CENTRAL THEOREMS FOR COHOMOLOGIES OF CERTAIN SOLVABLE GROUPS

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Abstract. We show that the group cohomology of torsion-free virtually polycyclic groups and the continuous cohomology of simply connected solvable Lie groups can be computed by the rational cohomology of algebraic groups. Our results are generalizations of certain results on the cohomology of solvmanifolds and infra-solvmanifolds. Moreover as an application of our results, we give a new proof of the surprising cohomology vanishing theorem given by Dekimpe-Igodt.

1. Main theorems

For a set $M$ and a group $G$, we suppose that $G$ acts on $M$. Then we denote by $M^G$ the set of the $G$-invariant elements. Let $k$ be a subfield of $\mathbb{C}$. A $k$-defined algebraic group $G$ is a Zariski-closed subgroup of $GL_n(\mathbb{C})$ which is defined by polynomials with coefficients in $k$. For a $k$-defined algebraic group $G$, we denote by $U(G)$ the unipotent radical of $G$ and we denote by $G(k)$ the group of $k$-points. In this paper, a $k$-vector space $V$ is a vector space over $\mathbb{C}$ with a $k$-structure $V_k$ as in [8]. Let $V$ be a $k$-vector space. We consider the following cohomologies.

1. For a group $\Gamma$, assuming that $V_k$ is a $\Gamma$-module, we define the group cohomology $H^*(\Gamma, V_k) = \text{Ext}^*_\Gamma(k, V_k)$ as in [6].

2. For a connected Lie group $G$, assuming that $k = \mathbb{R}$ or $\mathbb{C}$ and $V_k$ is a topological $G$-module, we define the continuous cohomology $H^c_c(G, V_k) = \text{Ext}^*_G(k, V_k)$ as in [9, Section IX].

3. For a $k$-defined algebraic group $G$, assuming that $V$ is a $k$-rational $G$-module, we define the rational cohomology $H^*_r(G, V_k) = \text{Ext}^*_G(k, V_k)$ as in [14] and [19].

We suppose that we have an inclusion $\Gamma \subset G(k)$ (resp. $G \subset G(k)$) as a subgroup (resp. Lie-subgroup). Let $V$ be a $k$-rational $G$-module. Then the inclusion induces the homomorphism $H^*_r(G, V_k) \to H^*(\Gamma, V_k)$ (resp. $H^*_r(G, V_k) \to H^*_c(G, V_k)$).

A group $\Gamma$ is polycyclic if it admits a sequence

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}$$

of subgroups such that each $\Gamma_i$ is normal in $\Gamma_{i-1}$ and $\Gamma_{i-1}/\Gamma_i$ is cyclic. We denote $\text{rank} \Gamma = \sum_{i=1}^k \text{rank} \Gamma_{i-1}/\Gamma_i$.

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In this paper we study the group cohomology \( H^*(\Gamma, V_k) \) of a torsion-free virtually polycyclic group \( \Gamma \), the continuous cohomology \( H^*_c(G, \mathbb{C}) \) of a simply connected solvable Lie group \( G \) and the rational cohomology \( H^*_r(\mathcal{G}, V_k) \) of an algebraic group \( \mathcal{G} \) which contains \( \Gamma \) or \( G \) as a Zariski-dense subgroup. We have:

**Lemma 1.1 (\cite{27} Lemma 4.36).** Let \( \Gamma \) (resp. \( G \)) be a torsion-free virtually polycyclic group (simply connected solvable Lie group). For a finite-dimensional representation \( \rho : \Gamma \to GL_n(\mathbb{C}) \) (resp. \( \rho : G \to GL_n(\mathbb{C}) \)), denoting by \( \mathcal{G} \) the Zariski-closure of \( \rho(\Gamma) \) (resp. \( \rho(G) \)) in \( GL_n(\mathbb{C}) \), we have \( \dim U(\mathcal{G}) \leq \text{rank} \Gamma \) (resp. \( \dim U(\mathcal{G}) \leq \dim G \)), where we denote by \( U(\mathcal{G}) \) the unipotent radical of \( \mathcal{G} \).

By this lemma, we consider the following definition.

**Definition 1.2.** Let \( \Gamma \) (resp. \( G \)) be a torsion-free virtually polycyclic group (resp. simply connected solvable Lie group). For an algebraic group \( \mathcal{G} \), a representation \( \rho : \Gamma \to \mathcal{G} \) (resp. \( \rho : G \to \mathcal{G} \)) is called full if the image \( \rho(\Gamma) \) (resp. \( \rho(G) \)) is Zariski-dense in \( \mathcal{G} \) and we have \( \dim U(\mathcal{G}) = \text{rank} \Gamma \) (resp. \( \dim U(\mathcal{G}) = \dim G \)).

In this paper, we prove the following theorem.

**Theorem 1.3.** Let \( \Gamma \) be a torsion-free virtually polycyclic group. We suppose that for a subfield \( k \subset \mathbb{C} \) we have a \( k \)-defined algebraic group \( \mathcal{G} \) and an inclusion \( \Gamma \subset \mathcal{G}(k) \) which is a full representation. Let \( V \) be a \( k \)-rational \( \mathcal{G} \)-module. Then the inclusion \( \Gamma \subset \mathcal{G}(k) \) induces a cohomology isomorphism

\[
H^*_r(\mathcal{G}, V_k) \cong H^*(\Gamma, V_k).
\]

We also prove the following theorem.

**Theorem 1.4.** Let \( G \) be a simply connected solvable Lie group. We suppose that for a field \( k = \mathbb{R} \) or \( \mathbb{C} \) we have a \( k \)-defined algebraic group \( \mathcal{G} \) and an inclusion \( G \subset \mathcal{G}(k) \) which is a full representation. Let \( V \) be a rational \( \mathcal{G} \)-module. Then the inclusion \( G \subset \mathcal{G}(k) \) induces a cohomology isomorphism

\[
H^*_c(\mathcal{G}, V_k) \cong H^*_c(G, V_k).
\]

By Theorem \[13\] for any torsion-free polycyclic group \( \Gamma \) and any finite-dimensional representation \( \rho : \Gamma \to GL(V_k) \), we can describe the group cohomology \( H^*(\Gamma, V_k) \) as a subspace of the cohomology of a certain nilpotent Lie algebra determined by \( \Gamma \) (Corollary \[7\], see also Theorem \[8\]). The above main theorems are generalizations of the classical results on the cohomology of nilmanifolds and solvmanifolds given by Nomizu and Mostow and Baues’ recent result on the cohomology of compact aspherical manifolds with torsion-free virtually polycyclic fundamental groups. We see that the theorems imply such results as corollaries (Section \[7\]).

Moreover, as an application of our theorem, we will give a new proof of the surprising cohomology vanishing theorem given by Dekimpe-Igodt in \[10\] (Section \[8\]). For constructing bounded polynomial crystallographic actions of polycyclic groups Dekimpe and Igodt prove the vanishing of certain cohomology. By our proof, we can regard such vanishing as a conclusion of a fundamental lemma on cohomology of algebraic groups.
2. Proof of main theorems

2.1. First step.

**Lemma 2.1.** Let $\Gamma = \mathbb{Z}$. We suppose that for a subfield $k \subset \mathbb{C}$ we have a $k$-defined algebraic group $G$ and an inclusion $\Gamma \subset G(k)$ which is a full representation. Let $V$ be a $k$-rational $G$-module. Then the inclusion $\Gamma \subset G(k)$ induces a cohomology isomorphism

$$H^r_\Gamma(G, V_k) \cong H^r(\Gamma, V_k).$$

**Proof.** For a rational $G$-module, we have $V = \bigcup_i V^i$ for finite-dimensional rational $k$-submodules $V^i$. Then by [7] we have

$$H^r(\Gamma, \bigcup_i V^i_k) \cong \varinjlim H^r(\Gamma, V^i_k),$$

and by [19] Part 1, Lemma 4.17] we have

$$H^r_\Gamma(G, \bigcup_i V^i_k) \cong \varinjlim H^r_\Gamma(G, V^i_k).$$

Hence we can assume that the rational $G$-module is finite-dimensional.

For $* \neq 0, 1$ we have $H^0(\Gamma, V_k) = 0$ and $H^*_\Gamma(G, V_k) = 0$. Since $\Gamma$ is a Zariski-dense subgroup in $G$, we have $H^0(\Gamma, V_k) = V^\Gamma = V^G_{0(k)} = H^0(G, V_k)$. Hence it is sufficient to show that the map $H^1_\Gamma(G, V_k) \to H^1(\Gamma, V_k^0)$ is an isomorphism.

