Symmetry properties of orthogonal and covariant Lyapunov vectors and their exponents

Harald A Posch

Computational Physics Group, Faculty of Physics, University of Vienna, Boltzmannngasse 5, A-1090 Vienna, Austria

E-mail: harald.posch@univie.ac.at

Received 29 June 2012
Published 4 June 2013
Online at stacks.iop.org/JPhysA/46/254006

Abstract

Lyapunov exponents are indicators for the chaotic properties of a classical dynamical system. They are most naturally defined in terms of the time evolution of a set of so-called covariant vectors, co-moving with the linearized flow in tangent space. Taking a simple spring pendulum and the Hénon–Heiles system as examples, we demonstrate the consequences of symplectic symmetry and of time-reversal invariance for such vectors, and study the transformation between different parameterizations of the flow.

This article is part of a special issue of Journal of Physics A: Mathematical and Theoretical devoted to ‘Lyapunov analysis: from dynamical systems theory to applications’.

PACS numbers: 05.45.–a, 05.40.–a, 05.20.–y, 05.45.Pq

(Some figures may appear in colour only in the online journal)

1. Introduction

The stability of the phase-space trajectory of a dynamical system is determined by the so-called Lyapunov exponents, which are the time-averaged rate constants for the norms of a set of perturbation vectors in tangent space, which grow or shrink exponentially with time. Various such sets have been introduced in connection with the algorithms for the computation of the Lyapunov exponents. The features of these sets and of their associated exponents are well known to mathematicians and theoretical physicists [1–4]. In view of an increasing number of applications to ever more sophisticated physical systems, it seems worthwhile to become familiar with the properties of these tangent-space objects for some simple models. This is the aim of this paper.

The most familiar set of perturbation vectors spanning the tangent space is the set of orthonormal Gram Schmidt (GS) vectors \( \{ g^\ell \} \), \( \ell \in \{ 1, \ldots, D \} \), commonly also referred to as backward Lyapunov vectors (since they depend on the history in the past). \( D \) is the dimension of phase space. For time-continuous systems they are most elegantly obtained...
as the forward-in-time solution (indicated here by an index (+)) of the linearized motion equations, augmented by a set of constraints, which continuously enforce the orthogonality and the norm conservation of these vectors. The latter gives rise to the GS-Lyapunov exponents along the way [5, 6]. Instead of constraints, the standard algorithms for the computation of Lyapunov exponents [7–9] employ a periodic GS re-orthogonalization scheme, which may also be easily adapted for many-dimensional systems involving time-discontinuous maps such as the dynamics of hard spheres [10].

A second and less familiar set of perturbation vectors are the covariant vectors \{+v_\ell\}, \ell \in \{1, \ldots, D\}, which are of unit length but generally not orthogonal to each other. They evolve in tangent space according to the non-constrained linearized motion equations. Still required is a periodic re-normalization, which also generates the corresponding Lyapunov exponents. These vectors constitute a practical realization of the Oseledec splitting [1–3] of the tangent space into a hierarchy of stable and unstable subspaces at any phase-space point visited by the trajectory.

In addition to these sets connected with the tangent flow forward in time, there exist corresponding sets, if the dynamics is followed backward in time. They will be distinguished by the index (−). Their application gives rise to the orthonormal forward GS Lyapunov vectors \{−g_\ell\}, which are conventionally called ‘forward’ since they depend on their history in the future. In general, they are not simply related to the set of backward GS vectors. Similarly, there exists a set of time-reversed covariant vectors \{−v_\ell\}, which, however, for time-reversal invariant dynamics agrees with its time-forward counterpart up to a simple reversal of the indices, \ell \rightarrow D + 1 − \ell.

While the GS vectors have been central to any algorithm since the pioneering days of the numerical stability analysis of dynamical systems [7–9] more than 30 years ago, the covariant vectors proved rather elusive due to the Lyapunov instability of the computational process itself. Practical schemes for the computation of covariant vectors have been developed only recently [11, 27]. A few studies of their properties for various systems have appeared since then [4, 13–20]. In the following sections we shall study these properties for two simple symplectic systems, the chaotic spring pendulum and the Hénon–Heiles system. To establish our notation, we first summarize the most important relations below.

