Network broadcast mode analysis and control of turbulent flows

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We present a network-based modal analysis that identifies the key dynamical paths along which perturbations amplify in an isotropic turbulent flow. This analysis is built upon the Katz centrality, which reveals the flow structures that can effectively spread perturbations over the time-evolving network of vortical elements that constitute the turbulent flow field. Motivated by the resolvent form of the Katz function, we take the singular value decomposition of the resulting communicability matrix, complementing resolvent analysis. The right-singular vector, referred to as the broadcast mode, gives insights into the sensitive regions where introduced perturbations can be effectively spread over the entire fluid-flow network as it evolves over time. The broadcast mode reveals that vortex dipoles are the important structures in spreading perturbations. By perturbing the flow with the broadcast mode, we demonstrate its capability to effectively modify the evolution of turbulent flows. The current network-inspired work presents a novel use of network analysis to guide flow control efforts, in particular for time-varying turbulent base flows.

1. Introduction

In a sea of vortices, a dense network of vortical interactions gives rise to their complex dynamics. For the characterization, modeling, and control of such vortical flow, the identification of flow structures influential to the overall governing dynamics is critical (Schlueter-Kuck & Dabiri 2017; Jiménez 2018, 2020; Taira et al. 2016; Gopalakrishnan Meena & Taira 2020). Due to the large degrees of freedom in describing complex vortical flows, tremendous efforts have been placed on utilizing modal analysis techniques to extract the key flow structures in a low-order representation (Holmes et al. 2012; Schmid 2010; Taira et al. 2017, 2020). By revealing the influential structures, the actuation can be applied wisely to modify the overall vortical flow dynamics efficiently and effectively, providing pathways to improve operations of fluid based engineering systems.

Analogous problems of information and disease transmissions over networks are studied in the field of network science (Newman 2018). The identification of critical nodes in a network for these problems is important from the standpoint of control, security, and public health. These nodes are often found using network centralities that quantify their connectivities in terms of their ability to broadcast or receive information (or disease) over the network. Naturally, these concepts from network science can be related to actuator and sensor placement problems in flow control.

In the present study, we identify influential structures in complex vortical flows by combining the toolsets from network analysis and modal analysis. For demonstration of the present approach, we consider the analysis and modification of two-dimensional (2D) isotropic turbulence, as shown in figure 1. Through the lens of a time-evolving network of vortical elements, network centralities reveal the sensitive regions where perturbation addition can effectively modify the flow. In particular, we extend the Katz centrality

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\[
\Delta x^n \omega (x_i, t) = A_{ji} \omega (x_j, t)
\]

Figure 1. Overview of the present study. A time-evolving network model is constructed along the time-trajectory of the turbulent flow. Broadcast mode analysis identifies sensitive regions to introduce forcing for effective spreading of perturbations over the evolving turbulent network.

(Katz 1953) to identify important nodes in time-evolving networks (Grindrod et al. 2011). Motivated by the mathematical formulation of Katz centrality, we combine it with concepts from resolvent analysis to characterize the input-output relationship over the time-evolving turbulent flow (Trefethen et al. 1993; Jovanović & Bamieh 2005; McKeon & Sharma 2010; Yeh & Taira 2019). We refer to this blended formulation as the broadcast mode analysis, which serves as a systematic approach for analyzing time-varying base flows that remain challenging to most modal analysis techniques.

In what follows, we present the broadcast mode analysis and its implications for studying time-varying networks/flows in section 2. The current formulation is then used to analyze and modify two-dimensional isotropic turbulence as an example in section 3. At last, concluding remarks are offered in section 4.

2. Broadcast mode analysis

2.1. Time-evolving network

A network is defined by a set of nodes connected by edges holding weights to quantify the strengths of the connections. The adjacency matrix \( A \in \mathbb{R}^{n \times n} \) uniquely describes a network of \( n \) nodes, with each entry \( A_{ij} \) being the edge weight that quantifies the influence of node \( j \) on node \( i \). For a time-evolving network, the connections between the nodes, or edge weights, vary in time, yielding a time-dependent \( A(t) \).

