BASE CHANGE OF TWISTED FONTAINE-FALTINGS
MODULES AND TWISTED HIGGS-DE RHAM FLOWS
OVER VERY RAMIFIED VALUATION RINGS

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Abstract. In this short notes, we prove a stronger version of Theorem 0.6 in our previous paper [5]: Given a smooth log scheme \((X \supset D)_{W(F_q)}\), each stable twisted \(f\)-periodic logarithmic Higgs bundle \((E, \theta)\) over the closed fiber \((X \supset D)_{\bar{F}_p}\) will correspond to a \(\text{PGL}_r(F_{p^f})\)-crystalline representation of \(\pi_1((X \setminus D)_{W(F_q)\bar{Q}_p}\big)\) such that its restriction to the geometric fundamental group is absolutely irreducible.

1. Introduction

In the previous paper [5], we prove the following results

**Theorem 1.1** (Theorem 0.6 in [5]). Let \(k\) be a finite field of characteristic \(p\). Let \(X\) be a smooth proper geometrically connected scheme over \(W(k)\) together with a smooth log structure \(\mathcal{D}/W(k)\). Assume that there exists a semistable graded logarithmic Higgs bundle \((E, \theta)/(X, \mathcal{D})_1\) with \(r := \text{rank}(E) \leq p - 1\), discriminant \(\Delta_H(E) = 0\), \(r\) and \(\text{deg}_H(E)\) are coprime. Let \(X^0 = X \setminus \mathcal{D}\) and \(K' = W(k \cdot F_{p^f})[1/p]\). Then, there exist a positive integer \(f\) and a \(\text{PGL}_r(F_{p^f})\)-crystalline representation \(\rho\) of \(\pi_1(X_{K'}^0)\), which is irreducible in \(\text{PGL}_r(F_p)\).

In the proof we use the so called twisted periodic Higgs-de Rham flow, which is a variant of the periodic Higgs-de Rham flow defined by Lan, Sheng and Zuo in [3].

Now we want to improve this result to a stronger version. The \(\text{PGL}_r(F_{p^f})\)-crystalline representation of \(\pi_1((X \setminus D)_{W(F_q)\bar{Q}_p}\big)\) corresponding to the stable Higgs bundle should have the following property: its restriction to the geometric fundamental group \(\pi_1((X \setminus D)_{\bar{Q}_p}\big)\) is absolutely irreducible. Here the "absolutely irreducible" means that the representation is still irreducible after extending the coefficient \(F_{p^f}\) to \(F_p\).

We outline the proof as follows. Fix a \(K_0\)-point in \(X_{K_0}\), one can pull back the representation \(\rho\) to a representation of the galois group, whose
image is finite. This finite quotient will give us a field extension $K/K_0$ such that the restriction of $\rho$ on $\text{Gal}(\bar{K}_0/K)$ is trivial. That means $\rho(\pi_1(X^o_K)) = \rho(\pi_1(X^o_{K_0}))$. So it suffices to prove the irreducibility of $\rho$ on $\pi_1(X^o_K)$, which gives us the chance to apply the method of twisted periodic Higgs-de Rham flows as in \([3]\). But the field extension $K/K_0$ is usually ramified. So we have to work out the construction in \([3]\) to the very ramified case.

2. Base change of twisted Fontaine-Faltings modules and Twisted Higgs-de Rham flows over very ramified valuation rings

2.1. Notations in the case of $\text{Spec} k$. In this notes, $k$ will always be a perfect field of characteristic $p > 0$. Let $\pi$ be a root of an Eisenstein polynomial $f(T) = T^e + \sum_{i=0}^{e-1} a_i T^i$ of degree $e$ over the Witt ring $W = W(k)$. Denote $K_0 = \text{Frac}(W) = W[[\frac{1}{p}]]$ and $K = K_0[\pi]$, where $K_0[\pi]$ is a totally ramified extension of $K_0$ of degree $e$. Denote by $W_\pi = W[\pi]$ the ring of integers of $K$, which is a complete discrete valuation ring with maximal ideal $\pi W_\pi$ and the residue field $W_\pi/\pi W_\pi = k$. Denote by $W_\pi[[T]]$ the ring of formal power-series over $W$. Then

$$W_\pi = W[[T]]/fW[[T]].$$

The PD-hull $B_{W_\pi}$ of $W_\pi$ is the PD-completion of the ring obtained by adjoining to $W[[T]]$ the divided powers $\frac{f^n}{m}$. More precisely

$$B_{W_\pi} = \left\{ \sum_{n=0}^{\infty} a_n T^n \in K_0[[T]] \mid a_n[n/e]! \in W \text{ and } a_n[n/e]! \to 0 \right\} .$$

A decreasing filtration is defined on $B_{W_\pi}$ by the rule that $F^q(B_{W_\pi})$ is the closure of the ideal generated by divided powers $\frac{f^n}{m}$ with $n \geq q$. Note that the ring $B_{W_\pi}$ only depends on the degree $e$ while this filtration depends on $W_\pi$ and $e$. One has

$$B_{W_\pi}/\text{Fil}^1 B_{W_\pi} \simeq W_\pi.$$

There is a unique continuous homomorphism of $W$-algebra $B_{W_\pi} \to B^+(W_\pi)$ which sends $T$ to $[\pi]$. Here $\pi = (\pi, \pi^\frac{1}{p}, \pi^\frac{1}{p^2}, \ldots) \in \lim \tilde{R}$. We denote

$$\tilde{B}_{W_\pi} = B_{W_\pi}[\frac{f}{p}]$$

which is a subring of $K_0[[T]]$. The idea $(\frac{f}{p})$ induces a decreasing filtration $\text{Fil} \tilde{B}_{W_\pi}$ such that

$$\tilde{B}_{W_\pi}/\text{Fil}^1 \tilde{B}_{W_\pi} \simeq W_\pi.$$
The Frobenius endomorphism on $W$ can be extended to an endomorphism $\varphi$ on $K_0[[T]]$, where $\varphi$ is given by $\varphi(T) = T^p$. Since $\varphi(f)$ is divided by $p$, we have $\varphi(B_{W_\pi}) \subset B_{W_\pi}$. Thus one gets two restrictions

$$
\varphi : \tilde{B}_{W_\pi} \to B_{W_\pi} \text{ and } \varphi : \tilde{B}_{W_\pi} \to B_{W_\pi}.
$$

Note that the ideal of $B_{W_\pi}$, generated by $Fil^1 B_{W_\pi}$ and $T$, is stable under $\varphi$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
B_{W_\pi} & \longrightarrow & B_{W_\pi}/(Fil^1 B_{W_\pi}, T) = k \\
\downarrow \varphi & & \downarrow (\cdot)^p \\
B_{W_\pi} & \longrightarrow & B_{W_\pi}/(Fil^1 B_{W_\pi}, T) = k
\end{array}
$$