2. Definitions and notation

If \( \Gamma(t) \) denotes the state of a dynamical system of dimension \( D \), its evolution equation and formal solution are given by

\[ \dot{\Gamma} = F(\Gamma), \quad \Gamma(t) = \phi^t(\Gamma(0)) \]

where \( F \) is a (generally nonlinear) vector-valued function of dimension \( D \), and the map \( \phi^t \) defines the phase flow. The linearized evolution equation and the formal solution for an arbitrary perturbation vector \( \delta \Gamma(t) \) in tangent space become

\[ \delta \dot{\Gamma} = J(\Gamma) \delta \Gamma, \quad \delta \Gamma(t) = D\phi^t_{\Gamma(0)} \delta \Gamma(0) \]

where \( J(\Gamma) = \partial F / \partial \Gamma \), and where \( D\phi^t_{\Gamma(0)} \) denotes the flow in tangent space. As already mentioned, the covariant vectors evolve—co-rotate in particular—according to the unconstrained tangent-space flow,

\[ v^\ell(t) = \frac{D\phi^t_{\Gamma(0)} v^\ell(\Gamma(0))}{\|D\phi^t_{\Gamma(0)} v^\ell(\Gamma(0))\|}. \]

Of course, the algorithm has to ascertain that the initial vector \( v^\ell(\Gamma(0)) \) is already properly oriented and normalized in order to qualify as covariant. The stretching factor in the
denominator of equation (3) provides the general definition for the (global) Lyapunov exponents:

\[
(\pm)\bar{\lambda}_\ell = \lim_{t \to \pm \infty} \frac{1}{|t|} \ln \|D\phi^t|_{\Gamma(0)} v' (\Gamma(0)) \|, \quad \ell \in \{1, \ldots, D\}.
\]

(4)

Under very mild conditions, the multiplicative ergodic theorem of Oseledec [1, 3] asserts that the global Lyapunov exponents are dynamical invariants and do not depend on the norm and, hence, the particular parameterization in phase space.

Two other limits of equation (4) are of special interest:

- \( \tau \) large but finite:

\[
\Lambda^\text{cov}_\ell (\tau) = \lim_{\tau \to \infty} \frac{1}{|\tau|} \ln \|D\phi^\tau|_{\Gamma(0)} v' (\Gamma(0)) \|,
\]

(5)

These objects depend on \( \tau \) and are referred to as finite-time Lyapunov exponents (FTLEs). For infinite time they converge to the global exponents \( \bar{\lambda}_\ell \). For finite \( \tau \) they are no dynamical invariants and depend, for example, on the coordinate system in use. However, it was recently shown by Kuptsov and Politi [20] that the fluctuations of these quantities, i.e. the linear growth rate of the covariances of the logarithmic expansion factors \( \tau \Lambda^\text{cov}_\ell (\tau) \)

\[
D^\text{cov}_\ell (\tau) = \lim_{\tau \to \infty} [\langle \Lambda^\text{cov}_\ell (\tau) \rangle - \bar{\lambda}_\ell \bar{\lambda}_\ell'] \tau,
\]

(6)

is a dynamical invariant for large-enough \( \tau \). Here, \( \langle \ldots \rangle \) denotes an average over (infinitely) many uncorrelated realizations along a trajectory. Below we shall demonstrate this property for the chaotic pendulum.

