Bound states for a stationary nonlinear Schrödinger-Poisson system with sign-changing potential in \(\mathbb{R}^3\) *

Yongsheng Jiang and Huan-Song Zhou†

Abstract: We study the following Schrödinger-Poisson system

\[
(P_\lambda) \begin{cases} 
-\Delta u + V(x)u + \lambda \phi(x)u = Q(x)|u|^{p-1}u, & x \in \mathbb{R}^3 \\
-\Delta \phi = u^2, & \lim_{|x| \to +\infty} \phi(x) = 0, u > 0,
\end{cases}
\]

where \(\lambda \geq 0\) is a parameter, \(1 < p < +\infty\), \(V(x)\) and \(Q(x)\) are sign-changing or non-positive functions in \(L^\infty(\mathbb{R}^3)\). When \(V(x) \equiv Q(x) \equiv 1\), D.Ruiz [19] proved that \((P_\lambda)\) with \(p \in (2, 5)\) has always a positive radial solution, but \((P_\lambda)\) with \(p \in (1, 2]\) has solution only if \(\lambda > 0\) small enough and no any nontrivial solution if \(\lambda \geq \frac{1}{4}\). By using sub-supersolution method, we prove that there exists \(\lambda_0 > 0\) such that \((P_\lambda)\) with \(p \in (1, +\infty)\) has always a bound state (\(H^1(\mathbb{R}^3)\) solution) for \(\lambda \in [0, \lambda_0]\) and certain functions \(V(x)\) and \(Q(x)\) in \(L^\infty(\mathbb{R}^3)\). Moreover, for every \(\lambda \in [0, \lambda_0]\), the solutions \(u_\lambda\) of \((P_\lambda)\) converges, along a subsequence, to a solution of \((P_0)\) in \(H^1\) as \(\lambda \to 0\).

1 Introduction

In this paper, we are concerned with the existence of positive solutions of the following nonlinear elliptic system

\[
\begin{cases} 
-\Delta u + V(x)u + \lambda \phi(x)u = Q(x)|u|^{p-1}u, & x \in \mathbb{R}^3 \\
-\Delta \phi = u^2, & \lim_{|x| \to +\infty} \phi(x) = 0,
\end{cases}
\]

where \(\lambda > 0\) is a parameter, \(p \in (1, +\infty)\), \(V(x)\) and \(Q(x)\) are functions in \(L^\infty(\mathbb{R}^3)\). This kind of problem is related to looking for solitary wave type solution of nonlinear Schrödinger equation for a particle in an electromagnetic field [12], for more physical background about this system we refer the reader to [5, 6, 12, 17, 18, 20, 22] and the references therein. Under variant assumptions

*This work was supported by NSFC and CAS-KJCX3-SYW-03. † Corresponding author.

2000 Mathematical Subject Classification: 35J60.

Key words: Schrödinger-Poisson system, sub-supersolutions, supercritical Sobolev exponent, sign-changing potential, bound state.

To appear in Acta Math. Sci., 29B(2009), No.4, p. 1095-
on $V(x)$ and $Q(x)$, problem (1.1) has been studied widely. For $Q(x) \equiv 0$ and
$V(x) \equiv constant$, this problem was studied as an eigenvalue problem in [4] on
bounded domain and in [6, 17] on $\mathbb{R}^3$. For $Q(x) \equiv 1$ with $p \in (1, 5)$, there has
been quite a lot of interest on problem (1.1) in recent years. For examples, the
existence of solutions to problem (1.1) with $V(x) \equiv constant$ and $\lambda = 1$ was
obtained in [12] if $p \in (3, 5)$ and in [11] if $p \in [3, 5)$, then in [19] for $p \in (1, 5)$ and
$\lambda$ may not be equal to 1. Moreover, the existence of multiple solutions of (1.1)
with $V(x) \equiv Q(x) \equiv 1$ and $p \in (1, 5)$ was proved by Ambrosetti-Ruiz in [2, 3].
If $V(x)$ is not a constant, some existence results on problem (1.1) were given in [1]
for $Q(x) \equiv 1$ with $p \in (3, 5)$, then in [25] for $p \in (2, 3]$, and in [21] [22] for a
general nonlinear term $f(x, u)$. If $V$ and $Q$ are radial, positive, and vanishing
at infinity, the existence and nonexistence of solutions to (1.1) were studied in [10]
for some $p \in (1, 5)$. The results obtained in all the papers mentioned above
are based on variational methods, this leads to the restriction on $p \in (1, 5)$. For
$V(x) \equiv Q(x) \equiv 1$ and $\lambda = 1$, it was proved in [10, 19] that problem (1.1) does
not possess any nontrivial solution if $p \leq 2$ or $p \geq 5$. What would happen if $V(x)$
and $Q(x)$ are not equal to 1? Is it possible to get a solution of problem (1.1)
for all $p \in (1, +\infty)$? To the authors’ knowledge, there seems no any results in
this direction. In this paper, we prove that for any $p \in (1, +\infty)$ and for certain
$V(x)$, there always exists $Q(x)$ such that problem (1.1) has a positive solution
if $\lambda > 0$ small. As it is known, if $p \in (1, +\infty)$, the variational approach is
no longer applicable and here we use sub-supersolution method instead. But
problem (1.1) is a coupled system, it seems not easy to construct a reasonable
sub- and supersolutions to ensure the existence of a solution to the problem.
Motivated by the paper of Edelson-Stuart [13] and based on an estimate for
the fundamental solution $\phi$ of the second equation in (1.1), we get the desired
sub- and supersolutions of (1.1) for some kinds of $V(x)$, $Q(x)$ and $\lambda \geq 0$ small.
Therefore, by an iterative procedure, we obtain a solution $u_\lambda$ of (1.1)
for each $\lambda \geq 0$ small enough. In particular, our results imply the existence of positive
solution to the following single equation

