On The Algebraic Characterization Of Aperiodic Tilings Related To $ADE$-Root Systems

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Abstract

The algebraic characterization of classes of locally isomorphic aperiodic tilings, being examples of quantum spaces, is conducted for a certain type of tilings in a manner proposed by A. Connes. These 2-dimensional tilings are obtained by application of the strip method to the root lattice of an $ADE$-Coxeter group. The plane along which the strip is constructed is determined by the canonical Coxeter element leading to the result that a 2-dimensional tiling decomposes into a cartesian product of two 1-dimensional tilings. The properties of the tilings are investigated, including selfsimilarity, and the determination of the relevant algebraic invariant is considered, namely the ordered $K_0$-group of an algebra naturally assigned to the quantum space. The result also yields an application of the 2-dimensional abstract gap labelling theorem.

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Introduction

Aperiodic (non-periodic) tilings have been considered in various contexts. The question of whether or not it is possible to find a finite set of prototiles (e.g. polytopes) which tile the plane only in a non-periodical way emerged from mathematical logic and has lead to the first construction of such a set in 1966. Whereas at that time the number of necessary prototiles had been quite large, R. Penrose achieved to reduce it to two, thereby yielding tilings which almost have a fivefold symmetry.

Amazingly enough it is a 3-dimensional generalization of R. Penrose’s tilings [23] which serves as an initial stage for a theoretical model for what are now called quasicrystals, which were first experimentally observed by D. Shechtman et al. in 1984 [32]. They are a crystalline substance yet having, in contradiction to the theory of crystal symmetries, Bragg-reflexes with 'forbidden' symmetries such as the fivefold symmetry. It is not just the discovery of substances having these symmetries, but also the fascinating properties of the aperiodic tilings, which serve to explain these symmetries, that has initiated the study of quantum mechanical and statistical systems, whose underlying structure is not a periodic lattice but an aperiodic tiling. In this context the work of Bellissard et al. [4] is of particular interest for us. Here the authors compute the labelling of the possible gaps in the spectrum of a discrete Hamiltonian, whose potential depends on the structure of a 1-dimensional aperiodic tiling, with the help of the $K$-theory of a certain non-commutative $C^*$-algebra related to that tiling.

Among the properties of aperiodic tilings mentioned above are local isomorphism and selfsimilarity, both being properties that involve non-classical mathematical methods. And indeed, in his book Géométrie Non-Commutative [9], A. Connes’ first example of what he calls an éspace quantique is the set of all Penrose tilings. The clue of Connes’ analysis is to use the selfsimilarity to construct a non-commutative $C^*$-algebra, certain invariants of which (in this case the ordered $K_0$-group) describe the set. This seems to be a promising ansatz, since, in the above example, the ordered $K_0$-group shares essential properties with an aperiodic tiling. The above non-commutative algebra is the $AF$-algebra obtained by the tower construction of V. Jones [18] applied to an inclusion of multi-matrix algebras, the inclusion graph of which is the Coxeter graph of $A_4$. Its ordered $K_0$-group is $\mathbb{Z}^2$ with order relation $z > 0$ whenever $(\nu, z) > 0$, hereby $(\nu, z)$ denoting the Euclidian scalar product with the Perron Frobenius vector $\nu$ of the inclusion matrix. On the other hand, among the initial data for the construction of 1-dimensional tilings related to $A_4$ are $\mathbb{Z}^2$ and the line which is spanned by $\nu$. But this line is not just any subspace of $\mathbb{R}^2$ having irrational slope with respect to $\mathbb{Z}^2$, but, if we identify, as is explained in subsection 1.3, $\mathbb{Z}^2$ with $\Lambda_1$ ("one half" of the root lattice of $A_4$), it is $P_1$, a line determined by the canonical Coxeter element of $A_4$.

The present work may be understood as a step towards a generalization of Connes’ result on Penrose tilings (which are related to $A_4$) to arbitrary $ADE$-Coxeter groups. It consists of two parts, the first one concerns the construction and description of the aperiodic tilings, the second one the algebraic characterization of the corresponding quantum spaces.

Commonly generalizations of the Penrose tilings are considered to be applications of the strip method to the lattice $\mathbb{Z}^N$ and a 2-dimensional subspace $P$, which is stable under the $N$-fold symmetry, $N = 5$ corresponding to Penrose tilings. However, as Connes’ analysis made clear, it is the root lattice of $A_4$ rather than $\mathbb{Z}^5$ which encodes the relevant information of the quantum space. Any $A_{N-1}$-root lattice may be embedded into $\mathbb{Z}^N$ in such a way that the $N$-fold symmetry acts, restricted to the root lattice, as a specific Coxeter element. Therefore it seems to be more appropriate to apply the projection method directly to these root lattices.
the role of the $N$-fold symmetry being taken over by a Coxeter element. Root lattices are also the framework of a different method to construct aperiodic tilings, the so-called dualization method being exposed in [1, 2]. But in contrast to that work, we use a different plane $P$ along which the strip is constructed. Choosing a different plane $P$ may be interpreted as using a different Coxeter element, which is guided by the reasoning that for general Coxeter groups, the above specification of a Coxeter element (through an $N$-fold symmetry of some bigger lattice $\mathbb{Z}^N$) would not exist. Instead, if the Coxeter graph is bipartite, it is replaced by one the choice of which is canonical for all these groups. The relation between these two apparently different tilings coming from one and the same Coxeter group is not yet worked out to a satisfactory extent, but in the case of $N = 5$ the latter projections are closely related to Ammann-quasicrystals of Penrose tilings [17, 22].

Our main result of the first part is that, under certain circumstances, 2-dimensional tilings decompose into 1-dimensional ones. This is very helpful, as it simplifies the construction of an appropriate non-commutative $\mathcal{C}^*$-algebra and allows for the determination of its $K_0$-group. Furthermore, several properties of the tilings under investigation are discussed including selfsimilarity, which is manifested by an inflation/deflation procedure, and which is important for an alternative description of the quantum space.

The second part is devoted to the algebraic characterization of the quantum spaces. After exposing the general philosophy according to which relevant information is contained in the ordered $K_0$-group of a certain $\mathcal{C}^*$-algebra being assigned to the quantum space, two different realizations of the quantum space are considered. The first one, which is directly obtained from the strip method, leads to the consideration of a twofold iterated crossed product with $\mathbb{Z}$, the $K$-group of which is computed with the help of the Pimsner Voiculescu exact sequence. Here we are not able to determine the order structure, but it is possible to determine the range of the tracial state on the $K_0$-group. This yields an application of the 2-dimensional abstract gap labelling theorem [3], namely (like in the 1-dimensional situation) the values of the tracial state are determined by the relative frequencies of patterns in the tilings. We briefly review the gap labelling in this context.

The second (alternative) realization makes contact with the description of Penrose tilings used by A. Connes, it is based on the selfsimilarity of the tilings [17]. Here the results are only partially rigorous and moreover restricted to 1-dimensional tilings. This perspective however has the advantage that one can use $AF$-algebras instead of crossed products with $\mathbb{Z}$. The $AF$-algebras in question are all given by path algebras over the graph (or some related version of it) of the Coxeter group defining the tilings. This yields, at the level of $K_0$-groups, a much more intuitive picture, which very much resembles an aperiodic tiling again. In fact, whereas the outcome of the investigation of the Pimsner Voiculescu exact sequence is still abstract, the latter perspective allows one to directly read off e.g. the values of the tracial state on the $K_0$-group.

We close this introduction with the remark that the mathematical methods used here are partly similar to those in the theory of super selection sectors of 2-dimensional quantum field theory. This seems to be due to the fact we mentioned above, namely that the ordered $K_0$-groups of $AF$-algebras appearing in that theory, see [14, 4, 14], have a lot in common with the aperiodic tilings discussed here. Moreover an approach to the fusion structure by means of these ordered $K_0$-groups has been obtained in [30]. For example Chebyshev polynomials of the second kind (which reflect the fusion structure of $SU(2)$-WZW-models) naturally appear in the analysis of the structure of the tilings.
1 Quantum Spaces Related To $ADE$-Root Systems

1.1 Classes of Locally Isomorphic Tilings

Classes of locally isomorphic aperiodic tilings (modulo translation) yield examples for - following the terminology of A. Connes - quantum spaces. Their definition is content of this section.

The aperiodic tilings considered here can be obtained by the strip method, a detailed description of which is given in [11] or, for the special case of the Penrose tilings, in [7]. We shall briefly describe some of their results. A set of aperiodic tilings may be defined with the help of

- an $N$-dimensional abelian lattice $\Lambda \subset \mathbb{R}^N$,
- an $n$-dimensional subspace $P \subset \mathbb{R}^N$,
- a chosen basis $L$ of $\Lambda$.

Given $L$ the subset $\gamma = \{ \sum_{\alpha \in L} \lambda \alpha | \lambda \in (0,1) \}$ is the interior of a fundamental domain of $\Lambda$. For any $a \in P^\perp$, the (Euclidian) orthogonal complement of $P$, one may construct the strip along $P$

$$S_{\gamma-a} = \{ x + y - a | x \in \gamma, y \in P \},$$

which is an open subset of $\mathbb{R}^N$. Its intersection with the lattice $\Lambda$ is orthogonally projected onto $P$ furnishing the points of an $n$-dimensional tiling, which shall be denoted by $T_{\gamma-a}$. We denote by $\pi$ resp. $\pi^\perp$ the orthogonal projection onto $P$ resp. $P^\perp$, hence

$$T_{\gamma-a} = \pi(S_{\gamma-a} \cap \Lambda).$$

The set $L$ indicates which of the lattice points are joined by a link, namely all pairs of the form $(\lambda, \lambda + \alpha)$, $\lambda \in \Lambda$, $\alpha \in L$. The links of the tiling are then given by the projection of all links in $\mathbb{R}^N$ which lie completely in $S_{\gamma-a}$. We denote by $T_{\gamma-a}$ the whole tiling, i.e. the vertex set $T_{\gamma-a}$ together with the links.

The best known examples of 2-dimensional tilings are given by the following projections to which we refer as $\mathbb{Z}^N$-projections. The lattice $\Lambda = \mathbb{Z}^N$ with standard basis $L = \{ e_1, \ldots, e_N \}$ is invariant under the $N$-fold symmetry $\omega : e_i \mapsto e_{i+1}$, $(e_{N+1} = e_1)$. This symmetry furnishes a decomposition of $\mathbb{R}^N$ into subspaces which are stable under its action

$$\mathbb{R}^N = \bigoplus_{0 \leq m \leq \frac{N}{2}} P^{(m)},$$

$m$ being an integer. Hereby $\omega$ acts on the planes $P^{(m)}$, $0 < m < \frac{N}{2}$, as a rotation around $\frac{2\pi m}{N}$, on $P^{(0)}$, which is the symmetry axis, as the identity, and, if $N$ is even, on $P^{(\frac{N}{2})}$ as a reflection. Now one takes $P = P^{(1)}$. In particular for $N = 5$ in this way all Penrose tilings are obtained as $T_{\gamma-a}$ with $a \in P^{(2)}$.

A tiling is called aperiodic, if there is no translation $y \in P$ such that $T_{\gamma-a} - y = T_{\gamma-a}$. In the cases being investigated in this work, in which $\Lambda$ is either $\mathbb{Z}^n$ or a root lattice, one may establish the following results [11]:


\begin{itemize}
  \item $T_{\gamma-a}$ is aperiodic, iff $P \cap \Lambda = \{\emptyset\}$.
  \item $\pi^\perp(\Lambda)$ is dense in $P^\perp$, iff $P^\perp \cap \Lambda = \{\emptyset\}$.
\end{itemize}

Comparing different values for the parameter $a$ yields the following:

1. If $a' - a = \pi^\perp(\lambda)$ for some $\lambda \in \Lambda$, then $T_{\gamma-a} = T_{\gamma-a'} - \pi(\lambda)$, and hence $T_{\gamma-a}$ and $T_{\gamma-a'}$ differ by an overall translation on $P$. It is therefore natural to identify them.

2. If $a' - a \in \overline{\pi^\perp(\Lambda)}$ (the closure of $\pi^\perp(\Lambda)$), then for any finite subset $X \subset T_{\gamma-a}$ there are infinitely many translations $\pi(\lambda)$ on $P$ such that $X \subset T_{\gamma-a'} - \pi(\lambda)$.

3. If however $a' - a \notin \overline{\pi^\perp(\Lambda)}$, then in general there are finite patterns in $T_{\gamma-a}$ which do not occur in $T_{\gamma-a'}$.

Due to the identification of tilings through overall translations on $P$ one is tempted to say that the elements of

$$\{T_{\gamma-a} \mid a \in \overline{\pi^\perp(\Lambda)}\} \mod \text{translation},$$

are locally indistinguishable. However, there is a subtle complication arising from the fact that this set contains singular tilings. To illustrate the notion of a singular tiling - which is the analog of a singular grid in de Bruijn’s approach [7] - let us for the moment simplify the situation by considering $\Lambda = \mathbb{Z}^n$ and $P$ being a 1-dimensional subspace spanned by a vector having positive entries with respect to the basis $\mathcal{L}$ such that $P^\perp \cap \Lambda = \{\emptyset\}$. Then for almost all $a \in \overline{\pi^\perp(\Lambda)} = P^\perp$ the points $T_{\gamma-a}$ divide $P$ completely into intervals of length $\pi(\alpha)$, $\alpha \in \mathcal{L}$, these intervals being the prototiles. Such a tiling shall be called regular. However, care has to be taken if the boundary $\partial S_{\gamma-a}$ of $S_{\gamma-a}$ has nonempty intersection with $\Lambda$. If $v \in \partial S_{\gamma-a} \cap \Lambda$, then by construction there is a unique $\alpha \in \mathcal{L}$ such that $v - \alpha \in S_{\gamma-a} \cap \Lambda$, but no $\alpha' \in \mathcal{L}$ satisfies $v - \alpha + \alpha' \in S_{\gamma-a}$. Hence $\pi(v - \alpha) \in T_{\gamma-a}$ is the boundary point of an interval which is not of length $\pi(\alpha')$ for some $\alpha' \in \mathcal{L}$. In fact at least one point is missing to make it a prototile; one says the tiling has a hole. Such a tiling is certainly locally distinguishable from regular tilings and is consequently called singular.

