Asymmetric Skyrmion lattice in helimagnets

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Intricate spin textures in helimagnets, identified as stable topological Skyrmions, were observed experimentally, where Skyrmie lattice was supposed to exhibit symmetric structures in the ground state. We show the possibility of asymmetric Skyrmions in a helimagnetic model, for individual Skyrmion as well as for the hexagonal Skyrmie crystal with higher charge, as perturbative deformation and stabilization of exact ferromagnetic solitons. Such nonsymmetric configurations for the Skyrmie lattice, predicted here theoretically, need to be verified in precision experiments.

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I. INTRODUCTION

Unusual magnetic behavior of helimagnet MnSi unusual, which triggered the suspicion, that the magnetic states in such helimagnetic materials might be of topological origin, is confirmed as Skyrmions in quantum Hall (QH) QHexperim, and small angle neutron scattering (SANS) experiments [7]. These experiments, though made pioneering observations, could give only indirect evidence for the existence of Skyrmion spin textures, due to their confinement in the momentum space. However, more recently, in a real space experiment with Lorentz transmission electron microscopy (TEM), direct photographic evidence of Skyrmion crystals was obtained in a thin film of helimagnet Fe\textsubscript{0.5}Co\textsubscript{0.5}Si, on a plane perpendicular to the applied magnetic field [8]. Skyrmion crystals were found to be in a beautiful hexagonal form, with the localized individual Skyrmions serving as its constituent molecules, extended over lattice spacing of 30 nm range.

For theoretical description of these fascinating experimental observations in helimagnets, the accepted basic model is given by a ferromagnetic spin exchange Hamiltonian in combination with a Dzyaloshinskii-Moriya (DM) interaction bogdanMnsi,LorentzExp,heliModel,heliModelPRL10. It is argued, that the occurrence of topological solitons in helimagnets is due to the competing forces between the ferromagnetic and the effective DM interactions, where the ferromagnetic spin exchange tends to align the neighboring spins parallel to each other, while the DM spin-orbital interaction with its broken inversion symmetry, orients them to be mutually perpendicular, resulting to a helical order of topological origin.

Since the experimentally observed Skyrmions are found to have extended nature, slowly varying in comparison with the lattice structure of the original magnetic crystals, the long wavelength limit is justified and at low energy and weak DM coupling we can consider the continuum approximation. At this semiclassical limit an intriguing topological property sets in, with a spatially dependent magnetization as a unit vector, wrapping around a 2-sphere, while the 2d coordinate space, due to the fixed orientation of the spins at space-infinities, compactifies to another 2-sphere, inducing a sphere to sphere mapping. The degree of this mapping, counting the number of times the Skyrmion magnetic field sweeps the target sphere, when the coordinate space is covered once, defines the integer valued topological charge \( Q = N \) [11].

It is a remarkable fact, that at the continuum limit, the ferromagnetic Hamiltonian itself, defined on a 2d plane, allows exact Skyrmion solutions with arbitrary integer charge \( N \), which can be linked to the holomorphic functions [12]. However, such solutions exhibiting in general a noncircular symmetry, posses a scale invariance property, which does not allow to fix the Skyrmion size and makes the Skyrmions in a ferromagnetic model unphysical metastable states. However, the addition of a DM type interaction with broken symmetry can change this picture significantly by providing the necessary scale through its coupling parameter and stabilizes the magnetic Skyrmions, that have been observed in several recent experiments in different helimagnetic materials. However, at the theoretical level, the addition of the DM Hamiltonian can not sustain the analytic solutions obtained in the original ferromagnetic model and at the same time introduces a high level of asymmetry to the resulting equations, due to explicitly broken inversion symmetry in the DM interaction. Therefore, the appearance of Skyrmion spin patterns with noncircular...
symmetry seems to be more generic in such systems, in particular for describing the Skyrme crystals. However, based on the experimental data, available so far, in observing the magnetic Skyrmions in helimagnets, any definitive statement about the symmetry and isotropy of the individual Skyrmions located in the Skyrme lattice seems to be difficult to make. In QH experiments the indirect evidence of Skyrmions was obtained in the form of an unusual magnetic field giving no details about the structure of the soliton. In SANS experiment, though the situation was improved with the detection of the hexagonal form of Skyrme crystals through a 6-fold symmetry, any symmetry statement about the individual Skyrmion molecules was beyond the scope of this experiment. Finally in recent thin film Lorentz TEM experiment images of extended Skyrme molecules inside helical Skyrmion crystals were obtained in real space, revealing much detailed structures of these objects appearing in a helimagnetic material. However, even from this best available data, one can not confirm with certainty about the nature of symmetry and isotropy of the individual Skyrmions, since the color-code of the images shows directional asymmetry and color inhomogeneity (see Fig. 1e-f of [8]). Therefore, for a clearer picture about the precise symmetry of the Skyrmions and the exact directional orientations of the magnetic field lines one has to look for further finer and precision experiments.