$$- \Delta u + V(x)u = Q(x)|u|^{p-1}u, \quad x \in \mathbb{R}^3,$$

(1.2)

where $V(x)$ and $Q(x)$ are functions in $L^\infty(\mathbb{R}^3)$. Moreover, we prove that, along
a subsequence, the solutions $u_\lambda$ of (1.1) for $\lambda \in (0, -2(2\alpha - 1)\Lambda)$ converges in
$H^1(\mathbb{R}^3)$ to a solution of (1.2), where $\alpha > \frac{3}{4}$ and $\Lambda < 0$ are given by $(H_3)$ and
$(H_4)$ below, respectively. (1.2) is essentially the special case of (1.1) as $\lambda = 0$,
and it has been studied by many authors, such as [7, 8, 9], etc. However, in those
papers, $p \in (1, 5)$, $V(x)$ is assumed to be of the form $\lambda h(x)$ and $Q(x)$ is required to
have a negative limit as $|x| \to +\infty$, $\int Q(x)\phi_1^{p+1}(x)dx < 0$ or $Q(x) \equiv 1$, where
$\phi_1 > 0$ is the first eigenfunction of the problem (see e.g. [1] Corollary 2)

$$- \Delta u = \mu V(x)u, \quad x \in \mathbb{R}^3 \text{ and } u \in D^{1,2}(\mathbb{R}^3).$$

(1.3)

In some sense, our result on (1.2) also generalizes that of [7, 8, 9]. Specially, in
our case, $p \in (1, +\infty)$ is allowed, and we do not require that $\{x \in \mathbb{R}^3 : Q(x) > 0\} \cap
\{x \in \mathbb{R}^3 : Q(x) < 0\} = \emptyset$. See our Examples 1.1 and 1.2.
Now, we give our assumptions on $V(x)$ and $Q(x)$.

(H1) $V(x)$, $Q(x)$ are nonpositive, or sign-changing, functions in $C^{0,\gamma}_{loc}(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3)$ with $\gamma \in (0,1)$.

(H2) There exists a constant $V_\infty \geq 0$ such that $\liminf_{|x| \to \infty} V(x) = V_\infty$.

(H3) $A := \inf\{\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 \, dx : u \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} u^2 \, dx = 1\} < 0$.

(H4) There exists $\alpha > 3/4$ such that
\[
Q(x) \leq (1 + r^2)^{\alpha(p-1)}|V(x) - H(r)| \quad \text{for } x \in \mathbb{R}^3, \tag{1.4}
\]
where $H(r) = 2\alpha[(2\alpha - 1)r^2 - 3](r^2 + 1)^{-2}$ and $r = |x|$. Moreover,
\[
\lim_{|x| \to \infty} |x|^2 V(x) > 2\alpha(2\alpha - 1) \quad \text{if } V_\infty = 0 \quad \text{and} \quad p \geq 1 + 1/\alpha. \tag{1.5}
\]

Remark 1.1 Note that (1.5) is only used to ensure that $Q(x)$ with property (H4) is not $-\infty$. Condition (H4) can be slightly weakened by assuming that
\[
(H4)' \quad \text{There exist } \alpha > 3/4, \, \theta > 0 \quad \text{and} \quad a > 0 \quad \text{such that}
\]
\[
Q(x) \leq a^{1-p}(1 + \theta r^2)^{\alpha(p-1)}|V(x) - \theta H(\theta r)|, \quad x \in \mathbb{R}^3.
\]

Remark 1.2 For $H(r)$ given by (H4), let $r_0 = \sqrt{\frac{3}{2\alpha-1}}$, we see that $H(r_0) = 0$, $H(r) < 0$ for $r \in (0, r_0)$ and $H(r) > 0$ for $r > r_0$.

Here are two examples on our assumptions. Example 1.1 satisfies (H1) – (H4) and the assumptions of [7]. Example 1.2 satisfies (H1) – (H4), but does not satisfy the assumptions of [7, 9].

Example 1.1 Let $\alpha > 3/4$, $b > 1$ and $\beta > 0$. For $H(r)$ given by (H4), let $V(x) = bH(r)$ and $Q(x) \leq (b - 1)(\frac{2\alpha - 1}{2\alpha})^{\alpha(p-1)}H(r)$ with $Q(x) \in C^{0,\gamma}_{loc}(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3)$ and \( \lim_{|x| \to \infty} Q(x) = -\beta \). Then, (H1) and (H2) with $V_\infty = 0$ are satisfied. For $r_0$ given by Remark 1.2 by taking $\varphi(x) \geq 0$ and $\varphi \in C^\infty_0(B_{r_0}(0)) \setminus \{0\}$ and we see that (H3) is satisfied if $b > 1$ large enough. By Remark 1.2 and a direct computation shows that (H4) is also satisfied. Moreover, for any $p > 1$, it follows from \( \lim_{|x| \to \infty} Q(x) = -\beta \) that there is $\beta_0 > 0$ such that $\int_{\mathbb{R}^3} Q(x)\phi_1^{p+1} < 0$ if $\beta > \beta_0$, where $\phi_1 > 0$ is the first eigenfunction of (1.3). Hence, the conditions of [7] are also satisfied.

