The Weyl–Cartan Gauss–Bonnet gravity

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Abstract

In this paper, we consider the generalized Gauss–Bonnet action in four-dimensional Weyl–Cartan spacetime. In this spacetime, the presence of a torsion tensor and Weyl vector implies that the generalized Gauss–Bonnet action will not be a total derivative in four-dimensional spacetime. It will be shown that the higher than two time derivatives can be removed from the action by choosing a suitable set of parameters. In the special case where only the trace part of the torsion remains, the model reduces to general relativity plus two vector fields, one of which is massless and the other is massive. We will then obtain the healthy region of the five-dimensional parameter space of the theory in some special cases.

Keywords: Weyl–Cartan theory, Gauss–Bonnet action, Proca action, Ostrogradski instability

1. Introduction

In 1918 Weyl proposed a new geometry to unify electromagnetism with Einstein’s general relativity \cite{1}. In Riemannian geometry one has an \textit{a priori} condition that the length of a vector should not change during the parallel transportation. In the Weyl geometry, this assumption is dropped and so a parallel transported vector has different length and direction with respect to the original vector. The gravitational theory which is built on the Weyl geometry is known as the Einstein–Weyl gravity \cite{1}. In Einstein–Weyl gravity the connection is no longer metric compatible, so, the covariant derivative of the metric is not zero. Instead one has the relation

\[ \tilde{\nabla}_\mu g_{\nu\rho} = Q_{\mu\nu\rho}, \]  

(1)

where the tensor $Q_{\mu\nu\rho}$ is symmetric with respect to its last two indices. Weyl proposed the special case $Q_{\mu\nu\rho} \propto w_\mu g_{\nu\rho}$ for his theory where $w_\mu$ is the Weyl vector. One of the important

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consequences of this geometry is that the unit vector changes through parallel transportation. Suppose that the length of an arbitrary vector field $A^\mu$ is $l$. During the parallel transportation, the variation of the length of $A^\mu$ can be written in terms of the Weyl vector as

$$dl = lw_\mu dx^\mu.$$  \hspace{1cm} (2)

For a closed curve, the length of the vector $A^\mu$ changes as

$$l \to l - \int_S lw_{\mu\nu} dS^\mu\nu,$$  \hspace{1cm} (3)

where $S$ is the area of the closed curve, $dS^\mu\nu$ is the infinitesimal element of the surface, and

$$w_{\mu\nu} = \partial_\nu w_\mu - \partial_\mu w_\nu,$$  \hspace{1cm} (4)

is called the Weyl’s length curvature which is the same as the electromagnetic field strength. This implies that one has the freedom to choose the unit length at each point, which is the Weyl gauge freedom [1]. A variety of works have been done in the Weyl geometry including the cosmology [2], relations to scalar–tensor [3] and teleparallel theories [4].

One can also restrict the form of Weyl vector to be a derivative of a scalar as $w_\mu = \partial_\mu \phi$ [5]. In this case the length curvature $W_{\mu\nu}$ vanishes and one can then define a fixed unit length at each point. We note that the unit length varies at different points. The resulting theory is known as the Weyl integrable theory [6].

Another generalization of Einstein gravity can be proposed by assuming the existence of an asymmetric connection on the spacetime manifold. The first attempt for this purpose was done by Eddington in 1921 in order to generalize Einstein’s general relativity to obtain some insight on microscopic physics [7]. The major attempt in this way was done by Cartan in 1922 where he defined the torsion tensor as the antisymmetric part of the general connection [8]. The theory based on this assumption is called the Einstein–Cartan theory. Cartan believed that the torsion tensor should be related in some way to the intrinsic angular momentum of the matter content of the universe. So, the torsion should vanish in the absence of matter [8]. However, this idea had been forgotten for a while. The concept of the asymmetric affine connection came back to the literature as a way of writing a unified field theory of general relativistic type [9]. The energy–momentum tensor of a spinning massive particle was known to be asymmetric [10, 11]. It was clear that these particles can not be a source of Einstein’s general relativity equation. In order to consider these particles in the context of general relativity, one should generalize Einstein’s equation to a theory with asymmetric energy–momentum tensor. The concept of torsion can then be used as a way to consider these types of particles [12]. Many works have been done in the context of theories with torsion, including teleparallel theories [13] and Weyl–Cartan theories [14, 15]. For a very good review on the subject of Cartan theory, see [16].

There is another way to generalize Einstein’s general relativity, by replacing the Ricci scalar with a general function of it, known as $f(R)$ gravity theories [17, 18]. But one can add to the action, any scalar combination of the Riemann tensor and its contractions. One of the first attempts was done by Kretschmann [19] in 1917 by introducing the action of the form

$$S_K = \int d^4x \sqrt{-g} R_{\rho\mu\nu\sigma} R^{\rho\mu\nu\sigma},$$  \hspace{1cm} (5)

instead of the Einstein–Hilbert action. The above action has higher than second order time derivatives of the metric in its field equation and hence contains ghost instabilities. It turns out that the unique combination of two Riemann tensors and its contractions which leads to at most second order time derivatives in the field equation is of the form
\[ S_{GB} = \int d^4x \sqrt{-g} \left( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right), \] \hspace{1cm} (6)

which is called the Gauss–Bonnet term. In four spacetime dimensions, this term can be written as a total derivative, and can be dropped from the field equations [18]. This leads to the conclusion that in 4D, the Riemannian geometry together with the condition of stability has a unique candidate for the gravitational action, which is the Einstein–Hilbert action.