Since $\varphi(\cdot)_p$ is invertible in $B_{W_\pi}$, we have $\varphi((Fil^1 \tilde{B}_{W_\pi}, T)) \not\subset (Fil^1 B_{W_\pi}, T)$. Hence the following diagram is not commutative

$$
\begin{array}{ccc}
\tilde{B}_{W_\pi} & \longrightarrow & \tilde{B}_{W_\pi}/(Fil^1 \tilde{B}_{W_\pi}, T) = k \\
\downarrow \varphi & & \downarrow (\cdot)^p \\
\tilde{B}_{W_\pi} & \longrightarrow & \tilde{B}_{W_\pi}/(Fil^1 \tilde{B}_{W_\pi}, T) = k
\end{array}
$$

2.2. **Base change in the small affine case.** For a smooth and small $W$-algebra $R$ (which means there exists an étale map

$$
W[T_1^{\pm 1}, T_2^{\pm 1}, \cdots, T_d^{\pm 1}] \to R,
$$

see [1]), Lan-Sheng-Zuo constructed categories $\mathcal{MIC}(R/pR)$, $\tilde{\mathcal{MIC}}(R/pR)$, $\mathcal{MF}(R/pR)$ and $\mathcal{MF}(R/pR)$. A Fontaine-Faltings module over $R/pR$ is an object $(V, \nabla, Fil)$ in $\mathcal{MF}(R/pR)$ together with an isomorphism $\varphi : (\tilde{V}, \tilde{\nabla}, \tilde{Fil}) \otimes_\Phi \tilde{R} \to (V, \nabla)$ in $\mathcal{MIC}(R/pR)$, where $\tilde{\cdot}$ is the Faltings’ tilde functor.

We generalize those categories over the $W_\pi$-algebra $R_\pi = R \otimes_W W_\pi$. In general, there does not exist Frobenius lifting on the p-adic completion of $\tilde{R}_\pi$. We lift the absolute Frobenius map on $R_\pi / \pi R_\pi$ to a map $\Phi : B_{R_\pi} \to B_{R_\pi}$

$$
\begin{array}{ccc}
\mathcal{B}_{R_\pi} & \longrightarrow & R_\pi / \pi R_\pi = R/pR \\
\downarrow \Phi & & \downarrow (\cdot)^p \\
\mathcal{B}_{R_\pi} & \longrightarrow & R_\pi / \pi R_\pi = R/pR
\end{array}
$$

where $\mathcal{B}_{R_\pi}$ is the p-adic completion of $B_{W_\pi} \otimes_W R$. This lifting is compatible with $\varphi : B_{W_\pi} \to B_{W_\pi}$. Denote $\tilde{B}_{R_\pi} = B_{R_\pi}[\frac{1}{p}]$. Then $\Phi$ can be extended to

$$
\Phi : \tilde{B}_{R_\pi} \to B_{R_\pi}
$$
uniquely, which is compatible with \( \varphi : \tilde{B}_W \to B_W \). The filtrations on \( B_{R_e} \) and \( \tilde{B}_{R_e} \) induce filtrations on \( B_{R_e} \) and \( \tilde{B}_{R_e} \) respectively, which satisfy
\[
B_{R_e} / \text{Fil}^1 B_{R_e} \simeq \hat{R}_π \simeq \tilde{B}_{R_e} / \text{Fil}^1 \tilde{B}_{R_e}.
\]

**Lemma 2.1.** Let \( n < p \) be a natural number and let \( b \) be an element in \( F^n B_{R_e} \). Then \( \frac{b}{p^n} \) is an element in \( F^n \tilde{B}_{R_e} \).

**Proof.** Since the filtrations on \( B_{R_e} \) and \( \tilde{B}_{R_e} \) are induced by those on \( B_W \) and \( \tilde{B}_W \) respectively, we have
\[
F^n B_{R_e} = \left\{ \sum_{i \geq n} a_i \frac{f_i}{i!} \mid a_i \in \hat{R}[[T]] \text{ and } a_i \to 0 \right\},
\]
and
\[
F^n \tilde{B}_{R_e} = \left\{ \sum_{i \geq n} a_i \frac{f_i}{p^i} \mid a_i \in B_{R_e} \text{ and } a_i = 0 \text{ for } i \gg 0 \right\}
\]
(4)
Assume \( b = \sum_{i \geq n} a_i \frac{f_i}{i!} \) with \( a_i \in \hat{R}[[T]] \) and \( a_i \to 0 \), then
\[
\frac{b}{p^n} = \sum_{i \geq n} \frac{p^i a_i}{p^n i!} \cdot \frac{f_i}{p^i},
\]
and the lemma follows. \( \square \)

**Lemma 2.2.** There are isomorphisms of free \( \hat{R}_π \)-modules of rank 1
\[
\begin{array}{ccc}
\text{Gr}^n \tilde{B}_{R_e} & \longrightarrow & \text{Gr}^n B_{R_e} \\
\frac{f^n}{p^n} & \longrightarrow & \frac{f^n}{\pi^n} \longrightarrow 1
\end{array}
\]
(5)

**Proof.** This can be checked directly. \( \square \)

Recall that \( B_{R_e} \) and \( \tilde{B}_{R_e} \) are \( \hat{R} \)-subalgebras of \( \hat{R}(\frac{1}{p})[[T]] \). We denote by \( \Omega^1_{B_{R_e}} \) (resp. \( \Omega^1_{\tilde{B}_{R_e}} \)) the \( B_{R_e} \)-submodule (resp. \( \tilde{B}_{R_e} \)-submodule) of \( \Omega^1_{\hat{R}(\frac{1}{p})[[T]]/W} \) generated by elements \( db \), where \( b \in B_{R_e} \) (resp. \( b \in \tilde{B}_{R_e} \)). There is a filtration on \( \Omega^1_{B_{R_e}} \) (resp. \( \Omega^1_{\tilde{B}_{R_e}} \)) given by \( F^n \Omega^1_{B_{R_e}} = F^n B_{R_e} \cdot \Omega^1_{B_{R_e}} \) (resp. \( F^n \Omega^1_{\tilde{B}_{R_e}} = F^n \tilde{B}_{R_e} \cdot \Omega^1_{\tilde{B}_{R_e}} \)). One gets the following result directly by Lemma 2.1.

**Lemma 2.3.** Let \( n < p \) be a natural number. Then \( \frac{F^n \Omega^1_{B_{R_e}}}{p^n} \subset F^n \Omega^1_{\tilde{B}_{R_e}} \).

