Galois theory of the canonical theta structure

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Abstract
In this article we give a Galois-theoretic characterization of the canonical theta structure. The Galois property of the canonical theta structure translates into certain $p$-adic theta relations which are satisfied by the canonical theta null point of the canonical lift. As an application we give a purely algebraic proof of some 2-adic theta identities which describe the set of theta null points of the canonical lifts of ordinary abelian varieties in characteristic 2. The latter theta relations are suitable for explicit canonical lifting.

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1 Introduction
The main result of this article is a proof of the characterizing Galois property of the canonical theta structure. The results of this article complement the discussion in [5, §2.2]. The proofs given there are valid for theta structures which are ‘away from $p$’. In this article we deal with the inseparable case, i.e. we focus on theta structures centered ‘at $p$’. Our main result forms an important ingredient of the method for the computation of canonical lifts that is given in [6].

In this section we briefly review our main theorem and give an application to the characterization of the theta null points of 2-adic canonical lifts. In the remaining sections of the introduction we show how the canonical theta structure can be used to give a theoretical foundation to Mestre’s AGM point counting algorithm.

Assume that we are given the canonical lift $A$ of an ordinary abelian variety over a perfect field of characteristic $p > 0$ and that $A$ admits an ample symmetric line bundle $\mathcal{L}$ of degree 1. It is proved in [4] that there exists a canonical theta structure $\Theta$ for the $p$-th power line bundle $\mathcal{L}^p$. We note that the canonical theta structure depends on the choice of a Lagrangian structure on the special fiber. The theta null point with respect to the canonical theta structure gives an arithmetic invariant which is attached to the triple $(A, \mathcal{L}, \Theta)$.

It is classically known that there exists a canonical lift $F : A \rightarrow A^{(p)}$ of the relative
$p$-Frobenius morphism. Assume that there exists a $p$-adic lift $\sigma$ of the $p$-th power Frobenius automorphism of the residue field. In the following we describe the interplay of the action of $F$ and $\sigma$ on the arithmetic invariant of the canonical lift which is induced by the canonical theta structure. First, consider the following construction using the Frobenius lift $F$. One can prove that there exists an induced ample symmetric line bundle $L^{(p)}$ of degree 1 on $A^{(p)}$ such that $F^*L^{(p)} \cong L^{(p)}$. The $p$-th power of the line bundle $L^{(p)}$ carries a natural theta structure $\Theta^{(p)}$ which one obtains by canonically descending the canonical theta structure $\Theta$ along the Frobenius lift $F$. As a consequence, one gets an arithmetic invariant of $A^{(p)}$ by taking the theta null point with respect to the triple $(A^{(p)}, (L^{(p)})^p, \Theta^{(p)})$. Another way of defining an arithmetic invariant of $A^{(p)}$ is by twisting the triple $(A, L, \Theta)$ along the automorphism $\sigma$. We denote the resulting triple by $(A^{(\sigma)}, (L^{(\sigma)})^p, \Theta^{(\sigma)})$. Since $A$ is assumed to be a canonical lift, it follows that $A^{(p)}$ and $A^{(\sigma)}$, and the $p$-th powers of the line bundles $L^{(p)}$ and $L^{(\sigma)}$, are isomorphic. In this article we prove the following equality.

**Theorem 1.1.** Up to canonical isomorphism one has

$$\Theta^{(p)} = \Theta^{(\sigma)}. \quad (1)$$

We have stated the above theorem in a slightly informal way to avoid technicalities. For more details regarding the formalism and a proof see Theorem 3.8 of Section 3. The above theorem tells us that the two constructions from above, namely the descent along $F$ and the twisting by $\sigma$, define the same arithmetic invariant of $A^{(p)} = A^{(\sigma)}$. The equality (1) implicitly gives a set of special $p$-adic theta relations which hold exclusively for the canonical theta null point, i.e. the theta null point with respect to the canonical theta structure.

Using the Galois-theoretic characterization of the canonical theta structure in the case of residue characteristic 2 one can prove some 2-adic theta relations which describe the set of theta null points of canonical lifts. Now let $p = 2$ and $d \geq 1$ an integer. Assume that the abelian scheme $A$ from above is defined over the unramified extension $\mathbb{Z}/2\mathbb{Z}$ of degree $d$ of the 2-adic integers. We denote the theta null point induced by the canonical theta structure $\Theta$ by $(a_u)_{u \in (\mathbb{Z}/2\mathbb{Z})^g}$. Combining Mumford’s 2-multiplication formula [13, §3] with the theta identities coming from the Galois theoretic characterization of the canonical theta structure (compare Theorem 1.1) we get the following 2-adic theta identities.

**Theorem 1.2.** There exists an $\omega \in \mathbb{Z}_q^*$ such that for all $u \in (\mathbb{Z}/2\mathbb{Z})^g$ the following equality holds

$$a_u^2 = \omega \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} a_{u+v}^\sigma a_v^\sigma. \quad (2)$$

This statement is a trivial corollary of Theorem 4.1 which is proved in Section 4. We illustrate the above results by examples in Section 5. We remark that the theta relations (2) form an essential ingredient of the quasi-quadratic point counting algorithm that is given in [10]. By applying a multivariate Hensel lifting algorithm to the theta identities
one obtains an explicit method for higher dimensional canonical lifting. Practical aspects of the lifting of the canonical theta null point on the basis of the equations are discussed in [9, §4.1]. We would like to point out that the algorithm given there produces completely integral recursions locally at 2, which contrasts with the classical analytic AGM recursions used by Mestre (see Section 1.1.1), that require division by $2^g$ in the 2-adic Witt ring. This in itself yields a more efficient and stable algorithm.

1.1 On Mestre’s generalized $p$-adic AGM sequence

The results of the preceding section can be used to give a theoretical foundation to Mestre’s AGM algorithm. In the year 2002 Mestre proposed an algorithm for point counting on ordinary hyperelliptic curves over finite fields of characteristic 2 which is based on the computation of the generalized 2-adic arithmetic geometric mean (AGM) sequence (see the informal notes [12]). The latter algorithm extends earlier results of his on the algorithmic aspects of the $p$-adic Gaussian AGM (see [8] [11]). Lercier and Lubicz have shown in [10] how Mestre’s algorithm can be modified such that its asymptotic complexity becomes quasi-quadratic in the degree of the finite field.

1.1.1 Elementary properties of the AGM sequence

In this section we recall a result of Mestre about the convergence of the generalized $p$-adic AGM sequence. By lack of a suitable reference we also provide a proof. Consider the multi-valued sequence of vectors $(x_u^{(n)})_{u \in (\mathbb{Z}/2\mathbb{Z})^g}$ which is given by the recursive law

$$x_u^{(n+1)} = \frac{1}{2^g} \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} \sqrt{x_{u+2^g v}^{(n)}}, \quad (3)$$

where $u \in (\mathbb{Z}/2\mathbb{Z})^g$. For $g = 1$ the above formulas resemble the classical AGM formulas of Gauss [7] [2]. The case $g = 2$ was first studied by Borchardt [1]. Under suitable assumptions, that are made precise in the following theorem, the sequence (3) is well defined over the $p$-adic numbers.

Let $g \geq 1$ and let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$ which has $q$ elements. We denote the Witt vectors with values in $\mathbb{F}_q$ by $\mathbb{Z}_q$.

**Lemma 1.3** (Mestre). *If one starts with $x_u^{(0)} \in 1 + 2^{g+1} p\mathbb{Z}_q$ and if one chooses all square roots $\equiv 1 \mod 2p$ then the recursion (3) gives a well-defined sequence in the ring $\mathbb{Z}_q$. Under the assumption

$$x_u^{(0)} \in 1 + 2^{g+2} p\mathbb{Z}_q, \quad \forall u \in (\mathbb{Z}/2\mathbb{Z})^g \quad (4)$$

the generalized AGM sequence (3) converges coordinatewise, and one has

$$\lim_{n \to \infty} x_u^{(n)} = \lim_{n \to \infty} x_0^{(n)}, \quad \forall u \in (\mathbb{Z}/2\mathbb{Z})^g. \quad (5)$$*
Proof. It is a basic fact that the squares in the ring $\mathbb{Z}_q$ that are congruent 1 modulo $2p$ are given by the subset $1 + 4p\mathbb{Z}_q$. In the following let $v_p : \mathbb{Z}_q^* \to \mathbb{Z}$ denote an exponential valuation such that $v_p(p) = 1$. We set $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{Z}_q^*$. If $x \in 1 + 4p\mathbb{Z}_q$, then there exists a unique square root $\sqrt{x}$ of $x$ which is congruent 1 modulo $2p$. Moreover, if $p = 2$ then $v_2(\sqrt{x} - 1) = v_2(x - 1) - 1$. Otherwise, if $p > 2$ then $v_p(x - 1) = v_p(\sqrt{x} - 1)$.

Now assume that $x_u(0) \in 1 + 2^{g+1}p\mathbb{Z}_q$ for all $u \in (\mathbb{Z}/2\mathbb{Z})^g$. By the above discussion the value $x_u(0)$ is a square. Let $y_u(0)$ be the unique square root of $x_u(0)$ such that $y_u(0) \equiv 1 \mod 2p$. Let now $x_u(1)$ be defined by the formulas (3), where the square roots are chosen $\equiv 1 \mod 2p$. The formula (3) reads as

$$x_u(1) = \frac{1}{2^g} \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} y_u(0) \cdot y_v(0), \quad u \in (\mathbb{Z}/2\mathbb{Z})^g.$$  (6)

Let $w \in (\mathbb{Z}/2\mathbb{Z})^g$ and let $S_w$ be a subset of $(\mathbb{Z}/2\mathbb{Z})^g$ which is maximal with respect to the property that for all $v \in S_w$ one has $v + w \not\in S_w$. By assumption we get

$$|x_u(1) - x_v(1)|_p \leq \frac{1}{2^g} \cdot \max_{v \in (\mathbb{Z}/2\mathbb{Z})^g} |y_v(0) - y_w(0)|_p^2.$$  (7)

It follows that

$$\max_{v \in (\mathbb{Z}/2\mathbb{Z})^g} |x_u(1) - x_v(1)|_p \leq \frac{1}{2^g} \cdot \max_{v \in (\mathbb{Z}/2\mathbb{Z})^g} |y_v(0) - y_w(0)|_p^2.$$  (8)

The latter equation implies that $x_u(1) \in 1 + 2^{g+1}p\mathbb{Z}_q$. This shows that the AGM sequence is well-defined.

Now let $y_u(1)$ be the unique square root of $x_u(1)$ such that $y_u(1) \equiv 1 \mod 2p$. One computes

$$|2|_p \cdot |y_u(1) - y_w(1)|_p = |y_u(1) + y_w(1)|_p \cdot |y_0(1) - y_w(1)|_p = |x_0(1) - x_w(1)|_p.$$  (8)

Suppose that we have $x_u(0) \in 1 + 2^{g+2}p\mathbb{Z}_q$. Under this assumption, by combining (8) and (7), we get

$$\max_{v \in (\mathbb{Z}/2\mathbb{Z})^g} |y_0(1) - y_v(1)|_p \leq \frac{1}{2} \cdot \max_{v \in (\mathbb{Z}/2\mathbb{Z})^g} |y_0(0) - y_v(0)|_p$$

We conclude by induction that the sequence of vectors $(y_0^{(n)} - y_u^{(n)})$, and hence also the sequence $(x_0^{(n)} - x_u^{(n)})$, converge to zero for all $u \in (\mathbb{Z}/2\mathbb{Z})^g$. The coordinatewise convergence now follows from the estimate

$$|x_0^{(n+1)} - x_0^{(n)}|_p = \frac{1}{|2|_p} \cdot \left| \left( \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} x_v^{(n)} \right) - 2^g \cdot x_0^{(n)} \right|_p$$

$$\leq \frac{1}{|2|_p} \cdot \max_{v \in (\mathbb{Z}/2\mathbb{Z})^g} |x_0^{(n)} - x_v^{(n)}|_p$$

$$\frac{1}{2} \cdot \max_{v \in (\mathbb{Z}/2\mathbb{Z})^g} |y_0^{(n)} - y_v^{(n)}|_p.$$
This finishes the proof of the lemma.

