Maximal Correlation Secrecy

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Abstract

This paper shows that the Hirschfeld-Gebelein-Rényi maximal correlation between the message and the ciphertext provides good secrecy guarantees for ciphers with short keys. We show that a small maximal correlation $\rho$ can be achieved via a randomly generated cipher with key length $\approx 2 \log(1/\rho)$, independent of the message length, and by a stream cipher with key length $2 \log(1/\rho) + \log n + 2$ for a message of length $n$. We provide a converse result showing that these ciphers are close to optimal. We then show that any cipher with a small maximal correlation achieves a variant of semantic security with computationally unbounded adversary. Finally, we show that a small maximal correlation implies secrecy with respect to several mutual information based criteria but is not necessarily implied by them. These results clearly demonstrate that maximal correlation is a stronger and more practically relevant measure of secrecy than mutual information.

Index Terms

Information-theoretic secrecy, Hirschfeld-Gebelein-Rényi maximal correlation, stream cipher, expander graph.

I. INTRODUCTION

Consider the symmetric-key cryptosystem setting in which Alice encrypts a message (plaintext) $M$ using a shared secret key $K$ into a ciphertext $C$ and sends it to Bob who recovers the message using the ciphertext and the key. The system is said to provide perfect secrecy if the eavesdropper Eve cannot gain any information about the message from the ciphertext $C$ alone, that is, if $M$ and $C$ are independent, or equivalently if their mutual information $I(M; C)$ is zero. Shannon [1] showed that achieving perfect secrecy requires the key to be as long as the message, which is impractical in most applications.

To analyze cryptosystems that use shorter keys, less stringent secrecy criteria have been developed. One popular criterion in computer science is semantic security [2], which restricts the eavesdropper to use only probabilistic polynomial-time algorithms. Although satisfied by short keys, the proofs of semantic security rely on unproven computational hardness assumptions. A secrecy criterion that is also satisfied by short keys but does not rely on computational hardness is to require that the mutual information $I(M; C)$ be smaller than some positive value. As pointed out by Maurer [3], mutual information is too loose a criterion if it is not required to approach zero. For example, consider a 2-bit message $M \sim \text{Unif}\{0, 1, 2, 3\}$ with the constraint that $I(M; C) \leq 1$. It is not difficult to see that $I(M; C) = 1$ is achieved by a one-bit key $K \sim \text{Bern}(1/2)$ and a simple encryption function that hides only the most significant bit of the message, i.e., $C_1 = M + 2K \mod 4$. Such encryption function is clearly unsatisfactory as Eve would know the least significant bit of the message perfectly. On the other hand, if we use the encryption function $C_2 = M + K \mod 4$, which also achieves $I(M; C) = 1$, Eve would not be able to correctly guess either bit of the message with probability greater than $3/4$ — a better secrecy guarantee than using $C_1$. Hence not all ciphers that achieve a given mutual information bound provide good secrecy guarantees.

A natural question to ask then is whether there is a measure of secrecy that allows for short keys and every cipher that achieves it provides good secrecy guarantees even when the eavesdropper has unlimited computational power.

In this paper we answer this question in the affirmative. We show that the Hirschfeld-Gebelein-Rényi maximal correlation [4]–[6] between the message and the ciphertext,

$$\rho_m(M; C) = \max_{f(m), g(c): \mathbb{E}(f(M)) = \mathbb{E}(g(C)) = 0, \quad \mathbb{E}(f'(M)) = \mathbb{E}(g'(C)) = 1} \mathbb{E}(f(M)g(C)),$$

(1)

satisfies all the properties of the desired measure of secrecy. We say that a cipher achieves $\rho$-maximal correlation secrecy if $\rho_m(M; C) \leq \rho$. While the extreme case of $\rho = 0$ is equivalent to perfect secrecy, $\rho < 1$ provides better secrecy guarantees than mutual information. While in the above example, the two encryption functions $C_1$ and $C_2$ yield the same mutual information, the maximal correlation for $C_2$, $\rho_m(M; C_2) = \sqrt{2}/2$, is lower than that for $C_1$, $\rho_m(M; C_1) = 1$, signifying that $C_2$ indeed provides better secrecy than $C_1$.

The relationship between maximal correlation and the probability of agreement between binary decisions made by two parties (which loosely corresponds to the probability that Eve can guess a one-bit function of $M$ correctly in the current setting) was investigated by Witsenhausen [7]. Recently, Calmon et al. [8] studied the use of principle intertias (which includes maximal correlation) in estimation problems and suggested that they can be used as measures of secrecy. They showed that unlike mutual information, a small maximal correlation guarantees that Eve cannot guess any function of $M$ with correct
probability substantially greater than if she did not know \( C \). Their result assumed that the pmf on \( M \) is fixed, which is not completely satisfactory since in practice the same cipher is used on different types of data with possibly different pmfs. Other information-theoretic secrecy measures that allow the key to be shorter than the message are proposed by Maurer [9], Cachin and Maurer [10], and Calmon et al. [11], [12].

In this paper, we show that maximal correlation is not only a stronger measure of secrecy than mutual information, but is also quite relevant to practical cryptosystems. The main contributions of this paper are as follows.

**Maximal correlation secrecy key length.** It is quite straightforward to show that for an \( n \)-bit message \( M \), the constraint \( I(M;C) \leq r \) is achieved with equality by many ciphers that use an \((n-r)\)-bit key, including those that protect only \((n-r)\) bits of the message and leave the rest completely unprotected. The story for the \( \rho \)-maximal correlation secrecy criterion is less straightforward and more interesting. In Section III, we show the surprising result that \( \rho \)-maximal correlation secrecy can be achieved by short keys of length independent of the message length. We first establish a converse result showing that every \( \rho \)-maximal correlation secure cipher must have a key length greater than \( 2 \log(1/\rho) - \log(1 + 2^{-n}\rho^{-2}) \) bits. We then show that a cipher constructed using expander graphs can achieve \( \rho \)-maximal correlation secrecy with a key length of \((2 + o(1)) \log(1/\rho) \) bits as \( \rho \to 0 \), independent of \( n \). We further show that \( \rho \)-maximal correlation secrecy can be achieved with high probability via a randomly generated binary additive stream cipher with a key length of \( 2 \log(1/\rho) + \log n + 2 \) bits. These results make maximal correlation secrecy relevant to practical cryptosystems, which typically use keys of fixed length. The proofs of these results are given in Section VI.

