The spaces of non-contractible closed curves in compact space forms

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Abstract. The rational equivariant cohomology of noncontractible loop spaces is calculated for compact space forms. It is also shown how to use these calculations to establish the existence of closed geodesics.

Bibliography: 18 titles.

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§ 1. Introduction

The study of the periodic problem for Finsler geodesics was initiated by Anosov in [1]. He explained that his interest in Finsler geometry arose because “it leads to a wide class of dynamical systems which lets us use geometric concepts and arguments in formulating and investigating problems”. For instance, in the early 1980s the study of variational problems for magnetic geodesic flows was started in [2] (see also [3]). Magnetic geodesic flows become particular cases of Finsler geodesic flows, provided certain conditions are satisfied.

In [1] Anosov claimed that, by contrast with the Riemannian case where, by the Lyusternik-Schnirelman theorem, there exist at least three non-self-intersecting closed geodesics on the two-sphere, for irreversible Finsler metrics “we can only guarantee the existence of two closed geodesics”. This remarkable result was proved only recently by Bangert and Long [4] and the proof relies heavily on the index iteration formulae derived by Long [5]. Katok’s example [6] shows that the estimate cannot be improved.

Recently Xiao and Long studied the topological structure of noncontractible loop spaces for odd-dimensional projective spaces. In particular, they calculated the equivariant cohomology with $\mathbb{Z}_2$-coefficients of the path spaces [7] and, together with Duan, applied these results to prove the existence of at least two geometrically distinct noncontractible closed geodesics for irreversible bumpy Finsler metrics on $\mathbb{R}P^3$ [8].

In this article we demonstrate how to use a result from [9] and [10] to calculate the rational equivariant cohomology of noncontractible loop spaces for compact space forms. We also show how to use these calculations to establish the existence of closed geodesics.
§ 2. The path spaces

Let $M^n$ be a closed Riemannian manifold. Suppose that $\Lambda(M^n) = H^1(S^1, M)$ denotes the space of $H^1$-maps

$$\gamma : [0,1] \to M^n, \quad f(0) = f(1),$$

of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ into $M^n$, $\Omega_x(M^n)$ denotes the subspace of $\Lambda(M^n)$ formed by loops starting and ending at $\gamma(0) = \gamma(1) = x \in M^n$, and that $\Pi^+(M^n)$ and $\Pi(M^n)$ denote the quotients of $\Lambda(M^n)$ with respect to the $SO(2)(= S^1)$-action:

$$\varphi \cdot \gamma(t) = \gamma(t + \varphi), \quad \varphi \in S^1 = \mathbb{R}/\mathbb{Z},$$

and the $O(2)$-action respectively. Here the $O(2)$ action is the extension of the $SO(2)$-action by the involution

$$\sigma \cdot f(t) = f(-t).$$

The space $\Lambda(M)$ is a Hilbert manifold and the fibration

$$H^1(S^1, TM) \to H^1(S^1, M) = \Lambda(M)$$

is the tangent bundle to it. A detailed description of the topology and the atlas of a Hilbert manifold on $\Lambda(M)$ can be found in [11] and [12]. The space $\Lambda$, which was introduced into the calculus of variations in the mid-1960s, has many functorial properties. In particular,

1) if $f : M \to N$ is a smooth map, then the induced map $\Lambda f : \Lambda(M) \to \Lambda(N)$ is a smooth map of Hilbert manifolds; moreover, if $f_s : M \times [a,b] \to N$ is a smooth homotopy, then $\Lambda f_s$ also is a smooth homotopy;

2) the map $g : \Lambda M \to \Lambda M$ which assigns to a curve the same geometrical curve parametrized proportionally to the arc-length, is continuous; further, we can assume that the condition $(g \cdot \gamma)(0) = \gamma(0)$ also holds;

3) the above map $g$, which satisfies $(g \cdot \gamma)(0) = \gamma(0)$, is homotopic to the identity and, moreover, such a homotopy can be chosen to be $SO(2)$- or $O(2)$-invariant.

The proofs of these statements are given in [12].

We let $L(M)$ denote the space

$$L(M) = g(\Lambda(M))$$

and $P^+(M)$ and $P(M^n)$ denote the following quotient-spaces:

$$P^+(M) = L(M)/SO(2) \quad \text{and} \quad P(M) = L(M)/O(2).$$

These spaces are deformation retracts of $\Lambda(M)$, $\Pi^+(M)$ and $\Pi(M)$, respectively, and therefore are homotopically equivalent to them.

