On the mod-Gaussian convergence of a sum over primes

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Abstract We prove mod-Gaussian convergence for a Dirichlet polynomial which approximates \( \text{Im} \log \zeta(1/2 + it) \). This Dirichlet polynomial is sufficiently long to deduce Selberg’s central limit theorem with an explicit error term. Moreover, assuming the Riemann hypothesis, we apply the theory of the Riemann zeta-function to extend this mod-Gaussian convergence to the complex plane. From this we obtain that \( \text{Im} \log \zeta(1/2 + it) \) satisfies a large deviation principle on the critical line. Results about the moments of the Riemann zeta-function follow.

Keywords Distribution of primes · Mod-Gaussian convergence · Riemann zeta-function · Selberg’s central limit theorem · Large deviations

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1 Introduction

In this paper we study the distribution of values taken by \( \log \zeta(1/2 + it) \). A breakthrough was achieved by Selberg who showed that as \( t \) varies in \([T, 2T]\), the distribution of \((\text{Re} \log \zeta(1/2 + it), \text{Im} \log \zeta(1/2+it))\) is approximately Gaussian, with independent components each having expectation 0 and variance \((\log \log T)/2\). More precisely, he proved a central limit theorem which, by the Lévy continuity theorem, is equivalent to the statement that

\[
\frac{1}{T} \int_T^{2T} e^{iu \frac{\text{Re} \log \zeta(1/2+it)}{\sqrt{\log \log T}/2} + iv \frac{\text{Im} \log \zeta(1/2+it)}{\sqrt{\log \log T}/2}} dt \to e^{-u^2/2-v^2/2},
\]

as \( T \to \infty \), for all real numbers \( u \) and \( v \). For the case of \( \text{Im} \log \zeta(1/2 + it) \) see [17,18], and also the work of Ghosh [7]. The general case is investigated for instance in the book of Joyner

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Some of Selberg’s more recent results, for example about the rate of convergence, can be found in [19] and the thesis of Tsang [21]. Initially, Selberg obtained the asymptotics of the joint moments which lead to (1.1) by the method of moments. A more effective approach, applied in our analysis, too, is treated in the work of Bombieri and Hejhal [2].

A central limit theorem for the sum over primes \( (1/\sqrt{\log x})/2 \sum_{p \leq x} p^{-1/2-iUT} \), \( U_T \) being random variables uniformly distributed on \([T, 2T] \), \( \log x = \log T / (\log \log T)^{1/4} \), follows from the mean value theorem of Montgomery and Vaughan and the method of moments. To complete the proof (see [2, Lemma 3 and Corollary]), they showed that the \( L^1 \)-norm of \( \log (1/2 + iUT) - \sum_{p \leq x} p^{-1/2-iUT} \) is sufficiently small.

The convergence in (1.1) is also a consequence of a conjecture on the behaviour of the moments of the Riemann zeta-function on the critical line (see, e.g., the work of Keating and Snaith [12] and the references therein). It asserts that

\[
e^{\left(\imath z_1^2 + \imath z_2^2\right) (\log \log T)/4} \frac{1}{T} \int_T^{2T} e^{\imath z_1 \Re \log \zeta(1/2+i\imath t) + i\imath z_2 \Im \log \zeta(1/2+i\imath t)} \, dt
\]

\[
\rightarrow \Phi_g(z_1, z_2) \Phi_a(z_1, z_2) \quad \text{as} \ T \rightarrow \infty \quad (1.2)
\]

locally uniformly for \( z_1, z_2 \in \mathbb{C} \) with \( \Re (iz_1) > -1 \) and analytic functions \( \Phi_g, \Phi_a \) (see [13, Conjecture 9] and also [9, Conjecture 1]). This type of convergence was introduced in [10] where it is called mod-Gaussian convergence.

A precise form of the function \( \Phi_g \) was conjectured by Keating and Snaith and is based on calculations in the theory of random matrices (see [12], [13, formula (18)]). The arithmetic factor \( \Phi_a \) can be explained, e.g., by computing the characteristic function of \( \sum_{n \leq x} \Lambda(n)/(n^{1/2+iUT} \log n) \) (see [9, Theorem 2]), where \( x \) has to be \( O((\log T)^{2-\epsilon}) \) or of the corresponding stochastic model (replace \( \{p^{iUT}\}_{p \in \mathbb{P}} \) by an independent sequence of random variables uniformly distributed on the unit circle, see [13, Example 4]).

In this paper we further investigate the distribution of the sum over primes \( \sum_{p \leq x} p^{-1/2-\imath t} \) as \( t \) varies in \([T, 2T] \) and its consequences on the distribution of values of the Riemann zeta-function on the critical line. Here, we will restrict ourselves to the case of \( \Im \log \zeta(1/2 + i\imath t) \).

Note that some of the arguments cannot be applied to the case of \( \Re \log \zeta(1/2 + i\imath t) \). It is our first aim to establish mod-Gaussian convergence if \( x \) fulfills certain conditions. Precisely, in Sect. 4 we prove the following:

**Theorem 1** Let \( x = e^{\log T / N} \) and \( N \) such that \( x \rightarrow \infty \) and \( N / \log \log T \rightarrow \infty \) as \( T \rightarrow \infty \). Then

\[
e^{u^2(\log \log x + \gamma)/4} \frac{1}{T} \int_T^{2T} e^{\imath u \sum_{p \leq x} \frac{\sin(\imath t \log p)}{\sqrt{p}}} \, dt \rightarrow \Phi(u) \quad \text{as} \ T \rightarrow \infty \quad (1.3)
\]

locally uniformly for \( u \in \mathbb{R} \). Here, \( \gamma \) denotes Euler’s constant and \( \Phi \) is the analytic function given by

\[
\Phi(u) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p}\right)^{-u^2/4} J_0\left(\frac{u}{\sqrt{p}}\right),
\]

where \( J_0 \) denotes the zeroth Bessel function (see, e.g., Sect. 3).

One interesting point of the result seems to be the size of \( x \). It can be chosen large enough to obtain Selberg’s central limit theorem with Selberg’s explicit error term (see [19, Theorem 2] and “Appendix 1”). Moreover, we obtain the following improvement of (1.1):
Corollary 1 Assume RH. For \( T \) sufficiently large, we have

\[
\frac{1}{T} \int_0^{2T} e^{i\frac{v}{\sqrt{\log \log T}} \log \log \log T} dt = e^{-v^2/2} + v^2 O\left(\frac{\log \log \log T}{\log \log T}\right) + O(1/\log T)
\]

uniformly for \(|v| \leq \sqrt{\log \log T/\log \log \log T}\).

In Sect. 5 we deal with the question if the convergence in Theorem 1 can be extended to the complex plane. Assuming the Riemann hypothesis, we prove such a result for a weighted sum over primes.

Theorem 2 Assume RH. Let \( x = e^{\log T/N} \) and \( N \) such that \( x \to \infty \) and \( N/\log \log T \to \infty \) as \( T \to \infty \). Furthermore, let \( f \) be the function \( f(u) = (\pi u/2) \cot(\pi u/2) \) and \( \gamma_f = -0.1080 \ldots \) be the constant defined by \( \prod_{p \leq x} (1 - f^2(\log p/\log x)/p) = (e^{-\gamma_f}/\log x)(1 + o(1)) \). Then

\[
e^{-(log \log x + \gamma_f)/4} \frac{1}{T} \int_0^{2T} e^{i \sum_{p \leq x} \sin\left(\frac{\log p}{\sqrt{p}}\right) \log \log \log T} dt \to \Phi(z) \text{ as } T \to \infty
\]

locally uniformly for \( z \in \mathbb{C} \), where \( \Phi \) is given by (1.4).

More general sums are possible as well (see [8, Lemma 1] and [2, Lemma 1]). For the evaluation of \( \gamma_f \) see [8, proof of Lemma 6].

The crucial step from Theorem 1 to Theorem 2 is an estimate of the exponential moments of the above sum. For this purpose let \( x \leq T^2 \) and \( h \in \mathbb{R} \). Assuming the Riemann hypothesis, we then show that there exist constants \( C, C' \), and \( C'' \) such that

\[
\frac{1}{T} \int_0^{2T} e^{h \sum_{n \leq x} \frac{\Delta(n) \sin(\log n)}{\sqrt{n}} \log \log \log T} dt \leq C' e^{C|h| \log T/\log x + C'' h^2 \log \log T}.
\]

Note that this inequality, which is almost a subgaussian bound, is valid beyond the range which is contained in Theorems 1 and 2.

We turn to the applications of Theorem 2. As described above, Theorem 1 can be used to obtain results in connection with the central limit theorem. In addition, Theorem 2 yields large deviations results. Applying the Gärtner-Ellis theorem and Theorem 2, one obtains a large deviation principle (see [4, chapter 1.2] or “Appendix 3” for the definition of the large deviation principle) from which we will deduce the following two Corollaries.

