Vectors and covectors in non-commutative setting

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Abstract

Following the guidelines of classical differential geometry the ‘building material’ for the tensor calculus in non-commutative geometry is suggested. The algebraic account of moduli of vectors and covectors is carried out.

Introduction

The main feature of the mathematics of quantum mechanics is captured in its non-commutativity. Thus, in order to build a quantum theory of spacetime it would be reasonable to implement an amount of non-commutativity into the classical differential geometry and general relativity. To do it, we ought to use their algebraic formulation [1, 4]. In particular, it was shown by R. Geroch [4] that the entire content of general relativity can be reformulated in mere terms of the algebra $C^\infty(M)$ of smooth functions on a spacetime manifold.

The direct attempt to substitute the commutative algebra $C^\infty(M)$ by a non-commutative one causes both mathematical and physical problems related with the ambiguity of the generalization of geometrical objects [6]. Besides that, there is a duality at the very starting point which can be briefly formulated as ‘what to begin with: covectors or vectors?’ The first opportunity was investigated by M.Karoubi [5] and A.Connes [2]. Rather, in our paper we shall deal with vectors as basic objects keeping ourselves closer to the conventional account of the differential geometry. It should pointed out that in the classical theory both accounts are equivalent while in the non-commutative environment this is no longer so and the resulting ‘non-commutative geometries’ are different.

We would like to outline the liaisons of our approach with the ‘French version’ of non-commutative geometry [2, 3, 5]. For us, the starting object is a dual to the bimodule $\Omega^1$ rather than $\Omega^1$ itself. However, the consequence of our construction is the appearance of a kind of ‘ghosts’, that is, the covectors which can not be expressed in terms of differential forms.

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1 Differential algebras

Let \( \mathcal{A} \) be an associative algebra, \( \text{Der}(\mathcal{A}) \) be the set of its derivatives, that is, the linear mappings \( v : \mathcal{A} \rightarrow \mathcal{A} \) for which the Leibniz rule holds:

\[
v(ab) = va \cdot b + a \cdot vb
\]  

(1)

\( \text{Der}(\mathcal{A}) \) is a vector space in the sense that any derivative multiplied by a number remains derivative. When \( \mathcal{A} = C^\infty(M) \) the space \( \text{Der}(\mathcal{A}) \) is formed by all smooth vector fields on \( M \).

Even in classical differential geometry \( \text{Der}(\mathcal{A}) \) considered vector space has infinite dimension. Enlarging the set of multiplicators from the numbers to the elements of \( \mathcal{A} \) we make \( \text{Der}(\mathcal{A}) \) left \( \mathcal{A} \)-module which is always finitely generated. This module may not be free being however projective due to Swan theorem \([7]\) (see the example below).

Sometimes subspaces \( V \subseteq \text{Der}(\mathcal{A}) \) of derivatives rather than the whole \( \text{Der}(\mathcal{A}) \) are considered. In the classical setting it happens in the two following situations. The first one is when theories with a symmetry group are considered and only the invariant vector fields are taken into account. The second one (gauge theories) is when fiber bundles are considered and the vector fields on the total space tangent to the fibers are specified.

To pass to the non-commutative setting we begin with the differential algebra being the couple \((\mathcal{A}, V)\) where \( \mathcal{A} \) is an associative algebra and \( V \subseteq \text{Der}(\mathcal{A}) \) is a linear subspace of \( \text{Der}(\mathcal{A}) \). As in the classical setting, to reduce the number of generators of \( V \) we attempt to endow \( V \) with the structure of a module. Let, for instance, \( V = \text{Der}(\mathcal{A}), v \in \mathcal{A} \). As in the classical setting, defining for an arbitrary element \( s \in \mathcal{A} \) the linear mapping \( sv : \mathcal{A} \rightarrow \mathcal{A} \) as \((sv)a = s \cdot va\). Checking the Leibniz rule \([1]\) for the mapping \( sv \)

\[
(sv)(ab) = sva \cdot b + a \cdot svb - [a, s] \cdot vb
\]

for non-commutative \( \mathcal{A} \) we see that it may not hold in general. However for any \( s \in Z \) (the center of \( \mathcal{A} \)) the mapping \( sv \) will be the element of \( \text{Der}(\mathcal{A}) \), therefore \( \text{Der}(\mathcal{A}) \) is always \( Z \)-module. That is why in the definition of the differential algebra we shall always require \( V \) to be a \( Z \)-submodule of \( \text{Der}(\mathcal{A}) \).

