On the ‘Section Conjecture’ in anabelian geometry*

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Dedicated to Alexander Prestel on the occasion of his 60th birthday

Abstract

Let $X$ be a smooth projective curve of genus $> 1$ over a field $K$ with function field $K(X)$, let $\pi_1(X)$ be the arithmetic fundamental group of $X$ over $K$ and let $G_F$ denote the absolute Galois group of a field $F$. The section conjecture in Grothendieck’s anabelian geometry says that the sections of the canonical projection $\pi_1(X) \rightarrow G_K$ are (up to conjugation) in one-to-one correspondence with the $K$-rational points of $X$, if $K$ is finitely generated over $\mathbb{Q}$. The birational variant conjectures a similar correspondence w.r.t. the sections of the projection $G_{K(X)} \rightarrow G_K$.

So far these conjectures were a complete mystery except for the obvious results over separably closed fields and some non-trivial results due to Sullivan and Huisman over the reals. The present paper proves — via model theory — the birational section conjecture for all local fields of characteristic 0 (except $\mathbb{C}$), disproves both conjectures e.g. for the fields of all real or $p$-adic algebraic numbers, and gives a purely group theoretic characterization of the sections induced by $K$-rational points of $X$ in the birational setting over almost arbitrary fields.

As a biproduct we obtain Galois theoretic criteria for radical solvability of polynomial equations in more than one variable, and for a field to be PAC, to be large, or to be Hilbertian.

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1 Introduction

1.1 The arithmetic fundamental group

Let $X$ be a smooth (always absolutely irreducible, i.e. geometrically connected) curve over a field $K$ with function field $K(X)$ of genus $g_X$. For any field extension $F/K$, write $X_F := X \otimes_K F$ for the curve considered over $F$ and denote the set of $F$-rational points of $X$ by $X(F)$. Fix an algebraic closure $\overline{K}$ of $K$ and the separable closure $K^{sep}$ and the perfect hull $K^{perf}$ of $K$ in $\overline{K}$. Let $\tilde{X}$ be the smooth completion of $X$. Then $\tilde{X}(\overline{K}) \setminus X(\overline{K})$ is a finite
set, say of cardinality \( n \). Let \( \widetilde{K}(X)^X \) denote the maximal Galois extension of \( K(X) \) which is unramified over \( X \). Then the **arithmetic fundamental group** \( \pi_1(X) \) of \( X/K \) is defined as the Galois group of \( \widetilde{K}(X)^X/K(X) \). Denoting the absolute Galois group of a field \( F \) by \( G_F = Gal(F^{sep}/F) \), one obtains the following canonical exact sequences (with commuting squares):

\[
1 \longrightarrow G_{K^{sep}(X)} \longrightarrow G_{K(X)} \xrightarrow{pr_{X/K}} G_K \longrightarrow 1
\]

\[
1 \longrightarrow \pi_1(X_{K^{sep}}) \longrightarrow \pi_1(X) \xrightarrow{pr_{X/K}} G_K \longrightarrow 1
\]

Passing from \( K \) to \( K^{perf} \) leaves the diagram unchanged (\( G_{K^{perf}} \cong G_K \) etc.). \( \pi_1(X_K^{perf}) \) (\( \cong \pi_1(X_{K^{sep}}) \)), the arithmetic fundamental group of \( X \) over \( K \), is called **geometric fundamental group**. If \( char K = 0 \), \( X \) may be regarded as curve over \( \mathbb{C} \) and then the geometric fundamental group is the profinite completion of the ‘algebraic fundamental group’ of \( X_{\mathbb{C}} \), i.e. the usual fundamental group of the corresponding \( n \)-fold punctured Riemann surface which is generated by elements \( \alpha_1, \beta_1, \ldots, \alpha_{gX}, \beta_{gX}, \gamma_1, \ldots, \gamma_{nX} \) subject to the single relation

\[
\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_{gX} \beta_{gX} \alpha_{gX}^{-1} \beta_{gX}^{-1} \gamma_1 \cdots \gamma_{nX} = 1.
\]

So if \( n_X \geq 1 \), then \( \pi_1(X_K^{perf}) \) is a free profinite group in \( 2g_X + n_X - 1 \) generators.

If \( char K > 0 \) and \( n_X \geq 1 \), then \( \pi_1(X_K^{perf}) \) is no longer free, but still projective ([MM], V. Theorem 5.3).

### 1.2 Grothendieck’s anabelian geometry

The fundamental conjecture of Grothendieck’s program called ‘anabelian geometry’ says in its simplest form that a smooth hyperbolic curve \( X \) over a finitely generated extension \( K \) of \( \mathbb{Q} \) is up to \( K \)-isomorphism determined by its fundamental group, or, more precisely, by the projection \( pr_{X/K}^1 \) (recall that \( X \) is **hyperbolic** if, in the notation above, \( \chi(X) := 2 - 2g_X - n_X < 0 \), i.e. if \( \pi_1(X) \) is non-abelian: hence the term ‘anabelian’). This conjecture was proved in a series of papers by Nakamura ([Na1]), Tamagawa ([Ta]) and Mochizuki ([Mo]), and generalised in many ways, for example in the following three respects: the constant field \( K \) may be any **sub-\( p \)-adic field**, i.e. any subfield of a finitely generated extension of \( \mathbb{Q}_p \), the fundamental group
may be replaced by the quotient obtained by passing to the maximal pro-$l$-quotient of the kernel of $pr_1^{X/K}$, and the isomorphism version is replaced by a much more general ‘Hom’-version. For an excellent survey see [MNT].

The birational version that $X$ be encoded in $G_{K(X)}$ up to $K$-birational equivalence was proved by Pop for arbitrary curves $X$ over finitely generated extensions of $\mathbb{Q}$ ([Po2]), where one should note that the result is ‘absolute’ in the sense that it suffices to look at $G_{K(X)}$ and get $pr_{X/K}: G_{K(X)} \to G_K$ for free.

In all these approaches it is clear that the Galois theoretic data encode, in particular, $K$-rational points. The section conjecture says that the group theoretic code for $K$-rational points of $X$ is as simple as possible: The $K$-rational points, according to the conjecture, should be in $1$-$1$ correspondence with the conjugacy classes of sections of $pr_1^{X/K}$ resp. certain canonical families of conjugacy classes of sections of $pr_{X/K}$. We will explain this now more precisely.

