Real simple symplectic triple systems

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Abstract
The simple symplectic triple systems over the real numbers are classified up to isomorphism, and linear models of all of them are provided. Besides the split cases, there are two kinds of non-split real simple symplectic triple systems with classical enveloping algebra, called unitarian and quaternionic types, and five non-split real simple symplectic triple systems with exceptional enveloping algebra.

Keywords Symplectic triple system · \( \mathbb{Z}/2 \)-graded simple Lie algebra · Weak isomorphism · Einstein manifold

Mathematics Subject Classification Primary 17A40 · Secondary 17B60

1 Introduction
Symplectic triple systems constitute a ternary algebraic structure that appears widely in the literature, although often hidden under a variety of algebraic and geometric objects. The goal of this paper is to provide a self-contained classification of the

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simple symplectic triple systems over the real numbers, while showing at the same
time very explicit models for all of them, using just tools from Linear Algebra.

Symplectic triple systems were introduced in 1975 in [34] by Yamaguti and Asano
in their attempt to understand the ternary structures involved in Freudenthal’s construc-
tion of the exceptional Lie algebras (see for instance, [20]). Thus, for \( T \) a symplectic
triangular system, then \( T \oplus T \) is a Lie triangular system, and, in general, symplectic triple
systems are variations on the Freudenthal triangular systems \([29, 30]\), or on the balanced
symplectic ternary algebras \([19]\), among others.

Although several of these ternary structures are related, each of them with its own set
of defining identities and most of them participating on the construction of Lie algebras,
the concrete point of view provided by symplectic triangular systems not only has algebraic
interest but also has many applications to Differential Geometry. For instance, complex
symplectic triangular systems appear naturally in the study of 3-Sasakian homogeneous
manifolds \( M = G/H \) \([9, Remark 4.11]\): the complexification of the reductive pair
\((\text{Lie}(G), \text{Lie}(H))\) coincides necessarily with the standard enveloping algebra and the
inner derivation algebra of a simple complex symplectic triangular system. The paper \([10]\)
takes advantage of such facts to write, for some convenient invariant affine connections,
the (complexification of the) curvature tensors of any 3-Sasakian homogeneous
manifold in terms of the related triangular system and to compute the corresponding holon-
omy algebras, supporting in this way the choice of some specific affine connections
suitable for studying (not necessarily homogeneous) 3-Sasakian manifolds. Inspired
by \([9]\), an Einstein manifold is constructed related to each simple real symplectic triangular
system in \([11]\). In fact, the proof in \([11]\) that the Ricci tensor is a multiple of the metric
is purely algebraic, making use only of the properties of the symplectic triangular systems.
Precisely this work is part of our motivation to classify completely the simple real
symplectic triangular systems, in order to obtain an algebraic description of such family of
Einstein manifolds (of their tangent spaces) which permits to do explicit computations
and to give a concrete algebraic treatment. These manifolds appear independently in
\([2]\), where Alekseevsky and Cortés study para-quaternionic Kähler manifolds from a
quite different approach based on some kind of twistor spaces.

Actually, thanks to the uniqueness in Corollary 3.4, the list of different possibilities
for the real simple symplectic triangular systems, \textit{up to weak isomorphism(!)} (see
Definition 2.3) can be read from Table 1.1 of \([6]\), where the pairs \((g, h)\) of Lie alge-
bras corresponding to the para-quaternion Kähler manifolds \( G/K \) with \( G \) simple are
given. Over an algebraically closed field, weak isomorphism is trivially equivalent to
isomorphism (see the paragraph before Definition 2.3), but this is much subtler over the
real numbers.

The symplectic triangular systems appear too as the 1-homogeneous components of
contact gradings. A grading on a semisimple Lie algebra \( g = \sum_{i=2}^{2} g_i \) is said to be
contact if \( \dim g_{-2} = 1 \) and the bracket \([ , ]\): \( g_{-1} \times g_{-1} \rightarrow g_{-2} \) is non-degenerate,
that is, \([x, g_{-1}] \neq 0 \) if \( 0 \neq x \in g_{-1} \). These gradings appear naturally in Differential
Geometry because they are related to parabolic geometries which have an underlying
contact structure (see \([5, §3.2.4]\) for the complex case and \([5, §3.2.10]\) for the real case).
Contact gradings can only exist on simple Lie algebras, and on each complex simple
Lie algebra of rank larger than one there is a unique (up to inner automorphism) contact
grading. For a nice survey on differential systems associated with simple graded Lie algebras including contact gradings, the reader may consult [33].

For some recent work on concrete expressions of triple products related to arbitrary 5-gradings (including contact gradings as a special case) and motivated by Differential Geometry, the reader may consult [4, 26].

Note that this strong relationship between symplectic triple systems and Differential Geometry does not mean that simple symplectic triple systems are well-known. On the contrary, their simple enveloping Lie algebras are well-known, while concrete models including expressions for the triple products are more necessary than ever, specially models suitable to be used in applications. In this work, we exhibit models based exclusively on Linear Algebra that do not make use of Jordan algebras or structurable algebras, with a view towards its potential applications.

A last word about applications to Projective Geometry. Given the contact grading of a complex simple Lie algebra \( g \), the associated \textit{Freudenthal variety} (in \( \mathbb{P}(g_1) \)) is just the projectivization of the set \( \{0 \neq x \in g_1 : \text{ad}(x)^2 g_{-2} = 0\} \) with its reduced structure. A systematic study of the projective geometry of Freudenthal varieties is developed in [24], making use precisely of the triple product provided by the symplectic triple system. Some applications to incidence geometries come from abelian inner ideals. Note that \( g_{-2} \) is an (obviously minimal, being one-dimensional) abelian inner ideal of \( g \), that is, \( \{g_{-2}, [g_{-2}, g]\} \subset g_{-2} \) (any abelian inner ideal of a simple complex Lie algebra \( g \) is the corner of a \( \mathbb{Z} \)-grading). The \textit{inner ideal geometry} of a Lie algebra \( g \) is the point-line geometry with the minimal inner ideals as point set, as lines the inner line ideals, and inclusion as incidence. This inner ideal geometry is a generalization of an extremal geometry. In fact, inner ideals are used to construct Moufang sets, Moufang triangles and Moufang hexagons in [8], which uses the structurable algebra related to symplectic triple systems coming from a Jordan division algebra.

Coming back to the algebraic origin of these ternary structures as pieces involved in the construction of simple (5-graded) Lie algebras, this role can be extended to arbitrary fields of characteristic different from 2 and 3, but in characteristic 3, symplectic triple systems are used surprisingly to construct simple Lie superalgebras (see [14] or the review [17]). Conversely, orthogonal triple systems, which provide constructions of Lie superalgebras if the characteristic of the field is different from 2 and 3, are used in characteristic 3 to construct simple Lie algebras. That is, symplectic triple systems are a source of nice algebraic constructions.

Also the interplay among several families of ternary structures provides different landscapes (for instance, Freudenthal triple systems constitute the ternary structure used in [11] to describe a tensor measuring how far is the manifold from being of constant sectional curvature equal to 1). Some relations between ternary non-associative structures in general (and symplectic triple systems in particular) and Physics are explained in [25].

There is a strong connection between symplectic triple systems and structurable algebras of skew-dimension one. Structurable algebras form a class of non-associative algebras with involution introduced by Bruce Allison in [3], which always gives rise to a 5-graded Lie algebra. (They include, for instance, Jordan algebras with trivial involution, and associative algebras with involution.) The structurable algebras of skew-dimension one have been constructed by means of non-linear isotopies of cubic
norm structures and also by means of hermitian cubic norm structures (see the review [7]), but it would be convenient to have easier models in the real case based on Linear Algebra, obtained from our models of symplectic triple systems.

The classification of symplectic triple systems for an arbitrary algebraically closed field of characteristic different from 2 and 3 is obtained in [14, Theorem 2.30]. This is not the case for the real simple symplectic triple systems, despite the large amount of information scattered among the papers mentioned above on Differential Geometry, where the concrete expression for the ternary product is rarely made explicit. Here very concrete models of all the possibilities and a purely algebraic self-contained classification will be given explicitly.

The structure of the paper is as follows. Section 2 deals with the necessary background on symplectic triple systems, including the concepts of isomorphism and weak isomorphism, which in the real case are not equivalent in an obvious way. Each simple symplectic triple system has two related Lie algebras, namely, the simple and \( \mathbb{Z}/2\)-graded standard enveloping algebra and the inner derivation algebra. Some results on \( \mathbb{Z}/2\)-graded Lie algebras will be stated in Sect. 3 over fields of zero characteristic, which will be crucial for the classification. Section 4 gives precise models of all the split simple symplectic triple systems, valid over any field of zero characteristic, in particular over the real field. Precisely, special, symplectic and orthogonal models are recalled, while for the exceptional cases, new models are given using only Linear Algebra tools, that is, vector spaces, exterior powers, symmetric or alternating bilinear forms and so on, instead of cubic norms or Jordan algebras, which appear usually in the exceptional descriptions. In Sect. 5 specific non-split models for the real case will be described: unitarian and quaternionic symplectic triple systems among the symplectic triple systems with classical envelope, as well as several symplectic triple systems with exceptional envelope. Again the provided exceptional examples make use only of Linear Algebra tools, and give some nice constructions of the corresponding real simple Lie algebras in the spirit of the models that appear in [1].

The rest of the work will show that the list of examples is exhaustive. Section 6 contains a purely algebraic and self-contained classification of the real simple symplectic triple systems up to weak isomorphism, and Sect. 7 is devoted to proving that two weakly isomorphic simple real symplectic triple systems are necessarily isomorphic. It must be remarked here that over an algebraically closed field, two weakly isomorphic simple systems are isomorphic in an obvious way, but this is not the case for arbitrary fields (see Definition 2.3 and the paragraph preceding it). The final result (Theorem 7.5) summarizes the classification of the simple real symplectic triple systems up to isomorphism.

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## 2 Symplectic triple systems

This material goes back essentially to [34], although we follow the approach in [10, 14].
Throughout this work $\mathbb{F}$ will always denote a field of zero characteristic, although some results are valid in a more general setting. All vector spaces and algebras considered will be assumed to be finite-dimensional.

**Definition 2.1** Let $T$ be a vector space over a field $\mathbb{F}$ endowed with a non-zero alternating bilinear form $\langle \cdot \mid \cdot \rangle : T \times T \to \mathbb{F}$, and a trilinear product $[\cdot, \cdot, \cdot] : T \times T \times T \to T$. The triple $(T, [\cdot, \cdot, \cdot], \langle \cdot \mid \cdot \rangle)$ is said to be a symplectic triple system if the following conditions are satisfied

\[
[x, y, z] = [y, x, z], \quad (2.1)
\]

\[
[x, y, z] - [x, z, y] = (x|z)y - (x|y)z + 2(y|z)x, \quad (2.2)
\]

\[
[x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]], \quad (2.3)
\]

\[
([x, y, u]|v) = -(u|[x, y, v]), \quad (2.4)
\]

for all $x, y, z, u, v, w \in T$.

An ideal of the symplectic triple system $T$ is a subspace $I$ of $T$ such that $[T, T, I] + [T, I, T] \subset I$; and the system is said to be simple if $[T, T, T] \neq 0$ and it contains no proper ideal. The simplicity of $T$ is equivalent to the non-degeneracy of the bilinear form (\cite[Proposition 2.4]{14}), and this implies that the simplicity is preserved under scalar extensions(!).

Of course two symplectic triple systems $(T, [\cdot, \cdot, \cdot], \langle \cdot \mid \cdot \rangle)$ and $(T', [\cdot, \cdot, \cdot]', \langle \cdot \mid \cdot \rangle')$ are said to be isomorphic if there is a bijective linear map $f : T \to T'$ such that $(x|y) = (f(x)|f(y))'$ and $f([x, y, z]) = [f(x), f(y), f(z)]'$ for any $x, y, z \in T$.

There is a close relationship between symplectic triple systems and a particular kind of $\mathbb{Z}/2$-graded Lie algebras. We denote by $d_{x, y}$ the linear map

\[
d_{x, y} := [x, y, \cdot] \in \text{End}_F(T),
\]

for $x, y \in T$. Observe that the above set of identities can be read in the following way. By (2.4), $d_{x, y}$ belongs to the symplectic Lie algebra

\[
\mathfrak{sp}(T, (\cdot, \cdot)) = \{d \in \text{gl}(T) : (d(u)|v) + (u|d(v)) = 0 \ \forall u, v \in T\},
\]

which is a subalgebra of the general linear algebra $\text{gl}(T) = (\text{End}_F(T), [\cdot, \cdot])$; and by (2.3), $d_{x, y}$ belongs to the Lie algebra of derivations of the triple too, i.e.,

\[
det(T, [\cdot, \cdot, \cdot]) := \{d \in \text{gl}(T) : d([u, v, w]) = [d(u), v, w] + [u, d(v), w] + [u, v, d(w)] \ \forall u, v, w \in T\},
\]

which is also a Lie subalgebra of $\text{gl}(T)$. The set of inner derivations is the linear span

\[
\text{in} \text{det}(T) := \left\{ \sum_{i=1}^{n} d_{x_i, y_i} : x_i, y_i \in T, \ n \geq 1 \right\},
\]
which is a Lie subalgebra of $\mathfrak{det}(T, [\cdot, \cdot, \cdot])$, again taking into account (2.3). Now, consider $(V, \langle \cdot | \cdot \rangle)$ a two-dimensional vector space endowed with a non-zero alternating bilinear form, and the vector space

$$g(T) := \mathfrak{sp}(V, \langle \cdot | \cdot \rangle) \oplus \text{indet}(T) \oplus (V \otimes T).$$

Then $g(T)$ is endowed with a $\mathbb{Z}/2$-graded Lie algebra structure ( [14,Theorem 2.9]) such that the even part is $g(T)_0 := \mathfrak{sp}(V, \langle \cdot | \cdot \rangle) \oplus \text{indet}(T)$, a direct sum of two ideals, and the odd part is $g(T)_1 := V \otimes T$. The anticommutative product is given by

- the usual bracket on $g(T)_0$, which in this way is a Lie subalgebra of $g(T)$;
- the natural action of $g(T)_0$ on $g(T)_1$, that is,

$$[\gamma + d, a \otimes x] = \gamma(a) \otimes x + a \otimes d(x),$$

for $\gamma \in \mathfrak{sp}(V, \langle \cdot | \cdot \rangle)$, $d \in \text{indet}(T)$, $a \in V$, $x \in T$; and
- the product of two odd elements as follows

$$[a \otimes x, b \otimes y] = (x|y)\gamma_{a,b} + \langle a|b\rangle dx + \langle a|b\rangle y \in g(T)_0,$$

for $a, b \in V$ and $x, y \in T$, where $\gamma_{a,b} \in \mathfrak{sp}(V, \langle \cdot | \cdot \rangle)$ is defined by

$$\gamma_{a,b} := \langle a|\cdot \rangle b + \langle b|\cdot \rangle a. \quad (2.6)$$

The Lie algebra $g(T)$ is called the standard enveloping algebra related to the symplectic triple system $T$. Moreover, the $\mathbb{Z}/2$-graded Lie algebra $g(T)$ is simple if and only if so is the symplectic triple system $(T, [\cdot, \cdot, \cdot], \langle \cdot | \cdot \rangle)$. Actually, the standard enveloping algebras of symplectic triple systems are $\mathbb{Z}/2$-graded Lie algebras of a very specific type.

We will need the next well-known result.

**Lemma 2.2** [32, Lemma 1.4] For any $i = 1, 2, 3$, take $X_i$ a trivial $L$-module and $U_i$ an irreducible $L$-module such that $\dim \text{Hom}_L(U_1 \otimes U_2, U_3) = 1$. Fix $p : U_1 \times U_2 \to U_3$ a non-zero $L$-invariant bilinear map. Then, for any $\varphi : (U_1 \otimes X_1) \times (U_2 \otimes X_2) \to (U_3 \otimes X_3)$ an $L$-invariant bilinear map, there exists a bilinear map $\eta : X_1 \times X_2 \to X_3$ such that $\varphi(u_1 \otimes x_1, u_2 \otimes x_2) = p(u_1, u_2)\eta(x_1, x_2)$.

In consequence, if $g = g_0 \oplus g_1$ is a $\mathbb{Z}/2$-graded Lie algebra such that there are a two-dimensional vector space endowed with a non-zero alternating bilinear form $(V, \langle \cdot | \cdot \rangle)$, an ideal $s$ of $g_0$, and an $s$-module $T$, such that

$$g_0 = \mathfrak{sp}(V, \langle \cdot | \cdot \rangle) \oplus s, \quad g_1 = V \otimes T \quad (2.7)$$

and the action of $g_0$ on $g_1$ is the natural one, then the invariance of the Lie bracket in $g$ under the $\mathfrak{sp}(V, \langle \cdot | \cdot \rangle)$-action provides the existence of

- an alternating bilinear form $\langle \cdot | \cdot \rangle : T \times T \to \mathbb{F}$, and
a symmetric bilinear map $d: T \times T \to \mathfrak{s}$,

such that (2.5) holds for any $x, y \in T$, $a, b \in V$. Under these assumptions, (see, for instance, the arguments in [16, Theorem 4.4] interpreting [34]), if $(\cdot|\cdot) \neq 0$, then $(T, [\cdot, \cdot, \cdot], (\cdot|\cdot))$ is a symplectic triple system for the triple product on $T$ defined by $[x, y, z] := d_{x,y}(z) \equiv d(x, y)z \in \mathfrak{s} T \subset T$. If $\mathfrak{g}$ is simple, then $T$ is a simple symplectic triple system with

$$\mathfrak{g} \cong \mathfrak{g}(T), \; \mathfrak{s} \cong \text{indet}(T).$$

From here it is not clear whether the pair $(\mathfrak{g}(T), \text{indet}(T))$ determines $T$ in some way.

If $(T, [\cdot, \cdot, \cdot], (\cdot|\cdot))$ is a symplectic triple system and $0 \neq \alpha \in \mathbb{F}$, then we can consider another symplectic triple system $T^{[\alpha]} := (T, [\cdot, \cdot, \cdot]^\alpha, (\cdot|\cdot)^\alpha)$ by means of

$$(x|y)^\alpha = \alpha(x|y); \quad [x, y, z]^\alpha = \alpha[x, y, z];$$

(2.8)

for any $x, y, z \in T$, which will be called the $\alpha$-shift of $T$. Observe that $\text{indet}(T^{[\alpha]}) = \text{indet}(T)$ and $\mathfrak{g}(T^{[\alpha]}) \cong \mathfrak{g}(T)$ because the inner derivations of $T^{[\alpha]}$ are trivially the maps $d_{x,y}^\alpha := [x, y, \cdot]^{\alpha} = \alpha d_{x,y}$.

It is now natural to introduce the notion of weak isomorphism:

**Definition 2.3** Two symplectic triple systems $T$ and $T'$ over a field $\mathbb{F}$ are said to be weakly isomorphic if there is a nonzero scalar $\alpha \in \mathbb{F}$ such that $T'$ is isomorphic to the $\alpha$-shift $T^{[\alpha]}$.

Note that the map $x \mapsto \beta x$ gives an isomorphism $T^{[\beta^2]} \to T$ for any $0 \neq \beta \in \mathbb{F}$. In particular, over any algebraically closed field $\mathbb{F}$ any shift of $T$ is isomorphic to $T$, while over the reals, any shift of $T$ is isomorphic either to $T$ or to $T^{[-1]}$. (But it is not clear whether $T$ and $T^{[-1]}$ are isomorphic.)

## 3 $\mathbb{Z}/2$-graded simple Lie algebras

The standard enveloping algebra of a symplectic triple system $T$ belongs to a very specific class of $\mathbb{Z}/2$-graded Lie algebras (see (2.7)). In this section, properties of $\mathbb{Z}/2$-graded Lie algebras will be exploited to ensure the existence, under certain conditions, of symplectic triple systems over non algebraically closed fields. (See Corollary 3.4.)

Our first result is well-known.

**Lemma 3.1** Let $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ be a $\mathbb{Z}/2$-graded finite-dimensional simple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0, with $\mathcal{L}_1 \neq 0$.

1. $\mathcal{L}_0$ is reductive and it acts faithfully and completely reducibly on $\mathcal{L}_1$, which is a sum of at most two irreducible $\mathcal{L}_0$-modules.
2. The center of $\mathcal{L}_0$: $Z(\mathcal{L}_0)$, is non-zero if and only if $\mathcal{L}_1$ is not irreducible as a module for $\mathcal{L}_0$. In this case $\mathcal{L}_1 = V \oplus W$ for irreducible $\mathcal{L}_0$-modules $V$ and $W$.
and the direct sum \( W \oplus L_0 \oplus V \) is a \( \mathbb{Z} \)-grading of \( L \), with \( L_{-1} = W \), \( L_0 = L_0 \), and \( L_1 = V \). Moreover, \( Z(L_0) = \mathbb{F} \mathbb{Z} \) with
\[
ad_z|_V = \text{id}, \quad \ad_z|_W = -\text{id}.
\]

(3) The space of symmetric, \( L_0 \)-invariant, bilinear forms \( L_1 \times L_1 \to \mathbb{F} \) has dimension one.

We include a proof, due to the lack of a suitable reference.

Note that the uniqueness, up to scalars, of invariant metrics, is important quite often in arguments in Differential Geometry.

**Proof** The adjoint representation restricts to a faithful representation \( \rho : L_0 \to \mathfrak{gl}(L) \). The subspaces \( L_0 \) and \( L_1 \) are orthogonal relative to the Killing form \( \kappa \) of \( L \). In particular, \( \kappa_0 = \kappa|_{L_0} \) is non-degenerate, and this is the trace form \( (x, y) \mapsto tr \rho(x)\rho(y) \).

By [13, p. 99], we conclude that \( L_0 = Z(L_0) \oplus \langle L_0, L_0 \rangle \), with \( Z(L_0) \) toral, \( [L_0, L_0] \) semisimple, and that \( L_1 \) is a completely reducible \( L_0 \)-module.

If \( \mathcal{V} \) is an \( L_0 \)-submodule of \( L_1 \) and \( [\mathcal{V}, \mathcal{V}] \neq 0 \), then \( [\mathcal{V}, \mathcal{V}] \oplus [\mathcal{V}, [\mathcal{V}, \mathcal{V}]] \) is an ideal of \( L \):

- It is \( L_0 \)-invariant.
- \( [L_1, [\mathcal{V}, [\mathcal{V}, \mathcal{V}]]] \subseteq [L_0, [\mathcal{V}, \mathcal{V}]] + [\mathcal{V}, [L_0, \mathcal{V}]] \subseteq [\mathcal{V}, \mathcal{V}] \).
- By complete reducibility, \( L_1 = \mathcal{V} \oplus \mathcal{W} \) for some \( L_0 \)-submodule \( \mathcal{W} \), and hence \( [L_1, [\mathcal{V}, \mathcal{V}]] = [\mathcal{V}, [\mathcal{V}, \mathcal{V}]] + [\mathcal{W}, [\mathcal{V}, \mathcal{V}]] \).

