NAHM’S, BASU-HARVEY-TERASHIMA’S EQUATIONS AND LIE SUPERALGEBRAS

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Abstract. We discuss the correspondence between Nahm’s equations, the Basu-Harvey-Terashima equations, and Lie superalgebras.

1. Introduction

This paper arose from the following observation. Consider $\text{Mat}_{n,m}(\mathbb{C}) \oplus \text{Mat}_{m,n}(\mathbb{C})$ with its flat $U(n) \times U(m)$-invariant hyperkähler structure. Let $\mu_1, \mu_2, \mu_3$ be the hyperkähler moment map for the $U(n)$-action and $\nu_1, \nu_2, \nu_3$ the hyperkähler moment map for the $U(m)$-action. Then, along the gradient flow of

$$F = |\mu_1|^2 - |\nu_1|^2,$$

$\mu_1, \mu_2, \mu_3$ and $-\nu_1, -\nu_2, -\nu_3$ satisfy Nahm’s equations.

This fact has several explanations and consequences. At the simplest level, it follows from the fact that

$$I_1(X_{\mu_1} - X_{\nu_1}) = I_2(X_{\mu_2} - X_{\nu_2}) = I_3(X_{\mu_3} - X_{\nu_3}),$$

where $X_\rho$ is the vector field generated by a $\rho$ in the Lie algebra of the symmetry group.

The function (1.1) is a quartic polynomial on $W = \text{Mat}_{n,m}(\mathbb{C}) \oplus \text{Mat}_{m,n}(\mathbb{C})$ and the gradient flow equations are

$$\dot{A} = \frac{1}{2}(ABB^* - B^*BA),$$
$$\dot{B} = \frac{1}{2}(A^*AB - BAA^*).$$

These equations are known as the ABJM version of the Basu-Harvey equations and are due to Terashima [7], and, consequently, we shall refer to them as the BHT-equations. We observe that they have a very natural interpretation as double superbracket equations on the odd part of the Lie superalgebra $\mathfrak{gl}_{n|m}(\mathbb{C})$:

$$\dot{C} = \frac{1}{2}[[J(C), C], C],$$

where $C = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ and $J$ is the quaternionic automorphism $J(A, B) = (-B^*, A^*)$.

This equation makes sense for any complex anti-Lie triple system [8] equipped with a quaternionic automorphism, and we observe that any solution of (1.4) in this general setting leads to a solution to Nahm’s equations (with values in an appropriate Lie algebra).

We give two more interpretations of equations (1.3). Firstly, there is a geometric interpretation as a gradient flow on a $GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$-orbit in $\text{Mat}_{n,m}(\mathbb{C}) \oplus \text{Mat}_{m,n}(\mathbb{C})$ for a quadratic function with respect to certain indefinite metric (§3).
Secondly, similarly to Nahm’s equations [3, 4], there is an interpretation as a linear flow on the Jacobian of an algebraic curve. This time, however, the spectral curve is a subscheme of \( \mathbb{P}^2 \) and the flow is restricted to line bundles equivariant with respect to certain involution \( \tau \) of the spectral curve.

2. Moment maps, Nahm’s and the Basu-Harvey-Terashima equations

We consider the vector space \( W_{n,m} = \text{Mat}_{n,m}(\mathbb{C}) \oplus \text{Mat}_{m,n}(\mathbb{C}) \) with its natural flat hyperkähler structure: the quaternionic structure \( J \) is given by \( J(A, B) = (-B^*, A^*) \) and the metric is

\[
\frac{1}{2} \text{Re} \left( dA \otimes dA^* + dB \otimes dB^* \right).
\]

This hyperkähler structure is invariant under the natural \( U(n) \times U(m) \)-action, given by

\[
(g, h)(A, B) = (gAh^{-1}, hBg^{-1}).
\]

The hyperkähler moment map for the \( U(n) \)-action is

\[
i\mu_1(A, B) = \frac{1}{2}(AA^* - B^*B), \quad (\mu_2 + i\mu_3)(A, B) = AB,
\]

while the moment map for the \( U(m) \)-action is

\[
i\nu_1(A, B) = -\frac{1}{2}(A^*A - BB^*), \quad (\nu_2 + i\nu_3)(A, B) = -BA.
\]

Here we identified Lie algebras with their duals using the Ad-invariant metrics \( \|X\|^2 = -\text{tr} X^2 \). A simple calculation shows that for \( i = 1, 2, 3 \) and any \( (A, B) \in W_{n,m} \)

\[
\|\mu_i(A, B)\|^2 - \|\nu_i(A, B)\|^2 = \frac{1}{2} \text{tr}(A^*ABB^* - B^*BAA^*).
\]

The fact that \( \|\mu_i\|^2 - \|\nu_i\|^2 \) is independent of \( i \) has the following consequence.

**Proposition 2.1.** Let \( m(t) \in W_{n,m} \) be a gradient flow curve of the function \( F = \frac{1}{2}\|\mu_1(A, B)\|^2 - \frac{1}{2}\|\nu_1(A, B)\|^2 \). Then the \( u(n) \)-valued functions \( T_i(t) = \mu_i(m(t)) \) satisfy Nahm’s equations

\[
\dot{T}_1 = [T_2, T_3], \quad \dot{T}_2 = [T_3, T_1], \quad \dot{T}_3 = [T_1, T_2].
\]

Similarly, the \( u(m) \)-valued functions \( S_i(t) = -\nu_i(m(t)) \) satisfy the Nahm equations.