Geometrically closed geodesics of the metric $g_{ik} \, dx^i \, dx^k$ are extremals of the energy functional

$$E(\gamma) = \frac{1}{2} \int_{\gamma} |\dot{\gamma}|^2 \, dt, \quad E : \Lambda(M) \to \mathbb{R},$$
and of the length functional

\[ S(\gamma) = \int |\dot{\gamma}| \, dt, \quad S : L(M) \to \mathbb{R}, \]

where

\[ |\dot{\gamma}| = \sqrt{g_{ik} \dot{\gamma}^i \dot{\gamma}^k}. \]

The Euler-Lagrange equations for the energy functional imply that the parameter on an extremal has to be proportional to the arc-length and that to every closed geodesic there corresponds a pair of \( S^1 \)-families (an \( O(2) \)-orbit) of extremals. For the length functional every reparameterization of an extremal is again an extremal. Therefore, we have to fix a parameter which is proportional to the arc-length to obtain a pair of \( S^1 \)-families of extremals in \( L(M) \).

A manifold \( M \) with a function \( F(x, \dot{x}) \) defined on its tangent bundle is called a Finsler manifold if

1) \( F(x, \dot{x}) > 0 \), with equality if and only if \( \dot{x} = 0 \);
2) \( F(x, \lambda \dot{x}) = \lambda F(x, \dot{x}) \) for all \( \lambda > 0 \);
3) the unit spheres \( \{F(x, \dot{x}) = 1\} \) are convex and their curvatures are positive with respect to the Euclidean metrics in the tangent spaces \( T_x(M) \).

For every smooth path \( \gamma(t), a \leq t \leq b \), on a Finsler manifold the Finsler length

\[ S(\gamma) = \int_a^b F(\gamma, \dot{\gamma}) \, dt \]

is defined. Since \( F(x, \dot{x}) \) is homogeneous of the first order in \( \dot{x} \), every reparameterization of an extremal of \( S \) is also extremal for \( S \). Therefore, the variational problem for closed Finsler geodesics is best posed for the functional

\[ S(\gamma) = \int \gamma F(\gamma, \dot{\gamma}) \, dt, \quad S : L(M) \to \mathbb{R}. \]

In this case to each extremal there corresponds an \( O(2) \)-orbit of extremals in \( L(M) \). If the Finsler metric is reversible, that is, \( F(x, \dot{x}) = F(x, -\dot{x}) \), in particular, if \( F(x, \dot{x}) = |\dot{x}| \) for some Riemannian metric, then each extremal that is not a single point generates an \( O(2) \)-orbit consisting of extremals.

To reduce the number of critical points which correspond geometrically to the same closed curve we have to consider the length functionals \( S \) on the spaces \( P^+(M) \) (for irreversible metrics) and \( P(M) = P^+(M)/\sigma \) (for reversible metrics). It is clear that to every extremal closed curve there corresponds a unique critical point of \( S \) in \( P^+(M) \) (for irreversible metrics) and in \( P(M) = P^+(M)/\sigma \) (for reversible metrics).

Two closed geodesics are called distinct if they are not both iterates \( \gamma^n \) of the same closed curve \( \gamma \) where \( n \in \{1, 2, \ldots\} \) for irreversible metrics and \( n \in \{\pm 1, \pm 2, \ldots\} \) for reversible metrics.

\[ \text{§ 3. Rational homotopy of path spaces} \]

Let \( h \in \pi_1(M, x_0) \) and let \([h]\) be the corresponding free homotopy class of closed curves: \([h] \in [S^1, M]\). We let

\[ \Lambda M[h] \subset \Lambda M \quad \text{and} \quad LM[h] \subset LM \]
denote the connected components of $\Lambda M$ and $LM$ consisting of curves in $[h]$.

Let $h$ be realized by a map $\omega: [0, 1] \rightarrow M$ with $\omega(0) = x_0$, and let $h_i$ be the automorphism

$$h_i: \pi_i(M, x_0) \rightarrow \pi_i(M, x_0)$$

corresponding to the standard action of $h \in \pi_1$ on $\pi_i$.

The following theorem was proved independently in [9] and [10].

**Theorem A** (see [9] and [10]). The mapping

$$\pi: \Lambda M \rightarrow M, \quad \pi(\gamma) = \gamma(0),$$

which assigns to a closed curve $\gamma$ the marked point $\gamma(0)$, is the Serre fibration with fibre $\Omega M$:

$$\Lambda M \xrightarrow{\Omega M} M.$$ The exact homotopy sequence for this fibration restricted to $\Lambda M[h]$ takes the form

$$\cdots \rightarrow \pi_i(\Lambda M[h], \omega) \xrightarrow{\pi_*} \pi_i(M, x_0) \xrightarrow{f_i} \pi_i-1(\Omega x_0(M), \omega) \rightarrow \cdots,$$

(3.1)

where

a) $\pi_*(\pi_i(\Lambda M[h], \omega)) = \text{St}(h_i)$, where $\text{St}(h_i)$ is the subgroup of $\pi_i(M, x_0)$ consisting of all elements fixed under $h_i$;

b) $f_i = h_i - \text{id}$ for $i \geq 2$.

