Bayesian Approach in Estimation of Shape Parameter of an Exponential Inverse Exponential Distribution

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Authors’ contributions

This work was carried out in collaboration between both authors. Author TMA designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author BOS managed the analyses of the study and managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

Aims: This study aimed to obtain the shape parameter of an Exponential Inverted Exponential distribution using different prior distributions under different loss functions.

Methodology: The Bayes’ theorem was adopted to obtain the posterior distribution of the shape parameter of an Exponential inverted Exponential distribution for both non-information prior (such as Jeffreys prior, Hartigen prior and Uniform prior) and an informative prior (such as Gamma distribution and chi-square distribution). Different loss functions (such as Entropy loss function, Square error loss function, Al-Bayyati’s loss function and Precautionary loss function) were employed to obtain the estimate parameter of the shape parameter with an assumption that the scale parameter is known.

Results: The posterior distribution of the shape parameter of an Exponential Inverted Exponential distribution follows a Gamma distribution for all the prior distribution in the study. Also the Bayes estimate for the simulated datasets and real life dataset were obtained.

Conclusion: The Bayes’ estimates for different prior distribution under different loss functions are close to the true parameter value of the shape parameter. The estimators are then compared in terms of their Mean Square Error (MSE) which is computed using R programming language. We deduce that the MSE reduces as the sample size (n) increases.

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1 Introduction

Exponential distribution is a special case of a Gamma distribution with parameter G(1,θ). The exponential distribution is commonly used as a lifetime model but at times it fails to model some real life phenomena whose failure rate are not constant properly [1]. This led to several modifications and generalization of the exponential distribution.

Among others, [2,3] and [4] focused on the Bayesian and classical estimations on the generalized Exponential Distribution using outliers, censored and grouped data, respectively. The Inverted Exponential distribution has an inverted bathtub hazard rate and it’s suitable for modeling real life phenomena with inverted bathtub failure rates. [5] described Inverted Exponential Distribution as a model that is useful in survival analysis. It is useful in survival analysis. [6] and [7] extended the Inverted Exponential (IE) distribution to Generalized Inverse Exponential Distribution and the Beta Inverted Exponential Distribution respectively.

The Exponential Inverse Exponential Distribution (EIED) is a specific family of an exponential family. [8] proposed the Exponential Inverse Exponential Distribution (EIED) with probability density function (PDF) given as

\[ f(x) = \frac{\alpha^\theta e^{-\theta x} \left(1-e^{-\theta x}\right)^{1-\alpha}}{x^\alpha \left(1-e^{-\theta x}\right)^{\alpha-1}} \quad x > 0, \alpha > 0, \theta > 0 \quad (1) \]

where \( \alpha \) is the shape parameter and \( \theta \) is the scale parameter and the Exponential Inverse Exponential Distribution is denoted as EIED(\( \alpha, \theta \)). The cumulative density function (cdf) is defined as

\[ F(x) = 1 - e^{-\alpha \left(1-e^{-\theta x}\right)} \quad x > 0, \alpha > 0, \theta > 0 \quad (2) \]

Fig. 1 shows the pdf of EIED for various values of both shape and scale parameters.

The survival function is given by

\[ S(x) = e^{-\alpha \left(1-e^{-\theta x}\right)} \quad x > 0, \alpha > 0, \theta > 0 \quad (3) \]

and the hazard function is

\[ h(x) = \frac{\alpha^\theta e^{-\theta x} \left(1-e^{-\theta x}\right)^{1-\alpha}}{x^\alpha \left(1-e^{-\theta x}\right)^{\alpha}} \quad x > 0, \alpha > 0, \theta > 0 \quad (4) \]

Oguntunde et al. [8] concluded that the EIED performs better than the IE distribution except In the case where the data is over-dispersed.
In this article, we propose the Bayes estimators of the shape parameter of an Exponential Inverse Exponential Distribution under the Squared Error loss function, precautionary loss function, entropy loss function and Al-bayatti’s loss function given that the scale parameter is known. In Sections 2 and 3, we discuss estimation of the shape parameter. In Section 4 numerical results are presented for both the simulated and the data on survival times of patients with breast cancer, and Section 5 contains the conclusion.

2 Methodology

2.1 Maximum likelihood function

Consider a random sample \( x = (x_1, x_2, ..., x_n) \) of size \( n \) drawn from Exponential Inverse Exponential Distribution. The likelihood function for the given random sample can be expressed as

\[
L(x/\alpha) = \alpha^n \times \theta^n \times \sum x^{-2} \times e^{-\theta \sum x^{-1}} \times \prod (1 - e^{-x})^2 \times e^{-\theta \sum (\frac{e^{-\theta x}}{1-e^{-x}})}
\]  
(5)

The log-likelihood function of (5) is

\[
lnL(x, \alpha) = nln\alpha + nln\theta - 2 \sum_{i=1}^{n} ln(x) - \theta \sum_{i=1}^{n} x^{-1} - 2 \sum_{i=1}^{n} ln(1 - e^{-x}) - \alpha \sum_{i=1}^{n} (e^{-\theta x} - \frac{e^{-\theta x}}{1-e^{-x}})
\]  
(6)

The maximum likelihood estimator of the shape parameter \( \alpha \) given that the scale parameter \( \theta \) is known is obtained by solving the

\[
\frac{\partial}{\partial \alpha} lnL(x/\alpha) = \frac{n}{\alpha} - \sum_{i=1}^{n} (\frac{e^{-\theta x}}{1-e^{-x}})
\]  
(7)
\[ \hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \left( \frac{y_i}{x_i} \right)} \]  

(8)

