On the construction of solutions to the Yang-Mills equations in higher dimensions

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1 Introduction

Let $M$ be a Riemannian manifold of dimension $n \geq 4$. A connection $A$ on a vector bundle over $M$ is a Yang-Mills connection if the curvature $F_A$ satisfies

$$D_A^* F_A = 0.$$ 

This is the Euler-Lagrange equation for the functional

$$E[A] = \int_M |F_A|^2.$$ 

Let $\{A_k\}$ be a sequence of Yang-Mills connections with uniformly bounded energy, i.e.

$$\sup_k E[A_k] < \infty.$$ 

Then the set

$$S = \left\{ x \in M : \lim_{k \to \infty} r^{4-n} \int_{B_r(x)} |F_{A_k}|^2 \geq \varepsilon_0 \text{ for all } r > 0 \right\}$$

is called the blow-up set of the sequence $\{A_k\}$. G. Tian [20] proved that the blow-up set $S$ is closed and $H^{n-4}$-rectifiable. Moreover, the energy densities satisfy

$$|F_{A_k}|^2 dvol \rightharpoonup |F_{A_\infty}|^2 dvol + 8\pi^2 \Theta dH^{n-4}|_S$$

as $k \to \infty$, where $A_\infty$ is the limiting connection defined on $M \setminus S$, $\Theta$ denotes the density function, and $H^{n-4}$ is the $(n-4)$-dimensional Hausdorff measure. Furthermore, if the limiting connection $A_\infty$ is admissible, then the generalized mean curvature of $S$ is equal to 0 (see [20] [16]). This result generalizes a theorem of K. Uhlenbeck in dimension 4.
In this paper, we consider a smooth minimal submanifold \( S \) of dimension \( n - 4 \). Our aim is to construct a sequence \( \{A_k\} \) of smooth Yang-Mills connections whose blow-up set is equal to \( S \).

In the first step, we construct a suitable family of approximate solutions. To this end, we assume that the normal bundle of \( S \) can be endowed with a complex structure \( J \) and a complex volume form \( \omega \). Each approximate solution is described by a set \( (v, \lambda, J, \omega) \), where \( v \) is a section of the normal bundle of \( S \), \( \lambda \) is a positive function on \( S \), and \( (J, \omega) \) is a \( SU(2) \)-structure on \( NS \).

For every point \( x \in S \), the normal components of an approximate solution \( A \) coincide with the basic instanton on the fibre \( NS_x \), i.e.

\[
A(e_1^\perp) = \frac{-(y - \varepsilon v)_2 i - (y - \varepsilon v)_3 j - (y - \varepsilon v)_4 k}{\varepsilon^2 \lambda^2 + |y - \varepsilon v|^2}
\]

\[
A(e_2^\perp) = \frac{(y - \varepsilon v)_1 i - (y - \varepsilon v)_4 j + (y - \varepsilon v)_3 k}{\varepsilon^2 \lambda^2 + |y - \varepsilon v|^2}
\]

\[
A(e_3^\perp) = \frac{(y - \varepsilon v)_4 i + (y - \varepsilon v)_1 j - (y - \varepsilon v)_2 k}{\varepsilon^2 \lambda^2 + |y - \varepsilon v|^2}
\]

\[
A(e_4^\perp) = \frac{-(y - \varepsilon v)_3 i + (y - \varepsilon v)_2 j + (y - \varepsilon v)_1 k}{\varepsilon^2 \lambda^2 + |y - \varepsilon v|^2},
\]

where \( \{e_\alpha^\perp: 1 \leq \alpha \leq 4\} \) denotes a \( SU(2) \) basis for the fibre \( NS_x \).

Our aim is to deform the connection \( A \) to a nearby connection \( \tilde{A} = A + a \) such that \( \tilde{A} \) is a solution of the Yang-Mills equations.

In the following, we denote by \( h \) the second fundamental form of the submanifold \( S \), and by \( R \) the Riemann curvature tensor of \( M \).

**Theorem 1.1.** Suppose that \( H^1(M) = 0 \). Then, for each \( \varepsilon > 0 \), there exists a mapping \( \Xi_\varepsilon \) which assigns to each set of glueing data \( (v, \lambda, J, \omega) \in C^{2,\gamma}(S) \) a section of the vector bundle \( NS \oplus \mathbb{R} \oplus \Lambda_+^2 NS \) of class \( C^\gamma(S) \) such that the following holds.

(i) If \( (v, \lambda, J, \omega) \) is a set of glueing data such that

\[
\|v\|_{C^{2,\gamma}(S)} \leq K,
\]

\[
\|\lambda\|_{C^{2,\gamma}(S)} \leq K, \quad \inf \lambda \geq 1,
\]

\[
\|(J, \omega)\|_{C^{2,\gamma}(S)} \leq K,
\]

\[

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then we have the estimate

\[
\left\| \Xi_\varepsilon(v, \lambda, J, \omega) \right\|_{C^2(S)} 
- \left( \Delta v_\rho + \sum_{i,j=1}^{n-4} \sum_{\rho,\sigma=1}^{4} h_{ij,\rho} h_{ij,\sigma} v_\sigma + \sum_{i=1}^{n-4} \sum_{\rho,\sigma=1}^{4} R_{i\rho\sigma} v_\sigma, \right.
\]

\[
\frac{1}{\lambda} \Delta \lambda + \frac{1}{4} \sum_{i,j=1}^{n-4} \sum_{\rho=1}^{4} h_{ij,\rho} h_{ij,\rho} + \frac{1}{4} \sum_{i=1}^{n-4} \sum_{\rho=1}^{4} R_{i\rho\rho} - \frac{1}{4} |\theta|^2,
\]

\[
\frac{1}{\lambda^2} \sum_{i=1}^{n-4} \nabla_i (\lambda^2 \theta_i,\rho) \right\| \leq C \varepsilon \frac{1}{\varepsilon^2}.
\]

(ii) If \( \Xi_\varepsilon(v, \lambda, J, \omega) = 0 \), then the approximate solution \( A \) corresponding to \( (v, \lambda, J, \omega) \) can be deformed to a nearby connection \( \tilde{A} \) satisfying \( D_\tilde{A} F_\tilde{A} = 0 \).

In Section 2, we recall some results about the linearized operator on \( \mathbb{R}^4 \). In particular, the kernel of the linearized operator on \( \mathbb{R}^4 \) is isomorphic to \( \mathbb{R}^4 \oplus \mathbb{R}^4 \oplus \Lambda^2 \mathbb{R}^4 \) (compare [3]).

In Section 3, we study the mapping properties of a model operator on the product manifold \( \mathbb{R}^{n-4} \times \mathbb{R}^4 \).

In Section 4, we construct a family of approximate solutions of the Yang-Mills equations. More precisely, given any set of glueing data \( (v, \lambda, J, \omega) \) satisfying

\[
||v||_{C^2(S)} \leq K,
\]

\[
||\lambda||_{C^2(S)} \leq K, \quad \inf \lambda \geq 1,
\]

\[
||(J, \omega)||_{C^2(S)} \leq K,
\]

we construct a connection \( A \) such that

\[
||D_\tilde{A}^* F_\tilde{A}||_{C^2(S)} \leq C \varepsilon^2.
\]

Here, the weighted Hölder space \( C^\gamma_\varepsilon(M) \) is defined as

\[
||u||_{C^\gamma_\varepsilon(M)} = \sup (\varepsilon + \text{dist}(p, S))^{\nu} |u(p)|
\]

\[
+ \sup_{\substack{4\text{dist}(p_1, p_2) \leq \varepsilon + \text{dist}(p_1, S) + \text{dist}(p_2, S) \varepsilon \text{dist}(p_1, p_2) \gamma}} \frac{|u(p_1) - u(p_2)|}{\text{dist}(p_1, p_2)^\gamma}.
\]

In Section 5, we derive uniform estimates for the operator \( \mathbb{L}_A = L_A + D_\tilde{A} D_\tilde{A}^* \). Here, \( L_A \) is the linearization of the Yang-Mills equations at an approximate
solution $A$. The additional term $D_A D_A^* a$ must be included because $L_A$ is not an elliptic operator.

To derive uniform estimates independent of $\varepsilon$, we need to restrict the operator $L_A$ to a subspace $E_\gamma(M) \subset C^\gamma(M)$. A 1-form $a$ belongs to $E_\gamma(M)$ if

$$\int_{NS_x} \sum_{\alpha=1}^4 \langle a(e_\alpha^\perp), F_A(X, e_\alpha^\perp) \rangle = 0$$

for all $x \in S$ and all vector fields of the form

$$X = \varepsilon w_\rho e_\rho^\perp + \mu (y - \varepsilon v)_\rho e_\rho^\perp + r_{\rho\sigma} (y - \varepsilon v)_\sigma e_\rho^\perp$$

with $w \in NS_x$, $\mu \in \mathbb{R}$, and $r \in \Lambda^2_{+} NS_x$.

In Section 6, we apply the contraction mapping principle to deform the approximate solution $A$ to a nearby connection $\tilde{A} = A + a$ such that

$$(I - P)(D_A^* F_A + D_A D_A^* a) = 0,$$

where $(I - P)$ is the fibrewise projection from $C^\gamma(M)$ to the subspace $E_\gamma(M)$. In particular, if the balancing condition

$$P(D_A^* F_A + D_A D_A^* a) = 0$$

is satisfied, then $\tilde{A}$ is a Yang-Mills connection.

In Section 7, we calculate the leading term in the asymptotic expansion of

$$P(D_A^* F_A + D_A D_A^* a) = 0.$$

This concludes the proof of Theorem 1.1.

An example is discussed in Section 8.

A related balancing condition occurs in the work of R. Schoen and D. Pollack [15] on the constant scalar curvature equation in conformal geometry. In this way, an infinite dimensional problem is reduced to solving a finite dimensional balancing condition. The balancing condition ensures that the energy is stationary with respect to variations of the glueing data. For the constant scalar curvature equation, an asymptotic expansion for the energy of a multi-peak solution was calculated in A. Bahri and J. M. Coron [2]. Similar results have been proved for constant mean curvature hypersurfaces (see [4, 5, 6]).
In the examples mentioned above, the blow-up set consists of isolated points and each approximate solution is characterized by a finite-dimensional set of gluing data. In our situation, the space of approximate solutions is infinite-dimensional. This leads to technical difficulties. A similar problem occurs in the work of F. Pacard and M. Ritoré [13] on the gradient theory of phase transitions, and in our earlier work on the Ginzburg-Landau equations in higher dimensions.

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2 The kernel of the linearized operator on $\mathbb{R}^4$

The basic instanton on $\mathbb{R}^4$ is given by the formula

\[
B_1 = \frac{-y_2 i - y_3 j - y_4 k}{\varepsilon^2 + |y|^2},
\]
\[
B_2 = \frac{y_1 i - y_4 j + y_3 k}{\varepsilon^2 + |y|^2},
\]
\[
B_3 = \frac{y_4 i + y_1 j - y_2 k}{\varepsilon^2 + |y|^2},
\]
\[
B_4 = \frac{-y_3 i + y_2 j + y_1 k}{\varepsilon^2 + |y|^2},
\]

where $\{i, j, k\}$ is the standard basis of $\mathfrak{su}(2)$, i.e.

\[
i(\frac{\partial}{\partial y_1}) = -\frac{\partial}{\partial y_2}, \quad i(\frac{\partial}{\partial y_2}) = \frac{\partial}{\partial y_1}, \quad i(\frac{\partial}{\partial y_3}) = \frac{\partial}{\partial y_4}, \quad i(\frac{\partial}{\partial y_4}) = -\frac{\partial}{\partial y_3},
\]
\[
j(\frac{\partial}{\partial y_1}) = -\frac{\partial}{\partial y_3}, \quad j(\frac{\partial}{\partial y_2}) = -\frac{\partial}{\partial y_4}, \quad j(\frac{\partial}{\partial y_3}) = \frac{\partial}{\partial y_1}, \quad j(\frac{\partial}{\partial y_4}) = \frac{\partial}{\partial y_2},
\]
\[
k(\frac{\partial}{\partial y_1}) = \frac{\partial}{\partial y_4}, \quad k(\frac{\partial}{\partial y_2}) = \frac{\partial}{\partial y_3}, \quad k(\frac{\partial}{\partial y_3}) = -\frac{\partial}{\partial y_2}, \quad k(\frac{\partial}{\partial y_4}) = -\frac{\partial}{\partial y_1}.
\]

Note that $[i, j] = 2k$, $[j, k] = 2i$, $[k, i] = 2j$. The curvature of $B$ satisfies

\[
F_{B,12} = -F_{B,34} = \frac{2\varepsilon^2 i}{(\varepsilon^2 + |y|^2)^2},
\]
\[
F_{B,13} = -F_{B,42} = \frac{2\varepsilon^2 j}{(\varepsilon^2 + |y|^2)^2},
\]
\[
F_{B,14} = -F_{B,23} = \frac{2\varepsilon^2 k}{(\varepsilon^2 + |y|^2)^2}.
\]

We first recall some well-known facts about this solution. In the first step, we construct a frame which is asymptotically parallel as $|y| \to \infty$. 

