UMD-VALUED SQUARE FUNCTIONS ASSOCIATED WITH BESSEL OPERATORS IN HARDY AND BMO SPACES

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Abstract. We consider Banach valued Hardy and BMO spaces in the Bessel setting. Square functions associated with Poisson semigroups for Bessel operators are defined by using fractional derivatives. If $B$ is a UMD Banach space we obtain for $B$-valued Hardy and BMO spaces equivalent norms involving $\gamma$-radonifying operators and square functions. We also establish characterizations of UMD Banach spaces by using Hardy and BMO-boundedness properties of $g$-functions associated to Bessel-Poisson semigroup.

1. Introduction

If $\{P_t\}_{t>0}$ denotes the classical Poisson semigroup, for every $k \in \mathbb{N}$, the $k$-th square (also called Littlewood-Paley-Stein) function $g_k(\{P_t\}_{t>0})$ is defined by

$$g_k(\{P_t\}_{t>0})(f)(x) = \left( \int_0^\infty |t^k \partial_t^k P_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

for every $f \in L^p(\mathbb{R}^n)$. It is well-known that, for every $k \in \mathbb{N}$ and $1 < p < \infty$, there exists $C > 0$ such that

$$\frac{1}{C} \|f\|_{L^p(\mathbb{R}^n)} \leq \|g_k(\{P_t\}_{t>0})(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n),$$

or, in other words, for every $k \in \mathbb{N}$, the norm $\|\cdot\|_{p,k}$ defined by

$$\|f\|_{p,k} = \|g_k(\{P_t\}_{t>0})(f)\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n),$$

is equivalent to the usual norm in $L^p(\mathbb{R}^n)$. These equivalent norms $\|\cdot\|_{p,k}$ are more suitable to establish $L^p$-boundedness properties of certain operators (for instance, Fourier multipliers ([37, p. 58])).

Square functions have been also defined for other semigroups of operators. In [37] it was developed the Littlewood-Paley theory for diffusion semigroups. If $\{T_t\}_{t>0}$ is a diffusion semigroup (in the sense of Stein) on the measure space $(\Omega,\Sigma,\mu)$ where $\mu$ is a $\sigma$-finite measure defined on the $\sigma$-algebra $\Sigma$ in $\Omega$, for every $k \in \mathbb{N}$, we define

$$g_k(\{T_t\}_{t>0})(f)(x) = \left( \int_0^\infty |t^k \partial_t^k T_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad x \in \Omega,$$

for every $f \in L^p(\Omega,\mu), 1 < p < \infty$.

We have that, for every $k \in \mathbb{N}$ and $1 < p < \infty$, there exists a constant $C > 0$ such that

$$\frac{1}{C} \|f - E_0(f)\|_{L^p(\Omega,\mu)} \leq \|g_k(\{T_t\}_{t>0})(f)\|_{L^p(\Omega,\mu)} \leq C \|f\|_{L^p(\Omega,\mu)},$$

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for every $f \in L^{p}(\Omega, \mu)$ \cite{mathematics}}. Here $E_0$ denotes the projector from $L^{p}(\Omega, \mu)$ to the fixed point space of $\{T_t\}_{t>0}$. In \cite[Corollary 3, p. 121]{mathematics} \cite{mathematics} was used to study $L^{p}$-boundedness of Laplace transform type multipliers associated to $\{T_t\}_{t>0}$. Spectral multipliers for a general class of operators were analyzed by Meda \cite{mathematics} by using $g$-functions.

Also, the square functions have been defined by using convolutions. Suppose that $\psi \in L^{2}(\mathbb{R}^{n})$. We define the square function $g_{\psi}$ as follows:

$$g_{\psi}(f)(x) = \left(\int_{0}^{\infty} |\psi_t * f(x)|^{2} \frac{dt}{t}\right)^{1/2}, \quad x \in \mathbb{R}^{n},$$

where $\psi_t(x) = t^{-n}\psi(x/t)$, $x \in \mathbb{R}^{n}$ and $t > 0$. Conditions on the function $\psi$ can be given in order that the norm $\| \cdot \|_{\psi}$ defined by

$$\|f\|_{\psi} = \|g_{\psi}(f)\|_{L^{p}(\mathbb{R}^{n})}, \quad f \in L^{p}(\mathbb{R}^{n}),$$

is equivalent to the usual norm in $L^{p}(\mathbb{R}^{n})$ \cite[Remark 2.3]{mathematics}. Note that the classical Poisson semigroup is a convolution semigroup.

Assume now that $\mathbb{B}$ is a Banach space and $\Omega \subset \mathbb{R}^{n}$ (for us, usually, $\Omega = \mathbb{R}$ or $\Omega = (0, \infty)$). For every $1 \leq p < \infty$, we denote by $L^{p}(\Omega, \mathbb{B})$ the $p$-th Bochner-Lebesgue $\mathbb{B}$-valued function space with respect to the Lebesgue measure. By $L^{1,\infty}(\Omega, \mathbb{B})$ we represent the weak $L^{1}$-Bochner-Lebesgue $\mathbb{B}$-valued function space.

Suppose that $T$ is a bounded operator from $L^{p}(\mathbb{R}^{n})$ into itself, for some $1 \leq p < \infty$. We can define the tensor operator $T \otimes I_{\mathbb{B}}$ on $L^{p}(\mathbb{R}^{n}) \otimes \mathbb{B}$ in the natural way. Here $I_{\mathbb{B}}$ denotes the identity operator in $\mathbb{B}$. We cannot ensure that $T \otimes I_{\mathbb{B}}$ can be extended to $L^{p}(\mathbb{R}^{n}, \mathbb{B})$ as a bounded operator in $L^{p}(\mathbb{R}^{n}, \mathbb{B})$. However, if $T$ is a positive operator, that is, $T(f) \geq 0$ when $f \geq 0$, then $T \otimes I_{\mathbb{B}}$ can be extended from $L^{p}(\mathbb{R}^{n}) \otimes \mathbb{B}$ to $L^{p}(\mathbb{R}^{n}, \mathbb{B})$ as a bounded operator from $L^{p}(\mathbb{R}^{n}, \mathbb{B})$ into itself.

If $\{T_t\}_{t>0}$ is a diffusion semigroup, $T_t$ is a positive operator in $L^{p}(\Omega)$, for every $t > 0$ and $1 < p < \infty$. Then, for every $t > 0$ the operator $T_t \otimes I_{\mathbb{B}}$ can be extended to $L^{p}(\Omega, \mathbb{B})$ as a bounded operator from $L^{p}(\Omega, \mathbb{B})$ into itself, for every $1 < p < \infty$. We continue denoting this extension by $T_t$.

In order to define square functions acting on $\mathbb{B}$-valued functions the more natural way is to replace the modulus in the definitions by the norm $\| \cdot \|_{\mathbb{B}}$ in $\mathbb{B}$. We consider, for every $k \in \mathbb{N}$ and $1 < p < \infty$,

$$g_{k,\mathbb{B}}(\{P_t\}_{t>0})(f)(x) = \left(\int_{0}^{\infty} \|t^{k}\partial_{x}^{k}P_t(f)(x)\|_{\mathbb{B}}^{2} \frac{dt}{t}\right)^{1/2}, \quad x \in \mathbb{R}^{n},$$

for every $f \in L^{p}(\mathbb{R}^{n}, \mathbb{B})$.

Kwapień \cite{mathematics} established that the Banach-valued version of \cite{mathematics} characterizes the Hilbert spaces in the following sense: $\mathbb{B}$ is isomorphic to a Hilbert space if, and only if, for some (equivalently, for every) $1 < p < \infty$, there exits $C > 0$ such that

$$\frac{1}{C} \|f\|_{L^{p}(\mathbb{R}^{n}, \mathbb{B})} \leq \|g_{1,\mathbb{B}}(\{P_t\}_{t>0})(f)\|_{L^{p}(\mathbb{R}^{n}, \mathbb{B})} \leq C \|f\|_{L^{p}(\mathbb{R}^{n}, \mathbb{B})},$$

for every $f \in L^{p}(\mathbb{R}^{n}, \mathbb{B})$. For this type of vector valued $g$-functions the question is to describe the Banach spaces $\mathbb{B}$ for which one of the two inequalities in (3) holds. This problem was considered in the first time by Xu \cite{mathematics} by using square functions defined by the Poisson semigroup for the torus. In \cite{mathematics} Xu introduced generalized square functions where the exponent 2 is replaced by $q \in (1, \infty)$ and he characterized the Banach spaces of $q$-martingale type and cotype as those for
which some of the inequalities for the $q$-square function in (3) holds. After Xu’s results, other authors have investigated this question for vector valued $q$-square functions associated with other semigroups of operators (see [1], [13], [31] and [41], amongst others).

Hytönen [26] and Kaiser and Weis ([27] and [28]) have introduced other definitions of square functions in vector valued settings. They obtained, by using these new square functions, equivalent norms in $L^p(\mathbb{R}^n, \mathcal{B})$, $1 < p < \infty$, and also in the Hardy space $H^1(\mathbb{R}^n, \mathcal{B})$ and in the bounded mean oscillation function space $BMO(\mathbb{R}^n, \mathcal{B})$, provided that $\mathcal{B}$ is a UMD Banach space.

Hytönen (in [26]) defined square functions associated with subordinated diffusion semigroups in terms of stochastic integrals. Kaiser and Weis ([27] and [28]) used $\gamma$-radonifying operators to get equivalent norms by employing convolution type square functions. Both approaches (stochastic integrals and $\gamma$-radonifying operators) are connected (see, for instance, [43]). In this paper, we will work with square functions involving Poisson semigroups associated with Bessel operators and we will use $\gamma$-radonifying operators.

UMD Banach spaces (as in [26], [27] and [28]) play an important role in our results. The Hilbert transform $\mathcal{H}(f)$ of $f$ is defined by

$$\mathcal{H}(f)(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy, \quad \text{a.e. } x \in \mathbb{R},$$

for every $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. It is a key result in harmonic analysis that the Hilbert transform is a bounded operator from $L^p(\mathbb{R})$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R})$ into $L^{1,\infty}(\mathbb{R})$.

It is clear that the Hilbert transform is not a positive operator in $L^p(\mathbb{R})$, $1 \leq p < \infty$. A Banach space $\mathcal{B}$ is said to be a UMD Banach space when the operator $\mathcal{H} \otimes I_{\mathcal{B}}$ can be extended from $L^p(\mathbb{R}) \otimes \mathcal{B}$ to $L^p(\mathbb{R}, \mathcal{B})$ as a bounded operator from $L^p(\mathbb{R}, \mathcal{B})$ into itself, for some $1 < p < \infty$. This extension property does not depend on $1 < p < \infty$ in the following sense: the extension property holds for some $1 < p < \infty$ if, and only if, it is true for every $1 < p < \infty$.

The main properties of UMD Banach spaces were established by Bourgain ([18]) and Burkholder (20). There exist a lot of characterizations for UMD Banach spaces (see, for instance, [18], [19], [20], [21], [25], [35], and [44]).

We recall now some definitions and properties concerning $\gamma$-radonifying operators. Suppose that $(\gamma_k)_{k \in \mathbb{N}}$ is a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$, $\mathcal{B}$ is a Banach space and $H$ is a Hilbert space. If $T: H \to \mathcal{B}$ is a linear operator we define $\|T\|_{\gamma(H, \mathcal{B})}$ as follows:

$$\|T\|_{\gamma(H, \mathcal{B})} = \sup \left( \mathbb{E} \left\| \sum_k \gamma_k T(h_k) \right\|_\mathcal{B}^2 \right)^{1/2},$$

where the supremum is taken over all the finite orthonormal sets $\{h_k\}$ in $H$. Here $\mathbb{E}$ denotes the expectation in $(\Omega, \Sigma, \mathbb{P})$. The space $\gamma(H, \mathcal{B})$ of $\gamma$-radonifying operators from $H$ to $\mathcal{B}$ is defined as the completion, with respect to $\| \cdot \|_{\gamma(H, \mathcal{B})}$, of the space of finite rank operators in $L(H, \mathcal{B})$, the space of bounded linear operators from $H$ into $\mathcal{B}$.

If $H$ is a separable Hilbert space and the Banach space $\mathcal{B}$ does not contain copies of $c_0$ then

$$\|T\|_{\gamma(H, \mathcal{B})} = \left( \mathbb{E} \left\| \sum_{k=1}^{\infty} \gamma_k T(h_k) \right\|_\mathcal{B}^2 \right)^{1/2}, \quad T \in \gamma(H, \mathcal{B}),$$
Thus, the Hankel transform can be extended from the last equality does not depend of the orthonormal basis \( \{ h_k \}_{k=1}^{\infty} \) of \( H \). If \( \mathbb{B} \) is a UMD Banach space, \( \mathbb{B} \) does not contain copies of \( c_0 \).

Suppose now that \( H = L^2(\mathbb{R}, \mu) \) where \( (\mathbb{R}, \mu) \) is a \( \sigma \)-finite measure space with countably generating \( \sigma \)-algebra \( \Gamma \) and let \( \mathbb{B} \) be a Banach space. If \( f : \mathbb{A} \to \mathbb{B} \) is weakly continuous in \( H \), that is, for every \( S \in \mathbb{B}^* \), the dual space of \( \mathbb{B} \), \( S \circ f \in H \), then there exists \( T_f \in L(H, \mathbb{B}) \) such that

\[
\langle S, T_f(h) \rangle_{\mathbb{B}^* \otimes \mathbb{B}} = \int_A \langle S, f(t) \rangle_{\mathbb{B}^* \otimes \mathbb{B}} h(t) d\mu(t), \quad h \in H.
\]

It is usual to write \( f \in \gamma(A, \mu, \mathbb{B}) \) when \( T_f \in \gamma(H, \mathbb{B}) \) and, to simplify, to identify \( f \) with \( T_f \). If \( \mathbb{B} \) does not contain copies of \( c_0 \), the space \( \{ T_f, f \in \gamma(A, \mu, \mathbb{B}) \} \) is dense in \( \gamma(H, \mathbb{B}) \). Throughout this paper we consider \( H = L^2((0, \infty), dt/t) \).

As a consequence of [28, Theorem 4.2] we can deduce the following result.

**Theorem A.** Let \( \mathbb{B} \) be a UMD Banach space and \( k \in \mathbb{N} \). We define

\[
G_{k, \mathbb{B}}(f)(t, x) = t^k \partial_t^k P_t(f)(x), \quad x \in \mathbb{R}^m \text{ and } t > 0,
\]

for every \( f \in \mathcal{S}(\mathbb{R}^m, \mathbb{B}) \), the \( \mathbb{B} \)-valued Schwartz function space. Then, there exists \( C > 0 \) such that

\[
\frac{1}{C} \| f \|_{E(\mathbb{R}^m, \mathbb{B})} \leq \| G_{k, \mathbb{B}}(f) \|_{E(\mathbb{R}^m, \gamma(H, \mathbb{B}))} \leq C \| f \|_{E(\mathbb{R}^m, \mathbb{B})},
\]

where \( E = L^p \), \( 1 < p < \infty \), \( E = H^1 \) or \( E = \text{BMO} \).

Note that, since \( \gamma(H, \mathbb{C}) = H \), [1] can be seen as a Banach valued extension of [1].

Motivated by Theorem A, our objective in this paper is to obtain equivalent norms for Hardy and BMO spaces defined via Bessel operators by using square functions involving Bessel Poisson semigroups in a Banach valued setting.

The study of harmonic analysis associated with Bessel operators was started by Muckenhoupt and Stein (33). We consider the Bessel operator

\[
\Delta_\lambda = -x^{-\lambda} \frac{d}{dx} x^{2\lambda} \frac{d}{dx} x^{-\lambda} \quad \text{on } (0, \infty),
\]

where \( \lambda > 0 \). The Hankel transform \( h_\lambda \) is defined by

\[
h_\lambda(f)(x) = \int_0^\infty \sqrt{xy} J_{\lambda-1/2}(xy) f(y) dy, \quad x \in (0, \infty),
\]

for every \( f \in L^1(0, \infty) \). By \( J_\nu \) we denote the Bessel function of the first kind and order \( \nu \). The Hankel transform can be extended from \( L^1(0, \infty) \cap L^2(0, \infty) \) to \( L^2(0, \infty) \) as an isometry in \( L^2(0, \infty) \).

The space \( \mathcal{S}_\lambda(0, \infty) \) is constituted by all those smooth functions \( \phi \) on \( (0, \infty) \) such that, for every \( m, k \in \mathbb{N} \),

\[
\eta^\lambda_{m, k}(\phi) = \sup_{x \in (0, \infty)} (1 + x^2)^m \left| \left( \frac{1}{x} \frac{d}{dx} \right)^k \left( x^{-\lambda} \phi(x) \right) \right| < \infty.
\]

\( \mathcal{S}_\lambda(0, \infty) \) is endowed with the topology generated by the system \( \{ \eta^\lambda_{m, k} \}_{m, k \in \mathbb{N}} \) of seminorms. Thus, \( \mathcal{S}_\lambda(0, \infty) \) is a Fréchet space and the Hankel transform \( h_\lambda \) is a bounded bijective operator from \( \mathcal{S}_\lambda(0, \infty) \) into itself and \( h_\lambda^{-1} = h_\lambda \) ([35, Theorem 5.4-1]).
The Bessel operator $\Delta_\lambda$ and the Hankel transform $h_\lambda$ are closely connected, as the following equality shows:

$$h_\lambda(\Delta_\lambda f)(x) = x^2 h_\lambda(f)(x), \quad x \in (0, \infty),$$

for every $f \in S_\lambda(0, \infty)$. The Hankel transform plays for the Bessel operator the same role as the Fourier transformation with respect to the Laplacian operator.

We define the operator $\sqrt{\Delta_\lambda}$ as follows:

$$\sqrt{\Delta_\lambda} f = h_\lambda(y h_\lambda(f)),$$

where $D(\sqrt{\Delta_\lambda}) = \{ f \in L^2(0, \infty) : y h_\lambda(f) \in L^2(0, \infty) \}$. The Poisson semigroup $\{P_t^\lambda\}_{t > 0}$ associated with the Bessel operator $\Delta_\lambda$, that is, generated by the operator $-\sqrt{\Delta_\lambda}$, is defined by

$$P_t^\lambda(f)(x) = \int_0^\infty P_t^\lambda(x, y)f(y)dy, \quad t, x, y \in (0, \infty),$$

for every $f \in L^p(0, \infty)$, $1 \leq p \leq \infty$, where

$$P_t^\lambda(x, y) = \frac{2\pi}{\Gamma(\lambda)} \int_0^\infty \frac{(\sin \theta)^{2\lambda - 1}}{[r^2 + 2 + 2ry(1 - \cos \theta)]^{\lambda + 1}} dr, \quad t, x, y \in (0, \infty).$$

Square functions defined by the Bessel Poisson semigroup have been studied in [6], [13], [14] and [39], amongst others. In [13] it was considered generalized Littlewood-Paley functions associated with $\{P_t^\lambda\}_{t > 0}$ in a Banach valued setting. If $\mathbb{B}$ is a Banach space and $1 < q < \infty$ we consider the $q$-square function defined by

$$g^\mathbb{B}_q(\{P_t^\lambda\}_{t > 0})(f)(x) = \left( \int_0^\infty \|\partial_t P_t^\lambda(f)(x)\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q}, \quad x \in (0, \infty).$$

By using these Littlewood-Paley functions, martingale type and cotype of the Banach space $\mathbb{B}$ can be characterized. Moreover, as in the classical case [29], according to [13] Theorems 2.4 and 2.5, if $\lambda > 0$, the Banach space $\mathbb{B}$ is isomorphic to a Hilbert space if, and only if, for some (equivalently, for every) $1 < p < \infty$, there exists $C > 0$ such that

$$\frac{1}{C} \|f\|_{L^p((0, \infty), \mathbb{B})} \leq \|g^\mathbb{B}_q(\{P_t^\lambda\}_{t > 0})(f)\|_{L^p((0, \infty), \mathbb{B})} \leq C \|f\|_{L^p((0, \infty), \mathbb{B})}, \quad f \in L^p((0, \infty), \mathbb{B}).$$

In order to extend the equivalence (5) to Banach spaces that are not Hilbert spaces we define, motivated by [28], new square functions involving Bessel Poisson semigroup and $\gamma$-radonifying operators.

In [36] Segovia and Wheeden introduced fractional derivatives as follows. Suppose that $F : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{C}$ is a nice enough function, $\beta > 0$ and $m \in \mathbb{N}$ is such that $m - 1 \leq \beta < m$. The $\beta$-th derivative $\partial_t^\beta F$ of $F$ with respect to $t$ is defined by

$$\partial_t^\beta F(t, x) = \frac{e^{-is(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_t^m F(t + s, x)s^{\beta - 1}ds, \quad x \in \mathbb{R} \text{ and } t > 0.$$

We consider the operator

$$G_{\mathbb{B}}^{\lambda, \beta}(f)(t, x) = t^\beta \partial_t^\beta P_t^\lambda(f)(x), \quad t, x \in (0, \infty),$$

for every $f \in L^p((0, \infty), \mathbb{B})$, $1 \leq p \leq \infty$.

The following result was established in [6, Theorem 1.2].

**Theorem B.** Let $\lambda > 0$, $\beta > 0$ and $1 < p < \infty$. Suppose that $\mathbb{B}$ is a UMD Banach space. Then, there exists $C > 0$ such that

$$\frac{1}{C} \|f\|_{L^p((0, \infty), \mathbb{B})} \leq \|G_{\mathbb{B}}^{\lambda, \beta}(f)\|_{L^p((0, \infty), \gamma(H, \mathbb{B}))} \leq C \|f\|_{L^p((0, \infty), \mathbb{B})}, \quad f \in L^p((0, \infty), \mathbb{B}).$$
Note that the Bessel Poisson semigroup \( \{P^\lambda_t\}_{t>0} \) is not a diffusion semigroup because it is not conservative. Then, Theorem [3] is not included in [26, Theorem 1.6].

Our objective in this paper is to study the operator \( G^\lambda_{\mathbb{B}} \) in Hardy and BMO spaces adapted to the Bessel setting.

Fridli [23], in his study about the local Hilbert transform considered the Hardy type space \( H^1_{\mathbb{B}}(0, \infty) \) which consists on all those functions \( f \in L^1(0, \infty) \) such that the odd extension function \( f_o \) of \( f \) to \( \mathbb{R} \) is in the classical Hardy space \( H^1(\mathbb{R}) \). In [23, Theorem 2.1] the space \( H^1_{\mathbb{B}}(0, \infty) \) was characterized by using the local Hilbert transform.

The maximal operator \( P^\lambda_t \) associated with the semigroup \( \{P^\lambda_t\}_{t>0} \) is defined by
\[
P^\lambda_t(f) = \sup_{t>0} |P^\lambda_t(f)|, \quad f \in L^p(0, \infty), \quad 1 \leq p \leq \infty.
\]
The space \( H^1_{\mathbb{B}}(0, \infty) \) can be described by \( P^\lambda_t \) as follows (see [11, Theorem 1.10]).

**Theorem C.** Let \( \lambda > 0 \) and \( f \in L^1(0, \infty) \). Then, \( f \in H^1_{\mathbb{B}}(0, \infty) \) if, and only if, \( P^\lambda_t(f) \in L^1(0, \infty) \).

Note that, according to Theorem C the Hardy space defined by the Bessel Poisson semigroup \( \{P^\lambda_t\}_{t>0} \) actually does not depend on \( \lambda \).

If \( \mathbb{B} \) is a Banach space, the \( \mathbb{B} \)-valued Hardy space \( H^1_{\mathbb{B}}((0, \infty), \mathbb{B}) \) is defined as in the scalar case. By considering the maximal operator \( P^\lambda_t \) on \( L^p((0, \infty), \mathbb{B}) \), \( 1 \leq p < \infty \), as follows
\[
P^\lambda_t(f) = \sup_{t>0} \|P^\lambda_t(f)\|_{\mathbb{B}},
\]
we introduce the space \( H^1((\Delta_\lambda), \mathbb{B}) \) constituted by all those functions \( f \in L^1((0, \infty), \mathbb{B}) \) such that \( P^\lambda_t(f) \in L^1(0, \infty) \). As in Theorem C a function \( f \in L^1((0, \infty), \mathbb{B}) \) is in \( H^1((0, \infty), \mathbb{B}) \) if, and only if, \( f \in H^1((\Delta_\lambda), \mathbb{B}) \) (see Proposition 2.1).

By \( BMO(\mathbb{R}) \) we denote as usual the space of all bounded mean oscillation functions on \( \mathbb{R} \). In [9] the space \( BMO_0(0, \infty) \) was introduced. A complex measurable function \( f \) on \( (0, \infty) \) is said to be in \( BMO_0(0, \infty) \) when there exists \( C > 0 \) such that:

\[
(Bi) \quad \frac{1}{|I|} \int_I |f(x)| dx \leq C, \quad \text{for every } I = (0, r], \quad r > 0,
\]

\[
(Bii) \quad \frac{1}{|I|} \int_I |f(x) - f_I| dx \leq C, \quad \text{for each } I = (r, s), \quad 0 < r < s < \infty.
\]

Here, \( |I| \) denotes the length of \( I \) and \( f_I = \frac{1}{|I|} \int_I f(y) dy \).

