THE CHOW RING OF THE NON-LINEAR GRASSMANNIAN

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0. Summary

Let \( \mathbb{C} \) be the ground field of complex numbers. Let \( 1 \leq k \leq r \) be integers. The Grassmannian \( \mathbb{G}(\mathbb{P}^k, \mathbb{P}^r) \) of projective \( k \)-planes in \( \mathbb{P}^r \) can be viewed as the moduli space of (unparameterized) regular maps from \( \mathbb{P}^k \) to \( \mathbb{P}^r \) of degree 1. Let \( M_{\mathbb{P}^k}(\mathbb{P}^r, d) \) be the coarse moduli space of (unparameterized) regular maps \( \mu : \mathbb{P}^k \to \mathbb{P}^r \) satisfying \( \mu^*(\mathcal{O}_{\mathbb{P}^r}(1)) \cong \mathcal{O}_{\mathbb{P}^k}(d) \). Two maps \( \mu : \mathbb{P}^k \to \mathbb{P}^r, \mu' : \mathbb{P}^k \to \mathbb{P}^r \) are equivalent for the moduli problem if there exists an element \( \sigma \in \text{PGL}_{k+1} \) satisfying \( \mu' \circ \sigma = \mu \). If \( \mu : \mathbb{P}^k \to \mathbb{P}^r \) is a non-constant regular map, it is easy to show that \( \text{dim}(\text{Im}(\mu)) = k \) and \( \mu : \mathbb{P}^k \to \text{Im}(\mu) \) is a finite morphism. The space \( M_{\mathbb{P}^k}(\mathbb{P}^r, d) \) is a natural non-linear generalization of the Grassmannian.

In section (1), \( M_{\mathbb{P}^k}(\mathbb{P}^r, d) \) will be constructed via Geometric Invariant Theory. \( M_{\mathbb{P}^k}(\mathbb{P}^r, d) \) is an irreducible, normal, quasi-projective variety with finite quotient singularities. Let \( A_i(M_{\mathbb{P}^k}(\mathbb{P}^r, d)) \otimes \mathbb{Q} \) be the Chow group (tensor \( \mathbb{Q} \)) of \( i \)-cycles modulo linear equivalence. Since the space \( M_{\mathbb{P}^k}(\mathbb{P}^r, d) \) has finite quotient singularities, the Chow groups \( \bigoplus(A_i \otimes \mathbb{Q}) \) naturally form a graded ring via intersection. Since \( \mathbb{Q} \)-coefficients are required for the intersection theory, all Chow groups considered here will be taken with \( \mathbb{Q} \)-coefficients. Let \( Ch(k, r, d) \) denote the Chow ring of \( M_{\mathbb{P}^k}(\mathbb{P}^r, d) \). The ring \( Ch(k, r, 1) \) is simply the Chow ring of the linear Grassmannian \( \mathbb{G}(\mathbb{P}^k, \mathbb{P}^r) \). The main result of this paper is a determination of \( Ch(k, r, d) \).

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Theorem 1. There is a canonical isomorphism of graded rings

\[ \lambda : Ch(k, r, d) \rightarrow Ch(k, r, 1). \]

Let \( \overline{M}_{0,n}(\mathbb{P}^r, d) \) be the coarse moduli space of \( n \)-pointed Kontsevich stable maps from a genus 0 curve to \( \mathbb{P}^r \). Let \( M_{0,n}(\mathbb{P}^r, d) \subset \overline{M}_{0,n}(\mathbb{P}^r, d) \) denote the non-empty open set corresponding to \( n \)-pointed, stable maps from \( \mathbb{P}^1 \) to \( \mathbb{P}^r \). The complement of \( M_{0,n}(\mathbb{P}^r, d) \) in \( \overline{M}_{0,n}(\mathbb{P}^r, d) \) consists of the stable maps with reducible domains. A foundational treatment of these moduli spaces of pointed stable maps of genus 0 curves can be found in [K], [KM], and [FP]. The spaces \( M_{0,0}(\mathbb{P}^r, d) \) and \( M_{\mathbb{P}^1}(\mathbb{P}^r, d) \) are identical. The following corollary is therefore a special case of Theorem (1).

Corollary 1. The Chow ring (with \( \mathbb{Q} \)-coefficients) of \( M_{0,0}(\mathbb{P}^r, d) \) is canonically isomorphic to the Chow ring of the Grassmannian \( G(\mathbb{P}^1, \mathbb{P}^r) \).

Corollary (I) is related by loose analogy to results and conjectures on the Chow ring of \( M_g \). C. Faber has studied the subring of the Chow ring of \( M_g \) generated by certain geometric classes. Faber has conjectured a presentation of this subring (which may be the entire Chow ring of \( M_g \)). The conjectured ring looks like the cohomology ring of a compact manifold – for example, it satisfies Poincaré duality. \( M_{0,0}(\mathbb{P}^r, d) \subset \overline{M}_{0,0}(\mathbb{P}^r, d) \) is a zero-pointed open cell analogous to \( M_g \subset \overline{M}_g \). Corollary (I), then, is analogous to Faber’s conjectures.

In [GP], the Poincaré polynomial of \( \overline{M}_{0,n}(\mathbb{P}^r, d) \) is computed. The virtual Poincaré polynomial of \( M_{0,0}(\mathbb{P}^r, d) \) is needed as a preliminary result. It was found the virtual Poincaré polynomial of \( M_{0,0}(\mathbb{P}^r, d) \) is essentially the Poincaré polynomial of \( G(\mathbb{P}^1, \mathbb{P}^r) \). This observation provided the starting point for Theorem (I). Thanks are especially due to E. Getzler for many discussions about the geometry of the space \( M_{0,0}(\mathbb{P}^r, d) \). The theory of equivariant Chow groups ([EG], [T]) plays an essential role in the proof of Theorem (I). The author wishes to thank D. Edidin, W. Graham, and B. Totaro for the long discussions in which this theory was explained. The author has also benefitted from conversations with P. Beloruski, W. Fulton, and H. Tamvakis.
1. $M_{P^k}(P^r, d)$

A family of degree $d$ maps of $P^k$ to $P^r$ consists of the data $(\pi : P \to S, \mu : P \to P^r)$ where:

(i.) $S$ is a noetherian scheme of finite type over $\mathbb{C}$.
(ii.) $\pi : P \to S$ is a flat projective morphism with geometric fibers isomorphic to $P^k$.
(iii.) The restriction of $\mu^*(O_{P^r}(1))$ to each geometric fiber of $\pi$ is isomorphic to $O_{P^k}(d)$.