**Constrained-distribution security.** In Section IV, we show that a \( \rho \)-maximal correlation secure cipher also achieves a variant of semantic security with computationally unbounded adversary. We show that Eve cannot guess the outcome of any one-bit function of the message \( M \) better than if she did not know the ciphertext \( C \), even if the pmf of \( M \) can be selected from a large class of pmfs characterized by their \( \chi^2 \)-divergence from the uniform pmf. Therefore we are able to provide secrecy guarantees for a cipher used on data with different pmfs, and even if some partial information about \( M \) is provided to Eve, she cannot guess the outcome of any one-bit function of \( M \) better than if she knew the partial information but not \( C \). In Section IV, we formally define the constrained-distribution security setting. The proof of the result in given in Section VI-F.

**Relationship to other secrecy criteria.** In Section V we discuss the relationship between \( \rho \)-maximal correlation and strong secrecy, weak secrecy, and leakage rate. We show that \( \rho \)-maximal correlation secrecy is stronger than these other secrecy criteria, in the sense that they are implied by \( \rho \)-maximal correlation secrecy with suitable choices of \( \rho \), but they do not imply \( \rho \)-maximal correlation secrecy for any \( \rho < 1 \). The proofs of these relationships are given in Section VI-G.

### II. Definitions and Notation

Throughout this paper, we denote the joint probability matrix of \( X \) and \( Y \) by \( P_{X,Y} \in \mathbb{R}^{\mathcal{X} \times \mathcal{Y}} \). We denote the spectral norm of the matrix \( A \in \mathbb{R}^{m \times n} \) as

\[
\|A\| = \max_{v \in \mathbb{R}^n, \|v\|_2 = 1} \|Av\|
\]

and its Frobenius norm as \( \|A\|_F \). We denote the \( m \times n \) matrix consisting of all ones by \( \mathbf{1}_{m \times n} \). The log function is base 2 and the entropy is measured in bits. We use the notation \([1:n] = \{1, 2, \ldots, n\}\) and \( \text{Unif}(A) \) to be the uniform probability mass function (pmf) over a finite set \( \mathcal{A} \).

We consider a cryptosystem that consists of

- a message \( M \in \mathcal{M} \), where \( \mathcal{M} = [1:2^n] \), i.e., \( M \) is an \( n \)-bit message, unless specified otherwise,
- a random secret key \( K \sim \text{Unif}(K) \), where \( K = [1:2^n] \) unless specified otherwise,
- an encryption function \( E(k,m) \) that maps every pair \((k,m) \in K \times M\) into a ciphertext \( c \in C \), where \( C = [1:2^n] \) unless specified otherwise, and
- a decryption function \( D(k,c) \) that maps every pair \((k,c) \in K \times C\) into a message \( m \in \mathcal{M} \) such that \( D(k,E(k,m)) = m \) for any \( m \)

The pair of encryption and decryption functions \( (E,D) \) is called a (block) cipher. We assume throughout the paper that the eavesdropper knows the ciphertext \( C \) but not the message \( M \) or the key \( K \). A cipher is said to be \( \rho \)-maximal correlation secure if \( \rho_m(M,E(K,M)) \leq \rho \) assuming that \( M \sim \text{Unif}(M) \), where \( \rho_m \) is as defined in (1). The encryption function can also be probabilistic. In this case, the ciphertext \( C = E(K,M,W) \) is also a function of a random variable \( W \), which is generated using the local randomness at the sender, and is unknown to the receiver and the eavesdropper. The cipher is assumed to be deterministic unless specified otherwise.

### III. Maximal Correlation Secrecy Key Length

We provide bounds on the key length of a \( \rho \)-maximal correlation secure cipher in terms of \( \rho \) and the message length \( n \). We first establish the following lower bound on the key length.
Theorem 1. If a cipher is $\rho$-maximal correlation secure, then its key length is lower bounded as

$$s \geq \log \left( \frac{1}{\rho^2 + 2^n} \right).$$

The proof of this theorem is given in Section VI-B. Note that when $\rho > 0$, this lower bound can be written as

$$s \geq 2 \log \frac{1}{\rho} - \log \left( 1 + \frac{1}{2^n \rho^2} \right),$$

which approaches $2 \log(1/\rho)$ as $n$ tends to infinity. Also note that this bound applies to any ciphertext length (not necessarily equal to message length) and to probabilistic encryption functions.

We now show a construction of a cipher with key length close to the lower bound using expander graphs. For an integer $d > 0$, let $\sigma_1, \ldots, \sigma_d$ be permutations of $[1 : 2^n]$ which satisfy $|\{i : \sigma_i = \sigma\}| = |\{i : \sigma_i = \sigma^{-1}\}|$ for any $\sigma$. These permutations induce a $d$-regular graph with vertex set $[1 : 2^n]$, edges $(i, \sigma_k(i))$ for $i \in [1 : 2^n]$ and $k \in [1 : d]$, and an adjacency matrix

$$A = \sum_{k=1}^{d} A_k,$$

where $A_k$ is the permutation matrix corresponding to $\sigma_k$. Such a graph is referred to as an expander graph if the magnitude of the second largest eigenvalue (in absolute value) of $A$

$$|\lambda_2(A)| = \left\| A - \frac{d}{2^n} I_{2^n \times 2^n} \right\|$$

is small. Such a graph can be constructed explicitly. For example, a non-bipartite Ramanujan graph [13] has a second eigenvalue $|\lambda_2(A)| \leq 2\sqrt{d} - 1$.

Given an expander graph, we can define a corresponding expander graph cipher with $M = C = [1 : 2^n]$, $K = [1 : d]$, $E(k, m) = \sigma_k(m)$, and $D(k, c) = \sigma^{-1}_k(c)$. We now find the maximal correlation for such an expander graph cipher.

Theorem 2. The cipher defined by an expander graph with adjacency matrix $A$ has maximal correlation $\rho_m(M; C) = \frac{1}{d} |\lambda_2(A)|$.

As a result, the cipher corresponding to a non-bipartite Ramanujan graph is $\rho$-maximal correlation secure if

$$\log d \geq 2 \log \frac{1}{\rho} + 2,$$

where $s = \log d$ corresponds to the key length if it is an integer.

The proof of this theorem is given in Section VI-C. It is a consequence of the characterization of maximal correlation in [7]. The relationship between maximal correlation and the second eigenvalue of a graph is also studied in [14]. A limitation of this construction is that there may not be constructions of Ramanujan graphs for a desired $n$ and $s$. Using the result in [15] on the second eigenvalue of random regular graphs, we can show the existence of maximal correlation secure ciphers with key lengths close to the lower bound for any large enough $n$ and $s$.

Theorem 3. There exists a $\rho$-maximal correlation secure cipher with message length $n \geq 2$ and key length $s \geq 2$ if

$$s \geq \left( 2 \log \frac{1}{\rho} \right) \left( 1 + \frac{\alpha}{\log n} \right) + \alpha,$$

where $\alpha > 0$ is a constant.

