Rectangular M-tensors and strong rectangular M-tensors

Jun He*, Yanmin Liu, Guangjun Xu

School of Mathematics, Zunyi Normal College, Zunyi, Guizhou 563006 China

*Corresponding author, e-mail: hejunfan1@163.com

ABSTRACT: In this paper, two new classes of rectangular tensors called rectangular M-tensors and strong rectangular M-tensors are introduced. It is shown that an even-order partially symmetric rectangular M-tensor is positive semidefinite and an even-order partially symmetric strong rectangular M-tensor is positive definite. As a generalization of rectangular M-tensors, we introduce the rectangular H-tensors. In addition, some properties of (strong) rectangular M-tensors are established.

KEYWORDS: rectangular tensor, H-rectangular tensor, V-singular value, positive definiteness

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INTRODUCTION

Let \( \mathbb{R}(\mathbb{C}) \) be the real (complex) field, \( p, q, m, n \) be positive integers, \( m, n \geq 2, [n] = \{1, 2, \ldots, n\} \). A \((p, q)\)-th order \((m \times n)\)-dimensional real rectangular tensor, denoted by \( \mathcal{A} = (a_{i_{1}i_{2}\ldots i_{p}j_{1}j_{2}\ldots j_{q}}) \in \mathbb{R}^{[p,q;m,n]} \), is defined as follows:

\[
a_{i_{1}i_{2}\ldots i_{p}j_{1}j_{2}\ldots j_{q}} \in \mathbb{R}, \quad i_{1}, \ldots, i_{p} \in [m], \quad j_{1}, \ldots, j_{q} \in [n].
\]

\( \mathcal{A} \) is called nonnegative if \( a_{i_{1}i_{2}\ldots i_{p}j_{1}j_{2}\ldots j_{q}} \geq 0 \), denoted by \( \mathcal{A} \in \mathbb{R}^{[p,q;m,n]}_{+} \). A rectangular tensor \( \mathcal{A} \) is called real partially symmetric, if \( a_{i_{1}i_{2}\ldots i_{p}j_{1}j_{2}\ldots j_{q}} \) is invariant under any permutation of indices among \( i_{1}, \ldots, i_{p} \), and any permutation of indices among \( j_{1}, \ldots, j_{q} \), i.e.,

\[
a_{\pi(i_{1}\ldots i_{p})\sigma(j_{1}\ldots j_{q})} = a_{i_{1}i_{2}\ldots i_{p}j_{1}j_{2}\ldots j_{q}}, \quad \pi \in S_{p}, \quad \sigma \in S_{q},
\]

where \( S_{r} \) is the permutation group of \( r \) indices. When \( p, q \) are even, \( \mathcal{A} \) is called even-order partially symmetric.

For any vectors \( x \in \mathbb{C}^{m}, y \in \mathbb{C}^{n} \), let \( \mathcal{A}x^{p-1}y^{q} \) be a vector in \( \mathbb{C}^{m} \) such that

\[
(\mathcal{A}x^{p-1}y^{q})_{j} = \sum_{i_{1}, \ldots, i_{p}=1}^{m} \sum_{j_{1}, \ldots, j_{q}=1}^{n} a_{i_{1}i_{2}\ldots i_{p}j_{1}j_{2}\ldots j_{q}}x_{i_{1}}\cdots x_{i_{p}}y_{j_{1}}\cdots y_{j_{q}},
\]

where \( i \in [m] \). Let \( \mathcal{A}x^{p}y^{q-1} \) be a vector in \( \mathbb{C}^{n} \) such that

\[
(\mathcal{A}x^{p}y^{q-1})_{j} = \sum_{i_{1}, \ldots, i_{p}=1}^{m} \sum_{j_{1}, \ldots, j_{q}=1}^{n} a_{i_{1}i_{2}\ldots i_{p}j_{1}j_{2}\ldots j_{q}}x_{i_{1}}\cdots x_{i_{p}}y_{j_{1}}\cdots y_{j_{q}},
\]

where \( j \in [n] \).

Definition 1 [1] Let \( \mathcal{A} \in \mathbb{R}^{[p,q;m,n]} \) be a partially symmetric rectangular tensor, if there exist a number \( \lambda \in \mathbb{C} \) and the vectors \( x \in \mathbb{C}^{m}\setminus\{0\}, y \in \mathbb{C}^{n}\setminus\{0\} \) such that

\[
\mathcal{A}x^{p-1}y^{q} = \lambda x^{l-1}, \quad \mathcal{A}x^{p}y^{q-1} = \lambda y^{l-1},
\]

where \( x^{[a]} = [x_{1}^{a}, \ldots, x_{n}^{a}]^{T} \) and \( l = p + q \), then \( \lambda \) is called the singular value of \( \mathcal{A} \), and \( (x, y) \) is the left and right eigenvectors pair of \( \mathcal{A} \), associated with \( \lambda \). If \( \lambda \in \mathbb{R}, x \in \mathbb{R}^{m}, y \in \mathbb{R}^{n} \), then \( \lambda \) is called the H-singular value of \( \mathcal{A} \), and \( (x, y) \) is the left and right H-eigenvectors pair associated with \( \lambda \).