The maps $h_i$ can be written uniformly in the simple form

$$f_k(g) = [h, g], \quad g \in \pi_k(M, x_0), \quad k \geq 1,$$

where $[h, g]$ is the Whitehead product of $h \in \pi_1$ and $g \in \pi_k$.

Consider the case when

$$M = S^n/\Gamma, \quad h \neq 1 \quad \text{in} \quad \pi_1(M, x_0),$$

where $\Gamma$ acts freely and isometrically on the $n$-sphere and therefore $M^n$ is diffeomorphic to a compact space form.

If $n = 2k$, then the only nontrivial group which acts freely on $S^{2k}$ is $\mathbb{Z}_2$ and $S^{2k}/\mathbb{Z}_2 = \mathbb{R}P^{2k}$.

Now consider the rational homotopy groups

$$\pi^Q_i(X) = \pi_i(X) \otimes \mathbb{Q}, \quad i \geq 2.$$ By the Cartan-Serre theorem

$$\pi^Q_i(S^{2k}) = \begin{cases} \mathbb{Q} & \text{for } i = 2k, 4k - 1, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \pi^Q_i(S^{2k+1}) = \begin{cases} \mathbb{Q} & \text{for } i = 2k + 1, \\ 0 & \text{otherwise} \end{cases},$$

and moreover,

$$\pi_n(S^n) = \mathbb{Z}, \quad n \geq 1.$$ Suppose that $n \geq 2$.

We have the following result.
Theorem 1. Let $M = S^n/\Gamma$, where $\Gamma$ acts freely and isometrically on $S^n$, and let $h \neq 1 \in \pi_1(M)$.

Then

1) for $i \geq 2$

$$
\pi_i^Q(\Lambda M[h]) = \begin{cases} 
\mathbb{Q} & \text{for } i = 4k - 2, 4k - 1, \\
0 & \text{otherwise}
\end{cases}
$$

for $M = S^{2k}/\mathbb{Z}_2 = \mathbb{R}P^{2k}$, and

$$
\pi_i^Q(\Lambda M[h]) = \begin{cases} 
\mathbb{Q} & \text{for } i = 2k, 2k + 1, \\
0 & \text{otherwise}
\end{cases}
$$

for $M = S^{2k+1}/\Gamma$;

2) if $n \geq 3$

$$
\pi_1(\Lambda M[h]) = C(h) = \mathbb{Z}_{r(h)},
$$

where $C(h) = \mathbb{Z}_{r(h)} \subset \Gamma$ is the centralizer of $h$ in $\Gamma$, and

$$
\pi_1(\Lambda \mathbb{R}P^2[h]) = \mathbb{Z}_4.
$$

Proof. For $h \in \pi_1(X)$ the action $h_\#$ is induced by the corresponding deck transformation of the universal covering $\tilde{X} \to X$. Therefore, if $h \neq 1$, then the following hold.

1. The action $h_{2k+1}$ on $\pi_{2k+1}(S^{2k+1}/\Gamma) = \mathbb{Z}$ is trivial: $h_{2k+1}(z) = z$, because the deck transformation of $S^{2k+1}$ is a rotation. Therefore, by Theorem A

$$
\pi_{2k}^Q(\Lambda(S^{2k+1}/\Gamma)[h]) = \pi_{2k+1}^Q(\Lambda(S^{2k+1}/\Gamma)[h]) = \mathbb{Q}.
$$

2. The action $h_{2k}$ of a nontrivial element $h \in \mathbb{Z}_2 = \pi_1(\mathbb{R}P^{2k})$ on $\pi_{2k}(\mathbb{R}P^{2k}) = \mathbb{Z}$ is multiplication by $-1$:

$$
h_{2k}(z) = -z,
$$

because the corresponding deck transformation is the reflection $x \to -x$, which changes the orientation of the sphere. It follows from Theorem A that

$$
\pi_{2k}^Q(\Lambda \mathbb{R}P^{2k}[h]) = 0
$$

and

$$
\pi_{2k-1}^Q(\Lambda \mathbb{R}P^{2k}[h]) = 0 \quad \text{for } k > 1.
$$

3. The action $h_{4k-1}$ on $\pi_{4k-1}(\mathbb{R}P^{2k})/\text{Torsion} = \mathbb{Z}$ is trivial: $h_{4k-1}(z) = z$, because $\pi_{4k-1}$ is generated by the Whitehead product $[i_{2k}, i_{2k}]$, where $i_{2k}$ is the generator of $\pi_{2k}$ and $h_{2k}(i_{2k}) = -i_{2k}$. By Theorem A, we have

$$
\pi_{4k-2}^Q(\Lambda \mathbb{R}P^{2k}[h]) = \pi_{4k-1}^Q(\Lambda \mathbb{R}P^{2k}[h]) = \mathbb{Q}.
$$

It is well known that every commutative subgroup of $\Gamma$ is cyclic (see, for instance, [13]). Since in addition $\Gamma$ is finite, the centralizer $C(h)$ is commutative and hence cyclic: $C(h) = \mathbb{Z}_{r(h)}$, where $r(h)$ is the order of the maximal cyclic subgroup of $\Gamma$ which contains $h \in \Gamma$. By Theorem A we have $\pi_1(\Lambda(S^n/\Gamma)[h]) = \mathbb{Z}_{r(h)}$ for $n \geq 3$. 