**Proof.** The gradient vector field of \( F \) is \( I_1X_{\mu_1} - I_1X_{\nu_1} \). Since \( F \) is also equal to \( \frac{1}{2}\|\mu_i(A, B)\|^2 - \frac{1}{2}\|\nu_i(A, B)\|^2 \) for \( i = 2, 3 \), we obtain

\[
I_1X_{\mu_1} - I_1X_{\nu_1} = I_2X_{\mu_2} - I_2X_{\nu_2} = I_3X_{\mu_3} - I_3X_{\nu_3}.
\]

We compute, using the fact that the moment map for a group action is invariant with respect to any commuting Lie group action,

\[
\dot{T}_1 = d\mu_1(\nabla F) = d\mu_1(I_2X_{\mu_2} - I_2X_{\nu_2}) = d\mu_1(I_2X_{\mu_2}) = d\mu_3(X_{\mu_2}) = [\mu_2, \mu_3] = [T_2, T_3],
\]

and similarly for \( T_2, T_3 \). The argument for the \( S_i \) is completely analogous.

The gradient flow equations for the function \( F = \text{tr}(A^*ABB^* - B^*BAA^*) \) are

\[
\dot{A} = \frac{1}{2}(ABB^* - B^*BA) \\
\dot{B} = \frac{1}{2}(A^*AB - BAA^*).
\]
One can also check directly that, for a solution $A, B$ of these equations, the functions $T_1 = \frac{1}{2}(AA^* - B^*B)$, $T_2 + iT_3 = AB$ satisfy Nahm’s equations, and similarly the functions $S_1 = \frac{1}{2}(A^*A - BB^*), S_2 + iS_3 = BA$.

For the reason mentioned in the introduction, we shall refer to equations \eqref{eq:BHT} as the Basu-Harvey-Terashima (BHT) equations.

Remark 2.2. Similarly to Nahm’s equations, there exists a gauge-dependent version of the BHT-equations. Introduce two more matrix valued functions $u(t) \in \mathfrak{u}(n)$ and $v(t) \in \mathfrak{u}(m)$ and consider the following equations:

$$\begin{align*}
\dot{A} + uA - Av &= \frac{1}{2}(ABB^* - B^*BA) \\
\dot{B} + vB - Bu &= \frac{1}{2}(A^*AB - BAA^*).
\end{align*}$$

These equations are invariant under the following $U(n) \times U(m)$-valued gauge group action:

$$
A \mapsto gAh^{-1}, \quad B \mapsto hBg^{-1}, \quad u \mapsto gug^{-1} - gg^{-1}, \quad v \mapsto hvh^{-1} - hh^{-1}.
$$

3. Lax pair interpretation

Proposition 3.1. Let $I$ be an interval and $n \geq m$ be two positive integers. Let $X : I \to \text{Mat}_{n,m}, Y : I \to \text{Mat}_{m,n}$ be of class $C^k$, $k \geq 1$, and of rank $m$ for all $t \in I$. Suppose that $Z = XY$ satisfies the Lax equation $\dot{Z} = [M, Z]$ for some $M : I \to \text{Mat}_{n,m}$ of class $C^{k-1}$. Then there exists a unique $N : I \to \text{Mat}_{m,n}$ of class $C^{k-1}$, such that the following equations are satisfied:

$$\begin{align*}
\dot{X} &= MX + XN \\
\dot{Y} &= -YM - NY.
\end{align*}$$

Conversely, if \eqref{eq:Lax} are satisfied, then $\dot{Z} = [M, Z]$, and, moreover, $W = YX$ satisfies the Lax equation $\dot{W} = [W, N]$.

Proof. A direct computation shows that if \eqref{eq:Lax} are satisfied, then both $Z$ and $W$ satisfy the relevant Lax equations. Conversely, suppose that $\dot{Z} = [Z, M]$, i.e. $XY + X\dot{Y} - MXY + XYM = 0$. We rewrite this as

$$(\dot{X} - MX)Y + X(\dot{Y} + YM) = 0.$$ 

Let $U = \dot{X} - MX$ and $V = -\dot{Y} - YM$. Then $UY = XV$. Since $X$ and $Y$ have rank $m$, there are unique $N_1, N_2$ such that $U = XN_1, V = N_2Y$. It follows that $X(N_1 - N_2)Y = 0$, and using the maximality of the rank of $X, Y$, we conclude that $N_1 = N_2$. The equations \eqref{eq:Lax} are satisfied with $N = N_1 = N_2$. The differentiability class of $N$ follows from its uniqueness. \hfill $\square$

Equations \eqref{eq:Lax} can also be written in the Lax form. Set

$$
C = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}, \quad C_+ = \begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix}.
$$