As \( L \) is simple, it follows that \( L = [\mathcal{V}, \mathcal{V}] \oplus [\mathcal{V}, [\mathcal{V}, \mathcal{V}]] \), and in particular \( L_0 = [\mathcal{V}, \mathcal{V}] \) and \( L_1 = \mathcal{V} \).

Now, if \( L_1 = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \) for non-zero \( L_0 \)-modules \( \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \), then the above shows that \( [\mathcal{V}_i, \mathcal{V}_j, \mathcal{V}_i \oplus \mathcal{V}_j] = 0 \) for any \( i \neq j \), but this gives \( [L_1, L_1] = 0 \), a contradiction. This proves the first assertion.

If \( L_1 \) is an irreducible \( L_0 \)-module, then Schur’s Lemma shows that any \( 0 \neq z \in Z(L_0) \) satisfies \( \ad_z|_{L_1} = \alpha \text{id} \) for some \( 0 \neq \alpha \in \mathbb{F} \). Hence we get \( 0 = \ad_z|_{L_0} = \ad_z|_{L_1, L_1} = 2\alpha \text{id} \), a contradiction. Hence the center of \( L_0 \) is trivial if \( L_1 \) is irreducible.

Otherwise, \( L_1 = \mathcal{V} \oplus \mathcal{W} \) for irreducible \( L_0 \)-modules \( \mathcal{V} \) and \( \mathcal{W} \). By the arguments above, \( [\mathcal{V}, \mathcal{W}] = 0 = [\mathcal{W}, \mathcal{W}] \), so that \( L_0 = [L_1, L_1] = [\mathcal{V}, \mathcal{W}] \), and the decomposition \( \mathcal{V} \oplus L_0 \oplus \mathcal{W} \) is a \( \mathbb{Z} \)-grading of \( L \) such that \( \mathcal{V}, L_0 \) and \( \mathcal{W} \) are the homogeneous components of degrees 1, 0 and -1 respectively. Thus the endomorphism \( d \) that acts trivially on \( L_0 \), as the identity on \( \mathcal{V} \), and as minus the identity on \( \mathcal{W} \), is a derivation of \( L \). But \( L \) is simple, so \( d \) is inner, and hence there exists an element \( z \in L \) with \( d = \ad_z \). It follows that \( z \in Z(L_0) \). Schur’s Lemma shows that \( Z(L_0) = \mathbb{F} \mathbb{Z} \). This proves part (2).

As \( \kappa(L_0, L_1) = 0 \), the restriction \( \kappa|_{L_1} : L_1 \times L_1 \to \mathbb{F} \) is a non-degenerate symmetric \( L_0 \)-invariant bilinear form and, in particular, \( L_1 \) is a self-dual \( L_0 \)-module. If
Corollary 3.2 Let $\mathcal{L}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic 0, let $\rho: \mathcal{L} \to \mathfrak{gl}(\mathcal{V})$ be a finite-dimensional, faithful, and completely reducible, representation of $\mathcal{L}$. Then, up to scalars, there is at most one $\mathcal{L}$-invariant bilinear map $\mu: \mathcal{V} \times \mathcal{V} \to \mathcal{L}$ such that $\mathcal{L} \oplus \mathcal{V}$, with multiplication given by:

$$[x + v, y + w] = ([x, y] + \mu(v, w)) + (\rho_x(w) - \rho_y(v))$$

for $x, y \in \mathcal{L}$ and $v, w \in \mathcal{V}$, is a central simple Lie algebra.

**Proof** If $\mu$ is such a map, $\mathfrak{g} = \mathcal{L} \oplus \mathcal{V}$ is a form of a simple $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g}_{\mathbb{F}} = \mathfrak{g} \otimes_{\mathbb{F}} \mathbb{F}$, where $\mathbb{F}$ is an algebraic closure of $\mathbb{F}$. Let $\kappa$ be the Killing form of $\mathfrak{g}$. For any $x, y \in \mathcal{L}$, $\kappa(x, y) = \text{tr}(\text{ad}_x \text{ad}_y) + \text{tr}(\rho_x \rho_y)$ depends only on $\mathcal{L}$ and $\rho$, so it gives a fixed non-degenerate invariant symmetric bilinear form on $\mathcal{L}$.

Note also that the restriction of $\kappa$ to $\mathcal{V}$ is a symmetric, non-degenerate, $\mathcal{L}$-invariant, bilinear form on $\mathcal{V}$. By Lemma 3.1, there is, up to scalars, a unique non-zero symmetric, $\mathcal{L}$-invariant, bilinear form $b: \mathcal{V} \times \mathcal{V} \to \mathbb{F}$, and hence there is a non-zero $\alpha \in \mathbb{F}$ such that $\kappa|_{\mathcal{V}} = \alpha b$.

But for any $v, w \in \mathcal{V}$ and $x \in \mathcal{L}$, we have

$$\kappa|_{\mathcal{L}}(x, \mu(v, w)) = \kappa|_{\mathcal{V}}(\rho_x(v), w) = \alpha b(\rho_x(v), w).$$

(3.2)

The non-degeneracy of $\kappa|_{\mathcal{L}}$ shows that $\mu$ is unique up to scalars. \qed

**Remark 3.3** In the conditions of Corollary 3.2, if there is an $\mathcal{L} \otimes_{\mathbb{F}} \mathbb{F}$-invariant bilinear map $\nu: (\mathcal{V} \otimes_{\mathbb{F}} \mathbb{F}) \times (\mathcal{V} \otimes_{\mathbb{F}} \mathbb{F}) \to \mathcal{L} \otimes_{\mathbb{F}} \mathbb{F}$ making $(\mathcal{L} \oplus \mathcal{V}) \otimes_{\mathbb{F}} \mathbb{F}$ a simple Lie algebra with the bracket as in (3.1), then the same is valid for $\mathcal{L}$ and $\mathcal{V}$.

**Proof** The existence of such a bilinear map $\nu$ forces that the space of symmetric $\mathcal{L}$-invariant bilinear forms on $\mathcal{V}$ is one-dimensional by item (3) in Lemma 3.1, since this dimension remains invariant under scalar extension. Also, since the dimension of $\text{Hom}(\mathcal{L} \otimes_{\mathbb{F}} \mathcal{V}, \mathcal{L})$ remains invariant too under scalar extension, this dimension is 1 by Corollary 3.2, and hence there is a bilinear $\mathcal{L}$-invariant map $\mu: \mathcal{V} \times \mathcal{V} \to \mathcal{L}$ such that its complexification is a scalar multiple of $\nu$. The Jacobi identity for the bracket in $(\mathcal{L} \oplus \mathcal{V}) \otimes_{\mathbb{F}} \mathbb{F}$ given by (3.1) for the map $\nu$ is also true for any scalar multiple of $\nu$. Therefore, the Jacobi identity for the bracket in $\mathcal{L} \oplus \mathcal{V}$ given by (3.1) follows from the Jacobi identity in $(\mathcal{L} \oplus \mathcal{V}) \otimes_{\mathbb{F}} \mathbb{F}$ using the complexification of $\mu$. \qed
Corollary 3.4. Let $T$ be a simple symplectic triple system, over an algebraic closure $\overline{F}$ of the field $F$ of characteristic 0, with inner derivation algebra $\text{ind}_F(T)$. Let $s$ be a form of $\text{ind}_F(T)$ and let $S$ be a module for $s$ such that $S_\overline{F} := S \otimes_F \overline{F}$ is isomorphic to $T$ as a module for $s \otimes_F \overline{F} \simeq \text{ind}_F(T)$. Then, up to scalars, there is a unique non-degenerate alternating bilinear form $S \times S \to \overline{F}$: $(x, y) \mapsto (x, y)$, and a unique triple product $S \times S \times S \to S$: $(x, y, z) \mapsto [x, y, z]$, that makes $S$ a symplectic triple system, with inner derivation algebra equal to the image in $\mathfrak{gl}(S)$ of $s$.

Proof. Let $V$ be a two-dimensional vector space over $\overline{F}$ endowed with a non-zero alternating bilinear form $\langle u | v \rangle$. Consider the Lie algebra $L = \mathfrak{sl}(V) \oplus s$, and its module $V = V \otimes_F S$. By Corollary 3.2 and Remark 3.3 there is a unique, $L$-invariant bilinear map $\mu: V \times V \to L$, up to scalars, making $L \oplus V$ into a central simple Lie algebra with bracket given by (3.1).

Recall that $\text{Hom}_{\mathfrak{sl}(V)}(V \otimes V, \overline{F}) = \overline{F} \langle \cdot | \cdot \rangle$ and that $\text{Hom}_{\mathfrak{sl}(V)}(V \otimes V, \mathfrak{sl}(V)) = \overline{F} \gamma$, with $\gamma(u \otimes v) = \gamma_{u,v}$ the zero trace map defined by $\gamma_{u,v}(w) = \langle u | w \rangle v + \langle v | w \rangle u$. This fact, together with the $\mathfrak{sl}(V)$-invariance of $\mu$, implies that this unique product has the form $\mu(u \otimes x, v \otimes y) = (x|y)\gamma_{u,v} + \langle u | v \rangle dx$, for an alternating bilinear form $(x|y)$ and a bilinear map $S \times S \to s$, $(x, y) \mapsto d_{x,y}$, both being completely determined by $\mu$.

Now it is enough to define $[x, y, z] = d_{x,y}(z)$ (the action of $d_{x,y} \in s$ on the element $z \in S$). \qed

The above corollary is a key tool for the classification of the real simple symplectic triple systems. We simply have to find, for each complex simple symplectic triple system $T$, the real forms of the Lie algebra $\text{ind}_F(T)$ which admit a representation whose complexification is $T$, and in such case the triple product is defined up to scalars. In other words, we look for real forms of the pairs $(\text{ind}_F(T), T)$, for $T$ any simple symplectic triple system over $\mathbb{C}$. Here a pair $(\mathfrak{h}, U)$ consisting of a real Lie algebra $\mathfrak{h}$ and an $\mathfrak{h}$-module $U$, with corresponding representation $\rho: \mathfrak{h} \to \mathfrak{gl}(U)$, is said to be a real form of the pair $(\mathfrak{g}, V)$ for a complex Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module $V$, with representation $\mu: \mathfrak{g} \to \mathfrak{gl}(V)$, if there is an isomorphism of complex Lie algebras $\varphi: \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{g}$, and a linear isomorphism of complex vector spaces $\Phi: U \otimes_{\mathbb{R}} \mathbb{C} \to V$ such that

$$\Phi(\rho_x(u) \otimes \alpha) = \alpha \mu_{\varphi(x \otimes 1)}(\Phi(u \otimes 1))$$

for any $x \in \mathfrak{h}$, $u \in U$ and $\alpha \in \mathbb{C}$.

To summarize, the uniqueness only up to scalars of the triple product in Corollary 3.4 tells us that the lists of the real forms in Sect. 6 of the pairs $(\text{ind}_F(T), T)$, for the simple symplectic triple systems over $\mathbb{C}$, complete the classification of the simple real symplectic triple systems up to weak isomorphism. Finally, in Sect. 7 it will be proven that two simple weakly isomorphic real symplectic triple systems are actually isomorphic. (This is not trivial at all!)
4 The split simple symplectic triple systems

The uniqueness in Corollary 3.4 shows that the classification of the simple symplectic triple systems over an algebraically closed field $\mathbb{F}$ of characteristic 0 is equivalent to the classification of order 2 automorphisms (i.e., gradings by $\mathbb{Z}/2$) of the simple Lie algebras over $\mathbb{F}$ such that the even part is the direct sum of a copy of $\mathfrak{sl}_2$ and a reductive Lie algebra $\mathfrak{s}$, and the odd part is, as a module for the even part, the tensor product of the two-dimensional module for $\mathfrak{sl}_2$ and a module $S$ for $\mathfrak{s}$. The possible pairs $(\mathfrak{s}, S)$, up to isomorphism, can be read from the classification of finite order automorphisms (see, e.g., [23, Chapter 8] and [14]), where two pairs $(\mathfrak{s}, S)$ and $(\mathfrak{s}', S')$ are isomorphic if and only if there is an isomorphism $\phi: \mathfrak{s} \rightarrow \mathfrak{s}'$ of Lie algebras and a linear isomorphism $\psi: S \rightarrow S'$ such that $\psi(s \cdot x) = \phi(s) \cdot \psi(x)$ for all $s \in \mathfrak{s}$ and $x \in S$.

The complete list of these pairs, up to isomorphism, is given in the following list, whose items are labeled according to the type of the envelope:

- **Special** $(\mathfrak{gl}(W), W \oplus W^*)$, for a non-zero vector space $W$.
- **Orthogonal** $(\mathfrak{sp}(V) \oplus \mathfrak{so}(W), V \otimes W)$, where $V$ is a vector space of dimension 2 endowed with a non-zero alternating bilinear form $\langle \cdot | \cdot \rangle$, and $W$ is a vector space of dimension $\geq 3$ endowed with a non-degenerate symmetric bilinear form.
- **Symplectic** $(\mathfrak{sp}(W), W)$ for a non-zero even-dimensional vector space $W$ endowed with a non-degenerate alternating bilinear form.
- **$G_2$-envelope** $(a_1, V(3\varpi_1))$.
- **$F_4$-envelope** $(c_3, V(\varpi_3))$.
- **$E_6$-envelope** $(a_5, V(\varpi_3))$.
- **$E_7$-envelope** $(d_6, V(\varpi_6))$.
- **$E_8$-envelope** $(e_7, V(\varpi_1))$.

In this list, the simple Lie algebra over $\mathbb{F}$ of type $L_n$ is denoted $l_n$, a Cartan subalgebra and system $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots are fixed, with the ordering used in [31], $\varpi_1, \ldots, \varpi_n$ are the corresponding fundamental weights, and for any dominant weight $\Lambda$, $V(\Lambda)$ denotes the irreducible module of highest weight $\Lambda$.

For each case, let us describe explicitly the triple product and the alternating bilinear form. Actually, this works over an arbitrary field of characteristic 0, thus providing the list of what will be called the split simple symplectic triple systems. Thus, in what follows, $\mathbb{F}$ will denote an arbitrary field of characteristic 0, $V$ will denote a two-dimensional vector space endowed with a non-zero alternating bilinear form $\langle u | v \rangle$. Details on the classical cases can be found in [14, Examples 2.26], while for the exceptional cases, instead of the models based on Jordan algebras in [14], new models based on Linear Algebra will be given.

4.1 Special type

Let $W$ be a non-zero finite-dimensional vector space over our ground field $\mathbb{F}$, and let $T$ be the direct sum of $W$ and its dual: $T = W \oplus W^*$. Then $(T, [\cdot, \cdot, \cdot], (\cdot | \cdot))$ is a
simple symplectic triple system with
\[
[x, f, y] = f(x)y + 2f(y)x, \quad [f, x, g] := -f(x)g - 2g(x)f, \\
[x, y, ·] = 0 = [f, g, ·], \\
(f|x) = -(x|f) = f(x), \quad (x|y) = 0 = (f|g)
\] (4.1)

for any \(x, y \in W\) and \(f, g \in W^*\).

The inner derivation algebra and the standard enveloping algebra are the following:
\[
\text{inder}(T) \cong \mathfrak{gl}(W), \quad \mathfrak{g}(T) \cong \mathfrak{sl}(V \oplus W).
\]

Actually, this symplectic triple system is related to the natural grading by \(\mathbb{Z}/2\) of \(\mathfrak{sl}(V \oplus W)\) obtained by splitting \(\mathfrak{gl}(V \oplus W)\) into blocks corresponding to \(V\) and \(W\). The even part consists of the block diagonal endomorphisms, and the odd part of the off block diagonal endomorphisms.

4.2 Orthogonal type

Let \((W, b)\) be a vector space of dimension at least 3 endowed with a non-degenerate symmetric bilinear form \(b\). Consider the vector space \(T = V \otimes W\). Then \((T, [·, ·, ·], (·|·))\) is a simple symplectic triple system with
\[
[u \otimes x, v \otimes y, w \otimes z] = \frac{1}{2}((u|w)v + (v|w)u) \otimes b(x, y)z \\
+ (u|v)w \otimes (b(x, z)y - b(y, z)x),
\] (4.2)

for any \(u, v, w \in V\) and \(x, y, z \in W\).

Moreover, the inner derivation algebra and the standard enveloping algebra are
\[
\text{inder}(T) \cong \mathfrak{sl}(V) \oplus \mathfrak{so}(W, b), \quad \mathfrak{g}(T) \cong \mathfrak{so}(V \otimes V \oplus W, \tilde{b}).
\]

Here \(\tilde{b}\) is the non-degenerate symmetric bilinear form on the direct sum \((V \otimes V) \oplus W\) that restricts to \(b\) on \(W\), satisfies \(\tilde{b}(V \otimes V, W) = 0\), and restricts to
\[
(u_1 \otimes v_1, u_2 \otimes v_2) \mapsto (u_1|u_2)(v_1|v_2)
\]
on \(V \otimes V\). Again, this corresponds to the natural \(\mathbb{Z}/2\)-grading on \(\mathfrak{so}(V \otimes V \perp W, \tilde{b})\) that is associated to the orthogonal sum (relative to \(\tilde{b}\)) \((V \otimes V) \perp W\).

4.3 Symplectic type

Let \(T\) be an even-dimensional vector space endowed with a non-degenerate alternating bilinear form \((·|·)\). Then \((T, [·, ·, ·], (·|·))\) is a simple symplectic triple system with the
triple product given by

\[ [x, y, z] = (x|z)y + (y|z)x, \] (4.3)

for \( x, y, z \in T \).

Moreover, the inner derivation algebra and the standard enveloping algebra are

\[ \text{inderv}(T) \cong \text{sp}(T, \langle \cdot | \cdot \rangle), \quad g(T) \cong \text{sp}(V \oplus T, \langle \cdot | \cdot \rangle'). \]

Here \( \langle \cdot | \cdot \rangle' \) is the non-degenerate alternating bilinear form on the direct sum \( V \oplus T \) that restricts to \( \langle \cdot | \cdot \rangle \) on \( V \), to \( \langle \cdot | \cdot \rangle \) on \( T \), and that satisfies \( V|T)' = 0 \). This corresponds to the natural \( \mathbb{Z}/2 \)-grading on \( \text{sp}(V \perp W, \langle \cdot | \cdot \rangle') \) that is associated to the orthogonal sum \( V \perp W \).

### 4.4 \( G_2 \)-type

Denote by \( V_n \) the vector space of the homogeneous polynomials of degree \( n \) in two variables \( X \) and \( Y \). For any \( f \in V_n, g \in V_m \), consider the transvection

\[ (f, g)_q = \frac{(n-q)!}{n!} \frac{(m-q)!}{m!} \sum_{i=0}^{q} (-1)^i \binom{q}{i} \frac{\partial^q f}{\partial X^{q-i} Y^i} \frac{\partial^q g}{\partial X^i Y^{q-i}} \in V_{m+n-2q}. \]

Then the vector space \( T = V_3 \) of the homogeneous polynomials of degree 3, endowed with the alternating form and the triple product given by:

\[ (f|g) = (f, g)_3, \quad [f, g, h] = 6((f,g)_2, h)_1. \] (4.4)

for any \( f, g, h \in T \), is a simple symplectic triple system, whose inner derivation algebra is isomorphic to \( \text{sl}_2(\mathbb{F}) \) and its standard enveloping Lie algebra is isomorphic to the split simple Lie algebra of type \( G_2 \).

There is a misprint in [14, p. 210] so, for the convenience of the reader, we will justify the assertions above.

The key is to use the classical Gordan identities for transvections [21, p. 56–57]. For \( f \in V_m, g \in V_n, h \in V_p \) and \( \alpha_1, \alpha_2, \alpha_3 \) non-negative integers such that \( \alpha_1 + \alpha_2 \leq p \), \( \alpha_2 + \alpha_3 \leq m \), \( \alpha_3 + \alpha_1 \leq n \), and such that either \( \alpha_1 = 0 \) or \( \alpha_2 + \alpha_3 = m \), one has

\[ \sum_{i \geq 0} \binom{n-\alpha_1-\alpha_3}{m+n-2\alpha_3-i+1} \binom{\alpha_2}{i} ((f, g)_{\alpha_3+i}, h)_{\alpha_1+\alpha_2-i} \]

\[ = (-1)^{\alpha_1} \sum_{i \geq 0} \binom{p-\alpha_1-\alpha_2}{m+p-2\alpha_2-i+1} \binom{\alpha_3}{i} ((f, h)_{\alpha_2+i}, g)_{\alpha_1+\alpha_3-i}. \]

This identity is usually denoted by \( \left( \begin{array}{ccc} f & g & h \\ m & n & p \\ \alpha_1 & \alpha_2 & \alpha_3 \end{array} \right) \).
Gordan identity \( \begin{pmatrix} f & g & h \\ 3 & 3 & 3 \\ \hline 0 & 1 & 2 \end{pmatrix} \) gives
\[
((f, g)_{2}, h)_{1} + \frac{1}{2}((f, g)_{3}h) = ((f, h)_{1}, g)_{2} + ((f, h)_{2}, g)_{1} + \frac{1}{3}(f, h)_{3}g. \quad (4.5)
\]

Replace \( f \) with \( h \) in (4.5), and sum both identities to get
\[
((f, g)_{2}, h)_{1} + ((h, g)_{2}, f)_{1} + \frac{1}{2}((f, g)_{3}h + (h, g)_{3}f) = 2((f, h)_{2}, g)_{1}. \quad (4.6)
\]

Replace \( f \) with \( g \) in (4.6), and subtract both identities to get
\[
2(f, g)_{3}h + (h, g)_{3}f - (h, f)_{3}g = 6 (((f, h)_{2}, g)_{1} - (g, h)_{2}, f)_{1}). \quad (4.7)
\]

Now, the only scalar \( \alpha \) such that the triple product \( [f, g, h] := \alpha((f, g)_{2}, h)_{1} \) satisfies the identities of a symplectic triple system in Definition 2.1 is \( \alpha = 0 \).

Over an algebraically closed field, this simple symplectic triple system corresponds to the unique order two automorphism, up to conjugacy, of the simple Lie algebra of type \( G_{2} \).

### 4.5 \( F_{4} \)-type

Let \( W \) be a six-dimensional vector space, endowed with a non-degenerate alternating bilinear form \( b_{a} \). Let \( \{u_{1}, u_{2}, u_{3}, v_{1}, v_{2}, v_{3}\} \) be a symplectic basis: \( b_{a}(u_{i}, v_{j}) = -b_{a}(v_{j}, u_{i}) = 1 \), and all the other values of \( b_{a} \) on basic elements are 0. Consider the corresponding symplectic Lie algebra \( \text{sp}(W, b_{a}) \), which is spanned by the endomorphisms \( \gamma_{u, v}: w \mapsto b_{a}(u, w)v + b_{a}(v, w)u \). The subalgebra \( h \) of diagonal elements, relative to the basis above, is a Cartan subalgebra, and the root system is \( \Phi = \{ \pm \epsilon_{i} \pm \epsilon_{j}, \pm 2 \epsilon_{i} : 1 \leq i \neq j \leq 3 \} \), where \( \epsilon_{i} \) denotes the weight of \( u_{i}, i = 1, 2, 3 \), under the natural action. A system of simple roots is given by \( \Pi = \{ \epsilon_{1} - \epsilon_{2}, \epsilon_{2} - \epsilon_{3}, 2\epsilon_{3} \} \).

