Divisorial extractions from singular curves in a smooth 3-fold, II: low codimension.

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Abstract
Following the first paper, we continue to study Mori extractions from singular curves centred in a smooth 3-fold. We treat the case where the divisorial extraction exists in relative codimension at most 3.

Introduction

The focus of this paper is on the classification of Mori extractions—that is 3-dimensional divisorial extractions over $\mathbb{C}$ with at worst terminal singularities. The first paper [2] introduced a method to construct the coordinate ring of a divisorial extraction from $C \subset X$, a singular curve $C$ contained in a smooth 3-fold $X$, using type I Gorenstein unprojection. This divisorial extraction is the unique Mori extraction from $C$, if such an extraction exists, and is isomorphic to the blowup of the symbolic power algebra

$$\sigma: Y \cong \text{Proj}_X \bigoplus I_{C/X}^n \to X$$

where $I_{C/X}$ is the ideal defining $C \subset X$ (see [2] Proposition 1.4). Thus, if $\bigoplus I_{C/X}^n$ is generated by $r + 1$ generators, we get a presentation of $Y$ as a codimension $r$ subvariety $Y \subset X \times w\mathbb{P}^r$ for some weighted projective space $w\mathbb{P}^r$ of dimension $r$.

This paper studies the cases for which a Mori extraction exists in codimension $\leq 3$. It may be unreasonable to expect an explicit classification in the completely general case. In fact we know that the divisorial extractions can have arbitrarily high codimension – see for example [3] §6.3 where one such family of examples is constructed by serial unprojection. Moreover, in general it seems hard to find an exact condition on $C \subset X$ for which the divisorial extraction has at worst terminal singularities.

Nevertheless, the hope is that this list will find useful applications in studying the explicit birational geometry of 3-folds, e.g. in the Sarkisov program. For instance, Prokhorov & Reid [7] used a Sarkisov link beginning with the simplest Mori extraction
in Table 2 (type $A_1^1$) to construct a new $\mathbb{Q}$-Fano 3-fold of index 2, and more complicated divisorial extractions can be used to construct other interesting examples [4].

0.1 Main result

The aim of the paper is to prove the following result:

**Theorem 0.1.** Suppose $P \in C \subset S_X \subset X$ is an inclusion of varieties, as in §1, and suppose $\sigma : Y \subset X \times \mathbb{P}^r \to X$ is a codimension $r$ Mori extraction from $C \subset X$ constructed by unprojection. Then:

1. If $r \leq 2$ then $C \subset X$ is a l.c.i. and $Y$ is the ordinary blowup of $C$. The possible numerical types for $C \subset S_X$ are given in Table 1, along with the curves extracted by $\sigma|_{S_Y}$. Moreover, every case in the table has a Mori extraction.

2. If $r = 3$ then $wP = \mathbb{P}(1,1,1,2)$ and $Y$ has a singularity of index 2. The possible numerical types for $C \subset S_X$ are given in Table 2, along with the curves extracted by $\sigma|_{S_Y}$.

(We do not claim that every case appearing in Table 2 has a Mori extraction.)

0.2 Notation and conventions

**Resolutions of Du Val singularities.** We fix the following numbering of the $ADE$ Dynkin diagrams:

$A_n$:  
1 2 \cdots n-1 n

$D_n$:  
1 2 \cdots n-2 n-1 n

$E_n$:  
1 2 3 4 \cdots n-1 n

Given the minimal resolution of a Du Val singularity $\mu : (E \subset \tilde{S}) \to (P \in S)$ we write $E_i$ for the exceptional divisor corresponding to the $i$th vertex in the Dynkin diagram, $\tilde{C}_i \subset \tilde{S}$ for a smooth curve transverse to $E_i$ and $C_i = \mu(\tilde{C}_i) \subset S$.

**Numerical types.** Given $C \subset S$, the numerical type of $C$ is $C \equiv_{\text{num}} \sum_{i=1}^{n} \alpha_i C_i$ where $\tilde{C} \equiv_{\text{num}} \sum_{i=1}^{n} \alpha_i \tilde{C}_i$ on $\tilde{S}$. (Note that $\tilde{C}$ may intersect $E$ nontransversely.)

**The cycle $\Delta_i$.** When $P \in S$ is of type $D_n$ we define the numerical type $\Delta_i \subset S$, for $i = 1, \ldots, n$, by the following formula:

$$\Delta_i = \begin{cases} 
C_i & 1 \leq i \leq n-2 \\
C_{n-1} + C_n & i = n-1 \\
2C_{n-1} \text{ or } 2C_n & i = n 
\end{cases}$$

At one point (see Table 2) we need to distinguish between the two different cases for $\Delta_n$, but otherwise we treat both cases simultaneously.
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1 The divisorial extraction

As in [2], we assume the following inclusion of algebraic varieties over $\mathbb{C}$:

$$P \in C \subset S_X \subset X$$

where $P \in X$ is a smooth point in a 3-fold, $P \in S_X$ is a Du Val singularity and $C$ is a curve with a singularity at $P$. We are interested in the existence of a Mori extraction $\sigma: (F \subset Y) \to (C \subset X)$, where $F$ is the exceptional divisor extracted from $C$. By [2] Proposition 1.4, $Y$ is isomorphic to the blowup of the symbolic power algebra of the ideal $I_{C/X}$ defining $C \subset X$.

The general elephant. The existence of the Du Val hypersurface $S_X$ is a consequence of the general elephant conjecture which states that, for a Mori extraction (or a Mori flipping contraction) $\sigma: Y \to X$, a general member $S_Y \in |-K_Y|$ and $S_X := \sigma(S_Y) \in |-K_X|$ should both have at worst Du Val singularities. Then, by adjunction, the restriction $\sigma|_{S_Y}: S_Y \to S_X$ is a partial crepant resolution.

Unlike [2], we do not assume that $S_X$ is a general hypersurface section containing $C$. Part of the information given in the description of a Mori extraction is which curves (if any) are extracted from $P \in S_X$ by $\sigma|_{S_Y}$.

1.1 Constructing the divisorial extraction

We briefly recall the method explained in [2] §2.4 to construct $Y$.

