Asymptotic normalization coefficient method for two-proton radiative capture

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\textbf{Abstract}

The method of asymptotic normalization coefficients is a standard approach for studies of two-body non-resonant radiative capture processes in nuclear astrophysics. This method suggests a fully analytical description of the radiative capture cross section in the low-energy region of the astrophysical interest. We demonstrate how this method can be generalized to the case of three-body 2p radiative captures. It was found that an essential feature of this process is the highly correlated nature of the capture. This reflects the complexity of three-body Coulomb continuum problem. Radiative capture $^{15}$O$p+p→^{17}$Ne+$γ$ is considered as an illustration.

\textbf{Keywords}: asymptotic normalization coefficient method; two-proton nonresonant radiative capture; E1 strength function; three-body hyperspherical harmonic method.

\textbf{Introduction}

In the asymptotic normalization coefficient (ANC) approach the nuclear wave function (WF) is characterized only by the behavior of its asymptotics. This asymptotics is defined in terms of the modified Bessel function of the second kind $K$ in the Coulomb case

$$ψ_{gs}(r→∞) = C_2 \sqrt{2qr/π} K_{l+1/2}(qr) \sim C_2 \exp[-qr],$$

or in terms of the Whittaker function $W$ in Coulombic case

$$ψ_{gs}(r→∞) = C_2 W_{-η,l+1/2}(2qr) \sim C_2 (2kr)^{-η} \exp[-qr],$$

where $η = Z_1 Z_2 e^2 M/q$ is the Sommerfeld parameter. Thus the asymptotics and, hence, the related observables are defined just by two parameters: the g.s. binding energy $E_b = q^2/(2M)$ and the 2-body ANC value $C_2$.

Such an approximation is valid for highly peripheral processes. The nonresonant radiative capture reactions at astrophysical energies are the main subject of interest here \cite{1,2,3,4,5}. An asymptotic normalization coefficient characterizes the virtual decay of a nucleus into clusters and, thus, it is equivalent to the coupling constant in particle physics \cite{6}. For that reason the ANC formalism naturally provides a framework for deriving the low-energy astrophysical information from peripheral reactions, such as direct transfer reactions, at intermediate energies (the so-called “Trojan horse” method \cite{7,8,9}). From the short list of references above, it can be seen that the ANC study is quite active and has a number of controversial unresolved issues.

For the network nucleosynthesis calculations in a thermalized stellar environment it is necessary to determine the astrophysical radiative capture rates $⟨σ_{part,γ}⟩$. The two-body resonant radiative captures

$$⟨σ_{part,γ}⟩(T) = \frac{1}{T^{3n/2}} \exp \left[ -\frac{E_r}{kT} \right] \frac{Γ_γ Γ_{part}}{Γ_{tot}}, \quad (1)$$

can be related to experimentally observable quantities \cite{10,11,12}: resonance position $E_r$, gamma $Γ_γ$ and $Γ_{part}$ particle widths ($n = 1$ for two-body and $n = 2$ for three-body captures).

The situation is much more complicated for nonresonant radiative capture rates. The direct measurements of the low-energy capture cross sections could be extremely difficult. For the three-body capture rates the direct measurements of the capture cross sections are not possible. Therefore, experimental approaches to three-body processes include studies of the photo and Coulomb dissociation, which are reciprocal processes for radiative captures. However, the “extrapolation” of three-body cross sections from experimentally accessible energies to the low energies, important for astrophysics, may require tedious theoretical calculations. This is because relatively simple “standard” quasiclassical sequential formalism \cite{11,11} may not work in essentially quantum mechanical cases \cite{12,13,14,15}.

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The 2n and 2p nonresonant radiative capture rates have been investigated in a series of papers Ref. [13][14][15] by the examples of the \(^{4}\text{He}+\text{n}+\text{n} \rightarrow ^{4}\text{He}+\gamma\) and \(^{1}\text{O}+\text{p}+\text{p} \rightarrow ^{1}\text{Ne}+\gamma\) transitions. This work also required the development of exactly solvable approximations to understand underlying physics of the process and achieve the accuracy needed for astrophysical calculations [16][17][18]. Some of the universal physical aspects observed in the papers mentioned above have motivated the search for simple analytic models. The following qualitative aspects of the low-energy E1 strength function (SF) behavior were emphasized in [18][14][15]: (i) sensitivity to the g.s. binding energy \(E_b\); (ii) sensitivity to the asymptotic weights of configurations determining the transition; (iii) importance of one of near-threshold resonances in the two-body subsystems (virtual state in \(n-n\) channel in the neutral case and lowest resonance in the core-p channel in the Coulombic case), which effect on SF is found to be crucial even at asymptotically low three-body energies. Points (i) and (ii) are the obvious motivation for ANC-like developments; point (iii) represents important and problematic difference from the two-body case.

This work to some extent summarize this line of research suggesting analytical framework for two-nucelar astrophysical capture processes. We demonstrate that it is possible to generalize the two-body ANC2 method to the ANC3 method in the situation of three-body radiative captures. While for the 2n capture the practical applicability of ANC3 method remains questionable, for the 2p captures it is established beyond any doubt. In this work we provide compact fully analytical framework for the processes, which previously could be considered only in bulky and unstable three-body calculations.

