Harmonic Cellular Maps which are not Diffeomorphisms

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0 Introduction

The use of harmonic maps has been spectacularly successful in proving rigidity (and super-rigidity) results for non-positively curved Riemannian manifolds. This is witnessed for example by results of Sui [31], Sampson [26], Corlette [6], Gromov and Schoen [18], Jost and Yau [22], and Mok, Sui and Yeung [23]. All of which are based on the pioneering existence theorem of Eells and Sampson [13] and the uniqueness theorem of Hartman [19] and Al’ber [1]. In light of this we believe that it is interesting to demarcate this technique. For example, it was shown in [15] that a harmonic homotopy equivalence between closed negatively curved Riemannian manifold is sometimes not a diffeomorphism, even when one of the manifolds has constant sectional curvature equal to $-1$ (i.e. is a real hyperbolic manifold). Later other examples were given in [16] and [17] where such a harmonic homotopy equivalence $f$ is not even a homeomorphism; even though the ones constructed in [17] are homotopic to diffeomorphisms. In this paper we construct a harmonic map $h$ between closed negatively curved Riemannian manifolds $M$ and $N$ which is not a diffeomorphism but is the limit of a 1-parameter family of diffeomorphisms; in particular, $h$ is a cellular map. In our example either $M$ or $N$ (but of course not both) can be a real hyperbolic manifold and the other have its sectional curvature pinched within $\epsilon$ of $-1$, where $\epsilon$ is any preassigned positive number. (We do not know whether such a harmonic map $h$ can ever be a homeomorphism. See our acknowledgment below.) This result is contained in Theorem 1 its Addendum and Theorem 2. We construct such examples in all dimensions $> 10$ and conjecture that this can be improved to all dimensions $\geq 6$.

We have also discovered a curious relationship between the Poincaré Conjecture in low dimensional topology and the existence of a certain type of harmonic map $k : M \to N$ between

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high dimensional (i.e. dim $M > 10$) closed negatively curved Riemannian manifolds. Recall that the Poincaré Conjecture asserts that the only simply connected closed 3-dimensional manifold is the 3-sphere. If this is true, then there exists such a harmonic map $k$ which is homotopic to a diffeomorphism but cannot be approximated by homeomorphisms; i.e. is not a cellular map. (See [7] for a discussion of cellular maps which are called cell like maps in that article.) In particular if $f : M \to N$ is any smooth map homotopic to $k$ and $f_t$, $t \geq 0$, denotes the heat flow from $f_0 = f$ to $f_\infty = k$ given by the Eells-Sampson Theorem [13], then $f_t$ is neither univalent (i.e. not one-to-one) nor an immersion for all $t$ sufficiently large (i.e. all $t \geq T_f$ for some $T_f \in \mathbb{R}$). Again either $M$ or $N$ (but not both) can, in this example, be real hyperbolic and the sectional curvatures of the other be pinched within $\epsilon$ of $-1$. This result is contained in the Addendum to Theorem 2.

The key to these two theorems and their addenda is a Proposition. We now describe this Proposition and outline its proof. Crucial use is made of the main result of [17]. In that paper a pair of homeomorphic but not PL homeomorphic closed negatively curved Riemannian manifolds $M$ and $\mathcal{M}$ are constructed satisfying:

1. $M$ is real hyperbolic.

2. $\mathcal{M}$ has a 2-sheeted cover $q : \hat{M} \to \mathcal{M}$ where $\hat{M}$ admits a real hyperbolic metric $\nu$.

Let $\mu$ be a given negatively curved Riemannian metric on $\mathcal{M}$ and $q^*(\mu)$ be the induced Riemannian metric on $\hat{M}$. We would like to find a 1-parameter family of negatively curved Riemannian metrics connecting $q^*(\mu)$ to $\nu$. But we don’t know how to do this. In fact this is in general an open problem; cf. [4, Question 7.1]. However by passing to a large finite sheeted cover $r : \hat{\mathcal{M}} \to \hat{M}$, we are able to connect $(q \circ r)^*(\mu)$ to the real hyperbolic metric $r^*(\nu)$ by a 1-parameter family of negatively curved Riemannian metrics; this is essentially the content of our Proposition in which $p = q \circ r$. To accomplish this, several results about smooth pseudo-isotopies are used; in particular, the main result of [14] concerning the space of stable topological pseudo-isotopies of real hyperbolic manifolds together with the comparison between the spaces of stable smooth and stable topological pseudo-isotopies contained in [3] and [20]. And finally we need Igusa’s fundamental result [21] comparing the spaces of pseudo-isotopies and stable pseudo-isotopies. We need that dim $M > 10$ in order to invoke Igusa’s result. We also need a formula calculating the sectional curvatures of doubly warped products. Although such a formula is probably known to experts, we sketch a proof of it in the Appendix to this paper for the sake of completeness.
Finally our two theorems and their addenda are derived from our Proposition by using the continuous dependence (in the $C^\infty$-topology) of the harmonic map homotopic to a homotopy equivalence $f : (M, \mu_M) \to (N, \mu_N)$ on the negatively curved Riemannian metrics $\mu_M$ and $\mu_N$. This dependence was proved by Sampson [26], Schoen and Yau [29], and Eells and Lemaire [10].

In addition the derivation of the Addenda to Theorem 2 depends on Scharlemann’s result [27] which is the key unlocking the connection to the Poincaré Conjecture.

**Acknowledgment.** We wish to thank David Gabai for asking whether a harmonic homeomorphism between negatively curved Riemannian manifolds must be a diffeomorphism.

1 Main results

In this section we state the main results of the paper which are Theorem 1 and 2, their Addenda, and the Proposition. Then we show how the Proposition implies the other main results.

**Theorem 1.** For every integer $m > 10$, there is a harmonic cellular map $h : M_1 \to M_2$, between a pair of closed negatively curved $m$-dimensional Riemannian manifolds, which is not a diffeomorphism.

**Addendum.** The map $h$ in Theorem 1 can be approximated by diffeomorphisms. Also, either $M_1$ or $M_2$ can be chosen to be a real hyperbolic manifold and the other chosen to have its sectional curvatures pinched within $\epsilon$ of -1; where $\epsilon$ is any preassigned positive number.

**Remark 1.** Siebenmann [30] showed that a continuous map $f : X \to Y$ between a pair of closed manifolds of dimension $\geq 5$ is cellular if and only if it is the limit of homeomorphisms.

**Remark 2.** Note that a smooth non-diffeomorphic cellular map $h$ between closed smooth manifolds cannot be a smooth immersion; i.e. $dh$ is not one-to-one on some tangent space. However we do not know whether the map $h$ we construct in proving Theorem 1 is univalent (i.e. one-to-one) or whether a harmonic homeomorphism between closed negatively curved Riemannian manifolds must always be a diffeomorphism.

**Theorem 2.** For every integer $m > 10$, and $\epsilon > 0$, there are an $m$-dimensional closed orientable smooth manifold $M$, and a $C^\infty$ family of Riemannian metrics $\mu_s$, on $M$, $s \in [0, 1]$, such that:
(a) \( \mu_1 \) is hyperbolic.

(b) The sectional curvatures of \( \mu_s, s \in [0, 1] \), are all in interval \((-1 - \epsilon, -1 + \epsilon)\).

(c) The maps \( k \) and \( l \) are both not univalent (i.e. not one-to-one) where \( k : (\mathcal{M}, \mu_0) \to (\mathcal{M}, \mu_1) \) and \( l : (\mathcal{M}, \mu_1) \to (\mathcal{M}, \mu_0) \) are the unique harmonic maps homotopic to \( \text{id}_\mathcal{M} \).

**Addendum.** Assuming that the Poincaré Conjecture is true, then the harmonic maps \( k \) and \( l \) (of Theorem 2) are not cellular. And consequently the maps \( k_t \) and \( l_t \) in the heat flow of \( \text{id} = k_0 \) to \( k = k_\infty \) and of \( \text{id} = l_0 \) to \( l = l_\infty \) are not univalent for all \( t \) sufficiently large.