### 1.3 The section property

If $X$ is a smooth curve over a perfect field $K$, then $K(X)^X$ is the intersection of all inertia subfields of $K(X)^{sep}/K(X)$ w.r.t. all (valuations on $K(X)$ corresponding to the) points $P \in X(K)$. Hence, the inertia subgroups of $\pi_1(X)$ w.r.t. points in $X(K)$ are trivial and so any $K$-rational point $P \in X(K)$ canonically induces a conjugacy class $[s_P]$ of sections $s_P$ of $pr_1^{X/K}$, where the image $s_P(G_K)$ is a decomposition subgroup of $\pi_1(X)$ w.r.t. $P$.

If $X$ is a smooth curve over an arbitrary field $K$, this also holds for any $P \in X(K^{perf})$ via the canonical restriction isomorphisms:

$$
\begin{align*}
\pi_1(X^{K^{perf}}) & \xrightarrow{pr_1^{X/K^{perf}}} G_{K^{perf}} \\
\downarrow \cong & \\
\pi_1(X_K) & \xrightarrow{pr_1^{X/K}} G_K
\end{align*}
$$

**Definition 1.1** For a smooth curve $X$ over a field $K$ we define the section property

$$
\text{SP}(X/K) : \sigma_{X/K} : X(K^{perf}) \to \{\text{non-branch sections of } pr_1^{X/K}\}/\text{conj.}
$$

such that $P \mapsto [s_P]$ is bijective.
where a section is called non-branch section, if the image is not in a decomposition subgroup of $\pi_1(X)$ w.r.t. a branch point, i.e. a point in $\tilde{X} \setminus X$.

The section property has been conjectured by Grothendieck for complete curves of genus $> 1$ over fields which are finitely generated over $\mathbb{Q}$ ([G], p.5).

**Remark 1.2**

1. If $K$ is separably closed then for any smooth complete curve $X/K$, $\sigma_{X/K}$ is obviously surjective, but not injective. Due to a result of Sullivan ([Su]) and Huisman ([Hu]), the same holds for $K = \mathbb{R}$: the sections of $pr^1_{X/K}$ correspond to the connected components of $X(\mathbb{R})$.

2. By Theorem 19.1 of [Mo], $\sigma_{X/K}$ is injective for any smooth complete curve $X$ of genus $> 1$ over a sub-$p$-adic field $K$ (i.e. a subfield $K$ of a finitely generated extension of $\mathbb{Q}_p$).

3. Any branch point $P \in \tilde{X}(K^{perf}) \setminus X(K^{perf})$ also induces sections of $pr^1_{X/K}$ induced by the corresponding sections of $pr_{X/K}$ which will be described in the next section.

**1.4 The birational section property**

For the birational version of the section property we may as well assume that the curves to be considered are complete. A point $P \in X(K^{perf})$ of a smooth complete curve $X$ over a field $K$ also induces sections $s$ of $pr_{X/K}$, where the image $s(G_K)$ is a complement of the inertia subgroup of a decomposition subgroup of $G_{K(X)}$ w.r.t. $P$. Such complements always exist ([KPR]), but they need not all be conjugate. We call such sections induced by points in $X(K^{perf})$ geometric.

**Observation 1.3** Let $X$ be a smooth complete curve over a field $K$ and let $s$ be a section of $pr_{X/K}$. Then $s$ is geometric iff $s(G_K)$ is contained in a decomposition subgroup of $G_{K(X)}$ w.r.t. some $P \in X(K)$. 

**Proof:** If $s$ is geometric then, by definition, $s(G_K)$ is contained in a decomposition subgroup of $G_{K(X)}$ w.r.t. some $P \in X(K^{perf}) \subseteq X(\overline{K})$.

For the converse, assume $s(G_K) \subseteq D_P$ for some decomposition subgroup $D_P$ of $G_{K(X)}$ w.r.t. some (place $\mathfrak{p}$ of $K(X)_{sep}$ above the place of $K(X)$ corresponding to) $P \in X(K)$. Write $F$ for the fixed field of $s(G_K) = G_F$. 

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Since $s$ is a section of $pr_{X/K} : G_{K(X)} \to G_K$, $K$ is relatively algebraically closed in $F$, and so, in particular, in the fixed field $F_P$ of $D_P$. As $P \in X(\overline{K})$ and as $F_P$ is henselian (w.r.t. $\mathfrak{p}$) this means $P \in X(K_{\text{perf}})$. The fixed field of the inertia subgroup of $D_P$ is then $K_{\text{sep}}F_P$, and $s(G_K) = G_F$ is a complement of $G_{K_{\text{sep}}F_P}$ in $D_P = G_{F_P}$. $F \cap K_{\text{sep}}F_P = F_P$ and $F(K_{\text{sep}}F_P) = F_{\text{sep}}P = F_{\text{sep}}$.\[\square\]

Note that, if $K$ is not separably closed, the pendant to the injectivity of $\sigma_{X/K}$ is always given in the birational setting: any conjugates of two decomposition subgroups of $G_{K(X)}$ corresponding to distinct points have trivial intersection.

**Definition 1.4** For a smooth complete curve $X$ over a field $K$ we define the birational section property

$$\text{BSP}(X/K) : \text{all sections of } pr_{X/K} \text{ are geometric.}$$

**1.5 The weak (birational) section property**

Note that the section property and the birational section property imply that sections can only exist when there are $K_{\text{perf}}$-rational points. We also name this weaker property:

**Definition 1.5** Let $X$ be a smooth resp. a smooth complete curve over a field $K$. Then the weak section property resp. the weak birational section property are:

$$\text{sp}(X/K) : \exists \text{ non-branch sections of } pr_{X/K}^1 \iff X(K_{\text{perf}}) \neq \emptyset$$

$$\text{bsp}(X/K) : \exists \text{ sections of } pr_{X/K} \iff X(K_{\text{perf}}) \neq \emptyset$$

**Remark 1.6** 1. Since sections of $pr_{X/K}$ induce sections of $pr_{X/K}^1$, $\text{sp}(X/K)$ always implies $\text{bsp}(X/K)$ for a smooth complete curve $X/K$.