We are left to show that $\pi_1(\Lambda \mathbb{R}P^2[h]) = \mathbb{Z}_4$. By Theorem A, we have the exact sequence

$$0 \to \pi_2(\mathbb{R}P^2) = \mathbb{Z} \xrightarrow{\times(-2)} \pi_2(\mathbb{R}P^2) = \mathbb{Z} \to \pi_1(\Lambda \mathbb{R}P^2[h]) \to \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2 \to 0,$$

which implies the exact splitting sequence

$$0 \to \mathbb{Z}_2 \to \pi_1(\Lambda \mathbb{R}P^2[h], \omega) \to \mathbb{Z}_2 \to 0.$$

We will describe the homomorphisms from the last sequence. We realize $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$ as the unit sphere in $\mathbb{R}^3$. Then $\mathbb{R}P^2$ is the quotient of the unit sphere with respect to the antipodal involution. Take the north and south poles of $S^2$: $x_3 = \pm 1$, which correspond to the same point in $\mathbb{R}P^2$, and consider the paths $\kappa_\phi$ of the form

$$x_1 = \cos \phi \sin \theta, \quad x_2 = \sin \phi \sin \theta, \quad x_3 = \cos \theta, \quad 0 \leq \theta \leq \pi.$$

They join the poles on $S^2$ and realize loops in $\mathbb{R}P^2$. This $\phi$-family of loops, where $0 \leq \phi \leq 2\pi$, forms a loop in $\Lambda \mathbb{R}P^2[h]$, which represents an element $[\kappa] \in \pi_1(\Lambda \mathbb{R}P^2[h], \kappa_0)$. By construction, this element generates the image of the homomorphism

$$\pi_1(\Omega \mathbb{R}P^2, \kappa_0) = \pi_2(\mathbb{R}P^2) = \mathbb{Z} \to \pi_1(\Lambda \mathbb{R}P^2[h], \kappa_0),$$

and therefore

$$2[\kappa] = 0.$$

Take another element $[\eta] \in \pi_1(\Omega \mathbb{R}P^2, \gamma_0)$, which is represented by the $\phi$-family of loops $\eta_\phi$ of the form

$$x_1 = \sin(\theta + \phi), \quad x_2 = 0, \quad x_3 = \cos(\theta + \phi), \quad 0 \leq \theta \leq \pi,$$

where $0 \leq \phi \leq \pi$. It is clear that the paths $\eta_\phi$ and $\eta_{\phi + \pi}$ determine the same loop in $\mathbb{R}P^2$ and $\eta_0 = \kappa_0$. By construction, the image of $[\eta]$ under the homomorphism

$$\pi_1(\Lambda \mathbb{R}P^2[h]) \to \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$$

is nontrivial. Therefore $\pi_1(\Lambda \mathbb{R}P^2[h])$ is generated by $[\eta]$ and $[\kappa]$.

Let us consider the family of paths $\eta_\phi$ with $0 \leq \phi \leq 2\pi$. It represents $2[\eta]$ and if we take the centre of each path $\eta_\phi$ and rotate the path about the axis going through its centre and the origin, we obtain an $S^1$-family of paths $\tilde{\eta}_\phi$, which is transformed into $\kappa_\phi$ by a rotation of the sphere. Hence

$$[\kappa] = 2[\eta]$$

and $\pi_1(\Lambda \mathbb{R}P^2[h]) = \mathbb{Z}_4$.

This finishes the proof of Theorem 1.
§ 4. Homotopy quotients of path spaces

There is a natural $\text{SO}(2)$-action on $\Lambda M$ which consists in changing the base points. It is described in §2 where the quotient space $\Pi^+(M) = L(M)/\text{SO}(2)$ is defined. However, this action is not free since iterated contours have nontrivial isotropy groups.

The homotopy quotient $X_G$ of the $G$-space $X$, where $G$ is a group, is the quotient of the product $X \times EG$ with respect to the diagonal action of $G$. Here

$$EG \xrightarrow{G} BG$$

is the universal $G$-bundle. For $G = \text{SO}(2)$ we have $BG = \mathbb{C}P^\infty$, and $EG$ is contractible by definition. The $G$-equivariant cohomology is defined by

$$H^*_G(X) = H^*(X_G).$$

In addition, we prefer to work with the spaces $LM$ which are formed by curves parameterized with respect to arc length and which are $\text{SO}(2)$-equivariant deformation retracts of $\Lambda M$ (see §2). The action functional

$$S: LM \to \mathbb{R}$$

satisfies a nice property: the closed extremals of $S$ form $\text{SO}(2)$-orbits, and therefore to every nonparameterized closed extremal of $S$ there corresponds just one critical point of the action functional

$$S: LM_{\text{SO}(2)} \to \mathbb{R},$$

here, and in the sequel, for brevity we denote $(LM)_{\text{SO}(2)}$ by $LM_{\text{SO}(2)}$.