We have the following categories
Since $ij \omega \in \mathcal{A}$ Fontaine-Faltings module over $\mathcal{V}$, where $\tilde{\mathcal{B}}$ BASE CHANGE OF TWISTED HIGGS-DE RHAM FLOWS OVER VERY RAMIFIED VALUATION RINGS 5

Lemma 2.4. The $\tilde{\mathcal{B}}/p\tilde{\mathcal{B}}$-module $\tilde{\mathcal{V}}$ equipped with the $p$-connection $\tilde{\nabla}$ is independent of the choice of the filtered basis $v_i$ up to a canonical isomorphism.

Proof. We write $v = (v_1, v_2, \cdots)$ and $\omega = (\omega_{ij})_{i,j}$. Then

$$\nabla(v) = v \otimes \omega \text{ and } \tilde{\nabla}([v]) = [v] \otimes (pQ\omega Q^{-1}),$$

where $Q = \text{diag}(p^{q_1}, p^{q_2}, \cdots)$ is a diagonal matrix. Assume that $v'_i$ is another filtered basis (of degree $q_i$, $0 \leq q_i \leq a$) and $(\tilde{\mathcal{V}}', \tilde{\nabla}')$ is the corresponding $\tilde{\mathcal{B}}/p\tilde{\mathcal{B}}$-module equipped with the $p$-connection. Similarly, we have

$$\nabla(v') = v' \otimes \omega' \text{ and } \tilde{\nabla}([v']) = [v'] \otimes (pQ\omega' Q^{-1}),$$

A Fontaine-Faltings module over $\mathcal{B}/p\mathcal{B}$ of weight $a$ ($0 \leq a \leq p - 2$) is an object $(\mathcal{V}, \nabla, \text{Fil})$ in the category $\mathcal{MF}_{[0,a]}(\mathcal{B}/p\mathcal{B})$ together with an isomorphism in $\mathcal{MIC}(\mathcal{B}/p\mathcal{B})$

$$\varphi : (\mathcal{V}, \nabla, \text{Fil}) \otimes \Phi \mathcal{B} \to (\mathcal{V}, \nabla),$$

where $(\cdot) : \mathcal{MF}(\mathcal{B}/p\mathcal{B}) \to \mathcal{MIC}(\mathcal{B}/p\mathcal{B})$ is an analogue of the Faltings’ tilde functor. For an object $(\mathcal{V}, \nabla, \text{Fil})$ in $\mathcal{MF}(\mathcal{B}/p\mathcal{B})$ with filtered basis $v_i$ (of degree $q_i$, $0 \leq q_i \leq a$), $\tilde{\mathcal{V}}$ is defined as a filtered free $\tilde{\mathcal{B}}/p\tilde{\mathcal{B}}$-module

$$\tilde{\mathcal{V}} = \bigoplus_i \tilde{\mathcal{B}}/p\tilde{\mathcal{B}} \cdot [v_i]_{q_i},$$

with filtered basis $[v_i]_{q_i}$ (of degree $q_i$, $0 \leq q_i \leq a$). Informally one can view $[v_i]_{q_i}$ as “$\frac{v_i}{p^{q_i}}$”. Since $\nabla$ satisfies the Griffiths transversality, there are $\omega_{ij} \in F^{q_j-1-q_i} \Omega^1_{\mathcal{B}/\mathcal{B}}$ satisfying

$$\nabla(v_j) = \sum_i v_i \otimes \omega_{ij}.$$ Since $q_j - 1 - q_i < a \leq p - 2$, $\frac{\omega_{ij}}{p^{q_j-1-q_i}} \in F^{q_j-1-q_i} \Omega^1_{\mathcal{B}/\mathcal{B}}$. We define a $p$-connection $\tilde{\nabla}$ on $\tilde{\mathcal{V}}$ via

$$\tilde{\nabla}([v_j]_{q_j}) = \sum_i [v_i]_{q_i} \otimes \frac{\omega_{ij}}{p^{q_j-1-q_i}}.$$
Assume \( v'_j = \sum_i a_{ij} v_i \) \((a_{ij} \in F^{n_i}B_{R_a})\). Then \( A = (a_{ij})_{i,j} \in \text{GL}_{\text{rank}(V)}(B_{R_a})\) and \( QAQ^{-1} = (a_{ij})_{i,j} \in \text{GL}_{\text{rank}(V)}(\overline{B}_{R_a})\). We construct an isomorphism from \( \widetilde{V}' \) to \( \widetilde{V} \) by

\[
\tau([v']) = [v] \cdot (QAQ^{-1}),
\]

where \([v] = ([v_1]_{q_1}, [v_2]_{q_2}, \ldots)\) and \([v'] = ([v'_1]_{q_1}, [v'_2]_{q_2}, \ldots)\). Now we only need to check that \( \tau \) preserve the \( p \)-connections. Indeed,

\[
\tau \circ \nabla'([v']) = [v] \otimes (QAQ^{-1} \cdot pQ'Q^{-1}) = [v] \otimes (pQ \cdot A'Q^{-1}) \quad (6)
\]

and

\[
\nabla \circ \tau([v']) = \nabla([v]QAQ^{-1}) = [v] \otimes (pQ \cdot A'Q^{-1} + p \cdot QdAQ^{-1}) \quad (7)
\]

Since \( \nabla(v') = \nabla(vA) = v \otimes dA + v \otimes \omega A = v' \otimes (A^{-1}dA + A^{-1} \omega A)\), we have \( \omega' = A^{-1}dA + A^{-1} \omega A\) by definition. Thus \( \tau \circ \nabla' = \nabla \circ \tau\).

The functor

\[
\mathcal{M}F_{[0,a]} : \mathcal{M}IC(\overline{B}_{R_a}/p\overline{B}_{R_a}) \to \mathcal{M}IC(B_{R_a}/pB_{R_a})
\]

is induced by base change under \( \Phi \). Note that the connection on \((\widetilde{V}, \nabla) \otimes \Phi B_{R_a}\) is given by

\[
d + \frac{d\Phi}{p}(\Phi^*\nabla)
\]

Denote by \( \mathcal{M}F_{[0,a]}(B_{R_a}/pB_{R_a})\) the category of all Fontaine-Faltings modules over \( B_{R_a}/pB_{R_a} \) of weight \( a \).