Let \( T \) be the kernel of the \( \text{sp}(W, b_{a}) \)-invariant map
\[
\bigwedge^{3} W \to W, \quad x_{1} \wedge x_{2} \wedge x_{3} \mapsto b_{a}(x_{1}, x_{2})x_{3} + b_{a}(x_{2}, x_{3})x_{1} + b_{a}(x_{3}, x_{1})x_{2}. \quad (4.8)
\]

The dimension of \( T \) is \( \binom{6}{3} - 6 = 14 \), and \( T \) is the irreducible module for \( \text{sp}(W, b_{a}) \) with highest weight \( \varpi_{3} = \epsilon_{1} + \epsilon_{2} + \epsilon_{3} \). The element \( u_{1} \wedge u_{2} \wedge u_{3} \) lies in \( T_{\varpi_{3}} \).

Because of Corollary 3.4, up to scalars, there are a unique \( \text{sp}(W, b_{a}) \)-invariant alternating form \( (\cdot | \cdot) \) on \( T \), and a bilinear map \( T \times T \to \text{sp}(W, b_{a}), (x, y) \mapsto d_{x, y} \), such that \( (T, [\cdot, \cdot], (\cdot | \cdot)) \) is a simple symplectic triple system, for \([x, y, z] = d_{x, y, z} \), that corresponds to the pair \((\epsilon_{3}, V(\varpi_{3})) \) \((F_{4}\text{-envelope})\).

Let us describe explicitly both \([\cdot, \cdot, \cdot]\) and \((\cdot | \cdot)\).

Note that the weights of \( W \) are \( \pm \epsilon_{i}, 1 \leq i \leq 3 \); and the weights of \( \bigwedge^{3} W \) are \( \pm \epsilon_{1} \pm \epsilon_{2} \pm \epsilon_{3}, \) with multiplicity 1, and \( \pm \epsilon_{i}, i = 1, 2, 3, \) with multiplicity 2. Hence the weights of \( T \) are \( \pm \epsilon_{1} \pm \epsilon_{2} \pm \epsilon_{3} \) and \( \pm \epsilon_{i}, i = 1, 2, 3, \) all with multiplicity 1.
The subspace $\mathfrak{sp}(W, b_a) \otimes T$ is generated, as a module for $\mathfrak{sp}(W, b_a)$, by
\[ \gamma_{u_1, u_2} \otimes (v_1 \wedge v_2 \wedge v_3), \]
which is the tensor product of a highest root vector: $\gamma_{u_1, u_2} \in \mathfrak{sp}(W, b_a)_{\epsilon_1}$, and a lowest weight vector: $v_1 \wedge v_2 \wedge v_3 \in T_{-\epsilon_3}$, in the ordering imposed by $\Pi$. The image of $\gamma_{u_1, u_2} \otimes (v_1 \wedge v_2 \wedge v_3)$ under any $\mathfrak{sp}(W, b_a)$-invariant linear map to $T$ lies in the weight space $T_{\epsilon_1-\epsilon_2-\epsilon_3}$, which is one-dimensional. We conclude that $\text{Hom}_{\mathfrak{sp}(W, b_a)}(\mathfrak{sp}(W, b_a) \otimes T, T)$ is one-dimensional, and hence spanned by the action of $\mathfrak{sp}(W, b_a)$ on $T$.

Fix the determinant map $\det: \bigwedge^6 W \to \mathbb{F}$ given by
\[ \det(u_1 \wedge u_2 \wedge u_3 \wedge v_1 \wedge v_2 \wedge v_3) = 1. \]

This map $\det$ determines an alternating bilinear form on $\bigwedge^3 W$:
\[ (x|y) = \det(x \wedge y), \quad (4.9) \]
which restricts to a non-degenerate alternating $\mathfrak{sp}(W, b_a)$-invariant bilinear map on $T$, also denoted by $(\cdot|\cdot)$. In particular, $T$ is self-dual and $\text{Hom}_{\mathfrak{sp}(W, b_a)}(T \otimes T, \mathbb{F})$ is spanned by $(\cdot|\cdot)$ by Schur’s Lemma.

Because of the self-duality as $\mathfrak{sp}(W, b_a)$-modules of both $T$ and $\mathfrak{sp}(W, b_a)$, and since the dimension of $\text{Hom}_{\mathfrak{sp}(W, b_a)}(\mathfrak{sp}(W, b_a) \otimes T, T)$ is 1, so is the dimension of $\text{Hom}_{\mathfrak{sp}(W, b_a)}(T \otimes T, \mathfrak{sp}(W, b_a))$. (Note that for a Lie algebra $\mathcal{L}$ and $\mathcal{L}$-modules $U$ and $V$, there are isomorphisms $\text{Hom}_{\mathcal{L}}(U, V) \cong (U^* \otimes V)^{\mathcal{L}} \cong \text{Hom}_{\mathcal{L}}(U \otimes V^*, \mathbb{F}) \cong \text{Hom}_{\mathcal{L}}(V^*, U^*)$, where $U^{\mathcal{L}} = \{ u \in U : x.u = 0 \forall x \in \mathcal{L} \}$.)

Up to scalars, the unique $\mathfrak{sp}(W, b_a)$-invariant linear map $T \otimes T \to \mathfrak{sp}(W, b_a)$, $x \otimes y \mapsto d_{x,y}$, is given by the formula
\[ \text{tr}(fd_{x,y}) = -2(f.x|y) \quad (4.10) \]
for any $f \in \mathfrak{sp}(W, b_a)$, where $f.x$ denotes the action of $f \in \mathfrak{sp}(W, b_a)$ on $x \in T$.

Consider the triple product $[x, y, z] = d_{x,y}z$ and the alternating form $(\cdot|\cdot)$ on $T$. By the uniqueness, up to scalars, given by Corollary 3.4, there is a scalar $0 \neq \alpha \in \mathbb{F}$ such that
\[ d_{x,y}z - d_{x,z}y = \alpha((x|z)y - (x|y)z + 2(y|z)x). \quad (4.11) \]
If we prove that $\alpha = 1$, then we will have proved that $(T, [\cdot, \cdot, \cdot], (\cdot|\cdot))$ is a symplectic triple system.

For $x = u_1 \wedge u_2 \wedge u_3 = y$, and $z = v_1 \wedge v_2 \wedge v_3$, which belong to $T$, we have $(x|y) = 0$, and $d_{x,y} = 0$, since $2(\epsilon_1 + \epsilon_2 + \epsilon_3) \notin \Phi$. Besides $d_{x,z}$ belongs to the weight space $\mathfrak{sp}(W, b_a)_0$, so $d_{x,z} = \sum_{i=1}^3 a_i \gamma_{u_i, v_i}$, for some scalars $a_i \in \mathbb{F}$. Note that $\gamma_{u_1, v_1}$ takes $u_1$ to $-u_1$, leaves $v_1$ fixed and vanishes on the remaining basic elements. Hence
\[ \text{tr}(\gamma_{u_1, v_1}d_{x,z}) = a_1 \text{tr}(\gamma_{u_1, v_1}^2) = 2a_1 \]
is equal to
\[ -2(\gamma_{u_1,v_1},x|z) = -2(\gamma_{u_1,v_1},(u_1 \wedge u_2 \wedge u_3)|v_1 \wedge v_2 \wedge v_3) \\
= -2(-u_1 \wedge u_2 \wedge u_3|v_1 \wedge v_2 \wedge v_3) = 2. \]

We conclude that \( \alpha_1 = 1 \), and analogously \( \alpha_2 = \alpha_3 = 1 \). Thus we get
\[ -d_{x,z}y = -\left(\sum_{i=1}^{3} \gamma_{u_i,v_i}\right)(u_1 \wedge u_2 \wedge u_3) = -(-1 - 1 - 1)(u_1 \wedge u_2 \wedge u_3) = 3y, \]
while \((x|z)y - (x|y)z + 2(y|z)x = y + 0 + 2x = 3y\) too. This shows that indeed \( \alpha = 1 \).

To summarize, \((T, [,\cdot,\cdot], (\cdot|\cdot))\) is a simple symplectic triple system, for \( T \) the kernel of the map in (4.8), the alternating form \((\cdot|\cdot)\) defined in (4.9) and the triple product \([x, y, z] = d_{x,y}z\) for \( d_{x,y} \) defined by (4.10). Its inner derivation algebra is isomorphic to \( \mathfrak{sp}(W, b_\lambda) \cong \mathfrak{sp}_6(\mathbb{F}) \) and its standard enveloping algebra is isomorphic to the split exceptional simple Lie algebra of type \( F_4 \).

### 4.6 \( E_6 \)-type

This case has been considered in [18, Lemma 6.45]. For the benefit of the reader, we give the details.

As for the \( F_4 \)-type, let \( W \) be a vector space of dimension 6. Fix \( \{e_1, e_2, e_3, e_4, e_5, e_6\} \) a basis of \( W \). The subalgebra \( \mathfrak{h} \) of diagonal elements of \( \mathfrak{sl}(W) \), relative to this basis, is a Cartan subalgebra, with root system \( \Phi = \{\pm(e_i - e_j) : 1 \leq i < j \leq 6\} \). Here \( e_i \) denotes the weight of \( e_i \) under the natural \( \mathfrak{sl}(W) \)-action on \( W \), for any \( i = 1, \ldots, 6 \). A system of simple roots of \( \Phi \) is given by \( \Pi = \{e_i - e_{i+1} : 1 \leq i \leq 5\} \).

The space \( T = \bigwedge^3 W \) is the irreducible module for \( \mathfrak{sl}(W) \) with highest weight \( \varpi_3 = \epsilon_1 + \epsilon_2 + \epsilon_3 \). The element \( e_{123} := e_1 \wedge e_2 \wedge e_3 \) lies in the weight space \( T_{\varpi_3} \).

We will use the notation \( e_{i_1 \ldots i_r} = e_{i_1} \wedge \cdots \wedge e_{i_r} \) throughout.

Fix the determinant map \( \det : \bigwedge^6 W \rightarrow \mathbb{F} \) given by \( \det(e_{123456}) = 1 \) and, as in (4.9), consider the non-degenerate alternating bilinear form \((\cdot|\cdot)\) on \( T = \bigwedge^3 W \) given by
\[ (x|y) = \det(x \wedge y). \quad (4.12) \]

This is the unique, up to scalars, \( \mathfrak{sl}(W) \)-invariant bilinear form on \( T \).

With the same arguments as for the \( F_4 \)-type, the unique, up to scalars, \( \mathfrak{sl}(W) \)-invariant linear map \( T \otimes T \rightarrow \mathfrak{sl}(W) : x \otimes y \mapsto d_{x,y} \), is given by the formula
\[ \text{tr}(fd_{x,y}) = -2(f.x|y). \quad (4.13) \]
(There is a misprint in [18, Lemma 6.45], where 24 appears instead of 2 in the formula above.)
Again, as for the $F_4$-type, there is a scalar $0 \neq \alpha \in \mathbb{F}$ such that (4.11) holds. For $x = e_{123} = y$ and $z = e_{456}$ we have $d_{x,y} = 0$, and $d_{x,z}$ is the endomorphism of $W$ with coordinate matrix (in the given basis) diag$((-1, -1, -1, 1, 1, 1)$ due to (4.13), so that $d_{x,z} y = -3y$. Also $(x|y) = 0$ and $(x|z) = 1 = (y|z)$. It follows that $\alpha = 1$.

Therefore $\left( T = \bigwedge^3 W, [\cdot, \cdot, \cdot], (\cdot|\cdot) \right)$ is a simple symplectic triple system, with inner derivation algebra isomorphic to $\mathfrak{sl}(W) \cong \mathfrak{sl}_6(\mathbb{F})$ and standard enveloping algebra isomorphic to the split exceptional simple Lie algebra of type $E_6$.

### 4.7 $E_7$-type

Let $W$ be a six-dimensional vector space and denote by $W^*$ its dual. Fix a basis $\{e_i : 1 \leq i \leq 6\}$ of $W$, and consider its dual basis $\{e^i : 1 \leq i \leq 6\}$ in $W^*$. The direct sum $W \oplus W^*$ is endowed with the non-degenerate quadratic form given by $q(u + f) = f(u)$ for all $u \in W$ and $f \in W^*$. The corresponding orthogonal Lie algebra $\mathfrak{so}(W \oplus W^*, q)$ is the split simple Lie algebra of type $D_6$. The subalgebra $\mathfrak{h}$ of diagonal elements, relative to the basis $\{e_1, \ldots, e_6, e^1, \ldots, e^6\}$ of $W \oplus W^*$, is a Cartan subalgebra. The corresponding root system is $\Phi = \{ \pm e_i \pm e_j : 1 \leq i < j \leq 6\}$, where $e_i$ is the weight of $e_i$ in the natural action of $\mathfrak{so}(W \oplus W^*, q)$ on $W \oplus W^*$. A system of simple roots is $\Pi = \{ e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5, e_5 - e_6, e_5 + e_6\}$. The fundamental dominant weight $\sigma_6$, relative to this system, is $\sigma_6 = \frac{1}{2}(e_1 + \cdots + e_6)$.

On the exterior algebra $\bigwedge W$ write, as above, $e_{i_1 \cdots i_r} = e_{i_1} \wedge \cdots \wedge e_{i_r}$ if each $1 \leq i_j \leq 6$, and shorten it to $e_I = e_{i_1 \cdots i_r}$ for any sequence $I = (i_1, \ldots, i_r)$ with $1 \leq i_1 < \cdots < i_r \leq 6$.

Consider the linear form det: $\bigwedge W \to \mathbb{F}$ which is trivial on $\bigwedge^I W$ for $i < 6$ and such that $\det(e_{123456}) = 1$. Also, let $s \mapsto \hat{s}$ be the involution of $\bigwedge W$ which is the identity on $W = \bigwedge^1 W$, and hence, for $I = (i_1, \ldots, i_r)$ with $1 \leq i_1 < \cdots < i_r \leq 6$, $\hat{s}^I = e_{i_r \cdots i_1} = (-1)^{i_1} e_I$.

Finally, consider the non-degenerate bilinear form

$$
\begin{align*}
b_a : \bigwedge W \times \bigwedge W & \to \mathbb{F} \\
(s, t) & \mapsto \det(\hat{s} \wedge t).
\end{align*}
$$

(4.14)

This is alternating, and the even and odd parts: $\bigwedge^0 W$ and $\bigwedge^1 W$, are orthogonal.

Denote by $\tau_{b_a}$ the involution of the endomorphism algebra $\text{End}_{\mathbb{F}}(\bigwedge W)$ induced by $b_a$:

$$
b_a(\varphi(s), t) = b_a(s, \tau_{b_a}(\varphi(t))),
$$

for all $s, t \in \bigwedge W$ and $\varphi \in \text{End}_{\mathbb{F}}(\bigwedge W)$.

Consider the linear map $W \oplus W^* \to \text{End}_{\mathbb{F}}(\bigwedge W)$ that sends any $u \in W$ to the left multiplication $l_u$ by $u$ on $\bigwedge W$, and any $f \in W^*$ to the odd superderivation $\delta_f$ of $\bigwedge W$ such that $\delta_f(u) = f(u)$ for all $u \in W$. (The fact that $\delta_f$ is an odd superderivation means that $\delta_f(s \wedge t) = \delta_f(s) \wedge t + (-1)^{\deg(s)} s \wedge \delta_f(t)$ for any homogeneous (even or odd) elements $s, t$ of $\bigwedge W$.) Note that $l_u^2 = 0 = \delta_f^2$, that $\delta_f(\hat{s}) = (-1)^{\deg(s)} \delta_f(s)$,
and that \( \tau_b(l_u) = l_u \) and \( \tau_b(\delta_f) = \delta_f \) for any \( u \in W \) and \( f \in W^* \). It follows that 
\[
 l_u \delta_f + \delta_f l_u = f(u) \text{id} \quad \text{for all } u \in W \text{ and } f \in W^*,
\]
and hence this linear map extends to a homomorphism of algebras with involution (see [27, §8] or [15]):

\[
\Lambda : \left( \mathfrak{Cl}(W \oplus W^*, q), \tau \right) \rightarrow \left( \text{End}_F(\bigwedge W), \tau_b \right),
\]

where \( \mathfrak{Cl}(W \oplus W^*, q) \) is the Clifford algebra associated to the quadratic form \( q \), and \( \tau \) is its canonical involution, that is, it restricts to the identity on \( W \oplus W^* \). By dimension count, \( \Lambda \) is an isomorphism. Moreover, \( \Lambda \) restricts to an isomorphism, also denoted by \( \Lambda \), of the even subalgebras:

\[
\Lambda : \left( \mathfrak{Cl}_0(W \oplus W^*, q), \tau \right) \rightarrow \left( \text{End}_F(\bigwedge_0 W) \times \text{End}_F(\bigwedge_1 W), \tau_b \right).
\]

The orthogonal Lie algebra \( \mathfrak{so}(W \oplus W^*, q) \) lives inside the even Clifford algebra \( \mathfrak{Cl}_0(W \oplus W^*, q) \): The endomorphism \( \sigma_{x,y} : z \mapsto q(x, z) - q(y, z) x \) in \( \mathfrak{so}(W \oplus W^*, q) \) corresponds to \( e_i \mapsto -\frac{1}{2}(x_i - y_i) \) in \( \mathfrak{Cl}_0(W \oplus W^*, q) \), for all \( x, y \in W \oplus W^* \), because in the Clifford algebra we have \([x, y] , z) = xyz - yxz - zxy + zyx = x(yz + zy) - y(xz + zx) - (xz + zx)y + (zy + yz)x = 2q(y, z)x - 2q(x, z)x = -2\sigma_{x,y}(z)\).

(Throughout this paper the product in the Clifford algebra is denoted by juxtaposition, and the polar form of \( q \) is defined by \( q \).) Note that the element \( \sigma_{e_i, e_i} \in \mathfrak{so}(W \oplus W^* q) \) satisfies \( \sigma_{e_i, e_i}(e_i) = e_i, \sigma_{e_i, e_i}(e_j) = 0, \sigma_{e_i, e_i}(e_i^\dagger) = -e_i^\dagger, \) and \( \sigma_{e_i, e_i}(e_j^\dagger) = 0 \) for \( j \neq i \).

Take the half-spin module \( T = \bigwedge_0 W \) of \( \mathfrak{so}(W \oplus W^*, q) \), that is, the representation of \( \mathfrak{so}(W \oplus W^*, q) \) obtained by first embedding it on \( \mathfrak{Cl}_0(W \oplus W^*, q) \) and then composing with \( \Lambda \). Recall, for any increasing sequence \( J \), that \( \sigma_{e_i, e_i} e_J = \frac{1}{2}(e_i \wedge \delta_{e_i} e_J - \delta_{e_i}(e_i \wedge e_J)) \) equals \( \frac{1}{2} e_J \) if \( i \in J \) and \( -\frac{1}{2} e_J \) if \( i \notin J \). Thus, for \( I = (1, 2, 3, 4, 5, 6) \), the element \( e_I \in T_2(\epsilon_1 + \cdots + \epsilon_6) \) is a highest weight vector in \( T \). Thus \( T \) is the irreducible module with highest weight \( a_6 \). (Note that had we swapped the last two elements in our system of simple roots \( \Pi \), we would have had to consider the other half-spin module: \( \bigwedge_1 W \), of highest weight \( \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_5 - \epsilon_6) \), as the irreducible module with highest weight \( a_6 \).

By irreducibility, the restriction of \( b_a \) is the unique, up to scalars, \( \mathfrak{so}(W \oplus W^*, q) \)-invariant non-zero bilinear form on \( T \), and the unique, up to scalars, \( \mathfrak{so}(W \oplus W^*, q) \)-invariant bilinear map \( T \times T \rightarrow \mathfrak{so}(W \oplus W^*, q) \) is given by the formula

\[
\text{tr}(\sigma d_{x,y}) = -4b_a(\sigma x, y), \quad (4.16)
\]

for \( \sigma \in \mathfrak{so}(W \oplus W^*, q) \) and \( x, y \in T \). (See, e.g., [15, Proposition 2.19].)

Again, as for the \( F_4 \) or \( E_6 \) types, with \((x|y) = b_6(x, y) \), there is a scalar \( 0 \neq \alpha \in \mathbb{F} \) such that (4.11) holds. For \( x = y = 1(= e_\emptyset) \) and \( z = e_{123456} \), we have \( d_{x,y} = 0 \), while \( d_{x, z} \), which belongs to the Cartan subalgebra \( h \), coincides with the endomorphism \( \sum_{i=1}^6 \sigma_{e_i, e_i} \) due to (4.16). Thus \( d_{x,z,y} = \sum_{i=1}^6 \sigma_{e_i, e_i} e_\emptyset = \sum_{i=1}^6 \frac{-1}{2} e_\emptyset = -3y \). Also \( (x|y) = 0 \) and \( (x|z) = 1 = (y|z) \). It follows that \( \alpha = 1 \).

Therefore, \(( T = \bigwedge_0 W, [\cdot, \cdot, \cdot, \cdot] )\) is a simple symplectic triple system, for \((x|y) = b_a(\sigma x, y) \) in (4.14) and \([x, y, z] = d_{x, y, z} \) in (4.16). Its inner derivation algebra
is isomorphic to $\mathfrak{so}(W \oplus W^*, q)$ and its standard enveloping algebra is isomorphic to
the split exceptional simple Lie algebra of type $E_7$.

4.8 $E_8$-type

Let $U$ be an eight-dimensional vector space. Fix a basis $\{e_i : 1 \leq i \leq 8\}$ and its
dual basis $\{e^i : 1 \leq i \leq 8\}$ in $U^*$. For any sequence $I = (i_1, \ldots, i_r)$ with $1 \leq i_1 < \cdots < i_r \leq 8$, write $e_I = e_{i_1} \wedge \cdots \wedge e_{i_r}$ in $\bigwedge U$, and $e^I = e^{i_1} \wedge \cdots \wedge e^{i_r}$ in $\bigwedge U^*$.

Consider the linear map $\det : \bigwedge U \rightarrow \mathbb{F}$ that annihilates $\bigwedge^i U, 0 \leq i \leq 7$, and such that $\det(e_{12345678}) = 1$. It induces a non-degenerate bilinear form $(\cdot | \cdot)_\wedge$ on $\bigwedge U$ by means of $(x | y)_\wedge = \det(x \wedge y)$ for all $x, y$. Its restriction to $\bigwedge^4 U$ is a (non-degenerate) symmetric bilinear form.