Nevertheless, at the theoretical level, most of the models tend to support the symmetric form of the individual Skyrmions and their isotropy inside a Skyrme crystal, neglecting the inter-Skyrmion interactions. For example, in a numerical simulation by minimizing the discretized model Hamiltonian using the Monte Carlo method, circular symmetric Skyrmion molecules forming a Skyrme crystal in symmetric form have been obtained [8]. Similarly, in the prevalent theoretical models bogdanMnsi,heliModelPRL10 a circular symmetric ansatz is assumed for the solution of individual Skyrmions. However, since the general nature of the governing Euler equations are highly asymmetric, such a symmetric assumption put strong restrictions on the corresponding solutions, confining it to the unit charge sector and imposing a fixed initial phase shift, which limits the spin pattern only to a particular orientation ($D_n$ type) bogdanMnsi,heliModelPRL10.

It is important to stress, that since the individual Skyrmions as well as the Skyrme crystals in general should be solutions of the energy minimizing equations, they should ideally be derived from the Euler equations dictated by the governing helimagnetic model without restriction of symmetry, orientation and charge. However, no satisfactory theoretical proposal seems to have been offered to this challenging problem of describing the hexagonal Skyrme crystal structure with noncircular symmetry, as a solution of the underlying equations with higher topological charges, which would induce nonlinear interactions between the constituent Skyrmion molecules. It is indeed a difficult problem to solve the governing equations, representing highly nonlinear coupled partial differential equations allowing no separation of variables.

Current studies on the helimagnetic model avoid this difficult problem by completely neglecting the inter-molecular interactions between the Skyrmions, by assuming a circular cell approximation bogdanMnsi,heliModelPRL10, though such a linear superposition of individual solutions does not represent a solution of the governing nonlinear equations. Moreover, such a configuration as a equidistant collection of individual Skyrmions can not be reproduced as a limiting case of a general solution with a higher charge.

Other investigations on the Skyrme lattice in the helimagnetic model focus usually on different other aspects like the effect of current and temperature gradients on the rotating of Skyrme lattice [13], pinning of Skyrmion through inhomogeneity of magnetic exchange [14], Skyrmion spin lattice coupling with electric dipole [15] etc., leaving aside the basic problem of the formation of Skyrme crystals as a topological solution.

Our aim here is to focus on the above posed problem of constructing a Skyrme crystal as a solution of the helimagnetic equations. However, we bypass the direct problem, which is hard to solve and obtain the solutions with higher topological charges linked to the Skyrme crystal, by looking first for the Skyrmions appearing in the pure ferromagnetic model (FM) and subject them subsequently, to the DM interaction perturbatively, by assuming the coupling to be weak. This stabilizes the solitons through DM coupling and goes beyond the limitations of the earlier studies by allowing Skyrmion solutions with higher charges and more general initial phase admitting different types of orientations for the spin texture. This would mean, however that one has to deal with noncircular symmetric fields without separation of variables, which unlike earlier proposals, is a more difficult task to handle. Nevertheless, this approach solves to some extend the problem of formation of hexagonal Skyrme crystals within the helimagnetic model with weak DM coupling, by finding a perturbed Skyrmion solution with charge $Q = -7$, and without the usual circular cell approximation.
II. THE MODEL