In general a regular tiling may be defined as follows: A tiling $T_{\gamma-a}$ shall be called regular if for any bounded subset $U \subset P$ there is an open neighborhood $V \subset \overline{\pi^\perp(\Lambda)}$ of $\emptyset$ such that for all $a' - a \in V$: $\pi^{-1}(U) \cap S_{\gamma-a} \cap \Lambda = \pi^{-1}(U) \cap S_{\gamma-a'} \cap \Lambda$, $\pi^{-1}(U)$ denoting the preimage of $U$. In other words, a regular tiling is locally stable against small perturbations of $a$ inside $\overline{\pi^\perp(\Lambda)}$. Non-regular tilings are also called singular tilings. We denote $K_{\gamma} = \{a \in \pi^\perp(\gamma) \cap \pi^\perp(\Lambda) \mid T_{\gamma-a} \text{ regular}\}$

Lemma 1 Assume $\overline{\pi^\perp(\Lambda)} = P^\perp$. Then $T_{\gamma-a}$ is regular iff $\partial S_{\gamma-a} \cap \Lambda = \emptyset$.

Proof: Let $\partial S_{\gamma-a} \cap \Lambda = \emptyset$ and $U \subset P$ bounded. Then there is an open neighborhood $V \subset P^\perp$ of $\emptyset$ such that $\pi^{-1}(U) \cup \partial S_{\gamma-a} \cap \Lambda = \emptyset$ for all $b \in V$. Hence $\pi^{-1}(U) \cap S_{\gamma-a-b} \cap \Lambda = \pi^{-1}(U) \cap S_{\gamma-a} \cap \Lambda$ and therefore $T_{\gamma-a}$ is regular. Now assume $v \in \partial S_{\gamma-a} \cap \Lambda$, in particular $v \notin S_{\gamma-a}$. There ought to be a $v_0 \in P^\perp$ such that $v - cv_0 \in S_{\gamma-a}$ for all $c \in (0,1)$, i.e. $v \in S_{\gamma-a} + cv_0 \cap \Lambda$. As any open neighborhood $V \subset P^\perp$ of $\emptyset$ has nonzero intersection with $\bigcup_{c \in (0,1)} \{cv_0\}$, $T_{\gamma-a}$ can not be regular.

Remark: The above lemma shows that in the case of $\overline{\pi^\perp(\Lambda)} = P^\perp$ regular tilings do not have holes. If this condition is not satisfied, holes may appear even in regular tilings, but then they occur periodically.
Translation of a regular tiling yields a regular one so that we may restrict \([\mathbb{I}]\) to regular tilings hereby obtaining tilings that are indeed locally indistinguishable:

**Proposition 1** Let \(T_{\gamma-a}\) and \(T_{\gamma-a'}\) be regular tilings with \(a, a' \in \pi^\perp(\Lambda)\). For any bounded subset \(U \subset P\) there are infinitely many translations \(\pi(\lambda)\) on \(P\), \(\lambda \in \Lambda\), such that \(U \cap T_{\gamma-a} = U \cap \{T_{\gamma-a'} - \pi(\lambda)\}\).

**Proof:** Let \(U \subset P\) be bounded and \(V \subset \pi^\perp(\Lambda)\) be an open neighborhood of \(\vec{0}\) such that \(\pi^{-1}(U) \cap S_{\gamma-a} \cap \Lambda = \pi^{-1}(U) \cap S_{\gamma-a-b} \cap \Lambda\) for all \(b \in V\). As \(a' - a \in \pi^\perp(\Lambda)\) there are infinitely many \(\lambda \in \Lambda\) such that \(a - a' - \pi^\perp(\lambda) \in V\). Taking these values for \(b\) shows \(U \cap T_{\gamma-a} = U \cap \{T_{\gamma-a'} - \pi(\lambda)\}\). \(\square\)

Two tilings satisfying the property that every finite pattern of the one occurs in the other (and vice versa) are called *locally isomorphic* \([17]\). This motivates the consideration of so called *LI-classes* \([23,33]\) of tilings, a concept which is designed for any tiling of \(P\) by prototiles. The *LI-class* represented by a tiling \(T_{\gamma-a}\) for \(a \in K_\gamma\) is given by all tilings \(T\) of \(P\) which are locally isomorphic to \(T_{\gamma-a}\) meaning that for any bounded \(U \subset P\) there is a translation \(y\) on \(P\), such that \(U \cap T = U \cap \{T_{\gamma-a} - y\}\). By the above proposition this *LI-class* contains all \(T_{\gamma-a'}\) with \(a' \in K_\gamma\) (values \(a' \notin \pi^\perp(\Lambda)\) will in general lead to different *LI-class*). Now one considers the quotient of the *LI-class* represented by the tilings with \(a \in K_\gamma\) modulo translation. It is convenient to restrict the total space to

\[
\Omega = \{T \mid \forall U \subset P \text{ bounded } \exists a \in K_\gamma : U \cap T = U \cap T_{\gamma-a}\}, \tag{5}
\]

for we shall see in section 2.1 that \(\Omega\) can be equipped with a topology with respect to which it is totally disconnected and compact. The above quotient then being expressed as

\[
\Psi = \Omega / \text{transl.} \tag{6}
\]

furnishes for aperiodic tilings an example of a *quantum space* as described in \(\Lambda\). Connes’ book \([3]\). This particular name originates in the failure of classical (commutative) geometry to describe it properly. In fact as \(\Psi\) is in that case the quotient of a topological space modulo a dense equivalence relation its quotient topology is not Hausdorff, namely the only closed sets are \(\Psi\) itself and the empty set, i.e. it appears, analyzed in terms of classical topology, as a single point \([3]\). However its structure is much richer. It may be revealed by assigning a certain non-commutative \(C^*\)-algebra to \(\Psi\) and investigating its invariants. This program has been carried out in \([\mathbb{I}]\) for the Penrose tilings. The final goal of this work shall be a generalization of it to classes of tilings which are defined by *ADE*-root systems.

### 1.2 Tilings Defined by Root Systems

We shall apply the strip method to the root lattice \(\Lambda\) of a root system. The subspace \(P\) along which the strips are constructed is characterized by its invariance under the action of a Coxeter element of the Weyl group of reflections in \(\Lambda\).

Any root \(\alpha\) of the root system defines an automorphism of the root lattice \(s_\alpha : \Lambda \mapsto \Lambda\)

\[
s_\alpha(\beta) := \alpha - 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \beta \tag{7}
\]

which may be considered as a reflection in the Euclidian space \(\mathbb{R}^N\), \((.,.)\) denoting its scalar product, in which \(\Lambda\) is embedded. Having chosen a system of simple (or fundamental) roots
\{\alpha_i\}_i$, the group generated by the elements $s_i = s_{\alpha_i}$ contains all the above reflections; it is called Weyl group of reflections. It is a Coxeter group, i.e. the relations of its generators are of the form

$$(s_is_j)^{m_{ij}} = 1,$$  \hspace{1cm} (8)

where $m_{ij} = m_{ji} \in \mathbb{Z}^\geq 2$ for $i \neq j$ and $m_{ii} = 1$. In particular all generators are of order 2 and $s_is_j$ commute, iff $m_{ij} = 2$. The relations (8) may be uniquely encoded in a so called Coxeter graph $\Gamma$ consisting of a set of vertices $\Gamma^{(0)}$ and a set of unoriented edges $\Gamma^{(1)}$. The vertices are associated to the simple roots, $\Gamma^{(0)} \equiv \{\alpha_1, \alpha_2, ..\}$, the $i$'th and $j$'th vertex are connected by

$$C_{ij} := m_{ij} - 2$$  \hspace{1cm} (9)

dges, and no edge has equal source and range: $C_{ii} = 0$. The matrix $C$ is called connectivity (or adjacency) matrix of the graph. A particular role in the analysis of Coxeter groups is played by elements which are products of all generators. They are called Coxeter elements. Two distinct Coxeter elements $\omega, \tilde{\omega}$ differ by the succession of the generators in the product, but they are conjugate, i.e. there is a Weyl-reflection $\delta$, such that

$$\tilde{\omega} = \delta \omega \delta^{-1}. \hspace{1cm} (10)$$

In particular all Coxeter elements have the same order $h (\omega^h = 1)$, called the Coxeter number of the group, and the same eigenvalues. The eigenvalues, coming, if complex, in conjugate pairs, are of the form $exp\left(\frac{2\pi im}{h}\right)$, the possible values for $m \in \mathbb{Z}_h$ being called Coxeter exponents here denoted by $Exp(\Gamma)$. 0 is never but 1 is always a Coxeter exponent. $\mathbb{R}^N$ may now be decomposed into invariant subspaces of $\omega$

$$\mathbb{R}^N = \bigoplus_{m \in Exp(\Gamma) \leq \frac{h}{2}} P^{(m)}_\omega, \hspace{1cm} (11)$$

$P^{(m)}_\omega$ denoting the subspace on which $\omega$ acts as a rotation around $\frac{2\pi m}{h}$ or, if $m = \frac{h}{2}$, as a reflection. Consequently the direct sum runs over roughly speaking half of the Coxeter exponents (one for any complex pair of eigenvalues) eventually including $\frac{h}{2}$.

Given a root system we now consider 2-dimensional tilings which are defined with the help of

- the root lattice $\Lambda$,
- $P_\omega = P^{(1)}_\omega$ being specified by the choice of a Coxeter element $\omega$,  
- a base $L$ of $\Lambda$.

How does the choice of a Coxeter element affect that construction? Whereas the set $Exp(\Gamma)$, is independent of the choice of $\omega$, the decomposition of $\mathbb{R}^N$ into invariant subspaces $P^{(m)}_\omega$ is not. If two Coxeter elements $\omega, \tilde{\omega}$ are related by $\delta$ as in (10), then

$$P^{(m)}_{\tilde{\omega}} = \delta P^{(m)}_\omega$$  \hspace{1cm} (12)

and in particular $\tilde{\pi} = \delta \pi \delta^{-1}$, $\tilde{\pi}$ resp. $\pi$ denoting the projection onto $P^{(1)}_{\tilde{\omega}}$ resp. $P^{(1)}_\omega$. Therefore

$$\tilde{S}_\gamma = \{\gamma + P_\omega\} = \delta \{\delta^{-1} \gamma + P_\omega\} = \delta S_{\delta^{-1} \gamma}$$  \hspace{1cm} (13)
as well as
\[ \tilde{T}_\gamma = \tilde{\pi}(\tilde{S}_\gamma \cap \Lambda) = \delta \pi \delta^{-1}(\tilde{S}_\gamma \cap \Lambda) = \delta T_{\delta^{-1}_{\gamma}}. \] (14)
Moreover if \((\lambda, \lambda + \alpha) \in \tilde{S}_\gamma\), i.e. \(\tilde{\pi}(\lambda, \lambda + \alpha)\) is a link on \(P_\omega\), then \(\delta^{-1}(\lambda, \lambda + \alpha) \in S_{\delta^{-1}_{\gamma}}\), i.e. \(\pi \delta^{-1}(\lambda, \lambda + \alpha)\) is a link on \(P_\omega\). Hence up to \(\delta\), which acts as an isometry on \(P\), a change of the Coxeter element by \(\omega \mapsto \delta \omega \delta^{-1}\) has the same effect as transforming the links by \(\mathcal{L} \mapsto \delta^{-1} \mathcal{L}\) (which amounts to \(\gamma \mapsto \delta^{-1}\gamma\)). We may therefore concentrate on a particular choice for the Coxeter element.

1.3 Decomposition into 1-Dimensional Tilings

For further discussions we have to restrict ourself to root systems of type \(ADE\), whose Coxeter graphs are simply laced and bipartite (bicoloured). A Coxeter graph \(\Gamma\) is called bipartite, if the canonical plane \(\omega\) is (up to inversion, \(\omega^{-1} = s_{(2)} s_{(1)}\)) canonical, since the above decomposition of \(\Gamma^{(0)}\) is unique. We simply denote \(P^{(m)} = P_\omega^{(m)}\), if they are determined by this canonical \(\omega\), and call \(P = P_\omega^{(1)}\) the canonical plane.

On the one hand we may now decompose the root lattice \(\Lambda\) as well as \(P^{(m)}\) into the sub-lattices \(\Lambda_\epsilon\), which are generated by the simple roots assigned to \(\Gamma^{(0)}\) as well as the subspaces \(P_\epsilon^{(m)} = P^{(m)} \cap \mathbb{R} \Lambda_\epsilon\). On the other hand \(\mathbb{R}^N\) may be decomposed into the eigenspaces of the connectivity matrix \(\mathcal{C} \in \text{End}(\Lambda) \subset \text{End}(\mathbb{R}^N)\). The latter may be written as
\[ \mathcal{C} = \sum_{m \in \text{Exp}(\Gamma)} \tau^{(m)} \pi^{(m)}, \] (17)
where \(\pi^{(m)}\) denotes the projection onto the eigenspace \(E^{(m)}\) of the eigenvalue
\[ \tau^{(m)} = 2 \cos\left(\frac{m \pi}{h}\right) \] (18)
of \(\mathcal{C}\), \(h\) the Coxeter number of \(\Gamma\). These two decompositions of \(\mathbb{R}^N\) are related as follows:

**Lemma 2** For \(m \neq \frac{h}{2}\): \(P^{(m)} = E^{(m)} \oplus E^{(h-m)}\).

**Proof:** The main part of the proof is given in the book of Carter [8]. Concerning this part we state only some details, which are needed later on. Remember that in the simply laced case \(C_{ij} = 2 - 2(\alpha_i, \alpha_j)\) (with normalization \((\alpha_i, \alpha_i) = 1\)). Let \(\vec{\nu}^{(m)} = \sum_i \nu_i^{(m)} \alpha_i\) be an eigenvector of \(\mathcal{C}\) to the eigenvalue \(\tau^{(m)}\), i.e.
\[ E^{(m)} = \mathbb{R} \vec{\nu}^{(m)}. \] (19)
Furthermore let \(\tilde{\nu}_\epsilon^{(m)} = \sum_{j \in \Gamma^{(0)}} \nu_j^{(m)} \alpha_j\). Clearly \(\tilde{\nu}_1^{(m)} + \tilde{\nu}_2^{(m)} = \tilde{\nu}^{(m)}\). In [8] the dual basis \(\{\tilde{\alpha}_i\}_{\epsilon}\) is used to define \(\tilde{\mu}_\epsilon^{(m)} = \sum_{j \in \Gamma^{(0)}} \nu_j^{(m)} \tilde{\alpha}_j\). They are shown to have the following properties:
1. Both, $\hat{\mu}_1^{(m)}$ and $\hat{\mu}_2^{(m)}$ have the same length and form an angle of $\frac{m\pi}{h}$.