However, in general, for fixed $X/K$, no implications can be made between $\text{SP}(X/K)$ and $\text{BSP}(X/K)$, since the projection $G_{K(X)} \to \pi_1(X)$ may, in general, not split.

Yet, since $pr_{X/K} = \lim_{\to} pr_{X'/K}^1$, where the inverse limit is taken over all Zariski open $X' \subseteq X$, $\text{SP}(X'/K)$ for all smooth curves $X'$ over $K$ implies $\text{BSP}(X/K)$ for all smooth complete curves $X$ over $K$. 

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2. \( \text{bsp}(X/K) \) holds for any smooth complete curve \( X \) of genus \( 0 \) over any field \( K \) of characteristic \( \neq 2 \), because any section \( s \) of \( \text{pr}_X^1 \) induces an embedding of the 2-torsion part of Brauer groups \( Br_2(K) \hookrightarrow Br_2(K(X)) \) (and, more general, an embedding of the corresponding cohomology groups \( H^n(G_K) \hookrightarrow H^n(G_K(X)) \) with coefficients in \( \mu(K) \) for any \( n: \text{pr}_X^1 \circ s = \text{id}_{G_K} \)), and because the element in \( Br_2(K) \) corresponding to a conic over \( K \) is nontrivial iff \( X(K) = \emptyset \).

We do not know what happens for fields of characteristic 2.

3. Non-existence of sections of \( \text{pr}_X^1 \) or of \( \text{pr} \) is an obstruction to the existence of points in \( X(K^\text{perf}) \). In [CS], Section 2.2, another such obstruction is studied, the so-called elementary obstruction saying that the canonical exact sequence of \( G_K \)-modules

\[
1 \to \mathbb{K}^\times \to \mathbb{K}(X)^\times \to \mathbb{K}(X)^\times / \mathbb{K}^\times \to 1
\]

does not split. For curves of genus 0, or, more generally, for Severi-Brauer varieties and for principal homogenous spaces of tori, this is, again, the only obstruction ([CS], Example 2.2.11). This elementary obstruction is closely related to the abelianization of our obstruction, i.e. to the exact sequence

\[
1 \to \pi_1(X)^{\text{ab}} \to \pi_1(X)^{\text{ab}} \to G_K \to 1
\]

being non-split, where \( \pi_1(X)^{\text{ab}} \) is the maximal abelian quotient of \( \pi_1(X) \) and \( \pi_1(X)^{\text{ab}} \) is the corresponding extension of \( G_K \) by \( \pi_1(X)^{\text{ab}} \). For details cf. [HS], Section 3.4.

4. If \( X/K \) is a smooth complete curve with \( X(K^\text{perf}) = \emptyset \) and if \( G_K \) is projective then both \( \text{sp}(X/K) \) and \( \text{bsp}(X/K) \) do not hold, as any epimorphism onto a projective group splits. As an example you may take \( K = \mathbb{C}(t) \) or \( K = \mathbb{C}((t)) \) and consider the curve defined by \( X^3 + tY^3 = t^2Z^3 \).

The following observation is implicit in [Ta], Prop. 2.8:

**Lemma 1.7** Let \( K \) be a finite extension of \( \mathbb{Q}_p \) or a field finitely generated over \( \mathbb{Q} \) or a finite field. Then:

\[
[\forall X/K \ (B)\text{SP}(X/K)] \iff [\forall X/K \ (b)\text{sp}(X/K)].
\]
where the quantification is over all smooth (complete) curves $X/K$. For $K = \mathbb{R}$ the same equivalence holds, but only in the birational version.

**Proof:** Let $X/K$ be a smooth complete curve, let $s$ be a section of $pr^1_{X/K}$ resp. $pr_{X/K}$, and let $F$ be the fixed field of $s(G_K)$. Assuming the right hand side we have to show that $s$ comes from a point in $X(K)$.

$F/K$ is a regular extension. So to any finite subextension $F'$ of $F/K(X)$ we may choose a smooth complete curve $X'$ over $K$ with function field $F'$. Denote the collection of these curves by $\mathcal{X}$. Then for each $X' \in \mathcal{X}$, $s$ is also a section of $pr^1_{X'/K}$ resp. $pr_{X'/K}$ and hence, by assumption, $X'(K) \neq \emptyset$. For $F' \subseteq F''$ there is a canonical projection $X'' \rightarrow X'$ inducing a projection $X''(K) \rightarrow X'(K)$.

If $K$ is a finite extension of $\mathbb{Q}_p$ or if $K = \mathbb{R}$ then $X'(K)$ is compact for each $X' \in \mathcal{X}$. And if $K$ is finite or finitely generated over $\mathbb{Q}$ then, by Faltings, $X'(K)$ is finite, provided $g_{X'} > 1$, which holds for sufficiently large $F'$. Thus, $\lim \rightarrow X'(K) \neq \emptyset$, i.e. there is a $K$-rational place of $F/K$.\bbox

We expect that the Lemma is also true over any non-large field.

### 2 \textit{p}-adically closed fields

Let us recall that a field $K$ is called **\textit{p}-adically closed** ($p$ a prime) if $\text{char } K = 0$ and if $K$ admits a valuation $v$ (with valuation ring $\mathcal{O}_v$) such that for some integer $n > 1$, $(K, v)$ is algebraically maximal with the property that $\sharp(\mathcal{O}_v/p\mathcal{O}_v) = n$. The following fact is well known (see [PR] and [Ko1]):

**Fact 2.1** For a field $K$ the following conditions are equivalent:

1. $K$ is \textit{p}-adically closed.

2. There is a (possibly trivial) henselian valuation with divisible value group on $K$ such that the residue field is a relatively algebraically closed subfield of some finite extension of $\mathbb{Q}_p$.

3. $K$ is elementarily equivalent (in the language of fields) to some finite extension of $\mathbb{Q}_p$.

4. $G_K \cong G_F$ for some finite extension $F$ of $\mathbb{Q}_p$. 

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We shall also use the following consequence of the Galois characterization [Ko1] of $p$-adic fields (which, in fact, already follows from the relative characterization in [Po1]):

**Fact 2.2** Let $K$ be a $p$-adically closed field, and let $F/K$ be a field extension. If $\text{res} : G_F \to G_K$ is an isomorphism, then $K$ is an elementary substructure of $F$.

