Pointwise multipliers on the weighted BMOA space

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Abstract. Let \( D = \{ z : |z| < 1 \} \) be the unit disk in the complex plane \( \mathbb{C} \), \( \phi : D \to \mathbb{C} \) is an analytic map. We study the multiplication operator \( M_\phi \) on the logarithmic weighted BMOA space \( \text{BMOA}_{\log} = \{ g \in H(D) : \sup_{b \in D} (\log 2^{-1} - |b|^2) \int_D |g'(z)|^2 (1 - |\phi_b(z)|^2) dm(z) < \infty \} \). We obtain that a sufficient condition for the operator \( M_\phi \) to be a bounded operator on \( \text{BMOA}_{\log} \). We also get that another necessary condition for the operator \( M_\phi \) to be bounded on \( \text{BMOA}_{\log} \).

1. Introduction
Suppose that \( D = \{ z : |z| < 1 \} \) be the open unit disk in the complex plane \( \mathbb{C} \), and the circular \( \partial D = \{ z : |z| = 1 \} \) is its boundary. Let \( H(D) \) be the set of all analytic functions on \( D \). Let \( dm(z) \) denote the norm Lebesgue measure on \( D \) ie., \( m(D) = 1 \). The bounded mean oscillation of analytic space, denoted by \( \text{BMOA} \), consists of \( g \in H(D) \) for which

\[
\| g \|_{\text{BMOA}} = \sup_{b \in D} \left\{ \int_D |g'(z)|^2 (1 - |\phi_b(z)|^2) \, dm(z) \right\}^{\frac{1}{2}} < \infty,
\]

where \( \phi_b(z) = (b - z)/(1 - \overline{b}z) \) is a Möbius transformation of \( D \).

Many scholars have extensively studied the functions in BMOA space, such as [1, 2, 3]. For example, Girela in [2] gave a very complete introduction to the theory of BMOA space containing not only the basic theorems but also several interesting topics such as Taylor coefficients, multipliers, inner functions, univalent functions and mean Lipschitz functions. The author in [4] characterized the cyclic vector in BMOA space. Brown and Sadek in [5] characterized the invariant subspaces of BMOA under the shift operator. Shamoyan in [6, 7] studied the coefficient multipliers, convolution and Toeplitz operators in the BMOA type spaces. Siskakis and Zhao in [8] studied a Volterra type operator \( T_g \) on BMOA space, and obtained this operator \( T_g \) is bounded iff \( g \) belongs to a BMOA type space. Now we will study the pointwise multiplier on this BMOA type space, which is called the logarithmic BMOA space \( \text{BMOA}_{\log} \).

The logarithmic BMOA space \( \text{BMOA}_{\log} \) is the set of all \( g \in H(D) \) such that

\[
\| g \|_{\text{LBMOA}} = \sup_{b \in D} \left\{ (\log 2^{-1} - |b|^2)^2 \int_D |g'(z)|^2 (1 - |\phi_b(z)|^2) \, dm(z) \right\}^{\frac{1}{2}} < \infty.
\]
The logarithmic vanishing mean oscillation of analytic space, denoted by $VMOA_{\log}$ space, is the subset of $BMOA_{\log}$ space consisting of those $g \in BMOA_{\log}$ for which

$$\lim_{|b| \to 1} (\log \frac{2}{1 - |b|^2})^2 \int_{\mathbb{D}} |g'(z)|^2(1 - |\varphi_b(z)|^2) \, dm(z) = 0.$$ 

It follows from [8] that $BMOA_{\log}$ is a Banach space under the norm $\|g\|_s = |g(0)| + \|g\|_{LBMOA}$. Siskakis and Zhao [8] have proved that $VMOA_{\log}$ is a closed subset of $BMOA_{\log}$, and equal to the closure of polynomials under this norm. The author and Lou [9] studied the cyclic vector and $g$ by Zhao in [11]. For more information about the logarithmic Carleson measure, see [11, 12].

Particularly, if $s = 2$, $\mu$ is called a vanishing logarithmic Carleson measure. This logarithmic Carleson is firstly introduced by Zhao in [11]. For more information about the logarithmic Carleson measure, see [11, 12].

In [8], Siskakis and Zhao have proved that $g \in BMOA_{\log}$ if and only if

$$\|g\| = \sup_{J \subset \partial \mathbb{D}} \frac{1}{|J|} \log^2(\frac{2}{|J|}) \int_{S(J)} |g'(z)|^2(1 - |z|^2) \, dm(z) \frac{1}{2} < \infty$$

and $g \in VMOA_{\log}$ iff

$$\lim_{|J| \to 0} \frac{1}{|J|} \log^2(\frac{2}{|J|}) \int_{S(J)} |g'(z)|^2(1 - |z|^2) \, dm(z) = 0.$$
In this case $\|g\|$ is comparable to $\|g\|_{LBMOA}$. Then we can easily prove that a function $g \in BMOA_{\log}$ when and only when the measure

$$\mu_g(z) = |g'(z)|^2 (1 - |z|^2) \, dm(z)$$

is a logarithmic Carleson, and $g \in VMOA_{\log}$ iff $\mu_g(z)$ is a vanishing logarithmic Carleson. In the case the logarithmic Carleson norm $N(\mu_g) \approx \|g\|^2_{LBMOA}$.

