Pull-Based Distributed Event-triggered Consensus for Multi-agent Systems with Directed Topologies

Xinlei Yi, Wenlian Lu and Tianping Chen

Abstract—This paper mainly investigates consensus problem with pull-based event-triggered feedback control. For each agent, the diffusion coupling feedbacks are based on the states of its in-neighbors at its latest triggering time and the next triggering time of this agent is determined by its in-neighbors’ information as well. The general directed topologies, including irreducible and reducible cases, are investigated. The scenario of distributed continuous monitoring is considered firstly, namely each agent can observe its in-neighbors’ continuous states. It is proved that if the network topology has a spanning tree, then the event-triggered coupling strategy can realize consensus for the multi-agent system. Then the results are extended to discontinuous monitoring, i.e., self-triggered control, where each agent computes its next triggering time in advance without having to observe the system’s states continuously. The effectiveness of the theoretical results are illustrated by a numerical example finally.

Keywords: Directed, irreducible and reducible, consensus, multi-agent systems, event-triggered, self-triggered.

I. INTRODUCTION

Consensus problem in multi-agent systems has been widely and deeply investigated. The basic idea of consensus lies in that each agent updates its state based on its own state and the states of its neighbors in such a way that the final states of all agents converge to a common value [3]. The model normally is of the following form:

\[ \dot{x}(t) = -Lx(t), \quad (1) \]

where the column vector \( x(t) \) consists of all nodes’ states and \( L \) is the corresponding weighted Laplacian matrix. There are many results reported in this field [3–8] and the references therein. In these researches, the network topologies vary from fixed topologies to stochastically switching topologies, and the most basic condition to realize a consensus is that the underlying graph of the network system has a spanning tree [3].

In recent years, with the development of sensing, communications, and computing equipment, event-triggered control [9–17] and self-triggered control [18–24] have been proposed and studied. Instead of using the continuous state to realize a consensus, the control in event-triggered control strategy is piecewise constant between the triggering times which need been determined. The event-triggered control strategy can be found in early papers [11] and [2]. The key point in event-triggered control is how to design the event-triggered controller and determine the corresponding triggering times. Self-triggered control is a natural extension of the event-triggered control since the derivative of the concern multi-agent system’s state is piecewise constant, which is very easy to work out solutions (agents’ states) of the system. Specifically, each agent predicts its next triggering time at the previous one. In [11], [9], the triggering times are determined when a certain error becomes large enough with respect to the norm of the state. In [14], under the condition that the graph is undirected and strongly connected, the authors provide event-triggered and self-triggered approaches in both centralized and distributed formulations. It should be emphasized that the approaches cannot be applied to directed graph. In [15], the authors investigate the average-consensus problem of multi-agent systems with directed and weighted topologies, but they need an additional assumption that the directed topology must be balanced. In [17], the authors propose a new combinational measurement approach to event design, which will be used in this paper.

In this paper, continuing with previous works, we study event-triggered and self-triggered consensus in multi-agent system with directed, reducible (irreducible) and weighted topology.

Consider the following continuous-time linear multi-agent system with discontinuous diffusions as follows

\[
\begin{align*}
\dot{x}_i(t) &= u_i(t) \\
u_i(t) &= -\sum_{j=1}^{m} L_{ij} x_j(t_k(t_i)), i = 1, \cdots, m
\end{align*}
\]

(2)

The increasing triggering event time sequence \( \{t_k\}_{k=1}^{\infty} \) (to be defined) are agent-wise and \( t_0 = 0 \), for all \( i \in I \), where \( I = \{1, 2, \cdots, m\} \). At each \( t \), each agent \( v_i \) “pulls” its in-neighbours’ states with respect to an identical time point \( t_k(t_i) \) with

\[ k(t_i) = \arg\max_{k'} \{ t_k(t_i) \leq t \}. \]

We highlight the basic idea behind the setup of the coupling term above as follows. Instead of using the continuous states from the neighbours to realize a consensus, which have many drawbacks as mentioned above, an alternative for the agent \( v_i \) is to pull its in-neighbours’ constant states at the nearest time point \( t_k(t_i) \) until some pre-defined event is triggered at time \( t_{k+1}(t_i) \), then after getting information from its in-neighbors, agent \( i \) updates its state at \( t_{k+1}(t_i) \) until the next event is triggered, and so on. We will show that the event is determined only by

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its in-neighbors’ states. This process goes on each agent in a parallel fashion.

Let us recall the model

\[ x^i(t+1) = f(x^i(t)) + c_i \sum_{j=1}^m a_{ij}(f(x^j(t))) \]

where \( \dot{s}(t) = f(s(t)) \) is a chaotic oscillator. It was proposed and investigated in [1] for synchronization of chaotic systems. It can also be considered as nonlinear consensus model.

As a special case, let \( f(x(t)) = x(t) \) and \( c_i = (t^i_{k+1} - t^i_k) \), then

\[ x^i(t^i_{k+1}) = x^i(t^i_k) + (t^i_{k+1} - t^i_k) \sum_{j=1}^m a_{ij} x^j(t^j_k) \]

which is just the event triggering (distributed) model for consensus problem, though the term “event triggering” was not used. In centralized control, the bound for \((t^i_{k+1} - t^i_k) = (t^i_k - t^i_k)\) to reach synchronization was given in that paper when the coupling graph is indirected (or in [2] for direct graph), too.

In this paper, the distributed continuous monitoring with pull-based feedback as the event-triggered controller is considered firstly, namely agent can observe its in-neighbors’ continuous states. This event-triggered principle is named as pull-based event-triggered principle. It is proved that if the directed network topology is irreducible, then the pull-based event-triggered coupling strategy can realize consensus for the multi-agent system. Then we generalize it to the irreducible case. By mathematical induction, it is proved that if the network topology has a spanning tree, then the pull-based event-triggered coupling strategy can realize consensus for the multi-agent system, too. Finally the results are extended to discontinuous monitoring, where each agent computes its next triggering time in advance without having to observe the system’s state continuously (self-triggered).