Since $\Gamma$ is Abelian, $G$ is also Abelian, and hence we have $G = G_s \times U(G)$ where $G_s = \{g \in G| g$ is semi-simple$\}$ (see [8] Theorem 4.7]). For the representation $\phi : G \to GL(V)$, the decomposition $G = G_s \times U(G)$ gives the decomposition $\phi = \phi_s \times \phi_u$. Let $V^0 = \{v \in V| \forall g \in G \phi_s(g)v = v\}$. Then $V^0$ is a $k$-rational submodule of $V$ for both the representations $\phi_s$ and $\phi$. We can take a complement $V^1$ such that we have a direct sum $V = V^0 \oplus V^1$ of a submodule for both $\phi_s$ and $\phi$.

We first show $H^1_\Gamma(G, V^1_k) = 0$ and $H^1(\Gamma, V^1_k) = 0$. It is sufficient to show the case $k = \mathbb{C}$. Then we have the sequence $V^1 = V_{1,0}^1 \supset V_{1,1}^1 \supset \cdots \supset V_{1,m}^1$ of submodules such that $\dim V_{1,i}/V_{1,i+1} = 1$. By induction, it is sufficient to show the case $V^1$ is a non-trivial $1$-dimensional module. In this case it is easy to show $H^1(\Gamma, V^1_k) = 0$.

Since $V^1$ is $1$-dimensional, the unipotent radical $U(G)$ acts trivially on $V^1$. Let $u_k$ be the $k$-Lie algebra of $U(G)$. Then we have

$$H^1_\Gamma(G, V^1_k) \cong H^1(u_k, V^1_k)^{G_s(k)};$$

see Theorem [13] Since $G_s$ acts non-trivially on $V^1, U(G)$ acts trivially on $V^1$ and the decomposition $G = G_s \times U(G)$ is a direct product, we have

$$H^1(u_k, V^1_k)^{G_s(k)} = u_k^* \otimes (V^1_k)^{G_s(k)} = 0.$$

We show that the inclusion $\Gamma \subset G(k)$ induces a cohomology isomorphism

$$H^1_\Gamma(G, V^0_k) \cong H^1(\Gamma, V^0_k).$$

Since $\phi$ is a unipotent representation on $V^0$, we have $V^0 = V^{0,0} \supset V^{0,1} \supset \cdots \supset V^{0,m}$ of submodules such that $V^{0,i}/V^{0,i+1}$ is the $1$-dimensional trivial module. By induction, it is sufficient to show the case $V^0_k = k$ as the trivial module. By the projection $G = G_s \times U(G) \to U(G)$, we have the Zariski-dense inclusion $\Gamma \subset U(G)(k)$.

By $\Gamma = \mathbb{Z}$, we have $U(G)(k) = k$ as the additive group and $\Gamma$ is embedded as $a\mathbb{Z}$ for
some $a \in k$. We can easily show that the inclusion $\Gamma \subset \mathcal{U}(\mathcal{G})$ induces a cohomology isomorphism

$$H_r^1(\mathcal{U}(\mathcal{G}), k) \cong H_r^1(\Gamma, k);$$

see [19] Part I, 4.21 Lemma. By the direct product $\mathcal{G} = \mathcal{G}_s \times \mathcal{U}(\mathcal{G})$, we have

$$H_r^1(\mathcal{G}, k) \cong H_r^1(\mathcal{U}(\mathcal{G}), k)^{\mathcal{G}_s}(k) \cong H_r^1(\mathcal{U}(\mathcal{G}), k);$$

see Theorem 4.1. By this isomorphism, we can prove that the inclusion $\Gamma \subset \mathcal{G}(k)$ induces a cohomology isomorphism

$$H_r^1(\mathcal{G}(k), V_k^0) \cong H_r^1(\Gamma, V_k^0),$$

and the lemma follows. \qed

By a similar proof, we have the following lemma. (See [9] Chapter IX for the commutativity of inductive limits for the continuous cohomology of Lie groups.)

**Lemma 2.2.** Let $G = \mathbb{R}$. We suppose that for a field $k = \mathbb{R}$ or $\mathbb{C}$ we have a $k$-defined algebraic group $\mathcal{G}$ and an inclusion $G \subset \mathcal{G}(k)$ which is a full representation. Let $V$ be a $k$-rational $\mathcal{G}$-module. Then the inclusion $G \subset \mathcal{G}(k)$ induces a cohomology isomorphism

$$H_r^*(\mathcal{G}, V_k) \cong H_r^*(G, V_k).$$

2.2. **Proof of main theorems.**

**Proof of Theorem 1.3** We proceed by induction on rank $\Gamma$. By Lemma 2.1 in the case rank $\Gamma = 1$ the statement follows.

$\Gamma$ admits a finite index normal polycyclic subgroup $\Gamma'$ such that $\Gamma'$ admits a normal subgroup $\Gamma''$ such that $\Gamma'/\Gamma''$ is infinite cyclic (see [27]). Denote by $\mathcal{G}'$ and $\mathcal{H}$ the Zariski-closures of $\Gamma'$ and $\Gamma''$ respectively in $\mathcal{G}$. We notice that $\Gamma'/\Gamma'' \to \mathcal{G}'/\mathcal{H}$ has a Zariski-dense image.

Since $\Gamma'$ is a finite index subgroup of $\Gamma$, we have rank $\Gamma = $ rank $\Gamma'$. By the fullness, we have dim $\mathcal{U}(\mathcal{G}) = $ rank $\Gamma$. Since $\Gamma/\Gamma'$ is finite, $\mathcal{G}/\mathcal{G}'$ is also finite, and so we have $\mathcal{U}(\mathcal{G}) = \mathcal{U}(\mathcal{G}')$. Hence we have dim $\mathcal{U}(\mathcal{G}') = $ rank $\Gamma'$ and so $\Gamma' \to \mathcal{G}'$ is also full.

We have

$$\text{rank } \Gamma' = \text{rank } \Gamma'' + \text{rank } \Gamma'/\Gamma''$$

and

$$\dim \mathcal{U}(\mathcal{G}') = \dim \mathcal{U}(\mathcal{H}) + \dim \mathcal{U}(\mathcal{G}'/\mathcal{H}).$$

By Lemma 1.1 we have dim $\mathcal{U}(\mathcal{H}) \leq $ rank $\Gamma''$ and dim $\mathcal{U}(\mathcal{G}'/\mathcal{H}) \leq $ rank $\Gamma'/\Gamma''$. By dim $\mathcal{U}(\mathcal{G}') = $ rank $\Gamma'$, these relations imply dim $\mathcal{U}(\mathcal{H}) = $ rank $\Gamma''$ and dim $\mathcal{U}(\mathcal{G}'/\mathcal{H}) = $ rank $\Gamma'/\Gamma''$. Hence both representations $\Gamma'' \to \mathcal{H}$ and $\Gamma'/\Gamma'' \to \mathcal{G}'/\mathcal{H}$ are full.

Take $\Delta = \Gamma' \cap \mathcal{H}$. By $\mathcal{G}'' \subset \Delta$, we have rank $\Gamma'/\Delta \leq $ rank $\Gamma'/\Gamma''$. Since $\Gamma'/\Delta$ is Zariski-dense in $\mathcal{G}'/\mathcal{H}$, by Lemma 1.1 we have dim $\mathcal{U}(\mathcal{G}'/\mathcal{H}) \leq $ rank $\Gamma'/\Delta$. By dim $\mathcal{U}(\mathcal{G}'/\mathcal{H}) = $ rank $\Gamma'/\Gamma''$, we obtain rank $\Gamma'/\Delta = $ rank $\Gamma'/\Gamma''$. Since we have $\Gamma'/\Delta \cong (\Gamma'/\Gamma'')/(\Delta/\Gamma'')$ and $\Gamma'/\Gamma''$ is infinite cyclic, we have $\Delta/\Gamma'' = 1$ and so $\Delta = \Gamma''$.

Now we have the commutative diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & \Delta & \longrightarrow & \Gamma' & \longrightarrow & \Gamma'/\Delta & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G}'/\mathcal{H} & \longrightarrow & 1
\end{array}
$$
such that $\Delta$, $\Gamma'$ and $\Gamma'/\Delta$ are Zariski-dense subgroups in $\mathcal{H}$, $\mathcal{G}'$ and $\mathcal{G}'/\mathcal{H}$ respectively. The groups $\Delta$, $\Gamma'$ and $\Gamma'/\Delta$ are torsion-free polycyclic, and the inclusions $\Delta \subset \mathcal{H}$, $\Gamma' \subset \mathcal{G}'$ and $\Gamma'/\Delta \subset \mathcal{G}'/\mathcal{H}$ are full representations.

