IWASAWA THEORY FOR SYMMETRIC SQUARE OF NON-\(p\)-ORDINARY EIGENFORMS

KÁZIM BÜYÜKBODUK, ANTONIO LEI, AND GUHAN VENKAT

Abstract. Our main goal in this article is to prove a divisibility statement in the Iwasawa main conjectures for symmetric squares of non-\(p\)-ordinary eigenforms (twisted by an auxiliary Dirichlet character). This task is carried out with the aid of Beilinson-Flach elements, which need to be suitably modified to obtain their integral counterparts. The key technical novelty is a significant improvement of the signed factorization procedure employed in the semi-ordinary Rankin-Selberg products, dwelling on ideas of Perrin-Riou on higher rank Euler systems.

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4.6. Analytic Main conjectures with $p$-adic $L$-functions

References

1. INTRODUCTION

1.1. Background. Throughout this article, we fix an odd prime $p \geq 7$ and embeddings $\iota_\infty : \mathbb{Q} \hookrightarrow \mathbb{C}$ and $\iota_p : \mathbb{Q} \hookrightarrow \mathbb{C}_p$. Let $f$ be a normalised, cuspidal new eigenform of weight $k + 2$, level $N$ and nebentype $\epsilon_f$. We assume that $p \nmid N$, $p > 2k + 2$ and $a_p(f) = 0$. We shall write $\pm \alpha$ for the roots of the Hecke polynomial $X^2 + \epsilon_f(p)p^{k+1}$ of $f$ at $p$.

Let $L/\mathbb{Q}$ be a number field containing the Hecke field of $f$ as well as $\alpha^2$. Let $p$ be a prime in $L$ above $p$. We denote by $E$ the completion of $L$ at $p$. Let $\mathcal{O}$ denote the ring of integers of $E$. We fix a Galois-stable $\mathcal{O}$-lattice $R_f$ inside Deligne’s $E$-linear representation $W_f$ of $G_Q$. Let $\Gamma = \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$. We write $\Gamma = \Gamma_{\text{tors}} \times \Gamma_1$, where $\Gamma_{\text{tors}}$ is a finite group of order $p - 1$ and $\Gamma_1 = \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$. We fix a topological generator $\gamma$ of $\Gamma_1$, which in turn determines an isomorphism $\Gamma_1 \cong \mathbb{Z}_p$. For a module $M$, we denote its Pontryagin dual, $\text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$, by $M^\vee$.

In [BLLV18], we studied the cyclotomic Iwasawa theory of the Rankin-Selberg convolution of two modular forms $f$ and $g$ that are non-ordinary at $p$, making use of the Beilinson-Flach Euler systems constructed by Loeffler and Zerbes in [LZ16]. In this paper, we concentrate on the case where $f = g$ and $a_p(f) = 0$. The results we obtain in this set up do not rely on the conjectural existence of a rank two Euler system, as some of our main results in [BLLV18] do. Our treatment naturally goes through the study of the symmetric square motive $\text{Sym}^2 f$. This extends the work of Loeffler and Zerbes [LZ15] in the ordinary case.

Let us write $W_f^*$ for the linear dual of $W_f$ and endow it with the natural Galois action. For $\lambda, \mu \in \{\pm \alpha\}$ and an integer $m$ that is coprime to $p$, recall from [LZ16] the Beilinson-Flach elements

\[
BF_m^\lambda \mu \in H^1(\mathbb{Q}(\mu_m), W_f^* \otimes W_f^* \otimes \mathcal{H}_{E, k+1}(\Gamma)) \cong \mathcal{H}_{E, k+1}(\Gamma) \otimes H^1(\mathbb{Q}(\mu_m), W_f^* \otimes W_f^* (1) \otimes \Lambda(\Gamma)^\vee) \cong \mathcal{H}_{E, k+1}(\Gamma) \otimes H^1_{\text{Iw}}(\mathbb{Q}(\mu_m), W_f^* \otimes W_f^* (1)),
\]

where $\mathcal{H}_{E, k+1}(\Gamma)$ denotes the ring of $E$-valued tempered distributions of order $k + 1$ on $\Gamma$, $\Lambda(\Gamma)^\vee$ denotes the free rank one $\Lambda(\Gamma)^\vee$-module on which $G_Q$ acts via the inverse of the canonical character $G_Q \twoheadrightarrow \Gamma \hookrightarrow \Lambda(\Gamma)^\vee$. Consider the decomposition

\[
W_f^* \otimes W_f^* = \text{Sym}^2 W_f^* \otimes \bigwedge^2 W_f^*.
\]

In Section 2.1, we explain that the twist of the Beilinson-Flach classes by an even Dirichlet character $\chi$ take values in the corresponding twist $\text{Sym}^2 W_f^* (1 + \chi)$. This equips us with a non-integral collection of cohomology classes that verify a close variant of the Euler system distribution relation. The non-integrality of these classes is the source of main difficulty in the non-ordinary set up. The main task we carry out here is to obtain an integral collection which we may plug in the Euler system machinery.

This goal has been partially achieved in [BLLV18], employing ideas from signed Iwasawa theory (expanding on [BL17] where the authors treated the semi-ordinary case), which are also inspired from Perrin-Riou’s theory of higher rank Euler systems. The theory of higher rank Euler systems suggests a signed factorization of the four collections Beilinson-Flach elements (see Theorem 1.2.1 below for the shape of this factorization). However, the interpolative properties of Beilinson-Flach classes cover only half of the critical range for the symmetric square motive and as a result, the standard techniques only enable us to prove a weaker form of this factorization (and resulting in still non-integral collections of cohomology classes). We develop a new method (see Sections 3.3 and 4.5 below) which allows us to improve this factorization statement to cover the full critical range.

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1.2. Main results. Our first result in this paper is the existence of integral Beilinson-Flach Euler systems when \( f = g \) and \( a_p(f) = 0 \). In this particular set up, we prove [BLV18 Conjecture 5.3.1] under suitable hypotheses.

Given a Dirichlet character \( \psi \) of conductor \( N_\psi \), we let \( \mathcal{R}_\psi \) denote the collection of square-free products of primes which are coprime to \( pN_\psi \). For any prime \( \ell \), let \( Q(\ell) \) denote the unique abelian \( p \)-extension in \( \mathbb{Q}(\mu_\ell) \). For an element \( r = \ell_1 \cdots \ell_s \in \mathcal{R}_\psi \), we define \( Q(r) \) to be the compositum of the (linearly disjoint) fields \( Q(\ell_1), \ldots, Q(\ell_s) \). We also set \( \Delta_r = \text{Gal}(Q(r)/\mathbb{Q}) \) and note that \( \Delta_r = \Delta_{\ell_1} \times \cdots \times \Delta_{\ell_s} \). For a factor \( \ell \) of \( r \), we shall think of \( \Delta_r \) both as a subgroup and as a quotient of \( \Delta_r \) through this identification. We finally let \( \Lambda_r \) denote the ring \( \mathcal{O}[[\Delta_r \times \Gamma]] \).

We fix forever an even Dirichlet character \( \chi \). For \( m \in N_\chi \) (where \( N_\chi \) is given in Definition [2.3.4]), we let \( \text{BF}_{m,\chi}^{\lambda,\mu} \) denote the image of the class \( \text{BF}_{m,\chi}^{\lambda,\mu} \) in \( \mathcal{H}_{E,k+1}(\Gamma) \otimes H^1_{\text{tw}}(\mathbb{Q}(m), W_f^* \otimes W'_f(1+\chi)) \). We consider the following hypotheses.

1. There exists \( u \in (\mathbb{Z}/N_\psi \mathbb{Z})^\times \) such that \( \epsilon_f \psi(u) \neq \pm 1 \) (mod \( p \)) and \( \psi(u) \) is a square modulo \( p \).
2. \( \epsilon_f \chi^{-1}(p) \neq 1 \) and \( \phi(N) \phi(\chi) \) is coprime to \( p \), where \( \phi \) is Euler’s totient function.
3. The prime \( p \over \mathbb{Z} \) in the Shimura field \( K_f \) has degree 1 and \( \text{im}(\chi) \subset \mathbb{Z}_p^* \).

Theorem 1.2.1 (Corollary 3.3.4, Proposition 3.3.5). Suppose that \( \chi \) verifies the hypotheses (1) and (2). Assume also that (3) holds true. Then for every \( m \in N_\chi \), there exist

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
\alpha^2 & \alpha^2 & -\alpha^2 & -\alpha^2 \\
2\alpha & -2\alpha & 0 & 0 \\
0 & 0 & -2\alpha & 2\alpha
\end{pmatrix}
\begin{pmatrix}
\text{BF}_{m,\chi}^{\alpha,\alpha} \\
\text{BF}_{m,\chi}^{\alpha,-\alpha} \\
\text{BF}_{m,\chi}^{-\alpha,\alpha} \\
\text{BF}_{m,\chi}^{-\alpha,-\alpha}
\end{pmatrix}
= \begin{pmatrix}
\log_{p,2k+2}^{\alpha,\alpha} \text{BF}_{m,\chi}^{\alpha,\alpha} \\
\log_{p,2k+2}^{\alpha,-\alpha} \text{BF}_{m,\chi}^{\alpha,-\alpha} \\
\log_{p,k+1}^{\alpha,\alpha} \text{BF}_{m,\chi}^{\alpha,\alpha} \\
\log_{p,k+1}^{\alpha,-\alpha} \text{BF}_{m,\chi}^{\alpha,-\alpha}
\end{pmatrix}.
\]

Here, \( \log_{p,2k+2} \) and \( \log_{p,k+1} \) are some explicit logarithmic functions defined in Section 3. Furthermore, there exists an integer \( C \) independent of \( m \) such that

\[
C \times \text{BF}_{m,\chi}^{\bullet} \in H^1_{\text{tw}}(\mathbb{Q}(m), R_f^* \otimes R'_f(1+\chi))
\]

for all four choices of \( \bullet \in \{+,-,\circ,\cdot\} \).

Under our assumption that \( \chi \) is even, it is easy to verify that all four signed Beilinson-Flach classes take values in \( \text{Sym}^2 R_f^*(1+\chi) \) (and in fact, also that \( \text{BF}_{m,\chi}^{\circ} = 0 \) for all \( m \)):

Corollary 1.2.2 (Corollary 3.3.6). In the setting of Theorem 1.2.1, the signed classes

\[
C \times \text{BF}_{m,\chi}^+, C \times \text{BF}_{m,\chi}^-, C \times \text{BF}_{m,\chi}^\circ, C \times \text{BF}_{m,\chi}^\cdot
\]

take values in \( H^1_{\text{tw}}(\mathbb{Q}(m), \text{Sym}^2 R_f^*(1+\chi)) \).

In particular, each one of four collections \( \{C \times \text{BF}_{m,\chi}^\bullet\} \), where \( \bullet \in \{+,-,\circ,\cdot\} \), form a (rank one) Euler system for \( \text{Sym}^2 R_f^*(1+\chi) \). As remarked above \( \text{BF}_{m,\chi}^\circ = 0 \). In order to apply the Euler system machinery, we need to ensure that at least one of the remaining three Euler systems in Corollary 1.2.2 is non-trivial. In order to do this, it suffices to prove that all four of the non-integral classes \( \{\text{BF}_{m,\chi}^{\alpha,\alpha}, \text{BF}_{m,\chi}^{\alpha,-\alpha}, \text{BF}_{m,\chi}^{-\alpha,\alpha}, \text{BF}_{m,\chi}^{-\alpha,-\alpha}\} \) are non-trivial.

To achieve this, we appeal to the reciprocity laws of Leoflter and Zerbes in [LZ16], which enable us to reduce the required non-vanishing to the non-triviality of the four Rankin-Selberg \( p \)-adic \( L \)-functions associated to \( f \otimes f \otimes \chi \). Note that the motive associated to \( f \otimes f \otimes \chi \) does not possess any critical values and as a result, one may not appeal to non-vanishing statements for complex \( L \)-values to deduce the required non-triviality. However, Arlandini’s work in progress (extending Dasgupta’s result [Das16 Theorem 1]) in the
$p$-ordinary case) shows that the $p$-adic $L$-functions in question factors as a product of the symmetric square $p$-adic $L$-function and a Kubota-Leopoldt p-adic $L$-function. The required non-triviality easily follows from generic non-vanishing statements for symmetric square $L$-values; see Section \ref{section:technical} for details.

Let us set $T := \text{Sym}^2 R^*_1(1 + \chi)$ to ease our notation. Our running hypothesis $a_p(f) = 0$ yields a $G_{Q_p}$-equivariant decomposition

$$T = R^*_1 \oplus R^*_2.$$

Exploiting this decomposition, we define signed Coleman maps as in \cite{Lei12}. More precisely, we define $\Lambda(\Gamma)$-morphisms

$$\text{Col}^{\clubsuit} : H^1_{Iw}(Q_p(\mu_{p^\infty}), T) \to \Lambda(\Gamma)$$

for $\clubsuit \in \{+, -, \bullet\}$ in Section \ref{section:signed Coleman}. For each $\mathfrak{S} = (\clubsuit, \bullet) \in \{(+, -), (+, \bullet), (-, \bullet)\}$, we define the 

doubly signed

Beilinson-Flach $p$-adic $L$-function in Section \ref{section:BF} by setting

$$\mathcal{L}_{\mathfrak{S}} := \text{Col}^{\clubsuit} \circ \text{res}_p(\text{BF}_{\mathfrak{S}, \chi}) \in \Lambda_p(\Gamma).$$

Still using the signed Coleman maps alluded to above, we define also doubly signed Selmer groups which we denote by $\text{Sel}_\mathfrak{S}(T^\vee(1)/Q(\mu_{p^\infty}))$ (where $\mathfrak{S}$ is as above). This allows us to formulate our Doubly Signed Iwasawa Main Conjecture (Conjecture \ref{conjecture:main conjecture} below), relating $\mathcal{L}_{\mathfrak{S}}$ and $\text{Sel}_\mathfrak{S}(T^\vee(1)/Q(\mu_{p^\infty}))^\vee$.

The following is one of our main results towards the Iwasawa main conjectures for non-ordinary symmetric squares. For a given integer $j$, we let $e_{\omega^j}$ denote the idempotent attached to the character $\omega^j$.

**Theorem 1.2.3** (Theorem \ref{theorem:main theorem}). Suppose that the Dirichlet character $\chi$ verifies the hypotheses $(\Psi_1)$, $(\Psi_2)$ and $(\Psi_3)$. Assume also that $(\text{Im})$ holds true.

i) There exists $j \in \{k + 2, \ldots, 2k + 2\}$ even and $\mathfrak{S} \in \{(+, -), (+, \bullet), (-, \bullet)\}$ such that $e_{\omega^j} \mathcal{L}_{\mathfrak{S}} \neq 0$.

ii) For $j$ and $\mathfrak{S}$ as in i), the $\Lambda(\Gamma)$-module $\text{Sel}_\mathfrak{S}(T^\vee(1)/Q(\mu_{p^\infty}))^\vee$ is torsion.

iii) For $j$ and $\mathfrak{S}$ as in i),

$$\text{char}_{\Lambda(\Gamma)}(e_{\omega^j} \text{Sel}_\mathfrak{S}(T^\vee(1)/Q(\mu_{p^\infty}))^\vee) \mid (e_{\omega^j} \mathcal{L}_{\mathfrak{S}})$$

as ideals of $\Lambda(\Gamma) \otimes \mathbb{Q}_p$.

**Remark 1.2.4.** Since the proof of our main technical ingredient (Theorem \ref{theorem:technical}) is rather involved, we would like to make a few comments.

To begin with, note that the shape of the factorization result presented therein is predicted by the existence of a rank 2 Euler system, whose rank reduction via the Perrin-Riou functionals conjecturally give rise to all four unbounded collections of Beilinson-Flach elements (see \cite{BLLV18} Conjecture 3.5.1 for a precise version of this conjecture). We can partially prove the claimed factorization result directly, using the interpolative properties of the Beilinson-Flach classes along the cyclotomic tower (Lemma \ref{lemma:interpolative} below). The readers should compare this portion to the origins of the signed Iwasawa theory, going back to Kobayashi and Pollack.

However, Lemma \ref{lemma:interpolative} alone does not give rise to integral classes and it is not enough for our purposes. The novelty in the present work is that we are able to prove the required factorization result even in the absence of a “sufficiently large range of interpolation” which compensates the growth properties of denominators that are involved. Note in contrast that in \cite{Kob03, Pol03, Lei11, BL17}, interpolative properties were the sole source for similar factorization statements.

The idea here is to keep track of the failure of \cite{BLLV18} Conjecture 3.5.1 in a systematic way. Section \ref{section:higher rank} is devoted to proving that this failure is accounted for by a collection of analytic Selmer groups, whose characteristic ideals are coprime to appropriate factors of logarithmic functions. This allows us to prove \cite{BLLV18} Conjecture 3.5.1 up to a controlled error and in turn, deduce Theorem \ref{theorem:higher rank}.

In a subsequent work, we will build on the results here and go around the circle in the opposite direction to prove the existence of a non-trivial rank 2 Euler system. To best of our knowledge, this is the first example of a higher rank Euler system, whose non-triviality is accounted by $L$-values.
We close the introduction with the consequences of our results towards the Pottharst-style (analytic) Iwasawa main conjectures for non-ordinary symmetric squares. The work of Arlandini allows us to define a pair of geometric $p$-adic $L$-function $L_p^{\text{geom}}(\text{Sym}^2 f \otimes \chi)$ (where $\lambda = \pm \alpha$), interpolating the $L$-values of the twisted symmetric square motive $\text{Sym}^2 f \otimes \chi$. On the other hand, the $(\varphi, \Gamma)$-module attached to the $\lambda$-eigenspace in the Dieudonné module of $W_f$ gives rise to a Pottharst-style analytic Selmer group $\tilde{H}_I^1(Q, V, \mathbb{D}_\lambda)$ (see \textsection 4.6 for details). We prove the following partial result towards the analytic main conjecture that relates these two objects.

**Theorem 1.2.5** (Theorem [4.6.9]). Suppose that the Dirichlet character $\chi$ verifies the hypotheses $(\Psi_1)$, $(\Psi_2)$ and $(\Psi_3)$. Assume also that $(\text{Im})$ holds true. For $f$ and $\mathcal{H} = \{\mathfrak{a}, \mathfrak{b}\}$ as in Theorem [LZ16 Theorem 5.4.2],

$$
\text{char}_H \left( e_{\omega, \tilde{H}_I^2(Q, V, \mathbb{D}_\lambda)} \right) = \text{char}(e_{\omega, \text{cokerCol}^\bullet}) e_{\omega, L_p^{\text{geom}}(\text{Sym}^2 f_\lambda \otimes \chi^{-1})} L_{p, N \chi}(\chi^{-1} e_f) \cdot \mathcal{H}.
$$

Here, $H := \lim_{\longrightarrow \, m} \mathcal{H}_{E, m}(\Gamma_1)$ and $L_{p, N \chi}(\chi e_f)$ is the Kubota-Leopoldt $p$-adic $L$-function attached to the Dirichlet character $\chi e_f$, with Euler factors at primes dividing $N \chi$ removed.

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2. Beilinson-Flach elements for Symmetric Square

2.1. Beilinson-Flach elements and its twists. For $\lambda, \mu \in \{\pm \alpha\}$, $c > 1$ coprime to $6\mathfrak{n}$, $m \geq 1$ coprime to $\mathfrak{c}$, and $a \in (\mathbb{Z}/mp\mathbb{Z})^\times$, let

$$
e_{BF}^{\lambda, \mu}_{m, a} \in H^1(Q(\mu_m), W_f^* \otimes W_f^* \otimes \mathcal{H}_{E, k+1}(\Gamma)^t)
$$

be the Beilinson–Flach element constructed in \textsection 4.6.9. By setting it as the image of $e_{BF}^{\lambda, \mu}$ under the map

$$H^1(Q(\mu_m), W_f^* \otimes \mathcal{H}_{E, k+1}(\Gamma)^t) \cong H^1(Q(\mu_m), W_f^* \otimes W_f^* (1 + \chi) \otimes \mathcal{H}_{E, k+1}(\Gamma)^t) \xrightarrow{\text{cor}} H^1(Q(m), W_f^* \otimes W_f^* (1 + \chi) \otimes \mathcal{H}_{E, k+1}(\Gamma)^t).
$$

**Proposition 2.1.3.** We have the dichotomy

$$
e_{BF}^{\lambda, \mu}_{m, \chi} \in \begin{cases} H^1(Q(m), \text{Sym}^2 W_f^* (1 + \chi) \otimes \mathcal{H}_{E, k+1}(\Gamma)^t) & \text{if } \chi(-1) = +1; \\ H^1(Q(m), \lambda^2 W_f^* (1 + \chi) \otimes \mathcal{H}_{E, k+1}(\Gamma)^t) & \text{if } \chi(-1) = -1. \end{cases}
$$

Moreover,

$$
e_{BF}^{\lambda, \mu}_{m, \chi} + e_{BF}^{\mu, \lambda}_{m, \chi} \in \begin{cases} H^1(Q(m), \text{Sym}^2 W_f^* (1 + \chi) \otimes \mathcal{H}_{E, k+1}(\Gamma)^t) & \text{if } \chi(-1) = +1; \\ H^1(Q(m), \lambda^2 W_f^* (1 + \chi) \otimes \mathcal{H}_{E, k+1}(\Gamma)^t) & \text{if } \chi(-1) = -1, \end{cases}
$$

$$
e_{BF}^{\lambda, \mu}_{m, \chi} - e_{BF}^{\mu, \lambda}_{m, \chi} \in \begin{cases} H^1(Q(m), \lambda^2 W_f^* (1 + \chi) \otimes \mathcal{H}_{E, k+1}(\Gamma)^t) & \text{if } \chi(-1) = +1; \\ H^1(Q(m), \text{Sym}^2 W_f^* (1 + \chi) \otimes \mathcal{H}_{E, k+1}(\Gamma)^t) & \text{if } \chi(-1) = -1. \end{cases}
$$
Proof. The proof of \cite{LZ15} Corollary 4.1.3 goes through verbatim. \qed

For the rest of the article, we shall fix an even Dirichlet character \( \chi \) of conductor \( N_\chi \), which is assumed to be coprime to \( p \). We note that we will not solely work with Beilinson-Flach classes twisted by \( \chi \) but by a range of Dirichlet characters.

### 2.2. Imprimitive \( L \)-functions and \( p \)-adic \( L \)-functions

For \( \chi \) as above, we let \( L^{\text{imp}}(\text{Sym}^2 f \otimes \chi, s) \) denote the imprimitive \( L \)-function given as in \cite{LZ16} Definition 2.1.3.

**Theorem 2.2.1** (Gelbart-Jacquet, Jacquet-Shalika, Schmidt). Suppose that \( f \) has minimal level among its twist by Dirichlet characters. Then for every integer \( j \geq k + 2 \), we have

\[
L^{\text{imp}}(\text{Sym}^2 f \otimes \chi^{-1}, j) \neq 0.
\]

**Proof.** The primitive \( L \)-function \( L(\text{Sym}^2 f \otimes \chi^{-1}, s) \) is non-zero at integers \( j > k + 2 \) since the Euler product defining these \( L \)-functions converges absolutely in that range. Considering the Jacquet-Gelbart lift of \( \text{Sym}^2_f \) to \( GL_3 \) and making use of a non-vanishing result for \( GL_n \) automorphic \( L \)-functions due to Jacquet and Shalika [JS77], it follows that \( L(\text{Sym}^2 f \otimes \chi^{-1}, k + 2) \neq 0 \) as well.

The desired non-vanishing for \( L^{\text{imp}}(\text{Sym}^2 f \otimes \chi^{-1}, j) \) for \( j \geq k + 2 \) now follows from [Sch88] Lemmas 1.5 and 1.6. More specifically, our hypothesis that \( f \) has minimal level among its twists by Dirichlet characters implies that the quotient \( L^{\text{imp}}(\text{Sym}^2 f \otimes \chi^{-1}, s)/L(\text{Sym}^2 f \otimes \chi^{-1}, s) \) is an entire function with zeroes only on the line \( \text{Re}(s) = k + 1 \). The result follows. \qed

We now discuss the ongoing work of Arlandini, which generalize Dasgupta’s factorization formula [Das10] Theorem 1.

