Predictions from warped flavordynamics based on the $T'$ family group

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We propose a realistic five-dimensional warped scenario where all standard model fields propagate in the bulk. The assumed $T'$ flavor symmetry is broken on the branes by flavon fields, providing a consistent scenario where fermion mass hierarchies are accounted for by adequate choices of the bulk mass parameters, while quark and lepton mixing angles are restricted by the flavor symmetry. Neutrino mixing parameters and the Dirac CP violation phase are all described in terms of just two independent parameters. This leads to predictions for the neutrino mixing angles and the Dirac CP phase, as well as a $0\nu\beta\beta$ decay rate within reach of upcoming experiments. The scheme provides a good global description of flavor observables also in the quark sector.

I. INTRODUCTION

Understanding flavor from first principles is one of the greatest challenges in particle physics. The coin has two sides. On the one hand there is the problem of understanding the observed hierarchies of quark and lepton masses, explaining why is the muon about 200 times heavier than the electron, or why does the top quark seem to play such a special role in being the heaviest.

On the other hand comes the problem of finding a rationale for the observed pattern of mixing parameters. This problem has only become trickier after the discovery of neutrino oscillations [1, 2] which implies not only the need for neutrino masses – and understanding their smallness with respect to the charged fermion masses – but also the need to understand why the pattern of neutrino mixing is so special when compared to that of quarks [3].

The Standard Model (SM) lacks an organizing principle to account for the observed flavor properties. The existence of flat extra dimensions has been suggested as a way to shed light on the possible nature of the family symmetry [4]. In particular, six-dimensional theories compactified on a torus have been suggested [5, 6] and a successful model has recently been proposed [7] in which fermions are nicely arranged within the framework of an $A_4$ family symmetry, with good predictions for fermion masses and mixings, including the “golden” quark-lepton unification formula [8–11]. Although intriguingly successful, this theory remains far from giving a complete description of mass hierarchies.

As a possible alternative to the flat-extra-dimensions approach here we turn to the possibility of warped extra dimensions. These have been proposed by Randall & Sundrum [12] in order to address the hierarchy problem without the need to invoke supersymmetry. The fundamental scale of gravity gets exponentially reduced with respect to the

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Planck scale by having the Higgs sector localized near the boundary of the extra dimensions. Here we assume the standard model fermions to propagate in the bulk, though peaked towards either brane. This allows us to address at once both aspects of the flavor problem: the fermion mass hierarchy problem, as well as their mixing pattern, due to the imposition of a family symmetry group. This follows the approach suggested in Ref. [13]. In such scenario fermion mass hierarchies are accounted for by adequate choices of the bulk mass parameters, while quark and lepton mixing angles are restricted by the assumed family symmetry, broken on the branes by flavon fields.

Our present scenario employs the $T'$-based family group and predicts the neutrino mixing parameters and the Dirac CP violation phase in terms of only two independent parameters at leading order. In contrast to Ref. [13] where neutrinos were Dirac particles, here a viable description of neutrino oscillations requires neutrinos to be Majorana particles. Moreover, given the predicted regions for the oscillation parameters it follows that there must be a lower bound on the neutrinoless double beta decay rate even if the spectrum is normal-ordered. We show that the model also provides a successful global description of flavor, consistent with the observed CKM quark mixing matrix, in which the successful Gatto-Sartori relation emerges in leading order.

The paper is organised as follows. In Sec. II we present the theoretical framework for the lepton sector, while in Sec. III we sketch the quarks sector, its field content and quantum numbers. In Sec. IV we give a numerical analysis of the resulting predictions and conclude in Sec. V.

II. LEPTON SECTOR

Here we study the implementation of a flavor symmetry within a warped extra dimensional theory context. For the flavor symmetry we choose the $T'$ group. The $T'$ flavor symmetry has been studied in the literature [14–22]. We introduce four flavon fields $\varphi_l$ and $\sigma_l$ localized on the UV brane, and flavons $\varphi_\nu$ and $\sigma_\nu$ localized on the IR brane. The fermion fields and Higgs field live in the bulk, and the profiles of their zero modes in the fifth dimension are displayed in Fig. 1. The transformation properties of the lepton and scalar fields under the standard model $SU(3)\times SU(2)\times U(1)_Y$ gauge symmetry and $T'\times Z_3\times Z_4$ flavor symmetry are summarized in table I. The vacuum expectation values (VEVs) of the flavon fields are

$$
\langle \varphi_l \rangle = (1,0)v_{\varphi_l}, \quad \langle \sigma_l \rangle = v_{\sigma_l}, \quad \langle \varphi_\nu \rangle = (1,-2\omega^2,-2\omega)v_{\varphi_\nu}, \quad \langle \sigma_\nu \rangle = v_{\sigma_\nu},
$$

where $\omega = e^{\frac{2\pi}{3}}$, $v_{\varphi_l}$, $v_{\sigma_l}$, $v_{\varphi_\nu}$ and $v_{\sigma_\nu}$ are arbitrary complex numbers. As shown in Appendix C, the alignment in Eq. (1) is the minimum of the scalar potential.

The leading order charged lepton Yukawa interactions respecting both gauge and flavor symmetries are of the
\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\text{Field} & \Psi_t & \Psi_e & \Psi_{\mu} & \Psi_{\tau} & \Psi_{\nu} & H & \varphi_t & \sigma_l & \varphi_{\nu} & \sigma_{\nu} \\
\text{SU}(3) \times \text{SU}(2) \times U(1)_Y & (1,2,-1/2) & (1,1,-1) & (1,1,-1) & (1,1,-1) & (1,1,0) & (1,2,1/2) & (1,1,0) & (1,1,0) & (1,1,0) & (1,1,0) \\
\hline
T' & 3 & 1' & 1'' & 1 & 3 & 1 & 2 & 1'' & 3 & 1 \\
\hline
Z_3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
Z_4 & i & i & i & i & i & i & 1 & 1 & 1 & 1 \\
\hline
\end{array}
\]

TABLE I. The transformation properties of the particles in lepton sector under the standard model \(SU(3) \times SU(2) \times U(1)_Y\) gauge group and the \(T' \times Z_3 \times Z_4\) flavor symmetry, where \(\omega = e^{2\pi i/3}\).

following form,

\[
\mathcal{L}_Y' = \sqrt{\frac{G}{L^2}} \left[ y_e (\varphi_t^2 \overline{\Psi}_t)_{1} H \Psi_e + y_\mu (\varphi_t^2 \overline{\Psi}_t)_{1} H \Psi_\mu + y_\tau (\varphi_t^2 \overline{\Psi}_t)_{1} H \Psi_\tau \right] \delta(y) + \text{h.c.},
\]

(2)

where \(G = e^{-8k y}\) is the determinant of the 5-D metric. Inserting the vacuum configuration of Eq. (1) into Eq. (2) and noticing that

\[
\langle \varphi_t \varphi_t \rangle_3 = (0,0,1)v_{\varphi_t}^2,
\]

(3)

one can read out the charged lepton mass matrix in the zero mode approximation as

\[
m_l = \frac{1}{\Lambda^2}v \begin{pmatrix}
\tilde{y}_e v_{\varphi_t}^2 & 0 & 0 \\
0 & \tilde{y}_\mu v_{\varphi_t}^2 & 0 \\
0 & 0 & \tilde{y}_\tau v_{\varphi_t}^2
\end{pmatrix},
\]

(4)

where \(v = \langle H \rangle\) is the vacuum expectation value of the Higgs field and

\[
\tilde{y}_{e,\mu,\tau} = \frac{y_{e,\mu,\tau}}{\sqrt{L^2 \Lambda^3}} f_L(0,c_\ell) f_R(0,c_{e,\mu,\tau}) f_H(0).
\]

(5)

Here \(f_{L,R}\) and \(f_H\) are the zero-mode wave functions of fermion and Higgs fields, their explicit forms are given in Appendix B. One sees that the charged lepton mass matrix is diagonal with

\[
m_e = \tilde{y}_e v_{\varphi_t}^2, \quad m_\mu = \tilde{y}_\mu v_{\varphi_t}^2, \quad m_\tau = \tilde{y}_\tau v_{\varphi_t}^2.
\]

(6)

The correct values of \(m_{e,\mu,\tau}\) can be naturally achieved via the wave function overlaps in the usual way. In our model, neutrino masses are generated by the type-I seesaw mechanism. The corresponding terms invariant under the flavor symmetry \(T' \times Z_3 \times Z_4\) are given by

\[
\mathcal{L}_Y'' = y_{\nu_1} \sqrt{\frac{G}{L^2}} (\overline{\Psi}_1 \nu_1 1 H) + \frac{y_{\nu_1}}{2 \Lambda} \left[ y_{\nu_1} (\overline{\nu}_1 \Psi_1)_{1} \sigma_\nu + y_{\nu_1} (\overline{\nu}_1 \Psi_1)_{0} \sigma_\nu^* \right]
\]

\[
+ y_{\nu_3} (\overline{\nu}_3 \Psi_3)_{1} + y_{\nu_3} (\overline{\nu}_3 \Psi_3)_{0} \right] \delta(y - L) + \text{h.c.},
\]