Example 1.2 In Example 1.2, we take $Q(x) = (b - 1)(\frac{2\alpha - 1}{2\alpha})^{\alpha(p-1)}H(r)$, and now $\beta = 0$. Then we still have that (H1) – (H4) are satisfied for $b > 1$ large. But the condition on $Q(x)$ in [7, 9] cannot be satisfied because here we have that $\int_{\mathbb{R}^3} Q(x)\phi_1^{p+1} > 0$ for some $p_0 > 1$. In fact, since $s > 3$ we know that
$V(x) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, it follows from [1, Corollary 2] that the first eigenvalue $\mu_1$ of (1.3) is positive and it has an positive eigenfunction $\phi_1 \in D^{1,2}(\mathbb{R}^3)$. Hence $\int V(x)\phi_1^2 = \frac{1}{\mu_1} \int |\nabla \phi_1|^2 > 0$ by (1.3), this implies that $\int_{\mathbb{R}^3} Q(x)\phi_1^2 > 0$. Moreover, by [14, Theorem 8.17] we see that $\phi_1 \in L^\infty(\mathbb{R}^3)$. Then the dominated convergence theorem shows that $\int_{\mathbb{R}^3} Q(x)\phi_p^{p+1} \rightarrow \int_{\mathbb{R}^3} Q(x)\phi_1^2 > 0$ as $p \rightarrow 1^+$. So, there is some $\delta > 0$ such that $\int_{\mathbb{R}^3} Q(x)\phi_p^{p+1} > 0$ for $p \in (1, 1+\delta)$.

Finally, we give the main results of the paper.

**Theorem 1.1** For any $p \in (1, +\infty)$, suppose that $(H_1)$ to $(H_4)$ are satisfied and $\Lambda < V_\infty$. Then, problem (1.1) has at least a positive solution $u_\lambda \in C^{2,\gamma}_{loc}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3)$ for any $\lambda \in [0, -2(2\alpha - 1)\Lambda]$ and all $q \in [2, +\infty)$. Moreover, $u_\lambda$ is a bound state with

$$0 < u_\lambda \leq \frac{1}{(1 + |x|^2)\alpha},$$

and

$$\|u_\lambda\|_{W^{2,q}(\mathbb{R}^3)} \leq C,$$

where $C$ is a constant independent of $\lambda$.

In particular, if $\lambda = 0$ in (1.1), Theorem 1.1 implies that

**Corollary 1.1** Under the assumptions of Theorem 1.1, problem (1.2) possesses a positive solution $u \in C^{2,\gamma}_{loc}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3)$ for all $q \in [2, +\infty)$.

**Theorem 1.2** For each $\lambda \in (0, -2(2\alpha - 1)\Lambda)$, let $u_\lambda$ denote a solution of problem (1.1) obtained by Theorem 1.1, then there exists $u_0 \in C^{2,\gamma}_{loc}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3)$ for all $q \in [2, +\infty)$ such that, along a subsequence,

$$\|u_\lambda - u_0\|_{W^{1,2}(\mathbb{R}^3)} \rightarrow 0, \text{ as } \lambda \rightarrow 0,$$

and $u_0$ is a positive solution of (1.2).

**Remark 1.3** We believe that our methods for proving the above results work also for the general case $N > 3$.

Throughout this paper, we denote the usual norm of $L^q(\mathbb{R}^3)$ and $W^{2,q}(\mathbb{R}^3)$ for $q \in [1, +\infty]$, respectively, by $|\cdot|_q$ and $\|\cdot\|_{2,q}$.

## 2 Subsolution and Supersolution

The aim of this section is to construct a subsolution and a supersolution of problem (1.1). Based on these sub- and supersolutions Theorem 1.1 is proved in Section 3. We begin this section by giving our definitions of sub- and supersolutions for system (1.1).
where $l$ is satisfied by lemmas. To construct the desired sub- and supersolutions, we need some preliminary lemmas.

**Definition 2.1** A positive function $\psi(x) \in C^2(\mathbb{R}^3)$ is said to be a supersolution of (1.1) if
\[
- \Delta \psi(x) + V(x)\psi(x) + \lambda \phi(x)\psi(x) \geq Q(x)\psi^p(x), \quad x \in \mathbb{R}^3,
\]
with $\phi(x)$ satisfies
\[
- \Delta \phi(x) = u^2(x), \quad \lim_{|x| \to +\infty} \phi(x) = 0,
\]
for $u \in W^{2,2}(\mathbb{R}^3)$ and $0 < u(x) \leq \psi(x)$ on $\mathbb{R}^3$. A positive function $\varphi(x) \in C^2(\mathbb{R}^3)$ is said to be a subsolution of (1.1) if the opposite inequality to (2.1) is satisfied by $\varphi(x)$, that is
\[
- \Delta \varphi(x) + V(x)\varphi(x) + \lambda \phi(x)\varphi(x) \leq Q(x)\varphi^p(x), \quad x \in \mathbb{R}^3,
\]
and (2.2) holds.

To construct the desired sub- and supersolutions, we need some preliminary lemmas.

**Lemma 2.1** Let $V(x)$ satisfy $(H_1) - (H_2)$, $\Lambda$ is defined by $(H_3)$. If $\Lambda < V_{\infty}$, then $\Lambda$ has a minimizer $\varphi(x) \in C_{loc}^{2,\gamma} \cap W^{2,q}(\mathbb{R}^3)$ for any $q \in (1, +\infty)$ with
\[
- \Delta \varphi(x) + V(x)\varphi(x) = \Lambda \varphi(x), \quad x \in \mathbb{R}^3, \tag{2.4}
\]
\[
0 < \varphi(x) < C|\varphi|_\infty e^{-l|x|}, \quad x \in \mathbb{R}^3, \tag{2.5}
\]
where $l \in (0, \sqrt{V_{\infty} - \Lambda})$ and $C = C(l, \Lambda) > 0$ is a constant.

**Proof.** It follows from Theorems 3.19 and 3.20 of [21] that there exists $\varphi(x) \in C^{2}(\mathbb{R}^3) \cap W^{2,2}(\mathbb{R}^3)$ such that (2.4) - (2.5) hold and $\varphi \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ by (2.5). Hence $(H_1)$ and our Lemma 3.1 in Section 3 show that $\varphi(x) \in W^{2,q}(\mathbb{R}^3)$ for any $q \in (1, +\infty)$. This and embedding theorem implies that $\varphi(x) \in C_{loc}^{1,\gamma}(\mathbb{R}^3)$. Thus, Theorem 9.19 in [14] gives that $\varphi(x) \in C_{loc}^{2,\gamma}(\mathbb{R}^3)$.