Condition (Bii) says that \( f \in BMO(0, \infty) \). It is not hard to see that \( f \in BMO_0(0, \infty) \) if, and only if, the odd extension \( f_o \) of \( f \) to \( \mathbb{R} \) is in \( BMO(\mathbb{R}) \). As in the classical case, the dual space of \( H^1_{\mathbb{B}}((0, \infty), \mathbb{B}) \) can be identified with the space \( BMO_0(0, \infty) \). In [5] harmonic analysis operators (maximal operators, \( g \)-functions, Riesz transforms, ... ) in the Bessel setting were studied on \( BMO_0(0, \infty) \) and in [9], \( BMO_0(0, \infty) \) was described by using Carleson measures involving Bessel Poisson semigroups.

If \( \mathbb{B} \) is a Banach space, the Banach valued function space \( BMO_0((0, \infty), \mathbb{B}) \) is defined as in the scalar case, replacing the modulus by the norm \( \| \cdot \|_{\mathbb{B}} \) in \( \mathbb{B} \). The dual space of \( H^1_{\mathbb{B}}((0, \infty), \mathbb{B}) \) can be identified with \( BMO_0((0, \infty), \mathbb{B}^*) \), provided that \( \mathbb{B} \) has the UMD property.

Our first result is the following.
Theorem 1.1. Let $\lambda \geq 1$ and $\beta > 0$. Suppose that $B$ is a UMD Banach space. Then, there exists $C > 0$ such that 
\[
\frac{1}{C} \| f \|_{E((0,\infty), B)} \leq \| G^\lambda_B(f) \|_{E((0,\infty), \gamma(H, B))} \leq C \| f \|_{E((0,\infty), B)}, \quad f \in E((0,\infty), B),
\]
where $E$ denotes $H^1_0$ or $BMO_o$.

The Bessel operator $\Delta_\lambda$ can be written as $\Delta_\lambda = D^*_\lambda D_\lambda$, where $D_\lambda = x^{\lambda} \frac{d}{dx} x^{-\lambda}$ and $D^*_\lambda = -x^{-\lambda} \frac{d}{dx} x^\lambda$ is the adjoint operator of $D_\lambda$. $D_\lambda$ is used to define the Riesz transform in the Bessel setting (see [2]). We consider the operator $G^\lambda_B$ acting on $B$-valued functions by 
\[
G^\lambda_B(f)(t, x) = t D^*_\lambda P^1_1(f)(x), \quad t, x \in (0, \infty).
\]
This operator will be useful to get characterizations of UMD Banach spaces.

Theorem 1.2. Let $\lambda \geq 1$. Assume that $B$ is a UMD Banach space. Then, the operator $G^\lambda_B$ is bounded from $H^1_o((0,\infty), B)$ into $H^1_o((0,\infty), \gamma(H, B))$ and from $BMO_o((0,\infty), B)$ into $BMO_o((0,\infty), \gamma(H, B))$.

We now establish new characterizations of the UMD Banach spaces by using our square functions.

Theorem 1.3. Let $B$ be a Banach space and $\lambda \geq 1$. The following assertions are equivalent.

(i) $B$ is UMD.
(ii) There exists $C > 0$ such that, for every $f \in E((0,\infty)) \otimes B$,
\[
\| f \|_{E((0,\infty), B)} \leq C \| G^\lambda_B(f) \|_{E((0,\infty), \gamma(H, B))}
\]
and
\[
\| G^\lambda_B(f) \|_{E((0,\infty), \gamma(H, B))} \leq C \| f \|_{E((0,\infty), B)},
\]
where $E$ denotes $H^1_0$ or $BMO_o$.
(iii) For a certain (equivalently, for every) $\beta > 0$ there exists $C > 0$ such that
\[
\frac{1}{C} \| f \|_{E((0,\infty), B)} \leq \| G^\lambda_B(f) \|_{E((0,\infty), \gamma(H, B))} \leq C \| f \|_{E((0,\infty), B)}, \quad f \in E((0,\infty), B),
\]
for $\delta = \beta$ and $\delta = \beta + 1$, where $E$ represents $H^1_0$ or $BMO_o$.

In the following sections we present proofs of Theorems 1.1, 1.2 and 1.3.

Throughout this paper by $c$ and $C$ we represent positive constants not necessarily the same in each occurrence.

2. Proof of Theorem 1.1

2.1. In this part we prove that the operator $G^\lambda_B$ is bounded from $H^1_o((0,\infty), B)$ into $H^1_o((0,\infty), \gamma(H, B))$.

As in [23] we say that a strongly measurable function $a : (0,\infty) \rightarrow B$ is a $q$-atom, where $1 < q \leq \infty$, when one of the following two conditions is satisfied:

(Ai) $a = \frac{b}{\delta} \chi_{(0,\delta)}$, where $\delta > 0$ and $b \in B$, being $\| b \|_B = 1$,
(Aii) there exists an interval $I \subset (0,\infty)$ such that $\text{supp} \, a \subset I$, $\| a \|_{L^q((0,\infty), B)} \leq |I|^{1/q - 1}$ and 
\[
\int_I a(x) dx = 0.
\]

The arguments presented in the proof of [11] Proposition 3.7 and [23] Theorem 2.1 allow us to get the following.
Proposition 2.1. Let $X$ be a Banach space, $\lambda > 0$ and $1 < q < \infty$. Suppose that $f : (0, \infty) \to X$ is a strongly measurable function. The following assertions are equivalent:

(i) $f \in H^1(\Delta_\lambda, X)$.

(ii) $f \in H^1_q(0, \infty, X)$.

(iii) For every $j \in \mathbb{N}$, there exist a q-atom $a_j$ and $\lambda_j \in \mathbb{C}$ such that $\sum_{j \in \mathbb{N}} |\lambda_j| < \infty$ and $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$ in $L^1((0, \infty), X)$.

Moreover, in this case, we have that

$$\|P^*_\lambda(f)\|_{L^1((0,\infty),X)} \sim \|f\|_{H^1((0,\infty),X)} \sim \inf_{j \in \mathbb{N}} |\lambda_j|,$$

where the infimum is taken over all the complex sequences $\{\lambda_j\}_{j \in \mathbb{N}}$ such that $\sum_{j \in \mathbb{N}} |\lambda_j| < \infty$ and, for a certain sequence of q-atoms $\{a_j\}_{j \in \mathbb{N}}$, $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$. Here $f_o$ denotes the odd extention to $\mathbb{R}$ of $f$.

In the sequel we write $H^1_q(0, \infty, \mathbb{B})$ to refer us $H^1(\Delta_\lambda, \mathbb{B})$, $\lambda > 0$, and we denote by $H^1((0, \infty), \mathbb{B})$ the classical $\mathbb{B}$-valued Hardy space on $(0, \infty)$, that is, a function $f \in L^1((0, \infty), \mathbb{B})$ is in $H^1((0, \infty), \mathbb{B})$ if, and only if, $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$, where $\lambda_j \in \mathbb{C}$, $j \in \mathbb{N}$, being $\sum_{j \in \mathbb{N}} |\lambda_j| < \infty$, and for some $q \in (1, \infty)$, $a_j$ satisfies the condition (Aii), for every $j \in \mathbb{N}$.

Our objective is to see that, for every $f \in H^1_q((0, \infty), \mathbb{B})$, $G^\lambda_\beta(f) \in H^1_q((0, \infty), \gamma(H, \mathbb{B}))$, and

$$\|G^\lambda_\beta(f)\|_{H^1_q((0,\infty),\gamma(H,\mathbb{B}))} \leq C\|f\|_{H^1_q((0,\infty),\mathbb{B})},$$

where $C > 0$ does not depend on $f$.

By taking into account Proposition 2.1 in order to show this, we can see that $G^\lambda_\beta$ is bounded from $H^1_q((0, \infty), \mathbb{B})$ into $L^1((0, \infty), \gamma(H, \mathbb{B}))$ (Lemma 2.2) and that $P^*_\lambda \circ G^\lambda_\beta$ is bounded from $H^1_q((0, \infty), \mathbb{B})$ into $L^1(0, \infty)$ (Lemma 2.3). We first need to establish the following boundedness property for $G^\lambda_\beta$.

Lemma 2.1. Let $\mathbb{B}$ be a UMD Banach space, $\beta > 0$ and $\lambda \geq 1$. The operator $G^\lambda_\beta$ is bounded from $L^1((0, \infty), \mathbb{B})$ into $L^1(0, \infty)$ and from $H^1((0, \infty), \mathbb{B})$ into $L^1((0, \infty), \gamma(H, \mathbb{B}))$.

Proof. Let us define

$$K^\beta_\lambda(t;x, y) = t^\beta \partial_\lambda^\beta P^*_{t\lambda}(x, y), \quad t, x, y \in (0, \infty),$$

and consider $m \in \mathbb{N}$ such that $m - 1 \leq \beta < m$.

According to [23 (4.6)], for every $k \in \mathbb{N}$, $\theta \in (0, \pi)$ and $t, s, x, y \in (0, \infty)$, we can write

$$\partial_\lambda^k \left[ \frac{t + s}{[(t + s)^2 + (x - y)^2 + 2xy(1 - \cos \theta)]^{\lambda+1}} \right] = -\frac{1}{2\lambda} \partial_\lambda^{k+1} \left[ \frac{1}{[(t + s)^2 + (x - y)^2 + 2xy(1 - \cos \theta)]^{\lambda+1}} \right]$$

where $E_{k+1, t} = \frac{\partial_\lambda^{k+1} - \partial_\lambda^k}{(k + 1)!}, \partial_\lambda^k = \frac{k!}{2^k k!} (k + 1)!/((t + s)!), \partial_\lambda^k \in (\mathbb{N}, 0 \leq \ell \leq \frac{(k + 1)!}{2^k k!}, 0 \leq \ell \leq \frac{(k + 1)!}{2^k k!})$.

Then, for each $k \in \mathbb{N}$, $\theta \in (0, \pi)$ and $t, s, x, y \in (0, \infty)$,

$$\left| \partial_\lambda^k \left[ \frac{t + s}{[(t + s)^2 + (x - y)^2 + 2xy(1 - \cos \theta)]^{\lambda+1}} \right] \right| \leq \frac{C}{(t + s + |x - y| + \sqrt{2xy(1 - \cos \theta)})^{2\lambda+1+k}},$$

where $C > 0$. It follows that, for $k \in \mathbb{N}$ and $t, s, x, y \in (0, \infty)$,

$$\partial_\lambda^k P^*_{t\lambda+s}(x, y) = \frac{2\lambda xy}{\pi} \int_0^\pi (\sin \theta)^{2\lambda-1} \partial_\lambda^k \left[ \frac{t + s}{[(t + s)^2 + (x - y)^2 + 2xy(1 - \cos \theta)]^{\lambda+1}} \right] d\theta.$$
Then, for \( k \in \mathbb{N} \), we have that
\[
|\partial_t^k P_{t+s}^\lambda(x, y)| \leq C \frac{(xy)\lambda}{(t + s + |x - y|)^{2\lambda+1+k}}, \quad t, s, x, y \in (0, \infty),
\]
and also
\[
|\partial_t^k P_{t+s}^\lambda(x, y)| \leq C(xy)\lambda \left( \int_0^{\pi/2} \frac{u^{2\lambda-1}}{(t + s + |x - y| + u)^{2\lambda+1+k}} du + \int_{\pi/2}^\pi \frac{(\sin \theta)^{2\lambda-1}}{(t + s + |x - y| + \sqrt{xy})^{2\lambda+1+k}} d\theta \right)
\]
\[
\leq C \left( \int_0^{\sqrt{\pi}/2} \frac{u^{2\lambda-1}}{(t + s + |x - y| + u)^{2\lambda+1+k}} du + \frac{(xy)\lambda}{(t + s + |x - y| + \sqrt{xy})^{2\lambda+1+k}} \right)
\]
\[
\leq C \frac{1}{(t + s + |x - y|)^{k+1}}, \quad t, s, x, y \in (0, \infty).
\]

Let \( g \in \mathcal{S}_\lambda(0, \infty) \). We can write
\[
t^\beta \partial_t^\beta P_{t+s}^\lambda(y)(x) = \int_0^\infty K_{\lambda}^\beta(t; x, y)g(y)dy, \quad t, x \in (0, \infty).
\]
Indeed, by \( (7) \) we can differentiate under the integral sign and write
\[
\partial_t^m P_{t+s}^\lambda(x, y) = \int_0^\infty \partial_t^m P_{t+s}^\lambda(x, y)g(y)dy, \quad t, s, x \in (0, \infty),
\]
and, again by \( (7) \),
\[
\int_0^\infty s^{m-\beta-1} \int_0^\infty |\partial_t^m P_{t+s}^\lambda(x, y)| |g(y)|dyds \leq Cx^\lambda \int_0^\infty \frac{s^{m-\beta-1}}{(t + s)^{2\lambda+1+m}} ds < \infty, \quad t, x \in (0, \infty).
\]
Hence,
\[
\partial_t^\beta P_{t}^\lambda(y)(x) = \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_t^m P_{t+s}^\lambda(x, y)s^{m-\beta-1}ds
\]
\[
= \int_0^\infty g(y) \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_t^m P_{t+s}^\lambda(x, y)s^{m-\beta-1}dsdy
\]
\[
= \int_0^\infty g(y)\partial_t^\beta P_{t}^\lambda(x, y)dy, \quad t, x \in (0, \infty),
\]
and \( (9) \) is proved.

On the other hand, by using Minkowski’s inequality and \( (8) \) we deduce that
\[
\left\| K_{\lambda}^\beta(\cdot; x, y) \right\|_H \leq C \int_0^\infty s^{m-\beta-1} \left( \int_0^\infty |t^\beta \partial_t^m P_{t+s}^\lambda(x, y)|^2 dt \right)^{1/2} ds
\]
\[
\leq C \int_0^\infty s^{m-\beta-1} \left( \int_0^\infty \frac{t^{2\beta-1}}{(t + s + |x - y|)^{2\lambda+1+m+2}} dt \right)^{1/2} ds
\]
\[
= C \int_0^\infty \frac{s^{m-\beta-1}}{(s + |x - y|)^{m-\beta+1}} ds = \frac{C}{|x - y|}, \quad x, y \in (0, \infty), \quad x \neq y.
\]
From \( (10) \) we infer that the integral in \( (9) \) converges as a \( H \)-Bochner integral, provided that \( x \notin \text{supp} \ g \). We define
\[
F(x) = \int_0^\infty K_{\lambda}^\beta(\cdot; x, y)g(y)dy, \quad x \notin \text{supp} \ g,
\]
where the integral is understood in the \( H \)-Bochner sense.
Then, it follows that
\[ G^\lambda_\mathcal{C}(g)(\cdot, x) = \int_0^\infty K_\mathcal{C}^\lambda(\cdot; x, y)g(y)dy, \quad x \notin \text{supp}_g, \]
where the integral is understood in the $H$-Bochner sense.

Suppose now that $g \in S_\mathcal{C}(0, \infty) \otimes \mathbb{R}$, that is, $g = \sum_{i=1}^{n} b_i g_i$, where $b_i \in \mathbb{R}$ and $g_i \in S_\mathcal{C}(0, \infty)$, $i = 1, \ldots, n \in \mathbb{N}$. Since $\gamma(H, \mathbb{C}) = H$, we have that $K_\mathcal{C}^\lambda(\cdot; x, y)g(y) \in \gamma(H, \mathbb{R})$, $x, y \in (0, \infty)$, $x \neq y$, and
\[
\|K_\mathcal{C}^\lambda(\cdot; x, y)g(y)\|_{\gamma(H, \mathbb{R})} \leq C\|K_\mathcal{C}^\lambda(\cdot; x, y)\|_{H}\|g(y)\|_{\mathbb{R}} \leq C\frac{\|g(y)\|_{\mathbb{R}}}{|x - y|}, \quad x, y \in (0, \infty), \quad x \neq y.
\]
Also,
\[
\int_0^\infty K_\mathcal{C}^\lambda(\cdot; x, y)g(y)dy = \sum_{i=1}^{n} b_i \int_0^\infty K_\mathcal{C}^\lambda(\cdot; x, y)g_i(y)dy, \quad x \notin \text{supp}_g,
\]
where the integral in the left hand side is understood in the $\gamma(H, \mathbb{R})$-Bochner sense and the ones in the right hand side are understood in the $H$-Bochner sense. Hence, we obtain
\[
G^\lambda_\mathbb{R}(g)(\cdot, x) = \int_0^\infty K_\mathcal{C}^\lambda(\cdot; x, y)g(y)dy, \quad x \notin \text{supp}_g,
\]
as elements in $\gamma(H, \mathbb{R})$, where the integral is understood in the $\gamma(H, \mathbb{R})$-Bochner sense.

On the other hand, we have that, for every $t, x, y \in (0, \infty)$, and $x \neq y$,
\[
\partial_x K_\mathcal{C}^\lambda(t; x, y) = \frac{t^\beta e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_x P_{t+\lambda}^\mu(x, y)s^{m-\beta-1}ds
\]
\[
= \frac{2\lambda^\beta e^{-i\pi(m-\beta)}}{\pi \Gamma(m-\beta)} \int_0^\infty s^{m-\beta-1} \int_0^{\pi} (\sin \theta)^{2\lambda-1} \partial_x P_{t+\lambda}^\mu \left[\frac{(xy)^{\lambda}(t + s)}{[(t + s) + (x - y)^2 + 2xy(1 - \cos \theta)]^{\lambda+1}}\right]d\theta ds.
\]
Derivations under the integral signs can be justified as above. By making a derivative with respect to $x$ in (6) for $k = m$, we get
\[
|\partial_x P_{t+\lambda}^\mu(x, y)| \leq C \int_0^{\pi} \left(\frac{(xy)^{\lambda} \partial_x P_{t+\lambda}^\mu(x, y)}{(t + s + |x - y| + \sqrt{2xy(1 - \cos \theta)})^{2\lambda + m + 2}}d\theta\right)
\]
\[
\leq C \int_0^{\pi} \left(\frac{(xy)^{\lambda} \partial_x P_{t+\lambda}^\mu(x, y)}{(t + s + |x - y| + \sqrt{2xy(1 - \cos \theta)})^{2\lambda + m + 2}}d\theta\right)
\]
\[
= C \left\{I_1(t, s; x, y) + I_2(t, s; x, y)\right\}, \quad t, s, x, y \in (0, \infty).
\]
By proceeding as in the proof of (5) for $k = m + 1$ we obtain
\[
(13) \quad I_1(t, s; x, y) \leq \frac{C}{(t + s + |x - y|)^{m + 2}}, \quad t, s, x, y \in (0, \infty).
\]
Since $\lambda \geq 1$, we also have that

\[
I_2(t, s; x, y) \leq C y \left\{ \int_0^{\pi/2} \frac{\theta^{2\lambda-1}(xy)^{\lambda-1}}{(t + s + |x - y| + \sqrt{xy})^{2\lambda+1}} d\theta + \frac{(xy)^{\lambda-1}}{(t + s + |x - y| + \sqrt{xy})^{2\lambda+1}} \right\}
\]

\[
\leq C y \left\{ \int_0^{\pi/2} \frac{\theta}{(t + s + |x - y| + \sqrt{xy})^{m+3}} d\theta + \frac{1}{(t + s + |x - y| + \sqrt{xy})^{m+3}} \right\}
\]

\[
\leq C \left\{ \int_0^{\pi/2} \frac{\theta y}{(t + s + |x - y| + y)^{m+3}} d\theta + \frac{1}{(t + s + |x - y| + y)^{m+2}} \right\}, \quad 0 < y \leq 2x, \ x \neq y,
\]

\[
\leq C \frac{1}{(t + s + |x - y|)^{m+2}}, \quad t, s, x, y \in (0, \infty).
\]

Hence, by Minkowski's inequality we get

\[
\left\| \partial_t K^\beta_H(\cdot; x, y) \right\|_H \leq C \int_0^\infty s^{m-\beta-1} \left( \int_0^\infty t^{2\beta-1} |\partial_t \partial_t^m P_{t+s}(x, y)|^2 dt \right)^{1/2} ds
\]

\[
\leq C \int_0^\infty s^{m-\beta-1} \left( \int_0^\infty t^{2\beta-1} \left( t + s + |x - y| \right)^{2m+4} dt \right)^{1/2} ds = \frac{C}{|x - y|^2}, \quad x, y \in (0, \infty), \ x \neq y.
\]

We conclude that

\[
\left\| \partial_t K^\beta_H(\cdot; x, y) \right\|_H \leq \frac{C}{|x - y|^2}, \quad x, y \in (0, \infty), \ x \neq y.
\]

By taking into account symmetries we also get

\[
\left\| \partial_y K^\beta_H(\cdot; x, y) \right\|_H \leq \frac{C}{|x - y|^2}, \quad x, y \in (0, \infty), \ x \neq y.
\]

Fix now $x, y \in (0, \infty), \ x \neq y$. We define the operator $L(x, y)$ by

\[
L(x, y) : \mathbb{B} \longrightarrow \gamma(H, \mathbb{B})
\]

\[
b \longmapsto L(x, y)b = K^\beta_H(\cdot; x, y)b.
\]

If \{h_j\}_{j \in \mathbb{N}} is an orthonormal basis in $H$, since $\gamma(H, C) = H$, we deduce from (10),

\[
\left\| L(x, y)b \right\|_{\gamma(H, \mathbb{B})} = \left\| K^\beta_H(\cdot, x, y)b \right\|_{\gamma(H, \mathbb{B})} = \left( \mathbb{E} \left\| \sum_{j=1}^\infty \gamma_j \int_0^\infty K^\beta_H(t; x, y)h_j(t) \frac{dt}{t} \right\|^2_\mathbb{B} \right)^{1/2}
\]

\[
= \|b\|_\mathbb{B} \left( \mathbb{E} \left\| \sum_{j=1}^\infty \gamma_j \int_0^\infty K^\beta_H(t; x, y)h_j(t) \frac{dt}{t} \right\|^2_\mathbb{B} \right)^{1/2} \leq C \|b\|_\mathbb{B} \left\| K^\beta_H(\cdot; x, y) \right\|_H
\]

\[
\leq C \|b\|_\mathbb{B} \frac{1}{|x - y|}, \quad b \in \mathbb{B}.
\]

Hence,

\[
\left\| L(x, y) \right\|_{\mathbb{B} \rightarrow \gamma(H, \mathbb{B})} \leq \frac{C}{|x - y|}.
\]

In a similar way, by identifying $\partial_x K^\beta_H(\cdot; x, y)$ and $\partial_y K^\beta_H(\cdot; x, y)$ with the corresponding operators in $L(\mathbb{B}, \gamma(H, \mathbb{B}))$, the space of linear and bounded operators from $\mathbb{B}$ into $\gamma(H, \mathbb{B})$, and by using (15) and (16), we can get that

\[
\left\| \partial_x K^\beta_H(\cdot; x, y) \right\|_{\mathbb{B} \rightarrow \gamma(H, \mathbb{B})} \leq \frac{C}{|x - y|^2},
\]
and

\[ \left\| \partial_y K^\beta_\lambda(\cdot; x, y) \right\|_{B \to \gamma(H, \mathbb{B})} \leq \frac{C}{|x-y|^2}. \]  

By Theorem 1, \( G^\lambda_\beta \) is a bounded operator from \( L^p((0, \infty), \mathbb{B}) \) into \( L^p((0, \infty), \gamma(H, \mathbb{B})) \), for every \( 1 < p < \infty \). Then, from (11), (17), (18) and (19) we conclude that the operator \( G^\lambda_\beta \) is a \((\mathbb{B}, \gamma(H, \mathbb{B}))\)-Calderón-Zygmund operator. Hence, \( G^\lambda_\beta \) can be extended from \( S_\lambda((0, \infty) \otimes \mathbb{B}) \) to \( L^1((0, \infty), \mathbb{B}) \) as a bounded operator from \( L^1((0, \infty), \mathbb{B}) \) into \( L^1((0, \infty), \gamma(H, \mathbb{B})) \) and to \( H^1((0, \infty), \mathbb{B}) \) as a bounded operator from \( H^1((0, \infty), \mathbb{B}) \) into \( L^1((0, \infty), \gamma(H, \mathbb{B})) \). We denote by \( \tilde{G}^\lambda_\beta \) the extension of \( G^\lambda_\beta \) to \( L^1((0, \infty), \mathbb{B}) \) as a bounded operator from \( L^1((0, \infty), \mathbb{B}) \) into \( L^{1,\infty}((0, \infty), \gamma(H, \mathbb{B})) \).

We now prove that

\[ \tilde{G}^\lambda_\beta(g)(x) = G^\lambda_\beta(g)(\cdot, x), \quad \text{a.e. } x \in (0, \infty), \]

as elements of \( \gamma(H, \mathbb{B}) \), for every \( g \in L^1((0, \infty), \mathbb{B}) \).