2. The canonical Coxeter element acts on the linear span of $\hat{\mu}_1^{(m)}$ and $\hat{\mu}_2^{(m)}$ as rotation around $\frac{2m\pi}{h}$, and hence this span equals $P^{(m)}$.

Now $\alpha_i = \sum_j (\delta_{ij} - \frac{1}{2} \mathcal{C}_{ij}) \hat{a}_j$ may be used to obtain $\hat{\nu}_i^{(m)} = \tilde{\mu}_i^{(m)} - \cos(\frac{m\pi}{h})\hat{\mu}_{i+1}^{(m)}$, which implies that $\hat{\nu}_1^{(m)}$ and $\hat{\nu}_2^{(m)}$ form an angle of $\frac{(h-m)\pi}{h}$ and span $P^{(m)}$, too. Therefore

$$P^{(m)}_\epsilon = \mathbb{R}\hat{\nu}_i^{(m)}.$$ (20)

On the other hand the bipartition of the graph implies $\mathcal{C}(\Lambda_\epsilon) \subset \Lambda_{\epsilon+1}$ and hence $\mathcal{C}\hat{\nu}_i^{(m)} = \tau^{(m)}\hat{\nu}_{\epsilon+1}^{(m)}$. Therefore

$$\mathcal{C}(\hat{\nu}_i^{(m)} - \hat{\nu}_2^{(m)}) = \tau^{(m)}(\hat{\nu}_2^{(m)} - \hat{\nu}_1^{(m)}) = \tau^{(h-m)}(\hat{\nu}_1^{(m)} - \hat{\nu}_2^{(m)}).$$

Hence $E^{(h-m)}$ is spanned by $\hat{\nu}_1^{(m)} - \hat{\nu}_2^{(m)}$. \qed

The first of the above decompositions furnishes a decomposition of the tilings, provided the links $\mathcal{L}$ decompose, too:

**Theorem 1** If $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, where $\mathcal{L}_\epsilon \subset \Lambda_\epsilon$, then any 2-dimensional tiling related to $(\Lambda, P, \mathcal{L})$ decomposes into a Cartesian product of two 1-dimensional tilings. More precisely

$$T_{\gamma-a} = T_{\gamma_1-a_1} \times T_{\gamma_2-a_2}$$ (21)

where $\gamma_\epsilon = \{\sum_{\alpha \in \mathcal{L}_\epsilon} \lambda_\alpha \alpha | \lambda_\alpha \in (0,1)\}$, $T_{\gamma-a}$ are the points of the 1-dimensional tiling related to $(\Lambda_\epsilon, P_\epsilon, \mathcal{L}_\epsilon)$ and $a_\epsilon$ is obtained by decomposing $a$ into the direct sum of vectors of $P_\epsilon = P^\perp \cap \mathbb{R}\Lambda_\epsilon$.

**Proof:** (21) is to be interpreted as follows: For every $t \in T_{\gamma-a} \subset P$ there is a unique decomposition $t = t_1 + t_2$ with $t_\epsilon \in T_{\gamma_\epsilon-a_\epsilon} \subset P_\epsilon$, and vice versa for every pair $t_\epsilon \in T_{\gamma_\epsilon-a_\epsilon}$, $\epsilon = 1,2$, the sum $t = t_1 + t_2$ lies in $T_{\gamma-a}$. This follows directly from the assumption (which implies that the elements of $\gamma$ have a unique decomposition too), if the projection onto $P$ preserves the decomposition, i.e. if $\pi(\Lambda_\epsilon) \subset P_\epsilon$.

Setting $\hat{\nu}_\epsilon = \frac{\hat{\nu}_1^{(\epsilon)}}{\|\hat{\nu}_2^{(\epsilon)}\|}$ we have for $a \in \mathbb{R}^N$

$$\pi(a) = \frac{(\hat{\nu}_1 + \hat{\nu}_2, a)}{2(1 - \cos \frac{\pi}{h})}(\hat{\nu}_1 + \hat{\nu}_2) + \frac{(\hat{\nu}_1 - \hat{\nu}_2, a)}{2(1 + \cos \frac{\pi}{h})}(\hat{\nu}_1 - \hat{\nu}_2).$$

Using

$$(\hat{\nu}_\epsilon, a) = \sum_{i,k} \hat{\nu}_{\epsilon i}(\delta_{ij} - \frac{1}{2} \mathcal{C}_{ij}) a_j = \sum_{i \in \Gamma^{(0)}_1} \hat{\nu}_{\epsilon i} a_i - \cos \frac{\pi}{h} \sum_{i \in \Gamma^{(0)}_{\epsilon+1}} \hat{\nu}_{\epsilon+1 i} a_i$$

one obtains

$$\pi(a) = \sum_{i \in \Gamma^{(0)}_1} \hat{\nu}_{\epsilon i} a_i \hat{\nu}_1 + \sum_{i \in \Gamma^{(0)}_2} \hat{\nu}_{\epsilon i} a_i \hat{\nu}_2.$$ (22)

and in particular $\pi(\Lambda_\epsilon) \subset P_\epsilon$. As $\mathcal{L}_\epsilon \subset \Lambda_\epsilon$ any link of the 2-dimensional tiling is either parallel to $P_1$ or to $P_2$, it is therefore either in the 1-dimensional tiling related to $P_1$ or in the one related to $P_2$. \qed
For application of this theorem we mainly consider what we call ADE-projections. These are tilings defined (next to the root lattice and the canonical plane) by the basis of simple roots \( \mathcal{L} = \{ \alpha_i \} \). The vector \( \vec{r}_i^2 \) spanning \( P_i \) is the Perron Frobenius vector of \( C^2|_\Lambda \), having only strictly positive (or strictly negative) entries. Decomposition of a regular tiling yields regular tilings and we may think of the latter as two-sided infinite sequences of intervals of the form \((\pi(x), \pi(x + \alpha_i))\), at least in case \( P^\perp \cap \Lambda_\epsilon = \{ \vec{0} \} \). For \( P^\perp \cap \Lambda_\epsilon \neq \{ \vec{0} \} \) holes may appear even in regular tilings, as mentioned earlier, but they occur periodically. In order to treat both cases in the same manner we shall fix these holes by adding some of the points that are on \( \partial S_{\gamma - \alpha_\epsilon} \cap \Lambda_\epsilon \) periodically.

Remark: It is noteworthy that in perturbed conformal field theories which are related to ADE-Lie algebras an explanation of the allowed mass triangles has been proposed using mathematical methods which have a substantial overlap to the one used here \([21],[10]\): By projection of a certain subset of three roots (satisfying what is called fusing condition in \([10]\)) onto \( P^{(m)} \) a reformulation of the bootstrap equations is obtained. In this framework \( \nu_{i}^{(m)} \) corresponds for \( m > 1 \) to the components of the conserved higher spin charges and for \( m = 1 \) to the particle masses.

### 1.4 Aperiodicity and Further

This and the following subsection are devoted to the study of the properties of the tilings related to ADE-root systems. As we are interested in aperiodic tilings, the computation of \( P \cap \Lambda \) will be treated first. Hereafter we turn into the analysis of the structure of \( P^\perp \cap \Lambda \), which is of relevance for the definition of the quantum space \( \Psi \). We determine a subspace \( E \) of \( \mathbb{R} \Lambda \), of which may be shown that

- \( P \subset E \) and the dimension of \( E \) resp. \( E^\perp \) as vector spaces equal the dimension of \( E \cap \Lambda \) resp. \( E^\perp \cap \Lambda \) as lattices.

Furthermore, under certain circumstances

- \( E^\perp \cap \Lambda = P^\perp \cap \Lambda \),

i.e. the connected component of \( \pi^\perp(\Lambda) \) that contains \( \vec{0} \) equals \( E^\perp \). We may concentrate onto the invariant eigenspaces \( P^{(m)} \) of the canonical Coxeter element, since \( P_\omega \cap \Lambda \) resp. \( P^\perp_\omega \cap \Lambda \) and \( P_\omega \cap \Lambda \) resp. \( P^\perp_\omega \cap \Lambda \) are related by the automorphism \( \delta \) as in \([12]\).

Concerning the aperiodicity of the tilings we have:

**Proposition 2** \( P^{(m)} \cap \Lambda = \{ \vec{0} \} \) iff \( (\tau^{(m)})^2 \notin \mathbb{Q} \).

**Proof:** Remember \([20]\) and \( \mathcal{C}(P^{(m)}_\epsilon) = P^{(m)}_{\epsilon + 1} \). Hence \( P^{(m)} \cap \Lambda \neq \{ \vec{0} \} \) iff \( P^{(m)}_1 \cap \Lambda_1 \neq \{ \vec{0} \} \) iff \( P^{(m)}_2 \cap \Lambda_2 \neq \{ \vec{0} \} \). Moreover \( \nu^{(m)}_i \) is an eigenvector of \( C^2 \) to the eigenvalue \( (\tau^{(m)})^2 \). Now assume \( \vec{0} \neq z \in P^{(m)} \cap \Lambda_\epsilon \). Then \( (\tau^{(m)})^2 z = C^2 z \in P^{(m)}_\epsilon \) implying \( (\tau^{(m)})^2 \notin \mathbb{Q} \). Conversely let \( a(\tau^{(m)}) \in \mathbb{Z} \) such that \( a(\tau^{(m)})^2 \in \mathbb{Z} \). This implies \( \{ \vec{0} \} \neq Ker \mathbb{Z}(aC^2 - (\tau^{(m)})^2) \subset P^{(m)} \cap \Lambda \).

As rationality of \( (\tau^{(1)})^2 \) does only occur for \( h = 2, 3, 4, 6 \), all ADE-root systems except \( A_2, A_3, A_5 \) and \( D_4 \) lead to aperiodic tilings.
To analyse the structure of $P^\perp \cap \Lambda$ we use the Chebyshev polynomials of the second kind: Consider the set of polynomials in one variable over $\mathbb{Z}$, $p_1(x) = 1$ and

$$p_k(x) = \begin{vmatrix} x & 1 & 0 & \cdots \\ 1 & x & 1 & \ddots \\ 0 & 1 & x & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{vmatrix}$$

for $k \geq 2$. In fact $p_k(-x)$ is the characteristic polynomial of the connectivity matrix of $A_{k-1}$, the latter being denoted by $A_k$. The polynomials satisfy the recursion relations

$$p_{k+2}(x) = xp_{k+1}(x) - p_k(x)$$

and have homogeneous $\mathbb{Z}_2$-degree

$$p_k(-x) = (-1)^{k-1}p_k(x).$$

For $|x| \leq 1$, $p_k(x) = U_{k-1}(\frac{x}{2})$, where $U_k(x)$ is the $k$'th Chebyshev polynomial of the second kind \cite{Ref} and

$$U_{k-1}(\cos\Theta) = \frac{\sin k\Theta}{\sin \Theta}.$$  

As long as $i + 1 \leq h - 1$ an equivalent way of writing \cite{Ref} is

$$xp_i(x) = (A_h)^i_j p_j(x),$$

which indicates that the vector $\vec{p}(x) = (p_1(x), p_2(x), \ldots, p_{h-1}(x))$ is an eigenvector of $A_h$ whenever $x$ is an eigenvalue, i.e. $p_h(x) = 0$. Hence for $m \in \text{Exp}(A_{h-1})$ the polynomials obey

$$p_i(\tau^{(m)}) p_k(\tau^{(m)}) = (p_i(A_h))^i_k p_l(\tau^{(m)})$$

(for any $i, k \leq h - 1$) and

$$p_k(\tau^{(m)}) = \frac{\sin \frac{k\pi m}{h}}{\sin \frac{\pi m}{h}}.$$  

Note that $\vec{p}(\tau^{(m)})$ yields an explicit realization of the vector spanning $E^{(m)}$ in the case of the Coxeter group $A_{h-1}$. As $|p_k(\tau^{(m)})| = |p_{k+h}(\tau^{(m)})|$ it is useful to regard the set of Coxeter exponents $\text{Exp}(\Gamma)$ as a subset of $\mathbb{Z}_h (= \mathbb{Z} \mod h)$, $h$ being the Coxeter number of $\Gamma$. It may be checked that the subset of $\text{Exp}(\Gamma)$ of elements which are invertible in $\mathbb{Z}_h$ does only depend on the Coxeter number, in fact

$$\{m \in \text{Exp}(\Gamma) \mid m \text{ is invertible in } \mathbb{Z}_h\} = \mathbb{Z}_h^*,$$

$\mathbb{Z}_h^*$ denoting the group of invertible elements of $\mathbb{Z}_h$. Furthermore as long as $k \in \mathbb{Z}_h^*$ one has $k \text{Exp}(\Gamma) = \text{Exp}(\Gamma)$ and $\frac{k}{2} = \frac{h}{2}$ if $\frac{h}{2} \in \text{Exp}(\Gamma)$.