Consensus problem of multi-agent systems by event-triggered strategy were studied by [13] for indirected and weighted but balanced graph topologies. Directed graph topology was considered by [24] and [25].

The main contributions of this paper are as follows: a) we investigate directed topologies, including irreducible and reducible cases, and we do not make assumption that they are balanced; b) we give new approaches that the updating of the triggered time points of each agent only depend on states of its in-neighbors at their triggered time points, i.e., we give self-triggered principle under directed topologies, and as far as we know this is studied first time; c) the event-triggered principles in this paper are distributed, i.e. each agent only needs the information of its neighbors and itself, and asynchronous, and all the agents are not required to be triggered at a synchronous way.

The paper is organized as follows: in Section II some necessary definitions and lemmas are given; in Section III the pull-based event-triggered consensus in multi-agent systems with directed topologies is discussed; in Section IV the self-triggered formulation of the frameworks provided in Section III is presented; in Section V one numerical example is provided to show the effectiveness of the theoretical results; the paper is concluded in Section VI.

II. Preliminaries

In this section we first review some relating notations, definitions and results on algebraic graph theory [27], [28] which will be used later in this paper.

Notions: \( \| \cdot \| \) represents the Euclidean norm for vectors or the induced 2-norm for matrices. \( \mathbf{1} \) denotes the column vector with each component 1 with proper dimension. \( \rho(\cdot) \) stands for the spectral radius of matrices and \( \rho_2(\cdot) \) indicates the smallest positive eigenvalue for matrices having nonnegative eigenvalues. Given two symmetric matrices \( M, N, M > N \) (or \( M \geq N \)) means \( M - N \) is a positive definite (or positive semi-definite) matrix.

For a weighted directed graph (or digraph) \( G = (\mathcal{V}, \mathcal{E}, \mathcal{A}) \) with \( m \) agents (vertices or nodes), the set of agents \( \mathcal{V} = \{v_1, \ldots, v_m\} \), set of links (edges) \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) and the weighted adjacency matrix \( \mathcal{A} = (a_{ij}) \) with nonnegative adjacency elements \( a_{ij} \geq 0 \). A link of \( G \) is denoted by \( e(i,j) = (v_i, v_j) \in \mathcal{E} \) if there is a directed link from agent \( v_i \) to agent \( v_j \) with weight \( a_{ij} > 0 \), i.e. agent \( v_j \) can send information to agent \( v_i \) while the opposite direction transmission might not exist or with different weight \( a_{ji} \). The adjacency elements associated with the links of the graph are positive, i.e., \( e(i,j) \in \mathcal{E} \iff a_{ij} > 0 \) for all \( i, j \in \mathcal{I} \). It is assumed that \( a_{ii} = 0 \) for all \( i \in \mathcal{I} \). Moreover, the in- and out-neighbors set of agent \( v_i \) are defined as

\[ N^\text{in}_i = \{v_j \in \mathcal{V} \mid a_{ij} > 0\}, \quad N^\text{out}_i = \{v_j \in \mathcal{V} \mid a_{ji} > 0\} \]

The in- and out-degree of agent \( v_i \) are defined as follows:

\[ \deg^\text{in}(v_i) = \sum_{j=1}^m a_{ij}, \quad \deg^\text{out}(v_i) = \sum_{j=1}^m a_{ji} \]

The degree matrix of digraph \( G \) is defined as \( D = \text{diag}[\deg^\text{in}(v_1), \ldots, \deg^\text{in}(v_m)] \). The weighted Laplacian matrix associated with the digraph \( G \) is defined as \( L = D - \mathcal{A} \). A directed path from agent \( v_0 \) to agent \( v_k \) is a directed graph with distinct agents \( v_0, \ldots, v_k \) and links \( e_0, \ldots, e_{k-1} \) such that \( e_i \) is a link directed from \( v_i \) to \( v_{i+1} \) for all \( i < k \).

**Definition 1**: We say a directed graph \( G \) is strongly connected if for any two distinct agents \( v_i, v_j \), there exits a directed path from \( v_i \) to \( v_j \).

By [28], we know that strongly connectivity of \( G \) is equivalent to the corresponding Laplacian matrix \( L \) is irreducible.

**Definition 2**: We say a directed graph \( G \) has a spanning tree if there exists at least one agent \( v_i \) such that for any other agent \( v_j \), there exits a directed path from \( v_i \) to \( v_j \).

By Perron-Frobenius theorem [22] (for more detail and proof, see [23]), we have

**Lemma 1**: If \( L \) is irreducible, then \( \text{rank}(L) = m - 1 \), zero is an algebraically simple eigenvalue of \( L \) and there is a positive vector \( \mathbf{\xi}^\top = [\xi_1, \ldots, \xi_m] \) such that \( \mathbf{\xi}^\top L = 0 \) and \( \sum_{i=1}^m \xi_i = 1 \).

Let \( \Xi = \text{diag}[\xi_1, \ldots, \xi_m] \), by Perron-Frobenius theorem and the results first given in [22], we have
Lemma 2: If $L$ is irreducible, then $\Xi L + L^T \Xi$ is a symmetric matrix with all row sums equal to zeros and has zero eigenvalue with algebraic dimension one.

Here we define some matrices, which will be used later. Let $R = [R_{ij}]_{i,j=1}^m$, where

$$R = (1/2)(\Xi L + L^T \Xi)$$

Obviously, $R$ is positive semi-definite. Denote the eigenvalue of $R$ by $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_m$, counting the multiplicities.