For \( \lambda \in \{ \pm \alpha \} \), let \( f_\lambda \) be the \( p \)-stabilization at \( \lambda \). Let \( (F, \epsilon_F) \) be the Coleman family, defined over some affinoid disc \( U \) in the weight space \( \mathcal{W} \), passing through \( f_\lambda \). Loeffler and Zerbes in [LZ16] Definition 9.1.1 define a three-variable geometric \( p \)-adic \( L \)-function, \( L^{\text{geom}}_p(F, F \otimes \chi^{-1}) \in \mathcal{O}(U \times U \times \mathcal{W}) \). On restricting \( L^{\text{geom}}_p(F, F \otimes \chi^{-1}) \) to the image of \( \text{U} \rightarrow \text{U} \times \text{U} \times \mathcal{W} \) induced by the diagonal embedding \( \Delta : U \hookrightarrow U \times U \), we will henceforth treat it as an element of \( \mathcal{O}(U \times \mathcal{W}) \).

**Definition 2.2.2.** Let \( L_{p,N\chi}(\chi^{-1} \epsilon_F) \in \mathcal{O}(\mathcal{W}) \) denote the Kubota-Leopoldt \( p \)-adic \( L \)-function that interpolates the values of the Dirichlet \( L \)-series \( L_{N\chi}(-) \) with the Euler factors at primes dividing \( NN_\chi \) removed. We define the geometric symmetric square \( p \)-adic \( L \)-function \( L^{\text{geom}}_p(\text{Sym}^2 F \otimes \chi^{-1}) \in \text{Frac}(\mathcal{O}(U \times \mathcal{W})) \) by setting

\[
L^{\text{geom}}_p(\text{Sym}^2 F \otimes \chi^{-1})(\kappa, \sigma) := \frac{L^{\text{geom}}_p(F, F \otimes \chi^{-1})(\kappa, \sigma)}{L_{p,N\chi}(\chi^{-1} \epsilon_F)(\sigma - \kappa)}.
\]

In particular, on restricting this definition to \( (k + 2, s) \in U \times \mathcal{W} \), we have

\[
L^{\text{geom}}_p(f_\lambda, f_\lambda \otimes \chi^{-1}, s) = L^{\text{geom}}_p(\text{Sym}^2 f_\lambda \otimes \chi^{-1}, s)\,L_{p,N\chi}(\chi \epsilon_f, s - k - 1).
\]

**Theorem 2.2.3** (Arlandini). We have \( L^{\text{geom}}_p(\text{Sym}^2 F \otimes \chi^{-1}) \in \mathcal{O}(U \times \mathcal{W}) \) and it verifies the following interpolation property.

(i) Let \( 1 \leq j \leq k + 1 \) be an even integer. Then

\[
L^{\text{geom}}_p(\text{Sym}^2 F \otimes \chi^{-1})(k + 2, j) = \frac{(-1)^{j-k-1}j!}{2^{k+4}a}\,\mathcal{E}_p(j)\,L^{\text{imp}}(\text{Sym}^2 f \otimes \chi^{-1}, j),
\]

where \( a = 0 \) if \( k \) is even and \( a = 1 \) if \( k \) is odd. The Euler factor \( \mathcal{E}_p(j) \) is given by

\[
1 - p^{j-1} \chi(p)^{-1}(1 + \chi^{-1}(p)\lambda^2 p^{-j})(1 - \chi^{-1}(p)\lambda^2 p^{-j}).
\]

(ii) Let \( k + 2 \leq j \leq 2k + 2 \) be an even integer. Then

\[
L^{\text{geom}}_p(\text{Sym}^2 F \otimes \chi^{-1})(k + 2, j) = \frac{(j-k-1)!j!}{2^{2j+1}}\,\mathcal{E}_p(j)\,L^{\text{imp}}(\text{Sym}^2 f \otimes \chi^{-1}, j).\]
where the Euler factor $\mathcal{E}_p(s)$ is given by

$$
(1 - p^{j-1} \chi(p) \lambda^{-2})(1 + p^{j-1} \chi(p) \lambda^{-2})(1 - \chi^{-1}(p) \lambda^2 p^{-j}).
$$

Corollary 2.2.4 (Exceptional Zeroes). $L_p^{\text{geom}}(\text{Sym}^2 f_\lambda \otimes \chi^{-1}, s)$ has an exceptional zero at $s$ if and only if $\epsilon_f \chi^{-1}(p) = 1$ and $s = k + 1$ or $s = k + 2$. Furthermore, $L_p^{\text{geom}}(\text{Sym}^2 f \otimes \chi^{-1}, s)$ is non-vanishing for all $k + 3 \leq s \leq 2k + 2$.

Proof. Since $\lambda^2 = -\epsilon_f(p)p^{k+1}$, we have

$$
1 + \frac{\chi^{-1}(p)\lambda^2}{p^{k+1}} = 1 - \chi^{-1}(p)\epsilon_f(p)
$$

and hence the Euler factor $\mathcal{E}_p(k+1)$ vanishes when $\epsilon_f \chi^{-1}(p) = 1$. Similarly, $\mathcal{E}_p(k+2)$ is non-zero only when $\epsilon_f \chi^{-1}(p) = 1$. Second part of the corollary follows from Theorems 2.2.1 and 2.2.3(iii).

Proposition 2.2.5. Let $\xi_{f,\lambda,\chi} \in \mathbb{D}_{\text{cris}}(\text{Sym}^2 W_f \otimes \chi^{-1})$ be the vector chosen as in [LZ15, Definition 4.2.4]. We then have,

$$
\left(\mathcal{L}(\text{BF}_{1,\chi}^\lambda), \xi_{f,\lambda,\chi}\right) = (-1)^s (c^2 - c^{2s-2k-2} \epsilon_f(c)^{-2}) G(\chi^{-1})^2 G(\epsilon_f^{-1})^2 L_p^{\text{geom}}(f_\lambda, f_\lambda \otimes \chi^{-1}, s)
$$

$$
= (-1)^s (c^2 - c^{2s-2k-2} \epsilon_f(c)^{-2}) G(\chi^{-1})^2 G(\epsilon_f^{-1})^2 
\times L_p^{\text{geom}}(\text{Sym}^2 f_\lambda \otimes \chi^{-1}, s) L_{p,NN}\chi(\chi^{-1} \epsilon_f, s-k-1). 
$$

Proof. This is the same as [LZ15, Theorem 4.2.5].

Corollary 2.2.6. Assume that $f$ has minimal level among its twists by Dirichlet characters and suppose $(\Psi_2)$ holds true. Then for all $k + 2 < j \leq 2k + 2$ even, the image of $\epsilon_f \chi^{-1}$ in $H^1(\mathbb{Q}_p, \text{Sym}^2 W_f^*(1 + \chi)(-j))$ is non-zero. In particular, the class $\text{res}_p(\epsilon_f \chi^{-1}) \in H^1_{\text{tw}}(\mathbb{Q}_p, \text{Sym}^2 W_f^*(1 + \chi))$ is non-zero.

Proof. By Proposition 2.2.6 and Lemma 2.2.7, we only have to verify the non-vanishing of the quantity $L_p^{\text{geom}}(\text{Sym}^2 f_\lambda \otimes \chi^{-1}, j)$. By Corollary 2.2.4, there are no exceptional zeroes for $j \in \{k + 3, \ldots, 2k + 2\}$ and the assertion now follows from Theorem 2.2.4.

Lemma 2.2.7. For all even integers $j \in \{k + 2, \ldots, 2k + 2\}$, there is a choice of $c > 1$ coprime to $6NN\chi p$ for which the product $(c^2 - c^{2j-2k-2} \epsilon_f(c)^{-2}) L_{p,NN}\chi(\chi^{-1} \epsilon_f, j - k - 1)$ is non-zero as long as the character $\chi \epsilon_f$ is not quadratic or $j \neq k + 2$.

Proof. This follows from [LZ15, Proposition 4.3.1].

Note that when $j = k + 2$ is even and if $\chi \epsilon_f$ is quadratic but non-trivial, then neither can we dispense off with the factor $(c^2 - c^{2j-2k-2} \epsilon_f(c)^{-2})$, nor can we use it to cancel a pole in the $p$-adic $L$-series (since no such pole exists).

Remark 2.2.8. By Lemma 2.2.7, we may choose an appropriate value of the auxiliary integer $c$ for which $(c^2 - c^{2j-2k-2} \epsilon_f(c)^{-2})$ is non-zero for $k + 2 < j \leq 2k + 2$. For the rest of the paper, we fix such a value and dispense with the factor $c$ from the notation.

We now proceed to show the non-triviality of the classes $\text{BF}^\lambda_{1,\chi}$ for $\lambda \in \{\pm \alpha\}$ using anti-symmetry relations in slight variations of Proposition 2.2.5.

Corollary 2.2.9. The class $\text{res}_p(\text{BF}^\lambda_{1,\chi}) \in H^1_{\text{tw}}(\mathbb{Q}_p, \text{Sym}^2 W_f^*(1 + \chi))$ is non-zero.

Proof. Let $v_{f,\lambda} \in \mathbb{D}_{\text{cris}}(W_f)$ be $\varphi$-eigenvector as chosen in [BLY15, Section 3.5]. Also set

$$
v_{\lambda,\lambda,\chi} := G(\chi^{-1}) v_{f,\lambda} \otimes v_{f,\lambda}, v_{\lambda,\lambda,\chi} := G(\chi^{-1}) v_{f,\lambda} \otimes v_{f,\lambda} \in \mathbb{D}_{\text{cris}}(W_f \otimes W_f \otimes \chi^{-1}).
$$
By [BLLV18 Theorem 3.6.5], we have

\begin{equation}
\langle L(BF_{1,x}^{\lambda,-\lambda}), v_{\lambda,-\lambda} \rangle = \frac{A_f \log_{p,k+1}^{(1)}}{2\lambda} L_p^{\text{geom}}(f_{\lambda}, f_{\lambda} \otimes \chi^{-1}, s),
\end{equation}

where $A_f$ is a non-zero constant independent of $\lambda$.

We have seen in the proof of Corollary [2.2.6] that $L_p^{\text{geom}}(f_{\lambda}, f_{\lambda} \otimes \chi^{-1}, s)$ is non-zero. Hence, (2.1) tells us that $\text{res}_{p(c)BF_{1,x}^{\lambda,-\lambda}} \in H^1_{\text{lp}}(\mathbb{Q}_p(\mu_p), \text{Sym}^2 W_f^*(1 + \chi))$ is non-zero.

\section{Structure of elementary Selmer modules.}
Throughout this section, we fix an even Dirichlet character $\psi$ of conductor $N_\psi$ co-prime to $p$. Enlarging $L$ if necessary, we shall assume that $\psi$ may be realized over $L$. We will take $\psi$ to be $\chi \nu$, where $\chi$ is the character fixed in Section 2.1 whereas $\nu$ is some Dirichlet character of conductor prime to $pNN_\chi$ and $p$-power order.

Throughout this section, we shall assume that the validity of the following large image hypothesis:

\begin{itemize}
    \item \textbf{(Im)} $\text{im}(G_\mathbb{Q} \rightarrow \text{Aut}(R_f \otimes \mathbb{Q}_p))$ contains a conjugate of $\text{SL}_2(\mathbb{Z}_p)$.
\end{itemize}

We will consider the following conditions on $\psi$ and $f$:

\begin{itemize}
    \item $\Psi_1$: There exists $u \in (\mathbb{Z}/NN_\psi \mathbb{Z})^\times$ such that $\epsilon_f \psi(u) \not\equiv \pm 1 \pmod{p}$ and $\psi(u)$ is a square modulo $p$.
    \item $\Psi_2$: $\epsilon_f \psi(p) \neq 1$.
\end{itemize}

\begin{lemma}
Suppose that $\chi$ satisfies the hypothesis ($\Psi_1$) and also that $\epsilon_f \chi(p) \neq 1$ and $\phi(N)\phi(N_\chi)$ is coprime to $p$, where $\phi$ is Euler’s totient function.

Then the conditions ($\Psi_1$) and ($\Psi'_2$) hold true for any $\psi = \chi \nu$ where $\nu$ is a Dirichlet character of conductor prime to $NN_\chi$ and $p$-power order.
\end{lemma}

\begin{proof}
Let $u$ be an integer satisfying ($\Psi_1$) with $\psi = \chi$ and that $u \equiv 1 \pmod{N_\nu}$ (such $u$ exists by Chinese Remainder Theorem). The chosen $u$ will verify ($\Psi_1$) with $\psi = \chi \nu$. We now check ($\Psi'_2$) for $\psi = \chi \nu$. If it was the contrary, we would then have that $\epsilon_f \chi(p) = \nu^{-1}(p)$. This would mean that $\epsilon_f \chi(p)$ is a $p$-power root of unity, contradicting ($\Psi_2$).
\end{proof}

We set $T_\psi := \text{Sym}^2 R_f^1(1) \otimes \psi$ (so that $T_{\psi}(1) = \text{Sym}^2 R_f \otimes \psi^{-1}$). Notice that the character $\chi_{\text{cyc}} \psi$ is odd. Choose an arbitrary integer $j \in [k + 2, 2k + 2]$. We shall work with the representation

\[ T_{\psi,j} := T_\psi(-j) \otimes \omega^j = \text{Sym}^2 R_f^1(1 - j) \otimes \omega^j \psi \]

(where $\omega$ is the Teichmüller character) in this section. We remark that the character $\chi_{\text{cyc}}^{-1} \omega^j \psi$ is always odd. We also set

\[ X_{\psi,j} := \bigwedge^2 R_f^1(1 - j) \otimes \omega^j \psi \cong \mathcal{O}(k + 2 - j) \otimes \omega^j \epsilon_f \psi \]

and observe that the character $\chi_{\text{cyc}}^{k+2-j} \omega^j \epsilon_f \psi$ is even.

\begin{proposition}
Suppose that $\psi$ satisfies ($\Psi_1$) and ($\Psi'_2$). Then there exists $\tau \in G_\mathbb{Q}$ with the following properties:

\begin{itemize}
    \item $\tau$ acts trivially on $\mu_{p^\infty}$.
    \item $T_{\psi,j}/(\tau - 1)T_{\psi,j}$ is free of rank one.
    \item $\tau - 1$ acts invertibly on $X_{\psi,j}$.
\end{itemize}
\end{proposition}

\begin{proof}
This is exactly [LZ15 Proposition 5.2.1].
\end{proof}

Recall that $\Lambda_\mathcal{O}(\Gamma)'$ is the free $\Lambda_\mathcal{O}(\Gamma)$-module of rank one on which $G_\mathbb{Q}$ acts via the inverse of the canonical character $G_\mathbb{Q} \rightarrow \Gamma \rightarrow \Lambda_\mathcal{O}(\Gamma)^\times$. 

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Definition 2.3.3. Let $K$ be any number field. Given an arbitrary free $\mathcal{O}$-module $M$ of finite rank that is endowed with a continuous $G_K$-action unramified outside a finite set of places of $K$, we let $\mathcal{F}_{\text{can}}$ denote the canonical Selmer structure on $M$ (or $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$), given as in [MR04, Definition 3.2.1]. We also let $\mathcal{F}_{\text{can}}^*$ denote the dual Selmer structure on $M^\vee(1)$ (or on $(M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^*(1)$), defined as in Section 2.3 of loc. cit.

We shall also write $\mathcal{F}_{\text{can}}$ for the Selmer structure on $M := M \otimes \Lambda(\Gamma)^\vee$, denoted by $\mathcal{F}_{\Lambda}$ in Section 5.3 of loc. cit. We shall write $\mathcal{F}_{\text{can}}^*$ for the Selmer structure $\mathcal{F}_{\Lambda}^*$ of loc. cit. on the Galois representation $M^\vee(1)$.

Remark 2.3.4. For $K$ and $M$ as above, we have

$$H^1_{\text{can}}(K,M) = H^1(K,M)$$

by [MR04] Lemma 5.3.1. This in turn means that $H^1_{\text{can}}(K,M^\vee(1))$ consists of classes which are locally trivial everywhere.

Proposition 2.3.5. $H^1_{\text{can}}(\mathbb{Q},\psi,j) = 0$.

Proof. We first prove the case when $j = k + 2$. We remark that already this much will be sufficient for our purposes. In this situation, $X_{\psi,k+2} = \mathcal{O}(\epsilon_f^j \omega^{k+2})$ and the conclusion follows from the validity of Leopoldt’s Conjecture for abelian number fields and the fact that $\epsilon_f^j \omega^{k+2}$ is even.

Suppose now that $j \geq k + 3$. To ease notation, we set $\eta = \omega^{k+2} \epsilon_f^j \psi$ and $\rho := \chi_{\psi,j}^{k+2-j} \omega^{-k-2+j}$. Notice that $\eta$ is an even character and $\rho$ is a character of $\Gamma$. Furthermore, we have an isomorphism

$$X_{\psi,j} \cong \mathcal{O}(\eta) \otimes \rho,$$

which, together with the twisting theorems of [Rub00, Section 6], control theorem for the canonical Selmer structure on $X_{\psi,j}$ and the truth of the Main Conjectures for abelian fields, reduces the desired vanishing of the Selmer group to the verification that

$$L \left( \omega^{(k+1)-j} \eta, k + 2 - j \right) \neq 0.$$

But this is well-known, since $(k + 2 - j) - (k + 1 - j) = 1$ is odd. □

Corollary 2.3.6. $H^1(\mathbb{Q},X_{\psi,j} \otimes \Lambda(\Gamma)^\vee) = 0$.

Proof. This follows from Proposition 2.3.5 and Nakayama’s lemma, since we have an injection

$$H^1(\mathbb{Q},X_{\psi,j} \otimes \Lambda(\Gamma)^\vee) \to H^1_{\text{can}}(\mathbb{Q},X_{\psi,j}) = 0.$$ □

Remark 2.3.7. One might give a direct proof of Corollary 2.3.6 without relying on the Iwasawa main conjectures (and using our assumptions (Ψ_1) and (Ψ_2) on $\psi$). We first note that since the character $\rho$ above factors through $\Gamma$, it suffices to prove that

$$H^1(\mathbb{Q},\mathcal{O}(\eta) \otimes \Lambda(\Gamma)^\vee) = 0$$

for $\eta = \omega^{k+2} \epsilon_f^j \psi$ also as above. Since $\eta$ is an even character, it follows from the weak Leopoldt conjecture for abelian fields (which we know to hold true) that the $\Lambda(\Gamma)$-module $H^1(\mathbb{Q},\mathcal{O}(\eta) \otimes \Lambda(\Gamma)^\vee)$ is torsion. Notice further that the character $\eta$ does not factor through the group $\Gamma$ under our running hypotheses and hence the module $H^1(\mathbb{Q},\mathcal{O}(\eta) \otimes \Lambda(\Gamma)^\vee)$ is torsion-free. The proof follows.

For $j$ and $\psi$ as above, we set $V_{\psi,j} := T_{\psi,j} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Corollary 2.3.8. For each $r \in \mathcal{R}_\psi$ we have

$$H^1_{\text{can}}(\mathbb{Q}(r),W^*_f \otimes W^*_j(1-j) \otimes \omega^j \psi) = H^1_{\text{can}}(\mathbb{Q}(r),V_{\psi,j}).$$

Proof. For $W := M \otimes \mathbb{Q}_p$ as in Definition 2.3.3 let us write $\mathcal{F}_{\text{can}}|_W$ for the canonical Selmer structure on $W$ to emphasize the dependence on $W$. Set $Y_{\psi,j} := X_{\psi,j} \otimes \mathbb{Q}_p$ and observe that we have

$$\mathcal{F}_{\text{can}}|_{Y_{\psi,j}} = \mathcal{F}_{\text{can}}|_{V_{\psi,j}} \oplus \mathcal{F}_{\text{can}}|_{Y_{\psi,j}}$$
where the direct sum of Selmer structures on the respective direct sum of Galois representations is defined in the obvious manner. This in turn implies (c.f., [MR04 Remark 3.1.4]) that
\[ H^1_{\text{can}}(\mathbb{Q}, W^*_j \otimes W^*_j (1 - j) \otimes \omega^j \psi) = H^1_{\text{can}}(\mathbb{Q}, V_{\psi,j}) \oplus H^1_{\text{can}}(\mathbb{Q}, Y_{\psi,j}) \]
The asserted identification follows on applying Proposition 2.3.3 with \( \psi \) replaced by \( \psi \nu \), as \( \nu \) runs through the characters of \( \Delta_r \) (note that since \( p \) is odd and \( \Delta_r \) is a \( p \)-group, all characters \( \nu \) on \( \Delta_r \) are necessarily even, which allows us to apply Proposition 2.3.5). \( \square \)

**Definition 2.3.9.** Let \( \mathcal{P}_\chi \) denote the set of primes \( \ell \mid pN_\chi \) for which we have
\begin{itemize}
  \item \( \ell \equiv 1 \mod p \),
  \item \( T_{\chi,j}/(\text{Frob}_\ell - 1)T_{\chi,j} \) is a free \( \mathcal{O} \)-module of rank one,
  \item \( \text{Frob}_\ell - 1 \) is bijective on \( X_{\chi,j} \).
\end{itemize}
We let \( N_\chi \) denote the set of square-free products of integers in \( \mathcal{P}_\chi \).

**Remark 2.3.10.** Since we insist that \( \ell \equiv 1 \mod p \) in Definition 2.3.9, the remaining conditions hold true for one \( j \) if and only if they hold for every \( j \). This justifies our choice to denote this set of primes by \( \mathcal{P}_\chi \).

**Remark 2.3.11.** Let \( T_{\chi,j} \) denote the residual representation of \( T_{\chi,j} \) and let \( \mathbb{Q}(T_{\chi,j}, \mu_p) \) denote the number field that is given as the fixed field of \( \ker(G_\mathbb{Q} \to \text{Aut}(T_{\chi,j} \oplus \mu_p)) \). Then any prime \( \ell \) whose Frobenius in \( \text{Gal}(\mathbb{Q}(T_{\chi,j}, \mu_p)/\mathbb{Q}) \) is conjugate to the image of \( \tau \) given as in Proposition 2.3.2 verifies the requirements of Definition 2.3.9. In particular, \( \mathcal{P}_\chi \) has infinite cardinality.

**Lemma 2.3.12.** For each \( r \in \mathcal{R}_\chi \) and integer \( j \) as above we have \( H^0(\mathbb{Q}(r)\mathbb{Q}_\infty, T_{\chi,j}) = 0 \).

**Proof.** If on the contrary \( T^{(0)} \subset H^0(\mathbb{Q}(r)\mathbb{Q}_\infty, T_{\chi,j}) \) were a rank-one \( G_\mathbb{Q} \)-stable \( \mathcal{O} \)-subquotient of \( T_{\chi,j} \), then \( G_\mathbb{Q} \) would act on \( T^{(0)} \) via \( \chi_{\mathbb{Q},r}^\ast \otimes \theta \) where \( s \in \mathbb{Z}, \theta \) is a character of \( p \)-power conductor and order, and \( \nu \) is a character of conductor dividing \( r \in \mathcal{R}_\chi \). Since \( N_\chi > 1 \) is prime to \( Npr \) by choice and \( \text{Sym}^2 R_j \) is unramified outside \( Np \), a subquotient \( T^{(0)} \) with these properties could not exist. \( \square \)

**Theorem 2.3.13.** Let \( \chi \) be an even Dirichlet character that satisfies \( (\Psi_1) \) and \( (\Psi_2) \) and let \( j \in [k + 2, 2k + 2] \) be an arbitrary integer. Then for each \( r \in N_\chi \), there exist two cohomology classes \( d^\alpha_r, d^{-\alpha}_r \in H^1(\mathbb{Q}(r), V_{\chi,j}) \) with the following properties:
\begin{itemize}
  \item[i)] \( d^\alpha_r, d^{-\alpha}_r \in H^1_{\text{can}}(\mathbb{Q}(r), V_{\chi,j}) \).
  \item[ii)] There exists a constant \( D \) (that does not depend on \( r \)) such that \( Dd^\alpha_r, Dd^{-\alpha}_r \in H^1(\mathbb{Q}(r), T_{\chi,j}) \).
  \item[iii)] For \( r \in \mathcal{P}_\psi \) and \( \mu \in \{\alpha, -\alpha\} \) we have \( \text{cor}_{\mathbb{Q}(r)\ell}/\mathbb{Q}(r) \) \( d^\mu_r = P_\ell(\ell^{-j} F_{\ell}^{-1}) \cdot d^\mu_r \) where \( P_\ell(X) \) is the Euler polynomial for \( L(\text{Sym}^2 f \otimes \omega^j \chi, s) \) at \( \ell \) and \( F_{\ell} \) is the arithmetic Frobenius.
  \item[iv)] The classes \( d^\alpha_1, d^{-\alpha}_1 \in H^1_{\text{can}}(\mathbb{Q}, V_{\chi,j}) \) are linearly independent.
\end{itemize}

We call the two collections of classes \( \{Dd^\alpha_r\}_{r \in N_\chi} \) horizontal Beilinson-Flach Euler systems.