Let \( g \in L^1((0, \infty), \mathbb{B}) \). We choose a sequence \((g_k)_{k \in \mathbb{N}} \subset S_\lambda((0, \infty) \otimes \mathbb{B})\) such that

\[ g_k \to g, \quad \text{as } k \to \infty, \text{ in } L^1((0, \infty), \mathbb{B}). \]

Then,

\[ G^\lambda_\beta(g_k) \to \tilde{G}^\lambda_\beta(g), \quad \text{as } k \to \infty, \text{ in } L^{1,\infty}((0, \infty), \gamma(H, \mathbb{B})), \]

and hence, there exist a set \( \Omega \subset (0, \infty) \), being \(|(0, \infty) \setminus \Omega| = 0\), and an increasing sequence \((n_k)_{k \in \mathbb{N}} \subset \mathbb{N}\) such that, for every \( x \in \Omega \),

\[ G^\lambda_\beta(g_{n_k})(\cdot, x) \to \tilde{G}^\lambda_\beta(g)(x), \quad k \to \infty, \text{ in } \gamma(H, \mathbb{B}). \]

Let \( \varepsilon > 0 \). By proceeding as in the proof of (10) we obtain

\[ \left\| K^\beta_\lambda(\cdot; x, y) \right\|_{L^2((\varepsilon, \infty), dt/t)} \leq C \int_0^\infty s^{-\beta-1} \left( \int_{\varepsilon}^{\infty} \left( \frac{(t+\varepsilon-s+|x-y|)^{2\beta-1}}{(t+\varepsilon+|x-y|)^{2\beta}+2^\beta+1} \right)^{1/2} ds \right) \leq C, \quad x, y \in (0, \infty). \]

Then, by using Minkowski’s inequality it follows that

\[ \left\| G^\lambda_\beta(g_{n_k})(\cdot, x) - \tilde{G}^\lambda_\beta(g)(\cdot, x) \right\|_{L^2((\varepsilon, \infty), dt/t, \mathbb{B})} \leq C \int_0^\infty \left\| g(y) - g_{n_k}(y) \right\|_B \left\| K^\beta_\lambda(\cdot; x, y) \right\|_{L^2((\varepsilon, \infty), dt/t)} dy \leq C \frac{1}{\varepsilon} \left\| g - g_{n_k} \right\|_{L^1((0, \infty), \mathbb{B})}, \quad x \in (0, \infty), \]

that is,

\[ G^\lambda_\beta(g_{n_k})(\cdot, x) \to G^\lambda_\beta(g)(\cdot, x), \quad k \to \infty, \text{ in } L^2((\varepsilon, \infty), dt/t, \mathbb{B}), \]

uniformly in \( x \in (0, \infty) \).

By taking into account that \( \gamma(H, \mathbb{B}) \) is continuously contained in \( L(H, \mathbb{B}) \), the space of bounded linear operators from \( H \) into \( \mathbb{B} \), it follows that, for every \( S \in \mathbb{B}^* \) and \( h \in C^\infty_c(0, \infty) \), the space
of smooth functions with compact support on \((0, \infty)\), we have that

\[
\langle S, \tilde{G}_{B}^{\lambda, \beta}(g)(x)[h]\rangle_{B^* , B} = \lim_{k \to \infty} \langle S, G_{B}^{\lambda, \beta}(g_{n_k})(\cdot, x)[h]\rangle_{B^* , B} = \lim_{k \to \infty} \int_{0}^{\infty} \langle S, G_{B}^{\lambda, \beta}(g_{n_k})(t, x)\rangle_{B^* , B} h(t) \frac{dt}{t}, \quad x \in \Omega,
\]
and then,

\[
\left| \int_{0}^{\infty} \langle S, G_{B}^{\lambda, \beta}(g)(t, x)\rangle_{B^* , B} h(t) \frac{dt}{t} \right| \leq \|S\|_{B^*} \cdot \|\tilde{G}_{B}^{\lambda, \beta}(g)(x)[h]\|_{B} \leq \|S\|_{B^*} \cdot \|G_{B}^{\lambda, \beta}(g)(x)\|_{H \to B} \cdot \|h\|_{H}, \quad x \in \Omega.
\]

Hence, \(\langle S, G_{B}^{\lambda, \beta}(g)(\cdot, x)\rangle_{B^* , B} \in H, \ x \in \Omega, \) and

\[
G_{B}^{\lambda, \beta}(g)(x) = G_{B}^{\lambda, \beta}(g)(\cdot, x), \quad x \in \Omega.
\]

We conclude that \(G_{B}^{\lambda, \beta}\) is bounded from \(L^1((0, \infty), B)\) into \(L^{1, \infty}((0, \infty), \gamma(H, B))\) and from \(H^{1}((0, \infty), B)\) into \(L^{1}((0, \infty), \gamma(H, B))\). \(\square\)

Next, we establish the behavior of \(G_{B}^{\lambda, \beta}\) on \(H^{1}_{0}((0, \infty), B)\).

**Lemma 2.2.** Let \(B\) be a UMD Banach space, \(\beta > 0\) and \(\lambda \geq 1\). The operator \(G_{B}^{\lambda, \beta}\) is bounded from \(H^{1}_{0}((0, \infty), B)\) into \(L^{1}((0, \infty), \gamma(H, B))\).

**Proof.** Let \(f \in H^{1}_{0}((0, \infty), B)\). By Proposition \(2.1\) we write \(f = \sum_{j \in \mathbb{N}} a_{j} j\), where \(a_{j}\) is a 2-atom and \(\lambda_{j} \in \mathbb{C}, \ j \in \mathbb{N}, \) being \(\sum_{j \in \mathbb{N}} |\lambda_{j}| < \infty\). Since the series \(\sum_{j \in \mathbb{N}} a_{j} j\) converges in \(L^{1}((0, \infty), B)\) and \(G_{B}^{\lambda, \beta}\) is bounded from \(L^{1}((0, \infty), B)\) into \(L^{1, \infty}((0, \infty), \gamma(H, B))\) (Lemma \(2.1\)), we have that

\[
G_{B}^{\lambda, \beta}(f) = \sum_{j \in \mathbb{N}} \lambda_{j} G_{B}^{\lambda, \beta}(a_{j}),
\]
where the series converges in \(L^{1, \infty}((0, \infty), \gamma(H, B))\).

If \(a\) is a 2-atom satisfying (Aii), since, by Lemma \(2.1\) \(G_{B}^{\lambda, \beta}\) is a bounded operator from \(H^{1}((0, \infty), B)\) into \(L^{1}((0, \infty), \gamma(H, B))\), we get

\[
\|G_{B}^{\lambda, \beta}(a)\|_{L^{1}((0, \infty), \gamma(H, B))} \leq C,
\]
being \(C > 0\) independent of \(a\).

Suppose now that \(a = b \chi_{(\delta, \delta)}/\delta\), for some \(\delta > 0\) and \(b \in B, \ |b|_{B} = 1\). By taking into account that \(G_{B}^{\lambda, \beta}\) is bounded from \(L^{2}((0, \infty), B)\) into \(L^{2}((0, \infty), \gamma(H, B))\) (Theorem \(B\)), we obtain

\[
\int_{0}^{\delta} \|G_{B}^{\lambda, \beta}(a)(\cdot, x)\|_{\gamma(H, B)} dx \leq (2\delta)^{1/2} \|G_{B}^{\lambda, \beta}(a)\|_{L^{2}((0, \infty), \gamma(H, B))} \leq C \delta^{1/2} \|a\|_{L^{2}((0, \infty), B)} \leq C,
\]
where \(C > 0\) does not depend on \(\delta\) or \(b\).

According to (11), since \(\gamma(H, C) = H\), we have that

\[
\|G_{B}^{\lambda, \beta}(a)(\cdot, x)\|_{\gamma(H, B)} \leq \frac{1}{\delta} \int_{0}^{\delta} \|K_{\lambda}^{\delta}(\cdot, x, y)\|_{H} dy, \quad x \geq 2\delta.
\]

By proceeding as in (10), and taking into account (7) we can write

\[
\left\| K_{\lambda}^{\delta}(\cdot, x, y) \right\|_{H} \leq C(xy)^{\lambda} \int_{0}^{\infty} s^{m-\beta-1} \left( \int_{0}^{\infty} \frac{t^{\beta-1}}{(t + s + |x - y|)^{4\lambda + 2m + 2}} dt \right)^{1/2} ds \leq C(xy)^{\lambda} \int_{0}^{\infty} \frac{s^{m-\beta-1}}{(s + |x - y|)^{2\lambda + 1 + m - \beta}} ds \leq C \frac{(xy)^{\lambda}}{|x - y|^{2\lambda + 1}}, \ x, y \in (0, \infty), \ x \neq y.
\]
Hence, it follows that
\[ \| G^\lambda_\beta (a, x) \|_{(H, B)} \leq C_1 \delta \int_0^\delta \frac{(xy)^\lambda}{|x - y|^{2\lambda + 1}} dy \leq C \frac{\delta^{\lambda}}{x^{\lambda + 1}}, \quad x \geq 2\delta, \]
and we get
\[ \int_0^\infty \| G^\lambda_\beta (a, x) \|_{(H, B)} dx \leq C \int_0^\infty \frac{\delta^{\lambda}}{x^{\lambda + 1}} dx \leq C, \]
where \( C > 0 \) does not depend on \( \delta \) and \( b \).

From [22] and [24] we deduce
\[ \| G^\lambda_\beta (a) \|_{L^1((0, \infty), (H, B))} \leq C, \]
where \( C > 0 \) is independent of \( \delta \) and \( b \).

By using [20], [21] and [25] we conclude
\[ \| G^\lambda_\beta (f) \|_{L^1((0, \infty), (H, B))} \leq C \sum_{j \in \mathbb{N}} |\lambda_j|. \]
Hence,
\[ \| G^\lambda_\beta (f) \|_{L^1((0, \infty), (H, B))} \leq C \| f \|_{H^1_B(\mathbb{R})}. \]

According to [16] Theorem 2.4] the maximal operator \( P_\lambda^a \) given by
\[ P_\lambda^a (g)(x) = \sup_{s > 0} \| P_\lambda^a (g)(x) \|_{(H, B)}, \]
for every \( g \in L^p((0, \infty), (H, B)) \), \( 1 \leq p \leq \infty \), is bounded from \( L^p((0, \infty), (H, B)) \) into \( L^p(0, \infty) \), for every \( 1 < p < \infty \), and from \( L^1((0, \infty), (H, B)) \) into \( L^{1, \infty}(0, \infty) \). Then, the operator \( P_\lambda^a \circ G^\lambda_\beta \) is bounded from \( H^1((0, \infty), B) \) into \( L^{1, \infty}(0, \infty) \).

We now show that \( P_\lambda^a \circ G^\lambda_\beta \) is a bounded operator from \( H^1_B((0, \infty), B) \) into \( L^1(0, \infty) \).

**Lemma 2.3.** Let \( B \) be a UMD Banach space, \( \beta > 0 \) and \( \lambda \geq 1 \). We have that \( P_\lambda^a \circ G^\lambda_\beta \) is a bounded operator from \( H^1_B((0, \infty), B) \) into \( L^1(0, \infty) \).

**Proof.** Note firstly that \( P_\lambda^a \circ G^\lambda_\beta \) is bounded from \( H^1_B((0, \infty), B) \) into \( L^{1, \infty}(0, \infty) \).

According to [6] Lemma 3.1], we have that, for every \( \phi \in S_A(0, \infty) \),
\[ h_\lambda \left( t^\beta \delta^\beta_t P_\lambda^a (\phi) \right) = e^{i\beta x} (ty)^\beta e^{-yt} h_\lambda (\phi)(y), \quad t > 0. \]
Since \( h_\lambda \) is an isometry in \( L^2(0, \infty) \), it follows that \( t^\beta \delta^\beta_t P_\lambda^a (\phi) \in L^2(0, \infty) \), for every \( \phi \in S_A(0, \infty) \) and \( t > 0 \), and by [22] §8.5 (19) , we get
\[ h_\lambda \left( P_\lambda^a \left[ t^\beta \delta^\beta_t P^\lambda_1 (\phi) \right] \right)(x) = e^{i\beta x} (tx)^\beta e^{-xt} h_\lambda (\phi)(x), \quad \phi \in S_A(0, \infty) \text{ and } t, s, x \in (0, \infty), \]
and then, for every \( \phi \in S_A(0, \infty) \) and \( t, s, x \in (0, \infty) \),
\[ P^a_\lambda \left[ t^\beta \delta^\beta_t P^\lambda_1 (\phi) \right] (x) = t^\beta \delta^\beta_t P^\lambda_{1+s} (\phi)(x) = \int_0^\infty t^\beta \delta^\beta_t P^\lambda_{1+s} (x, y) \phi(y) dy. \]
Also, for every \( \phi \in S_A(0, \infty) \otimes B \),
\[ P_\lambda^a (G^\lambda_\beta (\phi)(t, \cdot))(x) = P_\lambda^a \left[ t^\beta \delta^\beta_t P^\lambda_1 (\phi) \right] (x) = t^\beta \delta^\beta_t P^\lambda_{1+s} (\phi)(x), \quad t, s, x \in (0, \infty), \]
By defining the function
\[ K^\beta_\lambda (t, s, x, y) = t^\beta \delta^\beta_t P^\lambda_{1+s} (x, y), \quad t, s, x, y \in (0, \infty), \]
we have that
\[
\|K_\lambda^2(\cdot, \cdot; x, y)\|_{L^\infty((0, \infty), H)} \leq \frac{C}{|x - y|}, \quad x, y \in (0, \infty), \ x \neq y.
\]
Indeed, as in the proof of (10), by using (8) we get
\[
\|K_\lambda^2(\cdot, \cdot; x, y)\|_{L^\infty((0, \infty), H)} = \sup_{s > 0} \|\ell^\beta \partial_t^\beta P_{t+s}(x, y)\|_H
\leq C \sup_{s > 0} \int_0^\infty u^{m-\beta-1} \left( \int_0^\infty \frac{u^{2\beta-1}}{(t + s + u + |x - y|)^{2m+2}} dt \right)^{1/2} du
\leq C \sup_{s > 0} \frac{1}{s + |x - y|} \leq \frac{C}{|x - y|}, \quad x, y \in (0, \infty), \ x \neq y.
\]
Also, we get, for every $x, y \in (0, \infty)$, $x \neq y$,
\[
\|\partial_x K_\lambda^2(\cdot, \cdot; x, y)\|_{L^\infty((0, \infty), H)} + \|\partial_y K_\lambda^2(\cdot, \cdot; x, y)\|_{L^\infty((0, \infty), H)} \leq \frac{C}{|x - y|^2}.
\]
According to (26), for every $\phi \in S_\lambda(0, \infty)$, the integral
\[
\int_0^\infty K_\lambda^2(\cdot, \cdot; x, y)\phi(y)dy
\]
is convergent in the $L^\infty((0, \infty), H)$-Bochner sense provided that $x \notin \text{supp} \phi$.

Let $N \in \mathbb{N}$. We define the operator $Q_{\lambda, N}^2$ as follows:
\[
Q_{\lambda, N}^2(\phi)(x) = \int_0^\infty K_\lambda^2(\cdot, \cdot; x, y)\phi(y)dy, \quad \phi \in S_\lambda(0, \infty) \text{ and } x \notin \text{supp} \phi,
\]
where the integral is understood in the $C([1/N, N], H)$-Bochner sense. Here, by $C([1/N, N], H)$ we denote the space of $H$-valued continuous functions on $[1/N, N]$.

Let $\phi \in S_\lambda(0, \infty)$. We have that
\[
\left[Q_{\lambda, N}^2(\phi)(x)\right](s) = \int_0^\infty K_\lambda^2(\cdot, s; x, y)\phi(y)dy, \quad x \notin \text{supp} \phi \text{ and } s \in [1/N, N],
\]
where the integral is understood in the $H$-Bochner sense and the equality is understood in $H$.

Moreover, according to some properties of Bochner integration and by applying Fubini’s theorem, we get
\[
\int_0^\infty \left[\left[Q_{\lambda, N}^2(\phi)(x)\right](s)\right](t)\frac{dt}{t} = \int_0^\infty \phi(y) \int_0^\infty K_\lambda^2(t, s; x, y)h(t)\frac{dt}{t} dy,
\]
\[
= \int_0^\infty h(t) \int_0^\infty K_\lambda^2(t, s; x, y)\phi(y)dy\frac{dt}{t}, \quad x \notin \text{supp} \phi \text{ and } s \in [1/N, N].
\]
Then,
\[
\left[\left[Q_{\lambda, N}^2(\phi)(x)\right](s)\right](t) = \int_0^\infty K_\lambda^2(t, s; x, y)\phi(y)dy, \quad x \notin \text{supp} \phi \text{ and } s \in [1/N, N],
\]
as elements of $H$.

We define $Q_{\lambda, N}^2$ on $S_\lambda(0, \infty) \otimes B$ in the natural way. For every $\phi \in S_\lambda(0, \infty) \otimes B$ we have that
\[
\left[\left[Q_{\lambda, N}^2(\phi)(x)\right](s)\right](t) = P_s^\lambda \left(G_{\lambda, N}^{\alpha, \beta}(\phi)(t, \cdot)\right)(x), \quad x \notin \text{supp} \phi \text{ and } s \in [1/N, N],
\]
in the sense of equality in $\gamma(H, B)$.

Since $P_s^\lambda$ is bounded from $L^2((0, \infty), \gamma(H, B))$ into $L^2(0, \infty)$ and $G_{N, N}^{\alpha, \beta}$ is bounded from $L^2((0, \infty), B)$ into $L^2((0, \infty), \gamma(H, B))$ (Theorem [B]), the operator $P_s^\lambda \circ G_{\lambda, N}^{\alpha, \beta}$ is bounded from $L^2((0, \infty), B)$ into $L^2(0, \infty)$. Hence, the operator
\[
Z_{\lambda, N}^2(f)(t, s; x) = P_s^\lambda \left(G_{\lambda, N}^{\alpha, \beta}(f)(t, \cdot)\right)(x), \quad t, x \in (0, \infty), \ s \in [1/N, N],
\]
is bounded from $L^2((0, \infty), \mathbb{B})$ into $L^2((0, \infty), C([1/N, N], \gamma(H, \mathbb{B})))$. Moreover,

$$\sup_{M \in \mathbb{N}} \|Z^\beta_{\lambda,M}\|_{L^2((0, \infty), \mathbb{B}) \to L^2((0, \infty), C([1/M, M], \gamma(H, \mathbb{B})))} < \infty.$$ 

By taking into account (26), (27) and (28) and by using vector valued Calderón-Zygmund theory we conclude that $Z^\beta_{\lambda,N}$ can be extended to $L^1((0, \infty), \mathbb{B})$ as a bounded operator from $L^1((0, \infty), \mathbb{B})$ into $L^{1,\infty}((0, \infty), C([1/N, N], \gamma(H, \mathbb{B})))$ and to $H^1((0, \infty), \mathbb{B})$ as a bounded operator from $H^1((0, \infty), \mathbb{B})$ into $L^1((0, \infty), C([1/N, N], \gamma(H, \mathbb{B})))$. We denote by $\tilde{Z}^\beta_{\lambda,N}$ to this extension of $Z^\beta_{\lambda,N}$ to $L^1((0, \infty), \mathbb{B})$. It has that

$$\sup_{M \in \mathbb{N}} \|\tilde{Z}^\beta_{\lambda,M}\|_{L^1((0, \infty), \mathbb{B}) \to L^{1,\infty}((0, \infty), C([1/M, M], \gamma(H, \mathbb{B})))} < \infty,$$

and

$$\sup_{M \in \mathbb{N}} \|\tilde{Z}^\beta_{\lambda,M}\|_{H^1((0, \infty), \mathbb{B}) \to L^1((0, \infty), C([1/M, M], \gamma(H, \mathbb{B})))} < \infty.$$

Our objective now is to show that

$$\tilde{Z}^\beta_{\lambda,N}(f) = P_{s}^\lambda \left(G^\lambda_{\beta}(f)\right), \quad f \in L^1((0, \infty), \mathbb{B}),$$

as elements in $L^{1,\infty}((0, \infty), C([1/N, N], \gamma(H, \mathbb{B}))).$

Let $f \in L^1((0, \infty), \mathbb{B})$. We choose a sequence $(f_k)_{k \in \mathbb{N}} \subset S_\lambda(0, \infty) \otimes \mathbb{B}$ such that

$$f_k \rightarrow f, \quad \text{as } k \rightarrow \infty, \text{ in } L^1((0, \infty), \mathbb{B}).$$

Then,

$$Z^\beta_{\lambda,N}(f_k) \rightarrow \tilde{Z}^\beta_{\lambda,N}(f), \quad \text{as } k \rightarrow \infty, \text{ in } L^{1,\infty}((0, \infty), C([1/N, N], \gamma(H, \mathbb{B}))).$$

It is not hard to see that, for every $t, x \in (0, \infty)$ and $s \in [1/N, N]$,

$$Z^\beta_{\lambda,N}(f)(t, s; x) = \int_0^\infty K^\beta_\lambda(t, s; x, y)f(y)dy = G^\lambda_{\beta}(P^\lambda_s(f))(t, x).$$

We know that, for every $s \in (0, \infty)$, $P^\lambda_s$ is a bounded operator from $L^1((0, \infty), \mathbb{B})$ into itself, and that the operator $G^\lambda_{\beta}$ is bounded from $L^1((0, \infty), \mathbb{B})$ into $L^{1,\infty}((0, \infty), \gamma(H, \mathbb{B}))$ (Lemma 2.1).

Then, it follows that, for every $s \in [1/N, N]$,

$$Z^\beta_{\lambda,N}(f_k)(\cdot, s; \cdot) \rightarrow Z^\beta_{\lambda,N}(f)(\cdot, s; \cdot), \quad \text{as } k \rightarrow \infty, \text{ in } L^{1,\infty}((0, \infty), \gamma(H, \mathbb{B})).$$

Hence, we can find an increasing sequence $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ and a subset $\Omega$ of $(0, \infty)$ such that $|(0, \infty) \setminus \Omega| = 0$, and

$$Z^\beta_{\lambda,N}(f_{n_k})(\cdot, s; x) \rightarrow Z^\beta_{\lambda,N}(f)(\cdot, s; x), \quad \text{as } k \rightarrow \infty, \text{ in } \gamma(H, \mathbb{B}),$$

for every $x \in \Omega$ and $s \in Q \cap [1/N, N]$. Here $\Omega$ and $(n_k)_{k \in \mathbb{N}}$ do not depend on $N$.

Also, there exists an increasing sequence $(k_j)_{j \in \mathbb{N}} \subset \mathbb{N}$ and a subset $W$ of $\Omega$ with $|\Omega| = |W|$, such that

$$Z^\beta_{\lambda,N}(f_{k_j})(\cdot, s; x) \rightarrow [\tilde{Z}^\beta_{\lambda,N}(f)(x)](s), \quad \text{as } j \rightarrow \infty, \text{ in } \gamma(H, \mathbb{B}),$$

for every $x \in W$ and $s \in Q \cap [1/N, N]$. Again, $W$ and $(k_j)_{j \in \mathbb{N}}$ do not depend on $N$.

We conclude that

$$Z^\beta_{\lambda,N}(f)(\cdot, s, x) = [\tilde{Z}^\beta_{\lambda,N}(f)(x)](s), \quad x \in W, \ s \in Q \cap [1/N, N].$$

This equality is understood in $\gamma(H, \mathbb{B})$. 
Hence, we can write
\[
\left\{ x \in (0, \infty) : P_s^\lambda (G_B^{\lambda, \beta} (f))(x) > \alpha \right\} \\
= \bigcup_{M \in \mathbb{N}} \left\{ x \in (0, \infty) : \sup_{s \in [1/M, M]} \left\| P_s^\lambda (G_B^{\lambda, \beta} (f))(x) \right\|_{\gamma(H, B)} > \alpha \right\} \\
\leq \lim_{M \to \infty} \left\{ x \in (0, \infty) : \sup_{s \in [1/M, M]} \left\| P_s^\lambda (G_B^{\lambda, \beta} (f))(x) \right\|_{\gamma(H, B)} > \alpha \right\} \\
\leq \lim_{M \to \infty} \left\{ x \in W : \sup_{s \in [1/M, M]} \left\| Z_{\lambda, \beta} (f)(\cdot, s, x) \right\|_{\gamma(H, B)} > \alpha \right\} \\
\leq \frac{C}{\alpha} \| f \|_{L^1((0,\infty), B)}, \quad \alpha > 0.
\]
Thus, we prove that the operator \( Z_{\lambda}^\beta \) defined by
\[
Z_{\lambda}^\beta (f)(t, s, x) = P_s^\lambda \left( G_B^{\lambda, \beta} (f)(t, \cdot) \right)(x), \quad s, t, x \in (0, \infty),
\]
is bounded from \( L^1((0, \infty), B) \) into \( L^{1,\infty}((0, \infty), L^\infty((0, \infty), \gamma(H, B))) \).