**Lemma 2.2** For any measurable function $u(x)$ on $\mathbb{R}^3$ with
\[
0 < u(x) \leq \psi(x) = \frac{1}{(1 + |x|^2)^{\alpha}}, \quad \text{with} \quad \alpha > \frac{3}{4},
\]
Then $\psi(x) \in L^q(\mathbb{R}^3)$ for all $q \in [2, +\infty]$. Let $\phi(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|} dy$, we have that
\[
0 < \phi(x) \leq \frac{1}{2(2\alpha - 1)}, \quad \text{for any} \quad x \in \mathbb{R}^3. \tag{2.6}
\]

**Proof.** By the definition, $\psi(x) \in L^\infty(\mathbb{R}^3)$. Since $\alpha > \frac{3}{4}$ and $2q\alpha > 3$ for $q \in [2, +\infty)$, it is not difficult to see that $\psi(x) \in L^q(\mathbb{R}^3)$ for $q \in [2, +\infty)$.

Let $g(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy$. Theorem 9.9 of [14] shows that $g \in D^{2,q}(\mathbb{R}^3)$ and hence $g(x) \in C^2(\mathbb{R}^3)$ by Theorem 9.19 of [14], and
\[
- \Delta g(x) = \psi^2(x), \quad x \in \mathbb{R}^3. \tag{2.7}
\]
This shows that \( g(|x|) \) is strictly decreasing. Hence, for any \( x \in \mathbb{R}^3 \)

\[
\phi(x) \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\psi^2(y)}{|x-y|} dy \leq g(0) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\psi^2(y)}{|y|} dy = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|y|(1+|y|^2)^{2\alpha}} dy
\]

\[
= \int_{\mathbb{R}^3} \frac{r}{(1+r^2)^{2\alpha}} dr = \frac{1}{2(2\alpha-1)}.
\]

So, (2.6) is proved. □

The following lemma gives a pair of sub- and supersolutions of (1.1).

**Lemma 2.3** Under the assumptions of Theorem 1.1, let \( \varphi(x) \) and \( \psi(x) \) be given by Lemmas 2.1 and 2.2, respectively. Then, for each \( \lambda \in [0, -2(2\alpha-1)\Lambda] \), there exists \( \epsilon_0 \in (0, 1) \) such that \( \psi(x) \) and \( \epsilon_0 \varphi(x) \) are super- and subsolutions of (1.1), respectively. Moreover \( \epsilon_0 \varphi(x) < \psi(x) \) for any \( x \in \mathbb{R}^3 \).

**Proof:** For any \( u \in W^{2,2}(\mathbb{R}^3) \), let \( \phi(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|} dy \). Since \( u \in W^{2,2}(\mathbb{R}^3) \subset L^q(\mathbb{R}^3) \cap C^{0,\gamma_1}(\mathbb{R}^3) \) for any \( q \in [2, +\infty] \) and some \( \gamma_1 \in (0,1) \), and the Hardy-Littlewood-Sobolev inequality (see e.g. [15] Theorem 4.3) yields that \( \|\phi\|_6 \leq C\|u\|_{2^*}^2 \) and \( \phi \in L^1_{\text{loc}}(\mathbb{R}^3) \). Then, it follows from [14] Theorem 9.9 that

\[
|D^2\phi|_q \leq C\|u\|^2_{2^*} \quad \text{for each} \quad q \in (1, +\infty), \quad \text{and} \quad -\Delta \phi(x) = u^2(x) \quad \text{a.e.} \quad x \in \mathbb{R}^3.
\]

So \( u \in C^{0,\gamma_1}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3) \) by Sobolev embedding. Hence, Theorems 9.19 and 9.20 of [14] imply that \( \phi \in C^{2,\gamma_1}_{\text{loc}}(\mathbb{R}^3) \) and

\[
-\Delta \phi(x) = u^2(x), \quad x \in \mathbb{R}^3, \quad \lim_{|x| \to +\infty} \phi(x) = 0. \quad (2.8)
\]

On the other hand, the uniqueness of the solution of (2.8) implies that any solution of (2.8) with \( u \in W^{2,2}(\mathbb{R}^3) \) must have the form of

\[
\phi(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|} dy.
\]

Since (H4) and \( \Delta \psi(x) = \frac{2\alpha[(2\alpha-1)r^2-3]}{(r^2+1)^{2\alpha+1}}\psi(x) = H(r)\psi(x) \), it follows that

\[
\Delta \psi + Q(x)\psi^p = H(r)\psi + Q(x)\psi^p \leq H(r)\psi + (V(x) - H(r))\psi = V(x)\psi. \quad (2.9)
\]
This and \( \phi(x) \geq 0 \) follow that, for each \( \lambda \geq 0 \),
\[
- \Delta \psi(x) + V(x)\psi(x) + \lambda \phi(x)\psi(x) \geq Q(x)\psi^p(x), \quad \text{on } \mathbb{R}^3,
\]
where \( \phi(x) \) satisfies (2.8). So, \( \psi(x) \) is a supersolution of (1.1).