2.2 Bayes’ theorem

Bayes’ theorem is expressed as a conditional distribution of \( \alpha \) given \( x \) as

\[ \pi_\beta(\alpha|x) = \frac{f(x|\alpha)\pi_\beta(\alpha)}{\int_{\alpha} f(x|\alpha)\pi_\beta(\alpha) \, d\alpha} \]  

(9)

where the likelihood function of the distribution is \( f(x|\alpha) \), the prior distribution of the parameter \( \alpha \) is \( \pi_\beta(\alpha) \) and the posterior probability distribution is \( \pi_\beta(\alpha|x) \). In this research, (5) is the likelihood function of \( f(x|\alpha) \) and two types of prior information will be used in this work namely: non-informative prior distribution (Hartigen prior distribution, Jeffrey’s prior distribution and Uniform prior distribution) and Informative prior distribution (Gamma prior distribution and Chi-square prior distribution).

2.3 Prior distribution

2.3.1 Extension Jeffrey’s prior distribution

Following [9, 10] and [11], the prior probability distribution is assumed to follow an extension Jeffrey’s prior distribution which can be expressed as

\[ \pi_1 = \left( \frac{1}{\alpha} \right)^\kappa \quad \alpha > 0 \]  

(10)

**Remark 1:** If \( \kappa = 1 \), we have a Jeffrey’s prior. That is

\[ \pi_{11}(\alpha) \propto \frac{1}{\alpha} \quad \alpha > 0 \]  

(11)

**Remark 2:** If \( \kappa = \frac{3}{2} \), we have a Hartigen or modified Jeffrey’s prior. That is

\[ \pi_{12}(\alpha) \propto \left( \frac{1}{\alpha} \right)^{3/2} \quad \alpha > 0 \]  

(12)

**Remark 3:** If \( \kappa = 0 \), we have a Uniform prior. That is

\[ \pi_{13}(\beta) \propto 1 \]  

(13)

2.3.2 Gamma prior distribution

The parameter \( \alpha \) is assumed to follow an informative prior distribution Gamma \((a,b)\) distribution of the form

\[ \pi_2(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1}e^{-ba} \quad \alpha > 0, a > 0, b > 0 \]  

(14)

2.3.3 Chi-square prior distribution

The informative prior distribution for parameter \( \alpha \) is assumed to follow a Chi-square probability distribution with parameter \( \alpha_1 \)
2.4 Loss functions

The concept of loss function was reintroduced in statistics by Abraham Wald in the middle of 20th Century. Sound statistical practice requires selecting an estimator consistent with the actual acceptable variation experienced in the context of a particular applied problem. In this paper we considered the use squared error loss function, Al-Bayyati’s loss function, Entropy loss function and Precautionary loss function for better comparison of Bayes’ estimators.

2.4.1 Squared Error Loss Function (SELF)

The squared error loss function is given as

\[ l_{sq}(\hat{\alpha}, \alpha) = c(\hat{\alpha} - \alpha)^2 \]  

(16)

The risk function under squared-error loss function is written as

\[ R_{RSQ}(\hat{\alpha}) = c(\hat{\alpha} - \alpha)^2 \pi_i(\alpha|x) \, d\alpha \quad i = 1, 2, 3 \]  

(17)

2.4.2 Precautionary Loss Function (PLF)

The precautionary loss function is given as

\[ l_{PLF}(\hat{\alpha}, \alpha) = \frac{c(\hat{\alpha} - \alpha)^2}{\alpha} \]  

(18)

The risk function under the precautionary loss function is written as

\[ R_{RPLF}(\hat{\alpha}) = \int_0^\infty \frac{c(\hat{\alpha} - \alpha)^2}{\alpha} \pi_i(\alpha|x) \, d\alpha \quad i = 1, 2, 3 \]  

(19)

2.4.3 Al-Bayyati’s Loss Function (ALF)

The Al-Bayyati’s loss function is given as

\[ l_{ALF}(\hat{\alpha}, \alpha) = \alpha^c(\hat{\alpha} - \alpha)^2; \quad c \in R^+ \]  

(20)

The risk function under the Al-Bayyati’s loss function is written as

\[ R_{RALF}(\hat{\alpha}) = \int_0^\infty \alpha^c(\hat{\alpha} - \alpha)^2 \pi_i(\alpha|x) \, d\alpha \quad i = 1, 2, 3 \]  

(21)

where \( \hat{\alpha} \) is the estimated value of parameter \( \alpha \).

2.4.4 Entropy Loss Function (ALF)

The Entropy loss function is defined as

\[ R_{RELF}(\hat{\alpha}) = \left( \frac{\hat{\alpha}}{\alpha} \right) - \log \left( \frac{\hat{\alpha}}{\alpha} \right) - 1 \]  

(22)

where \( \hat{\alpha} \) is the parameter estimate of \( \alpha \).
3 Bayesian Estimation

3.1 Bayesian estimation assuming extension Jeffrey’s prior distribution

Following (9), the posterior distribution can be expressed as

\[ \pi(\alpha|x) = \frac{\alpha^n \theta^n \sum_{i=1}^{n} x^{-2} e^{-\theta \sum_{i=1}^{n} x^{-1}} \left( \frac{1}{1-e^{-\theta}} \right)^{2} \left( \frac{\theta}{1-e^{-\theta}} \right)^{\frac{\alpha}{\gamma}} }{ \int_{0}^{\infty} \alpha^n \theta^n \sum_{i=1}^{n} x^{-2} e^{-\theta \sum_{i=1}^{n} x^{-1}} \left( \frac{1}{1-e^{-\theta}} \right)^{2} \left( \frac{\theta}{1-e^{-\theta}} \right)^{\frac{\alpha}{\gamma}} \, d\alpha } \]

Following (9), the posterior distribution can be expressed as

\[ \pi(\alpha|x) = \frac{\theta_{n-k+1}^{n-k}}{\Gamma(n-k+1)} x^{-n-k} e^{-\theta_{n-k+1} \left( \frac{e^{-\theta}}{1-e^{-\theta}} \right)} \]

\[ \alpha > 0 \] (23)

which is the kernel density of gamma distribution parameters \((n - \kappa + 1, \theta_1)\) denoted as \(\Gamma(n - \kappa + 1, \theta_1)\).

where \(\theta_1 = \sum_{i=1}^{n} \left( \frac{e^{-\theta}}{1-e^{-\theta}} \right)\).

