Axial gravitational perturbations of an infinite static line source

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Received 4 August 2014, revised 18 November 2014
Accepted for publication 9 January 2015
Published 23 February 2015

Abstract

The Levi-Civita metric, which contains a naked singularity that has been interpreted as an infinite static line source, appears, for instance, as the possible end point in the collapse of cylindrically symmetric objects such as shells of dust. The analysis of its gravitational stability should therefore be relevant in the contexts of the cosmic censorship and hoop conjectures. In this paper we study axial gravitational perturbations of the Levi-Civita metric. The perturbations are restricted to axial symmetry but break the cylindrical symmetry of the background metric. We analyze the gauge issues that arise in setting up the appropriate form of the perturbed metric and show that it is possible to restrict the perturbations to diagonal terms but that this does not fix the gauge completely. We derive and solve the perturbation equations. The solutions contain gauge-trivial parts, and we show how to extract the gauge-nontrivial components. We impose appropriate boundary conditions on the solutions and show that these lead to a boundary value problem that determines the allowed functional forms of the perturbation modes. The associated eigenvalues determine a sort of ‘dispersion relation’ for the frequencies and corresponding ‘wave vector’ components. The central result of this analysis is that the spectrum of allowed frequencies contains one unstable (imaginary frequency) mode for every possible choice of the background metric. The completeness of the mode expansion in relation to the initial value problem and to the gauge problem is discussed in detail, and we show that the perturbations contain an unstable component for generic initial data and therefore that the Levi-Civita space times are gravitationally unstable. We also include, for completeness, a set of approximate eigenvalues and examples of the functional form of the solutions.

Keywords: axial symmetry, line source, perturbations
1. Introduction

In this paper we study axial gravitational perturbations of an infinite static line source, represented by a form of the Levi-Civita metric [1] given by Thorne [2]. The Levi-Civita metric, which contains a naked singularity, appears, for instance, as the possible end point in the collapse of a cylindrically symmetric shell of dust [3]. Even if one does not consider an infinite cylinder, on intuitive grounds one would expect that in the case where one has initially an almost static cylinder that is very long compared with its radius, the Levi-Civita metric would describe the gravitational field close to the cylinder for a time on the order of its length. Again on intuitive grounds, one would expect the cylinder to collapse also in the longitudinal direction, but, depending on details of the initial conditions, this might happen in such a way that only one, or several different, ‘clumps’ are formed, possibly creating horizons, in accordance with the cosmic censorship conjecture, and satisfying the hoop conjecture [3]. It is then of interest to analyze the effect of departures from perfect cylindrical symmetry on the stages close to the formation of the singularity. A first step in this direction, because of simplicity, is to analyze the stability of the singular space time itself, as carried out here.

The Levi-Civita metric has been used and studied in many instances, including also the possibility of time dependence, but keeping the cylindrical symmetry of the metric. In this case the gravitational perturbations depend only on time and the radial coordinate, and they correspond to a form of Einstein–Rosen waves [4]. Analyses where the cylindrical symmetry is broken have appeared only recently [5] but are restricted to the evolution of scalar, or Maxwell, test fields in the field of the static line source. The main scope of those studies was directed at elucidating the relationship between the classical and quantum nature of the naked singularity present in the background metric. Our aim here is different. What we want to study is if, given the presence of the naked singularity, even after imposing appropriate boundary conditions, the gravitational perturbations contain an unstable component, as has previously been shown to happen for the negative mass Schwarzschild [6] and for super-extreme Reissner–Nordstrom and Kerr black holes [7]. As is shown here, we also find in this case unstable modes. We remark again that, as is the case in both [6] and [7], the instability is a consequence of the shape of a related ‘potential’ away from the singularity, and therefore, it is possible that it remains even after the (curvature) singularity is smoothed out by spreading the source over a small region. Such spreading could correspond to the replacement of the line (singular) ‘source’ by a matter cylinder of small but finite radius. As shown, e.g., in [8], this ‘matching’ is possible for an arbitrary radius and matter satisfying reasonable energy conditions. Our current analysis, where we investigate the characteristics of the vacuum modes, would be a first step in an analysis of the stability of such a cylinder plus vacuum system. The study of that possibility is, however, outside the scope of the current research.

As indicated, the perturbations we consider here are restricted to axial symmetry but break the cylindrical symmetry of the background metric. In section 3 we analyze the gauge issues that arise in setting up the appropriate form of the perturbed metric and show that it is possible to restrict the perturbations to diagonal terms but that this does not fix the gauge completely. We analyze in detail the gauge ambiguity remaining after imposing the diagonal form, and show that some of this may be removed by restricting the perturbations to thoses that are effectively of compact support. We find several gauge invariants that are used later in the analysis. We derive the perturbation equations in section 4, and in section 5 we show that they can be solved by solving a third-order ordinary differential equation for an appropriately chosen function of the perturbed metric
coefficients. The set of solutions to this equation contains gauge-trivial parts, and we show how to extract the gauge-nontrivial components. We impose appropriate boundary conditions on the solutions in section 6 and show that these lead to a boundary value problem that determines the allowed functional forms of the perturbation modes. The corresponding eigenvalues determine a sort of ‘dispersion relation’ for the frequencies and corresponding ‘wave vector’ components. Since we not have analytic solutions, we use a numerical approach to determine approximate values for eigenvalues and the approximate functional form of the eigenfunctions, for a set of background metrics. The central result of this analysis is that the resultant spectrum of allowed frequencies contains one unstable (imaginary frequency) mode for every possible choice of background metric. The completeness of the mode expansion in relation to the initial value problem and the gauge problem is discussed in detail in section 7, and we show that the perturbations contain an unstable component for generic initial data. This result implies that the Levi-Civita space times are gravitationally unstable. We close the paper in section 8 with some final comments.

2. The metric of a static solution line source

The metric of a static solution line source may be written in the form [2],
\[ ds^2 = e^{2\psi} (dr^2 + r^2 d\phi^2) + e^{2\gamma} (dz^2 + d\tau^2) = e^{-2\psi} (dr^2 - d\psi^2) + e^{2\gamma} d\tau^2 + e^{-2\gamma} r^2 d\phi^2 \] (1)
where in general we have:
\[ \psi(r) = -\kappa \ln (r/R_0) \]
\[ \gamma(r) = \gamma_0 + \kappa^2 \ln (r/R_0) \] (2)
where \( \gamma_0 \) and \( R_0 \) are arbitrary constants that can be reset by changing the scales of \( t, r, z, \phi \), whereas \( \kappa \) is a positive constant related to the mass per unit length of the line source. Without loss of generality and mostly for simplicity we will set \( \gamma_0 = 0 \) and \( R_0 = 1 \) in the rest of this paper. The metric (1) may be thought of as the vacuum metric outside a massive cylinder in the limit where the radius of the cylinder goes to zero but the mass per unit length is kept fixed at some finite value. Notice that in the original form of the metric, analyzed, for instance, in [8], the metric contains a parameter \( \sigma \), for which the line source interpretation is possible only for \( \sigma \leq -1 \). This range corresponds to \( 0 \leq \kappa < \infty \). More explicitly, if we define
\[ \sigma = \frac{\kappa}{2(1 + \kappa)} \] (3)
and change to new coordinates \( (T, R, Z, \Phi) \) in accordance with
\[ r = K_1 R^{1+\sigma} \]
\[ t = N_1 T \]
\[ z = L_1 Z \]
\[ \phi = M_1 \Phi \] (4)
where \( K_1, L_1, M_1, \) and \( N_1 \) are arbitrary constants, the metric takes the general Levi-Civita form used in, e.g., [8]:
\[ dx^2 = -N^2 R^{4\sigma} d\tau^2 + R^{4(n-1)} \left( K^2 dR^2 + L^2 dZ^2 \right) + M^2 R^{2-4\sigma} d\Phi^2 \]  

(5)

where \( K_1, L_1, M_1, \) and \( N_1 \) are arbitrary constants. Thus (1) and (5) are isometric for all \( \kappa \). As indicated in [8], for any \( \sigma \) in \( 0 < \sigma < 1/2 \) and therefore \( 0 < \kappa < \infty \), one can match (5) to a matter cylinder of arbitrary nonvanishing radius, satisfying acceptable energy conditions. (Refer to [8] for details.)