For any \( \epsilon > 0 \), since \( \varphi \) satisfies (2.4), this yields
\[
- \Delta \epsilon \varphi(x) + V(x)\epsilon \varphi(x) = \Lambda \epsilon \varphi(x) \quad \text{for all } x \in \mathbb{R}^3.
\]  
(2.10)

For any \( \phi \) satisfying (2.8) with \( 0 < u(x) \leq \psi(x) \), if \( \lambda \in [0, -2(2\alpha - 1)\Lambda] \), it follows from (2.6) that
\[
\lambda \phi(x) + \Lambda \leq \frac{1}{2(2\alpha - 1)}[\lambda + 2(2\alpha - 1)\Lambda] := \delta_\lambda < 0, \quad \text{for any } x \in \mathbb{R}^3.
\]  
(2.11)

By (H1) and (2.5), \( Q(x)\varphi^{p-1} \in L^\infty(\mathbb{R}^3) \), then there exists a constant \( M \in (0, +\infty) \) such that
\[
Q(x)\varphi^{p-1} \geq -M \quad \text{for any } x \in \mathbb{R}^3.
\]  
(2.12)

From (2.11) and (2.12), there exists \( \epsilon_\lambda > 0 \) such that, for any \( \epsilon \in (0, \epsilon_\lambda) \)
\[
\lambda \phi(x) + \Lambda \leq \delta_\lambda \leq -\epsilon^{p-1}M \leq \epsilon^{p-1}Q(x)\varphi^{p-1}, \quad \text{for all } x \in \mathbb{R}^3.
\]  
(2.13)

Then for each \( \lambda \in [0, -2(2\alpha - 1)\Lambda] \), and \( \epsilon \in (0, \epsilon_\lambda) \), it follows from (2.10), (2.13) and \( \varphi(x) > 0 \) that
\[
- \Delta \epsilon \varphi(x) + V(x)\epsilon \varphi(x) + \lambda \phi(x)\epsilon \varphi(x) = (\Lambda + \lambda \phi(x))\epsilon \varphi(x) \leq Q(x)(\epsilon \psi)^p(x), \quad \text{on } \mathbb{R}^3.
\]

This means that \( \epsilon \varphi(x) \) is a subsolution if \( \epsilon \in (0, \epsilon_\lambda) \). Moreover, by (2.5) and the definition of \( \psi \), we know that there exits \( \epsilon_0 \in (0, \epsilon_\lambda) \) such that \( \epsilon_0 \varphi(x) < \psi(x) \) for any \( x \in \mathbb{R}^3 \). □

### 3 Proofs of the main Theorems

Now, we turn to showing our main Theorems 1.1 and 1.2. To prove Theorem 1.1, an iteration sequence is required, and it can be obtained by the sub- and supersolutions given by Lemma 2.3 as well as the following lemmas.

**Lemma 3.1** [13] **Proposition 1** Consider \( k > 0 \),

(i) For each \( f \in L^q(\mathbb{R}^3) \) with \( q \in [1, +\infty] \), there exists an unique \( u := Tf \in L^q(\mathbb{R}^3) \) satisfying \( -\Delta u + ku = f \) on \( \mathbb{R}^3 \) in the sense of distributions.

(ii) Let \( f \in L^q(\mathbb{R}^3) \) with \( q \in (1, +\infty) \), then \( Tf \in W^{2,q}(\mathbb{R}^3) \). Moreover, there exists a constant \( C = C(k, q) > 0 \) such that \( \|Tf\|_{2,q} \leq C|f|_q \). □

**Lemma 3.2** Let \( \epsilon_0 \varphi(x) \) and \( \psi(x) \) be given by Lemma 2.3. Consider the following problem
\[
- \Delta u(x) + ku(x) = f(x, w, v), \quad x \in \mathbb{R}^3,
\]  
(3.1)

where \( k \) is a positive constant, \( w \) and \( v \) are functions on \( \mathbb{R}^3 \), \( f: \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \). For \( \gamma \in (0, 1) \) and \( q \in (1, +\infty) \), we assume that
Then, by Lemma 3.1 (ii),

\[ \| \begin{align*} & \text{(F}_1\text{)} f(x, u, v) \in C^{0,\gamma}_{\text{loc}}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3) \text{ if } w, v \in C^{2,\gamma}_{\text{loc}}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3) \text{ and } \epsilon_0 \varphi(x) \leq w, v \leq \psi(x). \\
& \text{(F}_2\text{)} f(x, u(x), \epsilon_0 \varphi(x)) \leq f(x, u(x), u(x)) \leq f(x, u(x), \psi(x)) \text{ for any } u(x) \in C^{2,\gamma}_{\text{loc}}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3) \text{ with } \epsilon_0 \varphi(x) \leq u(x) \leq \psi(x). \\
& \text{(F}_3\text{)} -\Delta \epsilon_0 \varphi(x) + k \epsilon_0 \varphi(x) \leq f(x, u(x), \epsilon_0 \varphi(x)) \text{ and } -\Delta \psi(x) + k \psi(x) \geq f(x, u(x), \psi(x)) \text{ for any } u(x) \in C^{2,\gamma}_{\text{loc}}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3) \text{ with } \epsilon_0 \varphi(x) \leq u(x) \leq \psi(x). \\
\end{align*} \]

Then, there exists \( \{u_n\} \subset C^{2,\gamma}_{\text{loc}}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3) \) for any \( q \in (1, +\infty) \) such that

\[ -\Delta u_{n+1}(x) + k u_{n+1}(x) = f(x, u_n, u_n), \quad x \in \mathbb{R}^3, \quad (3.2) \]

\[ \epsilon_0 \varphi(x) \leq u_n(x) \leq \psi(x), \quad x \in \mathbb{R}^3, \quad (3.3) \]

\[ ||u_{n+1}||_{2,q} \leq C(k, q) |f(x, u_n, u_n)|_q. \quad (3.4) \]

**Proof:** Let \( u_0 = \epsilon_0 \varphi \) and Lemma 2.1 implies that \( u_0 \in C^{2,\gamma}_{\text{loc}}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3) \) for any \( q \in (1, +\infty) \). Then \( f(x, u_0, u_0) \in C^{0,\gamma}_{\text{loc}}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3) \) by (F1). Applying Lemma 3.1 to problem (3.1) with \( w = v = u_0 \), we get \( u_1(x) \in W^{2,q}(\mathbb{R}^3) \) such that

\[ -\Delta u_1(x) + k u_1(x) = f(x, u_0, u_0), \quad x \in \mathbb{R}^3, \quad (3.5) \]

and then \( u_1 \in C^{2,\gamma}_{\text{loc}}(\mathbb{R}^3) \) by Theorem 9.19 in [14]. Taking \( u = u_0 \) in (F2) (F3) and noting that \( \epsilon_0 \varphi < \psi \), we see that

\[ f(x, u_0(x), \epsilon_0 \varphi(x)) \leq f(x, u_0(x), u_0(x)) \leq f(x, u_0(x), \psi(x)), \]

\[ -\Delta \epsilon_0 \varphi(x) + k \epsilon_0 \varphi(x) \leq f(x, u_0(x), \epsilon_0 \varphi(x)), \quad x \in \mathbb{R}^3, \]

\[ -\Delta \psi(x) + k \psi(x) \geq f(x, u_0(x), \psi(x)), \quad x \in \mathbb{R}^3. \]