- \( \tau \to 0 \): The limit

\[
\Lambda^\text{cov}_\ell (\Gamma(0)) = \lim_{\tau \to 0} \frac{1}{|\tau|} \ln \|D\phi^\tau|_{\Gamma(0)} v' (\Gamma(0)) \|,
\]

(7)

provides a definition of the so-called local Lyapunov exponents (LLEs). They are point functions in phase space. Their time average over a long trajectory also converges to the global exponents \( \bar{\lambda}_\ell \). If the covariant vector \( v' (\Gamma) \) is known at a point \( \Gamma \) the corresponding covariant LLE follows from

\[
\Lambda^\text{cov}_\ell (\Gamma) = [v' (\Gamma)]^T J (\Gamma) v' (\Gamma),
\]

(8)

where \( T \) means transposition and \( J \) is the Jacobian of equation (2). This nicely underlines the local nature of the LLEs, which depend implicitly on time.

So far all definitions of Lyapunov exponents are in terms of covariant Lyapunov vectors. There are analogous definitions for the orthonormal GS vectors both forward and backward in time, which are summarized in [16] and are not repeated here. The indices GS and cov will be used in the following to distinguish between the quantities. The GS-FTLEs also converge to the global exponents with \( \tau \to \infty \), as do the GS-LLEs when time-averaged along a trajectory. It is interesting to note that the algorithm with continuously constrained orthonormality mentioned above [5, 6, 16] provides an expression for the GS-LLEs,

\[
\Lambda^\text{GS}_\ell (\Gamma) = [g' (\Gamma)]^T J (\Gamma) g' (\Gamma),
\]

(9)

which is analogous to equation (8) for the covariant case.

The classical algorithms involving GS re-orthonormalization keep track of the volume changes of \( d \)-dimensional volume elements in phase space \( (d \leq D) \). For symplectic systems, for which the phase volume is invariant, the GS-LLEs show the symplectic local pairing symmetry [21, 16, 17],

\[
^{(\pm)} \Lambda^\text{GS}_\ell (t) \equiv -^{(\pm)} \Lambda^\text{GS}_D+1-\ell (t)
\]

(10)
\[ (-)^{\ell} \Lambda_{G}^{GS}(t) = (-)^{\ell+1} \Lambda_{D}^{GS}(t), \]  
(11)
where \((-)^{\ell}\) and \((-)^{\ell+1}\) indicate whether the trajectory is followed forward or backward in time. Since the angle information is discarded by the GS-process, these exponents do not show time-reversal symmetry,

\[ (-)^{\ell} \Lambda_{G}^{GS}(t) \neq (-)^{\ell+1} \Lambda_{D}^{GS}(t). \]  
(12)
If the system is not symplectic, also equations (10) and (11) cease to exist. On the other hand, the orientation of the covariant vectors is not affected by any process such as re-orthogonalization, and the covariant LLEs display the time reversal symmetry [2, 16]:

\[ (-)^{\ell} \Lambda_{G}^{cov}(t) = (-)^{\ell+1} \Lambda_{D}^{cov}(t) ; \quad \ell = 1, \ldots, D. \]  
(13)
Similarly, the covariant vectors obey

\[ (-)^{\ell} v^{D+1-\ell}(t) = \pm (-)^{\ell+1} v^{D+1-\ell}(t). \]  
(14)
These relations apply whether or not the system is symplectic, as long as it is time reversible.

Another attractive property of the covariant vectors derives from the fact that they are spanning sets for the stable and unstable Oseledec subspaces associated with the respective Lyapunov exponents (one-dimensional without degeneracy, and \(m\)-dimensional for multiplicity \(m\)), and are defined without reference to a particular parameterization. This does not apply to the GS-exponents which suffer from the additional constraint of orthogonality. It has been shown recently by Yang and Radons that covariant vectors may be easily transformed from one coordinate system to another, and the same is true for the covariant finite-time and LLEs [4]. We shall demonstrate this property for the chaotic pendulum below by transforming from the Cartesian representation to a polar coordinate system.

In the following section we consider the chaotic spring pendulum, and in section 3 we discuss the Hénon–Heiles system. We close with a short survey of the merits and disadvantages of the covariant analysis.