In order to verify the positive definiteness of a \((p, q)\)-th order \((m \times n)\)-dimensional partially symmetric rectangular tensor, the definition of V-singular value is introduced as follows.

Definition 2 [2] Let \( \mathcal{A} \in \mathbb{R}^{[p,q;m,n]} \) be a partially symmetric rectangular tensor, \( p, q \geq 2 \). If there exist a number \( \lambda \in \mathbb{R}, x \in \mathbb{R}^{m}\setminus\{0\}, \) and
\[
y \in \mathbb{R}^n \setminus \{0\} \text{ such that }
\mathcal{A} x^{p-1} y^q = \lambda x^{(p-1)} , \quad \mathcal{A} x^p y^{q-1} = \lambda y^{(q-1)} ,
\sum_{i=1}^{m} x_i^p = 1 , \quad \sum_{j=1}^{n} y_j^q = 1 ,
\tag{2}
\]
then \( \lambda \) is called the V-singular value of \( \mathcal{A} \), and \((x, y)\) is the left and right eigenvectors pair of \( \mathcal{A} \), associated with \( \lambda \).

Suppose that \( \mathcal{A} \in \mathbb{R}^{[p; q; m; n]} \) is a partially symmetric rectangular tensor, \( p \) and \( q \) are even. Then,
\[
f(x, y) = \mathcal{A} x^p y^q = \sum_{i_1, \ldots, i_p=1}^{m} \sum_{j_1, \ldots, j_q=1}^{n} a_{i_1 \cdots i_p, j_1 \cdots j_q} x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q} > 0
\]
for all nonzero vectors \( x \in \mathbb{R}^m, y \in \mathbb{R}^n \) if and only if \( \mathcal{A} \) is positive definite. \( \mathcal{A} \) is called an elasticity tensor if \( p = q = 2 \), \( m = n = 2 \) or 3, and \( \mathcal{A} \) is a real partially symmetric rectangular tensor. When \( \mathcal{A} \) is an elasticity tensor, the strong ellipticity condition holds if and only if \( \mathcal{A} \) is positive definite, the strong ellipticity condition plays an important role in the theory of elasticity [3–5]. The following necessary and sufficient conditions for the positive definiteness of a partially symmetric rectangular tensor are provided in as follows.

**Theorem 1** [1, 2] Suppose that \( \mathcal{A} \in \mathbb{R}^{[p; q; m; n]} \) is a partially symmetric rectangular tensor, \( p \) and \( q \) are even. Then,
(a) \( \mathcal{A} \) is positive definite if and only if all of its H-singular values are positive.
(b) \( \mathcal{A} \) is positive definite if and only if all of its V-singular values are positive.

Eigenvalue problems of square tensor have been drew widespread attention [6–8]. In order to verify the positive definiteness of an \( m \)-th order \( n \)-dimensional real square symmetric tensor \( \mathcal{A} \), the definition of H-eigenvalue is introduced by Qi in [9].

**Definition 3** [9] Let \( \mathcal{A} \in \mathbb{R}^{[m; n]} \) be an \( m \)-th order \( n \)-dimensional real square tensor, if there exists a vector \( x \in \mathbb{R}^n \) and a number \( \lambda \in \mathbb{R} \) such that
\[
\mathcal{A} x^{m-1} = \lambda x^{(m-1)},
\]
where
\[
\mathcal{A} x^{m-1} = \left( \sum_{i_2, \ldots, i_m=1}^{n} a_{i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{i \in [n]},
\]
x\(^{(m-1)}\) = (x\(^{(m-1)}\)\(_i\))\(_{i \in [n]}\),
then \( \lambda \) is called an H-eigenvalue of \( \mathcal{A} \) and \( x \) is called an H-eigenvector of \( \mathcal{A} \) associated with \( \lambda \).

Recently, M-tensors and strong M-tensors are introduced as the generalizations of the well-known M-matrices [10–12], the authors proved that an even-order symmetric M-tensor is positive semidefinite and an even-order symmetric strong M-tensor is positive definite [12–14].