Then \eqref{eq:Lax} is equivalent to

$$\begin{align*}
\dot{C} &= [C_+, C].
\end{align*}$$

We now introduce a spectral parameter, and consider $X(\zeta) = A_0 + A_1\zeta, Y(\zeta) = B_0 + B_1\zeta$. We suppose that $Z(\zeta) = X(\zeta)Y(\zeta)$ satisfies the Lax equation

$$\dot{Z} = [Z_\#, Z],$$
where \( Z_\# = \frac{1}{2}(A_0 B_1 + A_1 B_0) + A_1 B_1 \zeta \). According to the previous proposition, we can find an \( N(\zeta) \), so that \( S_3 \) holds. We have then \( W = [-N, W] \), where \( W(\zeta) = Y(\zeta) X(\zeta) \). We try \( N(\zeta) \) of the form \( N = -W_\# \), i.e. \(-N = \frac{1}{2}(B_0 A_1 + B_1 A_0) + B_1 A_1 \zeta \). Substituting into the equations (3.1) we obtain:

\[
\begin{align*}
\dot{A}_0 &= \frac{1}{2} (A_1 B_0 A_0 - A_0 B_0 A_1) \\
\dot{A}_1 &= -\frac{1}{2} (A_1 B_1 A_0 - A_0 B_1 A_1) \\
\dot{B}_0 &= \frac{1}{2} (B_1 A_0 B_0 - B_0 A_0 B_1) \\
\dot{B}_1 &= -\frac{1}{2} (B_1 A_1 B_0 - B_0 A_1 B_1).
\end{align*}
\]

(3.3)

Thus, these equations are equivalent to

\[
(3.4) \quad \dot{Z} = [Z_\#, Z], \quad \dot{W} = [W_\#, W],
\]

where \( Z, Z_\#, W, W_\# \) are defined above. Now suppose that the \( A_i \) and \( B_i \) satisfy the reality condition: \( A_1 = -B_0^* \), \( B_1 = A_0^* \). We write simply \( A, B \) for \( A_0, B_0 \). It follows that the equations (3.4) are equivalent to Nahm’s equations for \( (T_1, T_2, T_3) \) and \( (S_1, S_2, S_3) \), where

\[
(3.5) \quad T_2 + iT_3 = AB, \quad iT_1 = \frac{1}{2} (A^* - B^* B),
\]

(3.6) \quad \quad S_2 + iS_3 = BA, \quad iS_1 = \frac{1}{2} (A^* A - BB^*).

On the other hand, (3.3) becomes the BHT-equations (2.4). Therefore we have a different proof of Proposition 2.1, the statement of which can be strengthened as follows:

**Corollary 3.2.** Let \( n \geq m \). The equations (2.4) are equivalent to Nahm’s equations for \( (T_1, T_2, T_3) \) defined by (3.5). In addition, they imply Nahm’s equations for \( (S_1, S_2, S_3) \) defined by (3.6). \( \Box \)

We now rewrite (2.4) in the form (3.2). Thus

\[
C = \begin{pmatrix} 0 & A - B^* \zeta \\ B + A^* \zeta & 0 \end{pmatrix}, \quad C_+ = \begin{pmatrix} \frac{1}{2} (AA^* - B^* B) - B^* A^* \zeta & 0 \\ 0 & \frac{1}{2} (A^* A - BB^*) - A^* B^* \zeta \end{pmatrix}.
\]

Let us write \( \mathcal{M}_0 \) for the block-diagonal part of \( \text{Mat}_{m+n,m+n} \) and \( \mathcal{M}_1 \) for the off-diagonal part. Thus \( C \in \mathcal{M}_1 \) and \( C_+ \in \mathcal{M}_0 \). Let \( J : \mathcal{M}_1 \to \mathcal{M}_1 \) be the canonical quaternionic structure on \( \mathcal{M}_1 = \text{Mat}_{n,m} \oplus \text{Mat}_{m,n} \), i.e.

\[
J \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & -B^* \\ A^* & 0 \end{pmatrix}.
\]

We have \( J^2 = -1 \) and equations (2.4) can be written as

\[
(3.7) \quad \dot{C} = \frac{1}{2} [C J(C) + J(C) C, C] = \frac{1}{2} [J(C), C^2].
\]

This equation implies that the flow of \( (A, B) \) remains in an orbit of \( GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \) (in fact \( S(GL_n(\mathbb{C}) \times GL_m(\mathbb{C})) \) and \( SL_n(\mathbb{C}) \times SL_m(\mathbb{C}) \) for \( n = m \)). We shall now interpret the flow as a gradient flow on such an orbit, with respect to certain metric. First of all, let us view \( \text{Mat}_{m+n,m+n} \) as the Lie superalgebra \( \mathfrak{gl}_{n|m}(\mathbb{C}) \) with
\( \mathcal{M}_0 \) being the even part and \( \mathcal{M}_1 \) the odd part. The Lie superbracket is defined as 
\[ [A, B] = AB - (-1)^{|A||B|}BA. \]
Equation (3.7) can be then written as
\[ (3.8) \quad \dot{C} = \frac{1}{2} [J(C), C]. \]

**Remark 3.3.** The map \( J \) is the restriction of the following antilinear map on \( \mathfrak{gl}_{n|m}(\mathbb{C}) \):
\[ (3.9) \quad \left( \begin{array}{cc} U & A \\ B & V \end{array} \right) \mapsto \left( \begin{array}{cc} -U^* & -B^* \\ A^* & -V^* \end{array} \right), \]
which we also denote by \( J \). It is the negative of complex conjugation followed by the supertranspose, and, hence, it commutes with the superbracket. One could therefore consider equation (3.8) on all of \( \mathfrak{gl}_{n|m}(\mathbb{C}) \), rather than just on the odd part.