A normal form for $C$. If $C \subset S_X$ is not a local complete intersection (l.c.i.) then by [2] Lemma 2.1 we can write the equations of $C$ as the minors of a $2 \times 3$ matrix $M$:

$$2 \bigwedge \left( \begin{array}{cc} \phi & -b \\ a & \end{array} \right) = 0$$

(1.1)

where $\phi$ is a $2 \times 2$ matrix such that $\det \phi$ is the equation of the Du Val singularity $P \in S_X$ and $a, b \in \mathbb{C}[x, y, z]$ are some functions on $X$. After a change of variables we can take $\phi$ to be one of the following matrices:

$$A_{n-1}^i: \begin{pmatrix} x \\ y^{n-i} \\ z \end{pmatrix} \quad D_n^i: \begin{pmatrix} x & y^2 + z^{n-2} \\ z & x \end{pmatrix} \quad D_{2k}^n: \begin{pmatrix} x & yz + z^k \\ y & x \end{pmatrix}$$

$$D_{2k+1}^n: \begin{pmatrix} x & yz \\ y & x + z^k \end{pmatrix} \quad E_6: \begin{pmatrix} x & y^2 \\ y & x + z^2 \end{pmatrix} \quad E_7: \begin{pmatrix} x & y^2 + z^3 \\ y & x \end{pmatrix}$$

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(E₈ does not appear in this list as it is factorial and has no such nontrivial matrix factorisation.) We can use row and column operations to ensure that \( b = b(y, z) \) and \( a = a(y, z) \) or \( a = a(x, y) \) in case \( A_{i-1}^i \).

We write \( I := I(M) \) for the ideal generated by the entries of \( M \) and \( I(\phi) \) for the ideal generated by those of \( \phi \). Clearly \( I(\phi) \subseteq I \) and it is also clear that \( C \subseteq X \) is a l.c.i. if (and only if) \( I \not\subseteq \mathfrak{m}_P \) (i.e. at least one of \( a, b \notin \mathfrak{m}_P \)).

**The blowup of \( C \).** We can write down the (ordinary) blowup of \( C \subseteq X \) as a complete intersection of codimension 2, with equations given by the two syzygies coming from Cramer’s rule:

\[
\sigma_0: Y_0 \subseteq X \times \mathbb{P}^2_{(\nu, \xi, \eta)} \to X \quad \left( \begin{array}{c} \phi \\ -b \\ a \end{array} \right) \left( \begin{array}{c} \nu \\ -\xi \\ \eta \end{array} \right) = 0
\]

(1.2)

In particular \( \eta \) corresponds to the equation \( \det \phi \) defining \( S_X \). If \( I \subseteq \mathfrak{m}_P \) the exceptional divisor of this blowup has two components \( F_0 \cup D_0 \), where \( F_0 \) is a reduced divisor dominating \( C \) and \( D_0 = V(I) \) is a (not necessarily reduced) codimension 1 subscheme dominating \( P \subseteq X \).

**Definition 1.1.** The width of \( D_0 \) is defined to be \( w(D_0) := \dim \mathbb{C}[x, y, z]/I \).

Since \( I(\phi) \subseteq I \), the width of \( D_0 \) is bounded by \( w_\phi := \dim \mathbb{C}[x, y, z]/I(\phi) \). In the different cases this bound is given by:

| Type | \( A_{i-1}^i \) | \( D_n^I \) | \( D_{2k}^r \) | \( D_{2k+1}^r \) | \( E_6 \) | \( E_7 \) |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( I(\phi) \) | \((x, y^i, z)\) | \((x, y^2, z)\) | \((x, y, z^k)\) | \((x, y, z^k)\) | \((x, y, z^2)\) | \((x, y, z^3)\) |
| \( w_\phi \) | \( i \) | \( 2 \) | \( k \) | \( k \) | \( 2 \) | \( 3 \) |

In particular \( D_0 \cong \mathbb{P}^2 \times \text{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^{w(D_0)}) \) is the largest possible subscheme contained in \( Y_0 \) supported on \( P \times \mathbb{P}^2 \) (c.f. [6] Definition 5.3).

**Unprojecting \( D_0 \).** Since \( Y_0 \) and \( D_0 \) are both complete intersections in \( X \times \mathbb{P}^2 \) (and hence Gorenstein) we can use the unprojection theorem ([5] Theorem 1.5) to unproject \( D_0 \subseteq Y_0 \). In this simple case we can obtain the unprojection variable \( \zeta \) by rewriting (1.2) as a \( 2 \times 3 \) matrix annihilating \( I \) and applying [8] Example 4.16. By construction \( \zeta \) satisfies \( \zeta I \subseteq I_C^{[2]} \). Hence unprojection gives us a Gorenstein variety \( Y_1 \subseteq X \times \mathbb{P}(1, 1, 1, 2) \) in codimension 3 with a birational map

\[
u: (D_0 \subseteq Y_0) \dashrightarrow (Q \in Y_1)
\]

\( Y_1 \) is defined by five equations and so, by a theorem of Buchsbaum and Eisenbud, we can (and will) write the equations as the maximal Pfaffians of a \( 5 \times 5 \) skew matrix. This is also the way the equations are presented in [8] Example 4.1.
Unprojection is a type of birational surgery that blows up \( D_0 \) to a Cartier divisor \( \tilde{D}_0 \) and contracts the birational transform. In the nicest case \( Y_0 \) has only isolated ordinary nodal singularities at the points where \( D_0 \) fails to be Cartier and blowing up \( D_0 \) makes a small resolution of these nodes. However unprojection can be more complicated in practice. For example, \( Y_0 \) may contain a line of ordinary nodes (or a line of transverse \( A_n \) singularities) along \( D_0 \), and then unprojection introduces a new divisor \( D_1 \subset Y_1 \) above \( D_0 \). This can lead to a chain of serial unprojections. One family of divisorial extractions given an arbitrarily long sequence of serial unprojections is constructed in [3] §6.3.

2 Codimension \( \leq 2 \) cases

We first treat the cases in which a Mori extraction \( \sigma: Y \to X \) exists in codim \( \leq 2 \). In all of these cases \( C \subset X \) is a l.c.i. and then, by a theorem of Cutkosky [1] (see also [2], Lemma 1.6), it is known that a necessary and sufficient condition for \( C \) to have a Mori extraction is that \( C \) is contained in a smooth hypersurface. Using this condition we give a classification of the possible numerical types for \( C \) and we also classify which curve (if any) is extracted from \( S_X \).

2.1 Summary

We summarise all the cases in Table 1. The table lists the \( ADE \) type of \( P \in S_X \), the format for \( C \subset S_X \), which exceptional curves (if any) are extracted by \( \sigma|_{S_Y}: S_Y \to S_X \) and the possible numerical types of \( C \).

