A Classes of Variational Inequality Problems Involving Multivalued Mappings

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Abstract: The main objective of the Variational inequality problem is to study some functional analytic tools, projection method and fixed point theorems and then exploiting these to study the existence of solutions and convergence analysis of iterative algorithms developed for some classes of Variational inequality problem. The main objective of this paper is to study the existence of solutions of some classes of Variational inequalities using fixed point theorems for multivalued and using Banach contraction theorem we prove the existence of a unique solution of multi value Variational inequality problem.

Keywords: Fixed Points Theorems, Variational Inequality Problems, Strongly Lipschitz Operator

1. Introduction

Variational inequalities and complementarity problem play equally important roles in applied mathematics, physics, control theory and optimization, equilibrium theory of transportation and economics, mechanics, and engineering sciences. We study the existence and convergence of solutions of some classes of Variational inequalities using fixed point theorem for multivalued mappings we develop an iterative algorithm for prove the approximate solution converges to solution of multi value Variational inequality problem.

Definition (1-1): Let $X$ be a metric space with metric $d$. A mapping $F: X \rightarrow X$ is called contraction mapping if:

$$d(F(a), F(b)) \leq \alpha d(a, b), \text{for all } a, b \in X$$

Definition (1-2): Let $a(\cdot, \cdot): H \times H \rightarrow R$ be a bilinear from, $K$ a nonempty closed convex set of $H$.

Definition (1-2): If $H = \mathbb{R}^2$ and project any point $x = (x_1, x_2)$ on the $x_1$ axis then the projection $P_K(x) = (x_1, 0)$.

Theorem (1-3): Let $H$ be a real Hilbert space, let $K \subset H$ be a nonempty closed convex set and let $P_K$ due the projection

$$\|P_K(x_1) - P_K(x_2)\| \leq (x_1 - x_2, P_K(x_1) - P_K(x_2))$$

which in particular implies the monotonicity of $P_K$, further if $K = H$ then $P_K = 1$, an identity mapping, and one has strict monotonicity, but in general for $K \neq H$ then $P_K$ is not injective and hence not strictly monotone. If $x \notin K$ then

$$\text{Proof: Let us first note that the characterization of the projection } (y - x, y - \eta) \leq 0, $$

$$\text{for all } y \in K \text{ can be written}$$

$$(P_K(x) - x, P_K(x) - y) \text{ for all } y \in K \quad (1)$$

To show that $P_K$ is monotone, let us fix $x_1$ and $x_2$ in $H$ and write

$$(P_K(x_1) - x_1, P_K(x_1) - y) \text{ for all } y \in K \quad (2)$$

$$(P_K(x_2) - x_2, P_K(x_2) - y) \text{ for all } y \in K \quad (3)$$

Putting $y = P_K(x_2)$ in (1.2) and $y = P_K(x_1)$ in (1), we have

$$(P_K(x_1) - x_1 - P_K(x_2) + x_2, P_K(x_1) - P_K(x_2)) \leq 0 \quad (4)$$

And therefore

$$(P_K(x_1) - x_1, P_K(x_1) - P_K(x_2) + x_2, P_K(x_1) - P_K(x_2)) \leq 0 \quad (5)$$
\(P_K(x) \neq 0\) but \(P_K(P_K(x)) = P_K\).

To show that \(P_K\) is non expansive it is enough to apply the Schwartz inequality to (1.5) and obtain thus
\[
\|P_K(x_1) - P_K(x_2)\| \leq \|x_1 - x_2\| \|P_K(x_1) - P_K(x_2)\| \quad (6)
\]

Or, dividing by \(\|P_K(x_1) - P_K(x_2)\| = 0\) then
\[
\|x_1 - x_2\| \leq \|x_1 - x_2\| \quad \text{the strong continuity follows immediately from (5)}.
\]

2. Preliminaries

Definition (2-1): Let \(H\) be a real Hilbert space and \(\|\cdot\|\) and \(\langle\cdot,\cdot\rangle\) denote norm and inner product on \(H\) respectively. Given multivalued mappings \(A, B : H \to 2^H\) is a power set of \(H\), a nonlinear mapping \(g : H \to H\) and a proper convex and lower.