Recall now the notion of the supertrace:
\[ \text{str} \left( \begin{array}{cc} U & A \\ B & V \end{array} \right) = \text{tr} U - \text{tr} V. \]
It has the following ad-invariance property:
\[ (3.10) \quad \text{str}[X, Y]Z + (-1)^{|X||Y|} \text{str} Y[X, Z] = 0. \]

We define the following symmetric form on \( \mathfrak{gl}_{n|m}(\mathbb{C}) \):
\[ (3.11) \quad \langle X, Y \rangle = -\frac{1}{2} \text{str}(J(X)Y + J(Y)X). \]
If we write \( X \) and \( Y \) in the block form as \( (X_{ij}) \) and \( (Y_{ij}) \), \( i, j = 0, 1 \), then
\[ \langle X, Y \rangle = \frac{1}{2} \sum_{i,j=0}^1 (-1)^{ij} \text{tr}(X_{ij}^*Y_{ij} + Y_{ij}^*X_{ij}). \]

In what follows \( G \) denotes \( SL_n(\mathbb{C}) \times SL_m(\mathbb{C}) \) for \( n \neq m \) and \( G = SL_n(\mathbb{C}) \times SL_m(\mathbb{C}) \) for \( n = m \), and \( \mathfrak{g} \) denotes its Lie algebra. In order to define an appropriate metric on an orbit of \( G \) in \( \mathcal{M}_1 = \text{Mat}_{n,m}(\mathbb{C}) \oplus \text{Mat}_{m,n}(\mathbb{C}) \) we adopt the following definition.

**Definition 3.4.** An element \( C \) of \( \mathcal{M}_1 \) is called \( \langle \cdot, \cdot \rangle \)-regular if \( \ker \text{ad} C \subset \mathfrak{g} \) is nondegenerate with respect to the form (3.11).

If \( C \) is \( \langle \cdot, \cdot \rangle \)-regular, then \( \ker \text{ad} C \) has an \( \langle \cdot, \cdot \rangle \)-orthogonal complement \( V_C \), which is also \( \langle \cdot, \cdot \rangle \)-nondegenerate. In this case, we can decompose uniquely any \( X \in \mathfrak{g} \) as \( X = X^C + X^0 \) with \( X^C \in V_C \) and \( X^0 \in \ker \text{ad} C \).

Let now \( \mathcal{O} \) be an orbit of \( G \) in \( \mathcal{M}_1 \) and \( C \in \mathcal{O} \) its \( J \)-regular element. For two vectors \([C, X]\) and \([C, Y]\) tangent to \( \mathcal{O} \) at \( C \) we define their inner product to be \( \langle X^C, Y^C \rangle \). We obtain a pseudo-Riemannian metric on the \( J \)-regular part of \( \mathcal{O} \), which we denote by \( \langle \cdot, \cdot \rangle_\mathcal{O} \).

**Theorem 3.5.** On the \( \langle \cdot, \cdot \rangle \)-regular part of \( \mathcal{O} \) the flow (3.8) is the gradient flow of the function \( H(C) = \frac{1}{4} \langle C, C \rangle \) with respect to the metric \( \langle \cdot, \cdot \rangle_\mathcal{O} \).
Proof. The function \( H \) can be written as \(-\frac{1}{4} \text{str} \ J(C) C\). By the definition of the gradient we have, for any tangent vector \([C, \rho]_s\),

\[
\langle \text{grad} \, H, [C, \rho]_s \rangle_C = -\frac{1}{4} \text{str} \left( J([C, \rho]_s) C + J(C) [C, \rho]_s \right) = -\frac{1}{2} \text{Re str} \ J(C) [C, \rho]_s.
\]

Setting \( \text{grad} \, H = [C, X]_s \), we can rewrite this as

\[
\langle X^C, \rho^C \rangle = \frac{1}{2} \text{Re str} \ J(C) [C, \rho]_s.
\]

Recalling (3.10) and using the fact that \([C] = [J(C)] = 1\) we obtain

\[
\langle X^C, \rho^C \rangle = \frac{1}{2} \text{Re str} \ J(C) [C]_s \rho = -\frac{1}{2} \langle [J(C), C]_s, \rho \rangle.
\]

Since \( \text{Ker ad} \, C \) is \( \langle , \rangle \)-orthogonal to \( \text{Im ad} \, J(C) \), we have \( \langle [J(C), C]_s, \rho \rangle = \langle [J(C), C]_s, \rho^C \rangle \) and \([J(C), C]_s \in \text{V}_C\). Since the metric \( \langle , \rangle \) is nondegenerate on \( \text{V}_C \), we can conclude that \( X^C = -\frac{1}{2} [J(C), C]_s \). Thus \( \text{grad} \, H = [C, X]_s = [C, X^C]_s = \frac{1}{2} [J(C), C]_s, C]_s \).

\[ \square \]

4. Nahm’s equations from anti-Lie triple systems

As observed in the previous section, the BHT-equation (1.3) have a natural interpretation as a double superbracket equation on the odd part of the Lie superalgebra \( \mathfrak{gl}_{n,m} (\mathbb{C}) \). We shall now generalise this to arbitrary Lie superalgebras, or, equivalently to the anti-Lie triple systems (3).