**Definition 4** [10, 12] A tensor \( \mathcal{A} \in \mathbb{R}^{[m; n]} \) is called an M-tensor, if there exist a nonnegative tensor \( \mathcal{B} \) and a positive real number \( s \geq \rho(\mathcal{B}) \) where
\[
\rho_H(\mathcal{A}) = \max \{ |\lambda| : \lambda \text{ is an H-eigenvalue of } \mathcal{A} \}
\]
such that
\[
\mathcal{A} = s \mathcal{A} - \mathcal{B},
\]
in which \( \mathcal{A} = (\delta_{i_1 \cdots i_m}) \) is the \( m \)-th order \( n \)-dimensional identity tensor with
\[
\delta_{i_1 \cdots i_m} = \begin{cases} 
1, & \text{if } i_1 = \cdots = i_m, \\
0, & \text{otherwise.}
\end{cases}
\]
Furthermore, if \( s > \rho_H(\mathcal{B}) \), then \( A \) is called a strong M-tensor.

Lately, Ding et al [5] introduced a structured partially symmetric tensor named elasticity M-tensors, and proved that a nonsingular elasticity M-tensor is positive definite.

**Definition 5** [5] Let \( \mathcal{A} \in \mathbb{R}^{[2; 2; m; n]} \), \( x = (x_i)_{i=1}^{m} \in \mathbb{R}^m \setminus \{0\} \), \( y = (y_i)_{i=1}^{n} \in \mathbb{R}^n \setminus \{0\} \) and \( \lambda \in \mathbb{R} \), such that
\[
\begin{align*}
\mathcal{A} x y y &= \lambda x, \\
\mathcal{A} x x y &= \lambda y, \\
x^T x &= 1, \quad y^T y = 1,
\end{align*}
\tag{3}
\]
where
\[
\begin{align*}
(\mathcal{A} x y y)_i &= \sum_{j=1}^{m} a_{ijkl} x_j y_k y_l, \\
(\mathcal{A} x x y)_i &= \sum_{j=1}^{m} a_{ijkl} x_j x_k y_l.
\end{align*}
\tag{4}
\]
Then \( \lambda \) is called an M-eigenvalue of \( \mathcal{A} \), the vectors \( x \) and \( y \) are called the corresponding M-eigenvectors.

**Definition 6** [5] A partially symmetric tensor \( \mathcal{A} \in \mathbb{R}^{[2; 2; m; n]} \) is called an elasticity M-tensor if there exist a nonnegative partially symmetric tensor \( \mathcal{B} \in \mathbb{R}^{[2; 2; m; n]} \) and a real number \( s \geq \rho_M(\mathcal{B}) \), where
\[
\rho_M(\mathcal{A}) = \max \{ |\lambda| : \lambda \text{ is an M-eigenvalue of } \mathcal{A} \}
\]
such that
\[
\mathcal{A} = s \mathcal{A} - \mathcal{B},
\]
in which \( \mathcal{S}_E = (e_{ijkl}) \in \mathbb{R}^{[2;2;n;n]} \) is the elasticity identity tensor with
\[
e_{ijkl} = \begin{cases} 
1, & \text{if } i = j \text{ and } k = l, \\
0, & \text{otherwise}.
\end{cases}
\]
Furthermore, if \( s > \rho_{V}(\mathcal{S}) \), then \( \mathcal{S} \) is called a nonsingular elasticity M-tensor.

NOTATION AND PRELIMINARIES

In this section, we shall introduce some definitions and important properties related to eigenvalue of a tensor, which are needed in the subsequent analysis.

Let \( \mathbb{R}_+^n \) denote the cone of nonnegative vectors. We use small letters \( a, b, \ldots \) for scalars, small letters \( x, y, \ldots \) for vectors, capital letters \( A, B, \ldots \) for matrices, calligraphic letters \( \mathcal{A}, \mathcal{B}, \ldots \) for tensors. The \( i \)-th entry of a vector \( x \) is denoted by \( x_i \), and the \((i, j)\)-th entry of a matrix \( A \) is denoted by \( a_{ij} \). For any rectangular tensor \( \mathcal{A} = (a_{i_1 \ldots i_p j_1 \ldots j_q}) \in \mathbb{R}^{[p; q; m; n]} \), denote \( |\mathcal{A}| = (|a_{i_1 \ldots i_p j_1 \ldots j_q}|) \).

