REGULARITY OF FRACTIONAL HEAT SEMIGROUP ASSOCIATED WITH
SCHRÖDINGER OPERATORS

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Abstract. Let \( L = -\Delta + V \) be a Schrödinger operator, where the potential \( V \) belongs to the reverse Hölder class. By the subordinative formula, we introduce the fractional heat semigroup \( \{ e^{-tL^\alpha} \}_{t>0}, \alpha > 0 \), associated with \( L \). By the aid of the fundamental solution of the heat equation:
\[
\partial_t u + Lu = \partial_t u - \Delta u + Vu = 0,
\]
we estimate the gradient and the time-fractional derivatives of the fractional heat kernel \( K_L^\alpha (\cdot, \cdot) \), respectively. This method is independent of the Fourier transform, and can be applied to the second order differential operators whose heat kernels satisfying Gaussian upper bounds. As an application, we establish a Carleson measure characterization of the Campanato type space \( BMO_\gamma (\mathbb{R}^n) \) via \( \{ e^{-tx^2} \}_{t>0} \).

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1. Introduction

The aim of this paper is to investigate the fractional heat semigroup of Schrödinger operators

\[
L := -\Delta + V(x),
\]

where \(-\Delta\) denotes the Laplace operator: \( \Delta = \sum_{i=1}^{n} \partial_i^2 / \partial x_i^2 \) and \( V \) is a nonnegative potential belonging to the reverse Hölder class \( B_q \).

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Definition 1.1. A nonnegative locally $L^q$ integrable function $V$ on $\mathbb{R}^n$ is said to belong to $B_q$, $1 < q < \infty$, if there exists $C > 0$ such that the reverse Hölder inequality
\begin{equation}
\left( \frac{1}{|B|} \int_B V^q(x)dx \right)^{1/q} \leq \frac{C}{|B|} \int_B V(x)dx
\end{equation}
holds for every ball $B$ in $\mathbb{R}^n$.

For the special case: $V = 0$, $L = -\Delta$, the fractional heat semigroup can be defined via the Fourier transform:
\begin{equation}
(e^{-t(-\Delta)^\alpha} f)(\xi) := e^{-t|\xi|^{2\alpha}} \hat{f}(\xi), \; \alpha \in (0, 1].
\end{equation}
In the literature, the fractional heat semigroup $\{e^{-t(-\Delta)^\alpha}\}_{t>0}$ has been widely used in the study of partial differential equations, harmonic analysis, potential theory and modern probability theory. For example, the semigroup $\{e^{-t(-\Delta)^\alpha}\}_{t>0}$ is usually applied to construct the linear part of solutions to fluid equations in the mathematical physics, e.g. the generalized Naiver-Stokes equation, the quasi-geostrophic equation, the generalized MHD equations. In the field of probability theory, the researchers use some kind of Markov processes with jumps. For further information and the related applications of fractional heat semigroups $\{e^{-t(-\Delta)^\alpha}\}_{t>0}$, we refer the reader to \cite{1, 2, 18, 33}.

Denote by $K_{\alpha,t}(\cdot, \cdot)$ the integral kernel of $e^{-t(-\Delta)^\alpha}$, i.e.,
\begin{equation}
K_{\alpha,t}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle - t|\xi|^{2\alpha}} d\xi
\end{equation}
and denote by $K_{\alpha,t}^\theta(\cdot)$ the integral kernel of $(-\Delta)^{\theta/2} e^{-t(-\Delta)^\alpha}$. In \cite{24}, by an invariant derivative technique and the Fourier analysis method, Miao-Yuan-Zhang concluded that the kernels $K_{\alpha,t}$ and $K_{\alpha,t}^\theta$ satisfy the following pointwise estimates, respectively (cf. \cite{24} Lemmas 2.1 and 2.2),
\begin{equation}
\begin{cases}
K_{\alpha,t}(x) \leq \frac{t}{(t^{1/2\alpha} + |x|)^{n+2\alpha}} \quad \forall \; (x, t) \in \mathbb{R}^{n+1}, \\
K_{\alpha,t}^\theta(x) \leq \frac{1}{(t^{1/2\alpha} + |x|)^{n+\theta}} \quad \forall \; (x, t) \in \mathbb{R}^{n+1}.
\end{cases}
\end{equation}

Compared with $-\Delta$, for arbitrary Schrödinger operator $L$ with the non-negative potential $V$, the fractional heat semigroup $\{e^{-tL^\alpha}\}_{t>0}, \alpha \in (0, 1)$, can not be defined via \cite{12}. Also, it is obvious that the methods in \cite{24} are invalid for the estimation of the integral kernels of $\{e^{-tL^\alpha}\}_{t>0}$. In this paper, by the functional calculus, we observe that the integral kernel of the Poisson semigroup associated with $L$ can be defined as
\begin{equation}
P^L_t(x, y) = \int_0^\infty e^{-u} \sqrt{u} K^L_{2/4\alpha}(x, y)du = \int_0^\infty \frac{te^{-t^2/4s}}{2s^{3/2}} K^L_s(x, y)ds,
\end{equation}
where $K^L_s(\cdot, \cdot)$ denotes the integral kernel of $e^{-sL}$, i.e.,
\begin{equation}
e^{-sL}(f)(x) := \int_{\mathbb{R}^n} K^L_s(x, y)f(y)dy.
\end{equation}
Recall that $K^L_s(\cdot, \cdot)$ is a positive, symmetric function on $\mathbb{R}^n \times \mathbb{R}^n$, and satisfies $\int_{\mathbb{R}^n} K^L_s(x, y)dy \leq 1$. Generally, for $\alpha > 0$, the subordinative formula (cf. \cite{12}) indicates that there exists a continuous function $\eta^\theta_t(\cdot)$ on $(0, \infty)$ such that
\begin{equation}
K^L_{\alpha,t}(x, y) = \int_0^\infty \eta^\theta_t(s) K^L_s(x, y)ds.
\end{equation}
The identity (1.5) enables us to estimate $K^L_{\alpha,t}(\cdot, \cdot)$ via the heat kernel $K^L_t(\cdot, \cdot)$. Let $\rho(\cdot)$ be the auxiliary function defined by (2.2) below. In Propositions 3.1 and 3.2, we can obtain the following pointwise
estimates of \(K_{\alpha,t}^L(\cdot,\cdot)\): for every \(N > 0\), there exists a constant \(C_N\) such that

\[
\left| K_{\alpha,t}^L(x,y) \right| \leq \frac{C_N t^{\frac{1}{1+2\alpha}}}{(t^{1/2\alpha} + |x-y|)^{n+2\alpha} \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)}\right)^N},
\]

and for every \(N > 0\), \(0 < \delta' < \min \{1, 2 - n\} / q\) and all \(|h| \leq 1\), there exists a constant \(C_N\) such that

\[
\left| K_{\alpha,t}^L(x+h,y) - K_{\alpha,t}^L(x,y) \right| \leq \frac{C_N t^{\frac{1}{1+2\alpha}}}{(t^{1/2\alpha} + |x-y|)^{n+2\alpha} \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)}\right)^N}.
\]

Based on the estimates (1.6) and (1.7), we consider the regularity properties of \(K_{\alpha,t}^L(\cdot,\cdot)\). Let \(\nabla_{x,t}\) denote the gradient operator on \(\mathbb{R}^n\), that is, \(\nabla_{x,t} = (\nabla_x, \partial/\partial t)\), where \(\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)\). We obtain an energy estimate of the solution to the equation:

\[
\partial_t u + Lu = \partial_t u - \Delta u + Vu = 0.
\]

By the fundamental solution of \(-\Delta\), we prove that, for any \(N > 0\), there exists a constant \(C_N\) such that

\[
|\nabla_x K_{\alpha,t}^L(x,y)| \leq \frac{C_N}{t^{1/2}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \quad \sqrt{t} \leq |x-y|;
\]

\[
|\nabla_x K_{\alpha,t}^L(x,y)| \leq \frac{C_N}{t^{1/2}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \quad \sqrt{t} \geq |x-y|;
\]

\[
|\nabla_x K_{\alpha,t}^L(x,y)| \leq \frac{C_N}{t^{1/2}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \quad \forall \ t, x, y.
\]

A direct computation, together with the subordinative formula, gives

\[
|\nabla_x K_{\alpha,t}^L(x,y)| \leq \frac{C_N t^{1/2}}{t^{1/2} + |x-y|^{n+2\alpha}} \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N} \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N},
\]

see Proposition 3.6. By a similar method, we obtain the Hölder regularity of \(\nabla_x K_{\alpha,t}^L(\cdot,\cdot)\), i.e., for \(|h| < |x-y|/4\) and \(\delta' = 1 - n/q\),

\[
|\nabla_x K_{\alpha,t}^L(x+h,y) - \nabla_x K_{\alpha,t}^L(x,y)| \leq C_N \frac{|h|}{t^{1/2}} \frac{t}{t^{1/2} + |x-y|^{n+2\alpha}},
\]

see Proposition 3.9.

In Section 3.3, we focus on the time-fractional derivatives of \(K_{\alpha,t}^L(\cdot,\cdot)\). Recently, there has been an increasing interest in fractional calculus since time-fractional operators are proved to be very useful for modeling purpose. For example, the following fractional heat equations

\[
\beta \partial_t^\beta u(x,t) = \Delta u(x,t)
\]

are used to describe heat propagation in inhomogeneous media. It is known that as opposed to the classical heat equation, the equation (1.9) is known to exhibit sub diffusive behaviour and is related with anomalous diffusions or diffusions in non-homogeneous media, with random fractal structures. Recall that the fractional derivative of \(K_{\alpha,t}^L(\cdot,\cdot)\) is defined as

\[
\partial_t^\beta K_{\alpha,t}^L(x,y) := \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_t^\alpha K_{\alpha,t}^L(x-u) u^{-\beta} \frac{du}{u}, \quad \beta > 0 \text{ and } m = [\beta] + 1.
\]

In Section 3.1, we first obtain the regularity estimates of \(\partial_t^\beta K_{\alpha,t}^L(\cdot,\cdot)\) denoted by \(D_{\alpha,t}^L(\cdot,\cdot)\), see Proposition 3.2. Then the desired estimates of \(\partial_t^\beta K_{\alpha,t}^L(\cdot,\cdot)\) can be deduced from (1.10) and Proposition 3.3; see Propositions 3.4, 3.5, respectively.

As an application, in Section 4, we characterize the Casamino type spaces associated with \(L\), denoted by \(BMO^L_{\alpha}(\mathbb{R}^n)\), via the fractional heat semigroup \(e^{-\Delta L^\alpha_{\cdot,t}}\). In the last decades, the characterizations of function spaces associated with Schrödinger operators via semigroups and Carleson measures have attracted the attentions of many authors. Let \(V \in B_q, q > n/2\). Using the family of operators \(\{t\partial_t e^{-\Delta L^\alpha_{\cdot,t}}\}_{t > 0},\)
the Carleson measure characterization of \( BMO_1(\mathbb{R}^n) \) was obtained by Dziubański-Garrigós-Martínez-Torreá-Zienkiewicz [9]. Replacing the potential \( V \) by a general Radon measure \( \mu \), in [29], Wu-Yan extended the result of [9] to generalized Schrödinger operators. The analogue in the setting of Heisenberg groups was obtained by Lin-Liu [21]. Ma-Stinga-Torreá-Zhang [23] characterized the Campanato type spaces associated with \( L \) via the fractional derivatives of the Poisson semigroup. For further information on this topic, we refer to [4,5,17,19,28,31,32] and the references therein. Assume that \( L = -\Delta + V \) with \( V \in Bq, q > n \). By the regularity estimates obtained in Section 3, we establish the following equivalent characterizations: for \( 0 < \gamma < \min\{2\alpha, 2\alpha\beta\} \),

\[
\begin{align*}
f \in BMO^\gamma_1(\mathbb{R}^n) & \sim \sup_B \frac{1}{|B|^{1+2\gamma/n}} \int_{\mathbb{R}^n} |f|^{\frac{\gamma}{2}} \int_B \left| \partial^\alpha_x \partial^\beta_t \right| e^{-tL^n}(f)(x)| \frac{dxdt}{t} < \infty \quad \text{for} \quad 0 < \gamma < \min\{2\alpha, 2\alpha\beta\}, \\
& \sim \sup_B \frac{1}{|B|^{1+2\gamma/n}} \int_{\mathbb{R}^n} |f|^{\frac{\gamma}{2}} \int_B \left| \partial^\alpha_x \partial^\beta_t \right| e^{-tL^n}(f)(x)| \frac{dxdt}{t} < \infty, \end{align*}
\]

where \( \nabla_{\alpha} := (\nabla_x, \partial^\alpha_t)^{1/2\alpha} \). See Theorems 4.7 and 4.8 respectively.

**Remark 1.2.** The regularity estimates obtained in this paper generalize several results on the regularities of Schrödinger operators. Letting \( \alpha = 1/2 \), \( K_{1/2}^L(\cdot, \cdot) = P_{1/2}^L(\cdot, \cdot) \), the Poisson kernel associated with the Schrödinger operator. For this case, Propositions 3.6 and 3.9 come back to [5, Lemma 3.9]. Also, the regularities of \( \partial_{\alpha}^\beta K_{\alpha,\beta}^L(\cdot, \cdot) \) obtained in Section 3.3 generalize [23, Proposition 3.6, (b), (c), (d)].

**Remark 1.3.** For the case of \( \alpha = 1/2 \), the regularities of \( \partial_{\alpha}^\beta K_{1/2}^L \) have been studied by Ma-Stinga-Torreá-Zhang [23]. We point out that our method is slightly different from that of [23]. In [23], via the Hermite polynomials \( H_m(\cdot) \), the authors convert the estimate of \( \partial_{\alpha}^\beta K_{1/2}^L(\cdot, \cdot) \) to the estimate of \( \partial_{\alpha}^\beta K_{1/2}^L(\cdot, \cdot) \), see [23, (3.12)] in Section 3.3, we estimate the time-fractional derivatives of \( K_{\alpha,\beta}^L(\cdot, \cdot) \) via \( \partial_{\alpha}^\beta K_{\alpha,\beta}^L(\cdot, \cdot) \) instead of the Hermite polynomials.

Some notations:

- \( U \cong V \) represents that there is a constant \( c > 0 \) such that \( c^{-1}V \leq U \leq cV \) whose right inequality is also written as \( U \leq V \). Similarly, one writes \( V \geq U \) for \( V \geq cU \).
- For convenience, the positive constants \( C \) may change from one line to another and usually depend on the dimension \( n \), \( \alpha \), \( \beta \) and other fixed parameters.
- Let \( B \) be a ball with the radius \( r \). In the rest of this paper, for \( c > 0 \), we denote by \( B_{cr} \) the ball with the same center and radius \( cr \).

2. Preliminaries

2.1. **The Schrödinger operator.** Let \( L = -\Delta + V \) be the Schrödinger differential operator on \( \mathbb{R}^n, n \geq 3 \). Throughout this paper, we will assume that \( V \) is a nonzero, nonnegative potential, and belongs to the reverse Hölder class \( Bq, q > n/2 \), which is defined in Definition 1.1. By Hölder’s inequality, we can get \( Bq_1 \subset Bq_2 \) for \( q_1 \geq q_2 > 1 \). One remarkable feature about the class \( Bq \) is that if \( V \in Bq \) for some \( q > 1 \), then there exists \( \varepsilon > 0 \), which depends only on \( n \) and the constant \( C \) in 1.1, such that \( V \in B_{q+\varepsilon} \). It is also well known that if \( V \in Bq, q > 1 \) then \( V(x)dx \) is a doubling measure. Namely, for any \( r > 0, x \in \mathbb{R}^n \),

\[
(2.1) \quad \int_{B(x,2r)} V(y)dy \leq C_0 \int_{B(x,r)} V(y)dy.
\]

The auxiliary function \( m(x, V) \) is defined by

\[
(2.2) \quad \frac{1}{m(x, V)} := \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1 \right\}.
\]
Clearly, $0 < m(x, V) < \infty$ for every $x \in \mathbb{R}^n$ and if $r = 1/m(x, V)$, then $\frac{1}{m(x, V)} \int_{B(x,r)} V(y)dy = 1$. For simplicity, we denote $1/m(x, V)$ by $\rho(x)$ in the proof sometime. We state some properties of $m(x, V)$ which will be used in the proofs of main results.

**Lemma 2.1.** ([25 Lemma 1.2]) There exists a constant $C > 0$ such that for every $0 < r < R < \infty$ and $y \in \mathbb{R}^n$, we have

$$\frac{1}{r^{n-2}} \int_{B(y,r)} V(x)dx \leq C \left(\frac{r}{R}\right)^{2-n/q} \frac{1}{R^{n-2}} \int_{B(y,R)} V(x)dx.$$  

**Lemma 2.2.** ([20 Lemma 3]) For every constant $C_1 > 1$, there exists a constant $C_2 > 1$ such that if

$$\frac{1}{C_1} \leq \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C_1,$$

then $C_2^{-1} \leq r m(x, V) \leq C_2$.

**Lemma 2.3.** ([25 Lemma 1.4]) For every constant $C_1 \geq 1$, there is a constant $C_2 \geq 1$ such that

$$\frac{1}{C_2} \leq \frac{m(x, V)}{m(y, V)} \leq C_2$$

for $|x - y| \leq C_1 \rho(x)$. Moreover, there exist constants $k_0, C, c > 0$ such that

$$\begin{cases}
m(y, V) \leq C(1 + |x - y|m(x, V))^{k_0} m(x, V); \\
m(y, V) \geq cm(x, V)(1 + |x - y|m(x, V))^{-k_0/(1+k_0)}.
\end{cases}$$

**Lemma 2.4.** ([25 Lemma 1.8]) There exist constants $k_0, C > 0$ such that for $R \geq m(x, V)^{-1}$,

$$\frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy \leq C(Rm(x, V))^{k_0}.$$  

**Lemma 2.5.** ([11 Lemma 1]) Suppose $V \in B_q$, $q > n/2$. Let $m_0 > \log_2 C_0 + 1$, where $C_0$ is the constant in [2.7]. Then for any $x_0 \in \mathbb{R}^n$, $R > 0$,

$$\frac{1}{\{1 + Rm(x_0, V)\}^{m_0}} \int_{B(x_0,R)} V(x)dx \leq CR^{n-2}.$$  

As a corollary of [8 Corollary 4.8], we have

**Lemma 2.6.** There exist constants $C$, $\delta$ and $l$ such that

$$\frac{1}{p^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{c|x-y|^2}{t}} V(y)dy \leq \begin{cases}
\frac{C}{l} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta}, & \sqrt{t} < m(x, V)^{-1}; \\
\frac{C}{l} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{l}, & \sqrt{t} \geq m(x, V)^{-1}.
\end{cases}$$

Since the potential $V$ is nonnegative, it follows from the Feynman-Kac formula that the kernels $K_t^L(\cdot, \cdot)$ have a Gaussian upper bound:

$$0 \leq K_t^L(x, y) \leq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}.$$  

Furthermore,

**Proposition 2.7.** ([8 Theorem 4.10]) For every $N > 0$, there exist constants $C_N$ and $c$ such that for all $x, y \in \mathbb{R}^n$,

$$0 \leq K_t^L(x, y) \leq \frac{C_N}{p^{n/2}} e^{-\frac{c|x-y|^2}{t}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$  

(2.3)
Proposition 2.8. ([8 Proposition 4.11]) For every $0 < \delta' < \delta_0 = \min\{1, 2 - n/q\}$ and every $N > 0$, there exist constants $C_N > 0$ and $c$ such that for $|h| < \sqrt{t}$,

$$|K^L_t(x + h, y) - K^L_t(x, y)| \leq \frac{C_N}{t^{n/2}} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta'} e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$ 

Let $Q^L_{t,m}(x, y) := t^m \partial^m_x K^L_t(x, y)$, $m \in \mathbb{Z}_+$. Then

Proposition 2.9. ([16 Proposition 3.3]) Let $m \in \mathbb{Z}_+$.

(i) For every $N > 0$, there exist constants $C_N > 0$ and $c > 0$ such that

$$|Q^L_{t,m}(x, y)| \leq \frac{C_N}{t^{n/2}} e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$ 

(ii) Let $0 < \delta' \leq \delta_0$, where $\delta_0$ appears in Proposition 2.8. For every $N > 0$, there exist constants $C_N > 0$ and $c$ such that for $|h| < t$,

$$|Q^L_{t,m}(x + h, y) - Q^L_{t,m}(x, y)| \leq \frac{C_N}{t^{n/2}} e^{-c|x-y|^2/t} \left(\frac{|h|}{\sqrt{t}}\right)^{\delta'} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$ 

(iii) For every $N > 0$ and $0 < \delta' \leq \delta_0$, there exists a constant $C_N > 0$ such that

$$\left| \int_{\mathbb{R}^n} Q^L_{t,m}(x, y) d\mu(y) \right| \leq C_N \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta'} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N}.$$ 

2.2. Fractional heat kernels associated with $L$. In this section, we first state some backgrounds on the fractional heat semigroup and the fractional heat kernel associated with $L$. For the case $V \neq 0$, the fractional heat semigroup associated with $L$ can not be defined via the Fourier multiplier method [112] as the Laplace operator. We strike out on a new path and introduce the fractional heat semigroup via the subordinative formula.

The Schrödinger operator $L$ can be seen as the generator of the semigroup $\{e^{-tL}\}_{t>0}$, i.e.,

$$L(f) := \lim_{t \to 0} \frac{f - e^{-tL}f}{t},$$

where the limit is in $L^2(\mathbb{R}^n)$. $L$ is a self-adjoint, positive definite operator. The integral kernels of the semigroups $\{e^{-tL}\}_{t>0}$ are denoted by $K^L_t(\cdot, \cdot)$. It is easy to verify that the kernel $K^L_t(\cdot, \cdot)$ satisfies

(i) $K^L_t(x, y) \geq 0$, $x, y \in \mathbb{R}^n$;

(ii) $K^L_t(x, y) = K^L_t(y, x)$;

(iii) $K^L_{t+s}(x, y) = \int_{\mathbb{R}^n} K^L_t(x, z) K^L_s(z, y) dz$;

(iv) $\lim_{t \to 0^+} \int_{\mathbb{R}^n} K^L_t(x, y) f(y) dy = f(x)$ $\forall f \in L^2(\mathbb{R}^n)$.