Each case appears in the table up to a symmetry of the corresponding Dynkin diagram, e.g. for the \( E_6 \) case the divisorial extraction from \( C_5 \) (with exc(\( \sigma|_{S_Y} \)) = \( E_1 \)) also includes the divisorial extraction from \( C_1 \) (with exc(\( \sigma|_{S_Y} \)) = \( E_5 \)).

Since they are contained in a smooth hypersurface section, all the cases in the table have a Mori extraction, even the degenerate cases in which \( \tilde{C} \cap E \) is nontransverse.

2.2 Proof of the classification

The proof follows from explicit calculations. First note that one of the following occurs:

(i) \( C \subset S_X \) is a l.c.i., in which case \( S_Y \cong S_X \).

We write \( C = V(f, g) \subset X \) where \( S_X = V(f) \) and \( g \) is the equation of a smooth hypersurface. The Mori extraction from \( C \) is given by the codimension 1 model:

\[
Y = V(f\xi - g\eta) \subset X \times \mathbb{P}^1_{(\xi, \eta)}.
\]

\( S_Y = V(\eta) \subset Y \) meets the central fibre \( Z = P \times \mathbb{P}^1 \) at the point \( P_\xi \), where all variables apart from \( \xi \) vanish, and \( P_\xi \in S_Y \) is a singularity with equation \( f = 0 \). So \( \sigma|_{S_Y} \) is an isomorphism \( S_Y \cong S_X \).
Table 1: Curves with codimension ≤ 2 Mori extraction.

| Type  | Format | exc($\sigma|_{S_Y}$) | Numerical types |
|-------|--------|----------------------|-----------------|
| $A_{n-1}$ | l.c.i. | $\emptyset$ | $\sum_{j=1}^{n-1} \alpha_j C_j : \sum_j j\alpha_j = n$ or $\sum_{j=1}^{n-1} (n-j)\alpha_j = n$ |
| $A'_{n-1}$ | $E_{n-i}$ | | $\sum_{j=1}^{n-1} \alpha_j C_j : \sum_j j\alpha_j = i$ or $\sum_{j=1}^{n-1} (n-j)\alpha_j = n-i$ |
| $D_n$ | l.c.i. | $\emptyset$ | $\Delta_{2i} (\forall i \leq \lfloor \frac{n}{2} \rfloor)$, $C_1 + \Delta_{2i-1} (\forall i \leq \lfloor \frac{n+1}{2} \rfloor)$ |
| $D_n^l$ | $E_1$ | | $\Delta_{2i-1} (\forall i \leq \lfloor \frac{n+1}{2} \rfloor)$ |
| $D_n^r$ | $E_n$ | | $C_n$, $C_1 + C_{n-1}$ |
| $E_6$ | l.c.i. | $\emptyset$ | $C_6$, $C_1 + C_5$, $C_3$, $C_1 + C_2$, $3C_1$, $C_4 + C_5$, $3C_5$ |
| $E_6$ | $E_1$ | | $C_5$, $C_2$, $2C_1$ |
| $E_7$ | l.c.i. | $\emptyset$ | $C_1$, $C_5$, $2C_6$, $C_2$, $C_6 + C_7$ |
| $E_7$ | $E_6$ | | $C_6$, $C_7$ |
| $E_8$ | l.c.i. | $\emptyset$ | $C_7$, $C_1$, $C_6$, $C_8$ |

Note: The cycles $\Delta_j$ appearing in the $D_n$ cases are defined in §0.2.

We can now classify the possible numerical types for $C$ by writing down a minimal resolution $\mu: (E \subset \tilde{S}_X) \to (P \in S_X)$ and calculating $\tilde{C} \cap E$, for all $C$ subject to the condition that $g \notin m_P^2$. (See §2.2.2 for an example of the kind of necessary calculation.)

(ii) $C \subset S_X$ is not a l.c.i., in which case $S_Y \not\cong S_X$.

We use (1.1) to write the equations of $C \subset X$ and (1.2) to write the equations of $Y = Y_0$. Of the two terms $a, b$ appearing in the format (1.2) we must have at least one of $a, b \notin m_P$, else $Y$ contains an unprojection divisor. The central fibre of $\sigma$ is $Z = P \times \mathbb{P}^1_{(\xi,\nu)}$ and $Z \subset S_Y = V(\eta)$. Hence exc($\sigma|_{S_Y}$) $\neq \emptyset$ and $S_Y \not\cong S_X$.

We can now check in each of the cases which curve is extracted from $P \in S_X$ and which numerical types are possible for $C$ by explicitly calculating $\tilde{C} \subset \tilde{S}_X$, subject to condition that at least one of $a, b \notin m_P$.

2.2.1 $A_{n-1}$ cases

In the $A_{n-1}$ case we can consider $S_X$ as the $\frac{1}{n}(1, n-1)$ cyclic quotient singularity

$$\pi: \mathbb{C}^2_{u,v} \to \mathbb{C}^2/\mu_n =: S_X$$

where $\mu_n = \{\varepsilon : \varepsilon^n = 1\}$, the cyclic group of the nth roots of unity, acts on $\mathbb{C}^2$ by $(u, v) \mapsto (\varepsilon u, \varepsilon^{n-1} v)$. We write $x, y, z = u^n, uv, v^n$ for the invariants of this action which satisfy the relation $xz = y^2$. We can pull back any curve $C \subset S_X$ to an invariant curve...
Γ := \pi^{-1}(C) \subset C_{u,v}^2 \text{ given by a semi-invariant orbifold equation } \Gamma = V(\gamma(u, v)). \text{ If } C \equiv \sum_{j=1}^{n-1} \alpha_j C_j \text{ then } \gamma \text{ factors (analytically) as a product } \\
\gamma(u, v) = \prod_{j=1}^{n-1} \gamma_j(u^j, v^{n-j}) \\
\text{where } \gamma_j(U, V) \in \mathbb{C}[x, y, z] \text{ is a homogeneous polynomial of degree } \alpha_j \text{ whose roots correspond to the intersection points of } \tilde{C} \cap E_j \text{ counted with multiplicity.}

By the normal form (1.1), the equations of } C \subset X \text{ are given by the following format for some } i: \\
\bigwedge^2 \begin{pmatrix} x & y^i \ y^{n-i} & -b \ z & a \ z \end{pmatrix} = 0 \\
\text{In this case the orbifold equation is given by } \gamma(u, v) = au^i + bv^{n-i} \text{ and the equations } ax + by^{n-i}, ay^i + bz \text{ are given by rendering the invariants } u^{n-i} \gamma, v^i \gamma \text{ in terms of } x, y, z.