**Remark 3.** The heat flow \( k_t \) mentioned in this Addendum refers to the solution to the initial value problem.

\[
\frac{\partial k_t}{\partial t} = \tau(k_t), \quad k_t|_{t=0} = \text{id}
\]

where \( \tau(k_t) \) is the tension field of \( k_t \). A fundamental result due to Eells and Sampson \[13\] is that this PDE has a unique solution \( k_t \) (for all \( t \geq 0 \)) and that \( \lim_{t \to \infty} k_t = k \); cf. \[12\] pp. 22-24.

These theorems and their addenda are a consequence of the following result.

**Proposition.** Given an integer \( m > 10 \) and a positive number \( \epsilon \), there exist a \( m \)-dimensional closed orientable real hyperbolic manifold \( M \) and a smooth manifold \( \mathcal{M} \) with the following properties:

(i) \( M \) is homeomorphic to \( \mathcal{M} \).

(ii) \( M \) is not PL homeomorphic to \( \mathcal{M} \).

(iii) \( \mathcal{M} \) admits a Riemannian metric \( \mu \), whose sectional curvatures are all in the interval \((-1 - \epsilon, -1 + \epsilon)\).

(iv) There is a finite sheeted cover \( p : \tilde{M} \to \mathcal{M} \) and a one-parameter \( C^\infty \) family of Riemannian metrics \( \mu_s \), on \( \tilde{M} \), \( s \in [0, 1] \), such that \( \mu_0 = p^* \mu \) and \( \mu_1 \) is hyperbolic. The sectional curvatures of \( \mu_s, s \in [0, 1] \), are all in the interval \((-1 - \epsilon, -1 + \epsilon)\).

Before deducing these two theorems and their addenda from the Proposition, we recall some results concerning harmonic maps and introduce some notation.

Let \( X \) and \( Y \) be closed negatively curved Riemannian manifolds with Riemannian metrics \( \mu_X \) and \( \mu_Y \), respectively. Let \( g : X \to Y \) be a homotopy equivalence. Then there is a unique...
harmonic map $k : X \to Y$ homotopic to $g$, given by the fundamental existence result of Eells and Sampson \cite{13} and uniqueness result by Hartman \cite{19} and Al’ber \cite{1}. $k$ depends on $\mu_X$, $\mu_Y$ and $g$. We write $k = \text{har}(\mu_X, \mu_Y, g)$. In fact, fixing $g$, the map

$$ \text{har}_g : \text{Met}^{(-)}(X) \times \text{Met}^{(-)}(Y) \to C^\infty(X, Y) $$

$$(\mu_X, \mu_Y) \mapsto \text{har}(\mu_X, \mu_Y, g)$$

is continuous because of \cite{26}, \cite{29} and \cite{10}; cf. \cite{12, §2.18}. Here $\text{Met}^{(-)}(\cdot)$ is the space of negatively curved smooth Riemannian metrics, with the $C^\infty$ topology. Note that $\text{Met}^{(-)}(\cdot)$ is an open set of the space $\text{Met}(\cdot)$ of smooth Riemannian metrics, with the $C^\infty$ topology.

**Proof of Theorems 7 and 14** We prove the theorems assuming the Proposition.

Let $M, \mathcal{M}, \bar{M}, \bar{p}, \mu$ and $\mu_s$ be as in the Proposition. Let $f : M \to \mathcal{M}$ be a homeomorphism. Write $\mu_M$ for the hyperbolic metric of $M$. Let $k : M \to \mathcal{M}$ be the unique harmonic map homotopic to $f$, where we consider $M$ with metric $\mu_M$ and $\mathcal{M}$ with metric $\mu$. Using the notation above, we have $k = \text{har}(\mu_M, \mu, f)$.

Let $q : \bar{M} \to M$ be the pullback of $p : \bar{\mathcal{M}} \to \mathcal{M}$ via $f : M \to \mathcal{M}$ and $\bar{f} : \bar{M} \to \bar{\mathcal{M}}$ be the lifting of $f$ occurring in the pullback diagram

\[
\begin{array}{ccc}
\bar{M} & \xrightarrow{\bar{f}} & \bar{\mathcal{M}} \\
\downarrow q & & \downarrow p \\
M & \xrightarrow{f} & \mathcal{M}
\end{array}
\]

Write $\mu_{\bar{M}}$ for the hyperbolic metric $q^*(\mu_M)$ on $\bar{M}$. On $\bar{\mathcal{M}}$ define

$$ h_s = \text{har}_f(\mu_{\bar{M}}, \mu_s) = \text{har}(\mu_{\bar{M}}, \mu_s). $$

Then $s \mapsto h_s$ is a continuous map from $[0,1]$ to $C^\infty(\bar{M}, \bar{\mathcal{M}})$.

Let $\bar{k} : \bar{M} \to \bar{\mathcal{M}}$ be the lifting of $k : M \to \mathcal{M}$. It is easily deduced from \cite{11}, 2.2.0 and 2.3.2 that $\bar{k}$ is also a harmonic map from $\bar{M}$, with metric $\mu_{\bar{M}} = q^*(\mu_M)$, to $\bar{\mathcal{M}}$, with metric $\mu_0 = p^*(\mu)$. Note that $\bar{k}$ is homotopic to $\bar{f}$.

But we also have that $h_0$ is the unique harmonic map homotopic to $\bar{f} : \bar{M} \to \bar{\mathcal{M}}$, where we consider $\bar{M}$ with metric $\mu_{\bar{M}}$ and $\bar{\mathcal{M}}$ with metric $\mu_0$. Hence $h_0 = \bar{k}$.

**Claim.** $h_0$ is not univalent.

We use the same argument as in \cite{17} pp. 229-230].
It is enough to prove that $k$ is not univalent. Suppose $k$ is univalent. Then $k$ is a $C^\infty$-homeomorphism between $M$ and $\mathcal{M}$, and hence $M$ and $\mathcal{M}$ are PL homeomorphic, by the $C^\infty$-Hauptvermutung proven by M. Scharlemann and L. Siebenmann [28]. This contradicts (ii) of the Proposition and proves the claim.

Note that $h_1$ is a harmonic map between the hyperbolic manifolds $\bar{M}$ and $\bar{\mathcal{M}}$ (with metrics $\mu_{\bar{M}}$ and $\mu_1$, respectively) homotopic to the homotopy equivalence $\bar{f} : \bar{M} \to \bar{\mathcal{M}}$. Hence, by Mostow’s Rigidity Theorem [24], $h_1$ is an isometry. In particular $h_1$ is a diffeomorphism.

We have proven that there is a continuous map $s \mapsto h_s$ from $[0,1]$ to $C^\infty(M,\mathcal{M})$, with the following properties:

(a) $h_0$ is not univalent.

(b) $h_1$ is a diffeomorphism.

(c) $h_s$ is a harmonic map between the hyperbolic manifold $\bar{M}$ and the negatively curved Riemannian manifold $\bar{\mathcal{M}}$ (with metric $\mu_s$).

Define

$$s_0 = \inf\{s \in [0,1] : h_s \text{ is a diffeomorphism}\}.$$ 

Because the space of diffeomorphisms, from $\bar{M}$ to $\bar{\mathcal{M}}$, is open in $C^\infty(\bar{M},\bar{\mathcal{M}})$, we have that $s_0 < 1$. Also, since $h_0$ is not univalent, $h_{s_0}$ is not a diffeomorphism. Moreover, $h_{s_0}$ is a cellular map, since it can be approximated by the diffeomorphisms $h_t$, $t \in (s_0,1]$.

Take $h = h_{s_0}$, $M_1 = \bar{M}$ with the hyperbolic metric $\mu_{\bar{M}}$, and $M_2 = \bar{\mathcal{M}}$ with the negatively curved metric $\mu_{s_0}$. Then the harmonic map $h : M_1 \to M_2$ is a cellular map which is not a diffeomorphism. The sectional curvatures of $M_2$ lie in the interval $(-1 - \epsilon, -1 + \epsilon)$. This proves Theorem [28] and part of its Addendum. To prove that we can take $M_2$ to be hyperbolic, just repeat the argument above with $f^{-1} : \mathcal{M} \to M$, with the obvious modifications. This completes the proof of Theorem [28] and its Addendum.