### 2. ANC3 in the hyperspherical harmonics (HH) approximation

The HH formalism for calculations of the E1 SF is provided in details in Ref. [19] and here we just give a sketch. Assume that the bound and continuum wave functions (WF) can be described in a three-cluster core+\(n+n\) approach by solving the three-body Schrödinger equation

\[
\hat{H}_3 - E_T \Psi = 0,
\]

where \(E_T\) is the energy relative to the three-cluster breakup threshold. See Fig. 1 for definition of coordinates used in this work. In the hyperspherical harmonics method this equation is reduced to a set of coupled differential equations

\[
\Psi(\rho_\gamma, \Omega_3) = \rho^{-5/2} \sum_{K\gamma} \chi_{K\gamma}(\rho_\gamma) \mathcal{J}_{K\gamma}(\Omega_3),
\]

\[
\left[ \frac{d^2}{dp^2} + \frac{L(L+1)}{\rho^2} + 2M(E_T - V_{K\gamma}, \rho_\gamma) \right] \chi_{K\gamma}(\rho_\gamma)
= \sum_{K\gamma' \neq K\gamma} 2MV_{K\gamma', \rho_\gamma},
\]

\[
\rho^2 = (A_1A_2r_{12}^2 + A_2A_3r_{23}^2 + A_3A_1r_{31}^2)/(A_1 + A_2 + A_3),
\]

depending on the collective coordinate — angular momentum \(\rho\). The “scaling” mass \(M\) is taken as an average nucleon mass in the system and \(\mathcal{J}_{K\gamma}(\Omega_3)\) is the hyperspherical harmonic. The three-body potentials are defined as

\[
V_{K\gamma', \rho_\gamma}(\rho_\gamma) = \langle \mathcal{J}_{K\gamma'}(r) | V_{ij}(r_{ij}) | \mathcal{J}_{K\gamma} \rangle.
\]

The effective orbital momentum \(L = K + 3/2\) is nonzero even for the lowest excitation \(K = 0\).

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**Fig. 1.** Coordinate systems and potential sets for “hyperspherical harmonics” HH and “simplified Hamiltonian” SH approaches to ANC3. (a) The complete 3-body Hamiltonian. (b) For core+p+p system the dynamical domination of lowest resonance in the core-p subsystem motivates the use of simplified Hamiltonian in the “Y” Jacobi system. (c) For core+n+n system the dynamical domination of the \(n-n\) final state interaction motivates the use of simplified Hamiltonian in the “T” Jacobi system.

The continuum three-body problem is solved using the same Eq. (5) but for continuum WF \(\chi_{JK\gamma, K'\gamma'}(\rho_\gamma)\) (square matrix of solutions) diagonalizing S-matrix. Hypermomentum \(K\) is defined as \(K = \sqrt{2ME_T}\). In the no-Coulomb case the WF is constructed by diagonalizing the 3 \(\to\) 3 elastic scattering S-matrix on asymptotics

\[
\chi_{JK\gamma, K'\gamma'}(\rho_\gamma) = \exp(i\delta_{K\gamma, K'\gamma'}\rho_\gamma - (K + 2)\pi/2
+ i\delta_{\gamma, \gamma'}) \delta_{K', K} = \exp(2i\delta_{K\gamma, K'\gamma'}),
\]

in analogy with the two-body case. The details of the method and its applications are well explained in the literature [20][21][22][23][19] and we will not dwell on that too much.

The form of hyperspherical equations (3) immediately provides the vision for the low-energy behavior of observables in E1 continuum since the only \(K = 1\) component with the lowest centrifugal barrier is important in the \(E_T \to 0\) limit.

The E1 transitions between three-body cluster core+\(n+n\) states are induced by the following operator

\[
\mathcal{O}_{E1, mn} = \sum_{i=1,3} Z_i r_i Y_{1m}(\rho_i),
\]

where \(D = \sum_{i=1,3} e_i^2 r_i\) is the dipole operator, and

\[
\mathcal{O}_{E1, m} = Z eff \rho \cos(\theta_\rho) Y_{1m}(\gamma),
\]

\[
Z_{eff} = \left\{ \frac{Z_i^2}{(Z_i - A_3)^2} \right\} A_3(A_1 + A_2 + A_3).
\]

The upper value in curly braces is for core+n+n and the lower one is for core+p+p three-body systems, taking into account the c.m. relation \(r_1 + r_2 = -A_3 r_3\) for the three-body system.

The 1\textsuperscript{st} dissociation SF in the HH approach is

\[
\frac{d\mathcal{B}_{E1}}{dE_T} = \sum_{J_1} J_1 f_{J_1} \sqrt{\frac{M}{2E_T}} \sum_{K\gamma} |M_{J_1, K\gamma}|^2,
\]

where \(J_1 f_{J_1} = (2J_1 + 1)/(2J_1 + 1)\) is a statistical factor, and the matrix element \(M_{J_1} \) is

\[
M_{J_1, K\gamma} = Z eff \sum_{K\gamma'} \sum_{K\gamma''} M_{\gamma'} M_{\gamma''} \int d\rho \sqrt{2/\pi} \chi_{J_1, K\gamma' \gamma''}(\rho_\gamma) \rho \chi_{J_1, K\gamma}(\rho_\gamma).
\]

For example, the reduced angular momentum matrix element \(M_{\rho} = 1/\sqrt{4\pi}\) for \(J = 0 \to J = 1\) transition and the hyper-angular matrix element \(M_{\theta} = 1/\sqrt{2}\) for \(K = 0 \to K = 1\) transition.
Fig. 2. Left axis: ground state WF components with $K = 0$ for $^4$He (solid curve) and $^{17}$Ne (dashed curve), matched to asymptotic Bessel (thick gray curves) and Whittaker (short dashed and dotted curves) functions. For $^4$He ANC3 value is $C_3 = 0.3 \text{ fm}^{-1/2}$. For $^{17}$Ne ANC3 values are $C_3 = 13400 \text{ fm}^{-1/2}$ with $Z_{hh} = 27.57$ (see Eq. (24)) and $C_3 = 5958 \text{ fm}^{-1/2}$ with fitted $Z_{hh} = 26.14$. The lines on the right axis show the ratio of $^{17}$Ne WF to Whittaker functions with mentioned $Z_{hh}$ and $C_3$.