By proceeding in a similar way, since \( L_\infty^c((0, \infty), \mathbb{B}) \) is a dense subset of \( H^1((0, \infty), B) \), we can see that \( Z_{\lambda}^\beta \) defines a bounded operator from \( H^1((0, \infty), B) \) into \( L^1((0, \infty), L^\infty((0, \infty), \gamma(H, B))) \).

Here \( L_\infty^c((0, \infty) \) represents the space of bounded measurable functions with compact support in \((0, \infty)\).

Thus, if \( a \) is a 2-atom satisfying (A\text{iii}) we get that
\[
\left\| P_s^\lambda \left( G_B^{\lambda, \beta} (a) \right) \right\|_{L^1((0, \infty), L^\infty((0, \infty), \gamma(H, B)))} \leq C,
\]
where \( C \) does not depend on \( a \).

On the other hand, by using (7) it follows that
\[
\| K_{\lambda}^\beta (\cdot, \cdot; x, y) \|_{L^\infty((0, \infty), H)} \leq C \frac{|xy|^\lambda}{|x-y|^{2\lambda+1}}, \quad x, y \in (0, \infty), \quad x \neq y.
\]
The, since the operator \( Z_{\lambda}^\beta \) is bounded from \( L^2((0, \infty), B) \) into \( L^2((0, \infty), L^\infty((0, \infty), \gamma(H, B))) \),
by proceeding as in the proof of (25) we can deduce that there exists \( C > 0 \) such that, for every \( \delta > 0 \), and \( b \in \mathbb{B}, \| b \|_B = 1, \)
\[
\left\| P_s^\lambda \left( G_B^{\lambda, \beta} (a) \right) \right\|_{L^1((0, \infty), L^\infty((0, \infty), \gamma(H, B)))} \leq C,
\]
when \( a = b\chi_{(0,\delta)}/\delta \).

We conclude that, for every \( f \in H^1_B((0, \infty), B), \)
\[
\| P_s^\lambda (G_B^{\lambda, \beta} (f)) \|_{L^1((0, \infty), B)} \leq C \| f \|_{H^1_B((0, \infty), B)}.
\]

\[\square\]

2.2. We show now that \( G_B^{\lambda, \beta} \) is bounded from \( BMO_0((0, \infty), B) \) into \( BMO_0((0, \infty), \gamma(H, B)) \).
This requires verifying the corresponding vector-valued conditions (B\text{i}) and (B\text{ii}) which we collected in Lemma 2.4 and Lemma 2.5, respectively.

**Lemma 2.4.** Consider \( B \) a UMD Banach space and \( \beta, \lambda > 0 \). There exists \( C > 0 \) such that, for every \( r > 0, \)
\[
\frac{1}{r} \int_0^r \| G_B^{\lambda, \beta} (f)(\cdot, x) \|_{\gamma(H, B)} dx \leq C \| f \|_{BMO_0((0, \infty), B)}, \quad f \in BMO_0((0, \infty), B).
\]
Proof. Assume that \( f \in BMO_\infty((0, \infty), \mathcal{B}) \). According to (7), for every \( k \in \mathbb{N} \), we have that

\[
|\partial_t^k P_t^\lambda(x, y)| \leq C \frac{(xy)^\lambda}{(t + |x - y|)^{2\lambda + 1} + k} \leq \frac{C}{t^k} \frac{(xy)^\lambda}{(t + |x - y|)^{2\lambda + 1}}, \quad t, x, y \in (0, \infty).
\]

Then, for every

\[
\int_0^\infty |\partial_t^k P_t^\lambda(x, y)| \|f(y)\|_\infty dy \
\leq C \frac{1}{t^k} \left\{ \int_0^{2x} \frac{x^{2\lambda}}{t^{2\lambda + 1}} \|f(y)\|_\infty dy + \int_{2x}^\infty \frac{x^\lambda}{y^{2\lambda + 1}} \|f(y)\|_\infty dy \right\}
\leq C \frac{1}{t^k} \left\{ \left( \frac{x}{t} \right)^{2\lambda + 1} \|f\|_{BMO_\infty((0, \infty), \mathcal{B})} + \sum_{j=1}^\infty \frac{x^\lambda}{(x/2)^{2\lambda + 1}} \int_0^{2x(j+1)^{2/\lambda}} \|f(y)\|_\infty dy \right\}
\leq C \frac{1}{t^k} \left\{ \left( \frac{x}{t} \right)^{2\lambda + 1} + \sum_{j=1}^\infty \frac{1}{j^2} \right\} \|f\|_{BMO_\infty((0, \infty), \mathcal{B})}, \quad t, x \in (0, \infty) \text{ and } k \in \mathbb{N}.
\]

We can write

\[
\partial_t^k P_t^\lambda(f)(x) = \int_0^\infty \partial_t^k P_t^\lambda(x, y)f(y)dy, \quad t, x \in (0, \infty) \text{ and } k \in \mathbb{N}.
\]

By using again (7), if \( m \in \mathbb{N} \) such that \( m - 1 \leq \beta < m \), we get

\[
\int_0^\infty s^{m-\beta-1} \int_0^\infty |\partial_t^m P_{t+s}^\lambda(x, y)| \|f(y)\|_\infty dy ds \leq C \|f\|_{BMO_\infty(\mathbb{R}, \mathcal{B})} \int_0^\infty s^{m-\beta-1} \left( \frac{x}{t+s} \right)^{2\lambda + 1} ds < \infty, \quad t, x \in (0, \infty).
\]

This leads to

\[
G_{\beta}^\lambda(f)(t, x) = \int_0^\infty K_t^\lambda(t; x, y)f(y)dy, \quad t, x \in (0, \infty),
\]

where \( K_t^\lambda(t; x, y) = t^\beta \partial_t^\beta P_t^\lambda(x, y), \quad t, x, y \in (0, \infty) \).

Let \( r > 0 \). We split \( G_{\beta}^\lambda(f)(t, x) \) as follows

\[
G_{\beta}^\lambda(f)(t, x) = G_{\beta, 1}^\lambda(f)(t, x) + G_{\beta, 2}^\lambda(f)(t, x), \quad t, x \in (0, \infty),
\]

being

\[
G_{\beta, 1}^\lambda(f)(t, x) = \int_0^{2r} K_t^\lambda(t; x, y)f(y)dy, \quad t, x \in (0, \infty).
\]

Since \( G_{\beta}^\lambda \) is a bounded operator from \( L^2((0, \infty), \mathbb{B}) \) into \( L^2((0, \infty), \gamma(H, \mathcal{B})) \) (Theorem B), we obtain

\[
\frac{1}{r} \int_0^r \|G_{\beta, 1}^\lambda(f)(\cdot, x)\|_{\gamma(H, \mathcal{B})}dx \leq \left( \frac{1}{r} \int_0^\infty \|G_{\beta}^\lambda(f)(\cdot, x)\|_{\gamma(H, \mathcal{B})}^2 dx \right)^{1/2} \leq C \left( \frac{1}{r} \int_0^{2r} \|f(y)\|_\infty^2 dy \right)^{1/2} \leq C \|f\|_{BMO_\infty((0, \infty), \mathcal{B})}.
\]

Note that the last inequality follows from John-Nirenberg’s property.
If \( h \in H \), by using (23) and proceeding as in (31) we have
\[
\int_0^\infty \int_{2r}^\infty |K_\lambda^\beta(t;x,y)||f(y)||_B dy|h(t)|\frac{dt}{t} \leq \|h\|_H \int_0^\infty \|K_\lambda^\beta(;x,y)\|_H |f(y)||_B dy
\]
\[
\leq C\|h\|_H \int_2^\infty \frac{(xy)^\lambda}{|x-y|^{2\lambda+1}} |f(y)||_B dy \leq C\|h\|_H x^\lambda \int_2^\infty \frac{1}{y^{\lambda+1}} |f(y)||_B dy
\]
\[
\leq C\|h\|_H \|f\|_BMO,((0,\infty),B), \quad x \in (0,r).
\]

Then, if \((h_j)_{j=1}^n\) is a set of orthonormal functions in \( H \), we can write
\[
\left( \mathbb{E} \left[ \sum_{j=1}^n \gamma_j \int_0^\infty h_j(t) \int_{2r}^\infty K_\lambda^\beta(t;x,y)f(y)dydt \right] \right)^{1/2}
\]
\[
= \left( \mathbb{E} \left[ \int_{2r}^\infty |f(y)|^2 \sum_{j=1}^n \gamma_j \int_0^\infty K_\lambda^\beta(t;x,y)h_j(t)dydt \right] \right)^{1/2}
\]
\[
\leq \int_{2r}^\infty |f(y)| \left( \mathbb{E} \left[ \sum_{j=1}^n \gamma_j \int_0^\infty K_\lambda^\beta(t;x,y)h_j(t)dy \right] \right)^{1/2} dy
\]
\[
\leq \int_{2r}^\infty |f(y)||_B \|K_\lambda^\beta(;x,y)\|_{\gamma(H,B)} dy.
\]

Since \( \gamma(H,C) = H \) and again by (23) we get, for each \( x \in (0,r) \),
\[
\|G_{\lambda,2}^\beta(f)||_{\gamma(H,B)} = \left\| \int_{2r}^\infty K_\lambda^\beta(;x,y)f(y)dy \right\|_{\gamma(H,B)} \leq C \left\| \int_{2r}^\infty |f(y)||_B \|K_\lambda^\beta(;x,y)\|_H dy \right\| \leq C \|f\|_BMO,((0,\infty),B).
\]

Hence,
\[
(33) \quad \frac{1}{r} \int_0^r |G_{\lambda,2}^\beta(f)(,x)||_{\gamma(H,B)} dx \leq C \|f\|_BMO,((0,\infty),B).
\]

From (32) and (33) we conclude the proof of this Lemma. \( \square \)

Note that (30) implies that \( \|G_{\lambda,2}^\beta(f)(,x)||_{\gamma(H,B)} < \infty \), a.e. \( x \in (0,\infty) \).

**Lemma 2.5.** Let \( B \) be a UMD Banach space and \( \beta, \lambda > 0 \). The operator \( G_{\lambda,2}^\beta \) is bounded from \( BMO,((0,\infty),B) \) into \( BMO((0,\infty),\gamma(H,B)) \).

**Proof.** Let \( f \in BMO,((0,\infty),B) \). We consider the odd extension function \( f_\circ \) of \( f \) to \( \mathbb{R} \) and
\[
G_{\lambda,2}^\beta(f_\circ)(t,x) = \int_{\mathbb{R}} K_\lambda^\beta(t;x,y)f_\circ(y)dy, \quad t \in (0,\infty) \text{ and } x \in \mathbb{R},
\]
where
\[
K_\lambda^\beta(t;x,y) = t^\beta \partial_t^\beta P_t(x-y), \quad t \in (0,\infty) \text{ and } x,y \in \mathbb{R}.
\]
Here \( P_t(z) = t/[\pi(t^2 + z^2)] \), \( t \in (0,\infty) \) and \( z \in \mathbb{R} \), is the classical Poisson semigroup.
We can write
\[ G_{20}^\beta(f_v)(t, x) = \int_0^\infty t^\beta \partial_t^\beta P_t(x - y) f(y) dy - \int_0^\infty t^\beta \partial_t^\beta P_t(x + y) f(y) dy \]
\[ = \int_{-\infty}^{x/2} t^\beta \partial_t^\beta P_t(x - y) f(y) dy - \int_{x/2}^{2x} t^\beta \partial_t^\beta P_t(x + y) f(y) dy \]
\[ + \int_{x/2}^{\infty} t^\beta \partial_t^\beta [P_t(x - y) - P_t(x + y)] f(y) dy + \int_{2x}^{\infty} t^\beta \partial_t^\beta [P_t(x - y) - P_t(x + y)] f(y) dy \]
\[ = \sum_{j=1}^{4} I_j(f)(t, x), \quad t, x \in (0, \infty). \]

In [5, Lemma 1] it was established that, if \( m \in \mathbb{N} \) is such that \( m - 1 \leq \beta < m, \)
\[ t^\beta \partial_t^\beta P_t(z) = \sum_{k \in \mathbb{N}, 0 \leq k \leq (m+1)/2} \frac{c_k}{t} \varphi^k \left( \frac{z}{t} \right), \quad t \in (0, \infty) \text{ and } z \in \mathbb{R}, \]
where, for every \( k \in \mathbb{N}, 0 \leq k \leq (m+1)/2, \ c_k \in \mathbb{C} \)
and
\[ \varphi^k(z) = \int_{0}^{\infty} \frac{(1 + v)^{m+1-2k} v^{m-\beta-1}}{((1 + v)^2 + z^2)^{m-k+1}} dv, \quad z \in \mathbb{R}. \]

Let \( k \in \mathbb{N} \) such that \( 0 \leq k \leq (m+1)/2. \) We have that
\[ \frac{1}{t} \varphi^k \left( \frac{x + y}{t} \right) - \frac{1}{t} \varphi^k \left( \frac{x - y}{t} \right) = t^{2(m-k)+1} \int_{0}^{\infty} (1 + v)^{m+1-2k} v^{m-\beta-1} \]
\[ \times \frac{1}{(1 + v)^2 + (x + y)^2} \int_{0}^{\infty} (1 + v)^{m+1-2k} v^{m-\beta-1} \]
\[ = t^{2(m-k)+1} \sum_{\ell=0}^{m-k+1} \binom{m-k+1}{\ell} \int_{0}^{\infty} (1 + v)^{m+1-2k} v^{m-\beta-1} \]
\[ \times \frac{((1 + v)t)^{2(m-k+1-\ell)}}{((1 + v)^2 + (x + y)^2) ((1 + v)^2 + (x - y)^2)^{m-k+1}} dv, \quad t, x, y \in (0, \infty). \]

We get
\[ \left| \frac{1}{t} \varphi^k \left( \frac{x + y}{t} \right) - \frac{1}{t} \varphi^k \left( \frac{x - y}{t} \right) \right| \leq C y t^{2(m-k)+1} \sum_{\ell=1}^{m-k+1} x^{2\ell-1} \int_{0}^{\infty} (1 + v)^{m+1-2k} v^{m-\beta-1} \]
\[ \times \frac{((1 + v)t)^{2(m-k+1-\ell)}}{((1 + v)t + x)^{m-k+1}} dv \]
\[ \leq C y t^{2(m-k)+1} \int_{0}^{\infty} (1 + v)^{m+1-2k} v^{m-\beta-1} \frac{((1 + v)t + x)^{2(m-k)+1}}{((1 + v)t + x)^{2(m-k)+1}} dv, \quad t \in (0, \infty), \ 0 < y < x/2. \]

Minkowski’s inequality leads to
\[ \left\| \frac{1}{t} \varphi^k \left( \frac{x + y}{t} \right) - \frac{1}{t} \varphi^k \left( \frac{x - y}{t} \right) \right\|_H \leq C y \int_{0}^{\infty} (1 + v)^{m+1-2k} v^{m-\beta-1} \]
\[ \times \left( \int_{0}^{\infty} \frac{t^{-2(m-k)+1}}{((1 + v)t + x)^{m-k+1}} dt \right)^{1/2} dv \]
\[ \leq C y \int_{0}^{\infty} \frac{v^{m-\beta-1}}{(1 + v)^{m}} dv \]
\[ \leq C y \frac{x^2}{x^2}, \quad 0 < y < x/2. \]

Hence, we obtain
\[ \| t^\beta \partial_t^\beta [P_t(x - y) - P_t(x + y)] \|_H \leq C y \frac{x^2}{x^2}, \quad 0 < y < x/2. \]
It follows that

\[ \|I_1(f)(\cdot, x)\|_{\gamma(H, B)} \leq \frac{C}{x^2} \int_0^{x/2} y \|f(y)\|_2 dy \leq C\|f\|_{BMO_h((0, \infty), B)}, \quad x \in (0, \infty). \]

By symmetries, we also get

\[ \|t^\beta \partial_t^2 [P_t(x - y) - P_t(x + y)]\|_H \leq \frac{C}{y^2}, \quad 0 < 2x < y, \]

and then

\[ \|I_4(f)(\cdot, x)\|_{\gamma(H, B)} \leq Cx \int_{2x}^{\infty} \frac{\|f(y)\|_2}{y^2} dy \leq Cx \sum_{k=1}^{\infty} \int_{2k^2x}^{(2k+1)^2x} \frac{\|f(y)\|_2}{y^2} dy \]

\[ \leq C \sum_{k=1}^{\infty} \frac{1}{xk^2} \int_0^{(2k+1)^2x} \|f(y)\|_2 dy \leq C\|f\|_{BMO_h((0, \infty), B)}, \quad x \in (0, \infty). \]

Note that to establish

\[ \|I_3(f)(\cdot, x)\|_{\gamma(H, B)} \leq C\|f\|_{BMO_h((0, \infty), B)}, \quad x \in (0, \infty), \]

it is enough to use that

\[ \|t^\beta \partial_t^2 [P_t(x - y) - P_t(x + y)]\|_H \leq \frac{C}{x}, \quad 0 < y < \frac{x}{2}. \]

However, the estimation

\[ \|t^\beta \partial_t^2 [P_t(x - y) - P_t(x + y)]\|_H \leq \frac{C}{y}, \quad 0 < 2x < y, \]

does not allow to show that

\[ \|I_4(f)(\cdot, x)\|_{\gamma(H, B)} \leq C\|f\|_{BMO_h((0, \infty), B)}, \quad x \in (0, \infty). \]

Also, we obtain

\[ \|t^\beta \partial_t^2 P_t(x + y)\|_H \leq C \sum_{k \in \mathbb{N}} \frac{1}{k} \int_0^{\infty} (1 + v)^{m+1-2k} v^{\mu-\beta-1} \left( \int_0^{\infty} \frac{t^{4(m-k)+1}}{(1+v)t+x+y} dt \right)^{1/2} dv. \]

\[ \leq \frac{C}{x+y}, \quad x, y \in (0, \infty). \]

Hence,

\[ \|I_2(f)(\cdot, x)\|_{\gamma(H, B)} \leq \frac{C}{x} \int_{x/2}^{2x} \|f(y)\|_2 dy \leq C\|f\|_{BMO_h((0, \infty), B)}, \quad x \in (0, \infty). \]

We conclude that

\[ G_{\mathbb{B}}(f_o)(t, x) - \int_{x/2}^{2x} t^\beta \partial_t^2 P_t(x - y)f(y)dy \in L^\infty ((0, \infty), \gamma(H, B)), \]

and we deduce that \( G_{\mathbb{B}}(f_o) \in BMO((0, \infty), \gamma(H, B)) \) if, and only if, \( G_{\mathbb{B}, loc}(f) \in BMO((0, \infty), \gamma(H, B)) \), where

\[ G_{\mathbb{B}, loc}(f)(t, x) = \int_{x/2}^{2x} t^\beta \partial_t^2 P_t(x - y)f(y)dy, \quad t, x \in (0, \infty). \]

We are going to prove that \( G_{\mathbb{B}}(f_o) \in BMO((0, \infty), \gamma(H, B)) \). Let \( 0 < r < s < \infty \). We define \( I = (r, s), x_I = (r + s)/2 \) and \( d_I = (s - r)/2 \). We decompose \( f_o \) as follows:

\[ f_o = (f_o - f_I)\chi_{2I} + (f_o - f_I)\chi_{(0, \infty)\setminus 2I} + f_I = f_1 + f_2 + f_3, \]
where $2I = (x_I - 2d_I, x_I + 2d_I)$. By \cite{23} Theorem 4.2 and \cite{28} proof of Theorem 1.2, it follows that the operator $G^\beta_B$ is bounded from $L^2(\mathbb{R}, \mathbb{B})$ into $L^2(\mathbb{R}, \gamma(H, \mathbb{B}))$. Then,

$$
\frac{1}{|I|} \int_I \|G^\beta_B(f_1)(\cdot, x)\|_{\gamma(H, \mathbb{B})}dx \leq \left( \frac{1}{|I|} \int_I \|G^\beta_B(f_1)(\cdot, x)\|_{\gamma(H, \mathbb{B})}^2dx \right)^{1/2}
$$

(36)

$$
\leq C \left( \frac{1}{|I|} \int_{2I} \|f_0(x) - f_I\|_B^2dx \right)^{1/2} \leq C\|f\|_{BMO_{\infty}(0, \infty, \mathbb{B})}.
$$

Hence,

$$
\|G^\beta_B(f_1)(\cdot, x)\|_{\gamma(H, \mathbb{B})} < \infty, \quad \text{a.e. } x \in (0, \infty).
$$

On the other hand, since

$$
\int_{\mathbb{R}} P_t(x - y)dy = 1, \quad t \in (0, \infty) \text{ and } x \in \mathbb{R},
$$

if $m \in \mathbb{N}$, being $m - 1 \leq \beta < m$,

(37)

$$
\partial^m_t \int_{\mathbb{R}} P_t(x - y)dy = \frac{e^{-\pi(x - \beta)} \Gamma(m - \beta)}{\Gamma(m)} \int_0^\infty s^{m-\beta-1} \partial^m_r \int_{\mathbb{R}} P_{r+s}(x - y)dyds = 0, \quad t \in (0, \infty) \text{ and } x \in \mathbb{R}.
$$

It follows that $G^\beta_B(f_3) = 0$.

In \cite{15} Section 3.1 it was proved that

$$
\|K^\beta_A(; x, y) - K^\beta_A(; x, y)\|_H \leq \frac{C}{x}, \quad x/2 < y < 2x, \quad x \in (0, \infty).
$$

Then,

$$
\left\| \int_{x/2}^{2x} \left( K^\beta_A(; x, y) - K^\beta_A(; x, y) \right) f(y)dy \right\|_{\gamma(H, \mathbb{B})} \leq \int_{x/2}^{2x} \left\| K^\beta_A(; x, y) - K^\beta_A(; x, y) \right\|_H \|f(y)\|_{\mathbb{B}}dy
$$

(38)

\begin{align*}
\leq \frac{C}{x} \int_{x/2}^{2x} \|f(y)\|_{\mathbb{B}}dy \leq C\|f\|_{BMO_{\infty}(0, \infty, \mathbb{B})}, \quad x \in (0, \infty).
\end{align*}

Moreover, by using \cite{23} and as it was seen in \cite{31},

(39)

$$
\left\| \int_{2x}^{\infty} K^\beta_A(; x, y)f(y)dy \right\|_{\gamma(H, \mathbb{B})} \leq Cx^\beta \int_{2x}^{\infty} \|f(y)\|_{\mathbb{B}} \frac{dy}{y^{\beta+1}} \leq C\|f\|_{BMO_{\infty}(0, \infty, \mathbb{B})}, \quad x \in (0, \infty),
$$

and

(40)

$$
\left\| \int_0^{x/2} K^\beta_A(; x, y)f(y)dy \right\|_{\gamma(H, \mathbb{B})} \leq C \int_0^{x/2} \|f(y)\|_{\mathbb{B}}dy \leq C\|f\|_{BMO_{\infty}(0, \infty, \mathbb{B})}, \quad x \in (0, \infty).
$$

Hence,

$$
\|G^\beta_B(f)(\cdot, x) - G^\beta_B(f_0)(\cdot, x)\|_{\gamma(H, \mathbb{B})} \leq C\|f\|_{BMO_{\infty}(0, \infty, \mathbb{B})}, \quad x \in (0, \infty).
$$

Since, $\|G^\beta_B(f)(\cdot, x)\|_{\gamma(H, \mathbb{B})} < \infty$ and $\|G^\beta_B(f_1)(\cdot, x)\|_{\gamma(H, \mathbb{B})} < \infty$, a.e. $x \in (0, \infty)$, also

$$
\|G^\beta_B(f_2)(\cdot, x)\|_{\gamma(H, \mathbb{B})} < \infty, \quad \text{a.e. } x \in (0, \infty).
$$

By proceeding as in the proof of \cite{15} we get

(41)

$$
\|\partial_x K^\beta(; x, y)\|_H \leq \frac{C}{|x - y|^2}, \quad x, y \in (0, \infty), \quad x \neq y.
$$
We choose $x_0 \in I$ such that $\|G^\beta_{B}(f_2)(\cdot, x_0)\|_{\gamma(H, B)} < \infty$. By using (41) we obtain
\[
\frac{1}{|I|} \int_I \|G^\beta_{B}(f_2)(\cdot, x) - G^\beta_{B}(f_2)(\cdot, x_0)\|_{\gamma(H, B)}dx
\leq \frac{C}{|I|} \int_I \int_{(0, \infty) \setminus 2I} \|f_0(y) - f_1\|_B \|K^\beta(x, y) - K^\beta(x_0, y)\|_H dydx
\leq \frac{C}{|I|} \int_I \int_{(0, \infty) \setminus 2I} \|f_0(y) - f_1\|_B \left| \int_x^{x_0} \|\partial_u K^\beta(t, u, y)\|_H du \right| dydx
\leq \frac{C}{|I|} \int_I \int_{(0, \infty) \setminus 2I} \|f_0(y) - f_1\|_B \frac{|x - x_0|}{|x - y|} dydx.
\]
By employing standard arguments (see, for instance, [12, (9)]) we deduce that
\begin{equation}
\frac{1}{|I|} \int_I \|G^\beta_{B}(f_2)(\cdot, x) - G^\beta_{B}(f_2)(\cdot, x_0)\|_{\gamma(H, B)}dx \leq C \|f\|_{BMO_a((0, \infty), B)}.
\end{equation}

By putting together (36), (37) and (42) we conclude that
\[
\frac{1}{|I|} \int_I \|G^\beta_{B}(f_0)(\cdot, x) - G^\beta_{B}(f_2)(\cdot, x_0)\|_{\gamma(H, B)}dx \leq C \|f\|_{BMO_a((0, \infty), B)},
\]
where $C > 0$ does not depend on $I$. Hence, $G^\beta_{B}(f_0) \in BMO((0, \infty), \gamma(H, B))$, and then $G^\beta_{B, \text{loc}}(f) \in BMO((0, \infty), \gamma(H, B))$. Finally by using (38), (39) and (40) we obtain that
\[
\|G^\lambda,\beta_{B}(f)(\cdot, x) - G^\beta_{B, \text{loc}}(f)(\cdot, x)\|_{\gamma(H, B)} \leq C \|f\|_{BMO_a((0, \infty), B)}, \quad x \in (0, \infty),
\]
and hence $G^\lambda,\beta_{B}(f) \in BMO((0, \infty), \gamma(H, B))$. $\square$

2.3. In this section we are going to show that
\[
\|f\|_{BMO_a((0, \infty), B)} \leq C \|G^\lambda,\beta_{B}(f)\|_{BMO_a((0, \infty), \gamma(H, B))}, \quad f \in BMO_a((0, \infty), B).
\]

To establish this property we need to prove some auxiliary results.

