$SU_3$ coherent state operators and invariant correlation functions and their quantum group counterparts

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Abstract

Coherent state operators (CSO) are defined as operator valued functions on $G = SL(n, C)$ being homogeneous with respect to right multiplication by lower triangular matrices. They act on a model space containing all holomorphic finite dimensional representations of $G$ with multiplicity 1. CSO provide an analytic tool for studying $G$ invariant 2- and 3-point functions, which are written down in the case of $SU_3$. The quantum group deformation of the construction gives rise to a non-commutative coset space. We introduce a “standard” polynomial basis in this space (related to but not identical with the Lusztig canonical basis) which is appropriate for writing down $U_q(s\ell_3)$ invariant 2-point functions for representations of the type $(\lambda, 0)$ and $(0, \lambda)$. General invariant 2-point functions are written down in a mixed Poncaré-Birkhoff-Witt type basis.

Key words : Quantum groups.
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1 Introduction

Extending the applications of the quantum universal enveloping algebra $U_q(s\ell_2)$ to rational conformal models with braid group statistics [1, 2, 3, 4, 5, 6] to the higher rank case encounters two types of difficulties. At the classical (undeformed) level the n-point invariants for higher rank groups are not known and appear hard to classify. At the q-deformed level we have to deal in addition with a non-commutative coset space.

A powerful analytic tool for writing down (classical and quantum) n-point invariants and $U_q$ exchange relations is provided by the notion of a coherent state operator (CSO). This is the underlying concept behind the polynomial realization of finite dimensional representations (used in this context in Refs. [7, 8]). It was spelled out and applied to the simplest rank 1 (quantum) algebra in Refs. [8, 9]. Here we extend this construction to the case of $SU_{r+1}$ and $U_q(s\ell_{r+1})$ ($r$ being the rank).

The next step, computing n-point invariants, reveals a complexity that increases with the rank. Therefore, we restrict attention to the rank 2 case, $SU_3$ and

$$U_q \equiv U_q(s\ell_3) .$$  

In order to define $U_q$ CSO we need the dual object, the quantum group $SL_q(3)$. More generally, $SL_q(n)$ is defined as a Hopf algebra with $n^2$ generators subject to ($n^2$) commutation relations (CR) and an n-linear normalization condition stating that the quantum determinant of the matrix $(T^{\alpha}_{\beta})$ is one. The CR are expressed in terms of the $U_q(s\ell_n)$ R-matrix computed for the coproduct of the defining (n-dimensional) representation of the quantum algebra -see Ref. [10]. The next step, the identification of a quantum Borel subgroup $B_q$ and of the corresponding q-coset space $SL_q(n)/B_q$, has also been sketched for an arbitrary $n$ -see Ref. [11]. In order to avoid unnecessary complications and bulky formulae we restrict from the outset attention to the case $n = 3$.

We start in Sec. 2 by introducing our version of coherent states (CS). We write down a closed form polynomial expression for the general $SU_3$ invariant 3-point function (Proposition 2.1).

The central object of this paper, the $U_q$ homogeneous space

$$N_q = SL_q(3)/B_q ,$$  

is introduced and studied in Sec. 3. Quantum CSO are defined in Sec. 4, where we write down $U_q$ invariant 2-point functions introducing on the way a new standard basis, which is a variant of the Lusztig canonical basis [12].
2 Classical coherent states

The concept of a coherent state, originally associated with a nilpotent Lie group, goes back to Hermann Weyl and has been the subject of a long evolution (see Ref. [13] and references therein). The most familiar example is given by the Fock space vector

$$\Phi(\zeta) = e^{\zeta a^* |0>},$$

where $a^*$ is a creation operator, its adjoint annihilating the Fock vacuum: $a|0>=0$. A characteristic property for such a vector valued function is provided by the equation

$$(a^* - \frac{d}{d\zeta})\Phi(\zeta) = 0.$$  

In the case of a simple compact Lie group G the role of $a^*$ is played by the Cartan-Weyl raising operators $E_\alpha$ of the complexified Lie algebra. For an irreducible finite dimensional representation of $G$, the counterpart of $\Phi$ appears as a polynomial function of its arguments.

2.1 CSO for $SU_3$. Transformation properties

To fix the ideas we begin by considering the induced representation of the complexification $SL(3, C)$ of the simple compact Lie group $SU_3$ with a Borel inducing subgroup B of lower triangular matrices. The homogeneous space $SL(3, C)/B$ has a dense open set isomorphic to the subgroup $N$ of upper triangular matrices with units on the diagonal:

$$N = \left\{ Z = \begin{pmatrix} 1 & \zeta_1 & \zeta_{12} \\ 0 & 1 & \zeta_2 \\ 0 & 0 & 1 \end{pmatrix} \right\}. \quad (2.1)$$

For each highest weight $\Lambda = (\lambda_1, \lambda_2)$ (where $\lambda_i = 0, 1, \ldots$) of $SU_3$ we define a CSO $\Lambda = \Lambda(g)$ as follows. Let $\chi_\Lambda$ be a character (i.e., a 1 dimensional representation) of the inducing subgroup B given by

$$\chi_\Lambda(b) = \beta_1^{-\lambda_2} \beta_2^{-\lambda_1} \quad \text{for} \quad b = \begin{pmatrix} \beta_1 & 0 & 0 \\ b_1 & \beta_1^{-1} \beta_2 & 0 \\ b_{12} & b_2 & \beta_2^{-1} \end{pmatrix}. \quad (2.2)$$

$\Lambda = \Lambda(g)$ is a function on $SU_3$ with values in an algebra of operators on a Hilbert space (to be introduced in Subsection 2), satisfying the homogeneity condition

$$\Lambda(gb) = \Lambda(g) \chi_\Lambda(b). \quad (2.3)$$
It follows that $\Lambda$ is determined by its values on the subgroup $N$, (2.1). Using Eqs. (2.2) and (2.3) and the Gauss decomposition

$$gZ = Z_gb(g, \zeta) \ , \ Z_g \in N \ , \ b(g, \zeta) \in B$$

(which assumes $g_{11} \neq 0$), we deduce the following transformation law for $\Lambda(\zeta_1, \zeta_{12}, \zeta_2)$ under the various subgroups of $SL(3, C)$:

$$Z' : \Lambda(\zeta_1, \zeta_{12}, \zeta_2) \rightarrow \Lambda(\zeta_1 + \zeta'_1, \zeta_{12} + \zeta'_2, \zeta_2 + \zeta'_2)$$

$$h_\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha^{-1}_1 \alpha_2 & 0 \\ 0 & 0 & \alpha^{-1}_2 \end{pmatrix} : \Lambda(\zeta_1, \zeta_{12}, \zeta_2) \rightarrow \Lambda(\alpha^2_1 \alpha^{-1}_2 \zeta_1, \alpha_1 \alpha_2 \zeta_{12}, \alpha^{-1}_1 \alpha_2 \zeta_2) \alpha^{-1}_1 \alpha_2^{-1} \lambda_1$$

$$b = \begin{pmatrix} 1 & 0 & 0 \\ b_1 & 1 & 0 \\ 0 & b_2 & 1 \end{pmatrix} : \Lambda(\zeta_1, \zeta_{12}, \zeta_2) \rightarrow \Lambda(\tilde{\zeta}_1, \tilde{\zeta}_{12}, \tilde{\zeta}_2) \beta^{-1}_1 \beta^{-1}_2 \lambda_1$$

$$\tilde{\zeta}_1 = \beta_1(\zeta_1 + b_2(\zeta_1 \zeta_2 - \zeta_{12}))$$

$$\tilde{\zeta}_2 = \beta_2(\zeta_2 + b_1 \zeta_{12})$$

$$\tilde{\zeta}_{12} = \beta_2 \zeta_{12}$$

Denoting the $s\ell_3$ Chevalley generators corresponding to the parameters $\zeta'_1, \zeta'_2, \ln \alpha_1, \ln \alpha_2, b_1, b_2$ by $E_1, E_2, H_1, H_2, F_1, F_2$ we can write the associated infinitesimal law:

$$[E_1, \Lambda] = (\partial_1 + \zeta_2 \partial_{12}) \Lambda \ , \ [E_2, \Lambda] = \partial_2 \Lambda \ (\partial_i \equiv \frac{\partial}{\partial \zeta_i})$$

$$[H_1, \Lambda] = (2\zeta_1 \partial_1 + \zeta_{12} \partial_{12} - \zeta_2 \partial_2 - \lambda_2) \Lambda$$

$$[H_2, \Lambda] = (2\zeta_2 \partial_2 + \zeta_{12} \partial_{12} - \zeta_1 \partial_1 - \lambda_1) \Lambda$$

$$[F_1, \Lambda] = (\lambda_2 \zeta_1 - \zeta'_1 \partial_1 + \zeta_{12} \partial_2) \Lambda$$

$$[F_2, \Lambda] = (\lambda_1 \zeta_2 + (\zeta_1 \zeta_2 - \zeta_{12}) \partial_1 - \zeta'_2 \partial_2 - \zeta_2 \zeta_{12} \partial_{12}) \Lambda$$
The action of the two remaining infinitesimal operators is derived from their definitions:

\[
E_{12} = [E_1, E_2] \Rightarrow [E_{12}, \Lambda] = [\partial_2, \partial_1 + \zeta_2 \partial_1] \Lambda = \partial_2 \Lambda , \\
F_{12} = [F_2, F_1] \Rightarrow [F_{12}, \Lambda] = [\zeta_1 \partial_1 - \zeta_2 \partial_2] + (\zeta_1 \zeta_2 - \zeta_{12}) (\partial_1 \Lambda) , \\
\]

(2.6d)

We recall that the Chevalley generators satisfy the commutation relations (CR) :

\[
[H_i, E_j] = c_{ij} E_j , \quad [H_i, F_j] = -c_{ij} F_j , \\
[E_i, F_j] = \delta_{ij} H_i , \quad [E_i, E_{12}] = 0 , \quad [F_i, F_{12}] = 0 , \\
\]

(2.7a)

where \((c_{ij})\) is the Cartan matrix

\[
(c_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} . \\
\]

(2.8)

2.2 A model Hilbert space. Coherent state vectors. 2-point functions

We now proceed to describing an operator realization of \(\Lambda(\zeta)\).

