More CFTs and RG Flows from Deforming M2/M5-Brane Horizon

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\textbf{abstract}

Near-horizon geometry of coincident M2-branes at a conical singularity is related to M-theory on $\text{AdS}_4$ times an appropriate seven-dimensional manifold $X_7$. For $X_7 = \mathbb{C}^{0,1,0}$, squashing deformation is known to lead to spontaneous (super)symmetry breaking from $\mathcal{N} = (3,0)$ to $\mathcal{N} = (0,1)$ in gauged $\text{AdS}_4$ supergravity. Via AdS/CFT correspondence, it is interpreted as renormalization group flow of strongly coupled three-dimensional field theory with $SU(3) \times SU(2)$ global symmetry. The flow interpolates between $\mathcal{N} = (0,1)$ fixed point in the UV to $\mathcal{N} = (3,0)$ fixed point in the IR. Evidences for the interpretation are found both from critical points of the supergravity scalar potential and from conformal dimension of relevant chiral primary operators at each fixed point. We also analyze cases with $X_7 = SO(5)/SO(3)_{\text{max}}, V_{5,2}(R), M^{1,1,1}, Q^{1,1,1}$ and find that there is no nontrivial renormalization group flows. We extend the analysis to Englert type vacua of M-theory. By analyzing de Wit-Nicolai potential, we find that deformation of $S^7$ gives rise to renormalization group flow from $\mathcal{N} = 8$, $SO(8)$ invariant UV fixed point to $\mathcal{N} = 1$, $G_2$ invariant IR fixed point. For $\text{AdS}_7$ supergravity relevant for near-horizon geometry of coincident M5-branes, we also point out a nontrivial renormalization group flow from $\mathcal{N} = 1$ superconformal UV fixed point to non-supersymmetric IR fixed point.

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1 Introduction

Few examples are known for three-dimensional interacting conformal field theories or for renormalization group flows among themselves, mainly due to strong coupling dynamics in the infrared limit. For example, $\mathcal{N} = 1$ superconformal field theories, which arise in various situations involving D-brane dynamics, do not have any continuous R-symmetries, holomorphy constraints or any known non-renormalization theorems. In the previous paper [1], as an alternative route, we have proposed to classify three-dimensional (super)conformal field theories by utilizing the AdS/CFT correspondence [2, 3, 4] and earlier, exhaustive study of the Kaluza-Klein supergravity [5].

The simplest spontaneous compactification of the eleven-dimensional supergravity [6] is the Freund-Rubin [7] compactification to a product of $AdS_4$ spacetime and an arbitrary compact seven-dimensional Einstein manifold $\mathcal{X}_7$ of positive scalar curvature. Continuous deformations among $\mathcal{X}_7$’s is of interest as they would be interpreted, in the strongly interacting $d = 3$ quantum field theory dual to the $AdS_4$ supergravity, to renormalization group flows among interacting (super)conformal fixed points. The best known example is provided by round and squashed $S^7$. The standard Einstein metric of the round $S^7$ yields a vacuum with $SO(8)$ gauge symmetry and $\mathcal{N} = 8$ supersymmetry. The $S^7$ also admits the second, squashed Einstein metric [8], yielding a vacuum with $SO(5) \times SO(3)$ gauge symmetry and $\mathcal{N} = 1$ or $\mathcal{N} = 0$ supersymmetry, depending on the orientation of the $S^7$ [9].

In [1], we have shown that the well-known spontaneous (super)symmetry breaking deformation from round- to squashed-$S^7$ is mapped to a renormalization group flow from $\mathcal{N} = 0$ or 1, $SO(5) \times SO(3)$ invariant fixed point in the UV to $\mathcal{N} = 8$, $SO(8)$ invariant fixed point in the IR. In particular, in [1], we have shown that (1) the squashing deformation corresponds to an irrelevant operator at the $\mathcal{N} = 8$ superconformal fixed point and a relevant operator at the $\mathcal{N} = 1$ or 0 (super)conformal fixed point, respectively, and (2) the renormalization group flow is described geometrically, as in $AdS_5$ supergravity [10], by a ‘static’ domain wall of the sort studied earlier in [11], which interpolates the two asymptotically $AdS_4$ spacetimes with $\mathcal{X}_7$ round and squashed $S^7$’s.

In this paper, we will be studying further known examples of Kaluza-Klein supergravity vacua and reinterpret them in terms of three-dimensional (super)conformal field theories and associated renormalization group flows. First, we will be exploring various Freund-Rubin type spontaneous compactifications on $AdS_4 \times \mathcal{X}_7$. For M2-branes on an eight-dimensional manifold, the near-horizon geometry $\mathcal{X}_7$ is expected to change as the branes are placed at or away a

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2 corresponding to the near-horizon geometry of coincident, infinitely planar M2-branes

3 We restrict our foregoing discussions only to deformations among $\mathcal{X}_7$’s of same topology. While examples of deformation between $\mathcal{X}_7$’s of distinct topology would be extremely interesting, we are not aware of any explicit example.
conical singularity of the manifold \([12, 13]\). More specifically, we will consider \(X_7\) being 1) 3-Sasaki, 2) Sasaki-Einstein or 3) weak \(G_2\) holonomy manifolds, describing near-horizon geometry of M2-branes at relevant conical singularities. Each of them preserves \((3,0), (2,0)\) and \((1,0)\) supersymmetries, respectively, where \((N_L, N_R)\) denotes chirality of supercharges on \(AdS_4\).

Explicit examples of the above \(X_7\)'s are
1) \(N^{0,1,0}_I\) space with \(\mathcal{N} = (3,0)\)
2) \(M^{1,1,1}, Q^{1,1,1}, V_{5,2}(R)\) spaces with \(\mathcal{N} = (2,0)\)
3-L) Squashed \(S^7, SO(5)/SO(3)_{\text{max}}\) and other \(N^{p,q,r}_I\) space with \(\mathcal{N} = (1,0)\)
3-R) all \(N^{p,q,r}_{II}\) space with \(\mathcal{N} = (0,1)\).

In section 2, by analyzing relevant scalar potentials developed by Yasuda \([14, 15]\) for each case, we will be finding further examples of supergravity dual to a three-dimensional quantum field theory exhibiting a nontrivial renormalization group flow.

The first example is provided by \(X_7 = N^{0,1,0}_I\). The manifold \(N^{p,q,r}_I\) has been studied originally by Castellani and Romans \([16]\), identified as a coset manifold of the form \([SU(3) \times U(1)]/[U(1) \times U(1)]\). Embedding of \([U(1) \times U(1)]\) in \([SU(3) \times U(1)]\) is specified by a choice of the integers \((p, q, r)\). Each choice of \(p, q, r\) leads to an Einstein manifold, yielding \(\mathcal{N} = (3,0)\) supersymmetry and \(SU(3) \times SU(2)\) gauge symmetry for \(N^{0,1,0}_I\), or \(\mathcal{N} = (1,0)\) supersymmetry and \(SU(3) \times U(1)\) gauge symmetry otherwise. Later, Page and Pope \([17]\) have completed the coset manifold construction by showing existence of another series of Einstein manifold, \(N^{p,q,r}_{II}\), except for those corresponding to \(q = 0\). The \(N^{p,q,r}_{II}\)'s, which can be obtained from geometric squashing of the \(N^{p,q,r}_I\)'s, retain the same gauge group as the latter but instead preserve \(\mathcal{N} = (0,1)\) supersymmetry.

As in the case of \(X_7 = S^7\), the existence of two Einstein metrics on \(N^{p,q,r}_I\)'s offers an interpretation in terms of spontaneous (super)symmetry breaking: scalar field corresponding to the squashing deformation acquires a nonzero vacuum expectation value, leading to (super)-Higgs mechanism. The symmetry breaking pattern depends on the choice of the \(X_7\) orientation. With one orientation, for \(N^{0,1,0}_I\)'s, the squashing interpolates between a \(\mathcal{N} = 3\) supersymmetric vacuum and another with \(\mathcal{N} = 0\) supersymmetry. With opposite orientation, it interpolates between a non-supersymmetric vacuum and a supersymmetry restored one with \(\mathcal{N} = 1\) supersymmetry. In both cases, however, the gauge group is locally \(SU(3) \times SU(2)\) and remains unbroken.