### 2.1 The section property for $p$-adic fields

Regarding the section property for $p$-adically closed fields we can, at the moment, only prove the following proposition. We conjecture, however, that the section property $\text{SP}(X/K)$ holds for all smooth complete curves of genus $> 1$ over any local $p$-adically closed field $K$ (i.e. any finite extension $K$ of $\mathbb{Q}_p$). Our proposition implies, conversely, that a $p$-adically closed field $K$ over which the section property holds for all such $X/K$ must be local:

**Proposition 2.3** Let $K$ be a $p$-adically closed field. Then:

1. $K$ is sub-$p$-adic iff $\sigma_{X/K}$ is injective for all smooth complete curves $X/K$ of genus $> 1$.

2. If $K$ is a proper relatively algebraically closed subfield of a finite extension of $\mathbb{Q}_p$ then $\sigma_{X/K}$ is not surjective for any smooth complete curve $X/K$ with $X(K) \neq \emptyset$.

**Proof:** 1. The direction ‘$\Rightarrow$’ is Mochizuki’s Theorem 19.1 [Mo] already mentioned in the introduction. For the other direction, assume $K$ is not sub-$p$-adic. Then the henselian valuation $w$ from Fact 2.1.2. is non-trivial. Let $K^{\text{alg}} = K \cap \overline{\mathbb{Q}}$ be the algebraic part of $K$ and observe that $\text{res} : G_K \to G_{K^{\text{alg}}}$ is an isomorphism and that $K^{\text{alg}}$ is also a relatively algebraically closed subfield of the residue field of $w$.

   Now let $X$ be a smooth complete curve over $K^{\text{alg}}$ with $X(K^{\text{alg}}) \neq \emptyset$, say $P \in X(K^{\text{alg}})$, consider $P$ as point of $X$ over the residue field of $w$ and lift $P$ in two different ways to points $P_1 \neq P_2 \in X(K)$ (via the place corresponding
to $w$). Then $P_1$ and $P_2$ induce the same section of $pr_{X/K}^1$:

\[
\begin{array}{ccc}
\pi_1(X_K) & \overset{pr_{X/K}^1}{\longrightarrow} & G_K \\
\downarrow \cong & & \downarrow \cong \\
\pi_1(X_{Kalg}) & \overset{pr_{X/Kalg}^1}{\longrightarrow} & G_{Kalg}
\end{array}
\]

2. If $K$ is a proper relatively algebraically closed subfield of a finite extension $F$ of $\mathbb{Q}_p$, and if $X$ is a curve over $\overline{K}$ with $X(K) \neq \emptyset$, then there are points in $X(F) \setminus X(\overline{K})$. By Mochizuki’s injectivity result, the sections of $pr_{X/K}^1 = pr_{F/K}^1$ induced by such points do not come from $K$-rational points. □

2.2 The birational section property for $p$-adic and for real closed fields

For the birational section property we can give the complete picture over $p$-adically closed (and real closed) fields:

**Proposition 2.4** Let $K$ be a $p$-adically closed or real closed field. Then:

1. $\text{bsp}(X/K)$ holds for all smooth complete curves $X/K$.

2. $\text{BSP}(X/K)$ holds for all smooth complete curves $X/K$ iff $K$ is a local field, i.e. $K$ is a finite extension of $\mathbb{Q}_p$ or $K = \mathbb{R}$.

**Proof:** 1. If $X$ is a smooth complete curve over $K$ and $s$ is a section of $pr_{X/K}^1$ : $G_{K(X)} \rightarrow G_K$, then $s$ is an isomorphism of $G_K$ onto $G_F$ for some algebraic extension $F$ of $K(X)$ in $\overline{K}(X)$. Hence, by Fact 2.2, $F$ is an elementary extension of $K$ ($s^{-1} = \text{res} : G_F \rightarrow G_K$). Since the $K$-curve $X$ has an $F$-rational point ($K(X) \subseteq F$), it, therefore, also has a $K$-rational point.

2. ‘$\Leftarrow$’ follows immediately from 1. and Lemma 1.7. For ‘$\Rightarrow$’, assume $K$ is not a local field and choose a smooth complete curve $X$ over $K^{alg}$ with $X(K^{alg}) \neq \emptyset$.

We distinguish two cases.

**Case 1:** $K$ is a proper relatively algebraically closed subfield of a local field. Then one can prolong the $p$-adic valuation resp. the ordering on $K$ to $K(X)$ in such a way that it remains a rank-1-valuation resp. an archimedean.
ordering. Thus, the corresponding $p$-adic resp. real closure $F$ of $K(X)$ in $\hat{K}(X)$ induces a non-geometric section of $pr_{X/K}$: $F$ cannot have a non-trivial henselian valuation which is trivial on $K$.

**Case 2:** If $K$ is not a subfield of a local field then $K$ admits a henselian valuation $w$ with non-trivial divisible value group $\Gamma_w$ and residue field of characteristic 0 (cf. Fact 2.1.2). Since $X(K^{\text{alg}}) \neq \emptyset$, $w$ can be prolonged to a valuation $u$ of $K(X)$ in such a way that $\Gamma_w$ is cofinal in $\Gamma_u$. The fixed field $F$ of a complement of the inertia subgroup of the decomposition subgroup of $G_{K(X)}$ w.r.t. $u$ then induces a non-geometric section of $pr_{X/K}$, for the same reason as in case 1. $\blacksquare$

**Remark 2.5** It is clear from the proof of the weak birational section property $\text{bsp}(X/K)$ for $p$-adically or real closed $K$ that this generalizes to smooth complete varieties $X$ over $K$ of arbitrary dimension.

However, the strong (birational) section property has no analogue in higher dimension: while for $\dim X = 1$ any valuation on $K(X)$ which is trivial on $K$ is geometric, there may be many non-geometric valuations on $K(X)$ which are trivial on $K$ and have residue field $K$ (e.g. with archimedean value group of rational rank $= \dim X$) if $\dim X > 1$. Such valuations induce non-geometric sections of the analogous $pr_{X/K}$.