These and (3.5) give that

\[ -\Delta (\epsilon_0 \varphi - u_1) + k (\epsilon_0 \varphi - u_1) \leq 0 \leq -\Delta (\psi - u_1) + k (\psi - u_1). \quad (3.6) \]

Hence the maximum principle implies that \( \epsilon_0 \varphi(x) \leq u_1(x) \leq \psi(x) \). On the other hand, Lemma 3.1 (ii) shows that

\[ ||u_1||_{2,q} \leq C(k, q) |f(x, u_0, u_0)|_q. \quad (3.7) \]

Inductively, given \( u_n \in C^{2,\gamma}_{\text{loc}}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3) \) \( (n = 1, 2, \cdots) \) with \( \epsilon_0 \varphi(x) \leq u_n(x) \leq \psi(x) \), by Lemma 3.1 (i) and Theorem 9.19 of [14], there exists \( u_{n+1} \in C^{2,\gamma}_{\text{loc}}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3) \) such that

\[ -\Delta u_{n+1}(x) + k u_{n+1}(x) = f(x, u_n, u_n), \quad x \in \mathbb{R}^3. \quad (3.8) \]

Taking \( u = u_n \) in (F2) and (F3), similar to the discussion of (3.6) and (3.7), it follows from (3.8) and the maximum principle that \( \epsilon_0 \varphi(x) \leq u_{n+1}(x) \leq \psi(x) \). Then, by Lemma 3.1 (ii), \( ||u_{n+1}||_{2,q} \leq C(k, q) |f(x, u_n, u_n)|_q. \) \( \Box \)
Proof of Theorem 1.1: For \( v, w \in W^{2,2}(\mathbb{R}^3) \), we denote
\[
\phi_w(x) = \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} w^2(y)dy,
\]
and for \( \lambda \in [0, -2(2\alpha - 1)\Lambda) \) and \( k > 0 \) large enough, define
\[
f(x, w, v) = Q(x)|v|^{p-2}u + kv - V(x)v - \lambda \phi_w(x)v. \tag{3.9}
\]
We prove now the theorem by the following steps. In what follows, \( \epsilon_0 \varphi \) and \( \psi \) are the sub- and supersolutions given by Lemma 3.2.

Step 1: There exists \( \{u_n\} \subset C_{loc}^{2,\gamma}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3) \) for any \( q \in (1, +\infty) \) such that
\[
- \Delta u_{n+1} + ku_{n+1} = f(x, u_n, u_n), \quad x \in \mathbb{R}^3, \tag{3.10}
\]
\[
|\epsilon_0 \varphi(x) - u_n(x)| \leqslant |\psi(x)|, \quad x \in \mathbb{R}^3, \tag{3.11}
\]
\[
\|u_{n+1}\|_{2,q} \leqslant C(k, q)|f(x, u_n, u_n)|_q, \tag{3.12}
\]
where \( f \) is defined by (3.9).

By Lemma 3.2, Step 1 is proved if the function \( f \) defined by (3.9) satisfies (F1) to (F3). By (H1), Lemmas 2.2 and 2.3, it is not difficult to know that (F1) and (F3) hold. For \( 0 < w, v \leqslant \psi \) and \( k > 0 \) large enough, it follows from (H1) and Lemma 2.2 that
\[
\frac{\partial f(x, w, v)}{\partial v} = pQ(x)|v|^{p-2}v + k - V(x) > 0, \quad \text{for any} \quad x \in \mathbb{R}^3. \tag{3.13}
\]
This implies that (F2) holds. Hence, Step 1 is complete.

Step 2: There exists \( u \in W^{2,q}(\mathbb{R}^3) \) such that, by passing to a subsequence, \( \{u_n\} \) converges to \( u \) weakly in \( W^{2,q}(\mathbb{R}^3) \) and strongly in \( L^q(\mathbb{R}^3) \) for all \( q \in [2, +\infty) \).

By Lemma 2.2, \( \psi \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \), then it follows from (3.9) and (H1) that
\[
|f(x, u_n, u_n)|_q \leqslant C|\psi|_q, \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad q \in [2, +\infty),
\]
where \( C > 0 \) is a constant independent of \( n \) and \( \lambda \). So, Lemma 3.2 implies that
\[
\|u_n\|_{2,q} \leqslant C(k, q)|f(x, u_n, u_n)|_q \leqslant C|\psi|_q, \quad q \in [2, +\infty). \tag{3.14}
\]
This means that \( \{u_n\} \) is bounded in \( W^{2,q}(\mathbb{R}^3) \) for each \( q \in [2, +\infty) \). So, passing to a subsequence, there is \( u \in W^{2,q}(\mathbb{R}^3) \) such that
\[
u_n \overset{n}{\rightharpoonup} u \quad \text{weakly in} \quad W^{2,q}(\mathbb{R}^3) \quad \text{and} \quad u_n \overset{n}{\rightharpoonup} u, \quad a.e \quad \text{on} \quad x \in \mathbb{R}^3.
\]
Therefore, \( u \in \cap_{2 \leq q < +\infty} W^{2,q}(\mathbb{R}^3) \). By Lemma 2.2, \( \psi \in \cap_{q=2}^{+\infty} L^q(\mathbb{R}^3) \), then \( 0 < u_n^q(x) \leqslant \psi^q(x) \) by (3.11). Thus, the dominated convergence theorem shows that
\[
\int_{\mathbb{R}^3} |u_n(x)|^q dx \overset{n}{\rightharpoonup} \int_{\mathbb{R}^3} |u(x)|^q dx, \quad \text{for all} \quad q \in [2, +\infty),
\]
and Lemma 1.32 in [23] implies that $|u_n(x) - u(x)|_q \to 0$.