Here we mention a fact about $BMOA_{\log}$, which we will use later. It is easy checked that, for any $b \in \mathbb{D}$, $g_b(z) = \log(\log e^{1 - bz}) \in VMOA_{\log}$ and

$$\|g_b\|_* \leq \|\log(\log \frac{e}{1 - z})\|_* < +\infty. \tag{1.1}$$

In presenting our proofs we will also refer to the logarithmic-weighted Bloch space. The logarithmic-weighted Bloch space $\beta_L$ is the set of analytic functions $g$ on $\mathbb{D}$ for which

$$\|g\|_{\beta_L} = \sup_{z \in \mathbb{D}} (1 - |z|^2) \log(\frac{e}{1 - |z|}) |g'(z)| < \infty.$$ 

2. Auxiliary results

Lemma 2.1 \cite{9} $BMOA_{\log} \subset \beta_L$.

Lemma 2.2 Suppose that $g \in \beta_L$. Then $|g(z)| \leq \log(\log \frac{e}{1 - |z|}) \|g\|_{\beta_L} + |g(0)|$.

Proof Assume that $g \in \beta_L$ and $z \in \mathbb{D}$. We obtain that

$$|g(z) - g(tz)| = |z| \int_t^1 g'(zt) \, dt \leq \|g\|_{\beta_L} \int_t^1 \frac{|z|}{(1 - |zt|^2) \log \frac{e}{1 - |zt|}} \, dt$$

$$\leq \|g\|_{\beta_L} \int_t^1 \frac{dx}{(1 - x) \log \frac{e}{1 - x}} = \|g\|_{\beta_L} (\log(\log \frac{e}{1 - |z|}) - \log(\log \frac{e}{1 - |tz|}))$$

$$= \log(\frac{\log \frac{e}{1 - |z|}}{\log \frac{e}{1 - |tz|}}) \|g\|_{\beta_L}.$$ 

Especially, $|g(z) - g(0)| \leq \|g\|_{\beta_L} \log(\log \frac{e}{1 - |z|})$, hence

$$|g(z)| \leq \log(\log \frac{e}{1 - |z|}) \|g\|_{\beta_L} + |g(0)|.$$ 

Lemma 2.3 Suppose that $g \in BMOA_{\log}$. Then exists absolutely positive constant $M > 0$ such that

$$|g(z)| \leq M \log(\log \frac{e}{1 - |z|}) \|g\|_*.$$
Lemma 2.4 [9] If $g \in BMOA_{\log}$, $0 < t < 1$, then $g_t \in VMOA_{\log}$ and $\|g_t\|_* \leq \|g\|_*$, where $g_t(z) = g(tz)$.

**Proof** Suppose $\phi \in M(VMOA_{\log})$ and $\forall g \in BMOA_{\log}$, we have $g_t \in VMOA_{\log}$ for all $\forall 0 < t < 1$. Then, owing to Lemma 2.4,

$$\|\phi g\|_* \leq \|M_\phi\|\|g_t\|_* \leq \|M_\phi\|\|g\|_* < +\infty.$$  

Hence $\|\phi g\|_* \leq \|M_\phi\|\|g\|_* < +\infty$, so $\phi \in M(BMOA)$, which shows $M(VMOA_{\log}) \subset M(BMOA_{\log})$.

Lemma 2.6 [11] A positive measure $\mu$ over the disc $\mathbb{D}$ is a logarithmic Carleson measure when and only when

$$\sup_{b \in \mathbb{D}} (\log(\frac{2}{1 - |b|^2}))^2 \int_{\mathbb{D}} \frac{1 - |b|^2}{|1 - \overline{b}z|^2} d\mu(z) < \infty.$$  

3. Main results

**Theorem 3.1** The following are equivalent.

(i) $M_\phi$ is a bounded operator on $BMOA_{\log}$;

(ii) $\phi \in H^\infty$ and $|g(z)|^2|\phi'(z)|^2(1 - |z|^2)dm(z)$ is a logarithmic Carleson measure for every $g \in BMOA_{\log}$.

