SHARP LOWER BOUNDS FOR MOMENTS OF QUADRATIC DIRICHLET \(L\)-FUNCTIONS

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Abstract. We establish sharp lower bounds for the \(k\)-th moment in the range \(0 \leq k \leq 1\) of the family of quadratic Dirichlet \(L\)-functions at the central point.

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1. Introduction

As moments of families of \(L\)-functions at the central point can be applied to address important issues such as non-vanishing results concerning these central values, they have been studied intensively in the literature. For the family of quadratic Dirichlet \(L\)-functions, asymptotic expressions are available only for the first four moments presently. These results are obtained by M. Jutila \[8\] for the first two moments, by K. Soundararajan \[16\] for the third moment and Q. Shen \[14\] for the fourth moment. See also \[4, 15, 16, 18–20\] for various improvements on the error terms. Besides the above, conjectured asymptotic formulas for various families of \(L\)-functions are made in the work of J. P. Keating and N. C. Snaith \[9\], building on the relation with random matrix theory. More precise formulas including lower order terms are further conjectured by J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith in \[2\].

Much progress has been made towards establishing bounds for moments of \(L\)-functions of the order of magnitude in agreement with the above conjectures. There are now several general approaches that allow one to achieve this. For upper bounds, one can apply a method due to K. Soundararajan in \[17\] together with its sharpened version by A. J. Harper \[5\] or a principle built by M. Radziwiłł and K. Soundararajan in \[11\]. For lower bounds, one can make use of a simple and powerful method developed by Z. Rudnick and K. Soundararajan in \[12, 13\], or a principle enunciated by W. Heap and K. Soundararajan in \[6\] which can be regarded as dual to the corresponding one of Radziwiłł and Soundararajan concerning upper bounds.

We now return to the case of quadratic Dirichlet \(L\)-functions. More specifically, we consider the family \(\{L(s, \chi_{8d})\}\) with \(\chi_{8d} = (8d)\) being the Kronecker symbol such that \(d\) is odd and square-free. For this family, it is conjectured by J. C. Andrade and J. P. Keating \[1\] that for all positive real \(k\),

\[
\sum^*_{0 < d < X, (d, 2) = 1} |L(\frac{1}{2}, \chi_{8d})|^k \sim C_k X (\log X)^{\frac{k(k+1)}{2}},
\]

where \(\sum^*\) denotes the sum over square-free integers and \(C_k\) are explicit constants. Owing to the work of \[5, 10, 12, 13, 17\], lower and upper bounds of the conjectured order of magnitude for the above family have been established for all real \(k \geq 1\) unconditionally and for all real \(k \geq 0\) under the generalized Riemann hypothesis, respectively.

In \[3\], the author applied the above mentioned upper bounds principle of Radziwiłł and Soundararajan to obtain sharp upper bounds for the \(k\)-th moment of the family \(\{L(\frac{1}{2}, \chi_{8d})\}\) unconditionally for \(0 \leq k \leq 2\). In this paper, based on the lower bounds principle of Heap and Soundararajan in \[6\] together with an idea in \[3\], we further explore the commensurate lower bounds for the same family. As we have indicated above, sharp bounds for the \(k\)-th moment are already known for any real \(k \geq 1\). Thus, we focus on the case \(0 \leq k \leq 1\) here, even though our approach in this paper can be extended to treat the case for \(k \geq 1\) as well. Our result is as follows.

Theorem 1.1. For any real number \(k\) such that \(0 \leq k \leq 1\), we have

\[
\sum^*_{0 < d < X, (d, 2) = 1} |L(\frac{1}{2}, \chi_{8d})|^k \gg_k X (\log X)^{\frac{k(k+1)}{2}}.
\]

2. Plan of Proof

We may assume that \(X\) is a large number and we denote \(\Phi\) for a smooth, non-negative function compactly supported on \([1/8, 7/8]\) such that \(\Phi(x) \leq 1\) for all \(x\) and \(\Phi(x) = 1\) for \(x \in [1/4, 3/4]\). For convenience, we replace \(k\) by \(2k\) and
assume that \( 0 < k < 1/2 \) in the rest of the paper. Upon dividing \( 0 < d < X \) into dyadic blocks, we see that in order to prove Theorem \([11]\) it suffices to show that
\[
\sum_{0<d<X \atop (d,2)=1} |L(\frac{1}{2}, \chi_{8d})|^{2k} \Phi\left( \frac{d}{X} \right) \gg_k X (\log X)^{2k/(2k+1)}.
\]
To achieve this, we recall the lower bounds principle of Heap and Soundararajan given in \([6]\) for our setting. This builds on the observation that a result of B. Hough \([7]\) on the distribution of \( \log |L(\frac{1}{2}, \chi_{8d})| \) shows that the behaviour on average of \( |L(\frac{1}{2}, \chi_{8d})| \) can be modeled by the quantity \( \log d \)^{1/2} \( \exp(\mathcal{P}(d)) \), where
\[
\mathcal{P}(d) = \sum_{p \leq z} \frac{1}{\sqrt{p}} \chi_{8d}(p),
\]
with \( z = X^{1/(\log \log X)^2} \). Now, the presence of \( \Phi \) restricts the size of \( \log d \) to be about \( \log X \), so we may ignore it to see that the expression
\[
\sum_{0<d<X \atop (d,2)=1} |L(\frac{1}{2}, \chi_{8d})|^{k_1} \exp(k_2 \mathcal{P}(d))
\]
should provide enough information towards our understanding of \( |L(\frac{1}{2}, \chi_{8d})|^{2k} \) on average as long as \( k_1 + k_2 = 2k \). As expanding \( \exp(k_2 \mathcal{P}(d)) \) into a Dirichlet polynomial would result in an expression too long to control, the next idea is to approximate \( \exp(k_2 \mathcal{P}(d)) \) by a suitable Taylor expansion (upon realizing that \( \mathcal{P}(d) \) is often small in size). Further noticing that larger values of primes should contribute significantly less to the value of \( \mathcal{P}(d) \) so that instead of approximating \( \exp(k_2 \mathcal{P}(d)) \) by a single Taylor expansion, it is desirable to split \( \mathcal{P}(d) \) into sums over primes of different ranges and approximate each corresponding exponential by tailoring a suitable Taylor polynomial.

