THE NASH PROBLEM ON ARCS FOR SURFACE SINGULARITIES

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ABSTRACT.1 Let (X,O) be a germ of a normal surface singularity, \( \pi : \tilde{X} \to X \) be the minimal resolution of singularities and let \( A = (a_{i,j}) \) be the \( n \times n \) symmetrical intersection matrix of the exceptional set of \( \tilde{X} \). In an old preprint Nash proves that the set of arcs on a surface singularity is a scheme \( \mathcal{H} \), and defines a map \( \mathcal{N} \) from the set of irreducible components of \( \mathcal{H} \) to the set of exceptional components of the minimal resolution of singularities of \( (X,O) \). He proved that this map is injective and ask if it is surjective. In this paper we consider the canonical decomposition \( \mathcal{H} = \cup_{i=1}^{n} \overline{\mathcal{N}}_{i} \):

- For any couple \( (E_{i},E_{j}) \) of distinct exceptional components, we define Numerical Nash condition \( (NN_{(i,j)}) \). We have that \( (NN_{(i,j)}) \) implies \( \overline{\mathcal{N}}_{i} \not\subseteq \overline{\mathcal{N}}_{j} \). In this paper we prove that \( (NN_{(i,j)}) \) is always true for at least the half of couples \( (i,j) \).
- The condition \( (NN_{(i,j)}) \) is true for all couples \( (i,j) \) with \( i \neq j \), characterizes a certain class of negative definite matrices, that we call Nash matrices. If \( A \) is a Nash matrix then the Nash map \( \mathcal{N} \) is bijective. In particular our results depends only on \( A \) and not on the topological type of the exceptional set.
- We recover and improve considerably almost all results known on this topic and our proofs are new and elementary.
- We give infinitely many other classes of singularities where Nash Conjecture is true.

The proofs are based on my old work [7] and in Plenat [9].

1 Introduction

Let \( (X,O) \) be a germ of a normal surface singularity. In an old preprint, published recently by Duke [8], Nash proved that the set of arcs on a surface singularity is a scheme \( \mathcal{H} \), and defined a map \( \mathcal{N} \) from the set of irreducible components of \( \mathcal{H} \) to the set of exceptional components of the minimal resolution of singularities of \( (X,O) \). He proved that this map is injective and ask if it is surjective.

Among the principal contributions to this subject we can cite Monique Lejeune-Jalabert [15], Ana Reguera [12], S. Ishii and J. Kollar [4], G. Gonzalez-Sprungberg and Monique Lejeune-Jalabert [3].

1first version july 2005, revised version december 2005
Camille Plenat, C. Plenat and P. Popescu-Pampu. The study of arcs spaces was further developed by Kontsevich, Denef and Loeser in the theory of motivic integration.

Let \( \pi : \tilde{X} \rightarrow X \) be the minimal resolution of singularities, and \( E_1, \ldots, E_n \) be the components of the exceptional divisor, Ana Reguera associates to every \( E_i \) the family of arcs \( N_i \) such that the proper transform cuts properly \( E_i \), the spaces \( \bar{N}_i \) are irreducible and give a decomposition of the space of arcs \( H = \bigcup \bar{N}_i \). In order to give an affirmative answer to the Nash problem it is sufficient to prove that for any \( i \neq j \) then \( \bar{N}_i \not\subset \bar{N}_j \).

Recently Camille Plenat, Proposition 2.2 gives the following criterion to separate two Nash components:

**Proposition 1** Let \( \pi : \tilde{X} \rightarrow X \) be the minimal resolution of singularities and \( E_1, \ldots, E_n \) be the components of the exceptional divisor, if there exist some \( f \in O_{\tilde{X},O} \) such that \( \text{ord}_{E_i}(f) < \text{ord}_{E_j}(f) \) then \( \bar{N}_i \not\subset \bar{N}_j \).

The following Theorem follows from my work Theorem 1.1 and Lemma 2.2. Remark that in C. Plenat and P. Popescu-Pampu have recently rediscover a similar condition.

**Theorem 1** Let \((X,O)\) be a germ of normal surface singularity, \( \pi : \tilde{X} \rightarrow X \) be the minimal resolution of singularities and \( E_1, \ldots, E_n \) be the components of the exceptional divisor. Let \( K_{\tilde{X}} \) the canonical divisor on \( \tilde{X} \). Let \( E \) be an exceptional effective divisor and \( Q = \pi_*O_{\tilde{X}}(-E) \),

1. If \( -E \cdot E_i \geq 2K_{\tilde{X}} \cdot E_i \) for all \( i = 1, \ldots, n \) then \( QO_{\tilde{X}} = O_{\tilde{X}}(-E) \)
2. For any general linear combination \( f \) of a set of generators of \( Q \) we have \( \text{div}(f \circ \pi) = \tilde{H} + E \), where \( \tilde{H} \) is the proper transform of the cycle defined by \( f \).

**Remark 1**

- For any irreducible component \( E_i \) of the exceptional divisor, we consider the adjunction formula for (eventually singular) curves

\[
p(E_i) = \frac{E_i \cdot (E_i + K_{\tilde{X}})}{2} + 1
\]

where \( p(E_i) \) is the genus of \( E_i \). Recall that \( p(E_i) \geq 0 \) and \( p(E_i) = 0 \) if and only if \( E_i \) is a curve of genus zero and self intersection equal to \(-1\), which is impossible by Castelnuovo theorem since we are assuming that \( \pi : \tilde{X} \rightarrow X \) is the minimal resolution of singularities of \( X \). As a consequence \( K_{\tilde{X}} \cdot E_i = 2(p(E_i) - 1) - E_i^2 \geq 0 \) for any \( i = 1, \ldots, n \).

- Since the graph of the resolution is connected we have that for any \( 1 \leq i < k \leq n \) the intersection number \( E_i \cdot E_k \geq 0 \) and for each index \( k \) there are at least one index \( i \) such that \( E_i \cdot E_k > 0 \).