The Perron-Frobenius theorem for nonnegative square tensors is introduced in [15], which states that the spectral radius of any nonnegative square tensor is an eigenvalue with a nonnegative eigenvector. Denote \( \lambda_{\max}(\mathcal{A}) \) as the maximal \( V \)-singular value of \( \mathcal{A} \in \mathbb{R}^{[p; q; m; n]} \). Then
\[
\lambda_{\max}(\mathcal{A}) = \max \left\{ |\lambda| : \lambda \in \sigma(\mathcal{A}) \right\}
\]
is proved. The identity tensor \( \mathcal{I} \) plays an important role in the definition of M-tensor, because
\[
\mathcal{I} x^{m-1} = x^{[m-1]}
\]
always holds for any nonzero vector \( x \). That is to say, 1 is an unique \( H \)-eigenvalue of the identity tensor \( \mathcal{I} \). Similarly, the elasticity identity tensor \( \mathcal{S}_E \) also plays an important role in the definition of elasticity M-tensor, because
\[
\mathcal{I}_E x^2 = x, \quad \mathcal{S}_E x^2 y = y
\]
always holds for any nonzero vectors \( x, y \). That is to say, 1 is an unique M-eigenvalue of the identity tensor \( \mathcal{I} \). We next introduce (strong) rectangular M-tensors based on the so-called rectangular identity tensor.

Definition 7 A tensor \( \mathcal{A} \in \mathbb{R}^{[p; q; m; n]} \) is called a rectangular M-tensor if there exist a partially symmetric nonnegative tensor \( \mathcal{B} \in \mathbb{R}^{[p; q; m; n]} \) and a real number \( s \geq \rho_{V}(\mathcal{B}) \), where
\[
\rho_{V}(\mathcal{B}) = \max \left\{ |\lambda| : \lambda \text{ is a } V \text{-singular value of } \mathcal{B} \right\}
\]
such that
\[
\mathcal{A} = s \mathcal{I}_E - \mathcal{B},
\]
in which \( \mathcal{I}_E = (\epsilon_{i_1 \ldots i_p j_1 \ldots j_q}) \in \mathbb{R}^{[p; q; m; n]} \) is the rectangular identity tensor with
\[
\epsilon_{i_1 \ldots i_p j_1 \ldots j_q} = \begin{cases} 
1, & \text{if } i_1 = \cdots = i_p \text{ and } j_1 = \cdots = j_q, \\
0, & \text{otherwise}.
\end{cases}
\]
Furthermore, if \( s > \rho_{V}(\mathcal{B}) \), then \( \mathcal{A} \) is called a strong rectangular M-tensor.

Remark 1 If \( \mathcal{A} \in \mathbb{R}^{[p; q; m; n]} \) is a partially symmetric rectangular tensor, \( p \) and \( q \) are even. Then, there exist \( V \)-singular value of \( \mathcal{A} \) and associated left and right eigenvectors [2].

Remark 2 From the definition of rectangular identity tensor, we can get, 1 is its unique \( V \)-singular value of \( \mathcal{I}_E \).

Remark 3 If \( \mathcal{A} \in \mathbb{R}^{[2;2;n;n]} \) and \( \mathcal{B} \in \mathbb{R}^{[2;2;n;n]} \) are partially symmetric, then, the definition of the rectangular M-tensor is the same as the definition of the elasticity M-tensor. Therefore, the rectangular M-tensor can be regarded as a generalization of the elasticity M-tensor.
Theorem 2 Let \( \mathcal{B} \in \mathbb{R}^{[p:q:m:n]} \) be a partially symmetric rectangular tensor, \( \mathcal{A} = a(\mathcal{B} + b\mathcal{R}) \), where \( a \) and \( b \) are two real numbers. Then \( \mu \) is a V-singular value of \( \mathcal{A} \) if and only if \( \mu = a(\lambda + b) \) and \( \lambda \) is a V-singular value of \( \mathcal{B} \). In this case, they have the same eigenvectors pair.

Proof: If \( \lambda \) is a V-singular value of \( \mathcal{B} \) with eigenvectors pair \((x, y)\), then
\[
\mathcal{B}x^{p-1}y^q = \lambda x^{[p-1]}, \quad \mathcal{B}x^py^{q-1} = \lambda y^{[q-1]},
\]
\[
\sum_{i=1}^{m} x_i^p = 1, \quad \sum_{j=1}^{n} y_j^q = 1.
\] (5)
Since \( \mathcal{R} \) is a rectangular identity tensor, then
\[
\mathcal{R}_x x^{p-1}y^q = x^{[p-1]}, \quad \mathcal{R}_x^py^{q-1} = y^{[q-1]} \] (6)
From (5) and (6), we have
\[
a(\mathcal{B} + b\mathcal{R})x^{p-1}y^q = a(\lambda + b)x^{[p-1]},
\]
a(\mathcal{B} + b\mathcal{R})x^py^{q-1} = a(\lambda + b)y^{[q-1]},
which means
\[
\mathcal{A}x^{p-1}y^q = \mu x^{[p-1]}, \quad \mathcal{A}x^py^{q-1} = \mu y^{[q-1]},
\]
i.e., \( \mu \) is a V-singular value of \( \mathcal{A} \) with eigenvectors pair \((x, y)\).
On the other side, if \( a = 0 \), the result is trivial. If \( a \neq 0 \), suppose \( \mu \) is a V-singular value of \( \mathcal{A} \) with eigenvectors pair \((x, y)\), then
\[
\mathcal{B}x^{p-1}y^q = \left( \frac{1}{a} \right)(\mu - ab)x^{[p-1]},
\]
\[
\mathcal{B}x^py^{q-1} = \left( \frac{1}{a} \right)(\mu - ab)y^{[q-1]},
\]
i.e., \( \lambda = \frac{1}{a}(\mu - ab) \) is a V-singular value of \( \mathcal{B} \) with eigenvectors pair \((x, y)\). \( \square \)