**Proof** First, we assume that $M_\phi$ is a bounded operator on $BMOA_{\log}$. By Proposition 3 in [14], we obtain that $\phi \in H^\infty$ and $\|g\phi\|_{L^{BMOA}} \leq \|g\phi\|_* \leq C\|g\|_*$ for every $g \in BMOA_{\log}$. It follows that

$$\sup_{b \in \mathbb{D}} (\log(\frac{2}{1 - |b|^2}))^2 \int_{\mathbb{D}} |g(z)|^2|\phi'(z)|^2(1 - |\phi_b(z)|^2) dm(z)$$

$$\leq C\|\phi\|_*^2 + \sup_{b \in \mathbb{D}} (\log(\frac{2}{1 - |b|^2}))^2 \int_{\mathbb{D}} |g'(z)|^2|\phi(z)|^2(1 - |\phi_b(z)|^2) dm(z)$$

$$\leq C\|\phi\|_*^2 + \|\phi\|_{L^{BMOA}}^2 \leq C\|\phi\|_*^2 < \infty.$$  

Thus $|g(z)|^2|\phi'(z)|^2(1 - |z|^2)dm(z)$ is a logarithmic Carleson measure for every $g \in BMOA_{\log}$.

Next, if $\phi \in H^\infty$ and $|g(z)|^2|\phi'(z)|^2(1 - |z|^2)dm(z)$ is a logarithmic Carleson measure for every $g \in BMOA_{\log}$, then, by Lemma 2.6,

$$\|g\phi\|_{L^{BMOA}}^2 \leq 2\|\phi\|_{L^{BMOA}}^2 \|g\|_{L^{BMOA}}^2$$

$$+ 2\sup_{b \in \mathbb{D}} (\log(\frac{2}{1 - |b|^2}))^2 \int_{\mathbb{D}} |g(z)|^2|\phi'(z)|^2(1 - |\phi_b(z)|^2) dm(z) < \infty,$$  

thus, $\phi \in M(BMOA_{\log})$.  


Theorem 3.2 (i) If $\varphi \in H^\infty$ and $|\varphi'(z)|^2(1-|z|^2)\log^2(\log \frac{e}{1-|z|})dm(z)$ is a logarithmic Carleson measure, then $M_\varphi$ is a bounded operator on $BMOA_{log}$.

(ii) If $M_\varphi$ is a bounded operator on $BMOA_{log}$, then $\phi \in H^\infty$ and

$$\sup_{b \in D}(\log \frac{e}{1-|b|^2})^2(\log(\log \frac{e}{1-|b|}))^2 \int_D |\varphi'(z)|^2(1-\varphi_b(z))^2 dm(z) < +\infty. \quad (3.1)$$

(iii) $M(VMOA_{log}) = M(BMOA_{log})$.

Proof (i) Let $g \in BMOA_{log}$. By Lemma 2.3 we have $|g(z)| \leq C \log(\log \frac{r}{1-|z|}) \|g\|_*$. Then we obtain that

$$\sup_{b \in D}(\log \frac{e}{1-|b|^2})^2(\log(\log \frac{e}{1-|b|}))^2 \int_D |\varphi'(z)|^2(1-|z|^2) dm(z) \leq C\|g\|^2 \sup_{b \in D}(\log \frac{e}{1-|b|^2})^2(\log(\log \frac{e}{1-|b|}))^2 \int_D |\varphi'(z)|^2(1-|z|^2) dm(z) < +\infty.$$

Thanks to Theorem 3.1, $M_\varphi$ is a bounded operator on $BMOA_{log}$.

(ii) Suppose that $M_\varphi$ is a bounded operator on $BMOA_{log}$. Then, by Proposition 3 in [14], we have $\phi \in H^\infty$ and $|\varphi(z)| \leq \|M_\varphi\|$. We take the test function

$$g_b(z) = \log(\log \frac{e}{1-\overline{a}z}), \quad b \in D$$

By (1.1), we have

$$\sup_{J}(\log \frac{2}{|J|})^2(\log(\log \frac{e}{|J|}))^2 \int_{S(J)} |(\varphi g_b)'(z)|^2(1-|z|^2) dm(z) \leq C\|\varphi g_b\|^2_{BMOA} \quad (3.2)$$

Let $J \subset \partial D$ be an arc and $\xi$ be the center of $J$. Let $b = (1-|J|)\xi$. We know that $|g_b(z)|$ is comparable to $\log(\log \frac{e}{|J|})$ for all $z \in S(J)$. Then, by (3.2), it follows that

$$\sup_{J}(\log \frac{2}{|J|})^2(\log(\log \frac{e}{|J|}))^2 \int_{S(J)} |\varphi'(z)|^2(1-|z|^2) dm(z) \leq \sup_{J}(\log \frac{2}{|J|})^2 \int_{S(J)} |g_b(z)|^2 \log^2(\log \frac{e}{1-|z|}) dm(z)$$

$$\leq \sup_{J}(\log \frac{2}{|J|})^2 \int_{S(J)} |g_b(z)|^2 \log^2(\log \frac{e}{1-|z|}) dm(z) + C\|M_\varphi\|^2 \log(\log \frac{e}{1-\xi})^2$$

$$\leq C\|M_\varphi\|^2 \log(\log \frac{e}{1-\xi})^2 + C\|M_\varphi\|^2 \log(\log \frac{e}{1-\xi})^2 < +\infty.$$
So,

\[ M = \sup_{J} \frac{1}{|J|} (\log \frac{2}{|J|})^2 (\log \log \frac{e}{|J|})^2 \int_{S(J)} |\phi'(z)|^2 (1 - |z|^2) dm(z) < +\infty \quad (3.3) \]