To prove Theorem 2, let the manifold $\mathcal{M}$ in it be the manifold $\bar{\mathcal{M}}$ of the Proposition and let the Riemannian metrics $\mu_s$ in Theorem [28] be those of the Proposition. Then notice that $l$ is $h_0 \circ h_t^{-1}$; which is harmonic since $h_t$ is an isometry. Also $k = \hat{h}_1^{-1} \circ \hat{h}_0$ where

$$\hat{h}_t = \text{har}_f^{-1}(\mu_t, \mu_{\bar{\mathcal{M}}}).$$

And as mentioned above, it can be shown analogously that $\hat{h}_1$ is an isometry while $\hat{h}_0$ is not univalent.
To prove the Addendum to Theorem 2, it suffices to show that $h_0$ and $\hat{h}_0$ are both not cellular. We will only explicitly show this for $h_0$ since the argument for $\hat{h}_0$ is analogous.

**Caveat.** In this argument we now revert to our earlier notation given on line 5 in the “Proof of Theorems 1 and 2” where

\[ k = har(\mu_M, \mu, f); \]

$k$ will no longer denote $har(\mu_0, \mu_1, \text{id})$.

Recall that we are now assuming that the Poincaré Conjecture is true. Under this assumption Scharlemann’s main result in [27] and his result with Siebenmann [28] routinely combine to the sharper statement that the smooth map $k$ is not cellular since $M$ and $\mathcal{M}$ are not PL-homeomorphic. Now recall that a continuous map between closed manifolds is cellular if and only if the inverse image, under it, of each contractible open subset of the range is contractible; cf. [7]. Hence there exists a contractible open subset $U$ of $\mathcal{M}$ such that $k^{-1}(U)$ is not contractible. Now consider the open subset $W$ of $\bar{M}$ defined by

\[ W = q^{-1}(k^{-1}(U)) = \bar{k}^{-1}(p^{-1}(U)). \]

The diagram

\[
\begin{array}{c}
W \xrightarrow{k} p^{-1}(U) \\
\downarrow q \quad \quad \quad \downarrow p \\
k^{-1}(U) \xrightarrow{k} U
\end{array}
\]

shows that $q : W \to k^{-1}(U)$ is the pullback of the covering space $p : p^{-1}(U) \to U$. But $p : p^{-1}(U) \to U$ is a trivial covering space since $U$ is contractible; consequently, $q : W \to k^{-1}(U)$ is also trivial. Now let $V$ be a sheet of $p : p^{-1}(U) \to U$, then $V$ is a contractible open subset of $\mathcal{M}$ since it is homeomorphic to $U$. And one easily sees that $\bar{k}^{-1}(V)$ is homeomorphic to $k^{-1}(U)$; consequently, $h_0 = \bar{k}$ is not cellular. This completes the proof of Theorem 2 and its Addendum.

We will prove the Proposition in section 3. In the next section (section 2) we give three lemmas which will be needed to prove the Proposition.

## 2 Preliminary Lemmas

The first lemma we state is similar to the Lemma of [17].
Lemma 2.1. Given an integer $m \geq 6$ and a positive number $r$, there exist closed connected oriented real hyperbolic manifolds $M$, $N$, $T$ and a pair of cohomology classes $\alpha \in H^1(M, \mathbb{Z}_2)$ and $\beta \in H^2(M, \mathbb{Z}_2)$ satisfying the following properties:

(1) $\dim(M) = m$ and $T$ is a totally geodesic codimension-one submanifold of $M$.

(2) $N$ is a totally geodesic framable codimension-two submanifold of $M$, whose normal geodesic tubular neighborhood has width $\geq r$.

(3) The isometry class of $N$ depends only on $m$ (not on $r$).

(4) $\alpha \cup \beta \neq 0$.

(5) $\alpha$ is the Poincare dual of the homology class represented by $T$ in $H_{m-1}(M, \mathbb{Z}_2)$.

(6) $\beta$ is the Poincare dual of the homology class represented by $N$ in $H_{m-2}(M, \mathbb{Z}_2)$.

Proof. Our proof is the same as the proof of the Lemma in [17], just interchange $n_1$ and $n_2$ at the beginning of that proof. 

We now give a geometric lemma, but first we introduce some notation and make some comments.

Let $M$ be a Riemannian manifold, with Riemannian metric $\sigma$. Let also $\phi : \mathbb{R} \to (0, \infty)$ be a smooth function. Consider the warped metric $\rho = \phi^2 \sigma + dt^2$ on $M \times \mathbb{R}$. A classic formula of Bishop and O’Neill [2], gives the sectional curvatures $K_\rho$, of the Riemannian metric $\rho$, in terms of $\phi$ and the sectional curvatures $K_\sigma$ of $\sigma$:

$$K_\rho(P) = -\frac{\phi''(t)}{\phi(t)} s^2 + \left( K_\sigma(u, v) - \frac{(\phi'(t))^2}{\phi^2(t)} \right) \|u\|^2$$

Here $P \subset T_{(x,t)}(M \times \mathbb{R}) = T_x M \oplus \mathbb{R}$ is the two-plane generated by the orthonormal basis $\{u + s \frac{\partial}{\partial t}, v\}$, where $u, v \in T_x M$. Note that $s^2 + \|u\|^2 = 1$, $\|v\|^2 = 1$ and $\langle u, v \rangle = 0$. It follows that $K_\rho(P)$ is a convex linear combination of $-\frac{\phi''(t)}{\phi(t)}$ and $\frac{K_\sigma(u, v) - (\phi'(t))^2}{\phi^2(t)}$.

We consider now doubly warped metrics. Let $M_1$ and $M_2$ be Riemannian manifolds with Riemannian metrics $\sigma_1$ and $\sigma_2$, respectively. Let also $\phi_i : \mathbb{R} \to (0, \infty)$ be smooth functions, $i = 1, 2$. Define the doubly warped metric $\rho$ on $M_1 \times M_2 \times \mathbb{R}$ by

$$\rho = \phi_1^2 \sigma_1 + \phi_2^2 \sigma_2 + dt^2$$
A generalization of Bishop-O’Neill’s formula gives the sectional curvatures $K_{\rho}$ of the Riemannian metric $\rho$ in terms of $\phi_i$ and the sectional curvatures $K_i$ of $\sigma_i$, $i = 1, 2$ (see the appendix for a proof):

$$
K_{\rho}(P) = -\frac{\phi_i''(t)}{\phi_i(t)} s^2 ||u_2||^2 - \frac{\phi_i'(t)}{\phi_i(t)} s^2 ||v_2||^2
+ \left( \frac{K_1(u_1, u_2) - (\phi_1'(t))^2}{\phi_1'(t)} \right) (||u_1||^2 ||u_2||^2 - \langle u_1, u_2 \rangle^2)
+ \left( \frac{K_2(v_1, v_2) - (\phi_2'(t))^2}{\phi_2'(t)} \right) (||v_1||^2 ||v_2||^2 - \langle v_1, v_2 \rangle^2)
- \frac{\phi_i'(t)\phi_i''(t)}{\phi_i(t)\phi_i'(t)} (||u_1||^2 ||v_2||^2 + ||v_1||^2 ||u_2||^2 - 2\langle u_1, u_2 \rangle \langle v_1, v_2 \rangle).
$$

Here $P \subset T(x_1, x_2, t)(M_1 \times M_2 \times \mathbb{R}) = T_{x_1}M_1 \oplus T_{x_2}M_2 \oplus \mathbb{R}$ is the two-plane generated by the orthonormal basis $\{u_1 + v_1 + s\frac{\partial}{\partial t}, u_2 + v_2\}$, where $u_1, u_2 \in T_{x_1}M_1$, $v_1, v_2 \in T_{x_2}M_2$. Note that $s^2 + ||u_1||^2 + ||v_1||^2 = 1$, $||u_2||^2 + ||v_2||^2 = 1$, and $\langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = 0$. It follows that $K_{\rho}(P)$ is a convex linear combination of $-\frac{\phi_i'(t)}{\phi_i(t)}$, $\frac{K_i(u_1, u_2) - (\phi_i'(t))^2}{\phi_i'(t)}$ and $-\frac{\phi_i'(t)\phi_i''(t)}{\phi_i(t)\phi_i'(t)}$, $i = 1, 2$.