The basic Helimagnetic model, as motivated above, is given by the Hamiltonian

\[ H = J H_{FM} + D H_{DM}, \]

with

\[ H_{FM} = \frac{1}{2} \int d^2 x (\nabla M)^2, \]

\[ H_{DM} = \int d^2 x (M \cdot [\nabla \times M]), \]

where \( M \) with \( M^2 = 1 \) is the magnetization vector and \( H_{FM}, H_{DM} \) are the competing ferromagnetic and DM interactions, respectively. Since the magnetization, taking values on a 2-sphere is restricted to a 2d plane, the system may be described by two spherical angles \( \beta(\rho, \alpha) \) and \( \gamma(\rho, \alpha) \) defined through polar coordinates \((\rho, \alpha)\) with the magnetization components given as

\[ M^1 \pm i M^2 = \sin \beta e^{\pm i \gamma}, \quad M^3 = \cos \beta. \]

The helimagnetic Hamiltonian (1) may be expressed through these angle variables for convenience as

\[ H_{FM} = \frac{1}{2} \int d^2 x (\nabla \beta)^2 + \sin^2 \beta (\nabla \gamma)^2, \]

\[ H_{DM} = \int d^2 x (\sin(\gamma - \alpha)(\partial_\rho \beta - \frac{1}{2\rho} \sin 2\beta \partial_\alpha \gamma) - \cos(\gamma - \alpha)(\frac{\partial_\alpha \beta}{\rho} + \frac{1}{2} \sin 2\beta \partial_\rho \gamma)). \]

The associated topological charge takes the explicit form

\[ Q = \frac{1}{4\pi} \int d^2 x \sin \beta |[\nabla \beta \times \nabla \gamma]|, \]

A. The Euler equations

Note that the energy minimizing Euler equations for the fields \( \beta \) and \( \gamma \), may be derived in the static case from the model Hamiltonian (4, 5) as

\[ \nabla^2 \beta - \frac{1}{2} \sin 2\beta (\nabla \gamma)^2 + 2\epsilon M_\beta(\beta, \gamma) = 0, \quad \epsilon = \frac{D}{J}, \]

for angle \( \beta(\rho, \alpha) \) and

\[ \sin 2\beta (\nabla \beta \cdot \nabla \gamma) + \sin^2 \beta (\nabla^2 \gamma) + 2\epsilon M_\gamma(\beta, \gamma) = 0, \]

and with respect to angle \( \gamma(\rho, \alpha) \), where the additional DM terms are

\[ M_\beta(\beta, \gamma) \equiv \sin^2 \beta (\partial_\rho \gamma \cos(\gamma - \alpha) + \frac{1}{\rho} \partial_\alpha \gamma \sin(\gamma - \alpha)) \]

and

\[ M_\gamma(\beta, \gamma) \equiv - \sin^2 \beta (\partial_\rho \beta \cos(\gamma - \alpha) + \frac{1}{\rho} \partial_\alpha \beta \sin(\gamma - \alpha)) = 0. \]

Note that these equations for finding the field configurations minimizing the Hamiltonian, represent highly nonlinear inhomogeneous coupled partial differential equations in two variables and are difficult to solve in general. A simple possibility to overcome this difficulty, as adopted in most of the earlier studies \[4, 5, 10\], is to focus on an individual Skyrmion with a circular symmetric ansatz

\[ \beta = \beta(\rho), \quad \gamma = N \alpha + \alpha_0, \]

with initial phase \( \alpha_0 \), as an arbitrary constant and \( N \) an arbitrary integer. Specific values of \( \alpha_0 \), represent precise crystallographic forms, describing the spin orientation in the related magnetic pattern bogdanMnsi. A closer look
However reveals that for the circular symmetric ansatz (11) to work, the equation (8) for the angle $\gamma$ must be zero, which due to $\partial_\alpha \beta = 0$, $\partial_\rho \gamma = 0$ reduces to the vanishing of $M_\rho$ (10) or in turn to the vanishing condition for the term