(7)

where \(\tilde{H} = i \sigma_2 H^*\) is the conjugate Higgs, \(\Lambda'\) denotes the cut-off scale at the IR brane, generally suppressed by the exponential warp factor \(e^{-kL}\) with respect to \(\Lambda\). This naturally accounts for the large values required for the seesaw scale. Notice that \(\Psi_i' = C \overline{\Psi_i'}\) is defined as in the same way as in four dimensions, and \(C\) is the charge conjugation matrix. Given the vacuum alignment of \(\sigma_\nu\) and \(\varphi_\nu\) in Eq. (1), we can read out the Dirac and Majorana neutrino mass matrices as follows

\[
m_D = \tilde{y}_{\nu_1} v \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(8)

\[
m_N = (\tilde{y}_{\nu_3} v_{\sigma_\nu} + \tilde{y}_{\nu_3} v_{\sigma_\nu}^*) \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} + \tilde{y}_{\nu_3} v_{\varphi_\nu} \begin{pmatrix}
2 & 2\omega & 2\omega^2 \\
2\omega & -4\omega^2 & -1 \\
2\omega^2 & -1 & -4\omega
\end{pmatrix} + \tilde{y}_{\nu_3} v_{\varphi_\nu}^* \begin{pmatrix}
2 & 2\omega^2 & 2\omega \\
2\omega^2 & -4\omega & -1 \\
2\omega & -1 & -4\omega^2
\end{pmatrix},
\]

(9)
with
\[ \bar{y}_\nu = \frac{y_{\nu}}{\sqrt{\Lambda}} \int_0^L f_L(y, c_\ell) f_R(y, c_\nu) f_H(y) dy, \quad \bar{y}_{\nu 2.3.4.5} = \frac{y_{\nu 2.3.4.5}}{\Lambda} f_{H}^2(L, c_\nu). \] (10)

By performing the seesaw diagonalization procedure [23, 24], one finds that the effective light neutrino mass matrix is given as
\[ m_{\nu} = -m_D^T m_N^{-1} m_D^T \]
\[ = m_0 \begin{pmatrix}
1-2y_1-2y_2-15y_3+18y_4y_5-15y_6^2 & -2\omega(y_4+y_5+3y_4^2+9\omega^2y_4y_5+3y_5^2) & -2\omega(y_4+y_5+3y_4^2+9\omega^2y_4y_5+3y_5^2) \\
3(y_4+y_5+1) & 3(y_4+y_5+1) & 3(y_4+y_5+1) \\
3(y_4+y_5+1) & 3(y_4+y_5+1) & 3(y_4+y_5+1)
\end{pmatrix}^{-1} \]
\[ \cdot \begin{pmatrix}
1+2y_2-6y_1^2+9y_3^2-18y_4y_5+y_6^2 & 1+y_1-6y_2+y_3-6y_4^2+y_5 & 1+y_1-6y_2+y_3-6y_4^2+y_5 \\
3(y_4+y_5+1) & 3(y_4+y_5+1) & 3(y_4+y_5+1) \\
3(y_4+y_5+1) & 3(y_4+y_5+1) & 3(y_4+y_5+1)
\end{pmatrix} \]
\[ \cdot \begin{pmatrix}
-2\omega(y_4+y_5+3y_4^2+9\omega^2y_4y_5+3y_5^2) & -2\omega(y_4+y_5+3y_4^2+9\omega^2y_4y_5+3y_5^2) & -2\omega(y_4+y_5+3y_4^2+9\omega^2y_4y_5+3y_5^2) \\
3(y_4+y_5+1) & 3(y_4+y_5+1) & 3(y_4+y_5+1) \\
3(y_4+y_5+1) & 3(y_4+y_5+1) & 3(y_4+y_5+1)
\end{pmatrix}^{-1}, \] (11)

where \( m_0 = \frac{y_{\nu}^2}{\bar{y}_{\nu}^2} \), \( y_4 = \frac{y_{\nu 2} v_{\nu 2} + y_{\nu 3} v_{\nu 3}}{y_{\nu 2} v_{\nu 2} + y_{\nu 3} v_{\nu 3}} \), and \( y_5 = \frac{y_{\nu 4} v_{\nu 4} + y_{\nu 5} v_{\nu 5}}{y_{\nu 4} v_{\nu 4} + y_{\nu 5} v_{\nu 5}} \). It is remarkable that, apart from an overall mass scale \( m_0 \), the mass matrix \( m_{\nu} \) only depends on two complex input parameters \( y_4, y_5 \). These will describe the three neutrino masses as well as the full lepton mixing matrix. We first perform a tri-bimaximal transformation on the neutrino fields. The resulting light neutrino mass matrix becomes
\[ m'_{\nu} = U_{TBM}^T m_{\nu} U_{TBM} \]
\[ = m_0 \begin{pmatrix}
-1 & 0 & 0 & -3\sqrt{2}(y_4-y_5) & -3\sqrt{2}(y_4-y_5) \\
0 & 1 & 0 & \sqrt{2}(y_4-y_5)^2+3(y_4+y_5)-1 & \sqrt{2}(y_4-y_5)^2+3(y_4+y_5)-1 \\
0 & 0 & 1 & \sqrt{2}(y_4-y_5)^2+3(y_4+y_5)-1 & \sqrt{2}(y_4-y_5)^2+3(y_4+y_5)-1 \\
\end{pmatrix}, \] (12)

where \( U_{TBM} \) is the well-known tri-bimaximal mixing matrix,
\[ U_{TBM} = \begin{pmatrix}
\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}. \] (13)

Since \( m'_{\nu} \) is a block-diagonal symmetric matrix, it can be exactly diagonalized as
\[ U_{\nu}' m'_{\nu} U_{\nu}'^T = \text{diag}(m_1, m_2, m_3), \] (14)

where \( U_{\nu}' \) can be generally denoted as
\[ U_{\nu}' = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \cos \theta_\nu & \sin \theta_\nu e^{i\delta_\nu} & 0 & 0 \\
0 & -\sin \theta_\nu e^{-i\delta_\nu} & \cos \theta_\nu & 0 & 0
\end{pmatrix}. \] (15)

Since the charged lepton mass matrix \( m_l \) is diagonal in this case, the lepton mixing matrix is determined to be\(^1\)
\[ U = U_{TBM} U_{\nu}', \]
\[ = \begin{pmatrix}
\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta_\nu & \sin \theta_\nu e^{i\delta_\nu} & 0 \\
0 & -\sin \theta_\nu e^{-i\delta_\nu} & \cos \theta_\nu & 0
\end{pmatrix}, \]
\[ = \begin{pmatrix}
\frac{\cos \theta_\nu}{\sqrt{3}} & \frac{\sin \theta_\nu e^{-i\delta_\nu}}{\sqrt{3}} & -\frac{\sin \theta_\nu e^{i\delta_\nu}}{\sqrt{3}} \\
\frac{\sin \theta_\nu e^{-i\delta_\nu}}{\sqrt{3}} & \frac{-\cos \theta_\nu}{\sqrt{3}} & \frac{\sin \theta_\nu e^{i\delta_\nu}}{\sqrt{3}} \\
\frac{\sin \theta_\nu e^{i\delta_\nu}}{\sqrt{3}} & \frac{-\cos \theta_\nu}{\sqrt{3}} & \frac{\sin \theta_\nu e^{-i\delta_\nu}}{\sqrt{3}}
\end{pmatrix}. \] (16)

\(^1\) We notice that the first column of the lepton mixing matrix is fixed to be \((2, -1, -1)^T/\sqrt{6} \).
A. Neutrino oscillations

From the lepton mixing matrix in Eq. (16), one can easily extract the following results for the neutrino mixing angles as well as the Jarlskog invariant,

\[ \sin^2 \theta_{13} = \frac{\sin^2 \theta_{\nu}}{3}, \]  
\[ \sin^2 \theta_{12} = 1 - \frac{4}{5 + \cos 2 \theta_{\nu}}, \]  
\[ \sin^2 \theta_{23} = \frac{1}{2} \frac{\sqrt{3} \sin 2 \theta_{\nu} \cos \delta_{\nu}}{5 + \cos 2 \theta_{\nu}}, \]  
\[ J_{CP} = \frac{\sin 2 \theta_{\nu} \sin \delta_{\nu}}{6\sqrt{6}}. \]