- It follows from the previous item that if \( E = \sum_{k=1}^{n} n_k E_k, n_k \in \mathbb{N} \) is an exceptional divisor such that \( E \cdot E_k \leq -2K_{\tilde{X}} \cdot E_k \leq 0 \), for all \( k = 1, \ldots, n \), then \( E \) has full support, i.e. \( n_k > 0 \) for all \( k = 1, \ldots, n \).
If \( E = \sum_{k=1}^{n} n_k E_k \) with \( n_k \in \mathbb{N} \) for \( k = 1, \ldots, n \), is an exceptional divisor such that \( E \cdot E_k \leq -2K_{\tilde{X}} \cdot E_k \), then for any \( \alpha \in \mathbb{N}^n \) we have \( (\alpha E) \cdot E_k \leq -2K_{\tilde{X}} \cdot E_k \).

**Definition 1** Let \((X, O)\) be a germ of normal surface singularity, \(\pi : \tilde{X} \rightarrow X\) be the minimal resolution of singularities, \(E_1, \ldots, E_n\) be the components of the exceptional divisor and \(A = (a_{i,j})\) with \(a_{i,j} = E_i \cdot E_j\), be the \(n \times n\) symmetrical intersection matrix of the exceptional set of \(\tilde{X}\). The dual graph \(\Gamma\) of the intersection matrix \(A\) is defined as follows:

- The vertices of the graph \(\Gamma\) are \(E_1, \ldots, E_n\),
- For \(i \neq j\) there is an edge between \(E_i\) and \(E_j\) if and only if \(a_{i,j} \neq 0\).

**Remark 2** The graph \(\Gamma\) is connected and conversely by a theorem due to Grauert, given a \(n \times n\) symmetrical negative definite matrix \(A = (a_{i,j})\) with a connected graph there exist a singularity with \(A\) as intersection matrix.

Now we introduce the definition of Nash numerical conditions, this is the central point of this work, in the other sections we will prove that Nash numerical conditions depend only on the intersection matrix of the exceptional set. A Nash matrix will be a matrix satisfying the Nash numerical conditions. In section 2, 3 we characterize some Nash matrix, in section 4 we consider like star shaped graphs and in section 5 we present some examples.

**Definition 2** Let \((X, O)\) be a germ of normal surface singularity, \(\pi : \tilde{X} \rightarrow X\) be the minimal resolution of singularities, \(E_1, \ldots, E_n\) be the components of the exceptional divisor. Let \(K_{\tilde{X}}\) the canonical divisor on \(\tilde{X}\). We say that \((X, O)\) satisfies numerical Nash condition for \((i, j)\) if the following condition is fulfilled

\[
(\text{NN}(i,j)) \quad \exists E = \sum_{k=1}^{n} n_k E_k, n_k \in \mathbb{N}^n \text{ with } n_i < n_j \text{ and } -E \cdot E_k \geq 2K_{\tilde{X}} \cdot E_k, \forall k = 1, \ldots, n
\]

We also say that \((X, O)\) satisfies numerical Nash condition, \((\text{NN})\), if \((\text{NN}(i,j))\) is true for all couples \((i, j)\), with \(i \neq j\).

As an immediate consequence of Proposition 1 and Theorem 1 we have:

**Corollary 1** With the above notations, if \((X, O)\) satisfy numerical Nash condition for \((i, j)\) then \(\tilde{N}_i \not\subset \tilde{N}_j\). In particular if \((\text{NN})\) is true then the Nash problem on arcs has a positive answer.

**Proposition 2** With the notations as above. Let \(\Gamma\) be the dual graph of the intersection matrix of the exceptional set. If \((\text{NN})\) is true for \(\Gamma\), then

- \((\text{NN})\) is true for any subgraph of \(\Gamma\)
- \((\text{NN})\) is true by decreasing the self intersection numbers.
• Let consider a subgraph $G$ of $\Gamma$ and let $I$ be its support. Since $(NN)$ is true for $\Gamma$, for any $i, j \in I, i \neq j$, there exist $E = \sum_{k=1}^{n} n_k E_k, n_k \in \mathbb{N}^n$ with $n_i < n_j$ such that $E \cdot E_k \leq -2K_X \cdot E_k, \forall k = 1, \ldots, n$

It then follows that for any $k \in I$,

$$(\sum_{i \in I} n_i E_i) \cdot E_k \leq -2K_X \cdot E_k - \sum_{i \notin I} n_i E_i \cdot E_k \leq -2K_{X'} \cdot E_k,$$

where $K_{X'}$ is the canonical divisor of the minimal resolution singularity $X'$, having $G$ as dual graph of the exceptional set. Remark that $K_{X'} \cdot E_k = K_X \cdot E_k$.

• In order to prove the second assertion it will be enough to consider one index $k \in \{1, \ldots, n\}$ and the intersection matrix $A' = (a'_{i,j})$ defined by $a'_{i,j} = a_{i,j}$ if $(i, j) \neq (k, k)$ and $a'_{k,k} = a_{k,k} - 1$. Let remark that the matrix $A'$ corresponds to a minimal resolution of some isolated singularity, $\pi' : \tilde{X}' \rightarrow X'$, call $E'_1, \ldots, E'_n$ the irreducible components of the exceptional set in $\tilde{X}'$ (In fact as a curve $E'_i = E_i$, but we need to distinguish them in $\tilde{X}$ and $\tilde{X}'$). Let $E = \sum_{k=1}^{n} n_k E_k, n_k \in \mathbb{N}^n$ with $n_i < n_j$ such that $E \cdot E_k \leq -2K_X \cdot E_k, \forall k = 1, \ldots, n$ and set $E' = \sum_{k=1}^{n} n_k E'_k$. By the Remark 1 we can assume that $n_k \geq 2$ for any $k = 1, \ldots, n$. It follows that

$$K_{\tilde{X}'} \cdot E'_i = K_X \cdot E_i \text{ for } i \neq k,$$

$$K_{\tilde{X}} \cdot E'_k = K_X \cdot E_k + 1,$$

$$E' \cdot E'_i = E \cdot E_i \leq -2K_X \cdot E_i = -2K_{\tilde{X}} \cdot E_i \text{ for } i \neq k,$$

$$E' \cdot E'_k = E \cdot E_k - n_k \leq -2K_{\tilde{X}} \cdot E_k - n_k = -2K_{\tilde{X}} \cdot E'_k - n_k + 2 \leq -2K_{\tilde{X}} \cdot E'_k.$$

This complete the proof of the second assertion.