2.1. No Coulomb case in HH approach

For the three-body plane-wave case the solution matrix is diagonal and expressed in terms of cylindrical Bessel functions

$$\chi_{Jf,K_f \gamma, K_f'}(\rho \rho') = \sqrt{\frac{2}{\pi}} \delta_{K_f, K_f'} \delta_{J_f, J_f'} \sqrt{\frac{\pi \rho \rho'}{2}} J_{K_f+2}(\rho \rho'),$$

with asymptotics for small $\rho \rho'$

$$\sqrt{\rho \rho'} J_{K_f+2}(\rho \rho') \sim (\rho \rho')^{K_f+5/2}/[(K_f + 2)! 2^{K_f+2}].$$

(7)

This expression can be used to separate the leading term of the low-energy dependence of the matrix element, labeled for simplicity only by the values of $K$ for the initial and final states

$$M_{K_i,K_f}(E_T) = \frac{1}{\pi \rho^{3/2}} \int d\rho \rho^{3/2} J_{K_f+2}(\rho \rho') \chi_{J_f,K_f \gamma}(\rho),$$

(8)

where the overlap integral $I_{K_i,K_f}$ tends to a constant at $E_T \to 0$ and weakly depends on energy in the range of interest.

Let us consider only the transition from the lowest bound state component $K_i = 0$ to the lowest $E_1$ continuum component with $K_f = 1$:

$$dB_{E_1}/dE_T = \frac{1}{\pi} G_{Jf,Z_f M_f^2} Z_{hh}^2 M_{hh}(2M)^4 E_T^2 I_{10}(E_T).$$

(10)

Now we replace the bound state WF $\chi$ in Eq. (9) by its long-range asymptotics expressed in terms of the three-body ANC value $C_3$ and cylindrical Bessel functions $K$

$$\chi_{K=0}(\rho) \to C_3 \sqrt{2\rho \rho'/\pi} K_2(\rho \rho'),$$

(11)

where the g.s. hypermoment $K = \sqrt{2ME_b}$ is defined via the binding energy $E_b$. This approximation is valid in a broad range of $\rho$ values, see Fig. 2. The $^4$He WF is taken from [19] The overlap integral now has simple analytical form

$$I_{10}(E_T) = 4C_3/[(2ME_b)^{1/4}(1 + E_T/E_b)^2].$$

(12)

It can be found that the ANC3 approximation of the overlap value (12) deviates within very reasonable $\sim 7\%$ limits from the directly calculated by Eq. (11) in a broad energy range ($E_T \leq 1 \text{ MeV}$).

2.2. Discussion of $^6$He case

The $E_1$ SF and the astrophysical capture rate for the $\alpha+n+n \to ^6\text{He}+\gamma$ was recently studied in Refs. [15, 19]. It can be found that Eq. (10) is not sufficient in this case for two reasons:

(i) In the $p$-shell $^6$He nucleus not only the $K_i = 0 \to K_f = 1$ transition is important, but also $K_i = 2 \to K_f = 1$. The asymptotics of the $K_i = 2$ WF component falls off much faster than that of the component $K_i = 0$. However, the weight of the $K_i = 2$ WF component corresponding to $[p^2]$ configuration is much larger ($\sim 80\%$), than the weight of the $K_i = 0$ WF component ($\sim 5\%$), due to Pauli-suppressed $[s^2]$ configuration. So, finally their contributions to the low-energy ME are comparable.

(ii) It was shown in [15, 19] that the low-energy part of the $E_1$ SF is highly sensitive to the final state $n-n$ interaction (an increase in SF when the $n-n$ interaction is taken into account is a factor of 8). The paper [13] is devoted to the study of this effect in the dynamic dineutron model. We do not currently see a method to consider this effect analytically.

Applicability of the approximation (11) to the other cases of $2n$ capture should be considered separately.

2.3. Coulomb case in HH approach

Let us consider the transition to the single $K_f = 1$ continuum final state. The low-energy behavior of continuum single channel WF in the Coulomb case is provided by the regular at the origin Coulomb WF

$$\chi_{K_f}(\rho \rho') \to \sqrt{2/\pi} F_{K_f+3/2}(\eta_{hh}, \rho \rho').$$

(13)

The suitable asymptotics of the Coulomb WFs are

$$F_{I}(\eta, kr) = \frac{(2l + 1)! C_I(\eta)}{(2\eta)^{l+1}} \sqrt{2\beta} l_{l+1}(2\sqrt{2\beta}),$$

(14)

$$G_{I}(\eta, kr) = \frac{2(2\eta)^l}{(2l + 1)! C_I(\eta)} \sqrt{2\beta} k_{l+1}(2\sqrt{2\beta}),$$

(15)

$$C_I(\eta) = 2^l \exp[-\eta/2] \left[ |l| + i|n| + 1 \right] / |l|! |l|! 1 + |l|),$$

(16)

$$C_I(\eta) = \frac{(2\eta)^{l+1/2}}{2(2l)!} \exp[-\eta],$$

(17)

$$D_{I}(\eta, k) = \frac{2(2l + 1)! C_I(\eta)}{(2\eta)^{l+1}} \sqrt{2\beta} = \sqrt{\pi} k \exp[-\eta],$$

(18)

where $I$ and $K$ are modified Bessel functions. Approximation (17) for the Coulomb coefficient (10) works for $\eta \gg 1$.