Corollary 2 Assume RH. Let \( U_T \) be random variables uniformly distributed on \([T, 2T]\). Then the family \((1/((\log \log T)/2))\Im \log \zeta(1/2 + iT)\) satisfies the large deviation principle with the speed \( 1/((\log \log T)/2) \) and the rate function \( I(h) = h^2/2 \). For instance,

\[
\frac{1}{(\log \log T)^2} \log \left( \frac{1}{T} \lambda\left(\{t \in [T, 2T] : \Im \log \zeta(1/2 + it) \geq h(\log \log T)/2\}\right) \right) \to -h^2/2 \text{ as } T \to \infty,
\]

where \( h > 0 \) and \( \lambda \) denotes the Lebesgue measure.
Corollary 3 Assume RH. Let \( h \in \mathbb{R} \). Then
\[
\frac{1}{(\log \log T)/2} \log \left( \frac{1}{T} \int_{T}^{2T} e^{h \Im \log \zeta(1/2+it)} dt \right) \to h^2/2 \quad \text{as } T \to \infty.
\]

Related papers which also discuss large deviations results are the work of Radziwiłł [15], who extended the range of Selberg’s central limit theorem for \( \Re \log \zeta(1/2 + it) \) and the work of Soundararajan [20], who proved large deviation bounds for \( \Re \log \zeta(1/2 + it) \). In fact, Soundararajan [20, Corollary A] completed the proof of Corollary 3 in the case of \( \Re \log \zeta(1/2 + it) \) by proving the upper bound. The result can be stated as follows. For all \( \epsilon > 0 \) and all \( h > 0 \) we have \( (\log T)^{h^2-\epsilon} \ll h, \epsilon \int_{T}^{2T} |\zeta(1/2 + it)|^{2h} dt \ll h, \epsilon \) \( (\log T)^{h^2+\epsilon} \).

Note that the proof of the upper bound also applies to the case of \( \Im \log \zeta(1/2 + it) \) and that we apply a slightly weaker upper bound in the proofs of Theorem 2 and Corollary 3.

Notation For \( y \geq 2 \) and a function \( g : [0, 1] \to [0, 1] \), we define
\[
\Sigma_{g,y}(t) = \sum_{p \leq y} \frac{1}{p^{1/2+it}} g\left( \frac{\log p}{\log y} \right),
\]
\[
\Sigma^{*}_{g,y}(t) = \sum_{n \leq y} \frac{\Lambda(n)}{\log n} \frac{1}{n^{1/2+it}} g\left( \frac{\log n}{\log y} \right),
\]
\[
r_{g,y}(t) = \log \zeta(1/2 + it) - \Sigma_{g,y}(t), \quad \text{and} \quad r^{*}_{g,y}(t) = \log \zeta(1/2 + it) - \Sigma^{*}_{g,y}(t).
\]

2 Moments of a sum over primes

Section 2 is devoted to some standard mean value calculations. In doing so, we will apply the following generalization of the mean value theorem of Montgomery and Vaughan contained in [16, Theorem 1.4.3] (see also [21, Lemma 3.1]). Let \( a_1, \ldots, a_M \) and \( b_1, \ldots, b_M \) be complex numbers, \( M \geq 2 \), and let \( T > 0 \). Then
\[
\frac{1}{T} \int_{T}^{2T} \left( \sum_{m \leq M} a_m m^{-it} \right) \left( \sum_{m \leq M} b_m m^{-it} \right) dt = \sum_{m \leq M} a_m b_m + \theta \frac{2D}{T} \sqrt{\sum_{m \leq M} m |a_m|^2} \sqrt{\sum_{m \leq M} m |b_m|^2},
\]
where \( \theta \) depends on the various parameters but satisfies \( |\theta| \leq 1 \) and \( D \) is the universal constant in [16, Theorem 1.4.3].

Proposition 1 Let \( x \geq 2 \) and \( T > 0 \) be real numbers, \( k \) be a nonnegative integer, and \( p_1, \ldots, p_n \) be the prime numbers not exceeding \( x \). Then
\[
\frac{1}{T} \int_{T}^{2T} \left( \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^{2k} dt = \frac{1}{2^k} \binom{2k}{k} \sum_{\lambda_1 + \cdots + \lambda_n = k} \left( \frac{k!}{\lambda_1! \cdots \lambda_n!} \right)^2 p_1^{-\lambda_1} \cdots p_n^{-\lambda_n} + \theta \frac{2D}{T} \sqrt{n^{2k}(2k)!}
\]
and \(|(1/T) \int_T^{2T} (\sum_{p \leq x} \sin(t \log p)/\sqrt{p})^{2k+1} dt| \leq (2D/T)\sqrt{n^{2k+1}(2k+1)!} \) with \(|\theta| \leq 1 \) and \(D\) the constant in (2.1). Furthermore, the main term in (2.2) is bounded by \((2k)!/2^{2k}k!(\sum_{p \leq x} 1/p)^k\).

Proof From \(\sin(t \log p) = (p^i - p^{-i})/2i\), we obtain

\[
\frac{1}{T} \int_T^{2T} \left( \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^k dt = \frac{1}{(2i)^k} \sum_{j=0}^{k} (k)! (-1)^j \int_T^{2T} \left( \sum_{p \leq x} \frac{1}{p^{1/2+it}} \right)^j \left( \sum_{p \leq x} \frac{1}{p^{1/2+it}} \right)^{k-j} dt. \tag{2.3}
\]

For \(j = 1, \ldots, k\) the multinomial theorem yields

\[
\left( \sum_{p \leq x} \frac{1}{p^{1/2+it}} \right)^j = \sum_{\lambda_1 + \cdots + \lambda_n = j} j! \frac{1}{\lambda_1! \cdots \lambda_n!} (p_1^{-\lambda_1} \cdots p_n^{-\lambda_n})^{1/2+it}. \tag{2.4}
\]

If we plug in (2.4) into (2.3) with \(k\) replaced by \(2k\), we obtain from (2.1) that

\[
\frac{1}{T} \int_T^{2T} \left( \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^{2k} dt = \frac{1}{2^{2k}k!} \sum_{\lambda_1 + \cdots + \lambda_n = k} \frac{k!}{\lambda_1! \cdots \lambda_n!} \frac{2}{p_1^{-\lambda_1} \cdots p_n^{-\lambda_n}} + \frac{\theta 2D}{2^{2k}T} \sum_{j=0}^{2k} \frac{1}{(2k)!} \sum_{\lambda_1 + \cdots + \lambda_n = j} j! \frac{1}{\lambda_1! \cdots \lambda_n!} \frac{2}{p_1^{-\lambda_1} \cdots p_n^{-\lambda_n}} \frac{(2k-j)!}{(\lambda_1! \cdots \lambda_n!)} \tag{2.5}
\]

with \(|\theta| \leq 1\). Applying \(j!/(\lambda_1! \cdots \lambda_n!) \leq j!\), \(j = 0, \ldots, 2k\), we bound the absolute value of the remainder by

\[
\frac{2D}{2^{2k}T} \sum_{j=0}^{2k} \left( \frac{2k}{j} \right) \sqrt{n^j j! n^{2k-j}(2k-j)!} \leq \frac{2D}{T} \sqrt{n^{2k}(2k)!}. \tag{2.6}
\]

The main term in (2.2) can be bounded similarly. As in (2.5) and (2.6), we also bound the \((2k+1)\)th moment. Note that there is no main term in this case. This completes the proof.

We want to compare these mean value estimates to some random variables expectations. Therefore, let \(X_1, X_2, \ldots\) be an i.i.d. sequence of random variables uniformly distributed on the unit circle and let \(p_1, \ldots, p_n\) be the primes not exceeding \(x\). Then

\[
\mathbb{E} \left[ \left( \sum_{i=1}^n \frac{\text{Im} X_i}{\sqrt{p_i}} \right)^{2k} \right] = \frac{1}{2^{2k}k!} \sum_{\lambda_1 + \cdots + \lambda_n = k} \frac{k!}{\lambda_1! \cdots \lambda_n!} \frac{2}{p_1^{-\lambda_1} \cdots p_n^{-\lambda_n}} \tag{2.7}
\]

and \(\mathbb{E} \left[ (\sum_{i=1}^n \text{Im} X_i / \sqrt{p_i})^{2k+1} \right] = 0\). To prove this, we replace \(\sin(t \log p)\) by \(\text{Im} X_i\) and integration by expectation in (2.3) and (2.4) and then apply the formula \(\mathbb{E}[X_1^{\lambda_1} \cdots X_n^{\lambda_n} X_1^{-\mu_1} \cdots X_n^{-\mu_n}] = 1\) if \(\lambda_j = \mu_j\) for all \(j = 1, \ldots, n\) and \(= 0\) else.
3 Bessel functions

The Bessel functions appear in the Fourier expansion of the function $e^{iz \sin \theta}$,

$$e^{iz \sin \theta} = \sum_{k=-\infty}^{\infty} J_k(z) e^{ik\theta}.$$  \hfill (3.1)

Explicitly the $k$th Bessel function $J_k(z)$ is given by

$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{k+2n}}{n!(k+n)!}$$  \hfill (3.2)

for $k \geq 0$ and given by the relation $J_k(z) = (-1)^k J_{-k}(z)$ for $k < 0$ (for these and more facts about Bessel functions see, e.g., the book of Andrews, Askey and Roy [1]). This section is devoted to the following mod-Gaussian convergence result (compare to [10, Proposition 4.1]).