3. The spring pendulum

We consider the planar chaotic motion of the spring pendulum [22, 23] as a simple example. It consists of a mass \(m\) attached to a harmonic spring with spring constant \(k\) and rest length \(R\), which is exposed to a homogeneous gravitational force of strength \(mg\) in the negative \(y\) direction. The spring is attached to a pivot at the origin. The Hamiltonian in Cartesian coordinates \((x, y)\) and with the conjugate momenta \((p_{x}, p_{y}) = m(\dot{x}, \dot{y})\) is given by

\[ H_{C} = \frac{p_{x}^{2} + p_{y}^{2}}{2m} + \frac{k}{2}(\sqrt{x^{2} + y^{2}} - R)^{2} + mg, \]  
(15)
from which follow the equations of motion for the reference trajectory in phase space,

\[ \dot{x} = p_{x}/m \]  
\[ \dot{y} = p_{y}/m \]  
\[ \dot{p}_{x} = k[(R/r) - 1]x \]  
\[ \dot{p}_{y} = k[(R/r) - 1]y - mg, \]  
(16)
and for the perturbation vectors in tangent space.

\[ \delta x = \delta p_{x}/m \]  
\[ \delta y = \delta p_{y}/m \]  
\[ \delta p_{x} = k[(R/r) - 1]\delta x - (kR/r^{3})(x\delta x + y\delta y) \]  
\[ \delta p_{y} = k[(R/r) - 1]\delta y - (kR/r^{3})(x\delta x + y\delta y), \]  
(17)
Here, \(r = \sqrt{x^{2} + y^{2}}\).
Introducing polar coordinates \((r, \phi)\) through

\[
x = r \sin \phi, \quad y = r \cos \phi,
\]

the Hamiltonian becomes

\[
H_P = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + \frac{k}{2}(r - R)^2 + mg \cos \phi,
\]

where \(p_r = mr\dot{r}\) and \(p_\phi = mr^2\dot{\phi}\) are the conjugate momenta. The equations of motion in phase space now read

\[
\begin{align*}
\dot{r} &= \frac{p_r}{m} \\
\dot{\phi} &= \frac{p_\phi}{mr^2} \\
\dot{p}_r &= \frac{2p_\phi}{mr^3} - \frac{3p_r^2}{mr^4} - k \delta r + mg \sin \phi \delta \phi \\
\dot{p}_\phi &= mg \sin \phi \delta r + mgr \cos \phi \delta \phi.
\end{align*}
\]

Let us consider the state point \(\Gamma_C = (x, y, p_x, p_y)\), which in the polar coordinate system becomes \(\Gamma_P = (r, \phi, p_r, p_\phi)\). This, according to equation (18), is accomplished by the transformation

\[
\Gamma_P = \mathcal{P}(\Gamma_C) = \left(\sqrt{x^2 + y^2}, \tan^{-1}\left(\frac{x}{y}\right), \frac{x p_y + y p_x}{\sqrt{x^2 + y^2}}, (p_x, y - x p_y)\right)^T.
\]

Any Cartesian covariant vector \(\mathbf{v}_C^\ell\) is transformed to the polar representation according to

\[
\delta \Gamma_P^\ell = \mathcal{M} \delta \Gamma_C^\ell,
\]

where \(\mathcal{M} = \partial \mathcal{P}/\partial \Gamma_C\) is the Jacobian of the transformation (22). A final normalization,

\[
\mathbf{v}_P^\ell = \pm \frac{\delta \Gamma_P^\ell}{\|\delta \Gamma_P^\ell\|} = \pm \frac{\mathcal{M} \mathbf{v}_C^\ell}{\|\mathcal{M} \mathbf{v}_C^\ell\|},
\]

yields the covariant vector in the polar representation. Although formulated in terms of a particular coordinate transformation, the expressions (23) and (24) are completely general [4].