One sees that the three neutrino mixing angles as well as the Dirac CP violation phase are all expressed in terms of just two parameters, \( \theta_{\nu} \) and \( \delta_{\nu} \). Therefore there are two relations between these mixing angles and the Dirac CP violation phase, that can be expressed analytically as

\[ \cos^2 \theta_{12} \cos^2 \theta_{13} = \frac{2}{3}, \quad \cos \delta_{CP} = \frac{(3 \cos 2\theta_{12} - 2) \cos 2\theta_{23}}{3 \sin 2\theta_{23} \sin 2\theta_{12} \sin \theta_{13}}. \]

In Fig. 2 we display the contour plots of \( \sin^2 \theta_{12}, \sin^2 \theta_{13}, \) and \( \sin^2 \theta_{23} \) in the \( \theta_{\nu} - \delta_{\nu} \) plane. The shaded regions are the ones allowed by individual measurements of the three mixing angles, according to the global oscillation analysis in Ref. [3]. One sees that the parameter \( \theta_{\nu} \) is constrained to lie within quite narrow regions around \( \theta_{\nu} \approx 0.082\pi \) and \( \theta_{\nu} \approx 0.918\pi \). The left panel in Fig. 3 shows the contour plots of \( \delta_{CP} \) in the \( \theta_{\nu} - \delta_{\nu} \) plane. The black bands denote the regions in which all the three lepton mixing angles vary within the experimentally allowed 3\( \sigma \) ranges [3]. In our model, the sign of \( \delta_{CP} \) can not be fixed uniquely, with the predicted correlation between \( |\delta_{CP}| \) and \( \sin^2 \theta_{23} \) shown in the right panel of Fig. 3.

![Fig. 2](image-url)  
**FIG. 2.** Contours of \( \sin^2 \theta_{12}, \sin^2 \theta_{13}, \) and \( \sin^2 \theta_{23} \) in the \( \theta_{\nu} - \delta_{\nu} \). The red, green and blue areas denote the 3\( \sigma \) regions of \( \sin^2 \theta_{13}, \sin^2 \theta_{12} \) and \( \sin^2 \theta_{23} \) respectively, and the dashed lines refer to their best fit values taken from [3].
FIG. 3. Contour plots of $\delta_{CP}$ in the $\theta_\nu - \delta_\nu$ plane (left) and correlation between $|\delta_{CP}|$ and $\sin^2 \theta_{23}$ (right). The black areas correspond to the $3\sigma$ allowed regions of lepton mixing angles [3]. The vertical solid and dashed lines in the right panel represent the best fit values of $\sin^2 \theta_{23}$ for NO and IO respectively.

B. Neutrinoless double beta decay

We start this section by noticing that, in the absence of the Majorana terms in Eq. (7), in this model neutrinos would be unmixed, since both charged lepton and Dirac mass terms are diagonal. Neutrinos would also be degenerate in mass. Hence neutrino mass differences, as well as mixing and CP violation, all result from the seesaw mechanism. This is in sharp contrast with the warped standard model extension proposed in Ref. [13]. This also implies that, in contrast to Ref. [13], in the present model neutrinos must be Majorana particles, implying the existence of neutrinoless double beta decay, or $0^{\nu}\beta\beta$ for short.

One can determine the expected ranges for the $0^{\nu}\beta\beta$ decay amplitude, taking into account the allowed neutrino oscillation parameters obtained from experiment [3]. In Fig. 4 we plot the expected values for the mass parameter $|m_{ee}|$ characterizing the $0^{\nu}\beta\beta$ amplitude. In a generic model the regions expected for inverted-ordered and normal-ordered neutrino masses are indicated by the broad shaded regions indicated in Fig. 4.

The current experimental bound from KamLAND-Zen [26] as well as the estimated experimental sensitivities are indicated by the horizontal lines [27–32]. We now show how, within our model, the predictions for the oscillation parameters imply important restrictions for the effective Majorana mass $|m_{ee}|$. In fact, the allowed ranges are quite narrow. If the neutrino mass spectrum is inverted-ordered (IO), the effective Majorana mass has a lower limit $|m_{ee}| \geq 0.0162$ eV, while the lightest neutrino mass satisfies $m_{\text{lightest}} \geq 0.0133$ eV. In contrast, in the case of normal-ordering (NO), the effective mass $|m_{ee}|$ lies in the narrow interval [5.2meV, 9.6meV], and the allowed region of $m_{\text{lightest}}$ is [4.8meV, 7.2meV]2. As indicated in the figure, we expect that these predictions will be tested by the next generation $0^{\nu}\beta\beta$ decay experiments.

2 As shown in Fig. 4, the neutrino mass spectrum could possibly be quasi-degenerate as well, however this region is disfavored by both KamLAND-Zen and Planck.
In fact, as indicated in table II, the predicted neutrino mass parameter in $\beta$ decay and cosmology are also interesting. These should be compared with the recent limits from the KATRIN experiment [33], and the 95% confidence limit for the sum of neutrino masses set by the Planck collaboration [25].

\[
\begin{array}{|c|c|c|}
\hline
\text{Parameter} & \text{Experimental results} & \text{Predictions} \\
\hline
m_\beta \text{ [meV] (NO)} & < 1100 & 10.62 \\
m_\beta \text{ [meV] (IO)} & & 52.07 \\
\Sigma_i m_i \text{ [meV] (NO)} & < 120 & 66.81 \\
\Sigma_i m_i \text{ [meV] (IO)} & & 123.34 \\
\hline
\end{array}
\]

TABLE II. The predictions for the effective neutrino mass $m_\beta$ in $\beta$ decay and the sum of neutrino masses. The latest experimental bounds on $m_\beta$ and $\Sigma_i m_i$ are taken from KATRIN [33] and Planck 2018 [25] respectively.

III. QUARK SECTOR

We now extend our model to the quark sector. The classification of the quark fields under the flavor symmetry $T' \times Z_3 \times Z_4$ is given in table III, and no new flavon fields are required. We show the profiles of the zero models of

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\text{Field} & \Psi_{UC} & \Psi_T & \Psi_u & \Psi_c & \Psi_t & \Psi_{ds} & \Psi_{sb} & H & \varphi_t & \sigma_t \\
\hline
SU(3) \times SU(2) \times U(1)_Y & (3, 2, 1/6) & (3, 2, 1/6) & (3, 1, 2/3) & (3, 1, 2/3) & (3, 1, -1/3) & (3, 1, -1/3) & (1, 2, 1/2) & (1, 1, 0) & (1, 1, 0) \\
T' & 2 & 1 & 1' & 1'' & 1' & 2' & 1'' & 1 & 2 & 1'' \\
Z_3 & \omega^2 & \omega & 1 & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega \\
Z_4 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 \\
\hline
\end{array}
\]

TABLE III. The transformation properties of the quark fields under the standard model gauge group $SU(3) \times SU(2) \times U(1)_Y$ and the flavor symmetry $T \times Z_3 \times Z_4$. Note that no new scalars are needed beyond those in table I.
the quark fields in Fig. 5. It is straightforward to read off the down-type quark Yukawa interactions

$$\mathcal{L}_Y^d = \sqrt{\frac{G}{\Lambda^7}} \left[ y_{ds_1}(\overline{H}\hat{H}\overline{\psi}_{ds})_{3\varphi_i}^2 + y_{ds_2}(\overline{H}\hat{H}\overline{\psi}_{ds})_{1\sigma_i}^2 + y_b'(\overline{H}\hat{H}\overline{\psi}_b)_{2\sigma_i}^2 + \delta(y) + \text{h.c.} + \cdots \right] \quad (22)$$

where dots stand for higher dimensional operators. Similarly the up-type quark Yukawa interactions take the form

$$\mathcal{L}_Y^u = \sqrt{\frac{G}{\Lambda^7}} \left[ y_u'(\overline{H}\hat{H}\overline{\psi}_u)_{1\varphi_i}^2 + y_u(\overline{H}\hat{H}\overline{\psi}_u)_{2\sigma_i}^2 + y_{u_b}(\overline{\psi}_{u_b})_{2\varphi_i}^2 + \delta(y) + \text{h.c.} + \cdots \right] \quad (23)$$

In the zero mode approximation, we integrate over the fifth dimension and then obtain the up- and down-type quark effective mass matrices as follows

$$m^d = v \begin{pmatrix} \tilde{y}_{ds_2} v_{\sigma_i}^2 / \Lambda^2 & 0 & 0 \\ 0 & \tilde{y}_{ds_1} v_{\sigma_i}^2 / \Lambda^2 & \tilde{y}_b v_{\sigma_i}^2 / \Lambda^2 \\ 0 & 0 & \tilde{y}_b v_{\sigma_i}^2 / \Lambda^2 \end{pmatrix} \quad (24)$$

$$m^u = v \begin{pmatrix} \tilde{y}_u v_{\varphi_i} \sigma_i / \Lambda^2 & 0 & 0 \\ 0 & \tilde{y}_u v_{\varphi_i} \sigma_i / \Lambda^2 & \tilde{y}_b v_{\varphi_i} \sigma_i / \Lambda^2 \\ \tilde{y}_u v_{\sigma_i} / \Lambda & \tilde{y}_u v_{\sigma_i} / \Lambda & \tilde{y}_b v_{\sigma_i} / \Lambda \end{pmatrix} \quad (25)$$

with

$$\tilde{y}_{u,t,ds_1} = \frac{y_{u,t,ds_1}}{\sqrt{L^1 \Lambda^3}} f_L(0, c_{UC}) f_R(0, c_{u}, t, ds_1) f_H(0), \quad (26)$$

$$\tilde{y}''_{u,c,t,b} = \frac{y''_{u,c,t,b}}{\sqrt{L^3 \Lambda^3}} f_L(0, c_T) f_R(0, c_{u}, c_T, b) f_H(0). \quad (27)$$