For $\alpha \in (0, 1)$, the fractional power of $L$, denoted by $L^\alpha$, is defined as

$$L^\alpha(f) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left( e^{-t \sqrt{L}} f(x) - f(x) \right) \frac{dt}{t^{1+2\alpha}} \quad \forall f \in L^2(\mathbb{R}^n).$$

Here $\{e^{-t\sqrt{L}}\}_{t>0}$ denotes the Poisson semigroup related to $L$ with the kernel $P^L_t(\cdot, \cdot)$ defined as

$$P^L_t(x, y) = \int_0^\infty \frac{e^{-u}}{\sqrt{u}} K^L_{\sqrt{u}/4t}(x, y) du = \int_0^\infty \frac{e^{-u^2/4t}}{2\pi} K^L_t(x, y) ds.$$
By the subordinative formula, we know that there exists a non-negative continuous function \( \eta_t^\alpha(\cdot) \) satisfying (cf. [12])

\[
\begin{align*}
\eta_t^\alpha(s) &= \frac{1}{t^{1/\alpha}} \eta_1^\alpha(s/t^{1/\alpha}); \\
\eta_t^\alpha(s) &\leq \frac{t}{s^{1+\alpha}} \quad \forall \ s, t > 0; \\
\int_0^\infty s^{-\gamma} \eta_t^\alpha(s) ds &< \infty, \quad \gamma > 0; \\
\eta_t^\alpha(s) &\approx \frac{t}{s^{1+\alpha}} \quad \forall \ s \geq t^{1/\alpha} > 0,
\end{align*}
\]

such that \( K_{\alpha,t}^L(\cdot, \cdot) \) can be expressed as

\[
K_{\alpha,t}^L(x, y) = \int_0^\infty \eta_t^\alpha(s) K_t^L(x, y) ds,
\]

see [12] for some examples of \( \eta_t^\alpha(\cdot) \). The function \( \eta_t^\alpha(\cdot) \) plays an important role in the estimate of the fractional heat kernel \( K_{\alpha,t}^L(\cdot, \cdot) \). Take \( \alpha = 1/2 \) for example: by (1.4), we can see that \( \eta_t^{1/2}(s) = \frac{1}{\sqrt{\pi t}} e^{-s^2/4t} \forall s, t > 0 \). It is easy to verify that such \( \eta_t^{1/2}(\cdot) \) satisfies conditions in (2.4). For the special case \( L = -\Delta \), a direct computation gives

\[
P_t^{-\gamma}(x, y) = \int_0^\infty \frac{e^{-s^2/4t}}{2\pi^{n/2}} s^{-n/2} e^{-|x-y|^2/s} ds = \frac{c_n t}{(t^2 + |x-y|^2)^{(n+1)/2}},
\]

which coincides with the classical Poisson kernel obtained via (1.3).

2.3. Campanato type spaces associated with \( L \). The Campanato type space associated with \( L \) is defined as follows.

**Definition 2.10.** The space \( BMO_\gamma^L(\mathbb{R}^n), 0 < \gamma \leq 1 \), is defined as the set of all locally integrable functions \( f \) satisfying there exists a constant \( C \) such that

\[
\sup_B \frac{1}{|B|^{1+\gamma/n}} \int_B |f(x) - f(B, V)| dx \leq C < \infty,
\]

where the supremum is taken over all balls \( B \) centered at \( x_B \) with radius \( r_B \), and

\[
f(B, V) := \begin{cases} f_{B}, & r_B < \rho(x_B); \\
0, & r_B \geq \rho(x_B).
\end{cases}
\]

The norm \( ||f||_{BMO_\gamma^L} \) is defined as the infimum of the constants \( C \) such that (2.6) above holds.

**Proposition 2.11.** ([23, Proposition 4.3]) Let \( B = B(x, r) \) with \( r < \rho(x) \). If \( f \in BMO_\gamma^L(\mathbb{R}^n), 0 < \gamma \leq 1 \), then there exists a constant \( C_\gamma \) such that \( |f|_B \leq C_\gamma (\rho(x))^\gamma ||f||_{BMO_\gamma^L} \).

The space \( BMO_\gamma^L(\mathbb{R}^n) \) is equivalent to the following Lipschitz type space related to \( L \).

**Definition 2.12.** For \( 0 < \gamma \leq 1 \), a continuous function \( f \) defined on \( \mathbb{R}^n \) belongs to the space \( C^{0,\gamma}_L(\mathbb{R}^n) \) if

\[
\sup_{x, y \in \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|^{\gamma}} < \infty \quad \text{and} \quad \sup_{x \in \mathbb{R}^n} \frac{|f(x)|}{\rho(x)^\gamma} < \infty,
\]

**Proposition 2.13.** ([23, Proposition 4.6]) If \( 0 < \gamma \leq 1 \), then the spaces \( BMO_\gamma^L(\mathbb{R}^n) \) and \( C^{0,\gamma}_L(\mathbb{R}^n) \) are equal and their norms are equivalent.
It is well known that Hardy spaces \( H^p(\mathbb{R}^n) \), \( 0 < p \leq 1 \), are the predual spaces of Campanato spaces (cf. [10]). In 2000s, such dual relationship was extended to function spaces associated with operators, see [5, 7, 9, 15, 22, 30]. For a Schrödinger operator \( L \), the following Hardy type spaces \( H^p_L(\mathbb{R}^n) \), \( 0 < p \leq 1 \), were introduced in [6, 8].

**Definition 2.14.** For \( 0 < p \leq 1 \), an integrable function \( f \) is an element of the Hardy type space \( H^p_L(\mathbb{R}^n) \) if the maximal function

\[
T^*(f)(x) := \sup_{t>0} |T^L_t(f)(x)|
\]

belongs to \( L^p(\mathbb{R}^n) \). The quasi-norm in \( H^p_L(\mathbb{R}^n) \) is defined by

\[
\|f\|_{H^p_L} := \|T^*(f)\|_{L^p}.
\]

Let \( \delta_0 = \min\{1, 2 - n/q\} \) and \( n/(n + \delta_0) < p \leq 1 \). An atom of \( H^p_L(\mathbb{R}^n) \) associated with a ball \( B(x_B, r_B) \) is a function \( a \) such that

\[
\begin{cases}
supp a \subseteq B(x_B, r_B), \quad r_B \leq \rho(x_B); \\ \|a\|_{L^{n/\delta}} \leq |B(x_B, r_B)|^{-1/p}; \\ \int_{\mathbb{R}^n} a(x) dx = 0, \quad r_B < \rho(x_B)/4.
\end{cases}
\]

In [8], Dziubański and Zienkiewicz obtained the following atomic characterization of \( H^p_L(\mathbb{R}^n) \).

**Proposition 2.15.** ([8 Theorem 1.13]) Let \( n/(n + \delta_0) < p \leq 1 \). \( f \in H^p_L(\mathbb{R}^n) \) if and only if \( f = \sum_j \lambda_j a_j \), where \( \{a_j\} \) are \( H^p_L \)-atoms and \( \sum_j |\lambda_j|^p < \infty \).

**Theorem 2.16.** ([23 Theorem 4.5]) Let \( 0 \leq \gamma < 1 \). Then the dual space of \( H_L^{n/(n+\gamma)}(\mathbb{R}^n) \) is \( \text{BMO}^\gamma_L(\mathbb{R}^n) \). More precisely, any continuous linear functional \( \Phi \) over \( H_L^{n/(n+\gamma)}(\mathbb{R}^n) \) can be represented as

\[
\Phi(a) = \int_{\mathbb{R}^n} f(x)a(x) dx
\]

for some function \( f \in \text{BMO}^\gamma_L(\mathbb{R}^n) \) and all \( H_L^{n/(n+\gamma)} \)-atoms \( a \). Moreover, the operator norm \( \|\Phi\|_{\text{op}} \sim \|f\|_{\text{BMO}^\gamma_L} \).

**Lemma 2.17.** ([9 Lemma 7]) Let \( q_t(x, y) \) be a function of \( x, y \in \mathbb{R}^n, \ t > 0 \). Assume that for every \( N > 0 \), there exists a constant \( C_N \) such that for some \( \gamma' \geq \gamma \),

\[
|q_t(x, y)| \leq C_N(1 + t/\rho(x) + t/\rho(y))^{-N} r^{-n}(1 + |x - y|/t)^{-(n+\gamma')}.
\]

Then for every \( H_L^{n/(n+\gamma')} \)-atom \( g \) supported on \( B(x_0, r) \), there exists \( C_{N,x_0,r} > 0 \) such that

\[
\sup_{r > 0} \left| \int_{\mathbb{R}^n} q_t(x, y) g(y) dy \right| \leq C_{N,x_0,r}(1 + |x|)^{-(n+\gamma')}, \ x \in \mathbb{R}^n.
\]

3. **Regularities on fractional heat semigroups associated with \( L \)**

The aim of this section is to estimate the regularities of the fractional heat kernel \( K_{\alpha,t}^L(\cdot, \cdot) \). By use of (1.3), we first estimate \( \partial_t^\alpha K_{\alpha,t}^L(\cdot, \cdot), m \geq 1 \). Then, via the solution to (1.8), we investigate the spatial gradient of \( K_{\alpha,t}^L(\cdot, \cdot) \). At last, we obtain the estimation of the time-fractional derivatives of \( K_{\alpha,t}^L(\cdot, \cdot) \).
3.1. Regularities of the fractional heat kernel. We first investigate the regularities of $K_{\alpha,t}^L(\cdot, \cdot)$.

Proposition 3.1. Let $\alpha \in (0, 1)$. For every $N > 0$, there exists a constant $C_N$ such that

$$
|K_{\alpha,t}^L(x,y)| \leq \frac{C_N t}{(t^{1/2\alpha} + |x-y|^{n+2\alpha})^{1/2}} \left( 1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N}.
$$

Proof. By Proposition 2.7 we use (2.3), (2.4) and (2.5) to obtain

$$
|K_{\alpha,t}^L(x,y)| \leq \int_0^\infty \frac{t}{s^{1+\alpha}} |K_s^L(x,y)| ds
$$

Taking the change of variable $s = t^{1/\alpha} u$, we have

$$
|K_{\alpha,t}^L(x,y)| \leq \frac{C_N}{t^{1/\alpha}} \left( \frac{t^{1/2\alpha}}{\rho(x)} \right)^{-N} \left( \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N} \int_0^\infty \frac{1}{u^{1+\alpha}} u^{-n/2-N} e^{-\frac{u^{1/2\alpha}}{\rho(x)}} e^{-\frac{u^{1/2\alpha}}{\rho(y)}} du.
$$

Let $\frac{|x-y|^2}{t^{1/\alpha}} = r^2$. Then

$$
|K_{\alpha,t}^L(x,y)| \leq \frac{C_N}{t^{1/\alpha}} \left( \frac{t^{1/2\alpha}}{\rho(x)} \right)^{-N} \left( \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N} \int_0^\infty \frac{(x-y)^2}{t^{1/\alpha} r^2} e^{-r^2} [x-y]^2 ds.
$$

On the other hand, using the change of variables again, we obtain

$$
|K_{\alpha,t}^L(x,y)| \leq \frac{C_N}{t^{1/\alpha}} \left( \frac{t^{1/2\alpha}}{\rho(x)} \right)^{-N} \left( \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N} \int_0^\infty \frac{1}{u^{1+\alpha}} u^{-n/2-N} e^{-\frac{u^{1/2\alpha}}{\rho(x)}} e^{-\frac{u^{1/2\alpha}}{\rho(y)}} du.
$$

The above estimate implies that

$$
|K_{\alpha,t}^L(x,y)| \leq \frac{C_N}{t^{1/\alpha}} \left( \frac{t^{1/2\alpha}}{\rho(x)} \right)^{-N} \left( \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N} \int_0^\infty \frac{1}{u^{1+\alpha}} u^{-n/2-N} e^{-\frac{u^{1/2\alpha}}{\rho(x)}} e^{-\frac{u^{1/2\alpha}}{\rho(y)}} du.
$$

Now, combining (3.1) and (3.2), we have

$$
\left( \frac{t^{1/2\alpha}}{\rho(x)} \right)^N \left( \frac{t^{1/2\alpha}}{\rho(y)} \right)^N |K_{\alpha,t}^L(x,y)| \leq C_N \min \left\{ \frac{t^{1+N/\alpha}}{|x-y|^{n+2\alpha}}, \frac{t^{1/2\alpha}}{|x-y|^{n/2}} \right\},
$$

which, together with the arbitrariness of $N$, indicates that

$$
|K_{\alpha,t}^L(x,y)| \leq \frac{C_N}{(t^{1/2\alpha} + |x-y|^{n+2\alpha})^{1/2}} \left( 1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N}.
$$
This completes the proof of Proposition 3.1.

**Proposition 3.2.** Let $\alpha \in (0, 1)$. For any $N > 0$, there exists a constant $C_N > 0$ such that for every $0 < \delta' < \delta_0 = \min(1, 2 - n/q)$ and all $|h| \leq t^{1/\alpha}$,

$$|K_{r,t}^L(x + h, y) - K_{r,t}^L(x, y)| \leq C_N(|h|/t^{1/\alpha} \rho_t^N) \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N}.$$ 

**Proof.** The proof is similar to that of Proposition 3.1. By the subordinated formula (2.5), we can use Proposition 2.8 to get

$$\left|K_{r,t}^L(x + h, y) - K_{r,t}^L(x, y)\right| \leq C_N \int_0^\infty s^{-n/2} e^{-\epsilon s} s^{n/2+\epsilon} \left(|h|/\sqrt{s}\right)^{\delta'} s^{1/2} \left(1 + \frac{\sqrt{s}}{\rho(x)}\right)^{-N} s^{1/2} \left(1 + \frac{\sqrt{s}}{\rho(y)}\right)^{-N} ds$$

$$\leq C_N \left(\frac{|h|}{t^{1/2\alpha}}\right)^{\delta'} \left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N} \left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N} \frac{t^{1+\max(1/\alpha, 1)}}{|x - y|^{2\alpha + 2N + \delta'}},$$

which implies

$$\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^N \left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^N \left|K_{r,t}^L(x + h, y) - K_{r,t}^L(x, y)\right| \leq C_N t^{-n/2\alpha} |h|^{\delta'}.$$ 

On the other hand, letting $\tau = s/t^{1/\alpha}$, we have

$$\left|K_{r,t}^L(x + h, y) - K_{r,t}^L(x, y)\right| \leq C_N \int_0^\infty s^{-n/2} \frac{1}{t^{1/\alpha} \eta_t^N(s/t^{1/\alpha})} \left(|h|/\sqrt{s}\right)^{\delta'} \left(1 + \frac{\sqrt{s}}{\rho(x)}\right)^{-N} s^{1/2} \left(1 + \frac{\sqrt{s}}{\rho(y)}\right)^{-N} ds$$

$$\leq C_N \int_0^\infty (t^{1/2\alpha})^{-n/2} \eta_t^N(\tau) \left(|h|/\sqrt{t^{1/2\alpha}}\right)^{\delta'} \left(1 + \frac{\sqrt{t^{1/2\alpha}}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{t^{1/2\alpha}}}{\rho(y)}\right)^{-N} d\tau$$

$$\leq C_N \left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N} \left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N} \int_0^\infty \tau^{-n/2 - N - \delta'/2} \eta_t^N(\tau) d\tau.$$ 

This gives

$$\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^N \left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^N \left|K_{r,t}^L(x + h, y) - K_{r,t}^L(x, y)\right| \leq C_N t^{-n/2\alpha} \left(|h|/t^{1/2\alpha}\right)^{\delta'}.$$ 

The estimates (3.3) and (3.4) indicate that

$$\left(\frac{t^{1/2\alpha}}{\rho(x)}\right)^N \left(\frac{t^{1/2\alpha}}{\rho(y)}\right)^N \left|K_{r,t}^L(x + h, y) - K_{r,t}^L(x, y)\right| \leq C_N \min \left\{ t^{1+\max(1/\alpha, 1)} |h|^{\delta'}, t^{-n/2\alpha} \left(|h|/t^{1/2\alpha}\right)^{\delta'} \right\}.$$ 

Due to the arbitrariness of $N$, we have

$$\left|K_{r,t}^L(x + h, y) - K_{r,t}^L(x, y)\right| \leq C_N t^{1+\max(1/\alpha, 1)} \left(|h|/t^{1/2\alpha}\right)^{\delta'} \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(x)}\right)^{-N}.$$ 

This proves Proposition 3.2.

For $m \in \mathbb{Z}^+$ and $t > 0$, define $\overline{D}_{r,t}^m(\cdot, \cdot) = t^m \rho_t^m K_{r,t}^L(\cdot, \cdot)$. We can obtain the following estimates about the kernel $\overline{D}_{r,t}^m(\cdot, \cdot)$.

**Proposition 3.3.** Let $\alpha \in (0, 1)$, $m \in \mathbb{Z}_+$ and $\delta = \min(2\alpha, \delta_0)$, where $\delta_0$ appears in Proposition 2.8.

(i) For any $N > 0$, there exists a constant $C_N > 0$ such that

$$|\overline{D}_{r,t}^m(x, y)| \leq \frac{C_N t^m}{(t^{1/2\alpha} + |x - y|)^{n+2am}} \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)}\right)^{-N}.$$
(ii) Let $0 < \delta' \leq \delta$. For any $N > 0$, there exists a constant $C_N > 0$ such that for all $|h| \leq t^{1/2a}$,
\[
|\overline{D}_{a,t}^{L,m}(x+h,y) - \overline{D}_{a,t}^{L,m}(x,y)| \leq C_N \left( \frac{|h|}{t^{1/2a}} \right)^{\delta'} \frac{t^{2m}}{(t^{1/2a} + |x-y|)^{\nu+2m}} \left( 1 + \frac{t^{1/2a}}{\rho(x)} + \frac{t^{1/2a}}{\rho(y)} \right)^{-N}.
\]

(iii) Let $0 < \delta' \leq \delta$. For any $N > 0$, there exists a constant $C_N > 0$ such that
\[
|\int_{\mathbb{R}^n} \overline{D}_{a,t}^{L,m}(x,y)dy| \leq C_N \frac{(t^{1/2a}/\rho(x))^{\delta'}}{(1 + t^{1/2a}/\rho(x))^N}.
\]

Proof. For (i), since $\eta_0^a(s) = \frac{1}{\rho(x)} \eta^0_1(s/t^{1/2a})$, we have
\[
K_{a,t}^{L}(x,y) = \int_0^\infty \frac{1}{t^{1/2a}} \eta^0_1(s/t^{1/2a}) K_{\frac{t}{t^{1/2a}}}(x,y)ds = \int_0^\infty \eta^0_1(\tau) K^{L}_{1/\alpha}((x,y)d\tau).
\]

Hence
\[
\left| \frac{\partial^m}{\partial t^m} K_{a,t}^{L}(x,y) \right| \leq \left| \frac{\partial^m}{\partial t^m} \left( \int_0^\infty \eta^0_1(\tau) K_{\frac{t}{t^{1/2a}}}(x,y) d\tau \right) \right| \leq C_N t^{-m} \int_0^\infty \eta_1^0(\tau) \left( t^{1/2a} \right)^{-N/2} e^{-c|x-y|^2/t^{1/2a}} \left( 1 + \frac{\sqrt{t^{1/2a}}}{\rho(x)} \right)^{-N} \left( 1 + \frac{\sqrt{t^{1/2a}}}{\rho(y)} \right)^{-N} d\tau.
\]

Notice that $\eta_0^a(\tau) \leq C/\tau^{1+a}$. By changing of variables, we obtain
\[
\left| \frac{\partial^m}{\partial t^m} K_{a,t}^{L}(x,y) \right| \leq \frac{C_N}{m^{n/2a}} \left( \frac{t^{1/2a}}{\rho(x)} \right)^{-N/2} \left( \frac{t^{1/2a}}{\rho(y)} \right)^{-N} \int_0^\infty \tau^{1-a-n/2-N} e^{-c|x-y|^2/t^{1/2a}} d\tau \leq \frac{C_N}{|x-y|^{2n+a+2N}} \left( \frac{t^{1/2a}}{\rho(x)} \right)^{-N} \left( \frac{t^{1/2a}}{\rho(y)} \right)^{-N}.
\]

On the other hand,
\[
\left| \frac{\partial^m}{\partial t^m} K_{a,t}^{L}(x,y) \right| = \left| \int_0^\infty \eta_1^0(\tau) \frac{\partial^m}{\partial t^m} K_{\frac{t}{t^{1/2a}}}(x,y) d\tau \right| \leq \frac{C_N m^{n/2a}}{t^{1/2a}} \left( \frac{t^{1/2a}}{\rho(x)} \right)^{-N} \left( \frac{t^{1/2a}}{\rho(y)} \right)^{-N} \int_0^\infty \eta_1^0(\tau) (\tau^{1/\alpha})^{-n/2-N} d\tau \leq \frac{C_N}{m^{n/2a}} \left( \frac{t^{1/2a}}{\rho(x)} \right)^{-N} \left( \frac{t^{1/2a}}{\rho(y)} \right)^{-N}.
\]

Finally, we have proved that for arbitrary $N > 0$,
\[
\left( \frac{t^{1/2a}}{\rho(x)} \right)^N \left( \frac{t^{1/2a}}{\rho(y)} \right)^N \left| \frac{\partial^m}{\partial t^m} K_{a,t}^{L}(x,y) \right| \leq C_N \min \left( \frac{t^{1+N/\alpha-a}}{|x-y|^{2n+2N}}, \frac{1}{m^{n/2a}} \right),
\]

which gives
\[
|\overline{D}_{a,t}^{L,m}(x,y)| \leq \frac{C_N m^{n/2a}}{(t^{1/2a} + |x-y|)^{\nu+2m}} \left( 1 + \frac{t^{1/2a}}{\rho(x)} + \frac{t^{1/2a}}{\rho(y)} \right)^{-N}.
\]

For (ii), via the subordination formula (2.5), we can complete the proof by using (ii) of Proposition 2.9. We omit the details.