In the ANC3 approximation the g.s. WF $\chi$ can be replaced by its long-range asymptotics

$$\chi_{K,0}(\rho) \to C_3 W_{-\eta_{hh} K,2}(2\rho \rho').$$

(19)

The Sommerfeld $\eta$ for continuum and bound states are

$$\eta_{hh} = Z_{hh}^2 e^2 M, \quad \eta_{hh} = \eta_{hh} / \sqrt{\kappa}, \quad \eta_{hh} = Z_{hh}^2 e^2 M / \kappa.$$ (20)

The effective charges for isolated hyperspherical channels can be defined as

$$Z_{hh}(Kl \ell_\nu) = \rho \left( K I_{2l} \rho \right)_{r_{l_{12}}} \left( Z_2 Z_3 \right)_{r_{23}} + \left( Z_3 Z_1 \right)_{r_{31}} \left( K I_{2l} \rho \right)_{r_{l_{31}}}.$$ (21)

For the $^{17}$Ne case the $K = 0$ and $K = 1$ effective charges are

$$Z_{hh}^{(000)} = 27.50, \quad Z_{hh}^{(101)} = 27.41.$$ (22)

Fig. 2 shows that the substitution Eq. (19) works well in a very broad range of radii (the $^{17}$Ne g.s. WF is from Ref. [24]).
The effective charge in Eq. (22) obtained for $K_i = 0$ is very reasonable. However, slightly different effective charge value $Z_{hh}^{(000)} = 26.14$ is required for an almost perfect match to the asymptotics. This is a clear indication of coupled-channel dynamics in this case. It acts as a nontrivial fact that all the complexity of this dynamics reduces to a simple renormalization of effective charges.

Using Eqs. (13) and (14) we can factorize the $E1$ matrix element as:

$$M_{K_i K_f} = \sqrt{2/\pi} D_{K_f+3/2}(\eta_{hh}, \chi) M_{hh} Z_{eff} I_{K_i K_f}(E_T),$$

$$I_{K_i K_f}(E_T) = \int d\rho \frac{F_{K_f+3/2}(\eta_{hh}, \rho, \chi)}{D_{K_f+3/2}(\eta_{hh}, \rho, \chi)} \rho^{3/2} \chi J_i K_f \gamma_i(\rho),$$

where the overlap integral $I_{K_i K_f}(E_T)$ weakly depends on the energy and in the limit $E_T \to 0$ has a rather simple form

$$I_{K_i K_f}(E_T) \approx \int d\rho I_{21+1}(2\sqrt{2/\pi} \rho, \chi) \rho^{3/2} \chi J_i K_f \gamma_i(\rho).$$

The overlaps (23) for $K_i = 0 \to K_f = 1$ transition are shown in Fig. 3. It can be found that in the ANC3 approximation the Eq. (18) is quite accurate: in this case the overlap increases just less than 6% compared the calculation with the real g.s. WF. It is also seen that the use of simple energy-independent overlap Eq. (24) instead of (23) gives almost perfect result below 10 keV and is reasonable below 100 keV. For the $E1$ SF we get:

$$\frac{dB_{E1}}{dE_T} = G_f J_i Z_{eff}^2 M_{hh}^2 M_{ij}^2 I_{10}^{(2)}(E_T) \exp[-2\pi \eta_{hh}].$$

The energy dependence of the derived expression at $E_T \to 0$ is pure Coulomb exponent $\exp[-2\pi \eta_{hh}]$. The SF calculation results are shown in Fig. 4. They strongly disagree with calculation results from Refs. [13] and [14]. The modification of the “effective continuum charge” $Z_{hh}^{(001)}$ from Eq. (22) does not save the situation since the energy dependence of the SF in Eq. (25) and that of the SF in [13,14] are too different. We demonstrate in the next section that the Eq. (25) is actually incorrect. However, the derivations of this section are still important for our further discussion.

3. ANC3 in the simplified Hamiltonian (SH) approximation

The approximation is based on the usage of a simplified three-body Hamiltonian for the $E1$ continuum instead of the real one

$$\hat{H}_3 \to \hat{H}_3^{\prime} = \hat{T}_3 + V_{\text{ex}}(X) + V_{\text{y}}(Y),$$

where $X \equiv r_{41}$ is the Jacobi vector in the “Y” Jacobi system, while $Y$ corresponds to the second Jacobi vector, see Fig. 1. Such a Hamiltonian is quite reliable since the nuclear interaction with a proton in a non-natural parity state is weak. The model was used for nonresonant astrophysical rate calculations in $^{17}$Ne in Ref. [13] and in $^4$He in Ref. [15]. A thorough check of the model is given in Ref. [16], and the detailed description of the formalism for complicated angular momentum couplings in Ref. [14].