3.1.1 Bayes’ estimation under Squared Error loss function using extension of Jeffrey’s prior probability

Following (17), the risk function can be define as

\[ R_{(SQ.E)}(\hat{\alpha}) = \int_{0}^{\infty} c(\hat{\alpha} - \alpha)^2 \frac{\theta_{n-k+1}^{n-k}}{\Gamma(n-k+1)} x^{-n-k} e^{-\theta_{n-k+1} \left( \frac{e^{-\theta}}{1-e^{-\theta}} \right)} \, d\alpha \]

\[ = c \frac{\theta_{n-k+1}^{n-k}}{\Gamma(n-k+1)} \left[ \hat{\alpha}^2 \frac{\Gamma(n-k+1) \theta_{n-k+1}^{n-k+1}}{\theta_{n-k+1}^{n-k+3}} + \frac{\Gamma(n-k+3) \theta_{n-k+1}^{n-k+3}}{\theta_{n-k+1}^{n-k+5}} - 2 \hat{\alpha} \frac{\Gamma(n-k+2) \theta_{n-k+1}^{n-k+2}}{\theta_{n-k+1}^{n-k+4}} \right] \] (24)

Now, on solving \(\frac{\partial R_{(SQ.E)}}{\partial \hat{\alpha}} = 0\), for \(\hat{\alpha}\) we will have required Bayes’ estimator given by

\[ \hat{\alpha}_{(SQ.E)} = \frac{n-k+1}{1} \] (25)

3.1.2 Bayes’ estimation under Precautionary loss function using extension of Jeffrey’s prior probability

Following (19), the risk function is given as

\[ R_{(PLE)}(\hat{\alpha}) = \int_{0}^{\infty} c(\hat{\alpha} - \alpha)^2 \frac{\theta_{n-k+1}^{n-k}}{\alpha} x^{-n-k} e^{-\theta_{n-k+1} \left( \frac{e^{-\theta}}{1-e^{-\theta}} \right)} \, d\alpha \]

\[ = c \frac{\theta_{n-k+1}^{n-k}}{\alpha} \left[ \hat{\alpha}^2 \frac{\Gamma(n-k+1) \theta_{n-k+1}^{n-k+1}}{\theta_{n-k+1}^{n-k+3}} + \frac{\Gamma(n-k+3) \theta_{n-k+1}^{n-k+3}}{\theta_{n-k+1}^{n-k+5}} - 2 \hat{\alpha} \frac{\Gamma(n-k+2) \theta_{n-k+1}^{n-k+2}}{\theta_{n-k+1}^{n-k+4}} \right] \] (26)

Now, on solving \(\frac{\partial R_{(PLE)}}{\partial \hat{\alpha}} = 0\), for \(\hat{\alpha}\) we will have required Bayes’ estimator given by

\[ \hat{\alpha}_{(PLE)} = \frac{\gamma(n-k+1)(n-k+2)}{\theta_1} \] (27)
3.1.3 Bayes’ estimation under Al-Bayyati’s loss function using extension of Jeffreys’ prior probability

Following (21), the risk function is given as

\[ R_{(A\text{L}E\text{F})} (\hat{\alpha}) = \int_0^\infty \alpha \epsilon_2 (\hat{\alpha} - \alpha)^2 \frac{\theta_1^{n-k+1}}{\Gamma(n-k+1)} \alpha^{n-k} e^{-\alpha \sum_{i=1}^n \left( \frac{\theta_i}{\theta_1} \right) \frac{e^{\theta_i/\theta} - 1}{1-e^{-\theta}}} \partial \alpha \]

Following (22), the risk function is given as

\[ R_{(E\text{L}E\text{F})} (\hat{\alpha}) = \int_0^\infty \alpha \epsilon_2 (\hat{\alpha} - \alpha)^2 \frac{\theta_1^{n-k+1}}{\Gamma(n-k+1)} \alpha^{n-k} e^{-\alpha \sum_{i=1}^n \left( \frac{\theta_i}{\theta_1} \right) \frac{e^{\theta_i/\theta} - 1}{1-e^{-\theta}}} \partial \alpha \]

Now, on solving \( \frac{\partial R_{(A\text{L}E\text{F})}}{\partial \hat{\alpha}} = 0 \), for \( \hat{\alpha} \) we will have required Baye’s estimator given by

\[ \hat{\alpha}_{(A\text{L}E\text{F})} = \frac{(n-k+\epsilon+1)}{\theta_1} \]

3.1.4 Bayes’ estimation under Entropy loss function using extension of Jeffreys’ prior probability

Following (22), the risk function is given as

\[ R_{(E\text{L}E\text{F})} (\hat{\alpha}) = \int_0^\infty \alpha \epsilon_2 (\hat{\alpha} - \alpha)^2 \frac{\theta_1^{n-k+1}}{\Gamma(n-k+1)} \alpha^{n-k} e^{-\alpha \sum_{i=1}^n \left( \frac{\theta_i}{\theta_1} \right) \frac{e^{\theta_i/\theta} - 1}{1-e^{-\theta}}} \partial \alpha \]