We also denote

$$U = \Xi - \xi \xi^T$$

It can also be seen that $U$ has a simple zero eigenvalue and its eigenvalues (counting the multiplicities) can be arranged as $0 = \mu_1 < \mu_2 \leq \ldots \leq \mu_m$. We also denote the eigenvalues of $L^T L$ by $0 = \gamma_1 < \gamma_2 \leq \ldots \leq \gamma_m = \rho(L^T L)$, where $\rho(L^T L)$ is the spectrum norm of $L^T L$. Then, for all $x \in R^m$ satisfying $x \perp 1$, we have

$$\lambda_2 e^T x \leq x^T Rx$$

and

$$x^T U U x \leq \mu_2^2 x^T x$$

Therefore, we have

$$\frac{\lambda_2}{\lambda_2} R \geq U U$$

(3)

and

$$\frac{\lambda_m}{\gamma_2} L^T L \geq R \geq \frac{\lambda_2}{\rho(L^T L)} L^T L$$

(4)

III. PULL-BASED EVENT-TRIGGERED PRINCIPLES

In this section, we consider event-triggered control for multi-agent systems with directed and weighted topology.

A. Directed and irreducible topology

Firstly, we consider the case of irreducible $L$.

Denote $q(t) = [q_1(t), \ldots, q_m(t)]^T$, and $f(t) = [f_1(t), \ldots, f_m(t)]^T$, where

$$q_k(t) = -\sum_{j=1}^m \lambda_{ij} x_j(t)$$

and

$$f_i(t) = q_i(t_k) - q_i(t), \quad t \in [t_k, t_{k+1}], \quad k = 0, 1, 2, \ldots$$

Obviously, in (2)

$$u_i(t) = q_i(t_k), \quad t \in [t_k, t_{k+1}], \quad k = 0, 1, 2, \ldots$$

To depict the event that trigger the next coupling time point, we consider the following candidate Lyapunov function:

$$V(t) = \frac{1}{2} \sum_{i=1}^m \xi_i (x_i(t) - \bar{x}(t))^2$$

$$= \frac{1}{2} (x(t) - \bar{X}(t))^T \Xi (x(t) - \bar{X}(t))$$

$$= \frac{1}{2} x^T U x(t)$$

(5)

where $\bar{x}(t) = \sum_{i=1}^m \xi_i x_i(t)$ and $\bar{X}(t) = [\bar{x}(t), \ldots, \bar{x}(t)]^T$.

Since $\xi^T L = 0$, for any $a > 0$, the derivative of $V(t)$ along (2) is

$$\frac{d}{dt} V(t) = \sum_{i=1}^m \xi_i (x_i(t) - \bar{x}(t)) \left\{ q_i(t_k) - \sum_{i=1}^m \xi_i q_i(t_k) \right\}$$

(6)

Thus, noting $\sum_{i=1}^m \xi_i (x_i(t) - \bar{x}(t)) = 0$, we have

$$\frac{d}{dt} V(t) = \sum_{i=1}^m \xi_i (x_i(t) - \bar{x}(t)) \left\{ f_i(t) + q_i(t_k) \right\}$$

$$= \sum_{i=1}^m \xi_i (x_i(t) - \bar{x}(t)) \left\{ f_i(t) - \sum_{j=1}^m L_{ij} x_j(t) \right\}$$

$$= -\sum_{i,j=1}^m x_i(t) \xi_i L_{ij} x_j(t) + \sum_{i=1}^m \xi_i (x_i(t) - \bar{x}(t)) f_i(t)$$

$$= -x^T(t) Rx(t) + x^T(t) U f(t)$$

$$\leq -x^T(t) Rx(t) + \frac{a}{2} x^T(t) U U x(t) + \frac{1}{2a} f^T(t) f(t)$$

$$\leq - (1 - \frac{a \mu_m^2}{2\lambda_2}) x^T(t) Rx(t) + \frac{1}{2a} f^T(t) f(t)$$

(7)

By (4), we have

$$\frac{d}{dt} V(t) \leq - (1 - \frac{a \mu_m^2}{2\lambda_2}) \frac{\lambda_2}{\rho(L^T L)} x^T(t) L x(t) + \frac{1}{2a} f^T(t) f(t)$$

$$= - (1 - \frac{a \mu_m^2}{2\lambda_2}) \frac{\lambda_2}{\rho(L^T L)} q^T(t) q(t) + \frac{1}{2a} f^T(t) f(t)$$

$$= \sum_{i=1}^m - (1 - \frac{a \mu_m^2}{2\lambda_2}) \frac{\lambda_2}{\rho(L^T L)} q_i^2(t) + \frac{1}{2a} (q_i(t_k) - q_i(t))^2$$

(8)

Therefore, we have

Theorem 1: Suppose that $G$ is strongly connected. For $i = 1, \ldots, m$, set

$$t_{k+1} = \max_{\tau \geq t_k} \left\{ \exists \tau : q_i(t_k) - q_i(t) \leq \sqrt{2ab\gamma} q_i(t), \quad \forall t \in [t_k, \tau] \right\}$$

(9)

with $\gamma \in (0, 1)$, $0 < a < \frac{2\lambda_2}{\mu_m^2}$, and $b = (1 - \frac{a \mu_m^2}{2\lambda_2}) \frac{\lambda_2}{\rho(L^T L)}$. Then, system (2) reaches a consensus; In addition, for all $i \in I$, we have $\lim_{t \to \infty} |x_i(t) - \sum_{j=1}^m \xi_j x_j(t)| = 0$ and $\lim_{t \to \infty} \sum_{j=1}^m \xi_j x_j(t) = 0$ exponentially.

