The paradox of infinitesimal granularity: Chaos and the reversibility of time in Newton’s theory of gravity

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Abstract. The fundamental laws of physics are time-symmetric, but our macroscopic experience contradicts this. The time reversibility paradox is partly a consequence of the unpredictability of Newton’s equations of motion. We measure the dependence of the fraction of irreversible, gravitational N-body systems on numerical precision and find that it scales as a power law. The stochastic wave packet reduction postulate then introduces fundamental uncertainties in the Cartesian phase space coordinates that propagate through classical three-body dynamics to macroscopic scales within the triple’s lifetime. The spontaneous collapse of the wave function then drives the global chaotic behavior of the Universe through the superposition of triple systems (and probably multi-body systems). The paradox of infinitesimal granularity then arises from the superposition principle, which states that any multi-body system is composed of an ensemble of three-body problems.

During the Great Plague of London in 1665, Isaac Newton went in quarantine for two years at his home in Woolsthorpe. There he laid down the foundation for his theory on gravitation. During last year’s COVID-19 pandemic, we have been working on one of the consequences of Newton’s theory of gravitation and it’s relation to chaos and time reversibility.

The chaotic problem of three bodies started with Henry Poincaré who argued in 1891 \cite{1} that the problem was unsolvable, but he was wrong. This led to the Kolmogorov-Arnold-Moser theorem, and a solution based on an infinite time-series \cite{2,3}, which therefore has little practical use \cite{4}. Montgomery \cite{5} discovered a family of periodic solutions, One of which is presented in fig. 1.

Newton could, in principle, have realized that stable periodic solutions existed. Such solutions, called braids, are found by laboriously searching parameter space using action minimizers under topological constraints \cite{6,7}. Braids are stable, regular, and once found, they can be integrated relatively easily \cite{8}. Due to a lack of computer resources, Newton would not have been able to calculate these solutions, even during his two years of social distancing.

Non-periodic democratic solutions, as depicted in fig. 1 on the other hand, are impossible to solve by hand \cite{10}, and even with modern digital computers they can only be approximated \cite{11}.

THE PARADOX OF INFINITESIMAL GRANULARITY

The intrinsic chaotic nature of the democratic N-body problem causes small errors introduced by computer round-off and integration time discretizations to grow exponentially \cite{12}. Still, solutions can be acquired by systematically reducing the time step and increasing the computer’s mantissa until the solution converges. Finding such a solution...
FIGURE 1. Two orbits of three-body problems, a periodic braid to the left, and right a partial solution of a democratic triple system. The orbit to the left is bounded and time reversible, it was found by [9]. The orbit to the right is unbounded and not time reversible.

is an elaborate iterative process in which the time step is reduced and the mantissa extended until the result becomes independent of these.

For this purpose, we developed Brutus, a direct N-body solver for self-gravitating systems of N point-masses to arbitrary precision [13]. In Brutus round-off is controlled by specifying the word-length \( L_w \) in which floating point arithmetic is performed using the MPFR arbitrary-precision library [14]. To assure that the method of numerical integration does not introduce a bias in our simulations we adopted the Gragg-Bulirsch-Stoer algorithm [15], which combines the modified midpoint method [16] for solving Newton’s ordinary differential equation [17] with Richardson extrapolation [18] to improve the rate of convergence. We can now control the algorithmic error by changing an energy tolerance parameter \( \epsilon \) and the length of the mantissa \( L_w \): Both parameters can be varied simultaneously as in [19]

\[
L_w = 4|\log_{10}(\epsilon)| + 32 \text{bits}. \tag{1}
\]

By repeating calculations with identical realizations of the initial conditions while systematically changing \( \epsilon \) and \( L_w \) we can achieve a solution that becomes independent of \( \epsilon \) and \( L_w \) to \( p \) decimal places.

Integrating converged solutions scales \( \propto N^2 \), but requires many iterations, which progressively become more expensive. Acquiring a converged solutions to an N-body problem is easily \( 10^4 \) times more expensive than regular integration techniques using IEEE 754 double precision arithmetic [20]. Any non-converged solution to a self-gravitating system under Newton’s equations of motion should be subject to considerable concern [21], although statistically, the phase-space coverage of the solutions may be indistinguishable from the true outcome space [22].

The measure of chaos, often quantified with the largest positive Lyapunov exponent expressed in a time scale, \( t_{ly} \), cannot be determined without performing extensive numerical analyses. We do not know how chaotic N-body systems are, but estimate values for \( t_{ly} \) by running the same initial realization twice until a converged solution is achieved: In the second calculation, a small relative displacement of a random particle by \( 10^{-10} \) is introduced in one of the principal Cartesian coordinates. We calculate the systems until a single star escapes. We then turn time backward in the computer, and calculate the final conditions back to their initial realization, recovering the \( 10^{-10} \) initial displacement of the perturbed solution with respect to the unperturbed solution [19]. When performing this operation for a large number of randomly generated triples, we can calculate how many of those converged solutions were successfully run backward as a function of \( \epsilon \). In figure 2 we present this fraction of irregular (or irreturnable) solutions \( f_{irr} \) as a function of \( \epsilon \).