### 2.3 Applications to number fields

#### 2.3.1 Global sections give local points

The following corollary is immediate from Proposition 2.4:

**Corollary 2.6** Let $X$ be a smooth complete curve over a number field $K$ and assume that $pr_{X/K}$ has a section. Then $X(\hat{K}) \neq \emptyset$ for all completions $\hat{K}$ of $K$.

**Proof:** Any section of $pr_{X/K}$ induces a section of $pr_{X/\hat{K}^{\text{alg}}}$. Hence, by Proposition 2.4.1, $X(\hat{K}^{\text{alg}}) \neq \emptyset$, and so $X(\hat{K}) \neq \emptyset$. $\square$

Recall that a field is called **pseudo $p$-adically** or **pseudo real closed** if it satisfies a local-global-principle for rational points on varieties w.r.t. all $p$-adic resp. all real closures.

**Corollary 2.7** If $K$ is a pseudo $p$-adically or a pseudo real closed field then $\text{bsp}(X/K)$ holds for any curve $X$ over $K$. 

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2.3.2 The local-global-principle for mere covers

In [D], Dèbes proved a local-global-principle for Galois-covers of curves defined over number fields. For so-called mere covers such a principle is expected to fail in general, yet no counterexamples could be found so far. We will show that the birational section property for all curves over number fields implies the existence of such counterexamples.

**Definition 2.8** Let $X$ be a smooth complete curve defined over a number field $K$. Then the **local-global-principle for mere covers** of $X$ is the following property:

\[ \text{LGP}(X/K) : \text{any } \overline{K}\text{-cover } Y \rightarrow X \text{ which is definable over all local closures } \hat{K} \text{ of } K \text{ (i.e. there is a } \hat{K}\text{-cover } Y_{\hat{K}} \rightarrow X \text{ with } Y_{\hat{K}} = Y_{\hat{K}} \otimes \overline{K} \text{ for any local closure } \hat{K} \text{ of } K) \text{.} \]

Here ‘local closure’ means a henselisation or a real closure of $K$.

**Proposition 2.9** Let $K$ be a number field. If the birational section conjecture $\text{BSP}(X/K)$ holds for any smooth complete curve $X/K$, then there is a counterexample to the local-global-principle for mere covers over $K$.

**Proof:** We prove the contraposition of the implication in the proposition, and assume $\text{LGP}(X/K)$ holds for all smooth complete curves $X$ over $K$. We have to find some $X_0/K$ with a non-geometric section for $\text{pr}_{X_0/K}$. The trick is to choose $X_0$ to be a smooth complete curve over $K$ with $X_0(K) = \emptyset$, but with $X_0(\hat{K}) \neq \emptyset$ for all local closures $\hat{K}$ of $K$. We shall construct a (necessarily non-geometric) section of $\text{pr}_{X_0/K} : G_{K(X_0)} \rightarrow G_K$.

Let

\[ \overline{K}(X_0) = L_0 \subseteq L_1 \subseteq L_2 \subseteq \ldots \subseteq \overline{K}(X_0) \]

be a tower of finite field extensions of $\overline{K}(X_0)$ such that each $L_i$ is Galois over $K(X_0)$ and such that $\overline{K}(X_0) = \bigcup_i L_i$ (this is possible since $\overline{K}(X_0)/K(X_0)$ is a Galois extension and so any extension of $\overline{K}(X_0)$ has only finitely many conjugates over $K(X_0)$).

We shall construct a sequence of $K$-covers

\[ X_0 \leftrightarrow X_1 \leftrightarrow X_2 \leftrightarrow \ldots \]

of curves (or, equivalently, of function fields $K(X_0) \subseteq K(X_1) \subseteq K(X_2) \subseteq \ldots$ over $K$) such that, for each $i$, $L_i = \overline{K}(X_i)$ and $X_i(\hat{K}) \neq \emptyset$ for any local
closure $\hat{K}$ of $K$. If this is achieved, we are done: take $F := \bigcup_{i=0}^{\infty} K(X_i)$ and observe that $\text{res} : G_F \to G_K$ is an isomorphism (surjective, since $F/K$ is regular, and injective, since $F\overline{K} = \bigcup_{i=0}^{\infty} L_i = \overline{K(X_0)} = \overline{F}$); hence $\text{res}^{-1}$ is a section of $pr_{X_0/K}$.

For the construction of the $X_i$, we start with the given $X_0$, and show how to obtain, for $i \geq 0$, $X_{i+1}$ from $X_i$. Since $L_{i+1}/K(X_0)$ is Galois, so is $L_{i+1}/\hat{K}(X_i)$ for any local closure $\hat{K}$ of $K$. As $X_i(\hat{K}) \neq \emptyset$, and as $\hat{K}$ is large, we may choose $\hat{P}_i \in X_i(\hat{K})$ such that $\hat{P}_i$ is unramified in $L_{i+1}$ and let $\hat{E}_{i+1}$ be a decomposition subfield of $L_{i+1}/\hat{K}(X_i)$ w.r.t. $\hat{P}_i$. Then $E_{i+1}/\hat{K}$ is a function field with $\hat{K}$-rational point and $L_i\hat{E}_{i+1} = L_{i+1}$, i.e. the $\overline{K}$-cover corresponding to the inclusion of $\overline{K}$-function fields $L_1 \subseteq L_{i+1}$ is definable over $\hat{K}$. Now apply $\text{LGP}(X_i/K)$ to obtain a $K$-cover $X_{i+1} \to X_i$ with $\hat{K}(X_{i+1}) = \hat{E}_{i+1}$ for each local closure $\hat{K}$ of $K$, and so, in particular, with $X_{i+1}(\hat{K}) \neq \emptyset$ and $\overline{K}(X_{i+1}) = L_{i+1}$.$\square$

### 3 Large countable fields

Recall that a field $K$ is called large (or ample) if any variety of dimension $\geq 1$ (or, equivalently, any curve) defined over $K$ with one smooth $K$-rational point has infinitely many $K$-rational points.

**Proposition 3.1** Let $K$ be a large countable field (e.g. $K = Q_p \cap \overline{Q}$ or $K = R \cap \overline{Q}$) and let $X$ be a smooth complete curve over $K$ with $X(K) \neq \emptyset$. Then $X/K$ has neither the section property $\text{SP}(X/K)$ nor the birational section property $\text{BSP}(X/K)$.