**Theorem 2.** Let $M = S^n/\Gamma$, where $\Gamma$ acts freely and isometrically on $S^n$, and $h \neq 1 \in \pi_1(M)$.

Then

1) for $i \geq 2$

$$\pi_i^Q(LM[h]_{\text{SO}(2)}) = \begin{cases} \mathbb{Q} & \text{for } i = 2, 4k - 2, 4k - 1, \\ 0 & \text{otherwise} \end{cases}$$

for $M = S^{2k}/\mathbb{Z}_2 = \mathbb{R}P^{2k}$ and

$$\pi_i^Q(LM[h]_{\text{SO}(2)}) = \begin{cases} \mathbb{Q} & \text{for } i = 2, 2k, 2k + 1, \\ 0 & \text{otherwise}, \end{cases}$$

for $M = S^{2k+1}/\Gamma$;

2) for odd $n \geq 3$ the spaces $LM[h]_{\text{SO}(2)}$ are homotopically simple and

$$\pi_1(LM[h]_{\text{SO}(2)}) = C(h)/\mathbb{Z}[h],$$

where $C(h) \subset \Gamma$ is the centralizer of $h$ in $\Gamma$ and $\mathbb{Z}[h]$ is the subgroup of $C(h)$ generated by $h$;

3) for $n \geq 1$

$$\pi_1(L\mathbb{R}P^{2n}[h]_{\text{SO}(2)}) = 0.$$
Proof. The proof of this theorem is an immediate consequence of the exact homotopy sequence of the fibration

$$LM[h] \times \text{ESO}(2) \xrightarrow{\text{SO}(2)} LM[h]_{\text{SO}(2)}.$$ 

The statement about homotopical simplicity needs clarification. This is done as follows: for odd $n$ the fundamental groups of $S^n/\Gamma$ act trivially on the higher homotopy groups because the deck transformations of the universal coverings are homotopy equivalent to the identity, and now, from the explicit description of the homotopy groups of the path spaces (Theorem A), we can conclude that the path spaces and their homotopy quotients are also homotopically simple.

The groups $H_{4k-1}(L\mathbb{R}P^{2k}[h];\mathbb{Q})$ and $H_{2k+1}(L(S^{2k+1}/\Gamma)[h];\mathbb{Q})$ are generated by the following homology classes:

1) take the $(4k-1)$-dimensional manifold $V^{4k-1}$ formed by the pairs $(x, v)$ where $x \in S^{2k} = \{ |y| = 1, y \in \mathbb{R}^{2k+1} \}$ and $v$ is a unit vector tangent to $S^{2k}$ at $x$. Define a map $F: V^{4k-1} \to L\mathbb{R}P^{2k}[h]$ which assigns to each such pair $(x, v)$ the semicircle $\gamma$ (in $S^{2k}$) starting at $x$ in the direction of $v$. It is easy to show that $V^{4k-1}$ is rationally homotopy equivalent to the $(4k-1)$-sphere and the image of the induced map in homology generates $H_{4k-1}(L\mathbb{R}P^{2k-1}[h];\mathbb{Q}) = \mathbb{Q}$;

2) since $h \in \Gamma$, its action on the unit sphere $S^{2k+1}$ has the form

$$(z_1, \ldots, z_{k+1}) \mapsto (e^{i\alpha_1}z_1, \ldots, e^{i\alpha_{k+1}}z_{k+1}),$$

where $S^{2k+1} = \{ z_1, \ldots, z_{k+1} \in \mathbb{C}, |z_1|^2 + \cdots + |z_{k+1}|^2 = 1 \}$.

To each $z \in S^{2k+1}$ we assign the path

$$\gamma(t) = (e^{i\alpha_1 t}z_1, \ldots, e^{i\alpha_{k+1} t}z_{k+1}), \quad 0 \leq t \leq 1,$$

which starts at $z$ and finishes at $h(z)$. This correspondence defines a map

$$F: S^{2k+1} \to LM[h]$$

in a natural way, and the image of the induced map in homology generates $H_{2k+1}(LM[h];\mathbb{Q}) = \mathbb{Q}$.