For (iii), it is easy to see that $e^{-cL_t^m}(f)(x) = \int_0^\infty \eta_1^0(\tau) e^{-\tau L_t^m}(f)(x)d\tau$. Hence
\[
\frac{\partial^m}{\partial t^m} K_{a,t}^{L}(x,y) = \frac{\partial^m}{\partial t^m} \left( \int_0^\infty \eta_1^0(\tau) K_{\frac{t}{t^{1/2a}}}(x,y) d\tau \right) = C_m \sum_{i=1}^{m} \int_0^\infty \eta_1^0(\tau) Q_{\frac{t}{t^{1/2a}}}(x,y) d\tau.
\]
It follows from (iii) of Proposition 2.9 that

\[
\left| \int_{\mathbb{R}^n} \tilde{D}_{\alpha,t}^m(x,y)dy \right| \leq \int_0^\infty \eta^\alpha_1(\tau) \left| \int_{\mathbb{R}^n} Q_{\sqrt{1/\alpha \tau}t}^L(x,y)dy \right| d\tau
\leq C_N \int_0^\infty \eta^\alpha_1(\tau) \left( \frac{\sqrt{1/\alpha \tau}/\rho(x)}{1 + \sqrt{1/\alpha \tau}/\rho(x)^N} \right)^{\delta'} d\tau.
\]

If \( t^{1/2} > \rho(x) \), then

\[
\left| \int_{\mathbb{R}^n} \tilde{D}_{\alpha,t}^m(x,y)dy \right| \leq C_N \rho(x)^N \delta' t^{(\delta'-N)/2} \int_0^\infty \eta^\alpha_1(\tau) (\sqrt{1/\alpha \tau}/\rho(x))^{\delta'} d\tau
\leq \frac{C_N(t^{1/2}/\rho(x))^{\delta'}}{(1 + t^{1/2}/\rho(x))^N}.
\]

If \( t^{1/2} \leq \rho(x) \), then

\[
\left| \int_{\mathbb{R}^n} \tilde{D}_{\alpha,t}^m(x,y)dy \right| \leq C_N \int_0^\infty \eta^\alpha_1(\tau) \left( \frac{\sqrt{1/\alpha \tau}/\rho(x)}{1 + \sqrt{1/\alpha \tau}/\rho(x)^N} \right)^{\delta'} d\tau
\leq \frac{C_N(t^{1/2}/\rho(x))^{\delta'}}{(1 + t^{1/2}/\rho(x))^N}.
\]

\[ \square \]

3.2. Estimation on the spatial gradient. In this section, we investigate the spatial gradient of \( K_{\alpha,t}^L(\cdot, \cdot) \), \( \alpha > 0 \). For the special case \( \alpha = 1/2 \), i.e., the Poisson kernel, the regularity estimates have been obtained in \([5\) Lemma 3.9\(]\)

**Lemma 3.4.** Suppose that \( V \in B_q \) for some \( q > n \). For every \( N > 0 \), there exist constants \( C_N > 0 \) and \( c > 0 \) such that for all \( x,y \in \mathbb{R}^n \) and \( t > 0 \), the kernels \( K_t^L(\cdot, \cdot) \) satisfy the following estimates:

\[
|\nabla_x K_t^L(x,y)| \leq \begin{cases} 
\frac{C_N}{t^{(\alpha+1)/2}} e^{-c|x-y|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}, & \sqrt{t} \leq |x-y|; \\
\frac{C_N}{|x-y|^n/2} e^{-c|x-y|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}, & \sqrt{t} > |x-y|.
\end{cases}
\]

**Proof.** Let \( \Gamma_0(\cdot, \cdot) \) be the fundamental solution of \( -\Delta \) in \( \mathbb{R}^n \), i.e.,

\[
\Gamma_0(x,y) = \frac{1}{n(n-2)\omega(n)} \frac{1}{|x-y|^{n-2}}, \quad n \geq 3,
\]

where \( \omega(n) \) denotes the area of the unit sphere in \( \mathbb{R}^n \). Fix \( t > 0 \) and \( x_0, y_0 \in \mathbb{R}^n \). Assume that \( u(\cdot, \cdot) \) is a solution to the equation

\[
\partial_t u + Lu = \partial_t u + (-\Delta) u + Vu = 0.
\]

Let \( \eta \in C_0^\infty(B(x_0, 2R)) \) with some \( R > 0 \) such that \( \eta = 1 \) on \( B(x_0, 3R/2) \), \( |\nabla \eta| \leq C/R \) and \( |\nabla^2 \eta| \leq C/R^2 \). Noticing that \( \partial_t u + Lu = 0 \), we can obtain

\[
-\Delta(\eta u) = -\sum_{i=1}^n \left( \frac{\partial^2 u}{\partial x_i^2} \cdot \eta + 2 \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_i} + u \cdot \frac{\partial^2 \eta}{\partial x_i^2} \right)
= -\Delta u \cdot \eta - 2\nabla u \cdot \nabla \eta - u \cdot \Delta \eta
= -\partial_t(u \eta) - Vu \eta - 2 \nabla u \nabla \eta - u \nabla \eta,
\]

which, together with integration by parts, gives

\[-\int_{\mathbb{R}^n} \Gamma_0(x, y) \nabla u(y, t) \cdot \nabla \eta(y) dy = -\sum_{i=1}^{n} \int_{\mathbb{R}^n} \Gamma_0(x, y) \frac{\partial u(y, t)}{\partial y_i} \frac{\partial \eta(y)}{\partial y_i} dy\]

\[= \sum_{i=1}^{n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial y_i} \Gamma_0(x, y) \right) \frac{\partial \eta(y)}{\partial y_i} u(y, t) dy + \sum_{i=1}^{n} \int_{\mathbb{R}^n} \Gamma_0(x, y) \left( \frac{\partial^2 \eta(y)}{\partial x_i^2} \right) u(y, t) dy\]

\[= \int_{\mathbb{R}^n} \nabla \eta \cdot \nabla \eta(y) u(y, t) dy + \int_{\mathbb{R}^n} \nabla \eta \cdot \Delta \eta(y) u(y, t) dy.\]

Then we can get

\[u(x, t) \eta(x) = \int_{\mathbb{R}^n} \Gamma_0(x, y) \left( -V(y) u(y, t) \eta(y) - \eta(y) \partial_t u(y, t) - 2 \nabla u(y, t) \cdot \nabla \eta(y) - u(y, t) \Delta \eta(y) \right) dy\]

\[= \int_{\mathbb{R}^n} \Gamma_0(x, y) \left( -V(y) u(y, t) \eta(y) - \eta(y) \partial_t u(y, t) + \Delta \eta(y) \cdot u(y, t) \right) dy\]

\[+ 2 \int_{\mathbb{R}^n} \nabla \eta \cdot \nabla \eta(y) u(y, t) dy.\]

Notice that it follows from Lemma [2.5] that

\[\int_{B(x_0, 2R)} \frac{|V(y)|}{|x-y|^{n-1}} dy \leq \frac{C}{R^{n-1}} \int_{B(x_0, 2R)} V(y) dy \leq \frac{C}{R} \left( 1 + \frac{R}{\rho(x_0)} \right)^{m_0}, \quad m_0 > 1.\]

Thus for \(x \in B(x_0, R)\), it holds

\[|\nabla x u(x, t)| \leq \int_{B(x_0, 2R)} \frac{V(y)|u(y, t)||\eta(y)|}{|x-y|^{n-1}} dy + \int_{B(x_0, 2R)} \frac{|\partial_t u(y, t)||\eta(y)|}{|x-y|^{n-1}} dy + \int_{B(x_0, 2R)} \frac{|u(y, t)|}{|x-y|^{n-1}} dy\]

\[\leq \frac{C}{R} \sup_{B(x_0, 2R)} |u(y, t)| \left( 1 + \frac{R}{\rho(x_0)} \right)^{m_0} + CR \sup_{B(x_0, R)} |\partial_t u(y, t)|.\]

Take \(u(x, t) = K^L_t(x, y_0)\) and \(R < \min(|x_0 - y_0|/8, \rho(x_0))\). We obtain that

\[|\nabla_x K^L_t(x_0, y_0)| \leq \frac{C}{R} \sup_{B(x_0, 2R)} |K^L_t(x, y_0)| \left( 1 + \frac{R}{\rho(x_0)} \right)^{m_0} + R \sup_{B(x_0, 2R)} |\partial_t K^L_t(x, y_0)|.\]

If \(x \in B(x_0, 2R)\), then \(|x - y_0| \sim |x_0 - y_0|\). Also \(\rho(x) \sim \rho(x_0)\) for \(|x - x_0| < 2R < 2\rho(x_0)\). It follows from Propositions [2.7] and [2.9] that for any \(N > 0\) there exists a constant \(C_N\) such that

\[\sup_{x \in B(x_0, 2R)} |K^L_t(x, y_0)| \leq \frac{C_N}{R^N} e^{-t|x_0-y_0|^2/|t|} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N};\]

\[\sup_{x \in B(x_0, 2R)} |\partial_t K^L_t(x, y_0)| \leq \frac{C_N}{R^N} e^{-t|x_0-y_0|^2/|t|} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N}.\]

Finally, it can be deduced from (3.5) that

\[|\nabla_x K^L_t(x_0, y_0)| \leq \frac{C_N}{R} t^{-n/2} e^{-t|x_0-y_0|^2/|t|} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N} \left( 1 + \frac{R^2}{t} \right).\]

The rest of the proof is divided into three cases.

**Case 1: \(R > \sqrt{t}\).** For this case, \(\sqrt{t} < R < \min(|x_0 - y_0|/8, \rho(x_0))\). We split

\[|\nabla_x K^L_t(x_0, y_0)| \leq C_N(M_1 + M_2),\]
where

\[
\begin{align*}
M_1 &:= \frac{\sqrt{t}}{R^{n+1}/2} e^{-c|x_0-y_0|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N}; \\
M_2 &:= \frac{\sqrt{t}}{R^{n+1}/2} e^{-c|x_0-y_0|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N} \frac{R^2}{t}.
\end{align*}
\]

It is obvious that

\[
M_1 \leq \frac{1}{R^{n+1}/2} e^{-c|x_0-y_0|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N}.
\]

Similarly, for the term \(M_3\), we can also get

\[
M_2 \leq \frac{1}{R^{n+1}/2} e^{-c|x_0-y_0|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N} \frac{R^2}{t}.
\]

\[\begin{align*}
\text{Case 2: } 0 < R &\leq \sqrt{t} < \min(|x_0 - y_0|/8, \rho(x_0)). \text{ We write} \\
|\nabla_x K^L_t(x_0, y_0)| &\leq \frac{C_N}{R^{n+1}/2} e^{-c|x_0-y_0|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N} \left(\frac{\sqrt{t}}{R} + \frac{R}{\sqrt{t}}\right).
\end{align*}\]

Because \(R < \sqrt{t} < \min(|x_0 - y_0|/8, \rho(x_0))\), taking the infimum for \(R\) yields

\[|\nabla_x K^L_t(x_0, y_0)| \leq \frac{C_N}{R^{n+1}/2} e^{-c|x_0-y_0|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N}.
\]

\[\begin{align*}
\text{Case 3: } 0 < R < \min(|x_0 - y_0|/8, \rho(x_0)) < \sqrt{t}. \text{ Similarly, we can see that} \\
|\nabla_x K^L_t(x_0, y_0)| &\leq \left(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N} \left(\frac{\sqrt{t}}{R} + \frac{R}{\sqrt{t}}\right).
\end{align*}\]

Since \(0 < R < \min(|x_0 - y_0|/8, \rho(x_0)) < \sqrt{t}\), the function \(\sqrt{t}/R + R/\sqrt{t}\) is decreasing and with the infimum at \(R = \min(|x_0 - y_0|/8, \rho(x_0))\). Then

\[|\nabla_x K^L_t(x_0, y_0)| \leq \frac{C_N}{R^{n+1}/2} e^{-c|x_0-y_0|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N} \times \left\{ \frac{\sqrt{t}}{\min(|x_0 - y_0|/8, \rho(x_0))} + \frac{\min(|x_0 - y_0|/8, \rho(x_0))}{\sqrt{t}} \right\}.
\]

\[\text{Case 3.1: } \rho(x_0) \leq |x_0 - y_0|/8. \text{ Since } N \text{ is arbitrary, we can deduce from (3.6) that} \\
|\nabla_x K^L_t(x_0, y_0)| &\leq \frac{C_N}{R^{n+1}/2} e^{-c|x_0-y_0|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N} \left(\frac{\sqrt{t}}{\rho(x_0)} + \frac{\rho(x_0)}{\sqrt{t}}\right) \\
&\leq \frac{C_N}{R^{n+1}/2} e^{-c|x_0-y_0|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N}.
\]
Case 3.2: $\rho(x_0) > |x_0 - y_0|/8$. For this case, by (3.6) again, it holds
\[
|\nabla_x K^L_t(x_0, y_0)| \leq \frac{C_N}{t^{(n+1)/2}} e^{-c|x_0-\gamma_0|^2/t} \left( \frac{1}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^N \left( \frac{\sqrt{t}}{|x_0 - y_0|} + \frac{1}{\sqrt{t}} \right).
\]
\[
\leq \frac{C_N}{t^{(n+1)/2}} e^{-c|x_0-\gamma_0|^2/t} \left( \frac{1}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^N \frac{C_N}{|x_0 - y_0|^{n/2}} e^{-c|x_0-\gamma_0|^2/t} \left( \frac{1}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^N.
\]
Finally, we obtain the following estimates:
\[
|\nabla_x K^L_t(x_0, y_0)| \leq \begin{cases} \frac{C_N}{t^{(n+1)/2}} e^{-c|x_0-\gamma_0|^2/t} \left( \frac{1}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^N, & \sqrt{t} \leq \min \left\{ \frac{|x_0 - y_0|}{8}, \rho(x_0) \right\}; \\
\frac{C_N}{t^{(n+1)/2}} e^{-c|x_0-\gamma_0|^2/t} \frac{1}{\sqrt{t}} + \frac{1}{|x_0 - y_0|} \left( \frac{1}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^N, & \sqrt{t} > \min \left\{ \frac{|x_0 - y_0|}{8}, \rho(x_0) \right\}. \end{cases}
\]
Then if $\sqrt{t} \geq |x_0 - y_0|/8$,
\[
|\nabla_x K^L_t(x_0, y_0)| \leq \frac{C_N}{|x_0 - y_0|^{n/2}} e^{-c|x_0-\gamma_0|^2/t} \left( \frac{1}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^N.
\]
This proves Lemma 3.4. 

Lemma 3.5. Suppose that $V \in B_q$ for some $q > n$. For every $N > 0$, there exists a constant $C_N > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$, the semigroup kernels $K^L_t(\cdot, \cdot)$ satisfy the following estimate:
\[
|\nabla_x K^L_t(x, y)| \leq \frac{C_N}{t^{(n+1)/2}} \left( \frac{1}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^N.
\]

Proof. Assume that $u(\cdot, \cdot)$ is a solution of the equation
\[
\partial_t u + Lu = \partial_x u + (-\Delta)u + Vu = 0.
\]
Similar to Lemma 3.4, we can prove that
\[
|\nabla_x u(x, t)| \leq \frac{C}{R} \sup_{B(x_0, 2R)} |u(y, t)| \left\{ \left( 1 + \frac{R}{\rho(x_0)} \right)^{n_0} + 1 \right\} + CR \sup_{B(x_0, R)} |\partial_x u(\gamma_0, t)|.
\]
Take $u(x, t) = K^L_t(x, y_0)$ for fixed $y_0$, and let $R \in (0, \min\{\rho(x_0), \sqrt{t}\})$. It can be deduced from Propositions 2.7 and 2.9 that
\[
\sup_{x \in B(x_0, 2R)} \left\{ |K^L_t(x, y_0)| + |\partial_x K^L_t(x, y_0)| \right\} \leq \frac{C_N}{t^{(n+1)/2}} \left( \frac{1}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^N.
\]
This, together with $R < \rho(x_0)$, implies that
\[
|\nabla_x K^L_t(x_0, y_0)| \leq \frac{C_N}{R^{n/2}} \left( \frac{1}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^N \left\{ 1 + \frac{R^2}{\sqrt{t}} \right\}
\]
\[
\leq \frac{C_N}{R^{n+1/2}} \left( \frac{1}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^N \left( \frac{\sqrt{t}}{R} + \frac{R}{\sqrt{t}} \right).
\]
If $\sqrt{t} \leq \rho(x_0)$, taking the infimum for $R$ on both sides of (3.7) reaches
\[
|\nabla_x K^L_t(x_0, y_0)| \leq \frac{C_N}{t^{(n+1)/2}} \left( \frac{1}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^N.
\]
If $\sqrt{t} > \rho(x_0)$, note that the function $h(t) := R/\sqrt{t} + \sqrt{t}/R$ is decreasing on $R \in (0, \rho(x_0))$. Taking the infimum again, we get

$$|\nabla_x K^L_{\alpha,f}(x_0, y_0)| \leq \frac{C_N}{f^{(n+1)/2}} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^{\alpha} \left( \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N}.$$ 

This completes the proof of Lemma 3.5.$\Box$

Now we give the gradient estimate of $K^L_{\alpha,f}(\cdot, \cdot)$.

**Proposition 3.6.** Suppose $\alpha > 0$ and $V \in B_q$ for some $q > n$. For every $N > 0$, there exists a constant $C_N > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,

$$|t^{1/2\alpha} \nabla_x K^L_{\alpha,f}(x, y)| \leq \frac{C_N t}{(t^{1/2\alpha} + |x - y|^{n+2\alpha}) \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{\alpha}} $$

for every $N > 0$, there exists a constant $C_N > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,

$$|t^{1/2\alpha} \nabla_x K^L_{\alpha,f}(x, y)| \leq \frac{C_N t}{(t^{1/2\alpha} + |x - y|^{n+2\alpha}) \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{\alpha}} $$

**Proof.** The subordinate formula gives

$$\nabla_x K^L_{\alpha,f}(x, y) = \int_0^{\infty} \eta^\alpha_t(s) \nabla_x K^L_{\alpha,f}(x, y) ds,$$

which, together with Lemma 3.4, implies that $|\nabla_x K^L_{\alpha,f}(x, y)| \leq C_N(L_1 + L_2)$, where

$$L_1 := \int_0^{\infty} \eta^\alpha_t(s) \frac{1}{s^{(n+1)/2}} e^{-c|x-y|^2/s} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} ds;$$

$$L_2 := \int_0^{\infty} \eta^\alpha_t(s) \frac{1}{s^{n/2}|x - y|^2} e^{-c|x-y|^2/s} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} ds.$$ 

For $L_1$, letting $s = t^{1/\alpha} u$, we can get

$$L_1 \leq \int_0^{\infty} \frac{t}{(t^{1/\alpha} u)^{1+\alpha}} (t^{1/\alpha} u)^{-n/2} e^{-\frac{c|x-y|^2}{t^{1/\alpha} u}} \left( 1 + \frac{t^{1/2\alpha} \sqrt{u}}{\rho(x)} \right)^{-N} \left( 1 + \frac{t^{1/2\alpha} \sqrt{u}}{\rho(y)} \right)^{-N} t^{1/\alpha} du$$

Similarly, for the term $L_2$, a change of variables yields

$$L_2 \leq \frac{1}{|x - y|} \int_0^{\infty} \frac{t}{(t^{1/\alpha} u)^{1+\alpha}} (t^{1/\alpha} u)^{-n/2} e^{-\frac{c|x-y|^2}{t^{1/\alpha} u}} \left( 1 + \frac{t^{1/2\alpha} \sqrt{u}}{\rho(x)} \right)^{-N} \left( 1 + \frac{t^{1/2\alpha} \sqrt{u}}{\rho(y)} \right)^{-N} t^{1/\alpha} du$$

The estimates for $L_1$ and $L_2$ indicate that

$$\frac{(t^{1/2\alpha})^N (1 + t^{1/2\alpha})^N}{|x - y|^{2\alpha+2N+n+1}}.$$
On the other hand, by Lemma 3.5 and changing variable \( \tau = s/t^{1/\alpha} \), we obtain
\[
|\nabla_x K^L_{\alpha,t}(x,y)| \leq C_N \int_0^{\infty} s^{-(n+1)/2} \frac{1}{t^{1/\alpha}} \eta^\alpha(s) \left( 1 + \frac{s}{\rho(x)} \right)^{-N} \left( 1 + \frac{\sqrt{s}}{\rho(y)} \right)^{-N} ds
\]
\[
\leq C_N \int_0^{\infty} (t^{1/\alpha} \tau)^{-(n+1)/2} \eta^\alpha(\tau) \left( 1 + \frac{t^{1/2\alpha} \sqrt{\tau}}{\rho(x)} \right)^{-N} \left( 1 + \frac{t^{1/2\alpha} \sqrt{\tau}}{\rho(y)} \right)^{-N} d\tau
\]
\[
\leq C_N \frac{t^{1/2\alpha}}{t^{(n+1)/2\alpha}} \left( \frac{t^{1/2\alpha}}{\rho(x)} \right)^{-N} \left( \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N}.
\]

Finally, we obtain
\[
\left( \frac{t^{1/2\alpha}}{\rho(x)} \right)^N \left( \frac{t^{1/2\alpha}}{\rho(y)} \right)^N |\nabla_x K^L_{\alpha,t}(x,y)| \leq C_N \min \left\{ t^{-(n+1)/2\alpha}, \frac{t^{1+N/\alpha}}{|x-y|^{n+1+2\alpha}} \right\}.
\]
The arbitrariness of \( N \) indicates that
\[
|\nabla_x K^L_{\alpha,t}(x,y)| \leq C_N \frac{t}{t^{1/2\alpha} (t^{1/2\alpha} + |x-y|^{n+2\alpha})} \left( 1 + \frac{t^{1/2\alpha}}{\rho(x)} \right)^{-N} \left( 1 + \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N}.
\]

\[\square\]

Below we estimate the Lipschitz continuity of \( |\nabla_x K^L_{\alpha}(\cdot, \cdot)| \).

