Free Field Representations
of Extended Superconformal Algebras

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Abstract

We study the classical and quantum $G$ extended superconformal algebras from the hamiltonian reduction of affine Lie superalgebras, with even subalgebras $G \oplus sl(2)$. At the classical level we obtain generic formulas for the Poisson bracket structure of the algebra. At the quantum level we get free field (Feigin-Fuchs) representations of the algebra by using the BRST formalism and the free field realization of the affine Lie superalgebra. In particular we get the free field representation of the $sl(2) \oplus sp(2N)$ extended superconformal algebra from the Lie superalgebra $osp(4|2N)$. We also discuss the screening operators of the algebra and the structure of singular vectors in the free field representation.
1 Introduction

The interplay of the world-sheet superconformal symmetry and the target space symmetry of related supersymmetric non-linear $\sigma$ models has attracted much attention in relation to the compactification of superstrings and topological field theories. In particular, the relation between $N = 2$ and 4 superconformal algebras and the supersymmetric non-linear $\sigma$ models on Calabi-Yau manifolds is well known and has been studied in great detail \cite{1},\cite{2}. It seems an interesting problem to study a further extension of the superconformal algebra as well as its geometrical structure, which might have further nontrivial topological properties.

There are several distinct approaches to the construction of extended superconformal algebras. Originally, the linear $o(N)$ supersymmetric extension of the Virasoro algebra, which was introduced by Ademollo et al. \cite{3}, was realized as the superconformal transformation on extended superspace with internal $o(N)$ symmetry. However, this formulation is valid only for $N \leq 4$ due to the appearance of negative conformal weight supercurrents \cite{3} and the absence of a central extension for $N \geq 5$ \cite{4}. A slightly different approach is to regard linear $N = 2, 4$ extended superconformal algebras as a symmetry of supersymmetric non-linear $\sigma$ models on group manifolds and on a coset space such as a hermitian symmetric space \cite{5} or the Wolf spaces \cite{6}.

Knizhnik and Bershadsky \cite{7} have proposed a non-linear extension of superconformal algebras with $u(N)$ and $so(N)$ Kac-Moody symmetries by considering the closure and the associativity of operator product expansions for the currents (the OPE method). This non-linearity means that these extended superconformal algebras belong to the class of Zamolodchikov’s $W$-algebras \cite{8}. It is by now well understood that the $W$-algebra associated with a simple Lie algebra can be obtained from the hamiltonian reduction \cite{9}-\cite{12} of the corresponding affine Lie algebra by considering the constraints on the phase space of currents using a gauge fixing procedure. Recently various types of extensions
of the $W$-algebra [13]-[17] including their supersymmetric generalization [18]-[20] have been considered by using the hamiltonian reduction technique. Among them, in refs. [16] and [17], the $u(N)$ bosonic superconformal algebras have been constructed from a certain hamiltonian reduction procedure of the affine $sl(N + 2)$ algebra.

In ref. [21], two of the present authors have proposed a procedure for constructing the classical extended superconformal algebras by the hamiltonian reduction of affine Lie superalgebras with even Lie subalgebras $G \oplus sl(2)$ (see table 1). Remarkably our classification corresponds to that of the reduced holonomy groups of non-symmetric Riemannian manifolds [22]. We expect that these extended superconformal algebras might correspond to non-linear $\sigma$ models with rich geometrical structures such as quaternionic or octonionic structures. Recently a similar classification based on the OPE method has been proposed in [23].

The purpose of the present paper is to develop the hamiltonian reduction in the quantum case and to study the representation theory of the extended superconformal algebra using free fields, which is useful for the analysis of singular vectors and the computation of correlation functions [24].

This paper is organized as follows: In sect. 2, after we review some basic properties of affine Lie superalgebras, we discuss the classical hamiltonian reduction for an affine Lie superalgebra $\hat{g}$ associated with a Lie superalgebra $g$ which has the even subalgebra $G \oplus sl(2)$ and we derive the classical $G$ extended superconformal algebra. In sect. 3, by using the BRST gauge fixing procedure and the Wakimoto realization [25] of the affine Lie superalgebra, we derive the free field realization of the extended superconformal algebra. In sect 4, some explicit examples of extended superconformal algebras corresponding to the non-exceptional Lie superalgebras are given. In sect. 5, we investigate the structure of screening operators and the null field structure of degenerate representations of the $G$ extended superconformal algebra.
2 The Classical Hamiltonian Reduction

2.1 Affine Lie Superalgebras

We start with explaining some definitions of basic classical Lie superalgebras \([26]\) and their affine extensions. Let \(g = g_0 \oplus g_1\) be a rank \(n\) basic classical Lie superalgebra with an even subalgebra \(g_0\) and an odd subspace \(g_1\). \(\Delta = \Delta^0 \cup \Delta^1\) is the set of roots of \(g\), where \(\Delta^0\) (\(\Delta^1\)) is the set of even (odd) roots. Denote the set of positive even (odd) roots by \(\Delta^0_+\) (\(\Delta^1_+\)). The superalgebra \(g\) has a canonical basis \(\{E_\alpha, e_\gamma, h^i\}\) \((\alpha \in \Delta^0, \gamma \in \Delta^1, i = 1, \ldots, n)\), which satisfies (anti-)commutation relations

\[
[E_\alpha, E_\beta] = \begin{cases} 
N_{\alpha,\beta} E_{\alpha+\beta}, & \text{for } \alpha, \beta, \alpha + \beta \in \Delta^0, \\
\frac{2\alpha \cdot h}{\alpha^2}, & \text{for } \alpha + \beta = 0,
\end{cases}
\]

\[
\{e_\gamma, e_\gamma'\} = N_{\gamma,\gamma'} E_{\gamma+\gamma'}, \text{ for } \gamma, \gamma' \in \Delta^1, \gamma + \gamma' \in \Delta^0,
\]

\[
\{e_\gamma, e_{-\gamma}\} = \gamma \cdot h, \text{ for } \gamma \in \Delta^1_+,
\]

\[
[e_\gamma, E_\alpha] = N_{\gamma,\alpha} e_{\gamma+\alpha}, \text{ for } \alpha \in \Delta^0, \gamma, \gamma + \alpha \in \Delta^1,
\]

\[
[h^i, E_\alpha] = \alpha^i E_\alpha, \quad [h^i, e_\gamma] = \gamma^i e_\gamma.
\] (2.1)

The even subalgebra \(g_0\) is generated by \(\{E_\alpha, h^i\}\). The odd subspace \(g_1\) is spanned by \(e_\gamma\). \(g_0\) acts on \(g_1\) as a faithful representation. The Killing form \(( , )\) on \(g\) is defined by

\[
(E_\alpha, E_\beta) = \frac{2}{\alpha^2} \delta_{\alpha+\beta,0}, \quad (e_\gamma, e_{-\gamma'}) = -(e_{-\gamma}, e_\gamma') = \delta_{\gamma,\gamma'}, \quad (h^i, h^j) = \delta_{ij},
\] (2.2)

for \(\alpha, \beta \in \Delta^0, \gamma, \gamma' \in \Delta^1_+, i, j = 1, \ldots, n\).

An affine Lie superalgebra \(\hat{g}\) at level \(k\) is the (untwisted) central extension of \(g\) and consists of the elements of the form \((X(z), x_0)\), where \(X(z)\) is a \(g\)-valued Laurent polynomial of \(z \in \mathbb{C}\) and \(x_0\) is a number \([27]\). The commutation relation for two elements \((X(z), x_0)\) and \((Y(z), y_0)\) is given by

\[
[ (X(z), x_0), (Y(z), y_0) ] = ( [ X(z), Y(z) ], k \oint \frac{dz}{2\pi i} (X(z), \partial Y(z)) ).
\] (2.3)

The dual space \(\hat{g}^*\) of \(\hat{g}\) is generated by the current \((J(z), a_0)\). Using the Killing form \(( , )\) on \(g\), we may identify \(\hat{g}^*\) with \(\hat{g}\). The inner product \(\langle , \rangle\) of \((J(z), a_0) \in \hat{g}^*\) and
\((X(z), x_0) \in \hat{g}\) is given by
\[
\langle (J, a_0), (X, x_0) \rangle = \oint \frac{dz}{2\pi i} (J(z), X(z)) + a_0 x_0.
\] (2.4)

Let us define the coadjoint action \(\text{ad}^*\) of \(\hat{g}\) on \(\hat{g}^*\) by
\[
\langle \text{ad}^*(X, x_0) (J, a_0), (Y, y_0) \rangle = -\langle (J, a_0), [ (X, x_0), (Y, y_0) ] \rangle.
\] (2.5)

Using (2.3), we get
\[
\text{ad}^*(X, x_0)(J, a_0) = ( \left[ X(z), J(z) \right] + k a_0 \partial X(z), 0 ).
\] (2.6)

Namely, the coadjoint action is nothing but the infinitesimal gauge transformation of the current \(J(z)\). Denote this gauge transformation with the gauge parameter \(\Lambda(z)\) as \(\delta_\Lambda\):
\[
\delta_\Lambda J(z) = [ \Lambda(z), J(z) ] + k \partial \Lambda(z),
\] (2.7)

In the following we take the number \(a_0\) to be 1. In terms of the canonical basis
\[
J(z) = \sum_{\alpha \in \Delta^0} \frac{\alpha^2}{2} J_\alpha(z) E_\alpha + \sum_{\gamma \in \Delta^1} j_\gamma(z) e_\gamma + \sum_{i=1}^n H^i(z) h^i.
\] (2.8)

and
\[
\Lambda(z) = \sum_{\alpha \in \Delta^0} \varepsilon_\alpha(z) E_\alpha + \sum_{\gamma \in \Delta^1} \xi_\gamma(z) e_\gamma + \sum_{i=1}^n \varepsilon^i(z) h^i,
\] (2.9)

we can express the gauge transformation (2.7) in terms of components:
\[
\delta_\Lambda J_\alpha = \sum_{\beta, \alpha-\beta \in \Delta^0} \frac{(\alpha - \beta)^2}{\alpha^2} N_{\beta, \alpha-\beta} \varepsilon_\beta J_{\alpha-\beta} + \frac{2(\alpha \cdot H)}{\alpha^2} \varepsilon_\alpha + \frac{2k}{\alpha^2} \partial \varepsilon_\alpha + (\alpha \cdot \varepsilon) J_\alpha
\]
\[
+ \sum_{\gamma, \alpha-\gamma \in \Delta^1} \frac{2}{\alpha^2} N_{\gamma, \alpha-\gamma} \xi_\gamma J_{\alpha-\gamma},
\]
\[
\delta_\Lambda j_\gamma = \sum_{\alpha \in \Delta^0} N_{\alpha, \gamma-\alpha} \varepsilon_\alpha j_\gamma + (\gamma \cdot \varepsilon) j_\gamma + \sum_{\alpha \in \Delta^0} \frac{\alpha^2}{2} N_{\gamma-\alpha, \alpha} \xi_\gamma J_\alpha - \xi_\gamma \gamma \cdot H + k \partial \xi_\gamma,
\]
\[
\delta_\Lambda H^i = \sum_{\alpha \in \Delta^0} \alpha^i \varepsilon_\alpha J_\alpha + \sum_{\gamma \in \Delta^1} \gamma^i (\xi_\gamma j_{\gamma-\gamma} + \xi_\gamma j_\gamma) + k \partial \varepsilon^i.
\] (2.10)
Writing the gauge transformations $\delta_A$ as

$$\delta_A = \oint \frac{dz}{2\pi i} (A(z), J(z))$$

$$= \oint \frac{dz}{2\pi i} \sum_{\alpha \in \Delta^0} \varepsilon_\alpha J_{-\alpha} + \sum_{\gamma \in \Delta^1_+} (j_{\gamma} \xi_{-\gamma} + \xi_{\gamma} j_{-\gamma}) + \sum_{i=1}^n \varepsilon^i H^i,$$  \hspace{1cm} (2.11)

one can introduce a canonical Poisson structure on the dual space $\mathfrak{g}^*$. This Poisson structure is expressed in the form of the operator product expansions:

$$J_\alpha(z)J_\beta(w) = \left\{ \begin{array}{ll}
\frac{N_{\alpha,\beta}J_{\alpha+\beta}(w)}{z-w} + \cdots, & \text{for } \alpha, \beta, \alpha + \beta \in \Delta^0, \\
\frac{\frac{2k}{2\alpha^2} \frac{2\alpha H(w)}{\alpha^2}}{(z-w)^2} + \frac{2\alpha H(w)}{\alpha^2} + \cdots, & \text{for } \alpha + \beta = 0,
\end{array} \right.$$  

$$j_{\gamma}(z)j_{\gamma'}(w) = \frac{N_{\gamma,\gamma'}J_{(\gamma+\gamma')}(w)}{z-w} + \cdots, \quad \text{for } \gamma, \gamma' \in \Delta^1_+, \ \gamma + \gamma' \in \Delta^0,$$  

$$j_{\gamma}(z)j_{-\gamma'}(w) = \left\{ \begin{array}{ll}
\frac{-N_{\gamma,\gamma'}J_{\gamma'-\gamma}(w)}{z-w} + \frac{-k}{(z-w)^2} + \frac{\gamma H(w)}{z-w} + \cdots, & \text{for } \gamma, \gamma' \in \Delta^1_+, \ \gamma - \gamma' \in \Delta^0, \\
\frac{-k}{(z-w)^2} + \frac{\gamma H(w)}{z-w} + \cdots, & \text{for } \gamma = \gamma' \in \Delta^1_+, \ \gamma \in \Delta^1, \ \alpha \in \Delta^0, \\
\frac{N_{\gamma,\gamma+\alpha}J_{\gamma+(\gamma+\alpha)}(w)}{z-w} + \cdots, & \text{for } \gamma \in \Delta^1, \ \alpha \in \Delta^0, \\
\frac{\alpha^2 J_{\alpha}(w)}{z-w} + \cdots, & \text{for } \gamma \in \Delta^1, \ \alpha \in \Delta^0, \\
\frac{\gamma \gamma'}{z-w} + \cdots. & \text{for } \gamma \in \Delta^1, \ \alpha \in \Delta^0.
\end{array} \right.$$  \hspace{1cm} (2.12)