Consider the fibrations

$$LM[h] \times \text{ESO}(2) \xrightarrow{S^1} LM[h]_{\text{SO}(2)}, \quad h \neq 1 \in \pi_1(M).$$

There are $\text{SO}(2)$-equivariant maps

$$f: V^{4k-1} \times \text{ESO}(2) \to L\mathbb{R}P^{2k}[h] \times \text{ESO}(2)$$

and

$$f: S^{2k+1} \times \text{ESO}(2) \to L(S^{2k+1}/\Gamma)[h] \times \text{ESO}(2)$$

and the corresponding induced maps of the spectral sequences:

$$f^*: E^n_{p,q} \to E^n_{p,q}.$$
which have the following form for $n = 2$:
\[
f^*: H^p(L\mathbb{R}P^{2k}[h]_{\text{SO}(2)}; H^q(S^1; \mathbb{Q})) \to H^p(V^{4k-1}/\text{SO}(2); H^q(S^1; \mathbb{Q})), \\
f^*: H^p(LM[h]_{\text{SO}(2)}; H^q(S^1; \mathbb{Q})) \to H^p(S^{2k+1}/\text{SO}(2); H^q(S^1; \mathbb{Q})).
\]

In both cases the $\text{SO}(2)$-actions are induced by changes of the base points on the paths $\gamma$ coming into the definitions of the mappings $f$. In both cases these actions are free and, in particular, we see that

1) (for $\dim M = 2k$) $E^2_{k-2,1} = H^{4k-2}(V^{4k-1}/\text{SO}(2); \mathbb{Q}) \otimes H^1(S^1; \mathbb{Q})$ is generated by $u^{k-1} \otimes v$, where $u$ is the generator of $H^2(V^{4k-1}/\text{SO}(2); \mathbb{Q})$ and $v$ is the generator of $H^1(S^1; \mathbb{Q})$ such that $d_2v = u$, $E^\infty_{k-2,1} = H^{4k-1}(V^{4k-1}; \mathbb{Q}) = \mathbb{Q}$ and therefore $d(u^{k-1} \otimes v) = u^{2k} = 0$;

2) (for $\dim M = 2k + 1$) $E^2_{k,1} = H^{2k}(S^{2k+1}/\text{SO}(2); \mathbb{Q}) \otimes H^1(S^1; \mathbb{Q})$ is generated by $u^k \otimes v$, where $u$ is the generator of $H^2(S^{2k+1}/\text{SO}(2); \mathbb{Q})$ and $v$ is the generator of $H^1(S^1; \mathbb{Q})$ such that $d_2v = u$, and $E^\infty_{2k,1} = H^{2k+1}(S^{2k+1}; \mathbb{Q}) = \mathbb{Q}$ and therefore $d(u^k \otimes v) = u^{k+1} = 0$.

We have
\[
u = f^*(u_2), \quad u_2 \in H^2(LM[h]_{\text{SO}(2)}; \mathbb{Q}),
\]
and since
\[
E^\infty_{k-2,1} = f^*(E^\infty_{4k-2,1}) = H^{4k-2}(LM[h]_{\text{SO}(2)}; \mathbb{Q}) \otimes H^1(S^1; \mathbb{Q})
\]
for $\dim M = 2k$ and
\[
E^\infty_{2k,1} = f^*(E^\infty_{2k,1}) = H^{2k}(LM[h]_{\text{SO}(2)}; \mathbb{Q}) \otimes H^1(S^1; \mathbb{Q})
\]
for $\dim M = 2k + 1$, we deduce from the spectral sequences for the fibrations $LM[h] \times \text{ESO}(2) \xrightarrow{S^1} LM[h]_{\text{SO}(2)}$ that
\[
u_2^{2k} = 0 \quad \text{for} \quad \dim M = 2k \quad \text{and} \quad \nu_2^{k+1} = 0 \quad \text{for} \quad \dim M = 2k + 1.
\]

Now consider the minimal models of $LM[h]_{\text{SO}(2)}$. The space $LM[h]_{\text{SO}(2)}$ is simply connected for $\dim M = 2k$ and is homotopically simple, with a finite cyclic fundamental group, for $\dim M = 2k + 1$. In both cases the minimal models (in the sense of Sullivan; see, for instance, [14] and [15]) are defined. We briefly recall only the simplest properties of minimal models:

1) the minimal model $\mathcal{M}(X)$ of $X$ is a free graded skew-commutative algebra $\sum_{k \geq 0} \mathcal{M}_k$ over $\mathbb{Q}$ such that its generators $\{u_\alpha\}$ (we assume that they are homogeneous: $u_\alpha \in \mathcal{M}_k$, that is, $\deg u_\alpha = k$) are in a one-to-one correspondence with the generators of $\pi^Q(X)$;

2) there is a differential $d: \mathcal{M}(X) \to \mathcal{M}(X)$ such that
\[
d^2 = 0, \quad d\mathcal{M}_k \subset \mathcal{M}_{k+1},
\]
and the differential $du_\alpha$ of each generator $u_\alpha$ is expressed in terms of generators of degree less than $\deg u_\alpha$;

3) the graded skew-commutative algebras $H^*(\mathcal{M}(X), d)$ and $H^*(X; \mathbb{Q})$ are isomorphic.
Now we compute the minimal models for $LM[h]_{SO(2)}$.