The following corollary provides a bound on $s$ which is independent of $n$.

Corollary 1. There exists a $\rho$-maximal correlation secure cipher with message length $n \geq 2$ and key length $s \geq 2$ if

$$s \geq \left( 2 \log \frac{1}{\rho} \right) \left( 1 + \frac{3\alpha/2}{\log(\log(1/\rho) + 1)} \right),$$

where $\alpha > 0$ is a constant.

The proofs of Theorem 3 and Corollary 1 are in Section VI-D. This corollary shows that for any $\rho$, a key length which depends only on $\rho$ is sufficient to achieve $\rho$-maximal correlation secrecy for any message length. This is in a strong contrast to perfect secrecy, which requires the key length to be at least the message length.
Maximal correlation secrecy can also be achieved by a simpler cipher with a slightly longer key length. Consider a binary additive stream cipher \( M = C = \{0, 1\}^n \), \( K = \{0, 1\}^s \), \( E(k, m) = m \oplus g(k) \), \( D(k, c) = c \oplus g(k) \), where \( g(k) = (g_1(k), g_2(k), \ldots, g_n(k)) \in \{0, 1\}^n \) is the keystream generator and \( \oplus \) is component-wise \( \mod 2 \) addition. The following theorem shows that most binary additive stream ciphers with slightly longer key than the lower bound in Theorem 1 are \( \rho \)-maximal correlation secure.

**Theorem 4.** Let \( G_i(k), i \in [1 : n], k \in [1 : 2^s] \) be i.i.d. \( \text{Bern}(1/2) \) random keystream components. Let \( \rho > 0, \epsilon > 0 \), then

\[
P\{\rho_m(M; M \oplus G(K)) \leq \rho\} > 1 - \epsilon,
\]

where the randomness of \( \rho_m(M; M \oplus G(K)) \) is induced by the random keystream generator, if the key length

\[
s \geq 2 \log \frac{1}{\rho} + \log n + \log \left(1 + \frac{1}{n} \log \frac{1}{\epsilon}\right) + 2.
\]

The proof of this theorem is given in Section VI-E. Substituting \( \epsilon = 1 \) in the theorem shows that there exists a binary additive stream cipher that is \( \rho \)-maximal correlation secure with a key length

\[
s \geq 2 \log \frac{1}{\rho} + \log n + 2.
\]

Hence for a constant \( \rho > 0 \), a key size of around \( \log n \) is sufficient.

Figure 1 plots the lower bound on the key length in Theorem 1, the key length achievable by the expander graph cipher using the Ramanujan graphs in Theorem 2, and the key length achievable by the random stream cipher in Theorem 4 versus \( \rho \) for \( n = 10000 \).

![Figure 1. Comparison of the lower bound and the achievable key lengths for \( n = 10000 \).](image-url)

**IV. CONSTRAINED-DISTRIBUTION SECURITY**

We have shown that the \( \rho \)-maximal correlation secrecy criterion with \( \rho > 0 \) can be satisfied by keys with much shorter lengths than the message. In this section we show that every \( \rho \)-maximal correlation secure cipher also satisfies a variant of semantic security.

Recall that a cipher is said to be semantically secure [16] if for any pmf \( p(m) \) on \( M \), any function \( f(m) \), and any partial information function \( h(m) \) of the message, if \( M \) is generated according to \( p(m) \), the eavesdropper who observes the ciphertext \( C \) and \( h(M) \) (and also knows the choices of \( n, p, f \) and \( h \)) cannot correctly guess \( f(M) \) using a probabilistic, polynomial-time algorithm with probability non-negligibly higher than the best probabilistic, polynomial-time algorithm for guessing \( f(M) \).
using only \( h(M) \) (and also the choices of \( n, p, f \) and \( h \)). In other words, the eavesdropper cannot improve the probability of guessing \( f(M) \) correctly by observing \( C \). Note that the definition in [16] allows \( p(m) \) to be a pmf on messages with different lengths. For simplicity, we consider \( p(m) \) to be a pmf on messages with the same length \( n \).

Constrained-distribution security is a variant of semantic security in which we remove the limitation on computational power but consider only one-bit functions \( f(m) \in \{0, 1\} \) and restrict the choice of the pmf \( p(m) \) to have a small \( \chi^2 \)-divergence [17] from the uniform pmf, that is,

\[
\chi^2(p \| \text{Unif}[1 : 2^n]) = 2^n \sum_m (p(m))^2 - 1 \leq \delta
\]

for some \( \delta \geq 0 \). The partial information \( h(m) \) will be addressed later. We say that a cipher is \((\delta, \epsilon)\)-constrained-distribution secure if for any pmf \( p(m) \) with \( \chi^2(p \| \text{Unif}[1 : 2^n]) \leq \delta \), any one-bit function \( f(m) \) of the message, and any eavesdropper’s guess \( \hat{f}(c) \) of \( f(m) \), when the message \( M \) is generated according to \( p(m) \), the advantage of the eavesdropper defined as

\[
P\{ f(M) = \hat{f}(C) \} - \max \{ P\{ f(M) = 0 \}, P\{ f(M) = 1 \} \}
\]

is less than or equal to \( \epsilon \). Note that the first term is the probability that the eavesdropper can guess \( f(M) \) correctly with knowledge of \( C \), and the second term is the probability that the eavesdropper can guess it correctly without \( C \) by fixing the outcome of \( \hat{f}(C) \).

We now show that maximal correlation secrecy implies constrained-distribution security.

**Theorem 5.** A \( \rho \)-maximal correlation secure cipher is \((\delta, \epsilon)\)-constrained-distribution secure for any \( \delta \geq 0 \) and

\[
\epsilon = \frac{1}{2} \rho \sqrt{\delta + 1}.
\]

The proof of this theorem is given in Section VI-F.

Next we present a generalization of constrained-distribution security in which the partial information \( h(m) \) is also available to Eve. We restrict the choices of \( p(m) \) and \( h(m) \) to satisfy the condition

\[
\mathbb{E} \left( \sqrt{\chi^2(p_{M|h(M)}(\cdot | h(M)) \| \text{Unif}[1 : 2^n]) + 1} \right) \leq \gamma,
\]

where \( \gamma \geq 1 \) is a constant and

\[
p_{M|h(M)}(m | a) = \frac{p(m) \mathbf{1}_{\{h(m) = a\}}(m)}{\sum_m p(m) \mathbf{1}_{\{h(m) = a\}}(m)},
\]

is the conditional pmf of \( M \) given \( h(M) \), which is a random pmf of \( M \) that depends on the value of \( h(M) \). The eavesdropper’s guess \( \hat{f}(c, h(m)) \) can depend on \( h(M) \), and the advantage is defined as

\[
\mathbb{P}\{ f(M) = \hat{f}(C, h(M)) \} - \mathbb{E} \left( \max \{ \mathbb{P}\{ f(M) = 0 \ | h(M) \}, \mathbb{P}\{ f(M) = 1 \ | h(M) \} \} \right),
\]

where the second term is the probability of guessing \( f(M) \) correctly using the maximum a posteriori estimation of \( f(M) \) given \( h(M) \). As a consequence of Theorem 5, for a \( \rho \)-maximal correlation secure cipher, the advantage is upper bounded by \( \epsilon = (1/2)\gamma \rho \).