If } a_0, b_0 \text{ are the constant terms of } a, b, \text{ then the condition that } C \text{ is contained in a smooth hypersurface is equivalent to asking that at least one of } a_0, b_0 \neq 0. \text{ But } a_0 \text{ is the coefficient of } u^i \text{ in } \gamma \text{ so, by the product expression above, } a_0 \neq 0 \text{ if and only if } i = \sum_{j=1}^{n-1} j \deg \gamma_j = \sum_{j=1}^{n-1} j \alpha_j. \text{ Similarly } b_0 \neq 0 \text{ if and only if } n - i = \sum_{j=1}^{n-1} (n - j) \alpha_j.

2.2.2 Example calculation

We explain the } E_6 \text{ case as an example of the calculations that can be used to verify the other type } D \text{ and } E \text{ cases appearing Table 1.}

The equations of } C. \text{ Using the normal form (1.1) in the } E_6 \text{ case we write the equations of } C \subset X \text{ as the } 2 \times 3 \text{ minors of the matrix:} \\
\bigwedge^2 \begin{pmatrix} x & y^2 \ y & x + z^2 \ z & a \end{pmatrix} \\
\text{for some choice of } a, b \in \mathbb{C}[y, z]. \text{ As before we write } a_0, b_0 \text{ for the constant terms of } a, b. \text{ Note that } C \text{ is contained in a smooth hypersurface if and only if at least one of } a_0, b_0 \neq 0.

Explicit resolution of } P \in S_X \text{ We resolve } S_X = V(x(x + z^2) - y^3) \text{ as follows. Let } b_y, b_z \text{ be the two following coordinate changes:} \\
b_y: (x, y, z) \mapsto (xy, y, yz), \quad b_z: (x, y, z) \mapsto (xz, yz, z). \\
\text{Then the minimal resolution } \mu: (E \subset \tilde{S}_X) \rightarrow (P \in S_X) \text{ can be covered by the five following affine charts:}
For example, we reach the chart (4) by the change of coordinates
\[ \mu_4: (x, y, z) \mapsto (xy^2z^4, y^2z^3, yz^2). \]
In this chart \( \tilde{S}_X = V(x(x+1) - y^2z) \) is a smooth surface and (up to a choice of relabelling) the exceptional curves are given by
\[ E_1 = V(x, y), \quad E_2 = V(x, z), \quad E_4 = V(x + 1, z), \quad E_5 = V(x + 1, y). \]

**Description of \( C \subset S_X \).** By calculating \( \tilde{C} \subset \tilde{S}_X \) we see that there are exactly three numerical types for curves satisfying the condition that one of \( a_0, b_0 \neq 0 \). We do the computation in chart (4) of the resolution given above and leave the rest of the calculation to the reader. In fact, in each of these three cases \( \tilde{C} \cap E = \emptyset \) outside of chart (4) (apart from possibly the point at \( \infty \) on either \( E_1 \) or \( E_5 \) contained in chart (5)).

(i) First assume \( b_0 \neq 0 \). Then in chart (4) \( \tilde{C} \) is given by
\[
\tilde{C} = V(a'xz + b'x(x+1) - y^2z)
\]
(where \( a' = \mu_4(a) \) and \( b' = \mu_4(b) \)). Since \( b_0 \neq 0 \), \( \tilde{C} \cap E_i = \emptyset \) for \( i = 1, 2, 4 \) and \( \tilde{C} \) intersects \( E_3 \) at one point transversely at the point \( x = y = a_0z - b_0 = 0 \) (if \( a_0 = 0 \) then this intersection point is the point at \( \infty \) in chart (5)). Hence \( C \equiv C_5 \).

(ii) If \( b_0 = 0 \) then we replace \( b \) by \( by + cz \) and now we assume \( a_0, c_0 \neq 0 \). Then in chart (4) \( \tilde{C} \) is given by
\[
\tilde{C} = V(a'y + b'(x+1)yz + c'(x+1), a'x + b'z^2 + c'yz)
\]
\( \tilde{C} \) intersects \( E_2 \) transversely at the point \( x = z = a_0y = c_0 = 0 \). Moreover, if \( c_0 \neq 0 \) then \( \tilde{C} \) does not intersect \( E_1 \) and if \( a_0 \neq 0 \) then \( \tilde{C} \) does not intersect \( E_i \) for \( i = 3, 4, 5, 6 \). Hence \( C \equiv C_2 \).

(iii) Lastly we assume that \( c_0 = 0 \) and \( a_0 \neq 0 \) in case (ii), so we replace \( by + cz \) by \( by + cz^2 \). Then in chart (4) \( \tilde{C} \) is given by
\[
\tilde{C} = V(a' + b'(x+1)z + c'(x+1)z^2, x(x+1) - y^2z)
\]
\( \tilde{C} \) intersects \( E \) twice according to the roots of \( c_0 z^2 + b_0 z + a_0 = 0 \) (again, if \( c_0, b_0 = 0 \) then these points may be the point at \( \infty \) in chart (5)). Moreover, if \( a_0 \neq 0 \) then \( \tilde{C} \) does not intersect any other component of \( E \), so \( C \equiv 2C_1 \).

In all other cases \( C \) is not contained in a smooth hypersurface and \( Y_0 \) contains an unprojection plane.

**The general elephant** \( S_Y \). The last thing to check is that \( \text{exc}(\sigma|_{S_Y}) = E_1 \) as claimed. By (1.2) we write \( Y = Y_0 \subset X \times P^2(\xi:\nu) \) as a complete intersection:

\[
Y = V(y^2 \xi - x\nu + b\eta, \ y\nu - (x + z^2)\xi + a\eta)
\]

with central fibre \( Z = P^1_{\xi:\nu} \subset S_Y \). Therefore exactly one curve is extracted from \( P \in S_X \).

It follows directly from these equations that \( S_Y \) is smooth apart from at the point \( P_\xi \), where all variables except \( \xi \) vanish. At \( P_\xi \) we can eliminate \( x \) by the equation \( x = y\nu - z^2 \) to be left with the \( D_5 \) singularity

\[
V(y^2 - y\nu^2 + \nu z^2) \subset \mathbb{C}^3_{y,z,\nu}.
\]

Hence \( \sigma|_{S_Y} \) extracts either \( E_5 \) or \( E_1 \) from \( P \in S_X \) and the extracted curve is independent of the choice of \( a, b \). To see which one it is we can consider \( C' = (\sigma|_{S_Y})^{-1}C \), the birational transform of \( C \) under \( \sigma|_{S_Y} \). For example take case (iii) above, we see that \( C' \) is the complete intersection of \( S_Y \) and \( y = a + b\nu + c\nu^2 \), so that \( C' \) intersects \( Z \) twice according to the two roots of \( a_0 + b_0\nu + c_0\nu^2 = 0 \). Therefore it is \( E_1 \) that is extracted.