**Proof:** We shall simultaneously prove the following two claims:

1. If $\sigma_{X/K}$ is injective then $pr_{X/K}^1$ has a non-geometric section.

resp.

2. $pr_{X/K}$ has a non-geometric section.

It is clear that this proves the proposition.

Replacing $K$ by $K^\text{perf}$ we may assume that $K$ is perfect. Let $\overline{K}(X) = L_0 \subseteq L_1 \subseteq \ldots$ be a tower of finite separable extensions of $\overline{K}(X)$ in $K(X)^x$ resp. in $K(X)^{\text{sep}}$ such that each $L_i$ is Galois over $K(X)$ and such that $\bigcup_{i=0}^{\infty} L_i = K(X)^x$ resp. $K(X)^{\text{sep}}$. Let $X(K) = \{P_1, P_2, \ldots\}$. 

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We shall construct a tower of function fields
\[ K(X) = K(X_0) \subseteq K(X_1) \subseteq \ldots \]
of smooth complete curves \( X_i \) over \( K \) in \( K(X)^X \) resp. in \( K(X)^{sep} \) such that, for each \( i > 0 \),
\[
\star_i \quad L_i \subseteq \overline{K}(X_i) \\
\star\star_i \quad X_i(K) \neq \emptyset \\
\star\star\star_i \quad X_i(K) \text{ contains no points above } P_i
\]
If this is achieved, then, as in the proof of the previous proposition, \( F := \bigcup_{i=0}^{\infty} K(X_i) \) does the job: \( F/K \) is regular, \( F\overline{K} = \bigcup_{i=0}^{\infty} \overline{K}(X_i) = \bigcup_{i=0}^{\infty} L_i = K(X)^X \) resp. \( K(X)^{sep} \), so \( res : Gal(K(X)^X/F) \to G_K \) resp. \( res : G_F \to G_K \) is an isomorphism, and \( res^{-1} \) is the desired section: it is non-geometric because no \( P \in X(K) \) has a \( K \)-rational prolongation to \( F \).

For the construction, we start with \( X = X_0 \) and assume, for \( i \geq 0 \), that \( K(X_0) \subseteq K(X_1) \subseteq \ldots \subseteq K(X_i) \) have been constructed according to \( \star_j - \star\star\star_j \) for each \( j \leq i \). Now choose \( P \in X_i(K) \) not above \( P_{i+1} \) and choose \( k > i \) such that \( P_{i+1} \) has no \( K \)-rational prolongation to a decomposition subfield \( K(X_{i+1}) \) of the Galois extension \( L_kK(X_i)/K(X_i) \) w.r.t. \( P \) (here the injectivity assumption enters), and such that \( P \) is unramified in \( L_{i+1}/L_i \).
Then \( X_{i+1} \) satisfies \( \star_{i+1} = \star\star\star_{i+1} \).

4 Galois characterization of rational points over almost arbitrary fields

4.1 Group theoretic description of geometric sections

We apply our characterization of decomposition subgroups of absolute Galois groups in [Ko2] to provide the ‘local theory’ for (1-dimensional) birational anabelian geometry over almost arbitrary fields (including all sub-\( p \)-adic fields, but also all finitely generated fields of positive characteristic, \( \mathbb{Q}^{ab} \), \( \mathbb{Q}^{solv} \), \( \mathbb{Q}^p \), \( \mathbb{C}(t) \) etc.):

**Theorem 4.1** Let \( K \) be a field such that

- \( K \) is not separably closed or real closed
• if $\text{char } K = p > 0$ then $G_K$ is not a pro-$p$ group

• $K$ either admits no non-trivial henselian valuation or $K$ admits a henselian rank-1-valuation of mixed characteristic $(0, p)$ and $p \mid \sharp G_K$.

Let $X$ be a smooth complete curve over $K$ and let $s$ be a section of $\text{pr}_{X/K} : G_{K(X)} \to G_K$.

Then $s$ is geometric iff $s(G_K)$ normalizes some pro-cyclic subgroup $C$ of $G_{K(X)}$ in $G_{K(X)}$ (i.e. $\langle C, s(G_K) \rangle = C \rtimes s(G_K) \leq G_{K(X)}$) with

$$C \cong \begin{cases} \hat{\mathbb{Z}} & \text{if } \text{char } K = 0 \\ \hat{\mathbb{Z}}/\mathbb{Z}_p = \prod_{q \neq p} \mathbb{Z}_q & \text{if } \text{char } K = p > 0. \end{cases}$$

Proof: We denote by $\rho : G_{K_{\text{perf}}(X)} \to G_{K(X)}$ the canonical restriction isomorphism. If $s$ is geometric, say induced by $P \in X(K_{\text{perf}})$, then $\rho^{-1}(s(G_K))$ is a complement of the inertia subgroup $I$ of some decomposition subgroup $D$ of $G_{K_{\text{perf}}(X)}$ w.r.t. $P$. Now choose a complement $C'$ of the ramification subgroup of $I$ ([KPR]) and let $C = \rho(C')$. Then $C$ is pro-cyclic of the indicated shape and is normalized by $s(G_K)$ in $G_{K(X)}$.

For the converse, assume $C \leq G_{K(X)}$ is a pro-cyclic subgroup of the indicated shape and normalized by $s(G_K)$ in $G_{K(X)}$. Then, by Theorem 1 of [Ko2], the fixed field $F$ of the subgroup $G_F = C \rtimes s(G_K)$ of $G_{K(X)}$ carries a tamely branching henselian valuation $v$: more precisely, for any prime $p$ with $p^2 \mid (\sharp C, \sharp G_K)$ there is a henselian valuation $v$ tamely branching at $p$, i.e. with non-$p$-divisible value group (so $v$ is non-trivial) and residual characteristic $\neq p$. Since $K$ is relatively algebraically closed in $F$, the restriction of $v$ to $K$ is henselian. So, if $K$ has no non-trivial henselian valuation, then $v$ is trivial on $K$, i.e. it comes from a geometric place on $K(X)$, and so $s$ is by Obs. 1.3 geometric. If $K$ has a henselian rank-1-valuation of mixed characteristic $(0, p)$ and $p \mid \sharp G_K$, then $v$ may be chosen to be tamely branching at $p$, and so the residual characteristic of $(F, v)$ is different from $p$. This, again, forces $v$ to be trivial on $K$, since a henselian rank-1-valuation of mixed characteristic $(0, p)$ allows no non-trivial henselian valuations of residual characteristic $\neq p$ on the same field. So we can proceed as above. □