Using these ingredients, Adams gives in [1, Chapter 12] the following explicit con-
struction of the split simple Lie algebra of type $E_7$ and of its irreducible 56-dimensional
representation. This 56-dimensional module is the vector space on which the last split
simple symplectic triple system, with $E_7$ as its inner derivation algebra, is built. More
precisely, the vector space $L = \mathfrak{sl}(U) \oplus \bigwedge^4 U$ is the split simple Lie algebra of type
$E_7$, with bracket given by

- the usual bracket in $\mathfrak{sl}(U)$,
- $[f, x] = f . x$, the natural action of $f \in \mathfrak{sl}(U)$ on $x \in \bigwedge^4 U$,
- for $x, y \in \bigwedge^4 U$, $[x, y]$ is the element in $\mathfrak{sl}(U)$ determined by the condition

$$\text{tr}(f [x, y]) = (f . x | y)_\wedge$$

for all $f \in \mathfrak{sl}(U)$.

The decomposition $L = \mathfrak{sl}(U) \oplus \bigwedge^4 U$ is a grading by $\mathbb{Z}/2$.

The Killing form of $L$ is $36(\cdot | \cdot)_L$, where $(\mathfrak{sl}(U) | \bigwedge^4 U)_L = 0$, $(f | g)_L = \text{tr}(fg)$, and $(x | y)_L = (x | y)_\wedge = \det(x \wedge y)$, for $f, g \in \mathfrak{sl}(U)$ and $x, y \in \bigwedge^4 U$.

Moreover, $(\cdot | \cdot)_\wedge$ gives $\mathfrak{sl}(U)$-invariant linear maps, for $0 \leq i \leq 8$:

$$\Phi_i : \bigwedge^i U \rightarrow \left(\bigwedge^i U^*\right)^* \cong \bigwedge^{8-i} U^* \quad (4.17)$$

$$x \mapsto (x | \cdot)_\wedge,$$

where $\bigwedge^i U^*$ is identified with $\left(\bigwedge^i U\right)^*$ naturally:

$$f_1 \wedge \cdots \wedge f_i \leftrightarrow \left(v_1 \wedge \cdots \wedge v_i \mapsto \det\left(f_k(v_j)\right)_{k,j}\right). \quad (4.18)$$

Finally, take $T = \bigwedge^2 U \oplus \bigwedge^2 U^*$, with the action of $L$ given by:

- the natural action of $\mathfrak{sl}(U)$ on both $\bigwedge^2 U$ and $\bigwedge^2 U^*$,
• \([x, p] = \Phi_6(x \wedge p) \in \bigwedge^2 U^*,\) for \(x \in \bigwedge^4 U\) and \(p \in \bigwedge^2 U,\)
• \([x, q] = \Phi_2^{-1}(\Phi_4(x) \wedge q) \in \bigwedge^2 U,\) for \(x \in \bigwedge^4 U\) and \(q \in \bigwedge^2 U^*.\)

Then \(T\) is the only 56-dimensional irreducible module for \(\mathcal{L},\) that is, the irreducible module with highest weight \(\varpi_1\) (once a Cartan subalgebra and a system of simple roots is chosen).

Using these previous results from [1], we proceed as follows to determine the structure of simple symplectic triple system on the irreducible \(\mathcal{L}\)-module \(T.\)

Given two disjoint increasing sequences \(I = (i_1, \ldots, i_r)\) and \(J = (j_1, \ldots, j_s),\) its union \(I \cup J\) gives another increasing sequence. Let \((-1)^{IJ}\) be 1 or \(-1\) according to the rule \(e_I e_J = (-1)^{IJ} e_{I \cup J}\) (that is, the sign of the permutation). Also denote by \(\overline{T}\) be the increasing sequence whose underlying set is \(\{1, \ldots, 8\} \setminus I.\) The size of a sequence \(I\) will be denoted by \(|I|\). The isomorphisms \(\Phi_i\) in (4.17) are then given by \(e_I \mapsto (-1)^{I\overline{T}} e_{\overline{T}}\) (where \(i = |I|).\)

With these notations, the action of \(\bigwedge^4 U\) on \(T\) works as follows, for \(I\) and \(J\) increasing sequences with \(|I| = 4\) and \(|J| = 2,\)
\[
e_{I} e_{J} = \begin{cases} 0 & \text{if } I \cap J \neq \emptyset, \\ (-1)^{I J} (-1)^{(I \cup J)(\overline{I \cup J})} e_{\overline{I \cup J}} & \text{otherwise; } \end{cases}
\]
and
\[
e_{I} e_{\overline{J}} = \begin{cases} 0 & \text{if } J \not\subseteq I, \\ (-1)^{I \overline{J}} (-1)^{(I \cup J)(\overline{I \cup J})} e_{I \cap J} & \text{otherwise. } \end{cases}
\]
(Note that \(\overline{I \cup J} = I \cap J).\)

Endow \(T = \bigwedge^2 U \oplus \bigwedge^2 U^*\) with the non-degenerate alternating bilinear form \((\cdot, \cdot) : T \times T \to \mathbb{F}\) such that \(\bigwedge^2 U\) and \(\bigwedge^2 U^*\) are totally isotropic subspaces, and such that
\[
(u_1 \wedge u_2) \omega_1 \wedge \omega_2 = \det(\omega_i(u_j)).
\] (4.19)

This alternating form is \(\mathcal{L}\)-invariant.

Adams also shows in [1,Theorem 12.4] that if the map \(\circ : T \times T \to \mathcal{L}, (x, y) \mapsto x \circ y\) is defined by
\[
(l|x \circ y)_\mathcal{L} = (l.x|y)
\]
for \(l \in \mathcal{L}\) and \(x, y \in T,\) then \(x \circ y\) is symmetric and
• \(x \circ y = -x \wedge y \in \bigwedge^4 U\) for \(x, y \in \bigwedge^2 U,\)
• \(x \circ y = \Phi_2^{-1}(x \wedge y) \in \bigwedge^4 U\) for \(x, y \in \bigwedge^2 U^*,\)
• for \(u_1, u_2 \in U\) and \(\omega_1, \omega_2 \in U^*,\)
\[
(u_1 \wedge u_2) \circ (\omega_1 \wedge \omega_2) = \omega_1(u_1) u_2 \otimes \omega_2 - \omega_1(u_2) u_1 \otimes \omega_2 - \omega_2(u_1) u_2 \otimes \omega_1 + \omega_2(u_2) u_1 \otimes \omega_1 - \frac{1}{4} \det(\omega_i(u_j)) \text{id}_U \in \text{sl}(U),
\]
where \( u \otimes \omega \in \mathfrak{gl}(U) \) denotes the map \( u' \mapsto \omega(u')u \), for \( u, u' \in U \) and \( \omega \in U^* \).

Define \( d_{x,y} \) for \( x, y \in T \) by \( d_{x,y} = -2x \circ y \), that is, \( d_{x,y} \) is defined by the equation

\[
\text{tr}(ld_{x,y}) = -2(l.x|y)
\]

for \( l \in \mathcal{L} \) and \( x, y \in T \). (Recall that \( \text{tr}(ld_{x,y}) = (l|d_{x,y})\mathcal{L} \).)

As in the previous cases, \((\cdot|\cdot)\) is the unique, up to scalars, non-zero \( \mathcal{L} \)-invariant bilinear form on \( T \), and \( d_{x,y} \) is the unique, up to scalars, non-zero \( \mathcal{L} \)-invariant bilinear map \( T \times T \to \mathcal{L} \). Therefore there is a scalar \( 0 \neq \alpha \in \mathbb{F} \) such that \((4.11)\) holds. Let us find it.

Take \( x = z = e_{12} \) and \( y = e_{12} \). On one hand, \( d_{x,z} = 0 \) and \( d_{x,y} = -2(e_2 \otimes e^2 + e_1 \otimes e^1 - \frac{1}{4} \text{id}) \), so \( d_{x,y} z = -3e_{12} \) (since \( \text{id}_U.z = 2z \)). On the other hand, \((x|z) = 0, (x|y) = (z|y) = 1\), so \((x|z) y - (x|y) z + 2(y|z)x = -3e_{12}\). We thus conclude that \( \alpha = 1 \), and hence, with \([x, y, z] = d_{x,y} z\) and \((\cdot|\cdot)\) in \((4.19)\), the triple \((T = \bigwedge^2 U \oplus \bigwedge^2 U^*, [\cdot, \cdot, \cdot], (\cdot|\cdot))\) is a simple symplectic triple system with inner derivation algebra the split exceptional simple Lie algebra of type \( E_7 \) (isomorphic to \( \mathfrak{sl}(U) \oplus \bigwedge^4 U \)), and standard enveloping algebra isomorphic to the split exceptional simple Lie algebra of type \( E_8 \).

**Remark 4.1** It turns out [14] that this same classification and list of split examples work, with minor modifications, over fields of characteristic \( \neq 2, 3 \).

## 5 Non-split simple real symplectic triple systems

Our aim is to exhibit a list of some real non-split symplectic triple systems that will be proved later on to exhaust all the possibilities.

For convenience, first we will relate the signatures of the Killing forms of the Lie algebras \( g(T) \) and \( \text{der}(T) \) associated to a real simple symplectic triple system \((T, [\cdot, \cdot, \cdot], (\cdot|\cdot))\).

### 5.1 Some remarks on signatures

Let us denote by \( \text{sign}(\alpha) \) the signature of the Killing form \( \kappa_\alpha \) of any semisimple Lie algebra \( \alpha \). The signature is understood here as the difference between the number of elements in an orthogonal basis with positive ‘norm’ and the number of them with negative ‘norm’.

**Lemma 5.1** Let \( T \) be a real simple symplectic triple system. If \( \kappa \) denotes the Killing form of \( g(T) \), then there exists a skew-symmetric bilinear map \( \eta: T \times T \to \mathbb{R} \) such that

\[
\kappa(a \otimes x, b \otimes y) = (a|b)\eta(x, y),
\]

for any \( a, b \in V \) and \( x, y \in T \). In particular, the signature of the restriction \( \kappa|_{g(T)_1} \) is zero.
Proof First note that $\dim_{\mathbb{R}} \text{Hom}_{\mathfrak{sp}(V)}(V \otimes V, \mathbb{R})$ equals 1, because this is the case after complexification: $\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}) = \dim_{\mathbb{C}} \text{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathfrak{gl}_2(\mathbb{C}), \mathbb{C}) = 1$. As $(\cdot | \cdot): V \times V \rightarrow \mathbb{R}$ is a non-zero $\mathfrak{sp}(V)$-invariant map, then Lemma 2.2 gives the existence of a bilinear map $\eta: T \times T \rightarrow \mathbb{R}$ satisfying (5.1). But it is easy to check that $\eta$ must be $\text{indet}(T)$-invariant, by using the invariance of $\kappa$. Also, as $\kappa$ is symmetric and $(\cdot | \cdot)$ is skew-symmetric, then $\eta$ is skew-symmetric, and we may think of it as an element in $\text{Hom}_{\text{indet}(T)}(\wedge^2 T, \mathbb{R})$.

This space of homomorphisms has dimension 1 and it is spanned by $(\cdot | \cdot)$. Indeed, each $\eta' \in \text{Hom}_{\text{indet}(T)}(\wedge^2 T, \mathbb{R})$ would give $(\cdot | \cdot) \otimes \eta'$, which would belong to $\text{Hom}_{\mathfrak{gl}_0}(S^2(\mathfrak{g}_T), \mathbb{R})$, and this is one-dimensional, because so is its complexification according to part (3) in Lemma 3.1.

Then there is a scalar $\alpha \in \mathbb{R}$ such that $\kappa(\alpha \otimes x, b \otimes y) = \alpha(a|b)(x|y)$. Now, if $\{x_i, y_i : i = 1, \ldots, n\}$ is a symplectic basis of $(T, (\cdot | \cdot))$, that is, $\kappa(x_i|y_i) = 1 = -(y_i|x_i)$, and $\kappa(x_i|y_j) = 0 = (y_j|x_i)$ for $i \neq j$, and if $\{e_1, e_2\}$ is a symplectic basis of $V$, then the family

$$\{e_1 \otimes x_i + e_2 \otimes y_i, e_1 \otimes y_i - e_2 \otimes x_i, e_1 \otimes x_i - e_2 \otimes y_i, e_1 \otimes y_i + e_2 \otimes x_i : i = 1, \ldots, n\}$$

is a $\kappa$-orthogonal basis of $\mathfrak{g}(T)_1 = V \otimes T$ such that the ‘length’ $\kappa(z, z)$ of any of the first $2n$ elements is $2\alpha$ and the ‘length’ of any of the last $2n$ elements is $-2\alpha$. Therefore, the signature of $\kappa|_{\mathfrak{g}(T)_1}$ is 0. \hfill \Box

A classical result will be useful for us too.

Lemma 5.2 If $L \subset \mathfrak{gl}(U)$ is a complex simple Lie algebra and $\kappa: L \times L \rightarrow \mathbb{C}$ denotes its Killing form, then there is a scalar $\alpha \in \mathbb{Q}$, $\alpha > 0$, such that $\kappa(f, g) = \alpha\text{tr}(fg)$ for all $f, g \in L$.

Proof The vector space $\text{Hom}_L(S^2(L), \mathbb{C})$ is contained in $\text{Hom}_L(L \otimes L, \mathbb{C})$, which is linearly isomorphic to $\text{Hom}_L(L, L^*)$, and hence to $\text{Hom}_L(L, L) = \text{Id}_L$, because the adjoint module $L$ is self-dual ($\kappa$ is non-degenerate). We conclude that $\text{Hom}_L(S^2(L), \mathbb{C})$ has dimension one and it is spanned by the Killing form $\kappa$.

The symmetric bilinear form $b: L \times L \rightarrow \mathbb{C}$, $b(f, g) = \text{tr}(fg)$, is $L$-invariant, so there is a scalar $\alpha \in \mathbb{C}$ such that $\kappa(f, g) = \alpha\text{tr}(fg)$ for all $f, g \in L$. Take now a subalgebra of $L$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and let $\{h, e, f\}$ be a standard basis. As $L$ is a completely reducible module, it is a sum of $\mathfrak{sl}_2(\mathbb{C})$-modules of type $V(n\varpi_1)$ so that $\text{ad}h$ acts diagonally on $L$ and all its eigenvalues are integer numbers. This means that $\kappa(h, h) = \text{tr}(\text{ad}h^2)$ is in $\mathbb{Z}_{\geq 0}$, in fact it is strictly positive. In the same vein, $U$ is completely reducible as a module for this subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, and all the eigenvalues of $h \in \mathfrak{gl}(U)$ are integers, so we get $b(h, h) \in \mathbb{Z}_{\geq 0}$ too. From $\kappa(h, h) = \alpha b(h, h)$ we obtain that the scalar $\alpha$ is a positive rational number. \hfill \Box

Proposition 5.3 Let $(T, [\cdot, \cdot, \cdot], (\cdot | \cdot))$ be a real simple symplectic triple system. Write $\mathfrak{g} = \mathfrak{g}(T)$ and $\mathfrak{h} = \text{indet}(T)$. Then, if $\mathfrak{h}$ is simple, the difference between the signatures of $\mathfrak{g}$ and $\mathfrak{h}$ is 1: $\text{sign}(\mathfrak{g}) - \text{sign}(\mathfrak{h}) = 1$.
**Proof** Denote by $\kappa$ the Killing form of $\mathfrak{g}$. Lemma 5.1 gives $\text{sign}(\mathfrak{g}) = \text{sign}(\kappa|_{\mathfrak{g}(T)_{\bar{0}}})$. Now note that $\mathfrak{g}(T)_{\bar{0}} = \mathfrak{sp}(V) \oplus \mathfrak{in}_\mathbb{C}(T)$ is an orthogonal sum, so that $\text{sign}(\mathfrak{g}) = \text{sign}(\kappa|_{\mathfrak{sp}(V)}) + \text{sign}(\kappa|_{\mathfrak{in}_\mathbb{C}(T)})$.

Let us prove first that $\kappa|_{\mathfrak{sp}(V)}$ is a positive multiple of the Killing form $\kappa_{\mathfrak{sp}(V)}$, which has signature $1$ ($\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{sp}(V)$ is the split algebra of type $A_1$). Indeed, if $\gamma, \gamma' \in \mathfrak{sp}(V)$, then $[\gamma, [\gamma', a \otimes t]] = \gamma \gamma'(a) \otimes t$, so that

$$\kappa(\gamma, \gamma') = \kappa_{\mathfrak{sp}(V)}(\gamma, \gamma') + \text{tr}(\gamma \gamma') \dim T.$$ 

But for $h = \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$, $\text{tr}(h^2) = 2$, while $\kappa_{\mathfrak{sp}(V)}(h, h) = 8$, so that we have $\text{tr}(\gamma \gamma') = \frac{1}{4}\kappa_{\mathfrak{sp}(V)}(\gamma, \gamma')$, and this implies $\kappa(\gamma, \gamma') = \left(1 + \frac{\dim T}{4}\right)\kappa_{\mathfrak{sp}(V)}(\gamma, \gamma')$. In particular, we get $\text{sign}(\kappa|_{\mathfrak{sp}(V)}) = 1$.

For $d, d' \in \mathfrak{in}_\mathbb{C}(T) = \mathfrak{h}$, we get $[d, [d', a \otimes t]] = a \otimes dd'(t)$ and

$$\kappa(d, d') = \kappa_{\mathfrak{h}}(d, d') + 2\text{tr}(dd').$$

Our $\mathfrak{h}$ is simple so, as in the lemma above, we have $\alpha\kappa_{\mathfrak{h}}(d, d') = \text{tr}(dd')$ for a positive $\alpha \in \mathbb{Q}$, because the same happens after complexification. It follows that $\kappa(d, d')$ equals $(1 + 2\alpha)\kappa_{\mathfrak{h}}(d, d')$ for all $d, d' \in \mathfrak{h}$ and that $\text{sign}(\kappa|_{\mathfrak{h}})$ equals $\text{sign}(\mathfrak{h})$. □

Let us now describe some non-split real symplectic triple systems with classical enveloping algebra. Namely, we will consider one family of triple systems with special envelope and another one with orthogonal envelope.

### 5.2 Unitarian type

Let $W$ be a non-zero complex vector space endowed with a non-degenerate hermitian form $h_W : W \times W \to \mathbb{C}$. Hence $h_W$ is linear in the first component, conjugate linear in the second, $h_W(x, y) = h_W(y, x)$ for all $x, y \in W$, and the kernel $\{x \in W : h_W(x, W) = 0\}$ is trivial. Recall that there are always orthogonal bases relative to $h_W$ and, as in Sect. 5.1, the signature of $h_W$ is defined as the difference between the number of elements $w$ in any of these basis with $h_W(w, w) > 0$ and the number of those with $h_W(w, w) < 0$.

For any $x, y \in W$, we may consider the real and imaginary parts:

$$h_W(x, y) = \{x|y\} + i(x|y). \quad (5.2)$$

Then $\{\cdot|\cdot\}$ is a non-degenerate symmetric real bilinear form on $W$ (considered as a real vector space), and $(\cdot|\cdot)$ is a non-degenerate alternating real bilinear form.

The $\mathbb{C}$-linear operators

$$\eta_{a,b} = h_W(\cdot, a)b - h_W(\cdot, b)a,$$

for $a, b \in W$, span (over $\mathbb{R}$) the unitary Lie algebra

$$\mathfrak{u}(W, h_W) = \{f \in \text{End}_\mathbb{C}(W) : h_W(f(a), b) = -h_W(a, f(b)) \forall a, b \in W\}.$$
Moreover, we have \( \eta_{aa,b} = \eta_{a,\overline{a}b} \) and \([\sigma, \eta_{a,b}] = \eta_{\sigma(a),b} + \eta_{a,\sigma(b)}\) for all \( a, b \in W \), \( \alpha \in \mathbb{C} \) and \( \sigma \in u(W, h_W) \).

Let now \( U \) be a two-dimensional vector space over \( \mathbb{C} \) endowed with a non-degenerate hermitian form \( h_U : U \times U \to \mathbb{C} \) of signature 0. Then there is a basis \( \{u, v\} \) of \( U \) with \( h_{U}(u, u) = 0 = h_{U}(v, v) \) and \( h_{U}(u, v) = i \). The special unitary Lie algebra \( su(U, h_U) \) is \( \mathbb{R} \)-spanned by \( \eta_{u,0}, \eta_{v,0}, \eta_{0,u}, \eta_{0,v} \) (note that \( \eta_{u,v} = -i \text{id} \)). It leaves invariant the two-dimensional vector space \( V = Ru \oplus Rv \), and it is isomorphic, by restriction to \( V \), to \( sl(V) \). Indeed, for any \( a, b \in V \), \( h_U(a, b) = i \langle a \mid b \rangle \), where \( \langle \cdot \mid \cdot \rangle : V \times V \to \mathbb{R} \) is alternating with \( \langle u \mid v \rangle = 1 \). For \( a, b, c \in V \) we have:

\[
\eta_{a,ib}(c) = h_U(c, a)i\overline{b} - h_U(c, ib)\overline{a}
= -\langle a \mid b \rangle \overline{b} - \langle c \mid b \rangle \overline{a}
= \langle a \mid c \rangle \overline{b} + \langle b \mid c \rangle \overline{a} = \gamma_{a,b}(c).
\]

This means that we have the isomorphism \( su(U, h_U) \cong sp(V, \langle \cdot \mid \cdot \rangle) = sl(V) \). Also, using that \( \langle a \mid b \rangle \overline{c} + \langle b \mid c \rangle \overline{a} + \langle c \mid a \rangle \overline{b} = 0 \), because \( \text{dim}_{\mathbb{R}} V = 2 \), we obtain

\[
\eta_{a,b}(c) = h_U(c, a)b - h_U(c, b)\overline{a}
= i\langle c \mid a \rangle \overline{b} - i\langle c \mid b \rangle \overline{a} = -i \langle a \mid b \rangle c,
\]

so we get

\[
\eta_{a,b} = -\langle a \mid b \rangle L_1,
\]

where \( L_1 \) denotes the scalar multiplication by the imaginary unit \( i \).

The direct sum \( U \oplus W \) is endowed with the non-degenerate hermitian form given by the ‘orthogonal sum’ \( h = h_U \perp h_W \) of \( h_U \) and \( h_W \). It is \( \mathbb{Z}/2 \)-graded with even part \( U \) and odd part \( W \), which induces a \( \mathbb{Z}/2 \)-grading on the endomorphism algebra and hence a \( \mathbb{Z}/2 \)-grading on the corresponding special unitary Lie algebra, where the even part consists of the endomorphisms preserving \( U \) and \( W \), and the odd part consists of the endomorphisms interchanging \( U \) with \( W \). In this way, such special unitary Lie algebra decomposes, with the natural embeddings, as follows:

\[
su(U \perp W, h) = (su(U, h_U) \oplus su(W, h_W) \oplus \mathbb{R} J) \oplus \eta_{U,W},
\]

where \( J \) is the trace zero endomorphism determined by:

\[
J(x) = \begin{cases} 
\text{(dim}_{\mathbb{C}} W)i\overline{x} & \text{for } x \in U, \\
-2i\overline{x} & \text{for } x \in W.
\end{cases}
\]

It must be remarked here that the space \( \eta_{U,W} \) is the same as \( \eta_{V,W} \) and this is isomorphic to \( V \otimes_{\mathbb{R}} W (u \otimes x \mapsto \eta_{v,x}) \). Also, \( su(W, h_W) \oplus \mathbb{R} J \) is isomorphic to the unitary Lie algebra \( u(W, h_W) \), as this is the direct sum of \( su(W, h_W) \) and a one-dimensional
center. Thus we can write, using the natural identifications:

$$\mathfrak{su}(U \perp W, h) = (\mathfrak{sp}(V, \langle \cdot, \cdot \rangle) \oplus \mathfrak{u}(W, h_W)) \oplus (V \otimes_{\mathbb{R}} W),$$

(5.6)

where the product of the even part with the odd one is given by the natural action.

