Last time: Magnetic systems

Easy model: \( \mathcal{H} = -J \sum c_{ij} \sigma_i \sigma_j - H \sum \sigma_i \)

\( \sigma_i \in \{ \pm 1 \} \quad (i = 1, 2) \)

Partition function: \( Z = \sum \exp \left( \frac{-\beta \mathcal{H}}{k_B T} \right) \)

Weisz MF approx: \( \sum c_{ij} \sigma_i \sigma_j = \sum \sigma_i \cdot \sum \sigma_j \frac{\beta J}{2} \delta_{ij} \)

\( -\beta \mathcal{H} = \beta \sum \sigma_i + \beta H \sum \sigma_i \)

\( = -\sum_i \beta \mathcal{H}_i (\sigma_i, \bar{\sigma}) \)

\( \bar{\sigma} = \langle \sigma_i \rangle = \frac{\sum_{\sigma_i = \pm 1} \sigma_i e^{-\beta \mathcal{H}_i}}{\sum_{\sigma_i = \pm 1} e^{-\beta \mathcal{H}_i}} = \tanh \left( \frac{J \bar{\sigma} + h}{T} \right) \)

\( h = 0 \). \( T = T_c (1 + \delta) \)

\( \delta = (T - T_c) / T_c < 1 \)

\( \frac{\beta J}{k_B T_c} = 1 - \frac{T}{T_c} \quad \left( T_c = \frac{J}{k_B} \right) \)

\( \bar{\sigma} = \tanh \left( \frac{h}{\delta} \right) \)

\( t > 0, \bar{\sigma} = 0 \)

\( t < 0, \bar{\sigma} = \frac{1}{2} \pm \sqrt{3t} \)

\( \delta \approx 0 \) soft phase lower free energy

\( F = -N k_B T \log \left[ e^{\delta\bar{\sigma}} + e^{-\delta\bar{\sigma}} \right] \)

\( \text{Spontaneous sym. breaking} \)

\( \bar{\sigma} \propto |T_c - T|^\beta \) with BMF = \( \frac{1}{2} \).

Spec. heat: \( C_H = \frac{3}{2} \frac{\partial^2 \langle \mathcal{H} \rangle}{\partial T^2} = \begin{cases} 3 g J / T_c & T > T_c \\ 3N k_B & T_c > T \end{cases} \)

Generally, \( C_H \propto |T - T_c|^\alpha \); \( \alpha_{BMF} = 0 \)
Energy:

\[
\frac{1}{N} \langle \mathcal{H} \rangle = \frac{M^e}{N^e} = \Phi J \bar{\sigma} \cdot \langle \mathcal{O} \rangle = \frac{3}{2} N^e J_{\tau c} \frac{\bar{\sigma}}{T} \quad t > 0
\]

\[
= -g J \bar{\sigma}^2 = \sum \frac{g J t}{t-c} \quad t < 0
\]

\[
C_H = \frac{3}{8} \langle \mathcal{H} \rangle = \begin{cases} 
3 N^e g J_{\tau c} & t > 0 \\
\frac{1}{2} N^e g J_{\tau c} & t < 0 
\end{cases}
\]

Generally, \( C_H \propto (T-T_c)^{-\alpha} \rightarrow [\alpha = 0] \)

\# effect of magnetic field: consider \( h > 0 \)

\( \bar{\sigma} = \tanh (J \bar{\sigma} + h) \)

\[ T > T_c \]

\[ T < T_c \]

\[ \text{lowest free energy} \quad (\text{lowest } \bar{\sigma}^2) \]

\[ \text{not selected} \]
\[ f n \ h < 0 , \ \bar{\sigma} \to -\bar{\sigma} . \]  

1st order transition across \( h = 0 \). \( fn T < T_c \)

Phase diagram:

\[ \begin{array}{c|c}
\hline
h & \bar{\sigma} > 0 \\
\hline
0 & \bar{\sigma} < 0 \\
\hline
\end{array} \]

2nd order transition

1st order transition

Cf. Liquid-gas transition

\[ \begin{array}{c}
P \uparrow \text{ gas} \\
\hline
P_c \to T \\
\hline
\end{array} \]

Order parameter: \( \text{fig} - \text{gas} \)

Small field \( h \) around crit. pt:

\[ \bar{\sigma} = \tanh (g\bar{\sigma} + h) = (1 - t)\bar{\sigma} - \frac{\bar{\sigma}^3}{3} + h \]  

(egn of 566)

\( \uparrow \) Taylor expand again

(Expecting small change \( \Delta T \))

\[ t\bar{\sigma} = -\frac{\bar{\sigma}^3}{3} + h \]  

(\( \text{small} \ h \))

\( t > 0 , \ \bar{\sigma} \approx \frac{h}{t} \)

\( t = 0 : \ \bar{\sigma} \approx B h^{1/3} \)

\[ S_{MF} = 3 \]
Magnetic susceptibility:

\[ \chi = \frac{2M}{\delta T} \Bigg|_{T \to 0} \propto N \frac{\delta \sigma}{\delta h} \Bigg|_{h \to 0} = \frac{1}{t} = \frac{1}{T - T_c} \text{ for } T > T_c \]

