2, ..., S_{2^k-1}$ the nonempty subsets of $\mathbb{N}_k \setminus \{1\}$. Moreover, we arrange them such that $S_1, S_2, ..., S_{k-1}$ are the 1-subsets, $S_k, S_{k+1}, ..., S_{k+(k-1)}$ are the 2-subsets, and so on. Note that $2^{k-1} - 1 \geq m$. Now, at each $v_i$, where $1 \leq i \leq m$, we append $|S_i| - 1$ pendant vertices; for each $i$, we denote the set of these pendant vertices by $U_i$. Note that $U_1 = U_2 = \cdots = U_{k-1} = \emptyset$. Moreover, for each $i$, we denote by $T_i$ the set of pendant edges...
incident with $v_i$. The resulting graph, which we denote by $G$, and its middle graph are shown in Figure 2.

![Graphs](image)

**Figure 2.** The graph $G$ (left) and its middle graph (right)

Note that, for each $i$, the vertices in $\{w_i, v_i\} \cup T_i$ form a clique of order $|S_i| + 1$ in $M(G)$. By Proposition 2.2, $\chi_s(M(G)) \geq k$. Moreover, it can be shown that the $k$-coloring $c$ of $M(G)$ defined by assigning

(a) the color 1 to every vertex in the vertex set $\{v_0, w_1, w_2, \ldots, w_m\} \cup \bigcup_{i=1}^{m} U_i$, and

(b) the colors in $S_i$ to the vertices in $\{v_i\} \cup T_i$ such that $c(v_i) = \min S_i$ and $c(T_i) = S_i \setminus \{c(v_i)\}$,

is a set coloring. Therefore, $\chi_s(M(G)) = k$, as desired.

3. The set chromatic number of stars, paths, cycles, and tadpoles

We begin by recalling the following from [4]. For integers $n \geq 2$ and $t \in \{0, 1, \ldots, n\}$, let $G_{n,t}$ denote the graph of order $n + t$ obtained from $K_n$ with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ by adding $t$ new vertices $u_1, u_2, \ldots, u_t$ (if $t \geq 1$) and joining each $u_i$ to $v_i$ for $1 \leq i \leq t$. The set chromatic number of such a graph is given below.

**Proposition 3.1 ([4]).** For $n \geq 2$ and $0 \leq t \leq n$, $\chi_s(G_{n,t}) = n$.

Noting that $M(K_{1,m}) \cong G_{m+1,m}$, we obtain the following.

**Observation 3.2.** For $m \geq 2$, $\chi_s(M(K_{1,m})) = m + 1$.

We now turn to the set chromatic number of the middle graph of cycles.

**Proposition 3.3.** For $n \geq 3$, $\chi_s(M(C_n)) = 3$.

**Proof.** Let $C_n$ be $v_1v_2 \cdots v_nv_1$. Since $M(C_n)$, for $n \geq 3$, always contains an odd cycle, we must have $\chi_s(M(C_n)) \geq 3$. Suppose, that in $M(C_n)$, $u_n$ is the vertex introduced by subdividing the edge $v_nv_1$ in $G$, $u_1$ is the vertex introduced by subdividing $v_1v_2$ in $G$, $u_2$ is the vertex introduced by subdividing $v_2v_3$ in $G$, and so on. Then it can be shown that the colorings defined below are set 3-colorings and the conclusion follows.

- If $n$ is even:
  
  \[
  c(y) = \begin{cases} 
  1 & \text{if } y = u_i \text{ for all } 1 \leq i \leq n, \\
  2 & \text{if } y = u_i \text{ and } i \text{ is odd,} \\
  3 & \text{if } y = u_i \text{ and } i \text{ is even.}
  \end{cases}
  \]

- If $n$ is odd:
  
  \[
  c(y) = \begin{cases} 
  1 & \text{if } y \in \{v_1, u_n\} \text{ for all } 1 \leq i \leq n, \\
  2 & \text{if } y = u_i \text{ and } i \neq n \text{ is odd,} \\
  3 & \text{if } y = u_i \text{ and } i \text{ is even.}
  \end{cases}
  \]

$\square$
The following proposition on the set chromatic number of the middle graph of paths can be proven similarly as Proposition 3.3.

**Proposition 3.4.** For \( n \geq 3 \), \( \chi_s(M(P_n)) = 3 \).

Now, we present the set chromatic number of the middle graph of tadpole graphs. For integers \( m \geq 3 \) and \( t \geq 1 \), the **tadpole graph** \( T_{m,t} \) is the graph created by concatenating the cycle \( C_m \) and the path \( P_t \) using an edge from any vertex of \( C_m \) to a pendant of \( P_t \).

