NEW INVARIANTS OF LEGENDRIAN KNOTS

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Abstract. We give new functions of Legendrian knots derived from Legendrian fronts. These are integer-valued linear functions that are alike the Arnold basic invariant of plane curves. Various generalizations of the Arnold basic invariant have been known. In this paper, we give another extension of Arnold’s idea.

1. Introduction

Arnold \cite{3} introduced integer-valued functions \(J^+, J^-,\) and \(St\) for plane curves, each of which is the image of a generic immersion \(S^1 \to \mathbb{R}^2\), where the self-intersections are transverse double points. A plane curve is regarded as an object called a Legendrian front that is an image of a projection of a Legendrian knot. Arnold showed that \(J^+\) is a Legendrian knot invariant. Nowadays, functions \(J^+, J^-,\) and \(St\) are called Arnold (basic) invariants. Arnold invariants are of much interest to researchers dealing with some aspects, and have been studied a lot: several versions of explicit formulae of them (Polyak \cite{10}, Shumakovitch \cite{11}, and Viro \cite{12}), generalizations to fronts (Aicardi \cite{1}, and Arnold \cite{3}, Polyak \cite{10}), and higher-order cases: Arakawa-Ozawa \cite{2} for \(St\), Goryunov \cite{4} for \(J^+\), and Viro \cite{12} for \(J^-\) with a setting that Arnold invariants are of lower orders.

Arnold showed his function \(J^+\) is a Legendrian knot invariant \cite{3}. In \cite{7}, Hayano and one of the author NI gave Fact 1. In Fact 1, a deformation of type strong RI I (RI II, resp.) is called \(iR_2\)-move (\(R_3\)-move, resp.). From here, the terminologies of this paper obey \cite{7}.

Fact 1 (Hayano-Ito \cite{7}). Let \(r\) be an integer and \(j^+\) an even integer. For a plane curve \(C\), let \(\text{rot}(C)\) be the rotation number and \(J^+(C)\) the Arnold invariant. Then, for any pair \((r, j^+)\), there exists an infinite family of plane curves \(\{C_\lambda \mid \lambda \in \mathbb{N}\}\) satisfying the following conditions:

- \(\text{rot}(C_\lambda) = r\) and \(J^+(C_\lambda) = j^+\) for any \(\lambda\).
- For any \(\lambda\) and \(\mu\), \(C_\lambda\) and \(C_\mu\) are not equivalent under \(iR_2\)-move and \(R_3\)-move.

In order to prove Fact 1 we may use plane curves as in Fig. 1 with the non-negative rotation number \(r \ (= a)\). This example is similar to \cite{7}. This sequence \(C(a, b, c)\) is parametrized by a tuple \((a, b, c)\) of integers \(a, b,\) and \(c\). Here, if \(C(a, b, c)\) has the number of parts of type \((a)\) \((b),\) (c), resp.) is \(a_0\) \((b_0, c_0),\) resp.), we say that \(C(a_0, b_0, c_0)\). Note that \(J^+\)(\(C(a, b, c)\)) = \(a - 2b + 2c\). Further, it is proved that for
fixed nonnegative integers $a_0$, $b_0$ and $c_0$ such that $a_0 - 2b_0 + 2c_0 = j^+$ and the rotation number $r$ ($= a_0$), any two plane curves in the family $\{C(r, b_0 + k, c_0 + k)\}_{k \in \mathbb{Z}_{>0}}$ are not equivalent under iR2-move and R3-move (cf. [7]).

Figure 1. $C(a, b, c)$ (upper) and parts (a), (b), and (c) (lower) with a nonnegative rotation number. The integer $a$ ($b$, $c$, resp.) indicates the number of appearances of type (a) ((b), (c), resp.). A plane curve $\tilde{C}(a, b, c)$ is the corresponding $C(a, b, c)$ with the base point at the position on which the arrow mark places.

Let $\tilde{C}(a, b, c)$ be the plane curve $C(a, b, c)$ with the base point at the position on which the arrow mark places as in Fig. 1. Let $J^+$ be the Arnold invariant of long curves, each of which is identified with a plane curve with a base point [8, Definition 4.3].

**Theorem 1.** Let $r$ be a nonnegative integer, $j^+$ an integer, rot the rotation number, and $J^+$ the Arnold invariant. There exist functions $I_{2,3,k}$ ($1 \leq k \leq 5$) that are invariant under iR2-move, R3-move preserving the base point such that the function detects any two plane curves in the family $\{\tilde{C}(r, b_0 + k, c_0 + k)\}_{k \in \mathbb{Z}_{>0}}$ as in Fig. 1, where $\text{rot}(\tilde{C}(r, b_0 + k, c_0 + k)) = r$ and $J^+(\tilde{C}(r, b_0 + k, c_0 + k)) = j^+$.