**Lemma 3.7.** Suppose that \( \alpha > 0 \) and \( V \in B_q \) for some \( q > n \). Let \( \delta' = 1 - n/q \). For every \( N > 0 \), there exist constants \( C_N > 0 \) and \( c > 0 \) such that for all \( x, y \in \mathbb{R}^n, t > 0 \) and \( |h| < |x-y|/4 \),
\[
|\nabla_x K^L_{\alpha}(x+h,y) - \nabla_x K^L_{\alpha}(x,y)|
\]
\[
\leq \left\{ \begin{array}{ll}
C_N \frac{|h|^{\delta'}}{\sqrt{\tau}} e^{-c|y-x|^2/\tau} \left( 1 + \frac{\sqrt{\tau}}{\rho(x)} + \frac{\sqrt{\tau}}{\rho(y)} \right)^{-N}, & \sqrt{\tau} \leq |x-y|; \\
C_N \frac{|h|^{\delta'}}{|x-y|^{n/2}} \left( 1 + \frac{\sqrt{\tau}}{\rho(x)} + \frac{\sqrt{\tau}}{\rho(y)} \right)^{-N}, & \sqrt{\tau} \geq |x-y|.
\end{array} \right.
\]

**Proof.** The proof is similar to that of Lemma 3.4. Let \( \Gamma_0(\cdot, \cdot) \) be the fundamental solution of \(-\Delta\) in \( \mathbb{R}^n \). Assume that \( \partial_t u + (-\Delta) u + Vu = 0 \). Let \( \eta \in C_0^\infty(B(x_0, 2R)) \) such that \( \eta = 1 \) on \( \bar{B}(x_0, 3R/2) \), \( |\nabla \eta| \leq C/R \) and \( |\nabla^2 \eta| \leq C/R^2 \). It is easy to see that
\[
(-\Delta)(u\eta) = (-\Delta)u\eta - 2\nabla u \cdot \nabla \eta + u \cdot (-\Delta \eta).
\]

Similar to the proof of Lemma 3.4 an integration by parts implies that
\[- \int_{\mathbb{R}^n} \Gamma_0(x,y) \nabla u(y,t) \cdot \nabla \eta(y)dy = \int_{\mathbb{R}^n} \nabla_x \Gamma_0(x,y) \nabla \eta(y)u(y,t)dy + \int_{\mathbb{R}^n} \Gamma_0(x,y) \Delta \eta(y)u(y,t)dy,\]

which yields
\[
u(x,t)\eta(x) = \int_{\mathbb{R}^n} \Gamma_0(x,y) \left[ (-\partial_t u(y,t))\eta(y) - V(y)u(y,t)\eta(y) + u(y,t)\Delta \eta(y) \right]dy
\]
\[+ 2\int_{\mathbb{R}^n} \nabla_x \Gamma_0(x,y)u(y,t) \cdot \nabla \eta(y)dy.\]

Then for \( x \in B(x_0, R), u(x,t) = u(x,t)\eta(x) \) and
\[
\nabla^2_x u(x,t) = \nabla^2_x \left\{ \int_{\mathbb{R}^n} \Gamma_0(x,y) \left[ (-\partial_t u(y,t))\eta(y) - V(y)u(y,t)\eta(y) + u(y,t)\Delta \eta(y) \right]dy
\]
\[+ 2\int_{\mathbb{R}^n} \nabla_x \Gamma_0(x,y)u(y,t) \cdot \nabla \eta(y)dy,\]
which gives \( \|\nabla^2_x u(\cdot, t)\|_q \leq \sum_{i=1}^{4} \|S_i(\cdot, t)\|_q \), where

\[
\begin{align*}
S_1(x, t) &:= \int_{\mathbb{R}^n} |\nabla^2_x \Gamma_0(x, y)| \cdot |\eta(y)| \cdot |\partial_1 u(y, t)| dy; \\
S_2(x, t) &:= \int_{\mathbb{R}^n} |\nabla^2_x \Gamma_0(x, y)| \cdot |\Delta \eta(y)| \cdot |u(y, t)| dy; \\
S_3(x, t) &:= \int_{\mathbb{R}^n} |\nabla^2_x \nabla_y \Gamma_0(x, y)| \cdot |\nabla \eta(y)| \cdot |u(y, t)| dy; \\
S_4(x, t) &:= \int_{\mathbb{R}^n} |\nabla^2_x \Gamma_0(x, y)| \cdot |\eta(y)| |V(y)| \cdot |u(y, t)| dy.
\end{align*}
\]

Now we estimate the terms \( \|S_i(\cdot, t)\|_q, i = 1, 2, 3, 4 \), separately. For the term \( \|S_1(\cdot, t)\|_q \), because \( \nabla^2_x \Gamma_0(\cdot, \cdot) \) is a Calderón-Zygmund kernel, we have

\[
\|S_1(\cdot, t)\|_q = \left\| \int_{\mathbb{R}^n} |\eta(y)| |\partial_1 u(y, t)| |\nabla^2_x \Gamma_0(\cdot, y)| dy \right\|_q \\
\leq \sup_{y \in B(x_0, 2R)} |\partial_1 u(y, t)| \left\| \int_{\mathbb{R}^n} |\eta(y)| |\nabla^2_x \Gamma_0(\cdot, y)| dy \right\|_q \\
\leq \|\eta\|_q \sup_{y \in B(x_0, 2R)} |\partial_1 u(y, t)|.
\]

The estimate of \( S_2 \) is similar to that of \( S_1 \). Noting that \( \eta = 1 \) on \( B(x_0, 3R/2) \), we can get

\[
S_2(x, t) \leq \int_{B(x_0, 2R) \setminus B(x_0, 3R/2)} \frac{|u(y, t)| |\Delta \eta(y)|}{|x - y|^n} dy \lesssim \sup_{y \in B(x_0, 2R)} |u(y, t)| \int_{B(x_0, 2R) \setminus B(x_0, 3R/2)} \frac{|\Delta \eta(y)|}{|x - y|^n} dy.
\]

For \( 3R/2 < |y - x_0| < 2R \) and \( x \in B(x_0, R) \), a direct computation gives \( |x - y| \sim R \) and

\[
S_2(x, t) \leq \sup_{y \in B(x_0, 2R)} |u(y, t)| \left\| \int_{B(x_0, 2R) \setminus B(x_0, 3R/2)} \frac{|\Delta \eta(y)|}{|x - y|^n} dy \right\|_q \\
\leq \sup_{y \in B(x_0, 2R)} |u(y, t)| \left\| \int_{B(x_0, 2R) \setminus B(x_0, 3R/2)} \frac{|\Delta \eta(y)|^q}{|x - y|^{qn+1}} dy \right\|^{1/q} \\
\leq R^{n/q+2} \sup_{y \in B(x_0, 2R)} |u(y, t)|.
\]

Following the same procedure, we apply the Young inequality to obtain

\[
\|S_3(\cdot, t)\|_q \leq \left\| \int_{B(x_0, 2R) \setminus B(x_0, 3R/2)} |\nabla^2_x \nabla_y \Gamma_0(\cdot, y)| \cdot |\nabla \eta(y)| dy \right\|_q \\
\leq \left\| \int_{B(x_0, 2R) \setminus B(x_0, 3R/2)} |\nabla^2_x \nabla_y \Gamma_0(\cdot, y)| \cdot |\nabla \eta(y)| dy \right\|_1 \\
\leq \left\| \sup_{y \in B(x_0, 2R)} |u(y, t)| |\nabla \eta(y)| \right\|_q \leq R^{n/q-2} \sup_{y \in B(x_0, 2R)} |u(y, t)|.
\]
At last, for the term $S_4$, by Lemma 2.5 and the condition $V \in B_q$, we can obtain, via the $L^p$-boundedness of the operator with the kernel $\nabla_2^2 \Gamma_q(\cdot, \cdot)$, that

\[
|S_4(x, \cdot)|_{L^q} \leq \left\| V(y) |\eta(y)\|_{L^q} \right\|
\]

\[
\leq \left\{ \sup_{y \in B(x_0, 2R)} |u(y, t)| \right\} \left( \int_{B(x_0, 2R)} V^q(y) dy \right)^{1/q}
\]

\[
\leq \left\{ \sup_{y \in B(x_0, 2R)} |u(y, t)| \right\} R^{n/q} \frac{1}{|B(x_0, R)|} \int_{B(x_0, 2R)} V^q(y) dy
\]

\[
\leq \left\{ \sup_{y \in B(x_0, 2R)} |u(y, t)| \right\} R^{n/q-1} \frac{1}{R^n} \int_{B(x_0, 2R)} V(y) dy
\]

\[
\leq \left\{ \sup_{y \in B(x_0, 2R)} |u(y, t)| \right\} R^{n/q-2} \left( 1 + \frac{R}{\rho(x_0)} \right)^{m_0}.
\]

The estimates for $|S_i(x, t)|_{L^q}, i = 1, 2, 3, 4$, indicate that

\[
|\nabla_2^2 u(x, t)| \leq R^{n/q-2} \left( \left( 1 + \frac{R}{\rho(x_0)} \right)^{m_0} + 1 \right) \left\{ \sup_{y \in B(x_0, 2R)} |u(y, t)| \right\} + R^{n/q} \left\{ \sup_{y \in B(x_0, 2R)} |\partial_t u(y, t)| \right\}.
\]

Let $u(x_0, t) = K^L_{t}(x_0, y_0)$. Then

\[
|\nabla_x K^L_{t}(x_0 + h, y_0) - \nabla_x K^L_{t}(x_0, y_0)|
\]

\[
\leq \left\{ \frac{|h|}{\sqrt{t}} \right\}^{1-n/q} \left\{ \int_{B(x_0, R)} \left| \nabla_2^2 K^L_{t}(x, y_0) \right|^q dx \right\}^{1/q}
\]

\[
\leq \left\{ \frac{|h|}{\sqrt{t}} \right\}^{1-n/q} \left\{ \left( 1 + \frac{R}{\rho(x_0)} \right)^{m_0} + 1 \right\} \left\{ \sup_{x \in B(x_0, 2R)} |K^L_{t}(x, y_0)| + \frac{R^2}{t} \sup_{y \in B(x_0, 2R)} |\partial_t K^L_{t}(x, y_0)| \right\}
\]

\[
\leq \frac{C_N}{R} \left\{ \frac{|h|}{\sqrt{t}} \right\}^{1-n/q} \left\{ \left( 1 + \frac{R}{\rho(x_0)} \right)^{m_0} + 1 \right\} \left\{ \sup_{x \in B(x_0, 2R)} \frac{1}{n^{1/2}} e^{-(x-y)^2/t} \right\} \left\{ \frac{1}{n^{1/2}} e^{-(x-y)^2/t} \right\} \left\{ 1 + \sqrt{\frac{t}{\rho(x^2)}} \right\}^{-N} \left\{ 1 + \sqrt{\frac{t}{\rho(y^2)}} \right\}^{-N}
\]

\[
= \frac{C_N}{R} \left\{ \frac{|h|}{\sqrt{t}} \right\}^{1-n/q} \left\{ \left( 1 + \frac{R}{\rho(x_0)} \right)^{m_0} + 1 \right\} \left\{ \sup_{x \in B(x_0, 2R)} \frac{1}{n^{1/2}} e^{-(x-y)^2/t} \right\} \left\{ 1 + \sqrt{\frac{t}{\rho(x^2)}} \right\}^{-N} \left\{ 1 + \sqrt{\frac{t}{\rho(y^2)}} \right\}^{-N}.
\]

Take $0 < R < \min[\rho(x_0), |x_0 - y_0|/8]$. If $x \in B(x_0, 2R)$, then $|x - x_0| < 2 \rho(x_0)$, that is, $\rho(x) \sim \rho(x_0)$. Also, if $x \in B(x_0, 2R)$, $|x - x_0| < 2R < |x_0 - y_0|/4$, which means that $|x - y_0| \sim |x_0 - y_0|$. We can get

\[
(3.8) \quad |\nabla_x K^L_{t}(x_0 + h, y_0) - \nabla_x K^L_{t}(x_0, y_0)|
\]

\[
\leq \frac{C_N}{n^{(n+1)/2}} \left\{ \frac{|h|}{\sqrt{t}} \right\}^{1-n/q} \left\{ \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N} \left( \frac{\sqrt{t}}{n^{1/2}} \right)^{1-n/q} \left( \frac{\sqrt{t}}{R^n} + \frac{R}{\sqrt{t}} \right). \right.
\]

Define a function $F(x) = x^{1-n/q}(x + 1/x), x > 0$. Then we can see that for $x > \sqrt{n/(2q - n)}$, $F'(x) > 0$, i.e., $F$ is increasing, which means that the function $f(R) := F(\sqrt{t}/R)$ is decreasing for $R \in (0, \sqrt{(2q - n)/n})$. Below we divide the rest of the proof into two cases.

Case 1: $0 < R < \min[\rho(x_0), |x_0 - y_0|] < \sqrt{t}$. We further divide the discussion into two subcases.

Case 1.1: $0 < R < \rho(x_0) < \sqrt{t} < \sqrt{(2q - n)/n}$, i.e., $\rho(x_0) \leq |x_0 - y_0|$. Taking the infimum on both sides of (3.8), we can get

\[
\left( \frac{\sqrt{t}}{R} \right)^{1-n/q} \left( \frac{\sqrt{t}}{R^n} + \frac{R}{\sqrt{t}} \right) \leq \left( \frac{\sqrt{t}}{\rho(x_0)} \right)^{1-n/q} \left( \frac{\sqrt{t}}{\rho(x_0)} + \frac{\rho(x_0)}{\sqrt{t}} \right) \leq \left( \frac{\sqrt{t}}{\rho(x_0)} \right)^{1-n/q} \left( \frac{\sqrt{t}}{\rho(x_0)} + 1 \right),
\]
where we have used the fact that $\rho(x_0) < \sqrt{t}$. This gives
\[
|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\
\leq \frac{C_N}{\rho(x_0)^{n+1/2}} \left( \frac{|h|}{\sqrt{t}} \right)^{1-n/q} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N} \left( \frac{\sqrt{t}}{\rho(x_0)} \right)^{1-n/q} \left( \frac{\sqrt{t}}{\rho(x_0)} + 1 \right) \\
\leq \frac{C_N}{\rho(x_0)^{n+1/2}} \left( \frac{|h|}{\sqrt{t}} \right)^{1-n/q} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N} \left( \frac{\sqrt{t}}{\rho(x_0)} \right)^{1-n/q} \left( \frac{\sqrt{t}}{\rho(x_0)} + 1 \right).
\]

Case 1.2: $0 < R < |x_0 - y_0| < \sqrt{t}$, i.e., $\rho(x_0) > |x_0 - y_0|$. By taking the infimum on $f(R)$, we can get
\[
\left( \frac{\sqrt{t}}{R} \right)^{1-n/q} \left( \frac{\sqrt{t}}{R} + \frac{R}{\sqrt{t}} \right) \leq \left( \frac{\sqrt{t}}{|x_0 - y_0|} \right)^{1-n/q} \left( \frac{\sqrt{t}}{|x_0 - y_0|} + \frac{|x_0 - y_0|}{\sqrt{t}} \right) \\
\leq \left( \frac{\sqrt{t}}{|x_0 - y_0|} \right)^{1-n/q} \left( \frac{\sqrt{t}}{|x_0 - y_0|} + 1 \right).
\]

Then we obtain
\[
|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\
\leq \frac{C_N}{\rho(x_0)^{n+1/2}} \left( \frac{|h|}{\sqrt{t}} \right)^{1-n/q} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N} \left( \frac{\sqrt{t}}{|x_0 - y_0|} \right)^{1-n/q} \left( \frac{\sqrt{t}}{|x_0 - y_0|} + 1 \right).
\]

It is easy to see that
\[
|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\
\leq \left( \frac{|h|}{|x_0 - y_0|} \right)^{1-n/q} \frac{1}{\rho(x_0)^{n+1/2}} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N} \left( \frac{\sqrt{t}}{|x_0 - y_0|} \right)^{1-n/q} \left( \frac{\sqrt{t}}{|x_0 - y_0|} + 1 \right) \\
\leq \left( \frac{|h|}{|x_0 - y_0|} \right)^{1-n/q} \frac{1}{\rho(x_0)^{n+1/2}} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N} \left( \frac{\sqrt{t}}{|x_0 - y_0|} \right)^{1-n/q} \left( \frac{\sqrt{t}}{|x_0 - y_0|} + 1 \right) \\
\leq \left( \frac{|h|}{|x_0 - y_0|} \right)^{1-n/q} \frac{1}{\rho(x_0)^{n+1/2}} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N} \left( \frac{\sqrt{t}}{|x_0 - y_0|} \right)^{1-n/q} \left( \frac{\sqrt{t}}{|x_0 - y_0|} + 1 \right).
\]

Case 2: $\sqrt{t} < \min(\rho(x_0), |x_0 - y_0|/8)$. Similar to Case 1, we divide the discussion into two subcases again.

Case 2.1: $0 < R < \sqrt{t} < \sqrt{2q-n}t/n < \min(\rho(x_0), |x_0 - y_0|/8)$. It follows from (3.8) that
\[
(3.9)
|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)| \\
\leq C_N \left( \frac{|h|}{\sqrt{t}} \right)^{1-n/q} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N} \left( \frac{\sqrt{t}}{R} \right)^{1-n/q} \left( \frac{\sqrt{t}}{R} + \frac{R}{\sqrt{t}} \right).
\]

Taking the infimum on both sides (3.9) reaches
\[
|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)| \leq \frac{C_N}{\rho(x_0)^{n+1/2}} \left( \frac{|h|}{\sqrt{t}} \right)^{1-n/q} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N} \left( \frac{\sqrt{t}}{|x_0 - y_0|} \right)^{1-n/q} \left( \frac{\sqrt{t}}{|x_0 - y_0|} + 1 \right).
\]
Case 2.2: \(0 < R < \sqrt{t} < \min \{\rho(x_0), |x_0 - y_0|/8\} < \sqrt{(2q-n)t/n}\). Similarly, taking the infimum on both sides of (3.9), we obtain

\[
|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)|
\leq C_N \left( \frac{|h|}{\sqrt{t}} \right)^{1-n/q} \frac{1}{t^{(n+1)/2}} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N}
\times \left( \frac{\sqrt{t}}{\min \{\rho(x_0), |x_0 - y_0|/8\}} \right)^{1-n/q} \left( \frac{\sqrt{t}}{\min \{\rho(x_0), |x_0 - y_0|/8\}} + \frac{\min \{\rho(x_0), |x_0 - y_0|/8\}}{\sqrt{t}} \right).
\]

If \(\rho(x_0) < |x_0 - y_0|/8\), then

\[
|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)|
\leq C_N \left( \frac{|h|}{\sqrt{t}} \right)^{1-n/q} \frac{1}{t^{(n+1)/2}} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N}
\times \left( \frac{\sqrt{t}}{\rho(x_0)} \right)^{1-n/q} \left( \frac{\sqrt{t}}{\rho(x_0)} + \frac{\rho(x_0)}{\sqrt{t}} \right)
\leq C_N \left( \frac{|h|}{|x_0 - y_0|} \right)^{1-n/q} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N} \left( \frac{\sqrt{t}}{\rho(x_0)} + \frac{|x_0 - y_0|}{\sqrt{t}} \right).
\]

If \(\sqrt{t} < |x_0 - y_0|\), then

\[
|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)|
\leq C_N \left( \frac{|h|}{\sqrt{t}} \right)^{1-n/q} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N}.
\]

If \(\sqrt{t} \geq |x_0 - y_0|\), then

\[
|\nabla_x K_t^L(x_0 + h, y_0) - \nabla_x K_t^L(x_0, y_0)|
\leq C_N \left( \frac{|h|}{|x_0 - y_0|} \right)^{1-n/q} e^{-c|x_0-y_0|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x_0)} + \frac{\sqrt{t}}{\rho(y_0)} \right)^{-N}.
\]

\[\square\]

**Lemma 3.8.** Suppose that \(V \in B_q\) for some \(q > n\). Let \(\delta' = 1 - n/q\). For every \(N > 0\), there exists a constant \(C_N > 0\) such that for all \(x, y \in \mathbb{R}^n\) and \(t > 0\), the semigroup kernels \(K_t^L(\cdot, \cdot)\) satisfy the following estimate: for \(|h| < |x - y|/4\),

\[
|\nabla_x K_t^L(x + h, y) - \nabla_x K_t^L(x, y)| \leq C_N \left( \frac{|h|}{\sqrt{t}} \right)^{\delta'} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}.
\]
Proof. Similar to Lemma 3.7, we take $R \in (0, \min(\rho(x_0), \sqrt{t}))$ and obtain that

\begin{equation}
|\nabla_{x} K^{L}_{t}(x_0 + h, y_0) - \nabla_{x} K^{L}_{t}(x_0, y_0)| \leq C_N \frac{\|h\|^l_{\ell(n+1)2}}{(\sqrt{t})^{1-n/k}} \left(1 + \frac{\sqrt{t}}{\rho(x_0)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(x_0)}\right)^{1-n/k} \left(1 + \frac{\sqrt{t}}{\rho(y_0)}\right)^{1-n/k} \left(\frac{\sqrt{t}}{\rho(x_0)} + R \frac{\sqrt{t}}{\rho(y_0)}\right)\left(\frac{\sqrt{t}}{R} + \frac{R}{\sqrt{t}}\right).
\end{equation}

Case 1: $\rho(x_0) \leq \sqrt{t}$. This implies $0 < R < \rho(x_0) < \sqrt{t} < (2q-n)t/n$. We can get

\begin{equation}
|\nabla_{x} K^{L}_{t}(x_0 + h, y_0) - \nabla_{x} K^{L}_{t}(x_0, y_0)| \leq C_N \frac{\|h\|^l_{\ell(n+1)2}}{(\sqrt{t})^{1-n/k}} \left(1 + \frac{\sqrt{t}}{\rho(x_0)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(x_0)}\right)^{1-n/k} \left(1 + \frac{\sqrt{t}}{\rho(y_0)}\right)^{1-n/k} \left(\frac{\sqrt{t}}{\rho(x_0)} + R \frac{\sqrt{t}}{\rho(y_0)}\right)\left(\frac{\sqrt{t}}{R} + \frac{R}{\sqrt{t}}\right).
\end{equation}