An anti-Lie triple system (ALTS) is a vector space with a triple (trilinear) product \([\cdot, \cdot, \cdot]\) satisfying the following identities

\[
[x, y, z] = [y, x, z]
\]

\[
[x, y, z] + [z, x, y] + [y, z, x] = 0,
\]

\[
[u, v, [x, y, z]] = [[u, v, x, y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]].
\]

The third equation can be rewritten as a condition on left multiplications \( L(\cdot, \cdot) \) (defined via \( L(x, y) z = [x, y, z] \)):

\[
[L(u, v), L(x, y)] = L(L(u, v)x, y) + L(x, L(u, v)y).
\]

A basic example is the vector space of \( k \times k \)-matrices with the triple product:

\[
[A, B, C] = ABC + BAC - CBA - BCA
\]

This triple product leaves invariant the subspace \( \text{Mat}_{n,m} \oplus \text{Mat}_{m,n} \) of off-diagonal blocks, and so the latter is also an ALTS.

We recall (3) the construction of a Lie superalgebra associated to an anti-Lie triple system \((V, [\cdot, \cdot, \cdot])\).

Let \( D(V) \) denote the Lie algebra of all left multiplications \( L(x, y) \) on \((V, [\cdot, \cdot, \cdot])\). Then \( D(V) \oplus V \) becomes a Lie superalgebra \( l(V) \) under the following bracket:

\[
[L(x, y), L(u, v)] = L(x, y) \circ L(u, v) - L(u, v) \circ L(x, y)
\]

\[
[L(x, y), z] = L(x, y) z
\]

\[
[x, y] = L(x, y).
\]

The even part of \( l(V) \) is \( l_0 = D(V) \) and the odd one is \( l_1 = V \). Conversely, given a Lie superalgebra \( l = l_0 \oplus l_1 \), the double superbracket defines an anti-Lie triple product on \( l_1 \):

\[
[x, y, z] = [[x, y], z].
\]
Example 4.1. Applying this construction to the ALTS Mat\(_{n,m}\) \(\oplus\) Mat\(_{m,n}\) with the triple product given by (4.3) produces the Lie superalgebra \(\mathfrak{gl}_{n|m}(\mathbb{C})\).

Let now \((V, [\cdot,\cdot,\cdot])\) be a complex ALTS and \(J\) a quaternionic automorphism, i.e. \(J\) preserves the triple product, is antilinear, and satisfies \(J^2 = -1\). We can extend \(J\) to an antilinear automorphism of \((V)\) by setting \(J(L(x,y)) = L(J(x),J(y))\). On \(l_0\) it satisfies \(J^2 = 1\), an so the Lie algebra \(l_0\) has a symmetric pair decomposition \(l_0 = \mathfrak{k} \oplus \mathfrak{m}\), where \(\mathfrak{k}\) is the +1-eigenspace and \(\mathfrak{m}\) the −1-eigenspace of \(J\). The antilinearity of \(J\) implies that, the following three functions

\[
T_1 = -\frac{i}{2}[C,J(C)] \quad T_2 = \frac{1}{2}[C,C]_\mathfrak{k} \quad T_3 = -\frac{i}{2}[C,C]_\mathfrak{m}
\]

take values in \(\mathfrak{k}\).

We consider the following ODE on \(V\):

(4.6) \[
\dot{C} = \frac{1}{2}[J(C),C,C],
\]

Proposition 4.2. \(C = C(t)\) is a solution of (4.6) if and only if \(T_1, T_2, T_3\) satisfy the Nahm equations.

Proof. The definition implies that \(T_2 = \frac{1}{2}([C,C] + J[C,C])\) and \(T_3 = -\frac{i}{2}([C,C] - J[C,C])\). Setting \(\alpha = iT_1\) and \(\beta = T_2 + iT_3\), we have

\[
\alpha = \frac{1}{2}[C,J(C)], \quad \beta = \frac{1}{2}[C,C].
\]

The super-Jacobi identity implies that if \(x\) is an odd element of a Lie superalgebra \(\mathfrak{g}\), then for any \(y \in \mathfrak{g}\)

(4.7) \[
[x, [x, y]] = \frac{1}{2}i[x, x, y].
\]

In particular, the equation (4.6) can be rewritten as

\[
\dot{C} = \frac{1}{4}[[J(C), C], C].
\]

We compute using (4.7):

\[
\dot{\alpha} = \frac{1}{2}[\dot{C}, J(C)] + \frac{1}{2}[C, J(\dot{C})] = \frac{1}{8}[[J(C), [C,C]],J(C)] - \frac{1}{8}[C,[C,[J(C),J(C)]]] = \frac{1}{8}[[J(C),[J(C),C]],C] = \frac{1}{2}[J(\beta),\beta] = \frac{1}{2}[T_2 - iT_3, T_2 + iT_3] = iT_2 T_3.
\]

Similarly:

\[
\dot{\beta} = [\dot{C},C] = \frac{1}{2}[[J(C),C],C] = \frac{1}{4}[[J(C), C], [C,C]] = [\alpha,\beta],
\]

which is equivalent to the remaining two Nahm equations. \(\square\)

Remark 4.3. We can also consider an arbitrary, real or complex, anti-Lie triple system \((V, [\cdot,\cdot,\cdot])\) equipped with an automorphism \(J\) such that \(J^2 = -1\). The even part of the Lie algebra \(l_0\) still has the symmetric decomposition \(l_0 = \mathfrak{k} \oplus \mathfrak{m}\) into the ±-eigenspaces of \(J\) extended to \(l_0\). We can define the three functions:

\[
R_1 = \frac{1}{2}[C,J(C)] \in \mathfrak{m}, \quad R_2 = \frac{1}{2}[C,C]_\mathfrak{k} \in \mathfrak{k}, \quad R_3 = \frac{1}{2}[C,C]_\mathfrak{m} \in \mathfrak{m}.
\]
Equation 1.6 implies that $R_1, R_2, R_3$ satisfy the *Nahm-Schmid* equations [6]:

\[ \hat{R}_1 = \frac{1}{2}[R_2, R_3], \quad \hat{R}_2 = \frac{1}{2}[R_1, R_3], \quad \hat{R}_3 = \frac{1}{2}[R_1, R_2]. \]