\section*{1.2 2-Adic approximation of canonical lifts}

In this section we give a scheme theoretic counterpart of the generalized AGM sequence. We use the notation of the previous section. Let $q = 2^d$ where $d \geq 1$ is an integer. As before, let $\mathbb{Z}_q$ denote the Witt vectors with values in a finite field $\mathbb{F}_q$ with $q$ elements. Suppose that $x_u(0) \in 1 + 2^{g+2}\mathbb{Z}_q$, but $x_u(0) \notin 1 + 2^{g+3}\mathbb{Z}_q$. In this case the generalized AGM sequence \cite{3} is well-defined but does not converge coordinatewise. Mestre observed that, under suitable initial conditions, the sequence of $\mathbb{Z}_q$-valued vectors \((x_u^{(dn)}_n)_{u \in (\mathbb{Z}/2\mathbb{Z})^g}\) converges for $n \to \infty$. In other words, the sequence of projective points \((x_u^{(dn)})_{u \in (\mathbb{Z}/2\mathbb{Z})^g}\) converges to a unique projective point of the projective space $\mathbb{P}^{2^g-1}$. This convergence forms a crucial ingredient of Mestre’s AGM algorithm. In the following we will give a theoretical reason for this phenomenon.

We define a scheme theoretic counterpart of the AGM sequence as follows. Let $A$ be an abelian scheme of relative dimension $g$ over $\mathbb{Z}_q$ which has ordinary reduction. It is well-known that there exists a unique isogeny $F : A \to A^{(2)}$ which lifts the relative Frobenius on the special fiber. One has $\ker(F) = A[2]_{\text{loc}}$, where the latter denotes the connected component of the 2-torsion subgroup $A[2]$. Successively dividing out the connected component of the 2-torsion gives a sequence of abelian schemes and Frobenius lifts

\begin{equation}
A \xrightarrow{F} A^{(2)} \xrightarrow{F} A^{(4)} \xrightarrow{F} A^{(8)} \to \ldots
\end{equation}

Let us fix the following notation. If $n \geq 1$ and $B$ is an abelian scheme over $\mathbb{Z}_q$, then we denote its reduction modulo $2^n$ by $B_{2^n}$. Furthermore, we set $A^{(1)} = A$. Obviously, there exists a canonical isomorphism $A_2 \simeq A_2^{(2^n)}$ for all $n \geq 0$. Let $\tilde{A}^{(2^n)}$ denote the canonical lift of $A_2^{(2^n)}$.

\textbf{Theorem 1.4.} \textit{The sequence $A^{(q^n)}$ converges 2-adically to the abelian scheme $\tilde{A}^{(1)}$. More precisely, for all $n \geq 0$ there exists an isomorphism $A_{2^{n+1}}^{(2^{n})} \simeq \tilde{A}_{2^{n+1}}^{(2^{n})}$}.

A more general version of the above theorem with detailed proof can be found in \cite[Th.2.1.2]{3}.

Now assume that $A$ comes equipped with an ample symmetric line bundle $\mathcal{L}$ of degree 1. Then by \cite[Th.5.1]{4} for all $n \geq 1$ the line bundle $\mathcal{L}^{2^n}$ canonically descends along the isogeny $F^n$ to an ample symmetric line bundle $\mathcal{L}^{(2^n)}$ of degree 1 on $A^{(2^n)}$. We set $\mathcal{L}^{(1)} = \mathcal{L}$. Assume that we are given an isomorphism

\begin{equation}
(\mathbb{Z}/2\mathbb{Z})^g \simeq A[2]_{\text{et}}
\end{equation}
where $A[2]^{et}$ denotes the maximal étale quotient of $A[2]$. By composition with $F^n$ the isomorphism $\Xi$ induces a trivialization $(\mathbb{Z}/2\mathbb{Z})^g \cong A^{(2^n)}[2]^{et}$ for all $n \geq 1$. By Th.2.1 for every natural number $n \geq 2$ there exists a canonical symmetric theta structure $\Theta^{(2^n)}$ of type $(\mathbb{Z}/2\mathbb{Z})^g$ for the 2-nd power of the line bundle $L^{(2^n)}$ which only depends on the choice of the trivialization $\Xi$. To the sequence of triples $(A^{(2^n)}, (L^{(2^n)})^2, \Theta^{(2^n)})$ one can associate a sequence of theta null points $(a^{(n)}_u)_{u \in (\mathbb{Z}/2\mathbb{Z})^g}$ in the projective space $\mathbb{P}_{\mathbb{Z}_q}^{2^n-1}$. The line bundle $L^{(1)}$ on $A$ induces an ample symmetric line bundle $L^{(1)}_2$ of degree 1 on the reduction of $A$. By general theory the line bundle $L^{(1)}_2$ lifts to an ample symmetric line bundle $\tilde{L}^{(1)}$ of degree 1 on $\tilde{A}^{(1)}$. We note that the class of the line bundle $(\tilde{L}^{(1)})^2$ does not depend on the chosen lift $\tilde{L}^{(1)}$ of $L^{(1)}_2$. By Th.2 there exists a canonical theta structure $\tilde{\Theta}^{(1)}$ of type $(\mathbb{Z}/2\mathbb{Z})^g$ for the 2-nd power of the line bundle $\tilde{L}^{(1)}$ which only depends on the reduction of the trivialization $\Xi$. We denote the theta null point with respect to the triple $\left(\tilde{A}^{(1)}, (\tilde{L}^{(1)})^2, \tilde{\Theta}^{(1)}\right)$ by $(\tilde{a}^{(0)}_u)$.

**Theorem 1.5.** The sequence of points $a^{(dn)}_u$ converges to the point $a^{(0)}_u$ in the projective space $\mathbb{P}_{\mathbb{Z}_q}^{2^n-1}$ in the following sense: We have $a^{(dn)}_0, \tilde{a}^{(0)}_0 \in \mathbb{Z}_q^*$ and

$$\lim_{n \to \infty} a^{(dn)}_u = \tilde{a}^{(0)}_u$$

for all $u \in (\mathbb{Z}/2\mathbb{Z})^g$.

**Proof.** By Theorem 1.4 we have an isomorphism $A^{(2dn)}_{2dn+1} \cong \tilde{A}^{(1)}_{2dn+1}$ for all $n \geq 0$. We claim that $(\tilde{L}^{(1)}_{2dn+1})^2 \cong (\tilde{L}^{(2dn)}_{2dn+1})^2$. By writing $\tilde{L}^{(1)}_{2dn+1}$ and $\tilde{L}^{(2dn)}_{2dn+1}$ we indicate that we are working over the quotient ring of $\mathbb{Z}_q$ modulo $2^{dn+1}$. By abuse of notation we consider the line bundle $\tilde{L}^{(2dn)}_{2dn+1}$ to be defined on the abelian scheme $\tilde{A}^{(1)}_{2dn+1}$. To prove our claim it suffices to show that the line bundles $\tilde{L}^{(1)}_{2dn+1}$ and $\tilde{L}^{(2dn)}_{2dn+1}$ differ by an element of $\text{Pic}^0_{\tilde{A}^{(1)}_{2dn+1}}[2]$. But this follows immediately from the fact that both line bundles are symmetric and coincide on the special fiber. Hence our claim is proved.

By abuse of notation we consider $\Theta^{(2dn)}_{2dn+1}$, the reduction of $\Theta^{(2dn)}$ modulo $2^{2dn+1}$, to be defined on the abelian scheme $\tilde{A}^{(1)}_{2dn+1}$. We claim that $\Theta^{(2dn)}_{2dn+1} = \tilde{\Theta}^{(1)}_{2dn+1}$. Our claim follows from the following considerations. As canonical theta structures, both theta structures are uniquely determined by the a trivialization, which in both cases equals the isomorphism $\Xi$. As a consequence, the equality of theta structures holds by construction (for more details see Th.2.1-2.2).

It remains to show that $a^{(dn)}_0$ and $\tilde{a}^{(0)}_0$ are units in $\mathbb{Z}_q$. By the above discussion we have $a^{(dn)}_0 \equiv \tilde{a}^{(0)}_0 \pmod{2}$. Applying Theorem 1.2 we deduce that for all $0 \neq u \in (\mathbb{Z}/2\mathbb{Z})^g$ we have $\tilde{a}^{(0)}_u \equiv 0 \pmod{2}$. By ampleness we have $\tilde{a}^{(0)}_0 \not\equiv 0 \pmod{2}$. This completes the proof of the theorem. \qed
Theorem 1.6. For all \( n \geq 2 \) there exists a \( \lambda_n \in \mathbb{Z}_q^* \) such that

\[
\left( a_u^{(n)} \right)^2 = \lambda_n \cdot \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} a_{u+v}^{(n+1)} \cdot a_v^{(n+1)}, \quad u \in (\mathbb{Z}/2\mathbb{Z})^g.
\]  

(12)

Proof. The proof is very similar to the one of Theorem 3.4, so we don’t give it here. The most important ingredient of the proof, namely the \( F \)-compatibility of the theta structures \( \Theta(2^n) \) and \( \Theta(2^{n+1}) \), is implied by slightly modified versions of Lemma 3.1 and of Lemma 3.4.

As an obvious consequence of Theorem 1.2 the points \( (a_u^{(n)}) \) lie in the neighborhood of the point \( (1,0,\ldots,0) \) while the points in Mestre’s AGM sequence are contained in a 2-adic disc around the point \( (1,\ldots,1) \). In the following we will give a transformation of the neighborhood of the point \( (1,0,\ldots,0) \) to the one of \( (1,\ldots,1) \). For every element \( u = (u_1, \ldots, u_g) \in (\mathbb{Z}/2\mathbb{Z})^g \) we define a multiplicative character

\[
l_u : (\mathbb{Z}/2\mathbb{Z})^g \to \{1, -1\}, \quad (v_1, \ldots, v_g) \mapsto \prod_{i=1}^g (-1)^{u_i v_i}.
\]

We note that \( l_u(v) = l_v(u) \) for all \( u, v \in (\mathbb{Z}/2\mathbb{Z})^g \). Now consider the linear transformation which is given by the matrix \( M = (m_{u,v})_{u,v \in (\mathbb{Z}/2\mathbb{Z})^g} \in \text{Mat}(2^g, \mathbb{Z}) \) with entries \( m_{u,v} = l_u(v) \). Obviously, the matrix \( M \) maps the point \( (1,0,\ldots,0) \) to the point \( (1,\ldots,1) \).

In the following we assume that we have normalized the sequence of theta null points \( (a_u^{(n)}) \) such that equation (12) holds with \( \lambda_n = 1 \) for all \( n \geq 2 \). The following lemma was communicated to the author by Lubicz. By lack of a suitable reference we give a proof in here.

Lemma 1.7. If we define \( x_u^{(n)} \in (\mathbb{Z}/2\mathbb{Z})^g \) and \( x_u^{(n+1)} \in (\mathbb{Z}/2\mathbb{Z})^g \) coordinatewise by the formulas

\[
x_u^{(n)} = \left( \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} m_{u,v} \cdot a_v^{(n)} \right)^2 \quad \text{and} \quad x_u^{(n+1)} = \left( \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} m_{u,v} \cdot a_v^{(n+1)} \right)^2
\]

then \( x_u^{(n)} \) and \( x_u^{(n+1)} \) satisfy the relation (3) for all \( u \in (\mathbb{Z}/2\mathbb{Z})^g \).

