Rational points on symmetric powers and categorical representability

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Abstract. In this paper we observe that for geometrically integral projective varieties \(X\), admitting a full weak exceptional collection consisting of pure vector bundles, the existence of a \(k\)-rational point implies \(\text{rdim}(X) = 0\). We also study the symmetric power \(S^n(X)\) of Brauer–Severi and involution varieties over \(\mathbb{R}\) and prove that the equivariant derived category \(D^b_{S^n(X^n)}\) admits a full weak exceptional collection. As a consequence, we find \(\text{rdim}(X) = 0\) if and only if \(\text{rdim}(D^b_{S^n(X^n)}) = 0\) for \(1 \leq n \leq 3\). If \(X\) is Brauer–Severi, the existence of a \(\mathbb{R}\)-rational point on \(X\) or \(S^3(X)\) is equivalent to \(\text{rdim}(D^b_{S^3(X^3)}) = 0\).

1. Introduction

When the base field \(k\) is not algebraically closed, the existence of \(k\)-rational points on \(X\) (being a necessary condition for rationality) is a major open problem in arithmetic geometry. In [9] Auel and Bernardara formulated the following question, actually posed by H. Esnault in 2009:

**Question 1.1.** Let \(X\) be a smooth projective variety over a field \(k\). Can the bounded derived category \(D^b(X)\) detect the existence of a \(k\)-rational point on \(X\)?

This question is now central for arithmetic aspects of the theory of derived categories, see [9], [2], [3], [7], [8], [21], [28] and [34]. In the present work we use a potential measure for rationality to answer Question 1.1 affirmatively in some special cases. In [12] Bernardara and Bolognesi introduced the notion of categorical representability. Below we use the definition given in [10]. So a \(k\)-linear triangulated category \(T\) is said to be representable in dimension \(m\) if there is a semiorthogonal decomposition \(T = \langle A_1, \ldots, A_n \rangle\) and for each \(i = 1, \ldots, n\) there exists a smooth projective connected variety \(Y_i\) with \(\text{dim}(Y_i) \leq m\), such that \(A_i\) is equivalent to an admissible subcategory of \(D^b(Y_i)\) (see [10]). We use the following notation

\[
\text{rdim}(T) := \min\{m \mid T\text{ is representable in dimension } m\},
\]

whenever such a finite \(m\) exists. Let \(X\) be a smooth projective \(k\)-variety. One says \(X\) is representable in dimension \(m\) if \(D^b(X)\) is representable in dimension \(m\). We will use the following notation:

\[
\text{rdim}(X) := \text{rdim}(D^b(X)).
\]

Quite recently, it has been shown that certain varieties \(X\) admit \(k\)-rational points if and only if \(\text{rdim}(X) = 0\), see [9], [34] and [35]. Among these varieties are certain Fano varieties having full weak exceptional collections. For arbitrary varieties admitting full weak exceptional collections, we observe the following:

**Theorem** (Theorem 6.3). Let \(X\) be a smooth projective variety over a field \(k\) satisfying \(H^0(X^*, O_{X^*}) = k^*\) and assume \(D^b(X)\) admits a full weak exceptional collection consisting of pure vector bundles. If \(X(k) \neq \emptyset\), then \(\text{rdim}(X) = 0\).

Recall that a Fano variety \(X\) of dimension 1 is a Brauer–Severi curve. In this case \(D^b(X)\) has a full weak exceptional collection consisting of pure vector bundles. Results
Theorem (Theorem 6.5) With the help of Theorem 6.5, we can prove the following:
Brauer–Severi and involution varieties over arbitrary fields to be central simple or matrix algebras over a field. This is quite hard to accomplish for any of these cases, and we make use of the endomorphism algebras of the irreducible representations of the symmetric group $S_n$. At some point in the proof we need the endomorphism algebras of the irreducible representations of the symmetric group $S_n$. We set $A := D^\text{b}_h(X)$. Suppose $X(k) \neq \emptyset$ if and only if $\text{rdim}(X) = 0$. Note that there are also Fano threefolds $X$, such as Brauer–Severi varieties or twisted forms of quadrics which admit full weak exceptional collections, respectively semiorthogonal decompositions of the form
$$D^\text{b}(X) = \langle D^\text{b}(l_1/k, A_1), ..., D^\text{b}(l_n/k, A_n) \rangle.$$  

The results in [34] show that $X(k) \neq \emptyset$ if and only if $\text{rdim}(X) = 0$. On the other hand, with the help of [13], Proposition 1.7 one can cook up examples of anisotropic quadrics admitting full exceptional collections which are not rational and have no $k$-rational points. So there are examples of Fano varieties $X$ for which $\text{rdim}(X) = 0$ does not imply the existence of a $k$-rational point. Motivated by these examples, we want to ask the following question.

**Question 1.2.** Let $X$ be a smooth Fano variety over a field $k$. Suppose $D^\text{b}(X)$ admits a full weak exceptional collection consisting of pure vector bundle, or more generally a semiorthogonal decomposition of the form
$$D^\text{b}(X) = \langle D^\text{b}(l_1/k, A_1), ..., D^\text{b}(l_n/k, A_n) \rangle,$$
where $l_i/k$ are field extensions and $A_i$ suitable central simple algebras over $l_i$.

Is there any characterization for which $X$, $\text{rdim}(X) = 0$ implies $X(k) \neq \emptyset$?

The main goal of the present paper is to modify Question 1.1 by studying singular projective varieties. However, one has to seek for the "right" analogue of the derived category $D^\text{b}(X)$. In Section 6 we study symmetric powers of Brauer–Severi varieties $X$ over $\mathbb{R}$ and its associated equivariant derived category $D^\text{b}_{S_n}(X^n)$. This consideration is motivated by a paper of Krashen and Saltman [26] in which they studied the question whether for Brauer–Severi varieties $X$ the rationality of the symmetric power $S^n(X)$ implies rationality of $X$. In this context we also want to mention a paper of Kollár [24] where products of symmetric powers of a Brauer–Severi variety are classified up-to stable birational equivalence. In the present work we want to shed light on the existence of rational points on $S^n(X)$ from a derived point of view by using the concept of categorical representability. Notice that in some cases the existence of a rational point on $S^n(X)$ forces $X$ to be rational and we can relate $\text{rdim}(X)$ to $\text{rdim}(D^\text{b}_{S_n}(X^n))$. We first show the following result which is not a surprise and follows from available results and techniques in the literature. We focus on Brauer–Severi varieties and involution varieties of orthogonal type over $\mathbb{R}$ since we make use of the endomorphism algebras of the irreducible representations of the symmetric group $S_n$. At some point in the proof we need the endomorphism algebras to be central simple or matrix algebras over a field. This is quite hard to accomplish for Brauer–Severi and involution varieties over arbitrary fields $k$.

**Theorem (Theorem 6.5).** Let $X$ be a Brauer–Severi variety over $\mathbb{R}$ or a twisted quadric associated to a central simple $\mathbb{R}$-algebra $(A, \sigma)$ with involution of orthogonal type. Then $D^\text{b}_{S_n}(X^n)$ admits a full weak exceptional collection.

With the help of Theorem 6.5, we can prove the following:

**Theorem (Theorem 6.7).** Let $X$ be a Brauer–Severi variety over $\mathbb{R}$ or a twisted quadric associated to a central simple $\mathbb{R}$-algebra $(A, \sigma)$ with involution of orthogonal type and $1 \leq n \leq 3$. We set $T := D^\text{b}_{S_n}(X^n)$. Then the following hold:
(i) \( \text{rdim}(X) = 0 \) if and only if \( \text{rdim}(T) = 0 \).
(ii) \( X(\mathbb{R}) \neq \emptyset \) if and only if \( \text{rdim}(T) = 0 \).

The proof of Theorem 6.7 in particular shows that the implication \( \text{rdim}(T) = 0 \Rightarrow \text{rdim}(X) = 0 \) holds for arbitrary positive integers \( n \). As a consequence of the latter result we find:

**Corollary 1.3** (Corollary 6.9). Let \( X \) be a Brauer–Severi variety over \( \mathbb{R} \) corresponding to \( A \) and assume \( \deg(A) > 3 \). Then \( S^3(X)(\mathbb{R}) \neq \emptyset \) if and only if \( \text{rdim}(T) = 0 \).

The existence of rational points seems, in general, not to be related to categorical representability in dimension zero. For instance, an elliptic curve over a number field is categorical representable in dimension one (see [10]) although it has rational points. In-representability in dimension zero. For instance, an elliptic curve over a number field is \( k \)-rational. Do we have \( \text{rcodim}(X) \geq 2 \)?