Let \(\mathcal{H}\) be a model Hilbert space for the finite dimensional (unitary) irreducible representations (IR) of \(SU_3\) :

\[
\mathcal{H} = \bigoplus_{\lambda} \mathcal{H}_\lambda , \quad \text{dim} \mathcal{H}_\lambda = \frac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2) , \\
\]

(2.9)

each \(\lambda\) entering with multiplicity 1. Let \(U(g)\) be the corresponding (infinite dimensional) fully reducible representation of \(SU_3\). The covariance law (2.5) for CSO can be summed up by

\[
U(g) \Lambda(Z) U(g)^{-1} = \Lambda(gZ) = \Lambda(Zg) \chi_\lambda(b(g, \zeta)) , \\
\]

(2.10)

where \(Z_g\) and \(b(g, \zeta)\) are defined by the Gauss decomposition (2.4) (and computed for the various subgroups of \(SL(3, C)\) in (2.5)).

We single out a unit vector \(|0\rangle\), the \(SU_3\) vacuum, in the 1-dimensional \(SL(3, C)\) invariant subspace \(\mathcal{H}_\lambda\) of \(\mathcal{H}\). The action of a CSO on \(|0\rangle\) is determined -up to normalization- by its transformation properties.

We recall the Gel’fand-Cetlin pattern notation \([4]\) for the \(SU_3 \supset U_2 \supset U_1\) chain :

\[
\begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_2 & 0 \\ m_1 & m_2 & m_3 \end{pmatrix} , \quad 0 \leq m_2 \leq \lambda_2 , \\
\lambda_2 \leq m_1 \leq \lambda_1 + \lambda_2 , \\
m_2 \leq m_3 \leq m_1 . \\
\]

(2.11)
Such a pattern represents an eigenvector of $H_1$ and $H_2$ with eigenvalues $2m_3 - m_1 - m_2$ and $2m_1 + 2m_2 - m_3 - \lambda_1 - 2\lambda_2$, respectively. The CSO $\Lambda(Z) \equiv \Lambda(\zeta)$ will be normalized by the condition

$$\Lambda(\zeta = 0)|0 >= | - \lambda_2, -\lambda_1 > ,$$  

(2.12a)

where the right hand side is a shorthand for the (unit) lowest weight vector

$$| - \lambda_2, -\lambda_1 >= \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_2 & 0 \\ \lambda_2 & 0 \\ 0 \end{pmatrix}$$  

(2.12b)

characterized by

$$(H_1 + \lambda_2)| - \lambda_2, -\lambda_1 >= 0 = (H_2 + \lambda_1)| - \lambda_2, -\lambda_1 >= F_i| - \lambda_2, -\lambda_1 > ,$$  

(i = 1, 2)  

(2.12c)

Under these conditions one can derive the expression for the $\zeta$ dependent coherent state

$$\Lambda(\zeta)|0 > = e(\zeta_2 E_2) e(\zeta_1 E_{12}) e(\zeta_1 E_1) | - \lambda_2, -\lambda_1 > ,$$  

(2.13)

where $e(X)$ is the exponential function.

Indeed, applying $E_i$ to both sides and using the definition (2.6d) of $E_{12}$ and the commutation property (2.7b), it is straightforward to recover the infinitesimal transformation law (2.6a). The remaining $s\ell_3$ transformation properties of the vector (2.13) are verified in a similar manner. These properties, together with the “initial condition” (2.12), fix completely the CS vector.

The CS (2.13) is actually a polynomial in $(\zeta_\alpha)$ which can be written in the following factorized form:

$$\Lambda(\zeta)|0 > = \sum_{m_2+n_2 \leq \lambda_2} \zeta_2^{m_2} \zeta_1^{n_2} \frac{E_2^{m_2} E_{12}^{n_2}}{m_2! n_2!} | - \lambda_2, 0 > \otimes \sum_{m_1+n_1 \leq \lambda_1} \zeta_1^{m_1} (\zeta_1 \zeta_2)^{n_1} \frac{E_1^{m_1} E_{12}^{n_1}}{m_1! n_1!} |0, -\lambda_1 > ;$$  

(2.14)

here we have used the identity

$$e(\zeta_2 E_2) e(\zeta_1 E_{12}) e(\zeta_1 E_1) = e(\zeta_1 E_1) e((\zeta_1 \zeta_2) E_{12}) e(\zeta_2 E_2) .$$  

(2.15)

The highest weight vector entering the expansion (2.14) is the coefficient to $\zeta_1^{\lambda_1} (\zeta_1 \zeta_2)^{\lambda_2}$:

$$|0, \lambda_2 >= \frac{E_2^{\lambda_2}}{\lambda_2!} | - \lambda_2, 0 > , \quad |\lambda_1, 0 >= \frac{E_1^{\lambda_1}}{\lambda_1!} |0, -\lambda_1 > ,$$  

(2.16a)
\[ |\lambda_1, \lambda_2> = |\lambda_1, 0 > \otimes |0, \lambda_2> = \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_2 & 0 \\ \lambda_2 & \lambda_1 + \lambda_2 & \lambda_2 \\ \lambda_1 + \lambda_2 & \lambda_2 & \lambda_1 + \lambda_2 \end{pmatrix}, \quad (2.16b) \]

\[ (H_1 - \lambda_1)|\lambda_1, \lambda_2> = (H_2 - \lambda_2)|\lambda_1, \lambda_2> = 0 = E_\alpha|\lambda_1, \lambda_2> . \quad (2.16c) \]

We shall also use the bra CS vector

\[ <0|\Lambda(\zeta) = <\lambda_2 \lambda_1|e(-\zeta_1 E_1)e(-\zeta_{12} E_{12})e(-\zeta_2 E_2) . \quad (2.17) \]

We find it convenient in what follows to use the notation \( \Lambda \) for both the CSO and the associated representation of \( SU_3 \) (interchanging in the second meaning \( \Lambda \) with \( \bar{\Lambda} \)).

The inner product of two CS vectors corresponding to the representations \( \Lambda \) and \( \Lambda' \) is only nonzero if \( \Lambda' \) is the conjugate representation to \( \Lambda \) : \( \Lambda' = (\lambda_2, \lambda_1) \ (\equiv \bar{\Lambda}) \); indeed,

\[ <0|\Lambda'(Z')\Lambda(Z)|0> = [P(\zeta', \zeta)]_{\lambda_1} [\bar{P}(\zeta, \zeta')]_{\lambda_2} \delta_{\lambda_1 \lambda_2} \delta_{\lambda_1' \lambda_2'} , \quad (2.18a) \]

where

\[ P(\zeta', \zeta) = \zeta_{12} - \zeta'_{12} - \zeta_1'(\zeta_2 - \zeta'_2) = \zeta_{12} + \zeta_1' - \zeta'_1 \zeta_2 \quad (2.18b) \]

with

\[ \zeta_{21} = \zeta_1 \zeta_2 - \zeta_{12} . \quad (2.18c) \]

To define \( \Lambda \) on a vector belonging to a nontrivial IR \( \Lambda^{(i)} \), one has to specify a nonvanishing matrix element of the type

\[ <\lambda_1^{(f)}, \lambda_2^{(f)}|\Lambda(\zeta)| - \lambda_2^{(i)}, -\lambda_1^{(i)}> . \quad (2.19) \]

(\text{In the case of } SU_2 \text{ it is enough to specify the weight } \lambda^{(f)} = 2I^{(f)} \text{ of the final state, the counterpart of (2.19) being then a multiple of } \zeta^{I^{(i)}+I-I^{(f)}}. \text{ On the other hand, one knows -see Subsection 3 below- that there could be two linearly independent } SU_3 \text{ invariant 3-point functions for a given triplet of } SU_3 \text{ weights.) The problem is thus to classify all 3-point invariants of } SU_3 \text{ (and then, hopefully, to get a hold on the } n\text{-point invariants).} \]
The general $SU_3$ invariant 2-point function is proportional to the right-hand side of Eq. (2.18a). Using invariance under $Z$ shifts and the relation

$$Z^{-1}Z' = \begin{pmatrix} 1 & -\zeta_1 & \zeta_1\zeta_2 - \zeta_{12} \\ 0 & 1 & -\zeta_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \zeta'_1 & \zeta'_{12} \\ 0 & 1 & \zeta'_2 \\ 0 & 0 & 1 \end{pmatrix},$$

the proof of this statement is reduced to an application of the factorization property (2.14) for $m_i = 0, n_i = \lambda_i$ ($i = 1, 2$). For a 3-point function we use the full $SU_3$ invariance

$$\begin{pmatrix} \Lambda^1 \\ Z_1^{(1)} \\ Z_2^{(1)} \end{pmatrix} \chi_1(b(g, \zeta^{(1)})) \chi_2(b(g, \zeta^{(2)})) \chi_3(b(g, \zeta^{(3)}))$$

(2.20)

$$(Z_g \text{ and } b(g, \zeta) \text{ being given by Eqs. (2.4)-(2.7)}), \text{ without recourse to specific properties of the operator realization.}$$

Proposition 2.1. For each decomposition

$$\lambda^{(i)} = \sum_{j=1}^{3} \mu_{ij} + \nu_1, \quad \lambda^{(i)} = \sum_{j=1}^{3} \mu_{ji} + \nu_2 \tag{2.21}$$

of the three weights $\Lambda^{(i)} = (\lambda^{(i)}_1, \lambda^{(i)}_2)$, $i = 1, 2, 3$, into nonnegative integers $\mu_{ij}$ and $\nu_\alpha$, there exists an invariant 3-point function given by the product

$$\prod_{i,j=1}^{3} P_{ij}^{\mu_{ij}} Q_{12}^{\nu_1} Q_{21}^{\nu_2}, \quad P_{ij} = P(\zeta^{(i)}, \zeta^{(j)}) \tag{2.22a}$$

where $Q_{12}(Q_{21})$ are invariant 3-point functions of three (anti)quark representations given by the determinant

$$Q_{\alpha\beta} = \begin{vmatrix} \zeta^{(1)}_{\alpha\beta} - \zeta^{(2)}_{\alpha\beta} & \zeta^{(2)}_{\alpha\beta} - \zeta^{(3)}_{\alpha\beta} \\ \zeta^{(1)}_{\beta\beta} - \zeta^{(2)}_{\beta\beta} & \zeta^{(2)}_{\beta\beta} - \zeta^{(3)}_{\beta\beta} \end{vmatrix}, \quad \alpha \neq \beta, \quad (\alpha, \beta = 1, 2) \tag{2.22b}$$
All polynomial 3-point invariants are linear combinations of the expressions (2.22); these expressions are linearly independent provided
\[ \nu_1 \nu_2 = 0 \quad . \]
\[ (2.23) \]

In particular, if \( \lambda^{(i)}_\alpha \) are not decomposable in the form \((2.21)\), then there exists no invariant 3-point function.

