Stable Gapless Bose Liquid Phases without Any Symmetry

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It is well-known that a stable algebraic spin liquid state (or equivalently an algebraic Bose liquid (ABL) state) with emergent gapless photon excitations can exist in quantum spin ice systems\[1\,\ldots\,9\] or in a quantum dimer model on a bipartite 3d lattice.\[10\] This photon phase is stable against any weak perturbation without assuming any symmetry. Further works concluded that certain lattice models give rise to more exotic stable algebraic Bose liquid phases with graviton-like excitations.\[11\,\ldots\,15\] In this paper we will show how these algebraic Bose liquid states can be generalized to stable phases with even more exotic types of gapless excitations and then argue that these new phases are stable against weak perturbations. We also explicitly show that these theories have an (algebraic) topological ground state degeneracy on a torus, and construct the corresponding topological invariants.

I. INTRODUCTION

It is now well-known that quantum disordered states of many-body systems can be fundamentally different from classical disordered states. Without assuming any symmetry, there is simply one type of trivial classical state, but there can be many stable quantum disordered states. Many of these nontrivial quantum disordered phases have a gapped spectrum and topological degeneracy on a manifold with nontrivial topology,\[10\,\ldots\,15\] such as fractional quantum Hall states. In this paper we consider another kind of stable quantum disordered phases without assuming any symmetry. These states are characterized by their bulk gapless bosonic modes that cannot be interpreted as Goldstone modes. Furthermore, physical quantities have power-law (or algebraic) correlations instead of short-range correlations found in gapped systems.

Although such gapless states are not rare at all in condensed matter systems, they usually occur at quantum critical points and are protected by certain symmetries. Generically, we would expect there to be relevant perturbations that will open the gap in these critical states. But the examples we will discuss in this paper all have very stable gapless bosonic modes, which are invulnerable to any weak perturbations. Thus, to establish that an algebraic Bose liquid (ABL) phase is stable, we must show that all potential gap-opening perturbations are irrelevant at the IR fixed point of the ABL phase. Drawing intuition from the \((2 + 1)d\) compact lattice U(1) gauge field, we must demonstrate not only a direct mass term of these gapless modes are forbidden, but also that the space-time topological defects in the dual picture must also be suppressed (or irrelevant).

A few examples of this type of states are already known. In Ref\[11\,\ldots\,13\] a stable ABL phase with photon like excitations were proposed, and it has attracted great interest.\[13\,\ldots\,15\] So far compelling experimental evidences for such liquid states have been found.\[14\,\ldots\,16\] Later, a different type of ABL phase with graviton like excitation was studied in Ref. \[11\,\ldots\,13\] It turns out that the graviton-ABL state has a close cousin with a different dispersion relation.\[14\,\ldots\,15\] So far these are the only three types of known stable ABL states with emergent gapless bosonic excitations without assuming any symmetry. The Bose metal phase proposed in Ref. \[14\,\ldots\,20\] rely on a special quasi one dimensional conservation, which is different from the scenarios we will focus on.

In this work, we expand these ideas even further, demonstrating that there are an infinite number of gapless phases that fit into this class of states. We provide several examples of these so-called “higher-rank” ABL theories. We also investigate the topological properties of these models, showing that they are “topologically ordered” in the same sense as the photon and graviton theories, even though they are gapless in the bulk. At finite system size \(L\), the emergent gauge bosons will lead to an energy splitting between different sectors that scales as a power law of \(1/L\).

II. REVIEW OF RANK-1 AND -2 THEORIES

A. The Rank-1 Case

We first review the essential facts about the well-known \(U(1)\) photon ABL phase in \(3 + 1d\). In order to connect to the more general construction, we will address the problem from a somewhat different (but physically equivalent) viewpoint than the original works.\[12\,\ldots\,14\] The gauge structure (and its duality) is of paramount importance, so we will omit some details in favor of a more easily generalizable procedure. For simplicity, we will consider the cubic lattice, where spins are defined on the links, i.e. the corner-sharing octahedra. The most important term of the Hamiltonian is simply an Ising antiferromagnetic interaction on each octahedron:

\[
H = \frac{J}{2} \sum_{\text{oct}} (S^z_{\text{oct}})^2,
\]

(1)

With a very large \(J\), this term will give rise to a locally conserved \(z\)-component of spin, which we will enforce as.
a constraint on the low energy Hilbert space:

\[ \sum_{i \in \text{oct}} S_i^z = 0. \quad (2) \]