Corollary 1 Suppose \( \mathcal{B} \in \mathbb{R}^{[p:q:m:n]} \) is partially symmetric, \( s \) is a real numbers and \( \mathcal{A} = s\mathcal{R} - \mathcal{B} \). Then for any V-singular value \( \eta \) of \( \mathcal{A} \) with eigenvectors pair \((x, y)\), there exists a V-singular value \( \eta = s - \eta \) of \( \mathcal{B} \) with same eigenvectors pair \((x, y)\).

It is showed that all H-eigenvalues of a M-tensor are nonnegative, and all H-eigenvalues of a strong M-tensor are positive [10, 12]. For rectangular M-tensors and strong rectangular M-tensors, the following spectral properties are presented.

Theorem 3 If \( \mathcal{A} = s\mathcal{R} - \mathcal{B} \) is a partially symmetric rectangular M-tensor and \( \eta \) is a V-singular value of \( \mathcal{A} \), then \( \eta \) is nonnegative. If \( \mathcal{A} = s\mathcal{R} - \mathcal{B} \) is a partially symmetric strong rectangular M-tensor and \( \eta \) is a V-singular value of \( \mathcal{A} \), then \( \eta \) is positive.

Proof: Let \( \rho_v(\mathcal{B}) \) be the largest V-singular value of \( \mathcal{B} \), according to Corollary 1, there exists a V-singular value \( \theta \) of \( \mathcal{B} \) such that
\[
\eta = s - \theta.
\]
Since \( \mathcal{A} = s\mathcal{R} - \mathcal{B} \) is a rectangular M-tensor, then
\[
\eta = s - \theta \geq s - \rho_v(\mathcal{B}).
\]
Similarly, the results about strong rectangular M-tensors can be obtained. \( \square \)

Theorem 4 Let a partially symmetric rectangular tensor \( \mathcal{B} \) be nonnegative, irreducible and \( \mathcal{A} = s\mathcal{R} - \mathcal{B} \) be a rectangular M-tensor. Then the smallest V-singular value of \( \mathcal{A} \) is nonnegative and its corresponding eigenvectors are positive. If \( \mathcal{A} \) is a strong rectangular M-tensor, then the smallest V-singular value of \( \mathcal{A} \) is positive and its corresponding eigenvectors are positive.

Proof: By Lemma 1, we know that \( \rho_v(\mathcal{B}) \) is a positive V-singular value with positive eigenvectors. By Corollary 1, we can see that \( c = s - \rho_v(\mathcal{B}) \geq 0 \) is a V-singular value of \( \mathcal{A} \) and they have the same eigenvectors. If \( \mathcal{A} \) is a strong rectangular M-tensor, note that \( c = s - \rho_v(\mathcal{B}) > 0 \). \( \square \)

The entries \( a_{i...j}(i \in [m], j \in [n]) \) are called diagonal, and other entries are called off-diagonal. A rectangular tensor in \( \mathbb{R}^{[p:q:m:n]} \) is called a rectangular Z-tensor if all its off-diagonal entries are nonpositive.

Theorem 5 Let \( \mathcal{A} \in \mathbb{R}^{[p:q:m:n]} \) be a partially symmetric rectangular Z-tensor. Then \( \mathcal{A} \) is a strong rectangular M-tensor if and only if \( a > \rho_v(\alpha\mathcal{R} - \mathcal{A}) \), where \( \alpha = \max_{i \in [m], j \in [n]} \{a_{i...j}\} \).

Proof: If \( a > \rho_v(\alpha\mathcal{R} - \mathcal{A}) \), by \( \mathcal{A} = \alpha\mathcal{R} - \mathcal{A} \) and the definition of strong rectangular M-tensors, then \( \mathcal{A} \) is a strong rectangular M-tensor.

