On extremal transitions of Calabi-Yau threefolds
and the singularity of the associated 7-space from rolling

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(In memory of Professor Frederick J. Almgren Jr.)

Abstract

M-theory compactification leads one to consider 7-manifolds obtained by rolling Calabi-Yau threefolds in the web of Calabi-Yau moduli spaces. The resulting 7-space in general has singularities governed by the extremal transition undergone. After providing some background in Sec. 1, the simplest case of conifold transitions is studied in Sec. 2. In Sec. 3 we employ topological methods, Smale's classification theorem of simply-connected spin closed 5-manifolds, and a computer code in the Appendix to understand the 5-manifolds that appear as the link of the singularity of a singular Calabi-Yau threefolds from a Type II primitive contraction of a smooth one. From this we obtain many locally admissible extremal transition pairs of Calabi-Yau threefolds, listed in Sec. 4. Their global realization will require further study. As a mathematical byproduct in the pursuit of the subject, we obtain a formula to compute the topology of the boundary of the tubular neighborhood of a Gorenstein rational del Pezzo surface embedded in a smooth Calabi-Yau threefold as a divisor.

Key words: M-theory, Calabi-Yau threefold, handle decomposition, extremal transition, del Pezzo surface, A-D-E singularity, Smale, simply-connected spin 5-manifold.

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“Anything that’s worth doing is worth doing badly.”

—— Frederick J. Almgren Jr. (1933-1997)

Notes from C.H.L. [In memory of a teacher]. I got some acquaintance with Prof. Almgren from three occasions at Princeton: firstly as a TA for a session of calculus course that he taught also; secondly in his joint course/seminar with Prof. Frank Morgan; and thirdly in an annual Mayday relay between the math departments of Princeton and Rutgers, in which he, his wife Prof. Jean Taylor, and their daughter all participated. After the relay we had a picnic at his home. Since he is one of the few professors at Princeton, whom I had more contact during my study there, I cannot help but feeling bad when knowing of his passing away the spring 1997. However, it was not until I read the above quote from Notices half a year later, as recalled by his son, that I realized that no wonder he has such a warmth that projects also to people around him. What a power both to work and to life that can be hidden in this motto! And what an encouragement this motto provides to a bewildered student who tries to understand the amazingly beautiful and yet bafflingly difficult Final Theory of Nature - Superstrings! Thus I like to dedicate this humble piece of work to him for an unforgettable memory and gratitudes.
0. Introduction and outline.

Introduction.

M-theory anticipates the space-time to be 11-dimensional compactified on a 7-dimensional space. If one requires this 7-space to be closed and satisfy the first order constraint from the string $\beta$-function, i.e. Ricci flatness, then recently Joyce has constructed a class of closed 7-manifolds that admit torsion-free $G_2$ structures and, hence, Ricci flat metrics ([Jo1], [Jo2], [D-T]). On the other hand, from the spirit of the work by Horava and Witten ([H-W1], [H-W2]), more likely this 7-space is compact yet with boundary that are Calabi-Yau complex threefolds. While this 7-space itself serves geometrically as an interpolating space of different Calabi-Yau threefolds, the M-theory built thereon serves physically as an interpolating theory for the string theories on the boundary Calabi-Yau threefolds. If one is willing to accept this point of view, then certainly a key question is that; What could this 7-space be?

First notice that the oriented cobordism ring is trivial in dimension 6 ([M-S]), which means that any two oriented closed 6-manifolds, in particular Calabi-Yau threefolds, bound an oriented 7-manifold. From this point view and following Joyce’s spirit, a possible kind of 7-space that could be relevant are Ricci-flat compact 7-manifolds with Calabi-Yau boundaries.

On the other hand, in view of the phenomenon of enhanced gauge symmetry for strings moving in a singular Calabi-Yau threefolds, one may allow this 7-space to have some reasonable kind of singularities, i.e. non-manifold points. Taking all these into account and balancing them, one leads to a very special class of 7-spaces that are closely related to Calabi-Yau threefolds: 7-space from rolling Calabi-Yau threefolds in the web of Calabi-Yau moduli spaces. Among many details one may like to understand for such 7-spaces, we focus in this paper on one single issue: its possible isolated singularities, which arise when the degenerate Calabi-Yau threefold that appears in the extremal transition (i.e. rolling through a shared boundary point of one Calabi-Yau moduli space and entering another) has an isolated singularity.

After providing some mathematical background in Sec. 1 for physicists, we discuss in Sec. 2 the resulting singularity in the 7-space from rolling when passing through a conifold. We then move on in Sec. 3 and 4 to consider the more complicated extremal transitions that involve pinching a del Pezzo surface embedded in a Calabi-Yau threefold to a point. After showing that the boundary 5-manifold of a tubular neighborhood of a certain class of del Pezzo surfaces embedded in a smooth Calabi-Yau threefold is simply-connected and working out its second integral homology, using Smale’s classification theorem of smooth simply-connected spin closed 5-manifolds, we can lay down many topologically locally admissible extremal transitions, each of which contributes to a possible isolated singularity in the 7-space from rolling. Exactly which of them will appear globally and, when it does, what is the relation of the 6-dimensional link of the singularity in the 7-space to the phenomenon of enhanced gauge symmetry will require further study in the future.
From a pure mathematical aspect, a main result that we obtain in this paper, which serves as a tool in the pursuit of the above issue, is the following:

**Proposition 3.2.8 [homology of boundary].** Let \( Z \) be a Gorenstein rational singular del Pezzo surface with \( \text{Pic}(Z) = \mathbb{Z} \) that is embedded in a smooth Calabi-Yau threefold \( X \) as a divisor. Denote the minimal resolution of \( Z \) by \( Z_r \), which is a smooth del Pezzo surface obtained by \( \mathbb{C}P^2 \) blown up at \( r \) many points, and the vanishing cycle of the resolution by \( E \) with irreducible components \( E_1, \ldots, E_r \). Then \( \partial \nu_X(Z) \) is a simply-connected spin 5-manifold with the second integral homology \( H_2 \) isomorphic to \( \mathbb{Z}^r / \{ \mathbb{Z}^r \cdot (A^t U^{-1} A) \} \), where \( U \) is the Seifert matrix associated to the singularity with respect to \( (E_1, \ldots, E_r) \) and \( A \) is the matrix whose \( i \)-th row is the coefficients of \( E_i \) with respect to any basis for \( K_{Zr}^\perp \) in \( H_2(Z_r; \mathbb{Z}) \). (In the above expression, elements in \( \mathbb{Z}^r \) are integral row vectors and the \( i \)-th column of \( U \) is the coefficients of \( U(E_i) \) with respect to \( (E_1, \ldots, E_r) \).)

**Convention.** Since both real and complex manifolds are involved in this article, to avoid confusion, a real \( n \)-dimensional manifold will be called an \( n \)-manifold while a complex \( n \)-dimensional manifold an \( n \)-fold.

**Outline.**

1. Essential mathematical backgrounds.
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   1.2 Essence of Calabi-Yau threefolds.
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   3.2 The case of a rational singular del Pezzo surface.
4. Topologically admissible singularities from pinching a del Pezzo surface.

Appendix. A computer code.
1 Essential mathematical backgrounds.

In Sec. 1.1 we collect some mathematical background involved in this article for the convenience of physicists. In Sec. 1.2 the essence of Calabi-Yau threefolds needed for the work is provided. Some expository articles or textbooks are also referred.

Caution. The term link has two different meanings in this paper: (1) a collection of $S^n$ embedded disjointly in $S^{2n+1}$; or (2) the base for a cone-type neighborhood of a cone point in a topological space, e.g. the link of any point in a closed $n$-manifold is an $(n-1)$-sphere. Both are standard and should be distinguishable from the context.

1.1 Geometric operations, Smale’s work on 5-manifold, and singularity.

- **General fundamentals.** Readers are referred to [B-T], [G-P], [Hirs], [Mu], [Sp], and [Sw] for algebraic and differential topology; and to [B-P-VV], [G-H], [Sh1], and [Sh2] for complex algebraic geometry.

- **Handle decompositions of manifolds.** ([G-S], [Ki], [Po], and [Sm1]; also [Mi] for the related Morse theory.) Let $D^r$ be the $r$-dimensional ball. Then a handle decomposition of a compact $n$-manifold $M^n$ is a sequence

$$D^n = M_0 \subset \cdots \subset M_{i-1} \subset M_i \subset \cdots \subset M_k = M^n,$$

where $M_i$ is obtained from $M_{i-1}$ by adding a $k_i$-handle, i.e. $M_i = M_{i-1} \cup f_i (D^{k_i} \times D^{n-k_i})$ where the attaching map $f_i : \partial D^{k_i} \times D^{n-k_i} \to \partial M_{i-1}$ is an embedding. In terms of Morse theory, this corresponds to a non-degenerate critical point with index $k_i$ of a Morse function on $M^n$. Its effect to the $(n-1)$-manifold $\partial M_{i-1}$ is that an embedded $(k_i-1)$-sphere - i.e. $f_i(\partial D^{k_i})$ - gets shrunk to a point and then gets expanded in transverse directions to an embedded $(n-k_i-1)$-sphere - i.e. $\partial D^{n-k_i}$ - in $\partial M_i$. (Figure 1-1).

![Figure 1-1](image)

Figure 1-1. In this illustration, a 1-handle (darkly shaded) is attached to $M_i$ on its boundary $\partial M_i$ (lightly shaded).

- **Fiber sum and plumbing.** ([Go], [G-S].) Let $\pi_i : E_i \to M_i^n$, $i = 1, 2$, be fibrations over $n$-manifolds with the same generic fibers $F$ and let $p_i \in M_i$ such that the fibers $\pi_i^{-1}(p_i)$
are generic. Take tubular neighborhoods $\nu_{E_i}(\pi_i^{-1}(p_i))$ of these fibers and a fiber-preserving homeomorphism $\rho$ between the boundary of $E_i - \nu_{E_i}(\pi_i^{-1}(p_i))$. One can then glue the latter spaces together by $\rho$ and construct a new space $E_1 \sharp f E_2$, which now admits a fibration over the connected sum $M_1^0 \sharp M_2^0$ with the same generic fiber $F$. This space $E_1 \sharp f E_2$ is called a fiber sum of $\pi_1 : E_1 \to M_1^0$ and $\pi_2 : E_2 \to M_2^0$. (Figure 1-2(a).)

Let $\pi : E \to M^n$ be a $D^n$-bundle over an $n$-manifold that may have several components. Let $(D_{11}, D_{12}), \ldots, (D_{k1}, D_{k2})$ be a disjoint collection of pairs of disjoint $n$-disks whose closure lie in the interior of $M^n$. Since $E$ restricted to these $n$-disks are the trivial bundle $D_j \times D^n$, one can identify $\pi^{-1}(D_{j1})$ with $\pi^{-1}(D_{j2})$ for each $j$ using a map that preserves the product structures but interchanges the factors. The resulting new manifold-with-boundary $W^{2n}$ is called a plumbing of $\pi : E \to M^n$ (Figure 1-2(b)). For example, when an $n$-manifold $M$ is immersed in a $2n$-manifold $N$ with only transverse crossings, its tubular neighborhood $\nu_{N}(M)$ is then a plumbing of the $D^n$-bundle associated to the normal bundle over $M$ from the immersion.

Associated to $W^{2n}$ is a diagram $\Gamma$, the plumbing diagram, whose vertices $v_i$ correspond to the connected components $M_i$ of $M^n$ and edges $e_j$ correspond to the occurrences of plumbing associated to the pairs $(D_{j1}, D_{j2})$. There is a natural continuous pinching map $\phi$ from $W^{2n}$ onto $\Gamma$ defined as follows: Let $\partial D_{js} \times [0, \epsilon]$ be a collar of $\partial D_{js}$ in $M^n - \cup_{j,s} D_{js}$. Define $\phi$ first on $M^n$ by sending $M_i - (all \ the \ n$-disks with the collar) to $v_i$, $D_{j1} \cup D_{j2}$ to the mid-point of $e_j$, and then extending the $\phi$ so far defined to the whole $M^n$ by the projection $\partial D_{js} \times [0, \epsilon] \to [0, \epsilon]$ with $[0, \epsilon]$ identified with the half-edge of $e_j$ at the vertex that corresponds to the component of $M^n$ that contains $D_{js}$. By pre-composition with the bundle projection $\pi$, one then obtains a map, still denoted by $\phi$, from $W^{2n}$ onto $\Gamma$. (Figure 1-2(b).)

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1-2.pdf}
\caption{In (a), a fiber sum of the fibrations $E_i \to M_i$ is shown. In (b), a plumbing of the two disk-bundles are shown. The associated plumbing diagram $\Gamma$ and the natural map $\phi$ are also indicated.}
\end{figure}
• Smale’s work on 5-manifolds. ([Sm2])

Fact 1.1.1 [Smale]. Smooth simply-connected spin closed 5-manifolds $M^5$ are classified by their second integral homology group $H_2(M^5; \mathbb{Z})$ and are realizable as the boundary of 6-dimensional handlebodies with only one 0-handle and some 3-handles. The detail follows:

(1) Let $H_2(M^5; \mathbb{Z}) = H_2^{\text{free}}(M^5; \mathbb{Z}) \oplus H_2^{\text{tor}}(M^5; \mathbb{Z})$ be a direct sum decomposition of $H_2(M^5; \mathbb{Z})$ into a free and torsion part. Then the torsion part is the direct sum of two identical groups. In notation,

$$H_2^{\text{tor}}(M^5; \mathbb{Z}) = \frac{1}{2} H_2^{\text{tor}}(M^{5}; \mathbb{Z}) \oplus \frac{1}{2} H_2^{\text{tor}}(M^{5}; \mathbb{Z}).$$

(2) The correspondence

$$M^5 \mapsto H_2^{\text{free}}(M^5; \mathbb{Z}) \oplus \frac{1}{2} H_2^{\text{tor}}(M^5; \mathbb{Z})$$

is bijective between the set of diffeomorphism classes of smooth simply-connected spin 5-manifolds and the set of isomorphism classes of finitely generated abelian groups.

(3) Let $H^6$ be the 6-dimensional handlebody obtained by one 0-handle attached with $k$ 3-handles. By handle sliding, one may assume that all the attaching $S^2$ are in the boundary $S^5$ of the 0-handle. Let $Q$ be the associated anti-symmetric linking matrix of these $S^2$’s in $S^5$. Recall that $Q$ can be transformed uniquely into the following block diagonal form ([Bou])

$$\text{Diag} \begin{bmatrix} 0 & \alpha_1 & \cdot \cdot \cdot & \cdot \cdot \cdot & 0 & \alpha_r & \cdot \cdot \cdot & \cdot \cdot \cdot & 0 \\ -\alpha_1 & 0 & \cdot \cdot \cdot & \cdot \cdot \cdot & -\alpha_r & 0 & \cdot \cdot \cdot & \cdot \cdot \cdot & 0 \end{bmatrix}, \quad \text{O}_{k-2r}$$

under the transformation $Q \rightarrow SQS^t$ for some $S$ in $GL(k, \mathbb{Z})$, where $\alpha_i$ are positive integers that satisfy $\alpha_1 | \alpha_2 | \cdot \cdot \cdot | \alpha_r$ and $\text{O}_{k-2r}$ is the $(k-2r) \times (k-2r)$ zero-matrix. Then

$$\partial H^6 \cong M_0^5 \sharp \cdot \cdot \cdot \sharp M_r^5 \sharp (k-2r) S^2 \times S^3,$$

where $M_0^5$ is the boundary of the 6-dimensional 3-handlebody obtained by one 0-handle attached with two 3-handles whose attaching 2-spheres form a link with linking matrix $\begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$. The 5-manifold $\partial H^6$ is simply-connected and spin; and conversely every closed simply-connected spin 5-manifold arises this way.

