A CHARACTERIZATION OF GROUPS OF TYPE $F_4$
ARISING FROM THE FIRST TITS CONSTRUCTION

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In memoriam Professor Jacques Tits

Abstract. We show that algebraic groups of type $F_4$ (or equivalently
Albert algebras) arising from the first Tits construction are determined
uniquely by their $g_3$ invariant.

1. Introduction

It is well-known that the set of isomorphism classes of Albert algebras over
a field $K$ is in one-to-one correspondence with the set of isomorphism classes
of algebraic groups over $K$ of type $F_4$. By (non-abelian) Galois cohomology
considerations there are canonical bijections between each these two sets and
the pointed set $H^1(K, G_0)$, were $G_0$ is the split $K$-group of type $F_4$.

A finer understanding of these algebras and groups can be obtained by
means of “cohomological invariants” first defined by J.-P. Serre (see [8] for a full
account of the theory). J.-P. Serre and M. Rost introduced three cohomological
invariants

\[ f_3 : H^1(K, G_0) \rightarrow H^3(K, \mathbb{Z}/2\mathbb{Z}), \quad f_5 : H^1(K, G_0) \rightarrow H^5(K, \mathbb{Z}/2\mathbb{Z}), \quad \text{and} \]
\[ g_3 : H^1(K, G_0) \rightarrow H^3(K, \mathbb{Z}/3\mathbb{Z}) \]

that are central to our work.

In 1991 J.-P. Serre asked the following question:

Do the cohomological invariants $f_3$, $f_5$ and $g_3$ classify Albert algebras over $K$
up to isomorphism?

This question is one of the main remaining open problems in the theory
of Jordan algebras. The main result of our paper answers Serre’s question
in the affirmative for Albert algebras arising from the first Tits construction.
The difference between Albert algebras arising from the first and second Tits
constructions notwithstanding, we hope that some of the ideas used in the
proof may shed light on a path forward towards a general affirmative answer
to the question.

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\[ ^1 \text{Recall that } G_0 = \text{Aut}(G_0). \]
The structure of the paper is as follows. §2 summarizes the basic notation and conventions regarding algebraic groups and algebras. A summary of known results about Albert algebras and their cohomological invariants is given in §3. This section also has the precise statement of our main theorem, and the next nine sections give its proof.

§4 appeals to a result of P. Gille to reduce the problem to the case when the base field $K$ is of characteristic 0. In §5 further properties of Albert algebras $A$ arising from the first Tits construction are given. This section also contains properties of a cubic Galois extension $E$ of $K$ contained in an Albert algebra $A$, and two very important algebraic groups attached to $A$ and $E$, namely the structure group $\text{Str}(A)$ and the group $\mathcal{H}$ of automorphisms of $A$ that fix $E$.

In §6 we construct a special maximal torus $T$ of $\mathcal{H}$ that is key to the proof. Indeed in §7 we use a result of Steinberg to reduce our problem to classes of $H^1(K, G_0)$ that come from $H^1(K, T)$.

§8 is long, very technical and at first glance out of context. It shows the existence of a very special embedding $T \hookrightarrow G_0$ whose image has the property that the classes of a cocycle and of its inverse have the same image in $H^1(K, G_0)$. This is needed later in the argument whereby an element of the Brauer group of $K$ is allowed to be replaced by its inverse. The key to the section is a result of Tits about the lifting to the normalizer $N_{G_0}(K)$ of elements of a Weyl group $W$ of type $F_4$.

The embedding identifies $T$ with a twisted torus $^{\text{tw}}T$ of $^{\text{tw}}G_0 = G_0$. Twisting of some intermediate groups leads to a sequence of $K$-groups $T \subset M_0 \subset H_0 \subset G_0$. Our ultimate goal, as explained above, is to show that two given classes of $H^1(K, G_0)$ with the same $g_3$ invariant coincide. We do not know how to do this directly, but can show that the two classes already coincide in $H^1(K, M_0)$. The explanation for this very strange phenomenon and the final path to the end of the proof is given at the end of the section.

In §9 we show that we may assume that the Galois group of $K$ is a pro-3-group. §10 shows that, assuming that $\text{Gal}(K)$ is pro-3-group, the classes of our two starting cocycles, not only agree on $H^1(K, G_0)$, but also in $H^1(K, H_0)$. In §11, we show that the map $H^1(K, M_0) \to H^1(K, H_0)$ has trivial kernel. The end of the proof is given in §12 by a delicate twisted argument on $M_0$ that utilizes Severi-Brauer varieties and the Rost invariant.

The final section §13 gives two applications of our main result that strengthen two important Theorems in the theory of Albert Algebras.

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2 As explained in the Introduction, to a $K$-Albert algebra $A$, it corresponds a class $[\xi_A] \in H^1(K, G_0)$. The main theorem states that if $A$ and $A'$ arise from the first Tits construction, then $g_3(A) = g_3(A')$ implies that $A$ and $A'$ are isomorphic. This amounts to $[\xi_A] = [\xi_A']$. The reduction in question shows that these classes may be assumed to come from $H^1(K, T)$.

3 Recall that we are assuming that our two classes come from $H^1(K, T)$. 
Throughout $K$ will denote a field. We fix once and for all a separable and algebraic closure $K \subset K^{\text{sep}} \subset \overline{K}$ of $K$. The Galois group of the extension $K^{\text{sep}}/K$ will be denoted by $\text{Gal}(K)$.

An algebraic group over $K$, or simply a $K$-group is an affine group scheme of finite type over $K$. If $R$ is a ring extension of $K$, we denote as usual by $G(R)$ the (abstract) group of $R$-points of $G$. This is the functor of points of $G$. The derived group of $G$ will be denoted by $G'$, and its centre by $Z(G)$. For a closed subgroup $H$ of $G$ the centralizer and normalizer of $H$ in $G$ will be denote by $N_G(H)$ and $Z_G(H)$ respectively.

Let $T \subset G$ be a $K$-subgroup. If $\xi \in Z^1(K, T)$ is a cocycle, we denote by $\xi_G$ its image in $Z^1(K, G)$. In addition if $L/K$ is a field extension and $[\xi] \in H^1(K, T)$ we denote by $[\xi]_L$ the image of $[\xi]$ in $H^1(L, G)$. 

### 3. Recollections of Albert algebras and cohomological invariants

In this section we recall with references some known results and state our main Theorem. Throughout this section we assume that the base field $K$ is of characteristic $\neq 2, 3$. For the concept and references on Albert algebras over fields of characteristic 2, 3 we refer to [15].

#### 3.1. Albert Algebras

A Jordan algebra over $K$ is a unital, commutative, not necessarily associative $K$-algebra $A$ in which the Jordan identity

$$(xy)(xx) = x(y(xx))$$
is satisfied. Given an associative unital algebra \( B \) with multiplication written by juxtaposition, the anticommutator \( x \cdot y = \frac{1}{2}(xy + yx) \) defines on the vector space \( B \) the structure of a Jordan algebra, denoted by \( B^+ \). A Jordan algebra \( A \) is said to be special if it is isomorphic to a Jordan subalgebra of \( B^+ \) for an associative algebra \( B \), and exceptional otherwise. An Albert algebra is by definition a finite dimensional simple, exceptional Jordan algebra.

**Reduced Albert algebras.** Let \( C \) be an octonion \( K \)-algebra and \( \gamma_1, \gamma_2, \gamma_3 \in K^\times \). Write \( \Gamma \) for the diagonal matrix whose coefficients are \( (\gamma_1, \gamma_2, \gamma_3) \).

On the matrix algebra \( M_3(C) \) we define the involution \( M \mapsto M^* = \Gamma^{-1} M^T \Gamma \) where \( M \) is obtained by applying the conjugation of \( C \) on each coefficient and \( T \) is the transpose. Let \( \text{Her}_3(C, \Gamma) \) be the vector space of \( * \)-symmetric elements endowed with the product \( M \cdot N = \frac{1}{2}(MN + NM) \) where the product on the right-hand side is the matrix product.

This is an Albert algebra.\(^4\) The case where \( C \) is the split octonion algebra and \( \gamma_1 = \gamma_2 = \gamma_3 = 1 \) is called the split Albert algebra.

**Theorem 3.1.**
(1) Over a separably closed field, the split Albert algebra is the only Albert algebra up to isomorphism.
(2) Albert \( K \)-algebras are exactly the \( K \)-forms of the split Albert \( K \)-algebra.

**Proof.** See [1]. \( \square \)

Albert algebras are thus of dimension 27.

3.2. **Albert algebras arising from the first Tits construction.** Let \( D \) be a central simple associative \( K \)-algebra of degree 3, and let \( \mu \in K^\times \). We define an Albert \( K \)-algebra \( A(D, \mu) \) as follows. As a vector space let
\[
A(D, \mu) = D \oplus D \oplus D.
\]

For \( a, b \in D \), we denote by \( ab \) the product of \( D \) and by \( a \cdot b = \frac{1}{2}(ab + ba) \) the product of \( D^+ \). If \( T \) the reduced trace of \( D \), write
\[
a \times b = 2a \cdot b - T(a)b - T(b)a + T(a)T(b) - T(a \cdot b)
\]
and
\[
\tilde{a} = \frac{1}{2}(T(a) - a).
\]

Then the product in \( A(D, \mu) \) of \( x = (a_1, a_2, a_3) \) and \( y = (b_1, b_2, b_3) \) is \( x \cdot y = (c_1, c_2, c_3) \) where
\[
c_1 = a_1 \cdot b_1 + \tilde{a_2} b_3 + b_3 \tilde{a_1},
\]
\[
c_2 = \tilde{a_1} b_2 + \tilde{b_1} a_2 + \frac{1}{2\mu} a_3 \times b_3,
\]
\[
c_3 = a_3 \tilde{b_1} + b_3 \tilde{a_1} + \frac{\mu}{2} a_2 \times b_2.
\]

---

\(^4\) Albert algebras obtained in this way up to isomorphism are called reduced.
The Jordan algebra $D^+$ is seen to be a subalgebra of $A(D, \mu)$ identified with the first component $D$ of $A$.

Albert algebra isomorphic to an $A(D, \mu)$ are said to be of, or arise from the first Tits construction.

3.3. **Algebraic groups attached to Albert algebras.** It is well known that the automorphism group $G_0 = \text{Aut}(A_0)$ of the split Albert algebra $A_0$ is a split simple $K$-group of type $F_4$ (see [13, Theorem 25.13]). By non-abelian Galois cohomology considerations we conclude that the isomorphism classes of Albert $K$-algebras are in one-to-one correspondence with the set $H^1(K, G_0)$ of isomorphism classes of $G_0$-torsors. On the other hand, since the adjoint map $G_0 \to \text{Aut}(G_0)$ is an isomorphism, we get that elements in $H^1(K, G_0)$ are in one-to-one correspondence with isomorphism classes of groups of type $F_4$ over $K$. We thus get the following result.

**Theorem 3.2.** There is a one-to-one correspondence between the sets of

- isomorphism classes of Albert $K$-algebras;
- isomorphism classes of semisimple $K$-group of type $F_4$;
- isomorphism classes of $G_0$-torsors.

If needed, this theorem allows us to freely identify an Albert $K$-algebra $A$ with the corresponding $G_0$-torsor or with its automorphism group $\text{Aut}(A)$ which is a simple $K$-group of type $F_4$. Furthermore, we deduce from the theorem that two Albert algebras are isomorphic if and only if their $K$-groups of automorphisms are.

Recall that an Albert $K$-algebra $A$ is equipped with a cubic form $N : A \to K$, called the norm of $A$. One of the most important $K$-groups attached to $A$ is its structure group $\text{Str}(A)$, whose functor of points is given by

$$\text{Str}(A)(R) = \{ x \in \text{GL}(A)(R) \mid N_R(x(a)) = \nu(x)N_R(a) \quad \forall a \in A_R \},$$

where $R$ is a ring extension of $K$, $N_R$ is the base change of $N$ to $A_R$, and the multiplier $\nu(x) \in R^\times$ depends only on $x$.\footnote{Note that $\nu(x)$ is “functorial on $R$.”}

The derived subgroup $\text{Str}(A)'$ of $\text{Str}(A)$ is known to be a strongly inner form of a split simple simply connected $K$-group of type $E_6$. Moreover, $\text{Str}(A)$ is an almost direct product of $G_m$ and $\text{Str}(A)'$, their intersection being the centre $\mu_3$ of $\text{Str}(A)'$. Note that $\text{Str}(A)$ is reductive and contains $G := \text{Aut}(A)$ since the elements of $G$ preserve the norm. Since $G$ is its own derived group, we have in fact $G \subset \text{Str}(A)'$.

3.4. **Cohomological invariants.** Let $K$ be a field of characteristic $\neq 2, 3$, and let $G_0$ be the simple split $K$-group of type $F_4$. J.-P. Serre [8, Theorems 22.4 and 22.5] constructed two cohomological invariants with coefficients in the group $\mathbb{Z}/2\mathbb{Z}$:

$$f_3 : H^1(-, G_0) \to H^3(-, \mathbb{Z}/2\mathbb{Z}) \quad \text{and} \quad f_5 : H^1(-, G_0) \to H^5(-, \mathbb{Z}/2\mathbb{Z}).$$
Furthermore, M. Rost [20] defined a cohomological invariant with coefficients in \( \mathbb{Z}/3\mathbb{Z} \):

\[
g_3 : H^1(-, G_0) \longrightarrow H^3(-, \mathbb{Z}/3\mathbb{Z}).
\]

The \( H^i \) above are functors from the category of field extensions of \( K \) to the category of sets, and \( f_3, f_5 \) and \( g_3 \) natural transformations. According to Theorem 3.2, to an Albert \( K \)-algebra \( A \) or a semisimple \( K \)-group \( G \) of type \( F_4 \) it corresponds a unique \( [\xi] \in H^1(K, G_0) \). By a harmless abuse of notation we will denote \( g_3([\xi]) \) by \( g_3(A) \). Similarly for \( f_3 \) and \( f_5 \).

