Stability of Regularized Hastings-Levitov Aggregation in the Subcritical Regime

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Bacterial growth in increasingly stressed conditions

Source:
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Let $D_0$ denote the exterior unit disk in the complex plane $\mathbb{C}$ and $P$ denote a particle.

There exists a unique conformal mapping $F : D_0 \to D_0 \setminus P$ that fixes $\infty$ in the sense that

$$F(z) = e^c z + O(1) \quad \text{as} \quad |z| \to \infty,$$

for some $c > 0$. We use $F$ as a mathematical description of the particle. The (log of the) capacity, $c$, is a measure of the size of the particle.
Suppose $P_1, P_2, \ldots$ is a sequence of particles, where $P_n$ has capacity $c_n$ and attachment angle $\Theta_n$, $n = 1, 2, \ldots$. Let $F_n$ be the particle map corresponding to $P_n$.

- Set $\Phi_0(z) = z$.
- Recursively define $\Phi_n(z) = \Phi_{n-1} \circ F_n(z)$, for $n = 1, 2, \ldots$.

This generates a sequence of conformal maps $\Phi_n : D_0 \to K_n^c$, where $K_{n-1} \subset K_n$ are growing compact sets, which we call clusters.
Cluster formed by iteratively composing mappings

\[ \Phi_{n-1} \]
Cluster formed by iteratively composing mappings

\[ \Phi_n = \Phi_{n-1} \circ F_n = F_1 \circ F_2 \circ \cdots \circ F_n \]
Parameter choices for physical models

- By varying the sequences \(\{\Theta_n\}\) and \(\{c_n\}\), it is possible to describe a wide class of growth models.
- For biological growth (Eden model)
  \[
P(\Theta_n \in (a, b)) \propto \int_a^b |\Phi_n' (e^{i\theta})| \, d\theta
  \]
  and
  \[
c_n \approx c |\Phi_n' (e^{i\Theta_n})|^{-2}
  \]
- For DLA, \(c_n\) is as above and
  \[
P(\Theta_n \in (a, b)) = P(\Phi_n^{-1}(B_\tau) \in (a, b)) \propto (b - a)
  \]
  where \(B_t\) is Brownian motion started from \(\infty\) and \(\tau\) is the hitting time of the cluster \(K_{n-1}\).
Aggregate Loewner Evolution, ALE($\alpha, \eta, \sigma$)

- $\Theta_n$ distributed $\propto |\Phi'_{n-1}(e^{\sigma+i\theta})|^{-\eta} d\theta$; 
  $c_n = c|\Phi'_{n-1}(e^{\sigma+i\Theta_n})|^{-\alpha}$. 

Conformal models for planar random growth

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Even after the arrival of a single slit particle, the map \( \theta \mapsto |\Phi'_n(e^{i\theta})| \) is badly behaved and takes the values 0 and \( \infty \).

For some values of \( \eta \),

\[
\int_{-\pi}^{\pi} |\Phi'_{n-1}(e^{i\theta})|^{-\eta} d\theta = \infty,
\]

so regularization is necessary to even define the measure.

A solution is to let \( \Theta_n \) have distribution

\[
\propto |\Phi'_{n-1}(e^{\sigma+i\theta})|^{-\eta} d\theta
\]

for \( \sigma > 0 \) and take the limit \( \sigma \to 0 \).

Models are very sensitive to the rate at which \( \sigma \to 0 \). Can be argued that \( \sigma \sim c^{1/2} \) is natural from a physical point of view.
Universality of particle shapes

The model depends on the choice of a family of basic particles \((P^{(c)} : c \in (0, \infty))\) with \(P^{(c)}\) of capacity \(c\). We will require that

\[
P^{(c_1)} \subset P^{(c_2)} \quad \text{for } c_1 < c_2
\]

and, for some \(\Lambda \in [1, \infty)\),

\[
\sup\{|z - 1| : z \in P^{(c)}\} \leq \Lambda \sup\{|z| - 1 : z \in P^{(c)}\} \quad \text{for all } c.
\]

For small \(c\), the second condition forces the particles to concentrate near the point 1 while never becoming too flat against the unit circle.
Phase transition

Open Problem:

Does ALE(\(\alpha, \eta, \sigma\)) exhibit a phase transition from disks to non-disks along the line \(\alpha + \eta = 1\) in the limit as \(c \to 0\) (for ‘broad’ choices of the regularization parameter \(\sigma\))?  