**Proof of Theorem 2.3.13.** This is essentially Theorem 8.1.4 of [LZ16] (combined with certain elements from [LZ15]) and we shall only explain why the line of reasoning in loc. cit. is sufficient to validate our theorem.

We first construct classes \( c^\alpha_r \in H^1(\mathbb{Q}(r), W^*_j \otimes W^*_j (1 - j) \otimes \omega^j \chi) \) as in the proof of Theorem 8.1.4 [LZ16] (the twisting with the appropriate characters may be carried out as in Definition 2.1.2 above). We shall construct \( d^\alpha_r \) using \( c^\alpha_r \). To avert any potential confusion, we remark that in place of the twist \( 1 - j \) we consider here, Loeffler and Zerbes in [LZ16] write \(-j\).
Notice that although the Assumption 3.5.6 of op. cit. does not hold in our case of interest, we still have
\[ H^0(Q(r)Q_\infty, W^*_f \otimes W_j^*(1-j) \otimes \omega^j \chi) = 0 \]
thanks to our running hypothesis on \( \chi \). Indeed, as we have observed as part of Remark 2.2.7, the Dirichlet character \( \eta = \omega^{k+1} \epsilon_f \chi \) that we have defined in the proof of Proposition 2.3.3 does not factor through \( \Gamma \). Since the conductor of \( \epsilon_f \chi \) is prime \( r \) (and non-trivial), it follows that \( H^0(Q(r)Q_\infty, X_{\psi,j}) = 0 \). Lemma 2.3.10 shows that \( H^0(Q(r)Q_\infty, T_{\psi,j}) = 0 \) as well. These two vanishing results conclude the proof of (2.2).

Thanks to (2.2), the proof of [LZ16, Theorem 3.5.9] goes through verbatim and allows one to obtain the interpolated Beilinson-Flach elements along a Coleman family. The desired classes \( e^\alpha, \pm \) are obtained on specializing these interpolated classes (and modifying them slightly (as in the proof of 8.1.4(iii), that in turn relies on the argument in [LZ14, §7.3]) in order to ensure that they verify the correct Euler system distribution relation). We remark that we work over the fields \( Q(r) \) (resp., \( Q_\infty \)) here instead of the full cyclotomic fields \( Q(\mu_r) \) (resp., \( Q(\mu_\infty) \)) as Loeffler and Zerbes in loc. cit. does. This is sufficient for our purposes.

The classes \( e^\alpha, \pm \) verify the conclusion of [LZ16, Theorem 8.1.4(i)], for the same reason that these classes extend in the cyclotomic directions and therefore Proposition 2.4.4 in op. cit. applies. In other words, we infer that
\[ e^\alpha, \pm \in H^1_{F_{\can}}(Q(r), W^*_f \otimes W^*_j(1-j) \otimes \omega^j \chi). \]

We now explain how to define \( d^\alpha, \pm \) using \( e^\alpha, \pm \). We follow [LZ13, proof of Theorem 5.3.3]. For each prime \( \ell \in \mathcal{P}_\chi \) such that \( r/\ell \in \mathcal{N}_\chi \), we let \( \varphi_\ell \in \Delta_r \), denote the unique class that maps to the pair \((s_\ell, 1)\) under the canonical isomorphism \( \Delta_r \cong \Delta_r/\ell \times \Delta_\ell \). As in op. cit., notice that \( \varphi_\ell \) is congruent to 1 modulo the radical of the ring \( O[\Delta_r] \) and hence
\[ 1 - \ell^{k+1-j} \epsilon_f \chi(\ell) \mathcal{F}_\ell^{-1} \in O[\Delta_r]^\times. \]

We now define \( d^\alpha, \pm \) to be the image of
\[ \prod_{\ell | r} (1 - \ell^{k+1-j} \epsilon_f \chi(\ell) \mathcal{F}_\ell^{-1})^{-1} e^\alpha, \pm \in H^1_{F_{\can}}(Q(r), W^*_f \otimes W^*_j(1-j) \otimes \omega^j \chi) \]
under the identification of Corollary 2.3.8. These cohomology classes verify i) by definition.

In order to check the validity of ii), we note that Proposition 2.4.7 of [LZ16] applies thanks to (2.2) and as in the proof of Theorem 8.1.4(ii) in op. cit., it yields the desired integrality result.

We now prove iii). Let \( Q_\ell(X) \) denote the Euler polynomial for \( L(f \otimes f \otimes \omega^j \chi) \). For \( r/\ell \in \mathcal{N}_\chi \), the classes \( e^\alpha, \pm \) enjoy the distribution property
\[ \text{cor}_{Q(\ell r)/Q(r)}(e^\alpha, \pm) = Q_\ell(\ell^{-j} \mathcal{F}_\ell^{-1}) \cdot e^\alpha, \pm \]
Since we have
\[ Q_\ell(\ell^{-j} \mathcal{F}_\ell^{-1}) = (1 - \ell^{k+1-j} \epsilon_f \psi(\ell) \mathcal{F}_\ell^{-1}) \cdot P_\ell(\ell^{-j} \mathcal{F}_\ell^{-1}) \]
thanks to the decomposition (1.1), the proof of iii) follows by our definition of the classes \( d^\alpha, \pm \).

We remark that \( d^\alpha, \pm = e^\alpha, \pm \) by definition and iv) is equivalent to the assertion of Corollary 1.2.4. \( \square \)

**Theorem 2.3.14.** Fix \( r \in \mathcal{N}_\chi \) and let \( \nu \) be a Dirichlet character of conductor \( r \). Set \( \psi = \chi \nu \). Suppose that \( f \) and \( \chi \) verify the hypotheses (Im), (Ψ_1) and (Ψ_2). Then,
\[ \dim_E H^1_{F_{\can}}(Q, V_{\psi,j}) = 2; \]
\[ H^1_{F_{\can}}(Q, V^*_\psi,j(1)) = 0. \]

**Proof.** We start with the observation that \( H^0(Q, T_{\psi,j}) = 0 \) due to weight considerations. Notice further that \( \dim_E V_{\psi,j}^- = 2 \). It therefore follows from [MR04, Theorem 5.2.15] that
\[ \dim_E H^1_{F_{\can}}(Q, V_{\psi,j}) - \dim_E H^1_{F_{\can}}(Q, V_{\psi,j}(1)) = 2. \]
As a result, the two assertions in the statement of our theorem are in fact equivalent and the latter follows from Theorem 2.3.13 (the existence of a non-trivial horizontal Euler system) and [Rub00, Theorem 2.2.3] (whose assumptions are modified via [Rub00, §9.1], by replacing the condition (ii) in the statement of [Rub00, Theorem 2.2.3] with (ii)' in §9.1 of loc.cit. so as to cover our case).

Corollary 2.3.15. In the setting of Theorem 2.3.14, the $\mathcal{O}$-module $H^1_{\text{can}}(\mathcal{O}, T_{\psi,j})$ is free of rank 2.

Proof. After Theorem 2.3.14 we only need to prove that $H^1_{\text{can}}(\mathcal{O}, T_{\psi,j})$ is torsion-free. This follows from the fact that $H^0(\mathcal{O}, T_{\psi,j}) = 0$ under our running hypotheses.

For integers $j \in [k + 2, 2k + 2]$, set $T_j := T_{\chi,j}$ and $V_j = V_{\chi,j}$.

Corollary 2.3.16. In the setting of Theorem 2.3.14, we have

$$\text{rank}_{\mathcal{O}} H^1_{\text{can}}(\mathcal{O}(r), T_j) = 2|\Delta_r|,$$

for each $r \in \mathcal{N}_\chi$. Furthermore, the module $H^1_{\text{can}}(\mathcal{O}(r), T_j^\vee(1))$ has finite cardinality.

Proof. For each character $\nu$ of $\Delta_r$, we infer from Theorem 2.3.14 (applied with the character $\psi = \chi \nu$) that $\dim H^1_{\text{can}}(\mathcal{O}(r), W_{\chi \nu,j}) = 2|\Delta_r|$ and the first assertion follows by Shapiro’s Lemma. The second assertion is an immediate consequence of the first and global duality.

Recall that $T_j := T_{\chi,j} = \text{Sym}^2 R_j(1 + \chi)$. Set $T_j := T_j \otimes \Lambda_\mathcal{O}(\Gamma)_r$ and recall that we have $H^1_{\text{can}}(\mathcal{O}, T_j) = H^1(\mathcal{O}, T_j)$ by [MR04, Lemma 5.3.1]. When $j = 0$, we shall drop $j$ from the notation and simply write $T$ in place of $T_0 = T \otimes \Lambda_\mathcal{O}(\Gamma)_r$.

Corollary 2.3.17. In the setting of Theorem 2.3.14, the $\Lambda_\mathcal{O}(\Gamma)$-module $H^1_{\text{can}}(\mathcal{O}(r), T_j^\vee(1))$ is cotorsion and the $\Lambda_r$-module $H^1(\mathcal{O}(r), T_j)$ has rank $2|\Delta_r|$ for each $r \in \mathcal{N}_\chi$.

Proof. The first assertion follows from control theorem [MR04, Lemma 3.5.3]

$$\left( H^1_{\text{can}}(\mathcal{O}(r), T_j^\vee(1)) \right)_r \xrightarrow{\sim} H^1_{\text{can}}(\mathcal{O}(r), T_j^\vee(1)) \right)_r$$

and Corollary 2.3.16. The second assertion follows from the first by Poitou-Tate global duality.

Corollary 2.3.18. In the setting of Theorem 2.3.14, the $\Lambda_\mathcal{O}(\Gamma)$-module $H^1(\mathcal{O}(r), T)$ has rank $2|\Delta_r|$ for every $r \in \mathcal{N}_\chi$.

Proof. This follows from Corollary 2.3.17 on noticing that

$$H^1(\mathcal{O}(r), T) \xrightarrow{\sim} H^1(\mathcal{O}(r), T_j) \otimes \chi_{\text{cyc}}^j \omega^{-j}.$$

Lemma 2.3.19. The $\Lambda_\mathcal{O}(\Gamma)$-module $H^1(\mathcal{O}(r), T)$ is torsion free.

Proof. This is immediate by Lemma 2.3.12.

Theorem 2.3.20. The $\Lambda_r$-module $H^1(\mathcal{O}(r), T)$ is free of rank 2.

Proof. We have a natural injection

$$H^1(\mathcal{O}(r), T)/(\mathcal{A}_r, \omega^{-j} \chi_{\text{cyc}}^j \gamma - 1) \hookrightarrow H^1_{\text{can}}(\mathcal{O}, T_j),$$

where $\mathcal{A}_r \subset \mathcal{O}[\Delta_r]$ is the augmentation ideal. It follows from Nakayama’s lemma and Corollary 2.3.16 that the $\Lambda_r$-module $H^1(\mathcal{O}(r), T)$ may be generated by at most 2 elements. Let $\{c_1, c_2\}$ be any set of such generators. To prove our theorem, it suffices to check that $c_1$ and $c_2$ do not admit a non-trivial $\Lambda_r$-linear relation.
Assume the contrary and suppose that there is a non-trivial relation
\[ \alpha_1 c_1 + \alpha_2 c_2 = 0, \quad \alpha_1, \alpha_2 \in \Lambda_r. \]
Write \( \mathcal{B} = \{ \delta c_j : \delta \in \Delta_r, j = 1, 2 \} \). Notice that \( \mathcal{B} \) generates \( H^1(Q(r), T) \) as a \( \Lambda_\mathcal{O}(\Gamma) \)-module and
\[ |\mathcal{B}| = 2|\Delta_r| = \dim_{\text{Frac}(\Lambda_\mathcal{O}(\Gamma))} \left( H^1(Q(r), T) \otimes_{\Lambda_\mathcal{O}(\Gamma)} \text{Frac}(\Lambda_\mathcal{O}(\Gamma)) \right), \]
where the final equality is Corollary 2.3.16. The equation (2.3) may be rewritten as
\[ \sum_{\delta, j} \lambda_{\delta, j} \cdot \delta c_j = 0 \]
with \( \lambda_{\delta, j} \in \Lambda_\mathcal{O}(\Gamma) \). Since \( H^1(Q(r), T) \) is \( \Lambda_\mathcal{O}(\Gamma) \)-torsion-free by Proposition 2.3.19, we have a canonical injection
\[ \iota : H^1(Q(r), T) \hookrightarrow H^1(Q(r), T) \otimes_{\Lambda_\mathcal{O}(\Gamma)} \text{Frac}(\Lambda_\mathcal{O}(\Gamma)) \]
and furthermore, as a \( \text{Frac}(\Lambda_\mathcal{O}(\Gamma)) \)-vector space, \( H^1(Q(r), T) \otimes_{\Lambda_\mathcal{O}(\Gamma)} \text{Frac}(\Lambda_\mathcal{O}(\Gamma)) \) is generated by the set \( \iota(\mathcal{B}) \). It follows from the relation (2.5) that
\[ \dim_{\text{Frac}(\Lambda_\mathcal{O}(\Gamma))} \left( H^1(Q(r), T) \otimes_{\Lambda_\mathcal{O}(\Gamma)} \text{Frac}(\Lambda_\mathcal{O}(\Gamma)) \right) \leq |\iota(\mathcal{B})| - 1 = 2|\Delta_r| - 1, \]
which contradicts (2.4) and concludes our proof. \( \square \)

### 3. Signed Iwasawa Theory

We shall show that the Beilinson-Flach elements in Section 2.1 enjoy a Kobayashi/Pollack-style plus/minus splitting. We recall Pollack’s plus and minus logarithms in [Pol03] are defined as follows. Let \( \chi_{\text{cyc}} \) denote the cyclotomic character on \( \Gamma \) and recall that \( \gamma \) is a fixed topological generator of \( \Gamma_1 \). For an integer \( r \geq 1 \), we define
\[
\log^+_p(r) = \prod_{j=0}^{r-1} \prod_{n=1}^{\infty} \frac{\Phi_{p^n}(\chi_{\text{cyc}}^{-j}(\gamma)\gamma)}{p},
\]
\[
\log^-_p(r) = \prod_{j=0}^{r-1} \prod_{n=1}^{\infty} \frac{\Phi_{p^{n-1}}(\chi_{\text{cyc}}^{-j}(\gamma)\gamma)}{p}.
\]
Recall from [Pol03] that \( \log^+_p(r) \in \mathcal{H}_{E,r/2}(\Gamma) \) (in fact, \( \log^+_p(r) \sim O(\log p^{r/2}) \)). We shall also write
\[ \log_p(r) = \prod_{j=0}^{r-1} \log_p(\chi_{\text{cyc}}^{-j}(\gamma)\gamma) \in \mathcal{H}_{E,r}(\Gamma). \]

If \( n \) is an integer, we write
\[ (3.1) \quad \text{Tw}_n : \mathcal{H}_{E,r}(\Gamma) \to \mathcal{H}_{E,r}(\Gamma) \]
for the \( E \)-linear map induced by \( \sigma \mapsto \chi_{\text{cyc}}^n(\sigma)\sigma \) for all \( \sigma \in \Gamma \).

\[ \text{Tw}_n \log^\gamma_p = \log^\gamma_p(r+n)/\log^\gamma_p(n), \quad \gamma = \emptyset, \pm. \]

As a shorthand for \( \text{Tw}_{-1} \log^\gamma_p = \log^\gamma_{p+1}/\log^\gamma_p(2), \) we write \( \log^\gamma_p(1) \).
3.1. **Local theory.** Let $D$ be the Dieudonné module of $W_f|G_{q_p}$. Recall that

$$\dim L \Fil^i D = \begin{cases} 2 & \text{if } i \leq 0, \\ 1 & \text{if } 1 \leq i \leq k+1, \\ 0 & \text{if } i \geq k+2. \end{cases}$$

Recall that we have assumed the Fontaine-Laffaille condition $p > 2k + 2$ holds. On combing the Wach module basis in [BLZ04 §3] with construction of integral Dieudonné module in [Ber04 §IV], there is an $O$-lattice $D_{\text{cris}}(R_f)$ inside $D$ that is generated by $\omega, p^{-k-1} \varphi(\omega)$, where $\omega$ is an $O$-basis of $\Fil^1 D_{\text{cris}}(R_f)$, which we fix from now on. (see also [LLZ17 Lemma 3.1] for a similar basis.)

**Lemma 3.1.1.** The filtered $\varphi$-module $\Sym^2 D$ is decomposable into $D_1 \oplus D_2$, where $D_i$ is of dimension $i$ for both $i = 1, 2$.

**Proof.** The filtration of $\Sym^2 D$ is given by:

$$\Fil^i \Sym^2 D = \begin{cases} \Sym^2 D & \text{if } i \leq 0, \\ \langle \omega \otimes \omega, \omega \otimes \varphi(\omega) + \varphi(\omega) \otimes \omega \rangle & \text{if } 1 \leq i \leq k + 1, \\ \langle \omega \otimes \omega \rangle & \text{if } k \leq i \leq 2k + 2, \\ 0 & \text{if } i \geq 2k + 3 \end{cases}$$

(c.f. [Lei12 §3.3] for the same calculations in the CM case).

We define

$$D_1 = \langle \omega \otimes \varphi(\omega) + \varphi(\omega) \otimes \omega \rangle;$$

$$D_2 = \langle \omega \otimes \omega, \varphi(\omega) \otimes \varphi(\omega) \rangle.$$  

The fact that $\varphi^2(\omega) = \alpha^2 \omega$ implies that both $D_1$ and $D_2$ are stable under $\varphi$. Furthermore, they both respect the filtration of $\Sym^2 D$. Hence, they are both filtered $\varphi$-modules as required. 

**Corollary 3.1.2.** The $G_{q_p}$-representation $\Sym^2 W_f|G_{q_p}$ splits into $W_1 \oplus W_2$, where $W_i$ is of dimension $i$.

**Proof.** This follows from the correspondence between $G_{q_p}$-representations and filtered $\varphi$-modules. 

**Remark 3.1.3.** We note that this decomposition was exploited in [Lei12] in the CM case. In fact, this decomposition holds as $G_\Q$-representations (not just $G_{q_p}$-representations) when $f$ is of CM type.

We define $R_1$ and $R_2$ the lattices inside $W_1$ and $W_2$ corresponding to the decomposition

$$\Sym^2 R_f|G_{q_p} = R_1 \oplus R_2.$$ 

The Dieudonné module of $R_1$ defined by [Ber04 §V] is generated by $p^{-k-1}(\omega \otimes \varphi(\omega) + \varphi(\omega) \otimes \omega)$, whereas that of $R_2$ is generated by $\omega \otimes \omega$ and $p^{-2k-2}\varphi(\omega) \otimes \varphi(\omega)$.

Let $\chi$ be a fixed Dirichlet character as in Section 2.1. In particular, $\chi$ is unramified at $p$. We may canonically identify $D_{\text{cris}}(W_f \otimes W_f(\chi^{-1}))$ with $D_{\text{cris}}(W_f \otimes W_f)$ with the action of $\varphi$ twisted by $\chi(p)^{-1}$. Consequently, the local representation $\Sym^2 W_f(\chi^{-1})|G_{q_p}$ splits into $W_{1,\chi} \oplus W_{2,\chi}$ as in Corollary 3.1.2. Similarly, we have the integral counterpart

$$\Sym^2 R_f(\chi^{-1})|G_{q_p} = R_{1,\chi} \oplus R_{2,\chi}.$$ 

We see from the proof of Lemma 3.1.1 that the Hodge-Tate weight of $W_{1,\chi}$ is $-1 - k$, whereas those of $W_{2,\chi}$ are $0$ and $-2 - 2k$. Furthermore, the characteristic polynomial of $\varphi$ on $D_{\text{cris}}(W_2)$ is $X^2 - \chi(p)^{-2} \alpha^2$ and the filtration on $D_{\text{cris}}(W_{2,\chi})$ is given by

$$\Fil^i D_{\text{cris}}(W_{2,\chi}) = \begin{cases} D_{\text{cris}}(W_{2,\chi}) & \text{if } i \leq 0, \\ \langle \omega \otimes \omega \rangle & \text{if } 1 \leq i \leq 2k + 2, \\ 0 & \text{if } i \geq 2k + 3. \end{cases}$$
By duality, we have the decomposition
\[ W_\chi^* \otimes W_\chi^*(\chi) = 2 W_\chi^*(\chi) \oplus W_{1,\chi}^* \oplus W_{2,\chi}^*, \quad R_f^* \otimes R_f^*(\chi) = 2 R_f^*(\chi) \oplus R_{1,\chi}^* \oplus R_{2,\chi}^*. \]
as $G_{\mathbb{Q}_p}$-representations. Furthermore, as $G_{\mathbb{Q}_p}$-representations, both $R_{1,\chi}^*$ and $A^2 R_f^*(\chi)$ are of the form $O(\psi + k + 1)$, where $\psi$ is an unramified character on $G_{\mathbb{Q}_p}$ sending $p$ to $\pm e_f(p)$ respectively. In particular, they both have a single Hodge-Tate weight, namely $k + 1$. For the representation $W_{2,\chi}^*$, we have the filtration
\[ \text{Fil}^i D_{\text{cris}}(W_{2,\chi}^*) = \begin{cases} D_{\text{cris}}(W_{2,\chi}^*) & i \leq -2k - 2, \\ \{ \omega' \otimes \omega' \} & -2k - 1 \leq i \leq 0, \\ 0 & i \geq 1, \end{cases} \]
for some basis $\omega' \in \text{Fil}^0 D_{\text{cris}}(W_f^*)$ (which we also identify with a basis of $\text{Fil}^0 D_{\text{cris}}(W_f^*)$ so that $\omega' \otimes \omega'$ is a basis of $\text{Fil}^0 D_{\text{cris}}(W_f^* \otimes W_f^*(\chi))$). We fix $\omega'$ so that it generates the $O$-module $\text{Fil}^0 D_{\text{cris}}(R_f^*)$. Then $\omega', \varphi(\omega')$ is a basis of $D_{\text{cris}}(R_f^*)$ and $D_{\text{cris}}(W_{2,\chi}^*)$ is generated by $\omega' \otimes \omega'$ and $\varphi(\omega') \otimes \varphi(\omega')$ (again after identifying $D_{\text{cris}}(W_f^* \otimes W_f^*(\chi))$ with $D_{\text{cris}}(W_f^* \otimes W_f^*)$).

Let $F/\mathbb{Q}_p$ be a finite unramified extension. Given a crystalline $E$-linear representation $W$ of $G_F$ whose Hodge-Tate weights are all non-negative, we write
\[ L_{W,F} : H_{1,\text{tw}}(F, W) \to F \otimes \Lambda_{O}(\Gamma) \otimes D_{\text{cris}}(W), \]
for the Perrin-Riou regulator map (c.f., [LLZ11, §3.1] and [LZ14, Appendix B]). Here, $r$ denotes the largest slope of the Frobenius on $D_{\text{cris}}(W)$.

**Lemma 3.1.4.** Let $W = W_{1,\chi}^*$ or $\wedge^2 W_f^*(\chi)$ and $F/\mathbb{Q}_p$ a finite unramified extension. For all $z \in H_{1,\text{tw}}^1(F, W)$, we have
\[ L_{W,F}(z) \in \log_{p,k+1} F \otimes \Lambda_{O}(\Gamma) \otimes D_{\text{cris}}(W). \]
Let $R = R_{1,\chi}^*$ or $\wedge^2 R_f^*$. If $z \in H_{1,\text{tw}}^1(F, R)$, then
\[ L_{W,F}(z) \in \log_{p,k+1} O_F \otimes \Lambda_{O}(\Gamma) \otimes D_{\text{cris}}(R), \]
where $D_{\text{cris}}(R)$ is the $O$-lattice inside $D_{\text{cris}}(W)$ as defined in [Ber04, §IV].