Proof. We set

\[
y_u^{(n)} = \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} m_{u,v} \cdot a_v^{(n)} \quad \text{and} \quad y_u^{(n+1)} = \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} m_{u,v} \cdot a_v^{(n+1)}
\]

One computes

\[
\frac{1}{2^g} \cdot \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} y_u^{(n)} \cdot y_v^{(n)} = \frac{1}{2^g} \cdot \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} \left( \sum_{s \in (\mathbb{Z}/2\mathbb{Z})^g} l_{u+v}(s) \cdot a_s^{(n)} \right) \cdot \left( \sum_{t \in (\mathbb{Z}/2\mathbb{Z})^g} l_v(t) \cdot a_t^{(n)} \right)
\]

(13)
Let \( w \in (\mathbb{Z}/2\mathbb{Z})^g \). Applying equation (13) one gets

\[
\sum_{u \in (\mathbb{Z}/2\mathbb{Z})^g} l_w(u) \cdot \left( \sum_{v \in (\mathbb{Z}/2\mathbb{Z})^g} y_{u+v}^{(n)} \cdot y_v^{(n)} \right) = \sum_{u \in (\mathbb{Z}/2\mathbb{Z})^g} l_u(u) \cdot \sum_{s \in (\mathbb{Z}/2\mathbb{Z})^g} l_u(s) \cdot (a_s^{(n)})^2 = \sum_{s \in (\mathbb{Z}/2\mathbb{Z})^g} (a_s^{(n)})^2 \cdot \sum_{u \in (\mathbb{Z}/2\mathbb{Z})^g} l_u(u) \cdot l_u(s) = \sum_{s \in (\mathbb{Z}/2\mathbb{Z})^g} (a_s^{(n)})^2 \cdot \sum_{u \in (\mathbb{Z}/2\mathbb{Z})^g} l_u(w + s) = 2^g \cdot (a_w^{(n)})^2
\]

Using equation (14) we compute

\[
\sum_{u \in (\mathbb{Z}/2\mathbb{Z})^g} l_w(u) \cdot \left( y_u^{(n+1)} \right)^2 = \sum_{u \in (\mathbb{Z}/2\mathbb{Z})^g} l_u(u) \cdot \left( \sum_{s,t \in (\mathbb{Z}/2\mathbb{Z})^g} l_u(s) \cdot a_s^{(n+1)} \cdot a_t^{(n+1)} \right) = \sum_{s,t \in (\mathbb{Z}/2\mathbb{Z})^g} a_s^{(n+1)} \cdot a_t^{(n+1)} \sum_{u \in (\mathbb{Z}/2\mathbb{Z})^g} l_w(u) \cdot l_u(s + t) = \sum_{s,t \in (\mathbb{Z}/2\mathbb{Z})^g} a_s^{(n+1)} \cdot a_t^{(n+1)} \sum_{u \in (\mathbb{Z}/2\mathbb{Z})^g} l_u(w + s + t) = 2^g \cdot \sum_{s \in (\mathbb{Z}/2\mathbb{Z})^g} a_s^{(n+1)} \cdot a_{w+s}^{(n+1)}
\]

By the equations (13) and (16) the equality (12) holds for \( \lambda_n = 1 \) if and only if the relation (3) is satisfied with the choice of \( x_u^{(n)} \) and \( x_u^{(n+1)} \) as in the lemma. \( \square \)
The Lemma 1.7 implies that, using the linear transformation $M$, one can compute the sequence $(a^{(n)}_u)_{n \geq 2}$ in terms of Mestre’s generalized AGM sequence. For $n \geq 2$ let $x^{(n)}_u$ and $x^{(n+1)}_u$ be defined as in Lemma 1.7. Then Theorem 1.5 implies that the sequence (9) converges coordinatewise for $n \to \infty$.

1.3 Perspectives

We use the notation of the preceding section. We note that the linear transformation which is induced by the matrix $M$ of Section 1.2 is not invertible over the ring $\mathbb{Z}_q$ because of the following fact.

Lemma 1.8. We have $\det(M) = \pm 2^{(2^g-1)g}$.

Proof. We claim that $M^2 = 2^g \cdot I_{2^g}$, where $I_{2^g}$ denotes the unit matrix of dimension $2^g$. Let $b_{u,v}$ denote the entries of the Matrix $M^2$, where $u, v \in (\mathbb{Z}/2\mathbb{Z})^g$. Then we have

$$b_{u,v} = \sum_{k=0}^{g} m_{u,k} \cdot m_{k,v} = \sum_{k=0}^{g} l_u(k) \cdot l_k(v)$$

$$= \sum_{k=0}^{g} l_{u+v}(k) = \begin{cases} 2^g, & u = v \\ 0, & \text{else} \end{cases}$$

This proves the claim. The lemma now follows from the the multiplicativity of the determinant.

We conclude that the two theories, on one hand the theory of canonical theta null points and on the other hand Mestre’s theory of the generalized AGM sequence, are not equivalent. The difference lies in the special fiber at the prime 2.

Now let $A$ be an ordinary abelian variety over a perfect field $k$ of characteristic 2. Let $g = \dim(A)$. Assume that we are given an ample symmetric line bundle $L$ of degree 1 on $A$ and a symmetric theta structure $\Theta$ of type $(\mathbb{Z}/2\mathbb{Z})^g$ for the line bundle $L^2$. Let $(a_u)_{u \in (\mathbb{Z}/2\mathbb{Z})^g}$ denote the theta null point with respect to the triple $(A, L^2, \Theta)$.

Lemma 1.9. The theta null point $(a_u)$ equals the point $(1, 0, \ldots, 0)$.

Proof. We denote $Z_2 = (\mathbb{Z}/2\mathbb{Z})^g$. Let $\hat{Z}_{2,k}$ denote the Cartier dual of the finite constant group $Z_{2,k}$ associated to the abstract group $Z_2$. Recall that a theta structure is given by a $\mathbb{G}_{m,k}$-equivariant isomorphism $\Theta : G(Z_2) \cong G(L^2)$, where $G(Z_2) = \mathbb{G}_{m,k} \times Z_{2,k} \times \hat{Z}_{2,k}$ is the standard Heisenberg group of type $Z_2$ and $G(L^2)$ the theta group of the line bundle $L^2$. For the definition of the theta group and the standard Heisenberg group see [13, §1] and Section 2.1. Let $G$ be the group of $k$-rational $\mathbb{G}_{m,k}$-equivariant automorphisms of the group $G(Z_2)$. The group $G$ acts by composition on the set of $k$-rational theta structures of type $Z_2$ for the line bundle $L^2$. Every element of the group $G$ is of the form $(\tau, \delta)$, where

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\[ \tau : \mathbb{Z}_2 \times \hat{\mathbb{Z}}_2 \rightarrow G_{m,k} \text{ is a character and } \delta : \mathbb{Z}_2 \times \hat{\mathbb{Z}}_2 \sim \mathbb{Z}_2 \times \hat{\mathbb{Z}}_2 \text{ is an automorphism which is compatible with the pairing} \]

\[ e_2 : (\mathbb{Z}_2 \times \hat{\mathbb{Z}}_2)^2 \rightarrow G_{m,k}, \quad ((x_1, l_1), (x_2, l_2)) \mapsto \frac{l_2(x_1)}{l_1(x_2)} \]

The given theta structure \( \Theta \) induces a trivialization

\[ \mathbb{Z}_2 \sim A[2]^\text{et} \]

where \( A[2]^\text{et} \) denotes the maximal étale quotient of \( A[2] \). By [4] Th.2.2 there exists a canonical theta structure \( \Theta^\text{can} \) of type \( \mathbb{Z}_2 \) for the line bundle \( \mathcal{L}^2 \) which depends on the trivialization (17). In the following we denote the canonical theta null point with respect to the canonical theta structure by \( (a^\text{can}_u)_{u \in \mathbb{Z}_2} \). The action of \( G \) on the set of \( k \)-rational theta structures of type \( \mathbb{Z}_2 \) for \( \mathcal{L}^2 \) is faithful and transitive. Hence there exists a unique \( g = (\tau, \delta) \in G \) such that \( g \ast \Theta^\text{can} \) equals the given theta structure \( \Theta \). Since we are over a field of characteristic 2, the character \( \tau \) has to be trivial. We note that, because \( k \) is assumed as perfect, every finite flat group over \( k \) splits uniquely in a product of its connected component with its maximal étale quotient. Hence the isomorphism \( \delta \) acts componentwise as a pair of automorphisms \((\delta_1, \delta_2)\), where \( \delta_2 \) is the inverse of the Cartier dual of \( \delta_1 \). By the choice of the trivialization (17) we conclude that \( \delta_1 = \delta_2 = \text{id} \). Thus we have \( \Theta = \Theta^\text{can} \).

Theorem 1.2 implies that up to normalization we have

\[ (a_u) = (a^\text{can}_u) = (1, 0, \ldots, 0). \]

This completes the proof of the lemma.

This shows that the points of Mestre’s AGM sequence cannot be 2-theta null points of abelian schemes since this is not so modulo the prime 2. Recall that the points in Mestre’s AGM sequence lie in a 2-adic neighborhood of the point \((1, \ldots, 1)\). We conjecture that Mestre’s AGM sequence has a moduli interpretation in terms of degenerate abelian varieties, or, more strongly, in terms of semi-abelian schemes. Our conjecture should follow from a moduli interpretation of the ‘boundary’ of Mumford’s moduli space. This ‘boundary’ is canonically given by the points in the projective closure of the moduli space of abelian varieties with theta structure under the canonical projective embedding.

**Leitfaden**

This article is structured as follows. We recall some classically known facts about algebraic theta functions in Section 2. In Section 3 we prove the Galois-theoretic property of the canonical theta structure. In Section 4 we prove theta relations which have as solutions the canonical theta null points of canonical lifts of ordinary abelian varieties over a perfect field of characteristic 2. We give some examples of canonical theta null points in Section 5.
Notation

Let $R$ be a ring, $X$ an $R$-scheme and $S$ an $R$-algebra. By $X_S$ we denote the base extended scheme $X \times_{\text{Spec}(R)} \text{Spec}(S)$. Let $\mathcal{M}$ be a sheaf on $X$. Then we denote by $\mathcal{M}_S$ the sheaf that one gets by pulling back via the projection $X_S \to X$. Let $I : X \to Y$ be a morphism of $R$-schemes. Then $I_S$ denotes the morphism that is induced by $I$ via base extension with $S$.

We use the same symbol for a scheme/module and the associated fppf-sheaf. By a group we mean a group object in the category of fppf-sheaves. If a group $G$ is representable by a scheme and the representing object has the property of being finite (flat, étale, connected, etc.) then we simply say that $G$ is a finite (flat, étale, connected, etc.) group. Similarly we will say that a morphism of groups is finite (faithfully flat, smooth, etc.) if the groups are representable and the induced morphism of schemes has the corresponding property.

A group (morphism of groups) is called finite locally free if it is finite flat and of finite presentation. The Cartier dual of a finite locally free commutative group $G$ will be denoted by $G^D$. The multiplication by an integer $n \in \mathbb{Z}$ on $G$ will be denoted by $n$. A finite locally free and surjective morphism between groups is called an isogeny. By an elliptic curve we mean an abelian scheme of relative dimension 1.

If $G$ and $H$ are groups then we denote by $\text{Hom}(G,H)$ the fppf-sheaf which is defined by $U \mapsto \text{Hom}(G(U),H(U))$, where the latter denotes the group homomorphisms $G(U) \to H(U)$.

2 Background on algebraic theta functions

In this section we recall some classically known facts about algebraic theta functions. We present Mumford’s results on an algebraic theory of theta functions in such a way that they apply to our situation. The theory of algebraic theta functions was developed by David Mumford in [13], [14] and [15].