Outline of proof. Verifying the invariance of Eq.(2.22) is straightforward. The completeness proof is based on the knowledge of \( SU_3 \) fusion coefficients \[15\], which allows one to determine the number of independent 3-point invariants as functions of \( \{ \Lambda^{(i)} \} \). Here are the main steps in the argument.

The possibility to impose the restriction \((2.23)\) comes from the relation
\[ Q_{12}Q_{21} = -P_{12}P_{31}P_{23} - P_{21}P_{13}P_{32} \quad . \]
\[ (2.24) \]
This is, in fact, the only relation among the 3-point blocks \((2.22)\). Thus, for given \( \Lambda^{(i)} \) all expressions of the type \((2.22)\) with \( \mu_{ij} \) and \( \nu_\alpha \) satisfying \((2.21)\) and \((2.23)\) are linearly independent.

Which of the two \( \nu_\alpha \) vanishes is determined by the sign of the difference
\[ \rho \equiv \frac{1}{3}(\lambda_1 - \lambda_2) = \nu_1 - \nu_2 \quad , \]
\[ (2.25a) \]
negative \( \nu_\alpha \) being excluded; here
\[ \lambda_\alpha = \sum_{i=1}^{3} \lambda^{(i)}_\alpha \), \( \alpha = 1, 2 \) ;
\[ (2.25b) \]
\( \rho \) should be an integer whenever \( \Lambda^{(i)} \) admit a nontrivial 3-point invariant. It follows that
\[ \ell_1 = \frac{1}{3}(2\lambda_1 + \lambda_2) = \lambda_1 - \rho \), \( \ell_2 = \frac{1}{3}(\lambda_1 + 2\lambda_2) = \lambda_2 + \rho \]
\[ (2.26) \]
are natural numbers in this case. Using the fact that the intermediate weights \( \mu_{ij} \) are nonnegative integers, we deduce that the number of independent 3-point invariants of the type \((2.22)\) is
\[ N(\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}) = 1 + \min(\ell_1, \ell_2) - \max_i(\lambda^{(i)}_1 + \lambda^{(i)}_2, \ell_1 - \lambda^{(i)}_1, \ell_2 - \lambda^{(i)}_2) \quad . \]
\[ (2.27) \]
Comparison with Ref.\[15\] shows that this number coincides with the number of independent Clebsch-Gordan coefficients for the triple \((\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)})\).
A simple example of multiplicity higher than 1 is provided by the case in which all \( \Lambda^{(i)} \) coincide with the adjoint representation: \( \Lambda^{(i)} = (1,1), i = 1,2,3 \). In this case there are two solutions \( \mu_{ij} \) and \( \bar{\mu}_{ij} \) of Eq.(2.21) that respect Eq.(2.23) (in accord with Eq. (2.27)):

\[
\mu_{12} = 0 = \mu_{23} = \mu_{31}, \quad \mu_{21} = 1 = \mu_{13} = \mu_{32}, \quad \bar{\mu}_{ij} = 1 - \mu_{ij} \quad (\nu_\alpha = 0).
\]

A natural basis of invariants in this case is provided by the \( S_3 \) symmetric combination (2.24) and the skew symmetric one

\[
A = A(\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}) = -P_{12}P_{31}P_{23} + P_{21}P_{13}P_{32}.
\] (2.28)

We note that the current \( J(z, \zeta) \), which generates the level \( k \) \( \widehat{su}_3 \) Kac-Moody algebra, being a local Bose field has an overall symmetric 3-point function that is the product of an \( S_3 \) odd function of \( z_i \) with \( A \):

\[
<0|J(z_1, \zeta^{(1)}),J(z_2, \zeta^{(2)}),J(z_3, \zeta^{(3)})|0> = \frac{KA}{z_{12}z_{23}z_{31}}, \quad z_{ij} = z_i - z_j.
\] (2.29)
3 \( U_q(s\ell_3) \) and its dual. A quantum homogeneous space

We define, following Ref. [10], \( SL_q(3) \), the \( q \)-deformed algebra of functions on the group \( SL(3, C) \), as an associative algebra generated by nine elements \( T_\alpha^\beta (\alpha, \beta = 1, 2, 3) \), including the identity 1, subject to the commutation relations:

\[
R_{\sigma\tau}^{\alpha\beta} T_\gamma^\sigma T_\delta^\tau = T_\gamma^\beta T_\delta^\alpha R_{\gamma\delta}^{\sigma\tau}, \quad (3.1)
\]

where \( R \) is the \( 9 \times 9 \) R-matrix for the product of two “quark” representations of the quantum universal enveloping algebra

\[
U_q := U_q(s\ell_3); \quad (3.2)
\]

\( T = (T_\beta^\alpha) \) is further restricted by the condition

\[
\text{det}_q T = 1, \quad (3.3)
\]

where the \( q \)-determinant is defined as a central element of the algebra of \( T_\alpha^\beta \) satisfying (3.1). We review the derivation of the expression for \( R \) in Subsection 1 (fixing on the way our notation and conventions for the quantum algebra \( U_q \)) and define in Subsection 2 the main object of interest in this paper, the \( q \)-deformation of the 3-dimensional homogeneous space \( SL(3, C)/B \) of Sec. 2.1.

3.1 \( U_q \): definition, \( R \)-matrix, exchange operator

In this synopsis on the Hopf algebra (3.2) we mostly follow the notation and conventions of Ref. [11].

The quantum universal enveloping algebra (3.2) has the same Chevalley-Cartan generators as \( s\ell_3 \) with the provision that one also uses \( q^{\pm H_i}, \ i = 1, 2, \) as elements of the algebra. The CR (2.7a) remain unchanged, while Eqs.(2.7b) are replaced by:

\[
[E_i, F_j] = [H_i] \delta_{ij} \quad (i, j = 1, 2), \quad [X] := \frac{q^X - \bar{q}^X}{q - \bar{q}} \quad (\bar{q} \equiv q^{-1}) \quad (3.4)
\]

and by the \( q \)-deformed Serre relations

\[
X_i^{(2)} X_j + X_j X_i^{(2)} = X_i X_j X_i, \quad X = E, F, \quad j = i \pm 1, \quad X^{(2)} = \frac{1}{2} X^2, \quad [2] = q + \bar{q}. \quad (3.5)
\]
The latter relations suggest the introduction of a pair of $q$-Weyl operators $E_{12}$ and $E_{21}$ and their conjugates by

$$E_{ij} = E_i E_j - q E_j E_i, \quad F_{ij} = F_j F_i - q F_i F_j$$

((ij)=(12) or (21)), such that

$$E_i E_{ij} = q E_{ij} E_i, \quad F_i F_{ij} = q F_{ij} F_i.$$  \[3.6a\]

These relations are respected (and indeed dictated) by the coproduct $\Delta : U_q \to U_q \otimes U_q$ which is fixed by its values on the generators:

$$\Delta(q^{\pm H_i}) = q^{\pm H_i} \otimes q^{\pm H_i} \quad \text{(or} \quad \Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i)$$ \[3.7a\]

and

$$\Delta(E_i) = E_i \otimes q^{H_i} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + q^{H_i} \otimes F_i,$$ \[3.7b\]

or, alternatively,

$$\Delta(\tilde{E}_i) = \tilde{E}_i \otimes 1 + q^{H_i} \otimes \tilde{E}_i, \quad \Delta(\tilde{F}_i) = \tilde{F}_i \otimes q^{H_i} + 1 \otimes \tilde{F}_i.$$ \[3.7c\]

Here, Eqs.\(3.7b\) and \(3.7c\) should be viewed as relations in the same quantum universal enveloping algebra with generators $X$ and $\tilde{X}$ related by

$$\tilde{E}_i = q^{H_i} E_i, \quad \tilde{F}_i = F_i q^{H_i}.$$ \[3.8\]

We note that $\tilde{E}_i$ and $\tilde{F}_j$ satisfy the same CR (3.4)-(3.6) as $E_i$ and $F_j$. We shall make use of both sets of variables noting that $\tilde{X}_i$ and $-X_i$ are related by the antipode $\gamma : U_q \to U_q$ defined as an algebra antihomomorphism such that

$$\gamma(q^{\pm H_i}) = q^{\mp H_i}, \quad \gamma(\tilde{E}_i) = -E_i, \quad \gamma(\tilde{F}_j) = -F_j, \quad \gamma(E_i) = -E_i q^{H_i}, \quad \text{etc.}.$$ \[3.9a\]

To complete our review of basic notions, we need the counit $\epsilon : U_q \to C$ which defines the trivial representation of $U_q$:

$$\epsilon(q^{\pm H_i}) = 1, \quad \epsilon(E_i) = 0 = \epsilon(F_i).$$ \[3.9b\]