Via AdS/CFT correspondence, this implies that the three-dimensional quantum field theory dual to the \(AdS_4 \times N^{0,1,0}\) supergravity exhibits two types of nontrivial renormalization group flow: one between \(\mathcal{N} = 3\) superconformal UV fixed point and non-supersymmetric IR fixed point, and another between nonsupersymmetric UV fixed point and \(\mathcal{N} = 1\) superconformal IR fixed point. For both, along the renormalization group flow trajectory, the global symmetry \(SU(3) \times SU(2)\) is always maintained. Analogously, for all other \(N^{p,q,r}_I\)'s except \(q = 0\), the dual
quantum field theory exhibits two renormalization group flows between $\mathcal{N} = 1$ superconformal fixed point and nonsupersymmetric one. We present these analysis in Section 2.1.

In Section 2.2, we will be studying other examples of $X_7$’s. We will find that $N_{I,II}^{p,q,r}$’s give rise to a nontrivial, $SU(3) \times U(1)$ invariant renormalization group flow between $\mathcal{N} = 1$ superconformal fixed point and nonsupersymmetric conformal fixed point. On the other hand, for $X_7 = M_{-1,1,1} Q_{1,1,1} V_{5,2}(R), SO(5)/SO(3)_{\text{max}}$, it turns out the squashing deformation does not lead to any nontrivial renormalization group flow.\footnote{Other aspects of three-dimensional superconformal field theories from Sasaki seven-manifold have been studied in [18].}

We next consider M-theory compactification vacua of Englert type. By generalizing compactification vacuum ansatz to the nonlinear level, solutions of the eleven-dimensional supergravity were obtained directly from the scalar and pseudo-scalar expectation values at various critical points of the $\mathcal{N} = 8$ supergravity potential [19]. This way, it was possible to reproduce all known Kaluza-Klein solutions of the eleven-dimensional supergravity: round $S^7$ [20], $SO(7)^-$ invariant, parallelized $S^7$ [21], $SO(7)^+$-invariant vacuum [22], $SU(4)^-$-invariant vacuum [23], and a new one with $G_2$ invariance. Among them, round $S^7$- and $G_2$-invariant vacua are stable, while $SO(7)^\pm$-invariant ones are known to be unstable [24]. In all these vacua, generically, either the metric is endowed with a nontrivial warp factor or the four-form magnetic flux $G_{abcd}$ is nonvanishing, corresponding to turning on vacuum expectation values for scalar or pseudoscalar field, respectively. Novelty of vacua with a nontrivial warp factor is that they corresponds to inhomogeneous deformations of $S^7$. In section 3, we will analyze the above vacua by investigating de Wit-Nicolai potential. We will be identifying a deformation which gives rise to a renormalization group flow associated with the symmetry breaking $SO(8) \rightarrow G_2$ (both of which are stable vacua) and find that the deformation operator is relevant at the $SO(8)$ fixed point but becomes irrelevant at the $G_2$ fixed point.

In Section 4, we study near-horizon geometry of coincident M5-branes, which are described by $AdS_7$ gauged supergravity. The geometrical interpretation of seven dimensional solutions from the eleven dimensional viewpoint has been obtained recently [25]. By taking a nonlinear Kaluza-Klein ansatz to the eleven-dimensional supergravity, they were able to obtain a consistent $S^4$ reduction to seven-dimensional, $\mathcal{N} = 1$ gauged supergravity. Field content of the latter includes a scalar field parametrizing inhomogeneous deformations of $S^4$, Yang-Mills gauge fields, and a topologically massive three-form potential. Turning on the scalar field induces again spontaneous (super)symmetry breaking, and we interpret it as being dual to a renormalization group flow from $\mathcal{N} = 1$ supersymmetric UV fixed point to a nonsupersymmetric IR fixed point of a putative $d = 6, \mathcal{N} = (1,0)$ quantum theory. The existence of stable, nonsupersymmetric $d = 6$ conformally invariant quantum theory should be of considerable interest [26], as the latter includes noncritical bosonic strings as part of the spectrum.
2 3d CFTs from Freund-Rubin Compactifications

Spontaneous compactification of M-theory to AdS$_4 \times X_7$ is obtained from near-horizon geometry of $N$ coincident M2-branes. The configuration is equivalent to Freund-Rubin compactification of the eleven-dimensional supergravity: through $X_7$, the M2-branes thread nonvanishing flux of four-form field strength

$$G_{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{-g_4}} Q e^{-7s} \epsilon_{\alpha\beta\gamma\delta}, \quad Q \equiv 96\pi^2 N_{\text{pl}}^6$$

The parameter $Q$ refers to so-called ‘Page’ charge $[27]$, $Q \equiv \pi^{-4} f_{X_7}(G + C \wedge G)$. Here, the $d = 11$ coordinates with indices $A, B, \cdots$ are decomposed into AdS$_4$ coordinates $x$ with indices $\alpha, \beta, \cdots$ and $X_7$ coordinates $y$ with indices $a, b, \cdots$. We will also adopt the eleven-dimensional metric convention as $(-, +, \cdots, +)$. A Weyl rescaling of AdS$_4$ and $X_7$, appropriate for M-theory description, is given by $g_{\alpha\beta} \rightarrow e^{7s} g_{\alpha\beta}$ and $g_{ab} \rightarrow e^{-2s} g_{ab}$.

In this section, we will be studying critical points of the scalar potential for various choices of $X_7$ and, in the corresponding three-dimensional conformal field theories, identifying an operator that gives rise to a renormalization group flow among the critical points.

2.1 $N^{0,1,0}$ space

We will begin with the case $X_7 = N^{p,q,r}$ space $[14]$, which is a homogeneous space $[SU(3) \times U(1)]/[U(1) \times U(1)]$. Consider a subset of all continuous deformation of $N^{p,q,r}$, in which the vielbein $B^a$ is given by

$$B^a = (\frac{1}{\alpha} \Omega^a, \beta (p \Omega^8 + q \Omega^3 + r \Omega^9), \frac{1}{\gamma} \Omega^A, \frac{1}{\delta} \Omega^{A'})$$

Here, $\Omega^a(a = 1, 2), \Omega^3, \Omega^A(A = 4, 5), \Omega^{A'}(A' = 6, 7), \Omega^8$ and $\Omega^9$ are the left invariant one-forms corresponding to the coset generators $\lambda^a, \lambda^3, \lambda^A, \lambda^{A'}, \lambda^8$ of $SU(3)$ and the generator of $U(1)$ factor respectively. The integer $p, q, r$ characterize the embedding of $U(1) \times U(1)$ in $SU(3) \times U(1)$. It is convenient to define

$$\alpha = e^{s+u/2-v}, \quad \beta = e^{s-3u}, \quad \gamma = e^{s+(u+v+w)/2}, \quad \delta = e^{s+(u+v-w)/2}$$