**Proposition 3.5.** For \( m \geq 3 \) and \( t \geq 1 \),

\[
\chi_s(M(T_{m,t})) = \begin{cases} 
4 & \text{if } m = 3 \text{ and } t = 1, \\
3 & \text{otherwise.}
\end{cases}
\]

**Proof.** Since \( T_{m,t} \), for \( m \geq 3 \) and \( t \geq 1 \), always contains an odd cycle, \( \chi_s(M(T_{m,t})) \geq 3 \). It can be verified that the exceptional case \( T_{3,1} \) has set chromatic number 4. To establish the result remaining results, it is sufficient to construct a set 3-coloring for \( T_{m,t} \). Label the vertices of \( M(T_{m,t}) \) as shown in Figure 3.

![Figure 3. The middle graph of the tadpole graph \( T_{m,t} \)](image)

It can then be verified that the following colorings are set colorings of \( T_{m,t} \) under different cases.

(i) Suppose \( m = 3 \) and \( t > 1 \). Define the coloring \( c : V(M(T_{3,t})) \rightarrow \{1, 2, 3\} \) as follows:

\[
c(y) = \begin{cases} 
3 & \text{if } y \in \{v_3, u_i\}, \text{ where } i \text{ is even}, \\
2 & \text{if } y \in \{v_2, w_i\}, \text{ where } i \text{ is odd}, \\
1 & \text{otherwise}.
\end{cases}
\]

(ii) Suppose \( m \equiv 2 \pmod{3} \). Define the coloring \( c : V(M(T_{m,t})) \rightarrow \{1, 2, 3\} \) as follows:

\[
c(y) = \begin{cases} 
3 & \text{if } y \in \{w_j, s_i, v_h\} \text{ such that } h, j \equiv 1(\text{mod } 3), h \neq 1, \\
& i \equiv 2(\text{mod } 3) \text{ and } i \neq m, \\
2 & \text{if } y \in \{w_j, s_i, v_h\} \text{ such that } h, j \equiv 2(\text{mod } 3) \text{ and } i \equiv 0(\text{mod } 3), \\
1 & \text{otherwise}.
\end{cases}
\]

(iii) Suppose \( m \equiv 1 \pmod{3} \). Define the coloring \( c : V(M(T_{m,t})) \rightarrow \{1, 2, 3\} \) as follows:

\[
c(y) = \begin{cases} 
3 & \text{if } y \in \{w_j, s_i, v_h\} \text{ such that } i, j \equiv 1(\text{mod } 3), i \notin \{1, m\}, \\
& h \equiv 0(\text{mod } 3), \text{ and } h \geq 6, \\
2 & \text{if } y \in \{w_j, s_i, v_2, v_h\} \text{ such that } i, j \equiv 2(\text{mod } 3), i \neq 2, \\
& h \equiv 1(\text{mod } 3), \text{ and } h \neq 1, \\
1 & \text{otherwise}.
\end{cases}
\]
Suppose $m \equiv 0 \pmod{3}$ and $m > 3$. Define the coloring $c : V(M(T_{m,t})) \to \{1, 2, 3\}$ as follows:

$$c(y) = \begin{cases} 
3 & \text{if } y \in \{w_j, s_2, s_1, v_h\} \text{ such that } j \equiv 1 \pmod{3}, i \equiv 0 \pmod{3}, 
6 \leq i < m, h \equiv 2 \pmod{3} \text{ and } h \geq 5, 
2 & \text{if } y \in \{w_j, s_1, v_2, v_h\} \text{ such that } j \equiv 2 \pmod{3}, i \equiv 1 \pmod{3}, i \geq 4, 
h \equiv 0 \pmod{3}, \text{ and } h \geq 6, 
1 & \text{otherwise.} 
\end{cases}$$

4. The set chromatic number of some trees of height 2

We begin this section by determining the set chromatic number of the middle graph of any double-star graph. By a double-star graph, we mean a tree containing exactly two non-pendant vertices; moreover, a double-star graph with degree sequence $(m + 1, n + 1, 1, \ldots, 1)$ is denoted by $S_{m,n}$.