**Proposition 2** Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables uniformly distributed on the unit circle and let $p_1, p_2, \ldots$ be the increasing sequence of all primes. Then

$$e^{z^2 (\log \log x + \gamma)/4} \mathbb{E} \left[ e^{i z \sum_{j=1}^{\pi(x)/\sqrt{p_j}} \Im X_j} \right] \to \Phi(z) \quad \text{as } x \to \infty$$  \hfill (3.3)

locally uniformly for $z \in \mathbb{C}$. Here, $\gamma$ denotes Euler’s constant, $\pi(x)$ denotes the number of primes not exceeding $x$, and $\Phi(z)$ is given by (1.4).

**Proof** By (3.1), we have

$$\mathbb{E}[e^{iz \Im X_1}] = \frac{1}{2\pi} \int_0^{2\pi} e^{iz \sin \theta} d\theta = J_0(z).$$  \hfill (3.4)

Applying the independence of the $X_j$’s, (3.4), and finally Mertens’ formula $\prod_{p \leq x} (1 - 1/p) = (e^{-\gamma}/\log x)(1 + o(1))$, we obtain that the left hand side of (3.3) is equal to

$$e^{z^2 (\log \log x + \gamma)/4} \prod_{p \leq x} J_0\left(\frac{z}{\sqrt{p}}\right) = (1 + o(1)) z^2/4 \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-z^2/4} J_0\left(\frac{z}{\sqrt{p}}\right).$$

It remains to show that the above product converges to $\Phi(z)$, locally uniformly for $z \in \mathbb{C}$. This follows from the fact that the product $\Phi(z)$ is normally convergent (see [6, Chapter IV.1, especially Remark IV.1.7]). This completes the proof. \hfill \qed

Consider the random variables $\Im \Sigma_{1,x}(-UT)$, $UT$ being random variables uniformly distributed on $[T, 2T]$. Note that we computed their moments in Proposition 1. As mentioned in the Introduction, one can use the method of moments to deduce that, as $x \to \infty$, $x = T^{o(1)}$, $(1/\sqrt{(\log \log x)/2})\Im \Sigma_{1,x}(-UT)$ converges in distribution to a Gaussian random variable with expectation 0 and variance 1 (see [2, Proof of Theorem B]). We will generalize this result by considering the cumulants of $\Im \Sigma_{1,x}(-UT)$.

If $Y$ is a real random variable such that $\mathbb{E}[e^{zY}]$ exists and is finite for all $z \in \mathbb{C}$, $\mathbb{E}[e^{zY}]$ is an analytic function and there exists a neighbourhood of 0 where $\log \mathbb{E}[e^{zY}] = \sum_{m=1}^{\infty} \kappa_m(Y) z^m / m!$. The coefficients $\kappa_m(Y)$, $m \geq 1$, are called the cumulants of $Y$. Thus, $\kappa_m(Y)$ is equal to the $m$th derivative of $\log \mathbb{E}[e^{zY}]$ evaluated at 0.
Corollary 4 Let \( x = e^{\log T/N} \) and \( N \) such that \( x \to \infty \) and \( N \to \infty \) as \( T \to \infty \) and let \( U_T \) be random variables uniformly distributed on \([T, 2T]\). Then, as \( T \to \infty \), 
\[ \kappa_2(\Im \Sigma_{1,x}(-U_T)) = \log(\log x + \gamma)/2 \to c_2 \] and for \( m \neq 2 \) \( \kappa_m(\Im \Sigma_{1,x}(-U_T)) = c_m, \) where the \( c_m \)'s are defined by the series expansion \( \log \Phi(-iz) = \sum_{m=1}^{\infty} c_m z^m / m! \), for \( z \) in a neighbourhood of 0.

Proof By the construction of \( \Phi \), there exists a real number \( 0 < r \leq 1 \) such that for \( |z| \leq r \), \( \log \Phi(z) = \sum_{p \in \mathbb{P}} (-z^2/4) \log(1 - 1/p) + \log J_0(z/\sqrt{p}) \). Hence, by Merten’s formula,
\[
(-z^2/4)(\log \log x + \gamma) + \sum_{p \leq x} \log J_0\left(-iz \sqrt{p}\right) \to \log \Phi(-iz) \text{ as } x \to \infty \tag{3.5}
\]
uniformly for \( |z| \leq r, z \in \mathbb{C} \). The uniform convergence implies (see [6, Theorem III.1.3]), that the \( m \)th derivative of the left hand side of (3.5) evaluated at 0 converges to \( c_m \). Hence, under the assumptions of Proposition 2, the cumulants of \( \sum_{j=1}^{\pi(x)} \Im X_j / \sqrt{p_j} \) satisfy the convergence described in Corollary 4, since \( \mathbb{E}[\exp(z \sum_{j=1}^{\pi(x)} \Im X_j / \sqrt{p_j})] = \prod_{p \leq x} \log J_0(-iz / \sqrt{p}) \).

It remains to show that for \( m \geq 1 \)
\[
\kappa_m \left( \Im \Sigma_{1,x}(-U_T) \right) - \kappa_m \left( \sum_{j=1}^{\pi(x)} \Im X_j / \sqrt{p_j} \right) \to 0 \text{ as } T \to \infty. \tag{3.6}
\]

To prove this, we use the fact that the cumulants can be expressed in terms of the moments, namely \( \kappa_m(Y) = \sum_{\lambda_1, \ldots, \lambda_m} \mathbb{E}[Y^{\lambda_1} \cdots Y^{\lambda_m}] \lambda_m \), where the sum is over all positive integers such that \( 1 \lambda_1 + 2 \lambda_2 + \cdots + m \lambda_m = m \), \( \lambda_1, \ldots, \lambda_m \) are integers, and \( Y \) is a random variable as above. If we plug in Proposition 1 and (2.7) into this formula, (3.6) follows from multiplying out since for \( k \leq m \) and \( x \geq 3 \) the main terms in (2.2) are \( O((\sum_{p \leq x} 1/p)^m) = O((\log \log x)^m) \) (see [3, (5) of chapter 7]), while for \( k \leq m \) the remainders in (2.2) are \( O(T^{(m/N)-1}) \) which is \( O(T^{-a}) \) for some \( 0 < a < 1 \) if \( T \) is sufficiently large. \( \square \)

4 Mod-convergence of a sum over primes

By means of Proposition 1 and (2.7), we can apply the method of moments for fixed \( x \) and obtain the following convergence
\[
\frac{1}{T} \int_T^{2T} e^{iu \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \, dt} \to \prod_{p \leq x} \log \left( \frac{u}{\sqrt{p}} \right) \text{ as } T \to \infty. \tag{4.1}
\]

Another proof of (4.1) is contained in [14, Theorem 5.1]. The techniques used therein can be applied to get Theorem 1 and Theorem 2 for the choice \( x = (\log T)^{2-\epsilon}, \epsilon > 0 \) arbitrary. The improvement of Theorem 1 follows from Proposition 3 combined with Proposition 2.

Proposition 3 Let \( c > 1 \) be a constant. Define \( x = e^{\log T/N} \) with \( N = (c' \log^2 / 4) \log \log T \), where \( c' > 1 \) is allowed to depend on \( T \) but such that \( x \to \infty \) as \( T \to \infty \). For \( T \geq 3 \), sufficiently large such that \( x \geq 2 \) and \( N \geq 1 \), we have
\[
\frac{1}{T} \int_T^{2T} e^{iu \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \, dt} = \prod_{p \leq x} \log \left( \frac{u}{\sqrt{p}} \right) + O\left((1/c')^{N-1} + (2c^2 / \log x)^N \right) \tag{4.2}
\]
uniformly for \( |u| \leq c, u \in \mathbb{R} \).
Proof of Theorem 1 We apply Proposition 3 with $x, N$ as in Theorem 1 and $c$ an arbitrary constant with $c > 1$. Since $c' \to \infty$ in that case, the remainder in (4.2) is $o(\exp(-c^2(\log \log T)/4)).$ If we multiply in (4.2) both sides by $\exp(2u^2(\log \log x + \gamma)/4)$ and then apply Proposition 2, we obtain (1.3) uniformly for $|u| \leq c$. Since $c > 1$ is arbitrary, this completes the proof.