2.1 Theta structures

Let $A$ be an abelian scheme over a ring $R$ and $\mathcal{L}$ a line bundle on $A$. Consider the morphism

$$\varphi_\mathcal{L} : A \to \text{Pic}_A^{0}, \ x \mapsto \langle T_x^* \mathcal{L} \otimes \mathcal{L}^{-1} \rangle$$

where $\langle \cdot \rangle$ denotes the class in $\text{Pic}_A^{0}$. We set $\check{A} = \text{Pic}_A^{0}$. Note that $\check{A}$ is the dual of $A$ in the category of abelian schemes. We denote the kernel of the morphism $\varphi_\mathcal{L}$ by $A[\mathcal{L}]$. A line bundle $\mathcal{L}$ on $A$ satisfies $A[\mathcal{L}] = A$ if and only if its class is in $\text{Pic}_A^{0}(R)$. Also it is well-known that if $\mathcal{L}$ is relatively ample then $\varphi_\mathcal{L}$ is an isogeny. In the latter case we say that $\mathcal{L}$ has degree $d$ if $\varphi_\mathcal{L}$ is fiberwise of degree $d$. Let $S$ be an $R$-algebra. We define the theta group of $\mathcal{L}$ as the functor

$$G(\mathcal{L})(S) = \left\{ (x, \varphi) \mid x \in A[\mathcal{L}](S), \ \varphi : \mathcal{L}_S \sim T_x^{*} \mathcal{L}_S \right\}.$$
The functor $G(\mathcal{L})$ has the structure of a group given by the group law

$$(y, \psi), (x, \varphi) \mapsto (x + y, T_x^* \psi \circ \varphi).$$

There are natural morphisms

$$G(\mathcal{L}) \to A[\mathcal{L}], \quad (x, \varphi) \mapsto x \quad \text{and} \quad \mathbb{G}_{m,R} \to G(\mathcal{L}), \quad \alpha \mapsto (0_A, \tau_\alpha)$$

where $0_A$ denotes the zero section of $A$ and $\tau_\alpha$ denotes the automorphism of $\mathcal{L}$ given by the multiplication with $\alpha$. The induced sequence of groups

$$0 \to \mathbb{G}_{m,R} \to G(\mathcal{L}) \xrightarrow{\pi} A[\mathcal{L}] \to 0 \quad (18)$$

is central and exact. Now let $\mathcal{L}$ be relatively ample of degree $d$. Then $A[\mathcal{L}]$ is finite locally free of order $d^2$.

Let $K$ be a finite constant group. The standard Heisenberg group of type $K$ is given by the scheme $\mathbb{G}_{m,R} \times K_R \times K_R^D$ with group operation

$$(\alpha_1, x_1, l_1) \star (\alpha_2, x_2, l_2) \overset{\text{def}}{=} (\alpha_1 \cdot \alpha_2 \cdot l_2(x_1), x_1 + x_2, l_1 \cdot l_2).$$

Here $K_R^D$ denotes the Cartier dual of $K_R$.

**Definition 2.1.** A theta structure of type $K$ for $\mathcal{L}$ is a $\mathbb{G}_{m,R}$-equivariant isomorphism of groups

$$G(K) \xrightarrow{\sim} G(\mathcal{L}).$$

For more details about theta structures we refer to \[13\, \S1].

### 2.2 Mumford’s Isogeny Theorem

Let $R$ be a noetherian local ring. Let $I : A \to A'$ be an isogeny of abelian schemes over $R$. Assume that we are given ample line bundles $\mathcal{L}$ and $\mathcal{L}'$ on $A$ and $A'$, respectively. Suppose we are given theta structures

$$\Theta_A : G(K_A) \xrightarrow{\sim} G(\mathcal{L}) \quad \text{and} \quad \Theta_{A'} : G(K_{A'}) \xrightarrow{\sim} G(\mathcal{L}')$$

where $K_A$ and $K_{A'}$ are finite constant groups. Let $\alpha : I^* \mathcal{L}' \xrightarrow{\sim} \mathcal{L}$ be an isomorphism of $\mathcal{O}_A$-modules. The existence of $\alpha$ implies that Ker$(I)$ is contained in $A[\mathcal{L}]$. By [13, §1,Prop.1] the morphism $\alpha$ induces a section Ker$(I) \to G(\mathcal{L})$ of the natural projection $G(\mathcal{L}) \to A[\mathcal{L}]$. Let $\tilde{K}$ denote the image of Ker$(I)$ in $G(\mathcal{L})$. Assume that

$$\left(\dagger\right) \quad \text{the image of } \tilde{K} \text{ under } \Theta_A^{-1} \text{ is of the form } \{1\} \times Z_1 \times Z_2$$

with subgroups $Z_1 \leq K_A$ and $Z_2 \leq K_{A'}^D$. 

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One defines

\[ Z_1^\perp = \{ x \in K_A \mid (\forall l \in Z_2) \ l(x) = 1 \} \]

and

\[ Z_2^\perp = \{ l \in K_A^D \mid (\forall x \in Z_1) \ l(x) = 1 \}. \]

Note that \( Z_1^\perp \times Z_2^\perp \) is the subgroup of points of \( K_A \times K_A^D \) being orthogonal to \( Z_1 \times Z_2 \) and that \( Z_1 \times Z_2 \subseteq Z_1^\perp \times Z_2^\perp \).

**Proposition 2.2.** Let \( \sigma : Z_1^\perp \to K_{A'} \) be a surjective morphism of groups having kernel \( Z_1 \). There exists a theta structure \( \Theta_A(\sigma) \) of type \( K_{A'} \) for the pair \((A', \mathcal{L}')\) depending on the theta structure \( \Theta \) and the morphism \( \sigma \).

**Proof.** Let \( G(\mathcal{L})^* \) denote the centralizer of \( \tilde{K} \) in \( G(\mathcal{L}) \). There exists a natural isomorphism of groups

\[ G(\mathcal{L})^*/\tilde{K} \cong G(\mathcal{L}') \]  

(compare [13, Prop.2]). The image of \( G(\mathcal{L})^* \) under the morphism \( \Theta_A^{-1} \) equals \( \mathbb{G}_{m,R} \times Z_1^\perp \times Z_2^\perp \). The isomorphism [19] composed with the induced isomorphism

\[ \mathbb{G}_{m,R} \times Z_1^\perp / Z_1 \times Z_2^\perp / Z_2 \cong G(\mathcal{L})^*/\tilde{K} \]

gives an isomorphism

\[ \mathbb{G}_{m,R} \times Z_1^\perp / Z_1 \times Z_2^\perp / Z_2 \cong G(\mathcal{L}). \]  

We claim that there exists a natural isomorphism

\[ Z_2^\perp / Z_2 \cong (Z_1^\perp / Z_1)^D. \]  

This follows from the Snake Lemma applied to the following commutative diagram of exact sequences

\[
\begin{array}{cccccc}
0 & \to & Z_1 & \to & K_A & \to & (Z_2^\perp)^D & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \text{id} & & \downarrow & & \\
0 & \to & Z_1^\perp & \to & K_A & \to & Z_2^D & \to & 0.
\end{array}
\]

Here the upper exact sequence is obtained by dualizing the exact sequence

\[ 0 \to Z_2^\perp \to K_A^D \xrightarrow{\text{res}} Z_1^D \to 0 \]

and \( e \) is given by \( x \mapsto e((x,1),(0,1)) \) where \( e : (K_A \times K_A^D)^2 \to \mathbb{G}_{m,R} \) denotes the commutator pairing. The left hand vertical morphism is the natural inclusion. The morphism \( \sigma \) induces an isomorphism \( \sigma_1 : K_{A'} \cong Z_1^\perp / Z_1 \). We denote by \( \sigma_2 : K_{A'}^D \cong Z_2^\perp / Z_2 \) the inverse of the composition of \( \sigma_1^D \) with the isomorphism [21]. Composing the isomorphism [20] with the isomorphism \( \text{id} \times \sigma_1 \times \sigma_2 \) we get a theta structure

\[ \Theta(\sigma) : G(K_{A'}) \cong G(\mathcal{L}). \]

This proves the proposition. \( \square \)
Note that Θ_A(σ) does not depend on the choice of α.

**Definition 2.3.** We say that Θ_A and Θ_A′ are I-compatible if there exists α as above, assumption (†) holds and there exists a morphism σ as above such that Θ_A′ = Θ_A(σ).

Let π_A and π_A′ denote the structure maps of A and A′, respectively. Since I is faithfully flat the natural morphism L′_τ → I_*I^*L′ is injective. As a consequence there exists an injective morphism ι: π′_L′ → π_*L of O_R-modules given by the composition

π′_L′ π′_τ −→ π′_I_*I^*L′ = π_*I_*L′ π∗α −→ π_*L.

The morphism ι identifies the sections of π′_*L′ with those sections of π_*L which are invariant under the translations with points in the kernel of the isogeny I. Assume that we have chosen theta group invariant isomorphisms β_A: π∗L ∼→ V(K_A) and β_A′: π′_*L′ ∼→ V(K_A′), where we set V(K) = Hom(K, O_R) for a finite group K. We define V_I: V(K_A′) → V(K_A) by setting V_I = β_A o ι o β_A−1.

**Theorem 2.4** (Isogeny Theorem). Suppose Θ_A and Θ_A′ are I-compatible. In particular we are given a morphism σ as above such that Θ_A′ = Θ_A(σ). There exists a λ ∈ R∗ such that for all f ∈ V(K_A′) we have

V_I(f)(x) = \begin{cases} 0 & x \notin Z_1^\perp \\ \lambda f(σ(x)) & x \in Z_1^\perp \end{cases}

where x ∈ K_A.

For a proof of Theorem 2.4 in the case where R is a field see [13, §1, Th.4]. The latter proof can easily be adapted to our situation.

**2.3 Mumford’s 2-multiplication formula**

Let R be a noetherian local ring and let A → Spec(R) be an abelian scheme and L an ample symmetric line bundle on A which is generated by global sections. Let n ≥ 2 be an integer. We set L_n = L^⊗n. Assume that we are given theta structures

Θ: G(K) ∼→ G(L) and Θ_n: G(K_n) ∼→ G(L_n).

One defines a morphism of groups ε_n : G(L) → G(L_n) by setting

(x, ψ) ↦ (x, ψ^⊗n).

Note that there is a natural inclusion A[L] ⊆ A[L_n] and the multiplication-by-n induces an epimorphism A[L_n] → A[L]. On the image of G_m,R the morphism ε_n equals the n-th
Next we define a morphism of groups \( \eta_n : G(L_n) \to G(L) \) using the symmetry of \( L \). Assume we are given \((x, \psi) \in G(L_n)\). Since \( L \) is symmetric there exists an isomorphism \( \gamma : L^\otimes n^2 \to \sim [n]^* L \). Consider the composed isomorphism \[ [n]^* L \overset{\gamma^{-1}}\to L^\otimes n^2 \overset{\psi^\otimes n}{\to} T_x^* L_n^\otimes n = T_x^* L^\otimes n^2 \overset{T_x^* \gamma}{\to} T_x^*[n]^* L = [n]^* T_{nx}^* L. \]

Since \( nx \) is a point of \( A[L] \) there exists an isomorphism \( \rho : L \to T_{nx}^* L \) inducing the above isomorphism. Since \([n]^* \) is faithfully flat the morphism \( \rho \) is uniquely determined. We set \( \eta_n(x, \psi) = (nx, \rho) \). One can check that this definition is independent of the choice of \( \gamma \).

We define morphisms \( E_n : G(K) \to G(K_n) \) and \( H_n : G(K_n) \to G(K) \) by setting\[ (\alpha, x, l) \mapsto (\alpha^n, x, l^n), \]
respectively. Here we consider points of \( K \times K^D \) as points of \( K_n \times K_n^D \) via the natural inclusion.

**Definition 2.5.** We say that the theta structures \( \Theta_n \) and \( \Theta \) are \( n \)-compatible if

(i) \( K \leq K_n \) and \( K = \{nx | x \in K_n\} \),

(ii) \( \delta_n \) restricted to \( K \times K^D \) equals \( \delta \),

(iii) \( \Theta_n \circ E_n = \epsilon_n \circ \Theta \) and \( \Theta \circ H_n = \eta_n \circ \Theta_n \).

Now assume that \( n = 2 \). Choose theta group invariant isomorphisms \( \beta : \pi_* L \to V(K) \) and \( \beta_2 : \pi_* L_2 \to V(K_2) \),

where \( V(K) \) and \( V(K_2) \) are defined as in the preceding section.

**Definition 2.6.** Let \( s \) and \( s' \) be sections of \( \pi_* L \). Set \( f = \beta(s) \) and \( f' = \beta(s') \). We define \( f \ast f' = \beta_2(s \otimes s') \).