Let $a, b \in \mathbb{R}$, $0 < a < b$. Let also $\phi_i : \mathbb{R} \times (0, \infty) \to (0, \infty)$, $i = 1, 2$ be two smooth functions. The first and second derivatives of $\phi_i$ with respect to the first variable, will be denoted by $\phi_i'$ and $\phi_i''$, respectively. In the next lemma we consider Riemannian metrics $\rho_\alpha$ on $M_1 \times M_2 \times [a, b]$:

$$
\rho_\alpha(x_1, x_2, t) = \phi_1^2(\alpha, \alpha)\sigma_1(x_1) + \phi_2^2(\alpha, \alpha)\sigma_2(x_2) + \alpha^2 dt^2
$$

**Lemma 2.2.** Let $M_1$ and $M_2$ be compact Riemannian manifolds with Riemannian metrics $\sigma_1$ and $\sigma_2$, respectively. Let $\phi_i : \mathbb{R} \times (0, \infty) \to (0, \infty)$, $i = 1, 2$ be two smooth functions. Suppose that

$$
\lim_{\alpha \to \infty} \frac{\phi_i'(\alpha, \alpha)}{\phi_i(\alpha, \alpha)} = \lim_{\alpha \to \infty} \frac{\phi_i''(\alpha, \alpha)}{\phi_i(\alpha, \alpha)} = 1
$$

and

$$
\lim_{\alpha \to \infty} \phi_i(\alpha, \alpha) = \infty
$$

$i = 1, 2$, uniformly for $t \in [a, b]$. Then, given $\epsilon > 0$, there is an $\alpha_0 \in \mathbb{R}$, such that, for all $\alpha > \alpha_0$, all sectional curvatures of $\rho_\alpha$ lie in the interval $(-1 - \epsilon, -1 + \epsilon)$.

**Proof.** The manifold $M_1 \times M_2 \times [a, b]$ with the Riemannian metric $\rho_\alpha$ is isometric to $M_1 \times M_2 \times [\alpha a, \alpha b]$ with the doubly warped metric

$$
\tilde{\rho}_\alpha(x_1, x_2, t) = \phi_1^2(t, \alpha)\sigma_1(x_1) + \phi_2^2(t, \alpha)\sigma_2(x_2) + dt^2.
$$
(The isometry is \( t \mapsto \alpha t, \ t \in [a,b] \).)

As mentioned above, the sectional curvatures of \( \bar{\rho}_\alpha \) are convex linear combinations of
\[
- \frac{\phi''(t,\alpha)}{\phi(t,\alpha)} K_{i(\cdot)} - \frac{(\phi'(t,\alpha))^2}{\phi(t,\alpha)} \quad \text{and} \quad - \frac{\phi'(t,\alpha)\phi'_2(t,\alpha)}{\phi_1(t,\alpha)\phi_2(t,\alpha)}, \ i = 1, 2.
\]

But \( M_1 \) and \( M_2 \) are compact, therefore the sectional curvatures \( K_1 \) and \( K_2 \) are bounded.
Consequently, choosing \( \alpha_0 \) sufficiently large, we can suppose that all the terms
\[
- \frac{\phi''(t,\alpha)}{\phi(t,\alpha)} K_{i(\cdot)} - \frac{(\phi'(t,\alpha))^2}{\phi(t,\alpha)} \quad \text{and} \quad - \frac{\phi'(t,\alpha)\phi'_2(t,\alpha)}{\phi_1(t,\alpha)\phi_2(t,\alpha)}, \ i = 1, 2,
\]
are within \( \epsilon \) of -1, for \( \alpha > \alpha_0 \) and \( t \in [\alpha a, \alpha b] \). It follows that all the sectional curvatures of \( \bar{\rho}_\alpha \) (and \( \rho_\alpha \)) lie in the interval \((-1 - \epsilon, -1 + \epsilon)\). This proves the lemma.

Finally, we will need the following lemma.

**Lemma 2.3.** Let \( N \) be a closed connected orientable real hyperbolic manifold of dimension \( \geq 10 \), and let \( f : N \times [0,1] \to N \times [0,1] \) be a diffeomorphism which is smoothly pseudo-isotopic to the identity, rel \( \partial \). Then there is a positive integer \( J \) and a finite number of non-conjugate elements \( a_1, \ldots, a_l \in \pi_1 N \) with the following property:

If \( \bar{p} : \bar{N} \to N \) is a connected finite cover such that no conjugate of \( a^J_i \) belongs to \( \bar{p}_* (\pi_1 \bar{N}) \subset \pi_1 N \), then \( \bar{f} : \bar{N} \times [0,1] \to N \times [0,1] \) is smoothly isotopic to the identity, rel \( \partial \).

**Remark.** Note that if \( p' : N' \to N \) is a connected finite cover that factors through \( \bar{p} : \bar{N} \to N \), then the lemma holds also for \( p' \) and \( N' \); that is, \( f' : N' \times [0,1] \to N \times [0,1] \) is also smoothly isotopic to the identity.

**Proof of Lemma 2.3.** Let \( n = \dim N \), \( P(\ ) \) denote the space of topological pseudo-isotopies and \( \mathcal{P}(\ ) \) denote the space of stable topological pseudo-isotopies. Also let \( P^{\text{diff}}(\ ) \) denote the space of smooth pseudo-isotopies and \( \mathcal{P}^{\text{diff}}(\ ) \) the space of stable smooth pseudo-isotopies.

Recall that we have canonical stabilization maps
\[
\iota^{\text{diff}} : P^{\text{diff}}(\ ) \to \mathcal{P}^{\text{diff}}(\ ) \quad \text{and} \quad \iota : P(\ ) \to \mathcal{P}(\ )
\]
such that the following square of maps commutes:
\[
\begin{array}{ccc}
P^{\text{diff}}(\ ) & \xrightarrow{\iota^{\text{diff}}} & \mathcal{P}^{\text{diff}}(\ ) \\
\downarrow & & \downarrow \\
P(\ ) & \xrightarrow{\iota} & \mathcal{P}(\ ).
\end{array}
\]

The vertical arrows in this square denote the natural forget structure maps. It is a consequence of work of Burghelea and Lashof \[3\] and Cerf \[5\] that the forgetful map \( \mathcal{P}^{\text{diff}}(\ ) \to \mathcal{P}(\ ) \)
induces an isomorphism on $\pi_0$; cf. [20, p. 12]. Furthermore $\iota$ and $\iota^{\text{diff}}$ induce isomorphisms on $\pi_0$ for all manifolds of dim $> 10$ by Igusa [21]. Consequently the square shows that the forgetful map $P^{\text{diff}}(\cdot) \to P(\cdot)$ also induces an isomorphism on $\pi_0$ for all smooth manifolds of dim $> 10$.

Now these four isomorphisms combined with [14, Theorem 6.0] show that we can assume that the pseudo-isotopy of $f$ (given in Lemma 2.3) is supported on the disjoint neighborhoods of a finite number of (non-conjugate and non-trivial) embedded loops in $N$. These neighborhoods are diffeomorphic to $D^n \times S^1$, where $D^n$ is the closed $n$-disc. Hence the lemma follows from the following claim.

**Claim.** For every smooth pseudo-isotopy $F : [0, 1] \times D^n \times S^1 \to [0, 1] \times D^n \times S^1$, there is a $J$ such that if $\tilde{F}$ is the lifting of $F$ by the connected $j$-sheeted cover $[0, 1] \times D^n \times S^1 \to [0, 1] \times D^n \times S^1$, with $j \geq J$, then $\tilde{F}$ is smoothly isotopic to the identity, rel($\{0\} \times D^n \times S^1 \cup [0, 1] \times \partial D^n \times S^1$).

**Proof of Claim.** Because of the above discussion it suffices to show that $\tilde{F}$ is topologically isotopic to the identity, rel($\{0\} \times D^n \times S^1 \cup [0, 1] \times \partial D^n \times S^1$).

Given $\epsilon > 0$, by taking $j$ sufficiently large, we have that $\tilde{F}$ becomes $\epsilon$-controlled in the $S^1$-direction. Then by appropriately shrinking inwards in the $[0, 1] \times D^n$ direction (an Alexander type isotopy) we get $\epsilon$-control in all directions. Hence, by [15], $\tilde{F}$ can be topologically isotoped to the identity, rel($\{0\} \times D^n \times S^1 \cup [0, 1] \times \partial D^n \times S^1$).

This proves the claim and thus completes the proof of Lemma 2.3.