We remind the reader that H. Petersson and M. Racine extended the construction of \( f_3, f_5, g_3 \) to fields of characteristic 2 and 3 (see [16], [17], [18]).

For a fixed \( K \), one consolidates these three cohomological invariants into a map

\[
\phi : H^1(K, G_0)_{g_3=0}^{(f_3, f_5, g_3)} \longrightarrow H^3(K, \mathbb{Z}/2\mathbb{Z}) \times H^5(K, \mathbb{Z}/2\mathbb{Z}) \times H^3(K, \mathbb{Z}/3\mathbb{Z}).
\]

One of the remaining fundamental open problems in the theory of Jordan algebras is to determine if \( \phi \) is injective. The following theorem contains a partial result in this direction. We will denote by \( H^1(K, G_0)_{g_3=0} \) the subset of cohomology classes having trivial \( g_3 \)-invariant.

**Theorem 3.3.** ([23]) The map

\[
H^1(K, G_0)_{g_3=0}^{(f_3, f_5)} \longrightarrow H^3(K, \mathbb{Z}/2\mathbb{Z}) \times H^5(K, \mathbb{Z}/2\mathbb{Z})
\]

is injective.

**Corollary 3.4.** The map \( \phi \) has trivial kernel.

Lastly, we recall the following two results.

**Theorem 3.5.** [15, Theorem 49] Let \( A \) be an Albert \( K \)-algebra. Then \( A \) is a division algebra if and only if \( g_3(A) \neq 0 \). □

**Theorem 3.6.** ([15, Theorem 58]) An Albert \( K \)-algebra \( A \) arises from the first Tits construction if and only if \( f_3(A) = 0 \). □

The main result of our paper is the following

**Theorem 3.7.** Let \( K \) be a field of an arbitrary characteristic. Then the invariant \( g_3 \) classifies Albert \( K \)-algebras arising from the first Tits construction.

Lastly we mention that for an arbitrary simple simply connected group \( G \) over a field \( K \) of arbitrary characteristic, M. Rost constructed a cohomological invariants

\[
R_G : H^1(-, G) \longrightarrow H^3(-, \mathbb{Q}/\mathbb{Z}(2)).
\]

For the details we refer the reader to [8].

\[6\] \( f_5 \) is the cup product of \( f_3 \) with an element of \( H^2(-, \mathbb{Z}/2\mathbb{Z}) \). In particular \( f_5(A) = 0 \) whenever \( A \) arises from the first Tits construction.
4. Reduction to characteristic 0

A result of P. Gille allows us to restrict the proof of the main Theorem to the case when $K$ is of characteristic 0.

Proposition 4.1. If the invariant $g_3$ determines (up to isomorphism) Albert algebras arising from the first Tits construction over fields of characteristic 0, then it does so over arbitrary fields.

Proof. Let $F$ be a field of positive characteristic. By [25, Lem. 10.160.6] there exists a complete discrete valuation ring $\mathcal{O}$ whose fraction field $K$ has characteristic 0, and whose residue field is $F$. Let $G_0$ be the Chevalley-Demazure $\mathbb{Z}$-group of type $F_4$. The map $H^1(\mathcal{O}, G_0) \to H^1(K, G_0)$ is injective and the map $H^1(\mathcal{O}, G_0) \to H^1(F, G_0)$ is a bijection (see [5, Section 3, Lemme 2]).

Let $R_{G_0}$ be the Rost invariant of $G_0$ over $K$ or $F$. Theorem 2 of [11] states that there exists an endomorphism $h$ of the group $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ such that the following diagram commutes:

$$
\begin{array}{ccc}
H^1(K, G_0) & \xrightarrow{R_{G_0}} & H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \\
| & | & |
\uparrow & h & \uparrow
\end{array}
$$

$$
\begin{array}{ccc}
H^1(\mathcal{O}, G_0) & \cong & H^1(F, G_0) \xrightarrow{R_{G_0}} H^3(F, \mathbb{Q}/\mathbb{Z}(2))
\end{array}
$$

The set of all normalized cohomological invariants of $G_0$ with values in $H^3(-, \mathbb{Q}/\mathbb{Z}(2))$ has a natural abelian group structure. According to Rost’s Theorem (see [8]), this abelian group is cyclic of order 6 and is generated by $R_{G_0}$.

On the other hand we have the cohomological invariants

$$
H^1(-, G_0) \xrightarrow{f_3} H^3(-, \mathbb{Z}/2\mathbb{Z}) \to H^3(-, \mathbb{Q}/\mathbb{Z}(2))
$$

and

$$
H^1(-, G_0) \xrightarrow{g_3} H^3(-, \mathbb{Z}/3\mathbb{Z}) \to H^3(-, \mathbb{Q}/\mathbb{Z}(2))
$$

of order 2 and 3 respectively. Their sum $f_3 + g_3$ in the group of normalized cohomological invariants is of order 6, so that $f_3 + g_3 = \pm R_{G_0}$. It follows that we can replace $R_{G_0}$ by $f_3 + g_3$ in (*) and the resulting diagram still commutes.

Let $x, y \in H^1(F, G_0)$ be such that $f_3(x) = f_3(y) = 0$ and $g_3(x) = g_3(y)$. Denote by $x'$ (resp. $y'$) the image of $x$ (resp. $y$) by

$$
H^1(F, G_0) \simeq H^1(\mathcal{O}, G_0) \to H^1(K, G_0).
$$
The commutativity of (⋆) for \( f_3 + g_3 \) implies that \( f_3(x') = f_3(y') = 0 \) and \( g_3(x') = g_3(y') \). Thus, if \( g_3 \) classifies Albert algebras arising from the first Tits constructions over \( K \), then \( x' = y' \). Since the map \( H^1(\mathcal{O}, G_0) \to H^1(K, G_0) \) is injective we have \( x = y \).

5. Further results on Albert Algebras

5.1. Assumptions and Conventions. Throughout \( A \) will denote an Albert \( K \)-algebra. In view of Proposition 4.1 to establish our main Theorem we may, and will henceforth assume that the base field \( K \) is of characteristic zero.

Throughout we assume that \( A \) arises from the first Tits construction or, what is equivalent (Theorem 3.6), \( f_3(A) = 0 \). We also assume that \( g_3(A) \neq 0 \). Then, by Theorem 3.5, \( A \) is a division algebra. In particular \( \text{Str}(A)' \), hence also \( G = \text{Aut}(A) \), are anisotropic. Indeed, we have the following.

**Proposition 5.1.** Let \( A \) be an Albert \( K \)-algebra such that \( f_3(A) = 0 \). Then \( A \) is a division algebra if and only if the derived group \( \text{Str}(A)' \) of \( \text{Str}(A) \) is anisotropic.

**Proof.** For convenience we denote \( \text{Str}(A)' \) by \( D \).

Assume that \( A \) is not a division algebra. Then by Theorem 3.5 \( g_3(A) = 0 \). Since in addition \( f_3(A) = 0 \), it follows from Corollary 3.4 that \( A \) is split. Then \( \text{Aut}(A) \subset D \) is split and therefore \( D \) is isotropic.

Conversely, assume that \( A \) is a division algebra and that \( D \) is isotropic. Fix a copy of \( G_m \) inside \( D \subset \text{Str}(A) \subset \text{GL}(A) \). Let \( B = \{v_1, \ldots , v_{27}\} \) be a \( K \)-basis of \( A \) in which \( G_m \) acts diagonally. Let \( \chi_i \) be the corresponding characters of \( G_m \), that is

\[
t \cdot v_i = \chi_i(t)v_i \quad \forall t \in G_m(K) = K^\times
\]

Let \( e_i \in \mathbb{Z} \) be such that

\[
\chi_i(t) = t^{e_i}.
\]

We have

\[
\nu(t)N(v_i) = N(t \cdot v_i) = N(t^{e_i}v_i) = t^{3e_i}N(v_i).
\]

Since \( A \) is a division algebra \( N(v_i) \neq 0 \). It follows that all the \( e_i \) coincide and therefore that the action of \( t \in G_m(K) \) on \( A = A \otimes_k K \) is given by scalar multiplication by \( \nu(t) = t^e \) for some \( e \in \mathbb{Z} \).

Recall that \( \text{Str}(A) = G_m \cdot D \), where this copy of \( G_m \) acts on \( \overline{A} \) by scalar multiplication. It follows that the two copies of \( G_m \) mentioned above produce the same elements of \( \text{Str}(\overline{A}) \). But their intersection is trivial or \( \mu_3 \) which contradicts that the scalar is arbitrary. This contradiction shows that there exists no copy of \( G_m \) in \( D \). \( \square \)

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7 We used here the equalities \( 3(f_3 + g_3) = f_3 \) and \( 2(f_3 + g_3) = 2g_3 \).
8 Many of the following results used in the proof hold with less restrictive assumptions on the characteristic of \( K \). We leave it to reader to look into the details should he or she find any of these results of interest over more general fields.
9 The norm of \( A \) is then an anisotropic cubic form.
Proposition 5.2. There exists a field extension \( E \subset A \) of \( K \) such that

(i) \( E/K \) is cubic Galois.

(ii) \( A_E \) and \( \text{Str}(A)_E \simeq \text{Str}(A_E) \) are split.

Proof. The algebra \( A \) as a \( K \)-space is the direct sum of three copies of a central simple \( K \)-algebra \( D \) of degree 3 (see \( \S 3.2 \)). It is well known that a cubic Galois extension \( E \) of \( K \) can be found inside the first copy of \( D \) (Wedderburn Theorem. See [13] Theorem 19.2). Since \( E \otimes_K E \) has zero divisors, \( A_E \) is not a division algebra. Thus \( g_3(A_E) = 0 \). By assumption we also have \( f_3(A_E) = 0 \).

Thus \( A_E \) is split by Corollary 3.4. The automorphism group of the split Albert algebras is split, hence also its structure group. \( \square \)

Let \( K \subset E \subset A \) be as above. Recall the following results from [2] and [3].

Let

\[ H := \text{Str}(A/E) = \text{Aut}(A/E) \subset G \subset \text{Str}(A) \]

be the subgroup of elements that fix \( E \) pointwise. It is a simple simply connected \( K \)-group of type \( D_4 \). Furthermore, let \( \text{Str}(A,E) \) be the subgroup of \( \text{Str}(A) \) that stabilizes \( E \). The \( K \)-group \( \text{Str}(A,E)^0 \) is reductive and

1. \( \text{Str}(A,E)^0 = R \cdot H \) where \( R \) is the central torus of \( \text{Str}(A,E)^0 \).
2. \( R \simeq R_{E/K}(G_{m,E}) \).
3. \( R \cap H = Z(H_E) \).
4. (\( Z_{\text{Str}(A)}(R) \))' = H.

Lemma 5.3. \( H_E \) is split.

Proof. Since \( R_E \) is a split torus and the reductive \( E \)-group \( \text{Str}(A_E) \) is split, \( Z_{\text{Str}(A_E)}(R_E) \) is split reductive. But then so is its derived group. By (iv) this derived group is \( H_E \). \( \square \)

6. The special maximal torus \( T \) of \( H \)

We maintain the notation and assumptions of \( \S 5 \).

Proposition 6.1. The group \( H \) contains a maximal torus \( T \) which is an almost direct product of two copies of \( R_{E/K}^{(1)}(G_{m,E}) \).

The proof will be given after a series of preparatory results. We mimic the proof of Lemma 6.28 in [19].

By Lemma 5.3 the \( E \)-group \( H_E \) is split. Fix a split maximal torus \( Q \) of \( H_E \) and consider the Dynkin diagram

\[
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{array}
\]\n
of \( H_E \) where the \( \alpha_i \) are the simple roots of a fixed epinglage of \( (H_E, Q) \). Let \( P \) be the parabolic subgroup of \( H_E \) associated to the set of simple roots

\[ \Delta = \{ \alpha_2, \alpha_3, \alpha_4 \}. \]

\[ ^{10} \text{The elements of } \text{Str}(A) \text{ that fix } 1_K = 1_E \text{ necessarily belong to } G \text{ by [24, 5.9.4].} \]
This is the smooth connected subgroup of $H_E$ of dimension $4 + 12 + 6 = 22$. The corresponding Levi subgroup $L$ is

$$L = A \cdot L'$$

where $L' \simeq SL_{4,E}$ corresponds to the root system of type $A_3$ attached to $\Delta$, and $A \simeq G_{m,E}$.

Recall that

$$P = U \rtimes L$$

where $U$ is the unipotent radical of $P$.