$\Delta, \gamma, \epsilon$ and the (associative) multiplication $m$ in $U_q$ are verified to satisfy

$$m(\gamma \otimes 1) \Delta(X) = \epsilon(X) = m(1 \otimes \gamma) \Delta(X).$$ \[3.10\]
The universal $R$ matrix is defined, for generic $q$, as an invertible element in the (topological) tensor product $U_q \otimes U_q$ which intertwines the permuted coproduct

$$\Delta'(X) = \sigma \Delta(X), \quad \sigma(X \otimes Y) = Y \otimes X \quad (3.11a)$$

with $\Delta$:

$$\Delta'(X) = R \Delta(X) R^{-1} \quad \text{for all } X \in U_q. \quad (3.11b)$$

It is known by now quite explicitly [16, 17, 11]:

$$R = Q W_1 W_2 W_3, \quad (3.12a)$$

where

$$Q = q^{e^{-1}} H_i \otimes H_j, \quad (e^{-1}) = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad Q^{-1}(1 \otimes E_i) Q = q^{H_i} \otimes E_i, \quad \text{etc.}, \quad (3.12b)$$

$$W_\alpha = e_+ (\rho E_\alpha \otimes F_\alpha), \quad \rho = \bar{q} - q, \quad \alpha = 1, 2, 12. \quad (3.12c)$$

The $q$-deformed exponents

$$e_\pm(X) = \sum_{n=0}^{\infty} \frac{X^n}{(n)_\pm!}, \quad (n)_\pm = \frac{q^{\pm 2n} - 1}{q^{\pm 2} - 1}, \quad (n)_\pm! = (n)_\pm (n-1)_\pm! \quad (3.13)$$

are well defined for generic $q$ (for which $q^{\pm 2n} \neq 1$) and satisfy

$$e_+(X)e_-(X) = 1. \quad (3.14)$$

For a finite dimensional representation of $U_q$ the series $W_\alpha$ reduce to finite sums and make also sense for $q^p = 1$ provided that the weight of the representation satisfies $\lambda_1 + \lambda_2 < p$.

We define the exchange operator $\hat{R} = R^{\pi_1 \pi_2'}$ for a given pair of representations $(\pi, \pi')$ of $U_q$ as the corresponding $R$ followed by a permutation $P$ (acting in the tensor product of representation spaces):

$$\hat{R} = P R^{\pi_1 \pi_2'}, \quad (3.15a)$$

where

$$R^{\pi_1 \pi_2'} = \sum_n \pi(X_n) \otimes \pi'(Y_n) \quad \text{for } R = \sum_n X_n \otimes Y_n. \quad (3.15b)$$

We note that the action of $U_q$ in the tensor product of $U_q$ modules is given by the coproduct:

if

$$\Delta(X) = \sum_i L_i \otimes R_i, \quad L_i = L_i(X), \quad R_i = R_i(X) \quad (3.16a)$$
then
\[ \Delta \pi'(X) = \sum_i \pi(L_i) \otimes \pi'(R_i). \] (3.16b)

The exchange operator intertwines between the products \( \pi \otimes \pi' \) and \( \pi' \otimes \pi \):
\[ \hat{R} \pi' \Delta \pi'(X) = \Delta \pi' \pi(X) \hat{R} \pi'. \] (3.17)

We shall reproduce the computation of the exchange operator for the fundamental representations of \( U_q \) (cf. Ref. [10]).

For \( \Lambda_1 = (1,0) \) the representation \( \chi_1 \) of \( U_q \) coincides with the 3 \( \times \) 3 matrix representation of the undeformed \( s\ell_3 \) Lie algebra; we have:
\[ \pi_1(E_i) = e_{ii+1} =: e_i, \quad \pi_1(F_i) = e_{i+1i} =: f_i, \] (3.18a)
\[ \pi_1(H_i) = e_{ii} - e_{i+1i+1} =: h_i = [e_i, f_i]. \] (3.18b)

Here \( e_{ij} \) are the Weyl matrices characterized by the product formula
\[ e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell} \quad (e_{ij} = e_{ji}^*). \] (3.19)

Noting that \( e_i^2 = 0 = f_i^2 \) we can write in this representation
\[ R_{12} = Q_{12} + q^{1/3} \rho \sum_{k<\ell} e_{k\ell}^1 e_{\ell k}^2, \] (3.20a)
where
\[ Q_{12} = q^{1/3} \sum_{ij} q^{\delta_{ij}} e_{ii}^1 e_{jj}^2, \] (3.20b)
(the superscript on \( e \) indicating the space in which it acts, so that \( e \)'s with different superscripts commute). The notation \( R_{ab} \), that involves a pair of labels \( (a,b) \) indicating the space in a multiple tensor product on which \( R \) acts nontrivially, is used systematically by Faddeev et al. [10]. The Yang-Baxter equation assumes in this notation the following compact form:
\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \] (3.21)

Here is a step in verifying Eq. (3.21) which decipheres on the way the operational meaning of the above notation. The left-hand side of Eq. (3.21) is expressed in terms of the Weyl generators as follows:
\[ \bar{q}R_{12}R_{13}R_{23} = \sum_{i,j,k} q^{\delta_{ij}+\delta_{jk}+\delta_{ik}}e_{ii}^1e_{jj}^2e_{kk}^3 \\
+ \rho \sum_{k<\ell} \left\{ q^{\delta_{ik}+\delta_{\ell k}} (e_{\ell k}^1e_{tk}^3 + e_{kt}^1e_{tk}^3) \right\} + q^{2\delta_{ik}} e_{kk}^3 e_{ii}^1 \\
+ e_{12}^1 e_{33}^2 e_{kk}^3 + \bar{q} \sum_{k<\ell} e_{kt}^1 e_{\ell k}^3 e_{kk}^3 + \rho^2 (3 e_{13}^1 e_{22}^2 e_{31}^3 \\
+ e_{12}^1 e_{33}^2 e_{kk}^3 + \bar{q} \sum_{k<\ell} e_{kt}^1 e_{\ell k}^3 e_{kk}^3) + \rho^3 e_{13}^1 e_{22}^2 e_{31}^3 \\
(\rho = \bar{q} - q) . \] (3.22)

(We note that Eq. (3.21) is not verified order by order in \( \rho \); only the sum of the terms proportional to \( \rho \) and \( \rho^2 \) cancel from both sides of the equation.)

We proceed to computing the exchange matrix \( \hat{R} \) for a pair of fundamental representations.

The permutation operator in \( C^3 \times C^3 \) is given by:

\[ P_{12} = \sum_{ij} e_{ij}^1 e_{ji}^2 \quad (P_{12}^2 = \sum_{i} e_{ii}^1 \sum_{j} e_{jj}^2 = 1_{112}) . \] (3.23)

It commutes with \( Q_{12} \),

\[ P_{12}Q_{12} = Q_{12}P_{12} = q^{1/3} \sum_{ij} q^{\delta_{ij}} e_{ij}^1 e_{ji}^2 , \] (3.24)

and symmetrizes the product of \( W \)'s:

\[ PW_1 W_{12} W_2 = P + \rho \sum_{k<\ell} e_{\ell\ell} \otimes e_{kk} . \] (3.25)

Inserting in the expression for \( \hat{R} \) we end up with

\[ \hat{R}_{12} = q^{1/3} \left\{ \sum_{ij} q^{\delta_{ij}} e_{ij}^1 e_{ji}^2 + \rho \sum_{k<\ell} e_{\ell\ell}^1 e_{kk}^2 \right\} . \] (3.26)

The exchange operator thus obtained has two eigenvalues: a 6-fold degenerate one, 
\( q^{-2/3} \), corresponding to a “\( q \)-symmetric tensor” expressed by products of “\( q \)-bosonic” variables:

\[ \hat{R}_{\alpha\beta} x^\gamma x^\delta = q^{-2/3} x^\alpha x^\beta \quad \text{or} \quad x^\alpha x^\beta = q x^\beta x^\alpha \quad \text{for} \quad \alpha < \beta ; \] (3.27)

and a 3-fold degenerate eigenvalue, \(-q^{4/3}\), associated with \( q \)-fermions [18, 19, 20, 21] :

\[ \hat{R}_{\alpha\beta} \xi^\gamma \xi^\delta = -q^{4/3} \xi^\alpha \xi^\beta \quad \text{or} \quad (\xi^\gamma)^2 = 0 = \xi^\alpha \xi^\beta + \bar{q} \xi^\beta \xi^\alpha \quad \text{for} \quad \alpha < \beta . \] (3.28)
3.2 The Hopf algebra $SL_q(3)$, a Borel subalgebra and a quantum coset space

There are three equivalent approaches to introducing the algebra $SL_q(3)$.

(i) The Kobyzev-Manin approach \cite{19} introduces the generators $T_\alpha^\beta$ of $GL_q(3)$ by demanding that the pair of transformations

$$x^\alpha \to T_\alpha^\beta x^\beta, \; \xi^\alpha \to T_\alpha^\beta \xi^\beta$$

be nonsingular and preserve the exchange relations (3.27) and (3.28), respectively. The $q$-determinant, defined by the relation

$$T_\alpha^1 \xi^\alpha T_\beta^2 \xi^\beta T_\gamma^3 \xi^\gamma = \det_q(T_\alpha^\beta) \xi^1 \xi^2 \xi^3,$$

is shown to commute with $T_\alpha^\beta$ and can hence be set equal to 1.

(ii) The Faddeev-Reshetikhin-Takhtajan approach \cite{10} starts with the exchange relations (3.1) for $T_\alpha^\beta$.

(iii) In Drinfeld’s framework \cite{22} $T_\alpha^\beta$ appear as linear functionals on $U_q$. Their algebraic properties are determined from the duality between the coalgebra structure in $U_q$ and the algebra structure in $SL_q(3)$ expressed by the relation

$$< A.B, X > = < A \otimes B, \Delta(X) > \quad \text{for} \quad X \in U_q.$$ 

We shall relate the first two approaches and will content ourselves with a remark concerning the third one.