2 Nash matrices, Gauss sequences

Let $\pi : \tilde{X} \rightarrow X$ be the minimal resolution of singularities and let $A = (a_{i,j})$ be the $n \times n$ symmetrical intersection matrix of the exceptional set of $X$, consider an exceptional effective divisor $E = x_1 E_1 + \ldots + x_n E_n$, then

$$E \cdot E_k = x_1 E_1 \cdot E_k + \ldots + x_n E_n \cdot E_k = x_1 a_{k,1} + \ldots + x_n a_{k,n}.$$

Set $'X = (x_1, \ldots, x_n)$ and $'C = (-2K_X \cdot E_1, \ldots, -2K_X \cdot E_n) and dc_i = -2K_X \cdot E_i$, then

1. Corollary 1 can be translated into linear algebra:

   If the inequality: $AX \leq C$ has a solution $(x_1, \ldots, x_n) \in \mathbb{N}^n$ such that $x_i < x_j$, then $\tilde{N}_i \not\subset \tilde{N}_j$

2. The condition $(NN_{(i,j)})$ is equivalent to the condition:

   the inequality : $AX \leq C$ has solutions $(x_1, \ldots, x_n) \in \mathbb{N}^n$ such that $x_i < x_j$.

Remark that since $\tilde{X}$ is the minimal resolution we have $K_{\tilde{X}} \cdot E_i \geq 0$ for any $i$. In what follows we allow the intersection matrix $A$ to have rational terms, remark that after multiplication by a convenient integer it will correspond to a singularity.
Lemma 1 Let \( (X,O) \) be a germ of a normal surface singularity, \( \pi: \tilde{X} \to X \) be the minimal resolution of singularities. Assume that \( \pi \) has only two exceptional components \( E_1, E_2 \). Let \( A = \begin{pmatrix} -a & c \\ c & -b \end{pmatrix} \) the intersection matrix of \( E_1, E_2 \). Then

1. \( c < a \) if and only if \((NN_{(1,2)})\) is true
2. \( c < b \) if and only if \((NN_{(2,1)})\) is true
3. \( c < \min\{a,b\} \) if and only if \((NN)\) is true.

In particular since the quadratic form associated to the matrix \( A \) is negative definite, we have \( c^2 < ab \), which implies that either \( \mathcal{N}_1 \not\subset \mathcal{N}_2 \) or \( \mathcal{N}_2 \not\subset \mathcal{N}_1 \).

Proof We are looking for solutions \((x,y) \in \mathbb{N}^*\) of the system:

\[
\begin{align*}
-ax + cy &\leq c_1 \leq 0 \\
ax - by &\leq c_2 \leq 0
\end{align*}
\]

let \( D_1 \) the line of equation \(-ax + cy = c_1\) and \( D_2 \) the line with equation \( ax - by = c_2 \), since \( A \) is negative definite we have \( c^2 < ab \), which implies \( c/b < a/c \), so the relative positions of the lines \( D_1, D_2 \), and the set of solutions of the system (*) are represented in figures below. Since these are the unique possible cases we are done.

Corollary 2 Let \( \pi: \tilde{X} \to X \) be the minimal resolution of singularities and \( E_1, \ldots, E_n \) be the components of the exceptional divisor, then for any \( i \neq j \) either \( \mathcal{N}_i \not\subset \mathcal{N}_j \) or \( \mathcal{N}_j \not\subset \mathcal{N}_i \). In any case if \( i \neq j \) then \( \mathcal{N}_i \neq \mathcal{N}_j \). In particular after considering numerical Nash conditions, in order to check if Nash is true, we will be reduced to check at most the half of non inclusion conditions.

We prove the Corollary by induction on \( n \). For \( n = 2 \) it was proved in lemma 1.

Assume \( n \geq 3 \), by changing the order in the set \( E_1, \ldots, E_n \), we can suppose that \( i = 1 \) and \( j = 2 \), now pick \( k \) a positive integer such that \( ka_{n,n} < c_n \) and put \(-a_{n,n}x_n = \sum_{1 \leq i \leq n-1} a_{n,i}x_i - ka_{n,n} \) in our system \( AX \leq C \), then we have the inequality: \( A'X \leq C' \) where \( a'_{i,j} = a_{n,i}a_{n,j} - a_{i,j}a_{n,n} \) for all \( i, j \).
and \( c'_j = (-c_j + ka_{n,j})a_{n,n} \). By induction hypothesis there exist a vector \( S = (s_1, \ldots, s_{n-1}) \in \mathbb{N}^n \) solution of the in-equation
\[
A'X \leq C'
\]
with \( s_1 \neq s_2 \). Let \( s_n = a_{n,1}s_1 + \ldots + a_{n,n-1}s_{n-1} + k \), then a simple computation shows that the

\[

\text{vector } T = (-a_{n,n}s_1, \ldots, -a_{n,n}s_{n-1}, s_n)
\]
is a solution of \( AX \leq C \) for \( k \) large enough.

Remark that by construction the vector \( T \) has strict positive components.

Now we consider the sequences appearing in the proof of the last Corollary.

**Definition 3** Let: \( a_{i,j}^{(n)} = a_{i,j} \) and for any \( 2 \leq l \leq n-1 \) set \( a_{i,j}^{(l)} = a_{i,j}^{(l+1)} - a_{i,j}^{(l+1)} \), \( 1 \leq i, j \leq l \).

Also for any \( 2 \leq l \leq n \) let \( C(A)_1^{(l)} = \sum_{j=1}^{l} a_{i,j}^{(l)} \). We will also use the notation \( C(A)_i = C(A)_i^{(n)} \).

**Lemma 2** The matrices \( A^{(l)} = (a_{i,j}^{(l)}) \) appear naturally when we use the Gauss method to decompose
the quadratic form associated to \( A \) into a sum of squares. In particular the matrix \( A^{(l)} \) are negative
definite. For this reason we will call the terms \( a_{i,j}^{(l)} \) the Gauss sequence associated to \( A \).