To find \( \chi \) for \( T < T_c \), take derivative of the equation of state:

\[ t \sigma = -\frac{\sigma^3}{3} + h \rightarrow t \frac{\delta \sigma}{\delta h} + \sigma^2 \frac{\delta \sigma}{\delta h} = 1 \]

\[ \chi \propto \frac{\delta \sigma}{\delta h} = \frac{1}{t + \sigma^2} = \begin{cases} \frac{1}{t} & t > 0 \\ -\frac{1}{2t} & t < 0 \end{cases} \]

Generally, \( \chi \propto |T - T_c|^{-\gamma} \rightarrow |\delta_{MF}| = 1 \)

Note: exponents \( \alpha, \beta, \gamma, \delta \) same as liquid-gas transition

\[ \Rightarrow \text{same "universality class"} \]
1) Bragg-William MFT

- Fix magnetization: \( m = \frac{N_+ - N_-}{N} \)
- Compute trial free energy

\[
\Psi(m) = E(m) - TS(m) \quad \text{for each } m
\]
- Obtain the ensemble avg in by minimizing \( \Psi(m) \), i.e., \( F(m) = \min_{m} \Psi(m) \)

\( H \) entropy: diff config of \( N_+ \) up and \( N_- \) down spins

\[
S = \frac{(N_+ + N_-)!}{N_+! \times N_-!} = \frac{N!}{\left(\frac{N}{2} \right) \left(1+m\right)! \times \left(\frac{N}{2} \right) \left(1-m\right)!}
\]

\[
\frac{S}{k_B} = \log S = N \log N - N - \frac{N}{2} \left(1+m\right) \log \left[ \frac{N}{2} \left(1+m\right) \right]
+ \frac{N}{2} \left(1-m\right) \log \left[ \frac{N}{2} \left(1-m\right) \right] + \frac{N}{2} \left(1-m\right)
\]

\[
= N \log N - N \log \left(\frac{N}{2}\right)
- \frac{N}{2} \left[ \left(1+m\right) \log \left(1+m\right) + \left(1-m\right) \log \left(1-m\right) \right]
\]

\[
\frac{S}{Nk_B} = \log 2 - \frac{1+m}{2} \log \left(1+m\right) - \frac{1-m}{2} \log \left(1-m\right)
\]

mixing entropy; exact up to Stirling approx
E (m) = \langle \hat{H} \rangle_m = \frac{\sum \varepsilon \psi e^{-\beta J m} \delta \left(m - \frac{\sum \psi \delta_j}{N} \right)}{\sum \varepsilon \psi e^{-\beta J m} \delta \left(m - \frac{\sum \psi \delta_j}{N} \right)}

B-W approx.: \quad E (m) = -J \sum \delta_{ij} m^2 = -\frac{\beta}{2} N \text{Tr} m^2

\Rightarrow \frac{\psi (m)}{N} = \frac{E (m) - TS (m)}{N}

= -\frac{8 \beta J}{2} m^2 - \log \left[ \log 2 - \frac{1 + m}{2} \log \left(1 + m\right) \frac{1 - \log \left(1 - m\right)}{2} \right]

Expect small m at vicinity of critical point

\beta \frac{\psi (m)}{N} \ll 1 \quad - \frac{8 \beta J}{2} m^2 + \frac{1 + m}{2} \left( m - \frac{1}{2} m^2 + \frac{1}{2} m^3 + \ldots \right)

\quad + 1 \frac{1 - m}{2} \left( -m - \frac{1}{2} m^2 - \frac{1}{2} m^3 + \ldots \right)

= -\frac{8 \beta J}{2} m^2 + \frac{1}{2} m^2 + \frac{1}{12} m^4 + \ldots

\text{with} \quad 8 \beta J = \delta = 1 - t \quad \text{where} \quad t = \frac{T - Tc}{Tc}

we have \quad \beta \frac{\psi (m)}{N} = \frac{t}{2} m^2 + \frac{1}{12} m^4

\quad t > 0 \quad (T > Tc)

\quad t < 0 \quad (T < Tc)

\Rightarrow B-W theory brings out explicitly the competing effects of energy and entropy which underlies phase transition.
Free energy minimization:
\[
\frac{\partial U}{\partial m}\bigg|_{m=\bar{m}} = 0 \implies \bar{m} + \frac{1}{3} \bar{m}^3 = 0
\]
Same as Weiss MFT!

Include magnetic field: \( \beta H \rightarrow \beta H - \beta \frac{m}{h} \)

\[
E(m) = -\frac{J}{2} \bar{m}^2 - N \bar{m}
\]

\[
\psi(m) = \frac{1}{2} \bar{m}^2 - \bar{m} + \frac{1}{12} \bar{m}^4 + h.o.t.(m)
\]

Note: Correspondence between B-W & Weiss MFT is exact

\[
\frac{2}{\bar{m}} \psi\bigg|_{\bar{m}} = 0 = -\bar{h} - \frac{1}{2} \bar{m}
\]

\[
+ \frac{1}{2} \frac{2}{\bar{m}} \left[ \left( \frac{1}{1-m} \right) \log \left( \frac{1}{1-m} \right) + \left( 1-m \right) \log (1-m) \right] \bigg|_{\bar{m}}
\]

\[
= -\bar{h} - \frac{1}{2} \bar{m} + \frac{1}{2} \log \left( \frac{1}{1-m} \right) + \tanh^{-1}(m)
\]

\[
\therefore \bar{m} = \tanh \left( \frac{1}{2} \bar{m} + \bar{h} \right)
\]