Assuming that $T_\alpha^\beta$ commute with both $x^\gamma$ and $\xi^\gamma$ we find that Eq.(3.1) is a necessary and sufficient condition, that the transformed (according to Eqs.(3.29)) $x$ and $\xi$ obey the same exchange relations (3.27) and (3.28) as the original ones. Indeed, requiring the CR

$$\hat{R}_{12} T_1 T_2 = T_1 T_2 \hat{R}_{12} \quad \text{(3.32a)}$$

we guarantee that $T_1 T_2$ maps eigenvectors of $\hat{R}_{12}$ (like $x^\alpha x^\beta$ or $\xi^\alpha \xi^\beta$) into eigenvectors corresponding to the same eigenvalue. On the other hand, Eq. (3.32a) is equivalent to

$$R_{12} T_1 T_2 = P_{12} T_1 T_2 P_{12} R_{12} = T_2 T_1 R_{12} \quad \text{(3.32b)}$$

which is a shorthand for Eq. (3.1).

In exploiting Eq. (3.1) it is convenient to write $R_{12}$ in tensor components:

$$\tilde{q}^{1/3} R_{\alpha \gamma}^{\beta \delta} = \tilde{q}^{\delta \alpha} \delta_\alpha^\delta \delta_\beta^\gamma + \rho \sum_{k < \ell} \delta_\kappa^\alpha \delta_\kappa^\delta \delta_\ell^\beta \delta_\ell^\gamma$$ 

\[ (3.33) \]
\( \rho = \bar{q} - q \). Inserting Eq. (3.33) into Eq. (3.1) we obtain (some of the relations below differ from those listed in Ref. [23]):

\[
\bar{q} T^\alpha \gamma^\gamma T^\gamma _\gamma = \bar{q} \delta^\beta_\gamma T^\alpha_\gamma T^\gamma _\gamma + \rho T^\alpha_\gamma T^\gamma _\gamma \theta_{\beta > \gamma},
\]

or

\[
T^\alpha_\gamma T^\gamma _\gamma = q T^\alpha_\gamma T^\gamma _\gamma \quad \text{for} \quad \beta < \gamma ;
\]

\[
\bar{q} T^\beta_\gamma T^\alpha_\gamma = \bar{q} \delta^\alpha_\beta T^\alpha_\gamma T^\gamma _\gamma + \rho T^\alpha_\gamma T^\gamma _\gamma \theta_{\beta > \alpha},
\]

or

\[
T^\alpha_\gamma T^\gamma _\gamma = q T^\alpha_\gamma T^\gamma _\gamma \quad \text{for} \quad \alpha > \beta ;
\]

\[
\bar{q} \delta^\alpha_\beta T^\alpha_\gamma T^\gamma _\gamma + \rho T^\beta_\gamma T^\alpha_\gamma \theta_{\beta > \alpha} = \bar{q} \delta^\alpha_\gamma T^\alpha_\gamma T^\gamma _\gamma + \rho T^\beta_\gamma T^\alpha_\gamma \theta_{\alpha > \gamma},
\]

or, assuming \( \beta \neq \alpha \neq \gamma \),

\[
[T^\alpha_\alpha, T^\beta_\gamma] = (q - \bar{q}) T^\alpha_\gamma T^\gamma _\gamma (\theta_{\beta > \alpha} - \theta_{\alpha > \gamma});
\]

for \( \beta = \gamma \) Eqs. (3.36) and the antisymmetry of the commutator imply

\[
[T^\alpha_\beta, T^\beta_\alpha] = 0,
\]

a relation that is also recovered as a special case from

\[
\bar{q} \delta^\alpha_\beta T^\alpha_\gamma T^\gamma _\gamma - \bar{q} \delta^\alpha_\gamma T^\alpha_\gamma T^\gamma _\gamma = (q - \bar{q}) T^\beta_\gamma T^\alpha_\gamma (\theta_{\beta > \alpha} - \theta_{\gamma > \alpha}).
\]

Since for \( \alpha, \beta, \gamma, \delta = 1, 2, 3 \) at least two indices always coincide the above relations (3.34)-(3.38) exhaust all possibilities.

Using Eq. (3.30) we compute the \( q \)-determinant of \( T \) with the result

\[
\det_q(T_\alpha^\alpha) = T_1^1 T_2^2 T_3^3 + q^2 T_1^1 T_3^2 T_1^3 + q^2 T_3^1 T_1^2 T_2^3 - q(T_1^1 T_2^2 T_3^3 + T_2^1 T_1^2 T_3^3 + q T_3^1 T_2^2 T_3^1).
\]

It is verified to commute with all \( T^\alpha_\beta \).

The algebra generated by \( T^\alpha_\beta \) can be equipped with a coproduct,

\[
\Delta(T^\alpha_\beta) = \sum_{\sigma=1}^{3} T^\sigma_\sigma \otimes T^\sigma_\beta,
\]
which respects the above commutation relations. One can also introduce a counit $\epsilon$ and an inverse $\gamma$, defined on the generators by:

$$\epsilon(T^\alpha_\beta) = \delta^\alpha_\beta, \quad \gamma(T^\alpha_\beta) = (T^{-1})^\alpha_\beta,$$  (3.41)

$T^{-1}$ being the inverse matrix of $T$, which has the form (in view of Eqs. (3.3) and (3.39)):

$$T^{-1} = \begin{pmatrix}
T^2_3T_3^3 - qT^2_3T_2^3 & T^1_3T_2^3 - qT^1_3T_3^3 & q^2T^1_2T_3^2 - qT^1_3T^2_2 \\
q^2T^2_3T_1^3 - qT^2_1T^3_3 & T^1_3T_3^3 - qT^1_3T^3_1 & T^1_3T^1_3 - qT^1_3T^2_2 \\
q^2T^2_1T_2^3 - qT^2_2T^3_3 & q^2T^1_2T_3^3 - qT^1_3T^3_2 & T^1_3T_2^3 - qT^1_3T^2_2
\end{pmatrix}. \quad (3.42)$$

$\Delta, \epsilon$ and $\gamma$ along with the ordinary multiplication $m$, can be verified to satisfy the defining conditions (3.10) for a Hopf algebra.

**Remark.** The elements of $SL_q(3)$ can be viewed as linear functionals on $U_q$, the values of which are computed as follows.

Consider the $3 \times 3$ matrix representation $\pi_1$ of weight (1,0) of $U_q$ defined by Eqs. (3.18). We then define the linear functional $T^\alpha_\beta$ on an arbitrary element $X$ of $U_q$ by setting

$$(T^\alpha_\beta, X) = \pi_1(X)^\alpha_\beta.$$  (3.43)

We then use Eq. (3.31) to extend this definition to products of $T^\alpha_\beta$ and verify that the products so obtained are noncommutative, but satisfy Eqs. (3.32).

It is straightforward to define the quantum Borel subalgebra $B_q$ generated by elements of lower triangular matrices (2.2) satisfying (in accord with Eqs. (3.34)-(3.38)):

$$[\beta_1, \beta_2] = 0, \quad \beta_i b_j \beta_i^{-1} = q^{\delta_{ij}} b_j, \quad \beta_i b_{12} = q b_{12} \beta_i \quad (i, j = 1, 2); \quad (3.44a)$$

$$[b_1, b_2] = (q - \bar{q})b_{12} \beta_1^{-1} \beta_2, \quad b_1 b_{12} = q b_{12} b_1, \quad b_{12} b_2 = q b_2 b_{12}. \quad (3.44b)$$

For the subset of generators for which

$$\beta_2 = (T^3_3)^{-1} \quad \text{and} \quad \beta_1 = (T^2_3T_3^3 - qT^2_3T_2^3)^{-1}$$

exist

the matrix $(T^\alpha_\beta)$ admits a Gauss decomposition

$$T^\alpha_\beta = \begin{pmatrix}
1 & u_1 & u_{12} \\
0 & 1 & u_2 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\beta_1 & 0 & 0 \\
b_1 & \beta_1^{-1} \beta_2 & 0 \\
b_{12} & b_2 & \beta_2^{-1}
\end{pmatrix}. \quad (3.45)$$

\[20\]
with $\beta_1$ and $\beta_2$ given by Eqs. (3.45) and

$$
\begin{align*}
 b_{12} &= T_1^3, \quad b_2 = T_2^3, \\
 b_{1\beta_1^{-1}} &= T_1^3 T_3^3 - q T_3^2 T_1^3;
\end{align*}
$$

(3.47)

$$
\begin{align*}
 u_{12} &= T_3^l \beta_2, \quad u_2 = T_3^2 \beta_2, \quad u_1 = (T_3^1 T_3^3 - q T_3^1 T_2^3) \beta_1.
\end{align*}
$$

(3.48)

By definition, $u_a$ generate the quantum coset space

$$
N_q = SL_q(3)/B_q
$$

(3.49)

(that can be viewed as an associative algebra, but not as a bialgebra).

As a consequence of Eqs. (3.48) and (3.45) which imply

$$
[\beta_1, \beta_2] = 0, \quad \beta_2 T_3^i = q T_3^i \beta_2 \quad (i = 1, 2),
$$

$$
\beta_1 T_3^j = q T_3^j \beta_1 \quad (j = 1, 2),
$$

(3.50a)

we obtain the CR:

$$
qu u_2 - u_2 u_1 = (q - \bar{q}) u_{12}, \quad u_1 u_{12} = q u_{12} u_1, \quad u_{12} u_2 = q u_2 u_{12}.
$$

(3.50b)

### 3.3 Quantum group action on the noncommutative coset space $N_q$

Thus, for $q^2 \neq 1$, the polynomial algebra, that defines the quantum coset space (3.49), has just two generators, $u_1$ and $u_2$. Following the pattern of Sec. 2 we can view $N_q$ as a quantum group orbit. We shall display the induced action of the two conjugate Borel subgroups $SL_q(3)$ on this noncommutative coset space.