Consider the spectral sequence $E_{r}^{*,*}(\Gamma', \Delta, V_{k})$ of the group extension

$$1 \longrightarrow \Delta \longrightarrow \Gamma' \longrightarrow \Gamma'/\Delta \longrightarrow 1,$$

as in [17] and the spectral sequence $rE_{r}^{*,*}(\mathcal{G}', \mathcal{H}, V_{k})$ of the $k$-defined algebraic group extension

$$1 \longrightarrow \mathcal{H} \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G}'/\mathcal{H} \longrightarrow 1$$

as in [19] Proposition 6.6. Then the commutative diagram

$$
\begin{array}{ccc}
1 & \longrightarrow & \Delta \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \mathcal{H} \\
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\longrightarrow & \longrightarrow & \longrightarrow \\
\end{array}
\begin{array}{ccc}
\Gamma' & \longrightarrow & \Gamma'/\Delta \\
\downarrow & & \downarrow \\
\mathcal{G}' & \longrightarrow & \mathcal{G}'/\mathcal{H} \\
\end{array}
$$

induces a homomorphism $rE_{2}^{*,*}(\mathcal{G}', \mathcal{H}, V_{k}) \rightarrow E_{2}^{*,*}(\Gamma', \Delta, V_{k})$ such that we have the commutative diagram

$$
\begin{array}{ccc}
rE_{2}^{*,*}(\mathcal{G}', \mathcal{H}, V) & \longrightarrow & E_{2}^{*,*}(\Gamma', \Delta, V_{k}) \\
\downarrow \cong & & \downarrow \cong \\
H_{r}^{*}(\mathcal{G}'/\mathcal{H}; H_{r}^{*}(\mathcal{H}, V_{k})) & \longrightarrow & H^{*}(\Gamma'/\Delta, H^{*}(\Delta, V_{k})).
\end{array}
$$

By induction hypothesis, the inclusion $\Delta \subset \mathcal{H}$ induces a cohomology isomorphism $H_{r}^{*}(\mathcal{H}, V_{k}) \cong H^{*}(\Delta, V_{k})$ and the inclusion $\Gamma' \subset \mathcal{G}'$ induces a cohomology isomorphism $H_{r}^{*}(\mathcal{G}'/\mathcal{H}, H_{r}^{*}(\mathcal{H}, V_{k})) \cong H^{*}(\Gamma'/\Delta, H_{r}^{*}(\mathcal{H}, V_{k}))$. Hence the homomorphism $rE_{2}^{*,*}(\mathcal{G}', \mathcal{H}, V_{k}) \rightarrow E_{2}^{*,*}(\Gamma', \Delta, V_{k})$ is an isomorphism. By [21] Theorem 3.5, we can show the isomorphism

$$H_{r}^{*}(\mathcal{G}', V_{k}) \cong H^{*}(\Gamma', V_{k}).$$

Since $\Gamma'/\Gamma'$ is a finite group, the map $\Gamma'/\Gamma' \rightarrow \mathcal{G}'/\mathcal{G}'$ is surjective by the Zariski-density. Hence $\mathcal{G}/\mathcal{G}'$ is also finite. We have isomorphisms

$$H_{r}^{*}(\mathcal{G}, V_{k}) \cong H_{r}^{*}(\mathcal{G}', V_{k})^{\mathcal{G}/\mathcal{G}'(k)} \quad \text{and} \quad H^{*}(\Gamma', V_{k})^{\Gamma'/\Gamma'} \cong H^{*}(\Gamma, V_{k}),$$

and the induced map

$$H_{r}^{*}(\mathcal{G}, V_{k}) \rightarrow H^{*}(\Gamma, V_{k})$$

is identified with the map

$$H_{r}^{*}(\mathcal{G}', V_{k})^{\mathcal{G}/\mathcal{G}'(k)} \rightarrow H^{*}(\Gamma', V_{k})^{\Gamma'/\Gamma'}$$

associated with the maps $\Gamma' \rightarrow \mathcal{G}'$ and $\Gamma/\Gamma' \rightarrow \mathcal{G}/\mathcal{G}'$. Since the map $\Gamma/\Gamma' \rightarrow \mathcal{G}/\mathcal{G}'$ is surjective and we have the isomorphism

$$H_{r}^{*}(\mathcal{G}', V_{k}) \cong H^{*}(\Gamma', V_{k})$$

as above, we can show that the map

$$H_{r}^{*}(\mathcal{G}', V_{k})^{\mathcal{G}/\mathcal{G}'(k)} \rightarrow H^{*}(\Gamma', V_{k})^{\Gamma'/\Gamma'}$$

is an isomorphism. The injectivity is obvious. By the surjectivity of $\Gamma/\Gamma' \rightarrow \mathcal{G}/\mathcal{G}'$, we can easily check that if $s(a) \in H^{*}(\Gamma', V_{k})^{\Gamma'/\Gamma'}$ for $a \in H_{r}^{*}(\mathcal{G}', V_{k})$, then we have $a \in H_{r}^{*}(\mathcal{G}', V_{k})^{\mathcal{G}/\mathcal{G}'(k)}$ where we denote by $s$ the isomorphism $H_{r}^{*}(\mathcal{G}', V_{k}) \cong H^{*}(\Gamma', V_{k})$. 

This implies the surjectivity of $H^*_r(G', V_k)^{G'/G''(k)} \to H^*(\Gamma', V_k)^{\Gamma'/\Gamma'}$. Hence the theorem follows. \hfill \Box

**Proof of Theorem 1.4.** We proceed by induction on $\dim G$. By Lemma 2.2, in the case $\dim G = 1$ the statement follows.

We have a normal subgroup $G'' \subset G$ such that $G/G'' = \mathbb{R}$. Denote by $H$ the Zariski-closure of $G''$ in $G$. We have

$$\dim G = \dim G'' + \dim G/G''$$

and

$$\dim \mathcal{U}(G) = \dim \mathcal{U}(H) + \dim \mathcal{U}(G/H).$$

By Lemma 1.1, we have $\dim \mathcal{U}(H) \leq \dim G''$ and $\dim \mathcal{U}(G/H) \leq \dim G/G''$. By the fullness, we have $\dim G = \dim \mathcal{U}(G)$. These relations imply $\dim \mathcal{U}(H) = \dim G''$ and $\dim \mathcal{U}(G/H) = \dim G/G''$. Hence both representations $G'' \to H$ and $G/G'' \to G/H$ are full.

Take $H = G \cap H$. Then $H$ is a closed normal subgroup in $G$ and hence $H$ and $G/H$ are simply connected (see [25]). By $G'' \subset H$, we have $\dim G/H \leq \dim G/G''$. Since $G/H$ is Zariski-dense in $G/H$, by Lemma 1.1, we have $\dim \mathcal{U}(G/H) \leq \dim G/H$. Hence, by $\dim \mathcal{U}(G/H) = \dim G/G''$, we obtain $\dim G/H = \dim G/G''$. Thus $\dim H = \dim G''$ and so $H = G''$.

We have the commutative diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 1
\end{array}
\]

such that $H$, $G$ and $G/H$ are Zariski-dense subgroups in $H$, $G$ and $G/H$ respectively. The groups $H$, $G$ and $G/H$ are simply connected solvable Lie groups, and the inclusions $H \subset H$, $G \subset G$ and $G/H \subset G/H$ are full representations.