The deformation considered above can be summarized compactly in terms of four-dimensional effective Lagrangian $[14]$

$$\mathcal{L}_4 = \frac{1}{2} \sqrt{-g_4} \left(-R_4 - \frac{63}{2} (\partial_\mu s)^2 - \frac{21}{2} (\partial_\mu u)^2 - 3 (\partial_\mu v)^2 - (\partial_\mu w)^2 - V(s, u, v, w)\right),$$

where

$$V(s, u, v, w) = e^{9s} \left(\frac{3}{2} e^u (e^{-2v} + e^{v+w} + e^{v-w}) + \frac{1}{4} e^u (e^{-2v+2w} + e^{-2v-2w} + e^{4w})
+ \frac{1}{4} e^{8u-4v} + \frac{(1 + x)^2}{16} e^{8u+2v+2w} + \frac{(1 - x)^2}{16} e^{8u+2v-2w}\right) + Q^2 e^{21s}$$
and \( Q \) is the aforementioned Page charge. Critical points of the deformation are determined by the stationarity condition \( \frac{\partial V}{\partial u} = \frac{\partial V}{\partial v} = \frac{\partial V}{\partial w} = 0 \). It turns out the solution \([16, 28]\) is specified by one parameter \( c \):

\[
\alpha^2 = \frac{64}{(c+2)^2} \left( \frac{5}{4}c^2 + 3c + 2 \right) e^2, \quad \beta = \pm \frac{16(\frac{5}{4}c^2 + 3c + 2)}{(c+2)(3c+2)} e, \\
\gamma^2 = \frac{64}{(c+2)^2} \left( c + \frac{d}{2} + \frac{3}{2} \right) e^2, \quad \delta^2 = \frac{64}{(c+2)^2} \left( c - \frac{d}{2} + \frac{3}{2} \right) e^2. 
\tag{3}
\]

with \(-1 \leq c \leq 1\) and \( d = \pm \sqrt{1-c^2} \) and \( c \) is related to

\[
x \equiv \frac{3p}{q} = -\frac{(5c + 6)d}{3c+2}. 
\tag{4}
\]

It is known \([16]\) that for a particular choice of \( p, q, r \) the isometry of \( N^{p,q,r} \) is \( SU(3) \times SU(2) \) in which \( \mathcal{N} = 3 \) supersymmetry survives. We call it type I solution. For all the other values of \( p, q, r \), supersymmetry is broken to \( \mathcal{N} = 1 \) and the isometry group is \( SU(3) \times U(1) \). Moreover, as we mentioned in the introduction, Page and Pope worked out a different construction of metrics on the \( N^{0,1,0} \) space and showed that there exists another Einstein metric, squashed from the first. The isometry is \( SU(3) \times SU(2) \) and the supersymmetry is \( \mathcal{N} = (0, 1) \). We call this type II solution. Strictly speaking, the analysis of Killing spinors in \([17]\) shows that a possible different root of squashing parameter gives different supersymmetry. In other words, Type I solutions are known to preserve \( (3, 0) \). Similarly, the case of Type II solutions have \( (0, 1) \) supersymmetry. Thus for the squashing with left-handed orientation, the renormalization group flow interpolates between the boundary conformal field theories with \( \mathcal{N} = 3 \) and \( \mathcal{N} = 0 \) supersymmetry while for the squashing with right-handed orientation, the flow interpolates between conformal field theories with \( \mathcal{N} = 0 \) and \( \mathcal{N} = 1 \). From now on we will restrict foregoing discussions to \((p, q, r) = (0, 1, 0)\) case.

- **Type I solution for the space** \( N^{0,1,0}_1 \)

Type I solution with \( \mathcal{N} = (3, 0) \), as studied by Castellani and Roman \([16]\), is defined by

\[-1 \leq c \leq -\frac{2}{\sqrt{5}}, \quad d = \sqrt{1-c^2}.\]

In this case, the Eq. (3) become

\[
\alpha^2 = 16e^2, \quad \beta = \pm 4e, \quad \gamma^2 = 32e^2, \quad \delta^2 = 32e^2, 
\tag{5}
\]

\(^5\) Actually this is equivalent to the requirement \([16]\) that the internal 7 manifold has to be an Einstein space. The rescalings are fixed by this condition \([16]\).

\(^6\) The normalization convention we adopt here is as follows: \( R^{\alpha}_{\beta} = -24e^2 \delta^\alpha_{\beta}, R^\alpha_h = 12e^2 \delta^\alpha_h \) and \( G_{\alpha\beta\gamma\delta} = e \epsilon_{\alpha\beta\gamma\delta} \).
Figure 1: $(u,v)$-subspace slice of the $N^{0,1,0}$ scalar potential over $u = [-0.45, +0.15]$, $v = [-0.5, +0.5]$. The Type I vacuum is the saddle point in the middle-left corner, and the Type II vacuum is the local minimum in the middle-right corner.

where we put $c = -1$ and $d = 0$ by plugging $p = 0$ into Eq. (3). Differentiating $V$ with respect to $u, v, w$ with $x = 0$, we obtain

$$V_{uu} = 21 \times 2^{3/7} e^{9s_1}, \quad V_{vv} = 6 \times 2^{3/7} e^{9s_1}, \quad V_{ww} = -2^{3/7} e^{9s_1},$$

$$V_{uv} = 0, \quad V_{uw} = 0, \quad V_{vw} = 0,$$

evaluated at $s_1, u_1, v_1$ and $w_1$ which are the extremum values corresponding to field $s, u, v,$ and $w$, respectively,

$$e^{12s_1} = \frac{9}{2^{18/7} Q^2}, \quad u_1 = \frac{2}{21} \ln 2, \quad v_1 = \frac{1}{3} \ln 2, \quad w_1 = 0$$

that can be obtained explicitly using Eq.(2) and Eq.(5).

Conformal dimension of the perturbation operator representing squashing is determined by fluctuation spectrum of the scalar fields. After rescaling $\sqrt{63} s = \bar{s}, \sqrt{21} u = \bar{u}, \sqrt{6} v = \bar{v}, \sqrt{2} w = \bar{w}$, one finds that the nonzero fluctuation spectrums for $u, v$ and $w$ fields around the $N^{0,1,0}_1$ take

$$M_{uu}^2(N^{0,1,0}_1) = \left[ \frac{\partial^2 V}{\partial u^2} \right]_{\bar{s}_1, \bar{u}_1, \bar{v}_1, \bar{w}_1} = +2^{3/7} e^{9s_1},$$

$$M_{vv}^2(N^{0,1,0}_1) = \left[ \frac{\partial^2 V}{\partial v^2} \right]_{\bar{s}_1, \bar{u}_1, \bar{v}_1, \bar{w}_1} = +2^{3/7} e^{9s_1},$$

$$M_{ww}^2(N^{0,1,0}_1) = \left[ \frac{\partial^2 V}{\partial w^2} \right]_{\bar{s}_1, \bar{u}_1, \bar{v}_1, \bar{w}_1} = -\frac{1}{2} \times 2^{3/7} e^{9s_1}.$$
The cosmological constant $\Lambda_I$ is

$$\Lambda_I = \frac{V_I}{2} = -\frac{9\sqrt{6}}{16} \left( \frac{1}{Q^2} \right)^{3/4} \equiv -\frac{3}{r_{\text{IR}}^2 \ell_{\text{pl}}^2}.$$ 

One finds that the fluctuation spectrum for $u, v, w$ fields around $N = 3$ fixed point takes two positive values and one negative value:

$$M_{uu}^2(N_I^{0,1,0}) = +4 \frac{1}{r_{\text{IR}}^2}, \quad M_{vv}^2(N_I^{0,1,0}) = +4 \frac{1}{r_{\text{IR}}^2}, \quad M_{ww}^2(N_I^{0,1,0}) = -2 \frac{1}{r_{\text{IR}}^2}.$$ 

Via AdS/CFT correspondence, one finds that in $d = 3$ conformal field theory with $N = 3$ supersymmetry, the squashing ought to be an irrelevant perturbation of conformal dimension $\Delta = 4$ for $u, v$ fields and an relevant perturbation of conformal dimension $\Delta = 2$ or $\Delta = 1$ for $w$ field. According to the result of [29], mass spectrum of the Lichnerowtiz scalar field $\phi$ can be obtained from that of transverse spinor $\lambda_T$ via $m^2 = m_{\lambda_T}(m_{\lambda_T} - 4)$ in the normalization convention, $(\Box - 32 + m^2 \phi) = 0$. Moreover, from [30], $m_{\lambda_T} = D + 3$, where $D$ is the Dirac operator of each spinor harmonics. Hence, canonically normalized mass of the Lichnerowitz scalar is given by $\overline{m}^2 = -(D + 3)(D - 1) + 32$. Unfortunately, at present, complete eigenvalues of the Dirac operator $D$ is not known for the transverse harmonics, even though there have been attempts recently to classify the full mass spectra [31]. We anticipate that the linear combination of masses of $u$ and $v$ fields should correspond to $\overline{m}^2$.