Assuming \( \text{dim}(X) \geq 2 \), Theorem 6.3 in particular implies that if \( X \) is \( k \)-rational, then \( \text{rcodim}(X) \geq 2 \). Moreover, in view of the latter question, we show that the dg enhancement of \( D^b_c(X^n) \) is a noncommutative resolution of singularities of the noncommutative scheme \( \text{perf}(S^3(X)) \) (see Theorem 6.14). Then Corollary 6.9 from above says:

**Corollary** (Corollary 6.15). Let \( X \) be a Brauer–Severi variety over \( \mathbb{R} \) corresponding to \( A \) and assume \( \deg(A) > 3 \). If \( S^3(X) \) is \( \mathbb{R} \)-rational, we have \( \text{rcodim}(S^3(X)) \geq 2 \).

**Acknowledgement.** I wish to thank Zinovy Reichstein for answering questions and providing me with literature for group actions on central simple algebras. I also like to thank Asher Auel, Marcello Bernardara and Michele Bolognesi, whose articles inspired me to deal with this subject. Finally, I would like to thank the Heinrich–Heine–University for financial support via the SFF-grant.

**Notations.** If \( X \) is a \( k \)-variety, we will denote by \( D^b(X) \) the bounded derived category of complexes of coherent sheaves on \( X \). Notice that \( D^b(X) \) is a \( k \)-linear category. Let \( B \) be an \( O_X \)-algebra, we will denote by \( D^b(X, B) \) the bounded derived category of complexes of \( B \)-modules, considered as a \( k \)-linear category. For \( X = \text{Spec}(K) \) and \( B \) associated to a \( K \)-algebra, we will write \( D^b(K/k, B) \). Also \( D^b(K, B) \) is shorthanded for \( D^b(K/K, B) \).

2. BRAUER–SEVERI AND INVOLUTION VARIETIES

A *Brauer–Severi variety* of dimension \( n \) is a \( k \)-variety \( X \) such that \( X \otimes_k L \simeq \mathbb{P}^n \) for a finite field extension \( L/k \). An extension \( L/k \) for which \( X \otimes_k L \simeq \mathbb{P}^n \) is called *splitting field* of \( X \). In fact, every Severi–Brauer variety always splits over a finite separable field extension of \( k \) (see [10], Corollary 5.1.4) and therefore over a finite Galois extension. It follows from descent theory that \( X \) is projective, integral and smooth over \( k \). Via Galois...
cohomology, isomorphism classes of $n$-dimensional Severi–Brauer varieties are in one-to-one correspondence with isomorphism classes of central simple $k$-algebras of degree $n + 1$. If $A$ is a central simple algebra, we will write $BS(A)$ for the corresponding Brauer–Severi variety.

Recall, a finite-dimensional $k$-algebra $A$ is called central simple if it is an associative $k$-algebra that has no two-sided ideals other than 0 and $A$ and if its center equals $k$. Note that $A$ is a central simple if and only if there is a finite field extension $L/k$, such that $A \otimes_k L \cong M_n(L)$ (see [19], Theorem 2.2.1). An extension $L/k$ such that $A \otimes_k L \cong M_n(L)$ is called splitting field for $A$. The degree of a central simple algebra $A$ is defined to be $\deg(A) := \sqrt{\dim_k A}$. According to the Wedderburn Theorem for any central simple $k$-algebra $A$ there is an integer $n > 0$ and a division algebra $D$, such that $A \cong M_n(D)$. The division algebra $D$ is also central and unique up to isomorphism. The degree of the unique central division algebra $D$ is called the index of $A$ and is denoted by $\text{ind}(A)$.

Two central simple $k$-algebras $A \cong M_n(D)$ and $B \cong M_m(D')$ are called Brauer-equivalent if $D \cong D'$. The Brauer group $\text{Br}(k)$ of a field $k$ is the group whose elements are equivalence classes of central simple $k$-algebras, with addition given by the tensor product of algebras. It is an abelian group with inverse of the equivalence class $[A]$ being $[A^{op}]$. The neutral element is $[k]$. It is a fact that the Brauer group of any field is a torsion group. The order of an equivalence class $[A] \in \text{Br}(k)$ is called the period of $[A]$ and is denoted by $\text{per}(A)$. For instance if $k = \mathbb{R}$, the Brauer group $\text{Br}(\mathbb{R})$ is cyclic of order two and generated by the equivalence class of the Hamilton quaternions $\mathbb{H}$.

To a central simple algebra $A$ of degree $n$ with involution $\sigma$ of the first kind over a field $k$ of $\text{char}(k) \neq 2$ one can associate the involution variety $\text{IV}(A, \sigma)$. This variety can be described as the variety of $n$-dimensional right ideals $I$ of $A$ such that $\sigma(I) = I = 0$. If $A$ is split so $(A, \sigma) \cong (M_n(k), q^*)$, where $q^*$ is the adjoint involution defined by a quadratic form $q$ one has $\text{IV}(A, \sigma) \cong V(q) \subset \mathbb{P}^{n-1}$, Here $V(q)$ is the quadric determined by $q$. By construction such an involution variety $\text{IV}(A, \sigma)$ becomes a quadric in $\mathbb{P}^{n-1}$ after base change to some splitting field $L$ of $A$. In this way the involution variety is a twisted form of a smooth quadric. Recall from [10] that a splitting field $L$ splits $A$ isotropically if $(A, \sigma) \otimes_k L \cong (M_n(L), q^*)$ with $q$ an isotropic quadratic form over $L$. Although the degree of $A$ is arbitrary, (when $\text{char}(k) \neq 2$), the case where degree of $A$ is odd does not give anything new, since central simple algebras of odd degree with involution of the first kind are split (see [24], Corollary 2.8). For details on the construction and further properties on involution varieties and the corresponding algebras we refer to [16].

3. Semiorthogonal decomposition and exceptional collections

Let $\mathcal{D}$ be a triangulated category and $A$ a triangulated subcategory. The subcategory $\mathcal{A}$ is called thick if it is closed under isomorphisms and direct summands. For a subset $\mathcal{M}$ of objects of $\mathcal{D}$ we denote by $(\mathcal{M})$ the smallest full thick subcategory of $\mathcal{D}$ containing the elements of $\mathcal{M}$. Furthermore, we define $\mathcal{M}^\perp$ to be the subcategory of $\mathcal{D}$ consisting of all objects $C$ such that $\text{Hom}_\mathcal{D}(E[i], C) = 0$ for all $i \in \mathbb{Z}$ and all elements $E$ of $\mathcal{M}$. We say that $\mathcal{M}$ generates $\mathcal{D}$ if $\mathcal{M}^\perp = 0$. Now assume $\mathcal{D}$ admits arbitrary direct sums. An object $F$ is called compact if $\text{Hom}_\mathcal{D}(F, -)$ commutes with direct sums. Denoting by $\mathcal{D}^c$ the subcategory of compact objects we say that $\mathcal{D}$ is compactly generated if the objects of $\mathcal{D}^c$ generate $\mathcal{D}$. Let $\mathcal{D}$ be a compactly generated triangulated category. Then a set of objects $\mathcal{A} \subset \mathcal{D}^c$ generates $\mathcal{D}$ if and only if $(\mathcal{A}) = \mathcal{D}^c$ (see [16], Theorem 2.1.2).

Let $G$ be a finite group acting on a smooth projective variety $X$ over a field $k$ and assume $\text{char}(k) \nmid \text{ord}(G)$. The equivariant derived category, denoted by $\mathcal{D}^b_G(X)$, is defined to be $\mathcal{D}^b(\text{Coh}_G(X))$. For details see for instance [33], Section 2. For any two objects $\mathcal{F}^*, \mathcal{G}^* \in \mathcal{D}^b_G(X)$ we write $\text{Hom}_G(\mathcal{F}^*, \mathcal{G}^*) := \text{Hom}_{\mathcal{D}^b_G(X)}(\mathcal{F}^*, \mathcal{G}^*)$. As $\text{char}(k) \nmid \text{ord}(G)$, the functor $(-)^G$ is exact and one has

$$\text{Hom}_G(\mathcal{F}^*, \mathcal{G}^*[i]) \cong \text{Hom}(\mathcal{F}^*, \mathcal{G}^*[i])^G.$$
Recall, that for every subgroup $H \subset G$, the restriction functor $\text{Res}: D^b_G(X) \to D^b_H(X)$ has the inflation functor $\text{Inf}: D^b_H(X) \to D^b_G(X)$ as a left and right adjoint, see [13], Section 3. It is given for $A \in D^b_H(X)$ by

$$\text{Inf}^G_H(A) = \bigoplus_{[g] \in H/G} g^*A.$$ 

**Definition 3.1.** Let $A$ be a division algebra over $k$, not necessarily central. An object $\mathcal{E}^i \in D^b_G(X)$ is called weak exceptional if $\text{End}_G(\mathcal{E}^i) = A$ and $\text{Hom}_G(\mathcal{E}^i, \mathcal{E}^j[r]) = 0$ for $r \neq 0$. If $A = k$ the object is called exceptional.

