NEUTRINO PHYSICS AND LEPTONIC WEAK BASIS INVARIANTS

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Abstract. In this talk we present a powerful tool applied to the study of Leptonic Physics. This tool is based on the construction of Weak Basis invariant relations associated to different properties of leptonic models. The rationale behind these constructions is the fact that fermion mass matrices related through weak basis transformations look different but lead to the same physics. Such invariants can be built, for instance, with the aim to test leptonic models for different types of CP violation. These invariants are also relevant beyond such tests and have been applied to the study of implications from zero textures appearing in the leptonic mass matrices. In this case an important question is, how can a flavour model corresponding to a set of texture zeros be recognised, when written in a different weak basis, where the zeros are not explicitly present. Another important application is the construction of invariants sensitive to the neutrino mass ordering and the $\theta_{23}$ octant.

1 Introduction

One of the major puzzles in Particle Physics is the origin of fermion masses, mixing and CP violation, the so-called Flavour Puzzle. It is by now established that at least two of the three active light neutrinos have non-zero masses. In the Standard Model neutrinos are strictly massless. Accounting for neutrino masses requires physics beyond the Standard Model and has profound phenomenological implications. Neutrino Physics is at present an important field of research with many different experiments taking data and future new facilities and upgrades being planned. Among the fundamental open questions in this field are whether or not neutrinos are Dirac or Majorana particles, whether or not there is CP violation in the leptonic sector, what is the absolute neutrino mass scale and what is the mass ordering, meaning, what is the sign of $m_3^2 - m_1^2$? Is the leptonic mixing matrix, $U_{PMNS}$ unitary? Are there sterile neutrinos? Do neutrinos have nonstandard interactions?

Attempts at solving the Flavour Puzzle are often based on the use of symmetries or else of special textures for the mass matrices. The fact that fermion mass matrices related through weak basis transformations look different while leading to the same physics raises the question of how to recognise the same model written in different bases where the symmetry (or the special texture) is not apparent. A special technique based on the use of weak basis (WB) invariants has been developed to tackle this problem. In this talk special WB

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Invited talk given at 18th Lomonosov Conference on Elementary Particle Physics (Moscow, Russia, August 24-30, 2017)
invariant conditions adapted to answer several different specific questions concerning leptonic physics are presented. The same technique has been applied to the quark sector. The use of Higgs basis invariants based on the same rationale are extremely useful in the study of the scalar potential for multi-Higgs models.

2 Building of weak basis invariant conditions

The technique employed in this talk was developed for the first time by the authors of Ref. [1] applied to the study of the CP properties of the quark sector in the Standard Model. After spontaneous symmetry breaking quark masses are generated and can be written as:

\[ \mathcal{L}_m(\text{quarks}) = -\bar{u}_L m_u u_R - \bar{d}_L m_d d_R^* + \text{h.c.} \]  

still in a weak basis where, by definition of weak basis, the charged current is diagonal. The charged current has the form:

\[ \mathcal{L}_W(\text{quarks}) = -\frac{g}{\sqrt{2}} W_\mu^+ \bar{u}_L \gamma^\mu d_R + \text{h.c.} \]  

Weak basis transformations are given by:

\[ \begin{align*}
  d_0^L &\rightarrow U'd_L^L, & \bar{u}_0^L &\rightarrow U'\bar{u}_L^L, & d_0^R &\rightarrow W'd_R^0, & \bar{u}_0^R &\rightarrow V'\bar{u}_R^0 \\
\end{align*} \]  

with \( U', V', W' \) arbitrary unitary matrices, whereas the most general CP transformation for fermion fields in a weak basis allows for the combination of the definition of the CP transformation of a single fermion with a weak basis transformation [2], this is so because the fermionic CP transformation must be defined taking into account the part of the Lagrangian that conserves CP, to wit, the fermion gauge couplings. Pure gauge theories together with fermions do not violate CP [3]. The most general CP transformation allowed by these couplings is then:

\[ \begin{align*}
  \text{CP}\bar{u}_L^0(\text{CP})^T &= U'\gamma^0 C \bar{u}_L^T; & \text{CP}\bar{d}_L^0(\text{CP})^T &= V'\gamma^0 C \bar{d}_L^T \\
  \text{CP}\bar{u}_R^0(\text{CP})^T &= U'\gamma^0 C \bar{u}_R^T; & \text{CP}\bar{d}_R^0(\text{CP})^T &= V'\gamma^0 C \bar{d}_R^T \\
\end{align*} \]  

In order for the Lagrangian to be CP invariant the following relations have to be verified:

\[ \begin{align*}
  U'^\dagger m_u V' &= m_u^* \\
  U'^\dagger m_d W' &= m_d^* \\
\end{align*} \]  

this means that there must be unitary matrices \( U', V', W' \) that obey these relations. For real mass matrices these relations are trivially satisfied with
identity matrices. Looking for such unitary matrices in more general cases is not simple, however by combining these relations in such a way as to produce similarity transformations it is possible to obtain simple conditions expressed only in terms of the mass matrices by applying traces and determinants. In this way one may obtain a simple condition \([1]\):

\[
\text{tr} [H_u, H_d]^3 = 0
\]

which is a necessary and sufficient condition for CP conservation in the SM. For three generations this condition is equivalent to

\[
\text{det} [H_u, H_d] = 0
\]

which was obtained in Ref. [4] for the particular weak basis where the quark mass matrices are Hermitian. However this choice of a particular basis is not necessary since both conditions are weak basis invariant.