If \( \mathcal{A} \) is a strong rectangular M-tensor, then \( \mathcal{A} \) can be written as \( \mathcal{A} = \alpha\mathcal{R} - \mathcal{B} \), where \( \mathcal{B} \) is a nonnegative rectangular tensor and \( \alpha > \rho_v(\mathcal{B}) \). Then \( \alpha\mathcal{R} - \mathcal{A} = (\alpha - \rho_v(\mathcal{B}))\mathcal{R} + \mathcal{B} \), which yields \( \alpha - \rho_v(\alpha\mathcal{R} - \mathcal{A}) = \rho_v(\mathcal{B}) \mathcal{R} - \mathcal{B} > 0 \), therefore \( \alpha > \rho_v(\alpha\mathcal{R} - \mathcal{A}) \). \( \square \)

Theorem 6 \( \mathcal{A} \in \mathbb{R}^{[p:q:m:n]} \) is a rectangular M-tensor if and only if \( \mathcal{A} + t\mathcal{R} \) is a strong rectangular M-tensor for any \( t > 0 \).

Proof: If \( \mathcal{A} + t\mathcal{R} \) is a strong rectangular M-tensor for any \( t > 0 \), when \( t \) approaches 0, then \( \mathcal{A} \) is a strong rectangular M-tensor.
If \(\mathcal{A}\) is a strong rectangular M-tensor, then \(\mathcal{A}\) can be written as \(\mathcal{A} = s\mathcal{R} - \mathcal{B}\), where \(\mathcal{B}\) is a nonnegative rectangular tensor and \(s > \rho_V(\mathcal{B})\). Then \(\mathcal{A} + t\mathcal{R} = (s + t)\mathcal{R} - \mathcal{B}\), which yields that, \(\mathcal{A} + t\mathcal{R}\) is a strong rectangular M-tensor. \(\square\)

**Theorem 7** When \(p, q\) are even, let \(\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}\) be a partially symmetric rectangular Z-tensor. Then \(\mathcal{A}\) is a strong rectangular M-tensor if and only if \(\mathcal{A}\) is positive definite, and \(\mathcal{A}\) is a rectangular M-tensor if and only if \(\mathcal{A}\) is positive semidefinite.

**Proof:** When \(p, q\) are even, if \(\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}\) is a strong rectangular M-tensor, by Theorem 3 and Theorem 4, then \(\mathcal{A}\) is positive definite.

If \(\mathcal{A}\) is positive definite, then for any vectors \(x, y \neq 0\), \(\mathcal{A}x^py^q > 0\). Denote \(\mathcal{A} = s\mathcal{R} - \mathcal{B}\), where \(\mathcal{B}\) is a nonnegative rectangular tensor, then \((s\mathcal{R} - \mathcal{B})x^py^q > 0\), which yields \(s > \rho_V(\mathcal{B})\) by \(\sum_{i=1}^{m} x_i^p = 1, \sum_{j=1}^{n} y_j^q = 1\). The result for rectangular M-tensors can be obtained similarly. \(\square\)

The following propositions can be obtained from the definitions of \(\mathcal{A}x^p\) and \(\mathcal{A}y^q\).

**Theorem 8** When \(p, q\) are even, \(\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}\) is a partially symmetric rectangular Z-tensor. Then \(\mathcal{A}\) is a strong rectangular M-tensor if and only if \(\mathcal{A}x^p\) is a strong M-tensor for each \(x \geq 0\), and \(\mathcal{A}y^q\) is a rectangular M-tensor if and only if \(\mathcal{A}y^q\) is a M-tensor for each \(y \geq 0\).

**Proof:** When \(p, q\) are even, if \(\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}\) is a strong rectangular M-tensor, by Theorem 3, then for any vectors \(y \neq 0\), \(\mathcal{A}x^py^q > 0\), which yields \(\mathcal{A}x^p\) is positive definite. And we find that, \(\mathcal{A}x^p\) is a Z-tensor for each \(x \geq 0\). Therefore, \(\mathcal{A}x^p\) is a strong M-tensor.

If \(\mathcal{A}x^p\) is a strong M-tensor, then for any vectors \(y \neq 0\), \(\mathcal{A}x^py^q > 0\). Denote \(\mathcal{A} = s\mathcal{R} - \mathcal{B}\), where \(\mathcal{B}\) is a nonnegative rectangular tensor, then \(s > \rho_V(\mathcal{B})\) for each \(x, y \geq 0\), which yields \(s > \rho_V(\mathcal{B})\) by \(\sum_{i=1}^{m} x_i^p = 1, \sum_{j=1}^{n} y_j^q = 1\) and Lemma 1. Therefore, \(\mathcal{A}\) is a strong rectangular M-tensor. The result for rectangular M-tensors can be obtained similarly. \(\square\)

**Theorem 9** When \(p, q\) are even, \(\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}\) is a partially symmetric rectangular Z-tensor. Then \(\mathcal{A}\) is a strong rectangular M-tensor if and only if \(\mathcal{A}y^q\) is a strong M-tensor for each \(y \geq 0\), \(\mathcal{A}\) is a rectangular M-tensor if and only if \(\mathcal{A}y^q\) is a M-tensor for each \(y \geq 0\).