Proof: Combining inequalities (8), (9) and (4), we have

$$-(1 - \gamma) b q^T(t) q(t) = -(1 - \gamma) b x^T(t) L x(t)$$

$$\leq - (1 - \gamma) \frac{\lambda_2}{\rho_m (L^T L)} x^T(t) U x(t)$$

$$= - (1 - \gamma) \frac{2\lambda_2}{\mu_m (L^T L)} V(t)$$
\[ \frac{d}{dt} V(t) \leq - (1 - \gamma) b g(t) q(t) \]
\[ = -(1 - \gamma) b g(T) L x(t) \]
\[ \leq -(1 - \gamma) \frac{\rho_2(L^T L)}{\mu_m} \]
\[ t x(t) \]
\[ = - (1 - \gamma) \frac{2 \rho_2(L^T L)}{\mu_m} V(t) \]

for all \( t \geq 0 \). It means
\[ V(t) = O \left( e^{\left( - (1 - \gamma) \frac{2 \rho_2(L^T L)}{\mu_m} t \right)} \right) \]

This implies that system (2) reaches a consensus and for all \( i = 1, \ldots, m \),
\[ x_i(t) - \sum_{j=1}^m \xi_j x_j(t) = O \left( e^{\left( - (1 - \gamma) \frac{2 \rho_2(L^T L)}{\mu_m} t \right)} \right) \]

and
\[ q_i(t) - q_i(t_{k_i}(t)) = O \left( e^{\left( - (1 - \gamma) \frac{2 \rho_2(L^T L)}{\mu_m} t \right)} \right) \]

Thus
\[ \lim_{t \to \infty} \sum_{i=1}^m \xi_i x_i(t) = \lim_{t \to \infty} \sum_{i=1}^m \xi_i q_i(t) = \lim_{t \to \infty} \sum_{i=1}^m \xi_i t_{k_i}(t) \]
\[ = \lim_{t \to \infty} - \sum_{i=1}^m \xi_i \sum_{j=1}^m L_{ij} x_j(t) \]
\[ = \lim_{t \to \infty} - \sum_{j=1}^m x_j(t) \sum_{i=1}^m \xi_i L_{ij} = 0 \]

This completes the proof.

As special cases, we have

Corollary 1: Suppose that \( G \) is strongly connected. Set
\[ t_{k+1}^i = \max_{\tau \geq t_k^i} \left\{ \tau : \left| q_i(t_{k}^i) - q_i(t) \right| \leq c q_i(t), \quad \forall t \in [t_k^i, \tau) \right\} \] (10)

or
\[ t_{k+1}^i = \max_{\tau \geq t_k^i} \left\{ \tau : \left| q_i(t_{k}^i) \right| \leq \frac{c}{1 + c} q_i(t), \quad \forall t \in [t_k^i, \tau) \right\} \] (11)

for some sufficient small constant \( c \). Then, system (2) reaches a consensus; in addition, for all \( i \in I \), we have \( \lim_{t \to \infty} |x_i(t) - \sum_{j=1}^m \xi_j x_j(t)| = 0 \) and \( \lim_{t \to \infty} \sum_{j=1}^m \xi_j x_j(t) = 0 \) exponentially.

Theorem 1 shows that a constant \( c \) does exist.

Now, we will show that under the condition and the event-triggered principle in Theorem 1, the Zeno behavior can be excluded (see [29]) by proving following theorem.

Theorem 2: For any initial condition, at any time \( t \geq 0 \), under the condition and the event-triggered principle in Theorem 1, there exists at least one agent \( v_j \), of which the next inter-event time is strictly positive before

**Proof:** Suppose that there is no trigger event when \( t > T \). Then, we have
\[ x_i(t) = \sum_{j=1}^m L_{ij} x_j(T_{k_i}(T)), \quad t > T, \quad i, j = 1, \ldots, m \] (12)
which implies
\[ x_i(t) - x_i(T) = \sum_{j=1}^m L_{ij} x_j(T_{k_i}(T)). \] (13)

By Theorem 1, we have \( x_i(t) - x_i(T) \to 0 \). Therefore, for all \( i, j = 1, \ldots, m \), we have \( T_{k_i}(T) = T_{k_j}(T) = T, x_i(T) = x_j(T), \) and \( \sum_{j=1}^m L_{ij} x_j(T_{k_i}(T)) = \sum_{j=1}^m L_{ij} x_j(T_{k_j}(T)), \) which implies \( x_i(t) = x_j(t) \) for all \( t > T \) and \( i, j = 1, \ldots, m \). It means that in case there is no triggering time for \( t > T \), the consensus has reached at time \( T \). This implies that Zeno behavior can be excluded.

**Remark 1:** It can be seen that in [29] the updating of the event times for agent \( v_j \) only depends on the states of its in-neighbors.

**B. Directed and reducible topology**

In this section, we consider the case \( L \) is reducible. The following mathematical methods are inspired by that given in [80]. By proper permutation, we rewrite \( L \) as the following Perron-Frobenius form:

\[ L = \begin{bmatrix} L_{1,1} & L_{1,2} & \cdots & L_{1,K} \\ 0 & L_{2,2} & \cdots & L_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_{K,K} \end{bmatrix} \] (14)

where \( L_{k,k} \) is with dimension \( n_k \) and associated with the \( k \)-th strongly connected component (SCC) of \( G \), denoted by \( SCC_k, \ k = 1, \ldots, K \). Accordingly, define \( x = [x^T, \ldots, x^T]^T \), where \( x^k = [x^1, \ldots, x^k]^T \).

For agent \( v_i \in SCC_k \), i.e., \( i = M_{k-1} + 1, \ldots, M_k \), where \( M_0 = 0, M_k = \sum_{i=1}^{K} n_i \), denote the combinational state measurement \( q_i^k(t) = - \sum_{j=M_{k-1}+1}^{M_k} L_{i,j} x_j(t) = - \sum_{j=1}^{M_k} L_{i,j} x_j(t) = q_{i,M_k}(t) \). And the combinational measurement error by \( f_{i}^k(t) = q_i^k(t) - q_i^k(T) \) and \( u_{i}^k(t) = q_i^k(T) \), \( t \in [t_{i}^k, t_{i}^k(T)) \), \( l = 0, 1, 2, \ldots \), moreover, write \( q_i^k(t) = [q_i^k(t), \ldots, q_i^k(t)]^T \) and \( f_i^k(t) = [f_i^k(t), \ldots, f_i^k(t)]^T \).