We were surprised to find that triples with finite angular momentum turn out to be more chaotic than their zero-angular momentum counterparts. The underlying reason hides in the triple’s lifetime: high-angular momentum triples have a lower rate of close encounters, and those encounters are typically at a larger mutual distance. As a consequence,
Figure 2. The fraction of irreversible systems of converged solutions for zero-angular momentum-triples ($J=0.0$ in red) and triples with the maximum amount of angular momentum giving virial initial conditions ($J=0.19$ in blue). Both curves can be fitted by a power law: $\log_{10} f_{irr} \simeq 0.036 \log_{10} \epsilon + 0.25$ for $J=0$, and $\log_{10} f_{irr} \simeq 0.016 \log_{10} \epsilon + 0.11$.

Low angular momentum systems tend to have a higher probability of violent, stellar ejecting encounters causing these systems to live shorter on average compared to their high angular-momentum equivalents. The shorter lifetime of low-angular momentum systems cause them to have less time for chaos to propagate, even at the same Lyapunov time scale.

Another remarkable aspect of figure 2 is that about $\sim 10\%$ of virialized triples are so chaotic that these trajectories are affected by perturbations on a relative scale of $10^{-70}$ [11]. When considering the size of the universe, a perturbation of $1.3 \cdot 10^{-34} \text{Å}$, or $\sim 8.1 \cdot 10^{-10} \text{lp}$ would then still drive exponential divergence to a level of making the system unpredictable at any scale.

Global Chaos Due to Newton’s Equations of Motion

Newton operates on all scales and independently of the intrinsic nature of the objects, either being black holes or the balls of a juggler. The interference of other physical processes is irrelevant so long as the dynamics is driven in part by Newton’s equations of motion. The fundamental irreversibility of time in Newton’s equations of motion propagates to all scales and affects every object [21]. Time irreversibility on macroscopic scales then is a consequence of the chaotic nature of Newton’s equation of motion, as the spontaneous collapse of the wave function [23] introduces stochastic
FIGURE 3. Estimate of the Lyapunov timescale as a function of the number of particles. Here the horizontal axis is not linear, but in \( \ln(\ln(N)) \) to illustrate the scaling proposed in [27]. The different symbols and colors represent different calculations (see legend). The vertical bars, plotted for Newton’s Hermite only, show the root-mean-square of the dispersion in the series of solutions. The error bars in the results obtained with Brutus are statistically indistinguishable from the presented bars. The blue horizontal dotted line accentuates the boundary where large \( N \) systems become on average as chaotic as the most chaotic 3-body systems in our adopted initial conditions.

After exploring the three-body problem, we continue with virialized equal-mass large \( N \)-systems in a homogeneous unit cube (we adopted dimension-less coordinates in units with \( G = 1 \) [24, 25]). We explore systems with up to \( N = 131072 \) (\( \equiv 128 \) k). The results are presented in fig.3, where we show the Lyapunov time scale for of ensembles of multi-body systems. These calculations are continued for about 3 or 4 dynamical times, but the measurements are carried out on the sub-sample of the data, from \( t = 1/\pi \) until growth of the phase-space distance between two converged solutions of a perturbed and a non-perturbed initial realization exceeds 0.1. In some cases, the exponential growth of the phase-space distance eventually flattens to a \( \log_{10}(\delta) \propto t \) scaling relation (consistent with the expectation for a relaxation-driven process, see [26]). We acquired converged solutions for \( N = 4, 8, 16, 32, 64, 128, \) and \( N = 1024 \), and representative [19] solutions for \( N \) up to 128k particles using a regular fourth-order Hermite integration algorithm using IEEE 754 double precision arithmetic.

The Lyapunov time-scale \( \propto \ln(\ln(N)) \) [27] with a discontinuity near \( N = 32 \) due to the transition from crossing-time dominated to relaxation time dominated dynamics. The distribution width of Lyapunov timescales for larger \( N \) is narrower than for small \( N \). For up to \( N \sim 128 \) k the majority of systems tend to be more regular than for \( N = 3 \). For larger \( N \), however, all systems tend to be even more chaotic than the most chaotic three-body systems.

We argue that if wave functions collapse spontaneously, as is the case in Ghirardi-Rimini-Weber theory [23], microscopic stochastic effects propagate non-linearly to macroscopic scales through Newtonian dynamics. The motion of a considerable fraction of triples of self-gravitating bodies, and possibly the vast majority of systems with \( \gtrsim 128 \) k
are affected by the spontaneous collapse of the wave function. As a consequence, the evolution of self-gravitating multi-body system is eventually stochastic, due to the proliferation of chaotic processes that propagate from microscopic scales to macroscopic scales. This connection between the microscopic quantum scale and macroscopic scales cause the dynamics to be fundamentally time irreversible: Newton’s equations of motion are incalculable (they cannot be calculated precisely in a deterministic nature) and inconsequential (there is no point in achieving such a result).

Changes in the macroscopic distribution of mass in the Galaxy are driven by chaos on microscopic scales. Due to the spontaneous collapse of the wave function, perturbations at Planck length scales are magnified exponentially by the chaotic dynamics of planetary systems, stellar multiples, star clusters and giant molecular clouds. This growth cannot continue indefinitely, and converges once the perturbation has approximately reached the size of the Galaxy \( \mathcal{L} \). The divergence in orbital energy then starts exponentially but eventually transforms into a much slower random walk, consistent with the global relaxation process \([26, 29]\). Hence, one may think of a Galactic butterfly effect, where fluctuations lead to unpredictable local events, but which only gradually affect the global Galactic climate.

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