**Proof** The quadratic form associated to the matrix \( A \) is:
\[
Q = \sum_{i=1}^{n} a_{i,i}x_i^2 + 2\sum_{1 \leq i < j \leq n} a_{i,j}x_ix_j
\]
we follow Gauss method to squaring a quadratic form:
\[
Q = \sum_{i=1}^{n-1} a_{i,i}x_i^2 + 2\sum_{1 \leq i < j \leq n-1} a_{i,j}x_ix_j + a_{n,n}x_n^2 + 2\sum_{i=1}^{n-1} a_{i,n}x_ix_n
\]
but
\[
a_{n,n}x_n^2 + 2\sum_{i=1}^{n-1} a_{i,n}x_ix_n = a_{n,n}(x_n + \sum_{i=1}^{n-1} a_{i,n}x_i)^2 + \sum_{i=1}^{n-1} a_{i,n}^2x_i^2 - 2\sum_{1 \leq i < j \leq n-1} a_{i,n}a_{j,n}x_ix_j
\]
Hence
\[
Q = a_{n,n}(x_n + \sum_{i=1}^{n-1} a_{i,n}x_i)^2 + \sum_{i=1}^{n-1} (a_{i,i} - a_{i,n}a_{n,n})x_i^2 + 2\sum_{1 \leq i < j \leq n-1} (a_{i,j} - a_{i,n}a_{j,n})x_ix_j
\]
and \( A \) is negative definite if and only if \( a_{n,n} < 0 \) and \( A^{(n-1)} \) is negative definite.

**Remark 3**

1. By multiplying by a convenient natural number the matrix \( A \) has integer coefficients and correspond to some singularities. Our definition does not depend on the topological type of the components of the exceptional divisor.

2. For \( l \geq 3 \) the operation \( A^{(l)} \rightarrow A^{(l-1)} \) consist to contract the exceptional component \( E_l \) in
the graph \( \Gamma_l \) corresponding to \( A^{(l)} \), it is an algebraic operation and this contraction has no
geometry meaning. In what follows we will use this notation.
We have immediately from lemma 1 and Corollary 2 that

**Proposition 3** Let \( \pi : \bar{X} \to X \) be the minimal resolution of singularities and let \( A = (a_{i,j}) \) be the \( n \times n \) symmetrical intersection matrix of the exceptional set of \( X \). Then

1. \( a^{(2)}_{1,2} < a_{0,1}^{(2)} \) if and only if \((NN(1,2))\) is true
2. \( a_{1,2}^{(2)} < a_{2,2}^{(2)} \) if and only if \((NN(2,1))\) is true
3. \( a_{1,2}^{(2)} < \min\{a_{1,1}^{(2)}, a_{2,2}^{(2)}\} \) if and only if both \((NN(1,2)), (NN(2,1))\) are true.

**Theorem 2** Let \( \pi : \bar{X} \to X \) be the minimal resolution of singularities, let \( A = (a_{i,j}) \) be the \( n \times n \) symmetrical intersection matrix of the exceptional set of \( X \) and let \( C(A)^{(l)} = \sum_{j=1}^{l} a_{i,j}^{(l)} \). For \( l \geq 1 \), we consider the property:

\[(*)_{l+1} C(A)^{(l+1)} < 0, \quad \text{for} \quad i = 1, \ldots, l+1.\]

If \((*)_{l+1}\) is true for some \( l \geq 2 \) then \((*)_1\) is true.

Let \( \sigma \in S_n \) any permutation of \( E_1, \ldots, E_n \), we denote by \( A^{\sigma} \) the corresponding intersection matrix obtained from \( A \) by permuting lines and columns. Then \((NN)\) is true if and only if there exist a natural integer \( l \geq 1 \) such that

\[(*)_{l+1} C(A^{\sigma})^{(l+1)} < 0, \quad \text{for} \quad i = 1, \ldots, l+1, \forall \sigma \in S_n.\]

In particular we recover the following result from [11]: if \( C(A)^{(a)} < 0, \quad \text{for} \quad i = 1, \ldots, n \) then the Nash map \( \mathcal{N} \) is bijective.

Note that condition \((*)_1\) has a meaning only if \( l \geq 2 \).

**Proof** Assume that \( C(A)^{(l+1)}_i < 0, \quad \text{for} \quad i = 1, \ldots, l+1, \) let \( i \leq l \), by definition

\[ C(A)^{(l)}_i = \sum_{j=1}^{l} a_{i,j}^{(l)} = \sum_{j=1}^{l} (a_{i,j}^{(l+1)} - \frac{a_{i+1,j}^{(l+1)}}{a_{l+1,i+1}} a_{i,j}^{(l+1)} a_{i,l+1}^{(l+1)} a_{l+1,j+1}^{(l+1)}) \]

\[ C(A)^{(l)}_i = \sum_{j=1}^{l} a_{i,j}^{(l+1)} - \frac{a_{i+1,j}^{(l+1)}}{a_{l+1,i+1}} \sum_{j=1}^{l} a_{i,j+1}^{(l+1)} \]

\[ C(A)^{(l)}_i = C(A)^{(l+1)}_i - \frac{a_{i+1,j}^{(l+1)}}{a_{l+1,i+1}} C(A)^{(l+1)}_i < 0 \]

The second assertion follows from Proposition 3. Remark that it is not necessary to consider all permutation of \( E_1, \ldots, E_n \).

**Definition 4** Let \( A = (a_{i,j}) \) be the \( n \times n \) symmetrical negative definite matrix with rational coefficients with \( a_{i,i} < 0, a_{i,j} > 0 \) for all \( i, j, i \neq j \). We say that \( A \) is a Nash matrix if for any permutation \( \sigma \) of the set \( \{1, \ldots, n\} \), \( C(A^{\sigma})^{(2)}_1 < 0, C(A^{\sigma})^{(2)}_2 < 0 \)
3 Trees, Cycles, Generalized Cycles

We look now for some necessary or sufficient conditions in order to have the condition \((NN)\) true. For the moment we need to recall some notation on graphs.