To obtain the $E1$ strength function in this approximation we solve the following inhomogeneous Schrödinger equation:

$$(\hat{H}_3 - E_T)\Psi_{M_{3 m}}^{J_f M_f^{(+)}} = \mathcal{O}_{E1 m} \Psi_{M_{3 m}}^{J_f M_f}. $$

The transition operator Eq. (11) depending on $r_3$ can be rewritten in $X$ and $Y$ coordinates:

$$r_3 = X A_2/(A_2 + A_3) - Y A_1/(A_1 + A_2 + A_3).$$

Since the factorized form of the Hamiltonian in Eq. (25) allows a semi-analytical expression for the three-body Green’s function,
3.1. Coulomb case in SH approach

A rather simple expression for the SF can be obtained

\[
\frac{dB_{E1}}{d\varepsilon} = G_{f_{1}} \frac{4}{\pi^{2}} E_{\ell} \int_{0}^{1} \frac{d\varepsilon}{k_{x}k_{y}} |A(E_{x}, E_{y})|^{2},
\]

(28)

where \( E_{x} = \varepsilon E_{\ell}, \quad E_{y} = (1 - \varepsilon) E_{\ell}, \quad k_{x,y} = \sqrt{2M_{x,y}E_{x,y}}, \)

where \( \varepsilon \) is the energy distribution parameter. The amplitude \( A \) is defined as

\[
A(E_{x}, E_{y}) = \int dX dY f_{i_{x}}(k_{x}X) f_{i_{y}}(k_{y}Y) \Phi(X, Y).
\]

(29)

where the “source function” \( \Phi \) is defined by the E1 operator acting on \( \Psi_{gs} \). The WF \( f_{i} \) and \( f_{i_{x}} \) are eigenfunctions of sub-Hamiltonians depending on \( X \) and \( Y \) Jacobi coordinates in S-matrix representation with asymptotics

\[
f_{i_{x}}(k) = e^{i\varepsilon_{i_{x}}x} \Phi_{i_{x}}(kx) \cos(\lambda b_{i_{x}}) + G_{i_{x}}(k) \sin(\lambda b_{i_{x}}).
\]

(30)

Eq. (29) is provided in a simplified form, neglecting angular momentum couplings, more details can be found in [14]. We skip this part of the formalism in this work. The calculations of the E1 strength function in the SH approximation without final state interactions in \( X \) and \( Y \) channels for the 2n capture are equivalent to calculations in the HH approximation. So, we skip no Coulomb case and proceed to the 2p capture.

3.1. Coulomb case in SH approach

With good accuracy, one can calculate the amplitude only for the \( Y \) coordinate and then double the result. This is not difficult to prove, but tedious, so we do not provide a proof here. The amplitude \( A \) for the \( Y \) coordinate [see Eq. (24)] from the transition operator Eq. (4) with extracted by Eqs. (11) and (13) low-energy dependence is written in terms of the overlap integral \( I^{(cc)} \) as

\[
A(E_{x}, E_{y}) = M_{x} D_{i_{x}}(\eta_{x}, k_{x}) D_{i_{y}}(\eta_{y}, k_{y}) I^{(cc)}_{i_{x}i_{y}}(\varepsilon, E_{\ell}),
\]

(31)

\[
I^{(cc)}_{i_{x}i_{y}}(\varepsilon, E_{\ell}) = \int dX dY F_{i_{x}}(k_{x}X) F_{i_{y}}(k_{y}Y) \frac{1}{D_{i_{x}}(\eta_{x}, k_{x}) D_{i_{y}}(\eta_{y}, k_{y})} \psi_{gs}(X, Y),
\]

\[
\eta_{x} = Z_{2}Z_{3}e^{2}M_{x}/k_{x}, \quad \eta_{y} = (Z_{2} + Z_{3})Z_{1}e^{2}M_{y}/k_{y}.
\]

The asymptotic form of this overlap, independent of energy, is

\[
I^{(cc)}_{i_{x}i_{y}} = \frac{1}{2} \sqrt{X} \sqrt{Y} \psi_{gs}(X, Y).
\]

(32)

The WF \( \psi_{gs} \) and the integrand of Eq. (31) on the \( \{X, Y\} \) plane are shown in Figs. 5 (a) and (b). Their comparison illustrates the extreme peripheral character of the low-energy E1 transition: the WF maximum is at a distance of \( \sim 3 \text{ fm} \), while sizable contributions to the transition ME can be found up to \( \sim 60 \text{ fm} \).

The Coulomb exponent in \( I \) has a very sharp energy dependence, see Fig. 6. The energy dependence of \( I^{(cc)} \) is shown in Fig. 5 (c): it is quite flat for \( \varepsilon \approx \varepsilon_{0} \). Thus, \( I^{(cc)} \) can be evaluated at the peak \( \varepsilon = \varepsilon_{0} \) and the \( \varepsilon \) integration can be performed by the saddle point method:

\[
I(E_{\ell}) = I^{(cc)}_{10}^{2}(\varepsilon_{0}, E_{\ell}) E_{\ell} \frac{\exp(-2\pi\varepsilon_{0})}{\sqrt{3\varepsilon_{0}^{2}2}},
\]

(33)

\[
\varepsilon_{0} = \frac{b_{x}}{b_{y}}, \quad Z_{sh} = \frac{b_{x} + b_{y}}{\sqrt{2}},
\]

(34)

For the \( \text{^{17}Ne} \rightarrow \text{^{15}O} + p + p \) transition

\[
\varepsilon_{0} = 0.48, \quad Z_{sh} = 23.282.
\]

(35)

The accuracy of the saddle point integration is \( \sim 2\% \) and \( \sim 6\% \) at 0.1 and 1 MeV, respectively.