**Lemma 2.5.** We have the following commutative diagram by extending the coefficient ring from \( R \) to \( B_{R_a} \) (or \( \overline{B}_{R_a} \))

\[
\begin{array}{ccc}
\mathcal{M}F(R/pR) & \xrightarrow{(\cdot)} & \mathcal{M}IC(R/pR) & \xrightarrow{-\otimes R} & \mathcal{M}IC(R/pR) \\
\downarrow{-\otimes R B_{R_a}} & & \downarrow{-\otimes \overline{B}_{R_a}} & & \downarrow{-\otimes \overline{B}_{R_a}} \\
\mathcal{M}F(B_{R_a}/pB_{R_a}) & \xrightarrow{(\cdot)} & \mathcal{M}IC(\overline{B}_{R_a}/p\overline{B}_{R_a}) & \xrightarrow{-\otimes \Phi B_{R_a}} & \mathcal{M}IC(B_{R_a}/pB_{R_a})
\end{array}
\]

In particular, we get a functor from the category of Fontaine-Faltings modules over \( R/pR \) to that over \( B_{R_a}/pB_{R_a} \)

\[
\mathcal{M}F_{[0,a]}(R/pR) \to \mathcal{M}F_{[0,a]}(B_{R_a}/pB_{R_a}).
\]

Those categories of Fontaine-Faltings modules are independent of the choice of the Frobenius lifting by the Taylor formula.

**Theorem 2.6.** For any two choices of \( \Phi_{B_{R_a}} \) there is an equivalence between the corresponding categories \( \mathcal{M}F_{[0,a]}(B_{R_a}/pB_{R_a}) \) with different \( \Phi_{B_{R_a}} \). These equivalences satisfy the obvious cocycle condition. Therefore, \( \mathcal{M}F_{[0,a]}(B_{R_a}/pB_{R_a}) \) is independent of the choice of \( \Phi_{B_{R_a}} \) up to a canonical isomorphism.
**Definition 2.7.** For an object \((V, \nabla, \text{Fil}, \varphi)\) in \(\mathcal{MF}_{[0,a]}(\mathcal{B}_{R_p}/p\mathcal{B}_{R_p})\), denote
\[
\mathbb{D}(V, \nabla, \text{Fil}, \varphi) = \text{Hom}_{B^+(R), \text{Fil}, \varphi}(V \otimes_{\mathcal{B}_{R_p}} B^+(R), B^+(R)/pB^+(R)).
\]

The proof of Theorem 2.6 in [1] works in this context. we can define an adjoint functor \(\mathbb{E}\) of \(\mathbb{D}\) as
\[
\mathbb{E}(L) = \lim_{\rightarrow}\{V \in \mathcal{MF}_{[0,a]}(\mathcal{B}_{R_p}/p\mathcal{B}_{R_p}) \mid L \rightarrow \mathbb{D}(V)\}.
\]

The proof in page 41 of [1] still works. Thus we obtain:

**Theorem 2.8.** i) The homomorphism set \(\mathbb{D}(V, \nabla, \text{Fil}, \varphi)\) is an \(\mathbb{F}_p\)-vector space with a linear \(\text{Gal}(\overline{R}_K/R_K)\)-action whose rank equals to the rank of \(V\).

ii) The functor \(\mathbb{D}\) from \(\mathcal{MF}_{[0,a]}(\mathcal{B}_{R_p}/p\mathcal{B}_{R_p})\) to the category of \(\text{W}_n(\mathbb{F}_p)\)-\(\text{Gal}(\overline{R}_K/R_K)\)-modules is fully faithful and its image on objects is closed under subobjects and quotients.

2.3. Categories and Functors on proper smooth variety over very ramified ring \(W_\pi\). Let \(X\) be a smooth proper \(W\)-scheme and \(X_\pi = X \otimes_W W_\pi\). Let \(\mathcal{X}_\pi\) be the formal completion of \(X \otimes_W \mathcal{B}_{W_\pi}\) and \(\widetilde{\mathcal{X}}_\pi\) be the formal completion of \(X \otimes_W \mathcal{B}_{W_\pi}\). Then \(\mathcal{X}_\pi\) is an infinitesimal thickening of \(X_\pi\) and the ideal defining \(X_\pi\) in \(\mathcal{X}_\pi\) has a nilpotent PD-structure which is compatible with that on \(F^1(\mathcal{B}_{W_\pi})\) and \((p)\)

\[
\begin{array}{ccc}
X_\pi & \longrightarrow & \mathcal{X}_\pi \\
\downarrow & & \downarrow \\
\text{Spec} W_\pi & \longrightarrow & \text{Spec} \mathcal{B}_{W_\pi} \\
\end{array}
\]

Let \(\{U_i\}_i\) be a covering of small affine open subsets of \(X\). By base change, we get a covering \(\{\mathcal{U}_i = U_i \times_X \mathcal{X}_\pi\}_i\) of \(\mathcal{X}_\pi\) and a covering \(\{\mathcal{U}_i = U_i \times_X \widetilde{\mathcal{X}}_\pi\}_i\) of \(\widetilde{\mathcal{X}}_\pi\). For each \(i\), we denote \(R_i = \mathcal{O}_{\mathcal{X}_\pi}(U_i \times_X X_\pi)\). Then \(\mathcal{B}_{R_i} = \mathcal{O}_{\mathcal{X}_\pi}(U_i)\) and \(\overline{\mathcal{B}}_{R_i} = \mathcal{O}_{\widetilde{\mathcal{X}}_\pi}(U_i)\) are the coordinate rings. Fix a Frobenius-lifting \(\Phi_i : \overline{\mathcal{B}}_{R_i} \rightarrow \mathcal{B}_{R_i}\), one gets those categories of Fontaine-Faltings modules
\[
\mathcal{MF}_{[0,a]}(\mathcal{B}_{R_i}/p\mathcal{B}_{R_i}).
\]

By the Theorem 2.6, these categories are glued into one category. Moreover those underlying modules are glued into a bundle over \(\mathcal{X}_\pi, 1 = \mathcal{X}_\pi \otimes_{\mathbb{Z}_p} \mathbb{F}_p\). We denote this category by \(\mathcal{MF}_{[0,a]}(\mathcal{X}_\pi, 1)\).

2.3.1. **Inverse Cartier functor and a description of \(\mathcal{MF}_{[0,a]}(\mathcal{X}_\pi, 1)\) via Inverse Cartier functor.** Let \(\overline{\Phi} : \mathcal{B}_{R_\pi}/p\mathcal{B}_{R_\pi} \rightarrow \mathcal{B}_{R_\pi}/p\mathcal{B}_{R_\pi}\) be the \(p\)-th power map. Then we get the following lemma directly.