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Lemma 2.1. Let
\[ u = \left( \varepsilon^2 + |y|^2 \right)^{-\frac{1}{2}} \left( \mu y_\rho + r_\rho y_\sigma \right) \frac{\partial}{\partial y_\rho} \]
for some \( \mu \in \mathbb{R} \) and \( r \in \Lambda_+^2 \mathbb{R}^4 \). Then
\[ D_{B,\alpha} u = \varepsilon^2 \left( \varepsilon^2 + |y|^2 \right)^{-\frac{3}{2}} \left( \mu \delta_\rho \alpha + r_\rho \alpha \right) \frac{\partial}{\partial y_\rho}. \]

Proof. We only consider the case \( \mu = 1, r = 0 \). By definition of \( B \), we have
\[
\left( \varepsilon^2 + |y|^2 \right)^{\frac{3}{2}} D_{B,1} u = \left( \varepsilon^2 + |y|^2 \right) \frac{\partial}{\partial y_1} - y_1^2 \frac{\partial}{\partial y_1} - y_1 y_2 \frac{\partial}{\partial y_2} - y_1 y_3 \frac{\partial}{\partial y_3} - y_1 y_4 \frac{\partial}{\partial y_4}
\]
\[
+ y_1 \left( y_2 \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_3} + y_4 \frac{\partial}{\partial y_4} \right)
\]
\[
+ y_2 \left( -y_2 \frac{\partial}{\partial y_1} + y_3 \frac{\partial}{\partial y_4} - y_4 \frac{\partial}{\partial y_3} \right)
\]
\[
+ y_3 \left( -y_2 \frac{\partial}{\partial y_4} - y_3 \frac{\partial}{\partial y_1} + y_4 \frac{\partial}{\partial y_2} \right)
\]
\[
+ y_4 \left( y_2 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_2} - y_4 \frac{\partial}{\partial y_1} \right)
\]
\[
= \varepsilon^2 \frac{\partial}{\partial y_1},
\]

\[
\left( \varepsilon^2 + |y|^2 \right)^{\frac{3}{2}} D_{B,2} u = \left( \varepsilon^2 + |y|^2 \right) \frac{\partial}{\partial y_2} - y_2 y_1 \frac{\partial}{\partial y_2} - y_2^2 \frac{\partial}{\partial y_1} - y_2 y_3 \frac{\partial}{\partial y_3} - y_2 y_4 \frac{\partial}{\partial y_4}
\]
\[
+ y_1 \left( -y_1 \frac{\partial}{\partial y_2} + y_4 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_4} \right)
\]
\[
+ y_2 \left( y_1 \frac{\partial}{\partial y_1} + y_4 \frac{\partial}{\partial y_4} + y_3 \frac{\partial}{\partial y_3} \right)
\]
\[
+ y_3 \left( y_1 \frac{\partial}{\partial y_4} - y_4 \frac{\partial}{\partial y_1} - y_3 \frac{\partial}{\partial y_2} \right)
\]
\[
+ y_4 \left( -y_1 \frac{\partial}{\partial y_3} - y_4 \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_1} \right)
\]
\[
= \varepsilon^2 \frac{\partial}{\partial y_2}.
\]
\[(\varepsilon^2 + |y|^2)^{\frac{1}{2}} D_{B,3} u = (\varepsilon^2 + |y|^2) \frac{\partial}{\partial y_3} - y_3 y_1 \frac{\partial}{\partial y_1} - y_3 y_2 \frac{\partial}{\partial y_2} - y_3 \frac{\partial}{\partial y_3} - y_3 y_4 \frac{\partial}{\partial y_4} + y_1 \left( -y_4 \frac{\partial}{\partial y_2} - y_1 \frac{\partial}{\partial y_3} + y_2 \frac{\partial}{\partial y_4} \right) \\
+ y_2 \left( y_4 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_4} - y_2 \frac{\partial}{\partial y_3} \right) \\
+ y_3 \left( y_4 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_4} + y_2 \frac{\partial}{\partial y_3} \right) \\
+ y_4 \left( -y_4 \frac{\partial}{\partial y_3} + y_1 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_1} \right) \\
= \varepsilon^2 \frac{\partial}{\partial y_3}, \]

\[(\varepsilon^2 + |y|^2)^{\frac{1}{2}} D_{B,4} u = (\varepsilon^2 + |y|^2) \frac{\partial}{\partial y_4} - y_4 y_1 \frac{\partial}{\partial y_1} - y_4 y_2 \frac{\partial}{\partial y_2} - y_4 y_3 \frac{\partial}{\partial y_3} - y_4 \frac{\partial}{\partial y_4} + y_1 \left( y_3 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_3} - y_1 \frac{\partial}{\partial y_4} \right) \\
+ y_2 \left( -y_3 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial y_4} + y_1 \frac{\partial}{\partial y_3} \right) \\
+ y_3 \left( -y_3 \frac{\partial}{\partial y_4} + y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2} \right) \\
+ y_4 \left( y_3 \frac{\partial}{\partial y_3} + y_2 \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_1} \right) \\
= \varepsilon^2 \frac{\partial}{\partial y_4}. \]

The remaining cases are left to the reader.

**Lemma 2.2.** For every \( r \in \Lambda^2_+ \mathbb{R}^4 \), we have the identity

\( D_B (r_{\rho \sigma} y_\sigma B_\rho) = -F_B (r_{\rho \sigma} y_\sigma \frac{\partial}{\partial y_\rho}) \).

**Proof.** By direct calculation, one can see that

\( r_{\rho \sigma} y_\sigma \partial_\rho B_\alpha + r_{\rho \alpha} B_\rho = 0 \)

for all \( r \in \Lambda^2_+ \mathbb{R}^4 \). This implies

\( r_{\rho \sigma} y_\sigma F_{B,\rho \alpha} = r_{\rho \sigma} y_\sigma (\partial_\rho B_\alpha - \partial_\alpha B_\rho + [B_\rho, B_\alpha]) \\
= r_{\rho \sigma} y_\sigma (\partial_\rho B_\alpha - D_{B,\rho \alpha} B_\rho) \\
= r_{\rho \sigma} y_\sigma \partial_\rho B_\alpha + r_{\rho \alpha} B_\rho - D_{B,\rho \alpha} (r_{\rho \sigma} y_\sigma B_\rho) \\
= -D_{B,\rho \alpha} (r_{\rho \sigma} y_\sigma B_\rho). \)
This proves the assertion.

In the second step, we study the linearized operator (cf. [17]), which we denote by $L_B$. The operator $L_B$ is given by the formula

$$L_B a = D^*_B D_B a - *[F_B, a].$$

Using the Weitzenböck formula

$$D^*_B D_B a + D_B D^*_B a = \nabla^*_B \nabla_B a - *[F_B, a],$$

we obtain

$$L_B a + D_B D^*_B a = \nabla^*_B \nabla_B a - 2 * [F_B, a].$$

In particular, $L_B + D_B D^*_B$ is an elliptic operator.

**Lemma 2.3.** For every $a \in \Omega^1(\mathbb{R}^4)$ we have

$$L_B a + D_B D^*_B a = 2 D^*_B P_B D_B a + D_B D^*_B a.$$

Hence, the operator $L_B + D_B D^*_B$ is positive semidefinite.

**Proof.** By definition of $L_B$, we have

$$L_B a = D^*_B D_B a - *[F_B, a]$$

$$= D^*_B D_B a - *D_B D_B a$$

$$= D^*_B D_B a + D^*_B * D_B a$$

$$= 2 D^*_B P_B D_B a.$$

From this the assertion follows.

Our aim is to describe the kernel of the operator $L_B + D_B D^*_B : C^{2, \gamma}_{1+\nu}(\mathbb{R}^4) \to C^{\gamma}_{3+\nu}(\mathbb{R}^4)$.

**Proposition 2.4.** Let

$$a = F_B \left( \varepsilon w_\rho \frac{\partial}{\partial y_\rho} + \mu y_\rho \frac{\partial}{\partial y_\rho} + r_{\rho\sigma} y_\sigma \frac{\partial}{\partial y_\rho}, \right)$$

for some $w \in \mathbb{R}^4$, $\mu \in \mathbb{R}$, and $r \in \Lambda^2_{\mathbb{R}} \mathbb{R}^4$. Then $a$ satisfies $P_B D_B a = 0$ and $D^*_B a = 0$. 

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Proof. Using the Bianchi identity, we obtain
\[ DBa = \varepsilon w_\rho D_{B,\rho} F_B + \mu y_\rho D_{B,\rho} F_B + r_{\rho\sigma} y_\sigma D_{B,\rho} F_B + 2\mu F_B. \]
Since \( F_B \in \Lambda_+^2 \mathbb{R}^4 \), this implies \( P_a D_B a = 0 \). Similarly, we obtain
\[ D_B^* a = (D_B^* F_B) \left( \varepsilon w_\rho \frac{\partial}{\partial y_\rho} + \mu y_\rho \frac{\partial}{\partial y_\rho} + r_{\rho\sigma} y_\sigma \frac{\partial}{\partial y_\rho} \right) = 0. \]
This proves the assertion.

Corollary 2.5. Let
\[ a = F_B \left( \varepsilon w_\rho \frac{\partial}{\partial y_\rho} + \mu y_\rho \frac{\partial}{\partial y_\rho} + r_{\rho\sigma} y_\sigma \frac{\partial}{\partial y_\rho} \right) \]
for some \( w \in \mathbb{R}^4 \), \( \mu \in \mathbb{R} \), and \( r \in \Lambda_+^2 \mathbb{R}^4 \). Then \( a \) satisfies \( L_B a + D_B D_B^* a = 0 \).

We now prove the converse statement. Due to the conformal invariance of the Yang-Mills equations in dimension 4, we may lift the problem on \( S^4 \). The kernel of the linearized operator on \( S^4 \) is described in the following result which is well-known.

Lemma 2.6. Let \( a \) be a 1-form on \( S^4 \) such that \( L_{B,g_{S^4}} a = 0 \). Then there exists some \( w \in \mathbb{R}^4 \), \( \mu \in \mathbb{R} \), and an infinitesimal gauge transformation \( u \) such that
\[ a = F_B \left( \varepsilon w_\rho \frac{\partial}{\partial y_\rho} + \mu y_\rho \frac{\partial}{\partial y_\rho} + r_{\rho\sigma} y_\sigma \frac{\partial}{\partial y_\rho} \right) + D_B u. \]

Proposition 2.7. Let \( 0 < \nu < 1 \). Assume that \( a \in C_{1+\nu,2}^\infty (\mathbb{R}^4) \) satisfies
\[ L_B a + D_B D_B^* a = 0. \]
Then \( a \) is of the form
\[ a = F_B \left( \varepsilon w_\rho \frac{\partial}{\partial y_\rho} + \mu y_\rho \frac{\partial}{\partial y_\rho} + r_{\rho\sigma} y_\sigma \frac{\partial}{\partial y_\rho} \right) \]
for some \( w \in \mathbb{R}^4 \), \( \mu \in \mathbb{R} \), and \( r \in \Lambda_+^2 \mathbb{R}^4 \).

Proof. Using integration by parts, we deduce that \( L_B a = 0 \) and \( D_B^* a = 0 \). We now lift the problem to \( S^4 \). The round metric on \( S^4 \) is given by
\[ g_{S^4} = \frac{4\varepsilon^2}{(\varepsilon^2 + |y|^2)^2} g_{\mathbb{R}^4}. \]
Using the estimate
\[ |a(y)|_{\mathbb{R}^4} = O(|y|^{-1-\nu}), \]
we obtain
\[ |a(y)|_{S^4} = \frac{\varepsilon^2 + |y|^2}{2\varepsilon} |a(y)|_{\mathbb{R}^4} = O(|y|^{1-\nu}). \]

The conformal invariance of the Yang-Mills equations implies that \( L_{B,\mathbb{R}^4} a = 0 \). From this it follows that
\[ a = F_B \left( \varepsilon w_\rho \frac{\partial}{\partial y_\rho} + \mu y_\rho \frac{\partial}{\partial y_\rho}, \cdot \right) + D_B u, \]

where \( w \in \mathbb{R}^4, \mu \in \mathbb{R} \), and \( u \) denotes an infinitesimal gauge transformation. The function \( u \) can be written as
\[ u = -r_{\rho\sigma} y_\sigma B_\rho + u_0, \]
where \( r \in \Lambda^2_+ \mathbb{R}^4 \) and \( u_0 = O(|y|^{-\nu}) \). Therefore, we obtain
\[ a = F_B \left( \varepsilon w_\rho \frac{\partial}{\partial y_\rho} + \mu y_\rho \frac{\partial}{\partial y_\rho} + r_{\rho\sigma} y_\sigma \frac{\partial}{\partial y_\rho}, \cdot \right) + D_B u_0. \]

Since \( D_B^* a = 0 \), it follows that \( D_B^* D_B u_0 = 0 \). Since \( u_0 \in C^3_0(\mathbb{R}^4) \), we conclude that \( u_0 = 0 \). Therefore, we obtain
\[ a = F_B \left( \varepsilon w_\rho \frac{\partial}{\partial y_\rho} + \mu y_\rho \frac{\partial}{\partial y_\rho} + r_{\rho\sigma} y_\sigma \frac{\partial}{\partial y_\rho}, \cdot \right). \]

This proves the assertion.