Consider the spectral sequence $E_{r,s}^*(G, H, V_k)$ of the Lie group extension

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1,$$

as in [9, Chapter IX, Theorem 4.3] or [16, Proof of Theorem 7.1], and the spectral sequence $E_{r,s}^*(G, H, V_k)$ of the $k$-defined algebraic group extension

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1,$$

as in [19, Proposition 6.6]. Then the commutative diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 1
\end{array}
\]
induces a homomorphism \( \rho_{E_2^{*,i}}(\mathcal{G}, \mathcal{H}, V_k) \rightarrow \rho_{E_2^{*,c}}(G, H, V_k) \) such that we have the commutative diagram

\[
\begin{array}{ccc}
\rho_{E_2^{*,i}}(\mathcal{G}, \mathcal{H}, V_k) & \xrightarrow{\cong} & \rho_{E_2^{*,c}}(G, H, V_k) \\
H^*_c(\mathcal{G}/\mathcal{H}, H^*_c(\mathcal{H}, V_k)) & \cong & H^*_c(G/H, H^*_c(H, V_k)).
\end{array}
\]

By induction hypothesis, the inclusion \( H \subset \mathcal{H} \) induces a cohomology isomorphism \( H^*_c(\mathcal{H}, V) \cong H^*_c(H, V) \), and the inclusion \( G/H \subset \mathcal{G}/\mathcal{H} \) induces a cohomology isomorphism \( H^*_c(\mathcal{G}/\mathcal{H}, H^*_c(\mathcal{H}, V_k)) \cong H^*_c(G/H, H^*_c(H, V_k)) \). Hence the homomorphism \( \rho_{E_2^{*,i}}(\mathcal{G}, \mathcal{H}, V_k) \rightarrow \rho_{E_2^{*,c}}(G, H, V_k) \) is an isomorphism. By [21, Theorem 3.5], we have

\[ H^*_c(G, V_k) \cong H^*_c(G, V_k). \]

\[ \square \]

3. Full representations and algebraic hulls

An algebraic group \( \mathcal{G} \) is minimal if the centralizer \( Z_\mathcal{G}(U(\mathcal{G})) \) of \( U(\mathcal{G}) \) is contained in \( U(\mathcal{G}) \) (see [1, Section 1] for the meaning of minimal).

Let \( \Gamma \) (resp. \( G \)) be a torsion-free virtually polycyclic group (resp. simply connected solvable Lie group).

**Theorem 3.1** ([2, Theorem A.1, Corollary A.3], [27, Proposition 4.40, Lemma 4.41]). There exists a \( \mathbb{Q} \)-defined (resp. \( \mathbb{R} \)-defined) minimal algebraic group \( \mathcal{G} \) with an inclusion \( \Gamma \subset \mathcal{G}(\mathbb{Q}) \) (resp. \( G \subset \mathcal{G}(\mathbb{R}) \)) which is a full representation.

Moreover such \( \mathcal{G} \) is unique up to isomorphism of \( \mathbb{Q} \)-defined (resp. \( \mathbb{R} \)-defined) algebraic groups.

We call the \( \mathbb{Q} \)-(resp. \( \mathbb{R} \)-defined) algebraic group as in this theorem the algebraic hull of \( \Gamma \) (resp. \( G \)). We denote it by \( \mathcal{A}_\Gamma \) (resp. \( \mathcal{A}_G \)).

Let \( k \) be a subfield \( k \subset \mathbb{C} \) (resp. \( k = \mathbb{R} \) or \( \mathbb{C} \)). Let \( \rho : \Gamma \rightarrow GL(V_k) \) (resp. \( \rho : G \rightarrow GL(V_k) \)) be a finite-dimensional representation.

**Proposition 3.2** ([2, Proposition A.6]). \( \rho \) can be extended to a \( k \)-rational representation of \( \mathcal{A}_\Gamma \) (resp. \( \mathcal{A}_G \)) if the Zariski-closure of the image of \( \rho \) is minimal.

In general, a finite-dimensional representation \( \rho \) cannot be extended to a \( k \)-rational representation of \( \mathcal{A}_\Gamma \). In this paper we consider the following construction.

**Definition 3.3.** Consider the inclusion \( i_\Gamma : \Gamma \hookrightarrow \mathcal{A}_\Gamma(\mathbb{Q}) \) (resp. \( i_G : G \hookrightarrow \mathcal{A}_G(\mathbb{R}) \)) and the representation \( i_\Gamma \oplus \rho : \Gamma \rightarrow \mathcal{A}_\Gamma(k) \times GL(V_k) \) (resp. \( i_G \oplus \rho : G \rightarrow \mathcal{A}_G(k) \times GL(V_k) \)). Then we denote by \( \mathcal{G}_k^\rho \) (resp. \( \mathcal{G}_k^\rho_G \)) the Zariski-closure of the image of \( i_\Gamma \oplus \rho \) (resp. \( i_G \oplus \rho \)) and we call it the \( \rho \)-relative algebraic hull of \( \Gamma \) (resp. \( G \)).

We can easily show the following properties.

**Proposition 3.4.**

- \( \rho : \Gamma \rightarrow GL(V_k) \) (resp. \( \rho : G \rightarrow GL(V_k) \)) can be extended to a \( k \)-rational representation \( \mathcal{G}_k^\rho \rightarrow GL(V) \) (resp. \( \mathcal{G}_k^\rho_G \rightarrow GL(V) \)).

- We have an isomorphism \( U(\mathcal{G}_k^\rho) \cong U(\mathcal{A}_k^\rho) \) of \( k \)-defined algebraic groups.

- \( i_\Gamma \oplus \rho : \Gamma \rightarrow \mathcal{G}_k^\rho \) (resp. \( i_G \oplus \rho : G \rightarrow \mathcal{G}_k^\rho_G \)) is an injective full representation.

- If the Zariski-closure of \( \rho(\Gamma) \) (resp. \( \rho(G) \)) is minimal, then \( \mathcal{G}^\rho = \mathcal{A}_\Gamma \) (resp. \( \mathcal{G}^\rho_G = \mathcal{A}_G \)).
Proof. The restriction $\mathcal{G}_G^0 \to GL(V)$ of the projection $A_G \times GL(V) \to GL(V)$ gives the first assertion.

Consider the restriction $\mathcal{G}_G^0 \to A_G$ of the projection $A_G \times GL(V) \to A_G$. Since the inclusion $i_G : G \to A_G$ has the Zariski-dense image, the map $\mathcal{G}_G^0 \to A_G$ is surjective and hence the restriction $U(\mathcal{G}_G^0) \to U(A_G)$ is also surjective. By Lemma 1.1 we have

$$\text{rank } \Gamma \geq \dim U(\mathcal{G}_G^0) \geq \dim U(A_G) = \text{rank } \Gamma$$

and so $\dim U(\mathcal{G}_G^0) = \dim U(A_G)$. Hence the map $U(\mathcal{G}_G^0) \to U(A_G)$ induces an isomorphism of Lie algebras. By using the exponential maps, we can show that the map $U(\mathcal{G}_G^0) \to U(A_G)$ is also an isomorphism. Hence the second and third assertions hold.

The fourth follows from Proposition 3.2.

For a simply connected solvable Lie group $G$, we can directly construct the algebraic hull $A_G$ of $G$ by using the Lie algebra $\mathfrak{g}$. Let $\mathfrak{n}$ be the nilradical (i.e. maximal nilpotent ideal) of $\mathfrak{g}$. There exists a subvector space (not necessarily Lie algebra) $V$ of $\mathfrak{g}$ so that $\mathfrak{g} = V \oplus \mathfrak{n}$ as the direct sum of vector spaces and for any $A, B \in V$ $(ad_A)_s(B) = 0$ where $(ad_A)_s$ is the semi-simple part of $ad_A$ (see [11] Proposition III.1.1). We define the map $ad_s : \mathfrak{g} \to D(\mathfrak{g})$ as $ad_{sA+X} = (ad_A)_s(B)$ for $A \in V$ and $X \in \mathfrak{n}$. Then we have $[ad_s(\mathfrak{g}), ad_s(\mathfrak{g})] = 0$ and $ad_s$ is linear (see [11] Proposition III.1.1). Since we have $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}$, the map $ad_s : \mathfrak{g} \to D(\mathfrak{g})$ is a representation, and the image $ad_s(\mathfrak{g})$ is Abelian and consists of semi-simple elements. We take the Lie group homomorphism $Ad_s : G \to \text{Aut}(\mathfrak{g})$ which corresponds to the Lie algebra homomorphism $ad_s$. We define the nilpotent Lie algebra

$$u = \{X - ad_sX \in \text{Im } ad_s \times \mathfrak{g}\}$$

which satisfies $\text{Im } ad_s \times \mathfrak{g} = \text{Im } ad_s \times u$. By the exponential map, we have the unipotent $\mathbb{R}$-defined algebraic group $U$ corresponding to the Lie algebra $u$. We have $Ad_s(G) \subset \text{Aut}(U)$ and $\text{Aut}(U)$ is an $\mathbb{R}$-defined algebraic group. Take the Zariski-closure $T$ of $Ad_s(G)$ in $\text{Aut}(U)$.

Proposition 3.5 ([18] Proposition 2.4). The $\mathbb{R}$-defined algebraic group $T \rtimes U$ is an algebraic hull of $G$.