Suppose firstly that $f \in BMO_a(0, \infty)$. According to [7, Theorem 6.1] we have that the measure
\[
d\mu_f(x, t) = \left| t^\beta \partial_t^\beta P^\lambda_t(f)(x) \right|^2 \frac{dxdt}{t}
\]
is Carleson on $(0, \infty)^2$. Then, by [7, Proposition 5.3], for every $a \in L^\infty_{\gamma}(0, \infty)$, where $L^\infty_{\gamma}(0, \infty)$ denotes the space of bounded measurable functions with upper bounded support on $(0, \infty)$,
\begin{equation}
\int_0^\infty \int_0^\infty t^\beta \partial_t^\beta P^\lambda_t(f)(x)t^\beta \partial_t^\beta P^\lambda_t(a)(x) \frac{dxdt}{t} = \frac{e^{2\pi i \beta}}{2^\frac{3\beta}{2}} \int_0^\infty f(x)a(x)dx.
\end{equation}

The following is a vector-valued version of (43).

**Proposition 2.2.** Let $\lambda, \beta > 0$. If $f \in BMO_a((0, \infty), B)$ and $a \in L^\infty_{\gamma}(0, \infty) \otimes \mathbb{R}^n$, then
\[
\int_0^\infty \int_0^\infty (G^\lambda,\beta_{B}(a)(t, x), G^\lambda,\beta_{B}(f)(t, x))_{B^{\frac{n}{\beta}}, B} \frac{dxdt}{t} = \frac{e^{2\pi i \beta}}{2^\frac{3\beta}{2}} \int_0^\infty (a(x), f(x))_{B^{\frac{n}{\beta}}, B}dx.
\]
Proof. Assume that $f \in BMO_0((0, \infty), \mathbb{B})$ and $a \in L^\infty_{+c}(0, \infty) \otimes \mathbb{B}^*$. If $a = \sum_{j=1}^n a_j b_j$, being $a_j \in L^\infty_{+c}(0, \infty)$ and $b_j \in \mathbb{B}^*$, $j = 1, \ldots, n$, by using (43) we can write

$$
\int_0^\infty \int_0^\infty (G_{\mathbb{B}}^{\lambda, \beta}(a)(t, x), G_{\mathbb{B}}^{\lambda, \beta}(f)(t, x))_{\mathbb{B}^* \mathbb{B}} \frac{dxdt}{t} = \sum_{j=1}^n \int_0^\infty \int_0^\infty G_{\mathbb{C}}^{\lambda, \beta}(a_j)(t, x)(b_j, G_{\mathbb{B}}^{\lambda, \beta}(f)(t, x))_{\mathbb{B}^* \mathbb{B}} \frac{dxdt}{t} = \sum_{j=1}^n \int_0^\infty G_{\mathbb{C}}^{\lambda, \beta}(a_j)(t, x)(b_j, f)_{\mathbb{B}^* \mathbb{B}}(t, x) \frac{dxdt}{t} = \frac{e^{2\pi i \beta} \Gamma(2\beta)}{2^{2\beta}} \sum_{j=1}^n \int_0^\infty a_j(x) b_j, f(x))_{\mathbb{B}^* \mathbb{B}} dx = \frac{e^{2\pi i \beta} \Gamma(2\beta)}{2^{2\beta}} \int_0^\infty \langle a(x), f(x) \rangle_{\mathbb{B}^* \mathbb{B}} dx,
$$

because, for every $j = 1, \ldots, n$, $\langle b_j, f \rangle_{\mathbb{B}^* \mathbb{B}} \in BMO_0(0, \infty).$ \hfill $\square$

Let $f \in BMO_0((0, \infty), \mathbb{B})$. We have that

$$
\|f\|_{BMO_0((0,\infty),\mathbb{B})} = \sup_{g \in \mathcal{A} \otimes \mathbb{B}^*, \|g\|_{H^1((0,\infty),\mathbb{B}^*)} \leq 1} \left| \int_0^\infty \langle g(x), f(x) \rangle_{\mathbb{B}^* \mathbb{B}} \right|,
$$

where $\mathcal{A} = \text{span}\{a : a \text{ is an } \infty - \text{atom}\}$. Note that, since $\mathbb{B}$ is a UMD space, $\mathbb{B}^*$ is also UMD, $\mathbb{B}$ is reflexive and $BMO_0((0, \infty), \mathbb{B})$ is the dual space of $H^1_0((0, \infty), \mathbb{B}^*)$. Moreover, since $\mathcal{A}$ is a dense subspace of $H^1_0((0, \infty))$, $\mathcal{A} \otimes \mathbb{B}^*$ is dense in $H^1_0((0, \infty), \mathbb{B}^*)$.

According to Proposition 2.2 we get, for every $a \in \mathcal{A} \otimes \mathbb{B}^*$,

$$
\int_0^\infty \int_0^\infty (G_{\mathbb{B}^*}^{\lambda, \beta}(a)(t, x), G_{\mathbb{B}}^{\lambda, \beta}(f)(t, x))_{\mathbb{B}^* \mathbb{B}} \frac{dxdt}{t} = \frac{e^{2\pi i \beta} \Gamma(2\beta)}{2^{2\beta}} \int_0^\infty \langle a(x), f(x) \rangle_{\mathbb{B}^* \mathbb{B}} dx.
$$

Also, for every $a \in \mathcal{A} \otimes \mathbb{B}^*$, we can write

$$
\int_0^\infty (G_{\mathbb{B}}^{\lambda, \beta}(a)(t, x), G_{\mathbb{B}}^{\lambda, \beta}(f)(t, x))_{\mathbb{B}^* \mathbb{B}} dt = \langle G_{\mathbb{B}}^{\lambda, \beta}(a)(\cdot, x), G_{\mathbb{B}}^{\lambda, \beta}(f)(\cdot, x) \rangle_{\gamma(H, \mathbb{B}^*) \gamma(H, \mathbb{B})}, \quad x \in (0, \infty).
$$

Indeed, we take $a = \sum_{j=1}^n a_j b_j$, where $a_j \in \mathcal{A}$ and $b_j \in \mathbb{B}^*$, $j = 1, \ldots, n$. By taking into account the results in Section 2.1 we have that

$$
G_{\mathbb{B}}^{\lambda, \beta}(a)(t, x) = \sum_{j=1}^n b_j G_{\mathbb{C}}^{\lambda, \beta}(a_j)(t, x) \in H^1_0((0, \infty), \gamma(H, \mathbb{B}^*)).$$
The dual \((\gamma(H, \mathbb{B}))^\ast\) of \(\gamma(H, \mathbb{B})\) can be identified with \(\gamma(H, \mathbb{B}^\ast)\) by using the trace functional. If \((h_m)_{m \in \mathbb{N}}\) is an orthonormal basis in \(H\) we can write

\[
\left\langle G_{B, \ast}^{\lambda, \beta}(a)(\cdot, x), G_{B}^{\lambda, \beta}(f)(\cdot, x) \right\rangle_{\gamma(H, \mathbb{B}^\ast), \gamma(H, \mathbb{B})} = \sum_{j=1}^{n} \left\langle b_j G_{C, \ast}^{\lambda, \beta}(a_j)(\cdot, x), G_{B}^{\lambda, \beta}(f)(\cdot, x) \right\rangle_{\gamma(H, \mathbb{B}^\ast), \gamma(H, \mathbb{B})}
\]

\[
= \sum_{j=1}^{n} \sum_{m \in \mathbb{N}} \int_{0}^{\infty} h_m(t) \int_{0}^{\infty} \left\langle b_j G_{C, \ast}^{\lambda, \beta}(a_j)(u, x), G_{B}^{\lambda, \beta}(f)(t, x) \right\rangle_{\mathcal{B}, \mathbb{B}} h_m(u) \frac{du}{u} \frac{dt}{t}
\]

\[
= \sum_{j=1}^{n} \sum_{m \in \mathbb{N}} \int_{0}^{\infty} h_m(t) \int_{0}^{\infty} G_{C, \ast}^{\lambda, \beta}(a_j)(u, x) G_{C}^{\lambda, \beta}(b_j, f)(t, x) h_m(u) \frac{du}{u} \frac{dt}{t}
\]

We have used that \(\gamma(H, \mathbb{C}) = H\).

By (45) we obtain

\[
\int_{0}^{\infty} \left\langle G_{B, \ast}^{\lambda, \beta}(a)(\cdot, x), G_{B}^{\lambda, \beta}(f)(\cdot, x) \right\rangle_{\gamma(H, \mathbb{B}^\ast), \gamma(H, \mathbb{B})} \, dx \leq \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \left| \frac{G_{C, \ast}^{\lambda, \beta}(a_j)(t, x)}{G_{C}^{\lambda, \beta}(b_j, f)(t, x)} \right| \, dt \, dx.
\]

Since \((b_j, f)_{\mathbb{B}^\ast, \mathbb{B}} \in BMO_{\mathbb{C}}(0, \infty)\), for every \(j = 1, \ldots, n\), by [4] Theorem 6.1, \(\|G_{C}^{\lambda, \beta}(b_j, f)_{\mathbb{B}^\ast, \mathbb{B}}(t, x)\|_{H, \mathbb{B}^\ast} \leq C_{\mathbb{B}, \mathbb{B}^\ast} \sqrt{t} dx/dt\) is a Carleson measure on \((0, \infty)^2\), for every \(j = 1, \ldots, n\). Then, according to [7] Propositions 5.1 and 5.2,

\[
\int_{0}^{\infty} \left\langle G_{B, \ast}^{\lambda, \beta}(a)(\cdot, x), G_{B}^{\lambda, \beta}(f)(\cdot, x) \right\rangle_{\gamma(H, \mathbb{B}^\ast), \gamma(H, \mathbb{B})} \, dx < \infty.
\]

From [4] Proposition 2.5 (adapted to this Bessel context), Proposition 2.2 (45), and by taking into account, as it has already been proved, that \(G_{B, \ast}^{\lambda, \beta}(a) \in H^1_{\mathbb{B}^\ast}(0, \infty, \gamma(H, \mathbb{B}^\ast))\) (Section 2.1) and \(G_{B}^{\lambda, \beta}(f) \in BMO_{\mathbb{C}}((0, \infty), \gamma(H, \mathbb{B}))\) (Section 2.2), we conclude that

\[
\left| \int_{0}^{\infty} \langle a(x), f(x) \rangle_{\mathbb{B}^\ast, \mathbb{B}} \, dx \right| \leq C \|G_{B, \ast}^{\lambda, \beta}(a)\|_{H^1_{\mathbb{B}^\ast}(0, \infty, \gamma(H, \mathbb{B}^\ast))} \|G_{B}^{\lambda, \beta}(f)\|_{BMO_{\mathbb{C}}((0, \infty), \gamma(H, \mathbb{B}))}
\]

\[
\leq C \|a\|_{H^1_{\mathbb{B}^\ast}(0, \infty, \mathbb{B}^\ast)} \|G_{B}^{\lambda, \beta}(f)\|_{BMO_{\mathbb{C}}((0, \infty), \gamma(H, \mathbb{B}))}.
\]

From [44] it follows that

\[
\|f\|_{BMO_{\mathbb{C}}((0, \infty), \mathbb{B})} \leq C \|G_{B}^{\lambda, \beta}(f)\|_{BMO_{\mathbb{C}}((0, \infty), \gamma(H, \mathbb{B}))}.
\]

2.4. Our objective is to prove that, for every \(a \in H^1_{\mathbb{B}}((0, \infty), \mathbb{B})\),

\[
\|a\|_{H^1_{\mathbb{B}}((0, \infty), \mathbb{B})} \leq C \|G_{B}^{\lambda, \beta}(a)\|_{H^1_{\mathbb{B}^\ast}(0, \infty, \gamma(H, \mathbb{B}))},
\]

We have that

\[
\|a\|_{H^1_{\mathbb{B}}((0, \infty), \mathbb{B})} = \sup_{\|f\|_{BMO_{\mathbb{B}^\ast}(0, \infty, \mathbb{B}^\ast)} \leq 1} \left| \int_{0}^{\infty} \langle f(x), a(x) \rangle_{\mathbb{B}^\ast, \mathbb{B}} \, dx \right|, \quad a \in H^1_{\mathbb{B}^\ast}((0, \infty), \mathbb{B}).
\]
Suppose that $a \in A \otimes B$, where $A$ is defined as in Section 2.3. By proceeding as in Section 2.3, we get
\[
\int_0^\infty \| f(x), a(x) \|_{B^*B} \leq C\| G_{\lambda,\beta}^y (f) \|_{BMO(\gamma(H,B^*))} \| G_{\lambda,\beta}^y (a) \|_{H^1(\gamma(H,B))}.
\]
Hence, (46) holds.

In order to see that (46) holds for every $a \in H^1_0((0, \infty), B)$ it is enough to take into account that $A \otimes B$ is a dense subset of $H^1_0((0, \infty), B)$ and that the operator $G_{\lambda,\beta}^y$ is bounded from $H^1_0((0, \infty), B)$ into $H^1_0((0, \infty), \gamma(H,B))$ (Section 2.1).

3. Proof of Theorem 1.2

In this section we prove that
\[
G_{\lambda}^y(f)(t,x) = tD^\lambda_{x,y}P_t^{\lambda+1}(f)(x), \quad t,x \in (0, \infty),
\]
is a bounded operator from $H^1_0((0, \infty), B)$ into $H^1_0((0, \infty), \gamma(H,B))$ and from $BMO(\gamma(H,B))$ into $BMO(\gamma(H,B))$. Here $D^\lambda_{x,y} = -x^{-\lambda} \frac{d}{dx} x^\lambda$.

3.1. We establish now the behavior of $G_{\lambda}^y$ between Hardy spaces. In [6, Theorem 1.3] we proved that the operator $G_{\lambda}^y$ is bounded from $L^p((0, \infty), B)$ into $L^p((0, \infty), \gamma(H,B))$, for every $1 < p < \infty$.

We show that $G_{\lambda}^y$ is a $(B, \gamma(H,B))$-Calderón-Zygmund operator.

Lemma 3.1. Let $B$ be a UMD Banach space and $\lambda \geq 1$. The operator $G_{\lambda}^y$ is a $(B, \gamma(H,B))$-Calderón-Zygmund operator.

Proof. We consider the function
\[
M^\lambda(t;x,y) = tD^\lambda_{x,y}P_t^{\lambda+1}(x,y), \quad t,x \in (0, \infty).
\]

$M^\lambda$ defines, as it will be specified, a standard $(B, \gamma(H,B))$-Calderón-Zygmund kernel. Indeed, we have that
\[
M^\lambda(t;x,y) = -tx^{-\lambda} \frac{d}{dx} \left[ 2(\lambda + 1)x^{2\lambda+1}y^{\lambda+1} \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+2}} d\theta \right]
\]
\[
= -\frac{2(\lambda + 1)}{\pi} t^2y^{\lambda+1} \left[ (2\lambda + 1)x^\lambda \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+2}} d\theta \right]
\]
\[
- 2(\lambda + 2)x^{\lambda+1} \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+3}} d\theta, \quad t,x \in (0, \infty).
\]

Then,
\[
|M^\lambda(t;x,y)| \leq C\left[ xy^{\lambda+1} \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+5}} d\theta \right]
\]
\[
+ y^{\lambda+1}x^\lambda \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+4}} d\theta
\]
\[
= C[\mathcal{I}_1(t;x,y) + \mathcal{I}_2(t;x,y)], \quad t,x \in (0, \infty).
\]

We observe that $\mathcal{I}_j(t;x,y) = t^2I_j(t/2,t/2;x,y)$, $t,x,y \in (0, \infty)$, $j = 1,2$, where $I_j$, $j = 1,2$, are the functions appearing in (12) for $m = 1$ and $\lambda + 1$ instead of $\lambda$. 

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Then, by (13) and (14), for \( j = 1, 2 \), we have that
\[
I_j(t; x, y) \leq C \frac{t^2}{(t + |x - y|)^3}, \quad t, x, y \in (0, \infty),
\]
and
\[
\|I_j(t; x, y)\|_H \leq C \left( \int_0^\infty \frac{t^3}{(t + |x - y|)^3} \, dt \right)^{1/2} = \frac{C}{|x - y|}, \quad x, y \in (0, \infty), x \neq y.
\]
Hence,
\[
\|M^\lambda(\cdot; x, y)\|_H \leq \frac{C}{|x - y|}, \quad x, y \in (0, \infty), x \neq y.
\]
We consider, for every \( x, y \in (0, \infty), x \neq y \), the operator
\[
M^\lambda(x, y) : B \rightarrow \gamma(H, B)
\]
\[
b \rightarrow M^\lambda(x, y)(b) : H \rightarrow B
\]
\[
h \rightarrow \int_0^\infty M^\lambda(t; x, y)h(t) \frac{dt}{t} b.
\]
We have that, for every \( b \in B \),
\[
\|M^\lambda(x, y)(b)\|_{\gamma(H, B)} \leq \|M^\lambda(\cdot; x, y)\|_H\|b\|_B \leq C \frac{\|b\|_B}{|x - y|}, \quad x, y \in (0, \infty), x \neq y.
\]
Then,
\[
\|M^\lambda(x, y)\|_{B \rightarrow \gamma(H, B)} \leq C \frac{1}{|x - y|}, \quad x, y \in (0, \infty), x \neq y.
\]
We can write, for every \( t, x, y \in (0, \infty) \),
\[
\partial_t M^\lambda(t; x, y) = -\frac{2(\lambda + 1)}{\pi} t^2 y^{\lambda+1} \left[ (2\lambda + 1)\lambda x^{-1} \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}}{(x - y)^2 + t^2 + 2xy(1 - \cos \theta)} \, d\theta \right.
\]
\[
-2(2\lambda + 1)(\lambda + 2)x^{\lambda} \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}(x - y) + y(1 - \cos \theta)}{(x - y)^2 + t^2 + 2xy(1 - \cos \theta)} \, d\theta
\]
\[
-2(\lambda + 1)(\lambda + 2)x^{\lambda} \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}(x - y) + y(1 - \cos \theta)}{(x - y)^2 + t^2 + 2xy(1 - \cos \theta)} \, d\theta
\]
\[
+4(\lambda + 2)(\lambda + 3)x^{\lambda+1} \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}(x - y) + y(1 - \cos \theta)^2}{(x - y)^2 + t^2 + 2xy(1 - \cos \theta)} \, d\theta
\]
\[
-2(\lambda + 2)x^{\lambda+1} \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}}{(x - y)^2 + t^2 + 2xy(1 - \cos \theta)} \, d\theta.
\]
It follows that
\[
|\partial_t M^\lambda(t; x, y)| \leq C t^2 \left[ y^{\lambda+1} x^{-1} \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}}{(x - y)^2 + t + \sqrt{2xy(1 - \cos \theta)}}^{2\lambda+4} \, d\theta \right.
\]
\[
+ y^{\lambda+1} x^\lambda \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}}{(x - y)^2 + t + \sqrt{2xy(1 - \cos \theta)}}^{2\lambda+6} \, d\theta
\]
\[
+ (yx)^{\lambda+1} \int_0^\pi \frac{(\sin \theta)^{2\lambda+1}}{(x - y)^2 + t + \sqrt{2xy(1 - \cos \theta)}}^{2\lambda+8} \, d\theta
\]
\[
= C[J_1(t; x, y) + J_2(t; x, y) + J_3(t; x, y)], \quad t, x, y \in (0, \infty).
\]
We note that \( J_2(t; x, y) = t^2 I_2(t/2, t/2; x, y) \) and \( J_3(t; x, y) = t^2 I_1(t/2, t/2; x, y) \), \( t, x, y \in (0, \infty) \), being \( I_j, j = 1, 2 \), the functions in (12) for \( m = 2 \) and \( \lambda + 1 \) instead of \( \lambda \). Hence, by (13) and (14), for \( \ell = 2, 3 \),
\[
J_\ell(t; x, y) \leq C \frac{t^2}{(t + |x - y|)^4}, \quad t, x, y \in (0, \infty).
\]
On the other hand, since $\lambda \geq 1$, by proceeding as in the estimation (51), we obtain

\begin{equation}
J_1(t; x, y) \leq C \frac{t^2}{|t + |x - y||^2}, \quad t, x, y \in (0, \infty).
\end{equation}

Thus, for $\ell = 1, 2, 3$,

\begin{equation}
\|J_\ell(\cdot, x, y)\|_{H} \leq C \left( \int_{0}^{\infty} \frac{t^3}{(t + |x - y|)^3} dt \right)^{1/2} \leq \frac{C}{|x - y|^2}, \quad x, y \in (0, \infty), x \neq y.
\end{equation}

Hence,

\begin{equation}
\|\partial_x M^\lambda(\cdot, x, y)\|_{H} \leq \frac{C}{|x - y|^2}, \quad x, y \in (0, \infty), x \neq y.
\end{equation}

We have also that

\begin{equation}
\|\partial_y M^\lambda(\cdot, x, y)\|_{H} \leq \frac{C}{|x - y|^2}, \quad x, y \in (0, \infty), x \neq y.
\end{equation}

Estimations (51), (56) and (57) show that $M^\lambda$ is a standard $(\mathbb{B}, \gamma(H, \mathbb{B}))$-Calderón-Zygmund kernel.

Let $f \in S_\lambda(0, \infty) \otimes \mathbb{B}$. According to (51) we have that

\begin{equation}
\int_{0}^{\infty} \|M^\lambda(x, y)\|_{\mathbb{B} \to \gamma(H, \mathbb{B})} \|f(y)\|_{\mathbb{B}} dy < \infty, \quad x \notin \text{supp } f.
\end{equation}

We define

\begin{equation}
Q^\lambda(f)(x) = \int_{0}^{\infty} M^\lambda(t; x, y) f(y) dy, \quad x \notin \text{supp } f,
\end{equation}

where the integral is understood in the $\gamma(H, \mathbb{B})$-Bochner sense.

We can differentiate under the integral sign to get

\begin{equation}
G^\lambda_\delta(f)(t, x) = \int_{0}^{\infty} M^\lambda(t; t, y) f(y) dy, \quad t, x \in (0, \infty),
\end{equation}

and by using Minkowski’s inequality and (51) we obtain

\begin{equation}
\|G^\lambda_\delta(f)(\cdot, x)\|_{L^2(0, \infty), \mathbb{B}, \text{at }, \mathbb{B})} \leq C \int_{0}^{\infty} \|f(y)\|_{\mathbb{B}} \frac{dy}{|x - y|}, \quad x \notin \text{supp } f.
\end{equation}

We consider, for every $x \notin \text{supp } f$, the operator

\begin{equation}
G^\lambda_\delta(f)(x) : H \to \mathbb{B}
\end{equation}

\begin{equation}
\text{h} \to [G^\lambda_\delta(f)(x)](h) = \int_{0}^{\infty} G^\lambda_\delta(f)(t, x) h(t) \frac{dt}{t}.
\end{equation}

We can write

\begin{align*}
[Q^\lambda(f)(x)](h) &= \int_{0}^{\infty} \left[ M^\lambda(x, y) f(y) \right](h) dy \\
&= \int_{0}^{\infty} \int_{0}^{\infty} M^\lambda(t; t, y) h(t) \frac{dt}{t} f(y) dy \\
&= \int_{0}^{\infty} G^\lambda_\delta(f)(t, x) h(t) \frac{dt}{t}, \quad h \in H \text{ and } x \notin \text{supp } f.
\end{align*}

The interchange of the order of integration is justified because

\begin{equation}
\int_{0}^{\infty} \int_{0}^{\infty} |M^\lambda(t; t, y)| h(t) \frac{dt}{t} \|f(y)\|_{\mathbb{B}} dy \leq C \|h\|_{H} \int_{0}^{\infty} \frac{\|f(y)\|_{\mathbb{B}} dy}{|x - y|} < \infty, \quad h \in H \text{ and } x \notin \text{supp } f.
\end{equation}

Hence, we deduce that

\begin{equation}
Q^\lambda(f)(x) = G^\lambda_\delta(f)(\cdot, x), \quad x \notin \text{supp } f,
\end{equation}

as elements of $\gamma(H, \mathbb{B})$. □
According to [6, Theorem 1.3], by using vector-valued Calderón-Zygmund theory (33) we infer that the operator defined by (58) can be extended from \( S(0, \infty) \otimes \mathbb{B} \) to \( L^1(0, \infty, \mathbb{B}) \) as a bounded operator from \( L^1((0, \infty), \mathbb{B}) \) into \( L^1((0, \infty), \gamma(H, \mathbb{B})) \) and from \( H^1((0, \infty), \mathbb{B}) \) into \( L^1((0, \infty), \gamma(H, \mathbb{B})) \). By proceeding as in the proof of Lemma 2.1 we can conclude that the operator \( G_a \) is bounded from \( L^1((0, \infty), \mathbb{B}) \) into \( L^1((0, \infty), \gamma(H, \mathbb{B})) \) and from \( H^1((0, \infty), \mathbb{B}) \) into \( L^1((0, \infty), \gamma(H, \mathbb{B})) \).