Two families of maps over $S$,

$$(\pi : P \to S, \mu), (\pi' : P' \to S, \mu')$$

are isomorphic if there exists an isomorphism of schemes $\sigma : P \to P'$ such that

$$\mu = \mu' \circ \sigma, \pi = \pi' \circ \sigma.$$ 

Let $M_{P^k}(P^r, d)$ be the contravariant functor from schemes to sets defined as follows. $M_{P^k}(P^r, d)(S)$ is the set of isomorphism classes of families over $S$ of degree $d$ maps from $P^k$ to $P^r$.

A coarse moduli space $M_{P^k}(P^r, d)$ is easily obtained via Geometric Invariant Theory. Care is taken here to exhibit $M_{P^k}(P^r, d)$ as a quotient of a proper $GL_{k+1}$-action with finite stabilizers. In section (3), the equivariant Chow groups of this $GL_{k+1}$-action will be analyzed.

Let $U(k, r, d) \subset \bigoplus_0^r H^0(P^k, O_{P^k}(d))$ be the Zariski open locus of basepoint free $(r+1)$-tuples of polynomials. There is a natural $GL_{k+1}$-action on $\bigoplus_0^r H^0(P^k, O_{P^k}(d))$ obtained from the naturally linearized action of $GL_{k+1}$ on $P^k$. This $GL_{k+1}$-action leaves $U(k, r, d)$ invariant. Note, since every regular map $\mu : P^k \to P^r$ is finite onto its image, $GL_{k+1}$ acts with finite stabilizers on $U(k, r, d)$.

Let $1 \cong \mathbb{C}$ be a 1 dimensional complex vector space with the trivial $GL_{k+1}$-action. Let $Det$ be the 1 dimensional determinant representation of $GL_{k+1}$. For convenience, let $Z$ denote $\bigoplus_0^r H^0(P^k, O_{P^k}(d))$.  

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There is a $GL_{k+1}$-equivariant inclusion
\[ U(k, r, d) \subset P\left(\text{Det} \otimes (1 \oplus \text{Sym}^q(Z) \oplus Z)\right) \]
obtained by the following equation:
\[ \xi \in U(k, r, d) \rightarrow [1 \otimes (1 \oplus (\xi \otimes \cdots \otimes \xi) \oplus \xi)]. \quad (1) \]
The representations $\text{Det}$ and $\text{Sym}^q(Z)$ in (1) occur to obtain the correct G.I.T. linearization. The final $Z$ factor occurs to insure (1) is an inclusion (consider scaling $U(k, r, d)$ by a constant $q$'th-root of unity).

**Lemma 1.** Consider the naturally linearized action of $GL_{k+1}$ on
\[ P\left(\text{Det} \otimes (1 \oplus \text{Sym}^q(Z) \oplus Z)\right). \]
Then, for $q > k + 1$,
\[ U(k, r, d) \subset P\left(\text{Det} \otimes (1 \oplus \text{Sym}^q(Z) \oplus Z)\right)^{\text{stable}}. \]

**Proof.** The Lemma is a consequence of the Numerical Criterion of stability. A development of Geometric Invariant Theory can be found in [MFK] and [N]. Let $V_{k+1}$ be a $(k + 1)$-dimensional $\mathbb{C}$-vector space such that $P^k = P(V_{k+1})$. Let $v = v_0, \ldots, v_k$ be a basis of $V_{k+1}$ with integer weights $w_0, \ldots, w_k$ (not all zero). Let $\xi \in U(k, r, d)$ correspond to a basepoint free map determined (in the basis $v$) by $[f_0, \ldots, f_r]$ where each $f_i$ is an an element of $\text{Sym}^d(V_{k+1}^*)$. The diagonal coordinates of
\[ \xi \in P\left(\text{Det} \otimes (1 \oplus \text{Sym}^q(Z) \oplus Z)\right) \]
with respect to the $\mathbb{C}^*$-action determined by the weights and basis $v$ are the following:
\[ 1 \otimes 1 \in \text{Det} \otimes 1, \]
\[ 1 \otimes (\xi \otimes \cdots \otimes \xi) \in \text{Det} \otimes \text{Sym}^q\left(\bigoplus_{0}^{r} \text{Sym}^d(V_{k+1}^*)\right), \quad (2) \]
\[ 1 \otimes \xi \in \text{Det} \otimes \left(\bigoplus_{0}^{r} \text{Sym}^d(V_{k+1}^*)\right). \]
The weight of $1 \otimes 1 \in \text{Det} \otimes 1$ is $\sum_{0}^{k} w_i$. Since the polynomials $\{f_i\}$ do not simultaneously vanish at $[1, 0, \ldots, 0] \in P^k$, one of the coefficients of $v_{0}^{*d}$ among the polynomials $\{f_i\}$ must be non-zero. Similarly non-zero coefficients of $v_{1}^{*d}, \ldots, v_{k}^{*d}$ can be found among the polynomials $\{f_i\}$. Therefore, the terms
\[ 1 \otimes (v_{j}^{*d} \otimes \cdots \otimes v_{j}^{*d}) \quad (3) \]
occur in (2) and have weight $-qd \cdot w_j + \sum_{0}^{k} w_i$. 
There are now two cases. First assume \( \sum_0^k w_i > 0 \), then \( 1 \otimes 1 \) has positive weight. If \( \sum_0^k w_i \leq 0 \), there must exist \( j \) such that \( w_j < 0 \). Let \( w_j \) be the negative weight of greatest absolute value. Hence, for all \( i \), if \( w_i < 0 \), then \( -w_j + w_i \geq 0 \). Finally, since \( q > k + 1 \),

\[
-q d \cdot w_j + \sum_0^k w_i > 0.
\]

The term (3) therefore has positive weight. The Numerical Criterion implies \( \xi \) is a stable point for the \( GL_{k+1} \)-action.

As a consequence of Lemma (1), \( U(k, r, d)/GL_{k+1} \) exists as a quasi-projective variety. Standard arguments show that the space

\[
M_{P_k}(P^r, d) \cong U(k, r, d)/GL_{k+1}
\]

has the desired functorial properties. Note: the family of maps

\[
(\pi : P \to S, \mu : P \to P^r)
\]

may not be a Zariski locally trivial \( P^k \)-bundle over \( S \). A Galois cover construction is required to obtain the canonical algebraic morphism

\[
S \to M_{P_k}(P^r, d)
\]

induced by the family (3). Alternatively, one can define a map to \( M_{P_k}(P^r, d) \) locally in the \( \acute{e}tale \) topology on \( S \). The morphism (3) is then obtained via descent. Since \( M_{P_1}(P^r, d) \) and \( M_{0,0}(P^r, d) \) coarsely represent the same functor, these spaces are canonically isomorphic.