Here we use some identities for the structure constants, which come from the Jacobi identities:

$$\frac{2N_{\alpha,\beta}}{(\alpha + \beta)^2} = \frac{2N_{\beta,-\alpha-\beta}}{\alpha^2} = \frac{2N_{-\alpha-\beta,\alpha}}{\beta^2}, \quad \text{for } \alpha, \beta \in \Delta^0,$$

$$\frac{2N_{\gamma,\gamma'}}{(\gamma + \gamma')^2} = -N_{\gamma,-\gamma-\gamma'} = N_{-\gamma-\gamma',\gamma},$$

$$\frac{2N_{\gamma,-\gamma'}}{(\gamma - \gamma')^2} = N_{-\gamma,-\gamma+\gamma'} = N_{-\gamma+\gamma',\gamma}, \quad \text{for } \gamma, \gamma' \in \Delta^1_+. \hspace{1cm} (2.13)$$

In the following we will study a class of Lie superalgebras, which include even algebras of the form $G \oplus A_1$. Using Kac’s notation, these algebras are classified as follows $A(n|1)$ $(n \geq 1)$, $A(1,0) = C(2)$, $B(n|1)$ $(n \geq 0)$, $B(1|n)$ $(n \geq 1)$, $D(n|1)$ $(n \geq 2)$, $D(2|n)$ $(n \geq 1)$,
\( D(2|1; \alpha), F(4) \) and \( G(3) \) (see table [I]). The embedding of \( A_1 \) in \( \mathfrak{g} \) carried by \( \mathfrak{g}_1 \) has spin \( \frac{1}{2} \), except for \( B(1|n) \). In the case of \( B(1|n) \) the embedding has spin one. In the spin \( \frac{1}{2} \) embedding case, the odd subspace \( \mathfrak{g}_1 \) belongs to the spin \( \frac{1}{2} \) representation with respect to the even subalgebra \( A_1 \):

\[
\mathfrak{g}_1 = (\mathfrak{g}_1)_+^{\frac{1}{2}} \oplus (\mathfrak{g}_1)_-^{\frac{1}{2}}. \tag{2.14}
\]

Based on the decomposition (2.14), the odd root space \( \Delta^1 \) can be divided into two parts

\[
\Delta^1 = \Delta^1_+^{\frac{1}{2}} \cup \Delta^1_-^{\frac{1}{2}}, \tag{2.15}
\]

where \( (\mathfrak{g}_1)_ \pm^{\frac{1}{2}} \) is spanned by the generators whose roots belong to \( \Delta^1_ \pm^{\frac{1}{2}} \). More explicitly, the sets \( \Delta^1_ \pm^{\frac{1}{2}} \) consist of the roots \( \gamma \in \Delta^1 \) satisfying

\[
\frac{\gamma \cdot \theta}{\theta^2} = \pm \frac{1}{2}, \tag{2.16}
\]

where \( \theta \) is the simple root of \( A_1 \). By choosing appropriate simple roots for \( \mathfrak{g} \), we can take the root space \( \Delta^1_+^{\frac{1}{2}} \) as the space of odd positive roots;

\[
\Delta^1_+ = \Delta^1_+^{\frac{1}{2}}. \tag{2.17}
\]

Moreover we find that the relation

\[
\Delta^1_+^{\frac{1}{2}} = \Delta^1_-^{\frac{1}{2}} + \theta, \tag{2.18}
\]

holds. We note that each odd space \( (\mathfrak{g}_1)_ \pm^{\frac{1}{2}} \) belongs to a fundamental representation of \( G \) of dimension \( |\Delta^1_\pm| \), this being the number of positive, odd roots. From the explicit expressions for the root system which is given in appendix A, we will find that the root system of the even subalgebra of \( G \) is expressed in the form of \( \gamma - \gamma' \), where \( \gamma \) and \( \gamma' \) are some positive odd roots. This fact turns out to be essential for the study of the structure of general \( G \) extended superconformal algebras, as will be discussed in the next subsection.
2.2 Classical hamiltonian reduction

We consider the hamiltonian reduction of the phase space of the currents $\hat{g}^*$ associated with the Lie superalgebra $g$ with the even subalgebra $G \oplus A_1$. Our main idea is to impose the constraint on the even subalgebra $A_1$ and to keep affine $G$ algebra symmetry. For $A(1,0)$, this reduction procedure has been discussed in ref. [18].

We start from the phase space $\mathcal{F}$ of the currents with the constraint

$$J_{-\theta}(z) = 1.$$  \hfill (2.19)

Let us consider the gauge group $\mathcal{N}$ which preserves the constraint (2.19). Putting $-\theta$ for $\alpha$ in the first formula of eqs. (2.10), we get

$$\delta_{\Lambda}J_{-\theta} = \frac{2(\theta \cdot H)}{\theta^2} \varepsilon_{-\theta} + \frac{2k}{\theta^2} \partial \varepsilon_{-\theta} - (\theta \cdot \varepsilon)J_{-\theta} + \sum_{\gamma \in \Delta^1_+} \frac{2}{\theta^2} N_{-\gamma, -\theta + \gamma} \xi_{-\gamma} J_{-\theta + \gamma}. \hfill (2.20)$$

Here we use the relation (2.18) and $N_{\alpha, -\theta - \alpha} = 0$ for $\alpha \in \Delta(G)$. Therefore the condition which preserves the constraints (2.19) $\delta_{\Lambda}J_{-\theta} = 0$ is equivalent to

$$\varepsilon_{-\theta} = \theta \cdot \varepsilon = 0, \quad \xi_{-\gamma} = 0, \quad \text{for } \gamma \in \Delta^1_+. \hfill (2.21)$$

Namely the Lie algebra $n$ of the gauge group $\mathcal{N}$ is equal to $\hat{G} \oplus n_1$, where $\hat{G}$ is the affine Lie algebra corresponding to the even subalgebra $G$ and $n_1$ is generated by the elements:

$$\Lambda(z) = \varepsilon_{\theta}(z) E_{\theta} + \sum_{\gamma \in \Delta^1_+} \xi_{\gamma}(z) e_{\gamma}. \hfill (2.22)$$

Let $\mathcal{N}_1$ be the gauge group corresponding to $n_1$. The reduced phase space $\mathcal{M}$ is defined as the quotient space $\mathcal{M} = \mathcal{F}/\mathcal{N}_1$. By the standard gauge fixing procedure, one chooses one or the other gauge slice as a representative of the reduced phase space. One typical gauge is the “Drinfeld-Sokolov (DS)”-type gauge [4]:

$$J_{\theta}(z) = T(z), \quad \theta \cdot H(z) = 0, \quad j_{\gamma}(z) = G_{\gamma}(z), \quad j_{-\gamma}(z) = 0,$$  \hfill (2.23)

for $\gamma \in \Delta^1_+$. The generic gauge transformation projected on the DS gauge slice becomes in general the transformation corresponding to the extended Virasoro algebra [8]-[13].
Hence the final step is to consider the generic gauge transformation preserving the gauge condition (2.23). In particular

\[
\delta_A J_\theta = - (\theta \cdot \varepsilon) + \frac{2k}{\theta^2} \partial \varepsilon_\theta, \\
\delta_A (\theta \cdot H) = \theta^2 \varepsilon_\theta - \theta^2 \varepsilon_{-\theta} T + k \partial (\theta \cdot \varepsilon) + \sum_{\gamma \in \Delta_+^1} \theta \cdot \gamma \xi_\gamma G_\gamma,
\]

(2.24)

\[
\delta_A j_{-\gamma} = N_{-\theta, \theta - \gamma \varepsilon - \theta} G_{\theta - \gamma} + \sum_{\gamma' \in \Delta_+^1} N_{\gamma, \gamma'} \xi_{-\gamma'} J_{\gamma' - \gamma} + N_{\theta, \gamma} \xi_{\theta - \gamma} + \xi_{-\gamma} (\gamma \cdot H) + k \partial \xi_{-\gamma},
\]

for \( \gamma \in \Delta_+^1 \). By solving the conditions which preserve the DS-gauge \( \delta J_{\theta} = 0 \), \( \delta (\theta \cdot H) = 0 \) and \( \delta j_{-\gamma} = 0 \), we can express the parameters \( \varepsilon_\theta, \theta \cdot \varepsilon \) and \( \xi_\gamma (\gamma \in \Delta_+^1) \) in terms of the other parameters, \( \varepsilon_{-\theta}, \xi_{-\gamma} (\gamma \in \Delta_+^1), \varepsilon_\alpha \) and \( \alpha \cdot \varepsilon (\alpha \in \Delta(G)) \):

\[
\theta \cdot \varepsilon = \frac{2k}{\theta^2} \partial \varepsilon_{-\theta}, \quad \varepsilon_\theta = \varepsilon_{-\theta} T - \frac{2k^2}{(\theta^2)^2} \partial^2 \varepsilon_{-\theta} - \frac{1}{2} \sum_{\gamma \in \Delta_+^1} \xi_{-\gamma} G_\gamma,
\]

(2.25)

\[
\xi_{\theta - \gamma} = -\frac{1}{N_{\theta, \theta - \gamma}} \left\{ N_{\theta, \theta - \gamma} G_{\theta - \gamma} \varepsilon_{-\theta} + \sum_{\gamma' \in \Delta_+^1} N_{\gamma, \gamma'} \xi_{-\gamma'} J_{\gamma' - \gamma} + \varepsilon_{-\gamma} (\gamma \cdot H) + k \partial \xi_{-\gamma} \right\}.
\]

In the DS gauge, the gauge transformation of \( J_\theta(z) = T(z) \) becomes

\[
\delta T = \sum_{\gamma \in \Delta_+^1} \frac{2}{\theta^2} N_{\gamma, \theta - \gamma} \xi_\gamma G_{\theta - \gamma} + (\theta \cdot \varepsilon) T + \frac{2k}{\theta^2} \partial \varepsilon_\theta.
\]

(2.26)

From (2.25) we get

\[
\delta T = -\frac{4k^3}{(\theta^2)^3} \partial^2 \varepsilon_{-\theta} + \frac{2k}{\theta^2} (2T \partial \varepsilon_{-\theta} + \varepsilon_{-\theta} T) + \frac{1}{\theta^2} \sum_{\gamma \in \Delta_+^1} \left\{ 3k \partial \xi_{-\gamma} G_\gamma + k \xi_{-\gamma} \partial G_\gamma + 2 \xi_{-\gamma} (\gamma \cdot H) G_\gamma + 2 \sum_{\gamma' \in \Delta_+^1, \gamma' \neq \gamma} N_{\gamma, \gamma'} \xi_{-\gamma'} G_{\gamma' - \gamma} \right\}.
\]

(2.27)

With respect to the parameter \( \varepsilon_{-\theta} \), \( T(z) \) behaves as an energy momentum tensor under conformal transformations. The \( G \) Kac-Moody currents \( J_\beta \) and \( \beta \cdot H \) (\( \beta \in \Delta(G) \)) transform as

\[
\delta J_\beta = \sum_{\alpha, \beta - \alpha \in \Delta(G)} N_{-\alpha, \beta} \varepsilon_\alpha J_{\beta - \alpha} - \frac{2\beta \cdot H}{\beta^2} \varepsilon_\beta + \frac{2k}{\beta^2} \partial \varepsilon_\beta + (\beta \cdot \varepsilon) J_\beta + \frac{2N_{\gamma, \beta + \gamma} \xi_{-\gamma} G_{\beta + \gamma}}{\beta^2},
\]

\[
\delta (\beta \cdot H) = \sum_{\gamma \in \Delta_+^1} (\beta \cdot \varepsilon) J_{-\alpha} + \frac{2N_{\gamma, \beta + \gamma} \xi_{-\gamma} G_{\beta + \gamma}}{\beta^2},
\]