For this purpose, we define for any non-negative integer \( \ell \) and any real number \( x \),
\[
E_\ell(x) = \sum_{j=0}^{\ell} \frac{x^j}{j!}.
\]
We also let \( N, M \) be two large natural numbers (depending on \( \ell \) only) and denote \( \{\ell_j\}_{1 \leq j \leq R} \) for a sequence of even natural numbers such that \( \ell_1 = 2 \lfloor N \log \log X \rfloor \) and \( \ell_{j+1} = 2 \lfloor N \log \ell_j \rfloor \) for \( j \geq 1 \), where \( R \) is defined to the largest natural number satisfying \( \ell_R > 10^M \). We may assume that \( M \) is so chosen so that we have \( \ell_j > \ell_{j+1}^2 \) for all \( 1 \leq j \leq R-1 \) and this further implies that we have
\[
R \ll \log \log \ell_1, \quad \sum_{j=1}^{R} \frac{1}{\ell_j} \leq \frac{2}{\ell_R}.
\]
We denote \( P_1 \) for the set of odd primes not exceeding \( X^{1/\ell_1^2} \) and \( P_j \) for the set of primes lying in the interval \( (X^{1/\ell_{j-1}^2}, X^{1/\ell_j^2}] \) for \( 2 \leq j \leq R \).

We set
\[
\mathcal{P}_j(d) = \sum_{p \in P_j} \frac{1}{\sqrt{p}} \chi_{8d}(p),
\]
and for any real number \( \alpha \),
\[
\mathcal{N}_j(d, \alpha) = E_{\ell_j}(\alpha \mathcal{P}_j(d)), \quad \mathcal{N}(d, \alpha) = \prod_{j=1}^{R} \mathcal{N}_j(d, \alpha).
\]
Here we notice that it follows from \([11]\), Lemma 1] that the quantities defined in \((2.4)\) are all positive. Instead of examining \((2.2)\), we may now look at
\[
\sum_{0<d<X \atop (d,2)=1} |L(\frac{1}{2}, \chi_{8d})|^{k_1} \prod_{2 \leq j \leq J} \mathcal{N}(d, k_j),
\]
for real numbers \( k_j \) satisfying \( \sum_{1 \leq j \leq J} k_j = 2k \). It remains to apply Hölder’s inequality to bound the expression in \((2.5)\) from above by averages of various quantities that are of the same form as the one appearing in \((2.5)\), one of which being \( |L(\frac{1}{2}, \chi_{8d})|^{2k} \). Thus, if one is able to bound those other quantities on average from above and the expression in \((2.5)\) from below, then one should be able to obtain a lower bound estimation for the \( 2k \)-th moment.
More concretely, we may adapt the choices for $k_j$ in (2.3) as those used in [3] to see that, via Hölder’s inequality,
\[
\sum_{(d, 2) = 1}^{s} |L(\frac{1}{2}, \chi_{8d})| N(d, k)N(d, k - 1) \Phi\left(\frac{d}{X}\right)
\leq \left( \sum_{(d, 2) = 1}^{s} |L(\frac{1}{2}, \chi_{8d})|^{2k} \Phi\left(\frac{d}{X}\right) \right)^{1/2} \left( \sum_{(d, 2) = 1}^{s} |L(\frac{1}{2}, \chi_{8d})| N(d, k - 1)|^{2} \Phi\left(\frac{d}{X}\right) \right)^{(1-k)/2} \left( \sum_{(d, 2) = 1}^{s} N(d, k)^{2/k} N(d, k - 1)^{2} \Phi\left(\frac{d}{X}\right) \right)^{k/2}.
\]

Although the above may allow one to obtain lower bounds estimation for the $2k$-th moment, it nevertheless requires estimations on averages of $|L(\frac{1}{2}, \chi_{8d})| N(d, k)N(d, k - 1)$ and $|L(\frac{1}{2}, \chi_{8d})| N(d, k - 1)^{2}$, which we find a little inconvenient. Thus, we incorporate an idea given in [3] by observing that it follows from [3] Lemma 4.1 that we have
\[
N(d, \alpha)N(d, -\alpha) \geq 1.
\]