Since \( U(k, r, d) \) is nonsingular, contained in the stable locus, and the \( GL_{k+1} \)-action has finite stabilizers, Luna’s \( \acute{e}tale \) Slice Theorem can be applied to conclude \( M_{P_k}(P^r, d) \) has finite quotient singularities (see [L]). Luna’s Theorem requires a characteristic zero hypothesis.

Finally, since \( U(k, r, d) \) is equivariant and contained in a G.I.T. stable locus, the group action

\[
GL_{k+1} \times U(k, r, d) \to U(k, r, d)
\]

is a proper action. This is established in [MFK], Corollary (2.5). The properness of the action (3) is needed in section (3).
2. The Homomorphism $\lambda: Ch(k, r, d) \to Ch(k, r, 1)$

Let $\nu: P^r \to P^r$ be a regular map satisfying $\nu^*(\mathcal{O}_{P^r}(1)) \cong \mathcal{O}_{P^r}(d)$. The map $\nu$ induces a canonical morphism $\tau^*_\nu: G(P^k, P^r) \to M_{P^k}(P^r, d)$ by the following considerations. Let $\pi: \mathcal{P} \to G(P^k, P^r)$ be the tautological $P^k$-bundle over the Grassmannian. Since

\[ P \subset G(P^k, P^r) \times P^r, \]  

there is a canonical projection $\eta: \mathcal{P} \to P^r$. Let $\mu: \mathcal{P} \to P^r$ be determined by $\mu = \nu \circ \eta$. The family

\[ (\pi: \mathcal{P} \to G(P^k, P^r), \mu: \mathcal{P} \to P^r) \]  

is a family over $G(P^k, P^r)$ of degree $d$ maps from $P^k$ to $P^r$. Since $M_{P^k}(P^r, d)$ is a coarse moduli space, the family induces a morphism from the base to moduli:

\[ \tau^*_\nu: G(P^k, P^r) \to M_{P^k}(P^r, d). \]

Let $\tau^*_\nu$ be the ring homomorphism induced by pull-back:

\[ \tau^*_\nu: Ch(k, r, d) \to Ch(k, r, 1). \]

Since $M_{P^k}(P^r, d)$ has finite quotient singularities, the pull-back map $\tau^*_\nu$ is well defined (see [V]).

**Proposition 1.** The homomorphism $\tau^*_\nu$ does not depend upon $\nu$ and is a graded ring isomorphism.

Let $\lambda: Ch(k, r, d) \to Ch(k, r, 1)$ be the ring isomorphism $\tau^*_\nu$ for any regular map $\nu$. Theorem (1) is a consequence of Proposition (1).

The proof of Proposition (1) will be undertaken in several steps. First the independence result will be established in Lemma (2). A surjectivity Lemma will be also be proven in this section. The injectivity of $\tau^*_\nu$ will be proven in section (3).

**Lemma 2.** The homomorphism $\tau^*_\nu$ does not depend upon $\nu$.

**Proof.** Let $U(r, r, d) \subset \bigoplus_{r} H^0(P^r, \mathcal{O}_{P^r}(d))$ be the Zariski open locus of basepoint free $(r + 1)$-tuples of polynomials as defined in section (1). There is a tautological morphism $\nu_U: U(r, r, d) \times P^r \to P^r$. 
The tautological family \( \mathcal{P}_U \) over the Grassmannian pulls-back to a tautological family \( \mathcal{P}_U \) over \( \mathbb{G}(\mathbb{P}^k, \mathbb{P}^r) \times U(r, r, d) \).

\( \mathcal{P}_U \) is equipped with a canonical projection
\[
\eta_U : \mathcal{P}_U \to U(\nabla, \nabla, [\cdot]) \times \mathbb{P}^r.
\]

Let \( \mu_U = \nu_U \circ \eta_U \). The map \( \mu_U \) defines a family of degree \( d \) maps from \( \mathbb{P}^k \) to \( \mathbb{P}^r \) over \( \mathbb{G}(\mathbb{P}^k, \mathbb{P}^r) \times U(r, r, d) \). There is an induced map
\[
\tau_U : \mathbb{G}(\mathbb{P}^k, \mathbb{P}^r) \times U(r, r, d) \to M_{\mathbb{P}^k}(\mathbb{P}^r, d).
\]

The morphism \( \tau_{\nu} \) is induced by the composition of the inclusion
\[
i_{\nu} : \mathbb{G}(\mathbb{P}^k, \mathbb{P}^r) \to \mathbb{G}(\mathbb{P}^k, \mathbb{P}^r) \times [\nu] \subset \mathbb{G}(\mathbb{P}^k, \mathbb{P}^r) \times U(r, r, d)
\]
with \( \tau_U \). Hence, \( \tau_{\nu}^* = i_{\nu}^* \circ \tau_U^* \). Since \( U(r, r, d) \) is an open set in affine space, \( i_{\nu}^* = i_{\nu'}^* \) for any two maps \( [\nu], [\nu'] \in U(r, r, d) \).

If \( k = r \), then \( \mathbb{G}(\mathbb{P}^k, \mathbb{P}^r) \) is a point and \( \tau_{\nu}^* \) is surjective. Assume \( k < r \). Let \( 1 \leq j \leq r - k \). Let \( H_j \subset \mathbb{P}^r \) be a linear subspace of codimension \( k+j \). Define an algebraic subvariety \( C(H_j) \subset M_{\mathbb{P}^k}(\mathbb{P}^r, d) \) by the following condition. \( C(H_j) \) is the set of maps that meet \( H_j \). \( C(H_j) \) is easily seen to be an irreducible subvariety of codimension \( j \) in \( M_{\mathbb{P}^k}(\mathbb{P}^r, d) \). There is a natural \( \mathbb{G}L_{r+1} \)-action on \( M_{\mathbb{P}^k}(\mathbb{P}^r, d) \) obtained from the symmetries of \( \mathbb{P}^r \). Let \( \xi \in \mathbb{G}L_{r+1} \). Certainly
\[
\xi(C(H_j)) = C(\xi(H_j)).
\]

Since \( \mathbb{G}L_{r+1} \) is a connected rational group, the class \( \sigma_j \) of \( C(H_j) \) in the Chow ring \( Ch(k, r, d) \) is well-defined (independent of \( H_j \)).