**Lemma 3** Let $\Gamma$ be an ADE-Coxeter group with Coxeter number $h$, $C$ the connectivity matrix of its graph. If $k \in \mathbb{Z}_h^*$, then $p_k(C)$ is an automorphism of $\Lambda$, whereas otherwise $p_k(C)$ is not even injective.
Proof: At first let $k \in \mathbb{Z}_h^*$. We have
\[
\prod_{m \in \text{Exp}(\Gamma)} |\sin\left(\frac{m\pi}{h}\right)| = \prod_{m \in k \text{Exp}(\Gamma)} |\sin\left(\frac{m\pi}{h}\right)| = \prod_{m \in \text{Exp}(\Gamma)} |\sin\left(\frac{km\pi}{h}\right)|
\]
which, in view of (29), implies
\[
\prod_{m \in \text{Exp}(\Gamma)} |p_k(\tau(m))| = 1.
\] (31)
But this implies for the determinant of $p_k(C)$ to be of modulus 1 (if $\frac{h}{2} \in \text{Exp}(\Gamma)$, then $|p_k(\tau(\frac{h}{2}))| = 1$, hence the statement is also true for $D_n$ with $n$ even) proving the first part of the lemma. The second one follows from (29) which implies $|p_k(\tau(m))| = 0$ iff $km = 0 \text{ mod } h$, and the determinant of $p_k(C)$ vanishes for $k \notin \mathbb{Z}_h^*$.

Define
\[
E = \bigoplus_{m \in \mathbb{Z}_h^*} E^{(m)},
\] (32)
and $E^\perp$, its orthogonal complement in $\mathbb{R}^N$. Clearly $P \subset E$ and $E^\perp \subset P^\perp$.

**Proposition 3** $E \cap \Lambda$ and $E^\perp \cap \Lambda$ have both maximal dimension, i.e. the dimension of $E \cap \Lambda$ resp. $E^\perp \cap \Lambda$ as a lattices equal the dimension of $E$ resp. $E^\perp$ as vector spaces.

Proof: Given any map $\phi \in \text{End}(\Lambda) \subset \text{End}(\mathbb{R}^N)$, $\text{Im } \phi \cap \Lambda$ as well as $\text{Ker } \phi \cap \Lambda$ have always maximal dimension. Set $\phi := \prod_{k \in \mathbb{Z}_h^* \setminus \{0\}} p_k(C) \in \text{End}(\Lambda)$. Viewed as an element of $\text{End}(\mathbb{R}^N)$,
\[
\text{Im } \phi = E
\]
and
\[
\text{Ker } \phi = E^\perp,
\]
following again from $|p_k(\tau(m))| = 0$ iff $km = 0 \text{ mod } h$. □

The remaining question is, whether $E \cap P^\perp \cap \Lambda = \{0\}$ or not. As well as any automorphism of $\Lambda$ which preserves $E \cap P^\perp$ and its orthogonal complement, $p_k(C)$ restricts, for $k \in \mathbb{Z}_h^*$, to an automorphism of $\Delta' = E \cap P^\perp \cap \Lambda$. Hence $\mathbb{R}\Delta'$ may be decomposed into eigenspaces of $C$, i.e. there is a subset $S \subset \mathbb{Z}_h^* \setminus \{1, -1\}$ such that
\[
\mathbb{R}\Delta' = \bigoplus_{m \in S} E^{(m)}.
\] (33)
Moreover $S = -S$, because of the fact that every element of $\Delta'$ decomposes into a sum of elements of $\Lambda_\epsilon$ and $C(P_\epsilon^{(m)}) = P_{\epsilon+1}^{(m)}$. Since $p_k(C)$ restricts for all $k \in \mathbb{Z}_h^*$ to an automorphism on $\Delta'$
\[
\prod_{m \in S} |p_k(\tau(m))| = 1,
\] (34)
which is
\[
\prod_{m \in S} |\sin\left(\frac{m\pi}{h}\right)| = \prod_{m \in S} |\sin\left(\frac{km\pi}{h}\right)|.
\] (35)
Formula (35) may be regarded as a necessary condition for the existence of $S$. By the construction of $S$, which in particular rules out $S = \mathbb{Z}_h^*$, it seems to be rather difficult to find such a set $S$. Let us already mention here that the existence of a solution of (43) (see below) contradicts (34). In that case $\Delta' = \{0\}$ and $P^\perp \cap \Lambda = E^\perp \cap \Lambda$. □
1.5 Selfsimilarity

Many aperiodic tilings furnish selfsimilar sets. Selfsimilarity, being manifested by an inflation / deflation procedure, is the basis of an alternative description of the quantum space which shall be discussed in the second part of this work.

Considering tilings related to arbitrary \( \Lambda, P, \mathcal{L} \) a map \( \sigma \in Aut(\Lambda) \), acting on the links by \( \mathcal{L} \mapsto \sigma(\mathcal{L}) \) and correspondingly on the vertices of the tiling by \( T_{\gamma-a} \mapsto T_{\sigma(\gamma-a)} \), is called an inflation for \( T_{\gamma-a} \), if there is a real number \( \tau > 1 \) such that

\[
T_{\sigma(\gamma-a)} \subset T_{\gamma-a} \quad \text{(36)}
\]

\[
\mathcal{T}_{\sigma(\gamma-a)} = \tau T_{\gamma-a} \quad \text{(37)}
\]

Hence the vertex set of a tiling which admits an inflation is a selfsimilar set, see \([11]\) for the the possibility of selfsimilarity up to symmetries of the lattice (e.g. reflections). The inverse of an inflation is a deflation.

Note that the above conditions are stated for a single tiling. An inflation for \( T_{\gamma-a} \) will in general not be at the same time an inflation for \( T_{\gamma-a'} \). Nevertheless \( T_{\gamma-a'} \) would for \( a' - a \in \mathbb{N} \) also be selfsimilar.

Specifying now to tilings related to ADE-root systems that are cartesian products of 1-dimensional tilings (i.e. \( \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \)), we require that the inflation decomposes, too, \( \sigma = \sigma_1 \oplus \sigma_2 \) where \( \sigma_\epsilon = \sigma|_{\Lambda_\epsilon} \) is now an inflation for \( T_{\gamma_\epsilon-a_\epsilon} \).

The important property of an inflation is given by:

**Proposition 4** Let \( \mathcal{L} = \{\alpha_i\} \), and \( \sigma \) be an inflation for some \( T_{\gamma-a} \) with decomposition as above. Then \( \sigma_\epsilon \) is a positive endomorphism (i.e. has entries in \( \mathbb{Z}^+ \)) preserving \( P_\epsilon \) and having \( \tau \) as largest and nondegenerate eigenvalue.

**Proof:** Consider a point \( x \in S_\sigma(\gamma_\epsilon-a_\epsilon) \cap \Lambda_\epsilon \) such that \( x' = x + \sigma_\epsilon^n(\alpha_i) \in S_\sigma(\gamma_\epsilon-a_\epsilon) \cap \Lambda_\epsilon \) for some \( \alpha_i \in \mathcal{L}_\epsilon, \ n \geq 1 \). Hence \( \pi(x) \) and \( \pi(x') \) are the boundary points of a prototile in \( S_\sigma(\gamma_\epsilon-a_\epsilon) \). As the latter appears as a twosided sequence of prototiles \( n \)-fold use of (36) implies that there is a sequence of \( k = \sum_i (\sigma_\epsilon^n)_{li} \) generators \( \alpha_{i_1}, \ldots, \alpha_{i_k} \), such that

\[
x' - x = \alpha_{i_1} + \cdots + \alpha_{i_k}, \quad \text{(38)}
\]

and for all \( 1 \leq j < k \)

\[
x + \alpha_{i_1} + \cdots + \alpha_{i_j} \in S_{\gamma_\epsilon-a_\epsilon} \setminus S_{\sigma_\epsilon^n(\gamma_\epsilon-a_\epsilon)}. \quad \text{(39)}
\]

Since the r.h.s. of (38) contains only + signs \( (\sigma_\epsilon)_li \) is positive. Now remember that the vector \( \vec{\nu}^{(1)} \) spanning \( P_\epsilon \) has strictly positive entries. Therefore, and because of \( \pi_\epsilon(x' - x) \) being bounded from above, all \( \alpha_i \in \mathcal{L}_\epsilon \) have to appear on the r.h.s. of (38), if \( n \) is large enough. Hence, for \( n \) large enough \( \sigma_\epsilon^n \) has strictly positive entries, and the Perron Frobenius theorem may be applied to establish that \( \sigma_\epsilon \) has a nondegenerate eigenvalue which exceeds all other eigenvalues in modulus. The corresponding eigenvector is called Perron Frobenius vector and has only strictly positive (or strictly negative) components. The Perron Frobenius theorem also applies to \( \sigma_\epsilon^1 \). Now by (37) \( (\sigma_\epsilon)\vec{\nu}, \alpha_i) = (\vec{\nu}, \sigma_\epsilon(\alpha_i)) = \tau(\vec{\nu}, \alpha_i) \) for all \( i \) and therefore \( \vec{\nu} \) is the Perron Frobenius eigenvector of \( \sigma_\epsilon^1 \) to the eigenvalue \( \tau \). In particular \( \tau \) is the largest eigenvalue not only of \( \sigma_\epsilon^1 \) but also of \( \sigma_\epsilon \). Finally observe that (37) implies \( \sigma_\epsilon \vec{\nu} = \tau \vec{\nu} + \vec{\mu} \) for some \( \vec{\mu} \in P_\epsilon^\perp \) which is compatible with the above only for \( \vec{\mu} = 0 \). \( \square \)

\[13\]
The above proposition asserts that the specific form of the inflation will not be of importance for our investigations in the last section, since there only the Perron Frobenius vector will be of importance. So we are left with the question of whether all tilings defined by root lattices allow for an inflation. For a general treatment of selfsimilarlarity in two dimensions we refer to \[24\]. But let us comment on a possible way to find one in our framework, in case \(P^\perp \cap \Lambda = E^\perp \cap \Lambda\).

Similar to the common method to obtain an inflation we search for an automorphism \(\sigma'\) of \(\Lambda\), which rescales the vectors of \(P\) by a factor > 1, preserves \(P^\perp\) and is strictly contracting on \(E \cap P^\perp\) \[13\] \[11\]. In fact, under suitable circumstances the requirements formulated above are sufficient, for the following reason: First observe that for \(\lambda \in \Delta = E^\perp \cap \Lambda\)

\[
\pi(S_{\gamma-a-\lambda} \cap \Lambda) = \pi(S_{\gamma-a} \cap \Lambda).
\]

Maximalitiy of the dimension of \(\Delta\) implies

\[
\pi(S_{\gamma-a} \cap \Lambda) = \sum_{b \in \pi_{E^\perp}(\Lambda) \cap \pi_{E^\perp}(\gamma-a)} \pi(S_{\gamma-a} \cap \{E + b\} \cap \Lambda),
\]

the sum being finite, where \(\pi_{E^\perp}\) denotes the orthogonal projection onto \(E^\perp\). Now \[36\] would be satisfied, if for all \(b \in \pi_{E^\perp}(\Lambda) \cap \pi_{E^\perp}(\gamma-a)\) there is a \(\lambda(b) \in \Delta\) such that

\[
\sigma(S_{\gamma-a} \cap \{E + b\}) - \lambda(b) \subset S_{\gamma-a} \cap \{E + \sigma(b) - \lambda(b)\}.
\]

As \(\pi_{E^\perp}(\gamma-a)\) contains the interior of a fundamental domain of \(\Delta\), \(\sigma(b) \in \pi_{E^\perp}(\gamma-a) - \Delta\) for almost all \(b \in \pi_{E^\perp}(\gamma-a)\). If \(\sigma'\) is strictly contracting on \(E \cap P^\perp\), \[12\] may be satisfied for some finite power \(\sigma = \sigma'^n\) provided all \(b \in \pi_{E^\perp}(\Lambda) \cap \pi_{E^\perp}(\gamma-a)\) are inner points of \(\gamma-a \cap \{E + b\}\).

Therefore automorphisms of the form \(\sigma = \sigma'^n\) with \(\sigma' = \prod_{k \in \mathbb{Z}_h} p_k^{n_k}(\mathcal{C})\), \(n_k \in \mathbb{Z}\), which satisfy

\[
\prod_{k \in \mathbb{Z}_h} |p_k^{n_k}(\tau^{(m)}_h)| < 1
\]

for all \(m \in \mathbb{Z}_h^* \setminus \{1, -1\}\) are possible candidates for inflations. Any solution of the inequality \[13\] has to restrict to an automorphism of \(E \cap P^\perp \cap \Lambda\), requiring therefore \(E \cap P^\perp \cap \Lambda = \{\vec{0}\}\). \[13\] is equivalent to

\[
\sum_{k \in \mathbb{Z}_h^* \setminus \{1, -1\}} n_k ln|p_k(\tau^{(m)}_h)| < 0
\]

and may be certainly obtained, if the determinant of the matrix having \(ln|p_i(\tau^{(i)}_h)|\) as entries \((i, j \in \mathbb{Z}_h^* \setminus \{1, -1\})\) does not vanish. Note that again \[43\] does only refer to the Coxeter number, not to the group itself. It is not difficult to construct solutions of \[43\] by hand as long as the number of elements in \(\mathbb{Z}_h^*\) is not too large. By numerical inversion of the above mentioned matrix a solution for \(h = 29\) was obtained.