$$\sin^2 \beta' \beta' \cos((N-1)\alpha + \alpha_0).$$

Clearly it becomes zero (for $\beta' \neq 0$), when the cosine function vanishes under the combined condition: $N = 1$ and $\alpha_0 = \pm \frac{\pi}{2}$. Therefore, for the validity of the symmetric ansatz (11) for the Skyrmions in helimagnets, one gets a solution restricted only to unit topological charge $Q = N = 1$ and with fixed value for the initial phase linked to $D_n$ type of magnetic pattern [4, 5, 10]. Therefore, though this is a convenient way to achieve separation of variables in a coupled nonlinear equation, it can not get generic Skyrmion solutions with higher topological charges $Q = N > 1$ and misses the possibility of finding other crystallographic forms for other values of $\alpha_0$, involving noncircular symmetric solutions. More importantly, such a symmetric ansatz can not describe the Skyrmion lattice structure having multiple centers and higher topological charge, including the Skyrmie crystals in hexagonal form, as a solution of the full set of governing equations (7-8). Nevertheless, a formal insertion of such circular symmetric Skyrmions at each lattice site, sometimes with anisotropic terms like $h M^3$, $K (M^3)^2$ etc. added to the helimagnetic model [1], was proposed theoretically for describing a Skyrme crystal [2], which however neglects completely the interaction between the Skyrme molecules, under circular cell approximation.

### III. THE APPROACH

Going beyond the conventional circular cell approximation, we intend to look for more general noncircular symmetric solutions by considering interactions between the individual Skyrme molecules, consistent with the coupled Euler equations (7-8), minimizing the helimagnetic model. However, since such asymmetric solutions are difficult to obtain in general, we adopt a bypass route by considering the deformation of Skyrmion solutions of pure ferromagnetic model [2], by switching on the DM interaction (3) perturbatively, by taking the DM coupling $D$ weaker in comparison with the ferromagnetic exchange coupling $J$ with $\epsilon \equiv D/J$ as a small parameter. The assumption of a weak relative coupling $D/J$, however has gone already in the justification of the continuum approximation for the present model (see e.g. LorentzExp). Note, that this approach also brings in the necessary scaling parameter for stabilizing the Skyrmions.

#### A. Skyrmions in ferromagnetic model

In the first step, we focus on the nonlinear Euler equations associated with the ferromagnetic model alone, without the DM interaction and look for the exact Skyrmions with higher topological charge, which in general would have noncircular symmetry. In the next step, this exact solution is subjected to the DM interaction perturbatively, where though the perturbing fields do not allow separation of variables, the governing equations become linear, allowing numerical solutions. Therefore, we start with suitable solutions $\beta_0(\rho, \alpha), \gamma_0(\rho, \alpha)$ for the ferromagnetic model (2) (or (4)), satisfying the related Euler equations (7-8), at $D = 0$. It is remarkable, that the topological Skyrmions for this model can be given by the celebrated exact solutions found way back by Belavin and Polyakov (BP) [BP]. A deep theoretical concept and beautiful geometrical picture go into this construction, where an important lower bound for the energy (11) is revealed through the topological charge $Q \geq 4\pi|Q|$. Note that due to the conservation of charge the lower bound guarantees, that the finite energy topological solitons can not decay into the vacuum or any other topological states. Interestingly, the lower bound is saturated under the so called self-duality condition

$$\partial_\rho \beta_0 = \pm \frac{2}{\rho} \sin \beta_0 \partial_\alpha \gamma_0, \quad \partial_\alpha \beta_0 = \mp 2\rho \sin \beta_0 \partial_\rho \gamma_0,$$

and since this is the minimizing condition for the energy functional, the corresponding field configuration also becomes a solution of the associated Euler equations (7-8 at $D = 0$), which are our current focus. Note, interestingly that these Euler equations are actually solved without solving them directly, but by solving the self-duality condition and therefore, for finding the Skyrmion solutions in pure ferromagnetic model, one has to consider only the solution of (12), which remarkably can be solved exactly. Deep reason behind this intriguing fact is that the relations (12) for attaining the energy minimum represent the well known Cauchy-Riemann (CR) condition.
(expressed in the polar coordinates), required for the analyticity of the complex function \( f(z) \), linked through the stereographic projection

\[
f(z) = \frac{M^1 + iM^2}{1 + M^3} = \tan \frac{\beta_0}{2} e^{i\gamma_0}.
\]  

(13)

Consequently, as shown in Ref. [12], any analytic function of the form

\[
f(z) = \prod_i (z - z_i)^{n_i} \prod_j (z - z_j)^{n_j}, \quad N = \sum_i n_i - \sum_j n_j,
\]

(14)

would be an exact Skyrmion solution of the ferromagnetic model with topological charge \( Q = N \), allowing a scale invariance: \( z \to \lambda z \).