1) For $\dim M = 2k$ the minimal model is generated by $u_2$, $u_{4k-2}$ and $u_{4k-1}$ such that $\deg u_l = l$, $l = 2, 4k - 2, 4k - 1$. It is clear that

$$du_2 = 0, \quad du_{4k-2} = 0,$$

and since it was shown above that $[u_2]^{2k} = 0$ in cohomology, we have

$$du_{4k-1} = u_2^{2k}.$$

2) For $\dim M = 2k + 1$ the fundamental group of $LM[h]_{SO(2)}$ is finite cyclic and therefore its rational fundamental group is zero. The minimal model of $LM[h]_{SO(2)}$ (which is a nilpotent space: see [14], [15]) coincides with the minimal model of its universal covering and it is generated by $u_2$, $u_{2k}$ and $u_{2k+1}$ with $\deg u_l = l$, $l = 2, 2k, 2k + 1$. For the same reasons as in the previous case,

$$du_2 = 0, \quad du_{2k} = 0, \quad du_{2k+1} = u_2^{k+1}.$$

Thus we have proved the following.

**Theorem 3.** Let $M = S^n/\Gamma$, where $\Gamma$ acts freely and isometrically on $S^n$, and let $h \neq 1 \in \pi_1(M)$.

Then

1) for $n = 2k$ the minimal model of $LM[h]_{SO(2)}$ is generated by $u_2$, $u_{2k}$ and $u_{2k+1}$ such that $\deg u_l = l$ for all $l$ and $du_2 = 0$, $du_{4k-2} = 0$ and $du_{4k-1} = u_2^{2k+1}$. The cohomology ring has the form

$$H^*(LM[h]_{SO(2)}; \mathbb{Q}) = \mathbb{Q}[w, z]/\{w^{2k} = 0\}, \quad \deg w = 2, \quad \deg z = 4k - 2;$$

2) for $n = 2k + 1$ the minimal model of $LM[h]_{SO(2)}$ is generated by $u_2$, $u_{2k}$ and $u_{2k+1}$ such that $\deg u_l = l$ for all $l$ and $du_2 = 0$, $du_{2k} = 0$ and $du_{2k+1} = u_2^{k+1}$. The cohomology ring has the form

$$H^*(LM[h]_{SO(2)}; \mathbb{Q}) = \mathbb{Q}[w, z]/\{w^{k+1} = 0\}, \quad \deg w = 2, \quad \deg z = 2k.$$

**§ 5. Noncontractible closed geodesics**

Morse theory describes how closed extremals contribute to the topology of the path space. We recall a result due to Bott [16].

Let $\gamma$ be a simple closed (Finsler or Riemannian) geodesic in $M$. Then there exists a function

$$I : S^1 = \{|z| = 1, z \in \mathbb{C}\} \to \mathbb{N},$$

such that

1) $\text{ind} \gamma^m = \sum_{z_m = 1} I(z)$, where ind is the Morse index of an extremal;

2) $I$ is piecewise constant and is discontinuous exactly at the points $\{\lambda_1, \ldots, \lambda_k\}$ which are eigenvalues of the complexified linearized Poincaré mapping for $\gamma$ that lie on the unit circle $|\lambda| = 1$.

Assertion 2) implies that the points of discontinuity $\lambda_1, \ldots, \lambda_k$ are invariant with respect to complex conjugation, and that there are at most $2n - 2$ of them, where $n = \dim M$ is the dimension of the (configuration) manifold.
The geodesic $\gamma^m$ is called nondegenerate if $\pm 1$ do not lie in the spectrum of the corresponding Poincaré mapping, which holds if $\lambda_i^m \neq \pm 1$ for all $i = 1, \ldots, k$.

A metric is called bumpy if all its closed geodesics that are distinct from one-point curves are nondegenerate.

It was observed by Schwarz that an iterate $\gamma^m$ of a simple closed geodesic $\gamma$ contributes to the rational homology of $L(M)/SO(2)$ if and only if

$$\text{the difference } (\text{ind } \gamma^m - \text{ind } \gamma) \text{ is even},$$

which, by Bott’s theorem, holds if $(I(\gamma^2) - I(\gamma))$ is even or $m$ is odd.