Suppose $G$ admits a lattice (i.e. cocompact discrete subgroup) $\Gamma$. A discrete subgroup $\Gamma$ of a simply connected solvable Lie group $G$ is torsion-free polycyclic with $\text{rank } \Gamma \leq \dim G$. It is known that $\Gamma$ is a lattice if and only if $\text{rank } \Gamma = \dim G$ (see [4] Lemma 3.4]). It is known that for the algebraic hull $A_G$ of $G$, the Zariski-closure of $\Gamma$ in $A_G$ is isomorphic to the algebraic hull $A_G$ of $\Gamma$ as an $\mathbb{R}$-defined algebraic group (see [27], Proof of Theorem 4.34)). But it does not coincide with $A_G$ in general.

Let $k = \mathbb{R}$ or $\mathbb{C}$ and let $\rho : G \to GL(V_k)$ be a finite-dimensional representation. Consider the $\rho$-relative algebraic hull $\mathcal{G}_G^\rho$. Since the Zariski-closure of $\Gamma$ in $A_G$ is the algebraic hull $A_G$ of $\Gamma$ as an $\mathbb{R}$-defined algebraic group, the Zariski-closure of $\Gamma$ in $\mathcal{G}_G^\rho$ is identified with the $\rho$-relative algebraic hull $\mathcal{G}_G^\rho$. Hence we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{G}_G^\rho & \longrightarrow & \mathcal{G}_G^\rho \\
\downarrow_{i_G+\rho} & & \downarrow_{i_G+\rho} \\
\Gamma & \longrightarrow & G
\end{array}
\]
and each map is an inclusion. We have $U(G_G) = U(G_{G_1})$, but we do not have $G_{G_1} = G_{G}$ in general.

**Proposition 3.6.** The following two conditions are equivalent.

- $G_{G_1} = G_{G}$.
- $\rho$ is $\Gamma$-admissible; i.e. for the representation $\text{Ad} \oplus \rho : G \to \text{Aut}(g \otimes \mathbb{C}) \times GL(V)$, $(\text{Ad} \oplus \rho)(G)$ and $(\text{Ad} \oplus \rho)(\Gamma)$ have the same Zariski-closure in $\text{Aut}(g \otimes \mathbb{C}) \times GL(V)$.

**Proof.** Let $F$ be the Zariski-closure of $\text{Ad}(G)$ in $\text{Aut}(g \otimes \mathbb{C})$. Then the Zariski-closure $T$ of $\text{Ad}_s(G)$ is a maximal torus of $F$ (see [18, Proof of Proposition 3.2]). Hence, for a splitting $F = T \times U(F)$ with the projection $p : F \to T$, the map $\text{Ad}_s : G \to T$ is considered as the composition $G \xrightarrow{\text{Ad} \times F} F \xrightarrow{p} T$.

On the other hand, by Proposition [8,5] $T$ is a maximal torus of the algebraic hull $\mathcal{A}_G = T \times U$, and hence $\text{Ad}_s$ is the composition $G \longrightarrow \mathcal{A}_G = T \times U \longrightarrow T$.

Let $F'$ be the Zariski-closure of $\rho(G)$ in $GL(V)$. Take a maximal torus $T'$ and a splitting $F' = T' \times U(F')$ with the projection $q : F' \to T'$. Consider $\text{Ad}_s \oplus q \circ \rho \to T \times T'$. Then the Zariski-closure of $(\text{Ad}_s \oplus q \circ \rho)(G)$ is a maximal torus of both the Zariski-closures of $(\text{Ad} \oplus \rho)(G)$ and $(\text{Ad} \oplus \rho)(\Gamma)$.

Let $\sigma : G \to GL(V)$ be a finite-dimensional representation. Take the Zariski-closures $G$ and $G'$ of $\sigma(G)$ and $\sigma(\Gamma)$. Then we have $U(G) = U(G')$ (see [27, Theorem 3.2]). Hence, for a maximal torus $S$ of $G$, $G = G'$ if and only if $S \subset G'$. Taking a splitting $G = S \times U(G)$ with the projection $r : G \to S$, $G = G'$ if and only if $r \circ \sigma(\Gamma)$ is Zariski-dense in $S$. By this argument, the two conditions in the proposition are equivalent to the condition that $(\text{Ad}_s \oplus q \circ \rho)(G)$ and $(\text{Ad}_s \oplus q \circ \rho)(\Gamma)$ have the same Zariski-closure. Hence the proposition holds. □

## 4. Cohomology of algebraic groups

Let $k$ be a subfield of $\mathbb{C}$. Let $G$ be a $k$-defined algebraic group and $\mathcal{H}$ a normal subgroup of $G$. Consider the functor $V \mapsto V^G$ from the category of $k$-rational $G$-modules to the category of $k$-vector spaces. Regarding this functor as the composition $V \to V^{\mathcal{H}} \to (V^{\mathcal{H}})^{G/\mathcal{H}}$, we can obtain the spectral sequence $E_r^{*,*}$ such that $E_2^{*,*} = H_r^*(G/\mathcal{H}, H_\ell^*(\mathcal{H}, V))$ and it converges to $H^*(G, V)$ (see [19, Proposition 6.6]). Considering the spectral sequence for the unipotent radical $U(G)$, we have the following result; see [13].

**Theorem 4.1.** For a $k$-rational $G$-module $V$, we have an isomorphism $H_r^*(G, V_k) \cong H_r^*(U(G), V_k)^{G/U(k)}$.

The extension $1 \to U(G) \to G \to G/U(G) \to 1$ splits (see [22]). Hence we have an affine action $G \to \text{Aut}(U(G)) \ltimes U(G)$.

Consider the coordinate ring $k[U(G)]$ as a $k$-rational $G$-module. Then by [19, Part I, 4.7], for
a $k$-rational $G$-module $V$, we have
\begin{equation*}
H^*(G, k[U(G)] \otimes V_k) \cong H^*(U(G), k[U(G)] \otimes V_k)^G/U(G)(k) \cong \left\{ \begin{array}{ll}
V_k^{G/U(G)(k)}, & * = 0, \\
0, & * > 0.
\end{array} \right.
\end{equation*}

Hence we have:

Corollary 4.2. For $* > 0$, we have
\begin{equation*}
H^*_r(G, V_k) = 0.
\end{equation*}

Let $u_k$ be the $k$-Lie algebra of $U(G)$. For a $k$-rational $G$-module $V$, we consider the Lie algebra cohomology $H^*(u_k, V_k)$.

Theorem 4.3 ([14 Theorem 5.2]). We have a natural isomorphism
\begin{equation*}
H^*_r(G, V_k) \cong H^*(u_k, V_k)^{G/U(G)(k)}.
\end{equation*}

5. de Rham cohomology of solvmanifolds and Lie algebra cohomology

Let $G$ be a simply connected solvable Lie group with a lattice $\Gamma$ and $V$ a finite-dimensional $G$-module for a Lie group representation $\rho: G \to GL(V)$ of a finite-dimensional complex vector space $V$. We consider the solvmanifold $\Gamma \backslash G$. Since we have $\pi_1(\Gamma \backslash G) \cong \Gamma$, we have a flat vector bundle $E$ with flat connection $D$ over $\Gamma \backslash G$ whose monodromy is $\rho: \Gamma \to GL(V)$. Denote by $A^*(\Gamma \backslash G, E)$ the cochain complex of $E$-valued differential forms on $\Gamma \backslash G$ with the differential $D$. Since the solvmanifold $\Gamma \backslash G$ is an Eilenberg–MacLane space with the fundamental group $\Gamma$, the de Rham cohomology $H^*(\Gamma \backslash G, E)$ is isomorphic to the group cohomology $H^*(\Gamma, V)$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\bigwedge \mathfrak{g}^* \otimes V$ be the cochain complex of the Lie algebra $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $V$. Then we regard $\bigwedge \mathfrak{g}^* \otimes V$ as the cochain complex of left-$G$-invariant differential forms on $\Gamma \backslash G$ and we consider the inclusion
\begin{equation*}
\iota: \bigwedge \mathfrak{g}^* \otimes V \hookrightarrow A^*(\Gamma \backslash G, E).
\end{equation*}

Theorem 5.1 ([27 Remark 7.30]). The inclusion
\begin{equation*}
\iota: \bigwedge \mathfrak{g}^* \otimes V \hookrightarrow A^*(\Gamma \backslash G, E)
\end{equation*}

induces an injection
\begin{equation*}
H^*(\mathfrak{g}, V) \hookrightarrow H^*(\Gamma \backslash G, E).
\end{equation*}

The well-known proof of this theorem is given by geometric techniques. As we consider $A^*(\Gamma \backslash G, E) = C^\infty(\Gamma \backslash G) \otimes \bigwedge \mathfrak{g}^* \otimes V$, by using a normalized Haar measure on $G$ and integration on $\Gamma \backslash G$, we can construct the cochain complex homomorphism
\begin{equation*}
\mu: A^*(\Gamma \backslash G, E) \to \bigwedge \mathfrak{g}^* \otimes V
\end{equation*}
such that $\mu \circ \iota = \text{id}$. In this paper, by using our theorems, we will give an algebraic proof of this theorem.