**Proof.** Since $k \geq 0$, we have the identification
\[ H_{1,\text{tw}}^1(F, R) = N_F(R)^{\psi = 1} \]
where $N_F(R) = O_F \otimes \mathbb{Q}_p N_{\mathbb{Q}_p}(R)$ denotes the Wach module of $R$ over $F$ for $R = R_{1,\chi}^*$ or $\wedge^2 R_f^*(\chi)$ (c.f. [LZ14, §2.7]). Similarly, we may identify $H_{1,\text{tw}}^1(F, W)$ with $N_F(W)^{\psi = 1}$.

We recall that the construction of $L_{W,F}$ can be realized as
\[ 1 - \varphi : N_F(W)^{\psi = 1} \to (\varphi^* N_F(W))^{\psi = 0}. \]
The right-hand side is contained inside $F \otimes (\text{Fil}^+_{\text{rig}, \mathbb{Q}_p})^{\psi = 0} \otimes D_{\text{cris}}(W)$, which in turn is isomorphic to $F \otimes \cup_{\psi} H_{Q, r}(\Gamma) \otimes D_{\text{cris}}(W)$ via the Mellin transform.

The image of $N_F(R)^{\psi = 1}$ under $1 - \varphi$ lies inside $(\varphi^* N_{\mathbb{Q}_p}(R))^{\psi = 0}$, which is a free $O_F \otimes \Lambda_{O}(\Gamma)$-module generated by $(1 + \pi) \varphi(n)$ for some $\mathcal{O} \otimes \mathbb{A}^+_{\mathbb{Q}_p}$-basis $n$ of $N_{\mathbb{Q}_p}(R)$ by [LLZ10, Theorem 3.5]. By [Ber04, proof of Proposition V.2.3], $v := n \mod \pi$ is a basis of $D_{\text{cris}}(R)$. Furthermore, [Ber04, Proposition III.2.1] tells us that $n$ and $(t/\pi)^{k+1} v$ agree up to a unit in $\mathbb{A}^+_{\text{rig}, \mathbb{Q}_p}$. But since both $v$ and $n$ are defined integrally, the aforementioned unit is in fact defined over $\mathbb{A}^+_{\text{rig}, \mathbb{Q}_p}$. Consequently, if $x \in N_F(R)^{\psi = 1}$, we have $(1 - \varphi)(x) \in \varphi((t/\pi)^{k} O_F \otimes (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi = 0} v$. On taking Mellin transform, this lies inside $\log_{p,k+1} O_F \otimes \Lambda_{O}(\Gamma) v$ by [LLZ10, Theorem 5.4] and [LLZ17, Theorem 2.1].

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Lemma 3.1.5. Let $F/\mathbb{Q}_p$ be a finite unramified extension and $z \in H^1_{Iw}(F, W^*_{\chi})$. We write
\[ \mathcal{L}_{W^*_{\chi},F}(z) = \mathcal{L}_{+,F}(z)\omega' \otimes \omega' + \mathcal{L}_{-,F}(z)\varphi(\omega') \otimes \varphi(\omega'), \]
where $\mathcal{L}_{\pm,F}(z) \in F \otimes \mathcal{H}_{E,k+1}(\Gamma)$. Then, $\mathcal{L}_{\pm,F}(z) \in \log^+_{p,2k+2} F \otimes \Lambda_\mathcal{O}(\Gamma)$. Furthermore, if $z \in H^1_{Iw}(F, R^*_{\chi})$, then $\mathcal{L}_{\pm,F}(z) \in \log^+_{p,2k+2} \mathcal{O}_F \otimes \Lambda_\mathcal{O}(\Gamma)$.

Proof. The first part follows from the same proof of [Lei11 Proposition 3.14]. For the integrality statement, we may argue as in Lemma 3.1.4 on using the Wach module basis of $\mathcal{N}_{\mathbb{Q}_p}(R^*_{\chi})$ as given in [Ber11 §A] and [LLZ10 §5.2].

Let $\omega'$ be a fixed basis of $\text{Fil}^0 \mathcal{D}_{cris}(R^*_{\chi})$ as above. The eigenvalues of $\varphi$ on $\mathcal{D}_{cris}(W^*_{\chi})$ are $\pm \frac{1}{\lambda}$ and we have the eigenvectors
\[ v_{\lambda} = \varphi(\omega') + \frac{1}{\lambda} \omega', \]
for $\lambda = \pm \alpha$. Then, the four vectors $v_{\lambda} \otimes v_{\mu}$ decomposes $\mathcal{D}_{cris}(W^*_{\chi} \otimes W^*_{\chi}) = \mathcal{D}_{cris}(W^*_{\chi} \otimes W^*_{\chi}(\chi))$ into a direct sum of one-dimensional subspaces.

Definition 3.1.6. For any finite unramified extension $F$ of $\mathbb{Q}_p$ and element $z \in H^1_{Iw}(F, W^*_{\chi} \otimes W^*_{\chi}(\chi))$, we write
\[ \mathcal{L}_{W^*_{\chi},F}(z) = \sum_{\lambda, \mu} \mathcal{L}_{\lambda,\mu,F}(z)v_{\lambda} \otimes v_{\mu} \]
\[ = \mathcal{L}_{o,F}(z)v_o + \mathcal{L}_{\bullet,F}(z)v_\bullet + \mathcal{L}_{+,F}(z)\omega' \otimes \omega' + \mathcal{L}_{-,F}(z)\varphi(\omega') \otimes \varphi(\omega'), \]
where $v_o = \omega' \otimes \varphi(\omega') - \varphi(\omega') \otimes \omega'$ and $v_\bullet = \omega' \otimes \varphi(\omega') + \varphi(\omega') \otimes \omega'$.

Lemma 3.1.7. For all $z \in H^1_{Iw}(F, R^*_{\chi} \otimes R^*_{\chi}(\chi))$, we have
\[ \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ \alpha^2 & \alpha^2 & -\alpha^2 & -\alpha^2 \\ 2\alpha & -2\alpha & 0 & 0 \\ 0 & 0 & -2\alpha & 2\alpha \end{array} \right) \left( \begin{array}{c} \mathcal{L}_{\alpha,\alpha,F}(z) \\ \mathcal{L}_{-\alpha,\alpha,F}(z) \\ \mathcal{L}_{-\alpha,-\alpha,F}(z) \\ \mathcal{L}_{\alpha,-\alpha,F}(z) \end{array} \right) \in \left( \begin{array}{c} \log^+_{p,2k+2} F \\ \log^+_{p,2k+2} \mathcal{O}_F \\ \log_{p,k+1} \mathcal{O}_F \\ \log_{p,k+1} \mathcal{O}_F \end{array} \right) \mathcal{H}_{E,k+1}(\Gamma). \]

Proof. We have the change of basis matrix
\[ \left( \begin{array}{cccc} \varphi(\omega') \otimes \varphi(\omega') & \omega' \otimes \omega' & \varphi(\omega') \otimes \varphi(\omega') & \varphi(\omega') \otimes \varphi(\omega') \\ \varphi(\omega') \otimes \omega' + \omega' \otimes \varphi(\omega') & \omega' \otimes \omega' & \varphi(\omega') \otimes \varphi(\omega') & \varphi(\omega') \otimes \varphi(\omega') \\ \varphi(\omega') \otimes \omega' - \omega' \otimes \varphi(\omega') & \omega' \otimes \omega' & \varphi(\omega') \otimes \varphi(\omega') & \varphi(\omega') \otimes \varphi(\omega') \end{array} \right) = 1/4 \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ \alpha^2 & \alpha^2 & -\alpha^2 & -\alpha^2 \\ 2\alpha & -2\alpha & 0 & 0 \\ 0 & 0 & -2\alpha & 2\alpha \end{array} \right) \left( \begin{array}{c} v_{\alpha} \otimes v_{\alpha} \\ v_{-\alpha} \otimes v_{-\alpha} \\ v_{\alpha} \otimes v_{-\alpha} \\ v_{-\alpha} \otimes v_{\alpha} \end{array} \right). \]

Hence, our result follows from Lemmas 3.1.4 and 3.1.5.

Definition 3.1.8. For any number field $K$ which unramified at all primes above $p$ and given an element
\[ z = z_1 \wedge z_2 \in \bigwedge^2 H^1(K, W^*_{\chi} \otimes W^*_{\chi}(\chi) \otimes \Lambda_{\mathcal{O}}(\Gamma)^n) \]
as well as $\lambda, \mu = \pm \alpha$, we define
\[ \text{pr}_{\lambda,\mu}(z) = \mathcal{L}_{\lambda,\mu,K} (\text{res}_p(z_1))z_2 - \mathcal{L}_{\lambda,\mu,K} (\text{res}_p(z_2))z_1 \in H^1_{Iw}(K, W^*_{\chi} \otimes W^*_{\chi}(\chi)) \otimes \mathcal{H}_{E,k+1}(\Gamma), \]
where $\mathcal{L}_{\lambda,\mu,K}$ is the shorthand for the sum $\sum_{\nu \mid p} \mathcal{L}_{\lambda,\mu,K_\nu}$. When $K = \mathbb{Q}(m)$ with $m \in N_{\chi}$, we will write $\mathcal{L}_{\lambda,\mu,m}$ in place of $\mathcal{L}_{\lambda,\mu,Q(m)}$. 

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Proposition 3.1.9. For $K$ as in Definition 3.1.8 and $z \in \bigwedge^2 H^1(K, R^+_f \otimes R^+_f(\chi) \otimes \Lambda_\mathcal{O}(\Gamma^+) \otimes \Lambda_\mathcal{O}(\Gamma^+))$, we have
\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
\alpha^2 & \alpha^2 & -\alpha^2 & -\alpha^2 \\
2\alpha & -2\alpha & 0 & 0 \\
0 & 0 & -2\alpha & 2\alpha
\end{pmatrix}
\begin{pmatrix}
pr_{\alpha,\alpha}(z) \\
pr_{-\alpha,\alpha}(z) \\
pr_{\alpha,-\alpha}(z) \\
pr_{-\alpha,-\alpha}(z)
\end{pmatrix}
\in
\begin{pmatrix}
\log_{p,2k+2}^- \\
\log_{p,2k+2}^+ \\
\log_{p,k+1}^-
\end{pmatrix}
H^1(K, R^+_f \otimes R^+_f(\chi) \otimes \Lambda_\mathcal{O}(\Gamma^+) \otimes \Lambda_\mathcal{O}(\Gamma^+)).
\]

Proof. This follows from Lemma 3.1.15.

Remark 3.1.10. We may define similar maps on the Tate twists of $W^+_f \otimes W^+_f(\chi)$ via:
\[
H^1_{Tw}(F, W^+_f \otimes W^+_f(\chi + j)) \to H^1_{Tw}(F, W^+_f \otimes W^+_f(\chi)) \to \mathcal{H}_{E,k+1}(\Gamma) \to \mathcal{H}_{E,k+1}(\Gamma),
\]
which gives maps on $H^1_{Tw}(\mathbb{Q}(m), W^+_f \otimes W^+_f(\chi + j))$ and
\[
\bigwedge^2 H^1_{Tw}(\mathbb{Q}(m), W^+_f \otimes W^+_f(\chi + j)) \to H^1_{Tw}(\mathbb{Q}(m), W^+_f \otimes W^+_f(\chi + j)) \otimes \mathcal{H}_{E,k+1}(\Gamma),
\]
for every integer $j$ and $m \in \mathcal{N}_\gamma$. By an abuse of notation, we shall denote these maps by $\mathcal{L}_{\lambda,\mu,m}$ and $pr_{\lambda,\mu}$. Note that this is different from the Perrin-Riou map attached to $W^+_f \otimes W^+_f(\chi + j)$ since the Hodge-Tate weights are shifted by $j$ compared to $W^+_f \otimes W^+_f(\chi)$.

3.2. Signed Coleman maps and Selmer conditions. Let $F$ be a finite unramified extension of $\mathbb{Q}_p$ and $\omega'$ a fixed basis of $Fil_0 \mathcal{O}_F(\mathcal{H}_{E,k+1}(\Gamma))$. Recall that we have set $T := \text{Sym}^2 R^+_f(1) \otimes \chi$ and $V := \text{Sym}^2 W^+_f(1) \otimes \chi$. Given any element $z \in H^1(F, V)$, we defined in Remark 3.1.10 the maps
\[
\mathcal{L}_{\pm,F}(z) \in \log_{p,2k+2}^+ F \otimes \Lambda_\mathcal{O}(\Gamma) \text{ and } \mathcal{L}_{\bullet,F}(z) \in \log_{p,k+1}^+ F \otimes \Lambda_\mathcal{O}(\Gamma).
\]
Furthermore, if $z \in H^1(F, T)$, then
\[
\mathcal{L}_{\pm,F}(z) \in \log_{p,2k+2}^+ \mathcal{O}_F \otimes \Lambda_\mathcal{O}(\Gamma) \text{ and } \mathcal{L}_{\bullet,F}(z) \in \log_{p,k+1}^+ \mathcal{O}_F \otimes \Lambda_\mathcal{O}(\Gamma)
\]
(c.f., Lemmas 3.1.2 and 3.1.3). We now define the Signed Coleman maps as follows.

Definition 3.2.1. For $\bullet \in \{+,-,\bullet\}$, we let $\log_{p}^\bullet$ stand for $\log_{p,2k+2}^+, \log_{p,2k+2}^-$ or $\log_{p,k+1}^+$. We define the signed Coleman maps $\text{Col}^\bullet_F$ by setting
\[
\text{Col}^\bullet_F : H^1_{Tw}(F,T) \to \mathcal{O}_F \otimes \Lambda_\mathcal{O}(\Gamma)
\]
\[
z \mapsto \mathcal{L}_{\bullet,F}(z)/\log_{p}^\bullet.
\]
for each $\bullet \in \{+,-,\bullet\}$

Let $\eta$ be a character on $\Gamma_{tors}$. We may identify $\epsilon_{p}\Lambda_\mathcal{O}(\Gamma)$ with the power series ring $\mathcal{O}[[X]]$, where $X$ is given by $\gamma - 1$. The images of the plus and minus Coleman maps for $\mathbb{Q}_p$ can be described as follows.

Proposition 3.2.2. For $\eta$ as above, $\epsilon_{p}\text{Im}(\text{Col}^\bullet_F)_{\mathbb{Q}_p}$ is pseudo-isomorphic to $\prod_{j \in S^+_p} (X - \chi_{\gamma}^1(p) - 1)\mathcal{O}[[X]],$ where $S^+_p$ is some subset of $\{0, 1, \ldots, 2k + 1\}$.

Proof. This follows from [LLZ11] Corollary 4.15.

Definition 3.2.3. Following [Kob03, Lei11], we define the signed Selmer conditions
\[
H^1_{Tw,\bullet}(F,T) = \ker(\text{Col}^\bullet_F).
\]
Fix an integer $m \in N_{\chi}$. We may combine the signed Coleman maps $\text{Col}_{\text{BF}(m)_e}$ for primes $\nu$ of $Q(m)$ above $p$ to obtain

$$\text{Col}_{\text{BF}(m)_e}: H^1(Q(m)_p, T) \longrightarrow \Lambda_{\chi}(\Gamma).$$

When $m = 1$, we will write $\text{Col}_{\text{BF}}$ in place of $\text{Col}_{\text{BF}(m)_e}$.

For $\bullet \in \{+,-,\text{•}\}$, we define the (compact) signed Selmer group $H^1_{\text{BF}(m)}(Q(m), T)$ by setting

$$H^1_{\text{BF}(m)}(Q(m), T) := \ker \left( H^1(Q(m), T) \longrightarrow \frac{H^1(Q(m)_p, T)}{\ker(\text{Col}_{\text{BF}(m)_e})} \right).$$

**Definition 3.2.4.** We let

$$H^1_{\text{BF}}(\mu_{p^\infty})_p, T^\vee(1)) \subset H^1(\mu_{p^\infty}, T^\vee(1))$$

denote the orthogonal complement of $\ker(\text{Col}_{\text{BF}(m)_e})$ under Tate duality. The discrete signed Selmer group $\text{Sel}_{\text{BF}}(T^\vee(1)/\mu_{p^\infty})$ is defined as the kernel of the restriction map

$$H^1(Q(\mu_{p^\infty}), T^\vee(1)) \longrightarrow \prod_v H^1(Q(\mu_{p^\infty}), T^\vee(1)) \times \prod_v H^1(Q(\mu_{p^\infty}), T^\vee(1)),$$

where $v$ runs through all primes of $Q(\mu_{p^\infty})$.

**3.3. Signed Beilinson-Flach Classes.** For $\lambda, \mu \in \{\pm \alpha\}$ and $m \in N_{\chi}$, let $\text{BF}_{m,\chi}^{\lambda,\mu}$ be the Beilinson-Flach element from [2,1]. Let us write

$$\text{BF}_{m,\chi}^{\lambda,\mu} = \sum_i F_i^{\lambda,\mu} z_i,$$

where $\sum F_i^{\lambda,\mu} \in H_{E,k+1}(\Gamma)$ and $\{z_i\}$ is some fixed basis of $H^1_{\text{BF}}(Q(\mu_{p^\infty}), W^* \otimes W^*_j(1 + \chi))$. We recall from [BL17, 3.1] and [BLL18, Proposition 4.3.1] that if $0 \leq j \leq k$ and $\theta$ is a Dirichlet character of conductor $p^n > 1$, then

$$F_i^{\lambda,\mu}(\chi^j \theta) = (\lambda \mu)^{-n} c_{n,i,j},$$

for some constant $c_{n,i,j}$ that is independent of the choice $\lambda$ and $\mu$. This implies the following lemma.

**Lemma 3.3.1.** There exists $\overline{\text{BF}}_{m,\chi}^{\lambda,\mu} \in H^1(Q(m), W^*_j \otimes W^*_j(1 + \chi) \otimes H_{E,k+1}(\Gamma)^{\vee})$ ($\lambda, \mu = \pm \alpha$) such that

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha^2 & \alpha^2 & -\alpha^2 & -\alpha^2 \\ 2\alpha & -2\alpha & 0 & 0 \\ 0 & 0 & -2\alpha & 2\alpha \end{pmatrix} \begin{pmatrix} \text{BF}_{m,\chi}^{\lambda,\mu} \\ \text{BF}_{m,\chi}^{\lambda,-\alpha} \\ \text{BF}_{m,\chi}^{-\alpha,\lambda} \\ \text{BF}_{m,\chi}^{-\alpha,-\alpha} \end{pmatrix} = \begin{pmatrix} \log_{p,k+1}^{+}(1) & \log_{p,k+1}^{-}(1) \\ \log_{p,k+1}^{+} \overline{\text{BF}}_{m,\chi}^{\alpha,\alpha} & \log_{p,k+1}^{-} \overline{\text{BF}}_{m,\chi}^{-\alpha,-\alpha} \\ \log_{p,k+1}^{+} \text{BF}_{m,\chi}^{\alpha,-\alpha} & \log_{p,k+1}^{-} \text{BF}_{m,\chi}^{-\alpha,\alpha} \\ \log_{p,k+1}^{+} \overline{\text{BF}}_{m,\chi}^{-\alpha,-\alpha} & \log_{p,k+1}^{-} \overline{\text{BF}}_{m,\chi}^{\alpha,\alpha} \end{pmatrix}.$$

**Remark 3.3.2.** On comparing denominators, we see that

$$\overline{\text{BF}}_{m,\chi}^{\alpha,\alpha} \overline{\text{BF}}_{m,\chi}^{-\alpha,-\alpha} \in H^1(Q(m), W^*_j \otimes W^*_j(1 + \chi) \otimes H_{E,k+1}(\Gamma)^{\vee}),$$

$$\text{BF}_{m,\chi}^{\alpha,-\alpha}, \overline{\text{BF}}_{m,\chi}^{-\alpha,-\alpha} \in H^1_{\text{BF}}(Q(m), W^*_j \otimes W^*_j(1 + \chi)).$$

**Theorem 3.3.3.** Suppose that the Dirichlet character $\chi$ verifies the conditions ($\Psi_1$) and ($\Psi_2$). Assume also that ($\text{Im}$) holds true. For $m \in N_{\chi}$ and $\eta \in \Delta_m$, we write $e_\eta$ for the corresponding idempotent. For all four choices of $\lambda, \mu \in \{\pm \alpha\}$, there exist $c_m \in Q[\Delta_m] \otimes \text{Frac}(H_{E})$ and $z_m \in \text{H}_{\text{BF}}^1(Q(m), W^*_j \otimes W^*_j(1 + \chi))$ (both of which depend only on $m$ and not on the choice of the pair $\lambda, \mu$) satisfying the following properties.

1. $\text{BF}_{m,\chi}^{\lambda,\mu} = \delta c_m = \text{pr}_{\lambda,\mu}(z_m)$, where $\delta \in \{\pm\}$ determined according to $\lambda \mu = \delta \alpha^2$.
2. For each $\eta \in \Delta_m$, the element $e_\eta c_m \in \text{Frac}(H_{E})$ is non-zero.
(3) For each $\eta \in \widehat{\Delta}_m$, we write $c_\eta = d_\eta/h_\eta$, where $d_\eta, h_\eta \in \mathcal{H}_E$ are coprime. Then, $h_\eta$ is coprime to $\log_{p,2k+3}/\log_{p,k+2}$.

The proof of this theorem is rather involved and it will be presented in Section 3.10.

**Corollary 3.3.4.** In the setting of Theorem 3.3.3, there exist

$$BF_{m,\chi}^+, BF_{m,\chi}^-, BF_{m,\chi}^\bullet, BF_{m,\chi}^{\circ} \in H^1_{Iw}(\mathbb{Q}(m), W^\bullet_f \otimes W^\bullet_f(1 + \chi))$$

such that

$$\left(\begin{array}{ccc} 1 & 1 & 1 \\ \alpha^2 & \alpha^2 & -\alpha^2 \\ 2\alpha & -2\alpha & 0 \end{array}\right) \left(\begin{array}{c} BF_{m,\chi}^{\alpha,\alpha} \\ BF_{m,\chi}^{\alpha,-\alpha} \\ BF_{m,\chi}^{-\alpha,-\alpha} \end{array}\right) = \left(\begin{array}{c} \log_{p,k+2}^+(1) \cdot BF_{m,\chi}^+ \\ \log_{p,k+2}^-(1) \cdot BF_{m,\chi}^- \\ \log_{p,k+1}^+(1) \cdot BF_{m,\chi}^{\bullet} \\ \log_{p,k+1}^+(1) \cdot BF_{m,\chi}^{\circ} \end{array}\right).$$

(3.3)

**Proof.** As discussed above, the result for the last two rows follow from Lemma 3.3.4. We shall prove the divisibility on the first row; that for the second row can be proved in a similar fashion.

It follows from Proposition 3.1.9 and Theorem 3.3.3 (3) that

$$\left(\frac{pr_{\alpha,\alpha} + pr_{-\alpha,-\alpha} - pr_{\alpha,-\alpha} - pr_{-\alpha,\alpha}}{(c_m z_m)}\right) \in \frac{\log_{p,2k+3}^+}{\log_{p,k+2}^+} H^1_{Iw}(\mathbb{Q}(m), W^\bullet_f \otimes W^\bullet_f(1 + \chi)) \otimes \mathcal{H}_{E,(k+1)/2}(\Gamma).$$

Therefore, on combining this with Theorem 3.3.3 (1), we deduce that the sum

$$BF_{m,\chi}^{\alpha,\alpha} + BF_{m,\chi}^{\alpha,-\alpha} + BF_{m,\chi}^{-\alpha,-\alpha} + BF_{m,\chi}^{-\alpha,\alpha}$$

is divisible by $\frac{\log_{p,k+2}^+(1)}{\log_{p,k+2}^+}$ in the module

$$\mathbb{H}_m := H^1_{Iw}(\mathbb{Q}(m), W^\bullet_f \otimes W^\bullet_f(1 + \chi)) \otimes \mathcal{H}_{E,(k+1)/2}(\Gamma).$$

Furthermore, Lemma 3.3.4 says that this class is also divisible by $\log_{p,k+1}^+(1)$. Note that

$$\log_{p,k+1}^+(1) \times \frac{\log_{p,2k+3}^+}{\log_{p,k+2}^+} = \log_{p,2k+2}^+(1).$$

The conclusion follows using the fact that $\mathbb{H}_m$ is free, along with growth order considerations. \qed

**Proposition 3.3.5.** In the setting of Theorem 3.3.3, there exists an integer $C$ independent of $m$ such that

$$C \times BF_{m,\chi}^\bullet \in H^1_{Iw}(\mathbb{Q}(m), R^\bullet_f \otimes R^\bullet_f(1 + \chi))$$

for all four choices of $\bullet \in \{+,-,\bullet,\circ\}$.