For our numerical work we use reduced units for which the mass \(m\), the rest length \(R\), and the gravitational acceleration \(g\) are unity. The spring constant \(k\) is set to two. For the initial condition of the reference trajectory we take \((x, y, p_x, p_y) = (0.00001, 1, 0, 0)\), which determines the energy. Care must be taken that the reference trajectory in the Cartesian and polar representations coincide. Therefore, only equations (16) are integrated, and the solutions of the polar equations (20) are obtained by the Cartesian-to-polar transformation (22). The computation of the covariant vectors and their associated LLEs has been outlined in [11, 15, 16], to which we refer for details. The integration is carried out with a fourth-order Runge–Kutta algorithm with a time step of 0.002. The global Lyapunov spectrum is found to be \([0.0565, 0, 0, -0.0565]\).

By the smooth (red) line in figure 1 we show a projection of a short trajectory into the three-dimensional subspace spanned by \(x, y, p_x\). The covariant vector \(\mathbf{v}^1\), which spans the unstable manifold, is indicated by short solid (green) lines. Similarly, the covariant vector \(\mathbf{v}^4\) spanning
Figure 1. Projection of a short spring-pendulum trajectory onto the \((x, y, p_x)\)-subspace. The short solid (green) and dashed (blue) lines represent projections of the covariant vectors \(v^1\) and \(v^4\), respectively, and give an impression of the respective unstable and stable manifolds along the trajectory. The time interval for this trajectory segment is 16 time units.

the stable manifold is shown by dashed (blue) lines. Of course, these projected vectors are not of unit length. The figure may provide an intuitive understanding of the directions of maximum stretching or contraction of perturbations in phase space.

The time evolution of the polar components of \(v^1\) is depicted in figure 2. This vector was chosen for display since it is the most time consuming to compute and has a non-vanishing global exponent. The smooth (red) lines are the results of a direct integration of equation (21) within the polar environment, whereas the dashed (green) lines are obtained by converting the Cartesian covariant vector \(v^4_C\) to its polar representation. As is indicated in equation (24), the sign is irrelevant since only the sense of direction counts. The same agreement is also obtained for the other covariant vectors (not shown).

The FTLEs (including the limiting case of local exponents) in the Cartesian and polar representations are also intimately connected. Let us consider a time element of duration \(t_{n+1} - t_n = \tau\). The Cartesian covariant vector \(v^\ell_C(t_n)\) evolves according to equations (3) and (5) into a vector \(\exp\{-\Lambda^\ell_C(\tau)\}v^\ell_C(t_{n+1})\), which—according to equation (23)—may be transformed to the polar representation to give

\[
\exp\{-\Lambda^\ell_P(\tau)\}M(t_{n+1})v^\ell_C(t_{n+1}).
\]

Alternatively, \(v^\ell_C(t_n)\) at \(t_n\) may be first transformed to the polar representation, \(M(t_n)v^\ell_C(t_n)\), and then be evolved in time to the end of the interval, which yields the vector

\[
\exp\{-\Lambda^\ell_P(\tau)\}M(t_n)v^\ell_C(t_n) = \exp\{-\Lambda^\ell_P(\tau)\}M(t_n)\left\|M(t_n)M(t_{n+1})\right\|v^\ell_P(t_{n+1}).
\]

Equating these two expressions and taking the absolute value on both sides yields

\[
\Lambda^\ell_P = \Lambda^\ell_C + \frac{1}{\tau} \ln \frac{\left\|M(t_{n+1})v^\ell_C(t_{n+1})\right\|}{\left\|M(t_n)v^\ell_C(t_n)\right\|}.
\]

This provides the relation between the covariant local exponents in the two coordinate systems. Although expressed here in terms of the Cartesian and polar coordinates, the expressions (24) and (25) are completely general.

Equation (25) clearly shows that the local exponents differ for different coordinate systems [4]. To test the last relation, we show by the smooth (red) lines in figure 3 the local (time-dependent) covariant exponents in the polar representation for the unstable manifold \((\ell = 1,\)
Figure 2. Time evolution along the trajectory segment depicted in figure 1 of the four polar components $\delta r, \delta \phi, \delta p_r, \delta p_\phi$ (from top left to bottom right) for the covariant vector $v^4_p$ spanning the stable manifold of the spring pendulum. The smooth (red) lines are obtained by direct integration with polar coordinates, whereas the dashed (green) curves were converted from the Cartesian representation.