Semi – continuous function \(j : H \to R \cup \{+\infty\} \) with \(\text{Im}g \cap \delta j \neq \emptyset\), where \(\delta j\) denote the subdifferential of \(j\). We consider the following Variational inequality problem (GVIP):

\[
\text{GVIP:} \quad \text{Find} \quad B \in A, x \in B \text{ such that} \quad \langle -u, C - x \rangle \geq j(x) - j(y), \forall y \in H.
\]

Definition (2-2): let \(j : H \to H\) is said to be:

(i) \(\eta\) - strongly monotone, if there exists a constant \(\eta > 0\) such that
\[
\langle j(x) - j(y), x - y \rangle \geq \eta \|x - y\|^2, \forall x, y \in H
\]

(ii) \(\rho\) - Lipschitz continuous, if there exists a constant \(\rho > 0\) such that
\[
\|j(x) - j(y)\| \leq \rho \|x - y\|, \forall x, y \in H
\]

Definition (2-3): A multivalued mapping \(T : H \to 2^H\) is said to be:

(i) \(\alpha\) - strongly monotone, if there exists a constant \(\alpha > 0\) such that
\[
\langle j(x) - j(y), x - y \rangle \geq \alpha \|x - y\|^2, \forall x, y \in H
\]

(ii) \(\delta\) - continuous, if there exists a constant \(\delta > 0\) such that
\[
\|j(x) - j(y)\| \leq \delta \|x - y\|, \forall x, y \in H
\]

Definition (2-4): \(H : \text{the} \) mapping defined on a metric space \(X\), into itself, find \(x \in X\) such that \(Tx = x\).

A point \(x\) is said to be fixed point \(T\) if \(x = x\).

Fixed Point Problem: Let \(T\) be a mapping defined on a metric space \((X, d)\) into itself, find \(x \in X\) such that \(Tx = x\).

Definition (2-5): If \(F\) is multivalued mapping on \(X\) into itself. Then a point \(x\) \(\in X\) is called a fixed point of \(F\) if
\[
F(x) = x.
\]

Lemma (2-6): A multivalued mapping \(F\) on \(X\) into \(Y\) is continuous at point \(x_0\) if and only if \(F(x_n) \to F(x_0)\) for all sequence \(\{x_n\}\) in \(X\) with \(x_n \to x_0\).

Proof: Suppose that \(x_n \to x_0\). Given \(\epsilon > 0\) there exists \(\delta > 0\) such that
\[
\|x_0 - x\| < \delta \Rightarrow \|u - v\| < \epsilon \text{ for all } u \in F(x_0)\text{ and } v \in F(x)
\]

And there exists a positive integer \(N\) such that
\[
\|x_0 - x_n\| < \delta \text{ for all } n \geq N
\]

Thus, if \(n \geq N\) we have
\[
\|u_n - u\| < \epsilon \text{ for all } u_n \in F(x_n)\text{ and } u \in F(x)
\]

Therefore \(F(x_n) \to F(x)\).

Conversely, suppose that \(F(x_n) \to F(x)\), i.e if \(n \geq N\)
\[
\|u_n - u\| < \epsilon \text{ for all } u_n \in F(x_n)\text{ and } u \in F(x)
\]

And \(\|x_n - x\| < \delta\). Suppose that \(F\) is not continuous at \(x_0\). Then there exists \(\epsilon > 0\) there exists \(\delta > 0\) there exists \(x \in X\) such that
\[
\|x_0 - x\| < \delta \Rightarrow \|u - v\| < \epsilon \text{ for all } u \in F(x_0)\text{ and } v \in F(x)
\]

In particular, for each positive integer \(n\) there exists \(x_n \in X\) such that
\[
\|x_0 - x\| < \frac{1}{n} \text{ and } \|u_n - u_0\| < \epsilon \text{ for all } u_n \in F(x_n)\text{ and } u_0 \in F(x_0)
\]

Clearly, \(x_n\) converges to \(x\) but \(F(x_n)\) does not converges to \(F(x_0)\). which is a contradiction, this prove the lemma #
Theorem (2-7): Let \( X \) be a Banach space. If \( F \) is multivalued contraction mapping on \( X \) into itself, then \( F \) has a fixed point.

***Proof:** Let \( \mu < 1 \) be contraction constant for \( F \) and let \( x_0 \in X \). Choose \( x_1 \in F(x_0) \). Since \( F(x_0) \) and \( F(x_1) \) are subsets of \( X \) and \( x_1 \in F(x_0) \) there is an \( x_2 \in F(x_1) \) such that

\[
\|x_1 - x_2\| \leq \mu \|x_0 - x_1\|.
\]

Now, since \( F(x_1) \) and \( F(x_2) \) are subsets of \( X \) and \( x_2 \in F(x_2) \), there is a point \( x_3 \in F(x_2) \) such that \( \|x_2 - x_3\| \leq \mu \|x_1 - x_2\| \leq \mu^2 \|x_0 - x_1\| \)

We produce a sequence \( \{x_n\} \) of points of \( X \) such that \( x_{n+1} \in F(x_n) \) and

\[
\|x_n - x_{n+1}\| \leq \mu \|x_{n-1} - x_n\| \leq \mu^2 \|x_0 - x_1\|, \text{ for all } n \geq 1.
\]

Now

\[
\|x_n - x_{n+m}\| \leq \|x_n - x_{n+1}\| \leq \|x_{n+1} - x_{n+2}\| + \cdots + \|x_{n+m-1} - x_{n+m}\| \leq \mu^n \|x_0 - x_1\| + \mu^{n+1} \|x_0 - x_1\| + \cdots + \mu^{n+m-1} \|x_0 - x_1\| = (\mu^n + \mu^{n+1} + \cdots + \mu^{n+m-1}) \|x_0 - x_1\| \leq \mu^n \left( \sum_{i=0}^{m} \mu^i \right) \|x_0 - x_1\| \text{ for all } n, m \geq 1.
\]

If \( n, m \rightarrow \infty \), then the sequence \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is Banach space.