- Type II solution for the space $N_{II}^{0,1,0}$

The Type II solution with $N = (0, 1)$ supersymmetry exists, as pointed out in [17], in the range of $c$:

$$\frac{2}{\sqrt{5}} \leq c \leq 1, \quad d = -\sqrt{1 - c^2}.$$ 

For $N_{II}^{0,1,0}$, it leads to

$$\alpha^2 = \frac{400}{9} e^2, \quad \beta = \pm \frac{20}{3} e, \quad \gamma^2 = \frac{160}{9} e^2, \quad \delta^2 = \frac{160}{9} e^2,$$

where we have put $c = 1$ and $d = 0$. One again obtains

$$V_{uu} = 189 \frac{2^{3/7}}{5^{10/7}} e^{9_{\text{II}}}, \quad V_{vv} = -6 \frac{2^{3/7}}{5^{10/7}} e^{9_{\text{II}}}, \quad V_{ww} = 11 \frac{2^{3/7}}{5^{10/7}} e^{9_{\text{II}}},$$

$$V_{uv} = -84 \frac{2^{3/7}}{5^{10/7}} e^{9_{\text{II}}}, \quad V_{uw} = 0, \quad V_{vw} = 0,$$

all evaluated at the extremum

$$e^{12_{\text{II}}} = \frac{81}{2^{48/7} \times 5^{10/7} Q^2}, \quad u_{II} = \frac{2}{21} \ln \frac{2}{5}, \quad v_{II} = \frac{1}{3} \ln \frac{2}{5}, \quad w_{II} = 0.$$ 

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From these, one calculates mass spectrum of the scalar fields straightforwardly:

\[
M_{\mu\nu}^2(N_{\mu}^{0,1,0}) = \left[ \frac{\partial^2 V}{\partial u_\mu \partial u_\nu} \right]_{\mu \neq \nu} = +9 \times \frac{2^{3/7}}{5^{10/7}} e^{9 s_{II}},
\]

\[
M_{\mu\nu}^2(N_{\mu}^{0,1,0}) = \left[ \frac{\partial^2 V}{\partial u_\mu \partial v} \right]_{\mu \neq \nu} = -2 \sqrt{14} \times \frac{2^{3/7}}{5^{10/7}} e^{9 s_{II}},
\]

\[
M_{\mu\nu}^2(N_{\mu}^{0,1,0}) = \left[ \frac{\partial^2 V}{\partial v^2} \right] = -2 \frac{3}{5^{10/7}} e^{9 s_{II}},
\]

\[
M_{\mu\nu}^2(N_{\mu}^{0,1,0}) = \left[ \frac{\partial^2 V}{\partial w^2} \right] = \frac{11}{2} \times \frac{2^{3/7}}{5^{10/7}} e^{9 s_{II}}.
\]

Diagonalizing the mass matrix, one obtains the mass eigenvalues

\[ M^2 = -5 \times \frac{2^{3/7}}{5^{10/7}} e^{9 s_{II}}, \quad +13 \times \frac{2^{3/7}}{5^{10/7}} e^{9 s_{II}}, \quad + \frac{11}{2} \times \frac{2^{3/7}}{5^{10/7}} e^{9 s_{II}}. \]

The eigenvector \( \sqrt{2 u + \sqrt{v^2}} \) (corresponding to the ‘tachyonic’ eigenvalue) in fact represents squashing of \( N_{0,1,0} \) manifold, whose magnitude is parametrized by \( \lambda^2 \):

\[ \lambda^2 = \frac{1}{4} e^{\frac{1}{2}(u+v)}. \]

For \( N_{1}^{0,1,0}, \lambda^2 = 1/2 \), while, for \( N_{II}^{0,1,0}, \lambda^2 = 1/10 \).

The cosmological constant of Type II solution \( \Lambda_{II} \) is

\[ \Lambda_{II} = \frac{V_{II}}{2} = -\frac{729}{2} \times 10^{3} \left( \frac{1}{Q^2} \right)^{3/4} \equiv -\frac{3}{r_{UV}^2 r_{pl}^2}. \]

One finds that the fluctuation spectrum for \( u, v, w \) fields around \( \mathcal{N} = 1 \) fixed point takes one negative value and two positive values:

\[
M_{uu}^2(N_{II}^{0,1,0}) = -\frac{20}{9} \frac{1}{r_{UV}^2}, \quad M_{uv}^2(N_{II}^{0,1,0}) = \frac{52}{9} \frac{1}{r_{UV}^2}, \quad M_{ww}^2(N_{II}^{0,1,0}) = \frac{22}{9} \frac{1}{r_{UV}^2},
\]

where \( \tilde{u} = \sqrt{\frac{14}{3}}(u + v) \) and \( \tilde{v} = \frac{1}{\sqrt{3}}(7u - 2v) \).

From the above mass spectrum, one finds that, in \( \mathcal{N} = 1 \) superconformal field theory, the squashing deformation ought to be a relevant perturbation of conformal dimension \( \Delta = 5/3 \) or \( 4/3 \). Scaling dimensions of other deformations are \( \Delta = 13/3 \) for \( \tilde{v} \) field and \( \Delta = 11/3 \) for \( w \) field, respectively.

### 2.2 \( SO(5)/SO(3)_{\text{max}}, V_{5,2}(R), M^{1,1,1}, Q^{1,1,1,1}, N^{p,q,r} \)

We will now study other homogeneous Einstein manifolds, for which four-dimensional scalar potential is known on an appropriate subspace.
As recalled above, Page and Pope have shown that, for $q \neq 0$, by geometric squashing, $N_{I,II}^{p,q,r}$ manifold leads, much as in $N_{I,II}^{0,1,0}$'s, to $N_{I,II}^{p,q,r}$. It is straightforward to show that the effective potential Eq.(4), which is valid for all $x$ hence $(p, q \neq 0, r)$, have two critical points. The supersymmetry preserved by $N_{I}^{p,q,r}$ and $N_{II}^{p,q,r}$ is $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (0, 1)$, respectively. On the other hand, depending on the value of $x$, relative magnitdue of the cosmological constants, $V_{I,II}$, changes. Hence, in three-dimensional conformal field theory with $SU(3) \times U(1)$ global symmetry dual to $AdS_{4} \times N_{I,II}^{p,q,r}$'s, there will be two classes of renormalization group flows: one flowing from a $\mathcal{N} = 1$ UV fixed point to a nonsupersymmetric IR fixed point, and another flowing from a nonsupersymmetric UV fixed point to a $\mathcal{N} = 1$ IR fixed point.

$SO(5)/SO(3)_{\text{max}}$

The $SO(5)/SO(3)_{\text{max}}$ seven-manifold is constructed from maximal embedding of $SO(3)$ in $SO(5)$. Let us consider squashing each coset direction as specified by the vielbein:

$$B^\alpha = \left( \frac{1}{\alpha} \Omega^a, \frac{1}{\beta} \Omega^{\hat{a}}, \frac{1}{\gamma} \Omega^0 \right),$$

where $\Omega^a (a = 1, 2, 3)$, $\Omega^{\hat{a}} (\hat{a} = 1, 2, 3)$ and $\Omega^0$ are the left invariant one-forms corresponding to the generators $X^a, X^{\hat{a}}, X^0$ of $SO(5)$, respectively. Parametrizing as

$$\alpha = e^{s+(u+v)/2}, \quad \beta = e^{s+(u-v)/2}, \quad \gamma = e^{s-3u},$$

the squashing deformation can be summarized compactly in terms of the following four-dimensional effective Lagrangian [13]:

$$\mathcal{L}_4 = \sqrt{-g_4} \left( R_4 - \frac{63}{2} (\partial_\mu s)^2 - \frac{21}{2} (\partial_\mu u)^2 - \frac{3}{2} (\partial_\mu v)^2 - V(s, u, v) \right).$$