Next, let us consider the $E$-subgroup $(P \cap \sigma(P) \cap \sigma^2(P))^0$ of $H_E$, where $\sigma$ is a generator of Gal($E/K$). It descends to a $K$-subgroup $N$ of $H$. Since $P$ is of codimension 6 in $H_E$ we have

$$\dim(N) = \dim(N_E) \geq \dim(H) - 3 \text{codim}(P) = 28 - 18 = 10.$$

Since $N_E \subset P$ we see $N$ has rank $\leq 4$. Moreover, since $K$ is of characteristic 0 and the group $H$ is anisotropic (Proposition 5.1), $N(K)$ does not contain unipotent elements and therefore $N$ is reductive. We have the standard decomposition

$$N = N'' \cdot N'$$

where $N'$ is the derived group of $N$, $N''$ is the central torus of $N$, and $N'' \cap N'$ is a finite $K$-group of multiplicative type.

**Lemma 6.2.** The group $N'$ is semisimple of type $A_2$.

**Proof.** Since $N$ is reductive it is well known that $N'$ is semisimple. The restriction of the canonical map

$$P \twoheadrightarrow P / U \simeq L$$

to $N_E$ has trivial kernel since $N_E \cap U = 1$. The resulting embedding $N_E \hookrightarrow L$ yields, after passing to the derived groups (see (1) above), an embedding $N'_E \hookrightarrow L' = SL_{4,E}$ [1]. Thus $N'$ is of rank $\leq 3$.

To establish the Lemma we may replace $N'$ by its simply connected cover. We may thus assume that $N'$ is a product of Weil restrictions $\mathcal{R}_{L_i/K}(G_i)$'s with $L_i/K$ finite field extensions, and $G_i$ absolutely simple semisimple $L_i$-groups $(i = 1, \ldots, n)$.

Since the rank of $N'$ is at most 3, the $G_i$ are of type $A_\ell$ or $B_\ell$ with $\ell \leq 3$, $G_2$ or $C_3$. Observe that $\mathcal{R}_{L_i/K}(G_i)$ has rank $[L_i : K] \cdot \text{rk}(G_i)$.

**Case 1:** $n > 1$. We claim that there exists $i$ such that $G_i$ is of type $A_1$ and $L_i = K$. Indeed, rank of $\text{rk}(N') \leq 3$. Hence by rank considerations one of the $G_i$ is of type $A_1$. Say $i = 1$. If $[L_1 : K] > 1$ then necessarily $G_2$ is of type $A_1$ and $L_2 = K$. The claim holds.

---

[1] An embedding $H \hookrightarrow G$ of $K$-groups is a $K$-group homomorphism that is also a closed immersion of schemes. Since we are in the affine case, this last amount to the induced $K$-ring homomorphism $\mathcal{O}(G) \rightarrow \mathcal{O}(H)$ being surjective.
This situation, however, cannot occur because an absolutely simple semisimple group of type $A_1$ becomes split over a field extension $L$ of $K$ of degree 2. But $A$ is still a division algebra over $L$ because $L$ cannot kill $g_3(A)$. This contradicts that $\text{Str}(A)$, hence $N'$, remain anisotropic under such field extension (Proposition 5.1).

Case 2: $n = 1$. By rank considerations $[L_1 : K] \leq 3$.

2(iii): Let $[L_1 : K] = 3$. Then $G_1$ is necessarily of rank 1, hence of type $A_1$. Thus up to isogeny $G_1 = \text{SL}_1(Q)$ for some quaternion algebra $Q$ over $L_1$. Therefore, there is an isogeny

$$\mathcal{R}_{L_1/K}(\text{SL}_1(Q))_E \longrightarrow N'_E \subset L' = \text{SL}_{4,E}.$$ 

Passing to $\overline{K}$ yields a group of type $A_1 \times A_1 \times A_1$ inside $\text{SL}_{4,\overline{K}}$. But the root system of type $A_3$ does not admit subroot systems of type $A_1 \times A_1 \times A_1$.

2(ii): Let $[L_1 : K] = 2$. We again necessarily have $G_1 = \text{SL}_1(Q)$ (up to isogeny) for some quaternion algebra $Q$ over $L_1$. Let $L$ be a quadratic extension of $L_1$ that splits $Q$. Then $[L : K] = 4$ and the $L$-group $\mathcal{R}_{L_1/K}(\text{SL}_1(D))_L$ is isotropic. From

$$\mathcal{R}_{L_1/K}(\text{SL}_1(Q))_L \longrightarrow N'_L \subset H_L \subset \text{Str}(A_L)'$$

it follows that $\text{Str}(A_L)'$ is isotropic. By Propositions 5.1 and Theorem 3.5, $A_L$ is not a division algebra, and therefore $g_3(A_L) = 0$. This shows that $g_3(A) \in \text{H}^3(K, \mathbb{Z}/3\mathbb{Z})$ has order dividing 4. This forces $g_3(A) = 0$. But $g_3(A) \neq 0$ since $A$ is a division algebra.

2(i): Let $[L_1 : K] = 1$. As already observed $G_1$ is of type $A_\ell$ or $B_\ell$ with $\ell \leq 3$, $G_2$ or $C_3$. It remains to rule out all of these types except $A_2$.

Type $A_1$ was ruled out in Case 1. The same consideration rules out types $A_3, G_2, B_2, B_3, C_3$ (all such groups split over an extension of $K$ of degree a power of 2).

Lemma 6.3. Recall the standard decomposition $N = N'' \cdot N'$ of $\mathfrak{N}$.

(i) The torus $N''$ is two-dimensional and splits over $E$. Furthermore, $N'' \simeq \mathcal{R}_{E/K}^{(1)}(G_{m,E})$.

(ii) $N'$ is an inner form of $\text{SL}_3$ that splits over $E$. The simply connected cover of $N'$ is of the form $\text{SL}_1(D)$ for some cubic division algebra $D$ over $K$.

Proof. We have observed that $N$ is of rank at most 4 and dimension at least 10. By the last Lemma $N'$ has dimension 8 and rank 2. This yields that $N''$ is a two-dimensional torus (which we recall is central in $N$).

To show that $N_E'$ is split we reason as follows. Let $D$ be a maximal torus of $N_E$ containing $S_E$. Since $N_E \hookrightarrow A \cdot \text{SL}_4$ and $N$ is of rank 4, it follows that $D$ is a maximal torus of $A \cdot \text{SL}_4$. Since $A$ is central, $A \subset D$ and $A \subset Z(N_E) = N_E''$.

Since $N_E'$ is two-dimensional and $A \simeq G_{m,E}$, if $N_E'$ is not split, then $A$ is the unique maximal split torus of $N_E''$. Let $\sigma$ be a generator of $\text{Gal}(E/K)$. Since $\sigma(A) \subset N_E''$ is a split torus we have $\sigma(A) = A$. Thus $A$ descends to a one-dimensional $K$-torus $\tilde{A}$. By the classification of one-dimensional tori
\[ \widetilde{A} \simeq \mathcal{R}_{L/K}^{(1)}(G_{m,L}) \] for some quadratic étale Galois extension \( L/K \) but such tori split over a quadratic field extension of \( K \) which leads to a contradiction by the anisotropic consideration explained in the proof of Lemma 5.2.

Finally, that \( N'' \simeq \mathcal{R}_{E/K}^{(1)}(G_{m,E}) \) follows from the classification of two-dimensional tori. More precisely, both \( N'' \) and \( \mathcal{R}_{E/K}^{(1)}(G_{m,E}) \) split over \( E \), so they correspond to an action of \( \text{Gal}(E/K) \) on their character groups \( \mathbb{Z}^2 \). Since both tori are anisotropic, the two actions of \( \text{Gal}(E/K) \) on \( \mathbb{Z}^2 \) are faithful, so we have two embeddings \( \text{Gal}(E/K) \hookrightarrow \text{GL}_2(\mathbb{Z}) \). To establish the Lemma it will suffice to show that these two copies of \( \text{Gal}(E/K) \) are conjugate in \( \text{GL}_2(\mathbb{Z}) \).

Now \( \text{GL}_2(\mathbb{Z}) \) has only two maximal finite subgroups up to conjugacy. One is of order 8 and the other, say \( H \), of order 12. Since \( \text{Gal}(E/K) \) is of order 3, the two copies of the Galois group are in \( H \), hence conjugate. This completes the proof of part (i).

Remark 7.1. Let \( S \) be a maximal torus of \( G_0 \). If \( L/K \) is the (unique up to isomorphism) minimal splitting field of \( S \), then \( L/K \) is finite Galois and the group \( \text{Gal}(L/K) \) acts naturally and faithfully as automorphisms of the root system \( \Sigma(G_0, S) \). Since the automorphism group of a root system of type \( F_4 \)

Construction of \( T \): Recall that the cubic division algebra \( D \) from the previous Lemma is split by \( E \), so \( D \) contains a \( K \)-subalgebra isomorphic to \( E \) (Th. 4.5.3 of [7]). This leads us to the fact that \( N' \) contains an anisotropic 2-dimensional torus \( S_2 \) which is split by the cubic Galois extension \( E/K \). As explained in the previous Lemma, \( S_2 \simeq \mathcal{R}_{E/K}^{(1)}(G_{m,E}) \). Let \( S_1 = N'' \). Then \( T = S_1 \cdot S_2 \) is a torus of \( H \). One checks that the central subgroup \( S_1 \cap S_2 \) of \( N' \) is nontrivial, hence \( S_1 \cap S_2 = \mu_3 \) (the centre of \( N' \)). Thus \( T \) is isomorphic to the almost direct product of two copies of \( \mathcal{R}_{E/K}^{(1)}(G_{m,E}) \). So \( T \) is of rank 4, hence a maximal torus of \( H \). This finishes the proof of Proposition 6.1.

7. INDEPENDENCE OF THE EMBEDDING OF \( T \) IN \( G_0 \)

Let \( G_0 \) be the split \( K \)-group of type \( F_4 \) and \( W \) its Weyl group.

Remark 7.1. Let \( S \) be a maximal torus of \( G_0 \). If \( L/K \) is the (unique up to isomorphism) minimal splitting field of \( S \), then \( L/K \) is finite Galois and the group \( \text{Gal}(L/K) \) acts naturally and faithfully as automorphisms of the root system \( \Sigma(G_0, S) \). Since the automorphism group of a root system of type \( F_4 \)

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12 The case of \( G_m \) corresponds to \( L = K \times K \).
coincides with its Weyl group we have an embedding of abstract groups
\[
\text{Gal}(L/K) \hookrightarrow W.
\]

Let \( T \) be the maximal torus of \( H \) constructed in the previous section. Our group \( G \) is a twisted form of \( G_0 \). Since \( G_0 \) coincides with its automorphism group, we have \( G \simeq \xi G_0 \) for some \( \xi \in H^1(K, G_0) \). We now recall the following result of R. Steinberg (see [26, §10], [19, Propositions 6.18, 6.19]).

**Theorem 7.2.** There exists an embedding \( \iota: T \hookrightarrow G_0 \) of \( K \)-groups such that \( [\xi] \) is in the image of the induced map of \( H^1(K, T) \) in \( H^1(K, G_0) \).

The existence of such an embedding, which we call a Steinberg embedding, is already remarkable. For our purposes we need even more.

**Proposition 7.3.** The image of \( H^1(K, T) \) in \( H^1(K, G_0) \) does not depend on an embedding \( T \hookrightarrow G_0 \).

**Proof.** Let \( \iota_1, \iota_2 : T \hookrightarrow G_0 \) be embeddings. We need to show that the images of the induced maps \( \iota_1^*, \iota_2^* : H^1(K, T) \to H^1(K, G_0) \) coincide.

For \( i = 1, 2 \), let \( T_i = \iota_i(T) \subset G_0 \), and \( X_i \) the character groups of \( \overline{T_i} \) (viewed as \( \text{Gal}(K) \)-modules). Let \( \Sigma_i = \Sigma(G_0, T_i) \subset X_i \) be the corresponding root systems.

**Lemma 7.4.** There exists an isomorphism \( \phi : T_1 \to T_2 \) such that the induced map \( \phi^* : X_2 \to X_1 \) maps \( \Sigma_2 \) onto \( \Sigma_1 \).\(^{13}\)

**Proof.** From [4] one knows that \( |W| = 2^73^2 \) and that the 3-Sylow subgroups are of the form \( \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \).

Let \( T_0 \) be a maximal split torus of \( G_0 \), and fix a basis \( \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \) of the corresponding root system \( \Sigma(G_0, T_0) \). We denote its character lattice by \( X \).

Consider the corresponding extended Coxeter-Dynkin diagram

\[
\begin{array}{cccccc}
-\tilde{\alpha} & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\end{array}
\]

There are two copies of \( \text{SL}_3 \) inside \( G_0 \) that are relevant to our construction, the one corresponding to \( \{-\tilde{\alpha}, \alpha_1\} \), and the one corresponding to \( \{\alpha_3, \alpha_4\} \). They will be denoted by \( \text{SL}_3^1 \) and \( \text{SL}_3^2 \) respectively. We have \( \text{SL}_3^1 \cap \text{SL}_3^2 = \mu_3 \). We denote the Weyl group of \( \text{SL}_3 \) by \( W^i \), which we view as subgroups of \( W \). The \( W^i \) are symmetric groups in three symbols, and we let \( \tau_i \) be elements of order 3 of \( W^i \). The \( \tau_i \) are Coxeter transformations of \( W^i \) and they are all conjugate. There are two of them: \( \tau_1 \) and \( \tau_2 \). The \( \tau_i \) clearly commute (their underlying root systems are orthogonal to each other) and \( < \tau_1, \tau_2 > \) is a 3-Sylow subgroup of \( W = W(G_0, T_0) \).