**Proof.** Let $\lambda, \mu \in \{\pm \alpha\}$ and fix $m$. Note that $\frac{1}{\rho_{\mu,\mu}}$ is a $p$-adic unit given that $v_p(\lambda) = v_p(\mu) = (k + 1)/2$. Write $x_{\lambda,\mu}^r \in H^1(\mathbb{Q}(m)(\mu_p), W^\bullet_f \otimes W^\bullet_f(1 + \chi))$ for the image of the Iwasawa theoretic Beilinson-Flach class $BF_{m,\chi}^{\lambda,\mu}$. Then by [LZ16, Theorem 8.1.4(ii)]

$$C_0 \times p^{(k+1)r}x_{\lambda,\mu}^r \in H^1(\mathbb{Q}(m)(\mu_p), R^\bullet_f \otimes R^\bullet_f(1 + \chi))$$

for some integer $C_0$ that is independent of $r$, $m$ and the choices of $\lambda$ and $\mu$.

Let $BF^\bullet$ any one of the four linear combinations of Beilinson-Flach classes on the left-hand side of (3.3) and expand $BF^\bullet$ with respect to a fixed basis $\{z_i\}$ of $H^1_{Iw}(\mathbb{Q}(m), W^\bullet_f \otimes W^\bullet_f(1 + \chi))$, say $BF^\bullet = \sum z_i F_i$. Let log $\bullet$ be the corresponding logarithm on the right-hand side of (3.3). Then,

- $\log^\bullet | F_i$ for all $i$;
- $F_i = O(\log_{p}^{k+1})$.
For all \( r \geq 1 \), we have \( p^{-(k+1)r}||F_i||_{p^r} \) is bounded independently of \( i, r, m \) and the choice of \( \bullet \).

Here \( ||\bullet||_{p^r} \) is the norm on power series defined as in [BL17, §2.1]. Consequently \( ||F_i/\log \bullet||_{p^r} \) is bounded independently of \( i, r, m \) and the choice of \( \bullet \). Hence, \( BF^\bullet/\log \bullet \) are bounded classes as required. \( \Box \)

Recall that \( T := \text{Sym}^2 R_f^!(1 + \chi) \) and \( T := T \otimes \Lambda_Q(\Gamma)' \).

**Corollary 3.3.6.** In the setting of Theorem 3.3.3 and for \( \bullet \in \{+, -, \bullet\} \), we have
\[
C \times BF^\bullet_{m, \chi} \in H^1(Q(m), T), \quad BF^\circ_{m, \chi} = 0.
\]

**Proof.** The first part of the corollary follows from Proposition 3.3.3, Corollary 3.3.3 and Proposition 3.3.4. For the second part of the corollary, note that
\[
BF^\circ_{m, \chi} = (BF^{-a,-a}_{m, \chi} - BF^{a,-a}_{m, \chi})/2a
\]
Under our assumption on the parity of the Dirichlet character \( \chi \), we show that
\[
BF^\lambda, \mu_{m, \chi} = BF^\mu, \lambda_{m, \chi}
\]
for all choices of \( \lambda, \mu \in \{\alpha, -\alpha\} \) in Section 4.5. In particular, see Remark 4.5.2. \( \Box \)

### 3.4. \( p \)-local properties of signed Beilinson-Flach classes

Throughout this section, we assume that the character \( \chi \) verifies \( (\Psi_1) \) and \( \Psi_2 \). Suppose also that \( (\text{Im}) \) holds true. We also fix an integer \( m \in \mathcal{N}_\chi \).

Recall from 3.3.3 that there exist signed Beilinson-Flach classes
\[
BF^+_{m, \chi} = \frac{1}{\log_{p,2k+2}(\zeta_{p,2k+2})} \left( BF^{a, a}_{m, \chi} + BF^{-a, -a}_{m, \chi} + BF^{a, -a}_{m, \chi} + BF^{-a, a}_{m, \chi} \right),
\]
\[
BF^-_{m, \chi} = \frac{a^2}{\log_{p,2k+2}(\zeta_{p,2k+2})} \left( BF^{a, a}_{m, \chi} + BF^{-a, -a}_{m, \chi} - BF^{a, -a}_{m, \chi} - BF^{-a, a}_{m, \chi} \right)
\]
\[
BF^\bullet_{m, \chi} = \frac{2a}{\log_{p,2k+2}(\zeta_{p,2k+2})} \left( BF^{a, a}_{m, \chi} - BF^{-a, -a}_{m, \chi} \right)
\]
The goal of this section is to study the image of these classes under the localization map at \( p \). Firstly, let us recall the following “geometric” property of the unbounded Beilinson-Flach classes.

**Proposition 3.4.1.** For an integer \( j \in [-k, 0] \) and \( \lambda, \mu \in \{\pm \alpha\} \), the natural image of \( \text{loc}_p \left( BF^\lambda, \mu_{m, \chi}(j) \right) \) in \( H^1(Q(m)(\mu_{p^r}), W_f^* \otimes W_f^*(1+j+\chi)) \) belongs to the Bloch-Kato subgroup \( H^1(Q(m)(\mu_{p^r}), W_f^* \otimes W_f^*(1+j+\chi)) \).

**Proof.** This is [KLZ17, Proposition 3.3.3], since \( H^1_1 = H^1_\mathbb{R} \) in this case (see [LZ16, Proposition 8.1.3]). \( \Box \)

**Proposition 3.4.2.** \( \text{res}_p \left( BF^\bullet_{m, \chi} \right) \in H^1_{1, w, \bullet}(Q(m), p, W_f^* \otimes W_f^*(1+\chi)). \)

**Proof.** We shall only consider the “+” case. Let us set \( F = Q(m)_v \) for some place \( v \) of \( Q(m) \) above \( p \) and write
\[
z = \text{res}_p \left( BF^{a, a}_{m, \chi} + BF^{-a, -a}_{m, \chi} + BF^{a, -a}_{m, \chi} + BF^{-a, a}_{m, \chi} \right).
\]
Proposition 3.4.1 tells us that the image of \( z \) in \( H^1(F(\mu_{p^r}), W_f^* \otimes W_f^*(1+\chi)) \) falls within the Bloch-Kato subgroup \( H^1(F(\mu_{p^r}), W_f^* \otimes W_f^*(1+\chi)) \) for all \( r \geq 0 \). By the interpolative properties of Perrin-Riou’s map, this implies that
\[
\mathcal{L}_{W_f^* \otimes W_f^*(1+\chi), F}(z) \in F \otimes \mathcal{H}_E \otimes \mathcal{D}_\text{cris}(W_f^* \otimes W_f^*(1+\chi))
\]
vanishes at all finite characters on \( \Gamma \). Let \( \mathcal{L}_{+, F} \) be the morphism given in 3.2. Then, both \( \mathcal{L}_{+, F}(z) \) and \( \mathcal{L}_{-, F}(z) \) vanish at all finite characters of \( \Gamma \).
By an abuse of notation, we shall denote $L_{+,F}$ (respectively $\text{Col}^+_F$) composed with the projection map $W^*_{\ell} \otimes W^*_{\ell}(\chi) \to W^*_{\ell}(1 + \chi)$ by the same symbol. Note that

$$L_{+,F}(z) = \left( \log_{p,2k+2}^{+(1)} \right)^2 \text{Col}^+_F \circ \text{loc}_p \left( \text{BF}_{m,\chi}^+ \right).$$

Therefore, $\text{Col}^+_F \circ \text{loc}_p \left( \text{BF}_{m,\chi}^+ \right)$ vanishes at infinitely many finite characters of $\Gamma$ (the ones that do not vanish at $\log_{p,2k+2}^{+(1)}$). This forces $\text{Col}^+_F \circ \text{loc}_p \left( \text{BF}_{m,\chi}^+ \right)$ to vanish, as required. □

Note that we have $C \times \text{BF}_{m,\chi}^+ \in H^1_{\ell,\bullet}(\mathbb{Q}(\mu_m), \mathbb{T})$ for $\bullet \in \{+, -, \bullet\}$ by Proposition 3.4.2.

3.5. Doubly signed Main conjectures. Recall that $T := \text{Sym}^2 R^*_f(1 + \chi)$ and $\mathbb{T} = T \otimes \Lambda'$. We now define doubly signed compact and discrete Selmer groups as well as doubly signed $p$-adic $L$-functions in the spirit of [BLLV18].

**Definition 3.5.1.** Let $S$ denote the set of pairs $\{(+, -), (+, \bullet), (-, \bullet)\}$. For $\mathcal{S} = (\bullet, \bullet) \in S$, we define the following objects

- A compact Selmer group $H^1_{\ell,\bullet}(\mathbb{Q}, \mathbb{T})$, given by

$$H^1_{\ell,\bullet}(\mathbb{Q}, \mathbb{T}) := \ker \left( H^1(\mathbb{Q}, \mathbb{T}) \to \frac{H^1(\mathbb{Q}_p, \mathbb{T})}{\ker(\text{Col}^\bullet) \cap \ker(\text{Col}^\bullet)} \right).$$

- A discrete Selmer group $\text{Sel}_\mathcal{S}(T^\vee(1)/\mathbb{Q}(\mu_p^\infty))$, given by the kernel of the restriction map

$$H^1(\mathbb{Q}(\mu_p^\infty), T^\vee(1)) \to \prod_{v \mid p} H^1_{\ell,\bullet}(\mathbb{Q}(\mu_p^\infty)_v, T^\vee(1)) \times \prod_{v \mid p} H^1(\mathbb{Q}(\mu_p^\infty)_v, T^\vee(1)),$$

where $v$ runs through all primes of $\mathbb{Q}(\mu_p^\infty)$, and for $v \mid p$ the local condition $H^1_{\ell,\bullet}(\mathbb{Q}(\mu_p^\infty)_v, T^\vee(1))$ is the orthogonal complement of $\ker(\text{Col}^\bullet) \cap \ker(\text{Col}^\bullet)$ under the local Tate pairing.

- In the setting of Theorem 3.3.3 we define the doubly-signed $p$-adic $L$-function by setting

$$L_{\mathcal{S}} := \text{Col}^\bullet \circ \text{res}_p \left( \text{BF}_{1,\chi}^\bullet \right) \in C^{-1} \Lambda_0(\Gamma).$$

**Remark 3.5.2.** Interchanging the roles of $\bullet$ and $\bullet$ has the effect of multiplying $L_{\mathcal{S}}$ by $-1$. This is the content of [BLLV18] Proposition 5.3.4] (see Proposition 4.6.3 for its incarnation in our setting). Since we are only interested in the ideal generated by $L_{\mathcal{S}}$, the ambiguity of sign is not an issue for us.

We are now in a position to formulate a doubly-signed Iwasawa main conjecture for the symmetric square representation of a non-$p$-ordinary eigenform.

**Conjecture 3.5.3.** For every $\mathcal{S} \in S$ and every character $\eta$ of $\Gamma_{\text{tors}}$, the module $e_\eta \text{Sel}_\mathcal{S}(T^\vee(1)/\mathbb{Q}(\mu_p^\infty))$ is $\Lambda(\Gamma_1)$-cotorsion and

$$\text{char}_{\Lambda_0(\Gamma_1)}(e_\eta \text{Sel}_\mathcal{S}(T^\vee(1)/\mathbb{Q}(\mu_p^\infty))^\vee) = (e_\eta L_{\mathcal{S}})$$

as ideals of $\Lambda(\Gamma_1) \otimes \mathbb{Q}_p$, with equality away from the support of $\ker(\text{Col}^\bullet)$ and $\ker(\text{Col}^\bullet)$.

**Proposition 3.5.4.** Suppose we are in the setting of Theorem 3.3.3 Then there exists a choice of $\mathcal{S} \in S$ such that $e_\omega L_{\mathcal{S}} \neq 0$.

**Proof.** By Lemma 3.3.7 and Definition 3.2.1, we have for all $z \in H^1_{\text{loc}}(\mathbb{Q}(m), R^*_f \otimes R^*_f(1 + \chi))$ the following identity:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
\alpha^2 & \alpha^2 & -\alpha^2 & -\alpha^2 \\
2\alpha & -2\alpha & 0 & 0 \\
0 & 0 & -2\alpha & 2\alpha
\end{pmatrix}
\begin{pmatrix}
L_{\alpha,1,F}(z) \\
L_{1,-\alpha,F}(z) \\
L_{1,-\alpha,F}(z) \\
L_{-\alpha,F}(z)
\end{pmatrix}
= 
\begin{pmatrix}
\log_{p,2k+2}^{+(1)} \text{Col}^+(z) \\
\log_{p,2k+2}^{-(1)} \text{Col}^-(z) \\
\log_{p,k+1}^{+(1)} \text{Col}^+(z) \\
\log_{p,k+1}^{-(1)} \text{Col}^-(z)
\end{pmatrix}.
$$
In particular,
\[ \mathcal{L}_{\alpha,\alpha}(z) = \frac{\log_{p^{2k+2}} p}{4} \operatorname{Col}^-(z) + \frac{\log_{p^{2k+2}} p}{4\alpha^2} \operatorname{Col}^+(z) + \frac{\log_{p^{k+1}} p}{4\alpha} \operatorname{Col}^*(z). \]

Similarly, by Corollary 3.3.4 we have
\[ \operatorname{BF}^{\alpha,-\alpha}_{i,x} = \frac{\log_{p^{2k+2}} p}{4} \operatorname{BF}^{+}_{i,x} - \frac{\log_{p^{2k+2}} p}{4\alpha^2} \operatorname{BF}^{-}_{i,x} - \frac{\log_{p^{k+1}} p}{4\alpha} \operatorname{BF}^{0}_{i,x}. \]

By Corollary 3.3.4 we know that \( \operatorname{BF}^{0}_{i,x} = 0 \). Hence, \( \mathcal{L}_{\alpha,\alpha}(\operatorname{BF}^{\alpha,-\alpha}_{i,x}) \) is a \( H_E \)-linear combination of the terms \( \operatorname{Col}^\bullet \circ \operatorname{res}_p(\operatorname{BF}^\bullet_{i,x}) \) for \( (\blacklozenge, \blacklozenge) \in S \). By (2.1), we know that \( \mathcal{L}_{\alpha,\alpha}(\operatorname{BF}^{\alpha,-\alpha}_{i,x}) \) is a non-zero multiple of the geometric \( p \)-adic \( L \)-function and hence is non-zero. We conclude that there exists at least one \( \mathcal{S} = (\blacklozenge, \blacklozenge) \in S \) such that \( \operatorname{Col}^\bullet \circ \operatorname{res}_p(\operatorname{BF}^\bullet_{i,x}) \) is non-zero.

**Theorem 3.5.5.** Suppose that the hypotheses \( (\Psi_i) \ (i = 1, 2, 3), (\operatorname{Im}) \) hold true. Then for every \( j \in \{k + 2, \ldots, 2k + 2\} \) even and \( \mathcal{S} \in S \) that validates the conclusion of Proposition 3.5.4 the \( \omega^j \)-isotypic component \( e_{\omega^j} \operatorname{Sel}_E(T^{\vee}(1)/\mathbb{Q}(\mu_{p^\infty})) \) is \( \Lambda(\Gamma_1) \)-cotorsion and we have
\[ \operatorname{char}_{\Lambda(\Gamma_1)} (e_{\omega^j} \operatorname{Sel}_E(T^{\vee}(1)/\mathbb{Q}(\mu_{p^\infty}))) = (e_{\omega^j} \mathcal{L}_E) \]
as ideals of \( \Lambda(\Gamma_1) \otimes \mathbb{Q}_p \).

**Proof.** The proof follows from the same proof of [BLLV18 Theorem 6.2.4], using the (rank 1) locally restricted Euler system machinery we have set up above. The quadruple sign \( \mathcal{S} = \{(\Delta, \square), (\bullet, \circ)\} \) therein corresponds to our double sign \( \mathcal{S} = (\blacklozenge, \blacklozenge) \in S \). The additional hypothesis \( (\Psi_3) \) ensures the big image condition on \( T \) in order to apply the Euler system machinery holds (c.f. [LZ15 §5.2]). \( \square \)

### 4. Analytic Main conjectures

Our goal in this section is twofold: To give proofs of Theorem 2.3.13 (iv) and Theorem 3.3.3. Both results are crucial for the construction of bounded Beilinson-Flach classes as well as to translate our results on the signed Iwasawas main conjectures into the analytic language of Pottharst [Pot12, Pot13] and Benois [Ben15].

Fix once and for all an integer \( r \in \mathbb{N}_r \) as in Definition 2.3.3 where the Dirichlet character \( \chi \) is given as in Section 2.3 which verifies \( (\Psi_1) \) and \( (\Psi_2) \). Fix also a character \( \nu \in \hat{\Delta}_r \) and set \( \psi := \chi \nu \). Also as in Section 2.3 we choose an arbitrary positive integer \( j \) from the interval \( \{k + 2, 2k + 2\} \). With those fixed choices, recall the symmetric square Galois representation
\[ V_{\psi,j} := \text{Sym}^2 W^*_f(1 - j) \otimes \psi^j \psi \]
as well as t
\[ W_{\psi,j} := W^*_f \otimes W^*_f (1 - j) \otimes \psi^j \psi = \left( \bigotimes^{2} W_f^*(1 - j) \otimes \psi^j \psi \right) \oplus V_{\psi,j}. \]

For our fixed choice of \( j \) and only in this section, we shall set \( \mathcal{V}_\psi := V_{\psi,j} \otimes \psi^{-j} \) and \( \mathcal{W}_\psi := W_{\psi,j} \otimes \psi^{-j} \).

We shall make use of the identification
\[ H^1_{\text{Iw}}(K, \mathcal{W}_\psi)^{-j} \cong H^1_{\text{Iw}}(K^\Delta, W_{\psi,j}) \]
(and likewise, for the representations \( \mathcal{V}_\psi \) and \( V_{\psi,j} \)) for any abelian extension \( K \) of \( \mathbb{Q} \) that contains \( \mu_p \), where \( \Delta = \text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \) and \( K^\Delta \) is the fixed field of \( \Delta \). This will be particularly useful when considering the crystalline Dieudonné modules of various of twists.
4.1. **Local Analysis.** We fix forever a generator \( \{ \varepsilon_{(n)} \}_n \) of \( \lim_{\mu \to \emptyset} \mu = \mathbb{Z}_p(1) \).

Recall from Corollary 3.1.2 the \( G_{Q_p} \)-invariant subspaces \( W_1 \) and \( W_2 \) of \( \text{Sym}^2 W_f \). We define

\[
D_i := \mathbb{D}_{\text{cris}} (W_i^* (1 - j) \otimes \psi)
\]

and also set

\[
D_0 := \mathbb{D}_{\text{cris}} \left( \bigwedge^2 W_j^* (1 - j) \otimes \psi \right)
\]

so that we have

\[
\mathbb{D}_{\text{cris}}(\mathcal{W}_\psi) = D_0 \oplus D_1 \oplus D_2, \quad \mathbb{D}_{\text{cris}}(V_\psi) = D_1 \oplus D_2.
\]

The crystalline Frobenius \( \varphi \) acts on \( D_1 \) by \( \alpha_{\psi,j} := p^{j-1} \psi^{-1}(p)/\alpha \), whereas it acts on \( D_0 \) by \( -\alpha_{\psi,j} = p^{j-1} \psi^{-1}(p)/\alpha \). Recall from [3.2] the \( \varphi \)-eigenvectors \( v_{\pm \alpha} \in \mathbb{D}_{\text{cris}}(W_f^*) \). If we fix a non-zero vector \( v_{j,\psi} \in \mathbb{D}_{\text{cris}}(E(1-j) \otimes \psi) \). We have the following \( \varphi \)-eigenvectors in \( \mathbb{D}_{\text{cris}}(\mathcal{W}_\psi) \): \( \omega_{\pm} := v_{\pm \alpha} \otimes v_{\pm \alpha} \otimes v_{n,\psi} \). It is easy to see that

- \( D_0 = \text{span}\{\omega_{-} - \omega_{+}\} \)
- \( D_1 = \text{span}\{\omega_{++} - \omega_{--}\} \)
- \( D_2 = \text{span}\{\omega_{++} + \omega_{--}, \omega_{+-} + \omega_{-+}\} \).

**Definition 4.1.1.** Given \( \lambda \in \{\alpha, -\alpha\} \), we let \( \delta_\lambda : \mathbb{Q}_p^- \to E^\times \) denote the character that is given by \( \delta_\lambda(p) = 1/\lambda \) and \( \delta(u) = 1 \) for \( u \in \mathbb{Z}_p^\times \). We also write \( \delta_{\psi,j} : \mathbb{Q}_p^- \to E^\times \) for the character which is given by \( \delta_{\psi,j}(p) = p^{j-1} \psi^{-1}(p) \) and \( \delta_{\psi,j}(u) = u^{1-j} \).

For each \( \varphi \)-eigen-subspace \( \mathbb{D}_{\text{cris}}(W_f^*)^{\varphi=1/\lambda} \) of \( \mathbb{D}_{\text{cris}}(W_f^*) \) (where \( \lambda = \pm \alpha \) as above), there is a unique rank-one \( \langle \varphi, \Gamma \rangle \)-submodule \( \mathbb{D}_\lambda \subset \mathbb{D}_f := D_{\text{rig}}^1(W_f^*) \), which is of the form \( \mathcal{R}(\delta_\lambda) \), where \( \mathcal{R} \) is the Robba ring over \( E \) (see [Ben14, §1.2.5]). More precisely, \( \mathbb{D}_\lambda \) is the free \( \mathcal{R}(\quad) \)-module generated by an element \( e_\lambda \in \mathbb{D}_\lambda \) for which we have

\[
\varphi(e_\lambda) = \delta_\lambda(p) \cdot e_\lambda, \quad \tau(e_\lambda) = \delta(\chi_{\text{cyc}}(\tau)) \cdot e_\lambda \quad (\forall \tau \in \Gamma)
\]

We also set \( \mathbb{D}_{\psi,j} \) to be \( \mathcal{R}(\delta_{\psi,j}) \) and define

\[
D_{f,j} := D_f \otimes \mathbb{D}_{\psi,j} \cong D_{\text{rig}}^1(W_f^* (1 - j) \otimes \psi), \quad \mathbb{D}_{\lambda,j} := \mathbb{D}_\lambda \otimes \mathbb{D}_{\psi,j}
\]

In what follows, the following \((\varphi, \Gamma)\)-subquotients (all of which are necessarily crystalline) of \( D_{\text{rig}}^1(\mathcal{W}_\psi) \) will play a crucial role. Let \( \lambda, \mu \in \{\alpha, -\alpha\} \).