**Lemma 3.** The pull-back of the class \( \sigma_j \) for \( 1 \leq j \leq r-k \) is determined by:
\[
\tau_{\nu}^*(\sigma_j) = d^{k+j} \cdot \sigma_j.
\]

**Proof.** Let \( \nu : \mathbb{P}^r \to \mathbb{P}^r \) be a fixed morphism satisfying \( \nu^* (\mathcal{O}_{\mathbb{P}^r}(1)) \cong \mathcal{O}_{\mathbb{P}^r}(d) \). Let \( H_j \subset \mathbb{P}^r \) be a general (with respect to \( \nu \)) linear space. By Bertini’s Theorem, \( \nu^{-1}(H_j) \) is a nonsingular complete intersection of \( k + j \) hypersurfaces of degree \( d \) in \( \mathbb{P}^r \). The set theoretic inverse image \( \tau_{\nu}^{-1}(C(H_j)) \) is the set of \( k \)-planes of \( \mathbb{P}^r \) meeting \( \nu^{-1}(H_j) \). A simple tangent space argument shows that the scheme theoretic inverse image \( \tau_{\nu}^{-1}(C(H_j)) \) is generically reduced. Hence,
\[
\tau_{\nu}^*(\sigma_j) = [\tau_{\nu}^{-1}(C(H_j))] \in Ch(k, r, 1).
\]
It remains to determine \([\tau_{\nu}^{-1}(C(H_j))] \in Ch(k, r, 1)\).
Recall $\pi : \mathcal{P} \to \mathbf{G}(\mathbb{P}^k, \mathbb{P}^r)$ is the tautological $\mathbb{P}^k$-bundle over the Grassmannian. Let $L$ be the Chern class of the line bundle $\eta^*(\mathcal{O}_{\mathbb{P}^r}(1))$ on $\mathcal{P}$. The following equations hold:

$$\pi^*(L^k_j + j) = \sigma_j,$$
$$\pi^*((d \cdot L)^{k+j}) = [\tau^{-1}_\nu(C(H_j))].$$

These equations imply $\tau^*_\nu(\sigma_j) = d^{k+j} \cdot \sigma_j$. □

Consider the $d = 1$ case, $\mathbf{G}(\mathbb{P}^k, \mathbb{P}^r) \cong \mathbb{M}_{\mathbb{P}^k}(\mathbb{P}^r, 1)$. There is a tautological bundle sequence on $\mathbf{G}(\mathbb{P}^k, \mathbb{P}^r)$:

$$0 \to S \to \mathbb{C}^{r+k} \to Q \to \mathbb{P}^r.$$

$Q$ is a bundle of rank $r-k$. For $1 \leq j \leq r-k$, let $c_j(Q) \in Ch(k, r, 1)$ be the $j^{th}$ Chern class of $Q$. It is well known that

$$c_j(Q) = \sigma_j.$$

Also, the classes $c_j(Q) \in Ch(k, r, 1)$ generate $Ch(k, r, 1)$ as ring. These facts can be found, for example, in [F]. Therefore, the following Lemma is a consequence of Lemma (3).

**Lemma 4.** The homomorphism $\tau^*_\nu : Ch(k, r, d) \to Ch(k, r, 1)$ is surjective.

In fact, the subring of $Ch(k, r, d)$ generated by $\sigma_1, \ldots, \sigma_{r-k}$ surjects onto $Ch(r, k, 1)$ via $\tau^*_\nu$.

### 3. Generation of $Ch(1, r, d)$

In order to complete the proof of Proposition (4), results on the generation of $Ch(k, r, d)$ are needed. In this section, a special argument in the $k = 1$ case is developed. In sections (4)-(6), a general generation argument using the theory of equivariant Chow groups is established. The general argument also covers the $k = 1$ case. The special method for the $k = 1$ case involves a natural stratification of $\mathbb{M}_{\mathbb{P}^1}(\mathbb{P}^r, d)$. Unfortunately, this stratification does not easily generalize when $k > 1$.

Let $0 \leq j \leq r-1$. Let $\sigma_0 \in Ch(1, r, d)$ be the unit (the fundamental class). Let $\sigma_{j \neq 0}$ be the class defined in section (2).

**Proposition 2.** The elements $\sigma_i \cdot \sigma_j$ ($0 \leq i \leq j \leq r-1$) span a $\mathbb{Q}$-basis of $Ch(1, r, d)$. 


The proof of Proposition (2) uses the 3-pointed moduli space of maps \( M_{0,3}(\mathbb{P}^r, d) \). Let \( 1, 2, \infty \in \mathbb{P}^1 \) be three marked points. There is a natural isomorphism:

\[
M_{0,3}(\mathbb{P}^r, d) \cong \mathbb{P}(U(1, r, d)/\mathbb{C}^*) \subset \mathbb{P}\left( \bigoplus_{0}^{r} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \right)
\]

(9)

where \( U(1, r, d) \) is the basepoint free locus (see section (1)). An element of \( \mathbb{P}(U) \) corresponds to a degree \( d \) map from \( \mathbb{P}^1 \) to \( \mathbb{P}^r \) with the three markings \( 1, 2, \infty \in \mathbb{P}^1 \). A map \([\mu]\) \( \in M_{0,3}(\mathbb{P}^r, d) \) corresponds to a point in \( \mathbb{P}(U) \) by identifying the three markings of \([\mu]\) with the points \( 1, 2, \infty \in \mathbb{P}^1 \). A tangent space argument shows this identification is an isomorphism of schemes (both are non-singular varieties).

The proof of Proposition (2) is a refinement of the methods that appear in [P]. For \( 0 \leq j \leq r - 1 \), let \( H_j \subset \mathbb{P}^r \) be a linear space of codimension \( 1 + j \). For \( 0 \leq a, b \leq r - 1 \), let \( C(H_a, H_b) \subset M_{0,0}(\mathbb{P}^r, d) \) be the subvariety of maps meeting \( H_a \) and \( H_b \) (where \( H_a \) and \( H_b \) are in general position). A simple argument shows the equation

\[
[C(H_a, H_b)] = \sigma_a \cdot \sigma_b
\]

holds in \( Ch(1, r, d) \). Note: intersection with the hyperplane \( H_0 \) imposes no condition on the maps. In particular, \( C(H_0, H_0') = M_{0,0}(\mathbb{P}^r, d) \).