For simplicity, we denote the $ij$ element of $m^u (m^d)$ as $m_{ij}^u (m_{ij}^d)$. The down-type quark mass matrix is block diagonal with $m_{11}^d = m_{22}^d$, and it can easily diagonalized by a unitary transformation $U_d$,

$$U_d = \begin{pmatrix} \cos \theta_d & -\sin \theta_d e^{i\varphi_d} & 0 \\ \sin \theta_d e^{-i\varphi_d} & \cos \theta_d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (28)$$

with

$$\tan 2\theta_d = |2m_{11}^d / m_{21}^d|, \quad \varphi_d = \arg(m_{11}^d m_{21}^*) . \quad (29)$$
The down-type quark masses are determined to be
\[ m_{d,s} = \sqrt{|m_{11}^d|^2 + |m_{21}^d|^2/2 \pm |m_{22}^d|/4} \quad m_b = |m_{33}^d|. \tag{30} \]

The resulting up-type diagonalization matrix can be parameterized as
\[ m^u m^{u\dagger} = \begin{pmatrix} |m_{11}^u|^2 & 0 & m_{10}^{u*} \\ 0 & |m_{22}^u|^2 & m_{21}^{u*} \\ m_{10}^{u*} m_{21}^{u*} & m_{21}^{u*} m_{31}^{u*} & |m_{11}^u|^2 + |m_{22}^u|^2 + |m_{33}^u|^2 \end{pmatrix}. \tag{31} \]

The down-type quark masses are determined to be
\[ m_{u_3} = \frac{(|m_{11}^u|^2 + |m_{33}^u|^2 - |m_{23}^u|^2) \pm \sqrt{(|m_{11}^u|^2 + |m_{33}^u|^2 - |m_{23}^u|^2)^2 - 4(|m_{11}^u|^2 + |m_{33}^u|^2)|m_{23}^u|^2}}{2|m_{11}^u|^2 - |m_{23}^u|^2}, \]
where
\[ 0 < \frac{2|m_{11}^u m_{33}^u|}{|m_{11}^u|^2 + |m_{33}^u|^2 - |m_{23}^u|^2}, \quad -m_{10}^{u*} m_{21}^{u*} = \frac{-m_{11}^u m_{33}^u}{m_{11}^u|^2 + |m_{33}^u|^2 + |m_{23}^u|^2 - |m_{11}^u|^2}, \quad \varphi_u = \text{arg}(m_{23}^u m_{33}^u). \tag{33} \]

We find the up-type quark mass eigenvalues are
\[ m_u \simeq |m_{11}^u| \sqrt{1 - \frac{|m_{23}^u m_{33}^u|^2}{m_{11}^u m_{33}^u}}, \quad m_{c,t} = \frac{1}{\sqrt{2}} \sqrt{X^+ \pm \sqrt{(X^-)^2 + 4|m_{11}^u m_{33}^u|^2}}, \tag{34} \]
with \[ X^+ = |m_{11}^u|^2 + |m_{33}^u|^2 + |m_{23}^u|^2 \pm |m_{21}^u|^2. \]

As a result, the quark mixing matrix is given by
\[ V_{\text{CKM}} = U_{u}^\dagger U_{d} \tag{35} \]
\[ \approx \begin{pmatrix} \cos \theta_d & e^{i\varphi_u} \sin \theta_d & \epsilon \\ -e^{-i\varphi_u} \cos \theta_d \sin \theta_u - e^{i\varphi_u} \sin \theta_d \cos \theta_u & \cos \theta_d \cos \theta_u - e^{i(\varphi_u + \varphi_d)} \sin \theta_u \cos \theta_u & -e^{i\varphi_u} \sin \theta_u \\ -e^{-i(\varphi_u + \varphi_u)} \sin \theta_d \sin \theta_u - e^{i\varphi_u} \cos \theta_d \cos \theta_u & -e^{-i\varphi_u} \cos \theta_d \sin \theta_u - e^{i\varphi_u} \cos \theta_u \cos \theta_u & \cos \theta_u \end{pmatrix}, \]
from which we can extract the expressions of CP violation phase and Jarlskog invariant in the quark sector as follows,
\[ \delta^q_{\text{CP}} = \pi - \arg(\epsilon) + \varphi_d + \varphi_u, \tag{36} \]
\[ J^q_{\text{CP}} \simeq \frac{1}{4} |\epsilon| \sin 2\theta_d \sin 2\theta_u \sin \delta^q_{\text{CP}}. \tag{37} \]

Besides, we can find that in this case $\theta_c \simeq \theta_d$. With the fact that the down quark mass matrix is block-diagonalized and it satisfies the relation $m_{11}^d = m_{22}^d$, we can obtain the celebrated Gatto-Sartori relation for the Cabibbo angle [34], i.e.
\[ \frac{m_d}{m_s} \simeq \tan^2 \theta_c. \tag{38} \]

**IV. GLOBAL FIT OF FLAVOR OBSERVABLES**

We have already discussed the predictions for the oscillation parameters in Eq. (21), shown in Figs. 2 and 3, as well as those for neutrinoless double beta decay and the quark sector prediction for the Cabibbo angle in Eq. (38). We now provide a global description of all flavor observables, including the quark and lepton mass parameters.
A. Global Flavour Fit

In our numerical analysis, we assume that the fundamental 5-D scale is \( k \simeq \Lambda \simeq M_{Pl} \), with \( M_{Pl} \simeq 2.44 \times 10^{18} \text{GeV} \) as the reduced Planck mass. In order to account for the hierarchy between the Planck and the electroweak scales we also set the scale \( \Lambda' = k e^{-kL} \simeq 1.5 \text{TeV} \). This allows for the lowest Kaluza-Klein gauge boson resonances (with masses \( m_{KK} = 3 \sim 4 \text{ TeV} \)) to be within reach of the LHC experiments. The Higgs VEV is identified with its standard model value \( v \simeq 174 \text{GeV} \), and the ratios \( v_{\varphi_1}/\Lambda, v_{\sigma_1}/\Lambda, v_{\varphi_2}/\Lambda', v_{\sigma_2}/\Lambda' \), are all fixed to 0.2, assuming real flavon VEVs. The Higgs localization parameter \( \beta \) is chosen as \( \beta = 3 \) in the following discussion. We now give a set of benchmark values for the bulk mass parameters and coupling constants of the model. In the lepton sector, we can choose

\[
c_l = 0.46, \quad c_e = -0.189, \quad c_{\mu} = -0.474, \quad c_{\tau} = -2.076, \quad |y_e| = 1, \quad |y_{\mu}| = 1, \quad |y_{\tau}| = 1,
\]

and

\[
\text{NO} : c_{\nu} = -0.153, \quad y_{\nu 1} = y_{\nu 2} = y_{\nu 3} = 1, \quad y_{\nu 4} = 0.235 + 0.0770i, \quad y_{\nu 5} = 0.340 + 0.0710i, \\
\text{IO} : c_{\nu} = -0.174, \quad y_{\nu 1} = y_{\nu 2} = y_{\nu 3} = 1, \quad y_{\nu 4} = -0.354 + 0.275i, \quad y_{\nu 5} = -0.562 + 0.270i.
\]

The resulting predictions for neutrino and charged lepton masses as well as lepton mixing parameters are given in table IV, and they reproduce very well current experimental data. For the quark sector we take

\[
c_{UC} = 0.676, \quad c_T = 1.800, \quad c_u = -0.487, \quad c_c = -0.460, \\
c_t = -2.491, \quad c_d = -0.37, \quad c_s = -0.506, \\
y_u = 0.1, \quad y_t = 0.0898, \quad y'_u = 2, \quad y'_c = 1, \quad y'_t = -3.247 - 1.974i, \\
y_{ds1} = 2, \quad y_{ds2} = 0.223 + 0.386i, \quad y'_b = 1.
\]

