Comparative study of positive feedbacks on linear and nonlinear coupled logistic maps

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Abstract. This is an extended work of the authors on coupled logistic maps with delayed linear, or non-linear nearest neighbor coupling. Here we investigate the effect of positive feedback in the coupled map lattices with update rules on (i) the phase diagrams showing non-zero persistence in the $\mu$-$\varepsilon$ parameter space, and (ii) on power law exponents of decay of persistence. We find that while feedback is increased the increased non-zero persistence regions in the phase diagram has no noticeable effect on the power law exponents. We also find that the role played by time lag in the absence of feedback, discussed in our previous work, remains intact even with the introduction of feedback. We offer an extensive comparison of without feedback and with feedback plots.

Keywords: delay, feedback, logistic map, persistence, universality

1. Introduction

Phase transitions, especially second order phase transitions, have been studied for many years now. Second order phase transitions in equilibrium thermodynamic systems do not show coexistence of phases [1] and also do not involve any latent heat. Magnetic transitions are the best example of second order phase transitions. The transition can be said to be associated with emergence of a global order described by an order parameter – the net magnetization of the phase. Studies in phase transitions have considered non-equilibrium transitions and focused on search for an order parameter, which would be non-zero in the emerging phase. The associated power laws and universality class [2] have been of concern in these studies.

An important class of models of non-equilibrium systems is coupled map lattice (CML) models. This class of models is computationally more efficient and economical compared the other models. CML [3] models have been extensively studied for the past few decades. These are basic models for time evolution of systems with non-linear interactions. They are discrete space, discrete time models, in which real values, or vectors are assigned to a lattice of points in space. Time evolution of the system is defined by dynamical equations of the model which are in the form of update rule for site vectors.

The present work is focused on phase transitions in one dimensional CML models with logistic map, defined by $f(x) = \mu x (1 - x)$ as local dynamics. Such models have been used for long for non-equilibrium phase transitions. The models studied here incorporate either linear, or non-linear nearest neighbor (NN) coupling and feedback with time lag. The logistic map parameter, strength of delayed NN-coupling and feedback, and the value of time delay, (or time lag) provide parameters of the CML. We begin by randomly initializing the lattice site values and evolve them using dynamical equations defining the CML model, examining them after each time step to assign a spin value by the following rule: Up spin for those values which are above the fixed point $x^*$ of the logistic map and down spin to those below it. The fixed
point is given by the equation $x^* = 1 - 1/\mu$. The sites, which retain their original spin even after certain number of time steps, are said to be persistent sites. Fraction of persistent sites at any time step defines the persistence probability at that time. We look for CML parameter sets for which asymptotic persistence probability is non-zero. For fixed values of time lag and feedback strength, non-zero persistence provides pairs of values of the remaining parameters – logistic parameter and coupling strength – making up a phase diagram for the system. We investigate existence of a power law for persistence in time for CML models on a linear part of boundary of the phase diagram lying within the region $0 \leq \epsilon \leq 0.17$.

The CML models we investigate have the following dynamics for updating site values:

$$x_i(t + 1) = \left((1 - \beta - \epsilon)f(x_i(t)) + \frac{\epsilon}{2}(x_{i-1}(t - \tau) + x_{i+1}(t - \tau)) + \beta x_i(t - \tau)\right) \mod 1$$

for linear feedback and NN Coupling, and

$$x_i(t + 1) = \left((1 - \beta - \epsilon)f(x_i(t)) + \frac{\epsilon}{2}\left(f(x_{i-1}(t - \tau)) + f(x_{i+1}(t - \tau))\right) + \beta f(x_i(t - \tau))\right) \mod 1$$

for the non-linear case. Thus, the non-linearity in feedback and coupling is defined by the same function $f$, which in our case is the logistic map. Also, the time lag in both the coupling and the feedback is taken to be the same. Here, $\beta$ and $\epsilon$ are respectively the feedback and the NN-coupling strengths and $\tau$, the delay, or time lag. The index $i$ ranges over 1 to $N$, the total number of lattice sites, or the lattice size. As is usual, we impose cyclic boundary conditions, where the last lattice point is the neighbor of the first. The variables $x_i(t) \in [0,1]$ are the real values attached at time-step $t$ to the $i^{th}$ lattice point of a one-dimensional lattice of size $N$.