In non-Riemannian geometries such as the Weyl and Cartan geometries, the above conclusion is no longer true and the Gauss–Bonnet term will not become total derivative. In [20] the Gauss–Bonnet combination was obtained in the context of Weyl geometry, and it turns out that the remaining term in 4D is the Weyl vector kinetic term

\[ S_{GB} \propto \int d^4x \sqrt{-g} W_\mu W^{\mu}. \] \hspace{1cm} (7)

It is the aim of the present paper to generalize the above argument to the case of Weyl–Cartan spacetime. Similar to the Einstein–Weyl spacetime, in the Weyl–Cartan spacetime the Gauss–Bonnet term will not be a total derivative. The theory is not in general Ostrogradski stable. In order to have a stable theory one should constrain the parameter space of the model as we will do in the following sections.

The Weyl–Cartan model has also been considered in the context of Weitzenboch gravity in [14]. The authors have added the kinetic terms for the Weyl vector and the torsion tensor by hand, using the trace of torsion tensor. The main purpose of the present work is to see if adding higher order gravity terms in the Lagrangian can show a way to have kinetic terms automatically. For this case we will just add the Gauss–Bonnet term which we know does not produce any ghost for the metric perturbations. Adding non-metricity as well as torsion has been studied in [21]. The author has shown that at the second order gravity level, connection has no dynamics which is same as the results in [14]. Higher order curvature terms in the presence of torsion (without non-metricity) has been studied in [22] where the authors have shown that chiral fermionic matter fields can live in a Riemann–Cartan geometry. Particularly, the Lovelock theory in the presence of the torsion has been studied in the \( d \)-dimensions [23] and in the context of stringy fluxes [24].

Before going ahead let us briefly state our approach. In general, we are working in the context of effective field theory. However there are two (equivalent) approaches which have been extensively studied, for example, in the context of inflation. In the first approach [25] all the possible perturbations have been assumed at the level of the Lagrangian with arbitrary coefficients. The equivalent alternative approach is to assume the possible terms at the level of background [26]. Here we will do more or less the same approach as in [26]. We first assume we have just a metric and try to have all the consistent possible terms at the level of the Lagrangian which is Ricci scalar and Gauss–Bonnet (in general Lovelock terms). Then we ask what happens to this model if one assumes non-metricity and torsion. The other approach, which is more similar to [25], is to write all the possible terms constructed by non-metricity and torsion in addition to the metric itself. This second approach was done in [21, 22].

We will see in this paper that considering the Gauss–Bonnet action can produce automatically all the kinetic terms of [14]. It is worth mentioning that the theory [14] has a potential ghost, noting that in the Weitzenboch gravity the torsion has some relation to the Ricci scalar, and as a result the torsion kinetic term has more than second time derivatives. However, the present paper is free from the aforementioned instability due to the absence of the Weitzenboch condition. One should note that the torsion self-interaction term \( \nabla_\mu T V^\mu T \) with \( T = T^\mu T_\mu \) and \( T_\mu = T^{\mu\nu} \) in [14] can not be produced in the present context, because it is
fourth order in the torsion and second order in derivatives. In order to produce such a term one should consider the higher order Lovelock terms in the action.

The present theory may in general have some tachyon instabilities but the analysis is very complicated because of the appearance of the torsion tensor. In section 2 the generalized Gauss–Bonnet action in Weyl–Cartan spacetime is introduced and shown that the higher than two time derivatives are removed in the action. In section 3 we will consider a restricted form for the torsion tensor and obtain the healthy region of the parameter space in which tachyonic instabilities are removed.

2. The model

The Weyl geometry proposal induces a new vector which results in non-metricity of the connection i.e. \( \hat{\nabla}_\mu g_{\nu\sigma} = 2w_\mu g_{\nu\sigma}, \) where \( w_\mu \) is the Weyl vector.

\[
\hat{\nabla}_\mu g_{\nu\sigma} = 2w_\mu g_{\nu\sigma},
\]

where \( w_\mu \) is the Weyl vector. So the Weyl connection can be obtained as

\[
\Gamma^\lambda_{\mu\nu} = \{ \lambda_{\mu\nu} \} + Q^\lambda_{\mu\nu},
\]

where

\[
Q^\lambda_{\mu\nu} = g_{\mu\nu}w^\lambda - \delta^\lambda_\mu w_\nu - \delta^\lambda_\nu w_\mu,
\]

and \( \{ \lambda_{\mu\nu} \} \) is the Christoffel symbol. In addition one may generalize the above connection by adding the effects of the torsion into it as

\[
\Gamma^\lambda_{\mu\nu} = \{ \lambda_{\mu\nu} \} + Q^\lambda_{\mu\nu} + C^\lambda_{\mu\nu}.
\]

Note that the third term is named the contortion tensor and is defined as

\[
C^\lambda_{\mu\nu} = T^\lambda_{\mu\nu} - g^\lambda_{\nu\sigma}T^\sigma_{\beta\nu} - g^\lambda_{\mu\sigma}T^\sigma_{\beta\mu},
\]

where we have defined the torsion tensor \( T^\lambda_{\mu\nu} \) as

\[
T^\lambda_{\mu\nu} = \frac{1}{2}\left( \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right).
\]

It is easy to show that the additional torsion does not affect the non-metricity relation i.e. the relation (8) is still valid. By using the metric one can build \( C^\mu_{\lambda\nu} = g_{\nu\sigma}C^\sigma_{\mu\nu} \) which is antisymmetric with respect to its two first indices by keeping in mind that the torsion tensor is antisymmetric with respect to its lower indices in \( T^\lambda_{\mu\nu} \).