**Definition 3.2.** A totally ordered set $\{\mathcal{E}^1, ..., \mathcal{E}^n\}$ of weak exceptional objects in $D^b_G(X)$ is called an weak exceptional collection if $\text{Hom}_G(\mathcal{E}^i, \mathcal{E}^j[r]) = 0$ for all integers $r$ whenever $i > j$. An weak exceptional collection is full if $\{\mathcal{E}^1, ..., \mathcal{E}^n\} = D^b_G(X)$ and strong if $\text{Hom}_G(\mathcal{E}^i, \mathcal{E}^j[r]) = 0$ whenever $r \neq 0$. If the set $\{\mathcal{E}^1, ..., \mathcal{E}^n\}$ consists of exceptional objects it is called exceptional collection.

A generalization of the notion of a full weak exceptional collection is that of a semiorthogonal decomposition of $D^b_G(X)$. Recall that a full triangulated subcategory $\mathcal{A}$ of $D^b_G(X)$ is called admissible if the inclusion $\mathcal{D} \to D^b_G(X)$ has a left and right adjoint functor.

**Definition 3.3.** Let $X$ be a smooth projective variety over $k$. A sequence $\mathcal{A}_1,...,\mathcal{A}_n$ of full triangulated subcategories of $D^b_G(X)$ is called semiorthogonal if all $\mathcal{A}_i \subset D^b_G(X)$ are admissible and $\mathcal{A}_j \subset \mathcal{A}_i^\perp = \{\mathcal{F}^i \in D^b_G(X) \mid \text{Hom}_G(\mathcal{G}^i, \mathcal{F}^j) = 0, \forall \mathcal{G}^i \in \mathcal{A}_i\}$ for $i > j$. Such a sequence defines a semiorthogonal decomposition of $D^b_G(X)$ if the smallest full thick subcategory containing all $\mathcal{A}_i$ equals $D^b_G(X)$.

For a semiorthogonal decomposition we write $D^b_G(X) = \langle \mathcal{A}_1,...,\mathcal{A}_n \rangle$.

4. Descent for vector bundles

Let $X$ be a proper $k$-variety and $W$ an indecomposable vector bundle on $X \otimes_k k^*$. A vector bundle $\mathcal{V}$ on $X$ is called pure of type $W$ if $\mathcal{V} \otimes_k k^* \simeq W^\otimes m$ (see [3]). We say $\mathcal{V}$ is pure if it is pure of type $\mathcal{L}$ for a line bundle $\mathcal{L}$. Note that $\mathcal{L}$ is $G_2 := \text{Gal}(k^*|k)$ invariant. Recall the Brauer obstruction for invariant line bundles on smooth proper geometrically integral $k$-varieties $X$. The sequence of low degree terms of the Leray spectral sequence is

$$0 \to \text{Pic}(X) \to \text{Pic}(X_{k^*})^G_2 \xrightarrow{d} \text{Br}(X) \to \text{Br}(X).$$

It is well-known that for proper varieties over $k$ one has $\text{Pic}(X_{k^*})^G_2 \simeq \text{Pic}(\text{Pic}(X_{k}(et)))(k)$. Recall from [29] that a field extension $L/k$ is called splitting field for $\mathcal{L} \in \text{Pic}(X_{\text{k}(et))}(k)$ if $\mathcal{L} \otimes_k L$ lies $\text{Pic}(X \otimes_k L)$. Moreover, the class $\mathcal{L}$ is called globally generated if there is a splitting field $K$ of $\mathcal{L}$ such that $\mathcal{L} \otimes_k L$ is globally generated. The following result will be applied in Section 6.

**Theorem 4.1** ([29], Theorem 1.1). Let $X$ be a proper variety over a field $k$ and $P$ a Brauer–Severi variety. Then there exists a morphism $\phi: X \to P$ if and only if there is a globally generated $\mathcal{L} \in \text{Pic}(X_{\text{k}(et))}(k)$ with $d(\mathcal{L}) = [P] \in \text{Br}(k)$.

5. Noncommutative motives of central simple and separable algebras

We refer to the book [13] (alternatively see [40] and [49] for a survey on noncommutative motives). Let $A$ be a small dg category. The homotopy category $H^0(A)$ has the same objects as $A$ and as morphisms $H^0(\text{Hom}_A(x,y))$. A source of examples is provided by schemes since the derived category of perfect complexes $\text{perf}(X)$ of any quasi-compact quasi-separated scheme $X$ admits a canonical dg enhancement $\text{perf}_{dg}(X)$ (for details see [21]). Denote by $\text{dgcat}$ the category of small dg categories. The opposite dg category $A^{op}$ has the same objects as $A$ and $\text{Hom}_{A^{op}}(x,y) := \text{Hom}_A(y,x)$. A right $A$-module is a dg functor $A^{op} \to C_{dg}(k)$ with values in the dg category $C_{dg}(k)$ of complexes of $k$-vector
functor \(D \) and a derived category \( \mathcal{A} \) is defined as follows: the set of objects is the cartesian product of the sets of objects in \( \mathcal{A} \) and \( \mathcal{B} \) and \( \text{Hom}_{\mathcal{A} \otimes \mathcal{B}}((x, w), (y, z)) : = \text{Hom}_{\mathcal{A}}(x, y) \otimes \text{Hom}_{\mathcal{B}}(w, z) \) (see \[20\]). Given two dg categories \( \mathcal{A} \) and \( \mathcal{B} \), let \( \text{rep}(\mathcal{A}, \mathcal{B}) \) be the full triangulated subcategory of \( D(\mathcal{A}^{op} \otimes \mathcal{B}) \) consisting of those \( \mathcal{A} \)-\( \mathcal{B} \)-bimodules \( M \) such that \( M(x, -) \) is a compact object of \( D(\mathcal{B}) \) for every object \( x \in \mathcal{A} \). The category \( \text{dgcat} \) of all (small) dg categories and dg functors carries a Quillen model structure whose weak equivalences are Morita equivalences. Let us denote by Hmo the homotopy category hence obtained and by \( \text{Hmo}_0 \) its additivization.

Now to any small dg category \( \mathcal{A} \) one can associate functorially its noncommutative motive \( U(\mathcal{A}) \) which takes values in \( \text{Hmo}_0 \). This functor \( U: \text{dgcat} \to \text{Hmo}_0 \) is proved to be the universal additive invariant (see \[43\]). Recall that an additive invariant is any functor \( E: \text{dgcat} \to \mathcal{D} \) taking values in an additive category \( \mathcal{D} \) such that

(i) it sends derived Morita equivalences to isomorphisms,

(ii) for any pre-triangulated dg category \( \mathcal{A} \) admitting full pre-triangulated dg subcategories \( \mathcal{B} \) and \( \mathcal{C} \) such that \( H^p(\mathcal{A}) = (H^p(\mathcal{B}), H^p(\mathcal{C})) \) is a semiorthogonal decomposition, the morphism \( E(\mathcal{B}) \oplus E(\mathcal{C}) \to E(\mathcal{A}) \) induced by the inclusions is an isomorphism.

For central simple \( k \)-algebras one has the following comparison theorem.

**Theorem 5.1** (\[12\], Theorem 2.19). Let \( A_1, \ldots, A_n \) and \( B_1, \ldots, B_m \) be central simple \( k \)-algebras, then the following are equivalent:

(i) There is an isomorphism

\[
\bigoplus_{i=1}^n U(A_i) \cong \bigoplus_{j=1}^m U(B_j).
\]

(ii) The equality \( n = m \) holds and for all \( 1 \leq i \leq n \) and all \( p \)

\[
[B_i^p] = [A_{\sigma_p(i)}^p] \in \text{Br}(k)
\]

for some permutations \( \sigma_p \) depending on \( p \).

Later, we also need the following result.

**Proposition 5.2** (\[15\], Proposition 4.5). Let \( A_1, \ldots, A_n \) and \( B_1, \ldots, B_m \) be central simple \( k \)-algebras, and \( NM \) a noncommutative motive. If

\[
\bigoplus_{i=1}^n U(A_i) \oplus NM \cong \bigoplus_{j=1}^m U(B_j) \oplus NM,
\]

then \( n = m \) and \( U(A_1) \oplus \cdots \oplus U(A_n) \cong U(B_1) \oplus \cdots \oplus U(B_n) \).

Recall the following definition (see for instance \[14\]).