For any $a, b \in V$ and $x, y \in W$, easy computations give:

$$[\eta_{a,x}, \eta_{b,y}] = \eta_{\eta_{a,x}(b),y} + \eta_{b,\eta_{a,x}(y)}$$

$$= \eta_{h_U(b,a)x,y} + \eta_{h_W(y,x)a,b}$$

$$= (\langle a|b \rangle \eta_{x,iy} + \{x|y\} \eta_{a,b}) + (x|y) \eta_{a,ib}$$

(5.7)

where we have used (5.2), (5.3), and (5.4), and where $L_i|U$ denotes the endomorphism that is trivial on $W$ and equals the scalar multiplication by $i$ on $U$.

Note that the trace of $\eta_{x,iy}$ is obtained as follows:

$$\text{tr}(\eta_{x,iy}) = h_W(iy, x) - h_W(x, iy) = i(h_W(y, x) + h_W(x, y)) = 2\{x|y\}i,$$

and hence $\eta_{x,iy} - \{x|y\}L_i|U$ is traceless so belonging to $\mathfrak{su}(W, h_W) \oplus \mathbb{R}J$. Moreover, for $c \in V$ and $z \in W$, we have:

$$[\eta_{x,iy} - \{x|y\}L_i|U, \eta_{c,z}] = \eta_{c,\eta_{x,iy}(z)} - \{x|y\} \eta_{c,z}$$

$$= \eta_{c,\eta_{x,iy}(z) + i\{x|y\}z}$$

$$= \eta_{c, h_W(z,x)iy - h_W(z,iy)x + i\{x|y\}z} = \eta_{c, [x,y,z]},$$

for the triple product on $W$ defined by

$$[x, y, z] = i(h(z, x)y + h(z, y)x + \{x|y\}z).$$

(5.8)

Thus we have checked that $[\text{proj}_{\mathfrak{su}(W, h_W) \oplus \mathbb{R}J}([\eta_{a,x}, \eta_{b,y}], \eta_{c,z})] = \eta_{[a|b]c, [x,y,z]}$.

(Here we denote by $\text{proj}_{\mathfrak{su}(W, h_W) \oplus \mathbb{R}J}$ the projection onto $\mathfrak{su}(W, h_W) \oplus \mathbb{R}J$ according to the decomposition in (5.5).)

Then, Equations (2.5), (5.6), and (5.7), give at once that $(T = W, [\cdot, \cdot, \cdot], \langle \cdot, \cdot \rangle)$ is a simple symplectic triple system, for $[x, y, z]$ in (5.8) and $(x|y)$ in (5.2), with inner derivation algebra isomorphic to $\mathfrak{u}(W, h_W)$, and standard enveloping algebra isomorphic to $\mathfrak{su}(U \perp W, h)$.

These symplectic triple systems will be said to be of unitarian type. The signatures of $h_W$ and $h$ coincide, because the signature of $h_U$ is 0. Hence we get the isomorphism $(g(T), \text{ind}eT(T)) \cong (\mathfrak{su}_{p+1,n+1-p}, \mathfrak{u}_{p,n-p})$ for $n = \dim \mathbb{C} W$ and some $p \geq \frac{n}{2}$.

### 5.3 Quaternionic type

Let here $U$ be a two-dimensional right $\mathbb{H}$-module, where $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ is the real division algebra of quaternions, endowed with a skew-hermitian non-degenerate
form \( h_U : U \times U \to \mathbb{H} \). That is, \( h_U \) is \( \mathbb{R} \)-bilinear, \( h_U(a, bq) = h_U(a, b)q, h_U(aq, b) = \overline{q}h_U(a, b) \), and \( h_U(a, b) = -h_U(b, a) \), for all \( a, b \in U \) and \( q \in \mathbb{H} \), where \( q \mapsto \overline{q} \) is the canonical conjugation in \( \mathbb{H} \). Note that non-degenerate skew-hermitian forms are unique up to ‘isometry’. Denote by \( \mathbb{H}_0 \) the subspace of zero trace quaternions: \( \mathbb{H}_0 = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k. \)

Pick a basis \{\( u, v \)\} of \( U \) with \( h_U(u, u) = 0 = h_U(v, v) \), \( h_U(u, v) = 1 \) and, as in the unitarian case, consider the two-dimensional vector space \( V = \mathbb{R}u \oplus \mathbb{R}v \). The restriction of \( h_U \) to \( V \) is a non-zero alternating bilinear form \( \langle \cdot | \cdot \rangle \).

Denote by \( \mathfrak{so}^*(U, h_U) \) the Lie algebra of skew-symmetric endomorphisms relative to \( h_U \):

\[
\mathfrak{so}^*(U, h_U) = \{ f \in \text{End}_\mathbb{H}(U) : h_U(f(a), b) = -h_U(a, f(b)) \; \forall a, b \in U \}.
\]

Note that \( \mathfrak{so}^*(U, h_U) \) is the \( \mathbb{R} \)-linear span of the operators

\[
\sigma_{a,b} = a \, h_U(b, \cdot) + b \, h_U(a, \cdot), \tag{5.9}
\]

for \( a, b \in U \).

**Lemma 5.4** The Lie algebra \( \mathfrak{so}^*(U, h_U) \) is the direct sum of two three-dimensional simple ideals:

\[
\mathfrak{so}^*(U, h_U) = \sigma_{V,V} \oplus \sigma_{u,v\mathbb{H}_0}.
\]

Moreover, \( \sigma_{V,V} \) is isomorphic to the split simple Lie algebra \( \mathfrak{sl}_2(\mathbb{R}) \), while \( \sigma_{u,v\mathbb{H}_0} \) is isomorphic to the compact Lie algebra \( \mathfrak{su}_2 \).

**Proof** For any \( f \in \mathfrak{so}^*(U, h_U) \), \( a, b \in U \) and \( q \in \mathbb{H} \), we have \( [f, \sigma_{a,b}] = \sigma_{f(a),b} + \sigma_{a,f(b)} \) and \( \sigma_{aq,b} = \sigma_{a,bq} \). Also, for \( a \in U \) and \( q \in \mathbb{H}_0 \):

\[
\sigma_{a,aq} = \begin{cases} 
\sigma_{aq,a} = -\sigma_{aq,a} & \text{as } \overline{q} = -q, \\
\sigma_{aq,a} & \text{as } \sigma \text{ is symmetric on its arguments.}
\end{cases}
\]

Hence \( \sigma_{a,aq} = 0 \). From here it is easy to obtain

\[
\sigma_{a,bq} = \langle a|b \rangle \sigma_{u,vq} \tag{5.10}
\]

for any \( a, b \in V \) and \( q \in \mathbb{H}_0 \). We then get

\[
\mathfrak{so}^*(U, h_U) = \sigma_{U,U} = \sigma_{V,V} \oplus \sigma_{V,V} = \sigma_{V,V} + \sigma_{V,V} = \sigma_{V,V} + \sigma_{u,v\mathbb{H}_0}.
\]

For \( a, b \in V \), we have \( \sigma_{a,b}|_V = \gamma_{a,b} \in \text{sp}(V, \langle \cdot | \cdot \rangle) \), and hence \( \sigma_{V,V} \) is a Lie subalgebra isomorphic to \( \text{sp}(V, \langle \cdot | \cdot \rangle) \cong \mathfrak{sl}_2(\mathbb{R}) \).
For \( a, b \in V \) and \( q, q' \in \mathbb{H}_0 \) we compute:

\[
[\sigma_{a,b}, \sigma_{u,vq}] = \sigma_{\sigma_{a,b}(u),vq} + \sigma_{u,\sigma_{a,b}(v)q} \\
= \left( \langle \sigma_{a,b}(u) | v \rangle + \langle u | \sigma_{a,b}(v) \rangle \right) \sigma_{u,vq} = 0,
\]

\[
\sigma_{u,vq}(u) = u h_U(vq, u) = -u \overline{q} = u q,
\]

\[
\sigma_{u,vq}(v) = v q h_U(u, v) = v q,
\]

\[
[\sigma_{u,vq}, \sigma_{u,vq}'] = \sigma_{\sigma_{u,vq}(u),vq'} + \sigma_{u,\sigma_{u,vq}(v)q'} \\
= \sigma_{uq,vq'} + \sigma_{u,vqq'} = \sigma_{u,v[q,q']}.
\]

This shows that \( \sigma_{V,V} \) and \( \sigma_{u,v\mathbb{H}_0} \) are two ideals of \( \mathfrak{so}^*(U, h_U) \), and that \( \sigma_{u,v\mathbb{H}_0} \) is isomorphic to the compact Lie algebra \( \mathbb{H}_0 \) (with the usual bracket).

Let now \( W \) be a non-zero right \( \mathbb{H} \)-module endowed with a non-degenerate skew-hermitian form \( h_W : W \times W \to \mathbb{H} \). As for the unitarian type, the direct sum \( U \oplus W \) is endowed with the non-degenerate skew-hermitian form \( h = h_U \perp h_W \) and there appears a natural \( \mathbb{Z}/2 \)-grading on the Lie algebra \( \mathfrak{so}^*(U \perp W, h) \), where the endomorphisms in the even part preserve the subspaces \( U \) and \( W \), and those in the odd part swap them:

\[
\mathfrak{so}^*(U \perp W, h) = (\mathfrak{so}^*(U, h_U) \oplus \mathfrak{so}^*(W, h_W)) \oplus \sigma_{U,W}
\]

(5.12)

(The notation used here for \( \sigma_{x,y} \) when \( x, y \in W \), extends the one in (5.9).) Besides, \( \sigma_{U,W} = \sigma_{V,W} \) is isomorphic to \( V \otimes_{\mathbb{R}} W \) by means of \( a \otimes x \mapsto \sigma_{a,x} \).

For any \( x, y \in W \), consider the real and imaginary parts:

\[
h_W(x, y) = (x|y) + \{x|y\},
\]

(5.13)

with \( (x|y) \in \mathbb{R} \) and \( \{x|y\} \in \mathbb{H}_0 \). Then \( (\cdot | \cdot) \) is a non-degenerate alternating (real) bilinear form on \( W \), while \( \{\cdot | \cdot\} : W \times W \to \mathbb{H}_0 \) is a symmetric (real) bilinear map.

For \( a, b \in V \) and \( x, y \in W \) we compute, using (5.10):

\[
[\sigma_{a,x}, \sigma_{b,y}] = \sigma_{\sigma_{a,x}(b),y} + \sigma_{b,\sigma_{a,x}(y)} \\
= \langle a|b \rangle \sigma_{x,y} + \langle b, a \rangle h_W(x,y) \\
= \langle a|b \rangle \sigma_{x,y} + \{x|y\} \sigma_{a,b} + \sigma_{b,a\{x|y\}} \\
= \langle a|b \rangle \left( \sigma_{x,y} - \sigma_{u,v\{x|y\}} \right) + \{x|y\} \sigma_{a,b}.
\]

Moreover, for \( a \in V \) and \( q \in \mathbb{H}_0 \), (5.11) gives \( \sigma_{u,vq}(a) = a q \), and hence

\[
[\sigma_{u,vq}, \sigma_{a,z}] = \sigma_{\sigma_{u,vq}(a),z} = \sigma_{aq,z} = -\sigma_{a,qz},
\]

for any \( z \in W \). It follows, for \( x, y, z \in W \) and \( a \in V \), that:

\[
[\sigma_{x,y} - \sigma_{u,v\{x|y\}}, \sigma_{a,z}] = \sigma_{a,\sigma_{x,y}(z)+z\{x|y\}} = \sigma_{a,[x,y,z]},
\]
for the triple product in $W$ defined by

$$[x, y, z] = \sigma_{x,y}(z) + z[x|y]. \quad (5.14)$$

Then Equation (5.12) gives at once that $(W, [\cdot, \cdot, \cdot], (\cdot|\cdot))$ is a simple symplectic triple system, for $(x|y)$ in (5.13), with inner derivation algebra isomorphic to $\mathfrak{su}_2 \oplus \mathfrak{so}^*(W, h_W)$, and standard enveloping algebra isomorphic to $\mathfrak{so}^*(U \perp W, h)$. These symplectic triple systems will be said to be of quaternionic type.

Let us proceed now with examples with exceptional envelopes. Our constructions are much in the spirit of [1] and provide models of some of the exceptional simple real Lie algebras.

### 5.4 Non-split $E_6$ types

As in Sect. 4.6, let $W$ be a six-dimensional vector space, now over $\mathbb{C}$ and endowed with a non-degenerate hermitian form $h : W \times W \to \mathbb{C}$. Changing $h$ by $-h$ if necessary, we will assume that the signature of $h$ is $\geq 0$. Fix then an orthogonal basis $\{e_i : 1 \leq i \leq 6\}$ of $W$, with $h(e_i, e_i) = 1$ for $1 \leq i \leq p$, and $h(e_i, e_i) = -1$ for $p + 1 \leq i \leq 6$, with $p \geq 3$. Use this basis to define the determinant map $\det : \wedge^6 W \to \mathbb{C}$ given by $\det(e_{123456}) = 1$, and consider the alternating bilinear form $(\cdot|\cdot)$ on $\wedge^3 W$ given by (4.12): $(x|y) = \det(x \wedge y)$. Consider too the dual basis $\{e^i : 1 \leq i \leq 6\}$ in $W^*$.

Analogously to (4.17), this induces an $\mathfrak{sl}(2W)$-invariant linear isomorphism $\wedge^3 W \to (\wedge^3 W)^*$, $x \mapsto (x|\cdot)$. As $\wedge^3 W^*$ is canonically isomorphic, similarly to (4.18), to $(\wedge^3 W)^*$, we obtain the following $\mathfrak{sl}(W)$-invariant linear isomorphism

$$\Phi_3 : \wedge^3 W \longrightarrow \wedge^3 W^*$$

$$e_{\sigma(1)\sigma(2)\sigma(3)} \mapsto (-1)^{\sigma(4)\sigma(5)\sigma(6)} e_{\sigma(4)\sigma(5)\sigma(6)}$$

for any permutation $\sigma$ of $\{1, \ldots, 6\}$, where $(-1)^{\sigma}$ denotes the signature of $\sigma$.

On the other hand, consider the conjugate linear $\mathfrak{su}(W, h)$-invariant map $W \to W^*$: $u \mapsto h(\cdot, u)$, which induces the following conjugate linear $\mathfrak{su}(W, h)$-invariant map:

$$\Psi : \wedge^3 W \longrightarrow \wedge^3 W^*$$

$$u_1 \wedge u_2 \wedge u_3 \mapsto h(\cdot, u_1) \wedge h(\cdot, u_2) \wedge h(\cdot, u_3).$$

For any $1 \leq i < j < k \leq 6$ one has

$$\Psi(e_{ijk}) = (-1)^{[i,j,k] \cap [p+1,6]} e_{ijk}.$$

The composition $\Gamma = \Phi_3^{-1} \Psi : \wedge^3 W \to \wedge^3 W$ is bijective, conjugate linear, and $\mathfrak{su}(W, h)$-invariant. For any permutation $\sigma$, we get

$$\Gamma(e_{\sigma(1)\sigma(2)\sigma(3)}) = (-1)^{\sigma} (-1)^{[\sigma(1),\sigma(2),\sigma(3)] \cap [p+1,6]} e_{\sigma(4)\sigma(5)\sigma(6)}.$$
Then $\Gamma^2$ is $\mathbb{C}$-linear and $\mathfrak{su}(W, h)$-invariant, so it is $\mathfrak{sl}(W)$-invariant. Schur’s Lemma shows then that $\Gamma^2$ is a scalar multiple of the identity. But we have:

$$
\Gamma(\epsilon_{123}) = -\epsilon_{456},
$$

$$
\Gamma(\epsilon_{456}) = (-1)^{|\{4,5,6\} \cap \{p+1,...,6\}} \epsilon_{123},
$$

which gives:

$$
\Gamma^2 = \begin{cases} 
\text{id} & \text{if } p = 3 \text{ or } 5, \\
-\text{id} & \text{if } p = 4 \text{ or } 6.
\end{cases}
$$

For the rest of the section, we assume $\Gamma^2 = \text{id}$, that is, $p = 3$ or 5.

As $\Gamma^2 = \text{id}$, the subspace of fixed elements, $((\bigwedge^3 W)^\Gamma)$, is an irreducible $\mathfrak{su}(W, h)$-module, and $\bigwedge^3 W = (\bigwedge^3 W)^\Gamma \oplus i(\bigwedge^3 W)^\Gamma$. As

$$
\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{sl}(W)}((\bigwedge^3 W \otimes_{\mathbb{C}} \bigwedge^3 W, \mathbb{C})) = 1,
$$

this complex vector space is spanned by the map $x \otimes y \mapsto (x|y)$. Hence the (real) dimension of $\text{Hom}_{\mathfrak{su}(W, h)}((\bigwedge^3 W)^\Gamma \otimes_{\mathbb{R}} (\bigwedge^3 W)^\Gamma, \mathbb{R})$ is also 1 and there is a complex number $\alpha$ such that $(x|y) \in \mathbb{R}\alpha$ for any $x, y \in (\bigwedge^3 W)^\Gamma$. The elements $\epsilon_{123} - \epsilon_{456}$ and $i(\epsilon_{123} + \epsilon_{456})$ are in $(\bigwedge^3 W)^\Gamma$, and

$$
(\epsilon_{123} - \epsilon_{456}|i(\epsilon_{123} + \epsilon_{456})) = 2i.
$$

We conclude that the restriction of $(\cdot|\cdot)$ to $(\bigwedge^3 W)^\Gamma$ takes values in $\mathbb{R}i$. In other words, the map $x \otimes y \mapsto i(x|y)$ spans $\text{Hom}_{\mathfrak{su}(W, h)}((\bigwedge^3 W)^\Gamma \otimes_{\mathbb{R}} (\bigwedge^3 W)^\Gamma, \mathbb{R})$.

Also, we know from Sect. 4.6 that the complex vector space $\text{Hom}_{\mathfrak{sl}(W)}((\bigwedge^3 W \otimes_{\mathbb{C}} \bigwedge^3 W, \mathfrak{sl}(W))$ is one-dimensional and spanned by the map $x \otimes y \mapsto d_{x,y}$ in (4.13). Therefore, the real vector space $\text{Hom}_{\mathfrak{su}(W, h)}((\bigwedge^3 W \otimes_{\mathbb{R}} (\bigwedge^3 W)^\Gamma, \mathfrak{su}(W, h))$ is one-dimensional too. We conclude that there is a complex number $\beta$ such that the $d_{x,y} \in \beta \mathfrak{su}(W, h)$ for all $x, y \in (\bigwedge^3 W)^\Gamma$. Equation (2.2) then shows that $d_{x,y} \in i\mathfrak{su}(W, h)$ for any $x, y \in (\bigwedge^3 W)^\Gamma$.

Therefore, $((\bigwedge^3 W)^\Gamma, i[\cdot, \cdot, \cdot], i(\cdot|\cdot))$ is a real simple symplectic triple system, for $(x|y)$ in (4.12) and $[x, y, z] = d_{x,y}(z)$ in (4.13), with inner derivation algebra isomorphic to $\mathfrak{su}(W, h)$.

In conclusion, we have constructed real simple symplectic triple systems with inner derivation algebras isomorphic to $\mathfrak{su}_3$ and $\mathfrak{su}_5$, depending on the signature of the hermitian form $h$ being 0 or 4, respectively, and with enveloping algebras necessarily real forms of the Lie algebra of type $E_6$. Proposition 5.3 shows that the signature of the Killing form of the enveloping algebra of a symplectic triple system is one plus the signature of the Killing form of its inner derivation algebra. The signature of the Killing form of $\mathfrak{su}_{p, q}$ is $1 - (p - q)^2$, so $\text{sign}(\mathfrak{su}_3, 3) = 1$ and $\text{sign}(\mathfrak{su}_5, 1) = -15$. 


We conclude that the standard enveloping algebras of the triple systems above are isomorphic to \( e_{6,2} \) and \( e_{6,-14} \), respectively.

For the case of standard enveloping algebra of signature \(-14\), the reader can also consult [12,§V.A.], where the related \( \mathbb{Z} \)-grading on \( e_{6,-14} \) is described in detail in order to look for the fine gradings obtained by refining the \( \mathbb{Z} \)-grading (gradings which are compatible with our linear model).

5.5 Non-split \( E_7 \) types

The half-spin modules for \( so_{12}(\mathbb{C}) \) descend to \( \mathbb{R} \) for \( so_{p,q}(\mathbb{R}) \) (\( p + q = 12, \ p \geq 6 \)), if and only if the even Clifford algebra \( Cl(p,q)(\mathbb{R}) \) is isomorphic to \( Mat_{32}(\mathbb{R}) \times Mat_{32}(\mathbb{R}) \), and this is the case if and only if the discriminant of the quadratic form with signature \( p - q \) and the Brauer class of \( Cl(p,q)(\mathbb{R}) \) are trivial. This happens if and only if \( p \) is even and \( p - q \) is congruent to 0 or 2 modulo 8 ([28,p. 125]). That is, if and only if \( p = 6 \) or \( p = 10 \). In case \( p = 6 \), \( so_{p,q}(\mathbb{R}) = so_{6,6}(\mathbb{R}) \) is the split simple Lie algebra of type \( D_6 \). Let us consider now the other case: \( p = 10 \).

Let then \( V \) be a real vector space of dimension 12 endowed with a non-degenerate quadratic form \( q_V \) of signature 8. Let \( \{w_1, \ldots, w_{12}\} \) be an orthogonal basis relative to \( q_V \) with \( q_V(w_i) = 1 \) for \( 1 \leq i \leq 10 \), and \( q_V(w_i) = -1 \) for \( i = 11, 12 \).

As in Sect. 4.7, let \( W \) be a six-dimensional complex space. The complexified space \( V^\mathbb{C} = V \otimes_{\mathbb{R}} \mathbb{C} \) can be isometrically identified with \( W \oplus W^* \) by means of

\[
\begin{align*}
  w_i &\leftrightarrow e_i + e^i & \text{for } i = 1, \ldots, 6, \\
  w_{6+i} &\leftrightarrow i(e_i - e^i) & \text{for } i = 1, \ldots, 4, \\
  w_{6+i} &\leftrightarrow e_i - e^i & \text{for } i = 5, 6.
\end{align*}
\]

In this way, the real Clifford algebra \( Cl(V, q_V) \) embeds into the complex Clifford algebra \( Cl(W \oplus W^*, q) \). Consider the isomorphism \( \Lambda \) in (4.15). The element \( x = w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} \) in \( Cl(V, q_V) \subset Cl(W \oplus W^*, q) \) commutes with \( w_i \), for \( i \neq 7, 8, 9, 10 \), anticommutes with \( w_i \), for \( i = 7, 8, 9, 10 \) and it satisfies \( x^2 = 1 \), so \( \Lambda(x) \in \text{End}_\mathbb{C}(\bigwedge W) \) is such that \( \Lambda(x)^2 = \text{id} \).