**Definition 5** Let \(A = (a_{i,j})\) be the \(n \times n\) symmetrical negative definite matrix with rational coefficients with \(a_{i,i} < 0, a_{i,j} \geq 0\) for all \(i, j, i \neq j\). Let \(\Gamma\) the dual graph associated to \(A\). We say that \(E_i\) is a leaf of \(\Gamma\) if \(a_{i,j} \neq 0\) for exactly one index \(j \neq i\) i.e. \(E_i\) is connected to only one other vertex of \(\Gamma\). A cycle of \(\Gamma\) is a subgraph \(C\) where every vertex is connected to exactly two others vertex of \(\Gamma\). A tree is a subgraph with no cycles. Finally a complete subgraph is a subset of \(\Gamma\), where every two points are connected.

**Lemma 3** Assume that for any point \(E_j\) of \(\Gamma\), we have \(C(A)_{j} \leq 0\).

1. For any \(l \leq n\) and \(j \leq l\) we have \(C(A)_{j}^{(l)} \leq 0\)
2. If \(C(A)^{(l+1)}_{i} < 0\) then \(C(A)^{(l)}_{i} < 0\)
3. Let consider a path \(E_{i_1}, E_{i_2}, \ldots, E_{i_k}\) in \(\Gamma\), and \(C(A)_{i_k} < 0\). After contracting \(E_{i_k}, E_{i_{k-1}}, \ldots, E_{i_2}\) we will have \(C(A)_{i_1}^{(2)} < 0\).

**Proof** The first two assertions follow immediately from the following formula, which is true for any \(l \geq 2\), and \(1 \leq i \leq l\):

\[
C(A)^{(l)}_{i} = C(A)^{(l+1)}_{i} - \frac{a_{i,(l+1)}^{(l+1)}}{a_{(l+1),(l+1)}} C(A)^{(l+1)}_{l+1} < 0
\]

We prove the third assertion by induction on \(k\) the length of the path, if \(k = 2\), by the above formula we get the answer. Now take any \(k \geq 3\), then using again the above formula we have that \(C(A)^{(n-1)}_{i} < 0\), by the induction hypothesis we get \(C(A)(n-k+1)_{i} < 0\), so by the assertion 1 we are done.

**Theorem 3** Let \(\pi : \tilde{X} \rightarrow X\) be the minimal resolution of singularities and let \(A = (a_{i,j})\) be the \(n \times n\) symmetrical intersection matrix of the exceptional set of \(X\). If \((NN)\) is true then \(C(A)_{i} < 0\) for any leaf \(E_i\) of \(\Gamma\).

**Proof** Suppose that \((NN)\) is true. Let \(E_i\) be a leaf of \(\Gamma\), we can assume that \(i = 1\) and \(E_2\) is the unique vertex connected to \(E_1\), by contracting all other vertex of \(\Gamma\), we will have \(a_{1,1}^{(2)} = a_{1,1}, a_{1,2}^{(2)} = a_{1,2}\). By Proposition 3 (or Theorem 2) we must have \(C(A)_{1} = C(A)_{1}^{(2)} < 0\). This concludes the proof.

**Theorem 4** Let \(\pi : \tilde{X} \rightarrow X\) be the minimal resolution of singularities and let \(A = (a_{i,j})\) be the \(n \times n\) symmetrical intersection matrix of the exceptional set of \(X\). Assume that \(\Gamma\) is a tree and \(C(A)_{i} \leq 0\) for any vertex \(E_i\) of \(\Gamma\). Then \(C(A)_{i} < 0\) for any leaf \(E_i\) of \(\Gamma\) if and only if \((NN)\) is true. In particular if the above conditions are satisfied the Nash map \(N\) is bijective.
Proof The necessary condition was proved before. The proof of the other implication is by induction on \( n \). If \( n = 2 \) the hypothesis implies that \((NN)\) is true by Lemma 1. So assume the case \( n - 1 \) is solved and we prove the case \( n \). Take any \( i \neq j \). We have two cases

1) Both \( E_i, E_j \) are leaves of \( \Gamma \), then \( C(A)^{(n)}_i < 0, C(A)^{(n)}_j < 0 \), by contracting all vertex in \( \Gamma \) except \( E_i, E_j \) and applying Lemma 2, we get that \( C(A)^{(n)}_i < 0, C(A)^{(n)}_j < 0 \), and we are done.

2) At most one of \( E_i, E_j \) is a leaf, then there exist a leaf \( E_k \), different from \( E_i, E_j \), so after changing the order of the exceptional components we can assume that \( i = 1, j = 2, k = n \), let \( E_i \) be unique component connected to \( E_n \). By contracting \( E_n \), we get the matrix \( A^{(n-1)} = (a^{(n-1)}_{i,j}) \), with \( a^{(n-1)}_{i,j} = a_{i,j} \) for any \( (i,j) \neq (l,l) \) and \( a^{(n-1)}_{l,l} = a_{l,l} - \frac{a^2_{l,n}}{a_{n,n}} < 0 \). It follows that \( C(A)^{(n-1)}_i = C(A)_i \leq 0 \) for \( i \neq l \) and \( C(A)^{(n-1)}_l = C(A)_l - \frac{a^2_{l,n}}{a_{n,n}} + a_{l,n} < C(A)_l \leq 0 \). Also the graph corresponding to the matrix \( A^{(n-1)} \) is a tree, so by induction hypothesis \((NN(1,2))\) and \((NN(2,1))\) are true, and we are done.

Remark 4 Inside the class of rational singularities, rational minimal singularities are exactly those for which the graph satisfies the hypothesis of the above theorem. Note that Nash problem’s on arcs for (rational) minimal singularities has a positive solution by the work of Ana Reguera [12], also C. Plenat [10] and Fernandez-Sanchez [2] gave different proofs. Our Theorem applies without any restriction on the topological type of the exceptional components and so extends to non rational singularities the mentioned results.

Theorem 5 Let \( \pi : \tilde{X} \to X \) be the minimal resolution of singularities and let \( A = (a_{i,j}) \) be the \( n \times n \) symmetrical intersection matrix of the exceptional set of \( X \). Assume that the graph \( \Gamma \) of the exceptional set is a cycle, with \( n \geq 3 \), and \( C(A)_i \leq 0 \) for all \( i \). Then \((NN)\) is true if and only if \( C(A)_i < 0 \) for at least two exceptional components. In particular if these conditions are fulfilled the Nash map \( N \) is bijective.