1.6 \(\mathbb{Z}^h\)- versus \(A_{h-1}\)-Projections

The set of classes of locally isomorphic tilings obtained from \(A_{h-1}\) on the one hand and from \(\mathbb{Z}^h\) on the other are intimately related. The mapping \(\alpha_i \mapsto e_i - e_{i+1}\) is an embedding of the root lattice \(\Lambda_{h-1}\) of \(A_{h-1}\) into \(\mathbb{Z}^h\) in such a way that \(\mathbb{Z}^h \cap \mathbb{R}\Lambda_{h-1} = \Lambda_{h-1}\). It is the symmetry axis \(P^{(0)}\) of the \(h\) fold symmetry which adds to \(\mathbb{R}\Lambda_{h-1}\) to yield the embedding space \(\mathbb{R}^h\) of
Moreover the $h$-fold symmetry of $\mathbb{Z}^h$ restricts to a Coxeter element $\omega'$ of $A_{h-1}$, although it is not the canonical one. Hence the plane along which the $\mathbb{Z}^h$-strip is constructed corresponds to $P_{\omega'}^{(1)} \subset \text{IRA}_{h-1}$. Let $\delta$ be the Weyl-reflection such that

$$\omega' = \delta \omega \delta^{-1}, \quad (45)$$

$\omega$ being the canonical Coxeter element. By transforming the links as $\mathcal{L} \mapsto \delta^{-1} \mathcal{L}$ we may compare $A_{h-1}$-projections directly with $\mathbb{Z}^h$-projections. We have

$$\pi^\perp(\mathbb{Z}^h) = \pi^\perp(\mathbb{Z}e_1) + \pi^\perp(A_{h-1}), \quad (46)$$

and from $\pi^\perp(he_1) \subset \pi^\perp(\sum_{i=1}^h e_i) + \pi^\perp(A_{h-1})$ and $\mathbb{Z} \sum_{i=1}^h e_i = P^{(0)} \cap \mathbb{Z}^h$ one concludes that $\pi^\perp(ne_1) + \pi^\perp(A_{h-1})$ and $\pi^\perp(me_1) + \pi^\perp(A_{h-1})$ have for $n \neq m$ at least Euclidian distance $\sqrt{h}$. Therefore

$$\pi^\perp(\mathbb{Z}^h) = \pi^\perp(\mathbb{Z}e_1) + \pi^\perp(A_{h-1}), \quad (47)$$

hence

$$\frac{\pi^\perp(\mathbb{Z}^h)}{\pi^\perp(\mathbb{Z}^h)} = \frac{\pi^\perp(A_{h-1})}{\pi^\perp(A_{h-1})}. \quad (48)$$

This shows that the difference between the quantum sets related to $\mathbb{Z}^h$ on the one hand and to $A_{h-1}$ on the other may have its origine only in the topolgy of the total space which is after all determined by the singular tilings.

As mentioned above the Coxeter element $\omega'$ is not the canonical one we discussed in (15). In fact for $A$-Coxeter groups there are two natural choices for Coxeter elements (up to inversion). The one here is $\omega' = s_1 s_2 \cdots s_{h-1}$ having numbered the simple roots, i.e. the vertices of the Coxeter graph in rising order from one end to the other (which is possible only for $A$-Coxeter groups). For this reason the $A_{h-1}$-projections look quite different from $\mathbb{Z}^h$-projections. In case of the Coxeter group $A_4$, there is an interesting connection with Ammann bars as they are described in [14] for Penrose tilings. Ammann bars are a decoration of the prototiles out of which one may construct a Penrose tiling. In fact there is an alternative procedure (actually the original one) to construct such a tiling, namely by fitting prototiles together, where one has to obey certain local matching conditions. Ammann bars may be understood as a way to formulate these conditions. If one keeps only the Ammann bar decoration of a complete tiling, one is left with a so called Ammann-quasicrystal [22], which consists of five 1-dimensional tilings of the sort we obtained from our decomposition of an $A_4$-tiling. However, according to a result of John H. Conway, which is stated in [13], these five 1-dimensional tilings are not completely independent but the whole Ammann-quasicrystal is determined, up to an uppermost threefold degeneracy, by just two of them. These two may be taken to be a 2-dimensional $A_4$ tiling projected onto the canonical plane (15) and taking the links to be the simple roots (an $A_4$-projection). Figure 1 is meant to illustrate this. It consists of the so-called "cartwheel tiling" [17] together with its Ammann bar decoration appearing here as five aperiodic sequences of parallels with distances 1 or $\tau$ (the golden ratio). Figure 1 has a fivefold symmetry, but it is incomplete. In the cartwheel tiling the so-called Conway worms are not filled out and correspondingly in each of the five sequences of parallels the line closest to the center is missing. If one now completes two of them (there are two choices for each one) one is left with only three (up to symmetry) instead of eight choices for completing the others.

For completeness let us finally mention how candidates for inflations for $A_{h-1}$-tilings may be used to define candidates for inflations for $\mathbb{Z}^h$-tilings.
The role of the square of connectivity matrix is played by the matrix $\mathcal{M} + 2id$, $\mathcal{M}$ having in the canonical basis entries $\mathcal{M}_{ij} = 1$ if $j = i - 1 \mod h$ or $j = i + 1 \mod h$ and otherwise 0. E.g. for $h = 5$

$$\mathcal{M} = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{pmatrix}. \quad (49)$$

Clearly this matrix commutes with the $h$-fold symmetry showing that it leaves $P^{(m)}_{\omega}$ invariant, $0 \leq m \leq \frac{1}{2}$. Note that $\mathcal{M}$ is the connectivity matrix of the affine extension of $A_{h-1}$. Its eigenvalues are $2\cos(\frac{2\pi m}{h}) = 4\cos^2(\frac{\pi m}{h}) - 2$ for all $m$. For this reason (and with the above identification $\alpha_i = e_i - e_{i+1}$)

$$(\mathcal{M} + 2id)|_{\Lambda_{h-1}} = \delta C^2 \delta^{-1}. \quad (50)$$

Now let $p = \prod_{k \in \mathbb{Z}_h} p_k^{n_k}$, $n_k \in \mathbb{Z}$, be a solution of (43) and for which in addition holds $p(2) = 1 \mod h$. Let $q$ be the polynomial defined by $q(x^2) = p(x)$ ($p$ has to be even). Then $q(\mathcal{M} + 2id)$ may be understood as an automorphism of $\mathbb{Z}^h/(P^{(0)} \cap \mathbb{Z}^h)$ which is strictly contracting on $E \cap P^{\perp}$, hence it is a candidate for an inflation for $\mathbb{Z}^h$-tilings. This follows from $\pi(S(\gamma - \lambda) \cap \mathbb{Z}^h) = \pi(S(\gamma) \cap \mathbb{Z}^h)$ for $\lambda \in P^{(0)} \cap \mathbb{Z}^h$ and the fact that 2 is the eigenvalue of $\mathcal{M}$ to $P^{(0)}$. Now as $p_k(2) = k$ and $\mathbb{Z}_h^*$ is finite, there is for $k \in \mathbb{Z}_h^*$ always a power $n'$ such that $p_k^{n'}(2) = 1 \mod h$. Hence a given even solution of (43) leads always to a candidate for an inflation of $\mathbb{Z}^h$. As an example, for $\mathbb{Z}^7$ the polynomial $p_2 p_4 = x^4 - 2x^2$ yields an inflation which scaling factor 4.049. It is given in figure 2.
2 Algebraic Characterization

Characterization of Quantum Spaces by $K_0$-Groups

The notion of a noncommutative space (or a quantum space), as it is investigated in A. Connes book [9], arose from the attempt to extend the duality between unital commutative $C^*$-algebras and compact Hausdorff spaces to noncommutative $C^*$-algebras. This duality is expressed in the theorem of Gelfand-Neumark stating that for a unital commutative $C^*$-algebra $A$

$$A = C(\text{spec}(A)), \quad (51)$$

i.e. $A$ coincides with the algebra of continuous complex functions over its spectrum $\text{spec}(A)$, which is given by the set of all maximal ideals of $A$ and which may be considered as a compact subspace (in the weak* topology) of $A'$, the dual vector space of $A$. Moreover for compact $X$, $\text{spec}(C(X)) = X$ as a topological space. Hence for a complete extension of the above duality to noncommutative algebras, an appropriate notion of their dual (generalizing the commutative case) has to be set up as well as a construction which, for a given space, yields an algebra such that the space appears as the dual of this algebra.

The notion of a spectrum of an algebra extending the commutative case which is used in [9] is defined with the help of the irreducible representations (irreps) of the algebra:

$$\hat{A} = \{\text{irreps of } A\} \text{ modulo unitary equivalence}. \quad (52)$$

An alternative notion could be the primitive spectrum $\check{A} = \{\text{Ker} \rho \mid \rho \text{irrep of } A\}$, however this turns out to be to small. In general $\check{A} \subset \hat{A}$ and both sets are equal if and only if $A$ is of type I (or postlimial) [26].

Instead of discussing the general theory let us specialize to spaces of the form that have been obtained in the first part of this article, namely spaces $X/R$, where $X$ is a compact totally disconnected Hausdorff space and $R$ is an equivalence relation. How can one construct an algebra $A$, such that $\hat{A} = X/R$?

An equivalence relation carries a groupoid structure. For elements $(x, y) \in R \subset X \times X$ composition and inversion are defined through the transitivity and the reflexivity: $(x, y)$ and $(y', z)$ are composable iff $y = y'$ yielding $(x, y)(y, z) = (x, z)$, and $(x, y)^{-1} = (y, x)$. A. Connes proposes to take $A$ to be the groupoid $C^*$-algebra $C^*(R)$ defined by $R$. To define it, one first considers the complex continuous functions over $R$ with compact support and (truncated) convolution product and involution, given in case the orbits of $R$ are discrete by

$$f \ast g(x, y) = \sum_{z \sim x} f(x, z)g(z, y), \quad (53)$$

$$f^*(x, y) = \overline{f(y, x)}. \quad (54)$$

This is a topological $*$-algebra. The norm of an element may be defined to be the supremum of its operator-norms in all bounded $*$-representations. Closure with respect to this norm yields the $C^*$-algebra $C^*(R)$ [31]. For more general situations, where one has to put a measure $R$, see [31]. Let us look at two simple examples:

1) If $R = \{(x, x) \mid x \in X\}$, i.e. $X/R = X$, then the product in (53) becomes the usual point multiplication of continuous functions over $X$ and we recover the commutative case.

2) Let $X = \{x_i^n\}_{i,a}$ with $i \in \{1, \cdots, k\}$ and $a \in \{1, \cdots, m_i\}$ be a finite set and $R$ be the equivalence relation $x_i^n \sim x_i^{n'}$ for $a, a' \in \{1, \cdots, m_i\}$. Then the product (53) coincides with
as we are interested in the quotient space only, and not its realization as a specific quotient, characterize the quotient space, among these being its invariants of stabilized algebras only, and this is reflected in example 2 above by the fact that has corresponding to a simple component of matrix multiplication in

\[ C^*(R) \cong \bigoplus_{i=1}^{k} M_{m_i}(\mathbb{C}). \]  

(55)

\( X/R \) has \( k \) points, each one may be identified with an irrep of \( C^*(R) \), i.e. each one corresponding to a simple component of \( C^*(R) \).

The second example already shows that \( C^*(R) \) contains not only information about the space \( X/R \), but also about the largeness of the orbits (equivalence classes) of \( R \). But as long as we are interested in the quotient space only, and not its realization as a specific quotient, only the invariants of \( C^*(R) \) which are insensitive against a change of such a realization characterize the quotient space, among these being its \( K_0 \)-group. In fact \( K_0 \)-groups are invariants of stabilized algebras only, and this is reflected in example 2 above by the fact that \( C^*(R) \) and \( M_n(C^*(R)) \cong C^*(R) \otimes M_n(\mathbb{C}) \) have the same spectrum. The stabilized algebra of a \( C^* \)-algebra \( A \) is by definition \( M_\infty(A) \cong A \otimes M_\infty(\mathbb{C}) \), \( M_\infty(\mathbb{C}) \) being the algebra of matrices with all but finitely many entries being non zero. It may be obtained by taking the direct (or induced) algebraic limit of the system \( M_n(\mathbb{C}) \subset M_{n+1}(\mathbb{C}) \subset \cdots \) of algebras, the embeddings being nonunital.

However, if \( X/R \) is a quantum space of tilings, \( C^*(R) \) will not be of type I, its primitive spectrum will reduce to one point and its spectrum will be much larger than \( X/R \), i.e. the "non-commutative" analog of (51) fails to hold true. Nevertheless \( C^*(R) \) - or better its \( K_0 \)-group - will still characterize \( X/R \). In a way the functor \( K_0 \) singles out certain irreps of \( C^*(R) \) which allow to recover \( X/R \) by restricting (52) to these irreps.

In general \( K_0(A) \), the \( K_0 \)-group of a unital \( C^* \)-algebra \( A \), is given by the Grothendieck completion of the following abelian monoid \( V(A) \): \( V(A) \) consists of equivalence classes of projections in the stabilized algebra \( M_\infty(A) \), namely \( p \sim q \) whenever there are elements \( x, y \in M_\infty(A) \) such that \( p = xy \) and \( q = yx \). The sum of projections \( p + q \) is again a projection, if they are orthogonal, i.e. \( pq = 0 \). However, by using the stabilized algebra \( M_\infty(A) \) instead of \( A \) itself, it is always possible to find, for given classes \([p], [q]\), orthogonal representatives \( \tilde{p}, \tilde{q} \) yielding a well defined addition \([p] + [q] = [\tilde{p} + \tilde{q}]\) on \( V(A) \). \( K_0(A) \) itself is given by formal differences in \( V(A) \), i.e. its elements are given by pairs \(([p], [q]) \in V(A) \times V(A) \) modulo the equivalence relation \(([p], [q]) \sim ([p'], [q']) \) whenever there is a \([r] \in V(A) \) such that \([p] + [q'] + [r] = [q] + [p'] + [r] \).

Not only the \( K_0 \)-group but also \( K_1(A) = \{(p, 0) | p \in V(A)\} \) characterizes the algebra, as the latter elements correspond to projections in \( M_\infty(A) \). To avoid cumbersome notation we shall write \([p]\) for \(([p], [0])\). In the cases which are of interest for us, \( K_1(A) \) defines a positive cone, i.e. an additively closed subset such that \( K_0^+(A) - K_0^+(A) = K_0(A) \) and \( K_0^+ \cap -K_0^+ = 0 \). In other words \( K_0(A) \) becomes an ordered group. The image of the unit in \( K_0^+(A) \) is called the order unit.

We will also have to use the \( K_1 \)-group of \( A \). For its definition consider \( GL_n(A) \), the group of invertible elements of \( M_n(A) \). \( GL_n(A) \) may be embedded into \( GL_{n+1}(A) \) by a group homomorphism yielding a directed system of groups the direct limit of which shall be denoted by \( GL(A) \). Let \( GL(A)_0 \) denote the connected component of the unit. Then \( K_1(A) = GL(A)/GL(A)_0 \). \( K_1(A) \) is an abelian group which is trivial if \( A \) is an \( AF \)-algebra.