We deduce from this that for any real $c$ such that $0 < c < 1$,
\[
\sum_{(d, 2) = 1}^{s} L(\frac{1}{2}, \chi_{8d}) N(d, 2k - 1) \Phi\left(\frac{d}{X}\right) \leq \sum_{(d, 2) = 1}^{s} |L(\frac{1}{2}, \chi_{8d})| N(d, 2k - 1) \Phi\left(\frac{d}{X}\right)
\leq \sum_{(d, 2) = 1}^{s} |L(\frac{1}{2}, \chi_{8d})| \cdot |L(\frac{1}{2}, \chi_{8d})|^{1-c}N(2k - 2, d)(1-\frac{1}{2}) \cdot N(d, 2k - 1)N(d, 2 - 2k)(1-\frac{1}{2}) \Phi\left(\frac{d}{X}\right).
\]

Applying Hölder’s inequality with exponents being $2k/c, 2/(1-c), ((1 + c)/2 - c/(2k))^{-1}$ to the last sum above, we deduce that
\[
\sum_{(d, 2) = 1}^{s} L(\frac{1}{2}, \chi_{8d}) N(d, 2k - 1) \Phi\left(\frac{d}{X}\right)
\leq \left( \sum_{(d, 2) = 1}^{s} |L(\frac{1}{2}, \chi_{8d})|^{2k} \Phi\left(\frac{d}{X}\right) \right)^{c/(2k)} \left( \sum_{(d, 2) = 1}^{s} |L(\frac{1}{2}, \chi_{8d})|^{2} \Phi\left(\frac{d}{X}\right) \right)^{(1-c)/2}
\times \left( \sum_{(d, 2) = 1}^{s} N(d, 2k - 1)^{(1+c)/2 - c/(2k)} \cdot N(d, 2 - 2k)^{(1-c)/2 - c/(2k)} \right)^{-1} \Phi\left(\frac{d}{X}\right)^{(1+c)/2 - c/(2k)}.
\]

We now set
\[
(1-c)/2 \cdot ((1 + c)/2 - c/(2k))^{-1} = 2
\]
which implies that $c = (\frac{3}{2} - 3)^{-1}$ and that
\[
((1 + c)/2 - c/(2k))^{-1} = \frac{2(2 - 3k)}{1 - 2k}.
\]

One checks that the above value of $c$ does satisfy that $0 < c < 1$ when $0 < k < 1/2$. We then deduce from (2.6) that in order to establish a lower bound estimation for the $2k$-th moment, it suffices to establish the following three propositions.

**Proposition 2.1.** With notations as above, we have
\[
\sum_{(d, 2) = 1}^{s} L(\frac{1}{2}, \chi_{8d}) N(d, 2k - 1) \Phi\left(\frac{d}{X}\right) \gg X(\log X)^{(2k)^2 + c}.
\]

**Proposition 2.2.** With notations as above, we have
\[
\sum_{(d, 2) = 1}^{s} N(d, 2k - 1)^{(2k-3k)} N(d, 2 - 2k)^{2} \Phi\left(\frac{d}{X}\right) \ll X(\log X)^{(2k)^2}.
\]

**Proposition 2.3.** With notations as above, we have
\[
\sum_{(d, 2) = 1}^{s} L(\frac{1}{2}, \chi_{8d})^{2} N(d, 2k - 2) \Phi\left(\frac{d}{X}\right) \ll X(\log X)^{(2k)^2 - c}.
\]

In fact, combining (2.6) with the above three propositions, we see that
\[
X(\log X)^{(2k)^2 + c} \ll \left( \sum_{(d, 2) = 1}^{s} |L(\frac{1}{2}, \chi_{8d})|^{2k} \Phi\left(\frac{d}{X}\right) \right)^{c/(2k)} \left( X(\log X)^{(2k)^2} \right)^{(1-c)/2} \left( X(\log X)^{(2k)^2} \right)^{(1+c)/2 - c/(2k)}.
\]

The desired lower bound given in (2.1) follows immediately from this.
Notice that Proposition 2.3 can be established similar to [14 Proposition 3.1] so that it remains to prove Propositions 2.1 and 2.2 in the rest of the paper.

3. Preliminaries

We include in this section a few lemmas that are needed in our proofs. In what follows, we denote the letter $p$ for a prime number and the symbol $\square$ for a perfect square. We define $\delta_n=\square$ to be 1 when $n=\square$ and 0 otherwise. For the function $\Phi$ described in Section 2, we associate its Mellin transform $\hat{\Phi}(s)$ for any complex number $s$ by

$$
\hat{\Phi}(s) = \int_0^\infty \Phi(x)x^{-s}dx.
$$

We first recall the following two lemmas that are given in [14] as Lemma 2.2 and 2.3 there, respectively.

**Lemma 3.1.** Let $x \geq 2$. We have, for some constant $b$,

$$
\sum_{p \leq x} \frac{1}{p} = \log \log x + b + O\left(\frac{1}{\log x}\right).
$$

Also, for any integer $j \geq 1$, we have

$$
\sum_{p \leq x} \frac{(\log p)^j}{p} = \frac{1}{j} (\log x)^j + O((\log x)^{j-1}).
$$

**Lemma 3.2.** For large $X$ and any odd positive integer $n$, we have

$$
\sum_{(d, 2)=1}^{\ast} \chi_{sd}(n) \Phi\left(\frac{d}{X}\right) = \delta_n=\square \hat{\Phi}(1) \frac{2X}{3\zeta(2)} \prod_{p|n} \left(\frac{p}{p+1}\right) + O(X^{\frac{\epsilon}{2}+\epsilon\sqrt{n}}).
$$

Next, similar to [3], Lemma 2.4, by setting $Y = X^{1/4}$, $M = 1$ in [16] Proposition 1.1.1.2, we have the following asymptotic formula concerning the twisted first moment of quadratic Dirichlet $L$-functions.