Proof Assume first that \( C(A)_i < 0 \) for at least two exceptional components We prove that \((NN)\) is true by induction on \( n \).

If \( n = 3 \), we contract the exceptional fiber \( E_3 \) and we get the matrix:

\[
A^{(2)} = \begin{pmatrix}
    a_{1,1} - \frac{a^2_{3,1}}{a_{3,3}} & a_{1,2} - \frac{a_{3,3}a_{2,3}}{a_{3,3}} \\
    a_{1,2} - \frac{a_{3,3}a_{1,3}}{a_{3,3}} & a_{2,2} - \frac{a^2_{3,3}}{a_{3,3}}
\end{pmatrix}
\]

It follows that \( C(A)^{(2)}_1 = C(A)_1 - \frac{a^2_{3,3}}{a_{3,3}}(C(A)_3) < 0 \) and \( C(A)^{(2)}_2 = C(A)_2 - \frac{a^2_{3,3}}{a_{3,3}}(C(A)_3) < 0 \) since by hypothesis two over the three numbers \( C(A)_1, C(A)_2, C(A)_3 \) are strictly negative. So the case \( n = 3 \) is over.

9
Consider now the case \( n \geq 4 \). By contracting \( E_n \), we get the matrix \( A^{(n-1)} = (a_{i,j}^{(n-1)}) \), with
\[
a_{i,j}^{(n-1)} = a_{i,j} \text{ if } i,j \in \{2,3,\ldots,n-2\} \text{ and } a_{1,1}^{(n-1)} = a_{1,1} - \frac{a_{1,n}^2}{a_{n,n}}, \quad a_{1,n-1}^{(n-1)} = -\frac{a_{1}a_{n-1,n}}{a_{n,n}} \quad \text{and} \quad a_{n-1,n-1}^{(n-1)} = -\frac{a_{n-1,n}^2}{a_{n,n}}.
\]
It follows that \( C(A)_i^{(n-1)} = C(A)_i \leq 0 \) for \( i \in \{2,3,\ldots,n-2\} \), \( C(A)_{n-1}^{(n-1)} = C(A)_{n-1} - \left(\frac{a_{n-1,n}}{a_{n,n}}(C(A)_n)\right) \) and \( C(A)_1^{(n-1)} = C(A)_1 - \left(\frac{a_{1,n}}{a_{n,n}}(C(A)_n)\right) \). We have to consider three cases:

i) \( C(A)_1 = C(A)_{n-1} = C(A)_n = 0 \) then there are two indexes \( i,j \in \{2,3,\ldots,n-2\} \) such that \( C(A)_i^{(n-1)} < 0, C(A)_j^{(n-1)} < 0 \).

ii) \( C(A)_n < 0 \), then \( C(A)_{n-1}^{(n-1)}, C(A)_1^{(n-1)} \) are strictly negative.

iii) at least one of \( C(A)_1 = 0 \) and \( C(A)_{n-1} \) and \( C(A)_n = 0 \), then either \( C(A)^{(n-1)}_{n-1} < 0 \) or \( C(A)_1^{(n-1)} < 0 \).

So the induction hypothesis is verified by \( A^{(n-1)} \) and we are done.

Conversely, if \((NN)\) is true and \( C(A)_i < 0 \) for at most one index \( i \), take any index \( j \neq i \), by contracting all other components \( E_k, k \neq i,j \) we will have \( C(A)^{(2)}_i = C(A)_i = 0, C(A)^{(2)}_j = C(A)_j < 0 \), this is a contradiction by Proposition 3. We can give a more general result that the preceding one, for this we need some definitions.

**Definition 6** We say that a subgraph \( G \) of \( \Gamma \) is a generalized cycle if any two vertex of \( G \) are connected by a cycle. Remark that a cycle or a complete graph are generalized cycles.

A generalized cycle is a leaf of \( \Gamma \) if at most one vertex of \( G \) is connected to one vertex of \( \Gamma \setminus G \).

The proof of the next Corollary is exactly the same as for a cycle, and we left it to the reader:

**Corollary 3** Let \( \pi: \bar{X} \twoheadrightarrow X \) be the minimal resolution of singularities of \( X \) and let \( A = (a_{i,j}) \) be the \( n \times n \) symmetrical intersection matrix of the exceptional set of \( X \). Suppose that \( \Gamma \) is a generalized cycle.

We assume that \( n \geq 3 \) and \( C(A)_i \leq 0 \), for any vertex \( E_i \). Then \((NN)\) is true if and only if \( C(A)_i < 0 \), for at least two vertex.

**Example 1** The following matrix and graph correspond to a generalized cycle, for which Nash’s problem has an affirmative answer.

\[
A = \begin{pmatrix}
-5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & -6 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & -5 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & -5 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & -3
\end{pmatrix}
\]
Theorem 6 Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities of $X$ and let $A = (a_{i,j})$ be the $n \times n$ symmetrical intersection matrix of the exceptional set of $X$.

We assume that $n \geq 3$, $\Gamma$ is not a generalized cycle and

1. $C(A)_i \leq 0$, for any vertex $E_i$.
2. $C(A)_i < 0$, for any leaf $E_i$ of $\Gamma$.

Then $(NN)$ is true if and only if for any generalized cycle $G$ of $\Gamma$ there is a vertex in $G$, not connected to one vertex of $\Gamma \setminus G$ such that $C(A)_i < 0$.

Proof Assume that $(NN)$ is true, we have seen that $C(A)_i < 0$, for any leaf $E_i$ of $\Gamma$, now consider any generalized cycle $G$, we contract all points outside this generalized cycle, so $(NN)$ is still true for this $G$, this implies that $C(A)_i < 0$ for at least one vertex in $G$ not connected to one vertex of $\Gamma \setminus G$. We have finished to prove the necessary condition.

We prove now the other implication. Take two vertex $E_i, E_j$ of $\Gamma$, since $\Gamma$ is connected, there is a path $C$ in $\Gamma$ connecting them. We must consider two cases,

1. $C$ cannot be extended to a cycle, then by contracting all the vertex not in $C$, we are reduced to the case of a tree, which was solved in Theorem 4.
2. $C$ can be extended to a cycle then $E_i, E_j$ are inside a generalized cycle, then by contracting all the vertex not in $C$, we are reduced to the case of a generalized cycle, which was solved just before.