**Lemma 2.9.** Let \(\Phi : \mathcal{B}_{R_\pi} \rightarrow \mathcal{B}_{R_\pi}\) and \(\Psi : \mathcal{B}_{R_\pi} \rightarrow \mathcal{B}_{R_\pi}\) be two liftings of \(\overline{\Phi}\) which are both compatible with the Frobenius map on \(\mathcal{B}_{W_\pi}\).
i). Since $\varphi(f)$ is divided by $p$, we extend $\Phi$ and $\Psi$ to maps on $\mathcal{B}_{R_e}$ via $\frac{f^p}{p} \mapsto \left(\frac{\varphi(f)}{p}\right)^n$ uniquely;

ii). the difference $\Phi - \Psi$ on $\mathcal{B}_{R_e}$ is still divided by $p$;

iii). the differentials $d\Phi : \Omega^1_{\mathcal{B}_{R_e}} \rightarrow \Omega^1_{\mathcal{B}_{R_e}}$ and $d\Psi : \Omega^1_{\mathcal{B}_{R_e}} \rightarrow \Omega^1_{\mathcal{B}_{R_e}}$ are divided by $p$.

From now on, we call the extension given by i) of Lemma 2.10 the Frobenius liftings of $\Phi$ on $\mathcal{B}_{R_e}$.

**Lemma 2.10.** Let $\Phi : \mathcal{B}_{R_e} \rightarrow \mathcal{B}_{R_e}$ and $\Psi : \mathcal{B}_{R_e} \rightarrow \mathcal{B}_{R_e}$ be two Frobenius liftings of $\Phi$ on $\mathcal{B}_{R_e}$. Then there exists a $\mathcal{B}_{R_e}/p\mathcal{B}_{R_e}$-linear morphism $h_{\Phi,\Psi} : \Omega^1_{\mathcal{B}_{R_e}/p\mathcal{B}_{R_e}} \otimes \Phi \mathcal{B}_{R_e}/p\mathcal{B}_{R_e} \rightarrow \mathcal{B}_{R_e}/p\mathcal{B}_{R_e}$ satisfying that:

i). we have $\frac{d\Phi}{p} - \frac{d\Psi}{p} = dh_{\Phi,\Psi}$ over $\Omega^1_{\mathcal{B}_{R_e}/p\mathcal{B}_{R_e}} \otimes \Phi 1$;

ii). the cocycle condition holds.

**Proof.** As $\Omega^1_{\mathcal{B}_{R_e}/p\mathcal{B}_{R_e}} \otimes \Phi \mathcal{B}_{R_e}/p\mathcal{B}_{R_e}$ is an $\mathcal{B}_{R_e}/p\mathcal{B}_{R_e}$-module generated by elements of the form $dg \otimes 1$ ($g \in \mathcal{B}_{R_e}/p\mathcal{B}_{R_e}$) with relations $d(g_1 + g_2) \otimes 1 - dg_1 \otimes 1 - dg_2 \otimes 1$ and $d(g_1g_2) \otimes 1 - dg_1 \otimes \Phi(g_2) - dg_2 \otimes \Phi(g_1)$. Since $\Phi - \Psi$ is divided by $p$, we can denote $h_{ij}(dg \otimes 1) = \frac{\Phi(g) - \Psi(g)}{p}$ (mod $p$) $\in \mathcal{B}_{R_e}/p\mathcal{B}_{R_e}$ for any element $g \in \mathcal{O}_U$ (the definition does not depend on the choice of the lifting $\tilde{g}$ of $g$ in $\mathcal{O}_U$). By direct calculation, we have

$$h_{ij}(dg_1 \otimes 1) = h_{ij}(dg_1 \otimes 1) + h_{ij}(dg_2 \otimes 1)$$

and

$$h_{ij}(dg_1g_2 \otimes 1) = \Phi(g_2) \cdot h_{ij}(dg_1 \otimes 1) + \Phi(g_1) \cdot h_{ij}(dg_2 \otimes 1)$$

Thus $h_{ij}$ can be $\mathcal{B}_{R_e}/p\mathcal{B}_{R_e}$-linearly extended. One checks i) and ii) directly by definition. \[\square\]

Let $(\tilde{\nabla}_i, \tilde{\nabla})$ be a locally filtered free sheaf over $\mathcal{X}_{\pi,1} = \mathcal{X}_{\pi} \otimes \mathbb{Z}_p \mathbb{F}_p$ with an integrable $p$-connection. Here a “filtered free” module over a filtered ring $R$ is a direct sum of copies of $R$ with the filtration shifted by a constant amount. The associated graded then has a basis over $gr_F(R)$ consisting of homogeneous elements (see [2]). Let $(\tilde{V}_i, \tilde{\nabla}_i) = (\tilde{V}, \tilde{\nabla}) |_{\mathcal{X}_{\pi,1}}$ be its restriction on the open subset $\mathcal{U}_{\pi,1} = \mathcal{U}_1 \otimes \mathbb{Z}_p \mathbb{F}_p$. By taking functor $\Phi_i^*$, we get bundles with integrable connections over $\mathcal{U}_{\pi,1} = \mathcal{U}_1 \otimes \mathbb{Z}_p \mathbb{F}_p$

$$\left(\Phi_i^*(\tilde{V}_i, d + \frac{d\Phi_i}{p}(\Phi_i^*\tilde{\nabla}))\right).$$
Lemma 2.11. Let \((\widetilde{V}, \widetilde{\nabla})\) be a locally filtered free sheaf over \(\mathcal{X}_{\pi,1}\) with an integrable \(p\)-connection. Then these local bundles with connections
\[
\left( \Phi_i^* \widetilde{V}_i, d + \frac{d \Phi_i}{p}(\Phi_i^* \widetilde{\nabla}) \right)
\]
can be glued into a global bundle with a connection on \(\mathcal{X}_{\pi,1}\) via transition functions
\[
G_{ij} = \exp \left( h_{\Phi_i, \Phi_j}(\Phi_i^* \widetilde{\nabla}) \right) : \Phi_i^* (\widetilde{V}_i) \to \Phi_j^* (\widetilde{V}_j).
\]
Denote this global bundle with connection by \(C^{-1}_{\mathcal{X}_{\pi,1}}(\widetilde{V}, \widetilde{\nabla})\). Then we can construct a functor
\[
C^{-1}_{\mathcal{X}_{\pi,1}} : \hat{\text{MIC}}(\mathcal{X}_{\pi,1}) \to \text{MIC}(\mathcal{X}_{\pi,1}).
\]
Proof. The cocycle condition easily follows from the integrability of the Higgs field. We show that the local connections coincide on the overlaps, that is
\[
(G_{ij} \otimes \text{id}) \circ \left( d + \frac{d \Phi_i}{p}(\Phi_i^* \widetilde{\nabla}) \right) = \left( d + \frac{d \Phi_j}{p}(\Phi_j^* \widetilde{\nabla}) \right) \circ G_{ij}.
\]
It suffices to show
\[
\frac{d \Phi_i}{p}(\Phi_i^* \widetilde{\nabla}) = G_{ij}^{-1} \circ d G_{ij} + G_{ij}^{-1} \circ \frac{d \Phi_j}{p}(\Phi_j^* \widetilde{\nabla}) \circ G_{ij}.
\]
Since \(G_{ij}^{-1} \circ d G_{ij} = d h_{\Phi_i, \Phi_j}(\Phi_i^* \widetilde{\nabla}) \) and \(G_{ij}\) commutes with \(\frac{d \Phi_j}{p}(\Phi_j^* \widetilde{\nabla})\) we have
\[
G_{ij}^{-1} \circ d G_{ij} + G_{ij}^{-1} \circ \frac{d \Phi_j}{p}(\Phi_j^* \widetilde{\nabla}) \circ G_{ij} = d h_{\Phi_i, \Phi_j}(\Phi_i^* \widetilde{\nabla}) + \frac{d \Phi_j}{p}(\Phi_j^* \widetilde{\nabla})
\]
by the integrability of the Higgs field. Thus we glue those local bundles with connections into a global bundle with connection via \(G_{ij}\). \(\square\)