Proof of Proposition 3 Let $N' = \lfloor N \rfloor$. From the Taylor expansion $e^{iu} = \sum_{k \leq 2N' - 1} (iu)^k/k! + \theta u^{2N'}/(2N')!$, $u \in \mathbb{R}$, with $|\theta| \leq 1$, we obtain

$$\frac{1}{T} \int_T^{2T} e^{iu} \sum_{p \leq x} \frac{\sin(t \log p)}{p^{1/2}} dt = \sum_{k \leq 2N' - 1} \frac{(iu)^k}{k!} \frac{1}{T} \int_T^{2T} \left( \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^k dt + \theta \frac{u^{2N'}/(2N')!}{T} \int_T^{2T} \left( \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^{2N'} dt (4.3)$$

with $|\theta| \leq 1$. By Proposition 1, the remainder is

$$O \left( \frac{c^{2N'}}{N!} \frac{1}{2N'} \left( \sum_{p \leq x} \frac{1}{p} \right)^{N'} + \frac{(c^2 \pi(x))^{N'}}{T} \right).$$

Using the bound $(N')! \geq (N'/e)^{N'}$, elementary results in the theory of primes, namely the formulas $\sum_{p \leq x} 1/p = \log \log x + c_1 + O(1/\log x)$ and $\pi(x) \leq 2x/\log x$, and finally $N' = \lfloor N \rfloor$, this is

$$O \left( \left( \frac{e e^2 \log \log T}{4N'} \right)^N + \frac{(c^2 \pi(x))^N}{T} \right) = O \left( \left( \frac{1}{c'} \right)^{N-1} + \frac{(2e^2/\log x)^N}{T} \right).$$

Now, let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables uniformly distributed on the unit circle. By Proposition 1 and (2.7), the moments in (4.3) are equal to those of the stochastic model plus a remainder which is bounded by $(2D/T) \sqrt{\pi(x)^N k!}$. The resulting remainders in (4.3), $k \leq 2N' - 1$, add up to $O((c^2 \pi(x))^N/T) = O((2e^2/\log x)^N)$. Hence, (4.3) is equal to

$$\sum_{k \leq 2N' - 1} \frac{(iu)^k}{k!} \left[ \left( \sum_{j=1}^{\pi(x)} \frac{\text{Im } X_j}{\sqrt{p_j}} \right)^k \right] + O \left( (1/c')^{N-1} + (2e^2/\log x)^N \right).$$

Applying the above Taylor expansion again, we obtain

$$\prod_{p \leq x} J_0 \left( \frac{u}{\sqrt{p}} \right) = \mathbb{E} \left[ e^{iu} \sum_{j=1}^{\pi(x)} \frac{\text{Im } X_j}{\sqrt{p_j}} \right] = \sum_{k \leq 2N' - 1} \frac{(iu)^k}{k!} \left[ \left( \sum_{j=1}^{\pi(x)} \frac{\text{Im } X_j}{\sqrt{p_j}} \right)^k \right] + \theta \frac{u^{2N'}/(2N')!}{T} \mathbb{E} \left[ \left( \sum_{j=1}^{\pi(x)} \frac{\text{Im } X_j}{\sqrt{p_j}} \right)^{2N'} \right]$$

with $|\theta| \leq 1$. The remainder already appeared in (4.3) and is $O(1/c')^{N-1}$. This completes the proof. □

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5 Mod-convergence in the complex plane

Section 5 is devoted to the proof of Theorem 2. Here, we will apply an explicit formula obtained by Goldston [8, Lemma 1] assuming RH. For $4 \leq x \leq t^2$ and $t \neq \gamma$, we have

$$\text{Im} \log \zeta(1/2 + it) = - \sum_{n \leq x} \frac{\Lambda(n) \sin(t \log n)}{\sqrt{n}} f \left( \frac{\log n}{\log x} \right) + \sum_{\gamma} \sin((t - \gamma) \log x) \int_0^\infty \frac{u}{u^2 + ((t - \gamma) \log x)^2} \sinh u \, du + O \left( \frac{1}{t (\log x)^2} \right),$$

(5.1)

where $f(u) = (\pi u/2) \cot(\pi u/2)$. We will also apply the following estimate obtained by Soundararajan assuming RH. For every $h \in \mathbb{R}$ there exist constants $C'$, $C'' > 0$ such that

$$\frac{1}{T} \int_T^{2T} e^{h \text{Im} \log \zeta(1/2 + it)} \, dt \leq C'' e^{C'h^2 \log \log T}.$$

(5.2)

Soundararajan [20] proved (5.2) for Re $\log \zeta(1/2 + it)$. However, by using [17, Theorem 1] instead of [20, Proposition], his arguments apply to Im $\log \zeta(1/2 + it)$, too. We prove (compare to [2, Lemma 3 and Corollary]):

**Proposition 4** Assume RH. Let $4 \leq x \leq T^2$ and $f(u) = (\pi u/2) \cot(\pi u/2)$. For every $h \in \mathbb{R}$ there exist constants $C$, $C'$, and $C''$ such that

$$\frac{1}{T} \int_T^{2T} e^{h \sum_{n \leq x} \frac{\Lambda(n) \sin(t \log n)}{\sqrt{n}} f \left( \frac{\log n}{\log x} \right)} \, dt \leq C'' e^{C|h| \log T + C'h^2 \log \log T}.$$

Proof of Theorem 2 The proof mainly differs from the proof of Theorem 1 and Proposition 3 in its estimation of the remainder term. Nevertheless, we will repeat the main steps. We assume that $|z| \leq c$, where $c > 1$ is an arbitrary constant and $z \in \mathbb{C}$, say $z = u - ih$ with $u, h \in \mathbb{R}$. Let $N' = \lfloor N/2 \rfloor$. From the Taylor expansion $e^{iz} = e^{h + it} = \sum_{k \leq 2N'-1} (iz)^k/k! + \theta e^h (|z|^{2N'}/(2N')!)$ with $|\theta| \leq 1$, we obtain

$$\frac{1}{T} \int_T^{2T} e^{iz \text{Im} \Sigma_{f,x}(-t)} \, dt = \sum_{k \leq 2N'-1} \frac{(iz)^k}{k!} \frac{1}{T} \int_T^{2T} (\text{Im} \Sigma_{f,x}(-t))^k \, dt + \theta \frac{e^{2N'}}{(2N')!} \frac{1}{T} \int_T^{2T} e^{h \text{Im} \Sigma_{f,x}(-t)} (\text{Im} \Sigma_{f,x}(-t))^{2N'} \, dt$$

(5.3)

with $|\theta| \leq 1$. To continue as in the proof of Proposition 3, we use that under the assumptions of Proposition 1 we have

$$\frac{1}{T} \int_T^{2T} (\text{Im} \Sigma_{f,x}(-t))^k \, dt = \mathbb{E} \left[ \left( \sum_{j=1}^{\pi(x)} \frac{\text{Im} X_j}{\sqrt{p_j}} f \left( \frac{\log p_j}{\log x} \right) \right)^k \right] + \theta \frac{2D}{T} \sqrt{(\pi(x))^{2k/3}}$$

(5.4)
with $|\theta| \leq 1$. Moreover, if we replace $k$ by $2k$, the main term is bounded by 
$\left( (2k)!/2^{2k}k! \right) (\sum_{p \leq x} 1/p)^k$. These estimates follow as in the proof of Proposition 1 and (2.7).