**Lemma 5.** Let \( 0 \leq a, b \leq r - 1 \). Assume \((a, b) \neq (r - 1, r - 1)\). Let \( H_a, H_b \subset \mathbb{P}^r \) be linear spaces of codimension \( 1 + a, 1 + b \) in general position. Let \( H_{a+1} \subset H_a, H_{b+1} \subset H_b \) be linear spaces of codimension 1. The natural map

\[
C(H_{a+1}, H_b) \cup C(H_a, H_{b+1}) \cup C(H_0, H_a \cap H_b) \to C(H_a, H_b)
\]

(10)

yields a surjection on Chow groups of proper codimension in \( C(H_a, H_b) \). If the linear spaces \( H_{a+1}, H_{b+1}, \) or \( H_a \cap H_b \) are empty, the corresponding cycle on the left in (10) is taken to be empty. By the assumption \((a, b) \neq (r - 1, r - 1), not all cycles are empty.

**Proof.** Let \( F \) be a hyperplane in general position with respect to \( H_a \) and \( H_b \). Let \( \overline{N} = \overline{M}_{0,3}(\mathbb{P}^r, d) \) and \( \overline{M} = \overline{M}_{0,0}(\mathbb{P}^r, d) \). Let \( N, M \) be the unbarred moduli spaces. Let \( e_i : \overline{N} \to \mathbb{P}^r \) be the natural evaluation maps for the markings \( 1 \leq i \leq 3 \). Let

\[
X = e_1^{-1}(F) \cap e_2^{-1}(H_a) \cap e_3^{-1}(H_b).
\]
$X$ is closed subvariety of $\overline{\mathcal{M}}$. The natural forgetful morphism $\rho : X \to \mathcal{M}$ is proper. Also $\rho(X) \cap M = C(H_a, H_b)$. Let $Z \subset C(H_a, H_b)$ be the open set of $\rho(X)$ corresponding to Kontsevich stable maps satisfying the following conditions:

(i.) The domain curve is $\mathbb{P}^1$.
(ii.) The map meets $H_a$ and $H_b$.
(iii.) The map does not pass through $F \cap H_a$, $F \cap H_b$, or $H_a \cap H_b$.

Let $[\mu] \in Z$ be an element. By condition (iii), the image $\text{Im}(\mu) \subset \mathbb{P}^r$ cannot be contained in $F$, $H_a$, or $H_b$. Moreover, by (iii), $\rho^{-1}(Z) \subset N$. Hence, the map $\rho^{-1}(Z) \to Z$ has finite fibers. Since $\rho^{-1}(Z) \to Z$ is a proper morphism with finite fibers, it is a finite morphism. Therefore, if $A_i(\rho^{-1}(Z)) \otimes \mathbb{Q} = 0$, then $A_i(Z) \otimes \mathbb{Q} = 0$.

The set $\rho^{-1}(Z) \subset N \cong \mathbb{P}(U)$ (see (9) above) is isomorphic to a quasi-projective variety in $\mathbb{P}(\bigoplus_0^r \mathcal{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)))$. The quasi-projective subvariety

$$\rho^{-1}(Z) \subset \mathbb{P}(\bigoplus_0^r \mathcal{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)))$$

can be identified as follows. Let $L_1 \subset \mathbb{P}(U)$ correspond to maps $\mu : \mathbb{P}^1 \to \mathbb{P}^r$ satisfying $\mu(1) \in F$. Let $L_2$, $L_\infty$ be the linear subspaces in $\mathbb{P}(U)$ where $\mu(2) \in H_a$, $\mu(\infty) \in H_b$. Let $L_1 \cap L_2 \cap L_\infty = I \subset \mathbb{P}(U)$. Let $D \subset I$ be the union of the three hypersurfaces of maps meeting the linear spaces $F \cap H_a$, $F \cap H_b$, and $H_a \cap H_b$ respectively. Since $(a, b) \neq (r-1, r-1)$, $F \cap H_a$ or $F \cap H_b$ is non-empty. Therefore, $D \subset I$ is a subvariety of codimension 1. Then

$$\rho^{-1}(Z) = I \setminus D.$$ 

$I$ is an open set of a linear subspace of $\mathbb{P}(\bigoplus_0^r \mathcal{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)))$. Since $D$ is of codimension 1 in $I$, all the Chow groups of $\rho^{-1}(Z)$ of proper codimension vanish. Hence all the Chow groups (tensor $\mathbb{Q}$) of $Z$ of proper codimension also vanish.

By definition, $Z \subset C(H_a, H_b)$. The complement of $Z$ in $C(H_a, H_b)$ is the set of maps meeting $F \cap H_a$, $F \cap H_b$, or $H_a \cap H_b$. Therefore, the complement of $Z$ in $C(H_a, H_b)$ is the union of three cycles:

$$C(F \cap H_a, H_b) \cup C(H_a, F \cap H_b) \cup C(H_0, H_a \cap H_b). \quad (11)$$
Since the Chow groups of $\mathbb{Z}$ vanish in proper codimension, the Chow groups of the union $\bigcup$ surject onto the Chow groups of $C(H_a, H_b)$ (in proper codimension).

A vanishing result is also required.

**Lemma 6.** Chow groups in proper codimension of $C(H_{r-1}, H'_{r-1})$ vanish.

**Proof.** Let $F$ be a hyperplane in general position with respect to two distinct points $p = H_{r-1}$ and $q = H'_{r-1}$. The notation $N \subset \overline{N}$, $M \subset \overline{M}$ of Lemma 6 will be used. Let

$$X = e_1^{-1}(F) \cap e_2^{-1}(p) \cap e_3^{-1}(q).$$

Let $\rho : X \to \overline{M}$ be the proper forgetful morphism. Again, $\rho(X) \cap M = C(p, q)$. Let $Z \subset C(p, q)$ be the open set of $\rho(X)$ corresponding to Kontsevich stable maps satisfying the following conditions:

(i.) The domain curve is $\mathbb{P}^1$.
(ii.) The map meets $p$ and $q$.

Note $F \cap p$, $F \cap q$, and $p \cap q$ are empty. By these conditions on $Z$, the map $\rho^{-1}(Z) \to Z$ is finite and proper. Therefore, if $A_i(\rho^{-1}(Z)) \otimes \mathbb{Q} = 0$, then $A_i(Z) \otimes \mathbb{Q} = 0$. Also, $\rho^{-1}(Z) \subset N$.

