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Abstract
Holomorphic functions are amazing because their values in an ever so small disk in the complex plane completely determine the function values at arbitrary points in their maximum possible domain. The process of extending such a function beyond its initial domain is called analytic continuation. We attempt to make this theoretic result tractable by computers. In the present article, we first prove that any algorithm for analytic continuation can generally not depend on finitely many function values only, without closer inspection of the function itself. We then derive a computable local bound on the step size between sampling points which yields an algorithm for analytic continuation of complex plane algebraic curves. Finally, we provide a numerical example demonstrating its practical use.

1. Introduction
Let $U \subset \mathbb{C}$ be an open set. A function
$$f : U \rightarrow \mathbb{C}, \quad f(x + iy) = u(x, y) + iv(x, y),$$
is complex differentiable at a point $z = x + iy \in U$, if its real part $u(x, y)$ and its imaginary part $v(x, y)$ satisfy the Cauchy–Riemann equations
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$Function $f$ is holomorphic, if it is complex differentiable at every point in its domain. Equivalently, around every point $z_0 \in U$, $f$ can be locally represented as a power series with complex coefficients,
$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$This power series has a positive radius of convergence $r_0$, such that it converges at every point whose distance from $z_0$ is smaller than $r_0$, and diverges at every point whose distance from $z_0$ is greater than $r_0$. Therefore, the representation of $f$ as a power series around $z_0$ is valid only inside the open disk of radius $r_0$ around $z_0$, denoted by $B_{r_0}(z_0)$.

However, for every point $z_1 \in B_{r_0}(z_0)$, we may rearrange the terms of the power series according to
$$z - z_0 = z - z_1 - (z_0 - z_1)$$to obtain another series which proceeds in powers of $z - z_1$,
$$f(z) = \sum_{n=0}^{\infty} b_n(z - z_1)^n,$$
and converges inside an open disk of convergence $B_{r_1}(z_1)$. Clearly, the two power series representations of $f$ agree on the intersection of $B_{r_0}(z_0)$ and $B_{r_1}(z_1)$. At points in $B_{r_1}(z_1)$ not in $B_{r_0}(z_0)$, we can use the latter power series representation to obtain values for $f$ which we cannot compute using the former power series representation. Those values are exactly the values which allow to consistently define $f$ as a holomorphic function on $B_{r_0}(z_0)$ and $B_{r_1}(z_1)$. This process, by which a holomorphic function can be extended beyond its initial domain, is called analytic continuation.

Since it carries all information about $f$ as a holomorphic function, a power series expansion of $f$ around a point $z_0$ with positive radius of convergence and complex coefficients $a_0, a_1, \ldots$ is called a (holomorphic) function germ of $f$. $z_0$ is called its base and $a_0$ its top (value at $z_0$). We will use the compact notation $(z_0, a_0, a_1, \ldots)$ for function germs.

Analytic continuation of a function germ may have interesting effects: For example, consider a function germ of the complex square root function with base $z_0 \in \mathbb{C}$, $z_0 \neq 0$, and value $+\sqrt{z_0}$ at $z_0$. The corresponding power series has radius of convergence $|z_0|$. If we continue the function germ analytically along the circle of radius $|z_0|$ around the origin — that is to say, we choose expansion points $z_j+1 \in B_{|z_0|}(z_j)$, $j = 0, 1, \ldots$, which lie on that circle — then after one round we return to $z_0$ with a different function germ. It is the function germ of the complex square root with base $z_0$ and value $-\sqrt{z_0}$ at $z_0$. This happens because the origin is a ramification point of the complex square root, that is a point at which the two branches $+\sqrt{z}$ and $-\sqrt{z}$ collapse into the branch point $0 = \pm \sqrt{0}$. The ramification point is also responsible for the radius of convergence around $z_0$ not being greater than $|z_0|$. The complex square root function can therefore not be defined as a holomorphic function on the whole complex plane.

Other singularities which delimit a disk of convergence are poles and essential singularities. If we perform analytic continuation along a curve containing a singularity as one of its points, we cannot continue beyond that point.

Another representation of the complex square root function, which emphasizes its multivaluedness, is as an algebraic function $f(x, y) = y^2 - x = 0$. We see that immediately that two $y$-values (counted with multiplicity) satisfy the equation for every $x$-value. In general, an algebraic function is a polynomial in $x$ and $y$ with complex coefficients, whose zero set describes a complex one-dimensional surface, a complex plane algebraic curve. To every $x$-value correspond one or more $y$-values. Around most points in the complex $x$-plane (with only finitely many exceptions), the different $y$-values are holomorphic functions.