Lemma (2-8): \( u \in H, x \in T(u), y \in A(u) \) is a solution of GVIP if and only if for some given \( \alpha > 0 \), the mapping \( F: H \rightarrow 2^H \) defined by

\[
F(u) = \bigcup_{x \in T(u)} \bigcup_{y \in A(u)} \{u - g(u) + p^1_\alpha(g(u) - \alpha(x - y))\}
\]

Has a fixed point \( u \), where \( p^1_\alpha = (I + \alpha\delta)^{-1} \) is called proximal mapping. \( I \) stands for the identity mapping on \( H \).

**Proof:** Let \( u \in H, x \in T(u), y \in A(u) \) be a solution of GVIP, i.e., \( u \in H, x \in T(u), y \in A(u) \) satisfy \( (x - y, v - g(u)) \geq j((g(u)) - j(v), \forall v \in H \). By definition of \( \delta \), we have \( x - y, \in (g(u)) \)

\[
\Rightarrow g(u) - \alpha(x - y) = g(u) + \alpha\delta j(g(u))
\]

\[
\Rightarrow g(u) - \alpha(x - y) \in (I + \alpha\delta j)((g(u))
\]

\[
\Rightarrow p^1_\alpha(g(u) - \alpha(x - y) = g(u)
\]

\[
\Rightarrow u = u - (g(u) + p^1_\alpha((g(u) - \alpha(x - y))
\]

\[
\Rightarrow u = u - g(u) + p^1_\alpha(g(u) - \alpha(x - y))
\]

Conversely, let \( u \) be a fixed point of \( F \), i.e., \( \exists x \in T(u), y \in A(u) \) such that

\[
u = u - g(u) + p^1_\alpha(g(u) - \alpha(x - y))
\]

\[
\Rightarrow g(u) = p^1_\alpha(g(u) - \alpha(x - y)
\]

\[
\Rightarrow g(u) - \alpha(x - y) \in (I + \alpha\delta j)((g(u))
\]

\[
\Rightarrow g(u) - \alpha(x - y) \in g(u) + \alpha\delta j((g(u))
\]

\[
\Rightarrow -\alpha(x - y) \in \alpha\delta j((g(u))
\]
\[ \Rightarrow y - x = \alpha d_j(g(u)), \text{since } \alpha > 0 \]

Hence, by definition of \( \partial j \)
\((x - y, v - g(u)) \geq j(g(u)) - j(v), \forall v \in H\). This complete the proof.

### 3. Main Result

Let \( T, A : H \to 2^H \) be multivalued mappings then multivalued Variational inequality problem is to find \( u \in K \) such that

\[ (p, v - u) \geq 0, \text{for all } v \in K \quad \text{(7)} \]

\[ (p, v - u) \geq (q, v - u), \text{for all } v \in K \quad \text{(8)} \]

**Lemma (3-1):** \((u, p, q)\) is a solution of multivalued Variational inequality problem (3-1) if and only if \( u \in H \) is a fixed point of mapping \( F : H \to 2^H \) defined as

\[ F(u) = \bigcup_{p \in T(u), q \in A(u)} (P(v - \xi(p - q))) \]

For some positive \( \xi \).

**Proof:**
Suppose \((u, p, q)\) satisfies if it satisfies \((p - q, v - u) \geq 0, \text{for all } v \in K \)
\[ u = P(v - \xi(p - q)) \]
\[ u = \bigcup_{q \in A(u)} (P(v - \xi(p - q))) \]

We prove the existence of a unique solution of multivalued Variational inequality problem (3-1)

**Theorem (3-2):** Let \( T : H \to 2^H \) be a \( \eta_1 \)-Lipschitz continuous and \( \lambda_1 \)-strongly monotone multivalued mapping and let \( A : H \to 2^H \) be \( \rho_1 \)-Lipschitz continuous multivalued mapping. Then multivalued Variational inequality problem (3-1) has a solution.