(2.28)
The gauge transformation of supercurrents \( G_\gamma(z) \) \((\gamma \in \Delta_+^1)\) in the DS gauge is given by

\[
\delta G_\gamma(z) = \sum_{\alpha \in \Delta(G)} N_{\alpha,\gamma-\alpha} \xi_\alpha G_{\gamma-\alpha} + (\gamma \cdot \varepsilon) G_\gamma
\]  
(2.29)

\[
- \sum_{\gamma' \in \Delta_+^1} N_{\gamma',\gamma} J_{\gamma'-\gamma} - N_{\gamma,\theta} \xi_\theta T - \xi_\gamma (\gamma \cdot H) + k \partial \xi_\gamma.
\]

By inserting (2.25) into this expression we get the transformation on the reduced phase space. The parts in (2.29) containing \( \varepsilon_- \), \( \varepsilon^i \) and \( \varepsilon_\alpha \) \((\alpha \in \Delta(G))\) are consistent with (2.27) and (2.28). The nontrivial gauge transformation is that containing the anticommuting gauge parameters \( \xi_{-\gamma'} \), which is denoted by \( \delta_\xi G_\gamma \). From (2.29) and (2.23), one finds that this transformation becomes

\[
\delta_\xi G_\gamma = \frac{-k^2}{N_{\gamma,\theta-\gamma}} \partial^2 \xi_{\gamma-\theta} + \sum_{\gamma' \in \Delta_+^1} \frac{2 N_{\gamma',\gamma}}{N_{\gamma,\theta-\gamma}} k \partial \xi_{\gamma'-\theta} J_{\gamma'-\gamma} + \frac{2}{N_{\gamma,\theta-\gamma}} k \partial \xi_{\gamma-\theta} (\gamma \cdot H)
\]

\[
- N_{\gamma,\gamma-\theta} \xi_{\gamma-\theta} T + \frac{k \xi_{\gamma-\theta} \partial (\gamma \cdot H)}{N_{\gamma,\theta-\gamma}} - \frac{\xi_{\gamma-\theta} (\gamma \cdot H)^2}{N_{\gamma,\theta-\gamma}}
\]

\[
+ \sum_{\gamma' \in \Delta_+^1} \frac{N_{\gamma',\gamma}}{N_{\gamma',\theta-\gamma'}} \xi_{\theta+\gamma' \gamma'} k \partial J_{\gamma'-\gamma} + \sum_{\gamma' \in \Delta_+^1} \frac{N_{\gamma',\gamma}}{N_{\gamma',\theta-\gamma'}} \xi_{\gamma'-\theta} J_{-\gamma'-\gamma} (\gamma' - \gamma) \cdot H
\]

\[
+ \sum_{\gamma' \in \Delta_+^1} \sum_{\gamma'' \in \Delta_+^1} \frac{N_{\gamma',\gamma''}}{N_{\gamma',\theta-\gamma'}} \xi_{\theta+\gamma' \gamma''} J_{-\gamma'-\gamma''} J_{\gamma'-\gamma''},
\]

(2.30)

Here we have used the identities for the structure constants eqs. (2.13), as well as

\[
\frac{N_{\theta-\gamma,\gamma'-\theta}}{N_{\gamma,\theta-\gamma}} = -\frac{N_{\gamma',\gamma}}{N_{\gamma',\theta-\gamma'}}.
\]

(2.31)

for \( \gamma, \gamma' \in \Delta_+^1 \) in order to simplify the formula. This formula can be checked by using explicit matrix representations of \( g \) in table I. If we denote the gauge transformation in the DS-gauge as

\[
\delta = \oint \frac{dz}{2\pi i} (\Lambda(z) \cdot J_{DS}(z)),
\]

(2.32)

where

\[
J_{DS}(z) = T(z) E_\theta + \sum_{\gamma \in \Delta_+^1} G_\gamma(z) e_\gamma + \sum_{\alpha \in \Delta(G)} J_\alpha(z) E_\alpha + \sum_{i=1}^{\text{rank}(G)} H^i h^i,
\]

(2.33)
we get the Poisson bracket structure on the reduced phase space. Here we take \( h^i (i = 1, \ldots, \text{rank}(G)) \) as the generators of the Cartan subalgebra of \( G \). After rescaling \( \tilde{T} = \frac{\theta^2}{2h} T \) and \( \tilde{G}_\gamma = \sqrt{\frac{N_{\nu, \gamma}}{\theta}} G_{\gamma} \) for \( \gamma \in \Delta^+_1 \), these classical Poisson brackets may be written as follows in the form of formal operator product expansions:

\[
\begin{align*}
\tilde{T}(z)\tilde{T}(w) &= \frac{-6k}{(z-w)^4} + \frac{2\tilde{T}(w)}{(z-w)^2} + \frac{\partial \tilde{T}(w)}{z-w} + \cdots, \\
\tilde{T}(z)\tilde{G}_\gamma(w) &= \frac{2G_\gamma(w)}{(z-w)^2} + \frac{\partial G_\gamma(w)}{z-w} \\
&- \frac{1}{k}\{((\gamma \cdot H)\tilde{G}_\gamma + \sum_{\gamma' \in \Delta^+_1} N_{\gamma,-\gamma'}\tilde{G}_{\gamma'}J_{\gamma-\gamma'}(w))\} \\
&- \frac{1}{k}\{((\gamma \cdot H)\tilde{G}_\gamma + \sum_{\gamma' \in \Delta^+_1} N_{\gamma,-\gamma'}\tilde{G}_{\gamma'}J_{\gamma-\gamma'}(w))\} \\
J_\beta(z)\tilde{G}_\gamma(w) &= -\frac{N_{\beta,-\gamma}\tilde{G}_{\beta+\gamma}(w)}{z-w} + \cdots, \quad H^i(z)\tilde{G}_\gamma(w) = \frac{\gamma^i\tilde{G}_\gamma(w)}{z-w} + \cdots,
\end{align*}
\]

(2.34)

where

\[
O^2_{\gamma', \gamma} = \begin{cases} -2N_{\gamma', \gamma}\sqrt{\frac{N_{\gamma, \gamma'}}{N_{\gamma', \gamma'}}}J_{\gamma-\gamma'}, & \text{for } \gamma - \gamma' \in \Delta(G) \\
-2\gamma \cdot H, & \text{for } \gamma' = \gamma, \\
\end{cases}
\]

(2.35)

\[
O^1_{\gamma', \gamma} = \begin{cases} -N_{\gamma', \gamma}\sqrt{\frac{N_{\gamma, \gamma'}}{N_{\gamma', \gamma'}}}\partial J_{\gamma-\gamma'} + \frac{1}{k}\sqrt{\frac{N_{\gamma', \gamma}}{N_{\gamma, \gamma'}}}N_{\gamma', \gamma'}J_{\gamma-\gamma'}(\gamma' - \gamma) \cdot H \\
+ \frac{1}{k}\sqrt{\frac{N_{\gamma', \gamma}}{N_{\gamma, \gamma'}}}\sum_{\gamma'' \in \Delta^+_1} N_{\gamma'', \gamma}N_{\gamma, \gamma''}J_{\gamma-\gamma''}J_{\gamma''+\gamma}, & \text{for } \gamma - \gamma' \in \Delta(G), \\
\frac{2}{k^2}N_{\theta+\gamma'',-\gamma}N_{\theta-\gamma, \gamma'}\tilde{T} - \partial(\gamma \cdot H) + \frac{1}{k}(\gamma \cdot H)^2 \\
+ \frac{1}{k}\sum_{\gamma'' \in \Delta^+_1} (N_{\gamma'', \gamma})^2 J_{\gamma-\gamma''}J_{\gamma''+\gamma}, & \text{for } \gamma' = \gamma.
\end{cases}
\]

(2.36)

Note that the supercurrents \( \tilde{G}_\gamma \) are not primary fields with respect to \( \tilde{T}(z) \). However, if we define the total energy-momentum tensor \( T_{\text{ESA}} \) by adding the Sugawara form \( T_{\text{Sugawara}} \) of the \( \tilde{G} \) affine Lie algebra: \( T_{\text{ESA}} = \tilde{T} + T_{\text{Sugawara}} \) where

\[
T_{\text{Sugawara}} = \frac{1}{2k}\left\{ \sum_{\alpha \in \Delta(G)} \frac{\alpha^2}{2} J_\alpha \right\}.
\]

(2.37)

we can check that the supercurrents have conformal weight 3/2 with respect to \( T_{\text{ESA}} \). The classical value of the central charge \( c_{\text{ESA}} \) is \(-12k/\theta^2\).
3 The Quantum Hamiltonian Reduction

3.1 BRST formalism

In this section we discuss the quantum hamiltonian reduction \[11\] for an affine Lie superalgebra \( \hat{g} \) at level \( k \), generated by \( J_\alpha(z) (\alpha \in \Delta^0) \), \( j_\gamma(z) (\gamma \in \Delta^1) \) and \( H^i(z) (i = 1, \ldots, n) \). Let \( T_{WZNW}(z) \) be the energy-momentum tensor of an affine Lie superalgebra \( \hat{g} \), defined by the Sugawara form:

\[
T_{WZNW} = \frac{1}{2(k + h^\vee)} \left( \sum_{\alpha \in \Delta^0} \frac{\alpha^2}{2} (J_\alpha J_{-\alpha} + J_{-\alpha} J_\alpha) + \sum_{\gamma \in \Delta^1} (j_\gamma j_{-\gamma} - j_{-\gamma} j_\gamma) + \sum_{i=1}^n H^i H^i : \right),
\]

where \( h^\vee \) is the dual Coxeter number of \( g \) and \( : \) denotes the normal ordering. In order to impose the constraint for the currents at the quantum level, we have to “improve” the energy-momentum tensor by a contribution from the Cartan currents \( H^i(z) \):

\[
T_{improved}(z) = T_{WZNW}(z) - \mu \cdot \partial H(z).
\]

Here \( \mu \) is an \( n \)-vector. With respect to the improved energy-momentum tensor \( T_{improved}(z) \), the currents corresponding to the roots \( \alpha \) have conformal weights \( 1 + \mu \cdot \alpha \). We are concerned with a class of Lie superalgebras whose even subalgebras are \( G \oplus A_1 \), where \( G \) is a semisimple Lie algebra. In the previous section we have considered the constraint \( J_\theta(z) = 1 \) but the \( G \) currents have conformal dimension one, so in order for the constraint to make sense, it must have improved conformal dimension, 0. This means that the vector \( \mu \) should satisfy

\[
\mu \cdot \theta = 1, \quad \mu \cdot \alpha = 0, \quad \text{for } \alpha \in \Delta(G).
\]

These equations (3.3) determine the vector \( \mu \) uniquely:

\[
\mu = \frac{\theta}{\theta^2} \quad (3.4)
\]

From (2.10), we find that the fermionic currents \( j_\gamma(z) (j_{-\gamma}(z)) \) for the positive roots \( \gamma \in \Delta^1_+ \) have conformal weight 3/2 (1/2). The current \( J_\theta(z) \) has conformal weight 2.
Now we use the BRST-gauge fixing procedure. In the previous section we took the Drinfeld-Sokolov gauge (2.23) and derived the Poisson bracket structure for the currents. In order to study the representation of the algebra, it is very useful to consider the free field representation. This is realized by taking the “diagonal” gauge

\[ J_\theta(z) = 0, \quad \theta \cdot H(z) = a_0 \partial \phi(z), \]
\[ j_{-\gamma}(z) = \sqrt{N_{-\gamma,-\theta+\gamma}} \chi_\gamma(z), \quad \text{for} \quad \gamma \in \Delta^1_+. \]  

(3.5)

in (2.8). Here \( \phi(z) \) is a free boson coupled to the world sheet curvature and \( a_0 \) is a constant. \( \chi_\gamma(z) \) are free fermions satisfying operator product expansions

\[ \chi_\gamma(z) \chi_{\theta-\gamma'}(w) = \frac{\delta_{\gamma,\gamma'}}{z-w} + \cdots, \quad \text{for} \quad \gamma, \gamma' \in \Delta^1_+. \]  

(3.6)