Lemma 2.12. To give an object in the category \(\mathcal{M}F(\mathcal{X}_{\pi,1})\) is equivalent to give a tuple \((V, \nabla, \text{Fil}, \varphi)\) satisfying
i). \(V\) is filtered local free sheaf over \(\mathcal{X}_{\pi,1}\) with local basis having filtration degrees contained in \([0, a]\);
ii). \(\nabla : V \to V \otimes \Omega^1_{\mathcal{X}_{\pi,1}}\) is an integrable connection satisfying the Griffiths transversality;
iii). \(\varphi : C^{-1}_{\mathcal{X}_{\pi,1}}(\widetilde{V}, \widetilde{\nabla}, \text{Fil}) \simeq (V, \nabla)\) is an isomorphism of sheaves with connections over \(\mathcal{X}_{\pi,1}\).

2.3.2. The functors \(\mathbb{D}\) and \(\mathbb{D}^P\). For an object in \(\mathcal{M}F_{[0, a]}(\mathcal{X}_{\pi,1})\), we get locally constant sheaves on \(\mathcal{U}_K\) by applying the local \(\mathbb{D}\)-functors. These locally constant sheaves can be expressed in terms of certain finite étale coverings. They can be glued into a finite covering of \(\mathcal{X}_{\pi,K} = \mathcal{X}_K\). We have the following result.
Theorem 2.13. Suppose that $X$ is a proper smooth and geometrically connected scheme over $W$. Then there exists a fully faithful contravariant functor $D$ from $\mathcal{M}_F_{[0,a]}(\mathcal{X}_{\pi,1})$ to the category of $\mathbb{F}_p$-representations of $\pi_1(X_K)$. The image of $D$ on objects is closed under subobjects and quotients. Locally $D$ is given by the same as in Lemma 2.7.

Again one can define the category $\mathcal{M}_F_{[0,a]}(\mathcal{X}_{\pi,1})$ in the logarithmic case, if one replaces all "connections" by "logarithmic connections" and "Frobenius lifting" by "logarithmic Frobenius lifting". We also have the version of $\mathcal{M}_F_{[0,a]}(\mathcal{X}_{\pi,1})$ with endomorphism structures of $\mathbb{F}_p$, which is similar as the Variant 2 discussed in section 2 of [3]. And the twisted versions $\mathcal{T}_M\mathcal{F}_{[0,a]}(\mathcal{X}_{\pi,1})$ can also be defined on $\mathcal{X}_{\pi,1}$ in a similar way as in [5]. More precisely, let $L$ be a line bundle over $\mathcal{X}_{\pi,1}$. The $L$-twisted Fontaine-Faltings module is defined as follows.

Definition 2.14. An $L$-twisted Fontaine-Faltings module over $\mathcal{X}_{\pi,1}$ with endomorphism structure is a tuple
\[
((V, \nabla, \text{Fil})_0, (V, \nabla, \text{Fil})_1, \ldots, (V, \nabla, \text{Fil})_{f-1}, \varphi)
\]
where $(V, \nabla, \text{Fil})_i$ are objects in $\mathcal{M}_F(\mathcal{X}_{\pi,1})$ equipped with isomorphisms in $\mathcal{MIC}(\mathcal{X}_{\pi,1})$
\[
\varphi_i : C^{-1}_{\mathcal{X}_{\pi,1}}(V, \nabla, \text{Fil})_i \simeq (V, \nabla)_{i+1} \text{ for } i = 0, 1, \ldots, f-2;
\]
and
\[
\varphi_{f-1} : C^{-1}_{\mathcal{X}_{\pi,1}}(V, \nabla, \text{Fil})_{f-1} \otimes (L^p, \nabla_{\text{can}}) \simeq (V, \nabla)_0.
\]

The proof of Theorem 0.4 in [5] works in this context. Thus we obtain the following result.

Theorem 2.15. Suppose that $\mathcal{X}$ is a proper smooth and geometrically connected scheme over $W$ equipped with a smooth log structure $\mathcal{D}/W(k)$. Suppose that the residue field $k$ contains $\mathbb{F}_p$. Then there exists an exact and fully faithful contravariant functor $D^P$ from $\mathcal{T}M\mathcal{F}_{a,f}(\mathcal{X}_{\pi,1})$ to the category of projective $\mathbb{F}_p$-representations of $\pi_1(X_K)$. The image of $D^P$ is closed under subobjects and quotients.

Recall that $\{U_i\}$ is an open covering of $\mathcal{X}$. A line bundle on $\mathcal{X}$ can be expressed by the transition functions on $\mathcal{U}_{ij}$.

Lemma 2.16. Let $L$ be a line bundle on $\mathcal{X}_{\pi,1}$ expressed by $(g_{ij})$. Denote by $\tilde{L}$ the line bundle on $\mathcal{X}_{\pi,1}$ defined by the same transition functions $(g_{ij})$. Then one has
\[
C^{-1}_{\mathcal{X}_{\pi,1}}(\tilde{L}, 0) = L^p.
\]

Proof. Since $g_{ij}$ is an element in $\mathcal{B}_{R_{ij}} \subset \mathcal{B}_{R_{ij}}$, by diagram (3), one has
\[
\Phi(g_{ij}) \equiv g_{ij}^p \pmod{p}.
\]
On the other hand, since the $p$-connection is trivial, one has

$$C^{-1}_{X, 1}(\mathcal{L}, 0) = (\Phi \mod p)^*(\mathcal{L}).$$

Thus one has $C^{-1}_{X, 1}(\mathcal{L}, 0) = (\mathcal{O}_{\mathcal{F}^{(i)}}, g^p_{ij}) = \mathcal{L}^p$. \hfill $\square$

In a similar way, one can define the Higgs-de Rham flow on $X_{\pi, 1}$ as a sequence consisting of infinitely many alternating terms of Higgs bundles over $\tilde{X}_{\pi, 1}$ and filtered de Rham bundles over $X_{\pi, 1}$

$$\{(E, \theta)_0, (V, \nabla, \text{Fil})_0, (E, \theta)_1, (V, \nabla, \text{Fil})_1, \cdots\}$$

with $(V, \nabla)_i = C^{-1}_{X, 1}(E, \theta)_i$ and $(E, \theta)_{i+1} = (\mathcal{L}^p, \nabla)_i$ for all $i \geq 0$.