Consider the conjugate-linear endomorphism \( \Delta \) of \( \bigwedge W \) that is the identity on the basic elements \( e_I \), for \( I \subset \{1, \ldots, 6\} \). Then \( \Delta^2 = \text{id}, \Delta \) commutes with \( \Lambda(e_i) = I_{e_i} \) and \( \Lambda(e^i) = \delta_{e_i} \) for all \( i \), and hence \( \Delta \) commutes too with \( \Lambda(w_i) \), for \( i \neq 7, 8, 9, 10 \), and anticommutes with \( \Lambda(w_i) \) for \( i = 7, 8, 9, 10 \). This shows that \( \Delta \) and \( \Lambda(x) \) commute.

Take the composition \( \Gamma = \Delta \Lambda(x) \). This is a conjugate-linear endomorphism of \( \bigwedge W \), it preserves its even and odd parts, its order is 2: \( \Gamma^2 = \text{id} \), and it commutes with \( \Lambda(w_i) \) for all \( i = 1, \ldots, 12 \). Hence \( \Gamma \) commutes with the action of \( Cl(V, q_V) \) on \( \bigwedge W \) through \( \Lambda \).

It turns out that the fixed subspace \( (\bigwedge W)^\Gamma = \{ z \in \bigwedge W : \Gamma(z) = z \} \), and its even and odd parts, are invariant under the action of \( Cl(V, q_V) \), and hence the even and odd parts are the half-spin modules of \( so(V, q_V) \).

Denote by \( T = \bigwedge_0 W \) the complex symplectic triple system as in Sect. 4.7. The unique, up to scalars, non-zero \( so(W \oplus W^*, q) \)-invariant bilinear form on \( T \) is the
restriction (\(\cdot | \cdot\)) of \(b\) in (4.14). Hence there is a scalar \(\alpha \in \mathbb{C}\) such that the restriction of (\(\cdot | \cdot\)) to the fixed subspace satisfies (\(T^\Gamma \mid T^\Gamma\)) \(\in \mathbb{R}\alpha\). Let us check that this scalar is real. Note that \(\Gamma(1) = e_{1234}, \Gamma(e_{56}) = e_{123456},\) and that \(1 + \Gamma(1)\) and \(e_{56} + \Gamma(e_{56})\) are in \(T^\Gamma\). We compute

\[
(1 + e_{1234}|e_{56} + e_{123456}) = (1|e_{123456}) + (e_{1234}|e_{56}) = 2 (\in \mathbb{R}).
\]

We conclude, as in Sect. 5.4, that \((T^\Gamma, [\cdot, \cdot, \cdot], (\cdot | \cdot))\) is a real symplectic triple system, with inner derivation algebra isomorphic to \(\mathfrak{so}(V, q_V) \cong \mathfrak{so}_{10,2}(\mathbb{R})\), and standard enveloping Lie algebra isomorphic to \(e_7, -25\) (using Proposition 5.3 and that the signature of the Killing form of \(\mathfrak{so}_{10,2}(\mathbb{R})\) is \(-26\)).

There is another example of real simple symplectic triple system with standard enveloping Lie algebra of type \(E_7\).

To describe it, let now \(\mathcal{U}\) be a right \(\mathbb{H}\)-module of dimension 6 endowed with a skew-hermitian non-degenerate form \(h: \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{H}\). Consider the natural copy of \(\mathbb{C} = \mathbb{R} + \mathbb{R}i\) contained in \(\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k\). By restriction of scalars, we may look at \(\mathcal{U}\) as a complex vector space. We will write \(\mathcal{U}_C\) when we do this. Then \(\mathcal{U}_C\) has dimension 12, and it is endowed with a non-degenerate quadratic form \(q_{\mathcal{U}}\) defined by:

\[
q_{\mathcal{U}}(w) = \frac{1}{2} \text{proj}_C(jh(w, w)), \quad (5.15)
\]

where we use the natural projection \(\text{proj}_C(a + bi + cj + dk) = a + bi\). Note that for all \(w \in \mathcal{U}\), \(q_{\mathcal{U}}(wj)\) is the (complex) conjugate of \(q_{\mathcal{U}}(w)\):

\[
q_{\mathcal{U}}(wj) = \overline{q_{\mathcal{U}}(w)}. \quad (5.16)
\]

Denote by \(b_{\mathcal{U}}\) the associated (bilinear, complex) polar form: \(b_{\mathcal{U}}(u, v) = q_{\mathcal{U}}(u + v) - q_{\mathcal{U}}(u) - q_{\mathcal{U}}(v)\). Note that for any \(u, v \in \mathcal{U}\) we have \(jh(u, v) = b_{\mathcal{U}}(u, v) + [u|v]j\) for some \([u|v] \in \mathbb{C}\), and hence:

\[
h(u, v) = \begin{cases} 
jh(uj, v) = b_{\mathcal{U}}(uj, v) + [uj|v]j, \\
-j^2h(u, v) = -j(b_{\mathcal{U}}(u, v) + [u|v]j) = [u|v] - b_{\mathcal{U}}(u, v)j.
\end{cases}
\]

because \(j\alpha = \overline{\alpha}j\) for all \(\alpha \in \mathbb{C}\). Thus we recover \(h\) as \(h(u, v) = b_{\mathcal{U}}(uj, v) - b_{\mathcal{U}}(u, v)j\).

The simple Lie algebra \(\mathfrak{so}^*(\mathcal{U}, h)\) appears now as follows, where \(R_j\) denotes the multiplication by \(j\) on \(\mathcal{U}\):

\[
\mathfrak{so}^*(\mathcal{U}, h) = \{ f \in \text{End}_\mathbb{H}(\mathcal{U}) : h(f(u), v) + h(u, f(v)) = 0 \forall u, v \in \mathcal{U}\} = \{ f \in \text{End}_C(\mathcal{U}) : fR_j = R_j f, b_{\mathcal{U}}(f(u), v) + b_{\mathcal{U}}(u, f(v)) = 0 \forall u, v \in \mathcal{U}\} = \mathfrak{so}(\mathcal{U}_C, q_{\mathcal{U}}) \cap \{ f \in \text{End}_C(\mathcal{U}) : fR_j = R_j f \}.
\]  

(5.17)

Pick a basis \(\{v_1, v_2, v_3, w_1, w_2, w_3\}\) of \(\mathcal{U}\) (over \(\mathbb{H}\)) with \(h(v_i, v_j) = 0 = h(w_i, w_j)\), and \(h(v_i, w_j) = \delta_{ij}\), for \(1 \leq i, j \leq 3\). As in Sect. 4.7, let \(W\) be a six-dimensional
The conjugate-linear map \( \pi : U_{123456} \) of dimension 16 and proves that it satisfies

\[
\pi \cdot \pi = -1.
\]

for \( i = 1, 2, 3 \). That is, \( W \) is identified with \( v_1 \mathbb{H} + v_2 \mathbb{H} + v_3 \mathbb{H} \) and \( W^* \) with \( w_1 \mathbb{H} + w_2 \mathbb{H} + w_3 \mathbb{H} \). In this way, the (complex) Clifford algebra \( \mathfrak{C}(U_\mathbb{C}, q_{\mathbb{C}}) \) is identified with \( \mathfrak{C}(W \oplus W^*, q) \) in Sect. 4.7.

The conjugate-linear map \( R_j \) on \( U_\mathbb{C} \) induces a conjugate-linear algebra automorphism \( \tilde{R}_j \) of \( \mathfrak{C}(U_\mathbb{C}, q_{\mathbb{C}}) \): \( \tilde{R}_j(u_1 \cdots u_r) = (u_1 j) \cdots (u_r j) \) for any \( r \geq 0 \) and \( u_1, \ldots, u_r \in U \). In the same vein, it induces a conjugate-linear algebra automorphism \( \tilde{R}_j \) of the exterior algebra \( \bigwedge W \): \( \tilde{R}_j(u_1 \wedge \cdots \wedge u_r) = (u_1 j) \wedge \cdots \wedge (u_r j) \) for any \( r \geq 0 \) and \( u_1, \ldots, u_r \in W \). Note that \( \tilde{R}_j \) has order 2 in the even Clifford algebra \( \mathfrak{C}_0(U_\mathbb{C}, q_{\mathbb{C}}) \), and order 4 in the odd part. Also, \( \tilde{R}_j \) has order 2 in \( T = \bigwedge \mathbb{R} \), and order 4 in \( \bigwedge \mathbb{H} \).

Using (5.16), it follows that the algebra isomorphism \( \Lambda \) in (4.15) satisfies

\[
\Lambda \tilde{R}_j(x) \tilde{R}_j = \tilde{R}_j \Lambda(x)
\]

for all \( x \in \mathfrak{C}(U_\mathbb{C}, q_{\mathbb{C}}) \), that is, \( \Lambda \tilde{R}_j = \text{Ad} \tilde{R}_j \Lambda \). Thus we have

\[
\tilde{R}_j(x.s) = \tilde{R}_j(x) \tilde{R}_j(s)
\]

for \( x \in \mathfrak{C}(U_\mathbb{C}, q_{\mathbb{C}}) \) and \( s \in T = \bigwedge \mathbb{R} \). We conclude that the space of fixed elements \( T \tilde{R}_j \) is a module for the fixed subalgebra \( \tilde{C} = (\mathfrak{C}_0(U_\mathbb{C}, q_{\mathbb{C}})) \tilde{R}_j \), whose complexification is the half-spin module \( T = \bigwedge \mathbb{R} \).

Note that \( \text{Ad} \tilde{R}_j(\sigma_{x,y}) = \sigma_{x j y j} \), and this coincides with \( \tilde{R}_j(\sigma_{x,y}) \) if we see the element \( \sigma_{x,y} \in \mathfrak{so}(U_\mathbb{C}, q_{\mathbb{C}}) \) as the element \( \frac{-1}{2} [x, y] \) in \( \mathfrak{C}_0(U_\mathbb{C}, q_{\mathbb{C}}) \). Hence \( \mathfrak{so}^*(U, h) \) is contained in \( \tilde{C} \) by (5.17), and acts in \( T \tilde{R}_j \).

The unique, up to scalars, non-zero \( \mathfrak{so}(U_\mathbb{C}, q_{\mathbb{C}}) \)-invariant bilinear form on \( T \) is the restriction \( (\cdot | \cdot) \) of \( \beta_0 \) in (4.14). Hence there is a scalar \( \alpha \in \mathbb{C} \) such that the restriction of \( (\cdot | \cdot) \) to the fixed subspace satisfies \( (T \tilde{R}_j | T \tilde{R}_j) \subseteq \mathbb{R} \alpha \). Note that the element \( e_{123456} = v_1 \wedge v_2 \wedge v_3 \wedge v_1 j \wedge v_2 j \wedge v_3 j \) is fixed by \( \tilde{R}_j \), as well as the element 1. Since \( (1 | e_{123456}) = 1 \), it follows that \( \alpha \) belongs to \( \mathbb{R} \), and we conclude, as above, that \( (T \tilde{R}_j, [\cdot, \cdot, \cdot, \cdot], (\cdot | \cdot)) \) is a real symplectic triple system, with inner derivation algebra isomorphic to \( \mathfrak{so}^*(U, h) \cong \mathfrak{so}^*_{12} \), and standard enveloping Lie algebra isomorphic to \( \mathfrak{e}_{7,-5} \) (since the signature of the Killing form of \( \mathfrak{so}^*_{12} \) is \( -6 \)).

**Remark 5.5** The conjugate-linear map \( \tilde{R}_j \) squares to \( -\text{id} \) on \( \bigwedge \mathbb{H} U \), commuting with the action of the fixed subalgebra \( \tilde{C} \). This gives \( \bigwedge \mathbb{H} W \) the structure of an \( \mathbb{H} \)-module of dimension 16 and proves that \( \tilde{C} \) is isomorphic to \( \text{Mat}_{16}(\mathbb{H}) \).

(Even) Clifford algebras can be defined for any finite-dimensional central simple algebra endowed with an orthogonal involution over a field of characteristic not two. According to [22, §3], \( \tilde{C} \) is the even Clifford algebra of the pair \( (\text{End}_{\mathbb{H}}(U), \ast) \), where \( \ast \) denotes the involution obtained as the conjugation associated to the skew-hermitian form \( h \).
5.6 Non-split $E_8$ type

We will follow here the notation and results in Sect. 4.8 over $\mathbb{C}$. Thus $U$ is an eight-dimensional complex vector space, but this time endowed with a non-degenerate hermitian form $h : U \times U \rightarrow \mathbb{C}$. Again, changing $h$ by $-h$ if necessary, we assume the signature of $h$ is $\geq 0$. We fix an orthogonal basis $\{e_i : 1 \leq i \leq 8\}$ with $h(e_i, e_i) = 1$ for $1 \leq i \leq p$, and $h(e_i, e_i) = -1$ for $p + 1 \leq i \leq 8$, with $p \geq 4$. Use this basis to define the determinant map $\det : \bigwedge U \rightarrow \mathbb{C}$, so that $\det(\bigwedge^i U) = 0$ for $i < 8$, and $\det(e_12345678) = 1$.

The hermitian form $h$ induces a conjugate-linear automorphism $\Upsilon$ of $L = sl(U) \oplus \bigwedge^4 U$ as follows:

- $\Upsilon(l) = -l^*$ for $l \in sl(U)$, where $l^*$ is the adjoint of $l$ relative to $h$: $h(l(v), w) = h(v, l^*(w))$ for all $v, w \in U$.
- $h$ induces a conjugate-linear map for any $s = 0, \ldots, 8$:

$$\hat{h}_s : \bigwedge^s U \rightarrow \bigwedge^s U^* \quad w_1 \wedge \cdots \wedge w_s \mapsto h(\cdot, w_1) \wedge \cdots \wedge h(\cdot, w_s).$$

Then for all $x \in \bigwedge^4 U$, we define, using (4.17),

$$\Upsilon(x) := (\hat{h}_4)^{-1} \Phi_4(x).$$

In particular, for any increasing sequence $I$ of size 4, we get:

$$\Upsilon(e_I) = (-1)^{|I_1|}(-1)^{|I \cap I_p|}e_I,$$

where $I_p = \{p + 1, \ldots, 8\}$, because $h(\cdot, e_i)$ is $e^i$ for $i \leq p$ and $-e^i$ for $i > p$.

**Lemma 5.6** The map $\Upsilon : L \rightarrow L$ thus defined is a Lie algebra automorphism over $\mathbb{R}$ with $\Upsilon^2 = \text{id}$ if and only if $p$ is even.

**Proof** Since $l \mapsto l^*$ is an involution of $\text{End}_\mathbb{C}(U)$, the restriction $\Upsilon|_{sl(U)}$ is a Lie algebra automorphism whose square is the identity. For any increasing sequence $I$ of size 4, we have

$$\Upsilon^2(e_I) = (-1)^{|I_1|}(-1)^{|I \cap I_p|}\Upsilon(e_I) = (-1)^{|I \cap I_p|}(-1)^{|I \cap I_p|}e_I = (-1)^{|I \cap I_p|}e_I = (-1)^8 e_I,$$

so that $\Upsilon^2 = \text{id}$ if and only if $p$ is even.
Assume hence from now on that \( p \) is even. For \( l \in \mathfrak{sl}(U) \) and \( w_1, \ldots, w_4 \in U \), we get
\[
\hat{h}_4(l_4(w_1 \wedge \cdots \wedge w_4)) = h(\cdot, l \cdot w_1) \wedge h(\cdot, w_2) \wedge h(\cdot, w_3) \wedge h(\cdot, w_4) + \cdots \\
+ h(\cdot, w_1) \wedge h(\cdot, w_2) \wedge h(\cdot, w_3) \wedge h(\cdot, l \cdot w_4) \\
= h(l^*(\cdot), w_1) \wedge \cdots \wedge h(\cdot, w_4) + \cdots \\
= (-l^* h(\cdot, w_1)) \wedge \cdots \wedge h(\cdot, w_4) + \cdots \\
= \Upsilon(l).\hat{h}_4(w_1 \wedge \cdots \wedge w_4).
\]
This proves \( \hat{h}_4(\Upsilon(l).\Upsilon(x)) = l.(\hat{h}_4 \Upsilon(x)) = l.\Phi_4(x) \), and hence, for \( l \in \mathfrak{sl}(U) \) and \( x \in \bigwedge^4 U \), we get:
\[
\Upsilon([l, x]) = \Upsilon(l \cdot x) = (\hat{h}_4)^{-1} \Phi_4(l \cdot x) \\
= (\hat{h}_4)^{-1} (l.\Phi_4(x)) \quad \text{(as \( \Phi_4 \) is \( \mathfrak{sl}(U) \) - invariant)} \\
= \Upsilon(l).\Upsilon(x) = [\Upsilon(l), \Upsilon(y)].
\]

Note that for any increasing sequences \( I, J \) with \(|I| = 4 = |J|\), (5.18) gives
\[
(\Upsilon(e_I)|\Upsilon(e_T))^\wedge = (-1)^{|I|}(-1)^{|T|} (e_I|e_T)^\wedge = (e_I|e_T)^\wedge
\]
and \( (\Upsilon(e_I)|\Upsilon(e_J))^\wedge = 0 = (e_I|e_J)^\wedge \) if \( J \neq T \). Hence we get
\[
(\Upsilon(x)|\Upsilon(y))^\wedge = (x|y)^\wedge
\]
for all \( x, y \in \bigwedge^4 U \). Finally, for \( l \in \mathfrak{sl}(U) \) and \( x, y \in \bigwedge^4 U \), we get
\[
\text{tr}\left(\Upsilon(l)\Upsilon([x, y])\right) = \text{tr}(l^*[x, y]^*) = \text{tr}([l^* x, y]) = \text{tr}(\bigwedge^4 x, y)^\wedge \\
= \text{tr}([l^* x, y]) = (l.x|y)^\wedge = (\Upsilon(l \cdot x)|\Upsilon(y))^\wedge \\
= (\Upsilon(l).\Upsilon(x)|\Upsilon(y))^\wedge = \text{tr}(\Upsilon(l)[\Upsilon(x), \Upsilon(y)]).
\]
so that \( \Upsilon([x, y]) = [\Upsilon(x), \Upsilon(y)] \), and \( \Upsilon \) is a conjugate-linear bijection and an automorphism of \( \mathcal{L} \) as a real Lie algebra. \( \square \)

**Proposition 5.7** With the notations above and even \( p \), the fixed subalgebra of \( \Upsilon \):
\( S = \mathcal{L}^\Upsilon := \{ l \in \mathcal{L} : \Upsilon(l) = l \} \) is a form of \( \mathcal{L} \) (i.e., \( S \) is a real subalgebra and \( \mathcal{L} = S \oplus iS \)). Moreover,

- If \( p = 4 \) or \( 8 \), then \( S \) is, up to isomorphism, the simple exceptional real Lie algebra \( \mathfrak{e}_7,7 \) (the split simple real Lie algebra of type \( E_7 \)).
- If \( p = 6 \), then \( S \) is, up to isomorphism, the simple exceptional real Lie algebra \( \mathfrak{e}_7,25 \).
Proof It is clear that $S$ is a real form of $\mathcal{L}$, since it is the fixed subspace by an order 2 conjugate-linear automorphism. The bilinear form $(\cdot, \cdot)_{\mathcal{L}}$ is $\frac{1}{36}$ times the Killing form of $\mathcal{L}$. Hence the restriction of $(\cdot, \cdot)_{\mathcal{L}}$ to $S$ is $\frac{1}{36}$ times the Killing form of $S$ and, in particular, it takes values in $\mathbb{R}$.

Taking into account that the signature of the Killing form of $\text{su}(p, 8-p)$ is $1 - p - (8-p))^2 = 1 - (2p - 8)^2$, and that the number of increasing sequences $I = (1, i, j, k)$ with even $|I \cap I_p|$ is:

- $1 + 3\binom{4}{2} = 19$ if $p = 4$ (so $I_p = \{5, 6, 7, 8\}$,
- $\binom{5}{3} + 5 = 15$ if $p = 6$,
- $\binom{6}{3} = 35$ if $p = 8$,

we conclude that the signature of $\kappa_S$ is:

- $1 + (38 - 32) = 7$ for $p = 4$,
- $-15 + (30 - 40) = -25$ for $p = 6$,
- $-63 + 70 = 7$ for $p = 8$.

This finishes the proof. \hfill \Box

Remark 5.8 We may consider too the real forms of $\mathcal{L}$ obtained as $\overline{S} = \text{su}(U, h) \oplus i\left(\bigwedge^4 U\right)^\top$. The corresponding signatures are:

- $1 - (38 - 32) = -5$ for $p = 4$,
- $-15 - (30 - 40) = -5$ for $p = 6$,
- $-63 - 70 = -133$ for $p = 8$.

Therefore we obtain precise linear models of the other two real simple Lie algebras of type $E_7$: $\mathfrak{e}_{7,-5}$ and the compact one, $\mathfrak{e}_{7,-133}$. This is a nice complement to [1, Chapter 12].

From now on, we restrict to the case $p = 6$ (as for $p = 4$ or $p = 8$, $S$ is the split algebra $\mathfrak{e}_{7,7}$, and the attached symplectic triple system is, up to weak isomorphism, the split one in Sect. 4.7). The aim is to get a symplectic triple system with inner derivation algebra $\mathfrak{e}_{7,-25}$.

On the $\mathcal{L}$-module $T = \bigwedge^2 U \oplus \bigwedge^2 U^*$ consider the order 2 conjugate-linear isomorphism $\Upsilon_T$ given by:

- $\Upsilon_T(x) = \widehat{h}_2(x) \in \bigwedge^2 U^*$ for $x \in \bigwedge^2 U$,
- $\Upsilon_T(y) = (\widehat{h}_2)^{-1}(y) \in \bigwedge^2 U$ for $y \in \bigwedge^2 U^*$.

As in (5.19), we have, for $l \in \mathfrak{sl}(U)$ and $x \in \bigwedge^2 U$,

$$\Upsilon_T(l.x) = \widehat{h}_2(l.x) = \Upsilon(l).\widehat{h}_2(x) = \Upsilon(l).\Upsilon_T(x),$$

and, similarly, $\Upsilon_T(l.y) = \Upsilon(l).\Upsilon_T(y)$ for $y \in \bigwedge^2 U^*$. Also, for increasing sequences $I$ of size 4 and $J$ of size 2, $I \cap J = \emptyset$,

$$\Upsilon_T(e_I, e_J) = (\widehat{h}_2)^{-1}((-1)^{IJ}(-1)^{(I \cup J)(I \cup J)^{\top}}e_{I \cup J}) = (-1)^{IJ}(-1)^{(I \cup J)(I \cup J)^{\top}}(-1)^{(I \cup J)^{\top}\cap 0}e_{I \cup J},$$
while
\[
\Upsilon(e_I).\Upsilon_T(e_J) = (-1)^{|I\cap J|}e_I\Upsilon((-1)^{|I\cap J|}e_J)
= (-1)^{|I\cap J|}(1|\cup J\cap h|)e_I\Upsilon e_J
= (-1)^{|I\cap J|}(1|\cup J\cap h|)\Upsilon((-1)^{|I\cap J|}(1|\cup J\cap h|)e_{I\cap J}.
\]
so we get \(\Upsilon_T(e_I.e_J) = \Upsilon(e_I).\Upsilon_T(e_J)\), and similarly for \(e_I.e_J\). Therefore we obtain
\[
\Upsilon_T(l.x) = \Upsilon(l).\Upsilon_T(x)
\]
for any \(l \in \mathcal{L}\) and \(x \in T\), and hence the subspace of fixed elements \(T^\Upsilon := \{x \in T : \Upsilon_T(x) = x\}\) is invariant under the action of \(S\), with \(T = T^\Upsilon \oplus iT^\Upsilon\).