---

\(^{13}\) Strictly speaking the map is \( \overline{\phi}^* \).
Since $E$ is the minimal splitting field of $T_i$, we have by Remark 7.1 the embeddings

$$\text{Gal}(E/K) \hookrightarrow \text{Aut}(\Sigma_i) := W_i.$$ 

Since the $T_i$ are conjugate to $T_0$ in $G_0$, there exist automorphisms $\psi_i$ of $G_0$ that by restriction yields an isomorphism (also denoted) $\psi_i : T_i \to T_0$, hence abstract group isomorphisms

$$\psi_i^W : W_i = \mathcal{N}_{G_0(K)}(T_i(K))/T_i(K) \to \mathcal{N}_{G_0(K)}(T_0(K))/T_0(K) = W.$$ 

Fix $i = 1, 2$. Recall that $\psi_i^* : X \to X_i$ is given by $\chi \mapsto \chi \circ \psi_i$. We claim that if $w_i \in W_i$ and we set $w = \psi_i^W(w_i) \in W$, then for all $\chi \in X$

$$(4) \quad \psi_i^*(w \cdot \chi) = w_i \cdot \psi_i^*(\chi).$$ 

Let us recall that if $t \in T_0(K)$, then $(w \cdot \chi)(t) = \chi(w^{-1}(t))$. Now let $g \in \mathcal{N}_{G_0(K)}(T_0(K))$ be such that $w(t) = gtg^{-1}$, and let $g_i \in \mathcal{N}_{G_0(K)}(T_i(K))$ be such that $\psi_i(g_i) = g$. Then for $t_i \in T_i(K)$ we have

$$(\psi_i^*(w \cdot \chi))(t_i) = (w \cdot \chi)((\psi_i(t_i)) = \chi(w^{-1}(\psi_i(t_i))) = \chi(g^{-1}(\psi_i(t_i))g) = \chi(\psi_i(g_i^{-1})\psi(t_i)\psi_i(g_i)) = \chi(\psi_i(g_i^{-1}t_i g_i)) = \psi_i^*(\chi)(g_i^{-1}t_i g_i) = (w_i \cdot \psi_i^*(\chi))(t_i).$$

The claim follows.

Let $\sigma$ be a generator of $\text{Gal}(E/K)$, and let $\sigma_i$ be its image in $W_i$ given by Remark 7.1. Since $\sigma$ is of order 3 we may assume up to conjugacy (in $W_i$) that $\psi_i^W(\sigma_i) \in <\tau_1, \tau_2>$. Since $T_1$ is anisotropic the action of $\sigma_i$ on $X_i$ has no fixed points. Because of (4) $\psi_i^W(\sigma_i)$ can have no fixed points in $X$. This forces

$$\psi_i(\sigma_i) \in \{\tau_1\tau_2, \tau_1^2\tau_2, \tau_1\tau_2^2, \tau_1^2\tau_2^2\}.$$ 

Since as observed above $\tau_i$ and $\tau_i^2$ are conjugate in $W_i$, there exists bases $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ of $\Sigma_1$ and $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ of $\Sigma_2$ such that $\psi_i^W(\sigma_i) = \tau_1\tau_2$. Since the roots span the character lattice of type $F_4$ there exists a (unique) group isomorphism $\psi^* : X_2 \to X_1$ such that $\psi^*(\gamma_i) = \beta_i$. By applying (4) twice we see that

$$\psi^*(\sigma_2 \cdot \chi) = \sigma_1 \cdot \psi^*(\chi) \quad \forall \chi \in X_2.$$ 

It follows that $\psi^*$ commutes with the action of $\text{Gal}(E/K)$.

Recall that since $T_1$ and $T_2$ split over $E$, to give an $E$-group isomorphism $\psi : T_1_K \to T_2_K$ “is the same” as to give an abstract $\mathbb{Z}$-module isomorphism $\psi^* : X_2 \to X_1$. Furthermore, this $\psi^*$ commutes with the action of $\text{Gal}(E/K)$ if and only if $\psi$ descends to a $K$-group isomorphism $\phi : T_1 \to T_2$. But we have constructed such $\psi^*$. This finishes the proof of Lemma 7.4. \hfill \Box

We now finish the proof of Proposition 7.3. The isomorphism $\phi : T_1 \to T_2$ of Lemma 7.3 can be extended to an automorphism of $G_0$ (see [12, Theorem 32.1]). This extension is of the form int($g$) for some $g \in G_0(K)$. We claim that $g^{-1}\gamma g \in T_1(K)$ for all $\gamma \in \text{Gal}(K)$. Indeed, for $t \in T_1(K)$ one has

$$(g^{-1}\gamma g) t (g^{-1}\gamma g)^{-1} = g^{-1}(\gamma gt \gamma g^{-1}) g = g^{-1}(\gamma g \gamma t \gamma g^{-1}) g$$
We used here the fact that the restriction of \( \text{int}(g) \) to \( T_1 \) is the \( K \)-group isomorphism \( \phi \). Thus \( g^{-1} \gamma g \) commutes with \( T_1(K) \). Since \( K \) is of characteristic 0, \( T_1(K) \) is dense in \( T_1(K) \). Therefore, \( g^{-1} \gamma g \) commutes with \( T_1(K) \) and the claim holds.

Let \([\eta_2] \in H^1(K, T_2)\). We want to show that there is \([\eta_1] \in H^1(K, T_1)\) such that the classes of \( \eta_1 \) and \( \eta_2 \) in \( H^1(K, G_0) \) coincide. Say \( \eta_2 = (t_\gamma)_{\gamma \in \text{Gal}(K)} \) with \( t_\gamma \in T_2(K) \). The class \( \eta_2 \) given by \( g^{-1} t_\gamma \gamma g \) coincides with that of \( \eta_2 \) in \( H^1(K, G_0) \). But \( g^{-1} t_\gamma \gamma g \in T_1(K) \). Indeed

\[
g^{-1} t_\gamma \gamma g = (g^{-1} t_\gamma g)(g^{-1} \gamma g)
\]

and, as we know, both \( g^{-1} t_\gamma g \) and \( g^{-1} \gamma g \) are elements of \( T_1(K) \). \( \square \)

**Corollary 7.5.** Let \( A, H, T \) be as in \( \S 6 \). Let \( A' \) be an Alberta algebra arising from the first Tits construction such that \( g_3(A) = g_3(A') \). To show that \( A \cong A' \) there is no loss of generality in assuming that the class of \( A' \) in \( H^1(K, G_0) \) comes from \( H^1(K, T) \) where \( T \hookrightarrow G_0 \) is an arbitrary embedding.

**Proof.** Let \( T \subset G = \text{Aut}(A) \) be the maximal torus constructed in \( \S 6 \) By construction the field extension \( E/K \) splits \( T \). Therefore, \( E/K \) kills \( g_3(A) = g_3(A') \), hence it splits \( A' \). Arguing as above we conclude that \( T \) admits embedding into \( G' = \text{Aut}(A') \). Then the assertion follows from the Steinberg Theorem \( 7.2 \) and Proposition \( 7.3 \) \( \square \)

8. A Special Maximal Torus of \( G_0 \)

Let \( T \) be an almost direct product of two copies of \( R_{E/K}^{(1)}(G_{m,E}) \) as above. The main result of this section is the following

**Proposition 8.1.** There exists an embedding \( \iota : T \hookrightarrow G_0 \) such that if \([\xi] \in H^1(K, T)\) then the classes in \( H^1(K, G_0) \) corresponding to \( \xi \) and \( \xi^{-1} \) coincide.\(^{15}\)

The proof is very technical but essential for establishing the main Theorem.

We maintain all the notation from the previous section. Recall that the quotient \( N_{G_0}(T_0)/T_0 \) is a constant \( K \)-group \( W \) corresponding to the (abstract finite) group \( W \).

By a result of Tits \([27]\) there exists an exact sequence of finite (abstract) groups

\[
1 \to I \to J \to W \to 1
\]

where \( J \subset N_{G_0}(T_0)(K) \) and \( I = T_0(K\gamma_0) \) (the 2-torsion subgroup of \( T_0(K) \)). Note that for the corresponding exact sequence of constant \( K \)-group schemes

\[^{14}\] The restriction of \( \text{int}(g) \) to \( T_1 \) is “\( K \)-defined”.

\[^{15}\] Note that \( \xi = (\xi_\gamma)_{\gamma \in \text{Gal}(K)} \) where the \( \xi_\gamma \) are elements of the commutative group \( T(K) \).

It is clear that \( \xi^{-1} = ((\xi_\gamma)^{-1})_{\gamma \in \text{Gal}(K)} \) is also a cocycle.
we have well defined commutative diagram

\[
\begin{array}{ccccccccc}
1 & \rightarrow & T_0 & \rightarrow & \mathcal{N}_{G_0}(T_0) & \rightarrow & W & \rightarrow & 1 \\
1 & \rightarrow & \mathcal{I} & \rightarrow & \mathcal{J} & \rightarrow & \mathcal{W} & \rightarrow & 1 \\
\end{array}
\]

of group schemes. We used here the fact that \( K \) is of characteristic 0, hence \( \mu_2 \) and the constant \( K \)-group corresponding to \( \mathbb{Z}/2\mathbb{Z} \) are isomorphic.

Recall that in §7 we constructed two commuting copies \( \text{SL}_3^1 \) and \( \text{SL}_3^2 \) of \( \text{SL}_3 \) inside \( G_0 \), and the elements \( \tau_1, \tau_2 \) of order 3 of their respective Weyl groups \( W^i \). Clearly, one of the matrices

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

of \( \text{SL}_3^i \), say \( g_i \), maps to \( \tau_i \) under the map \( \mathcal{N}_{G_0}(T_0)(K) \to W(K) = W \). Note that the \( g_i \) are of order 3. Without any risk of confusion we henceforth denote \( g_i \) by \( \tau_i \). One easily checks that \( \tau_1, \tau_2 \in J \).

Let \( E/K \) be our cubic Galois extension. The isomorphism \( \text{Gal}(E/K) \to \langle \tau_1 \tau_2 \rangle \) of abstract groups that maps \( \sigma \) to \( \tau_1 \tau_2 \) yields a cocycle \( \eta_J \in Z^1(K, J) \), and a class \( [\eta_J] \in H^1(K, J) \). Since \( H^1(K, \text{SL}_3^1) = H^1(K, \text{SL}_3^2) = 1 \), this class viewed in \( H^1(K, \text{SL}_3^1 \cdot \text{SL}_3^2) \), and hence in \( H^1(K, G_0) \), is trivial.

The inclusion \( J = J(K) \subset \mathcal{N}_{G_0}(T_0)(K) \) allows us also to view \( \eta_J \) as a cocycle with values in \( \mathcal{N}_{G_0}(T_0)(\overline{K}) \) that we denote by \( \eta^0 \) with corresponding class \( [\eta^0] \in H^1(K, \mathcal{N}_{G_0}(T_0)) \).

**Lemma 8.2.** The twisted \( K \)-torus \( \eta^0 T_0 \) is a subgroup of \( G_0 \) isomorphic to \( T \).

**Proof.** Since \( [\eta^0] \) viewed as an element of \( H^1(K, G_0) \) is trivial we have

\[
\eta^0 T_0 \subset \eta^0 G_0 \simeq G_0.
\]

To show that \( \eta^0 T_0 \simeq T \) it suffices to show that the two actions of \( \sigma \in \text{Gal}(E/K) \) on their character groups \( X_0 \) and \( X \) coincide. But this is clear since the action of \( \sigma \) in both cases is given by \( \tau_1 \tau_2 \).

We need one more property of the twisted torus \( \eta^0 T_0 \). Recall (see [6, Exp. XXIV, Prop. 3.16.2, p. 355]) that \(-1 \in W\) and that there exists a preimage \( c \in \mathcal{N}_{G_0}(T_0)(K) \) of \(-1\) such that

(i) \( c^2 = 1 \).

(ii) \( c^{-1}zc = z^{-1} \) for all \( z \in T_0(K) \).
Lemma 8.3. There exists an element $\tilde{c} \in J$ with the following properties

(i) $\tilde{c}$ is of order 2.
(ii) $\tilde{c}$ is a lift of $-1 \in W$
(iii) $\tilde{c}$ is fixed by the twisted action of $\eta^0$.
(iv) $\tilde{c}^{-1} z \tilde{c} = z^{-1}$ for all $z \in T_0(K)$.

Proof. Consider the exact sequence

$$(7) \quad 1 \to \eta^0 I \to \eta^0 J \to \eta^0 W \to 1$$

By passing to cohomology this yields

$$(8) \quad (\eta^0 I)(K) \to (\eta^0 J)(K) \to (\eta^0 W)(K) \xrightarrow{\delta} H^1(K, \eta^0 I)$$

Note that $-1 \in W$ is central, hence stable under twisting by $\eta^0$. In other words $-1 \in (\eta^0 W)(K)$. Furthermore, since $\eta^0$ is trivial over $E$ the exact sequence (8) shows that the connecting map $\delta$ factors through $H^1(E/K, \eta^0 I(E))$ and this $H^1$ vanishes since $\text{Gal}(E/K)$ is of order 3 and $\eta^0 I$ is 2-torsion. This shows that there exists a lifting $\tilde{c} \in (\eta^0 J)(K) \subset J(K) = J$ of $-1 \in (\eta^0 W)(K)$. Since both $\tilde{c}$ and $c$ map to $-1$ we have $\tilde{c} = tc$ for some $t \in T_0(K)$. Thus

$$\tilde{c}^2 = ttc = tc^{-1}tc = tt^{-1} = 1.$$

Since $\tilde{c} \in \eta^0 J(K)$, it is by definition stable under the $\text{Gal}(K)$ action twisted by $\eta^0$.