The quasi-projective subvariety

$$\rho^{-1}(Z) \subset \mathbb{P}(\bigoplus_{0}^{r} \mathbb{H}^{0}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)))$$

can be identified as follows. Let $L_1 \subset \mathbb{P}(U)$ correspond to maps $\mu : \mathbb{P}^1 \to \mathbb{P}^r$ satisfying $\mu(1) \in F$. Let $L_2$, $L_\infty$ be the linear subspaces in $\mathbb{P}(U)$ where $\mu(2) \in p$, $\mu(\infty) \in q$. Let $L_1 \cap L_2 \cap L_\infty = I \subset \mathbb{P}(U)$. Then

$$\rho^{-1}(Z) = I$$

$I$ is an open set of a linear subspace of $\mathbb{P}(\bigoplus_{0}^{r} \mathbb{H}^{0}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)))$. Let $\overline{I}$ be the closure of $I$. It will be shown that $\overline{I} \setminus I$ has codimension 1 in $\overline{I}$. The Chow groups of $\rho^{-1}(Z)$ of proper codimension therefore vanish. Hence all the Chow groups of $Z$ of proper codimension also vanish.

Let $[A_0, \ldots, A_r]$ be homogeneous coordinates on $\mathbb{P}^r$. Let

$$F = (A_0 - A_r), \ p = [1, 0, \ldots, 0], \ q = [0, \ldots, 0, 1].$$
Let $[S, T]$ be homogeneous coordinates on $\mathbf{P}^1$. Let $1, 2, \infty \in \mathbf{P}^1$ be the points $[1, 1], [1, 0], [0, 1]$ respectively. An element $[\mu] \in \mathbf{P}(\bigoplus H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)))$ is given by an $r$-tuple of degree $d$ homogeneous polynomials in $S$ and $T : [f_0, \ldots, f_r]$. The element $[\mu]$ is in $I$ if and only if

(i.) $f_0, \ldots, f_r$ span a basepoint free linear system on $\mathbf{P}^1$.
(ii.) $f_0(1, 1) = f_r(1, 1)$.
(iii.) $T$ divides $f_1, \ldots, f_r$.
(iv.) $S$ divides $f_0, \ldots, f_{r-1}$.

The additional condition

$S$ divides $f_r$

is a codimension 1 condition contained in the set $T \setminus I$. Hence $T \setminus I$ has codimension 1 in $I$. \hfill \Box

Repeated application of Lemma (3) shows the ring $Ch(1, r, d)$ is generated (as a $\mathbb{Q}$-vector space) by the classes $[C(H_a, H_b)]$ and the Chow groups of $C(H'_{r-1}, H'_{r-1})$. Lemma (3) shows the Chow groups of $C(H_{r-1}, H'_{r-1})$ vanish in proper codimension. Hence the classes $[C(H_a, H_b)] = \sigma_a \cdot \sigma_b$ generate $Ch(1, r, d)$.

Via the classical Schubert calculus, the classes $\sigma_a \cdot \sigma_b$ for $0 \leq a, b \leq r - 1$ are easily seen to span a basis of the Chow ring of the linear Grassmannian $Ch(1, r, 1)$. Consider the ring homomorphism

$\tau^*: Ch(1, r, d) \rightarrow Ch(1, r, 1)$

defined in section (2). By Lemma (3),

$\tau^*(\sigma_0) = \sigma_0,$

$\forall a > 0, \ \tau^*(\sigma_a) = d^{1+a}\sigma_a,$

$\forall a, b > 0, \ \tau^*(\sigma_a \cdot \sigma_b) = d^{2+a+b}\sigma_a \cdot \sigma_b.$

Therefore, the elements $\sigma_a \cdot \sigma_b$ for $0 \leq a, b \leq r - 1$ are independent in $Ch(1, r, d)$. Since generation was established above, Proposition (2) is proven. In case $k = 1$, the injectivity of $\tau^*$ has been proven.
4. Equivariant Chow Groups

Let $G$ be a group. Let $G \times X \rightarrow X$ be a left group action. In topology, the $G$-equivariant cohomology of $X$ is defined as follows. Let $EG$ be a contractable topological space equipped with a free left $G$-action and quotient $EG/G = BG$. Consider the left action of $G$ on $X \times EG$ defined by:

$$g(x, b) = (g(x), g(b)).$$

$G$ acts freely on $X \times EG$. Let $X \times_G EG$ be the (topological) quotient. The $G$-equivariant cohomology of of $X$, $H^*_G(X)$, is defined by:

$$H^*_G(X) = H^*_{sing}(X \times_G EG).$$

If $X$ is a locally trivial principal $G$-bundle, then $X \times_G EG$ is a locally trivial fibration of $EG$ over the quotient $X/G$. In this case, $X \times_G EG$ is homotopy equivalent to $X/G$ and

$$H^*_G(X) = H^*_{sing}(X \times_G EG) \simeq H^*_{sing}(X/G).$$

For principal bundles, computing the equivariant cohomology ring is equivalent to computing the cohomology of the quotient.

There is an analogous equivariant theory of Chow groups developed by D. Edidin, W. Graham, and B. Totaro in [EG], [T]. Let $G$ be a linear algebraic group. Let $G \times X \rightarrow X$ be a linearized, algebraic $G$-action. The algebraic analogue of $EG$ is attained by approximation. Let $V$ be a $\mathbb{C}$-vector space. Let $G \times V \rightarrow V$ be an algebraic representation of of $G$. Let $W \subset V$ be a $G$-invariant open set satisfying:

(i.) The complement of $W$ in $V$ is of codimension greater than $q$.
(ii.) $G$ acts on $W$ with trivial stabilizers.
(iii.) There exists a geometric quotient $W \rightarrow W/G$.

$W$ is an approximation of $EG$ up to codimension $q$. By (iii) and the assumption of linearization, a geometric quotient $X \times_G W$ exits as an algebraic variety. Let $d = dim(X)$, $e = dim(X \times_G W)$. The equivariant Chow groups are defined by:

$$A^G_{d-j}(X) = A_{e-j}(X \times_G W)$$

for $0 \leq j \leq q$. An argument is required to check these equivariant Chow groups are well-defined (see [EG]). The basic functorial properties of
equivariant Chow groups are established in [EG]. In particular, if $X$ is nonsingular, there is a natural intersection ring structure on $A^G_i(X)$.

Let $Z$ be a variety of dimension $z$. For notational convenience, a superscript will denote the Chow group codimension:

$$A^G_{z-j}(Z) = A^j_G(Z), \quad A_{z-j}(Z) = A^j(Z).$$

In particular, equation (12) becomes:

$$\forall 0 \leq j \leq q, \quad A^j_G(X) = A^j(X \times_G W).$$

The following result of [EG] will be used in section (6).