For a reference to general operator \( K \)-theory see \[13\]. In the context of \( AF \)-algebras, which is discussed in more detail below, the above abstract definition yields an intuitive picture which may be encoded in a Bratteli diagram. According to a theorem of Elliot \[13\], the \( K_0 \)-group together with an order structure is a complete invariant for the stabilized algebra of an \( AF \)-algebra, and two \( AF \)-algebras having the same stabilized algebra are isomorphic,
if their order units coincide.

### 2.1 Application to $ADE$-tilings

The strip method suggested a realization of the quantum space of tilings by

$$
\Psi = \Omega_{\text{transl}},
$$

but here an orbit of the equivalence relation, i.e., an equivalence class of a tiling, contains not all translates of that tiling but only those which are integer linear combinations of links.

We denote the groupoid defined by the equivalence relation by $R_\Omega$. Taking, as at the end of subsection 12, $\mathcal{L} = \{\alpha_i\}$, application of theorem 1 leads to a decomposition, first of the total space, $\Omega = \Omega_1 \times \Omega_2$, and second of the equivalence classes, $R_\Omega = R_{\Omega_1} \times R_{\Omega_2}$, so that

$$
\Psi = \Psi_1 \times \Psi_2,
$$

where $\Psi_\epsilon$ is related to $(\Lambda_\epsilon, P_\epsilon, \mathcal{L}_\epsilon)$.

Let us first consider one single factor $\Psi_\epsilon$. Such a situation has been investigated by J. Bellissard [3]. Remember that the 1-dimensional tilings appear as twosided sequences of prototiles, which are in this case intervals of the form $(\pi(x), \pi(x+\alpha_i))$, $\alpha_i \in \mathcal{L}_\epsilon$, $x \in \Lambda_\epsilon$. Such a prototile shall be abbreviated by $i$ and referred to as a letter. A natural topology on $\Omega_\epsilon$ is given by the following subbase [29]: Its sets are labelled by a letter $i$ and $k \in \mathbb{Z}$:

$$
U_i^{(k)} = \{ \mathcal{T} \mid \text{the } k\text{'th letter of } \mathcal{T} \text{ is } i \}.
$$

Here we say that the first letter is the one to the right of $\bar{\epsilon}$. Since $U_i^{(k)} \cap U_j^{(k)} = \emptyset$ if $i \neq j$, and $\Omega_\epsilon \setminus \bigcup_{i \neq j} U_i^{(k)}$ these sets are open and closed so that $\Omega_\epsilon$ becomes a totally disconnected space. In fact this topology is inherited from the product topology on the set of all twosided sequences of the above letters. That set is compact and metrizable. A distance between two $\mathcal{T}$, $\mathcal{P}$ is related to (\Lambda_\epsilon, P_\epsilon, \mathcal{L}_\epsilon).

Now any orbit of the groupoid $R_{\Omega_\epsilon}$ may be identified with $\mathbb{Z}$, since translation here amounts to shifting by a certain number of letters. A shift by one letter to the right is a topologically transitive homeomorphism on $\Omega_\epsilon$. We denote this shift action by $\hat{\varphi}_\epsilon$ and identify $R_{\Omega_\epsilon}$ with $\Omega_\epsilon \times \mathbb{Z}$ through $(\mathcal{T}, \hat{\varphi}_\epsilon^{-k}(\mathcal{T})) \cong (\mathcal{T}, k)$. Then $\Omega_\epsilon$ is the closure of one single $\mathbb{Z}$-orbit.

The groupoid $C^*$-algebra defined by $\Omega_\epsilon \times \mathbb{Z}$ is a crossed product of $C(\Omega_\epsilon)$ with $\mathbb{Z}$, namely becomes

$$
f * g (\mathcal{T}, k) = \sum_{m \in \mathbb{Z}} f(\mathcal{T}, m) g(\hat{\varphi}_\epsilon^{-m}(\mathcal{T}), k - m),
$$

so that the functions $\hat{f} : \mathbb{Z} \rightarrow C(\Omega_\epsilon)$ given by $\hat{f}(k)(\mathcal{T}) = f(\mathcal{T}, k)$ are elements of

$$
C(\Omega_\epsilon) \times_{\hat{\varphi}_\epsilon} \mathbb{Z}
$$

with involution now given by $\hat{f}^* (k) = \hat{\varphi}_\epsilon^k (\hat{f}(\bar{k}))$. Here the action $\varphi_\epsilon$ on $C(\Omega_\epsilon)$ is the pull back of $\hat{\varphi}_\epsilon$, i.e., $\varphi_\epsilon(f) = \hat{f} \circ \hat{\varphi}_\epsilon^{-1}$. Bellissard’s work contains also an approach to this algebra which is physically motivated. The single tiling is then understood as a quasicrystal and the above algebra is related to the algebra of observables. The $K$-groups of $C(\Omega_\epsilon) \times_{\hat{\varphi}_\epsilon} \mathbb{Z}$ have
ever, a similar result as in the purely 1-dimensional case, namely that the relative frequencies of 2-patterns are just given by products of relative frequencies of 1-patterns. How-

That the quantum space in the 2-dimensional case is a cartesian product implies that relative frequencies it does not belong to the positive cone, but it is shown e.g. in [19] that a sequence exists for any $C^*$-algebra above, but it is argued that, if $E_{\varphi_\epsilon}$ acts transitively on $\Omega_\epsilon$, this consideration does not lead, at least not directly, to the determination of the positive cone, namely $\|p - q\| < 1$ implies $[p] = [q]$. This explains the stability of the gap labelling, because the spectral gaps having lower energy $E$ may be labelled by $tr(\chi_H) = tr_*(\chi_H \in C^*(R_\Omega))$. Hereby $tr_*: K_0(C^*(R_\Omega)) \to \mathbb{R}$ is the tracial state induced by the normalized trace on $C^*(R_\Omega)$. The latter restricts to a $\hat{\varphi}_\epsilon$-invariant trace on $C(\Omega_\epsilon)$, i.e. it is determined by an invariant measure on $\Omega_\epsilon$. Therefore the elements of $tr_*(K_0(C^*(R_\Omega))) \cap [0, 1]$ are given by integer linear combinations of relative frequencies of patterns (words) in the tiling. In certain cases (see below) these values already determine the ordered $K_0$-group completely. One expects that, if the potential term in $H$ depends upon the structure (e.g. the prototiles) of the tiling, most of the values of $tr_*$ between 0 and 1 do actually correspond to labels of spectral gaps of $H$.

That the quantum space in the 2-dimensional case is a cartesian product implies that relative frequencies of 2-patterns are just given by products of relative frequencies of 1-patterns. However, a similar result as in the purely 1-dimensional case, namely that the relative frequencies
of patterns determine the gap labelling or even the ordered \(K_0\)-group, is a priori not clear. The groupoid being \((\Omega_1 \times \mathbb{Z}) \times (\Omega_2 \times \mathbb{Z})\) its \(C^*\)-algebra is now a crossed product with \(\mathbb{Z} \oplus \mathbb{Z}\),

\[
C(\Omega) \times _{\varphi_1 \times \varphi_2} \mathbb{Z} \oplus \mathbb{Z},
\]

where \(\varphi_1\) and \(\varphi_2\) are the pull back action of the first resp. second \(\mathbb{Z}\). Here \(\hat{\varphi}_\epsilon\) acts trivially on \(\Omega_{\epsilon+1}\). We may as well express \((\ref{eq:1})\) as

\[
(C(\Omega) \times _{\varphi_1} \mathbb{Z}) \times _{\varphi_2} \mathbb{Z},
\]

\(\varphi_2(b)(m) = \varphi_2(b(m))\) for \(b \in C(\Omega) \times _{\varphi_1} \mathbb{Z}\) and therefore apply the Pimsner Voiculescu exact sequence twice to obtain the \(K\)-groups.

**Theorem 2** Let \(\Omega_\epsilon\), \(\epsilon = 1, 2\), be a totally disconnected compact metrizable spaces and \(\varphi_\epsilon\) topologically transitive homeomorphisms on \(\Omega_\epsilon\). Then, with \(\Omega = \Omega_1 \times \Omega_2\),

\[
K_0((C(\Omega) \times _{\varphi_1} \mathbb{Z}) \times _{\varphi_2} \mathbb{Z}) \equiv C(\Omega_1, \mathbb{Z})/E_{\varphi_1} \otimes C(\Omega_2, \mathbb{Z})/E_{\varphi_2} \oplus \mathbb{Z},
\]

and, if \(C(\Omega_1, \mathbb{Z})/E_{\varphi_1}\) is free,

\[
K_1((C(\Omega) \times _{\varphi_1} \mathbb{Z}) \times _{\varphi_2} \mathbb{Z}) \equiv C(\Omega_1, \mathbb{Z})/E_{\varphi_1} \oplus C(\Omega_2, \mathbb{Z})/E_{\varphi_2}.
\]

**Proof:** We first apply the Pimsner Voiculescu exact sequence to \(A = C(\Omega)\) and \(\alpha = \varphi_1\). As \(\Omega\) is totally disconnected compact and metrizable, one obtains, analogous to the purely 1-dimensional case, \(K_0(C(\Omega)) \equiv C(\Omega, \mathbb{Z})\) and \(K_1(C(\Omega)) = \{0\}\). Hence \((\ref{eq:2})\) yields an exact sequence

\[
0 \rightarrow K_1(C(\Omega) \times _{\varphi_1} \mathbb{Z})) \rightarrow C(\Omega, \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0
\]

and under the above identifications \(\varphi_{\epsilon*}\) becomes the pull back of \(\hat{\varphi}_\epsilon\). Consequently

\[
K_0(C(\Omega) \times _{\varphi_1} \mathbb{Z}) \cong C(\Omega, \mathbb{Z})/im(id - \varphi_{1*}) \cong C(\Omega_1, \mathbb{Z})/E_{\varphi_1} \otimes C(\Omega_2, \mathbb{Z})
\]

\[
K_1(C(\Omega) \times _{\varphi_1} \mathbb{Z}) \cong ker(id - \varphi_{1*}) = C(\Omega_2, \mathbb{Z}).
\]

Hereby use has been made of \(C(\Omega) \equiv C(\Omega_1) \otimes C(\Omega_2)\) \((C(\Omega)\) is an AF-algebra). Moreover \(\hat{\varphi}_1\) being topologically transitive implies that every \(\hat{\varphi}_1\)-invariant function on \(\Omega_1\) is constant which leads to \((\ref{eq:3})\). Note that \((i = 0, 1) K_i(C(\Omega) \times _{\varphi_1} \mathbb{Z}) \cong K_i(C(\Omega_1) \times _{\varphi_1} \mathbb{Z}) \otimes C(\Omega_2, \mathbb{Z})\). Application of the Pimsner Voiculescu sequence to \(A = C(\Omega) \times _{\varphi_1} \mathbb{Z}\) and \(\alpha = \varphi_2\) now gives

\[
\begin{array}{c}
\uparrow \\
C(\Omega_1, \mathbb{Z})/E_{\varphi_1} \otimes C(\Omega_2, \mathbb{Z}) \xrightarrow{id-\varphi_{2*}} C(\Omega_1, \mathbb{Z})/E_{\varphi_1} \otimes C(\Omega_2, \mathbb{Z}) \xrightarrow{i*} K_0(A \times _{\varphi_2} \mathbb{Z}) \\
K_1(A \times _{\varphi_2} \mathbb{Z}) \xleftarrow{i*} C(\Omega_2, \mathbb{Z}) \xrightarrow{id-\varphi_{2*}} C(\Omega_2, \mathbb{Z})
\end{array}
\]

The kernel of \(id - \varphi_{2*}\) as an endomorphism on \(C(\Omega_2, \mathbb{Z})\) is \(\mathbb{Z}\), a free abelian group, so that the exact sequence of abelian groups

\[
C(\Omega_1, \mathbb{Z})/E_{\varphi_1} \otimes C(\Omega_2, \mathbb{Z}) \xrightarrow{id-\varphi_{2*}} C(\Omega_1, \mathbb{Z})/E_{\varphi_1} \otimes C(\Omega_2, \mathbb{Z}) \xrightarrow{i*} K_0(A \times _{\varphi_2} \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow 0
\]

splits thus yielding \((\ref{eq:3})\). By the same reasoning one obtains from

\[
C(\Omega_2, \mathbb{Z}) \xrightarrow{id-\varphi_{2*}} C(\Omega_2, \mathbb{Z}) \xrightarrow{i*} K_1(A \times _{\varphi_2} \mathbb{Z}) \rightarrow C(\Omega_1, \mathbb{Z})/E_{\varphi_1} \rightarrow 0
\]
the result (88), if \( C(\Omega_1, \mathbb{Z})/E_{\varphi_1} \) is free (to guarantee the split property).