Thus the numerical fitted results of quark mass matrices are given by

\[
m_u = \begin{pmatrix} 0.109 & 0 & 0 \\ 0 & 0 & 7.407 \\ 29.532 & 0.589 & -145.433 - 88.418i \end{pmatrix}, \quad m_d = \begin{pmatrix} 0.0198 & 0 & 0 \\ 0.0443 + 0.0769i & 0.0198 & 0 \\ 0 & 0 & 4.180 \end{pmatrix},
\]

in GeV units. The fitted values of fermion masses and the mixing parameters are summarized in table IV. In particular the fitted CKM matrix is

\[
V_{\text{CKM}} \simeq \begin{pmatrix} 0.974 + 0.0175i & -0.0331 + 0.223i & -0.00367 \\ 0.0329 + 0.222i & 0.973 - 0.0176i & -0.0359 + 0.0219i \\ -0.00010 + 0.00879i & 0.0353 + 0.0215i & 0.999 \end{pmatrix}.
\]

The fitted value for the Jarlskog invariant is

\[
J^0_{CP} = 3.14 \times 10^{-5}.
\]

V. SUMMARY AND CONCLUSIONS

We have proposed a realistic five-dimensional warped extension of the standard model where all leptons and quarks propagate in the bulk, see Figs. 1 and 5. We have assumed a \( T' \otimes Z_3 \otimes Z_4 \) family symmetry broken on the branes by flavon fields. We have shown that it provides a consistent scenario where fermion mass hierarchies are accounted for by adequate choices of the bulk mass parameters, while quark and lepton mixing angles are restricted by the flavor symmetry. Neutrino masses are generated by the type-I seesaw mechanism, with the seesaw scale determined
| parameter | best-fit ± 1σ | Predictions |
|-----------|---------------|-------------|
| \sin^2 \theta_{12} | 0.22500±0.00100 | 0.22503 |
| \sin^2 \theta_{13} | 0.003675±0.000095 | 0.003668 |
| \sin^2 \theta_{23} | 0.04200±0.00059 | 0.04205 |
| \delta_{CP}' | 66.9±2 | 68.2 |
| \mu_\nu [MeV] | 2.16^{+0.49}_{-0.26} | 2.16 |
| \mu_e [GeV] | 1.27±0.02 | 1.27 |
| \mu_\tau [GeV] | 172.9±0.4 | 172.90 |
| \mu_d [MeV] | 4.6^{+19.48}_{-17.17} | 4.21 |
| \mu_s [MeV] | 93^{+11}_{-5} | 93.00 |
| \mu_\tau [GeV] | 4.18^{+0.03}_{-0.02} | 4.18 |
| \sin^2 \theta_{12}/10^{-1} (NO) | 3.20^{+0.20}_{-0.16} | 3.19 |
| \sin^2 \theta_{12}/10^{-1} (IO) | 3.20^{+0.20}_{-0.16} | 3.18 |
| \sin^2 \theta_{23}/10^{-1} (NO) | 5.4^{+10.20}_{-6.30} | 5.47 |
| \sin^2 \theta_{23}/10^{-1} (IO) | 5.5^{+10.18}_{-6.30} | 5.51 |
| \sin^2 \theta_{13}/10^{-2} (NO) | 2.160^{+0.083}_{-0.069} | 2.160 |
| \sin^2 \theta_{13}/10^{-2} (IO) | 2.220^{+0.074}_{-0.076} | 2.220 |
| \delta_{CP}/\pi (NO) | 1.33^{+0.21}_{-0.15} | 1.567 |
| \delta_{CP}/\pi (IO) | 1.56^{+0.13}_{-0.15} | 1.571 |
| \mu_\nu [MeV] | 0.511 ± 3.1 × 10^{-9} | 0.511 |
| m_\mu [MeV] | 105.658 ± 2.4 × 10^{-6} | 105.658 |
| m_\tau [MeV] | 1776.86 ± 0.12 | 1776.86 |
| \Delta m^2_{21} [10^{-5}eV^2] (NO) | 7.55^{+0.20}_{-0.16} | 7.55 |
| \Delta m^2_{21} [10^{-5}eV^2] (IO) | 7.55^{+0.20}_{-0.16} | 7.55 |
| |\Delta m^2_{31} [10^{-3}eV^2] (NO) | 2.50±0.03 | 2.50 |
| |\Delta m^2_{31} [10^{-3}eV^2] (IO) | 2.42^{+0.03}_{-0.04} | 2.42 |
| \chi^2 (NO) | 7.65 | 7.65 |
| \chi^2 (IO) | 7.66 | 7.66 |

TABLE IV. Global warped flavordynamics fit: the neutrino oscillation parameters are taken from the global analysis in [3], while the quark parameters are obtained from the PDG [35].

by the cut-off scale at the IR brane, generally suppressed by the exponential warp factor $e^{-kL}$ with respect to $\Lambda$, the fundamental UV scale. This naturally accounts for the large values required for the seesaw mechanism. Neutrino mixing parameters and the Dirac CP violation phase are all described in terms of just two independent parameters. The resulting predictions for the neutrino oscillation parameters are summarized in Figs. 2 and 3. In addition to these oscillation results we predict a $0\nu\beta\beta$ decay rate within reach of the upcoming generation of experiments, as seen in Fig. 4. Our scheme also provides a good description of the quark sector, as seen in table IV and Eqs. 44 and 45, recovering the successful Gatto-Sartori relation.

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APPENDIX

Appendix A: Group Theory of $T'$

The $T'$ group is the double covering of the tetrahedral group $A_4$. It has 24 elements which can be generated by three generators $S$ and $T$ and $R$ obeying the relations\(^3\),

$$S^4 = (ST)^3 = T^3 = 1, \quad S^2T = TS^2. \quad (A.1)$$

The $T'$ group has seven inequivalent irreducible representations: three singlets $1$, $1'$, and $1''$, three doublets $2$, $2'$ and $2''$, and one triplet $3$. The representations $1'$, $1''$ and $2$, $2''$ are complex conjugated to each other respectively. The two-dimensional representations $2$, $2'$ and $2''$ are faithful representations of $T'$ group while the odd dimensional representations $1$, $1'$, $1''$ and $3$ coincide with those of $A_4$. In the present work we shall adopt the basis of [36, 37].

For the singlet representations, we have

$$1 : S = T = 1, \quad 1 : S = 1, \quad T = \omega, \quad 1'' : S = 1, \quad T = \omega^2, \quad (A.2)$$

with $\omega = e^{i2\pi/3}$. In the doublet representations, the generators $S$ and $T$ are given by

$$2 : S = -\frac{1}{\sqrt{3}} \begin{pmatrix} i & \sqrt{2}e^{i\pi/12} \\ -\sqrt{2}e^{-i\pi/12} & -i \end{pmatrix}, \quad T = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix},$$

$$2' : S = -\frac{1}{\sqrt{3}} \begin{pmatrix} i & \sqrt{2}e^{i\pi/12} \\ -\sqrt{2}e^{-i\pi/12} & -i \end{pmatrix}, \quad T = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix},$$

$$2'' : S = -\frac{1}{\sqrt{3}} \begin{pmatrix} i & \sqrt{2}e^{i\pi/12} \\ -\sqrt{2}e^{-i\pi/12} & -i \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \end{pmatrix}. \quad (A.3)$$

For the triplet representation $3$, the generators are

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \quad (A.4)$$

Notice that due to the choice of complex representation matrices for the real representation $3$ the conjugate $a^*$ of $a \sim 3$ does not transform as $3$, but rather ($a_1^*, a_2^*, a_3^*$) transforms as triplet under $T'$. The reason for this is that $T^* = U_3^T T U_3$ and $S^* = U_3^T S U_3 = S$ where $U_3$ is the permutation matrix which exchanges the 2nd and 3rd row and column. Similarly, notice that the irreducible representations $2$ and $2''$ are complex conjugated to each other by a unitary transformation $U_2$ with

$$U_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

i.e, $T_2^* = U_2^T T_2 U_2$ and $S_2^* = U_2^T S_2 U_2$. Besides, the real doublet representation $2'$ and its complex conjugation are also related by the unitary transformation $U_2$, i.e, $T_2^* = U_2^T T_2 U_2$ and $S_2^* = U_2^T S_2 U_2$. Thus we have

$$\begin{align*}
    b = (b_1, b_2)^T \sim 2, & \quad \rightarrow \quad (-b_1^*, b_2^*)^T \sim 2'' \\
    b = (b_1, b_2)^T \sim 2', & \quad \rightarrow \quad (-b_2^*, b_1^*)^T \sim 2 \\
    b = (b_1, b_2)^T \sim 2, & \quad \rightarrow \quad (-b_2^*, b_1^*)^T \sim 2'.
\end{align*} \quad (A.5)$$

\(^3\) The $T'$ group can also be equivalently expressed in terms of three generators $S$, $T$ and $R$ with $S^2 = R$, $RT = TR$ and $(ST)^3 = T^3 = R^2 = 1$[16, 18, 36, 37].
In the following, we collect the Clebsch-Gordan coefficients for the decomposition of product representations in our basis, all the results are taken from [36, 37]. We use $\alpha_i$ to indicate the elements of the first representation of the product, $\beta_i$ to indicate those of the second representation. For convenience, we shall denote $1 \equiv 1^0$, $1' \equiv 1^1$, $1'' \equiv 1^2$ for singlet representations and $2 \equiv 2^0$, $2' \equiv 2^1$, $2'' \equiv 2^2$ for the doublet representations.