**Proposition 4.1.** Let $m$, $n$ be positive integers with $m \geq n$. Then

$$\chi_s(M(S_{m,n})) = \begin{cases} 
3 & \text{if } m = n = 1, 
4 & \text{if } m = n = 2, 
5 & \text{if } m = n = 3, 
m + 1 & \text{otherwise.} 
\end{cases}$$

**Proof.** We label the vertices of $M(S_{m,n})$ as shown in Figure 4.

![Figure 4](image-url)

**Figure 4.** The middle graph $M(S_{m,n})$ of the double star graph $S_{m,n}$.

The proofs for the exceptional cases $M(S_{1,1}), M(S_{2,2})$, and $M(S_{3,3})$ are left to the reader. Now, suppose $(m,n) \not\in \{(1,1),(2,2),(3,3)\}$. By Lemma 2.1 and since $m \geq n$, we have $\chi_s(M(S_{m,n})) \geq m + 1$. Define the coloring $c$ of $S_{m,n}$:

For $1 \leq n \leq 3$:

$$c(y) = \begin{cases} 
i + 1 & \text{if } y = v_i \text{ for all } i \in \mathbb{N}_m, 
i + 2 & \text{if } y \in \{w_0, s_1, s_2, s_3\}, 
1 & \text{otherwise.} 
\end{cases}$$

For $n \geq 4$:

$$c(y) = \begin{cases} 
m + 1 & \text{if } y = v_1, 
m & \text{if } y = v_2, 
i - 1 & \text{if } y = u_i \text{ for } 3 \leq i \leq m, 
i + 1 & \text{if } y = w_i \text{ for } 1 \leq i \leq 2, 
m + 1 - (n - i) & \text{if } y = s_i \text{ for } 3 \leq i \leq n, 
1 & \text{otherwise.} 
\end{cases}$$

It can be verified that, in either case, $c$ is an $(m+1)$-set coloring. Hence, $\chi_s(M(S_{m,n})) = m + 1$ for $(m,n) \not\in \{(1,1),(2,2),(3,3)\}$. 
Finally, we determine the set chromatic number of some trees of height 2. We make use of an algorithm to generate the optimal set coloring for these trees.

**Theorem 4.2.** Let $T$ be a tree of height 2 rooted at a vertex $v_0$ with $\deg(v_0) \geq 4$. Let $W$ be the set of internal vertices of $T$ not equal to $v_0$. If there is a vertex $w \in W$ with $\deg(w) = \max \{ \deg(v) : v \in W \} \geq \deg(v_0) + 1$, then $\chi_s(M(T)) = \deg(w)$.

**Proof.** We will label the vertices (or sets of vertices) in $M(T)$ as shown in Figure 5.

Figure 5. The middle graph of $T$.

For all $v \in W$, $v$ has one nonpendant neighbor which is $v_0$ and has $\deg(v) - 1$ pendant neighbors. Hence, if $w \in W$ is a vertex satisfying the condition $\deg(w) = \max \{ \deg(v) : v \in W \} \geq \deg(v_0) + 1$, then by Lemma 2.1,

$$\chi_s(M(T)) \geq \max \{ (\deg(v) - 1) + 1 \} = (\deg(w) - 1) + 1 = \deg(w).$$

Now, we show that $\chi_s(M(T)) \leq \deg(w)$. Without loss of generality, assume that $v_{\deg(v_0)}$ is a vertex $w$ satisfying the condition $\deg(w) = \max \{ \deg(v) : v \in W \} \geq \deg(v_0) + 1$. Also, let $\deg(w) = k$. Using Algorithm 1, we construct a vertex coloring $c : V(M(T)) \to \mathbb{N}_k$ that uses exactly $k$ colors. In this algorithm, for $i \in \mathbb{N}_{\deg(v_0)}$, we let $S_i = \{ s_{i,1}, s_{i,2}, \ldots, s_{i,|S_i|} \}$, if $S_i$ is non-empty.

Since for the coloring $c$ constructed by Algorithm 1, we have $c(v_{\deg(v_0)}) = k$, $c(r) = k - 1$ where $r \in R_{\deg(v_0)}$, and $c(S_{\deg(v_0)} \setminus \{ s \}) = \mathbb{N}_{k-2}$, we see that $c$ uses exactly $k$ colors. Hence, $c(M(T)) = \mathbb{N}_k$.

Table 1 shows that for every pair of adjacent vertices $\gamma$ and $\delta$ of $M(T), NC(\gamma) \neq NC(\delta)$. Let $i, j \in \{ 1, 2, \ldots, \deg(v_0) \}$ such that $i \neq j$. Since any two adjacent vertices of $M(T)$ have different neighborhood color sets, we see that $c$ is a set coloring. Hence, $\chi_s(M(T)) \leq m + 1$ and the result follows.