### 3 Proof of the Proposition

Let $M$, $N$ and $T$ be as in Lemma 2.1 relative to a sufficiently large positive real number $r$. (How large is sufficient, will presently become clear.) Define $P = N \cap T$. Since $\alpha \cup \beta \neq 0$, $N$ and $T$ intersect transversally and $P$ is a codimension-three totally geodesic submanifold of $M$. Moreover $\alpha \cap \beta$ is the Poincare dual of the cycle represented by $P$ in $H_{m-3}(M, \mathbb{Z}_2)$.

Since the trivial normal geodesic tubular neighborhood of $N$ has width $\geq r$, we can identify the tubular neighborhood $V$, of width $r$, with $N \times B$, where $B \subset \mathbb{R}^2$ is the open ball, centered at the origin, of radius $r$. This identification is a metric identification on

$$V - N = (N \times B) - (N \times \{0\}) = N \times (B - \{0\}) = N \times S^1 \times (0, r),$$
where we consider $N \times S^1 \times (0, r)$ with the doubly warped Riemannian metric

$$\rho(x, u, t) = \cosh^2(t)\sigma_N(x) + \sinh^2(t)\sigma_{S^1}(u) + dt^2$$

Here $\sigma_N$ is the hyperbolic metric on $N$ and $\sigma_{S^1}$ is the canonical Riemannian metric on $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

Define $N_0$, $P_0$, $Q$ and $R$ by

$$N_0 = N \times \{(1, 0)\} \times \left\{ \frac{r}{2} \right\}$$
$$\subseteq N \times S^1 \times (0, r) = (V - N)$$
$$\subseteq M$$

$$P_0 = N_0 \cap T$$

$$Q = N \times S^1 \times \left\{ \frac{r}{2} \right\}$$

$$R = N \times S^1_+ \times \left\{ \frac{r}{2} \right\}$$

where

$$S^1_+ = \{(x, y) \in S^1 \mid x > 0\}.$$ 

It follows from these definitions that

$$P_0 = P \times \{(1, 0)\} \times \left\{ \frac{r}{2} \right\}$$

$$P_0 \subseteq N_0 \subseteq Q \subseteq V \subseteq M$$

$R$ is diffeomorphic to $N_0 \times [0, 1]$.

The smooth manifold $\mathcal{M}$ of the statement of the Proposition is constructed by cutting $M$ apart along $Q$ and gluing back with a twist $f : Q \to Q$. For the details of this construction see the “Proof of Corollary” in [17, pp. 230-233] with the following modifications:

- Replace $N$ in [7] by $Q$.

- Replace $N$ by $T$ in the definition of $\hat{\alpha}$ at the top of [7, p. 232].

Then we have that this smooth manifold $\mathcal{M}$, constructed as above, satisfies the following properties:

1. There is a homeomorphism $g : \mathcal{M} \to M$.

2. $\mathcal{M}$ is not $PL$ homeomorphic to $M$. 

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3. There is a two-sheeted connected double cover $q' : \tilde{M} \to M$ such that the lifting $\tilde{g} : \tilde{M} \to \tilde{M}$, of $g$, is (topologically) pseudo-isotopic to a diffeomorphism.

Here $p' : \tilde{M} \to M$ is the double cover induced from $q'$ via $g$ which occurs in the pullback diagram

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{g}} & \tilde{M} \\
\downarrow{p'} & & \downarrow{q'} \\
M & \xrightarrow{g} & M
\end{array}
$$

Let $\tilde{N}_0$, $\tilde{R}$, $\tilde{Q}$, be the liftings, by $q'$, of $N_0$, $R$, $Q$, respectively. It is clear from the construction of $q'$ that $\tilde{Q}$ is connected. Note that $\tilde{R}$ is also diffeomorphic to $\tilde{N}_0 \times [0,1]$. Let $\tilde{f} : \tilde{Q} \to \tilde{Q}$ be the lifting of $f : Q \to Q$.

**Claim.** We can choose $f$ such that:

1. $f$ is the identity outside $R \subset Q$.
2. $\tilde{f}$ is smoothly pseudo-isotopic to the identity.

**Proof of this Claim.** To prove (1) just note that the map $\gamma$ in [7] can be chosen to be constant outside a small neighborhood of $P_0$ in $M$. Hence $f$ can chosen to be the identity outside a small neighborhood of $P_0$ in $Q$. Since $P_0 \subset \text{int}(R)$, it follows that $f$ can be chosen to satisfy (1) of the claim.

To prove (2) consider the following diagram

$$
\begin{array}{ccc}
\tilde{Q} \times I & \xrightarrow{\tilde{\sigma}} & \tilde{M} \\
\downarrow{q'} & & \downarrow{q} \\
Q \times I & \xrightarrow{\sigma} & M
\end{array}
\quad
\begin{array}{ccc}
S^2 \times S^1 & \xrightarrow{\psi} & S^3 \\
\downarrow{1_{S^2} \times p} & & \downarrow{\phi} \\
S^2 \times S^1 & \xrightarrow{\psi} & S^3
\end{array}
\quad
\begin{array}{c}
\tilde{\eta} \\
\downarrow{\text{Top/0}}
\end{array}
$$

Here $Q \times I$ is a tubular neighborhood of $Q$ and $\sigma : Q \times I \to M$ is the inclusion. The diagram above is the diagram of [17] p. 233] except for the first column and that the second vertical arrow is now denoted $q'$ instead of $q$. Since $\tilde{\eta} \varphi \psi \xi \tilde{\sigma}$ is null homotopic, we have that the differentiable structure on $\tilde{Q} \times I$, rel $\partial$, induced by the inclusion in $\tilde{M}$ is smoothly concordant, rel $\partial$, to the one induced by the inclusion in $\tilde{M}$. It follows that $\tilde{f}$ is smoothly pseudo-isotopic to the identity. This proves the Claim.

Since the fundamental group of $\tilde{R}$ injects into the fundamental group of $\tilde{M}$, and $\pi_1\tilde{M}$ is residually finite, we can apply Lemma [25] to $\tilde{f}|_{\tilde{R}} : \tilde{R} \to \tilde{R}$ to obtain a finite cover.
\( \tilde{q} : \tilde{M} \to \tilde{M} \) with \( f : \tilde{Q} \to \tilde{Q} \) smoothly isotopic to the identity. Write \( q = q' \circ \tilde{q} : \tilde{M} \to M \) and let \( p : \tilde{M} \to M \) be the covering space induced by \( g \) from \( q \) via the following pullback diagram:

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{g}} & \tilde{M} \\
p & & q \\
\downarrow & & \downarrow \\
\tilde{M} & \xrightarrow{\tilde{q}} & M
\end{array}
\]

So far we have obtained the following:

- \( M \) is a closed connected orientable real hyperbolic manifold of dimension \( > 10 \).
- \( \mathcal{M} \) is obtained from \( M \) by cutting along the hypersurface \( Q \) and gluing back with the twist \( f : Q \to Q \).
- \( \mathcal{M} \) is homeomorphic, but not \( PL \) homeomorphic, to \( M \).
- There is a connected finite sheeted cover \( p : \tilde{M} \to M \) such that \( f : \tilde{Q} \to \tilde{Q} \) is smoothly isotopic to the identity \( 1_{\tilde{Q}} \). In particular, \( \tilde{M} \) is diffeomorphic to \( \tilde{M} \).

Note that \( \tilde{f} \) and \( p \) depend only on \( f \). This ends the topological part of the proof of the Proposition.

Let \( \epsilon > 0 \). We now show how to construct the Riemannian metrics \( \mu \) and \( \mu_s \), of the statement of the Proposition, with sectional curvatures in the interval \( (-1 - \epsilon, -1 + \epsilon) \).