5. Flows on Jacobians

It is well-known [11, 5] that Nahm’s equations correspond to a linear flow on the Jacobian of an algebraic curve embedded in $T\mathbb{P}^1$, i.e. in the total space $|\mathcal{O}(2)|$ of the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$. Similarly, as we shall shortly see, the Basu-Harvey-Terashima equations (2.4) correspond to a linear flow on the *equivariant* Jacobian of a curve in $\mathbb{P}^2 \setminus \mathbb{P}^1$, i.e. in the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$.

In this section we aim to make precise the correspondence between the Nahm flow and the Basu-Harvey-Terashima flow on the Jacobians. We consider first the purely holomorphic picture in the spirit of Beauville [2]. Thus the Nahm matrices are replaced by a quadratic matrix polynomial $X(\zeta) = X_0 + X_1\zeta + X_2\zeta^2$ with $X_i \in \mathfrak{gl}_n(\mathbb{C})$. Such a polynomial corresponds to an acyclic 1-dimensional sheaf $\mathcal{F}$ on $T = |\mathcal{O}(2)|$ defined via

\[ (5.1) \quad 0 \to \mathcal{O}_T(-3)^{\oplus n} \xrightarrow{\eta - X(\zeta)} \mathcal{O}_T(-1)^{\oplus n} \to \mathcal{F} \to 0. \]

The support of $\mathcal{F}$ is the 1-dimensional scheme $S$ cut out by $\det(\eta - X(\zeta))$. As long as this polynomial is irreducible, $S$ is integral and $\mathcal{F}$ is a line bundle on $S$. More generally, $\mathcal{F}$ is a line bundle (i.e. an invertible sheaf) on $S$ as long as $X(\zeta)$ is a regular element of $\mathfrak{gl}_n(\mathbb{C})$ for each $\zeta \in \mathbb{P}^1$ (with $X(\infty) = X_2$). In fact, we have the following result of Beauville:

**Theorem 5.1** (Beauville [2]). Let $d$ be a positive integer and $P(\zeta, \lambda) = \lambda^k + a_1(\zeta)\lambda^{k-1} + \cdots + a_k(\zeta)$ a polynomial with $\deg a_i(\zeta) = \text{id}$, $i = 1, \ldots, k$. Consider the variety

\[ M(P) = \{ X(\zeta) \in \mathfrak{gl}_n(\mathbb{C})|\zeta| ; \deg X(\zeta) = d, \; \deg(\lambda - X(\zeta)) = P(\zeta, \lambda), \} \]

and its subvariety $M(P)^{\text{reg}}$ consisting of $X(\zeta)$ which are regular for each $\zeta \in \mathbb{P}^1$.

The following exact sequence on $T = |\mathcal{O}(d)|$

\[ (5.2) \quad 0 \to \mathcal{O}_T(-d-1)^{\oplus k} \xrightarrow{\lambda - X(\zeta)} \mathcal{O}_T(-1)^{\oplus k} \to \mathcal{F} \to 0 \]

induces a 1-1 correspondence between $M(P)^{\text{reg}}/GL_k(\mathbb{C})$ and $\text{Jac}^g(\mathcal{O}(d))$, where $S \subset |\mathcal{O}(d)|$ is the curve of (arithmetic) genus $g = (k-1)(dk-2)/2$ defined by the equation $P(\zeta, \lambda) = 0$.

We shall call this correspondence the *Beauville isomorphism*.

5.1. $\tau$-sheaves. We now replace $W_{n,m}$ by its complexification, i.e. the vector space $R_{n,m}$ of quadruples of complex matrices $(A_0, A_1, B_0, B_1)$ with $A_0, A_1$ of size $n \times m$, $B_0, B_1$ of size $m \times n$. $R_{n,m}$ a biquaternionic vector space, i.e. a module over $\text{Mat}_{2,2}(\mathbb{C})$, and comes equipped with a 2-sphere of complex symplectic structures:

\[ (5.3) \quad \omega : \text{tr} \; d(A_0 + A_1\zeta) \wedge d(B_0 + B_1\zeta), \]

\footnote{This is true if the curve is smooth or integral; in general, the flow is on the generalised Jacobian or on the moduli space of higher rank vector bundles.}
where \( \zeta \) denotes the affine coordinate on \( \mathbb{P}^1 \). We can view \( \omega \) itself as an \( \mathcal{O}(2) \)-twisted symplectic form. It is clearly \( GL(m, \mathbb{C}) \times GL(n, \mathbb{C}) \)-invariant. The (twisted) moment map for the \( GL(n) \)-action is given by:

\[
(5.4) \quad \mu : (A_0, A_1, B_0, B_1) \mapsto A_0B_0 + (A_0B_1 + A_1B_0)\zeta + A_1B_1\zeta^2,
\]

while the one for the \( GL(m) \)-action is:

\[
(5.5) \quad \nu : (A_0, A_1, B_0, B_1) \mapsto B_0A_0 - (B_0A_1 + B_1A_0)\zeta - B_1A_1\zeta^2.
\]