3. Perturbations along the symmetry axis I: the gauge problem

In this paper we consider perturbations that preserve the axial symmetry. These correspond to terms in the metric that depend only on \( t, r, \) and \( z \). If we call \( g_{0}^{ab} \) the static metric given by (1) and (2), and \( g_{1}^{ab} \) the perturbation, the perturbed metric \( g \) is given by

\[ g_{ab} = g_{0}^{ab} + \epsilon g_{1}^{ab} \]  

(6)

where \( \epsilon \) is the perturbation parameter, and we will consider the entire expression only up to the first order in \( \epsilon \).

We restrict the perturbations \( g_{1}^{ab} \) to axial symmetry and set \( g_{ab}^{1} = 0 \) if one (but not both) of the indices corresponds to \( \phi \).\(^1\) This implies in principle that we have seven independent functions for \( g_{ab}^{1} \), but we may use the freedom to change the \( (r, z, t) \) coordinates also to the first order in \( \epsilon \) to restrict the perturbations to only diagonal terms. In more detail, we consider a (linearized) set of coordinates of the form,

\[ t = T + \epsilon t_1(\rho, T, Z) \]
\[ r = \rho + \epsilon r_1(\rho, T, Z) \]
\[ z = Z + \epsilon z_1(\rho, T, Z) \]  

(7)

Under this transformation, the off-diagonal terms in (6) become

\[ g_{\rho T}(\rho, T, Z) = \left( g_{\rho T}^{0}(\rho, T, Z) + g_{\rho T}^{1}(\rho) \left( \frac{\partial h_1}{\partial T} - \frac{\partial h_1}{\partial \rho} \right) \right) \]
\[ g_{\rho Z}(\rho, T, Z) = \left( g_{\rho Z}^{0}(\rho, T, Z) + g_{\rho Z}^{1}(\rho) \left( \frac{\partial h_1}{\partial Z} + \frac{\partial h_1}{\partial \rho} \right) \right) \]
\[ g_{TZ}(\rho, T, Z) = \left( g_{TZ}^{0}(\rho, T, Z) - g_{TT}^{0}(\rho) \left( \frac{\partial h_1}{\partial T} - \frac{\partial h_1}{\partial \rho} \right) \right) \]  

(8)

Setting all the left-hand sides in (8) to zero, we get a set of equations for \( t_1, r_1, \) and \( z_1 \) that can be straightforwardly solved for any choice of \( g_{\rho T}^{1}, g_{\rho Z}^{1}, \) and \( g_{TZ}^{1} \). Therefore, any off-diagonal terms in (6) can be eliminated by a suitable coordinate transformation. Consequently, we take for the perturbed metric the form,

\[ dx^2 = -e^{2\gamma_2-2\gamma_1}(1 + e h_1(r, t, z))dr^2 + e^{2\gamma_2-2\gamma_1}(1 + e h_{rr}(r, t, z))dz^2 \]
\[ + e^{2\gamma_2}(1 + e h_{zz}(r, t, z))dz^2 + e^{-2\rho_2}(1 + e h_{\Phi\Phi}(r, t, z))d\Phi^2 \]  

(9)

\(^1\) One can verify that the linearized equations for the set \( \{ h_0^{\rho T}, h_0^{\rho Z}, h_0^{TZ} \} \) actually decouple from the rest, and therefore the solutions correspond to modes independent of those analyzed here. They will be considered elsewhere.
Unfortunately, imposing the vanishing of the off-diagonal terms does not fix the gauge completely, and therefore, (9) is not unique. It is easy to get the most general transformation of the form (7) that preserves the diagonal form (9). The corresponding coordinate transformation functions can be written in the form,

\[ t_1(\rho, T, Z) = \frac{\partial W_1}{\partial T} + e^{3\psi-\gamma} \frac{\partial W_2}{\partial T} \]

\[ \eta(\rho, T, Z) = \frac{\partial W_1}{\partial \rho} + \left( 2 \frac{\partial \psi}{\partial \rho} - \frac{\partial y}{\partial \rho} \right) e^{3\psi-\gamma} W_2 + e^{4\psi-2\gamma} \frac{\partial W_1}{\partial \rho} \]

\[ z_1(\rho, T, Z) = e^{-2\psi+\gamma} \frac{\partial W_2}{\partial Z} - \frac{\partial W_3}{\partial Z} \]  

(10)

where \( W_1 = W_1(\rho, T), W_2 = W_2(T, Z), \) and \( W_3 = W_3(\rho, Z) \) are arbitrary independent functions of the indicated arguments. Under this restricted set of transformations the metric coefficients are modified in accordance with

\[ h_{rr} \rightarrow h_{rr} - 2e^{2\psi-\gamma} \left[ 2 \left( \frac{\partial \psi}{\partial r} \right)^2 - 3 \frac{\partial \psi}{\partial r} \frac{\partial y}{\partial r} + \left( \frac{\partial y}{\partial r} \right)^2 \right] W_2 + 2 \frac{\partial^2 W_1}{\partial r^2} + 2e^{2\psi-\gamma} \frac{\partial^2 W_2}{\partial r^2} \]

\[ - 2 \left( \frac{\partial \psi}{\partial r} - \frac{\partial y}{\partial r} \right) \left( \frac{\partial W_1}{\partial r} + e^{4\psi-2\gamma} \frac{\partial W_3}{\partial r} \right) \]

\[ h_{rr} \rightarrow h_{rr} + 2 \left[ 2 \left( \frac{\partial \psi}{\partial r} \right)^2 - \frac{\partial \psi}{\partial r} \frac{\partial y}{\partial r} + 2 \frac{\partial^2 \psi}{\partial r^2} - \frac{\partial^2 y}{\partial r^2} \right] W_2 + 2e^{4\psi-2\gamma} \left( 3 \frac{\partial \psi}{\partial r} - \frac{\partial y}{\partial r} \right) \frac{\partial W_3}{\partial r} \]

\[ - 2 \left( \frac{\partial \psi}{\partial r} - \frac{\partial y}{\partial r} \right) \frac{\partial W_1}{\partial r} + 2e^{4\psi-2\gamma} \frac{\partial^2 W_3}{\partial r^2} + 2 \frac{\partial^2 W_1}{\partial r^2} \]

\[ h_{zz} \rightarrow h_{zz} + 2 \frac{\partial \psi}{\partial r} \frac{\partial W_1}{\partial r} + 2e^{2\psi+\gamma} \frac{\partial^2 W_3}{\partial r^2} - 2 \frac{\partial^2 W_3}{\partial r^2} \]

\[ + 2e^{-2\psi+\gamma} \frac{\partial y}{\partial r} \frac{\partial \psi}{\partial r} \frac{\partial W_1}{\partial r} W_2 + 2e^{4\psi-2\gamma} \frac{\partial \psi}{\partial r} \frac{\partial W_3}{\partial r} \]

\[ h_{\phi\phi} \rightarrow h_{\phi\phi} - \frac{2}{r} e^{2\psi-\gamma} \left[ - \frac{\partial \psi}{\partial r} + \frac{\partial y}{\partial r} + 2r \left( \frac{\partial \psi}{\partial r} \right)^2 - r \frac{\partial \psi}{\partial r} \frac{\partial y}{\partial r} \right] W_2 \]

\[ - \frac{2}{r} \left( \frac{\partial \psi}{\partial r} - \frac{\partial y}{\partial r} \right) \left( \frac{\partial W_1}{\partial r} + e^{4\psi-2\gamma} \frac{\partial W_3}{\partial r} \right) \]  

(11)

Notice that in all quantities that are already first order in \( \epsilon \), the arguments may be taken as \((\rho, T, Z)\) or \((r, t, z)\) indistinctly, as this leads to differences that are second order in \( \epsilon \).