By (48) we have that
\[
|M(t;x,y)| \leq C[I_1(t;x,y) + I_2(t;x,y)], \quad t, x, y \in (0, \infty),
\]
where
\[
I_1(t;x,y) = t^2(xy)^{1/2 + 1} \int_0^\pi \frac{(\sin \theta)^{1/3}}{(y + t + \sqrt{2xy(1 - \cos \theta)})^{1/2 + \delta}} d\theta, \quad t, x, y \in (0, \infty),
\]
\[
I_2(t;x,y) = t^2 y^{1/2 + 1} x^{1/3} \int_0^\pi \frac{(\sin \theta)^{1/3}}{(y + t + \sqrt{2xy(1 - \cos \theta)})^{1/2 + \delta}} d\theta, \quad t, x, y \in (0, \infty),
\]
and \( M(t;x,y), t, x, y \in (0, \infty), \) is the kernel introduced in (17).

We obtain (see the proof of (7) with \( \lambda + 1 \) instead of \( \lambda \) and \( k = 2 \))
\[
|I_1(t;x,y)| \leq C \frac{t^2(xy)^{1/2 + 1}}{(y + t + \sqrt{xy})^{1/2 + \delta}}, \quad t, x, y \in (0, \infty),
\]
and
\[
|I_2(t;x,y)| \leq C \frac{t^2 y^{1/2 + 1} x^{1/3}}{(y + t + \sqrt{xy})^{1/2 + \delta}}, \quad t, x, y \in (0, \infty),
\]
Then,
\[
||I_1(:x,y)||_H \leq C \frac{(xy)^{1/3}}{(y + t + \sqrt{xy})^{1/2 + \delta}}, \quad x, y \in (0, \infty), \quad x \neq y,
\]
and
\[
||I_2(:x,y)||_H \leq C \frac{y^{1/2 + 1} x^{1/3}}{(y + t + \sqrt{xy})^{1/2 + \delta}}, \quad x, y \in (0, \infty), \quad x \neq y.
\]
Let \( \delta > 0 \) and \( b \in \mathbb{B}, \|b\|_\mathbb{B} = 1 \). We define \( a = \frac{b}{2} \chi_{(0, b)} \). Since \( G_a^1 \) is bounded from \( L^2((0, \infty), \mathbb{B}) \) into \( L^2((0, \infty), \gamma(H, \mathbb{B})) \) ([3, Theorem 1.3]), we deduce that
\[
\int_0^{2\delta} ||G_a^1(a)(x)||_{\gamma(H, \mathbb{B})} dx \leq C\delta^{1/2} ||G_a^1(a)||_{L^2((0, \infty), \gamma(H, \mathbb{B}))} \leq C\delta^{1/2} ||a||_{L^2((0, \infty), \mathbb{B})} \leq C,
\]
where \( C > 0 \) does not depend on \( \delta \) or \( b \). Also, by (61) and (62), we get
\[
\int_{2\delta}^{\infty} ||G_a^1(a)(x)||_{\gamma(H, \mathbb{B})} dx \leq \frac{C}{\delta} \int_{2\delta}^{\infty} \int_0^{2\delta} \left( \frac{y^{1/2 + 1} x^{1/3}}{(y + t + \sqrt{xy})^{1/2 + \delta}} + \frac{(xy)^{1/3}}{(y + t + \sqrt{xy})^{1/2 + \delta}} \right) dy dx \leq \frac{C}{\delta} \int_{2\delta}^{\infty} \frac{dx}{x^{1/2 + \delta}} \int_0^{2\delta} \frac{y^{1/2 + 1} dy}{(y + t + \sqrt{xy})^{1/2 + \delta}} \leq C,
\]
where \( C > 0 \) does not depend on \( \delta \) or \( b \). We conclude that \( ||G_a^1(a)||_{L^1((0, \infty), \gamma(H, \mathbb{B}))} \leq C \), where \( C > 0 \) does not depend on \( \delta \) or \( b \).

Also, since \( G_a^1 \) is bounded from \( H^1((0, \infty), \mathbb{B}) \) into \( L^1((0, \infty), \gamma(H, \mathbb{B})) \), there exists \( C > 0 \) such that, for every 2-atom \( a \) satisfying (Aii),
\[
||G_a^1(a)||_{L^1((0, \infty), \gamma(H, \mathbb{B}))} \leq C.
\]
Then, by taking into account that \( G_a^1 \) is bounded from \( L^1((0, \infty), \mathbb{B}) \) into \( L^1((0, \infty), \gamma(H, \mathbb{B})) \), we conclude that \( G_a^1 \) is bounded from \( H^1((0, \infty), \mathbb{B}) \) into \( L^1((0, \infty), \gamma(H, \mathbb{B})) \).
Since the maximal operator $P^{\lambda+1}$ is bounded from $H^1_{\lambda}((0, \infty), \mathbb{B})$ into $L^1((0, \infty), \gamma(H, \mathbb{B}))$ it is enough to show that $P^{\lambda+1} \circ G^\lambda_B$ is bounded from $H^1_{\lambda}((0, \infty), \mathbb{B})$ into $L^1((0, \infty), \mathbb{B})$, where

$$P^{\lambda+1}(f) = \sup_{s > 0} \|P^{\lambda+1}(f)\|_{\gamma(H, \mathbb{B})}, \quad f \in L^p((0, \infty), \gamma(H, \mathbb{B})), \quad 1 \leq p < \infty.$$ 

Since the maximal operator $P^{\lambda+1}$ is bounded from $L^1((0, \infty), \gamma(H, \mathbb{B}))$ into $L^{1, \infty}(0, \infty)$, the operator $P^{\lambda+1} \circ G^\lambda_B$ is bounded from $H^1_{\lambda}((0, \infty), \mathbb{B})$ into $L^{1, \infty}(0, \infty)$. By using vector-valued Calderón-Zygmund theory as in the proof of Lemma 2.3 we can see that $P^{\lambda+1} \circ G^\lambda_B$ is bounded from $H^1((0, \infty), \mathbb{B})$ into $L^1(0, \infty)$.

Moreover, as above, we can prove that there exists $C > 0$ such that, for every $\delta > 0$ and $b \in \mathbb{B}$, $\|b\|_{\mathbb{B}} = 1$,

$$\|P^{\lambda+1}(G^\lambda_B(a))\|_{L^1(0, \infty)} \leq C,$$

being $a = \frac{b}{\delta} \chi((0, \delta))$. Then, we conclude that $P^{\lambda+1} \circ G^\lambda_B$ is bounded from $H^1_{\lambda}((0, \infty), \mathbb{B})$ into $L^1(0, \infty)$.

Thus the proof of Theorem 3.2 related to Hardy spaces is complete.

3.2. Our objective is to show that $G^\lambda_B$ is bounded from $BMO_o((0, \infty), \mathbb{B})$ into $BMO_o((0, \infty), \gamma(H, \mathbb{B}))$. As in Section 2.2 this is naturally divided into the following two lemmas.

**Lemma 3.2.** Assume that $\mathbb{B}$ is a UMD Banach space and $\lambda > 0$. We can find $C > 0$ such that, for every $r > 0$,

$$\frac{1}{r} \int_0^r \|G^\lambda_B(f)(s)\|_{\gamma(H, \mathbb{B})} ds \leq C \|f\|_{BMO_o((0, \infty), \mathbb{B})}, \quad f \in BMO_o((0, \infty), \mathbb{B}).$$

**Proof.** Let $f \in BMO_o((0, \infty), \mathbb{B})$. According to (48), (59) and (60) we get

$$|D^*_\lambda P^{\lambda+1}(x, y)| \leq C t \left(\frac{(xy)^{\lambda+1}}{(|x - y| + t)^{2\lambda+5}} + \frac{y^{\lambda+1}x^\lambda}{(|x - y| + t)^{2\lambda+4}}\right), \quad t, x, y \in (0, \infty).$$

Since $\int_0^\infty (1 + y^2)^{-1} \|f(y)\|_{\mathbb{B}} dy < \infty$, we can write

$$G^\lambda_B(f)(t, x) = \int_0^\infty tD^*_\lambda P^{\lambda+1}(x, y) f(y) dy, \quad t, x \in (0, \infty).$$

Let $r > 0$. Since $G^\lambda_B$ is bounded from $L^2((0, \infty), \mathbb{B})$ into $L^2((0, \infty), \gamma(H, \mathbb{B}))$ (Theorem 1.3) we get

$$\frac{1}{r} \int_0^r \|G^\lambda_B(f(x, 2r))(\cdot, x)\|_{\gamma(H, \mathbb{B})} dx \leq \frac{1}{r} \int_0^\infty \|G^\lambda_B(f(x, 2r))\|_{L^2((0, \infty), \gamma(H, \mathbb{B}))} dx \leq C \|f\|_{BMO_o((0, \infty), \mathbb{B})}.$$ 

Moreover, by using (61) and (62), as in (31) we obtain

$$\frac{1}{r} \int_0^r \|G^\lambda_B(f(x, 2r))(\cdot, x)\|_{\gamma(H, \mathbb{B})} dx \leq \frac{1}{r} \int_0^r \int_2^\infty \|M^\lambda(\cdot ; x, y)\|_H \|f(y)\|_{\mathbb{B}} dy dx \leq \frac{1}{r} \int_0^r \int_2^\infty \left(\frac{(xy)^{\lambda+1}}{(|x - y| + t)^{2\lambda+3}} + \frac{y^{\lambda+1}x^\lambda}{(|x - y| + t)^{2\lambda+2}}\right) \|f(y)\|_{\mathbb{B}} dy dx \leq \frac{1}{r} \int_0^r \int_2^\infty \frac{x^\lambda}{y^{\lambda+1}} \|f(y)\|_{\mathbb{B}} dy dx \leq C \|f\|_{BMO_o((0, \infty), \mathbb{B})}.$$ 

Hence, (63) holds with $C$ independent of $f$ and $r$, and then, $G^\lambda_B(f)(\cdot, x) \in \gamma(H, \mathbb{B})$, a.e. $x \in (0, \infty)$. 

\[ \square \]
Lemma 3.3. Let $\mathbb{B}$ be a UMD Banach space and $\lambda > 0$. $G^\lambda_\mathbb{B}$ is a bounded operator from $BMO_\mathbb{B}((0, \infty), \mathbb{B})$ into $BMO((0, \infty), \gamma(H, \mathbb{B}))$.

Proof. Let $f \in BMO_\mathbb{B}((0, \infty), \mathbb{B})$. We consider the even extension $f_e$ of the function $f$ to $\mathbb{R}$.

We define

$$G_\mathbb{B}(f_e)(t, x) = -\int_\mathbb{R} t\partial_x P_t(x - y)f_e(y)dy, \quad t \in (0, \infty) \text{ and } x \in \mathbb{R},$$

where $P_t(z) = \frac{1}{\pi} \frac{t}{t^2 + z^2}$, $t \in (0, \infty)$ and $z \in \mathbb{R}$.

We have that

$$G_\mathbb{B}(f_e)(t, x) = -\int_0^{\infty} t\partial_x [P_{t}(x - y) + P_{t}(x + y)]f(y)dy, \quad t \in (0, \infty) \text{ and } x \in \mathbb{R}.$$  

By using mean value theorem we obtain

$$|\partial_x [P_{t}(x - y) + P_{t}(x + y)]| = \frac{2t}{\pi} \left| \frac{y - x}{(t^2 + (x - y)^2)^2} - \frac{x + y}{(t^2 + (x + y)^2)^2} \right| \leq C \frac{tx}{(t + |x - y|)^2}, \quad t, x, y \in (0, \infty).$$  

We split $G_\mathbb{B}(f_e)$ as follows:

$$G_\mathbb{B}(f_e)(t, x) = -\int_0^{x/2} t\partial_x [P_{t}(x - y) + P_{t}(x + y)]f(y)dy \quad \text{with} \quad x > 0$$

$$= -\int_0^{x/2} t\partial_x P_{t}(x - y)f(y)dy - \int_{x/2}^{x} t\partial_x P_{t}(x + y)f(y)dy - \int_0^{x/2} t\partial_x P_{t}(x + y)f(y)dy + \int_0^{x/2} t\partial_x P_{t}(x - y)f(y)dy$$

(65)

We also have that

$$\|I_3(f)\|_{\gamma(H, \mathbb{B})} \leq C \int_{x/2}^{x} \|\partial_x P_{t}(x + y)\|_{H} \|f(y)\|_{\mathbb{B}}dy \leq C \int_{x/2}^{x} \frac{t^3}{(t + x + y)^6} dt \left( \int_{x/2}^{x} \frac{t^3}{(t + x + y)^6} dt \right)^{1/2} \|f\|_{BMO_\mathbb{B}((0, \infty), \mathbb{B})}, \quad x \in (0, \infty),$$

and, as in \[33\],

$$\|I_4(f)\|_{\gamma(H, \mathbb{B})} \leq C \int_{x/2}^{x} \frac{t^3}{(t + y)^6} dy \leq C \|f\|_{BMO_\mathbb{B}((0, \infty), \mathbb{B})}, \quad x \in (0, \infty).$$

We also have that

$$\|I_5(f)\|_{\gamma(H, \mathbb{B})} \leq C \int_{x/2}^{x} \frac{tx}{(t + x + y)^2} \|f(y)\|_{\mathbb{B}}dy \leq C \int_{x/2}^{x} \frac{t^3}{(t + x + y)^6} dt \left( \int_{x/2}^{x} \frac{t^3}{(t + x + y)^6} dt \right)^{1/2} \|f\|_{BMO_\mathbb{B}((0, \infty), \mathbb{B})}, \quad x \in (0, \infty).$$
We are going to show that $G_\mathcal{B}(f_\mathcal{B}) \in BMO((0, \infty), \gamma(H, \mathbb{B})).$ In order to do this we proceed as in Section 2.2. Let $0 < r < s < \infty$. We define $I = (r, s)$ and $2I = (x_I - 2d_I, x_I + 2d_I)$, where $x_I = (r + s)/2$ and $d_I = (s - r)/2$, and we decompose $f_\mathcal{B}$ as follows:

$$f_\mathcal{B} = (f_\mathcal{B} - f_I)\chi_I + (f_\mathcal{B} - f_I)\chi_{(0, \infty)}\parallel I + f_I = f_1 + f_2 + f_3.$$ 

Since $G_\mathcal{B}$ is a bounded operator from $L^2(\mathbb{R}, \mathbb{B})$ into $L^2(\mathbb{R}, \gamma(H, \mathbb{B}))$ ([23, Theorem 4.2]) we deduce that

$$\frac{1}{|I|} \int_I \|G_\mathcal{B}(f_1)(\cdot, x)\|_{\gamma(H, \mathbb{B})} dx \leq C\|f\|_{BMO,((0, \infty), \mathbb{B})}$$

and then, $G_\mathcal{B}(f_1)(\cdot, x) \in \gamma(H, \mathbb{B})$, a.e. $x \in I$. Moreover, since $\int_\mathbb{R} P_t(x - y)dy = 1$, $t \in (0, \infty)$ and $x \in \mathbb{R}$, $G_\mathcal{B}(f_3) = 0$. Finally, we study $G_\mathcal{B}(f_2)$. We have that

$$G_\mathcal{B}^3(f)(t, x) - G_\mathcal{B}(f_\mathcal{B})(t, x) = \sum_{j=1}^3 J_3^j(f)(t, x) - \sum_{j=1}^3 I_j(f)(t, x), \quad t, x \in (0, \infty),$$

where

$$J_3^1(f)(t, x) = \int_0^{x/2} M^\lambda(t; x, y)f(y)dy, \quad t, x \in (0, \infty),$$

$$J_3^2(f)(t, x) = \int_{2x}^{\infty} M^\lambda(t; x, y)f(y)dy, \quad t, x \in (0, \infty),$$

and

$$J_3^3(f)(t, x) = \int_{x/2}^{2x} \left[M^\lambda(t; x, y) + t\partial_x P_t(x - y)\right]f(y)dy, \quad t, x \in (0, \infty).$$

Here, $I_j$, $j = 1, 2, 3$, are defined as in [65] and $M^\lambda$ is the kernel in [47].

According to [61] and [62] we get

$$\|M^\lambda(\cdot; x, y)\|_{H} \leq C \left(\frac{(xy)^{\lambda+1}}{|x-y|^{2\lambda+3}} + \frac{y^{\lambda+1}x^\lambda}{|x-y|^{2\lambda+2}}\right), \quad x, y \in (0, \infty), \quad x \neq y.$$ 

It follows that, for every $x \in (0, \infty)$

$$\|J_3^1(f)(\cdot, x)\|_{\gamma(H, \mathbb{B})} \leq \int_0^{x/2} \|M^\lambda(\cdot; x, y)\|_{H} \|f(y)\|_{\mathbb{B}} dy \leq C_x \int_0^{x/2} \|f(y)\|_{\mathbb{B}} dy \leq C\|f\|_{BMO,((0, \infty), \mathbb{B})},$$

and, as in [31],

$$\|J_3^2(f)(\cdot, x)\|_{\gamma(H, \mathbb{B})} \leq C_x \lambda \int_{2x}^{\infty} \frac{\|f(y)\|_{\mathbb{B}}}{y^{\lambda+1}} dy \leq C\|f\|_{BMO,((0, \infty), \mathbb{B})}, \quad x \in (0, \infty).$$

We write

$$M^\lambda(t; x, y) = -\frac{2(\lambda + 1)}{\pi} t^2 y^{\lambda+1} \left[2(\lambda + 1)x^\lambda \int_0^{\pi} \frac{(\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+2}} d\theta \right. \right.$$

$$- 2(\lambda + 2)x^{\lambda+1} \int_0^{\pi} \frac{(\sin \theta)^{2\lambda+1}((x-y) + y(1-\cos \theta))}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+3}} d\theta \left. \right]$$

$$= M_1^\lambda(t; x, y) + M_2^\lambda(t; x, y), \quad t, x, y \in (0, \infty),$$

where

$$M_1^\lambda(t; x, y) = -\frac{2(\lambda + 1)}{\pi} t^2 y^{\lambda+1} \left[2(\lambda + 1)x^\lambda \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+2}} d\theta \right. \right.$$

$$- 2(\lambda + 2)x^{\lambda+1} \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda+1}((x-y) + y(1-\cos \theta))}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+3}} d\theta \left. \right], \quad t, x, y \in (0, \infty).$$
We have that
\[
\|M_2^\lambda(\cdot, x, y)\|_H \leq C \left[ x^{2\lambda+1} \left( \int_0^\infty \frac{t^3}{(|x-y| + t + \sqrt{x^2 + y^2})^{4(\lambda+2)}} dt \right)^{1/2} \right] + x^{2\lambda+2} \left( \int_0^\infty \frac{t^3}{(|x-y| + t + \sqrt{x^2 + y^2})^{4(\lambda+5/2)}} dt \right)^{1/2} \leq \frac{C}{x}, \quad 0 < \frac{x}{2} < y < 2x. \tag{67}
\]

We now write
\[
M_1^\lambda(t; x, y) = -2(\lambda + 1)\pi t^2 y^{\lambda+1} \left[ (2\lambda + 1)x^\lambda \int_0^{\pi/2} \left( \frac{(\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+2}} \right) d\theta 
- 2(\lambda + 2)x^{\lambda+1} \int_0^{\pi/2} \left( \frac{(x-y)\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+3}} \right) d\theta 
- 2(\lambda + 2)x^{\lambda+1} \int_0^{\pi/2} \left( \frac{(x-y)(\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+3}} \right) d\theta \right] = 3 \sum_{j=1}^3 M_{1,j}^\lambda(t; x, y), \quad t, x, y \in (0, \infty),
\]

We obtain that
\[
|M_{1,2}^\lambda(t; x, y)| \leq C |M_{1,1}^\lambda(t; x, y)| \leq C t^2 x^\lambda y^{\lambda+1} \int_0^{\pi/2} (\sin \theta)^{2\lambda+1} \left( \frac{(x-y)^2 + t^2}{(x-y)^2 + t^2 + 2xy(1-\cos \theta)} \right)^{\lambda+4} d\theta, \quad t, x, y \in (0, \infty).
\]

Then,
\[
\|M_{1,2}^\lambda(\cdot; x, y)\|_H \leq C \|M_{1,1}^\lambda(t; x, y)\|_H \leq C x^\lambda y^{\lambda+1} \int_0^{\pi/2} (\sin \theta)^{2\lambda+1} \left( \int_0^\infty \frac{t^3}{(|x-y| + t + \sqrt{x^2 + y^2})^{4\lambda+8}} dt \right)^{1/2} d\theta 
\leq C x^\lambda y^{\lambda+1} \int_0^{\pi/2} (\sin \theta)^{2\lambda+1} \left( \frac{(x-y)^2 + t^2}{(x-y)^2 + t^2 + 2xy(1-\cos \theta)} \right)^{\lambda+2} d\theta 
\leq C \sqrt{\frac{y}{x}} \int_0^{\pi/2} \frac{d\theta}{x-y + \sqrt{x^2 + y^2}} \leq C \frac{\log \left( 1 + \frac{\sqrt{x^2 + y^2}}{|x-y|} \right)}{x}, \quad x, y \in (0, \infty).
\tag{68}
\]

We put
\[
J_3^\lambda(f) = M_{1,1}^\lambda(f) + M_{1,2}^\lambda(f) + M_2^\lambda(f) + H^\lambda(f),
\]

being
\[
M_{1,j}^\lambda(t; x, y) = \int_{x/2}^{2x} M_{1,j}^\lambda(t; x, y) f(y) dy, \quad t, x \in (0, \infty), \quad j = 1, 2,
\]
\[
M_2^\lambda(f)(t; x, y) = \int_{x/2}^{2x} M_2^\lambda(t; x, y) f(y) dy, \quad t, x \in (0, \infty),
\]

and
\[
H^\lambda(f)(t; x, y) = \int_{x/2}^{2x} [M_{1,3}^\lambda(t; x, y) + \theta \partial_y P_t(x-y)] f(y) dy, \quad t, x \in (0, \infty).
\]

According to (67), we obtain that
\[
\|M_2^\lambda(f)(x)\|_{\tau(H, B)} \leq C \int_{x/2}^{2x} \|M_2^\lambda(\cdot; x, y)\|_H \|f(y)\|_\mathbb{B} dy 
\leq \frac{C}{x} \int_{x/2}^{2x} \|f(y)\|_\mathbb{B} dy \leq C \|f\|_{\text{BMO}_x((0, \infty), B)}, \quad x \in (0, \infty).
\]
Also, by taking into account \[68\] we can write

\[
\|M^j_{1,3}(f)(x,y)\|_{(H,B)} \leq \frac{C}{x} \int_{x/2}^{2x} \log \left(1 + \frac{\sqrt{xy}}{|x-y|}\right) \|f(y)\|_B \, dy
\]

\[
\leq C \left( \frac{1}{x} \int_{x/2}^{2x} \left(\log \left(1 + \frac{\sqrt{xy}}{|x-y|}\right)\right)^2 \, dy \right)^{1/2} \left( \frac{1}{x} \int_{x/2}^{2x} \|f(y)\|_B^2 \, dy \right)^{1/2}
\]

\[
\leq C \left( \frac{1}{\lambda} \int_{1/2}^2 \left(\log \left(1 + \frac{\sqrt{u}}{|1-u|}\right)\right)^2 \, du \right)^{1/2} \|f\|_{BMO_{1,(0,\infty)}}
\]

\[
\leq C \|f\|_{BMO_{1,(0,\infty),B}}, \quad x \in (0,\infty) \quad \text{and} \quad j = 1, 2.
\]

We now consider

\[
H^j(t;x,y) = M^j_{1,3}(t;x,y) + t\partial_x P_t(x-y), \quad t, x, y \in (0,\infty).
\]