This is consistent with the OPE’s of \( \hat{g} \) together with the constraint. Let us introduce fermionic ghosts \((b_\theta(z), c_\theta(z))\) with conformal weights \((0,1)\) and bosonic ghosts \((\tilde{b}_\gamma(z), \tilde{c}_\gamma(z))\) of weight \((\frac{1}{2}, \frac{1}{2})\) for \( \gamma \in \Delta^1_+ \). The BRST current \( J_{BRST}(z) \) is defined as (cf. ref. [19])

\[ J_{BRST}(z) = c_\theta (J_{-\theta} - 1) + \sum_{\gamma \in \Delta^1_+} \tilde{c}_\gamma (j_{-\gamma} - \sqrt{N_{-\gamma,-\theta+\gamma}} \chi_\gamma) + \frac{1}{2} \sum_{\gamma \in \Delta^1_+} N_{\gamma,\theta-\gamma} : \tilde{c}_\gamma \tilde{c}_{\theta-\gamma} b_\theta : . \]  

(3.7)

We can easily show that the BRST charge \( Q_{BRST} = \oint \frac{dz}{2\pi i} J_{BRST}(z) \) satisfies the nilpotency condition \( Q^2_{BRST} = 0 \). The total energy-momentum tensor is expressed as

\[ T_{total}(z) = T_{improved}(z) + T_\chi + T_{ghost}(z), \]  

(3.8)

where

\[ T_{ghost}(z) = : (\partial b_\theta) c_\theta + \frac{1}{2} \sum_{\gamma \in \Delta^1_+} (\tilde{b}_\gamma \partial \tilde{c}_\gamma - (\partial \tilde{b}_\gamma) \tilde{c}_\gamma) :, \]
\[ T_\chi(z) = -\frac{1}{2} \sum_{\gamma \in \Delta^1_+} : \chi_{\theta-\gamma} \partial \chi_\gamma :. \]  

(3.9)

The central charge \( c \) of the total system is computed to be

\[ c_{total} = c_{WZNW} - 12 k \mu^2 + \frac{1}{2} |\Delta^1_+| - 2 - |\Delta^1_+|, \]  

(3.10)
(where $|\Delta^1_+|$ is still the number of positive odd roots). The last two terms are contributions from ghost fields. The central charge $c_{WZNW}$ of the WZNW models on the Lie supergroup at level $k$ is given by the formula:

$$c_{WZNW} = \frac{k \text{sdimg}}{k + h^\vee},$$

where the super dimension $\text{sdimg}$ of a Lie superalgebra $g$ is defined as $\text{dim}g_0 - \text{dim}g_1$. The list for the corresponding central charges is shown in table 2. The results here are in complete agreement with previous calculations using a variety of different techniques. Thus, the results for $A(n|1), B(n|1), D(n|1)$ were obtained in ref. [7]; the one for $D(2|1; \alpha)$ agrees with ref. [6, 30] (see also, [31]), the result for $D(2|n)$ and $G(3)$ agrees with ref. [23], and the result for $F(4)$ with that of [32]. Our treatment here, however, provides a unifying framework.

In the classical limit $k \to \infty$, the expression (3.17) becomes $-12k/\theta^2$, which agrees with the result (2.34) obtained in the previous section.

### 3.2 The Free Field Representation

We consider the free field representations of $G$ extended superconformal algebra based on the Wakimoto construction [25] of the affine Lie superalgebra $\hat{g}$ [33]. Let us introduce bosonic ghosts $(\beta_\alpha(z), \gamma_\alpha(z))$ for even positive roots $\alpha \in \Delta^0_+$ with conformal dimensions $(1,0)$, fermionic ghosts $(\eta_\gamma(z), \xi_\gamma(z))$ for odd positive roots $\gamma \in \Delta^1_+$ with (unimproved) conformal dimensions $(1,0)$ and $n$ free bosons $\varphi(z) = (\varphi_1(z), \ldots, \varphi_n(z))$ coupled to the world-sheet curvature, satisfying the operator product expansions:

$$\begin{align*}
\beta_\alpha(z)\gamma_{\alpha'}(w) &= \frac{\delta_{\alpha,\alpha'}}{z - w} + \cdots, & \eta_\gamma(z)\xi_{\gamma'}(w) &= \frac{\delta_{\gamma,\gamma'}}{z - w} + \cdots, \\
\varphi_i(z)\varphi_j(w) &= -\delta_{ij}\ln(z - w) + \cdots.
\end{align*}$$

Using these free fields the energy-momentum tensor is expressed as

$$T_{WZNW}(z) = \sum_{\alpha \in \Delta^0_+} :\beta_\alpha \partial \gamma_\alpha : - \sum_{\gamma \in \Delta^1_+} :\eta_\gamma \partial \xi_\gamma : - \frac{1}{2} : (\partial \varphi)^2 : -\frac{i\varphi \cdot \partial^2 \varphi}{\alpha_+},$$

(3.13)
where $\alpha_+ = \sqrt{k + h}$ and $\rho = \rho_0 - \rho_1$. $\rho_0$ ($\rho_1$) is half the sum of positive even (odd) roots:

$$\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta^0_+} \alpha, \quad \rho_1 = \frac{1}{2} \sum_{\gamma \in \Delta^1_+} \gamma.$$  \hspace{1cm} (3.14)

The Cartan part currents $H^i(z)$ are given by

$$H^i(z) = -i \alpha_+ \partial \phi^i + \sum_{\alpha \in \Delta^0_+} \alpha^i : \gamma_\alpha \beta_\alpha : + \sum_{\gamma \in \Delta^1_+} \gamma^i : \xi_\gamma \eta_\gamma :.$$  \hspace{1cm} (3.15)

After improvement of the energy momentum tensor, the ghost systems acquire conformal dimensions, $(0, 1)$ for $(\beta_\theta, \gamma_\theta)$ and $(\frac{1}{2}, \frac{1}{2})$ for $(\eta_\gamma, \xi_\gamma)$. In the BRST gauge fixing procedure it is natural to assume that the multiplets $(\beta_\theta, \gamma_\theta, b_\theta, c_\theta)$ and $(\eta_\gamma, \xi_\gamma, \tilde{b}_\gamma, \tilde{c}_\gamma)$ for $\gamma \in \Delta^1_+$ form Kugo-Ojima quartets [34], and that the corresponding energy-momentum tensor is BRST-exact. In the case of the $\widehat{sl}(N)$ affine Lie algebra, this has been proven both by homological techniques [33], and by a more direct method [36]. This ansatz enables us to write the total energy-momentum tensor in the form:

$$T_{total}(z) = T_{ESA}(z) + \{Q_{BRST}, * \},$$  \hspace{1cm} (3.16)

where $T_{ESA}$ is the energy-momentum tensor of $G$ extended superconformal algebra:

$$T_{ESA}(z) = -\frac{1}{2} : (\partial \phi)^2 : -i \left( \frac{\rho}{\alpha_+} - \alpha_+ \mu \right) \cdot \partial^2 \phi + \sum_{\alpha \in \Delta^1(G)} : \beta_\alpha \partial \gamma_\alpha : + \frac{1}{2} \sum_{\gamma \in \Delta^1_+} : (\partial \chi_\gamma) \chi_{\theta-\gamma} :.$$  \hspace{1cm} (3.17)

The central charge of $T_{ESA}$ is given by the formula

$$c = n - 12 \left( \frac{\rho}{\alpha_+} - \alpha_+ \mu \right)^2 + 2|\Delta_+(G)| + \frac{1}{2}|\Delta^1_+|.$$  \hspace{1cm} (3.18)

This may be shown to be equal to $c_{total}$ (3.10).

In the case that $\widehat{G}$ is expressed as a direct sum of simple affine Lie algebras $\oplus_i \widehat{G}_i$, the relation between the level $k$ of the affine Lie superalgebra $\widehat{g}$ and that, $K_i$, of the $i$’th affine Lie algebra $\widehat{G}_i$, is given by considering the decomposition of the vector $\rho/\alpha_+ - \alpha_+ \mu$ into the roots of the even subalgebras $(\oplus_i G_i) \oplus A_1$:

$$\frac{\rho}{\alpha_+} - \alpha_+ \mu = \sum_i \frac{\rho G_i}{\alpha_+} + (\frac{1}{2} - \frac{|\Delta^1_+|}{4}) \frac{\alpha_+}{\theta^2} \theta.$$  \hspace{1cm} (3.19)
Here $\rho_{G_i}$ is half the sum of positive roots of the subalgebra $G_i$. The above decomposition means that

$$k + h^\vee = \frac{\alpha_{Li}^2}{2}(K_i + h_i^\vee),$$

(3.20)

where $h_i^\vee$ is the dual Coxeter number of the even subalgebra $G_i$ and $\alpha_{Li}$ is a long root of the subalgebra $G_i$.

The $n$ bosons $\varphi(z) = (\varphi_1(z), \ldots, \varphi_n(z))$ coupled to the world sheet curvature, can be divided into two classes, due to the decomposition of the Cartan subalgebra $h = H_G \oplus H_{A_1}$, where $H_G$ and $H_{A_1}$ are the Cartan subalgebras of the even subalgebras $G$ and $A_1$, respectively. A boson $\theta \cdot \varphi$ in the $\theta$ direction of the root space of the even subalgebra $A_1$, commutes with the bosons lying along the root space of the even subalgebra $G$. The remaining $n - 1$ free bosons are used for the free field representation of the $\hat{G}$ affine Lie algebra, combined with $(\beta_\alpha, \gamma_\alpha)$-systems ($\alpha \in \Delta_+(G)$). If we define a Feigin-Fuchs boson $\phi$:

$$\phi(z) = \frac{\theta \cdot \varphi(z)}{\sqrt{\theta^2}},$$

(3.21)

the energy-momentum tensor (3.17) becomes $T_{ESA}(z) = T_\phi(z) + T_G(z) + T_\chi(z)$, where

$$T_\phi = -\frac{1}{2}(\partial \phi)^2 - \frac{iQ}{2} \partial^2 \phi, \quad Q = \sqrt{\theta^2}(\frac{2 - |\Delta_+|}{2\alpha_+} - \frac{2\alpha_+}{\theta^2}).$$

(3.22)

$T_G(z)$ is the Sugawara energy-momentum tensor of the affine Lie algebra $\hat{G}$.

So far we have discussed the energy-momentum tensor at the quantum level using the free field representation. In the following we will give some examples of the free field representation of $G$ extended superconformal algebras. Several of those have been provided previously in the literature. This is true for $osp(N|2)$ and for $A(n|1)$, $n > 1$, \cite{37}, and for $A(1|1)$, \cite{38}. We repeat the results here from our point of view, partly for illustration and completeness, partly because we shall need them for the construction of screening operators in sect. 5. Our results for $D(2|n)$, however, are new. The same is true for our results for $D(2|1; \alpha)$ presented in ref. \cite{31}.
4 Examples: Non-exceptional extended superconformal algebras

In this section we discuss the free field representation of specific extended superconformal algebras. The free field representations of $so(N)$ and $u(N)$ extended superconformal algebras were found in ref. [37] from the viewpoint of the $osp(N|2)$ KdV equations [39]. In ref. [21] two of us gave the classical free field representation by connecting the DS-gauge and the diagonal gauge (the Miura transformation). In the quantum case, we need some modification of the coefficients due to double contractions in the operator product expansions. In the present paper we treat the non-exceptional type Lie superalgebras $A(n|1)$, $B(n|1)$, $D(n|1)$ and $D(2|n)$. An example of an exceptional type Lie superalgebra is $D(2|1;\alpha)$, from which one gets the doubly extended superconformal algebra $\tilde{A}_\gamma$. This free field representation is treated in a separate paper [31]. Although we may treat the other exceptional type Lie superalgebras $F(4)$ and $G(3)$ in a similar way, we defer explicit formulas to a separate publication.