Recall that \( Q \) has the normal geodesic tubular neighborhood in \( M \),

\[ V - N = N \times S^1 \times (0, r) \]

which is equipped with the doubly warped Riemannian metric

\[ \rho(x, u, t) = \cosh^2(t)\sigma_N(x) + \sinh^2(t)\sigma_{S^1}(u) + dt^2 \]

where \( \sigma_N \) is the hyperbolic metric on \( N \) and \( \sigma_{S^1} \) is the canonical Riemannian metric on \( S^1 \). Note that \( N \times S^1 \times (0, r) \) with the Riemannian metric \( \rho \) is isometric to \( N \times S^1 \times (0, 6) \) with the Riemannian metric

\[ \rho_r(x, u, t) = \cosh^2(\alpha t)\sigma_N(x) + \sinh^2(\alpha t)\sigma_{S^1}(u) + \alpha^2 dt^2 \]

where \( \alpha = r/6 \).
Let
\[ \delta_1 : \mathbb{R} \to [-1, 1] \]
\[ \delta_2 : \mathbb{R} \to [0, 1] \]
\[ \delta_3 : \mathbb{R} \to [-1, 1] \]
be smooth functions such that:
\[ \delta_1(t) = \begin{cases} -1 & t \leq 2 \\ 1 & t \geq 3 \end{cases} \]
\[ \delta_2(t) = \begin{cases} 0 & t \leq 3 \\ 1 & t \geq 4 \end{cases} \]
\[ \delta_3(t) = \begin{cases} 1 & t \leq 4 \\ -1 & t \geq 5 \end{cases} \]
and all \( \delta_i \) are constant near 1, 2, 3, 4, 5.

Notice that \( Q \) has the (smooth) tubular neighborhood \( [N \times S^1 \times (0, r)]_f \) in \( \mathcal{M} \) obtained from \( N \times S^1 \times (0, r) \) by cutting along \( Q = N \times S^1 \times \{ \frac{r}{2} \} \) and gluing back with \( f : Q \to Q \).

Also note that \( [N \times S^1 \times (0, r)]_f \) is diffeomorphic to \( [N \times S^1 \times (0, 6)]_f \) which is obtained from \( N \times S^1 \times (0, 6) \) by cutting along \( Q = N \times S^1 \times \{ 3 \} \) and gluing back with \( f : Q \to Q \).

On \( [N \times S^1 \times (0, 6)]_f \), consider the following Riemannian metric:
\[
\lambda_r(x, u, t) = \begin{cases} 
\rho_r(x, u, t) & t \leq 2, 5 \leq t \\
\cosh^2(\alpha t)\sigma_N(x) + \left( \frac{e^{\alpha t} + \delta_1(t)e^{-\alpha t}}{2} \right)^2\sigma_{S^1}(u) + \alpha^2 dt^2 & 2 \leq t \leq 3 \\
\cosh^2(\alpha t)\left\{ (1 - \delta_2(t)) f^*[\sigma_N(x) + \sigma_{S^1}(u)] \\
+ \delta_2(t)[\sigma_N(x) + \sigma_{S^1}(u)] \right\} + \alpha^2 dt^2 & 3 \leq t \leq 4 \\
\cosh^2(\alpha t)\sigma_N(x) + \left( \frac{e^{\alpha t} + \delta_3(t)e^{-\alpha t}}{2} \right)^2\sigma_{S^1}(u) + \alpha^2 dt^2 & 4 \leq t \leq 5 
\end{cases}
\]
where \( \alpha = r/6 \).

It can be verified from Lemma 2.2 that, taking \( r \) large enough, \( \lambda_r(x, u, t) \) has sectional curvatures within \( \epsilon \) of \( -1 \), for \( t \leq 3 \) and \( 4 \leq t \). Also, for \( 3 \leq t \leq 4 \), by taking \( r \) large enough, \( \lambda_r(x, u, t) \) has sectional curvatures within \( \epsilon \) of \( -1 \). (See [25, pp. 11-13].)

We now define the metric \( \mu \) on \( \mathcal{M} \) in the following way (see [25] for more details): \( \mu(p) \) is the hyperbolic metric, for \( p \notin [N \times S^1 \times (r/3, 5r/6)]_f \), and \( \mu(p) \) is the pullback of the metric \( \lambda_r \) by the map \( t \mapsto t/\alpha \), where \( \alpha = r/6 \), and \( p = (x, u, t) \in [N \times S^1 \times (0, r)]_f \subset \mathcal{M} \).
Let $s \mapsto f_s$, $s \in [0, 1/2]$ be a smooth isotopy of $\bar{f}$, with $f_0 = \bar{f}$ and $f_{1/2} = 1_Q$. We assume that this isotopy is constant near 0 and 1/2. Let $\mathcal{M}_s$, be the smooth manifold obtained from $\mathcal{M}$ by cutting along $\bar{Q}$ and gluing back with $f_s$. We construct a family of Riemannian metrics $\mu'_s$, $s \in [0, 1/2]$, on $\mathcal{M}_s$. This construction is identical to that of $\mu$, just repeat all the definitions and arguments above, writing a “bar” above each symbol and replacing $f$ by $f_s$. For instance, on $[N \times S^1 \times (0, 6)]_{f_s}$ the Riemannian metric $(\bar{\lambda}_r)_s$ is given by the following formula:

$$(\lambda_r)_s(x, u, t) = \begin{cases} \\
    \bar{\rho}_r(x, u, t) & t \leq 2, 5 \leq t \\
    \cosh^2(\alpha t) \sigma_N(x) + (\frac{e^{\alpha t + \delta_1(t) e^{-\alpha t}}}{2})^2 \sigma_S^1(u) + \alpha^2 dt^2 & 2 \leq t \leq 3 \\
    \cosh^2(\alpha t) \left\{ (1 - \delta_2(t)) f_s^* [\sigma_N(x) + \sigma_S^1(u)] + \delta_2(t)[\sigma_N(x) + \sigma_S^1(u)] \right\} + \alpha^2 dt^2 & 3 \leq t \leq 4 \\
    \cosh^2(\alpha t) \sigma_N(x) + (\frac{e^{\alpha t + \delta_3(t) e^{-\alpha t}}}{2})^2 \sigma_S^1(u) + \alpha^2 dt^2 & 4 \leq t \leq 5 \\
\end{cases}$$

where $\alpha = r/6$.

In this way we obtain Riemannian metrics $\mu'_s$ on $\mathcal{M}_s$.

Since $f$, $\bar{f}$ (hence $f_s$) do not depend on $r$, we can choose $r$ large enough so that all sectional curvatures of $\mu'_s$ are are within $\epsilon$ of -1. Note that the constructions of the Riemannian metrics $\mu$ and $\mu'_0$ above commute with the the cover $p : \mathcal{M} \to \mathcal{M}$, that is, $p^* \mu = \mu'_0$.

Since $f_{1/2}$ is the identity, $\mathcal{M}_{1/2} = \bar{M}$ and $(\lambda_r)_{1/2}$ is given by:

$$(\lambda_r)_{1/2}(x, u, t) = \begin{cases} \\
    \bar{\rho}_r(x, u, t) & t \leq 2, 5 \leq t \\
    \cosh^2(\alpha t) \sigma_N(x) + (\frac{e^{\alpha t + \delta_1(t) e^{-\alpha t}}}{2})^2 \sigma_S^1(u) + \alpha^2 dt^2 & 2 \leq t \leq 3 \\
    \cosh^2(\alpha t) \sigma_N(x) + \cosh^2(\alpha t) \sigma_S^1(u) + \alpha^2 dt^2 & 3 \leq t \leq 4 \\
    \cosh^2(\alpha t) \sigma_N(x) + (\frac{e^{\alpha t + \delta_3(t) e^{-\alpha t}}}{2})^2 \sigma_S^1(u) + \alpha^2 dt^2 & 4 \leq t \leq 5 \\
\end{cases}$$

where $\alpha = r/6$.

We now define Riemannian metrics $\mu'_s$, $s \in [1/2, 1]$. Let $\eta : [1/2, 1] \to [1, 2]$ be a smooth function such that $\eta(1/2) = 1$, $\eta(1) = 0$ and $\eta$ is constant near 1/2 and 1. For $s \in [1/2, 1]$,
define
\[
(\lambda_r)_s(x, u, t) = \begin{cases}
\bar{\rho}_r(x, u, t) & t \leq 2, 5 \leq t \\
\cosh^2(\alpha t)\sigma_N(x) + \left(\frac{e^{\alpha t} + \left[\eta(s)(1+\delta_1(t))\right] e^{-\alpha t}}{2}\right)^2 \sigma_{\Sigma^1}(u) + \alpha^2 dt^2 & 2 \leq t \leq 3 \\
\cosh^2(\alpha t)\sigma_N(x) + \left(\frac{e^{\alpha t} + \left[\eta(s)(1+\delta_1(t))\right] e^{-\alpha t}}{2}\right)^2 \sigma_{\Sigma^1}(u) + \alpha^2 dt^2 & 3 \leq t \leq 4 \\
\cosh^2(\alpha t)\sigma_N(x) + \left(\frac{e^{\alpha t} + \left[\eta(s)(1+\delta_1(t))\right] e^{-\alpha t}}{2}\right)^2 \sigma_{\Sigma^1}(u) + \alpha^2 dt^2 & 4 \leq t \leq 5
\end{cases}
\]
where \(\alpha = r/6\).