Figure 6 shows a set 6-coloring, generated using the algorithm, of the middle graph $M(T)$ of a tree $T$ satisfying the assumptions of Theorem 4.2.
Algorithm 1 Constructing an Optimal Set Coloring $c$ of $M(T)$

1: $c(v_0) \leftarrow 3$, $c(v_1) \leftarrow 1$
2: for $i \leftarrow 1$ to $\text{deg}(v_0)$ do
3:   if $i < \text{deg}(v_0)$ then
4:     $c(u_i) \leftarrow 1$
5:   if $i \neq 1$ then
6:     $c(v_i) \leftarrow i + 2$
7:   end if
8: for all $r \in R_i$ do
9:   $c(r) \leftarrow 1$
10: end for
11: $j \leftarrow 1$
12: for $k \leftarrow 1$ to $|S_i|$ do
13:   if $j = c(v_i)$ then
14:     $j \leftarrow j + 1$
15:   end if
16: $c(s_{i,k}) \leftarrow j$, $j \leftarrow j + 1$
17: end for
18: else
19:   $c(u_i) \leftarrow 2$, $c(v_i) \leftarrow k$, $\text{temp} \leftarrow 0$
20: for all $r \in R_i$ do
21:   if $\text{temp} = 0$ then
22:     $c(r) \leftarrow k - 1$, $\text{temp} \leftarrow 1$, $r^* \leftarrow r$
23:   else
24:     $c(r) \leftarrow 2$
25: end if
26: end for
27: $h \leftarrow 1$
28: for $k \leftarrow 1$ to $|S_i|$ do
29:   if $\{r^*, s_{i,k}\} \in E(M(T))$ then
30:     $c(s_{i,k}) \leftarrow 2$
31:   else
32:     $c(s_{i,k}) \leftarrow h$, $h \leftarrow h + 1$
33: end if
34: end for
35: end if
36: end for

Figure 6. A set 6-coloring of a middle graph of a tree of height 2.
In this paper, we considered the set chromatic number in relation to the operation of taking the middle graph of graphs and considered the sharpness of these bounds. Then we determined the exact set chromatic numbers of the middle graph of stars, paths, cycles, tadpoles, double stars, and some trees of height 2. For future work, we recommend the following possible topics:

(i) Let $T$ be a tree of height 2 rooted at a vertex $v_0$ with $\text{deg}(v_0)$. Let $W$ be the set of internal vertices of $T$ not equal to $v_0$. Suppose there is a vertex $w \in W$ with $\text{deg}(w) = \max\{\text{deg}(v) : v \in W\} < \text{deg}(v_0) + 1$. The goal is to determine the set chromatic number of $M(T)$.

In Theorem 4.2, we were able to determine the exact set chromatic number of the middle graph of a graph $T$ belonging to a family of trees of height 2. This illustrates the importance of Lemma 2.1 that provides a convenient lower bound for $\chi_s(M(T))$. The proof of the theorem is completed by constructing an optimal set coloring of $M(T)$ using Algorithm 1.

### Table 1. Reason(s) for $NC(\gamma) \neq NC(\delta)$, where $\gamma$ and $\delta$ are adjacent vertices in $M(T)$.

| $\gamma$ | $\delta$ | $NC(\gamma)$ | $NC(\delta)$ | Reason(s) |
|-----------|----------|---------------|---------------|-----------|
| $v_0$     | $u_i$    | $\{1, 2\}$   | $\mathbb{N}_3$ $\cup \{i + 2\}$ $\mathbb{N}_3$ $\cup \{k - 1, k\}$ | $|NC(\gamma)| \neq |NC(\delta)|$ |
| $u_i$     | $u_j$    | $\mathbb{N}_3$ $\cup \{i + 2\}$ $\mathbb{N}_3$ $\cup \{k - 1, k\}$ | $|NC(\gamma)| \neq |NC(\delta)|$ |
| $u_i$     | $v_i$    | $\mathbb{N}_3$ $\cup \{i + 2\}$ $\mathbb{N}_3$ $\cup \{k - 1, k\}$ | $\{2, k - 1\}$ | $|NC(\gamma)| \neq |NC(\delta)|$ |
| $u_i$     | $p \in R_i$ | $\mathbb{N}_3$ $\cup \{i + 2\}$ | $\{1, c(q)\}$ $\{1, i + 2, c(q)\}$ | $|NC(\gamma)| \neq |NC(\delta)|$ (since $i \geq 2$ in this case) |
| $p \in R_i$ | $s \in S_i$ | $\mathbb{N}_3$ $\cup \{k - 1, k\}$ | $\{1, c(q)\}$ $\{1, i + 2, c(q)\}$ | $|NC(\gamma)| \neq |NC(\delta)|$ (since $i \geq 2$ in this case) |
| $p \in R_i$ | $s \in S_i$ | $\mathbb{N}_3$ $\cup \{k - 1, k\}$ | $\{2, k\}$ $\{2, k - 1, k, c(q)\}$ | $|NC(\gamma)| \neq |NC(\delta)|$ (since $i \geq 2$ in this case) |