2 The coupling procedure

A coupling procedure binding the vectors and covectors is the necessary condition to introduce such basic geometrical entities as, say, curvature. The notions of vectors and covectors are introduced in this section based on the coupling procedure borrowed from the classical differential geometry.

Let \((\mathcal{A}, V)\) be a differential algebra. As in the classical setting, we shall call the elements of the module \( V \) VECTORS. Rather, the covectors are yet to
be defined and there are \textit{a priori} several ways to do it. Clearly, the coupling procedure needs at least an object to be coupled with. In classical differential geometry this object is uniquely defined being the dual $\mathcal{A}$-module to $V$. In the non-commutative case it can not be carried out since $V$ is not an $\mathcal{A}$-module (as it was shown above) and a thoroughful analysis of the available opportunities is needed.

Recall that $V$ is nevertheless $\mathbb{Z}$-module and consider its standard algebraic dual

$$V^* = \text{Hom}_\mathbb{Z}(V, \mathbb{Z})$$

(2)

where $\text{Hom}_\mathbb{Z}$ means the set of all $\mathbb{Z}$-linear mappings. Thus the coupling is automatically defined for any $\omega \in V^*$, $v \in V$

$$< \omega, v > = \omega(v)$$

(3)

It is by virtue of the definition of $< \omega, v >$ that this form is $\mathbb{Z}$-linear by the second argument. Note that in the classical setting the introduced object is exactly the module of covectors. Thus it seems natural to call the elements of $V^\dagger$ \textit{covectors} as well.

However, in order to follow the geometrical guideline, it would be reasonable to require the differentials $da$ to be covectors, where as usually,

$$da(v) = va$$

for any $a \in \mathcal{A}, v \in V$. This requirement is not in general compatible with the definition (3), since the values of $va$ may occur beyond the center $\mathbb{Z}$.

To gather it we shall expand the range of the values of the form (3) to the whole $\mathcal{A}$. Thus the following dual object $V^\dagger$ is suggested

$$V^\dagger = \text{Hom}_\mathbb{Z}(V, \mathcal{A})$$

(4)

keeping the same definition of the coupling bracket [3].

**Proposition 1** The linear space $V^\dagger$ is $\mathcal{A}$-bimodule with respect to the following action of $\mathcal{A}$:

$$(a \omega b)(v) = a \cdot \omega(v) \cdot b$$

(5)

where $a, b \in \mathcal{A}, \omega \in V^\dagger, v \in V$

**Proof.** It suffices to prove that $a \omega b$ is $\mathbb{Z}$-linear $(a \omega b)(zv) = a \cdot z\omega(v) \cdot b = z \cdot a \cdot \omega(v) \cdot b$ since $z$ commutes with any element of $\mathcal{A}$. \hfill \Box

**Corollary.** 1. This proposition enables the Leibniz rule to hold for all differentials: $d(ab) = da \cdot b + a \cdot db$.
2. The bilinear form $< \cdot, \cdot >$ [3] is $\mathcal{A}$-linear with respect to the first argument and $\mathbb{Z}$-linear with respect to the second one.
Proposition 2  The bilinear form (3) is nondegenerate:
\[
< \omega, v > = 0, \quad \forall v \in V \Rightarrow \omega = 0
\]
\[
< \omega, v > = 0, \quad \forall \omega \in V^\dagger \Rightarrow v = 0
\]  (6)

Sketch of proof. The first implication holds by the definition of $V^\dagger$. To prove the second it suffices to check it for all differentials $da$. \hfill \square

3  The second dual

The non-commutativity of the basic algebra $\mathcal{A}$ breaks the symmetry between the vectors and covectors: the vectors form a $\mathcal{Z}$-module while the covectors are $\mathcal{A}$-bimodule. Moreover, the transition from $V$ to $V^\dagger$ is not the conjugation of $\mathcal{Z}$-moduli. It is asymmetry that compensates asymmetry. So, we take
\[
V^{\dagger\dagger} = \text{Hom}_\mathcal{A}(V^\dagger, \mathcal{A})
\]  (7)
(the set of all homomorphisms of $\mathcal{A}$-bimoduli) as the second dual to $V$. Now the symmetry is restored which is corroborated by the following proposition.

Proposition 3  $V^{\dagger\dagger}$ is $\mathcal{Z}$-module with respect to the standard action of $\mathcal{Z}$:
\[
(zw)(\omega) = z \cdot w(\omega)
\]
for $z \in \mathcal{Z}, w \in V^{\dagger\dagger}, \omega \in V^\dagger$.