The value of \( \rho \) directly corresponds to the eavesdropper advantage and the correct probability of the eavesdropper’s guess. For example, if the message \( M \) is uniformly distributed, (i.e., \( \delta = 0 \)), then the eavesdropper cannot correctly guess any one-bit function such that \( \mathbb{P}\{ f(M) = 1 \} = 1/2 \) with probability larger than \((1 + \rho)/2\). As another example, if \( M \) is uniformly distributed and \( l \) bits of \( M \) (at fixed positions) are provided to the eavesdropper via the partial information \( h(m) \), then the advantage of the eavesdropper is upper bounded by \( 2^{l/2-1} \rho \).

To illustrate our results, suppose we wish to protect a message of length \( n = 8 \times 10^9 \) (i.e., 1GB) with a key of length \( s = 512 \). By Theorem 4, we can achieve \( \rho \)-maximal correlation secrecy for \( \rho = 1.55 \cdot 10^{-72} \) using a binary additive stream cipher. As a result, if \( M \) is uniformly distributed, then the advantage of the eavesdropper is upper bounded by \( 7.73 \cdot 10^{-73} \). If \( l = 100 \) bits of \( M \) are provided to the eavesdropper, then the advantage is bounded by \( 8.70 \cdot 10^{-58} \). We can see that a cipher with key length much shorter than the message length can provide good security guarantees.

V. RELATIONSHIP TO OTHER SECRECY MEASURES

We compare maximal correlation secrecy to strong secrecy [3], weak secrecy [18], and leakage rate [19]. We first show that \( \rho \)-maximal correlation secrecy guarantees a small mutual information.

**Proposition 1.** Let \( X \) and \( Y \) be two discrete random variables, then

\[
I(X; Y) \leq \log \left( (\min\{|X|, |Y|\} - 1) \cdot \rho_m^2(X; Y) + 1 \right).
\]

(2)
The proof of this proposition is given in Section VI-G. In the following we assume that $M \sim \text{Unif}([1 : 2^n])$, which reduces (2) to
\[
I(M; C) \leq \log \left( (2^n - 1) \rho_{\text{m}}^2 (M; C) + 1 \right) .
\]
We now use the above proposition to compare $\rho$-maximal correlation secrecy to secrecy criteria that use the mutual information.

**Strong secrecy.** This criterion requires that $\lim_{n \to \infty} I(M; C) = 0$. From (3) this is implied by $\rho$-maximal correlation secrecy for
\[
\rho = o(2^{-n/2}).
\]

**Weak secrecy.** This criterion requires that $\lim_{n \to \infty} I(M; C)/n = 0$. From (3), this is implied by $\rho$-maximal correlation secrecy for
\[
\rho = 2^{-n/2 + o(n)}.
\]

**Leakage rate.** Note that both weak and strong secrecy require the key rate $\lim_n s/n = 1$. By requiring that $\lim_n I(M; C)/n \leq R_L$ for some leakage rate $R_L$, a key rate of $1 - R_L$ can be achieved. From (3), this is implied by $\rho$-maximal correlation secrecy for
\[
\rho = 2^{-(1 - R_L)n/2 + o(n)}.
\]

Note that Theorem 4 implies that such $\rho$ can be achieved also by a key rate of $1 - R_L$. Hence maximal correlation secrecy provides a better security guarantee than leakage rate with no penalty on the key rate.

The above results show that $\rho$-maximal correlation secret implies secrecy criteria involving mutual information. We now show that a small $I(M; C)$ does not necessarily imply $\rho$-maximal correlation secrecy. Consider the following cipher: Let $M = C = K = [0 : 2^n - 1]$, and the encryption and decryption functions be
\[
E(k, m) = \begin{cases} 
  m + k \mod 2^n - 1 & \text{if } m < 2^n - 1 \\
  2^n - 1 & \text{if } m = 2^n - 1,
\end{cases}
\]
and $D(k, c) = E(-k, c)$. Direct computation yields $I(M; C) = 2^{-n}(n + 2 - 2^{-(n-1)})$, which goes to zero as $n$ tends to infinity, and thus the cipher satisfies strong secrecy. However, since one can determine if $M = 2^n - 1$ or not by observing $C$, $\rho_{\text{m}}(M; C) = 1$. Hence $\rho$-maximal correlation secrecy is a strictly stronger secrecy criterion than criteria that use mutual information.

**VI. PROOF OF THE RESULTS**

**A. Properties of Maximal Correlation**

We first establish a characterization of maximal correlation using the spectral norm, which will be useful in proving the main results of the paper.

**Lemma 1.** Let $(X, Y) \sim p(x, y)$ be discrete random variables with marginals $p(x)$ and $p(y)$. Define the matrix $B \in \mathbb{R}^{|X| \times |Y|}$ with entries
\[
B_{xy} = \frac{p(x, y)}{\sqrt{p(x)p(y)}} - \sqrt{p(x)p(y)}.
\]
Then,
\[
\rho_{\text{m}}(X; Y) = \|B\|.
\]

**Proof:** The lemma follows directly from the singular value characterization of maximal correlation [7]. Alternatively, it can be shown directly as follows:
\[
\rho_{\text{m}}(X; Y) = \max_{E(f(X)) = E(g(Y)) = 0, E(f^2(X)) = E(g^2(Y)) = 1} \mathbb{E} \left( f(X) g(Y) \right)
= \max_{E(f^2(X)) = E(g^2(Y)) = 1} \mathbb{E} \left( f(X) g(Y) \right) - \mathbb{E} \left( f(X) \right) \mathbb{E} \left( g(Y) \right)
= \sum_x u(x) v^2(x) = \sum_y u^2(y) \sum_x p(x) v(y) = \sum_{x, y} u(x) v(y) (p(x, y) - p(x)p(y))
= \sum_x \hat{u}^2(x) = \sum_y \hat{v}^2(y) = \|B\|.
\]
Consider the following elementary result relating maximal correlation and the \( \chi^2 \)-divergence between the joint pmf and the product of the marginal pmfs, also known as \( \chi^2 \) measure of correlation. We include the proof for the sake of completeness.