There are certainly other terms on the lattice that involve \( S^\pm \), but their specific forms are not important, as long as they are all dominated by the \( J \) term. Under the standard change of variables \( S^\pm \sim n - 1/2 \) and \( S^\pm \sim e^{\pm i\theta} \), this model becomes a boson rotor model on the links of a cubic lattice. Noting that the locally conserved integer \( S_{\text{oct}}^z \) generates a \( U(1) \) gauge symmetry, after we change variables again to \( E_{rr'} \sim (-1)^{n_{rr'}} \) and \( A_{rr'} \sim (-1)^{\theta_{rr'}} \). Note that \( A_{rr'} \) is only defined modulo \( 2\pi \).

The operators \( E \) and \( A \) are defined on the links of the lattice, and this endows them with a vector structure. We can thus identify a vector of operators \( \mathbf{E}(\vec{x}) \) and \( \mathbf{A}(\vec{x}) \) at each site of the cubic lattice, along with lattice derivatives \( \partial_i E_j(\vec{x}) = E_j(\vec{x} + \hat{i}) - E_j(\vec{x}) \). They satisfy the normal commutation relations \( [\mathbf{A}(\vec{x}), \mathbf{E}(\vec{y})] = i \delta_{jk} \delta^3(\vec{x} - \vec{y}) \).

When phrased in terms of these new variables, the low energy effective Hamiltonian is bound to take the following form:

\[ H = U \sum_r \mathbf{E}(r)^2 - K \sum_{\Box} \cos[\text{curl}(\mathbf{A})_{\Box}] \quad (3) \]

Once we project all the physics down to the low energy subspace of the Hilbert space that obeys the constraint imposed by the \( J \) term, the low energy effective Hamiltonian must have the gauge symmetry \( \mathbf{A} \to \mathbf{A} + \nabla f \), which is generated by the local constraint which can now be written as

\[ \partial_i E_i = 0. \quad (4) \]

Tentatively ignoring the fact that \( \mathbf{A} \) is compactly defined, we can expand the low energy effective Hamiltonian at the minimum of the cosine function (spin wave expansion):

\[ H = U \sum_r E_i^2 + \frac{K}{2} \sum_r (\epsilon_{ijk} \partial_j A_k)^2 \quad (5) \]

Equation (5) is the effective low energy Hamiltonian for \((3 + 1)d\) quantum electrodynamics (QED) in its deconfined phase. Solving the equation of motion of Eq. (5) from the Heisenberg equation directly, we will obtain a gapless photon excitation with linear dispersion relation \( \omega \sim c|\vec{k}| \), where the speed of light \( c \sim \sqrt{UK} \). However, we know that in \( 2+1d \), the compact QED suffers from the instanton effect: proliferation of magnetic monopoles in the space time opens up the photon gap, but that effect is only made clear in terms of the dual variables. Thus, we will also consider the dual theory to ascertain if there is a similar gap-opening effect.

We see that the solution of \( E_i \) to the local constraint Eq. (4) can be written as the curl of another vector field \( h_i, \ E_i = \epsilon_{ijk} \partial_j h_k \). This new field \( h_i \) is defined on each plaquette center and it is canonically conjugate to the magnetic field \( B_i \). We can now rewrite the Hamiltonian Eq. (5) as

\[ H = U \sum_r (\epsilon_{ijk} \partial_j h_k)^2 + \frac{K}{2} \sum_r B_i^2 \quad (6) \]

In contrast to the \( 2 + 1d \) case, this new Hamiltonian has the same form as the original Eq. (5), and formally \( h_i \) has the same gauge symmetry as \( A_i \):

\[ h_i \to h_i + \nabla_i f. \quad (7) \]

We thus say that the system (at least in the photon phase) is self-dual. This is an emergent feature in the infrared.