We define the curvature tensor as

\[
K^\lambda_{\mu\nu\sigma} = \partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\alpha_{\mu\nu} \Gamma^\lambda_{\alpha\sigma} - \Gamma^\alpha_{\mu\sigma} \Gamma^\lambda_{\alpha\nu}.
\]

One can decompose the curvature tensor into four parts as

\[
K^\lambda_{\mu\nu\sigma} = R^\lambda_{\mu\nu\sigma} + C^\lambda_{\mu\nu\sigma} + Q^\lambda_{\mu\nu\sigma} + I^\lambda_{\mu\nu\sigma},
\]

where the first term in the right hand side of the above relation is the Riemann curvature tensor defined by the Christoffel symbol and we have defined
where $\nabla_\mu$ is the covariant derivative with respect to the Christoffel symbol and $I_{\mu\nu\sigma}^\lambda$ represents interaction between non-metricity and torsion parts. It is possible to rewrite the purely non-metricity part \((17)\) as

$$\frac{1}{2}Q^{\lambda}_{\mu\nu\sigma} = -\delta^{\lambda}_{\mu}(\nabla_{[\nu}w_{\sigma]} - \delta^{\lambda}_{\nu}(\nabla_{[\sigma}w_{\mu]} - g_{\nu[\sigma}(\nabla_{\mu]}w^{\lambda} + \delta^{\lambda}_{\nu}w_{[\sigma)w_{\mu]}) + g_{\nu}[\sigma}w_{\mu]w^{\lambda}) \tag{19}$$

where $w^{2} = w_{\mu}w^{\mu}$ and the interaction part \((18)\) as

$$\frac{1}{2}I^{\lambda}_{\mu\nu\sigma} = -w^{\mu}C_{\alpha\mu[\sigma}g_{\nu]}^{\lambda} - w^{\nu}C_{\alpha\nu[\sigma}g_{\mu]}^{\lambda} - w^{\lambda}C_{\mu[\nu}g_{\sigma]} - w_{\mu}C_{\lambda[\nu}^{\lambda}g_{\sigma]} \tag{20}$$

Note that the curvature tensor $K^{\nu}_{\mu\sigma}$ is still antisymmetric wrt its last two indices.

In order to construct a higher order gravity model e.g. Gauss–Bonnet action, one should multiply the curvature tensor to itself. There are seven different ways to do this

$$K_{\lambda\mu\nu}K^{\lambda\mu\nu}, K_{\lambda\mu\nu}K^{\mu\lambda\sigma}, K_{\lambda\mu\nu}K^{\rho\nu\mu\lambda\sigma}, K_{\lambda\mu\nu}K^{\rho\sigma\lambda\mu}, K_{\lambda\mu\nu}K^{\rho\mu\sigma}, K_{\lambda\mu\nu}K^{\rho\sigma\lambda\nu}, K_{\lambda\mu\nu}K^{\rho\sigma\lambda\nu}. \tag{21}$$
One should note that in the case of vanishing Weyl and torsion, only the first three terms of equation (21) can be reduced to \( K_{\mu \nu \alpha \beta} K^{\lambda}_{\mu \nu \alpha \beta} \) which is present in the standard Gauss–Bonnet Lagrangian. Adding the rest of these combinations to the Lagrangian will produce higher than second order time derivatives of the metric which makes our theory unstable. So, we will ignore them in the following and only add the first three terms to the action.

Now consider the contractions of different parts of the curvature tensor. It is well-known that the Riemann tensor has only one independent contraction. The Weyl part of the above curvature tensor has two independent contractions

\[
\begin{align*}
Q_{\lambda \mu \nu} &= -4W_{\mu \nu}, \\
Q_{\mu \lambda \nu} &= -\nabla_{\mu}w_{\nu} + 3\nabla_{\nu}w_{\mu} + g_{\mu \nu}w_{\lambda}w^{\lambda} + 2w_{\mu}w_{\nu} - 2g_{\mu \nu}w^{2},
\end{align*}
\]

(22)

where we have defined

\[
W_{\mu \nu} = \nabla_{\mu}w_{\nu} - \nabla_{\nu}w_{\mu}.
\]

(23)

The contortion part of the curvature tensor has only one independent contraction

\[
\begin{align*}
C_{\lambda \mu \nu} &= 0, \\
C_{\mu \lambda \nu} &= \nabla_{\lambda}C_{\mu \nu} + \nabla_{\nu}C_{\mu \lambda},
\end{align*}
\]

(24)

where we have defined \( C_{\mu \nu} = C^{\alpha \mu \nu} \).