4 Like Star graphs

We can improve the above result in the some special situations:

Theorem 7 Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities and let $A = (a_{i,j})$ be the $n \times n$ symmetrical intersection matrix of the exceptional set of $X$. Assume that $X$ has a polygon singularity, i.e. the graph of the exceptional set is a star with root $E_n$ and all other vertex are leaves.

Then $(NN)$ is true if and only if we have the following conditions:

- $\forall i = 1, \ldots, n - 1 \quad a_{i,i} + a_{i,n} < 0$
- $\forall i, j = i, \ldots, n - 1 \quad a_{i,i}a_{j,j}\Delta_n + a_{j,n}(a_{i,i}a_{j,n} - a_{j,j}a_{i,n}) < 0$
- $\forall i = i, \ldots, n - 1 \quad \frac{a_{i,n}}{a_{i,i}}(a_{i,i} + a_{i,n}) + \Delta_n < 0$

where $\Delta_n = a_{n,n} - \sum_{i=1}^{n-1} \frac{a_{i,n}}{a_{i,i}}$. We note that $A$ is negative definite if and only if $\Delta_n < 0$. 

Proof It is enough to compare any two leaves and any leaf with the root. So we consider the following order: \(E_1, E_2, E_n, E_3, ..., E_{n-1}\). After applying the construction above we are reduced to the matrix

\[
A^{(3)} = \begin{pmatrix}
    a_{1,1} & 0 & a_{1,n} \\
    0 & a_{2,2} & a_{2,n} \\
    a_{1,n} & a_{2,n} & a_{n,n}
\end{pmatrix}
\]

where \(a_{n,n}^{(3)} = a_{n,n} - \sum_{i=3}^{n-1} \frac{a_{i,i}^2}{a_{i,i}}\). We are reduced to consider two cases:

first case: Comparison of \(E_1, E_2\)

Again by the construction above we are reduced to the matrix:

\[
A^{(2)} = \begin{pmatrix}
    a_{1,1} - \frac{a_{1,n}^2}{a_{n,n}} & -\frac{a_{1,n}a_{2,n}}{a_{n,n}} \\
    -\frac{a_{1,n}a_{2,n}}{a_{n,n}} & a_{2,2} - \frac{a_{2,n}^2}{a_{n,n}}
\end{pmatrix}
\]

So \((NN)_{(1,2)}\) and \((NN)_{(2,1)}\) are true if and only if:

\[a_{1,1} - \frac{a_{1,n}^2}{a_{n,n}} - \frac{a_{1,n}a_{2,n}}{a_{n,n}} < 0\]
\[a_{2,2} - \frac{a_{2,n}^2}{a_{n,n}} - \frac{a_{1,n}a_{2,n}}{a_{n,n}} < 0\]

by simple computations these are equivalent to:

\[a_{i,i}a_{j,j}\Delta_n + a_{j,n}(a_{i,i}a_{j,n} - a_{3,j}a_{i,n}) < 0\]

for \(\{i, j\} = \{1, 2\}\).

Let consider now the second case: Comparison of \(E_1, E_n\)

By the construction above we are reduced to the matrix:

\[
\begin{pmatrix}
    a_{1,1} & a_{1,n} \\
    a_{1,n} & a_{n,n} - \frac{a_{2,n}^2}{a_{2,2}}
\end{pmatrix}
\]

So \((NN)_{(1,2)}\) and \((NN)_{(2,1)}\) are true if and only if: \(a_{1,1} + a_{1,n} < 0\) and \(a_{1,n} + a_{n,n} = a_{n,n}^{(3)} - \frac{a_{2,n}^2}{a_{2,2}} < 0\).

After simple computations these are equivalent to: \(a_{1,1} + a_{1,n} < 0\) and \(\frac{a_{1,n}}{a_{1,1}}(a_{1,1} + a_{1,n}) + \Delta_n < 0\).

Since the choice of the leaves were arbitrary, we are done.

The next corollary follows immediately from the theorem.

**Corollary 4** Let \(\pi: \tilde{X} \rightarrow X\) be the minimal resolution of singularities and let \(A = (a_{i,j})\) be the \(n \times n\) symmetrical intersection matrix of the exceptional set of \(X\). Assume that \(X\) has a polygon singularity, the graph of the exceptional set is a star shaped with root \(E_n\), and \(a_{i,n} = 1, a_{i,i} = -2\), for \(i = 1, ..., n - 1\). Then the matrix \(A\) is negative definite if and only if \(-a_{n,n} > \frac{n - 1}{2}\) and \((NN)\) is true if and only if \(-a_{n,n} > \frac{n}{2}\). So if \(n\) is odd \((NN)\) is always true, but if \(n\) is even it remains open the case \(-a_{n,n} = \frac{n}{2}\). By the above theorem only the cases \((NN)_{(n,i)}\) for \(i = 1, ..., n - 1\) are not true.
Example 2  Our theorem cannot be applied to the following graph of a sandwich singularity where \( a \geq 3 \). In fact it follows from the theorem that only \((NN)_{1,3}\) is not true. Note that Nash problem’s on arcs for (rational) sandwich singularities has a positive solution by the work of Monique Lejeune-Jalabert and Ana Reguera [6].

\[
A = \begin{pmatrix}
-a & 0 & 1 & 0 \\
0 & -2 & 1 & 0 \\
1 & 1 & -2 & 1 \\
0 & 0 & 1 & -2 \\
\end{pmatrix}
\]

Theorem 8  Let \( \pi : \tilde{X} \rightarrow X \) be the minimal resolution of singularities and let \( A = (a_{i,j}) \) be the \( n \times n \) symmetrical intersection matrix of the exceptional set of \( X \). Assume that the singularity is like a star, i.e. the graph of the exceptional set is a star with root \( E_0 \) having \( s \geq 3 \) branches.