### 3 Codimension 3 cases

Now we treat the cases in which a Mori extraction \( \sigma: Y \to X \) exists in codimension 3. As described in §1.1, such an extraction has a model

\[
\sigma: Y := Y_1 \subset X \times F(1,1,1,2) \to X
\]

where \( Y_1 \) is the unprojection of \( D_0 \subset Y_0 \). We consider each format \( A^n, \ldots, E_7 \) in turn and split into subcases depending on \( w(D_0) \), the width of the unprojection divisor \( D_0 \subset Y_0 \) appearing in the construction of \( Y \).

Using the equations of \( Y \) we give conditions for the reduced central fibre \( Z = \sigma^{-1}(P)_{\text{red}} \) to be small (i.e. purely 1-dimensional) and conditions for \( Y \) to have isolated singularities. Then we use these conditions to give a classification of the possible numerical types for \( C \subset S_X \). We also check directly which curves, if any, are extracted by \( \sigma|_{S_Y} \) from \( P \in S_X \).

We are not claiming at present that every case in the list gives rise to a Mori extraction. A full analysis of the singularities appearing in each case would be very involved, and we would have to consider a lot of exceptional cases in which \( \tilde{C} \) degenerates to a curve intersecting \( E \) nontransversely. However, once we establish that \( \sigma|_{S_Y}: S_Y \to S_X \) is a partial crepant resolution of a Du Val singularity then the only non-terminal singularities of \( Y \) must have centre in \( Y \setminus S_Y \), so it would be enough just to check the singularities in this open set.
3.1 Summary

We summarise all the cases in Table 2. In this table we list the format of $C \subset S_X$, the width $w = w(D_0)$ of the unprojection divisor $D_0 \subset Y_0$, which exceptional curves (if any) are extracted by $\sigma|_{S_Y} : S_Y \to S_X$ and the possible numerical types of $C$.

As before, each case appears in the table only up to a symmetry of the corresponding Dynkin diagram.

In the $A_{n-1}^i$ case the table only contains the generic case for the numerical type of $C$. For a more detailed description of the possible numerical types see §3.2.1.

Table 2: Curves with codimension 3 Mori extraction.

| Format | $w(D_0)$ | exc$(\sigma|_{S_Y})$ | Numerical types |
|--------|----------|---------------------|-----------------|
| $A_{n-1}^i$ | $w < i$ | $E_{n-i} + E_{n-i+w}$ | generically $C_w + C_i + C_{n-w}$ |
|        | $i$ ($n > 2i$) | $E_{n-2i}$ | generically $2C_i + C_{n-i}$ |
|        | $i$ ($n = 2i$) | $\emptyset$ | generically $3C_i$ |
| $D_n^i$ | 1 | $E_2$ | $C_1 + \Delta_{2i}$ ($\forall i \leq \lceil \frac{n}{2} \rceil$) |
|        | 2 | $\emptyset$ | $2C_1 + \Delta_{2i-1}$ ($\forall i \leq \lfloor \frac{n+1}{2} \rfloor$) |
| $D_n^r$ | $w \leq \lceil \frac{n}{2} \rceil$ | $E_{n-2w}$ ($\emptyset$ if $n = 2w$) | $C_n + \Delta_{2w}$ $(\ast)$, $C_1 + C_n + \Delta_{2w-1}$, $C_{n-1} + \Delta_{2w+1}$ $(\ast\ast)$ |
| $E_6$ | 1 | $E_2$ | $C_5 + C_6$, $C_1 + C_4$ |
|        | 2 | $E_5$ | $C_1 + 2C_5$, $C_3 + C_5$, $2C_4$ |
| $E_7$ | 1 | $E_5$ | $C_1 + C_6$, $C_4$ |
|        | 2 | $E_1$ | $C_5 + C_6$, $C_2 + C_6$ |
|        | 3 | $\emptyset$ | $3C_6$, $2C_6 + C_7$ |

Note: There are two exceptions in the $D_n^r$ case:

($\ast$) $C_n + \Delta_{2w}$ has $w(D_0) = w$ unless $2w = n$, in which case $3C_n$ has $w(D_0) = w$ and $2C_{n-1} + C_n$ has $w(D_0) = w - 1$.

($\ast\ast$) $C_{n-1} + \Delta_{2w+1}$ has width $w(D_0) = w$ unless $2w + 1 = n$, in which case $3C_{n-1}$ has $w(D_0) = w - 1$ and $C_{n-1} + 2C_n$ has $w(D_0) = w$. (If $w = 2k$ then $2w + 1 > n$ and we ignore this case.)

3.2 Proof of the classification

We now prove the classification by dividing into cases according to the different formats $A_{n-1}^i, \ldots, E_7$. 

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3.2.1 Type $A^i_{n-1}$

In this section we assume that our curve $C \subset S_X \subset X$ is of type $A^i_{n-1}$. In this case $w(D_0) \leq i$ and we split into the following three subcases:

(i) $w(D_0) < i$,

(ii) $w(D_0) = i$ and $2i < n$,

(iii) $w(D_0) = i$ and $2i = n$.

The equations of $C$ and $Y$. Our curve $C \subset X$ and the variety $Y \subset X \times \mathbb{F}(1,1,1,2)$, given by the unprojection of $D_0 \subset Y_0$, are defined by the following equations:

$$2 \left( \frac{x}{y^{n-i}} y^i - (cy^w + dz) \right) \text{Pf} \left( \begin{array}{ccc} \zeta & \nu & y^{i-w} \xi + c \eta & -d \eta \\ -a \eta & y^{n-i-w} \nu + b \eta & \xi \\ z & y^w \\ x \end{array} \right)$$

where the five equations defining $Y$ are written as the Pfaffians of a skew matrix, as in [8] Example 4.1. (Only the strict upper diagonal part of the matrix is written). Without loss of generality, we collect terms together so that

$$(\text{ii}) \quad \text{exc}(\sigma) \subset \mathbb{P}^1_{(\xi, \zeta)} \cup \mathbb{P}^1_{(\nu, \zeta)}$$

corresponding to $E_{n-i+w}$ and $E_{n-i-w}$ respectively.