We know that both the (complex) dimensions of Hom\(_\mathcal{L}(T \otimes_{\mathbb{C}} T, \mathcal{L})\) and Hom\(_\mathcal{L}(T \otimes_{\mathbb{C}} T, \mathbb{C})\) are equal to 1. Therefore, the (real) dimensions of Hom\(_S(T^\Upsilon \otimes_{\mathbb{R}} T^\Upsilon, S)\) and Hom\(_S(T^\Upsilon \otimes_{\mathbb{R}} T^\Upsilon, \mathbb{R})\) are equal to 1 too.

It follows, as in Sect. 5.4, that there is a non-zero scalar \(\alpha \in \mathbb{C}\) such that \(d_{x,y} \in \alpha S\) for any \(x, y \in T^\Upsilon\), where \(d_{x,y}\) is given by (4.20). For \(x = e_{12} + e^{12} = e_{12} + \Upsilon_T(e_{12}) \in T^\Upsilon\), we get
\[
d_{x,x} = d_{e_{12} + e^{12}, e_{12} + e^{12}} = 2d_{e_{12}, e^{12}} = -4(e_2 \otimes e^2 + e_1 \otimes e^1 - \frac{1}{4}\text{id}_U),
\]
that is, \(d_{x,x}\) is the element of \(\mathfrak{sl}(U)\) whose coordinate matrix in our basis is
\[
\text{diag}(-3, -3, 1, 1, 1, 1, 1, 1),
\]
so that \(d_{x,x} \in \mathfrak{i}\mathfrak{su}(U, h)\). We conclude that \(d_{T^\Upsilon, T^\Upsilon} \subseteq iS\) and, as in Sect. 5.4, that \((T^\Upsilon, [\cdot, \cdot, \cdot], i(\cdot|\cdot))\) is a real simple symplectic triple system with inner derivation algebra isomorphic to \(\mathfrak{e}_{7,-25}\). Moreover, by Proposition 5.3, its standard enveloping Lie algebra is, up to isomorphism, the exceptional Lie algebra \(\mathfrak{e}_{8,-24}\).

6 The classification of the real simple symplectic triple systems, up to weak isomorphism

According to the classification of the simple symplectic triple systems over \(\mathbb{C}\), reviewed in Sect. 4, the pairs \((\text{indeter}(T), T)\) for a simple symplectic triple system \(T\) over \(\mathbb{C}\) with classical envelope \(\mathfrak{g}(T)\) are, up to isomorphism, those in the following list:

- **Special** \((\mathfrak{gl}(W), W \oplus W^*)\), for a non-zero vector space \(W\).
- **Orthogonal** \((\mathfrak{sp}(V) \oplus \mathfrak{so}(W), V \otimes W)\), where \(V\) is a vector space of dimension 2 endowed with a non-zero alternating bilinear form \(\langle \cdot|\cdot \rangle\), and \(W\) is a vector space of dimension \(\geq 3\) endowed with a non-degenerate symmetric bilinear form.
- **Symplectic** \((\mathfrak{sp}(W), W)\) for a non-zero even-dimensional vector space \(W\) endowed with a non-degenerate alternating bilinear form.
Recall that, by Corollary 3.4, in order to classify the real forms up to weak isomorphism (Definition 2.3), it is enough to classify the real forms of the pairs in the list above. We will do it now according to the type of \( T \).

The results on this section are summarized in the following theorem. A word of caution is needed here. All the simple real symplectic triple systems in Sect. 4 are called \textit{split}, even though, in the orthogonal case in Sect. 4.2, the real vector space with a nondegenerate symmetric bilinear form \((W, b)\) of dimension \( \geq 3 \) is arbitrary (that is, its Witt index is not necessarily maximal).

\textbf{Theorem 6.1} Any simple real symplectic triple system is, up to weak isomorphism, one of the \textit{split} simple real symplectic triple systems in Sect. 4: special, orthogonal, symplectic, \( G_2 \), \( F_4 \), \( E_6 \), \( E_7 \), \( E_8 \) types, or one of the non-split simple real symplectic triple systems in Sect. 5: unitarian and quaternionic types, plus the non-split \( E_6 \)-types (two possibilities), the non-split \( E_7 \)-types (two possibilities), and the non-split \( E_8 \)-type.

The proof of this result is achieved in the next sections.

\textbf{6.1 Special type}

If \((\mathfrak{h}, U)\) is a real form of a pair of special type, then \( \mathfrak{h} = Z(\mathfrak{h}) \oplus [\mathfrak{h}, \mathfrak{h}] \), with the center \( Z(\mathfrak{h}) \) of dimension 1, and \([\mathfrak{h}, \mathfrak{h}]\) simple (or 0). Moreover, \( Z(\mathfrak{h}) = \mathbb{R}z \) for an element \( z \) such that \( \rho_{\mathfrak{h}}^2 = \pm \text{id} \) (by part (2) of Lemma 3.1, for any \( 0 \neq x \in Z(\mathfrak{h}) \), there is a scalar \( \alpha \in \mathbb{C} \) such that, in the complexification of the pair, we have \( \rho_{\mathfrak{h}}^2 \alpha x = \text{id} \), so that \( \alpha \) is either real or purely imaginary, and the existence of \( z \) above follows).

If \( \rho_{\mathfrak{h}}^2 = \text{id} \), then \( \rho_{\mathfrak{h}} \) is diagonalizable and we can consider the eigenspace decomposition \( U = U_+ \oplus U_- \). Then \( U_+ \) and \( U_- \) are dual modules for the action of \( \mathfrak{h} \), because this is the case after extending scalars to \( \mathbb{C} \). Besides \([\mathfrak{h}, \mathfrak{h}]\) embeds in \( \mathfrak{sl}(U_+) \) and \( \mathfrak{h} \) in \( \mathfrak{gl}(U_+) \). By dimension count \( \mathfrak{h} \) is, up to isomorphism, \( \mathfrak{gl}(U_+) \), and we get the pair \( (\mathfrak{gl}(U_+), U_+ \oplus U_+^*) \), for a non-zero real vector space \( U_+ \). (\textit{Special type}.)

Otherwise \( \rho_{\mathfrak{h}}^2 = -\text{id} \) and \( U \) becomes a complex vector space by defining \( iu = \rho_{\mathfrak{h}}(u) \) for any \( u \in U \). There is a unique, up to scalars, non-zero \( \mathfrak{h} \)-invariant symmetric bilinear form \( b \) on \( U \), because this is so after scalar extension. Define the \( \mathbb{R} \)-bilinear map \( h : U \times U \to \mathbb{C} \) by

\[ h(u, v) = b(u, v) + ib(u, \rho_{\mathfrak{h}}(v)). \]

The \( \mathfrak{h} \)-invariance of \( b \) shows that \( h \) is hermitian and non-degenerate. Moreover, it is also \( \mathfrak{h} \)-invariant and, by dimension count, it follows that \( \mathfrak{h} \) is, up to isomorphism, the unitary Lie algebra \( \mathfrak{u}(U, h) \). (\textit{Unitarian type}.)

\textbf{6.2 Symplectic type}

If \((\mathfrak{h}, U)\) is a real form of a pair of symplectic type, then \( U \) is endowed with a unique, up to scalars, non-degenerate \( \mathfrak{h} \)-invariant alternating form \( (\cdot|\cdot) \) (because this is the case after scalar extension). Thus \( \mathfrak{h} \) embeds in \( \mathfrak{sp}(U, (\cdot|\cdot)) \), and by dimension count \( \rho \) gives an isomorphism \( \mathfrak{h} \simeq \mathfrak{sp}(U, (\cdot|\cdot)) \). (\textit{Symplectic type}.)
6.3 Orthogonal type

If \((\mathfrak{h}, U)\) is a real form of a pair of orthogonal type, then \(\mathfrak{h}_C = \mathfrak{h} \otimes_R \mathbb{C}\) is isomorphic to \(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}_n(\mathbb{C})\), with \(n \geq 3\), and the module \(U \otimes_R \mathbb{C} \cong T\) is isomorphic to the tensor product of the natural (two-dimensional) module for \(\mathfrak{sl}_2(\mathbb{C})\) and the natural \((n\text{-dimensional})\) module for \(\mathfrak{so}_n(\mathbb{C})\). The space \(\text{Hom}_\mathfrak{h}(U \otimes \mathbb{C} U, \mathbb{R})\) is one-dimensional, and its elements are skew-symmetric, due to Lemma 3.1 and its proof. Indeed, the same happens after complexification, and each element \(\eta \in \text{Hom}_{\mathfrak{h}_C}(T \otimes \mathbb{C} U, \mathbb{C})\) provides \(\langle \cdot | \cdot \rangle \otimes \eta : g(T)_1 \otimes g(T)_1 \rightarrow \mathbb{C}\) a \((T)_0\)-invariant map, which is symmetric if \(\eta\) is skew-symmetric.

If \(n \geq 5\), then \(\mathfrak{so}_n(\mathbb{C})\) is simple and not isomorphic to \(\mathfrak{sl}_2(\mathbb{C})\). Hence \(\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2\), with \(\mathfrak{h}_1\) a real form of \(\mathfrak{sl}_2(\mathbb{C})\) and \(\mathfrak{h}_2\) a real form of \(\mathfrak{so}_n(\mathbb{C})\). Besides, the centralizer \(\text{End}_{\mathfrak{h}_2}(U)\) is a real form of \(M_2(\mathbb{C})\), so it is a quaternion algebra. Also, \(\mathfrak{h}_1\) embeds naturally in \(\text{End}_{\mathfrak{h}_2}(U)\) and hence it embeds isomorphically in its derived subalgebra.

Two possibilities appear. If \(\text{End}_{\mathfrak{h}_2}(U)\) is the split quaternion algebra, then \(\mathfrak{h}_1\) is isomorphic to \(\mathfrak{sl}_2(\mathbb{R})\), and \(U\), as a module for \(\text{End}_{\mathfrak{h}_2}(U)\), is a direct sum of copies of its unique irreducible module (the natural module for \(2 \times 2\)-matrices). It follows that \(U = U_1 \otimes_R U_2\), with \(U_1\) the natural module for \(\mathfrak{h}_1 \cong \mathfrak{sl}_2(\mathbb{R})\), and \(U_2\) an irreducible module for \(\mathfrak{h}_2\). Besides, Lemma 2.2 shows that \(U_2\) is endowed with a unique, up to scalars, \(\mathfrak{h}_2\)-invariant non-degenerate symmetric bilinear form \(b\) and, by dimension count, the action of \(\mathfrak{h}_2\) fills \(so(U_2, b)\).

Therefore, up to isomorphism, \(\mathfrak{h}\) is the direct sum of \(\mathfrak{sl}_2(\mathbb{R})\) and of the special orthogonal Lie algebra \(so(U_2, b)\), and \(U\) is the tensor product of the natural modules for \(\mathfrak{sl}_2(\mathbb{R})\) and \(so(U_2, b)\). (Orthogonal type.)

However, if \(\text{End}_{\mathfrak{h}_2}(U)\) is the real division algebra of quaternions \(\mathbb{H} = \mathbb{R}1 + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k\), then \(\mathfrak{h}_1\) embeds isomorphically into \(\mathbb{R}i + \mathbb{R}j + \mathbb{R}k\). In other words, \(\mathfrak{h}_1\) is isomorphic to \(su_2\). The action of the centralizer \(\text{End}_{\mathfrak{h}_2}(U)\) will be considered on the right (note that \([\mathfrak{h}_1, \mathfrak{h}_2]\) equals \(0\)). Thus \(U\) is a (free) right \(\mathbb{H}\)-module. Let \(b_a\) be a non-zero element in \(\text{Hom}_{\mathfrak{h}}(U \otimes_R U, \mathbb{R})\), so that \(b_a\) is a non-degenerate alternating bilinear form on \(U\), and in the same vein as for the special case, consider the \(\mathbb{R}\)-bilinear map \(h : U \times U \rightarrow \mathbb{H}\) given by:

\[
h(u, v) = b_a(u, v) - b_a(u, vi) - b_a(u, vj)j - b_a(u, vk)k.
\]

As we have \(b_a(u, vi) = -b_a(ui, v) = b_a(v, ui)\) for any \(u, v \in U\) by \(\mathfrak{h}_1\)-invariance, it follows that \(h\) is a non-degenerate \(\mathfrak{h}_2\)-invariant skew-hermitian form. By dimension count, \(\mathfrak{h}_2\) is isomorphic to the Lie algebra of skew-adjoint endomorphisms of \(\text{End}_\mathbb{H}(U)\) relative to \(h\), which is the Lie algebra \(so^*(U, h) \cong so_{2n}^*\).

Therefore, in this case, \(\mathfrak{h}\) is, up to isomorphism the direct sum of \(su_2\) and \(so_{2n}^*\), and \(U\) is the natural module for \(so_{2n}^*\), with the natural action of \(su_2\) commuting with the action of \(so_{2n}^*\). (Quaternionic type.)

Now, for \(n = 4\), \(so_4(\mathbb{C})\) is isomorphic to \(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})\), while the tensor product of the natural modules for \(\mathfrak{sl}_2(\mathbb{C})\) and \(so_4(\mathbb{C})\) is, up to isomorphism, the tensor product of the natural modules for each of the three copies of \(\mathfrak{sl}_2(\mathbb{C})\) in \(\mathfrak{sl}_2(\mathbb{C}) \oplus so_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})\). The centroid of \(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})\) is \(\mathbb{C} \times \mathbb{C} \times \mathbb{C}\), and hence the centroid of \(\mathfrak{h}\) is either \(\mathbb{R} \times \mathbb{R} \times \mathbb{R}\) or \(\mathbb{R} \times \mathbb{C}\). Therefore \(\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2\) for
a three-dimensional simple Lie algebra and a six-dimensional semisimple Lie algebra \( h_2 \) which is either the direct sum of two central simple ideals, or a simple Lie algebra with centroid \( C \). Thus the arguments for \( n \geq 5 \) work here, and we get either the orthogonal or the quaternionic type.

Finally, for \( n = 3 \), \( so_3(\mathbb{C}) \) is isomorphic to \( sl_2(\mathbb{C}) \), while the tensor product of the natural modules for \( sl_2(\mathbb{C}) \) and \( so_3(\mathbb{C}) \) is the tensor product of the natural module for the first copy of \( sl_2(\mathbb{C}) \) and the adjoint module for the second copy of \( sl_2(\mathbb{C}) \) in \( sl_2(\mathbb{C}) \oplus so_3(\mathbb{C}) \simeq sl_3(\mathbb{C}) \oplus sl_2(\mathbb{C}) \). There are two possibilities for \( h \). Either \( h \) is the direct sum of two three-dimensional simple Lie algebras, so that the arguments for \( n \geq 5 \) work, or \( h \) is simple with centroid \( C \), and hence it is isomorphic to \( sl_2(\mathbb{C}) \) considered as a real Lie algebra. Note that this is, up to isomorphism, the Lorentz Lie algebra \( so_3,1(\mathbb{R}) \). But the real Lie algebra \( sl_2(\mathbb{C}) \) has no six-dimensional representation whose complexification is the tensor product above. (See, for instance, [31,Example 8.2].) Therefore this last situation cannot occur, and again, for \( n = 3 \), we are in the orthogonal type. (The quaternionic type is impossible here as 3 is odd.)

### 6.4 Exceptional envelope

From the classification of the simple symplectic triple systems over \( \mathbb{C} \) with exceptional envelope reviewed in Sect. 4, and with the notation used there, the pairs \((\mathfrak{ind}(T), T)\) are, up to isomorphism, those in the next list:

- **G2-envelope**: \((a_1, V(\varpi_1))\),
- **F4-envelope**: \((c_3, V(\varpi_3))\),
- **E6-envelope**: \((a_5, V(\varpi_3))\),
- **E7-envelope**: \((d_6, V(\varpi_6))\),
- **E8-envelope**: \((e_7, V(\varpi_1))\).

Note that in all cases, \( T \) is an irreducible module for \( \mathfrak{ind}(T) \).

Let \((h, U)\) be a real form of one of the pairs \((g, V(\Lambda))\) in the list above. This means that \( h \) is a real form of \( g \), and \( U \) is the unique absolutely irreducible module for \( h \) whose complexification is \( V(\Lambda) \).

Given a real form \( h \) of \( g \), such an absolutely irreducible module \( U \) exists if and only if (see [31,§8])

\[
s_0(\Lambda) = \Lambda \quad \text{and} \quad \varepsilon(h, \Lambda) = 1,
\]

where \( s_0 \) is the automorphism of the Dynkin diagram attached to the real form \( h \), and \( \varepsilon(h, \Lambda) \) the Cartan index of the representation of \( h \) on \( U \otimes_{\mathbb{R}} \mathbb{C} \simeq V(\Lambda) \).

For \( a_1, c_3, \) and \( e_7 \), the Dynkin diagram has no proper automorphisms, and for \( a_5 \), its fundamental weight \( \varpi_3 \) is invariant under the non-trivial automorphism of the Dynkin diagram, so only for the \( E_7 \)-envelope we must worry about the first restriction in (6.1).

The information contained in [31,Table 5] gives at once the classification of these real forms \((h, U)\), up to isomorphism. The corresponding simple real Lie algebra \( h \) are given in the following list. The corresponding modules \( U \) are uniquely determined from \( V(\Lambda) \) and have been explicitly constructed in Sects. 4 and 5.

- **G2-envelope**: \( h = sl_2(\mathbb{R}) \), so only the split case appears;
**F₄-envelope:** \( \mathfrak{h} = \mathfrak{sp}_6(\mathbb{R}) \), again only the split case appears, because

\[
\varepsilon(\mathfrak{sp}_{1,2}, \sum_{i=1}^{3} \Lambda_i \varpi_i) = \varepsilon(\mathfrak{sp}_3, \sum_{i=1}^{3} \Lambda_i \varpi_i) = (-1)^{\Lambda_1 + \Lambda_3}
\]

are the Cartan indexes of the irreducible representations of the other real forms of \( \mathfrak{sp}_6(\mathbb{C}) \) (denoted by \( \mathfrak{sp}_{1,2} \) and \( \mathfrak{sp}_3 \) in [31]);

**E₆-envelope:** \( \mathfrak{h} \) is isomorphic to either \( \mathfrak{sl}_6(\mathbb{R}) \) (split case), \( \mathfrak{su}_3 \), or \( \mathfrak{su}_5 \), because we have

\[
\varepsilon(\mathfrak{su}_6, \sum_{i=1}^{5} \Lambda_i \varpi_i) = (-1)^{\Lambda_3},
\]

\[
\varepsilon(\mathfrak{su}_{4,2}, \sum_{i=1}^{5} \Lambda_i \varpi_i) = (-1)^{5\Lambda_3},
\]

\[
\varepsilon(\mathfrak{sl}_3(\mathbb{H}), \sum_{i=1}^{5} \Lambda_i \varpi_i) = (-1)^{\Lambda_1 + \Lambda_3 + \Lambda_5}.
\]

**E₇-envelope:** \( \mathfrak{h} \) is isomorphic to either \( \mathfrak{so}_p,12-p(\mathbb{R}) \), where \( p \) and \( 6-p \) are even, or to \( \mathfrak{so}_6^* \). Hence we obtain three different possibilities: \( \mathfrak{so}_{6,6}(\mathbb{R}) \) (split case), \( \mathfrak{so}_{10,2}(\mathbb{R}) \), and \( \mathfrak{so}_{12}^* \) (denoted by \( \mathfrak{u}_6^*(\mathbb{H}) \) in [31]), because we have

\[
\varepsilon(\mathfrak{so}_{12}^*, \sum_{i=1}^{6} \Lambda_i \varpi_i) = (-1)^{\Lambda_1 + \Lambda_3 + \Lambda_5}
\]

(which equals 1 for \( \varpi_6 \)), while

\[
\varepsilon(\mathfrak{so}_{12}(\mathbb{R}), \sum_{i=1}^{6} \Lambda_i \varpi_i) = (-1)^{3(\Lambda_5 + \Lambda_6)},
\]

\[
\varepsilon(\mathfrak{so}_{4,8}(\mathbb{R}), \sum_{i=1}^{6} \Lambda_i \varpi_i) = (-1)^{\Lambda_5 + \Lambda_6}.
\]

**E₈-envelope:** \( \mathfrak{h} \) is isomorphic to either \( \mathfrak{e}_{7,7} \) (split case) or to \( \mathfrak{e}_{7,-25} \), because we have

\[
\varepsilon(\mathfrak{e}_{7,-133}, \sum_{i=1}^{7} \Lambda_i \varpi_i) = \varepsilon(\mathfrak{e}_{7,-5}, \sum_{i=1}^{7} \Lambda_i \varpi_i) = (-1)^{\Lambda_1 + \Lambda_3 + \Lambda_7}.
\]

**7 The classification of the real simple symplectic triple systems**

In this final section, we will prove that weakly isomorphic simple real symplectic triple systems are actually isomorphic, so the list in Theorem 6.1 gives the classification up to isomorphism too.

Over any field \( \mathbb{F} \) with \( \mathbb{F} = \mathbb{F}^2 \), it is trivial to check that weakly isomorphic systems are isomorphic (see the argument prior to Definition 2.3). However, over \( \mathbb{R} \) the situation is much subtler although, eventually, the same result holds: weakly isomorphic simple symplectic triple systems are isomorphic.

Our proof is based on the following preliminary result.

**Lemma 7.1** Let \( T \) be a real symplectic triple system endowed with a \( \mathbb{Z}/4 \)-grading with support \( \{1, 3\} \). Then \( T \) is isomorphic to \( T^{[1]} \).
**Proof** Let \( T = T_1 \oplus T_3 \) be the given \( \mathbb{Z}/4 \)-grading, i.e., \( \{ T_i, T_j, T_k \} \subset T_{i+j+k} \). Then the linear map \( T \rightarrow T \) given by \( x \mapsto x \) for \( x \in T_1 \) and \( x \mapsto -x \) for \( x \in T_3 \) is an isomorphism \( T \rightarrow T[-1] \). □

**Remark 7.2** Using that \([x, y, z] - [x, z, y] = (x|y)z - (x|z)y + 2(y|z)x\), it is easy to see that any grading by an abelian group of \( \langle T, [\cdot, \cdot, \cdot] \rangle \) satisfies \( T_g \) isomorphic to \( T[-1] \).