The action of the 4-parameter $q$-subgroup of upper triangular matrices is determined from

$$
\left(\begin{array}{ccc}
\alpha_1 & t_1 \alpha_1^{-1} \alpha_2 & 0 \\
0 & \alpha_1^{-1} \alpha_2 & t_2 \alpha_2^{-1} \\
0 & 0 & \alpha_2^{-1}
\end{array}\right)
\left(\begin{array}{cc}
1 & u_1 \\
0 & 1 & u_2 \\
0 & 0 & 1
\end{array}\right) =
\left(\begin{array}{ccc}
\alpha_1 & \bar{u}_1 \alpha_1^{-1} \alpha_2 & \bar{u}_1 \alpha_2^{-1} \\
0 & \alpha_1^{-1} \alpha_2 & \bar{u}_2 \alpha_2^{-1} \\
0 & 0 & \alpha_2^{-1}
\end{array}\right) \equiv n,
$$

(3.51a)

where the right-hand side has the Gauss decomposition

$$
n = n(\alpha, \bar{u}) =
\left(\begin{array}{ccc}
1 & \bar{u}_1 & \bar{u}_1 \\
0 & 1 & \bar{u}_2 \\
0 & 0 & 1
\end{array}\right)
\left(\begin{array}{ccc}
\alpha_1 & 0 & 0 \\
0 & \alpha_1^{-1} \alpha_2 & 0 \\
0 & 0 & \alpha_2^{-1}
\end{array}\right);
$$

(3.51b)
\[
\tilde{u}_1 = \alpha_1 u_1 \alpha_2^{-1} + t_1, \quad \tilde{u}_2 = \alpha_1^{-1} \alpha_2 u_2 \alpha_2 + t_2, \quad \tilde{u}_{12} = \alpha_1 u_{12} \alpha_2 + t_1 \alpha_1^{-1} \alpha_2 u_2 \alpha_2.
\]

(3.51c)

The CR among group parameters and coset space coordinates are dictated by the requirement that both the first factor in Eq. (3.51a) and the product \(n(\alpha, \tilde{u})\) belong to \(SL_q(3)\). The former implies

\[
\alpha_i t_j = q^{\delta_{ij}} t_j \alpha_i, \quad t_2 t_1 = q t_1 t_2,
\]

(3.52a)

while the latter gives :

\[
\alpha_i u_j = q^{\delta_{ij}} u_j \alpha_i, \quad [u_i, t_j] = 0 \quad (i, j = 1, 2).
\]

(3.52b)

Similarly, lower diagonal \(SL_q(3)\) transformations are deduced from the product formula

\[
\begin{pmatrix}
\beta_1 & 0 & 0 \\
b_1 & \beta_1^{-1} \beta_2 & 0 \\
0 & b_2 & \beta_2^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & u_1 & u_{12} \\
0 & 1 & u_2 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
\beta_1 & \beta_1 u_1 & \beta_1 u_{12} \\
b_1 & b_1 u_1 + \beta_1^{-1} \beta_2 & b_1 u_{12} + \beta_1^{-1} \beta_2 u_2 \\
0 & b_2 & \beta_2^{-1} + b_2 u_2
\end{pmatrix}
\equiv g,
\]

(3.53a)

where \(g\) also admits a Gauss decomposition

\[
g = g(b, \alpha; v) = \begin{pmatrix}
1 & v_1 & v_{12} \\
0 & 1 & v_2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_1^{-1} \alpha_2 & 0 \\
0 & 0 & \alpha_2^{-1}
\end{pmatrix},
\]

(3.53b)

with

\[
\alpha_2^{-1} = \beta_2^{-1} + b_2 u_2, \quad \alpha_1^{-1} = \beta_1^{-1} + b_1 u_1 \beta_2^{-1} + b_1 b_2 u_{21},
\]

\[
u_{21} = q(u_1 u_2 - u_{12}) = \frac{q u_2 u_1 - u_1 u_2}{q - \bar{q}},
\]

\[
v_1 = \beta_1 u_1 \alpha_1 \alpha_2^{-1}, \quad v_2 = (b_1 u_{12} + \beta_1^{-1} \beta_2 u_2) \alpha_2,
\]

(3.54a)

The requirement that \(g(b, \alpha; v)\) belongs to \(SL_q(3)\) and so does the first factor in the left-hand side of Eq. (3.53a) yields the relations :

\[
\beta_i b_j = q^{\delta_{ij}} b_j \beta_i, \quad [b_1, b_2] = 0 = [\beta_1, \beta_2],
\]

(3.55a)
\[ b_i u_i = q u_i b_i, \quad u_2 b_2 = q b_2 u_1, \quad [b_1, u_2] = 0 = [b_2, u_{12}], \quad (3.55b) \]

The correct CR among the elements of the two matrices in the right-hand side of Eq. (3.53a) is then a consequence. It is instructive to verify, for instance, the CR

\[ [\alpha_i^{-1}, \alpha_j^{-1}] = 0, \quad \alpha_i b_j = q^{b_j} b_j \alpha_i. \quad (3.55c) \]

In establishing the first one we use the relations

\[ u_2 u_{21} = q u_{21} u_2, \quad (3.56a) \]

\[ [b_1 u_1 \beta_2^{-1}, b_2 u_2] = [\beta_2^{-1}, b_1 b_2 u_{21}] = (\bar{q} - q) b_1 b_2 \beta_2^{-1} u_{21}. \quad (3.56b) \]

The noncommutative version of the transformation law (2.5) for \( U_q \) CSO is defined to map polynomials into polynomials (of the same maximal degree \( \lambda_1 + \lambda_2 \) in either \( u_1 \) or \( u_2 \)).
4 Quantum CSO. $U_q$ invariant 2-point function

4.1 PBW and canonical bases on the quantum coset space

There are two Poincaré-Birkhoff-Witt (PBW) type bases (cf. Ref. [16]) in the polynomial algebra of $u_1, u_2$:

$$B^{(1)}_{\ell mn} = u_1^{(\ell)}(u_2 u_1 - \bar{q} u_1 u_2)^{(m)} u_2^{(n)},$$ (4.1a)

$$B^{(2)}_{\ell mn} = u_2^{(m)}(u_1 u_2 - \bar{q} u_2 u_1)^{(m)} u_1^{(\ell)},$$ (4.1b)

where we are using the shorthand notation:

$$x^{(n)} = \frac{x^n}{[n]!}, \quad [n]! = [n][n-1]!, \quad [0]! = 1.$$ (4.2)

Clearly the middle factors are expressible in terms of the variables $u_{ij}$ defined in Eqs. (3.50b) and (3.54a):

$$u_1 u_2 - \bar{q} u_2 u_1 = (1 - \bar{q}^2) u_{12}, \quad u_2 u_1 - \bar{q} u_1 u_2 = (1 - \bar{q}^2) u_{21}.$$ (4.3)

The form (4.1) has the advantage of exhibiting the “zero temperature”, $\bar{q} \to 0$, limit in which the two bases are simply related to the unique canonical (“crystal”) basis of Lusztig [12] and Kashiwara [24], defined by:

$$L^{(1)}_{\ell mn} = u_1^{(\ell)} u_2^{(m)} u_1^{(n)}, \quad L^{(2)}_{\ell mn} = u_2^{(n)} u_1^{(m)} u_2^{(\ell)}$$ for $\ell + n \leq m$. (4.4)

The consistency condition for $m = \ell + n$,

$$L^{(1)}_{\ell \ell + nn} = L^{(2)}_{\ell \ell + nn},$$ (4.5)

is a consequence (albeit not an easy one) of the Serre relations

$$u_1^{(2)} u_2 + u_2^{(2)} u_1 = u_1 u_2 u_1, \quad u_1^{(2)} u_2^{(2)} + u_2^{(2)} u_1 = u_2 u_1 u_2.$$ (4.6)

It follows from Lusztig’s expansion formulae

$$L^{(1)}_{\ell mn} = \sum_{k=0}^{n} \bar{q}^{(m-k)(n-k)} \left[ \begin{array}{c} \ell + n - k \\ \ell \\ \end{array} \right] B^{(1)}_{\ell + n - k k m - k}$$

$$= \sum_{k=0}^{\ell} \bar{q}^{(\ell-k)(m-k)} \left[ \begin{array}{c} \ell + n - k \\ n \\ \end{array} \right] B^{(2)}_{m - k k \ell + n - k},$$ (4.7a)

24
\[
L_{\ell mn}^{(2)} = \sum_{k=0}^{\ell} q^{(\ell-k)(m-k)} \left[ \frac{\ell + n - k}{n} \right] B_{\ell+n-k}^{(2)} k_{m-k} \\
= \sum_{k=0}^{n} q^{(m-k)(n-k)} \left[ \frac{\ell + n - k}{\ell} \right] B_{m-k}^{(1)} k_{\ell+n-k}, \tag{4.7b}
\]
which also demonstrate that the two PBW bases coincide in the \(\bar{q} \to 0\) limit. Here we have used the \(q\)-binomial coefficients
\[
\left[ \begin{array}{c} m \\ \ell \end{array} \right] = \frac{[m]!}{[\ell]![m-\ell]!}. \tag{4.7c}
\]

Preparing for the study of invariant 2-point functions we shall introduce a variant of the canonical basis written in terms of one of the variables \(u_i\) and the commuting pair \((u_{12}, u_{21})\) [Eqs. (4.3)] which allows us to exclude the products \(u_i u_j\) :

\[
u_1 u_2 = u_{12} + \bar{q} u_{21}, \quad u_2 u_1 = u_{21} + \bar{q} u_{12}, \quad [u_{12}, u_{21}] = 0. \tag{4.8}
\]
The new standard basis also consists of polynomials homogeneous with respect to each \(u_i\) and splits into two parts depending on whether the degree in \(u_1\) exceeds the one in \(u_2\) or vice versa :

\[
S_{mn\mu}^{(1)} = u_{12}^{(m)} u_1^{(n)} u_{21}^{(\mu)}, \quad S_{mn\nu}^{(2)} = u_{21}^{(n)} u_2^{(\nu)} u_{12}^{(m)}. \tag{4.9a}
\]

\(L^{(i)}\) and \(S^{(i)}\), i=1,2, span the same subspace of polynomials and can be expressed in terms of each other. The counterpart of Eq. (4.5),

\[
S_{mn0}^{(1)} = S_{mn0}^{(2)}, \tag{4.9b}
\]
is a trivial consequence of the commutativity of \(u_{ij}\) [Eq. (4.8)]. It allows us, in turn, to derive Eq. (4.5), since

\[
u_1^m u_2^{m+n} u_1^n = \prod_{j=1}^{m} (q^{j-1}u_{12} + \bar{q}^j u_{21}) \prod_{i=1}^{n} (q^{i-1}u_{21} + \bar{q}^i u_{12})
\]

\[
= u_2^n u_1^{m+n} u_2^m. \tag{4.10}
\]