- Longstanding conjectures:
  - HL(\(\alpha\)) has a phase transition at \(\alpha = 1\).
  - DBM(\(\eta\)) has a phase transition at \(\eta = 0\).
ALE(0,0,0) cluster with 25,000 particles for $c = 10^{-4}$

Simulation by Alan Sola

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Stability of Regularized Hastings-Levitov Aggregation in the Subcritical Regime
ALE(1,0,0.02) cluster with 25,000 particles for $c = 10^{-4}$
ALE(1.5,0,0.02) cluster with 25,000 particles for $c = 10^{-4}$

Simulation by Alan Sola
ALE(2,0,0.02) cluster with 25,000 particles for $c = 10^{-4}$
Define the driving measure \( \mu_t = \delta_{e^{i\xi_t}} \), where
\[
\xi_t = \sum_{k=1}^{N} \Theta_k 1(c_{k-1}, c_k)(t),
\]
with \( C_k = \sum_{j=1}^{k} c_k \), for angles \( \{\Theta_k\} \) and capacities \( \{c_k\} \) as above.

Consider the solution to the Loewner equation
\[
\partial_t \Psi_t(z) = z \Psi_t'(z) \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} d\mu_t(e^{i\theta}),
\]
with initial condition \( \Psi_0(z) = z \).

Then (for slit particles, but general case similar)
\[
\Phi_n = \Psi_{C_n}, \quad n = 0, 1, 2, \ldots .
\]
Continuity properties of the Loewner equation

- Solutions to the Loewner equation are close if the driving measures are close in some suitable sense.
  - Suppose $\mu^n = \{\mu^n_t\}_{t \geq 0}$, $n = 1, 2, \ldots$, and $\mu = \{\mu_t\}_{t \geq 0}$ are families of measures on the unit circle $\mathbb{T}$.
  - Let $\Psi^n_t$ be the solution to the Loewner equation corresponding to $\mu^n$ and $\Psi_t$ be the solution corresponding to $\mu$.
  - To show that $\Psi^n_t \to \Psi_t$ uniformly on compact subsets of $D_0$, it is enough to show that

$$\int_{\mathbb{T} \times [0, \infty)} f(e^{i\theta}, t) d\mu^n_t(e^{i\theta}) dt \to \int_{\mathbb{T} \times [0, \infty)} f(e^{i\theta}, t) d\mu_t(e^{i\theta}) dt$$

for all continuous functions $f$ in $\mathbb{T} \times [0, \infty)$ with compact support.
Example: Anisotropic Hastings-Levitov

- Suppose \( \Theta_n \) are i.i.d. with density \( h(\theta) \) on \([0, 2\pi)\).
- Suppose \( c_n = cg(\Theta_n) \), for some bounded continuous function \( g \) on \([0, 2\pi)\).
- Let \( \Psi_t \) solve

  \[
  \partial_t \Psi_t(z) = z\Psi_t'(z) \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}g(\theta)h(\theta)} d\theta,
  \]

  with initial condition \( \Psi_0(z) = z \).

**Theorem (Viklund, Sola, T. ’12):** Fix \( T > 0 \). As \( c \to 0 \), 
\( \Phi_{\lfloor T/c \rfloor} \to \Psi_T \) in probability.
Clusters with non-uniform attachment angles

Simulations by Alan Sola

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Loewner chain analysis

Heuristic for ALE scaling limit

- ALE does not fit into the framework above as the attachment densities and capacities are random and depend on the cluster.

- Nevertheless, the same heuristic suggests that a candidate scaling limit for $\Phi_{\lfloor T/c \rfloor}$ is the solution $\Psi_t$ to

$$
\partial_t \Psi_t(z) = \frac{z \Psi_t'(z)}{Z_t} \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} |\Psi_t'(e^{i\theta})|^{-(\alpha+\eta)} d\theta,
$$

with initial condition $\Psi_0(z) = z$, where

$$
Z_t = \int_0^{2\pi} |\Psi_t'(e^{i\theta})|^{-\eta} d\theta.
$$

- It is straightforward to check that $\Psi_t(z) = (1 + \alpha t)^{1/\alpha} z$. 