These are complexifications of the moment maps defined in [2]. As in section 4.3 we can view \( R_{n,m} \) as the following subset of \( \text{gl}(m+n) \otimes \mathbb{C}^2 \):

\[
(5.6) \quad C_0 = \begin{pmatrix} 0 & A_0 \\ B_0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & A_1 \\ B_1 & 0 \end{pmatrix}.
\]

For any pair \( C_0, C_1 \) of quadratic matrices, say of size \( k \times k \), we can define an acyclic 1-dimensional sheaf on \( \tilde{T} = |\mathcal{O}(1)| \) via the exact sequence \( (5.2) \) with \( d = 1 \) and \( X(\zeta) = C_0 + C_1\zeta \). We are interested in the structure of these sheaves and their supports for \( C_0, C_1 \) of the form \( (5.6) \), and in their relation to sheaves on \( |\mathcal{O}(2)| \) defined via maps \( (5.4) \) and \( (5.5) \). Observe that \( |\mathcal{O}(2)| \) is the quotient of \( |\mathcal{O}(1)| \) by the following involution on \( |\mathcal{O}(1)| \):

\[
(5.7) \quad \tau(\zeta, \lambda) = (\zeta, -\lambda).
\]

For an element \( (A_0, A_1, B_0, B_1) \) of \( R_{n,m} \) with \( n \geq m \), the polynomial \( \det(\lambda - C_0 - C_1\zeta) \) is \( \tau \)-invariant and of the form

\[
(5.8) \quad P(\zeta, \lambda) = \lambda^{n-m}(\lambda^{2m} + a_1(\zeta)\lambda^{2m-2} + \cdots + a_{m-1}(\zeta)\lambda^2 + a_m(\zeta)), \quad \deg a_i(\zeta) = 2i.
\]

\( GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \)-orbits of elements of \( R_{n,m} \) correspond to acyclic \( \tau \)-sheaves on \( \tilde{S} = \{(\zeta, \lambda); P(\zeta, \lambda) = 0\} \), i.e. sheaves equivariant with respect to the action of \( \tau \). In the case of a line (or vector) bundle \( \mathcal{F} \) on \( \tilde{S} \) this means that \( \tau \) lifts to an involutive bundle map on the total space of \( \mathcal{F} \).

Let us write \( R_{n,m}(P) \) for \( R_{n,m} \cap M(P) \) and \( R_{n,m}(P)_{\text{reg}} \) for \( R_{n,m} \cap M(P)_{\text{reg}} \). We have

**Proposition 5.2.** The Beauville isomorphism induces a 1−1 correspondence between \( R_{n,m}(P)_{\text{reg}}/GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \) and the isomorphism classes of acyclic \( \tau \)-line bundles on \( \tilde{S} \).

**Proof.** Let \( (A_0, A_1, B_0, B_1) \) with the corresponding \( C(\zeta) = C_0 + C_1\zeta \) given by \( (5.6) \) belong to \( R_{n,m}(P)_{\text{reg}} \). Then \( g_0C(\zeta)g_0^{-1} = -C(\zeta), \) where \( g_0 = \begin{pmatrix} \text{Id}_n & 0 \\ 0 & -\text{Id}_m \end{pmatrix} \). The commutative diagram on \( |\mathcal{O}(1)| \) (with \( k = n + m \))

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}(-2)^{\oplus k} \\
\downarrow g_0 & & \downarrow g_0 \\
0 & \longrightarrow & \mathcal{O}(-2)^{\oplus k}
\end{array}
\]

\[
\begin{array}{ccc}
\longrightarrow & \mathcal{O}(-1)^{\oplus k} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
\lambda + C(\zeta) & \lambda - C(\zeta) & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
\]

defines a lift \( \tilde{\tau} : \mathcal{F} \rightarrow \mathcal{F} \) of \( \tau \). Conjugating \( C(\zeta) \) by an element \( T \) of \( GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \) commutes with \( g_0 \) and so \( C(\zeta) \) and \( TC(\zeta)T^{-1} \) induce isomorphic \( \tau \)-sheaves. Conversely, suppose that we are given a lift \( \tilde{\tau} \) of \( \tau \) on an acyclic line bundle \( \mathcal{F} \), satisfying \( \tilde{\tau}^2 = 1 \). We obtain the corresponding involution \( \tilde{\tau} \) on \( \mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m \).
$H^0(\hat{S}, \mathcal{F}(1))$. We can choose a basis of $H^0(\hat{S}, \mathcal{F}(1))$ so that $\hat{t}$ is represented by the matrix $g_0 = \left( \begin{array}{cc} \text{Id}_n & 0 \\ 0 & -\text{Id}_m \end{array} \right)$. It follows from the above commutative diagram that $g_0C(\zeta)g_0^{-1} = -C(\zeta)$, so that $(A_0, A_1, B_0, B_1)$ belongs to $R_{n,m}$. □

5.2. **The case** $n = m$. In this case the quotient of the curve $\hat{S}$ by the involution $\tau$ is (as a scheme) a curve $S$ in $T = |\mathcal{O}(2)|$. The maps $[54]$ and $[56]$ induce, via the above Proposition and Theorem $[5.3]$, correspondences between acyclic $\tau$-line bundles on $\hat{S}$ and acyclic line bundles on $S$. We wish to understand these correspondences.