Remark 1.1.2. Note that $M_0^5 = S^2 \times S^3$, $M_1^5 = S^5$, and, for $\alpha > 1$, $H_2(M_\alpha^5; \mathbb{Z}) = \mathbb{Z}_\alpha \oplus \mathbb{Z}_\alpha$.

• Rational surface singularities. ([A-GZ-V], [B-P-VV], [Di], [Hi-K-K], [Mi2], [Sl1], and [Sl2].) These are exactly the quotient singularities $(\mathbb{C}^2/G, 0)$ for finite groups $G \subset SU(2)$
and hence their links are homeomorphic to $S^3/G$. Their resolution graphs coincide with the Dynkin diagrams of the simple complex Lie algebras of type $A_k$, $D_k$, $E_6$, $E_7$, and $E_8$ respectively. For this reason, they are also called the $A-D-E$ singularities. Such singularity is special in that its minimal resolution is the same as its smooth deformation ([H-K-K]), which is the Milnor fiber $F$ appears in the Milnor fibration associated to the singularity. The topology of $F$ is a plumbing of a collection of the tangent $D^2$-bundle of $S^2$ with the plumbing diagram the above mentioned Dynkin diagram. The boundary $\partial F$ of $F$ is the same as the link $S^3/G$ of the singularity. Recall that the total space of the Milnor fibration in the spherical form is a mapping torus $F \times [0,2\pi]/(F \times \{0\} \sim F \times \{2\pi\})$, by which the monodromy homeomorphism $\tau$ is defined. It is noteworthy that, up to homotopy, the restriction of $\tau$ to the boundary $\partial F$ is the identity map for these singularities. Readers are referred particularly to [Di] for more details.

**1.2 Essence of Calabi-Yau threefolds.**

- **Calabi-Yau threefolds.** ([A-G-M], [A-L], [B-C-dlO], [C-dlO], [C-G-H], [C-H-S-W], [Gr], [Hü], [Ti], [Vo], [Wil1]), and [Wil2]) A Calabi-Yau threefold $X$ is a compact Kähler threefold with $c_1 = 0$ (or equivalently trivial canonical line bundle $K_X$). They have moduli that form a space which itself is Kähler. Two different points in the Calabi-Yau moduli space $M_X$ associated to the smooth manifold $X$ can be connected by deformations of the complex and Kähler structures on $X$. The boundary points of $M_X$ correspond to singular Calabi-Yau threefolds $\overline{X}$ that can be obtained from $X$ by such deformations. Readers are referred particularly to [Gr] and [Hü] for more details.

- **Extremal transitions.** ([A-G-E], [C-G-G-K], [Gr], [G-M-S], [G-M-V], [Hü], [M-V1], [M-V2], and [Wil2].) Two Calabi-Yau moduli spaces $M_X$ and $M_{X'}$ can be connected with each other if they happen to share some common boundary points, which means that $X$ and $X'$ have a same degeneration $\overline{X}$. Indeed it has been conjectured that all the Calabi-Yau moduli spaces associated to different smooth topologies are related this way and form a connected web $M_{CY}$ of Calabi-Yau moduli spaces. Thus one may roll a smooth Calabi-Yau threefold $X$, degenerating it to a singular $\overline{X}$ and then rolling on to another smooth $X'$ with possibly different topology - all happening in $M_{CY}$. From mirror symmetry, the corresponding conformal field theory on these Calabi-Yau spaces that occur in the rolling may still be well-defined and smoothly deformed accordingly. Such transmutation is called an extremal transition of Calabi-Yau threefolds.

There are six generic types of degenerations of Calabi-Yau threefolds whose combinations lead to various extremal transitions discussed in the literature. To explain them, let $(X,Z) \rightarrow (\overline{X},\overline{Z})$ be a degeneration of a smooth $X$ to $\overline{X}$ by collapsing a subspace $Z$ of $X$ to $\overline{Z}$. In all these six cases, $Z$ is a fibration over $\overline{Z}$ and the collapsing comes from pinching all the fibers:

- From the deformation of the complex structures on $X$: C
(C₀) \( Z \) is an embedded 3-sphere \( S^3 \) in \( X \) and \( \overline{Z} \) is a point in \( \overline{X} \).

- From the deformation of the Kähler structures on \( X \) to one that sits in the codimension-one wall of the Kähler cone ([Wil2]):

(C₁) (Type I primitive contraction or small contraction): \( Z \) is a finite disjoint union of a collection of embedded \( \mathbb{CP}^1 \)'s in \( X \) and \( \overline{Z} \) is a finite set of points in \( \overline{X} \).

(C₂) (Type II primitive contraction): \( Z \) is a generalized del Pezzo surface in \( X \) and \( \overline{Z} \) is a point in \( \overline{X} \).

(C₃) (Type III primitive contraction): \( Z \) is a conic bundle in \( X \) over a complex curve \( C \) and \( \overline{Z} \) is the curve \( C \) embedded in \( \overline{X} \).

(C₄) \( X \) admits a K3 or \( T^4 \) fibration over a complex curve \( C \); \( Z = X \) and \( \overline{Z} = C \).

(C₅) \( X \) admits an elliptic fibration over a complex surface \( S \); \( Z = X \) and \( \overline{Z} = S \).

- 7-spaces from rolling Calabi-Yau threefolds. Continuing the above discussion, let \( X_t \to \overline{X} = X_0, \ t \in [-\epsilon, 0] \) be a rolling of Calabi-Yau threefolds that degenerate \( X = X_{-\epsilon} \) to \( \overline{X} \) by shrinking the fibers of a fibration \( \pi: Z \to \overline{Z} \) of a subspace \( Z \) in \( X \) over a subspace \( \overline{Z} \) in \( \overline{X} \). Naturally associated to this shrinking is a cone bundle \( \Lambda_{\overline{Z}} \) over \( \overline{Z} \), whose fiber over \( \overline{z} \in \overline{Z} \) is the cone over the restriction of a tubular neighborhood \( \nu_X(Z) \subset X \) on \( \pi^{-1}(\overline{z}) \). The resulting 7-space \( Y^-_7 \) is homeomorphic to \( X \times [-\epsilon, 0] \) with \( \Lambda_{\overline{Z}} \) attached canonically to \( X \times \{0\} \) along the common subspace \( \nu_X(Z) \) of both \( X \) and \( \Lambda_{\overline{Z}} \); in notation, \( Y^-_7 = X \times [-\epsilon, 0] \cup \nu_X(Z) \Lambda_{\overline{Z}} \) (Figure 1-3). When an extremal transition is obtained by a rolling \( (X, Z) \to (\overline{X}, \overline{Z}) \leftarrow (X', Z') \), parametrized by \( t \in [-\epsilon, \epsilon] \), where both the degenerations \( (X, Z), (X', Z') \to (\overline{X}, \overline{Z}) \) are of the six types, then the resulting 7-space, \( Y^-_7 \) from \( (X, Z) \to (\overline{X}, \overline{Z}) \) and \( Y^+_7 \) from \( (X', Z') \to (\overline{X}, \overline{Z}) \), can be sewed along the shared \( X \) and form a 7-space \( Y^+_7 \). This is our 7-space from rolling. From the discussion, it is clear that

**Lemma 1.2.1 [topological matching condition].** A necessary condition for an extremal transition \( (X, Z) \to (X', Z') \) to be possible is that the two 5-manifolds \( \partial \nu_X(Z) \) and \( \partial \nu_{X'}(Z') \) are homeomorphic.

When \( \overline{Z} \) is a point \( p_0 \), in general it induces then an isolated singularity, still denoted by \( p_0 \), in the 7-space \( Y^-_7 \), whose link is a 6-manifold \( \nu_X(Z) \cup \partial \nu_{X'}(Z') \), obtained by pasting \( \nu_X(Z) \) and \( \nu_{X'}(Z') \) along their homeomorphic boundary.

## 2 Lessons from conifold transitions.

A special case of examples of extremal transitions of Calabi-Yau threefolds that has been discussed quite extensively in physics literature is the conifold transition (cf. [C-G-G-K],
Figure 1-3. The 7-space $Y^7$ that arises from degenerating Calabi-Yau threefolds is topologically realizable as $X \times [-\epsilon, 0] \cup_{\nu_X(Z)} \Lambda Z$.

(In general, the fibration $\pi : Z \to \mathcal{Z}$ may have singular fibers.)

[Gr], [G-M-S], [Hü], and [Str]). Hence let us start with this example and try to extract some lessons from it.

2.1 Conifold transition I.

A complex 3-dimensional conifold is a threefold except at some ordinary double points (nodes). Each of these nodes is modelled on the germ of the quadric hypersurface $xy - uv = 0$ in $\mathbb{C}^3$ at the origin. This is an $A_1$-singularity and can be realized as a real cone over $S^2 \times S^3$. Such degeneration arises from either process (C0) or (C1). Let $\overline{Z} = \{p_0\}$ and $Y$ be the 7-space associated to the transition $(C_0C_1^{-1})$ that deforms first the complex structure of $X$ to shrink down an $S^3$ to get a conifold $X$ and then resolves the singularity by a small resolution that blows up an $S^2$ transverse to the blown-down $S^3$ to obtain a new non-singular $X'$. Then $p_0$ is at worst a cone-point in $Y$ with the link coming from the pasting of $\nu_X(S^3)$ in $X$ and $\nu_{X'}(S^2)$ in $X'$ along their boundary. From [At], the local topology around the node $p_0$ in $\overline{X}$ indicates that the former is homeomorphic to $S^3 \times D^3$, the latter homeomorphic to $S^2 \times D^4$, and the pasting homeomorphism $h$ from $\partial \nu_X(S^3)$ to $\partial \nu_{X'}(S^2)$ is given by

$$h : S^3 \times S^2 \longrightarrow S^2 \times S^3$$

$$(a, b) \mapsto (b, a)$$

This shows that indeed the link of $p_0$ is a 6-sphere $S^6$ since

$$S^6 = \partial D^7 = \partial(D^4 \times D^3) = S^3 \times D^3 \cup_{S^3 \times S^2} D^4 \times S^2.$$
Furthermore, comparing with the setting of handle-decomposition/Morse-theory of manifolds, one notices that the blown-down \( S^3 \) may be regarded as the attaching/descending sphere while the blown-up \( S^2 \) as the belt/ascending sphere of a 4-handle \( D^4 \times D^3 \) in 7 dimensions (cf. [G-S], [Ki]). In other words, \( Y \) is simply \( X \times [-\epsilon, 0] \) attached with a 4-handle to the boundary \( X \times \{0\} \).

Similarly the reverse process \( (C_1C_0^{-1}) \) from \( X' \) to \( X \) corresponds to attaching a 3-handle \( D^3 \times D^4 \) to the boundary \( X' \times \{0\} \) of \( X' \times [0, \epsilon'] \). In summary

**Lemma 2.1.1 [conifold I].** The conifold transition of type either \( (C_0C_1^{-1}) \) or \( (C_1C_0^{-1}) \) for Calabi-Yau threefolds does not lead to non-manifold points on the resulting 7-space \( Y \).

**Remark 2.1.2.** As we shall see that this is likely the only case of extremal transitions that does lead to 7-manifolds without any singularities. In view of the rich variations of the topologies of Calabi-Yau threefolds, this already provides us with a rich class of 7-manifolds that could play roles in M-theory.

### 2.2 Conifold transition II: flops.

On the other hand, the neighborhood of a node on a complex 3-conifold can also be realized as a complex cone over \( \mathbb{C}P^1 \times \mathbb{C}P^1 \). One may deform the Kähler structure of \( X \) in such a way that first blows down \( \mathbb{C}P^1 \) to get a conifold \( \overline{X} \) and then blows up another \( \mathbb{C}P^1 \) transverse to the previous \( \mathbb{C}P^1 \). Such procedure is called a *flop* and corresponds to a nontrivial process \( (C_1C_1^{-1}) \). Let \( \overline{Z} = \{p_0\} \) and \( Y \) the resulting 7-space. Then again \( p_0 \) is at worst a cone-point in \( Y \). Its link \( \text{Lk}(p_0) \) in \( Y \) now comes from the pasting of \( \nu_X(S^2) \) and \( \nu_X'(S^2) \) along their boundary \( S^2 \times S^3 \). To see what this 6-manifold is, we need to recall some details of this flop from [At].

Let \( L \) be the tautological line bundle over \( \mathbb{C}P^1 \) and \( V \) be the quadric surface \( \{xy - uv = 0\} \). The variety \( V \) contains two transverse families of complex lines. Each of the families is parametrized by \( \mathbb{C}P^1 \) and each gives a ruling for \( V \). Let us denote the first family by \( \mathbb{C}P^1_\alpha \) and call the lines it parameterizes by *\( \alpha \)-lines*. Similarly for \( \mathbb{C}P^1_\beta \) and *\( \beta \)-lines* ([G-H], [W-W]). This gives an isomorphism between \( V \) and \( \mathbb{C}P^1_\alpha \times \mathbb{C}P^1_\beta \). In terms of these, the line bundle \( L \) is represented by \((-1, -1)\) in the Picard group \( \text{Pic}(V) = H_2(V; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \) with the canonical basis given by the pair of homology classes \( ([\mathbb{C}P^1_\alpha], [\mathbb{C}P^1_\beta]) \). If one pinches each of the \( \beta \)-lines in the zero-section of \( L \) to a point, then the result is a smooth complex threefold \( E_\alpha \) isomorphic to the total space of \( L_1 \oplus L_1 \) over \( \mathbb{C}P^1_\alpha \). On the other hand, if one pinches each of the \( \alpha \)-lines in the zero-section of \( L \) to a point, then the result is a smooth complex threefold \( E_\beta \) isomorphic to the total space of \( L_1 \oplus L_1 \) over \( \mathbb{C}P^1_\beta \). Pinching further the zero-section of \( E_\alpha \) or the zero-section of \( E_\beta \) is the same as pinching the zero-section of \( L \) altogether to a point. The resulting space is exactly the complex cone \( OV \) in \( \mathbb{C}^3 \) over \( V \), which describes the neighborhood of the node \( p_0 \) in \( \overline{X} \).