The following theorem can be obtained by Theorem 8 and Theorem 9.

**Theorem 10** When \(p, q\) are even, \(\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}\) is a partially symmetric rectangular Z-tensor. Then \(\mathcal{A}\) is a strong rectangular M-tensor if and only if one of the following conditions satisfies:

1. For each \(x \geq 0\), there exists \(y \geq 0\) such that \(\mathcal{A}x^py^q > 0\);
2. For each \(x \geq 0\), there exists \(y > 0\) such that \(\mathcal{A}x^py^{q-1} > 0\);
3. For each \(y \geq 0\), there exists \(x \geq 0\) such that \(\mathcal{A}x^{p-1}y^q > 0\);
4. For each \(y \geq 0\), there exists \(x > 0\) such that \(\mathcal{A}x^{p-1}y^q > 0\).

**RECTANGULAR H-TENSOR AND STRONG RECTANGULAR H-TENSOR**

\(\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}\) is a copositive rectangular tensor, if for any \(x \in \mathbb{R}_+^n, y \in \mathbb{R}^n, \mathcal{A}x^py^q \geq 0, \mathcal{A} \in \mathbb{R}^{[p;q;m;n]}\) is a strictly copositive rectangular tensor, if for any \(x \neq 0 \in \mathbb{R}_+^n, 0 \neq y \in \mathbb{R}^n, \mathcal{A}x^py^q > 0\) [16]. The definition of H-tensor was introduced in [10]. In this section, we extend rectangular M-tensors to rectangular H-tensors as follows.

**Definition 8** Let \(\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}\) be a partially symmetric rectangular Z-tensor. Then \(M(\mathcal{A}) = (m_{i_1...i_p,j_1...j_q}) \in \mathbb{R}^{[p,q;m,n]}\) is called comparison rectangular tensor of \(\mathcal{A}\), whose entries are defined as:

\[
m_{i_1...i_p,j_1...j_q} = \begin{cases} +|a_{i_1...i_p,j_1...j_q}|, & \text{if } i_1 = \cdots = i_p, j_1 = \cdots = j_q, \\ -|a_{i_1...i_p,j_1...j_q}|, & \text{otherwise}, \end{cases}
\]

A rectangular tensor is called a rectangular H-tensor, if its comparison tensor is a rectangular M-tensor, and a rectangular tensor is called a strong rectangular H-tensor, if its comparison tensor is a strong rectangular M-tensor.
Theorem 11 ([16]) Let $A \in \mathbb{R}^{[p;q;m;n]}$ be a partially\symmetric rectangular tensor. Then $A$ is copositive if and only if

$$N_{\min}^1(A) \equiv \min \left\{ A x^p y^q : x \in \mathbb{R}^m_+, y \in \mathbb{R}^n_+ \right\} \geq 0,$$

and only if

$$N_{\min}^2(A) \equiv \min \left\{ A x^p y^q : x \in \mathbb{R}^m_+, y \in \mathbb{R}^n_+, \sum_{i=1}^m x_i^p = 1, \sum_{j=1}^n y_j^q = 1 \right\} \geq 0.$$ 

$A$ is strictly copositive if and only if

$$N_{\min}^1(A) > 0$$

and

$$N_{\min}^2(A) > 0.$$ 

A general case of above theorem is given as follows.

Theorem 12 Let $A \in \mathbb{R}^{[p;q;m;n]}$ be a partially\symmetric rectangular tensor. Then $A$ is copositive if and only if

$$N_{\min}^1(A) \equiv \min \left\{ A x^p y^q : x \in \mathbb{R}^m_+, y \in \mathbb{R}^n_+ \right\} \geq 0,$$

and

$$N_{\min}^2(A) \equiv \min \left\{ A x^p y^q : x \in \mathbb{R}^m_+, y \in \mathbb{R}^n_+, \sum_{i=1}^m x_i^p = 1, \sum_{j=1}^n y_j^q = 1 \right\} > 0.$$ 

$A$ is strictly copositive if and only if

$$N_{\min}^1(A) > 0$$

and

$$N_{\min}^2(A) > 0.$$ 

Proof: For any $0 \neq x \in \mathbb{R}^m_+, 0 \neq y \in \mathbb{R}^n_+$, let

$$\tilde{x} = \frac{x}{\left( \sum_{i=1}^m x_i^p \right)^{\frac{1}{p}}}, \quad \tilde{y} = \frac{y}{\left( \sum_{j=1}^n y_j^q \right)^{\frac{1}{q}}}.$$

then

$$\sum_{i=1}^m \tilde{x}_i^p = 1, \quad \sum_{j=1}^n \tilde{y}_j^q = 1,$$

and

$$A \tilde{x}^p \tilde{y}^q = \frac{A x^p y^q}{\sum_{i=1}^m x_i^p \sum_{j=1}^n y_j^q}.$$

Therefore, $N_{\min}^1(A) > 0$ if and only if $N_{\min}^2(A) > 0$. The second conclusion is obtained similarly.