Finally, let $z \in T_0(K)$. Recall that $\tilde{c} = tc$ with $t \in T_0(K)$. Then

$$\tilde{c}^{-1} z \tilde{c} = c^{-1}t^{-1}ztc = c^{-1}zc = z^{-1},$$

as required. \qed

We now turn to the proof of the Proposition. We may assume by Lemma 8.2 that

$$\xi = (\xi_\gamma)_{\gamma \in \text{Gal}(K)} \in Z^1(K, \eta^0 T_0).$$

Let $\tilde{c}$ be as above. Consider the cocycle $\xi' \in Z^1(K, \eta^0 G_0)$ given by $\tilde{c}^{-1} \xi_\gamma \gamma' \tilde{c}$ (where $\gamma'$ is the twisted action by $\eta^0$). Then $[\xi_{\eta^0 G_0}] = [\xi']$ in $H^1(K, \eta^0 G_0) = H^1(K, G_0)$. But $\gamma' \tilde{c} = \tilde{c}$ by Lemma 8.3(iii). Thus

$$\tilde{c}^{-1} \xi_\gamma \gamma' \tilde{c} = \tilde{c}^{-1} \xi_\gamma \tilde{c} = \xi_\gamma^{-1}.$$

This shows that $[\xi'] = [\xi_{\eta^0 G_0}]$. The proof of Proposition 8.1 is complete.

The maximal torus $\eta^0 T_0$ of $G_0$ that we have constructed is very special and crucial for the proof of our main result. The construction depends only the cubic Galois extension $E/K$ and a choice of a maximal split torus $T_0$ of $G_0$. If we now fix a $K$-group isomorphism $T \simeq \eta^0 T_0$, we obtain what we call a special embedding $\iota : T \hookrightarrow G_0$. Let us summarizes a list of important properties which easily follows from our construction. Let $T_0 \subset G_0$ be a maximal split torus as in the proof of Lemma 7.3. Recall that we have two commuting copies $SL_3^1, SL_3^2 \subset G_0$ of $SL_3$ containing $T_0$. Let $T_0^1 = T_0 \cap SL_3^1$ and $T_0^2 = T_0 \cap SL_3^2$. Clearly, $T_0^1 \cap T_0^2 = \mu_3$ and so $T_0 = T_0^1 \cdot T_0^2$. 
We remind the reader that $\SL_3^1$ corresponds to the long roots $-\alpha, \alpha_1$ and $\SL_3^2$ corresponds to the short roots $\alpha_3, \alpha_4$. Let $\widetilde{M}_0 = \SL_3^1 \cdot T_0^2$. It is a split reductive group with central torus $T_0^2$ and derived subgroup $\SL_3^1$. Finally, let $\widetilde{H}_0 \subset G_0$ be the subgroup generated by all long roots of the root system of $(G_0, T_0)$. It is a split simple simply connected group of type $D_4$.

By construction, we have the following chain of split reductive/semisimple subgroups of $G_0$:

$$T_0 \subset \widetilde{M}_0 \subset \widetilde{H}_0 \subset G_0.$$ 

Note that all members of this chain are stable with respect to conjugation by $\tau_1$ and $\tau_2$. Since twisting by $\eta^0$ is given by conjugation by $\tau_1 \tau_2$ all of them are stable with respect to the twisted action of $\text{Gal}(K)$ by $\eta^0$. We can thus consider the twisted $K$-groups $\eta^0 T_0, \eta^0 \widetilde{M}_0, \eta^0 \widetilde{H}_0, \eta^0 G_0$.

**Lemma 8.4.** We have the following:

1. $\eta^0 T_0 = \iota(T)$;
2. $\eta^0 G_0 \simeq G_0$;
3. $M_0 := \eta^0 M_0$ is a reductive group whose derived subgroup is isomorphic to $\SL_3$ and its central torus $S := \eta^0 (T_0^3)$ is isomorphic to $\mathcal{R}_{E/K}(G_{m,E})$;
4. $H_0 := \eta^0 \widetilde{H}_0$ is a quasi-split group of type $3D_4$ that splits over $E$.

**Proof.** (1) was proved in Lemma 8.2. (2) follows from the fact that the class of $\eta^0$ viewed in $H^1(K, \SL_3; \SL_3^2)$ is trivial. As for (3), one needs to observe that the action of $\sigma$ on the root system of $\SL_3^2$ is given by the Coxeter transformation $\tau_2$. For (4), one notes that the chain of subgroups

$$\SL_3 \simeq M'_0 \subset M_0 \subset H_0$$

shows that $H_0$ contains a 2-dimensional split torus. Choose such a torus, say $Q \subset M'_0 \subset M_0$. We claim that it is maximal split in $H_0$. Indeed, it suffices to show that $Q$ is a maximal split torus in $Z_{H_0}(Q)$. But the derived subgroup of $Z_{G_0}(Q)$ corresponds to the short roots $\alpha_3, \alpha_4$ of $G_0$. Therefore,

$$Z_{H_0}(Q) = Z_{G_0}(Q) \cap H_0 = Q \cdot S$$

and we are done because $S$ is anisotropic. $\square$

All together, we have the following chain of $K$-subgroups:

$$T \subset M_0 \subset H_0 \subset G_0.$$ 

where we have identified $T$ with $\iota(T) = \eta^0 T_0$

***

Our main Theorem can be restated as follows.

**Theorem 8.5.** Let $[\xi_1], [\xi_2] \in H^1(K, T)$ be as above (see Corollary 7.5). Assume that $[\xi_1 G_0], [\xi_2 G_0] \in H^1(K, G_0)$ are such that

$$f_3([\xi_1 G_0]) = f_3([\xi_2 G_0]) = 0 \text{ and } g_3([\xi_1 G_0]) = g_3([\xi_2 G_0]).$$

Then $[\xi_1 G_0] = [\xi_2 G_0]$. 

---
We now outline its proof.

Let $A_1, A_2$ be the Albert $K$-algebras corresponding to $\xi_{1G_0}, \xi_{2G_0}$. Choose a cubic Galois extension $E/K$ splitting $A_1$. As we discussed above, $E$ splits $A_2$ as well (see Corollary 7.5). Then, as we know, the torus

$$T = R^{(1)}_{E/K}(G_{m,E}) \cdot R^{(1)}_{E/K}(G_{m,E})$$

admits embeddings into $\text{Aut}(A_1)$ and $\text{Aut}(A_2)$. By Steinberg’s Theorem 7.2 there are closed embeddings $i_1 : T \hookrightarrow G_0$ and $i_2 : T \hookrightarrow G_0$ such that, up to equivalence, $\xi_{1G_0}$ and $\xi_{2G_0}$ take values in $i_1(T)$ and $i_2(T)$ respectively. Furthermore, according to Proposition 7.3 we may assume without loss of generality that $i_1(T) = i_2(T) = \varphi^0 T_0$, where the maximal torus $\varphi^0 T_0$ of $G_0$ is as above. As it is customary we can also identify the images of $i_1(T) = i_2(T)$ with $T$. This simplifies the notation greatly.

To be precise, henceforth our torus $T$ is the torus $\varphi^0 T_0$ of $G_0$, and the special embedding $i$ the inclusion map. Furthermore, $[\xi_1], [\xi_2] \in H^1(K, T)$.

Our starting point then is two classes $[\xi_1], [\xi_2] \in H^1(K, T)$ with trivial $f_3$ invariant and with the same $g_3$ invariant. We also have the chain of groups

$$T \subset M_0 \subset H_0 \subset G_0$$

constructed in Lemma 8.4. For $i = 1, 2$, let $[\xi_{iM_0}], [\xi_{iH_0}]$ and $[\xi_{iG_0}]$ be the images of $[\xi_i]$ in $H^1(K, M_0), H^1(K, H_0)$ and $H^1(K, G_0)$ respectively. By taking the above discussion into consideration, Theorem 8.5 follows immediately from the much stronger

**Theorem 8.6.** Let $[\xi_1], [\xi_2] \in H^1(K, T)$. Assume these two classes have trivial $f_3$ invariant and the same $g_3$ invariant. Then $[\xi_{1M_0}] = [\xi_{2M_0}]$.

The reader may find surprising that we prove that our two classes coincide in $H^1(K, G_0)$ (the statement of the main Theorem) by showing that the two classes already coincide in a much smaller subgroup!

Leaving aside that we do not know a direct proof, let us shed some light behind the reason for which this “strengthening” reduction actually takes place. Under the assumption that $\text{Gal}(K)$ is a pro-3-group, the work of Rost [21] shows that the two classes in $G_0$ are indeed the same (Proposition 10.4). We show even more, namely that the classes already agree in $H_0$ (Theorem 10.5) and even, and this is the crucial step, in $M_0$ (see Theorem 11.1). We do not know how to remove the pro-3-group assumption at the level of $G_0$ or $H_0$, but we can prove that this assumption is superfluous at the level of $M_0$ (Theorem 9.1).

***

We finish this section by showing how the use of “twisting” allows us to deal with a single class $[\xi]$ instead of the two classes $[\xi_1], [\xi_2]$.

Let

$$T = \xi_1 T, \quad M := \xi_{1M_0} M_0, \quad H := \xi_{1H_0} H_0, \quad G := \xi_{1G_0} G_0.$$

We have a new twisted sequence of $K$-groups
where $T = \eta^0 T_0$ is still our special torus.

Observe that $M$ is a reductive group whose central torus $S$ isomorphic to $R_{E/K}(G_m,E)$, and whose derived subgroup is an inner form of type $A_2$ that splits over $E$ (because $T$ is split over $E$), hence it is of the form $SL_1(D)$ for some cubic division $K$-algebra $D$. The group $H$ is a trialitarian group and $G$ is a group of type $F_4$. Both $H$ and $G$ split over $E$.

Consider now the commutative diagram

$$
\begin{array}{ccc}
H^1(K, T) & \rightarrow & H^1(K, M_0) \\
\downarrow & & \downarrow \\
H^1(K, T) & \rightarrow & H^1(K, M) \\
\downarrow & & \downarrow \\
H^1(K, T) & \rightarrow & H^1(K, H) \\
\downarrow & & \downarrow \\
H^1(K, T) & \rightarrow & H^1(K, G) \\
\end{array}
$$

where the vertical arrows are the torsion bijections constructed in [22]. They take the classes $[\xi_1], [\xi_{1M_0}], [\xi_{1H_0}]$ and $[\xi_{1G_0}]$ into the trivial classes $[1]$ of the corresponding $H^1$ in the bottom row. Let $[\xi]$ be the image of $[\xi_2]$ under the left vertical arrow. Note that our torsion bijection maps are compatible with shift in $H^3(\mathbb{Q}/\mathbb{Z}(2))$ by $-R_{G_0}(\xi_1)$. Therefore, the equalities $f_3([\xi_1]) = f_3([\xi_2]) = 0$ and $g_3([\xi_1]) = g_3([\xi_2])$ translates into $R_G([\xi_G]) = 1$. All together we obtain that Theorem 8.6 is equivalent to the following statement.

**Theorem 8.7.** Let $[\xi] \in H^1(K, T)$ be such that $R_G([\xi_G]) = 1$. Then $[\xi_M] = [1] \in H^1(K, M)$.

9. **Reduction to the pro-3-group case**

**Theorem 9.1.** Let $L/K$ be a finite field extension of degree prime to 3. Then the natural map

$$H^1(K, M) \rightarrow H^1(L, M)$$

has trivial kernel.

**Proof.** Recall the commutative diagram with exact rows

$$
\begin{array}{ccc}
1 & \rightarrow & \mu_3 \\
\downarrow & & \downarrow \\
S \times SL_1(D) & \rightarrow & M \\
\downarrow & & \downarrow \\
1 & \rightarrow & \mu_3 \times \mu_3 \\
\end{array}
$$

where the map $\mu_3 \rightarrow \mu_3 \times \mu_3$ is given by $\lambda \mapsto (\lambda, \lambda^{-1})$.

We have $H^1(K, \mu_3) \simeq K^\times /K^\times 3$. If $a \in K^\times$ we denote by $\overline{a}$ the corresponding element of $H^1(K, \mu_3)$.

**Lemma 9.2.** The map $H^1(K, \mu_3 \times \mu_3) \rightarrow H^1(K, S \times SL_1(D))$ induced by the above diagram is surjective.
Proof. Since $H^1(K, S) \simeq K^\times/N_{E/K}(E)^\times$, the group $H^1(K, \mu_3)$ surjects onto $H^1(K, S)$. Similar considerations apply to the second component since

$$H^1(K, SL_1(D)) \simeq K^\times / \text{Nrd}(D)^\times.$$ 

The proof is complete. \hfill \qed

Consider the commutative diagram with exact Galois cohomology rows

$$
\begin{array}{cccccc}
H^1(K, \mu_3) & \longrightarrow & H^1(K, S) \times H^1(K, SL_1(D)) & \longrightarrow & H^1(K, M) & \longrightarrow & H^2(K, \mu_3) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(L, \mu_3) & \longrightarrow & H^1(L, S) \times H^1(L, SL_1(D)) & \longrightarrow & H^1(L, M) & \longrightarrow & H^2(L, \mu_3)
\end{array}
$$

The righthand vertical map is injective since the $H^2$ in question are abelian 3-groups and the map is given by multiplication by $[L : K]$. It follows that if $[\xi] \in H^1(K, M)$ is in the kernel of our map, then $[\xi]$ maps to $[1] \in H^2(K, \mu_3)$, and therefore $[\xi]$ has a preimage $[\xi'] = ([\alpha], [\beta]) \in H^1(K, S) \times H^1(K, SL_1(D))$.