Applying the Cauchy–Schwarz inequality, the absolute value of the remainder in (5.3) can be bounded by

$$
\frac{c^{2N'}}{(2N')!} \left| \frac{1}{T} \int_{T}^{2T} (\text{Im} \sigma_{f,x}(-t))^{2N'} dt \right|
$$

For $x \geq 2$, we have $|\text{Im} \sigma_{f,x}(-t) - \text{Im} \sigma_{f,x}^p(-t)| \leq (\log \log x)/2 + O(1)$, say $\leq CN$ for $T$ sufficiently large. Hence, by (5.4) and Proposition 4, the above is

$$
O \left( \frac{c^{2N'}}{(2N')!} \left| \frac{(4N')!(\sum_{p \leq x} 1/p)^{2N'}}{2^{4N'}(2N')!} \right| + \frac{\sqrt{(4N')!(\pi(x))^{4N'}}}{T} \right)
$$

for $T$ sufficiently large. Applying $(4N')/(2^{4N'}(2N')!) \leq (2N')!$, $(2N')! \geq (2N'/e)^{N'}$, $\pi(x) \leq 2x/\log x$, and $N' = [N/2]$, there exists a constant $c'' > 0$ (depending on $c$, $C$, and $C'$) such that this is

$$
O \left( \left( \frac{c'' \sum_{p \leq x} 1/p}{N'} \right)^{N'} + \left( \frac{c''}{\log x} \right)^{N/2} \right)
$$

Since $N'/\sum_{p \leq x} 1/p$ and $\log x$ tend to infinity, this is $o(\exp(-c^2(\log \log x)/4))$. Hence, applying (5.4) to the other terms, (5.3) is equal to

$$
\sum_{k \leq 2N'-1} \frac{(iz)^k}{k!} \left[ \pi(x) \left( \frac{\text{Im} X_j}{\sqrt{p_j}} f \left( \frac{\log p_j}{\log x} \right) \right)^k \right] + O(e^{-c^2(\log \log T)/4})
$$

uniformly for $|z| \leq c$. If we replace $\sin(t \log p)$ by $\text{Im} X_i$ and integration by expectation in (5.3), we can bound the resulting remainder as above. In doing so, we apply (5.5) instead of Proposition 4. The result is that

$$
\frac{1}{T} \int_{T}^{2T} e^{iz \text{Im} \sigma_{f,x}(-t)} dt = \mathbb{E} \left[ e^{iz \sum_{j=1}^{\pi(x)} \frac{\text{Im} X_j}{\sqrt{p_j}} f \left( \frac{\log p_j}{\log x} \right)} \right] + O(e^{-c^2(\log \log T)/4})
$$

uniformly for $|z| \leq c$. If we multiply both sides by $\exp(z^2(\log \log x + \gamma_f)/4)$ and then apply the formula

$$
e^{-z^2(\log \log x + \gamma_f)/4} \mathbb{E} \left[ e^{iz \sum_{j=1}^{\pi(x)} \frac{\text{Im} X_j}{\sqrt{p_j}} f \left( \frac{\log p_j}{\log x} \right)} \right] \rightarrow \Phi(z) \quad \text{as } x \rightarrow \infty \quad (5.5)
$$

locally uniformly for $z \in \mathbb{C}$, the statement of Theorem 2 follows by the same argument as in the proof of Theorem 1. (5.5) follows as in the proof of Proposition 2 by using the additional fact that

$$
\prod_{p \leq x} \left( 1 - \frac{1}{p} f^2 \left( \frac{\log p}{\log x} \right) \right)^{-z^2/4} J_0 \left( z \sqrt{p} f \left( \frac{\log p}{\log x} \right) \right) \rightarrow \Phi(z) \quad \text{as } x \rightarrow \infty
$$
locally uniformly for \( z \in \mathbb{C} \). We conclude by a brief argument why this holds. Split the product in \( p \leq y \) and \( y < p \leq x \). The product over \( y < p \leq x \) converges locally uniformly to 1 if \( y \to \infty \), while one can show that the product over \( p \leq y \), say \( y = \log x \), converges locally uniformly to \( \phi(z) \), by using, e.g., \( f^2(\log p / \log x) - 1 = O((\log \log x) / \log x) \) if \( p \leq \log x \). This completes the proof.

**Proof of Proposition 4** From (5.1), (5.2), and the Cauchy–Schwarz inequality, we obtain

\[
\frac{1}{T} \int_{-T}^{T} e^{h \sum_{n \leq x} A(n) \sin(t \log n) - \log n} f\left(\frac{\log n}{\log x}\right) dt \leq C'''' e^{2C'h^2 \log \log T} \sqrt{\frac{1}{T} \int_{-T}^{T} e^{2h \sum_{y < p \leq x} \sin((t-\gamma) \log x) \int_{0}^{\infty} f^{\infty}(\frac{u}{u^2 + (t-\gamma) \log x}) \frac{du}{u^2 + (t-\gamma) \log x} dt} \right]
\]

where \( C'''' \) is a constant. The absolute value of the sum over zeros is bounded by a constant times

\[
\sum_{|((t-\gamma) \log x| \leq 1} 1 + \sum_{|((t-\gamma) \log x| > 1} \frac{1}{((t-\gamma) \log x)^2}
\]

and therefore it suffices to deal with the exponential moments of (5.6) with \( h \geq 0 \). Using the Cauchy–Schwarz inequality again, we obtain

\[
\frac{1}{T} \int_{-T}^{T} e^{h \sum_{|((t-\gamma) \log x| \leq 1} 1 + h \sum_{|((t-\gamma) \log x| > 1} \frac{1}{((t-\gamma) \log x)^2} dt \leq \left[ \frac{1}{T} \int_{-T}^{T} e^{2h \sum_{|((t-\gamma) \log x| \leq 1} 1 dt \right] \left[ \frac{1}{T} \int_{-T}^{T} e^{2h \sum_{|((t-\gamma) \log x| > 1} \frac{1}{((t-\gamma) \log x)^2}} dt. \right.
\]

We start with the first term, using the following fact on the number of zeros (see [3, (1) of Ch. 15])

\[
N(t) = \frac{t}{2\pi} \log t - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O(1/t),
\]

where \( t \neq \gamma \) and \( S(t) = (1/\pi) \text{Im} \log \xi(1/2 + it) \). We compute (note that \( h \geq 0 \))

\[
\frac{1}{T} \int_{-T}^{T} e^{h \sum_{|((t-\gamma) \log x| \leq 1} 1 dt = \frac{1}{T} \int_{-T}^{T} e^{h \left( N(t + \frac{1}{\log x}) - N(t - \frac{1}{\log x}) \right) dt} \leq \frac{1}{T} \int_{-T}^{T} e^{Ch \log t + h \left( S(t + \frac{1}{\log x}) - S(t - \frac{1}{\log x}) \right) dt}
\]

\[\square\]
\[ \leq e^{Ch \log \frac{T}{\log x}} \sqrt{\frac{1}{T}} \int_{T}^{2T} e^{2hS(t + \frac{1}{\log x})} dt \sqrt{\frac{1}{T}} \int_{T}^{2T} e^{-2hS(t - \frac{1}{\log x})} dt \]

\[ = O \left( e^{Ch \log \frac{T}{\log x} + 4C' h/\pi)^2 \log \log T} \right). \]  

(5.8)

In the last step we used (5.2). Next, we divide the sum over \(|(t - \gamma) \log x| > 1\) into \(|t - \gamma| \geq T\), \(1 < |t - \gamma| < T\), and \(1/ \log x < |t - \gamma| \leq 1\).

For \(t \in [T, 2T]\), we have

\[ \sum_{|t - \gamma| \geq T} \frac{1}{((t - \gamma) \log x)^2} = O \left( \sum_{\gamma} \frac{1}{\gamma^2(\log x)^2} \right) = O \left( \frac{1}{(\log x)^2} \right). \]

The last step results from [3, (4) of Ch. 12]. For the second sum we use the fact that \(N(t + 1) - N(t) = O(1 + \log^{+} |t|)\) (see [3, (2) of Ch. 15]). For \(t \in [T, 2T]\), we obtain

\[ \sum_{1 < |t - \gamma| < T} \frac{1}{((t - \gamma) \log x)^2} \leq \sum_{k=1}^{[T] - 1} \frac{N(t + k + 1) - N(t + k)}{k^2(\log x)^2} + \sum_{k=1}^{[T] - 1} \frac{N(t - k) - N(t - k - 1)}{k^2(\log x)^2} \]

\[ = O \left( \sum_{k=1}^{[T] - 1} \frac{\log T}{k^2(\log x)^2} \right) = O \left( \frac{\log T}{(\log x)^2} \right). \]

Next, we consider the sum over \(1/ \log x < \gamma - t \leq 1\). We have

\[ \sum_{1/ \log x < \gamma - t \leq 1} \frac{1}{((t - \gamma) \log x)^2} \leq \sum_{j=1}^{M} \frac{N \left( t + \frac{k_j x}{\log x} \right)}{k_j^2} - N \left( t + \frac{k_{j-1} x}{\log x} \right) \]

(5.9)

where \(1 = k_0 < k_1 < \cdots < k_M\) with \(k_M - 1 < \log x \leq k_M\). By (5.7), this is bounded by, recall \(t \in [T, 2T]\),

\[ \sum_{j=1}^{M} \left( \frac{C(k_j - k_{j-1}) \log T}{(\log x)k_j^2} + \frac{S \left( t + \frac{k_j x}{\log x} \right) - S \left( t + \frac{k_{j-1} x}{\log x} \right)}{k_j^2} \right). \]

We choose \(k_j = 2^{j/2}\) and bound the left hand side of (5.9) by

\[ \sqrt{2C} \frac{\log T}{\log x} + \sum_{j=1}^{M} \frac{S \left( t + \frac{2^{j/2}}{\log x} \right) - S \left( t + \frac{2^{(j-1)/2}}{\log x} \right)}{2^{j-1}}. \]