As in Proposition 3.6, we call a representation $\rho$ $\Gamma$-admissible if for the representation $\text{Ad} \oplus \rho: G \to \text{Aut}(\mathfrak{g} \otimes \mathbb{C}) \times GL(V)$, $(\text{Ad} \oplus \rho)(G)$ and $(\text{Ad} \oplus \rho)(\Gamma)$ have the same Zariski-closure in $\text{Aut}(\mathfrak{g} \otimes \mathbb{C}) \times GL(V)$.
Theorem 5.2 ([23, 27 Theorem 7.26]). If $\rho$ is $\Gamma$-admissible, then the inclusion

$$\iota : \bigwedge g^* \otimes V \hookrightarrow A^*(\Gamma \setminus G, E)$$

induces a cohomology isomorphism

$$H^*(g, V) \cong H^*(\Gamma \setminus G, E).$$

We will see that our theorems can be regarded as generalizations of this theorem. We note that we have an isomorphism

$$Corollary 6.2.$$ state the following fact.

By Theorem 6.1, since $M_\Gamma$ is an aspherical manifold with $\pi_1(M_\Gamma) = \Gamma$, we can state the following fact.

Corollary 6.2. We have an isomorphism

$$H^*_r(\mathcal{A}_\Gamma, \mathbb{R}) \cong H^*(\Gamma, \mathbb{R}).$$

Our main theorems give this fact without using the geometry of an infra-solvmanifold and generalize this isomorphism for the cohomology with values in modules.
7. Corollaries of Theorems 1.3 and 1.4

7.1. Cohomologies of torsion-free virtually polycyclic groups and simply connected solvable Lie groups. Let \( k \) be a subfield of \( \mathbb{C} \) (resp. \( k = \mathbb{R} \) or \( \mathbb{C} \)). Let \( \Gamma \) (resp. \( G \)) be a torsion-free virtually polycyclic group (resp. simply connected solvable Lie group) and \( \rho : \Gamma \to GL(V_k) \) (resp. \( \rho : G \to GL(V_k) \)) a finite-dimensional representation. We consider the algebraic hull of \( A_\Gamma \) (resp. \( A_G \)) and the \( \rho \)-relative algebraic hull \( G_\Gamma^\rho \) (resp. \( G_G^\rho \)). Let \( u_k \) be the \( k \)-Lie algebra of \( U(A_\Gamma) \) (resp. \( U(A_G) \)). It is also the Lie algebra of \( U(G_\Gamma^\rho) \) (resp. \( U(G_G^\rho) \)) by Proposition 3.4.

By Theorem 1.3, Proposition 3.4 and Proposition 4.3 we have the following results.

**Corollary 7.1.** For a \( k \)-rational \( G_\Gamma^\rho \)-module (resp. \( G_G^\rho \)-module) \( W \) (e.g. \( W = V \)), we have isomorphisms

\[
H^*(\Gamma, W_K) \cong H^*_r(G_\Gamma^\rho, W_K) \cong H^*_r(u_k, W_K)G_\Gamma^\rho/\mathcal{U}(G_\Gamma^\rho)(k)
\]

(resp.

\[
H^*_r(G, W_K) \cong H^*_r(G_G^\rho, W_K) \cong H^*_r(u_k, W_K)G_G^\rho/\mathcal{U}(G_G^\rho)(k).
\]

By the fourth assertion of Proposition 3.4 under the good condition, we can simplify the statement and imply Baues’ isomorphism.

**Corollary 7.2.** If the Zariski-closure of \( \rho(\Gamma) \) (resp. \( \rho(G) \)) is minimal, we have isomorphisms

\[
H^*(\Gamma, V_K) \cong H^*_r(A_\Gamma, V_K) \cong H^*_r(u_k, V_K)A_\Gamma/\mathcal{U}(A_\Gamma)(k)
\]

(resp.

\[
H^*_r(G, V_K) \cong H^*_r(A_G, V_K) \cong H^*_r(u_k, V_K)A_G/\mathcal{U}(A_G)(k).
\]

**Remark 1.** This statement gives an isomorphism

\[
H^*(\Gamma, \mathbb{Q}) \cong H^*_r(A_\Gamma, \mathbb{Q}) \cong H^*_r(u_{\mathbb{Q}}, \mathbb{Q})A_\Gamma/\mathcal{U}(A_\Gamma)(\mathbb{Q}).
\]

For the standard \( \Gamma \)-manifold \( M_\Gamma \), in [3, Proposition 13.6], Baues and Grunewald claim the isomorphism

\[
H^*(M_\Gamma, \mathbb{Q}) \cong H^*_r(u_{\mathbb{Q}}, \mathbb{Q})A_\Gamma/\mathcal{U}(A_\Gamma)(\mathbb{Q}),
\]

but they do not give an explicit proof.

**Remark 2.** The isomorphism in Corollary 7.1 is absolutely algebraic. In Theorem 8.3, we will prove an extension of Baues’ isomorphism from a geometric viewpoint.

**Remark 3.** By Proposition 3.5 the Lie algebra of \( U(A_G) \) is given by

\[
u = \{X - \text{ad}_sX \in \text{Im} \text{ad}_s \times \mathfrak{g}\}
\]

as in Section [3]. Since \( U(A_G) \) is unipotent, a rational \( U(A_G) \)-module is unipotent and hence it is a nilpotent \( u \)-module. Hence this corollary gives “nilpotentizations” of cohomology of solvable Lie algebras.

**Corollary 7.3.** Let \( \mathfrak{g} \) be a solvable real Lie algebra. Take the nilpotent Lie algebra

\[
u = \{X - \text{ad}_sX \in \text{Im} \text{ad}_s \times \mathfrak{g}\}
\]

as in Section [8]. Then for a finite-dimensional real \( \mathfrak{g} \)-module \( V \), taking a certain nilpotent \( u \)-module structure on \( V \), the Lie algebra cohomology \( H^*(\mathfrak{g}, V) \) is isomorphic to a subspace of Lie algebra cohomology \( H^*(u, V) \).
7.2. Cohomology of lattices in simply connected solvable Lie groups. In this subsection, \( k = \mathbb{R} \) or \( \mathbb{C} \). Let \( G \) be a simply connected solvable Lie group with a lattice \( \Gamma \). Let \( \rho : G \to GL(V_k) \) be a finite-dimensional representation. We consider the commutative diagram

\[
\begin{array}{ccc}
G^\rho_\Gamma & \longrightarrow & G^\rho_G \\
\downarrow i_G \oplus \rho & & \downarrow i_G \oplus \rho \\
\Gamma & \longrightarrow & G
\end{array}
\]

as in Section 3. Then by our results, the inclusions \( i_G \oplus \rho : \Gamma \to G^\rho_\Gamma \) and \( i_G \oplus \rho : G \to G^\rho_G \) induce isomorphisms

\[ H^*(\Gamma, V_k) \cong H^*_c(G^\rho_\Gamma, V_k) \cong H^*_c(U(G^\rho_\Gamma), V_k)_{G^\rho_\Gamma / U(G^\rho_\Gamma)}(k) \]

and

\[ H^*_c(G, V_k) \cong H^*_c(G^\rho_G, V_k) \cong H^*_c(U(G^\rho_G), V_k)_{G^\rho_G / U(G^\rho_G)}(k). \]

Since we have \( U(G^\rho_\Gamma) = U(G^\rho_G) \), the induced map \( H^*_c(G, V_k) \to H^*(\Gamma, V_k) \) is identified with the inclusion

\[ H^*_c(U(G^\rho_G), V_k)_{G^\rho_G / U(G^\rho_G)}(k) \subset H^*_c(U(G^\rho_\Gamma), V_k)_{G^\rho_\Gamma / U(G^\rho_\Gamma)}(k). \]

Hence our theorem gives the following result without using the geometry of the solvmanifold \( \Gamma \backslash G \).