Case 2: $\rho(x_0) > \sqrt{t}$. For this case, $0 < R < \sqrt{t}$. Then the following two cases are considered.

\begin{align*}
\text{Case 2.1:} & \quad 0 < R < \rho(x_0) < (2q-n)t/n; \\
\text{Case 2.2:} & \quad 0 < R < \sqrt{(2q-n)t/n} < \rho(x_0).
\end{align*}

It is obvious that Case 2.1 comes back to Case 1. For Case 2.2, letting $R \to \sqrt{(2q-n)t/n}$ on the right-hand side of (3.10), we have

\begin{equation}
|\nabla_{x} K^{L}_{t}(x_0 + h, y_0) - \nabla_{x} K^{L}_{t}(x_0, y_0)| \leq C_N \frac{\|h\|^l_{\ell(n+1)2}}{(\sqrt{t})^{1-n/k}} \left(1 + \frac{\sqrt{t}}{\rho(x_0)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(y_0)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(x_0)}\right)^{1-n/k} \left(1 + \frac{\sqrt{t}}{\rho(y_0)}\right)^{1-n/k} \left(\frac{\sqrt{t}}{\rho(x_0)} + R \frac{\sqrt{t}}{\rho(y_0)}\right)\left(\frac{\sqrt{t}}{R} + \frac{R}{\sqrt{t}}\right).
\end{equation}

\[\square\]

Proposition 3.9. Suppose that $\alpha > 0$ and $V \in B_{q}$ for some $q > n$. Let $\delta' = 1 - n/k$. For every $N > 0$, there exists a constant $C_{N} > 0$ such that for all $x, y \in \mathbb{R}^{n}$ and $t > 0$, the fractional heat kernels $K^{L}_{t}(x, y)$ satisfy the following estimate: for $|h| < |x - y|/4$,

\begin{equation}
|\nabla_{x} K^{L}_{t}(x + h, y) - \nabla_{x} K^{L}_{t}(x, y)| \leq C_{N} \frac{\|h\|^l_{\ell(n+1)2}}{(\sqrt{t})^{1-n/k}} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{1-n/k} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{1-n/k} \left(\frac{\sqrt{t}}{\rho(x)} + R \frac{\sqrt{t}}{\rho(y)}\right)\left(\frac{\sqrt{t}}{R} + \frac{R}{\sqrt{t}}\right).
\end{equation}

Proof. By the subordinative formula and Lemma 3.7, we can get

\begin{equation}
|\nabla_{x} K^{L}_{t}(x + h, y) - \nabla_{x} K^{L}_{t}(x, y)| \leq C_{N} \int_{0}^{t} |\nabla_{x} K^{L}_{s}(x + h, y) - \nabla_{x} K^{L}_{s}(x, y)| ds \leq C_{N}(M_{6} + M_{7}),
\end{equation}

where

\begin{align*}
M_{6} & := \int_{0}^{4|x-y|} \frac{t}{s^{1+\alpha}} \frac{\|h\|^l_{\ell(n+1)2}}{(\sqrt{s})^{1-n/k}} \left(1 + \frac{\sqrt{s}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{s}}{\rho(y)}\right)^{-N} ds; \\
M_{7} & := \int_{4|x-y|}^{\infty} \frac{t}{s^{1+\alpha}} \frac{\|h\|^l_{\ell(n+1)2}}{(\sqrt{s})^{1-n/k}} \left(1 + \frac{\sqrt{s}}{\rho(x)}\right)^{-N} \left(1 + \frac{\sqrt{s}}{\rho(y)}\right)^{-N} ds.
\end{align*}
We first estimate $M_6$ and apply change of variables to obtain

$$M_6 \leq \int_0^\infty \frac{t}{s^{1+n/2}} \left( \frac{|h|}{\sqrt{s}} \right)^\delta e^{-c|x-y|/\sqrt{s}} \left( 1 + \frac{\sqrt{s}}{\rho(x)} \right)^{-N} \left( 1 + \frac{\sqrt{s}}{\rho(y)} \right)^{-N} ds$$

$$\leq \tau^{-(n+1)/2a} \int_0^\infty \frac{t^{1/2a}}{\rho(x)} \left( \frac{|h|}{\rho(y)} \right)^\delta e^{-c|x-y|^2/\rho(y)^{1/2a}} ds$$

Similarly, for $M_7$, we have

$$M_7 \leq \frac{1}{|x-y|} \int_0^\infty \frac{t}{s^{1+n/2}} \left( \frac{|h|}{\sqrt{s}} \right)^\delta e^{-c|x-y|/\sqrt{s}} \left( 1 + \frac{\sqrt{s}}{\rho(x)} \right)^{-N} \left( 1 + \frac{\sqrt{s}}{\rho(y)} \right)^{-N} ds$$

which gives

$$|\nabla_x K_{\alpha,t}^L(x+h,\cdot) - \nabla_x K_{\alpha,t}^L(x,\cdot)| \leq C_N \left( \frac{|h|}{|x-y|^{2a+2N+n+1}} \right)^\delta \left( t^{1/2a} \rho(x) \right)^{-N} \left( t^{1/2a} \rho(y) \right)^{-N}.$$ 

On the other hand, we can deduce from Lemma 3.8 that

$$|\nabla_x K_{\alpha,t}^L(x+h,\cdot) - \nabla_x K_{\alpha,t}^L(x,\cdot)| \leq C_N \left( \frac{|h|}{|x-y|^{2a+2N+n+1}} \right)^\delta \left( t^{1/2a} \rho(x) \right)^{-N} \left( t^{1/2a} \rho(y) \right)^{-N}.$$ 

Finally, the arbitrariness of $N$ indicates that

$$\left( 1 + \frac{t^{1/2a}}{\rho(x)} \right)^N \left( 1 + \frac{t^{1/2a}}{\rho(y)} \right)^N \left| \nabla_x K_{\alpha,t}^L(x+h,\cdot) - \nabla_x K_{\alpha,t}^L(x,\cdot) \right|$$

$$\leq C_N \min \left\{ \left( \frac{|h|}{|x-y|} \right)^\delta \left( t^{1+2N/a} \right)^{-N}, \left( \frac{|h|}{t^{1/2a}} \right)^\delta \left( \frac{1}{t^{(n+1)/2a}} \right)^{-N} \right\},$$

which proves Proposition 3.9.

\[\square\]

**Proposition 3.10.** Assume that $V \in B_q$ for some $q > n$. Let $\alpha \in (0, 1/2 - n/2q)$. For every $N > 0$,

$$\left| t^{1/2a} \nabla_x e^{-H^a}(1)(x) \right| \leq \min \left\{ \left( \frac{t^{1/2a}}{\rho(x)} \right)^{1+2a} \left( \frac{1}{\rho(x)} \right)^{-N}, \left( \frac{t^{1/2a}}{\rho(x)} \right)^{-N} \right\}.$$

**Proof.** We divide the proof into two cases.

**Case 1:** $t^{1/2a} > \rho(x)$. By Proposition 3.6 we use a direct computation to obtain

$$\left| t^{1/2a} \nabla_x e^{-H^a}(1)(x) \right| \leq \int_{\mathbb{R}^n} \frac{t}{(t^{1/2a} + |x-y|)^{n+2a}} \left( 1 + \frac{t^{1/2a}}{\rho(x)} \right)^{-N} \left( 1 + \frac{t^{1/2a}}{\rho(y)} \right)^{-N} dy$$

$$\leq \left( \frac{t}{\rho(x)} \right)^{-N} \int_{\mathbb{R}^n} \frac{t}{(t^{1/2a} + |x-y|)^{n+2a}} dy$$

$$\leq \left( \frac{t}{\rho(x)} \right)^{-N}.$$
Because $t^{1/2a} > \rho(x)$, then
\[
|t^{1/2a} \nabla_x e^{-tL^\alpha}(1)(x)| \leq \min\left\{ \left(\frac{t^{1/2a}}{\rho(x)}\right)^\delta, \left(\frac{t^{1/2a}}{\rho(x)}\right)^{-N} \right\}.
\]

Case 2: $t^{1/2a} \leq \rho(x)$. It follows from (1.5) that
\[
t^{1/2a} \nabla_x e^{-tL^\alpha}(1)(x) = t^{1/2a} \nabla_x \int_{\mathbb{R}^n} K^L_{\alpha,s}(x,y)dy = I_1 + I_2,
\]
where
\[
\begin{aligned}
I_1 &= t^{1/2a} \int_{\rho^2(x)}^{\infty} \eta^\iota_t(s) \left( \int_{\mathbb{R}^n} \nabla_x K^L_{\alpha,s}(x,y)dy \right) ds; \\
I_2 &= t^{1/2a} \int_0^{\rho^2(x)} \eta^\iota_t(s) \left( \int_{\mathbb{R}^n} \nabla_x K^L_{\alpha,s}(x,y)dy \right) ds.
\end{aligned}
\]
We claim that
\[
L_\delta(x) := \int_{\mathbb{R}^n} \nabla_x K^L_{\alpha,s}(x,y)dy \leq \frac{1}{\sqrt{s}}.
\]
In fact, by Lemma 3.4, we have $L_\delta(x) \leq L_{s,1}(x) + L_{s,2}(x)$, where
\[
\begin{aligned}
L_{s,1}(x) &= \int_{|y|: \sqrt{s} \leq |x-y|} \frac{1}{s^{(a+1)/2}} \frac{e^{-\varepsilon|x-y|^2/s}}{\rho(x) + \rho(y)^{-N}} dy; \\
L_{s,2}(x) &= \int_{|y|: |x-y| < \sqrt{s}} \frac{1}{s^{(2/2)|x-y|^2/s}} \frac{e^{-\varepsilon|x-y|^2/s}}{\rho(x) + \rho(y)^{-N}} dy.
\end{aligned}
\]
Taking $N$ large enough, it is easy to see that
\[
L_{s,1}(x) \leq \int_{\mathbb{R}^n} \frac{1}{s^{(a+1)/2}} \frac{e^{-\varepsilon|x-y|^2/s}}{\rho(x)} dy \leq \frac{1}{\sqrt{s}}
\]
Similarly, a direct calculus gives, together with changing variable: $|x-y|/\sqrt{s} = u$,
\[
L_{s,2}(x) \leq \frac{1}{\sqrt{s}} \int_{\mathbb{R}^n} \frac{1}{s^{(2)/2}|x-y|/\sqrt{s}} e^{-\varepsilon|x-y|^2/s} \frac{e^{-\varepsilon|x-y|^2/s}}{\rho(x)} dy \leq \frac{1}{\sqrt{s}} \int_0^{\infty} u^{n-2} e^{-cu^2} du \leq \frac{1}{\sqrt{s}}
\]
Then we can deduce from (3.11) that
\[
I_1 \leq t^{1/2a} \int_{\rho^2(x)}^{\infty} \eta^\iota_t(s) \frac{ds}{\sqrt{s}} \leq t^{1+1/2a} \int_{\rho^2(x)}^{\infty} \frac{1}{s^{a+3/2}} ds \leq \left(\frac{t^{1/2a}}{\rho(x)}\right)^{1+2a}.
\]
For $I_2$, it follows from the formula
\[
h_u(x-y) - K^L_{\alpha,s}(x,y) = \int_0^u \int_{\mathbb{R}^n} h_3(x-z)V(z)K_{\alpha,s}(z,y)dzds
\]
that for $\delta = 2 - n/q > 1$,
\[
|\sqrt{u} \nabla_x e^{-uL}(1)(x)| \leq \int_0^u \sqrt{\frac{u}{s}} \frac{\sqrt{s}}{\rho(x)} \frac{ds}{s} \leq \left(\frac{\sqrt{u}}{\rho(x)}\right)^{\delta}.
\]
Therefore, noting that $0 < 2\alpha < 1 - n/q$, we can use the change of variables to obtain

$$
I_2 \leq t^{1/2}\int_0^{\eta^2(x)} \eta^\rho(s) \frac{1}{\sqrt{s}} \int_{\mathbb{R}^n} \nabla x K^T_t(x, y) dy \, ds
$$

$$
\leq t^{1/2}\int_0^{\eta^2(x)} \eta^\rho(s) \frac{1}{\sqrt{s}} \left( \frac{\sqrt{s}}{\rho(x)} \right)^\theta \, ds
$$

$$
\leq t^{1/2\alpha - 1/\alpha} \int_0^{\eta^2(x) t^{1/\alpha}} \eta^\rho(\tau) \frac{1}{\sqrt{t^{1/\alpha} \tau}} \left( \frac{\sqrt{t^{1/\alpha} \tau}}{\rho(x)} \right)^\theta \, d\tau
$$

$$
\leq \left( \frac{t^{1/2\alpha}}{\rho(x)} \right)^{1+2\alpha}.
$$

The above estimates (3.13) and (3.12) imply that

$$
|t^{1/2\alpha} \nabla x e^{-tL^\alpha} (1)(x)| \leq \left( \frac{t^{1/2\alpha}}{\rho(x)} \right)^{1+2\alpha}.
$$

\[\Box\]

3.3. Estimation on time-fractional derivatives. In this section we give some gradient estimate for the fractional heat kernel associated with the variable $t$. Define an operator

$$
D^{\alpha, \beta}_{a,i}(f) = \partial^\alpha_\xi \rho^\beta e^{-tL^\alpha} f, \quad \alpha \in (0, 1) \& \beta > 0.
$$

Denote by $D^{\alpha, \beta}_{a,i}(\cdot, \cdot)$ the integral kernel of $D^{\alpha, \beta}_{a,i}$. Then we can get the following proposition.

**Proposition 3.11.** Let $\alpha \in (0, 1)$ and $\beta > 0$. For every $N > 0$, there exists a constant $C_N > 0$ such that

$$
|D^{\alpha, \beta}_{a,i}(x, y)| \leq \frac{C_N^\beta}{(t^{1/2\alpha} + |x - y|)^{\alpha + 2\beta}} \left( 1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N}.
$$

**Proof.** The following two cases are considered.

Case I: $\beta \in (0, 1)$. It is easy to see that

$$
\rho^\beta \partial^\alpha_\xi \rho^\beta e^{-tL^\alpha} = c_\beta^\rho \int_0^\infty \partial^\alpha_\xi e^{-{(t+s)\rho^\alpha}^{1+2\beta}} ds = c_\beta^\rho \int_0^\infty (-L)^\alpha e^{-{(t+s)\rho^\alpha}^{1+2\beta}} ds = \frac{c_\beta^\rho}{(t+s)^\beta} \int_0^\infty (t+s)^{1-\rho^\alpha \rho^\beta} ds
$$

which, together with Proposition [3.3] gives

$$
|D^{\alpha, \beta}_{a,i}(x, y)| \leq C_N^\rho \int_0^\infty \frac{1}{(t+s)^{1/2\alpha}} \frac{1}{|x - y|^{\alpha + 2\beta}} \left( 1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N} ds
$$

$$
\leq C_N^\rho \int_0^\infty \frac{1}{(t+s)^{1/2\alpha}} \frac{1}{(t+s)^{1/2\alpha}} \left( 1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N} ds
$$

$$
\leq C_N^\rho \rho(x)^N \rho(y)^N \int_0^\infty (t+s)^{-n/2\alpha - n/\alpha - 1} s^{-\beta} ds
$$

$$
\leq \frac{C_N}{\rho^{n/2\alpha}} \left( \frac{t^{1/2\alpha}}{\rho(x)} \right)^{-N} \left( \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N}.
$$
One the other hand, since $e^{-tL^a} = \int_0^\infty \eta_1^0(\tau)e^{-\tau^1/\alpha\tau^L}d\tau$, 

\[
\beta \frac{\partial^\beta}{\partial t^\beta} e^{-tL^a} = \beta \int_0^\infty \partial_t^\beta \left( \int_0^\infty \eta_1^0(\tau)e^{-(t+r)^{1/\alpha\tau^L}}d\tau \right) \frac{dr}{\rho^\beta} \\
= \beta \int_0^\infty \left( \int_0^\infty \eta_1^0(\tau)(t+r)^{1/\alpha}\tau^L e^{-(t+r)^{1/\alpha\tau^L}}d\tau \right) \frac{dr}{\rho^\beta} \\
= C_\beta \frac{\partial^\beta}{\partial t^\beta} \int_0^\infty \left( \int_0^\infty \frac{Q^L}{\sqrt{(t+r)^{1/\alpha\tau^L}1}} \frac{dr}{(t+r)^{\rho^\beta}} \right) \eta_1^0(\tau)d\tau.
\]

By Proposition 2.9, we can get

\[
\left| D_{a,t}^{L,\beta}(x,y) \right| \leq C_N \beta \int_0^\infty \eta_1^0(\tau) \left( \int_0^\infty \left( (t+r)^{1/\alpha}\tau^L \right)^{-n/2} e^{-\sqrt{|x-y|^2/(t+r)^{1/\alpha\tau}}} \right) \frac{dr}{\rho^\beta(t+r)}d\tau \\
\leq C_N \int_0^\infty \eta_1^0(\tau) \left( \int_0^\infty \frac{dr}{\rho^\beta(t+r)} \right) d\tau \\
\leq C_N \beta \frac{\partial^\beta}{\partial t^\beta} \left( \int_0^\infty \frac{Q^L}{\sqrt{|x-y|^2+2\beta}} \frac{dr}{\rho^\beta} \right) \eta_1^0(\tau).
\]

By the arbitrariness of $N$, we obtain

\[
\left| D_{a,t}^{L,\beta}(x,y) \right| \leq C_N \min \left\{ \frac{1}{p^{1/2\alpha}}, \frac{\beta}{|x-y|^{2\alpha^2+2\alpha\beta}} \right\} \left( \int_0^\infty \frac{dr}{\rho^\beta(t+r)} \right) \eta_1^0(\tau) \\
\leq \frac{C_N \beta}{(1^{1/2\alpha} + |x-y|)^{2\alpha^2+2\alpha\beta}} \left( \frac{1}{p(x)} + \frac{1/2\alpha}{p(y)} \right)^{-N}.
\]

Case 2: $\beta \geq 1$. Let $m = [\beta] + 1$. We can get

\[
\beta \frac{\partial^\beta}{\partial t^\beta} e^{-tL^a} = c_{\beta} \beta \int_0^\infty \partial_t^\beta \left( \int_0^\infty (-L)^m e^{-(t+s)L^a} \frac{ds}{s^{1+\beta-m}} \right) \frac{ds}{(t+s)^m s^{1+\beta-m}} \\
= c_{\beta} \beta \int_0^\infty \left( (t+s)^m (-L)^m e^{-(t+s)L^a} \frac{ds}{s^{1+\beta-m}} \right) \frac{ds}{(t+s)^m s^{1+\beta-m}} \\
= c_{\beta} \beta \int_0^\infty \left( (t+s)^m (-L)^m e^{-(t+s)L^a} \frac{ds}{s^{1+\beta-m}} \right) \frac{ds}{(t+s)^m s^{1+\beta-m}}
\]

It follows from Proposition 3.3 that

\[
\left| D_{a,t}^{L,\beta}(x,y) \right| \leq \frac{C_N \beta}{(t+s)^m s^{1+\beta-m}} \int_0^\infty \left( (t+s)^m (-L)^m e^{-(t+s)L^a} \frac{ds}{s^{1+\beta-m}} \right) \frac{ds}{(t+s)^m s^{1+\beta-m}} \\
\leq \frac{C_N \beta\rho(x)^N}{p^{1/\rho\alpha}} \int_0^\infty \left( (t+s)^m (-L)^m e^{-(t+s)L^a} \frac{ds}{s^{1+\beta-m}} \right) \frac{ds}{(t+s)^m s^{1+\beta-m}} \\
\leq \frac{C_N \beta}{(t+s)^m s^{1+\beta-m}} \left( 1 + \frac{1/2\alpha}{\rho(y)} \right)^{-N}.
\]
On the other hand, we obtain
\[ |D_{\alpha,\beta}^L(x,y)| \leq \beta \int_0^\infty \left\{ \int_0^\infty \left( \int_0^\infty (t+r)^{-m/\alpha} e^{-c|x-y|^2/(t+r)^{1/\alpha}} \right) \frac{dr}{r^{\beta+1-m}} \right\} \eta_1^\alpha(\tau) d\tau \]
\[ \leq C_N \beta \int_0^\infty \left( \int_0^\infty (t+r)^{-m/\alpha} \right) \frac{dr}{r^{\beta+1-m}} \eta_1^\alpha(\tau) d\tau \]
\[ \leq C_N \beta \rho(x)^N \rho(y)^N \int_0^\infty \left( \int_0^\infty (1+u)^{-m-\beta-N} du \right) \eta_1^\alpha(\tau) \tau^{\alpha-N} d\tau \]
\[ \leq C_N \beta \frac{\rho(x)^N \rho(y)^N}{|x-y|^{n+2\alpha}} \int_0^\infty \left( \int_0^\infty (1+u)^{-m-\beta-N} du \right) \eta_1^\alpha(\tau) \tau^{\alpha-N} d\tau \]
which indicates \( \text{(3.14)} \) holds. \( \square \)

In the next proposition, we give the Lipschitz continuity of \( D_{\alpha,\beta}^L(\cdot,\cdot) \).