Let us consider a simple example in figure 2 to analyze a time-evolving network and show how the influential nodes can be drastically different when we take time evolution into account. This example models a communication network with directed edges between the nodes being one-way information transfers. For this evolving network, we seek the node to seed a message at \( t = t_1 \) such that it can be received by the most nodes at \( t = t_3 \).

The propagation of the message from nodes A and C at \( t = t_1 \) are shown in figure 2 (a) and (b), respectively. If we consider the network at \( t = t_1 \) as static, node A is the most influential node in spreading the message, as it possesses the most outgoing edges, or the highest out degree (Newman 2018). The out degree serves as an effective centrality measure for static networks. However, when considering the time-evolving network, the nodes that receive the message (marked in blue) at \( t = t_3 \) are in fact fewer than those when the message was initially seeded at node C, which has no connections at \( t_1 \). This observation motivates the use of an alternative measure for time-evolving networks.

We also note that, since node C has no connections at \( t_1 \), the seeded message only stays at node C without being passed. The option for the information to stay at a node until better connections appear at a later time is important in determining effective nodes for...
broadcasting information (Grindrod et al. 2011). This option is fulfilled with the use of the Katz centrality for identifying important nodes in a time-evolving network.

2.2. Katz centrality and walk downweighting

The concept of Katz centrality is based on the combinatorics of walks via which a distributed information, $f \in \mathbb{R}^n$, can be spread over the network (Katz 1953; Grindrod et al. 2011). Here, a walk is defined as a single pass of the information from one node to another if there is an edge connecting them. Hence, the propagation of the distributed information after one walk can be represented by $q = Af$, where $q \in \mathbb{R}^n$ is the amount of information aggregated to each node after the walk. For the fluid-flow network, the information $f$ and $q$ can be respectively interpreted as a perturbation to be spread and the global response over the flow field.

The extraction of Katz centrality considers the process where $f$ is passed over the nodes by any amount of walks, namely $A^p f$ with $p \in \mathbb{N}$. Along each walk, the amount of information transferred from node $j$ to $i$ is scaled by the edge weight $A_{ij}$ and a walk-downweighting parameter $\alpha$, whose value can be chosen based on the physics of the problem. Accounting for all paths comprised of any amount of walks, this process of information transfer can be expressed as

$$q = (I + \alpha A + \alpha^2 A^2 + \cdots) f,$$

where we note that the identity matrix represents the option for the information to stay at the same node without being passed to others. Moreover, when $\alpha$ satisfies $\alpha < 1/\rho(A)$, where $\rho(A)$ is the spectral radius of $A$, the infinite series in (2.1) converges such that

$$q = (I - \alpha A)^{-1} f.$$

By taking the column sum of the Katz function $(I - \alpha A)^{-1}$, the traditional Katz broadcast centrality is computed as $b_K = 1(I - \alpha A)^{-1}$ with $1 = (1, 1, \ldots, 1)$, which quantifies the ability of each node to broadcast information to all others in the network.

We note that the Katz function $K(\alpha, A) \equiv (I - \alpha A)^{-1}$ is in the resolvent form of the adjacency matrix $A$. It also captures the input–output process between $q$ and $f$ as shown in equation (2.2). Therefore, instead of taking the column sum to compute the broadcast centrality, we follow the resolvent analysis formulation (Trefethen et al. 1993) and perform the singular value decomposition (SVD) of the Katz function as

$$(I - \alpha A)^{-1} = R \Sigma B,$$

where we refer to the leading right-singular vector $b$ in $B$ as the broadcast mode. This broadcast mode also complements the concept of forcing mode in resolvent analysis, as it identifies the most influential nodes or sensitive regions in the flow that can effectively
spread perturbations over the vortical network. Furthermore, the leading left-singular vector $r$ in $R$ can be interpreted as the receive modes, which account for the information aggregation to each node when distributed by $b$. Compared to the traditional Katz centrality, the use of SVD not only reveals the important nodes in broadcasting or receiving information, but also identifies the optimal way to distribute information over the nodes with amplification corresponding to the singular value $\sigma$ when being spread through the operation of $K(\alpha, A)$.