**Proof:** By Lemma (3-1), it is enough to prove that multivalued mapping \( F \) is contraction mapping.

Let \( w_1 \in F(u_1) \) and \( w_2 \in F(u_2) \), we have
\[ w_1 = P_r(u_1 - \xi(p_1 - q_1)) \text{ for } p_1 \in T(u_1) \text{ and } q_1 \in A(u_1) \]
\[ w_2 = P_r(u_2 - \xi(p_2 - q_2)) \text{ for } p_2 \in T(u_2) \text{ and } q_2 \in A(u_2) \]

Now
\[ ||w_1 - w_2|| \leq \|P_r(u_1 - \xi(p_1 - q_1)) - P_r(u_2 - \xi(p_2 - q_2))\| \]
\[ \leq \|(u_1 - \xi(p_1 - q_1)) - (u_2 + \xi(p_2 - q_2))\| \]
\[ \leq \|u_1 - u_2 - \xi(p_1 - p_2)\| + \xi\|q_1 - q_2\| \]
\[ \leq \|u_1 - u_2 - \xi(p_1 - p_2)|| + \xi\rho_1\|q_1 - q_2\| \]

By \( \lambda_1 \)-strongly monotonicity and \( \eta_1 \)-Lipschitz continuous of \( T \), we have
\[ ||u_1 - u_2 - \xi(p_1 - p_2)||^2 \leq ||u_1 - u_2||^2 - 2\xi||p_1 - p_2|| + \xi^2||p_1 - p_2||^2 \]
\[ \leq ||u_1 - u_2||^2 - 2\xi\lambda_1||u_1 - u_2||^2 + \xi^2\eta_1^2||u_1 - u_2||^2 \]
\[ \leq (1 - 2\xi\lambda_1 + \xi^2\eta_1^2)||u_1 - u_2||^2 \]

Therefore:
\[ ||w_1 - w_2|| \leq \sqrt{1 - 2\xi\lambda_1 + \xi^2\eta_1^2}||u_1 - u_2|| + \xi\rho_1\|u_1 - u_2\| \]
\[ = \xi\rho_1 + \sqrt{1 - 2\xi\lambda_1 + \xi^2\eta_1^2}||u_1 - u_2|| \]
\[ = \theta||u_1 - u_2|| \]

Where \( \theta = \xi\rho_1 + \sqrt{1 - 2\xi\lambda_1 + \xi^2\eta_1^2} < 1 \)
\[ \xi \leq \frac{2(\lambda_1 - \rho_1)}{\eta_1^2 - \rho_1^2}, \rho_1 < \lambda_1 \text{ and } \rho_1 < \lambda_1 \]

Hence \( F \) is contraction multivalued mapping. By Theorem (3-2), \( F \) has a fixed point, say \( u, \text{ i.e, } u \in F(u) \) then

\[ u = P_K(u - \xi(p - q)) \text{ and } p \in T(u), q \in A(u) \]

This completes the proof.

Iterative Algorithm (3-3):
For any given \( u_0 \in K \), compute \( u_{n+1} \) defined as

\[ u_{n+1} = P_K(u_n - \xi(p_n - q_n)) \]

\[ p_{n+1} \in T(u_{n+1}) \text{ and } q_{n+1} \in A(u_{n+1}) \text{ for some constant } \xi \]

Theorem (3-4): Let \( T: H \to 2^H \) be a \( \eta_1 \)-Lipschitz continuous and \( \lambda_1 \)-strongly monotonicity multivalued mapping and let \( A: H \to 2^H \) be \( \rho_1 \)-Lipschitz continuous

\[ \|u_{n+1} - u\| = \|P_K(u_n - \xi(p_n - q_n)) - P_K(u - \xi(p - q))\| \]

\[ \leq \left( \xi \rho_1 + \sqrt{1 - 2\xi\lambda_1 + \xi^2\eta_1^2}\|u_1 - u_2\| \right) \]

\[ = \theta\|u_1 - u_2\| \]

By theorem

Where \( \theta = \left( \xi \rho_1 + \sqrt{1 - 2\xi\lambda_1 + \xi^2\eta_1^2} \right) < 1 \)

For \( \xi < \frac{2(\lambda_1 - \rho_1)}{\eta_1^2 - \rho_1^2}, \rho_1 < \lambda_1 \text{ and } \rho_1 < \lambda_1 \)

Then by iteration, we have

\[ \|u_{n+1} - u\| \leq \theta^n\|u_1 - u_2\| \]

Since \( \theta < 1 \), we have \( u_{n+1} \) converges to \( u \) strongly in \( H \) and \( p_{n+1} \to p \) strongly in \( H \) and \( q_{n+1} \to q \) strongly in \( H \).

This completes the proof.

4. Conclusions

We study the existence of solutions of some classes of Variational inequalities using fixed point theorems for multivalued and using Banach contraction theorem we prove the existence of a unique solution of multi value Variational inequality problem discussed in the article research.

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Statement of the Problem

Find solutions of some classes of Variational inequalities.

Research Objectives

Study the existence of solutions of some classes of

Variational inequalities using fixed point theorems for multivalued and using Banach contraction theorem

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