Ref. [1] was the starting point for the development of an extremely powerful technique to test for CP violation in many different scenarios.

### 2.1 Testing for CP violation, leptonic sector, low energies

At low energies, assuming that the lepton number is violated, one can write an effective Majorana neutrino mass matrix and the leptonic mass terms are then of the form:

\[
\mathcal{L}_{\text{mass}} = -\frac{1}{2} \nu^T \nu C^{-1} m_\nu \nu^0 - \ell^T m_\ell \ell^0_R + \text{h.c.}
\]

the charged currents have a form similar to the one of Eq. (2) written in terms of leptons. The WB transformations in the leptonic sector are given by:

\[
\nu^0_L \rightarrow V \nu^0_L, \quad \ell^0_L \rightarrow V \ell^0_L, \quad \ell^0_R \rightarrow W \ell^0_R
\]

with \(V\) and \(W\) unitary \(3 \times 3\) matrices. Leptonic mixing and CP violation in the leptonic sector is described by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS), \(U_{PMNS}\) matrix. Using the standard parametrisation [5] this matrix can be written:

\[
U_{PMNS} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & s_{23}c_{13}
\end{pmatrix} \cdot P
\]

\[
P = \text{diag} (1, e^{i\alpha_{21}}, e^{i\alpha_{31}})
\]

where \(c_{ij} \equiv \cos \theta_{ij}\) and \(s_{ij} \equiv \sin \theta_{ij}\). There are three CP violating phases, \(\delta, \alpha_{21},\) and \(\alpha_{31}\) in Eq. (11). In Ref. [6] a set of necessary and sufficient WB
Invariant conditions for CP invariance were derived, valid in the case of three generations for nonzero and nondegenerate masses:

\[
\text{Im} \text{ Tr } [h_\ell \cdot m_\nu^* \cdot m_\nu \cdot h_\ell^* \cdot m_\nu] = 0 \tag{12}
\]

\[
\text{Im} \text{ Tr } [h_\ell \cdot (m_\nu^* \cdot m_\nu)^2 \cdot (m_\nu^* \cdot h_\ell^* \cdot m_\nu)] = 0 \tag{13}
\]

\[
\text{Im} \text{ Tr } [h_\ell \cdot (m_\nu^* \cdot m_\nu)^2 \cdot (m_\nu^* \cdot h_\ell^* \cdot m_\nu) (m_\nu^* \cdot m_\nu)] = 0 \tag{14}
\]

\[
\text{Im} \text{ Tr } [(m_\nu \cdot h_\ell \cdot m_\nu^*) + (h_\ell^* \cdot m_\nu \cdot m_\nu^*)] = 0 \tag{15}
\]

In Ref. [7], a minimal set of necessary and sufficient conditions for CP invariance was given:

\[
\text{Tr } [m_\nu^* \cdot m_\nu^T \cdot h_\ell] = 0 \tag{16}
\]

\[
\text{Tr } [m_\nu \cdot h_\ell \cdot m_\nu^* \cdot h_\ell^*] = 0 \tag{17}
\]

\[
\text{ImTr } (h_\ell \cdot m_\nu^* \cdot m_\nu \cdot h_\ell^* \cdot m_\nu) = 0 \tag{18}
\]

Eq. (16) is similar to Eq. (7), which was derived for the quark sector; this invariant is only sensitive to the Dirac-type phase, $\delta$. The other two invariants are sensitive both to Dirac and Majorana type phases. The invariant of Eq. (17) was first derived in Ref [8] applied to the study of CP violation in the context of three degenerate neutrinos which still allows for Majorana-type CP violation. The invariant of Eq. (18) coincides with Eq. (12) and was derived before in Ref. [6], applied to the case of two generations, which also allows for Majorana-type CP violation.

### 2.2 Testing for CP violation, leptonic sector, Leptogenesis

In the minimal seesaw framework [9,10,11,12,13], one introduces righthand neutrino fields which are singlets of $SU(2) \times U(1)$. The most general leptonic mass term may then be written as:

\[
\mathcal{L}_m = -[\nu_R^0 \cdot M_D \nu_R^0] + \frac{1}{2} \nu_R^0 \cdot C M_R \nu_R^0 + \overline{\nu_L^0 \cdot m_L^0 \nu_R^0}] + h.c. \tag{19}
\]