In writing down CS vectors we shall need the \(U_q\) counterpart of Eq. (2.15) :

\[
e_+ (u_1 \tilde{E}_1) e_+ (u_{21} \tilde{E}_{21}) e_+ (u_2 \tilde{E}_2) = e_+ (u_2 \tilde{E}_2) e_+ (u_{12} \tilde{E}_{12}) e_+ (u_1 \tilde{E}_1) =: E_+ (u_1, u_2), \tag{4.11}
\]
which is an identity between the two PBW bases valid for all values of \(q\). In the small \(\bar{q}\) limit it reduces to the relations (4.8). Here we have used the Chevalley generators (3.8) and the counterpart of Eq. (3.6a) for \(\tilde{E}_i\) :

\[
\tilde{E}_{ij} = \tilde{E}_i \tilde{E}_j - \bar{q} \tilde{E}_j \tilde{E}_i \quad ((i, j) = (1, 2) \text{ or } (2, 1)). \tag{4.12}
\]
4.2 Quantum CS vectors. Raising operators as q-derivatives

Equation (4.11) suggests the following $U_q$ analogue of the CS ket (2.13):

$$\Lambda(u)|0> : (\begin{array}{c} \lambda_1 \\ u_1 \\ \lambda_2 \\ u_2 \end{array}) |0>= E_+(u_1, u_2)|-\lambda_2, -\lambda_1> ;$$

(4.13a)

the conjugate bra is written as a highest weight vector times the antipode (3.9) of $E_+$:

$$< 0| (\begin{array}{c} \lambda_2 \\ u_1 \\ \lambda_1 \\ u_2 \end{array}) =< \lambda_1, \lambda_2 | \gamma(E_+(u_1, u_2)) .$$

(4.13b)

We note that the order of $u_\alpha$ should not change under the map $\gamma$:

$$\gamma\{e_+(u_j \tilde{E}_j)e_+(u_{ij}\tilde{E}_{ij})e_+(u_i\tilde{E}_i)\} =: e_+(-u_i E_i)e_+(u_{ij}E_{ji})e_+(-u_j\tilde{E}_j) : ,$$

(4.14a)

where the “normal product” of $u$-factors is given in Eq. (4.14b) by:

$$: u_i^\ell u_{ij}^m u_j^n := u_j^n u_{ij}^m u_i^\ell .$$

(4.14b)

One can verify that the $U_q$ action on CS vectors agrees with the global transformation law (3.51), (3.53) in the limit of infinitesimal group parameters $t_1$ (or $b_1$). We shall display the result for the raising operators $\tilde{E}_i$ for which it is particularly simple (and hence, useful).

The 2-parameter $SL_q(3)$ subgroup $(\alpha_1, t_1)$ acting on $u_\alpha$ according to Eq. (3.51c) (with $t_1 = 0 = \alpha_2 - 1$) gives rise to the following transformation law for ordered monomials:

$$(\alpha_1, t_1) : u_1^\ell u_{12}^m u_1^n \rightarrow (\alpha_1^{-1} u_2)^\ell (\alpha_1 u_{12} + t_1\alpha_1^{-1} u_2)^m (\alpha_1 u_1 \alpha_1 + t_1)^n \alpha_1^{-\lambda_2} .$$

Keeping only the linear term in $t_1$ (for $t_1 \rightarrow 0$) and moving $t_1$ to the left, using the CR (3.52), we find:

$$\delta t_1 u_2^\ell u_{12}^m u_1^n = t_1 D_1^+ u_2^\ell u_{12}^m u_1^n ,$$

(4.15)

where we have let $\alpha_1 \rightarrow 1$ (after commuting with $t_1$) and introduced the $q$-derivative $D_1^+$ satisfying :

$$D_1^+ u_2^\ell u_{12}^m u_1^n = q^{m-\ell}(n)_+ u_2^\ell u_{12}^m u_1^{n-1} + q^{-\ell}(m)_+ u_2^{\ell+1}u_{12}^{m-1}u_1^n , \quad (n)_+ = q^+(n-1)[n] .$$

(4.16)

A similar procedure for the subgroup $(1/2, t_2)$ of transformations (3.51c) yields:

$$(\delta t_2 - t_2 D_2^+) u_2^\ell u_{12}^m u_1^n = 0$$

(4.17)
with
\[ D_2^+ u_2^\ell u_1^m u_1^n = (\ell)_+ u_2^{\ell-1} u_1^m u_1^n. \] (4.18)

Equations (4.16) and (4.18) are verified if we assume the CR
\[ D_i^+ u_j = q^{c_{ij}} u_j D_i^+ + \delta_{ij}, \] (4.19)
where \((c_{ij})\) is the \(su_3\) Cartan matrix (2.8); as a consequence
\[ D_i^+ u_{ij} = q u_{ij} D_i^+, \quad D_i^+ u_{ji} = q u_{ji} D_i^+. \] (4.20)

It is straightforward to verify, starting from (4.16) and (4.18), that
\[ (\tilde{E}_i - D_i^+) e_+ (u_2 \tilde{E}_2) e_+ (u_1 \tilde{E}_1) e_+ (u_1 \tilde{E}_1) | - \lambda_2, -\lambda_1 >= 0. \] (4.21)

Moreover, strictly speaking, we have to rely on Eq. (4.21) to fix some remaining ambiguity hidden in the above derivation of Eq. (4.15). Indeed, we have made some arbitrary choices, which affect the overall power of \(q\), in choosing the place of the multiplier \(\alpha^{-\lambda_j}\) and in moving the infinitesimal variables \(t_i\) to the left (rather than to the right).

We end up with a remark concerning the definition of the \(q\)-derivatives.

Starting from a purely algebraic set-up we are looking for an associative algebra with four generators \(u_i, D_j, \ i, j = 1, 2\), subject to the (cubic) Serre relations (4.6) and
\[ D_i^{(2)} D_j + D_j D_i^{(2)} = D_i D_j D_i \quad (i \neq j = 1, 2) \] (4.22)
(the latter reflecting the fact that the map \(\tilde{E}_i \rightarrow D_i^+\) is an antihomomorphism of algebras) and to a quadratic relation of the type (4.19) (with undetermined \(c_{ij}\)).

**Proposition.** Under the above assumptions \((\epsilon_i \epsilon_j c_{ij})\) is the \(su_3\) Cartan matrix for some choice of the sign factors \(\epsilon_1\) and \(\epsilon_2\).

**Proof.** Applying \(D_i\) to both sides of Eq. (4.6) we find:

\[ 1 + q^{c_{11}} = [2]q^{c_{11}+c_{12}}, \quad q^{c_{12}} (1 + q^{c_{11}}) = [2], \]
\[ 1 + q^{c_{22}} = [2]q^{c_{22}+c_{21}}, \quad q^{c_{21}} (1 + q^{c_{22}}) = [2]. \]

The proposition follows.

Our choice, \(D_i^+\) (corresponding to \(\epsilon_i = 1\)), is dictated by Eq. (4.20). The \(e_+\)-exponent is an eigenfunction of \(D_i^+\):
\[ (D_i^+ - a) e_+ (au_i) = 0; \] (4.23)
this guarantees the validity of Eq. (1.21). It is coupled to the expression (3.7c) of the coproduct for the raising operators $\tilde{E}_i$ (3.8); we have the factorization property:

$$e_+ (u_i \Delta (\tilde{E}_i)) (| - \lambda_2, - \lambda_1 > \otimes | - \lambda'_2, - \lambda'_1 >) =$$

$$e_+ (u_i \tilde{E}_i) | - \lambda_2, - \lambda_1 > \otimes e_+ (q^{-\lambda_3} u_i \tilde{E}_i) | - \lambda'_2, - \lambda'_1 > .$$

(4.24)

We note finally, that the choice (4.11) for, say, $\tilde{E}_{12}$ is associated with the CR

$$u_{12} \tilde{E}_{12} u_2 \tilde{E}_2 = q^2 u_2 \tilde{E}_2 u_{12} \tilde{E}_{12} ,$$

$$u_1 \tilde{E}_1 u_{12} \tilde{E}_{12} = q^2 u_{12} \tilde{E}_{12} u_1 \tilde{E}_1 ,$$

(4.25)

which ensure the validity of the relations

$$e_+ (u_2 \tilde{E}_2) e_+ (u_{12} \tilde{E}_{12}) = e_+ (u_2 \tilde{E}_2 + u_{12} \tilde{E}_{12}) ,$$

$$e_+ (u_{12} \tilde{E}_{12}) e_+ (u_1 \tilde{E}_1) = e_+ (u_{12} \tilde{E}_{12} + u_1 \tilde{E}_1) .$$

(4.26)

These relations are used in deriving a factorizing property of the type (1.24) for CS.