In the dual theory, we might expect relevant "vertex operators" \( \alpha \cos(2\pi Nh_i) \), whose analogue in \((2 + 1)d\) plays the role of the flux creation. In \((3 + 1)d\), this vertex operator corresponds to hopping of the magnetic monopole of the compact \( U(1) \) gauge field. Whether this vertex operator is important or not, can be determined by evaluating its correlation function in the limit where \( \alpha = 0 \). However, in \((3 + 1)d\), the correlation function between two such terms in the limit \( \alpha = 0 \) is

\[ \langle \cos(2\pi Nh_i(\vec{x})) \cos(2\pi Nh_j(\vec{y})) \rangle_0 \sim \delta_{ij} \delta^3(\vec{x} - \vec{y}), \quad (8) \]

because this is not a gauge invariant correlation function under gauge transformation Eq. (7). Thus at the Gaussian fixed point, the vertex operators \( \cos(2\pi Nh_i) \) are irrelevant - at least perturbatively. When the vertex operator is strong enough, it will induce magnetic monopole condensation and drive the system into the confined phase. Thus, the gapless photon is perturbatively protected by the gauge symmetry of both the original and the dual theory, i.e. the self-duality protects the stability of the photon phase.

Our review in this subsection is no more than restating the known fact that the \((3 + 1)d\) compact \( U(1) \) gauge field has a deconfined phase, which corresponds to the phase where neither the charge nor the magnetic monopole condenses. In this subsection, we identified the photon phase where the magnetic monopole is gapped as the phase where the vertex operator is irrelevant. The language and logic used in this subsection can be conveniently generalized to other ABL phases.

**B. The Rank-2 Case**

In this section we will review the construction of another ABL phase with rank-2 tensor gapless bosonic excitations that are analogous to gravitons. We will omit the exact details of the microscopic derivation; interested readers are referred to the original papers Ref. [11,13].

The \( 3 + 1d \) microscopics in this system give rise to a symmetric rank-2 tensor field, in contrast to the rank-1 tensor field in the previous example. Once again, the system can be simply phrased in terms of the boson number.
and its canonical conjugate phase variables $\theta_{ij}$. We can again define gauge field variables $E_{ij} \sim n_{ij}$ ($i \neq j$), $E_{ii} \sim 2n_{ii}$, and $A_{ij} \sim \theta_{ij}$, noting that $A_{ij}$ is compactly defined with modulo $2\pi$.

The low-energy subspace has the local constraint

$$\partial_i E_{ij} = 0,$$  \hspace{1cm} (9)

which is imposed by a large local term similar to Eq. [1]. This constraint generates the gauge transformation

$$A_{ij} \rightarrow A_{ij} + \frac{1}{2} (\partial_i \lambda_j + \partial_j \lambda_i)$$  \hspace{1cm} (10)

This gauge transformation is the same as that of linearized gravity, if we were to treat $A_{ij}$ as the fluctuation of a background metric $n_{ij}$. Hence, we term the gauge boson a “graviton”. The original works \[21-23\] use the language of general relativity to write the Hamiltonian in terms of the curvature tensor for $A_{ij}$. We will avoid that notation here while noting that it has very nice connections to the Lifshitz gravity proposed independently in Ref. [21-23].

We want to establish the simplest Hamiltonian possible that is gauge-invariant. To do this, we will again write down a gauge-invariant quantity $B_{ij}$ which should be thought of as the “2-curl” of $A_{ij}$:

$$B_{ij} = \epsilon_{iab} \epsilon_{jcd} \partial_a \partial_c A_{bd}$$  \hspace{1cm} (11)

The low energy effective Hamiltonian, or the Hamiltonian after the “spin-wave expansion”, then takes the simple form

$$H = U \sum_r E_{ij}^2 + K \sum_r B_{ij}^2,$$  \hspace{1cm} (12)

where $U$ and $K$ may, in general, take different values for the $\sum_i X_{ij}^2$ and $\sum_i X_{ii}^2$ terms, if an ordinary cubic lattice symmetry is assumed; however, lattice symmetry is not essential to our work here.

The spin-wave expanded Hamiltonian above already gave us a gapless “graviton-like” bosonic mode with a quadratic dispersion. In order to guarantee that this gapless mode is not ruined by the compactness of the gauge field, we must once again consider the dual theory. The dual variables solve the constraint equation (9), and we can write $E$ as the 2-curl of a new field $h$:

$$E_{ij} = \epsilon_{iab} \epsilon_{jcd} \partial_a \partial_c h_{bd}.$$  \hspace{1cm} (13)

We see that $h$ transforms under the same gauge transformation as tensor $A$ and is canonically conjugate to the tensor field $B$. The vertex operators will take the form $\cos(2\pi N h_{ij})$, and just as before the gauge-dependence makes them irrelevant at the infrared Gaussian fixed point because it violates the gauge symmetry of the Gaussian fixed point field theory. This will once again guarantee the gaplessness of the graviton mode, and we see from the above Hamiltonian that $\omega \sim k^2$.