Let us assume that three righthanded neutrinos are introduced, this does not need to be the case, one needs at least two. The scale of $M_R$ can be much higher than the electroweak scale and in this case the seesaw mechanism operates. In the context of seesaw, it is possible to generate a lepton number asymmetry through the decays of the heavy Majorana neutrinos. This is the so-called Leptogenesis mechanism [14,15,16] and requires CP violation at high energies. In the general seesaw framework it is not possible to establish a connection between leptonic CP violation at low energies and CP violation at high energies [17,18]. Such a relation can only be established in the context of a flavour theory.
For Leptogenesis in the single flavour approximation, i.e., in the case when washout effects are not sensitive to the different flavours of charged leptons into which the heavy neutrino decays, the possibility of having Leptogenesis can be probed by means of the following invariant conditions [17]:

\[ I_1 \equiv \text{ImTr}[m_D m_D^\dagger M_R^\dagger M_R^T m_D^\dagger M_R] = 0 \]  
\[ I_2 \equiv \text{ImTr}[m_D m_D^\dagger (M_R^\dagger M_R)^2 M_R^T m_D^\dagger M_R] = 0 \]  
\[ I_3 \equiv \text{ImTr}[m_D m_D^\dagger (M_R^\dagger M_R)^2 M_R^T m_D^\dagger M_R M_R^\dagger M_R] = 0 \]

The first one of these invariants was discussed in Ref. [19]. For a detailed discussion of these invariant conditions see Ref. [17]. In the case of flavoured Leptogenesis additional CP-odd weak basis invariant conditions are required. A simple choice of additional invariant conditions is obtained by replacing \( m_D \) in the above equations by \( m_D h_l \), where \( h_l = m_l m_l^\dagger \) [17].

2.3 Beyond tests for CP violation. Texture zeros

The imposition of texture zeros in the Yukawa couplings allows to establish a connection between low energy physics and physics at high energies, for instance Leptogenesis. In Ref. [20] we addressed the question of how to recognise flavour models corresponding to a set of texture zeros when written in a different weak basis where the zeros are not explicitly present. We considered texture zeros in \( m_D \) in the seesaw framework, appearing in the weak basis where \( M_R \) and \( m_l \) are diagonal and real and we found invariants that vanish for different classes of textures. One such example is:

\[ I = \text{Tr}[m_D M_R^\dagger M_R m_D^\dagger h_l] \]

Implications for two texture zeros in \( m_D \) in the case of two righthanded neutrinos were studied in [21]. In Ref [22] we classified all allowed four zero textures in \( m_D \) with three righthanded neutrinos and we showed that in general CP may be violated both at low and high energies. Furthermore, in all these cases the parameters relevant for Leptogenesis can be fully specified in terms of light and heavy neutrino masses and low energy leptonic mixing.

2.4 Beyond tests for CP violation. The octant of \( \theta_{23} \) together with the neutrino mass ordering

In Ref. [23] we have built weak basis invariants that provide a clear indication of whether a particular lepton flavour model leads to normal or inverted hierarchy for neutrino masses and what is the octant of \( \theta_{23} \), of the standard parametrisation of \( U_{PMNS} \). It was shown in Ref. [23] that the sign of the invariant:

\[ \tilde{I}_1 \equiv \text{Tr}[H_\ell H_\nu] - \frac{1}{3} \text{Tr}[H_\ell] \text{Tr}[H_\nu] \]

(24)
indicates the ordering of the neutrino masses and that the invariant:
\[ \tilde{I}_2 \equiv Tr[H_1^\dagger] Tr[H_2^2 H_\nu] - Tr[H_2^2] Tr[H_1 H_\nu] \] (25)
is sensitive to the \( \theta_{23} \) octant. In Table 1 we show how to combine the information provided by these two invariants. In Ref. [23] these invariants were applied to

Table 1: Combination of the two invariants. NO stands for normal ordering, IO for inverted ordering

| \( \tilde{I}_2 \) | \( \tilde{I}_1 \) | NO | \( \theta_{23} \) | IO | \( \theta_{23} \) |
|----------------|----------------|----|----------------|----|----------------|
| > 0            | > 0            | NO | \( \theta_{23} \leq \pi/4 \) | NO | \( \theta_{23} > \pi/4 \) |
| < 0            | < 0            | IO | \( \theta_{23} > \pi/4 \) | IO | \( \theta_{23} < \pi/4 \) |

specific Ansätze studied in the literature [24]. For a different strategy see [25].

3 Conclusions

This talk presents a very powerful tool based on the derivation of weak basis invariant conditions that are extremely useful for model building. Such conditions have been widely used and derived in the literature in different contexts by many different authors. Ref. [26] provides a long list of references for conditions relevant to the study of CP violation both in the quark and in the leptonic sector, as well as in the Higgs sector, in several extensions of the Standard Model.

Acknowledgments

The author thanks the Organisers of the 18th Lomonosov Conference for the very fruitful scientific meeting and the warm hospitality. This work was partially supported by Fundação para a Ciência e a Tecnologia (FCT, Portugal) through the projects CERN/FIS-NUC/0010/2015, and CFTP-FCT Unit 777 (UID/FIS/00777/2013) which are partially funded through POCTI (FEDER), COMPETE, QREN and EU.

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