The interaction part also has one independent contraction which can be written as

\[
\begin{align*}
I_{\lambda \mu \nu} &= 0, \\
I_{\mu \lambda \nu} &= -w^{\alpha}(C_{\alpha \mu \nu} + C_{\nu \alpha \mu}).
\end{align*}
\]

(25)

For the Riemann curvature tensor, we have \( R_{\lambda \mu \nu} = 0 \) and \( R_{\mu \lambda \nu} = R_{\mu \nu} \), where \( R_{\mu \nu} \) is the standard Ricci tensor. For the contracted curvature tensor, the two independent contractions are

\[
K_{\mu \nu} \equiv K_{\lambda \mu \nu}, \quad K_{\alpha \mu} \equiv K_{\lambda \mu \alpha}.
\]

There are four independent combinations of them as follows

\[
K_{\mu \nu}K^{\mu \nu}, \quad K_{\mu \nu}K^{\nu \mu}, \quad K_{\mu \nu}K^{\nu \mu}, \quad K_{\mu \nu}K^{\nu \mu}.
\]

(26)

In the case of vanishing Weyl and torsion, the first two terms of equation (26) vanish, and the rest becomes identical to \( R_{\mu \nu}R^{\mu \nu} \).

There is only one independent curvature scalar of the tensor \( K_{\mu \nu} \) which can be defined by contracting the tensor \( K_{\mu \nu} \) with the metric

\[
K = R + 6\nabla_{\nu}w^{\mu} - 6w^{2} + 2\nabla_{\nu}C - C_{\alpha \beta}C_{\alpha \beta} + C_{\alpha \beta \lambda}C^{\alpha \beta \lambda} - 4w^{\alpha}C_{\alpha}.
\]

(27)

Let us propose the following action

\[
S = \frac{1}{2n^{2}} \int d^{n}x \sqrt{-g}K + S_{G},
\]

(28)

where \( S_{G} \) is the Gauss–Bonnet action defined as

\[
S_{G} = \int d^{n}x \sqrt{-g} \left[ \alpha_{1}K^{\alpha \beta \gamma}K_{\alpha \beta \gamma \delta} + \alpha_{2}K^{\alpha \beta \gamma \delta}K_{\beta \alpha \delta \gamma} - \alpha_{3}K^{\alpha \beta \gamma \delta}K_{\delta \alpha \beta \gamma} - 4\beta_{1}K_{\alpha \beta}K^{\alpha \beta} - 4\beta_{2}K_{\alpha \beta}K^{\alpha \beta} - 4\beta_{3}K_{\alpha \beta}K^{\alpha \beta} - 4\beta_{4}K_{\alpha \beta}K^{\alpha \beta} + K^{2} \right],
\]

(29)

where \( \alpha \) and \( \beta \) are arbitrary constants. To get the standard Gauss–Bonnet action in the absence of torsion and non-metricity we need to impose the following constraints on the
The coefficients \( \beta_3 \) and \( \beta_4 \) do not enter to the above conditions, because in the case of vanishing Weyl vector, the tensor \( K_{\mu\nu} \) vanishes. We should note that the above action is the most general action for the second order higher gravity in the Weyl–Cartan theory which reduces to the standard Gauss–Bonnet action in the limit of zero Weyl and torsion.

The potentially dangerous terms which can produce an Ostrogradski ghost can be written as

\[
S_G \supset 4 \int d^4x \sqrt{-g} \left( -2(\beta_1 + \beta_2)R_{\mu\nu}\nabla^\mu C^\nu + R^\mu C_{\mu} \\
-2 \left[ 2(\beta_1 + \beta_2) - (\alpha_1 + \alpha_2 + \alpha_3) \right] R_{\mu\nu} \nabla^\mu W_{\rho} R_{\mu\nu} \nabla^\rho W_{\nu} \\
+ \left[ 3 - 2(\beta_1 + \beta_2) \right] R^{\mu\nu} w_{\mu} - 2(\beta_1 + \beta_2) R_{\mu\nu} \nabla^\alpha C^\mu_{\alpha} \\
+ (\alpha_1 + \alpha_2 + \alpha_3) R_{\alpha(\beta\gamma)} \nabla^\beta C^{\gamma\delta} \right).
\]

Using equations (30), one can write the RHS of (31) as

\[
-8 \int d^4x \sqrt{-g} \left( C_{\mu\nu} \nabla^\mu \left( w_{\nu} + C^\nu \right) + R_{\mu\nu} \nabla^\alpha C^\mu_{\alpha} - \frac{1}{2} R_{\alpha(\beta\gamma)} \nabla^\beta C^{\gamma\delta} \right).
\]
\[ S_C = \int d^4x \sqrt{-g} \left\{ -4 R^{\alpha \beta \gamma \delta} C_{\alpha \beta}^{\gamma} C_{\delta \gamma} - 8 R^{\alpha \beta} C^{\gamma} C_{\alpha \beta} + 8 R^{\alpha \beta} C_{\alpha}^{\gamma} C_{\beta \delta} \right. \]
\[ + 2 R C^{\alpha \beta} C_{\alpha \beta} + 4 G^{\alpha \beta} C_{\alpha \beta} \]
\[ + (2 - 2 \alpha_2) \left( \nabla^\alpha C^{\beta \gamma} \nabla_{\beta \gamma} - \nabla^\alpha C^{\beta \gamma} \nabla_{\beta \gamma} \right) \]
\[ - 4 \nabla^\alpha C^{\beta \gamma} C_{\alpha \beta}^{\gamma} C_{\delta \gamma} + 4 \alpha_2 \nabla^\alpha C^{\beta \gamma} \nabla_{\beta \gamma} \nabla_{\delta \gamma} \]
\[ + 8 \alpha_2 \nabla^\alpha C^{\beta \gamma} C_{\alpha \beta}^{\gamma} C_{\delta \gamma} + 2 \alpha_2 \nabla^\alpha C_{\alpha \beta}^{\gamma} C_{\gamma \beta}^{\mu} C_{\mu \delta} \]
\[ - 2 \alpha_2 C^{\alpha \beta} C_{\alpha \beta}^{\gamma} C_{\gamma \beta}^{\mu} C_{\mu \delta} + (4 - 4 \beta_2) \left( \nabla^\alpha C_{\alpha \beta}^{\gamma} \nabla_{\delta \gamma} \nabla_{\mu \delta} \right) \]