- \( D_{\text{rig}}^1(\mathcal{W}_\psi)^{\lambda,\mu} := \mathbb{D}_\lambda \otimes \mathbb{D}_\mu \)
- \( D_{\text{rig}}^1(\mathcal{W}_\psi)^{\lambda,\circ} := \mathbb{D}_\lambda \otimes \mathbb{D}_{f,j} \)
- \( D_{\text{rig}}^1(\mathcal{W}_\psi)/_{\lambda,\circ} := D_{\text{rig}}^1(\mathcal{W}_\psi) / D_{\text{rig}}^1(\mathcal{W}_\psi)^{\lambda,\circ} \)
- \( D_{\text{rig}}^1(\mathcal{W}_\psi)/_{\lambda,\mu} := (\mathbb{D}_f / \mathbb{D}_\lambda) \otimes \mathbb{D}_{\mu,j} \subset D_{\text{rig}}^1(\mathcal{W}_\psi)/_{\lambda,\circ} \)
- \( D_{\text{rig}}^1(\mathcal{W}_\psi)^{\lambda,\mu,+} := (\mathbb{D}_\lambda \otimes \mathbb{D}_f + D_f \otimes \mathbb{D}_\mu) \otimes \mathbb{D}_{\psi,j} \)
- \( D_{\text{rig}}^1(\mathcal{W}_\psi)^{\lambda,\mu,-} := D_{\text{rig}}^1(\mathcal{W}_\psi) / D_{\text{rig}}^1(\mathcal{W}_\psi)^{\lambda,\mu,+} \)

Since the \((\varphi, \Gamma)\)-module \( D_{\text{rig}}^1(\mathcal{W}_\psi) \) is crystalline, it follows that

\[
\mathbb{D}_{\text{cris}}(\mathcal{W}_\psi)^{\lambda,\circ} := \mathbb{D}_{\text{cris}}(D_{\text{rig}}^1(\mathcal{W}_\psi)^{\lambda,\circ}) \subset \mathbb{D}_{\text{cris}}(D_{\text{rig}}^1(\mathcal{W}_\psi)) = \mathbb{D}_{\text{cris}}(\mathcal{W}_\psi),
\]

We define the map (of \( \varphi \)-modules)

\[
\mathcal{D}/_{\lambda,\circ} : \mathbb{D}_{\text{cris}}(\mathcal{W}_\psi) \longrightarrow \mathbb{D}_{\text{cris}}(\mathcal{W}_\psi) / \mathbb{D}_{\text{cris}}(\mathcal{W}_\psi)^{\lambda,\circ} =: \mathbb{D}_{\text{cris}}(\mathcal{W}_\psi)/_{\lambda,\circ}.
\]

The exact sequence

\[
0 \longrightarrow D_{\text{rig}}^1(\mathcal{W}_\psi)/_{\lambda,\mu} \longrightarrow D_{\text{rig}}^1(\mathcal{W}_\psi)/_{\lambda,\circ} \longrightarrow D_{\text{rig}}^1(\mathcal{W}_\psi)^{\lambda,\mu,-} \longrightarrow 0
\]

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of crystalline \((\phi, \Gamma)\)-modules gives rise to the following exact sequence of \(\phi\)-modules:

\[
\begin{align*}
0 & \longrightarrow \mathbb{D}_{\text{cris}}(D_{\text{rig}}^1(W_{\phi}/\lambda, \mu)) \overset{\partial^*}{\longrightarrow} \mathbb{D}_{\text{cris}}(W_{\phi}/\lambda, \mu) \overset{\partial}{\longrightarrow} \mathbb{D}_{\text{cris}}(D_{\text{rig}}^1(W_{\phi})^\lambda_{\mu^-}) \longrightarrow 0
\end{align*}
\]

Lemma 4.1.2. \(\text{ker}(\partial_{\lambda, \mu}) = (\mathbb{D}_{\text{cris}}(W_{\phi}/\lambda, \mu))^{\phi = \frac{\mu - 1 - 1(p)}{-\lambda\mu}}\)

Proof. Note that \(\mathbb{D}_{\text{cris}}(D_{\text{rig}}^1(W_{\phi}/\lambda, \mu))\) is one dimensional, and a consideration involving the \(\phi\)-eigenvalues shows that this \(\phi\)-module is forced to be isomorphic to \(\mathbb{D}_{\text{cris}}(\mathbb{D}_{\text{cris}} \otimes \mathbb{D}_{\mu})\). The proof now follows from the exact sequence (4.1), as the image of \(\partial^*\) may be identified with the (one dimensional) \(p^{1-1}q^{1(p)}(-\lambda\mu)\) eigenspace for the \(\phi\)-action on \(\mathbb{D}_{\text{cris}}(W_{\phi}/\lambda, \mu)\).

Given an integer \(m \in \{1, \ldots, p - 1\}\), we let \([m] \in \mathbb{Q}_p^\times\) denote its Teichmüller lift. We also let \(\text{Tr}_j : \mathbb{Q}_p(\mu_p)^{\omega^{-j}} \rightarrow \mathbb{Q}_p\) denote the twisted trace map induced by

\[
\frac{1}{p - 1} \sum_{r=1}^{p-1} [r] \varepsilon^{\omega}_{(1)} \rightarrow 1.
\]

Definition 4.1.3. For any crystalline \(G_{\mathbb{Q}_p}\)-representation or a \((\phi, \Gamma)\)-module \(D\), we denote the composition of the arrows

\[
H^1_{\omega^{-j}}(\mathbb{Q}_p(\mu_p^n), D) \overset{\text{exp}^*}{\longrightarrow} \mathbb{Q}_p(\mu_p)^{\omega^{-j}} \otimes \mathbb{D}_{\text{cris}}(D) \overset{\text{Tr}_j \otimes 1}{\longrightarrow} \mathbb{D}_{\text{cris}}(D)
\]

by \(\omega^{-j} \circ \exp^*\). Once we identify \(\mathbb{Q}_p(\mu_p)^{\omega^{-j}}\) with \(\mathbb{Q}_p[\Delta]^{\omega^{-j}}\) via the generator \(\varepsilon_{(1)}\) of \(\mu_p\), note that the map \(\text{Tr}_j\) agrees with the map \(\omega^{-j} : \mathbb{Q}_p[\Delta]^{\omega^{-j}} \rightarrow \mathbb{Q}_p\), which justifies the notation we have chosen for the composition (4.2). More generally, if \(\theta\) is a character of \(\Gamma_n := \Gamma/\Gamma^n\) that does not factor through \(\Gamma_{n-1}\) (where \(n\) is a positive integer), we may define a map

\[
\omega^{-j} \circ \exp^* : H^1(\mathbb{Q}_p(\mu_p^n), D)^{\omega^{-j}} \theta \longrightarrow \mathbb{D}_{\text{cris}}(D)
\]

starting off with the map \(\omega^{-j} \theta : \mathbb{Q}_p[\Delta \times \Gamma_n]^{\omega^{-j}} \rightarrow \mathbb{Q}_p\) and identifying it (via the generator \(\varepsilon_{(n)}\) of \(\mu_p^n\)) with a twisted trace map \(\mathbb{Q}_p(\mu_p^n) \rightarrow \mathbb{Q}_p\).

Definition 4.1.4. Given a finite extension \(K\) of \(\mathbb{Q}_p\) and a crystalline \(G_K\)-representation \(V\), we set

\[
H^1_{\text{res}}(K, V) := H^1(K, V)/H^1_{\text{rig}}(K, V)
\]

and call it the singular quotient. For each positive integer \(n\), we further set

\[
\text{res}_n : H^1(\mathbb{Q}(\mu_p^n), V) \longrightarrow H^1(\mathbb{Q}(\mu_p^n), V)
\]

(the singular projection) to denote the compositum of the arrows

\[
H^1(\mathbb{Q}(\mu_p^n), V) \overset{\text{res}_n}{\longrightarrow} H^1(\mathbb{Q}(\mu_p^n), V) \overset{\text{res}}{\longrightarrow} H^1_{\text{res}}(\mathbb{Q}(\mu_p^n), V).
\]

Lemma 4.1.5. \(\text{Let } x, y \in H^1(\mathbb{Q}_p(\mu_p, W_{\phi})^{\omega^{-j}}\) be two classes with non-trivial singular projection, for which we have

\[
\begin{align*}
\partial_{\lambda, \mu}(x) &= 0, \\
\partial_{\lambda, \mu} \circ \omega^{-j} \circ \exp^*_{\varepsilon(1)}(x) &\in (\mathbb{D}_{\text{cris}}(W_{\phi}/\lambda, \mu))^{\phi = \frac{\mu - 1 - 1(p)}{-\lambda\mu}}, \\
\partial_{\lambda, \mu} \circ \omega^{-j} \circ \exp^*_{\varepsilon(1)}(y) &\in (\mathbb{D}_{\text{cris}}(W_{\phi}/\lambda, \mu))^{\phi = \frac{\mu - 1 - 1(p)}{-\lambda\mu}}.
\end{align*}
\]

Then the classes \(s(x)\) and \(s(y)\) are linearly independent in \(H^1_{\text{res}}(\mathbb{Q}_p(\mu_p, W_{\phi})\).\)

Proof. This is clear, as the images of \(x\) and \(y\) under the dual exponential map (that factors through the singular cohomology) composed with the map \(\partial_{\lambda, \mu} \circ \omega^{-j}\) fall within different eigenspaces. \(\square\)
4.2. Proof of Theorem 2.3.13 (iv). For $\lambda, \mu \in \{\alpha, -\alpha\}$, the Beilinson-Flach classes $\text{BF}_{r,\chi}^{\lambda,\mu} \in H^1(Q(r), W(I \otimes H_{E,+1}(\Gamma)))$ from [241] give rise to the classes in $\text{BF}^{\lambda,\mu(j)}_{\psi} \in H^1_{Iw}(Q(p), \mathcal{W}) \otimes \mathcal{H}))$ of $\lambda, \mu$,

$$f \ast_{\lambda, \mu} \in H^1_{Iw}(Q(p), \mathcal{W})$$

To ease notation, we shall write below $H^1_{Iw}(Q(p), \mathcal{W})$ of the following maps

$$H^1_{Iw}(Q(p), \mathcal{W}) \ast \mathcal{H} \to H^1(Q(p), \mathcal{W}) \to H^1(Q(p), \mathcal{W})$$

$$\ast_{\lambda, \mu}$$

Remark 4.2.1. The classes $c^1_{\lambda, \mu} \in H^1(Q, W_{\psi})$ that we have introduced as part of Theorem 2.3.13 maps to the class $\text{bf}^{\lambda,\mu}$ under the canonical isomorphism

$$H^1(Q, W_{\psi}) \to H^1(Q(p), \mathcal{W})$$

Theorem 4.2.2. The classes $\text{res}_{/f}(\text{bf}^{\lambda,\mu})$ and $\text{res}_{/f}(\text{bf}^{\lambda,-\mu})$ are non-trivial in $H^1_{Iw}(Q(p), \mathcal{W})$.

Proof. By Lemmas 4.1.1 and 4.1.4, the theorem will follow once we verify the following two properties.

(i) $\partial_{/\lambda, o} \circ \omega_{/f} \circ \text{exp}_{\psi} \circ \text{res}_p(\text{bf}^{\lambda,\mu}) \in \ker(\partial_{/\lambda, o})$ for $\mu \in \{\alpha, -\alpha\}$.

(ii) $\text{res}_p(\text{bf}^{\lambda,\mu})$ and $\text{res}_p(\text{bf}^{\lambda,-\mu})$ are non-trivial.

The property (i) is immediate by the commutativity of the following diagram

$$
\begin{array}{ccc}
H^1_{Iw}(Q(p), \mathcal{W}) & \to & H^1_{Iw}(Q(p), \mathcal{W}) \\
\downarrow & & \downarrow \\
H^1(Q(p), \mathcal{W}) & \to & H^1(Q(p), \mathcal{W})
\end{array}
$$

and the fact that $\text{BF}^{\lambda,\mu(j)}_{\psi}$ belongs to the kernel of the top horizontal arrow by [LZ16, Theorem 7.1.2]. We now prove (ii), by arguing as in the proof of Theorem 8.2.1(v) in [LZ16].

To ease notation, we shall write below $H^1_{Iw}(X)$ (resp., $H^1(X)$) instead of writing $H^1_{Iw}(Q(p), \mathcal{W})$ (resp., $H^1(Q(p), \mathcal{W})$) in the following commutative diagram:

$$
\begin{array}{ccc}
H^1_{Iw}(\mathcal{W}) & \to & H^1_{Iw}(\mathcal{W}) \\
\downarrow & & \downarrow \\
H^1(\mathcal{W}) & \to & H^1(\mathcal{W})
\end{array}
$$

We remind the reader that for the étale $(\varphi, \Gamma)$-module $D_{\mathcal{W}}^i$, we have identified its cohomology with the cohomology of $\mathcal{W}$ in order to define the horizontal arrows on the left.

It follows from Theorem 7.1.2 in op.cit. that the image

$$\partial_{/\lambda, o} \circ \text{res}_p(\text{bf}^{\lambda,\mu(j)}_{\psi}) \in H^1_{Iw}(Q(p), D_{\mathcal{W}}^i)$$

of the Iwasawa theoretic Beilinson-Flach class in fact falls in the image of

$$H^1_{Iw}(Q(p), D_{\mathcal{W}}^i) \to H^1_{Iw}(Q(p), D_{\mathcal{W}}^i)$$
We let \( d_{1w} \in H^1_{\text{crys}}(\mathbb{Q}_p(\mu_p), D^1_{\text{rig}}(\mathcal{W}_\psi)/\lambda, \mu)^{\omega^{-j}} \) denote the unique element that maps to \( \partial_{/\lambda, \alpha} \circ \text{res}_p(\mathbf{B}^{\lambda, \mu}_\psi(j)) \). The commutative diagram above shows that

\[
\partial_{/\lambda, \alpha} \circ \text{res}_p \left( \mathbf{b}^{\lambda, \mu}_n \right) = a(d_{1w})
\]

It therefore suffices to prove that \( a(d_{1w}) \) is non-trivial. This, however, is equivalent by Theorem 7.1.5 of loc. cit. (the explicit reciprocity law) to the verification that \( L_p(f, f, \omega^{-j}, j) \neq 0 \). Notice that in place of the variable \( j \) in op. cit., we have used \( j - 1 \) and furthermore, we have projected to the \( \omega^{-j} \)-eigenspaces. As a result, the relevant \( p \)-adic \( L \)-value in the current work is \( L(f, f, \omega^{-j}, j) \) in place of \( L_p(f, f, 1 + j) \) in op. cit.

The desired non-vanishing follows from the factorization of the \( p \)-adic \( L \)-function associated to symmetric Rankin-Selberg convolutions (Theorem 4.3.2), the interpolation formulae for the geometric \( p \)-adic \( L \)-function and the non-vanishing of the values \( L(\text{Sym}^2 f \otimes \omega^j \psi, j) \) (Theorem 4.3.1) whenever \( j \in [k + 2, 2k + 2] \).

\[\square\]

**Remark 4.2.3.** Attentive reader will realize that we could have in fact worked over the fields \( \mathbb{Q}(\mu_{p^n}) \) in place of \( \mathbb{Q}(\mu_p) \) above and considered the \( \omega^{-j} \theta \) invariants (for characters of \( \Gamma_n := \Gamma/\Gamma^{p^n} \), where \( n \) is an arbitrary non-negative integer). The same proof would apply and prove for \( \mu \in \{\alpha, -\alpha\} \)

\[
\partial_{/\lambda, \alpha} \circ \omega^{-j} \theta \circ \exp^* \circ \text{res}_p \left( \mathbf{b}^{\lambda, \mu}_n \right) \in \left( \mathcal{D}_{\text{crys}}(\mathcal{W}_\psi)/\lambda, \alpha \right)^{\varphi = -\frac{p - 1 - \psi^{-1}(p)}{p - 1}},
\]

where \( \mathbf{b}^{\lambda, \mu}_n \in H^1_{\text{crys}}(\mathcal{Q}(\mu_{p^n}), \mathcal{W}_\psi, \omega^{-j} \theta) \) is the image of \( B^{\lambda, \mu}_\psi(j) \) and the morphism \( \omega^{-j} \theta \circ \exp^* \) is given as in Definition 4.1.3. This allows us to conclude that the classes \( \text{res}_f(\mathbf{B}^{\lambda, \mu}_n) \) and \( \text{res}_f(\mathbf{b}^{\lambda, \mu}_n) \) are linearly independent.

**Corollary 4.2.4.** The classes \( e_{1_1}^{\lambda, \alpha}, e_{1_1}^{\alpha, -\alpha} \in H^1_{\text{crys}}(\mathbb{Q}, W_{\psi, j}) \) are linearly independent.

### 4.3. Analytic Selmer groups.

Recall from [4.3.1] the decompositions

\[
D_0 \oplus D_1 \oplus D_2 = \mathcal{D}_{\text{crys}}(\mathcal{W}_\psi) \supset \mathcal{D}_{\text{crys}}(\mathcal{Y}_\psi) = D_1 \oplus D_2
\]

definition of filtered \( \varphi \)-modules. We fix throughout this section a choice of \( \lambda, \mu \in \{\alpha, -\alpha\} \). We define the \( (\varphi, \Gamma) \)-modules \( \mathbb{D}^+ \subset \mathbb{D}^+ \) by setting

\[
\mathbb{D}_\psi := D^1_{\text{rig}}(\mathcal{W}_\psi)^{\lambda, \mu, +} \cap D^1_{\text{rig}}(\mathcal{Y}_\psi),
\]

\[
\mathbb{D}^+ := \mathbb{D} \otimes \mathbb{D} \otimes \mathbb{D}_{\psi, j}.
\]

**Lemma 4.3.1.** The crystalline \((\varphi, \Gamma)\)-submodule \( \mathbb{D}^+ \subset D^1_{\text{rig}}(\mathcal{Y}_\psi) \) is a saturated \((\varphi, \Gamma)\)-submodule of rank 2. Likewise, the submodule \( \mathbb{D}^+ \subset \mathbb{D}^+ \) is saturated of rank one.

**Proof.** Notice that

\[
\mathbb{D}^+ := \ker \left( D^1_{\text{rig}}(\mathcal{Y}_\psi) \rightarrow D^1_{\text{rig}}(\mathcal{W}_\psi) \rightarrow D^1_{\text{rig}}(\mathcal{W}_\psi)^{\lambda, \mu, -} \right)
\]

The first claim follows since

\[
e_\lambda \otimes e_{-\lambda} - e_\mu \otimes e_{-\mu} \not\in \ker (\mathbb{D}_f \otimes \mathbb{D}_f \rightarrow \mathbb{D}_f/\mathbb{D}_f \otimes \mathbb{D}_f/\mathbb{D}_f)
\]

in case \( \mu = -\lambda \), and

\[
e_\lambda \otimes e_{\lambda} - e_{-\lambda} \otimes e_{-\lambda} \not\in \ker (\mathbb{D}_f \otimes \mathbb{D}_f \rightarrow \mathbb{D}_f/\mathbb{D}_f \otimes \mathbb{D}_f/\mathbb{D}_f)
\]

in case \( \lambda = \mu \), so that \( \mathbb{D}^+ \subset D^1_{\text{rig}}(\mathcal{Y}_\psi) \). The second part follows from the exactness of the following sequence of \((\varphi, \Gamma)\)-modules:

\[
0 \rightarrow \mathbb{D}^+ \rightarrow \mathbb{D}^+ \rightarrow (\mathbb{D}_f/\mathbb{D}_f \otimes \mathbb{D}_f/\mathbb{D}_f) \otimes \mathbb{D}_{\psi, j} \rightarrow 0 .
\]

\[\square\]

**Lemma 4.3.2.** \( \mathbb{D}^+ \cap D^1_{\text{rig}}(D_1) = \begin{cases} D^1_{\text{rig}}(D_1) & \text{if } \mu = -\lambda \\ 0 & \text{if } \mu = \lambda \end{cases} \)
Proposition 4.3.7. \( E \)

Proof. This is an immediate consequence of Liu’s local Euler characteristic formula proved in [Liu08].

Definition 4.3.6. For any Dirichlet character \( \lambda \)

Lemma 4.3.5. \( R \)

\( R \) (4.4)

Corollary 4.3.3. \( D \)

Proof. We have the identification

\[
D_{rig}^1(D_1) = \mathcal{R}_E(e_\alpha \otimes e_\alpha - e_{-\alpha} \otimes e_{-\alpha}) \otimes \mathcal{D}_{\psi,j}
\]

for \( \alpha \) (\( \varphi, \Gamma \))-modules. When \( \lambda = \mu \), the conclusion follows on noting once again that

\[
e_\alpha \otimes e_\alpha - e_{-\alpha} \otimes e_{-\alpha} \not\in \ker(D_f \otimes D_f \longrightarrow D_f/D_\lambda \otimes D_f/D_\mu)
\]

for either of the choices for \( \lambda = \mu \). When \( \lambda = \mu^* \),

\[
e_\alpha \otimes e_\alpha - e_{-\alpha} \otimes e_{-\alpha} \in \ker(D_f \otimes D_f \longrightarrow D_f/D_\lambda \otimes D_f/D_\mu)
\]

and our claim is valid also in this case. \( \square \)

We may now verify that the submodule \( D_{\psi}^{+o} \subset D_{rig}^1(\mathcal{Y}_\psi) \) is regular (in the sense of Perrin-Riou and Pottharst).

Corollary 4.3.3. \( D_{cris}(\mathcal{D}_{\psi}^{+o}) \cap \text{Fil}^0(\mathcal{D}_{cris}(\mathcal{Y}_\psi)) = 0. \)

Proof. Since \( D_{cris}(\mathcal{D}_{\psi}^{+o}) \) is one-dimensional, it follows that the intersection of the vector space \( D_{cris}(\mathcal{D}_{\psi}^{+o}) \) and \( \text{Fil}^0 D_{cris}(\mathcal{Y}_\psi) \) is either trivial or the whole of \( D_{cris}(\mathcal{D}_{\psi}^{+o}) \). In case of the latter, it would follow that \( D_{cris}(\mathcal{D}_{\psi}^{+o}) \subset \text{Fil}^0 D_{cris}(\mathcal{Y}_\psi) \) is a \( \varphi \)-stable subspace. On the other hand, the unique \( \varphi \)-stable subspace of

\[
\text{Fil}^0 D_{cris}(\mathcal{Y}_\psi) = D_1 \oplus \text{Fil}^0 D_{cris}(D_2)
\]

is \( D_1 \) and we therefore are forced to have the identification \( D_{cris}(\mathcal{D}_{\psi}^{+o}) = D_1 \). When \( \lambda = \mu \), this would contradict Lemma 4.3.2 and prove that \( D_{cris}(\mathcal{D}_{\psi}^{+o}) \) and \( \text{Fil}^0 D_{cris}(\mathcal{Y}_\psi) \) could not have a non-trivial intersection in that situation. When \( \mu = -\lambda \), the conclusion still follows as \( D_{rig}^1(D_1) \) does not fit in the exact sequence 4.3.

Definition 4.3.4. For \( \mathcal{D} = \mathcal{D}_{\psi}^{+o} \) or \( \mathcal{D}_{\psi}^{+} \), we let \( S^\bullet(\mathcal{Y}_\psi, \mathcal{D}) \) denote the (analytic) Selmer complex, given as in [BB15, Definition 2.4] (with base field \( \mathbb{Q}(\mu_p) \) in place of \( \mathbb{Q} \)) and let \( \text{R}\Gamma(\mathcal{Y}_\psi, \mathcal{D}) \) denote the class that the complex \( S^\bullet(\mathcal{Y}_\psi, \mathcal{D}) \) represents in the derived category of \( E \)-vector spaces.

We also define the Iwasawa theoretic (analytic) Selmer complex \( S^\bullet_{\text{Iw}}(\mathcal{Y}_\psi, \mathcal{D}) \) and the corresponding class \( \text{R}\Gamma_{\text{Iw}}(\mathcal{Y}_\psi, \mathcal{D}) \) in the derived category of \( H \)-modules as at the end of Section 3 in op. cit. (these are still defined over the number field \( \mathbb{Q}(\mu_p) \) in the current situation).

We set

\[
\tilde{H}^1(\mathbb{Q}(\mu_p), \mathcal{Y}_\psi, \mathcal{D}) := \text{R}\Gamma^1(\mathcal{Y}_\psi, \mathcal{D}) \text{ for } ? = 0, \text{Iw}
\]

and call them analytic Selmer groups.

We have the following control theorem in the context of analytic Selmer complexes:

\[
\text{(4.4)} \quad \text{R}\Gamma_{\text{Iw}}(\mathcal{Y}_\psi, \mathcal{D}) \otimes^\mathbb{L}_{\mathbb{Q}} E \sim \text{R}\Gamma(\mathcal{Y}_\psi, \mathcal{D}).
\]

Lemma 4.3.5. Let \( D \) be a \( (\varphi, \Gamma) \)-module over \( \mathcal{R}_E \) such that \( H^0(\mathbb{Q}_p, D) = H^2(\mathbb{Q}_p, D) = 0. \) Then \( H^1(\mathbb{Q}_p, D) \) is an \( E \)-vector space of dimension \( \text{rank } D \).