Assume that we have chosen a rigidification of \( L \). The latter choice implies the existence of finite theta functions \( q_L \in V(K) \) and \( q_{L_2} \in V(K_2) \) which interpolate the coordinates of the corresponding theta null points (cf. [13 §1]).
Theorem 2.7 (2-Multiplication Formula). Suppose $\Theta$ and $\Theta_2$ are 2-compatible theta structures. There exists a $\lambda \in \mathbb{R}^*$ such that for all $f, f' \in V(K)$ we have

$$(f \star f')(x) = \lambda \sum_{y \in x + K} f(x + y)f'(x - y)q_{L_2}(y)$$

where $x \in K_2$.

For a proof of Theorem 2.7 over a field see [13, §3]. Without major changes the proof given there applies to our situation.

Assume that we have chosen an embedding $\mathbb{Z}_2 \subseteq K_2$ where $\mathbb{Z}_2 = (\mathbb{Z}/2\mathbb{Z})^g$ and $g = \dim_R(A)$. Applying the above theorem to the Dirac-basis of $V(K)$ and evaluating at the zero section of the abelian scheme $A$ gives the following relation for theta null values.

Corollary 2.8. Suppose $\Theta$ and $\Theta_2$ are 2-compatible theta structures. Let $u, v \in K_2$ such that $u + v, u - v \in K$. There exists a $\lambda \in \mathbb{R}^*$ such that

$q_{L}(u + v)q_{L}(u - v) = \lambda \sum_{z \in \mathbb{Z}_2} q_{L_2}(u + z)q_{L_2}(v + z)$.

3 The Galois property of the canonical theta structure

In this section we prove the characterizing pull back property of the canonical theta structure (introduced in [4]). More precisely, we discuss the Galois theoretic properties of the canonical theta structure in the case where the theta structure is centered ‘at $p$’, i.e. the underlying level structure is non-étale and the corresponding polarization is inseparable. For the étale case see [5, §2.2].

3.1 Compatibility of the canonical theta structure

In the following we use the notation and the definitions of Section 2. Let $R$ denote a complete noetherian local ring with perfect residue field $k$ of characteristic $p > 0$. Assume that $R$ admits a lift of the $p$-th power Frobenius automorphism of $k$. Let $A \rightarrow \text{Spec}(R)$ be an abelian scheme of relative dimension $g$ having ordinary reduction and let $\mathcal{L}$ be an ample symmetric line bundle of degree 1 on $A$. Let $F : A \rightarrow A^{(p)}$ denote the unique lift of the relative $p$-Frobenius. We denote the structure map $A^{(p)} \rightarrow \text{Spec}(R)$ by $\pi^{(p)}$. By [4, Th.5.1] there exists an ample symmetric line bundle $\mathcal{L}^{(p)}$ of degree 1 on $A^{(p)}$ and an isomorphism $F^*\mathcal{L}^{(p)} \sim \mathcal{L}^{\otimes p}$ such that $(\mathcal{L}^{(p)})_k$ is the $p$-Frobenius twist of $\mathcal{L}_k$. For $i \geq 0$ we set

$$\mathcal{L}_i = \mathcal{L}^{\otimes p^i}, \quad \mathcal{M}_i = (\mathcal{L}^{(p)})^{\otimes p^i} \quad \text{and} \quad K_i = (\mathbb{Z}/p^i\mathbb{Z})^g_R.$$

Let $r \geq 1$ and assume that we are given an isomorphism

$$K_r \sim A[p^r]^{et},$$

(22)
where \( A[p'^r]^{et} \) denotes the maximal étale quotient of \( A[p^r] \). The lift of the relative \( p \)-Frobenius \( F \) induces an isomorphism \( F[p'^r]^{et} : A[p'^r]^{et} \cong A(p)^{[p'^r]^{et}} \). Composing \( F[p'^r]^{et} \) with the isomorphism (22) we get an isomorphism

\[
K_r \cong A^{(p)}[p'^r]^{et}.
\]

(23)

Now assume that \( A \) is the canonical lift of \( A_k \). As a consequence \( A^{(p)} \) is also a canonical lift. By [11, Th.2.2] there exist for all \( 0 \leq j \leq r \) canonical theta structures \( \Theta_j : G(K_j) \cong G(L_j) \) and \( \Sigma_j : G(K_j) \cong G(M_j) \) depending on the isomorphisms (22) and (23), respectively. For the definition of a theta structure see Definition 2.1.

Let us fix some notation that will be used in proofs of the following lemmas. By \( V_i : A \to A_i \), where \( i \geq 1 \), we denote the \( i \)-fold application of the lift of the \( p \)-Verschiebung. For \( 1 \leq i \leq r \) let \( \delta_i \) and \( \kappa_i \) denote the Lagrangian structures induced by the theta structures \( \Theta_i \) and \( \Sigma_i \), respectively. We note that by the definition of the canonical theta structure we have \( A_i = A/\delta_i(K_i) \) for \( i \leq r \). Let

\[
v_{j+1} : K_{j+1} \to G(L_{j+1}) \quad \text{and} \quad v_j : K_j \to G(L_j)
\]

be the liftings induced by the canonical theta structures \( \Theta_{j+1} \) and \( \Theta_j \), respectively. By descent the sections \( v_{j+1} \) and \( v_j \) correspond to line bundles \( L^{(j+1)} \) and \( L^{(j)} \) on \( A_{j+1} \) and \( A_j \), respectively, equipped with isomorphisms

\[
\beta_{j+1} : V_{j+1}^*L^{(j+1)} \cong L_{j+1} \quad \text{and} \quad \beta_j : V_j^*L^{(j)} \cong L_j.
\]

Also, we set \( A_0 = A \) and \( L^{(0)} = L \).

**Lemma 3.1.** For \( 0 \leq j < r \) the theta structures \( \Theta_{j+1} \) and \( \Sigma_{j+1} \) are \( p \)-compatible with \( \Theta_j \) and \( \Sigma_j \), respectively.

**Proof of Lemma 3.1.** It suffices to prove the lemma for the theta structures \( \Theta_j \) where \( 0 \leq j \leq r \). For trivial reasons the theta structure \( \Theta_1 \) is compatible with the theta structure \( \Theta_0 \). Assume that \( j \geq 1 \). Obviously, the theta structures \( \Theta_{j+1} \) and \( \Theta_j \) satisfy the conditions (i) and (ii) of Definition 2.5. It remains to show that condition (iii) of the latter definition is satisfied. We will proceed in two steps.

**Claim 3.2.** One has

\[
\Theta_{j+1} \circ E_p = \epsilon_p \circ \Theta_j,
\]

(24)

where \( E_p \) and \( \epsilon_p \) are defined as in Section 2.3.

**Proof of Claim 3.2.** We verify equation (24) for points which lie above \( K_j \). As the proof for points lying over \( K_j^D \) is analogous we do not present it here. Let \( x \) be a point of \( K_j \). One has

\[
v_{j+1}(x) = (\delta_{j+1}(x), T_{\delta_{j+1}(x)}^{s} \beta_{j+1} \circ \beta_{j+1}^{-1}) \quad \text{and} \quad v_j(x) = (\delta_j(x), T_{\delta_j(x)}^{s} \beta_j \circ \beta_j^{-1}).
\]
Here we consider $x$ as a point of $K_{j+1}$ by the given inclusion $K_j \subseteq K_{j+1}$. By the definition of the canonical theta structure there exists an isomorphism

$$\beta : \text{V}^*L^{(j+1)} \xrightarrow{\sim} (\text{L}^{(j)})^\otimes p,$$

where $\text{V} : A_j \to A_{j+1}$ denotes the $p$-Verschiebung. Consider the isomorphism $\psi$ given by the composition

$$V_{j+1}^*L^{(j+1)} = V_{j}^*V^*L^{(j+1)} \xrightarrow{V_j^*\beta} V_{j}^*(L^{(j)})^\otimes p \xrightarrow{\beta^\otimes p} \text{L}_j^\otimes p = \text{L}_{j+1}.$$

Since $\delta_j(x)$ lies in the kernel of $V_j$ we have $V_j \circ T_{\delta_j(x)} = V_j$ and hence $T_{\delta_j(x)}^*V_j^*\beta = V_j^*\beta$. It follows that

$$\epsilon_p(v_j(x)) = \epsilon_p(\delta_j(x), T_{\delta_j(x)}^*\beta_j \circ \beta_j^{-1}) = (\delta_{j+1}(x), T_{\delta_{j+1}(x)}^*\beta_j^\otimes p \circ \beta_j^{-p})$$

$$= (\delta_{j+1}(x), T_{\delta_{j+1}(x)}^*\psi \circ \psi^{-1}) = (\delta_{j+1}(x), T_{\delta_{j+1}(x)}^*\beta_{j+1} \circ \beta_{j+1}^{-1}) = v_{j+1}(x).$$

The latter equality sign follows from the fact that $\beta_{j+1}$ and $\psi$ differ by a unit. This proves our claim.

**Claim 3.3.** One has

$$\Theta_j \circ H_p = \eta_p \circ \Theta_{j+1}$$

(25)

where $H_p$ and $\eta_p$ are defined as in Section 2.3

**Proof of Claim 3.3.** We verify the equality (25) for points of $G(K_{j+1})$ which lie above $K_{j+1}$. The proof for points lying over $K_{j+1}$ is analogous. Consider a point $(1, x, 1)$ in $G(K_{j+1}).$ We have

$$\Theta_j(H_p(1, x, 1)) = v_j(px) = (\delta_j(px), \tau_j)$$

and

$$\Theta_{j+1}(1, x, 1) = v_{j+1}(x) = (\delta_{j+1}(x), \tau_{j+1})$$

where

$$\tau_{j+1} = T_{\delta_{j+1}(x)}^*\beta_{j+1} \circ \beta_{j+1}^{-1} \quad \text{and} \quad \tau_j = T_{\delta_j(px)}^*\beta_j \circ \beta_j^{-1}.$$

Choose an isomorphism $\gamma : L_j^{\otimes p^2} \xrightarrow{\sim} [p]^*L_j$. Consider the composed isomorphism

$$[p]^*L_j \xrightarrow{\gamma^{-1}} L_j^{\otimes p^2} = L_j^{\otimes p} \xrightarrow{T_{\delta_{j+1}(x)}^*} (T_{\delta_{j+1}(x)}^*L_{j+1})^\otimes p$$

$$= T_{\delta_{j+1}(x)}^*L_j^{\otimes p^2} \xrightarrow{T_{\delta_{j+1}(x)}^*\gamma} T_{\delta_{j+1}(x)}^*[p]^*L_j = [p]^*T_{\delta_{j+1}(x)}^*[p]^*L_j.$$

We claim that the latter isomorphism is induced by $\tau_j$. By the definition of the canonical theta structure there exists an isomorphism

$$\xi : F^*L^{(j)} \xrightarrow{\sim} (L^{(j+1)})^\otimes p$$
where $F$ denotes the lift of the relative $p$-Frobenius. The composed isomorphism

$$V_{j+1}^*(\mathcal{L}^{(j+1)}) \overset{\otimes \theta}{\longrightarrow} V_{j+1}^* F^* \mathcal{L}^{(j)} = V_j^* [p]^* \mathcal{L}^{(j)} = [p]^* V_j^* \mathcal{L}^{(j)} \overset{[p]^* \delta}{\longrightarrow} [p]^* \mathcal{L}_j.$$  

differs from $\gamma \circ \beta_{j+1}^{\otimes p}$ by a unit. We conclude that

$$T_{\delta_{j+1}(x)}^* \gamma \circ \tau_{j+1}^{\otimes p} \circ \gamma^{-1} = T_{\delta_{j+1}(x)}^*(\gamma \circ \beta_{j+1}^{\otimes p}) \circ (\gamma \circ \beta_{j+1}^{\otimes p})^{-1}$$

$$= T_{\delta_{j+1}(x)}^* ([p]^* \beta_j \circ V_{j+1}^* \xi^{-1}) \circ ([p]^* \beta_j \circ V_{j+1}^* \xi^{-1})^{-1} = T_{\delta_{j+1}(x)}^* [p]^* \beta_j \circ [p]^* \beta_j^{-1}$$

$$= [p]^* T_{\delta_j(py)}^* \beta_j \circ [p]^* \beta_j^{-1} = [p]^* (T_{\delta_j(py)}^* \beta_j \circ \beta_j^{-1}) = [p]^* \tau_j.$$

Note that $T_{\delta_{j+1}(x)}^* V_{j+1}^* \xi^{-1} = V_{j+1}^* \xi^{-1}$ since $x$ is in the kernel of $V_{j+1}$. \hfill \Box

This completes the proof of the lemma. \hfill \Box

**Lemma 3.4.** For all $0 \leq j < r$ the theta structures $\Theta_{j+1}$ and $\Sigma_j$ are $F$-compatible.

**Proof of Lemma 3.4.** As the lemma is trivial for $j = 0$ we may assume that $j > 1$. By assumption there exists an isomorphism

$$\gamma_{j+1} : F^* \mathcal{M}_j \simto \mathcal{L}_{j+1}.$$  

Obviously assumption (†) of Section 2.2 holds with $Z_1 = 0$ and $Z_2 = K_j^D$. It follows that $Z_1^\perp = K_{j+1}^D$. By duality we conclude that $Z_1^\perp$ coincides with the image of $K_j$ in $K_{j+1}$ under the given inclusion.