\[ + 2 \nabla^\gamma C^\delta \nabla_{\delta \gamma} C_{\mu \delta} - 2 \nabla^\gamma C_{\mu \delta} C^\delta \nabla_{\delta \gamma} \]
\[ + 2 \nabla^\gamma C_{\mu \delta} C_{\gamma \delta} - 2 \nabla^\gamma C_{\gamma \delta} C_{\mu \delta} + 2 \nabla^\gamma C^\delta \nabla_{\delta \gamma} C_{\gamma \delta} \]
\[ - C^{\alpha \beta} C_{\alpha \beta}^{\gamma} C_{\gamma \delta} C_{\delta \gamma} + 2 \nabla^\gamma C_{\mu \delta} C_{\gamma \delta} C_{\delta \gamma} \]
\[ + \nabla^\gamma C_{\mu \delta} C_{\gamma \delta} C_{\delta \gamma} + 4 \nabla^\gamma C_{\gamma \delta} C_{\mu \delta} + 4 \nabla^\gamma C_{\gamma \delta} C_{\mu \delta} \]
\[ + C^{\alpha \beta} C_{\gamma \delta} C_{\mu \delta} C_{\gamma \delta} \left( \frac{1}{2} C^{\alpha \beta} C_{\gamma \delta} \right) \right\} \]
\[ + 8 \beta_2 \nabla^\gamma C^\delta \nabla_{\delta \gamma} C_{\mu \delta} - 8 \beta_2 \nabla^\gamma C^\delta \nabla_{\delta \gamma} C_{\mu \delta} \]
\[ - 8 \beta_2 \nabla^\gamma C_{\mu \delta} C_{\gamma \delta} C_{\delta \gamma} \]
\[ + 8 \beta_2 \nabla^\gamma C_{\mu \delta} C_{\gamma \delta} C_{\delta \gamma} + 8 \beta_2 \nabla^\gamma C_{\mu \delta} C_{\gamma \delta} C_{\delta \gamma} - 4 \beta_2 C^{\alpha \beta} C_{\alpha \beta} C_{\gamma \delta} \]
\[ - 8 \beta_2 C^{\alpha \beta} C_{\gamma \delta} C_{\mu \delta} + 4 \beta_2 C^{\alpha \beta} C_{\gamma \delta} C_{\mu \delta} \]
\[ - 4 \nabla^\gamma C_{\mu \delta} + 4 \nabla^\gamma C_{\gamma \delta} C_{\mu \delta} + C^2 - 2 C^{\alpha \beta} C_{\alpha \beta} \]
\[ + C^{\alpha \beta} C_{\alpha \beta} C_{\gamma \delta} C_{\mu \delta} \].

where we have defined $C^2 = C_{\mu \nu}$ and $C_{\mu \nu} = \nabla_\mu C_\nu - \nabla_\nu C_\mu$.

One should note that the tensor $C_{\mu \nu}$ is proportional to the tensor $T_{\mu \nu}$ constructed similarly with the torsion tensor. One can see that the term $T_{\mu \nu} T^{\mu \nu}$ is produced naturally in this model, which is also the kinetic term assumed in [14].

The remaining terms of the action contain a variety of possible interactions between the Weyl vector and the contortion tensor.
It is worth mentioning that the term $\nabla_{\alpha} C^{\alpha \beta} j_{\beta}$ contains an interaction term between Weyl vector and torsion tensor which was assumed in [15].

Finally, the full action of the theory can be written as

$$S = \int d^4x \sqrt{-g} \left[ R - 6w^2 - C^2 + C_{\alpha \beta \lambda} C^{\alpha \beta \lambda} - 4 w^2 C_\alpha \right] + S_W + S_C + S_f. \tag{38}$$

Let us discuss the coupling of the ordinary matter field to our theory. Note that we have three independent fields in the theory. The metric tensor couples to the energy–momentum tensor of the matter and is responsible for the gravitational force. The torsion tensor should be coupled to the spin tensor of the matter [8]. Due to the work of Weyl to unify the electromagnetic field with gravity [1], one can imagine that the Weyl vector can be coupled to the electric current of the matter. In this sense, one can add the matter Lagrangian of the form

$$S_m = \int d^4x \sqrt{-g} \, \mathcal{L}_m \left( \psi_i, g_{\mu \nu}, T_{\rho \sigma \tau}, w_\lambda \right), \tag{39}$$

where $\psi_i$ presents the matter fields. By varying the matter Lagrangian wrt the metric one can obtain the energy–momentum tensor of the ordinary matter as

$$T_{\mu \nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \left( \sqrt{-g} \, \mathcal{L}_m \right)}{\delta g^{\mu \nu}}. \tag{40}$$