Construct Riemannian metrics \(\nu'_s\), \(s \in [1/2, 1]\) in the same way as before: pull back the metrics \((\lambda_r)_s\) and fit them in \(\bar{M}\). Note that \((\lambda_r)_1 = \bar{\rho}_r\); hence \(\nu'_1\) is the hyperbolic metric on \(\bar{M}\). Also, as before, we can suppose, taking \(r\) sufficiently large, that all sectional curvatures of \(\nu'_s\) are are within \(\epsilon\) of -1.

Finally, the smooth isotopy \(f_s\), \(s \in [0, 1/2]\), induces a (top) isotopy \(g_t : \bar{M} \to \bar{M}\), with \(g_0 = 1_{\bar{M}}\) and \(g_s : \bar{M} \to \bar{M}\) a diffeomorphism. Define \(\nu_s = g_s^*\nu'_s\), for \(s \in [0, 1/2]\), and \(\nu_s = \nu'_s\), for \(s \in [1/2, 1]\). It is straightforward to verify that we can choose the isotopy \(g_s\) in such a way that \(\nu_s\) satisfies (iv) of the Proposition.

**Appendix**

Here we sketch how to deduce the formula for the sectional curvature of a doubly warped metric used to prove Lemma 2.2.

We have three steps. First we calculate the Levi-Civita connection. Then the curvature operator, and finally the sectional curvatures.

As in section 2, let \(M_1\) and \(M_2\) be Riemannian manifolds with Riemannian metrics \(\sigma_1\) and \(\sigma_2\), respectively. Let also \(\phi_i : \mathbb{R} \to (0, \infty)\) be smooth functions, \(i = 1, 2\), and define the doubly warped metric \(\rho\) on \(M = M_1 \times M_2 \times \mathbb{R}\):
\[
\rho = \phi_1^2 \sigma_1 + \phi_2^2 \sigma_2 + dt^2
\]
or, equivalently,
\[
\langle u_1 + v_1 + s_1 \partial, u_2 + v_2 + s_2 \partial \rangle = \phi_1^2 \langle u_1, v_1 \rangle_1 + \langle u_2, v_2 \rangle_2 + s_1 s_2
\]
where \(\langle , \rangle = \langle , \rangle_\rho, \langle , \rangle_i = \langle , \rangle_{\sigma_i}, \partial = \frac{\partial}{\partial t}, u_1 + v_1 + s_1 \partial \in T_{x_1} M_1 \oplus T_{x_2} M_2 \oplus \mathbb{R} = T_{(x_1, x_2, t)} M\).
Note that if \( u_1, u_2 \) are vector fields on \( M \) which are zero in the \( M_2 \) and \( \mathbb{R} \) directions, then 
\[
\langle u_1, u_2 \rangle = \phi_1^2 \langle u_1, u_2 \rangle_1.
\]
Analogously, for vector fields \( v_1, v_2 \) on \( M \) which are zero in the \( M_1 \) and \( \mathbb{R} \) directions, we have 
\[
\langle v_1, v_2 \rangle = \phi_2^2 \langle v_1, v_2 \rangle_2.
\]

**The Connection.**

We use the Koszul formula that relates the Levi-Civita connection \( D \) with the Riemannian metric \( (\ , \ ) \) of a Riemannian manifold:

\[
2(Z, D_D Y) = X(Y, Z) + Y(X, Z) - Z(X, Y) \tag{1}
\]

Here we are assuming that the vector fields \( X, Y, Z \) commute.

Now \( u, u_1, u_2, \ldots \) will denote tangent vectors in \( TM_1 \subset TM \), or vector fields on \( M \) which are constant in the \( M_2 \) and \( \mathbb{R} \) directions. Analogously, \( v, v_1, v_2, \ldots \) will denote tangent vectors in \( TM_2 \subset TM \), or vector fields on \( M \) which are constant in the \( M_1 \) and \( \mathbb{R} \) directions. We assume that all vector fields commute. Let \( \nabla, \nabla^1, \nabla^2 \) denote the Levi-Civita connections of the Riemannian manifolds \( M, M_1, M_2 \), respectively. Write \( \partial = \frac{\partial}{\partial t} \).

**Claim 1.**

(i) \( \nabla_\partial \partial = 0 \)

(ii) \( \nabla_\partial u = \nabla_u \partial = \frac{\phi'_1}{\phi_1} u \)

(iii) \( \nabla_\partial v = \nabla_v \partial = \frac{\phi'_2}{\phi_2} v \)

(iv) \( \nabla_{u_1} u_2 = -\frac{\phi'_1}{\phi_1} \langle u_1, u_2 \rangle \partial + \nabla_{u_1}^1 u_2 \)

(v) \( \nabla_{v_1} v_2 = -\frac{\phi'_2}{\phi_2} \langle v_1, v_2 \rangle \partial + \nabla_{v_1}^2 v_2 \)

(vi) \( \nabla_u v = \nabla_v u = 0 \)

**Proof.**

(i) By (I) above, we have \( 2 \langle u, \nabla_\partial \partial \rangle = 2 \langle v, \nabla_\partial \partial \rangle = 2 \langle \partial, \nabla_\partial \partial \rangle = 0 \). Hence \( \nabla_\partial \partial = 0 \).

(ii) By (I) we have \( \langle v, \nabla_u \partial \rangle = 0 \). Since \( \langle \partial, \partial \rangle = 1 \), also by (I), we have \( \langle \partial, \nabla_u \partial \rangle = 0 \).

Finally, again by (I), \( 2 \langle u_1, \nabla_u \partial \rangle = \partial \langle u_1, u \rangle \).

Hence \( \langle u_1, \nabla_u \partial \rangle = \frac{1}{2} \partial \langle u_1, u \rangle = \frac{1}{2} \partial [\phi_1^2 \langle u_1, u \rangle_1] = \phi'_1 \phi_1 \langle u_1, u \rangle_1 = \frac{\phi'_1}{\phi_1} \langle u_1, u \rangle = \langle u_1, \frac{\phi'_1}{\phi_1} u \rangle \).

It follows that \( \nabla_u \partial = \frac{\phi'_1}{\phi_1} u \).

(iii) Same as (ii).

(iv) Since \( v \langle u_1, u_2 \rangle = 0 \), we have, by (I), that \( \langle v, \nabla_{u_1} u_2 \rangle = 0 \), for all \( v \).

Also, as in the proof of (ii), we have \( \langle \partial, \nabla_{u_1} u_2 \rangle = \frac{\phi'_1}{\phi_1} \langle u_1, u_2 \rangle \).

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Finally, for all \( u_3 \) we have \( 2\langle u_3, \nabla_u u_2 \rangle = u_2\langle u_1, u_3 \rangle + u_1\langle u_2, u_3 \rangle - u_3\langle u_1, u_2 \rangle = u_2\phi_1^2(u_1, u_3) + u_1\phi_1^2(u_2, u_3) - u_3\phi_1^2(u_1, u_2) \)

\[ \phi_1^2 \{ u_2\langle u_1, u_3 \rangle + u_1\langle u_2, u_3 \rangle - u_3\langle u_1, u_2 \rangle \} = \phi_1^2 \{ 2\langle u_3, \nabla_u^1 u_2 \rangle \} = 2\langle u_3, \nabla_u^1 u_2 \rangle. \]

This proves (iv).

(v) Same as (iv).

(vi) Since \( u\langle v, v_1 \rangle = v\langle u, v_1 \rangle = 0 \), by (i) we have \( \langle u_1, \nabla_u v \rangle = \langle v_1, \nabla_u v \rangle = \langle \partial, \nabla_u v \rangle = 0 \).

It follows that \( \nabla_u v = 0 \). This proves the claim.