Transforming from \( E_\alpha \) to \( OV \) and then to \( E_\beta \) is precisely the type-II conifold transition described above.
From this picture, one has
\[ \text{Lk}(p_0) \cong \partial \left( \nu_{\mathbb{L}_3}|_V \text{ (zero-section)} \right) \times [0, 1] / \sim, \]

where the relation \( \sim \) is defined by first regarding \( \partial \left( \nu_{\mathbb{L}_3}|_V \text{ (zero-section)} \right) \), denoted by \( M^5 \) in the following discussion, naturally as an \( S^1 \)-bundle over \( V \) and then performing the following pinchings:

1. Pinch each of the \( S^1 \)-fibers of \( M^5 \times \{0\} \) to a point, which transforms \( M^5 \times \{0\} \) to \( \mathbb{CP}^1_\alpha \times \mathbb{CP}^1_\beta \); and then pinch \( \mathbb{CP}^1_\beta \). These together transform \( M^5 \times \{0\} \) to \( \mathbb{CP}^1_\alpha \).

2. Pinch each of the \( S^1 \)-fibers of \( M^5 \times \{1\} \) to a point, which transforms \( M^5 \times \{1\} \) to \( \mathbb{CP}^1_\alpha \times \mathbb{CP}^1_\beta \); and then pinch \( \mathbb{CP}^1_\alpha \). These together transform \( M^5 \times \{1\} \) to \( \mathbb{CP}^1_\beta \).

With these geometric pictures in mind, we can now show that

**Lemma 2.2.1 [conifold II].** The conifold transition for Calabi-Yau threefolds via a flop leads to a non-manifold point \( p_0 \) on the resulting \( T \)-space \( Y \). The link of \( p_0 \) in \( Y \) is homeomorphic to \( \mathbb{CP}^3 \).

**Proof.** Let \((z_1, z_2, z_3, z_4)\) be the coordinates for \( \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2 \); \( S^3 \) (resp. \( S^{3'} \)) be the unit sphere in the first (resp. second) \( \mathbb{C}^2 \) factor of \( \mathbb{C}^4 \), and \( S^{\tau}_t \) be the sphere of radius \( t \) at the origin in \( \mathbb{C}^2 \). Consider the Hopf fibration and the compatible foliation of \( S^7 \) with generic leaf \( S^3 \times S^3 \) from the realization of \( S^7 \) as the join \( S^3 \ast S^{3'} ([\text{Ro}], [\text{Sp}]) \):

\[
\begin{align*}
S^1 & \rightarrow S^7 = S^3 \ast S^{3'} \leftrightarrow S^3_{\sqrt{1-\tau}}_{\sqrt{1-\tau}} \times S^3_t \\
& \downarrow \quad \downarrow \\
\mathbb{CP}^3 & \leftrightarrow M^5_t,
\end{align*}
\]

where \( M^5_t \) is by definition the image of \( S^3_{\sqrt{1-\tau}} \times S^3_t \) in \( \mathbb{CP}^3 \) under the Hopf map. Then the \( T^2 \)-action on \( S^7 \), given by

\[ \tau_{\theta, \theta'} : (z_1, z_2, z_3, z_4) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta'} z_3, e^{i\theta'} z_4), \]

leaves the Hopf fibration and leaves of the \( S^3 \times S^3 \)-foliation invariant. The restriction of the action to the diagonal subgroup \( \Delta \) gives indeed the Hopf map. Since this action is free, each \( S^3_{\sqrt{1-\tau}} \times S^3_t \) is thus realized as a \( T \)-bundle over the quotient \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) with fiber the orbit. Consequently, its image \( M^5_t \) in \( \mathbb{CP}^3 \) is the total space of an \( S^1 \)-bundle over \( \mathbb{CP}^1 \times \mathbb{CP}^1 \). The symmetry of the construction implies that this \( S^1 \)-bundle must correspond to a line bundle \((k, k), k \in \mathbb{Z}, \) in \( \text{Pic}(\mathbb{CP}^1 \times \mathbb{CP}^1) \).

Let us consider the leaf, say, \( S^3_{\sqrt{1/2}} \times S^3_{\sqrt{1/2}} \). Since the image in \( \mathbb{CP}^3 \) of \( S^3 \cup S^{3'} \) is a disjoint union \( \mathbb{CP}^1 \cup \mathbb{CP}^1 \), by construction the complement of \( M^5_t \) in \( \mathbb{CP}^3 \) is the
Consequently, $M_k^5$ is bundle-isomorphic to $S^2 \times S^3$ since $\nu_{\mathbb{CP}^1}(\mathbb{CP}^1)$ is bundle-isomorphic to $\mathbb{LP}_1 \oplus \mathbb{LP}_1$ by the adjunction formula and the latter is trivial as a real vector bundle due to its vanishing second Stiefel-Whitney class. Now let $(k,k)_{S^1}$ be the total space of the $S^1$-bundle associated to the complex line bundle $(k,k)$ in $\text{Pic}(\mathbb{CP}^1 \times \mathbb{CP}^1)$. Then in the smooth category ([Di])

- $(0,0)_{S^1} \cong S^2 \times S^2 \times S^1$;
- $(-1,-1)_{S^1} \cong (1,1)_{S^1} \cong S^2 \times S^3$; and
- For $k > 1$, $(-k,-k)_{S^1} \cong (k,k)_{S^1} \cong (1,1)_{S^1} / \mathbb{Z}_k$, whose fundamental group is $\mathbb{Z}_k$.

Consequently, $M_k^5$ is bundle-isomorphic to $(-1,-1)_{S^1}$. In particular, $M_5^{5 \sqrt{1/2}}$ is homeomorphic to $\partial \left( \nu_{\mathbb{LP}_3} \text{(zero-section)} \right)$.

Starting with $M_5^{5 \sqrt{1/2}}$ if we let $t \searrow 0$, then the whole space shrinks to $\mathbb{CP}^1$ in exactly the same way as $M_5 \times \{0\}$ is pinched to $\mathbb{CP}^1_0$. Similarly, when $t \nearrow 1$, $M_5^{5 \sqrt{1/2}}$ to $\mathbb{CP}^{1'}$ is the same as $M_5 \times \{1\}$ to $\mathbb{CP}^{1'}_\beta$. Consequently, $\text{Lk}(p_0)$ as described is indeed a $\mathbb{CP}^3$. This concludes the proof. \hfill \Box

**Remark 2.2.2.** Notice also that the trivial $(C_1 C_1^{-1})$ process that blows down an $S^2$ and blow the same one up again leads to a cone-point in $Y$ with link $S^2 \times S^4$. Such fake transitions can be removed by perturbing the corresponding path in the Calabi-Yau moduli space a little bit.

**Remark 2.2.3** [Conifold transition, real and complex join]. Given two complex manifolds $M_1$ and $M_2$, and a complex line bundle $L$ over $M_1 \times M_2$. One can do the one-point compactification for each fiber to get a $\mathbb{C}$-bundle $\hat{L}$ over $M_1 \times M_2$, where $\mathbb{C} = \mathbb{C} \cup \{\infty\}$. $\hat{L}$ has two distinguished sections: the zero-section $\sigma_0$ and the infinity-section $\sigma_\infty$. Both sections are naturally isomorphic to $M_1 \times M_2$. If one pinches $\sigma_0$ to $M_1$ and $\sigma_\infty$ to $M_2$, then one get a new space $M_1 *_\mathbb{C} M_2$, a complex join of $M_1$ and $M_2$. This generalizes the construction for the real join $\ast$. In terms of these, one obtains a unified picture for $\text{Lk}(p_0)$ in conifold transitions: (1) For Type-I, an $S^2$ and an $S^3$ are transmuted to each other; and $\text{Lk}(p_0)$ is the join $S^2 \ast S^3 = S^6$. (2) For Type-II, a transverse pair of $\mathbb{CP}^1$'s are transmuted to each other; and $\text{Lk}(p_0)$ is a complex join $\mathbb{CP}^1 *_{\mathbb{C}} \mathbb{CP}^1 = \mathbb{CP}^3$.

### 3 The type II primitive Kähler deformations.

Recall the topological matching condition in Sec. 1.2, which in the current case states that, if both $(X, Z)$ and $(X', Z')$ degenerate to a same $(Z, p_0)$, then the two 5-manifolds $\partial \nu_{X}(Z)$ and $\partial \nu'_{X}(Z')$ have to be homeomorphic. One can use this to rule out many impossible transitions and only focuses on those transitions that are topologically admissible in order
to make deeper and more effective study. For this reason and due to Wilson’s classification of the Kähler deformation associated to the wall of codimension 1 of the Kähler cone of a Calabi-Yau threefold ([Wil2]), in this section we shall spend our efforts in understanding the topology of $\partial \nu_X(Z)$ for $(X, Z)$, where $Z$ a del Pezzo surface embedded in $X$.

3.1 The case of smooth del Pezzo surfaces.

A del Pezzo surface $Z$ is characterized by the feature that its anticanonical bundle is ample ([De], [Wa3]). Such surfaces, when smooth, can only be the exceptional one $Z_e = \mathbb{CP}^1 \times \mathbb{CP}^1$ or the blow-up $Z_r$ of $\mathbb{CP}^2$ at $r$, $0 \leq r \leq 8$, distinct points in general positions (i.e. no three of them are collinear, no six of them lie in a conic, and no cubic through seven of them with a double point at the eighth). Such a realization provides a standard basis $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r)$ for $H_2(Z_r; \mathbb{Z}) = \text{Pic}(Z_r)$, where $\varepsilon_0$ is the canonical generator $[\mathbb{CP}^1]$ of $H_2(\mathbb{CP}^2; \mathbb{Z})$ lifted to $Z_r$ and $\varepsilon_i$, which is also a $\mathbb{CP}^1$ with self-intersection number $(-1)$, are from the exceptional divisors of the blow-up. With respect to this basis, the intersection form $Q|_{Z_r}$ of $Z_r$ is given by

$$Q|_{Z_r} = \langle 1 \rangle \oplus r \langle -1 \rangle ;$$

and the canonical divisor of $Z_r$ is known to be

$$K_{Z_r} = -3 \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_r .$$

In the following discussion, we shall not distinguish these classes with either the embedded $\mathbb{CP}^1$ (or their union) in $Z_r$ or the complex line bundle (and its real $\mathbb{R}^2$- or $D^2$-bundle) over $Z_r$ they correspond.

When $Z$ embeds in a Calabi-Yau threefold $X$, since $K_Z = K_X|_Z \otimes \nu_X(Z)$ and $K_X$ is trivial for Calabi-Yau manifolds, the tubular neighborhood $\nu_X(Z)$ of $Z$ in $X$ has to be isomorphic to $K_Z$, which restricts the possible embeddings. For $Z = Z_r$, the structure of its canonical divisor implies that $\nu_X(Z_r)$ is indeed a fiber sum of $\mathbb{L}_2^{\otimes 3}$ and $r$ many $\mathbb{L}_2^{-1}$s, where recall that $\mathbb{L}_2$ is the tautological line bundle over $\mathbb{CP}^2$. Notice that the total space of the $S^1$-bundle associated to $\mathbb{L}_2^{-1}$ is $S^5$ and the total space of the $S^1$-bundle associated to $\mathbb{L}_2^{\otimes 3}$ is the lens space $S^5/\mathbb{Z}_3$, where $\mathbb{Z}^3$ is the cyclic subgroup of order three contained in $U(1)$ that gives the Hopf fibration of $S^5$. Hence the boundary 5-manifold $\partial \nu_X(Z_r)$ is the fiber sum $(S^5/\mathbb{Z}_3) \sharp_f r S^5$ over $\mathbb{CP}^2 \sharp r \mathbb{CP}^2$. With the fact that $S^5 = S^3 \ast S^1$, one has the following two collections of $S^1$-bundles, which contain all the spaces involved in doing the fiber sum:

$$S^1 \to S^5 - \nu_{S^5}(S^1) \cong S^3 \times D^2 \xrightarrow{\partial} S^3 \times S^1 \downarrow$$

$$\mathbb{CP}^2 - D^4 = \nu_{\mathbb{CP}^2}(\mathbb{CP}^1) \xrightarrow{\partial} S^3$$

and

$$S^1 \to S^5 - \nu_{S^5}(S^3) = \nu_{S^5}(S^1) = D^4 \times S^1 \xrightarrow{\partial} S^3 \times S^1 \downarrow$$

$$\mathbb{CP}^2 - \nu_{\mathbb{CP}^2}(\mathbb{CP}^1) = D^4 \xrightarrow{\partial} S^3$$
Comparing with the decomposition
\[ S^5 = S^3 \times D^2 \cup_{S^3 \times S^1} D^4 \times S^1, \]
one recognizes that the fiber sum with \( r \)-many \( S^5 \)'s is indeed the surgery on \( r \) distinct \( S^1 \)-fibers of \( S^5/\mathbb{Z}_3 \) with respect to the local trivialization of their tubular neighborhood from the bundle structure. Together with the homotopy sequence
\[
\begin{align*}
\pi_1(S^1) & \to \pi_1(\partial \nu_X(Z_r)) \to \pi_1(\mathbb{CP}^2 \times r \mathbb{CP}^2) \\
\mathbb{Z} & \to \mathbb{Z} \to 0
\end{align*}
\]
it also follows that, for \( r \neq 0 \), \( \partial \nu_X(Z_r) \) is simply-connected since the only possible generator for \( \pi_1(\partial \nu_X(Z_r)) \) comes from an \( S^1 \)-fiber of \( \partial \nu_X(Z_r) \), which can be homotoped into the fiber over, say, \( \nu_{Z_r}(\varepsilon_1) \) and then homotoped to a point.

**Lemma 3.1.1.** For \( r \neq 0 \), \( \partial \nu_X(Z_r) \) is homeomorphic to the connected sum \( \natural r \, S^2 \times S^3 \).

**Proof.** Since \( \pi_1(\partial \nu_X(Z_r)) = 0 \), \( H_2(\partial \nu_X(Z_r); \mathbb{Z}) = \pi_2(\partial \nu_X(Z_r)) = \mathbb{Z}^r \) by the homotopy sequence. On the other hand, since \( c_1(X) = 0 \) and the normal bundle of \( \partial \nu_X(Z_r) \) in \( X \) is trivial, \( \partial \nu_X(Z_r) \) is a spin 5-manifold. The lemma then follows from the classification of closed simply-connected spin 5-manifolds by Smale [Sm2] (cf. Sec. 1.1). \( \square \)

**Remark 3.1.2.** For \( r = 0 \), \( \partial \nu_X(\mathbb{CP}^2) = S^5/\mathbb{Z}_3 \) since \( K_{\mathbb{CP}^2} = \mathcal{O}(-3) = \mathbb{L}_{\mathbb{CP}^2}^3 \). For the exceptional del Pezzo, \( \partial \nu_X(\mathbb{CP}^1 \times \mathbb{CP}^1) = (S^2 \times S^3)/\mathbb{Z}_2 \) (cf. proof of Lemma 2.2.1).

**Remark 3.1.3.** Notice that, for \( r \neq 0 \), \( H_2(\partial \nu_X(Z_r); \mathbb{Z}) \) is isomorphic to the kernel of the homomorphism from \( \pi_2(\mathbb{CP}^2 \times r \mathbb{CP}^2) \) to \( \pi_1(S^1) \) in the homotopy sequence. Consequently, \( H_2(\partial \nu_X(Z_r); \mathbb{Z}) = \{ \xi \in H_2(Z_r; \mathbb{Z}) \mid K_{Z_r} \cdot \xi = 0 \} = K_{Z_r}^+ \) with respect to \( Q_{Z_r} \).