5. Conclusion

In this paper, we considered the set chromatic number in relation to the operation of taking the middle graph of a graph. We established some bounds on the set chromatic number of the middle graph of graphs and considered the sharpness of these bounds. Then we determined the exact set chromatic numbers of the middle graph of stars, paths, cycles, tadpoles, double stars, and some trees of height 2. For future work, we recommend the following possible topics:
(ii) Determine the set chromatic numbers of the middle and total graph of rooted trees with heights of at least 3.

(iii) Investigate set coloring in relation to other unary graph operations like central graph, graph complement, line graph, power of graph, dual graph and others.

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References
[1] Aruldoss J and Mary S 2016 Harmonious coloring of middle and central graph of some special graphs *International Journal of Mathematics and its Application* 4 187-191
[2] Arumugam S, Premalatha K, Bača M and Semaničová-Feňovčíková A 2017 Local antimagic vertex coloring of a graph *Graphs and Combinatorics* 33 275–285
[3] Chartrand G, English S and Zhang P 2017 Kaleidoscopic colorings of graphs *Discussiones Mathematicae Graph Theory* 37 3
[4] Chartrand G, Okamoto F, Rasmussen C W and Zhang P 2009 The set chromatic number of a graph *Discussiones Mathematicae, Graph Theory* 29 545-561
[5] Chartrand G, Okamoto F and Zhang P 2009 The metric chromatic number of a graph *The Australasian Journal of Combinatorics* 44 273–286
[6] Chartrand G, Okamoto F and Zhang P 2010 Neighbor-distinguishing vertex colorings of graphs *Journal of Combinatorial Mathematics and Combinatorial Computing* 74
[7] Chartrand G, Okamoto F and Zhang P 2010 The sigma chromatic number of a graph *Graphs and Combinatorics* 26 755–773
[8] Chartrand G and Zhang P 2009 *Chromatic Graph Theory* (Boca Raton : Chapman & Hall/CRC)
[9] Felipe B C, Garciano A and Tolentino M A 2019 On the set chromatic number of the join and comb product of graphs *J. Phys.: Conf. Ser.* 1538 012009
[10] Guangrong L and Limin Z 2006 Total chromatic number of one kind of join graphs *Discrete Mathematics* 306 18951905
[11] Hamada T and Yoshimura I 1976 Traversability and connectivity of the middle graph of a graph *Discrete Mathematics* 14 247-255
[12] Hanna F 2006 equitable coloring of graph products *Opuscula Mathematica* 26 1
[13] Harjito L, Daﬁk, Kristiana A, Alfors R and Prihandini R 2020 On r-dynamic vertex coloring of line, middle, total and line graph *Journal of Physics: Conference Series* 1465 012014
[14] Muthuramakrishnanand D and Jayaraman G 2018 Total chromatic number of middle and total graph of path and sunlet *Graph International Journal of Scientific and Innovative Mathematical Research (IJSIMR)* 4 1-9
[15] Nihei M 1998 On the chromatic number of middle graph of a graph *PI MU EPSILON JOURNAL* 10
[16] Noga A and Bojan M 1993 The chromatic number of graph powers *Combinatorics, Probability and Computing* 11 110
[17] Okamoto F, Rasmussen C W and Zhang P 2009 Set vertex colorings and joins of graphs *Czechoslovak Mathematical Journal* 59 (134) 929-941
[18] Praveena K, Venkatachalam M, Rohini A and Daﬁk 2019 Equitable coloring of prism graph and its central, total and line graph *International Journal of Scientific and Technology Research* 8 2277-8616
[19] Rohini A and Venkatachalam M 2019 On irregular coloring of wheel related graphs *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* 2 1462-1472
[20] Rohini A and Venkatachalam M 2019 On irregular coloring of triple star graph families *Journal of Discrete Mathematical Sciences & Cryptography* 6 983-988
[21] Vivin J, Kowsalya V and Kumar S 2019 On the star chromatic number of prism graph families *TWMS J. App. Eng. Math.* 3 687-692
[22] Vivin J, Venkatachalam M and N Mohanapriya 2016 On b-chromatic number of some line, middle and total graph families *International J. Math. Combin.* 1 116125