Figure 3. Comparison of time dependent local covariant exponents for the unstable ($\ell = 1$, left panel) and stable ($\ell = 4$, right panel) manifolds of the spring pendulum. The smooth (red) lines are obtained by numerical integration of the polar equations of motion, the dotted (blue) lines by integration of the Cartesian equations of motion, followed by a conversion to the polar representation. The agreement is very good. For comparison, the corresponding Cartesian exponents, which are the basis of the conversion, are shown by the dashed (green) lines.

left panel) and for the stable manifold ($\ell = 4$, right panel) of the spring pendulum. They are obtained by direct integration of the motion equation (21) with polar coordinates. For comparison, the corresponding Cartesian exponents are plotted by the dashed (green) lines. If
Figure 4. Chaotic pendulum: $\tau$ dependence of the growth rate of the covariance matrix for the logarithmic expansion $\tau \Lambda (\tau)$. Only matrix elements not involving vanishing global exponents are shown. The panel on the right-hand side is a magnification of the small $\tau$ regime. Red full points: Cartesian coordinates and GS exponents (case GS-C); blue full squares: polar coordinates and GS exponents (case GS-P); green crosses: Cartesian coordinates and covariant exponents (case COV-C); violet open circles: polar coordinates and covariant exponents (case COV-P). In all cases the evolution is in the direction of the positive time axis.

the latter are transformed to the polar representation, the dotted (blue) lines are obtained. The perfect match with the smooth (red) lines of the strictly polar approach asserts the validity of equation (25).

In figure 4 we show the (linear) growth rate of the covariance matrix for the logarithmic expansion $\tau \Lambda (\tau)$ as a function of the averaging interval $\tau$ for various representations. For the GS-vectors this rate matrix is given by [20]

$$D_{\ell\ell'}^{\text{GS}}(\tau) = \lim_{\tau \to \infty} \left[ \langle \Lambda_{\ell}^{\text{GS}}(\tau) \Lambda_{\ell'}^{\text{GS}}(\tau) \rangle - \bar{\lambda}_{\ell} \bar{\lambda}_{\ell'} \right] \tau, \quad (26)$$

where we still have to distinguish the cases with Cartesian or polar coordinates. For covariant vectors the definition is given by equation (6). For clarity we restrict ourselves to matrix elements, which do not involve any of the vanishing Lyapunov exponents $\bar{\lambda}_2$ and $\bar{\lambda}_3$. The panel on the right-hand side is a magnification of the small-$\tau$ regime of the left panel. The elements for GS FTLEs with Cartesian coordinates are indicated by red full points (case GS-C), those with polar coordinates by the blue full squares (case GS-P). Similarly, the elements involving covariant FTLEs with Cartesian coordinates are indicated by green crosses (case COV-C), those with polar coordinates by violet open circles (case COV-P). The following observations are made:

(i) There exists the general symmetry $D_{11}(\tau) = D_{44}(\tau)$ and $D_{14}(\tau) = D_{41}(\tau)$ for any coordinate system and for any vector set used, be it GS or covariant. Whereas the latter equality is trivial, the former is not in view of the fact that neither $v^1, v^4$ nor $g^1, g^4$ are simply related to each other.

(ii) $D_{11}(\tau)$ for GS-C and COV-C agree for all $\tau$, and similarly $D_{11}(\tau)$ for GS-P and COV-P. This is to be expected, since the covariant vector $v_1$ and the GS vector $g_1$ associated with the maximum exponent are identical by construction.

(iii) For small averaging intervals $\tau$, the fluctuations of the finite-time exponents as measured by $D_{11}(\tau)$ and $D_{14}(\tau)$ significantly differ for the cases GS-C and GS-P, as has been observed before [22]. A similar observation is made for the cases COV-C and COV-P. These differences gradually disappear for averaging times $\tau > 10$.