### 3.2 The case of a rational singular del Pezzo surface.

We shall start with a review of some necessary facts on a certain rational singular del Pezzo surfaces \( Z \) and then turn to show that for such \( Z \), \( \partial \nu_X(Z) \) is simply-connected when it is embedded in a Calabi-Yau threefold \( X \). Since it is also spin, we then compute \( H_2(\partial \nu_X(Z); \mathbb{Z}) \), which, by Smale’s classification theorem [Sm2] in Fact 1.1.1, completely determines the topology of \( \partial \nu_X(Z) \). For the clarity of notation, in many places, we shall denote \( \partial \nu_X(Z) \) by \( M^5(Z, X) \).

**The setup and the simply-connectedness of \( \partial \nu_X(Z)(= M^5(Z, X)) \).**

Let \( Z \) be a Gorenstein rational singular del Pezzo surface with \( \text{Pic}(Z) = \mathbb{Z} \). For such \( Z \), it is known ([Fu], [K-MK], [M-Z]) that \( K_Z^2 = 9 - r \), \( r \geq 3 \), and that the minimal resolution of \( Z \) is a smooth del Pezzo surface of topology \( Z_r \). The types and combinations of the singularities that can appear together on \( Z \) are very restricted; the details are given in the following list[9].

---

13
$r = 3 \quad A_1 + A_2$

$\quad A_4$

$\quad 2A_1 + A_3, \quad D_5$

$\quad A_1 + A_5, \quad 3A_2, \quad E_6$

$\quad A_1 + 2A_3, \quad A_2 + A_5, \quad A_7, \quad 3A_1 + D_4, \quad A_1 + D_6, \quad E_7$

$\quad 2A_1 + 2A_3, \quad A_1 + A_2 + A_5, \quad A_1 + A_7, \quad 4A_2, \quad 2A_4, \quad A_8, \quad 2A_1 + D_6, \quad A_3 + D_5, \quad A_1 + E_7, \quad A_2 + E_6, \quad 2D_4, \quad D_8, \quad E_8$

where, for example, $2A_1 + A_3$ means that $Z$ has three isolated surface singularities: two of type $A_1$ and one of type $A_3$. Let $E$ be the exceptional divisor in $Z_r$ associated to the resolution and $E_i$ be the irreducible components of $E$. Then each $E_i$ is a $(-2)$ rational curve in $Z_r$ and their intersection diagram is the Dynkin diagram as indicated by the label $A_j, D_j,$ or $E_j$. The tubular neighborhood $\nu_X(E)$ of $E$ in $Z_r$ is thus a plumbing of a collection of the tangent $D_2$-bundle of $S^2$ with the same Dynkin diagram as the plumbing diagram. For $Z$ Gorenstein, the intersection number of these exceptional divisors in $Z_r$ with the canonical divisor $K_{Z_r}$ are all zero. Since our discussion is in topological category, by tubing if necessary, one can represent the homology class of the divisor $K_{Z_r}$ by a surface in $Z_r$ that is truly disjoint from $E$. For the following discussions, $K_{Z_r}$ will always be represented by such a surface when needed. To save notations, we also use $K_{Z_r}$ for the canonical line bundle of $Z_r$, which should be clear from the context.

When $Z$ embeds in a smooth Calabi-Yau threefolds $X$ as a divisor, its tubular neighborhood $\nu_X(Z)$ is still well-defined - though no longer a regular bundle over $Z$, for example, by the union of all small enough balls with center at $Z$ with respect to some Riemannian metric on $X$. The complement $\nu_X(Z) - Z$ remains having the product structure $\partial \nu_X(Z) \times (0, e)$. Though, due to the singularity of $Z$, the canonical line bundle $K_Z$ of $Z$ is no longer homeomorphic to the entire $\nu_X(Z)$, one may notice that the proof of the adjunction formula in [G-H] applies without difficulty to $Z - \nu_Z(Sing(Z))$ in $X - \nu_X(Sing(Z))$ (cf. [Di]). Thus one can conclude that $\nu_X(Z) - \nu_Z(Sing(Z)) (Z - \nu_Z(Sing(Z)))$ remains the same as the restriction $K_Z|Z - \nu_Z(Sing(Z))$. Furthermore, since $Z$ is Gorenstein and hence the restriction of $K_Z$ on $\nu_Z(Sing(Z))$ is trivial, the restriction $K_Z|Z - \nu_Z(Sing(Z))$ is bundle isomorphic to $K_{Z_r}|_{Z_r - \nu_{Z_r}(E)}$. On the other hand, since $Z$ is a divisor in the smooth $X$ and hence is locally principal, in a small neighborhood $U$ of $q \in Sing(Z)$ in $X$, $Z$ is the same as a germ $\psi$ of the corresponding isolated analytic hypersurface singularity in $\mathbb{C}^3$ at the origin, described by some polynomial equation $\psi = 0$. Let $\Delta$ be a small enough disk around the

---

3Readers may notice that the complete list given in [Fur] and [M-Z] differ slightly: Assuming that the $(r = 8, A_2 + E_7)$ in [Fur] is a typo for $(r = 8, A_2 + E_6)$, [Fur] has $(r = 7, A_3 + D_4)$ that does not appear in [M-Z] while [M-Z] has $(r = 1, A_1)$ and $(3A_1 + D_4)$ that do not appear in [Fur]. On the other hand, as will be clear below, there is an abelian group naturally associated to a singular del Pezzo surface, independent of whether it is embeddable in a smooth Calabi-Yau threefold or not, whose order is an integer square (cf. Proposition 3.2.8 and Corollary 3.2.9). Following the discussion below, this order is 8 for $(r = 7, A_3 + D_4)$ and is 16 for $(r = 7, 3A_1 + D_4)$. For this reason, we favor [M-Z] with $(r = 1, A_1)$ deleted. If there is going to be any change of the list we adapt here, it may influence the lists or tables based on it but not the abstract discussions below. I would like to thank Prof. Seán Keel for a discussion of this.
origin in \( C \). Then, as a function on \( U \),
\[
\psi : \psi^{-1}(\Delta) \subset U \rightarrow \Delta
\]
gives a fibration of \( \nu_X(q) \) over \( \Delta \), which is the same as the Milnor fibration associated to the singularity \( q \). Furthermore, since \( \psi = 0 \) describes \( \nu_Z(q) \) in \( X \), one can identify this \( \Delta \) as the fiber \( D^2 \) of the bundle \( \nu_X - \nu_X(\text{Sing}(Z))(Z - \nu_Z(\text{Sing}(Z))) \) over \( X - \nu_X(\text{Sing}(Z)) \) restricted to \( \partial(X - \nu_X(\text{Sing}(Z))) \). (Indeed these \( \psi \) together give a trivialization of the bundle restricted to a neighborhood of \( \partial(X - \nu_X(\text{Sing}(Z))) = \partial\nu_X(\text{Sing}(Z)) \).) Consequently, \( \nu_X(Z) \) is obtained from \( K_Z \) by removing its restriction to \( \nu_Z(E) \) and then pasting in a collection of the total space of the Milnor fibration associated to the isolated hypersurface singularities \( \text{Sing}(Z) \) in \( X \) (cf. Figure 3-1). To understand the boundary of \( \nu_X(Z) \), we need to rephrase this picture in terms of associated circle bundles.

Let \( \Xi^5 \) be the associated circle bundle of \( K_Z \) over \( Z_r \). For the clarity of notation, let us denote \( \partial\nu_X(Z) \) by \( M^5(Z,X), \Xi^5|_{Z_r - \nu_{Z_r}(E)} \) by \( W^5_{\text{reg}}(Z,X) \), and the complement of \( W^5_{\text{reg}}(Z,X) \) in \( M^5(Z,X) \) by \( W^5_{\text{Sing}}(Z,X) \). Then, following the previous discussions and notations, each component of \( W^5_{\text{Sing}}(Z,X) \) corresponds to a singularity of \( Z \) and is homeomorphic to the total space of the Milnor fibration in the slightly modified spherical form:
\[
\varphi : S^5 - \nu_{S^5}(\{\psi = 0\}) \rightarrow S^1, \quad \varphi = \psi/|\psi|.
\]

It is known (cf. Sec. 1.1) that the intersection of \( \{\psi = 0\} \) with \( S^5 \) is homeomorphic to a quotient \( S^3/G \), where \( G \) is a finite subgroup in \( SU(2) \) associated to the singularity in \( Z \). Hence the boundary of \( S^5 - \nu_{S^5}(\{\psi = 0\}) \) is homeomorphic to \( S^3/G \times S^1 \).

Now let \( [S^3/G; S^1] \) be the set of homotopy classes of maps from \( S^3/G \) to \( S^1 \). Then observe that
\[
[S^3/G; S^1] = H^1(S^3/G; \mathbb{Z}) = \text{Hom}(H_1(S^3/G; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_0(S^3/G; \mathbb{Z}), \mathbb{Z}) = 0
\]
since \( H_1(S^3/G; \mathbb{Z}) \) is torsion and \( H_0(S^3/G; \mathbb{Z}) \) is free. Thus all maps from \( S^3/G \) to \( S^1 \) are homotopic to each other. Consequently, no matter how we trivialize \( W^5_{\text{reg}}(Z,X) \) over \( Z_r - \nu_{Z_r}(K_Z) \), we can always homotope the trivialization so that over \( \partial(Z_r - \nu_{Z_r}(E)) \) it coincides with the product structure \( S^3/G \times S^1 \) from the Milnor fibration of each component of \( \partial W^5_{\text{Sing}}(Z,X) \) (Figure 3-1). Consequently, one has the following lemma:

**Lemma 3.2.1.** \( M^5(Z,X) \) can be obtained from \( \Xi^5 \) by the following cut-and-paste: First cut \( \Xi^5 \) along a lifting \( \nu_{Z_r}(E) \) of \( \nu_{Z_r}(E) \) into \( \Xi^5 \). The resulting manifold has a boundary which is the double \( D\nu_{Z_r}(E) \) of \( \nu_{Z_r}(E) \). Next close up this boundary by the orientation-reversing automorphism of \( D\nu_{Z_r}(E) \) induced by the monodromy homeomorphism \( \tau \) on \( \nu_{Z_r}(E) \) to itself (Figure 3-2).

**Corollary 3.2.2.** \( M^5(Z,X) \) is simply-connected.

**Proof.** Let \( \hat{E} \) be the lifting of \( E \) into \( \nu_{Z_r}(E) \). Notice that \( \hat{E} \) is a 2-cycle in \( M^5(Z,X) \) that consists of a collection of embedded \( S^2 \) that intersect at worst transversely at some
Figure 3-1. The tubular neighborhood $\nu_X(Z)$ of a singular del Pezzo surface $Z$ embedded in a smooth Calabi-Yau threefold $X$ is indicated. The $W^5_{\text{Sing}}(Z, X)$ part of the boundary $\partial \nu_X(Z)$ is shaded. The way it is realized as the total space of the Milnor fibration is shown on the right. While $W^5_{\text{reg}}(Z, X)$ is an $S^1$-bundle over the open complex surface $Z - \nu_Z(\text{Sing}(Z))$, each connected component of $W^5_{\text{Sing}}(Z, X)$ is an open surface bundle over $S^1$ with fiber $F$ homeomorphic to some connected component of $\nu_Z(E)$. Up to homotopy, both give the same product structure $S^3/G \times S^1$ for the components of their shared boundary 4-manifold. Living in the Milnor fiber $F$ is the exceptional divisor $E$ from the resolution of the singularity of $Z$. Its irreducible components $E_i$ are indicated by thick line segments due to the suppression of dimensions. Shown in the figure is a $D_6$-chain. The monodromy homeomorphism $\tau$ and the Milnor fibration map $\varphi$ are also indicated. If $\tau$ is replaced by the identity map, then the topology of $\partial \nu_X(Z)$ is changed and becomes $\Xi^5$, the $S^1$-bundle associated to the canonical line bundle $K_{Z_r}$ of the minimal resolution $Z_r$ of $Z$. 
isolated points. Thus, given a loop \( \gamma \) in \( M^5(Z,X) \), up to homotopy we may assume that \( \gamma \) is disjoint from \( \hat{E} \) and hence from \( \nu_{Z_r}(E) \). Lemma 3.2.1 implies that we may now regard \( \gamma \) as a loop in \( \Xi^5 \). Since \( \Xi^5 \) is simply-connected from the discussion in Sec. 3.1, \( \gamma \) bounds an embedded 2-disk \( D^2 \) in \( \Xi^5 \). By dimension count, we may assume that \( D^2 \) is disjoint from \( \hat{E} \) and hence from \( \nu_{Z_r}(E) \) by a further homotopy if necessary. This shows that \( \gamma \) in \( M^5(Z,X) \) indeed bounds a 2-disk and hence concludes the proof.

\( \square \)

Furthermore, since it bounds the spin 6-manifold \( \nu_X(Z) \), \( M^5(Z,X) \) is also spin. By Fact 1.1.1 in Sec. 1.1, its topology is determined by the group structure of its second integral homology \( H_2(M^5(Z,X);\mathbb{Z}) \), which we shall now turn to.

The integral homology \( H_2(M^5(Z,X);\mathbb{Z}) \).

The fact that \( M^5(Z,X) \) is obtained from \( \Xi^5 - \nu_{Z_r}(E) \) by closing up the boundary using the monodromy homeomorphism \( \tau \) suggests that one may relate \( H_2(M^5(Z,X);\mathbb{Z}) \) to \( H_2(\Xi^5 - \nu_{Z_r}(E);\mathbb{Z}) \) and the monodromy operator \( T \). We shall now work out the detail of this relation.