Historically, the desire to handle such “multivalued” functions like the complex square root gave rise to the theory of Riemann surfaces. We will take a slightly different approach and define a multivalued evaluation function. For the complex square root function we obtain for example the function

$$[\sqrt{\cdot}] : \mathbb{C} \to \mathbb{C}^2, \quad z \mapsto \begin{pmatrix} +\sqrt{z} \\ -\sqrt{z} \end{pmatrix}.$$ More formally, this leads to the following definition:

**Definition** Multivalued evaluation function. Let $U \subset \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ a holomorphic function, and $x_1 \in U$. Let $(x_1, a_0, a_1, \ldots)$ be a function germ of $f$ at $x_1$. Consider the space $X$ of all function germs which result from $(x_1, a_0, a_1, \ldots)$ by analytic continuation. Let $p_1 : X \to \mathbb{C}$, $(x, a_0, a_1, \ldots) \mapsto x$, be a projection map, which maps any function germ in $X$ to its base $x$. Furthermore, let $p_2 : X \to \mathbb{C}$, $(x, a_0, a_1, \ldots) \mapsto a_0$, be a projection map, which maps any function germ in $X$ to its top $a_0$, that is to its value at $x$. Suppose that the cardinality of the preimage of any point in the complex plane under $p_1$ is
finite and constant

\[ |p_1^{-1}(x)| \equiv n, \quad \forall x \in \mathbb{C}. \]

Let

\[ [f]: \mathbb{C} \to \mathbb{C}^n, \quad x \mapsto (p_2 \circ p_1^{-1})(x) \]

be a function which maps a point \( x \) in the complex plane to a vector of the values at \( x \) of the function germs in the preimage of \( x \) under \( p_1 \). The vector shall contain the values in a fixed, but otherwise arbitrary order, for instance in lexicographic order according to the decimal representation of real and imaginary part of the values. Then we call \([f]\) multivalued evaluation function of \( f \).

**Problem.** Suppose that we know the values of a multivalued evaluation function \([f]\) at two distinct points, \( z_0 \in \mathbb{C} \) not a ramification point of \( f \), and \( z_1 \in \mathbb{C} \). If we continue a function germ of \( f \) with base \( z_0 \) and a certain value at \( z_0 \) analytically along the segment between \( z_0 \) and \( z_1 \), how can we decide algorithmically which of the values in \([f](z_1)\) the resulting function germ will have at \( z_1 \)?

The above problem shall concern us in the following sections.

### 2. An undecidability result

A fundamental question in designing an algorithm is, on which input it ought to operate. Ideally, the input to an algorithm is so sparse that the algorithm becomes most generally applicable, but not too sparse so that the problem it addresses can still be solved efficiently. Our goal is to assign the function values of a multivalued evaluation function at one point to the function values at another point according to analytic continuation along the segment between these points. The problem statement includes the following data: a multivalued evaluation function, two points in the complex plane, and the function values at those points.

Therefore, it seems natural to consider, as the input to an algorithm, the two points in the complex plane and the possibility to evaluate the multivalued evaluation function at arbitrary points. Due to continuity, unless the segment contains a singularity, we can expect to solve the problem, if we take small steps along the segment and assign closest function values to each other in every step. However, it is difficult to estimate how small the step width must be in practice. Indeed we can show that no algorithm to solve our problem exists which relies solely on function evaluation. In particular, it is impossible to determine a sufficiently small step width without closer inspection of the analytic function itself.

**Theorem 1.** Let \( U \subset \mathbb{C} \) be an open set, \( f: U \to \mathbb{C} \) a holomorphic function, and \( x_1 \in U \) and \( x_2 \in \mathbb{C} \) distinct points. Let \([f]: \mathbb{C} \to \mathbb{C}^n\) be a multivalued evaluation function of \( f \). Let \( \pi \in S_n \) be a permutation of \( \{1, 2, \ldots, n\} \). Consider the following decision problem:

Is \([f](x_2)_{\pi(k)}\) the value at \( x_2 \) which results from analytic continuation of the function germ yielding value \([f](x_1)_k\) at \( x_1 \) along the segment between \( x_1 \) and \( x_2 \), for all \( k = 1, 2, \ldots, n \)?

The problem cannot be decided based only on \( x_1, x_2, n, \pi \), and the values of \([f]\) at finitely many points.
Proof. It suffices to show that after evaluating \([f]\) at \(k\) distinct points, \(k = 2, 3, \ldots\), there always exist two algebraic functions

\[
f_1(x, y) = 0, \quad f_2(x, y) = 0
\]

which produce the known values of \([f]\) but whose branches differ in their behaviour regarding analytic continuation along the segment between \(x_1\) and \(x_2\). Then branches of both algebraic functions represent a possible choice of \(f\) which explains the observed function values, and the problem must remain undecided, independent of the number \(k\) of evaluations of \([f]\).