Now, on solving \( \frac{\partial R_{(E\text{L}E\text{F})}}{\partial \hat{\alpha}} = 0 \), for \( \hat{\alpha} \) we will have required Baye’s estimator given by

\[ \hat{\alpha}_{(E\text{L}E\text{F})} = \frac{n-k}{\theta_1} \]

3.2 Bayesian estimation assuming gamma prior distribution

Following (9), the posterior distribution can be expressed as

\[ \pi (\alpha | x) = \frac{\alpha^n \theta^n \Sigma_{i=1}^n x_i^{-2} e^{-\theta \Sigma_{i=1}^n x_i^{-1}} \Sigma_{i=1}^n \left( \frac{1}{1-e^{-\theta}} \right) e^{-\alpha \sum_{i=1}^n \left( \frac{\theta_i}{\theta} \right) \frac{e^{\theta_i/\theta} - 1}{1-e^{-\theta}}} \partial \alpha}{\Gamma (\alpha)} \alpha^{n+a-1} e^{-ba} \]

\[ \pi (\alpha | x) = \frac{\alpha^n \theta^n \Sigma_{i=1}^n x_i^{-2} e^{-\theta \Sigma_{i=1}^n x_i^{-1}} \Sigma_{i=1}^n \left( \frac{1}{1-e^{-\theta}} \right) e^{-\alpha \sum_{i=1}^n \left( \frac{\theta_i}{\theta} \right) \frac{e^{\theta_i/\theta} - 1}{1-e^{-\theta}}} \partial \alpha}{\Gamma (\alpha)} \alpha^{n+a-1} e^{-ba} \]

\[ \pi (\alpha | x) = \frac{\theta_1^{n+a}}{\Gamma (n+a)} \alpha^{n+a-1} e^{-\alpha \sum_{i=1}^n \left( \frac{\theta_i}{\theta} \right) \frac{e^{\theta_i/\theta} - 1}{1-e^{-\theta}}} \quad \alpha > 0 \]

(32)

which is the density kernel of gamma distribution having parameters \((n + a, \theta_2)\) denoted as \( G(n + a, \theta_2) \).

Where \( \theta_2 = -\alpha \left[ \sum_{i=1}^n \left( \frac{\theta_i}{\theta} \right) \frac{e^{\theta_i/\theta} - 1}{1-e^{-\theta}} \right] + b \)

3.2.1 Bayes’ estimation under squared error loss function using gamma prior probability

Following (17), the risk function can be define as

\[ \pi (\alpha | x) = \frac{\theta_1^{n+a}}{\Gamma (n+a)} \alpha^{n+a-1} e^{-\alpha \sum_{i=1}^n \left( \frac{\theta_i}{\theta} \right) \frac{e^{\theta_i/\theta} - 1}{1-e^{-\theta}}} \quad \alpha > 0 \]

(33)
Following (22), the risk function is given as

$$R_{(SQ,GP)}(\hat{\alpha}) = \int_0^{\infty} c(\hat{\alpha} - \alpha)^2 \frac{\theta_{2}^{n-a}}{\Gamma(n + a)} e^{-a \left[ \sum_{i=1}^{n} \left( \frac{e^{\frac{-1}{\alpha}} - 1}{\alpha + 1} \right) + b \right]} \, d\alpha$$

$$= c \frac{\theta_{2}^{n-a}}{\Gamma(n+a)} \alpha^{n+a-1} \left[ \hat{\alpha}^2 \frac{\Gamma(n+a)}{\theta_{2}^{n+a}} + \frac{\Gamma(n+a+2)}{\theta_{2}^{n+a+2}} - 2\hat{\alpha} \frac{\Gamma(n+1+a)}{\theta_{2}^{n+a+1}} \right]$$

(33)

Now, on solving $\frac{\partial R_{(SQ,GP)}}{\partial \hat{\alpha}} = 0$, for $\hat{\alpha}$ we will have required Baye’s estimator given by

$$\hat{\alpha}_{(SQ,GP)} = \frac{n+a}{\theta_{2}}$$

(34)

### 3.2.2 Bayes’ estimation under precautionary loss function using gamma distribution prior probability

Following (19), the risk function is given as

$$R_{(PL,GP)}(\hat{\alpha}) = \int_0^{\infty} c(\hat{\alpha} - \alpha)^2 \frac{\theta_{2}^{n-a}}{\alpha \Gamma(n + a)} e^{-a \left[ \sum_{i=1}^{n} \left( \frac{e^{\frac{-1}{\alpha}} - 1}{\alpha + 1} \right) + b \right]} \, d\alpha$$

$$= c \frac{\theta_{2}^{n-a}}{\Gamma(n+a)} \left[ \hat{\alpha}^2 \frac{\Gamma(n+a)}{\theta_{2}^{n+a}} + \frac{1}{\alpha} \frac{\Gamma(n+a+2)}{\theta_{2}^{n+a+2}} - 2\hat{\alpha} \frac{\Gamma(n+1)\Gamma(n+a+1)}{\theta_{2}^{n+a+1}} \right]$$

(35)

Now, on solving $\frac{\partial R_{(PL,GP)}}{\partial \hat{\alpha}} = 0$, for $\hat{\alpha}$ we will have required Baye’s estimator given by

$$\hat{\alpha}_{(PL,GP)} = \frac{\sqrt{n(a+1)(n+a)}}{\theta_{2}}$$

(36)