**Definition 5.3.** Let \( \mathcal{T} \) be a triangulated category. A dg-enhancement of \( \mathcal{T} \) is a pair \( (\mathcal{A}, \epsilon) \), where \( \mathcal{A} \) is a pretriangulated dg category and \( \epsilon: H^0(\mathcal{A}) \to \mathcal{T} \) an equivalence of triangulated categories. Assume \( \mathcal{T} \) has a dg-enhancement. Then \( \mathcal{T} \) admits a unique dg-enhancement if for any two dg-enhancements \( (\mathcal{A}, \epsilon) \) and \( (\mathcal{A}', \epsilon') \) there is a dg-functor \( \Psi: \mathcal{A} \to \mathcal{A}' \), inducing an equivalence \( H^0(\Psi): H^0(\mathcal{A}) \to H^0(\mathcal{A}') \).
6. PROOFS OF THE RESULTS

Lemma 6.1. Let $X$ be a proper variety over a field $k$ with $H^0(X^s, O_{X^s}) = k^s$. If $V$ is a pure vector bundle on $X$, then $\text{End}(V)$ is a central simple $k$-algebra.

Proof. As $V$ is pure, there is a line bundle $L \in \text{Pic}(X \otimes_k k^s)$ such that $V \otimes_k k^s \simeq L^{\oplus m}$. The assertion then follows from the following isomorphisms

$$\text{End}(V) \otimes_k k^s \simeq \text{End}(L^{\oplus m}) \simeq \text{End}(O_{X^s}^{\oplus m}) \simeq M_m(k^s).$$

$\square$

Proposition 6.2. Let $X$ be a projective variety over a field $k$ with $H^0(X^s, O_{X^s}) = k^s$. Suppose $V$ is a pure vector bundle on $X$. Then there is a morphism $X \rightarrow \text{BS}(\text{End}(V))$.

Proof. We have $V \otimes_k k^s \simeq L^{\oplus m}$ for some line bundle $L \in \text{Pic}(X \otimes_k k^s)$ and Lemma 6.1 shows that $\text{End}(V)$ is isomorphic to a central simple $k$-algebra $A$. Let $E/k$ be a finite Galois extension within $k^s$ over which $V$ is defined, i.e. over which there exists a line bundle $L'$ such that $L \simeq L' \otimes_E k^s$. Then [26], Lemma 2.3 implies $V \otimes_k E \simeq N^{\oplus m}$. So we restrict to $X \otimes_k E$. On $X \otimes_k E$ we can find an ample line bundle $M$. Then $M' = \bigotimes g \in G \mathcal{O}^*M$ is a $G := \text{Gal}(E/k)$-equivariant ample line bundle on $X \otimes_k E$. Note that for a suitable $n > 0$, the line bundle $L' := N \otimes M^{\otimes n}$ is globally generated. From Section 4 we know $L' \in \text{Pic}(X(k)(\Sigma))(k)$. Since $M^{\otimes n}$ descents to a line bundle $\mathcal{R}$ on $X$, we conclude $(V \otimes \mathcal{R}) \otimes_k k^s \simeq L^{\oplus m}$. Now [26], Lemma 2.3 shows that if $W$ is another vector bundle satisfying $W \otimes_k E \simeq L^{\oplus m}$, then $V \otimes \mathcal{R} \simeq W$. Hence $\text{End}(W) \simeq \text{End}(V) \simeq A$. Theorem 4.1 provides us with a morphism $X \rightarrow \text{BS}(A)$.

$\square$

Theorem 6.3. Let $X$ be a smooth projective variety over a field $k$ with $H^0(X^s, O_{X^s}) = k^s$ and assume $D^h(X)$ admits a full weak exceptional collection of pure vector bundles. If $X(k) \neq \emptyset$, then $\text{rdim}(X) = 0$. In particular, assuming $\text{dim}(X) \geq 2$, if $X$ is $k$-rational, then $\text{rcodim}(X) \geq 2$.

Proof. Denote by $V_1, \ldots, V_r$ the full weak exceptional collection for $D^h(X)$. As all $V_i$ are pure, Proposition 6.2 provides us with morphisms $X \rightarrow \text{BS}(A_i)$, where $A_i = \text{End}(V_i)$. If $X$ admits a $k$-rational point, the Lang–Nishimura Theorem implies the existence of $k$-rational points on $\text{BS}(A_i)$. Therefore all $\text{BS}(A_i)$ are split. From this we conclude that $V_i = L^{\oplus m_i}$ for some line bundles $L_i \in \text{Pic}(X)$. But then $\{L_1, \ldots, L_r\}$ forms a full exceptional collection in $D^h(X)$. Finally, Proposition 6.1.6 in [10] yields $\text{rdim}(X) = 0$.

$\square$

Proposition 6.4. Let $X$ be a Brauer–Severi variety over a field $k$ of characteristic zero corresponding to $A$ and $S^l(X)$ its symmetric power. Denote by $X_l$ the generalized Brauer–Severi variety associated to $A$ and let $l < \text{deg}(A)$ be arbitrary. If $\text{dim}(X_l) = 0$, then $S^l(X)(k) \neq \emptyset$.

Proof. According to [29], Theorem 1.5 there is a birational map $X_i \times \mathbb{P}^{l(l-1)} \dashrightarrow S^l(X)$. If $\text{dim}(X_i) = 0$, then [29], Theorem 6.5 implies $X_i$ is a Grassmannian. Therefore $X_i(k) \neq \emptyset$. From the Lang–Nishimura Theorem we conclude $S^l(X)(k) \neq \emptyset$.

$\square$

Theorem 6.5. Let $X$ be a Brauer–Severi variety over $\mathbb{R}$ or a twisted quadric associated to a central simple $\mathbb{R}$-algebra $(A, \sigma)$ with involution of orthogonal type. Then $D^h_{\Sigma_n}(X^n)$ admits a full weak exceptional collection.