It is not known to us what the positive cone \( K^+_{00}(C(\Omega) \times_{\varphi_1} \mathbb{Z} \times_{\varphi_2} \mathbb{Z}) \) is. However, for application to the 2-dimensional gap labelling, refering to selfadjoint elements \( h : R_\Omega \rightarrow \mathbb{C} \) of \( C^*(R_\Omega) \cong (C(\Omega) \times_{\varphi_1} \mathbb{Z} \times_{\varphi_2} \mathbb{Z}) \) which play the role of Hamiltonians in representations on \( \ell^2(\mathbb{Z}^2) \), only the values of the tracial states on the \( K_0 \)-group need to be known. These may be obtained by application of theorem 3 of [27]: It states that, given a trace \( tr \) on \( A \times_{\alpha} \mathbb{Z} \), the map \( \Delta^0_{tr} : ker(id - \alpha) \subset K_1(A) \rightarrow \mathbb{R}/tr_*(K_0(A)) \) defined by

\[
\Delta^0_{tr}(u) = \frac{1}{2\pi i} \int_0^1 tr(\hat{t}(t)\xi^{-1}(t)) \, dt / tr_*(K_0(A)),
\]

\( \xi : [0, 1] \rightarrow GL(A) \) being a piecewise continuous path from 1 to \( u \alpha(u^{-1}) \), is a well defined group homomorphism, and moreover that the sequence

\[
0 \rightarrow tr_*(K_0(A)) \rightarrow tr_*(K_0(A \times_{\alpha} \mathbb{Z})) \xrightarrow{q} \Delta^0_{tr}(ker(id - \alpha)) \rightarrow 0,
\]

\( q : \mathbb{R} \rightarrow \mathbb{R}/tr_*(K_0(A)) \) being the canonical projection, is exact. Hereby the restriction of \( tr \) to \( A \) is also denoted by \( tr \). This enables us to show

**Theorem 3** Let \( \Omega_\epsilon \) and \( \varphi_\epsilon \) be as in the theorem [3] and \( tr \) be a trace on \( C(\Omega) \times_{\varphi_1} \mathbb{Z} \). Then

\[
tr_*(K_0((C(\Omega) \times_{\varphi_1} \mathbb{Z}) \times_{\varphi_2} \mathbb{Z})) = tr_*(C(\Omega, \mathbb{Z})).
\]

**Proof:** We apply the abovementioned theorem to \( A = C(\Omega) \times_{\varphi_1} \mathbb{Z} \) and \( \alpha = \varphi_2 \). \( ker(id - \varphi_2) \subset K_1(A) = C(\Omega_2, \mathbb{Z}) \) was already determined to be \( \mathbb{Z} \) and we need to identify a nontrivial element. Consider the constant function \( u : \mathbb{Z} \rightarrow C(\Omega) \) given by \( u(n) = \delta_{1n} \) (Kronecker’s \( \delta \)). It is the unitary in \( C(\Omega) \times_{\varphi_1} \mathbb{Z} \) conjugation by which yields the action of \( \varphi_1 \):

\[
u * f * u^{-1} = \varphi_1(f),
\]

\( f \in C(\Omega) \times_{\varphi_1} \mathbb{Z} \). Moreover \( u \notin GL(A)_0 \), because otherwise \( \varphi_1 \) would be homotopic to the identity. Since \( \varphi_2(u) = u \), \( [u] \) is indeed a nontrivial element of \( ker(id - \varphi_2) \subset K_1(A) \). Now \( u \varphi_2(u) = 1 \) implies \( \Delta^{\varphi_2}_{tr}([u]) = 0 \) so that

\[
tr_*(K_0((C(\Omega) \times_{\varphi_1} \mathbb{Z}) \times_{\varphi_2} \mathbb{Z})) = tr_*(K_0(C(\Omega) \times_{\varphi_1} \mathbb{Z})).
\]

The restriction of \( tr \) to \( C(\Omega) \times_{\varphi_1} \mathbb{Z} \) has to be invariant under \( \varphi_2 \). A second application of the above sequence with \( A = C(\Omega) \), \( \alpha = \varphi_1 \) directly implies \( tr_*(K_0(C(\Omega) \times_{\varphi_1} \mathbb{Z})) = tr_*(K_0(C(\Omega))) \), the restriction of \( tr \) to \( C(\Omega) \) now being \( \varphi_1 \times \varphi_2 \)-invariant.

Therefore (73) is determined by a \( \varphi_1 \times \varphi_2 \)-invariant probability measure on \( \Omega \). Moreover \( tr_*(K_0(C(\Omega))) = tr_*(C(\Omega_1, \mathbb{Z})) tr_*(C(\Omega_2, \mathbb{Z})) \), i.e. the invariant probability measure on \( \Omega \) is the product of the invariant measures on the single components (results of [23] for the 1-dimensional case show that it is unique). Like in the 1-dimensional case the values of the tracial state are given by integer linear combinations of relative frequencies of 2-patterns.

In [4], in case \( \Omega_\epsilon \) is given by substitution sequences, such an invariant measure is determined using special technics related to these substitutions. In the next section a different approach to their determination is proposed, which has the benefit of making the ordered \( K_0 \)-group itself more transparent.
2.2 Alternative description of the quantum space

In his treatment of the class of locally isomorphic Penrose tilings, i.e. the quantum space related to \((\mathbb{Z}^2, P^{(1)})\), A. Connes uses an alternative description which leads to another realization of that space as a quotient, see also \([13]\) for details. It is based on local inflation/deflation rules. By use of these rules to any Penrose tiling together with a chosen (starting-) point a sequence \(\{f_i\}_{i \geq 0}\) may be assigned, which has values in \(\{0, 1\}\) and satisfies the constraint \(f_i = 1 \Rightarrow f_{i+1} = 0\). Translation of that tiling, i.e. a change of the starting point, amounts to a change of the sequence yet only up to finitely many elements. Therefore the quantum space of Penrose tilings is as well given by the set of all 0,1-sequences with the above constraint modulo the equivalence relation which identifies sequences, if they differ only up to finitely many elements. The above sequences may be visualized as paths over the graph \(A_4/\mathbb{Z}_2\), or paths on the Bratteli diagram the inclusion graph of which is \(A_4\). The groupoid-\(C^*\)-algebra is in this case the \(AF\)-algebra which is defined by the Bratteli diagram. Ordered \(K_0\)-groups of this type of algebras are well known and relatively easy to compute (see below).

Comparing the approach of Connes with the statements made above, one is led to the question:

- Does this alternative way of describing the quantum space of tilings carry over to the general case?

A step towards an answer to that question is given in \([15]\), however only in the one dimensional context. The \(\Omega_c\) together with the homeomorphism \(\varphi_c\) furnishes a minimal topological dynamical system, and \(C(\Omega_c) \times_{\varphi_c} \mathbb{Z}\) is the algebra naturally assigned to it. Herman et al. found a way to formulate such a dynamical system on a Bratteli diagram, the \(\mathbb{Z}\)-action being encoded in an order on that diagram, thereby specifying one point in \(\Omega_c\) (more precisely one has to consider equivalence classes of pointed topological essential minimal dynamical systems on totally disconnected compact spaces and correspondingly about equivalence classes of essentially simple ordered Bratteli diagrams). \(C(\Omega_c) \times_{\varphi_c} \mathbb{Z}\) may roughly be understood as being generated by the \(AF\)-algebra defined by the Bratteli diagram and one additional element. Moreover it is shown that the ordered \(K_0\)-group of the crossed product equals the ordered \(K_0\)-group of that \(AF\)-algebra.

However, the construction of an ordered Bratteli diagram corresponding to a pointed topological dynamical system as it is described by Herman et al. is quite involved. On the other hand, given an inflation \(\sigma\) for some representative \(T \in \Psi_c\), it naturally defines a Bratteli diagram, and moreover it is possible - very similar to the Penrose-case - to map at least the whole class of \(T\), i.e. the \(\mathbb{Z}\)-orbit through \(T\) onto that diagram. This encourages us to conjecture that the the ordered \(K_0\)-groups of \(C(\Omega_c) \times_{\varphi_c} \mathbb{Z}\) and the \(AF\)-algebra being defined now by means of an inflation are equal.

Let us first clarify the notions. The inflation is a positive map and may therefore, like the matrix \(C\), be interpreted as a connectivity matrix of some graph \(\Sigma\) (which will in general not be a Coxeter graph). Again the vertices of \(\Sigma\) may be identified with the simple roots, the \(i\)'th and the \(j\)'th vertex yet being now joint by \(\sigma_{ji}\) oriented edges, which are to be distinguished. By saying that an edge \(f\) is oriented, one means that it has a source and a range denoted by the maps \(s\) and \(r\), namely in the above example \(s(f) = i\) and \(r(f) = j\). A path of length \(k\) over \(\Sigma\) is given by a sequence of \(k\) oriented edges \(f_1^{l_1}, f_2^{l_2}, \ldots, f_k^{l_k}\) such that \(r(f_n) = s(f_{n+1}^{l_{n+1}})\). We use extra labels \(l_k\) in order to distinguish between the \(\sigma_{ji}\) edges having source \(i\) and range \(j\). The set of all infinite paths over \(\Sigma\) shall be denoted by \(\mathcal{P}(\Sigma)\). We did require that
the inflation decomposes into two parts \( \sigma_1, \sigma_2 \) (its restrictions to \( \Lambda_1, \Lambda_2 \)), which then have to be irreducible. Hence \( \Sigma \) has two connected components \( \Sigma_1, \Sigma_2 \). As long as we discuss the 1-dimensional case we consider one component only. To better visualize an infinite path over a connected component \( \Sigma_e \) it may be unfolded, the set of all infinite paths yielding a Bratteli diagram.

Bratteli invented these diagrams in order to describe \( AF \)-algebras (approximately finite algebras). An \( AF \)-algebra is the \( C^* \)-hull of the direct limit of a directed system

\[
A_0 \xrightarrow{h_2} A_1 \xrightarrow{h_1} \cdots \tag{76}
\]

of finite dimensional \( C^* \)-algebras \( A_n \) and \(*\)-homomorphisms \( h_n \); we denote it by \( A_\infty \). For our purposes it suffices to consider unital embeddings only, i.e. \( h_n \) embeds \( A_n \) into \( A_{n+1} \) preserving the unit. A Bratteli diagram may be understood as a \( K \)-theoretic description of that system together with an order unit in terms of an infinite weighted graph. It determines the \( AF \)-algebra up to \(*\)-isomorphisms. The \( K_0 \)-group \( K_0(A_\infty) \) of \( A_\infty \) is the direct limit of the directed system of the \( K_0 \)-groups of \( A_n \), the positive homomorphisms \( h_{n*} \) being induced by the above embeddings \( h_n \). Every finite dimensional \( C^* \)-algebra is semisimple, i.e. its elements may be identified with block-diagonal matrices

\[
A_n \cong \bigoplus_{k=1}^{k_n} M_{m_k}(\mathbb{C}), \tag{77}
\]

\( M_{m_k}(\mathbb{C}) \) denoting the blocks consisting of \( m_k \times m_k \)-matrices. Then \( K_0(A_n) = \mathbb{Z}^{k_n} \) the generators standing for the distinct blocks or minimal central projections of \( A_n \), and \( h_{n*} \in \text{Hom}(\mathbb{Z}^{k_n}, \mathbb{Z}^{k_{n+1}}) \) is the \( k_{n+1} \times k_n \) matrix with entries in \( \mathbb{Z}^+ \) its \( ji \)-coefficient telling how often the block \( M_{m_j} \) of \( A_n \) is embedded into the block \( M_{m_j} \) of \( A_{n+1} \) by \( h_n \). This is graphically encoded in a Bratteli diagram as follows: The set of its vertices \( V \) is grouped into floors \( V_n \), \( V = \bigcup_{n \geq 1} V_n \), each vertex of \( V_n \) representing a generator of \( K_0(A_n) \). Its (oriented) edges are given through the homomorphisms \( h_{n*} \): The \( i \)'th vertex of \( V_n \) is joint with the \( j \)'th vertex of \( V_{n+1} \) by \((h_{n*})_{ji}\) edges all having their source at that \( i \)'th vertex of \( V_n \) and their range at that \( j \)'th vertex of \( V_{n+1} \). In the context of unital embeddings all information about the isomorphism class of the directed system is encoded in the graph except of the dimensions of the blocks of the first algebra \( A_0 \). These may be encoded by weights put on the vertices of \( V_0 \). It is convenient to enlarge the directed system (76) by adding \( A_{-1} = \mathbb{C} \subset A_0 \) to the left, for then it is understood that the dimension of the block of the first algebra (now being \( A_{-1} \)) is 1. In terms of the Bratteli diagrams one adds a floor \( V_{-1} \) having one vertex representing \( K_0(\mathbb{C}) = \mathbb{Z} \) and \( m_i \) edges from that vertex to the \( i \)'th vertex of \( V_0 \), if the \( i \)'th block of \( A_0 \) is of size \( m_i \times m_i \). The weight at the vertex of \( V_{-1} \) may then be left away, since it is always 1.

The set of paths over \( \Sigma_e \) yields such a Bratteli diagram: \( V_n = \Sigma_e^{(n)} \), for \( n \geq 0 \), and the edges connecting the \( i \)'th vertex of \( V_n \) with the \( j \)'th vertex of \( V_{n+1} \) are given by the edges of \( \Sigma_e^{(1)} \) having their source resp. range at \( i \) resp. \( j \). Adding the floor \( V_{-1} \) amounts to indicating the starting vertex of a single path over \( \Sigma_e \), namely by the edge that has source at the single vertex of \( V_{-1} \) and range equal the starting vertex. The directed system given now by the set of paths over \( \Sigma_e \) is obtained by putting for all \( n \geq 0 \)

\[
K_0(A_n) = \Lambda_e = \mathbb{Z}^r, \tag{78}
\]

\[
h_{n*} = \sigma_e, \tag{79}
\]

\( K_0(A_{-1}) = \mathbb{Z} \) and \( h_{-1*} = (11 \cdots 1)^T \), an \( r \times 1 \)-matrix, \( r = \text{dim}(\Lambda_e) \). Therefore

\[
K_0(A_\infty) \cong \Lambda_e, \tag{80}
\]

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following from $\sigma_\epsilon$ being an automorphism. Note that this group is free. In terms of algebras, this means that $A_\infty$ is (the $C^*$-closure of) the path algebra over the graph $\Sigma_\epsilon$; we also denote it by $AF(\Sigma_\epsilon)$. If $\sigma_\epsilon$ is symmetric, it is approximated by the tower of the inclusion $\Phi^r = A_0 \subset A_1$ with inclusion matrix $\sigma_\epsilon$. The order relation on the $K_0$-group is calculated as follows: We may use the elements of $K_0(A_0)$ as representatives for elements of the direct limit so that the positive cone is given by

$$K_0^+(A_\infty) = \bigcup_{n \geq 0} \sigma_\epsilon^{-n}(K_0^+(A_0)).$$

Hence $x \in K_0^+(A_\infty)$ iff there is an $n \geq 0$ such that $\sigma_\epsilon^n(x) \in K_0^+(A_0) = (\mathbb{Z}^r)^\epsilon$. Clearly if the latter holds for $n$ it holds as well for $n+1$ as $\sigma_\epsilon$ is positive. For that reason it is the projection of $x$ along the eigenspace to largest eigenvalue of the inflation which is crucial, namely

$$K_0^+(AF(\Sigma_\epsilon)) = \{x \in \Lambda(\nu_{\epsilon}, x) > 0\} \cup \{0\},$$

where $\nu_{\epsilon}$ is the Perron-Frobenius vector of the inflation. In particular, there is only one state on the ordered $K_0$-group $[12]$, which therefore has to be $tr_{\epsilon}$, the homomorphism which is induced by the unique normalized trace $tr : A_\infty \to \mathbb{C}$. It is, up to normalization, given by the orthogonal projection onto $P_\epsilon$. If $P_\perp \cap \Lambda_\epsilon = \emptyset$, $K_0(A_\infty)$ is totally ordered and any element of it is uniquely determined by its value under $tr_{\epsilon}$. As we already mentioned, the $AF$-algebra is determined by its ordered $K_0$-group together with an order unit. The latter is necessary for the definition of states (which involves a normalization) but it is not invariant under a change of a realization of $\Psi$. Note that not only the ordered $K_0$-group is independent of a choice for the inflation, as it should be, but even the above described $AF$-algebra as $(1, 1, \cdots, 1) \in \mathbb{Z}^r$ may in all cases be taken to be the order unit.