Proof  is obtained by direct checking the $\mathcal{A}$ linearity of $zw$ using the commutativity of the elements of $\mathcal{Z}$. \hfill \square

In the general theory of moduli there is the canonical homomorphism from a module to its second dual. In our setting this homomorphism is even injective:

Proposition 4  The canonical homomorphism $v \mapsto \hat{v}$
\[
\hat{v}(\omega) = < \omega, v > = \omega(v)
\]  (8)
is the embedding $V \rightarrow V^{\dagger\dagger}$.

Proof  follows immediately from the nondegeneracy of the coupling form (6). \hfill \square

Recall that in classical differential geometry $V \simeq V^{\dagger\dagger} = V^{**}$ so the reflexivity always takes place.
4 Projectivity and reflexivity

To build the working tensor calculus including the trace (which is necessary, in particular, to form the Ricci tensor) we have to deal with reflexive moduli $V \simeq V^{\dagger\dagger}$. For general differential algebras $(\mathcal{A}, V)$ this may not hold. In this section we show that in the case when $V$ is a projective $\mathbb{Z}$-module (as it is always in classical differential geometry according to Swan’s theorem [7]) the reflexivity of the module of vectors is guaranteed.

**Theorem 1** Let $(\mathcal{A}, V)$ is a differential algebra such that the module $V$ is a projective finitely generated $\mathbb{Z}$-module. Then the canonical embedding $(\mathcal{A}) V \to V^{\dagger\dagger}$ is isomorphism, that is, the module $V$ is reflexive.

**Proof.** We shall use the following definition of projectivity: there exist a set of generators $\{v_1, \ldots, v_n\}$ in $V$, and a set of cogenerators $\{\omega^1, \ldots, \omega^n\}$ in $V^* = \text{Hom}_\mathbb{Z}(V, \mathbb{Z})$ (sic!) such that for any $v \in V$

$$v = \omega^1(v)v_1 + \ldots + \omega^n(v)v_n$$

(9)

First prove that for any $\omega \in V^\dagger$ (rather than from $V^*$)

$$\omega = \omega^1\omega(v_1) + \ldots + \omega^n\omega(v_n)$$

which is obtained by calculation of the value of $\omega$ on arbitrary $v \in V$ decomposed by (9).

Now consider an arbitrary $w \in V^{\dagger\dagger}$ and find such $v \in V$ that $\hat{v} = w$. Introduce the following mapping $v : \mathcal{A} \to \mathcal{A}$:

$$v(a) = \sum w(\omega^i)v_ia$$

which is the element of $V$ since the coefficients $w(\omega^i)$ are always in $\mathbb{Z}$; $\omega^i(u) \in \mathbb{Z}$ for any $u \in V$, hence $\omega^i(u) = a\omega^i$ for all $a \in \mathcal{A}$, therefore $w(\omega^i)a = aw(\omega^i)$ since $w$ is homomorphism of $\mathcal{A}$-bimoduli, thus $v \in V$.

Finally, calculating directly the value of $\hat{v}$ on arbitrary $\omega \in V^\dagger$ we obtain $\hat{v}(\omega) = w(\omega)$, which completes the proof. \[Q.E.D.\]

**Concluding remarks**

Some conditions for developing the tensor calculus in non-commutative setting were studied in this paper in order to make it applicable for the quantization of gravity. That is, we had to follow the guidelines provided by the Einstein’s theory based on the classical differential geometry. In particular, we should take care of the correspondence principle so that our construction would really be a generalization of classical differential geometry. Passing to the non-commutative
setting yielded us a sort of ‘ghosts’ not existing in the classical theory: it turns out that the module of vectors may not be reflexive, and the dual space contains something more than vectors. Although, it was shown (theorem [1]) that in the situations similar to classical ones these ghosts do not exist and there are no interpretational problems in building the non-commutative version of differential geometry.

References

[1] Chevalley, K., Théorie des groupes de Lie, Hermann, Paris, 1955
[2] Connes A., Noncommutative differential geometry, Hermann, Paris, 1989
[3] Dubois-Violette, M., Dérivations et calcul différentiel non-commutatif, Comptes Rendus de l’Academie des Sciences de Paris, ser. I, 307, 403, (1988)
[4] Geroch, R. Einstein Algebras, Communications in Mathematical Physics, 26, 271, 1972
[5] Karoubi, M., Homologie cyclique et K-théorie, Astérisque, 149, 1, (1987)
[6] G.N.Parfionov, R.R.Zapatrin, Pointless Spaces in General Relativity, International Journal of Theoretical Physics, 34, 737, 1995 (eprint gr-qc/9503048)
[7] Swan, R.G., Vector fields and projective modules, Transactions of the AMS, 105, 264, (1962)