Here, the scalar potential is given by

$$V(s, u, v) = e^{9s} \left( -12e^{u+v} - 7e^{u-v} - 2e^{-6u} + e^{8u} + e^{-6u+2v} + e^{-6u-2v} + e^{u-3v} 
- 30e^u + 16e^{-(5u+v)/2} \right) + Q^2 e^{21s}.$$ 

Again, $Q$ is the conserved ‘Page’ charge. It is straightforward to check that there exists only one critical point, $u = v = 0$. In the subspace of the perturbations considered above, one finds that the strongly interacting $d = 3$ quantum field theory dual to $AdS_4 \times SO(5)/SO(3)_{\text{max}}$ has only one isolated (super)conformal fixed point, hence, no nontrivial renormalization group flow.
The $V_{5,2}(R)$ is a real Stiefel manifold, constructed by embedding $SO(3)$ in $SO(5)$ in such a way that 5 of $SO(5)$ branches into 3 of $SO(3)$ plus two singlets. The isometry of $V_{5,2}(R)$ is $SO(5) \times SO(2)$. Consider deformations rescaling each coset direction as

$$B^\alpha = \left( \frac{1}{\alpha} \Omega^m, \frac{1}{\beta} \Omega^{\hat{m}}, \frac{1}{\gamma} \Omega^0 \right),$$

where $\Omega^m (m = 1, 2, 3), \Omega^{\hat{m}} (\hat{m} = 1, 2, 3)$ and $\Omega^0$ are the left invariant one-forms corresponding to the generators $X^m = T^4_m, X^{\hat{m}} = T^5_{\hat{m}}, X^0 = T^45$ where $T^{ij} (i, j = 1, 2, 3, 4, 5)$ are generators of $SO(5)$. In terms of parametrization

$$\alpha = e^{s+(u+v)/2}, \beta = e^{s+(u-v)/2}, \gamma = e^{s-3u},$$

the four-dimensional effective Lagrangian \[14\] can be written as

$$L = \sqrt{-g_4} \left( R_4 - \frac{63}{2} \left( \partial_\mu s \right)^2 - \frac{21}{2} \left( \partial_\mu u \right)^2 - \frac{3}{2} \left( \partial_\mu v \right)^2 - V(s, u, v) \right),$$

where

$$V(s, u, v) = e^{9s} \left( -6e^{u+v} - 6e^{u-v} - 2e^{-6u} + e^{-6u+2v} + e^{-6u-2v} + e^{8u} \right) + Q^2 e^{21s}$$

and $Q$ is the Page charge. The potential exhibits only one critical point at $u = \frac{1}{7} \ln \frac{3}{2}, v = 0$. As in $SO(5)/SO(3)_{\text{max}}$ case, there is no renormalization group flow in the corresponding three-dimensional conformal field theory.

$\bullet M^{1,1,1}$

The manifold $M^{1,1,1}$ is a homogeneous space $[SU(3) \times SU(2) \times U(1)]/[SU(2) \times U(1) \times U(1)]$. For the vielbein $B^\alpha$, we rescale each coset direction as

$$B^\alpha = (\frac{1}{b} \Omega^m, \frac{1}{c} (\sqrt{\gamma} \Omega^8 + \Omega^3 + 2\Omega^{\lambda'}), \frac{1}{a} \Omega^A)$$

where $\Omega^m (m = 1, 2), \Omega^3, \Omega^8, \Omega^A (A = 4, 5, 6, 7)$, and $\Omega^{\lambda'}$ are the left invariant one-forms corresponding to the coset generators $\sigma^m$ and $\sigma^3$ of $SU(2)$, $\lambda^8$ and $\lambda^A$ of $SU(3)$, and the generator of $U(1)$ factor respectively. In terms of parametrization

$$a = e^{s+(u-v)/2}, \quad b = e^{s+u/2+v}, \quad c = e^{s-3u},$$

the four-dimensional effective Lagrangian \[14\] can be written as

$$L = \frac{1}{2} \sqrt{-g_4} \left( -R_4 - \frac{63}{2} \left( \partial_\mu s \right)^2 - \frac{21}{2} \left( \partial_\mu u \right)^2 - 3 \left( \partial_\mu v \right)^2 - V(s, u, v) \right).$$
The scalar potential is given by

\[ V(s, u, v) = e^{9s} \left( -3e^{u-v} - e^{u+2v} + \frac{9}{8} e^{8u-2v} + \frac{1}{4} e^{8u+4v} \right) + Q^2 e^{21s}, \]

where again \( Q \) is the conserved Page charge. The stationary conditions \( \frac{\partial V}{\partial s} = \frac{\partial V}{\partial u} = \frac{\partial V}{\partial v} = 0 \) lead to a set of cubic equations for \( t \equiv 3e^{-3v} \):

\[ t^3 - 3t^2 + 4t - 4 = (t - 2)(t^2 - t + 2) = 0. \]

One finds that there is no extra critical point except \( t = 2 \). Hence, one concludes that there ought to be no nontrivial renormalization group flow in the dual, \( d = 3 \) (super)conformal field theory.

\( \bullet Q^{1,1,1} \)

The manifold \( Q^{1,1,1} \) is a homogeneous space \([SU(2) \times SU(2) \times SU(2)]/[U(1) \times U(1) \times U(1)]\).

For the vielbein \( B^\alpha \), we rescale each coset direction as

\[ B^\alpha = (\frac{1}{a} \Omega^i, \frac{1}{d} (\Omega^0 + \Omega^0' + \Omega^0''), \frac{1}{b} \Omega^i', \frac{1}{c} \Omega^{i''}) \]

where \( \Omega^i(i = 1, 2), \Omega^0, \Omega^0'(i' = 4, 5), \Omega^0', \Omega^{i''}(i'' = 6, 7) \) and \( \Omega^{0''} \) are the left invariant one-forms corresponding to the coset generators \( \sigma^i \) and \( \sigma^3 \) of \( SU(2), SU(2) \) and \( SU(2) \) respectively. By parametrizing

\[ a = e^{s+(u+v+w)/2}, \quad b = e^{s+(u+v-w)/2}, \quad c = e^{s+u/2-v}, \quad d = e^{s-3u}, \]

the four dimensional effective Lagrangian \([14]\) can be written as

\[ L = \frac{1}{2 \sqrt{-g_4}} \left( -R_4 - \frac{63}{2} (\partial_\mu s)^2 - \frac{21}{2} (\partial_\mu u)^2 - 3 (\partial_\mu v)^2 - (\partial_\mu w)^2 - V(s, u, v, w) \right), \]

where

\[ V(s, u, v, w) = e^{9s} \left( -e^{u+v+w} - e^{u+v-w} - e^{-2v} + \frac{1}{4} e^{8u+2v+2w} + \frac{1}{4} e^{8u+2v-2w} + \frac{1}{4} e^{8u-4w} \right) + Q^2 e^{21s}. \]

The stationary conditions \( \frac{\partial V}{\partial u} = \frac{\partial V}{\partial v} = \frac{\partial V}{\partial w} = 0 \) yield a set of simple algebraic equations

\[ \alpha(1 - \alpha) = \beta(1 - \beta) = \gamma(1 - \gamma) = \alpha^2 + \beta^2 + \gamma^2, \]

where \( \alpha \equiv \frac{a^2}{2d^2}, \beta \equiv \frac{b^2}{2d^2}, \gamma \equiv \frac{c^2}{2d^2} \). The only solution is

\[ 2\alpha = 1 - \sqrt{1 - t} = 2\beta = 2\gamma, \quad 3 - 3\sqrt{1 - t} - 2t = 0. \]
It is easy to see that there exists only one extremum value when \( t = 3/4 \). One again concludes that there is no nontrivial renormalization group flow in the dual, \( d = 3 \) (super)conformal field theory.