### 3 The model problem on \( \mathbb{R}^{n-4} \times \mathbb{R}^4 \)

Let \( B \) be a connection on \( \mathbb{R}^{n-4} \times \mathbb{R}^4 \) which is invariant under translations along the \( \mathbb{R}^{n-4} \) factor and agrees with the basic instanton along the \( \mathbb{R}^4 \) factor. This implies
\[
B(e_1^+) = \frac{-y_2 i - y_3 j - y_4 k}{\varepsilon^2 + |y|^2},
\]
\[
B(e_2^+) = \frac{y_1 i - y_4 j + y_3 k}{\varepsilon^2 + |y|^2},
\]
\[
B(e_3^+) = \frac{y_4 i + y_1 j - y_2 k}{\varepsilon^2 + |y|^2},
\]
\[
B(e_4^+) = \frac{-y_3 i + y_2 j + y_1 k}{\varepsilon^2 + |y|^2},
\]

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where
\[ i(e_1^+) = -e_2^+, \quad i(e_2^+) = e_1^+, \quad i(e_3^+) = e_4^+, \quad i(e_4^+) = -e_3^+, \]
\[ j(e_1^+) = -e_3^+, \quad j(e_2^+) = -e_4^+, \quad j(e_3^+) = e_1^+, \quad j(e_4^+) = e_2^+, \]
\[ \mathfrak{t}(e_1^+) = -e_4^+, \quad \mathfrak{t}(e_2^+) = e_3^+, \quad \mathfrak{t}(e_3^+) = -e_2^+, \quad \mathfrak{t}(e_4^+) = e_1^+. \]
Furthermore, \( B(e_i) = 0 \) for \( 1 \leq i \leq 4 \).

The linearized operator satisfies
\[ L_B a = D_B^* D_B a + (-1)^n \ast [F_B, a] \]

Using the Weitzenböck formula
\[ D_B^* D_B a + D_B D_B^* a = \nabla_B^* \nabla_B a + (-1)^n \ast [F_B, a], \]
we obtain
\[ L_B a + D_B D_B^* a = \nabla_B^* \nabla_B a + (-1)^n 2 \ast [F_B, a], \]
For abbreviation, let \( \mathbb{L}_B = L_B + D_B D_B^* \).

We define the weighted Hölder space \( C_\nu^\gamma(\mathbb{R}^{n-4} \times \mathbb{R}^4) \) by
\[
\| u \|_{C_\nu^\gamma(\mathbb{R}^{n-4} \times \mathbb{R}^4)} = \sup (\varepsilon + |y|)^\nu |u(x, y)| + \sup_{4(|x_1 - x_2| + |y_1 - y_2|) \leq |x_1 - x_2| + |y_1 - y_2|} \varepsilon + |y_1| + |y_2|)^\nu + \gamma \left[ u(x_1, y_1) - u(x_2, y_2) \right],
\]
More generally, we define
\[
\| u \|_{C_{\nu, \gamma}^{k, \gamma}(\mathbb{R}^{n-4} \times \mathbb{R}^4)} = \sum_{l=0}^{k} \| \nabla^l u \|_{C_\nu^\gamma(\mathbb{R}^{n-4} \times \mathbb{R}^4)}.
\]

Let \( \mathcal{E}_{\nu, \gamma}^{k, \gamma}(\mathbb{R}^{n-4} \times \mathbb{R}^4) \) be the set of all \( a \in \Omega^1(\mathbb{R}^{n-4} \times \mathbb{R}^4) \) such that \( a \in C_{\nu, \gamma}^{k, \gamma}(\mathbb{R}^{n-4} \times \mathbb{R}^4) \) and
\[
\int_{\{x\} \times \mathbb{R}^4} \sum_{\alpha=1}^{4} \langle a(e_\alpha^+) \cdot F_B(X, e_\alpha^+) \rangle = 0
\]
for all \( x \in \mathbb{R}^{n-4} \) and all vector fields of the form
\[ X = \varepsilon w_\rho e_\rho^+ + \mu y_\rho e_\rho^+ + r_{\rho\sigma} y_\sigma e_\rho^+ \]
with \( w \in \mathbb{R}^4, \ \mu \in \mathbb{R}, \) and \( r \in \Lambda^2_+ \mathbb{R}^4. \)
Proposition 3.1. The operator $L_B$ maps $\mathcal{E}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$ into $\mathcal{E}^{\gamma}_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$.

Proof. It is obvious from the definition that $L_B$ maps $\mathcal{C}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$ into $\mathcal{C}^{\gamma}_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$. We now assume that $a \in \mathcal{C}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$ satisfies

$$\int_{\{x\} \times \mathbb{R}^4} \sum_{\alpha=1}^{4} a(e_{\alpha}^\perp), F_B(X, e_{\alpha}^\perp) = 0$$

for all $x \in \mathbb{R}^{n-4}$ and all vector fields of the form

$$X = \varepsilon w_\rho e_\rho^\perp + \mu y_\rho e_\rho^\perp + r_{\rho\sigma} y_\sigma e_\rho^\perp$$

with $w \in \mathbb{R}^4$, $\mu \in \mathbb{R}$, and $r \in \Lambda^2_+ \mathbb{R}^4$. Taking derivatives in horizontal direction, we obtain

$$\int_{\{x\} \times \mathbb{R}^4} \sum_{\alpha=1}^{4} \sum_{j=1}^{n-4} (\partial_j \partial_j a(e_{\alpha}^\perp), F_B(X, e_{\alpha}^\perp)) = 0.$$ 

Furthermore, integration by parts gives

$$\int_{\{x\} \times \mathbb{R}^4} \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} (D_{B,e_\beta^\perp} D_{B,e_\beta^\perp} a(e_{\alpha}^\perp), F_B(X, e_{\alpha}^\perp))$$

$$+ 2 \int_{\{x\} \times \mathbb{R}^4} \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} ([F_B(e_{\alpha}^\perp, e_{\beta}^\perp), a(e_{\alpha}^\perp)], F_B(X, e_{\alpha}^\perp))$$

$$= \int_{\{x\} \times \mathbb{R}^4} \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} (a(e_{\alpha}^\perp), D_{B,e_\beta^\perp} D_{B,e_\beta^\perp} F_B(X, e_{\alpha}^\perp))$$

$$+ 2 \int_{\{x\} \times \mathbb{R}^4} \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} (a(e_{\alpha}^\perp), [F_B(e_{\alpha}^\perp, e_{\beta}^\perp), F_B(X, e_{\alpha}^\perp)])$$

$$= 0.$$ 

Thus, we conclude that

$$\int_{\{x\} \times \mathbb{R}^4} \sum_{\alpha=1}^{4} \langle (L_B a)(e_{\alpha}^\perp), F_B(X, e_{\alpha}^\perp) \rangle = 0$$

for all $x \in \mathbb{R}^{n-4}$ and all vector fields of the form

$$X = \varepsilon w_\rho e_\rho^\perp + \mu y_\rho e_\rho^\perp + r_{\rho\sigma} y_\sigma e_\rho^\perp$$

with $w \in \mathbb{R}^4$, $\mu \in \mathbb{R}$, and $r \in \Lambda^2_+ \mathbb{R}^4$. 

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Proposition 3.2. Let $0 < \nu < 1$, $b \in C^\gamma_{3+\nu}(\mathbb{R}^4)$, and $\eta \in S(\mathbb{R}^{n-4})$. Moreover, assume that the Fourier transform of $\eta$ satisfies $\hat{\eta}(\xi) = 0$ for $|\xi| \leq \delta$ for some $\delta > 0$. Then there exists a 1-form $a \in C_{1+\nu}^{2,\gamma}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$ such that

$$L_B a = \eta(x) b(y).$$

Proof. We perform a Fourier transformation in the $\mathbb{R}^{n-4}$ variables. Let

$$\eta(x) = \int_{\mathbb{R}^{n-4}} e^{ix\xi} \hat{\eta}(\xi) \, d\xi.$$ 

For every $\xi \in \mathbb{R}^{n-4}$, there exists a 1-form $\hat{a}(\xi, \cdot) \in C^{2,\gamma}_{1+\nu}(\mathbb{R}^4)$ such that

$$\sum_{\beta=1}^4 D_{B,\beta} D_{B,\beta} \hat{a}_\alpha(\xi, y) + 2 \sum_{\beta=1}^4 [F_{B,\alpha\beta}, \hat{a}_\beta(\xi, y)] - |\xi|^2 \hat{a}_\alpha(\xi, y) = -b_\alpha(y)$$

for $1 \leq \alpha \leq 4$ and

$$\sum_{\beta=1}^4 D_{B,\beta} D_{B,\beta} \hat{a}_i(\xi, y) - |\xi|^2 \hat{a}_i(\xi, y) = -b_i(y)$$

for $1 \leq i \leq n-4$. We now define a 1-form $a$ by

$$a_\alpha(x, y) = \int_{\mathbb{R}^{n-4}} e^{ix\xi} \hat{\eta}(\xi) \hat{a}_\alpha(\xi, y) \, d\xi$$

for $1 \leq \alpha \leq 4$ and

$$a_i(x, y) = \int_{\mathbb{R}^{n-4}} e^{ix\xi} \hat{\eta}(\xi) \hat{a}_i(\xi, y) \, d\xi$$

for $1 \leq i \leq n-4$. Then the 1-form $a$ satisfies

$$\sum_{j=1}^{n-4} \partial_j \partial_j a_\alpha + \sum_{\beta=1}^4 D_{B,\beta} D_{B,\beta} a_\alpha + 2 \sum_{\beta=1}^4 [F_{B,\alpha\beta}, a_\beta] = -\eta(x) b_\alpha(y)$$

for $1 \leq \alpha \leq 4$ and

$$\sum_{j=1}^{n-4} \partial_j \partial_j a_i + \sum_{\beta=1}^4 D_{B,\beta} D_{B,\beta} a_i = -\eta(x) b_i(y)$$

for $1 \leq i \leq n-4$. From this we deduce that $L_B a = \eta(x) b(y)$.

Proposition 3.3. Let $0 < \nu < 1$, and suppose that $a \in C^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$ satisfies $L_B a = 0$. Then $a = 0$.  

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Proof. Let \( b \in C^\gamma_{3+\nu}(\mathbb{R}^4) \) and \( \zeta \in S(\mathbb{R}^{n-4}) \) be given. We define a function \( \eta \in S(\mathbb{R}^{n-4}) \) by \( \eta(x) = \zeta(x + x_0) - \zeta(x) \). Then the Fourier transform of \( \eta \) satisfies \( \hat{\eta}(0) = 0 \). We approximate \( \eta \) by functions \( \eta_\delta \) such that

\[
\|D^{n-4}(\hat{\eta} - \hat{\eta}_\delta)\|_{L^p(\mathbb{R}^{n-4})} \leq C \delta^{\frac{n-4}{p}},
\]

for all \( p \geq 2 \). From this it follows that

\[
\|\eta - \eta_\delta\|_{L^1(\mathbb{R}^{n-4})} \leq \left\| (1 + |x|)^{-n-4} \right\|_{L^p(\mathbb{R}^{n-4})} \left\| (1 + |x|)^{n-4} (\eta - \eta_\delta) \right\|_{L^p(\mathbb{R}^{n-4})} \leq C \delta^{\frac{n-4}{p}}
\]

for all \( p \geq 2 \). This implies

\[
\|\eta - \eta_\delta\|_{L^1(\mathbb{R}^{n-4})} \to 0
\]
as \( \delta \to 0 \).

For each \( \delta > 0 \), the 1-form \( \eta_\delta(x) b(y) \) lies in the image of \( \mathbb{L}_B \). Since \( a \) belongs to the kernel of \( \mathbb{L}_B \), we obtain

\[
\int_{\mathbb{R}^{n-4} \times \mathbb{R}^4} \langle a(x, y), \eta_\delta(x) b(y) \rangle = 0.
\]

Letting \( \delta \to 0 \), we obtain

\[
\int_{\mathbb{R}^{n-4} \times \mathbb{R}^4} \langle a(x, y), \eta(x) b(y) \rangle = 0,
\]

hence

\[
\int_{\mathbb{R}^{n-4} \times \mathbb{R}^4} \langle a(x, y), \zeta(x) b(y) \rangle = \int_{\mathbb{R}^{n-4} \times \mathbb{R}^4} \langle a(x - x_0, y), \zeta(x) b(y) \rangle.
\]

Since \( b \) and \( \zeta \) are arbitrary, we conclude that \( a(x, y) = a(x - x_0, y) \). Therefore, \( a(x, y) \) is constant in \( x \). Using Proposition 2.7, we obtain

\[
a = F_B(X, \cdot),
\]

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where $X$ is a vector field of the form

$$X = \varepsilon w \rho e_\rho + \mu y \rho e_\rho + r \rho \sigma y \sigma e_\rho = 0$$

for suitable $w \in \mathbb{R}^4$, $\mu \in \mathbb{R}$, and $r \in \Lambda^2_+ \mathbb{R}^4$. This proves the assertion.

**Proposition 3.4.** Let $0 < \nu < 1$. Then we have the estimate

$$\|a\|_{C^{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)} \leq C \|L_B a\|_{C^{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)}$$

for all $a \in \mathcal{E}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$.