The contraction rules involving singlets representations in the product are as follows,

\[
1^a \otimes 1^b = 1^{a+b \ (\text{mod} \ 3)} \sim \alpha \beta, \tag{A.6}
\]
\[
1^a \otimes 2^b = 2^{a+b \ (\text{mod} \ 3)} \sim \begin{pmatrix} \alpha \beta_2 \\ \alpha \beta_1 \end{pmatrix}, \tag{A.7}
\]
\[
1' \otimes 3 = 3 \sim \begin{pmatrix} \alpha \beta_3 \\ \alpha \beta_1 \\ \alpha \beta_2 \end{pmatrix}, \tag{A.8}
\]
\[
1'' \otimes 3 = 3 \sim \begin{pmatrix} \alpha \beta_2 \\ \alpha \beta_3 \\ \alpha \beta_1 \end{pmatrix}, \tag{A.9}
\]

where $a, b = 0, 1, 2$. The contraction rules for the products of two doublet representations are

\[
2 \otimes 2' = 2' \otimes 2'' = 3 \oplus 1' \quad \text{with} \quad \begin{cases} 
1' \sim \alpha \beta_2 - \alpha \beta_1 \\
3 \sim \frac{1}{\sqrt{2}} e^{i\pi/6} \alpha \beta_2 \\
1'' \sim \frac{1}{\sqrt{2}} e^{i\pi/12} (\alpha \beta_2 + \alpha \beta_1) \\
\alpha \beta_1 
\end{cases} \tag{A.10}
\]
\[
2 \otimes 2'' = 2'' \otimes 2' = 3 \oplus 1'' \quad \text{with} \quad \begin{cases} 
1 \sim \alpha \beta_2 - \alpha \beta_1 \\
3 \sim \frac{1}{\sqrt{2}} e^{i\pi/6} \alpha \beta_2 \\
1' \sim \frac{1}{\sqrt{2}} e^{i\pi/12} (\alpha \beta_2 + \alpha \beta_1) \\
\alpha \beta_1 
\end{cases} \tag{A.11}
\]
\[
2 \otimes 2' = 2' \otimes 2' = 3 \oplus 1 \quad \text{with} \quad \begin{cases} 
1 \sim \alpha \beta_2 - \alpha \beta_1 \\
3 \sim \frac{1}{\sqrt{2}} e^{i\pi/6} \alpha \beta_2 \\
1' \sim \frac{1}{\sqrt{2}} e^{i\pi/12} (\alpha \beta_2 + \alpha \beta_1) \\
\alpha \beta_1 
\end{cases} \tag{A.12}
\]
The products of doublet and triplet representations are decomposed as follows,

\[ 2 \otimes 3 = 2 \oplus 2' \oplus 2'' \quad \text{with} \quad 2 \sim \begin{cases} \alpha_1 \beta_1 & -\alpha_2 \beta_1 \\ \alpha_1 \beta_2 & -\alpha_2 \beta_2 \\ \alpha_1 \beta_3 & -\alpha_2 \beta_3 \end{cases}, \quad 2' \sim \begin{cases} \alpha_1 \beta_1 & -\alpha_2 \beta_1 \\ \alpha_1 \beta_2 & -\alpha_2 \beta_2 \\ \alpha_1 \beta_3 & -\alpha_2 \beta_3 \end{cases}, \quad 2'' \sim \begin{cases} \alpha_1 \beta_1 & -\alpha_2 \beta_1 \\ \alpha_1 \beta_2 & -\alpha_2 \beta_2 \\ \alpha_1 \beta_3 & -\alpha_2 \beta_3 \end{cases} \]

(A.14)

\[ 2' \otimes 3 = 2 \oplus 2' \oplus 2'' \quad \text{with} \quad 2 \sim \begin{cases} \alpha_1 \beta_2 & -\alpha_2 \beta_1 \\ \alpha_1 \beta_3 & -\alpha_2 \beta_3 \end{cases}, \quad 2' \sim \begin{cases} \alpha_1 \beta_1 & -\alpha_2 \beta_1 \\ \alpha_1 \beta_2 & -\alpha_2 \beta_2 \\ \alpha_1 \beta_3 & -\alpha_2 \beta_3 \end{cases}, \quad 2'' \sim \begin{cases} \alpha_1 \beta_1 & -\alpha_2 \beta_1 \\ \alpha_1 \beta_2 & -\alpha_2 \beta_2 \\ \alpha_1 \beta_3 & -\alpha_2 \beta_3 \end{cases} \]

(A.15)

\[ 2'' \otimes 3 = 2 \oplus 2' \oplus 2'' \quad \text{with} \quad 2 \sim \begin{cases} \alpha_1 \beta_1 & -\alpha_2 \beta_1 \\ \alpha_1 \beta_2 & -\alpha_2 \beta_2 \\ \alpha_1 \beta_3 & -\alpha_2 \beta_3 \end{cases}, \quad 2' \sim \begin{cases} \alpha_1 \beta_1 & -\alpha_2 \beta_1 \\ \alpha_1 \beta_2 & -\alpha_2 \beta_2 \\ \alpha_1 \beta_3 & -\alpha_2 \beta_3 \end{cases}, \quad 2'' \sim \begin{cases} \alpha_1 \beta_1 & -\alpha_2 \beta_1 \\ \alpha_1 \beta_2 & -\alpha_2 \beta_2 \\ \alpha_1 \beta_3 & -\alpha_2 \beta_3 \end{cases} \]

(A.16)

Finally the contractions of two triplets are given by

\[ 3 \otimes 3 = 3_S \oplus 3_A \oplus 1 \oplus 1' \oplus 1'' \quad \text{with} \quad 3_S \sim \begin{cases} 2 \alpha_1 \beta_1 - \alpha_2 \beta_3 & - \alpha_3 \beta_2 \\ 2 \alpha_3 \beta_3 - \alpha_1 \beta_1 - \alpha_2 \beta_1 \\ 2 \alpha_2 \beta_2 - \alpha_1 \beta_3 - \alpha_3 \beta_1 \end{cases}, \quad 3_A \sim \begin{cases} \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ \alpha_1 \beta_2 - \alpha_2 \beta_1 \\ \alpha_3 \beta_1 - \alpha_1 \beta_3 \end{cases}, \quad 1 \sim \alpha_1 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_2, \quad 1' \sim \alpha_3 \beta_3 + \alpha_1 \beta_2 + \alpha_2 \beta_1, \quad 1'' \sim \alpha_2 \beta_2 + \alpha_1 \beta_3 + \alpha_3 \beta_1 \]

(A.18)

Appendix B: 5-D profiles of Higgs and fermion fields

We formulate our model in the framework of Randall-Sundrum model [12], assuming the bulk of our model to be a slice of AdS$_5$ with curvature radius $1/k$. The extra dimension $y$ is compactified, and the two 3-branes with opposite tension are located at $y = 0$, the UV brane, and $y = L$, the IR brane. The bulk metric is non-factorizable,

\[ ds^2 = e^{-2ky} \eta_{\mu \nu} dx^\mu dx^\nu - dy^2. \quad \text{(B.1)} \]

We have assumed the Higgs field to be in the bulk, so it has the standard Kaluza-Klein decomposition as

\[ H(x^\mu, y) = H(x^\mu) \frac{f_H(y)}{\sqrt{L}} + \text{heavy KK Modes}. \quad \text{(B.2)} \]
where $f_H(y)$ is the zero mode profile. In this paper we adopt the zero mode approximation (ZMA) which identifies standard model fields with zero modes of corresponding 5-D fields. As in Ref. [38] we take the profile $f_H(y)$ to be of the form

$$f_H(y) = \sqrt{\frac{2kL(1-\beta)}{1-e^{-2(1-\beta)kL}}} e^{kL e^{(2-\beta)k(y-L)}}, \quad \text{(B.3)}$$

with $\beta = \sqrt{4 + m_H^2/k^2}$ where $m_H$ is the bulk mass parameter of the Higgs field. For 5-D fermion fields, the three families of leptons and quarks are given as

$$\text{leptons : } \Psi_{Li} = \left( \frac{\nu^{[++]}}{e^{[++]}} \right), \quad \Psi_{ei} = e^{[-]}, \quad \Psi_{\nu i} = \nu^{[-]}, \quad \text{(B.4)}$$

$$\text{quarks : } \Psi_{Qi} = \left( \frac{u^{[++]}}{d^{[++]}} \right), \quad \Psi_{ui} = u^{[-]}, \quad \Psi_{di} = d^{[-]}.$$  