For small values of $\epsilon$ and $\beta$, the dominant part of dynamics is given by the logistic map. It is known that for much of the unit interval the map sends a value $x < x^*$ ($x > x^*$) to another $> x^*$ ($< x^*$), i.e., it sends up (down) spin to down (respectively, up) spin. Since we are looking for local spin-persistence, it is natural to define a single time-step in our CML dynamics by a double application of the update rules above. We call this modulo-2 dynamics.

2. The Plots

We present the numerical results in three parts: phase plots, persistence plots and long-range order plots.

2.1. Phase Plots

In this section we present results of numerical computation with several specific cases of CML dynamics. The first phase plot is Figure 1 (a) is for linear case with $\beta = 0.0$ and $\tau = 0$, i.e., without feedback. Here, we extend the study to cases with non-zero feedback of $\beta = 0.0125$ Figure 1 (b) and $\beta = 0.025$ Figure 1 (c) with zero delay and linear NN coupling. Figure shows the phase plots in a part of the $\mu$-$\epsilon$ plane defined by $3.6 \leq \mu \leq 4; 0.0 \leq \epsilon \leq 1.0$. Computed for the CMLs of Eq(1a), for Eq(1b) the same set of parameters for nonlinear NN coupling is shown in Fig 2 without feedback (a) and delay zero has been extensively studied by Gade and Sahasrabudhe [4], and with feedback of $\beta = 0.0125$ as in Fig 2 (b) and with feedback of $\beta = 0.025$ as in Fig 2 (c) with coupling with delays still $\tau = 0$. Although all three plots show no variation in the lower critical line. Critical line is the boundary of non-zero asymptotic persistence. The zero persistence regions are white. We note that a large part of the $\mu$-$\epsilon$ region shown allows non-zero persistence. Moreover, in case of non-linear coupling delay seems to have little effect on the phase plots, whereas, odd delays seem to wipe out a considerable part of the non-zero persistence region when the coupling is linear. Lastly, we note a more, or less well-defined line in the $3.65 < \mu \leq 4, 0.0 \leq \epsilon < 0.17$ region in all the phase plots separating zero and non-zero persistence.
Figure 1. Phase plots for both linear NN coupling cases without feedback with $\tau = 0$, $\beta = 0.0$ (a) and with feedback (b) $\tau = 0$, $\beta = 0.0125$, (c) $\tau = 0$, $\beta = 0.025$. All cases without symmetry breaking parameters.
Figure 2. Phase plots for both nonlinear NN coupling cases without feedback with $\tau = 0$, $\beta = 0.0$ (a) and with feedback (b) $\tau = 0$, $\beta = 0.0125$, (c) $\tau = 0$, $\beta = 0.025$.

The next set of phase plots is Figure 3 (a) is for linear case with $\beta = 0.0$ and $\tau = 1$, i.e., without feedback. Here, we extend the study to cases with non-zero feedback of $\beta = 0.0125$ Figure 3 (b) and $\beta = 0.025$ Figure 3 (c) with delay of $\tau = 1$ and for nonlinear NN coupling is also computed for the in Fig 4 again without feedback (a) and delay $\tau = 1$ and with feedback of $\beta = 0.0125$ as in Fig 2 (b) and with feedback of $\beta = 0.025$ as in Figure 2 (c) with coupling with delays still $\tau = 1$. 
Figure 3. Phase plots for both linear NN coupling cases without feedback with $\tau = 1, \beta = 0.0$ (a) and with feedback (b) $\tau = 1, \beta = 0.0125$, (c) $\tau = 1, \beta = 0.025$. The shift in the lower critical line is clearly visible in (b) and (c).
Figure 4. Phase plots for both nonlinear NN coupling cases without feedback with \( \tau = 1, \beta = 0.0 \) (a) and with feedback (b) \( \tau = 1, \beta = 0.0125 \), (c) \( \tau = 1, \beta = 0.025 \). The shift in the lower critical line is clearly visible in (b) and (c).