It follows that

\[ \frac{1}{T} \int_{T}^{2T} e^{h \sum_{1/ \log x < \gamma - t \leq 1} \frac{1}{((t - \gamma) \log x)^2}} dt \]

\[ \leq e^{\sqrt{2C} \log \frac{T}{\log x} + \frac{1}{T} \int_{T}^{2T} e^{\frac{h}{2^{j-1}} \left( S \left( t + \frac{2^{j/2}}{\log x} \right) - S \left( t + \frac{2^{(j-1)/2}}{\log x} \right) \right)} dt. \]
Using \( \mathbb{E}[e^{h\sum_{j=1}^{M}X_j/2}] \leq \prod_{j=1}^{M}(\mathbb{E}[e^{hX_j}])^{1/2} \), which follows from repeated application of the Cauchy–Schwarz inequality, this is

\[
\leq e^{\sqrt{2}Ch_{\log T}/\log x} \prod_{j=1}^{M} \left( \frac{1}{T} \int_{T}^{2T} e^{2h(S(t+\frac{y^j/2}{\log x})-S(t+\frac{y^{j+1}/2}{\log x}))} dt \right)^{1/2}.
\]

Applying again the Cauchy–Schwarz inequality and then (5.2) [as in (5.8)], this is

\[
O \left( e^{\sqrt{2}Ch_{\log T}/\log x} e^{16C(h/\pi)^2 \log \log T} \right).
\]

The same bound is true for the sum over \( 1/\log x < t - \gamma \leq 1 \). The claim now follows from putting together all these estimates. \( \square \)

6 Proof of Corollary 1

Let \( T, c, c', x, \) and \( N \) be as in Proposition 3, \( T \geq 3 \) sufficiently large such that \( x \geq 2 \) and \( N \geq 2 \). Assume further that \( c' > 4 \) is a constant such that the bound \( (\log \log T)^{1/2}(c'/4)^{-N/2} = O(1/\log T) \) holds and that \( T \) is so big that the bound \( (\log T)(2e^2/\log x)^{N/2} = O(1/\log T) \) holds, too. Then we show that

\[
\frac{1}{T} \int_{T}^{2T} e^{iu \text{Im} \log \xi(1/2+it)} dt = \prod_{p \leq x} J_0 \left( \frac{u}{\sqrt{p}} \right) - \sum_{p \leq x} \frac{u}{k\sqrt{p^k}} J_k \left( \frac{u}{\sqrt{p^k}} \right) \prod_{q \leq x, q \neq p} J_0 \left( \frac{u}{\sqrt{q}} \right) + u^2 O(\log \log T) + O(1/\log T)
\]

(6.1)

uniformly for \( |u| \leq c, u \in \mathbb{R} \). One can deduce Corollary 1 from (6.1) as follows. Replace \( u \) by \( u/\sqrt{\log \log T}/2 \) with \( |v| \leq \sqrt{\log \log T}/\log \log T \) and let \( T \) be sufficiently large. Then, by (3.2) and the formulas \( \sum_{p \leq x} 1/p = \log \log x + c_1 + O(1/\log x) \) and \( \log x/\log T = 1 + O(\log \log T/\log \log T) \), the first term on the right side of (6.1) is equal to

\[
\exp \left( \sum_{p \leq x} \log J_0 \left( \frac{v}{\sqrt{p(\log \log T)/2}} \right) \right) = e^{-v^2/2} \left( 1 + v^2 O \left( \frac{\log \log T}{\log \log T} \right) \right)
\]

and, by using \( |J_k(u)| \leq (|u|/2)^k/k! \) and \( |J_0(u)| \leq 1, u \in \mathbb{R} \), the second term is \( v^4 O(1/(\log \log T)^2) \) which is smaller than \( v^2 O(\log \log T^2) \).

Hence, it remains to prove (6.1). From \( \text{Im} \log \xi(1/2 + it) = \text{Im} \Sigma_{1,x}(t) + \text{Im} r_{1,x}(t) \) and Taylor’s theorem, we obtain

\[
\frac{1}{T} \int_{T}^{2T} e^{iu \text{Im} \log \xi(1/2+it)} dt = \frac{1}{T} \int_{T}^{2T} e^{iu \text{Im} \Sigma_{1,x}(t)} dt + iu \frac{1}{T} \int_{T}^{2T} \text{Im} r_{1,x}(t) e^{iu \text{Im} \Sigma_{1,x}(t)} dt
\]

\[
+ \frac{\theta \cdot u^2}{2} \frac{1}{T} \int_{T}^{2T} (\text{Im} r_{1,x}(t))^2 dt
\]
with $|\theta| \leq 1$. By Proposition 3 and the above assumptions, the first term is equal to $\prod_{p \leq x} J_0(u/\sqrt{p}) + O(1/\log T)$ and by [21, Corollary of Theorem 5.1], the third term is $u^2 O(\log \log \log T)$. It remains to consider the second term. We start showing that

$$
\frac{1}{T} \int_T^{2T} \Im \log \zeta(1/2 + it) e^{iu \Im \Sigma_{1,x}(t)} \, dt
$$

$$
= \sum_{p \leq x} \frac{i}{k \sqrt{p}} J_k \left( \frac{u}{\sqrt{p}} \right) \prod_{q \leq x} J_0 \left( \frac{u}{\sqrt{q}} \right) + O(1/\log T)
$$

(6.2)

uniformly for $|u| \leq c$. Let $N' = \lfloor N/2 \rfloor$. From the Taylor expansion $e^{iu} = \sum_{k \leq 2N'-1} (iu)^k/k! + \theta u^{2N'}/(2N')!$, $u \in \mathbb{R}$, with $|\theta| \leq 1$, we obtain that the left hand side of (6.2) is equal to

$$
\sum_{k \leq 2N'-1} \frac{(iu)^k}{k!} \frac{1}{T} \int_T^{2T} \Im \log \zeta(1/2 + it)(\Im \Sigma_{1,x}(t))^k \, dt
$$

$$
+ \theta \frac{e^{2N'}/(2N')!}{T} \int_T^{2T} |\Im \log \zeta(1/2 + it)(\Im \Sigma_{1,x}(t))^{2N'}| \, dt
$$

(6.3)

with $|\theta| \leq 1$. Applying the Cauchy–Schwarz inequality, the estimates in the proof of Proposition 3, and $(1/T) \int_T^{2T} (\Im \log \zeta(1/2 + it))^2 \, dt = (\log \log T)/2 + O(1)$ (see [18, Theorem 3]), the remainder is $O((\log \log T)^{1/2}((c'/4)^{-N'/2} + (2c^2/\log x)^{N'/2}) = O(1/\log T)$. The remaining moments can be computed by using the following lemma which is a modification of [17, Lemma 5] and [8, equation (6.3)] and serves as a substitute for the mean value theorem of Montgomery and Vaughan in Sect. 2.

**Lemma 1** Assume RH. Let $k, h \leq T$ be two positive integers with $(k, h) = 1$. Then

$$
\int_T^{2T} \log \zeta(1/2 + it) \left( \frac{k}{h} \right)^{it} \, dt = \frac{T \Lambda(k)}{\sqrt{k} \log k} + O(\sqrt{kh} \log T), \quad h = 1
$$

$$
O(\sqrt{kh} \log T), \quad h \neq 1,
$$

$$
\int_T^{2T} \Im \log \zeta(1/2 + it) \left( \frac{k}{h} \right)^{it} \, dt = \frac{-iT \Lambda(k)}{2\sqrt{k} \log k} + O(\sqrt{kh} \log T), \quad h = 1
$$

$$
= \frac{iT \Lambda(h)}{2\sqrt{h} \log h} + O(\sqrt{kh} \log T), \quad k = 1
$$

$$
O(\sqrt{kh} \log T), \quad h, k \neq 1.
$$

(6.4)

Denote by $p_1, p_2, \ldots, p_n$ the prime numbers not exceeding $x$. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables uniformly distributed on the unit circle. Furthermore, let $k, h \leq T$ be positive integers with $k/h = p_1^{-k_1} \ldots p_n^{-k_n}$. Then (6.4) can be written as

\[ \text{Springer} \]
\[ \frac{1}{T} \int_T^{2T} \Im \log \zeta(1/2 + it) \left( p_{1}^{-k_1} \cdots p_{n}^{-k_n} \right)^it \, dt \]

\[ = \mathbb{E} \left[ - \sum_{j=1}^{n} \Im \log(1 - X_j/\sqrt{p_j}) X_j^{k_1} \cdots X_j^{k_n} \right] + O \left( \frac{1}{T} \sqrt{p_1^{k_1} \cdots p_n^{k_n}} \log T \right) . \]  