First notice that, if one denotes the lifting of \( E \) to \( \nu_{Z_r}(E) \) by \( \hat{E} \), then, from the fact that

\[
\nu_{\Xi^5}(\hat{E}) - \nu_{Z_r}(E) \cong \nu_{Z_r}(E) \times ((-\epsilon, 0) \cup (0, \epsilon)) \cong \nu_{M^5(Z,X)}(\hat{E}) - \nu_{Z_r}(E),
\]

\[
\nu_{\Xi^5}(\hat{E}) - \hat{E} \cong D\nu_{Z_r}(E) \times (0, \epsilon) \cong M^5(Z,X) - \hat{E},
\]

and that the boundary \( D\nu_{Z_r}(E) \) of \( \Xi^5(Z,X) - \nu_{Z_r}(E) \) has a collar, one has

\[
\Xi^5 - \nu_{Z_r}(E) \cong \Xi^5 - \nu_{\Xi^5}(\hat{E}) \cong \Xi^5 - \hat{E}
\]

\[
M^5(Z,X) - \nu_{Z_r}(E) \cong M^5(Z,X) - \nu_{M^5(Z,X)}(\hat{E}) \cong M^5(Z,X) - \hat{E}.
\]
Thus by the Lefschetz duality one obtains

\[ H_2(\Xi^5 - \nu_{\mathcal{Z}},(\hat{E}); \mathbb{Z}) = H_2(\Xi^5 - \hat{E}; \mathbb{Z}) = H^3(\Xi^5, \hat{E}; \mathbb{Z}). \]

Now recall the exact sequence associated to the pair \((\Xi^5, \hat{E})\):

\[ \cdots \to H^2(\Xi^5; \mathbb{Z}) \to H^2(\hat{E}; \mathbb{Z}) \xrightarrow{\delta^*} H^3(\Xi^5, \hat{E}; \mathbb{Z}) \to H^3(\Xi^5; \mathbb{Z}) \to H^3(\hat{E}; \mathbb{Z}) \xrightarrow{\delta^*} \cdots \]

where \(p.d.\) stands for the Poincaré duality, one then has the following short exact sequence

\[ 0 \to H^2(\hat{E}; \mathbb{Z})/\ker \delta^* \to H_2(\Xi^5 - \hat{E}; \mathbb{Z}) \to H_2(\Xi^5; \mathbb{Z}) \to 0. \]

**Lemma 3.2.3.** The above short exact sequence splits.

**Proof.** Recall that \(E \cdot K_{\mathbb{Z}} = 0\), \(H_2(\Xi^5; \mathbb{Z}) = K^2_{\mathbb{Z}},\) and that \(\Xi^5\) over \(Z_r - K_{\mathbb{Z}}\) is trivial. Thus there exists a partial section of \(\Xi^5\), defined only over \(Z_r - K_{\mathbb{Z}}\) and whose image is disjoint from \(\hat{E}\), that lifts \(K^2_{\mathbb{Z}}\) isomorphically onto \(H_2(\Xi^5; \mathbb{Z})\). Since the image of this section is contained in \(\Xi^5 - \hat{E}\), the composition of the section with the inclusion map of its image into \(\Xi^5 - \hat{E}\) induces then a homomorphism from \(K^2_{\mathbb{Z}}\) into \(H_2(\Xi^5 - \hat{E}; \mathbb{Z})\). By construction, the composition of this homomorphism with that induced by the projection map from \(\Xi^5\) onto \(Z_r\) is the identity map on \(K^2_{\mathbb{Z}}\). This shows that the above short exact sequence splits.

\[ \square \]

Consequently,

\[ H_2(\Xi^5 - \hat{E}; \mathbb{Z}) = H_2(\Xi^5; \mathbb{Z}) \oplus H^2(\hat{E}; \mathbb{Z})/\ker \delta^* \]

and we shall now study the second factor of this decomposition.

The universal coefficient theorem implies that

\[ H^2(\hat{E}; \mathbb{Z}) = \text{Hom}(H_2(\hat{E}; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_1(\hat{E}; \mathbb{Z}), \mathbb{Z}) = \text{Hom}(H_2(\hat{E}; \mathbb{Z}), \mathbb{Z}) \]

since \(H_1(\hat{E}; \mathbb{Z}) = 0\). Thus

\[ H^2(\hat{E}; \mathbb{Z})/\ker \delta^* = \text{Hom}(H_2(\hat{E}; \mathbb{Z}), \mathbb{Z})/\text{Im}(H^2(\Xi^5; \mathbb{Z}) \to H^2(\hat{E}; \mathbb{Z})). \]

Since \(H_2(\hat{E}; \mathbb{Z})\) is freely generated by the lifting \(\hat{E}\) to \(E\) of all the irreducible components \(E_i\) of \(E\), \(H^2(\hat{E}; \mathbb{Z}) = \text{Hom}(H_2(\hat{E}; \mathbb{Z}), \mathbb{Z})\) is freely generated by the dual basis \(\hat{E}_i^*\), \(i = 1, \ldots, r\), that sends \(\hat{E}_i\) to 1 and all other \(\hat{E}_j\) to 0. Under the Lefschetz duality, the image \(\delta^* \hat{E}_i^*\) in \(H^3(\Xi^5, \hat{E}; \mathbb{Z})\) is turned into the class in \(H_3(\Xi^5 - \hat{E}; \mathbb{Z})\) represented by the meridian 2-sphere \(\hat{S}_i\) of \(\hat{E}_i\) in \(\Xi^5 - \hat{E} \subset \Xi^5\). (Cf. Figure 3-3.)
The universal coefficient theorem implies also that
\[ H^2(\Xi^5; \mathbb{Z}) = \text{Hom}(H_2(\Xi^5; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_1(\Xi^5; \mathbb{Z}), \mathbb{Z}) = \text{Hom}(H_2(\Xi^5; \mathbb{Z}), \mathbb{Z}) \]
since \( H_1(\Xi^5; \mathbb{Z}) = 0 \). Taken into account the Poincaré duality, which identifies \( H^2(\Xi^5; \mathbb{Z}) \) with \( H_3(\Xi^5; \mathbb{Z}) \) by considering the intersection number of a 3-cycle with all the 2-cycles, one can work out a set of generators for \( H^2(\Xi^5; \mathbb{Z}) \) represented by a set of 3-cycles and hence the image of \( H^2(\Xi^5; \mathbb{Z}) \) in \( H^2(\hat{E}; \mathbb{Z}) \) as follows:

- Let \( e_1, \ldots, e_r \) be a basis for \( K_{Z_r}^2 \) and \( \xi_1, \ldots, \xi_r \) be a dual collection of 2-cycles in \( Z_r \) represented by maps, still denoted by \( \xi_i \), from \( S^2 \) into \( Z_r \), such that \( \xi_i \cdot e_j = \delta_{ij} \). By the unimodularness of the intersection form \( Q_{Z_r} \) of \( Z_r \), such a dual collection always exists but not unique; the non-uniqueness comes exactly by an arbitrary multiple of \( K_{Z_r} \). The natural bundle homomorphism \( \xi_i \) from the total space of the pullback \( S^1 \)-bundle \( \xi_i^* \Xi \) over \( S^2 \) into \( \Xi^5 \) gives then a collection of 3-cycles \( \tilde{\xi}_1, \ldots, \tilde{\xi}_r \) that form a basis for \( H_3(\Xi^5; \mathbb{Z}) \). By construction, \( \tilde{\xi}_i \cdot \hat{E}_j = \tilde{\xi}_i \cdot E_j = \delta_{ij} \).

- Since \( E_i \cdot K_{Z_r} = 0 \), \( E_i \) can be expressed uniquely as \( \sum_{j=1}^r a_{ij} e_j \) for some integers \( a_{ij} \). In terms of this, using the intersection property of \( \tilde{\xi}_i \) with \( \hat{E}_j \) above, one has

\[ \tilde{\xi}_i \cdot \hat{E}_j = a_{ji} \quad \text{and hence} \quad \tilde{\xi}_i = \sum_{j=1}^r a_{ji} \hat{E}_j^* . \]

The image \( \text{Im}(H^2(\Xi^5; \mathbb{Z}) \to H^2(\hat{E}; \mathbb{Z})) \) is then spanned by \( \tilde{\xi}_1, \ldots, \tilde{\xi}_r \) over \( \mathbb{Z} \).

Since \( \hat{E}_i^* \) corresponds to \( \hat{S}_i \) in \( H_2(\Xi^5 - \nu_{Z_r}(E); \mathbb{Z}) \), with the notations introduced above, one has thus proved

**Lemma 3.2.4.**

\[ H_2(\Xi^5 - \hat{E}; \mathbb{Z}) = \text{Span}_\mathbb{Z}\{ \hat{e}_1, \ldots, \hat{e}_r, \hat{S}_1, \ldots, \hat{S}_r \} / \sim = \mathbb{Z}^r \oplus \mathbb{Z}^r/(\mathbb{Z}^r \cdot A^t) , \]
where $\sim$ is generated by $\sum_{j=1}^r a_{ji} \hat{S}_j$, $i = 1, \ldots, r$, $\mathbb{Z}^r$ consists of the $r$-dimensional integral row vectors, and $A^t$ is the transpose of the coefficient matrix $A = (a_{ij})_{ij}$.

Remark 3.2.5. The above $\sim$ contains exactly all the combinations of $\hat{S}_i$ that bound a 3-cycle in $\Xi^5 - \hat{E}$.

Now the exact sequence
\[ \cdots \to H^2(\hat{E}; \mathbb{Z}) \xrightarrow{\delta^*} H^3(M^5(Z, X), \hat{E}; \mathbb{Z}) \to H^3(M^5(Z, X); \mathbb{Z}) \to H^3(\hat{E}; \mathbb{Z}) \xrightarrow{\delta^*} \cdots \]
implies that
\[ H_2(M^5(Z, X); \mathbb{Z}) = H_2(\Xi^5 - \hat{E}; \mathbb{Z})/\hat{\delta}^*(H^2(\hat{E}; \mathbb{Z})) \]
and hence it is generated by $\hat{e}_1, \ldots, \hat{e}_r, \hat{S}_1, \ldots, \hat{S}_r$ as well though the coboundary homomorphism $\hat{\delta}^*$ is expected to be different from $\delta^*$ in the exact sequence for $(\Xi^5, \hat{E})$ due to the twist by $\tau$. We shall now work out the relations of these generators in the quotient.

Since $E$ is a deformation retract of $\nu_{Z, r}(E)$,
\[ H_2(\hat{E}; \mathbb{Z}) = H^2(\nu_{Z, r}(E); \mathbb{Z}) \overset{\text{L.P.D.}}{=} H_2(\nu_{Z, r}(E), \partial \nu_{Z, r}(E); \mathbb{Z}), \]
where L.P.D. stands for the Lefschetz-Poincaré duality. Consider the relative 2-cycle in $(\nu_{Z, r}(E), \partial \nu_{Z, r}(E))$ represented by 2-disk $D_i$ that intersects $E_i$ at one point and is disjoint from all other $E_j$ (cf. the shaded disks in Figure 3-3 with the ambient space $N$ now being $Z_r$). They form a basis for $H_2(\nu_{Z, r}(E), \partial \nu_{Z, r}(E); \mathbb{Z})$. Their lifting to $\hat{E}$ shall be denoted by $\hat{D}_i$. Let $\hat{S}_i^\sim$ be a meridian 2-sphere of $\hat{E}_i$ in $M^5(Z, X)$. Then
\[ \delta^*(\hat{D}_i) = [\hat{S}_i], \quad \hat{\delta}^*(\hat{D}_i) = [\hat{S}_i^\sim] \quad \text{in} \quad H_2(\Xi^5 - \hat{E}; \mathbb{Z}) \]
after the various dualities. Furthermore, let $T^*$ be the dual monodromy operator of $T$ on $H^2(\hat{E}; \mathbb{Z})$; then by construction the difference
\[ \hat{S}_i^\sim - \hat{S}_i = T^* \hat{D}_i - \hat{D}_i = (T - \text{Id})^*(\hat{D}_i) \]
lies in $H_2(\nu_{Z, r}(E); \mathbb{Z}) = H_2(\hat{E}; \mathbb{Z})$ since the restriction of the monodromy homeomorphism $\tau$ to $\partial \nu_{Z, r}(E)$ is the identity map.

Lemma 3.2.6. Let $Q_{\nu_{Z, r}(E)}$ be the intersection matrix of $H_2(\nu_{Z, r}(E); \mathbb{Z})$ with respect to the basis $(E_1, \ldots, E_r)$. Then
\[ \hat{S}_i^\sim = \hat{S}_i + \sum_{j=1}^r \left( (T - \text{Id}) Q_{\nu_{Z, r}(E)}^{-1} \right)_{ij} \hat{E}_j, \quad i = 1, \ldots, r. \]
and they generate $\tilde{\delta}^*(H^2(\hat{E}; \mathbb{Z}))$ in $H_2(\mathbb{C}^5 - \hat{E}; \mathbb{Z})$.

Proof. The second statement is clear. For the first one, let $(T - Id)^* (\hat{D}_i) = \sum_{j=1}^r b_{ij} \hat{E}_j$; then, by intersecting both sides with $\hat{E}_1, \cdots, \hat{E}_r$ respectively, one has $T - Id = BQ_{\nu_{Zr}(E)}$, where $B$ is the coefficient matrix $(b_{ij})_{ij}$. This concludes the lemma.

Putting Lemma 3.2.4 and Lemma 3.3.6 together, one can now conclude that

$$H_2(M^5(Z, X); \mathbb{Z}) = \text{Span}_\mathbb{Z}\{ \hat{e}_1, \cdots, \hat{e}_r, \hat{S}_1, \cdots, \hat{S}_r \} / \sim,$$

where $\sim$ is generated by

$$\sum_{j=1}^r a_{ji} \hat{S}_j \quad \text{and} \quad \hat{S}_i + \sum_{j=1}^r \left( (T - Id) Q^{-1}_{\nu_{Zr}(E)} A \right)_{ij} \hat{e}_j, \quad \text{for } i = 1, \ldots, r.$$

These relators can be combined together to eliminate the generators $\hat{S}_i$ and give

$$H_2(M^5(Z, X); \mathbb{Z}) = \text{Span}_\mathbb{Z}\{ \hat{e}_1, \cdots, \hat{e}_r \} / \sim,$$

where $\sim$ is generated by

$$\eta_i = \left( A^t (T - Id) Q^{-1}_{\nu_{Zr}(E)} A \right)_{ik} \hat{e}_k.$$

To give the last step of simplification, one has to recall the following facts from [A-GZ-V] and [Di]:

**Fact 3.2.7 [Seifert matrix]** ([A-GZ-V] and [Di]) For an isolated hypersurface singularity in $\mathbb{C}^{n+1}$, the intersection matrix $Q$ for the vanishing cycles in the fiber of the Milnor fibration can be put into the form after choosing appropriate basis: $Q = -(U + (-1)^n U^t)$, where $U$ is upper triangular, unimodular, and is called the Seifert matrix with respect to this basis. The monodromy operator $T$ is then related to $U$ by $T = (-1)^{n+1} U^{-1} U^t$.

Consequently, $(T - Id) Q_{E}^{-1} = U^{-1}$ in our case and we can conclude and summarize the whole discussion by the following proposition.

**Proposition 3.2.8 [homology of boundary].** Let $Z$ be a Gorenstein rational singular del Pezzo surface with $\text{Pic}(Z) = \mathbb{Z}$ that is embedded in a smooth Calabi-Yau threefold $X$. Denote the minimal resolution of $Z$ by $Z_r$, which is a smooth del Pezzo surface obtained by $\mathbb{C}P^2$ blown up at $r$ many points, and the vanishing cycle of the resolution by $E$ with irreducible components $E_1, \cdots, E_r$. Then $\partial_{\nu_X}(Z)$ is a simply-connected spin $5$-manifold with the second integral homology $H_2$ isomorphic to $\mathbb{Z}^r / \{ r \cdot (A^t U^{-1} A) \} = \text{Span}_\mathbb{Z}\{ \hat{e}_1, \cdots, \hat{e}_r \}$, where $U$ is the Seifert matrix associated to the singularity with respect to $(E_1, \cdots, E_r)$ and $A$ is the matrix whose $i$-th row is the coefficients of $E_i$ with respect to any basis for $K_{\mathbb{C}P^2}$ in $H_2(\mathbb{C}P^2; \mathbb{Z})$. (In
the above expression, elements in $\mathbb{Z}^r$ are integral row vectors and the $i$-th column of $U$ is the coefficients of $U(E_i)$ with respect to $(E_1, \cdots, E_r)$.