We have that

\[
H^j(t;x,y) = -\frac{2t^2}{\pi} \frac{x-y}{(t^2 + (x-y)^2)^2} + \frac{4(\lambda + 1)(\lambda + 2)}{\pi} t^2 (xy)^{\lambda+1} \int_0^{\pi/2} \frac{(x-y)(\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+3}} \, d\theta
\]

\[
+ 2(\lambda + 1)(\lambda + 2)(xy)^{\lambda+1} \int_0^{\pi/2} \theta^{2\lambda+1} \left(\frac{1}{((x-y)^2 + t^2 + xy\theta^2)^{\lambda+3}} - \frac{1}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+3}}\right) \, d\theta
\]

\[
+ 2(\lambda + 1)(\lambda + 2)(xy)^{\lambda+1} \int_0^{\pi/2} \frac{\theta^{2\lambda+1} - (\sin \theta)^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+3}} \, d\theta
\]

\[
= \sum_{j=1}^3 H^j(t;x,y), \quad t, x, y \in (0,\infty).
\]

By using mean value theorem we get

\[
|H^j(t;x,y)| \leq C t^2 |x-y|(xy)^{\lambda+1} \int_0^{\pi/2} \frac{\theta^{2\lambda+1} x y |1 - \cos \theta - \theta^2/2|}{((x-y)^2 + t^2 + xy\theta^2)^{\lambda+4}} \, d\theta
\]

\[
\leq C t^2 |x-y|(xy)^{\lambda+2} \int_0^{\pi/2} \frac{\theta^{2\lambda+5}}{((x-y)^2 + t + \sqrt{xy}\theta)^{2\lambda+8}} \, d\theta
\]

\[
\leq C t^2 |x-y|(xy)^{\lambda+1} \int_0^{\pi/2} \frac{\theta^{2\lambda+3}}{((x-y)^2 + t + \sqrt{xy}\theta)^{2\lambda+6}} \, d\theta, \quad t, x, y \in (0,\infty),
\]

and

\[
|H^j(t;x,y)| \leq C t^2 |x-y|(xy)^{\lambda+1} \int_0^{\pi/2} \frac{\theta^{2\lambda+3}}{((x-y)^2 + t + \sqrt{xy}\theta)^{2\lambda+6}} \, d\theta, \quad t, x, y \in (0,\infty).
\]

Then

\[
\|H^j(t;x,y)\|_H \leq C (xy)^{\lambda+1} \int_0^{\infty} \frac{\theta^{2\lambda+3}}{((x-y)^2 + t + \sqrt{xy}\theta)^{2\lambda+10}} \, dt
\]

\[
\leq C (xy)^{\lambda+1} \int_0^{\pi/2} \frac{\theta^{2\lambda+3}}{((x-y)^2 + t + \sqrt{xy}\theta)^{2\lambda+3}} \, d\theta
\]

\[
\leq C \frac{(xy)^{\lambda+1}}{(\sqrt{xy})^{2\lambda+3}} \leq \frac{C}{x}, \quad 0 < x < 2, \quad j = 2, 3.
\]
On the other hand, after straightforward change of variables we get
\[
H_x^λ(t; x, y) = -\frac{2t^2}{\pi} (x - y) \left[ \frac{1}{(t^2 + (x - y)^2)^{\lambda+1}} - 2(\lambda + 1)(\lambda + 2)(xy)^{\lambda+1} \right.
\times \left( \int_0^\infty - \int_{x/2}^\infty \frac{1}{(x - y)^2} + t^2 + xy\theta^2)^{\lambda+3} \right) d\theta \]
\[
= -\frac{2t^2}{\pi} (x - y) \left[ 1 - 2(\lambda + 1)(\lambda + 2) \int_0^\infty \frac{\theta^{2\lambda+1}}{(1 + \theta^2)^{\lambda+3}} d\theta \right.
\]
\[
+ 2(\lambda + 1)(\lambda + 2)(xy)^{\lambda+1} \int_{\pi/2}^\infty \frac{\theta^{2\lambda+1}}{(x - y)^2 + t^2 + xy\theta^2)^{\lambda+3}} d\theta \]
\[
= -\frac{4(\lambda + 1)(\lambda + 2)}{\pi} t^2(x - y)(xy)^{\lambda+1} \int_{\pi/2}^\infty \frac{\theta^{2\lambda+1}}{(x - y)^2 + t^2 + xy\theta^2)^{\lambda+3}} d\theta, \quad t, x, y \in (0, \infty).
\]

It follows that
\[
\|H_x^λ(t; x, y)\|_H \leq C(xy)^{\lambda+1} \int_{\pi/2}^\infty \theta^{2\lambda+1} \left( \int_0^\infty \frac{t^3}{(|x - y| + t + \sqrt{x\theta^2})^{\lambda+3}} dt \right)^{1/2} d\theta
\]
\[
\leq C(xy)^{\lambda+1} \int_{\pi/2}^\infty \frac{\theta^{2\lambda+1}}{|x - y| + \sqrt{x\theta^2})^{\lambda+3}} d\theta
\]
\[
\leq C \frac{(xy)^{\lambda+1}}{\sqrt{xy}} \int_{\pi/2}^\infty \frac{d\theta}{\theta^{\lambda+3}} \leq C \frac{1}{x}, \quad 0 < \frac{x}{2} < y < 2x < \infty.
\]

We conclude that
\[
\|H^λ(\cdot; x, y)\|_H \leq C \frac{1}{x}, \quad 0 < \frac{x}{2} < y < 2x < \infty,
\]

and hence, for each \( x \in (0, \infty) \),
\[
\|H^λ(f)\|_{γ(H, B)} \leq C \int_{\pi/2}^2 \|H^λ(\cdot; x, y)\|_H \|f(y)\|_{BMO} dy \leq C \frac{1}{x} \int_{0}^2 \|f(y)\|_{\text{BMO}} dy \leq C\|f\|_{B^{\text{MO}}_0((0, \infty), B)}.
\]

By putting together the above estimates we obtain that
\[
\|G_B(f_2)(\cdot, x) - G_B^λ(f)(\cdot, x)\|_{γ(H, B)} \leq C\|f\|_{B^{\text{MO}}_0((0, \infty), B)}, \quad x \in (0, \infty).
\]

Since \( G_B(f_1)(\cdot, x) \in γ(H, B) \), a.e. \( x \in (0, \infty) \), \( G_B^λ(f)(\cdot, x) \in γ(H, B) \), a.e. \( x \in (0, \infty) \), and \( G_B(f_3)(t, x) = 0, t, x \in (0, \infty) \), we deduce that \( G_B(f_2)(\cdot, x) \in γ(H, B) \), a.e. \( x \in (0, \infty) \).

On the other hand,
\[
\|∂_{x} P(t, y - x)\|_H \leq C \left( \int_0^\infty \frac{t^3}{(t + |x - y|)^3} dt \right)^{1/2} \leq C \frac{1}{|x - y|^2}, \quad x, y \in (0, \infty), \ x \neq y.
\]

We have established all the properties needed in order to show, by using the arguments developed in Section 2.2, that
\[
\frac{1}{|I|} \int_I \|G_B(f_3)(\cdot, x) - G_B(f_2)(\cdot, x)\|_{BMO} dx \leq C\|f\|_{B^{\text{MO}}_0((0, \infty), B)},
\]
where \( x_0 \in I \) is chosen such that \( G_B(f_2)(\cdot, x_0) \in γ(H, B) \) and the constant \( C > 0 \) does not depend on \( I \).

From [66], [59] and [70] we deduce that there exists \( C > 0 \) such that, for every interval \( I \subset (0, \infty) \), we can find \( α_1 \in γ(H, B) \) such that
\[
\frac{1}{|I|} \int_I \|G_B^λ(f)(\cdot, x) - α_1\|_{γ(H, B)} dx \leq C\|f\|_{B^{\text{MO}}_0((0, \infty), B)}.
\]
4. Proof of Theorem 4.3

Properties (i) $\implies$ (ii) and (ii) $\implies$ (iii) have been established in Theorem 4.1 and Theorem 4.2.

4.1. We now prove that (ii) $\implies$ (i). In order to do this we use Riesz transforms in the Bessel setting. The Riesz transform $R_\lambda$ is defined by

$$R_\lambda(f)(x) = \lim_{\varepsilon \to 0^+} \int_{0, |x-\eta| > \varepsilon} R_\lambda(x, \eta) f(\eta) d\eta, \text{ a.e. } x \in (0, \infty),$$

for every $f \in L^p(0, \infty), 1 \leq p < \infty$. Here the kernel $R_\lambda$ is given by

$$R_\lambda(x, \eta) = \int_0^\infty D_{\lambda, \eta} P_\lambda(x, \eta) dt, \quad x, \eta \in (0, \infty), \ x \neq \eta.$$

$R_\lambda$ is a Calderón-Zygmund operator that is bounded from $L^p(0, \infty)$ into itself and from $L^1(0, \infty)$ into $L^{1, \infty}(0, \infty)$ ([3, Theorem 4.2]).

We define the Riesz transform on $L^p(0, \infty) \otimes \mathbb{B}, 1 \leq p < \infty$, in the natural way, and the new operator is also denoted by $R_\lambda$. In [33, Theorem 2.1] it was proved that the Banach space $\mathbb{B}$ is UMD if, and only if, $R_\lambda$ can be extended from $L^p(0, \infty) \otimes \mathbb{B}$ to $L^p((0, \infty), \mathbb{B})$ as a bounded operator from $L^p((0, \infty), \mathbb{B})$ into itself, for some (equivalently, for any) $1 < p < \infty$.

We consider the operator $R_\lambda^*$ defined by

$$R_\lambda^*(f)(x) = \lim_{\varepsilon \to 0^+} \int_{0, |x-\eta| > \varepsilon} R_\lambda(\eta, x) f(\eta) d\eta, \text{ a.e. } x \in (0, \infty),$$

for every $f \in L^p(0, \infty), 1 \leq p < \infty$. This operator has the same $L^p$-boundedness properties of $R_\lambda$. Also, the Banach space $\mathbb{B}$ is UMD if and only if $R_\lambda^*$ can be extended from $L^p(0, \infty) \otimes \mathbb{B}$ to $L^p((0, \infty), \mathbb{B})$ as a bounded operator from $L^p((0, \infty), \mathbb{B})$ into itself, for some (equivalently, for any) $1 < p < \infty$.

In [33 (16.6)] Cauchy-Riemann type equations in the Bessel setting were given. Motivated by these equations and using Hankel transform we can see that

$$\partial_t P_\lambda^* (R_\lambda^*(f)) = D_\lambda^* P_\lambda^{*+1}(f), \quad f \in S_\lambda(0, \infty).$$

In other words, we have that

$$G_\lambda^{*+1}(R_\lambda^*(f)) = G_\lambda^{1}(f), \quad f \in S_\lambda(0, \infty).$$

Since $S_\lambda(0, \infty)$ is a dense subspace of $L^p(0, \infty), 1 < p < \infty$, by using [3] Theorems 1.2 and 1.3 we obtain

$$G_\lambda^{*+1}(R_\lambda^*(f)) = G_\lambda^{1}(f), \quad f \in L^p(0, \infty), 1 < p < \infty. \tag{72}$$

Now, we need to know the behaviour of $R_\lambda^*$ on $H^1_0(\mathbb{R})$ and on $BMO_\alpha(\mathbb{R})$.

**Proposition 4.1.** Let $\lambda > 0$. The Riesz transform $R_\lambda^*$ is a bounded operator from $E(0, \infty)$ into itself, where $E$ denotes $H^1_0$ or $BMO_\alpha$.

**Proof of Proposition 4.1** the case of $E = H^1_0$. Since $R_\lambda^*$ is a Calderón-Zygmund operator, $R_\lambda^*$ is bounded from $H^1(0, \infty)$ into $L^1(0, \infty)$, and there exists $C > 0$ such that, for every $\infty$-atom $a$ satisfying the (Aii) property,

$$\|R_\lambda^*(a)\|_{L^1(0, \infty)} \leq C.$$
According to [2, (3.16)],

\[ |R_\lambda(y, x)| \leq C y^{\lambda+1} x^{-\lambda - 2}, \quad 0 < y < \frac{x}{2} < \infty. \]

Let \( \delta > 0 \). If \( a = \frac{1}{\delta} \lambda(0, \delta) \), being \( R_\lambda^a \) bounded from \( L^2(0, \infty) \) into itself, by [73] we have that

\[
\|R_\lambda^a(a)\|_{L^1(0, \infty)} = \int_0^{2\delta} |R_\lambda^a(a)(x)|dx + \int_{2\delta}^\infty \int_0^\delta |R_\lambda(y, x)| \frac{dy}{\delta} dx \\
\leq \left( \delta \int_0^\infty |R_\lambda^a(a)(x)|^2 dx \right)^{1/2} + C \int_{2\delta}^\infty \frac{1}{x^{\lambda+2}} \int_0^\delta y^{\lambda+1} dy dx \\
\leq C \left( 1 + \left( \delta \int_0^\infty |a(x)|^2 dx \right)^{1/2} \right) \leq C.
\]

Here \( C > 0 \) does not depend on \( \delta \).

Moreover, since \( R_\lambda^a \) is a bounded operator from \( L^1(0, \infty) \) into \( L^{1, \infty}(0, \infty) \) we obtain that \( R_\lambda^a \) is bounded from \( H^1_{\lambda, \infty}(0, \infty) \) into \( L^1(0, \infty) \).

On the other hand, it can be seen that \( R_\lambda(f) = h_{\lambda+1}(h_\lambda f), f \in L^2(0, \infty) \) ([33] §16]). Then,

\[ R_\lambda^a(f) = h_\lambda(h_{\lambda+1}(f)), \quad f \in L^2(0, \infty). \]

Then, by using Hankel transforms we get, for every \( f \in L^2(0, \infty) \),

\[ P_s^\lambda(R_\lambda^a(f)) = h_\lambda(e^{-sy} h_{\lambda+1}(f)) = h_\lambda h_{\lambda+1}(P_s^{\lambda+1}(f)) = R_\lambda^a(P_s^{\lambda+1}(f)), \quad s \in (0, \infty). \]

We consider the operators

\[ \mathcal{H}_s^\lambda(f)(x) = \sup_{s > 0} |P_s^\lambda(R_\lambda^a(f))(x)|, \quad x \in (0, \infty), \]

and, for every \( N \in \mathbb{N} \),

\[ \mathcal{H}_{N, s}^\lambda(f)(x) = \sup_{s \in [1/N, N]} |P_s^\lambda(R_\lambda^a(f))(x)|, \quad x \in (0, \infty), \]

where \( f \in L^p(0, \infty), 1 < p < \infty \). \( \mathcal{H}_s^\lambda \) and \( \mathcal{H}_{N, s}^\lambda, N \in \mathbb{N} \), are bounded operators from \( L^p(0, \infty) \) into itself, for each \( 1 < p < \infty \).

Since \( D_{\lambda, y} P_t^\lambda(x, y) = -D_{\lambda, x} P_t^{\lambda+1}(x, y), t, x, y \in (0, \infty) \), [74] leads to

\[ P_s^\lambda(R_\lambda^a(f))(x) = \int_0^\infty \mathcal{H}^\lambda(s; x, y) f(y) dy, \quad \text{a.e. } x \notin \text{supp } f, \]

for every \( f \in L^2(0, \infty) \), where

\[ \mathcal{H}^\lambda(s; x, y) = -\int_0^\infty D_{\lambda, x}^s P_{t+s}^{\lambda+1}(x, y) dt, \quad s, x, y \in (0, \infty). \]

By [48] and [49] we have that

\[
\int_0^\infty |D_{\lambda, x}^s P_{t+s}^{\lambda+1}(x, y)| dt \leq C \int_0^\infty \frac{t + s}{(t + s + |x - y|)^3} dt \leq C \int_0^\infty \frac{dt}{(t + s + |x - y|)^2} \\
\leq \frac{C}{s + |x - y|}, \quad s, x, y \in (0, \infty). \]

Then

\[ \|\mathcal{H}^\lambda(\cdot; x, y)\|_{L^\infty(0, \infty)} \leq \frac{C}{|x - y|}, \quad x, y \in (0, \infty), x \neq y. \]

On the other hand, from [52], [53] and [54] it follows that

\[ \int_0^\infty |\partial_x D_{\lambda, x}^s P_{t+s}^{\lambda+1}(x, y)| \leq C \int_0^\infty \frac{t + s}{(t + s + |x - y|)^4} dt \leq \frac{C}{(s + |x - y|)^2}, \quad s, x, y \in (0, \infty). \]
Hence,
\begin{equation}
\| \partial_y \mathcal{H}^\lambda(\cdot; x, y) \|_{L^\infty((0, \infty), (0, \infty), x \neq y)} \leq \frac{C}{|x - y|^2}, \quad x, y \in (0, \infty), x \neq y. \tag{77}
\end{equation}
The same arguments lead to
\begin{equation}
\| \partial_y \mathcal{H}^\lambda(\cdot; x, y) \|_{L^\infty((0, \infty), (0, \infty), x \neq y)} \leq \frac{C}{|x - y|^2}, \quad x, y \in (0, \infty), x \neq y. \tag{78}
\end{equation}
Let \( N \in \mathbb{N} \). We consider the operator
\[ \mathcal{H}^N_\lambda(f)(s, x) = P^N_\lambda(R^*_\lambda(f))(x), \quad x \in (0, \infty) \text{ and } s \in \left[ \frac{1}{N}, N \right]. \]
For every \( f \in S_\lambda(0, \infty) \), we have that
\begin{equation}
\mathcal{H}^N_\lambda(f)(\cdot, x) = \int_0^\infty \mathcal{H}_\lambda(\cdot; x, y) f(y) dy, \quad \text{a.e. } x \notin \text{supp } f, \tag{79}
\end{equation}
where the integral is understood in the \( C[1/N, N] \)-Bochner sense, the space of continuous functions in \([1/N, N]\). Indeed, let \( f \in S_\lambda(0, \infty) \). According to \( \mathcal{H}^N_\lambda \), the integral in \( \mathcal{H}^N_\lambda \) is convergent in the \( C[1/N, N] \)-Bochner sense, for almost all \( x \notin \text{supp } f \). Assume that \( \mu \in \mathcal{M}(\mathbb{C}[1/N, N]) \), the space of complex measures supported on \([1/N, N]\). Since \( \mathcal{M}(\mathbb{C}[1/N, N]) = (C[1/N, N])^* \), we can write
\[
\int_0^\infty \left( \int_0^\infty \mathcal{H}^\lambda(\cdot; x, y) f(y) dy \right) d\mu(s) = \int_0^\infty \int_0^\infty \mathcal{H}^\lambda(s, x, y) d\mu(s) f(y) dy = \int_0^\infty \left( \int_0^\infty \mathcal{H}^\lambda(s, x, y) f(y) dy \right) d\mu(s), \quad x \notin \text{supp } f.
\]
Note that the interchange of the order of integration is justified by \( \mathcal{H}^N_\lambda \).

According to the vector-valued Calderón-Zygmund theory, since \( \mathcal{H}^\lambda_\lambda \) is bounded from \( L^2(0, \infty) \) into itself, estimations \( \mathcal{H}^N_\lambda \), \( \mathcal{H}^N_\lambda \) and \( \mathcal{H}^N_\lambda \) imply that the operator \( \mathcal{H}^\lambda_\lambda \) can be extended to \( L^1(0, \infty) \) as a bounded operator from \( L^1(0, \infty) \) into \( L^{1, \infty}((0, \infty), C[1/N, N]) \) and from \( H^1(0, \infty) \) into \( L^1((0, \infty), C[1/N, N]) \). Moreover, if we denote by \( \hat{\mathcal{H}}^\lambda_\lambda \) to this extension we have that
\[
\sup_{N \in \mathbb{N}} \| \hat{\mathcal{H}}^\lambda_\lambda \|_{L^1(0, \infty) \rightarrow L^{1, \infty}((0, \infty), C[1/N, N])} < \infty,
\]
and
\[
\sup_{N \in \mathbb{N}} \| \mathcal{H}^\lambda_\lambda \|_{H^1(0, \infty) \rightarrow L^1((0, \infty), C[1/N, N])} < \infty.
\]
We now show that \( \hat{\mathcal{H}}^\lambda_\lambda(f) = \mathcal{H}^\lambda_\lambda(f), f \in L^1(0, \infty) \). Indeed, let \( f \in L^1(0, \infty) \). We choose a sequence \( (f_n)_{n \in \mathbb{N}} \subset S_\lambda(0, \infty) \) such that \( f_n \to f \) as \( n \to \infty \), in \( L^1(0, \infty) \). Then
\[
\mathcal{H}^\lambda_\lambda(f_n) \to \hat{\mathcal{H}}^\lambda_\lambda(f), \quad n \to \infty,
\]
in \( L^{1, \infty}((0, \infty), C[1/N, N]) \). There exists an increasing sequence \( (n_k)_{k \in \mathbb{N}} \) such that
\[
\| \mathcal{H}^\lambda_\lambda(f_{n_k})(\cdot, x) - \hat{\mathcal{H}}^\lambda_\lambda(f)(x)\|_{C[1/N, N]} \to 0, \quad as \ k \to \infty,
\]
for almost all \( x \in (0, \infty) \). Moreover by \( \mathcal{H}^N_\lambda \),
\[
\mathcal{H}^\lambda_\lambda(f_{n_k})(s, x) \to \mathcal{H}^\lambda_\lambda(f)(s, x), \quad as \ k \to \infty, \quad s, x \in (0, \infty).
\]
Hence, \( \mathcal{H}^\lambda_\lambda(f)(s, x) = [\mathcal{H}^\lambda_\lambda(f)(x)](s), \quad s \in [1/N, N] \) and almost all \( x \in (0, \infty) \).

We conclude that \( \mathcal{H}^\lambda_\lambda \) is bounded from \( L^1(0, \infty) \) into \( L^{1, \infty}((0, \infty), C[1/N, N]) \) and from \( H^1(0, \infty) \) into \( L^1((0, \infty), C[1/N, N]) \). Moreover
\[
\sup_{N \in \mathbb{N}} \| \mathcal{H}^\lambda_\lambda \|_{L^1(0, \infty) \rightarrow L^{1, \infty}((0, \infty), C[1/N, N])} < \infty,
\]
and

$$\sup_{N \in \mathbb{N}} \| \mathcal{H}_N^\lambda \|_{L^1(0,\infty) \rightarrow L^1((0,\infty),C[1/N, N])} < \infty.$$ 

Then, $\mathcal{H}_N^\lambda$ is bounded from $L^1(0,\infty)$ into $L^1(0,\infty)$ and from $H^1(0,\infty)$ into $L^1(0,\infty)$.

In order to see that $\mathcal{H}_N^\lambda$ is bounded from $\mathcal{H}_N^\lambda(0,\infty)$ into $L^1(0,\infty)$ it is enough to show that there exists $C > 0$ such that, for every $\infty$-atom $a$,

$$\| \mathcal{H}_N^\lambda(a) \|_{L^1(0,\infty)} \leq C. \tag{80}$$

Since $\mathcal{H}_N^\lambda$ is bounded from $H^1(0,\infty)$ into $L^1(0,\infty)$, for a certain $C > 0$, (80) holds provided that $a$ is a an $\infty$-atom satisfying (Ari). Also, by using (48), (59) and (60), and by proceeding as in (75) we deduce that

$$\| \mathcal{H}^\lambda(\cdot; x, y) \|_{L^\infty(0,\infty)} \leq C(xy)^{\lambda} \left( \frac{xy}{|x-y|^{2\lambda+3}} + \frac{y}{|x-y|^{2\lambda+2}} \right) \leq C^{\lambda+1} \frac{2^{\lambda+1}}{x^{1/2}} 0 < y < \frac{x}{2} < \infty. \tag{81}$$

Also $\mathcal{H}_N^\lambda$ is bounded from $L^2(0,\infty)$ into itself. Then, for every $a = \frac{1}{\delta} \chi_{(0,\delta)}$, with $\delta > 0$, by (81) we get

$$\| \mathcal{H}_N^\lambda(a) \|_{L^1(0,\infty)} \leq \int_0^{2\delta} |\mathcal{H}_N^\lambda(a)(x)| \, dx + \int_0^\infty \int_0^\delta \| \mathcal{H}^\lambda(\cdot; x, y) \|_{L^\infty(0,\infty)} |a(y)| \, dy \, dx \tag{82}$$

$$\leq C \left( \delta^{1/2} \| a \|_{L^2(0,\infty)} + \frac{1}{\delta} \int_0^{2\delta} \frac{1}{x^{\lambda+2}} \int_0^\delta y^{\lambda+1} \, dy \right) \leq C,$$

where $C > 0$ does not depend on $\delta$.