**Corollary 7.4.** The inclusion \( \Gamma \to G \) induces an injection

\[ H^*_c(G, V_k) \hookrightarrow H^*(\Gamma, V_k). \]

Moreover, as the following result, by Proposition 3.6 our theorems imply the Mostow theorem.

**Corollary 7.5.** If \( G^\rho_\Gamma = G^\rho_G \) (equivalently \( \rho \) is \( \Gamma \)-admissible), then we have

\[ H^*_c(U(G^\rho_G), V_k)_{G^\rho_G / U(G^\rho_G)}(k) = H^*_c(U(G^\rho_\Gamma), V_k)_{G^\rho_\Gamma / U(G^\rho_\Gamma)}(k), \]

and hence \( \Gamma \to G \) induces an isomorphism

\[ H^*_c(G, V_k) \cong H^*(\Gamma, V_k). \]

7.3. Nilpotent case. Let \( \Gamma \) be a torsion-free nilpotent group. Then we have (see [27, Chapter II] and [26, Chapter 2]):

- \( \Gamma \) is a lattice of some simply connected nilpotent Lie group \( N \).
- \( N \) can be the group \( U(\mathbb{R}) \) of real points of a \( \mathbb{Q} \)-defined algebraic unipotent group \( U \).
- \( \Gamma \subset U(\mathbb{Q}) \) and \( \Gamma \) is Zariski-dense in \( U \).

We denote by \( u_k \) the \( k \)-Lie algebra of \( U \). Then \( u_\mathbb{R} \) is the Lie algebra of \( N \). By our results, we have:

**Corollary 7.6.** For a \( k \)-rational \( U \)-module \( V \), the inclusion \( \Gamma \subset U(\mathbb{Q}) \) induces cohomology isomorphisms

\[ H^*(u_k, V_k) \cong H^*(U, V_k) \cong H^*(\Gamma, V_k). \]

**Remark 4.** This corollary is an algebraic analogy with the results given by Nomizu [24] and Lambe-Priddy [20]. An isomorphism

\[ H^*(u_\mathbb{R}, \mathbb{R}) \cong H^*(\Gamma, \mathbb{R}) \]
was given by the de Rham cohomology of the nilmanifold $\Gamma \backslash N$ in [24]. Moreover an isomorphism

$$H^*(u_\mathbb{Q}, \mathbb{Q}) \cong H^*(\Gamma, \mathbb{Q})$$

was given by the simplicial rational de Rham cohomology of the simplicial classifying complex of $\Gamma$ in [20].

8. New proof of the Dekimpe-Igodt surprising cohomology vanishing theorem

Let $\Gamma$ (resp. $G$) be a torsion-free virtually polycyclic group (resp. simply connected solvable Lie group) and $\rho : \Gamma \to GL(V_k)$ (resp. $\rho : G \to GL(V_k)$) a finite-dimensional representation. Consider the $\rho$-relative algebraic hull $G^\rho_\Gamma$ (resp $G^\rho_\Gamma$). Take a splitting $G^\rho_\Gamma = G^\rho_\Gamma / U(G^\rho_\Gamma) \times U(G^\rho_\Gamma)$ (resp. $G^\rho_\Gamma = G^\rho_\Gamma / U(G^\rho_\Gamma) \times U(G^\rho_\Gamma)$). We simply write $T = G^\rho_\Gamma / U(G^\rho_\Gamma)$ (resp. $G^\rho_\Gamma / U(G^\rho_\Gamma)$) and $U = U(G^\rho_\Gamma)$ (resp. $G^\rho_\Gamma / U(G^\rho_\Gamma)$). Then, considering $W_k = k[\mathcal{U}] \otimes V_k$ in Corollary 7.1 by Corollary 4.2 we have the following result.

**Corollary 8.1.** For $* > 0$, we have

$$H^*(\Gamma, k[\mathcal{U}] \otimes V_k) = 0$$

(resp.

$$H^*(G, k[\mathcal{U}] \otimes V_k) = 0$$

Since the exponential map $\exp : u \to \mathcal{U}$ is an isomorphism of $k$-defined algebraic variety, as we regard $u_k = k^r$ with $r = \text{rank} \Gamma$ (resp. $\text{dim} G$), the coordinate ring $k[\mathcal{U}]$ can be regarded as the space $P(k^r)$ of the $k$-polynomial functions on $u_k = k^r$.

We assume $k = \mathbb{R}$. A group action on $\mathbb{R}^r$ is called bounded polynomial if for some integer $d$ the action is represented by $d$-bounded degree polynomial diffeomorphisms of $\mathbb{R}^r$ with $d$-bounded degree polynomial inverse. For the $\mathbb{R}$-defined algebraic group $G^\rho_\Gamma = T \times U$, considering $u_\mathbb{R} = \mathbb{R}^r$, we have the algebraic group action of $G(\mathbb{R})$ on $\mathbb{R}^r$. By the fundamental theory of algebraic groups, this action is bounded polynomial. Hence the $\Gamma$-action on $u_\mathbb{R} = \mathbb{R}^r$ is also bounded polynomial. Take $S = Z(\mathcal{U}) \cap T$ where $Z(\mathcal{U})$ is the centralizer of $\mathcal{U}$. Then $G(\mathbb{R}) / S(\mathbb{R}) = (T(\mathbb{R}) / S(\mathbb{R})) \times U(\mathbb{R})$ is the $\mathbb{R}$-points of the algebraic hull of $\Gamma$ with the Zariski-dense inclusion $\Gamma \to (T(\mathbb{R}) / S(\mathbb{R})) \times U(\mathbb{R})$ (see [2] Appendix A). Hence the $\Gamma$-action on $U(\mathbb{R})$ is the action for the construction of Baues’ standard $\Gamma$-manifold as in Section 6. This implies that the $\Gamma$-action on $u_\mathbb{R} = \mathbb{R}^r$ is bounded polynomial and crystallographic (i.e. properly discontinuous and cocompact).

In [5], Benoist and Dekimpe showed that a bounded polynomial crystallographic action of $\Gamma$ is unique up to conjugation by a bounded polynomial diffeomorphism. Hence Corollary 8.1 states the following.

**Theorem 8.2.** Let $\Gamma$ be a torsion-free virtually polycyclic group. Suppose $\Gamma$ admits a bounded polynomial crystallographic action on $\mathbb{R}^r$. Consider the vector space $P(\mathbb{R}^r)$ of the polynomial functions on $\mathbb{R}^r$ as a $\Gamma$-module. Then for any representation $\rho : \Gamma \to GL(V_\mathbb{R})$ of a finite-dimensional $\mathbb{R}$-vector space $V$, for $* > 0$ we have

$$H^*(\Gamma, P(\mathbb{R}^r) \otimes V_\mathbb{R}) = 0.$$
Remark 5.

(1) If a virtually polycyclic group $\Gamma$ is not torsion-free, then we cannot construct the standard $\Gamma$-manifold. But, for every virtually polycyclic group $\Gamma$, there exists a finite index normal torsion-free subgroup $\Gamma'$ in $\Gamma$. Hence, assuming that $\Gamma$ admits a bounded polynomial crystallographic action on $\mathbb{R}^r$, for $* > 0$, by the vanishing $H^*(\Gamma', P(\mathbb{R}^r) \otimes V_{\mathbb{R}}) = 0$ for the torsion-free case, we can easily see $H^*(\Gamma, P(\mathbb{R}^r) \otimes V_{\mathbb{R}}) = 0$ (see [6, Chapter III 10]).

(2) Denote by $P(\mathbb{R}^r, V_{\mathbb{R}})$ the vector space of polynomial maps from $\mathbb{R}^r$ to $V_{\mathbb{R}}$. It is known that the map $P(\mathbb{R}^r) \otimes V_{\mathbb{R}} \to P(\mathbb{R}^r, V_{\mathbb{R}})$ such that $f \otimes v \mapsto (\mathbb{R}^r \ni a \mapsto f(a)v)$ is an isomorphism (cf. [15, p. 16]). Denote by $P(\mathbb{R}^r)$ the group of polynomial diffeomorphisms of $\mathbb{R}^r$. Since the map $P(\mathbb{R}^r) \otimes V_{\mathbb{R}} \to P(\mathbb{R}^r, V_{\mathbb{R}})$ is $P(\mathbb{R}^r) \times GL(V_{\mathbb{R}})$-equivariant, for a polynomial $\Gamma$-action on $\mathbb{R}^r$, we can identify the $\Gamma$-module $P(\mathbb{R}^r) \otimes V_{\mathbb{R}}$ with the $\Gamma$-module $P(\mathbb{R}^r, V_{\mathbb{R}})$.