| \( X_7 \)       | Isometry          | Holonomy | Supersymmetry | RG flow |
|------------------|-------------------|----------|---------------|---------|
| Round \( S^7 \)  | \( SO(8) \)       | 1        | (8, 8)        | Yes\(^5\) |
| Squashed \( S^7 \) | \( SO(5) \times SU(2) \) | \( G_2 \) | 1             | Yes\(^5\) |
| \( SO(5)/SO(3)_{\text{max}} \) | \( SO(5) \)       | \( G_2 \) | (1, 0)        | No      |
| \( V^5_{,2}(R) \) | \( SO(5) \times U(1) \) | \( SU(3) \) | (2, 0)        | No      |
| \( M^{1,1,1} \)  | \( SU(3) \times SU(2) \times U(1) \) | \( SU(3) \) | (2, 0)        | No      |
| \( Q^{1,1,1} \)  | \( SU(2)^3 \times U(1) \) | \( SU(3) \) | (2, 0)        | No      |
| \( Q^{p,q,r} \)  | \( SU(2)^3 \times U(1) \) | \( SU(7) \) | 0             |         |
| \( N_i^{0,1,0} \) | \( SU(3) \times SU(2) \) | \( SU(2) \) | (3, 0)        | Yes\(^*\) |
| \( N_{II}^{0,1,0} \) | \( SU(3) \times SU(2) \) | \( G_2 \) | (0, 1)        | Yes\(^*\) |
| \( N_i^{p,q\neq0,r} \) | \( SU(3) \times U(1) \) | \( G_2 \) | (1, 0)        | Yes\(^\dagger\) |
| \( N_{II}^{p,q\neq0,r} \) | \( SU(3) \times U(1) \) | \( G_2 \) | (0, 1)        | Yes\(^\dagger\) |

Table 1: Classification of Einstein spaces \( X_7 \) and Renormalization Group Flows. The flow found in [1] is marked \(^\#\), while the ones found in Section 2.1 and 2.2 are marked \(^\star\) and \(^\dagger\).

We have summarized our result of this Section in Table 1. In the subspace of squashing deformations considered, among various known \( X_7 \)'s, we have found that only \( N_{I,\text{II}}^{0,1,0} \)'s turn out to be dual to nontrivial renormalization group flows of strongly coupled three-dimensional field theories, connecting two (super)conformal fixed points. One novelty of these conformal field theories is that, even though the superconformal symmetry is changed, the global symmetry is not changed at all, in contrast to the deformation interpolating between round- and squashed-\( S^7 \) [1].

Although we have not considered nonsupersymmetric seven-manifolds \( M^{p,q,r} \) or \( Q^{p,q,r} \) in this paper, it is known that \( M^{p,q,r} \) remains stable for the specific region of \( 98/243 \leq p^2/q^2 \leq 6358/4563 \) and \( Q^{p,q,r} \) solutions are stable in a certain region containing the point \( p = q = r = 1 \). We therefore anticipate that, if there exists more than one critical points of the corresponding scalar potential, they will provide gravity dual to strongly interacting, stable, nonsupersymmetric field theories in three dimensions, whose renormalization group flows interpolate interacting conformal fixed points [26].

3 3d CFTs from Englert Compactifications

So far, we have deduced existence of three-dimensional conformal field theories out of Freund-Rubin compactification of M-theory. Let us now consider more general compactifications by
relaxing the restriction that the eleven-dimensional metric is a product space and that there is no magnetic four-form field strength flux threaded on $X_7$. The first solution of this sort has been found by Englert [21]. In contrast to the Freund-Rubin compactifications, the symmetry of the vacuum is no longer given by the isometry group of $X_7$ but rather by the group which leaves invariant both the metric $g_{ab}$ and four-form magnetic field strength $G_{abcd}$. In the Englert compactification [21], nonvanishing $G_{abcd}$ on the round $S^7$ breaks $SO(8)$ down to $SO(7)$. Precise interpretation of Englert compactification in terms of microscopic configuration of coincident M2-branes is still lacking. Nevertheless, AdS/CFT correspondence implies that there ought to be a three-dimensional (super)conformal field theory for each Englert-type compactification as well.

In Kaluza-Klein supergravity, it is well-known that the four-dimensional $\mathcal{N} = 8$ gauged supergravity can be embedded consistently into the eleven-dimensional supergravity. As shown in [32], the 70 scalars of $\mathcal{N} = 8$ supergravity live on the coset space $E_7/SU(8)$ and are described by an element $\mathcal{V}(x)$ of the fundamental 56-dimensional representation (56-bein) of $E_7$:

$$\mathcal{V}(x) = \begin{bmatrix} u_{ij}^I(x) & v_{ijKL}(x) \\ v_{IJ}(x) & u_{K}(x) \end{bmatrix},$$

where $SU(8)$ index pairs $[ij], \cdots$ and $SO(8)$ index pairs $[IJ], \cdots$ are antisymmetrized and therefore $u$ and $v$ are $28 \times 28$ matrices. Complex conjugation is done by raising or lowering indices, for example, $(u_{ij}^I)^* = u_{ij}^I$. Under local $SU(8)$ and local $SO(8)$, the matrix $\mathcal{V}(x)$ transforms as $\mathcal{V}(x) \to U(x)\mathcal{V}(x)O^{-1}(x)$, where $U(x) \subset SU(8)$ and $O(x) \subset SO(8)$ and matrices $U(x), O(x)$ are elements of the 56-dimensional representation. By appropriate gauge fixing of the local $SU(8)$ symmetry, the 56-bein $\mathcal{V}(x)$ can be brought into the following form:

$$\mathcal{V}(x) = \exp \left[ \begin{array}{cc} 0 & -\sqrt{2} \phi_{ijkl} \\ \frac{\sqrt{2}}{4} \phi_{ijkl} & 0 \end{array} \right],$$

where $\phi_{ijkl}$ is a complex self-dual tensor describing the 35 scalars $35_c$ (the real part of $\phi^{ijkl}$) and 35 pseudoscalar fields $35_p$ (the imaginary part of $\phi^{ijkl}$) of the $\mathcal{N} = 8$ supergravity. Note that, after gauge fixing, there is no distinction between $SO(8)$ and $SU(8)$ indices.

The scalar potential of the gauged $\mathcal{N} = 8$ supergravity is known to possess four critical points with at least $G_2$ invariance [33]. The maximally supersymmetric vacuum with $SO(8)$ symmetry, $S^7$, is where expectation value of both scalar and pseudoscalar fields vanish. Let us denote self-dual and anti-self-dual tensors of $SO(8)$ tensor as $C_{+}^{IJKL}$ and $C_{-}^{IJKL}$, respectively, satisfying

$$C_{\pm}^{IJMNC_{\pm}^{MNKL}} = 12\delta_{KL}^{IJ} \pm 4C_{\pm}^{IJKL}.$$ 

Turning on the scalar fields proportional to $C_{+}^{IJKL}$ yields an $SO(7)^+$ invariant vacuum. Likewise, turning on pseudoscalar fields proportional to $C_{-}^{IJKL}$ yields $SO(7)^-$ invariant vacuum.
Both $SO(7)^\pm$ vacua are nonsupersymmetric. However, simultaneously turning on both scalars and pseudoscalar fields proportional to $C_+^{IJKL}$ and $C_-^{IJKL}$, respectively, one obtains $G_2$-invariant vacuum with $\mathcal{N} = 1$ supersymmetry\footnote{The $G_2$ is the common subgroup of $SO(7)^+$ and $SO(7)^-$.}. The most general vacuum expectation value of 56-bein retaining $G_2$-invariance can be parametrized as

$$\langle \phi_{IJKL} \rangle = \frac{\lambda}{2\sqrt{2}} \left( \cos \alpha \ C_+^{IJKL} + i \sin \alpha \ C_-^{IJKL} \right).$$

In this case, the elements of 56-bein $V(x)$ can be written as:

$$u_{JK}^{IJ}(\lambda) = 2p^3 \delta_{IJ}^{KL} + \frac{1}{2} (1 + \cos 2\alpha) pq^2 \ C_+^{IJKL} + \frac{1}{2} (1 - \cos 2\alpha) pq^2 \ C_-^{IJKL}$$

$$-i pq^2 \sin 2\alpha \ D_+^{IJKL},$$

$$v_{IJKL}^{JJ}(\lambda) = \frac{1}{2} (3e^{ia} + e^{-3ia}) q^3 \delta_{IJ}^{KL} + p^2 q \cos \alpha \ C_+^{IJKL} - ip^2 q \sin \alpha \ C_-^{IJKL}$$

$$+ \frac{1}{2} (e^{ia} - e^{-3ia}) q^3 \ D_+^{IJKL},$$

where $D_{IJKL}^{\pm} \equiv \frac{1}{2} \left( C_+^{IJMN} C_{MNKL} \pm C_-^{IJMN} C_+^{MNKL} \right), p \equiv \cosh(\lambda/2\sqrt{2})$ and $q \equiv \sinh(\lambda/2\sqrt{2}).$

Figure 2: Scalar potential $V(\alpha, \lambda)$. The left axis corresponds to $v = \cos \alpha$ and right one does $\lambda$. The extremum value $V = -7.19$ for $G_2$ occurs around $v = 0.56$ and $\lambda = 0.73$ while the extremum value $V = -6$ for $SO(8)$ appears around $\lambda = 0$. We take $g^2$ as 1 for simplicity.