### 3.2.3 Bayes’ estimation under Al-Bayyati’s loss function using gamma prior probability

Following (21), the risk function is given as

$$R_{(AL,GP)}(\hat{\alpha}) = \int_0^{\infty} \alpha^2 c(\hat{\alpha} - \alpha)^2 \frac{\theta_{2}^{n-a}}{\Gamma(n + a)} e^{-a \left[ \sum_{i=1}^{n} \left( \frac{e^{\frac{-1}{\alpha}} - 1}{\alpha + 1} \right) + b \right]} \, d\alpha$$

$$= \frac{\theta_{2}^{n-a}}{\Gamma(n+a)} \left[ \hat{\alpha}^2 \frac{\Gamma(n+a+c)}{\theta_{2}^{n+a+c+e}} + \frac{\Gamma(n+a+c+2)}{\theta_{2}^{n+a+c+2+e}} - 2\hat{\alpha} \frac{\Gamma(n+a+c+1)}{\theta_{2}^{n+a+c+1}} \right]$$

(37)

Now, on solving $\frac{\partial R_{(AL,GP)}}{\partial \hat{\alpha}} = 0$, for $\hat{\alpha}$ we will have required Baye’s estimator given by

$$\hat{\alpha}_{(AL,GP)} = \frac{n+a+c}{\theta_{2}}$$

(38)

### 3.2.4 Bayes’ estimation under entropy loss function using gamma prior probability

Following (22), the risk function is given as

$$R_{(EL,GP)}(\hat{\alpha}) = \int_0^{\infty} c \left( \frac{\hat{\alpha}}{\alpha} - \log(\hat{\alpha}) - 1 \right) \frac{\theta_{2}^{n-a}}{\alpha \Gamma(n + a)} e^{-a \left[ \sum_{i=1}^{n} \left( \frac{e^{\frac{-1}{\alpha}} - 1}{\alpha + 1} \right) + b \right]} \, d\alpha$$

$$= c \frac{\theta_{2}^{n-a}}{\Gamma(n+a)} \left[ \hat{\alpha} \left( \frac{\Gamma(n-a-1)}{\theta_{2}^{n-a-1}} - \frac{\Gamma(n+a)}{\theta_{2}^{n+a}} \right) \log(\hat{\alpha}) + \frac{\Gamma(n+a)}{\theta_{2}^{n+a}} - \frac{\Gamma(n+a)}{\theta_{2}^{n+a}} \right]$$

(39)
Now, on solving $\frac{\partial R_{(ELGP)}}{\partial \hat{a}} = 0$, for $\hat{a}$ we will have required Baye’s estimator given by

$$\hat{a}_{(ELGP)} = \frac{n+a-1}{\theta_2}$$  

(40)

### 3.3 Bayesian estimation assuming chi-square prior distribution

Following (9), the posterior distribution can be expressed as

$$\pi(\alpha|x) = \frac{\alpha^n \theta^n \sum_{i=1}^n x^{-2} e^{-\theta \sum_{i=1}^n x^{-1}} \sum_{i=1}^n \left( \frac{1}{1-e^{1/n}} \right)^2 e^{-\theta \sum_{i=1}^n \left( \frac{1}{1-e^{1/n}} \right)^2} e^{-\theta \sum_{i=1}^n \left( \frac{e^{\frac{\theta}{x}}}{1-e^{1/n}} \right)^2}}{\Gamma(n+\frac{\theta}{2}) \alpha^\frac{n+a-1}{2} e^{\frac{\theta}{2}}}$$

(3.3.2)

Now, on solving

$$\pi(\alpha|x) = \frac{\theta_3^{n+\frac{a}{2}} \alpha^{n+\frac{a-1}{2}} e^{-\theta \sum_{i=1}^n \left( \frac{e^{\frac{\theta}{x}}}{1-e^{1/n}} \right)^2}}{\Gamma(n+\frac{\theta}{2})}$$

which is the density kernel of gamma distribution having parameters $(n + \frac{a}{2}, \theta_3)$ denoted as $G(n + \frac{a}{2}, \theta_3)$.

Where $\theta_3 = \sum_{i=1}^n \left( \frac{e^{\frac{\theta}{x}}}{1-e^{1/n}} \right)^2 + \frac{1}{2}$

### 3.3.1 Bayes’ estimation under squared error loss function using chi-square prior probability

Following (17), the risk function can be defined as

$$R_{(SQCP)}(\hat{a}) = \int_0^\infty c(\hat{a} - \alpha)^2 \frac{\theta_3^{n+\frac{a}{2}} \alpha^{n+\frac{a-1}{2}} e^{-\theta \sum_{i=1}^n \left( \frac{e^{\frac{\theta}{x}}}{1-e^{1/n}} \right)^2}}{\Gamma(n+\frac{\theta}{2})} d\alpha$$

$$= c \frac{\theta_3^{n+\frac{a}{2}}}{\Gamma(n+\frac{\theta}{2})} \left[ (n+\frac{a}{2}) \theta_3^{n+\frac{a}{2}} + \frac{\Gamma(n+\frac{a+2}{2})}{\theta_3^{n+\frac{a}{2}+2}} - 2 \theta_3 \right]$$

(42)

Now, on solving $\frac{\partial R_{(SQCP)}}{\partial \hat{a}} = 0$, for $\hat{a}$ we will have required Bayes’ estimator given by

$$\hat{a}_{(SQCP)} = \frac{n+\frac{a}{2}}{\theta_3}$$

(43)