Hence, suppose that for \(k\) distinct \(x\)-values, \(x_1, x_2, \ldots, x_k\), we know \(n\) corresponding \(y\)-values

\[
[f](x_j)_1, [f](x_j)_2, \ldots, [f](x_j)_n, \quad j = 1, 2, \ldots, k.
\]

Without loss of generality, we restrict ourselves to the case \(n = 2\). We interpret \(f_1(x, y)\) and \(f_2(x, y)\) as polynomials in \(y\) whose coefficients are polynomials in \(x\), that is

\[
f_1(x, y) = a_2(x)y^2 + a_1(x)y + a_0(x),
\]

\[
f_2(x, y) = b_2(x)y^2 + b_1(x)y + b_0(x).
\]

Thus the following equations should be satisfied for \(j = 1, 2, \ldots, k:\)

\[
f_1(x_j, [f](x_j)_1) = a_2(x_j) ([f](x_j)_1)^2 + a_1(x_j) [f](x_j)_1 + a_0(x_j) = 0
\]

\[
f_1(x_j, [f](x_j)_2) = a_2(x_j) ([f](x_j)_2)^2 + a_1(x_j) [f](x_j)_2 + a_0(x_j) = 0
\]

\[
f_2(x_j, [f](x_j)_1) = b_2(x_j) ([f](x_j)_1)^2 + b_1(x_j) [f](x_j)_1 + b_0(x_j) = 0
\]

\[
f_2(x_j, [f](x_j)_2) = b_2(x_j) ([f](x_j)_2)^2 + b_1(x_j) [f](x_j)_2 + b_0(x_j) = 0
\]

The existence of suitable polynomials \(a_2(x), a_1(x),\) and \(a_0(x)\) follows, if we prescribe

\[
\begin{bmatrix}
  a_2(x_j) \\
  a_1(x_j) \\
  a_0(x_j)
\end{bmatrix} = \begin{bmatrix}
  ([f](x_j)_1)^2 \\
  [f](x_j)_1 \\
  1
\end{bmatrix} \times \begin{bmatrix}
  ([f](x_j)_2)^2 \\
  [f](x_j)_2 \\
  1
\end{bmatrix}
\]

for \(j = 1, 2, \ldots, k\), where \(\times\) denotes the cross product of two vectors, and apply polynomial interpolation.

We want to construct a family of polynomials \(b_2(x_j), b_1(x_j),\) and \(b_0(x_j)\) with the following properties:

(i) \(b_2(x_j) = a_2(x_j), b_1(x_j) = a_1(x_j), b_0(x_j) = a_0(x_j)\) for \(j = 1, 2, \ldots, k\)

(ii) Every finite ramification point of \(y\) w.r.t. \(x\) in the algebraic curve described by \(f_1(x, y) = 0\) is a ramification point of \(y\) w.r.t. \(x\) in the algebraic curve described by \(f_2(x, y) = 0\).

(iii) The algebraic curve described by \(f_2(x, y) = 0\) has another finite ramification point of \(y\) w.r.t. \(x\) at an arbitrary, but fixed point \(z \in \mathbb{C}\).

The finite ramification points of \(y\) w.r.t. \(x\) in the algebraic curve described by \(f_1(x, y) = 0\) are exactly the zeros of the discriminant of \(f_1(x, y)\) with respect to \(y\),

\[
\Delta_y(f_1(x, y)) = a_1(x)^2 - 4a_0(x)a_2(x).
\]

The existence of suitable polynomials \(b_2(x), b_1(x),\) and \(b_0(x)\) follows, if we prescribe

\[
b_2(x_j) = a_2(x_j), \quad b_1(x_j) = a_1(x_j), \quad b_0(x_j) = a_0(x_j) \quad \text{for} \quad j = 1, 2, \ldots, k,
\]

\[
b_2(x) = a_2(x), \quad b_1(x) = a_1(x), \quad b_0(x) = a_0(x) \quad \text{for all} \quad x \text{ with} \quad \Delta_y(f_1(x, y)) = 0,
\]

\[
b_2(z) = 1, \quad b_1(z) = 0, \quad b_0(z) = 0,
\]

and apply polynomial interpolation.

It remains to be shown that \(z \in \mathbb{C}\) can be chosen such that the algebraic functions \(f_1(x, y) = 0\) and \(f_2(x, y) = 0\) with coefficients \(a_2(x), a_1(x), a_0(x)\) respectively \(b_2(x), b_1(x), b_0(x)\) as above, which produce the same known values of \([f]\), differ in their behaviour regarding analytic continuation along the segment between \(x_1\) and \(x_2\).
To that end, consider the segment in the complex $x$-plane with end points $x_1$ and $x_2$. According to the above considerations, we cannot be sure if the known values of $[f]$ come from an algebraic function $f_1(x, y) = 0$ for which analytic continuation of a branch of $y$ w.r.t. $x$ along the segment is possible or from an algebraic function $f_2(x, y) = 0$ with an additional ramification point of $y$ w.r.t. $x$ on the segment. Therefore, the question of our decision problem cannot be answered since it is unclear whether analytic continuation along the segment is at all possible.