(3) In [10], for the special bounded polynomial action (called “canonical type”) of a virtually polycyclic group $\Gamma$, Dekimpe and Igodt showed the cohomology vanishing $H^*(\Gamma, P(\mathbb{R}^r, V_{\mathbb{R}})) = 0$ for $* > 0$. They said such cohomology vanishing is surprising. The proof which is given by Dekimpe and Igodt is very hard (see [10, Section 3]). Now we obtain a new proof of such a vanishing theorem.

In this paper we give a new application of the vanishing theorem. Take the standard $\Gamma$-manifold $\Gamma \setminus \mathcal{U}(\mathbb{R})$. We consider the space 

\[
\left( \bigwedge u_{\mathbb{R}}^* \otimes V_{\mathbb{R}} \right)^{\mathcal{T}(\mathbb{R})}
\]

of $\mathcal{T}(\mathbb{R})$-invariant elements of the cochain complex of the $\mathbb{R}$-Lie algebra $u_{\mathbb{R}}$ of $\mathcal{U}$ with values in the $\mathbb{R}$-rational $\mathcal{U}$-module $V$. Then, as in Section 6, we have the natural inclusion

\[
\left( \bigwedge u_{\mathbb{R}}^* \otimes V_{\mathbb{R}} \right)^{\mathcal{T}(\mathbb{R})} \subset A^*(M_{\Gamma}, E)
\]

where $A^*(M_{\Gamma}, E)$ is the de Rham complex with values in the flat bundle $E$ corresponding to $\rho : \Gamma \to GL(V_{\mathbb{R}})$. We prove an extension of Theorem 6.1 for any finite-dimensional representation $\rho$ without using the geometry of an infra-solvmanifold. (On the other hand, we use polynomial geometry.)

**Theorem 8.3.** The inclusion

\[
\left( \bigwedge u_{\mathbb{R}}^* \otimes V_{\mathbb{R}} \right)^{\mathcal{T}(\mathbb{R})} \subset A^*(M_{\Gamma}, E)
\]

induces a cohomology isomorphism

\[
H^* (u_{\mathbb{R}}^*, V_{\mathbb{R}})^{\mathcal{T}(\mathbb{R})} \cong H^*(M_{\Gamma}, E).
\]

**Proof.** Let $A^*(\mathcal{U}(\mathbb{R}), V)$ be the cochain complex of $V$-valued $C^\infty$-differential forms on $\mathcal{U}(\mathbb{R})$ and $A^*_{pol}(\mathcal{U}(\mathbb{R}), V) \subset A^*(\mathcal{U}(\mathbb{R}), V)$ the subcomplex of $V$-valued $\mathbb{R}$-polynomial differential forms. Then as an $\mathbb{R}$-rational $G_{\rho}^\ell$-module, we have

\[
A^*_{pol}(\mathcal{U}(\mathbb{R}), V_{\mathbb{R}}) = \bigwedge u_{\mathbb{R}}^* \otimes V_{\mathbb{R}} \otimes \mathbb{R}[\mathcal{U}].
\]
Since $\Gamma$ is Zariski-dense in $G^\rho$, we have

$$A^\ast_{\text{poly}}(\mathcal{U}(\mathbb{R}), V)^\Gamma = A^\ast_{\text{poly}}(\mathcal{U}(\mathbb{R}), V_\mathbb{R})^{\mathcal{G}_\Gamma} = \left( \bigwedge u_\mathbb{R}^* \otimes V_\mathbb{R} \right)^{\mathcal{T}(\mathbb{R})}. $$

By Corollary 8.1 for any $s$, we have

$$H^*(\Gamma, A^\ast_{\text{poly}}(\mathcal{U}(\mathbb{R}), V_\mathbb{R})) = 0 \quad \text{for } * > 0.$$ 

for $* > 0$. Since $\mathcal{U}(\mathbb{R}) = \mathbb{R}^r$ by the exponential map, we have

$$H^*(A^\ast_{\text{poly}}(\mathcal{U}(\mathbb{R}), V_\mathbb{R})) = 0 \quad \text{for } * > 0.$$

Let $\mathcal{A}^\ast(M_\Gamma, E_\Gamma)$ be the complex of sheaves of differential forms with values in $E$. Consider the subcomplex $A^\ast_{\text{poly}}(M_\Gamma, E)$ of $\mathcal{A}^\ast(M_\Gamma, E)$ such that for an open set $U \in M_\Gamma$, the section $A^\ast_{\text{poly}}(M_\Gamma, E)(U)$ of $U$ consists of the forms $\omega \in \mathcal{A}^\ast(M_\Gamma, E)(U)$ so that $\pi^*\omega$ is a polynomial form on $\pi^{-1}(U)$ where $\pi : \mathcal{U}(\mathbb{R}) \to M_\Gamma$ is the quotient map. Then, for each $s$, for small $U$ we have $A^\ast_{\text{poly}}(M_\Gamma, E)(U) = A^\ast_{\text{poly}}(\mathcal{U}(\mathbb{R}), V_\mathbb{R})$, and the sheaf $A^\ast_{\text{poly}}(M_\Gamma, E)$ is the locally constant sheaf corresponding to the $\Gamma$-module $A^\ast_{\text{poly}}(\mathcal{U}(\mathbb{R}), V_\mathbb{R})$. Hence by the above argument, the complex $A^\ast_{\text{poly}}(M_\Gamma, E)$ is an acyclic resolution of the locally constant sheaf $\mathcal{E}$ defined by $E$. We have the diagram

$$
\begin{align*}
\mathcal{E} \longrightarrow & A^0_{\text{poly}}(M_\Gamma, E) \longrightarrow A^1_{\text{poly}}(M_\Gamma, E) \longrightarrow \ldots \\
\downarrow & \downarrow \downarrow \downarrow \\
\mathcal{E} \longrightarrow & A^0(M_\Gamma, E) \longrightarrow A^1(M_\Gamma, E) \longrightarrow \ldots
\end{align*}
$$

Let $A^\ast_{\text{poly}}(M_\Gamma, E)$ be the global section of $A^\ast_{\text{poly}}(M_\Gamma, E)(M)$. By the standard argument of sheaf cohomology and the de Rham theorem (see e.g. [28, Section 4]), the inclusion

$$A^\ast_{\text{poly}}(M_\Gamma, E) \subset A^\ast(M_\Gamma, E)$$

induces a cohomology isomorphism. By

$$\left( \bigwedge u_\mathbb{R}^* \otimes V_\mathbb{R} \right)^{\mathcal{T}(\mathbb{R})} = A^\ast_{\text{poly}}(\mathcal{U}(\mathbb{R}), V_\mathbb{R})^\Gamma = A^\ast_{\text{poly}}(M_\Gamma, E),$$

the theorem follows. \hfill \Box

By Corollary 8.1 we can also give the continuous cohomology version of the cohomology vanishing on a bounded polynomial simply transitive action of a simply connected solvable Lie group $G$. For the $\mathbb{R}$-defined algebraic group $G^\rho = \mathcal{T} \ltimes \mathcal{U}$ with a Zariski-dense inclusion $G \subset \mathcal{T} \ltimes \mathcal{U}$, the restriction of the projection $p : \mathcal{T} \ltimes \mathcal{U} \to \mathcal{U}$ on $G$ is a diffeomorphism onto $\mathcal{U}(\mathbb{R})$, and by this diffeomorphism, the restricted action of $G$ on $\mathcal{U}(\mathbb{R})$ can be identified with the action of left translation on $G$ (see [2, Section 2]). Hence as in the above argument, the action of $G$ on $u_\mathbb{R} = \mathbb{R}^r$ is bounded polynomial and simply transitive. In [5], Benoist and Dekimpe showed that a bounded polynomial simply transitive action of $G$ is unique up to conjugation by a bounded polynomial diffeomorphism. Hence we have the following result.
Theorem 8.4. Let $G$ be a simply connected solvable Lie group. Suppose $G$ admits a bounded polynomial simply transitive action on $\mathbb{R}^r$. Consider the vector space $P(\mathbb{R}^r)$ of the polynomial functions on $\mathbb{R}^r$ as a continuous $G$-module. Then, for any representation $\rho : G \to GL(V_{\mathbb{R}})$ of a finite-dimensional $\mathbb{R}$-vector space $V$, for $\ast > 0$, we have

$$H^\ast_c(G, P(\mathbb{R}^r) \otimes V_{\mathbb{R}}) = 0.$$ 

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