**Proposition 3.12.** Let \( \alpha \in (0,1) \) and \( \beta > 0 \). Let \( 0 < \delta' \leq \delta = \min\{2\alpha, \delta_0\} \). For every \( N > 0 \), there exists a constant \( C_N > 0 \) such that for all \( |h| \leq t^{1/2\alpha} \),

\[ |D_{\alpha,\beta}^L(x+h,y) - D_{\alpha,\beta}^L(x,y)| \leq C_N \left( \frac{|h|}{t^{1/2\alpha}} \right)^{\delta'} \frac{\beta}{(t^{1/2\alpha} + |x-y|^{n+2\alpha})} \left( 1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N} \]

**Proof.** It is equivalent to verify

\[ |D_{\alpha,\beta}^L(x+h,y) - D_{\alpha,\beta}^L(x,y)| \leq C_N \left( \frac{|h|}{t^{1/2\alpha}} \right)^{\delta'} \min\left\{ \frac{1}{|h|^{n/2\alpha}}, \frac{\beta}{|x-y|^{n+2\alpha}} \right\} \left( 1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N} \]

Without loss of generality, for \( m = \lfloor \beta \rfloor + 1 \), it holds

\[ \beta^m \int_0^\infty (t+s)^{m-1} e^{-(t+s)\Lambda} ds \leq C_{\alpha,\beta} \int_0^\infty (t+s)^{m-1} e^{-(t+s)\Lambda} \frac{ds}{(t+s)^m s^{1+\beta-m}} \]

By Proposition \( \text{3.3} \) we can get

\[ \left| D_{\alpha,\beta}^L(x+h,y) - D_{\alpha,\beta}^L(x,y) \right| \]
\[ \leq C_N t^{n/2\alpha} \left( \frac{t^{1/2\alpha}}{\rho(x)} \right)^{-N} \left( \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N} \left( \frac{|h|}{t^{1/2\alpha}} \right)^{\delta'} \]

On the other hand, we obtain
\[
\left| D_{a,t}^{\alpha,\beta}(x+y) - D_{a,t}^{\alpha,\beta}(x,y) \right| \\
\leq C_N t^\beta \int_0^\infty \left\{ \int_0^\infty (t+r)^{-m-n/2\alpha} r^{-n/2} \left( \frac{|h|}{\sqrt{(t+r)^{1/\alpha} r}} \right)^{\delta'} \times e^{-c\|x-y\|^2/(t+r))^{1/\alpha} r^{-N}} \left( \frac{\sqrt{(t+r)^{1/\alpha} r}^{1/2\alpha}}{p(y)} \right)^{-N} dr \right\} d\tau \eta_1^\delta(\tau) d\tau \\
\leq C_N t^\beta |h|^\delta \rho(x)^N \rho(y)^N \int_0^\infty \left\{ \int_0^\infty (t+r)^{-m+\beta-\delta}\|x-y\|^{2\alpha} \rho^{-\beta-1} dr \right\} d\tau \eta_1^\delta d\tau \\
\leq C_N t^\beta \frac{|h|^\delta}{|x-y|^{n+2\alpha}} \left( \frac{1}{t^{1/\alpha} \rho(x)} \right)^{\delta'} \left( 1 + \frac{1}{t^{1/\alpha} \rho(x)} \right)^{-N},
\]
which implies (3.13).

\[\square\]

**Proposition 3.13.** Let \( \alpha \in (0,1) \), \( \beta > 0 \) and \( 0 < \delta' \leq \min\{2\alpha, \delta_0\} \). For every \( N > 0 \), there exist constants \( C_N > 0 \) such that
\[
\left| \int_{\mathbb{R}^n} D_{a,t}^{\alpha,\beta}(x,y) dy \right| \leq C_N \frac{(t^{1/\alpha}/\rho(x))^{\delta'}}{(1 + t^{1/\alpha}/\rho(x))^N}.
\]

**Proof.** Let \( m = \lceil \beta \rceil + 1 \). By (iii) of Proposition 3.3, we change the order of integrations to obtain
\[
\left| \int_{\mathbb{R}^n} D_{a,t}^{\alpha,\beta}(x,y) dy \right| = \left| \int_{\mathbb{R}^n} \{ \int_0^\infty \partial_m^{\rho_m(x)} D_{a,t}^{\alpha,\beta}(x,y) ds \} dy \right| \\
\leq t^\beta \int_0^\infty \int_{\mathbb{R}^n} |\partial_m^{\rho_m(x)} D_{a,t}^{\alpha,\beta}(x,y)| dy ds \\
\leq C_N t^\beta \int_{\mathbb{R}^n} \frac{((t+s)^{1/\alpha}/\rho(x))^{\delta'}}{(1 + (t+s)^{1/\alpha}/\rho(x))^N} ds.
\]

If \( t^{1/\alpha} > \rho(x) \), then
\[
\left| \int_{\mathbb{R}^n} D_{a,t}^{\alpha,\beta}(x,y) dy \right| \leq C_N t^\beta \rho(x)^{N-\delta} \int_0^\infty (t+s)^{\delta'/2\alpha - N/2\alpha - m} s^{m-\beta-1} ds \\
\leq C_N \frac{(t^{1/\alpha}/\rho(x))^{\delta'}}{(1 + t^{1/\alpha}/\rho(x))^N}.
\]

If \( t^{1/\alpha} \leq \rho(x) \), then
\[
\left| \int_{\mathbb{R}^n} D_{a,t}^{\alpha,\beta}(x,y) dy \right| \leq C_N t^\beta \int_0^\infty (t+s)^{1/\alpha}/\rho(x) \right)^{\delta'} \frac{ds}{s^{1+\beta-m(t+s)^m}} \\
\leq C_N \frac{(t^{1/\alpha}/\rho(x))^{\delta'}}{(1 + t^{1/\alpha}/\rho(x))^N}.
\]

which completes the proof of Proposition 3.13. \(\square\)

4. **Characterization of Campanato-Morrey spaces associated with \( L \)

Firstly, we deduce a reproducing formula.
Lemma 4.1. Let $\alpha \in (0, 1)$ and $\beta > 0$. The operator $t^\alpha \partial_t^\beta e^{-tL^\alpha}$ defines an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{n+1}, dxdt/t)$. Moreover, in the sense of $L^2(\mathbb{R}^n)$, it holds

$$f(x) = c_{\alpha, \beta} \lim_{N \to \infty} \lim_{\epsilon \to 0} \int_\epsilon^N (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 (f)(x) \frac{dt}{t}.$$ 

Proof. Note that for $dE(\lambda)$ the spectral resolution of the operator $L$, it follows from

$$e^{-tL^\alpha} = \int_0^\infty e^{-t\epsilon^\alpha} dE(\lambda)$$

that

$$t^\alpha \partial_t^\beta e^{-tL^\alpha} = \alpha \int_0^\infty \partial_t^\alpha \left( \int_0^\infty e^{-(t+s)x^\alpha} dE(\lambda) \right) \frac{ds}{s^{1+\beta-m}}
= \alpha \int_0^\infty (s^{-1}) \lambda^{\alpha m} e^{-(t+s)x^\alpha} dE(\lambda) \frac{ds}{s^{1+\beta-m}}
= \alpha \int_0^\infty (t\lambda^\alpha) e^{tL^\alpha} dE(\lambda).$$

Then for $f \in L^2(\mathbb{R}^n)$, we have

$$\|t^\alpha \partial_t^\beta e^{-tL^\alpha} (f)(x)\|_{L^2(\mathbb{R}^{n+1}, dxdt/t)}^2 = \int_0^\infty \left( \int_{\mathbb{R}^n} |t^\alpha \partial_t^\beta e^{-tL^\alpha} (f)(x)|^2 dx \right) \frac{dt}{t}
= \int_0^\infty \left( \int f (t^\beta \partial_t^\beta e^{-tL^\alpha} (f))(x) dx \right) \frac{dt}{t}
= \int_0^\infty \left( \int (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 (f)(x) dx \right) \frac{dt}{t}
= \int_0^\infty \int_0^\infty \lambda^{2\alpha} \lambda^{2\alpha} e^{-2\lambda t} dE_{f, f}(\lambda) \frac{dt}{t}
= C_{\alpha, \beta} \|f\|_{L^2}^2.$$

Below we only prove that for every pair of sequences $n_k \uparrow \infty$ and $\epsilon_k \downarrow 0$ as $k \to \infty$,

$$\lim_{k \to \infty} \sum_{n_k}^{n_{k+1}} (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f(x) \frac{dt}{t} = \lim_{k \to \infty} \sum_{\epsilon_k}^{\epsilon_{k+1}} (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f(x) \frac{dt}{t} = 0. \quad (4.1)$$

If (4.1) holds, there exists a function $h \in L^2(\mathbb{R}^n)$ such that

$$\lim_{k \to \infty} \int_{\epsilon_k}^{\epsilon_{k+1}} (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f(x) \frac{dt}{t} = h(x),$$

which implies that for all $g \in L^2(\mathbb{R}^n)$,

$$\langle h, g \rangle = \left\langle \lim_{k \to \infty} \int_{\epsilon_k}^{\epsilon_{k+1}} (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f \frac{dt}{t}, g \right\rangle
= \lim_{k \to \infty} \int_{\epsilon_k}^{\epsilon_{k+1}} \left( (t^\beta \partial_t^\beta e^{-tL^\alpha})^2 f \right) g \frac{dt}{t}
= \lim_{k \to \infty} \int_{\epsilon_k}^{\epsilon_{k+1}} \left( (t^\beta \partial_t^\beta e^{-tL^\alpha}, t^\beta \partial_t^\beta e^{-tL^\alpha} g) \frac{dt}{t}
= C_{\alpha, \beta} \langle f, g \rangle.$$
This means $h = C_{\alpha, \beta} f$. Now we verify (4.1). As $k \to \infty$,
\[
\left\| \int_{n_k}^{n_{k+m}} (\partial_t^\alpha \partial_x^\beta e^{-t L^\alpha})^2 f(x) \frac{dt}{t} \right\|_{L^2}^2 \leq \int_{n_k}^{n_{k+m}} \left\| (\partial_t^\alpha \partial_x^\beta e^{-t L^\alpha})^2 f(x) \right\|_{L^2}^2 \frac{dt}{t}
= \int_0^\infty \int_{n_k}^{n_{k+m}} t^{2\beta} \lambda^{2\alpha} \alpha^{-2\alpha t^\alpha} \frac{dt}{t} dE_f(\lambda) \to 0
\]
since
\[
\lim_{k \to \infty} \left| \int_{n_k}^{n_{k+m}} t^{2\beta} \lambda^{2\alpha} \alpha^{-2\alpha t^\alpha} \frac{dt}{t} \right| = 0.
\]

The integral
\[
\lim_{k \to \infty} \int_{n_k}^{n_{k+m}} (\partial_t^\alpha \partial_x^\beta e^{-t L^\alpha})^2 f(x) \frac{dt}{t}
\]
can be dealt with similarly.

\[\square\]

The following inequality was established by Harboure-Salinas-Viviani [13].

**Lemma 4.2.** ([13] (5.3)) Let $0 < \gamma \leq 1$. For any pair of measurable functions $F$ and $G$ on $\mathbb{R}^{n+1}_+$, we have
\[
\iint_{\mathbb{R}^{n+1}_+} |F(x, t)| |G(x, t)| \frac{dxdt}{t} \leq C \sup_B \left\{ \frac{1}{|B|^{1+2\gamma/n}} \iint_B |F(x, t)|^2 \frac{dxdt}{t} \right\}^{1/2} \left\{ \int_{\mathbb{R}^n} \left( \int_0^{\infty} \int_{|x-y|<t} |G(y, t)|^2 \frac{dxdt}{t^{n+1}} \right)^{n/(n-\gamma)} dx \right\}^{1+\gamma/n}.
\]

In Lemma 4.2, letting
\[
\begin{align*}
F(x, t) &:= t^{2\alpha \beta} \partial_t^\alpha \partial_x^\beta e^{-t L^\alpha} \big|_{x=t} (f)(x) = Q_{\alpha, \beta}^L (f)(x); \\
G(x, t) &:= t^{2\alpha \beta} \partial_t^\alpha \partial_x^\beta e^{-t L^\alpha} \big|_{x=t} (g)(x) = Q_{\alpha, \beta}^L (g)(x),
\end{align*}
\]
we have
\[
\iint_{\mathbb{R}^{n+1}_+} |Q_{\alpha, \beta}^L (f)(x)| \cdot |Q_{\alpha, \beta}^L (g)(x)| \frac{dxdt}{t} \leq C \sup_B \left\{ \frac{1}{|B|^{1+2\gamma/n}} \iint_B |Q_{\alpha, \beta}^L (f)(x)|^2 \frac{dxdt}{t} \right\}^{1/2} \times \left\{ \int_{\mathbb{R}^n} \left( \int_0^{\infty} \int_{|x-y|<t} |Q_{\alpha, \beta}^L (g)(x)|^2 \frac{dxdt}{t^{n+1}} \right)^{n/(n-\gamma)} dx \right\}^{1+\gamma/n}.
\]

On the left-hand side of (4.2), since
\[
\begin{align*}
Q_{\alpha, \beta}^L (f)(x) &= t^{2\alpha \beta} L^{\alpha \beta} e^{-t^{\alpha} L^\alpha} (f); \\
Q_{\alpha, \beta}^L (g)(x) &= t^{2\alpha \beta} L^{\alpha \beta} e^{-t^{\alpha} L^\alpha} (g),
\end{align*}
\]
we can get, via the change of variables,

\[
\int_{\mathbb{R}^n} |Q_{\alpha,t}^{L_{a,t}}(f)(x)| \cdot |Q_{\alpha,t}^{L_{a,t}}(g)(x)| \frac{dxdt}{t}
\]

\[
= \int_{\mathbb{R}^n} |2^{\alpha t} L_{a,t}^{\alpha t} e^{-2\alpha t L_{a,t}}(f)(x)| \cdot |2^{\alpha t} L_{a,t}^{\alpha t} e^{-2\alpha t L_{a,t}}(g)(x)| \frac{dxdt}{t}
\]

\[
= \int_{\mathbb{R}^n} |s^{\alpha t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(f)(x)| \cdot |s^{\alpha t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(g)(x)| \frac{dxds}{s}
\]

On the right-hand side of (4.2), using change of variables again, we obtain

\[
\sup_B \left( \frac{1}{|B|^{1+2y/n}} \int_{B} |2^{\alpha t} L_{a,t}^{\alpha t} e^{-2\alpha t L_{a,t}}(f)(x)|^2 \frac{dxdt}{t} \right)^{1/2}
\]

\[
\leq \sup_B \left( \frac{1}{|B|^{1+2y/n}} \int_{B} |s^{\alpha t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(f)(x)|^2 \frac{dxds}{s} \right)^{1/2}
\]

\[
= \sup_B \left( \frac{1}{|B|^{1+2y/n}} \int_{B} |s^{\alpha t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(f)(x)|^2 \frac{dxds}{s} \right)^{1/2}
\]

meanwhile,

\[
\left\{ \int_{\mathbb{R}^n} \left( \int_{0}^{\infty} \int_{|x-y|<s} |2^{\alpha t} L_{a,t}^{\alpha t} e^{-2\alpha t L_{a,t}}(g)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{n/(2(n+y))} \right\}^{1+y/n}
\]

\[
= \left\{ \int_{\mathbb{R}^n} \left( \int_{0}^{\infty} \int_{|x-y|<s} |s^{\alpha t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(g)(y)|^2 \frac{dyds}{s^{n+1}} \right)^{n/(2(n+y))} \right\}^{1+y/n}
\]

\[
\leq \left\{ \int_{\mathbb{R}^n} \left( \int_{0}^{\infty} \int_{|x-y|<s/2^{\alpha t}} |s^{\alpha t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(g)(y)|^2 \frac{dyds}{s^{(n+1)/2^{\alpha t}}} \right)^{n/(2(n+y))} \right\}^{1+y/n}
\]

\[
\leq \left\{ \int_{\mathbb{R}^n} \left( \int_{0}^{\infty} \int_{|x-y|<s/2^{\alpha t}} |s^{\alpha t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(g)(y)|^2 \frac{dyds}{s^{(n+1)/2^{\alpha t}}} \right)^{n/(2(n+y))} \right\}^{1+y/n}
\]

Finally, we have

\[
(4.3) \quad \int_{\mathbb{R}^n} |s^{\alpha t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(f)(x)| \cdot |s^{\alpha t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(g)(x)| \frac{dxds}{s}
\]

\[
\leq \sup_B \left( \frac{1}{|B|^{1+2y/n}} \int_{B} |s^{\alpha t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(f)(x)|^2 \frac{dxds}{s} \right)^{1/2}
\]

\[
\times \left\{ \int_{\mathbb{R}^n} \left( \int_{|x-y|<s/2^{\alpha t}} |s^{\alpha t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(g)(y)|^2 \frac{dyds}{s^{n+1}} \right)^{n/(2(n+y))} \right\}^{1+y/n}
\]

For \( \alpha \in (0, 1) \) and \( \beta > 0 \), define an area function \( S_{\alpha,\beta}^{L} \) as follows:

\[
S_{\alpha,\beta}^{L}(h)(x) := \left( \int_{\Gamma_{\alpha}(x)} |\beta \partial_{t} L_{a,t}^{\alpha t} e^{-s L_{a,t}}(h)(y)|^2 \frac{dydt}{t^{n/(2\alpha t + 1)}} \right)^{1/2},
\]

where \( \Gamma_{\alpha}(x) \) denotes the cone \( \{(y, t) : |x - y| < t^{1/2\alpha}\} \).

**Lemma 4.3.** Let \( \alpha \in (0, 1) \) and \( \beta > 0 \). The area function \( S_{\alpha,\beta}^{L} \) is bounded on \( L^2(\mathbb{R}^n) \).
Proof. Let
\[ g^L_{\alpha,\beta}(h)(x) := \left( \int_0^\infty |\partial_t^\alpha \partial_x^\beta e^{-tL^n} h(x)|^2 \frac{dt}{t} \right)^{1/2}. \]
We can get
\[ (4.4) \]
\[ \|g^L_{\alpha,\beta}(h)\|^2_2 = \int_{\mathbb{R}^n} \left( \int_0^\infty \left| \int_{\mathbb{R}^n} |\partial_t^\alpha \partial_x^\beta e^{-tL^n} h(x)|^2 \frac{dt}{t} \right| dx \right)^{1/2} \]
\[ = \int_0^\infty \left( \int_{\mathbb{R}^n} \left| \partial_t^\alpha \partial_x^\beta e^{-tL^n} h(x) \right|^2 dx \right)^{1/2} \frac{dt}{t} \]
\[ = \int_0^\infty \left( \int_{\mathbb{R}^n} \left| \partial_t^\alpha \partial_x^\beta e^{-tL^n} \right|^2 h(x) \right)^{1/2} \frac{dt}{t} \]
\[ = \int_0^\infty \int_0^\infty t^{2\beta} L^{2\gamma} e^{-tL^n} dE_{h,h}(\lambda) \frac{dt}{t} \]
\[ \leq \|h\|^2_2. \]
Hence, it follows from (4.4) that
\[ \|S^L_{\alpha,\beta}(h)\|^2_2 = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left| \partial_t^\alpha \partial_x^\beta e^{-tL^n} \right|^2 dy dt \frac{1}{2\pi^n} \right)^{1/2} dx \]
\[ = \int_0^\infty \left( \int_{\mathbb{R}^n} \left| \partial_t^\alpha \partial_x^\beta e^{-tL^n} \right|^2 dy dt \frac{1}{2\pi^n} \right)^{1/2} dx \]
\[ \leq \int_0^\infty \int_{\mathbb{R}^n} \left| \partial_t^\alpha \partial_x^\beta e^{-tL^n} \right|^2 dy dt \frac{1}{2\pi^n + 1} \]
\[ \leq \int_0^\infty \int_{\mathbb{R}^n} \left| \partial_t^\alpha \partial_x^\beta e^{-tL^n} \right|^2 dy dt \frac{1}{2\pi^n} \leq \|h\|^2_2. \]

**Theorem 4.4.** Assume that \( \alpha \in (0, 1) \), \( \beta > 0 \) and \( 0 < \gamma \leq \min\{2\alpha, 2\alpha\beta\} \). Let \( f \) be a linear combination of \( H^{n/(n+\gamma)}_L \)-atoms. There exists a constant \( C \) such that
\[ \|S^L_{\alpha,\beta}(f)\|_{L^{n/(n+\gamma)}} \leq C\|f\|_{H^{n/(n+\gamma)}_L}. \]

**Proof.** Let \( a \) be an \( H^{n/(n+\gamma)}_L \)-atom associated with a ball \( B = B(x_0, r) \). Then we write
\[ \|S^L_{\alpha,\beta}(a)\|_{L^{n/(n+\gamma)}} \leq I + II, \]
where
\[ I := \int_{8B} |S^L_{\alpha,\beta}(a)(x)|^{n/(n+\gamma)} dx; \]
\[ II := \int_{(8B)^c} |S^L_{\alpha,\beta}(a)(x)|^{n/(n+\gamma)} dx. \]
We use Lemma 4.3 and Hölder’s inequality to obtain
\[ I \leq \left( \int_{8B} |S^L_{\alpha,\beta}(a)(x)|^2 dx \right)^{n/(2(n+\gamma))} |B|^{(n+2\gamma)/(2(n+\gamma))} \leq \|a\|_2^{n/(n+\gamma)} |B|^{(n+2\gamma)/(2(n+\gamma))} \leq 1. \]
Now we deal with \( II \) in the following two cases.
We can use Propositions 3.11 and 3.12 to deduce there exists \( r \) such that

\[
\| \mathcal{F} \|^2 \lesssim r^{2(\delta' - \gamma)}
\]

which, via a direct computation gives

\[
\int_{(8B_y)^c} |I_1(x)|^{\frac{\mu}{2(n+\delta')}} dx \lesssim \int_{(8B_y)^c} \left( \frac{r^{\delta' - \gamma}}{|x-x_0|^{\mu+\delta'}} \right)^{\frac{n}{n+\gamma}} dx \lesssim C.
\]

Let us continue with \( I_2 \). Similarly, it follows from Proposition 3.12 that

\[
|I_2(x)| \lesssim \int_{|x-x_0|^{2n+2\delta'}} \int_{|y|<t^{1/2\alpha}} \left\{ \int_{B} \left| \frac{|x-x_0|}{t^{1/2\alpha}} \right|^{\delta'} \left( \frac{r}{t^{1/2\alpha}} \right)^2 \frac{dx'}{B} \right\}^2 \frac{dydt}{t^{\mu/2\alpha+1}} \lesssim \int_{|x-x_0|^{2n+2\delta'}} \int_{|y|<t^{1/2\alpha}} \left( \frac{r}{t^{1/2\alpha}} \right)^{2\delta'} \frac{1}{t^{\mu/2\alpha+1}} \frac{dydt}{t^{\mu/2\alpha+1}} \lesssim \frac{r^{2(\delta' - \gamma)}}{|x-x_0|^{2n+2\delta'}}.
\]