**Proof.** By Schauder estimates, it suffices to prove that

$$\sup (\varepsilon + |y|)^{1+\nu} |a(x, y)| \leq C \sup (\varepsilon + |y|)^{3+\nu} |L_B a(x, y)|.$$

Suppose that this estimate fails. Then there exists a sequence of 1-forms $a^{(j)} \in \mathcal{E}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$ such that

$$\sup (\varepsilon + |y|)^{1+\nu} |a^{(j)}(x, y)| = 1$$

and

$$\sup (\varepsilon + |y|)^{3+\nu} |L_B a^{(j)}(x, y)| \to 0.$$

Then there exists a sequence of points $(x_j, y_j) \in \mathbb{R}^{n-4} \times \mathbb{R}^4$ such that

$$\sup (\varepsilon + |y_j|)^{1+\nu} |a^{(j)}(x_j, y_j)| \geq \frac{1}{2}.$$

There are two possibilities:

(i) Suppose that the sequence $|y_j|$ is bounded. After passing to a subsequence, we may assume that the sequence $a^{(j)}$ converges to a 1-form $a \in \mathcal{E}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$ such that

$$\sup (\varepsilon + |y|)^{1+\nu} |a(x, y)| \leq 1$$

and

$$L_B a = 0.$$

Using Proposition 3.3, we conclude that $a = 0$. This is a contradiction.

(ii) We now assume that $|y_j| \to \infty$. Let

$$\tilde{a}^{(j)}(x, y) = |y_j|^{1+\nu} a^{(j)}(x + x_j, |y_j| y).$$


After passing to a subsequence, we may assume that the sequence $\tilde{a}^{(j)}$ converges to a 1-form $\tilde{a}$ such that

$$\sup |y|^{1+\nu} |\tilde{a}(x,y)| \leq 1$$

and

$$d^*d\tilde{a} + dd^*\tilde{a} = 0.$$ 

Thus, we conclude that $\tilde{a} = 0$. This is a contradiction.

**Proposition 3.5.** Let $0 < \nu < 1$. Assume that $b \in C^\gamma_{3+\nu}(\mathbb{R}^4)$ satisfies

$$\int_{\mathbb{R}^4} \langle b, F_B(X, \cdot) \rangle = 0$$

for all $x \in \mathbb{R}^{n-4}$ and all vector fields of the form

$$X = \varepsilon \rho \ e_{\rho}^\perp + \mu y_{\rho} \ e_{\rho}^\perp + r_{\rho\sigma} y_{\sigma} \ e_{\rho}^\perp$$

with $w \in \mathbb{R}^4$, $\mu \in \mathbb{R}$, and $r \in \Lambda^2_+ \mathbb{R}^4$. Moreover, let $\eta \in S(\mathbb{R}^{n-4})$. Then there exists a 1-form $a \in \mathcal{E}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$ such that

$$L_B a = \eta(x) b(y).$$

**Proof.** Let

$$\eta(x) = \int_{\mathbb{R}^{n-4}} e^{ix\xi} \tilde{\eta}(\xi) \ d\xi.$$ 

For every $\xi \in \mathbb{R}^{n-4}$, there exists a 1-form $\hat{a}(\xi, \cdot) \in C^{2,\gamma}_{1+\nu}(\mathbb{R}^4)$ such that

$$\sum_{\beta=1}^{4} D_{B,\beta} D_{B,\beta} \hat{a}_\alpha(\xi,y) + 2 \sum_{\beta=1}^{4} [F_{B,\alpha\beta}, \hat{a}_\beta(\xi,y)] - |\xi|^2 \hat{a}_\alpha(\xi,y) = -b_\alpha(y)$$

for $1 \leq \alpha \leq 4$ and

$$\sum_{\beta=1}^{4} D_{B,\beta} D_{B,\beta} \hat{a}_i(\xi,y) - |\xi|^2 \hat{a}_i(\xi,y) = -b_i(y)$$

for $1 \leq i \leq n-4$. Furthermore, $\hat{a}(\xi, \cdot)$ satisfies

$$\int_{\mathbb{R}^4} \langle \hat{a}(\xi, \cdot), F_B(X, \cdot) \rangle = 0$$

for all $x \in \mathbb{R}^{n-4}$ and all vector fields of the form

$$X = \varepsilon \rho \ e_{\rho}^\perp + \mu y_{\rho} \ e_{\rho}^\perp + r_{\rho\sigma} y_{\sigma} \ e_{\rho}^\perp = 0.$$
with \( w \in \mathbb{R}^4, \mu \in \mathbb{R}, \) and \( r \in \Lambda^2_+ \mathbb{R}^4. \) We now define a 1-form \( a \in \mathcal{E}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4) \) by

\[
a(\alpha)(x, y) = \int_{\mathbb{R}^{n-4}} e^{ix\xi} \tilde{\eta}(\xi) \tilde{a}(\xi, y) d\xi
\]

for \( 1 \leq \alpha \leq 4 \) and

\[
a_i(x, y) = \int_{\mathbb{R}^{n-4}} e^{ix\xi} \tilde{\eta}(\xi) \tilde{a}_i(\xi, y) d\xi
\]

for \( 1 \leq i \leq n - 4. \) Then the 1-form \( a \) satisfies

\[
\sum_{j=1}^{n-4} \partial_j \partial_j a(\alpha)(x, y) + \sum_{\beta=1}^4 D_{B,\beta} D_{B,\beta} a(\alpha) + 2 \sum_{\beta=1}^4 [F_{B,\alpha\beta}, a(\beta)] = -\eta(x) b(\alpha)(y)
\]

for \( 1 \leq \alpha \leq 4 \) and

\[
\sum_{j=1}^{n-4} \partial_j \partial_j a_i(x, y) + \sum_{\beta=1}^4 D_{B,\beta} D_{B,\beta} a_i = -\eta(x) b_i(y)
\]

for \( 1 \leq i \leq n - 4. \) Thus, we conclude that \( a \in \mathcal{E}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4) \) and \( \mathbb{L}_B a = \eta(x) b(y). \)

**Corollary 3.6.** Let \( 0 < \nu < 1, \) and suppose that \( b \in \mathcal{E}^\gamma_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4) \) has compact support. Then there exists a 1-form \( a \in \mathcal{E}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4) \) such that

\[
\|a\|_{C^{1,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)} \leq C \|b\|_{\mathcal{C}^\gamma_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)}
\]

and

\[
\mathbb{L}_B a = b.
\]

**Proof.** It follows from Proposition 3.4 that the range of the operator \( \mathbb{L}_B: \mathcal{E}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4) \to \mathcal{E}^\gamma_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4) \) is a closed subspace of the Banach space \( \mathcal{E}^\gamma_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4). \) By Proposition 3.5, it contains all 1-forms of the form \( \eta(x) b(y), \) where \( \eta \in \mathcal{S}(\mathbb{R}^{n-4}) \) and \( b \in \mathcal{C}^\gamma_{3+\nu}(\mathbb{R}^4) \) satisfies

\[
\int_{\mathbb{R}^4} \langle b, F_B(X, \cdot) \rangle = 0
\]

for all \( x \in \mathbb{R}^{n-4} \) and all vector fields of the form

\[
X = \varepsilon w_\rho e_\rho^\perp + \mu y_\rho e_\rho^\perp + r_{\rho\sigma} y_\sigma e_\rho^\perp
\]

with \( w \in \mathbb{R}^4, \mu \in \mathbb{R}, \) and \( r \in \Lambda^2_+ \mathbb{R}^4. \) The assertion follows now by approximation.
Proposition 3.7. Let $0 < \nu < 1$. Suppose that $b \in \mathcal{E}^{2,\gamma}_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$ is supported in the set $\{(x,y) \in \mathbb{R}^{n-4} \times \mathbb{R}^4 : |x| \leq \delta, |y| \leq 2\delta^4\}$. Then there exists a 1-form $a \in C^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$ such that $a$ is supported in $\{(x,y) \in \mathbb{R}^{n-4} \times \mathbb{R}^4 : |x| \leq 2\delta, |y| \leq 2\delta^2\}$, and

$$\|a\|_{C^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)} \leq C \|b\|_{C^{2,\gamma}_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)}$$

and

$$\|\mathbb{L}_Ba - b\|_{C^{2,\gamma}_{3+\nu}(\{(x,y) \in \mathbb{R}^{n-4} \times \mathbb{R}^4 : |y| \leq 2\delta^4\})} \leq C \delta \|b\|_{C^{2,\gamma}_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)}$$

and

$$\|\mathbb{L}_Ba - b\|_{C^{2,\gamma}_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)} \leq C \log \delta^{-1} \|b\|_{C^{2,\gamma}_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)}.$$

Proof. By Corollary 3.6, there exists a 1-form $a \in \mathcal{E}^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)$ such that

$$\|a\|_{C^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)} \leq C \|b\|_{C^{2,\gamma}_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)}$$

and

$$\mathbb{L}_Ba = b.$$

Let $\zeta$ be a cut-off function on $\mathbb{R}^{n-4}$ such that $\zeta(x) = 1$ for $|x| \leq \delta$, $\zeta(x) = 0$ for $|x| \geq 2\delta$, and

$$\sup \delta |\nabla \zeta| + \sup \delta^2 |\nabla^2 \zeta| \leq C.$$

Furthermore, let $\eta$ be a cut-off function on $\mathbb{R}^{n-4}$ satisfying $\eta(y) = 1$ for $|y| \leq 2\delta^4$, $\eta(y) = 0$ for $|y| \geq 2\delta^2$, and

$$\sup |y| |\nabla \eta| + \sup |y|^2 |\nabla^2 \eta| \leq C |\log \delta|^{-1}.$$

Then we have the estimates

$$\|\eta \zeta a\|_{C^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)} \leq C \|b\|_{C^{2,\gamma}_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)}$$

and

$$\|\mathbb{L}_B(\zeta a) - b\|_{C^{2,\gamma}_{3+\nu}(\{(x,y) \in \mathbb{R}^{n-4} \times \mathbb{R}^4 : |y| \leq 2\delta^4\})}$$

$$= \|\mathbb{L}_B(\zeta a) - \zeta \mathbb{L}_Ba\|_{C^{2,\gamma}_{3+\nu}(\{(x,y) \in \mathbb{R}^{n-4} \times \mathbb{R}^4 : |y| \leq 2\delta^4\})}$$

$$\leq C \delta \|a\|_{C^{2,\gamma}_{1+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)}$$

$$\leq C \delta \|b\|_{C^{2,\gamma}_{3+\nu}(\mathbb{R}^{n-4} \times \mathbb{R}^4)}.$$
and

\[ \|L_B(\eta \zeta a) - b\|_{C^3_{\eta \zeta}([\mathbb{R}^{n-4} \times \mathbb{R}^4])} = \|L_B(\eta \zeta a - \eta \zeta L_B a\|_{C^3_{\eta \zeta}([\mathbb{R}^{n-4} \times \mathbb{R}^4])} \leq C |\log \delta|^{-1} \|a\|_{C^1_{\eta \zeta}([\mathbb{R}^{n-4} \times \mathbb{R}^4])} \leq C |\log \delta|^{-1} \|b\|_{C^3_{\eta \zeta}([\mathbb{R}^{n-4} \times \mathbb{R}^4])}. \]

From this the assertion follows.

### 4 Construction of the approximate solutions

In this section, we describe the construction of the approximate solutions. To this end, we assume that the normal bundle \( NS \) can be endowed with a \( SU(2) \)-structure \((J, \omega)\). Here, \( J \) is a complex structure and \( \omega \) is a complex volume form on \( NS \).

Let \( \nabla' = \nabla + \theta \) be a connection on the normal bundle \( NS \) such that \( \theta \) is a 1-form with values in the Lie algebra \( \Lambda^2_{\eta \zeta} NS \) and \((J, \omega)\) is parallel with respect to the connection \( \nabla' \). The 1-form \( \theta \) is uniquely determined by the covariant derivative of the pair \((J, \omega)\) with respect to the Levi-Civita connection \( \nabla \). Since \((J, \omega)\) is parallel with respect to \( \nabla' \), the connection induced by \( \nabla' \) on the bundle \( \Lambda^2_{\eta \zeta} NS \) is flat.

The connection \( \nabla' \) induces a splitting of the tangent space \( TNS \) into horizontal and vertical subspaces. Let \( \{e_i' : 1 \leq i \leq n-4\} \) be an orthonormal basis for the horizontal subspace with respect to \( \nabla' \), and let \( \{e_{\alpha}^\perp : 1 \leq \alpha \leq 4\} \) be a \( SU(2) \) basis for the vertical subspace \( V \).

In the first step, we define a connection on the pull-back bundle \( \pi^*NS \) of the normal bundle under the natural projection \( \pi : NS \to S \). Since we may identify a neighborhood of \( S \) in \( M \) with a neighborhood of the zero section in \( NS \), this gives a connection on a small neighborhood of \( S \) in \( M \). In the second step, we show that this connection can be extended to the whole of \( M \) using suitable cut-off functions.