Let $Q_E$ (resp. $Q_{K_{\mathbb{Z}_r}^+}$) be the intersection matrix of $E_1, \cdots, E_r$ (resp. $K_{\mathbb{Z}_r}^+$) in $H_2(Z_r; \mathbb{Z})$. Their determinant, whose value will be given in the explicit computations below, are all non-zero. Since $U$ is unimodular and hence $\det(A U^{-1} A) = (\det A)^2 = \det Q_E/\det Q_{K_{\mathbb{Z}_r}^+}$, one has

**Corollary 3.2.9.** $H_2(\nu_X(Z); \mathbb{Z})$ is a finite abelian group of order $|\det Q_E/\det Q_{K_{\mathbb{Z}_r}^+}|$.

$H_2(\partial\nu_X(Z); \mathbb{Z})$ and $\partial\nu_X(Z)$ for our examples.

With Proposition 3.2.8, we can now work out $H_2(\partial\nu_X(Z); \mathbb{Z})$ explicitly for the above kind of singular del Pezzo surface $Z$ when it is embedded in a smooth Calabi-Yau threefold $X$. This then determines the topology of $\partial\nu_X(Z)$. All the notations follow from previous part

(a) Basic ingredients. The intersection matrix $Q$, monodromy operator $T$, and the inverse of the Seifert matrix $U$ with respect to the set of vanishing cycles for an $A$-$D$-$E$ singularity in a complex surface are listed below: (Note: $\cdots$ indicates the pattern and the blank indicates 0.)

- The $A_k$-singularity:

  $$Q_{A_k} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ \vdots & \vdots & \vdots \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix}_{k \times k}, \quad T_{A_k} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ \vdots & \vdots & \vdots \\ 1 & -1 & -1 \end{bmatrix}_{k \times k},$$

  and $U_{A_k}^{-1} = \begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{bmatrix}_{k \times k}$.

- The $D_k$-singularity, $k \geq 4$:

  $$Q_{D_k} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ \vdots & \vdots & \vdots \\ 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix}_{k \times k}, \quad T_{D_k} = \begin{bmatrix} 0 & 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \end{bmatrix}_{k \times k},$$
and \[ U_{D_k}^{-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \end{bmatrix}_{k \times k} \] .

- The \( E_k \)-singularity, \( k = 6, 7, 8 \): (Only the matrices for \( k = 8 \) are shown; for \( k = 6 \) or 7, choose the corresponding last \( 6 \times 6 \) or \( 7 \times 7 \) submatrix.)

\[
Q_{E_8} = \begin{bmatrix} -2 & 1 & 1 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 1 \\ \end{bmatrix}, \quad T_{E_8} = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 1 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ \end{bmatrix},
\]

and \[ U_{E_8}^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \end{bmatrix} \] .

(b) The list of all possibilities. With respect to the canonical basis \( \varepsilon_0, \cdots, \varepsilon_r \) for \( H_2(Z_r; \mathbb{Z}) \), one can choose the basis \( e_1, \cdots, e_r \) for \( K_{Z_r}^{1/2} \) to be

\[
e_1 = \varepsilon_1 - \varepsilon_r, \quad \cdots, \quad e_{r-1} = \varepsilon_{r-1} - \varepsilon_r, \quad e_r = \varepsilon_0 - 3 \varepsilon_r.
\]

With respect to this,

\[
Q_{K_{Z_r}^{1/2}} = \begin{bmatrix} -2 & -1 & \cdots & -1 & -3 \\ -1 & -2 & \cdots & -1 & -3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & -2 & -3 \\ -3 & -3 & \cdots & -3 & -8 \\ \end{bmatrix}_{r \times r}
\]

where all the missing entries are \(-1\).

Its determinant is

\[
\det Q_{K_{Z_r}^{1/2}} = -6 \quad 5 \quad -4 \quad 3 \quad -2 \quad 1.
\]

The determinant of the intersection matrix \( Q \) for the connected components of \( E \) involved in our problem is

\[
\begin{array}{cccccc}
A_k & D_k & E_6 & E_7 & E_8 \\
(-1)^k (k + 1) & (-1)^k 4 & 3 & -2 & 1
\end{array}
\]
When $E$ has several connected components $E_1, \ldots$, the corresponding $Q_E$ is the block diagonal matrix $Q_{E_1} \oplus \cdots$. From the given data, we now give a list of the order of $H_2(\partial \nu_X(Z); \mathbb{Z})$ below:

$$
\begin{array}{cccccc}
\text{order} & 1 & 4 & 9 & 16 & 25 \\
\text{group} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_3 \oplus \mathbb{Z}_3 & (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) & \mathbb{Z}_5 \oplus \mathbb{Z}_5 & (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \\
\end{array}
$$

By the classification theorem for abelian groups [Ja] and the fact that the torsion part of the second integral homology of a simply-connected spin closed 5-manifold is the direct sum of two identical subgroups (Fact 1.1.1 in Sec. 1.1), the list of all abelian groups whose order appear above and have such splitting property are:

$$
\begin{array}{cccccc}
\text{order} & 1 & 4 & 9 & 16 & 25 & 36 & 64 & 81 \\
\text{group} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_3 \oplus \mathbb{Z}_3 & (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) & \mathbb{Z}_5 \oplus \mathbb{Z}_5 & (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \\
\end{array}
$$

These are the only groups, to which $H_2(\nu_X(Z); \mathbb{Z})$ can be isomorphic.

(c) **Conclusion.** From the tables given in (b) and comparing with Smale’s theorem, we can now conclude the possible topology of $\partial \nu_X(Z)$ in each case as follows: (See Fact 1.1.1 in Sec. 1.1 for the notation of manifolds.)

$$
\begin{array}{cccccc}
\text{order} & 1 & 4 & 9 & 16 & 25 & 36 & 64 & 81 \\
\text{group} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_3 \oplus \mathbb{Z}_3 & (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) & \mathbb{Z}_5 \oplus \mathbb{Z}_5 & (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \\
\end{array}
$$

(See Fact 1.1.1 in Sec. 1.1 for the notation of manifolds.)
Using this, one can write a computer code to first sort out all the chains of elements in \( K \), determined by the order of \( \partial \nu \). These are ruled out. This determines \( \partial \nu(X) \) associated to each combination of A-D-E singularities uniquely; and the discussion is thus complete.

**Remark 3.2.10 [computer check].** Let \( \xi = a_0 \varepsilon_0 + a_1 \varepsilon_1 + \cdots + a_r \varepsilon_r \) in \( K_2 \), such that \( \xi \cdot \xi = -2 \). Using the inequality \( \frac{a_1^2 + \cdots + a_r^2}{r} \geq \left( \frac{a_0 + \cdots + a_r}{r} \right)^2 \), one can show that \( a_0^2 \leq \frac{2r}{9-r} \). Thus, due to the symmetry of the equations for the coefficients \( a_1, \cdots, a_r \), one concludes after some elementary manipulation that the following elements generate all \( (-2) \)-norm elements in \( K_2 \), by overall sign change and permutations of the last \( r \) coordinates:

\[
\begin{align*}
& \quad r = 3: & (0; -1, -1, 0), & (1; -1, -1, -1). \\
& \quad r = 4: & (0; -1, 0, 0, 0), & (1; -1, -1, -1, -1). \\
& \quad r = 5: & (0; -1, 0, 0, 0, 0), & (1; -1, -1, -1, -1, -1). \\
& \quad r = 6: & (0; -1, 0, 0, 0, 0, 0), & (1; -1, -1, -1, 0, 0, 0), & (2; -1, -1, -1, -1, -1, -1). \\
& \quad r = 7: & (0; -1, 0, 0, 0, 0, 0, 0), & (1; -1, -1, -1, 0, 0, 0, 0), & (2; -1, -1, -1, -1, -1, -1, 0). \\
& \quad r = 8: & (0; -1, 0, 0, 0, 0, 0, 0, 0), & (1; -1, -1, -1, 0, 0, 0, 0, 0), & (2; -1, -1, -1, -1, -1, -1, -1, 0), & (3; -2, -1, -1, -1, -1, -1, -1, -1).
\end{align*}
\]

Using this, one can write a computer code to first sort out all the chains of \( (-2) \)-norm elements in \( K_2 \), whose intersection matrix is as indicated in the allowed combinations of singularities, and then computing \( A^t U^{-1} A \) and simplifying it by integral row and column operations to render \( A^t U^{-1} A \) an integral diagonal form

\[
\text{Diag}(1, \cdots, 1, d_1, d_1, \cdots, d_s, d_s),
\]

where \( d_i \neq 0, 1 \) and \( d_i | d_j \) for \( i \leq j \). This will conclude that

\[
H_2(\partial \nu_X(Z); \mathbb{Z}) = (\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_s}) \oplus (\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_s}) \quad \text{and} \quad \partial \nu_X(Z) = M_2^{\tilde{d}_1} \cdots M_2^{\tilde{d}_s}
\]

and hence give the definite answer to the question of which of the candidate 5-manifolds for \( \partial \nu_X(Z) \) would truly occur. The detail of the code is in the Appendix.
4 Topologically admissible isolated singularities.

Having studied the topology of $\partial \nu_X(Z)$ for a class of del Pezzo surface $Z$ embedded in a Calabi-Yau threefold $X$, we can now lay down many locally topologically admissible pairs $(X, Z)$ and $(X', Z')$ for some Calabi-Yau threefolds $X$, $X'$. Though we do not know at the moment what $X$ and $X'$ could be, we do know from the discussion in Sec. 3 that, if such extremal transition exists, then the topology $\nu_X(Z) \cup \partial \nu_{X'}(Z')$ of the link of the resulting isolated singularity $p_0$ in the 7-space $Y^7$ is independent of what $X$ and $X'$ exactly are. (See Remark 4.4 at the end for some notes on these 6-manifolds.)

**The transition:** $(X, Z) \xrightarrow{\text{Kähler deformation}} (X, p_0) \xleftarrow{\text{Complex deformation}} (X', Z')$.

Del Pezzo surfaces of the same group in the following list of examples when embedded in some Calabi-Yau threefolds have the chance to be pinched down to a point and puffed up to another while this is impossible for those in different groups:

**Example 4.1.**

- **Group-$(S^5)$:** $(r = 3 : A_1 + A_2)$, $(r = 4 : A_4)$, $(r = 5 : D_5)$, $(r = 6 : E_6)$, $(r = 7 : E_7)$, $(r = 8 : E_8)$.
- **Group-$(M^5_2)$:** $(r = 5 : 2A_1 + A_3)$, $(r = 6 : A_1 + A_5)$, $(r = 7 : A_7)$, $(r = 7 : A_1 + D_6)$, $(r = 8 : A_1 + E_7)$, $(r = 8 : D_8)$.
- **Group-$(M^5_2#M^5_2)$:** $(r = 7 : 3A_1 + D_4)$, $(r = 8 : 2A_1 + D_6)$, $(r = 8 : 2D_4)$.
- **Group-$(M^5_3)$:** $(r = 6 : 3A_2)$, $(r = 7 : A_2 + A_5)$, $(r = 8 : A_8)$, $(r = 8 : A_2 + E_6)$.
- **Group-$(M^5_4)$:** $(r = 7 : A_1 + 2A_3)$, $(r = 8 : A_1 + A_7)$, $(r = 8 : A_3 + D_5)$.

**The transition:** $(X, Z) \xrightarrow{\text{Kähler deformation}} (X, p_0) \xleftarrow{\text{Complex deformation}} (X', Z')$.

Recall (e.g. [A-L], [B-G], [B-C-dlO], and [Gr]) that, if one lets $b_3 = \dim H_3(X'; Z)$ be the third Betti number of $X'$ and $\{b_{ij}\}_{i,j=1}^{b_3}$ be a canonical basis for $H_3(X'; Z)$ such that

$$\alpha_i \cdot \beta_j = -\beta_j \cdot \alpha_i = \delta_{ij} \quad \text{and} \quad \alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0.$$ 

Then the complex structure on $X'$ is parametrized projectively by the periods

$$\int_{\alpha_i} \Omega \quad \text{(the \(\alpha\)-periods)} \quad \text{and} \quad \int_{\beta_j} \Omega \quad \text{(the \(\beta\)-periods)},$$

where $\Omega$ is the unique holomorphic 3-form (determined by the complex structure) on $X'$ up to a constant multiple. It is known ([B-G]) that indeed the $\alpha$-periods and the $\beta$-periods determine each other. For $X'$ to degenerate to a conifold via deformation of complex
structures, it is known that some disjoint collection of 3-cycles, realized as embedded $S^3$'s in $X'$, get pinched into a collection of isolated points, which is quite similar to the situation for Riemann surfaces. If one chooses the canonical basis appropriately, these conifolds may correspond to points in the moduli space of complex structures on $X'$, for which some of the periods vanish. This suggests that there may exist severer deformations of complex structures, for which a more complicated chain of $S^3$'s get shrunk to points. Unfortunately, though there is some work in this direction (e.g. [C-G-G-K]), there seems not much that is known.

On the other hand, since any vector bundle over $S^3$ is trivial, if more complicated union $Z'$ of $S^3$ can get shrunk to points, assuming that all the intersections are transverse, then the tubular neighborhood $\nu_X(Z')$ of $Z'$ in $X'$ is a plumbing of a collection of trivial $D^3$-bundle $S^3 \times D^3$, which depends only on the intersection pattern and the sign of intersections. For this reason, let us boldly press on a little bit and see locally how some isolated singularity of a singular Calabi-Yau threefold from previous Kähler deformations may be resolved by 3-cycles.

In view that most of $\partial \nu_X(Z)$ we have seen in this article is simply-connected, one may notice the following lemma:

**Lemma 4.2.** Let $W^6$ be the 6-manifold-with-boundary obtained by plumbing a collection of trivial $D^3$-bundle $S^3 \times D^3$ over $S^3$ (including possibly self-plumbing). Let $\Gamma$ be the plumbing diagram. If $\Gamma$ is not simply-connected, then $\partial W^6$ cannot be simply-connected either.

**Proof.** By construction, the closed 5-manifold $\partial W^6$ is obtained from a collection of 5-manifold-with-boundary of the form $(S^3 - \nu_{S^3}\{intersection points\}) \times S^2$ by pasting their boundary $S^2 \times S^2$ with a twist that interchanges the two $S^2$ factors. Recall from Sec. 1.1 the natural continuous pinching map $\phi$ from the plumbing $W^6$ to $\Gamma$. By construction, when restricted to $\partial W^6$, $\phi^{-1}(p)$ is homeomorphic to $(S^3 - \nu_{S^3}\{intersection points\}) \times S^2$ for $p$ a vertex of $\Gamma$ and to $S^2 \times S^2$ for $p$ an interior point of an edge of $\Gamma$. In particular, $\phi^{-1}(p)$ is path-connected for all $p$. Consequently, one can lift any non-null-homotopic loop in $\Gamma$ to a loop in $\partial W^6$, which must be non-null-homotopic either. This concludes the proof.