Proof. We start with $\text{dim}(X) = 1$. Note that in this case the twisted quadric is a Brauer–Severi curve over $\mathbb{R}$. Then $S^n(X)$ is smooth for any integer $n > 0$ and $S^n(X) \otimes_k k^s \simeq S^n(X \otimes_k k^s) \simeq S^n(\mathbb{R}^3) \simeq \mathbb{P}^n$. Therefore $S^n(X)$ is a Brauer–Severi variety which has a full weak exceptional collection over any field $k$ (see [32], Example 1.17). We first prove the assertion for $X$ being a non-split Brauer–Severi variety. We will see that the split case can be proved in the same way.
So let $X$ be a Brauer–Severi variety of dimension $2r - 1 \geq 2$ corresponding to the central simple algebra $M_r(\mathbb{H})$. Note that the period of $X$ is two. Therefore $O_X(2)$ exists in Pic($X$) (see [3]). From (32), Section 6 we know that there is an indecomposable vector bundle $\mathcal{V}_1$ on $X$ such that $\mathcal{V}_1 \otimes_k k^2 \simeq O_{X^r}(1)^{\otimes 2}$. If $X$ is split, $\mathcal{V}_1 \simeq O_{tX}(1)$. This vector bundle is unique up to isomorphism and $\text{End}(\mathcal{V}_1)$ is isomorphic to the central division algebra $D$ for which $M_r(D)$ corresponds to $X$. So if $X$ is split, $\text{End}(\mathcal{V}_1) \simeq \mathbb{R}$. Otherwise, we have $\text{End}(\mathcal{V}_1) \simeq \mathbb{H}$. Moreover, it is well-known that the collection\begin{equation}
abla = \{O_X, \mathcal{V}_1, O_X(2), \mathcal{V}_1 \otimes O_X(2), \ldots, O_X(2(r - 1)), \mathcal{V}_1 \otimes O_X(2(r - 1))\}
abla
\end{equation}
is a full weak exceptional collection in $D^b(X)$ (see [39], Example 1.17). For simplicity, let us denote the collection (1) by\begin{equation}\nabla_0 = \nabla \setminus \{\mathcal{V}_1\} = \{O_X, \mathcal{V}_2, \ldots, O_{X^r}(2(r - 1)), \mathcal{V}_2 \mathcal{V}_{2(r - 1)}, \mathcal{V}_{2(r - 1)}\}.
abla
\end{equation}
For every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1, \ldots, 2r - 1\}^n$ we write\begin{equation}\mathcal{V}(\alpha) = \mathcal{V}_{\alpha_1} \boxtimes \cdots \boxtimes \mathcal{V}_{\alpha_n}.
\end{equation}
This is a vector bundle on $X^n$ and it is easy to check that the collection consisting of the $\mathcal{V}(\alpha)$, ordered by the lexicographical order, generates $D^b(X^n)$. Now we want to reorder the sequence consisting of these $\mathcal{V}(\alpha)$. So for a multi-index $\alpha$, we follow [27], Section 4 and write for the unique non-decreasing representative of its $S_n$ orbit $\text{nd}(\alpha)$. Then one can define a total order $<_{\text{lex}}$ on $\{0, 1, \ldots, 2r - 1\}^n$ by\begin{equation}\alpha <_{\text{lex}} \beta \Leftrightarrow \begin{cases} \text{nd}(\alpha) = \text{nd}(\beta) \\ \text{nd}(\alpha) < \text{nd}(\beta) \end{cases} \text{ or } \alpha <_{\text{lex}} \beta \end{equation}
where $<_{\text{lex}}$ stands for the lexicographical order on $\{0, 1, \ldots, 2r - 1\}^n$. For details we refer to [27]. Now the group $S_n$ acts transitively on the blocks consisting of all $\mathcal{V}(\alpha)$ with fixed $\text{nd}(\alpha)$ because $\sigma^* \mathcal{V}(\alpha) \simeq \mathcal{V}(\sigma^{-1} \cdot \alpha)$. Furthermore, any $\mathcal{V}(\alpha)$ has a canonical $\text{Stab}(\alpha)$-linearization given by permutation of the factors in the box product. If $\alpha$ is a non-decreasing multi-index and $V_i^{(\alpha)}$ an irreducible representation of $H_n := \text{Stab}(\alpha)$, we can get a full weak exceptional collection out of the collection consisting of the vector bundles $\text{Inf}_{H_n}^{S_n}(\mathcal{V}(\alpha) \otimes V_i^{(\alpha)})$. To get this exceptional collection, we first consider $\text{End}_{S_n}(\text{Inf}_{H_n}^{S_n}(\mathcal{V}(\alpha) \otimes V_i^{(\alpha)}))$. In particular, the proof of Theorem 2.12 in [37] and the Künneth-formula show that there are isomorphisms\begin{equation}\text{End}_{S_n}(\text{Inf}_{H_n}^{S_n}(\mathcal{V}(\alpha) \otimes V_i^{(\alpha)})) \simeq \text{Hom}_{H_n}(\text{Res}_{H_n}^{S_n} \text{Inf}_{H_n}^{S_n}(\mathcal{V}(\alpha) \otimes V_i^{(\alpha)}), \mathcal{V}(\alpha) \otimes V_i^{(\alpha)}) \simeq \text{Hom}_{H_n}(\mathcal{V}(\alpha) \otimes V_i^{(\alpha)}, \mathcal{V}(\alpha) \otimes V_i^{(\alpha)}) \simeq (\text{End}(\mathcal{V}_{\alpha_1}) \otimes \cdots \otimes \text{End}(\mathcal{V}_{\alpha_n}) \otimes \text{End}(V_i^{(\alpha)}))^{H_n}.
\end{equation}
The case $n = 1$ is clear, since the Brauer–Severi variety $X$ admits a full weak exceptional collection (see [36], Example 1.17). For $n > 1$ we notice that the endomorphism ring of any irreducible representation of a finite group over $\mathbb{R}$ is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ according to Schur’s Lemma and a theorem of Frobenius. From the construction of the collection (1) we see that $\text{End}(\mathcal{V}_{\alpha_1}) \otimes \cdots \otimes \text{End}(\mathcal{V}_{\alpha_n})$ must be isomorphic to either $M_r(\mathbb{R})$ or $M_r(\mathbb{H})$ for suitable positive integers $s$ and $t$. Therefore, the finite-dimensional $\mathbb{R}$-algebra \begin{equation}\text{End}(\mathcal{V}_{\alpha_1}) \otimes \cdots \otimes \text{End}(\mathcal{V}_{\alpha_n}) \otimes \text{End}(V_i^{(\alpha)})\end{equation} must be isomorphic to $M_r(\mathbb{R})$, $M_r(\mathbb{C})$ or $M_r(\mathbb{H})$ for suitable positive integers $s', r'$ and $t'$. Since $S_n$, and hence $H_n$, acts on $\text{End}(\mathcal{V}_{\alpha_1}) \otimes \cdots \otimes \text{End}(\mathcal{V}_{\alpha_n}) \otimes \text{End}(V_i^{(\alpha)})$ by automorphism and any automorphism of a matrix-ring over a unique factorization domain is inner, [31], Corollary 2.13 implies that there are simple rings $A_1, \ldots, A_t(i, \alpha)$ such that \begin{equation}(\text{End}(\mathcal{V}_{\alpha_1}) \otimes \cdots \otimes \text{End}(\mathcal{V}_{\alpha_n}) \otimes \text{End}(V_i^{(\alpha)}))^{H_n} \simeq A_1 \times \cdots \times A_t(i, \alpha)\end{equation} Clearly, the rings $A_1, \ldots, A_t(i, \alpha)$ must be finite dimensional $\mathbb{R}$-algebras. Below we have to deal with positive integers $l(\alpha, i), m(\alpha, i), h(\alpha, i)$, depending on $\alpha$ and $i$. For a
better readability, we simply write $l$, $m$, and $h$. As $\text{Inf}^{S_n}_{H^a_n}(\mathcal{V}(\alpha) \otimes \mathcal{V}(\alpha))$ is a $S_n$-equivariant vector bundle, we apply the Krull–Schmidt Theorem to decompose it into a direct sum of indecomposables in the category of equivariant coherent sheaves $\text{Coh}_{S_n}(X^n)$. Let

$$\text{Inf}^{S_n}_{H^a_n}(\mathcal{V}(\alpha) \otimes \mathcal{V}(\alpha)) = T(\alpha, i)^{\oplus m_1}_1 \oplus \cdots \oplus T(\alpha, i)^{\oplus m_k}_h$$

be this decomposition. We have seen above that

$$\text{End}_{S_n}(T(\alpha, i)^{\oplus m_1}_1 \oplus \cdots \oplus T(\alpha, i)^{\oplus m_k}_h) \simeq A_1 \times \cdots \times A_l.$$ 

This implies $h = l$ and $\text{End}_{S_n}(T(\alpha, i)^{\oplus m_1}_1 \oplus \cdots \oplus T(\alpha, i)^{\oplus m_k}_h) \simeq A_1$. Combining a theorem of Frobenius with the Wedderburn Theorem, we obtain that $A_j$ is isomorphic to a matrix algebra over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We claim that for non-decreasing $\alpha$, the collection of blocks $\{T(\alpha, i)_1, ..., T(\alpha, i)_l\}$ forms a full weak exceptional collection. That the collection of blocks $\{T(\alpha, i)_1, ..., T(\alpha, i)_l\}$ generates $D^b_{S_n}(X^n)$ follows from the fact that the collection consisting of the vector bundles $\text{Inf}^{S_n}_{H^a_n}(\mathcal{V}(\alpha) \otimes \mathcal{V}(\alpha))$ generates $D^b_{S_n}(X^n)$. The argument for this fact is part of the proof of Theorem 2.12 in [17]. For convenience of the reader, we recall the argument.

Take any $\mathcal{F} \in D^b_{S_n}(X^n)$, $\mathcal{F} \neq 0$. As mentioned before, the collection consisting of the $\mathcal{V}(\alpha)$ generates $D^b(X^n)$. So for some $p$ and $\alpha$ we will have $\text{Hom}^p(\mathcal{F}, \mathcal{V}(\alpha)) \neq 0$. Hence $\text{RHom}(\mathcal{F}, \mathcal{V}(\alpha)) \neq 0$. Denote by $V$ the object $\text{RHom}(\mathcal{F}, \mathcal{V}(\alpha))^*$. Then because the functors $\text{Hom}(\mathcal{V}(\alpha)^*, \mathcal{V}(\alpha))$ and $F_{\mathcal{V}(\alpha)} := \mathcal{V}(\alpha) \otimes -$ are adjoint, we find

$$\text{Hom}_{S_n}(\mathcal{F}, \text{Inf}^{S_n}_{H^a_n}(\mathcal{V}(\alpha) \otimes V)) \simeq \text{Hom}_{S_n}(\mathcal{F}, \mathcal{V}(\alpha) \otimes V)$$

$$\simeq \text{Hom}_{S_n}(\text{RHom}(\mathcal{F}, \mathcal{V}(\alpha))^*, V)$$

$$\simeq \text{Hom}_{S_n}(V, V) \neq 0.$$

This proves that the collection of vector bundles $\text{Inf}^{S_n}_{H^a_n}(\mathcal{V}(\alpha) \otimes \mathcal{V}(\alpha))$, and therefore the collection of blocks $\{T(\alpha, i)_1, ..., T(\alpha, i)_l\}$, generates $D^b_{S_n}(X^n)$. So it remains to show that any $T(\alpha, i)_j$ is a weak exceptional object and the the collection of blocks $\{T(\alpha, i)_1, ..., T(\alpha, i)_l\}$ forms a weak exceptional collection. Note that $\text{End}_{S_n}(T(\alpha, i)_j)$ is a division algebra by construction.