From the linearity of the equations and the independence of the functions \( W_i \), we may consider separately the effect and consequences of their existence. We notice that they are functions of only two of the three coordinates \((t, r, z)\), and therefore, they are relevant in different contexts. In the following subsections we explore these features in detail.
3.1. Case $W_1(\rho, T) \neq 0$, $W_2(T, Z) = 0$ and $W_3(\rho, Z) = 0$

This case is relevant when we have perturbations that preserve the cylindrical symmetry of the unperturbed metric, i.e., both $\partial_t$ and $\partial_\phi$ remain Killing vectors, and therefore, the perturbations may be restricted to depend only on $(r, t)$. In this case, we need to consider only transformations where $W_1(\rho, T) \neq 0$, whereas we must set $W_2(\rho, Z) = 0$ and $W_3(T, Z) = 0$. One can verify that for this type of perturbation the gauge-independent part corresponds to cylindrical gravitational waves (see, e.g., [2]) and should be excluded here, as we are interested only in perturbations that break the cylindrical symmetry. From the point of view of the initial value problem, such perturbations correspond to initial data independent of $z$, and they can therefore be excluded, as will be carried out in what follows, by simply restricting the perturbations to initial data that is compactly supported in both $z$ and $r$ or more generally, that has a nontrivial dependence on $z$—for instance, an overall $z$-dependent factor, such as $\exp(ikz)$, as in the derivations in the following sections.

3.2. Case $W_1(\rho, T) = 0$, $W_2(T, Z) = 0$ and $W_3(\rho, Z) \neq 0$: 'zero' modes

This case is relevant when we consider axially symmetric, static perturbations, i.e., those for which both $\partial_t$ and $\partial_\phi$ remain Killing vectors. Since the background metric is independent of $t$, we may restrict the perturbations to those that depend only on $(r, z)$ and are also independent of $t$. When acceptable, they are usually called 'zero' modes. To analyze this case we first explore the general solution to the linearized Einstein equations by assuming for the perturbations the form,

$$
\begin{align*}
    h_\tau(r, t, z) &= F_1(r, z) \\
    h_\rho(r, t, z) &= F_2(r, z) \\
    h_\phi(r, t, z) &= K_1(r, z) \\
    h_\phi(r, t, z) &= K_2(r, z)
\end{align*}
$$

Substituting into the linearized Einstein equations we find that\(^2\)

\[ F_2 = \frac{r}{(1 + \kappa)^2} \frac{\partial F_1}{\partial r} + \frac{r}{(1 + \kappa)^2} \frac{\partial K_2}{\partial r} + \kappa(2 + \kappa) \frac{F_1}{(1 + \kappa)^2} + \frac{1}{(1 + \kappa)^2} + \frac{(1 + 2\kappa)}{(1 + \kappa)^2} \]

and

\[
\frac{\partial K_1}{\partial r} = - \frac{\kappa^2}{(1 + \kappa)^2} \frac{\partial K_2}{\partial r} - \frac{1}{(1 + \kappa)^2} \frac{\partial F_1}{\partial r} - \frac{1}{(1 + \kappa)^2} r^{2\kappa + 4\kappa + 1} \left( \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 K_2}{\partial z^2} \right)
\]

There is only one further independent equation, which can be written in the form,

\[
\frac{\partial^2 Q}{\partial r^2} + \frac{1}{r} \frac{\partial Q}{\partial r} + r^{2\kappa + 2} \frac{\partial^2 Q}{\partial z^2} = 0
\]

where

\[ Q(r, z) = F_1(r, z) - \kappa K_2(r, z) \]

Thus, the general $t$-independent solution is determined by a solution to (15) plus an arbitrary function of $(r, z)$. To see the relationship of this result to the gauge ambiguities we notice that under a coordinate transformation where only $W_3(\rho, Z) \neq 0$, we have that the relevant

\(^2\) Actually, equation (13) is a first-integral, derived from $G_{1}^{1} = 0$ and $G_{1}^{2} = 0$, where $G_{1}^{ab}$ is the first-order Einstein tensor and we have set an integration constant equal to zero.
functions $h_{ab}$ transform as

\begin{align}
F_1(r, z) & \rightarrow F_1(r, z) + 2\kappa (1 + \kappa) r^{-2\kappa^2-4\kappa-1} \frac{\partial W_3}{\partial r} \\
F_2(r, z) & \rightarrow F_2(r, z) - 2\kappa (3 + \kappa) r^{2\kappa^2-4\kappa-1} \frac{\partial W_3}{\partial r} + 2 r^{-2\kappa^2-4\kappa-1} \frac{\partial^2 W_3}{\partial r^2} \\
K_1(r, z) & \rightarrow K_1(r, z) - 2\kappa r^{-2\kappa^2-4\kappa-1} \frac{\partial W_3}{\partial r} - 2 \frac{\partial^2 W_3}{\partial r^2} \\
K_2(r, z) & \rightarrow K_2(r, z) + 2(1 + \kappa) r^{-2\kappa^2-4\kappa-1} \frac{\partial W_3}{\partial r} 
\end{align}

(17)

Notice that this implies that when $Q(r, z) = 0$, we can find a function $W_3(r, z)$, determined up to the addition of an arbitrary function of $z$, such that the transformed $F_1$ and $K_2$ are both zero, and therefore, on account of (13), we have $F_2 \rightarrow 0$ and $\partial K_1/\partial r \rightarrow 0$. Finally, using the arbitrary function of $z$ in $W_3$, on account of (17), we may also set the transformed $K_1 = 0$. Thus the perturbations are trivial unless $Q(r, z) \neq 0$.

The transformation law for $F_1$ and $K_2$ implies that the function $Q(r, z)$ is gauge invariant, and therefore, as indicated, it represents a gauge-nontrivial perturbation. To analyze this in more detail, we notice that if we write

$$Q(r, z) = \int e^{-ik\rho} \tilde{Q}(k, r) dk ,$$

(18)

$\tilde{Q}(k, r)$ satisfies the equation,

$$\frac{d^2 \tilde{Q}}{dr^2} + \frac{1}{r} \frac{d \tilde{Q}}{dr} - k^2 r^{2\kappa^2+2\kappa} \tilde{Q} = 0 ,$$

(19)

whose general solution is

$$\tilde{Q}(k, r) = C_1 \text{I}_0\left( k r^{\kappa^2+1}\right) \left(1 + \kappa\right) + C_2 \text{K}_0\left( k r^{\kappa^2+1}\right) \left(1 + \kappa\right)$$

(20)

where $C_1$ and $C_2$ are constants and $\text{I}_0(x)$ and $\text{K}_0(x)$ are modified Bessel functions of the first and second kind, respectively. But given the properties of these functions, this result implies that there are no gauge-invariant, time-independent perturbations that are finite in both the limits $r \rightarrow 0$ and $r \rightarrow \infty$. We may again exclude them by restricting the perturbations to initial data either of compact support in both $(r, z)$ or satisfying appropriate boundary conditions for $r \rightarrow 0$ and $r \rightarrow \infty$.