Variation of $\mathcal{L}_m$ wrt the torsion is the spin tensor

$$\tau_{\mu \nu} = \frac{2}{\delta T_{\mu \nu}}. \tag{41}$$

In the special case where the matter field can be described by a scalar field, the spin tensor is zero and the torsion equation of motion is source-free. In other cases, one can define the spin tensor of the matter field from Noether’s theorem and use it as a source of the torsion equation of motion. The variation of $\mathcal{L}_m$ wrt the Weyl vector is
which can be considered as an electric current of ordinary matter. In the case of an uncharged matter field the current $J^\mu$ will become zero and the Weyl equation of motion will become source-free.

In the following we will assume that the matter Lagrangian vanishes for simplicity. The effect of matter fields and the interaction of torsion tensor and the Weyl vector with it will be postponed to future works.

3. Special case for the contortion tensor

The torsion tensor can be decomposed irreducibly into

$$T_{\mu\nu\rho} = \frac{2}{3}(t_{\mu\nu\rho} - t_{\mu\rho\nu}) + \frac{1}{3}(Q_\mu g_{\nu\rho} - Q_\rho g_{\nu\mu}) + \epsilon_{\mu\nu\rho\sigma} S^\sigma,$$

where $Q_\mu$ and $S^\mu$ are two vector fields. The vector $Q_\mu$ is actually the trace of torsion over its first and third indices. The tensor $t_{\mu\nu\rho}$ is symmetric with respect to $\mu$ and $\nu$ and has the following properties

$$t_{\mu\nu\rho} + t_{\nu\rho\mu} + t_{\rho\mu\nu} = 0, \quad g_{\mu\nu} t^{\mu\nu} = 0 = g_{\mu\rho} t^{\mu\rho}.$$

One can decompose the contortion tensor according to the above relation as

$$C_{\mu\nu\rho} = \frac{4}{3}(t_{\mu\nu\rho} - t_{\mu\rho\nu}) + \frac{2}{3}(Q_\mu g_{\nu\rho} - Q_\rho g_{\nu\mu}) + \epsilon_{\mu\nu\rho\sigma} S^\sigma.$$

Let us assume that the contortion tensor has the following simple form

$$C_{\mu\nu\rho} = \hat{Q}_\mu g_{\nu\rho} - \hat{Q}_\rho g_{\nu\mu},$$

where $t_{\mu\nu\rho} = 0, S^\nu = 0$ and $\hat{Q}_\mu = \frac{2}{3} Q_\mu$.

The action can then be expanded as

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R - 6w^2 - 6\hat{Q}^2 + 12w^\alpha \hat{Q}_\alpha \right) - 4(1 + \alpha_2 - 2\beta_2) \hat{Q}_{\mu\nu} \hat{Q}^{\mu\nu} \\
+ 8(2 + \alpha_2 + 2\beta_4 - 4\beta_2) \hat{Q}_{\mu\nu} W^{\mu\nu} \\
- 4(3 + 2\alpha_2 + 2\alpha_3 - 8\beta_2 + 16\beta_3 + 8\beta_4) W_{\mu\nu} W^{\mu\nu} \right].$$

In general, the above action may have some ghost and/or tachyon instabilities. In order to examine this issue, we first diagonalize the kinetic and potential terms for $\hat{Q}_\mu$ and $w_\mu$ with the result

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{4} X_{\mu\nu} X^{\mu\nu} - \frac{1}{4} Y_{\mu\nu} Y^{\mu\nu} - \frac{1}{2} m^2 X_\mu X^\mu \right].$$

where $X_{\mu\nu}$ and $Y_{\mu\nu}$ are strength tensors respectively according to vectors $X_\mu$ and $Y_\mu$ which will be defined below. As one can see from the above action, the theory contains one massless and one massive vector field with mass

$$m^2 = \frac{1}{2\kappa^2} \frac{3(A + 2B + C)}{B^2 - AC}.$$
where we have defined
\[ A = -4 - 4\alpha + 8\beta, \]  
\[ B = 8 + 8\beta + 4\alpha - 16\beta, \]  
\[ C = -12 - 32\beta - 64\alpha - 8\alpha^2 + 32\beta^2 - 8\alpha^3. \]