(6.5)

Expanding \((\Im \Sigma_{1,x}(t))^k\) as in (2.3) and (2.4), we deduce from (6.5) that

\[ \frac{1}{T} \int_T^{2T} \Im \log \zeta(1/2 + it)(\Im \Sigma_{1,x}(t))^k \, dt \]

\[ = \mathbb{E} \left[ \left( - \sum_{j=1}^{n} \Im \log(1 - X_j/\sqrt{p_j}) \right) \left( \sum_{j=1}^{n} \frac{\Im X_j}{\sqrt{p_j}} \right)^k \right] \]

\[ + O \left( \frac{\log T}{2^k T} \sum_{k=0}^{k} \binom{k}{l} \frac{l!}{\lambda_1! \cdots \lambda_n!} \sum_{\lambda_1 + \cdots + \lambda_n = k-l} \frac{(k-l)!}{\lambda_1! \cdots \lambda_n!} \right) . \]

The remainder is \(O((\log T)n^k/T)\) and the resulting remainders in (6.3), \(k \leq 2N' - 1\), add up to \(O((\log T)(2c/\log x)^N) = O(1/\log T)\). Hence, (6.3) is equal to

\[ \sum_{k \leq 2N' - 1} \frac{(iu)^k}{k!} \mathbb{E} \left[ \left( - \sum_{j=1}^{n} \Im \log(1 - X_j/\sqrt{p_j}) \right) \left( \sum_{j=1}^{n} \frac{\Im X_j}{\sqrt{p_j}} \right)^k \right] + O(1/\log T) \]

\[ = \mathbb{E} \left[ \left( - \sum_{j=1}^{n} \Im \log(1 - X_j/\sqrt{p_j}) \right) e^{iu \sum_{j=1}^{n} \Im X_j/\sqrt{p_j}} \right] \]

\[ + \theta \frac{e^{2N'}}{(2N')!} \mathbb{E} \left[ \left( - \sum_{j=1}^{n} \Im \log(1 - X_j/\sqrt{p_j}) \right) \left( \sum_{j=1}^{n} \frac{\Im X_j}{\sqrt{p_j}} \right)^{2N'} \right] + O(1/\log T) . \]  

(6.6)

The last equality follows from applying Taylor’s theorem as in (6.3). If we treat the first remainder in the last row as the corresponding one in (6.3), using \(\mathbb{E}[(\sum_{j=1}^{n} \Im \log(1 - X_j/\sqrt{p_j}))^2] = (\log \log x)/2 + O(1)\) this time, one can show that it is also \(O(1/\log T)\). By plugging in (3.1) and expanding the logarithm, we obtain that (6.6) is equal to

\[ \sum_{\substack{p \leq x \\kappa \geq 1 \text{ odd}}} \frac{i}{k \sqrt{p}} J_k \left( \frac{u}{\sqrt{p}} \right) \prod_{q \leq x \atop q \neq p} J_0 \left( \frac{u}{\sqrt{q}} \right) + O(1/\log T) \]

which completes the proof of (6.2). The last step in the proof of (6.1) is to show that

\[ \frac{1}{T} \int_T^{2T} \Im \Sigma_{1,x}(t) e^{iu \Im \Sigma_{1,x}(t)} \, dt \]
Next, consider $\text{Im } f$. Using Theorem 3, we obtain that the family holds for the other family.

It remains to show (7.2). Therefore, let $h_{\delta}$ be as in Theorem 2 with the additional property that

$$\frac{1}{(\log \log T)/2} \log \left( \frac{1}{T} \int_{T}^{2T} e^{h \text{Im } \Sigma_{f,\delta}(t)} dt \right) \to h_{\delta}^{2}/2 \quad \text{as } T \to \infty. \quad (7.1)$$

Using Theorem 3, we obtain that the family (1/((log log T)/2))Im $\Sigma_{f,\delta}(U_{T})$ satisfies the large deviation principle with the speed 1/((log log T)/2) and the rate function $I(h) = h_{\delta}^{2}/2$. Next, consider $\text{Im } r_{f,\delta}(U_{T})$. We will show that there exists a constant $C > 0$ (the constant in (7.4)) such that for each $\delta > 0$

$$(1/T)\lambda(|t \in [T, 2T] : |\text{Im } r_{f,\delta}(t)| \geq C\delta \log \log T) \leq e^{-(1-o(1))[\delta \log \log T]}.$$ (7.2)

We postpone the proof of (7.2) to the end of this section. From (7.2) we deduce that for each $\delta > 0$

$$\frac{1}{(\log \log T)/2} \log \left( \frac{1}{T} \lambda(|t \in [T, 2T] : |\text{Im } r_{f,\delta}(t)| \geq \delta \log \log T) \right) \leq -2(\delta/C)(1 - o(1))(\log \log T + \log(\delta/C)).$$

As $T \to \infty$, the right hand side goes to $-\infty$. Hence, by Definition 1, the families (1/((log log T)/2))Im $\log \xi(1/2 + iU_{T})$ and (1/((log log T)/2))$\Sigma_{f,\delta}(U_{T})$ are exponentially equivalent. To obtain the statement of the theorem, we finally apply [4, Theorem 4.2.13], which states that if two families of random variables are exponentially equivalent, and one of them satisfies the large deviation principle with good rate function $I$, then the same large deviation principle holds for the other family.

It remains to show (7.2). Therefore, let $V = \delta \log \log T$ and decompose

$$\text{Im } r_{f,\delta} = \text{Im } \left( \sum_{g,T_{1/\delta}}^{*} + \left( \sum_{g,T_{1/\delta}}^{*} - \sum_{g,T_{1/\delta}} \right) + \left( \sum_{g,T_{1/\delta}} - \sum_{g,T_{1/\delta}}^{*} \right) + \sum_{g,T_{1/\delta}} \right).$$

If $|\text{Im } r_{f,\delta}(t)| \geq CV$, there exists a summand on the right hand side whose absolute value is greater or equal to $CV/4$. Applying the union bound, we obtain

$$(1/T)\lambda(|t \in [T, 2T] : |\text{Im } r_{f,\delta}(t)| \geq CV) \leq (1/T)\lambda(|t \in [T, 2T] : |\text{Im } r_{f,\delta}^{*}(t)| \geq CV/4) + (1/T)\lambda(|t \in [T, 2T] : |\text{Im } \sum_{g,T_{1/\delta}}^{*}(t) - \text{Im } \sum_{g,T_{1/\delta}}(t)| \geq CV/4))$$

$$= \mathbb{E} \left[ \left( \sum_{j=1}^{n} \text{Im } X_{j} \right) e^{i u \sum_{j=1}^{n} \text{Im } X_{j} / \sqrt{p_{j}}} \right] + O(1/\log T)$$

$$= \sum_{p \leq \sqrt{T}} \frac{i}{\sqrt{p}} J_{1} \left( \frac{u}{\sqrt{p}} \right) \prod_{q \leq \sqrt{T}} J_{0} \left( \frac{u}{\sqrt{q}} \right) + O(1/\log T)$$

uniformly for $|u| \leq c$. The first equality follows as above or as in the proof of Proposition 1, the second equality again by plugging in (3.1). This completes the proof.
For $T \to \infty$, we can apply Markov’s inequality and (9.1) to bound the last term by
\[
+(1/T)\lambda(\{t \in [T, 2T] : |\text{Im} \Sigma_{g,T/v}(t) - \text{Im} \Sigma_{g,x}(t)| \geq CV/4\})
\]
\[+(1/T)\lambda(\{t \in [T, 2T] : |\text{Im} \Sigma_{g,f,x}(t)| \geq CV/4\}).
\]
(7.3)

If we choose Selberg’s function $g(t) = e^{-2t} \min(1, 2(1-u))$, we can apply [17, Theorem 1], which says that, assuming RH, there exists constants $C, C' > 0$ such that for $2 \leq y \leq t^2$ and $t \geq 2$,
\[
|\text{Im} r^{*}_{g,y}(t)| \leq \left| \frac{C'}{\log y} \sum_{n \leq y} \Lambda(n) \frac{\log n}{\log y} \right| + \frac{C}{16} \log t.
\]
(7.4)

If we choose $y = T^{1/V}$ and $t \in [T, 2T]$, $T \geq 2$, we have $(C/16)(\log t/\log y) \leq CV/8$. For $T \geq 2$, sufficiently large such that $2 \leq T^{1/V} \leq T^2$, we obtain
\[
\frac{1}{T} \lambda(\{t \in [T, 2T] : |\text{Im} r^{*}_{g,T^{1/V}}(t)| \geq CV/4\})
\]
\[\leq \frac{1}{T} \lambda(\left\{t \in [T, 2T] : \left| \frac{C'}{\log T^{1/V}} \sum_{n \leq T^{1/V}} x_{n} \Lambda(n) \frac{\log n}{\log T^{1/V}} \right| \geq CV/8\}).
\]