Consequently, for most of the generic Kähler deformations we have seen that pinch either an $S^2$ or a del Pezzo surface $Z$ to a point $p_0$, if it is to be resolved by a smooth complex deformation that puffs $p_0$ to a chain of immersed $S^3$'s that may intersect among themselves transversely, then indeed all these $S^3$ have to be embedded and the associated intersection diagram has to be a tree. Let us give some examples below:

**Example 4.3.** (Each $C_i^3$ below is an embedded $S^3$ and all the intersections are transverse.)

- Group-$(S^5)$: $Z' = C_1^3 \cup \cdots \cup C_{2k}^3$ with the intersection diagram the Dynkin diagram $A_{2k}$, where $k$ is any positive integer.
\((S^2 \times S^3)\): \(Z\) is the del Pezzo surface \(Z_1\) and \(Z' = C_1^3 \cup \cdots \cup C_{2k-1}^3\) with the intersection diagram the Dynkin diagram \(A_{2k-1}\), where \(k\) is any positive integer.

These follow from the decomposition \(S^5 = (D^3 \times S^2) \cup \partial S^2 \times D^3\) induced by the realization of \(S^5\) as the join \(S^2 \ast S^2\).

Let us conclude this paper with the following remark:

**Remark 4.4 [link of isolated singularity].** The nature of the singularity \(p_0\) in \(Y^7\) certainly depends on its link, the 6-manifold \(\nu_X(Z) \cup_{\partial} \nu_{X'}(Z')\). When \(Z\) and \(Z'\) are both simply-connected as most of the examples discussed in this article turn out to be, so are \(\nu_X(Z)\) and \(\nu_{X'}(Z')\). Since the common boundary of the latter manifolds is connected, \(\nu_X(Z) \cup_{\partial} \nu_{X'}(Z')\) is also simply-connected by the van Kampen theorem. It has been known ([Wa2], [Wil2]) that the structure of a simply-connected closed 6-manifold \(M^6\) is closely coded in the following triple of data: (1) a cubic form on \(H^2(M^6; \mathbb{Z})\) given by the cup-product, (2) a linear form on \(H^2(M^6; \mathbb{Z})\) given by the cup product with the first Pontrjagin class \(p_1\) or the second Chern class \(c_2\), and (3) the middle cohomology \(H^3(M^6; \mathbb{Z})\). In view of this, first one likes to know which locally admissible examples constructed in this paper are globally admissible, and then, for those globally admissible ones, whether the above triple of data provides us with some simple description of the link of the corresponding isolated singularity in \(Y^7\) and its relations to the phenomenon of enhanced gauge symmetry in string theory. These will require further much more demanding work in the future.
Appendix. Computer code.

The problem is to find all $r$-tuples $(E_1, \cdots, E_r)$ of $(-2)$-norm elements $E_i \in K_{\mathbb{Z}_r}^\perp$ that have the right intersection matrix and then compute the smith normal form of $A^t U^{-1} A$. Such a work can be achieved by a computer program. Even so, checking all $r$-tuples straightforwardly is too time-consuming for $r = 8$, where we have (including permutations and sign change) 240 elements in $K_{\mathbb{Z}_8}^\perp$ with norm $(-2)$, which leads to $\frac{240!}{(240-8)!} \approx 10^{19}$ possibilities. Instead, we use backtracking with some simple optimizations. Let us explain in less technical terms.

The recursive way to assemble all $r$-tuples of a set $\{\xi_1, \cdots, \xi_n\}$ is to think of it as a tree where each vertex is a $k$-tuple, $k \leq r$ and each edge corresponds to appending a certain $\xi_i$ to the $k$-tuple:

This can be implemented by a procedure that takes the current $k$-tuple and the next element, appends the element to the list and then calls itself with the new list and all possible choices for the new next element. And of course the procedure has to save the tuple and return for $k = r$. Backtracking means that we only allow edges that lead to the “right” $(k+1)$-tuple, here those with the right intersection numbers. In our code this is performed by the procedure try.E. For $r = 7, 8$ we optimize this algorithm further by starting not with the empty list but with selected triplets, where we exclude triplets that follow from previous ones by permutation or overall sign change. Then the calculation finishes in each case within a few days on a modern (year 1998) computer.

For completeness, the code in Maple is given below.

```maple
# cat ADE.m | nice maple -f -q > /tmp/run.log
# you can run the program like this under unix
# this is the path where the result will be saved
savepath:="/u4/vrbraun/ade/";

with(linalg): with(combinat):
with(share): readshare(normform, linalg):

# constructs all possible permutations / sign change
find_permutations:=
proc(a_1,a_list) local i,a,result;
    result:=permute(a_list);
end:
```
result:=map( (i,a)->[a,op(i)], result, a_1);
result:=[op(result), op(map(i->map(j->-j,i),result)) ];
convert(convert(result,set),list);
end:

# find all
try_E:=
proc(E_list,E_next) local i,k,good,E_list_new; global A_searchfor,Q;

k:=nops(E_list)+1;
good:=true;
# compare the k-th row of A_searchfor with the corresponding elements in Q
# This checks that the new divisor has the right intersections.
for i from 1 to nops(E_list) while good do
    good:=good and A_searchfor[i,k]=Q[E_list[i],E_next];
od;
if good then
    # the new element leads to the desired intersection pattern
    E_list_new:=[op(E_list),E_next];
    if k=rowdim(A_searchfor) then # we found a solution!
        add_solution(E_list_new);
        RETURN();
    fi;
    for i from 1 to rowdim(Q) do # try adding E[i] to the list
        if not member(i,E_list_new) then # simple necessary condition
            try_E(E_list_new,i); # do the next recursion step
        fi;
od;
fi;
end:

find_d:=
proc(E_index_list) local n,this_E,A,At,M,M_diag,i,d,M1,M2;
    global U,all_E_newbasis,e,evalm,examples;
    n:=nops(E_index_list);
    At:=blockmatrix(1,n,all_E_newbasis[E_index_list[i]]$i=1..n);
    A:=transpose(At);
    M:=multiply(At,U,A);
    M_diag:=ismithex(M,M1,M2);
    examples:=[evalm(A),evalm(M1),evalm(M2)];
    d:=map(abs,M_diag)
end:

# this is called if we have found a configuration that has
# the right intersections.
add_solution:=
proc(E_list) local d; global searchfor_d,result,examples;
    d:=find_d(E_list); # The diagonal in the integer smith normal form
    if not member(d,result) then
        result:=[op(result),d];
        print('New solution: ',d);
        examples:=[op(examples),E_list];
    fi;
end:

# simple optimisation: Find triplets to start with, remove permutations
find_good_triplets:=
proc() local i1,i2,i3,firstdigits,old_triplets,N,i,t,encoded_t,sort_under;
global good_triplets,all_E,Q,A_searchfor;

result:=map( (i,a)->[a,op(i)], result, a_1);
result:=[op(result), op(map(i->map(j->-j,i),result)) ];
convert(convert(result,set),list);
end:
good_triplets:=[];

firstdigits:=map(abs,convert(map(i->i[1],all_E),set));
old_triplets:=table();
N:=max(op(firstdigits));
for i1 from -N to N do
  for i2 from -N to N do
    for i3 from -N to N do
      old_triplets[i1,i2,i3]:=[];
    od:od:od:
for i1 from 1 to nops(all_E) do
  for i2 from 1 to nops(all_E) do
    for i3 from 1 to nops(all_E) do
      # check for right intersections at this level
      if Q[i1,i2]=A_searchfor[1,2] and
        Q[i1,i3]=A_searchfor[1,3] and
        Q[i2,i3]=A_searchfor[2,3] then
        t:=[all_E[i1],all_E[i2],all_E[i3]];
        encoded_t:=encode_tuple(t);
        sort_under:=op(encoded_t[1]);
        if member(encoded_t, old_triplets[sort_under] ) then next; fi;
        encoded_t:=encode_tuple( map(i->evalm(-i),t) );
        if member(encoded_t,old_triplets[sort_under]) then next; fi;
        # new triplet, permutations dont occur
        old_triplets[sort_under]:=[op(old_triplets[sort_under]),encoded_t];
        good_triplets:=[op(good_triplets),[i1,i2,i3]];
      fi;
    od:od:od:
end:

# returns a unique representative (same for all permutations)
encode_tuple:=
proc(t) local result,i,j;
    result:=seq( ['t[i][j]'$'i'=1..nops(t) ], j=1..vectdim(t[1]) );
    result:=[result[1],{result[2..nops(result)]} ];
end:

# This is the main loop
find_all_d:=
proc() local i,b,r; global good_triplets,Q,result,examples,A_searchfor;
    result:=[ ]; examples:=[ ];
    r:=rowdim(A_searchfor);
    if r>=7 then
      # we need the optimisation of starting with triplets instead of empty list
      find_good_triplets();
      for b in good_triplets do
        print('testing ',b,' found so far ',result);
        for i from 1 to rowdim(Q) do
          try_E(b,i);
        od;
      od;
    else
      # r<7, computer is fast enough as it is
      for i from 1 to rowdim(Q) do
        print('testing ',b,' found so far ',result);
        try_E([],i);
      od;
      fi;
    end:

# to speed things up fill some tables:
calc_intersection:=
proc() local i; global n,e,Q,all_E,all_E_newbasis;
    n:=vectdim(all_E[1]);
e:=array(sparse,1..n-1,1..n):
for i from 1 to n-2 do e[i,i+1]:=1; e[i,n]:=-1; od:
e[n-1,1]:=e[n-1,n]:=-3:
Q:=matrix(nops(all_E),nops(all_E),
(i,j)->2*all_E[i][1]*all_E[j][1]-innerprod(all_E[i],all_E[j])):
all_E_newbasis:=map( i->linsolve(transpose(e), i), all_E):
RETURN(); # we changed global variables, dont need to return data end:

# set up A, D, E, U (U is $U^{-1}$) matrices.
for i from 1 to 8 do #set up matrices for A series
QA.i:=array(symmetric,sparse,1..i,1..i):
for j from 1 to i do QA.i[j,j]:=-2; od:
for j from 1 to i-1 do QA.i[j,j+1]:=1; od:
UA.i:=array(sparse,1..i,1..i):
for j from 1 to i do for k from j to i do UA.i[j,k]:=1; od; od:
for i from 4 to 8 do #set up matrices for D series
QD.i:=array(symmetric,sparse,1..i,1..i):
for j from 1 to i do QD.i[j,j]:=-2; od:
for j from 1 to i-2 do QD.i[j,j+1]:=1; od:
UD.i:=array(sparse,1..i,1..i):
for j from 1 to i do for k from j to i do UD.i[j,k]:=1; od; od:
for i from 6 to 8 do #set up matrices for E series
QE.i:=array(symmetric,sparse,1..i,1..i):
for j from 1 to i do QE.i[j,j]:=-2; od:
for j from 1 to i-3 do QE.i[j,j+1]:=1; od:
UE.i:=array(sparse,1..i,1..i):
for j from 1 to i do for k from j to i do UE.i[j,k]:=1; od; od:

# now calculate what we are interested in

# r=3
all_E:=map(vector,map(op,[
find_permutations( 0,[1,-1,0]), find_permutations( 1,[1,-1,-1]) ])):
calc_intersection();
A_searchfor:=BlockDiagonal(QA1,QA2); U:=BlockDiagonal(UA1,UA2);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A1A2.dat'));

# r=4
all_E:=map(vector,map(op,[
find_permutations( 0,[1,-1,0,0]), find_permutations( 1,[1,-1,-1,0]) ])):
calc_intersection();
A_searchfor:=BlockDiagonal(QA4); U:=BlockDiagonal(UA4);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A4.dat'));

# r=5
all_E:=map(vector,map(op,[
find_permutations( 0,[1,-1,0,0,0]), find_permutations( 1,[1,-1,-1,0,0]) ])):
calc_intersection();
A_searchfor:=BlockDiagonal(QA1,QA1,QA3); U:=BlockDiagonal(UA1,UA1,UA3);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A1A1A3.dat'));
A_searchfor:=BlockDiagonal(QD5); U:=BlockDiagonal(UD5);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_D5.dat'));

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# r=6
all_E:=map(vector,map(op,[
find_permutations( 0,[1,-1,0,0,0,0,0]),
find_permutations( 1,[1,-1,-1,0,0,0,0]),
find_permutations( 2,[1,-1,-1,-1,-1,-1]) ]));
calc_intersection();

A_searchfor:=BlockDiagonal(QA1,QA5); U:=BlockDiagonal(UA1,UA5);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A1A5.dat'));

A_searchfor:=BlockDiagonal(QA2,QA2,QA2); U:=BlockDiagonal(UA2,UA2,UA2);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A2A2A2.dat'));

A_searchfor:=BlockDiagonal(QA7); U:=BlockDiagonal(UA7);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A7.dat'));

# r=7
all_E:=map(vector,map(op,[
find_permutations( 0,[1,-1,0,0,0,0,0,0]),
find_permutations( 1,[1,-1,-1,0,0,0,0,0]),
find_permutations( 2,[1,-1,-1,-1,-1,0,0,0]) ]));
calc_intersection();

A_searchfor:=BlockDiagonal(QA1,QA3,QA3); U:=BlockDiagonal(UA1,UA3,UA3);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A1A3A3.dat'));

A_searchfor:=BlockDiagonal(QA1,QA2,QA5); U:=BlockDiagonal(UA1,UA2,UA5);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A1A2A5.dat'));

A_searchfor:=BlockDiagonal(QA2,QA2,QA2); U:=BlockDiagonal(UA2,UA2,UA2);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A2A2A2A2.dat'));

A_searchfor:=BlockDiagonal(QA4,QA4); U:=BlockDiagonal(UA4,UA4);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A4A4.dat'));

A_searchfor:=BlockDiagonal(QA8); U:=BlockDiagonal(UA8);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A8.dat'));

A_searchfor:=BlockDiagonal(QA1,QA1,QD6); U:=BlockDiagonal(UA1,UA1,UD6);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A1A1D6.dat'));

A_searchfor:=BlockDiagonal(QA3,QD5); U:=BlockDiagonal(UA3,UD5);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A3D5.dat'));

A_searchfor:=BlockDiagonal(QA1,QE7); U:=BlockDiagonal(UA1,UE7);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A1E7.dat'));

A_searchfor:=BlockDiagonal(QA2,QE6); U:=BlockDiagonal(UA2,UE6);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_A2E6.dat'));

A_searchfor:=BlockDiagonal(QD4,QD4); U:=BlockDiagonal(UD4,UD4);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_D4D4.dat'));

A_searchfor:=BlockDiagonal(QD8); U:=BlockDiagonal(UD8);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_D8.dat'));

A_searchfor:=BlockDiagonal(QE8); U:=BlockDiagonal(UE8);
find_all_d(); save(result,examples,all_E,cat(savepath,'ade_E8.dat'));
References

[Al] O. Alvarez, *Strings, branes, and the structure of space-time*, talk given at the AMS meeting at Chattanooga, Tennessee, October 1996.