The proof of Theorem 2.12 in [17] shows

$$\text{Ext}^d(\text{Inf}^{S_n}_{H^a_n}(\mathcal{V}(\alpha) \otimes \mathcal{V}(\alpha)), \text{Inf}^{S_n}_{H^a_n}(\mathcal{V}(\alpha) \otimes \mathcal{V}(\alpha))) = 0, \text{ for } d > 0.$$ 

Therefore $\text{Ext}^d(T(\alpha, i)_j, T(\alpha, i)_j) = 0$ for $d > 0$. This yields that $T(\alpha, i)_j$ is a weak exceptional object. From (2) we can also conclude that within a block $\{T(\alpha, i)_1, ..., T(\alpha, i)_l\}$, the following holds:

$$\text{Ext}^d(T(\alpha, i)_a, T(\alpha, i)_b) = 0, \text{ for } d > 0$$

if $a > b$. Moreover, the proof of Theorem 2.12 in [17] also shows

$$\text{Ext}^d(\text{Inf}^{S_n}_{H^a_n}(\mathcal{V}(\beta) \otimes \mathcal{V}(\alpha)), \text{Inf}^{S_n}_{H^a_n}(\mathcal{V}(\alpha) \otimes \mathcal{V}(\alpha))) = 0, \text{ for } d > 0$$

whenever $\alpha \prec \beta$. But this implies

$$\text{Ext}^d(T(\beta, j)_a, T(\alpha, i)_b) = 0, \text{ for } d > 0$$

whenever $\alpha \prec \beta$. This shows that the collection of blocks $\{T(\alpha, i)_1, ..., T(\alpha, i)_l\}$ forms a full weak exceptional collection.

If $X$ is split, i.e. isomorphic to $\mathbb{P}^m$, one can repeat the above argument with the collection

$$\{\mathcal{O}, \mathcal{O}(1), ..., \mathcal{O}(m - 1), \mathcal{O}(m)\}.$$ 

Denote this collection by $\{\mathcal{E}_0, ..., \mathcal{E}_m\}$ and consider multi-indices $\alpha \in \{0, 1, ..., m\}^n$. Again, we write

$$\mathcal{E}(\alpha) := \mathcal{E}_{\alpha_1} \boxtimes \cdots \boxtimes \mathcal{E}_{\alpha_n}.$$
Since the collection (3) is a full exceptional collection for $D^b(\mathbb{P}^n)$ and $(-)^{H_n}$ is exact, we have

$$\text{End}_{S_n}(\text{Inf}^n_{H_n}(\mathcal{E}(\alpha) \oplus V_i^{(\alpha)})) \simeq (\text{End}(V_i^{(\alpha)}))^{H_n} \simeq \text{End}_{H_n}(V_i^{(\alpha)}).$$

Note that $\text{End}_{H_n}(V_i^{(\alpha)})$ is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Now repeat the above arguments to conclude that the collection of vector bundles $\text{Inf}^n_{H_n}(\mathcal{E}(\alpha) \oplus V_i^{(\alpha)})$ forms a full weak exceptional collection for $D^b_{S_n}(\mathbb{C}^n)$. Now consider the case where $X$ is a twisted quadric associated to a central simple $\mathbb{R}$-algebra $(A, \sigma)$ with involution of orthogonal type. Let $n$ be the degree of $A$. According to [15], there is a semiorthogonal decomposition with exactly $n-1$ components

$$D^b_{\alpha}(\mathbb{C}) = (D^b(\mathbb{R}), D^b(A), ..., D^b(\mathbb{R}), D^b(A), D^b(C(A, \sigma))).$$

Note that this semiorthogonal decomposition is induced by vector bundles $\text{Inf}^n_{H_n}(\mathcal{E}(\alpha) \oplus V_i^{(\alpha)})$ forms a full weak exceptional collection for $D^b_{S_n}(\mathbb{P}^n)$ and $(-)^{H_n}$ is exact, we have

$$\text{End}_{S_n}(\text{Inf}^n_{H_n}(\mathcal{E}(\alpha) \oplus V_i^{(\alpha)})) \simeq (\text{End}(V_i^{(\alpha)}))^{H_n} \simeq \text{End}_{H_n}(V_i^{(\alpha)}).$$

Now the proof of Theorem 6.5 shows that the collection consisting of the vector bundles $\text{Inf}^n_{H_n}(\mathcal{E}(\alpha) \oplus V_i^{(\alpha)})$ forms a full weak exceptional collection and gives therefore rise to a semiorthogonal decomposition. Since $\text{rdim}(D^b(\mathbb{C})) = \text{rdim}(D^b(\mathbb{R})) = 0$ (see [10], Proposition 6.1.6), we finally obtain $\text{rdim}(\mathcal{T}) = 0$. For $X$ a twisted quadric associated to a central simple algebra $(A, \sigma)$ with involution of orthogonal type one can repeat the arguments from above. In this case one uses the full exceptional collection from Kapranov [22]. \[\square\]
Theorem 6.7. Let \( X \) be a Brauer–Severi variety over \( \mathbb{R} \) or a twisted quadric associated to a central simple \( \mathbb{R} \)-algebra \((A, \sigma)\) with involution of orthogonal type and \( 1 \leq n \leq 3 \). We set \( \mathcal{T} := D_{S_n}^b(X^n) \). Then the following hold:

(i) \( \text{rdim}(X) = 0 \) if and only if \( \text{rdim}(\mathcal{T}) = 0 \).

(ii) \( X(\mathbb{R}) \neq \emptyset \) if and only if \( \text{rdim}(\mathcal{T}) = 0 \).

Proof. For \( n = 1 \), (i) and (ii) is the content of \([34]\), Theorem 6.3 and Proposition 6.10. Now we assume \( \dim(X) = 2 \), we consider \( \mathcal{T} = D^b(X) \) and hence \( \text{rdim}(X) = \text{rdim}(\mathcal{T}) \). So we can consider \( 2 \leq n \leq 3 \). If \( \dim(X) = 1 \), \( S^n(X) \otimes_k k' \cong \mathbb{P}^n \) and therefore \( S^n(X) \) is a Brauer–Severi variety. Again, (i) and (ii) follows from \([34]\), Theorem 6.3. Now we assume \( \dim(X) > 1 \). We prove the statement only in the Brauer–Severi case and use the full weak exceptional collection (1). For \( n = 2 \), we consider \( \alpha = (1, 2) \) and see Stab(\(\alpha\)) = \{id\}. Analogously, for \( n = 3 \) we consider \( \alpha = (0, 1, 2) \) and observe Stab(\(\alpha\)) = \{id\}. So in both of these cases we have

\[
\text{End}_{S_n}(\text{Inf}_{\{id\}}^S(V(\alpha) \otimes V^{(\alpha)})) \simeq \text{End}(V_1).
\]

Recall, that \( \text{End}(V_1) \) is isomorphic to the central division algebra \( D \) for which \( M_r(D) \) corresponds to \( X \). Now we prove (i) for \( \dim(X) > 1 \) and \( 2 \leq n \leq 3 \). Assume \( \dim(X) = 0 \). Then Corollary 6.6 gives \( \text{rdim}(\mathcal{T}) = 0 \). On the other hand, if \( \dim(T) = 0 \), the derived category \( \mathcal{T} = D_{S_n}^b(X^n) \) must have a semiorthogonal decomposition of the form

\[
D_{S_n}^b(X^n) = \langle A_1, ..., A_e \rangle
\]

with \( A_i \cong D^b(\mathbb{R}, K_i) \) and \( K_i \) being étale \( \mathbb{R} \)-algebras (see \([10]\), Proposition 6.1.6). We remark that \( K_i \cong \mathbb{R}^{\times n_i} \times \mathbb{C}^{m_i} \). Now \([9]\), Lemma 1.17 implies

\[
D^b(\mathbb{R}, K_i) \simeq D^b(\mathbb{R}, \mathbb{R})^{\times n_i} \times D^b(\mathbb{R}, \mathbb{C})^{\times m_i}.
\]

Therefore we get a semiorthogonal decomposition given by

\[
D_{S_n}^b(X^n) = \langle g_{i}^{(1)}, ..., g_{i}^{(n_i)}, \tilde{f}_{i}^{(1)}, ..., \tilde{f}_{i}^{(m_i)} \rangle_{i=1, ..., e}
\]

where \( \text{End}_{S_n}(g_{i}^{(l)}) \cong \mathbb{R} \) and \( \text{End}_{S_n}(\tilde{f}_{i}^{(l)}) \cong \mathbb{C} \). Now Theorem 6.5 states that \( \mathcal{T} \) admits a full weak exceptional collection and its proof in particular shows that the endomorphism algebras of the weak exceptional vector bundles involved are isomorphic to \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \), considered as (simple) \( \mathbb{R} \)-algebras. Notice that one of the vector bundles occurring in the full weak exceptional collection is \( \text{Inf}_{\{id\}}^S(V(\alpha) \otimes V^{(\alpha)}) \) of (4). It is indecomposable, since its endomorphism algebra is a central division algebra over \( \mathbb{R} \).