3.3. Case $W_1(\rho, T) = 0$, $W_2(T, Z) \neq 0$, and $W_3(\rho, Z) = 0$

This type of transformation is relevant in the general problem where the perturbations effectively depend on $(r, t, z)$, with the restrictions previously indicated, to eliminate $W_1$ and $W_3$ (see sections 4 and 5 for more details). The result of applying it to the perturbed metric (9) is
Since this preserves the general form (9), any result obtained by solving Einstein’s equations for $h_{ab}$ will be subject to a gauge ambiguity. Nevertheless, we may extract from (21) the following gauge-invariant quantities:

\[
\begin{align*}
\mathcal{G}_1(r, t, z) &= h_{zz} + \frac{\kappa}{\kappa + 1} h_{\phi\phi} + \frac{r^{2(1 + \kappa)} t^2}{\kappa(1 + \kappa)(2 + \kappa)} \frac{\partial^2 h_{\phi\phi}}{\partial t^2} \\
\mathcal{G}_2(r, t, z) &= \frac{1}{1 + \kappa} h_{\phi\phi} + \frac{r^{2(\kappa + 1)} t^2}{\kappa(1 + \kappa)(2 + \kappa)} \frac{\partial^2 h_{\phi\phi}}{\partial t^2} + \frac{r^2}{\kappa(1 + \kappa)^2(2 + \kappa)} \frac{\partial^2 h_{\phi\phi}}{\partial t^2} \\
\mathcal{G}_3(r, t, z) &= h_{tt} - \frac{r^2}{\kappa(1 + \kappa)(2 + \kappa)} \frac{\partial^2 h_{\phi\phi}}{\partial t^2}
\end{align*}
\]

(22)

Actually, these three functions are not independent, and we have

\[
\mathcal{G}_1 - \mathcal{G}_2 - \frac{1}{1 + \kappa} \mathcal{G}_3 = 0
\]

(23)

The implications for deriving relevant physical conclusions will be considered in the following sections.

4. Perturbations along the symmetry axis II: the linearized equations

As discussed in the preceding sections, we may restrict the perturbations to diagonal terms. Furthermore, since the unperturbed metric depends only on $r$, and the equations are linear, it is appropriate to assume that the dependence on $t$ and $z$ is only through a factor $\exp(i (\Omega t - kz))$, with the most general solution a linear combination of the resultant solutions. Thus we take for the perturbed metric the form,

\[
\begin{align*}
\text{d}s^2 &= -\ddot{\epsilon}^2 r^{-2} (1 + \epsilon e^{i(\Omega t - kz)} F_1(r)) \text{d}t^2 + \ddot{\epsilon}^2 r^{-2} (1 + \epsilon e^{i(\Omega t - kz)} F_2(r)) \text{d}r^2 \\
&\quad + \epsilon e^{2i(\Omega t - kz)} K_1(r) \text{d}z^2 + e^{-2\epsilon r^2} (1 + \epsilon e^{i(\Omega t - kz)} K_2(r)) \text{d}\phi^2
\end{align*}
\]

(24)

where, as usual, it is understood that one should take only the real part of the full complex expression.

We next impose that (24) satisfies the vacuum Einstein equations to the first order in $\epsilon$. This leads in general to a set of coupled second-order ordinary differential equations for the functions $F_1$ and $K_1$, but it is easy to show that we must have

\[
F_2(r) = -K_2(r)
\]

(25)

and one can reduce the rest of the system to the following set of coupled first-order equations:
What we have in mind here is that, with appropriate boundary conditions, the set of solutions to \( (26) \), together with the \( e^{-ikz} \) factors, will provide a basis for an expansion of the arbitrary functions of \((r, z)\), leading to the formal solution to the initial value problem for the perturbations. In light of this, we may look at \( (26) \) as a boundary value problem that determines the allowed values of \( \Omega \) such that the boundary conditions are satisfied. As we show in the next section, it turns out to be convenient for this purpose to change \( (26) \) to an equation for a single function that satisfies a third-order ordinary differential equation. An issue that will also require further discussion is that of the gauge dependence of the solutions.

5. The general solution

Considering again the system \( (26) \), we notice that, if we are interested in the initial value problem for the perturbations, in particular if we are considering the possibility of unstable modes, then the question is, given \( \kappa \) and \( k \), are there solutions with acceptable values of \( \Omega \)? That is, are there values of \( \Omega \) such that the solutions satisfy appropriate boundary conditions? This is a standard boundary value problem, but it is not easy to handle if given in the form of \( (26) \). A simpler problem is obtained if we introduce two new functions, \( G_1(r) \) and \( G_2(r) \), such that

\[
\begin{align*}
G_1(r) &= (G_1(r) + G_2(r))/2; \\
G_2(r) &= (G_1(r) - G_2(r))/2
\end{align*}
\]

Then, using \( (26) \) repeatedly, we find

\[
\begin{align*}
G_2(r) &= -\frac{\kappa^2 + \kappa - 1}{1 + \kappa} G_1(r) + \frac{r}{1 + \kappa} \frac{dG_1}{dr} \\
F_1(r) &= \frac{r^2}{k^2 r^2 (1 + \kappa)^2 + 2 \kappa + \kappa^2} \frac{d^2 G_1}{dr^2} + \frac{r \left( 6 + 8 \kappa + \kappa^2 + k^2 r^2 (1 + \kappa)^2 \right)}{2 (1 + \kappa) \left( k^2 r^2 (1 + \kappa)^2 + 2 \kappa + \kappa^2 \right)}
\end{align*}
\]
\[ -\frac{4\kappa + 10\kappa^2 + 6\kappa^3 + \kappa^4 - 2r^2(1 + \kappa)\Omega^2 + \kappa(2 + \kappa)k^2r^2(1 + \kappa)^2}{2(1 + \kappa)(k^2r^2(1 + \kappa)^2 + 2\kappa + \kappa^2)}G_1(r) \] (29)

while for \( G_1 \) we find the equation,

\[
\frac{d^3G_1}{dr^3} = k^2A \left( k^2A + 6\kappa^2 + 12\kappa + 3 \right) - \left( k^2A + 2\kappa + \kappa^2 \right) r^2\Omega^2 - \kappa \left( \kappa^3 + 4\kappa + 6 + 7\kappa \right) \frac{dG_1}{dr} \]

\[ + \frac{(\kappa + 3)(\kappa - 1)k^2A - \kappa^4 - 4\kappa^3 - 9\kappa^2 - 10\kappa d^2G_1}{r \left( k^2A + 2\kappa + \kappa^2 \right)} \frac{dr^2}{dr} \]

\[ + \frac{\kappa(2 + \kappa) \left( \left( k^2A - \kappa^2 - 2\kappa - 2 \right) r^2\Omega^2 - k^2A \left( k^2A + 2 + 6\kappa + 3\kappa^2 \right) \right)}{r^3 \left( k^2A + 2\kappa + \kappa^2 \right)} G_1 \]
(30)

where

\[ A = r^2(1 + \kappa)^2 \]
(31)

It may appear rather unexpected that (30) has an exact solution given by

\[ G_1(r) = \frac{\kappa(2 + 2) + k^2r^2(1 + \kappa)^2}{r^2 + 2\kappa + \kappa^2} \]
(32)