4.1 $A(n|1)$

Let $\hat{J}_{ij}$ ($i, j = 1, \ldots, N = n + 1$) be the $\hat{sl}(N)$ currents at level $K$, satisfying

$$\hat{J}_{ij}(z)\hat{J}_{kl}(w) = \frac{K(\delta_{jk}\delta_{il} - \delta_{ij}\delta_{kl})}{(z-w)^2} + \frac{-\delta_{jk}\hat{J}_{il}(w) + \delta_{li}\hat{J}_{kj}(w)}{z-w} + \ldots. \quad (4.1)$$

The generators of the $u(N)$ extended superconformal algebras are the $u(N)$ currents $\hat{J}_{ij}(z)$, and $2N$ supercurrents $G_i(z)$, $\bar{G}_i(z)$ as well as the energy-momentum tensor $T(z)$. Their free field representations [37] are given by

$$J_{ij} = \hat{J}_{ij} - \chi_i\bar{\chi}_j + \sqrt{\frac{2}{N(N-2)}}\alpha_+\delta_{ij}\partial\hat{\phi},$$

$$G_i = i\partial\phi\chi_i - Q\partial\chi_i - \frac{i\sqrt{2}}{\alpha_+}(\hat{J}_{ik} - \chi_i\bar{\chi}_k - \sqrt{\frac{N-2}{2N}}\alpha_+\partial\hat{\phi}\delta_{ik})\chi_k,$$

$$G_i = i\partial\phi\bar{\chi}_i - Q\partial\bar{\chi}_i + \sqrt{\frac{2}{N-2}}\bar{\chi}_k(\hat{J}_{ki} - \chi_k\bar{\chi}_i - \sqrt{\frac{N-2}{2N}}\alpha_+\partial\hat{\phi}\delta_{ik}), \quad (4.2)$$
\[ T = -\frac{1}{2}(\partial \phi)^2 - \frac{iQ}{2} \partial^2 \phi - \frac{1}{2}(\bar{\chi}_i \partial \chi_i + \chi_i \partial \bar{\chi}_i) - \frac{1}{2}(\partial \hat{\phi})^2 + \frac{1}{2(K+N)} : \hat{J}_{ij} \hat{\phi} : , \]

where \( \alpha_+ = \sqrt{K+N} \) and \( Q = \frac{i\sqrt{2(1-N+\alpha_+^2)}}{\alpha_+} \). \( \hat{\phi}(z) \) is a free boson, which represents the \( u(1) \) direction of the even subalgebra, defined as \( \hat{\phi}(z) = \nu \cdot \varphi / \sqrt{\nu^2} \) with \( \nu = \frac{1}{N-2} \{ 2 (\sum_{i=1}^{N} e_i) - N(\delta_1 + \delta_2) \} \). Vectors, \( e_i, \delta_i \) are introduced in Appendix A. The \( u(1) \) current \( U(z) \) is given by the trace of a matrix \( J = (J_{ij}) \):

\[ U = J_{ii} = \bar{\chi}_i \chi_i + \sqrt{\frac{2N}{N-2}} \alpha_+ \partial \hat{\phi}. \quad (4.3) \]

One finds that \( J_{ij}(z) \) satisfies the operator product expansions:

\[ J_{ij}(z)J_{kl}(w) = \frac{(K+1)\delta_{jk}\delta_{il} - \frac{(K+2)\delta_{ij}\delta_{kl}}{N-2}}{(z-w)^2} + \frac{-\delta_{jk}J_{il}(w) + \delta_{li}J_{kj}(w)}{z-w} + \cdots. \quad (4.4) \]

The supercurrents \( G_i(z) \) and \( \bar{G}_i(z) \) belong to the \( N \)-dimensional representation of \( u(N) \) Kac-Moody algebra and its conjugate representation, respectively:

\[ J_{ij}(z)G_k(w) = \frac{-\delta_{jk}G_i(w)}{z-w} + \cdots, \quad J_{ij}(z)\bar{G}_k(w) = \frac{\delta_{ik}\bar{G}_j(w)}{z-w} + \cdots. \quad (4.5) \]

The nontrivial operator product expansions for the supercurrents are

\[ G_i(z)\bar{G}_j(w) = \frac{\delta_{ij}2(K+N+2)}{K+N} \frac{(z-w)^3}{(z-w)^3} + \frac{-2(2K+N+2)}{K+N} \frac{J_{ij}(w)}{(z-w)^2} + \frac{O_{ij}(w)}{z-w} + \cdots, \quad (4.6) \]

where

\[ O_{ij} = \delta_{ij}2T - \frac{2K+2+N}{K+N} \partial J_{ij} + \delta_{ij} \frac{K+2}{K+N} \partial U \]

\[ + \frac{1}{K+N} \left\{ 2(J_{ik}J_{kj})_{S} : UJ_{ij} : + \delta_{ij}(\frac{1}{2} : J_{ik}J_{lk} : + \frac{1}{2} : U^2 :) \right\}. \quad (4.7) \]

where we define the symmetric normal ordering \((AB)_{S}(z)\) of two fields \( A(z) \) and \( B(w) \), by \( \frac{1}{2}( : AB : + : BA :) (z) \). Note that for \( A(1|1) \) one gets \( N = 4 \) \( sl(2) \) extended superconformal algebra since the \( u(1) \) current decouples from the algebra. Recently the free field representation has been developed in ref. [38].
4.2 \( osp(N|2) \)

In the definition of the Lie superalgebra \( osp(N|2) \), we use a negative metric for the root space of the even subalgebra \( so(N) \), which makes the level of the affine \( so(N) \) negative (cf. Appendix A). In order to discuss the representation theory, it is convenient to change the sign of the metric.

The free field representation of \( so(N) \) extended superconformal algebras, was first given in ref. \[37\]. Following this, it is convenient to introduce the anti-symmetric basis for the \( so(N) \) currents \( \hat{J}_{ij}(z) \) \((1 \leq i, j \leq N)\) at level \( K \), satisfying \( \hat{J}_{ij} = -\hat{J}_{ji} \). The operator product expansions are

\[
\hat{J}_{ij}(z)\hat{J}_{lm}(w) = \frac{-K(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})}{(z-w)^2} + \frac{-\delta_{il}\hat{J}_{jm}(w) + \delta_{ji}\hat{J}_{lm}(w) + \delta_{jm}\hat{J}_{li}(w) - \delta_{im}\hat{J}_{lj}(w)}{z-w} + \cdots. \tag{4.8}
\]

Let \( \psi_i (i = 1, \ldots, N) \) be real fermions satisfying \( \psi_i(z)\psi_j(w) = \delta_{ij}z-w + \cdots \). The generators of the \( so(N) \) extended superconformal algebras are given by

\[
\begin{align*}
T &= -\frac{1}{2} \left( \frac{\partial^2 \phi}{2} - \frac{iQ}{2} \partial^2 \phi \right) - \frac{1}{2} : \psi_i \partial \psi_i : - \frac{1}{2(K+N-2)} : \hat{J}_{ij} \hat{J}_{ji} : , \\
G_i &= i\partial \phi \psi_i - Q \partial \psi_i + \frac{i}{\alpha_+} \hat{J}_{ij} \psi_j , \\
J_{ij} &= \hat{J}_{ij} + \psi_i \psi_j ,
\end{align*}
\]

where \( \alpha_+ = \sqrt{K+N-2} \) and \( Q = \frac{i(2-N-\alpha_+^2)}{\alpha_+} \). The currents \( J_{ij} \) satisfy the level \( K + 1 \) \( so(N) \) Kac-Moody algebra. Other nontrivial operator product expansions of generators are

\[
\begin{align*}
J_{ij}(z)G_k(w) &= \frac{\delta_{ik}G_i(w) - \delta_{ik}G_j(w)}{z-w} + \cdots , \\
G_i(z)G_j(w) &= \frac{2K^2 + 2K + N - 2}{K+N-2} \frac{\delta_{ij}}{(z-w)^3} + \frac{2K+N-2}{K+N-2} J_{ij}(w) + \frac{O_{ij}(w)}{z-w} + \cdots , \tag{4.10}
\end{align*}
\]

where

\[
O_{ij} = \delta_{ij} 2T + \frac{1}{2} \frac{12K + N - 2}{K+N-2} \partial J_{ij} - \frac{1}{K+N-2} \left\{ (J_{ik}J_{kj})S - \delta_{ij} : J_{lm}J_{ml} : \right\} . \tag{4.11}
\]
4.3 \( D(2|n) \)

The Lie superalgebra \( D(2|n) \) has the even subalgebra \( so(4) \oplus sp(2n) \). Since \( so(4) \) is isomorphic to \( sl(2) \oplus sl(2) \), we can use the hamiltonian reduction to find an \( sl(2) \oplus sp(2n) \) extended superconformal algebra. In this case we use a negative metric for the root space of the \( sl(2) \) subalgebra, and a positive metric for the root space of the \( sp(2n) \) subalgebra, so the levels of the \( sl(2) \) and of the \( sp(2n) \) algebras have opposite signs. Let \( \hat{J} \) and \( \hat{I} \) be the \( sp(2n) \) currents with level \( K \) and the \( sl(2) \) currents with level \( \tilde{K} \). From (3.20) we find that the level \( \tilde{K} \) is equal to \( -K - 2n - 2 \). An element of \( sp(2n) \) may be represented by a matrix of the form

\[
M = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},
\]

(4.12)

where \( A, B \) and \( C \) are \( n \times n \) matrices, the matrix \( 'A \) is defined as \( 'A_{i,j} = A_{n+1-j,n+1-i} \), and \( B = 'B, C = 'C \). We denote the currents in the \( sp(2n) \) matrix \( \hat{J} \) as \( (i,j \leq n) \)

\[
\hat{J}_{ij} = \hat{J}_{i,j}, \\
\hat{J}_{2n+1-i,j} = \hat{J}_{i,j}, \\
\hat{J}_{i,2n+1-j} = \hat{J}_{i,j}.
\]

(4.13)

The Cartan algebra element \( H^i = \hat{J}_{ii} \) is written as \( \hat{J}_{i,-i} \) in this notation. Note that \( \hat{J}_{i+j} = \hat{J}_{i,j} \) and \( \hat{J}_{i-j} = \hat{J}_{-i,j} \). We write the \( sl(2) \) currents as \( \hat{I}^\pm \) and \( \hat{I}^3 \). Introduce \( 4n \) free fermions \( \psi_{+i}^+ \) and \( \psi_{-i}^- \) with the OPE’s \( \psi_{+i}^-(z)\psi_{+j}^+(w) = \frac{\delta_{ij}}{z-w} + \cdots \). We can then write the \( sp(2n) \oplus sl(2) \) Kac-Moody currents of the extended superconformal algebra as

\[
I^3 = \hat{I}^3 - \frac{1}{2} : \psi_{+m}^- \psi_{-m}^- : - \frac{1}{2} : \psi_{-m}^- \psi_{+m}^- : , \quad I^\pm = \hat{I}^\pm + \psi_{+m}^\pm \psi_{-m}^\pm ,
\]

(4.14)

and

\[
J_{i-j} = \hat{J}_{i,-j} + : \psi_{+i}^+ \psi_{-j}^- : + : \psi_{+i}^- \psi_{-j}^+ : ,
\]

\[
J_{i+j} = \hat{J}_{i,j} + \psi_{+i}^+ \psi_{-j}^- - \psi_{+i}^- \psi_{-j}^+ ,
\]

\[
J_{-i-j} = \hat{J}_{i,-j} + \psi_{-i}^+ \psi_{-j}^- - \psi_{-i}^- \psi_{-j}^+ ,
\]

(4.15)
and the free field representation of the supercurrents are given by

\[
G_{+i}^\pm = -\frac{Q}{\sqrt{2}} \partial \psi_{+i}^\pm + \frac{i}{\sqrt{2}} \psi_{+i}^\pm \partial \phi + \frac{i}{\alpha_+} \left( \mp: \psi_{-m}^\pm J_{m+i}^+: -: \psi_{+m}^\pm J_{i-m}^+: \pm \psi_{+i}^{\pm} \hat{I}^3 + \psi_{+i}^{\pm} \hat{I}^\pm \right),
\]

\[
G_{-i}^\pm = -\frac{Q}{\sqrt{2}} \partial \psi_{-i}^\pm + \frac{i}{\sqrt{2}} \psi_{-i}^\pm \partial \phi + \frac{i}{\alpha_+} \left( \mp: \psi_{+m}^\pm J_{i-m}^-:+: \psi_{-m}^\pm J_{+m-i}^-: \pm \psi_{-i}^{\pm} \hat{I}^3 - \psi_{-i}^{\pm} \hat{I}^\pm \right),
\]

(4.16)

where \(\alpha_+ = \sqrt{K + 2n}\) and \(Q = -i\sqrt{2} \alpha_+^{2-2n+1}\). The superscript \(\pm\) corresponds to an \(sl(2)\) isospin of \(+1\) for the supercurrents and fermions, and \(-\) corresponds to isospin \(-\frac{1}{2}\), while the subscripts \(\pm i\) corresponds to a numbering of the weights of the \(2n\)-dimensional vector representation of \(sp(2n)\). With these representations, we get the standard OPE’s for the Kac-Moody currents, but with levels \(K + 2\) for \(sp(2n)\) and \(-K - n - 2\) for \(sl(2)\):

\[
J_{+i-j}(z)J_{+k-l}(w) = \frac{(K + 2)\delta_{il}\delta_{jk} + \delta_{kj}J_{+i-l}(w) - \delta_{il}J_{+j-k}(w)}{(z-w)^2} + \ldots,
\]

\[
J_{+i-j}(z)J_{+k+l}(w) = \frac{\delta_{ij}J_{+i+k}(w) + \delta_{jk}J_{z+i+l}(w)}{z-w} + \ldots,
\]

\[
J_{+i-j}(z)J_{-k-l}(w) = \frac{-\delta_{ik}J_{j-i}(w) + \delta_{il}J_{-j-k}(w)}{z-w} + \ldots,
\]

\[
J_{-i-j}(z)J_{+k+l}(w) = \frac{(K + 2)(\delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl})}{(z-w)^2} - \frac{\delta_{kj}J_{l-i}(w) + \delta_{ik}J_{l-j}(w) + \delta_{il}J_{k-j}(w) + \delta_{jl}J_{k-i}(w)}{z-w} + \ldots,(4.17)
\]

\[
I^3(z)I^\pm(w) = \frac{\pm I^\pm(w)}{z-w} + \ldots, \quad I^3(z)I^3(w) = \frac{-(K + n + 2)/2}{(z-w)^2} + \ldots,
\]

\[
I^+(z)I^-(w) = \frac{-K - n - 2}{(z-w)^2} + \frac{2I^3(w)}{z-w} + \ldots. \quad (4.18)
\]