**Lemma 2.** Let \( (X,Y) \sim p(x,y) \) be discrete random variables with marginals \( p(x) \) and \( p(y) \). Then,

\[
\frac{1}{\min \{|X|, |Y|\} - 1} \leq \frac{\rho_m^2(X;Y)}{\chi^2(p(x,y) \| p(x)p(y))} \leq 1,
\]

where

\[
\chi^2(p(x)\|q(x)) = \sum_m \frac{(p(x))^2}{q(x)} - 1
\]
is the \( \chi^2 \)-divergence between \( p(x) \) and \( q(x) \).

**Proof:** Define \( B \) as in Lemma 1. Let \( u \in \mathbb{R}^{|X|}, u_x = \sqrt{p(x)} \). Note that

\[
(u^T B)_y = \sum_x \left( \frac{p(x,y)}{\sqrt{p(x)p(y)}} - \sqrt{p(x)p(y)} \right) \sqrt{p(x)}
\]

\[
= \sum_x \left( \frac{p(x,y)}{\sqrt{p(y)}} - p(x) \sqrt{p(y)} \right)
\]

\[
= 0.
\]

Hence \( \text{rank}(B) \leq |X| - 1 \). Similarly we have \( \text{rank}(B) \leq |Y| - 1 \). Consider

\[
\|B\|_F^2 = \sum_{x,y} \left( \frac{p(x,y)}{\sqrt{p(x)p(y)}} - \sqrt{p(x)p(y)} \right)^2
\]

\[
= \sum_{x,y} \left( \frac{(p(x,y))^2}{p(x)p(y)} - 2p(x,y) + p(x)p(y) \right)
\]

\[
= \sum_{x,y} \frac{(p(x,y))^2}{p(x)p(y)} - 1
\]

\[
= \chi^2(p(x,y) \| p(x)p(y)).
\]

The result follows from \( \|B\|_F^2 / \text{rank}(B) \leq \|B\|^2 \leq \|B\|_F^2 \). \( \blacksquare \)

**B. Proof of Theorem 1**

Here we assume the ciphertext length is arbitrary, and the encryption function can be randomized, i.e., the ciphertext is \( C = E(K,M,W) \) where \( W \sim p(w) \) is the local randomness at the sender. We require that \( D(k, E(k,m,w)) = m \) for any \( k, m, w \). From Lemma 2,

\[
\rho^2 \geq \rho_m^2(M;C)
\]

\[
\geq 2^{-n} \chi^2 (p(m,c) \| p(m)p(c))
\]

\[
= 2^{-n} \left( \sum_{m,c} \frac{(p(m,c))^2}{p(m)p(c)} - 1 \right)
\]

\[
= \sum_{m,c} \frac{(p(m,c))^2}{p(c)} - 2^{-n}
\]

\[
= \sum_c \frac{\sum_m (p(m,c))^2}{p(c)} - 2^{-n}
\]

\[
\geq \sum_c \frac{|\{m : p(m,c) > 0\}|^{-1} (\sum_m p(m,c))^2}{p(c)} - 2^{-n}
\]

\[
= \mathbb{E} \left( |\{m : p(m,C) > 0\}|^{-1} \right) - 2^{-n}
\]

\[
\geq \mathbb{E} \left( |\{D(k,C) : k \in [1:2^n]\}|^{-1} \right) - 2^{-n}
\]

\[
\geq 2^{-s} - 2^{-n}.
\]
Hence,
\[ s \geq \log \left( \frac{1}{\rho^2 + 2^{-n}} \right) = 2 \log \frac{1}{\rho} - \log \left( 1 + \frac{1}{2^n \rho^2} \right). \]
This completes the proof.

C. Proof of Theorem 2

Theorem 2 is a direct consequence of Lemma 1. Since \( M, C \sim \text{Unif}[1 : 2^n] \), we have
\[
\rho_m(M; C) = 2^n \left\| P_{MC} - 2^{-2n} 1_{2^n \times 2^n} \right\| \\
= 2^n \left\| \frac{1}{d} A - 2^{-2n} 1_{2^n \times 2^n} \right\| \\
= \frac{1}{d} \left\| A - \frac{d}{2^n} 1_{2^n \times 2^n} \right\| \\
= \frac{1}{d} |\lambda_2(A)|.
\]
Ramanujan graphs have second eigenvalue \(|\lambda_2(A)| \leq 2\sqrt{d-1}\), hence their maximal correlation is
\[
\rho_m(M; C) \leq \frac{2\sqrt{d-1}}{d} \leq 2\sqrt{d}.
\]
As a result, if \( \log d \geq 2 \log(1/\rho) + 2 \), we have \( \rho_m(M; C) \leq \rho \).

D. Proofs of Theorem 3 and Corollary 1

We first prove a lemma on the composition of two ciphers with the same message length \( n \) but with possibly different key lengths \( s_1 \) and \( s_2 \), which yields a cipher with message length \( n \) and key length \( s_1 + s_2 \).

Lemma 3. Let \((E_1, D_1)\) and \((E_2, D_2)\) be two ciphers with key lengths \( s_1 \) and \( s_2 \), respectively, and the same message length \( n \). Define the composition of these two ciphers to be the cipher \( K = \{1 : 2^n\} \times \{1 : 2^n\}, M = C = \{1 : 2^n\} \).
\[
E(k_1, k_2, m) = E_2(k_2, E_1(k_1, m)), \quad D(k_1, k_2, m) = D_1(k_1, D_2(k_2, m)).
\]
Then we have
\[
\rho_m(M; E(K_1, K_2, M)) \leq \rho_m(M; E_1(K_1, M)) \cdot \rho_m(M; E_2(K_2, M)).
\]

Proof:
Consider the following alternate characterization of maximal correlation in [6]
\[
\rho_m(X; Y) = \max_{f(x): E(f(X)) = 0, E(f^2(X)) = 1} \sqrt{E \left( \left( E(f(X)) \left| Y \right. \right)^2 \right)}.
\]
Let \( M_1 \sim \text{Unif}[1 : 2^n], M_2 = E_1(K_1, M_1), C = E_2(K_2, M_2) \). Note that for any \( f, g : [1 : 2^n] \rightarrow \mathbb{R} \) with \( E(f(M_1)) = E(g(C)) = 0, E(f^2(M_1)) = E(g^2(C)) = 1 \), by the alternate characterization,
\[
E \left( f(M_1)g(C) \right) = E \left( E(f(M_1) \mid M_2) \cdot E(g(C) \mid M_2) \right) \leq \sqrt{E \left( \left( E(f(M_1) \mid M_2) \right)^2 \right) \cdot E \left( \left( E(g(C) \mid M_2) \right)^2 \right)} \leq \rho_m(M_1; M_2) \cdot \rho_m(C; M_2).
\]
The result follows.