The glueing data consist of a set \((v, \lambda, J, \omega)\), where \( v \) is a section of the normal bundle \( NS \), \( \lambda \) is a positive function on \( S \), and \((J, \omega)\) is a \( SU(2) \) structure on the normal bundle \( NS \). As in Section 3, let \( \{i, j, k\} \) be a basis
for the Lie algebra $\mathfrak{su}(NS)$ such that

$$
i(e_1) = -e_2, \quad i(e_2) = e_1, \quad i(e_3) = e_4, \quad i(e_4) = -e_3,$$

$$
j(e_1) = -e_3, \quad j(e_2) = -e_4, \quad j(e_3) = e_1, \quad j(e_4) = e_2,$$

$$
\mathfrak{k}(e_1) = -e_4, \quad \mathfrak{k}(e_2) = e_3, \quad \mathfrak{k}(e_3) = -e_2, \quad \mathfrak{k}(e_4) = e_1.
$$

We consider a connection of the form $D_A = \nabla' + A$. The vertical components of $A$ are defined by

$$
A(e_1^+) = \frac{-(y - \varepsilon v)_2 i - (y - \varepsilon v)_3 j - (y - \varepsilon v)_4 k}{\varepsilon^2 \lambda^2 + |y - \varepsilon v|^2},
$$

$$
A(e_2^+) = \frac{(y - \varepsilon v)_1 i - (y - \varepsilon v)_4 j + (y - \varepsilon v)_3 k}{\varepsilon^2 \lambda^2 + |y - \varepsilon v|^2},
$$

$$
A(e_3^+) = \frac{(y - \varepsilon v)_4 i + (y - \varepsilon v)_1 j - (y - \varepsilon v)_2 k}{\varepsilon^2 \lambda^2 + |y - \varepsilon v|^2},
$$

$$
A(e_4^+) = \frac{-(y - \varepsilon v)_3 i + (y - \varepsilon v)_2 j + (y - \varepsilon v)_1 k}{\varepsilon^2 \lambda^2 + |y - \varepsilon v|^2}.
$$

Since the basic instanton on $\mathbb{R}^4$ is $SU(2)$-equivariant, this definition is independent of the choice of $SU(2)$-frame $\{e_\alpha^+: 1 \leq \alpha \leq 4\}$. Furthermore, the horizontal components of $A$ are defined by

$$
A(e_\alpha^+) = -\varepsilon \nabla'_\alpha A(e_\rho^+) - \lambda^{-1} \nabla_i \lambda (y - \varepsilon v)_\rho A(e_\alpha^+)
$$

for $1 \leq i \leq n - 4$.

**Lemma 4.1.** The curvature of $A$ is given by

$$
F_A(e_i^+, e_\alpha^+) = -\left(\varepsilon \nabla'_i v_\rho + \lambda^{-1} \nabla_i \lambda (y - \varepsilon v)_\rho\right) F_A(e_\rho^+, e_\alpha^+)
$$

and

$$
F_A(e_i^+, e_j^+) = \left(\varepsilon \nabla'_i v_\rho + \lambda^{-1} \nabla_i \lambda (y - \varepsilon v)_\rho\right) \cdot \left(\varepsilon \nabla'_j v_\sigma + \lambda^{-1} \nabla_j \lambda (y - \varepsilon v)_\sigma\right) F_A(e_\rho^+, e_\sigma^+)
$$

$$
+ C_{ij} + A(C_{ij} (y - \varepsilon v)),
$$

where $C_{ij} \in \Lambda^2 NS$ is the curvature of the connection $\nabla'$.

**Proof.** By definition of $A$, we have

$$
F_A(e_i^+, e_\alpha^+) = \nabla^i e_\alpha^+ A(e_\alpha^+) - \nabla_{e_\alpha^+} A(e_i^+) + [A(e_i^+), A(e_\alpha^+)]
$$

$$
= -\varepsilon \nabla'_i v_\rho \left(\nabla^i e_\rho^+ A(e_\alpha^+) - \nabla_{e_\rho^+} A(e_\alpha^+) + [A(e_\rho^+), A(e_\alpha^+)]\right)
$$

$$
- \lambda^{-1} \nabla_i \lambda (y - \varepsilon v)_\rho \left(\nabla^i e_\rho^+ A(e_\alpha^+) - \nabla_{e_\rho^+} A(e_\alpha^+) + [A(e_\rho^+), A(e_\alpha^+)]\right)
$$

$$
= -\varepsilon \nabla'_i v_\rho F_A(e_\rho^+, e_\alpha^+)
$$

$$
- \lambda^{-1} \nabla_i \lambda (y - \varepsilon v)_\rho F_A(e_\rho^+, e_\alpha^+).
$$
Using Lemma 2.1, we obtain the following result. The assertion follows now from Lemma 4.1. 

**Proof.**

\[
F_A(e'_i, e'_j) = C_{ij} + \nabla_{e'_i}A(e'_j) - \nabla_{e'_j}A(e'_i) + [A(e'_i), A(e'_j)] - A([e'_i, e'_j]) \\
= \nabla_{e'_i}A(e'_j) - \nabla_{e'_j}A(e'_i) + [A(e'_i), A(e'_j)] + C_{ij} + A(C_{ij} y) \\
= \varepsilon \nabla_{i \rho} v \cdot \nabla_j v \sigma \left( \nabla_{\rho} A(e_{\sigma}) - \nabla_{\sigma} A(e_{\rho}) + [A(e_{\rho}), A(e_{\sigma})] \right) \\
+ \varepsilon \nabla_{i \rho} \lambda^{-1} \nabla_j \lambda (y - \varepsilon v)_{\sigma} \left( \nabla_{\rho} A(e_{\sigma}) - \nabla_{\sigma} A(e_{\rho}) + [A(e_{\rho}), A(e_{\sigma})] \right) \\
+ \lambda^{-1} \nabla_i \lambda (y - \varepsilon v)_{\rho} \varepsilon v_{\sigma} \left( \nabla_{\rho} A(e_{\sigma}) - \nabla_{\sigma} A(e_{\rho}) + [A(e_{\rho}), A(e_{\sigma})] \right) \\
+ C_{ij} + A(C_{ij} (y - \varepsilon v)) \\
= \varepsilon \nabla_{i \rho} v \cdot \nabla_j v \sigma F_A(e_{\rho}^+, e_{\sigma}^+) \\
+ \varepsilon \nabla_{i \rho} \lambda^{-1} \nabla_j \lambda (y - \varepsilon v)_{\sigma} F_A(e_{\rho}^+, e_{\sigma}^+) \\
+ \lambda^{-1} \nabla_i \lambda (y - \varepsilon v)_{\rho} \varepsilon v_{\sigma} F_A(e_{\rho}^+, e_{\sigma}^+) \\
+ C_{ij} + A(C_{ij} (y - \varepsilon v)).
\]

This proves the assertion.

Let \( \{ e_i : 1 \leq i \leq n - 4 \} \) be an orthonormal basis for the horizontal subspace with respect to the Levi-Civita connection \( \nabla \). Then we have the following result:

**Lemma 4.2.** The curvature of \( A \) satisfies

\[
F_A(e_i, e_{\alpha}^+) = -\left( \varepsilon \nabla_i v + \lambda^{-1} \nabla_i \lambda (y - \varepsilon v)_{\rho} + \theta_{i, \rho \sigma} (y - \varepsilon v)_{\sigma} \right) F_A(e_{\rho}^+, e_{\sigma}^+)
\]

and

\[
F_A(e_i, e_j) = \left( \varepsilon \nabla_i v + \lambda^{-1} \nabla_i \lambda (y - \varepsilon v)_{\rho} + \theta_{i, \rho \alpha} (y - \varepsilon v)_{\alpha} \right) \\
\cdot \left( \varepsilon \nabla_j v + \lambda^{-1} \nabla_j \lambda (y - \varepsilon v)_{\sigma} + \theta_{j, \sigma \beta} (y - \varepsilon v)_{\beta} \right) F_A(e_{\rho}^+, e_{\sigma}^+) \\
+ C_{ij} + A(C_{ij} (y - \varepsilon v)),
\]

where \( C_{ij} \in \Lambda^2 NS \) is the curvature of \( \nabla' \).

**Proof.** Since \( \nabla' = \nabla + \theta \), the orthonormal basis \( \{ e_i : 1 \leq i \leq n - 4 \} \) is related to the orthonormal basis \( \{ e'_i : 1 \leq i \leq n - 4 \} \) by

\[
e'_i = e_i + \theta_{i, \rho \sigma} y_{\sigma} e_{\rho}^+.
\]

The assertion follows now from Lemma 4.1.

Using Lemma 2.1, we obtain the following result.
Lemma 4.3. Suppose that $\mu$ is constant and $r$ is a section of the vector bundle $\Lambda^2 NS$ such that $\nabla' r = 0$. Let

$$u = (\varepsilon^2 \lambda^2 + |y - \varepsilon v|^2)^{-\frac{1}{2}} \left( \mu (y - \varepsilon v)_\rho + r_\rho \sigma (y - \varepsilon v)_\sigma \right) e_\rho$$

Then the covariant derivative of $u$ satisfies the estimate

$$\|D_A u\|_{C^1,\gamma(M)} \leq C \varepsilon^2.$$ 

Hence, as we move away from the submanifold $S$, the connection $A$ approaches a flat connection. Therefore, we can extend $A$ trivially to $M$.

Our aim is to derive estimates for $D_A^* F_A$ in $C^3(M)$. To this end, we assume that the glueing data $(v, \lambda, J, \omega)$ satisfy the estimates

$$\|v\|_{C^2,\gamma(S)} \leq K,$$

$$\|\lambda\|_{C^2,\gamma(S)} \leq K, \quad \inf \lambda \geq 1,$$

$$\|(J, \omega)\|_{C^2,\gamma(S)} \leq K$$

for some $K > 0$. All implicit constants will depend on $K$.

Proposition 4.4. If the set $(v, \lambda, J, \omega)$ is admissible, then we have the estimate

$$\|D_A^* F_A\|_{C^3(M)} \leq C \varepsilon^2.$$ 

Proof. Since $B$ is a Yang-Mills connection on $\mathbb{R}^4$, we have

$$\sum_{\beta=1}^4 D_{A,e_\beta^\perp} F_A(e_\alpha^\perp, e_\beta^\perp) = 0.$$ 

Using Proposition 2.4, we obtain

$$\sum_{\beta=1}^4 D_{A,e_\beta^\perp} (\nabla_i v_\rho F_A(e_\rho^\perp, e_\beta^\perp)) = 0,$$

$$\sum_{\beta=1}^4 D_{A,e_\beta^\perp} (\lambda^{-1} \nabla_i \lambda (y - \varepsilon v)_\rho F_A(e_\rho^\perp, e_\beta^\perp)) = 0,$$

$$\sum_{\beta=1}^4 D_{A,e_\beta^\perp} (\theta_i, \rho \sigma (y - \varepsilon v)_\sigma F_A(e_\rho^\perp, e_\beta^\perp)) = 0.$$ 

From this it follows that

\[ \sum_{\beta=1}^{4} D_{A,e_{\beta}} F_A(e_i, e_{\beta}^\perp) = 0. \]

Therefore, we obtain

\[ \|D^{*0}_A F_A\|_{C^2_3(M)} \leq C \varepsilon^2. \]

Here, \( g_0 \) denotes the product metric on \( NS \), i.e.

\[
\begin{align*}
g_0(e_i, e_j) &= \delta_{ij} \\
g_0(e_i, e_{\alpha}^\perp) &= 0 \\
g_0(e_{\alpha}^\perp, e_{\beta}^\perp) &= \delta_{\alpha\beta}.
\end{align*}
\]

Let \( g \) be the pull-back of the Riemannian metric on \( M \) under the exponential map \( \exp : NS \rightarrow M \). Then the metric \( g \) satisfies an asymptotic expansion of the form

\[
\begin{align*}
g(e_i, e_j) &= \delta_{ij} + 2 \sum_{\rho=1}^{4} h_{ij,\rho} y_{\rho} + O(|y|^2) \\
g(e_i, e_{\alpha}^\perp) &= O(|y|^2) \\
g(e_{\alpha}^\perp, e_{\beta}^\perp) &= \delta_{\alpha\beta} + O(|y|^2),
\end{align*}
\]

where \( h \) denotes the second fundamental form of \( S \). In particular, the volume form of \( g \) is related to the volume form of \( g_0 \) by

\[
\left( \frac{\det g}{\det g_0} \right)^{\frac{1}{2}} = 1 + H_{\rho} y_{\rho} + O(|y|^2),
\]

where \( H \) is the mean curvature vector of \( S \). Since the mean curvature of \( S \) is 0, we obtain

\[
\left( \frac{\det g}{\det g_0} \right)^{\frac{1}{2}} = 1 + O(|y|^2).
\]

Thus, we conclude that

\[ \|D^{*}_A F_A\|_{C^3_3(M)} \leq C \varepsilon^2 \]
5 Estimates for the linearized operator in weighted Hölder spaces

Our aim in this section is to analyze the mapping properties of the linearized operator $\mathbb{L}_A : \Omega^1(M) \to \Omega^1(M)$.