**Lemma 3.3.** With notations as above and writing any odd $l$ as $l = l_1l_2$ with $l_1$ square-free, we have for any $\epsilon > 0$,

$$
\sum_{(d, 2)=1}^{\ast} L\left(\frac{l}{X}, \chi_{sd}\right) \chi_{sd}(l) \Phi\left(\frac{d}{X}\right) = C\hat{\Phi}(1) \frac{\sqrt{X}}{l_1g(l)} X\left(\log\frac{\sqrt{X}}{l_1} + C_2 + \sum_{p|l} \frac{C_2(p)}{p} \log p\right) + O\left(X^{\frac{1}{4}+\epsilon}\right),
$$

where $C = \frac{1}{2} \prod_{p \geq 3} \left(1 - \frac{1}{p(p+1)}\right)$ and $g(l) = \prod_{p|l} \left(\frac{p+1}{p}\right) \left(1 - \frac{1}{p(p+1)}\right)$. Also, $C_2$ is a constant depending only on $\Phi$ and $C_2(p) \ll 1$ for all $p$.

Lastly, we present a result which is analogue to [3] Lemma 1 and is needed in the proof of Proposition 2.2.

**Lemma 3.4.** For $1 \leq j \leq R$, we have

$$
N_j(d, 2k-1) \ll N_j(d, 2k)^2 \leq N_j(d, 2k)^2 \left(1 + O\left(e^{-\ell_j}\right)\right) + Q_j(d),
$$

where the implied constants are absolute, and

$$
Q_j(d) = \left(\frac{24P_j(d)}{\ell_j}\right)^{2r_k\ell_j},
$$

with $r_k = 1 + (2 - 3k)/(1 - 2k)$.

**Proof.** We note first that when $|z| \leq aK/20$ for any constant $0 < a < 2$, we have

$$
\left|\sum_{r=0}^{K} \frac{z^r}{r!} - e^z\right| \leq \frac{|az|K}{K!} \leq \left(\frac{ae}{20}\right)^K.
$$

We deduce this that when $|P_j(d)| \leq \ell_j/20$,

$$
N_j(d, 2k-1) = \exp\left((2k-1)P_j(d)\right) \left(1 + O\left(\exp((2k-1)|P_j(d)|)\left(\frac{1-2k}{20}\right)\right)\right) = \exp\left((2k-1)P_j(d)\right) \left(1 + O\left((1-2k)e^{-\ell_j}\right)\right),
$$

Similarly, we have

$$
N_j(d, 2k-2) = \exp\left((2k-2)P_j(d)\right) \left(1 + O\left(e^{-\ell_j}\right)\right).
$$
The above estimations then allow us to see that when $|P_j(d)| \leq \ell_j/20$,

$$N_j(d, 2k - 1) = \frac{2^d + 1}{2^{d+1}} N_j(d, 2 - 2k)^2 = \exp(2kP_j(d)) \left( 1 + O(\ell_j) \right) = N_j(d, 2k) \left( 1 + O(\ell_j) \right).$$

Next, notice that when $|P_j(d)| \geq \ell_j/20$, we have

$$|N_j(d, 2 - 2k)| \leq \sum_{r=0}^{\ell_j} \frac{|2P_j(d)|^r}{r!} \leq |P_j(d)|^{\ell_j} \sum_{r=0}^{\ell_j} \frac{20}{r!} \ell_j^{-r} 2^r \leq \left( \frac{24|P_j(d)|}{\ell_j} \right)^{\ell_j}.$$

The assertion of the lemma now follows from (4.2) together with the observation that the same bound above holds for $|N_j(d, 2k - 1)|$ as well.

4. Proof of Proposition 2.1

Denote $\Omega(n)$ for the number of distinct prime powers dividing $n$ and $w(n)$ for the multiplicative function such that $w(p^a) = a!$ for prime powers $p^a$. Let $b_j(n)$, $1 \leq j \leq R$ be functions such that $b_j(n) = 1$ when $n$ is composed of at most $\ell_j$ primes, all from the interval $P_j$. Otherwise, we define $b_j(n) = 0$. We use these notations to see that

$$N_j(d, 2k - 1) = \sum_{n_j} \frac{1}{\sqrt{n_j}} \frac{(2k - 1)^{\Omega(n)}_j}{w(n_j)} b_j(n_j) \chi_{sd}(n_j), \quad 1 \leq j \leq R.$$

Note that each $N_j(d, 2k - 1)$ is a short Dirichlet polynomial since $b_j(n_j) = 0$ unless $n_j \leq (X^{1/\ell_j})^{\ell_j} = X^{1/\ell_j}$. It follows from this that $N(d, 2k - 1)$ is also a short Dirichlet polynomial of length at most $X^{1/\ell_1 + \cdots + 1/\ell_R} < X^{2/\log X}$ by (2.3).