If \( G \) has spanning trees, then each \( L_{i,j}^{k} \) is irreducible or has one dimension and for each \( k < K, L_{i,j}^{k} \neq 0 \) for at least one \( q > k \).

Define an auxiliary matrix \( \hat{L}_{i,j}^{k} = [L_{i,j}^{k}]_{i,j=1}^{l} \) as
\[ \hat{L}_{i,j}^{k} = \begin{bmatrix} L_{i,j}^{k} \\ - \sum_{p \neq i, p \neq j} L_{j,i}^{k} \end{bmatrix} \]

Then, let \( D_k = L_k^{k} - \hat{L}_{i,j}^{k} = d^{i}_{a} a_{i}^{j} (D_{i}^{k}, \ldots, D_{n}^{k}) \), which is a diagonal semi-positive definite matrix and has at least one diagonal positive (nonzero). Keep the following property in mind [31]:

**Property 1:** \( D_k \neq 0 \) if and only if there exists \( v_j \in \bigcup_{i=k}^{K} SCC_i \) such that there exists an directed link from \( v_j \) to \( v_i, M_{k-1} > 0 \) for some \( j \) and \( l > k \).
Let $\xi^k\top$ be the positive left eigenvector of the irreducible $\hat{L}^{k,k}$ corresponding to the eigenvalue zero and has the sum of components equaling to 1.

Denote $\Xi^k = \text{diag}(\xi^k)$. Then, we have

**Property 2:** Under the setup above, $\Xi^k L^{k,k} + L^{k,k}\top \Xi^k$ is positive definite for all $k < K$.

Here we define some matrices which will be used later. By the structure, it can be seen that $R^{K-1} = 1/2(\Xi^{K-1}\hat{L}^{K-1,K-1} + (\Xi^{K-1}\hat{L}^{K-1,K-1})\top) = [R_{ij}^{K-1}]_{i,j=1}^{n_K}$ has zero row sums and has zero eigenvalue with algebraic dimension one, and $Q^{K-1} = 1/2(\Xi^{K-1}L^{K-1,K-1} + (\Xi^{K-1}L^{K-1,K-1})\top) = [Q_{ij}^{K-1}]_{i,j=1}^{n_K} = R^{K-1} + \Xi^{K-1}D^{K-1}$ is positive definite. Let $\hat{U}^k = \Xi^k\Xi^k$, $k = 1, 2, \ldots, K - 1$, $\hat{U}_k = [\Xi^k - \xi^k(\xi^k)\top][\Xi^k - \xi^k(\xi^k)\top]$. Similar to (3), we have

$$\Xi^{K-1} < I \leq \frac{1}{\rho_2(Q^{K-1})}Q^{K-1}$$

$$\hat{U}_k \leq \frac{\rho(\hat{U}_k)}{\rho_2(Q^k)}Q^k$$

Now we are going to determine the triggering times for the system (3) to reach consensus. Firstly, applying Theorem 1 to the $K$-th SCC, we can conclude that the $K$-th SCC can reach a consensus with the agreement value $\nu(t) = \sum_{p=1}^{n_K} \xi_p^n x_p(t)$ and $\lim_{t\to\infty} \hat{\nu}(t) = 0$ exponentially.

Then, inductively, consider the $K$-th SCC. We will prove that $\lim_{t\to\infty} |x_p(t) - \nu(t)| = 0$, for all $p = 1, \ldots, n_K$.

Construct a candidate Lyapunov function as follows

$$V_{K-1}(t) = \frac{1}{2}(x^{K-1}(t) - \nu(t))^\top\Xi^{K-1}(x^{K-1}(t) - \nu(t))$$

Differentiate $V_{K-1}(t)$ along (3), we have

$$\frac{d}{dt} V_{K-1}(t) = (x^{K-1}(t) - \nu(t))^\top\Xi^{K-1}\left\{ f^{K-1}(t) + q^{K-1}(t) - \hat{\nu}(t)1 \right\}$$

$$= (x^{K-1}(t) - \nu(t))^\top\Xi^{K-1}\left\{ -L^{K-1,K-1}(x^{K-1}(t) - \nu(t)) - L^{K-1,K-1}(x^{K}(t) - \nu(t)) \right\}$$

$$= (x^{K-1}(t) - \nu(t))^\top\Xi^{K-1}\left\{ -f^{K-1}(t) + \hat{\nu}(t)1 \right\}$$

$$= W^{K-1}_1(t) + W^{K-1}_2(t) + W^{K-1}_3(t)$$

where

$$W^{K-1}_1(t) = (x^{K-1}(t) - \nu(t))^\top\Xi^{K-1}L^{K-1,K-1}(x^{K-1}(t) - \nu(t))$$

$$= (x^{K-1}(t) - \nu(t))^\top\Xi^{K-1}Q^{K-1,K-1}(x^{K-1}(t) - \nu(t))$$

$$= W^{K-1}_2(t) = -(x^{K-1}(t) - \nu(t))^\top\Xi^{K-1}(\nu(t))$$

$$= W^{K-1}_3(t) = -(x^{K-1}(t) - \nu(t))^\top\Xi^{K-1}(\nu(t))$$

By Cauchy inequality, for any $\nu_{2K} > 0$, $\nu_{3K} > 0$, we have

$$W^{K-1}_2(t) \leq \nu_{2K} V_{K-1}(t) + F^{K-1}_2(t)$$

$$W^{K-1}_3(t) \leq \nu_{3K} V_{K-1}(t) + F^{K-1}_3(t)$$

where

$$F^{K-1}_2(t) = \frac{1}{2\nu_{2K}} \sum_{i=1}^{n_K} \xi_i^{K-1} \left\{ \sum_{p=1}^{n_K} L_{i,p}^{K-1,K} [x_p(t) - \nu(t)] \right\}^2$$

$$F^{K-1}_3(t) = \frac{1}{2\nu_{3K}} \sum_{i=1}^{n_K} \xi_i^{K-1} [\hat{\nu}(t)]^2 = \frac{1}{2\nu_{3K}} [\hat{\nu}(t)]^2$$

According to the discussion of SCC and Theorem 1 for all $p = 1, \ldots, n_K$, we have

$$\lim_{t\to\infty} x_p(t) - \nu(t) = 0, \lim_{t\to\infty} \hat{\nu}(t) = 0$$

exponentially. Thus

$$\lim_{t\to\infty} F^{K-1}_2(t) = 0, \lim_{t\to\infty} F^{K-1}_3(t) = 0$$

exponentially.