**Proposition 5.1.** Suppose that $b \in C^{3+\gamma}_{3+\nu}(M)$ is supported in the set $\{p \in M : \text{dist}(p, S) \leq 2\delta^4\}$ and satisfies

$$\int_{NS} \sum_{\alpha=1}^4 \langle b(e^\perp_{\alpha}), F_A(X, e^\perp_{\alpha}) \rangle = 0$$

for all $x \in S$ and all vector fields of the form

$$X = \varepsilon w_\rho e^\perp_\rho + \mu (y - \varepsilon v)_\rho e^\perp_\rho + r_{\rho\sigma} (y - \varepsilon v)_\sigma e^\perp_\rho$$

with $w \in NS$, $\mu \in \mathbb{R}$, and $r \in \Lambda^2_\perp NS$. Then there exists a 1-form $a \in C^{2\gamma}_{1+\nu}(M)$ which is supported in the region $\{p \in M : \text{dist}(p, S) \leq 2\delta^2\}$ such that

$$\|a\|_{C^{2\gamma}_{1+\nu}(M)} \leq C \|b\|_{C^{3+\gamma}_{3+\nu}(M)}$$

and

$$\|\mathbb{L}_A a - b\|_{C^{2\gamma}_{1+\nu}((p \in M : \text{dist}(p, S) \leq 2\delta^4))} \leq C \delta \|b\|_{C^{3+\gamma}_{3+\nu}(M)}$$

and

$$\|\mathbb{L}_A a - b\|_{C^{2\gamma}_{1+\nu}(M)} \leq C \log \delta^{-1} \|b\|_{C^{3+\gamma}_{3+\nu}(M)}.$$

**Proof.** Let $\{\zeta^{(j)} : 1 \leq j \leq j_0\}$ be a partition of unity on $S$ such that each function $\zeta^{(j)}$ is supported in a ball $B_{\delta}(p_j)$, and

$$|\{1 \leq j \leq j_0 : x \in B_{4\delta}(p_j)\}| \leq C$$

for all $x \in S$ and some uniform constant $C$. For each $1 \leq j \leq j_0$, there exists a 1-form $a^{(j)} \in C^{2\gamma}_{1+\nu}(M)$ which is supported in the region $\{(x, y) \in NS : x \in B_{2\delta}(p_j), |y| \leq 2\delta^2\}$ such that

$$\|a^{(j)}\|_{C^{2\gamma}_{1+\nu}(M)} \leq C \|\zeta^{(j)} b\|_{C^{3+\gamma}_{3+\nu}(M)}$$

and

$$\|\mathbb{L}_A a^{(j)} - \zeta^{(j)} b\|_{C^{2\gamma}_{1+\nu}((p \in M : \text{dist}(p, S) \leq 2\delta^4))} \leq C \delta \|\zeta^{(j)} b\|_{C^{3+\gamma}_{3+\nu}(M)}.$$
and
\[ \|L_A a^{(j)} - \zeta^{(j)} b\|_{C^{2+\nu}(M)} \leq C \log \delta^{-1} \|b\|_{C^{3+\nu}(M)}. \]

We now define
\[ a = \sum_{j=1}^{j_0} a^{(j)}. \]

Then we have the estimates
\[ \|a\|_{C^{2,\gamma}_{1+\nu}(M)} \leq C \sup_{1 \leq j \leq j_0} \|a^{(j)}\|_{C^{2,\gamma}_{1+\nu}(M)} \]
\[ \leq C \sup_{1 \leq j \leq j_0} \|\zeta^{(j)} b\|_{C^{3+\nu}(M)} \]
\[ \leq C \|b\|_{C^{3+\nu}(M)}. \]

\[ \|L_A a - b\|_{C^{3+\nu}_{\{p \in M: \text{dist}(p,S) \leq 2\delta^4}\}} \leq C \sup_{1 \leq j \leq j_0} \|L_A a^{(j)} - \zeta^{(j)} b\|_{C^{3+\nu}(\{p \in M: \text{dist}(p,S) \leq 2\delta^4\})} \]
\[ \leq C \delta \sup_{1 \leq j \leq j_0} \|\zeta^{(j)} b\|_{C^{3+\nu}(M)} \]
\[ \leq C \delta \|b\|_{C^{3+\nu}(M)}. \]

\[ \|L_A a - b\|_{C^{3+\nu}(M)} \leq C \sup_{1 \leq j \leq j_0} \|L_A a^{(j)} - \zeta^{(j)} b\|_{C^{3+\nu}(M)} \]
\[ \leq C |\log \delta|^{-1} \sup_{1 \leq j \leq j_0} \|\zeta^{(j)} b\|_{C^{3+\nu}(M)} \]
\[ \leq C |\log \delta|^{-1} \|b\|_{C^{3+\nu}(M)}. \]

This proves the assertion.

**Proposition 5.2.** For every \( b \in C^{3+\nu}(M) \), there exists a 1-form \( a \in C^{2,\gamma}_{1+\nu}(M) \) such that
\[ \|a\|_{C^{2,\gamma}_{1+\nu}(M)} \leq C \|b\|_{C^{3+\nu}(M)} \]

and
\[ d^* da + dd^* a = b. \]

**Proof.** Since \( H^1(M) = 0 \), the operator \( d^*d + dd^* : \Omega^1(M) \to \Omega^1(M) \) is invertible. Hence, there exists a 1-form \( a \) such that
\[ d^* da + dd^* a = b. \]
Therefore, it remains to show that
\[ \|a\|_{C^2_{\gamma,\nu}(M)} \leq C \|d^* da + dd^* a\|_{C^2_{\gamma,\nu}(M)}. \]

By Schauder estimates, it suffices to show that
\[ \sup (\varepsilon + \text{dist}(p, S))^{1+\nu} |a| \leq C \sup (\varepsilon + \text{dist}(p, S))^{3+\nu} |d^* da + dd^* a|. \]

Suppose that this estimate fails. Then there exists a sequence of positive real numbers \( \varepsilon_j \) and a sequence of 1-forms \( a^{(j)} \in C^2_{\gamma,\nu}(M) \) such that
\[ \sup (\varepsilon_j + \text{dist}(p, S))^{1+\nu} |a^{(j)}| = 1 \]
and
\[ \sup (\varepsilon_j + \text{dist}(p, S))^{3+\nu} |d^* da^{(j)} + dd^* a^{(j)}| \to 0. \]

Then there exists a sequence of points \( p_j \in M \) such that
\[ \sup (\varepsilon_j + \text{dist}(p, S))^{1+\nu} |a^{(j)}(p_j)| \geq \frac{1}{2}. \]

There are two possibilities:

(i) Suppose that \( \text{dist}(p_j, S) \) is bounded from below. After passing to a subsequence, we may assume that the sequence \( a^{(j)} \) converges to a 1-form \( a \in \Omega^1(M) \) such that
\[ \sup \text{dist}(p, S)^{1+\nu} |a| \leq 1 \]
and
\[ d^* da + dd^* a = 0. \]

From this it follows that \( a \) is smooth. Since the operator \( d^* d + dd^* : \Omega^1(M) \to \Omega^1(M) \) has trivial kernel, it follows that \( a = 0 \). This is a contradiction.

(ii) We now assume that \( \text{dist}(p_j, S) \to 0 \). After rescaling and taking the limit, we obtain a 1-form \( \tilde{a} \in \Omega^1(\mathbb{R}^{n-4} \times \mathbb{R}^4) \) such that
\[ \sup |y|^{1+\nu} |\tilde{a}| \leq 1 \]
and
\[ d^* d\tilde{a} + dd^* \tilde{a} = 0. \]

Thus, we conclude that \( \tilde{a} = 0 \). This is a contradiction.
Proposition 5.3. Suppose that $b \in C^{\gamma}_{3+\nu}(M)$ is supported in the region \{ $p \in M : \text{dist}(p, S) \geq \delta^4$ \}. Then there exists a 1-form $a \in C^{2, \gamma}_{1+\nu}(M)$ which is supported in the region \{ $p \in M : \text{dist}(p, S) \geq \delta^8$ \} such that
\[
\| a \|_{C^{2, \gamma}_{1+\nu}(M)} \leq C \| b \|_{C^{\gamma}_{3+\nu}(M)}
\]
and
\[
\| \mathbb{L}_A a - b \|_{C^{\gamma}_{3+\nu}(M)} \leq C \left( | \log \delta |^{-1} + \delta^{-16} \varepsilon^2 \right) \| b \|_{C^{\gamma}_{3+\nu}(M)}.
\]

Proof. By Proposition 5.2, there exists a 1-form $a \in C^{2, \gamma}_{1+\nu}(M)$ such that
\[
\| a \|_{C^{2, \gamma}_{1+\nu}(M)} \leq C \| b \|_{C^{\gamma}_{3+\nu}(M)}
\]
and
\[
d^* da + dd^* a = b.
\]

Let now $\eta$ be a cut-off function such that $\eta(p) = 0$ for $\text{dist}(p, S) \leq \delta^8$, $\eta(p) = 1$ for $\text{dist}(p, S) \geq \delta^4$ and
\[
\sup \text{dist}(p, S) | \nabla \eta | + \sup \text{dist}(p, S)^2 | \nabla^2 \eta | \leq C | \log \delta |^{-1}.
\]

Then the 1-form $\eta a$ is supported in the region \{ $p \in M : \text{dist}(p, S) \geq \delta^8$ \} and satisfies
\[
\| \mathbb{L}_A(\eta a) - b \|_{C^{\gamma}_{3+\nu}(M)}
\]
\[
\begin{aligned}
&= \| D_A^* D_A(\eta a) + D_A D_A^*(\eta a) + (-1)^n \ast [F_A, \eta a] - \eta (d^* da + dd^* a) \|_{C^{\gamma}_{3+\nu}(M)} \\
&\leq \| D_A^* D_A(\eta a) + D_A D_A^*(\eta a) + (-1)^n \ast [F_A, \eta a] - d^* d(\eta a) - dd^* (\eta a) \|_{C^{\gamma}_{3+\nu}(M)} \\
&\quad + \| d^* d(\eta a) + dd^* (\eta a) - \eta (d^* da + dd^* a) \|_{C^{\gamma}_{3+\nu}(M)} \\
&\leq \| D_A^* D_A(\eta a) + D_A D_A^*(\eta a) + (-1)^n \ast [F_A, \eta a] - d^* d(\eta a) - dd^* (\eta a) \|_{C^{\gamma}_{3+\nu}(M)} \\
&\quad + \| d^* d(\eta a) + dd^* (\eta a) - \eta (d^* da + dd^* a) \|_{C^{\gamma}_{3+\nu}(M)} \\
&\leq C \delta^{-16} \varepsilon^2 \| a \|_{C^{1, \gamma}_{3+\nu}(M)} + C | \log \delta |^{-1} \| a \|_{C^{1, \gamma}_{3+\nu}(M)} \\
&\leq C \delta^{-16} \varepsilon^2 \| b \|_{C^{\gamma}_{3+\nu}(M)} + C | \log \delta |^{-1} \| b \|_{C^{\gamma}_{3+\nu}(M)}.
\end{aligned}
\]

This proves the assertion.

In the following, we will choose $\delta = \varepsilon^{1/4}$. Let $\kappa$ be a cut-off function such that $\kappa(p) = 1$ for $\text{dist}(p, S) \leq \varepsilon^{1/4}$ and $\kappa(p) = 0$ for $\text{dist}(p, S) \geq 2 \varepsilon^{1/4}$.

Let $\mathcal{E}^k_{\nu, \gamma}(M)$ be the set of all $b \in \Omega^1(M)$ such that $b \in C^{k, \gamma}_\nu(M)$ and
\[
\int_{NS_\varepsilon} \kappa \sum_{\alpha=1}^{4} \langle b(e^*_\alpha), F_A(X, e^*_\alpha) \rangle = 0
\]

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for all $x \in S$ and all vector fields of the form

$$X = \varepsilon w_{\rho} \rho_{\rho}^\perp + \mu (y - \varepsilon v)_{\rho} \rho_{\rho}^\perp + r_{\rho\sigma} (y - \varepsilon v)_{\sigma} \rho_{\rho}^\perp$$

with $w \in NS_x$, $\mu \in \mathbb{R}$, and $r \in \Lambda^2 NS_x$.

We denote by $I - P$ the fibrewise projection from $C^\gamma_\nu(M)$ to the subspace $E^\gamma_\nu(M)$. Hence, if $b$ is a 1-form, then the projection $Pb$ is of the form

$$Pb(e_\alpha^\perp) = \kappa (\varepsilon w_{\rho} + \mu (y - \varepsilon v)_{\rho} + r_{\rho\sigma} (y - \varepsilon v)_{\sigma}) F_A(e_{\rho}^\perp, e_{\alpha}^\perp)$$

for some $w \in NS$, $\mu \in \mathbb{R}$, and $r \in \Lambda^2 NS$. Let $\Pi$ be the linear operator which assigns to every 1-form $b$ the triplet

$$\Pi b = (w, \mu, r) \in NS \oplus \mathbb{R} \oplus \Lambda^2 NS.$$  

We shall need the following estimate for the operator norm of the projection operator $P$.