Furthermore, we may as well argue as follows, which shows that the decision problem remains undecidable even if we are guaranteed that analytic continuation along the segment is indeed possible: Suppose that we actually evaluated an algebraic function $f_1(x, y) = 0$. Let $\triangle ABC$ be a triangle in the complex $x$-plane whose closure does not contain any ramification point of $y$ w.r.t. $x$ in the algebraic curve described by $f_1(x, y) = 0$. According to the above considerations we cannot after finitely many evaluations of $[f]$ be sure that $\triangle ABC$ has this property. The finitely many function values of $[f]$ could just as well stem from a different algebraic function $f_2(x, y) = 0$ which produces exactly one ramification point of $y$ w.r.t. $x$ in the interior of $\triangle ABC$. Actually, by the monodromy theorem, analytic continuation of a function germ at $A$ with a certain $y$-value along the edges of $\triangle ABC$ must return to $A$ with the same $y$-value. On the other hand, if $\triangle ABC$ contained exactly one ramification point of $y$ w.r.t. $x$ in its interior, the $y$-values at $A$ would permute under analytic continuation along its edges. Hence, there must be an edge, say $BC$, for which the $y$-values at one end point correspond to different $y$-values at the other end point among the two situations. Since we cannot after finitely many evaluations of $[f]$ distinguish between the two situations and because we might have $B = x_1, C = x_2$, we can neither give an answer to the decision problem.

We have seen that any algorithm for analytic continuation of a holomorphic function germ can generally not depend on finitely many function values only, without closer inspection of the holomorphic function itself. Furthermore, we have shown that this statement still holds, if we know that the function to be analytically continued is an algebraic function and/or if we know that no singularities lie on the segment along which we would like to perform analytic continuation.

3. An algorithm for analytic continuation of plane algebraic curves

Recall the problem posed at the end of the first section: We want to assign the function values of a multivalued evaluation function $[f]$ of a holomorphic function $f$ at point $x_1 \in \mathbb{C}$, not a ramification point of $f$, to the function values at another point $x_2 \in \mathbb{C}$ according to analytic continuation along the segment between these points. In order to attack the problem algorithmically, we will as before pursue the idea that, due to continuity, we can perform analytic continuation along a segment not containing singularities, if we take small enough steps along that segment and assign closest function values to each other. The argument of the previous section has shown that to determine a feasible step size, we must take information about the holomorphic function beside function values into account. We will see that for algebraic functions $f(x, y) = 0$, we can derive a local upper bound for the step size which guarantees that the assignment of function values based on proximity is correct. Additionally, we will explain how all quantities which the bound involves can be computed (at least to arbitrary precision).

To that end, let $f(x, y) = 0$ be an algebraic function of degree $n$ in $y$. Without loss of generality, let $f(x, y)$ be irreducible (otherwise consider each of its irreducible factors separately). Consider a point $x_1 \in \mathbb{C}$, not a ramification point of $y$ w.r.t. $x$, and another point $x_2 \in \mathbb{C}$. Furthermore, let $y_j(x)$, $j = 1, \ldots, n$, denote the holomorphic function germs of $y$ w.r.t.
\[ |y_j(x_1) - y_j(x_2)| < \frac{1}{2} \min_{k \neq j} |y_j(x_1) - y_k(x_1)| =: \frac{1}{2} \delta \quad \text{for all } j = 1, 2, \ldots, n. \tag{3.1} \]

Under these circumstances, an assignment of \( y \)-values based on proximity produces the correct result. Taylor expansion of \( y_j(x) \) around \( x_1 \) yields
\[
y_j(x_2) = y_j(x_1) + (x_2 - x_1)y_j'(x_1) + (x_2 - x_1)^2 R_j(x_2),
\]
where the remainder \( R_j(x_2) \) can be expressed as a contour integral
\[
R_j(x_2) = \frac{1}{2\pi i} \oint_{\gamma} \frac{y_j(\xi)}{(\xi - x_1)^2(\xi - x_2)} d\xi,
\]
along a curve
\[
\gamma: [0, 2\pi] \to \mathbb{C}, \quad \gamma(t) = x_1 + \rho \cdot e^{it},
\]
for sufficiently small \( \rho > 0 \) such that \( 0 < |x_2 - x_1| < \rho \) (cf. [1] p. 124–126). If \( \rho \) is smaller than the minimum distance between \( x_1 \) and any singularity of \( y_j(x) \), then
\[
|y_j(x_1) - y_j(x_2)| < \frac{1}{2} \delta \iff |x_2 - x_1||y_j'(x_1) + (x_2 - x_1)R_j(x_2)| < \frac{1}{2} \delta.
\]