**Claim 3.5.** One has

$$\Sigma_j = \Theta_{j+1}(id)$$

where the notation is as in Proposition 2.2.

**Proof of Claim 3.5.** Checking the claim amounts to prove the commutativity of the diagram

$$\begin{array}{ccc}
G(\mathcal{L}_{j+1})/\tilde{K} & \overset{\Sigma_j}{\longrightarrow} & G(\mathcal{M}_j) \\
\downarrow\text{can} & & \downarrow\text{id} \times \sigma_1 \times \sigma_2 \\
G(\mathcal{L}_{j+1})/Z_1^\perp / Z_1 \times Z_2^\perp / Z_2 & & G(\mathcal{M}_j) / K_j \times K_j^D
\end{array}$$

where the upper horizontal arrow is induced by $\Theta_{j+1}$, the morphism $\sigma_1$ equals the ‘identity’ and the morphism $\sigma_2$ equals its dual (compare proof of Proposition 2.2). In the following we verify the commutativity of the above diagram for points of the form $(1, x, 1) \in G_{m,R} \times K_j \times K_j^D$. An analogous proof exists for points of the form $(1, 0, l)$. Via the morphisms $\sigma_1$ and $\sigma_2$ we can consider $(1, x, 1)$ as a point of $G(K_{j+1})$. Here we consider $x$ as an element
of $K_{j+1}$ via the given inclusion $K_j \subseteq K_{j+1}$. By definition its image under $\Theta_{j+1}$ is given by $v(x)$ where

$$v : K_{j+1} \to G(L_{j+1})$$

is the lifting which is induced by the existence of an isomorphism

$$\alpha_{j+1} : V_{j+1}^* L^{(j+1)} \cong L_{j+1}.$$ 

The class of $v(x)$ in $G(L_{j+1})/\bar{K}$ can be represented by an element of the form

$$\left(\delta_{j+1}(y), T_{\delta_{j+1}(y)}^* \alpha_{j+1} \circ \alpha_{j+1}^{-1}\right)$$

where $y \in K_{j+1}$ is chosen such that $F(\delta_{j+1}(y)) = \kappa_{j+1}(x)$. On the other hand we have

$$\Sigma_j(1, x, 1) = w(x)$$

where $w : K_j \to G(M_j)$ is the lifting that is induced by the canonical theta structure. The lifting $w$ corresponds to the line bundle $L^{(j-1)}$ on $A_{j-1}$ with isomorphism

$$\beta_j : V_j^* L^{(j-1)} \cong M_j.$$ 

We have

$$w(x) = (\kappa_j(x), T_{\kappa_j(x)}^* \beta_j \circ \beta_j^{-1}).$$

By the definition of the canonical theta structure there exist isomorphisms

$$\xi_1 : F^* L^{(j-1)} \cong (L^{(j)})^{\otimes p} \quad \text{and} \quad \xi_2 : V^* L^{(j+1)} \cong (L^{(j)})^{\otimes p}.$$ 

Let $\xi = \xi_1^{-1} \circ \xi_2$. The isomorphism $V_j^* \xi$ induces an isomorphism

$$V_{j+1} L^{(j+1)} = V_j^* V_j^* L^{(j+1)} \xrightarrow{V_j^* \xi} V_j^* F^* L^{(j-1)} = F^* V_j^* L^{(j-1)}.$$ 

The composed isomorphism $\gamma_{j+1} \circ F^* \beta_j \circ V_j^* \xi$ differs from $\alpha_{j+1}$ by a unit. By the definition of the canonical isomorphism

$$G(L_{j+1})/\bar{K} \cong G(M_j)$$

(see proof of [13, §1, Prop.2]) the element in $G(L_{j+1})/\bar{K}$ that corresponds to $w(x)$ is given by

$$\left(\delta_{j+1}(y), T_{\delta_{j+1}(y)}^* \gamma_{j+1} \circ F^* (T_{\delta_{j+1}(y)} \beta_j \circ \beta_j^{-1}) \circ \gamma_{j+1}^{-1}\right)$$

$$= \left(\delta_{j+1}(y), T_{\delta_{j+1}(y)}^* \gamma_{j+1} \circ F^* T_{\delta_{j+1}(y)}^* \beta_j \circ F^* \beta_j^{-1} \circ \gamma_{j+1}^{-1}\right)$$

$$= \left(\delta_{j+1}(y), T_{\delta_{j+1}(y)}^* \gamma_{j+1} \circ T_{\delta_{j+1}(y)}^* F^* \beta_j \circ F^* \beta_j^{-1} \circ \gamma_{j+1}^{-1}\right)$$

$$= \left(\delta_{j+1}(y), T_{\delta_{j+1}(y)}^* \gamma_{j+1} \circ F^* \beta_j \circ V_j^* \xi \circ (\gamma_{j+1} \circ F^* \beta_j \circ V_j^* \xi)^{-1}\right)$$

$$= \left(\delta_{j+1}(y), T_{\delta_{j+1}(y)}^* \alpha_{j+1} \circ \alpha_{j+1}^{-1}\right) = v(x).$$

Hence our claim is proved. 

This completes the proof of the lemma.
3.2 The pull back of the canonical theta null point

Let \( R \) be a complete noetherian local ring with perfect residue field \( k \) of characteristic \( p > 0 \). The following lemma forms the key ingredient of the theorem that is proved below.

**Lemma 3.6.** One has

\[
\mathrm{Hom}\left(\left(\mathbb{Q}_p/\mathbb{Z}_p\right)_R, \mu^\infty_{p,R}\right)(R) = 0.
\]

**Proof.** It suffices to prove that

\[
\mathrm{Hom}\left(\left(\mathbb{Q}_p/\mathbb{Z}_p\right)_R, \mu^\infty_{p,R}\right)(R) = 0. \tag{26}
\]

The elements of the group \( \mathrm{Hom}\left(\left(\mathbb{Q}_p/\mathbb{Z}_p\right)_R, \mu^\infty_{p,R}\right)(R) \) correspond to the compatible systems \((a_1, a_2, a_3, \ldots)\) of \( p \)-power roots of unity in \( R \), i.e. \( a_{i+1}^p = a_i \) for \( i \geq 1 \). Such a compatible system necessarily has to be trivial. The latter is obvious on the level Artin local quotients of \( R \) by powers of the maximal ideal. This completes the proof of the remark. \( \square \)

Suppose that we are given an abelian scheme \( A \) over \( R \) which has ordinary reduction. Let \( \mathcal{L} \) be an ample symmetric line bundle of degree 1 on \( A \). We set \( q = p^d \) where \( d \geq 1 \) is an integer. Assume that there exists a \( \sigma \in \text{Aut}(R) \) lifting the \( p \)-th power Frobenius automorphism of \( k \). Recall that there exists a canonical lift \( F : A \to A^{(p)} \) of the relative \( p \)-Frobenius morphism and a canonical ample symmetric line bundle \( \mathcal{L}^{(p)} \) of degree 1 on \( A^{(p)} \) such that \( F^* \mathcal{L}^{(p)} \cong \mathcal{L}^g \) (see [1, \S5]). We set \( Z_n = (\mathbb{Z}/n\mathbb{Z})^g \) for \( n \geq 1 \) where \( g = \dim_R(A) \). Assume that we are given an isomorphism

\[
Z_{q,R} \sim A[q]^{et}, \tag{27}
\]

where \( A[q]^{et} \) denotes the maximal étale quotient of \( A[q] \). The Frobenius lift \( F \) induces an isomorphism

\[
F[q]^{et} : A[q]^{et} \to A^{(p)}[q]^{et}.
\]

Composing \( F[q]^{et} \) with the isomorphism (27) gives an isomorphism

\[
Z_{q,R} \sim A^{(p)}[q]^{et}. \tag{28}
\]

Assume that \( A \) is the canonical lift of \( A_k \). Our assumption implies that the abelian scheme \( A^{(p)} \) is a canonical lift. By [4, Th.2.2] there exist canonical theta structures \( \Theta_q \) and \( \Theta_{q}^{(p)} \) of type \( Z_q \) for the line bundles \( \mathcal{L}^g \) and \( (\mathcal{L}^{(p)})^g \), which depend on the isomorphisms (27) and (28), respectively.

Let \( A^{(\sigma)} \) be defined by the Cartesian diagram

\[
\begin{array}{ccc}
A^{(\sigma)} & \xrightarrow{pr} & A \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \xrightarrow{\text{Spec}(\sigma)} & \text{Spec}(R),
\end{array} \tag{29}
\]
where the right hand vertical arrow is the structure morphism. Let \( \mathcal{L}^{(\sigma)} \) be the pull back of \( \mathcal{L} \) along the morphism \( \text{pr} : A^{(\sigma)} \to A \) which is defined by the diagram (29). We obtain a theta structure \( \Theta^{(\sigma)}_q \) for \( (\mathcal{L}^{(\sigma)})^q \) by extension of scalars along \( \text{Spec}(\sigma) \) applied to the theta structure \( \Theta_q : G(Z_q) \sim G(\mathcal{L}^q) \) and by chaining with the natural isomorphism \( Z_q \sim Z_{q, \sigma} \) times the inverse of its dual. In the following we will assume that the special fibers of \( A^{(p)} \) and \( A^{(\sigma)} \) are in fact equal. By uniqueness of the canonical lift there exists a canonical isomorphism \( \tau : A^{(p)} \sim A^{(\sigma)} \) over \( R \) lifting the identity on special fibers. We set \( \mathcal{L}_n = \mathcal{L}^n, \mathcal{L}_n^{[p]} = (\mathcal{L}^{(p)})^n \) and \( \mathcal{L}_n^{(\sigma)} = (\mathcal{L}^{(\sigma)})^n \) for \( n \geq 1 \).

**Lemma 3.7.** We have \( \tau^* \mathcal{L}^{(\sigma)}_q \sim \mathcal{L}^{(p)}_q \).

**Proof.** We set \( \mathcal{M} = (\tau^* \mathcal{L}^{(\sigma)}) \otimes (\mathcal{L}^{(p)})^{-1} \). It follows by the definition of \( \mathcal{L}^{(p)} \) that the class of \( \mathcal{M} \) reduces to the trivial class. By assumption the line bundle \( \tau^* \mathcal{L}^{(\sigma)} \) is symmetric. By [4, Th.5.1] also the line bundle \( \mathcal{L}^{(p)} \) is symmetric. As a consequence, the line bundle \( \mathcal{M} \) is symmetric and gives an element of \( \text{Pic}^{0}(A^{(p)/R}[2]) \). If \( p = 2 \) then clearly \( \mathcal{M} \) gives the trivial class. Suppose that \( p > 2 \). We observe that by assumption the group \( \text{Pic}^{0}(A^{(p)/R}[2]) \) is finite étale. We conclude by the connectedness of the ring \( R \) and by the fact that \( \mathcal{M} \) gives the trivial class modulo \( p \) that the class of \( \mathcal{M} \) is trivial. This proves the lemma.