The OPE’s of the \(sp(2n)\) Kac-Moody currents with the supercurrents \(G\) are

\[
J_{-i+j}(z)G_{+k}^\pm(w) = \frac{\delta_{ik}G_{+j}^\pm(w)}{z-w} + \ldots,
\]

\[
J_{-i+j}(z)G_{-k}^\pm(w) = \frac{-\delta_{jk}G_{-i}^\pm(w)}{z-w} + \ldots,
\]

\[
J_{+i+j}(z)G_{-k}^\pm(w) = \frac{\pm(\delta_{ik}G_{+j}^\pm(w) + \delta_{jk}G_{+i}^\pm(w))}{z-w} + \ldots,
\]

\[
J_{-i-j}(z)G_{+k}^\pm(w) = \frac{\pm(\delta_{ik}G_{+j}^\pm(w) + \delta_{jk}G_{+i}^\pm(w))}{z-w} + \ldots, \quad (4.19)
\]
and the OPE’s of the $sl(2)$ Kac-Moody currents with the supercurrents are

\[ I^3(z) G^\pm_{\pm i}(w) = \frac{\pm G^\pm_{\pm i}(w)}{z-w} + \cdots, \quad I^3(z) G^{+}_{\pm i}(w) = \frac{\frac{1}{2}G^\pm_{\pm i}(w)}{z-w} + \cdots, \]

\[ I^{-}(z) G^+_i(w) = \frac{\pm G^+_i(w)}{z-w} + \cdots, \quad I^{+}(z) G^-_i(w) = \frac{\pm G^-_i(w)}{z-w} + \cdots. \tag{4.20} \]

Finally we get the slightly more complicated expression for the operator product of two supercurrents

\[ G^\pm_{\pm i}(z) G^{\pm}_{\pm j}(w) = \frac{\pm 2 \alpha^2 J_{i+} J^\pm(w)}{z-w} + \cdots, \]

\[ G^\pm_{\pm i}(z) G^\pm_{\mp j}(w) = \frac{\pm 2 \alpha^2 J_{i-} J^\pm(w)}{z-w} + \cdots, \]

\[ G^{\pm}_{\pm i}(z) G^{\pm}_{\pm j}(w) = \frac{-\frac{2}{\alpha^2} \delta_{ij} K I^\pm(w) + \frac{1}{\alpha^2} \delta_{ij} K \partial I^\pm(w) + \frac{2}{\alpha^2} J_{i-} J^\pm(w)}{(z-w)^2} + \cdots, \]

\[ G^{\pm}_{\pm i}(z) G^+_{\pm j}(w) = \frac{\frac{2}{\alpha^2} J_{i+} J^+(w) - \frac{2K}{\alpha^2} \delta_{ij} I^3(w) + \frac{1}{\alpha^2} \delta_{ij} (J_{i-} J_{i+} I^3(w))}{(z-w)^2} + \frac{O^{\pm}_{\pm i+}(w)}{z-w} + \cdots, \tag{4.21} \]

where

\[ O^{\pm}_{\pm i+j} = \frac{K + n + 2}{\alpha^2} \partial J_{\pm(i+j)} + \frac{2}{\alpha^2} J_{\pm(i+j)} I^3 \]

\[ + \frac{1}{\alpha^2} (J_{k\pm i} J_{\pm k}) s - \frac{1}{\alpha^2} (J_{i-} J_{i+k}) s, \]

\[ O^{\pm}_{\pm i+j} = \pm \left[ \frac{(K + n + 2)}{\alpha^2} \partial J_{\pm(i-j)} + \frac{2}{\alpha^2} J_{\pm(i-j)} I^3 \right] \]

\[ + \frac{1}{\alpha^2} [(J_{i-k}(J_{i-j}) s + (J_{i+k}(J_{i-j}) s)] + \delta_{ij} \left[ -\frac{1}{4\alpha^2} \text{tr} : J^2 : + \frac{1}{\alpha^2} \{ K \partial I^3 + 2 : (I^3)^2 : + 2(I^3) s \} + T \right]. \tag{4.22} \]

Here the energy momentum tensor $T$ is $T = T_\phi + T_{sl(2)} + T_{sp(2n)} + T_\psi$, and

\[ T_\phi = -\frac{1}{2} : (\partial \phi)^2 : - \frac{iQ}{2} \partial^2 \phi, \]

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\[ T_{sl(2)} = -(\alpha_+)^{-2} \left( (\hat{I}^3)^2 + \hat{I}^- \hat{I}^+ \right)_S, \]
\[ T_{sp(2n)} = (\alpha_+)^{-2} \frac{1}{4} \text{tr} \hat{j}^2, \]
\[ T_\psi = \frac{1}{2} \left( : \partial \psi_m^+ \psi_m^- : + : \partial \psi_m^- \psi_m^+ : + : \partial \psi_m^+ \psi_m^- : \right). \]

These OPE’s are identical with the recent results obtained by a different approach in ref. [23]. In the next section we shall discuss the structure of screening operators for \( G \) extended superconformal algebras.

5 Degenerate Representation of \( G \) extended superconformal algebras

In this section we discuss the representation theory of an extended superconformal algebra with \( \hat{G} \) affine Lie algebra symmetry using free fields. The Fock space of the \( G \) extended superconformal algebras is a tensor product of ones for \( |\Delta_+^1| \) fermions \( \chi_\gamma \), free fields for the affine Lie algebra \( \hat{G} \), and a free boson \( \phi(z) \) coupled to the world sheet curvature.

The free field representations of affine Lie algebras \( \hat{G} \) are studied in refs. [25]. Let \( \tilde{\alpha}_i \) \((i = 1, \ldots, r)\) be simple roots of the even subalgebra \( G \). \( \tilde{\lambda}_i \) \((i = 1, \ldots, r)\) the fundamental weights of \( G \) satisfying \( \frac{2\tilde{\lambda}_i \cdot \tilde{\alpha}_j}{\tilde{\alpha}_j^2} = \delta_{ij} \). \( \Phi^{\tilde{\lambda}}_\lambda(z) \) is a primary field of the affine Lie algebra \( \hat{G} \) at level \( K \), with weight \( \tilde{\lambda} \) in the highest weight module with highest weight \( \tilde{\Lambda} \). In the free field representation, this field can be expressed as \( p^{\tilde{\lambda}}_\lambda(z) \text{exp} \left( \frac{i\tilde{\lambda} \cdot \phi(z)}{\alpha_+} \right) \), where \( p^{\tilde{\lambda}}_\lambda \) is a polynomial consisting of terms, \( \gamma_{\alpha_i} \cdots \gamma_{\alpha_k}(z) \) \((\alpha_i \in \Delta_+(G))\) such that \( \tilde{\lambda} = -\tilde{\Lambda} + \alpha_1 + \cdots + \alpha_k \). Note that in the present prescription, the vertex operator \( \text{exp} \left( \frac{i\tilde{\lambda} \cdot \phi(z)}{\alpha_+} \right) \) represents the lowest weight state \( \Phi^{-\tilde{\lambda}}_{-\lambda}(z) \).

Denote the total Fock space as \( F_{\chi, \lambda, p} = F^\chi \otimes F^G_{\tilde{\lambda}} \otimes F^\phi_p \), where \( F^\chi \) is a fermionic Fock space constructed from \( \chi_\gamma \) \((\gamma \in \Delta_+^1)\). \( F^G_{\tilde{\lambda}} \) is a Fock space of the algebra \( \hat{G} \) built on a primary field \( \Phi^{\tilde{\lambda}}_{-\lambda} = e^{i\tilde{\lambda} \cdot \phi(z)} \). \( F^\phi_p \) is a Fock space built on a vertex operator \( V_p(z) = e^{i\nu \sqrt{p} \phi(z)} \).

The dual spaces \( (F^G_{\tilde{\lambda}})^* \) and \( (F^\phi_p)^* \) are isomorphic to \( F^{-2p_{\hat{G}} - \tilde{\lambda}} \) and \( F^{-\phi - p} \), respectively.

A primary field of a \( G \) extended superconformal algebra is expressed as the products
of three fields:
\[ V_{\gamma_1, \ldots, \gamma_l, \lambda}(z) = \chi_{\gamma_1}(z) \cdots \chi_{\gamma_l}(z) \Phi_{\lambda}(z) e^{ip\sqrt{\theta^2} \phi(z)}, \]
where \( \gamma_i \) are positive odd roots. The conformal weight of (5.1) is given by
\[ \Delta = \frac{l}{2} + \frac{\Lambda(\Lambda + 2\rho_G)}{2\alpha^2_+} + \frac{1}{2}(p^2 + Qp)\theta^2. \]

5.1 Screening operators

In order to study the representation of the algebra using free fields, we must specify the screening operators which commute with the generators of the extended superconformal algebra. We consider screening operators which correspond to the simple roots of the Lie superalgebra \( g \). These screening operators are BRST-equivalent to those of the affine Lie superalgebra \( \hat{g} \) [11]. In the present choice of the simple root system of the Lie superalgebra in table [1], the simple roots of the even subalgebra \( G \) are a subset of those of \( g \) (see Appendix A). Thus we will get the screening operators corresponding to the simple roots of the affine Lie algebra \( \hat{G} \). Since the remaining simple roots are odd, we will obtain fermionic type screening operators. In addition, we will find another screening operator, which is necessary to characterize the \( A_1 \) even subalgebra corresponding to the root \( \theta \).

5.1.1 Affine screening operators.

First we can take the screening operators \( S_{\bar{\alpha}}(z) \) of the affine Lie algebra \( \hat{G} \) as those of \( G \) extended superconformal algebra:
\[ S_{\bar{\alpha}}(z) = s_{\bar{\alpha}}(z)e^{i\bar{\alpha}_+ \phi(z) \alpha_+}, \]
where \( s_{\bar{\alpha}} \) consists of terms like \( \beta_{\bar{\alpha}} \) and \( \gamma_{\alpha_1} \cdots \gamma_{\alpha_k} \beta_{\alpha_1 + \cdots + \alpha_k + \bar{\alpha}} \) with \( \alpha_1, \ldots, \alpha_k \in \Delta_+(G) \). These screening operators are used for the characterization of singular vectors in the Fock modules of the affine Lie algebra \( \hat{G} \).
5.1.2 Fermionic screening operators.

Next, we consider a class of screening operators, which corresponds to the odd simple roots. In the diagonal gauge (free field realization) we have fermions for every negative odd root. We expect that the resulting screening operators are expressed as some linear combinations of the odd root fermions $\chi_\gamma$. Remember that the negative odd root space $(\mathfrak{g}_1)_{-\lambda}$ belong to the fundamental representation of the even subalgebra $G$ of dimensions $|\Delta_1^{\perp}|$. Denote the highest weight of these representations as $\bar{\Lambda}^*$. To each odd positive root corresponds a weight vector in the representation with the highest weight $\bar{\Lambda}^*$. Let $\gamma(\bar{\lambda})$ be an odd positive root associate with the weight vector $\bar{\lambda}$. In table 3 we list such highest weights for every $\mathfrak{g}$. In order that this type of screening operator commutes with the $\hat{G}$ currents, this should be a $\hat{G}$ singlet operator, i.e. the operator product expansion with the $\hat{G}$ currents should be regular. These observations lead to fermionic screening operators of the following type:

$$S_f(z) = \sum_\lambda \chi_\gamma(\bar{\lambda}) \Phi_{\bar{\lambda}}^* \exp(-\frac{i\sqrt{2}\phi(z)}{2\alpha_+}),$$

where $\bar{\lambda}$ runs over the weights of the representation with the highest weight $\bar{\Lambda}^*$.