We shall write $A(\zeta)$ for $A_0 + A_1\zeta$ and $B(\zeta)$ for $B_0 + B_1\zeta$, so that the map $\mu$ gives the quadratic matrix polynomial $A(\zeta)B(\zeta)$ and $\nu$ the polynomial $B(\zeta)A(\zeta)$. Let us write $\hat{P}(\zeta, \lambda)$ for the polynomial $\det(\lambda - C(\zeta))$ and $P(\zeta, \eta)$ for the polynomial $\det(\eta - A(\zeta)B(\zeta)) = \det(\eta - B(\zeta)A(\zeta))$. Denote by $\hat{S}$ the curve in $\mathbb{P}^2$ cut out by $\hat{P}$ and by $S$ the curve cut out by $P$ in $T\mathbb{P}^1$. We have

$$\hat{P}(\zeta, \lambda) = P(\zeta, \lambda^2),$$

so that $\hat{S}$ is a double cover of $S$ ramified over $\eta = 0$. The genus of $S$ is equal to $(n - 1)^2$ and the genus of $\hat{S}$ is equal to $(n - 1)(2n - 1)$.

We shall denote by $\mathcal{L}$ the acyclic $\tau$-sheaf on $\hat{S}$ defined by $C(\zeta)$ and by $\mathcal{F}$ (resp. $\mathcal{G}$) the acyclic sheaf on $S$ defined by $A(\zeta)B(\zeta)$ (resp. $B(\zeta)A(\zeta)$). We shall assume that the zeros of $\det A(\zeta)$ are distinct from the zeros of $\det B(\zeta)$, and we shall write $\Delta_A$ (resp. $\Delta_B$) for the divisor $\det A(\zeta) = 0$, $\lambda = 0$ (resp. $\det B(\zeta) = 0$, $\lambda = 0$) on $\hat{S}$. Thus $\Delta_A + \Delta_B$ is the ramification divisor of the projection $\tau$.

**Proposition 5.3.** With the above assumptions $\mathcal{L} \simeq \pi^* \mathcal{F} \otimes [\Delta_B] \simeq \pi^* \mathcal{G} \otimes [\Delta_A]$, where $\pi : \hat{S} \to S$ is the projection.

**Proof.** We consider the sheaves $\mathcal{L}(1)$, $\mathcal{F}(1)$ and $\mathcal{G}(1)$ which are cokernels of $\lambda - C(\zeta) : \mathcal{O}(-1)^{\oplus 2n} \to \mathcal{O}^{\oplus 2n}$, $\eta - A(\zeta)B(\zeta) : \mathcal{O}(-2)^{\oplus 2n} \to \mathcal{O}^{\oplus 2n}$, and of $\eta - B(\zeta)A(\zeta) : \mathcal{O}(-2)^{\oplus 2n} \to \mathcal{O}^{\oplus 2n}$, respectively. Any vector $u \in \mathbb{C}^{2n}$ defines a global section $s_u$ of $\mathcal{F}(1)$ via $[5.1]$. We choose $u$ so that the zeros of $s_u$ are disjoint from $\eta = 0$ and from the singular locus of $S$. In other words $u \notin \text{Im} A(\zeta)B(\zeta)$ if $\det A(\zeta)B(\zeta) = 0$ and $u \notin \text{Im}(\eta - A(\zeta)B(\zeta))$ for a singular point $(\zeta, \eta) \in S$. Consider the vector $(u, 0) \in \mathbb{C}^{2n}$ which defines a global section $\hat{s}_u$ of $\mathcal{L}(1)$. It is then easy to check that $(u, 0) \in \text{Im}(\lambda - C(\zeta))$ if either $\lambda \neq 0$ and $u \in \text{Im}(\lambda^2 - A(\zeta)B(\zeta))$ or $\lambda = 0$, $\det B(\zeta) = 0$ and $u \in \text{Im} A(\zeta)$. Since $\Delta_A$ and $\Delta_B$ are assumed to be disjoint, the condition $u \in \text{Im} A(\zeta)$ follows from $\lambda = 0$ and $\det B(\zeta) = 0$. Thus the divisor $(\hat{s}_u)$ of $\hat{s}_u$ is $\pi^{-1}(s_u) + \Delta_B$ and the first isomorphism follows. The proof of $\mathcal{L} \simeq \pi^* \mathcal{F} \otimes [\Delta_A]$ is completely analogous. □

5.3. **Flows.** The Beauville correspondence implies that the flow of matrices satisfying the Nahm equations corresponds to a flow on $J^{n-1}(S) - \Theta$. It is well-known $[52]$ that this latter flow is the linear flow $\mathcal{F} \mapsto \mathcal{F} \otimes L^t$, where $L$ is the line bundle with transition function $\exp(\eta/\zeta)$. Similarly the BHT-flow corresponds to a linear flow on the moduli space of acyclic $\tau$-line bundles on $\hat{S}$ in the direction of the line bundle with transition function $\exp(\lambda^2/\zeta)$. In addition, in order to obtain the Basu-Harvey-Terashima equations, rather than purely holomorphic equations $[53]$, one needs to restrict the flow further to $\sigma$-line bundles on $\hat{S}$, i.e. line bundles equipped with a lift of the quaternionic structure of $|\mathcal{O}_{\mathbb{P}^2(1)}|$. 
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