Hence, under the above assumptions, the condition
\[
|x_2 - x_1||y_j'(x_1)| + |x_2 - x_1||R_j(x_2)| < \frac{1}{2} \delta
\]
is sufficient for [Inequality 3.1] to hold. We have
\[
|x_2 - x_1||y_j'(x_1)| + |x_2 - x_1||R_j(x_2)| < \frac{1}{2} \delta
\]
\[ \iff |R_j(x_2)||x_2 - x_1|^2 + |y_j'(x_1)||x_2 - x_1| - \frac{1}{2} \delta < 0. \tag{3.2} \]

Since \( |x_2 - x_1| > 0 \), this is equivalent to
\[
|x_2 - x_1| < \frac{\sqrt{|y_j'(x_1)|^2 + 2\delta |R_j(x_2)|} - |y_j(x_1)|}{2|R_j(x_2)|}. \tag{3.3}
\]

Note that although we only possess an implicit description of \( y_j(x) \), we can actually compute \( |y_j'(x_1)| \) using a technique called implicit differentiation. By the chain rule, the total differential of \( f(x, y_j(x)) = 0 \) with respect to \( x \) is
\[
Df(x, y_j(x)) = \frac{\partial}{\partial x} f(x, y_j(x)) + \frac{\partial}{\partial y_j} f(x, y_j(x)) \cdot y_j'(x) = 0.
\]
Therefore
\[
|y_j'(x_1)| = \left| \frac{f_x(x_1, y_j(x_1))}{f_y(x_1, y_j(x_1))} \right|,
\]
and the denominator does not vanish since by assumption \( x_1 \) is not a ramification point of \( y_j(x) \).

The right-hand side of [Inequality 3.3] is strictly decreasing in \( |R_j(x_2)| \), as one can easily check. It would therefore be desirable to determine an upper bound of \( |R_j(x_2)| \).
We have
\[
|R_j(x_2)| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{y_j(\xi)}{(\xi - x_2)^2} d\xi \right| = \left| \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{y_j(x_1 + \rho e^{it}) \rho e^{it}}{(\rho e^{it})^2(x_1 + \rho e^{it} - x_2)} dt \right|
\]
\[
= \left| \frac{1}{2\pi} \int_{0}^{2\pi} \frac{y_j(x_1 + \rho e^{it})}{\rho e^{it}(x_2 - x_1 - \rho e^{it})} dt \right| \leq \left| \frac{1}{2\pi} \int_{0}^{2\pi} \frac{y_j(x_1 + \rho e^{it})}{\rho e^{it}(x_2 - x_1 - \rho e^{it})} dt \right| \leq \frac{1}{2\pi} \cdot (2\pi - 0) \cdot \frac{M}{\rho(\rho - |x_2 - x_1|)}
\]
\[
= \frac{M}{\rho(\rho - |x_2 - x_1|)}
\]
where \( M = \max_{t \in [0,2\pi]} |y_j(x_1 + \rho e^{it})| \) (cf. [1, p. 126]).

Plugging into Inequality 3.2 yields
\[
\frac{M}{\rho(\rho - |x_2 - x_1|)} |x_2 - x_1|^2 + |y_j'(x_1)||x_2 - x_1| - \frac{1}{2}\delta < 0
\]
\[
\Leftrightarrow M|x_2 - x_1|^2 + \rho(\rho - |x_2 - x_1|)(|y_j'(x_1)||x_2 - x_1| - \frac{1}{2}\delta) < 0
\]
\[
\Leftrightarrow (M - \rho|y_j'(x_1)||x_2 - x_1|^2 + \rho(\rho|y_j'(x_1)| + \frac{1}{2}\delta)|x_2 - x_1| - \frac{1}{2}\delta\rho^2 < 0. \tag{3.5}
\]

First case: \( M - \rho|y_j'(x_1)| > 0 \) The left-hand side of Inequality 3.5 describes a smile parabola in \( |x_2 - x_1| \) with a positive and a negative root. Since \( |x_2 - x_1| > 0 \) by definition, we need only bound \( |x_2 - x_1| \) from above by the positive root, i.e.
\[
|x_2 - x_1| < \frac{-\rho|y_j'(x_1)| + \frac{1}{2}\delta + \sqrt{\rho^2|y_j'(x_1)| + \frac{1}{2}\delta)^2 + 2(M - \rho|y_j'(x_1)|)\delta\rho^2}{2(M - \rho|y_j'(x_1)|)}
\]
\[
= \frac{\rho \left( \sqrt{(\rho|y_j'(x_1)| - \frac{1}{2}\delta)^2 + 2\delta M - (\rho|y_j'(x_1)| + \frac{1}{2}\delta) \right)}{2(M - \rho|y_j'(x_1)|)}.
\]