By Lemma 9.2 we may assume that $[\xi'] = (\bar{a}, \bar{b})$ for some $a, b \in K^\times$. In the commutative group $H^1(K, \mu_3 \times \mu_3)$ we have $(\bar{a}, \bar{b}) = (\bar{a}, \bar{a}^{-1})(1, \bar{a}b)$.

Since $(\bar{a}, \bar{a}^{-1})$ comes from $H^1(K, \mu_3)$ its image in $H^1(K, M)$ vanishes. We may thus assume that $[\xi']$ has trivial first component, that is $[\xi'] = (\bar{1}, \bar{c})$ for some $c \in K^\times$ where $\bar{c}$ is the class of $c$ in $K^\times / \text{Nrd}(D)^\times$. To finish the proof it suffices to show that $c \in \text{Nrd}(D)$.

Consider the image $[\xi']_L$ of $[\xi']$ in $H^1(L, S) \times H^1(L, SL_1(D))$. We have $[\xi']_L = (\bar{1}, \bar{c}_L)$ where $c_L$ is $c$ viewed as an element of $L$, and $\bar{c}_L$ is the class of $c_L$ in $L^\times / \text{Nrd}(D_L)^\times$. By assumption $[\xi']_L$ maps to the trivial class of $H^1(L, M)$, and therefore it comes from an element $\bar{d} \in H^1(L, \mu_3)$ with $d \in L^\times$. Thus $[\xi']_L = (\bar{1}, \bar{c}_L) = (\bar{d}, \bar{d}^{-1})$ and therefore $d$, hence also $d^{-1}$ and $c_L$, are in $N_{E:L/D}(E \cdot L)^\times$. In particular $c_L \in \text{Nrd}(D_L)$.

Recall that the “Norm Principle” applied to $D = D_K \subset D_L$ states that

$$N_{L/K}(\text{Nrd}(D_L)) \subset \text{Nrd}(D).$$

Since $N_{L/K}(c_L) = c^{[L:K]}$, the norm principle shows that $c^{[L:K]} \in \text{Nrd}(D)$. But since $[L : K]$ is not divisible by 3 we have $c \in \text{Nrd}(D)$. Thus $\bar{c} = \bar{1}$ and therefore $[\xi'] = 1$. Since $[\xi']$ is a preimage of $[\xi]$ we have $[\xi] = 1$. \hfill \qed

**Corollary 9.3.** Let $\Gamma \subset \text{Gal}(K)$ be a Sylow 3-subgroup and $F = \overline{K}^\Gamma$. Then $[\xi_M] = 1$ if and only if $[\xi_{M,F}] = 1$.

Thus, without loss of generality we may assume that $\text{Gal}(K)$ is a pro-3-group. An application of Rost’s result [21] then shows $[\xi_G] = 1$. 

10. Reduction to $H_0$.

The sequence of $K$-groups $T \subset M \subset H \subset G$ is as in (10). By construction, $G$ is the automorphism group of the Albert algebra $A$ corresponding to $[\xi_1]$. Recall that $A$ comes from the first Tits construction because $f_3(A) = f_3([\xi_1]) = 0$. Our next aim is to show that the subgroup $H$ constructed above as the twist $H = \xi H_0$ admits a geometric description as $\text{Str}(A/E)$ for an embedding $E \hookrightarrow A$.

**Proposition 10.1.** Let $\hat{H}$ be a subgroup of $G$ of type $D_4$. Then

(i) $\hat{E} := A^{\hat{H}(K)} \subset A$ is a 3-dimensional Galois field extension of $K$.

(ii) $\hat{H} = \text{Str}(A/\hat{E})$.

**Proof.** (i) Since $\xi_1$ is split over $E$ so is $A$. Consider the embedding $i : E \hookrightarrow A$ of Proposition 5.2. We know that $\text{Str}(A/E)$ is a simple simply connected group of type $D_4$. Let $T' \subset \text{Str}(A/E)$ be the maximal torus constructed in Proposition 6.1 (note that $T' \simeq T$). There is no loss of generality in assuming that $K = \overline{K}$ for the proof that $\hat{E}$ has dimension 3. Let $\hat{T}$ be a maximal torus of $\hat{H}$. There exists $g \in G(K)$ such that $gT'g^{-1} = \hat{T}$. The roots of $(\text{Str}(A/E), T')$ are also roots of $(G, T')$. One easily sees that the roots involved are precisely the long roots of $(G, T')$. Similar considerations apply to $\hat{T} \subset \hat{H} \subset G$. Since $\text{int}(g)$ maps roots with respect to $T'$ to roots with respect to $\hat{T}$ we get $g \text{Str}(A/E)g^{-1} = \hat{H}$.

We know that the fixed points of $\text{Str}(A/E)(K)$ in $A$ is $E$, i.e. $A^{\text{Str}(A/E)(K)} = E$, which is 3-dimensional. It is straightforward to check that by restriction $g \in G(K) = \text{Aut}(A)(K)$ yields a $K$-space isomorphism $A^{\text{Str}(A/E)(K)} \simeq A^{\hat{H}(K)}$.

We now return to our original base field $K$ (which is not necessarily algebraically closed). Let $\hat{E} = A^{\hat{H}(K)}$. If $x, y \in \hat{E}$. Then for $h \in \hat{H}(K) \subset \text{Aut}(A)(K)$ we have $h(x \cdot y) = h(x) \cdot h(y) = x \cdot y$. Thus $A^{\hat{H}(K)}$ is a ring. Since $A$ is a division algebra, $A^{\hat{H}(K)}$ is in fact a division ring. Since its dimension is 3, it is commutative. If the degree 3 field extension $\hat{E}/K$ is not Galois, then its Galois closure has Galois group isomorphic to the symmetric group in three symbols. This contradicts that $\text{Gal}(K)$ is a pro-3-group.

(ii) We know that $\text{Str}(A/\hat{E})$ is a subgroup of $G$ of type $D_4$. By construction $\hat{H} \subset \text{Str}(A/\hat{E})$. Since $\hat{H}$ is also of type $D_4$ we get $\hat{H} = \text{Str}(A/\hat{E})$ as desired.

**Corollary 10.2.** $\hat{R} := Z_{\text{Str}(A)}(\hat{H})$ is a 3-dimensional $K$-torus isomorphic to $R_{\hat{E}/K}(G_{m, \hat{E}})$ and $\text{Str}(A, \hat{E})^0 = \hat{R} \cdot \text{Str}(A/\hat{E}) = \hat{R} \cdot \hat{H}$.

**Proof.** See Proposition 5.2 infra and references therein.

Proposition 10.1 applied to the group $H$ of (10) shows that $H = \text{Str}(A/\hat{E})$ for some cubic Galois field $\hat{E}$ in $A$.

---

This is a root system of type $F_4$. 
Corollary 10.3. \( \hat{E} \) and \( E \) are isomorphic field extensions of \( K \).

Proof. We know that in the decomposition \( \text{Str}(A, \hat{E})^o = \hat{R} \cdot \text{Str}(A/\hat{E}) \) the central torus \( \hat{S} \simeq \mathcal{R}_{\hat{E}/K}(G_{m, \hat{E}}) \). In particular the minimal splitting field of \( \hat{R} \) is \( \hat{E} \). Now \( \hat{R} \) is also the centralizer in \( \text{Str}(A) \) of any maximal torus of \( \text{Str}(A/\hat{E}) \). Since \( H = \text{Str}(A/\hat{E}) \) splits over \( E \), so does \( \hat{R} \). It follows that \( \hat{E} \) and \( E \) are isomorphic.

We continue to consider the \( K \)-group \( H \) of (10). Proposition 10.1 applied to \( H \) and Corollary 10.3 yield that there is an embedding \( E \hookrightarrow A \) such that \( H = \text{Str}(A/E) \).

Proposition 10.4. \( [\xi_G] = 1 \).

Proof. Recall that we are given two classes \( [\xi_1], [\xi_2] \in H^1(K, T) \) with trivial \( f_3 \) and with the same \( g_3 \)-invariant. Consider the corresponding classes \( [\xi_{1G_0}], [\xi_{2G_0}] \in H^1(K, G_0) \). By [21] there exist field extensions \( L \) and \( F \) of \( K \) such that \( [L : K] \) is prime to 3, \( [F : K] \) divides 3 such that

\[ [\xi_{1G_0}]_L = [\xi_{2G_0}]_L \quad \text{and} \quad [\xi_{1G_0}]_F = [\xi_{2G_0}]_F. \]

Since by assumption \( \text{Gal}(K) \) is a pro-3-group, \( L = K \). Thus

\[ [\xi_{1G_0}] = [\xi_{2G_0}] \in H^1(K, G_0). \]

Recall that \( G = \xi_{1G_0} G_0 \), and that under the twisted bijection between \( H^1(K, T) \) and \( H^1(K, \xi_1 T) \) we have \( [\xi_1] \mapsto [1] \) and \( [\xi_2] \mapsto [\xi] \). It follows that under the twisted bijection between \( H^1(K, G_0) \) and \( H^1(K, G) \) we have \( [\xi_{1G_0}] \mapsto [1] \) and \( [\xi_{2G_0}] \mapsto [\xi_G] \).

The final aim of this section is to prove

Theorem 10.5. Let the notation be as above. Then \( [\xi_H] = [1] \).

Write \( \xi = (\xi_\gamma)_{\gamma \in \text{Gal}(K)} \) where \( \xi_\gamma \in T(K) \). By the last proposition there exists \( g \in G(K) \) such that

\[ \xi_\gamma = g^{-1} \gamma g. \]

(11)

If we replace \( g \) by \( ag \) with \( a \in \text{Str}(A)(K) \), then \( \xi \) does not change. Indeed, since \( \gamma a = a \)

\[ (ag)^{-1} \gamma (ag) = g^{-1} a^{-1} \gamma a \gamma g = g^{-1} a^{-1} a \gamma g = g^{-1} \gamma g \]

Proposition 10.6. Recall \( \text{Str}(A, E)^o = R \cdot H \). There exists a reductive \( K \)-group \( F \subset \text{Str}(A) \) such that \( F \simeq \text{int}(g)(R \cdot H) = \text{int}(g)(R \cdot H) \). Furthermore \( F \simeq \xi(R \cdot H) \).

Proof. That \( R \cdot H = R \cdot H \) is clear. To show that a \( K \)-group \( F \) exists such that \( F \simeq \text{int}(g)(R \cdot H) \) amounts to showing that

\[ g R(K) \cdot H(K) g^{-1} \subset \text{Str}(A)(K) \]
is stable under the action of Gal($K$). Let $x \in g \mathbf{R}(\overline{K}) \cdot \mathbf{H}(\overline{K})g^{-1}$. Since both $\mathbf{R}$ and $\mathbf{H}$ are $K$-groups, for $\gamma \in \text{Gal}(K)$ we have

$$\gamma x \in \gamma g \mathbf{R}(\overline{K}) \cdot \gamma \mathbf{H}(\overline{K}) \gamma g^{-1} = \gamma g \mathbf{R}(\overline{K}) \cdot \mathbf{H}(\overline{K}) \gamma g^{-1}$$

$$= gg^{-1}\gamma g \mathbf{R}(\overline{K}) \cdot \mathbf{H}(\overline{K}) \gamma g^{-1} = g \xi_{\gamma} \mathbf{R}(\overline{K}) \cdot \mathbf{H}(\overline{K}) \xi_{\gamma}^{-1} g^{-1} \text{ by (11).}$$

But $\xi_{\gamma} \in T(\overline{K}) \subset \mathbf{H}(\overline{K})$. This shows that $\xi_{\gamma}$ commutes with $\mathbf{R}(\overline{K})$ since this is central in $\mathbf{R}(\overline{K}) \cdot \mathbf{H}(\overline{K})$. It follows that

$$g \xi_{\gamma} \mathbf{R}(\overline{K}) \cdot \mathbf{H}(\overline{K}) \xi_{\gamma}^{-1} g^{-1} = g \mathbf{R}(\overline{K}) \cdot \xi_{\gamma} \mathbf{H}(\overline{K}) \xi_{\gamma}^{-1} g^{-1} = g \mathbf{R}(\overline{K}) \cdot \mathbf{H}(\overline{K}) g^{-1}.$$
Lemma 10.9. $g_1(E \otimes_K \overline{K}) = E \otimes_K \overline{K}$.

Proof. By construction, we have $g \overline{H} g^{-1} = \overline{H}$. Hence $g(A^H) = A^{\overline{H}}$. But $A^H = E$ and $A^{\overline{H}} = \overline{E}$. Thus $g(E) = \overline{E}$ and the assertion follows. \hfill $\square$

Lemma 10.10. $g_1(E) = E$.