**Step 3**: $u \in C^2(\mathbb{R}^3)$ is a solution of (1.1).

Multiplying (3.10) by $\eta(x) \in C_0^\infty(\mathbb{R}^3)$, then integrating by parts over $\mathbb{R}^3$, it yields that

$$
\int_{\mathbb{R}^3} \{-\Delta u_{n+1}(x) + ku_{n+1}(x)\} \eta(x) dx = \int_{\mathbb{R}^3} u_{n+1}(x) \{-\Delta \eta(x) + k\eta(x)\} dx
$$

$$
= \int_{\mathbb{R}^3} f(x, u_n, u_n) \eta(x) dx, \tag{3.15}
$$

Since $u_n \to u$ a.e. on $\mathbb{R}^3$, noting (3.11) and (H1), the dominated convergence theorem shows that

$$
\lim_{n \to \infty} \int_{\mathbb{R}^3} u_n(x) [-\Delta \eta(x) + k\eta(x)] dx = \int_{\mathbb{R}^3} u(x) [-\Delta \eta(x) + k\eta(x)] dx, \tag{3.16}
$$

$$
\lim_{n \to \infty} \int_{\mathbb{R}^3} u_n(x) [k - V(x) - Q(x) u_n^{p-1}] \eta(x) dx = \int_{\mathbb{R}^3} u(x) [k - V(x) - Q(x) u^{p-1}] \eta(x) dx. \tag{3.17}
$$

Letting $\phi_n = \phi_{u_n}$. By Step 2, $|u_n(x) - u(x)|_q \to 0$ for $q \in [2, +\infty)$, it follows from [19] Lemma 2.1 that $\phi_n \to \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ and hence in $L^q(\mathbb{R}^3)$. For any $\eta(x) \in C_0^\infty(\mathbb{R}^3)$, noting that $\|\eta u_n\|_{L^q}$ is bounded, it follows from the Hölder inequality and Sobolev embedding that

$$
\lim_{n \to \infty} \left| \int_{\mathbb{R}^3} [\phi_n(x)u_n(x)\eta(x) - \phi_u(x)u(x)\eta(x)] dx \right|
\leq \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^3} |\phi_n - \phi_u| \|\eta\| u_n dx + \int_{\mathbb{R}^3} \phi_u \|\eta\| |u_n - u| dx \right\}
\leq \lim_{n \to \infty} |\phi_n - \phi_u| \|\eta u_n\|_{L^q} + o(1)
\leq \lim_{n \to \infty} C|\nabla(\phi_n - \phi_u)| \|\eta u_n\|_{L^q}. \tag{3.18}
$$

By (3.17) (3.18) and the definition of $f$ (3.9), we see that

$$
\lim_{n \to \infty} \int_{\mathbb{R}^3} \{f(x, u_n, u_u) - f(x, u, u)\} \eta(x) dx = 0, \text{ for any } \eta(x) \in C_0^\infty(\mathbb{R}^3). \tag{3.19}
$$

Thus, (3.15), (3.16) and (3.19) yield

$$
\int_{\mathbb{R}^3} \{-\Delta u(x) + ku(x)\} \eta(x) dx = \int_{\mathbb{R}^3} u(x) \{-\Delta \eta(x) + k\eta(x)\} dx = \int_{\mathbb{R}^3} f(x, u, u) \eta(x) dx, \tag{3.20}
$$

for any $\eta(x) \in C_0^\infty(\mathbb{R}^3)$. By the definition of $f$, it gives that

$$
\int_{\mathbb{R}^3} \{-\Delta u(x) + V(x)u(x)\} \eta(x) dx + \int_{\mathbb{R}^3} \lambda \phi_u(x)u(x) \eta(x) dx = \int_{\mathbb{R}^3} Q(x) u^p \eta(x) dx. \tag{3.21}
$$

Since $u \in \cap_{2 \leq q < +\infty} W^{2,q}(\mathbb{R}^3)$, and the embedding theorem shows that $u \in C^{1,\gamma}(\mathbb{R}^3)$. By (H1) and (3.9), $f(x, u, u) \in C_0^{0,\gamma}(\mathbb{R}^3)$. Then (3.20) and the Theorem 9.19 of [14] imply $u \in C_0^{0,\gamma}(\mathbb{R}^3)$.
Step 4: \( 0 < u(x) \leq \psi(x) \) and \( \|u\|_{2,q} \leq C \), where \( C > 0 \) is a constant and independent of \( \lambda \).