From (16), for any $a_k > 0$, (15) can be rewritten as follows

$$\frac{d}{dt} V_{K-1}(t) \leq \frac{a_{K-1}}{2} [x^{K-1}(t) - \nu(t)]^\top \hat{U}_1^{K-1} [x^{K-1}(t) - \nu(t)] + W_1^{K-1}(t)$$

$$+ \frac{1}{2a_{K-1}} [f^{K-1}(t)]^\top f^{K-1}(t) + W_2^{K-1}(t) + W_3^{K-1}(t) \leq \left(1 - \frac{a_{K-1}\rho(\hat{U}_1^{K-1})}{2\rho_2(Q^{K-1})}\right) W_1^{K-1}(t)$$

$$+ \frac{1}{2a_{K-1}} [f^{K-1}(t)]^\top f^{K-1}(t) + W_2^{K-1}(t) + W_3^{K-1}(t).$$

Thus, we have

**Theorem 3:** Suppose that $G$ has spanning trees and $L$ is written in the form of (4). For SCCs, the event time sequence $t^{p+M_{k-1}}_l$ for $v_{p+M_{k-1}} \in SCC_k$ is given by

$$t^{p+M_{k-1}}_{l+1} = \max_{\tau \geq t^{p+M_{k-1}}_l} \left\{ \tau : |q^k_p(t^{p+M_{k-1}}_l) - q^k_p(\tau)| \leq \sqrt{2} a_k b_k \gamma |q^k_p(t^{p+M_{k-1}}_l)|, t^{p+M_{k-1}}_l \leq \tau \right\}$$

for some fixed $\gamma \in (0, 1)$, $0 < a_k < 2\rho_2(Q^{k})/\rho(\hat{U}_1^{k})$, and $b_k = (1 - a_k \rho(\hat{U}_1^{k})/2\rho_2(Q^{k})) \rho_2(Q^{k})/\rho_2(Q^{k})$, $k = 1, 2, \ldots, K$. Then, system
reaches a consensus; In addition, for all $i \in I$, we have
$$\lim_{t \to \infty} \left| x_i(t) - \sum_{p=1}^{n_K} \xi^K_{ip} x^K_p(t) \right| = 0$$
and
$$\lim_{t \to \infty} \sum_{p=1}^{n_K} \xi^K_{ip} x^K_p(t) = 0$$

Proving: For the $K$-th SCC, the event-triggered rule (23) is the same as (9) in Theorem 1 since $L$ is written in the form of (14). By Theorem 1 we can conclude that under the updating rule of $\{t_j^{+M+1}\}$ for all $j = 1, \ldots, n_K$ and $\lim_{t \to \infty} \nu(t) = 0$, the subsystem restricted in $SCC_K$ reaches a consensus. Additionally, $\lim_{t \to \infty} |x^K_i(t) - \sum_{p=1}^{n_K} \xi^K_{ip} x^K_p(t)| = 0$ for all $i = 1, \ldots, n_K$ and $\lim_{t \to \infty} \nu(t) = 0$ as well.

In the following, we are to prove that the state of the agent $v_{p+M+1} \in SCC_{K-1}$ converges to $\nu(t)$. The remaining can be proved similarly by induction.

From (23), (15) and the inequality (23), we have
$$\frac{d}{dt} V_{K-1}(t) \leq \left(1 - \frac{a_{K-1} \rho(U^{K-1})}{2 \rho^2(-Q^{K-1})}\right) W_1^{K-1}(t)$$
$$+ \left(1 - \frac{a_{K-1} \rho(U^{K-1})}{2 \rho^2(-Q^{K-1})}\right) \frac{\gamma \rho^2(-Q^{K-1})}{\rho((L^{K-1}.K-1) \top L^{K-1}.K-1)}$$
$$\times \left| L^{K-1}.K-1[x^{K-1}(t) - \nu(t)1]\right|^2 + W_2^{K-1}(t) + W_3^{K-1}(t)$$
$$\leq \left(1 - \frac{a_{K-1} \rho(U^{K-1})}{2 \rho^2(-Q^{K-1})}\right) (1 - \gamma) W_1^{K-1}(t)$$
$$+ W_2^{K-1}(t) + W_3^{K-1}(t) + W_4^{K-1}(t)$$
where
$$W_4^{K-1}(t) = b_{K-1} \gamma \left\{2[x^{K-1}(t) - \nu(t)1] \top (L^{K-1}.K-1) \top L^{K-1}.K$$
$$[x^{K}(t) - \nu(t)1] + \left\| L^{K-1}.K [x^{K}(t) - \nu(t)1] \right\|^2 \right\}$$

Noting
$$\rho((L^{K-1}.K-1) \top L^{K-1}.K-1) I \geq (L^{K-1}.K-1) \top L^{K-1}.K-1$$
and (15), for any $\nu_4^{K-1} > 0$, we have
$$W_4^{K-1}(t) \leq -\nu_4^{K-1} \left(1 - \frac{a_{K-1} \rho(U^{K-1})}{2 \rho^2(-Q^{K-1})}\right) W_1^{K-1}(t) + F_4^{K-1}(t)$$
where
$$F_4^{K-1}(t) = b_{K-1} \gamma \left\{ \frac{1}{4 \nu_4^{K-1}} + 1 \right\} \left\| L^{K-1}.K [x^{K}(t) - \nu(t)1] \right\|^2$$