Now consider the following result from [15]. Let \( A_1, \ldots, A_d \in \mathbb{R}^{N \times N} \) be i.i.d. random permutation matrices uniformly distributed in the set of permutations of \( \{1, \ldots, N\} \). Then we have
\[
E \left( \left\| \lambda_2 \left( \sum_{i=1}^d (A_i + A_i^T) \right) \right\| \right) \leq 2\sqrt{2d-1} \left( 1 + \frac{\ln d}{\sqrt{2d}} + O \left( d^{-1/2} \right) \right) + O \left( \frac{d^{3/2} \ln \ln N}{\ln N} \right).
\] (4)
We use the above result to construct a cipher as follows. Generate \( d = 2^{s-1} \) permutations on \([1 : 2^n] \), namely \( \sigma_1, \ldots, \sigma_d \), i.i.d. uniformly at random. Let \( \sigma_{i+d} = \sigma_i^{-1} \) for \( i = 1, \ldots, d \). The cipher is defined as \( K = [1 : 2^n], M = C = [1 : 2^n], E(k, m) = \sigma_k(m), D(k, c) = \sigma_k^{-1}(c) \). By Lemma 1,

\[
\rho_m(M; C) = \left\| 2^n P_{MC} - \frac{1}{2^n} 1_{2^n \times 2^n} \right\|
\]

\[
= \left\| \frac{1}{2^d} \sum_{i=1}^{d} (A_i + A_i^T) - \frac{1}{2^n} 1_{2^n \times 2^n} \right\|
\]

\[
= \frac{1}{2^d} \left\| \lambda_2 \left( \sum_{i=1}^{d} (A_i + A_i^T) \right) \right\|.
\]

Hence by (4), there exist fixed \( \sigma_1, \ldots, \sigma_d \) and a constant \( \eta > 0 \) (that does not depend on \( s \) or \( n \)) such that

\[
\rho_m(M; C) \leq \frac{1}{2^d} \left( 2\sqrt{2d} - 1 \right) \left( 1 + \frac{\ln d}{\sqrt{2d}} + \eta \cdot \frac{d^{3/2} \log n}{n} \right)
\]

\[
\leq \frac{2}{\sqrt{2d}} \left( 1 + \frac{\ln d}{\sqrt{2d}} + \eta \left( \frac{d^{-1/2} + d \log n}{n} \right) \right)
\]

\[
\leq \frac{2}{\sqrt{2d}}
\]

\[
= 2^{-s/2 + 2}
\]

(5)

if \( d \geq 16\eta^2 \) and \( n/\log n \geq 4\eta d \), or equivalently,

\[
2 \log \eta + 5 \leq s \leq \log n - \log \log n - \log \eta - 1.
\]

(6)

Note that this construction only works for very short key lengths. We now provide a construction for general key length \( s \) by the composition of several ciphers with short key lengths. Let

\[
t = \left\lceil \frac{s}{\log n - \log \log n - \log \eta - 2} \right\rceil, \quad \tilde{s} = \left\lfloor \frac{s}{t} \right\rfloor, \quad a = t \left( \left\lfloor \frac{s}{t} \right\rfloor + 1 \right) - s, \quad b = s - t \left\lfloor \frac{s}{t} \right\rfloor,
\]

then we have \( a + b = t \) and \( s = a\tilde{s} + b(\tilde{s} + 1) \). Consider the composition of \( a \) ciphers with key length \( \tilde{s} \) and \( b \) ciphers of key length \( \tilde{s} + 1 \), which gives a cipher with key length \( s \). Let \( s_0 \) be an integer satisfying

\[
s_0 \geq \max \{ 4 \log \eta + 12, \ 2^{20} \}
\]

and

\[
\log s_0 - \log \log s_0 \geq 5 \log \eta + 14.
\]

Consider any \( s \geq s_0 \). If \( n \leq s \), then perfect secrecy can be achieved. Hence we assume \( n > s \geq s_0 \). To check the conditions in (6) for \( \tilde{s} \) and \( \tilde{s} + 1 \),

\[
\tilde{s} = \left\lceil \frac{s}{s (\log n - \log \log n - \log \eta - 2)^{-1}} \right\rceil
\]

\[
\geq \frac{s}{s (\log n - \log \log n - \log \eta - 2)^{-1}} - 1
\]

\[
= \frac{1}{(\log n - \log \log n - \log \eta - 2)^{-1} + s^{-1}} - 1
\]

\[
\geq \frac{1}{(4 \log \eta + 12)^{-1} + (4 \log \eta + 12)^{-1}} - 1
\]

\[
= 2 \log \eta + 5.
\]

And also

\[
\tilde{s} + 1 = \left\lceil \frac{s}{s (\log n - \log \log n - \log \eta - 2)^{-1}} \right\rceil + 1
\]

\[
\leq \log n - \log \log n - \log \eta - 1.
\]
Hence by (5) and Lemma 3, the maximal correlation of the resultant cipher is

\[ \rho_m(M; C) \leq \left(2^{-s/2+2}\right)^a \left(2^{-s/2+2}\right)^b = 2^{-s/2+2t}, \]

where

\[ 2t = 2 \left[ \log n - \log \log n - \log \eta - 2 \right] \frac{s}{2s} \leq \log n - \log \log n - \log \eta - 2 + 2 \]

\[ \leq \log n - \log \log n - (\log n - \log \log n)/5 + 2 \]

\[ = \log n - \log \log n + 2 \]

\[ \leq \frac{4s}{\log n} + 2, \]

where the last inequality is due to \(\log n \geq \log s_0 \geq 14\). Therefore,

\[ 2 \log \frac{1}{\rho_m(M; C)} \geq s \left(1 - \frac{8}{\log n}\right) - 4. \]

Rearranging, we have

\[ s \leq \left(2 \log \frac{1}{\rho_m(M; C)} + 4\right) \left(1 - \frac{8}{\log n}\right)^{-1} \]

\[ \leq \left(2 \log \frac{1}{\rho_m(M; C)} \right) \left(1 + \frac{16}{\log n}\right) + 16. \]

Hence if

\[ s \geq \left(2 \log \frac{1}{\rho}\right) \left(1 + \frac{\alpha}{\log n}\right) + \alpha. \]

where \(\alpha = \max\{16, s_0\}\), then \(s \geq s_0\), and \(\rho_m(M; C) \leq \rho\). This completes the proof of Theorem 3.