Second case: \( M - \rho|y_j'(x_1)| < 0 \) The left-hand side of Inequality 3.5 describes a frown parabola in \( |x_2 - x_1| \) with one root greater than \( \rho \) and one root between 0 and \( \rho \). Since \( |x_2 - x_1| < \rho \) by definition, we need only bound \( |x_2 - x_1| \) from above by the smaller root, i.e.
\[
|x_2 - x_1| < \frac{\rho|y_j'(x_1)| + \frac{1}{2}\delta + \sqrt{(\rho|y_j'(x_1)| + \frac{1}{2}\delta)^2 - 2(\rho|y_j'(x_1)| - M)\delta}}{2(\rho|y_j'(x_1)| - M)}
\]
\[
= \frac{\rho \left( \sqrt{(\rho|y_j'(x_1)| - \frac{1}{2}\delta)^2 + 2\delta M - (\rho|y_j'(x_1)| + \frac{1}{2}\delta) \right)}{2(M - \rho|y_j'(x_1)|)}.
\]

Third case: \( M - \rho|y_j'(x_1)| = 0 \) The left-hand side of Inequality 3.5 reduces to
\[
\rho|y_j'(x_1)| + \frac{1}{2}\delta |x_2 - x_1| - \frac{1}{2}\delta\rho^2 < 0
\]
\[
\Leftrightarrow |x_2 - x_1| < \frac{\delta\rho}{2\rho|y_j'(x_1)| + \delta}.
\]
This bound is asymptotically equivalent to the previous bounds for \( M \to \rho |y'_f(x_1)| \).

Altogether, we thus arrive at the sufficient bound

\[
|x_2 - x_1| < \rho \left( \frac{\sqrt{\rho |y'_f(x_1)| - \frac{1}{2} \delta}^2 + 2\delta M - (\rho |y'_f(x_1)| + \frac{1}{2} \delta)}{2(M - \rho |y'_f(x_1)|)} \right). \tag{3.6}
\]

The right-hand side of Inequality 3.6 has the expected qualitative behaviour: It is strictly increasing in \( \delta \) and \( \rho \), and strictly decreasing in \( M \) and \( |y'_f(x_1)| \).

Therefore, as a next step, \( M \) is to be computed or bounded from above. To that end, we can use Fujiwara’s bound (cf. [3, Inequality 3 on p. 168]). We interpret \( f(x, y) \) as polynomial in \( y \) with coefficients \( a_k(x) \) which are polynomials in \( x \):

\[
f(x, y) = \sum_{k=0}^{n} a_k(x)y^k.
\]

Fujiwara’s bound applied to \( f(x, y) \) in our notation reads

\[
|y| < 2 \max_k \left\{ \left| \frac{a_k(x)}{a_n(x)} \right|^\frac{1}{n-k} \right\} \quad |k = 0, 1, \ldots, n-1 \}.
\]

Consequently,

\[
M < 2 \max_{t \in [0, 2\pi]} \left\{ \left| \frac{a_k(x_1 + \rho e^{it})}{a_n(x_1 + \rho e^{it})} \right|^\frac{1}{n-k} \right\} \quad |k = 0, 1, \ldots, n-1 \}.
\]

We will use upper bounds \( \tilde{a}_k \) of \( \max_{t \in [0, 2\pi]} |a_k(x_1 + \rho e^{it})| \) and a lower bound \( \tilde{a}_n > 0 \) of \( \min_{t \in [0, 2\pi]} |a_n(x_1 + \rho e^{it})| \), which are easier to compute than these extreme values. For the former bounds, let the coefficients of the polynomial \( a_k(x) \) be denoted by \( a_{k,l} \), that is

\[
a_k(x) = \sum_{l=0}^{m_k} a_{k,l} x^l.
\]

Then

\[
|a_k(x_1 + \rho e^{it})| = \left| \sum_{l=0}^{m_k} a_{k,l} \cdot (x_1 + \rho e^{it})^l \right| \leq \sum_{l=0}^{m_k} |a_{k,l}| |(x_1 + |\rho|)^l| =: \tilde{a}_k.
\]

For the latter bound, let \( a_{n,m_n} \) denote the leading coefficient of \( a_n(x) \), and let \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{m_n} \) denote the zeros of \( a_n(x) \), that is

\[
a_n(x) = a_{n,m_n} \cdot (x - \bar{x}_1) \cdot (x - \bar{x}_2) \cdot \ldots \cdot (x - \bar{x}_{m_n}).
\]