Notice that (32) is independent of \( \Omega \). One can show, however, following the discussion in section 3.3, that this solution is a consequence of the gauge ambiguity intrinsic in the system (26). To see how this happens we recall that \( G_1 \) is defined up to a constant and write,

\[ G_1(r) = w(\Omega, k) \frac{\kappa(2 + 2) + k^2r^2(1 + \kappa)^2}{r^2 + 2\kappa + \kappa^2} \]
(33)

where \( w(\Omega, k) \) is an arbitrary function of \( \Omega \) and \( k \). Substituting into (27) and (28), we find (recall that \( F_2 = -K_2 \))

\[ F_1(r) = \kappa^2(1 + \kappa)(2 + \kappa)r^{-2 - 2\kappa - \kappa^2}w(\Omega, k) + \Omega^2r^{-\kappa(2 + \kappa)}w(\Omega, k) \]

\[ K_1(r) = -\kappa^2(2 + \kappa)r^{-2 - 2\kappa - \kappa^2}w(\Omega, k) + k^2r^{\kappa(2 + \kappa)}w(\Omega, k) \]

\[ K_2(r) = \kappa(1 + \kappa)(2 + \kappa)r^{-2 - 2\kappa - \kappa^2}w(\Omega, k) \]
(34)

Upon multiplication by \( e^{(i(kr - \kappa z))} \) and integration on \( \Omega \) and \( k \), we recover the full \( (r, t, z) \) dependence, and we get

\[ F_1(r, t, z) = \kappa^2(1 + \kappa)(2 + \kappa)r^{-2 - 2\kappa - \kappa^2}W(t, z) - r^{-\kappa(2 + \kappa)}\frac{\partial^2W}{\partial t^2} \]

\[ K_1(r, t, z) = -\kappa^2(2 + \kappa)r^{-2 - 2\kappa - \kappa^2}W(t, z) - r^{\kappa(2 + \kappa)}\frac{\partial^2W}{\partial z^2} \]

\[ K_2(r, t, z) = \kappa(1 + \kappa)(2 + \kappa)r^{-2 - 2\kappa - \kappa^2}W(t, z) \]
(35)
where

\[ W(t, z) = \int \int e^{i(\mathbf{r} \cdot \mathbf{k})} W(\Omega, k) \, d\Omega \, dk \]  

But a comparison with (21) immediately shows that this solution is pure gauge and should be discarded. Nevertheless, we may use it to look for solutions to (30) of the form,

\[ G_1(r) = \frac{\kappa(\kappa + 2) + k^2r^2(1 + \kappa)^2}{r^2 + 2\kappa + \kappa^2} H_1(r) \]  

Substituting into (30), we find that \( H_1(r) \) satisfies the equation,

\[ -\frac{d^2H_1}{dr^2} - \left( \frac{3 + 4\kappa + 2\kappa^2}{r} \right) k^2A - \kappa \left( 9\kappa + 2 + 2\kappa^3 + 8\kappa^2 \right) \frac{d^2H_1}{dr^2} \\
+ \left[ \frac{k^4A^2 \left( k^2A + 8\kappa - 4\kappa^3 + 3 - \kappa^4 \right) - \kappa(\kappa + 2) \left( 5\kappa(2 + \kappa)(3 + 2\kappa^2 + 4\kappa) + 6 \right) k^2A}{r^2 \left( k^2A + \kappa^2 + 2\kappa \right)^2} \right] \frac{dH_1}{dr} \\
- \frac{\kappa^2(2 + \kappa)^2(1 + \kappa)^4}{r^2 \left( k^2A + \kappa^2 + 2\kappa \right)^2} \left[ \frac{dH_1}{dr} \right] \\
= \Omega^2 \frac{dH_1}{dr} \]  

(38)

To simplify the treatment further, we introduce a new function \( H_2(r) \) such that

\[ \frac{dH_1(r)}{dr} = \mathcal{K}(r) H_2(r) \]  

(39)

where

\[ \mathcal{K}(r) = \frac{\sqrt{r}e^{2\kappa x}}{\kappa(\kappa + 2) + k^2r^2(1 + \kappa)^2} \]  

(40)

Substituting into (35) we get an equation for \( H_2(r) \) of the form,

\[ -\frac{d^2H_2}{dr^2} + V_2 H_2(r) = \Omega^2 H_2(r) \]  

(41)

where \( V_2 \) is a function of \( r, \kappa, \) and \( k \). It will be convenient to introduce a further change of variable and a new function given by

\[ x = \frac{1}{k^2r^2 + \kappa} r \]

\[ H_2(r) = H_3(x(r)) \]  

(42)

and we find that \( H_3(x) \) satisfies the equation,

\[ -\frac{d^2H_3}{dx^2} + V_3(x) H_3(x) = \lambda H_3(x) \]  

(43)
where
\[ \lambda = \frac{\Omega^2}{|k|^2} \]  
(44)

and
\[ V_3(x) = \left[ -\kappa^2 (2 + \kappa)^2 + 4 \chi^{6(1+\kappa)^2} + \left( 48 \kappa + 15 + 24 \kappa^2 \right) x^{4(1+\kappa)^2} \right. \]
\[ \left. - 2 \kappa (2 + \kappa) \left( 16 \kappa^4 + 64 \kappa^3 + 86 \kappa^2 + 44 \kappa + 9 \right) x^{2(1+\kappa)^2} \right] \times \left[ 4 x^2 \left( x^{2(1+\kappa)^2} + \kappa (2 + \kappa) \right)^2 \right]^{-1} \]  
(45)

We cannot fail to notice that (44) has the form of a ‘dispersion relation’ \( \Omega = |k|^2 \sqrt{\lambda} \) that changes with \( \lambda \). We remark, however, that \( k \) is not the modulus of the ‘wave vector’. In principle, one can understand (44) as resulting from the interference between a traveling wave part in the \( z \) direction and a standing wave in the radial direction that characterize the modes described by \( H_3 \).

Equation (43) has the typical form of an eigenvalue–eigenfunction equation. We first need to establish the general behavior of its solutions for both \( x \to 0 \) and \( x \to \infty \). Since (43) is a second-order ordinary differential equation, it has two independent solutions. Let us consider first \( x \to 0 \). It is not difficult to show that near \( r = 0 \), the general solution to (43) admits an asymptotic expansion to the form,
\[ H_3(x) = \sqrt{\kappa} \sum_{j=0}^\infty a_{2j}^{(2)} x^{2j} + \ln (x) \sum_{j=0}^\infty b_{2j}^{(2)} x^{2j} \]  
(46)

or, in more detail, up to the leading-order terms in \( x \),
\[ H_3(x) = \sqrt{\kappa} \left[ a_0^{(0)} + \frac{\lambda}{4} \left( b_0^{(0)} - a_0^{(0)} \right) x^2 + \ldots + \ln (x) \left( b_0^{(0)} - a_0^{(0)} \right) x^2 + \ldots \right] \]
\[ + x^{2(1+\kappa)^2} \left[ a_0^{(2)} + a_2^{(2)} x^2 + \ldots + \ln (x) \left( b_0^{(2)} + b_2^{(2)} x^2 + \ldots \right) \right] + \ldots \]  
(47)

where the dots indicate higher terms. All the \( b_{2j}^{(2)} \) coefficients are proportional to \( b_0^{(0)} \), and the \( a_{2j}^{(2)} \) coefficients are linear homogeneous in \( a_0^{(0)} \) and \( b_0^{(0)} \). This implies that we have two sets of independent solutions that can be selected by choosing either \( b_0^{(0)} = 0 \) or \( a_0^{(0)} = 0 \). In the first case \( b_0^{(0)} = 0 \), the solutions behave as \( \sqrt{\kappa} + \mathcal{O}(x^{5/2}) \) as \( x \to 0 \); and in the second \( a_0^{(0)} = 0 \), the solutions behave as \( \ln (x) \left( \sqrt{\kappa} + \mathcal{O}(x^{5/2}) \right) \) as \( x \to 0 \).