Hence (80) holds and we obtain that $\mathcal{H}_N^\lambda$ is bounded from $\mathcal{H}_N^\lambda(0,\infty)$ into $L^1(0,\infty)$. We conclude that $R_N^\lambda$ is bounded from $\mathcal{H}_N^\lambda(0,\infty)$ into itself. \(\square\)

**Proof of Proposition 4.1** the case of $E = BMO_0(0,\infty)$. Suppose that $f \in BMO_0(0,\infty)$. Let $r > 0$. Since $R_N^\lambda$ is bounded from $L^2(0,\infty)$ into itself and that (see [2 (3.15)])

$$|R_N^\lambda(y, x)| \leq C \frac{x^{\lambda}}{y^{\lambda+1}}, \quad 0 < 2x < y < \infty, \tag{83}$$

we obtain

$$\frac{1}{r} \int_0^r |R_N^\lambda(f)(x)| \, dx \leq \frac{1}{r} \left( \int_0^r |R_N^\lambda(f\chi_{(0,2r)})| \, dx + \int_0^r |R_N^\lambda(f\chi_{(2r,\infty)})| \, dx \right) \leq C \left[ \frac{1}{r} \int_0^r |R_N^\lambda(f\chi_{(0,2r)})|^2 \, dx \right]^{1/2} + \frac{1}{r} \int_0^r \int_{2r}^\infty |R_N^\lambda(y, x)| |f(y)| \, dy \, dx \leq C \left[ \frac{1}{r} \int_0^{2r} |f(y)|^2 \, dy \right]^{1/2} + r^\lambda \int_{2r}^\infty \frac{|f(y)|}{y^{\lambda+1}} \, dy \leq C \| f \|_{BMO_0(0,\infty)},$$

where $C > 0$ does not depend on $r$.

The next step is to show that $R_N^\lambda(f) \in BMO(0,\infty)$.

We consider the even extension $f_e$ of the function $f$ to $\mathbb{R}$. Then, $f_e \in BMO(\mathbb{R})$. According to [10] p. 294 the Hilbert transform $\mathcal{H}(f_e)$ given by

$$\mathcal{H}(f_e)(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|x-y| > \varepsilon} \left( \frac{1}{x-y} - \frac{\chi_{\mathbb{R} \setminus (-1,1)}}{y} \right) f_e(y) \, dy, \quad \text{a.e. } x \in \mathbb{R},$$

is in $BMO(\mathbb{R})$. 


In [8] (26) and (27) it was seen that

$$\mathcal{H}(f)(x) - \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{x/2, \varepsilon}^{2x} \left( \frac{1}{x - y} + \frac{\chi(1, \infty)(y)}{\lambda} \right) f(y)dy \in L^\infty(0, \infty),$$

and

\begin{equation}
\left\| \mathcal{H}(f)(x) - \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{x/2, \varepsilon}^{2x} \left( \frac{1}{x - y} + \frac{\chi(1, \infty)(y)}{\lambda} \right) f(y)dy \right\|_{L^\infty(0, \infty)} \leq C\|f\|_{BMO_\infty(0, \infty)}.
\end{equation}

We now show that

\begin{equation}
\left\| R^*_f(x) + \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{x/2, \varepsilon}^{2x} \left( \frac{1}{x - y} + \frac{\chi(1, \infty)(y)}{\lambda} \right) f(y)dy \right\|_{L^\infty(0, \infty)} \leq C\|f\|_{BMO_\infty(0, \infty)}.
\end{equation}

By using [8] (5), (73) and (83), as in [8] pp. 319 and 320 we deduce that

$$\left\| R^*_f(x) + \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{x/2, \varepsilon}^{2x} \left( \frac{1}{x - y} + \frac{\chi(1, \infty)(y)}{\lambda} \right) f(y)dy \right\| \leq C\|f\|_{BMO_\infty(0, \infty)}, \quad x \in (0, \infty),$$

where $\log_{+} z = \max\{0, \log z\}, \ z \in (0, \infty)$. Thus, (85) is established. From (84) and (85), since $\tilde{\mathcal{H}}(f_c) \in BMO(\mathbb{R})$, we deduce that $R^*_f(f) \in BMO(0, \infty)$. 

Let $f \in L^\infty_c((0, \infty) \cap \mathbb{B})$. Since $L^\infty_c((0, \infty) \cap \mathbb{B}) \subset H^1_\alpha((0, \infty) \cap \mathbb{B})$, by Proposition 4.1 $R^*_\alpha(f) \in H^1_\alpha((0, \infty) \cap \mathbb{B})$.

Assume that (ii) holds for $H^1_\alpha(\mathbb{R}, \mathbb{B})$. By (72) it follows that

$$\|R^*_f(f)\|_{H^1_\alpha((0, \infty) \cap \mathbb{B})} \leq C\|G_{\alpha, 1}(R^*_f(f))\|_{H^2_\alpha((0, \infty), \gamma, (0, \infty) \cap \mathbb{B})} \leq C\|G_{\alpha, 1}(f)\|_{H^2_\alpha((0, \infty), \gamma, (0, \infty) \cap \mathbb{B})} \leq C\|f\|_{H^2_\alpha((0, \infty), \gamma, (0, \infty) \cap \mathbb{B})}.$$ 

Since $H^1((0, \infty), \mathbb{B}) \subset H^1_\alpha((0, \infty) \cap \mathbb{B}) \subset L^p((0, \infty), \mathbb{B})$, according to [31] Theorem 4.1, $R^*_\alpha$ can be extended to $L^p((0, \infty), \mathbb{B})$ as a bounded operator from $L^p((0, \infty), \mathbb{B})$ into itself. Then, we conclude that $\mathbb{B}$ is UMD.

When it is assumed that (ii) holds for $BMO_\infty((0, \infty), \mathbb{B})$, by proceeding in a similar way, we can prove that $\mathbb{B}$ is UMD.

Thus, the proof of (ii) $\implies$ (i) is complete.

4.2. We prove in this section that (iii) $\implies$ (i). In order to see this, we use that the Banach space $\mathbb{B}$ is UMD if, and only if, for every $\gamma \in \mathbb{R} \setminus \{0\}$, the imaginary power $\Delta^\gamma_\alpha$ can be extended to $L^p((0, \infty), \mathbb{B})$ as a bounded operator from $L^p((0, \infty), \mathbb{B})$ into itself, for some (equivalently, for every) $1 < p < \infty$ (see [3] Proposition 5.1).

Let $\gamma \in \mathbb{R} \setminus \{0\}$. We recall that the imaginary power $\Delta^\gamma_\alpha$ of $\Delta_\alpha$ is defined by

$$\Delta^\gamma_\alpha(f) = h_\gamma(y^{2\gamma}h_\alpha(f)), \quad f \in L^2(0, \infty).$$
Since $h_\lambda$ is an isometry in $L^2(0,\infty)$, $\Delta^{\gamma}_\lambda$ is bounded from $L^2(0,\infty)$ into itself. Also, $\Delta^{\gamma}_\lambda$ is a Laplace transform type multiplier for the Bessel operator $\Delta_\lambda$, because

$$g^{2\gamma} = y^2 \int_0^\infty e^{-y^2 u} \frac{u^{-\gamma}}{\Gamma(1-i\gamma)} \, du, \quad y \in (0,\infty).$$

Laplace transform type Hankel multipliers have been studied in [3] and [15]. According to the results established in [3] and [15] the imaginary power $\Delta^{\gamma}_\lambda$ is a Calderón-Zygmund operator defined by the kernel

$$(86) \quad K^{\gamma}_\lambda(x,y) = -\frac{1}{\Gamma(1-i\gamma)} \int_0^\infty t^{-i\gamma} \partial_t W^\lambda_t(x,y) \, dt, \quad x, y \in (0,\infty), \quad x \neq y,$$

where $W^\lambda_t(x,y)$ is the heat kernel for the Bessel operator $\Delta_\lambda$, and it is given by

$$W^\lambda_t(x,y) = \frac{(xy)^{1/2}}{2t^{\lambda-1/2}} e^{-\left(\frac{x^2+y^2}{4t}\right)} \mathbb{I}_{\frac{1}{2}} \left(\frac{xy}{t}\right), \quad t, x, y \in (0,\infty).$$

Here $I_\nu$ represents the modified Bessel function of the first class and order $\nu$.

Moreover, the operator $\Delta^{\gamma}_\lambda$ can be extended to $L^p(0,\infty)$ as a bounded operator from $L^p(0,\infty)$ into itself, for every $1 < p < \infty$, from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$, and from $H^1(0,\infty)$ into $L^1(0,\infty)$. This extension, that we continue denoting by $\Delta^{\gamma}_\lambda$, has the following integral representation, for every $f \in L^p(0,\infty)$, $1 \leq p < \infty$,

$$(87) \quad \Delta^{\gamma}_\lambda(f)(x) = \lim_{\varepsilon \to 0^+} \left( \alpha(\varepsilon) f(x) + \int_0^\infty K^{\gamma}_\lambda(x,y) f(y) \, dy \right), \quad \text{a.e. } x \in (0,\infty),$$

where $\alpha$ is a measurable bounded function.

The operator $\Delta^{\gamma}_\lambda$ is defined on $L^p(0,\infty) \otimes \mathbb{B}$ in the natural way.

Let $\gamma \in \mathbb{R} \setminus \{0\}$ and $\beta > 0$. We define the operator $T_{\gamma,\beta}$ by

$$T_{\gamma,\beta}(h)(t) = \frac{1}{t^\beta} \int_0^t (t-s)^{\beta-1} h(t-s) \psi_\gamma(s) \, ds, \quad t > 0,$$

for every $h \in H$, where $\psi_\gamma(s) = s^{-2\gamma} \Gamma(1-2\gamma i)$, $s \in (0,\infty)$. $T_{\gamma,\beta}$ is a bounded operator from $H$ into itself.

If $f \in S_\lambda(0,\infty)$ we proved in [6] (55) that

$$(88) \quad G_{C_\gamma}^{\lambda,\beta}(\Delta^{\gamma}_\lambda(f))(\cdot, x) = -G_{C_\gamma}^{\lambda,\beta+1}(f)(\cdot, x) \circ T_{\gamma,\beta}, \quad \text{a.e. } x \in (0,\infty),$$

as elements of $\gamma(H,\mathbb{C}) = H$. Moreover, the two sides of (88) define bounded operators from $L^p(0,\infty)$ into $L^p((0,\infty),H)$ (see Theorem 4.1 and [12] Theorem 6.2), for every $1 < p < \infty$. Since $S_\lambda(0,\infty)$ is dense in $L^p(0,\infty)$, we deduce that (88) holds, for every $f \in L^p(0,\infty)$, $1 < p < \infty$.

In the following proposition we establish the behaviour of $\Delta^{\gamma}_\lambda$ on $H^1_0(0,\infty)$ and $BMO_0(0,\infty)$.

**Proposition 4.2.** Let $\gamma \in \mathbb{R} \setminus \{0\}$ and $\lambda > 0$. The operator $\Delta^{\gamma}_\lambda$ is bounded from $E(0,\infty)$ into itself, where $E$ denotes $H^1_0$ or $BMO_0$.

**Proof of Proposition 4.2** the case of $E = H^1_0$. In order to show that $\Delta^{\gamma}_\lambda$ is bounded from $H^1_0(0,\infty)$ into itself we consider the function $\varphi_\gamma(x) = x^{2\gamma}$, $x \in (0,\infty)$. It is not hard to see that

$$\sup_{t>0} \|\eta(\cdot) \varphi_\gamma(\cdot)\|_{L_2^1(0,\infty)} < \infty,$$

where $\eta \in C^\infty_c(0,\infty)$, the space of smooth functions with compact support on $(0,\infty)$, and $\|\cdot\|_{L_2^1(0,\infty)}$ denotes the Sobolev norm of order 1 and exponent 2. According to [11] Theorem 4.11] we deduce that $\Delta^{\gamma}_\lambda$ can be extended from $A = \{a \infty$-atom for $H^1_0(0,\infty)\}$ to $H^1_0(0,\infty)$ as a bounded operator from $H^1_0(0,\infty)$ into itself. We denote by $\tilde{\Delta}^{\gamma}_\lambda$ to this extension.
Suppose that $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$, where, for every $j \in \mathbb{N}$, $a_j$ is a $\infty$-atom and $\lambda_j \in \mathbb{C}$ such that $\sum_{j \in \mathbb{N}} |\lambda_j| < \infty$. We have that
\[
\Delta^{(i)\gamma}_x(f) = \sum_{j \in \mathbb{N}} \lambda_j \Delta^{(i)\gamma}_x(a_j), \text{ in } H^1_x(0, \infty).
\]
Since $H^1_x(0, \infty)$ is continuously contained in $L^1(0, \infty)$, the series $\sum_{j=0}^{\infty} \lambda_j \Delta^{(i)\gamma}_x(a_j)$ converges also in $L^1(0, \infty)$, and then in $L^{1,\infty}(0, \infty)$. Also, the operator $\Delta^{(i)\gamma}_x$ is bounded from $L^1(0, \infty)$ into $L^{1,\infty}(0, \infty)$. These facts lead to $\Delta^{(i)\gamma}_x(f) = \Delta^{(i)\gamma}_x(f)$.

We conclude that $\Delta^{(i)\gamma}_x$ is bounded from $H^1_x(0, \infty)$ into itself. \hfill \square

**Proof of Theorem 4.2:** the case of $E = BMO_o$. In a first step we prove that
\[
|K^\lambda(x,y)| \leq C \frac{(xy)^\lambda}{|x-y|^{2\lambda+1}}, \quad x, y \in (0, \infty), \quad x \neq y,
\]
being $K^\lambda$ the kernel in (86).

We will use the following properties of Bessel functions $I_\nu, \nu > -1$ (see [30, pp. 108, 110 and 123]):
\[
d\frac{\partial}{dz}(z^{-\nu} I_\nu(z)) = z^{-\nu} I_{\nu+1}, \quad z \in (0, \infty),
\]
\[
I_\nu(z) \sim \frac{1}{2^\nu \Gamma(\nu+1)} z^\nu, \quad \text{as } z \to 0^+,
\]
and
\[
e^{-z/\sqrt{z}} I_\nu(z) = O\left(\frac{1}{z}\right).
\]
According to [89] we have that
\[
\partial_t W^\lambda_t(x,y) = \partial_t \left[ \frac{(xy)^\lambda}{(2t)^{\lambda+1/2}} \left( \frac{xy}{2t} \right)^{-\lambda+1/2} I_{\lambda-1/2} \left( \frac{xy}{2t} \right) e^{-\left( x^2 + y^2 \right)/4t} \right]
\]
\[
= e^{-\left( x^2 + y^2 \right)/4t} \left[ -2(\lambda+1)(2t)^{-\lambda-3/2}(xy)^\lambda \left( \frac{xy}{2t} \right)^{-\lambda+1/2} I_{\lambda-1/2} \left( \frac{xy}{2t} \right) 
\]
\[- \frac{(xy)^\lambda}{(2t)^{\lambda+1/2}} \left( \frac{xy}{2t} \right)^{-\lambda+1/2} I_{\lambda+1/2} \left( \frac{xy}{2t} \right) 
\]
\[+ \frac{(xy)^\lambda}{(2t)^{\lambda+1/2}} \left( \frac{xy}{2t} \right)^{-\lambda+1/2} I_{\lambda-1/2} \left( \frac{xy}{2t} \right) \frac{x^2 + y^2}{4t^2} \right], \quad t, x, y \in (0, \infty).
\]

From (91) we deduce that
\[
|\partial_t W^\lambda_t(x,y)| \leq C e^{-(x^2+y^2)/4t} \left( \frac{(xy)^\lambda}{t^{\lambda+3/2}} + \frac{(xy)^{\lambda+2}}{t^{\lambda+7/2}} + \frac{(xy)^\lambda}{t^{\lambda+5/2}} (x^2 + y^2) \right)
\]
\[
\leq C e^{-c(x^2+y^2)/t} \left( \frac{(xy)^\lambda}{t^{\lambda+3/2}} \right), \quad t, x, y \in (0, \infty) \text{ and } xy \leq t,
\]
and by (92) we get
\[
|\partial_t W^\lambda_t(x,y)| \leq C e^{-c|x-y|^2/t} \left( \frac{1}{t^{\lambda/2}} + \frac{|x-y|^2}{t^{\lambda/2}} \right)
\]
\[
\leq C e^{-c|x-y|^2/t} \left( \frac{1}{t^{\lambda/2}} \right), \quad t, x, y \in (0, \infty) \text{ and } xy \geq t.
\]

Hence,
\[
|\partial_t W^\lambda_t(x,y)| \leq C e^{-c|x-y|^2/t} \left( \frac{xy)^\lambda}{t^{\lambda+3/2}} \right), \quad t, x, y \in (0, \infty).
\]
and then,

\[
|K_\lambda^\gamma(x, y)| \leq C(xy)^{\lambda} \int_0^\infty e^{-c|x-y|^2/t} \frac{dt}{t^{1+3/2}} \leq C \frac{(xy)^{\lambda}}{|x-y|^{2\lambda+1}}, \quad x, y \in (0, \infty), \ x \neq y.
\]

We define the operator $\Delta_\lambda^\gamma$ on $BMO_0(0, \infty)$ as follows: if $f \in BMO_0(0, \infty)$ and $\delta > 0$,

\[
\Delta_\lambda^\gamma(f)(x) = \Delta_\lambda^\gamma(f\chi_{(0,2\delta)})(x) + \int_{2\delta}^\infty K_\lambda^\gamma(x, y)f(y)dy, \text{ a.e. } x \in (0, \delta).
\]

This definition does not depend on $\delta$ (in the suitable sense). By (94) we have that, for every $\delta > 0$,

\[
\frac{1}{\delta} \int_0^\delta |\Delta_\lambda^\gamma(f)(x)|dx \leq \left( \frac{1}{\delta} \int_0^\delta |\Delta_\lambda^\gamma(f\chi_{(0,2\delta)})(x)|^2dx \right)^{1/2} + C \int_0^\delta \int_{28}^\infty \frac{(xy)^{\lambda}}{|x-y|^{2\lambda+1}} |f(y)|dydx
\]

\[
\leq \left( \frac{1}{\delta} \int_0^\delta |f(y)|^2dy \right)^{1/2} + C \int_0^\delta x^{\lambda} \int_{28}^\infty \frac{|f(y)|}{y^{\lambda+1}}dydx
\]

\[
\leq C\|f\|_{BMO_0(0, \infty)}, \quad f \in BMO_0(0, \infty).
\]

Hence, $|\Delta_\lambda^\gamma(f)(x)| < \infty$, a.e. $x \in (0, \infty)$, for every $f \in BMO_0(0, \infty)$.

Let $f \in BMO_0(0, \infty)$. We are going to show that $\Delta_\lambda^\gamma(f) \in BMO(0, \infty)$. It is well-known that $BMO(0, \infty) = (H^1(0, \infty))^*$. We denote by $\mathcal{A}(0, \infty) = \text{span } \{a \text{ is } \infty\text{-atom satisfying (A)}\}$. Let $g \in \mathcal{A}$. We choose $\delta > 0$ such that $\text{supp } g \subset (0, \delta)$. Then,

\[
\Delta_\lambda^\gamma(f)(x) = \Delta_\lambda^\gamma(f\chi_{(0,2\delta)})(x) + \int_{2\delta}^\infty K_\lambda^\gamma(x, y)f(y)dy, \text{ a.e. } x \in (0, \delta).
\]

Since $\Delta_\lambda^\gamma$ is bounded from $L^2(0, \infty)$ into itself, (94) leads to

\[
\int_0^\infty |\Delta_\lambda^\gamma(f)(x)||g(x)|dx \leq \int_0^\delta |\Delta_\lambda^\gamma(f\chi_{(0,2\delta)})(x)||g(x)|dx
\]

\[
+ \int_0^\delta \int_{2\delta}^\infty |K_\lambda^\gamma(x, y)||f(y)|dy|g(x)|dx
\]

\[
\leq \left( \int_0^\delta |\Delta_\lambda^\gamma(f\chi_{(0,2\delta)})(x)|^2dx \right)^{1/2} \left( \int_0^\delta |g(x)|^2dx \right)^{1/2}
\]

\[
+ C \int_0^\delta \int_{2\delta}^\infty \frac{(xy)^{\lambda}}{|x-y|^{2\lambda+1}} |f(y)|dy|g(x)|dx
\]

\[
\leq C \left( \int_0^\delta |f(x)|^2dx \right)^{1/2} + \int_0^\delta x^{\lambda} \int_{2\delta}^\infty \frac{|f(y)|}{y^{\lambda+1}}dydx
\]

\[
\leq C\|f\|_{BMO_0(0, \infty)},
\]

because $g \in L^\infty(0, \infty)$.

We define the functional $\mathcal{L}$ on $\mathcal{A}$ by

\[
\mathcal{L}(g) = \int_0^\infty \Delta_\lambda^\gamma(f)(x)g(x)dx, \quad g \in \mathcal{A}.
\]

Let $g \in \mathcal{A}$ such that $\text{supp } g \subset (0, \delta)$. We can write

\[
\mathcal{L}(g) = \int_0^\infty \Delta_\lambda^\gamma(f\chi_{(0,2\delta)})(x)g(x)dx + \int_0^\delta g(x) \int_{2\delta}^\infty K_\lambda^\gamma(x, y)f(y)dydx.
\]

Since, as it was seen,

\[
\int_0^\infty |g(x)| \int_{2\delta}^\infty |K_\lambda^\gamma(x, y)||f(y)|dydx < \infty,
\]
according to \[ \text{[87]} \] we get
\[
\int_0^\infty g(x) \int_{25}^\infty K_\lambda^\gamma(x, y) f(y) dy dx = \int_{25}^\infty f(y) \int_0^\infty K_\lambda^\gamma(x, y) g(x) dx dy = \int_{25}^\infty f(y) \Delta_\lambda^\gamma(g)(y) dy.
\]
By using Plancherel’s equality for Hankel transforms it follows that
\[
\int_0^\infty g(x) \Delta_\lambda^\gamma(f(\chi_{(0, 25)}))(x) dx = \int_0^\infty f(y) \Delta_\lambda^\gamma(g)(y) dy.
\]
Hence,
\[
\mathcal{L}(g) = \int_0^\infty f(y) \Delta_\lambda^\gamma(g)(y) dy.
\]
We also have that
\[
\int_0^\infty |f(y)\Delta_\lambda^\gamma(g)(y)| dy \leq \int_0^{2\delta} |f(y)| |\Delta_\lambda^\gamma(g)(y)| dy + \int_0^\infty |f(y)| \int_0^\delta |K_\lambda^\gamma(x, y)||g(x)| dx dy
\]
\[
\leq \left( \int_0^{2\delta} |f(y)|^2 dy \right)^{1/2} ||\Delta_\lambda^\gamma(g)||_{L^2(0, \infty)} + C \int_{2\delta}^\infty \frac{\delta^{\lambda+1}}{y^{\lambda+1}} |f(y)| dy
\]
\[
\leq C(\sqrt{\int |f||BMO_\delta(0, \infty)|^2} + 1)\int |f||BMO_\delta(0, \infty) \leq C\int |f||BMO_\delta(0, \infty).
\]
According to \[ \text{[4]} \] Proposition 2.5] adapted to this Bessel setting (see also \[ \text{[17]} \]), since \( \Delta_\lambda^\gamma \) is bounded from \( H^1_\delta(0, \infty) \) into itself, we deduce
\[
|\mathcal{L}(g)| \leq C\int |f||BMO_\delta(0, \infty)||\Delta_\lambda^\gamma(g)||_{H^1_\delta(0, \infty)} \leq C\int |f||BMO_\delta(0, \infty)||g||_{H^1_\delta(0, \infty)}.
\]
Here \( C > 0 \) does not depend on \( g \).

We conclude that \( \Delta_\lambda^\gamma(f) \in BMO_\delta(0, \infty) \) and \( \|\Delta_\lambda^\gamma(f)\|_{BMO_\delta(0, \infty)} \leq C\int |f||BMO_\delta(0, \infty). \)

Suppose now that \( (iii) \) holds in the Hardy setting. Let \( f \in H^1_\delta(0, \infty) \otimes \mathcal{B} \). According to Proposition \[ \text{[4,2]} \] \( \Delta_\lambda^\gamma(f) \in H^1_\delta(0, \infty) \otimes \mathcal{B} \). Then, by using \[ \text{[88]} \] and by taking into account \[ \text{[12]} \] Theorem 6.2] we get
\[
\|\Delta_\lambda^\gamma(f)\|_{H^1_\delta((0, \infty), \mathcal{B})} \leq C\|G^\lambda_\delta(\Delta_\lambda^\gamma(f))\|_{H^1_\delta((0, \infty), \mathcal{B})} \leq C\|G^\lambda_\delta(\Delta_\lambda^{\lambda+1}(f))\|_{H^1_\delta((0, \infty), \mathcal{B})} \leq C\|f\|_{H^1_\delta((0, \infty), \mathcal{B})}.
\]
According to \[ \text{[31]} \] Theorem 4.1] we deduce that \( \Delta_\lambda^\gamma \) can be extended to \( L^p((0, \infty), \mathcal{B}) \) into itself, for every \( 1 < p < \infty \).

Hence, by \[ \text{[6]} \] Proposition 5.1] we conclude that \( \mathcal{B} \) is a UMD space.

By proceeding in a similar way we can prove that \( \mathcal{B} \) is a UMD space provided that \( (iii) \) holds in the BMO situation.

Thus, the proof of this theorem is finished.

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