For example, in the case of $A(n|1)$, there are two highest weight vectors $\bar{\lambda}_1 \oplus \nu$ and $\bar{\lambda}_n \oplus (-\nu)$. The latter weight vector is the conjugate representation of the former. $\nu$ denotes the $u(1)$ charge. We find two fermionic screening operators

$$S_f(z) = \chi_i \Theta_i e^{-\sqrt{\frac{N}{2}}\alpha_+ \phi(z) + \frac{\phi(z)}{\sqrt{2}2\alpha_+}},$$

$$S_f(z) = \bar{\chi}_i \bar{\Theta}_i e^{\sqrt{\frac{N}{2}}\alpha_+ \phi(z) + \frac{\phi(z)}{\sqrt{2}2\alpha_+}},$$

where $\Theta_i(z)$ and $\bar{\Theta}_i(z)$ $(i = 1, \ldots, N)$ are primary fields of $J_{ij}$ satisfying

$$J_{ij}(z) \Phi_k(w) = \frac{\delta_{ik} \Phi_j(w) - \frac{1}{N} \delta_{ij} \Phi_k(w)}{z - w} + \cdots,$$

$$J_{ij}(z) \bar{\Phi}_k(w) = \frac{-\delta_{ik} \bar{\Theta}_j(w) + \frac{1}{N} \delta_{ij} \bar{\Phi}_k(w)}{z - w} + \cdots.$$

We can easily check that $S_f(z)$ and $\bar{S}_f(z)$ are $u(N)$ singlets and have conformal dimension
one. Moreover $S_f(z)$ satisfy

$$G_i(z)S_f(w) = \text{regular}, \quad \bar{G}_i(z)S_f(w) = \frac{\partial}{\partial w} \left( \frac{i\sqrt{2}\alpha_+ \Phi_i e^{-\frac{1}{\alpha_+} \phi(w)}}{z-w} \right) + \cdots, \quad (5.7)$$

by virtue of the Knizhnik-Zamolodchikov equations [40]:

$$\partial \Phi_i = \frac{1}{K+N} : J_{ki} \Phi_k :. \quad (5.8)$$

Similar relations hold for $\bar{S}_f(z)$.

In the case of $so(N)$ extended superconformal algebras, we find the following fermionic screening operator:

$$S_f(z) = \psi_i(z) \Phi_i(z) e^{-\frac{1}{\alpha_+} \phi(z)}, \quad (5.9)$$

where $\Phi_i(z)$ belongs to the $N$ dimensional representation of $\widehat{so(N)}$:

$$\hat{J}_{ij}(z) \Phi_k(w) = \frac{\delta_{jk} \Phi_i(w) - \delta_{ik} \Phi_j(w)}{z-w} + \cdots. \quad (5.10)$$

$S_f(z)$ is an $so(N)$ singlet operator and has conformal dimension one. The operator product expansions with the supercurrents are

$$G_i(z)S_f(w) = \frac{\partial}{\partial w} \left( \frac{i\alpha_+ \Phi_i(w) e^{-\frac{1}{\alpha_+} \phi(w)}}{z-w} \right) + \cdots, \quad (5.11)$$

We can now also construct fermionic screening operators for the $D(2|n)$ case. Let $\Phi_{\pm k}$ be primary fields in the $2n \times 2$ dimensional representation of $sp(2n) \oplus sl(2)$, which transform as $G$ under $\hat{J}$ and $\hat{I}$, i.e. we have

$$J_{-i+j}(z) \Phi_{\pm k}^\pm(w) = \frac{\delta_{ik} \Phi_{\pm j}^\pm(w)}{z-w} + \cdots, \quad J_{-i+j}(z) \Phi_{\mp k}^\pm(w) = \frac{-\delta_{jk} \Phi_{\pm i}^\pm(w)}{z-w} + \cdots,$$

$$J_{+i-j}(z) \Phi_{\pm k}^{\mp}(w) = \frac{\pm (\delta_{ik} \Phi_{\pm j}^{\mp}(w) + \delta_{jk} \Phi_{\pm i}^{\mp}(w))}{z-w} + \cdots,$$

$$J_{-i-j}(z) \Phi_{\pm k}^{\pm}(w) = \frac{\pm (\delta_{ik} \Phi_{\pm j}^{\pm}(w) + \delta_{jk} \Phi_{\pm i}^{\pm}(w))}{z-w} + \cdots,$$

$$I^3(z) \Phi_{\mp i}^{-}(w) = \frac{-\frac{1}{2} \Phi_{\mp i}^{\pm}(w)}{z-w} + \cdots, \quad I^3(z) \Phi_{\pm i}^{+}(w) = \frac{\frac{1}{2} \Phi_{\pm i}^{-}(w)}{z-w} + \cdots,$$

$$I^-(z) \Phi_{\pm i}^{+}(w) = \frac{\pm \Phi_{\pm i}^{\mp}(w)}{z-w} + \cdots, \quad I^+(z) \Phi_{\pm i}^{-}(w) = \frac{\pm \Phi_{\pm i}^{\pm}(w)}{z-w} + \cdots. \quad (5.12)$$
We can then write the fermionic screening operator as
\[ S_f(z) = (\psi^+_k \Phi^-_{-k} + \psi^-_k \Phi^+_{-k} + \psi^+_k \Phi^-_{+k} + \psi^-_k \Phi^+_{+k})(z) e^{-\frac{1}{\sqrt{2\alpha^+}}\phi(z)}. \] (5.13)

The OPE’s of the Kac-Moody currents with this screening operator is regular, and the OPE’s with the supercurrents are
\[ G^\pm_{+i}(z)S_f(w) = \frac{\partial}{\partial w} \left( i\alpha_+ \Phi^\pm_{+i}(w)e^{-\frac{1}{\sqrt{2\alpha^+}}\phi(w)} \right) + \cdots, \]
\[ G^\pm_{-i}(z)S_f(w) = \frac{\partial}{\partial w} \left( i\alpha_+ \Phi^\pm_{-i}(w)e^{-\frac{1}{\sqrt{2\alpha^+}}\phi(w)} \right) + \cdots. \] (5.14)

A similar construction is possible for other G extended superconformal algebras.

### 5.1.3 Screening operators in the θ-direction.

Finally we need a screening operator which is necessary to characterize the θ direction. Denote this screening operator as \( S_\theta(z) \). We assume that this operator takes the form:
\[ S_\theta(z) = s_\theta(z) \exp(\frac{2i\alpha_+ \phi(z)}{\sqrt{\theta^2}}), \] (5.15)
where \( s_\theta(z) \) is a G singlet operator with conformal dimension \( |\Delta^1_+|/2 \) containing a term \( \prod_{\gamma \in \Delta^1_+} \chi_\gamma \). This assumption is justified in part by the following list of results:

In the case of \( N = 4 \) sl(2) superconformal algebra this type of screening operator has been obtained in ref. [38]. When we consider the osp(\( N|2 \)) case, for \( N = 1 \) and 2, the screening operators are given as \( S_\theta = \psi_1 \exp(-\alpha_+ \phi(z)) \) and \( S_\theta = (\psi_1 \psi_2 - \frac{1}{K} J_{12}) \exp(-\alpha_+ \phi(z)) \), respectively. These are nothing but the screening operators of \( N = 1 \) and 2 minimal models. For the \( N = 3 \) case, this kind of screening operator has been found in ref. [41], the result of which becomes in our notation:
\[ S_\theta(z) = [\psi_1 \psi_2 \psi_3 - \frac{1}{K} (J_{12} \psi_3 + J_{32} \psi_1 + J_{13} \psi_2)](z) e^{\alpha_+ \phi(z)}. \] (5.16)
We expect that this procedure can be generalized to any $N$. In particular, we find that for $N = 4$, the screening operator takes the form:

$$S_\theta(z) = \{\psi_1 \psi_2 \psi_3 \psi_4 - \frac{1}{K} (J_{12} \psi_3 \psi_4 - J_{13} \psi_2 \psi_4 + J_{14} \psi_2 \psi_3 - J_{24} \psi_1 \psi_3 + J_{34} \psi_1 \psi_2 + J_{23} \psi_1 \psi_4)$$

$$+ \frac{1}{K(K + 2)} [(J_{12} J_{34})_S - (J_{13} J_{24})_S - (J_{14} J_{23})_S]\}(z)e^{\alpha_+ \phi(z)}. \quad (5.17)$$

The operator product expansions of the supercurrents with this screening operator is

$$G_1(z) S_\theta(w) = \frac{\partial}{\partial w} \left( \frac{i}{\alpha_+} \frac{O(w) \exp(\alpha_+ \psi(w))}{z - w} \right) + \cdots, \quad (5.18)$$

where

$$O = -\psi_2 \psi_3 \psi_4 + \frac{2}{K(K + 2)} (J_{23} \psi_4 + J_{34} \psi_2 + J_{42} \psi_3). \quad (5.19)$$

Similar expressions hold for the other supercurrents. For general $N$, we have a conjecture for the form $s_\theta(z)$. However due to the complicated operator product expansions we have not yet succeeded in a complete verification. As an example of a further nontrivial case, we have constructed the screening operator for $D(2|1; \alpha) \ [31]$. For generic $G$ we can easily show that the field $(5.15)$ has conformal dimension one. In the following we assume the existence of this kind of screening operator for any $G$.

### 5.2 Structure of null fields

Based on the above observations on the screening operators, we discuss the structure of singular vectors of $G$ extended superconformal algebras. We consider the Neveu-Schwarz sector for simplicity.

Firstly we consider the singular fields corresponding to the affine Lie algebra $\hat{G} \ [23]$. These are given by the following screened vertex operators:

$$\Psi^\Lambda_{\beta_1, \ldots, \beta_m}(z) = \oint du_1 \cdots du_m S_{\beta_1}(u_1) \cdots S_{\beta_m}(u_m) \Phi^\Lambda_{-\Lambda}(z), \quad (5.20)$$

where $\beta_a \ (a = 1, \ldots, m)$ are simple roots of the algebra $G$: i.e. $\beta_a = \bar{\alpha}_{i_a}$ for some $i_a = 1, \ldots, r$. The contours of integrations are taken as in ref. [12]. If the above integral
exists and is nonzero, this gives the null fields in the Fock space $F^G_{\Lambda^+ \Sigma_{a=1}^m \beta_a}$. The condition for existence of the null fields is given by the “on-shell” conditions \[42\]:

$$\frac{1}{\alpha_+^2} \sum_{a<b} \beta_a \cdot \beta_b - \frac{1}{\alpha_+} \sum_{a=1}^m \beta_a \cdot \bar{\Lambda} = -M,$$ \hspace{1cm} (5.21)

with a positive integer $M$. If $\sum_{a=1}^m \beta_a = n'\bar{\alpha}$ and $M = nn'$ for a positive root $\bar{\alpha} \in \Delta_+ (G)$ and positive integers $n$ and $n'$, we get the Kac-Kazhdan formula \[43\]:

$$(\bar{\Lambda} + \rho_G) \cdot \bar{\alpha} = -n\alpha_+^2 + n'\bar{\alpha}_+^2.$$ \hspace{1cm} (5.22)

This formula and its dual, which is obtained by replacing $\bar{\Lambda}$ by $-2\rho_G - \bar{\Lambda}$, characterize the singular vectors of the Fock module of the affine Lie algebra $\hat{G}$.

The fermionic singular vectors are given by considering a screened vertex operator of the form:

$$\Psi^{\Lambda,p}(z) = \oint du S_f(u) V_{\Lambda,p}(z),$$ \hspace{1cm} (5.23)

where $V_{\Lambda,p}(z) = \Phi_{\Lambda} \exp(ip\sqrt{\theta^2} \phi(z))$. The non-zero existence of the above contour integral requires the condition:

$$\frac{\bar{\Lambda} \cdot \bar{\Lambda}}{\alpha_+^2} - \frac{p\theta^2}{2\alpha_+} = -M,$$ \hspace{1cm} (5.24)

where $M$ is a positive integer. In this case $\Psi^{\Lambda,p}(z)$ is a singular vector in the Fock module $F^G_{\Lambda^+ \Lambda^*} \otimes F^{\phi}_{\frac{1}{2\alpha_+}}$ at level $M - \frac{1}{2}$.

The null fields in the $\theta$ direction can be obtained from the screened vertex operators:

$$\Psi^p(z) = \oint du_1 \cdots du_r S_\theta (u_1) \cdots S_\theta (u_r) V_p(z),$$ \hspace{1cm} (5.25)

where $V_p(z) = e^{ip\sqrt{\theta^2} \phi(z)}$. The on-shell condition becomes

$$\frac{r(r-1)}{2} \left( \frac{2\alpha_+}{\theta^2} \right)^2 \theta^2 + 2r p \alpha_+ = -M.$$ \hspace{1cm} (5.26)

with an positive integer $M$. Writing $M$ as $\frac{r(s - \mid \Delta_1 \mid + 2)}{2}$, we find that $p$ is given by

$$p_{r,-s} = -\frac{r - 1}{\theta^2} \alpha_+ - \frac{s - \mid \Delta_1 \mid + 2}{4\alpha_+}.$$ \hspace{1cm} (5.27)
In this case $\Psi^p_r(z)$ is a null field in the Fock module $F^\phi_{-Q-p_{r,s}}$ at level $\frac{rs}{2}$, where
\[ p_{r,s} = \frac{1 - r}{\theta^2} \alpha_+ + \frac{s + |\Delta^1_+| - 2}{4\alpha_+}. \] (5.28)

The precise range for $s$ cannot be identified unambiguously until the multiplicities of zeros of the Kac determinant (or something equivalent) has been dealt with.