### 3.3.2 Bayes’ estimation under precautionary loss function using chi-square distribution prior probability

Following (19), the risk function is given as

$$R_{(PLCP)}(\hat{a}) = \int_0^\infty c(\hat{a} - \alpha)^2 \frac{\theta_3^{n+\frac{a}{2}} \alpha^{n+\frac{a-1}{2}} e^{-\theta \sum_{i=1}^n \left( \frac{e^{\frac{\theta}{x}}}{1-e^{1/n}} \right)^2}}{\Gamma(n+\frac{\theta}{2})} d\alpha$$

$$= c \frac{\theta_3^{n+\frac{a}{2}}}{\Gamma(n+\frac{\theta}{2})} \left[ (n+\frac{a}{2}) \theta_3^{n+\frac{a}{2}} + 1 \frac{\Gamma(n+\frac{a+2}{2})}{\theta_3^{n+\frac{a}{2}+2}} - 2 \theta_3 \right]$$

(44)
Now, on solving $\frac{\partial R_{(PLCP)}}{\partial \hat{a}} = 0$, for $\hat{a}$ we will have required Baye’s estimator given by

$$\hat{a}_{(PLCP)} = \frac{\sqrt{(a_2^2 + a_1^2) + a_1^2}}{\theta_3}$$

(45)

### 3.3.3 Bayes’ estimation under Al-Bayyati’s loss function using chi-square prior probability

Following (21), the risk function is given as

$$R_{(ALCP)}(\hat{a}) = \int_0^\infty c \left( \frac{\hat{a}}{\alpha} - \log\left(\frac{\hat{a}}{\alpha}\right) - 1 \right) \frac{\theta_3^{n+\frac{a}{2}}}{\Gamma(n + \frac{a}{2})} \alpha^{n+\frac{a}{2}-1} e^{-\alpha\sum_{i=1}^n \left(\frac{e^{\frac{\theta}{\alpha} x_i}}{x_i}\right)^{x_i}} \, d\alpha$$

$$= \frac{\theta_3^{n+\frac{a}{2}}}{\Gamma(n + \frac{a}{2})} \left[ \hat{a}^2 \frac{\Gamma(n + \frac{a}{2} + c)}{\theta_3^{n+\frac{a}{2}+c}} + \frac{\Gamma(n + \frac{a}{2} + c + 2)}{\theta_3^{n+\frac{a}{2}+c+2}} - 2\hat{a} \frac{\Gamma(n + \frac{a}{2} + c + 1)}{\theta_3^{n+\frac{a}{2}+c+1}} \right]$$

(46)

Now, on solving $\frac{\partial R_{(ALCP)}}{\partial \hat{a}} = 0$, for $\hat{a}$ we will have required Baye’s estimator given by

$$\hat{a}_{(ALCP)} = \frac{n + \frac{a}{2} + c}{\theta_3}$$

(47)

### 3.3.4 Bayes’ estimation under entropy loss function using chi-square prior probability

Following (22), the risk function is given as

$$R_{(ELCP)}(\hat{a}) = \int_0^\infty c \left( \frac{\hat{a}}{\alpha} - \log\left(\frac{\hat{a}}{\alpha}\right) - 1 \right) \frac{\theta_3^{n+\frac{a}{2}}}{\Gamma(n + \frac{a}{2})} \alpha^{n+\frac{a}{2}-1} e^{-\alpha\sum_{i=1}^n \left(\frac{e^{\frac{\theta}{\alpha} x_i}}{x_i}\right)^{x_i}} \, d\alpha$$

$$= \frac{\theta_3^{n+\frac{a}{2}}}{\Gamma(n + \frac{a}{2})} \left[ \hat{a}^2 \frac{\Gamma(n + \frac{a}{2} + 1)}{\theta_3^{n+\frac{a}{2}+1}} - \frac{\Gamma(n + \frac{a}{2} + 2)}{\theta_3^{n+\frac{a}{2}+2}} \log(\hat{a}) + \frac{\Gamma(n + \frac{a}{2})}{\theta_3^{n+\frac{a}{2}}} - \frac{\Gamma(n + \frac{a}{2})}{\theta_2^{n+\frac{a}{2}}} \right]$$

(48)

Now, on solving $\frac{\partial R_{(ELCP)}}{\partial \hat{a}} = 0$, for $\hat{a}$ we will have required Baye’s estimator given by

$$\hat{a}_{(ELCP)} = \frac{n + \frac{a}{2} - 1}{\theta_2}$$

(49)

### 4 Analysis

#### 4.1 Monte Carlo simulation

In this section, our simulation study we have simulated a sample of sizes $n = 30$, $50$, $100$ and $200$ from an Exponential Inverse Exponential Distribution with parameters $\alpha = 0.5$, $1.5$ and $2$, $c = 1$ and $2$ and $\theta = 1$. Monte Carlo method is any computational approach pseudo-random number solve a mathematical problems as defined by [12]. Thus the numerical approach follows

1. For known parameters values ($\alpha, \theta$), we simulated a random sample of size of $n$ from EIE using the quantile function of the EIE distribution is given by

$$Q(u) = \frac{\theta}{\log(\gamma^{-1} \log(1-u))^{-1} + 1}$$

(50)
2. We then estimate the shape parameters $\alpha$ using Bayesian approach given that $\theta$ is known.
3. Perform 10,000 repetitions of step 1-2
4. We compute the MSE

The results are replicated 10,000 times and the average result was presented in the tables. To examine the performance of Bayesian estimates for shape parameter of Exponential Inverse Exponential distribution under different loss functions, estimates are presented along with their MSE in the below tables.