Hence we still have \( \int_{(8B_y)^c} |I_2(x)|^{\frac{\mu}{2(n+\gamma)}} dx \lesssim 1 \).
Case 2: $\rho(x_0)/4 < r < \rho(x_0)$. For this case, the atom $a$ has no canceling condition. We have $(S_{\alpha\beta}(a)(x))^2 \leq I_3(x) + I_4(x) + I_5(x)$, where

\[
\begin{align*}
I_3(x) &:= \int_0^{r^2/4\alpha} \int_{|x-y|<r^{1/2\alpha}} \left\| \int_{\mathbb{R}^n} \frac{\partial^{\beta}}{\partial y} e^{-iL_{x'}^y} (y, x') a(x') dx' \right\|^2 dydt \frac{r^{2\beta}}{n/2\alpha + 1}; \\
I_4(x) &:= \int_{2\alpha/r}^{r^2/4\alpha} \int_{|x-y|<r^{1/2\alpha}} \left\| \int_{\mathbb{R}^n} \frac{\partial^{\beta}}{\partial y} e^{-iL_{x'}^y} (y, x') a(x') dx' \right\|^2 dydt \frac{r^{2\beta}}{n/2\alpha + 1}; \\
I_5(x) &:= \int_{|x-y|>r^2/4\alpha} \int_{|x-y|<r^{1/2\alpha}} \left\| \int_{\mathbb{R}^n} \frac{\partial^{\beta}}{\partial y} e^{-iL_{x'}^y} (y, x') a(x') dx' \right\|^2 dydt \frac{r^{2\beta}}{n/2\alpha + 1}.
\end{align*}
\]

Because $x \in (8B)^c$ and $x' \in B$, then $|x-x_0| < |x-x_0|/8$. On the other hand, for $t \in (0, r^{2\alpha}/4\alpha)$, $|y-x| < r^{1/2\alpha} \leq r/2 < |x-x_0|/8$. This means that $|y-x'| \geq c|x-x_0|$. We can get

\[
|I_3(x)| \leq \int_0^{r^2/4\alpha} \int_{|x-y|<r^{1/2\alpha}} \left( \int_{\mathbb{R}^n} \frac{\partial^{\beta}}{\partial y} e^{-iL_{x'}^y} (y, x') dx' \right)^2 dydt \frac{r^{2\beta}}{n/2\alpha + 1}.
\]

which indicates that

\[
\int_{(8B)^c} \frac{r^{2\alpha\beta}}{n^{n/2\alpha + 2\beta}} \leq \int_{(8B)^c} \frac{r^{2\alpha\beta}}{n^{n/2\alpha + 2\beta}} dy \leq C.
\]

Similarly,

\[
|I_4(x)| \leq \int_{2\alpha/r}^{r^2/4\alpha} \int_{|x-y|<r^{1/2\alpha}} \left( \int_{\mathbb{R}^n} \frac{\partial^{\beta}}{\partial y} e^{-iL_{x'}^y} (y, x') dx' \right)^2 dydt \frac{r^{2\beta}}{n/2\alpha + 1}.
\]

Notice that $r/2 \leq r^{1/2\alpha} \leq |x-x_0|/8$ for $t \in (r^{2\alpha}/4\alpha, |x-x_0|^{2\alpha}/4\alpha^2)$. It can be deduced from the triangle inequality that $|y-x'| \sim |x-x_0|$. Then

\[
|I_4(x)| \leq \int_{2\alpha/r}^{r^2/4\alpha} \int_{|x-y|<r^{1/2\alpha}} \frac{1}{n^{n/2\alpha + 2\beta}} \frac{r^{2\beta}}{n^{n/2\alpha + 2\beta}} \frac{r^{2\beta}}{n^{n/2\alpha + 2\beta}} dy \leq \int_{2\alpha/r}^{r^2/4\alpha} \frac{r^{2\beta}}{n^{n/2\alpha + 2\beta}} \frac{r^{2\beta}}{n^{n/2\alpha + 2\beta}} dy \leq \frac{r^{2\beta}}{n^{n/2\alpha + 2\beta}} \frac{r^{2\beta}}{n^{n/2\alpha + 2\beta}}.
\]
The estimate for $I_5$ is similar to that of $I_4$. In fact, due to $r \sim \rho(x_0)$,

$$|I_5(x)| \lesssim \int_{|x-x_0|^{4/2n}}^{\infty} \int_{|y-y_0|<t^{1/2}a} \left\{ \int_B \frac{\rho}{(t^{1/2}a)^{n+2\alpha}} \, dx \left( \frac{\rho(x_0)}{t^{1/2}a} \right)^{N} \right\}^2 \, dy \, dt.$$

Then for any $H^{12(n+\gamma)}_L$-atom supported on $B(x_0, r)$, there exists a constant $C_{x_0, r}$ such that

$$\sup_{t>0} \left| \int_{\mathbb{R}^n} q_t(x, y) a(y) \, dy \right| \leq C_{x_0, r} (1 + |x|)^{-\gamma/r}, \quad x \in \mathbb{R}^n.$$

**Proof.** If $x \in B(x_0, 2r)$, then $1 + |x| \leq 1 + |x - x_0| + |x_0| \leq 1 + 2r + |x_0|$. It follows from the condition $|a| \leq |B(x_0, r)|^{1-\gamma/r}$ that

$$\left| \int_{\mathbb{R}^n} q_t(x, y) a(y) \, dy \right| \leq \int_{B(x_0, r)} |q_t(x, y)| a(y) \, dy \lesssim \int_{B(x_0, r)} t^{-n/2a} \left( \frac{|x-y|}{t^{1/2}a} \right)^{-\gamma} r^{-\gamma} \, dy \lesssim C_{x_0, r} (1 + |x|)^{-\gamma/r}.$$

If $x \notin B(x_0, 2r)$, then for any $y \in B(x_0, r)$, $|x-y| \sim |x-x_0|$. On the other hand, $\rho(y) \sim \rho(x_0)$ since $r < \rho(x_0)$ and $|y-x_0| < r$. By Proposition 3.11, we have

$$|q_t(x, y)| \lesssim t^{-n/2a} \left( \frac{|x-y|}{t^{1/2}a} \right)^{-\gamma} \left( \frac{\rho(x)}{t^{1/2}a} \right)^{-N} \lesssim t^{-n/2a} \left( \frac{|x-y|}{t^{1/2}a} \right)^{-\gamma} \left( \frac{\rho(x)}{t^{1/2}a} \right)^{-N} = (\rho(x_0))^{\theta} r^{-\gamma} |x-x_0|^{-\gamma} \lesssim C_{x_0, r} |x-x_0|^{-\gamma}.$$

Because $x \notin B(x_0, 2r)$, set $x = x_0 + 2r \epsilon$, where $|\epsilon| \geq 1$. Then $1 + |x| \leq 1 + |x_0| + 2r|\epsilon|$ and $\frac{1 + |x_0| + 2r}{2r}|x-x_0| = (1 + |x_0| + 2r)|\epsilon| \geq 1 + |x_0| + 2r|\epsilon|,$

which implies that $|x_0 - x| \geq (1 + |x|)C_{x_0, r}$. This completes the proof of Lemma 4.5. \hfill \Box
Lemma 4.6. Given $\alpha \in (0, 1)$, $\beta > 0$ and $0 < \gamma \leq \min(2\alpha, 2\alpha\beta)$. Let $f \in L^1(\mathbb{R}^n, (1 + |x|)^{-n+2\alpha+2\alpha\beta}dx)$ for any $\epsilon > 0$ and let $a$ be an $H_L^{n/(n+\gamma)}$-atom. Then for

$$
\begin{align*}
F(x, t) := & \beta^\alpha \partial_\xi^\alpha e^{-tL^\alpha}(f)(x); \\
G(x, t) := & \beta^\alpha \partial_\xi^\alpha e^{-tL^\alpha}(a)(x),
\end{align*}
$$

there exists a constant $C_{\alpha, \beta}$ such that

$$
C_{\alpha, \beta} \int_{\mathbb{R}^n} f(x)a(x)dx = \int \int_{\mathbb{R}^{n+1}} F(x, t)\overline{G(x, t)}\frac{dt}{t}.
$$

Proof. Assume that $a$ is an $H_L^{n/(n+\gamma)}$-atom associated to a ball $B(x_0, r)$. By Lemma 4.2 and Theorem 4.4, we get

$$
I = \int \int_{\mathbb{R}^{n+1}} F(x, t)\overline{G(x, t)}\frac{dt}{t}
= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} D^\beta_{\alpha, \beta}(f)D^\beta_{\alpha, \beta}(a)dxdt
= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \sup_{t>0} |D^\beta_{\alpha, \beta}(a)(x)| \{ \int_{\mathbb{R}^n} |D^\beta_{\alpha, \beta}(f)(x)|dx \}.
$$

The inner integration satisfies

$$
\left| \int_{\mathbb{R}^n} D^\beta_{\alpha, \beta}(f)(x)D^\beta_{\alpha, \beta}(a)(x)dx \right| \leq \int_{\mathbb{R}^n} |D^\beta_{\alpha, \beta}(f)(x)| \cdot |D^\beta_{\alpha, \beta}(a)(x)|dx
\leq \left\{ \sup_{t>0} |D^\beta_{\alpha, \beta}(a)(x)| \right\} \left\{ \int_{\mathbb{R}^n} |D^\beta_{\alpha, \beta}(f)(x)|dx \right\}.
$$

By Proposition 3.11 we can see that

$$
|D^\beta_{\alpha, \beta}(x, y)| \leq \frac{\beta^\alpha}{(t^{1/2\alpha} + |x-y|)^{n+2\alpha\beta}} \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N}
\leq r^{-n/2\alpha} \frac{1}{(1 + |x-y|/t^{1/2\alpha})^{n+2\alpha\beta}} \left(1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N}.
$$

If $x \in B(x_0, 2r)$, then $1 + |x| \leq 1 + |x - x_0| + |x_0| \leq 1 + 2r + |x_0|$. It follows from the condition $||a||_\infty \leq |B(x_0, r)|^{-1-\gamma/n}$ that

$$
\left| \int_{\mathbb{R}^n} D^\beta_{\alpha, \beta}(x, y)a(y)dy \right| \leq \int_{B(x_0, r)} |D^\beta_{\alpha, \beta}(x, y)||a(y)|dy
\leq \int_{B(x_0, r)} r^{-n/2\alpha} \left(1 + \frac{|x-y|}{t^{1/2\alpha}} \right)^{-n-2\alpha\beta} r^{-\gamma} dy
\leq r^{-n-\gamma} (1 + 2r + |x_0|)^{n+2\alpha\beta}
\frac{(1 + 2r + |x_0|)^{n+2\alpha\beta}}{(1 + 2r + |x_0|)^{n+2\alpha\beta}}
\leq C_{N, x_0, \gamma} (1 + |x|)^{-n-2\alpha\beta}.
$$
If \( x \notin B(x_0, 2r) \), then for any \( y \in B(x_0, r) \), \(|x - y| \sim |x - x_0|\). On the other hand, \( \rho(y) \sim \rho(x_0) \) since \( r < \rho(x_0) \) and \(|y - x_0| < r\). By Proposition 3.11, we have

\[
|D_{\alpha, \beta}^L(x, y)| \leq \frac{\rho^\beta}{(t^{1/2\alpha} + |x - y|)^{n+2\beta}} \left( 1 + \frac{t^{1/2\alpha}}{\rho(x)} + \frac{t^{1/2\alpha}}{\rho(y)} \right)^{-N}
\]

which implies that

\[
\left| \int_{\mathbb{R}^n} D_{\alpha, \beta}^L(x, y)a(y)dy \right| \leq \frac{(t^{1/2\alpha})^{-2\beta}}{\rho(x_0)} \int_{\mathbb{R}^n} |a(y)|dy \leq \rho(x_0)^{2\beta} r^{-\gamma} |x - x_0|^{-(n+2\beta)} := C_{\gamma, x_0, r}|x - x_0|^{-(n+2\beta)}.
\]

Because \( x \notin B(x_0, 2r) \), set \( x = x_0 + 2rz \), where \( |z| \geq 1 \). Then \( 1 + |x| \leq 1 + |x_0| + 2r|z| \) and

\[
\frac{1}{2r}|x - x_0| = (1 + |x_0| + 2r|z|) \geq 1 + |x_0| + 2r|z|,
\]

which implies that \(|x_0 - x| \geq (1 + |x|)/C_{x_0, r}\). The above estimate indicate that \( D_{\alpha, \beta}^L(\cdot, \cdot) \) satisfies (4.5) with \( \theta = 2\alpha \beta \). On the other hand,

\[
\int_{\mathbb{R}^n} |D_{\alpha, \beta}^L(f(x))|D_{\alpha, \beta}^L(a(x))dx \leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{|f(y)|\rho^\beta}{(t^{1/2\alpha} + |x - y|)^{n+2\beta}}dy \right) \frac{dxdy}{(1 + |x|)^{n+2\beta}} \leq I_1 + I_2,
\]

where

\[
I_1 := \int_{|x| + 1/2} \left( \int_{|y| \geq |x| + 1/2} \frac{|f(y)|\rho^\beta}{(t^{1/2\alpha} + |x - y|)^{n+2\beta}}dy \right) \frac{dxdy}{(1 + |x|)^{n+2\beta}}.
\]

If \(|x - y| < |y|/2\), then \(|y| \leq |x - y| + |x| \leq |y|/2 + |x|, i.e., \(|y| \leq 2|x|\).

\[
I_2 := \int_{|x| + 1/2} \left( \int_{|y| \leq |x| + 1/2} \frac{|f(y)|\rho^\beta}{(t^{1/2\alpha} + |x - y|)^{n+2\beta}}dy \right) \frac{dxdy}{(1 + |x|)^{n+2\beta}} < \infty.
\]

For \(I_1\), we have \(|x| \geq |y|/2\) since \(|x - y| \leq |y|/2\).

\[
I_1 \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)|\rho^\beta}{(t^{1/2\alpha} + |x - y|)^{n+2\beta}}dy \frac{dxdy}{(1 + |x|)^{n+2\beta}} < \infty.
\]

Notice that

\[
\int_{\mathbb{R}^n} D_{\alpha, \beta}^L(f(x))D_{\alpha, \beta}^L(a(x))dx = \int_{\mathbb{R}^n} f(x)(D_{\alpha, \beta}^L)^2(a(x))dx = \int_{\mathbb{R}^n} f(x)\partial^\beta_t e^{-2tL^\alpha}(a(x))dx,
\]

which, together with the Fubini theorem, indicates that

\[
(4.5) \quad I = \lim_{\epsilon \to 0} \int_{\epsilon}^{1/\epsilon} \left\{ \int_{\mathbb{R}^n} f(y)\partial^\beta_t e^{-2tL^\alpha}(a(y))dy \right\} \frac{dt}{t} = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} f(y)\left\{ \int_{\epsilon}^{1/\epsilon} \partial^\beta_t e^{-2tL^\alpha}(a(y))\frac{dt}{t} \right\}dy.
\]
For the term
\[
\int_{\epsilon}^{1/\epsilon} t^2 \frac{\partial^2 e^{-2tL^u}(a(y))}{\partial t^2} dt.
\]
we can see that
\[
| \int_{\epsilon}^{1/\epsilon} t^2 \frac{\partial^2 e^{-2tL^u}(a(y))}{\partial t^2} dt | \leq | \int_{\epsilon}^{1/\epsilon} t^2 \frac{\partial^2 e^{-2tL^u}(a(y))}{\partial t^2} dt | + | \int_{1/\epsilon}^{\infty} t^2 \frac{\partial^2 e^{-2tL^u}(a(y))}{\partial t^2} dt |
\]
\[
= \left| \int_{\epsilon}^{1/\epsilon} t^2 \frac{\partial^2 e^{-2tL^u}(a(y))}{\partial t^2} dt \right| a(x) dx \bigg| + \left| \int_{1/\epsilon}^{\infty} t^2 \frac{\partial^2 e^{-2tL^u}(a(y))}{\partial t^2} dt \right| a(x) dx.
\]
By the change of variables, we obtain
\[
\left| \int_{\epsilon}^{1/\epsilon} t^2 \frac{\partial^2 e^{-2tL^u}(x,y)}{\partial t^2} dt \right| = \left| C_\beta \int_{\epsilon}^{1/\epsilon} t^2 \int_0^{\infty} t^2 \frac{\partial^2 \left| e^{-2tL^u} \right|}{\partial t^2} dt \right| dx
\]
\[
= \left| C_\beta \int_{\epsilon}^{1/\epsilon} t^2 \int_0^{\infty} L^m e^{-2tL^u} \frac{ds}{s^{1+2\beta-m}} dt \right| dx
\]
\[
\approx \left| C_\beta \int_{\epsilon}^{1/\epsilon} t^2 \int_0^{\infty} \frac{ds}{s^{1+2\beta-m}} dt \right| dx
\]
The rest of the proof is divided into three cases.

**Case 1:** $2\beta < 1$. For this case, $[2\beta] + 1 = 1$. Then a change of variable reaches
\[
\int_{2\epsilon}^{\infty} t^2 \frac{\partial^2 e^{-2tL^u}(x,y)}{\partial t^2} dt = \int_{2\epsilon}^{\infty} t^2 \int_0^{\infty} \partial_t e^{-(t+u)L^u} (x,y) \frac{du}{u^{1+2\beta}} dt
\]
\[
= \int_{2\epsilon}^{\infty} t^2 \int_0^{\infty} \partial_u e^{-uL^u} (x,y) \frac{du}{(u-t)^{2\beta}} dt
\]
\[
= \int_{2\epsilon}^{\infty} \partial_u e^{-uL^u} (x,y) \left( \int_{2\epsilon}^{\infty} \left( \frac{t}{u-t} \right)^{2\beta} dt \right) du
\]
\[
= \int_{2\epsilon}^{\infty} \partial_u e^{-uL^u} (x,y) \left( \int_{2\epsilon/u}^{1} \left( \frac{w}{1-w} \right)^{2\beta} dw \right) du.
\]
Notice that
\[
e^{-L^u} (x,y) \int_{2\epsilon/2}^{1} \left( \frac{w}{1-w} \right)^{2\beta} dw = 0,
\]
and as $u \to \infty$,
\[
|e^{-uL^u} (x,y)| \leq \frac{u}{(u^{1/2\alpha} + |x-y|)^{\alpha+2\alpha}} \leq \frac{1}{u^{\alpha/2\alpha}} \to 0.
\]
An application of integration by parts gives
\[
\int_{2\epsilon}^{\infty} t^2 \frac{\partial^2 e^{-2tL^u}(x,y)}{\partial t^2} dt = -\int_{2\epsilon}^{\infty} e^{-uL^u}(x,y) \frac{\partial}{\partial u} \left( \int_{2\epsilon/u}^{1} \left( \frac{w}{1-w} \right)^{2\beta} dw \right) du
\]
\[
= \int_{2\epsilon}^{\infty} e^{-uL^u}(x,y) \left( \frac{2\epsilon}{u-2\epsilon} \right)^{2\beta} du
\]
\[
= I + II.
\]
where

\[
\begin{align*}
I & := \int_{2\epsilon}^{\infty} e^{-uL^u} (x,y) \left( \frac{2\epsilon}{u - 2\epsilon} \right)^{2\beta} \chi_A(u) \frac{du}{u}; \\
II & := \int_{2\epsilon}^{\infty} e^{-uL^u} (x,y) \left( \frac{2\epsilon}{u - 2\epsilon} \right)^{2\beta} \chi_{A'}(u) \frac{du}{u},
\end{align*}
\]

where \( A := \{ u : u - 2\epsilon \leq \epsilon + |x-y|^{2\alpha} \} \). By Proposition 3.1

\[
|I| \leq \int_{2\epsilon}^{\infty} \frac{u}{(u^{1/2\alpha} + |x-y|)^{n+2\alpha}} \left( 1 + \frac{u^{1/2\alpha}}{\rho(x)} + \frac{u^{1/2\alpha}}{\rho(y)} \right)^{-N} \left( \frac{2\epsilon}{u - 2\epsilon} \right)^{2\beta} \chi_A(u) \frac{du}{u}
\]

\[
\leq \frac{1}{\epsilon^{n/2\alpha}} \frac{1}{(1 + |x-y|/\epsilon^{1/2\alpha})^{n+4\beta}} \left( 1 + \frac{\epsilon^{1/2\alpha}}{\rho(x)} + \frac{\epsilon^{1/2\alpha}}{\rho(y)} \right)^{-N}.
\]

For \( II \),

\[
|II| \leq \int_{3\epsilon+|x-y|^{2\alpha}}^{\infty} \frac{u}{(u^{1/2\alpha} + |x-y|)^{n+2\alpha}} \left( 1 + \frac{u^{1/2\alpha}}{\rho(x)} + \frac{u^{1/2\alpha}}{\rho(y)} \right)^{-N} \left( \frac{2\epsilon}{u - 2\epsilon} \right)^{2\beta} \chi_{A'}(u) \frac{du}{u}
\]

\[
\leq \frac{1}{\epsilon^{n/2\alpha}} \frac{1}{(1 + |x-y|/\epsilon^{1/2\alpha})^{n+4\beta}} \left( 1 + \frac{\epsilon^{1/2\alpha}}{\rho(x)} + \frac{\epsilon^{1/2\alpha}}{\rho(y)} \right)^{-N}.
\]

Case 2: \( 2\beta = 1 \). A direct computation gives

\[
\left| \int_{2\epsilon}^{\infty} t \partial_r e^{-uL^u} (x,y) dt \right| \leq |e^{-uL^u} (x,y)|
\]

\[
\leq \frac{1}{\epsilon^{n/2\alpha}} \frac{1}{(1 + |x-y|/\epsilon^{1/2\alpha})^{n+2\alpha}} \left( 1 + \frac{\epsilon^{1/2\alpha}}{\rho(x)} + \frac{\epsilon^{1/2\alpha}}{\rho(y)} \right)^{-N}.
\]