**Proposition 7.3** Any classical simple real symplectic triple system \( T \) is isomorphic to \( T[-1] \).

**Proof** We will check that Lemma 7.1 applies in all cases.

**Special type:** \( T = W \oplus W^* \) and \( \text{indet}(T) = \mathfrak{gl}(W) \) for a vector space \( W \). Then \( T \) is \( \mathbb{Z}/4 \)-graded with \( T_1 = W \) and \( T_3 = W^* \) according to (4.1).

**Orthogonal type:** In this case, \( T = V \otimes \mathbb{R} W \), where \( V \) is a two-dimensional vector space endowed with a non-zero skew-symmetric bilinear form, and \( W \) is a vector space endowed with a non-degenerate symmetric bilinear form. Moreover, \( \text{indet}(T) = \mathfrak{so}(V) \oplus \mathfrak{so}(W) \). Let \( h \) be a diagonalizable element of \( \mathfrak{so}(V) \) with eigenvalues \( \pm 1 \). Then \( V = \mathbb{R} u \oplus \mathbb{R} v \), with \( hu = u \) and \( hv = -v \). Again \( T \) is \( \mathbb{Z}/4 \)-graded with \( T_1 = u \otimes W \) and \( T_3 = v \otimes W \), due to (4.2).

**Symplectic type:** Here \( T \) is a real vector space endowed with a non-degenerate skew-symmetric form \( (\cdot | \cdot) \). Take maximal complementary isotropic subspaces \( T_1 \) and \( T_3 \). These give a \( \mathbb{Z}/4 \)-grading of \( T \) as a vector space. Then \((\cdot | \cdot)\) is a homogeneous map of degree 0, and hence this grading is a grading as a triple system.

**Unitarian type:** Here \( T \) is a complex vector space, endowed with a non-degenerate hermitian form \( h: T \times T \rightarrow \mathbb{C} \). The skew-symmetric form and triple product on \( T \) are given by the following formulas, from (5.2) and (5.8):

\[
(x|y) = -\frac{i}{2}(h(x, y) - h(y, x)),
\]

\[
[x, y, z] = (x|y)z + i(h(z, x)y + h(z, y)x + h(x, y)z).
\]

Consider the \( \mathbb{Z}/4 \)-grading on \( \mathbb{C} \) with \( \mathbb{C}_0 = \mathbb{R}1, \mathbb{C}_2 = \mathbb{R}i \). Take an ‘orthonormal \( \mathbb{C} \)-basis’ \( \{ u_1, \ldots, u_n \} \), so \( h(u_r, u_s) = 0 \) if \( r \neq s \), and \( h(u_r, u_r) \) is either 1 or \(-1\). Then \( T \) is a \( \mathbb{Z}/4 \)-graded module over \( \mathbb{C} \) with \( T = T_1 \oplus T_3 \), where \( T_1 = \sum_{r=1}^n \mathbb{R} u_r \) and \( T_3 = i T_1 \). It follows that \( h(\cdot, \cdot) \), considered as a bilinear form over \( \mathbb{R} \), is homogeneous of degree \( 2 \), and hence both \( ih(\cdot, \cdot) \) and \((\cdot | \cdot)\) are homogeneous of degree 0. Therefore, this \( \mathbb{Z}/4 \)-grading on \( T \) is a grading as a triple system.

**Quaternionic type:** In this case, \( T \) is a right \( \mathbb{H} \)-vector space, endowed with a non-degenerate skew-hermitian form \( h: T \times T \rightarrow \mathbb{H} \). The skew-symmetric form and triple product are given by the following formulas, from (5.14) and (5.13):

\[
(x|y) = \frac{1}{2}(h(x, y) - h(y, x)),
\]

\[
[x, y, z] = xh(y, z) + yh(x, z) + zh(x, y) - z(x|y).
\]
Consider the \( \mathbb{Z}/4 \)-grading on \( \mathbb{H} \) with \( \mathbb{H}_0 = \mathbb{R} l + \mathbb{R} j, \mathbb{H}_2 = \mathbb{R} i + \mathbb{R} k \). Take an \( \mathbb{H} \)-basis \( \{ u_1, \ldots, u_n \} \) with \( h(u_r, u_s) = 0 \) if \( r \neq s \), and \( h(u_r, u_r) = 1 \). (This is always possible.) Then \( T \) is a \( \mathbb{Z}/4 \)-graded module over \( \mathbb{H} \) with \( T = T_1 \oplus T_3 \), for the real subspaces \( T_1 = \sum_{r=1}^n u_r \mathbb{H}_0 \) and \( T_3 = \sum_{r=1}^n u_r \mathbb{H}_2 \). It follows that \( h(\cdot, \cdot) \), considered as a bilinear form over \( \mathbb{R} \), is homogeneous of degree 0, and so is \( (\cdot, \cdot) \), and hence this \( \mathbb{Z}/4 \)-grading on \( T \) is a grading as a triple system.

\[ \square \]

**Proposition 7.4** Any exceptional simple real symplectic triple system \( T \) is isomorphic to \( T^{[-1]} \).

**Proof** Again we will check that Lemma 7.1 applies in all cases.

**Split types:** In the \( G_2 \)-type, \( \text{indet}(T) \) is the Lie algebra \( \mathfrak{sl}_2(\mathbb{R}) \) and \( T \) is its four-dimensional irreducible module. Let \( h \) be a diagonalizable element of \( \mathfrak{sl}_2(\mathbb{R}) \), chosen as usual so that the eigenvalues of \( \text{ad}_h \) are 0 and \( \pm 2 \). Then the eigenvalues of the action of \( h \) on \( T \) are \( \pm 1 \) and \( \pm 3 \): \( T = T_{-3} \oplus T_{-1} \oplus T_1 \oplus T_3 \). Then \( T \) is \( \mathbb{Z}/4 \)-graded with \( T_1 = T_{-3} \oplus T_1 \) and \( T_3 = T_{-1} \oplus T_3 \).

In the \( F_4 \)-type, \( \text{indet}(T) = \text{sp}(W, b) \) is the symplectic Lie algebra of a six-dimensional vector space \( W \) with an alternating form \( b \), and \( T \) is the kernel of the \( \text{sp}(W) \)-invariant linear map \( \bigwedge^3 W \rightarrow W : x_1 \wedge x_2 \wedge x_3 \mapsto b_n(x_1, x_2)x_3 + b_n(x_2, x_3)x_1 + b_n(x_3, x_1)x_2 \). As in Sect. 4.5, once we fix a Cartan subalgebra of \( \text{indet}(T) \), the weights of \( W \) relative to it are \( \pm \epsilon_i, 1 \leq i \leq 3 \), and, with the natural ordering with \( \epsilon_1 > \epsilon_2 > \epsilon_3 > 0 \), \( T \) is the irreducible module with highest weight \( \epsilon_1 + \epsilon_2 + \epsilon_3 \), whose weights are \( \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \) and \( \pm \epsilon_i, 1 \leq i \leq 3 \), all with multiplicity 1. The homomorphism \( \mathbb{Z} \epsilon_1 \oplus \mathbb{Z} \epsilon_2 \oplus \mathbb{Z} \epsilon_3 \rightarrow \mathbb{Z}/4 \) given by \( \epsilon_i \mapsto \bar{1} \) gives a \( \mathbb{Z}/4 \)-grading of \( T \) with support \( \{ \bar{1}, \bar{3} \} \).

In the \( E_6 \)-type, \( \text{indet}(T) = \mathfrak{sl}(W) \) is the six-dimensional vector space \( W \), and \( T \) is its module \( \bigwedge^3 W \). Fix a basis of \( W \) and take the element \( h \in \mathfrak{sl}(W) \) whose coordinate matrix is \( \text{diag}(1, 1, 1, -1, -1, -1) \). The eigenspaces of the action of \( h \) give a \( \mathbb{Z} \)-grading of \( T \) with support \( \{ -3, -1, 1, 3 \} \). As for the \( G_2 \)-type, \( T \) is \( \mathbb{Z}/4 \)-graded with \( T_1 = T_{-3} \oplus T_1 \) and \( T_3 = T_{-1} \oplus T_3 \).

In the \( E_7 \)-type, \( \text{indet}(T) = \mathfrak{so}(W \oplus W^*) \) for a six-dimensional vector space \( W \), and \( T = \bigwedge^3 W = \bigoplus_{i=0}^2 \bigwedge^i W \). Now \( \text{indet}(T) \) is graded by \( \mathbb{Z}/4 \) with \( \text{indet}(T)_{\bar{0}} = \sigma_{W, W^*} \) and \( \text{indet}(T)_{\bar{2}} = \sigma_{W, W} \oplus \sigma_{W^*, W^*} \). Then \( T \) is a \( \mathbb{Z}/4 \)-graded vector space with

\[ T_1 = \bigwedge^0 W \oplus \bigwedge^4 W \quad \text{and} \quad T_3 = \bigwedge^2 W \oplus \bigwedge^6 W. \quad (7.1) \]

The alternating bilinear form \( b_n \) in (4.14) is homogeneous of degree 0 on \( T \), and (4.16) shows that this \( \mathbb{Z}/4 \)-grading on \( T \) is a grading as a triple system.

Finally, in the \( E_8 \)-type, \( \text{indet}(T) \) is the Lie algebra of type \( E_7 \), which appears as \( \mathfrak{sl}(U) \oplus \bigwedge^4 U \), for an eight-dimensional vector space \( U \). Our symplectic triple system is \( T = \bigwedge^2 U \oplus \bigwedge^2 U^* \). This decomposition is a \( \mathbb{Z}/4 \)-grading with \( T_1 = \bigwedge^2 U \) and \( T_3 = \bigwedge^2 U^* \).

We will deal now with the remaining non-split cases.

**\( E_6 \)-types:** Use here the notations in Sect. 5.4. Thus \( \{ \epsilon_i : 1 \leq i \leq 6 \} \) is an orthogonal basis of the six dimensional complex vector space \( W \) relative to a non-degenerate hermitian form \( h : W \times W \rightarrow \mathbb{C} \), with \( h(\epsilon_i, \epsilon_i) = 1 \) if \( i \leq p \) and \( -1 \) otherwise, with
p either 3 or 5. As a real vector space, \( W = \mathbb{Z}/4 \)-graded with \( W_1 = \sum_{i=1}^{6} \mathbb{R} e_i \), and \( W_3 = iW_1 \). This grading on \( W \) induces a grading by \( \mathbb{Z}/4 \) on \( \bigwedge^3 W \) with support \( \{ 1, 3 \} \).

We had fixed the ‘determinant map’ \( \det: \bigwedge^6 W \to \mathbb{C} \) by imposing \( \det(e_{123456}) = 1 \). Then the associated skew-symmetric bilinear form \( (\cdot, \cdot): \bigwedge^3 W \times \bigwedge^3 W \to \mathbb{C} \) is homogeneous of degree \( \bar{2} \) (again the \( \mathbb{Z}/4 \)-grading on \( \mathbb{C} \) is given by \( \bar{C}_0 = \mathbb{R} 1, \mathbb{C}_2 = \mathbb{R} i \)). Hence the triple product \( [\cdot, \cdot, \cdot] \) on \( \bigwedge^3 W \) is also homogeneous of degree \( \bar{2} \). Moreover, the grading on \( \bigwedge^3 W \) induces one on its dual \( (\omega^1 \in (\bigwedge^3 W^*)_3 \) and \( i\omega^1 \in (\bigwedge^3 W^*)_1 \), and the (real) linear isomorphisms \( \Phi_3 \) and \( \Psi \) are homogeneous of degree \( \bar{2} \). It turns out that the composition \( \Gamma = \Phi_3^{-1} \Psi: \bigwedge^3 W \to \bigwedge^3 W \) is homogeneous of degree \( \bar{1} \), so the fixed subspace \( T = \bigwedge^3 W \) is a graded subspace: \( T = T_1 \oplus T_3 \). But \( i[\cdot, \cdot, \cdot] \) and \( i(\cdot, \cdot, \cdot) \) are both homogeneous of degree \( \bar{0} \), because multiplication by \( i \) has degree \( \bar{2} \). Hence this \( \mathbb{Z}/4 \)-grading of the (real) vector space \( T \) is in fact a grading of the symplectic triple system \( (T, i[\cdot, \cdot, \cdot], i(\cdot, \cdot, \cdot)) \).

First \( E_7 \)-type: The first non-split simple real symplectic triple system with envelope of type \( E_7 \), denoted by \( T^\Gamma \) in Sect. 5.5, satisfies that \( \text{in} \det(T^\Gamma) \) is isomorphic to \( \mathfrak{s}_0 \mathfrak{o}_{10,2}(\mathbb{R}) \). Thus \( \text{in} \det(T^\Gamma) \) is the orthogonal Lie algebra of a real vector space \( V \) of dimension 12, endowed with a symmetric bilinear form \( b \) such that there is a basis \( \{ u_1, u_2, v_1, v_2, w_1, \ldots, w_8 \} \) of \( V \), with \( b(u_1, v_1) = b(u_2, v_2) = 1 = b(w_r, w_r) \), \( 1 \leq r \leq 8 \), and the remaining values of \( b \) on vectors of the basis are either 0 or obtained from the above by symmetry. The linear endomorphism \( h \) with \( h(u_1) = u_2, h(v_1) = -v_1, h(u_2) = h(v_2) = h(w_r) = 0 \) for all \( r \), lies in the orthogonal Lie algebra \( \mathfrak{o}(V, b) \). The diagonalizable element \( h \) acts with eigenvalues 1, \(-1\) and \( 0 \) both on \( V \) and on the \( \mathfrak{o}(V, b) \), and with eigenvalues \( \frac{1}{2} \) and \( -\frac{1}{2} \) on the half-spin modules. To see this, note that after complexification, it is easy to get a Cartan subalgebra containing \( h \), with corresponding root system \( \Phi = \{ \pm \epsilon_r, \pm \epsilon_s: 1 \leq r \neq s \leq 6 \} \) with \( \epsilon_1(h) = 1 \), \( \epsilon_r(h) = 0 \) for \( r \neq 1 \), where \( \pm \epsilon_r, 1 \leq r \leq 6 \), are the weights of the natural module.

Then we get the following \( \mathbb{Z}/4 \)-grading of \( T^\Gamma: T^\Gamma = T_1^\Gamma \oplus T_3^\Gamma \), with \( T_1^\Gamma = \{ x \in T^\Gamma : h.x = \frac{1}{2} x \} \), \( T_3^\Gamma = \{ x \in T^\Gamma : h.x = -\frac{1}{2} x \} \).

‘Quaternionic’ \( E_7 \)-type: The other non-split simple real symplectic triple system with envelope of type \( E_7 \) is related to a skew-hermitian non-degenerate form on a quaternionic vector space.

In this case, let \( \mathcal{U} \) be a right \( \mathbb{H} \)-module of rank 6 endowed with a skew-hermitian non-degenerate form \( h: \mathcal{U} \times \mathcal{U} \to \mathbb{H} \) as in Sect. 5.5. Using the notation there, our simple real symplectic triple system is \( T^\mathcal{R} \), where \( T = \bigwedge^0 W \) with \( W = v_1 \mathbb{H} \oplus v_2 \mathbb{H} \oplus v_3 \mathbb{H} \). Recall from Sect. 5.5 that \( v_1, v_2, v_3 \) span a maximal isotropic subspace for \( h \).

Now, \( T \) is \( \mathbb{Z}/4 \)-graded as in (7.1). The conjugate-linear map \( T^\mathcal{R} \) preserves the homogeneous components \( T_1^\mathcal{R} = \bigwedge^0 W \oplus \bigwedge^4 W \) and \( T_3^\mathcal{R} = \bigwedge^2 W \oplus \bigwedge^6 W \), and therefore this \( \mathbb{Z}/4 \)-grading on \( T \) is inherited by its real form \( T^\mathcal{R} \).

\( E_8 \)-type: We use the notations in Sect. 5.6. Thus \( U \) denotes a complex eight-dimensional vector space, endowed with a non-degenerate hermitian form \( h: U \times U \to \mathbb{C} \) and an orthogonal basis \( \{ e_r : 1 \leq r \leq 8 \} \), with \( h(e_r, e_r) = \pm 1 \) for all \( r \). Recall that \( L = \mathfrak{sl}(U) \oplus \bigwedge^4 U \) is the algebra of inner derivations of the complex symplectic triple system \( T = \bigwedge^2 U \oplus \bigwedge^2 U^* \).
As a real vector space, $\bigwedge^2 U$ is $\mathbb{Z}/4$-graded with

$$(\bigwedge^2 U)_1 = \text{span}_\mathbb{R} \{ e_{rs} : 1 \leq r < s \leq 8 \},$$

and $(\bigwedge^2 U)_3 = i(\bigwedge^2 U)_1$. The same happens with $\bigwedge^2 U^*$, and hence $T$ is $\mathbb{Z}/4$-graded.

The $\mathbb{Z}/4$-grading on $U$ given by $U_0 = \bigoplus_{r=1}^{8} \mathbb{R} e_r$ and $U_2 = iU_0$ induces naturally a $\mathbb{Z}/4$-grading on $\mathfrak{sl}(U)$ with support $\{0, 2\}$. Also $\bigwedge^4 U$ is $\mathbb{Z}/4$-graded with

$$(\bigwedge^4 U)_0 = \text{span}_\mathbb{R} \{ e_{r_1r_2r_3r_4} : 1 \leq r_1 < r_2 < r_3 < r_4 \leq 8 \}$$

and $(\bigwedge^4 U)_2 = i(\bigwedge^4 U)_0$. So $\mathcal{L}$ is $\mathbb{Z}/4$-graded.

Now, the action $\mathcal{L} \times T \to T$ gives a homogeneous map of degree $\bar{0}$, the bilinear form $\langle \cdot | \cdot \rangle_{\mathcal{L}}$ is also homogeneous of degree $\bar{0}$, and the bilinear form $\langle \cdot | \cdot \rangle$ on $T$ is homogeneous of degree $\bar{2}$. It follows that the map $d_{\cdot \cdot} : T \times T \to \mathcal{L}$, determined by $(l|dx,y)_{\mathcal{L}} = -2(l|x|y)$ is homogeneous of degree $\bar{2}$.

Finally, $\Upsilon_T$ is homogeneous of degree $\bar{0}$, so the symplectic triple system $T^{\Upsilon}$, given by the elements in $T$ fixed by $\Upsilon_T$, inherits the $\mathbb{Z}/4$-grading on $T$, as a real vector space, with support $\{\bar{1}, \bar{3}\}$. As the triple product on $T^{\Upsilon}$ is given by $[x, y, z]_{T^{\Upsilon}} = id_{x,y}(z)$, for all $x, y, z \in T^{\Upsilon}$, and the map $id_{\cdot \cdot}$ is homogeneous of degree $\bar{0}$, so the $\mathbb{Z}/4$-grading on $T^{\Upsilon}$ is actually a $\mathbb{Z}/4$-grading as a (real) triple system, not just as a vector space. \hfill $\Box$

As a consequence, Theorem 6.1 may be reformulated, replacing the words ‘up to weak isomorphism’ by ‘up to isomorphism’, and this completes the classification of the simple real symplectic triple systems up to isomorphism.

We conclude the paper by stating this classification in a different way as follows:

**Theorem 7.5** If $T$ is a real simple symplectic triple system, then $(\mathfrak{g}(T), \text{ind} \mathfrak{e}(T))$ belongs to the following list, where information on $T$ as a module for $\text{ind} \mathfrak{e}(T)$ is given:

1. **Symplectic**: $(\mathfrak{sp}_{2n+2}(\mathbb{R}), \mathfrak{sp}_{2n}(\mathbb{R}))$, $T$ is the natural module for $\mathfrak{sp}_{2n}(\mathbb{R})$ (Sect. 4.3),
2. **Orthogonal**: $(\mathfrak{so}_{p+2,q+2}(\mathbb{R}), \mathfrak{so}_{p,q}(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R}))$, $T$ is the tensor product of the natural modules for $\mathfrak{so}_{p,q}(\mathbb{R})$ and $\mathfrak{sl}_2(\mathbb{R})$ (Sect. 4.2),
3. **Quaternionic**: $(\mathfrak{so}^*_n, \mathfrak{so}^*_n \oplus \mathfrak{su}_2)$, $T$ is the natural module for $\mathfrak{so}^*_n$ (Sect. 5.3),
4. **Special**: $(\mathfrak{sl}_{n+2}(\mathbb{R}), \mathfrak{gl}_n(\mathbb{R}))$, $T$ is the direct sum of the natural module for $\mathfrak{gl}_n(\mathbb{R})$ and its dual (Sect. 4.1),
5. **Unitarian**: $(\mathfrak{sp}_{p+1,n+1-p} \oplus \mathfrak{su}_{p,n-p})$, $T$ is the natural module for $\mathfrak{sp}_{p+1,n+1-p}$ (Sect. 5.2),
6. **Exceptional**:
   - $(\mathfrak{g}_{2,2}, \mathfrak{sl}_2(\mathbb{R}))$, $T$ is the 4-dimensional irreducible module for $\mathfrak{sl}_2(\mathbb{R})$ (Sect. 4.4),
   - $(\mathfrak{f}_{4,4}, \mathfrak{sp}_6(\mathbb{R}))$, $T$ is the kernel of the map $\bigwedge^3 W \to W$ in (4.8), where $W$ is the natural module for $\mathfrak{sp}_6(\mathbb{R})$ (Sect. 4.5),
   - $(\mathfrak{e}_{6,6}, \mathfrak{sl}_6(\mathbb{R}))$, $T = \bigwedge^3 W$ for $W$ the natural module for $\mathfrak{sl}_6(\mathbb{R})$ (Sect. 4.6),
   - $(\mathfrak{e}_{6,2}, \mathfrak{su}_{3,3})$ and $(\mathfrak{e}_{6,-14}, \mathfrak{su}_{5,1})$, $T$ lives inside the third exterior power of the natural module for $\mathfrak{su}_{3,3}$ or $\mathfrak{su}_{5,1}$ (Sect. 5.4),
• \((e_7, so_6, 6(R))\), \(T\) is the half-spin module for \(so_{6,6}(R)\) (Sect. 4.7),
• \((e_7, -25, so_{10,2}(R))\), \(T\) is the half-spin module for \(so_{10,2}(R)\) (Sect. 5.5),
• \((e_7, -5, so_{12}^*)\), \(T\) is an irreducible module for the even Clifford algebra of the pair \((\text{End}_H(U), *)\) in Sect. 5.5, where \(U\) is a right \(H\)-module of dimension 6 endowed with a skew-hermitian nondegenerate form, and \(\ast\) is the associated involution,
• \((e_8, e_7)\), \(T\) is the irreducible 56-dimensional module for \(e_7\) (Sect. 4.8),
• \((e_8, -24, e_7, -25)\), \(T\) is the irreducible 56-dimensional module for \(e_7, -25\) (Sect. 5.6).

Moreover, if \(T\) and \(T'\) are real simple symplectic triple systems with isomorphic inner derivation algebras and isomorphic enveloping algebras, then \(T\) and \(T'\) are isomorphic too.

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