Similar to (21), we have
$$\lim_{t \to \infty} F_4^{K-1}(t) = 0$$

exponentially. From (15) and (19), we have
$$V_{K-1}(t) \leq \left(1/(2 \rho^2(Q^{K-1}))\right)(-W_4^{K-1}(t))$$

Picking sufficiently small $\nu_4^{K-1}, \nu_3^{K-1}$, and $\nu_4^{K-1}$, there exists some $\varepsilon_{K-1} > 0$ such that
$$\frac{d}{dt} V_{K-1}(t) \leq -\varepsilon_{K-1} V_{K-1}(t) + F_2^{K-1}(t) + F_3^{K-1}(t) + F_4^{K-1}(t)$$
Thus
$$V_{K-1}(t) \leq e^{-\varepsilon_{K-1} t} \left\{ V_{K-1}(0) + \int_0^t e^{\varepsilon_{K-1}s} \sum_{i=2}^4 F_i^{K-1}(s)ds \right\}$$

From (21), we have $\lim_{t \to \infty} V_{K-1}(t) = 0$ exponentially. This implies that system (2) reaches a consensus and
$$\lim_{t \to \infty} \left|x_p^{K-1}(t) - \nu(t)\right| = 0$$
exponentially for all $p = 1, \ldots, n_{K-1}$. Then, we can complete the proof by induction to $SCC_k$ for $k < K - 1$.

Remark 2: It can be seen that in (23) the updating of the event times for agent $v_j$ only depends on the states of its in-neighbors. In other words, if agent $v_j \in SCC_k$ then the updating of the event times for agent $v_j$ only depends on the states of its in-neighbors in $SCC_k, \ldots, SCC_{K-1}$.

Remark 3: If graph $G$ only has one strongly connected component, i.e., graph $G$ itself is strongly connected, then Theorem 3 becomes Theorem 1.

Remark 4: Similar to the proof of Theorem 2, we can prove that the Zeno behavior can be excluded in above event-triggered rule. We omit the proof here.

IV. DISTRIBUTED SELF-TRIGGERED PRINCIPLES

In this section, we extend the pull-based event-triggered principles discussed in Section III to self-triggered case in order to avoid continuous monitoring of the system’s state. This idea can be traced to the early papers [1, 2].

The monitoring principles used in Theorem 1 and Theorem 3 may be costly since the state of the system should be observed continuously. An alternative strategy is to predict the time when inequality (9) or (23) does not hold and update the event timing accordingly. However, when agent $v_j$ updates its event timing, the timing predictions of the related agents, including $v_j$’s out-neighbors will be affected. So, each agent should recalculate their predictions whenever any of its in-neighbors renews its event timing.

For agent $v_p$, according to the current event timing $t_{kp}^p\nu(t)$, its state can be formulated as:
$$x_p(t) = x_p(t_{kp}^p\nu(t)) + (t - t_{kp}^p\nu(t)) q_p(t_{kp}^p\nu(t))$$
where $t_{kp}^p\nu(t)$ is the newest timing of the events of all its in-neighbors agents
$$t_{kp}^p\nu(t) = \max_{v_j \in N_p^\top} t_{kj}^p\nu(t)$$

(26)
Based on this timing, letting $\zeta_p = t - t^p_k(t)$, we can rewrite all states of $v_i$ with $v_i \in N^m_k$, as

$$x_i(t) = x_i(\zeta_i) = x_i(t^p_k(t_i)) + \zeta_i q_i(t^p_k(t_i))$$  \hfill (27)

Thus, in order to specify (9), from (25) and (27), we can rewrite

$$q_p(t) = q_p(\zeta_p) = \sum_{i} L_{pi} x_i(\zeta_i)$$

For agent $v_p$, solve the following inequality to maximize $\zeta_p$ so that

$$\tau^p_{t+1} = \max \left\{ \zeta_p : |q_p(t^p_k(t_i)) - q_p(\zeta_p)| \leq \sqrt{2\alpha b\gamma}|q_p(\zeta_p)| \right\}$$  \hfill (28)

Then, we have the following result

**Theorem 4:** Suppose that $G$ is strongly connected. Using the following triggering strategy:

1) For any agent $v_p$, $p = 1, \ldots, m$, initialize $t^0_p = 0$;
2) Pick $\gamma \in (0, 1)$ and $0 < a < \frac{2\alpha b}{\mu^2}$; assume $t^p_i = t^p_k(t_i)$, search $\tau^p_{t+1}$ by the rule (28);
3) In case that any of $v_p$’s in-neighbors does not trigger during $(t^p_i, t^p_{k}(t_i) + \tau^p_{t+1})$, i.e., the agent $v_p$ does not receive any renewed information form its in-neighbors during $(t^p_i, t^p_k(t_i) + \tau^p_{t+1})$, then $v_p$ triggers at time $t^p_{t+1} = t^p_k(t_i) + \tau^p_{t+1}$. The agent $v_p$ renews its state at $t = t^p_{t+1}$ and sends the renewed information (including the latest triggering time point $t^p_{t+1}$, state value $x_p(t^p_{t+1})$ and the latest control input value $q_p(t^p_{t+1})$) to all its out-neighbours immediately;
4) In case that some in-neighbors of agent $v_p$ triggers at time $t \in (t^p_i, t^p_k(t_i) + \tau^p_{t+1})$, i.e., agent $v_p$ received the renewed information form some of its in-neighbors, then updating $t^p_k(t_i)$ in (26) and go to step (2).

Then, system (2) reaches a consensus; In addition, $\lim_{t \to \infty} |x_i(t) - \sum_{p=1}^{m} \xi_p x_p(t)| = 0$ for all $i \in \mathcal{I}$ and $\lim_{t \to \infty} \sum_{p=1}^{m} \xi_p^2 x_p^2(t) = 0$ exponentially.