$f$-periodic $\mathcal{L}$-twisted Higgs-de Rham flow over $X_{\pi, 1}$ of level in $[0, a]$ is a Higgs-de Rham flow over $X_{\pi, 1}$

$$\{(E, \theta)_0, (V, \nabla, \text{Fil})_0, (E, \theta)_1, (V, \nabla, \text{Fil})_1, \cdots\}$$

equipped with isomorphisms $\phi_{f+i} : (E, \theta)_f \otimes (\mathcal{L}^{p^i}, 0) \rightarrow (E, \theta)_i$ of Higgs bundles for all $i \geq 0$

And for any $i \geq 0$ the isomorphism

$$C^{-1}_{X, 1}(\phi_{f+i}) : (V, \nabla)_f \otimes (\mathcal{L}^{p^{i+1}}, \nabla_{\text{can}}) \rightarrow (V, \nabla)_i,$$

strictly respects filtrations $\text{Fil}_{f+i}$ and $\text{Fil}_i$. Those $\phi_{f+i}$’s are related to each other by formula

$$\phi_{f+i+1} = \text{Gr} \circ C^{-1}_{X, 1}(\phi_{f+i}).$$

Just taking the same construction as in [3], we obtain the following result.

**Theorem 2.17.** There exists an equivalent functor $\mathcal{IC}_{X_{\pi, 1}}$ from the category of twisted periodic Higgs-de Rham flows over $X_{\pi, 1}$ to the category of twisted Fontaine-Faltings modules over $X_{\pi, 1}$ with a commutative diagram

$$\begin{array}{ccc}
\mathcal{THDF}(X_1) & \xrightarrow{\mathcal{IC}_{X_1}} & \mathcal{TMF}(X_1) \\
\downarrow \circ X_1 \circ X_{\pi, 1} & & \downarrow \circ X_1 \circ X_{\pi, 1} \\
\mathcal{THDF}(X_{\pi, 1}) & \xrightarrow{\mathcal{IC}_{X_{\pi, 1}}} & \mathcal{TMF}(X_{\pi, 1}).
\end{array}$$
2.4. degree and slope. Recall that \( \mathcal{X}_\pi \) is a smooth formal scheme over \( \mathcal{B}_{W_\pi} \). Then \( \mathcal{X}_{\pi,1} \) and \( X_1 \) are the modulo-\( p \) reductions of \( \mathcal{X}_\pi \) and \( X \) respectively. Also note that \( X_1 \) is the closed fiber of \( Y_1 = X_\pi \otimes_{\mathbb{Z}_p} \mathbb{F}_p \), \( \mathcal{X}_{\pi,1} = \mathcal{X}_{\pi} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \), \( X_\pi \) and \( \mathcal{X}_\pi \).

\[
\begin{array}{c}
\xymatrix{ X_1 \ar[r] & X_{\pi,1} & \mathcal{X}_{\pi,1} \\
X_1 \ar[r] \ar[u] & X_{\pi} \ar[u] & \mathcal{X}_{\pi} \ar[u] \\
\Spec k \ar[r] & \Spec W_\pi \\
\Spec B_\pi \ar[u] & \Spec B_W \ar[u] & \Spec W \ar[u]
\end{array}
\]

For a line bundle \( V \) on \( \mathcal{X}_{\pi,1} \) (resp. \( \mathcal{X}_{\pi,1} \)), \( V \otimes_{\mathcal{O}_{\mathcal{X}_{\pi,1}}} \mathcal{O}_{X_1} \) (resp. \( V \otimes_{\mathcal{O}_{\mathcal{X}_{\pi,1}}} \mathcal{O}_{X_1} \)) forms a line bundle on the special fiber \( X_1 \) of \( \mathcal{X} \). We denote \( \deg(V) := \deg(V \otimes_{\mathcal{O}_{\mathcal{X}_{\pi,1}}} \mathcal{O}_{X_1}) \).

For any bundle \( V \) on \( \mathcal{X}_{\pi,1} \) (resp. \( \mathcal{X}_{\pi,1} \)) of rank \( r > 1 \), we denote

\[
\deg(V) := \deg(\bigwedge^r V).
\]

By Lemma 2.9, the modulo-\( p \) reduction of the Frobenius lifting is globally well-defined. We denote it by \( \Phi_1 : \mathcal{X}_{\pi,1} \to \mathcal{X}_{\pi,1} \). Since \( \mathcal{X}_{\pi,1} \) and \( \mathcal{X}_{\pi,1} \) have the same closed subset \( X_1 \), we have the following diagram

\[
\begin{array}{c}
\xymatrix{ X_1 \ar[r]^-{\tau} & \mathcal{X}_{\pi,1} \ar[d]^-{\Phi_1} \\
X_1 \ar[r]^-{\tau} & \mathcal{X}_{\pi,1} \\
\end{array}
\]

Here \( \tau \) and \( \tau \) are closed embeddings and \( \Phi_{X_1} \) is the absolute Frobenius lifting on \( X_1 \). We should remark that the diagram above is not commutative, because \( \Phi_1 \) does not preserve the defining ideal of \( X_1 \).

Lemma 2.18. Let \( (V, \nabla, \Fil) \) be an object in \( \mathcal{MF}(\mathcal{X}_{\pi,1}) \) of rank 1. Then there is an isomorphism

\[
\Phi_{X_1}^* \circ \tau^*(\tilde{V}) \sim \tau^* \circ \Phi_1^*(\tilde{V}).
\]

Proof. Recall that \( \{ \mathcal{U}_i \} \) is an open covering of \( \mathcal{X} \). We express the line bundle \( V \) by the transition functions \( (g_{ij}) \), where \( g_{ij} \in (\mathcal{B}_{R_{ij}}/p\mathcal{B}_{R_{ij}})^\times \). Since
$V$ is of rank 1, the filtration Fil is trivial. Then by definition $\tilde{V}$ can also be expressed by $(g_{ij})$. Since $g_{ij} \in B_{R_{ij}}/pB_{R_{ij}}$, one has

$$(\Phi_{X_1}|_{U_{i,1}})^* \circ (\tau|\mathbb{G}_m)^* (g_{ij}) = (\tau|\mathbb{G}_m)^* \circ (\Phi_1|_{\mathbb{G}_m})^* (g_{ij}),$$

by diagram (3). This gives us the isomorphism $\Phi_{X_1}^* \circ \tau^*(\tilde{V}) \xrightarrow{\sim} \tau^* \circ \Phi_1^*(\tilde{V})$.