Proof. We identify $A$ inside $A_{\overline{K}}$ via $x \mapsto x \otimes 1$. Let $x \in E$. By the previous lemma $g_1(x) \in \overline{E}$ so we need only to show that $g_1(x)$ is Gal($\overline{K}$)-stable. For $\gamma \in \text{Gal}(\overline{K})$ we have

$$\gamma(g_1(x)) = \gamma g_1(\overline{x}) = \gamma g_1(x) = (g_1 g_1^{-1} \gamma g_1)(x) = g_1(\xi_\gamma(x)) = g_1(x)$$

where the last equality follows from $\xi_\gamma \in H(\overline{K})$ and the fact that $H(\overline{K})$ fixes $E_{\overline{K}}$. \hfill $\square$

Recall that by definition $R \subset \text{Str}(A, E)$, so $R(\overline{K})$ acts on $E^\times$.

Lemma 10.11. Let $e \in E^\times$. There exists $r \in R(\overline{K})$ such that $r(1) = e$.

Proof. Recall (see [2]) that $R \cap H = Z(H)$ is a 2-group since $H$ is of type $D_4$. Let $E^\times$ be the affine $K$-scheme “of units” of $E$, that is $E^\times(R)$ is the set of units of the multiplicative monoid $E_R$. According to [2] the $K$-morphism $\phi : R \to E^\times$ given by $r \mapsto r(1)$ induces an exact sequence

$$1 \to Z(H) \to R \to R / Z(H) \simeq E^\times \to 1.$$ 

Since $\text{Gal}(K)$ is a pro-3-group $H^1(K, Z(H)) = 1$. It follows that the map $r \mapsto r(1)$ is surjective at the level of $K$-points. \hfill $\square$

Let us return to our element $g_1 \in G(\overline{K})$ that induces our cocycle $\xi$, that is $\xi_\gamma = g_1^{-1} \gamma g_1$. By Lemma 10.10 we have $g_1(1) = e \in E$, while by Lemma 10.11 $e = r(1)$ for some $r \in R(K)$. Let $g_2 = r^{-1} g_1$. Since $r \in \text{Str}(A)(K)$ the element $g_2$ also creates the cocycle $\xi$, that is we still have $\xi_\gamma = g_2^{-1} \gamma g_2$ and by construction $g_2(1) = 1$. Recall that the stabilizer of $1_K = 1_E \in A$ is $H$. Hence $g_2 \in H(\overline{K})$ and this implies $\xi$ is trivial in $H$. \hfill $\square$

11. Triviality of $[\xi_M]$: Case $M = M_0$

Recall that in Lemma 8.4 we constructed a quasi-split group $H_0$ of type $D_4$ that splits over $E$, and its subgroup

$$M_0 = \text{SL}_3 \cdot S \subset H_0$$

where $S = R^{(1)}_{E/K}(G_{m,E})$.

Theorem 11.1. $\text{Ker}[H^1(K, M_0) \to H^1(K, H_0)] = 1$.

Proof. Let $[\lambda]$ belong to the kernel in question. By assumption there exists $h \in H_0(\overline{K})$ such that

$$\lambda_\gamma = h^{-1} \gamma h$$

(14)
Let \( \mathcal{N} := hM_0h^{-1} \). It is easy to verify that this \( \mathcal{K} \)-group is \( \text{Gal}(\mathcal{K}) \)-stable, hence descends to a \( \mathcal{K} \)-group, say \( \mathcal{N} \). Clearly \( \mathcal{N} \simeq \lambda M_0 \). Since \( S \) is central in \( M_0 \) we obtain that \( \mathcal{N}' \simeq \lambda SL_3 \).

Case 1: \( \lambda SL_3 \simeq SL_3 \).

Consider the exact sequence of \( \mathcal{K} \)-groups

\[
1 \rightarrow S \rightarrow M_0 \rightarrow SL_3/(SL_3 \cap S) \simeq SL_3/\mu_3 \simeq PGL_3 \rightarrow 1.
\]

If \([\lambda] \mapsto [\lambda] \) under the map \( H^1(K, M_0) \rightarrow H^1(K, PGL_3) \) we have \( PGL_3 \simeq PGL_3 \). This implies that \( [\lambda] = 1 \) and therefore

\[
[\lambda] \in \text{Im} \left[ H^1(K, S) \rightarrow H^1(K, M_0) \right].
\]

Observe that in \( M_0 \) we have \( \mu_3 = SL_3 \cap S \), and that the natural map \( H^1(K, \mu_3) \rightarrow H^1(K, S) \) is surjective. It follows that \( [\lambda] \) is in the image of the composed map

\[
H^1(K, \mu_3) \rightarrow H^1(K, SL_3) \rightarrow H^1(K, M_0).
\]

Since \( H^1(K, SL_3) = 1 \), we conclude that \( [\lambda] = 1 \).

Case 2: \( \lambda SL_3 \not\simeq SL_3 \).

Then \( \lambda SL_3 = SL_1(D) \) where \( D \) is a cubic division algebra over \( K \). Since the torus \( S \) is central in \( M_0 \), we see that \( \mathcal{N} = \lambda M_0 \) is of the form

\[
\mathcal{N} = S \cdot SL_1(D).
\]

Our aim is to show that \( \lambda H_0 \simeq H_0 \) does not admit any subgroups of this type.

Recall that the Tits index of \( H_0 \) is

\[
15
\]

We visually see that \( H_0 \) contains a \( \mathcal{K} \)-subgroup of the form \( \mathcal{R}_{E/K}(SL_{2, E}) \). This semisimple group is of type \( A_1 \times A_1 \times A_1 \) and corresponds to the the simple roots \( \alpha_1, \alpha_3 \) and \( \alpha_4 \). The centralizer in \( H_0 \) of this group is the copy of \( SL_2 \) corresponding to the highest root \( \tilde{\alpha} = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 \). Choose a torus \( G_m \subseteq \mathcal{L} \).

We let \( \mathcal{L} \) be the reductive subgroup of \( H_0 \) generated by \( \mathcal{R}_{E/K}(SL_{2, E}) \) and this \( G_m \). By construction, \( \mathcal{L} \) is the Levi subgroup of a parabolic subgroup \( P = U^+ \cdot \mathcal{L} \) of \( H_0 \) corresponding to the roots \( \{ \alpha_1, \alpha_3, \alpha_4 \} \). We now look at the subgroup \( (P \cap N)^0 \) of \( H_0 \).

**Lemma 11.2.** The \( \mathcal{K} \)-group \( (P \cap N)^0 \) is a two dimensional anisotropic torus. Furthermore the canonical embedding

\[
(P \cap N)^0 \hookrightarrow P / U^+ \simeq L = G_m \cdot \mathcal{R}_{E/K}(SL_{2, E})
\]
induces an isomorphism

\[(P \cap N)^0 \simeq \mathcal{R}^{(1)}_{E/K}(G_{m,E}) \subset \mathcal{R}_{E/K}(\text{SL}_{2,E})\]

where \(G_{m,E} \subset \text{SL}_{2,E}\) is a split torus. In particular, \((P \cap N)^0\) is a subgroup of the solvable \(K\)-group \(U^+ \rtimes \mathcal{R}^{(1)}_{E/K}(G_{m,E})\).

Proof. We proceed in steps.

(i) \(\text{dim } P = 19\) and \(\text{dim } N = 10\). Thus \(\text{dim}(P \cap N)^0 \geq 1\).

(ii) \((P \cap N)^0\) is anisotropic, hence reductive. Indeed, \((P \cap N)^0 \subset N\) and \(N\) is anisotropic by the assumption of Case 2.

(iii) The derived group \(D\) of \((P \cap N)^0\) is trivial. Otherwise \(D\) would be a semisimple subgroup of \(N' = \text{SL}_1(D)\). Since \(\text{SL}_1(D)\) does not have proper semisimple subgroups, we have \(D = \text{SL}_1(D)\), which is a group of type \(A_2\).

On the other hand \((P \cap N)^0 \hookrightarrow P / U^+ = L\). But \(L'\), which is of type \(A_1 \times A_1 \times A_1\), does not have subgroups of type \(A_2\).

(iv) The \(K\)-group \((P \cap N)^0\) is an anisotropic torus. This follows from (ii) and (iii).

(v) \(\text{dim}(P \cap N)^0 \leq 2\). Recall that

\[(P \cap M)^0 \hookrightarrow P / U^+ = G_m \cdot \mathcal{R}_{E/K}(\text{SL}_{2,E}).\]

By (iv) we get an induced embedding \((P \cap N)^0 \hookrightarrow \mathcal{R}_{E/K}(\text{SL}_{2,E})\). To establish (v) we need to rule out that \(\text{dim}(P \cap N)^0 = 3\). Now \((P \cap N)^0\) is a subgroup of \(N = \text{SL}_1(D) \cdot S\), a reductive group of rank 4. Let \(Z\) be a maximal torus of \(N\) containing \((P \cap N)^0\). Then \(Z / (P \cap N)^0\) is a one dimensional torus, hence split by a quadratic extension \(L\) of \(K\). Thus \(Z_L\) is isotropic. On the other hand \(D_L\) is still a cubic division \(L\)-algebra and \(S_L\) is still anisotropic. Thus the \(L\)-group \(\text{SL}_1(D_L) \cdot S_L\) is anisotropic, and this contradicts that \(Z_L\) is isotropic.

(vi) \((P \cap N)^0\) splits over \(E\). Consider the quotient map

\[\tilde{\phi} : N \to N / \text{SL}_1(D) \simeq S / \mu_3\]

and the induced restriction

\[\phi : (P \cap N)^0 \to N / \text{SL}_1(D) = S / \mu_3\,.

If \(\phi\) is an isogeny we are done (because \(S\) is split over \(E\)).

Since \(S\) does not admit any one dimensional subtori the remaining option is that \(\phi\) is trivial. Then \((P \cap N)^0\) is a subtorus of \(\text{SL}_1(D)\), hence of the form \(\mathcal{R}^{(1)}_{E'/K}(G_{m,E'})\) for some maximal subfield \(K \subset E'\) of \(D\). If \(E' \simeq E\) we are done. Assume otherwise. Then \(((P \cap N)^0)_E\) is anisotropic. On the other hand

\[((P \cap N)^0)_E \hookrightarrow (\mathcal{R}_{E/K}(\text{SL}_{2,E}))_E = \text{SL}^1_{2,E} \times \text{SL}^1_{2,E} \times \text{SL}^1_{2,E}\,.

But this last group does not contain any 2-dimensional subtorus of the form \(\mathcal{R}^{(1)}_{E'/E}(G_{m,E'}).\)

(vii) To finish the proof we observe that all 2-dimensional tori in \(\mathcal{R}_{E/K}(\text{SL}_{2,E})\) that split over \(E\) are of the required form. \(\square\)
Lemma 11.3. The subgroup \((P \cap N)^0\) of \(U^+ \rtimes \mathcal{R}^{(1)}_{E/K}(G_m) \subset U^+ \rtimes L\) given in Lemma 11.2 is conjugate to \(\mathcal{R}^{(1)}_{E/K}(G_{m,E})\).

Proof. \((P \cap N)^0\) and \(\mathcal{R}^{(1)}_{E/K}(G_{m,E})\) are two maximal tori of the solvable \(K\)-group \(U^+ \rtimes \mathcal{R}^{(1)}_{E/K}(G_{m,E})\). It follows that after passing to \(\overline{K}\) the two groups are conjugate by an element \(g_1 \in U^+(\overline{K})\). We write \(g_1 \mathcal{R}^{(1)}_{E/K}(G_{m,E}) g_1^{-1} = (P \cap N)^0\).

The same holds if we replace \(g_1\) by \(\gamma g_1\) for all \(\gamma \in \text{Gal}(K)\) (both corresponding \(\overline{K}\)-groups are “defined over \(K\)).

It follows that \(g_1^{-1} \gamma g_1 \in \mathcal{N}_{U^+}(\mathcal{R}^{(1)}_{E/K}(G_{m,E})) = Z_{U^+}(\mathcal{R}^{(1)}_{E/K}(G_{m,E}))\), where the last equality is a property of maximal tori of solvable algebraic groups (see [12, Section 19.4]).

For convenience let us denote \(Z_{U^+}(\mathcal{R}^{(1)}_{E/K}(G_{m,E}))\) by \(Z\). The \(K\)-group \(Z\) is connected and unipotent. Since \(K\) is of characteristic 0 it is split, (it has a decreasing sequence of subgroups with the additive group \(G_{a}\) as quotients). In particular \(H^1(K, Z) = 1\).

Let \(g_2 \in Z(\overline{K})\) be such that \(g_2^{-1} \gamma g_2 = g_1^{-1} \gamma g_1\). Then \(g_1 g_2^{-1} = \gamma g_1 \gamma(g_2^{-1})\).

Thus \(g = g_1 g_2^{-1} \in U^+(K)\).

Since \(g_2 \in Z(\overline{K})\) we have

\[
g \mathcal{R}^{(1)}_{E/K}(G_{m,E}) g^{-1} = g_1 g_2^{-1} \mathcal{R}^{(1)}_{E/K}(G_{m,E}) g_2 g_1^{-1} = g_1 \mathcal{R}^{(1)}_{E/K}(G_{m,E}) g_1^{-1} = (P \cap N)^0,
\]

as required. \(\square\)

We now return to the proof of Theorem 11.1. Up to conjugation in \(H_0\) we have the commutative square

\[
\begin{array}{ccc}
N = S \cdot SL_1(D) & \xleftarrow{\sim} & H_0 \\
\downarrow & & \downarrow \\
(P \cap N)^0 & \xleftarrow{\sim} & G_m \cdot \mathcal{R}_{E/K}(SL_{2,E})
\end{array}
\]

where, by the last Lemma, the bottom arrow lands in \(\mathcal{R}_{E/K}(SL_{2,E})\) and the derived group of the centralizer of its image inside \(H_0\) is the \(SL_3\) corresponding to the roots \(\alpha_2\) and \(-\tilde{\alpha}\) (see diagram (15)).