Now we consider the Neq. (3.1). Here we use a similar proof to that in [8]. If |b| \leq \frac{3}{4}, it is easy to see that (3.1) holds. Assume that |b| > \frac{3}{4} and let

\[ X_0 = \emptyset, \quad X_n = \{ z \in \mathbb{D} : |z - \frac{b}{|b|}| < 2^{n-1} (1 - |b|) \} \quad n = 1, 2, \ldots, N, \]

here N = N(b) is the smallest integer with 2^{N-1} (1 - |b|) \geq 1. From a simple calculation we see that N \approx \log(\frac{1}{1-|b|})/\log 2. By the geometric properties we obtain that there exists a constant C for which

\[ \frac{1 - |b|^2}{|1 - \bar{b} z|^2} \leq \frac{C}{2^{2n}(1 - |b|)}, \quad z \in X_n \setminus X_{n-1}, \quad n = 1, 2, \ldots N. \]

Also, let |J_n| = 2^{n-1} (1 - |b|) for n = 1, 2, \ldots, N. We get that the Carleson box S(J_n) with X_n \subset S(J_n). It follows that

\[
\int_{\mathbb{D}} |\phi'(z)|^2 (1 - |\varphi_{\mathbb{D}}(z)|^2) dm(z) \leq \sum_{n=1}^{N} \int_{X_n \setminus X_{n-1}} |\phi'(z)|^2 (1 - |\varphi_{\mathbb{D}}(z)|^2) dm(z)
\leq \sum_{n=1}^{N} \frac{C}{2^{2n}(1 - |b|)} \int_{X_n \setminus X_{n-1}} |\phi'(z)|^2 (1 - |z|^2) dm(z)
\leq \sum_{n=1}^{N} \frac{C}{2^{2n}(1 - |b|)} \int_{S(J_n)} |\phi'(z)|^2 (1 - |z|^2) dm(z)
\leq CM \sum_{n=1}^{N} \frac{1}{2^n (\log \frac{2}{2^{n-1} (1 - |b|)})^2 (\log \log \frac{1}{2^{n-1} (1 - |b|)})^2}
\leq \frac{CM}{(\log \frac{2}{1-|b|})^2 (\log \log \frac{1}{1-|b|})^2}.
\]

So,

\[
\sup_{b \in \mathbb{D}} \frac{1}{2^{1 - |b|}} (\log \log \frac{e}{1 - |b|})^2 \int_{\mathbb{D}} |\phi'(z)|^2 (1 - |\varphi_{\mathbb{D}}(z)|^2) dm(z) \leq CM.
\]

Hence (3.1) holds.

(iii) Suppose \( \phi \in M(BMO_{A_{\log}}) \), then (3.3) holds. It implies that

\[
\lim_{|J| \to 0} \frac{1}{|J|} \log^2 \left( \frac{2}{|J|} \right) \int_{S(J)} |\phi'(z)|^2 (1 - |z|^2) dm(z) = 0.
\]

Then \( \phi \in H^\infty \cap VMO_{A_{\log}} \). By a calculation we show that \( M_{\phi}(z^n) \in VMO_{A_{\log}} \) for every \( n = 0, 1, 2, \ldots \). Then, for each polynomial p, we get that \( M_{\phi}(p) \in VMO_{A_{\log}} \).
Next, given \( \forall f \in VMOA_{\log} \), there exists a sequence \( \{p_n\} \) of polynomials with \( \|f - p_n\|_* \to 0 \), and we easily obtain that

\[
0 \leq \|M_\phi(f) - M_\phi(p_n)\|_* = \|M_\phi(f - p_n)\|_* \leq \|M_\phi\| \|f - p_n\|_* \to 0
\]
as \( n \to +\infty \). This illustrates that \( M_\phi(f) \) can be approximated by a sequence of functions in \( VMOA_{\log} \) in the \( \| \cdot \|_* \). Since \( VMOA_{\log} \) is a closed set, we obtain that \( \phi \in M(VMOA_{\log}) \). Hence \( M(BMOA_{\log}) = M(VMOA_{\log}) \) by Lemma 2.5.

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