C. Additional constraints

For $n \geq 2$, the rank-$n$ theories have additional structure because they can accommodate several types of local constraints. Interestingly, we can also enforce more than one local constraint simultaneously. For example, we can take the above theory and additionally require that

$$E_i = \sum_i E_{ii} = 0$$  \hspace{1cm} (14)

This generates the gauge transformation

$$A_{ij} \rightarrow A_{ij} + \delta_{ij} \lambda$$  \hspace{1cm} (15)

We now ask a modified question - what is the simplest theory that is invariant under the gauge transformations generated by constraints (9) and (14) simultaneously? We see that our definition of $B_{ij}$ in Eq. (11) is not good enough. However, we can use the quantity $B = \sum_i B_{ii}$ to define a new tensor:

$$Q_{ij} = \epsilon_{ikl} \left(B_{jl} - \frac{1}{2} \delta_{jl} B\right),$$  \hspace{1cm} (16)

which is invariant under both gauge transformations. The new effective low energy Hamiltonian is now (15):

$$H = U \sum_r E_{ij}^2 + K \sum_r Q_{ij}^2.$$  \hspace{1cm} (17)

The new dual fields are defined in the same way, where $E$ and $h$ have the same functional relation as $Q$ and $A$. Thus this theory is again “self-dual” with identical gauge symmetries on the two sides of the duality. This theory (Eq. [17]) is again gapless, though it has a different dispersion: because there are now three spatial derivatives of $A$ in the $Q$ tensor Eq. [16], the dispersion of the low energy excitation is $\omega \sim k^2$.

III. GENERAL PROCEDURE

To generalize these arguments to higher rank tensor fields, we need to first establish which types of gauge transformations will be allowed. To simplify our discussion, we want the field theory to be rotationally symmetric, though it is possible that the lattice regularization may possess irrelevant rotation-breaking terms. Additionally, the gauge constraint should depend only on $E_{ijk...}$ and no other locally defined tensor fields.

These two requirements restrict the constraints that we will consider to higher-dimensional versions of the Gauss law and traceless conditions. These constraints are “rotationally” symmetric in the correct way to respect lattice symmetries (again, we stress that the states we construct should be insensitive to weak lattice symmetry breaking).

We enumerate the allowed gauge transformations in Table 1 for rank one through three. To simplify notation, we denote the symmetrizing operation...
An important generic question is the number of gapless modes in the system. This is determined by switching to a Lagrangian formulation and thinking of the $\lambda$ tensor as a Lagrange multiplier. Each degree of freedom of $\lambda$ will reduce the number of gapless modes by one (though there is a subtlety to this counting, which is detailed in the appendix). For example, for the familiar photon phase,

$$T_{ijk} = \frac{1}{3!} (T_{ij} + T_{jk} + \text{sym})$$  (18)

$E_i$ has three components initially, so the one free component of a scalar $\lambda$ reduces the number of gapless modes to the familiar two of the photon. For higher rank cases, though it quickly becomes tedious to count the number of free components of an arbitrary rank symmetric tensor, the idea is straightforward. Indeed, it is also possible to diagonalize the Hamiltonian directly, and this reproduces the previous results.

The essential component of many ABL theories is the process by which gap-opening perturbations are prohibited. Generically, any relevant term in the Lagrangian should open a gap, and so to eliminate all such terms places strict requirements on the theory. In the theories we consider in this paper, we use gauge-invariance and self-duality to protect the photon gap from perturbations at a Gaussian IR fixed point, just like the examples reviewed in the previous section.

The gauge structure in all of the theories we consider is emergent in the IR. Indeed, it is due to a constraint on the low-energy Hilbert space of the microscopic model. This means that the gapless phase is not stable to arbitrarily strong perturbations, since moving out of the constrained subspace generically destroys the gauge structure. As an example, consider the gauge charge excitation in the rank-1 theory. The low-energy subspace is that of the charge vacuum, but if we tune the charge gap to zero, the gauge charges condense and gap out the gauge boson through the Higgs mechanism.

Additionally, the gauge structure will constrain the form of the Hamiltonian. As we have seen above, we want to use $A$ to construct two gauge-invariant tensors $E$ and $B$ which play the usual roles in electromagnetism. Given that there is a direct relation between the gauge transformations on $A$ and the constraints on $E$, it is a straightforward task to build the most relevant terms. In this case, “most relevant” means that $B$ has the fewest number of spatial derivatives of $A$, but it must be gauge invariant still.