To any branch of the star we associate a continuous fractions expansion:

\[
q_i := a(i)_{1,1} - \frac{a(i)_{1,2}^2}{a(i)_{2,2} - \frac{a(i)_{2,3}^2}{a(i)_{3,3} - \frac{a(i)_{3,4}^2}{...}}}
\]

Then \((NN)\) is true if we have the following condition:

- for any leaf \( E_i \) we have \( C(A)_i < 0 \)
- for any vertex \( E_i \) which is not the root \( C(A)_i \leq 0 \)
- \( \forall i, j = 1, ..., s, i \neq j, a_{0,0} + a_{0,i} + a_{0,j} - \sum_{k=1, k \neq i,j}^{n} \frac{a_{0,k}^2}{q_k} \leq 0 \)

Proof  Our first step consist to contract a whole branch \( G_k \) of the star \( \Gamma \). We reorder the irreducible components of the exceptional set by letting \( E_n \) to be the leaf of the branch \( G_k \), \( E_{n-1} \) be the unique vertex connected to \( E_n \), \( E_{n-2} \) be the unique vertex connected to \( E_{n-1} \) but distinct from \( E_n \), and so on until we arrive to the root named always by \( E_0 \), the order in the other branches are arbitrary.
We also denote \( a_{0,k} := a(k)_{0,1} \). By contracting \( E_n \) we will get again a like star graph and a new matrix \( A^{(n-1)} = (a_{i,j}^{(n-1)}) \) given by

\[
a_{i,j}^{(n-1)} = a_{i,j}^{(n)} - \frac{a_{n,n}^{(n)} a_{n,j}^{(n)}}{a_{n,n}^{(n)}},
\]

regarding that our graph is a star we get:

\[
a_{i,j}^{(n-1)} = a_{i,j} \quad \text{if} \quad (i, j) \neq (n-1, n-1) \quad \text{and} \quad a_{n-1, n-1}^{(n-1)} = a_{n-1, n-1}^{(n)} - \frac{a_{n-1, n-1}^{2}}{a_{n,n}^{(n)}},
\]

Proceeding in this way we can contract all the vertices of \( G_k \), then we will get again a like star graph and a new matrix \( A' = (a'_{i,j}) \) given by

\[
a'_{i,j} = a_{i,j} \quad \text{if} \quad (i, j) \neq (0, 0) \quad \text{and} \quad a'_{0,0} = a_{0,0} - \frac{a_{0,k}^{2}}{q_k}.
\]

Now we are ready to prove the claim: we need to compare any two elements in the graph \( \Gamma \), these elements are in at most two branches \( G_{\alpha}, G_{\beta} \) of the star, we contract \( s-2 \) branches (indexed by a set \( I \)) of the star distinct from \( G_{\alpha}, G_{\beta} \) and we get a new graph of type \( A_n \) and a new matrix \( A'' = (a''_{i,j}) \) given by

\[
a''_{i,j} = a_{i,j} \quad \text{if} \quad (i, j) \neq (0, 0) \quad \text{and} \quad a''_{0,0} = a_{0,0} - \sum_{k \in I} \frac{a_{0,k}^{2}}{q_k},
\]

the hypothesis of the theorem imply that this special tree satisfies the hypothesis of Theorem 4 and we are done.

### 5 Examples

We discuss some examples, some of them are obtained by direct application of the results above. Numerical examples were computed with my software. Following the ideas developed in this paper I have written a program that given in entry the intersection matrix \( A \) of the exceptional set in the minimal resolution, compute all the matrices \( (A^\sigma)^{(l)} \) and check if the numerical Nash condition \( (NN(i,j)) \) is true or not, the output is a \( n \times n \) square matrix \( N \), such that :

\[
n_{ij} = \begin{cases} 
1 & \text{if} \ (NN(i,j)) \ \text{is true} \\
0 & \text{if} \ (NN(i,j)) \ \text{is false} \\
1 & \text{if} \ i = j
\end{cases}
\]
Example 3 Let $\pi : \tilde{X} \rightarrow X$ be the minimal resolution of singularities and let $A = (a_{i,j})$ be the $n \times n$ symmetrical intersection matrix of the exceptional set of $X$, let $d = \max_{i \neq j} \{a_{i,j}\}$ and $m = \min_i \{-a_{i,i}\}$. If $m > (n - 1)d$ then the Nash map $N$ is bijective.

It is an immediate consequence of Lemma 2. Remark that if $m, d$ are two strictly positive integers such that $m > (n - 1)d$. The quadratic form associated to the matrix $M$ such that $m_{i,i} = -m$ for any $i$, and $m_{i,j} = d$ for any $i \neq j$ is negative definite.

Example 4 If $n = 3$ Nash’s problem has a positive answer if for any distinct numbers $i, j, k$, we have $a_{k,i}a_{k,j} - a_{i,j}a_{k,k} < \min \{-a_{k,i}^2 + a_{i,i}a_{k,k}, -a_{k,j}^2 + a_{j,j}a_{k,k}\}$. For example if $a_{i,j} \in \{0, 1\}$ for any $i \neq j$ and $-a_{1,1} \geq 2, -a_{2,2} \geq 3, -a_{3,3} \geq 3$ then the Nash’s problem has a positive answer.

Example 5 $(NN)$ is true for Rational double points $A_n$ but no true for $D_n$ neither $E_6, E_7, E_8$.

Remark that recently C. Plenat has proved that the Nash map $N$ is bijective for the singularities $D_n$.

• By Proposition 2. it is enough to consider the singularity $D_4$, in this case we have

$$A = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix} \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

• Consider the singularity $E_6$, in this case we have

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

• Consider the singularity $E_7$, in this case we have

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

• Consider the singularity $E_8$, in this case we have
Example 6  The following two graphs are like star shaped, and condition \((NN)\) is not true.

\[
A = \begin{pmatrix}
  -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 \\
  0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
N = \begin{pmatrix}
  1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
  0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
  0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
  0 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

Example 7  The following graphs are like star shaped, and condition \((NN)\) is true.

\[
A = \begin{pmatrix}
  -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & -3 & 1 & 0 & 0 & 1 & 0 & 1 \\
  0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & -2 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & -2
\end{pmatrix}
\]

\[
N = \begin{pmatrix}
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

Example 8  In this example the graph of the singularity is a tree, \((NN)\) is true but we can’t apply Theorem 4.
\[ A = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix} \]

\[ N = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \]

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