The new fields can be related to the original fields \( \tilde{Q}_\mu \) and \( w_\mu \) as
\[
X_\mu = \frac{2}{\sqrt{2\beta^2 - 1}} \left[ \left( \alpha \lambda_+ - \lambda_- \sqrt{(1 - \alpha^2)(\beta^2 - 1)} \right) \tilde{Q}_\mu 
+ \left( \beta \lambda_+ \sqrt{1 - \alpha^2} + \alpha \lambda_- \sqrt{\beta^2 - 1} \right) w_\mu \right],
\]
\[
Y_\mu = -\frac{2}{\sqrt{2\beta^2 - 1}} \left[ \left( \beta \lambda_+ \sqrt{1 - \alpha^2} + \alpha \lambda_- \sqrt{\beta^2 - 1} \right) \tilde{Q}_\mu 
- \left( \alpha \lambda_+ - \lambda_- \sqrt{(1 - \alpha^2)(\beta^2 - 1)} \right) w_\mu \right],
\]
where
\[
\alpha = \left[ \frac{1}{2} \left( 1 + \frac{A - C}{\sqrt{4B^2 + (A - C)^2}} \right) \right]^{1/2},
\]
\[
\beta = -\left[ \frac{1}{2} \left( \frac{(A + 2B + C)\sqrt{4B^2 + (A - C)^2}}{4B^2 + (A - C)^2 + 2B(A + C)} + 1 \right) \right]^{1/2},
\]
and
\[
\lambda_\pm^2 = -\frac{1}{2} \left( A + C \pm \sqrt{4B^2 + (A - C)^2} \right).
\]

In order to have a ghost and tachyon free theory we should have \( m^2 > 0 \) and the new fields (51) and (52) should be meaningful. We thus conclude that the parameters \( \alpha_2, \alpha_3, \beta_2 \) and \( \beta_4 \) should satisfy the relations
\[
m^2 > 0, \quad \lambda_\pm^2 > 0,
\]

and
\[
\text{together with the reality of square roots.}
\]

The action (48) can be seen as a gravitational theory minimally coupled to a Proca field and a Maxwell field, i.e. the Einstein–Maxwell–Proca theory. Geometrization of the Einstein–Proca (EP) action has been done before with different approaches. In [20], the EP action was obtained by writing the Gauss–Bonnet action in the Weyl spacetime. However, in the Weyl spacetime, adding the contraction of the antisymmetric part of the Ricci tensor with the Ricci tensor itself is sufficient to produce the EP action, which was done in [27]. One can also obtain the EP action by writing the \( f(R) \) and also \( f(R, R_{\mu\nu}R^{\mu\nu}) \) action in the Palatini formalism [28]. In the case of \( f(R) \) Palatini gravity the equation of motion for the connection can be solved for the connection itself, leading to an expression which is equivalent to the Weyl connection with the Weyl vector \( w_\mu = \frac{1}{f} \partial_\mu f' \) where the prime is taking derivative with respect to the function’s argument. The connection of this theory and the resulting Ricci tensor is asymmetric. The antisymmetric part of the Ricci tensor can then produce a kinetic term for the Weyl vector \( w_\mu \).
In this paper, we have used the Weyl–Cartan spacetime for writing the Gauss–Bonnet action. In the special case where only the trace part of the torsion tensor is non-vanishing, the theory reduces to a bi-vector–tensor theory, which can not be obtained in the Weyl spacetime. Note that writing the Gauss–Bonnet action in the Cartan spacetime and keeping only the trace part of the torsion will produce the EP action.\footnote{This can be seen from equation (47) by assuming \( \omega_0 = 0 \).}

The theory presented here can be seen as a gravitational theory coupled to two vector Galilean fields [29]. Because we have added only a Gauss–Bonnet term (which is the second Lovelock term) to the action, only the second vector Galilean term appears. One may expect that adding higher order Lovelock terms to the action may produce higher order vector Galilean terms.\footnote{One may also expect that the \( S^0 \) term in the decomposition of torsion can produce a vector Galilean term because of the appearance of Levi-Civita tensor. This possibility will be considered in a separate work.}

One should note that in the vector Galilean theories, the helicity-0 part plays the role of Galilean fields [30]. Galileans are some scalar fields which have higher than second order time derivatives in the action, but produce at most second order field equations. The resulting action is known as the Horndeski action [31]. Note that in the action (48) the helicity-0 part vanishes due to the special form of the kinetic term. However, considering the theory in higher dimensions and then performing the Kaluza–Klein reduction, one can obtain the Horndeski theory [20].

Our theory can be seen as a generalization of the Horndeski action, in the sense that we have also added some higher order curvature tensor terms to the action. However, we have used the Gauss–Bonnet combination which causes the action to have no higher than second order time derivatives.

Conditions (56) can be solved analytically in terms of \( A, B \) and \( C \). The solution implies that \( C < 0 \) and \( A \in (C, 0) \). The parameter space is divided into three parts as follows

\[
C < A < \frac{1}{3}C \quad \Rightarrow \quad \begin{cases} 
0 < B < -\frac{1}{4}(A + C + \sqrt{(3C - A)(3A - C)}) \\
-\frac{1}{4}(A + C - \sqrt{(3C - A)(3A - C)}) < B < \sqrt{AC} 
\end{cases}
\]

\[
A = \frac{1}{3}C \quad \Rightarrow \quad 0 < B < -\frac{\sqrt{3}}{3}C, \quad B = -\frac{1}{3}C, \\
\frac{1}{3}C < A < 0 \quad \Rightarrow \quad 0 < B < \sqrt{AC}.
\]

Solving the above conditions in terms of \( \alpha_i \) and \( \beta_i \) is very difficult to obtain. So, in the following we will study some special cases.