Having chosen a single tiling $T = T_{\gamma \rightarrow -a_\epsilon}$ which allows for an inflation $\sigma_\epsilon$ we can map the orbit through $T$ onto the Bratteli diagram, which has been described above. As $T$ appears as a twosided sequence of letters, its selfsimilarity may be interpreted by saying that its letters are grouped into words, each one corresponding to the deflation (as the inverse process of inflation) of some letter of the twosided sequence $T_{\sigma_\epsilon(\gamma \rightarrow -a_\epsilon)}$. More precisely consider, like in the proof of proposition $[4]$, points $x, x' \in S_{\sigma_\epsilon(\gamma \rightarrow -a_\epsilon)} \cap \Lambda_\epsilon$ such that $x' - x = \sigma_\epsilon(\alpha_1) = \alpha_1 + \cdots + \alpha_k$ for some $\alpha_i \in \mathcal{L}_\epsilon$. Through $(81)$ the order of the letters in the word $i_1 \cdots i_k$ is determined, and $(\sigma_\epsilon)^l_i$ equals the number of times the letter $l$ appears in that word. Since the word arose by the process of deflation from $i$, we call it an $i$-deflation. Strictly speaking the terminus letter-deflation is a little bit imprecise, as the order of the letters in the word described above does not only depend on the letter but also on the point $\pi(x)$. In case this order is independent of that point, the above letter-deflation furnishes what is called a substitution, the tilings being substitution sequences, see $[23], [4]$. In fact most of what follows may be applied directly to substitutions.

To any letter of $T$ step by step a path over $\Sigma_\epsilon$ shall be assigned as follows:

**Step:** Let $i_0$ be the letter to start with. It belongs to a unique letter-deflation say of the letter $i_1$. If the $i_0$ we have picked out is the $l_1$’s letter $i_0$ in that $i_1$-deflation, then we encode this by the $l_1$’th edge $f_1^{(l_1)}$ having $s(f_1^{(l_1)}) = i_0$ and $r(f_1^{(l_1)}) = i_1$.

This step may be repeated, if one first uses $(17)$ to identify $T_{\sigma(\gamma \rightarrow -a_\epsilon)}$ with $T_{\gamma \rightarrow -a_\epsilon}$, hereby keeping track of the letter $i_1$ the deflation of which yielded $i_0$. This $i_1$, now as a letter of $T_{\gamma \rightarrow -a_\epsilon}$, shall be the letter to start with the next step leading to an edge $f_2^{(l_2)}$ with $s(f_2^{(l_2)}) = r(f_1^{(l_1)})$. $f_1^{(l_1)}, f_2^{(l_2)}$ defines a path of length 2 on $\Sigma_\epsilon$. The infinite path assigned to $i_0$ is obtained by infinite repetition of that step.
There are exactly two paths obtained by this procedure which are constant, namely the one assigned to the letter to the right as well as the one to the left of $\mathbf{0}$. We denote them by $\xi_+$ and $\xi_-$ respectively. Any other path obtained from a letter of $T$ will up to finitely many elements agree with $\xi_+$ or $\xi_-$ depending on whether it has been chosen right or left of $\mathbf{0}$. Moreover, different letters are mapped onto different paths, and any path ultimately agreeing with $\xi_+$ or $\xi_-$ can be obtained from a letter of $T$ (a preimage is easily found by inversion of the above process). To summarize, since the letters of $T$ are in one to one correspondence of the elements of the orbit through $T$, the above procedure defines a mapping $\phi$ from that orbit to the subspace of paths which ultimately agree with $\xi_+$ or $\xi_-$. To make contact with the results of [19] the following questions have to be clarified (the natural topology of $P(\Sigma_e)$, turning it into a totally disconnected compact space, is given below).

1. Is it possible to extend this mapping to the closure of that orbit, i.e. to all representatives of $\Psi_e$?

2. Let $R(\Sigma_e) = \phi(R\Omega_e)$. Does $\phi$ induce a homeomorphism between the quotients $\Omega_e/R\Omega_e$ and $P(\Sigma_e)/R(\Sigma_e)$?

These questions remain open in this work, but let us comment on them:

1) The first problem may possibly be attacked as follows: For $T' \in \Omega_e$ consider an increasing net of finite parts $\eta_1 \subset \eta_2 \subset \cdots \subset T'$ representing words which all contain the letter right of $\mathbf{0}$, such that $T'$ is approximated by these parts, i.e.

$$T' = \bigcup_{n=1}^{\infty} \eta_n.$$  \hfill (83)

As $T'$ is locally isomorphic to $T$, for each $n$ there is a $\lambda_n \in \Lambda_e$, such that

$$\eta_n \subset T - \pi(\lambda_n),$$  \hfill (84)

and we assign to the pair $(\eta_n, \lambda_n)$ the path which is given by the letter to the right of $\pi(\lambda_n)$ in $T$. Although that specific letter will be the same for each $\lambda_n$ satisfying (84), the letter-deflation to which it belongs may depend on $\lambda_n$. If for each $n$ there is a finite number $b_n$ such that the first $n$ elements of the path determined by $(\eta_n, \lambda_n)$ are independent of $\lambda_n$ as long as $\eta_n$ contains at least $b_n$ letters to the right and to the left of $\mathbf{0}$, then the path assigned to $T'$ may be taken to be the limit of the above paths in the topology a base of which is given by following the open-closed sets: Each set is defined by a path of finite length $\xi$ over $\Sigma_e$, namely it contains all (infinite) paths the beginning of which coincide with $\xi$. This way $\phi$ becomes a continous mapping. By construction of the path the $b_n$’s exist if $b_1$ does ($b_1$ has to grow exponentially with $n$). For substitution sequences the existence of such a $b_1$ is guaranteed if they are recognizable, for the definition of recognizability see [23]. For $A_4$, $D_5$, $A_7$ and $E_6$-projections the existence of a $b_1$ follows from $\partial(\pi(\mathbf{0}) \cap \{ E + a \}) \subset \pi(\mathbf{0})$.

2) Provided the first problem is solved, surjectivity of $\phi$ follows from the compactness of $\Omega_e$: Any path $\{f_i^j\}_{i \geq 1}$ over $\Sigma_e$ may be approximated by a sequence of paths $\xi_n$, where the first $n$ elements of $\xi_n$ coincide with $\{f_i^j\}_{1 \leq i \leq n}$ whereas the rest coincides ultimately with $\xi_+$. Let $T - \pi(\lambda_n)$ be a sequence of tilings such that $\phi(T - \pi(\lambda_n)) = \xi_n$. As $\Omega_e$ is compact, this sequence has a convergent subsequence, the limit of which is a preimage of $\{f_i^j\}_{i \geq 1}$.

Despite the abovementioned unsolved questions it seems to us quite plausible that the ordered $K_0$-group of $C(\Omega_e)\times_{\varphi} \mathbb{Z}$ coincides - similar to the case in [19] - with $\{8, 82\}$, the one of $AF(\Sigma_e)$.
As in [19] we expect $R(\Sigma_c)$ to be generated by $R_0(\Sigma_c) = \{(\xi, \xi') : \exists n \in \mathbb{N} : \xi_i = \xi'_i\}$ together with the element $(\xi_+, \xi_-)$, so that $AF(\Sigma_c) \cong C^*(R_0(\Sigma_c))$ and $C(\Omega_e) \times \varphi_e \mathbb{Z} \cong C^*(R(\Sigma_c))$. This is also supported by the observation that the (1-dimensional) aperiodic tiling contains already all elements needed for the construction of the ordered $K_0$-group. In fact the $K_0$-group corresponds to the lattice $\Lambda_e$ and the order relation may be expressed with the help of the orthogonal projection onto $P_e$, the latter being the subspace along which the strip is constructed.

Our final remark concerns the question after a connection between the crossed product of the 2-dimensional system (65) and $AF(\Sigma_1 \times \Sigma_2) \cong AF(\Sigma_1) \otimes AF(\Sigma_2)$, the path algebra over $\Sigma_1 \times \Sigma_2$. The latter $AF$ algebra is expected to be related to the alternative description of the 2-dimensional quantum space (57) analogously to the 1-dimensional case. The directed system given by the Bratteli diagram of paths over $\Sigma_1 \times \Sigma_2$ is, for $n \geq 0$

$$K_0(A_n) = \Lambda_1 \otimes \Lambda_2 \cong \mathbb{Z}^{r_1 r_2}, \quad h_{n*} = \sigma_1 \otimes \sigma_2,$$

and

$$K_0(A_{-1}) = \mathbb{Z} \quad \text{and} \quad h_{-1*} = (11 \cdots 1)^T,$$  

an $r_1 r_2 \times 1$-matrix, $r_e = \text{dim}(\Lambda_e)$. Hence

$$K_0(AF(\Sigma_1 \times \Sigma_2)) \cong \Lambda_1 \otimes \Lambda_2,$$

and

$$K_0'(AF(\Sigma_1 \times \Sigma_2)) = \{x_1 \otimes x_2 \in \Lambda_1 \otimes \Lambda_2 | (\vec{v}_1, x_1)(\vec{v}_2, x_2) > 0\} \cup \{0\}. \quad (88)$$

Again $tr_* : K_0(AF(\Sigma_1 \times \Sigma_2)) \to \mathbb{R}$ is given up to normalization by the scalar product of $x$ with the Perron Frobenius vector $\vec{v}_1 \otimes \vec{v}_2$ of $\sigma_1 \otimes \sigma_2$. Hence, although the $K_0$-groups of the the double crossed product (65) related to the tiling on the one hand and the above $AF$-algebra on the other do not coincide, their ranges of $tr_*$ would do so (compare with (75) and the sequel), provided the the above discussed equality $K_0(C(\Omega_e) \times \varphi_e \mathbb{Z}) = K_0(AF(\Sigma_c))$ as ordered groups holds.

Normalizing $\vec{u}_e$ by $\vec{v} = \vec{p}(\tau) \quad (\tau = 2\cos^2 \frac{x}{b})$, it follows from (25) that for $A_{2n}$ (for which $(\vec{v}_1, \Lambda_1) = (\vec{v}_2, \Lambda_2)$) the elements of $(\vec{v}_1, \Lambda_1)$ form a subring of $\mathbb{R}$, i.e. $(\vec{v}_1, z)(\vec{v}_1, z') \in (\vec{v}_1, \Lambda_1)$. In case of $A_{2n+1}$ this holds only for the even polynomials, namely only $(\vec{v}_1, \Lambda_1)$ forms a subring of $\mathbb{R}$. This ring structure depends upon the normalization, and, if one has in mind that the elements $(\vec{v}, \alpha, \alpha \in \Lambda_e)$, yield relative frequencies of prototiles, one might prefer the normalization to be in a way that the sum of the entries of $\vec{v}_e$ equals 1. For $A_{2n}$ this amounts to setting $\vec{v} = (2-\tau)\vec{p}(\tau)$ which leads to the same subring of $\mathbb{R}$. We conclude that, concerning the values of $tr_*$, Connes’ approach to the Penrose tilings and this approach to 2-dimensional $A_4$-projections lead to the same result. For $A_{2n+1}$ however the latter normalization yields $\vec{v}_2k = \frac{4r^2}{27} \vec{p}(\tau)$ and $\vec{v}_{2k+1} = \frac{4r^2}{9} \vec{p}(\tau)$ both of which do not generate $(\vec{v}_1, \Lambda_1)$ resp. $(\vec{v}_2, \Lambda_2)$ as a ring (2 is not invertible).

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**Figure 1:** The above figure is obtained by the strip method as follows: The incomplete cartwheel tiling corresponds to a $\mathbb{Z}^5$-projection onto $P_{\omega'}$ with $\mathcal{L} = \{\epsilon_i\}_{1 \leq i \leq 5}$ taking $a = \vec{0}$ (it is singular). Two of the above parallels are obtained by projection out of the lattice $< \mathcal{L}' > = -\frac{1}{2} \rho$ onto the same plane $P_{\omega'}$, where $< \mathcal{L}' >$ is the sublattice of $\mathbb{Z}^5$, which is generated by $\mathcal{L}' = \{C^2 \delta^{-1}(\epsilon_i - \epsilon_{i+1})\}_{1 \leq i \leq 4}$, $C$ is the connectivity matrix of $A_4$ and $\rho = C^2 \delta^{-1}(\epsilon_5 - \epsilon_1)$. Again (the singular choice) $a = \vec{0}$ is taken. Now the Ammann-quasicrystal is up to its asymmetric part completed by rotating around $\frac{2\pi}{5}$.

**Figure 2:** Selfsimilarity of the $\mathbb{Z}^7$-projection.