In accordance with (34), (36), (40), and (42), we have that the general solution of (30) may be written in the form,
\[ G_1(r) = \frac{k^2 \gamma x^{2(1+\kappa)^2} + \kappa (\kappa + 2)}{2^2 \sqrt{2} \sqrt{\kappa}^{2x^2} + \kappa (\kappa + 2)^2} \times \int_0^\infty \sqrt{\eta} \frac{r_1^{2x^2}}{k^2 r_1^{2x^2} + \kappa (\kappa + 2)} \left[ |k|^{2(1+\kappa)^2} \eta \right] d\eta + C_{G_1} \]  
(48)

where \( H_3(x) \) is the general solution to (43) and \( C_{G_1} \) is a constant. With this result and the expansion (47), we conclude that besides the solution provided by (32), \( G_1(r) \) has two other
linearly independent solutions, which, near \( r = 0 \), and up to the terms of order \( r^2 \), may be written in the form,

\[
G_1(r) \sim b_0 \left( 1 + b_2 r^2 + \ldots \right) + c_0 \left[ 1 + c_2 r^2 + \ldots + \ln(r) \left( d_0 + d_2 r^2 + \ldots \right) \right]
\]  

(49)

where \( b_2, c_2, d_0, \) and \( d_2 \) depend on \( \Omega \) and \( \kappa \) and are finite. Thus we get one solution that is finite for \( r = 0 \) by setting \( b_0 \neq 0, \ c_0 = 0 \), and an independent solution that diverges as \( \ln(r) \) by setting \( b_0 = 0, \ c_0 \neq 0 \).

The behavior of \( H_3(x) \) for large \( x \) is more difficult to establish, and we have not been able to obtain asymptotic expansions similar to (46) or (47). We nevertheless notice that for large \( x \) we have

\[
V_3(x) \sim x^{2\kappa(2+\kappa)}
\]  

(50)

and we may approximate (43) by

\[
-\frac{d^2 H_3}{dx^2} + x^{2\kappa(2+\kappa)} H_3(x) = 0
\]  

(51)

The general solution to this equation is

\[
H_3(x) = C_1 \sqrt{x} I_{2(1+\kappa)} \left( \frac{x^{(1+\kappa)^2}}{(1 + \kappa)^2} \right) + C_2 \sqrt{x} K_{2(1+\kappa)} \left( \frac{x^{(1+\kappa)^2}}{(1 + \kappa)^2} \right)
\]  

(52)

where \( I_\nu(x) \) and \( K_\nu(x) \) are modified Bessel functions and \( C_1 \) and \( C_2 \) are constants. Actually (52) is consistent only if we keep the leading terms Therefore, for \( x \to \infty \) we have two linearly independent solutions for \( H_3(x) \), whose leading terms are

\[
H_3(x) = C_3 \frac{1}{\sqrt{\pi} x^\kappa} \exp \left( \frac{x^{(1+\kappa)^2}}{(1 + \kappa)^2} \right) + C_4 \frac{1}{\sqrt{\pi} x^\kappa} \exp \left( -\frac{x^{(1+\kappa)^2}}{(1 + \kappa)^2} \right)
\]  

(53)

where \( C_3 \) and \( C_4 \) are constants.

We need to establish now the boundary conditions that should be considered acceptable. In principle, we expect a perturbation to be finite everywhere and that it should be possible to make it arbitrarily smaller than the background. On this account, considering (48), the only acceptable solutions for \( r \to 0 \) are those where we take the solution for \( H_3 \) that vanishes as \( x \), and \( C_1 = 0 \).

To analyze the behavior for large \( r \) we write (48) (with \( C_1 = 0 \)) in the form,

\[
G_1(r) = -\frac{r^{2(1+\kappa)^2 + \kappa(\kappa + 2)}}{r^{2+2\kappa+\kappa^2}} \int_r^\infty \frac{\sqrt{\eta} \tau_{1(1+\kappa)^2}}{\tau_{1(1+\kappa)^2} + \kappa(\kappa + 2)} H_3 \left( |k|^{(1+\kappa)^2} \eta \right) \eta d\eta
\]

\[
+ \frac{r^{2(1+\kappa)^2 + \kappa(\kappa + 2)}}{r^{2+2\kappa+\kappa^2}} I_3
\]  

(54)

where

\[
I_3 = \int_0^\infty \frac{\sqrt{\eta} \tau_{1(1+\kappa)^2}}{\tau_{1(1+\kappa)^2} + \kappa(\kappa + 2)} H_3 \left( |k|^{(1+\kappa)^2} \eta \right) \eta d\eta
\]  

(55)

Therefore, if we impose the condition that \( H_3(x) \) should vanish for \( x \to \infty \), the integral \( I_3 \) will be finite, and as \( x \to \infty \), the first term in (54) vanishes, and the remaining term, in accordance with our previous discussion, is pure gauge. Thus, we conclude that the
appropriate boundary conditions for $H_3(x)$ are

$$H_3(x) \sim \sqrt{x}, \quad \text{for } x \to 0$$

$$H_3(x) \to 0, \quad \text{for } x \to \infty$$

Note that the last condition corresponds to setting $C_3 = 0$ in (53).

The consequences of these boundary conditions for $H_3$ on its spectrum, and therefore on the behavior of the perturbations, are analyzed in the next section.

6. The spectrum of $\Omega$ and unstable solutions

As indicated in the preceding section, we need to impose boundary conditions on $G_\ell(r)$ to restrict the perturbations to geometrically acceptable ones. This, in turn, imposes boundary conditions on $H_3(x)$. These conditions are that $H_3$ should go to zero for $x \to \infty$ and that it should vanish as $\sqrt{x}$ for $x \to 0$. We recall that $H_3$ is a solution to (43). With the stated boundary conditions the operator on the left to (43) is self-adjoint, and, since $V_3 \to +\infty$ for $x \to +\infty$, its spectrum (i.e., the set of allowed values of $\lambda$) is fully discrete [11]. We may classify the corresponding eigenfunctions by the number of nodes $j$, that is, the zeros of $H_3$ for $x \neq 0$, and indicate them as $H_3^{(j)}(x)$. We then have

$$-\frac{d^2H_3^{(j)}}{dx^2} + V_3(x)H_3^{(j)}(x) = \lambda_j H_3^{(j)}(x)$$

where $\lambda_j$ is the corresponding eigenvalue.

Since the spectrum is discrete, the eigenfunctions $H_3^{(j)}$ are real and can be normalized so that they satisfy the relations,

$$\int_0^\infty H_3^{(j)}(x)H_3^{(\ell)}(x)dx = \delta_{j\ell}$$

With this normalization the set $\{H_3^{(j)}(x): j = 0, 1, 2 \ldots\}$ provides a complete orthonormal basis for the expansion of functions of $x$ in $0 < x < \infty$.