Just to limit the variety of states, we require rotational invariance in this paper, which also constrains the form of the Hamiltonian (as does gauge invariance). But we want to stress that weakly breaking the rotational invariance will not destroy the states we construct, namely it will not gap out the bosonic modes of the ABL phase. For example, the low energy photon excitations of the ABL phase studied in [1210] have a rotational invariant dispersion at low energy, but we know that breaking the rotational invariance will not destroy the photon excitations. The local gauge constraints are similarly influenced by the requirement of rotational invariance, as was noted above. We can then consider tensor representations of rotational group $SO(3)$, and it turns out that we will only be interested in the symmetric pieces.

For example, to construct the gauge invariant rank-3 magnetic field $Q_{ijk}$ with both a derivative constraint and a trace constraint on $E_{ijk}$, the resulting theories will involve $B_{ijk}$ which is a 3-curl of $A$ and $B_{i} = B_{i}$. Because $Q_{ijk}$ carries three vector indices, it can be constructed with three vector representation of $SO(3)$. The standard expansion of a tensor defined over three copies of the fundamental representation of $SO(3)$ is

$$1 \otimes 1 \otimes 1 = 3 \oplus 2 \oplus 1 \oplus 1 \oplus 1 \oplus 0$$

The spin-2 and spin-0 pieces here are antisymmetric in at least two indices. Requiring overall symmetrization will reduce the expansion to a fully symmetric spin-3 part $T_{ijkl}$ and a symmetric spin-1 part $T_{ijk}^\prime = \delta_{ij} T_{ik}$. Thus, we can understand connection between the allowed constraints and how they will involve traces of the curls of $A$ by considering which parts of the tensor representation are symmetric.

However, as was noted previously, gauge structure is not enough to guarantee the gaplessness of the photon. In 2+1d this manifests as the so-called “instanton effect”, which is to say that the magnetic flux insertion operator is always relevant at the Gaussian fixed point. Thus, the instantons proliferate and open a gap for the photons. Thus in general in our $(3+1)d$ ABLs, we need to argue that all of the vertex operators that generically take the form $\cos(2\pi N h_{\alpha \beta \gamma \dots})$ for the dual gauge field $h$ are irrelevant.

| Rank of theory | Local constraint | Gauge transformation |
|---------------|-----------------|---------------------|
| $n = 1$       | $\partial_i E_i = 0$ | $A_i \rightarrow A_i + \partial_i \lambda$ |
| $n = 2$       | $\partial_i E_{ij} = 0$ | $A_{ij} \rightarrow A_{ij} + \partial_i \lambda_{ij}$ |
|               | $E_{ij} = 0$     | $A_{ij} \rightarrow A_{ij} + \partial_j \lambda_{ij}$ |
| $n = 3$       | $\partial_i E_{ijk} = 0$ | $A_{ijk} \rightarrow A_{ijk} + \partial_i \lambda_{ijk}$* |
|               | $E_{ijk} = 0$    | $A_{ijk} \rightarrow A_{ijk} + \partial_j \lambda_{ijk}$ |

* These gauge transformations are not totally independent - $\lambda_{jk}$ should be made traceless.

TABLE I: Allowed gauge transformations which are rotationally invariant and do not depend on an auxiliary tensor field.
IV. EXAMPLES

In this section we will discuss a few examples of new ABL phases. The first example is similar to the graviton theories detailed in the previous section, except that it has a different local constraint. This is an interesting property of rank-$n$ theories for $n \geq 2$ which greatly enhances the variety of gapless gauge theories. There are roughly $n$ different constraints involving only derivatives for a given rank-$n$ theory in addition to the various types of traceless conditions.

The original graviton model had as its local constraint Eq. [9] We can instead contract another derivative on $E_{ij}$ to get a different theory:

$$\partial_i \partial_j E_{ij} = 0 \quad (20)$$

Compared to the theory governed by Eq. [9] this theory has a scalar (as opposed to vector) charge and has five total degrees of freedom (up from three). Even before determining the simplest possible Hamiltonian, we see that the gapless excitations are distinct in character from the original gravitons:

$$A_{ij} \to A_{ij} + \partial_i \partial_j \lambda \quad (21)$$

We can construct the Hamiltonian of this ABL state using the following symmetrized gauge invariant tensor field $B$:

$$B_{ij} = \frac{1}{2} (\epsilon_{iab} \partial_a A_{kj} + \epsilon_{jcd} \partial_c A_{ai}) \quad (22)$$

The corresponding low energy Hamiltonian again takes the schematic form of $E^2 + B^2$, as before, and this theory is again self-dual, but it now has a linear dispersion $\omega \sim k$.