**Lemma 2.19.** Let $(V, \nabla, \text{Fil})$ be an object in $\mathcal{MCF}(\mathcal{X}_{\pi,1})$. Then we have

$$\text{deg}(\tilde{V}) = \text{deg}(V) \text{ and } \text{deg}(C^{-1}_{\mathcal{X}_{\pi,1}}(\tilde{V})) = p \text{deg}(\tilde{V}).$$

**Proof.** Since the tilde functor and inverse Cartier functor preserve the wedge product and the degree of a bundle is defined to be that of its determinant, we only need to consider the rank 1 case. Now let $(V, \nabla, \text{Fil})$ be of rank 1. The reductions of $V$ and $\tilde{V}$ on the closed fiber $X_1$ are the same, by the proof of Lemma 2.18. Then we have

$$\text{deg}(\tilde{V}) = \text{deg}(V).$$

Since the filtration is trivial, the $p$-connection $\tilde{V}$ is also trivial. In this case, the transition functions $G_{ij}$ in Lemma 2.11 are identities. Thus

$$C^{-1}_{\mathcal{X}_{\pi,1}}(\tilde{V}) = \Phi_1^*(\tilde{V}).$$

Recall that $\text{deg}(\Phi_1^*(\tilde{V})) = \text{deg}(\tau^* \circ \Phi_1^*(\tilde{V}))$ and $\text{deg}(\tilde{V}) = \text{deg}(\tau^*(\tilde{V}))$. Lemma 2.18 implies $\text{deg}(\tau^* \circ \Phi_1^*(\tilde{V})) = \text{deg}(\Phi_{X_1}^* \circ \tau^*(\tilde{V}))$. Since $\Phi_{X_1}$ is the absolute Frobenius, one has $\text{deg}(\Phi_{X_1}^* \circ \tau^*(\tilde{V})) = p \text{deg}(\tau^*(\tilde{V}))$. Composing above equalities, we get $\text{deg}(C^{-1}_{\mathcal{X}_{\pi,1}}(\tilde{V})) = p \text{deg}(\tilde{V})$.

**Theorem 2.20.** Let $\mathcal{E} = \{(E, \theta)_0, (V, \nabla, \text{Fil})_0, (E, \theta)_1, (V, \nabla, \text{Fil})_1, \cdots \}$ be an $L$-twisted $f$-periodic Higgs-de Rham flow with endomorphism structure and log structure over $X_1$. Suppose that the degree and rank of the initial term $E_0$ are coprime. Then the projective representation $\mathbb{D}^P \circ \mathcal{IC}_{\mathcal{X}_{\pi,1}}(\mathcal{E})$ of $\pi_1(X_{K_0}^o)$ is still irreducible after restricting to the geometric fundamental group $\pi_1(X_{K_0}^o)$, where $K_0 = W[1/p]$.

**Proof.** Let $\rho : \pi_1(X_{K_0}^o) \to \text{PGL}(\mathbb{D}^P \circ \mathcal{IC}_{\mathcal{X}_{\pi,1}}(\mathcal{E}))$ be the projective representation. Fix a $K_0$-point in $X_{K_0}$, which induces a section $s$ of the surjective map $\pi_1(X_{K_0}^o) \to \text{Gal}(\overline{K_0}/K_0)$. We restrict $\rho$ on $\text{Gal}(\overline{K_0}/K_0)$ by this section $s$, whose image is finite. And there is a finite field extension $K/K_0$ such that the restriction of $\rho$ by $s$ on $\text{Gal}(\overline{K_0}/K)$ is trivial. Thus

$$\rho(\pi_1(X_{K_0}^o)) = \rho(\pi_1(X_{K_0}^o)).$$

It is sufficient to show that the restriction of $\rho$ on $\pi_1(X_{K_0}^o)$ is irreducible. Suppose that the restriction of $\mathbb{D}^P \circ \mathcal{IC}_{\mathcal{X}_1}(\mathcal{E})$ on $\pi_1(X_{K_0}^o)$ is not irreducible. Since the functors $\mathbb{D}^P$ and $C^{-1}_{\mathcal{X}_{\pi,1}}$ are compatible with those over $X_1$, the projective representation $\mathbb{D}^P \circ \mathcal{IC}_{\mathcal{X}_{\pi,1}}(\mathcal{E} \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{\mathcal{X}_{\pi,1}}) = \mathbb{D}^P \circ \mathcal{IC}_{X_1}(\mathcal{E})$ is also
not irreducible. Thus there exists a non-trivial quotient, which is the image of some nontrivial sub \( \mathcal{L} = L \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_{x,1}} \)-twisted \( f \)-periodic Higgs-de Rham flow of \( \mathcal{E} \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_{x,1}} \)

\[
\{(E', \theta')_0, (V', \nabla', \text{Fil}')_0, (E', \theta')_1, (V', \nabla', \text{Fil}')_1, \ldots \},
\]

under the functor \( \mathbb{D}^P \circ \mathcal{I} \mathcal{C}_{x_1} \) according to Theorem 2.15 and Theorem 2.17. Since \( E'_0 \) is a sub bundle of \( E_0 \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_{x,1}} \), we have \( 1 \leq \text{rank}(E'_0) < \text{rank}(E_0) \). By Theorem 4.17 in [4], \( \deg(E_{i+1}) = p \deg(E_i) \) for \( i \geq 0 \) and \( \deg(E_0) = p \deg(E_{f-1}) + \text{rank}(E_0) \times \deg(L) \). Thus

\[
\frac{\deg(E_0)}{\text{rank}(E_0)} = \frac{\deg(L)}{1 - p^f}.
\]

Similarly, by Lemma 2.19 one gets

\[
\frac{\deg(E'_0)}{\text{rank}(E'_0)} = \frac{\deg(L)}{1 - p^f}.
\]

Since \( \deg(L) = \deg(L) \), one has \( \deg(E_0) \cdot \text{rank}(E'_0) = \deg(E'_0) \cdot \text{rank}(E_0) \). Since \( \deg(E_0) \) and \( \text{rank}(E_0) \) are coprime, \( \text{rank}(E_0) \mid \text{rank}(E'_0) \). This contradicts to \( 1 \leq \text{rank}(E'_0) < \text{rank}(E_0) \). Thus the projective representation \( \mathbb{D}^P \circ \mathcal{I} \mathcal{C}_{X_1}(\mathcal{E}) \) is irreducible.

\[\square\]

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