Table 1. Approximate values for $\lambda_0$ and $\lambda_1$ for several choices of $\kappa$.

| $\kappa$ | $\lambda_0$   | $\lambda_1$   |
|----------|---------------|---------------|
| 0.001    | -1.00446265  | 1.01856808    |
| 0.005    | -1.0062283   | 1.0784925     |
| 0.01     | -1.01146     | 1.14635835    |
| 0.05     | -1.06132     | 1.653817      |
| 0.1      | -1.137215    | 2.303944      |
| 0.5      | -2.38340712  | 5.65762       |
| 1.0      | -6.928205    | 6.928165      |
| 2.0      | -44.50382    | 9.153668      |
| 2.5      | -91.590175   | 5.331185      |

A pedagogical discussion relevant to the problem of the self-adjoint extensions of a potential behaving as $-a/x^2$ near $x = 0$, including the case $a = 1/4$, is contained, e.g., in [9]. For a more technical discussion one may refer, e.g., to [10].
We recall that
\[ \Omega = \kappa + k(59) \]
Therefore, a negative \( \lambda \) corresponds to a pure imaginary value of \( \Omega \) and this, in turn, to unstable perturbations that diverge exponentially over time. Because of the rather involved dependence of \( V_3 \) on \( x \), it is not easy to determine analytically the details of the spectrum of \( \lambda \).

We may, nevertheless, resort to numerical methods to try to determine at least the lowest values of \( \lambda \), remembering that they still depend on \( \kappa \).

As a simple numerical approach we imposed on \( H_x(3) \) the boundary condition (47), with \( b_0(0) = b_0(0) \), and, using a 'shooting' method, we solved the boundary value problem for (57) for the lowest eigenvalues, finding approximate values for \( \lambda \) and corresponding functional forms for \( H_x(3) \) for several choices of \( \kappa \). The results for \( \lambda_0 \) and \( \lambda_1 \) are shown in table 1.

Some examples of the functional forms of \( H_x(3) \) are given in figure 1, and the potential \( V_3(x) \) is plotted in figure 2 for several values of \( \kappa \). It is apparent from these plots that \( V_3(x) \) is strongly dependent on \( \kappa \), and this, in turn is reflected in the dependence of \( \lambda_j \) on \( \kappa \).

The most important feature of these results is that, in all cases considered, \( \lambda_0 < 0 \), and therefore, we have shown that there is one unstable solution set for the equations of motion for the perturbations for every value of \( \kappa > 0 \) in the range of \( \kappa \) considered. Although we do not have analytic results, we can prove that this result must hold for any value of \( \kappa \) as follows.

Suppose there is a solution to \( H^{(0)}_j(x) \) of (57) with \( \lambda_j = 0 \). This implies that \( \Omega_j = 0 \) for all \( k \), and therefore, substituting into (48) we would obtain a gauge-nontrivial, time-independent solution that is finite in both the limits \( r \to 0 \) and \( r \to \infty \). But we have already shown that there are no such solutions, and therefore, (57) cannot have solutions with \( \lambda = 0 \) satisfying the boundary conditions (56). But all \( \lambda_j \), and in particular, \( \lambda_0 \) and \( \lambda_1 \), are continuous functions of \( \kappa \), and we have shown numerically that for a range of values of \( \kappa \) we have \( \lambda_0 < 0 < \lambda_1 \); and therefore, since they cannot vanish for any \( \kappa \), for all \( \kappa \) we must have \( \lambda_0 < 0 \) and \( \lambda_1 > 0 \), and we conclude that for all \( \kappa \) we have one and only one negative eigenvalue \( \lambda \).

We recall that
\[ \Omega^2 = |k|^{-1/2} \lambda \]  
Therefore, a negative \( \lambda \) corresponds to a pure imaginary value of \( \Omega \) and this, in turn, to unstable perturbations that diverge exponentially over time. Because of the rather involved dependence of \( V_3 \) on \( x \), it is not easy to determine analytically the details of the spectrum of \( \lambda \).

As a simple numerical approach we imposed on \( H_x(3) \) the boundary condition (47), with \( b_0(0) = b_0(0) \), and, using a 'shooting' method, we solved the boundary value problem for (57) for the lowest eigenvalues, finding approximate values for \( \lambda \) and corresponding functional forms for \( H_x(3) \) for several choices of \( \kappa \). The results for \( \lambda_0 \) and \( \lambda_1 \) are shown in table 1.

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Suppose there is a solution to \( H^{(0)}_j(x) \) of (57) with \( \lambda_j = 0 \). This implies that \( \Omega_j = 0 \) for all \( k \), and therefore, substituting into (48) we would obtain a gauge-nontrivial, time-independent solution that is finite in both the limits \( r \to 0 \) and \( r \to \infty \). But we have already shown that there are no such solutions, and therefore, (57) cannot have solutions with \( \lambda = 0 \) satisfying the boundary conditions (56). But all \( \lambda_j \), and in particular, \( \lambda_0 \) and \( \lambda_1 \), are continuous functions of \( \kappa \), and we have shown numerically that for a range of values of \( \kappa \) we have \( \lambda_0 < 0 < \lambda_1 \); and therefore, since they cannot vanish for any \( \kappa \), for all \( \kappa \) we must have \( \lambda_0 < 0 \) and \( \lambda_1 > 0 \), and we conclude that for all \( \kappa \) we have one and only one negative eigenvalue \( \lambda \).
7. The initial value problem

In this section we consider the initial value problem for the perturbed system, but, before we carry out this analysis we need to see the way in which the general problem of evolution is solved after we determine the allowed functions $H_3$ and the corresponding spectrum of values of $\lambda$. Let us consider again the functions $F_r(x)^1$, $F_r(x)^2$, $K_r(x)^1$, and $K_r(x)^2$. We have found that for every choice of $k$, after imposing appropriate boundary conditions, we have an infinite set of solutions, each one characterized by a function $H_3^j(x)$ and a value of $\lambda_j$. The allowed values of $\Omega$ follow from (59).

\[
\Omega_j(k) = \pm k \sqrt{\varepsilon_j + \kappa^2} \quad (60)
\]

In what follows we will call $\Omega_j(k)$ the quantity corresponding to taking the plus sign in (60) and will add a minus sign explicitly when necessary.

If we identify the corresponding metric coefficients as $F_j^1(k, r)$, $F_j^2(k, r)$, $K_j^1(k, r)$, and $K_j^2(k, r)$, where we remark that these functions depend only on $k^2$, we can construct more-general solutions by taking linear combinations. For example, we would have

\[
\sum_j \int \left( C_j^+(k) e^{i(\Omega_j k r - \kappa z)} + C_j^-(k) e^{i(\Omega_j k r + \kappa z)} \right) K_j^1(k, r) dk
\]

and similar expressions, with the same coefficients $C_j^{(\pm)}(k)$, for $h_{rr}(r, t, z)$ and $h_{rt}(r, t, z)$. We may partially invert (61) multiplying by $e^{ikz}$ and integrating over $z$. Furthermore, we may set $t = 0$ in both the functions and their $t$ derivatives, and finally combine them as follows:

\[
\int e^{ikz} \left( h_{zz}(r, 0, z) + h_{\phi\phi}(r, 0, z) \right) dz
\]

\[
= 2\pi \sum_j \left( C_j^+(k) + C_j^-(k) \right) \left( K_j^1(k, r) + K_j^2(k, r) \right) \quad (62)
\]
and

$$\int e^{ikz} \left( \frac{\partial h_{zz}}{\partial t} \bigg|_{t=0} + \frac{\partial h_{\phi\phi}}{\partial r} \bigg|_{r=0} \right) dz = 2\pi i \sum_j \Omega_j(k) \left( C_j^{(+)\ast}(k) - C_j^{(-)}(k) \right) \left( K^{(j)}_1(k, r) + K^{(j)}_2(k, r) \right)$$

(63)

Then, if we define

$$I_1(k, r) = \int e^{ikz} \left( h_{zz}(r, 0, z) + h_{\phi\phi}(r, 0, z) \right) dz$$

$$I_2(k, r) = \int e^{ikz} \left( \frac{\partial h_{zz}}{\partial t} \bigg|_{t=0} + \frac{\partial h_{\phi\phi}}{\partial r} \bigg|_{r=0} \right) dz$$