Claim: \(SL_1(D)\) splits over \(E\). Recall that \(SL_1(D)\) is the twisted group \(\lambda SL_3\). It will thus suffice to show that \(\lambda_E\) is trivial. Consider the exact sequence

\[1 \to SL_3 \to S \cdot SL_3 \to S / \mu_3 \to 1.\]
Both $E$-groups $SL_{3,E}$ and $(S/\mu_3)_E$ have trivial $H^1$ (the latter since $S$ splits over $E$.) Thus $H^1(E, S \cdot SL_3) = 1$. This establishes our claim.

Since $(P \cap N)^0$ is split over $E$, it sits inside a maximal torus $T$ of $S \cdot SL_1(D)$ that splits over $E$. Necessarily $T = S \cdot T_1$ where $T_1$ is a maximal torus of $SL_1(D)$ that splits over $E$. As we have observed $T_1 \simeq R_{E/K}(G_m,E)$.

Now
\[
Z_{H_0}(R_{E/K}(G_m,E)) = Z_{H_0}((P \cap N)^0) = (P \cap N)^0 \cdot SL_3
\]

Since $T$ commutes with $(P \cap N)^0$ we have $T \subset (P \cap N)^0 \cdot SL_3$.

The torus $T$ therefore has the following properties:

(a) $T$ is anisotropic and a maximal torus of $H_0$ split by $E$.
(b) There is an embedding $T \hookrightarrow (P \cap N)^0 \cdot SL_3$.
(c) There is an embedding $T \hookrightarrow S \cdot SL_1(D)$.

Recall that $E/K$ is the minimal splitting field of $T$, and that $Gal(E/K)$, which is a cyclic group of order 3, acts on the root system $\Sigma = \Sigma(H_0,T)$. So we have a canonical embedding $Gal(E/K) \hookrightarrow Aut(D_4)$. Any 3-Sylow subgroup of the last group is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$. One easily checks that for every element $a \in Aut(D_4)$ of order 3 there exists a unique subroot system $\Delta$ of $\Sigma$ such that $\Delta$ is of type $A_2$ and stable under the action of $a$. Thus, there exists a unique subsystem $\Delta \subset \Sigma$ of type $A_2$ stable with respect to $Gal(E/K)$. This $\Delta$ corresponds to a $K$-subgroup, say $F$, of $H_0$ which is of type $A_2$. By (b) we see that $F$ corresponds to $SL_3$, while by (c) it corresponds to $SL_1(D)$. Since $SL_3 \not\simeq SL_1(1,D)$ the proof of the Theorem is complete. \[\square\]

12. Triviality of $[\xi_M]$. End of the proof

Recall that $[\xi_1],[\xi_2] \in H^1(K,T)$ and by Theorem [10.5] they have the same image in $H_0$. We want to show that they agree in $M_0 = S \cdot SL_3$. Consider the exact sequence of $K$-groups

\[1 \to \mu_3 \to S \times SL_3 \to M_0 \to 1\]

and the induced Galois cohomology sequence

\[H^1(K,SL_3) \xrightarrow{\partial_0} H^1(K,M_0) \xrightarrow{\psi_0} H^2(K,\mu_3)\]
Lemma 12.1. The map $\psi_0$ above has trivial kernel.

Proof. Apply the same argument as in the proof of Lemma 9.2. □

For $i = 1, 2$, let $[\xi^0_i]$ be the image of $[\xi_i]$ in $H^1(K, M_0)$, and $D_i$ central simple $K$-algebras of degree 3 such that $\psi_0: [\xi^0_i] \mapsto [D_i] \in H^2(K, \mu_3) \subset Br(K)$.

Proposition 12.2. $[D_1]$ and $[D_2]$ generate the same subgroup of $Br(K)$.

Proof. Let $L$ be the function field of the Severi-Brauer variety of $D_1$. Then $[D_1 \otimes_K L] = [1]$.

We have

$$
\begin{array}{c}
\xymatrix{
H^1(L, M_0) \ar[r]^-{\psi_0L} & H^2(L, \mu_3) \\
H^1(K, M_0) \ar[u] & H^2(K, \mu_3) \ar[u]
}
\end{array}
$$

By Lemma 12.1, $[\xi^0_1]_L = [1]$. Since $[\xi_1]$ and $[\xi_2]$ have the same image in $H^1(K, H_0)$, say $[\zeta] \in H^1(K, H_0)$, we obtain $[\zeta]_L = [1] \in H^1(L, H_0)$. Consider the canonical commutative diagram

$$
\begin{array}{c}
\xymatrix{
H^1(L, M_0) \ar[r] & H^1(L, H_0) \\
H^1(K, M_0) \ar[u] & H^1(K, H_0) \ar[u]
}
\end{array}
$$

Since $[\zeta]_L = [1]$ and by Theorem 11.1 the top row has trivial kernel, we get that $[\xi^0_2]_L = [1]$. Then $[D_2 \otimes_K L] = [1]$ and the Proposition follows from Amitsur’s Theorem (see [10, Theorem 5.4.1]). □

Corollary 12.3. For proving $[\xi^0_1] = [\xi^0_2]$ we may assume without loss of generality that $D_1 = D_2$.

Proof. Case 1: $[D_1] = [1]$. Then $[\xi^0_1] = [1]$ by Lemma 12.1, hence $[\xi^0_2] = 1$ by Theorem 11.1.

Case 2: $[D_1] \neq [1]$ and $[D_2] \neq [1]$. By the last Proposition either $[D_1] = [D_2]$ or $[D_1] = [D_2]^2$. Recall that if $[\lambda] \in H^1(K, T)$, then $[\lambda]$ and $[\lambda^{-1}]$ have the same image in $H^1(K, G_0)$ by Proposition 8.1. Thus, at the very beginning of the proof of our main theorem, we may replace $[\xi_2]$ by $[\xi_2^{-1}]$ if necessary. But $\psi: ([\xi_2^{-1}])^0 := [\xi_2^{-1} M_0] \mapsto [D_2]^{-1} = [D_2]^2$. □

Proposition 12.4. Let the notation be as above. Then $[\xi^0_1] = [\xi^0_2]$. 
Proof. Recall that the twisted $K$-group $\xi^0_{\Delta} M_0$ was denoted by $M$. As we mentioned before the derived subgroup of $M$ is a $K$-group of type $A_2$ that splits over $E$ (because so does $T$), hence of the form $\text{SL}_1(D)$ where $D$ is a cubic division $K$-algebra. So twisting (16) by $[\xi^0_1]$ we have an exact sequence

$$1 \to \mu_3 \to S \times \text{SL}_1(D) \to M \to 1$$

which gives rise to

$$1 \to H^1(K, \mu_3) \to H^1(K, S) \times H^1(K, \text{SL}_1(D)) \xrightarrow{\phi^0} H^1(K, M) \xrightarrow{\psi} H^2(K, \mu_3).$$

Furthermore, we have the commutative diagram

$\begin{array}{ccc}
H^1(K, T) & \longrightarrow & H^1(K, M) \\
\uparrow \cong & & \uparrow \cong \\
H^1(K, T) & \longrightarrow & H^1(K, M_0)
\end{array}$

where the vertical upper arrows are the twisting bijections. Under these maps

$$[1], [\xi] \longrightarrow H^1(K, M)$$
$$\uparrow \cong \\
[\xi_1], [\xi_2] \longrightarrow H^1(K, M_0)$$

We need to show that $[\xi_M] = [1] \in H^1(K, M)$.

The commutativity of the diagram

$\begin{array}{ccc}
H^1(K, M) & \xrightarrow{\psi} & H^2(K, \mu_3) \\
\uparrow \cong & & \uparrow \cong \\
H^1(K, M_0) & \xrightarrow{\psi_0} & H^2(K, \mu_3)
\end{array}$

where the right vertical upper arrow is the shift by $-\psi_0([\xi^0_1])$ and the fact that $D_1 \cong D_2$ show that $\psi([\xi_M]) = 1$. Then using a reasoning similar to that of Lemma 9.2 we find that $[\xi_M]$ comes from $H^1(K, \text{SL}_1(D))$, say $[\xi_M] = \phi([\delta])$.

Recall the following property of the Rost invariant (see [8]):

**Proposition 12.5.** Let $G_1 \hookrightarrow G_2$ be an embedding of simple simply connected algebraic $K$-groups. There exists a positive integer $n$ (called the Rost multiplier of the embedding) such that the diagram

$\begin{array}{ccc}
H^1(K, G_2) & \xrightarrow{R_{G_2}} & H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \\
\uparrow & & \uparrow r_n \\
H^1(K, G_1) & \xrightarrow{R_{G_1}} & H^3(K, \mathbb{Q}/\mathbb{Z}(2))
\end{array}$

commutes, where $r_n : x \mapsto nx$. 

We apply the above to the case $G_1 = \text{SL}_1(D)$ and $G_2 = H$. By Prop. 7.9 of [8] pg. 122 we see that in our case $n = 1$. Since $[\delta_H] = [\xi_H] = [1]$, we conclude that $R_{\text{SL}_1(D)} : [\delta] \mapsto 1$. On the other hand, we have

$$H^1(K, \text{SL}_1(D)) \simeq K^\times / \text{Nrd}(D^\times).$$

If under this bijection $[\delta] \mapsto \overline{a}$ with $a \in K^\times$, then

$$R_{\text{SL}_1(D)} : [\delta] \mapsto [D] \cup (a).$$

By a result of Merkurjev-Suslin (see [14, Theorem 12.1]), $[D] \cup (a) = 1$ implies that $a \in \text{Nrd}(D^\times)$. Thus $\overline{a} = 1$ so that $[\delta] = [1]$. It follows that $[\xi_M] = \phi([\delta]) = [1]$. □

13. Applications

In this section we provide two applications of our result that strengthen two of the central known results (related to the J. P. Serre question) in the theory of Albert algebras. We maintain the notation of the previous sections.

13.1. A Theorem of M. Rost. Let $K$ be a field of characteristic $\neq 2, 3$ and $G_0$ be a split $K$-group of type $F_4$. Let $[\xi_1]$ and $[\xi_2]$ in $H^1(K, G_0)$ be classes corresponding to $K$-Albert algebras $A_1$ and $A_2$. Assume that $A_1$ and $A_2$ have the same $g_3$, $f_3$ and $f_5$ invariants. Then, as we saw, Rost [21] showed that there exist field extensions $L$ and $F$ of $K$ where $[L : K]$ is prime to 3, $[F : K]$ divides 3, and such that

$$[\xi_1]_L = [\xi_2]_L \text{ and } [\xi_1]_F = [\xi_2]_F.$$

We claim that the assumption $\text{char}(K) \neq 2, 3$ can be dropped and that $L/K$ can be chosen such that $[L : K] = 2$. Indeed, let $K$ be an arbitrary field and $[\xi_1], [\xi_2] \in H^1(K, G_0)$ have the same invariants. Since $[\xi_1], [\xi_2]$ have the same $f_3$ and $f_5$ invariant, the Albert algebras $A_1, A_2$ have the same reduced models (see [15, §12]). Take a separable quadratic field extension $L/K$ killing this reduced model. According to [15, Theorem 58], $A_{1L}$ and $A_{2L}$ arise from the first Tits construction. Since $A_{1L}$ and $A_{2L}$ have the same $g_3$ invariant, our Theorem applies and shows that they are isomorphic over $L$.

13.2. A Theorem of T. A. Springer. According to Theorem 3.3 and Corollary 3.4 (which are due to T. A. Springer) the Rost map

$$R_{G_0} : H^1(K, G_0) \to H^3(K, \mathbb{Q}/\mathbb{Z}(2))$$

has trivial kernel. Here $G_0$ is a split group of type $F_4$.

We claim that the same holds for an arbitrary $K$-group $G$ of type $F_4$ arising from the first Tits construction. To see this we reason as follows. Let $\zeta \in H^1(K, G_0)$ be such that $G = ^\zeta G_0$. Let $[\xi] \in \text{Ker} R_G$. There exists a unique $[\xi'] \in H^1(K, G_0)$ such that

$$[\xi'] = \zeta[\xi].$$
where the right vertical arrow is the twisting bijection. Furthermore, the twisting bijection map is compatible with the shift in $H^3(\mathbb{Q}/\mathbb{Z}(2))$ by $-R_{G_0}([\xi])$, so that we have a commutative diagram

\[
\begin{array}{ccc}
[\xi], [1] & \longrightarrow & H^1(K, G) \\
 & | & \\
[\xi'], [\zeta] & \longrightarrow & H^1(K, G_0)
\end{array}
\]

Since $R_G([1]) = R_G([\xi]) = 0$ we obtain $R_{G_0}([\xi']) = R_{G_0}([\zeta])$. Note that $f_3(\zeta) = 0$ since $G$ arises from the first Tits construction. This, together with $R_{G_0} = \pm(f_3 + g_3)$ yields $f_3(\xi') = 0$ and $g_3(\xi') = g_3(\zeta)$. By our main theorem $[\xi'] = [\zeta]$ and this implies $[\xi] = 1$.

\[\square\]

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