Some properties of new hesitant fuzzy operators

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Abstract. Hesitant fuzzy sets are tools that appear and are useful for dealing with vagueness and uncertainty. Interestingly, we can determine two new operators inspired by operators which have been defined. Some properties associated with binary operations of hesitant fuzzy elements that were introduced by previous researchers are constructed. We obtain several relations between the new operators and such binary operations. Here we also give a correction of a result in previous related article.

1. Introduction

The study of fuzzy sets has appeared in various fields both in terms of theory and application. The concept of fuzzy sets continues to grow as shown in intuitionistic fuzzy sets [1] which is useful in various applications in real life problems, fuzzification in implicative semigroups [2] which is included in fuzzy algebraic studies, hesitant fuzzy sets [3,4] as a generalization of fuzzy sets.

The concept of hesitant fuzzy sets continues to be developed by many researchers. In 2011, it has been introduced a detailed study on hesitant fuzzy sets related to distance similarity measures and correlation measures [5]. Nazra et al. [6,7] have studied on hesitant intuitionistic fuzzy soft sets and its generalization as combination between the concept of hesitant fuzzy sets, intuitionistic fuzzy sets and soft sets.

In 2014, Thakur et al. [8] proposed four operators on hesitant fuzzy sets and studied its properties with respect to such operators. This result motivates us to introduce two new operators on hesitant fuzzy sets and obtain the properties with respect to both the two new operators and the operators introduced by Thakur et al. [8]. The research contributes on extension of properties of operators on hesitant fuzzy sets which is associated with well-known previous operations.

2. Preliminaries

This section is provided to look back the existing extensions of fuzzy sets, namely hesitant fuzzy sets introduced by Rodríguez et al. [4], Torra [3] and restated by Xia and Xu [9] by using a mathematical symbol in order to be easily understood.

Definition 1. [9] Suppose that \( X = \{x_1, x_2, ..., x_n\} \) is fixed set. A hesitant fuzzy set (HFS) on \( X \) is a set \( H = \{\langle x, h_H(x) \rangle \mid x \in X\} \), where \( h_H(x) \) is a collection of some values in \([0,1]\), denoting the possible membership degrees of the element \( x \in X \) to the set \( H \). Here \( h_H(x) \) is called a hesitant fuzzy element (HFE) of \( x \) in \( X \).

The collection of all hesitant fuzzy elements in \( X \) is denoted by \( HFE(X) \).

Related to HFEs, Torra [3], Xia and Xu [9] defined several operations on HFEs stated as follows.

Definition 2. [3] Given two HFEs \( h \) and \( g \). Defined union \( \cup \) and intersection \( \cap \) operations as follows.
\[ h \cup g = \bigcup_{\tau_1, \tau_2 \in g} \max \{ \tau_1, \tau_2 \}, \quad h \cap g = \bigcup_{\tau_1, \tau_2 \in g} \min \{ \tau_1, \tau_2 \}. \]

**Definition 3.** [9] Let \( h, g \in \text{HFE}(X) \). Defined operations \( \oplus \) and \( \otimes \) as follows.

\[
\begin{align*}
    h \oplus g &= \bigcup_{\tau_1, \tau_2 \in g} \{ \tau_1 + \tau_2 - \tau_1 \tau_2 \}, \\
    h \otimes g &= \bigcup_{\tau_1, \tau_2 \in g} \{ \tau_1 \tau_2 \}. 
\end{align*}
\]

3. **Main Results**

First of all, we propose two new operators \( O_5 \) and \( O_6 \). Furthermore, we establish some relationships between such operators, union, intersection, \( \oplus \) and \( \otimes \) operations.

**Definition 4.** Let \( h, g \in \text{HFE}(X) \). Defined operators \( O_5 \) and \( O_6 \) as follows:

\[
\begin{align*}
    h O_5 g &= \bigcup_{\tau_1, \tau_2 \in g} \{ \frac{\tau_1 \tau_2}{1 + (\tau_1 \tau_2)} \}, \\
    h O_6 g &= \bigcup_{\tau_1, \tau_2 \in g} \{ \frac{\tau_1 \tau_2}{1 + 2(\tau_1 \tau_2)} \}. 
\end{align*}
\]

**Theorem 5.** Let \( h_{1,2} \in \text{HFE}(X) \). Then the following relationships hold.

1. \((h \oplus g) \cap (h O_5 g) = (h O_5 g)\).
2. \((h \oplus g) \cup (h O_5 g) = (h \oplus g)\).
3. \((h \otimes g) \cap (h O_5 g) = (h O_5 g)\).
4. \((h \otimes g) \cup (h O_5 g) = (h \otimes g)\).