Our strategy in constructing the Skyrmions for the Helimagnetic model, is to take the ferromagnetic soliton solutions \([13]\) as the unperturbed Skyrms \( \beta_0, \gamma_0 \) through mapping \([13]\), giving the angle variables as inverse functions

\[
\beta = 2 \tan^{-1}(|f|), \quad \gamma = \arg f.
\]

(15)

These are guaranteed to satisfy the self-duality (CR condition) \([12]\) and consequently, the Euler-equations \([7,8]\) for \( D = 0 \), as an exact solution. Therefore, for describing an individual Skyrmion we may consider the simplest solution does not hold. Therefore any solution for the Skyrme crystal with higher charges must satisfy the energy

\[
m_3^\alpha = \cos \beta = \frac{1 - |f_1|^2}{1 + |f_1|^2} = \frac{\rho^2 - 1}{\rho^2 + 1},
\]

(16)

having a perfect circular symmetry is shown in Fig. 1a in the graphical form.

For constructing Skyrmion lattice in a ferromagnetic model, a naive way would be to insert the same circular symmetric BP Skyrmion with \( Q = -1 \), constructed above, at each Skyrme lattice site on the 2d plane, separated uniformly from its neighbors at a distance of \( a_s \). This would give the magnetization in the form \( M^3 = \cos \beta_0 = \sum_j \frac{1 - |f_j(z_j)|^2}{1 + |f_j(z_j)|^2} \), neglecting the nonlinear interactions between the Skyrme molecules, induced by the governing Euler equations. Note, that a similar construction was adopted for the Skyrme crystals, but for the helimagnetic models, in most of the current studies assuming a circular cell approximation. However, unfortunately such a formal construction of Skyrme lattice using superposition of individual Skyrme solitons is not a solution of the energy minimizing equations dictated by the model Hamiltonian, both for the pure ferromagnetic as well as for the helimagnetic models, since due to nonlinearity of the governing equations the superposition principle of individual solutions does not hold. Therefore any solution for the Skyrme crystal with higher charges must satisfy the energy

\[
M^3 = \cos \beta_0 = \frac{1 - |f_7|^2}{1 + |f_7|^2}, \quad \text{and} \quad \gamma_0 = -\arg f_7(z),
\]

(18)

This exact solution may be linked to the hexagonal Skyrme crystal pattern for the angle variables \( \beta_0, \gamma_0 \) through the magnetization component

\[
M^3 = \cos \beta_0 = \frac{1 - |f_7|^2}{1 + |f_7|^2}, \quad \text{and} \quad \gamma_0 = -\arg f_7(z),
\]
which is obviously noncircular symmetric and expressed through BP solution (17), as graphically represented in Fig. 1b. This topological soliton solution with higher charge describing the hexagonal Skyrme crystal in a ferromagnetic model is guaranteed to satisfy the required energy minimizing equations, where parameter $a_s$ may serve as the Skyrme lattice constant fixing the size of the crystal. However this parameter remains arbitrary due to the scale invariance $\rho \to \rho_0 \rho$ of the solution, which also makes the Skyrmion to vanish at $\rho_0 \to 0$, inducing the well known unphysical metastable Skyrmionic states in a pure ferromagnetic model. However, the additional DM interaction (5), which is included in the helimagnetic model could stabilize the Skyrmion solutions by introducing the required scaling in the system through coupling parameter $D$, as we will see below. Note, that though in Fig 1a an isolated Skyrmion with unit charge exhibits perfect isotropy and circular symmetry, such individual symmetries no longer remain intact in the Skyrme solution of crystalline form with higher charge $Q = -7$, as shown in Fig 1b, due to interactions between the Skyrme molecules induced unavoidably by the underlying nonlinear equations. This characteristic behavior of the Skyrme crystals with higher charges exhibiting anisotropy and noncircular symmetry, as obtained here in the ferromagnetic model, is expected to remain prevalent also in the helimagnetic model with the inclusion of an additional symmetry breaking DM term (3), at least in the excited states. This is the emphasis of our proposal in what follows.