Now, we can apply Markov’s inequality and (9.1) to bound the last term by
\[
\left( \frac{8C'}{CV} \right)^{2[V]} 32V (2AV)^V + O(1)^V = e^{-(1-o(1))V \log V}.
\]

Similarly, by using the other bounds in “Appendix 2”, we can bound the three other terms in (7.3) by exp$(-(1-o(1))V \log V)$. Hence, (7.2) follows. This completes the proof.

\[\square\]

**Proof of Corollary 3** The asserted formula is exactly content of Varadhan’s integral lemma (see Theorem 4). The assumptions of the theorem are satisfied by Corollary 2 and Eq. (5.2).

\[\square\]

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**8 Appendix 1: Selberg’s result**

In this appendix we briefly discuss Selberg’s result about the rate of convergence in the central limit theorem of $\text{Im} \log \zeta(1/2 + it)$ (see [19, Theorem 2] and [21, Theorem 6.2]). From Theorem 1 we deduce:

**Lemma 2** Let $x = e^{\log T/N}$ and $N$ such that $x \to \infty$ and $N/\log \log T \to \infty$ as $T \to \infty$. Suppose further that $N/\log \log T = O(\log \log T)$. Then
\[
\sup_{a < b} \left( \frac{1}{T} \lambda(\{t \in [T, 2T] : \frac{1}{(\log \log x + y)/2} \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \in [a, b]\})
\]
\[- \int_{a}^{b} e^{-t^2/2} \frac{dt}{\sqrt{2\pi}} \right) = O(1/\sqrt{\log \log T}).
\]
(8.1)
Proof We denote by $\Phi_n(u)$ the left hand side of (1.3). Using [5, XVI.3, formula 3.13] we can bound the left hand side of (8.1) by

$$\frac{2}{\pi} \int_{-c \sqrt{\log \log x}}^{c \sqrt{\log \log x}} e^{-u^2/2}|(\Phi_n(u/\sqrt{(\log \log x + y)/2}) - 1)/u| du + O\left( \frac{1}{c \sqrt{\log \log x}} \right).$$

An inspection of the proof of Proposition 2 combined with (4.2) shows that $\Phi_n(u)$ is bounded by

$$\Phi_n(u) = \Phi_n(1 + O(1/\log x)) + O(1/\log T), \quad |u| \leq c.$$ If we choose $c > 0$ such that $\Phi(u)$ has no zeros for $|u| \leq c$, we obtain $\Phi_n(u) = \Phi(u)(1 + O(1/\log x))$, $|u| \leq c$. On the other hand, we have $\Phi(u/\sqrt{(\log \log x + y)/2}) = 1 + O(u^2/\log \log x)$, $|u| \leq c$. These estimates give that (8.2) is $O(1/\sqrt{\log \log x})$. From $N/\log \log T = O(\log \log T)$ we conclude that $\log \log T/\log \log x \to 1$ and this completes the proof.

This lemma combined with the bound (see [21, Lemma 6.2])

$$||t \in [T, 2T]: |r_1(t)| \geq c' \log \log \log T|| = O(1/\sqrt{\log \log T}),$$

where $c' > 0$ is a constant, yields Selberg’s result

$$\sup_{a < b} \left( \frac{1}{T} \lambda \left( \left\{ t \in [T, 2T] : \frac{\text{Im } \zeta(1/2 + it)}{\sqrt{\log \log T}} \in [a, b] \right\} \right) - \int_a^b e^{-t^2/2} \frac{dt}{\sqrt{2\pi}} \right) = O\left( \frac{\log \log \log T}{\sqrt{\log \log T}} \right).$$

9 Appendix 2: Mean value estimates

For completeness we present some standard mean value estimates which we applied in the proof of Corollary 2 (see [17, Lemma 3] and [20, Lemma 3]). For this purpose let $x$ and $y$ be positive real numbers, $a_p$ and $b_p$ be complex numbers with $|a_p| \leq 1$ and $|b_p| \leq \log p/\log x$, and $k$ be a nonnegative integer. By repeating the arguments in the proof of Proposition 1, we obtain

$$\frac{1}{T} \int_T^{2T} \left| \sum_{p \leq x} \frac{a_p}{p^{1+2it}} \right|^{2k} dt \leq k! \left( \sum_{p \leq x} \frac{1}{p^2} \right)^k + 2Dk!(\pi(x))^k/T,$$

$$\frac{1}{T} \int_T^{2T} \left| \sum_{y < p \leq x} \frac{a_p}{p^{1/2+it}} \right|^{2k} dt \leq k! \left( \sum_{y < p \leq x} \frac{1}{p} \right)^k + 2Dk!(\pi(x) - \pi(y))^k/T,$$

$$\frac{1}{T} \int_T^{2T} \left| \sum_{p \leq x} \frac{b_p}{p^{1/2+it}} \right|^{2k} dt \leq k! \left( \sum_{p \leq x} \frac{\log p}{p} \right)^k + 2Dk!(\pi(x))^k/T.$$ If $x \leq T^{1/k}$, the first and the third term are bounded by $(Ak)^k$ and the second by $(k(\log \log x - \log \log y + A))^k$, $A > 0$ some constant.
For example, we obtain for a function \(|g(u)| \leq 1\)

\[
\frac{1}{T} \int_T^{2T} \left| \frac{1}{\log T^{1/V}} \sum_{n \leq T^{1/V}} A(n) \frac{n}{n^{1+2it}} g \left( \frac{\log n}{\log T^{1/V}} \right) \right|^{2[V]} \ dt
\]

\[
= \frac{1}{T} \int_T^{2T} \left| \sum_{\rho \leq T^{1/V}} \frac{b_p}{\rho^{1+2it}} + \sum_{\rho^2 \leq T^{1/V}} \frac{a_p}{\rho^{1+2it}} + O(1) \right|^{2[V]} \ dt
\]

\[
\leq 32V((AV)^V + (AV)^V + O(1)^V). \quad (9.1)
\]

10 Appendix 3: Large deviation theory

In this appendix we give the definition of the large deviation principle and state two important results which we used in the proofs of Corollary 2 and 3 (see [4]).

A function \(I : \mathbb{R} \to [0, \infty]\) is called a rate function (resp. good rate function), if for all \(\alpha \in [0, \infty)\), the sets \(\{x : I(x) \leq \alpha\}\) are closed (resp. compact). A family \(\{Z_\epsilon\}\) of real-valued random variables satisfies the large deviation principle with the speed \(\epsilon\) and the rate function \(I\), if

(a) For any closed set \(F \subseteq \mathbb{R}\)

\[
\limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}(Z_\epsilon \in F) \leq -\inf_{x \in F} I(x).
\]

(b) For any open set \(G \subseteq \mathbb{R}\)

\[
\liminf_{\epsilon \to 0} \epsilon \log \mathbb{P}(Z_\epsilon \in G) \geq -\inf_{x \in G} I(x).
\]

**Theorem 3** (Gärtner-Ellis, see Theorem 2.3.6 or 4.5.20 in [4]) Suppose that for each \(\lambda \in \mathbb{R}\)

\[
\Lambda(\lambda) := \lim_{\epsilon \to 0} \epsilon \log \mathbb{E}[e^{Z_\epsilon/\epsilon}]
\]

exists and that \(\Lambda\) is differentiable. Then the family \(\{Z_\epsilon\}\) satisfies the large deviation principle with the good rate function \(I(x) = \sup_{\lambda \in \mathbb{R}} (\lambda x - \Lambda(\lambda))\).

**Theorem 4** (Varadhan, see Theorem 4.3.1 in [4]) Suppose that \(\{Z_\epsilon\}\) satisfies the large deviation principle with a good rate function \(I\) and let \(h \in \mathbb{R}\). Assume further that for some \(\gamma > 1\)

\[
\limsup_{\epsilon \to 0} \epsilon \log \mathbb{E}[e^{\gamma h Z_\epsilon/\epsilon}] < \infty. \quad (10.1)
\]

Then

\[
\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}[e^{h Z_\epsilon/\epsilon}] = \sup_{x \in \mathbb{R}} (xh - I(x)).
\]

**Definition 1** (see Definition 4.2.10 in [4]) Let \(\{Z_\epsilon\}\) and \(\{\bar{Z}_\epsilon\}\) be two families of real-valued random variables, defined on the same probability space. Then \(\{Z_\epsilon\}\) and \(\{\bar{Z}_\epsilon\}\) are called exponentially equivalent if for each \(\delta > 0\),

\[
\limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}(|Z_\epsilon - \bar{Z}_\epsilon| > \delta) = -\infty. \quad (10.2)
\]
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