### 2.3. Communicability matrix and age downweighting

On an evolving network the information can spread by taking any amount of walks over the time-frozen $A_k$ at a time slice $t_k$ before it propagates to the next time slice where the network evolves to $A_{k+1}$. The propagation of information can thus be evaluated as

$$q(t_m) = (I + \alpha A_m + \alpha^2 A_m^2 + \cdots) (\cdots) \cdots (\cdots) (I + \alpha A_0 + \alpha^2 A_0^2 + \cdots) f(t_0).$$

With an appropriate choice of the walk-downweighting parameter $\alpha$, we can use the Katz function for each time slice and describe the propagation of information over the time horizon of $t \in [t_0, t_m]$ as $q(t_m) = S_m f(t_0)$, where

$$S_m = (I - \alpha A_m)^{-1} \cdots (I - \alpha A_1)^{-1} (I - \alpha A_0)^{-1}$$

is referred to as the communicability matrix for the evolving network. It provides knowledge to the effective dynamical paths in the direction of time advancement, along which information propagates via the ordered operations of $K(\alpha, A_k)$ (Grindrod et al. 2011).

Similar to the concept of the walk-downweighting parameter $\alpha$, Grindrod & Higham (2013) considered the use of age-downweighting in the construction of the communicability matrix to further account for the decay of information intensity. Their formulation extends the product form in (2.5) to compute the communicability matrix as

$$S_{k+1} = (I - \alpha A_{k+1})^{-\Delta t_{k+1}} \left[ I + e^{-\gamma \Delta t_k} (S_k - I) \right],$$

with $S_k = (I - \alpha A_k)^{-\Delta t_k}$. Here, $\Delta t_k = t_{k+1} - t_k$ is the time interval between two consecutive time slices and $\gamma$ is the age-downweighting parameter. The product form in (2.5) can be recovered with $\Delta t_k = 1$ and $\gamma = 0$. In this study, we use (2.6) to compute the communicability matrix $S_m$ that accounts for the network evolution over $t \in [t_0, t_m]$ through the recurrence of $k = 0, 1, 2, \ldots, m$. Finally, we take the SVD of

$$S_m \equiv \tilde{R} \tilde{S} \tilde{B}$$

(2.7)
to find the broadcast mode $\tilde{b}$ by extracting the leading right-singular vector in $\tilde{B}$. With $S_m$ tracking all weighted dynamic paths, the mode $\tilde{b}$ gives insights into the gateways of those dynamic paths along which the seeded information can be effectively spread over time. We apply this broadcast mode analysis to time-evolving network of 2D isotropic turbulence. Leveraging the shapes of the broadcast modes, we add vortical perturbations to the turbulent flow to explore how these broadcast modes can be used to target the sensitive regions in the turbulence for flow modification.

### 3. Application to 2D decaying isotropic turbulence

#### 3.1. Model problem setup

To demonstrate the strength of the present approach, we consider its application to a complex vortical flow with high levels of unsteadiness. In particular, we select 2D decaying isotropic turbulence. The time-evolving turbulent flow field is obtained through direct numerical simulation (DNS) that solves the 2D vorticity transport equation

Model problem setup
on a square bi-periodic box using the Fourier spectral method (Taira et al. 2016). Here, \( \omega \) and \( u \) denote the vorticity and velocity, respectively, and \( \nu \) is the kinematic viscosity. We follow Taira et al. (2016) and Jiménez (2018) to define the Taylor length scale \( \lambda \equiv u^*/\omega^* \) and eddy turnover time \( t^* \equiv 1/\omega^* \), where \( \omega^* \equiv |\omega|^1/2 \) and \( u^* \equiv |u_0|^1/2 \) are respectively the spatial root-mean-square values for the vorticity and velocity at the initial condition. The Reynolds number is \( Re \equiv u^*/\nu = 184 \) and the size of the square domain is \( L = 20\lambda \), uniformly discretized with \( \Delta x = 0.079\lambda \). The time integration is performed with \( \Delta t = 0.0145t^* \) using the fourth-order Runge–Kutta scheme.