Setting $x, y, z, \eta = 0$ in the equations defining $Y$ we see that $\text{exc}(\sigma|_{S_Y})$ consists of the two irreducible components $\mathbb{P}^1_{(\xi, \zeta)} \cup \mathbb{P}^1_{(\nu, \zeta)}$. The restriction of $\xi, \nu, \zeta$ to $S_X$ can be written in terms of $u, v$ and $\gamma$ (the orbinates on $S_X$ introduced in §2.2.1) as follows:

$$\xi = [u^{n-i} \gamma], \quad \nu = [v^i \gamma], \quad \zeta = \frac{\xi \nu}{y^w} = [u^{n-i-w} v^i \gamma^2]$$

Then coordinates along $\mathbb{P}^1(1,2)_{(\xi, \zeta)}$ are given by the ratio

$$(\xi^2 : \zeta) = (u^{2(n-i)} \gamma^2 : u^{n-i-w} v^i \gamma^2) = (u^{n-i+w} : v^i-w)$$

so this component corresponds to the exceptional divisor $E_{n-i+w}$ above $P \in S_X$. Similarly $\mathbb{P}^1(1,2)_{(\nu, \zeta)}$ corresponds to $E_{n-i-w}$. Indeed, we can also use the equations to see that $S_Y$ is smooth apart from an $A_{i-w-1}$ singularity at $P_\xi$, an $A_{2w-1}$ singularity at $P_\zeta$ and an $A_{n-i-w-1}$ singularity at $P_\nu$, just as expected.

By a similar calculation in the other two cases:

(ii) $\text{exc}(\sigma|_{S_Y}) = \mathbb{P}^1_{(\nu, \zeta)}$ corresponding to $E_{n-2i}$,

(iii) $\sigma|_{S_Y}$ is an isomorphism.

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Small central fibre. In all cases $Z$ is small and has $\leq 3$ irreducible components, unless one of the following three conditions hold:

- $b_0 = c_0 = 0$ in case (i): but this contradicts $w = w(D_0)$.
- $a_0 = b_0 = 0$ in either case (i) or (ii): then $Z$ contains a new unprojection divisor $D_1 = V(x, y, z, \xi)$.
- $c_0 = d_0 = 0$ in case (i) or $b_0 = c_0 = d_0 = 0$ in case (ii): then $Z$ contains a new unprojection divisor $D_1 = V(x, y, z, \nu)$.

In particular, in case (iii) $Z$ is always small.

Isolated singularities. In all of these cases the index 2 point $Q \in Y$ is a (hyper)quotient singularity: 

$$(\xi \nu = y^w + ad\eta^2) \subset \mathbb{C}^4_{\xi, \nu, \eta, y} / \mathbb{Z}(1, 1, 1, 0)$$

If $w = 1$ then this is a $\mathbb{Z}(1, 1, 1)$ singularity and $Y$ must have isolated singularities. A necessary condition for $Y$ to be terminal in this case is that not all of $a_0, b_0, c_0, d_0 = 0$, else $Z = \mathbb{P}^1_{(\eta, \varsigma)}$ and $Y$ has an index 1 singularity at $P_0$ with embedding dimension $\geq 5$, as in [2] Lemma 2.3.

If $w > 1$ then this is a $cA/2$ singularity, although possibly not isolated. Indeed if both $a_0, d_0 = 0$ then $Z$ contains $\mathbb{P}^1_{(\eta, \varsigma)}$ as a component and $Y$ is singular along this line.

Description of $C \subset S_X$. We summarise the necessary conditions for $Y$ to be the Mori extraction from $C$ in each of the three cases (i)–(iii).

(i) At least one of $a_0, b_0 \neq 0$, at least one of $b_0, c_0 \neq 0$, at least one of $c_0, d_0 \neq 0$ and, if $w > 1$, at least one of $a_0, d_0 \neq 0$.

(ii) At least one of $a_0, b_0, c_0 \neq 0$, at least one of $c_0, d_0 \neq 0$ and, if $w > 1$, at least one of $a_0, d_0 \neq 0$.

(iii) At least one of $a_0, b_0, c_0, d_0 \neq 0$ and, if $w > 1$, at least one of $a_0, d_0 \neq 0$.

In the case where $a, b, c, d$ are chosen generically then $C \subset S_X$ is the curve:

(i) $C_w + C_i + C_{n-w}$
(ii) $2C_i + C_{n-i}$
(iii) $3C_i$

For example, in case (i) $C$ is given by the orbifold equation

$$\gamma(u, v) = au^{n+i} + bu^{i+w}v^w + cu^wv^{n-i+w} + dv^{2n-i}$$

which, if all $a_0, b_0, c_0, d_0 \neq 0$, has initial term

$$\gamma(u, v) = a_0(u^{n-w} + \frac{b_0}{a_0}v^w)(u^i + \frac{c_0}{b_0}v^{n-i})(u^w + \frac{d_0}{c_0}v^{n-w}) + \cdots$$
3.2.2 Type $D_n^i$

Now we assume that our curve $C \subset S_X \subset X$ is of type $D_n^i$. In this case $w \leq 2$ so we split into the two subcases: (i) $w = 1$, (ii) $w = 2$.

The equations of $C$ and $Y$. Our curve $C \subset X$ and the variety $Y \subset X \times \mathbb{P}(1, 1, 1, 2)$ given by the unprojection of $D \subset Y_0$ are given by the following equations:

$$
\begin{align*}
\wedge_2 \left( \begin{array}{ccc}
x & y^2 + z^{n-2} & -(cy^w + dz) \\
x & ay^w + bz \\
z & x
\end{array} \right)
\end{align*}

\text{Pf}\left( \begin{array}{cccc}
\zeta & \nu & y^{2-w}\xi + c\eta & z^{n-3}\xi + d\eta \\
\xi & a\eta & \nu + b\eta \\
-\zeta & y^w & x
\end{array} \right)
$$

where we choose $w = w(D_0)$.

The general elephant $S_Y$. In case (i) $\text{exc}(\sigma|_{S_Y}) = \mathbb{P}^1_{(\zeta, \xi)}$ corresponding to $E_2$ and in case (ii) $\sigma|_{S_Y}$ is an isomorphism.

Small central fibre. In case (i) $Z$ is small unless either $a_0 = c_0 = 0$ (which contradicts out choice of $w = w(D_0)$) or $c_0 = d_0 = 0$, in which case $Y_1$ contains a new unprojection divisor $D_1 = V(x, y, z, \nu)$. In case (ii) $Z$ is always small.