By the above discussion there exists an isomorphism \( \gamma : \tau^* \mathcal{L}^{(\sigma)}_q \sim \mathcal{L}^{(p)}_q \). We define a \( \mathbb{G}_{m,R} \)-invariant morphism of theta groups \( \tau^! : G(\mathcal{L}^{(\sigma)}_q) \to G(\mathcal{L}^{(p)}_q) \) by setting

\[
(x, \varphi) \mapsto (y, \tau^* \gamma \circ \tau^* \varphi \circ \gamma^{-1})
\]

where \( y = \tau^{-1}(x) \) and \( \varphi : \mathcal{L}^{(\sigma)}_q \sim T_q \mathcal{L}^{(\sigma)}_q \) is a given isomorphism. Obviously, our definition is independent of the choice of \( \gamma \). For trivial reasons the morphism \( \tau^! \) gives an isomorphism. By what has been said before, the isomorphism \( \tau^! \) is canonical.

**Theorem 3.8.** One has

\[
\tau^! \circ \Theta^{(\sigma)}_q = \Theta^{(p)}_q.
\]

**Proof of Theorem 3.8** In the following we fix the notation that we will use during the course of the proof. Let \( \delta^{(\sigma)}_q ; \delta^{(p)}_q \) and \( \delta^{(\sigma)}_q \) be the Lagrangian structures which are induced by \( \Theta_q, \Theta^{(p)}_q \) and \( \Theta^{(\sigma)}_q \), respectively. The isomorphism \( \tau \) induces isomorphisms \( \tau_q : A^{(p)}[q] \sim A^{(\sigma)}[q] \) and \( \tau^{\text{et}}_q : A^{(p)}[q]^{\text{et}} \sim A^{(\sigma)}[q]^{\text{et}} \), where \( A^{(p)}[q]^{\text{et}} \) and \( A^{(\sigma)}[q]^{\text{et}} \) denote the maximal étale quotients of \( A^{(p)}[q] \) and \( A^{(\sigma)}[q] \), respectively. Let

\[
\pi : A[q] \to A[q]^{\text{et}}, \quad \pi^{(\sigma)}_q : A^{(\sigma)}[q] \to A^{(\sigma)}[q]^{\text{et}} \quad \text{and} \quad \pi^{(p)}_q : A^{(p)}[q] \to A^{(p)}[q]^{\text{et}}
\]

denote the natural projections on maximal étale quotients. We denote the natural inclusion \( Z_{q,R} \to Z_{q,R} \times Z_{q,R} \) by \( i \).

Furthermore, let \( r^{(p)}_q : A^{(p)}[q]^{\text{et}} \to A^{(p)}[q] \) and \( r^{(\sigma)}_q : A^{(\sigma)}[q]^{\text{et}} \to A^{(\sigma)}[q] \) be the sections of
the natural projections \( \pi_q^{(p)} \) and \( \pi_q^{(\sigma)} \), corresponding to the theta structures \( \Theta_q^{(p)} \) and \( \Theta_q^{(\sigma)} \), respectively, such that

\[
\begin{align*}
\tau_q^{(p)} \circ \pi_q^{(p)} \circ \delta_q^{(p)} \circ i &= \delta_q^{(p)} \circ i \\
\tau_q^{(\sigma)} \circ \pi_q^{(\sigma)} \circ \delta_q^{(\sigma)} \circ i &= \delta_q^{(\sigma)} \circ i
\end{align*}
\]

and

The proof of the theorem is divided into several lemmas. We start the proof by checking that the equality (30) holds for the induced Lagrangian structures.

**Lemma 3.9.** One has the equality

\[
\delta_q^{(\sigma)} = \tau_q^{(p)}.
\]  

**Proof of Lemma 3.9.** First we verify the equality (31) on points which lie in the image of the morphism \( i \). Thus we have to show that

\[
\delta_q^{(\sigma)} \circ i = \tau_q^{(p)} \circ \delta_q^{(p)} \circ i.
\]  

**Claim 3.10.** There is an equality

\[
\pi_q^{(\sigma)} \circ \delta_q^{(\sigma)} \circ i = \pi_q^{(\sigma)} \circ \tau_q \circ \delta_q^{(p)} \circ i = \tau_q^{(\sigma)} \circ \pi_q^{(p)} \circ \delta_q^{(p)} \circ i.
\]  

**Proof of Claim 3.10.** The right hand equality of the equation (33) is obvious. In the following we will show the left hand equality of (33). Let \( F_q^{\text{et}} : A[\eta_q]^{\text{et}} \to A[\eta_q]^{\text{et}} \) denote the isomorphism which is induced by the Frobenius lift \( F \). By the definition of \( \delta_q^{(p)} \) we have

\[
\pi_q^{(p)} \circ \delta_q^{(p)} \circ i = F_q^{\text{et}} \circ \pi_q \circ \delta_q \circ i,
\]

and hence, by means of the right hand equality of (33), the left hand equation of (33) is equivalent to

\[
\pi_q^{(\sigma)} \circ \delta_q^{(\sigma)} \circ i = \tau_q^{(\sigma)} \circ F_q^{\text{et}} \circ \pi_q \circ \delta_q \circ i.
\]  

By the theory of finite étale groups it suffices to check that the latter equality holds on the special fiber. We claim that this indeed is the case. Our claim follows from the fact that by assumption the morphism \( \tau_q^{\text{et}} \) equals the identity on the special fiber. \( \square \)

**Claim 3.11.** One has

\[
\tau_q^{(\sigma)} \circ r_q^{(p)} = r_q^{(\sigma)} \circ \tau_q^{\text{et}}.
\]  

**Proof of Claim 3.11.** The sections \( r_q^{(\sigma)} \) and \( r_q^{(p)} \) are, by the definition of the canonical theta structure, the truncations to \( A^{(\sigma)}[\eta_q]^{\text{et}} \) and \( A^{(p)}[\eta_q]^{\text{et}} \) of sections \( r^{(\sigma)}_{\beta^{\infty}} \) and \( r^{(p)}_{\beta^{\infty}} \), respectively, of the connected-étale sequences of Barsotti-Tate groups

\[
0 \longrightarrow A^{(\sigma)}[\beta^{\infty}]^{\text{loc}} \longrightarrow A^{(\sigma)}[\beta^{\infty}] \longrightarrow A^{(\sigma)}[\beta^{\infty}]^{\text{et}} \longrightarrow 0
\]  

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The uniqueness follows from the following observation. Passing to the strict Henselization of \( R \) where \( g \) reduce to the \( q \) form a Hom

\[
\text{detailed proof of the equality (31) on the points of the connected component and refer to [4, §6.3] for details of the construction of the canonical level structure. The proof essentially is a consequence of the fact that the ‘connected factor’ of the canonical level structure equals (up to canonical isomorphism) the dual of the ‘étale factor’ of the canonical level structure. This completes the proof of the lemma.}
\]

Combining the Claims 3.10 and 3.11 we conclude that the equality (32) holds. We omit a detailed proof of the equality (31) on the points of the connected component and refer to [4, §6.3] for details of the construction of the canonical level structure. The proof essentially is a consequence of the fact that the ‘connected factor’ of the canonical level structure equals (up to canonical isomorphism) the dual of the ‘étale factor’ of the canonical level structure. This completes the proof of the lemma.

It remains to show, on top of Lagrangian structures, that the equality of theta structures holds, as claimed in the theorem. By [4 Prop.4.2] and [4 Prop.4.5] the verification of the equality of the theta structures \( \tau^1 \circ \Theta_{q}^{(\sigma)} \) and \( \Theta_{q}^{(p)} \) comes down to checking that certain descent line bundles are isomorphic. Let \( F_q : A^{(p)} \to A^{(pq)} \) and \( V_q : A^{(p)} \to A^{(p/q)} \) be the \( q \)-fold application of the lifts of the relative \( p \)-Frobenius and the Verschiebung, respectively. By construction we have

\[
\text{Ker}(F_q) = \delta^{(p)}(\{0\} \times Z_{q,R}^D) \quad \text{and} \quad \text{Ker}(V_q) = \delta^{(p)}(Z_{q,R} \times \{0\}).
\]

We first consider the descent of \( (L^{(p)})^q \) along \( F_q \). The theta structures \( \tau^1 \circ \Theta_{q}^{(\sigma)} \) and \( \Theta_{q}^{(p)} \) correspond to line bundles \( M^{(\sigma)} \) and \( M^{(p)} \) on \( A^{(pq)} \) such that

\[
(L^{(p)})^q \cong F_q^* M^{(\sigma)} \cong F_q^* M^{(p)}.
\]

By definition of the canonical theta structure both of the line bundles \( M^{(\sigma)} \) and \( M^{(p)} \) reduce to the \( q \)-Galois twist of \( L_k^{(p)} \). It follows by uniqueness [4 Th.5.1] that \( M^{(\sigma)} \cong M^{(p)} \).

Now consider the case of descent along Verschiebung \( V_q \). The symmetric theta structures \( \Theta_{q}^{(p)} \) and \( \tau^1 \circ \Theta_{q}^{(\sigma)} \) correspond to symmetric line bundles \( H^{(p)} \) and \( H^{(\sigma)} \) of degree 1, respectively, such that

\[
(L^{(p)})^q \cong V_q^* H^{(p)} \cong V_q^* H^{(\sigma)}.
\]

We claim that \( H^{(p)} \cong H^{(\sigma)} \). If \( p > 2 \) then by symmetry \( H^{(p)} \otimes (H^{(\sigma)})^{-1} \) gives an element of \( \text{Pic}^0_{A^{(n/p)}/R}[2](R) \) which lies in the kernel of the dual \( V_q^* \) of \( q \)-Verschiebung \( V_q \). The kernel
of $V_q^*$ has $p$-power order. Because of the assumption $(p, 2) = 1$ the claim follows. More
delicate is the case where $p = 2$. Recall that the line bundles $H^{(p)}$ and $H^{(\sigma)}$ are uniquely
determined by the choice of sections $s^{(p)}$ and $s^{(\sigma)}$ of the connected-étale sequence

$$0 \to A^{(2)}[2]^{loc} \to A^{(2)}[2]^{et} \to 0.$$  

(cf. [14 Th.5.2]). The sections $s^{(p)}$ and $s^{(\sigma)}$, which correspond to the theta structures $\Theta_q^{(p)}$
and $\tau^{\dagger} \circ \Theta_q^{(\sigma)}$, by construction both can be prolonged to sections of the connected-étale
sequence

$$0 \to A^{(2)}[2^\infty]^{loc} \to A^{(2)}[2^\infty] \to A^{(2)}[2^\infty]^{et} \to 0.$$ 

But there exists a unique section of the latter exact sequence of Barsotti-Tate groups. By
Lemma 3.6 we conclude that $s^{(p)} = s^{(\sigma)}$. This completes the proof of the theorem. \qed

4 Theta null points of 2-adic canonical lifts

Let $R$ be a complete noetherian local ring with perfect residue field $k$ of characteristic 2.
Assume that $R$ admits a lift $\sigma$ of the 2nd power Frobenius automorphism of $k$. Let $A$ be an
abelian scheme over $R$ of relative dimension $g$. Let $L$ be an ample symmetric line bundle
of degree 1 on $A$. We set $Z_n = (\mathbb{Z}/n\mathbb{Z})^g$ for $n \geq 1$. Assume that we are given $q = 2^j$
where $j \geq 1$ and an isomorphism

$$Z_{q,R} \cong A[q]^{et}$$ (38)

where $A[q]^{et}$ denotes the maximal étale quotient of $A[q]$. Suppose that $A$ is the canonical
lift of $A_k$. By [14 Th.2.2] there exists a canonical theta structure $\Theta_q$ of type $Z_q$ for the pair
$(A, L^q)$ depending on the isomorphism (38). Let $(x_u)_{u \in Z_q}$ denote the theta null point of $A
with respect to the canonical theta structure $\Theta_q$.