For the construction of the vortical network, we collect vorticity snapshots at a constant time interval \( \Delta t_k = 10\Delta t \) over a downsampled mesh with the cell size \( \Delta x_n = 2\Delta x \), for which we observed no significant difference in the broadcast modes for downsampling up to \( \Delta x_n = 4\Delta x \). We treat the vortical element residing in each Cartesian cell as a node in the network and the interactions between the vortical elements as the edges, as shown in figure 1 (Taira et al. 2016; Gopalakrishnan Meena & Taira 2020). The time-varying nature of the decaying turbulence is characterized by the varying interactions amongst the vortical elements, resulting in a time-evolving vortical network. By quantifying these interactions, a series of adjacency matrices \( A_k \) for vorticity snapshots \( \omega(x,t_k) \) can be established, as discussed below.

### 3.1.1. Edge weights: Biot–Savart network and Navier–Stokes network

We consider two definitions for \( A_{ij} \) that quantifies the interaction between the vortical elements at \( x_i \) and \( x_j \). The first one defines the edge weight according to the induced velocity magnitude at \( x_i \) imposed by the vortical element at \( x_j \) as

\[
A_{ij} = \frac{|\omega(x_j,t)\Delta x_n^2/(2\pi|x_i-x_j|)|}{\epsilon},
\]

with \( A_{ii} = 0 \) since there is no induced velocity by an vortical element on itself. This Biot–Savart edge weight has been used for network-theoretic models of vortical flows (Nair & Taira 2015; Taira et al. 2016; Gopalakrishnan Meena & Taira 2020).

The second edge-weight definition considers the change of vorticity evolution at \( x_i \) due to a spatial pulse perturbation introduced at \( x_j \). This is computed using

\[
A_{ij} = \left|\frac{[\mathcal{N}(\omega + \epsilon\delta(x_j)) - \mathcal{N}(\omega)]_{x_i}}{\epsilon}\right|,
\]

where \( \mathcal{N}(\omega) \equiv -(u \cdot \nabla)\omega + \nu\nabla^2\omega \) is the right-hand-side of the vorticity transport equation, and \( \epsilon\delta(x_j) \) is a vorticity pulse at \( x_j \) in the shape of a Taylor vortex

\[
\epsilon\delta(x_j) = \epsilon \left(\frac{2}{r_\delta} - \frac{|x - x_j|^2}{r_\delta^3}\right) \exp\left[-\frac{|x - x_j|^2}{2r_\delta^2}\right],
\]

with the amplitude \( \epsilon/u^* = 0.001 \) and radius \( r_\delta = 1.5\Delta x \). Using equation (3.3), we compute each \( A_{ij} \) by evaluating the difference between the two right-hand-side operations at \( x_i \), which accounts for the perturbation received at \( x_i \) due to the added pulse at \( x_j \). In what follows, we refer to the network defined by the first edge weight as the Biot–Savart (B–S) network and the second as the Navier–Stokes (N–S) network.

With the adjacency matrices constructed, we can compute the Katz function and the communicability matrix with appropriate choices of the walk- and age-downweighting parameters. In this study, we utilize the eddy turnover time \( t^* \) to choose the agedownweighting parameter \( \gamma = 1/t^* \) and set the walk-downweighting parameter \( \alpha = e^{-\Delta t_k/t^*} \) such that the information intensity downweighted by a walk within the time slice \( t_k \) is equal to that due to the march to the next time slice \( t_{k+1} \) in (2.6). This choice of \( \alpha \) also satisfies \( \alpha < 1/\rho(A_k) \) for all \( k \), allowing for the Katz function to be reduced to the resolvent form. Extracting the broadcast mode from the SVD in (2.3) and (2.7), we investigate its use for modifying the evolution of the turbulent flow.
3.1.2. Broadcast-mode-based perturbation

Since the broadcast mode \( b(x) \) identifies effective nodes to initialize perturbations, we use it to perturb the initial condition of the turbulent vorticity field as

\[
\omega_p(x, t_0) = \omega_0(x) + a \left[ b(x) - \langle b(x) \rangle \right].
\] (3.5)

Here, the removal of the spatial mean \( \langle b(x) \rangle \) ensures zero circulation in the periodic domain (Jiménez 2018), and the perturbation amplitude is chosen such that \( a \| b - \langle b \rangle \|_2 / \| \omega_0 \|_2 = 0.001 \). We evolve the turbulent flow from this perturbed initial condition \( \omega_p(x, t_0) \) and track the flow modification (Jiménez 2018, 2020) with

\[
\Delta \omega(x, t) = \omega_p(x, t) - \omega(x, t).
\] (3.6)

This modification is assessed with respect to the broadcast modes extracted from the B–S and N–S networks, and for varied time horizon \( t_m \).