**The Curvature Operator.**

Let \( R \) denote the curvature tensor (of type (3,1)) of the Riemannian manifold \( M \); that is \( R_{abc} = \nabla_b \nabla_a - \nabla_a \nabla_b - \nabla_{[b,a]}c \). A straightforward calculation, using (2), shows:

\[
R_{\partial u} \partial = -\frac{\phi''}{\phi_1} u \\
R_{\partial v} \partial = -\frac{\phi''}{\phi_2} v \\
R_{u_1 u_2} \partial = 0 \\
R_{u_1 v_2} \partial = 0 \\
R_{u_2 v} \partial = 0
\]

and

\[
R_{v \partial u} = 0 \\
R_{v \partial v} = 0 \\
R_{v_1 v_2} u = 0 \\
R_{v_1 v_2} v = 0 \\
R_{u_1 \partial u_2} = -\frac{\phi''}{\phi_1} \langle u_1, u_2 \rangle \partial \\
R_{v_1 \partial v_2} = -\frac{\phi''}{\phi_2} \langle v_1, v_2 \rangle \partial \\
R_{u_1 v_2} u_2 = -\frac{\phi'}{\phi_1 \phi_2} \langle u_1, u_2 \rangle v \\
R_{u_1 v_2} v_2 = -\frac{\phi'}{\phi_1 \phi_2} \langle v_1, v_2 \rangle u
\]

Note that (i) implies:

\[
\begin{align*}
&\langle u_2, u_1, u_3 \rangle + \langle u_2, \nabla_u u_3 \rangle = u_1 \langle u_2, u_3 \rangle + \langle u_1, \nabla_u u_3 \rangle \\
&\langle v_2, v_1, v_3 \rangle + \langle v_2, \nabla_v v_3 \rangle = v_1 \langle v_2, v_3 \rangle + \langle v_1, \nabla_v v_3 \rangle
\end{align*}
\]

From this and (2) it follows that

\[
R_{u_1 u_2} u_3 = R_{u_1 u_2}^1 u_3 - \frac{(\phi_1')^2}{\phi_1^2} (\langle u_1, u_3 \rangle u_2 - \langle u_2, u_3 \rangle u_1) \\
R_{v_1 v_2} v_3 = R_{v_1 v_2}^1 v_3 - \frac{(\phi_2')^2}{\phi_2^2} (\langle v_1, v_3 \rangle v_2 - \langle v_2, v_3 \rangle v_1)
\]

Let \( \mathcal{R} \) denote the curvature operator on two-forms for \( M \), defined by \( \langle \mathcal{R}(a \wedge b), c \wedge d \rangle = \langle R_{ab} c, d \rangle \). Also, let \( \mathcal{R}^1 \) denote the curvature operator on two-forms for \( M_i \). Recall that the scalar product on \( \wedge^2 TM \) is given by \( \langle a \wedge b, c \wedge d \rangle = \langle a, c \rangle \langle b, d \rangle - \langle a, d \rangle \langle b, c \rangle \).
Claim 2.

(i) \( \mathcal{R}(\partial \wedge u) = -\frac{\phi''}{\phi_1} \partial \wedge u \)

(ii) \( \mathcal{R}(\partial \wedge v) = -\frac{\phi''}{\phi_2} \partial \wedge v \)

(iii) \( \mathcal{R}(u_1 \wedge u_2) = \mathcal{R}^1(u_1 \wedge u_2) - \frac{(\phi'_1)^2}{\phi_1^2} u_1 \wedge u_2 \) \hspace{1cm} (6)

(iv) \( \mathcal{R}(v_1 \wedge v_2) = \mathcal{R}^2(v_1 \wedge v_2) - \frac{(\phi'_2)^2}{\phi_2^2} v_1 \wedge v_2 \)

(v) \( \mathcal{R}(u \wedge v) = -\frac{\phi'_1 \phi'_2}{\phi_1 \phi_2} u \wedge v \)

Proof. (i) From (3) or (4) we have \( \langle \mathcal{R}(\partial \wedge u), \partial \wedge v \rangle = 0 \). From (4) we have \( \langle \mathcal{R}(\partial \wedge u), v_1 \wedge v_2 \rangle = \langle R(\partial \wedge u), u_1 \wedge u_2 \rangle = \langle R(\partial \wedge u), u_1 \wedge v \rangle = 0 \). Hence \( \mathcal{R}(\partial \wedge u) \) is a linear combination of two-vectors of the form \( \partial \wedge u_1 \). From (3) or (4) we have that, for all \( u_1 \), \( \langle \mathcal{R}(\partial \wedge u), \partial \wedge u_1 \rangle = -\frac{\phi'}{\phi_1} \langle u, u_1 \rangle = -\frac{\phi'}{\phi_1} \partial \wedge u, \partial \wedge u_1 \rangle \). This proves (i).

(ii) Same as (i).

(iii) By (3), \( \langle \mathcal{R}(u_1 \wedge u_2), \partial \wedge v \rangle = \langle R(u_1 \wedge u_2), \partial \wedge u \rangle = 0 \). By (4), \( \langle \mathcal{R}(u_1 \wedge u_2), v_1 \wedge v_2 \rangle = \langle R(u_1 \wedge u_2), u \wedge v \rangle = 0 \). Hence \( \mathcal{R}(u_1 \wedge u_2) \) is a linear combination of two-vectors of the form \( u_1 \wedge u_2 \). But, by (5), we have

\[
\langle \mathcal{R}(u_1 \wedge u_2), u_3 \wedge u_4 \rangle = \langle \mathcal{R}^1(u_1 \wedge u_2), u_3 \wedge u_4 \rangle - \frac{(\phi'_1)^2}{\phi_1^2} (\langle u_1, u_3 \rangle \langle u_2, u_4 \rangle - \langle u_2, u_3 \rangle \langle u_1, u_4 \rangle)
\]

\[= \langle \mathcal{R}^1(u_1 \wedge u_2), u_3 \wedge u_4 \rangle - \frac{(\phi'_1)^2}{\phi_1^2} \langle u_1 \wedge u_2, u_3 \wedge u_4 \rangle
\]

\[= \langle \left\{ \mathcal{R}^1(u_1 \wedge u_2) - \frac{(\phi'_1)^2}{\phi_1^2} u_1 \wedge u_2 \right\}, u_3 \wedge u_4 \rangle
\]

for all \( u_3, u_4 \). This proves (iii).

(iv) Same as (iii).

(v) By (3) and (4), \( \mathcal{R}(u \wedge v) \) is a linear combination of two-vectors of the form \( u_i \wedge v_j \). From (4) we have that

\[\langle \mathcal{R}(u \wedge v), u_1 \wedge v_1 \rangle = -\frac{\phi'_1 \phi'_2}{\phi_1 \phi_2} \langle u, u_1 \rangle \langle v, v_1 \rangle = \langle -\frac{\phi'_1 \phi'_2}{\phi_1 \phi_2} u \wedge v, u_1 \wedge v_1 \rangle
\]

for all \( u_1, v_1 \). This proves (v) and the claim.

The Sectional Curvature.

Let \( P \subset T_{(x_1,x_2,t)}(M_1 \times M_2 \times \mathbb{R}) = T_{x_1}M_1 \oplus T_{x_2}M_2 \oplus \mathbb{R} \) be the two-plane generated by the orthonormal basis \( \{ u_1 + v_1 + \phi_1 \partial_t, u_2, v_2 \} \), where \( u_1, u_2 \in T_{x_1}M_1, v_1, v_2 \in T_{x_2}M_2 \).
Write \( a = u_1 + v_1 + s\partial \) and \( b = u_2 + v_2 \). Then the sectional curvature \( K(P) \) is given by 
\[
\langle R(a, b), a \wedge b \rangle. 
\]
Note that
\[
\begin{align*}
R(a \wedge b) &= R(u_1 \wedge u_2) + R(u_1 \wedge v_2) + R(v_1 \wedge u_2) + R(v_1 \wedge v_2) + sR(\partial \wedge u_2) + sR(\partial \wedge v_2) \\
\wedge b &= u_1 \wedge u_2 + u_1 \wedge v_2 + v_1 \wedge u_2 + v_1 \wedge v_2 + s\partial \wedge u_2 + s\partial \wedge v_2
\end{align*}
\]
Multiplying these terms, a straightforward calculation using (6) shows that our formula for the sectional curvature of a doubly warped metric holds.

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