**Theorem 4.1.** There exists an $\omega \in R^*$ such that for all $u, v \in Z_q$ one has

$$x_{u+v}x_{u-v} = \omega \sum_{z \in \mathbb{Z}_2} x_u^z x_v^z.$$ (39)

**Proof.** We can assume that we have chosen an isomorphism

$$Z_{2q,R} \cong A[2q]^{et}$$ (40)

which extends the given trivialization (38). We remark that in order to choose the iso-
morphism (40) one may has to extend the base locally-étale. Our assumption is justified
by the fact that the resulting formulas are already defined over the given ring $R$. By [4,
there exists a canonical theta structure $\Theta_{2q}$ for the line bundle $L^{2q}$ of type $\mathbb{Z}_{2q}$ which depends only on the isomorphism (40).

Assume that we are given a rigidification of the line bundle $L$. For an abstract finite constant commutative group $K$ we denote $V(K) = \text{Hom}(K, \mathcal{O}_R)$. Let $\pi : A \to \text{Spec}(R)$ denote the structure morphism of the abelian scheme $A$. By general theory one can choose theta group equivariant isomorphisms $\mu_q : \pi^* L^q \sim \to V(\mathbb{Z}_q)$ and $\mu_{2q} : \pi^* L^{2q} \sim \to V(\mathbb{Z}_{2q})$ which are uniquely determined up to scalar. As explained in [13, §1], together with the chosen rigidification, these isomorphisms determine finite theta functions $q_{L^q}$ and $q_{L^{2q}}$ in $V(\mathbb{Z}_q)$ and $V(\mathbb{Z}_{2q})$, respectively, which give the coordinates of the induced theta null points. It follows from Lemma 3.1 that the theta structures $\Theta_q$ and $\Theta_{2q}$ are 2-compatible, and hence satisfy the compatibility assumptions of Theorem 2.7. As a consequence, by Corollary 2.8 there exists a $\lambda \in R^*$ such that

$$q_{L^q}(u + v)q_{L^q}(u - v) = \lambda \sum_{z \in \mathbb{Z}_2} q_{L^{2q}}(u + z)q_{L^{2q}}(v + z)$$ (41)

for all $u, v \in \mathbb{Z}_q$. By Theorem 3.8, which describes the action of $\sigma$ on the canonical theta null point, by Lemma 3.4 and by [5, Lem.3.5] we deduce from equation (41) that

$$q_{L^q}(u + v)q_{L^q}(u - v) = \lambda \sum_{z \in \mathbb{Z}_2} q_{L^{2q}}^\sigma(u + z)q_{L^{2q}}^\sigma(v + z).$$

This completes the proof of the theorem.

We remark that one has $x_0 \in R^*$. For the case $q = 2$ this follows immediately from the specific shape of the equations (2). The case $q > 2$ can be reduced to the case $q = 2$ by descending along the relative $\frac{q}{2}$-Frobenius and by applying [5, Lem.3.5].

Assume now that $k$ is a finite field of characteristic 2 and that we have normalized the theta null point such that $x_0 = 1$. The resulting $\omega$ then is expected to be equal to the inverse of the determinant of relative Verschiebung (up to norm one unit). Evidence is given by the example of Section 5.1.

5 Examples of canonical theta null points

In the following we will illustrate Theorem 4.1 by some examples. The following lemma relates the theta null points of elliptic curves to the classical $\lambda$- and $j$-invariants. The lemma will be of use in the following sections. Let $K$ be a field of characteristic unequal to 2. Assume that we are given an elliptic curve $E$ over $K$. We denote the ample line bundle associated to the zero section of $E$ by $L$. Assume that we are given a theta structure $\Theta$ of type $\mathbb{Z}/2\mathbb{Z}$ for the line bundle $L^2$. We denote the theta null point of $E$ with respect to the triple $(E, L^2, \Theta)$ by $(a_0, a_1)$. 

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Lemma 5.1. The curve $E$ can be given by the equation

$$y^2 = x(x-1)(x-\lambda) \quad \text{where} \quad \lambda = \left(\frac{a_0^2 - a_1^2}{a_0^2 + a_1^2}\right)^2.$$ 

Proof. For simplicity we assume that $(a_0, a_1) = (1, \mu)$ with $\mu \in K$. The proof in the case $(a_0, a_1) = (\mu, 1)$ is analogous. The morphism $\tau: E \to \mathbb{P}^1_K$ induced by the theta structure $\Theta$ for $\mathcal{L}^2$ is surjective of degree 2. The group $G(\mathcal{L}^2)/\mathbb{G}_{m,K} \cong A[\mathcal{L}^2]$ acts on $E$ by translation. The theta structure $\Theta$ induces a Lagrangian structure $(\mathbb{Z}/2\mathbb{Z}) \times \mu_{2,K} \sim A[\mathcal{L}^2]$. We claim that the group elements $(1,1)$ and $(0,-1)$ act on $\mathbb{P}^1_K$ by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

We set

$$\delta_z(x) = \begin{cases} 1, & x = z \\ 0, & x \neq z \end{cases}$$

for $x, z \in \mathbb{Z}/2\mathbb{Z}$. Using the definition of the action of the standard theta group $G(\mathbb{Z}/2\mathbb{Z})$ on the module of finite theta functions one computes

$$(1,x,1)\delta_z = \delta_{z-x} \quad \text{and} \quad (1,0,l)\delta_z = l(z)\delta_z.$$ 

This proves our claim. By construction the theta null point $(1,\mu)$ induces a ramification point of the morphism $\tau$. The orbit of $(1,\mu)$ under the action of the group $A[\mathcal{L}^2]$ is given by $(1,\mu), (1,-\mu), (\mu,1)$ and $(-\mu,1)$. Clearly, the points in the orbit of $(1,\mu)$ give rise to ramification points of $\tau$. By Hurwitz’s theorem there are exactly 4 ramification points of $\tau$. We map $(1,\mu) \mapsto (0,1), (1,-\mu) \mapsto (1,0)$ and $(\mu,1) \mapsto (1,1)$ by the linear transformation

$$\begin{pmatrix} \mu^2+1 \\ \mu^2+1 \\ \mu^2+1 \end{pmatrix} \begin{pmatrix} 1 \\ \mu \\ \mu \end{pmatrix}.$$ 

The latter transformation maps the point $(-\mu,1)$ to $(1,\lambda)$. This completes the proof of the lemma.

5.1 Elliptic curves with 2-theta structure

Let $E$ be an elliptic curve over $\mathbb{Z}_q$ where $\mathbb{Z}_q$ denotes the Witt vectors with values in a finite field $\mathbb{F}_q$ with $q = 2^d$ elements. By $\sigma$ we denote the canonical lift of the absolute 2-Frobenius of $\mathbb{F}_q$ to $\mathbb{Z}_q$. Assume that $E$ has ordinary reduction and that $E$ is the canonical lift of $E_{\mathbb{F}_q}$. Let $\mathcal{L}$ denote the ample symmetric line bundle associated to the Weil divisor given by the zero section $0_E$ of $E$. There exists a unique isomorphism

$$(\mathbb{Z}/2\mathbb{Z})_{\mathbb{Z}_q} \sim E[2]^{\text{et}}.$$

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By [4, Th.2.2] there exists a canonical theta structure of type $\mathbb{Z}/2\mathbb{Z}$ for the pair $(E, L^\otimes 2)$. By Theorem 4.1 there exists an $\omega \in \mathbb{Z}_q^*$ such that the coordinates of the theta null point $(x_0, x_1)$ with respect to the canonical theta structure satisfy the equations
\[
x_0^2 = \omega((x_0^\sigma)^2 + (x_1^\sigma)^2) \quad \text{and} \quad x_1^2 = 2\omega x_0^\sigma x_1^\sigma.
\]
(42)
The equations (42) imply that $x_1 \equiv 0 \mod 2$. Hence $x_0 \in \mathbb{Z}_q^*$. We set $\mu = x_1/x_0$. Rewriting the equations (42) in terms of $\mu$ we get
\[
\mu^2((\mu^\sigma)^2 + 1) = 2\mu^\sigma.
\]
(43)
We set $L = \text{End}_{\mathbb{Z}_q}(E) \otimes \mathbb{Q}$.

Case $d = 1$. Equation (43) implies that
\[
0 = \mu^3 + \mu - 2 = (\mu - 1)(\mu^2 + \mu + 2).
\]
A short calculation shows that $L = \mathbb{Q}(\sqrt{-7})$ which has class number 1. The polynomial $x^2 + x + 2$ is reducible over $L$. We remark that $\mu = 2\omega$. Note that as expected $\omega$ is the inverse of the invertible eigenvalue of the 2-Frobenius endomorphism of $E_{\mathbb{F}_2}$. Lemma 5.1 implies that the $j$-invariant of the elliptic curve $E$, which corresponds to $\mu$, equals $-15^3$.

Case $d = 2$. Equation (43) implies that
\[
0 = (\mu^2 + \mu + 2)(\mu^4 + 4\mu^3 + 5\mu^2 + 2\mu + 4).
\]
Assume that the $j$-invariant of $E_{\mathbb{F}_4}$ is not equal to 1. Then $L = \mathbb{Q}(\sqrt{-15})$ which has class number 2. We remark that the minimal polynomial $x^4 + 4x^3 + 5x^2 + 2x + 4$ of $\mu$ generates the Hilbert class field of $L$. Using the right hand equation (42) and the relation $\mu^{\sigma^2} = \mu$ one deduces that $\mu^3 = 8\omega^2\omega^\sigma$. We conclude that $\mu = 2\zeta\omega$ where $\zeta$ is a norm one unit in $\mathbb{Z}_q$. This gives evidence for our conjecture that $\omega$ is the inverse of the determinant of the relative Verschiebung morphism (up to norm one unit).

5.2 Elliptic curves with 4-theta structure

Let $\mathbb{Z}_q$, $\mathbb{Q}_q$, $\sigma$, $E$ and $\mathcal{L}$ be as in Section 5.1. Assume that we are given an isomorphism
\[
(\mathbb{Z}/4\mathbb{Z})_R \cong E[4]^{\text{et}}.
\]
(44)
By [4, Th.2.2] there exists a canonical theta structure of type $(\mathbb{Z}/4\mathbb{Z})$ for the pair $(A, \mathcal{L}^{\otimes 4})$ depending on the trivialization (44). The canonical theta structure induces a closed immersion $\tau : E \to \mathbb{P}^3_{\mathbb{Z}_q}$. Let $(x_0, x_1, x_2, x_3)$ denote the image of the zero section of $E$ under $\tau$. According to Mumford the image of $\tau_{\mathbb{Q}_q}$ in $\mathbb{P}^3_{\mathbb{Q}_q}$ is the intersection of the quadratic hyper-surfaces
\[
y_1^2 + y_3^2 = 2\lambda y_0 y_2 \quad \text{and} \quad y_0^2 + y_2^2 = 2\lambda y_1 y_3
\]
(45)
where \( \lambda = \frac{x_1}{x_0^2} \). By symmetry the theta null point lies in the plane \( y_1 = y_3 \). For more details see [13, §5]. We can assume that \( x_0 = 1 \). By Theorem 4.1 there exists a unique \( \omega \in \mathbb{R}^\ast \) such that

\[
\begin{align*}
1 &= \omega (1 + (x_2^\sigma)^2) \quad (46) \\
x_1^2 &= \omega x_1^\sigma (1 + x_2^\sigma) \quad (47) \\
x_2^2 &= 2\omega x_2^\sigma \quad (48) \\
x_2 &= 2\omega (x_1^\sigma)^2. \quad (49)
\end{align*}
\]

Now assume that \( d = 1 \). By equation (48) we have \( x_2 = 2\omega \). It follows by the equation (47) that \( x_1 = x_3 = \omega (1 + 2\omega) \). As a consequence we have

\[
\lambda = \frac{\omega}{2} (1 + 2\omega)^2
\]

where \( \lambda \) is as above. Finally we conclude by equation (46) that

\[
4\omega^3 + \omega - 1 = (2\omega - 1)(2\omega^2 + \omega + 1) = 0.
\]

The latter factor makes clear that \( \omega \) is a reciprocal eigenvalue of Frobenius in an order of discriminant \(-7\).

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