In the above parametrization, the scalar potential is given by \cite{19}

$$V(\alpha, \lambda) = 2g^2 \left( (7v^4 - 7v^2 + 3)c^3 s^4 + (4v^2 - 7)v^5 s^7 + c^5 s^2 + 7v^3 c^2 s^5 - 3c^3 \right) \quad (7)$$
where \( c \equiv \cosh(\lambda/\sqrt{2}) \), \( s \equiv \sinh(\lambda/\sqrt{2}) \) and \( v \equiv \cos \alpha \). The potential is plotted in Figure 2 and the four critical points are summarized in Table 2.

| Gauge symmetry | Supersymmetry | \( \cos \alpha \) | \( c^2 \) | \( V \) |
|----------------|--------------|----------------|-------|-------|
| \( SO(8) \)   | \( \mathcal{N} = 8 \) | \( 0 \) | \( 1 \) | \( -6g^2 \) |
| \( SO(7)^- \)  | \( \mathcal{N} = 0 \) | \( -\frac{5}{4} \) | \( -\frac{25\sqrt{5}}{8}g^2 = -6.99g^2 \) |
| \( SO(7)^+ \)  | \( \mathcal{N} = 0 \) | \( 1 \) | \( \frac{1}{2} \left( \frac{4}{\sqrt{5}} + 1 \right) \) | \( -2 \cdot 5^{3/4}g^2 = -6.69g^2 \) |
| \( G_2 \)      | \( \mathcal{N} = 1 \) | \( \frac{1}{2}\sqrt{(3 - \sqrt{3})} = 0.56 \) | \( \frac{1}{2}(3 + 2\sqrt{3}) \) | \( -\frac{21\sqrt{2}}{25\sqrt{5}}3^{3/4}g^2 = -7.19g^2 \) |

### Table 2. Summary of four critical points: symmetry group, supersymmetry, vacuum expectation values of scalar and pseudoscalar fields, and cosmological constants.

In this section, we will be identifying a renormalization group flow associated with global symmetry breaking \( SO(8) \rightarrow G_2 \) in a three-dimensional strongly coupled field theory \( \mathcal{N} = 8 \). We will show that the perturbation operator is relevant at the \( SO(8) \) invariant UV fixed point but becomes irrelevant at the \( G_2 \) invariant IR fixed point. To identify conformal field theory operator corresponding to the perturbation while preserving \( G_2 \) symmetry, we will consider harmonic fluctuations of spacetime metric and \( \lambda \) scalar field around \( AdS_4 \times S^7 \). From the scalar potential Eq. (7), one finds that the cosmological constant \( \Lambda \) is given by

\[
\Lambda_{SO(8)} = -6g^2 = -\frac{3}{r_{\text{UV}}^2 \ell_{\text{pl}}^2},
\]

where \( r_{\text{UV}} \) is the radius of \( AdS_4 \) and \( \ell_{\text{pl}} \) is the eleven-dimensional Planck scale. Conformal dimension of the perturbation operator representing this deformation is calculated by fluctuation spectrum of the scalar fields. The kinetic term \(^34\) for \( \lambda \) field can be obtained from \(-|A^{ijkl}_\alpha|^2/96\), where

\[
A^{ijkl}_\alpha = -2\sqrt{2}(u^{ij}_K \partial_\alpha v^{klij} - v^{ijkl} \partial_\alpha u^{ij}_K).
\]

Then, from the explicit forms of \( u^{ij}_K \) and \( v^{ijkl} \) in Eq. (8), one obtains \(|A^{ijkl}_\alpha|^2 = 84(\partial_\alpha \lambda)^2\), where we have kept to quadratic order in the fluctuation of \( \lambda \). The resulting kinetic term is \(-\frac{7}{8}(\partial_\alpha \lambda)^2\). After rescaling the \( \lambda \) field as \( \lambda = \sqrt{\frac{7}{8}}\lambda \), one finds that the mass spectrum of the \( \lambda \) field around \( SO(8) \) fixed point is given by:

\[
\partial_\lambda^2 V(SO(8)) \bigg|_{\lambda = 0} = -2g^2 \ell_{\text{pl}}^2 = -2\frac{1}{r_{\text{UV}}^2}.
\]  

Via AdS/CFT correspondence, one finds that in the corresponding \( \mathcal{N} = 8 \) superconformal field theory, the \( G_2 \) symmetric deformation ought to be a relevant perturbation of conformal dimension \( \Delta = 1 \) or \( \Delta = 2 \). Recall that, on \( S^7 \), mass spectrum of the representation corresponding

\(^8\)As the \( SO(7)^\pm \) vacua are unstable \(^{24}\), we will not consider in the foregoing discussions.
to \textit{SO}(8) Dynkin label \((n, 0, 2, 0)\) is given by

\[ \tilde{M}^2 = ((n + 1)^2 - 9) m^2 \]

where \(m^2\) is mass-squared parameter of a given \textit{AdS}_4 spacetime and a scalar field \(S\) is defined by \((\Delta \text{AdS} + \tilde{M}^2)S = 0\). This follows from the known mass formula [35] \(M^2 = ((n + 1)^2 - 1)m^2\) for \(O^{-(1)}\) and the fact that \(M^2\) is traditionally defined according to \((\Delta \text{AdS} - 8m^2 + M^2) = 0\).

For \(35_c\) corresponding to \(n = 0\), \(\tilde{M}_{35_c}^2 = -8m^2\) and this ought to equal to Eq. (8). Recalling that \(r_{\text{UV}}^2 = r_{\text{IR}}^2/4 = 1/4m^2\),

\[ \partial^2 \mathcal{V}(\text{SO}(8)) \bigg|_{\lambda = 0} = -2 \frac{1}{r_{\text{UV}}^2} = \tilde{M}_{35_c}^2 \]

On the other hand, the mass spectrum of the representation corresponding to \(\text{SO}(8)\) Dynkin label \((n + 2, 0, 0, 0)\) is given by \(\tilde{M}^2 = ((n - 1)^2 - 9)m^2\). This follows from the known mass formula [35] \(M^2 = ((n - 1)^2 - 1)m^2\) for \(O^{+(1)}\). For \(35_v\) corresponding to \(n = 0\), \(\tilde{M}_{35_v}^2 = -8m^2\) and this ought to equal to Eq. (8).

Let us next consider the conformal fixed point corresponding to the \(G_2\) symmetry. Again, from the scalar potential Eq. (7), one finds that cosmological constant \(\Lambda\) is given by

\[ \Lambda_{G_2} = -\frac{216\sqrt{2}}{25\sqrt{5}} 3^{1/4} g^2 \equiv -\frac{3}{r_{\text{IR}}^2 \ell_{\text{pl}}^2}. \]

The mass spectrum for the \(\tilde{\lambda}\) field around \(G_2\) fixed point takes a positive value:

\[ \partial^2 \mathcal{V}(G_2) \bigg|_{c^2 = \frac{1}{(3 + 2\sqrt{3})}} = 15.446 g^2 \ell_{\text{pl}}^2 = 6.443 \frac{1}{r_{\text{IR}}^2}. \]

One finds that in the corresponding three-dimensional conformal field theory with \(N = 1\) supersymmetry, the \(G_2\) symmetric deformation ought to be an irrelevant perturbation of conformal dimension \(\Delta = 4.448\ldots\). We thus conclude that the perturbation operator dual to the \(\lambda\) field induces nontrivial renormalization group flow from \(N = 8\) superconformal UV fixed point with \(\text{SO}(8)\) symmetry to \(N = 1\) superconformal IR fixed point with \(G_2\) symmetry.