**Proposition 5.4.** For every 1-form $b \in C^\gamma_3(M)$, we have the estimates

$$\|\Pi b\|_{C^\gamma(S)} \leq C \varepsilon^{-2-\nu-\gamma} \|b\|_{C^\gamma_3(M)}$$

and

$$\|Pb\|_{C^\gamma_3(M)} \leq C \varepsilon^{-\nu-\gamma} \|b\|_{C^\gamma_3(M)}.$$  

**Proof.** Without loss of generality, we may assume that

$$\|b\|_{C^\gamma_3(M)} \leq 1.$$  

Consider a point $x \in S$ and let $X$ be a vector field of the form

$$X = \varepsilon w_{\rho} \rho_{\rho}^\perp + \mu (y - \varepsilon v)_{\rho} \rho_{\rho}^\perp + r_{\rho\sigma} (y - \varepsilon v)_{\sigma} \rho_{\rho}^\perp.$$  

Using the estimate

$$\sup (\varepsilon + |y|)^{3+\nu} |b(x, y)| \leq 1,$$  

we obtain

$$\left| \int_{NS_x} \kappa \sum_{\alpha=1}^4 \langle b(e_{\alpha}^\perp), F_A(X, e_{\alpha}^\perp) \rangle \right| \leq C \varepsilon^{-\nu} (|w| + |\mu| + |r|).$$  

From this it follows that

$$\sup \|\Pi b(x)\| \leq C \varepsilon^{-2-\nu}.$$  

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This implies
\[
\sup_{4|x_1-x_2|\geq \varepsilon} \frac{\|\Pi b(x_1) - \Pi b(x_2)\|}{|x_1-x_2|^{\gamma}} \leq C \varepsilon^{-2-\nu-\gamma}.
\]

Using the estimate
\[
\sup_{4|x_1-x_2|\leq \varepsilon} (\varepsilon + |y|)^{3+\nu} \frac{|b(x_1, y) - b(x_2, y)|}{|x_1-x_2|^{\gamma}} \leq \varepsilon^{-\gamma},
\]
we deduce that
\[
\sup_{4|x_1-x_2|\leq \varepsilon} \frac{\|\Pi b(x_1) - \Pi b(x_2)\|}{|x_1-x_2|^{\gamma}} \leq C \varepsilon^{-2-\nu-\gamma}.
\]

Thus, we conclude that
\[
\|\Pi b\|_{C^\gamma(S)} \leq C \varepsilon^{-2-\nu-\gamma},
\]

hence
\[
\|\mathbb{P} b\|_{C^{1+\nu}(M)} \leq C \varepsilon^{-\nu-\gamma}.
\]

This proves the assertion.

**Proposition 5.5.** For every \(b \in C^{1+\nu}(M)\) there exists a 1-form \(a \in C^{2,\gamma}_{1+\nu}(M)\) such that
\[
\|a\|_{C^{2,\gamma}_{1+\nu}(M)} \leq C \|b\|_{C^{1+\nu}(M)}
\]
and
\[
\|L_A a - b\|_{C^{1+\nu}(\{p \in M: \text{dist}(p, S) \leq \frac{\varepsilon}{16}\})} \leq C \varepsilon^{\frac{3}{16}} \|b\|_{C^{1+\nu}(M)}
\]
and
\[
\|L_A a - b\|_{C^{1+\nu}(M)} \leq C |\log \varepsilon|^{-1} \|b\|_{C^{1+\nu}(M)}.
\]

**Proof.** Apply Proposition 5.1 to \(\kappa b\) and Proposition 5.3 to \((1-\kappa) b\).

**Proposition 5.6.** For every \(b \in C^{1+\nu}(M)\) there exists a 1-form \(a \in C^{2,\gamma}_{1+\nu}(M)\) such that
\[
\|a\|_{C^{1+\nu}(M)} \leq C \|b\|_{C^{1+\nu}(M)}
\]
and
\[
(I - \mathbb{P}) L_A a = b.
\]

Furthermore, \(a\) satisfies the estimate
\[
\|L_A a\|_{C^{\gamma}(S)} \leq C \varepsilon^{-2+\frac{1}{16}} \|b\|_{C^{1+\nu}(M)}.
\]

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Proof. By Proposition 5.5, there exists an operator $S : \mathcal{E}_{3+\nu}^\gamma(M) \to \mathcal{C}_{1+\nu}^{2+\gamma}(M)$ such that

$$\|Sb\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(M)} \leq C \|b\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(M)}$$

and

$$\|L_A Sb - b\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(\{p \in M : \text{dist}(p, S) \leq \varepsilon^\frac{1}{2}\})} \leq C \varepsilon^\frac{1}{16} \|b\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(M)}$$

and

$$\|L_A Sb - b\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(M)} \leq C \log \varepsilon^{-1} \|b\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(M)}.$$  

This implies

$$\|\Pi L_A Sb\|_{\mathcal{C}^\gamma(S)} = \|\Pi(L_A Sb - b)\|_{\mathcal{C}^\gamma(S)} \leq C \varepsilon^{-2+\frac{1}{16}-\nu-\gamma} \|b\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(M)}.$$  

From this it follows that

$$\|(I - P) L_A Sb - b\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(M)} \leq C \log \varepsilon^{-1} \|b\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(M)}.$$  

Therefore, the operator $(I - P) L_A S : \mathcal{E}_{3+\nu}^\gamma(M) \to \mathcal{E}_{3+\nu}^\gamma(M)$ is invertible. Hence, if we define

$$a = S \left[(I - P) L_A S\right]^{-1} b,$$

then $a$ satisfies

$$\|a\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(M)} \leq C \|b\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(M)}$$

and

$$(I - P) L_A a = b.$$  

This proves the assertion.

6 The nonlinear problem

Proposition 6.1. For every approximate solution $A$, there exists a nearby connection $\tilde{A} = A + a$ such that

$$\|a\|_{\mathcal{C}_{1+\nu}^{2+\gamma}(M)} \leq C \varepsilon^{2-\nu-\gamma}$$

and

$$(I - P) \left(D^*_A F_{\tilde{A}} + D_{\tilde{A}} D^*_A a\right) = 0.$$  

Furthermore, $a$ satisfies the estimate

$$\|\Pi L_A a\|_{\mathcal{C}^\gamma(S)} \leq C \varepsilon^{\frac{1}{16}}.$$
Proof. The connection $\tilde{A} = A + a$ satisfies

$$D^{*}_{\tilde{A}}F_{\tilde{A}} + D_{A}D^{*}_{A}a = D^{*}_{A}F_{A} + L_{A}a + D_{A}D^{*}_{A}a + Q(a),$$

where $Q(a)$ contains only quadratic and cubic terms. This implies

$$D^{*}_{\tilde{A}}F_{\tilde{A}} + D_{A}D^{*}_{A}a = D^{*}_{A}F_{A} + L_{A}a + Q(a).$$

According to Proposition 5.6, there exists an operator $G : C^{2,\gamma}_{3+\nu}(M) \rightarrow C^{2,\gamma}_{1+\nu}(M)$ such that

$$\|Gb\|_{C^{2,\gamma}_{1+\nu}(M)} \leq C \|b\|_{C^{2,\gamma}_{3+\nu}(M)}$$

and

$$(I - P) L_{A} G = I.$$

We now define a mapping $\Phi : C^{2,\gamma}_{1+\nu}(M) \rightarrow C^{2,\gamma}_{1+\nu}(M)$ by

$$\Phi(a) = -G (I - P) (D^{*}_{A}F_{A}) - G (I - P) Q(a).$$

Then we have the estimate

$$\|\Phi(a)\|_{C^{2,\gamma}_{1+\nu}(M)} \leq C \|(I - P) (D^{*}_{A}F_{A})\|_{C^{2,\gamma}_{3+\nu}(M)} + C \|(I - P) Q(a)\|_{C^{2,\gamma}_{3+\nu}(M)}$$

$$\leq C \varepsilon^{-\nu-\gamma} \|D^{*}_{A}F_{A}\|_{C^{2,\gamma}_{3+\nu}(M)} + C \varepsilon^{-\nu-\gamma} \|Q(a)\|_{C^{2,\gamma}_{3+\nu}(M)}$$

$$\leq C \varepsilon^{-\nu-\gamma} \|D^{*}_{A}F_{A}\|_{C^{2,\gamma}_{3+\nu}(M)} + C \varepsilon^{-2\nu-\gamma} \|a\|_{C^{2,\gamma}_{1+\nu}(M)}^{2} + C \varepsilon^{-3\nu-\gamma} \|a\|_{C^{2,\gamma}_{1+\nu}(M)}^{3}$$

$$\leq C \varepsilon^{-2-\nu-\gamma}$$

for all $a \in C^{2,\gamma}_{1+\nu}(M)$ satisfying

$$\|a\|_{C^{2,\gamma}_{1+\nu}(M)} \leq \varepsilon^{\frac{3}{2}}.$$

Moreover, we have

$$\|\Phi(a) - \Phi(a')\|_{C^{2,\gamma}_{1+\nu}(M)} \leq C \varepsilon^{-\nu-\gamma} \|Q(a) - Q(a')\|_{C^{2,\gamma}_{3+\nu}(M)}$$

$$\leq C \varepsilon^{\frac{3}{2} - 2\nu-\gamma} \|a - a'\|_{C^{2,\gamma}_{1+\nu}(M)}$$

for all $a, a' \in C^{2,\gamma}_{1+\nu}(M)$ satisfying

$$\|a\|_{C^{2,\gamma}_{1+\nu}(M)}, \|a'\|_{C^{2,\gamma}_{1+\nu}(M)} \leq \varepsilon^{\frac{3}{2}}.$$

Hence, it follows from the contraction mapping principle that there exists a 1-form $a \in C^{2,\gamma}_{1+\nu}(M)$ such that

$$\|a\|_{C^{2,\gamma}_{1+\nu}(M)} \leq C \varepsilon^{2-\nu-\gamma}$$
and
\[ \Phi(a) = a. \]
From this it follows that
\[ \mathcal{G} (I - \mathcal{P}) (D_A^* F_A) + a + \mathcal{G} (I - \mathcal{P}) Q(a) = 0, \]
hence
\[ (I - \mathcal{P}) (D_A^* F_A) + (I - \mathcal{P}) \mathbb{L}_A a + (I - \mathcal{P}) Q(a) = 0. \]
Thus, we conclude that
\[ (I - \mathcal{P}) (D_A^* F_A + D_A D_A^* a) = 0. \]
This proves the assertion.

**Corollary 6.2.** If \( \tilde{A} \) satisfies
\[ \mathcal{P} (D_{\tilde{A}}^* F_{\tilde{A}} + D_{\tilde{A}} D_{\tilde{A}}^* a) = 0, \]
then \( \tilde{A} \) is a solution of the Yang-Mills equations, i.e.
\[ D_{\tilde{A}}^* F_{\tilde{A}} = 0. \]

**Proof.** By definition of \( \tilde{A} \), we have
\[ (I - \mathcal{P}) (D_{\tilde{A}}^* F_{\tilde{A}} + D_{\tilde{A}} D_{\tilde{A}}^* a) = 0. \]
Hence, if \( \tilde{A} \) satisfies
\[ \mathcal{P} (D_{\tilde{A}}^* F_{\tilde{A}} + D_{\tilde{A}} D_{\tilde{A}}^* a) = 0, \]
then we obtain
\[ D_{\tilde{A}}^* F_{\tilde{A}} + D_{\tilde{A}} D_{\tilde{A}}^* a = 0. \]
Using the Bianchi identity, we obtain
\[ D_{\tilde{A}}^* D_{\tilde{A}} D_{\tilde{A}}^* a = D_{\tilde{A}}^* D_{\tilde{A}}^* F_{\tilde{A}} + D_{\tilde{A}} D_{\tilde{A}} D_{\tilde{A}}^* a = 0. \]
Integrating over \( M \), we conclude that
\[ D_{\tilde{A}} D_{\tilde{A}}^* a = 0, \]
hence
\[ D_{\tilde{A}}^* F_{\tilde{A}} = 0. \]
This proves the assertion.
7 The balancing condition

**Proposition 7.1.** Let $g_0$ be the product metric on the normal bundle $NS$ (cf. Section 4). Then the fibrewise projection $\Pi(D_{A}^{g_0} F_A)$ is given by

\[
\Pi(D_{A}^{g_0} F_A) = \left(\Delta v, \frac{1}{\lambda} \Delta \lambda - \frac{1}{4} |\theta|^2, \frac{1}{\lambda^2} \sum_{i=1}^{n-4} \nabla_i (\lambda^2 \theta_i)\right).
\]

**Proof.** Using the results from Section 4, we obtain

\[
\sum_{\beta=1}^{4} D_{A, e_{\beta}^\perp} F_A(e_i, e_{\beta}^\perp) = 0.
\]

Moreover, the Bianchi identity implies that

\[
D_{A, e_{\beta}^\perp} F_A(e_i, e_{\alpha}^\perp) - D_{A, e_{\alpha}^\perp} F_A(e_i, e_{\beta}^\perp) + D_{A, e_i} F_A(e_{\alpha}^\perp, e_{\beta}^\perp) = 0.
\]

From this it follows that

\[
\sum_{i=1}^{n-4} \sum_{\alpha, \beta=1}^{4} (\nabla_{e_{\beta}^\perp} (F_A(e_i, e_{\beta}^\perp), F_A(e_i, e_{\alpha}^\perp)) - \frac{1}{2} \nabla_{e_{\alpha}^\perp} (F_A(e_i, e_{\beta}^\perp), F_A(e_i, e_{\beta}^\perp)) + \nabla_{e_i} (F_A(e_i, e_{\beta}^\perp), F_A(e_{\alpha}^\perp, e_{\beta}^\perp)) X^\alpha
\]

\[
= \sum_{i=1}^{n-4} \sum_{\alpha, \beta=1}^{4} \langle D_{A, e_i} F_A(e_i, e_{\beta}^\perp), F_A(e_{\alpha}^\perp, e_{\beta}^\perp) \rangle X^\alpha
\]

\[
= \langle D_{A}^{g_0} F_A, F_A(X, \cdot) \rangle.
\]