**Proposition 3.** Let $\mathbb{C}$ be the ground field of complex numbers. Let $X$ be a quasi-projective variety. Let $G$ be a reductive group. Let $G \times X \to X$ be a linearized, proper, $G$-action with finite stabilizers. Let $X \to X/G$ be a quasi-projective geometric quotient. Then, there are natural isomorphisms for all $j$:

$$A^j_G(X) \otimes \mathbb{Q} \cong A^j(X/G) \otimes \mathbb{Q}.$$ 

5. The Chow Ring of the Grassmannian and $A^G_{\ast}(pt)$

Let $0 \to S \to \mathbb{C}^{r+k} \to \mathbb{Q} \to \mathcal{F}$ be the tautological sequence on $G(P^k, P^r)$. The following presentation of the Chow ring will be used in section (6). Let

$$c_1, \ldots, c_{k+1} \in Ch(k, r, 1)$$

be the Chern classes of the rank $k+1$ bundle $S$. These classes generate $Ch(k, r, 1)$. Let

$$c(S) = 1 + c_1 t + c_2 t^2 + \cdots + c_{k+1} t^{k+1},$$

$$\frac{1}{c(S)} = 1 + p_1(c_1) t + p_2(c_1, c_2) t^2 + p_3(c_1, c_2, c_3) t^3 + \cdots$$

where the latter is the formal inverse in the formal power series ring $\mathbb{C}[[t]][[\gamma]]$. The ideal of relations among $c_1, \ldots, c_{k+1}$ in $Ch(k, r, 1)$ is generated by the polynomials

$$\{p_j \mid j > r - k\}.$$ 

Geometrically, these relations are obtained from the vanishing of the $j^{th}$ Chern class of the rank $r - k$ bundle $Q$ for $j > r - k$.

In section (6), a basic result on push-forwards is needed.
Lemma 7. Let \( \pi : P(S) \to G(P^k, P^r) \) be the canonical projection. Let \( \mathcal{O}_{P(S)}(1) \) be the canonical line bundle on \( P(S) \). Then, for \( l \geq k \),

\[
\pi_* \left( c_1(\mathcal{O}_{P(S)}(1)) \right) = p_{l-k} \in Ch(k, r, 1). \tag{13}
\]

Proof. Let \( \xi = c_1(\mathcal{O}_{P(S)}(1)) \). Certainly, \( \pi_* (\xi^k) = 1 = p_0 \). The equation

\[
\xi^{k+1} + c_1 \xi^k + \cdots + c_k \xi + c_{k+1} = 0
\]

recursively yields (13).

The equivariant Chow ring \( A^*_{GL_{k+1}}(pt) \) is computed to motivate the construction in (1). The notation of section (1) will be used here. Let \( V_{k+1} \) be a fixed \( k + 1 \)-dimensional complex vector space such that \( P(V_{k+1}) = P^k \). Let \( U(k, n, 1) \subset \bigoplus_0^n V_{k+1}^* \) be the basepoint free locus. The codimension of the complement of \( U(k, n, 1) \) is easily found to be \( n - k + 1 \). \( GL(V_{k+1}) \) acts on \( U(k, n, 1) \) with trivial stabilizers. As determined in section (1), there is a geometric quotient

\[
U(k, n, 1)/GL(V_{k+1}) \cong G(P^k, P^n).
\]

By the definition of the equivariant Chow ring,

\[
A^j_{GL_{k+1}}(pt) = A^j(G(P^k, P^n))
\]

for \( 0 \leq j \leq n - k \). By the presentation of the Chow ring of \( G(P^k, P^n) \) given above, the relations among the generators \( c_1, \ldots, c_{k+1} \) start in codimension \( n - k + 1 \). Hence, \( A^*_{GL_{k+1}}(pt) \) is freely generated (as a ring) by \( c_1, \ldots, c_{k+1} \) where \( c_j \in A^j_{GL_{k+1}}(pt) \).

6. The Generation Argument

Again, let \( U(k, n, 1) \subset \bigoplus_0^n V_{k+1}^* \) be the basepoint open set. As \( n \to \infty \), \( U(k, n, 1) \) approximates \( EGL_{k+1} \). By the definitions,

\[
A^j_{GL_{k+1}}(U(k, r, d)) \cong A^j \left( U(k, r, d) \times_{GL_{k+1}} U(k, n, 1) \right)
\]

for \( 0 \leq j \leq n - k \). Recall

\[
U(k, r, d) \subset \bigoplus_0^r \text{Sym}^d(V_{k+1}^*)
\]
is the basepoint free locus. There is a natural $\text{GL}(V_{k+1})$-equivariant open inclusion,

$$U(k, r, d) \times U(k, n, 1) \subset \bigoplus_0^r \text{Sym}^d(V^*_{k+1}) \times U(k, n, 1),$$

which yields an open inclusion

$$U(k, r, d) \times_{\text{GL}_{k+1}} U(k, n, 1) \subset \bigoplus_0^r \text{Sym}^d(V^*_{k+1}) \times_{\text{GL}_{k+1}} U(k, n, 1).$$

Let $0 \to S \to \mathbb{C}^{k+p} \to \mathbb{Q} \to \mathcal{L}$ be the tautological sequence on $G(\mathbb{P}^k, \mathbb{P}^n)$. It is routine to verify

$$\bigoplus_0^r \text{Sym}^d(V^*_{k+1}) \times_{\text{GL}_{k+1}} U(k, n, 1) = \bigoplus_0^r \text{Sym}^d(S^*)$$

where the latter is the total space of the bundle $\bigoplus_0^r \text{Sym}^d(S^*)$ over $G(\mathbb{P}^k, \mathbb{P}^n)$. Let $D$ be the complement of $U(k, r, d) \times_{\text{GL}_{k+1}} U(k, n, 1)$ in $\bigoplus_0^r \text{Sym}^d(S^*)$.

The Chow ring of $\bigoplus_0^r \text{Sym}^d(S^*)$ is isomorphic to $Ch(k, n, 1)$ via pull back. Let $\text{dim}$ be the dimension of the variety $\bigoplus_0^r \text{Sym}^d(S^*)$. Let

$$i_D : A_{\text{dim}-j}(D) \to A^j(\bigoplus_0^r \text{Sym}^d(S^*))$$

be the map obtained by the inclusion $D \subset \bigoplus_0^r \text{Sym}^d(S^*)$. There are exact sequences of Chow groups

$$A_{\text{dim}-j}(D) \to A^j(\bigoplus_0^r \text{Sym}^d(S^*)) \to A^j(U(k, r, d) \times_{\text{GL}_{k+1}} U(k, n, 1)) \to 0.$$ 

Let $c_1, \ldots, c_{k+1}$ be the classes of $Ch(k, n, 1)$ defined in section (3).