[As] P.S. Aspinwall, *K3 surfaces and string duality*, hep-th/9611137.

[At] M.F. Atiyah, *On analytic surfaces with double points*, Proc. Royal Soc. A247 (1958), pp. 237 - 244.

[A-G-M] P.S. Aspinwall, B.R. Greene, and D.R. Morrison, *Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory*, Nucl. Phys. B416 (1994), pp. 414 - 480.

[A-GZ-V] V.I. Arnold, S.M. Gusein-Zade, and A.N. Varchenko, *Singularities of differentiable maps, Vol. II: Monodromy and asymptotics of integrals*, Monographs Math. 83, Birkhäuser, 1988.

[A-L] P.S. Aspinwall and C.A. Lütken, *Quantum algebraic geometry of superstring compactifications*, Nucl. Phys. B355 (1991), pp. 482 - 510.

[Ba] D. Barden, *Simply connected five-manifolds*, Ann. Math. 82 (1965), pp. 365 - 385.

[Bou] N. Bourbaki, *Eléments de Mathématique, Livre II, Algèbre*, Hermann, 1959.

[B-B-S] K. Becker, M. Becker, and A. Strominger, *Fivebranes, membranes and non-perturbative string theory*, Nucl. Phys. B456 (1995), pp. 130 - 152.

[B-C-D] D. Berenstein, R. Corrado, and J. Distler, *On the moduli spaces of M(atrix)-theory compactifications*, hep-th/9704087.

[B-C-dlO] P. Berglund, P. Candelas, and X.C. de la Ossa, *Periods for Calabi-Yau and Landau-Ginzburg vacua*, Nucl. Phys. B419 (1994), pp. 352 - 403.

[B-G] R.L. Bryant and P.A. Griffiths, *Some observations on the infinitesimal period relations for regular threefolds with trivial canonical bundle*, in *Arithmetic and geometry - vol II: Geometry*, M. Artin and J. Tate eds. Progr. Math. 36, Birkhäuser, 1983.

[B-I-K-S-V] M. Bershadsky, K. Intriligator, S. Kachru, D.R. Morrison, V. Sadov, and C. Vafa, *Geometric singularities and enhanced gauge symmetries*, Nucl. Phys. B481 (1996), pp. 215 - 252.

[B-P-VV] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Ser. Modern Surveys Math. 4, Springer-Verlag, 1984.

[B-T] R. Bott and L.W. Tu, *Differential forms in algebraic topology*, GTM 82, Springer-Verlag, 1982.

[B-V-S1] M. Bershadsky, C. Vafa, and V. Sadov, *D-strings on D-manifolds*, Nucl. Phys. B463 (1996), pp. 398 - 414.

[B-V-S2] ——*, *D-branes and topological field theory*, Nucl. Phys. B463 (1996), pp. 420 - 434.

[C-dlO] P. Candelas and X.C. de la Ossa, *Moduli space of Calabi-Yau manifolds*, Nucl. Phys. B463 (1996), pp. 420 - 434.

[C-G-G-K] T.-M. Chiang, B.R. Greene, M. Gross, and Y. Kanter, *Black hole condensation and the web of Calabi-Yau manifolds*, hep-th/9511204.

[C-G-H] P. Candelas, P.S. Green, and T. Hubsch, *Rolling among Calabi-Yau vacua*, Nucl. Phys. B330 (1990), pp. 49 - 102.

[C-H-S-W] P. Candelas, G.T. Horowitz, A. Strominger, and E. Witten, *Vacuum configurations for superstrings*, Nucl. Phys. B258 (1985), pp. 46 - 74.

[C-P-R] P. Candelas, E. Perevalov, and G. Rajesh, *Toric geometry and enhanced gauge symmetry of F-theory/heterotic vacua*, hep-th/9704097.

[Da] V.I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys 33 (1978), pp. 97 - 154.

[De] M. Demazure, *Surfaces de Del Pezzo*, in *Séminaire sur les singularités des surfaces*, M. Demazure, H. Finkham, and B. Teissier eds., Lect. Notes Math. 777, Springer-Verlag, 1980.
[Di] A. Dimca, *Singularities and topology of hypersurfaces*, Springer-Verlag, 1992.

[D-G-M] J. Distler, B.R. Greene, and D.R. Morrison, *Resolving singularities in (0, 2) models*, Nucl. Phys. B481 (1996), pp. 289 - 312.

[D-G-W] R. Donagi, A. Grassi, and E. Witten, *A non-perturbative superpotential with Esymmetry*, Mod. Phys. Lett. A11 (1996), pp. 2199 - 2212.

[D-K-L] M.J. Duff, R. Khuri, and J.X. Lu, *String solitons*, Phys. Reports 259 (1995), pp. 213 - 326.

[D-M] M.R. Douglas and G. Moore, *D-branes, quivers, and ALE instantons*, [hep-th/9603167](https://arxiv.org/abs/hep-th/9603167).

[D-T] S.K. Donaldson and R.P. Thomas, *Gauge theory in higher dimensions*, Oxford preprint, 1997.

[Fr] R. Friedman, *Simultaneous resolution of threefold double points*, Math. Ann. 274 (1986), pp. 671 - 689.

[Ful] W. Fulton, *Introduction to toric varieties*, Ann. Math. Study 131, Princeton Univ. Press, 1993.

[Fur] M. Furushima, *Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space C3*, Nagoya Math. J. 104 (1986), pp. 1 - 28.

[F-K-M] S. Ferrara, R.R. Khuri, and R. Minasian, *M-theory on a Calabi-Yau manifold*, Phys. Lett. B375 (1996), pp. 81 - 88.

[Go] R.E. Gompf, *4-dimensional manifolds*, course given at the Department of Mathematics, U.T. Austin, fall 1997; also many private discussions after class.

[Gr] B.R. Greene, *String theory on Calabi-Yau manifolds*, lectures given at TASI-96 summer school on Strings, Fields, and Duality, [hep-th/9702155](https://arxiv.org/abs/hep-th/9702155).

[G-H] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, Inc., 1978.

[G-M-S] B.R. Greene, D.R. Morrison, and A. Strominger, *Black hole condensation and the unification of string vacua*, Nucl. Phys. B451 (1995), pp. 109 - 120.

[G-M-V] B.R. Greene, D.R. Morrison, and C. Vafa, *A geometric realization of confinement*, Nucl. Phys. B481 (1996), pp. 513 - 538.

[G-Mu-V] M. Green, J. Murre, and C. Voisin, *Algebraic cycles and Hodge theory*, Lect. Notes Math. 1594, Springer-Verlag, 1994.

[G-P] V. Guillemin and A. Pollack, *Differential topology*, Prentice-Hall, 1974.

[G-S] R.E. Gompf and A.I. Stipsicz, *An introduction to 4-manifolds and Kirby calculus*, preliminary version - July 1997, to be published in book form by the American Mathematical Society.

[Ha] A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. 36 (1961), pp. 47 - 82.

[Hiro] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I & II*, Ann. Math. 79 (1964), pp. 109 - 203 and pp. 205 - 326.

[Hirs] M.W. Hirsch, *Differential topology*, GTM 33, corrected 4th printing, Springer-Verlag, 1991.

[Hü] T. Hubsch, Calabi-Yau manifolds – a bestiary for physicists, World Scientific, 1992.

[H-K-K] J. Harer, A. Kas, and R. Kirby, *Handlebody decompositions of complex surfaces*, Memoirs Amer. Math. Soc. no. 350, Amer. Math. Soc., 1986.

[H-W1] P. Hořava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. B460 (1996), pp. 506 - 524.

[H-W2] ———, *Eleven-dimensional supergravity on a manifold with boundary*, Nucl. Phys. B475 (1996), pp. 94 - 114.

[I-M-S] K. Intriligator, D.R. Morrison, N. Seiberg, *Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces*, Nucl. Phys. B497 (1997), pp. 56 - 100.

[Ja] N. Jacobson, *Basic algebra I*, W.H. Freeman & Co., 1974.
[Jo1] D.D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2. I*, J. Diff. Geom. 43 (1996), pp. 291 - 328.

[Jo2] ———, *Compact Riemannian 7-manifolds with holonomy G_2. II*, J. Diff. Geom. 43 (1996), pp. 329 - 375.

[Ki] R.C. Kirby, *The topology of 4-manifolds*, Lect. Notes Math. 1374, Springer-Verlag, 1989.

[Kl] A. Klemm, *On the geometry behind N = 2 supersymmetric effective actions in four dimensions*, lectures presented at the Trieste Summer School 1996 and the 33rd Karpacz School on String Duality 1997. [hep-th/9705131]

[Kod] K. Kodaira, *Complex manifolds and deformations of complex structures*, translated by K. Akao, Grund. Math. Wiss. 283, Springer-Verlag, 1986.

[Kol] J. Kollár, *Flops*, Nagoya Math. J. 113 (1989), pp. 15 - 36.

[K-L-R-Y] A. Klemm, B. Lian, S.-S. Roan, and S.T. Yau, *Calabi-Yau fourfolds for M- and F-theory compactifications*, hep-th/9701023.

[K-MK] S. Keel and J. McKernan, *Rational curves on quasi-projective surfaces*, alg-geom/9707016.

[Le] W. Lerche, *Introduction to Seiberg-Witten theory and its stringy origin*, contribution to the Proceedings of the "Spring School and Workshop on String Theory, Gauge Theory and Quantum Gravity", I.C.T.P., Trieste, Italy, March 18 - 29, 1996. [hep-th/9611190]

[Ma1] E. Martinec, *Geometric structures of M-theory*, hep-th/9608017.

[Ma2] ———, *M-theory and N = 2 strings*, talk given at the NATO Advanced Study Institute on Strings, Branes, and Duality, Cargese France, summer 1997. [hep-th/9710122]

[MI1] J.W. Milnor, *Morse theory*, Ann. Math. Study 51, Princeton Univ. Press, 1963.

[MI2] ———, *Singular points of complex hypersurfaces*, Ann. Math. Study 61, Princeton Univ. Press, 1968.

[Mu] J.R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publ. Co., 1984.

[M-S] J.W. Milnor and J.D. Stasheff, *Characteristic classes*, Ann. Math. Study 76, Princeton Univ. Press, 1974.

[M-V1] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds (I)*, Nucl. Phys. B473 (1996), pp. 74 - 92.

[M-V2] ———, *Compactifications of F-theory on Calabi-Yau threefolds (II)*, Nucl. Phys. B476 (1996), pp. 437 - 469.

[M-Z] M. Miyanishi and D.Q. Zhang, *Gorenstein log del Pezzo surfaces of rank one*, J. Algebra, 118 (1988), pp. 63 - 84.

[Od] T. Oda, *Convex bodies and algebraic geometry - an introduction to the theory of toric varieties*, Springer-Verlag 1988.

[Pu] V. Poénaru, *On the geometry of differentiable manifolds*, in *Studies in modern topology*, P.J. Hilton ed., MAA Studies in Math., vol. 5, Math. Asso. Amer. & Prentice-Hall Inc., 1968.

[Re1] M. Reid, *Canonical threefolds*, in *Géométrie algébrique*, Angers 1979, A. Beauville ed., pp. 273 - 310, Sijthoff and Noordhoff, Alphen aan den Rijn, 1980.

[Re2] ———, *Young person's guide to canonical singularities*, in *Algebraic geometry*, Bowdoin 1985, S.J. Bloch with H. Clemens, D. Eisenbud, W. Fulton, D. Gieseker, J. Harris, R. Hartshorne, and S. Mori ed., Proc. Symp. Pure Math. vol. 46 - part 1, pp. 345 - 414, Amer. Math. Soc. 1987.

[Ro] D. Rolfsen, *Knots and links*, 2nd printing, Publish or Perish, 1990.

[Sa] S. Salamon, *Riemannian geometry and holonomy group*, Pitman Research Notes Math. Ser. 201, Longman Scientific & Technical, 1989.
[Sh1] I.R. Shafarevich, Basic algebraic geometry, vol I: varieties in projective space, vol II: schemes and complex manifolds, Springer-Verlag, 1994.

[Sh2] ——– ed., Algebraic geometry II: Cohomology of algebraic varieties by V.I. Danilov; Algebraic surfaces by V.A. Iskovskikh and I.R.S., Springer-Verlag, 1996.

[SI1] P. Slodowy, Simple singularities and simple algebraic groups, Lect. Notes Math. 815, Springer-Verlag, 1980.

[Sl2] ——–, Platonic solids, Kleinian singularities, and Lie groups, in Algebraic geometry, I. Dolgachev ed., pp. 102 - 138, Lect. Notes Math. 1088, Springer-Verlag, 1983.

[Sm1] S. Smale, Generalized Poincaré conjecture in dimensions greater than four, Ann. Math. 74 (1961), pp. 391 - 406.

[Sm2] ——–, On the structure of 5-manifolds, Ann. Math. 75 (1962), pp. 38 - 46.

[Sp] E.H. Spanier, Algebraic topology, Springer-Verlag, 1966.

[Sto] R.E. Stong, Notes on cobordism theory, Princeton Math. Notes, Princeton Univ. Press, 1958.

[Str] A. Strominger, Massless black holes and conifolds in string theory, Nucl. Phys. B451 (1995), pp. 96 - 108.

[Sw] R.M. Switzer, Algebraic topology - homotopy and homology, Grund. Math. Wiss. 212, Springer-Verlag, 1975.

[Ti] G. Tian, Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersen-Weil metric, in Mathematical aspects of string theory, S.T. Yau ed., pp. 629 - 646, World Scientific, 1987.

[To] P.K. Townsend, Four lectures on M-theory, To appear in the Proceedings of the 1996 ICTP Summer School in High Energy Physics and Cosmology (Trieste, June 10 - 26), hep-th/9612121.

[Vo] C. Voisin, Symétrie miroir, Panoramas et Synthèses 2, Soc. Math. de France 1996, distributed by Amer. Math. Soc.

[Wa1] C.T.C. Wall, Determination of the cobordism ring, Ann. Math. 72 (1960), pp. 292 - 311.

[Wa2] ——–, Classification problems in differential topology. V: On certain 6-manifolds, Invent. Math. 1 (1966), pp. 355 - 374.

[Wa3] ——–, Real forms of smooth del Pezzo surfaces, J. Reine Angew. Math. 375/376 (1987), pp. 47 - 66.

[Wil1] P.M.H. Wilson, Calabi-Yau manifolds with large Picard number, Invent. Math. 98 (1989), pp. 139 - 155.

[Wil2] ——–, The Kähler cone on Calabi-Yau threefolds, Invent. Math. 107 (1992), pp. 561 - 583.

[Wit1] E. Witten, String theory dynamics in various dimensions, Nucl. Phys. B443 (1995), pp. 85 - 126.

[Wit2] ——–, Phase transitions in M-theory and F-theory, Nucl. Phys. B471 (1996), pp. 195 - 216.

[Wit3] ——–, Non-perturbative superpotentials in string theory, Nucl. Phys. B474 (1996), pp. 343 - 360.

[W-W] R.S. Ward and R.O. Wells Jr. Twistor geometry and field theory, Cambridge Univ. Press, 1990.