If $X$ is a vector field of the form

\[
X = \varepsilon w_{\rho} e_{\rho}^\perp + \mu (y - \varepsilon v)_{\rho} e_{\rho}^\perp + r_{\rho \sigma} (y - \varepsilon v)_{\sigma} e_{\rho}^\perp,
\]

then we have

\[
\int_{NS^x} \sum_{i=1}^{n-4} \sum_{\alpha, \beta=1}^{4} (\nabla_{e_{\beta}^\perp} (F_A(e_i, e_{\beta}^\perp), F_A(e_i, e_{\alpha}^\perp)) - \frac{1}{2} \nabla_{e_{\alpha}^\perp} (F_A(e_i, e_{\beta}^\perp), F_A(e_i, e_{\beta}^\perp)) + \nabla_{e_i} (F_A(e_i, e_{\beta}^\perp), F_A(e_{\alpha}^\perp, e_{\beta}^\perp)) X^\alpha
\]

\[
= -\int_{NS^x} \sum_{i=1}^{n-4} \sum_{\alpha, \beta=1}^{4} (\langle F_A(e_i, e_{\beta}^\perp), F_A(e_i, e_{\alpha}^\perp) \rangle \nabla_{e_{\beta}^\perp} X^\alpha - \frac{1}{2} \langle F_A(e_i, e_{\beta}^\perp), F_A(e_i, e_{\beta}^\perp) \rangle \nabla_{e_i} X^\alpha)
\]

\[
= \varepsilon^2 (4\pi^2 |\nabla v|^2 + 8\pi^2 |\nabla \lambda|^2 + 2\pi^2 |\theta|^2) \mu.
\]

Similarly, we obtain

\[
\int_{NS^x} \sum_{\alpha, \beta=1}^{4} \langle F_A(e_i, e_{\beta}^\perp), F_A(e_{\alpha}^\perp, e_{\beta}^\perp) \rangle X^\alpha
\]

\[
= -\varepsilon^2 (4\pi^2 \langle \nabla_i v, w \rangle + 8\pi^2 \lambda \nabla_i \lambda \mu + 2\pi^2 \lambda^2 \langle \theta_i, r \rangle).
\]
Differentiating this identity, we obtain

\[
\int_{NS_x} \sum_{\alpha, \beta = 1}^4 \nabla e_i \langle F_A (e_i, e^\perp_\beta) , F_A (e^\perp_\alpha, e^\perp_\beta) \rangle X^\alpha \nabla X^\alpha
\]

\[
= \sum_{i=1}^{n-4} \nabla e_i \int_{NS_x} \sum_{\alpha, \beta = 1}^4 \langle F_A (e_i, e^\perp_\beta) , F_A (e^\perp_\alpha, e^\perp_\beta) \rangle X^\alpha \nabla X^\alpha
\]

\[
- \int_{NS_x} \sum_{i=1}^{n-4} \sum_{\alpha, \beta = 1}^4 \langle F_A (e_i, e^\perp_\beta) , F_A (e^\perp_\alpha, e^\perp_\beta) \rangle \nabla X^\alpha
\]

\[
= -\varepsilon^2 \left( 4\pi^2 \langle \Delta v , w \rangle + 8\pi^2 \lambda \Delta \lambda \mu + 8\pi^2 |\nabla \lambda|^2 \mu + 4\pi^2 |\nabla v|^2 \mu + 2\pi^2 \left( \sum_{i=1}^{n-4} \nabla (\lambda^2 \theta_i), r \right) \right).
\]

Thus, we conclude that

\[
\int_{NS_x} \langle D^{*}_A F_A , F_A (X, \cdot) \rangle
\]

\[
= -\varepsilon^2 \left( 4\pi^2 \langle \Delta v , w \rangle + 8\pi^2 \lambda \langle \Delta \lambda , \mu \rangle - 2\pi^2 \lambda^2 |\theta|^2 \mu + 2\pi^2 \left( \sum_{i=1}^{n-4} \nabla (\lambda^2 \theta_i), r \right) \right).
\]

From this the assertion follows.

**Proposition 7.2.** The fibrewise projection \( \Pi(D^{*}_A F_A) \) satisfies the estimate

\[
\left\| \Pi(D^{*}_A F_A) \right\| \leq C \varepsilon.
\]

**Proof.** The Riemannian metric satisfies the asymptotic expansion of the
\[ g(e_i, e_j) = \delta_{ij} + 2 \sum_{\rho=1}^{n-4} h_{ij,\rho} y_{\rho} + \sum_{k=1}^{n-4} \sum_{\rho,\sigma=1}^{4} h_{ik,\rho} h_{jk,\sigma} y_{\rho} y_{\sigma} \]
\[ \quad - \sum_{\rho,\sigma=1}^{4} R_{i\rho\sigma j} y_{\rho} y_{\sigma} + O(|y|^3) \]
\[ g(e_i, e^\perp_\alpha) = O(|y|^2) \]
\[ g(e^\perp_\alpha, e^\perp_\beta) = \delta_{\alpha\beta} - \frac{1}{3} \sum_{\rho,\sigma=1}^{4} R_{\alpha\rho\sigma\beta} y_{\rho} y_{\sigma} + O(|y|^3). \]

Using this asymptotic expansion, Proposition 7.2 can be deduced from Proposition 7.1. The details are left to the reader.

**Proposition 7.3.** The fibrewise projection of \( D^*_A F_A + D_A D^*_A a \) satisfies

\[ \| \Pi(D^*_A F_A + D_A D^*_A a) - \left( \Delta v_{\rho} + \sum_{i,j=1}^{n-4} \sum_{\rho,\sigma=1}^{4} h_{ij,\rho} h_{ij,\sigma} v_{\sigma} + \sum_{i=1}^{n-4} \sum_{\rho,\sigma=1}^{4} R_{i\rho\sigma i} v_{\sigma}, \frac{1}{\lambda} \Delta \lambda + \frac{1}{4} \sum_{i,j=1}^{n-4} \sum_{\rho=1}^{4} h_{ij,\rho} h_{ij,\rho} + \frac{1}{4} \sum_{i=1}^{n-4} \sum_{\rho=1}^{4} R_{i\rho i}, - \frac{1}{4} |\theta|^2, \sum_{i=1}^{n-4} \sum_{\rho=1}^{4} \nabla_i (\lambda^2 \theta_{i,\rho\sigma}) \right) \|_{C^\gamma(S)} \leq C \varepsilon^{\frac{1}{2}}. \]

**Proof.** Using the estimate

\[ \| a \|_{C^2_{1+\gamma}(M)} \leq C \varepsilon^{2-\nu-\gamma}, \]

we obtain

\[ \| \Pi Q(a) \|_{C^\gamma(S)} \leq C \varepsilon^{-2-\nu-\gamma} \| Q(a) \|_{C^2_{1+\nu}(M)} \]
\[ \quad \leq C \varepsilon^{-2-2\nu-\gamma} \| a \|_{C^2_{1+\nu}(M)}^2 + C \varepsilon^{-2-3\nu-\gamma} \| a \|_{C^3_{1+\nu}(M)}^3 \]
\[ \quad \leq C \varepsilon^{2-4\nu-3\gamma}. \]

Moreover, we have

\[ \| \Pi L_A a \|_{C^\gamma(S)} \leq C \varepsilon^{\frac{1}{2}}. \]

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Hence, the assertion follows from Proposition 7.2.

**Proof of Theorem 1.1.** Let
\[ \Xi_{\varepsilon}(v, \lambda, J, \omega) = \Pi(D^*_A F_A + D_A^* D_A^* a). \]

The first part of Theorem 1.1 follows from Corollary 6.2, the second part from Proposition 7.3.

### 8 An example

Suppose that the normal bundle $NS$ can be endowed with a $SU(2)$-structure $(J, \omega)$ which is parallel with respect to the Levi-Civita connection $\nabla$. This implies that $\theta = 0$. Moreover, suppose that $\lambda$ is a positive function on $S$ which satisfies the linear PDE
\[
\Delta \lambda + \frac{1}{4} \sum_{i,j=1}^{n-4} \sum_{\rho=1}^{4} h_{ij,\rho} h_{ij,\rho} \lambda + \frac{1}{4} \sum_{i=1}^{n-4} \sum_{\rho=1}^{4} R_{ippi} \lambda = 0.
\]

In addition, we assume that the following non-degeneracy conditions hold:

(i) The Jacobi operator of $S$ is invertible.

(ii) The kernel of the operator
\[
\Delta + \frac{1}{4} \sum_{i,j=1}^{n-4} \sum_{\rho=1}^{4} h_{ij,\rho} h_{ij,\rho} + \frac{1}{4} \sum_{i=1}^{n-4} \sum_{\rho=1}^{4} R_{ippi}
\]

is spanned by the function $\lambda$.

**Proposition 8.1.** Let $(w, \mu, r)$ be a section of the vector bundle $NS \oplus \mathbb{R} \oplus \Lambda^2 NS$ such that
\[
\|w\|_{C^2,\gamma(S)} \leq \varepsilon \frac{1}{\sqrt{t}},
\]
\[
\|\mu\|_{C^2,\gamma(S)} \leq \varepsilon \frac{1}{\sqrt{t}},
\]
\[
\|s\|_{C^2,\gamma(S)} \leq \varepsilon \frac{1}{\sqrt{t}}.
\]

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Then the connection $\tilde{A} = A + a$ corresponding to $(w, \mu, r)$ satisfies the estimate

$$\left\| \Pi(D^*_\tilde{A} F_{\tilde{A}} + D_\tilde{A} D^*_\tilde{A} a) \right\| - \left( \Delta w_\rho + \sum_{i,j=1}^{n-4} \sum_{\rho,\sigma=1}^{4} h_{ij,\rho} h_{ij,\sigma} w_\sigma + \sum_{i=1}^{n-4} \sum_{\rho,\sigma=1}^{4} R_{i\rho\sigma i} w_\sigma, \right.$$  

$$\left. \frac{1}{\lambda} \Delta (\lambda \mu) + \frac{1}{4} \sum_{i,j=1}^{n-4} \sum_{\rho=1}^{4} h_{ij,\rho} h_{ij,\rho} \mu + \frac{1}{4} \sum_{i=1}^{n-4} \sum_{\rho=1}^{4} R_{i\rho\rho i} \mu, \right.$$  

$$\left. \frac{1}{\lambda^2} \sum_{i=1}^{n-4} \nabla_i (\lambda^2 \nabla_i r) \right\|_{C^\gamma(S)} \leq C \varepsilon^{\frac{1}{32}}.$$

**Proof.** This follows immediately from Proposition 7.3.

For abbreviation, let

$$J(w, \mu, r) = \left( \Delta w_\rho + \sum_{i,j=1}^{n-4} \sum_{\rho,\sigma=1}^{4} h_{ij,\rho} h_{ij,\sigma} w_\sigma + \sum_{i=1}^{n-4} \sum_{\rho,\sigma=1}^{4} R_{i\rho\sigma i} w_\sigma, \right.$$  

$$\left. \frac{1}{\lambda} \Delta (\lambda \mu) + \frac{1}{4} \sum_{i,j=1}^{n-4} \sum_{\rho=1}^{4} h_{ij,\rho} h_{ij,\rho} \mu + \frac{1}{4} \sum_{i=1}^{n-4} \sum_{\rho=1}^{4} R_{i\rho\rho i} \mu, \right.$$  

$$\left. \frac{1}{\lambda^2} \sum_{i=1}^{n-4} \nabla_i (\lambda^2 \nabla_i r) \right).$$

If $(w, \mu, r)$ is a section of the vector bundle $NS \oplus \mathbb{R} \oplus \Lambda^2_+ NS$ of class $C^{2\gamma}$, then $J(w, \mu, r)$ is a section of the vector bundle $NS \oplus \mathbb{R} \oplus \Lambda^2_+ NS$ of class $C^\gamma$.

The kernel of $J$ is a vector space of dimension 4. It consists of all triplets $(0, \mu, r)$, where $\mu$ is constant on $S$ and $r$ is a parallel section of the vector bundle $\Lambda^2_+ NS$.

**Proposition 8.2.** There exists a section $(w, \mu, r)$ of the vector bundle $NS \oplus \mathbb{R} \oplus \Lambda^2_+ NS$ such that

$$\Pi(D^*_A F_A + D_A D^*_A a) \in V,$$

where $\tilde{A} = A + a$ is the connection corresponding to $(w, \mu, r)$.

**Proof.** By Proposition 8.1, we may write

$$\Pi(D^*_A F_A + D_A D^*_A a) = J(w, \mu, r) + R(w, \mu, r),$$

where $R(w, \mu, r)$ is a linear map.
where \( \|R(w, \mu, r)\|_{C^\gamma(S)} \leq C \varepsilon^{\frac{1}{32}} \) for \( \|(w, \mu, r)\|_{C^{2, \gamma}(S)} \leq \varepsilon^{\frac{1}{64}} \). Hence, the operator \(-J^{-1} R\) maps a ball of radius \( \varepsilon^{\frac{1}{64}} \) in the Banach space \( C^{2, \gamma}(S) \) into a ball of radius \( C \varepsilon^{\frac{1}{32}} \) in \( C^{2, \gamma}(S) \). Using an appropriate sequence of smoothing operators, we may approximate the mapping \(-J^{-1} R\) by a sequence of compact mappings. Each of these mappings has a fixed point in \( C^{2, \gamma}(S) \) by Schauder’s fixed point theorem. Taking limits, we obtain a fixed point of the original mapping \(-J^{-1} R\) in the Banach space \( C^{2, \gamma}(S) \). With this choice of the glueing data \((w, \mu, r)\), we obtain \((w, \mu, r) + J^{-1} R(w, \mu, r) = 0\), hence \( J(w, \mu, r) + R(w, \mu, r) \in V \).

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