**Lemma 8.** $p_j(c_1, \ldots, c_{k+1}) \in Im(i_D) \subset A^j(\bigoplus_0^r \text{Sym}^d(S^*))$ for all $j > r - k$.

**Proof.** The proof involves an auxiliary construction. Let $\pi : \mathbb{P}(S) \to G(\mathbb{P}^k, \mathbb{P}^n)$ be the projective bundle associated to $S$. Let $T = \pi^*(\bigoplus_0^r \text{Sym}^d(S^*))$. Denote the total space of the bundle $T$ also by $T$. There is a commutative diagram.

$$
\begin{array}{ccc}
T & \longrightarrow & \mathbb{P}(S) \\
\pi \downarrow & & \pi \downarrow \\
\bigoplus_0^r \text{Sym}^d(S^*) & \longrightarrow & G(\mathbb{P}^k, \mathbb{P}^n)
\end{array}
$$
There is a tautological rational evaluation map

$$\gamma : T \rightarrow P^r.$$ 

A point $$\tau \in T$$ is a triple

$$\tau = (v, V \subset C^k, \mathcal{O}_\nu, \mathcal{O}_{\mathfrak{z}_1}, \ldots, \mathcal{O}_{\mathfrak{z}_r})$$

where $$v$$ is element of the $$k$$-dimensional projective space $$P(V)$$ and $$f_i \in Sym^d(V^*)$$. The rational map $$\gamma$$ is obtained by

$$\gamma(\tau) = [f_0(v), \ldots, f_r(v)].$$

Let $$D$$ be the set of elements $$\tau \in T$$ such that all the $$f_i$$ vanish at $$v$$. $$D$$ is the locus where $$\gamma$$ is undefined. The important fact is

$$\pi(D) = D \subset \bigoplus_{0}^{r} Sym^d(S^*)$$

and $$\pi : D \rightarrow D$$ is a projective, birational morphism. The Lemma will be proved by finding the class of $$D$$ in $$A^*(T)$$ and pushing forward.

Let $$L$$ be the line bundle $$\mathcal{O}_{P(S)}(d)$$ on $$P(S)$$. Let $$L$$ also denote the pull-back of $$\mathcal{O}_{P(S)}(d)$$ to $$T$$. The rational map $$\gamma$$ is obtained from $$r + 1$$ tautological sections of $$L$$ on $$T$$. There is a natural equivalence

$$H^0(G(P^k, P^n), Sym^d(S^*)) \cong H^0(P(S), L).$$

Also, there is a natural inclusion

$$H^0(G(P^k, P^n), \bigotimes_{0}^{r} Sym^d(S) \subset H^0(T, L).$$

Since the bundle $$Sym^d(S^*) \otimes Sym^d(S)$$ has a canonical identity section, the bundle $$Sym^d(S^*) \otimes \bigoplus_{0}^{r} Sym^d(S)$$ has $$r + 1$$ canonical sections. It is straightforward to verify these $$r + 1$$ sections $$z_0, \ldots, z_r$$ of $$L$$ on $$T$$ yield the rational map $$\gamma$$. The base locus $$\mathcal{D}$$ is the common zero locus of the sections $$z_0, \ldots, z_r$$. In fact, $$\mathcal{D}$$ is a nonsingular variety of pure codimension $$r + 1$$. Explicit equations show $$\mathcal{D}$$ is nonsingular complete intersection. Hence $$[\mathcal{D}] = c_1(L)^{r+1} \in A_*(T)$$.

Certainly $$\pi_*(c_1(L)^{r+1}) \in Im(i_D)$$. Also, for all $$\alpha \geq 0$$,

$$\pi_*(c_1(L)^{r+1+\alpha}) = \pi_*(\mathcal{D} \cap c_1(L)^{\alpha}) \in Im(i_D).$$

It remains to compute $$\pi_*(c_1(L)^{r+1+\alpha}) \in A_*(\bigoplus_{0}^{r} Sym^d(S^*))$$. But since push-forward commutes with flat pull-back, it suffices to consider $$c_1(L)^{r+1+\alpha} \in A_*(P(S))$$ and compute $$\pi_*(c_1(L)^{r+1+\alpha}) \in Ch(k, n, 1)$$. By Lemma (13), since $$c_1(L) = d \cdot c_1(\mathcal{O}_{P(S)}(1))$$,

$$\pi_*(c_1(L)^{r+1+\alpha}) = d^{r+1+\alpha} \cdot p_{r-k+1+\alpha}.$$
Hence $p_j \in \text{Im}(i_D)$ for all $j > r - k$. 

By Lemma (8) and the presentation of $Ch(k, r, 1)$ given in section (5), the following inequality is obtained:

$$\dim_{\mathbb{Q}} A^j_{\text{GL}_{k+1}}(U(k, r, d)) \leq \dim_{\mathbb{Q}} A^j(G(P^k, P^r)).$$  \hspace{1cm} (14)

Recall $\text{GL}_{k+1} \times U(k, r, d) \to U(k, r, d)$ is a proper group action with finite stabilizers and geometric quotient $M_{P^r}(d)$. Hence, by Proposition (3) and inequality (14),

$$\dim_{\mathbb{Q}} A^j(M_{P^r}(d)) \leq \dim_{\mathbb{Q}} A^j(G(P^k, P^r)).$$

The surjectivity of $\tau^*_\nu : Ch(k, r, d) \to Ch(k, r, 1)$ obtained in section (2) implies

$$\dim_{\mathbb{Q}} A^j(M_{P^r}(d)) \geq \dim_{\mathbb{Q}} A^j(G(P^k, P^r)).$$

Therefore $\tau^*_\nu$ is injective. The proofs of Proposition (3) and Theorem (1) are complete. Since the subring of $Ch(r, k, d)$ generated by the classes $\sigma_1, \ldots, \sigma_{r-k}$ surjects onto $Ch(r, k, 1)$ via $\tau^*_\nu$, $Ch(r, k, d)$ is generated (as a ring) by the classes $\sigma_1, \ldots, \sigma_{r-k}$.

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