Now we prove Corollary 1. If

\[ s \geq \left(2 \log \frac{1}{\rho}\right) \left(1 + \frac{3\alpha/2}{\log(\log(1/\rho) + 1)}\right), \]

then

\[ \log \frac{1}{\rho} + 1 \leq \frac{s}{2} \left(1 + \frac{3\alpha/2}{\log(\log(1/\rho) + 1)}\right)^{-1} + 1 \]

\[ \leq \frac{s}{2} + 1 \]

\[ \leq s \]

due to the assumption that \(s \geq 2\). Hence,

\[ s \geq \left(2 \log \frac{1}{\rho}\right) \left(1 + \frac{3\alpha/2}{\log(\log(1/\rho) + 1)}\right) \]

\[ = \left(2 \log \frac{1}{\rho}\right) \left(1 + \frac{\alpha}{\log(\log(1/\rho) + 1)}\right) + \frac{\alpha \log(1/\rho)}{\log(\log(1/\rho) + 1)} \]

\[ \geq \left(2 \log \frac{1}{\rho}\right) \left(1 + \frac{\alpha}{\log(\log(1/\rho) + 1)}\right) + \alpha \]

\[ \geq \left(2 \log \frac{1}{\rho}\right) \left(1 + \frac{\alpha}{\log s}\right) + \alpha \]

\[ \geq \left(2 \log \frac{1}{\rho}\right) \left(1 + \frac{\alpha}{\log n}\right) + \alpha, \]

where the last step is due to the assumption that \(n > s\). This complete the proof of Corollary 1.
E. Proof of Theorem 4

We first compute the maximal correlation of a binary additive stream cipher.

**Proposition 2.** A binary additive stream cipher has a maximal correlation

\[
\rho_m(M; C) = \max_{v \in \{0, 1\}^n} \left\| \frac{1}{|K|} \sum_{k \in K} (-1)^n v_i G_i(k) \right\|.
\]

**Proof:** From Lemma 1,

\[
\rho_m(M; C) = 2^n \left\| P_{MC} - 2^{-2n} 1_{2^n \times 2^n} \right\|
\]

where we assume the bit sequences in \( \mathcal{M} \) are treated as binary representations of integers. Note that \( P_{MC} \) can be written as

\[
P_{MC} = \frac{1}{2^n |K|} \sum_{k \in K} \sum_{m=0}^{2^n-1} e_m e_{m \oplus G(k)},
\]

where \( e_i \) denotes the \( i \)-th standard basis vector.

Note that \( P_{MC} \) can be diagonalized by the Hadamard matrix

\[
H_{ij} = 2^{-n/2} (-1)^{\sum_{l=0}^{n-1} i_l j_l},
\]

where \( i_l \) and \( j_l \) denotes the \( l \)-th digit of \( i \) and \( j \) in their binary representation, by observing that

\[
HP_{MC}H = \frac{1}{2^n |K|} \sum_{k \in K} \sum_{m=0}^{2^n-1} H e_m e_{m \oplus G(k)} H,
\]

where each entry can be expressed as

\[
(HP_{MC}H)_{ij} = \frac{1}{2^n |K|} \sum_{k \in K} \sum_{m=0}^{2^n-1} \left( H e_m e_{m \oplus G(k)} H \right)_{ij}
\]

We have

\[
\rho_m(M; C) = 2^n \left\| P_{MC} - 2^{-2n} 1_{2^n \times 2^n} \right\|
\]

\[
= 2^n \left\| H P_{MC} H - 2^{-2n} H 1_{2^n \times 2^n} H \right\|
\]

\[
= 2^n \left\| H P_{MC} H - \text{diag} \left( 2^{-n}, 0, \ldots, 0 \right) \right\|
\]

\[
= 2^n \max_i \left\| (HP_{MC}H)_{ii} - 2^{-n} \delta_{i0} \right\|
\]

\[
= 2^n \max \left\{ \left\| 2^{-n} - 2^{-n} \right\|, \max_{i \neq 0} \left\| \frac{1}{|K|} \sum_{k \in K} (-1)^n i G_i(k) \right\| \right\}
\]

\[
= \max_{i \neq 0} \left\| \frac{1}{|K|} \sum_{k \in K} (-1)^n i G_i(k) \right\|.
\]

We now proceed to prove Theorem 4. Assume we generate \( G_i(k) \) i.i.d. Bern(1/2) across \( k \) and \( i \). For each fixed \( v \neq 0^n \), consider

\[
\frac{1}{|K|} \sum_{k \in K} (-1)^n v_i G_i(k).
\]
The terms \((-1)^{\sum_{i=0}^{n-1} v_i G_i(k)}\) are i.i.d. Rademacher. By the Chernoff bound,

\[
\Pr \left\{ \left| \frac{1}{|K|} \sum_{k \in K} (-1)^{\sum_{i=0}^{n-1} v_i G_i(k)} \right| \geq \rho \right\} \leq 2^{1-|K| \cdot D_{\text{KL}} \left( \frac{1+\rho}{2} \right) \frac{1}{2}}.
\]

By the union bound on all possible \(v \in \{0,1\}^n \setminus \{0^n\}\) and observing that \((\ln 2) D_{\text{KL}} \left( \frac{1+\rho}{2} \right) > \rho^2 / 2\) for \(\rho > 0\),

\[
\Pr \{ \rho_m(M; C) \geq \rho \} \leq 2^{(n+1)-|K| \cdot D_{\text{KL}} \left( \frac{1+\rho}{2} \right)} \leq 2^{(n+1)-|K| \rho^2 / (2 \ln 2)}.
\]

Hence if

\[
s \geq 2 \log \frac{1}{\rho} + \log n + \log \left(1 + \frac{1}{n} \log \frac{1}{\epsilon}\right) + 2,
\]

then

\[
2^n \geq 4 \rho^{-2} n \left(1 + \frac{1}{n} \log \frac{1}{\epsilon}\right) \geq 4 (\ln 2) \rho^{-2} \left(n + \log \frac{1}{\epsilon}\right).
\]

Therefore,

\[
\Pr \{ \rho_m(M; C) \geq \rho \} \leq 2^{(n+1)-2 \rho^2 / (2 \ln 2)} \leq 2^{(n+1)-2(n - \log \epsilon)} \leq 2^{1-n + \log \epsilon} \leq \epsilon.
\]

This completes the proof of Theorem 4.

**F. Proof of Theorem 5**

We prove Theorem 5, which shows that maximal correlation secrecy implies constrained-distribution security. Note that the theorem is implied by the following more general result.