For the equivariant cohomology we have the following: if an extremal $y$ is non-degenerate in the Morse sense, has index is equal to $k$, and if $y$ corresponds either to a simple closed geodesic or to $\gamma^m$, where $\gamma$ is a simple closed geodesic and $(\text{ind } \gamma^m - \text{ind } \gamma)$ is even, then

$$H^*_{SO(2)}(Y^{a-\varepsilon} \cup U(y), Y^{a-\varepsilon}) = H^{*+k}_{SO(2)}(S^1),$$

where $Y = LM_{SO(2)}$, $U(y)$ is a small neighbourhood of $y$ in $Y$, $S(y) = a$, $\varepsilon$ is positive and sufficiently small, and $U(y)$ contains only one critical point of $S$. For more detail we refer the reader to [17], where the systematic application of equivariant cohomology to closed geodesics was started, and to the survey [18] on the type numbers of closed geodesics.

In [7] the equivariant cohomology of $\mathbb{RP}^{2k+1}[h]$ with coefficients in $\mathbb{Z}_2$ was computed and in [8] it was shown for $\mathbb{RP}^3$ using some number-theoretical arguments that such cohomology cannot be generated by the iterates of a single closed geodesic of a bumpy irreversible Finsler metric.

Here we use Theorem 3 to prove the analogous result for $\mathbb{RP}^2$.

**Theorem 4.** For each bumpy irreversible Finsler metric on $\mathbb{RP}^2$ there exists at least two distinct noncontractible closed geodesics.

**Proof.** Let $c$ be a minimal noncontractible closed geodesic. By definition, it is simple and $\text{ind } c = 0$. Its iterates $c^{2k+1}$, $k = 0, 1, \ldots$, are noncontractible. Suppose that there are no other noncontractible closed geodesics. Then by Bott’s theorem the Bott function $I(z)$ has $2 = 2n-2$ points of discontinuity; we denote them by $e^{i\lambda}$ and $e^{-i\lambda}$. Without loss of generality, we assume that $0 < \lambda < \pi$. Since $\text{ind } c = 0$, we have $I(z) = 0$ for $z = e^{i\mu}$ with $-\lambda < \mu < \lambda$. By Theorem 3,

$$\sum_{z^{2k+1} = 1} I(z) = 2l, \quad k = 1, 2, \ldots,$$

and each even integer $2l$ is presented in this form in exactly two ways. This implies that $I(z) = 1$ for $z = e^{i\mu}$ with $\lambda < \mu < 2\pi - \lambda$, and, since $I(z) = 0$ outside the closure of this arc, this implies that $\lambda = \pi/2$ and therefore the geodesic $c^2$ is degenerate. Thus we arrive at a contradiction, which proves the theorem.

The following theorem shows how the fundamental group of $LM[h]$ being non-trivial can be used to prove the existence of closed geodesics.
Theorem 5. Let $M = S^{2n+1}/\Gamma$ and let $h$ be a nontrivial element in $\pi_1(M)$. Let $\pi_1(LM[h]_{SO(2)} \neq 1$ and assume that $h$ has even order in $\pi_1(M)$ and the elements of $C(h)$ (the centralizer of $h$) are pairwise nonconjugate.

Then every bumpy Finsler metric on $M$ has at least two distinct closed geodesics of the class $[h]$.

Proof. 1) Assume that the metric is irreversible. Let $c$ be a minimal closed geodesic of $[h]$. We have $c = \gamma^k$ where $\gamma$ is a simple closed geodesic. If all closed geodesics in $[h]$ are iterates of $c$, then they are of the form $\gamma^{k+2pl}$, where $l = 0, 1, 2, \ldots$ and $2p$ is the order of $h$ in $\pi_1(M)$. By Bott’s theorem, the Morse indices of these iterates are even, as is $\text{ind} c = 0$. Therefore, the handle decomposition of $LM[h]_{SO(2)}$ corresponding to the action functional $S$, which is a Morse function, contains only even-dimensional cells. That contradicts the nontriviality of $\pi_1(LM[h]_{SO(2)})$ and proves the theorem for irreversible metrics.

2) For reversible metrics, there is the possibility that $\gamma^{-m}$ is also a minimal closed geodesic for some positive $m$. However its iterates have the form $\gamma^{-m-2pl}$, $l = 1, 2, \ldots$, and their Morse indices are also even. Hence the handle decomposition of the space still contains only even-dimensional cells, which contradicts the nontriviality of $\pi_1(LM[h]_{SO(2)})$.

Remark. For reversible bumpy Finsler metrics on $\mathbb{R}P^{2n+1}$, $n = 1, 2, \ldots$, the existence of two distinct noncontractible closed geodesics was established in [8]. The following argument allows us to generalize this result to all projective spaces. If $c$ is a minimal geodesic, then $c^{-1}$ is also minimal. However, the space $L\mathbb{R}P^n[h]/SO(2)$ is connected and hence there is a saddle-type closed geodesic $c'$ with $\text{ind} c' = 1$. Since both geodesics $c$ and $c'$ contribute to the homology of the space and they have indices of different parity, by (5.1) they are distinct.

We dedicate this article to the memory of Dmitry Victorovich Anosov.

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