Each case, whether it is of linear or nonlinear coupling is iterated for \( 10^4 \) sites over \( 10^6 \) or more time steps to confirm the power law. The lower portions of the \( \mu - \epsilon \) phase plots define a critical line separating the zero and the nonzero persistence. Although these plots show variation with feedback and time lag values above the critical line, the line itself more, or less persists. The phase plots for linearly coupled CML fill the plot region more extensively than the non-linearly coupled one. However, this does not
seem to affect the critical line much for very small values of feedback strength. This changes for higher values of feedback. We see a rightward shift of the critical line along the μ-axis - the higher the feedback, the larger the shift and the smaller the μ-range over which it extends.

Figure 1 to 4 shows a complete comparative study of phase plots with $\tau = 0$ for linear NN coupling (Figure 1) nonlinear NN coupling (Figure 2) and with $\tau = 1$ for linear NN coupling (Figure 3) nonlinear NN coupling (Figure 4). The shift in the lower critical line is clearly seen. While Figure 5 effect of higher feedback with delay on lower critical line. Here the amount of persistence points for study will be reduced so higher feedback is not suitable for obtaining exponents over persistence region.

**Figure 5.** Phase plots for linear, $\tau = 3$, $\beta = 0.085$ showing vanishing lower critical line for higher values of feedback with delay.

Apart from the features mentioned in the text for zero feedback case, for non-zero delays here we notice a certain apparently $\beta$-dependent right-ward shift in the critical line. For both linear and non-linear coupling the shift is larger for odd delays.

### 2.2. Persistence Plots

We refer to the points on the critical line (the lower boundary separating zero and non-zero persistence regions) as critical points – each point being a $\mu$-$\epsilon$ pair of values. These pairs are then used in a persistence probability $P(t)$ calculation over $10^6$, or more time steps to generate a log-log plots of persistence points Vs. time in order to ascertain a power law $P(t) \propto t^\theta$, leading to a linear log-log plot, defines persistence exponent $\theta$[5]. As noted above, feedback leads to a rightward shift of the critical line shortening the $\mu$-range over which it extends. We find that this effect of feedback notwithstanding, the log-log plots show the same exponents as have been reported in earlier work by the same authors [5, 6] with zero feedback and zero initialization of time-lagged lattice values. The work here also shows that random initialization of time-lagged lattice values does not change the delay dependent exponents, or even the emergent long-range order [5, 6] of the system. Our computation shows that odd and even delays lead to two different values of $\theta$. (Figure 6) shows the results for linear NN-coupling. With different values of time delay (lag) as 0, 1, 2, 3 but zero feedback for first four and 0.025 feedback with delay 0 and 1 for last two plots. (Figure 7) is for nonlinear NN coupling.
Figure 6. Persistence plots for linear NN coupling with zero feedback of \( \beta = 0.0 \) and delays \( \tau = 0 \) (even) for \((\mu, \varepsilon) = (3.9, 0.12)\) (purple); \( \tau = 2 \) (even), \((\mu, \varepsilon) = (3.9, 0.12)\) (light blue) and with feedback of \( \beta = 0.025 \) and delays \( \tau = 0 \) (even) for \((\mu, \varepsilon) = (3.9, 0.12)\) (blue) showing exponent \( P(t) \approx t^{2/7} \) (red line of slope) and for delays \( \tau = 1 \) (odd), \((\mu, \varepsilon) = (3.9, 0.125)\) (green) and \( \tau = 3 \), \((\mu, \varepsilon) = (3.9, 0.12)\) (brown) and for nonzero feedback \( \beta = 0.025 \) and delay \( \tau = 1 \) for \((\mu, \varepsilon) = (3.9, 0.08)\) (yellow) showing exponent \( P(t) \approx t^{3/8} \) (black line of slope).