3.2. Biot–Savart vs. Navier–Stokes broadcast modes

We compute the broadcast modes using equation (2.3) for the static vortical network extracted from the instantaneous vorticity field shown in figure 3(a). The broadcast modes for the B–S and N–S networks are presented in figure 3(b) and (c), respectively. The B–S broadcast mode suggests that the vortex cores, featured with high levels of vorticity, are the regions of high broadcast strength. The study of Taira et al. (2016) identified these regions as the network ‘hubs’, which are characterized by high levels of vortical interaction occurring in turbulent flow. In a similar problem setting, Jiménez (2018) also labelled the vortices as the influential structures that dominate the evolution of the 2D turbulence.

The N–S broadcast mode paints a different picture. We find that the volumes of high broadcast strength occupy the regions between opposite-sign vortex pairs, as highlighted in the magenta boxes in figure 3(c). This observation aligns with Jiménez (2020), where these vortex ‘dipoles’ are identified as the influential structures in the 2D turbulence through a Monte–Carlo based search over all subvolumes in the flow. These vortex dipoles, acting as local ‘jets,’ locally build up shear layers that are sensitive to perturbations. Moreover, contrary to the B–S broadcast mode, the N–S broadcast mode reveals that the large vortex cores are the regions of the lowest broadcast strength. In 2D unforced turbulence, the only mechanism for vortex deformation is through a strain field of comparable strength to the vortex (Pullin & Saffman 1998). For large isolated vortices in 2D isotropic turbulence, such deformations are rare unless they form dipoles or merge with one another (McWilliams 1990). This is reflected in the N–S edge weight (3.3),
which can be expanded as $A_{ij} \propto u \cdot \nabla \delta \omega + \delta u \cdot \nabla \omega$ with $\delta \omega$ and $\delta u$ being the introduced vorticity and velocity perturbations, respectively. Here, we do not include the viscous term $\nu \nabla^2 \delta \omega$ since it is equal for all nodes and contribute no relative difference to the centrality measure. When the small Gaussian vorticity pulse is introduced at the core of large vortices, the induced $\nabla \delta \omega$ and $\delta u$ are respectively perpendicular to the local base flow $u$ and $\nabla \omega$, resulting in weak perturbations to all vortical elements in the 2D turbulence and hence resulting in the low broadcast strength.

Next, let us perturb the initial vorticity field using these N–S and B–S broadcast modes following equation (3.5) and track in time the modifications of the turbulent flow with $\Delta \omega$. The modifications achieved by the broadcast modes are shown in figure 4 and compared to that by a perturbation of superposed Taylor vortices of random strengths, core sizes, and locations, which results in low levels of $\Delta \omega$ over time. While the perturbations of both broadcast modes show much better capability for flow modification than random perturbation, the N–S broadcast mode produces even higher $\Delta \omega$ than that made by the B–S broadcast mode. The flow modification using the B–S broadcast mode remains in the vortex cores shortly after the initial condition at $t/t^* = 0.29$. At the same time, the perturbation based on the N–S broadcast mode has lost its initial shape due to the interactions with turbulence. Evolving in time, the modification achieved by N–S broadcast mode spreads over space with amplifying magnitude, while in the B–S case the regions of high $\Delta \omega$ remain in the center of the domain at even $t/t^* = 7.25$. We also note that even though the perturbation profiles are initially different for the two cases, the structures of $\Delta \omega$ share some similar signatures, particularly around the main vortex dipole region which gradually moves from the right to the center of the domain. This shows that the vortex dipole is not only the main driver for spreading and amplifying perturbation, but also a highly receptive structure to perturbations, forming an internal feedback loop to continuously amplify existing perturbations. Such receiving characteristics can be further investigated through a receiving mode analysis that focuses on the left-singular vectors $R$ in equations (2.3) and (2.7).