$$\hat{\alpha} = \frac{1}{10000} \sum_{i=1}^{10000} \hat{\alpha}_i$$  \hspace{1cm} (51)

### Table 1. Posterior estimates and MSE under Square Error Loss Function (SELF)

| n   | $\alpha$ | Jeffrey’s prior | Hartigan prior | Uniform prior | Gamma prior | Chi-square prior |
|-----|----------|-----------------|----------------|--------------|-------------|------------------|
|     |          | $\kappa = 1$    | $\kappa = 3$  | $\kappa = 0$ | $a = 1.5$, $b = 2$ | $a = 1$, $b = 0.5$ | $a = 1.5$, $b = 2$ |
| 30  | 2        | 2.1089          | 2.0548         | 2.1071       | 1.8363      | 2.0552           | 2.0476           | 2.1084           |
|     |          | (0.1489)        | (0.1229)       | (0.1353)     | (0.0926)    | (0.1192)         | (0.1393)         | (0.1417)         |
|     | 1.5      | 1.5440          | 1.5160         | 1.5995       | 1.4695      | 1.5464           | 1.5388           | 1.5565           |
|     |          | (0.1151)        | (0.0906)       | (0.0737)     | (0.0640)    | (0.0946)         | (0.0665)         | (0.0657)         |
|     | 0.5      | 0.5366          | 0.4963         | 0.5196       | 0.5152      | 0.5352           | 0.5041           | 0.5597           |
|     |          | (0.0096)        | (0.0108)       | (0.0076)     | (0.0070)    | (0.0105)         | (0.0093)         | (0.0051)         |
| 50  | 2        | 2.0511          | 1.9766         | 2.0573       | 1.9907      | 2.0243           | 2.0030           | 2.0673           |
|     |          | (0.0876)        | (0.0792)       | (0.0994)     | (0.0857)    | (0.0753)         | (0.0804)         | (0.0747)         |
|     | 1.5      | 1.5414          | 1.5420         | 1.5166       | 1.5282      | 1.5644           | 1.5321           | 1.5384           |
|     |          | (0.0435)        | (0.0658)       | (0.0377)     | (0.0477)    | (0.0379)         | (0.0620)         | (0.0444)         |
|     | 0.5      | 0.5069          | 0.5073         | 0.5220       | 0.5119      | 0.5278           | 0.5210           | 0.5300           |
|     |          | (0.0057)        | (0.0052)       | (0.0045)     | (0.0055)    | (0.0050)         | (0.0051)         | (0.0052)         |
| 100 | 2        | 2.0404          | 1.9887         | 2.0374       | 1.9456      | 2.0478           | 2.0153           | 2.0417           |
|     |          | (0.0418)        | (0.0442)       | (0.0372)     | (0.0411)    | (0.0335)         | (0.0431)         | (0.0482)         |
|     | 1.5      | 1.5193          | 1.5416         | 1.5210       | 1.5015      | 1.5111           | 1.5300           | 1.4988           |
|     |          | (0.0261)        | (0.0285)       | (0.0244)     | (0.0219)    | (0.0218)         | (0.0199)         | (0.0247)         |
|     | 0.5      | 0.5178          | 0.5047         | 0.5115       | 0.5088      | 0.5100           | 0.5046           | 0.5034           |
|     |          | (0.0032)        | (0.0024)       | (0.0025)     | (0.0050)    | (0.0026)         | (0.0026)         | (0.0022)         |
| 200 | 2        | 2.0152          | 2.0063         | 1.9963       | 1.9861      | 1.9891           | 2.0058           | 1.9829           |
|     |          | (0.0225)        | (0.0170)       | (0.0157)     | (0.0188)    | (0.0200)         | (0.0205)         | (0.0171)         |
|     | 1.5      | 1.4522          | 1.5073         | 1.5282       | 1.4994      | 1.5109           | 1.5049           | 1.5014           |
|     |          | (0.0139)        | (0.0124)       | (0.0135)     | (0.0111)    | (0.0130)         | (0.0118)         | (0.0094)         |
|     | 0.5      | 0.5058          | 0.4985         | 0.5029       | 0.5048      | 0.5088           | 0.5032           | 0.5015           |
|     |          | (0.0017)        | (0.0013)       | (0.0014)     | (0.0015)    | (0.0008)         | (0.0011)         | (0.001)          |