We use (4.1) to expand the term $N(d, 2k - 1)$ and apply Lemma 3.2 to evaluate it. As $N(d, 2k - 1)$ is a short Dirichlet polynomial, we may ignore the error term in Lemma 3.2 to consider only the main term contribution. Upon writing $n_j = (n_j)_1 (n_j)_2$ with $(n_j)_1$ being square-free, we see that

$$\sum_{(d,2)=1} L \left( \frac{1}{2}, \chi_{sd} \right) N(d, 2k - 1) \Phi \left( \frac{d}{X} \right) \gg X \sum_{n_1, \ldots, n_R} \left( \prod_{j=1}^R \frac{1}{\sqrt{n_j}} \frac{(2k - 1)^{\Omega(n)}_j}{w(n_j)} b_j(n_j) \frac{1}{g(n_j)} \right) \left( \log \frac{\sqrt{X}}{(n_1)_1 \cdots (n_R)_1} \right) + C_2 + \sum_{p|n_1 \cdots n_R} \frac{C_2(p)}{p} \log p.$$

We may further ignore the contribution from the terms involving $C_2 + \sum_{p|n_1 \cdots n_R} \frac{C_2(p)}{p} \log p$ as one may show using our arguments in the paper that the contribution is $\ll X \log X \frac{(\log X)^2}{2^{\ell+1}}$. Thus we deduce that

$$\sum_{(d,2)=1} L \left( \frac{1}{2}, \chi_{sd} \right) N(d, 2k - 1) \Phi \left( \frac{d}{X} \right) \gg X \sum_{n_1, \ldots, n_R} \left( \prod_{j=1}^R \frac{1}{\sqrt{n_j}} \frac{(2k - 1)^{\Omega(n)}_j}{w(n_j)} b_j(n_j) \frac{1}{g(n_j)} \right) \left( \log \frac{\sqrt{X}}{(n_1)_1 \cdots (n_R)_1} \right) = S_1 - S_2,$$

where

$$S_1 = \frac{1}{2} X \log X \sum_{n_1, \ldots, n_R} \left( \prod_{j=1}^R \frac{1}{\sqrt{n_j}} \frac{(2k - 1)^{\Omega(n)}_j}{w(n_j)} b_j(n_j) \frac{1}{g(n_j)} \right),$$

$$S_2 = X \sum_{n_1, \ldots, n_R} \left( \prod_{j=1}^R \frac{1}{\sqrt{n_j}} \frac{(2k - 1)^{\Omega(n)}_j}{w(n_j)} b_j(n_j) \frac{1}{g(n_j)} \right) \log \left( \prod_{i=1}^R (n_i)_1 \right).$$

It remains to bound $S_1$ from below and $S_2$ from above. We bound $S_1$ first by restating it as

$$S_1 = \frac{1}{2} X \log X \prod_{j=1}^R \sum_{n_j} \left( \frac{1}{\sqrt{n_j}} \frac{(2k - 1)^{\Omega(n)}_j}{w(n_j)} b_j(n_j) \frac{1}{g(n_j)} \right).$$

We consider the sum above over $n_j$ for a fixed $1 \leq j \leq R$. Note that the factor $b_j(n_j)$ restricts $n_j$ to have all prime factors in $P_j$ such that $\Omega(n_j) \leq \ell_j$. If we remove the restriction on $\Omega(n_j)$, then the sum becomes

$$\prod_{p \in P_j} \left( \sum_{i=0}^\infty \frac{1}{p^i} \frac{(2k - 1)^{2i}}{(2i)!} g(p^{2i}) \right) + \sum_{i=0}^\infty \frac{1}{p^{2i+1}} \frac{(2k - 1)^{2i+1}}{(2i+1)!} \frac{1}{g(p^{2i+1})} = \prod_{p \in P_j} \left( 1 - \frac{(2k - 1)^2}{2} \frac{1}{p} \right)^{-1} D(p),$$

where $D(p)$ is a function depending on the properties of $P_j$.
where
\[ D(p) = \left(1 - \left(\frac{(2k-1)^2}{2} + 2k-1\right) \frac{1}{p}\right) \left(1 + \frac{(2k-1)^2}{2p} + 2k - 1\right) \frac{1}{pg(p)} + \frac{(2k-1)^3}{6p^2 g(p)} \].

Note that for \( i \geq 2 \), we have
\[ \frac{1}{p^i} \left(\frac{(2k-1)^2}{2} + 2k-1\right) + \frac{1}{p^{i+1}} \left(\frac{(2k-1)^2}{2} + 2k - 1\right) \frac{1}{g(p^{2i+1})} > 0. \]

Observe further the estimation that
(4.3)
\[ 1 - \frac{1}{p} \leq \frac{1}{g(p)} \leq 1. \]

We then deduce that
\[ D(p) \geq \left(1 - \left(\frac{(2k-1)^2}{2} + 2k-1\right) \frac{1}{p}\right) \left(1 + \frac{(2k-1)^2}{2p} + 2k - 1\right) \frac{1}{pg(p)} + \frac{(2k-1)^3}{6p^2 g(p)} \]
\[ \geq \left(1 - \left(\frac{(2k-1)^2}{2} + 2k-1\right) \frac{1}{p}\right) \left(1 + \frac{(2k-1)^2}{2p} + 2k - 1\right) \frac{1}{p} + \frac{(2k-1)^3}{6p^2} \]
\[ \geq 1 - \frac{1}{4p^2} - \frac{5}{24p^2}, \]

where the last estimation above follows by noting that
\[ 1 - \left(\frac{(2k-1)^2}{2} + 2k-1\right) \frac{1}{p} = 1 + \frac{1 - 4k^2}{2p} \leq 1 + \frac{1}{4}. \]