**Remark 1.** For long curves, the Arnold invariant $J^+$ is formulated by Gusein-Zade [6] and Zhou-Zou-Pan [13], independently.

## 2. Preliminaries

### 2.1. Legendrian knots and fronts.** The reader who is familiar with [7] may skip this section; here we pick some definitions in [7].

**Definition 1.** A plane curve is the image of a generic immersion $S^1 \to \mathbb{R}^2$, where the self-intersections are transverse double points.

**Definition 2** (Legendrian knot $K_C$ associated with a plane curve $C$). Let $C$ be a plane curve. For an given oriented $C$, we take a generic immersion $f : S^1 \to \mathbb{R}^2$ so that $f(S^1) = C$ and the orientation of $C$ is induced by that of $S^1$. Let $df$ be the derivative $TS^1 \to T\mathbb{R}^2$. Since $f$ is an immersion, $f$ implies that $df(p) \neq 0$ for every $p$. Since there exists the projection $\pi : T\mathbb{R}^2 \setminus \mathbb{R}^2 \to UT\mathbb{R}^2$, and the 0-section of $T\mathbb{R}^2$ is identified with $\mathbb{R}^2$, the map $\pi \circ df|_{S^1} : S^1 \to UT\mathbb{R}^2$ gives a knot in $UT\mathbb{R}^2$. This knot is denoted by $K_C$ and called the Legendrian knot associated with $C$.  

Fact 2. If two plane curves $C_0$ and $C_1$ are equivalent under iR2-move and R3-move, then there exists an ambient isotopy in $S^3$ that deforms $K_{C_0}$ to $K_{C_1}$ and keeps the $(+2)$-framed unknot fixed. In particular, $K_{C_0}$ to $K_{C_1}$ are isotopic as framed knots in $S^3$.

3. Explicit relationship between plane curves and Legendrian knots

The realization of the Legendrian knot [7] gives one to one correspondence between an iR2 (R3, resp.)-move of plane curves and the Reidemeister move $\Omega_2$ ($\Omega_3$, resp.) as in Fig. 2 (Fig. 3, resp.) if the tangent vector of a branch of a self-tangency (triple point crossing, resp.) is not a horizontal direction vector oriented from right to left ($\ast$). Note that the condition ($\ast$) can be excluded from the neighborhood of a digon (triangle, resp.) by small isotopy for plane curves and for knots, respectively. Given a plane curve $C$, we have a diagram $D_{K_C}$ of a knot $K_C$ keeping the $(+2)$-framed unknot fixed [7] Section 3.2 just after Prop. 3.2 to Prop. 3.4] (Fig. 4).
4. Proof of Theorem

We apply the same argument as [9] with relators [9, Section 2.2, Definition 7]. Although the meaning of “the same argument” may be clear, we pose some comments. (Construction) First of all, if the reader does not know the definition of Gauss diagrams, see, e.g., [5, Page 1046, Fig. 1]. Second, though we have a Legendrian knot diagram $D_{KC}$ from $KC$, we use relators [9, Section 2.2, Definition 7] given by positive knot diagrams derived from plane curves. Third, we apply the same argument as [9] to these relators. Then we have invariants as in Theorem [1].

(Detection of Legendrian knots) Essentially, we count sub-arrow diagrams, each of which is isomorphic to $\square$. For every $(2,n)$-torus knot $T_n$ such that $n = 2m + 1$, the number of sub-arrow diagrams of type $\square$ is

$$\frac{1}{6}m(m + 1)(2m + 1).$$

Then the invariant detects two $T_n$ and $T'_n$ corresponding to $KC$ and $KC'$ respectively.

5. New functions of Legendrian fronts

Since we give our formulas with simpler presentations, we slightly change the notation as in [9]. More precisely, we switch each arrow presentation to the signed chord in the way of Fig. [7]

Figure 4. An example $KC$ (right) of Legendrian knots derived from a plane curve $C$ (left)

Figure 5. Switching each arrow presentation to the signed chord.

In the following, we list six invariants $I_{2,1}$ and $I_{3,i}(C)$ ($1 \leq i \leq 5$) for a given plane curve $C$: 
Finally, we would like to mention Proposition 1.

\textbf{Proposition 1.} $I_{2,1}$ is the Arnold invariant of long curve.

\textit{Proof.} For Arnold $J^+$ invariant of long curves, the known formula \cite[Proposition 4.6 (4.6)]{8} plus \cite[Lemma 4.5 (4.4)]{8} implies that $I_{2,1}$ equals $\frac{r^2 + c^2}{2}$ up to signs. \hfill \square

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