Then

\[
|a_n(x_1 + \rho e^{it})| = |a_{n,m_n}| \cdot |x_1 + \rho e^{it} - \bar{x}_1| \cdot |x_1 + \rho e^{it} - \bar{x}_2| \cdot \ldots \cdot |x_1 + \rho e^{it} - \bar{x}_{m_n}|
\]

\[
\geq |a_{n,m_n}| \cdot |\rho| \cdot |x_1 - \bar{x}_1| \cdot |\rho| \cdot |x_1 - \bar{x}_2| \cdot \ldots \cdot |\rho| \cdot |x_1 - \bar{x}_{m_n}|
\]

\[
= |a_{n,m_n}| \cdot |(x_1 - \bar{x}_1) - \rho| \cdot |(x_1 - \bar{x}_2) - \rho| \cdot \ldots \cdot |(x_1 - \bar{x}_{m_n}) - \rho| =: \tilde{a}_n.
\]

Note that \( \tilde{a}_n \) is positive since \( \rho \) is chosen smaller than the distance between \( x_1 \) and the closest singularity of \( y \) w.r.t. \( x \), particularly smaller than the distance between \( x_1 \) and any zero of \( a_n(x) \). The zeros of \( a_n(x) \) are exactly the poles of \( y \) w.r.t. \( x \). The remaining finite singularities of \( y \) w.r.t. \( x \) are exactly the finite ramification points of \( y \) w.r.t. \( x \). For irreducible \( f(x, y) \), these are exactly the zeros of the discriminant of \( f(x, y) \) w.r.t. \( y \). Once all finite singularities are computed, we may choose \( \rho \) smaller than the distance between \( x_1 \) and the closest singularity.

We have thus determined (bounds for) all quantities in Inequality 3.6 and arrive at the following result.
Theorem 2. Let \( f(x,y) = 0 \) be an irreducible algebraic function,
\[
f(x,y) = \sum_{k=0}^{n} \left( \sum_{l=0}^{m_k} a_{k,l} x^l \right) y^k,
\]
of degree \( n \) in \( y \). Let
\[
a_n(x) = \sum_{l=0}^{m_n} a_{n,l} x^l,
\]
and let \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{m_n} \) be its zeros. Let \( x_1 \) be a point in the complex \( x \)-plane, not a ramification point of \( y \) w.r.t. \( x \), and \( y_j(x), j = 1, \ldots, n \), the branches of \( y \)-values at \( x_1 \). Let \( x_2 \) be another point in the complex \( x \)-plane such that
\[
|x_2 - x_1| < \frac{\rho}{2} \left( \sqrt{(\rho \tilde{y} - \frac{1}{2}\delta)^2 + 2\delta M - (\rho \tilde{y} + \frac{1}{2}\delta)} \right),
\]
where
\[
\delta := \min_{k \neq j} |y_j(x_1) - y_k(x_1)|,
\]
\[
\rho < \min\{|x_1 - x| : a_n(x) \cdot \Delta_y(f(x,y))(x) = 0\},
\]
\[
M := 2 \max_k \left( \frac{\sum_{l=0}^{m_k} |a_{k,l}|(|x_1| + |\rho|)^l}{|a_{n,m_n}| \prod_{l=1}^{m_n} (|x_1 - \bar{x}_l| - \rho)} \right)^{\frac{1}{n-k}},
\]
\[
\tilde{y} := \max_j \left| \frac{f_x(x_1, y_j(x_1))}{f_y(x_1, y_j(x_1))} \right|.
\]
Then analytic continuation of \( y_j(x), j = 1, \ldots, n \), along the segment from \( x_1 \) to \( x_2 \) is possible. Furthermore, if \( y_j(x_2), j = 1, \ldots, n \), denote the values which result at \( x_2 \) under analytic continuation of \( y_j(x) \) along that segment, then for all \( k \neq j \)
\[
|y_j(x_2) - y_j(x_1)| < |y_k(x_2) - y_j(x_1)|.
\]

Do not let yourself be intimidated by these formulae; they are actually less unwieldy than they appear. On the contrary, they offer an algorithm for analytic continuation without explicit computation of a monodromy group. A numerical example which demonstrates the practical use will be provided in the following section.

On the other hand, there is of course still ample room for improvement. We propose to investigate whether there is, to the same effect, a simpler bound of the form
\[
|x_2 - x_1| \leq |a_n(x_1) \cdot \Delta_y(f(x,y))(x_1) \cdot \chi(?)|
\]
for some non-constant characteristic factor \( \chi(?) \).

4. A numerical example

Consider the algebraic function
\[
f(x,y) = y^5 - y + x = 0.
\]
A corresponding multivalued evaluation function is not expressible in terms of radicals. However, if we interpret \( f(x,y) \) as a polynomial in \( y \) with polynomial coefficients in \( x \), we
can evaluate \([f](x)\) numerically by approximating the roots of \(f(x, y)\), for example using the Durand–Kerner method (cf. [2], [4]).