It can be found in Fig. 8 that the analytical energy dependence of Eq. (12) obtained for the same system without Coulomb interaction is still a good approximation in the considered Coulomb case,

\[
I^{(cc)}_{10}(\varepsilon_{0}, E_{\ell}) = I^{(cc)}_{10}\frac{\sqrt{1 + E_{\ell}/E_{cy}}}{(1 + E_{\ell}/E_{cy})^{2}}, \quad E_{cy} = \frac{2M_{p}\beta_{0}^{2}}{1 - \varepsilon_{0}},
\]

(36)

Eq. (38) contains additional Coulomb correction for \( l_{y} = 1 \) motion in \( Y \) coordinate (with \( E_{cy} = 3.67 \text{ MeV} \)) and we use it later for astrophysical rate derivation.

The results of the SF calculation in the SH approximation are shown in Fig. 4 by the red dashed curve. Now there is no significant disagreement for \( E_{\ell} \to 0 \) of the SH model results with calculation results from Refs. [13] and [14]. In the next Sections 4.2 and 3.3 we answer the following questions: (i) what is the reason for the difference between HH and SH ANC3 methods and (ii) can we get a better “fit” of the complicated three-body results in the SH approximation?
## 3.2. Correlated 2p emission/capture

In the HH Eq. (25) and SH Eq. (33) approximations we get SF expressions with the low-energy asymptotics

$$
\frac{dB_{E1}}{dE_T} \propto \exp(-2\pi \eta_{ph}), \quad \frac{dB_{E1}}{dE_T} \propto \exp(-2\pi \eta_{ph})^{3/4}, \quad (39)
$$

which are qualitatively different. There are two main points:

(i) Effective charge, entering the Coulomb exponent is significantly lower in SH case, see Eq. (37) compared to Eq. (22): 27.41 vs. 23.282.

(ii) There is an additional power dependence on energy $E_T^{3/4}$ which, evidently, cannot be compensated, for example, by some modification of the effective charge. What is the source of qualitative difference between Eqs. (25) and (33)?

The answer is actually provided in Fig. 6 in the SH approach the emission of two protons is highly correlated process, which produces the narrow bell-shaped $\varepsilon$ distributions. In the approximation $K_f = 1$ used in Eq. (25) the momentum distribution is described by the phase space (thick gray curve in Fig. 6). In the correlated calculation Eq. (33) this distribution is drastically modified by the three-body Coulombic effect. The momentum distribution which “shrinks” to the proximity of the $\varepsilon_0$ value allows an easier penetration, which is reflected also in the smaller effective charge $\beta_f$ in the Coulomb exponent in Eq. (25).

So, what is wrong with Eq. (25)? Formally the transition by the dipole operator from $K_f = 0$ g.s. occurs only to $K_f = 1$ continuum, as we assumed. This means that the substitution of Eq. (33) is incorrect. This substitution is based on the assumption that for $E_T \rightarrow 0$ only the component with minimal possible $K_f = 1$ and, hence, the smallest centrifugal barrier, contributes to the penetrability. Now it is clear that for the three-body continuum Coulomb problem this “evident” argument is incorrect. Within the complete HH couple-channel problem the $K_f = 1$ channel should be affected by an infinite sum of the other channels in such a way that their cumulative effect does not vanish even in the limit $E_T \rightarrow 0$.

Analogous energy correlation effect is well known for the 2p radioactivity process. It was predicted by Goldansky in his pioneering work on 2p radioactivity [25]: in the Coulomb-correlated emission of two protons the energies of the protons tend to be equal in the limit of infinitely strong Coulomb interaction in the core+$p$ channel. This effect for two-proton radioactivity and resonant “true” two-proton emission is now well studied experimentally and understood in details in theoretical calculations [23,26]. It is proved that the approximations like Eq. (33) represent well the underlying physics of the phenomenon.

### 3.3. Effect of a resonant state in a two-body subsystem

It was shown in calculations of [13-14] (see Figs. 3 and 4-5 in these works) that the resonant state in the core-$p$ sub-

system with “natural parity” quantum numbers significantly affects both the profile of the E1 strength function in a wide range of energies and the asymptotic behavior at low $E_T$ values. To evaluate the influence of resonance on the asymptotics analytically, let us consider the two-body resonant scattering WF in the quasistationary approximation:

$$
f_l(kr) = e^{i\delta_l} F_l(kr) \cos(\delta_l) + \sqrt{\varepsilon_0} \Gamma(E)/2 \frac{\Gamma(E)/2}{E_r - E - i\Gamma(E)/2} \tilde{\psi}_l(r). \quad (40)
$$

This expression can be easily connected with the asymptotics Eq. (20) by using the resonant R-matrix formulas:

$$
\tan(\delta_l) = \frac{\Gamma(E)/2}{E_r - E - i\Gamma(E)/2}. \quad (41)
$$

The low-energy behavior of the overlap integrals Eq. (31) with the resonant continuum WF (10) in $X$ coordinate is then

$$
I_{l_4l_5}^{(cc)}(\varepsilon_0, E_T) = I_{l_4l_5}^{(cc)}(\varepsilon_0, E_T) + B(\varepsilon_0, E_T) \frac{\Gamma(E)/2}{E_r - E - i\Gamma(E)/2}. \quad (43)
$$

$$
\tilde{\psi}_l(r_c) \propto G_l(k_r r_c), \quad G_l(k_r r_c) = \int_0^{r_c} dr |\tilde{\psi}_l(r)|^2 = 1. \quad (42)
$$