Figure 7. Persistence plots for nonlinear NN coupling with zero feedback of \( \beta = 0.025 \) and delays \( \tau = 0 \) (even) for \((\mu, \varepsilon) = (3.9, 0.105)\) (purple) and for delays \( \tau = 2 \) (even) for \((\mu, \varepsilon) = (3.9, 0.095)\) (blue) (a) showing exponent \( P(t) \approx t^{3/8} \) (blue, and yellow lines of slope); while \( \tau = 1 \) (odd), \((\mu, \varepsilon) = (3.9, 0.095)\) (purple) with feedback of \( \beta = 0.025 \); and \( \tau = 3 \) (odd), \((\mu, \varepsilon) = (3.9, 0.091)\) (blue) with feedback of \( \beta = 0.025 \) both showing the exponent \( P(t) \approx t^{2/7} \) (blue, and yellow lines of slope).
The above figure shows the same with nonlinear NN coupling. We find that change in nature of coupling causes an exchange of exponents between even and odd delay cases. The exponent 2/7, obtained for even delays now gets associated with the odd delays, whereas exponent 3/8 is now obtained for even delays. Also, to ascertain whether these assertions hold for stronger feedbacks too, persistence computations were done for non-zero delays and increasing feedback strengths β ranging up to values limited only by the eventual disappearance of critical line in the phase plots (see Figure 5) signifying spread of persistence region right up to zero NN-coupling (μ-axis). The persistence plots in the Figure 6 and 7 shows no change in the exponents. The linear NN coupling case shows the same exponent of 2/7, while the nonlinear coupling case shows 3/8. Thus, persistence exponent is governed only by the type of coupling and the value of delay, with no dependence on feedback strength.

3. Conclusion

The conclusions of the work on linearly, or non-linearly NN coupled logistic map lattices studied are (i) that introduction of feedback leads to increase in the incidence of asymptotic persistence, thus adding regions to phase diagrams in the μ-ε plane, (ii) exponential power law is invariant with feedback, honoring their dependence on time delay, which was discovered in our earlier work [5, 6]. Persistence exponents are determined and dependent only on the type (linear, or non-linear) of NN-coupling and by parity (even /odd nature) of time delay; these persistence exponents are unaffected by feedbacks.

We earlier have interpreted and published the effect of time delay and reciprocity between even-valued (or odd-valued) lags in linear coupling and odd-valued (respectively, even-valued) lags in non-linear coupling in terms of a “spin-flipping fraction” identifiable at each modulo-2 dynamical step in evolution of the lattice endowed with coarse-grained association of spin with lattice site values. The spin-flipping fraction, which we may identify with \( P(t) - P(t+1) \) at time \( t \), is essentially the fraction of site values crossing \( x^* \) in the \( t \)th modulo-2 dynamical update of the lattice. It is responsible for the power-law decay in persistence. Earlier, we have argued that the CML models studied have features displayed by Glauber dynamics on Ising model [7] with unequal probabilities of up-to-down and down-to-up spin-flips [4]. Due to the nature of logistic map definition of dynamics as a double application of update rules of Eq(1) at each time step, a unit delay in coupling in the CML studied here affects the spin-flipping fraction in such a manner as to convert a positive coupling to a negative. This switches the long-range order emerging asymptotically from ferromagnetic to antiferromagnetic. The switch is repeated with each unit increase in delay. A major feature of dynamics thrown up by the present work is that feedback has no such effect on the spin-flipping fraction, thus leaving both persistence exponent and emergent long-range order unaffected, and otherwise determined only by coupling type and parity of time lag. However, feedback does affect the incidence of persistence itself, bringing larger regions of phase space under the critical line into the phase diagram.

4. References

[1] Jaeger G 1998 Arch Hist Exact Sc. 51
[2] Jabeen Z and Gupte N 2007 Physica A: Statistical Mechanics and its Applications 384 59
[3] Kaneko K 1993 Theory and Applications of Coupled Map Lattices (Nonlinear Science: Theory and Applications)
[4] Gade P M and Sahasrabudhe G G 2013 Physical Review E 87 052905
[5] Rajvaidya B P, Sahasrabudhe G G and Gade P M 2019 AIP Conference Proceedings 2104 030025
[6] Rajvaidya B P, Deshmukh A D, Gade P M and Sahasrabudhe G G 2020 Chaos, Solitons and Fractals 139 110301
[7] Glauber R J 1963 Journal of Mathematical Physics 4(2) 294