![Magnetization component $M^2 = \cos \beta_0$ in pure ferromagnetic model (2) for a) individual Skyrmion with $Q = -1$ and b) hexagonal Skyrmion crystal with $Q = -7$.](image)

**B. Skyrmions in helimagnetic model**

Our next step is to consider the helimagnetic model (1) by switching on the DM interaction (5) to the ferromagnetic model (4) investigated above. For finding the spin texture in this helimagnetic model described by the Skyrmion solution, one has to find the energy minimizing configuration given by the the solutions of the associated Euler equations for the fields $\beta$ and $\gamma$ derived from (4,5). However, as discussed above, since these coupled highly nonlinear partial differential equations (7,8) for the two fields involving both the polar coordinates without separation of variables, are difficult to solve in general for $D \neq 0$, our strategy would be to treat the system perturbatively by considering $\epsilon = D/J$, the relative coupling between the DM and the ferromagnetic exchange interaction, as a small parameter. The perturbative solution therefore may be given by

$$
\beta(\rho, \alpha) = \beta_0(\rho, \alpha) + \epsilon \beta_1(\rho, \alpha), \quad \gamma(\rho, \alpha) = \gamma_0(\rho, \alpha) + \epsilon \gamma_1(\rho, \alpha),
$$

in the first order of approximation, where $\beta_0(\rho, \alpha), \gamma_0(\rho, \alpha)$ are the unperturbed solutions induced by the ferromagnetic Hamiltonian (4), while $\beta_1(\rho, \alpha), \gamma_1(\rho, \alpha)$ are the deformations suffered, when the DM interaction (5) is switched on, perturbatively. The parameter $D$ also serves as the scaling parameter, breaking the scale invariance of the unperturbed Skyrmions and thus providing the required stability to the soliton solutions. Recall, that we have derived already the partial differential equations (7,8) for the two fields involving both the polar coordinates without separation of variables, are difficult to solve in general for $D \neq 0$, our strategy would be to treat the system perturbatively by considering $\epsilon = D/J$, the relative coupling between the DM and the ferromagnetic exchange interaction, as a small parameter. The perturbative solution therefore may be given by

$$
\nabla^2 \beta_1 - \sin 2\beta_0 (\nabla \gamma_1 \cdot \nabla \gamma_1) - \beta_1 \cos 2\beta_0 (\nabla \gamma_0)^2 + \frac{1}{2} M_\beta(\beta_0, \gamma_0) = 0,
$$

$$
\sin 2\beta_0 (\nabla \beta_0 \cdot \nabla \gamma_1 + \nabla \beta_1 \cdot \nabla \gamma_0 + \beta_1 \nabla^2 \gamma_0) + \beta_1 \cos 2\beta_0 (\nabla \beta_0 \cdot \nabla \gamma_0) + \sin^2 \beta_0 (\nabla^2 \gamma_1) + \frac{1}{2} M_\gamma(\beta_0, \gamma_0) = 0
$$

as shown in Fig 1. Therefore, it remains now to extract the deforming solutions $\beta_1, \gamma_1$. For this we insert the perturbative ansatz (19) in the full set of helimagnetic equations (7,8) and derive the corresponding linear set of equations in the first order of approximation $O(\epsilon)$:

$$
\nabla^2 \beta_1 - \sin 2\beta_0 (\nabla \gamma_1 \cdot \nabla \gamma_1) - \beta_1 \cos 2\beta_0 (\nabla \gamma_0)^2 + \frac{1}{2} M_\beta(\beta_0, \gamma_0) = 0,
$$

$$
\sin 2\beta_0 (\nabla \beta_0 \cdot \nabla \gamma_1 + \nabla \beta_1 \cdot \nabla \gamma_0 + \beta_1 \nabla^2 \gamma_0) + \beta_1 \cos 2\beta_0 (\nabla \beta_0 \cdot \nabla \gamma_0) + \sin^2 \beta_0 (\nabla^2 \gamma_1) + \frac{1}{2} M_\gamma(\beta_0, \gamma_0) = 0
$$