(64)

we have

$$I_1(k, r) = 2\pi \sum_j \left( C_j^{(+)\ast}(k) + C_j^{(-)}(k) \right) G^{(j)}_1(k, r)$$

$$I_2(k, r) = 2\pi i \sum_j \Omega_j(k) \left( C_j^{(+)\ast}(k) - C_j^{(-)}(k) \right) G^{(j)}_1(k, r)$$

(65)

The functions $G^{(j)}_1(k, r)$ do not satisfy orthonormality or other immediately useful conditions. Nevertheless, if we go back to (48), with $C_{\ell i} = 0$, we have

$$H_3^{(j)} \left( |k|^{2(1+\kappa)^2} \right) = \frac{k^{2r}\Omega_2^{(1-\kappa)^2}(k) + \kappa(2 + \kappa) \partial}{\sqrt{r^{2k+\kappa^2}}} \left( \frac{r^{2+2x+k^2}}{k^{2r}\Omega_2^{(1-\kappa)^2}(k) + \kappa(2 + \kappa)} I_1(k, r) \right)$$

(66)

Therefore, if we define

$$J_1(k, r) = \frac{k^{2r}\Omega_2^{(1-\kappa)^2}(k) + \kappa(2 + \kappa) \partial}{\sqrt{r^{2k+\kappa^2}}} \left( \frac{r^{2+2x+k^2}}{k^{2r}\Omega_2^{(1-\kappa)^2}(k) + \kappa(2 + \kappa)} I_1(k, r) \right)$$

$$J_2(k, r) = \frac{k^{2r}\Omega_2^{(1-\kappa)^2}(k) + \kappa(2 + \kappa) \partial}{\sqrt{r^{2k+\kappa^2}}} \left( \frac{r^{2+2x+k^2}}{k^{2r}\Omega_2^{(1-\kappa)^2}(k) + \kappa(2 + \kappa)} I_2(k, r) \right)$$

(67)

we have

$$J_1(k, r) = 2\pi \sum_j \left( C_j^{(+)\ast}(k) + C_j^{(-)}(k) \right) H_3^{(j)} \left( |k|^{2(1+\kappa)^2} \right)$$

$$J_2(k, r) = 2\pi i \sum_j \Omega_j(k) \left( C_j^{(+)\ast}(k) - C_j^{(-)}(k) \right) H_3^{(j)} \left( |k|^{2(1+\kappa)^2} \right)$$

(68)

and we may use the orthonormality of the $H_3^{(j)}(x)$ to obtain

$$\int_0^\infty H_3^{(j)} \left( |k|^{2(1+\kappa)^2} \right) J_1(k, r) dr = \frac{2\pi}{k^{1/2(1+\kappa)^2}} \left( C_j^{(+)\ast}(k) + C_j^{(-)}(k) \right)$$

$$\int_0^\infty H_3^{(j)} \left( |k|^{2(1+\kappa)^2} \right) J_2(k, r) dr = \frac{2\pi}{k^{1/2(1+\kappa)^2}} \Omega_j(k) \left( C_j^{(+)\ast}(k) - C_j^{(-)}(k) \right)$$

(69)

which can be used to solve for the coefficients $C_j^{(\pm)}(k)$. Therefore, we may invert (61) and, given $h_{zz}$ and $h_{\phi\phi}$, obtain the corresponding expansion coefficients. In fact, we only need...
This immediately suggests that, given any solution \( h_{\phi\phi}(r, t, z) \) to the perturbation equations, or rather, to the initial value problem, we may use (69) to compute the coefficients \( C_l^{(\pm)}(k) \) and find the evolution of the system that results from that initial data. Although this is conceptually correct, it requires closer examination. If we go back to (67), we notice that the right-hand sides act as projectors, since any portion of the initial data such that

\[
\kappa \kappa \kappa \kappa = \kappa \kappa \kappa \kappa + \kappa \kappa \kappa \kappa \kappa \kappa \kappa \kappa
\]

\[
(\kappa + 2) r^2 + \kappa \kappa \kappa \kappa + \kappa \kappa \kappa \kappa \kappa \kappa \kappa \kappa
\]

\[
(70)
\]

where \( q_1 \) and \( q_2 \) are constants, will be deleted from the computation of the coefficients \( C_l^{(\pm)}(k) \). But we have already noticed that this type of functional dependence is pure gauge and can be eliminated by an appropriate coordinate transformation. We can understand this feature by noticing that for a single mode, the gauge invariant \( \sim \mathcal{G}_l \) is given by,

\[
\mathcal{G}_l(r, t, z) = e^{i(\omega_l t - kr)} \left( K_1(r) + \frac{\kappa}{1 + \kappa} K_2(r) - \frac{k^2 r^2 (1 + \kappa)^2}{\kappa (1 + \kappa)(2 + \kappa)} K_2(r) \right)
\]

\[
e^{i(\omega_l t - kr)} \frac{(\kappa (2 + \kappa) + k^2 r^2 (1 + \kappa)^2)}{2\kappa(2 + \kappa)(1 + \kappa)^2} \frac{d}{dr} \left( \frac{r^2 + 2\kappa r^2}{\kappa (2 + \kappa) + k^2 r^2 (1 + \kappa)^2} G_l(r) \right)
\]

\[
e^{i(\omega_l t - kr)} \left( \frac{(\kappa (2 + \kappa) + k^2 r^2 (1 + \kappa)^2)}{2\kappa(2 + \kappa)(1 + \kappa)^2} r \right) H_{3l}^{(0)}(\sqrt{\kappa (1 + \kappa)^2} r)
\]

and, therefore, the functions \( H_{3l}^{(0)}(\sqrt{\kappa (1 + \kappa)^2} r) \) are themselves gauge invariant, and an expansion of the form of (61), even if obtained from arbitrary initial data, evolves only the gauge-nontrivial part of the perturbation. In particular, any initial data that has nonvanishing projection on \( H_{3l}^{(0)}(\sqrt{\kappa (1 + \kappa)^2} r) \) will lead to a gauge-nontrivial unstable evolution, diverging exponentially over time.

8. Final comments

In this paper we analyzed the axial gravitational perturbations of an infinite line source, and, after imposing boundary conditions at the symmetry axis and at radial infinity, such that the perturbations, in the sense of initial data, can be made arbitrarily smaller than the background, we found the general solution to the corresponding linearized Einstein equations. We analyzed also the problem of the gauge invariance of the solutions and found a complete set of gauge-nontrivial solutions with which we can describe the evolution of arbitrary initial data for the perturbations. The main result of our analysis is that the evolution will contain generically unstable components and therefore that the space time considered here is gravitationally unstable. Since this space time contains a naked singularity, one would be tempted to ascribe the instability to the presence of that singularity. We remark, however, that here, as in the cases considered in [6], and [7], the unstable mode is related to the form of a ‘potential’ away from the singularity, indicating the possibility that the instability might remain even after smoothing the (curvature) singularity by considering an extended source. In the current
case, we would have to consider a cylindrical regular source, such as an infinite cylinder of some kind of matter. The problem in this construction is that the resultant system is considerably more complex than the vacuum case considered here since, aside from conditions that must be imposed at the matter–vacuum boundary, we must incorporate an equation of state for the matter as well as the resultant appropriate boundary conditions. The question, nevertheless, is interesting, but it is outside the scope of this research.

Acknowledgments

This work was supported in part by CONICET (Argentina). I am grateful to Gustavo Dotti, Guido Raggio, and the anonymous referees for their helpful comments, suggestions, and criticisms.

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