We can also consider enforcing the constraint Eq. [14] in addition to Eq. [20]. However, in this case, $B_{ij}$ given by Eq. [22] is already invariant under both gauge transformations. In fact, in conjunction with the graviton theory discussed previously, we have now characterized all rank-2 symmetric gauge theories whose gauge transformations satisfy our criteria above.

While the rank-1 and rank-2 systems have nice interpretations as “photons” and “gravitons” due to the familiarity with known systems, there is no such nice identification for the rank-3 case. We cannot leverage any analogy to linearized gravity nor electromagnetism, and instead we will proceed using our general method.

To illustrate this case, we consider two canonically conjugate symmetric rank-3 tensor fields $A_{ijk}$ and $E_{ijk}$ where $A$ is defined modulo $2\pi$. We then impose the local constraint

$$\partial_i E_{ijk} = 0 \quad (23)$$

which generates the gauge transformation

$$A_{ijk} \to A_{ijk} + \partial_i \lambda_{jk} \quad (24)$$

The corresponding lattice system is given in the appendix. As before, we seek a “magnetic” field that is gauge-invariant and of lowest number of derivatives of $A$. Additionally, it should be symmetric. We see that

$$B_{ijk} = \epsilon_{iab} \epsilon_{jcd} \epsilon_{kef} \partial_a \partial_c \partial_e A_{bdf} \quad (25)$$

is the simplest tensor that fits the requirements. From this tensor we can construct a state with the following low energy effective Hamiltonian

$$H = U \sum_r E^2 + K \sum_r B^2, \quad (26)$$

where the coefficients of the $\sum X^2_{ij}$, $\sum X^2_{ij}$, and $\sum X^2_{ijkl}$ terms may in general be different. This system is self-dual in the same way as before, by defining the dual variable $h_{ijk}$ as

$$E_{ijk} = \epsilon_{iab} \epsilon_{jcd} \epsilon_{kef} \partial_a \partial_c \partial_e h_{bdf} \quad (27)$$

and requiring that $h_{ijk}$ transform in the same way as $A_{ijk}$ under a change of gauge. The vertex operators of the dual variables $\cos (2\pi Nh_{ijk})$ are easily seen to be gauge dependent, and thus irrelevant in the same way as before. This is a new gapless Bose liquid with $\omega \sim k^3$ with four independent modes for each momentum $k$.

We can then ask what happens when another local constraint is imposed.

$$\delta_{ij} E_{ijk} = 0. \quad (28)$$

This constraint gives rise to the gauge transformation

$$A_{ijk} \to A_{ijk} + \delta(ij) \lambda_{l}, \quad (29)$$

which provides a nice example of the “mode overcounting” discussed in the appendix. In particular, the above constraint gives rise to new physics only when the 1-form field $\lambda_k$ is not exact, i.e. $\lambda_k \neq \partial_i \Gamma$. If $\lambda_k$ is a total derivative of some scalar function, then this constraint Eq. [28] is not independent of the transformation Eq. [24] and the system as described by $B_{ijk}$ given before in Eq. [25] is invariant under both.

If $\lambda_k \neq \partial_i \Gamma$, then we have to construct a new “magnetic field” that is invariant under both gauge transformations. To do so, we need to define two quantities:

$$D_{ij} = \delta_{ij} \partial^2 - \partial_i \partial_j \quad (30)$$

$$B_k = B_{iik} \quad (31)$$

Using these quantities, the new low energy effective Hamiltonian is schematically $E^2 + Q^2$ where we have defined

$$Q_{ijk} = \partial^2 B_{ijk} - \frac{3}{4} D_{(ij} B_{k)}. \quad (32)$$

This theory has a rather soft dispersion $\omega \sim k^5$, and only one single mode at each momentum $k$. And just like all the examples before, this theory is also self-dual.
Continuing this procedure to higher rank theories generates an entire infinite family of ABL phases. The procedure is exactly the same, though the precise enumeration of possible gauge transformations (and, indeed, even the number of degrees of freedom) becomes tedious quickly. However, by leveraging the gauge structure in addition to the self-duality at the IR fixed point, we are able to in all cases derive the appropriate low-energy effective Hamiltonian for a given local constraint.