We further note that we have \((2k-1)^2/2 + 2k - 1 < 0\) when \(0 < k < 1/2\) and that \(-\log(1 + x) > -x\) for all \(x > 0\). This implies that
\[ \prod_{p \in P_j} \left(1 - \left(\frac{(2k-1)^2}{2} + 2k-1\right) \frac{1}{p}\right)^{-1} \geq \exp\left(\left(\frac{(2k-1)^2}{2} + 2k - 1\right) \sum_{p \in P_j} \frac{1}{p}\right). \]

We then deduce from the above estimations that the left side expression in (4.2) is
(4.4)
\[ \geq \exp\left(\left(\frac{(2k-1)^2}{2} + 2k - 1\right) \sum_{p \in P_j} \frac{1}{p}\right) \prod_{p \in P_j} (1 - \frac{1}{p^2}). \]

On the other hand, using Rankin’s trick by noticing that \(2^{\Omega(n_1) - \ell_1} \geq 1\) if \(\Omega(n_1) > \ell_1\), we see that the error introduced this way does not exceed
\[ \sum_{n_j} \frac{1}{\sqrt{n_j(w(n_j))}} \frac{|2k-1|^{\Omega(n_j)}}{g(n_j)} \frac{1}{g(n_j)} \]
\[ \leq 2^{-\ell_j} \prod_{p \in P_j} \left(1 + \sum_{i=1}^{\infty} \frac{1}{p^i} \frac{(2k-1)^2}{2i} \frac{1}{g(p^{2i})} + \sum_{i=0}^{\infty} \frac{1}{p^{i+1}} \frac{|2k-1|^{2i+1} 2^{2i+1}}{(2i+1)!} \frac{1}{g(p^{2i+1})}\right) \]
\[ \leq 2^{-\ell_j} \prod_{p \in P_j} \left(1 + \sum_{i=1}^{\infty} \frac{1}{p^i} \frac{(2k-1)^2}{2i} \frac{1}{g(p^{2i})} + \sum_{i=0}^{\infty} \frac{1}{p^{i+1}} \frac{|2k-1|^{2i+1} 2^{2i+1}}{(2i+1)!} \frac{1}{g(p^{2i+1})}\right) \]
\[ \leq 2^{-\ell_j} \exp\left(\left(2(2k-1)^2 + 2(1 - 2k)\right) \sum_{p \in P_j} \frac{1}{p} + \sum_{p \in P_j} R(p)\right), \]
where

\[ R(p) = \sum_{i=2}^{\infty} \frac{1}{p^i} \frac{(2k-1)^{2i+2}}{(2i)!} + \sum_{i=1}^{\infty} \frac{1}{p^{i+1}} \frac{|2k-1|^{2i+1}2^{2i+1}}{(2i+1)!} \]

\[ \leq \frac{1}{p^2} \sum_{i=2}^{\infty} \frac{2^{2i}}{(2i)!} + \frac{2}{p^2} \sum_{i=1}^{\infty} \frac{2^{2i}}{(2i+1)!} \]

\[ \leq 3 \frac{2}{p^2} \sum_{i=1}^{\infty} \frac{2^{i}}{i!} \leq 3e^2 \frac{2}{p^2}. \]

It follows that the error is hence

\[ \leq 2^{-\ell_j} \exp \left( (2(2k-1)^2 + 2(1-2k)) \sum_{p \in P_j} \frac{1}{p} + \sum_{n \geq 1} \frac{3e^2}{n^2} \right) \]

\[ \leq 2^{-\ell_j} \exp \left( \frac{(\pi)^2}{2} \right) \exp \left( \frac{3}{2} (2k-1)^2 + 3(1-2k) \right) \sum_{p \in P_j} \frac{1}{p} \times \exp \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_j} \frac{1}{p} \right). \]

We notice that the expression given in (4.4) satisfies

\[ \geq \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_j} \frac{1}{p} \right) \prod_{p \in P_j} (1 - \frac{1}{p}) = \frac{6}{\pi^2} \exp \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_j} \frac{1}{p}. \]

We may take \( N \) large enough so that by Lemma 3.1 we have that for all \( 1 \leq j \leq R \),

\[ \frac{1}{2N} \ell_j \leq \sum_{p \in P_j} \frac{1}{p} \leq \frac{2}{N} \ell_j. \]

We may also take \( M \) large enough to ensure that every \( \ell_j, 1 \leq j \leq R \) is large. We then deduce from (4.9) and (4.10) that the error introduced above is

\[ \leq 2^{-\ell_j/2} \exp \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_j} \frac{1}{p} \prod_{p \in P_j} (1 - \frac{1}{p^2}). \]

Combining (4.4) and (4.8), we see that the sum over \( n_j \) for each \( j, 1 \leq j \leq R \) in the expression of \( S_1 \) is

\[ \geq (1 - 2^{-\ell_j/2}) \exp \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_j} \frac{1}{p} \prod_{p \in P_j} (1 - \frac{1}{p^2}). \]

It follows from this that we have

\[ S_1 \geq \frac{1}{2} X \log X \prod_{j=1}^{R} \left( 1 - 2^{-\ell_j/2} \right) \exp \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_j} \frac{1}{p} \prod_{p \in P_j} (1 - \frac{1}{p^2}) \]

\[ \geq \frac{1}{2} X \log X \prod_{p} \left( 1 - \frac{1}{p^2} \right) \prod_{j=1}^{R} \exp \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \frac{1}{p} \prod_{p \in P_j} \frac{1}{p} \]

\[ \geq \frac{1}{2 \zeta(2)} X \log X \left( 1 - \sum_{j=1}^{R} 2^{-\ell_j/2} \right) \prod_{j=1}^{R} \exp \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \frac{1}{p} \prod_{p \in P_j} \frac{1}{p}, \]

where the last estimation above follows by noting that \( \prod_{i=1}^{R} (1 - x_i) \geq 1 - \sum_{i=1}^{R} x_i \) for positive real numbers \( x_i \) satisfying \( \sum_{i=1}^{R} x_i \leq 1 \).