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Stability of Regularized Hastings-Levitov Aggregation in the Subcritical Regime
Stability of the dynamics

- The randomness in ALE introduces perturbations around the disk solution.
- Depending on the stability of the Loewner equation, these perturbations can be suppressed or amplified by the PDE dynamics.
- The factor $Z_t$ just induces a time-change, so does not affect the stability.
- Stability therefore depends only on $\alpha + \eta$.
- Simulations suggest a transition between stable and unstable dynamics at $\alpha + \eta = 1$. 
Analysis of the stability

Set

\[ a(\phi)(z) = \frac{z\phi'(z)}{2\pi} \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} |\phi'(e^{i\theta})|^{-\zeta} \, d\theta. \]

Then

\[ a(\phi + \varepsilon\psi)(z) = a(\phi) + \varepsilon (z\psi'(z)h(z) - \zeta z\phi'(z)g(z)) + o(\varepsilon) \]

where

\[ h(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} |\phi'(e^{i\theta})|^{-\zeta} \, d\theta \]

and, setting \( \rho = \psi'/\phi' \),

\[ g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} |\phi'(e^{i\theta})|^{-\zeta} \Re \rho(e^{i\theta}) d\theta. \]

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Suppose $\phi_t(z) = e^{\tau_t}z$ where $\tau_t = \zeta^{-1} \log(1 + \zeta t)$. Then $\phi_t$ solves $\partial_t \phi_t = a(\phi_t)$ with initial condition $\phi_0(z) = z$. The integrals $h$ and $g$ can be explicitly evaluated as $h(z) = e^{-\zeta \tau_t}$ and $g(z) = e^{-(1+\zeta)\tau_t} \psi'(z)$ so, by equating coefficients of $\varepsilon$ in $\partial_t(\phi_t + \varepsilon \psi_t) = a(\phi_t + \varepsilon \psi_t)$, the first order variations $\psi_t$ around the solution $\phi_t$ can be expected to satisfy the linearized equation

$$\partial_t \psi_t = (1 - \zeta)z \psi'_t(z)e^{-\zeta \tau_t} = (1 - \zeta)z \psi'_t(z)\dot{\tau}_t.$$
Linear stability of disk solutions in the subcritical case

- Formally, this has solution

\[ \psi_t(z) = \psi_0 \left( e^{(1-\zeta)t} z \right). \]

- In the case \( \zeta > 1 \), \( \psi_t \) can be holomorphic in \( \{ |z| > 1 \} \) only if \( \psi_0 \) extends to a holomorphic function in the larger domain \( \{ |z| > e^{-(\zeta-1)t} \} \).

- In particular, if \( \psi_0 \) has singularities on the boundary \( \{ |z| = 1 \} \), then the variation blows up immediately.

- When \( \zeta \in [0, 1] \), the variation \( \psi_t \) is holomorphic in \( \{ |z| > 1 \} \) for all \( t \) and, when \( \zeta < 1 \), gets more regular as \( t \) increases.
Disk theorem for $\text{ALE}(\alpha, \eta, \sigma)$ when $\alpha + \eta < 1$

**Theorem (Norris, Silvestri, T.):**

For all $T \in [0, \infty)$, $\epsilon \in (0, 1/2)$ and $e^\sigma \geq 1 + c^{1/2-\epsilon}$, there exists a constant $C$ such that, with high probability, for all $n \leq T/c$ and $|z| \geq 1 + c^{1/2-\epsilon}$,

$$|\Phi_n(z) - (1 + \alpha cn)^{1/\alpha}z| \leq \frac{C}{|z|} \left( c^{1/2-\epsilon} + \frac{c^{1-\epsilon}}{(e^\sigma - 1)^2} \right).$$
Theorem (Norris, Silvestri, T.):

For all $T \in [0, \infty)$, $\epsilon \in (0, 1/5)$ and $e^\sigma \geq 1 + c^{1/5-\epsilon}$, there exists a constant $C$ such that, with high probability, for all $n \leq T/c$ and $|z| \geq 1 + c^{1/5-\epsilon}$,

$$|\Phi_n(z) - (1+\alpha cn)^{1/\alpha} z| \leq \frac{C}{|z|} \left( c^{1/2-\epsilon} \left( \frac{|z|}{|z|-1} \right)^{1/2} + \frac{c^{1-\epsilon}}{(e^\sigma - 1)^3} \right).$$
Fluctuations for $\text{ALE}(\alpha, \eta, \sigma)$ when $\alpha + \eta \leq 1$

Set

$$\mathcal{F}_n^{(c)}(z) = c^{-1/2}((1 + \alpha cn)^{-1/\alpha} \Phi_n(z) - z)$$

and let $n(t) = \lfloor t/c \rfloor$.