Proof. This is an immediate consequence of Liu’s local Euler characteristic formula proved in [Liu08]. \( \square \)

Definition 4.3.6. For any Dirichlet character \( \eta : G_{\mathbb{Q}} \rightarrow E^\times \), we let \( E_\eta \) to denote the one dimensional \( E \)-vector space on which \( G_{\mathbb{Q}} \) acts via \( \eta \).

Proposition 4.3.7. Fix an arbitrary non-negative integer \( n \) and let \( \theta \) be a character of the quotient group \( \Gamma_n \). Suppose at least one of the following three conditions holds:

(i) \( \psi(p) \neq \pm 1. \)
(ii) \( \omega^{-j}\theta \) is non-trivial.
(iii) \( j \neq k + 2. \)
The analytic Selmer groups fit in the following exact sequence:

\[
0 \to \tilde{H}^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi, \mathcal{D})^{\omega^{-j}\theta} \to H^1_{\text{can}}(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi)^{\omega^{-j}\theta} \to \left( H^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi) / H^1(\mathbb{Q}(\mu_p^n), \mathcal{D}) \right)^{\omega^{-j}\theta}.
\]

The Selmer group \( \tilde{H}^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi, \mathcal{D})^{\omega^{-j}\theta} \) is canonically isomorphic to the Bloch-Kato Selmer group \( H^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi)^{\omega^{-j}\theta} \), whereas \( H^2(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi, \mathcal{D}^{\omega^{-j}\theta}) \) is isomorphic to \( H^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi(1))^{\omega^{-j}\theta} \).

The Selmer group \( \tilde{H}^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi, \mathcal{D}^{\omega^{-j}\theta}) \) is one dimensional, whereas the Selmer group \( H^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi)^{\omega^{-j}\theta} \) is trivial.

**Proof.** The first one follows by the definition of the Selmer complex as a mapping cone, once we verify that \( H^0(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi)^{\omega^{-j}\theta} = 0 \). Noticing that

\[
p^{-1}(\psi(p)) = 1 \quad \text{or} \quad -p^{-1}(\psi(p)) = 1
\]

if and only if \( j = k + 2 \) and \( \psi(p) = \pm 1 \) (recall that \( \alpha^2 = p^{k+1} \)), the desired vanishing follows under our running hypotheses. The first portion of the second part is immediate by [Pot13, Proposition 3.7(3)] and its second portion by global duality (Theorem 1.15 of op.cit.). Note that the conditions of this proposition are valid thanks to Lemma 4.3.2 and our running hypotheses.

We follow the proof of [LZ16, Theorem 8.2.1] in order verify the final portion. We start off with the following exact sequence:

\[
0 \to H^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi)^{\omega^{-j}\theta} \to \left( H^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi) / \mathbb{Z} \right)^{\omega^{-j}\theta} \to H^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi)^{\omega^{-j}\theta}.
\]

We have verified that \( \left( H^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi) / \mathbb{Z} \right)^{\omega^{-j}\theta} \) is of dimension 2 and moreover, the image of \( \text{res}_{\mathbb{Q}E} \) is also two dimensional thanks to Theorem 4.2.2 and Remark 4.2.3. This proves that \( H^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi)^{\omega^{-j}\theta} = 0 \), as desired.

The exact sequence of (i) (used with \( \mathcal{D} = \mathbb{D}^+ \)) and Lemma 4.3.2 (used with \( D = \left( \mathbb{D}^+_{\text{rig}}(\mathcal{V}_\psi) / \mathbb{D}^+_{\psi} \right) \otimes \mathbb{D}^+_{\text{rig}}(E_{\omega^{-j-1}}) \) and \( K = \mathbb{Q}_p \)) shows that \( \tilde{H}^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi, \mathcal{D}^{\omega^{-j}\theta}) \) is at least one dimensional. Furthermore, the exact sequence

\[
0 \to \tilde{H}^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi, \mathcal{D}^{\omega^{-j}\theta}) \to \tilde{H}^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi, \mathcal{D}^{\omega^{-j}\theta}) \to H^1(\mathbb{Q}(\mu_p^n), \mathcal{D}^{\omega^{-j}\theta})
\]

put together with (ii), the vanishing of the Bloch-Kato Selmer group and Lemma 4.3.2 (used with \( D = \left( \mathbb{D}^+_{\psi} / \mathbb{D}^+_{\psi} \right) \otimes \mathbb{D}^+_{\text{rig}}(E_{\omega^{-j-1}}) \)) implies that \( \tilde{H}^1(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi, \mathcal{D}^{\omega^{-j}\theta}) \) is at most one dimensional.

**Corollary 4.3.8.** In the setting of Proposition 4.3.7, we have

\[
\tilde{H}^2(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi, \mathcal{D}^{\omega^{-j}\theta}) = 0.
\]

**Proof.** The definition of the Selmer complex as a mapping cone gives rise to a canonical surjection

\[
\tilde{H}^2(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi, \mathcal{D}^{\omega^{-j}\theta}) \to \tilde{H}^2(\mathbb{Q}(\mu_p^n), \mathcal{V}_\psi, \mathcal{D}^{\omega^{-j}\theta})
\]

and the asserted vanishing is immediate after Proposition 4.3.7(ii)-(iii). □

#### 4.4. Zeros of characteristic ideals

Throughout the remaining portion of this section, we shall assume that \( \psi(p) \neq \pm 1 \). This means precisely that we are avoiding the exceptional zeros phenomenon for symmetric squares. This is only to ensure that we may keep the length of this article within reasonable limits and we shall revisit this case in a complementary note.

Throughout Section 4.4, we continue to work with our fixed choice of a pair \( \lambda, \mu \in \{ \alpha, -\alpha \} \). We shall write \( \mathcal{H} \) for \( \lim_{\rightarrow m} \mathcal{H}_{E,m}(\Gamma) \).  

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Lemma 4.4.1. Let $D$ be a $(\varphi, \Gamma)$-module of rank $d$ over $\mathcal{R}_E$ such that $H^0(\mathbb{Q}_p(\mu_{p^\infty}), D) = H^2(\mathbb{Q}_p(\mu_{p^\infty}), D) = 0$. Then $H^1_{Iw}(\mathbb{Q}_p, D)$ is a projective $\mathcal{H}$-module whose rank equals $d$.

Proof. Only in this proof, we let $\psi$ denote the left inverse for the Frobenius operator $\varphi$, and not a Dirichlet character $\psi$ that is unramified at $p$. The proof of the lemma follows the fact that the complex $C_\psi^\bullet(D)$ is a perfect complex of $\mathcal{H}$-modules, which may be represented by a single projective module concentrated in degree 1 thanks to our running hypotheses. \hfill $\Box$

Proposition 4.4.2. The $\mathcal{H}$-module $\tilde{H}^1_{Iw}(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta}$ is a saturated rank one submodule of $H^1_{Iw}(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi) \otimes \mathcal{H} \cong H^1_{Iw}(\mathbb{Q}, V_{\psi,j} \otimes \theta^{-1}) \otimes \mathcal{H}$.

Proof. The definition of the Selmer complex as a mapping cone (and the fact that Iwasawa cohomology classes are unramified) yields the exact sequence

$$0 \to \tilde{H}^1_{Iw}(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta} \to H^1_{Iw}(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi) \otimes \mathcal{H} \to H^1_{Iw}(\mathbb{Q}(\mu_{p^n}), D_{\text{rig}}^1(\mathcal{V}_\psi)/\mathbb{D}^{+}_\psi)^{\omega^{-j}\theta} \to \tilde{H}^2_{Iw}(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta}.$$

The proposition will follow once we verify that

(i) The $\mathcal{H}$-module $H^1_{Iw}(\mathbb{Q}(\mu_{p^n}), D_{\text{rig}}^1(\mathcal{V}_\psi)/\mathbb{D}^{+}_\psi)^{\omega^{-j}\theta} \cong H^1_{Iw}(\mathbb{Q}, (D_{\text{rig}}^1(\mathcal{V}_\psi)/\mathbb{D}^{+}_\psi) \otimes D_{\text{rig}}^1(E_{\psi,j}^{-1})).$

is projective of rank one;

(ii) $\tilde{H}^2_{Iw}(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta}$ is torsion.

(i) follows from Lemma 4.4.1 thanks to our running hypotheses on $\psi$, so it remains to verify (ii). To do so, we first consider the exact sequence

$$0 \to \tilde{H}^1(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta} \to \tilde{H}^1(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta} \to H^1(\mathbb{Q}(\mu_{p^n}), \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta} \to \tilde{H}^2(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta} \to 0$$

of $E$-vector spaces. By Proposition 4.3.1(ii)-(iii) and global duality, it follows that

$$\tilde{H}^2(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta} \cong H^1(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi(1))^{\omega^{-j}\theta} = 0$$

and we conclude by the long exact sequence above that

$$\tilde{H}^2(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta} = 0.$$

On the other hand, control theorem for Selmer complexes [4.4] yields an injection

$$\left(\tilde{H}^2_{Iw}(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta}\right)_\Gamma \to \tilde{H}^2(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta},$$

which shows (thanks to the structure theory of admissible $\mathcal{H}$-modules, proved in [Pot12 Proposition 1.1]) that $\tilde{H}^2_{Iw}(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}\theta}$ could not have positive $\mathcal{H}$-rank. \hfill $\Box$

Recall from [Pot12 §1] the definition of the characteristic ideal $\text{char}_{\mathcal{H}}(M)$ of an admissible $\mathcal{H}$-module $M$.

Recall that $\chi_{\text{cyc}}$ denotes the cyclotomic character on $\Gamma$, $\Phi_{p^n}$ denotes the $p^n$-cyclotomic polynomial and $\text{Tw}_n$ is a twisting map defined in [3.1]. We define $\text{Tw}_{(m)}$ to be the twisting map that acts as Tw$_n$ on elements in $\Gamma_1$ and the identity on $\Gamma_{\text{tors}}$.

Theorem 4.4.3. For any $m \in [k + 1, 2k + 1]$ and any positive integer $n$, the element $\text{Tw}_{j-m-1}\Phi_{p^n}(\gamma)$ does not divide in $\mathcal{H}$ the ideal $\text{char}_{\mathcal{H}}\left(\tilde{H}^2_{Iw}(\mathbb{Q}(\mu_{p^n}), \mathcal{V}_\psi, \mathbb{D}^{+}_\psi)^{\omega^{-j}}\right)$.
Proof. In order to ease our notation, we shall adopt the following convention in this proof.

**Convention.** For any twist \( \mathcal{V} \) of \( \mathcal{V}_\psi \) by a power of the cyclotomic character that appears in the proof below, we denote the \((\varphi, \Gamma)\)-submodule of \( D_{\text{rig}}^!(\mathcal{V}) \) corresponding to \( D_{\psi}^+ \) also by \( D_{\psi}^+ \).

It follows by the twisting formalism (c.f., [Rub00, §6]) that
\[
\text{char}_H \left( \tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_\psi, D_{\psi}^+)^{\omega^{-1}} \right) = \text{Tw}_{(j-1)} \text{char}_H \left( \tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_\psi(j-1), D_{\psi}^+)^{\omega^{-1}} \right),
\]
hence our assertion is equivalent to the claim that \( \text{char}_H \tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_\psi(j-1), D_{\psi}^+)^{\omega^{-1}} \) is not divisible by any linear factor of the product
\[
\Phi_{p^n}(\chi_{cyc}^{-m}(\gamma)\gamma) := \prod_{\eta} (\chi_{cyc}^{-m}(\gamma)\eta^{-1}(\gamma)\gamma - 1)
\]
for any \( r \in [k+1, 2k+2] \) and any positive integer \( n \), where \( \eta \) runs through primitive character of \( \Gamma_{n-1} \).

Assume the contrary, so that there exists a positive integer \( n \) and an \( E \)-valued primitive character \( \theta \) of \( \Gamma_{n-1} \) (we enlarge \( E \) if necessary) such that
\[
(\chi_{cyc}^{-m}(\gamma)\theta^{-1}(\gamma)\gamma - 1) \mid \text{char}_H \left( \tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_\psi(j-1), D_{\psi}^+)^{\omega^{-1}} \right)
\]
for some \( m \in [k+1, 2k+1] \).

Set \( \mathcal{V}_m := \mathcal{V}_\psi(j - m - 1) \cong \text{Sym}^2 W^*_+(\mathbb{Q}) \otimes \psi \). Since
\[
\text{Tw}_{(m)} (\chi_{cyc}^{-m}(\gamma)\theta^{-1}(\gamma)\gamma - 1) = \theta^{-1}(\gamma)\gamma - 1,
\]
observe that the divisibility above is equivalent to
\[
(\theta^{-1}(\gamma)\gamma - 1) \mid \text{char}_H \left( \tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_\psi(j-1), D_{\psi}^+)^{\omega^{-1}} \right)
\]
\[
= \text{char}_H \left( \tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_\psi(j-1), D_{\psi}^+)^{\omega^{-1}} \otimes (\chi_{cyc})^{-m} \right)
\]
By the structure theory of admissible \( H \)-modules, this is equivalent to asking that the quotient
\[
\tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m} / (\theta^{-1}(\gamma)\gamma - 1)
\]
be a non-trivial \( E \)-vector space. Since the Iwasawa theoretic Selmer complex has no cohomology in degree 3, we have the following canonical isomorphism thanks to the control theorem for Selmer complexes:
\[
(4.5) \quad \tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m} / (\gamma^{p^n-1} - 1) \cong \tilde{H}^2(\mathbb{Q}(\mu_p^n), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m}
\]

Furthermore, as the element \( \gamma^{p^n-1} - 1 \) belongs to the ideal of \( \mathcal{H} \) generated by \( \theta^{-1}(\gamma)\gamma - 1 \), the surjective map
\[
\tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m} \twoheadrightarrow \tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m} / (\theta^{-1}(\gamma)\gamma - 1)
\]
factors as
\[
\begin{array}{ccc}
\tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m} & \twoheadrightarrow & \tilde{H}^2_{Iw}(\mathbb{Q}(\mu_p), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m} / (\theta^{-1}(\gamma)\gamma - 1) \\

\downarrow \quad \theta^{-1}(\gamma)\gamma - 1 & & \downarrow \quad (\theta^{-1}(\gamma)\gamma - 1)
\end{array}
\]

This shows that the \( E \)-vector space \( \tilde{H}^2(\mathbb{Q}(\mu_p^n), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m} / (\theta^{-1}(\gamma)\gamma - 1) \) is non-trivial. On the other hand, the exactness of the sequence
\[
0 \longrightarrow \tilde{H}^2(\mathbb{Q}(\mu_p^n), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m} \longrightarrow \tilde{H}^2(\mathbb{Q}(\mu_p^n), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m}
\]
\[
\theta^{-1}(\gamma)\gamma - 1 \tilde{H}^2(\mathbb{Q}(\mu_p^n), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m} \longrightarrow \tilde{H}^2(\mathbb{Q}(\mu_p^n), \mathcal{V}_m, D_{\psi}^+)^{\omega^{-1} - m} / (\theta^{-1}(\gamma)\gamma - 1)
\]

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shows that $\tilde{H}^2(\mathbb{Q}(\mu_{p^n}), \mathcal{V}'_m, \mathbb{D}^+_{\psi})^{\omega^{-1-m}\theta}$ is non-trivial as well. This contradicts Corollary 4.3.8 (used with the choice $j = m + 1$).

Recall that we have set $V_\psi := \text{Sym}^2 W_f^*(1) \otimes \psi$ to ease our notation.

**Corollary 4.4.4.** The characteristic ideal of the $\mathcal{H}$-module $\tilde{H}^2_{\text{iw}}(\mathbb{Q}, V_\psi, \mathbb{D}^+_{\psi})$ is prime to $\log_{2k+3} / \log_{k+2}$.

**Proof.** Note that we have

$$\text{char}_\mathcal{H} \tilde{H}^2_{\text{iw}}(\mathbb{Q}, V_\psi) = \text{Tw}_{(-j)} \text{char}_\mathcal{H} \tilde{H}^2_{\text{iw}}(\mathbb{Q}, V_\psi) \otimes (\chi_{\text{cyc}})^{-j}$$

$$= \text{Tw}_{(-j)} \text{char}_\mathcal{H} \tilde{H}^2_{\text{iw}}(\mathbb{Q}, \mathcal{V}'_\psi \otimes \omega^j)$$

$$= \text{Tw}_{(-j)} \text{char}_\mathcal{H} \tilde{H}^2_{\text{iw}}(\mathbb{Q}(\mu_p), \mathcal{V}'_\psi)^{\omega^{-j}}$$

where the final isomorphism is deduced from the Shapiro’s Lemma proved in [KPX14, Lemma 2.3.5] in the context of $(\phi, \Gamma)$-modules and their cohomology. The assertion follows from Theorem 4.4.3.

Recall the positive integer $r \in \mathcal{R}_\chi$ and $\eta \in \hat{\Delta}_\tau$ we have fixed at the start of this section (so that $\psi = \chi \eta$).

**Definition 4.4.5.** We denote by $h^\eta_{\lambda, \mu} \in \text{char}_\mathcal{H} \left( \tilde{H}^2_{\text{iw}}(\mathbb{Q}, V_\psi, \mathbb{D}^+_{\psi}) \right)$ any fixed generator. We also set

$$h^{\lambda, \mu} := \sum_{\eta \in \hat{\Delta}_\tau} e_\eta h^\eta_{\lambda, \mu} \in \mathcal{H}[\Delta_r],$$

where $e_\eta$ is the idempotent associated to $\eta$.

### 4.5. Proof of Theorem 3.3.3

We remark that all Hodge-Tate weights of $V = \text{Sym}^2 W_f^*(1) \otimes \chi$ are positive, which is crucial for our purposes.

**Definition 4.5.1.** Let $r \in \mathcal{R}_\chi$ and let $\eta \in \hat{\Delta}_\tau$ be a character. We denote by $\{F_r, i\}$; the completions of $\mathbb{Q}(r)$ at primes above $p$. We let

$$\mathcal{L}_{\lambda, \mu, r} : H^1_{\text{iw}}(\mathbb{Q}(r)_p, V) \to \mathbb{Q}(r) \otimes \mathcal{H} \otimes \mathbb{D}_{\text{cris}}(\text{Sym}^2 W_f^* \otimes \mathbb{D}_{\text{cris}}(L(1) \otimes \chi))$$

denote the Perrin-Riou map whose restriction to $H^1_{\text{iw}}(F_r, i, V_\psi)$ is the corresponding twist of the morphism $\mathcal{L}_{\lambda, \mu, F_r}$ we have introduced in Definition 2.1.7. For each $\eta \in \hat{\Delta}_\tau$ and for $\psi = \chi \eta$ we write

$$\mathcal{L}_{\lambda, \mu, \eta} : H^1(\mathbb{Q}_p, V_\psi) \to \mathcal{H} \otimes \mathbb{D}_{\text{cris}}(\text{Sym}^2 W_f^* \otimes \mathbb{D}_{\text{cris}}(L(1) \otimes \psi))$$

for the $\eta^{-1}$-component of $\mathcal{L}_{\lambda, \mu, r}$.

Recall that for our fixed Dirichlet character $\chi$, we have set $W := W_f^* \otimes W_f^*(1) \otimes \chi$ and $V := \text{Sym}^2 W_f^*(1) \otimes \chi$. As in 4.3.3 we have the Beilinson-Flach element

$$BF^\lambda_{F_r, \chi} \in H^1_{\text{iw}}(\mathbb{Q}(r), W) \otimes \mathcal{H} \cong H^1_{\text{iw}}(\mathbb{Q}(r), V) \otimes \mathcal{H}$$

for each positive integer $r \in \mathcal{R}_\chi$.

**Remark 4.5.2.** The equality ($\ast$) in (4.6) follows from Corollary 2.3.6 applied with $\psi = \eta \chi$ (where $\eta$ runs through characters of $\Delta_r$) and twisted by the character $\omega^{-1} \chi_{\text{cyc}}$ of $\Gamma$. We also have

$$\mathcal{L}_{\lambda, \mu, r} \big|_{H^1_{\text{iw}}(\mathbb{Q}(r)_p, V)} = \mathcal{L}_{\mu, \lambda} \big|_{H^1_{\text{iw}}(\mathbb{Q}(r)_p, V)}$$

for the restriction of the Perrin-Riou maps to the semi-local cohomology for the symmetric square. This combined with ($\ast$) in turn implies that

$$\mathcal{L}_{\mu, \lambda} \circ \text{resp} = \mathcal{L}_{\mu, \lambda} \circ \text{resp}.$$
Based on these remarks, we may easily go back and forth between the cohomological invariants of V and W.

We recall from Definition 3.1.8 the projectors

\[ \Lambda^2 H^1_{Iw}(\mathbb{Q}(r), W) \xrightarrow{pr_{\lambda,\mu}} H^1(\mathbb{Q}, W) \]

\[ \Lambda^2 H^1_{Iw}(\mathbb{Q}(r), V) \xrightarrow{} H^1(\mathbb{Q}, V) \]

Definition 4.5.3. For \( h^\lambda_{\eta,\mu} \) as in Definition 4.4.9 we set \( h_\eta := \prod_{\lambda,\mu} h^\lambda_{\eta,\mu} \in \mathcal{H} \) and define \( h_r := \sum_{\eta \in \hat{\Delta}} e_\eta h_{\eta-1} \in \mathcal{H}[\Delta_r] \).

Note that \( h_\eta \) and \( \log_{2k+3}/\log_{5k+2} \) have no common factor thanks to Corollary 4.4.3.

Proposition 4.5.4. For any choice of \( \lambda, \mu \in \{\alpha, -\alpha\} \), we have

\[ h_r \mathcal{H}[\Delta_r] \subset \mathcal{L}_{\lambda,\mu,r} \circ \text{res}_p(H^1_{Iw}(\mathbb{Q}(r), V)) \]

for the image of \( H^1_{Iw}(\mathbb{Q}, V) \) under the Perrin-Riou map.

Proof. It suffices to prove this for each isotopic component. Namely, once we verify that

\[ h_\eta \mathcal{H} \subset \mathcal{L}_{\lambda,\mu,\eta} \circ \text{res}_p(H^1_{Iw}(\mathbb{Q}, V)) \]

for each character \( \eta \in \hat{\Delta}_r \) with \( \psi = \chi\eta \), the proof shall follow.

By the definition of the Selmer complex as a mapping cone, we have the following exact sequence:

\[ 0 \rightarrow H^1_{Iw}(\mathbb{Q}, V) \otimes \mathcal{H} \xrightarrow{\text{res}_p} H^1_{Iw}(\mathbb{Q}, V) \otimes {D^\dagger_{\text{rig}}}(V)/{D^\dagger_{\psi}} \rightarrow H^2_{Iw}(\mathbb{Q}, V) \otimes {D^\dagger_{\psi}} \]

where we recall our convention that for twists \( \psi' \) of \( \mathcal{Y}_\psi \) by a character of \( \Gamma \) (such as our representation \( V_\psi \) here), we denote the \( (\varphi, \Gamma) \)-submodule of \( D^\dagger_{\text{rig}}(\mathcal{Y}) \) corresponding to \( D^\dagger_{\psi} \) also by \( D^\dagger_{\psi} \). Recall also that the \( (\varphi, \Gamma) \)-submodule \( D^\dagger_{\psi} \) depends on our choice of the pair \( \lambda, \mu \). Observe further that the map \( \mathcal{L}_{\lambda,\mu,\eta} \) factors as

\[ H^1_{Iw}(\mathbb{Q}, V) \otimes \mathcal{H} \xrightarrow{\mathcal{L}_{\lambda,\mu,\eta}} H^1_{Iw}(\mathbb{Q}, V) \otimes {D^\dagger_{\text{rig}}}(V)/{D^\dagger_{\psi}} \]

by its very definition. As the Perrin-Riou map

\[ \mathcal{L}_{\lambda,\mu,\eta} : H^1_{Iw}(\mathbb{Q}, V) \otimes \mathcal{H} \rightarrow \mathcal{H} \]

is surjective, the proof follows from the exact sequence (4.7) and the choice of \( h_\eta \).

Theorem 4.5.5. \( h_r BF_{r,\chi}^\lambda_{\eta,\mu} \in \text{pr}_{\lambda,\mu}(H^1_{Iw}(\mathbb{Q}(r), V) \otimes \mathcal{H}) \).