Next, we estimate \( S_2 \) by writing \( \log \left( \prod_{i=1}^{R} (n_i)_{1} \right) \) as a sum of logarithms of primes dividing \( \prod_{i=1}^{R} (n_i)_{1} \) to see that

\[ S_2 \leq X \sum_{q \in \bigcup P_j} \sum_{l \geq 0} \log q \left( \frac{1}{2l+1} - \frac{1}{g(q^{2l+1})} \right) \prod_{i=1}^{R} \left( \sum_{(n_i,q)=1} \frac{1}{\sqrt{n_i(n_i)_1}} \frac{\vartheta_{i}(n_i)}{g(n_i)} \right) \]

where we define \( \tilde{b}_{i,j}(n_i) = b_i(n_i q^j) \) for the unique index \( i \) \((1 \leq i \leq R)\) such that \( b_i(q) \neq 0 \) and \( \tilde{b}_{i,j}(n_i) = b_i(n_i) \) otherwise.
As above, if we remove the restriction of \( \tilde{b}_{i,l} \) on \( \Omega(n_i) \), then the sum over \( n_i \) becomes

\[
\prod_{p \in P_\gamma, (p,q)=1} \left( \sum_{m=0}^{\infty} \frac{1}{p^m} \frac{(2k-1)^{2m}}{(2m)! g(p^{2m})} + \sum_{m=0}^{\infty} \frac{1}{p^{m+1}} \frac{(2k-1)^{2m+1}}{(2m+1)! g(p^{2m+1})} \right).
\]

Note that for \( m \geq 1 \), we have

\[
\frac{1}{p^m} \frac{(2k-1)^{2m}}{(2m)! g(p^{2m})} + \frac{1}{p^{m+1}} \frac{(2k-1)^{2m+1}}{(2m+1)! g(p^{2m+1})} \leq 0.
\]

It follows from this that the expression in (4.11) is

\[
\prod_{p \in P_\gamma, (p,q)=1} \left( 1 + \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \frac{1}{pg(p)} \right) \leq \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_\gamma} \frac{1}{p} \right),
\]

where the last estimation above follows from the observation that \( 1 + x \leq e^x \) for any real \( x \).

Using further the estimation given in (4.13), we deduce from (4.12) that the expression in (4.11) is

\[
\leq \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_\gamma} \frac{1}{p} + \frac{1}{2} \sum_{p \in P_\gamma} \frac{1}{p^2} \right).
\]

Similar to our discussions above, we see that the error introduced in this process is

\[
\leq 2^{-\ell_j/2} \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_\gamma} \frac{1}{p} \right).
\]

We deduce from this that

\[
\prod_{i=1}^{R} \left( \sum_{(n_i,q)=1}^{\infty} \frac{1}{\sqrt{n_i} \Omega(n_i) w(n_i)} \tilde{b}_{i,l}(n_i) \frac{1}{g(n_i)} \right) \leq A \times \prod_{i=1}^{R} \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_\gamma} \frac{1}{p} \right),
\]

where \( A \leq e^{10} \) is a constant.

It follows from this and (4.10) that

\[
S_2 \leq A \prod_{i=1}^{R} \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_\gamma} \frac{1}{p} \right) \times \sum_{q \in \cup P_j}^{\log q} \frac{(1 - 2k)^{2l+1}}{(2l+1)! g(q^{2l+1})} \frac{1}{q^{2l+1}}
\]

\[
(4.13)
\]

\[
\leq A \prod_{i=1}^{R} \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_\gamma} \frac{1}{p} \right) \times \sum_{q \in \cup P_j} \left( \frac{\log q}{q} \right)
\]

\[
\leq A \prod_{i=1}^{R} \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_\gamma} \frac{1}{p} \right) \times \left( \frac{\log X}{102M} + O(1) \right),
\]

where the last estimation above follows from Lemma 3.4.

Combining (4.10) and (4.13), we deduce that, by taking \( M \) large enough,

\[
S_1 - S_2 \gg X \log X \prod_{i=1}^{R} \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_\gamma} \frac{1}{p} \right).
\]

Applying Lemma 3.1 again, we see that the proof of the proposition now follows from above.
It follows from Lemma\[3,2\] that we have

\[
\sum_{(d,2)=1}^{*} \mathcal{N}(d, 2k - 1)^{\frac{2(2k+1)}{4k+1}} \mathcal{N}(d, 2 - 2k)^{2} \Phi(\frac{d}{X}) \leq \sum_{(d,2)=1}^{*} \left( \prod_{j=1}^{R} \left( \mathcal{N}_j(d, 2k) \left( 1 + O(e^{-\ell_j}) \right) + \mathcal{Q}_j(d) \right) \right) \Phi(\frac{d}{X}).
\]

We now use the notations in Section 2 to write for\( 1 \leq j \leq R \),

\[
\mathcal{N}_j(d, 2k) = \sum_{n_j} \frac{1}{\sqrt{n_j}} \frac{(2k)^{\Omega(n_j)}}{w(n_j)} b_j(n_j) \chi_d(n_j),
\]

\[
\mathcal{P}_j(d)^{2r_k \ell_j} = \sum_{\Omega(n_j)=2r_k \ell_j, p|n_j \implies p \in P_j} \frac{1}{\sqrt{n_j}} \frac{(2r_k \ell_j)!}{w(n_j)} \chi_d(n_j),
\]

where\( r_k = 1 + (2 - 3k)/(1 - 2k) \).