Under the assumptions above (but with slightly stronger restrictions on $\sigma$), $\mathcal{F}_n^{(c)}(z) \to \mathcal{F}_t(z)$ where

$$\dot{\mathcal{F}}_t(z) = \frac{1}{1 + \alpha t} \left( (1 - \alpha - \eta)z\mathcal{F}'_t(z) - \mathcal{F}_t(z) + \sqrt{2}\xi_t(z) \right).$$

Here $\xi_t(z)$ is complex space-time white noise on the circle, analytically continued to the exterior unit disk.
Fluctuations for $\text{ALE}(\alpha, \eta, \sigma)$ when $\alpha + \eta \leq 1$

Specifically

$$F_t(z) = \sum_{m=0}^{\infty} (A^m_t + iB^m_t)z^{-m}$$

where

$$dA^m_t = -\frac{(m(1 - \alpha - \eta) + 1) A^m_t}{1 + \alpha t} dt + \frac{\sqrt{2}}{1 + \alpha t} d\beta^m_t$$

$$dB^m_t = -\frac{(m(1 - \alpha - \eta) + 1) B^m_t}{1 + \alpha t} dt + \frac{\sqrt{2}}{1 + \alpha t} d\beta'_m t.$$ 

Here $\beta^m_t, \beta'_m$ are i.i.d. Brownian motions for $m = 0, 1, \ldots$, so

$$A^m_t, B^m_t \sim \mathcal{N}\left(0, \frac{1 - e^{-2(m(1-\alpha-\eta)+1)\tau} t}{m(1 - \alpha - \eta) + 1}\right).$$
The map $z \mapsto F_t(z)$ is determined (by analytic extension) by the boundary process $\theta \mapsto F_t(e^{i\theta})$.

When $\alpha = \eta = 0$, these boundary fluctuations are the same as for internal diffusion limited aggregation (IDLA).

As $t \to \infty$, $F_t(e^{i\theta})$ converges to a Gaussian field.

- When $\alpha + \eta = 0$, $F_\infty(e^{i\theta})$ is known as the augmented Gaussian Free Field.
- When $\alpha + \eta < 1$, $\text{Cov} \left( F_\infty(e^{ix}), F_\infty(e^{iy}) \right) \asymp \log |x - y|$.
- When $\alpha + \eta = 1$, $F_\infty(e^{i\theta})$ is complex white noise.
Proofs

Idea behind the proof

Using the particle assumptions one can show that

\[
\log \frac{F_n(z)}{z} = c_n \frac{ze^{-i\Theta_n} + 1}{ze^{-i\Theta_n} - 1} + O(c_n^{3/2}).
\]

Therefore

\[
\Phi_n(z) = \Phi_{n-1}(F_n(z))
= \Phi_{n-1}(z) + z\Phi'_{n-1}(z)(\log F_n(z) - \log z) + \cdots
= \Phi_{n-1}(z) + cz\Phi'_{n-1}(z)|\Phi'_{n-1}(e^{\sigma+i\Theta_n})|^{-\alpha} \frac{ze^{-i\Theta_n} + 1}{ze^{-i\Theta_n} - 1} + O(c^{3/2}).
\]
Idea behind the proof

Now

\[
\mathbb{E} \left( z \Phi'_{n-1}(z) \Phi'_{n-1}(e^{\sigma+i\Theta_n}) \left| -\alpha \frac{ze^{-i\Theta_n} + 1}{ze^{-i\Theta_n} - 1} \right| \mathcal{F}_{n-1} \right)
\]

\[
= \frac{z \Phi'_{n-1}(z)}{Z_n} \int_0^{2\pi} |\Phi'_{n-1}(e^{\sigma+i\theta})| - (\alpha + \eta) \frac{ze^{-i\theta} + 1}{ze^{-i\theta} - 1} d\theta
\]

\[
:= a_\sigma(\Phi_{n-1})(z)
\]

so

\[
\frac{\Phi_n(z) - \Phi_{n-1}(z)}{c} = a_\sigma(\Phi_{n-1})(z) + M_n(z) + O(c^{1/2})
\]

where \( M_n(z) \) is a martingale difference term.
Proofs

Idea behind the proof

For \( \phi_t(z) = (1 + \alpha t)^{1/\alpha} z = e^{\tau t} z \), write

\[
\Phi_n(z) = \phi_{nc}(z) + M_n(z).
\]

Using the variational solution from earlier with \( \zeta = \alpha + \eta \) gives

\[
M_n(z) \approx M_{n-1} \left( e^{(1-\zeta)(\tau_{nc}-\tau_{(n-1)c})} \right) + M_n(z)
\]

\[
= \sum_{k=1}^{n} M_k \left( e^{(1-\zeta)(\tau_{nc}-\tau_{kc})} z \right).
\]

The scaling limit and fluctuation results follow from an analysis of the martingale and estimates on the errors.
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