### 4.3 Invariant 2-point functions

The invariant 2-point functions

$$< 0 | \tilde{\Lambda}(u) \Lambda(v)|0 > , \quad \Lambda = (\lambda_1 , \lambda_2) , \quad \tilde{\Lambda} = (\lambda_2 , \lambda_1)$$

can be determined from the expressions (4.13)-(4.14) for CS vectors. It is however convenient to first take into account the implications of $H_i$ and $\tilde{E}_i$ invariances. $H_i$ invariance implies the homogeneity condition:

$$< 0 | \tilde{\Lambda}(\rho_1 \rho_2^{-1} u_1 , \rho_1^{-1} \rho_2^2 u_2) \Lambda(\rho_1^2 \rho_2^{-1} v_1 , \rho_1^{-1} \rho_2^2 v_2)|0 > =$$

$$(\rho_1 \rho_2)^{\lambda_1 + \lambda_2} < 0 | \tilde{\Lambda}(u_1 , u_2) \Lambda(v_1 , v_2)|0 > , \quad \rho_i > 0 .$$

(4.27)

$\tilde{E}_i$ invariance yields a pair of $q$-differential equations:

$$D^+_v \Lambda (u_i) q^{H_i} |0 > = < 0 | q^{H_i} \tilde{\Lambda} (u_i) q^{H_i} D^+_v \Lambda (v) |0 > = 0 ,$$

(4.28)

where

$$q^{H_i} \tilde{\Lambda} (u_i , u_j) q^{H_i} = \tilde{q}^{\lambda_i} \Lambda (q^2 u_i , \tilde{q} u_j) , \quad i (\neq j) = 1 , 2 .$$

(4.29)

The invariance conditions should be supplemented by information about the degree of the CS vectors with respect to each of the variables $v_i$ (and $u_i$). These degrees can be extracted from Eqs. (4.13) and from the following properties of the raising operators:

$$\tilde{E}_1^{\lambda_2 + 1} | - \lambda_2 , - \lambda_1 > = 0 = \tilde{E}_2^{\lambda_1 + 1} | - \lambda_2 , - \lambda_1 > = < \lambda_1 , \lambda_2 | E_i^{\lambda_i + 1} .$$

(4.30)
Adding to this the homogeneity condition Eq. (4.27) we look for a general $U_q$-invariant 2-point function of the form

$$
< 0 | \tilde{A}(u) | A(v) | 0 > = \sum_{0 \leq m_i, n_i \leq \lambda_i}^{0 \leq m_i, n_i \leq \lambda_i} \left\{ A_{m_1m_2n_1n_2} u_{12}^{(m_2)} u_1^{(m_1)} v_2^{(n_2)} v_1^{(n_1)} + B_{m_1m_2n_1n_2} u_{12}^{(m_1)} u_2^{(m_2)} v_1^{(n_1)} v_2^{(n_2)} + C_{m_1m_2n_1n_2} u_{12}^{(m_1)} u_2^{(m_2)} v_1^{(n_1)} v_2^{(n_2)} \right\}, \quad (4.31a)
$$

where $x^{(n)}$ is defined by Eq. (4.2) and

$$
1 \leq \nu_i = \lambda_1 + \lambda_2 - m_1 - m_2 - n_1 - n_2 \leq \min(\lambda_i - m_i, \lambda_i - n_i), \quad i = 1, 2, \quad (4.31b)
$$

while in the last term the sum of indices is $\lambda_1 + \lambda_2$.

Invariance under $\tilde{E}_1$ gives the relations

$$
q^\nu ([m_1] A_{m_1m_2n_1n_2} + [\nu + m_2] A_{m_1m_2n_1n_2}) + q^{\lambda_2} (A_{m_1m_2n_1+1n_2} + \delta_{\nu_1} C_{m_1m_2n_1+1n_2}) = 0, \quad (4.32a)
$$

$$
q^\nu ([n_2] B_{m_1m_2n_1+1n_1-1} + [n_1 + \nu] B_{m_1m_2n_1n_2}) + q^{\lambda_1+2} (B_{m_1m_2+1n_1n_2} + \delta_{\nu_1} C_{m_1m_2+1n_1n_2}) = 0, \quad (4.32b)
$$

where

$$
\nu = \lambda_1 + \lambda_2 - m_1 - m_2 - n_1 - n_2. \quad (4.32c)
$$

$\tilde{E}_2$ invariance implies:

$$
q^\nu ([n_1] A_{m_1m_2n_1-1n_1n_2+1} + [n_2 + \nu] A_{m_1m_2n_1n_2}) + q^{\lambda_2+2} (A_{m_1+1m_2n_1n_2} + \delta_{\nu_1} C_{m_1+1m_2n_1n_2}) = 0, \quad (4.33a)
$$

$$
q^\nu ([m_2] B_{m_1m_2-1n_1n_2} + [\nu + m_1] B_{m_1m_2n_1n_2}) + q^{\lambda_1} (B_{m_1m_2n_1n_2+1} + \delta_{\nu_1} C_{m_1m_2n_1n_2+1}) = 0, \quad (4.33b)
$$

where $\nu$ is again given by Eq. (4.32c).

The basis (4.9) used above is particularly appropriate for $q$-symmetric tensor representations (of the type $(\lambda_1, 0)$ or $(0, \lambda_2)$). Using the normalization of (4.13) and the choice of phase

$$
< \lambda, 0 | \tilde{E}_{12}^{(\lambda)} | 0, -\lambda > = q^\lambda = < 0, \lambda | \tilde{E}_{21}^{(\lambda)} | -\lambda, 0 >, \quad (4.34)
$$
we end up with:

\[
<0| \begin{pmatrix} 0 & \lambda \\ u & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ v & 0 \end{pmatrix} |0> = [\lambda]! \sum_{0 \leq m,n, m+n \leq \lambda} \frac{(-u_1 v_2)^{\lambda-m-n}}{(\lambda - m - n)!} (\bar{q} u_{12})^{(m)} (q v_{12})^{(n)},
\]

(4.35a)

\[
<0| \begin{pmatrix} \lambda & 0 \\ u & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ v & 0 \end{pmatrix} |0> = [\lambda]! \sum_{0 \leq m,n, m+n \leq \lambda} \frac{(-u_2 v_1)^{\lambda-m-n}}{(\lambda - m - n)!} (\bar{q} u_{12})^{(m)} (q v_{21})^{(n)}.
\]

(4.35b)

(The two expressions are obtained from one another by just exchanging the subscripts 1 and 2.)

The structure of the invariant 2-point function for \( \lambda_1 \lambda_2 > 0 \) is more complicated. In the simplest example of this type, for the adjoint representation \( \Lambda = (1,1) = \bar{\Lambda} \), the above recurrence relations and normalization condition give:

\[
-<0| \begin{pmatrix} 1 & 1 \\ u & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ v & 0 \end{pmatrix} |0> = (\bar{q} u_{12} + q v_{12})(\bar{q} u_{21} + q v_{21})
\]

\(+ [2](u_{12} v_{12} + u_{21} v_{21}) - u_{12} u_{12} v_{2} - u_{21} u_{22} v_{1} - q^2 (u_{21} v_{21} + u_{12} v_{12})
\)

In order to write down the most general 2-point invariant we recur to a modified PBW basis looking for an expansion of the form:

\[
<0| \bar{\Lambda}(u) \Lambda(v) |0> = \sum_{r+s=\lambda_1+\lambda_2, m,n,p} C^{rs}_{mnp} \left[ \begin{array}{c} r \\ m \\ \lambda_2 \\ n \\ \lambda_1 \\ p \end{array} \right] \times u_1^m u_{12}^n v_2^r v_{12}^s v_1^{r-p},
\]

(4.37)

where we are using the \( q \)-binomial coefficients (4.7c) and the constants \( C^{rs}_{mnp} \) will be determined from \( \bar{E}_n \) invariance. The form (4.37) is suggested by the classical expression (2.18) for which the unknown coefficients give a sign factor, \((-1)^{m+n+r}\). In the \( q \)-deformed case it is consistent to demand that each such sign is multiplied by a power of \( q \).

The condition of \( \bar{E}_2 \) invariance is simpler to take into account. Using Eq.(4.29) and the relations

\[
D_2^+ u_1^p u_{12}^m u_2^n = q^{m+n-p-1}[m] u_1^p u_{12}^m u_2^n,
\]

\[
D_2^+ v_2^{-m} v_{12}^s v_1^{r-p} = q^{-r-1}[r-m] v_2^{-m} v_{12}^s v_1^{r-p},
\]

we find

\[
C^{rs}_{mnp} = q^{-s-n} C^{rs}_{m-1np} = (-1)^m q^{(r-\lambda_2-1)m} C^{rs}_{0np}.
\]

(4.38)
$\tilde{E}_1$ invariance yields a 4-term relation; using the relations
\[ D_1^+ u_1^p u_{12}^n u_2^m = q^{p-1}[p] u_1^{p-1} u_{12}^n u_2^m + q^{2p}[n] u_1^p u_{12}^{n-1} u_2^{m+1} \]
and
\[ D_1^+ v_2^{r-m} v_{12}^{s-n} v_1^{r-p} = q^{m+s-n-p-1} [r - p] \quad v_2^{r-m} v_{12}^{s-n} u_1^{r-p-1} \]
\[ + q^{m+s-r-n-1} [s - n] v_2^{r-m+1} v_{12}^{s-n-1} v_1^{r-p} \]
we find:
\[ [r] \left\{ [\lambda_1 - p] C_{mnp+1}^{rs} + q^{s-\lambda_1-1} [\lambda_2 + p - r + 1] C_{mnp}^{rs} \right\} \]
\[ + q^p [s + 1] \left\{ [m] C_{m-1n+1p}^{r-1s+1} + q^{s+1-r-\lambda_1} [r - m] C_{mnp}^{r-1s+1} \right\} = 0. \]

Demanding that $C_{mnp+1}^{rs} = q^x C_{mnp}^{rs}$ with $x$ chosen in such a way that the first pair of terms in the left-hand side is proportional to $[s + 1]$ we find $x = -\lambda_1 - 2$. Demanding similarly that the second pair of terms be proportional to $[r]$ and using Eq. (4.38) and the previous result we find $C_{mn+1p}^{rs} = -q^{-1} C_{mnp}^{rs}$. Imposing finally Eq. (4.39) we end up with the phase factors:
\[ C_{mnp}^{rs} = (-1)^{r+m+n} q^{r(\lambda_1 + (3-r)/2) + \lambda_1 - n - (\lambda_2 + 2) r + (\lambda_2 + 1 - r) m} \quad (s = \lambda_1 + \lambda_2 - r). \]

It is a straightforward exercise to verify, using
\[ u_2^{(n)} = \sum_{\ell=0}^{n} q^{n \ell} \frac{1}{(\ell)!} u_1^{(n-\ell)} u_2^{\ell}, \]
that the 2-point functions (4.35) and (4.36) are then reproduced as special cases of Eqs. (4.37) and (4.40).

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