### 3.3. Time-evolving broadcast mode: effects of time horizon

We examine the broadcast modes extracted from the communicability matrix using equation (2.7) for the time-evolving network in figure 5. Here, we focus on the N–S broadcast modes, since they are more effective in modifying the turbulent flow compared to the B–S broadcast mode. We vary the time horizon $t \in [0, t_m]$ over which the communicability matrices are constructed to examine its effect on the shape of the

![Figure 4](image_url)
Figure 5. (a) Broadcast modes $\tilde{b}(t_m)$ for the evolving network computed for $t \in [0, t_m]$. (b) Broadcast modes $b(t)$ computed for the static network at the same instant of $t = t_m$. (c) Flow modifications achieved by modes $\tilde{b}(t_m)$ computed for different time horizons $t_m$. The broadcast modes used here are for the N–S network. Instantaneous vorticity fields are superposed on broadcast modes with representative contour lines.

We have examined the capability of broadcast-mode-based perturbation to modify the evolution of turbulent flow. To this point, the modification is assessed according to the change in the flow field without a target state. Next, we also explore the use of the broadcast mode for modifying the flow along a prescribed direction.

### 3.4. Feedforward control

To modify the flow in a prescribed direction, we consider the signed N–S edge weight,

$$A_{ij} = \text{sign}(\omega(x_i)) \left[ N(\omega + \epsilon \delta(x_j)) - N(\omega) \right] x_i / \epsilon,$$

(3.7)

to account for the alignment in the directions of the base flow $\omega(x_i)$ and the perturbation received at $x_i$. Hence, a positive-valued $A_{ij}$ implies that a favorable perturbation to the base flow $\omega(x_i)$ is received at $x_i$ when introducing a vorticity pulse in the sign of $\epsilon$ at $x_j$. We can set this perturbation at $x_i$ into an opposing effort by reversing the sign of $\epsilon$ for the vorticity perturbation at $x_j$, since (3.7) is approximately linear for small $\epsilon$.

To perform feedforward control, we use (2.3) to compute the broadcast modes $b$ in real-time for the instantaneous vorticity field. Since the goal here is to modify the flow in a prescribed direction rather than simply achieving a high-level of modification, in the SVD we extract the broadcast mode $b$ that is associated with the highest gain-scaled mean value of the receive mode $\sigma |\langle r \rangle|$, where $\sigma$ is the corresponding singular value. With the definition of the signed edge, this $b$ generates the response $r$ of the highest global alignment to the direction of the base flow. According to this $b$, we design the broadcast-mode-based forcing $b_f$ by adding Gaussian vorticity pulses to the nodes in the top 1%
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4. Conclusion

We introduced the broadcast mode analysis and applied it to 2D decaying isotropic turbulence to identify the influential structures for the modification of turbulent flow. Due to the time-varying nature of 2D turbulence, we model it as an evolving network of vortical elements. By blending the formulations of Katz centrality and resolvent analysis, the broadcast mode analysis performs the SVD of the Katz function and the communicability matrix. The broadcast modes are extracted from the leading right-singular vectors, enabling us to pinpoint the vortical elements that effectively spread perturbations over the evolving turbulent vortical network.

For the vortical network, we considered two definitions for the edge weights to quantify the strength of the vortical interactions, yielding the B–S and N–S broadcast modes. We leveraged the insights given by the broadcast modes to design distributive forcing input for the turbulent flow with the goal of modifying its time evolution. While both B–S and N–S broadcast modes are shown to be able to effectively modify the flow evolution, we observe that the N–S broadcasting-mode-based perturbation is more effective in...
modifying flow evolution by focusing the actuation efforts towards the vortex dipoles. By constructing the communicability matrix over a longer time horizon to account for the longer history of the turbulent evolution, we show that the extracted broadcast modes are able to achieve higher levels of flow modification. The broadcast mode can also be extracted from a signed network, guiding flow modification in a desired direction in a feedforward control setup. The present network-inspired approach serves as a novel tool to analyze time-varying turbulent base flows and guide flow control efforts.

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