The low-energy behavior of the overlap integrals Eq. (31) with the resonant continuum WF (10) in $X$ coordinate is then

$$
I_{l_4l_5}^{(cc)}(\varepsilon_0, E_T) = I_{l_4l_5}^{(cc)}(\varepsilon_0, E_T) + B(\varepsilon_0, E_T) \frac{\Gamma(E)/2}{E_r - E - i\Gamma(E)/2}. \quad (43)
$$

$$
B(\varepsilon_0, E_T) = B_c \int dX dY \tilde{\psi}_l(X) G_l(k_r r_c) Y \psi_\varepsilon(X, Y), \quad (44)
$$

$$
B_c = \theta_x [4M_4 r c K_{2l+1} (2\sqrt{2\beta_c r_c}) E_r]^{-1}. \quad (44)
$$

Here we use the R-matrix width definition

$$
\Gamma(E) = \frac{\theta^2}{M^2} P_l(E, r_c), \quad P_l(E, r_c) = \frac{k r_c}{F_l^2(k_r r_c) + G_l^2(k_r r_c)}, \quad (45)
$$

which is simplified in the low-energy region using Eq. (15).

The integrand of Eq. (44) is shown in Fig. 5 (d) and it has quite peripheral character compared to the g.s. WF Fig. 6 (a). The “resonance correction function” $B$ is shown in Fig. 5 (d) demonstrating very weak dependence on energy. It is evaluated with function $\psi_\varepsilon(X)$ approximated by Hulten Ansatz with rms $X$ value 3.5 fm. Parameters $\theta_x = 1$ and $r_c = 3.7$ fm allows to reproduce correctly the experimental width $\Gamma = 25(5)$ keV of the $^{16}$O F 0$^-$ ground state at $E_r = 535$ keV. So, in the whole energy range of interest we can approximate $B$ as

$$
\tilde{B} = B(\varepsilon_0, E_T) \rightarrow 0. \quad (46)
$$

The blue dotted curve in Fig. 6 shows nice agreement of the “resonance corrected” E1 SF with complete three-body calculations up to $\sim 600$ keV. At this energy the two-body resonance well enters the “energy window” for three-body capture $E_T < E_r$ and turnover to sequential capture mechanism is taking place.
### 3.4. Astrophysical rate calculations

The nonresonant radiative capture rate is expressed via the E1 SF as [14]:

\[
\langle \sigma_{2p, \gamma} v \rangle = \left( \sum_{\alpha} \frac{A_{\alpha}}{A_1 A_2} \right)^{3/2} \left( \frac{2\pi}{mkT} \right)^3 \frac{2J_1 + 1}{2(2J_1 + 1)} I_E(T),
\]

\[
I_E(T) = \int dE \frac{16\pi^2 E^2}{9} \frac{dB_{E1}(E)}{dE} \exp \left[ -\frac{E}{kT} \right],
\]

where \( J_1 \) and \( J_f \) are spins of the \( ^{16}\)O and \( ^{17}\)Ne g.s., respectively.

The energy dependence of Eq. (33) is too complex to allow a direct analytical calculation of the astrophysical capture rate. However, using Eqs. (33), (35), and (40), the main analytical terms can be obtained by the saddle point calculation near the Gamow peak energy \( E_G \):

\[
I_E(T) \propto \int dE_T (E_b + E_T)^3 I_s(E_T) \exp \left[ -\frac{E_T}{kT} \right] = \frac{2\pi^2 E_b^3 E_G^3}{3\gamma \sqrt{kT}} \times \left[ 1 + \frac{E_G/E_b}{1 + \frac{E_G/E_b}{\gamma, I_{10}} \left( \frac{1 + E_G/E_b}{1 - \epsilon_0 E_G/E_b} \gamma \right)^{2/3}} \exp \left[ -\frac{3\gamma^{2/3}}{(kT)^{1/3}} \right] \right],
\]

\[
E_G = (\gamma kT)^{2/3}, \quad \gamma = \pi Z_{sh} e^2 \sqrt{M/2}, \quad \pi_{sh} = \gamma / \sqrt{E_T}.
\]

The Gamow peak energy can be found as \( \{0.05, 0.21, 1\} \) MeV at \( \{0.01, 0.1, 1\} \) GK. Comparison of the rates calculated in a model of the paper Ref. [14] and in this work is given in Fig. 7. It can be seen that even the very crude energy-dependent approximation Eq. (32) \( I^{(cc)}(\varepsilon, E_T) \rightarrow I^{(cc)} \) for \( T < 0.5 \) GK works well within a factor of 2. The energy-dependent calculations Eq. (33) are nearly perfect for \( T < 0.2 \) GK and the analytical expression Eq. (35) for the rate is a very good approximation to numerically computed rate for \( T > 0.4 \) GK.

The low-temperature dependence of the nonresonant 2p capture rate is

\[
\langle \sigma_{2p, \gamma} v \rangle \propto C_2^2 T^{5/3} \exp \left[ -\left( T_{eff}/T \right)^{1/3} \right], \quad kT_{eff} \approx 193 \text{ MeV}.
\]

An interesting formal result is related to the problem of the three-body Coulomb interaction in the continuum. We demonstrate that ANC3 method developed completely in three-body hyperspherical harmonics representation (named “HH approximation”) is not valid in the Coulomb case as it gives incorrect low-energy asymptotics of SF and hence incorrect low-temperature asymptotics of the astrophysical rate. The reason for this is the highly correlated nature of the 2p capture. The correct asymptotics can be obtained using the Coulomb-correlated SH approximation based on a simplified three-body Hamiltonian. The latter approximation also allows one to determine the correction related to the low-lying two-body resonant state in the core-nucleon channel.