(iv) The cross correlation $D_{14}(\tau)$ for the covariant and GS vector sets differ enormously for small $\tau$, regardless what coordinate system is used. The difference is even larger between
Figure 5. LLEs for the Hénon–Heiles system with energy $H = 1/6$. Left panel: illustration of the symplectic local pairing symmetry, equations (10) and (11), for the GS exponents $\Lambda_{2}^{GS}$ and $\Lambda_{3}^{GS}$ (smooth red lines). The dimension of phase space $D = 4$. The inequality equation (12) is demonstrated by the dashed green line for $\Lambda_{3}^{GS}$. Right panel: verification of the time-reversal invariance property (13) for the local covariant exponents specified.

COV-P and GS-P than between COV-C and GS-C. By increasing $\tau$ these differences very slowly disappear.

(v) From the left panel of figure 4 one infers that for $\tau > 200$ all elements of the covariance matrix for the FTLEs become independent of the choice of the set of perturbation vectors and of the coordinate system. Thus, $\lim_{\tau \to \infty} D(\tau)$ becomes a dynamical invariant independent of the metric and parameterization [20]. For high-dimensional chaotic systems such a result has been interpreted as evidence for effective hyperbolicity of the dynamics and of the statistical insignificance of hyperbolic tangencies.

4. The Hénon–Heiles system

As a second example, we consider the familiar Hénon–Heiles system [24] with a Hamiltonian in Cartesian coordinates

$$H_C = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3. \tag{27}$$

For an energy $1/6$, the system is known to be chaotic (with a global Lyapunov spectrum $\{0.127, 0, 0, -0.127\}$), where the trajectory visits most of the accessible phase space [21, 25]. The LLEs display the symplectic pairing symmetry equations (10) and (11), and the time reversal symmetry of equation (13), as is demonstrated in the respective left and right panels of figure 5.

In polar coordinates defined by $x = r \cos \phi$ and $y = r \sin \phi$ the Hamiltonian becomes

$$H_P = \frac{p_r^2}{2} + \frac{p_\phi^2}{2r^2} + \frac{r^2}{2} + \frac{r^3}{3} \sin 3\phi, \tag{28}$$

where, as before, $p_r = \dot{r}$ and $p_\phi = r^2 \dot{\phi}$ are the conjugate momenta. With an analysis completely analogous to that in the previous section for the spring pendulum, the covariant vectors in the polar representation may be obtained. Here, it suffices to present in figure 6 only results for the polar covariant vector $v_1^P$ associated with the maximum exponent. One observes that the covariant vectors are unique (up to the sign as the example in the previous section testifies), and that it does not matter what parameterization is used for their computation.
5. Résumé

There are many recent applications of Lyapunov vectors, which range from weather forecasting [28], geophysical applications [27], and studies of transport in turbulent flow [29, 30], to identifying and probing rare chaotic events by Lyapunov-weighted dynamics [31] and/or path sampling [32].

From the two sets of perturbation vectors commonly used—the orthonormal GS vectors and the co-moving covariant vectors—the latter are more attractive from a physics point of view. They reflect time-reversal invariance and, as is shown here, they are unique and independent of the choice of the coordinate system. By uniqueness we mean that they are properties of the flow in tangent space, and when they are known in a particular frame, they may be easily converted to another. They provide a spanning set for the Oseledec splitting of the tangent space into a hierarchy of subspaces, each characterized by a well-defined Lyapunov exponent. In this sense, they are a property of the physical system. This is not the case for the more artificial orthonormal GS vectors, which are a mathematical device for computing volume changes of \(d\)-dimensional volume elements, \(d \leq D\), in the \(D\)-dimensional phase space. Their advantage is that they are easier and faster to compute. According to the algorithm of Ginelli et al [11], the computation of the covariant vectors at a phase-point \(\Gamma\) requires the knowledge of the GS vectors for all points on the trajectory through \(\Gamma\) both in the past and the future [2]. Due to the availability of fast computers and programming techniques, rather high-dimensional systems have already been examined [4, 14, 15, 33].
For many problems only the first Lyapunov vector associated with the maximum (global) exponent is required. Since the first GS and the first covariant vectors always agree, one gets away with the fast computation of the first GS vector, which is covariant. This advantage may dissolve again if the application requires the knowledge of the maximum local exponent together with its covariant vector. Due to possible exponent entanglement [13, 16] the maximum local exponent at a particular instant may belong, say, to the fifth vector. In such a case more than one covariant vectors need to be computed.