Now let \( d \) be the number of vector bundles within the set of full weak exceptional collection with endomorphism algebra being isomorphic to \( \mathbb{C} \) and \( r \) the number of the remaining exceptional vector bundles. We denote the full weak exceptional collection given by Theorem 6.5 simply by

\[
D_{S_n}^b(X^n) = \langle \mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_{r+d} \rangle.
\]

The rank of the Grothendieck group \( K_0(\mathcal{T}) \) equals \( r + d \), i.e. \( K_0(\mathcal{T}) \cong \mathbb{Z}^{\mathbb{Z}^{r+d+1}} \). Note that \( r + d = \sum_{i=1} (n_i + m_i) \). For a \( S_n \)-equivariant object \( \mathcal{V} \in D_{S_n}^b(X^n) \) with \( \text{End}_{S_n}(\mathcal{V}) \cong \mathbb{C} \), considered as an \( \mathbb{R} \)-algebra, we obtain after base change \( \text{End}_{S_n}(\mathcal{V} \otimes_{\mathbb{R}} \mathbb{C}) \cong \text{End}_{S_n}(\mathcal{V} \otimes_{\mathbb{R}} \mathbb{C}) \) and hence

\[
\langle \text{End}_{S_n}(\mathcal{V} \otimes_{\mathbb{R}} \mathbb{C}) \rangle \simeq D^b(\mathbb{C}, \mathbb{C}) \times D^b(\mathbb{C}, \mathbb{C}).
\]

For \( \mathcal{E} \in D_{S_n}^b(X^n) \), we write \( \tilde{\mathcal{E}} := \mathcal{E} \otimes_{\mathbb{R}} \mathbb{C} \in D_{S_n}^b(X^n) \) for the equivariant object after scalar extension. We obtain semiorthogonal decompositions

\[
\mathcal{T}' := D_{S_n}^b(X^n) = \langle \tilde{g}_{i}^{(1)}, ..., \tilde{g}_{i}^{(n_i)}, \tilde{f}_{i}^{(1)}, ..., \tilde{f}_{i}^{(m_i)} \rangle_{i=1, ..., e}
\]

and

\[
\mathcal{T}' = D_{S_n}^b(X^n) = \langle \tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2, ..., \tilde{\mathcal{E}}_{r+d} \rangle.
\]
Now (7) and (9) tell us that after base change to \( \mathbb{C} \) the Grothendieck group \( K_0(T^\vee) \) has rank \( r + 2d \). The semiorthogonal decomposition (8) then implies \( r + 2d = \sum_{i=1}^e (n_i + 2m_i) \). Since \( r + d = \sum_{i=1}^e (n_i + m_i) \), we find \( d = \sum_{i=1}^e m_i \). Note that \( D^b_{\text{S}_{\mathcal{S}}}(X^n) \) admits a unique dg-enhancement, denoted by \( d_g D^b_{\text{S}_{\mathcal{S}}}(X^n) \). In fact this follows from the results in [18] and the well-known fact that \( D^b(X) \) admits a unique dg-enhancement. Alternatively see [11], since \([X^n/S_3]\) is a Deligne–Mumford stack and \( D^b([X^n/S_3]) \simeq D^b_{\text{S}}(X^n) \). As explained in (4), there is at least one bundle within the full weak exceptional collection in \( D^b_{\text{S}}(X^n) \) whose endomorphism algebra is isomorphic to the central simple \( \mathbb{R} \)-algebra corresponding to \( X \). The above semiorthogonal decompositions (5) and (6) show that the noncommutative motive \( U(d_g D^b_{\text{S}_{\mathcal{S}}}(X^n)) \) decomposes as

\[
\bigoplus_{j=1}^r U(A_j) \oplus U(\mathbb{C}) \oplus U(\mathbb{R}) \simeq U(d_g D^b_{\text{S}_{\mathcal{S}}}(X^n)) \simeq \bigoplus_{i=1}^e \left( U(\mathbb{R}) \oplus U(\mathbb{C}) \oplus U(\mathbb{R}) \right)^{\oplus m_i},
\]

with \( A_j \) being central simple \( \mathbb{R} \)-algebras. As mentioned before, there exists a \( j_0 \in \{1, ..., r\} \) such that \( A_{j_0} \) is the central simple algebra corresponding to \( X \). Since \( d = \sum_{i=1}^e m_i \), Proposition 5.2 implies

\[
\bigoplus_{j=1}^r U(A_j) \simeq \bigoplus_{i=1}^e U(\mathbb{R})^{\oplus m_i}.
\]

Then Theorem 5.1 yields that \( X \) is split, i.e. \( X \simeq \mathbb{R}^{\dim(X)} \). From the well-known fact that the projective space admits a full exceptional collection we conclude \( \dim(X) = 0 \).

Now we prove (ii). If \( X(\mathbb{R}) \neq \emptyset \), [34], Theorem 6.3 implies \( \dim(X) = 0 \). Now (i) gives \( \dim(T) = 0 \). Now, if \( \dim(T) = 0 \), we conclude from (i) and [34], Proposition 5.1 that \( X \) admits a full exceptional collection. But then [34], Theorem 6.3 implies \( \dim(X) \neq 0 \). In the case \( X \) is a twisted quadric associated to a central simple algebra \((A, \sigma)\) with involution of orthogonal type one can use the full weak exceptional collection from [18]. Then repeat the arguments from above to conclude that \( \dim(T) \) implies \( \dim(X) = 0 \). The other implication is the content of Corollary 6.6. This shows (i). To prove (ii) one can use [34], Theorem 5.6 and Proposition 6.10.

**Remark 6.8.** The proof of Theorem 6.7 in particular shows that the implication \( \dim(T) = 0 \Rightarrow \dim(X) = 0 \) holds for arbitrary positive integers \( n \). We believe that the other implication does not hold for arbitrary \( n \). The proofs of Theorem 6.5, Corollary 6.6 and Theorem 6.7 need \( \text{End}(V_{n_1}) \otimes \cdots \otimes \text{End}(V_{n_e}) \otimes \text{End}(V_1^{(a)}) \) to be isomorphic to matrix algebras over \( \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H} \). To make the proof work over arbitrary fields, we must ensure that any automorphism of \( \text{End}(V_{n_1}) \otimes \cdots \otimes \text{End}(V_{n_e}) \otimes \text{End}(V_1^{(a)}) \) is inner. We do not know if this is indeed true. We also do not know whether Theorem 6.5 for instance holds for twisted flags of type \( A_n \).

It is worth to mention that if \( X \) is split (and admits a full exceptional collection), \( D^b_{\text{S}}(X^n) \) cannot have a full exceptional collection. Indeed, as mentioned in the proof of Corollary 6.6 there is at least one bundle \( \text{Inf}^{S^2}_{\text{S}}(E(\alpha) \otimes V_1^{(a)}) \) within the full weak exceptional collection such that \( \text{End}_{\text{S}}(\text{Inf}^{S^2}_{\text{S}}(E(\alpha) \otimes V_1^{(a)})) \simeq \mathbb{C} \). The existence of a full exceptional collection in \( D^b_{\text{S}}(X^n) \) would give a decomposition of the noncommutative motive as

\[
\bigoplus_{j=1}^r U(\mathbb{R}) \oplus U(\mathbb{C}) \oplus U(\mathbb{R}) \simeq U(d_g D^b_{\text{S}}(X^n)) \simeq \bigoplus_{i=1}^e U(\mathbb{R}),
\]

which is impossible. This follows from [72], Corollary 2.13.

**Corollary 6.9.** Let \( X \) be a Brauer–Severi variety over \( \mathbb{R} \) corresponding to \( A \) and assume \( \text{deg}(A) > 3 \). Then \( S^3(X)(\mathbb{R}) \neq \emptyset \) if and only if \( \dim(T) = 0 \).
Proof. Let $X_n$ be the generalized Brauer–Severi variety associated to $A$. From [25], Theorem 1.5 we know that $S^3(X)$ is birational to $X_3 \times \mathbb{P}^n$. Now $X_n$ admits a $\mathbb{R}$-rational point if and only if $\text{ind}(A)$ divides $n$ (see [23], Proposition 1.17). Our assumption $S^3(X)(\mathbb{R}) \neq \emptyset$ therefore implies $\text{ind}(A) = 1$ and hence $X_3$ must be a Grassmannian over $\mathbb{R}$. One can show that this implies $X(\mathbb{R}) \neq \emptyset$ and hence $A$ must be split. But then $\text{rdim}(X) = 0$ and Theorem 6.7 gives $\text{rdim}(T) = 0$. On the other hand, $\text{rdim}(T) = 0$ implies $\text{rdim}(X) = 0$. From [34], Proposition 5.1 we conclude that $X$, and so $A$, must be split. Therefore $X_3$ is a Grassmannian and the Lang–Nishimura Theorem provides us with a $\mathbb{R}$-rational point on $S^3(X)$. □

Proposition 6.10. Let $C$ be a non-split Brauer–Severi curve over the field $\mathbb{R}$ and $2 \leq n \leq 3$. We set $T := D^b_{\text{ac}}(C^n)$. Then $\text{rdim}(T) = 1$.