Remark 4.2 It is clear that, if a pro-cyclic subgroup $C$ of $G_{K(X)}$ satisfies the condition in the theorem, then $C$ is the inertia subgroup of $G_F$, i.e. $C$ is contained in a (complement of the ramification subgroup of the) inertia
subgroup of some geometric decomposition subgroup of $G_{K(X)}$: the fixed field of $s(G_K)$ is a purely tamely ramified extension of $F$ and $K(X)^{sep}/F$ is purely inert w.r.t. $v$. In particular, $C \leq G_{K^{sep}(X)}$. If $X$ is a smooth curve over a field $K$ which is finitely generated over $\mathbb{Q}$, then Nakamura proves a similar sufficient inertia condition for subgroups of $\pi_1(X)$ (Theorem 3.4 of [Na1]): If $C$ is a non-trivial pro-cyclic subgroup of $\pi_1(X_{\overline{K}})$ such that there exists a section $s$ of $pr_{X/K}$ with $s(G_K)$ normalizing $C$ and acting on it via the cyclotomic character of $G_K$, then $C$ is contained in an inertia subgroup of $\pi_1(X)$ w.r.t. a $K$-rational point in $\overline{X} \setminus X$.

Define a profinite group $G$ to be a **hensel group** if there is some Sylow subgroup $P$ of $G$ containing some non-trivial normal abelian subgroup $N \triangleleft P$, but no procyclic open subgroup. Theorem 1 of [Ko2] says that a field whose absolute Galois group is a hensel group, admits a tamely branching henselian valuation. This allows us to give a Galois characterization of rational points:

**Corollary 4.3** Let $K$ and $X$ be as in Theorem 4.1. Then the map

$$X(K^{perf}) \rightarrow \left\{ \text{conjugacy classes of maximal hensel subgroups} \right\}$$

$$P \mapsto [D_P]$$

is bijective. Here $D_P$ denotes a decomposition subgroup of $G_{K(X)}$ w.r.t. $P$.

**Proof:** By the proof of Theorem 4.1, any hensel subgroup $D$ of $G_{K(X)}$ with $pr_{X/K}(D) = G_K$ is the absolute Galois group of a henselian algebraic extension $(F, v)$ of $(K(X), v_P)$ for some $P \in X(K^{perf})$ (with corresponding valuation $v_P$): note that $pr_{X/K}(D) = G_K$ implies that $K$ is algebraically closed in $F$. So $D \subseteq D_P$ for some decomposition subgroup $D_P$ of $G_{K(X)}$ w.r.t. $P$. Moreover, if for some $P' \in X(K^{perf})$, $D_{P'} \subseteq D$ then $D_{P'} \subseteq D_P$, and so, by a well-known theorem of F.K.Schmidt, $P' = P$ and $D_{P'} = D = D_P$. So the decomposition subgroups of $G_{K(X)}$ w.r.t. $K^{perf}$-rational points are indeed the maximal hensel subgroups $D$ of $G_{K(X)}$ with $pr_{X/K}(D) = G_K$, and distinct points have non-conjugate decomposition subgroups. □

Note that if $K$ satisfies the conditions of Theorem 4.1, then so does any finite extension $L/K$. So we get a Galois characterization of $L^{perf}$-rational points for any finite extension $L/K$ and, thus, a Galois characterization of all $P \in X(\overline{K})$. 

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4.2 Back to the roots: solving equations by radicals

If $L/K$ is a (possibly infinite) Galois extension, if $L$ (and hence $K$) satisfies the conditions of Theorem 4.1, and if $L$ is defined by a Galois theoretic property (e.g. $L = K^{ab}$ or $L = K^{solv}$ etc.) then for any smooth complete curve $X/K$, Corollary 4.3 provides a Galois characterization of $L$-rational points.

As a special instance, let us consider the original question of Galois theory whether any polynomial equation (over $\mathbb{Q}$) in one variable can be solved by radicals, and ask the same question for polynomial equations in two variables, where we now have to assume that the corresponding curve be geometrically irreducible. The answer to this question is still unknown (it is equivalent to the question whether $\mathbb{Q}^{solv}$ is a PAC-field, cf. [FJ], Problem 10.16(a)). Yet, following the steps of Galois, we can at least give a group theoretic criterion:

**Corollary 4.4** Let $X/\mathbb{Q}$ be a smooth complete curve. Then $X(\mathbb{Q}^{solv}) \neq \emptyset$ iff there is a hensel subgroup $D \leq G_{\mathbb{Q}(X)}$ with $G_{\mathbb{Q}^{solv}} \subseteq \text{pr}_{X/\mathbb{Q}}(D)$.

Let us mention that the problem of finding rational points over $\mathbb{Q}^{solv}$ is by no means out of date (cf. e.g. Section 4.5 of [SW]).

4.3 Relating the two fundamental conjectures

Mochizuki’s Theorem A in [Mo] implies both the fundamental conjecture and the birational fundamental conjecture for smooth hyperbolic curves over sub-$p$-adic fields (and even for higher-dimensional varieties). It is, however, not obvious whether, in general, one conjecture implies the other. The fact that $G_{K(X)}$ can be obtained as inverse limit of the $\pi_1(X')$’s for all Zariski-open $X' \subseteq X$ does not mean that $G_{K(X)}$ has to remember this genealogy. Yet, for almost all constant fields, it does:

**Corollary 4.5** Let $K$ be as in Theorem 4.1 and let $X/K$ be a smooth complete curve over $K$. Then there is a purely group-theoretic characterization of the quotients of $G_{K(X)}$ which are $\pi_1(X')$’s for some Zariski-open $X' \subseteq X$.

In particular, the fundamental conjecture for some $X' \subseteq X$ over $K$ implies the birational fundamental conjecture for $X$ over $K$.