Proof. We may once again prove this one character at a time. For each \( \eta \in \hat{\Delta}_r \) and \( \psi = \chi\eta \), we shall verify that

\[ h_\eta BF_{r,\chi}^\lambda_{\eta,\mu} \in \text{pr}_{\lambda,\mu}(H^1_{Iw}(\mathbb{Q}, V) \otimes \mathcal{H}) \, . \]

Here, \( BF_{\eta}^\lambda_{\mu} \in H^1_{Iw}(\mathbb{Q}, V) \otimes \mathcal{H} \) is the image of the class \( BF_{r,\chi}^\lambda_{\eta,\mu} \), on projection to the \( \eta^{-1} \)-isotypic component.

Recall that the \( \mathcal{H} \)-module \( H^1_{Iw}(\mathbb{Q}, V) \otimes \mathcal{H} \) is free of rank 2 thanks to Theorem 2.3.20. We fix a basis \( \{Y_1, Y_2\} \) of this module and observe that

\[ \text{pr}_{\lambda,\mu}(H^1_{Iw}(\mathbb{Q}(r), V) \otimes \mathcal{H}) = \text{span}_\mathcal{H}(\mathcal{L}_{\lambda,\mu,\eta}(Y_1) Y_2 - \mathcal{L}_{\lambda,\mu,\eta}(Y_2) Y_1) \, . \]

Using the fact that

\[ H^1_{Iw}(\mathbb{Q}, V) \otimes \mathcal{H}/\tilde{H}^1_{Iw}(\mathbb{Q}, V, D^\dagger_{\psi}) \rightarrow H^1_{Iw}(\mathbb{Q}, D^\dagger_{\text{rig}}(V)/D^\dagger_{\psi}) \]
for some $i \in \{1, 2\}$. The exact sequence (4.7) and factorization (4.8) yields the following containments:

$$\text{span}_H (\mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p(Y_1)Y_2 - \mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p(Y_2)Y_1) = \text{pr}_{\lambda, \mu} \left( H^1_{\text{Iw}}(\mathbb{Q}(r), V) \otimes \mathcal{H} \right)$$

$$\subseteq H^1_{\text{Iw}}(\mathbb{Q}, V, D^+_\psi)$$

(4.9)

$$\iff (H^1_{\text{Iw}}(\mathbb{Q}, V, \mathcal{H}) / \mathcal{H} \cdot Y_i).$$

**Case 1.** $Y_1 \in \tilde{H}^1_{\text{Iw}}(\mathbb{Q}, V, D^+_\psi)$. In this case, it follows from (4.10) applied with $i = 2$ that

$$\mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p(Y_2) = \text{pr}_{\lambda, \mu} \left( H^1_{\text{Iw}}(\mathbb{Q}, V, \mathcal{H}) \right)$$

for every class $c \in \tilde{H}^1_{\text{Iw}}(\mathbb{Q}, V, D^+_\psi)$; in particular, this holds true with the choice $c = \text{BF}_{\eta}^{\lambda, \mu}$. The proof in this case is complete on noticing that

$$\mathcal{H} \cdot \mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p(Y_1) = \mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p \left( H^1_{\text{Iw}}(\mathbb{Q}, V, \mathcal{H}) \right) \overset{\text{Prop. 4.5.4}}{\equiv} h_{\eta}$$

since $\mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p(Y_1) = 0$.

**Case 2.** $Y_2 \in \tilde{H}^1_{\text{Iw}}(\mathbb{Q}, V, D^+_\psi)$. The proof of Case 1 carries over verbatim.

**Case 3.** $Y_1, Y_2 \not\in \tilde{H}^1_{\text{Iw}}(\mathbb{Q}, V, D^+_\psi)$. In this case, it follows from (4.10) applied with both choices of $i \in \{1, 2\}$ yields

$$(r_1 \mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p(Y_1) + r_2 \mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p(Y_2)) c \in \text{pr}_{\lambda, \mu} \left( H^1_{\text{Iw}}(\mathbb{Q}(r), V) \otimes \mathcal{H} \right)$$

for every $r_1, r_2 \in \mathcal{H}$ and for every $c \in \tilde{H}^1_{\text{Iw}}(\mathbb{Q}, V, D^+_\psi)$. Since we have

$$\mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p \left( H^1_{\text{Iw}}(\mathbb{Q}, V, \mathcal{H}) \right) = \text{span}_H \left\{ \mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p(Y_1), \mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p(Y_2) \right\}$$

this amounts to saying that

$$\mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p \left( H^1_{\text{Iw}}(\mathbb{Q}, V, \mathcal{H}) \right) c \subset \text{pr}_{\lambda, \mu} \left( H^1_{\text{Iw}}(\mathbb{Q}(r), V) \otimes \mathcal{H} \right).$$

The proof follows on noting by Proposition 4.5.4 that

$$h_{\eta}c \in \mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p \left( H^1_{\text{Iw}}(\mathbb{Q}, V, \mathcal{H}) \right) c$$

and setting $c$ to be the class $\text{BF}_{\eta}^{\lambda, \mu}$.

**Definition 4.5.6.** For a fixed positive integer $r \in \mathcal{R}_\chi$ and each character $\eta \in \hat{\Delta}_r$, let $\{Y_1, Y_2\}$ be a basis of $H^1_{\text{Iw}}(\mathbb{Q}, V)$. Let $d_{\eta}^{\lambda, \mu} \in \mathcal{H}$ be the unique element with the property that

$$h_{\eta} \text{BF}_{\eta}^{\lambda, \mu} = d_{\eta}^{\lambda, \mu} \cdot (\mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p(Y_1)Y_2 - \mathcal{L}_{\lambda, \mu, \eta} \circ \text{res}_p(Y_2)Y_1).$$

Note that the existence of $d_{\eta}^{\lambda, \mu}$ is guaranteed by Theorem 4.5.4 and the description of $\text{pr}_{\lambda, \mu} \left( H^1_{\text{Iw}}(\mathbb{Q}(r), V) \otimes \mathcal{H} \right)$ in (4.9).

**Proposition 4.5.7.** The elements $d_{\eta}^{\lambda, \mu} \in \mathcal{H}$ are constant as $\lambda, \mu$ varies.

We define $d_{\eta} \in \mathcal{H}$ to denote this constant value $d_{\eta}^{\lambda, \mu}$ takes as $\lambda, \mu$ varies.

**Proof of Proposition 4.5.7** To ease notation, we fix $\eta$ and drop it from the notation we use for the Perrin-Riou maps. With a slight abuse, we shall also write $\mathcal{L}_{\lambda, \mu}$ in place of $\mathcal{L}_{\lambda, \mu} \circ \text{res}_p$ to ease our notation here.

Notice that we have

$$h_{\eta} \mathcal{L}_{\lambda, \mu} \left( \text{BF}_{\eta}^{\lambda, \mu} \right) = h_{\eta} \mathcal{L}_{\lambda, \mu} \left( \text{BF}_{\eta}^{\lambda, \mu}^* \right)$$

by the explicit reciprocity law for Beilinson-Flach elements. On the other hand,

$$h_{\eta} \mathcal{L}_{\lambda, \mu} \left( \text{BF}_{\eta}^{\lambda, \mu} \right) = d_{\eta}^{\lambda, \mu} \cdot (\mathcal{L}_{\lambda, \mu} (Y_1) \mathcal{L}_{\lambda, \mu}^* (Y_2) - \mathcal{L}_{\lambda, \mu} (Y_2) \mathcal{L}_{\lambda, \mu}^* (Y_1))$$

(4.10)
and
\[(4.12) \quad h_\eta \mathcal{L}_{\lambda,\mu} \left( BF_0^{\lambda,\mu} \right) = d_0^{\lambda,\mu} \cdot (\mathcal{L}_{\lambda,\mu} \cdot (\mathcal{Y}_1) \mathcal{L}_{\lambda,\mu} (\mathcal{Y}_2) - \mathcal{L}_{\lambda,\mu} (\mathcal{Y}_2) \mathcal{L}_{\lambda,\mu} (\mathcal{Y}_1)) \]

On comparing (4.11), (4.11) and (4.12), we conclude that \( d_0^{\lambda,\mu} = d_0^{\lambda,\mu} \). The proof follows using in addition the fact that \( d_0^{\lambda,\mu} = d_0^{\lambda,\mu} \), which we have thanks to Proposition 4.5.3 and Remark 4.5.2. \( \square \)

**Definition 4.5.8.** We set \( c_r := \frac{d_0}{h_\eta} \in \text{Frac}(\mathcal{H}) \) and
\[
c_r = \sum_{\eta \in \Delta_r} e_0 c_{\eta-1} \in \mathbb{Q}_p [\Delta_r] \otimes \text{Frac}(\mathcal{H}).
\]

Theorem 3.3.3 is clearly valid with this choice of \( c_r \).

4.6. **Analytic Main conjectures with \( p \)-adic \( L \)-functions.** We will begin with stating the analytic Iwasawa main conjecture formulated by Benois and Pottharst in our particular set up. Recall that \( V := \text{Sym}^2 W^p (1 + \chi) \) and \( \lambda, \mu \in \{ \pm \alpha \} \).

**Definition 4.6.1.** Let \( \delta_\chi : \mathbb{Q}_p^\times \to E^\times \) be the character defined by \( \delta_\chi(p) := p \chi^{-1}(p) \) and \( \delta_\chi(u) := u \) for \( u \in \mathbb{Z}_p^\times \). Let \( D_\chi \) denote the rank one \((\varphi, \Gamma)\)-module \( \mathcal{R}_E (\delta_\chi) \). We set
\[
\bullet \quad D_\chi^\lambda := \left( D_\chi \otimes D_f + D_f \otimes D_\mu \right) \otimes D_\chi \cap D_\chi^f (V),
\]
\[
\bullet \quad D_\chi := D_\chi \otimes D_\chi \otimes D_\chi.
\]

**Conjecture 4.6.2** (Analytic Iwasawa main conjecture). For \( j \in \{ k + 2, \ldots, 2k + 2 \} \) even, the \( \mathcal{H} \)-module \( e_\omega, H_\mathcal{H}^2 (\mathbb{Q}, V, D_\chi) \) is torsion and
\[
\text{char}_\mathcal{H} e_\omega, H_\mathcal{H}^2 (\mathbb{Q}, V, D_\chi) = e_\omega, L_\mathcal{H}^\text{geom} \left( \text{Sym}^2 f_\lambda \otimes \chi^{-1} \right) \cdot \mathcal{H}. \]

We will explain how our results in 3.5 hold on the signed Iwasawa main conjectures lead to partial results towards Conjecture 4.6.2. To this end, we assume until the end of this article that the hypothesis of Theorem 3.5.3 hold. Fix also an even integer \( j \in \{ k + 2, \ldots, 2k + 2 \} \) and \( \mathcal{S} = (\spadesuit, \heartsuit) \in \mathcal{S} \) as in Proposition 3.5.4

**Proposition 4.6.3.** \( e_\omega, \text{Col}^{\spadesuit} \circ \text{res}_p (BF_{1,\chi}^{\heartsuit}) = -e_\omega, \text{Col}^{\heartsuit} \circ \text{res}_p (BF_{1,\chi}^{\spadesuit}) \neq 0 \).

**Proof.** The proof of the asserted equality is identical to the proof of [BLV13 Proposition 5.3.4], on replacing reference to Theorem 3.9.1 in op. cit. to Theorem 4.5.5 here (with \( r = 1 \)). The non-vanishing follows thanks to the choice of \( \mathcal{S} \). \( \square \)

Let \( \{ \xi_1, \xi_2 \} \) be an \( \Lambda_{\mathcal{G}} (\Gamma_1) \)-basis of \( e_\omega, H_1 (\mathbb{Q}, \mathbb{T}) \). Recall that is free of rank two by Theorem 2.3.20

**Definition 4.6.4.** Let \( \mathcal{D}, \mathcal{E}_1, \mathcal{E}_2 \in \Lambda_{\mathcal{G}} (\Gamma_1) \) be non-zero and satisfy
\[
\mathcal{D} \cdot e_\omega, BF_{1,\chi}^{\spadesuit} = \mathcal{E}_1 (\text{Col}^{\spadesuit} \circ \text{res}_p (\xi_1)) \xi_2 - \text{Col}^{\spadesuit} \circ \text{res}_p (\xi_2) \xi_1,
\]
\[
\mathcal{D} \cdot e_\omega, BF_{1,\chi}^{\heartsuit} = \mathcal{E}_2 (\text{Col}^{\heartsuit} \circ \text{res}_p (\xi_1)) \xi_2 - \text{Col}^{\heartsuit} \circ \text{res}_p (\xi_2) \xi_1.
\]
The first relation holds in \( e_\omega, H_1 (\mathbb{Q}, \mathbb{T}) \) while the second holds in \( e_\omega, H_1 (\mathbb{Q}, \mathbb{T}) \).

Note that it is always possible to find such non-zero elements since the modules \( e_\omega, H_1 (\mathbb{Q}, \mathbb{T}) \) has rank one for \( ? \in \{ \spadesuit, \heartsuit \} \), as a consequence of the locally restricted Euler system machinery (used as in the proof of Theorem 3.5.5).

**Proposition 4.6.5.** \( \mathcal{D} \cdot \text{char}_\mathcal{H} \left( e_\omega, H_2 (\mathbb{Q}, \mathbb{T}) \right) \) divides \( \mathcal{E}_1 \cdot \text{char}_\mathcal{H} \left( e_\omega, \text{coker} (\text{Col}^{\spadesuit}) \right) \).
Proof. The proof proceeds as in the proof of [BLLV18 Proposition 7.4.4]. We set 
\[ H^1_{\mathbf{\Delta}}(\mathbb{Q}_p, \mathcal{T}) := H^1(\mathbb{Q}_p, \mathcal{T})/H^1_{\mathbf{\Delta}}(\mathbb{Q}_p, \mathcal{T}) \]
By the Poitou-Tate global duality, we have the following long exact sequence
\[
0 \rightarrow H^1_{\mathbf{\Delta}}(\mathbb{Q}, \mathcal{T})/(\Lambda_{\varepsilon}(\Gamma) \cdot BF_{1,\chi}^\bullet) \rightarrow H^1(\mathbb{Q}, \mathcal{T})/(BF_{1,\chi}^\bullet, BF_{1,\chi}^\bullet) \rightarrow H^1_{\mathbf{\Delta}}(\mathbb{Q}_p, \mathcal{T}) \rightarrow H^1_{\mathbf{\Delta}}(\mathbb{Q}_p, \mathcal{T})^\vee \rightarrow H^2(\mathbb{Q}, \mathcal{T}) \rightarrow 0,
\]
where \( \text{res}_{\mathbf{\Delta}} \) denotes the compositum
\[ H^1(\mathbb{Q}, \mathcal{T}) \xrightarrow{\text{res}_{\mathbf{\Delta}}} H^1(\mathbb{Q}_p, \mathcal{T}) \rightarrow H^1_{\mathbf{\Delta}}(\mathbb{Q}_p, \mathcal{T}). \]

The locally restricted Euler system machinery shows that
\[
e_{\omega} \cdot \text{char}_H(\text{Sel}_{\mathbf{\Delta}}(\mathcal{T}^\vee(1))) \mid \text{char}_H \left( e_{\omega} H^1_{\mathbf{\Delta}}(\mathbb{Q}, \mathcal{T})/\Lambda_{\varepsilon}(\Gamma_1) \cdot e_{\omega} \cdot BF_{1,\chi}^\bullet \right).
\]

Combining (4.13) and (4.14), we deduce that
\[
\text{char}_H \left( e_{\omega} H^1_{\mathbf{\Delta}}(\mathbb{Q}, \mathcal{T}) \right) \left( \text{char}_H \left( e_{\omega} \cdot \text{coker}(\text{Col}^\bullet) \right)^{-1} e_{\omega} \cdot \text{Col}^\bullet \circ \text{res}_p(BF_{1,\chi}^\bullet) \right) \mid \text{char}_H \left( e_{\omega} H^1(\mathbb{Q}, \mathcal{T}) \right) / e_{\omega}(BF_{1,\chi}^\bullet, BF_{1,\chi}^\bullet).
\]

We set
\[ \text{det} \circ \text{Col}(\mathbf{\Delta}, \mathbf{\Delta}) := \text{det} \begin{pmatrix} \text{Col}^\bullet \circ \text{res}_p(\varepsilon_1) & \text{Col}^\bullet \circ \text{res}_p(\varepsilon_2) \\ \text{Col}^\bullet \circ \text{res}_p(\varepsilon_1) & \text{Col}^\bullet \circ \text{res}_p(\varepsilon_2) \end{pmatrix}. \]

Note that
\[ \text{Col}^\bullet \circ \text{res}_p(e_{\omega} BF_{1,\chi}^\bullet) = \mathcal{D}^{-1} \mathcal{E}_2 \circ \text{Col}(\mathbf{\Delta}, \mathbf{\Delta}). \]

By Definition 4.6.4 we have that
\[ \text{char}_H \left( e_{\omega} H^1(\mathbb{Q}, \mathcal{T}) \right) / e_{\omega}(BF_{1,\chi}^\bullet, BF_{1,\chi}^\bullet) = \mathcal{D}^{-1} \mathcal{E}_2 \circ \text{Col}(\mathbf{\Delta}, \mathbf{\Delta}). \]

Combining (4.15), (4.16) and (4.17), we deduce that
\[ \mathcal{D} \cdot \text{char}_H \left( e_{\omega} H^2(\mathbb{Q}, \mathcal{T}) \right) \mid \mathcal{E}_1 \cdot \text{char}_H \left( e_{\omega} \cdot \text{coker}(\text{Col}^\bullet) \right), \]
as claimed. \qed

We will now use the bounds for the characteristic ideal of \( e_{\omega} H^2(\mathbb{Q}, \mathcal{T}) \) we obtained in Proposition 4.6.5 to bound characteristic ideals of the analytic Selmer groups.

Definition 4.6.6. Let \( a, b_1, b_2 \in \mathcal{H} \setminus \{0\} \) be elements satisfying
\[ a \cdot BF_{1,\chi}^{\lambda,\lambda} = b_1 \left( \mathcal{L}_{\lambda,\lambda} \circ \text{res}_p(\varepsilon_1) \right) - \mathcal{L}_{\lambda,\lambda} \circ \text{res}_p(\varepsilon_2) \]
\[ a \cdot BF_{1,\chi}^{-\lambda,-\lambda} = b_2 \left( \mathcal{L}_{\lambda,-\lambda} \circ \text{res}_p(\varepsilon_1) \right) - \mathcal{L}_{\lambda,-\lambda} \circ \text{res}_p(\varepsilon_2) \]

Lemma 4.6.7. \( \mathcal{E}_1 = \mathcal{E}_2 \) and \( b_1 = b_2 \). Moreover, \( a \mathcal{E}_1 = b_1 \mathcal{D} \).

Proof. First equality follows from Proposition 4.6.3 and second from Proposition 4.5.7. The final assertion is immediate by definitions of \( \mathcal{D}, \mathcal{E}_1, a, b_1 \) and Theorem 4.5.5 (applied with \( r = 1 \)). \qed

Proposition 4.6.8. We have the following divisibility of \( \mathcal{H} \)-ideals
\[ b \mathcal{D} \text{char}_H \left( e_{\omega} \tilde{H}_w^2(\mathbb{Q}, V, BF_{1,\chi}^{\lambda,\lambda}) \right) \mid a \mathcal{E}_1 \text{char}_H(\text{cokerCol}^\bullet) \text{char}_H \left( \frac{e_{\omega} \tilde{H}_w^1(\mathbb{Q}, V, BF_{1,\chi}^{\lambda,\lambda})}{\mathcal{H} \cdot e_{\omega} BF_{1,\chi}^{\lambda,\lambda}} \right). \]
Theorem 4.6.9. As ideals of $\mathcal{H}$,
\[\text{char}_\mathcal{H} e_\omega \tilde{H}^1_{t,w}(Q, V, \mathcal{D}_V^{\lambda,\lambda}) \mid \text{char}_\mathcal{H}(e_\omega \operatorname{coker} \mathrm{Col}^\bullet) e_\omega L_p, N_{\mathcal{D}}(\chi^{-1} \epsilon_f) L_p^{\text{geom}}(\operatorname{Sym}_f \chi^{-1})].\]

Proof. The argument very closely follows the proof of [BLLV18 Proposition 7.4.6]. Note that we have the following five term exact sequence of $\mathcal{H}$-modules:
\[
0 \to e_\omega \tilde{H}^1_{t,w}(Q, V, \mathcal{D}_V^{\lambda,\lambda}) \to e_\omega H^1_{t,w}(Q, V) \otimes \mathcal{H} \to e_\omega H^1_{/\lambda,\lambda}(Q_p, V) \to e_\omega H^2_{/\lambda,\lambda}(Q_p, V) \to 0
\]
where $H^1_{/\lambda,\lambda}(Q_p, V) := H^1_{t,w}(Q_p, V) \otimes \mathcal{H}(\Gamma)/H^1_{t,w}(Q_p, \mathcal{D}_V^{\lambda,\lambda})$ and $\operatorname{res}_{/\lambda,\lambda}$ is the compositum
\[
\text{res}_{/\lambda,\lambda} : H^1_{t,w}(Q, V) \otimes \mathcal{H} \to H^1_{t,w}(Q_p, V) \otimes \mathcal{H}(\Gamma) \to H^1_{/\lambda,\lambda}(Q_p, V).
\]

We note that the first injection in (4.18) is a special case of Proposition 4.3.7(iii) (which tells us that $\tilde{H}^1_{t,w}(Q, V, \mathcal{D}_V^{\lambda,\lambda}) = 0$). The asserted divisibility now follows on combining Proposition 4.6.8 and the final asserted identity in Lemma 4.6.7 together with the definition of the geometric $p$-adic $L$-function. □

We finally conclude with the following divisibility towards analytic main conjectures.

Theorem 4.6.9. As ideals of $\mathcal{H}$,

\[\text{char}_\mathcal{H} e_\omega \tilde{H}^1_{t,w}(Q, V, \mathcal{D}_V^{\lambda,\lambda}) \mid \text{char}_\mathcal{H}(e_\omega \operatorname{coker} \mathrm{Col}^\bullet) e_\omega L_p, N_{\mathcal{D}}(\chi^{-1} \epsilon_f) L_p^{\text{geom}}(\operatorname{Sym}_f \chi^{-1}).\]

Proof. We start off with the following four-term exact sequence induced by the definition of corresponding Selmer complexes:

\[
(4.18) \quad 0 \to e_\omega \tilde{H}^1_{t,w}(Q, V, \mathcal{D}_V^{\lambda,\lambda}) \to e_\omega H^1_{t,w}(Q_p, \mathcal{D}_V^{\lambda,\lambda}) \to e_\omega H^2_{t,w}(Q_p, \mathcal{D}_V^{\lambda,\lambda}) \to 0
\]
where $\operatorname{res}_p$ is the compositum of the arrows
\[
\tilde{H}^1_{t,w}(Q, V, \mathcal{D}_V^{\lambda,\lambda}) \to H^1_{t,w}(Q_p, \mathcal{D}_V^{\lambda,\lambda}) \to H^1_{t,w}(Q_p, \mathcal{D}_V^{\lambda,\lambda})/H^1_{t,w}(Q_p, \mathcal{D}_V^{\lambda,\lambda}).
\]

We note that the first injection in (4.18) is a special case of Proposition 4.3.7(iii) (which tells us that $\tilde{H}^1_{t,w}(Q, V, \mathcal{D}_V^{\lambda,\lambda}) = 0$). The asserted divisibility now follows on combining Proposition 4.6.8 and the final asserted identity in Lemma 4.6.7 together with the definition of the geometric $p$-adic $L$-function. □

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Kâzım Büyükboduk
UCD School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland

E-mail address: kazim.buyukboduk@ucd.ie

Antonio Lei
Département de Mathématiques et de Statistique, Université Laval, Pavillon Alexandre-Vachon, 1045 Avenue de la Médecine, Québec, QC, Canada G1V 0A6

E-mail address: antonio.lei@mat.ulaval.ca

Guhan Venkat
Département de Mathématiques et de Statistique, Université Laval, Pavillon Alexandre-Vachon, 1045 Avenue de la Médecine, Québec, QC, Canada G1V 0A6

E-mail address: guhanvenkat.harikumar.1@ulaval.ca