**Proof.** Using Definition 2, 3 and 4, all the above relationships are proved one by one as follows.

1. \((h \oplus g) \cap (h O_5 g) = \bigcup_{\tau_1, \tau_2 \in g} \{ \tau_1 + \tau_2 - \tau_1 \tau_2 \} \cap \bigcup_{\tau_1, \tau_2 \in g} \left\{ \frac{\tau_1 \tau_2}{1 + (\tau_1 \tau_2)} \right\} = \bigcup_{\tau_1, \tau_2 \in g} \min \left\{ \frac{\tau_1 \tau_2}{1 + (\tau_1 \tau_2)} \right\} = \bigcup_{\tau_1, \tau_2 \in g} \left\{ \frac{\tau_1 \tau_2}{1 + 2(\tau_1 \tau_2)} \right\} = (h O_5 g)\).

One can see that \( \min \left\{ \frac{\tau_1 \tau_2}{1 + (\tau_1 \tau_2)} \right\} = \left\{ \frac{\tau_1 \tau_2}{1 + 2(\tau_1 \tau_2)} \right\} \). For this, if \( \tau_1 + \tau_2 - \tau_1 \tau_2 < \frac{\tau_1 \tau_2}{1 + (\tau_1 \tau_2)} \),

\[
\begin{align*}
    \tau_1 + \tau_2 &= \frac{\tau_1 \tau_2}{1 + (\tau_1 \tau_2)}, \\
    \tau_1 + \tau_2(1 - \tau_1) &= \frac{\tau_1 \tau_2}{1 + (\tau_1 \tau_2)}, \\
    \tau_1 < \tau_1 + \tau_2(1 - \tau_1) &= \frac{\tau_1 \tau_2}{1 + (\tau_1 \tau_2)} < \tau_1 \tau_2 \\
    \tau_1 < \tau_1 \tau_2 .
\end{align*}
\]

We get a contradiction.

2. \((h \oplus g) \cup (h O_5 g) = (h \oplus g)\).

3. \((h \otimes g) \cap (h O_5 g) = (h O_5 g)\).
\[(h \otimes g) \cap (h O_5 g) = \left( \bigcup_{t_1 \in h, t_2 \in g} \{t_1 t_2\} \right) \cap \left( \bigcup_{t_1 \in h, t_2 \in g} \frac{t_1 t_2}{1 + (t_1 t_2)\}} \right) \]
\[= \bigcup_{t_1 \in h, t_2 \in g} \min \left\{\frac{t_1 t_2}{1 + (t_1 t_2)}\right\} \]
\[= \bigcup_{t_1 \in h, t_2 \in g} \frac{t_1 t_2}{1 + (t_1 t_2)} \]
\[= (h O_5 g).\]

(4). \( (h \otimes g) \cup (h O_5 g) = (h \otimes g) \).

\[(h \otimes g) \cup (h O_5 g) = \left( \bigcup_{t_1 \in h, t_2 \in g} \{t_1 t_2\} \right) \cup \left( \bigcup_{t_1 \in h, t_2 \in g} \frac{t_1 t_2}{1 + (t_1 t_2)\}} \right) \]
\[= \bigcup_{t_1 \in h, t_2 \in g} \max \left\{\frac{t_1 t_2}{1 + (t_1 t_2)}\right\} \]
\[= \bigcup_{t_1 \in h, t_2 \in g} \frac{t_1 t_2}{1 + (t_1 t_2)} \]
\[= (h O_5 g).\]

(5). \( (h \oplus g) \cap (h O_6 g) = (h O_6 g) \).

\[(h \oplus g) \cap (h O_6 g) = \left( \bigcup_{t_1 \in h, t_2 \in g} \{t_1 + t_2 - t_1 t_2\} \right) \cap \left( \bigcup_{t_1 \in h, t_2 \in g} \frac{t_1 t_2}{1 + 2(t_1 t_2)\}} \right) \]
\[= \bigcup_{t_1 \in h, t_2 \in g} \min \left\{t_1 + t_2 - t_1 t_2, \frac{t_1 t_2}{1 + 2(t_1 t_2)}\right\} \]
\[= \bigcup_{t_1 \in h, t_2 \in g} \frac{t_1 t_2}{1 + 2(t_1 t_2)} \]
\[= (h O_6 g).\]

Using the argument in proof (1), it is clear that \( t_1 + t_2 - t_1 t_2 \geq \frac{t_1 t_2}{1 + 2(t_1 t_2)} \).