It is known that, in \(N = 8\) supergravity, there also exists a \(N = 2\) supersymmetric, \(SU(3) \times U(1)\) invariant vacuum [36]. To reach this critical point, one has to turn on expectation values of both scalar and pseudoscalar fields as

\[ \langle \phi_{IJKL} \rangle = \frac{1}{2\sqrt{2}} \left( \lambda X_{IJKL}^+ + i \lambda' X_{IJKL}^- \right), \]

where

\[
X_{ijkl}^+ = +[(\delta_{ijkl}^{1234} + \delta_{ijkl}^{5678}) + (\delta_{ijkl}^{1256} + \delta_{ijkl}^{3478}) + (\delta_{ijkl}^{1278} + \delta_{ijkl}^{3456})]
\]

\[
X_{ijkl}^- = -[(\delta_{ijkl}^{1357} + \delta_{ijkl}^{2468}) + (\delta_{ijkl}^{1268} + \delta_{ijkl}^{2457}) + (\delta_{ijkl}^{1458} + \delta_{ijkl}^{2367}) + (\delta_{ijkl}^{1467} + \delta_{ijkl}^{2358})]
\]
The scalar potential is

\[ V = \frac{1}{2}g^2 \left( s'^4 \left( (x^2 + 3)c^3 + 4x^2v^3s^3 - 3v(x^2 - 1)s^3 + 12xv^2cs^2 - 6(x - 1)cs^2 + 6(x + 1)c^2sv \right) \\
+ 2s'^2 \left( 2(c^3 + v^3s^3) + 3(x + 1)vs^3 + 6xv^2cs^2 - 3(x - 1)cs^2 - 6c \right) - 12c \right) \]

where

\[ c \equiv \cosh(\lambda/\sqrt{2}), \quad s \equiv \sinh(\lambda/\sqrt{2}), \quad c' \equiv \cosh(\lambda'/\sqrt{2}), \quad s' \equiv \sinh(\lambda'/\sqrt{2}) \]

\[ v \equiv \cos \alpha, \quad x = \cos 2\phi. \]

At the critical point, \( s = 1/\sqrt{3}, s' = 1/\sqrt{2} \) and \( \alpha = 0 \), one finds that the cosmological constant is given by \( V = -\frac{2\sqrt{3}}{2}g^2 \). Hence, one expects that there ought to exist a renormalization group flow between \( \mathcal{N} = 8 \) SO(8) fixed point and \( \mathcal{N} = 2 \) SU(3) × U(1) fixed point.

## 4 6d CFTs from Inhomogeneous Compactification

We finally study the case of near-horizon geometry of coincident M5-branes, as described by AdS7 gauged supergravity. Recently, in [25], geometrical interpretation for vacua in seven-dimensional \( \mathcal{N} = 1 \) gauged supergravity has been given in terms of Englert type compactification of eleven-dimensional supergravity, having nonvanishing electric or tilted magnetic four-form fluxes and inhomogeneous metric deformation, by a set of consistent nonlinear ansatz. The inhomogeneous deformations of the \( S^4 \) is parametrized by a scalar field \( \phi \), while the SU(2) gauge fields, which are the surviving subgroup of the SO(5) Yang-Mills fields of the maximal gauged supergravity after the deformation, are associated with the right translations under the SU(2). The full two parameter potential of the resulting seven-dimensional gauged supergravity turns out equal to [37] (See also [38]):

\[ V(\phi) = 16 h^2 e^{-\frac{8}{\sqrt{5}}\phi} - 8\sqrt{2} h g e^{-\frac{2}{\sqrt{5}}\phi} - g^2 e^{\frac{2}{\sqrt{5}}\phi}, \quad (9) \]

where \( h \) and \( g \) are arbitrary real constants and \( e^{-\frac{2}{\sqrt{5}}\phi} \) represents the SU(2) gauge coupling constant. As shown in Figure 3, provided \( h/g > 0 \), the scalar potential has two extrema.

A local maximum of the potential Eq.(9) is located at

\[ \phi_1 = \frac{1}{\sqrt{5}} \ln \frac{8\sqrt{2} h}{g}, \]

whose curvature equals to

\[ \left[ \frac{\partial^2 V}{\partial \phi^2} \right]_{\phi = \phi_1} = -2 \times 2^{1/5} g^2 \left( \frac{\sqrt{2} h}{g} \right)^{2/5} \equiv -8 \frac{1}{r_{UV}}. \quad (10) \]
Figure 3: Schematic shape of the scalar potential of $d = 7, \mathcal{N} = 1$ gauged supergravity.

One then infers the radius of $AdS_7$ as

$$r_{UV}^2 = \frac{4}{(2h/g)^{2/5}g^2} = -15/V_I \quad \text{where} \quad V_I = -\frac{15}{2 \times 2^{4/5}g^2} \left( \frac{\sqrt{2}h}{g} \right)^{2/5}.$$  

According to the analysis of [39], if scalar field fluctuation about the maximum of the potential $V$ satisfy $(\Box + \frac{\partial^2}{r_{UV}^2})\phi = 0$, then perturbative stability is guaranteed provided $\alpha \leq (d-1)^2/4$. In the present case, the stability bound yields $\alpha_s = 9$ when $d = 7$. Eq. (10) is less than the bound, hence, is stable. In fact, the vacuum is $\mathcal{N} = 1$ supersymmetric. Conformal dimension of the perturbation operator representing the squashing is calculated by fluctuation spectrum of the scalar fields. From Eq. (10), via AdS/CFT correspondence, one finds that, in six-dimensional $\mathcal{N} = (1, 0)$ superconformal field theory, the perturbation operator representing the scalar field deformation ought to be a relevant perturbation of conformal dimension $\Delta = 2$ or $\Delta = 4$.

A local minimum of the scalar potential Eq. (9) is located at

$$\phi_{II} = \frac{1}{\sqrt{5}} \ln \frac{4\sqrt{2}h}{g},$$

around which

$$\left[ \frac{\partial^2 V}{\partial \phi^2} \right]_{\phi = \phi_{II}} = 2 \times 2^{4/5}g^2 \left( \frac{\sqrt{2}h}{g} \right)^{2/5} \equiv +12 \frac{1}{r_{IR}^2}.$$ 

One finds that the radius of $AdS_7$ is changed to:

$$r_{IR}^2 = -15/V_{II} \quad \text{where} \quad V_{II} = -\frac{5}{2^{4/5}g^2} \left( \frac{\sqrt{2}h}{g} \right)^{2/5}.$$  

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9 The radius of $AdS_7$ space is defined by $R_{\alpha \beta \gamma \delta} = -\frac{1}{r_{UV}^2}(g_{\alpha \gamma}g_{\beta \delta} - g_{\alpha \delta}g_{\beta \gamma})$. 

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Dual to the critical point is a six-dimensional conformal field theory with no supersymmetry, for which the perturbation associated with the squashing deformation is an irrelevant perturbation of conformal dimension $\Delta = 3 + \sqrt{21}$.

One thus finds that, in the subspace representing inhomogeneous deformation of $S^4$, there ought to be a six-dimensional quantum theory which interpolates between $\mathcal{N} = (1,0)$ superconformal fixed point in the UV and nonsupersymmetric fixed point in the IR. Note that the coformal field theory around the IR fixed point is a stable one, despite absence of any supersymmetry, which follows from the general argument of [26]. Existence of a stable, nonsupersymmetric $d = 6$ conformal field theory would be of considerable interest, as the theory would contain stable, noncritical bosonic strings as part of the spectra.

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