Proof. This follows from Theorem 6.5 as the endomorphism algebras of the weak exceptional vector bundles involved are isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Since $\text{rdim}(D^b(\mathbb{R})) = \text{rdim}(D^b(\mathbb{C})) = 0$ according to [10], Proposition 6.1.6 and $\text{rdim}(D^b(\mathbb{H})) = 1$ according to [10], Proposition 6.1.10, we obtain $\text{rdim}(T) \leq 1$. Notice that $\text{rdim}(T) = 0$ would imply $\text{rdim}(C) = 0$ by Theorem 6.7 which is impossible for $C$ being non-split. □

Remark 6.11. The implication in Proposition 6.10 is not an equivalence. Take for instance a non-split Brauer–Severi curve $C$ over $\mathbb{R}$. Then $S^2(C)$ is birational to $\mathbb{P}^2$. Therefore, $S^2(C)(\mathbb{R}) \neq \emptyset$, whereas $\text{rdim}(D^b_{\text{ac}}(C^2)) = 1$.

We recall Orlov’s definition of a noncommutative $k$-scheme [36].

Definition 6.12. A noncommutative scheme over $k$ is a pretriangulated dg category $A$ of the form $\text{perf}(E)$ for some cohomologically bounded dg $k$-algebra $E$. A noncommutative $k$-scheme $A$ is called geometric if there is a smooth and projective $k$-scheme $X$ such that $A$ is an admissible subcategory of $\text{perf}(X)$.

For noncommutative schemes one can define noncommutative resolution of singularities. We recall the definition given in [14].

Definition 6.13. Let $A$ be a geometric noncommutative $k$-scheme. A noncommutative resolution of $A$ is a smooth noncommutative $k$-scheme $B$ with a functor $\Phi: A \to B$ inducing a fully faithful functor $H^0(A) \to H^0(B)$. If $A = \text{perf}(X)$ for some scheme $X$, we say that $B$ is a noncommutative resolution of $X$.

There is also another definition, called categorical resolution of singularities, which slightly differs from the definition given above and which was introduced by Lunts and Kuznetsov [25]. Recall the following facts on Drinfeld and Verdier quotients. If $A$ is an abelian category, we write $C_{\text{ac}}(A)$ for the full subcategory of $C(A)$ consisting of acyclic objects. The Verdier quotient $D(A) = [C(A)]/[C_{\text{ac}}(A)]$ is the derived category of $A$. According to [25], Theorem 3.8, one has $[C(A)]/[C_{\text{ac}}(A)] = [C(A)/C_{\text{ac}}(A)]$ and hence $D(A) \simeq [C(A)/C_{\text{ac}}(A)]$. For a separated noetherian scheme $X$ over an arbitrary field $k$, let $\text{Qcoh}(X)$ be the abelian category of quasi-coherent sheaves. Denote by $\text{com}(X)$ the category of unbounded complexes and by $\text{com}_{\text{ac}}(X)$ the subcategory of all acyclic complexes. Note that there are enough $h$-flat complexes in $\text{com}(X)$ for any separated quasi-compact scheme. Hence the quotient $h$-$\text{flat-com}(X)/h$-$\text{flat-com}_{\text{ac}}(X)$ is a dg enhancement of $D(\text{Qcoh}(X))$ (see [25]). Let us denote by $\text{Perf}_X$ the dg subcategory of $h$-$\text{flat-com}(X)/h$-$\text{flat-com}_{\text{ac}}(X)$ consisting of all perfect complexes. The next theorem shows that if $G$ is a finite group of automorphisms acting on a smooth projective $k$-variety $X$, the dg enhancement $\text{Perf}_{G}$-$X$ is a noncommutative resolution of the quotient variety $Y := X/G$. That $D^b_{G}(X)$ is a categorical resolution of singularities in the sense of [25] is proved in [1].
Theorem 6.14. Let $X$ be a smooth projective $k$-variety and $G \subset \text{Aut}(X)$ a finite group acting on $X$. Then $\text{Perf}_{G^*-}\text{-X}$ is a noncommutative resolution of the quotient variety $Y = X/G$.

Proof. Recall that for any morphism $f : X \to Y$ the componentwise pullback $f^*$ preserves $h$-flat complexes and $h$-flat acyclic complexes (see [29], Lemma 3.10). Thus we obtain a dg-functor

$$f^* : \text{h-flat-com}(Y)/\text{h-flat-com}_{ac}(Y) \to \text{h-flat-com}(X)/\text{h-flat-com}_{ac}(X),$$

inducing the derived pullback $Lf^*$ between the derived categories. The dg-functor also induces a dg-functor

$$f^* : \text{Perf}^*_Y \to \text{Perf}^*_X.$$

For $X$ a smooth projective $k$-variety, we have $[\text{Perf}^*_X] \simeq D^b(X)$. Now let $G$ be a finite group acting on $X$. For $A = \text{Qcoh}_G(X)$ we denote by $\text{Perf}_{G^*-}\text{-X}$ the subcategory of $C(A)/C_{ac}(A)$ consisting of equivariant perfect complexes. It is easy to see that $[\text{Perf}_{G^*-}\text{-X}] \simeq D^b_G(X)$. We see that $\text{Perf}_{G^*-}\text{-X}$ is a dg enhancement of $D^b_G(X)$. Note that $d_\eta D^b_G(X) \simeq \mathcal{P} \text{erf}_{G^*-}\text{-X}$. The quotient map $\pi : X \to X/G$ induces a dg-functor

$$\pi^* : \text{Perf}^*_X/G \to \text{Perf}^*_G.X.$$

Therefore, we obtain the derived functor

$$L\pi^* : \text{perf}(X/G) \to D^b_G(X) = \text{perf}_G^*_X.$$

From [39], we conclude that $\text{Perf}_{G^*-}\text{-X}$ is smooth and geometric. Denote by $\pi : X \to X/G$ the quotient map. Then there is a functor

$$\pi^G_\pi : D_G^b(X) \to D(Y),$$

where $Y \mapsto \mathcal{F}$ is the functor of $G$-invariants. The left adjoint to $\pi^G_\pi$ is $\mathbb{L}\pi^*$, where the $G$-action on the object $\mathbb{L}\pi^*\mathcal{F}$ in $D(X)$ is the trivial one. From [38], we have $\pi^G_\pi \mathcal{O}_Y = \mathcal{O}_Y$ and projection formula implies

$$\pi^G_\pi \mathbb{L}\pi^* \simeq \text{id}.$$

Again by [38], $\omega_X = \pi^*\omega_Y$. Thus by Grothendieck duality, we find

$$\text{Hom}_{D^b(Y)}(\pi^*\mathcal{F}, \mathcal{G}) = \text{Hom}_{D^b(X)}(\mathcal{F}, L\pi^*\mathcal{G}).$$

for all $\mathcal{F} \in D^b(X)$ and all $\mathcal{G} \in \text{perf}(Y)$. As a consequence, for any $\mathcal{F} \in D^b(X)$, we have

$$\pi_* \mathbb{R}\text{Hom}(\mathcal{F}, \mathcal{O}_X) = \mathbb{R}\text{Hom}(\pi_*\mathcal{F}, \mathcal{O}_Y).$$

If $\mathcal{F}$ is assumed to be $G$-equivariant, we furthermore have

$$\mathbb{R}\text{Hom}(\pi_*\mathcal{F}, \mathcal{O}_Y)_G^X = \mathbb{R}\text{Hom}(\pi^G_\pi\mathcal{F}, \mathcal{O}_Y).$$

From the above isomorphisms, we finally obtain (see [1], p.684):

$$\text{Hom}_{D^b(Y)}(\pi^G_\pi\mathcal{F}, \mathcal{G}) \simeq \text{Hom}_{D^b_G(X)}(\mathcal{F}, L\pi^*\mathcal{G}).$$

Hence $L\pi^*$ is right adjoint to $\pi^G_\pi$ and well defined on $\text{perf}(Y)$. And since $\pi$ is proper and finite and $Y$ is locally noetherian, we conclude $\pi^G_\pi D^b_G(X) \subset \text{perf}(Y)$. Moreover, $\text{perf}(Y) \subset D^b(Y)$ is a full subcategory. Therefore,

$$\text{Hom}_{\text{perf}(Y)}(\pi^G_\pi\mathcal{F}, \mathcal{G}) \simeq \text{Hom}_{D^b_G(X)}(\mathcal{F}, L\pi^*\mathcal{G}).$$

Note that this implies that $L\pi_*$ is fully faithfull, showing that $\text{Perf}_{G^*-}\text{-X}$ is a noncommutative resolution of $Y$. }

Corollary 6.15. Let $X$ be a Brauer–Severi variety over $\mathbb{R}$ corresponding to $A$ and assume $\text{deg}(A) > 3$. If $S^3(X)$ is $\mathbb{R}$-rational, we have $\text{recdim}(S^3(X)) \geq 2$. 


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