Acknowledgments

The author acknowledges stimulating discussions with Hadrien Bosetti [26] and William G Hoover and some very constructive remarks by an anonymous referee.

References

[1] Oseledec V I 1968 Tr. Moskow Mat. Obshch. 19 179
   Oseledec V I 1968 Trans. Moscow Math. Soc. 19 197 (Engl. trans.)
[2] Ruelle D 1979 Publ. Math. l’IHÉS 50 27
[3] Eckmann J-P and Ruelle D 1985 Rev. Mod. Phys. 57 617
[4] Yang H-I and Radons G 2010 Phys. Rev. E 82 046204
[5] Posch H A and Hoover W G 1988 Phys. Rev. A 38 473
[6] Goldhirsch I, Sulem P-L, and Orszag S A 1987 Physica D 27 311
[7] Benettin G, Galgani L, Giorgilli A and Strelcyn J-M 1980 Meccanica 15 21
[8] Shimada I and Nagashima T 1979 Prog. Theor. Phys. 61 1605
[9] Wolf A, Swift J B, Swinney H L and Vastano J A 1985 Physica D 16 285
[10] Dellago C, Posch H A and Hoover W G 1996 Phys. Rev. E 53 1485
[11] Ginelli F, Poggi P, Turchi A, Chaté H, Livr R and Politi A 2007 Phys. Rev. Lett. 99 130601
[12] Wolfe C L and Samelson R M 2007 Tellus A 59 355
[13] Yang H-I and Radons G 2008 Phys. Rev. Lett. 100 024101
[14] Yang H-I, Takeuchi K A, Ginelli F, Chaté H and Radons G 2009 Phys. Rev. Lett. 102 074102
[15] Bosetti H and Posch H A 2010 Chem. Phys. 375 206
[16] Bosetti H, Posch H A, Dellago C and Hoover W G 2010 Phys. Rev. E 82 046218
[17] Posch H A and Bosetti H 2011 AIP Conf. Proc. 1332 230
[18] Hoover W G and Hoover C 2011 Commun. Nonlinear Sci. Numer. Simul. 17 1043
[19] Yang K A, Takeuchi H-I, Ginelli F, Radons G and Chaté H 2011 Phys. Rev. E 84 046214
[20] Kuptsov P V and Politi A 2011 Phys. Rev. Lett. 107 114101
[21] Ramaswamy R 2002 Eur. Phys. J. B 29 339
[22] Posch H A, Hoover W G and Hoover C G 1990 Phys. Rev. A 41 2999
[23] Dellago C and Hoover W G 2000 Phys. Lett. A 268 330
[24] Hénon M 1976 Commun. Math. Phys. 50 69
[25] Lichtenberg A J and Lieberman M A 1992 Regular and Stochastic Motion (New York: Springer)
[26] Bosetti H 2011 On the microscopic dynamics of particle systems in and out of thermal equilibrium PhD Thesis
   University of Vienna
[27] Wolfe C L and Samelson R M 2007 Tellus A 59 355
[28] Legras B and Vautard R 1996 Predictability vol 1, ed T Palmer (European Centre for Medium-Range Weather
   Forecasts) pp 135–46
[29] Balkovsky E and Fouxon A 1999 Phys. Rev. E 60 4164
[30] Fouxon A and Lebedev V 2003 Phys. Fluids 15 2060
[31] Tailleur J and Kurchan J 2007 Nature Phys. 3 203
[32] Geiger P and Dellago C 2010 Chem. Phys. 375 309
[33] Bosetti H and Posch H A 2013 J. Phys. A: Math. Theor. 46 254011