The formulas (5.22), (5.24) and (5.28) characterize the whole singular vector structure of the $G$ extended superconformal algebras.

6 Conclusions and Discussion

In the present paper we have studied $G$ extended superconformal algebras from the viewpoint of classical and quantum hamiltonian reductions of an affine Lie superalgebra $\hat{g}$, with even Lie subalgebras $\hat{G} \oplus \hat{sl}(2)$. At the classical level we have derived generic formulas for the commutation relations of the ensuing $G$ extended superconformal algebras. At the quantum level we have obtained the quantum expression of the energy momentum tensor for general types by adopting free field representations for the affine Lie superalgebra. We have given explicit free field expressions for the supercurrents in the case of the non-exceptional type Lie superalgebras $sl(N|2)$, $osp(N|2)$ and $osp(4|2N)$. For the exceptional Lie superalgebra $D(2|1;\alpha)$, we obtain the $N = 4$ $sl(2) \oplus sl(2)$ doubly extended superconformal algebra $\hat{A}_{\gamma}$ ($\gamma = \frac{\alpha}{1-\alpha}$), which admits one continuous parameter $\alpha$.[31]. A detailed analysis of the free field representations is presented in a separate publication.[31]. For the other exceptional type Lie superalgebras $F(4)$ and $G(3)$, we can construct a Poisson bracket structure of $N = 8$ $spin(7)$ and $N = 7$ $G_2$ extended superconformal algebras, respectively, by introducing a pseudo matrix representation.[29]. Although we do not present their free field representation of supercurrents in this paper, we expect that similar formulas hold as in the non-exceptional case. This subject will be treated elsewhere. We have introduced a set of screening operators for $G$ extended superconformal algebras. Using the null field construction, we have identified the primary fields.
corresponding to degenerate representations of these algebras.

In order to make further progress on the representation theory, one should for example investigate the Kac determinant formula.

It is now in principle possible to calculate correlation functions and characters for these models. They will then be expressed as products of those of \( \tilde{G} \) affine Lie algebras, Virasoro minimal models and free fermions.

Compared to the linearly extended superconformal algebras, a geometrical interpretation of the present non-linear algebras is unclear. It seems an interesting problem to try to find a way of interpreting these non-linear symmetries in terms of non-linear \( \sigma \)-models on non-symmetric Riemannian manifolds. This problem is important in order to clarify the geometrical meaning of the \( W \)-algebras.

The present construction of the extended superconformal algebra is based on the hamiltonian reduction of the affine Lie superalgebras. It is well understood that in the case of \( W \)-algebras associated with simple Lie algebras, the hamiltonian reduction \([9]-[12]\) provides a connection to various integrable systems such as Toda field theory \([44]\) and the generalized KdV hierarchy \([45]\). In the present case it is natural to expect the super Liouville model coupled to Wess-Zumino-Novikov-Witten (WZNW) models or the KdV hierarchy coupled to affine Lie algebras to arise. In the bosonic case, this kind of integrable system has been partially studied \([16]\).

One might further generalize the hamiltonian reduction procedure to any Lie superalgebra. This would then give rise to a super \( W \)-algebra coupled to WZNW models.
Appendix A. The root system

In this appendix, we describe the root systems of the Lie superalgebras with the even subalgebra $G \oplus A_1$ as given in table. $\theta$ is the simple root of $A_1$. We use the orthonormal basis $e_i$ ($i \geq 1$) with positive metric and $\delta_j$ ($j \geq 1$) with negative metric:

$$e_i \cdot e_j = \delta_{ij}, \quad \delta_i \cdot \delta_j = -\delta_{ij}, \quad e_i \cdot \delta_j = 0.$$

(\text{A.1})

1. $A(n|1)$ ($n \geq 1$), (rank $n + 2$, the dual Coxeter number $h^\vee = n - 1$)

Simple roots: $\alpha_1 = \delta_1 - e_1$, $\alpha_i = e_{i-1} - e_i$, ($i = 2, \ldots, n + 1$), $\alpha_{n+2} = e_{n+1} - \delta_2$.

Positive even roots: $e_i - e_j$, ($1 \leq i < j \leq n + 1$), $\theta = \delta_1 - \delta_2$.

Positive odd roots: $\delta_1 - e_j$, $e_j - \delta_2$, ($j = 1, \ldots, n + 1$).

2. $B(n|1)$ ($n \geq 0$), (rank $n + 1$, $h^\vee = 3 - 2n$)

Simple roots: $\alpha_1 = e_1 - \delta_1$, $\alpha_{i+1} = \delta_i - \delta_{i+1}$, ($i = 1, \ldots, n - 1$), $\alpha_{n+1} = \delta_n$.

Positive even roots: $\theta = 2e_1$, $\delta_i \pm \delta_j$, ($1 \leq i < j \leq n$), $\delta_i$, ($i = 1, \ldots, n$)

Positive odd roots: $e_1$, $e_1 \pm \delta_j$, ($j = 1, \ldots, n$).

3. $D(n|1)$ ($n \geq 2$), (rank $n + 1$, $h^\vee = 4 - 2n$)

Simple roots: $\alpha_1 = e_1 - \delta_1$, $\alpha_{i+1} = \delta_i - \delta_{i+1}$, ($i = 1, \ldots, n - 1$), $\alpha_{n+1} = \delta_{n-1} + \delta_n$.

Positive even roots: $\theta = 2e_1$, $\delta_i \pm \delta_j$, ($1 \leq i < j \leq n$).

Positive odd roots: $e_1 \pm \delta_j$, ($j = 1, \ldots, n$).

4. $D(2|n)$ ($n \geq 1$) (rank $n + 2$, $h^\vee = 2n - 2$

Simple roots: $\alpha_1 = -\delta_2 - \delta_1$, $\alpha_2 = \delta_1 - e_1$, $\alpha_{i+2} = e_i - e_{i+1}$, ($i = 1, \ldots, n - 1$), $\alpha_{n+2} = 2e_n$.

Positive even roots: $-\delta_2 - \delta_1$, $\theta = \delta_1 - \delta_2$, $e_i \pm e_j$, ($1 \leq i < j \leq n$).

Positive odd roots: $\delta_1 \pm e_j$, $-\delta_2 \pm e_j$, ($j = 1, \ldots, n$).

5. $D(2|1; \alpha)$ ($\alpha \neq 0, -1, \infty$), (rank 3, $h^\vee = 0$)

Simple roots: $\alpha_1 = \frac{1}{2}(\sqrt{2}\gamma \delta_1 + \sqrt{2}(1-\gamma)\delta_2 + \sqrt{2}e_3)$, $\alpha_2 = -\sqrt{2}\gamma \delta_1$, $\alpha_3 =$
\[-\sqrt{2(1-\gamma)}\delta_2, \text{ where } \alpha = \gamma/(1-\gamma).\]

Positive even roots: \(\alpha_2, \quad \alpha_3, \quad \theta = 2\alpha_1 + \alpha_2 + \alpha_3.\)

Positive odd roots: \(\alpha_1, \quad \alpha_1 + \alpha_2, \quad \alpha_1 + \alpha_3, \quad \alpha_1 + \alpha_2 + \alpha_3.\)

6. \(F(4) \ (\text{rank:4, } h^\vee = -3)\)

Simple roots: \(\alpha_1 = \frac{1}{2}(\sqrt{3}e_1+\delta_1+\delta_2+\delta_3), \quad \alpha_2 = -\delta_1, \quad \alpha_3 = \delta_1-\delta_2, \quad \alpha_4 = \delta_2-\delta_3.\)

Positive even roots: \(\alpha_2, \quad \alpha_3, \quad \alpha_4, \quad \alpha_2+\alpha_3, \quad \alpha_3+\alpha_4, \quad 2\alpha_2+\alpha_3, \quad \alpha_2+\alpha_3+\alpha_4, \quad 2\alpha_2+\alpha_3+\alpha_4, \quad 2\alpha_2+2\alpha_3+\alpha_4, \quad \theta = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4.\)

Positive odd roots: \(\alpha_1, \quad \alpha_1+\alpha_2, \quad \alpha_1+\alpha_2+\alpha_3, \quad \alpha_1+2\alpha_2+\alpha_3, \quad \alpha_1+\alpha_2+\alpha_3+\alpha_4, \quad \alpha_1+2\alpha_2+\alpha_3+\alpha_4, \quad \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4.\)

7. \(G(3) \ (\text{rank:3, } h^\vee = 2)\)

Simple roots: \(\alpha_1 = \sqrt[3]{\frac{2}{3}}\delta_1 + \frac{2\epsilon_1-\epsilon_2-\epsilon_3}{3}, \quad \alpha_2 = \frac{-\epsilon_1+2\epsilon_2-\epsilon_3}{3}, \quad \alpha_3 = -\epsilon_2 + \epsilon_3.\)

Positive even roots: \(\alpha_2, \quad \alpha_3, \quad \alpha_2+\alpha_3, \quad 2\alpha_2+\alpha_3, \quad 3\alpha_2+\alpha_3, \quad 3\alpha_2+2\alpha_3, \quad \theta = 2\alpha_1 + 4\alpha_2 + 2\alpha_3.\)

Positive odd roots: \(\alpha_1, \quad \alpha_1+\alpha_2, \quad \alpha_1+\alpha_2+\alpha_3, \quad \alpha_1+2\alpha_2+\alpha_3, \quad \alpha_1+3\alpha_2+\alpha_3, \quad \alpha_1+3\alpha_2+2\alpha_3, \quad \alpha_1 + 4\alpha_2 + 2\alpha_3.\)
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Table 1: Lie Superalgebras with an even subalgebra $A_1$

| $g$          | sdim      | rank | $h^\vee$ | $g_0$       | $G$            |
|--------------|-----------|------|----------|-------------|----------------|
| $A(n|1)$     | $n^2 - 2n$ | $n + 2$ | $n - 1$  | $A_n \oplus A_1 \oplus u(1)$ | $A_n \oplus u(1)$ |
| $B(n|1)$     | $(n - 1)(2n - 1)$ | $n + 1$ | $3 - 2n$ | $B_n \oplus A_1$ | $B_n$ |
| $D(n|1)$     | $(n - 1)(2n - 3)$ | $n + 1$ | $4 - 2n$ | $D_n \oplus A_1$ | $D_n$ |
| $D(2|n)$    | $(n - 2)(2n - 3)$ | $n + 2$ | $2n - 2$ | $A_1 \oplus A_1 \oplus C_n$ | $A_1 \oplus C_n$ |
| $D(2|1; \alpha)$ | 1         | 3    | 0        | $A_1 \oplus A_1 \oplus A_1$ | $A_1 \oplus A_1$ |
| $F(4)$      | 8         | 4    | $-3$     | $B_3 \oplus A_1$ | $B_3$ |
| $G(3)$      | 3         | 3    | 2        | $G_2 \oplus A_1$ | $G_2$ |
| $B(1|n)$    | $(n - 1)(2n - 3)$ | $n + 1$ | $2n - 1$ | $A_1 \oplus C_n$ | $C_n$ |

Table 2: Central charges for $G$ extended superconformal algebras

| $g$          | $c_{total}$ |
|--------------|-------------|
| $A(n|1)$     | $\frac{6k^2 + k(n^2 + 3n - 9) - n^2 - 2n + 3}{k(n - 1)}$ |
| $B(n|1)$     | $\frac{(k+1)(4n^2 + 4n - 15 - 6k)}{2(k+3 - 2n)}$ |
| $D(n|1)$     | $\frac{(k+1)(4n^2 - 16 - 6k)}{2(k+4 - 2n)}$ |
| $D(2|n)$    | $\frac{6k^2 + k(2n^2 + 3n - 8) + 4 - 4n^2}{k + 2n - 2}$ |
| $D(2|1; \alpha)$ | $-3 - 6k$ | $\frac{-k}{2}$ |
| $F(4)$      | $\frac{2(-2k^2 + 7k + 9)}{k + 3}$ |
| $G(3)$      | $\frac{9k^2 + 13k - 2}{2(k + 2)}$ |

Table 3:

| $g$          | $\Lambda^\epsilon$ |
|--------------|----------------------|
| $A(n|1)$     | $(\lambda_1 \oplus \nu), \lambda_n \oplus (-\nu)$ |
| $B(n|1)$     | $\tilde{\lambda}_1$ |
| $D(n|1)$     | $\tilde{\lambda}_1$ |
| $D(2|n)$    | $\lambda_1(A_1) \oplus \tilde{\lambda}_1(C_n)$ |
| $D(2|1; \alpha)$ | $\tilde{\lambda}_1(A_1) \oplus \tilde{\lambda}_1(A_1)$ |
| $F(4)$      | $\tilde{\lambda}_2 (= \tilde{\lambda}_s)$ : spin representation of $so(7)$ |
| $G(3)$      | $\tilde{\lambda}_1$ |