V. TOPOLOGICAL ORDER

The $U(1)$ spin liquid in $3 + 1d$, in addition to its stability, also possesses a curious type of topological order, which was discussed in detail in Ref. 1. When the system is put on a three dimensional torus $T^3$ with size $L^3$, it is possible to thread electric flux around each of the noncontractible loops. The flux integrals each commute with the low-energy Hamiltonian and each other, so they constitute constants of motion. When the flux spreads out over the whole thermodynamically large system, the energy cost goes to zero as $1/L$. An identical picture holds for the magnetic flux, so topological order is characterized by six integers. The system is stable due to the gap in both electric and magnetic charges, which makes it exponentially unlikely for a “particle-hole” pair to be created and propagate all the way around the torus to change the flux.

There is a similar construction for the graviton ABL discussed in Ref. 11. However, one must be more careful in the selection of which fluxes of $E_{ij}$ are used. In principle, there are twenty seven different fluxes - three orientations of the flux surface and nine components of $E_{ij}$. Upon calculation, one can show that fifteen of these are zero, and of the remaining twelve only nine are independent. The same result holds for the magnetic fluxes, meaning that the graviton ABL has topological order characterized by eighteen integers. It is similar to the $U(1)$ case in that the ground states are split by $1/L$ and are exponentially unlikely to mix.

We claim that similar arguments hold for the whole infinite family of theories constructed in the previous sections with only derivative constraints. The topological order is characterized by $6k$ integers corresponding to the electric and magnetic fluxes, where $k$ the number of independent components of the charge tensor. Since the Hamiltonian densities for these theories are generically $E^2 + B^2$, we expect that in all cases the ground state degeneracy closes as $1/L$ in the thermodynamic limit.

To understand the origin of the $6k$, we consider a generic local constraint written as

$$\left(\partial_i E_{ijk...} - \tilde{\rho}_{ijk...}\right) |\text{Phys}\rangle = 0 \quad (33)$$

It is natural to interpret the violations of the local constraints as “charges.” The particular choice of constraint endows the charges with a tensor structure, and the underlying symmetry of $E_{ijk...}$ is reflected in that structure.

Going back to the lattice model, these charges can also be thought of as the open ends of strings. The constraint is then interpreted as the condition that strings do not end on sites. Due to the all-important electromagnetic duality protecting these phases, there is a corresponding magnetic charge tensor with exactly the same structure as the electric charge.

Using these charge tensors, we can then create “particle-hole” pairs of a given type of charge and wind them around a noncontractible loop of $T^3$. Three dimensions times two species of charge gives the factor of six, and there are $k$ independent charges depending on the particular constraint. We see that $k = 1$ for the ordinary QED, while $k = 3$ for the graviton ABL which has a vector charge.

To extend these ideas to constraints with more derivatives, we see that the constraint $\partial_i \partial_j E_{ij} = 0$ can be rewritten as $\partial_i F_i = 0$ for a vector field $F_i = \partial_j E_{ij}$. This constraint has a scalar charge, and it can be shown easily that the fluxes of $F_i$ commute with each other and the Hamiltonian. This extra step is needed to invoke the divergence theorem, since we need to work with the divergence of a vector field. Importantly, the characterization is still the same - since the underlying charge is a scalar, this theory is characterized by six winding numbers.

Finally, we consider the second type of constraint detailed above, such as $\delta_{ij} E_{ij} = 0$. In the rank-2 case, we imagine threading a flux of $E_{xx}$ around the noncontractible loop in the $x$-direction while simultaneously threading a flux of $E_{yy}$ in the $y$-direction. The two “strings” involved in this threading process need not intersect, but in the ground state the fluxes spread out over the whole system. Once this occurs, we see that the traceless constraint fixes the flux of $E_{zz}$ through the $z$-direction so that the three integers sum to zero. This new phase is characterized by 16 integers. Extending these constraints to higher-rank theories is straightforward but tedious, and simply removes topological degrees of freedom from the diagonal fluxes.