### Table 2. Posterior estimates and MSE under Precautionary Loss Function (PLF)

| n   | $\alpha$ | Jeffrey’s prior | Hartigan prior | Uniform prior | Gamma prior | Chi-square prior |
|-----|----------|-----------------|----------------|--------------|-------------|------------------|
|     |          | $\kappa = 1$    | $\kappa = 3$  | $\kappa = 0$ | $a = 1.5$, $b = 2$ | $a = 1$, $b = 0.5$ | $a = 1.5$, $b = 2$ |
| 30  | 2        | 2.1438          | 2.0893         | 2.1408       | 1.8652      | 2.1392           | 2.0801           | 2.1421           |
|     |          | (0.1539)        | (0.1271)       | (0.1397)     | (0.0956)    | (0.1231)         | (0.1438)         | (0.1463)         |
|     | 1.5      | 1.5695          | 1.5414         | 1.6251       | 1.4927      | 1.5711           | 1.5636           | 1.6067           |
|     |          | (0.1189)        | (0.0937)       | (0.0761)     | (0.0661)    | (0.0976)         | (0.0686)         | (0.0678)         |
|     | 0.5      | 0.5455          | 0.5046         | 0.5279       | 0.5234      | 0.5434           | 0.5123           | 0.5483           |
|     |          | (0.0099)        | (0.0111)       | (0.0078)     | (0.0073)    | (0.0108)         | (0.0096)         | (0.0138)         |
| n  | α   | Jeffrey’s prior | Hartigan prior | Uniform prior | Gamma prior | Chi-square prior |
|----|-----|-----------------|----------------|--------------|-------------|-----------------|
|    |     | κ = 1           | κ = 3/2, c = 2 | κ = 0, c = 2 | a = 1.5, b = 2, c = 1 | a = 1, b = 0.5, c = 2 | a = 1.5, c = 2 | a = 2 |
| 50 | 2   | 2.0715          | 1.9965         | 2.0774       | 2.0100      | 2.0441          | 2.0226         | 2.0875 |
|    |     | (0.0947)        | (0.0808)       | (0.1014)     | (0.0874)    | (0.077)         | (0.081)        | (0.0762) |
|    | 1.5 | 1.5567          | 1.5575         | 1.5314       | 1.5430      | 1.5797          | 1.5471         | 1.5534 |
|    |     | (0.0044)        | (0.0671)       | (0.0384)     | (0.0486)    | (0.0387)        | (0.0632)       | (0.0452) |
|    | 0.5 | 0.5120          | 0.5125         | 0.5271       | 0.5168      | 0.5330          | 0.5261         | 0.53518 |
|    |     | (0.0058)        | (0.0054)       | (0.0046)     | (0.0056)    | (0.0051)        | (0.0052)       | (0.0051) |
| 100| 2   | 2.0505          | 1.9987         | 2.0475       | 1.9551      | 2.0579          | 2.0253         | 2.0518 |
|    |     | (0.0422)        | (0.0446)       | (0.0378)     | (0.0415)    | (0.0338)        | (0.0435)       | (0.0487) |
|    | 1.5 | 1.5268          | 1.5493         | 1.5285       | 1.5089      | 1.5185          | 1.5376         | 1.5062 |
|    |     | (0.0263)        | (0.0289)       | (0.0247)     | (0.0221)    | (0.0221)        | (0.0201)       | (0.0249) |
|    | 0.5 | 0.5204          | 0.5073         | 0.5141       | 0.5113      | 0.5125          | 0.5071         | 0.5059 |
|    |     | (0.0032)        | (0.0025)       | (0.0025)     | (0.0022)    | (0.0027)        | (0.0026)       | (0.0023) |
| 200| 2   | 2.0202          | 2.0114         | 2.0012       | 1.9910      | 1.9941          | 2.0108         | 1.9878 |
|    |     | (0.0226)        | (0.0171)       | (0.0158)     | (0.0189)    | (0.0201)        | (0.0206)       | (0.0172) |
|    | 1.5 | 1.4960          | 1.5111         | 1.5320       | 1.5031      | 1.5147          | 1.5184         | 1.5057 |
|    |     | (0.0140)        | (0.0124)       | (0.0112)     | (0.0113)    | (0.0131)        | (0.0104)       | (0.0077) |
|    | 0.5 | 0.5071          | 0.4997         | 0.5042       | 0.5060      | 0.5010          | 0.5045         | 0.5027 |
|    |     | (0.0017)        | (0.0013)       | (0.0011)     | (0.0016)    | (0.0008)        | (0.0011)       | (0.0012) |

Table 3. Posterior estimates and MSE under Al-bayyti’s Loss Function (ALF)
4.2 Application to patients with breast cancer

This data represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929-1938 used by [13] and [14]. The observations are as follows: 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 40.0, 40.0, 40.0, 41.0, 41.0, 42.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 110.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 139.0, 154.0.

Table 5. Posterior estimates for the shape parameter (α) for survival times of patients with breast cancer

|                | SELF | ALF | PLF | ELF |
|----------------|------|-----|-----|-----|
| Jeffrey’s prior (c=1.5) | 0.0688 | 0.0070 | 0.0069 | 0.0068 |
| Hartigan prior (c=2) | 0.0068 | 0.0069 | 0.0068 | 0.0067 |
| Uniform prior (c=3) | 0.0068 | 0.0070 | 0.0069 | 0.0068 |
| Gamma Prior (α=1.5,b=2, c=1) | 0.0069 | 0.0070 | 0.0069 | 0.0069 |
| Chi-Square Prior (α=1, c=2) | 0.0069 | 0.0070 | 0.0069 | 0.0068 |
5 Conclusion

This work emphasis on the importance of Bayesian approximation using loss functions. We employed the Bayesian approximation to obtain the posterior estimates of an Exponential inverted Exponential distribution using different prior distribution under different loss functions. Fig. 1 shows that the shape of the EIE distribution at varying parameter values were unimodal. Tables 1-4 shows the posterior estimates with MSE for different prior distribution under different loss functions for the simulated datasets. Table 5 shows the posterior estimate on survival times of patients with breast cancer under different prior distributions and loss functions. Table 5, shows the bayes’ estimate for the shape parameter with the assumption that the scale parameter equals to 0.350733 given by [8].

Based on the results displayed in Tables 1-3, we observed that all the posterior estimates of the shape parameter are close to the true values of parameters of an EIE distribution. Also, we observed that the mean squares error based on different priors tends to decrease with the increase in sample size. It implies that the estimators obtained are consistent. Also, it can be observed that the Bayesian estimates of the shape parameter under the informative prior distribution perform better than that of the non-informative prior distribution. The results obtained under Gamma and Chi-Square distribution were quite more efficient than others especially under the Entropy loss function, Square error loss function and Precautionary loss function.

Competing Interests

Authors have declared that no competing interests exist.

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