**Proof:** Following steps 1-3, solving the maximization problem (28), and by the same arguments as in the proof of Theorem 1 one can prove this theorem. 

**Remark 5:** It can be seen that each inequality in (28) is of the following form

$$|d_1 t + d_2| \leq |d_3 t + d_4|,$$  \hfill (29)

where $d_1, d_2, d_3, d_4$ are constants relating to $L_{ip}, x_i(t^p_k(t_i)), t^p_k(t_i)$, etc. It is easy to solve (29), since it is a polynomial-type inequality. And also for the following inequality (30).

Similarly, we can specify (29). For agent $v_{p+M-k-1}$ solve the following inequality to maximize $\zeta_p$ so that

$$\tau^p_{t+1} = \max \left\{ \zeta_p : |q_p(t^p_{k+p-M-k-1}(t)) - q_p(\zeta_p)| \leq \sqrt{2\alpha b\gamma}|q_p(\zeta_p)| \right\}$$  \hfill (30)

Then, we have the following result

**Theorem 5:** Suppose that $G$ has spanning tree and $L$ is written in the form of (14). Using the following triggered strategy:

1) For any agent $v_{p+M-k-1}$, $k = 1, \ldots, K$, $p = 1, \ldots, n_k$, initializing $t^0_{p+M-k-1} = 0$;
2) Pick $\gamma \in (0, 1)$ and $0 < a_k < \frac{2\alpha b}{\mu^2}$, letting $t^p_{p+M-k-1} = t^p_{k+p-M-k-1}(t)$, searching $\tau^p_{p+M-k-1}$ by the rule (30);
3) In case that no triggering events occur in all $v_{p+M-k-1}$’s in-neighbors during $(t^p_{p+M-k-1}, t^p_{k+p-M-k-1}(t) + \tau^p_{p+M-k-1})$, i.e., the agent $v_{p+M-k-1}$ does not receive any renewed information form its in-neighbors during $(t^p_{p+M-k-1}, t^p_{k+p-M-k-1}(t) + \tau^p_{p+M-k-1})$, then $v_{p+M-k-1}$ triggers at time $t^p_{p+M-k-1} = t^p_{k+p-M-k-1}(t) + \tau^p_{p+M-k-1}$.

The agent $v_{p+M-k-1}$ refreshes its state at $t = t^p_{p+M-k-1}$ and sends the renewed information (including the latest triggering time point $t^p_{p+M-k-1}$, state value $x_p(t^p_{p+M-k-1})$ and the latest control input value $q_p(t^p_{p+M-k-1})$) to all its out-neighbours immediately.

4) In case that some in-neighbors of agent $v_{p+M-k-1}$ triggers at time $t \in (t^p_{p+M-k-1}, t^p_{k+p-M-k-1}(t) + \tau^p_{p+M-k-1})$, i.e., agent $v_{p+M-k-1}$ received the renewed information from some its in-neighbors, then updating $t^p_{k+p-M-k-1}(t)$ in (26) and go to step (2).

Then, system (2) reaches a consensus; In addition, $\lim_{t \to \infty} |x_i(t) - \sum_{p=1}^{m} \xi_p x_p(t)| = 0$ for all $i \in \mathcal{I}$ and $\lim_{t \to \infty} \sum_{p=1}^{m} \xi_p^2 x_p^2(t) = 0$ exponentially.

**Proof:** Following steps 1-3, under the maximisation process (30), by the same arguments as in the proof of Theorem 1 one can prove this theorem.

**Remark 6:** Both event-triggered principles and self-triggered principles in this paper are distributed, asynchronous, and independent.

**V. EXAMPLES**

In this section, one numerical example is given to demonstrate the effectiveness of the presented results.

Consider a network of seven agents with a directed reducible Laplacian matrix

$$L = \begin{bmatrix}
-12 & 0 & 5 & 2 & 5 & 0 & 0 \\
3 & -8 & 3 & 0 & 0 & 0 & 2 \\
0 & 4 & -12 & 3 & 0 & 5 & 0 \\
0 & 0 & 0 & -11 & 1 & 4 & 0 \\
0 & 0 & 0 & 0 & -7 & 2 & 5 \\
0 & 0 & 0 & 0 & 5 & -6 & 1 \\
0 & 0 & 0 & 0 & 0 & 8 & -8
\end{bmatrix},$$

with a spanning tree described by Figure 1. The seven agents can be divided into two strongly connected components, i.e. the first four agents form a strongly connected component and the rest form another. The initial value of each agent is also randomly selected within the interval $[-5, 5]$ in our simulations. Figure 2 shows how the first agent evolves under the triggered principles provided
in Theorem 5 with $\gamma = 0.9$; $a_1 = \frac{\rho_2(-Q^T)}{\rho(U^2)} = 5.7247$, $a_2 = \frac{\rho_2(-Q^2)}{\rho(U^2)} = 20.2719$ and initial value $[2.6670, -2.8660, -2.9300, 0.4230, -4.0490, -2.1500, -2.1900]$ comparing with continuous control, i.e., evolving under (1). The symbol $\cdot$ indicates the agent’s triggering times.

Fig. 1. The communication graph.

Fig. 2. The agents evolve under the event-triggered principle provided in Theorem 5.

VI. CONCLUSIONS

In this paper, we present distributed event-triggered and self-triggered principles in for multi-agent systems with general directed topologies. We derive pull-based event-triggered principles: a) In case the graph is irreducible, the triggering time of each agent given by the inequality (22) depends on the states of each agent’s in-neighbours only; b) In case the graph is irreducible with a spanning tree, the triggering time of each agent given by the inequality (23) only depend on the states of each agent’s in-neighbors, too. It is shown that with those principles, consensus can be reached exponentially, and Zeno behavior can be excluded. The results then are extended to discontinuous monitoring, where each agent computes its next triggering time in advance without having to observe the systems state continuously. Moreover, it is pointed out that it is easy to give next triggering time. The effectiveness of the theoretical results are verified by one numerical example.

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