VI. SUMMARY AND DISCUSSION

In this work we have demonstrated that there is an infinite family of strongly-correlated gapless boson systems whose low-energy Hilbert space does not break any symmetries with gapless excitations stable with respect to small arbitrary perturbations. The gaplessness is protected by a combination of emergent gauge invariance (enforced by a local constraint on the low-energy Hilbert space) and a generalized electromagnetic duality. Within some limitations, the dispersion and representation of the emergent gauge boson can be tuned. Additionally, these theories have an interesting type of topological order characterized by $6k$ integers, depending on the exact underlying local constraint.

Although we have made heavy use of the gauge structure in constructing these ABL phases, we have not made
a careful analysis of the associated gauge groups. Apart from the simplest $U(1)$ spin liquid, the higher-rank gauge fields are not algebra-valued 1-forms, and therefore do not fit into the standard Yang-Mills architecture.

Our rank-2 model with constraint $\partial \bar{E}_{ij} = 0$ can potentially be thought of as $U(1) \times U(1) \times U(1)$ gauge theory after an (unusual) symmetrization between the space index and the flavor index; this provides a possible realization of our rank-2 theory starting with three copies of the photon phase. We also note that this connection between linearized gravity and the Yang-Mills gauge theory was already observed in the loop quantum gravity literature.[21,22] Further study of these models will hopefully elucidate these connections.

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Appendix A - Rank-3 Lattice Hamiltonian

For concreteness, we will present a lattice Hamiltonian for one of the rank-3 cases. In line with the previous work by Xu, this Hamiltonian has two pieces: a generic boson hopping term and a density-density repulsion term.

The unit cell for this lattice consists of a face-centered cubic lattice that also has a site at the center (see Figure 1). The boson occupation at the corners of the fcc unit cell are three-fold degenerate, labeled $n_{xxx}$, $n_{yyy}$, and $n_{zzz}$. The faces are two-fold degenerate with labels $n_{xyx}$, $n_{xzy}$, $n_{yxz}$, $n_{zxy}$, $n_{zyx}$, and so on, and the center is labeled $n_{xyz}$.

The hopping term of the Hamiltonian $H = H_v + H_o$ is generic, and in principle contains all 45 exchanges. The potential takes the form for average boson density $\bar{n}$

$$H_v = H_{xx} + H_{yy} + H_{zz} + H_{xy} + H_{yz} + H_{xz}$$  \hspace{1cm} (34)$$

$$H_{xx} = V(n_{xxx}, \bar{r} + \hat{z}/2 + \hat{y}/2 + \bar{n})^2$$
$$H_{xy} = V(n_{xyx}, \bar{r} + \hat{y}/2 + \bar{n})^2$$
$$H_{yz} = V(n_{zyx}, \bar{r} + \hat{x}/2 + \bar{n})^2$$
$$H_{xz} = V(n_{xzx}, \bar{r} + \hat{z}/2 - \bar{n})^2$$

with similar expressions for the other four terms. We define $E_{ijk} = (-1)^r(n_{ijk} - \bar{n})$ and use the usual lattice derivative to see that the low-energy subspace of this Hamiltonian has the local constraint $\partial_i E_{ijk} = 0$.

Appendix B - Mode Overcounting

There is a subtle point that needs to be addressed about the definitions of the above gauge transformations for rank greater than or equal to three. In particular, the two gauge transformations below are not totally independent:

$$A_{ijk} \rightarrow A_{ijk} + \partial_i (\lambda_{ijk})$$ \hspace{1cm} (37)$$
$$A_{ijk} \rightarrow A_{ijk} + \delta_{ij} \partial_k \lambda$$ \hspace{1cm} (38)$$

The first contains the second as a special case. This is understood in terms of tensor representations of $SO(3)$ by noting that a symmetric rank-2 tensor (six degrees of freedom) has a single scalar trace mode in addition to the five spin-2 modes. As such, a more correct accounting would require tracelessness of $\lambda_{ij}$, which is achieved via

$$\tilde{\lambda}_{ij} = \lambda_{ij} - \frac{1}{3} \delta_{ij} \lambda_{kk}$$ \hspace{1cm} (39)$$

This type of overcounting of trace modes persists into higher rank, and becomes increasingly complicated as the number of trace modes increases.

FIG. 1: The unit cell for the simplest rank-3 model. The red site is three-fold degenerate ($n_{xxx}$), the green sites are two-fold degenerate ($n_{xyx}$) and the blue site is nondegenerate ($n_{xyz}$).
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