Case 3: \( 2\beta > 1 \). Let \( k \in \mathbb{Z}_+ \) such that \( k - 1 < 2\beta \leq k, k \geq 2 \). We obtain

\[
M = \int_{2\epsilon}^{\infty} \int_0^{\infty} \partial_r e^{-uL^u} (x,y) \frac{dt}{t}
\]

\[
= \int_{2\epsilon}^{\infty} \int_0^{\infty} \partial_k e^{-((t+s)L^u) (x,y)} \frac{ds}{s^{1+2\beta-\alpha}} \frac{dt}{t}
\]

\[
= \int_{2\epsilon}^{\infty} \int_0^{\infty} L^{k-1} e^{-uL^u} (x,y) \frac{du}{(u^{1/2\alpha} + |x-y|)^{n+2\alpha}} \frac{dt}{t}
\]

\[
= \int_{2\epsilon}^{\infty} \int_0^{\infty} \partial_k e^{-uL^u} (x,y) \frac{dt}{t}
\]

\[
= \int_{2\epsilon}^{\infty} \partial_k e^{-uL^u} (x,y) \left( \int_{2\epsilon}^{\infty} t^{2\beta} (u-t)^{k-2\beta-1} \frac{dt}{t} \right) du
\]

\[
= \int_{2\epsilon}^{\infty} \partial_k e^{-uL^u} (x,y) \left( \int_{2\epsilon}^{\infty} \frac{\partial_k e^{-uL^u} (x,y) \int_{2\epsilon}^{\infty} u^{2\beta} (1-w)^{k-2\beta-1} \frac{dw}{w} \right) du,
\]
where in the last step we have used the change of variables: $w = t/u$. Notice that
\[
    u^{k-1} \partial_u^k e^{-uL^a}(x, y) = \partial_u [u^{k-1} \partial_u^{k-1} e^{-uL^a}(x, y)] - (k-1) \partial_u [u^{k-2} \partial_u^{k-2} e^{-uL^a}(x, y)] + \cdots + (-1)^{k-1} (k-1)! e^{-uL^a}(x, y).
\]
Then the integration by parts yields $M = \sum_{m=1}^{k} C_m I_m$, where $C_m = (-1)^{m-1}(k-1)/(k-m+1)$ and
\[
    I_m := \int_{2\epsilon}^{\infty} u^m \partial_u^m e^{-uL^a}(x, y) \frac{(2\epsilon)^{2\beta} u^{1-k} \rho(n)}{(u-2\epsilon)^{1+2\beta-k}} \, du.
\]
We obtain
\[
    |I_m| \leq \int_{2\epsilon}^{\infty} \frac{u^m}{(u^{1/2\alpha} + |x-y|)^{\gamma+2am}} (1 + \frac{u^{1/2\alpha}}{\rho(x)} + \frac{u^{1/2\alpha}}{\rho(y)})^{-N} \frac{(2\epsilon)^{2\beta} u^{1-k} \rho(n)}{(u-2\epsilon)^{1+2\beta-k}} \, du \leq I_m^{(1)} + I_m^{(2)},
\]
where
\[
    \begin{align*}
    I_m^{(1)} &:= \frac{e^{2\beta}}{(e^{1/2\alpha} + |x-y|)^{\gamma+2am}} (1 + \frac{e^{1/2\alpha}}{\rho(x)} + \frac{e^{1/2\alpha}}{\rho(y)})^{-N} \int_{2\epsilon}^{3\epsilon} \frac{1}{(u-2\epsilon)^{1+2\beta-k}} \, du, \\
    I_m^{(2)} &:= \frac{e^{2\beta}}{(e^{1/2\alpha} + |x-y|)^{\gamma+2am}} (1 + \frac{e^{1/2\alpha}}{\rho(x)} + \frac{e^{1/2\alpha}}{\rho(y)})^{-N} \int_{3\epsilon}^{\infty} \frac{1}{(u-2\epsilon)^{1+2\beta-k}} \, du.
    \end{align*}
\]
For $I_m^{(1)}$, since $2\epsilon < u < 3\epsilon$, we get
\[
    I_m^{(1)} \leq \frac{e^{2\beta}}{(e^{1/2\alpha} + |x-y|)^{\gamma+2am}} (1 + \frac{e^{1/2\alpha}}{\rho(x)} + \frac{e^{1/2\alpha}}{\rho(y)})^{-N} \frac{1}{e^{k-m}} \int_{2\epsilon}^{3\epsilon} \frac{1}{(u-2\epsilon)^{1+2\beta-k}} \, du \leq \frac{1}{e^{n/2\alpha}} (1 + \frac{1}{|x-y|/e^{1/2\alpha}})^{\gamma+2am} (1 + \frac{e^{1/2\alpha}}{\rho(x)} + \frac{e^{1/2\alpha}}{\rho(y)})^{-N}.
\]
Similarly, for $I_m^{(2)}$, because $u \in (3\epsilon, \infty)$, then $1/u \leq 1/(u-2\epsilon)$. Noticing that $m < 2\beta$, we obtain
\[
    I_m^{(2)} \leq \frac{e^{2\beta}}{(e^{1/2\alpha} + |x-y|)^{\gamma+2am}} (1 + \frac{e^{1/2\alpha}}{\rho(x)} + \frac{e^{1/2\alpha}}{\rho(y)})^{-N} \int_{3\epsilon}^{\infty} \frac{1}{(u-2\epsilon)^{1+2\beta-m}} \, du \leq \frac{1}{e^{n/2\alpha}} (1 + |x-y|/e^{1/2\alpha})^{\gamma+2am} (1 + \frac{e^{1/2\alpha}}{\rho(x)} + \frac{e^{1/2\alpha}}{\rho(y)})^{-N}.
\]
By Lemma 4.5, the above estimates in Cases 1-3 indicate that
\[
    \sup_{\epsilon > 0} \left| \int_{\epsilon}^{1/\epsilon} t^{2\beta} \partial_t^{2\beta} e^{-2L^a}(a)(y) \, dt \right| \leq (1 + |y|)^{-(n+\gamma+\epsilon)},
\]
Therefore we can use Lemma 4.1 completes the proof. \hfill \Box

Finally, we can obtain the following characterization of $BMO^\gamma_L(\mathbb{R}^n)$ corresponding to the time-fractional derivative.

**Theorem 4.7.** Let $V \in B_q, q > n$. Assume that $\alpha \in (0, 1)$, $\beta > 0$, $0 < \gamma < 1$ with
\[
    0 < \gamma < \min(2\alpha, 2\beta \alpha).
\]
Let $f$ be a function such that
\[
    \left( \int_{\mathbb{R}^n} \frac{|f(x)|}{(1 + |x|)^{n+\gamma+\epsilon}} \, dx \right) < \infty
\]
for some $\epsilon > 0$. The following statements are equivalent:

(4.6)
(i) $f \in \text{BMO}^\gamma_L(\mathbb{R}^n)$;
(ii) There exists $C_{\alpha, \beta}$ such that $\|D^L_{\alpha,t}(f)\|_\infty \leq C_{\alpha, \beta} r^{1/2\alpha}$;
(iii) For all $B = B(x_B, r_B) \subset \mathbb{R}^n$,

\begin{equation}
\left( \frac{1}{|B|} \int_0^{r_B^\alpha} \int_B |D^L_{\alpha,t}(f)(x)|^2 \frac{dxdt}{t} \right)^{1/2} \lesssim \|f\|_{\text{BMO}^\gamma_L} |B|^{\gamma/n}.
\end{equation}

Proof. (i)$\implies$(ii). If $f \in \text{BMO}^\gamma_L(\mathbb{R}^n)$, then $|\beta^\beta \delta_r e^{-\text{LT}_n} f(x)| \leq 1 + II$, where

\begin{align*}
II &: = f(x) \int_{\mathbb{R}^n} D^L_{\alpha,t}(x, y)|dy|
\end{align*}

For $I$, we have

\begin{equation*}
I \leq \|f\|_{\text{BMO}^\gamma_L} \int_{\mathbb{R}^n} \frac{r^\beta|x-y|^\gamma}{(1/t^{1/2\alpha} + |x-y|)^{\alpha+2\beta}} dy \lesssim t^{1/2\alpha} \|f\|_{\text{BMO}^\gamma_L}.
\end{equation*}

We further divide the estimation of $II$ into the following two cases.

Case 1: $\rho(x) \leq 1^{1/2\alpha}$. By Proposition 3.11

\begin{align*}
II &\leq \|f\|_{\text{BMO}^\gamma_L} \rho(x)^\gamma \int_{\mathbb{R}^n} D^L_{\alpha,t}(x, y) dy \\
&\leq \|f\|_{\text{BMO}^\gamma_L} \rho(x)^{1/2\alpha} \int_{\mathbb{R}^n} \frac{r^\beta}{(1/t^{1/2\alpha} + |x-y|)^{\alpha+2\beta}} dy \\
&\lesssim \|f\|_{\text{BMO}^\gamma_L} t^{1/2\alpha}.
\end{align*}

Case 2: $\rho(x) > 1^{1/2\alpha}$. We use Proposition 3.13 to obtain that, there exists $\delta' > \gamma$ such that

\begin{align*}
II &\leq \|f\|_{\text{BMO}^\gamma_L} \rho(x)^\gamma \int_{\mathbb{R}^n} D^L_{\alpha,t}(x, y) dy \\
&\leq \|f\|_{\text{BMO}^\gamma_L} \rho(x)^{1/2\alpha} \frac{(1/t^{1/2\alpha} \rho(x))^{\delta'}}{(1 + t^{1/2\alpha} \rho(x))^{\gamma}} \\
&\lesssim \|f\|_{\text{BMO}^\gamma_L} t^{1/2\alpha}.
\end{align*}

(ii)$\implies$(iii). Assume that (ii) holds. Then

\begin{equation*}
\left( \frac{1}{|B|} \int_0^{r_B^\alpha} \int_B |D^L_{\alpha,t}(f)(x)|^2 \frac{dxdt}{t} \right)^{1/2} \leq \|f\|_{\text{BMO}^\gamma_L} \left( \frac{1}{|B|} \int_0^{r_B^\alpha} \int_B \frac{dxdt}{t} \right)^{1/2} \leq \|f\|_{\text{BMO}^\gamma_L} |B|^{\gamma/n}.
\end{equation*}

(iii)$\implies$(i). Assume that (4.7) holds. Let $a$ be an $H^\gamma_L^{(n+\gamma)}$-atom associated with $B = B(x_B, r_B)$. Then by Lemma 4.6

\begin{equation*}
\int_{\mathbb{R}^n} f(x)a(x)dx \approx \int_{\mathbb{R}^{n+1}} r^\beta \delta_r^\beta e^{-\text{LT}_n} f(x) \delta_r^\beta e^{-\text{LT}_n} (a)(x) \frac{dt}{t}.
\end{equation*}
which, together with (4.3) and Theorem 4.4 gives
\[
\left| \int_{\mathbb{R}^n} f(x) \alpha(x) dx \right| \leq \sup_B \left( \frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B} \int_B |\partial_t^{1/2} e^{-tL^n}(f)(x)|^2 \frac{dxdt}{t} \right)^{1/2} \\
\times \left\{ \int_{\mathbb{R}^n} \left( \int_{|x-y|<1/2r} |\partial_t^{1/2} e^{-tL^n}(a)(y)|^2 \frac{1}{\rho(t)^{\alpha/n+1}} \right)^{n/(2n+\gamma)} dy \right\}^{1+\gamma/n} \\
\leq ||S_{L,\alpha,\beta}(a)||_{L^{n/(n+\gamma)}} \sup_B \left( \frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B} \int_B |\partial_t^{1/2} e^{-tL^n}(f)(x)|^2 \frac{dxdt}{t} \right)^{1/2} \\
\leq ||a||_{H^\gamma_L}^{n/(n+\gamma)},
\]

Hence
\[
T(g) := \int_{\mathbb{R}^n} f(x)g(x)dx, \quad g \in H^\gamma_L(\mathbb{R}^n),
\]
is a bounded linear functional on \(H^\gamma_L(\mathbb{R}^n)\), equivalently, \(f \in (H^\gamma_L(\mathbb{R}^n))^* = BMO^\gamma_L(\mathbb{R}^n)\).

Below we consider the characterization of \(BMO^\gamma_L(\mathbb{R}^n)\) via the the spatial gradient. Define a general gradient as \(\nabla a := (\nabla x, \partial_t^{1/2\alpha})\).

**Theorem 4.8.** Let \(V \in B_q, q > n\). Assume that \(\alpha \in (0, 1/2 - n/2q), \beta > 0\) and \(0 < \gamma \leq 1\) with \(0 < \gamma < \min(2\alpha, 2\alpha\beta)\).

Let \(f\) be a function satisfying (4.6). The following statements are equivalent:

(i) \(f \in BMO^\gamma_L(\mathbb{R}^n)\);

(ii) There exists a constant \(C > 0\) such that
\[
||1^{1/2\alpha} \nabla a e^{-tL^n} f||_\infty \leq Ct^{\gamma/2\alpha},
\]

(iii) \(u(x, t) = e^{-tL^n} f(x)\) satisfies that, for any balls \(B = B(x_B, r_B)\),
\[
\frac{1}{|B|^{1+2\gamma/n}} \int_0^{r_B} \int_B |1^{1/2\alpha} \nabla a e^{-tL^n}(f)(x)|^2 \frac{dxdt}{t} \leq C.
\]

**Proof.** (i) \(\Rightarrow\) (ii). Let \(f \in BMO^\gamma_L(\mathbb{R}^n)\). By Theorem 4.7, \(||1^{1/2\alpha} \partial_t^{1/2\alpha} e^{-tL^n}(f)||_\infty \leq C\alpha t^{\gamma/2\alpha}\). One writes
\[
1^{1/2\alpha} \nabla x e^{-tL^n} f(x) = \int_{\mathbb{R}^n} 1^{1/2\alpha} \nabla_x K^L_{\alpha,\beta}(f(z) - f(x))dz + f(x) 1^{1/2\alpha} \nabla x e^{-tL^n}(1)(x) := I(x) + II(x).
\]

We first estimate the term I. Because \(f \in BMO^\gamma_L(\mathbb{R}^n)\), then \(|f(x) - f(z)| \leq ||f||_{BMO^\gamma_L}|x - z|\). Since
\[
|1^{1/2\alpha} \nabla x K^L_{\alpha,\beta}(x, z)| \leq \frac{t}{(t^{1/2\alpha} + |x - z|)^{n+2\alpha}},
\]
a direct computation gives
\[
|I(x)| \leq ||f||_{BMO^\gamma_L} \int_{\mathbb{R}^n} |1^{1/2\alpha} \nabla x K^L_{\alpha,\beta}(x, z)| |x - z|\gamma dz \\
\leq ||f||_{BMO^\gamma_L} \int_{\mathbb{R}^n} \frac{t|x - z|\gamma}{(t^{1/2\alpha} + |x - z|)^{n+2\alpha}} dz \\
\leq t^{\gamma/2\alpha} ||f||_{BMO^\gamma_L}.
\]

By Proposition 3.10, we have
\[
|1^{1/2\alpha} \nabla x e^{-tL^n}(1)| \leq \min\left\{t^{1/2\alpha}/\rho(x))^{1+2\alpha}, (t^{1/2\alpha}/\rho(x))^{-N}\right\}.
\]
The estimate of II is divided into two cases.
Case 1: $\rho(x) \leq t^{1/2\alpha}$. If $f \in BMO^\gamma_L(\mathbb{R}^n)$ implies that $|f(x)| \leq \rho^\gamma(x)\|f\|_{BMO^\gamma_L}$. Then
\[
\begin{align*}
H(x) & \leq \|f\|_{BMO^\gamma_L} \rho^\gamma(x)|t^{1/2\alpha} \nabla x e^{-tL^\alpha}(1)(x)| \\
& \leq \|f\|_{BMO^\gamma_L} \rho^\gamma(x)\left(\frac{1/2\alpha}{\rho(x)}\right)^N \gamma^{-N} \\
& \leq \|f\|_{BMO^\gamma_L} \gamma^{-2\alpha} \|f\|_{BMO^\gamma_L}.
\end{align*}
\]

Case 2: $\rho(x) > t$. We can get
\[
\begin{align*}
H(x) & \leq \rho^\gamma(x)\|f\|_{BMO^\gamma_L} |t^{1/2\alpha} \nabla x e^{-tL^\alpha}(1)(x)| \\
& \leq \rho^\gamma(x)\|f\|_{BMO^\gamma_L} \left(\frac{1/2\alpha}{\rho(x)}\right)^{1+2\alpha} \\
& \leq \rho^\gamma(x)\|f\|_{BMO^\gamma_L} \left(\frac{1/2\alpha}{\rho(x)}\right)^{1+2\alpha-\gamma} \\
& \leq \rho^\gamma(x)\|f\|_{BMO^\gamma_L}.
\end{align*}
\]

(ii) $\implies$ (iii). For every ball $B = B(x_B, r_B)$,
\[
\int_0^{r_B^2} \int_B \left|t^{1/2\alpha} \nabla x e^{-tL^\alpha} f(x)\right|^2 \frac{dxdt}{t} \leq \int_0^{r_B^2} \int_B \left|t^{1/2\alpha} \nabla x e^{-tL^\alpha} (f(x))\right|^2 \frac{dxdt}{t} < \infty.
\]
which implies that (4.8) holds.

(iii) $\implies$ (i). Assume that (4.8) holds. For any ball $B = B(x_B, r_B)$, it holds
\[
\sup_B r_B^{-(n+2\gamma)} \int_0^{r_B^2} \int_B \left|t^{1/2\alpha} \partial_t^{1/2\alpha} e^{-tL^\alpha} (f(x))\right|^2 \frac{dxdt}{t} < \infty.
\]
It is a corollary of Theorem 4.7 that $f \in BMO^\gamma_L(\mathbb{R}^n)$ with
\[
\|f\|_{BMO^\gamma_L} \leq \sup_B r_B^{-(n+2\gamma)} \int_0^{r_B^2} \int_B \left|t^{1/2\alpha} \partial_t^{1/2\alpha} e^{-tL^\alpha} (f(x))\right|^2 \frac{dxdt}{t} < \infty.
\]

A positive measure $\nu$ on $\mathbb{R}^{n+1}_+$ is called a $\kappa$-Carleson measure if
\[
\|\nu\|_C := \sup_{x \in \mathbb{R}^n, r > 0} \frac{\nu(B(x, r) \times (0, r))}{|B(x, r)|^\kappa} < \infty.
\]

The following result can be deduced from Theorem 4.8 immediately.

**Theorem 4.9.** Let $V \in B_q, q > n$. Assume that $\alpha \in (0, 1/2 - n/2q), \beta > 0$ and $0 < \gamma \leq 1$ with
\[0 < \gamma < \min\{2\alpha, 2\alpha\beta\}.
\]

Let $d\nu_\alpha$ be a measure defined by
\[
d\nu_\alpha(x, t) := \left|t^{1/2\alpha} \partial_t^{1/2\alpha} e^{-tL^\alpha} (f(x))\right|^2 \frac{dxdt}{t}, \ (x, t) \in \mathbb{R}^{n+1}.
\]

(i) If $f \in BMO^\gamma_L(\mathbb{R}^n)$, then $d\nu_\alpha$ is a $(1 + 2\gamma/n)$-Carleson measure;
(ii) Conversely, if $f \in L^1((1 + |x|)^{-n-1} dx)$ and $d\nu_\alpha$ is a $(1 + 2\gamma/n)$-Carleson measure, then $f \in BMO^\gamma_L(\mathbb{R}^n)$.

Moreover, in any case, there exists a constant $C > 0$ such that
\[
C^{-1}\|f\|^2_{BMO^\gamma_L} \leq \|d\nu_\alpha\|_C \leq C\|f\|^2_{BMO^\gamma_L}.
\]
Proof. (i). In Theorem 4.7, letting $\beta = 1$, we obtain for $f \in BMO^\gamma_L(\mathbb{R}^n)$,

$$
\frac{1}{|B|^{1+2\gamma/n}} \int_B |\partial_t e^{-tL_n^\alpha}(f)(x)|^2 \frac{dxdt}{t} \leq \|f\|_{BMO^\gamma_L}^2,
$$

which, together with a change of variable, gives

$$
\frac{1}{|B|^{1+2\gamma/n}} \int_B |\partial_t e^{-tL_n^\alpha}(f)(x)|^2 \frac{dxdt}{t} = \frac{1}{|B|^{1+2\gamma/n}} \int_B |2^{2\gamma} L_n^\alpha e^{-2^{2\gamma} L_n^\alpha}(f)(x)|^2 \frac{dxdt}{t}
$$

$$
= \frac{1}{|B|^{1+2\gamma/n}} \int_B |\partial_t e^{-tL_n^\alpha}(f)(x)|^2 \frac{dxdt}{t} \leq \|f\|_{BMO^\gamma_L}^2.
$$

The estimation

$$
\frac{1}{|B|^{1+2\gamma/n}} \int_B |\nabla e^{-tL_n^\alpha} f(x)|^2 \frac{dxdt}{t} \leq \|f\|_{BMO^\gamma_L}^2
$$

can be obtained in the manner of Theorem 4.7.

(ii). Assume that $d\nu_\alpha$ is a $(1 + 2\gamma/n)$-Carleson measure, i.e., for any ball $B(x,r)$,

$$
\sup_B \frac{1}{|B|^{1+2\gamma/n}} \int_B |\nabla e^{-tL_n^\alpha} f(x)|^2 \frac{dxdt}{t} < \infty.
$$

Subsequently,

$$
\sup_B \frac{1}{|B|^{1+2\gamma/n}} \int_B |\partial_t e^{-tL_n^\alpha} (f)(x)|^2 \frac{dxdt}{t} < \infty.
$$

It can be deduced from Theorem 4.8 that $f \in BMO^\gamma_L(\mathbb{R}^n)$. \qed

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