where the two signs in the bracket indicate Neumann (+) or Dirichlet (−) BCs for the left-handed component of the corresponding field on UV and IR branes respectively. The Kaluza-Klein decomposition for the two different BCs are

$$\Psi^{[++]}(x^\mu, y) = e^{2ky/\sqrt{L}} \left\{ \psi_L(x^\mu) f_L(y, c_L) + \text{heavy KK modes} \right\}, \quad \text{(B.5)}$$

$$\Psi^{[-]}(x^\mu, y) = e^{2ky/\sqrt{L}} \left\{ \psi_R(x^\mu) f_R(y, c_R) + \text{heavy KK modes} \right\}, \quad \text{(B.6)}$$

with $\psi = \nu, e, u, d$. The zero modes 5-D fields with [++] BCs only have left-handed zero modes, and those with [++] BCs only have right-handed zero modes. $f_L(y, c_L)$ and $f_R(y, c_R)$ are the zero mode profiles [39–41]

$$f_L(y, c_L) = \sqrt{\frac{(1-2c_L)kL}{e^{1-2c_L}kL-1}} e^{-c_L ky}, \quad f_R(y, c_R) = \sqrt{\frac{(1+2c_R)kL}{e^{1+2c_R}kL-1}} e^{c_R ky}, \quad \text{(B.7)}$$

where $c_L$ and $c_R$ are the bulk mass parameters of the 5-D fermion fields in units of $k$.

### Appendix C: Vacuum Alignment

In this section, we will investigate the vacuum alignment of the flavon fields $\varphi_i$, $\sigma_1$, $\varphi_\nu$ and $\sigma_\nu$. At the UV brane $y = 0$, the scalar potential invariant under the flavor symmetry $T'$ and the auxiliary symmetries takes the following form

$$V_{\text{UV}} = M_2^2 \varphi_1 \varphi_1^* \varphi_1 + M_2^2 \sigma_1 \sigma_1^* \varphi_1 + f_1 \varphi_1 \varphi_1 + f_2 \varphi_1 \varphi_1 + f_3 \varphi_1 \varphi_1 + f_4 \varphi_1 \varphi_1 + f_5 \varphi_1 \varphi_1 + f_6 \varphi_1 \varphi_1 + f_7 \varphi_1 \varphi_1 + f_8 \varphi_1 \varphi_1 + f_9 \varphi_1 \varphi_1 + f_{10} \varphi_1 \varphi_1 + f_{11} \varphi_1 \varphi_1 + f_{12} \varphi_1 \varphi_1,$$

where the parameters $M_2^2, M_2^2, f_1, f_2, f_3$ and $f_4$ are real free parameters. For the field configuration

$$\langle \varphi_i \rangle = (1, 0) \nu, \quad \langle \sigma_1 \rangle = v_\sigma,$$

the minimum conditions of the UV potential read

$$\frac{\partial V_{\text{UV}}}{\partial \varphi_{i1}} = v_{\varphi_1} (M_2^2 + 2 f_1 v_{\varphi_1} v_{\varphi_1}^* + f_4 v_{\sigma_1} v_{\sigma_1}^*) = 0,$$

$$\frac{\partial V_{\text{UV}}}{\partial \varphi_{i2}} = 0,$$

$$\frac{\partial V_{\text{UV}}}{\partial \sigma_1^*} = v_{\sigma_1} (M_2^2 + 2 f_1 v_{\sigma_1} v_{\sigma_1}^* + f_4 v_{\varphi_1} v_{\varphi_1}^*) = 0.$$
and the solution is

$$|v_{\nu_1}|^2 = \frac{M_{f_2}^2 f_4 - 2M_{f_3}^2 f_3}{4f_1 f_3 - f_4^2}, \quad |v_{\sigma_1}|^2 = \frac{M_{f_2}^2 f_4 - 2M_{f_1}^2 f_1}{4f_1 f_3 - f_4^2}. \quad \text{(C.4)}$$

At the IR brane $y = L$, the most general renormalizable scalar potential $V_{IR}$ involving the flavon fields $\varphi_\nu$, $\sigma_\nu$ and $\sigma_\nu^*$ is given as

$$V_{IR} = M_1 (\varphi_\nu \varphi_\nu^*)_1 + M_2 (\sigma_\nu \sigma_\nu^*)_1 + g_1 [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu^* \varphi_\nu^*)_1]_1 + g_2 [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu^* \varphi_\nu^*)_1]_1$$

$$+ g_3 [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu^* \varphi_\nu^*)_1]_1 + g_4 [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu^* \varphi_\nu^*)_1]_1 + g_5 [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu^* \varphi_\nu^*)_1]_1 + g_6 \sigma_\nu^2 \sigma_\nu^*$$

$$+ (M_3 (\varphi_\nu \varphi_\nu)_1 + M_4 (\sigma_\nu \sigma_\nu)_1) + g_7 [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_8 [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_9 [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1$$

$$+ g_{10} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_{11} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_{12} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_{13} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1$$

$$+ g_{14} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_{15} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_{16} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_{17} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1$$

$$+ g_{18} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_{19} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_{20} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_{21} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1$$

$$+ g_{22} [(\varphi_\nu \varphi_\nu)_1 (\varphi_\nu \varphi_\nu)_1]_1 + g_{23} \sigma_\nu^4 + g_{24} \sigma_\nu^2 \sigma_\nu^* + h.c). \quad \text{(C.5)}$$

where the coupling parameters $M_{1,2}$ and $g_{1,2,3,4,5,6}$ are real, while the remaining coupling parameters are complex. For the desired flavon vacuum alignments

$$\langle \varphi_\nu \rangle = (1, -2\omega^2, -2\omega)v_{\varphi_\nu}, \quad \langle \sigma_\nu \rangle = v_{\sigma_\nu},$$

the minimization conditions read

$$\frac{\partial V_{IR}}{\partial \varphi_\nu^*} = A_1 = 0,$$

$$\frac{\partial V_{IR}}{\partial \varphi_\nu^*} = -2\omega^2 A_1 = 0,$$

$$\frac{\partial V_{IR}}{\partial \varphi_\nu^*} = -2\omega A_1 = 0,$$

$$\frac{\partial V_{IR}}{\partial \sigma_\nu^*} = A_2 = 0, \quad \text{(C.7)}$$

with

$$A_1 = 9(g_{10} + 4g_{13})v_{\varphi_\nu}^3 + 18(g_{19} + 4g_1)v_{\varphi_\nu}^2 - 6(g_{17}v_{\varphi_\nu} + g_{18}v_{\sigma_\nu})v_{\varphi_\nu} + 27(g_{10} + 4g_1)v_{\varphi_\nu}^2 v_{\sigma_\nu}^*$$

$$-12(g_{15}v_{\varphi_\nu} + g_{17}v_{\sigma_\nu}^*)v_{\varphi_\nu}^4 + (g_{22}v_{\varphi_\nu}^2 + g_{35}v_{\varphi_\nu}^2 + g_{22}v_{\varphi_\nu}^2 v_{\sigma_\nu} + v_{\varphi_\nu}^2 v_{\sigma_\nu}^* + 36(g_{15} + 4g_1)v_{\varphi_\nu}^4$$

$$-18(g_{16}v_{\varphi_\nu} + g_{15}v_{\varphi_\nu}^*)v_{\varphi_\nu}^2 + 2(g_{21}v_{\varphi_\nu} + g_{36}v_{\varphi_\nu}^*)v_{\varphi_\nu}^2 + 15v_{\varphi_\nu}^2 v_{\sigma_\nu} + 2M_1 v_{\varphi_\nu} + 2M_2 v_{\sigma_\nu} \quad \text{(C.8)}$$

$$A_2 = -54g_{15}v_{\varphi_\nu} + 9g_{20}v_{\varphi_\nu}^2 + 9(-g_{18}v_{\varphi_\nu} + g_{19}v_{\varphi_\nu} + 2g_{22}v_{\varphi_\nu}^*)v_{\varphi_\nu}^2 - 54g_{15}v_{\varphi_\nu}^3$$

$$+ 9(-g_{17}v_{\varphi_\nu} + g_{20}v_{\varphi_\nu} + 2g_{19}v_{\varphi_\nu}^*)v_{\varphi_\nu}^2 + 3g_{24}v_{\varphi_\nu}^2 v_{\sigma_\nu}^* + 4g_{23}v_{\varphi_\nu}^2 v_{\sigma_\nu}^* + g_{24}v_{\varphi_\nu}^3$$

$$+ 2g_{6}v_{\varphi_\nu}^2 v_{\sigma_\nu}^* + M_2 v_{\varphi_\nu} + 18g_{21}v_{\varphi_\nu}^2 v_{\sigma_\nu}^* + 2M_2 v_{\sigma_\nu} \quad \text{(C.9)}$$

One sees that the assumed vacuum alignment of the flavon fields can, indeed, be achieved within certain regions of parameter space.