Isolated singularities. In case (i) $Q \in Y$ is a $\frac{1}{2}(1, 1, 1)$ singularity and $Y$ must have isolated singularities. In case (ii) $Q \in Y$ is the hyperquotient singularity:

$$
\left( y^2 + \xi(z^{n-3}\xi + d\eta) = \nu(\nu + b\eta) \right) \subset \mathbb{C}^4_{x,y,\nu,\eta} / \frac{1}{2}(1, 1, 1, 0)
$$

If both $b_0, d_0 = 0$ then $\mathbb{P}^1_{(\nu, \zeta)} \subseteq Z$ and $Y$ becomes singular along this line.

Description of $C \subset S_X$. Using the restrictions obtained above on the curves for which $Y$ is a Mori extraction, we can now give an explicit description of $C \subset S_X$.

(i) At least one of $a_0, c_0 \neq 0$ and at least one of $b_0, d_0 \neq 0$. Assume for the moment that $d_0 \neq 0$. Resolving the $D_n$ singularity we see that $\tilde{C}$ intersects $E_1 \cong \mathbb{P}^1_{(u_1:v_1)}$ according to $a_0u_1 + c_0v_1 = 0$ and $E_2 \cong \mathbb{P}^1_{(u_2:v_2)}$ according to $c_0u_2 + d_0v_2 = 0$. Moreover $\tilde{C}$ does not meet any other exceptional curve. (Note that by our assumptions both these linear polynomials are nonzero and $\tilde{C}$ meets the intersection point $E_1 \cap E_2$ when $c_0 = 0$). Hence in this generic case $C \equiv C_1 + C_2$, as represented in the following diagram (where the white node is the curve extracted from $S_X$):

$$
generic case \quad d_0 \neq 0 \\
C = C_1 + C_2 \quad (a_0 : c_0) (c_0 : d_0)$$

1 1 ...
If \( d_0 = 0 \) then necessarily \( c_0 \neq 0 \), else \( Z \) has a 2-dimensional component. We can check that making the replacement

\[
\begin{cases}
  dz \mapsto dz^i \\
  dz \mapsto (dz \pm \sqrt{-1}c)z^{i-1}
\end{cases}
\quad i < \frac{n-1}{2}
\quad i = \frac{n-1}{2}
\quad \text{gives } \quad C = \begin{cases}
  C_1 + C_{2i} & i = 0 \\
  C_1 + C_{n-1} + C_n & i = \frac{n}{2} \\
  C_1 + 2C_{n-1} \text{ or } C_1 + 2C_n & i = \frac{n+1}{2}
\end{cases}
\]

Moreover these are the only numerical types satisfying \( c_0 \neq 0 \). Using the cycle \( \Delta_i \) defined in §0.2 we can write this family of curves as \( C_1 + \Delta_{2i} \), for \( i \leq \frac{n}{2} \).

(ii) By a similar calculation we see that \( C \equiv 3C_1 \) if \( d_0 \neq 0 \), intersecting \( E_1 \cong \mathbb{P}^1_{(u_1:v_1)} \) according to the roots of the cubic \( a_0u_1^3 + c_0u_1^2v_1 + b_0u_1v_1^2 + d_0v_1^3 = 0 \). If \( d_0 = 0 \) then \( b_0 \neq 0 \) else \( Y \) has non-isolated singularities. Then \( C \) can degenerate to any of the curves \( 2C_1 + \Delta_{2i} \), for \( i \leq \frac{n+1}{2} \).

### 3.2.3 Type \( D_n^r \)

The cases \( D_{2k}^r \) and \( D_{2k+1}^r \) turn out to be very similar, even though the formats for \( \phi \) in (1.1) initially look quite different. Therefore we only consider the even case \( D_{2k}^r \). Both cases are summarised in Table 2.

We assume that our curve \( C \subset S_X \subset X \) is of type \( D_{2k}^r \). In this case \( w \leq k \) and we split into the two subcases: (i) \( w < k \), (ii) \( w = k \).

#### The equations of \( C \) and \( Y \)

Our curve \( C \subset X \) and the variety \( Y \subset X \times \mathbb{P}(1,1,1,2) \) given by the unprojection of \( D \subset Y_0 \) are given by the following equations:

\[
\begin{align*}
2 & \left( \begin{array}{c}
  x \\
  y \\
  z
\end{array} \right) \begin{array}{c}
  yz + z^k & -(cy + dz^w) \\
  ay + bz^w & 0
\end{array} \\
\text{Pf} \left( \begin{array}{cccc}
  \zeta & \nu & z^w & z^k-w \xi + \eta \\
  \xi & \nu + a \eta & b \eta & 0 \\
  -z^w & y & x & 0
\end{array} \right)
\end{align*}
\]

where we choose \( w = w(D_0) \).

#### The general elephant \( S_Y \)

In case (i) \( \text{exc}(\sigma|_{S_Y}) = \mathbb{P}^1_{(\xi,\zeta)} \) corresponding to \( E_{2(k-w)} \) and in case (ii) \( \sigma|_{S_Y} \) is an isomorphism.

#### Small central fibre

In case (i) \( Z \) is small unless either \( b_0 = d_0 = 0 \) (which contradicts out choice of \( w = w(D_0) \)) or \( c_0 = d_0 = 0 \), in which case \( Y_1 \) contains a new unprojection divisor \( D_1 = V(x,y,z,\nu) \). In case (ii) \( Z \) is always small.

#### Isolated singularities

\( Q \in Y \) is the hyperquotient singularity:

\[
(z^w + \nu(\nu + a \eta) = \xi(z^w + c \eta)) \subset \mathbb{C}^4_{\xi,\nu,\eta,z} / \frac{1}{2}(1,1,1,0)
\]

If \( w > 1 \) and both \( a_0, c_0 = 0 \) then \( \mathbb{P}^1_{(\nu,\zeta)} \subset Z \) and \( Y \) becomes singular along this line.
Description of $C \subset S_X$. By explicitly resolving $S_X$ and computing $\widetilde{C} \subset \widetilde{S}_X$ we see that, for $w < k$, the three possible numerical equivalence classes for $C$ listed in Table 2 correspond to:

- $C_n + \Delta_{2w} : c_0, d_0 \neq 0$ (the generic case)
- $C_1 + C_n + \Delta_{2w-1} : c_0 = 0$ and $a_0, d_0 \neq 0$
- $C_{n-1} + \Delta_{2w+1} : d_0 = 0$ and $b_0, c_0 \neq 0$

In all other cases either $c_0 = d_0 = 0$ (so that $Z$ is not small), $b_0 = d_0 = 0$ (so that $w(D_0) > w$) or $a_0 = c_0 = 0$ (so that $Y$ has non-isolated singularities). Similarly for $w = k$ we have two cases to consider: the generic case $3C_n$, given by $c_0 \neq 0$, and a special case $C_1 + C_{n-1} + 2C_n$ given by $c_0 = 0$ and $a_0 \neq 0$.