Theorem 13 When $p, q$ are even, then a partially\symmetric rectangular $M$-tensor is copositive, and a partially\symmetric strong rectangular $M$-tensor is strictly copositive.

Proof: If $A$ is a partially\symmetric rectangular $M$-tensor, when $p, q$ are even, from Theorem 7 in [2], we have

$$N_{\min}^2(A) \geq \lambda_{\min}(A)$$

$$= \min \{ A x^p y^q : \sum_{i=1}^m x_i^p = 1, \sum_{j=1}^n y_j^q = 1 \} \geq 0,$$

which yields that, $A$ is copositive. The second conclusion can be obtained similarly.

Theorem 14 When $p, q$ are even, $A$ is a partially\symmetric rectangular $H$-tensor with nonnegative diagonal entries, then $A$ is positive semidefinite. If $A$ is a strong partially\symmetric rectangular $H$-tensor with positive diagonal entries, then $A$ is positive definite.

Proof: Let $A = D - B$, where $D$ is the diagonal part of $A$. Then its comparison tensor $M(A) = D - |B|$ is a partially\symmetric rectangular $M$-tensor. By Theorem 12, $M(A)$ is copositive, which yields

$$M(A) \tilde{x}^p \tilde{y}^q = D \tilde{x}^p \tilde{y}^q - |B| \tilde{x}^p \tilde{y}^q \geq 0,$$

where $\tilde{x} \in \mathbb{R}^m_+, \tilde{y} \in \mathbb{R}^n_+$, $\sum_{i=1}^m \tilde{x}_i^p = 1, \sum_{j=1}^n \tilde{y}_j^q = 1$. Then

$$A x^p y^q = D x^p y^q - B x^p y^q \geq D x^p y^q - |B||x|^p|y|^q \geq 0,$$

where $x \in \mathbb{R}^m, y \in \mathbb{R}^n, \sum_{i=1}^m x_i^p = 1, \sum_{j=1}^n y_j^q = 1$. Therefore, $A$ is positive semidefinite. The second conclusion can be obtained similarly.

RECTANGULAR TENSOR COMPLEMENTARITY PROBLEMS

Let $A \equiv (a_{i_1 j_1 \ldots i_p j_p \ldots i_q j_q}) \in \mathbb{R}^{[p;q;m;n]}$, $q_m \in \mathbb{R}^m$ and $q_n \in \mathbb{R}^n$. The rectangular tensor complementarity problem [17], denoted by RTCP($A$, $q_m$, $q_n$), is to find vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ such that

$$q_m + A x^{p-1} y^q \geq 0, \quad x \geq 0, \quad x^T(q_m + A x^{p-1} y^q) = 0,$$

$$q_n + A x^p y^{q-1} \geq 0, \quad y \geq 0, \quad y^T(q_n + A x^p y^{q-1}) = 0.$$ 

Vectors $x$ and $y$ are said to be feasible if and only if $x$ and $y$ satisfy the following inequalities:

$$q_m + A x^{p-1} y^q \geq 0, \quad x \geq 0,$$

$$q_n + A x^p y^{q-1} \geq 0, \quad y \geq 0.$$ 

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A rectangular tensor $\mathcal{A} = (a_{i_1i_2\ldots i_pj_1j_2\ldots j_q}) \in \mathbb{R}^{[p,q;m,n]}$ is called a rectangular S-tensor if and only if there exists $0 < x \in \mathbb{R}^m$, $0 < y \in \mathbb{R}^n$ such that
\[ \mathcal{A} x^{p-1} y^q > 0, \quad \mathcal{A} x^p y^{q-1} > 0. \] (13)

Then, a strong rectangular M-tensor is a rectangular S-tensor [17]. From Theorem 11 in [17], the following conclusion can be obtained easily.

**Corollary 2** Let $\mathcal{A} \in \mathbb{R}^{[p,q;m,n]}$ be a strong rectangular M-tensor. Then, the RTCP($\mathcal{A}, q_m, q_n$) is feasible for all $q_m \in \mathbb{R}^m$, $q_n \in \mathbb{R}^n$.

**CONCLUSION**

In this paper, based on the definition of V-singular value for rectangular tensors, we extend elasticity M-tensors to rectangular M-tensors. Some properties of rectangular M-tensors are also presented. Finally, we prove that, an even-order partially symmetric rectangular H-tensor with nonnegative diagonal entries is positive semidefinite and an even order partially symmetric rectangular H-tensor with positive diagonal entries is positive definite.

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