**Proposition 3.** Consider any two pmfs \(p_M(m)\) and \(\tilde{p}_M(m)\) on \(M\), and a Markov kernel \(p_{C|M}(c|m)\). The two pmfs induce the joint probability measures \(P\) and \(\tilde{P}\) on \((M, C)\), respectively. Let \(\rho_m(M; C)\) be the maximal correlation in \(P\). For any one-bit functions \(f : M \to \{0,1\}\) and \(\tilde{f} : C \to \{0,1\}\), we have

\[
\Pr \left\{ \tilde{f}(M) = \tilde{f}(C) \right\} - \frac{1}{2} \leq \sqrt{\frac{1}{4} \rho_m^2 (M; C) (\chi^2 (\tilde{p}_M || p_M) + 1) + (1 - \rho_m^2 (M; C)) \left( \Pr \{ f(M) = 0 \} - \frac{1}{2} \right)^2}.
\]

**Proof:** All expectations, variances and covariances in this proof are in \(P\). Let \(g(m) = (-1)^f(m) \tilde{p}_M(m)/p_M(m)\) and \(\tilde{g}(c) = (-1)^{\tilde{f}(c)}\). Write \(\chi^2 = \chi^2 (\tilde{p}_M || p_M), \tilde{p}_f(i) = \Pr \{ f(M) = i \}, p_f(i) = \Pr \{ \tilde{f}(C) = i \}\) and \(\tilde{p}_c = \Pr \{ f(M) \neq \tilde{f}(C) \}\). It is straightforward to check that Proposition 3 is true if \(\tilde{p}_c < \min\{\tilde{p}_f(0), \tilde{p}_f(1)\}\). Hence we assume \(\tilde{p}_c < \min\{\tilde{p}_f(0), \tilde{p}_f(1)\}\).

Observe that

\[
\mathbb{E}(g(M)) = \tilde{p}_f(0) - \tilde{p}_f(1),
\]

\[
\mathbb{E}(\tilde{g}(C)) = p_f(0) - p_f(1),
\]

\[
\text{Var}(g(M)) = \sum_m (\tilde{p}_M(m))^2 - (\tilde{p}_f(0) - \tilde{p}_f(1))^2
\]

\[
= (\chi^2 + 1) - (\tilde{p}_f(0) - \tilde{p}_f(1))^2,
\]

\[
\text{Var}(\tilde{g}(C)) = 1 - (p_f(0) - p_f(1))^2,
\]

\[
\text{Cov}(g(M), \tilde{g}(C)) = \mathbb{E}[(g(M)\tilde{g}(C)) - \mathbb{E}(g(M))\mathbb{E}(\tilde{g}(C))]
\]

\[
= \sum_m p(m)g(m) \sum_c p(c|m)\tilde{g}(c) - (\tilde{p}_f(0) - \tilde{p}_f(1))(p_f(0) - p_f(1))
\]

\[
= 1 - 2\tilde{p}_c - (\tilde{p}_f(0) - \tilde{p}_f(1))(p_f(0) - p_f(1)).
\]
We have
\[
\rho_m^2(M; C) \geq \frac{\text{Cov}(g(M), \hat{g}(C))^2}{\text{Var}(g(M))\text{Var}(\hat{g}(C))} = \frac{(1 - 2\hat{p}_e - (\hat{p}_f(0) - \hat{p}_f(1)) (p_f(0) - p_f(1)))^2}{(\chi^2 + 1) - (\hat{p}_f(0) - \hat{p}_f(1))^2} \geq \frac{(1 - 2\hat{p}_e)^2 - (1 - 2\hat{p}_f(0))^2}{(\chi^2 + 1) - (1 - 2\hat{p}_f(0))^2},
\]
where the last inequality is due to \( \hat{p}_e < \min\{\hat{p}_f(0), \hat{p}_f(1)\} \) and
\[
\frac{(a - bx)^2}{1 - x^2} = \left( \frac{a}{\sqrt{1 - x^2}} - \frac{bx}{\sqrt{1 - x^2}} \right)^2 \geq a^2 - b^2
\]
for any \( a, b \) such that \( |a| > |b| \) and \(-1 < x < 1\). Hence,
\[
(1 - 2\hat{p}_e)^2 \leq \rho_m^2(M; C) \left( (\chi^2 + 1) - (1 - 2\hat{p}_f(0))^2 \right) + (1 - 2\hat{p}_f(0))^2
= \rho_m^2(M; C) (\chi^2 + 1) + (1 - \rho_m^2(M; C)) (1 - 2\hat{p}_f(0))^2.
\]
The result follows.

To prove Theorem 5, note that by Proposition 3,
\[
\hat{P} \left\{ f(M) = \hat{f}(C) \right\} - \frac{1}{2} \leq \sqrt{\frac{1}{4} \rho_m^2(M; C) \left( \chi^2 (\hat{p}_M \| p_M) + 1 \right) + (1 - \rho_m^2(M; C)) \left( \hat{P} \left\{ f(M) = 0 \right\} - \frac{1}{2} \right)^2}
\]
\[
\leq \sqrt{\frac{1}{4} \rho_m^2(M; C) \left( \chi^2 (\hat{p}_M \| p_M) + 1 \right) + \left( \hat{P} \left\{ f(M) = 0 \right\} - \frac{1}{2} \right)^2}
\]
\[
\leq \sqrt{\frac{1}{4} \rho_m^2(M; C) \left( \chi^2 (\hat{p}_M \| p_M) + 1 \right) + \left( \hat{P} \left\{ f(M) = 0 \right\} - \frac{1}{2} \right)^2}
\]
\[
= \frac{1}{2} \rho_m(M; C) \sqrt{\chi^2 (\hat{p}_M \| p_M) + 1} + \max \left\{ \hat{P} \left\{ f(M) = 0 \right\}, \hat{P} \left\{ f(M) = 1 \right\} \right\} - \frac{1}{2}.
\]
This completes the proof.

G. Proof of Proposition 1

From Lemma 2, we know that
\[
\rho_m^2(X; Y) \geq \frac{1}{\min \{ |\mathcal{X}|, |\mathcal{Y}| \} - 1} \chi^2 (p(x, y) \| p(x)p(y))
\]
\[
= \frac{1}{\min \{ |\mathcal{X}|, |\mathcal{Y}| \} - 1} \left( \sum_{x,y} \frac{(p(x, y))^2}{p(x)p(y)} - 1 \right)
\]
\[
= \frac{1}{\min \{ |\mathcal{X}|, |\mathcal{Y}| \} - 1} \left( \mathbb{E} \left( \frac{p(x, y)}{p(x)p(y)} - 1 \right) \right).
\]
By Jensen’s inequality,
\[
\log \left( (\min \{ |\mathcal{X}|, |\mathcal{Y}| \} - 1) \cdot \rho_m^2(X; Y) + 1 \right) \geq \log \left( \mathbb{E} \left( \frac{p(x, y)}{p(x)p(y)} \right) \right)
\]
\[
= \mathbb{E} \left( \log \left( \frac{p(x, y)}{p(x)p(y)} \right) \right)
= I(X; Y).
\]
This completes the proof.
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