Let us, step by step, apply Theorem 2 to perform analytic continuation of \([f]\) along the circle of radius 0.5 around 0.75 from \(x_1 = 1.25\) until we return to \(x_1\). The algebraic function \(f(x, y) = 0\) does not have poles at finite \(x\)-values, and ramification points of \(y\) w.r.t. \(x\) at the roots of

\[
\Delta_y(f(x, y)) = 3125x^4 - 256,
\]

that is at \(x \approx 0.534992, x \approx 0.534992i, x \approx -0.534992, \) and \(x \approx -0.534992i\). Therefore, for all the points of the circle along which we continue analytically the closest singularity is the one at \(x \approx 0.534992\), and thus we must choose \(\rho\) smaller than \(|x_j - 0.534992|, j = 1, 2, \ldots\). The complicated expression for \(M\) reduces to

\[
\tilde{y} = \max_k \left| \frac{1}{5y_k(x_j)^4 - 1} \right|.
\]

Table 1 gives an overview of the approximate values of our computation. The result is depicted in Figure 1.

**Figure 1.** Computational analytic continuation of \(f(x, y) = y^5 - y + x = 0\) along a circle of radius 0.5 around 0.75. A plot of \(x_j\) and \(y_{j,k} = [f](x_j)_k\) in the complex plane shows how two branches of \(y\)-values permute. The assignment of \(y\)-values according to analytic continuation performed by our algorithm is indicated using arrows. Non-trivial assignments are highlighted in red. The ramification points of \(y\) w.r.t. \(x\) are drawn in white.
| \(j\) | \(x_j\) | \([f(x_j)]\) | \(\delta\) | \(\rho\) | \(M\) | \(\hat{y}\) | bound |
|---|---|---|---|---|---|---|---|
| 1 | 1.25 | \[
\begin{bmatrix}
-1.19588 \\
-0.20644 - 1.10954i \\
-0.20644 + 1.10954i \\
0.804379 - 0.416675i \\
0.804379 + 0.416675i
\end{bmatrix}
\] | 0.83335 | 0.7150(08) | 2.2893 | 0.261794 | 0.2317(91) |
| 2 | 1.0183 + 0.421918i | \[
\begin{bmatrix}
-1.17426 - 0.0407934i \\
-0.235648 + 1.04324i \\
-0.146469 - 1.13258i \\
0.667636 + 0.462142i \\
0.888743 - 0.323006i
\end{bmatrix}
\] | 0.815687 | 0.6415(62) | 2.23525 | 0.333464 | 0.2059(32) |
| 3 | 0.566602 + 0.465151i | \[
\begin{bmatrix}
-1.11857 - 0.0687226i \\
-0.19325 + 0.961577i \\
-0.0875389 - 1.10914i \\
0.475478 + 0.41891i \\
0.923806 - 0.020628i
\end{bmatrix}
\] | 0.766465 | 0.4662(24) | 2.07402 | 0.558022 | 0.1470(13) |
| 4 | 0.291096 + 0.198512i | \[
\begin{bmatrix}
-1.0654 - 0.0365796i \\
-0.0891239 + 0.965236i \\
-0.0572109 - 1.05202i \\
0.235087 + 0.199033i \\
0.925749 - 0.0756277i
\end{bmatrix}
\] | 0.696246 | 0.3144(71) | 2 | 0.94076 | 0.09456(91) |
| 5 | 0.26169 - 0.107487i | \[
\begin{bmatrix}
-1.05781 + 0.0204517i \\
-0.0716423 - 0.98478i \\
-0.0562173 + 1.03279i \\
0.260968 - 1.01024i \\
0.9247 + 0.0406907i
\end{bmatrix}
\] | 0.680435 | 0.2936(79) | 2 | 0.990049 | 0.08748(91) |
| 6 | 0.40776 - 0.364516i | \[
\begin{bmatrix}
-1.09104 + 0.0603336i \\
-0.144912 - 0.945816i \\
-0.0690808 + 1.08616i \\
0.377943 - 0.345706i \\
0.927094 + 0.14503i
\end{bmatrix}
\] | 0.73647 | 0.3860(83) | 2 | 0.746218 | 0.1197(43) |
| 7 | 0.726344 - 0.49944i | \[
\begin{bmatrix}
-1.14109 + 0.0673258i \\
-0.219742 - 0.988554i \\
-0.108662 + 1.1226i \\
0.551292 - 0.449437i \\
0.916406 + 0.248069i
\end{bmatrix}
\] | 0.787288 | 0.5348(42) | 2.14417 | 0.447683 | 0.1699(95) |
| 8 | 1.10951 - 0.347498i | \[
\begin{bmatrix}
-1.18317 + 0.0395944i \\
-0.233456 - 1.06218i \\
-0.161759 + 1.13142i \\
0.705622 - 0.45724i \\
0.87332 + 0.388409i
\end{bmatrix}
\] | 0.823032 | 0.6714(36) | 2.25794 | 0.30795 | 0.2161(17) |
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