3.1. Case I: \( \beta_3 = \beta_2 = \alpha_3 = 0 \) and \( \alpha_2 = 1 \)

In this case the the action \( S_G \) reduces to

\[
S_G = \int d^4x \sqrt{-g} \left[ K_{\gamma \beta \rho \sigma} K_{\gamma \beta \rho \sigma} - 4K_{\beta \gamma} K^{\beta \gamma} - 4\beta_4 K_{\alpha \beta \gamma} K^{\alpha \beta \gamma} + K^2 \right].
\]

and the constraints (56) satisfy if

\[
-\frac{1}{2}(1 - \sqrt{2}) < \beta_4 < \frac{1}{2}(1 + \sqrt{2}).
\]
3.2. Case II: $\beta_2 = 0 = \alpha_3$ and $\beta_3 = 0 = \beta_4$

In this case the the action $S_G$ reduces to

$$S_G = \int d^4x \sqrt{-g} \left[ (1 - \alpha_2)K^{\alpha\beta\gamma\delta}K_{\alpha\beta\gamma\delta} + \alpha_2 K^{\alpha\beta\gamma\delta}K_{\gamma\delta\alpha\beta} - 4K_{\alpha\beta}K^{\alpha\beta} + K^2 \right],$$

and the constraints (56) satisfy if

$$\frac{2}{1 + \sqrt{5}} < \alpha_2 < 2.$$ 

3.3. Case III: $\beta_2 = 0 = \alpha_3$ and $\beta_3 = -\beta_4$.

In this case the the action $S_G$ reduces to

$$S_G = \int d^4x \sqrt{-g} \left[ (1 - \alpha_2)K^{\alpha\beta\gamma\delta}K_{\alpha\beta\gamma\delta} + \alpha_2 K^{\alpha\beta\gamma\delta}K_{\gamma\delta\alpha\beta} - 4K_{\alpha\beta}K^{\alpha\beta} + K^2 \right] + 4\beta_4 \left( K_{\alpha\beta}K^{\alpha\beta} - K_{\alpha\beta}K^{\alpha\beta} \right).$$

In figure 1 we have plotted the allowed region of parameter space ($\alpha_2, \beta_4$) in order to have a ghost and tachyon free bi-vector theory.

4. Conclusion

In this paper, we have introduced a ghost-free modified theory of gravity by generalizing the geometry to be the Weyl–Cartan spacetime. Using the standard Einstein–Hilbert term for this geometry, the action reduces to the Ricci scalar, plus possible mass terms for the Weyl vector and the torsion tensor. In this case, kinetic terms for these two new fields are not produced. In order to make the Weyl vector and the torsion tensor dynamical, one can add some kinetic terms by hand, which was done in [14].

In this paper, in order to produce kinetic terms for the Weyl and torsion fields, we have generalized the action to be of Gauss–Bonnet type. In 4D Riemannian geometry the Gauss–Bonnet term becomes a total derivative and dropped from the action. However in the Weyl–Cartan geometry, this term produces a bunch of interaction and kinetic terms for Weyl and torsion fields. In the Weyl–Cartan geometry the curvature tensor has fewer symmetries than the Riemann tensor. So one can write more than three quadratic terms according to the curvature. In general, the resulting action does not reduce to the standard Gauss–Bonnet action. However, demanding that the action should reduce to the standard Gauss–Bonnet action in the case of vanishing Weyl and Cartan, all of the higher than two time derivatives of the action become a total derivative and do not contribute to the action.

For further considerations, we have studied a special case of the theory where only the trace part of the torsion tensor is non-zero. In this case, the theory is reduced to general relativity plus one massive and one massless vector field. This theory can be seen as an Einstein–Maxwell–Proca system which has 7 d.o.f., two for the graviton, two for the Maxwell field and three for the Proca vector field. The Einstein–Proca system was also geometrized in the context of Weyl gravity in [20, 27] and also in the context of Palatini-f(R) theories [28]. However, our theory in its reduced form considered in section 3, has two vector fields minimally coupled to gravity. One can see that the trace of the torsion tensor can also play the role of a Proca field.
It will be very interesting to note that, in order to obtain a kinetic term for a vector field by adding higher order curvature terms to the action, the Ricci tensor should be asymmetric. In Weyl spacetime considered in [20, 27] and also in the Weyl–Cartan spacetime considered here, the Ricci tensor is asymmetric. In [28], the Palatini assumption is sufficient to take the connection asymmetric, and as a result the Ricci tensor will be asymmetric.

In this paper we showed that the absence of an Ostrogradski ghost reduces the dimension of parameter space of the theory to five. However, this will not prove that the theory is healthy. The full theory may have some Boulware–Deser ghosts or tachyonic instabilities. In section 3 we have considered some special cases of the theory and obtained the healthy region of the parameter space of the theory.

It is worth mentioning that higher order Lovelock terms can also contribute to the action in 4D in the context of Weyl–Cartan theory. In this paper we have only studied the second order term to show that the formalism is capable of producing kinetic terms for Weyl vector and torsion tensor. Higher order terms can be responsible for higher order self-interaction terms for the Weyl and torsion which may resemble vector Galileans and its generalizations. This issue should be considered carefully in more detail, which should be studied in a separate work.

A question which should be answered in future works relates to the physics of our model. This can be done by solving for both cosmological and static solutions at the level of background to see if self-accelerating solutions exist and the solar system tests are satisfied. The other issue is coupling to matter which definitely affects predictions of our model and which is beyond the scope of the current work.

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