[1] A. B. McDonald, “Nobel Lecture: The Sudbury Neutrino Observatory: Observation of flavor change for solar neutrinos,” Rev. Mod. Phys. 88 no. 3, (2016) 030502.

[2] T. Kajita, “Nobel Lecture: Discovery of atmospheric neutrino oscillations,” Rev. Mod. Phys. 88 no. 3, (2016) 030501.

[3] P. de Salas et al., “Status of neutrino oscillations 2018: 3σ hint for normal mass ordering and improved CP sensitivity,” Phys.Lett. B782 (2018) 633–640, arXiv:1708.01186 [hep-ph].
[4] G. Altarelli and F. Feruglio, “Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions,” Nucl.Phys. B720 (2005) 64–88.

[5] F. J. de Anda and S. F. King, “An $S_4 \times SU(5)$ SUSY GUT of flavour in 6d,” JHEP 1807 (2018) 057, arXiv:1803.04978 [hep-ph].

[6] F. J. de Anda and S. F. King, “$SU(3) \times SO(10)$ in 6d,” JHEP 1810 (2018) 128, arXiv:1807.07078 [hep-ph].

[7] F. J. de Anda, J. W. F. Valle, and C. A. Vaquera-Araujo, “Flavour and CP predictions from orbifold compactification,” Phys.Lett. B801 (2020) 135195, arXiv:1910.05605 [hep-ph].

[8] S. Morisi et al., “Relating quarks and leptons without grand-unification,” Phys.Rev. D84 (2011) 036003, arXiv:1104.1633 [hep-ph].

[9] S. King et al., “Quark-Lepton Mass Relation in a Realistic $A_4$ Extension of the Standard Model,” Phys.Lett. B724 (2013) 68–72, arXiv:1301.7065 [hep-ph].

[10] S. Morisi et al., “Quark-Lepton Mass Relation and CKM mixing in an $A_4$ Extension of the Minimal Supersymmetric Standard Model,” Phys.Rev. D88 (2013) 036001, arXiv:1303.4394 [hep-ph].

[11] A. Aranda, C. D. Carone, and R. F. Lebed, “$U(2)$ flavor physics without $U(2)$ symmetry,” Phys.Lett. B747 (2015) 99–106, arXiv:1411.4883 [hep-ph].

[12] L. Randall and R. Sundrum, “A Large mass hierarchy from a small extra dimension,” Phys.Rev.Lett. 83 (1999) 3370–3373.

[13] P. Chen et al., “Warped flavor symmetry predictions for neutrino physics,” JHEP 1601 (2016) 007, arXiv:1509.06683 [hep-ph].

[14] A. Aranda, C. D. Carone, and R. F. Lebed, “$U(2)$ flavor physics without $U(2)$ symmetry,” Phys.Lett. B474 (2000) 170–176.

[15] A. Aranda, C. D. Carone, and R. F. Lebed, “Maximal neutrino mixing from a minimal flavor symmetry,” Phys.Rev. D62 (2000) 016009.

[16] F. Feruglio, C. Hagedorn, Y. Lin, and L. Merlo, “Tri-bimaximal Neutrino Mixing and Quark Masses from a Discrete Flavour Symmetry,” Nucl.Phys. B775 (2007) 120–142.

[17] M.-C. Chen and K. Mahanthappa, “$CKM$ and Tri-bimaximal MNS Matrices in a $SU(5) \times T'$ Model,” Phys.Lett. B652 (2007) 34–39, arXiv:0705.0714 [hep-ph].

[18] G.-J. Ding, “Fermion Mass Hierarchies and Flavor Mixing from $T'$ Symmetry,” Phys.Rev. D78 (2008) 036011, arXiv:0803.2278 [hep-ph].

[19] P. H. Frampton, T. W. Kephart, and S. Matsuzaki, “Simplified Renormalizable $T'$-prime Model for Tribimaximal Mixing and Cabibbo Angle,” Phys.Rev. D78 (2008) 073004, arXiv:0807.4713 [hep-ph].

[20] M.-C. Chen, K. Mahanthappa, and F. Yu, “A Viable Randall-Sundrum Model for Quarks and Leptons with $T'$ Family Symmetry,” Phys.Rev. D81 (2010) 036004, arXiv:0907.3963 [hep-ph].

[21] A. Meroni, S. Petcov, and M. Spinrath, “A SUSY $SU(5) \times T'$ Unified Model of Flavour with large $\theta_{13}$,” Phys.Rev. D86 (2012) 113003, arXiv:1205.5241 [hep-ph].

[22] I. Girardi, A. Meroni, S. Petcov, and M. Spinrath, “Generalised geometrical CP violation in a $T'$ lepton flavour model,” JHEP 1402 (2014) 056, arXiv:1312.1966 [hep-ph].

[23] J. Schechter and J. W. F. Valle, “Neutrino Masses in SU(2)$\times$U(1) Theories,” Phys. Rev D22 (1980) 2227.

[24] J. Schechter and J. W. F. Valle, “Neutrino Decay and Spontaneous Violation of Lepton Number,” Phys.Rev. D25 (1982) 774.

[25] Planck Collaboration, N. Aghanim et al., “Planck 2018 results. VI. Cosmological parameters,” arXiv:1807.06209 [astro-ph.CO].

[26] KamLAND-Zen Collaboration, A. Gando et al., “Search for Majorana Neutrinos near the Inverted Mass Hierarchy Region with KamLAND-Zen,” Phys.Rev.Lett. 117 (2016) 082503, arXiv:1605.02889 [hep-ex].

[27] CUORE Collaboration, C. Alduino et al., “First Results from CUORE: A Search for Lepton Number Violation via $0\nu\beta\beta$ Decay of $^{136}Te$,” Phys.Rev.Lett. 120 (2018) 132501, arXiv:1710.07988 [nucl-ex].

[28] EXO Collaboration, J. Albert et al., “Search for Neutrinoless Double-Beta Decay with the Upgraded EXO-200 Detector,” Phys.Rev.Lett. 120 (2018) 072501, arXiv:1707.08707 [hep-ex].

[29] GERDA Collaboration, M. Agostini et al., “Improved Limit on Neutrinoless Double-\(\beta\) Decay of \(^{76}\)Ge from GERDA Phase II,” Phys.Rev.Lett. 120 (2018) 132503, arXiv:1803.11100 [nucl-ex].

[30] SNO+ Collaboration, S. Andringa et al., “Current Status and Future Prospects of the SNO+ Experiment,” Adv.High
Energy Phys. 2016 (2016) 6194250, arXiv:1508.05759 [physics.ins-det].

[31] LEGEND Collaboration, N. Abgrall et al., “The Large Enriched Germanium Experiment for Neutrinoless Double Beta Decay (LEGEND),” vol. 1894, p. 020027. 2017. arXiv:1709.01980 [physics.ins-det].

[32] nEXO Collaboration, J. Albert et al., “Sensitivity and Discovery Potential of nEXO to Neutrinoless Double Beta Decay,” Phys.Rev. C97 (2018) 065503, arXiv:1710.05075 [nucl-ex].

[33] KATRIN Collaboration, M. Aker et al., “An improved upper limit on the neutrino mass from a direct kinematic method by KATRIN,” Phys. Rev. Lett. 123 (Nov, 2019) 221802. https://link.aps.org/doi/10.1103/PhysRevLett.123.221802.

[34] R. Gatto, G. Sartori, and M. Tonin, “Weak Selfmasses, Cabibbo Angle, and Broken SU(2) x SU(2),” Phys.Lett. B28 (1968) 128–130.

[35] Particle Data Group Collaboration, M. Tanabashi et al., “Review of Particle Physics,” Phys.Rev. D98 (2018) 030001.

[36] X.-G. Liu and G.-J. Ding, “Neutrino Masses and Mixing from Double Covering of Finite Modular Groups,” JHEP 1908 (2019) 134, arXiv:1907.01488 [hep-ph].

[37] J.-N. Lu, X.-G. Liu, and G.-J. Ding, “Modular symmetry origin of texture zeros and quark lepton unification,” arXiv:1912.07573 [hep-ph].

[38] G. Cacciapaglia, C. Csaki, G. Marandella, and J. Terning, “The Gaugephobic Higgs,” JHEP 0702 (2007) 036.

[39] T. Gherghetta and A. Pomarol, “Bulk fields and supersymmetry in a slice of AdS,” Nucl.Phys. B586 (2000) 141–162.

[40] Y. Grossman and M. Neubert, “Neutrino masses and mixings in nonfactorizable geometry,” Phys.Lett. B474 (2000) 361–371.

[41] S. J. Huber and Q. Shafi, “Neutrino oscillations and rare processes in models with a small extra dimension,” Phys.Lett. B512 (2001) 365–372.