The two-dimensional overlap integrals involved in the ANC3 approximation in the correlated Coulomb case are rather complicated compared to those in the ANC2 case. However, their calculation is a task that is incomparably simpler than any complete three-body calculation. The whole formal framework is compact and elegant and requires only two overlap calculations: \( I^{(cc)} \) and \( \bar{B} \). Thus we find that ANC3 approximation in a three-body case is a valuable development providing robust tool for estimates of the three-body nonresonant capture rates in the low-temperature \( T < 0.3 \) – 1 GK.

**Acknowledgments.** — LVG, YLP, and NBS were supported in part by the Russian Science Foundation grant No. 17-12-01367.

### REFERENCES

1. H. M. Xu, C. A. Gagliardi, R. E. Tribble, A. M. Mukhamedzhanov, N. K. Timofeyuk, Phys. Rev. Lett. 73 (1994) 2027–2030.
2. N. K. Timofeyuk, R. C. Johnson, A. M. Mukhamedzhanov, Phys. Rev. Lett. 91 (2003) 232501.
3. D. Y. Pang, F. M. Nunes, A. M. Mukhamedzhanov, Phys. Rev. C 75 (2007) 024601.
4. J. Okołowicz, N. Michel, W. Nazarewicz, M. Płoszajczak, Phys. Rev. C 85 (2012) 064320.
5. A. M. Mukhamedzhanov, Phys. Rev. C 99 (2019) 024311.
6. L. D. Blokhintsev, I. Borbely, E. I. Dolinskii, Sov. J. Part. Nucl. 8 (1977) 485.
7. Mukhamedzhanov, A. M., Blokhintsev, L. D., Brown, B. A., Burjany, V., Cherubini, S., Gagliardi, C. A., Irgaziev, B. F., Kroha, V., Nunes, F. M., Pirlepesov, F., Pizzone, R. G., Romano, S., Spitaleri, C., Tang, X. D., Trache, L., Tribble, R. E., Tumino, A., Eur. Phys. J. A 27 (2006) 205–215.
8. A. M. Mukhamedzhanov, G. V. Rogachev, Phys. Rev. C 96 (2017) 045811.
9. A. M. Mukhamedzhanov, D. Y. Pang, A. S. Kadyrov, Phys. Rev. C 99 (2019) 064618.
10. W. Fowler, G. Caughlan, B. Zimmerman, Annual Review of Astronomy and Astrophysics 5 (1967) 525.
11. C. Angulo, M. Arnould, M. Rayet, P. Descouvemont, D. Baye, C. Leclercq-Willain, A. Coc, S. Barhoumi, P. Aguer, C. Rolfs, R. Kunz, J. Hammer, A. Mayer, T. Paradellis, S. Kossionides, C. Chronidou, K. Spyrou, S. del’Innocenti, G. Fiorentini, B. Ricci, S. Zavatarelli, C. Providencia, H. Wolters, J. Soares, C. Granata, J. Rahighi, A., Shotter, M. L. Racht, Nucl. Phys. A 656 (1999) 3–183.
12. L. V. Grigorenko, M. V. Zhukov, Phys. Rev. C 72 (2005) 015803.
13. L. Grigorenko, K. Langanke, N. Shulgina, M. Zhukov, Physics Letters B 641 (3) (2006) 254–259.
14. Y. L. Parfenova, L. V. Grigorenko, I. A. Egorova, N. B. Shulgina, J. S. Vaagen, M. V. Zhukov, Phys. Rev. C 98 (2018) 034608.
15. L. Grigorenko, N. Shulgina, M. Zhukov, Physics Letters B 807 (2020) 135557.
16. L. V. Grigorenko, M. V. Zhukov, Phys. Rev. C 76 (2007) 014008.
17. L. V. Grigorenko, M. V. Zhukov, Phys. Rev. C 76 (2007) 014009.
18. L. V. Grigorenko, J. S. Vaagen, M. V. Zhukov, Phys. Rev. C 97 (2018) 034605.
19. L. V. Grigorenko, N. B. Shulgina, M. V. Zhukov, Phys. Rev. C 102 (2020) 014611.
20. M. V. Zhukov, B. Danilin, D. Fedorov, J. Bang, I. Thompson, J.S.Vaagen, Phys. Rep. 231 (1993) 151–199.
21. L. V. Grigorenko, R. C. Johnson, I. G. Mukha, I. J. Thompson, M. V. Zhukov, Phys. Rev. C 64 (2001) 054002.
22. L. V. Grigorenko, T. D. Wiser, K. Mercurio, R. J. Charity, R. Shane, L. G. Sobotka, J. M. Elseon, A. H. Wuosmaa, A. Banu, M. McCleskey, L. Trache, R. E. Tribble, M. V. Zhukov, Phys. Rev. C 80 (2009) 034602.
23. M. Pfützner, M. Karny, L. V. Grigorenko, K. Rüisager, Rev. Mod. Phys. 84 (2012) 567–619.
24. L. Grigorenko, I. Mukha, M. Zhukov, Rev. Mod. Phys. 19 (1960) 482–495.
25. K. W. Brown, R. J. Charity, L. G. Sobotka, L. V. Grigorenko, T. A. Golubkova, S. Bedoor, W. W. Buhro, Z. Chajeeck, J. M. Elseon, W. G. Lynch, J. Manfredi, D. G. McNeel, W. Reviol, R. Shane, R. H. Showalter, M. B. Tsang, J. R. Winkelbauer, A. H. Wuosmaa, Phys. Rev. C 92 (2015) 034329.