As\( (\frac{n}{e})^n \leq n! \leq n(\frac{n}{e})^n \), we deduce that

\[
\left( \frac{24}{\ell_j} \right)^{2r_k \ell_j} (2r_k \ell_j)! \leq 2r_k \ell_j \left( \frac{48r_k}{e} \right)^{2r_k \ell_j}.
\]

It follows from this that we can write\( \mathcal{N}_j(d, 2k) \left( 1 + O(e^{-\ell_j}) \right) + \mathcal{Q}_j(d) \) as a Dirichlet polynomial of the form

\[
D_j(d) = \sum_{n_j \leq X^{2r_k / \ell_j}} a_{n_j} \chi_d(n_j)
\]

where for some constant\( B(k) \) depending on\( k \) only,

\[
|a_{n_j}| \leq B(k)^{\ell_j}.
\]

We then apply Lemma\[3,2\] to evaluate the right side expression in\( 6.1 \) above and deduce from estimations in\( 2.3 \) that for large\( M, N \), the contribution arising from the error term in\( 6.1 \) is

\[
\ll B(k)^{R^2} X^{1/2+\varepsilon} X^{2r_k R^2} \ll B(k)^{R^2} X^{1/2+\varepsilon} X^{2r_k} \ll X^{1-\varepsilon}.
\]

We may thus focus on the main term contributions, which implies that the right side expression of\( 5.1 \) is

\[
\ll X \times \prod_{j=1}^{R} D_j(d) \times \text{a corresponding factor}
\]

\[
= X \times \prod_{j=1}^{R} \sum \left( \text{square term in } D_j(d) \times \text{a corresponding factor} \right)
\]

where by “a square term” in a Dirichlet polynomial\( \sum_n \frac{a_n}{n^s} \) we mean a term corresponding to\( n = \square \) and

\[
a \text{ corresponding factor } = \prod_{p|n} \left( \frac{p}{p+1} \right).
\]

We first note that

\[
\sum \text{square term in } \mathcal{N}_j(d, 2k) \times \text{a corresponding factor} = \sum_{n_j = \square} \frac{1}{\sqrt{n_j}} \frac{(2k)^{\Omega(n_j)}}{w(n_j)} b_j(n_j) \prod_{p|n_j} \left( \frac{p}{p+1} \right)
\]

\[
\leq \prod_{p \in P_j} \left( 1 + \frac{(2k)^2}{2p} \left( \frac{p}{p+1} \right) \right) + \sum_{i \geq 1} \frac{(2k)^{2i}}{2^{2i}i!} \left( \frac{p}{p+1} \right) \leq \prod_{p \in P_j} \left( 1 + \frac{(2k)^2}{2p} \left( \frac{p}{p+1} \right) + \frac{1}{p^2} \right)
\]

\[
\leq \exp \left( \frac{(2k)^2}{2p} \sum_{p \in P_j} \frac{1}{p} + \sum_{p \in P_j} \frac{1}{p^2} \right),
\]

where we apply the relation that\( 1 + x \leq e^x \) for any real\( x \) again to obtain the last estimation above.
Next, we notice that
\[
\sum \text{square term in } Q_j(d) \times \text{a corresponding factor} \quad \ll \left( \frac{24}{\ell_j} \right)^{2r_k \ell_j} \left( \frac{2r_k \ell_j}{(r_k \ell_j)!} \right) \sum_{\substack{n_j \equiv 0 \pmod{p} \\ \Omega(n_j)=2r_k \ell_j}} \frac{1}{w(n_j)} \prod_{p | n_j} \left( \frac{p}{p+1} \right) \ll \left( \frac{24}{\ell_j} \right)^{2r_k \ell_j} \left( \frac{2r_k \ell_j}{(r_k \ell_j)!} \right) \left( \frac{1}{p^r} \right).
\]

We apply (4.7) and (5.2) to estimate the right side expression above to see that for some constant \( B_1(k) \) depending on \( k \) only,
\[
\sum \text{square term in } Q_j(d) \times \text{a corresponding factor} \quad \ll B_1(k)^{\ell_j} e^{-r_k \ell_j \log(r_k \ell_j)} \left( \sum_{p \in P_j} \frac{1}{p} \right)
\]
\[
\ll e^{-\ell_j} \exp \left( \frac{(2k)^2}{2} \sum_{p \in P_j} \frac{1}{p} \right).
\]

Combining the above with (5.3) and (5.4), we see that the right side expression in (5.1) is
\[
\ll X \prod_{j=1}^{R} \left( 1 + e^{-\ell_j} \right) \times \exp \left( \frac{(2k)^2}{2} \sum_{p \in P_j} \frac{1}{p} + \sum_{p \in P_j} \frac{1}{p^2} \right) \ll X (\log X)^{(2k)^2}.
\]

where we deduce the last estimation above using Lemma 5.1 The assertion of the proposition now follows from this.

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