3.2.4 Type $E_6$

Now we assume that our curve $C \subset S_X \subset X$ is of type $E_6$. In this case $w \leq 2$ and we split into the two subcases: (i) $w = 1$, (ii) $w = 2$.

The equations of $C$ and $Y$. Our curve $C \subset X$ and $Y \subset X \times \mathbb{P}(1,1,1,2)$ are given by the following equations:

$$
\begin{align*}
\bigwedge^2 \begin{pmatrix} x & y^2 & -(cy+dz^w) \\ y & x+z^2 & ay+bz^w \end{pmatrix} & \quad \text{Pf} \begin{pmatrix} \zeta & \nu & y\xi + c\eta \\ \xi & \nu + a\eta & z^{2-w}\xi - b\eta \\ z^w & & y \end{pmatrix}
\end{align*}
$$

where we choose $w = w(D_0)$.

The general elephant $S_Y$. In both cases $\text{exc}(\sigma|_{S_Y}) = \mathbb{P}^1_{(\xi,\zeta)}$, corresponding to $E_2$ in case (i) and to $E_5$ in case (ii).

Small central fibre. In case (i) $Z$ is small unless either $b_0 = d_0 = 0$ (contradicting $w = w(D_0)$) or $c_0 = d_0 = 0$, in which case $Z$ contains a new unprojection divisor $D_1 = V(x,y,z,\nu)$. In case (ii) $Z$ is small unless either $c_0 = d_0 = 0$, in which case $Z$ contains the unprojection divisor $D_1 = V(x,y,z,\nu)$, or $c_0 = a_0 - d_0 = b_0 = 0$, in which case $Z$ contains the unprojection divisor $D_2 = V(x,y,z,\nu + a\eta)$.

Isolated singularities. In case (i) $Q \in Y$ is a $\frac{1}{2}(1,1,1)$ singularity and $Y$ must have isolated singularities. In case (ii) $Q \in Y$ is a $cA/2$ hyperquotient singularity. If both $a_0, c_0 = 0$ then $Z \supseteq \mathbb{P}^1_{(\eta,\zeta)}$ and $Y$ becomes singular along this line.
Description of $C \subset S_X$. By explicitly resolving the $E_6$ Du Val singularity we see that, for case (i), the two numerical types of Table 2 are given by (i.a) $C_5 + C_6$ if $d_0 \neq 0$ and (i.b) $C_1 + C_4$ if $d_0 = 0$ and both $b_0, c_0 \neq 0$.

Similarly for case (ii) the three numerical types are given by (ii.a) $C_1 + 2C_5$ if $c_0 \neq 0$, (ii.b) $C_3 + C_5$ if $c_0 = 0$ and all of $a_0, a_0 - d_0, d_0 \neq 0$ and (ii.c) $2C_4$ if $c_0 = a_0 - d_0 = 0$ and $b_0, d_0 \neq 0$.

In case (ii) a fourth numerical type is possible, given by $C_5 + 2C_6$ if $a_0 = c_0 = 0$ and $d_0 \neq 0$. The central fibre of $Y$ is small for this choice of $C$, so that $Y$ is a divisorial extraction, however $Y$ has non-isolated singularities.

3.2.5 Type $E_7$

We assume that $C \subset S_X \subset X$ is of type $E_7$. In this case $w \leq 3$ and we split into the subcases: (i) $w = 1$, (ii) $w = 2$, (iii) $w = 3$.

The equations of $C$ and $Y$. Our curve $C \subset X$ and $Y \subset X \times \mathbb{P}(1,1,1,2)$ are given by the following equations:

$$2 \left( \begin{array}{ccc} x & y^2 + z^3 & -(cy + dz^w) \\ y & x & ay + bz^w \end{array} \right) \quad \text{Pf} \left( \begin{array}{cccc} \zeta & \nu & y\xi + c\eta & z^{3-w}\xi + d\eta \\ \xi & \nu + a\eta & b\eta & y \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & x \end{array} \right)$$

where we choose $w = w(D_0)$.

The general elephant $S_Y$. In cases (i) and (ii) $\text{exc}(\sigma|_{S_Y}) = \mathbb{P}^1_{(\xi, \zeta)}$ corresponding to $E_5$ in case (i) and to $E_1$ in case (ii). In case (iii) $\sigma|_{S_Y}$ is an isomorphism.

Small central fibre. In cases (i) and (ii) $Z$ is small unless either $b_0 = d_0 = 0$ (contradicting $w = w(D_0)$) or $c_0 = d_0 = 0$, in which case $Z$ contains an unprojection divisor $D_1 = V(x, y, z, \nu)$. In case (iii) $Z$ is always small.

Isolated singularities. In case (i) $Q \in Y$ is a $\frac{1}{2}(1,1,1)$ singularity and $Y$ must have isolated singularities. In case (ii) $Q \in Y$ is a hyperquotient singularity. If both $a_0, c_0 = 0$ then $Z \supset \mathbb{P}^1_{(a, c)}$ and $Y$ becomes singular along this line.

Description of $C \subset S_X$. By explicitly resolving the $E_7$ Du Val singularity we see that the two possible numerical types for case (i) listed in Table 2 are given by (i.a) $C_1 + C_6$ if $d_0 \neq 0$ and (i.b) $C_4$ if $d_0 = 0$ and $b_0, c_0 \neq 0$. The two types in case (ii) are given by (ii.a) $C_5 + C_6$ if $c_0 \neq 0$ and (ii.b) $C_2 + C_6$ if $c_0 = 0$ and $a_0, d_0 \neq 0$. Lastly, the two types in case (iii) are given by (iii.a) $3C_6$ if $c_0 \neq 0$ and (iii.b) $2C_6 + C_7$ if $c_0 = 0$ and $a_0 \neq 0$.

A third numerical type is possible in case (ii), given by $2C_1 + C_6$ if $a_0 = c_0 = 0$ and $d_0 \neq 0$. The central fibre of $Y$ is small in this case, so $Y$ is a divisorial extraction, however $Y$ has non-isolated singularities.
3.3 Concluding remarks.

In the $E_6$ and $E_7$ cases it is possible to take these calculations further and exhaust all of the possible cases by repeated serial unprojection. In both cases all Mori extractions exist in relative codimension $\leq 5$ and the number of additional numerical types to consider is at most 12 for type $E_6$ and at most 5 for type $E_7$.

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