(6). \( (h \oplus g) \cup (h O_6 g) = (h \oplus g) \).

\[(h \oplus g) \cup (h O_6 g) = \left( \bigcup_{t_1 \in h, t_2 \in g} \{t_1 + t_2 - t_1 t_2\} \right) \cup \left( \bigcup_{t_1 \in h, t_2 \in g} \frac{t_1 t_2}{1 + 2(t_1 t_2)\}} \right) \]
\[= \bigcup_{t_1 \in h, t_2 \in g} \max \left\{t_1 + t_2 - t_1 t_2, \frac{t_1 t_2}{1 + 2(t_1 t_2)}\right\} \]
\[= \bigcup_{t_1 \in h, t_2 \in g} \frac{t_1 t_2}{1 + 2(t_1 t_2)} \]
\[= (h O_6 g).\]

(7). \( (h \otimes g) \cap (h O_6 g) = (h O_6 g) \).

\[(h \otimes g) \cap (h O_6 g) = \left( \bigcup_{t_1 \in h, t_2 \in g} \{t_1 t_2\} \right) \cap \left( \bigcup_{t_1 \in h, t_2 \in g} \frac{t_1 t_2}{1 + 2(t_1 t_2)\}} \right) \]
\[= \bigcup_{t_1 \in h, t_2 \in g} \min \left\{t_1 t_2, \frac{t_1 t_2}{1 + 2(t_1 t_2)}\right\} \]
\[= \bigcup_{t_1 \in h, t_2 \in g} \frac{t_1 t_2}{1 + 2(t_1 t_2)} \]
\[= (h O_6 g).\]

(8). \( (h \otimes g) \cup (h O_6 g) = (h \otimes g) \).

\[(h \otimes g) \cup (h O_6 g) = \left( \bigcup_{t_1 \in h, t_2 \in g} \{t_1 t_2\} \right) \cup \left( \bigcup_{t_1 \in h, t_2 \in g} \frac{t_1 t_2}{1 + 2(t_1 t_2)\}} \right) \]
\[
\tau_1 \cup \tau_2 = \bigcup_{\tau_1 \in h, \tau_2 \in g} \max \left\{ \tau_1 \tau_2, \frac{\tau_1 \tau_2}{1 + 2(\tau_1 \tau_2)} \right\} = \bigcup_{\tau_1 \in h, \tau_2 \in g} \{ \tau_1 \tau_2 \} = (h \otimes g).
\]

The results in Theorem 5 above are similar to some results of Theorem 1 in [8] even though we use different operators. These results certainly will enrich the properties of HFEs with respect to the operators given both in this paper and in [8]. On the other hand Verma and Sharma [10] and Xia et al. [11] also defined some operators and proposed its properties which are different with our results.

4. Counterexample

In this section we provide a counterexample to show that assertion (i), (ii), (iii) and (iv) in Theorem 3 proposed by Thakur et al. [8] are not true in general.

Example 6. Let \( X = \{ x \} \) is a fixed set. Given \( h_1, h_2 \) and \( h_3 \) hesitant fuzzy elements (HFE) of \( x \) in \( X \) defined by \( h_1(x) = \{0.2, 0.3\} \), \( h_2(x) = \{0.1, 0.4\} \) and \( h_3(x) = \{0.2, 0.7\} \). One can check that \( (h_1 \cup h_2) O_1 h_3 \neq (h_1 O_1 h_3) \cup (h_2 O_1 h_3) \). For assertion (ii), (iii) and (iv) give the similar conclusion.

5. Conclusion

In this article we have proposed two new operators on hesitant fuzzy sets in conjunction with its properties which are related to operators defined by previous researchers. On the other hand we also have given a correction of an article associated with this topic. This certainly will enrich the concepts in the field of hesitant fuzzy sets especially operators on hesitant fuzzy sets.

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