**Proof:** The previous corollary does not only imply a group-theoretic characterization of decomposition subgroups $D_P$ of $G_{K(X)}$ w.r.t. points $P \in X(\overline{K})$, 

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but also of the corresponding inertia subgroups $I_P = D_P \cap \ker \text{pr}_{X/K}$. The characterization is this: Any subgroup $I \leq G_{K(X)}$ is an inertia subgroup of $G_{K(X)}$ w.r.t. some $\overline{K}$-rational point of $X$ iff $I = D \cap \ker \text{pr}_{X/K}$ for some maximal hensel subgroup $D$ of $G_{K(X)}$ containing the image of a section of $\text{pr}_{X/L} : G_{L(X)} \to G_L$ for some open subgroup $G_L$ of $K$.

So if $\mathcal{I}$ is the set of conjugacy classes $[I]$ (in $G_{K(X)}$) of these group-theoretically described inertia subgroups $I$ of $G_{K(X)}$ one obtains a 1-1 correspondence

$$X(\overline{K}) \longleftrightarrow \mathcal{I}$$

$$P \quad \mapsto \quad [I_P]$$

Note that any two inertia groups belong to the same point iff they are conjugate.

Now the group-theoretic characterization of fundamental groups is easy: Let $N < G_{K(X)}$ be a normal subgroup. Then $G_{K(X)}/N \cong \pi_1(X')$ for some Zariski-open $X' \subseteq X$ iff $N = \langle I \mid [I] \in \mathcal{I} \setminus \mathcal{I}_0 \rangle$ for some finite $\mathcal{I}_0 \subseteq \mathcal{I}$.

Finally, assume the fundamental conjecture holds for some Zariski-open $X' \subseteq X$ over $K$, and let $Y/K$ be a smooth curve over $K$ with $G_{K(Y)} \cong_{G_K} G_{K(X)}$. We may as well assume that $Y$ is also complete. Let $N$ be the kernel of $G_{K(X)} \to \pi_1(X')$ and let $N'$ be the isomorphic copy of $N$ in $G_{K(Y)}$. Then $G_{K(Y)}/N' \cong \pi_1(Y')$ for some Zariski-open $Y' \subseteq Y$ and $\pi_1(Y') \cong_{G_K} \pi_1(X')$.

By assumption this implies $Y' \cong_K X'$ and so $X$ and $Y$ are birationally equivalent over $K$. □

### 4.4 Describing arithmetic properties in Galois-theoretic terms: the PAC-property, largeness and Hilbertianity

An immediate consequence of Corollary 4.3 is the following group theoretic characterization of PAC-fields and of large fields:

**Corollary 4.6** Let $K$ be a perfect field satisfying the hypothesis of Theorem 4.1 and let $\text{pr} = \text{pr}_{P/K} : G_{K(t)} \to G_K$ be the canonical projection. Then

1. $K$ is PAC iff every open subgroup $H$ of $G_{K(t)}$ with $\text{pr}(H) = G_K$ contains a hensel subgroup $D$ with $\text{pr}(D) = G_K$.
2. $K$ is large iff every open subgroup $H$ of $G_{K(t)}$ with $\text{pr}(H) = G_K$ contains either no or infinitely many pairwise non-conjugate maximal hensel subgroups $D$ with $\text{pr}(D) = G_K$.

Unlike large fields, Hilbertian fields always satisfy the hypothesis of Theorem 4.1. Recall that a field $K$ is separably Hilbertian if Hilbert’s Irreducibility Theorem holds for separable polynomials. In characteristic 0 this is equivalent to Hilbertianity, and in characteristic $p > 0$, $K$ is Hilbertian iff $K$ is imperfect and separably Hilbertian. If $K$ is separably Hilbertian then so is $K^{perf}$ (cf. [FJ], Section 11.3). It is not known whether the converse holds (cf. [J], Problem 13). So our Galois theoretic criterion for separable Hilbertianity can at the moment only be stated for perfect fields:

**Corollary 4.7** Let $K$ be a perfect field satisfying the hypothesis of Theorem 4.1 and let $\text{pr} : G_{K(t)} \to G_K$ be the canonical projection. Then $K$ is separably Hilbertian iff for every open subgroup $H$ of $G_{K(t)}$ there are infinitely many pairwise non-conjugate maximal hensel subgroups $D$ of $G_{K(t)}$ with $\text{pr}(D) = G_K$ such that

$$[D : D \cap H] = [G_{K(t)} : H].$$

**Proof:** $\Rightarrow$: Let $K$ be separably Hilbertian and let $H \leq G_{K(t)}$ be open. Then the fixed field $F$ of $H$ is of the form $F = K(t, \alpha)$, where the irreducible polynomial $f(t, Y)$ of $\alpha$ over $K(t)$ is separable and can be chosen in $K[t, Y]$. Hence there are infinitely many $\alpha \in K$ such that $f(a, Y) \in K[Y]$ is irreducible over $K$. For each such $\alpha$ the $(t - a)$-adic henselisation $L_\alpha$ of $K(t)$ has the property that

$$[L_\alpha(\alpha) : L_\alpha] = [F : K(t)].$$

$D := G_{L_\alpha}$ is then a maximal hensel subgroup of $G_{K(t)}$ with $\text{pr}(D) = G_K$ such that $D \cap H = G_{L_\alpha(\alpha)}$ and hence $[D : D \cap H] = [G_{K(t)} : H]$. Finally, distinct $\alpha$’s induce non-conjugate $D$’s, and the right hand side follows as $K$ is infinite.

$\Leftarrow$: Assume the right hand side and let $f \in K[t, Y]$ be separable and irreducible. Choose $\alpha \in K(t)$ with $f(t, \alpha) = 0$ and let $H = G_{K(t, \alpha)}$. Choose one of the infinitely many maximal hensel groups $D$ guaranteed by our assumption avoiding those (up to conjugation) finitely many corresponding to ramification points of $K(t, \alpha)/K(t)$ or to zeros of the discriminant of $f$. Then $f(t, Y)$ remains irreducible over the fixed field of $D$ which, by Corollary 4.3
is the henselisation of some \((t - a)\)-adic valuation of \(K(t)\) with \(a \in K\). By the choice of \(D\), \(f(a, Y)\) is still separable. So \(f(a, Y)\) is, by Hensel’s lemma, irreducible over \(K\). □

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