By (3.14) and the weakly lower semicontinuity of \( \|\cdot\|_{2,q} \), we see that \( \|u\|_{2,q} \leq C|\psi|_q \), where \( C \) is a constant and independent of \( \lambda \). By \( u_n \rightharpoonup u \) a.e. \( \mathbb{R}^3 \), and \( 0 < u_n(x) \leq \psi(x) \), we have \( 0 < u(x) \leq \psi(x) \) a.e \( \mathbb{R}^3 \). \( \square \)

**Proof of Theorem 1.2:** By Theorem 1.1, for each \( \lambda \in (0, -2(2\alpha - 1)\Lambda) \), (1.1) has a solution \( u_\lambda \) satisfying (3.21), and

\[
\|u_\lambda\|_{2,q} \leq C, \quad \text{for each } q \in [2, +\infty),
\]

where \( C \) is a constant independent of \( \lambda \). Then, passing to a subsequence, there exists \( u_0 \in W^{2,q}(\mathbb{R}^3) \) such that

\[
u_\lambda \rightharpoonup u_0, \text{ weakly in } W^{2,q}(\mathbb{R}^3).
\]

Similar to Step 2 in the proof of Theorem 1.1, it follows from the dominated convergence theorem and Lemma 1.32 of [23] that

\[
\int_{\mathbb{R}^3} |u_\lambda(x) - u_0(x)|^q dx \rightarrow 0. \tag{3.22}
\]

Finally, as Step 3 in the proof of Theorem 1.1, we know that, for any \( \eta(x) \in C_0^\infty(\mathbb{R}^3) \),

\[
\int_{\mathbb{R}^3} \{ -\Delta u_0(x) + V(x)u_0(x) \} \eta(x) dx = \int_{\mathbb{R}^3} Q(x)u_0^p \eta(x) dx, \tag{3.23}
\]

and \( u_0 \in C^{2,\gamma}_{loc}(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3) \) for all \( q \in [2, +\infty) \), so \( u_0 \) is a classical solution of (1.2). Moreover, using (3.21) to (3.23), we know that

\[
\|u_\lambda - u_0\|_{W^{1,2}(\mathbb{R}^3)} \rightarrow 0, \text{ as } \lambda \rightarrow 0. \square
\]

**References**

[1] W. Allegretto and Y.X. Huang. Eigenvalues of the indefinite-weight \( p \)-Laplacian in weighted spaces. *Funkcial. Ekvac.*, 38(2):233–242, 1995.

[2] A. Ambrosetti. On Schrodinger-Poisson Systems. *Milan J. Math.*, 76(1):257–274, DEC 2008.

[3] A. Ambrosetti and D. Ruiz. Multiple bound states for the Schrödinger-Poisson problem. *Commun. Contemp. Math.*, 10(3):391–404, 2008.

[4] A. Azzollini and A. Pomponio. Ground state solutions for the nonlinear Schrödinger-Maxwell equations. *J. Math. Anal. Appl.*, 345(1):90–108, 2008.

[5] V. Benci and D. Fortunato. An eigenvalue problem for the Schrödinger-Maxwell equations. *Topol. Methods Nonlinear Anal.*, 11(2):283–293, 1998.
[6] K. Benmlih. A note on a 3-dimensional stationary Schrödinger-Poisson system. *Electron. J. Differential Equations*, 2004(26):1–5, 2004.

[7] J. Chabrowski and D.G. Costa. On a class of Schrödinger-type equations with indefinite weight functions. *Comm. Partial Differential Equations*, 33(7-9):1368–1394, 2008.

[8] J.Q. Chen and S.J. Li. Existence and multiplicity of nontrivial solutions for an elliptic equation on $\mathbb{R}^N$ with indefinite linear part. *Manuscripta Math.*, 111(2):221–239, 2003.

[9] D.G. Costa and H. Tehrani. Existence of positive solutions for a class of indefinite elliptic problems in $\mathbb{R}^N$. *Calc. Var. Partial Differential Equations*, 13(2):159–189, 2001.

[10] T. D’Aprile and D. Mugnai. Non-existence results for the coupled Klein-Gordon-Maxwell equations. *Adv. Nonlinear Stud.*, 4(3):307–322, 2004.

[11] T. D’Aprile and D. Mugnai. Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 134(5):893–906, 2004.

[12] P. d’Avenia. Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations. *Adv. Nonlinear Stud.*, 2(2):177–192, 2002.

[13] A.L. Edelson and C.A. Stuart. The principal branch of solutions of a nonlinear elliptic eigenvalue problem on $\mathbb{R}^N$. *J. Differential Equations*, 124(2):279–301, 1996.

[14] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[15] E.H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.

[16] C. Mercuri. Positive solutions of nonlinear Schrödinger-Poisson systems with radial potentials vanishing at infinity. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 19(3):211–227, 2008.

[17] F. Nier. Schrödinger-Poisson systems in dimension $d \leq 3$: the whole-space case. *Proc. Roy. Soc. Edinburgh Sect. A*, 123A(Part 6):1179–1201, 1993.

[18] D. Ruiz. On the Schrödinger-Poisson-Slater system: Behavior of minimizers, radial and nonradial cases. *preprint*.

[19] D. Ruiz. The Schrödinger-Poisson equation under the effect of a nonlinear local term. *J. Funct. Anal.*, 237(2):655–674, 2006.
[20] Ó. Sánchez and J. Soler. Long-time dynamics of the Schrödinger-Poisson-Slater system. *J. Statist. Phys.*, 114(1-2):179–204, 2004.

[21] C.A. Stuart. 'An introduction to elliptic equations on $\mathbb{R}^N$. *Nonlinear functional analysis and applications to differential equations* (ed. A.Ambrosetti, K.-C.Chang and I.Ekeland, World Scientific, Singapore, 1998), p. 237–285.

[22] Z.P. Wang and H.S. Zhou. Positive solution for a nonlinear stationary Schrödinger-Poisson system in $\mathbb{R}^3$. *Discrete Contin. Dyn. Syst.*, 18(4):809–816, 2007.

[23] M. Willem. *Minimax theorems*. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston Inc., Boston, MA, 1996.

[24] M.B. Yang, Z.F. Shen and Y.H. Ding. Multiple semiclassical solutions for the nonlinear Maxwell-Schrodinger system. *Nonli. Anal.*, in press.

[25] L.G. Zhao and F.K. Zhao. On the existence of solutions for the Schrödinger-Poisson equations. *J. Math. Anal. Appl.*, 346(1):155–169, 2008.

**Wuhan Institute of Physics and Mathematics**
**Chinese Academy of Sciences**
**P.O.Box 71710, Wuhan 430071, China**
Email: flymath@163.com and hszhou@wipm.ac.cn