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where $M_β(β_0, γ_0), M_γ(β_0, γ_0)$ are as in (1, 10). However, due to the anisotropic and symmetry breaking contribution of the DM interaction the overall symmetry of the unperturbed Skyrmions gets broken generically, even for the isolated Skyrmions and these coupled set of equations, unlike in the pure ferromagnetic model, can not be solved analytically. Nevertheless, the linearity of these equations allows numerical solutions for the fields $β_1, γ_1$ (see Supplementary material for Mathematica 10 code). Adding now these perturbing solutions for the DM interaction to the corresponding analytic solutions $β_0, γ_0$ found above for the ferromagnetic model, which also stabilizes the solutions, we finally obtain the Skyrmions for the helimagnetic model in the form (19) as shown in Fig. 2, for both isolated and crystal solutions.

**FIG. 2:** Magnetization component $M^3 = \cos β$ in helimagnetic model for a) individual Skyrmion with $Q = -1$ and b) hexagonal Skyrmion crystal with $Q = -7$. These Skyrmion solutions are perturbations of ferromagnetic Skyrmions obtained in Fig. 1 for weak DM coupling, with small parameter $ε = 0.2$.

It is evident by comparing the Skyrmion solutions for the Helimagnetic model presented in Fig. 2 with those for the pure ferromagnetic model shown in Fig. 1, that the inclusion of the symmetry breaking DM interaction deforms the solutions and introduces further anisotropy and noncircular symmetry to the individual Skyrmion molecules as well as to the Skyrmion crystals appearing in the helimagnetic model. This result generalizes therefore the known circular symmetric Skyrmions and Skyrmion crystals obtained in the helimagnetic model to the asymmetric case.

### IV. CONCLUDING REMARKS

Appearance of Skyrmion crystals in the hexagonal form has been observed experimentally in the magnetization pattern in helimagnetic materials. However a full theoretical description for the formation of crystalline structures as a general topological Skyrmion solution of the governing equations minimizing the model Hamiltonian, which is difficult to achieve due to high asymmetry and nonlinearity inherent to the system is not yet available in the literature. Assuming weaker DM interaction relative to the ferromagnetic exchange coupling, which is also a requirement for the conventional continuum approximation, we present here such solutions for Skyrmion crystals with higher topological charge and for individual Skyrmions with unit charge in the helimagnetic model. Our approach of perturbing and stabilizing the Skyrmions of the ferromagnetic model with the symmetry breaking DM interaction, bypasses the major difficulty of nonlinearity and solves the energy minimizing equations without assuming any separation of variables or circular cell approximation, for the first time (see the Supplementary material). Note that, though in recent experiments the evidence of Skyrmion and Skyrmion crystals was found, the details about the circular symmetry or isotropy of Skyrmions in their individual state or when they appear as interacting molecules in the Skyrmion crystal are not much clear (see color-code of experimental images in [8]). Therefore, the theoretical possibility of asymmetric Skyrmion crystals found here, can not be ruled out without verifying in precision experiments, where deformation of symmetries and isotropy in the magnetic pattern could be detected in finer details. To tune with the validity region of the present solutions obtained for low values of $ε$, care should be taken in choosing the helimagnetic materials, so that their spin-orbital DM interaction is weaker in comparison with the ferromagnetic spin exchange. Though the helimagnetic crystals MnSi and Fe_{1−x}Co_{x}Si share similar magnetic features, they differ significantly in their electronic structures, which is likely to make the later system more suitable for investigating the nature of Skyrmion lattices over a large temperature range, with the possibility of revealing more refined structures. The recent Lorentz TEM experiment LorentzExp might well be revisited and rescrutinized for detecting the possible symmetry breaking in the observed magnetic pattern in Skyrmie crystals, in the light of the present result, at least in the excited states and in the higher temperature
range.

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